Max-Cut Parameterized Above the Edwards-Erdős Bound

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Abstract

We study the boundary of tractability for the Max-Cut problem in graphs. Our main result shows that Max-Cut above the Edwards-Erdős bound is fixed-parameter tractable: we give an algorithm that for any connected graph with n vertices and m edges finds a cut of size

\[ \frac{m}{2} + \frac{n - 1}{4} + k \]

in time \(2^{O(k)} \cdot n^4\), or decides that no such cut exists.

This answers a long-standing open question from parameterized complexity that has been posed a number of times over the past 15 years.

Our algorithm is asymptotically optimal, under the Exponential Time Hypothesis, and is strengthened by a polynomial-time computable kernel of polynomial size.

Keywords. Algorithms and data structures, maximum cuts, combinatorial bounds, fixed-parameter tractability.

1 Introduction

The study of cuts in graphs is a fundamental area in theoretical computer science, graph theory, and polyhedral combinatorics, dating back to the 1960s. A cut of a graph is an edge-induced bipartite subgraph, and its size is the number of edges it contains. Finding cuts of maximum size in a given graph was one of Karp’s famous 21 NP-complete problems [19].

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Since then, the MAX-CUT problem has received considerable attention in the areas of approximation algorithms, random graph theory, combinatorics, parameterized complexity, and others; see the survey \[28\].

As a fundamental NP-complete problem, the computational complexity of MAX-CUT has been intensively scrutinized. We continue this line of research and explore the boundary between tractability and hardness, guided by the question: *Is there a dichotomy of computational complexity of MAX-CUT that depends on the size of the maximum cut?*

This question was already studied by Erdős \[11\] in the 1960s, who gave a randomized polynomial-time algorithm that in any \(n\)-vertex graph with \(m\) edges finds a cut of size at least \(m/2\). Erdős \[11, 12\] also (erroneously) conjectured that the value \(m/2\) can be raised to \(m/2 + \varepsilon m\) for some \(\varepsilon > 0\); only much later it was shown \[16, 20\] that finding cuts of size \(m/2 + \varepsilon m\) is NP-hard for every \(\varepsilon > 0\). Furthermore, the MAX-CUT GAIN problem—maximize the gain compared to a random solution that cuts \(m/2\) edges—does not allow constant approximation \[20\] under the Unique Games Conjecture, and the best one can hope for is to cut a \(1/2 + \Omega(\varepsilon/\log(1/\varepsilon))\) fraction of edges in graphs in which the optimum is \(1/2 + \varepsilon\) \[6\].

However, the lower bound \(m/2\) can be increased, by a sublinear function: Edwards \[9, 10\] in 1973 proved that a cut of size

\[
\frac{m}{2} + \frac{1}{8}(\sqrt{8m} + 1 - 1) \tag{1}
\]

always exists, and for connected graphs this can be further increased to

\[
\frac{m}{2} + (n - 1)/4, \tag{2}
\]

which is always at least as large as \(1\). Thus, any graph with \(n\) vertices, \(m\) edges and \(t\) connected components has a cut of size at least \(m/2 + (n - t)/4\). The lower bound \(2\) is famously known as the Edwards-Erdős bound, and it is tight for cliques of every odd order \(n\). The bound has been proved several times \[1, 7, 13, 26, 27\], with some proofs yielding polynomial-time algorithms to attain it. As \(2\) is tight for infinitely many non-isomorphic graphs, and finding maximum cuts is NP-hard, raising the lower bound \(2\) even further requires a new approach: a fixed-parameter algorithm, that for any connected graph with \(n\) vertices and \(m\) edges, and integer \(k \in \mathbb{N}\), finds a cut of size at least \(m/2 + (n - 1)/4 + k\) (if such exists) in time \(f(k) \cdot n^c\), where \(f\) is an arbitrary function dependent only on \(k\) and \(c\) is an absolute constant independent of \(k\). The point here is to confine the combinatorial explosion to the (small) parameter \(k\). But at first sight, it seems not even
clear how to find a cut of size \( m/2 + (n - 1)/4 + k \) in time \( n^{f(k)} \), for an arbitrary function \( f \).

In 1997, Mahajan and Raman \[24\] gave a fixed-parameter algorithm for the variant of this problem with Erdős’ lower bound \( m/2 \), and showed how to decide existence of a cut of size \( m/2 + k \) in time \( 2^{O(k)} \cdot n^{O(1)} \). Their result was strengthened by Bollobás and Scott \[4\] who replaced \( m/2 \) by the stronger bound \( 1 \). It remained an open question (\[7, 15, 24, 25, 29\]) whether this result could be strengthened further by replacing \( 1 \) with the stronger bound \( 2 \).

**Main Results**

We settle the computational complexity of \textsc{Max-Cut} above the Edwards-Erdős bound \( 2 \).

**Theorem 1.** There is an algorithm that computes, for any connected graph \( G \) with \( n \) vertices and \( m \) edges and any integer \( k \in \mathbb{N} \), in time \( 2^{O(k)} \cdot n^{4} \) a cut of \( G \) with size at least \( m/2 + (n - 1)/4 + k \), or decides that no such cut exists.

Theorem 1 answers a question posed several times over the past 15 years \[24, 25, 7, 15, 24\]. In particular, instances with \( k = O(\log m) \) can be solved in polynomial time, thereby enlarging the realm of tractability.

The running time of our algorithm is likely to be optimal, as the following theorem shows.

**Theorem 2.** No algorithm can find cuts of size \( m/2 + (n - 1)/4 + k \) in time \( 2^{o(k)} \cdot n^{O(1)} \) given a connected graph with \( n \) vertices and \( m \) edges, and integer \( k \in \mathbb{N} \), unless the Exponential Time Hypothesis fails.

The Exponential Time Hypothesis was introduced by Impagliazzo and Paturi \[18\], and states that \( n \)-variable SAT formulas cannot be solved in subexponential time.

Fixed-parameter tractability of \textsc{Max-Cut} above Edwards-Erdős bound \( m/2 + (n - 1)/4 \) implies the existence of a so-called kernelization, which efficiently compresses any instance \((G, k)\) into an equivalent instance \((G', k')\), the kernel, whose size \( g(k) = |G'| + k' \) itself depends on \( k \) only. Alas, the size \( g(k) \) of the kernel for many fixed-parameter tractable problems is enormous, and in particular many fixed-parameter tractable problems do not admit kernels of size polynomial in \( k \) unless \( \text{coNP} \subseteq \text{NP/poly} \) \[3\]. We prove the following.
**Theorem 3.** There is a polynomial-time algorithm that compresses any connected graph $G = (V, E)$ with integer $k \in \mathbb{N}$ to a connected graph $G' = (V', E')$ of order $O(k^5)$, such that $G$ has a cut of size $|E|/2 + (|V| - 1)/4 + k$ if and only if $G'$ has a cut of size $|E'|/2 + (|V'| - 1)/4 + k'$, for some $k' \leq k$.

Bollobás and Scott [4] proved fixed parameter tractability for the weighted version of Max Cut parameterized above (1). They give a $2^{O(k^4)} + n + w(G)$ time algorithm to find a cut of weight $w(G)/2 + \frac{1}{8}(\sqrt{8w(G)} + 1 - 1) + k$ if such a cut exists, or else an optimal cut, where $w$ is an edge-weighting on the graph $G$. The proof in [4] can easily be seen to give a kernel of size $O(k^4)$ (although it is not described as such in [4], as kernelization has only recently begun to attract significant attention). We improve this to a kernel of size $O(k^3)$.

**Theorem 4.** There is a linear-time algorithm that, for any integer $k \in \mathbb{N}$, compresses any connected graph $G = (V, E)$ with edge-weighting $w$ to a connected graph $G' = (V', E')$ of size $O(k^3)$ with edge-weighting $w'$, such that $G$ has a cut of size $w(G)/2 + \frac{1}{8}(\sqrt{8w(G)} + 1 - 1) + k$ if and only if $G'$ has a cut of size $w'(G')/2 + \frac{1}{8}(\sqrt{8w'(G')} + 1 - 1) + k'$, for some $k' \leq k$.

The proof is a slight modification of the proof of Theorem 22 in Bollobás and Scott [4]. Note that Theorems 1 and 3 only hold for unweighted graphs; the weighted versions remain open.

**Our Techniques and Related Work**

Our results are based on algorithmic as well as combinatorial arguments. To prove Theorem 1 we design a Turing reduction to a special class of chordal graphs, on which we show how to solve the problem efficiently. Theorem 2 is established by a combinatorial reduction. Theorem 3 is proven by a careful analysis of random cuts via the probabilistic method, whereas the proof of Theorem 4 is through a characterization of graphs for which the lower bound is nearly tight, and weighted graph decompositions as “edge-sums”.

A number of the standard approaches that have been developed for “above-guarantee” parameterizations of other problems are unavailable for this problem; hence, our algorithm differs significantly from others in the area. The most common approach is to use probabilistic analysis of a random variable whose expected value corresponds to a solution matching the guarantee. However, there is no simple randomized procedure known giving a cut of size $m/2 + (n - 1)/4$.

Another approach is to make use of approximation algorithms that give a factor-$c$ approximation, when the problem is parameterized above the bound
$c \cdot n$; here, $n$ is the maximum value of the objective function. But there is no such approximation algorithm for this problem. The famous Goemans-Williamson SDP algorithm gives an approximation factor of $\gamma = 0.878$ for the Max-Cut value: Kim and Williams \cite{21} (in an early version of their paper) gave an $2^{O(k)}m$-time fixed-parameter algorithm for Max-Cut above $\gamma \cdot \text{opt}(G)$, where $\text{opt}(G)$ denotes the size of the maximum cut in $G$. But Theorem \ref{main} is stronger than Kim and Williams’ result for all graphs $G$ with $\text{opt}(G) \leq 1.13822(m/2 + (n - 1)/4)$.

Our paper also differs in its use of reduction rules. Most reduction rules for above-guarantee problems remove certain subgraphs of constant size, on which the bound is tight. But for our problem the bound is tight on cliques. Thus our reduction rules remove maximal cliques from the graph, which may contain a large fraction of the vertices in $G$. Moreover, rather than using reduction rules to reduce to an equivalent instance, which is then solved quickly, our reduction rules do not produce an equivalent instance. Instead, they either reduce to a ‘yes’-instance or we can determine useful restrictions on the structure of the original instance, which can then be used to solve the original instance in fixed-parameter tractable time.

The “boundary of tractability” for (variants of) Max-Cut is evident in dozens of approximation algorithms and parameterized algorithms. For Max-Bisection, where we seek a cut such that the number of vertices in both sides of the bipartition is as equal as possible, the tight lower bound on the bisection size is only $m/2$, and fixed-parameter tractability of Max-Bisection above $m/2$ was shown by Gutin and Yeo \cite{15}.

2 Preliminaries

With the exception of the term “tree of cliques” (see below), we use standard graph theory terminology and notation. Given a graph $G$, let $V(G)$ be the vertices of $G$ and let $E(G)$ be the edges of $G$. For disjoint sets $S, T \subseteq V(G)$, let $E(S, T)$ denote the set of edges in $G$ with one vertex in $S$ and one vertex in $T$. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph induced by the vertices of $S$, and let $G - S$ denote the graph $G'[V(G) \setminus S]$. We say that $G$ has a cut of size $t$ if there exists an $S \subseteq V(G)$ such that $|E(S, V(G) \setminus S)| = t$. The graph $G$ is connected if any two of its vertices are connected by a path, and it is 2-connected if $G - v$ is connected for every $v \in V(G)$. A connected component of $G$ is a connected subgraph $G'$ of $G$ that is maximal with respect to vertex inclusion, and we often identify $G'$ with its vertex set $V(G')$.

We study the following formulation of Max-Cut parameterized above
Edwards-Erdős bound:

**MAX-Cut above Edwards-Erdős (MAX-Cut-AEE)**

*Instance:* A connected graph $G$ with $n = |V(G)|$ vertices and $m = |E(G)|$ edges, and an integer $k \in \mathbb{N}$.

*Parameter:* $k$.

*Question:* Does $G$ have a cut of size at least $\frac{m}{2} + \frac{n-1}{4} + \frac{k}{4}$?

We ask for a cut of size $\frac{m}{2} + \frac{n-1}{4} + \frac{k}{4}$, rather than the more usual $\frac{m}{2} + \frac{n-1}{4} + k$, so that we may treat $k$ as an integer at all times. Note that this does not affect the existence of a fixed-parameter algorithm or polynomial-size kernel. A pair $(G, k)$ is called a “yes”-instance if $G$ has a cut of size at least $\frac{m}{2} + \frac{n-1}{4} + \frac{k}{4}$, and a “no”-instance otherwise.

An *assignment* or *coloring* on $G$ is a function $\alpha : V(G) \to \{\text{red}, \text{blue}\}$, and an edge is *cut* or *satisfied* by $\alpha$ if one of its vertices is colored red and the other vertex is colored blue. Note that a graph has a cut of size $t$ if and only if it has an assignment that satisfies at least $t$ edges. A partial assignment on $G$ is a function $\alpha : X \to \{\text{red}, \text{blue}\}$, where $X$ is a subset of $V(G)$.

A *clique* (in $G$) is a set of vertices $X \subseteq V(G)$ any two of which are adjacent in $G$, and it is *maximal* for $G$ if $X \cup \{v\}$ is not a clique for any $v \in V(G) \setminus X$. We define a class of chordal graphs, *trees of cliques*, as follows. A complete graph is a tree of cliques. For $G', G''$ trees of cliques, the graph that is formed by identifying one vertex of $G'$ with one vertex of $G''$ is a tree of cliques. Equivalently, a connected graph $G$ is a tree of cliques if the vertices of every cycle induce a cliques.

To arrive at a realistic analysis of the required computational effort, throughout our model of computation is the random access machine with the restriction that arithmetic operations are considered unit-time only for constant-size integers; in this model, two $b$-bit integers can be added, subtracted, and compared in $O(b)$ time.

### 3 Fixed-Parameter Algorithm for Max-Cut above the Edwards-Erdős bound

In this section, we prove Theorem 1. To this end, we prove the following lemma, which also forms the basis of our kernel in Theorem 3.

**Lemma 5.** Given a connected graph $G$ with $n$ vertices and $m$ edges and an integer $k$, we can in polynomial time decide that either $G$ has a cut of size
at least $\frac{m}{2} + \frac{n-1}{4} + \frac{k}{4}$, or find a set $S$ of at most $3k$ vertices in $G$ such that every connected component in $G - S$ is a tree of cliques.

The algorithm starts by applying the following rules to the given connected graph $G$. These rules are such that if an instance $(G', k')$ is reduced from $(G, k)$ and $(G', k')$ is a “yes”-instance, then $(G, k)$ is also a “yes”-instance. The converse does not necessarily hold – a “yes”-instance may be reduced to a “no”-instance. In some rules, we mark certain vertices, that will be collected in the set $S$ of Lemma 5.

| Rule 1: | Apply to a connected graph $G$ with $v \in V(G), X \subseteq V(G)$ such that $X$ is a connected component of $G - v$ and $X \cup \{v\}$ is a clique. |
| Remove: | All vertices in $X$ and incident edges. |
| Mark: | Nothing. |
| Parameter: | Reduce $k$ by 1 if $|X|$ is odd, otherwise leave $k$ the same. |

| Rule 2: | Apply to a connected graph $G$ reduced by Rule 1 with $v \in V(G)$ such that for all connected components $X$ of $G - v$, except possibly one, $X$ is a clique. |
| Remove: | The vertex $v$ and all incident edges; all vertices in $X$ and incident edges, for every connected component $X$ of $G - v$ which is a clique. |
| Mark: | $v$. |
| Parameter: | Reduce $k$ by $2t - 1$, where $t$ is the number of connected components of $G - v$ removed. (Note: Only apply this rule if $t \geq 1$) |

| Rule 3: | Apply to a connected graph $G$ with $a, b, c \in V(G)$ such that $\{a, b\}, \{b, c\} \in E(G), \{a, c\} \notin E(G)$, and $G - \{a, b, c\}$ is connected. |
| Remove: | The vertices $a, b, c$ and incident edges. |
| Mark: | $a, b, c$. |
| Parameter: | Reduce $k$ by 1. |
**Rule 4:** Apply to a connected graph $G$ with $x, y \in V(G)$ such that \{x, y\} $\notin E(G)$, and for all connected components $X$ of $G - \{x, y\}$, except possibly one, $X \cup \{x\}$ and $X \cup \{y\}$ are cliques.

**Remove:** Vertices $\{x, y\} \cup X$ for any clique $X$ satisfying above conditions.

**Mark:** $x, y$.

**Parameter:** Reduce $k$ by $3t - 2$, where $t$ is the number of connected components of $G - \{x, y\}$ removed.

(Note: Only apply this rule if $t \geq 1$)

These rules can be applied exhaustively in polynomial time, as each rule reduces the number of vertices in $G$, and for each rule we can check for any applications of that rule by trying every set of at most three vertices in $V(G)$ and examining the connected components of the graph when those vertices are removed.

**Lemma 6.** Let $(G, k)$ and $(G', k')$ be instances of Max-Cut-AEE such that $(G', k')$ is reduced from $(G, k)$ by an application of Rules 1, 2, 3 and 4. Then $G'$ is connected, and if $(G', k')$ is a “yes”-instance of Max-Cut-AEE then so is $(G, k)$.

**Proof.** First, we show that $G'$ is connected. For Rule 1 observe that for $s, t \in V(G) \setminus X$, no path between $s$ and $t$ passes through $X$, so $G - X$ is connected. For Rules 2 and 3 observe that we remove some vertices together with all but at most one of the connected components in the resulting graph, so we are left with a single component. For Rule 4 the conditions explicitly state that we only apply the rule if the resulting graph is connected.

Second, we prove separately for each rule the following claim, in which $n'$ denotes the number of vertices and $m'$ the number of edges removed by the rule.

Any assignment to the vertices of $G'$ can be extended to an assignment on $G$ that cuts an additional $\frac{m'}{2} + \frac{n'}{4} + \frac{k-k'}{4}$ edges. (**)

**Rule 1** Since $v$ is the only vertex connecting $X$ to the rest of the graph, any assignment to $G'$ can be extended to one which is optimal on $X \cup \{v\}$. (Let $\alpha$ be an optimal coloring of $G[X \cup \{v\}]$, and let $\alpha'$ be the $\alpha$ with all colors reversed. Both $\alpha$ and $\alpha'$ are optimal colorings of $G[X \cup \{v\}]$, and one of these will agree with the coloring we are given on $G'$ since the only overlap is $v$.) Observe that $n' = |X|$ and $m' = \frac{|X||(|X|+1)}{2}$, since the edges we
remove form a clique including \( v \), and all vertices in the clique except \( v \) are removed.

If \(|X|\) is even then the maximum cut of the clique \( X \cup \{v\} \) has size 
\[
\frac{|X|}{2} + \frac{|X|+2}{4} = \frac{|X|}{4} + |X| + \frac{|X|+1}{4} = n' + m' + n',
\]
which is what we require as \( k \) is unchanged in this case.

If \(|X|\) is odd then the maximum cut of the clique \( X \cup \{v\} \) has size
\[
\frac{|X|+1}{4} + 1 = \frac{|X|+1}{4} + \frac{|X|+1}{4} = \frac{|X|}{4} + \frac{|X|+1}{4} = n' + m' + n' + \frac{|X|+1}{4},
\]
which is what we require as we reduce \( k \) by 1 in this case.

\textbf{Rule 2} \quad \text{Given an assignment to } G', \text{ color } v \text{ so that at least half the edges between } G' \text{ and } v \text{ are satisfied. Let } X_1, \ldots, X_t \text{ be the connected components of } G - (G' \cup \{v\}). \text{ Now for every } i \in \{1, \ldots, t\}, \text{ let } n'_i = |X_i| \text{ and let } m'_i \text{ be the number of edges incident with vertices in } X_i \text{ (including edges between } X_i \text{ and } v). \text{ Order the vertices of } X_i \text{ as } x_1, x_2, \ldots, x_{n'_i}, \text{ such that there exists an } r \text{ so that } x_j \text{ is adjacent to } v \text{ for all } j \leq r \text{, and } x_j \text{ is not adjacent to } v \text{ for all } j > r. \text{ Since } G \text{ is connected and reduced by Rule 1}, \text{ and } r \geq 2 \text{ (if there is only one vertex } x \text{ in } X_i \text{ adjacent to } v \text{ then Rule 1 applies)}, \text{ and since } X_i \text{ is a clique but } X_i \cup \{v\} \text{ is not (otherwise Rule 1 applies), there exists } x \in X_i \text{ not adjacent to } v, \text{ and so } r \leq n'_i - 1. \text{ Color the vertices of } X_i \text{ such that } x_j \text{ is the opposite color to } v \text{ if } j \leq \left\lceil \frac{n'_i}{2} \right\rceil, \text{ and } x_j \text{ is the same color as } v \text{ otherwise. Observe that the total number of satisfied edges incident with } X_i \text{ is } \left\lceil \frac{n'_i}{2} \right\rceil \cdot \left\lceil \frac{n'_i}{2} \right\rceil + \min \left\{ r, \left\lceil \frac{n'_i}{2} \right\rceil \right\}.

Since \( m'_i = \frac{n'_i(n'_i-1)}{2} + r \), this means that the total number of satisfied edges incident with } X_i \text{ is }
\[
\frac{m'_i}{2} + \frac{n'_i}{4} - \frac{n'_i^2}{4} + \left\lceil \frac{n'_i}{2} \right\rceil \cdot \left\lceil \frac{n'_i}{2} \right\rceil + \min \left\{ r, \left\lceil \frac{n'_i}{2} \right\rceil \right\} - \frac{r}{2},
\]
which is at least \( \frac{m'_i}{2} + \frac{n'_i}{4} + \frac{r}{2} \) when \(|X_i|\) is even, and at least \( \frac{m'_i}{2} + \frac{n'_i}{4} + \frac{3}{2} \) when \(|X_i|\) is odd. Hence, the number of satisfied edges incident with } X_i \text{ is at least } \frac{m'_i}{2} + \frac{n'_i}{4} + \frac{1}{2}. \text{ Now } m' = |E(G', \{v\})| + \sum_{i=1}^{t} m_i' \text{ and } n' = \sum_{i=1}^{t} n_i' + 1, \text{ and the total number of removed edges satisfied is at least } \frac{|E(G', v)|}{2} + \sum_{i=1}^{t} \left( \frac{m'_i}{2} + \frac{n'_i}{4} + \frac{1}{2} \right) = \frac{m'}{2} + \frac{n'-1}{4} + \frac{t}{2} = \frac{m'}{2} + \frac{n'}{4} + \frac{2t-1}{4}.

\textbf{Rule 3} \quad \text{Observe that } n' = 3 \text{ and } m' = 2 + |E(G', \{a, b, c\})|. \text{ Consider two colorings } \alpha, \alpha' \text{ of } \{a, b, c\}: \alpha(a) = \alpha(c) = \text{red}, \alpha(b) = \text{blue}, \text{ and } \alpha'(a) = \alpha'(c) = \text{blue}, \alpha'(b) = \text{red}. \text{ Both these colorings satisfy edges } \{a, b\} \text{ and } \{b, c\}, \text{ and at least one of them will satisfy at least half the edges between } \{a, b, c\} \text{ and } G'. \text{ Therefore, the number of satisfied edges incident with } \{a, b, c\} \text{ is at least } 2 + \frac{|E(G', \{a,b,c\})|}{2} = \frac{m'}{2} + \frac{n'}{4} + \frac{1}{4}.\]

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Rule 4  Given an assignment to $G'$, color $x, y$ either both red or both blue, such that at least half of the edges between $G'$ and $\{x, y\}$ are satisfied. Assume, without loss of generality, that $x$ and $y$ are both colored red. Let $X_1, \ldots, X_t$ be the connected components of $G - (G' \cup \{x, y\})$. Now for every $i \in \{1, \ldots, t\}$, let $n'_i = |X_i|$ and let $m'_i$ be the number of edges incident with vertices in $X_i$ (including edges between $X_i$ and $\{x, y\}$). Observe that $m'_i = \frac{n'_i(n'_i - 1)}{2} + 2n'_i$.

If $n'_i$ is odd, color $\frac{n'_i + 1}{2}$ of the vertices in $X_i$ with red and $\frac{n'_i - 1}{2}$ vertices with red. Then the total number of satisfied edges incident with $X_i$ is $\frac{n'_i + 1}{2} \cdot \frac{n'_i - 1}{2} + 2 \cdot \frac{n'_i + 1}{2} = \frac{n'_i(n'_i - 1)}{4} + \frac{n'_i - 1}{4} + n'_i + 1 = m'_i + \frac{n'_i - 1}{4} + 1 = n'_i + \frac{n'_i + 3}{4}$.

If $n'_i$ is even, color $\frac{n'_i}{2} + 1$ of the vertices in $X_i$ with blue and $\frac{n'_i}{2} - 1$ vertices with red. Then the total number of satisfied edges incident with $X_i$ is $(\frac{n'_i}{2} + 1)(\frac{n'_i}{2} - 1) + 2(\frac{n'_i}{2} + 1) = \frac{n'_i^2}{4} - 1 + n'_i + 2 = m'_i + \frac{n'_i}{4} + 1$.

Observe that $m' = |E(G', \{x, y\})| + \sum_{i=1}^t m'_i$ and $n' = \sum_{i=1}^t n'_i + 2$, and hence the total number of removed edges satisfied is at least $\frac{|E(G', \{x, y\})|}{2} + \frac{m'}{2} + \frac{n'}{4} + \frac{3}{4} = \frac{m'}{2} + \frac{n'}{4} + \frac{3}{4} = \frac{n'}{2} + \frac{n'}{4} + \frac{3}{4} = \frac{n'}{2} + \frac{n'}{4} + \frac{3}{4}.

This concludes the proof of the claim (3).

We now know that any assignment on $G'$ can be extended to an assignment on $G$ that cuts an additional $\frac{m'}{2} + \frac{n'}{4} + \frac{k-k'}{4}$ edges. Hence, if $G'$ has a cut of size $\frac{|E(G')|}{2} + \frac{|V(G')|-1}{4} + \frac{k'}{4}$, then $G$ has a cut of size $\frac{m}{2} + \frac{n-n'-1}{4} + \frac{k'}{4} + \frac{m'}{2} + \frac{n'}{4} + \frac{k-k'}{4} = \frac{m}{2} + \frac{n-1}{4} + \frac{k}{4}$. Therefore, if $(G', k')$ is a “yes”-instance then so is $(G, k)$.

Lemma 7. To any connected graph $G$ with at least one edge, at least one of Rules 1, 2, and 3 applies.

Proof. Suppose that $G$ is reduced by Rules 1, 2, and 3. We show that there exist $a, b, c \in V(G)$ such that $\{a, b\}, \{b, c\} \in E(G)$ but $\{a, c\} \notin E(G)$ and $G - \{a, b, c\}$ is connected, that is, Rule 3 applies.

We first find a vertex $r$ and $X \subseteq V(G)$ such that $X$ is a connected component of $G - r$ and $X \cup \{r\}$ is 2-connected. If $G$ is 2-connected, let $r$ be an arbitrary vertex of $G$ and let $X = V(G) \setminus \{r\}$. Otherwise, consider a spanning tree of $G$ and let $R$ be the set of vertices $x$ for which $G - x$ is disconnected. Let $r$ be a vertex in $R$ which is ‘furthest out’, i.e. $r$ has the longest path in the tree from some arbitrarily selected root vertex. Then $X \cap R = \emptyset$, otherwise $r$ was not ‘furthest out’. Therefore, $G[X \cup \{r\}]$ is 2-connected.
Next we find three vertices \(a, b, c\) in \(X\) such that \(\{a, b\}, \{b, c\} \in E(G), \{a, c\} \notin E(G)\), and every connected component of \(G - \{a, b, c\}\) except the one containing \(r\) is a clique. Such a set \(\{a, b, c\}\), if it exists, can clearly be found in polynomial time. To see that such a set exists, first observe that since \(X\) is connected, if there is no set \(a, b, c\) in \(X\) such that \(\{a, b\}, \{b, c\} \in E(G), \{a, c\} \notin E(G)\), then \(X\) is a clique, and so either Rule 1 or 2 applies. Therefore, there exist \(a, b, c\) in \(X\) such that \(\{a, b\}, \{b, c\} \in E(G), \{a, c\} \notin E(G)\). If \(G - \{a, b, c\}\) is connected then Rule 3 applies and we are done. Otherwise, \(G - \{a, b, c\}\) is disconnected. If every connected component of \(G - \{a, b, c\}\) except the one containing \(r\) is a clique, we are done. Otherwise, pick a connected component \(X'\) of \(G - \{a, b, c\}\) not containing \(r\) which is not a clique, and find \(a', b', c' \in X'\) such that \(\{a', b'\}, \{b', c'\} \in E(G), \{a', c'\} \notin E(G)\), and repeat. Note that each time the component containing \(r\) expands, and so this process must eventually terminate.

Given \(a, b, c\), let \(G'\) denote the connected component of \(G - \{a, b, c\}\) containing \(r\). We now have the following properties:

1. \(\{a, b\}, \{b, c\} \in E(G)\) and \(\{a, c\} \notin E(G)\).
2. \(V(G')\) induces a connected component of \(G - \{a, b, c\}\).
3. For every vertex \(v\) not in \(G'\), \(G - v\) is connected.
4. Every connected component of \(G - \{a, b, c\}\), except possibly \(G'\), is a clique.

We now show that Rule 3 applies. Observe that for every connected component \(C\) of \(G - \{a, b, c\}\) that is a clique, at least two vertices from \(\{a, b, c\}\) are adjacent to a vertex in \(C\): if this were not the case, then there would be a vertex \(d \in \{a, b, c\}\) such that \(G - d\) is disconnected, which contradicts point 3. Similarly, at least two vertices from \(\{a, b, c\}\) are adjacent to a vertex in \(G'\).

Let \(\mathcal{C} = \{C_1, \ldots, C_t\}\) be the collection of connected components of \(G - (G' \cup \{a, b, c\})\) (which are all cliques). We may assume that \(\mathcal{C}\) is non-empty, since otherwise \(G - \{a, b, c\}\) is connected and Rule 3 applies. There are four cases to consider, one of which splits into multiple subcases:

1. \(\mathcal{C}\) consists of a single clique \(C\): Then \(C\) is adjacent to at least two vertices in \(\{a, b, c\}\), and since \(G - b\) is connected, at least one of \(a, c\) is adjacent to a vertex in \(G'\).

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(a) **C is adjacent to b and to exactly one of \{a, c\}:** By relabeling we may assume that C is adjacent to both b and c. If c is not adjacent to a vertex in \(G'\), then observe that \(G - b\) is disconnected, a contradiction. Therefore, assume that c is adjacent to a vertex in \(G'\). Let \(x\) be a vertex in C with \(\{x, b\} \in E(G)\) such that (if \(|C| > 1\) \(x\) is not the only vertex in C adjacent to c. (Such a vertex exists, as otherwise the only vertex adjacent to c is also the only vertex adjacent to b, and removing this vertex would disconnect the graph, a contradiction). Then \(G - \{a, b, x\}\) is connected, and we have an application of Rule 3.

(b) **Both a and c are adjacent to C:** Without loss of generality, assume that a is adjacent to a vertex in \(G'\).

i. **There exists an \(x \in C\) such that \(\{x, b\} \in E(G), \{x, c\} \notin E(G)\):** If \(G - \{x, b, c\}\) is connected, we have an application of Rule 3. Otherwise, x is the only vertex in C which is adjacent to a. Let y be a vertex in \(C - x\). If \(G - \{a, x, y\}\) is connected, we have an application of Rule 3. Otherwise, x, y are the only vertices in C which are adjacent to vertices in \(a, b, c\). Since \(\{c, x\} \notin E(G)\) but c is adjacent to C, we must have that c is adjacent to y. Let z be a vertex in \(C - \{x, y\}\); then \(G - \{c, y, z\}\) is connected (as we preserve the edge \(ax\)), and so we have an application of Rule 3.

ii. **There exists an \(x \in C\) such that \(\{x, c\} \in E(G), \{x, b\} \notin E(G)\):** If \(G - \{b, c, x\}\) is connected, we have an application of Rule 3. Otherwise, x is the only vertex in C which is adjacent to a. Let y be a vertex in \(C - x\). If \(G - \{a, x, y\}\) is connected, we have an application of Rule 3. Otherwise x, y are the only vertices in C which are adjacent to vertices in \(a, b, c\). Since \(x\) is the only vertex adjacent to a and \(G - x\) is connected, y must be adjacent to at least one of \(b, c\). Let z be a vertex in \(C - \{x, y\}\). If y is adjacent to b then \(\{b, y, z\}\) gives us an application of Rule 3. (We have connectedness, as edges \(\{c, x\}\) and \(\{x, a\}\) are preserved.) Otherwise, y is adjacent to c and \(\{b, c, y\}\) gives us application of Rule 3.

iii. **For every \(x \in C\), \(\{x, b\} \in E(G)\) if and only if \(\{x, c\} \in E(G)\):**

A. **There exist \(x, y \in C\) with \(\{c, x\} \in E(G), \{c, y\} \notin E(G)\):** If \(G - \{c, x, y\}\) is connected, we have an application of Rule 3. Otherwise, \(x, y\) are the only vertices in
C which are adjacent to \( a, b, c \). Since \( x \) is the only vertex which is adjacent to \( c \) (and therefore the only vertex which is adjacent to \( b \)) and \( G - x \) is connected, \( y \) must be adjacent to \( a \). Let \( z \) be a vertex in \( C - \{x, y\} \). Then \( G - \{a, y, z\} \) is connected (because \( G - a \) is connected, so \( b \) or \( c \) must be connected to \( G' \), and everything in \( C \) is connected to \( b \) and \( c \) via \( x \)), and we have an application of Rule 3.

B. No vertex in \( C \) is adjacent to \( c \): Then no vertex in \( C \) is adjacent to \( b \) either, and so \( G - a \) is disconnected, a contradiction.

C. Every vertex in \( C \) is adjacent to \( c \): Then every vertex in \( C \) is adjacent to \( b \) as well. If there exists \( x \in C \) not adjacent to \( a \), then let \( y \) be a vertex in \( C \) that is adjacent to \( a \). Then \( G - \{a, y\} \) is connected, and we have an application of Rule 3. Otherwise every vertex in \( C \) is adjacent to \( a \). If \( b \) is not adjacent to \( G' \) then we have an application of Rule 4. Otherwise, \( G - \{a, x, c\} \) is connected for any \( x \in C \), and we have an application of Rule 3.

2. There exist \( i \neq j \) such that \( a \) is adjacent to vertices in \( C_i \) and \( C_j \): We show that there exist \( v_i \in C_i, v_j \in C_j \) such that \( \{a, v_i\}, \{a, v_j\} \in E(G) \), and \( G - \{a, v_i\} \) and \( G - \{a, v_j\} \) are both connected. It follows that \( G - \{a, v_i, v_j\} \) is connected, and since \( \{v_i, v_j\} \notin E(G) \) this means we can apply Rule 3. If \( |C_i| = 1 \) then let \( v_i \) be the single vertex in \( C_i \), and observe that \( G - \{a, v_i\} \) is connected. If \( |C_i| \geq 2 \), then let \( x \) be a vertex in \( C_i \) with \( \{a, x\} \in E(G) \). Since \( G - x \) is connected, there exists another vertex \( y \in C_i \) which is adjacent to some vertex in \( \{a, b, c\} \). If \( y \) is adjacent to \( b \) or \( c \), then let \( v_i = x \) and observe that \( G - \{a, x\} \) is connected. Otherwise \( y \) must be adjacent to \( a \), but since \( G - a \) is connected there exists some vertex \( z \in C_i, z \neq y \) (possibly \( z = x \)) which is adjacent to either \( b \) or \( c \). Then observe that \( G - \{a, y\} \) is connected, and let \( v_i = y \). By an identical argument, we can find \( v_j \in C_j \) such that \( \{a, v_j\} \in E(G) \) and \( G - \{a, v_j\} \) is connected.

3. There exist \( i \neq j \) such that \( c \) is adjacent to vertices in \( C_i \) and \( C_j \): This case is handled identically to the previous case.

4. There exist \( i \neq j \) such that \( b \) is adjacent to vertices in \( C_i \) and
Let $x_i \in C_i$ be such that $\{x_i, b\} \in E(G)$. If $G - \{x_i, b\}$ is connected, then let $v_i = x_i$. Otherwise, $x_i$ is the only vertex in $C_i$ adjacent to $a$. Since $G - x_i$ is connected, there exist $z_i \in C_i, z_i \neq x_i$ such that $z_i$ is adjacent to $b$. Then let $v_i = z_i$, and note that $G - \{b, v_i\}$ is connected since $C_i$ is connected to $a$ via $x_i$. By an identical argument, we can find $v_j \in C_j$ such that $\{b, v_j\} \in E(G)$ and $G - \{b, v_j\}$ is connected.

\[ \blacksquare \]

**Lemma 8.** Let $G$ be a connected graph and let $S \subseteq V(G)$ be the set of vertices that are marked after applying Rules 1–4 exhaustively to $G$; then $G - S$ is a disjoint union of trees of cliques.

**Proof.** By Lemma 7, a graph to which Rules 1, 2, 3 and 4 have been applied exhaustively contains no edges, and is therefore a disjoint union of trees of cliques. (In fact by Lemma 6 such a graph is also connected, and therefore consists of a single vertex.) We now proceed by induction. Let $G'$ be a graph derived from $G$ by an application of Rule 1, 2, 3 or 4, and assume that $G' - S$ is a disjoint union of trees of cliques. We will show that $G - S$ is a disjoint union of trees of cliques.

If $G'$ is derived from $G$ by an application of Rule 1, observe that $G - S$ can be formed from $G - S$ by adding a disjoint clique and identifying one of its vertices with the vertex $v$ in $G'$ (unless $v$ is in $S$, in which case $G - S$ is just formed by adding a disjoint clique), where $v$ is the vertex referred to in Rule 1. Therefore, $G - S$ is a disjoint union of trees of cliques.

For every other rule, observe that $G - (G' \cup S)$ is a disjoint union of cliques, and that $S$ disconnects $G - (G' \cup S)$ from $G'$. Therefore, $G - S$ can be formed from $G' - S$ by adding a disjoint union of cliques, and so $G - S$ is a disjoint union of trees of cliques.

\[ \blacksquare \]

Putting Lemmas 6 and 8 together, we can now prove Lemma 5.
Proof of Lemma 5. Apply Rules 1, 2, 3 and 4 exhaustively, and let \( S \) be the set of vertices which are marked after doing this. By Lemma 8, every connected component in \( G - S \) is a tree of cliques. Therefore, if \( |S| < 3k \) we are done. It remains to show that if \( |S| \geq 3k \) then \( G \) has an assignment that satisfies at least \( m_2 + n - 1 + k \) edges.

So suppose that \( |S| \geq 3k \). Let \((G', k')\) be the instance obtained from \((G, k)\) by exhaustively applying Rules 1, 2, 3 and 4. Observe that every time \( k \) is reduced, at most three vertices are marked. Therefore since at least \( 3k \) vertices are marked, we have \( k' \leq 0 \). But since the Edwards-Erdős bound holds for all connected graphs, \( G' \) is a “yes”-instance. Therefore, by Lemma 6 \( G \) is a “yes”-instance with parameter \( k \), as required. 

We now show that, for a given assignment to \( S \), we can efficiently find an optimal extension to \( G - S \). For this, we consider the following generalisation of Max-Cut where each vertex has an associated weight for each part of the partition. These weights may be taken as an indication of how much we would like the vertex to appear in each part.

| Max-Cut-with-Weighted-Vertices |
|-------------------------------|
| **Instance**: A graph \( G \) with weight functions \( w_0 : V(G) \to \mathbb{N}_0 \) and \( w_1 : V(G) \to \mathbb{N}_0 \), and an integer \( t \in \mathbb{N} \). |
| **Question**: Does there exist an assignment \( f : V \to \{0, 1\} \) such that \( \sum_{xy \in E} |f(x) - f(y)| + \sum_{f(x)=0} w_0(x) + \sum_{f(x)=1} w_1(x) \geq t? \) |

Now Max-Cut is the special case of Max-Cut-with-Weighted-Vertices in which \( G \) is connected and \( w_0(x) = w_1(x) = 0 \) for all \( x \in V(G) \).

**Lemma 9.** Max-Cut-with-Weighted-Vertices is solvable in polynomial time when \( G \) is a disjoint union of trees of cliques.

**Proof.** We provide a polynomial-time transformation that replaces an instance \((G, w_0, w_1, t)\) with an equivalent instance \((G', w'_0, w'_1, t')\) such that \( G' \) has fewer vertices than \( G \). By applying the transformation at most \( |V(G)| \) times to get a trivial instance, we have a polynomial-time algorithm to decide \((G, w_0, w_1, t)\).

We may assume that \( G \) is connected, as otherwise we can handle each connected component of \( G \) separately. While \( G \) is non-empty, find a vertex \( v \in V(G) \) and set of vertices \( X \subseteq V(G) \) such that \( X \) is a clique and a connected component of \( G - v \). Such a pair \((v, X)\) must exist unless \( G \) contains no edges, by definition of a tree of cliques. For each possible assignment to
v, we will in polynomial time calculate the optimal extension to the vertices in X. (This optimal extension depends only on the assignment to v, since no other vertices are adjacent to vertices in X.) We can then remove all the vertices in X, and change the values of \(w_0(v)\) and \(w_1(v)\) to reflect the optimal extension for each assignment.

Suppose we assign \(v\) the value 1. Let \(\varepsilon(x) = w_1(x) - w_0(x)\) for each \(x \in X\). Now arrange the vertices of X in order \(x_1, x_2, \ldots x_{n'}\) (where \(n' = |X|\)), such that if \(i < j\) then \(\varepsilon(x_i) \geq \varepsilon(x_j)\). Observe that there is an optimal assignment for which \(x_i\) is assigned 1 for every \(i \leq t\), and \(x_i\) is assigned 0 for every \(i > t\), for some \(0 \leq t \leq n'\). (Consider an assignment for which \(f(x_i) = 0\) and \(f(x_j) = 1\), for \(i < j\), and observe that switching the assignments of \(x_i\) and \(x_j\) will not make the assignment worse.) Therefore we only need to try \(n' + 1\) different assignments to the vertices in \(X\) in order to find the optimal coloring when \(f(v) = 1\). Let \(A\) be the value of this optimal assignment (over \(X \cup \{v\}\)).

By a similar method we can find the optimal coloring when \(v\) is assigned 0. Let the number of satisfied edges in this coloring be \(B\). Now remove the vertices in \(X\) and incident edges, and let \(w_1(v) = A\), and let \(w_0(v) = B\).

We are ready to prove Theorem 1 and show that Max-Cut-AEE is fixed-parameter tractable.

**Proof of Theorem 1.** By Lemma 5, we can in polynomial time decide that either \(G\) has an assignment that satisfies at least \(\frac{m}{2} + \frac{n-1}{4} + \frac{k}{2}\) edges, or find a set \(S\) of at most \(3k\) vertices in \(G\) such that every connected component of \(G - S\) is a tree of cliques. So assume we have found such an \(S\). Then we transform our instance into at most \(2^{3k}\) instances of Max-Cut-with-Weighted-Vertices, such that the answer to our original instance is “yes” if and only if the answer to at least one of the instances of Max-Cut-with-Weighted-Vertices is “yes”, and in each Max-Cut-with-Weighted-Vertices instance the graph is a disjoint union of trees of cliques. As each of these instances can be solved in polynomial time by Lemma 5, we have a fixed-parameter tractable algorithm.

For every possible assignment to the vertices in \(S\), we construct an instance of Max-Cut-with-Weighted-Vertices as follows. For every vertex \(x \in G - S\), let \(w_0(x)\) equal the number of vertices in \(S\) adjacent to \(x\) which are colored blue, and let \(w_1(x)\) equal the number of vertices in \(S\) adjacent to \(x\) which are colored red. Then remove the vertices of \(S\) from \(G\). By Lemma 5, the resulting graph \(G'\) is a disjoint union of trees of cliques. Let \(m'\) be the number of edges in \(G - S\), let \(n'\) be the number of vertices in
$G - S$, and let $p$ be the number of edges within $S$ satisfied by the assignment to $S$. Then for an assignment to the vertices of $G - S$, the total number of satisfied edges in $G$ would be exactly $\sum_{x,y \in E(G - S)} |f(x) - f(y)| + p + \sum_{f(x) = 0} w_0(x) + \sum_{f(x) = 1} w_1(x)$, where $f : V(G) \setminus S \to \{0, 1\}$ is such that $f(x) = 0$ if $x$ is colored red, and $f(x) = 1$ if $x$ is colored blue. Thus, the assignment to $S$ can be extended to one that cuts at least $m/2 + n - 1/4 + k + p$ edges in $G$ if and only if the instance of Max-Cut-with-Weighted-Vertices is a “yes”-instance with $t = \frac{m}{2} + \frac{n-1}{4} + \frac{k}{4} - p$.

4 Algorithmic Lower Bounds

We now prove Theorem 2 by a reduction from the Max-Cut problem with parameter the size $k$ of the cut. By a result of Cai and Juedes [5], the Max-Cut problem with parameter the size $k$ of the cut cannot be solved in time $2^{o(k)} \cdot n^{O(1)}$ unless FPT = W[1]. Given the graph $G$ with $m$ edges and $n$ vertices and the integer $k \in \mathbb{N}$, find the value $k'$ such that $G$ has a cut of size $k$ if and only if $G$ has a cut of size $m/2 + (n - 1)/4 + k'$. Using the algorithm by Ngoe and Tuza [26], finding $k'$ takes polynomial time. Then use a hypothetical algorithm to solve Max-Cut AEE in time $2^{o(k')} \cdot n^{O(1)}$ on instance $(G, k')$, return the answer of the algorithm for the pair $(G, k)$ and the Max-Cut problem. Thus, since $k' \leq k$, an algorithm of time complexity $2^{o(k)} \cdot n^{O(1)}$ for Max-Cut AEE forces FPT = W[1], which in turn forces the Exponential Time Hypothesis to fail [8]. This completes the proof of Theorem 2.

5 Polynomial Kernel for Max-Cut Above Edwards-Erdős

In this section, we prove Theorem 3. By Lemma 5 in polynomial time we can either decide that $(G, k)$ is a “yes”-instance, or find a set $S$ of vertices in $G$ such that $|S| < 3k$ and every connected component in $G - S$ is a tree of cliques. In what follows we assume we have found such a set $S$.

Observe that we can find all maximal cliques in $G - S$ in polynomial time. Indeed, if $X$ is a clique on at least 2 vertices then any vertex not in $X$ which is adjacent to two or more members of $X$ is part of a maximal clique containing $X$, and there is only one such clique. Therefore, we can find all the maximal cliques by expanding greedily from each edge in $G - S$.

We define a directed forest $F^*$ to express intersections between maximal cliques in $G - S$. Each vertex $v^*_i$ in $F^*$ is associated with a maximal clique
Let $C_i$ be an arbitrary maximal clique in $G - S$, and let $v_i^*$ be the root vertex of a tree in $F^*$. For each $r$ in turn, if $\{C_1, \ldots, C_r\}$ is the set of maximal cliques in $G - S$, we are done. Otherwise, check whether there exists a maximal clique not in $\{C_1, \ldots, C_r\}$ which shares a vertex with $C_j$, for some $1 \leq j \leq r$. If such a clique exists, let $C_{r+1}$ be such a clique, let $v_{r+1}^*$ be a new vertex in $F^*$, and draw an arc from $v_j^*$ to $v_{r+1}^*$. (Observe that there is at most one $C_j$ for $1 \leq j \leq r$ which shares a vertex with $C_{r+1}$, as otherwise $G - S$ contains a cycle which is not contained within a clique.) Otherwise, let $C_{r+1}$ be an arbitrary maximal clique not in $C_1, \ldots, C_r$, and let $v_{r+1}^*$ be the root of a new tree in $F^*$. Observe that $F^*$ is a forest, and that two vertices $x, y \in G - S$ are only adjacent if there exists $i, j$ such that $x \in C_i, y \in C_j$, and either $i = j$ or $v_i^*$ and $v_j^*$ are adjacent in $F^*$.

Let $n^*$ be the number of vertices in $F^*$ (which equals the number of maximal cliques in $G - S$) and let $m^*$ be the number of edges in $F^*$. Let $L^*$ be the set of vertices in $F^*$ of degree 1, let $B^*$ be the set of vertices in $F^*$ of degree 2, and let $I^*$ be the set of vertices in $F^*$ of degree at least 3, where the degree of a vertex equals the sum of its in- and out-degree. Let $J$ be the set of vertices in $G - S$ which occur in two or more maximal cliques. For each $i \in \{1, \ldots, n^*\}$ let $A_i = C_i - J$. Note that $\{A_1, \ldots, A_n^*, J\}$ is a partition of the vertices of $G - S$. Note also that if $C$ is a connected component of $G - S$ containing $C_i$, then $C - A_i$ is connected.

We first apply rules to eliminate problematic cases in the lemmas that follow.

| Rule 5: | Apply if there exists a vertex $x \in G - S$ and a set of vertices $X \subseteq V(G) \setminus S$ such that $X \cup \{x\}$ is a clique and $X$ is a connected component of $G - (S \cup \{x\})$, and no vertex in $X$ is adjacent to any vertex in $S$. |
| Remove: | All vertices in $X$ and incident edges. |
| Parameter: | Reduce $k$ by 1 if $|X|$ is odd, otherwise leave $k$ the same. |

| Rule 6: | Apply if there exists a vertex $x \in G - S$ and a set of vertices $X \subseteq V(G) \setminus S$ such that $X \cup \{x\}$ is a clique and $X$ is a connected component of $G - (S \cup \{x\})$, and there is exactly one vertex $s \in S$ which is adjacent to $X$, and $X \cup \{s\}$ is a clique. |
| Remove: | All but one vertex of $X$ and incident edges. |
| Parameter: | Reduce $k$ by 1 if $|X|$ is even, otherwise leave $k$ the same. |
Rule 7: Apply if there exist vertex sets $X, Y \subseteq G - S$ such that $X$ and $Y$ are maximal odd cliques, with vertices $x \in X, y \in Y, \{z\} = X \cap Y$, such that $x, z$ are the only vertices in $X$ adjacent to a vertex in $G - X$, and $y, z$ are the only vertices in $Y$ adjacent to a vertex in $G - Y$.

Remove: All vertices in $(X \cup Y) - \{x, y, z\}$ and incident edges.

Add: New vertices $u, v$ and edges such that $\{x, y, z, u, v\}$ is a clique.

Parameter: No change.

Rule 8: Apply if for any maximal clique $C_i$ in $G - S$, there exists $X \subseteq A_i$ such that $|X| > \frac{|A_i| + |J| + |S|}{2}$ and for all $x, y \in X, x$ and $y$ have exactly the same neighbors in $S$.

Remove: Two arbitrary vertices from $X$ and incident edges.

Parameter: No change.

We now establish properties of graphs to which Rules 5–8 have been exhaustively applied.

Lemma 10. Let $(G, k)$ and $(G', k')$ be instances of MAX-CUT-AEE such that every connected component of $G - S$ is a tree of cliques, and $(G', k')$ is reduced from $(G, k)$ by an application of Rules 5, 6, 7 and 8. Then $G'$ is connected, every connected component of $G - S$ is a tree of cliques, and $(G', k')$ is a “yes”-instance if and only if $(G, k)$ is a “yes”-instance.

Proof of Lemma 10. It is easy to observe that each of the rules preserves connectedness, and that every connected component of $G' - S$ is a tree of cliques.

We now show for each rule that $(G', k')$ is a “yes”-instance if and only if $(G, k)$ is a “yes”-instance.

Rule 5: Let $n'$ be the number of vertices and $m'$ the number of edges removed. Note that $m' = \frac{n'(n'+1)}{2}$. Observe that whatever we assign to the rest of the graph, we can always find an assignment to $X$ that satisfies the largest possible number of edges within $X \cup \{x\}$. If $|X|$ is odd this is $\frac{(n'+1)(n'+1)}{4} = \frac{n'}{2} + \frac{n'+1}{4}$, and if $|X|$ is even this is $\frac{n'(n'+2)}{4} = \frac{n'}{2} + \frac{n'}{4}$. Therefore if $|X|$ is odd, we can satisfy $\frac{m}{2} + \frac{n-1}{4} + \frac{k}{4}$ edges in $G$ if and only if we can satisfy $\frac{m-m'}{2} + \frac{n-n'-1}{4} + \frac{k-1}{4}$ edges in $G'$, and if $|X|$ is even, we can satisfy $\frac{m}{2} + \frac{n-1}{4} + \frac{k}{4}$ edges in $G$ if and only if we can satisfy $\frac{m-m'}{2} + \frac{n-n'-1}{4} + \frac{k}{4}$ edges in $G'$.

Rule 6: Let $n' = |X|$ and let $m'$ be the number of edges within $X \cup \{s, x\}$. 19
First, when \( x \) and \( s \) are adjacent, then \( m' = \frac{(n'+2)(n'+1)}{4} \). Observe that whatever \( x \) and \( s \) are assigned, it is possible to find an assignment to \( X \) that satisfies the maximum possible number of edges within \( X \cup \{ s, x \} \). This is \( \frac{(n'+3)(n'+1)}{4} = \frac{m'}{2} + \frac{n'+1}{4} \) if \( |X| \) is odd, and \( \frac{(n'+2)(n'+2)}{4} = \frac{m'}{2} + \frac{n'+2}{4} \) if \( |X| \) is even.

Second, when \( x \) and \( y \) are not adjacent, then \( m' = \frac{(n'+2)(n'+1)}{2} - 1 \). Observe that if \( x \) and \( s \) are colored differently, the maximum number of edges within \( X \cup \{ x \} \cup \{ s \} \) we can satisfy is \( \frac{(n'+3)(n'+1)}{4} - 1 = \frac{m'}{2} + \frac{n'+1}{4} - \frac{1}{2} \) if \( |X| \) is odd, and \( \frac{(n'+2)(n'+2)}{4} - 1 = \frac{m'}{2} + \frac{n'+2}{4} - \frac{1}{2} \) if \( |X| \) is even. If \( x \) and \( s \) are colored the same, the maximum number of edges within \( X \cup \{ x \} \cup \{ s \} \) we can satisfy is \( \frac{(n'+3)(n'+1)}{4} = \frac{m'}{2} + \frac{n'+1}{4} + \frac{1}{2} \) if \( |X| \) is odd, and \( \frac{(n'+2)(n'+2)}{4} = \frac{m'}{2} + \frac{n'+2}{4} + \frac{1}{2} \) if \( |X| \) is even. In either case, given an assignment to \( x \) and \( s \), the amount we can get above the Edwards-Erdős bound remains the same regardless of \(|X|\), except that we have \( \frac{1}{2} \) extra if \( |X| \) is even. Therefore, if \( |X| \) is odd we can keep \( k \) the same and if \( |X| \) is even we reduce \( k \) by 1.

**Rule 7:** Let \( n' = |X \cup Y| \) and let \( m' \) be the number of edges within \( X \cup Y \). Let \( n'_1 = |X| \) and \( n'_2 = |Y| \), and let \( m'_1 \) and \( m'_2 \) be the number of edges within \( X \) and \( Y \), respectively. Observe that whatever \( x, y \) and \( z \) are assigned, we can always satisfy the maximal possible number of edges within \( X \cup Y \), which is \( \frac{(n'_1+1)(n'_1-1)}{4} + \frac{(n'_2+1)(n'_2-1)}{4} \). Let \( n'_3 = 5 = |\{ x, y, z, u, v \}| \) and let \( m'_3 = 10 \) be the number of edges within \( \{ x, y, z, u, v \} \). Then in \( G' \), whatever \( x, y \) and \( z \) are assigned, the maximum number of edges within \( \{ x, y, z, u, v \} \) we can satisfy is \( 6 = \frac{m'_3}{2} + \frac{n'_3-1}{4} \). Thus, the amount we gain above the Edwards-Erdős bound remains the same.

**Rule 8:** Let \( n' = |A_i| \). Assume some assignment to the vertices in \( S \cup J \). For each \( x \in A_i \), let \( \varepsilon_R(x) \) be the number of neighbors of \( x \) in \( S \cup J \) which are assigned red, and let \( \varepsilon_B(x) \) be the number of neighbors of \( x \) in \( S \cup J \) which are assigned blue. Let \( \varepsilon(x) = \varepsilon_B(x) - \varepsilon_R(x) \). Let the vertices of \( A_i \) be numbered \( x_1, x_2, \ldots, x_n \) such that \( \varepsilon(x_1) \geq \varepsilon(x_2) \geq \ldots \geq \varepsilon(x_n) \). Observe that the optimal assignment to \( A_i \) will be one in which \( x_j \) is assigned red for \( j \leq \frac{n'+x}{2} \), and blue otherwise, for some integer \( r \). Observe that the optimal value of \( r \) will be between \( -(|J|+|S|) \) and \( (|J|+|S|) \). Indeed, if \( r > |J|+|S| \), then switching one of the vertices from red to blue will gain at least \( |J|+|S| \) edges within \( A_i \), and lose at most \( |J|+|S| \) edges between \( A_i \) and \( J \cup S \). (A similar argument holds when \( r < -|J| - |S| \).)

Now suppose there exists \( X \subseteq A_i \) such that \( |X| > \frac{|A_i| + |J| + |S|}{2} \) and for all \( x, y \in X \), \( x \) and \( y \) have exactly the same neighbors in \( S \). Observe that
because the optimal value of $r$ is between $-(|J| + |S|)$ and $(|J| + |S|)$, there
will exist $x, y \in X$ such that $x$ is assigned blue and $y$ is assigned red. Then
note that if we remove $x$ and $y$, we lose $m'' = 2|N(x) \cap (S \cup J)| + 2(n' - 2) + 1$
edges and $n'' = 2$ vertices. Of the edges removed, exactly $|N(x) \cap (S \cup J)| + (n' - 2) + 1 = \frac{m''}{2} + \frac{n''}{4}$ were satisfied. Thus, the amount we gain over the
Edwards-Erdős bound remains the same. Note that this happens whatever
the assignment to $S \cup J$, and that we may assume without loss of generality
that the same $x$ and $y$ are colored differently for any assignment to $S \cup J$. □

We now assume that $G$ is reduced by Rules 5, 6, 7 and 8.

**Lemma 11.** If $|L^*| \geq 4|S|^2 + 2|S| + 2k - 2$ then $(G, k)$ is a “yes”-instance.

**Proof.** Suppose $|L^*| \geq 4|S|^2 + 2|S| + 2k - 2$. For $v_i^* \in L^*$, let $E_i$ be set of
dges within $G[A_i]$ together with the set of edges between $A_i$ and $S$, and
let $E_L = \bigcup_{i \in L^*} E_i$. Let $V_i$ be the total set of vertices in $\bigcup_{i \in L^*} A_i$, and
observe that $|V_L| = \sum_{i \in L^*} |A_i|$. Let $m_i = |E_i|$ and $n_i = |A_i|.$

For a partial assignment $\alpha$ and a set of edges $E$, let $\chi_{\alpha}(E)$ be the maximum possible number of edges we can satisfy in $E$, given $\alpha$. We will show via the probabilistic method that there exists a partial assignment $\alpha$ on $S$
such that $\chi_{\alpha}(E_L) \geq \frac{|E_L|}{2} + \frac{|V_L|}{4} + \frac{|L^*|}{4}$. Consider a random assignment $\alpha$
on $S$, in which each vertex is assigned red or blue with equal probability. For
any vertex $x \in G - S$, let $\varepsilon^a_B(x)$ be the number of neighbors of $x$ which
are assigned red, let $\varepsilon^a_R(x)$ be the number of neighbors of $x$ which are
assigned blue, and let $\varepsilon(x) = \varepsilon^a_B(x) - \varepsilon^a_R(x)$. Now consider $\chi_{\alpha}(E_i)$ for some $i$
where $v_i^* \in L^*$. Let $x_1, \ldots, x_{n_i}$ be an ordering of the vertices of $A_i$ such
that $\varepsilon(x_1) \geq \varepsilon(x_2) \geq \ldots \geq \varepsilon(x_{n_i}).$

First, suppose that $|A_i|$ is odd. Observe that if there exists $x \in A_i$ with
$|\varepsilon^a(x)| \geq r$, then $\chi_{\alpha}(E_i) \geq \frac{m_i}{2} + \frac{n_i - 1}{4} + \frac{r}{2}$. Indeed, suppose $\varepsilon^a(x) = r > 0$. Then color $x_j$ red if $j \leq \frac{m_i + 1}{2}$ and blue otherwise. Then the total number
of satisfied edges in $E_i$ is $\frac{m_i + 1}{2} \cdot \frac{n_i - 1}{4} + \sum_{j=1}^{n_i-1} \varepsilon^a_B(x_j) + \sum_{j=n_i+1}^{2n_i-1} \varepsilon^a_R(x_j) = \frac{m_i}{2} + \frac{n_i - 1}{4} + \frac{1}{2} \sum_{j=1}^{n_i-1} \varepsilon^a(x_j) - \sum_{j=n_i+1}^{2n_i} \varepsilon^a(x_j))$. Observe that this is at least
$\frac{m_i}{2} + \frac{n_i - 1}{4} + \frac{1}{2} \varepsilon^a(x_1) \geq \frac{m_i}{2} + \frac{n_i - 1}{4} + \frac{r}{2}$. A similar argument applies when
$\varepsilon(x_j) = r < 0$. Therefore, if there exists $x \in A_i$ with an odd number of
neighbors in $S$, then $|\varepsilon^a(x)| \geq 1$ and so $\chi_{\alpha}(E_i) \geq \frac{m_i}{2} + \frac{n_i}{4} + \frac{1}{4}$. This is
ture whatever $\alpha$ assigns to $S$, and therefore $\mathbb{E}_\alpha[\chi_{\alpha}(E_i)] \geq \frac{m_i}{2} + \frac{n_i}{4} + \frac{1}{4}$. If
there exists $x \in A_i$ with a non-zero even number of neighbors in $S$, then
observe that either $\varepsilon^a(x) = 0$ or $|\varepsilon^a(x)| \geq 2$. Furthermore, the probability
that $\varepsilon^a(x) = 0$ is at most $\frac{1}{2}$, since given an assignment to all but one of
the neighbors in $S$ of $x$, at most one of the possible assignments to the remaining neighbor will lead to $e^α(x)$ being 0. Therefore, $E_α[χ_α(E_i)] ≥ \frac{m_i}{2} + \frac{n_i}{4} + \frac{1}{2}$. We know that one of the above two cases must hold, since otherwise no vertex in $A_i$ has any neighbors in $S$, and Rule 5 applies. Therefore, if $|A_i|$ is odd, $E_α[χ_α(E_i)] ≥ \frac{m_i}{2} + \frac{n_i}{4} + \frac{1}{2}$.

Second, suppose that $|A_i|$ is even. Observe that if there exist $x, y \in A_i$ with $e^α(x) > e^α(y)$, then $χ_α(E_i) ≥ \frac{m_i}{2} + \frac{n_i}{4} + \frac{1}{2}$. Indeed, then $\sum_{j=1}^{n_i} e^α(x_j) - \sum_{j=\frac{n_i}{2}+1}^{n_i} e^α(x_j) ≥ 1$. So color $x_j$ with red if $j ≤ \frac{n_i}{2}$ and blue otherwise.

Then the total number of satisfied edges in $E_i$ is $\frac{m_i}{2} + \frac{n_i}{4} + \frac{1}{2}$. Indeed, suppose $e^α(x) > e^α(y)$ for all $x \in A_i$, then $χ_α(E_i) ≥ \frac{m_i}{2} + \frac{n_i}{4} + 1$. Then color $x_j$ with red if $j ≤ \frac{n_i}{2}$ and blue otherwise. Then the total number of satisfied edges in $E_i$ is $\frac{m_i}{2} + \frac{n_i}{4} + \frac{1}{2} + \sum_{j=\frac{n_i}{2}+1}^{n_i} e^α(x_j) + \sum_{j=\frac{n_i}{2}+2}^{n_i} e^α(x_j) = \frac{m_i}{2} + \frac{n_i}{4} + 1 + \frac{1}{2}(\sum_{j=1}^{n_i} e^α(x_j) - \sum_{j=\frac{n_i}{2}+1}^{n_i} e^α(x_j)) = \frac{m_i}{2} + \frac{n_i}{4} + 1 - 1 + \frac{1}{2}r ≥ \frac{m_i}{2} + \frac{n_i}{4} + 1$.

A similar argument applies when $e^α(x) = r ≥ 2$ for all $x \in A_i$. Finally observe that if $e^α(x) = 0$ or 1 for all $x \in A_i$, then by coloring $x_j$ with red if $j ≤ \frac{n_i}{2}$ and blue otherwise, $χ_α(E_i) ≥ \frac{m_i}{2} + \frac{n_i}{4}$.

Therefore, if there exist $x, y \in A_i$ such that $x$ has a neighbor in $S$ which is not adjacent to $y$, then the probability that $e^α(x) = e^α(y)$ is at most $\frac{1}{2}$ and so $E_α[χ_α(E_i)] ≥ \frac{m_i}{2} + \frac{n_i}{4} + \frac{1}{2} ∙ 0 + \frac{1}{2} ∙ \frac{1}{2} = \frac{m_i}{2} + \frac{n_i}{4} + \frac{1}{2}$.

Otherwise, all vertices in $A_i$ have the same neighbors in $S$. There must be at least two vertices in $S$ which are adjacent to the vertices in $A_i$, as otherwise Rule 5 or 6 would apply. Then the probability that $e^α(x) = 0$ for every $x \in A_i$ is at most $\frac{1}{2}$, and the probability that $|\varepsilon(x)| ≥ 2$ is at least $\frac{1}{4}$. (Consider any assignment to all but two of the neighbors of $x$ in $S$, and observe that of the four possible assignments to the remaining two vertices, at most two will lead to $e^α(x) = 0$, and at least one will lead to $|\varepsilon(x)| ≥ 2$.) Therefore, $E_α[χ_α(E_i)] ≥ \frac{m_i}{2} + \frac{n_i}{4} + \frac{1}{2} ∙ 0 + \frac{1}{2} ∙ \frac{1}{2} = \frac{m_i}{2} + \frac{n_i}{4} + \frac{1}{4}$.

By linearity of expectation and the fact that $χ_α(E_L) = \sum_{v^*_i ∈ L^*} χ_α(E_i)$, it holds $E_α[χ_α(E_L)] ≥ \sum_{v^*_i ∈ L^*} E_α[χ_α(E_i)] = \sum_{v^*_i ∈ L^*} (\frac{m_i}{2} + \frac{n_i}{4} + \frac{1}{4} = \frac{|E_L|}{2} + \frac{|V_L|}{4} + \frac{|L^*|}{4}).$ Therefore, there exists a partial assignment $α$ on $S$ such that $χ_α(E_L) ≥ \frac{|E_L|}{2} + \frac{|V_L|}{4} + \frac{|L^*|}{4}$, as required.

Now let $t'$ be the number of connected components in $G − (S ∪ V_L)$, and observe that $t' ≤ \frac{|L^*|}{2}$. Indeed, if $C_j$ is a connected component in $G − S$ then
\( C_j - V_L \) is connected if not empty. Furthermore, every connected component in \( F^* \) contains at least one leaf, and if a connected component in \( F^* \) contains only one leaf, then the corresponding connected component \( C_j \) in \( G - S \) consists of a single maximal clique, and so \( C_j - V_L \) is empty. Therefore, every connected component in \( G - (S \cup V_L) \) corresponds to a connected component in \( F^* \) with at least two leaves.

Let \( C_1, \ldots, C_{t'} \) be the connected components of \( G - (S \cup V_L) \). For each \( j \in \{1, \ldots, t'\} \) let \( n_j' \) be the number of vertices in \( C_j \) and \( m_j' \) the number of edges incident with vertices in \( C_j \) (including edges between \( C_j \) and \( S \cup V_L \)). Now let \( S \cup V_L \) be colored such that the number of satisfied edges in \( E_L \) is at least \( |E_L| + \frac{|V_L|}{4} + \frac{|L^*|}{4} \). Note that this might mean that none of the edges in \( G[S] \) are satisfied, but that there are at most \( |S|^2 \) of these. Now for each \( j \in \{1, \ldots, t'\} \) let \( C_j \) be colored so that at least half the edges in \( G[C_j] \) plus \( \frac{n_j' - 1}{4} \) are satisfied, which can be done as this is the Edwards-Erdős bound. By reversing all the colors in \( C_j \) if needed, we can ensure that at least half the edges between \( C_j \) and the rest of the graph are satisfied, and therefore we can ensure at least \( \frac{m_j'}{2} + \frac{n_j' - 1}{4} \) of the edges incident with \( C_j \) are satisfied.

We now have a complete assignment of colors to vertices, which satisfies at least \( \frac{|E_L|}{2} + \frac{|V_L|}{4} + \frac{|L^*|}{4} + \sum_{j=1}^{t'} \left( \frac{m_j'}{2} + \frac{n_j' - 1}{4} \right) = \frac{|E_L|}{2} + \sum_{j=1}^{t'} \frac{m_j'}{2} + \frac{|V_L|}{4} + \sum_{j=1}^{t'} \frac{|L^*| - |I^*|}{4} \geq \frac{m - |S|^2}{2} + \frac{n - |S|}{4} + \frac{|L^*|}{8} \) edges. Since \( |L^*| \geq 4|S|^2 + 2|S| + 2k - 2 \), this is at least \( \frac{m}{2} + \frac{n}{4} - \frac{4|S|^2 + 2|S|}{8} + \frac{4|S|^2 + 2|S| + 2k - 2}{8} = \frac{m}{2} + \frac{n - 1}{4} + \frac{k}{4} \). \( \Box \)

**Lemma 12.** If \( n^* \geq 4|L^*| + 2|S|^2 + |S| + k - 2 \) then \( (G, k) \) is a “yes”-instance.

**Proof.** Observe that since \( F^* \) is a forest, \( |L^*| \geq 2\lfloor |I^*| \rfloor + 1 \). We will first produce a partial coloring on \( G - S \) which satisfies \( \frac{|E(G - S)|}{2} + \frac{|V - S|}{4} \) edges, together with a set of vertices \( R \subseteq V - S \) such that we can change the color of any vertex in \( R \) without changing the number of satisfied edges in \( G - S \).

Recall that the vertices of \( F^* \) (and the corresponding maximal cliques in \( G - S \)) are numbered such that if there is an arc from \( v_i^* \) to \( v_j^* \) then \( i < j \). It follows that for any \( i \) the maximal clique \( C_i \) shares at most one vertex with \( \bigcup_{j<i} C_j \). So color the vertices of \( C_1, C_2, \ldots, C_{n^*} \) in turn such that if \( |C_i| \) is even then half the vertices of \( C_i \) are colored red and half are colored blue, and if \( |C_i| \) is odd then \( \frac{|C_i| + 1}{2} \) of the vertices are colored red and \( \frac{|C_i| - 1}{2} \) are colored blue. Furthermore, if \( |C_i| \) is odd and \( A_i \) contains a vertex which is adjacent to \( S \), then we ensure that at least one such vertex \( x \) is colored red, and we add \( x \) to \( R \).

Observe that the number of satisfied edges in a connected component \( C_j \) of \( G - S \) is \( \frac{|E(G[C_j])|}{2} + \frac{|C_j| + e_j - 1}{4} \), where \( e_j \) is the number of \( i \) for which
$C_i \subseteq C_j$ and $|C_i|$ is even. Therefore, the total number of satisfied edges within $G - S$ is $\frac{|E(G-S)|}{2} + \frac{|V-S|+|L^*-e|-t}{4}$, where $e$ is the number of maximal cliques in $G - S$ with an even number of vertices, and $t$ is the number of connected components in $G - S$. Note also that $t \leq |L^*|$. Observe that if we change the color of any vertex in $R$, the number of satisfied edges within $C_j$ is unchanged.

First, suppose that $|R| \geq \frac{2|S|^2+|S|+|L^*|-e+k-1}{2}$; we show that we have a "yes"-instance. Now color all the vertices in $S$ red, or all blue, whichever satisfies the most edges in $E(S, G - S)$. If this satisfies at least $|E(G-S)| + 2|S|^2+|S|+|L^*|-e+k-1$ edges in $E(S, G - S)$, then we are done, as the total number of satisfied edges is at least $|E(G-S)| + |E(S, G-S)| + |V(G-S)|+e-1 + 2|S|^2+|S|+|L^*|-e+k-1 \geq \frac{m-|S|^2}{2} + \frac{n-|S|}{4} + \frac{|e-|L^*||}{4} + \frac{2|S|^2+|S|+|L^*|-e+k-1}{4} \geq \frac{n-1}{2} + k$. Otherwise, the number of satisfied edges in $E(S, G - S)$ is between $|E(S, G-S)|$ and $\frac{2|S|^2+|S|+|L^*|-e+k-1}{4}$. Now change the colors of all the vertices in $R$ from red to blue, and recall that this does not affect the number of satisfied edges within $G - S$. If the vertices of $S$ were all colored red, then the number of satisfied edges in $E(S, G - S)$ is increased by $|R|$ to at least $|E(S, G-S)| + \frac{2|S|^2+|S|+|L^*|-e+k-1}{4}$, and we are done. Otherwise, the vertices of $S$ were all colored blue and we lose $|R|$ satisfied edges, so the number of satisfied edges in $E(S, G - S)$ is at most $|E(S, G-S)| - \frac{2|S|^2+|S|+|L^*|-e+k-1}{4}$. But then by changing the color of $S$ from blue to red, we satisfy at least $\frac{|E(S, G-S)|}{2} + \frac{2|S|^2+|S|+|L^*|-e+k-1}{4}$ edges in $E(S, G - S)$, and we are done.

Second, suppose that $|R| < \frac{2|S|^2+|S|+|L^*|-e+k-1}{2}$. Let $B'$ be the set of vertices $v_i^*$ in $F^*$ such that either $|C_i|$ is even or $C_i$ contains an element of $R$. Note that since $|R| < \frac{2|S|^2+|S|+|L^*|-e+k-1}{2}$, we have $|B'| \leq e + |R| \leq 2|B| + e \leq 2|S|^2 + |S| + |L^*| + k - 1$. Let $B'' = B^* - B'$. Note that $B''$ consists of vertices $v_i^*$ in $F^*$ such that the degree of $v_i^*$ is exactly 2; $|C_i|$ is odd, and no vertices in $C_i$ are adjacent to $S$ except possibly the two vertices in $C_i$ that appear in other maximal cliques (as otherwise we would have added a vertex from $C_i$ to $R$). Note further that no two vertices in $B''$ can be adjacent in $F^*$, as otherwise we would have an application of Rule 7. Therefore, every element of $B''$ appears in $F^*$ between two elements of $(L^* \cup I^* \cup B')$. Since $F^*$ is a forest, the maximum number of elements of $B''$ is therefore at most $|L^*| + |I^*| + |B'| - 1$. It follows that the total number of vertices in $F^*$ is at most $|L^*| + |I^*| + |B'| + |B''| = 2(|L^*| + |I^*| + |B'|) - 1 \leq 2(|L^*| + 2)\frac{|L^*|-1}{2} + 2|S|^2 + |S| + |L^*| - 1 \leq 5|L^*| + 4|S|^2 + 2|S| + 2k - 4$. \[\square\]
Lemma 13. If \(|A_i| \geq 2|S|^3 + 5|S|^2 + (|L^*| + k - 3)|S| - 2|L^*| - |J| - 2k\) for some \(i \in \{1, \ldots, n^*\}\) then \((G, k)\) is a “yes”-instance.

Proof. Let \(i\) be such that \(|A_i| \geq 2|S|^3 + 5|S|^2 + (|L^*| + k - 3)|S| - 2|L^*| - |J| - 2k\). Let \(n' = |A_i|\) and let \(m'\) be the number of edges within \(A_i\) and between \(A_i\) and \(S\). Observe first that if we can find an assignment to \(S \cup A_i\) that satisfies at least \(\frac{m'}{2} + \frac{n' - 1}{4} + \frac{2|S|^2 + |S|^2 + |L^*| + k}{4}\) of these edges, then we have a “yes”-instance: indeed, the connected component of \(G - S\) containing \(A_i\) is still connected in \(G - (S \cup A_i)\) unless it is empty. Therefore, the number of connected components in \(G - (S \cup A_i)\) is no more than in \(G - S\) and is therefore at most \(t \leq |L^*|\). Let \(C_1, C_2, \ldots, C_t\) be the connected components of \(G - (S \cup A_i)\). Color each connected component \(C_j\) optimally so that at least \(\frac{E(G[C_j])}{2} + \frac{|C_j| - 1}{4}\) of the edges within \(C_j\) are satisfied, and then reverse the colors of \(C_j\) if necessary to ensure that at least half the edges between \(C_j\) and \(S \cup A_i\) are satisfied. Then the total number of satisfied edges is at least \(\frac{m - |E(G[S \cup A_i])|}{2} + \frac{n - |S \cup A_i| - t}{4} + \frac{m'}{2} + \frac{n' - 1}{4} + \frac{2|S|^2 + |S|^2 + |L^*| + k}{4} \geq \frac{m - m' - |S|^2}{2} + \frac{n - n' - |S| - |L^*|}{4} + \frac{m' + |S|^2}{2} + \frac{n' + |S| + |L^*| + k - 1}{4} = m + \frac{n - 1}{4} + \frac{k - 2}{4}\), and so we have a “yes”-instance.

We now show that if \(n' \geq 2|S|^3 + 5|S|^2 + (|L^*| + k - 3)|S| - 2|L^*| - |J| - 2k\) then we can find a partial assignment to \(S \cup A_i\) that satisfies at least \(\frac{m'}{2} + \frac{n' - 1}{4} + \frac{2|S|^2 + |S|^2 + |L^*| + k}{4}\) of the edges within \(G[A_i]\) and between \(A_i\) and \(S\), completing the proof. For a given partial assignment \(\alpha\) on \(S\), and for any vertex \(x \in A_i\), let \(\varepsilon_R^B(x)\) be the number of neighbors of \(x\) which are assigned \(\text{red}\), let \(\varepsilon_B^R(x)\) be the number of neighbors of \(x\) which are assigned \(\text{blue}\), and let \(\varepsilon^\alpha(x) = \varepsilon_R^B(x) - \varepsilon_B^R(x)\). Let \(x_1, \ldots, x_t\) be the vertices of \(A_i\), ordered so that \(\varepsilon^\alpha(x_1) \geq \varepsilon^\alpha(x_2) \geq \ldots \geq \varepsilon^\alpha(x_t)\) (the ordering will depend on \(\alpha\)). Suppose for some \(r \in \{1, \ldots, n^*\}\) we assign \(x_r\) red for \(j \leq \frac{n' - r}{2}\), and the assign the remaining \(\frac{n' - r}{2}\) vertices in \(A_i\) blue. Then the number of satisfied edges is

\[
\frac{m'}{2} + \frac{n' - r}{4} + \frac{1}{2}(\sum_{j=1}^{\frac{n' - r}{2}} \varepsilon^\alpha(x_j) - \sum_{j=\frac{n' - r}{2} + 1}^{n'} \varepsilon^\alpha(x_j)).
\]

Suppose there exist \(X_1, X_2 \subseteq A_i\) with \(|X_1|, |X_2| \geq \frac{2|S|^2 + |S|^2 + |L^*| + k}{2}\) and such that \(\varepsilon^\alpha(x) > \varepsilon^\alpha(y)\) for all \(x \in X_1, y \in X_2\). Observe that by setting \(r = 0\) (if \(n'\) is even) or \(r \in \{\pm 1\}\) (if \(n'\) is odd), we get that \(\frac{1}{2}(\sum_{j=1}^{\frac{n' - r}{2}} \varepsilon^\alpha(x_j) - \sum_{j=\frac{n' - r}{2} + 1}^{n'} \varepsilon^\alpha(x_j))\) is at least \(\frac{2|S|^2 + |S|^2 + |L^*| + k}{2}\) and so we are done.

It remains to show that there is some partial assignment \(\alpha\) on \(S\) such that \(X_1, X_2\) exist. For the sake of contradiction, suppose this were not the case. Then for any assignment \(\alpha\) there exist a set of at least \(n' - 2|S|^2 - |S| - |L^*| - k\) vertices \(x\) in \(A_i\) for which \(\varepsilon^\alpha(x)\) is the same. Consider first the partial
assignment $\alpha$ for which every vertex in $S$ is assigned red. Then for every $x \in A_i$, $\varepsilon^\alpha(x) = |N(x) \cap S|$. It follows that $|N(x) \cap S|$ is the same for at least $n' - 2|S|^2 - |S| - |L^*| - k$ vertices $x$ in $A_i$. Let $A'_i$ be this set of vertices. Now for each vertex $s \in S$, consider the partial assignment $\alpha$ for which $s$ is assigned blue and every other vertex in $S$ is assigned red. Let $X_1$ be the set of vertices in $A'_i$ which are adjacent to $s$, and let $X_2$ be the set of vertices in $A'_i$ which are not adjacent to $s$. Then $\varepsilon^\alpha(x) = \varepsilon^\alpha(y)$ if $x, y \in X_1$ or $x, y \in X_2$, and $\varepsilon^\alpha(x) = \varepsilon^\alpha(y) - 2$ if $x \in X_1, y \in X_2$. Therefore by our assumption either $|X_1| < \frac{2|S|^2 + |S| + |L^*| + k}{2}$ or $|X_2| < \frac{2|S|^2 + |S| + |L^*| + k}{2}$. It follows that for each $s \in S$ there are at most $\frac{2|S|^2 + |S| + |L^*| + k}{2}$ vertices in $A'_i$ which are adjacent to $s$, or at most $\frac{2|S|^2 + |S| + |L^*| + k}{2}$ vertices in $A'_i$ which are not adjacent to $s$. Therefore, there is a set of at least $|A'_i| - |S| - 2|S|^2 - |S| - |L^*| - k - |S| \geq n' - 2|S|^2 - |S| - |L^*| - k - |S|^2 + |S| + |L^*| + k$ vertices in $A_i$ which are all adjacent to exactly the same vertices in $S$. If $2|S|^2 - |S| - |L^*| - k + |S| \geq n' - 2|S|^2 - |S| - |L^*| - k + |S| \geq \frac{2|S|^2 + |S| + |L^*| + k}{2}$ then we would have an application of Rule $\S$. Therefore, $|A_i| \leq 4|S|^2 - 2|S| - 2|L^*| - 2k + |S| + (|L^*| + k - 3)|S| - 2|L^*| - |J| - |S| = 2|S|^3 + 5|S|^2 + (|L^*| + k - 3)|S| - 2|L^*| - |J| - 2k$.

To complete the proof of Theorem 3 we prove Lemma 14 from which Theorem 3 follows.

**Lemma 14.** For a connected graph $G$ that is reduced by Rules 5, 6, 7 and $\S$ and satisfies $|V(G)| > 26244k^3 - 5832k^4 - 6966k^3 - 378k^2 + 513k - 51$, the pair $(G, k)$ is a “yes”-instance.

**Proof.** Observe that for every vertex in $J$ there is at least one edge in $F^*$, and that therefore $|J| \leq n^* - 1$. By the preceding three lemmas, we may assume that

1. $|L^*| < 4|S|^2 + 2|S| + 2k - 2$, and
2. $n^* < 4|L^*| + 2|S|^2 + |S| + k - 2$, and
3. for any $i \in \{1, \ldots, n^*\}$, $|A_i| < 2|S|^3 + 5|S|^2 + (|L^*| + k - 3)|S| - 2|L^*| - |J| - 2k$,

for otherwise we have a “yes”-instance. Furthermore, we know that $|S| < 3k$. Putting everything together, we have that $|L^*| < 4|S|^2 + 2|S| + 2k - 2 < 36k^2 + 8k - 2$. This in turn implies that $n^* < 4|L^*| + 2|S|^2 + |S| + k - 2 < 162k^2 + 36k - 10$, and so $|J| < 162k^2 + 36k - 11$. For every $i \in \{1, \ldots, n^*\}$, we have $|A_i| < 2|S|^3 + 5|S|^2 + (|S| - 2)|L^*| + (k - 3)|S| - |J| - 2k < 54k^3 +$
45k^2 + (3k - 2)(36k^2 + 8k - 2) + 3k^2 - 9k - |J| - 2k \leq 162k^3 - 33k + 4. \ (\text{Here we assume that } |S| \geq 2 \text{ and and } k \geq 3, \text{ as otherwise the problem can be solved in polynomial time.}) \ Observe that V = \bigcup_{i \in n^*} A_i \cup J \cup S. \ Therefore, the number of vertices in G is at most \((162k^2 + 36k - 10)(162k^3 - 33k + 4) + (162k^2 + 36k - 11) + 3k = 26244k^5 - 5832k^4 - 6966k^3 - 540k^2 + 474k - 40 + 162k^2 + 39k - 11 = 26244k^5 - 5832k^4 - 6966k^3 - 540k^2 + 474k - 40\). 

\[ \Box \]

6 Cubic Kernel for Max-Cut Above Edge Lower Bound

In this section, we prove Theorem 4.

The proofs in this section are based on a slight modification of the proofs of Theorems 21 and 22 in Bollobás and Scott \[4\]. The main difference is that our Lemma 23 produces a cut of weight $\Omega\left(\frac{w(G)}{1^{1/3}}\right)$ above the bound (or else a decomposition of the graph), as compared to a cut of weight $\Omega\left(\frac{w(G)}{1^{1/4}}\right)$ above the bound (or else a decomposition of the graph) in Theorem 21 in \[4\]. This allows us to improve the kernel size from $O(k^4)$ (in Theorem 22 in \[4\]) to $O(k^3)$ (in our Theorem 4).

We consider graphs whose edges are weighted by a polynomially-bounded integer-valued function $w$, which can take both positive and negative weights. For subgraphs $G'$ of $G$, define $w(G') = \sum_{e \in E(G')} w(e)$, and for disjoint vertex subsets $V_1, V_2 \subseteq V(G)$ let $w(V_1, V_2) = \sum_{e \in E(V_1, V_2)} w(e)$. For a graph $G$ with edge-weighting $w$, let $f_w(G)$ be the maximum weight of a cut of $G$. For every $W \in \mathbb{N}$, let $f(W)$ be the minimum value of $f_w(G)$ over all graphs $G$ with non-negative edge weights $w$ and total weight $w(G) = W$. We make use of the following recursive behaviour of $f_w(G)$.

**Proposition 15** (Bollobás and Scott \[4\]). Let $W_0 = 2.5 \times 10^{17}$, and let $G$ be a graph with edge-weighting $w$ and $w(G) > W_0$. Then for $z \in \mathbb{N}$ defined by $\left(\frac{z^2}{2}\right) \leq w(G) < \left(\frac{z^2}{2} + 1\right)$, it holds

\[
\begin{align*}
f_w(w(G)) &= \min \left\{ \left\lfloor \frac{(z + 1)^2}{4} \right\rfloor , \left\lfloor \frac{z^2}{4} \right\rfloor + f_w\left(\frac{w(G) - \left(\frac{z^2}{2}\right)}{2}\right) \right\}.
\end{align*}
\]

We further need the following lower bounds on the maximum weight of a cut.

**Proposition 16** (Alon \[1\], Andersen et al. \[2\], Lehel and Tuza \[22\], Locke \[23\]). Given a graph $G$ with $m$ edges and edge-weighting $w$, and a proper vertex-coloring of $G$ with $c$ colors, in time $O(m \log w(G))$ a cut in $G$ can be found of weight at least $w(G)/2 + w(G)/(2c)$. 

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Proposition 17 (Hofmeister and Lefmann [17]). Let $G$ be a graph with $m$ edges and edge-weighting $w$.

(i) Given $G$, $w$ and a matching $M$ of $G$, in time $O(m \log w(G))$ a cut in $G$ can be found of weight at least $w(G)/2 + w(M)/2$.

(ii) Given $G$, $w$ and a partial partition $V_1 \sqcup V_2$ of $V(G)$, in time $O(m \log w(G))$ a cut in $G$ can be found of weight at least $w(V_1, V_2) + \frac{1}{2} (w(G) - w(V_1 \cup V_2))$.

Proposition 18 ([4]). Given a graph $G$ with edge-weighting $w$, in time $O(m \log w(G))$ a cut in $G$ can be found of weight at least

$$w(G)/2 + \sqrt{\left( \sum_{e \in E(G)} |w(e)| \right)/8} + 1/64 - 1/8,$$

and if $G$ is connected then in time $O(m \log w(G))$ a cut in $G$ can be found of weight at least

$$w(G)/2 + (|V(G)| - 1)/4.$$ 

For a graph $G$ with edge-weighting $w$ and an edge $e = \{u_1, u_2\}$ of $G$, the contraction of $e$ in $G$ is the edge-weighted graph $G/e$ obtained from $G$ by identifying $u_1$ and $u_2$ to a new vertex $u$ and setting $w(\{u, v\}) = w(\{u_1, v\}) + w(\{u_2, v\})$ for all $v \in V(G) \setminus \{u_1, u_2\}$. A graph $G'$ is a contraction of $G$ if, for some $t \geq 0$, there are edges $e_1, \ldots, e_t$ of $G = G_0$ and graphs $G_1, \ldots, G_t$ such that $G_i = G_{i-1}/e_i$ for $i = 1, \ldots, t$ and $G_t = G'$.

Proposition 19 ([4]). There is an algorithm that, given a graph $G$ with $m$ edges and edge-weighting $w$, finds in time $O(m \log w(G) + \sqrt{w(G)})$ a contraction of $G$ to a complete graph $H$, in which each edge has positive weight and $w(H) \geq w(G)$.

Proposition 20 ([4]). There is an algorithm that, given a graph $G$ with $m$ edges and edge-weighting $w$, and a bipartition $\{V_1, V_2\}$ of $V(G)$ with $V_2$ an independent set of $G$, finds a cut in $G$ of weight at least

$$\frac{w(G)}{2} + \frac{1}{2} \sum_{v \in V_2} \sqrt{\sum_{u \in V_1} |w(\{v, u\})|}. $$

If $w(e) \neq 0$ for all $e \in E(G)$ and $G$ contains no isolated vertices, then the algorithm runs in time $O(|V_1|^2 \sum_{v \in V_2} \sum_{u \in V_1} |w(\{v, u\})| + m)$. 

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Definition 21. [Edge Sum] Let \( G_1 = (V, E_1), G_2 = (V, E_2) \) be graphs with edge-weightings \( w_1, w_2 \), respectively. The edge sum of \( G_1 \) and \( G_2 \) is the graph \( G_1 \oplus G_2 = (V, E_1 \cup E_2) \) with edge weighting \( w(e) = w_1(e) + w_2(e) \) for all \( e \in E_1 \cap E_2 \) and \( w(e) = w_1(e) \) for all \( e \in E_i \setminus E_{3-i} \).

The following special case of edge sums will be of particular relevance. First, let \( K_t \) be a complete graph on \( t \) vertices and edge-weighting \( w_1 \) with \( w_1(e) = 1 \) for all \( e \in E(K_t) \); then \( f_{w_1}(K_t) = \lfloor t^2/4 \rfloor \) and any maximum-weight cut of \( K_t \) is an equipartition of \( V(K_t) \). Now let \( G \) be a graph with edge-weighting \( w \) that contains \( K_t \) as a subgraph, for some \( t > |V(G)|/2 \), and such that \( w(V(K_t), V(G) \setminus V(K_t)) < |V(G)|/4 \). Then \( G = K_t \oplus J \) for some graph \( J \) with edge-weighting \( w_2 \), and the restriction of any maximum-weight cut of \( G \) to \( K_t \) is an equipartition of \( K_t \). Therefore,

\[
f_w(G) = f_{w_1}(K_t) + f_{w_2}(J). \tag{5}
\]

It remains to state a lemma about independent Bernoulli random variables.

Proposition 22 (\[^4\]). Let \( U \in \mathbb{N}, \) let \( s = \{s_1, \ldots, s_p\} \) be a partition of \( U \) and let \( s' = \{s'_1, \ldots, s'_q\} \) be a refinement of \( s \). Then for independent Bernoulli random variables \( \varepsilon_1, \ldots, \varepsilon_p \) and \( \varepsilon'_1, \ldots, \varepsilon'_q \) taking values \( +1 \) and \( -1 \) with probability \( 1/2 \) each, \( \mathbb{E}[|\sum_{i=1}^p \varepsilon_i s_i|] \geq \mathbb{E}[|\sum_{i=1}^q \varepsilon'_i s'_i|] \).

Lemma 23. Let \( G \) be a graph with \( m \) edges and edge-weighting \( w \), and let \( z \in \mathbb{N} \) be defined by \( \binom{z}{2} \leq w(G) < \binom{z+1}{2} \). There is an algorithm that, given \( G \) and \( w \), in time \( O(m \log w(G)) \) either finds a cut in \( G \) of weight at least \( w(G)/2 + \sqrt{w(G)/8 + w(G)^{1/3}/52} \), or a decomposition of \( G \) as an edge sum \( G = K^* \oplus J \) satisfying \( \[^3\] \), where \( K^* \) is obtained from a complete graph \( K_t \) of order \( t = z + O(1) \) with all edges of unit weight by contracting \( O(w(G)^{1/3}) \) edges and \( J \) is a graph such that \( |V(J) \cap V(K^*)| = O(z^{2/3}) \).

Proof: We first delete all edges of zero weight. Identifying one vertex from each connected component, we may assume that \( G \) is connected. Let \( W \) be large enough, so that, if \( w(G) \geq W \), \( \[^3\] \) holds and

\[
\frac{w(G)^2}{2} + \frac{\sqrt{w(G)/8}}{8} - 1 \leq f_w(w(G)) \leq \frac{w(G)^2}{2} + \frac{\sqrt{w(G)/8} + 2w(G)^{1/3}}{52} \leq \frac{w(G)^2}{2} + \frac{z}{4} + \frac{1 + 2\sqrt{z}}{208} z^{2/3}. \tag{6}
\]
First, suppose that \( w(G) \leq W \), and let \( U = \sum_{e \in E(G)} |w(e)| \). If \( U > 64W \) then Proposition \[18\] provides the required partition. Thus we may assume that \( U \leq 64W \), and since \( G \) has at most \( U \) edges we can examine all partitions explicitly in constant time.

Second, suppose that \( w(G) > W \). Now we may assume that
\[
n > z - \frac{3 + 2\sqrt{2}}{52} z^{2/3},
\]
for otherwise we can apply the algorithm of Proposition \[16\] with \( c = n \), and by \[9\] and \( w(G) \geq (z^2 - z)/2 \) obtain a cut of weight at least \( w(G)/2 + \sqrt{w(G)/8} + w(G)^{1/3}/52 \).

Next, we may assume that \( w(G) = O(m) \): otherwise, we may apply Proposition \[19\] to obtain a contraction of \( G \) that violates \( \text{(7)} \).

Further, the algorithm of Proposition \[18\] allows us to obtain a bipartite subgraph of weight at least \( w(G)/2 + (n - 1)/4 \), and so we can halt unless
\[
n < z - \frac{2 + 2\sqrt{2}}{52} z^{2/3}.
\]

Now we show that we may assume that \( G \) is almost complete. We run the algorithm from Proposition \[19\] on input \( (G, w') \) with edge-weighting \( w' \) defined by \( w'(e) = \max\{w(e), 0\} \) for all \( e \in E(G) \), to obtain a contraction of \( G \) to a complete graph \( H' \) with all edges of strictly positive weight. If \( H' \) satisfies
\[
|V(H')| \leq z - \frac{2 + 2\sqrt{2}}{52} z^{2/3},
\]
then applying the same sequence of contraction operations to \( (G, w) \) gives a graph \( H \) with weight \( w(H) \geq w(G) \) that satisfies \( \text{(9)} \), which we can partition with the algorithm of Proposition \[16\].

Otherwise, if \( H' \) violates \( \text{(9)} \), let \( Y^- \subseteq V(G) \) be the set of vertices of \( G \) that are identified with some other vertex in the contraction from \( G \) to \( H' \). Then \( Y^- \) covers all edges of \( G \) with weight at most 0, and together with \( \text{(8)} \)
\[
|Y^-| \leq 2(n - |V(H')|) \leq \frac{8 + 8\sqrt{2}}{52} z^{2/3}.
\]

Let \( Y^+ = V(G) \setminus Y^- \). Then \( G[Y^+] \) is a complete graph in which all edges have positive weight, and it follows from \( \text{(7)} \) and \( \text{(10)} \) that
\[
|Y^+| > z - \frac{3 + 2\sqrt{2}}{52} z^{2/3} - \frac{8 + 8\sqrt{2}}{52} z^{2/3} = z - \frac{11 + 10\sqrt{2}}{52} z^{2/3}.
\]
We claim that $G[Y^+]$ cannot contain too many edges with weight greater than 1. In a greedy fashion, compute a maximal matching $M$ of $G[Y^+]$ that maximizes the number of edges with weight strictly greater than 1. On the one hand, using the algorithm of Proposition 17(i), we obtain in time $O(m)$ a cut of $G[Y^+]$ with weight at least $w(G[Y^+])/2 + \frac{1}{2} \sum_{e \in M} w(e)$. By Proposition 17(ii), we can extend this cut of $G[Y^+]$ to a cut of $G$ with weight at least $w(G)/2 + \frac{1}{2} \sum_{e \in M} w(e)$. Since every edge in $M$ has weight at least 1, it follows from (6) that either we can halt the algorithm or
\[
\sum_{e \in M} w(e) \leq \frac{z}{2} + \frac{2 + 4\sqrt{2}}{208} z^{2/3}.
\] (12)

On the other hand, maximality of $M$ for $G[Y^+]$ implies that $|M| > |Y^+| - 1)/2$, and so
\[
w(M) \geq (|Y^+| - 1)/2 + |M'|
\]
where $M' \subseteq M$ is the set of edges in $M$ with weight strictly more than 1. Together with (11) and (12), this implies that
\[
|M'| \leq w(M) - (|Y^+| - 1)/2 \leq z/2 + \frac{2 + 4\sqrt{2}}{208} z^{2/3} - \left( z - \frac{11 + 10\sqrt{2}}{52} z^{2/3} - 1 \right)/2
\]
\[
\leq \frac{2 + 4\sqrt{2} + 22 + 20\sqrt{2} + 104}{208} z^{2/3} = \frac{128 + 24\sqrt{2}}{204} z^{2/3}.
\] (13)

As the edges of $M$ were chosen greedily, the edges of $M'$ meet every edge of $E(G[Y^+])$. Therefore, by (13) the set $V(M') \subseteq V(G)$ of vertices of $M'$ covers every edge in $E(G[Y^+])$ and satisfies
\[
|V(M')| \leq \frac{256 + 48\sqrt{2}}{208} z^{2/3}.
\] (14)

Let $Y = Y^- \cup V(M')$ and $X = V(G) \setminus Y$. Then by (10) and (14), $|Y| \leq \frac{8 + 8\sqrt{2}}{52} z^{2/3} + \frac{256 + 48\sqrt{2}}{208} z^{2/3} = \frac{72 + 20\sqrt{2}}{52} z^{2/3}$, whereas by (7),
\[
|X| \geq z - \frac{2 + 2\sqrt{2}}{52} z^{2/3} - \frac{72 + 20\sqrt{2}}{52} z^{2/3} = z - \frac{74 + 22\sqrt{2}}{52} z^{2/3}.
\] (15)

Note that $G[X]$ is a complete graph in which all edges have weight equal to 1. For each $y \in Y$, order the edges from $y$ to $X$ into non-decreasing order of weight, which takes time $O(z \log z) = O(m)$. Then for each $y \in Y$ partition $X \cup \{y\}$ into sets $X_1 \cup \{y\}$ and $X_2$, where $\{\{y, x_1\} | x_1 \in X_1\}$ are
the lightest \(|X|/2\) edges and \(\{y, x_2\} \mid x_2 \in X_2\) are the heaviest \(|X|/2\) edges. Then with the algorithm of Proposition 17(ii) extend the partition \((X_1 \cup \{y\}, X_2)\) of \(X \cup \{y\}\) to a cut of \(G\) with weight at least
\[
\frac{w(G)}{2} + \frac{(|X| - 1)}{4} + \frac{w(\{y\}, X_2) - w(\{y\}, X_1)}{2},
\]
and so by (15) we are done unless
\[
w(\{y\}, X_2) - w(\{y\}, X_1) < \frac{76 + 24\sqrt{2}}{52 \cdot 2} z^{2/3}.
\]
It follows that all but at most \(\frac{38 + 12\sqrt{2}}{52} z^{2/3}\) edges between \(y\) and \(X\) have the same weight, say \(t(y)\). (Checking (15) is easy, and \(t(y)\) can be determined for all \(y \in Y\) in total time \(O(|Y| \log z) = O(m)\).) Setting \(t(x) = 1\) for all \(x \in X\), we may assume that \(t(v)\) has been defined for all \(v \in V(G)\). Now for all \(e = \{v_1, v_2\} \in E(G)\) define a weighting
\[
u(\{v_1, v_2\}) = w(\{v_1, v_2\}) - t(v_1)t(v_2).
\]
It follows from (17) that, for each \(y \in Y\),
\[
\sum_{x \in X} |u(\{x, y\})| < \frac{38 + 12\sqrt{2}}{52} z^{2/3}.
\]
Set \(U' = \sum_{(x,y) \in X \times Y} |u(\{x, y\})|\), and consider the cases \(U' \geq z/4\) and \(U' < z/4\) separately. If \(U' \geq z/4\) we can find a set \(Y' \subseteq Y\) with
\[
z/4 \leq \sum_{(x,y) \in X \times Y'} |u(\{x, y\})| < z/4 + \frac{38 + 12\sqrt{2}}{52} z^{2/3}
\]
by choosing vertices one at a time from \(Y\) until both inequalities are satisfied.
Let \(X'\) be the set of vertices \(x \in X\) for which there exists a vertex \(y \in Y'\) with \(u(\{x, y\}) \neq 0\). Then the number of vertices in \(X'\) is at most \(z/4 + \frac{38 + 12\sqrt{2}}{52} z^{2/3}\), which is at most \(z/2\) since \(w(G)\) is sufficiently large. Let \(\{Y'_1, Y'_2\}\) be a random bipartition of \(Y'\), where each \(y \in Y'\) is in \(Y'_1\) or \(Y'_2\) independently with probability \(1/2\). Let \(R'_1 = Y'_1 \cup \{x \in X' \mid u(\{x\}, Y'_1) < u(\{x\}, Y'_2)\}\) and let \(R'_2 = Y'_2 \cup \{x \in X' \mid u(\{x\}, Y'_1) \geq u(\{x\}, Y'_2)\}\). Then use the algorithm of Proposition 17(iii) to extend the partition \(\{R'_1, R'_2\}\) of \(X' \cup Y'\) to a cut \(\{C_1, C_2\}\) of \(X \cup Y'\) by adding the vertices from \(X \setminus X'\) such that
\[
\sum_{v \in C_1} t(v) \leq \sum_{v \in C_2} t(v) \leq \sum_{v \in C_1} t(v) + 1,
\]

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which is possible since \( R_1' \cup R_2' \) contains at most \( z/2 \) elements of \( X \). Then for each \( x \in X \) and \( U_x = \sum_{y \in Y'} |u(\{x, y\})| \), Proposition 22 yields

\[
\mathbb{E}[|u(\{x, Y'_1\}) - u(\{x, Y'_2\})|] \geq \mathbb{E} \left[ \left| \sum_{i=1}^{U_x} \pm 1 \right| \right] \geq \frac{1}{\sqrt{3}} \sqrt{U_x},
\]

where we used a consequence of Hölder’s inequality for random variable \( X \):

\[
\mathbb{E}[|X|] \geq \frac{\mathbb{E}[X^{2}]^{3/2}}{\mathbb{E}[X^{4}]^{1/2}}.
\]

Set \( t(X \cup Y') = \sum_{v \in X \cup Y'} t(v) \). Then, together with

\[
\max\{u(\{x, Y'_1\}), u(\{x, Y'_2\})\} = \frac{u(\{x, Y'_1\}) + u(\{x, Y'_2\})}{2} + \frac{|u(\{x, Y'_1\}) - u(\{x, Y'_2\})|}{2},
\]

(18), (19), (20), and (21) imply

\[
\mathbb{E}[w(C_1, C_2)] \geq \mathbb{E} \left[ \frac{t(X \cup Y')^2}{4} + u(G[Y'])/2 + \sum_{x \in X} \max\{u(\{x, Y'_1\}), u(\{x, Y'_2\})\} \right]
\]

\[
\geq \frac{1}{2} w(X \cup Y') + \frac{|X| - 1}{4} + \mathbb{E} \left[ \sum_{x \in X} \frac{|u(\{x, Y'_1\}) - u(\{x, Y'_2\})|}{2} \right]
\]

\[
\geq \frac{1}{2} w(X \cup Y') + \frac{|X| - 1}{4} + \frac{1}{\sqrt{3}} \sum_{x \in X} \left( \sum_{y \in Y'} |u(\{x, y\})| \right)^{1/2}
\]

\[
\geq \frac{1}{2} w(X \cup Y') + \frac{|X| - 1}{4} + \frac{z/4}{\sqrt{3} \sqrt{38} + 12 \sqrt{2} z^{1/3}}
\]

\[
> w(X \cup Y')/2 + |X|/4 + z^{2/3}/52.
\]

Then, by applying the algorithm of Proposition 20 with \( V_1 = Y', V_2 = X \) and edge set \( E(Y') \cup E(Y, X) \), we extend the partition \( \{C_1, C_2\} \) of \( X \cup Y' \) to a cut of \( G \) with weight at least \( w(G)/2 + |X|/4 + z^{2/3}/52 > w(G)/2 + z/4 + \frac{14+2\sqrt{2}}{208} z^{2/3} \), which suffices by (6). By the running time stated in Proposition 20, this step takes time \( O(|Y'|^2 |U'| + m \log m) = O((z^{2/3})^2 z^{2/3} + m \log m) = O(m \log m) \).

Otherwise, \( U' < z/4 \) and we may decompose \( G \) as the edge sum of a graph \( J \) with edge weights given by \( u \) and a graph \( K^* = G[X] \) with edge weights given by \( t \). Further observe that \( |V(K^*)| = \sum |V(K^*)| = z + O(z^{2/3}) = z + O(w(G)^{1/3}) \), while a calculation shows that if \( f(G) \leq w(G)/2 + \sqrt{w(G)/8} + w(G)^{1/3}/52 \) then \( t = z + O(1) \). Finally, note that \( K^* \) is obtained from a complete subgraph of \( G \) by a sequence of \( O(w(G)^{1/3}) \) contractions. \( \square \)
We are ready to complete the proof of Theorem 4.

**Proof of Theorem 4.** Given a graph $G$ with $m$ edges and edge-weighting $w$, and $k \in \mathbb{N}$, the algorithm executes the following steps. First, if $\sum_{e \in E(G)} |w(e)| > 4(w(G) + 8k^2)$, then Proposition 18 gives us a large enough cut. Second, if $k > w(G)^{1/3}/52$, then $m \leq w(G) \leq 140608k^3$, and we return the graph $G$ as a kernel. Third, we apply Lemma 23: either we obtain a cut of weight at least $w(G)/2 + \sqrt{w(G)/8} + k$ (in which case we can return a trivial constant-size “yes”-instance), or we obtain a decomposition $G = K^* \oplus J$ where $f(G) = f(K^*) + f(J)$ and $K^*, J$ have edge weightings $w_1, w_2$, respectively.

We can calculate $f(K^*)$ exactly; so if $w_2(J) > 0$, define

$$k' := \frac{w(G)}{2} + \sqrt{\frac{w(G)}{8}} + k - f(K^*) - \frac{w_2(J)}{2} - \sqrt{\frac{w_2(J)}{8}}$$

and note that $k' \leq k$. Then repeat the algorithm on input $(J, w_2, k')$.

Otherwise, if $w_2(J) \leq 0$, define an edge weighting $u$ of $J$ by $u(e) = |w_2(e)|$ for all $e \in E(J)$. If $|u(J)| > 8k^2$ then Proposition 18 gives a cut of $J$ with weight at least $w_2(J)/2 + k$, which yields a cut of $G$ with weight at least $w_1(K^*)/2 + \sqrt{w_1(K^*)/8} + w_2(J)/2 + k \geq w(G)/2 + \sqrt{w(G)/8} + k$, and we can return a trivial “yes”-instance of constant size. Finally, if $|u(J)| \leq 8k^2$, then $J$ has at most $k^2$ edges, and since $f(G) = f(K^*) + f(J)$ we can simply return $J$ as a kernel. This completes the proof of Theorem 4. \qed

7 Discussion and Open Problems

We showed fixed-parameter tractability of MAX-CUT parameterized above the Edwards-Erdős bound $m/2 + (n - 1)/4$, and thereby resolved an open question from [7, 15, 24, 25, 29]. Furthermore, we showed that the problem has a kernel with $O(k^3)$ vertices and the “edge version” of the bound admits a kernel of size $O(k^3)$. We have not attempted to optimize running time or kernel size, and indeed we conjecture that MAX-CUT has a kernel with $O(k^3)$ vertices and the edge version admits a linear kernel.

It remains an open problem whether the weighted version of MAX-CUT above the Edwards-Erdős bound is fixed-parameter tractable; our conjecture is that this problem is also fixed-parameter tractable with a polynomial kernel.

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