MORITA BUNDLE GERBES

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ABSTRACT. The aim of this paper is to give a survey of the theory of bundle gerbes. In our approach we especially emphasize the unifying role of Morita equivalences in this theory. We also discuss a higher analog of Morita bundle gerbes called Morita 2-bundle gerbes.

1. INTRODUCTION

The aim of this paper is to give a survey of the theory of bundle gerbes and some of their generalizations. This paper does not contain new results excepting probably some proofs. In our approach we emphasize the role of Morita equivalence, which plays the unifying role in this theory. In particular, (see section 3.4) we show (after M. Karoubi [13]) that every module $E$ over a bundle gerbe $L$ defines a (Morita) equivalence between the categories of $L$- and $\text{End}(E)$-modules.

Probably, the main application of bundle gerbes is in the twisted $K$-theory. The general idea of twisted cohomology is the following: if this cohomology theory is represented by an $\Omega$-spectrum $E$, then the untwisted cohomology of a space $X$ with coefficients $E$ is given by homotopy classes of sections of the trivial bundle over $X$ with fiber $E$ (namely by $[X, E]$). The twists are then the (possibly non-trivial) bundles $B$ over $X$ with fiber $E$. These have morphisms: the suitably defined bundle automorphisms, and pullback makes this a functor on the category of spaces. The twisted cohomology for a given twist $B$ is defined as the homotopy classes of sections of the bundle $B$. Obviously, the details are a bit messy and probably best carried out in the context of higher categories. Details, in the context of $K$-theory, of such an approach are given in [1, 2, 3] in the context of $\infty$-categories, and in a more classical setting in [15].

This general approach lacks direct geometric interpretations. Therefore, often for subclasses of twists, other (equivalent) descriptions of twisted generalized cohomology, in particular of twisted $K$-theory, have been given.

An important remark has to be made here: Twisted cohomology requires much more precise data than just an equivalence class of twists. Indeed, an axiomatic framework might be given as follows: twists for $K$-theory on $X$ are given as the objects of a (higher) groupoid $Tw(X)$. The above-mentioned equivalence classes are the isomorphism classes of objects in the groupoid, but the morphisms are equally important. In particular, twists in general have non-trivial automorphisms. One would then require that $X \to Tw(X)$ forms a contravariant functor from spaces to groupoids. Twisted $K$-theory would then be a functor from $Tw(X)$ to abelian groups which is also functorial in $X$ in the evident way, and which satisfies further axioms of a cohomology theory. In particular, the automorphisms of a twist act (usually non-trivially) on the corresponding twisted $K$-theory. In light of this, it does not really make sense to talk about the twisted $K$-theory group for an equivalence class of twists: only the isomorphism type of this group is well defined. A more detailed description of this setup is given e.g. in [6, Section 3.1].

Twistings of $K(X)$ (where $X$ is a compact space) are classified by homotopy classes of maps to the “classifying space of bundles with fiber the $K$-theory spectrum”, i.e. by

$$X \to B(Z/2Z \times BU_{\otimes}) \simeq K(Z/2Z, 1) \times BU_{\otimes}.$$  

Because of the isomorphism $BU_{\otimes} \cong K(Z, 2) \times BSU_{\otimes}$ of spectra [14, 20], twistings are classified by elements of the group $H^1(Z/2Z, 1) \times H^3(X, Z) \times [X, BSU_{\otimes}]$.

Twistings corresponding to the first two factors $H^1(Z/2Z, 1) \times H^3(X, Z)$ were studied by Karoubi [12], Donovan and Karoubi [3] in the finite order case and by Rosenberg [19], Atiyah and Segal [4] in the general case. There is also the approach due to Bouwknegt, Carey, Mathai, Murray and Stevenson [8] via bundle gerbes and modules over them which we are based on. Note that, in line with the above comment, twists in all these approaches are always some kind of explicit “cocycle representatives” of the cohomology classes in question, to allow for a functorial construction and the internal structure of automorphisms. In particular, morphisms between bundle gerbes are precisely Morita equivalences, this indicates their important role once again.

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Partially supported by the RFBR (grant 14-01-00007-a “Analytic methods in noncommutative geometry and topology”).
Twisted K-theory is of particular relevance as it appears naturally in string theory: for space-times with background Neveu-Schwarz H-flux, the so-called Ramond-Ramond charges of an associated field theory are rather classified by twisted K-theory. This has been studied a lot in the context of T-duality, where isomorphisms of twisted K-theory groups have been constructed. The topological aspects of this are described e.g. in [3 7].

Acknowledgments. I would like to express my deepest gratitude to Professor Doctor Thomas Schick for numerous inspirational discussions and valuable contributions to some parts of this text.

2. Bundle gerbes

Bundle gerbes over a base space $X$ form a weak monoidal 2-groupoid. It is a categorification of the group $H^3(X, \mathbb{Z})$ in the sense that there is a natural isomorphism between this group and the group of equivalence classes of its objects (the group operation is induced by the monoidal structure). Our treatment of the higher versions of bundle gerbes generalizes the one of (common) bundle gerbes and modules over them, so we start the paper with a reminder of the corresponding results in a form suitable for our purposes. For details compare [3 16 17]. This section does not contain new results not to be found in these references.

The aim of this section is to define the 2-category of bundle gerbes over $X$. First, we define its objects, then its 1- and 2-morphisms and finally describe some of its properties.

2.1. Definition of bundle gerbes. Let $X$ be a compact Hausdorff space, $U = \{U_\alpha\}$ an open cover of $X$ indexed by a set $\{\alpha\}$.

2.1. Definition. A bundle gerbe $(L, \theta, U)$ is a collection of (complex) line bundles $L_{\alpha\beta} \to U_{\alpha\beta,\gamma}$ together with isomorphisms $\theta_{\alpha\beta\gamma} : L_{\alpha\beta} \otimes L_{\beta\gamma} \to L_{\alpha\gamma}$ over $U_{\alpha\beta\gamma}$ with associativity condition over four-fold overlaps, i.e. such that the diagrams

$$ L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\delta} \xrightarrow{\theta_{\alpha\beta\gamma} \otimes \text{id}} L_{\alpha\beta} \otimes L_{\beta\delta} \xrightarrow{\theta_{\alpha\beta\delta}} L_{\alpha\delta} $$

commute over $U_{\alpha\beta\gamma\delta}$.

The composite maps

$$ L_{\alpha\alpha} \cong L_{\alpha\alpha} \otimes \mathbb{C} \xrightarrow{id \otimes c} L_{\alpha\alpha} \otimes L_{\alpha\alpha} \otimes L_{\alpha\alpha}^* \xrightarrow{\theta_{\alpha\alpha\alpha} \otimes \text{id}} L_{\alpha\alpha} \otimes L_{\alpha\alpha}^* \xrightarrow{c} U_{\alpha} \times \mathbb{C}, $$

where $c$ is the contraction, define isomorphisms $\tau_\alpha : L_{\alpha\alpha} \to U_{\alpha} \times \mathbb{C}$. It is easy to verify that they make the following diagrams commutative

$$ L_{\alpha\beta} \otimes L_{\beta\beta} \cong L_{\alpha\beta} \otimes L_{\alpha\beta} \otimes L_{\beta\beta}^* \xrightarrow{\theta_{\alpha\beta\beta} \otimes \text{id}} L_{\alpha\beta} \otimes L_{\beta\beta}^* \xrightarrow{\tau_\alpha \otimes \text{id}} \mathbb{C} \otimes L_{\beta\beta}^* \cong L_{\beta\beta}^* $$

hence the identifications $\tau_\alpha$ agree with the bundle gerbe structure.

Analogously, the composite maps

$$ L_{\alpha\beta} \cong L_{\alpha\beta} \otimes \mathbb{C} \xrightarrow{id \otimes c} L_{\alpha\beta} \otimes L_{\beta\beta} \otimes L_{\beta\beta}^* \xrightarrow{\theta_{\alpha\beta\beta} \otimes \text{id}} L_{\alpha\beta} \otimes L_{\beta\beta}^* \xrightarrow{\tau_\beta \otimes \text{id}} \mathbb{C} \otimes L_{\beta\beta}^* \cong L_{\beta\beta}^* $$

allow us to coherently identify $L_{\alpha\beta}$ with $L_{\beta\beta}^*$.

2.2. Remark. Let us explain the heuristic behind this definition. Let $\text{Pic} := \text{Pic}(\mathbb{C})$ be the Picard 2-group of the field $\mathbb{C}$. Thus $\text{Pic}$ is a weak 2-category with a unique object $\bullet_\mathbb{C}$ (corresponding to the field $\mathbb{C}$) whose 1-morphisms are $(\mathbb{C}, \mathbb{C})$-bimodules (the composition law is defined by the tensor product of bimodules) and bimodule isomorphisms serve as 2-morphisms (see subsection 2.4). We also have the (topological) Čech groupoid $\check{C}(U)$ associated with the open cover $U$. Then a bundle gerbe is a weak 2-functor $\check{C}(U) \to \text{Pic}$ to the Picard 2-group. Indeed, to any object of $\check{C}(U)$ we associate the unique object $\bullet_\mathbb{C}$ in $\text{Pic}$. To morphisms $U_{\alpha\beta}$ in $\check{C}(U)$ we associate 1-morphisms in $\text{Pic}$ that form a line bundle $L_{\alpha\beta}$. Since our functor is weak, it does not preserve the composition of morphisms on the nose, but only up to 2-morphisms. In other words, the “discrepancy” between composition of 1-morphisms $U_{\alpha\beta}$ with $U_{\beta\gamma}$ and $U_{\alpha\gamma}$ corresponds to the isomorphism

\footnote{\text{U}_{\alpha_0...\alpha_r} := U_{\alpha_0} \cap ... \cap U_{\alpha_r}.}

\footnote{Since every covering $U$ has a “good” refinement (i.e. all nonempty finite overlaps are contractible) and therefore the bundles $L_{\alpha\beta}$ are trivial, the main data of a bundle gerbe $(L(g), \theta, U)$ is encoded by $\theta$.}
\[ \theta_{\alpha\beta\gamma} : L_{\alpha\beta} \otimes L_{\beta\gamma} \to L_{\alpha\gamma} \] that is a family of 2-morphisms in \( \mathcal{P}ic \). Thus, a bundle gerbe actually a cocycle with values in \( \mathcal{P}ic \).

Note that this heuristic is also helpful when we define 1-morphisms between bundle gerbes (see the next subsection) which are precisely natural transformations between 2-functors.

2.2. The category of bundle gerbes. We can regard bundle gerbes over \( X \) as objects of some weak monoidal 2-category \( BG(X) \) as follows. Objects of \( BG(X) \) are bundle gerbes over \( X \).

2.3. Definition. A 1-morphism \( M : L \to L' \) (where \( L = (L, \theta, U) \), \( L' = (L', \theta', U) \), \( U = \{ U_\alpha \}_{\alpha} \)) is a collection of line bundles \( \{ M_\alpha \} \to U_\alpha \) together with isomorphisms \( \varphi_{\alpha\beta} : L_{\alpha\beta} \otimes M_{\beta} \overset{\cong}{\to} M_{\alpha} \otimes L_{\alpha\beta} \) over \( U_{\alpha\beta} \) such that the diagram

\[
\begin{array}{ccc}
L_{\alpha\beta} \otimes M_{\beta} \otimes L_{\beta\gamma}' & \xrightarrow{1 \otimes \varphi_{\beta\gamma}} & L_{\alpha\beta} \otimes L_{\beta\gamma}' \otimes M_{\gamma} \\
\varphi_{\alpha\beta} \otimes 1 & \downarrow & \theta_{\alpha\beta\gamma} \otimes 1 \\
M_{\alpha} \otimes L_{\alpha\beta}' \otimes L_{\beta\gamma}' & \xrightarrow{1 \otimes \varphi_{\alpha\beta}^{-1}} & M_{\alpha} \otimes L_{\alpha\beta}' \otimes L_{\alpha\beta}
\end{array}
\]

commutes.

Note that we have given the definition of 1-morphisms between bundle gerbes over the same open cover \( U \), but there is no problem with the general case because any two covers \( U \) and \( V \) have the common refinement \( W = \{ W_{\alpha,\lambda} \} \). \( W_{\alpha,\lambda} := U_\alpha \cap V_\lambda \) and a bundle gerbe \( L = (L, \theta, U) \) defines the corresponding bundle gerbe over \( W \) by pullback (i.e. the restriction).

The composition of 1-morphisms is defined by tensor product. More precisely, let \( (N, \psi) : L' \to L'' \) be a second 1-morphism, where \( N = \{ N_{\alpha\beta} \} \), \( L'' = (L'', \theta'', U) \), \( \psi_{\alpha\beta} : L_{\alpha\beta}' \otimes N_{\beta} \overset{\cong}{\to} N_{\alpha} \otimes L_{\alpha\beta}' \). Then the compositions of isomorphisms

\[
L_{\alpha\beta} \otimes M_{\beta} \otimes N_{\beta} \overset{\varphi_{\alpha\beta} \otimes 1}{\longrightarrow} M_{\alpha} \otimes L_{\alpha\beta}' \otimes N_{\beta} \overset{1 \otimes \psi_{\alpha\beta}}{\longrightarrow} M_{\alpha} \otimes N_{\alpha} \otimes L_{\alpha\beta}'
\]

shows that

\[
\{ P_\alpha := M_{\alpha} \otimes N_\alpha \}, \{ \chi_{\alpha\beta} := (1 \otimes \psi_{\alpha\beta}) \circ (\varphi_{\alpha\beta} \otimes 1) \}
\]

defines the required composition.

By definition, a 2-morphism \( \omega : M \to M' \) between 1-morphisms \( (M, \varphi), (M', \varphi') : L \to L' \) is a collection \( \{ \kappa_\alpha \} \) of isomorphisms of bundles \( \{ M_\alpha \} \to \{ M'_\alpha \} \) that make all diagrams like

\[
\begin{array}{ccc}
L_{\alpha\beta} \otimes M_{\beta} & \xrightarrow{\varphi_{\alpha\beta}} & M_{\alpha} \otimes L_{\alpha\beta}' \\
\downarrow_{1 \otimes \kappa_\beta} & & \downarrow_{\kappa_\alpha \otimes 1} \\
L_{\alpha\beta} \otimes M_{\beta}' & \xrightarrow{\varphi_{\alpha\beta}^{-1}} & M_{\alpha}' \otimes L_{\alpha\beta}'
\end{array}
\]

commutative.

Note that the composition of 1-morphisms is not strictly associative but only up to 2-morphisms. Analogously, \( M \circ \text{id}_L \) and \( \text{id}_{L'} \circ N \) and \( N \) are not equal but only equivalent up to 2-morphisms. All these 2-morphisms form coherent families. Thus we have defined the weak 2-category \( BG(X) \).

2.3. 2-groupoid of bundle gerbes. Note that every 1-morphism is invertible (up to 2-morphism). Indeed, for a 1-morphism \( (M, \varphi) : (L, \theta, U) \to (L', \theta', U) \) as above define the inverse morphism \( (N, \psi) : (L', \theta', U) \to (L, \theta, U) \) as follows. Put \( N_\alpha := M_\alpha^* \) and define \( \psi_{\alpha\beta} : L_{\alpha\beta}' \otimes N_{\beta} \to N_{\alpha} \otimes L_{\alpha\beta} \) as the composite map

\[
L_{\alpha\beta}' \otimes M_{\beta}^* \cong M_{\alpha}^* \otimes M_{\alpha} \otimes L_{\alpha\beta}' \otimes M_{\beta}^* \overset{1 \otimes \varphi_{\alpha\beta}^{-1} \otimes 1}{\longrightarrow} M_{\alpha}^* \otimes L_{\alpha\beta} \otimes M_{\beta} \otimes M_{\alpha}^* \cong M_{\alpha}^* \otimes L_{\alpha\beta}
\]

(where the isomorphisms \( \cong \) are induced by canonical contractions).

The relations \( N \circ M \cong 1_{L'} \) and \( M \circ N \cong 1_L \) follow from the definition. Thereby we have defined a morphism \( N : L' \to L \) and proven the following proposition.

2.4. Proposition. Bundle gerbes with respect to above defined 1-morphisms and 2-morphisms form a weak 2-groupoid \( BG(X) \).
2.4. Weak 3-group of bundle gerbes. There is yet another operation on bundle gerbes, their tensor product, which equips the category $\mathcal{B}G(X)$ with the structure of a monoidal category. More precisely, for two bundle gerbes $(L, \theta, U)$, $(L', \theta', V)$ over $X$ their tensor product $(L \otimes L', \theta \otimes \theta', W)$, where $W = \{W_{\alpha}\}$, $W_{\alpha\lambda} := U_{\alpha} \cap V_{\lambda}$, is defined by $(L \otimes L')_{\alpha\lambda, \beta\mu} := L_{\alpha\beta} \otimes L'_{\lambda\mu}$.

This way, we have defined a monoidal 2-category $\mathcal{B}G(X)$ of bundle gerbes. In particular, its unit object is the strictly trivial bundle gerbe $(T, \tau, T)$, where the open cover $T$ consists of one element $X$, $T = X \times \mathbb{C}$ and $\tau: T \otimes T \rightarrow T$ is induced by the multiplication $\mathbb{C} \otimes \mathbb{C} \rightarrow \mathbb{C}$, $1 \otimes 1 \mapsto 1$ on complex numbers.

One can say even more about the monoidal 2-category $\mathcal{B}G(X)$: every object is invertible up to 1-morphism in the sense that for every bundle gerbe $(L, \theta, U)$ there is a bundle gerbe $(L, \theta, U)$ such that $L \otimes L'$ and $L' \otimes L$ are equivalent to $(T, \tau, T)$ as bundle gerbes. In order to construct $(L, \theta, U)$, put $L'_{\alpha\beta} := L_{\beta\alpha}$, then we define $\theta'_{\alpha\beta\gamma}(L'_{\alpha\beta} \otimes L'_{\gamma\delta}) := L_{\gamma\delta} \otimes L'_{\alpha\beta}$ as

$$L_{\beta\alpha} \otimes L_{\gamma\beta} \cong L_{\alpha\beta} \otimes L_{\gamma\beta} \theta_{\alpha\beta\gamma}. $$

Then we have isomorphisms

$$L_{\beta\alpha}|_{U_{\alpha}} \cong L_{\beta\alpha} \otimes L_{\alpha\beta} \cong L_{\alpha\beta} \otimes L_{\alpha\beta} \cong L_{\alpha\beta} \otimes L'_{\alpha\beta} \cong L_{\alpha\beta} \otimes L_{\beta\alpha} \cong L_{\alpha\beta}|_{U_{\alpha}}.$$

Now using a standard trick [5], this monoidal 2-category $\mathcal{B}G(X)$ can be reinterpreted as a weak 3-groupoid with one object, i.e. a weak 3-group whose 1-morphisms are bundle gerbes (with strictly trivial gerbe as the unit and tensor product as the composition), 2-morphisms are 1-morphisms between bundle gerbes and 3-morphisms are 2-morphisms in the previous sense.

2.5. Functoriality. For a map $f: X \rightarrow Y$ and a bundle gerbe $(L', \theta', V)$, on $Y$ where $V = \{V_{\lambda}\}$ is an open covering of $Y$, one can define the pullback $f^\ast(L', \theta', V)$ which is a bundle gerbe on $X$ in the obvious way. One can show that $f^\ast$ defines a weak monoidal 2-functor $\mathcal{B}G(Y) \rightarrow \mathcal{B}G(X)$ (cf. [6]).

2.6. A counterpart from Algebra: Brauer-Picard 3-group. In fact, $\mathcal{B}G(X)$ is a topological analog of the following monoidal 2-category $\mathcal{P}icBr(R)$ of a commutative unital ring $R$. Recall its definition [5]. Consider the monoidal 2-category $\mathcal{A}lg(R)$. Its objects $A$ are associative algebras over $R$, the monoidal structure is given by their tensor product over $R$. Its 1-morphisms $M: A \rightarrow B$ are $(A, B)$-bimodules $M$. The composition of 1-morphisms $M: A \rightarrow B$, $N: B \rightarrow C$ is given by the tensor product $M \otimes_{B} N$ of bimodules over $B$. Its 2-morphisms $f: M \rightarrow M'$ are homomorphisms of $(A, B)$-bimodules. Thereby we have defined the monoidal 2-category $\mathcal{A}lg(R)$.

A 2-morphism $f: M \rightarrow M'$ is an equivalence if and only if it is an $(A, B)$-bimodule isomorphism. A 1-morphism $M: A \rightarrow B$ is an equivalence if and only if it is invertible up to isomorphisms, i.e. $\exists N: B \rightarrow A$ such that $M \otimes_{B} N \cong A$, $N \otimes_{A} M \cong B$ as $(A, A)$- and $(B, B)$-bimodules respectively. That is $M$ is a Morita-equivalence bimodule.

Consider the subcategory $\mathcal{P}icBr(R) \subset \mathcal{A}lg(R)$ whose objects are Azumaya algebras over $R$ [4] - 1-morphisms are Morita-equivalences and 2-morphisms are bimodule isomorphisms. Then $\mathcal{P}icBr(R)$ is a monoidal 2-groupoid. Its group of equivalence classes of objects (i.e. the group of Azumaya algebras up to Morita equivalence) is the Brauer group $Br(R)$. The group of equivalence classes of 1-morphisms $R \rightarrow R$ (where $R$ is regarded as an associative algebra over $R$), i.e. the group of Morita equivalences from $R$ to $R$, is the Picard group $Pic(R)$. The group of equivalence classes of 2-morphisms $R \rightarrow R$ (where this time $R$ is regarded as an $(R, R)$-bimodule) is the unit group of $R$.

Again, using the monoidal structure on $\mathcal{P}icBr(R)$ we can reinterpret it as a weak 3-group with Azumaya algebras as 1-morphisms (and the $R$-algebra $R$ as the unit object), etc.

For example, one can take $R = C(X)$ for compact $X$ and obtain the corresponding contravariant functor $X \mapsto \mathcal{P}icBr(C(X)) := \mathcal{P}icBr(C) \mapsto$ from the homotopy category to the category of weak 2-groupoids (or weak 3-groups).

We see that for a space $X$ the monoidal 2-category $\mathcal{B}G(X)$ is an analog of $\mathcal{P}icBr(R)$. Indeed, as we have shown, its objects $L$, the bundle gerbes over $X$, are invertible (up to 1-morphisms) because for every bundle gerbe $L$ there exists a bundle gerbe $L'$ such that $L \otimes L'$ and $L' \otimes L$ are equivalent to the strictly trivial bundle gerbe. So 1-morphisms of $\mathcal{B}G(X)$ are akin to Morita equivalences.

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3 An associative unital $R$-algebra $A$ is an Azumaya algebra if there is an associative unital $R$-algebra $B$ such that $A \otimes_{R} B$ and $B \otimes_{R} A$ are Morita-equivalent to $R$ as associative algebras over $R$. 

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5 Consider the monoidal 2-category $\mathcal{A}lg(R)$. Its objects $A$ are associative algebras over $R$, the monoidal structure is given by their tensor product over $R$. Its 1-morphisms $M: A \rightarrow B$ are $(A, B)$-bimodules $M$. The composition of 1-morphisms $M: A \rightarrow B$, $N: B \rightarrow C$ is given by the tensor product $M \otimes_{B} N$ of bimodules over $B$. Its 2-morphisms $f: M \rightarrow M'$ are homomorphisms of $(A, B)$-bimodules. Thereby we have defined the monoidal 2-category $\mathcal{A}lg(R)$.
2.7. Classification of bundle gerbes. Dixmier-Douady class. It is well known that bundle gerbes up to equivalence are classified by their Dixmier-Douady class. We recall its definition: take \( U \) a good cover, then choose sections \( \sigma_{\alpha\beta} \) of the Hermitian line bundles \( L_{\alpha\beta} \to U_{\alpha\beta} \) whose modulus is equal to 1 in each fiber. Then over \( U_{\alpha\beta\gamma} \) we have:
\[
\theta_{\alpha\beta\gamma} (\sigma_{\alpha\beta} \otimes \sigma_{\beta\gamma}) = \lambda_{\alpha\beta\gamma} \sigma_{\alpha\beta}
\]
for some functions \( \lambda_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to \mathbb{U}(1) \), and the associativity condition (2) implies that \( \lambda = \{ \lambda_{\alpha\beta\gamma} \} \) is a Čech 2-cocycle with coefficients in \( \mathbb{U}(1) \), the sheaf of germs of continuous \( \mathbb{U}(1) \)-valued functions. Consider the coboundary homomorphism
\[
\delta : H^2(X, \mathbb{U}(1)) \to H^3(X, \mathbb{Z})
\]
of the long exact cohomology sequence associated with the short exact sequence of sheaves
\[
0 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{\exp(2\pi i \cdot \cdot \cdot)} \mathbb{U}(1) \to 1.
\]
In fact, \( \delta \) is an isomorphism because \( \mathbb{R} \) is a fine sheaf, and hence \( H^i(X, \mathbb{R}) = 0 \) for \( i \geq 1 \). We define the Dixmier-Douady class \( DD(L(g), \theta, U) \) as \( \delta([\lambda]) \), where \( [\lambda] \in H^2(X, \mathbb{U}(1)) \) is the cohomology class of the cocycle \( \lambda \).

It follows from diagram (3) that an equivalence between two bundle gerbes induces an equivalence between their Čech cocycles. Indeed, if \( \varphi_{\alpha\beta}(\sigma_{\alpha\beta} \otimes s_{\beta}) = \mu_{\alpha\beta}s_{\alpha} \otimes \sigma_{\alpha\beta}' \) (where \( s_{\alpha} \) is a section of \( M_{\alpha} \)) for some functions \( \mu_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{U}(1) \), then two ways in diagram (3) give equality
\[
\mu_{\alpha\beta} \lambda'_{\alpha\beta\gamma} s_{\alpha} \otimes \sigma_{\alpha\beta}' = \mu_{\beta\gamma}^{-1} \lambda_{\alpha\beta\gamma} \mu_{\alpha\gamma} s_{\alpha} \otimes \sigma_{\alpha\gamma}'.
\]
Moreover, bundle gerbes are equivalent if and only if they have the same Dixmier-Douady class.

So for the monoidal 2-category \( BG(X) \) we have:
(i) the group of equivalences classes of objects is the topological Brauer group \( Br(X) \cong H^3(X, \mathbb{Z}) \);
(ii) the group of equivalences classes of 1-isomorphisms of the strictly trivial bundle gerbe \( T \) is the Picard group \( Pic(X) \cong H^2(X, \mathbb{Z}) \). Indeed, it is easy to see that a 1-morphism \( T \to T \) is just a line bundle \( M \to X \).

2.5. Remark. Thus we see that for the strictly trivial bundle gerbe \( T \) we have \( Aut(T) \cong Pic(X) \cong H^2(X, \mathbb{Z}) \). But this is true for any bundle gerbe \( L \). Indeed, for a given 1-morphism \( M : T \to T \) (i.e. a line bundle) the tensor product \( M \otimes id_L \) is a morphism \( L \cong T \otimes L \to T \otimes L \cong L \). Conversely using subsection 2.4 to a morphism \( N : L \to L \) we assign a morphism \( T \cong L \otimes L^{-1} \to L \otimes L^{-1} \cong T \), namely \( N \otimes id_{L^{-1}} \), and show that these two correspondences are inverse to each other.

2.8. Trivializations.

2.6. Definition. A right trivialization of a bundle gerbe \( L = (L, \theta, U) \) is a 1-morphism \( \eta : L \to T \) to a strictly trivial bundle gerbe \( T = (T, \tau, T) \). Similarly, a left trivialization of \( L \) is a 1-morphism \( \kappa : T \to L \).

It immediately follows from the definition that such a right trivialization \( (\eta, \varphi, U) \) consists of a collection of line bundles \( \eta_{\alpha} \to U_{\alpha} \) and isomorphisms \( \varphi_{\alpha\beta} : L_{\alpha\beta} \otimes \eta_{\beta} \to \eta_{\alpha} \) over \( U_{\alpha\beta} \) such that diagrams
\[
\begin{array}{ccc}
L_{\alpha\beta} \otimes L_{\beta\gamma} & \xrightarrow{id \otimes \varphi_{\beta\gamma}} & L_{\alpha\beta} \otimes \eta_{\beta} \\
\theta_{\alpha\beta\gamma} \otimes id & \downarrow \cong & \cong \\
L_{\alpha\gamma} \otimes \eta_{\gamma} & \xrightarrow{\varphi_{\alpha\gamma}} & \eta_{\alpha}
\end{array}
\]
commute over \( U_{\alpha\beta\gamma} \). Similarly for a left trivialization.

Assume now that there are right \( \eta : L \to T \), \( \eta := (\eta, \varphi, U) \) and left \( \kappa : T \to L \), \( \kappa := (\kappa, \psi, U) \) trivializations of \( L := (L, \theta, U) \). We have isomorphisms over \( U_{\alpha\beta} \)
\[
id \otimes \varphi_{\alpha\beta} : \kappa_{\alpha} \otimes L_{\alpha\beta} \otimes \eta_{\beta} \cong \kappa_{\alpha} \otimes \eta_{\alpha}
\]
and
\[
\psi_{\alpha\beta} \otimes id : \kappa_{\alpha} \otimes L_{\alpha\beta} \otimes \eta_{\beta} \cong \kappa_{\beta} \otimes \eta_{\beta}.
\]
Over threefold overlaps $U_{\alpha\beta\gamma}$ we have commutative diagrams

\[\begin{array}{c}
\kappa_{\beta} \otimes L_{\beta\gamma} \otimes \eta_{\gamma} \\
\psi_{\alpha\beta} \otimes \id \\
\kappa_{\alpha} \otimes L_{\alpha\beta} \otimes \eta_{\gamma} \\
\psi_{\alpha\gamma} \otimes \id \\
\kappa_{\alpha} \otimes L_{\alpha\gamma} \otimes \eta_{\alpha} \\
\psi_{\beta\gamma} \otimes \id \\
\kappa_{\beta} \otimes \eta_{\beta} \\
\psi_{\alpha\beta} \otimes \id \\
\kappa_{\alpha} \otimes \eta_{\alpha}.
\end{array}\]

Hence the line bundles $\kappa_{\alpha} \otimes \eta_{\alpha}$ together with the isomorphisms $\chi_{\alpha\beta} := (\psi_{\alpha\beta} \otimes \id) \circ (\id \otimes \varphi_{\alpha\beta})^{-1}$ descend to a ("global") line bundle on $X$. In other words, two trivializations of the same bundle gerbe differ by a line bundle. Note that the obtained result agrees with the previous category-theoretic arguments: the composition $\eta \circ \kappa : T \to T$ is a 1-automorphism of the strictly trivial bundle gerbe $T$, i.e., a line bundle.

2.7. Remark. The obtained connection between trivializations and line bundles can also be illustrated by Čech cohomology as follows. Note that a bundle gerbe admits a trivialization iff its Dixmier-Douady class $\kappa$ has a trivialization $\eta$. Hence the line bundles $\kappa_{\alpha} \otimes \eta_{\alpha}$ to a ("global") line bundle on $X$. Note that the obtained result agrees with the previous category-theoretic arguments: the composition $\eta \circ \kappa : T \to T$ is a 1-automorphism of the strictly trivial bundle gerbe $T$, i.e., a line bundle.

2.9. Morita bundle gerbes. There is a generalization of the notion of a bundle gerbe related to the Brauer-Picard 2-groupoid (whose objects are Azumaya algebras, 1-morphisms Morita equivalences (bimodules) between them and 2-morphisms are isomorphisms of bimodules). More precisely, we have the following definition.

2.8. Definition. A Morita bundle gerbe (MBG for short) $(A, M, \theta, U)$ is the following collection of data. First, we have matrix algebra bundles $A_{\alpha} \to U_{\alpha}$, then invertible $(A_{\beta}, A_{\alpha})$-bimodules $\beta M_{\alpha}$ (Morita-equivalences between $A_{\alpha}|_{U_{\alpha\beta}}$ and $A_{\beta}|_{U_{\alpha\beta}}$), then $(A_{\gamma}, A_{\alpha})$-bimodule isomorphisms $\theta_{\alpha\beta\gamma} : \gamma M_{\alpha} \otimes \beta M_{\alpha} \to \alpha M_{\alpha}$ corresponding to diagrams

\[\begin{array}{c}
A_{\alpha} \\
\downarrow \theta_{\alpha\beta\gamma} \\
A_{\gamma} \\
\downarrow \theta_{\alpha\beta\gamma} \\
A_{\gamma}.
\end{array}\]

which satisfy relations

$\theta_{\alpha\gamma\delta} \circ (1 \otimes \theta_{\alpha\beta\gamma}) = \theta_{\alpha\beta\delta} \circ (\theta_{\beta\gamma\delta} \otimes 1)$

over four-fold overlaps. The last relations correspond to diagrams

\[\begin{array}{c}
\delta M_{\gamma} \otimes \beta M_{\alpha} \\
\theta_{\alpha\beta\gamma} \otimes 1 \\
\alpha M_{\alpha}.
\end{array}\]

2.9. Definition. A 1-morphism $(N, \varphi)$ between MBG’s $(A, M, \theta, U)$ and $(A', M', \theta', U)$ consists of $(A_{\alpha}, A'_{\alpha})$-bimodules $N_{\alpha}$ and $(A'_{\beta}, A_{\alpha})$-bimodule isomorphisms $\varphi_{\alpha\beta} : \beta M'_{\alpha} \otimes N_{\alpha} \cong N_{\beta} \otimes \beta M_{\alpha}$ corresponding to diagrams

\[\begin{array}{c}
A_{\alpha} \\
\downarrow \beta M_{\alpha} \\
A_{\beta}.
\end{array}\]

\[\begin{array}{c}
A'_{\alpha} \\
\downarrow \beta M'_{\alpha} \\
A'_{\beta}.
\end{array}\]
and such that diagrams

\[
\begin{align*}
\gamma M'_{A_0} \otimes_{A_0} M'_{A_0} & \overset{1 \otimes \psi_{\alpha \beta}}{\longrightarrow} \gamma M'_{A_0} \otimes_{A_0} M_{A_0} \\
& \overset{1 \otimes \phi_{\alpha \beta}}{\longrightarrow} \gamma M'_{A_0} \otimes_{A_0} M_{A_0} \\
N_{A_0} & \overset{\varphi_{\alpha \beta}}{\longrightarrow} N_{A_0} \otimes_{A_0} \gamma M_{A_0} \\
& \overset{\varphi_{\beta \gamma} \otimes 1}{\longrightarrow} N_{A_0} \otimes_{A_0} \gamma M_{A_0} \\
\end{align*}
\]

commute.

2-morphisms \( \psi: (N, \varphi) \Rightarrow (N', \varphi') \) between 1-morphisms \( (N, \varphi), (N', \varphi'): (A, M, \theta, U) \rightarrow (A', M', \theta', U) \) are bimodule isomorphisms which make all structure diagrams commutative. More precisely, for all \( \alpha \) we have isomorphisms \( \psi_\alpha: N_\alpha \rightarrow N'_\alpha \) of \( (A'_\alpha, A_\alpha) \)-bimodules such that diagrams

\[
\begin{align*}
\beta M'_\alpha \otimes N_\alpha & \overset{\varphi_{\alpha \beta}}{\longrightarrow} N_\beta \otimes_{A_\beta} \beta M_\alpha \\
& \overset{\varphi_{\beta \gamma} \otimes 1}{\longrightarrow} N_\beta \otimes_{A_\beta} \beta M_\alpha \\
\end{align*}
\]

commute.

By \( \text{MBG}(X) \) denote the monoidal 2-groupoid (3-group) of Morita bundle gerbes over \( X \).

Note that \( \text{MBG}(X) \) is a monoidal category with monoidal structure induced by the tensor product of MBG’s. Its unit object is the strictly trivial bundle gerbe.

Note also that the 2-groupoid \( \text{BG}(X) \) is a full subcategory in \( \text{MBG}(X) \). Moreover, this inclusion is an equivalence of 2-categories, because the natural inclusion of the Picard 2-group \( \text{Pic} \) to the Brauer-Picard 2-groupoid is an equivalence of 2-categories. But we can give an independent proof of this result.

2.10. Proposition. The inclusion \( \text{BG}(X) \rightarrow \text{MBG}(X) \) of the category of “common” bundle gerbes to the category of Morita bundle gerbes is an equivalence.

Proof. We must show that any MBG \((A, M, \theta, U)\) is equivalent to a “common” bundle gerbe \((L, \theta', U)\).

Assume that the cover \( U \) is good. Fix Morita-equivalences \( \xi_\alpha: A_\alpha \\rightarrow C_\alpha \), where \( C_\alpha := U_\alpha \times \mathbb{C} \) and also their inverse \( \xi^{-1}_\alpha \) together with isomorphisms \( i_\alpha: \xi^{-1}_\alpha \otimes C_\alpha \\rightarrow \text{id}_{A_\alpha} \). Put \( L_{\alpha \beta} := \xi_{\beta \alpha} \otimes \beta M_\alpha \otimes \xi^{-1}_\alpha \). Then \( \theta'_{\alpha \beta \gamma} \) is the only isomorphism which makes the diagram

\[
L_{\alpha \beta} \otimes L_{\beta \gamma} = \xi_{\alpha} \otimes_{A_\alpha} \beta M_{A_\beta} \otimes_{A_\beta} \xi_{\beta} \otimes_{A_\beta} \gamma M_{A_\alpha} \otimes_{A_\alpha} \xi_{\gamma} \otimes_{A_\gamma} \alpha M_{A_\beta} \otimes_{A_\beta} \xi_{\alpha} \otimes_{A_\alpha} \beta M_{A_\alpha} \otimes_{A_\alpha} \xi_{\alpha} \otimes_{A_\alpha} \beta M_{A_\alpha} \otimes_{A_\alpha} \xi_{\alpha} \otimes_{A_\alpha} \\
\theta'_{\alpha \beta \gamma} \otimes 1 \]

commutative. Now the identity

\[
\theta'_{\alpha \beta \gamma} \circ (1 \otimes \theta'_{\alpha \beta \gamma}) = \theta'_{\alpha \beta \gamma} \circ (\theta'_{\beta \gamma} \otimes 1)
\]

follows from the counterpart for \( \theta' \)’s.

Note that the collection \( \{\xi_\alpha\} \) of bimodules with obvious isomorphisms \( \varphi_{\alpha \beta}: L_{\alpha \beta} \otimes \xi_\alpha \rightarrow \xi_{\beta} \otimes \beta M_\alpha \) define a morphism \((A, M, \theta, U) \rightarrow (L, \theta', U)\).

Note that a global matrix algebra (“Azumaya”) bundle \( A \rightarrow X \) can be considered as a Morita bundle gerbe \((A, M, \theta, U)\) with respect to any open cover \( U \) where \( A_\alpha = A|_{U_\alpha} \) and \( M_{\alpha} = A|_{U_\alpha} \otimes \beta M_\alpha \). The assignment to a matrix algebra bundle \( A \) the equivalence class of the corresponding MBG corresponds to the map \( \text{BPU}(k) \rightarrow \text{K}(\mathbb{Z}, 3), A \mapsto DD(A) \). In order to define a lift of \( X \rightarrow K(\mathbb{Z}, 3) \) we need the concept of a bundle gerbe module (see subsection 3.3).

Note that the concept of a Morita bundle gerbe allows to treat a global matrix algebra bundle over \( X \) and the corresponding bundle gerbe with the same Dixmier-Douady class (of finite order in \( H^3(X, \mathbb{Z}) \)) as equivalent cocycles (cf. subsection 3.3).
The following Proposition is obvious.

2.11. Proposition. An MBG \((A, M, \theta, \mathcal{U})\) is equivalent to a global matrix algebra bundle over \(X\) (as an MBG) iff \(DD(A, M, \theta, \mathcal{U}) \in H^3_{\text{tors}}(X, \mathbb{Z})\).

2.10. Classifying space for bundle gerbes. Let \(\mathcal{H}\) be a separable Hilbert space, \(PU(\mathcal{H}) = U(\mathcal{H})/U(1)\) the corresponding projective unitary group (considered as a topological group with the norm topology). Let (5)
\[
\phi_1 = U(\mathcal{H}) \times \mathbb{C} \mapsto PU(\mathcal{H})
\]
be the canonical line bundle over \(PU(\mathcal{H})\) (also universal as \(U(\mathcal{H})\) is contractible and hence \(PU(\mathcal{H})\) is a model of \(BU(1))\), associated with the principal \(U(1)\)-bundle (6)
\[
U(1) \to U(\mathcal{H}) \xrightarrow{\pi} PU(\mathcal{H}).
\]

The following construction assigns a bundle gerbe to any projective cocycle. Let \((g, \mathcal{U})\) be a \(PU(\mathcal{H})\)-valued 1-cocycle \(\{g_{\alpha \beta}\}, g_{\alpha \beta} : U_{\alpha \beta} \to PU(\mathcal{H})\). The projective cocycle \((g, \mathcal{U})\) gives rise to a bundle gerbe \((L(g), \theta, \mathcal{U})\), where the line bundles \(L_{\alpha \beta} := g_{\alpha \beta}^* \phi_1 \to U_{\alpha \beta}\) are defined as pullbacks of the canonical line bundle \(\phi_1\), and where the product
\[
\theta_{\alpha \beta \gamma} : L_{\alpha \beta} \otimes L_{\beta \gamma} \xrightarrow{\cong} L_{\alpha \gamma}
\]
over three-fold overlaps \(U_{\alpha \beta \gamma}\) is defined by the group multiplication
\[
\mu_1 : U(\mathcal{H}) \times U(\mathcal{H}) \to U(\mathcal{H})
\]
(cf. [2]). Here we use the isomorphism (7)
\[
\mu_1^*(\phi_1) \cong \phi_1 \boxtimes \phi_1,
\]
where
\[
\mu_1 : PU(\mathcal{H}) \times PU(\mathcal{H}) \to PU(\mathcal{H})
\]
is the group multiplication and \(\boxtimes\) denotes the exterior tensor product. Then the commutative diagram
\[
\begin{array}{ccc}
U_{\alpha \beta} & \xrightarrow{\text{diag}} & U_{\alpha \beta} \times U_{\alpha \beta} \\
\downarrow{g_{\alpha \beta}} & & \downarrow{g_{\alpha \beta} \times g_{\alpha \beta}} \\
PU(\mathcal{H}) \times PU(\mathcal{H}) & \xrightarrow{\mu_1} & PU(\mathcal{H})
\end{array}
\]
gives us isomorphisms \(\theta_{\alpha \beta \gamma}\) between \((g_{\alpha \beta} g_{\alpha \beta})^* (\phi_1) = L_{\alpha \beta} \otimes L_{\beta \gamma}\) and \(g_{\alpha \beta \gamma}^* (\phi_1) = L_{\alpha \gamma}\) over \(U_{\alpha \beta \gamma}\).

Clearly, the product \(\theta = \{\theta_{\alpha \beta \gamma}\}\) is associative over four-fold overlaps, i.e. the diagrams [2] commute over \(U_{\alpha \beta \gamma \delta}\).

Moreover, equivalent cocycles give rise to equivalent bundle gerbes. So we have the natural transformation of homotopy functors \(\Phi : H^1(X, PU(\mathcal{H})) \to BG(X)\), where \(X \mapsto BG(X)\) denotes the functor which assigns to \(X\) the group equivalence classes of bundle gerbes over \(X\).

2.12. Theorem. \(\Phi\) is a natural isomorphism. In other words, any bundle gerbe over \(X\) is equivalent to a bundle gerbe of the form \((L(g), \theta(g), \mathcal{U})\).

2.13. Remark. An alternative explanation of this isomorphism: exact sequence of groups
\[
1 \to U(1) \to U(\mathcal{H}) \to PU(\mathcal{H}) \to 1
\]
gives rise to the isomorphism \(H^1(X, PU(\mathcal{H})) \cong H^2(X, U(1))\) and the last group is isomorphic to \(H^3(X, \mathbb{Z})\) which is isomorphic to the group \(BG(X)\), as we have seen in subsection [2.7]. So the standard proof of this result uses the Dixmier-Douady class which classifies equivalence classes of bundle gerbes. But we give a sketch of an independent proof which is more appropriate for generalizations we have in mind.

Proof. First note that \(X \mapsto BG(X)\) is a homotopy functor which satisfies the condition of the Brown representability theorem. Therefore it is represented by some CW-complex \(T\) which is unique up to homotopy equivalence. Next, according to the Yoneda lemma, the natural transformation \(\Phi\) defines a map \(\phi : BPU(\mathcal{H}) \to T\). As \(BPU(\mathcal{H})\) has the homotopy type of a CW-complex, by the Whitehead theorem it is sufficient to show that \(\phi\) induces isomorphisms on homotopy groups, i.e. \(\Phi\) induces isomorphisms for spheres.

So consider a bundle gerbe \((L, \theta, \mathcal{U})\) over \(X = S^n\). By \(X_0\) or \(X_1\) denote the (thickened) upper or lower open hemisphere, respectively, and let \(V := U \cap X_0 = \{U_0 \cap X_0\}_0\) and \(W := U \cap X_1\) be the corresponding cover of \(X_0\) or \(X_1\). Restricting, which is a particular case of the pullback \((L, \theta, \mathcal{U})\) to \(V \) and \(W\) we obtain bundle gerbes \((L_0, \theta_0, V)\) and \((L_1, \theta_1, W)\) over \(X_0\) or \(X_1\). Because of contractibility of \(X_0\) and \(X_1\) there are left and right trivializations \((\eta, \varphi, V)\) and \((\kappa, \psi, W)\) of \((L_0, \theta_0, V)\) or \((L_1, \theta_1, W)\) and these are unique up to the tensor product with a trivial line bundle.
Put \( X_{01} := X_0 \cap X_1 \simeq S^{n-1} \). We see that the restriction of \((L, \theta, U)\) to \(X_{01}\) has two trivializations (namely the restrictions of \((\eta, \varphi, V)\) and of \((\kappa, \psi, W)\)) and their “difference” \(\{\eta_\alpha \otimes \kappa_\alpha\}/ \sim\) is a global line bundle \(\zeta \to X_{10}\). If this bundle is trivial, it is easy to see that the trivializations \((\eta, \varphi, V)\) and \((\kappa, \psi, W)\) can be used to define a global trivialization of \((L, \theta, U)\). Therefore if \(n \neq 3\), the bundle gerbe \((L, \theta, U)\) over \(S^n\) is stably trivial.

On the other hand, for \(n = 3\) the isomorphism class of \(\zeta\) is the only invariant of the equivalence class of \((L, \theta, U)\), i.e. the equivalence class of a bundle gerbe \((L, \theta, U)\) over \(S^3\) is determined by the isomorphism class of a line bundle over \(S^2\) and hence by a \(\text{PU}(\mathcal{H})\)-cocycle \(g_{\chi \alpha} : S^{n-1} \to \text{PU}(\mathcal{H})\). \(\blacksquare\)

Note that the previous proof implies that there is an isomorphism

\[
BG(\Sigma X) \cong Pic(X)
\]
natural on \(X\).

2.14. Corollary. There is the natural isomorphism of functors

\[
\Phi' : BG(\ldots) \cong [\ldots, \text{PU}(\mathcal{H})].
\]

2.15. Corollary. There is a universal bundle gerbe over \(\text{BPU}(\mathcal{H})\) such that every bundle gerbe is equivalent to its pullback via some map (unique up to homotopy).

Proof. This follows from the Brown representability theorem. \(\blacksquare\)

2.16. Definition. Note that the tensor product of bundle gerbes induces a group operation on \(BG(X)\) and the above isomorphism \(\Phi'\) is an isomorphism of functors with values in abelian groups. Recall that \(\text{BPU}(\mathcal{H}) \cong K(\mathbb{Z}, 3)\) as \(H\)-spaces, therefore \(BG(X) \cong H^3(X, \mathbb{Z})\). The group \(BG(X)\) is called the Brauer group \(\text{Br}(X)\).

This isomorphism coincides with the one given by the Dixmier-Douady class \([8]\).

2.11. Finite order case. If we consider \(\text{PU}(k)\)-cocycles \((g, U)\) in place of \(\text{PU}(\mathcal{H})\)-cocycles, we obtain a particular (“finite order”) case of bundle gerbes. More precisely, fix a positive integer \(k > 1\) and consider the projective unitary group \(\text{PU}(k) := U(k)/U(1)\), i.e. the quotient of \(U(k)\) by its center. Let

\[
\tilde{\theta}_{k,1} = U(k) \times \mathbb{C} \to \text{PU}(k)
\]

be the canonical line bundle over \(\text{PU}(k)\) associated with the principal \(U(1)\)-bundle

\[
U(1) \to U(k) \xrightarrow{\pi} \text{PU}(k).
\]

Choose a projective cocycle \((g, U) := \{g_{\alpha\beta}\}, g_{\alpha\beta} : U_{\alpha\beta} \to \text{PU}(k)\).

The projective cocycle \((g, U)\) gives rise to a bundle gerbe \((L(g), \theta, U)\), where the line bundles \(L_{\alpha\beta} := g_{\alpha\beta}^* \tilde{\theta}_{k,1} \to U_{\alpha\beta}\) are defined as pullbacks of the canonical line bundle \(\tilde{\theta}_{k,1}\), and the product

\[
\theta_{\alpha\beta\gamma} : L_{\alpha\beta} \otimes L_{\beta\gamma} \xrightarrow{\sim} L_{\alpha\gamma}
\]

over three-fold overlaps \(U_{\alpha\beta\gamma}\) is defined by the group multiplication

\[
\tilde{\mu}_{k,1} : U(k) \times U(k) \to U(k)
\]

(cf. \([9]\)). In particular,

\[
\mu_{k,1}(\tilde{\theta}_{k,1}) \cong \tilde{\theta}_{k,1} \boxtimes \tilde{\theta}_{k,1},
\]

where

\[
\mu_{k,1} : \text{PU}(k) \times \text{PU}(k) \to \text{PU}(k)
\]

is the group multiplication and \(\boxtimes\) denotes the exterior tensor product.

We also have the group homomorphism

\[
\varphi : \text{PU}(k) \to \text{PU}(k) \cong \text{PU}(\mathcal{H}), \quad a \mapsto a \otimes \text{id}_{\mathcal{H}}
\]

which is the classifying map for \(\tilde{\theta}_{k,1}\), \(\varphi^*(\tilde{\theta}_{1}) \cong \tilde{\theta}_{k,1}\). Therefore we can consider the above equivalence relation on finite order bundle gerbes. Then their equivalence classes correspond to the image of the map \([X, \text{BPU}(k)] \to [X, \text{BPU}(\mathcal{H})]\).
2.17. Remark. Note that a PU($k$)-valued cocycle (and its PU($k$)-equivalence class) contains some additional information compared to the PU($\mathcal{H}$)-cocycle. More precisely, a PU($k$)-cocycle $[g_{\alpha\beta}]$ defines a principal PU($k$)-bundle over $X$, and in this way one obtains a one-to-one correspondence between the set $H^1(X, \text{PU}(k))$ and the set of isomorphism classes of principal PU($k$)-bundles over $X$. There is also a homotopy description of the previous set: each principal PU($k$)-bundle over $X$ is classified by some map $X \to \text{BPU}(k)$ which is unique up to homotopy, i.e. there exists a natural in $X$ bijection $H^1(X, \text{PU}(k)) \cong [X, \text{BPU}(k)]$, where $[X, Y]$ denotes the set of homotopy classes of maps $X \to Y$.

We also have the exact sequence of sheaves
\begin{equation}
1 \to U(1) \to U(k) \to \text{PU}(k) \to 1
\end{equation}
corresponding to the exact sequence of groups [10] and the corresponding coboundary homomorphism $\delta_k: H^1(X, \text{PU}(k)) \to H^2(X, U(1))$.

It is easy to prove that every element of finite order in $H^2(X, U(1)) \cong H^3(X, \mathbb{Z})$ belongs to the image of $\delta_k$ for some $k$. In other words, any bundle gerbe with Dixmier-Douady class of finite order is stably equivalent to some bundle gerbe given by the previous construction applied to a PU($k$)-projective cocycle.

The tensor product of finite order bundle gerbes corresponds to the homomorphisms $\text{PU}(k^n) \times \text{PU}(k^n) \to \text{PU}(k^{n+n})$ giving the Kronecker product of matrices. The corresponding finite Brauer group is
\[ Br_k(X) := \text{im}([X, \text{BPU}(k^\infty)] \to [X, \text{BPU}(\mathcal{H})] = Br(X), \]
the $k$-torsion subgroup in $Br(X)$ (this justifies the name “finite order”).

3. Bundle gerbe modules

As we stated in the Introduction, twisted $K$-theory is a functor from the groupoid of twists $Tw(X)$ ($\mathcal{B}G(X)$ in our case) to abelian groups. Here we shall define it (first as a functor to abelian semigroups).

3.1. Definition of a bundle gerbe module.

3.1. Definition. A (right) module $(E, \varepsilon, U)$ over a bundle gerbe $(L, \theta, U)$ is a collection of vector bundles $E_\alpha \to U_\alpha$ equipped with isomorphisms $\varepsilon_{\alpha\beta}: E_\alpha \otimes L_{\alpha\beta} \to E_\beta$ over $U_{\alpha\beta}$ such that diagrams
\[
\begin{array}{c}
E_\alpha \otimes L_{\alpha\beta} \otimes L_{\beta\gamma} \\
\downarrow \varepsilon_{\alpha\beta \gamma} \\
E_\beta \otimes L_{\beta\gamma}
\end{array} \xymatrix{
\ar[r]^-{\varepsilon_{\alpha\beta}} & E_\alpha \otimes L_{\alpha\gamma} \\
\ar[r]_-{\varepsilon_{\beta\gamma}} & E_\gamma
}
\]
over $U_{\alpha\beta\gamma}$ commute.

By $\text{Mod}(L)$ denote the set of all isomorphism classes of bundle gerbe modules over $(L, \theta, U)$. Given two modules $(E, \varepsilon, U)$ and $(E', \varepsilon', U)$ over the same $(L, \theta, U)$ one can define their direct sum $(E \oplus E', \varepsilon \oplus \varepsilon', U)$ which is an $(L, \theta, U)$-module again. Therefore $\text{Mod}(L)$ is an abelian semigroup. Thereby we have defined the functor $\text{Mod}$ on objects of $\mathcal{B}G(X)$. Note that $\text{Mod}(T) = \text{Bun}(X)$, where $T$ and $\text{Bun}(X)$ are the strictly trivial bundle gerbe and the semigroup of vector bundles over $X$.

3.2. Proposition. A 1-morphism $(M, \varphi)$ from $(L, \theta, U)$ to $(L', \theta', U)$ gives rise to a semigroup homomorphism $\text{Mod}(M): \text{Mod}(L) \to \text{Mod}(L')$ such that
\[
\text{Mod}(N) \circ \text{Mod}(M) = \text{Mod}(N \circ M) \quad \text{and} \quad \text{Mod}(\text{id}_L) = \text{id}_{\text{Mod}(L)}
\]
for a 1-morphism $N: (L', \theta', U) \to (L'', \theta'', U)$. As a corollary, $\text{Mod}(M)$ is an isomorphism for all 1-morphisms $M: L \to L'$.

Proof. For an $(L, \theta, U)$-module $(E, \varepsilon, U)$ and a morphism $(M, \varphi): L \to L'$ consider the collection of bundles $F_\alpha := E_\alpha \otimes M_\alpha \to U_\alpha$. There are isomorphisms
\[
E_\beta \otimes M_\beta \xymatrix{
\ar[r]^-{\varphi_{\alpha\beta,1}} & E_\alpha \otimes L_{\alpha\beta} \otimes M_\beta \\
\ar[r]_-{\varphi_{\alpha\beta,2}} & E_\alpha \otimes M_\alpha \otimes L'_{\alpha\beta}
}
\]
Define isomorphisms
\[
\zeta_{\alpha\beta}: F_\alpha \otimes L'_{\alpha\beta} \to F_\beta \quad \text{as} \quad \zeta_{\alpha\beta} := (\varepsilon_{\alpha\beta} \otimes 1) \circ (1 \otimes \varphi_{\alpha\beta}^{-1}).
\]
Now the commutative diagram

\[
\begin{array}{c}
E_\alpha \otimes M_\alpha \otimes L'_\alpha \otimes L'_\beta \gamma \\ \downarrow^{1 \otimes \varphi_{\alpha}\gamma^{-1} \otimes 1} \\
E_\alpha \otimes L_{\alpha\beta} \otimes M_{\beta\gamma} \otimes L'_\gamma \gamma \\ \downarrow^{1 \otimes \varphi_{\alpha}\beta^{-1} \otimes 1} \\
E_\beta \otimes M_{\beta\gamma} \otimes L'_\gamma \gamma \\
\end{array}
\]

shows that \((F, \zeta, U)\) is an \((L', \theta', U')\)-module. □

In particular, a trivialization \((\eta, \varphi)\): \((L, \theta, U) \rightarrow (T, \tau, T)\) determines isomorphisms \(\text{Mod}(\eta): \text{Mod}(L) \rightarrow \text{Bun}(X) (= \text{Mod}(T))\) of the corresponding semigroups. Indeed, for an \((L, \theta, U)\)-module \((E, \epsilon, U)\) put \(F_\alpha := E_\alpha \otimes \eta_\alpha\). Then we have isomorphisms

\[
E_\beta \otimes \eta_\beta \xleftarrow{\epsilon_{\alpha\beta} \otimes 1} E_\alpha \otimes L_{\alpha\beta} \otimes \eta_\beta \xrightarrow{1 \otimes \varphi_{\alpha}\beta} E_\alpha \otimes \eta_\alpha
\]

over \(U_{\alpha\beta}\) and commutative diagrams of isomorphisms

\[
F_\beta = E_\beta \otimes \eta_\beta
\]

\[
\begin{array}{c}
E_\beta \otimes L_{\beta\gamma} \otimes \eta_\gamma \\
\downarrow^{\epsilon_{\alpha\beta} \otimes 1} \\
E_\alpha \otimes L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes \eta_\gamma \\
\downarrow^{1 \otimes \varphi_{\alpha}\beta} \\
E_\alpha \otimes \eta_\alpha
\end{array}
\]

\[
F_\gamma = E_\gamma \otimes \eta_\gamma
\]

\[
\begin{array}{c}
E_\beta \otimes M_{\beta\gamma} \\
\downarrow^{\epsilon_{\alpha\beta} \otimes 1} \\
E_\alpha \otimes L_{\alpha\gamma} \otimes \eta_\gamma \\
\downarrow^{1 \otimes \varphi_{\alpha}\beta} \\
E_\alpha \otimes \eta_\alpha
\end{array}
\]

\[
E_\alpha \otimes \eta_\alpha = F_\alpha
\]

over \(U_{\alpha\beta}\). We see that this data indeed gives rise to a vector bundle \(F\) over \(X\).

In order to define the inverse map \(\text{Bun}(X) \rightarrow \text{Mod}(L)\) note that \(\epsilon_{\alpha\beta}: L_{\alpha\beta} \otimes \eta_\beta \rightarrow \eta_\alpha\) give isomorphisms \(\eta_\alpha^*: L_{\alpha\beta} \otimes \eta_\beta \rightarrow \eta_\alpha\). For a vector bundle \(F : X \rightarrow X\) define the collection \(\{E_\alpha = F \otimes \eta_\alpha^*\}\) together with isomorphisms \(F \otimes \eta_\alpha^* \otimes L_{\alpha\beta} \rightarrow F \otimes \eta_\beta^*\), i.e. with \(E_\alpha \otimes L_{\alpha\beta} \rightarrow E_\beta\). It is easy to see that we obtain a left \(L\)-module.

Note that different choices of trivializations give rise to the action of the Picard group \(\text{Pic}(X)\) (which is the group of equivalence classes of automorphisms of the trivial twist) on \(\text{Bun}(X)\).

3.2. Bundle gerbe \(K\)-theory. We have the following result [3].

3.3. Proposition. If \((L, \theta, U)\) has a module \((E, \epsilon, U)\) of finite rank \(r\) then its Dixmier-Douady class \(DD(L) \in H^3(X, \mathbb{Z})\) satisfies \(r \cdot DD(L) = 0\).

Given a bundle gerbe \((L, \theta, U)\) whose Dixmier-Douady class is of finite order we define its \(K\)-group, \(K(L)\), as the Grothendieck group of the semigroup \(\text{Mod}(L)\). Then an equivalence between \((L, \theta, U)\) and \((L', \theta', U')\) gives rise to a particular isomorphism between \(K(L)\) and \(K(L')\), i.e. \(K(L)\) up to isomorphism (not canonical) depends only on the class \(DD(L, \theta, U)\) of \((L, \theta, U)\) in \(B_{rk}(X)\).

The following properties of \(K(L)\) can also be easily verified [3].

3.4. Theorem. (i) If \(DD(L, \theta, U) = 0\), then \(K(L) \cong K(X)\).

(ii) \(K(L)\) is a \((L, \theta, U)\)-module).

(iii) There are homomorphisms \(K(L) \otimes K(L') \rightarrow K(L \otimes L')\) which satisfy the expected associativity.

(iv) For \(f: X \rightarrow Y\) and a bundle gerbe \((L, \theta, U)\) over \(Y\) we have a homomorphism \(K(L) \rightarrow K(f^*(L))\), making \(K(L)\) a functor.

3.3. Relation between bundle gerbe modules and Azumaya bundles. Assume that \(L := (L, \theta, U)\) is a bundle gerbe with a torsion Dixmier-Douady class. Then it admits some module \((E, \epsilon, U)\). Note that \((E, \epsilon, U)\) gives rise to a global matrix algebra bundle \(\text{End}(E) \rightarrow X\) (and every matrix algebra bundle can be obtained in this way). Indeed, isomorphisms \(\epsilon_{\alpha\beta}: E_\alpha \otimes L_{\alpha\beta} \rightarrow E_\beta\) give rise to isomorphisms \(\epsilon_{\alpha\beta}: \text{End}(E_\alpha) \rightarrow \text{End}(E_\beta)\) which satisfy the cocycle condition. (More precisely, \(\epsilon_{\alpha\beta}: E_\beta \rightarrow E_\alpha \otimes L^*\alpha\beta\) give rise to maps \(L_{\alpha\beta} \otimes E_\beta \rightarrow E_\alpha\) which allow to define isomorphisms

\[
E_\beta \otimes E_\beta^* \leftarrow E_\alpha \otimes L_{\alpha\beta} \otimes E_\beta^* \rightarrow E_\alpha \otimes E_\alpha^*
\]
on twofold overlaps etc.). The obtained global bundle \( \text{End}(E) \) can be regarded as a Morita bundle gerbe (cf. the definition of a strictly trivial bundle gerbe). Then the bundle gerbe module \((E, \varepsilon, \mathcal{U})\) is nothing but a 1-morphism \( \text{End}(E) \to (L, \theta, \mathcal{U}) \) of Morita bundle gerbes. Indeed, isomorphisms

\[
\text{End}E|_{U_\beta} \otimes \text{End}E|_{U_\beta} \xrightarrow{\text{can}} E_\beta \xleftarrow{\varepsilon_{\alpha \beta}} E_\alpha \otimes L_{\alpha \beta}
\]

play exactly the role of isomorphisms \( \varphi_{\alpha \beta} \) in Definition 2.3. (Let us remark the analogy with trivialization: like a stably trivial BG, \( L \) (with \( DD(L) \) of finite order) is Morita-equivalent to a global matrix algebra bundle \( \text{End}(E) \), but this time not necessarily 1-dimensional or even trivial).

If \((E, \varepsilon)\) has rank \( r \), then it is nothing but a fiberwise homotopy class of lifts \( f_L \) of the classifying map \( f_L : X \to K(\mathbb{Z}, 3) \) of the bundle gerbe \((L, \theta, \mathcal{U})\) in the fibration

\[
\begin{array}{ccc}
\text{BU}(r) & \longrightarrow & \text{BPU}(r) \\
\downarrow & & \downarrow DD \\
X & \longrightarrow & K(\mathbb{Z}, 3).
\end{array}
\]

This gives another proof of Proposition 3.3.

3.4. **An isomorphism between bundle gerbe K-theory and Azumaya algebra bundle K-theory.**

We have seen that a trivialization of a bundle gerbe \( L \) determines a semigroup isomorphisms between \( \text{Mod}(L) \) and \( \text{Bun}(X) \). One can expect that an \( L \)-module \( E \) gives rise to an isomorphism \( E_* : \text{Mod}(L) \to \text{Mod}(\text{End}(E)) \), where \( \text{Mod}(\text{End}(E)) \) is the semigroup of projective modules over the global Azumaya bundle \( \text{End}(E) \to X \) in the common sense.

Let \( L := (L, \theta, \mathcal{U}) \) be a bundle gerbe with a torsion Dixmier-Douady class, let \((E, \varepsilon, \mathcal{U})\) be a left module over \( L \) of finite rank. We are going to describe the explicit additive isomorphism between the category of \( \text{End}(E) \)-modules and the category of \( \text{End}(E) \)-modules and thereby between the \( K \)-theory of \( L \) and the \( K \)-theory of the matrix algebra (Azumaya) bundle \( \text{End}(E) \) (note that \( \text{End}(E) \) has the same Dixmier-Douady class as \( L \) given by \((E, \varepsilon, \mathcal{U})\)).

Let \( F := (F, \rho, \mathcal{U}) \) be a right \( L \)-module. The left \( \text{End}(E)|_{U_\alpha} \)-module \( E_\alpha \otimes F_\alpha \to U_\alpha \) denote by \( H_\alpha \). We must show that \( H_\alpha \)’s give rise to a global \( \text{End}(E) \)-module \( H \to X \). First, we start with isomorphisms

\[
H_{\alpha|_{U_{\alpha \beta}}} = E_\beta \otimes F_\beta|_{U_{\alpha \beta}} \xleftarrow{\varepsilon_{\alpha \beta} \otimes 1} E_\alpha \otimes L_{\alpha \beta} \otimes F_\beta \xrightarrow{1 \otimes \rho_{\alpha \beta}} E_\alpha \otimes F_\alpha|_{U_{\alpha \beta}} = H_\alpha|_{U_{\alpha \beta}}
\]

of \( \text{End}(E)|_{U_{\alpha \beta}} \)-modules (because this isomorphisms are obviously compatible with isomorphisms (14)).

Secondly, there are commutative diagrams of such isomorphisms

\[
\begin{array}{ccc}
E_\beta \otimes L_{\beta \gamma} \otimes F_\gamma & \xrightarrow{\varepsilon_{\alpha \beta} \otimes 1} & E_\alpha \otimes L_{\alpha \beta} \otimes F_\beta \\
\downarrow 1 \otimes \rho_{\alpha \beta} \otimes 1 & & \downarrow 1 \otimes \rho_{\alpha \beta} \\
E_\alpha \otimes L_{\alpha \beta} \otimes L_{\beta \gamma} \otimes F_\gamma & \xrightarrow{1 \otimes \rho_{\alpha \beta}} & E_\alpha \otimes L_{\alpha \beta} \otimes F_\beta
\end{array}
\]

over \( U_{\alpha \beta \gamma} \). We see that this data indeed gives rise to a left \( \text{End}(E) \)-module \( H \) over \( X \). So \( E \) plays the role of an \( (\text{End}(E), L) \)-bimodule.

In order to define the inverse map, for an \( \text{End}(E) \)-module \( H \) put \( F_\alpha := E_\alpha^* \otimes \text{End}(E)|_{U_\alpha} 
\). Then using \( L_{\alpha \beta} \otimes E_{\beta}^* \to E_\alpha^* \) we define maps \( L_{\alpha \beta} \otimes F_\beta \to F_\alpha \) providing \( \{F_\alpha\} \) with the structure of a left \( L \)-module.

Finally, we see that bundle gerbes with their modules lead to the same \( K \)-theory as matrix algebra bundles with the same Dixmier-Douady class of finite order.

3.5. **Theorem.** (cf. [13], Theorem 3.5) For any \( L \)-module \((E, \varepsilon, \mathcal{U})\) the above construction defines the equivalence \( E_* \) between the category of \( L \)-modules and the category of \( \text{End}(E) \)-modules, hence an isomorphism between their \( K \)-theories.

This generalizes the equivalence between modules over stably trivial bundle gerbe and vector bundles given by any trivialization. In particular, any choice of \( L \)-module \( E \) gives rise to a particular equivalence between \( L \)-modules and \( \text{End}(E) \)-modules.
3.5. Morita bundle gerbe modules.

3.6. Definition. A (left) module $(E, \varepsilon, U)$ over a Morita bundle gerbe $(A, M, \theta, U)$ is a collection of $A_\alpha$-modules $E_\alpha \to U_\alpha$ equipped with isomorphisms $\varepsilon_{\alpha\beta} : \beta M_\alpha \otimes E_\alpha \to E_\beta$ over $U_{\alpha\beta}$ such that diagrams

\[
\begin{array}{c}
\gamma M_\beta \otimes_\beta M_\alpha \otimes E_\alpha \xrightarrow{\varepsilon_{\alpha\beta}} \gamma M_\beta \otimes E_\beta \\
\theta_{\alpha\beta\gamma} \otimes \varepsilon_{\alpha\beta}
\end{array}
\]

over $U_{\alpha\beta\gamma}$ commute.

4. Morita 2-bundle gerbes

4.1. Definition. A Morita 2-bundle gerbe (2-MBG for short) $(A, M, \theta, U)$ is the following collection of data. First, over all $U_{\alpha\beta}$ we have matrix algebra bundles $A_{\alpha\beta} \to U_{\alpha\beta}$. Second, over triple overlaps there are $(A_{\alpha\gamma}|U_{\alpha\gamma}|A_{\beta\gamma}|U_{\alpha\gamma}|A_{\beta\gamma}|U_{\alpha\gamma})$-bimodules $M_{\alpha\beta\gamma} \to U_{\alpha\beta\gamma}$ that are Morita equivalences $A_{\alpha\beta} \otimes A_{\beta\gamma} \to A_{\alpha\gamma}$.

Then over fourfold overlaps we have diagrams

\[
\begin{array}{c}
A_{\alpha\beta} \otimes A_{\beta\gamma} \otimes A_{\gamma\delta} \xrightarrow{A_{\alpha\beta} \otimes M_{\beta\gamma}} A_{\alpha\beta} \otimes A_{\beta\delta} \\
M_{\alpha\beta\gamma} \otimes A_{\gamma\delta} \xrightarrow{M_{\alpha\beta\gamma}} A_{\alpha\beta\delta}
\end{array}
\]

which commutes up to isomorphisms $\vartheta_{\alpha\beta\gamma\delta}$, i.e.,

\[
\vartheta_{\alpha\beta\gamma\delta} : A_{\alpha\beta} M_{\alpha\beta\gamma} \otimes A_{\gamma\delta} \xrightarrow{(A_{\alpha\beta} \otimes A_{\beta\gamma} \otimes A_{\gamma\delta}) A_{\alpha\beta} \otimes A_{\gamma\delta}}
\]

are $(A_{\alpha\delta}, A_{\alpha\beta} \otimes A_{\beta\gamma} \otimes A_{\gamma\delta})$-bimodule isomorphisms. At last, over fivefold overlaps $\vartheta$'s satisfy the pentagon identity

\[
\vartheta_{\beta\gamma\delta\epsilon} \vartheta_{\alpha\beta\gamma\delta} \vartheta_{\alpha\beta\gamma\delta} = \vartheta_{\alpha\beta\gamma\delta} \vartheta_{\alpha\gamma\delta\epsilon}.
\]

Note that

\[
(A_{\alpha\beta} \otimes A_{\beta\gamma}) \otimes A_{\gamma\delta} \xrightarrow{\vartheta_{\alpha\beta\gamma\delta}} A_{\alpha\beta} \otimes (A_{\beta\gamma} \otimes A_{\gamma\delta})
\]

(different order of performing the tensor product), so the last identity corresponds to the diagram

\[
\begin{array}{c}
(A_{\alpha\beta} \otimes A_{\beta\gamma}) \otimes A_{\gamma\delta} \xrightarrow{\vartheta_{\alpha\beta\gamma\delta}}
\end{array}
\]

4.2. Remark. Let us explain the heuristics behind this definition. One may think about a Morita 2-bundle gerbe as a cocycle with values in the Brauer-Picard 3-group (or, equivalently, as a functor from the Čech groupoid associated with the open cover $U$ to this 3-group).

Note also that in case of 2-MBG's the role of dual vector space and dual linear isomorphisms are played by opposite algebra and dual bimodules respectively.

There are also some consequences from the definition that are counterparts for the ones for bundle gerbes which allows us to coherently identify $A_{\alpha\alpha}$, $A_{\alpha\beta}$ and $M_{\alpha\beta\gamma}$ with $U_{\alpha\alpha} \times \mathbb{C}$, $A_{\beta\alpha}$, and $M_{\beta\alpha\gamma}$, respectively ($A^\circ$ denotes the opposite algebra and $M^*$ the dual bimodule). More precisely, put $A := A_{\alpha\beta} \otimes A_{\beta\gamma}$, $B := A_{\alpha\gamma}$, $M := M_{\alpha\beta\gamma}$. Then $BM_A : A \to B$. By definition, $AN_B := M^* \cong \text{Hom}_A(M_A, A_A)$. Then

\[
AN_B = B^\circ N_A : A^\circ \to B^\circ
\]

4here and below we shall omit annoying explicit indication for restrictions to subsets.
and we have:
\[ A^0 = A^0_{\alpha\beta} \otimes A^0_{\beta\gamma} \cong A^0_{\beta\alpha} \otimes A^0_{\gamma\beta} \cong A_{\gamma\beta} \otimes A_{\beta\alpha} M_{\beta\alpha} A_{\alpha\gamma} \cong A^0_{\gamma\alpha} = B^0. \]

4.2. The category of Morita 2-bundle gerbes. 2-MBG’s over \( X \) form a weak monoidal 3-groupoid, \( 2\text{-MBG}(X) \). Let us define its 1-, 2- and 3-morphisms.

4.3. Definition. A 1-morphism \((A, M, \vartheta, U) \rightarrow (A', M', \vartheta', U)\) is the following collection of data \((B, N, \varphi)\).

First, we have matrix algebra bundles \( B_\alpha \rightarrow U_\alpha \). Second, over twofold overlaps we have \((B_\alpha \otimes A_{\alpha\beta}', A_{\alpha\beta} \otimes B_\beta)\)-bimodules \( N_{\alpha\beta} \rightarrow U_{\alpha\beta} \) which are Morita equivalences

\[ N_{\alpha\beta} : A_{\alpha\beta} \otimes B_\beta \rightarrow B_\alpha \otimes A_{\alpha\beta}'. \]

Third, over threefold overlaps we have diagrams

\[
\begin{array}{c}
\xymatrix{ & A_{\alpha\beta} \otimes B_\beta \otimes A_{\beta\gamma}' \ar[rr]_{1 \otimes N_{\gamma\alpha}} \ar[dd]_{N_{\alpha\beta} \otimes 1} & & A_{\alpha\beta} \otimes A_{\beta\gamma} \otimes B_{\gamma} \ar[rr]_{1 \otimes M_{\beta\alpha}} \ar[dd]_{M_{\beta\alpha} \otimes 1} & & A_{\alpha\gamma} \otimes B_{\gamma} \ar[dd]_{N_{\alpha\gamma}} \\
B_\alpha \otimes A_{\alpha\beta}' \otimes A_{\beta\gamma}' & & B_\alpha \otimes A_{\alpha\gamma}' & & B_\alpha \otimes A_{\alpha\gamma}' \\
& & & & \\
\end{array}
\]

and an isomorphism of \((A_{\alpha\beta} \otimes A_{\beta\gamma} \otimes B_\gamma, B_\alpha \otimes A_{\alpha\gamma}')\)-bimodules

\[ \varphi_{\alpha\beta\gamma} : (B_\alpha \otimes M_{\alpha\beta\gamma})_{B_{\alpha} \otimes A_{\alpha\beta}' \otimes A_{\beta\gamma}'} \rightarrow (N_{\alpha\beta} \otimes A_{\beta\gamma}')_{A_{\alpha\beta} \otimes B_\beta \otimes A_{\gamma\beta}} \rightarrow (A_{\alpha\beta} \otimes N_{\beta\gamma}) \]

satisfying the obvious relations over four-fold overlaps.

Note that the definition of 1-morphisms is nothing but the definition of equivalent cocycles (with values in the Brauer-Picard 3-group in our case).

There is the obvious definition of the composition of 1-morphisms and one can verify that it is well defined. In particular, for \((B, N, \varphi) : (A, M, \vartheta, U) \rightarrow (A', M', \vartheta', U)\) and \((C, P, \psi) : (A', M', \vartheta', U) \rightarrow (A'', M'', \vartheta'', U)\) we have

\[ A_{\alpha\beta} \otimes B_\beta \otimes C_\beta \xrightarrow{N_{\alpha\beta} \otimes 1} B_\alpha \otimes A_{\alpha\beta}' \otimes C_\beta \xrightarrow{1 \otimes P_{\alpha\beta}} B_\alpha \otimes C_\alpha \otimes A_{\alpha\beta}'. \]

and the composition has the form \((D, Q, \chi)\), where

\[ D = \{D_\alpha\}, \quad D_\alpha := B_\alpha \otimes C_\alpha, \quad Q = \{Q_{\alpha\beta}\}, \quad Q_{\alpha\beta} := (B_\alpha \otimes P_{\alpha\beta})_{B_{\alpha} \otimes A_{\alpha\beta}' \otimes C_{\beta}} \otimes (N_{\alpha\beta} \otimes C_{\beta}) . \]

4.4. Definition. A 2-morphism \((P, \chi) : (B, N, \varphi) \Rightarrow (B', N', \varphi')\), where \((B, N, \varphi)\), \((B', N', \varphi')\) are 1-morphisms \((A, M, \vartheta, U) \rightarrow (A', M', \vartheta', U)\) consists of \((B_\alpha', B_{\alpha\beta})\)-bimodules \( P_\alpha \) such that diagrams

\[
\begin{array}{c}
\xymatrix{ A_{\alpha\beta} \otimes B_\beta \ar[rr]^{N_{\alpha\beta}} \ar[d]_{1 \otimes P_{\beta}} & & B_\alpha \otimes A_{\alpha\beta}' \ar[d]_{P_{\alpha} \otimes 1} \\
A_{\alpha\beta} \otimes B_{\beta}' \ar[r]_{N_{\alpha\beta}} & B_{\alpha}' \otimes A_{\alpha\beta}' \\
\end{array}
\]

commute up to isomorphisms \( \chi_{\alpha\beta} \), i.e.

\[ \chi_{\alpha\beta} : N'_{\alpha\beta} \otimes A_{\alpha\beta}' P_{\alpha\beta} B_{\alpha\beta} \cong (P_{\beta} \otimes A_{\alpha\beta}') B_{\alpha\beta} A_{\alpha\beta}' \]

is an isomorphism of \((B_\alpha' \otimes A_{\alpha\beta}', A_{\alpha\beta} \otimes B_\beta)\)-bimodules. There are further relations which are obvious.

3-morphisms between 2-morphisms \( A \rightarrow A' \) are isomorphisms commuting with all structure maps. So every 3-morphism is invertible by definition. Clearly that every 2-morphism is invertible up to 3-morphism.

The composition of 1-morphisms is associative only up to 2-morphisms and we obtain a weak 3-category \( 2\text{-MBG}(X) \) of Morita 2-bundle gerbes over \( X \).
4.3. 3-groupoid of Morita 2-bundle gerbes. Note that any 1-morphism is invertible (up to 2-morphism). Let us briefly describe the construction of weak inverse \((C, P, \psi)\) for

\[(B, N, \varphi): (A, M, \vartheta, U) \to (A', M', \vartheta', U).\]

Put \(C_\alpha := B_\alpha^\alpha\), \(P_{\alpha\beta} := N_{\beta\alpha}\). Note that

\[N_{\beta\alpha} : A_{\beta\alpha} \otimes B_{\alpha} \to B_{\beta} \otimes A_{\beta\alpha}^\alpha,\]

i.e.

\[N_{\beta\alpha} : B_\beta^\alpha \otimes A_{\beta\alpha}^\alpha \to A_{\beta\alpha}^\alpha \otimes B_{\beta},\]

i.e.

\[N_{\beta\alpha} : A_{\beta\alpha}^\alpha \otimes B_{\beta} \to B_{\alpha} \otimes A_{\beta\alpha}.\]

So we have

\[A_{\alpha\beta} \otimes B_{\beta} \otimes B_{\beta}^\alpha \xrightarrow{N_{\alpha\beta\gamma}} B_{\alpha} \otimes A_{\alpha\beta} \otimes B_{\beta} \xrightarrow{1 \otimes P_{\alpha\beta}} B_{\alpha} \otimes B_{\alpha}^\alpha \otimes A_{\alpha\beta}.\]

Now put \(Q_{\alpha\beta} := (B_{\alpha} \otimes P_{\alpha\beta}) \otimes_{B_{\alpha} \otimes A_{\alpha\beta} \otimes C_{\beta}} (N_{\alpha\beta} \otimes C_{\beta}).\) Then we have the diagram

\[
\begin{array}{ccc}
A_{\alpha\beta} \otimes (B_{\beta} \otimes B_{\beta}^\alpha) & \xrightarrow{Q_{\alpha\beta}} & (B_{\alpha} \otimes B_{\alpha}^\alpha) \otimes A_{\alpha\beta} \\
1 \otimes R_{\alpha\beta} & & R_{\alpha\beta} \otimes 1 \\
\end{array}
\]

where \(R_{\alpha}, R_{\beta}\) are canonical Morita equivalences, etc.

Thus we see that the 3-category 2-MBG\((X)\) is a weak 3-groupoid.

4.4. Weak 4-group of Morita 2-bundle gerbes. There is yet another obvious operation on Morita 2-bundle gerbes, their tensor product, which equips the category 2-MBG\((X)\) with the structure of a monoidal category.

This way, we have defined a monoidal 3-category 2-MBG\((X)\) of Morita 2-bundle gerbes. In particular, its unit object is the obvious strictly trivial Morita 2-bundle gerbe \(T\).

One can say even more about the monoidal 3-category 2-MBG\((X)\): every its object is invertible up to 1-morphism.

Now using a standard trick \([5]\), this monoidal 3-category 2-MBG\((X)\) can be reinterpreted as a weak 4-groupoid with one object, i.e. a weak 4-group whose 1-morphisms are Morita 2-bundle gerbes (with strictly 1-morphism).

4.5. Commutative Morita 2-bundle gerbes. Consider a particular case when all algebras are one-dimensional, i.e. isomorphic to \(C\). So over all double overlaps we have trivial bundle with fiber the field \(C\). Then over threefold overlaps we have line bundles ("bimodules") \(L_{\alpha\beta} \to U_{\alpha\beta}\) and over fourfold overlaps we have isomorphisms

\[\vartheta_{\alpha\beta\gamma\delta} : L_{\alpha\beta\gamma} \otimes L_{\alpha\beta\gamma} \to L_{\alpha\beta\gamma} \otimes L_{\alpha\beta\gamma},\]

satisfying pentagon identity over fivefold overlaps. So this is nothing but a 2-bundle gerbe.

Such commutative Morita 2-bundle gerbes over \(X\) form a full subcategory 2-BG\((X)\) in 2-MBG\((X)\). One can show that this inclusion is an equivalence of categories.

Imitating the construction of Dixmier-Douady class (see subsection 2.7) one can see that such commutative Morita 2-bundle gerbes over \(X\) are classified up to equivalence (= 1-morphisms) by the group \(H^4(X, \mathbb{Z})\) (in particular, the cocycle condition follows from the pentagon identity). This also gives the classification of Morita 2-bundle gerbes up to equivalence.

4.6. The group of self-equivalences of the trivial ABG. It follows from definition \([13]\) that a 1-morphism from the strictly trivial 2-MBG \(T\) to itself is the following collection of data. First, we have algebra bundles \(B_{\alpha} \to U_{\alpha}\), then \((B_{\alpha}, B_{\beta})\)-bimodules \(N_{\alpha\beta}\), then bimodule isomorphisms \(\varphi_{\alpha\beta\gamma} : N_{\alpha\beta} \otimes N_{\beta\gamma} \to N_{\alpha\gamma}\) corresponding to diagrams

\[
\begin{array}{ccc}
B_{\gamma} & \xrightarrow{N_{\alpha\gamma}} & B_{\beta} \\
\downarrow_{N_{\alpha\gamma}} & & \downarrow_{N_{\beta\gamma}} \\
B_{\beta} & \xleftarrow{N_{\alpha\beta}} & B_{\alpha} \\
\end{array}
\]

which satisfy relations

\[\varphi_{\alpha\beta\delta} \circ (1 \otimes \varphi_{\beta\gamma\delta}) = \varphi_{\alpha\gamma\delta} \circ (\varphi_{\alpha\beta\gamma} \otimes 1)\]
over four-fold overlaps. The last relations correspond to diagrams
\[
\begin{align*}
N_{\alpha\beta} \otimes N_{\beta\gamma} \otimes N_{\gamma\delta} & \xrightarrow{1 \otimes \varphi_{\beta\gamma}} N_{\alpha\beta} \otimes N_{\beta\delta} \\
\varphi_{\alpha\beta\gamma} & \\
N_{\alpha\gamma} \otimes N_{\gamma\beta} & \xrightarrow{1 \otimes \varphi_{\beta\gamma}} N_{\alpha\beta}
\end{align*}
\]
This is exactly a Morita bundle gerbe (cf. subsection 2.9). Moreover, it follows from definition 4.4 that 2-morphisms between 1-automorphisms of the strictly trivial 2-MBG coincide with 1-morphisms between Morita bundle gerbes. So the group of autoequivalences of the trivial object is the 3-group of Morita bundle gerbes. But for any such a gerbe there is a 2-morphism to a “common” bundle gerbe which is unique up to equivalence. Thus we see that the group of equivalence classes (up to 2-morphisms) of 1-morphisms of the strictly trivial 2-MBG is isomorphic to the Brauer group \( Br(X) \cong H^3(X, \mathbb{Z}) \).

4.7. Trivializations.

4.5. Definition. A right trivialization of an 2-MBG \( A = (A, M, \vartheta, U) \) is a 1-morphism \((B, N, \varphi): A \rightarrow T\) to a strictly trivial 2-MBG \( T \). Similarly, a left trivialization of \( A \) is a 1-morphism \((C, P, \psi): T \rightarrow A\).

It immediately follows from the definition that such a right trivialization \((B, N, \varphi)\) consists of a collection of algebra bundles \( B_{\alpha} \rightarrow U_{\alpha}, (B_{\alpha}, A_{\alpha\beta} \otimes B_{\beta}) \)-bimodules \(N_{\alpha\beta}: A_{\alpha\beta} \otimes B_{\beta} \rightarrow B_{\alpha}\) and isomorphisms
\[
\varphi_{\alpha\beta\gamma}: N_{\alpha\beta} \otimes (A_{\alpha\beta} \otimes N_{\beta\gamma}) \Rightarrow N_{\alpha\gamma} \otimes (M_{\alpha\beta\gamma} \otimes B_{\gamma})
\]
over \( U_{\alpha\beta\gamma} \) corresponding to the diagram
\[
\begin{align*}
A_{\alpha\beta} \otimes A_{\beta\gamma} \otimes B_{\gamma} & \xrightarrow{id \otimes N_{\beta\gamma}} A_{\alpha\beta} \otimes B_{\beta} \\
M_{\alpha,\beta,\gamma} \otimes id & \\
A_{\alpha\gamma} \otimes B_{\gamma} & \xrightarrow{N_{\alpha,\gamma}} B_{\alpha}
\end{align*}
\]
and satisfying the obvious relations over four-fold overlaps. Similarly for a left trivialization.

Assume now that there are right \((B, N, \varphi): A \rightarrow T\) and left \((C, P, \psi): T \rightarrow A\) trivializations of \( A := (A, M, \vartheta, U) \). Over \( U_{\alpha\beta} \) we have Morita equivalences
\[
C_{\beta} \otimes B_{\beta} \xleftrightarrow{P_{\alpha,\beta} \otimes id} C_{\alpha} \otimes A_{\alpha\beta} \otimes B_{\beta} \xrightarrow{1 \otimes N_{\beta}} C_{\alpha} \otimes B_{\alpha}.
\]

Over threefold overlaps \( U_{\alpha\beta\gamma} \) we have diagrams
\[
\begin{align*}
C_{\beta} \otimes B_{\beta} & \xrightarrow{id \otimes N_{\beta\gamma}} C_{\beta} \otimes A_{\beta\gamma} \otimes B_{\gamma} \xrightarrow{id \otimes N_{\beta\gamma}} C_{\alpha} \otimes A_{\alpha\beta} \otimes B_{\beta} \\
P_{\alpha,\beta,\gamma} \otimes id & \\
C_{\gamma} \otimes B_{\gamma} & \xrightarrow{id \otimes M_{\alpha,\beta,\gamma} \otimes id} C_{\alpha} \otimes A_{\alpha\beta} \otimes B_{\beta} \xrightarrow{id \otimes N_{\beta}} C_{\alpha} \otimes B_{\alpha}
\end{align*}
\]
which are commutative up to isomorphisms.

Hence the algebra bundles \( C_{\alpha} \otimes B_{\alpha} \) together with the Morita-equivalences \( Q_{\alpha,\beta} := (P_{\alpha,\beta} \otimes id) \circ (id \otimes N_{\alpha\beta})^{-1} \) and isomorphisms form a Morita bundle gerbe over \( X \). In other words, two trivializations of the same Morita 2-bundle gerbe differ by a Morita bundle gerbe. Note that the obtained result agrees with the previous category-theoretic arguments: the composition \( B \circ C: T \rightarrow T \) is a 1-automorphism of the strictly trivial Morita 2-bundle gerbe \( T \), i.e. a Morita bundle gerbe as we have already seen in subsection 4.7.
Let $2\text{-}MBG(X)$ be the group of equivalence classes of 2-MBG’s over $X$ (with respect to the tensor product). Clearly, the homotopy functor $X \mapsto 2\text{-}MBG(X)$ is representable. One can repeat the arguments in the proof of Theorem 2.12 and show that $2\text{-}MBG(\Sigma X) \cong Br(X)$. Clearly, $2\text{-}MBG(X) \cong [X, K(\mathbb{Z}, 4)]$.

Thus we see that the theory of Morita (2)-bundle gerbes is equivalent to the theory of conventional (2)-bundle gerbes. The explanation of this result comes from the fact that the automorphism group of an invertible $(M_k(\mathbb{C}), M_l(\mathbb{C}))$-bimodule is the commutative group $\mathbb{C}^\ast$ and therefore our cocycles $\vartheta$’s take values in it.

4.6. Remark. It is not difficult to formally define the notion of a module over a 2-MBG. But it seems that they can not be implemented by finite-dimensional bundles (excepting trivial cases).

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