ON THE BLASCHKE-PETKANTSCHIN FORMULA
AND DRURY’S IDENTITY

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Abstract. The Blaschke-Petkantschin formula is a variant of the polar decomposition of the $k$-fold Lebesgue measure on $\mathbb{R}^n$ in terms of the corresponding measures on $k$-dimensional linear subspaces of $\mathbb{R}^n$. We suggest a new elementary proof of this formula and discuss its connection with the celebrated Drury’s identity that plays a key role in the study of mapping properties of the Radon-John $k$-plane transforms. We give a new derivation of this identity and provide it with precise information about constant factors and the class of admissible functions.

1. Introduction

The classical Blaschke-Petkantschin formula gives decomposition of the Euclidean measure on $k$ copies of $\mathbb{R}^n$ into the corresponding measures on $k$-dimensional subspaces of $\mathbb{R}^n$ with the relevant Jacobian; see the case $q = k$ in (2.5) below. After the pioneering works by Blaschke [3] and Petkantschin [21], this formula and its modifications arise in different aspects of Analysis, Integral Geometry, Multivariate Statistics, and Probability; see, e.g., [1, 5, 15, 17, 18, 25, 26, 27], to mention a few. The books [26, 27] contain a nice history of the subject.

The simplest case of the Blaschke-Petkantschin formula corresponds to $k = 1$ when the standard polar decomposition yields

$$\int_{\mathbb{R}^n} f(x) \, dx = \int_{S^{n-1}} d\theta \int_0^\infty f(r\theta) r^{n-1} \, dr = \frac{1}{2} \int_{S^{n-1}} d\theta \int_{-\infty}^\infty f(r\theta) |r|^{n-1} \, dr. \quad (1.1)$$

This gives

$$\int_{\mathbb{R}^n} f(x) \, dx = \frac{1}{2\sigma_{n-1}} \int_{G_{n,1}} dl \int_{\ell} f(x)|x|^{n-1} \, dx. \quad (1.2)$$

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where $G_{n,1}$ is the Grassmann manifold of lines $\ell$ through the origin, $d\ell$ is the standard probability measure on $G_{n,1}$, $\sigma_{n-1}$ denotes the area of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$, and $d\ell x$ stands for the Lebesgue measure on $\ell$.

The present article is influenced by intimate connection between the Blaschke-Petkantschin formula and its modification, known as "Drury's identity", which was independently discovered by Drury [7] in his study of norm estimates for the $k$-plane Radon-John transform; see the case $k + \ell + 1 = n$ in (3.10). This remarkable connection was pointed out by Baernstein II and Loss [1]. A number of breakthrough applications of Drury's identity to difficult problems related to $L^p$-$L^q$ estimates for $k$-plane transforms can be found in the works by Christ [4], Drouot [6], Flock [9], Bennett, Bez, Flock, Gutierrez, and Iliopoulou [2].

Another motivation for writing this article was Drury's observation [7] that his formula has a lot in common with analytic continuation of Riesz distributions on matrix spaces. These distributions and the corresponding potential operators were studied by Stein [28], Gelbart [13], Khekalo [16], Raïs [22], Rubin [24], where one can find further references.

Most of known proofs of the Blaschke-Petkantschin formula employ either a super-powerful machinery of differential forms (see, e.g., [25, 17, 18]) or a clever inductive argument. In Section 2 we present an elementary proof of a slightly more general version of this formula, following the same idea as in (1.1)-(1.2) and using the well-known polar decomposition of matrices. As a consequence, in Section 3 we formulate the affine version of this formula previously known under more restrictive assumptions, and obtain a slight generalization of Drury's identity with sharp constant and precise information about the class of admissible functions. This constant was not specified in [7, 1]. It was later obtained in [2] in a conceptually more complicated way than we do.

After finishing the first version of the paper, the author became aware of close works by Moghadasi [19] and Forrester [10] devoted to application of the matrix polar decomposition to derivation of the Blaschke-Petkantschin formula. Our reasoning essentially differs from [10, 19].

2. Derivation of the Blaschke-Petkantschin Formula

The reasoning in (1.1)-(1.2) extends to functions $F(x) = F(x_1, \ldots, x_k)$ on $(\mathbb{R}^n)^k$ if the latter is treated as the space $\mathcal{M}_{n,k}$ of real matrices having
n rows and k columns and the polar decomposition \(dx = r^{n-1} dr d\theta\) is replaced by its analogue for \(d\mathbf{x}\). Below we recall basic facts.

If \(\mathbf{x} = (x_1, \ldots, x_k) = (x_{i,j}) \in \mathfrak{M}_{n,k}\), then \(d\mathbf{x} = \prod_{i=1}^n \prod_{j=1}^k dx_{i,j}\) is the elementary volume in \(\mathfrak{M}_{n,k}\). In the following \(\mathbf{x}^T\) denotes the transpose of \(\mathbf{x}\) and \(I_k\) is the identity \(k \times k\) matrix. Given a square matrix \(\mathbf{a}\), we denote by \(|\mathbf{a}|\) the absolute value of the determinant of \(\mathbf{a}\); \(\text{tr}(\mathbf{a})\) stands for the trace of \(\mathbf{a}\).

Let \(\mathcal{S}_k\) be the space of \(k \times k\) real symmetric matrices \(\mathbf{s} = (s_{i,j}), s_{i,j} = s_{j,i}\). It is a measure space isomorphic to \(\mathbb{R}^{k(k+1)/2}\) with the volume element \(d\mathbf{s} = \prod_{i \leq j} ds_{i,j}\). We denote by \(P_k\) the cone of positive definite matrices in \(\mathcal{S}_k\). For \(n \geq k\), let \(V_{n,k} = \{\mathbf{v} \in \mathfrak{M}_{n,k} : \mathbf{v}^T\mathbf{v} = I_k\}\) be the Stiefel manifold of orthonormal \(k\)-frames in \(\mathbb{R}^n\). If \(n = k\), then \(V_{n,n} = O(n)\) is the orthogonal group in \(\mathbb{R}^n\). The group \(O(n)\) acts on \(V_{n,k}\) transitively by the rule \(g : \mathbf{v} \rightarrow g\mathbf{v}, \ g \in O(n)\), in the sense of matrix multiplication. We fix the corresponding invariant measure \(d\mathbf{v}\) on \(V_{n,k}\) normalized by

\[
\sigma_{n,k} \equiv \int_{V_{n,k}} d\mathbf{v} = \frac{2^k \pi^{nk/2}}{\Gamma_k(n/2)}, \tag{2.1}
\]

where

\[
\Gamma_k(\alpha) = \int_{P_k} \exp(-\text{tr}(\mathbf{r})) |\mathbf{r}|^{\alpha-(k+1)/2} d\mathbf{r} = \pi^{k(k-1)/4} \prod_{j=0}^{k-1} \Gamma(\alpha-j/2) \tag{2.2}
\]

is the Siegel gamma function associated to the cone \(P_k\) [20, p. 62]. This integral converges absolutely if and only if \(\Re \alpha > (k-1)/2\).

**Lemma 2.1.** (polar decomposition). Let \(\mathbf{x} \in \mathfrak{M}_{n,k}, \ n \geq k\). If \(\text{rank}(\mathbf{x}) = k\), then \(\mathbf{x}\) is uniquely decomposed as

\[\mathbf{x} = \mathbf{v} \mathbf{r}, \quad \mathbf{v} \in V_{n,k}, \quad \mathbf{r} = (\mathbf{x}^T\mathbf{x})^{1/2} \in P_k,\]

and for \(F \in L^1(\mathfrak{M}_{n,k})\) we have

\[
\int_{\mathfrak{M}_{n,k}} F(\mathbf{x}) \, d\mathbf{x} = 2^{-k} \int_{V_{n,k}} d\mathbf{v} \int_{P_k} F(\mathbf{v}\mathbf{r}) |\mathbf{r}|^{n-k-1} \, d\mathbf{r}. \tag{2.3}
\]

This statement and its generalizations can be found in different sources, see, e.g., [14, p. 482], [20, pp. 66, 591], [8, p. 130], [29, Lemma 3.1]. For \(\mathbf{x} \in \mathfrak{M}_{n,k}\), we set

\[
|\mathbf{x}|_k = (\det(\mathbf{x}^T\mathbf{x}))^{1/2}, \tag{2.4}
\]

which is the volume of the parallelepiped spanned by the column-vectors \(x_1, \ldots, x_k\) of the matrix \(\mathbf{x}\) [11, p. 251].
Our aim is to replace integration over the Stiefel manifold in (2.3) by integration over the Grassmann manifold $G_{n,k}$ of $k$-dimensional linear subspaces of $\mathbb{R}^n$. We perform this replacement in a slightly more general fashion. Given $\xi \in G_{n,k}$, we denote by $d\xi$ the $O(n)$-invariant probability measure on $G_{n,k}$ and write $\xi^q$ for the collection of $q$ copies of $\xi$.

**Theorem 2.2.** Let $1 \leq q \leq k \leq n$. If $F \in L^1(\mathfrak{m}_{n,q})$, then

$$
\int_{\mathfrak{m}_{n,q}} F(x) \, dx = \frac{\sigma_{n,q}}{\sigma_{k,q}} \int_{G_{n,k}} d\xi \int_{\xi^q} F(x) \left| x_q^{n-k} d\xi x \right|,
$$

where $x = (x_1, \ldots, x_q)$ and $d\xi x = d\xi x_1 \ldots d\xi x_q$ stands for the usual Lebesgue integration over $\xi^q$.

**Proof.** The case $q = k$ in (2.5) is the classical Blaschke-Petkantschin formula. To prove the theorem, we denote by $I$ the left-hand side of (2.5). By Lemma 2.1,

$$
I = 2^{-q} \int_{V_{n,q}} dv \int_{P_q} F(vr) |r|^{n-q-1} dr
$$

$$
= 2^{-q} \sigma_{n,q} \int_{O(n)} d\alpha \int_{P_q} F\left(\alpha \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} r\right) |r|^{n-q-1} dr.
$$

We replace $\alpha$ by

$$
\alpha \begin{bmatrix} \beta & 0 \\ 0 & I_{n-k} \end{bmatrix}, \quad \beta \in O(k),
$$

and integrate in $\beta$ with respect to the standard probability measure on $O(k)$. Changing the order of integration, we get

$$
I = 2^{-q} \sigma_{n,q} \int_{O(n)} d\alpha \int_{O(k)} d\beta \int_{P_q} F\left(\alpha \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix} r\right) |r|^{k+q-k-1} |r|^{n-k} dr
$$

$$
= \frac{2^{-q} \sigma_{n,q}}{\sigma_{k,q}} \int_{O(n)} d\alpha \int_{V_{k,q}} d\beta \int_{P_q} F\left(\alpha \begin{bmatrix} ur & 0 \\ 0 & 0 \end{bmatrix} r\right) |r|^{k+q-k-1} |r|^{n-k} dr.
$$
Now we apply Lemma 2.1 again, but in the opposite direction, to obtain

\[ I = \sigma_{n,q} \int_\mathcal{O} \left( n \right) d\alpha \int M_{k,q} F(\alpha) |y|^{n-k} dy, \quad (2.6) \]

\[ = \sigma_{n,q} \int \frac{d\xi}{\sigma_{k,q} \mathcal{O}(n)} \left( \int \cdots \int \right) F(\xi) |x|^{n-k} \, d\xi_1 \cdots d\xi_q, \quad (2.7) \]

which gives (2.5).

\[ \square \]

Remark 2.3. For the case \( q = k \), a certain analogue of (2.6) was obtained by Forrester [10, Proposition 4] in different notation, including real, complex, and quaternionic cases. However, Forrester’s result is somewhat incomplete because it does not contain the transition from the integration over square matrices to integration over subspaces that should be properly defined in the complex and quaternionic cases; cf. the transition from (2.6) to (2.7). In the real case and \( q = k \), this transition is performed by Moghadasi [19, p. 323].

3. Affine Blaschke-Petkantschin Formula and Drury’s Identity

The affine Blaschke-Petkantschin formula (see Theorem 3.1 below) can be easily derived from (2.5). For the sake of completeness, we reproduce this derivation in Appendix, following the reasoning from Gardner [12, Lemma 5.5] and removing unnecessary restrictions that were made in [12]. Once (2.5) is established, the derivation of its affine version needs actually nothing but Fubini’s Theorem.

We first observe that the volume \( |x|_q \) of the parallelepiped in (2.5) can be replaced by the volume \( \Delta_q(x) \) of the convex hall of \( \{0, x_1, \ldots, x_q\} \) by the known formula

\[ \Delta_q(x) = \frac{1}{q!} |x|_q. \quad (3.1) \]

Given \( \tilde{x} = (x_0, x_1, \ldots, x_q) \in \mathfrak{m}_{n,q+1} \), let

\[ \Delta_q(\tilde{x}) \equiv \Delta_q(x_0, x_1, \ldots, x_q) \quad (3.2) \]

be the \( q \)-dimensional volume of the convex hall of \( \{x_0, x_1, \ldots, x_q\} \). We denote by \( \mathcal{A}_{n,k} \) the Grassmannian bundle of affine \( k \)-planes \( \tau \) with the standard measure \( d\tau \); see, e.g., [23].
Theorem 3.1. Let \( 1 \leq q \leq k \leq n \). If \( F \in L^1(\mathfrak{M}_{n,q+1}) \), then
\[
\int_{\mathfrak{M}_{n,q+1}} F(\tilde{x}) \, d\tilde{x} = c \int_{\mathcal{A}_{n,k}} \int_{\tau^{q+1}} F(\tilde{x}) \Delta_q^{n-k}(\tilde{x}) \, d_\tau \tilde{x},
\]
where \( \tau^{q+1} \) is the collection of \( q+1 \) copies of \( \tau \),
\[
c = \frac{(q!)^{n-k} \sigma_{n,q}}{\sigma_{k,q}},
\]
d_\tau \tilde{x} = d_\tau x_0 \ldots d_\tau x_q \) stands for the usual Lebesgue integration over \( \tau^{q+1} \).

Changing notation for the function \( F \) in (2.5) and (3.3), we obtain the following corollary.

Corollary 3.2. Let \( 1 \leq q \leq k \leq n \). If \( F(x)|x|^{k-n} \in L^1(\mathfrak{M}_{n,q}) \), then
\[
\int_{G_{n,k}} d\xi \int F(x) \, d_\xi x = \frac{\sigma_{k,q}}{\sigma_{n,q}} \int_{\mathfrak{M}_{n,q}} F(x)|x|^{k-n} \, dx.
\]

If \( F(\tilde{x}) \Delta_q^{k-n}(\tilde{x}) \in L^1(\mathfrak{M}_{n,q+1}) \), then
\[
\int_{\mathcal{A}_{n,k}} \int_{\tau^{q+1}} F(\tilde{x}) \, d_\tau \tilde{x} = \frac{(q!)^{k-n} \sigma_{k,q}}{\sigma_{n,q}} \int_{\mathfrak{M}_{n,q+1}} F(\tilde{x}) \Delta_q^{k-n}(\tilde{x}) \, d\tilde{x}.
\]

An alternative proof of (3.5) and (3.6) for nonnegative \( F \), but without explicit constant and the explicit assumption for \( F \), was given in [1, Section 5].

The right-hand sides of (3.5) and (3.6) can be treated as particular cases of the corresponding Riesz type distributions on matrix spaces. For example, the right-hand side of (3.5) agrees with the Riesz distribution
\[
\zeta_F(\alpha) = \int_{\mathfrak{M}_{n,q}} F(x)|x|^{a-n} \, dx.
\]

If \( F \) is smooth and rapidly decreasing as a function on \( \mathbb{R}^{nq} \), the integral (3.7) is absolutely convergent provided \( Re \alpha > q - 1 \) and extends meromorphically to all complex \( \alpha \); see [24, Lemma 4.2] for details.

Important additional features of (3.5) and (3.6) can be revealed if we choose \( F(\tilde{x}) = f_1(x_0), \ldots, f_q(x_q) \) and write the right-hand sides as the corresponding multilinear forms. Then (3.6) can be written in terms of the \( k \)-plane transforms of \( f_j \) defined by \( R_k f_j = \int_{\tau} f_j \) with
integration against the usual Lebesgue measure on \( \tau \). Specifically, for \( 1 \leq q \leq k \leq n \),
\[
\int_{A_{n,k}} \left[ \prod_{j=0}^{q} (R_k f_j)(\tau) \right] d\tau = \frac{(q!)^{k-n} \sigma_{k,q}}{\sigma_{n,q}} \left( \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \Delta^{k-n}_{q}(x_0, x_1, \ldots, x_q) \right)
\times \left\{ \prod_{j=0}^{q} f_j(x_j) \right\} dx_0 \ldots dx_q.
\]
(3.8)

This equality agrees with the formula (2.11) in Baernstein II and Loss [1]. If \( q = k \), it agrees with Lemma 1 of Drury [7]. However, in both works, the constant on the right-hand side and conditions for \( f_j \) are not specified.

Another consequence of (3.6) can be obtained if we set \( q = k < n \) and choose \( F(\tilde{x}) \equiv F(x_0, \ldots, x_k) \) in the form
\[
F(x_0, \ldots, x_k) = \left[ \prod_{j=0}^{k} f(x_j) \right] \left[ \prod_{j=k+1}^{k+\ell} \int_{\tau(x_0, \ldots, x_k)} f(y_j) d\tau y_j \right],
\]
(3.9)
where \( \tau(x_0, \ldots, x_k) \) is a \( k \)-plane containing the points \( x_0, x_1, \ldots, x_k \), \( f \) is a function on \( \mathbb{R}^n \), and \( \ell \) is a positive integer. The \( k \)-plane \( \tau(x_0, \ldots, x_k) \) is unique if \( x_0, x_1, \ldots, x_k \) are in general position. For every fixed \( \tau \in A_{n,k} \) and \( F \) of the form (3.9) we have
\[
\int_{\tau^{k+1}} F(\tilde{x}) d_{\tau} \tilde{x} = \left[ \prod_{j=0}^{k} \int_{\tau} f(x_j) d_{\tau} x_j \right] \left[ \prod_{j=k+1}^{k+\ell} \int_{\tau} f(y_j) d_{\tau} y_j \right] = \left[ (R_k f)(\tau) \right]^{k+\ell+1}.
\]
Hence (3.6) yields
\[
\int_{A_{n,k}} [(R_k f)(\tau)]^{k+\ell+1} d\tau = \frac{(k!)^{k-n} \sigma_{k,k}}{\sigma_{n,k}} \left( \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} f(x_0) \ldots f(x_k) \right) \Delta^{k-n}_{k}(x_0, \ldots, x_k) dx_0, \ldots, dx_k.
\]
(3.10)

This formula is well-justified provided that the right-hand side of it exists in the Lebesgue sense; cf. the condition \( F(\tilde{x}) \Delta^{k-n}_{q}(\tilde{x}) \in L^1(\mathbb{R}^n) \) for (3.6).

The case \( k + \ell + 1 = n \) in (3.10) is known as Drury’s identity; cf. formula (4) in [7], which is understood in the sense of analytic continuation according to Gelbart [13].
Concluding Remark. An analogue of the polar decomposition in Lemma 2.1 is known for complex and quaternionic matrices and in the context of formally real Jordan algebras; see, e.g., Zhang [29, Lemma 3.1], Moghadasi [19, Theorem 2.5], Forrester [10], Faraut and Trava glini [8, Section 4]. We believe that our proof of Theorem 2.2 and its consequences extend to the corresponding more general settings.

4. Appendix: Proof of Theorem 3.1

Fix any $x_0 \in \mathbb{R}^n$ and set $\tilde{x} = (x_0, x) = (x_0, x_1, \ldots, x_q) \in \mathfrak{M}_{n,q+1}$. We replace $F(x) \equiv F(x_1, \ldots, x_q)$ by $F(x_0, x_1 + x_0, \ldots, x_q + x_0)$ in (2.5) to get

$$\int_{\mathfrak{M}_{n,q}} F(x_0, x) \, dx = \int_{\mathfrak{M}_{n,q}} F(x_0, x_1 + x_0, \ldots, x_q + x_0) \, dx = \frac{\sigma_{n,q}}{\sigma_{k,q}} \int_{G_{n,k}} d\xi \int_{\mathfrak{G}_{q}} F(x_0, x_1 + x_0, \ldots, x_q + x_0) |x_q^{n-k}(x)| \, d\xi x.$$

Note that

$$\frac{1}{q!} |x_q| = \Delta_q(x) \equiv \Delta_q(0, x_1, \ldots, x_q) = \Delta_q(x_0, x_1 + x_0, \ldots, x_q + x_0);$$

see (3.1) and (3.2). Hence

$$\int_{\mathfrak{M}_{n,q}} F(x_0, x) \, dx = c \int_{G_{n,k}} d\xi \int_{\mathfrak{G}_{q}} F(x_0, x_1 + x_0, \ldots, x_q + x_0) \times [\Delta_q(x_0, x_1 + x_0, \ldots, x_q + x_0)]^{n-k} d\xi x,$$

where

$$c = \frac{(q!)^{n-k} \sigma_{n,q}}{\sigma_{k,q}}.$$

Now we integrate (4.1) in the $x_0$-variable and change the order of integration. This gives

$$\int_{\mathfrak{M}_{n,q+1}} F(\tilde{x}) \, d\tilde{x} = c \int_{G_{n,k}} d\xi \int_{\mathfrak{G}_{q}} d\tilde{x}_0 \int_{\mathfrak{G}_{q}} F(x_0, x_1 + x_0, \ldots, x_q + x_0) \times [\Delta_q(x_0, x_1 + x_0, \ldots, x_q + x_0)]^{n-k} d\xi x_1 \cdots d\xi x_q.$$
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Let \( x_0 = y_0 + z_0 \), where \( y_0 \in \xi \) and \( z_0 \in \xi^\perp \), so that \( \tau = \xi + z_0 \) is an affine \( k \)-plane. Then the right-hand side becomes

\[
c \int_{G_{n,k}} d\xi \int_{\xi^\perp} d\xi_0 \int_{\xi} d\eta_0 \int_{\xi^q} F(y_0 + z_0, x_1 + y_0 + z_0, \ldots, x_q + y_0 + z_0) \times [\Delta_q(y_0 + z_0, x_1 + y_0 + z_0, \ldots, x_q + y_0 + z_0)]^{n-k} d\xi x_1 \cdots d\xi x_q.
\]

Setting \( y_0 = \eta_0 \), \( x_1 + y_0 = \eta_1 \), \ldots, \( x_q + y_0 = \eta_q \), we write this expression as

\[
c \int_{G_{n,k}} d\xi \int_{\xi^\perp} d\xi_0 \int_{\xi} d\eta_0 \int_{\xi^q} F(\eta_0 + z_0, \eta_1 + z_0, \ldots, \eta_q + z_0) \times [\Delta_q(\eta_0 + z_0, \eta_1 + z_0, \ldots, \eta_q + z_0)]^{n-k} d\xi \eta_1 \cdots d\xi \eta_q
\]

\[
= c \int_{A_{n,k}} d\tau \int_{\tau^{q+1}} F(x_0, x_1, \ldots, x_q) [\Delta_q(x_0, x_1, \ldots, x_q)]^{n-k} d\tau x_1 \cdots d\tau x_q,
\]

which gives (3.3).

\[\square\]

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