A CLASS OF COUNTER-EXAMPLES TO THE HYPERSECTION PROBLEM BASED ON FORCING EQUATIONS

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Abstract. We give a class of three-dimensional Stein spaces W together with a hypersurface H, such that the complement W − H is not Stein, but such that for every analytic surface S ⊂ W the complement S − S ∩ H is Stein. This class is constructed using forcing equations and gives new counter-examples to the hypersection problem.

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0. Introduction. Let W be a complex Stein space of dimension ≥ 3 and let H ⊂ W be an analytic hypersurface, U = W − H. Suppose that for every analytic hypersurface S ⊂ W the intersection U ∩ S is Stein, is then U itself Stein? This question is called the hypersection problem, see [2] for a general treatment and related problems. The first counter-example to this question was given by Coltoiu and Diederich in [1], using the affine cone over the complement of two sections on some ruled surface over an elliptic curve. In this way they get a normal three-dimensional isolated singularity.

In this paper we present another class of three-dimensional Stein spaces W together with a hypersurface H fulfilling the assumptions in the hypersection problem, but not its conclusion. The class is constructed in the following way: we start with a two-dimensional normal affine cone X over a smooth projective curve and the vertex point P ∈ X. Suppose that we have three homogeneous functions f_1, f_2 and f_0 on X. Then, under suitable conditions, W = V(f_1 t_1 + f_2 t_2 + f_0) ⊂ X × ℂ^2 and the hypersurface H = p^{-1}(P) have the desired properties, see Theorem 7. These conditions reduce to numerical conditions, which are easily to verify, see Corollary 8 and Example 9.

1. Forcing equations. Let X be an irreducible normal complex Stein space of dimension d together with a point P ∈ X. Let f_1, . . . , f_n be holomorphic functions on X such that the common zero set of these functions is exactly P. Let f_0 be another holomorphic function vanishing at P. Then we consider the complex space W ⊂ X × ℂ^n defined by the equation f_1 t_1 + . . . + f_n t_n + f_0 = 0,

W = \{(x, t_1, . . . , t_n) ∈ X × ℂ^n : f_1(x) t_1 + . . . + f_n(x) t_n + f_0(x) = 0\}.

The equation f_1 t_1 + . . . + f_n t_n + f_0 = 0 is called a forcing equation, since it forces f_0 to lie in the ideal generated by f_1, . . . , f_n. Forcing equations and the algebras defined by them play an important role in the theory of closure operations for ideals, e.g. tight closure and solid closure, see [3]. Let p : W → X be the projection. If d ≥ 2, then n ≥ 2 and W is an irreducible Stein space of dimension d + n − 1. Let U_i = {x ∈ X : f_i(x) = 0} and U = \bigcup_{i=1}^n U_i = X − P. Resolving t_i shows that p^{-1}(U_i) ∼= U_i × ℂ^{n−1}. The transition mappings however are only affine-linear, not
linear, hence $W|_U$ is not a vector bundle. $H := p^{-1}(P) \cong \mathbb{C}^n$ is a closed subset in $W$, which is a hypersurface in case $d = 2$. The existence of a section $X \to W$ is equivalent with $f_0 \in (f_1, \ldots, f_n)$ over $X$.

**Lemma 1.** Let $X$ be a normal irreducible Stein space together with a point $P$ and let $f_1, \ldots, f_n, f_0 \in \Gamma(X, \mathcal{O}_X)$ be holomorphic functions on $X$. Let $X'$ be another irreducible Stein space of the same dimension and let $\psi : X' \to X$ be a holomorphic mapping such that $\psi^{-1}(P)$ contains isolated points. Suppose that $f_0 \circ \psi \in (f_1 \circ \psi, \ldots, f_n \circ \psi)$ in $\Gamma(X', \mathcal{O}_{X'})$. Then $f_0 \in (f_1, \ldots, f_n)$ in $\mathcal{O}_{X,P}$.

**Proof.** Let $Q$ be an isolated point over $P$. Then there exist open neighborhoods $Q \subseteq U$ and $P \subseteq V$ such that $\psi : U \to V$ is finite, see [4], Ch. 3.2. Due to the finite mapping theorem, $\psi_*(\mathcal{O}_{X'})$ is a coherent analytic algebra on $P \in V \subseteq X$. Furthermore it is torsionfree due to the assumptions on the dimension. Since $X$ is normal, we have the trace map $\text{tr} : \psi_*(\mathcal{O}_U) \to \mathcal{O}_V$, which gives the result. \hfill $\square$

**Corollary 2.** Let $X$ be a normal irreducible Stein space of dimension $d$ together with a point $P$ and let $f_1, \ldots, f_n, f_0 \in \Gamma(X, \mathcal{O}_X)$ be holomorphic functions such that $f_0 \notin (f_1, \ldots, f_n)\mathcal{O}_P$. Let

$$W = V(f_1t_1 + \ldots + f_nt_n + f_0) \subseteq X \times \mathbb{C}^n \xrightarrow{p} X.$$ 

and $H = p^{-1}(P)$. Let $T$ be an irreducible complex space of dimension $d$ and let $\varphi : T \to W$ be a holomorphic map. Then $\varphi^{-1}(H) \subseteq T$ contains no isolated points and the codimension of $\varphi^{-1}(H)$ is $\leq d - 1$.

**Proof.** We look at the composed mapping $\psi = p \circ \varphi : T \xrightarrow{\varphi} W \xrightarrow{p} X$. Since it factors through $W$ it follows that $f_0 \circ \psi \in (f_1 \circ \psi, \ldots, f_n \circ \psi)$ in $\Gamma(T, \mathcal{O}_T)$. Due to the Lemma $\psi^{-1}(P) = \varphi^{-1}(H)$ cannot contain isolated points. \hfill $\square$

With this result we can establish the hypothesis of the hypersection problem in a broad class of example where the base space $X$ is two-dimensional, $n = 2$ and $H = p^{-1}(P) \cong \mathbb{C}^2$ is a hypersurface in three-dimensional $W$.

**Proposition 3.** Let $X$ be a normal irreducible two-dimensional Stein space together with a point $P$. Let $f_1, f_2, f_0 \in \Gamma(X, \mathcal{O}_X)$ be holomorphic functions such that $P = \{ f_1 = f_2 = 0 \}$ and suppose that $f_0 \notin (f_1, f_2)\mathcal{O}_P$. Let

$$W = Z(f_1t_1 + f_2t_2 + f_0) \subseteq X \times \mathbb{C}^2 \xrightarrow{p} X.$$ 

and let $H = p^{-1}(P) \subset W$. Then for every analytic surface $S \subset W$ the complement of $S \cap H \subset S$ is Stein.

**Proof.** We may assume that $S$ is irreducible, let $\tilde{S}$ be its normalization and let $\varphi : \tilde{S} \to W$ be the corresponding mapping. Due to Cor. \hfill $\square$ we know that $\varphi^{-1}(H) \subseteq \tilde{S}$ contains no isolated points. Hence $\varphi^{-1}(H)$ is a pure curve on a normal Stein surface and due to \hfill $\square$ its complement is Stein. But then also $S - \tilde{S} \cap H$ itself is Stein. \hfill $\square$

2. The graded situation. We have to look for examples of the type described in Proposition \hfill $\square$ where $W - H$ is not Stein. To this end we look at the graded situation. Let $Y \subseteq \mathbb{P}^N$ be a smooth projective variety with the very ample line bundle $H_Y \to Y$ (which is the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ to $Y$) and let $X \subseteq \mathbb{C}^{N+1}$ be the corresponding affine cone. Let $P$ be the vertex of the cone and assume that $X$ is normal. Recall that we have an action of $\mathbb{C}^*$ on $X$, which is free on $U = X - P$. 


Proposition 5. Let $Y$ be the quotient of this action and $U \to Y$ is a $\mathbb{C}^*$-principal bundle. A number $e \in \mathbb{Z}$ defines the action on $X \times \mathbb{C}$ by $\lambda(x, t) := (\lambda x, \lambda^e t)$, this action is free over $U$ and the quotient is the line bundle $H_Y^e \to Y$.

Suppose that the holomorphic functions $f_i$ are homogeneous of degree $d_i$, i.e. $f_i(\lambda x) = \lambda^{d_i} f(x)$, $x \in X$, $\lambda \in \mathbb{C}^*$. We may consider a homogeneous holomorphic function $f$ of degree $d$ as a section $Y \to H_Y^d$ and as a mapping of line bundles $H_Y^d \to Y \times \mathbb{C}$ or $H_Y^d \to H_Y^{d+d}$.

Proposition 4. Let $X$ be a normal affine cone over a smooth projective variety $Y$ and let $P$ be the vertex point. Let $f_1, \ldots, f_n$ be homogeneous functions such that $P = \{ f_1 = \ldots = f_n = 0 \}$, $U = X - P$. Let $d_i$ be the degrees of $f_i$ and let $e_i$ numbers such that $m = d_i + e_i$ is constant. Then the following hold.

(i) There is an exact sequence of vector bundles

\[ 0 \to V_m \to H_Y^{d_1} \times_Y \ldots \times_Y H_Y^{d_n} \to \sum_{i=1}^n f_i \to H_Y^m \to 0. \]

(ii) $V_m$ is the quotient of $V(f_1 t_1 + \ldots + f_n t_n)|_U$ by the action of $\mathbb{C}^*$ given by $\lambda(x, t_1, \ldots, t_n) = (\lambda x, \lambda^{e_1} t_1, \ldots, \lambda^{e_n} t_n)$.

(iii) We have $\text{Det} V_m \cong H_Y^{k - n}$, where $k = \sum_{i=1}^n e_i - m = -\sum_{i=1}^n d_i + (n-1)m$.

(iv) $V_m' = V_m \otimes H_Y^{m'}$. The projective bundle $\mathbb{P}(V_m)$ is independent of the chosen degree $m$.

Proof. (i). We consider the $f_i$ as morphisms $H_Y^d \to H_Y^{d_i}$. The morphism of vector bundles $H_Y^{d_1} \times_Y \ldots \times_Y H_Y^{d_n} \to H_Y^m$ over $Y$ is surjective, because the $f_i$ do not have a common zero on $Y$. Hence the kernel is a vector bundle $V_m$ on $Y$ of rank $n-1$.

(ii). The pull back under $q : U \to Y$ of the exact sequence in (i) gives

\[ 0 \to q^* V_m \to U \times \mathbb{C}^n \to \sum_{i=1}^n f_i \to U \times \mathbb{C} \to 0 \]

together with the described action, and $q^* V_m = V(f_1 t_1 + \ldots + f_n t_n)|_U$. (iii) and (iv) follow.

Proposition 5. Let $X$ be a normal affine cone over a smooth projective variety $Y$ and let $P$ be the vertex point, $U = X - P$. Let $f_1, \ldots, f_n$ be homogeneous functions such that $P = \{ f_1 = \ldots = f_n = 0 \}$. Let $f_0$ be another homogeneous function, $d_i = \deg(f_i)$, and let $e_i$ be numbers such that $m = d_i + e_i$ is constant for $i = 0, \ldots, n$. Let $V_m$ (resp. $V_m'$) be the vector bundle on $Y$ defined in Proposition 4 with respect to $f_1, \ldots, f_n$ (resp. $f_0, f_1, \ldots, f_n$). Then the following hold.

(i) There is an exact sequence of vector bundles on $Y$:

\[ 0 \to V_m \to V_m' \to H_Y^m \to 0. \]

(ii) The corresponding embedding $\mathbb{P}(V_m) \hookrightarrow \mathbb{P}(V_m')$ is independent of $m$, $\mathbb{P}(V_m)$ is a divisor on $\mathbb{P}(V_m')$.

(iii) Let $e_0 = 0$. The normal bundle for $\mathbb{P}(V) \to \mathbb{P}(V)'$ on $\mathbb{P}(V)$ is $H_{\mathbb{P}(V)}$, where $H_{\mathbb{P}(V)}$ denotes the relative very ample line bundle on $\mathbb{P}(V)$.

(iv) Let $e_0 = 0$. $W|_U \to \mathbb{P}(V)' - \mathbb{P}(V)$ is a quotient of the action on $W = V(f_1 t_1 + \ldots + f_n t_n + f_0)$ given by $\lambda(x, t_1, \ldots, t_n) \mapsto (\lambda x, \lambda^{e_1} t_1, \ldots, \lambda^{e_n} t_n)$.
Proof. (i). The mappings in the sequence follow from the defining sequences for $V_m$ and $V_m'$. The exactness of the sequence follows from diagram chasing. (ii) is clear.

(iii). Since we assume $e_0 = 0, t_0$ is a global function on $V'$ and it is a global section in the relative very ample line bundle $H_{\bar{P}(V')}$ on $\bar{P}(V')$, and $\bar{P}(V)$ is the corresponding divisor. Therefore the normal bundle of this embedding is $i^*H_{\bar{P}(V')} \cong H_{\bar{P}(V)}$.

(iv). First we may identify the closed subset $\{Q \in V' : t_0(Q) = 1\}$ with $\mathbb{P}(V') - \mathbb{P}(V)$. The described action on $W$ respects the forcing equation, for $f_1(\lambda x)\lambda^{s_1}t_1 + \ldots + f_n(\lambda x)\lambda^{s_n}t_n + f_0(\lambda x) = \lambda^{d_1}f_1(x)t_1 + \ldots + \lambda^{d_n}f_n(x)t_n + \lambda^{d_0}f_0(x) = \lambda^m(f_1(x)t_1 + \ldots + f_n(x)t_n + f_0(x)) = 0$. This action on $W|_U = W - p^{-1}(P)$ is the same action as the action on the vector bundle $V(f_1t_1 + \ldots + f_0t_0)|_U$ restricted to $t_0 = 1$ described in Proposition \textcolor{blue}{3}(iii). Its quotient is $\{Q \in V' : t_0(Q) = 1\}. \quad \Box$

Now we specialize to the two-dimensional situation.

Corollary 6. Let $X$ be a normal affine two-dimensional cone over a smooth projective curve $Y$ and let $P$ be the vertex point. Let $f_1, f_2$ be homogeneous functions such that $P = \{f_1 = f_2 = 0\}$. Let $f_0$ be another homogeneous function, $d_i = \deg(f_i)$, and let $e_i$ numbers such that $m = d_i + e_i$ is constant for $i = 0, 1, 2$. Let $V_m$ $(V_m')$ be the corresponding vector bundles on $Y$. Then the following hold.

(i) $\mathbb{P}(V')$ is a ruled surface and $\mathbb{P}(V) \subset \mathbb{P}(V')$ is a section (independent of $m$).

(ii) We have $V_m \cong H^{e_0+d_0-d_1-d_2}_{Y'}$ and the exact sequence

$$0 \rightarrow H^{e_0+d_0-d_1-d_2}_{Y'} \rightarrow V' \rightarrow H^{e_0}_{Y'} \rightarrow 0.$$

(iii) Let $e_0 = 0$. The normal bundle of the embedding $Y \cong \mathbb{P}(V) \subset \mathbb{P}(V')$ is $H^{d_1+d_2-d_0}_{Y'}$.

(iv) The self intersection number of $Y \cong \mathbb{P}(V) \hookrightarrow \mathbb{P}(V')$ is $(d_1 + d_2 - d_0) \deg H_Y$.

Proof. (i) is clear due to Proposition \textcolor{blue}{3}.

(ii) From the defining sequence in Proposition \textcolor{blue}{3} it follows that we have $V_m \cong H^{e_1}_{Y'} \otimes H^{e_2}_{Y'} \otimes H^{e_0}_{Y} = H^{e_1+e_2-m}_{Y'} = H^{e_0+d_0-d_1-d_2}_{Y'}$.

(iii). The normal bundle on $\mathbb{P}(V)$ is $H_{\bar{P}(V)}$ due to Proposition \textcolor{blue}{3}. But for a line bundle this is just the negative tautological bundle $-V$, therefore $N = -V = H^{d_1+d_2-d_0}_{Y'}$.

(iv). The self intersection number is $\mathbb{P}(V)^2 = \deg_Y N = \deg_Y H^{d_1+d_2-d_0}_{Y'} = (d_1 + d_2 - d_0) \deg H_Y. \quad \Box$

3. A class of examples.

Theorem 7. Let $X$ be a normal affine two-dimensional cone with vertex point $P$ over a smooth projective curve $Y$, let $f_1, f_2$ and $f_0$ be homogeneous holomorphic functions on $X$ with degrees $d_1, d_2, d_0$ such that

1. $V(f_1, f_2) = P$ 2. $f_0 \not\in (f_1, f_2)\mathcal{O}_{X,P}$ and 3. $d_1 + d_2 - d_0 < 0$.

Let $W = V(f_1t_1 + f_2t_2 + f_0) \subset X \times \mathbb{C}^2$ and $H = p^{-1}(P) \subset W$. Then $W - H \subset W$ is not Stein, but for every analytic surface $S \subset W$ the intersection $(W - H) \cap S$ is Stein.

Proof. We have to show that $W - H$ is not Stein. Since $W - H = W|_U, U = X - P$, the quotient of this open subset under the action of $\mathbb{C}^*$ is $\mathbb{P}(V') - \mathbb{P}(V)$. Due to \textcolor{blue}{3} it is enough to show that this complement of the section in the ruled surface $\mathbb{P}(V')$
We have to show that
\[ f \] is due to \cite{3} contractible and the complement cannot be Stein.

\textbf{Corollary 8.} Let \( h \) be an irreducible homogeneous polynomial of degree \( r \) in the three variables \( x, y, z \) and suppose that \( x, y \) are homogeneous parameters (i.e. that \( h = cz^r + \) other terms, \( c \neq 0 \)) and let \( X = V(h) \subset \mathbb{C}^3 \). Suppose that \( X \) is normal (hence \( Y \) is smooth). Let \( d_1, d_2 \geq 1 \) and \( d_0 \) be degrees such that \( d_1 + d_2 < d_0 < r \). Then \( f_1 = x^{d_1}, f_2 = y^{d_2} \) and \( f_0 = z^{d_0} \) fulfill the conditions of the theorem.

\textbf{Proof.} We have to show that \( z^{d_0} \notin (x^{d_1}, y^{d_2}) \mathcal{O}_P \). For this we look at the completion of the local ring, \( \hat{\mathcal{O}_P}/(x^{d_1}, y^{d_2}) = \mathbb{C}[x, y, z]/(h, x^{d_1}, y^{d_2}) \). Since \( d_0 < r = \deg(h) \) we see that \( z^{d_0} \neq 0 \) in this residue class ring. On the other hand, the self intersection number is \( (d_1 + d_2 - d_0) \deg H_Y < 0 \).

\textbf{Example 9.} Let \( X = V(x^r + y^r + z^r) \subset \mathbb{C}^3, r \geq 4 \) be a Fermat type hypersurface, let
\[ W = V(x^r + y^r + z^r, xt_1 + yt_2 + z^4) \subset \mathbb{C}^5, 3 \leq s < r \text{ and } H = V(x, y). \]

Then the conditions in the corollary are fulfilled.

The easiest example of this type is the Fermat quartic \( x^4 + y^4 + z^4 \) together with \( f_1 = x, f_2 = y \) and \( f_0 = z^3 \). Therefore
\[ W = V(x^4 + y^4 + z^4, xt_1 + yt_2 + z^3) \text{ and } H = V(x, y) \]
gives an counter-example to the hypersection problem.

\textbf{Remarks 10.} The hypersurface \( H \) in our example is the singular locus of \( W \). Since the normalization does not change the complement of \( H \) and since its preimage is still a hypersurface due to Corollary \cite{3}, we also may get normal examples.

The condition \( f_0 \notin (f_1, f_2) \mathcal{O}_P \) in Corollary \cite{3} ensures that the divisor \( \mathbb{P}(V) \subset \mathbb{P}(V') \) intersects every curve \( C \subset \mathbb{P}(V') \) positively. For a disjoined curve would yield a closed punctured surface (its cone) inside \( W - H \). If additionally \( d_1 + d_2 - d_0 > 0 \), then \( \mathbb{P}(V) \) is an ample divisor and its complement is affine, hence Stein. What happens if \( d_1 + d_2 - d_0 = 0 \)? Then the complement is not affine, but it may be Stein. For \( h = x^r + y^r + z^r = 0, f_1 = x, f_2 = y \) and \( f_0 = z^2 \) we get an instance of the classical construction of Serre of a Stein, but non-affine variety, see \cite{3}.

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