THE PROPAGATION OF CHAOS FOR A RAREFIED GAS OF HARD SPHERES IN THE WHOLE SPACE

RYAN DENLINGER

ABSTRACT. We discuss old and new results on the mathematical justification of Boltzmann’s equation. The classical result along these lines is a theorem which was proven by Lanford in the 1970s. This paper is naturally divided into three parts.

I. Classical. We give new proofs of both the uniform bounds required for Lanford’s theorem, as well as the related bounds due to Illner & Pulvirenti for a perturbation of vacuum. The proofs use a duality argument and differential inequalities, instead of a fixed point iteration.

II. Strong chaos. We introduce a new notion of propagation of chaos. Our notion of chaos provides for uniform error estimates on a very precise set of points; this set is closely related to the notion of strong (one-sided) chaos and the emergence of irreversibility.

III. Supplemental. We announce and provide a proof (in Appendix A) of propagation of partial factorization at some phase-points where complete factorization is impossible.

1. Introduction

We are interested in the system of $N$ identical elastic hard spheres of diameter $\varepsilon > 0$, which move through $d$-dimensional Euclidean space according to the laws of Newtonian mechanics. This is an important model in mathematical physics because the rules are relatively simple and yet they capture in a realistic way the macroscopic behavior of many physical systems. Usually the number of particles is quite large, say $N = 10^{23}$, so it seems hopeless to follow the microscopic dynamics directly. An alternative strategy, pioneered by Maxwell and Boltzmann, is to assign probabilities to the possible microscopic configurations of the system and study the evolution of these probabilities subject to mechanistic laws (e.g., conservation of mass, momentum and energy). Given a suitable choice of spatial and temporal scales, the equation one formally arrives at through this line of reasoning is known as Boltzmann’s equation.

Half a century after Boltzmann’s work, H. Grad used precise physical reasoning in an attempt to give Boltzmann’s equation a firm physical footing. He devised a special scaling limit, known today as the Boltzmann-Grad limit, in which the microscopic dynamics heuristically reduce to the Boltzmann equation under a “molecular chaos” assumption (the mathematical nature of the chaos assumption would not be clarified fully until the 1970s).
However, this did not resolve the question of deriving Boltzmann’s equation because there was no mathematical argument linking the microscopic Liouville equation to the Boltzmann equation. C. Cercignani gave a non-rigorous but fairly precise description of the convergence process. O.E. Lanford provided the first rigorous convergence proof for Boltzmann, by describing the reduced dynamics arising from low-order correlations, and showing that the high-order correlations have negligible influence on the behavior of the gas, at least for a short time. More recently, a careful quantitative analysis of Lanford’s theorem has been provided by I. Gallagher, L. Saint-Raymond and B. Texier.

We remark on several related developments. The major limitation in Lanford’s theorem is the short time of validity, which so far has not been lifted except in very restrictive perturbative regimes. R. Illner and M. Pulvirenti were able to overcome the time restriction and prove global convergence for a highly rarefied gas near vacuum, using inequalities related to the dispersive nature of the system. Different perturbative regimes can be obtained in bounded domains, most notably a periodic box (equipped with a Gibbs measure which is invariant under the dynamics). Perturbing only in the initial distribution of a single particle leads naturally to the (non-conservative) linear Boltzmann equation; perturbing all the particles in a symmetric way leads to the linearized Boltzmann equation. Both possibilities have been studied in the literature, most notably by H. van Beijeren, O. E. Lanford, J. Lebowitz and H. Spohn, and in a separate contribution by J. Lebowitz and H. Spohn. These perturbative settings have been studied more recently by T. Bodineau, I. Gallagher and L. Saint-Raymond, who proved quantitative error estimates on diverging timescales $T_N \approx (\log \log N)^r$ for some known $r > 0$, leading to hydrodynamic limits (namely Brownian motion, and the Stokes-Fourier equations). The perturbation of equilibrium is an extremely difficult problem (particularly on large time intervals) and we will not have more to say about it in this work.

There are several other important results which are not directly related to Lanford’s theorem but are nevertheless foundational in kinetic theory.

• **Stochastic models.** All models we have mentioned so far have been fully deterministic; this means that randomness is allowed in the choice of initial data, but the evolution for each initial state is fully determined. However, there is an important class of models in kinetic theory where the dynamics itself introduces randomness. We specifically mention the Kac model; in this model, the position coordinates are treated as hidden variables, and in particular the impact parameter for each collision is a random variable with some specified law. When the number of particles tends to infinity, the evolution is seen to converge to the (nonlinear) space-homogeneous Boltzmann equation with the appropriate collision kernel. These models were first analyzed in a couple of influential papers by M. Kac and H.
McKean. [20,24] There have been many papers dealing with similar models in the intervening years, and a very complete treatment has been given by S. Mischler and C. Mouhot. [26]

- **Lorentz gases.** We refer to a class of models first studied by G. Gallavotti. [12] In these Lorentz gas-type models, the dynamics is indeed deterministic, but they differ from the case of Lanford in that all the particles but one are considered stationary obstacles, distributed like Poisson scatterers. The dynamics is much simpler in this case because the background particles never move out of place; in the Boltzmann-Grad limit one recovers the linear Boltzmann equation for the evolution of the tagged particle. Note that it is not possible to enforce momentum conservation in a Lorentz gas, so these models are only physically realistic if the tagged particle is much lighter than the background particles.

- **Vlasov-type mean field limit.** Physical limits in which each particle feels the influence of the entire gas are generally called mean-field limits; these models can be fully deterministic, or they can possess some stochasticity. Mean field limits tend to have a relatively pleasant mathematical structure because a typical particle’s trajectory is governed by the average of the other particles’ trajectories; this property is very helpful in controlling the correlations generated by the dynamics. Whereas the Boltzmann-Grad scaling leads to Boltzmann’s kinetic equation, the Vlasov-type mean-field models lead to Vlasov-type equations in the limit $N \to \infty$. The study of Vlasov-type mean field limits is a vast field in its own right and we provide only a small sampling of the relevant literature. [10, 19, 25]

Henceforth in this work we will not be concerned with stochastic models, Lorentz gases, or mean field limits.

**The goals of the present work are twofold.** First, we shall present a new proof of the uniform bounds which are central to Lanford’s theorem. We use differential inequalities and a duality argument, instead of a fixed point argument, to control the growth of correlations in the BBGKY hierarchy. We will apply this method to prove both the short-time result of Lanford, as well as the global near-vacuum result of Illner & Pulvirenti. [17,18,22]

Our second goal is to thoroughly address the issue of uniform convergence of the marginals in the limit $N \to \infty$. The motivation is the notion of strong (one-sided) chaos and the appearance of irreversibility from an underlying reversible dynamics. The issue of irreversibility is tied to convergence properties along very singular sets in phase space; for this reason, uniform convergence

\[1\text{Note that the rigorous link between irreversibility and strong chaos in general requires application of the Hewitt-Savage theorem; see subsection [20] below for some discussion of the connection.}\]
convergence (on a sufficiently large set) becomes a central question in the
discussion of irreversibility.

Uniform convergence has been addressed by a number of authors going
back to the 1970s. (See [5, 21, 27], and Appendix A of [31].) The label
**strong chaos** is reserved for any notion of chaos for which the convergence
at positive times is strong enough to allow the re-application of the the con-
vergence theorem taking the evolved solution as initial data. See Appendix
A of [31], or [5], for examples of strong chaos results (note however that
the basic technique yielding uniformity is actually due to F. King [21]). By
definition, a strong chaos result *must* account for the directionality of time
due to the fact that Newton’s laws are time-reversible whereas Boltzmann’s
equation is irreversible; iteration can only be performed forwards in time,
not backwards, so it is a one-sided notion. (The term *strong one-sided chaos*
is actually redundant in the context of Lanford’s theorem but some authors
use the term *one-sided* in isolation to emphasize the fact that convergence is
occurring only at “pre-collisional” points in phase space.) Unlike previous re-
sults (except [5], which represents independent concurrent work), our strong
chaos result implies uniform error estimates arbitrarily close to the
boundary of the reduced phase space, which is significant because the physical inter-
action is confined to the boundary. Our error estimates are quantitative,
as in [11, 27], though for simplicity of presentation we will state our main
theorems without explicit rates (the estimates in the proof itself are also
much larger than necessary, again for simplicity of presentation).

**Remark.** A very clear exposition on the topic of strong chaos is found in [5];
we feel that the authors have brought great clarity to the topic and we make
no attempt to replicate their exposition.

**Remark.** Note that in the *original manuscript* of Lanford [22] (neglecting
his follow-up works) the stated result is a *weak chaos* result because the
assumptions on the initial data are much stronger than what is proven at
positive times, hence iteration in time is impossible. Lanford clearly ac-
knowledged this shortcoming and understood the technical steps required
to prove a strong chaos result (the details being filled in by his own student
King at roughly the same time).

A novel aspect of our analysis is that, given suitably prepared initial data,
we can propagate partial factorization even at phase points where complete
factorization necessarily fails (i.e. “post-collisional” configurations with \( t > 0 \)). As an application of our result, one obtains the existence of positive
measure sets, parameterized by \( \varepsilon \) in a natural way, with measure tending
to zero as \( \varepsilon \to 0 \), upon which \( f_N^{(3)} \approx f_N^{(2)} \otimes f_N^{(1)} \) but further factorization

---

\[ \text{We would like to thank L. Saint-Raymond and H. Spohn (private communications) for}
\text{insightful discussions and comments regarding the connection to irreversibility.} \]
Partial factorization should be viewed as complementary to results on correlations, such as [28]. Indeed whereas [28] gives remarkably precise estimates on the size of correlations, there was no characterization of the sets on which correlations were concentrated. On the flip side, we are able to say something about the structure of correlations but very little about their size. (The partial factorization result holds under the assumption of perfect factorization at \( t = 0 \), but this is a standard assumption in the field.) The proof of partial factorization draws significant inspiration from [3][28], though our methods are somewhat different. Partial factorization is easily generalized to include non-chaotic initial data, in the spirit of the Hewitt-Savage theorem. [15] We emphasize that non-chaotic initial states have been discussed in the context of irreversibility; see, e.g., [5].

**Organization of the paper.** In Section 2, we describe the ideas behind our proof, and we present our main convergence result. Section 3 gives the precise physical setting for our problem, along with a crucial comparison principle. Section 4 briefly introduces the BBGKY and dual BBGKY hierarchies. Section 5 & 6 give proofs of *a priori* bounds on the BBGKY hierarchy by a duality argument; bounds are proven both locally in time for large data, and globally in time for data sufficiently close to vacuum. (These *a priori* estimates are not new, but we use a different approach for the proofs.) Sections 7, 8, 9, 10 & 11 introduce a number of important technical tools and results; our main technical contribution is the stability result in Section 8. The detailed convergence proof (part (i) of Theorem 2.1) is given in Section 12. A proof of part (ii) of Theorem 2.1 is presented in Appendix A.

### 2. Statement of main results

#### 2.1. Uniform bounds via duality

We begin by briefly describing the role that duality plays in our proof. Throughout this work we will rely on the BBGKY hierarchy (Bogoliubov-Born-Green-Kirkwood-Yvon), which is a sequence of equations describing the evolution of marginals \( f_{N}^{(s)}(t) \) under the hard sphere flow. One of the key steps in the proof of Lanford’s theorem is to bound a weighted \( L^{\infty} \) norm of the sequence of marginals, uniformly in \( N \), in terms of the initial data. Lanford proves the uniform bound by rewriting the BBGKY hierarchy using Duhamel’s formula and then setting up a fixed point argument. [6][11][22] We have approached the uniform bounds

---

5Note carefully that \( f_{N}^{(2)} \approx f_{N}^{(1)} \otimes f_{N}^{(1)} \) at *most* phase points, but the theorem holds even at some points where \( f_{N}^{(2)} \) does not factorize.

4We are not aware of any satisfactory explanation for the physical relevance of a perfectly factorized initial state. There are well-known arguments based on minimization of entropy, but there is a problem of topologies: an entropically small perturbation will not be a uniformly small perturbation in general.

5In fact Lanford wrote out a series expansion for which he proved \( L^{\infty} \) bounds uniformly in \( N \); this is effectively equivalent to the fixed point argument, and he had to prove the same collision operator bounds as in [11].
from a somewhat different point of view. Our starting point is the dual BBGKY hierarchy, which is (formally) the semigroup whose generator is the (formal) adjoint of the semigroup generator for the BBGKY hierarchy. We refer to [7, 13] for background and results concerning the dual BBGKY hierarchy.

Physically, the dual BBGKY hierarchy describes the evolution of observables. We are able to bound the growth of observables in a weighted $L^1$ space; then, the classical $L^\infty$ bounds on the marginals follow by duality. Using the same strategy, with slight revisions, we are able to prove uniform bounds *globally in time* in the physical regime considered by Illner & Pulvirenti. [17, 18] We emphasize that all of our results concerning uniform bounds are classical; the only novelty lies in the method of proof. Note that certain very special observables, such as the kinetic energy, exhibit cancellation properties (e.g., conservation). However, our proof concerns generic observables; in particular, there seems to be no simple way to account for cancellations. Hence we cannot report any improvements beyond the perturbative regime (small time or large mean free path).

It should be noted that, in the context of Lanford’s original theorem [22], the duality argument does not seem to gain us anything new. However, there are some technical reasons to prefer the dual point of view. Most fundamentally, it is always possible to consider weak-* limits of solutions of the dual BBGKY hierarchy (which will converge to solutions of the dual Boltzmann hierarchy). This limit process will work for any observable which is not concentrated on certain submanifolds of high codimension (see Remark 2.1 below). By contrast, passing to the limit from the BBGKY hierarchy to the Boltzmann hierarchy is an incredibly delicate process, which is difficult to characterize using standard functional spaces. One would hope to use duality to simplify certain technical questions concerning the BBGKY hierarchy itself, by characterizing solutions of BBGKY in terms of their action on well-chosen families of observables. Note that the differential inequalities we use to prove Lanford’s uniform bounds (at the level of observables) can be adapted to give more precise information about observables (this is itself a topic of ongoing research).

Another advantage of duality (and the one which served as the original motivation for this project) is that it gives a somewhat unique proof of uniform bounds on the BBGKY hierarchy, globally in time, for a small perturbation of vacuum. [17, 18] Note that the correct proof given in [15] (as opposed to the incorrect proof in [17]) relies on a series expansion; indeed, the proof of [18] cannot really be viewed as a fixed point argument in the traditional sense. In particular, certain dispersive-type bounds for moving

---

6By contrast the corresponding global-in-time estimate for the Boltzmann hierarchy is a fixed point argument but it requires intertwining the free transport and collision terms. Intertwining in Lanford’s proof is fine on the short time (see the erratum of [11] for instance); unfortunately, for the global estimate, intertwining will ruin the dispersive property because one of the needed inequalities is false.
Maxwellian distributions must be propagated along an arbitrary sequence of particle creations. Our goal was to have a proof which could be explained using calculus alone, without a technical induction process. The duality argument trades one complication for another since we have to be extremely careful in manipulating weighted norms for observables. Nevertheless, in our view, the proof presented here is more elegant than that given in [18] and it seems to us the most novel aspect of this paper.

Remark. One can ask whether it is possible to treat the Boltzmann hierarchy using duality, in a manner similar to the BBGKY hierarchy. The answer is “yes, but...” The problem is that, whereas the dual BBGKY hierarchy propagates $L^1$ regularity, solutions of the dual Boltzmann hierarchy are measures even if the data is smooth. Unfortunately, the dual Boltzmann hierarchy isn’t well-defined for measure data due to the possibility of simultaneous collisions of three or more particles. Most likely it is possible to work with the dual Boltzmann hierarchy by restricting one’s attention to measures that assign zero weight to manifolds of sufficiently high codimension. However, we prefer not to confront these technical issues; instead, we prove uniform bounds for the Boltzmann hierarchy using the standard fixed-point argument.

2.2. Strong convergence. We now turn to the content of Theorem 2.1 (especially part (i)), which is our main new result. Essentially the result states that if a priori bounds are known then chaoticity is propagated forwards in time; the novelty of the result lies in the strength of the notion of convergence we employ at positive times. The direction of time is built into our notion of chaoticity, so the theorem cannot be applied to prove propagation of chaos backwards in time. Our convergence result is a type of strong chaos result; this means that we can take the evolved state at a time $t > 0$ and use this state as initial data in order to iterate the convergence to an even later time. The iteration can be continued as long as uniform bounds are known. We emphasize that strong chaos results are known in both the classical and very recent literature [5, 31]; however, to our knowledge, the present convergence result is the only one which extends to all distance scales $\{|x_i - x_j| > \varepsilon\}$ as long as the backwards trajectories of all $s$ particles are free (minus a small set in the velocity variables only, but see Remark 2.2 below for a discussion of ways to refine the sets of convergence). Moreover, as we will see, our proof extends without too much difficulty to obtain a pointwise description of two-particle correlations (higher-order correlations and better error estimates are topics of ongoing research).

---

7This is related to the fact that the Boltzmann hierarchy is not well-posed on (weighted) $L^\infty$ (without at least an assumption like factorization or exchangeability) but it is well-posed for continuous data.

8The strong chaos result in [5] requires $|x_i - x_j| \gtrsim \varepsilon \log 1/\varepsilon$.

9By contrast, the authors of [28] have provided a very precise but averaged (not pointwise) description of correlations.
Remark. The notion of chaos that Lanford originally proved (at positive times) states that the marginals \( f_N^{(s)}(t) \) converge pointwise almost everywhere to tensor products. It can be shown (see [22]) that this notion of chaos (combined with certain uniform estimates) implies that for any box \( \Delta \subset \mathbb{R}^d \times \mathbb{R}^d \), the occupation fraction

\[
\frac{1}{N} \sum_{i=1}^{N} 1_{(x_i(t),v_i(t)) \in \Delta} \to \text{constant}
\]

converges in probability to a constant depending only on \( t \) and \( \Delta \) when \( \varepsilon \to 0 \). The physical interpretation of Lanford’s result is that fluctuations tend to zero as \( \varepsilon \to 0 \). We emphasize that it is not true that if the marginals converge pointwise almost everywhere (at \( t = 0 \), or even for all \( t \in [0,T] \)), then the evolution is governed by Boltzmann’s equation. The classical counter-example uses the reversibility of Newton’s laws combined with the irreversibility of Boltzmann’s equation. [8] An even more striking counter-example has been constructed by T. Bodineau, I. Gallagher, L. Saint-Raymond and S. Simonella; these authors found an initial data such that the marginals converge pointwise almost everywhere to tensor products at \( t = 0 \) (indeed they obtained uniform convergence off explicit small sets), whereas the evolution is given by free transport. [5]

We will need to introduce several sets before we can state our main result; to this end, we will borrow notation from Section 3. We will view \( \eta > 0 \) as a small velocity cut-off, and \( R > 0 \) as a large velocity cutoff. The most important sets we will require are defined as follows:

\[
\mathcal{K}_s = \left\{ Z_s = (X_s, V_s) \in \mathcal{D}_s \mid \psi_{s}^{-\tau} Z_s = (X_s - V_s \tau, V_s) \ \forall \ \tau > 0 \right\} \subset \mathbb{R}^{2ds} \tag{2}
\]

\[
\mathcal{U}^\eta_s = \left\{ Z_s = (X_s, V_s) \in \mathcal{D}_s \mid \inf_{1 \leq i < j \leq s} |v_i - v_j| > \eta \right\} \subset \mathbb{R}^{2ds} \tag{3}
\]

Our main result will concern uniform convergence on the set \( \mathcal{K}_s \cap \mathcal{U}^\eta_s \) (with \( \eta(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \)); it is in proving uniform convergence on such a precise set that a strong chaos result is obtained. The condition \( Z_s \in \mathcal{K}_s \) means that particles never collide under the backwards particle flow (but, crucially, they are allowed to collide under the forwards flow). The condition \( Z_s \in \mathcal{U}^\eta_s \) is a technical condition which forces particles to disperse at an \( \eta \)-dependent rate; see Remark [22] for a few words on how to relax the definition of \( \mathcal{U}^\eta_s \). The remaining sets \( \mathcal{G}_s, \mathcal{V}^\eta_s, \hat{\mathcal{U}}^\eta_s \), to be defined next, are required only for stating a partial factorization result, and can be safely skipped.

\[
\mathcal{G}_s = \left\{ Z_s = (X_s, V_s) \in \mathcal{D}_s \mid \forall \tau > 0, \ \forall 3 \leq i \leq s, \ (\psi_{s}^{-\tau} Z_s)_i = (x_i - v_i \tau, v_i) \right\} \tag{4}
\]
the set $K$ holds only at phase points which possibly involve collisions in the
Remark.

of the set $G$ marginals based norm of convergence is
cially the convergence
Remark.

Observe that the sets $1$ very "thin" sets of points are necessarily excluded via the i ndicator function
The term
Remark.

An earlier version of this manuscript contained an incorrec t state-
Definition 2.2. which did not even imply weak chaoticit y; we are
Remark.

where
Remark.

nonuniform chaoticity
We write
Remark.

that
Remark.

$\limsup_{N\to\infty} \left\| \left( f_N^{(s)}(0, Z_s) - f_0^{(s)}(Z_s) \right) 1_{Z_s \in K_s \cap \mathcal{U}_s^{(c)}} 1_{E_s(Z_s) \leq R^2} \right\|_{L_{2,s}^{\infty}} = 0 \quad (7)$
where $\eta(\varepsilon) = \varepsilon^\kappa$ and $N\varepsilon^{d-1} = \ell^{-1}$.

Definition 2.2. The sequence of initial data $\{F_N(0) \mid N \in \mathbb{N}\}$ is nonuni-
formly $f_0$-chaotic if, for some $\kappa \in (0, 1)$, we have for all $s \in \mathbb{N}$ and all $R > 0$
that

$$
\limsup_{N\to\infty} \left\| \left( f_N^{(s)}(0, Z_s) - f_0^{(s)}(Z_s) \right) 1_{Z_s \in K_s \cap \mathcal{U}_s^{(c)}} 1_{E_s(Z_s) \leq R^2} \right\|_{L_{2,s}^{\infty}} = 0 \quad (8)
$$
and we have for all $s \in \mathbb{N}$ with $s \geq 3$ and all $R > 0$
that

$$
\limsup_{N\to\infty} \left\| \left( f_N^{(s)}(0, Z_s) - (f_N^{(2)}(0) \otimes f_0^{(s-2)}) (Z_s) \right) 1_{Z_s \in G_s \cap \mathcal{U}_s^{(c)}} 1_{E_s(Z_s) \leq R^2} \right\|_{L_{2,s}^{\infty}} = 0 \quad (9)
$$
where $\eta(\varepsilon) = \varepsilon^\kappa$ and $N\varepsilon^{d-1} = \ell^{-1}$.

Remark. An earlier version of this manuscript contained an incorrect state-
ment of Definition 2.2 which did not even imply weak chaoticity; we are
indebted to one of the anonymous referees for bringing this to our attention.

Remark. The term nonuniform chaoticity is motivated by the fact that the norm of convergence is based on the $L^{\infty}$ (uniform) norm in $\mathbb{R}^{2ds}$, yet crucially the convergence is not uniform across the whole phase space. Indeed, very “thin” sets of points are necessarily excluded via the indicator function $1_{Z_s \in K_s \cap \mathcal{U}_s^{(c)}}$.

Remark. Observe that the sets $G_s$ appearing in Definition 2.2 are not sym-
metric under particle interchange. Nevertheless, since we assume that the marginals $f_N^{(s)}$ are symmetric, the uniform error estimates hold on the image of the set $G_s \cap \mathcal{U}_s^{(c)}$ under any permutation of particle labels.

Remark. The key difference between Definition 2.1 and Definition 2.2 is that the set $K_s \cap \mathcal{U}_s^{(c)}$ is replaced by $G_s \cap \mathcal{U}_s^{(c)}$ in (9). Hence, the estimate (7) holds only at phase points which possibly involve collisions in the future,
but not the past. On the other hand, the estimate (9) holds even at points where at most two of the particles have collisions in the past. Also note that the structure of the set $\tilde{U}_s^{\eta(\varepsilon)}$ is more complicated than that of $U_s^{\eta(\varepsilon)}$ due to its dependence on the particle flow $\psi_s^{-t}$. Let us point out that the usual formulation of Lanford’s theorem says nothing at all about correlations (except to give a large set where, in a functional sense, correlations are asymptotically negligible). By contrast, we can apply part (ii) of Theorem 2.1 (stated below) to provide definite information about how the fine structure of the second marginal (beyond its being asymptotically factorized) affects the third marginal.

Remark. It is important to realize that complete factorization is allowed even at positive times along all of $K_s \cap U_s^{\eta(\varepsilon)}$, but only partial factorization is possible at some points of $G_s \cap \tilde{U}_s^{\eta(\varepsilon)}$ when $t > 0$; this is due to the fact that collisions generate correlations. In this sense, 2-nonuniform chaoticity captures (very crudely) the fine-scale structure of correlations at positive times. This is of interest even for the almost-perfectly factorized data of Section 11 because the dynamics will create correlations no matter how perfect the initial data happens to be. There has been some recent interest in precisely characterizing the size of correlations in the Boltzmann-Grad limit; we refer to [3, 28] for some results along these lines. Compared to these previous results, the main difference with our result is that we draw a connection between correlations and strong chaos.

Recall the Boltzmann equation

\[
(\partial_t + v \cdot \nabla_x) f(t) = \ell^{-1} Q(f(t), f(t))
\]

\[
Q(f, f) = \int_{\mathbb{R}^d \times S^{d-1}} [\omega \cdot (v_1 - v)]_+(f(x, v^*)f(x, v_1^*) - f(x, v)f(x, v_1)) \, d\omega dv_1
\]

We are able to show:

**Theorem 2.1.** Suppose that the Boltzmann equation (10) has a non-negative solution $f(t)$ for $t \in [0, T]$, with $\int f(t) \, dx dv = 1$, and further suppose that there exists $\beta > 0$ such that

\[
\sup_{0 \leq t \leq T} \sup_{x,v} e^{\frac{1}{2} \beta t} |v|^2 f(t, x, v) < \infty
\]

and $f(t) \in W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)$ for $t \in [0, T]$. Let $F_N(t)$ solve the hard sphere BBGKY hierarchy, under the Boltzmann-Grad scaling $N^{d-1} = \ell^{-1}$, and suppose that there is a $\tilde{\beta} > 0$, $\tilde{\mu} \in \mathbb{R}$ such that

\[
\sup_{N \in \mathbb{N}} \sup_{0 \leq t \leq T} \sup_{1 \leq s \leq N} \sup_{Z_s \in \mathcal{D}_s} e^{\frac{1}{2} \beta t E_s(Z_s)} e^{\tilde{\mu} t s} \left| f^{(s)}_N(t, Z_s) \right| < \infty
\]

Then the following holds:

(i) If $\{F_N(0)\}_N$ is nonuniformly $f_0$-chaotic, then for all $t \in [0, T]$, $\{F_N(t)\}_N$ is nonuniformly $f(t)$-chaotic (with the same $\kappa$).
(ii) If \( \{F_N(0)\}_N \) is 2-nonuniform \( f_0 \)-chaotic, then for all \( t \in [0, T] \), \( \{F_N(t)\}_N \) is 2-nonuniform \( f(t) \)-chaotic (with the same \( \kappa \)).

We will prove in full detail part (i) of Theorem 2.1. The proof of part (ii) is similar to the proof of part (i); the main differences are the use of an intermediate (Boltzmann-Enskog) hierarchy, as in [28], combined with a refined analysis of pseudo-trajectories. We supply the necessary ideas and all of the key technical estimates for part (ii) in Appendix A.

Remark. The time \( T \) in Theorem 2.1 is not necessarily the time in Lanford’s original theorem. For instance, in the case of a sufficiently small perturbation of vacuum [17, 18], it is permissible to take \( T \) arbitrarily large. More generally, if the \( a \) \( p \) \( r \) \( i \) \( o \) \( r \) estimate (13) is known for a specific (factorized) solution of the BBGKY hierarchy up to time \( T \), then we can propagate (2-)nonuniform chaoticity up to time \( T \). Note that \( T \) is necessarily smaller than the existence time for classical solutions of the Boltzmann equation.

Remark. There is a reasonable question to be asked about the optimality of the sets we have defined. Certainly the set \( K_s \) can be improved by specifying a “horizon” into the past beyond which collisions between particles are allowed (indeed this trivial refinement would be necessary when working in a bounded or periodic domain). More significantly, the condition defined by \( U_{\eta}^s \) is clearly not optimal (we thank the anonymous referees for bringing this issue to our attention). The set \( U_{\eta}^s \) was specifically chosen to simplify the inductive arguments, but it turns out that the same arguments apply while allowing some particles to have the same velocity, if they are far apart from each other. For example, let us define

\[
\iota(x, v) = \inf_{\tau \in \mathbb{R}} |x - v\tau|
\]

and introduce the sets (for \( 0 < \eta < 1 \))

\[
\tilde{U}_{\eta}^s = \left\{ Z_s = (X_s, V_s) \in D_s \left| \inf_{1 \leq i < j \leq s} \left( \frac{|v_i - v_j|}{\eta} + \frac{\iota(x_i - x_j, v_i - v_j)}{\eta \log \frac{1}{\eta}} \right) > 1 \right\}
\]

Then if \( Z_s \in \tilde{U}_{\eta}^s \) then for any \( i \neq j \) there are only two possibilities: either (i) \( |v_i - v_j| > \frac{1}{2} \eta \), or (ii) \( |v_i - v_j| \leq \frac{1}{2} \eta \) in which case we have

\[
\inf_{\tau \in \mathbb{R}} |(x_i - x_j) - (v_i - v_j)\tau| > \frac{1}{2} \eta \log \frac{1}{\eta}
\]

which implies that the two particles \( i, j \) can never get close enough to possibly prevent convergence. Creating particles is easy: when the choice can be made, we always choose to enforce \( |v_i - v_j| > \eta \). To summarize our argument, we find that if \( U_{\eta}^{s(\varepsilon)} \) is replaced by \( \tilde{U}_{\eta}^{s(\varepsilon)} \) in Definition 2.1 then Theorem 2.1 is still true (specifically part (i) of the theorem). The proof is unchanged, apart from replacing \( U_{\eta}^s \) by \( \tilde{U}_{\eta}^s \) throughout and choosing new constants where necessary. The sets of convergence can be refined even further (e.g. it is
possible to require simply \(\inf_{i\neq j} \left( |v_i - v_j| + \left( \log \frac{1}{\eta} \right)^{-1} |x_i - x_j| \right) \gtrsim \eta \) but we would pay a price, both in terms of readability of the proof and the error estimates themselves (since a loss is required at each step of induction). A similar situation holds with part \((ii)\) of Theorem 2.1 but we omit the details.

2.3. Non-chaotic data. So far we have drawn a connection between a particular notion of chaos and irreversibility. However, chaoticity is not a necessary condition for irreversible behavior. There is no novelty here. The relation between particle systems and the Hewitt-Savage theorem is a classical observation of H. Spohn [29], which was also implicit for instance in the work of O. E. Lanford and others through the use of the Boltzmann hierarchy. We refer to [5, 8, 30] for expository accounts of the connection between propagation of chaos and the Hewitt-Savage theorem.

Remark. The results of this subsection are, in some sense, not really more general than Theorem 2.1 due to the Hewitt-Savage theorem and the linearity of the BBGKY and Boltzmann hierarchies. This purpose of this short discussion is simply to emphasize that a good notion of chaos leads naturally to a good notion of convergence, even for a broad class of non-chaotic initial conditions.

Let us suppose that \(f_N^{(s)}(t)\) are the marginals of an underlying \(N\)-particle probability density \(f_N(t)\) which is symmetric under particle interchange. Assume that as \(N \to \infty\), the marginals \(f_N^{(s)}(t)\) converge to functions \(f_\infty^{(s)}(t)\) which satisfy the properties of non-negativity, normalization and consistency (respectively): (these are all true at finite \(N\) in any case)

\[
f_\infty^{(s)}(t) \geq 0 \tag{17}
\]

\[
\int_{\mathbb{R}^{2d}} f_\infty^{(s)}(t) dZ_s = 1 \tag{18}
\]

\[
f_\infty^{(s)}(t, Z_s) = \int_{\mathbb{R}^{2d}} f_\infty^{(s+1)}(t, Z_{s+1}) dZ_{s+1} \tag{19}
\]

If the functions \(\left\{f_\infty^{(s)}(t)\right\}_{s \in \mathbb{N}}\) are non-negative, normalized, and consistent, and symmetric under particle interchange, then the Hewitt-Savage theorem [15] tells us that there exists a time-dependent probability measure \(\pi_t \in \mathcal{P}(\mathcal{P}(\mathbb{R}^{2d}))\)\(^{10}\) such that

\[
f_\infty^{(s)}(t) = \int_{\mathcal{P}(\mathbb{R}^{2d})} h^{\otimes s}(Z_s) d\pi_t(h) \tag{20}
\]

Hence, in very great generality, we are free to assume that the limiting distribution is a convex combination of factorized distributions. If the convergence of \(\left\{f_N^{(s)}(t)\right\}_{1 \leq s \leq N}\) to \(\left\{f_\infty^{(s)}(t)\right\}_{s \in \mathbb{N}}\) is sufficiently strong, and we have

---

\(^{10}\)Here \(\mathcal{P}(X)\) is the set of Borel probability measures on the Polish space \(X\).
sufficient control on solutions to Boltzmann’s equation, then it is possible to explicitly characterize the measure \( \pi_t \).

It is possible to show the following result through a slight refinement of the proof of Theorem \[2,21\]

**Theorem 2.2.** Let \( \pi \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d)) \). Suppose that for \( \pi - \text{a.e.} \) \( h_0 \) there exists a non-negative solution \( h(t) \) of Boltzmann’s equation on \([0, T]\) with \( h(0) = h_0 \), and with \( \int h(t) dx dv = 1 \), and further suppose that there exist \( C_T, \beta_T > 0 \) (which are constants on a full \( \pi \)-measure) such that

\[
\sup_{0 \leq t \leq T} \sup_{x,v \in \mathbb{R}^d} e^{\frac{1}{\beta_T} |v|^2} h(t, x, v) \leq C_T \tag{21}
\]

\[
\sup_{0 \leq t \leq T} \| h(t) \|_{W^{1, \infty}([\mathbb{R}^d \times \mathbb{R}^d])} \leq C_T \tag{22}
\]

Let \( F_N(t) = \left\{ f_N^{(s)}(t) \right\}_{1 \leq s \leq N} \) solve the hard sphere BBGKY hierarchy, under the Boltzmann-Grad scaling \( N \epsilon^d - 1 = \ell^{-1} \). Assume that there is a \( \tilde{\beta}_T > 0 \), \( \tilde{\mu}_T \in \mathbb{R} \) such that

\[
\sup_{N \in \mathbb{N}} \sup_{0 \leq t \leq T} \sup_{1 \leq s \leq N, Z_s \in \mathcal{D}_s} e^{\tilde{\beta}_T E_s(Z_s) \ell^{-1} T} | f_N^{(s)}(t, Z_s) | < \infty \tag{23}
\]

Suppose that for some \( \kappa \in (0, 1) \), we have for all \( s \in \mathbb{N} \) and all \( R > 0 \) that

\[
\limsup_{N \to \infty} \left\| f_N^{(s)}(0) - \int \mathcal{P}(\mathbb{R}^d) h_0^{\otimes s} d\pi(h_0) \right\|_{L_{Z_s}^\infty} = 0 \tag{24}
\]

where \( \eta(\varepsilon) = \varepsilon^\kappa \). Then for all \( t \in [0, T] \), all \( s \in \mathbb{N} \), and all \( R > 0 \) we have:

\[
\limsup_{N \to \infty} \left\| f_N^{(s)}(t) - \int \mathcal{P}(\mathbb{R}^d) h(t)^{\otimes s} d\pi(h_0) \right\|_{L_{Z_s}^\infty} = 0 \tag{25}
\]

**Remark.** To see why Theorem \[2.2\] is true, it is enough to realize that the proof of Theorem \[2.1\] is through a comparison between the BBGKY and Boltzmann hierarchies (similar to \[11\,22\]). The Boltzmann hierarchy is linear, and therefore convex combinations of solutions are again solutions; uniqueness of the Boltzmann hierarchy follows from the estimates of \[22\] which are recalled in the present work.

Theorem \[2.2\] is a generalization of the propagation of nonuniform chaoticity when there is some uncertainty in the initial data \( h_0 \) for Boltzmann’s equation. It is similarly possible to generalize the propagation of 2-nonuniform chaoticity to the situation where \( h_0 \) is random. However, one must be quite careful when dealing with 2-nonuniform chaoticity because the representation formula

\[
\int \mathcal{P}(\mathbb{R}^d) h(t)^{\otimes s} d\pi(h_0) \tag{26}
\]
fails in general (when \( t > 0 \)) at phase points for which a collision has occurred in the past. We have, by slight refinements (the details being mostly notational in nature) of the proof of Theorem 2.1, the following result:

**Theorem 2.3.** Under the assumptions of Theorem 2.1, let us further suppose that for \( \pi - \text{a.e.} \ h_0 \) we have sequences \( H_N(t; h_0) = \{h_N^{(s)}(t; h_0)\}_{s \leq N} \) such that \( H_N(t; h_0) \) solves the hard sphere BBGKY hierarchy for \( \pi - \text{a.e.} \ h_0 \) fixed, and \( \{H_N(t; h_0)\}_N \) is \( 2 \)-nonuniformly \( h(t) \)-chaotic (with \( \kappa \) fixed once and for all) for each \( t \in [0, T] \). (The existence of such sequences \( \{H_N(t; h_0)\}_N \) can be proven on a short time interval using Theorem 2.1 and Lanford’s uniform bounds.) Assume as in the statement of Theorem 2.2 that

\[
\sup_{N \in \mathbb{N}} \sup_{0 \leq t \leq T} \sup_{1 \leq s \leq N} Z_s \in D_s e^{\tilde{\beta}_T E_s(Z_s)} e^{\tilde{\mu}_T s} \left| h_N^{(s)}(t, Z_s; h_0) \right| < C
\]

where \( C, \tilde{\beta}_T, \tilde{\mu}_T \) are constant on a set of full \( \pi \)-measure. Assume that

\[
\limsup_{N \to \infty} \left\| \left( f_N^{(s)}(0) - \int_{\mathcal{P}(\mathbb{R}^2d)} h_N^{(s)}(0; h_0) d\pi(h_0) \right) 1_{Z_s \in G_s \cap \tilde{U}_s^{\eta(\varepsilon)} 1_{E_s(Z_s) \leq R^2} } \right\|_{L_{Z_s}^\infty} = 0
\]

where \( \eta(\varepsilon) = \varepsilon^\kappa \). Then for all \( t \in [0, T] \) we have

\[
\limsup_{N \to \infty} \left\| \left( f_N^{(s)}(t) - \int_{\mathcal{P}(\mathbb{R}^2d)} h_N^{(s)}(t; h_0) d\pi(h_0) \right) 1_{Z_s \in G_s \cap \tilde{U}_s^{\eta(\varepsilon)} 1_{E_s(Z_s) \leq R^2} } \right\|_{L_{Z_s}^\infty} = 0
\]

### 3. Notation and a Comparison Principle

We will work in the spatial domain \( \mathbb{R}^d \) for some \( d \geq 2 \). Let \( \varepsilon > 0 \) and \( N \in \mathbb{N} \) satisfy the Boltzmann-Grad scaling \( N \varepsilon^{d-1} = \ell^{-1} \) for some fixed parameter \( \ell > 1 \); we will henceforth suppress the implicit dependence on \( \varepsilon, \ell \) in our notation, though they will be retained in formulas and estimates. If \( 1 \leq s \leq N \) then we define the phase space

\[
D_s = \left\{ Z_s = (X_s, V_s) \in \mathbb{R}^{ds} \times \mathbb{R}^{ds} \left| x_i - x_j > \varepsilon \quad \forall 1 \leq i < j \leq s \right. \right\}
\]

Suppose \( Z_s \in \partial D_s \), with \( x_j = x_i + \varepsilon \omega, \omega \in \mathbb{S}^{d-1}, \omega \cdot (v_j - v_i) \neq 0, i = j, \) and \( |x_{j'} - x_{j''}| > \varepsilon \) whenever \( j' < j'' \) and \( (j', j'') \neq (i, j) \); then the image point \( Z_s^* = (x_1, v_1, \ldots, x_i, v_i^*, \ldots, x_j, v_j^*, \ldots, x_s, v_s) \) is defined by the following rule:

\[
\begin{align*}
v_i^* &= v_i + \omega \omega \cdot (v_j - v_i) \\
v_j^* &= v_j - \omega \omega \cdot (v_j - v_i)
\end{align*}
\]

The parameter \( \ell \) is of order the mean free path length, insofar as the mean free path is well-defined.
Note that the map \( Z_s \mapsto Z^*_s \) is a measurable involution of \( \partial D_s \); and, in the identity \( Z^*_s = Z_s \) a.e. \( Z_s \in \partial D_s \), we use the same \( \omega \in \mathbb{S}^{d-1} \) for each transformation.

Let us denote by \( \psi^t(Z_s) \) the image of \( Z_s \) under the forward time evolution of \( s \) hard spheres at time \( t \); that is, if \( Z_s = Z_s(0) \), and the function \( Z_s(t) = (X_s(t), V_s(t)) \) is piecewise differentiable and has left and right limits at all points \( t \in \mathbb{R} \), and then holds

\[
\begin{cases}
  \frac{d}{dt} Z_s(t) = (V_s(t), 0) & \text{if } Z_s(t) \notin \partial D_s \\
  Z_s(t^+) = (Z_s(t^-))^* & \text{if } Z_s(t) \in \partial D_s
\end{cases}
\]

for all \( t \in \mathbb{R} \) then we write \( \psi^t_s Z_s = Z_s(t) \). This “definition,” unfortunately, does not define \( \psi^t_s Z_s \) uniquely in general, since there is no way to continuously extend the map \( Z_s \mapsto Z^*_s \) to all of \( \partial D_s \). Indeed, discontinuities will be observed whenever one particle simultaneously collides with at least two other particles. Nevertheless, up to deletion of a Lebesgue measure zero subset of initial phase points \( Z_s \in D_s \), we may assume that \( \psi^t_s Z_s \) is defined for all \( t \in \mathbb{R} \), that all collisions are non-grazing, and that all collisions are binary and linearly ordered in time (i.e. disjoint pairs of particles do not simultaneously collide). One can then show that, for each \( t \in \mathbb{R} \), \( \psi^t_s \) may be viewed as a measurable map \( D_s \to D_s \) preserving the induced Lebesgue measure. On bounded time intervals, the map \( (t, Z_s) \mapsto \psi^t_s Z_s \) is actually \textit{jointly continuous} away from certain higher codimension submanifolds of the domain, provided that one chooses to identify \( Z_s \in \partial D_s \) with its image \( Z^*_s \). However, we will not make such an identification; instead, we choose to enforce the convention that, for a.e. \( Z_s \in D_s \), there holds for all \( t \in \mathbb{R} \) that \( \psi^t_s Z_s = \psi^{t+}_s Z_s \). We will say that a point \( Z_s \in \partial D_s \) is a \textit{pre-collisional configuration} if \( Z_s = \psi^t_s Z_s \); or, we will call it a \textit{post-collisional configuration} if \( Z_s = \psi^{t+}_s Z_s \). Note in particular that, according to our conventions, \( Z_s \neq \psi^0_s Z_s \) for a.e. pre-collisional \( Z_s \in \partial D_s \).

Suppose \( f_N(0) \) is a probability measure supported on \( \overline{D_N} \) and absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^{2dN} \); by abuse of notation, we call the corresponding density \( f_N(0, Z_N) \). We will denote by \( S_N \) the symmetric group on \( N \) indices; if \( Z_N \in D_N \) then \( \sigma \in S_N \) acts on \( Z_N \) by permutation of particle indices: \( \sigma(z_1, \ldots, z_N) = (z_{\sigma(1)}, \ldots, z_{\sigma(N)}) \). We will always assume that \( f_N(0) \) is \textit{symmetric}, i.e. for any \( \sigma \in S_N \) there holds \( f_N(0, \sigma Z_N) = f_N(0, Z_N) \). Then for \( t \in \mathbb{R} \) we will define \( f_N(t, Z_N) = f_N(0, \psi^t_N Z_N) \); equivalently, since \( \psi^t_N \) preserves Lebesgue measure on \( \mathbb{R}^{2dN} \), we can say that \( f_N(t) \) is the pushforward of \( f_N(0) \) under \( \psi^t_N \). We will denote

\[
Z_{s,a+k} = (z_s, z_{s+1}, \ldots, z_{s+k}), Z^{(i)}_{s,a+k} = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_s), \text{ and similarly } Z^{(i)}_{s,a+k}
\]

in the case \( s \leq i \leq s + k \). We extend \( f_N(t) \) by zero so that it is defined on \( \mathbb{R}^{2dN} \); then the marginals \( f^{(s)}_N(t, Z_s) \) are defined by \( f^{(s)}_N(t, Z_s) = \int_{\mathbb{R}^{2d(N-s)}} f_N(t, Z_N) dZ_{(s+1):N} \). Each \( f^{(s)}_N(t) \) is a symmetric probability density

\[\text{classically differentiable on the complement of a closed set of isolated points}\]
supported on \( \overline{D_s} \); and, since \( f_N^{(s)}(t) \) is the marginal of \( f_N^{(s+1)}(t) \) for each \( 1 \leq s < N \), we say that the sequence \( \left\{ f_N^{(s)}(t) \right\}_{1 \leq s \leq N} \) is consistent. We also define the energy \( E_s(Z_s) = \frac{1}{2} \sum_{i=1}^s |v_i|^2 \), and we will also let \( I_s(Z_s) = \frac{1}{2} \sum_{i=1}^s |x_i|^2 \), and additionally \( \mathcal{Y}_s(Z_s) = \sum_{i=1}^s x_i \cdot v_i \).

**Remark.** Sometimes we will want to consider sequences \( \left\{ f_N^{(s)} \right\}_{1 \leq s \leq N} \) which are not consistent, and not necessarily normalized nor even non-negative. We will only point out this distinction when it is important for the analysis. For the remainder of this section, we will assume that \( \left\{ f_N^{(s)} \right\}_{1 \leq s \leq N} \) is a consistent sequence of symmetric probability densities.

We now turn to a comparison principle; this result is due to Illner & Pulvirenti [16–18] and is specific to the whole space case.

**Lemma 3.1.** For a.e. \( Z_s = (X_s, V_s) \in D_s \) and all \( t \geq 0 \),

\[
\mathcal{Y}_s(\psi_t^t Z_s) \geq 2tE_s(Z_s) + \mathcal{Y}_s(Z_s) \tag{33}
\]

**Proof.** Fix \( Z_s \in D_s \) such that \( \psi_t^t Z_s \) is defined for all \( t \in \mathbb{R} \), with all collisions binary and non-grazing. Let \( r(t) = \mathcal{Y}_s(\psi_t^t Z_s) - 2tE_s(\psi_t^t Z_s); \) then \( r(0) = \mathcal{Y}_s(Z_s) \). Between collisions we have \( \frac{d}{dt} r(t) = 0 \), and \( r \) can only increase across collisions. We use the energy conservation identity \( E_s(\psi_t^t Z_s) = E_s(Z_s) \) to conclude. \( \square \)

**Lemma 3.2.** For a.e. \( Z_s = (X_s, V_s) \in D_s \) and all \( t \in \mathbb{R} \),

\[
I_s(\psi_t^t Z_s) \geq I_s((X_s + V_s t, V_s)) \tag{34}
\]

**Proof.** Due to time reversibility, it suffices to consider the case \( t \geq 0 \). Fix \( Z_s \in D_s \) such that \( \psi_t^t Z_s \) is defined for all \( t \in \mathbb{R} \), with all collisions binary and non-grazing. Let \( b(t) = I_s(\psi_t^t Z_s) - I_s((X_s + V_s t, V_s)); \) observe that \( b(0) = 0 \), and \( b(t) \) is continuous and piecewise smooth. Between collisions we have

\[
\frac{d}{dt} b(t) = \mathcal{Y}_s(\psi_t^t Z_s) - 2tE_s(Z_s) - \mathcal{Y}_s(Z_s) \geq 0 \tag{35}
\]

where we have used Lemma 3.1. Therefore \( b(t) \geq 0 \) for all \( t > 0 \), and the result follows. \( \square \)

4. **The BBGKY and dual BBGKY hierarchies**

The BBGKY hierarchy is a sequence of equations which describe the evolution of the marginals \( f_N^{(s)}(t) \) of a solution \( f_N(t) \) of Liouville’s equation. The BBGKY hierarchy is one of the classical tools in the mathematical analysis of many-particle systems. Many derivations of the BBGKY hierarchy have been devised; we refer to [11], which will be the approach most convenient for us. We give a slightly generalized version of the weak form of the BBGKY hierarchy derived in [11], since it will enable us to easily read off the dual BBGKY hierarchy. The dual BBGKY hierarchy is the
sequence of equations whose semigroup generator is the adjoint of that of the BBGKY hierarchy. We will be using the dual BBGKY hierarchy in order to derive uniform bounds in Sections 5 and 6. The main advantage of the dual BBGKY hierarchy is that the semigroup generator makes sense without strong regularity assumptions; this is useful because the BBGKY hierarchy does not propagate smoothness of the marginals.

Suppose we are given a sequence of functions \( \{ f_N^{(s)}(t, Z_s) \}_{1 \leq s \leq N} \), with \( f_N^{(s)} \) defined on \([0, \infty) \times \mathcal{D}_s\) and \((\partial_t + V_s \cdot \nabla X_s) f_N^{(s)} \in L^1(\mathcal{O})\) for any bounded open set \( \mathcal{O} \subset [0, \infty) \times \mathcal{D}_s\). Further suppose the marginals satisfy permutation symmetry and the boundary condition \( f_N^{(s)}(t, Z_s^*) = f_N^{(s)}(t, Z_s) \) for a.e. \((t, Z_s) \in [0, \infty) \times \partial \mathcal{D}_s\). Then we will say that the sequence \( \{ f_N^{(s)}(t, Z_s) \}_{1 \leq s \leq N} \) solves the weak form of the BBGKY hierarchy provided that for every test function \( \varphi_s(t, Z_s) \in C_c^1([0, \infty) \times \mathcal{D}_s) \), satisfying permutation symmetry, there holds:

\[
\int_0^\infty \int_{\mathcal{D}_s} \left[ (\partial_t + V_s \cdot \nabla X_s) \varphi_s(t, Z_s) \right] f_N^{(s)}(t, Z_s) dZ_s dt = \\
= \int_{\mathcal{D}_s} \varphi_s(0, Z_s) f_N^{(s)}(0, Z_s) dZ_s \\
- \varepsilon^{d-1} \sum_{1 \leq i < j \leq s} \int_0^\infty \int_{\mathbb{R}^d \times \mathbb{R}^{d(s-1)} \times \mathbb{R}^{d-1}} 1_{Z_s \in \partial \mathcal{D}_s} \omega \cdot (v_j - v_i) \times \\
\times (\varphi_s f_N^{(s)}) (t, \ldots, x_i, v_i, \ldots, x_i + \varepsilon \omega, v_j, \ldots) d\omega dX_s^{(j)} dV_s dt \\
- (N-s) \varepsilon^{d-1} \sum_{1 \leq i < s} \int_0^\infty \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d(s-1)} \times \mathbb{R}^{d-1}} 1_{Z_{s+1} \in \partial \mathcal{D}_{s+1}} \omega \cdot (v_{s+1} - v_i) \times \\
\times \varphi_s(t, Z_s) f_N^{(s+1)}(t, Z_s, x_i + \varepsilon \omega, v_{s+1}) d\omega dv_{s+1} dX_s dV_s dt
\]

(36)

If \( f_N(0) \in C_0^\infty(\mathcal{D}_N) \) and \( f_N(t) \) satisfies Liouville’s equation, then the sequence of marginals \( \{ f_N^{(s)}(t) \}_{1 \leq s \leq N} \) solves the weak form of the BBGKY hierarchy. However, note that it is also possible to have solutions of the BBGKY hierarchy which are not sequences of marginals. Under suitable re-scalings, such solutions may have physical interpretations in the grand canonical ensemble, where the total number of particles is considered random. In our treatment, however, we will always be working in the canonical ensemble, since the total number of particles is just \( N \).

We now turn to the dual BBGKY hierarchy. Given a pair of densities \( F_N = \{ f_N^{(s)} \}_{1 \leq s \leq N} \) and test functions \( \Phi_N = \{ \varphi_N^{(s)} \}_{1 \leq s \leq N} \), with each \( f_N^{(s)}, \varphi_N^{(s)} \) symmetric under particle interchange, we define a duality bracket
\[ \langle \Phi_N, F_N \rangle = \sum_{s=1}^{N} \frac{1}{s!} \int_{D_s} \varphi_N^{(s)}(Z_s) f_N^{(s)}(Z_s) dZ_s \]  

(37)

We would like to define the dual BBGKY hierarchy by the following duality relation:

\[ \langle \Phi_N(t), F_N(0) \rangle = \langle \Phi_N(0), F_N(t) \rangle \]  

(38)

which should hold whenever \( F_N(t) \) solves the BBGKY hierarchy and \( \Phi_N(t) \) solves the dual BBGKY hierarchy. Applying (38) and considering arbitrary weak solutions \( F_N(t) \) of the BBGKY hierarchy, one can show that observables evolve according to the following hierarchy of equations (this is equivalent to equation 15 in [13], up to trivial re-scaling):

\[ (\partial_t - V_s \cdot \nabla X_s) \varphi_N^{(s)}(t, Z_s) = 0 \quad (Z_s \in D_s, s = 1, \ldots, N) \]  

(39)

\[ \frac{\varphi_N^{(s)}(t, Z_s)}{N - s + 1} + \varphi_N^{(s-1)}(t, (Z_s)^{(i)}) + \varphi_N^{(s-1)}(t, (Z_s)^{(j)}) = \]  

\[ = \frac{\varphi_N^{(s)}(t, Z_s)}{N - s + 1} + \varphi_N^{(s-1)}(t, Z_s^{(i)}) + \varphi_N^{(s-1)}(t, Z_s^{(j)}) \]  

(40)

where

\[ \Sigma_s(i, j) = \left\{ X_s \in \mathbb{R}^{d_s} \mid |x_i - x_j| = \varepsilon \right\} \]  

(41)

Given an initial data \( \varphi_N^{(s)}(0) \), \( 1 \leq s \leq N \), we can solve this hierarchy recursively. The nonzero observable of lowest order (at the initial time, and therefore all time) simply evolves via the backwards Liouville flow. Once \( \varphi_N^{(s-1)}(t) \) is known for all \( t \geq 0 \), it is possible to determine \( \varphi_N^{(s)}(t) \) by integrating along characteristics. One uses the knowledge of \( \varphi_N^{(s-1)} \) to determine the amount by which \( \varphi_N^{(s)} \) “jumps” at particle collisions. Let us point out that as \( Z_s \) ranges over an open subset of \( (\Sigma_s(i, j) \times \mathbb{R}^{d_s}) \cap \partial D_s \), the coordinates \( Z_s^{(i)} \), \ldots, cover an open subset of \( D_{s-1} \). Thus the source terms arising from \( \varphi_N^{(s-1)} \) are always well-defined functions on the set \( \partial D_s \). Note that, by a density argument involving a Duhamel-type formula, it is possible to use initial data \( \Phi_N(0) \) which does not satisfy the boundary condition (40).

5. Local a priori bounds on observables

We will prove weighted \( L^1 \) bounds on observables which are independent of \( N \); the stylized \( L \) is intended to distinguish the spaces in which we bound observables. The proof is a dualization of the classical proof of a priori bounds on the marginals \( f_N^{(s)} \) in weighted \( L^\infty \) spaces, originally due to Lanford. [11]22 As in the case of Lanford’s theorem, the a priori bounds will
only hold on a short time interval. Let us fix weight parameters $\beta > 0, \mu \in \mathbb{R}$, and define the norms

$$
\| \Phi_N \|_{L^1_{\beta,\mu}} = \sum_{s=1}^{N} \frac{1}{s!} \int_{D_s} \left| \varphi_N^{(s)}(Z_s) \right| e^{-\beta E_s(Z_s)} e^{-\mu s} dZ_s
$$

(42)

$$
|F_N|_{L^\infty_{\beta,\mu}} = \sup_{1 \leq s \leq N} \sup_{Z_s \in D_s} \left| f_N^{(s)}(Z_s) \right| e^{\beta E_s(Z_s)} e^{\mu s}
$$

(43)

where $E_s(Z_s) = \frac{1}{2} \sum_{i=1}^{s} |v_i|^2$. Then we have

$$
\langle \Phi_N, F_N \rangle \leq \| \Phi_N \|_{L^1_{\beta,\mu}} |F_N|_{L^\infty_{\beta,\mu}}
$$

(44)

Since $\varphi_N^{(s)}$ is transported along characteristics within $D_s$, $|\varphi_N^{(s)}(t, Z_s)|$ is transported as well. Therefore we have

$$
\frac{\partial}{\partial t} \int_{D_s} \left| \varphi_N^{(s)}(t, Z_s) \right| e^{-\beta E_s(Z_s)} e^{-\mu s} dZ_s =
$$

$$
= \int_{D_s} V_s \cdot \nabla X_s \left| \varphi_N^{(s)}(t, Z_s) \right| e^{-\beta E_s(Z_s)} e^{-\mu s} dZ_s
$$

$$
= \sum_{1 \leq i < j \leq s} \int_{\mathbb{R}^d} \int_{\Sigma_s(i,j)} n^{ij} \cdot V_s \left| \varphi_N^{(s)}(t, Z_s) \right| e^{-\beta E_s(Z_s)} e^{-\mu s} d\sigma^{ij} dV_s
$$

$$
= \frac{1}{2} \sum_{1 \leq i < j \leq s} \int_{\mathbb{R}^d} \int_{\Sigma_s(i,j)} n^{ij} \cdot V_s \times
$$

$$
\times \left( \left| \varphi_N^{(s)}(t, Z_s) \right| - \left| \varphi_N^{(s)}(t, Z^{*}_s) \right| \right) e^{-\beta E_s(Z_s)} e^{-\mu s} d\sigma^{ij} dV_s
$$

$$
\leq \frac{1}{2} \sum_{1 \leq i < j \leq s} \int_{\mathbb{R}^d} \int_{\Sigma_s(i,j)} \left| n^{ij} \cdot V_s \right| \times
$$

$$
\times \left| \varphi_N^{(s)}(t, Z_s) - \varphi_N^{(s)}(t, Z^{*}_s) \right| e^{-\beta E_s(Z_s)} e^{-\mu s} d\sigma^{ij} dV_s
$$
Now we employ the boundary condition to write
\[
\frac{\partial}{\partial t} \int_{D_s} \varphi_N^{(s)}(t, Z_s) e^{-\beta E_s(Z_s)} e^{-\mu s} dZ_s \leq \\
\frac{N}{2} \sum_{1 \leq i < j \leq s} \int_{\Sigma_s(i,j)} \int_{\mathbb{R}^d} |n^{i,j} \cdot V_s| \times \\
\times \left| \varphi_N^{(s-1)}(t, (Z_s^*)^{(i)}) + \varphi_N^{(s-1)}(t, (Z_s^*)^{(j)}) - \varphi_N^{(s-1)}(t, Z_s^{(i)}) - \varphi_N^{(s-1)}(t, Z_s^{(j)}) \right| \\
\times e^{-\beta E_s(Z_s)} e^{-\mu s} d\sigma^{i,j} dV_s \\
= \frac{N}{4} \sum_{i \neq j=1}^s \int_{\mathbb{R}^d} \int_{\Sigma_s(i,j)} |n^{i,j} \cdot V_s| \times \\
\times \left| \varphi_N^{(s-1)}(t, (Z_s^*)^{(i)}) + \varphi_N^{(s-1)}(t, (Z_s^*)^{(j)}) - \varphi_N^{(s-1)}(t, Z_s^{(i)}) - \varphi_N^{(s-1)}(t, Z_s^{(j)}) \right| \\
\times e^{-\beta E_s(Z_s)} e^{-\mu s} d\sigma^{i,j} dV_s \\
\leq \frac{N}{4} \sum_{i \neq j=1}^s \int_{\mathbb{R}^d} \int_{\Sigma_s(i,j)} |n^{i,j} \cdot V_s| \times \\
\times \left( \left| \varphi_N^{(s-1)}(t, (Z_s^*)^{(i)}) \right| + \left| \varphi_N^{(s-1)}(t, (Z_s^*)^{(j)}) \right| + \\
+ \left| \varphi_N^{(s-1)}(t, Z_s^{(i)}) \right| + \left| \varphi_N^{(s-1)}(t, Z_s^{(j)}) \right| \right) \\
\times e^{-\beta E_s(Z_s)} e^{-\mu s} d\sigma^{i,j} dV_s \\
= N \sum_{i \neq j=1}^s \int_{\mathbb{R}^d} \int_{\Sigma_s(i,j)} |n^{i,j} \cdot V_s| \times \\
\times \left| \varphi_N^{(s-1)}(t, Z_s^{(i)}) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s \\
+ \int_{D_s} \left| \varphi_N^{(s)}(t, Z_s) \right| \left\{ -\beta'(t) E_s(Z_s) - \mu'(t)s \right\} e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s
\]

We can generalize this inequality to the case of time-dependent weights.
\[
\frac{\partial}{\partial t} \int_{D_s} \varphi_N^{(s)}(t, Z_s) e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s \leq \\
\leq N \sum_{i \neq j=1}^s \int_{\mathbb{R}^d} \int_{\Sigma_s(i,j)} |n^{i,j} \cdot V_s| \times \\
\times \left| \varphi_N^{(s-1)}(t, Z_s^{(i)}) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} d\sigma^{i,j} dV_s + \\
+ \int_{D_s} \left| \varphi_N^{(s)}(t, Z_s) \right| \left\{ -\beta'(t) E_s(Z_s) - \mu'(t)s \right\} e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s
\]

Note that in the case \( s = 1 \) the first term on the RHS vanishes (there are no source terms at the boundary).

Let us estimate just the first term. The integral over the hypersurface \( \Sigma_s(i,j) = \{ X_s \in \mathbb{R}^d \mid |x_i - x_j| = \varepsilon \} \) brings down a factor of \( \varepsilon^{d-1} \), which is
then eliminated by virtue of the scaling $N \epsilon^d = \ell^{-1}$.

\[
N \sum_{i \neq j=1}^{s} \int_{\mathbb{R}^d} \int_{\Sigma_{s}(i,j)} |n^i \cdot V_s| \left| \varphi^{(s-1)}_{N}(t, Z_s(i)) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \, d\sigma_{i,j} \, dV_s \leq \\
\leq \ell^{-1} \sum_{i=1}^{s} \int_{\mathbb{R}^d} \int_{d(s-1)} \int_{d^d-1} \left( \sum_{j \neq i} |\omega \cdot (v_j - v_i)| \right) \left| \varphi^{(s-1)}_{N}(t, Z_s(i)) \right| \times \\
\times e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \, d\omega \, dX^{(i)}_s \, dV_s \\
\leq \ell^{-1} \sum_{i=1}^{s} \int_{\mathbb{R}^d} \int_{d(s-1)} \int_{d^d-1} \left| \varphi^{(s-1)}_{N}(t, Z_s(i)) \right| \times \\
\times \left( \sqrt{2}(s-1) \frac{1}{3} E_{s-1}(Z_s(i))^{\frac{3}{2}} + (s-1) |v_i| \right) e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \, d\omega \, dX^{(i)}_s \, dV_s \\
\leq C_d \ell^{-1} e^{-\mu(t)\beta(t)} \beta(t)^{-\frac{d}{2}} \int_{d(s-1) \times d(s-1)} \left| \varphi^{(s-1)}_{N}(t, Z_{s-1}) \right| \times \\
\times \left( (s-1) \frac{1}{2} E_{s-1}(Z_{s-1})^{\frac{1}{2}} + (s-1) \beta(t)^{-\frac{1}{2}} \right) \times \\
\times e^{-\beta(t)E_{s-1}(Z_{s-1})} e^{-\mu(t)(s-1)} \, dX_{s-1} \, dV_{s-1} \\
\]

We may sum over $s$ to obtain:

\[
\frac{\partial}{\partial t} \left\| \Phi_{N}(t) \right\|_{L^{1}_{\beta(t), \mu(t)}} \leq \\
\leq \sum_{s=2}^{N} \frac{1}{s!} C_d \ell^{-1} e^{-\mu(t)\beta(t)} \beta(t)^{-\frac{d}{2}} \int_{D_{s-1}} \left| \varphi^{(s-1)}_{N}(t, Z_{s-1}) \right| \times \\
\times \left( (s-1) \frac{1}{2} E_{s-1}(Z_{s-1})^{\frac{1}{2}} + \frac{(s-1)}{\beta(t)^{\frac{1}{2}}} \right) e^{-\beta(t)E_{s-1}(Z_{s-1})} e^{-\mu(t)(s-1)} \, dZ_{s-1} + \\
+ \sum_{s=1}^{N} \frac{1}{s!} \int_{D_s} \left| \varphi^{(s)}_{N}(t, Z_s) \right| \left\{ -\beta'(t) E_s(Z_s) - \mu'(t)s \right\} e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \, dZ_s \\
\]

(46)

We re-index the first term and combine; we furthermore assume that $\beta'(t), \mu'(t) > 0$ (this is opposite the usual convention because of duality). Then we have:

\[
\frac{\partial}{\partial t} \left\| \Phi_{N}(t) \right\|_{L^{1}_{\beta(t), \mu(t)}} \leq \\
\leq \sum_{s=1}^{N-1} \frac{1}{s!} \int_{D_s} \left| \varphi^{(s)}_{N}(t, Z_s) \right| \times \\
\times \left[ C_d \ell^{-1} e^{-\mu(t)\beta(t)} \beta(t)^{-\frac{d}{2}} \left( s \frac{1}{2} E_s(Z_s)^{\frac{3}{2}} + s \beta(t)^{-\frac{1}{2}} \right) - \beta'(t) E_s(Z_s) - \mu'(t)s \right] \times \\
\times e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \, dZ_s \\
\]

(47)
It is now apparent that $\Phi_N(t)$ is controlled as long as the quantity in brackets is everywhere nonpositive, for $0 \leq t \leq T$ and $Z_s \in D_s$. For instance, let us suppose that $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ are given. Then as long as $T_L > 0$ is chosen so that

$$T_L \leq C_d \ell e^{\mu_0 \beta_0^{d+1}}$$

then we shall have

$$\sup_{0 \leq t \leq T_L} \| \Phi_N(t) \|_{L^1_{\beta_0,\mu_0}} \leq \| \Phi_N(0) \|_{L^1_{\beta_0,\mu_0^{-1}}}$$

which implies by duality

$$\sup_{0 \leq t \leq T_L} | F_N(t) |_{L^\infty_{\beta_0,\mu_0^{-1}}} \leq | F_N(0) |_{L^\infty_{\beta_0,\mu_0}}$$

since the initial observable $\Phi_N(0)$ is arbitrary. Hence we obtain:

**Theorem 5.1.** Suppose $F_N(t) = \{ f_N^{(s)}(t) \}_{1 \leq s \leq N}$ is a solution of the BBGKY hierarchy [30], subject to the Boltzmann-Grad scaling $N e^{d-1} = \ell^{-1}$, and with each function $f_N^{(s)}(t, Z_s)$ symmetric under particle interchange. Further suppose that for some $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$,

$$\sup_{1 \leq s \leq N} \sup_{Z_s \in D_s} \left| f_N^{(s)}(0, Z_s) \right| e^{\beta_0 E_s(Z_s) e^{\mu_0 s}} \leq 1$$

Then there is a constant $C_d > 0$, depending only on $d$, such that if $T_L < C_d \ell e^{\mu_0 \beta_0^{d+1}}$ then there holds

$$\sup_{0 \leq t \leq T_L} \sup_{1 \leq s \leq N} \sup_{Z_s \in D_s} \left| f_N^{(s)}(t, Z_s) \right| e^{\frac{1}{2} \beta_0 E_s(Z_s) e^{\mu_0^{-1} s}} \leq 1$$

**Remark.** Theorem [5.1] does not require the functions $f_N^{(s)}$ to be non-negative, nor does it require that they form a consistent sequence of marginals.

The bound [50] is just the classical a priori bound of Lanford [11, 22]; note that the same argument based on observables would have worked in a periodic domain as well. Moreover, for any fixed initial datum, the Lanford time $T_L$ increases in direct proportion to the mean free path, which is $O(\ell)$.

### 6. Global a priori bounds on observables

Our goal is to extend the a priori bounds from the previous section to the entire time interval, $t \in [0, \infty)$, as soon as the mean free path $O(\ell)$ is sufficiently large. The relevant estimates were first proved by Illner & Pulvirenti [18], using the dispersive inequalities we have stated in Lemmas 3.1, 4.2. Our approach is slightly different, in that we will be working with the dual hierarchy. Note that once the correct weights are chosen, the rest amounts to a computation, plus one application of Lemma 3.1.

Let us be given a time $T > 0$, and smooth increasing functions $\beta(t) : [0, T] \to \mathbb{R}^+$, $\mu(t) : [0, T] \to \mathbb{R}$. The spaces $L^1_{\beta,\mu}$, $L^\infty_{\beta,\mu}$ are as defined in
the previous section. We are given functions $\Phi_N(t) = \{ \varphi_{N}^{(s)}(t) \}_{1 \leq s \leq N}$, with each $\varphi_{N}^{(s)} : [0, T] \times \mathcal{D}_s \rightarrow \mathbb{R}$ symmetric under particle interchange, such that $\Phi_N$ satisfies the dual hierarchy \cite{39,40} for $t \in [0, T]$. Define the functions
\[
\tilde{\varphi}_{N}^{(s)}(t, Z_s) = \varphi_{N}^{(s)}(t, Z_s) e^{-\beta(t)I_s((X_s - (T-t)V_s, V_s))}
\] (53)

We will be estimating $\|\tilde{\Phi}_N(t)\|_{\mathcal{L}_{\mu(t)}^{\beta(t)}}$ for $t \in [0, T]$.

Observe first that $(\partial_t - V_s \cdot \nabla X_s) I_s((X_s - (T-t)V_s, V_s)) = 0$ on any open subset of $\mathcal{D}_s$. On the other hand, for $Z_s = (X_s, V_s) \in \mathcal{D}_s$ we have
\[
I_s((X_s - (T-t)V_s, V_s)) = I_s(Z_s) - (T-t)Y_s(Z_s) + (T-t)^2 E_s(Z_s)
\] (54)
Clearly if $Z_s \in \partial \mathcal{D}_s$ then $I_s(Z_s^*) = I_s(Z_s)$, and $E_s(Z_s^*) = E_s(Z_s)$. Hence by Lemma 3.1,
\[
I_s((X_s - (T-t)V_s, V_s)) \geq I_s((X_s - (T-t)V_s^*, V_s^*))
\] (55)
whenever $t \in [0, T]$ and $Z_s = (X_s, V_s) \in \partial \mathcal{D}_s$ is pre-collisional.

The restriction $t \leq T$ is crucial; without this restriction the inequality could go the wrong way where we need it in the proof.

On any open subset of $\mathcal{D}_s$ we have
\[
\left(\frac{\partial}{\partial t} - V_s \cdot \nabla X_s\right) |\varphi_{N}^{(s)}(t, Z_s)| = 0
\] (56)
and likewise
\[
\left(\frac{\partial}{\partial t} - V_s \cdot \nabla X_s\right) I_s((X_s - (T-t)V_s, V_s)) = 0
\] (57)

Therefore by the divergence theorem we obtain the equality:
\[
\frac{\partial}{\partial t} \int_{\mathcal{D}_s} |\varphi_{N}^{(s)}(t, Z_s)| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s =
\]
\[
= \frac{1}{2} \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^{ds}} \int_{\Sigma_s(i,j)} n^{i \cdot j} \cdot V_s \left| \varphi_{N}^{(s)}(t, Z_s) \right| \times
\]
\[
\times e^{-\beta(t)|I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s))|} e^{-\mu(t)s} d\sigma^{i \cdot j} dV_s +
\]
\[
+ \int_{\mathcal{D}_s} |\tilde{\varphi}_{N}^{(s)}(t, Z_s)| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \times
\]
\[
\times \{-\beta'(t) [I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s)] - \mu'(t)s \} dZ_s
\] (58)

The boundary term can be re-written as an integral over pre-collisional configurations. Recall that, according to our conventions, $n^{i \cdot j} \cdot V_s = -\frac{x_j - x_i}{\varepsilon \sqrt{2}} \cdot (v_j - v_i)$ along $\Sigma_s(i,j) \times \mathbb{R}^{ds}$; therefore, $n^{i \cdot j} \cdot V_s > 0$ for pre-collisional
configurations. We have:

\[
\frac{\partial}{\partial t} \int_{\mathcal{D}_s} \left| \bar{\varphi}_N^{(s)} (t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s =
\]

\[
= \frac{1}{2} \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^d} \int_{\Sigma_{\text{inc}}^{(i,j)}} \left| n^{i,j} \cdot V_s \right| \left| \bar{\varphi}_N^{(s)} (t, Z_s) \right| \times
\]

\[
\times e^{-\beta(t)\left| I_s((X_s - (T - t)V_s, V_s)) + E_s(Z_s) \right|} e^{-\mu(t)s} d\sigma^{i,j} dV_s
\]

\[
- \frac{1}{2} \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^d} \int_{\Sigma_{\text{inc}}^{(i,j)}} \left| n^{i,j} \cdot V_s \right| \left| \bar{\varphi}_N^{(s)} (t, Z_s) \right| \times
\]

\[
\times e^{-\beta(t)\left| I_s((X_s - (T - t)V_s, V_s)) + E_s(Z_s) \right|} e^{-\mu(t)s} d\sigma^{i,j} dV_s
\]

\[
+ \int_{\mathcal{D}_s} \left| \bar{\varphi}_N^{(s)} (t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \times
\]

\[
\times \left\{ -\beta'(t) \left| I_s((X_s - (T - t)V_s, V_s)) + E_s(Z_s) \right| - \mu'(t)s \right\} dZ_s
\]

According to the boundary condition (40), for any \( Z_s \in \partial \mathcal{D}_s, \)

\[
\left| \bar{\varphi}_N^{(s)} (t, Z_s) \right| \leq \left| \bar{\varphi}_N^{(s)} (t, Z_s^*) \right| + N \left| \varphi_N^{(s-1)} (t, Z_s^{(i)}) \right| + N \left| \varphi_N^{(s-1)} (t, Z_s^{(j)}) \right|
\]

\[
+ N \left| \varphi_N^{(s-1)} (t, (Z_s^{(i)})^i) \right| + N \left| \varphi_N^{(s-1)} (t, (Z_s^{(j)})^j) \right|
\]

Therefore,

\[
\frac{\partial}{\partial t} \int_{\mathcal{D}_s} \left| \bar{\varphi}_N^{(s)} (t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s \leq
\]

\[
\leq \frac{1}{2} \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^d} \int_{\Sigma_{\text{inc}}^{(i,j)}} \left| n^{i,j} \cdot V_s \right| \left| \bar{\varphi}_N^{(s)} (t, Z_s^*) \right| \times
\]

\[
\times e^{-\beta(t)\left| I_s((X_s - (T - t)V_s, V_s)) + E_s(Z_s) \right|} e^{-\mu(t)s} d\sigma^{i,j} dV_s
\]

\[
+ N \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^d} \int_{\Sigma_{\text{inc}}^{(i,j)}} \left| n^{i,j} \cdot V_s \right| \left| \varphi_N^{(s-1)} (t, Z_s^{(i)}) \right| \times
\]

\[
\times e^{-\beta(t)\left| I_s((X_s - (T - t)V_s, V_s)) + E_s(Z_s) \right|} e^{-\mu(t)s} d\sigma^{i,j} dV_s
\]

\[
+ N \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^d} \int_{\Sigma_{\text{inc}}^{(i,j)}} \left| n^{i,j} \cdot V_s \right| \left| \varphi_N^{(s-1)} (t, (Z_s^{(i)})^i) \right| \times
\]

\[
\times e^{-\beta(t)\left| I_s((X_s - (T - t)V_s, V_s)) + E_s(Z_s) \right|} e^{-\mu(t)s} d\sigma^{i,j} dV_s
\]

\[
- \frac{1}{2} \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^d} \int_{\Sigma_{\text{inc}}^{(i,j)}} \left| n^{i,j} \cdot V_s \right| \left| \bar{\varphi}_N^{(s)} (t, Z_s^*) \right| \times
\]

\[
\times e^{-\beta(t)\left| I_s((X_s - (T - t)V_s, V_s)) + E_s(Z_s) \right|} e^{-\mu(t)s} d\sigma^{i,j} dV_s
\]

\[
+ \int_{\mathcal{D}_s} \left| \bar{\varphi}_N^{(s)} (t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \times
\]

\[
\times \left\{ -\beta'(t) \left| I_s((X_s - (T - t)V_s, V_s)) + E_s(Z_s) \right| - \mu'(t)s \right\} dZ_s
\]
We apply (55) to the first and third terms on the right hand side, for $0 \leq t \leq T$.

\[
\frac{\partial}{\partial t} \int_{D_s} \left| \varphi_{N}^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s \leq \]

\[
\leq \frac{1}{2} \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^{2ds}} \int_{\Sigma_{inc}^{i}(i,j)} \left| n^{i,j} \cdot V_s \right| \left| \varphi_{N}^{(s)}(t, Z_s^i) \right| \times e^{-\beta(t)\left[ I_s((X_s - (T-t)V_s, V_s^i)) + E_s(Z_s^i) \right]} e^{-\mu(t)s} d\sigma^{i,j} dV_s \]

\[
+ N \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^{2ds}} \int_{\Sigma_{inc}^{i}(i,j)} \left| n^{i,j} \cdot V_s \right| \left| \varphi_{N}^{(s-1)}(t, Z_s^{(i)}) \right| \times e^{-\beta(t)\left[ I_s((X_s - (T-t)V_s, V_s^i)) + E_s(Z_s^i) \right]} e^{-\mu(t)s} d\sigma^{i,j} dV_s \]

\[
- \frac{1}{2} \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^{2ds}} \int_{\Sigma_{inc}^{i}(i,j)} \left| n^{i,j} \cdot V_s \right| \left| \varphi_{N}^{(s)}(t, Z_s^i) \right| \times e^{-\beta(t)\left[ I_s((X_s - (T-t)V_s, V_s^i)) + E_s(Z_s^i) \right]} e^{-\mu(t)s} d\sigma^{i,j} dV_s \]

\[
+ \int_{D_s} \left| \tilde{\varphi}_{N}^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \times \{ -\beta'(t) \left[ I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s) \right] - \mu'(t)s \} dZ_s \] (62)

Now the first term precisely cancels the fourth term, whereas the second and third terms combine to yield an integral over all of $\Sigma_s(i,j)$.

\[
\frac{\partial}{\partial t} \int_{D_s} \left| \varphi_{N}^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s \leq \]

\[
\leq N \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^{2ds}} \int_{\Sigma_{inc}^{i}(i,j)} \left| n^{i,j} \cdot V_s \right| \left| \varphi_{N}^{(s-1)}(t, Z_s^{(i)}) \right| \times e^{-\beta(t)\left[ I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s) \right]} e^{-\mu(t)s} d\sigma^{i,j} dV_s \]

\[
+ \int_{D_s} \left| \tilde{\varphi}_{N}^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \times \{ -\beta'(t) \left[ I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s) \right] - \mu'(t)s \} dZ_s \] (63)

The following inequality is immediate and holds for all $Z_s \in \mathbb{R}^{2ds}$ and $t \in \mathbb{R}$:

\[
I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s) \geq \]

\[
\geq \frac{1}{2} \left( |x_i - (T-t)v_i|^2 + |v_i|^2 \right) + E_{s-1}(Z_s^{(i)}) \] (64)
We may eliminate $x_i$ from the right-hand side of (64) whenever $Z_s \in \Sigma_s(i,j) \times \mathbb{R}^{d_s}$, due to the condition $x_j = x_i + \varepsilon \omega$. Combining this fact with the Boltzmann-Grad scaling $N \varepsilon^{d-1} = \ell^{-1}$, we obtain the following from (63):

$$\frac{\partial}{\partial t} \int_{D_s} |\zeta_N^{(s)}(t, Z_s)| e^{-\beta(t)E_s(Z_s)}e^{-\mu(t)s} dZ_s \leq$$

$$\leq \ell^{-1} \sum_{i=1}^{s} \int_{\mathbb{R}^{2d(s-1)}} |\zeta_N^{(s-1)}(t, Z_s^{(i)})| \times$$

$$\times \sum_{j=1}^{s} \int_{\mathbb{R}^{d} \times \mathbb{R}^{d-1}} |\omega \cdot (v_j - v_i)| e^{-\frac{1}{2} \beta(t) [x_j - \varepsilon \omega - (T-t)v_i]^2 + |v_i|^2} d\omega dv_i \times$$

$$\times e^{-\beta(t) E_s(Z_s^{(i)})} e^{-\mu(t)s} dZ_s^{(i)}$$

$$+ \int_{D_s} |\zeta_N^{(s)}(t, Z_s)| e^{-\beta(t)E_s(Z_s)}e^{-\mu(t)s} \times$$

$$\times \{-\beta'(t) \left[ I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s) \right] - \mu'(t)s \} dZ_s$$

The integral in brackets is controlled using the classical dispersive inequality [2]:

$$\| \zeta(x - vt, v) \|_{L^d_{x,v} L^1_{t}} \leq |t|^{-d} \| \zeta(x, v) \|_{L^1_{x,v} L^\infty_{t}} \quad (65)$$

Hence,

$$\frac{\partial}{\partial t} \int_{D_s} |\zeta_N^{(s)}(t, Z_s)| e^{-\beta(t)E_s(Z_s)}e^{-\mu(t)s} dZ_s \leq$$

$$\leq \ell^{-1} s \int_{\mathbb{R}^{2d(s-1)}} |\zeta_N^{(s-1)}(t, Z_{s-1})| \times$$

$$\times \left[ C_d [1 + (T-t)]^{-d} \beta(t)^{-\frac{1}{2}} \left( (s-1)^\frac{1}{2} E_{s-1}(Z_{s-1})^\frac{1}{2} + (s-1) \beta(t)^{-\frac{1}{2}} \right) \right] \times$$

$$\times e^{-\beta(t) E_{s-1}(Z_{s-1})} e^{-\mu(t)s} dZ_{s-1}$$

$$+ \int_{D_s} |\zeta_N^{(s)}(t, Z_s)| e^{-\beta(t)E_s(Z_s)}e^{-\mu(t)s} \times$$

$$\times \{-\beta'(t) \left[ I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s) \right] - \mu'(t)s \} dZ_s$$

(67)
We can sum over $s$ to obtain, for $0 \leq t \leq T$,
\[
\frac{\partial}{\partial t} \left\| \Phi_N(t) \right\|_{L^1_{\beta(t),\mu(t)}} \leq \\
\leq \sum_{s=1}^{N-1} \frac{1}{s!} \int_{D_s} \left| \phi_N^{(s)}(t, Z_s) \right| e^{-\beta(t) E_s(Z_s)} e^{-\mu(t)s} \times \\
\times \left[ \frac{C_d e^{-\mu(t)} \beta(t)^{-\frac{d}{2}}}{\ell (1 + (T-t))^d} \left( s^{\frac{1}{2}} E_s(Z_s)^{\frac{1}{2}} + s \beta(t)^{-\frac{1}{2}} \right) - \beta'(t) E_s(Z_s) - \mu'(t)s \right] dZ_s
\]
(68)

Suppose $\beta_0 > 0$ and $\mu_0 \in \mathbb{R}$ are given. Then fixing any $T > 0$ we define
\[
\beta(t) = \beta_0 - \frac{1}{2} \beta_0 \left( 1 - [1 + (T-t)]^{-(d-1)} \right) 
\]
(69)
\[
\mu(t) = \mu_0 - \left( 1 - [1 + (T-t)]^{-(d-1)} \right) 
\]
(70)

We have $\beta(T) = \beta_0$, $\mu(T) = \mu_0$, $\inf_{0 \leq t \leq T} \beta(t) \geq \frac{1}{2} \beta_0$, $\inf_{0 \leq t \leq T} \mu(t) \geq (\mu_0 - 1)$, and
\[
\beta'(t) = \frac{1}{2} \beta_0 (d-1)[1 + (T-t)]^{-d} 
\]
(71)
\[
\mu'(t) = (d-1)[1 + (T-t)]^{-d} 
\]
(72)

Then if we assume further that $\ell^{-1} e^{-\mu_0} \beta_0^{-\frac{d+1}{2}}$ is sufficiently small (depending only on $d$), then
\[
\sup_{0 \leq t \leq T} \left\| \Phi_N(t) \right\|_{L^1_{\beta(t),\mu(t)}} \leq \left\| \Phi_N(0) \right\|_{L^1_{\beta_0(\mu_0-1)}} 
\]
(73)

In particular,
\[
\left\| \Phi_N(T) \right\|_{L^1_{\beta_0,\mu_0}} \leq \left\| \Phi_N(0) \right\|_{L^1_{\beta_0(\mu_0-1)}} 
\]
(74)

Since $T > 0$ is arbitrary, recalling the definition of $\Phi_N$ and using duality we obtain:

**Theorem 6.1.** (Illner & Pulvirenti 1989) Suppose $F_N(t) = \left\{ f_N^{(s)}(t) \right\}_{1 \leq s \leq N}$ is a solution of the BBGKY hierarchy (30), subject to the Boltzmann-Grad scaling $N e^{d-1} = \ell^{-1}$, and with each function $f_N^{(s)} : [0, \infty) \times \mathbb{R}^{2ds} \to \mathbb{R}$ symmetric under particle interchange. Further suppose that for some $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$,
\[
\sup_{1 \leq s \leq N} \sup_{Z_s \in D_s} \left| f_N^{(s)}(0, Z_s) \right| e^{\beta_0 [E_s(Z_s) + I_s(Z_s)]} e^{\mu_0 s} \leq 1 
\]
(75)

Then if $\ell^{-1} e^{-\mu_0} \beta_0^{-\frac{d+1}{2}}$ is sufficiently small (depending only on $d$) then we have
\[
\sup_{t \geq 0} \sup_{1 \leq s \leq N} \sup_{Z_s \in D_s} \left| f_N^{(s)}(t, Z_s) \right| e^{\frac{1}{2} \beta_0 [E_s(Z_s) + I_s((X_s - V_s t, V_s))]} e^{(\mu_0-1)s} \leq 1 
\]
(76)
Remark. Theorem 6.1 does not require the functions $f^{(s)}_N(t)$ to be non-negative, nor does it require that they form a consistent sequence of marginals.

7. Representation of marginals via pseudo-trajectories

We recall that any solution $f^{(s)}_N(t)$ of the BBGKY hierarchy may be decomposed in terms of the initial data propagated along “pseudo-trajectories.” This technique is first due to Lanford, and is now a standard tool in the analysis of the Boltzmann-Grad limit for hard spheres. To begin, observe that if \( \{ f^{(s)}_N(t, Z_s) \}_{1 \leq s \leq N} \) solves (36), then by considering test functions which vanish along \([0, \infty) \times \partial D_s\), it follows that the densities $f^{(s)}_N(t, Z_s)$ solve the following equation in the sense of distributions:

\[
\left( \frac{\partial}{\partial t} + V_s \cdot \nabla X_s \right) f^{(s)}_N(t, Z_s) = (N - s) \varepsilon^{d-1} C_{s+1} f^{(s+1)}_N(t, Z_s)
\]

where $f^{(s)}_N(t, Z_s) = f^{(s)}_N(t, Z_s^*)$ a.e. \((t, Z_s) \in [0, \infty) \times \partial D_s\), and $C_{s+1}$ is the collision operator

\[
C_{s+1} = \sum_{i=1}^{s} C_{i, s+1}
\]

(78)

\[
C_{i, s+1} = C_{i, s+1}^+ - C_{i, s+1}^-
\]

(79)

\[
C_{i, s+1}^+ f^{(s+1)}_N(t, Z_s) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} Z_{s+1} \in \partial D_{s+1} [\omega \cdot (v_{s+1} - v_i)]_+ \times
\]

\[
\times f^{(s+1)}_N(t, x_1, v_1, \ldots, x_i, v_i^*, \ldots, x_s, v_s, x_i + \varepsilon \omega, v_{s+1}^*) d\omega dv_{s+1}
\]

(80)

\[
C_{i, s+1}^- f^{(s+1)}_N(t, Z_s) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} Z_{s+1} \in \partial D_{s+1} [\omega \cdot (v_{s+1} - v_i)]_- \times
\]

\[
\times f^{(s+1)}_N(t, x_1, v_1, \ldots, x_i, v_i^*, \ldots, x_s, v_s, x_i + \varepsilon \omega, v_{s+1}) d\omega dv_{s+1}
\]

(81)

where

\[
\begin{align*}
  v_i^* &= v_i + \omega \omega \cdot (v_j - v_i) \\
  v_j^* &= v_j - \omega \omega \cdot (v_j - v_i)
\end{align*}
\]

(82)

We can re-write (77) by means of Duhamel’s formula, using the transport operator $T_s(t)$ defined by \((T_s(t)g_s)(Z_s) = g_s(\psi_s^{-1} Z_s)\) for any $g_s \in L^1(D_s)$. The operators $T_s(t)$ form a strongly continuous semigroup on $L^1(D_s)$, with generator given by $-V_s \cdot \nabla X_s$ and specular reflection at the boundary $\partial D_s$. We have

\[
f^{(s)}_N(t) = T_s(t) f^{(s)}_N(0) + (N - s) \varepsilon^{d-1} \int_0^t T_s(t - t_1) C_{s+1} f^{(s+1)}_N(t_1) dt_1
\]

(83)
Now by iterating this formula we can write the marginal $f_{N}^{(s)}(t)$ as a finite sum of terms, each of which depends only on the initial data:

$$f_{N}^{(s)}(t) = \sum_{k=0}^{N-s} a_{N,k,s} \times \int_{0}^{t} \int_{0}^{t_{1}} \cdots \int_{0}^{t_{k-1}} T_s(t-t_1)C_{s+1} \cdots T_{s+k}(t_k)f_{N}^{(s+k)}(0)dt_k \cdots dt_1$$  \tag{84}

where

$$a_{N,k,s} = \frac{(N - s)!}{(N - s - k)!} \epsilon^{k(d-1)} \tag{85}$$

Since we enforce the Boltzmann-Grad scaling $N_\epsilon^{d-1} = \epsilon^{-1}$, we have $0 \leq a_{N,k,s} \leq \ell^{-k}$ and $a_{N,k,s} \ell^k \to 1$ as $N \to \infty$ with $k, s$ fixed.

The Duhamel series (83) may be interpreted as a way of describing the solution $F_N(t)$ in terms of the data $F_N(0)$ propagated along a family of artificial trajectories, or “pseudo-trajectories.” [11][22][27] Given $Z_s \in \mathcal{D}_s$, along with times $0 \leq t_k \leq \cdots \leq t_1 \leq t$, velocities $v_{s+1}, \ldots, v_{s+k}$, impact parameters $\omega_1, \ldots, \omega_k$, and indices $i_1, \ldots, i_k$, we will define

$$Z_{s,s+k}[Z_s; t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k] \in \mathcal{D}_{s+k} \tag{86}$$

We assume $i_1 \in \{1, \ldots, s\}, i_2 \in \{1, \ldots, s, s + 1\}, \ldots, i_j \in \{1, 2, \ldots, s + j - 1\}$; we will also need to assume that certain “exclusion conditions” are satisfied, as will become clear. To begin the induction, for $Z_s \in \mathcal{D}_s$ and $t > 0$ we define

$$Z_{s,s}[Z_s, t] = \psi_s^{-t}Z_s \tag{87}$$

More generally, if the symbol

$$Z_{s,s+k}[Z_s; t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k] \in \mathcal{D}_{s+k} \tag{88}$$

is defined, then for $\tau > 0$ we define

$$Z_{s,s+k}[Z_s, t + \tau; t_1 + \tau, \ldots, t_k + \tau; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k] = \psi_{s+k}^{-\tau}Z_{s,s+k}[Z_s; t, t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k] \tag{89}$$

Similarly, if the symbol

$$Z_{s,s+k}[Z_s; t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k] = (X_{s+k}', V_{s+k}') \in \mathcal{D}_{s+k} \tag{90}$$

is defined (including the possibility $k = 0$) then for any given velocity $v_{s+k+1} \in \mathbb{R}^d$, any index $i_{k+1} \in \{1, \ldots, s, s+1, \ldots, s+k\}$, and any “suitable” choice of impact parameter $\omega_{k+1} \in S^{d-1}$, if $\omega_{k+1} \cdot (v_{s+k+1} - v_{i_{k+1}}') \leq 0$
then we define
\[ Z_{s,s+k+1}(Z_s, t; t_1, \ldots, t_k, 0; v_{s+1}, \ldots, v_{s+k}, v_{s+k+1}; \]
\[ \omega_1, \ldots, \omega_k, \omega_{k+1}; i_1, \ldots, i_k, i_{k+1} = \]
\[ = (x'_1, v'_1, \ldots, x'_{i_k+1}, v'_{i_k+1}, \ldots, x'_s, v'_s, x'_{i_k+1} + \varepsilon \omega_{k+1}, v_{s+k+1}) \]

whereas if \( \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_k+1}) > 0 \) then we define
\[ Z_{s,s+k+1}(Z_s, t; t_1, \ldots, t_k, 0; v_{s+1}, \ldots, v_{s+k}, v_{s+k+1}; \]
\[ \omega_1, \ldots, \omega_k, \omega_{k+1}; i_1, \ldots, i_k, i_{k+1} = \]
\[ = (x'_1, v'_1, \ldots, x'_{i_k+1}, v'_{i_k+1} + \omega_{k+1} \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_k+1}), \]
\[ \ldots, x'_s, v'_s, x'_{i_k+1} + \varepsilon \omega_{k+1}, v_{s+k+1} - \omega_{k+1} \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_k+1})) \] (91)

Here a “suitable” impact parameter \( \omega \) is one for which \( |x'_{i_k+1} + \varepsilon \omega - x'_j| > \varepsilon \) for each \( j \in \{1, \ldots, s, s+1, \ldots, s+k\} \setminus \{i_{k+1}\} \); note that the set of suitable impact parameters may be empty.

**Remark.** The physical interpretation of the above construction is that \( s \) particles begin in configuration \( Z_s \in D_s \) at time \( t \), then evolve under the backwards hard sphere flow for a time \( t - t_1 \); at time \( t_1 \), the \( (s+1) \)st particle is created adjacent to the \( i_1 \)st particle with velocity \( v_{s+1} \). If the pair \((i_1, s+1)\) is in a post-collisional configuration, then we perform an instantaneous collision to place the particles in a pre-collisional configuration. To continue the flow, we push the system through the backwards flow of \( (s+1) \) hard spheres for a time \( t_1 - t_2 \), and so forth. The state of the process at time \( 0 \) is then \( Z_{s,s+k}(Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k) \).

**Remark.** As a matter of convenience, we have enforced a convention whereby particles are always in a pre-collisional configuration at the moment that a new particle is created. Keep in mind, however, that the backwards flow can subsequently place particles into a post-collisional configuration, though this can only happen between particle creations.

We will also require an iterated collision kernel
\[ b_{s,s+k}(Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k) \] (93)
in order to account for each added particle. First we define
\[ b_{s,s}(Z_s, t) = 1_{Z_s \in D_s} \] (94)
If we have defined
\[ b_{s,s+k}(Z_s, t; t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k) \] (95)
then there are two cases: (i) \( Z_{s,s+k} \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_j=1^k \right] = (X'_{s+k}, V'_{s+k}) \in D_{s+k} \) is well-defined as above, in which case

\[
Z_{s,s+k} \left[ Z_s, t + \tau; t_1 + \tau, \ldots, t_k + \tau; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k \right] = b_{s,s+k} \left[ Z_s, t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k \right]
\]

\[b_{s,s+k+1} \left[ Z_s, t_1, \ldots, t_k, 0; v_{s+1}, \ldots, v_{s+k}, v_{s+k+1}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k, i_{k+1} \right] = \omega_{k+1} \left( v_{s+k+1} - v'_{i_{k+1}} \right) \times \prod_{j \in \{1, \ldots, s, s+1, \ldots, s+k\} \setminus \{i_{k+1}\}} 1_{[x'_{i_{k+1}} + \varepsilon \omega_{k+1} - x'_j > \varepsilon]} \times b_{s,s+k} \left[ Z_s, t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k \right]
\] (97)

(ii) otherwise,

\[
b_{s,s+k} \left[ Z_s, t + \tau; t_1 + \tau, \ldots, t_k + \tau; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k \right] = b_{s,s+k} \left[ Z_s, t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k \right] = 0
\] (98)

\[
b_{s,s+k+1} \left[ Z_s, t_1, \ldots, t_k, 0; v_{s+1}, \ldots, v_{s+k}, v_{s+k+1}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k, i_{k+1} \right] = 0
\] (99)

Then the Duhamel series \((83)\) becomes

\[
f_N^{(s)}(t, Z_s) = \sum_{k=0}^{N-s} a_{N,k,s} \times \sum_{i_1=1}^{s} \cdots \sum_{i_k=1}^{s+k-1} \int_0^t \cdots \int_0^{t_{k-1}} \int_{B^d} \left( \prod_{m=1}^k d\omega_m dv_{s+m} dt_m \right) \times \left( b_{s,s+k} \left[ \cdot \right] \right) f_N^{(s+k)}(0, Z_s, s+k \left[ \cdot \right]) \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_j=1^k \right]
\] (100)

Remark. The collision kernel \( b_{s,s+k} \left[ \cdot \right] \) vanishes automatically whenever \( Z_{s,s+k} \left[ \cdot \right] \) fails to be well-defined. This convention is convenient because it allows us to specify a fixed \( N \)-independent domain of integration in \((100)\).

8. Stability of pseudo-trajectories

The purpose of this section is to prove that typical pseudo-trajectories are stable with respect to the creation of a new particle, at least outside a small set of creation times, velocities, and impact parameters. The main novelty of this stability result, compared to previous results \((11)\), is that we are able to allow particles to pass arbitrarily close to each other in space under the backwards flow, as long as they do not collide. The price we pay for this improvement is that we must make explicit use of the time integrals
appearing in the Duhamel series \[^{100}\] \[^{100}\], and employ an unusual cut-off for nearby velocities. This proof is inspired in part by the ideas from \[^{27}\] \[^{27}\]; note, however, that there the authors required more sophisticated cut-offs to deal with rather general physical interactions.

We will require the following elementary geometrical fact (the proof is trivial):

**Lemma 8.1.** Fix $v \in \mathbb{R}^d \setminus \{0\}$, and for $\omega \in S^{d-1} \subset \mathbb{R}^d$ (where $S^{d-1}$ is the unit sphere centered on the origin) define

$$u_\omega = |v|^{-1} (2\omega \cdot v - v)$$

then $u_\omega \in S^{d-1}$ for each $\omega \in S^{d-1}$. If $S^d_v = \{ \omega \in S^{d-1} | \omega \cdot v > 0 \}$ then the map $\omega \mapsto u_\omega$ restricts to a diffeomorphism $S^d_v \rightarrow S^{d-1} \setminus \{-|v|^{-1}v\}$.

We will also need:

**Lemma 8.2.** Let $L \subset \mathbb{R}^d$ ($d \geq 2$) be a line, and for $\rho > 0$ consider the solid cylinder

$$C_\rho = \left\{ u \in \mathbb{R}^d | \text{dist} (u, L) \leq \rho \right\}$$

Then

$$\int_{S^{d-1}} 1_{\omega \in C_\rho} d\omega \leq C_d \rho^{(d-1)/2}$$

where the constant $C_d$ does not depend on the choice of line $L$.

**Proof.** There are two cases: either $C_\rho$ contains a point which is within distance $1 - 3\rho$ of the sphere’s center, or it does not. In the first case, we clearly have

$$\int_{S^{d-1}} 1_{\omega \in C_\rho} d\omega \leq C_d \rho^{d-3}$$

In the second case, we can estimate the size of a spherical cap to obtain that

$$\int_{S^{d-1}} 1_{\omega \in C_\rho} d\omega \leq C_d \rho^{(d-1)/2}$$

Since $d \geq 2$, we can take the maximum of these two bounds to obtain \[^{103}\] \[^{103}\]. □ □

We now turn to the main result for this section. To state the proposition, we must fix a parameter $\eta > 0$ and introduce the following sets:

$$K_s = \{ Z_s = (X_s, V_s) \in D_s | \psi_s^{-1} Z_s = (X_s - V_s \tau, V_s) \forall \tau > 0 \}$$

$$U^\eta_s = \left\{ Z_s = (X_s, V_s) \in D_s | \inf_{1 \leq i < j \leq s} |v_i - v_j| > \eta \right\}$$

**Remark.** The condition $Z_s \in U^\eta_s$ is meant to force particles to disperse away from each other under the action of the free flow.
Proposition 8.3. There is a constant $c_d > 0$, depending only on the dimension $d$, such that all the following holds: Assume that

$$Z_{s,s+k}[Z_s,t; t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k] = (X'_{s+k}, V'_{s+k}) \in K_{s+k} \cap \mathcal{U}^0_{s+k}$$

and $E_{s+k}(Z'_{s+k}) \leq 2R^2$; then,

(i) for all $\tau \geq 0$ we have

$$Z_{s,s+k}[Z_s,t + \tau; t_1 + \tau, \ldots, t_k + \tau; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k] \in K_{s+k} \cap \mathcal{U}^0_{s+k}$$

(ii) for any $i_{k+1} \in \{1, \ldots, s, s + 1, \ldots, s + k\}$, and for any $\theta, \alpha, y > 0$ such that $\sin \theta > c_d y^{-1} \varepsilon$, there exists a measurable set $\mathcal{B} \subset [0, \infty) \times \mathbb{R}^d \times S^{d-1}$, which may depend on $Z_s, t$, and $\{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k$, such that

$$\forall \ \eta < R, \ \forall \ T > 0,$$

$$\int_0^T \int_{B^d_{2R}} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}} \ d\omega_{k+1} dv_{s+k+1} d\tau \leq C_d (s + k) TR^d \left[ \alpha + \frac{y}{\eta T} + C_{d, \alpha} \left( \frac{\eta}{R} \right)^{d-1} + C_{d, \alpha} \theta^{(d-1)/2} \right]$$

and

$$Z_{s,s+k+1}[Z_s, t + \tau; t_1 + \tau, \ldots, t_k + \tau, 0; v_{s+1}, \ldots, v_{s+k}, v_{s+k+1}; \omega_1, \ldots, \omega_k, \omega_{k+1}; i_1, \ldots, i_k, i_{k+1}] \in K_{s+k+1} \cap \mathcal{U}^0_{s+k+1}$$

whenever $(\tau, v_{s+k+1}, \omega_{k+1}) \in [0, \infty) \times \mathbb{R}^d \times S^{d-1} \setminus \mathcal{B}$.

Remark. The important conclusion from (108) is that $\mathcal{B}$ is a set of small measure; on the complement of this small-measure set, the inductive hypothesis (“we are in $K_s \cap \mathcal{U}^0_s$”) is propagated due to (109). To see why $\mathcal{B}$ is of small measure, assume that $R$ is a large velocity cut-off, either constant or diverging very gently as $\varepsilon \to 0^+$. The parameter $\eta > 0$ represents the minimal velocity between particles and therefore we will always have $\eta \ll R$. Similarly $y$ is a minimal distance between particles at any time of particle creation; since particles are moving relatively with speed at least $\eta$, we will eventually require $y \ll \eta T$ so that particles are only nearby for a short time. The angle $\alpha$ is a technical cutoff on near-grazing collisions and is therefore small. The small angle $\theta$ is the opening angle of a cone inside of which particles may “recollide” (recollisions are the geometric mechanism by which correlations are generated). The purely geometric condition $\sin \theta > c_d y^{-1} \varepsilon$ forces particles to be widely separated (compared to their diameter) at the time of particle creation.
Proof. Claim (i) is trivial. For claim (ii), we distinguish between two possibilities for the added particle: either \((\tau, v_{s+k+1}, \omega_{k+1})\) is such that 
\[ \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) > 0, \]
or else \(\omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) \leq 0\). We introduce two sets,
\[
A^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \subset [0, \infty) \times \mathbb{R}^d \times S^{d-1} \text{ such that } \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) > 0 \right\}
\]
\[
A^- = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \subset [0, \infty) \times \mathbb{R}^d \times S^{d-1} \text{ such that } \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) \leq 0 \right\}
\]
then we write \(B = B^+ \cup B^-\) where \(B^+ \subset A^+\) and \(B^- \subset A^-\).

**Construction of \(B^-\).** We first eliminate creation times \(\tau\) which could result in spatial concentrations of particles. This is where we use the property that \(Z'_{s+k} \in U^{\eta}_{s+k}\), since this condition guarantees that two particles can only be close to each other for a short time (as long as the \((s+k)\) particles evolve under the free flow). We introduce the set
\[
B^-_I = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in A^- \text{ such that } \inf_{i \in \{1, \ldots, s-s_{s+k+1}, \ldots, s+k\}\setminus\{i_{k+1}\}} \left| (x'_{i_{k+1}} - x'_{i}) - \tau (v'_{i_{k+1}} - v'_{i}) \right| \leq y \right\}
\]
then we have
\[
\int_0^T \int_{B^d_{2R}} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in B^-_I} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_d (s+k-1) R^d \eta^{-1} y
\]
As a technical matter, we must also guarantee that the \((s+k+1)\)-particle state lives in \(U^{\eta}_{s+k+1}\) at the time of particle creation. Hence, we will define
\[
B^-_{II} = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in A^- \text{ such that } \inf_{1 \leq i \leq s+k} |v_{s+k+1} - v'_{i}| \leq \eta \right\}
\]
then we have
\[
\int_0^T \int_{B^d_{2R}} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in B^-_{II}} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_d (s+k) T \eta^d
\]
Lastly, we will guarantee (with high probability) that the created particle does not “recollide” under the backwards flow; that is, the \((s+k+1)\)-particle state must live in \(K_{s+k+1}\) at the time of particle creation. To this end, for
at the time of particle creation. Note that the (s) at a distance of $\epsilon$, the flow, as long as $(\text{is a "cone condition" whose complementary event prevents the newly created particle at the time of the particle creation. On the other hand, $(\text{is the relative velocity between the $(s, v_i$, whenever $(\text{is just the relative displacement between the $i$ and we let $B_{III,i}^-$ = \bigcup_{i \in \{1, \ldots, s+k\} \setminus \{i_{k+1}\}} B_{III,i}$; then we have

\[ \int_0^T \int_{B_{2R}^d} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, v_{s+k+1}) \in B_{III}^-} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_d (s + k - 1) T R^d \theta^{d-1} \]

(117)

Remark. The vector

\[ \left( x'_{i_{k+1}} - x'_i \right) - \tau \left( v'_{i_{k+1}} - v'_i \right) \]

is just the relative displacement between the $i_{k+1}$st particle and the $i$th particle at the time of the particle creation. On the other hand, $(v_{s+k+1} - v'_i)$ is the relative velocity between the $(s + k + 1)$st particle and the $i$th particle at the time of particle creation. Note that the $(s + k + 1)$st particle is created at a distance of $\epsilon$ from the $i_{k+1}$st particle. Hence the formula defining $B_{III,i}$ is a “cone condition” whose complementary event prevents the newly created $(s + k + 1)$st particle from colliding with the $i$th particle under the backwards hard sphere flow, as long as $\theta$ is not too small.

To conclude, we let $B^- = B_+ \cup B_{II}^- \cup B_{III}^-; then we have

\[ \int_0^T \int_{B_{2R}^d} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, v_{s+k+1}) \in B^-} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_d (s + k) T R^d \left[ \frac{y}{\eta} + \left( \frac{\eta}{R} \right)^d + \theta^{d-1} \right] \]

(118)

Then again, by assumption, $\sin \theta > c_d y^{-1} \epsilon$; by choosing $c_d$ sufficiently large we may guarantee that

\[ Z_{s, s+k+1} |Z_s; t + \tau; t_{1} + \tau; \ldots, t_k + \tau, 0; v_{s+1}, \ldots, v_{s+k}, v_{s+k+1}; \]

\[ \omega_1, \ldots, \omega_{k+1}; i_1, \ldots, i_k, i_{k+1} \]

\[ \in K_{s+k+1} \cap \mathcal{U}_{s+k+1}^{\eta} \]

(119)

whenever $(\tau, v_{s+k+1}, v_{s+k+1}) \in A^- \setminus B^-.$

Construction of $B^+$. The construction of $B^+$ is very similar to the construction of $B^-$; the main difference is that we have to account for the change of variables arising from one collision. We will find it helpful to define the following notation:

\[ v_{s+k+1}^* = v_{s+k+1} - \omega_{k+1} \omega_{k+1} \cdot \left( v_{s+k+1} - v'_{i_{k+1}} \right) \]

(120)
\begin{equation}
    v_{ik+1}^{s} = v_{ik+1}^t + \omega_{k+1} \omega_{k+1} \cdot (v_{s+k+1}^t - v_{ik+1}^t)
\end{equation}

Note that \( Z'_{s+k} \) is fixed as in the statement of the proposition, whereas \((\tau, v_{s+k+1}, \omega_{k+1}) \in A^+\) are considered free parameters. We eliminate creation times \(\tau\) for which particles are too concentrated in space:

\begin{equation}
    \mathcal{B}_I^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in A^+ \text{ such that } \inf_{i \in \{1, \ldots, s, s+1, \ldots, s+k\} \setminus \{i_{k+1}\}} \left| x_{ik+1}^t - x_i^t \right| - \tau \left| v_{ik+1}^t - v_i^t \right| \leq y \right\}
\end{equation}

then we have

\begin{equation}
    \int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_I^+} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_d (s + k - 1) R^d \eta^{-1} y
\end{equation}

We find it convenient to eliminate collisions which are too close to grazing; therefore, we define

\begin{equation}
    \mathcal{B}_{II}^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in A^+ \text{ such that } \left| \omega_{k+1} \cdot (v_{s+k+1}^t - v_{ik+1}^t) \right| \leq (\sin \alpha) \left| v_{s+k+1}^t - v_{ik+1}^t \right| \right\}
\end{equation}

then we have

\begin{equation}
    \int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_{II}^+} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_d T R^d \alpha
\end{equation}

We introduce the next three sets to guarantee that the \((s+k+1)\)-particle states lives in \( U_{s+k+1}^d \). In this instance we must impose \textit{multiple} conditions, since both the \((s+k+1)\)st particle and the \( i_{k+1} \)th particle are modified by the collision. Note that \( v_{s+k+1}^t - v_{ik+1}^t = \left| v_{s+k+1}^t - v_{ik+1}^t \right| \).

\begin{equation}
    \mathcal{B}_{III}^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in A^+ \setminus \mathcal{B}_I^+ \text{ such that } \inf_{i \in \{1, \ldots, s, s+1, \ldots, s+k\} \setminus \{i_{k+1}\}} \left| v_{s+k+1}^t - v_i^t \right| \leq \eta \right\}
\end{equation}

\begin{equation}
    \mathcal{B}_{IV}^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in A^+ \setminus \mathcal{B}_{II}^+ \text{ such that } \inf_{i \in \{1, \ldots, s, s+1, \ldots, s+k\} \setminus \{i_{k+1}\}} \left| v_{ik+1}^t - v_i^t \right| \leq \eta \right\}
\end{equation}

\begin{equation}
    \mathcal{B}_V^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in A^+ \left| v_{s+k+1}^t - v_{ik+1}^t \right| \leq \eta \right\}
\end{equation}

Then using Lemma 8.1 and the definition of \( \mathcal{B}_{II}^+ \), we obtain:

\begin{equation}
    \int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_{II}^+} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_d \alpha (s + k - 1) T R^d \eta^{d-1}
\end{equation}
Then using Lemmas 8.1 and 8.2, and the definition of $B_i$, we define

We will now show that, with high probability, the particle creation yields an $(s+k+1)$-particle state in $\mathcal{K}_{s+k+1}$, hence the backwards hard sphere flow coincides with the free flow. For $i \in \{1, \ldots, s, s+1, \ldots, s+k\} \setminus \{i_{k+1}\}$, we define

\[
B_{V,I,i}^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in A^+ \setminus B_{II}^+ \text{ such that } \begin{align*}
\left| \frac{\left((x'_{i+1}-x'_i)-\tau(v'_{i+1}-v'_i)\right) \cdot (v^*_{s+k+1}-v'_i)}{\left|x'_{i+1}-x'_i\right|-\tau\left(v'_{i+1}-v'_i\right)|v^*_{s+k+1}-v'_i|} \right| \geq \cos \theta
\end{align*} \right\}
\]

\[
B_{V,II,i}^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in A^+ \setminus B_{II}^+ \text{ such that } \begin{align*}
\left| \frac{\left((x'_{i+1}-x'_i)-\tau(v'_{i+1}-v'_i)\right) \cdot (v^*_{s+k+1}-v'_i)}{\left|x'_{i+1}-x'_i\right|-\tau\left(v'_{i+1}-v'_i\right)|v^*_{i+1}-v'_i|} \right| \geq \cos \theta
\end{align*} \right\}
\]

Then using Lemmas 8.1 and 8.2 and the definition of $B_{II}^+$, we have

\[
\int_0^T \int_{B_{2R}^d} \int_{S_{d-1}} 1(\tau, v_{s+k+1}, \omega_{k+1}) \in B_{V,I}^+ d\omega_{k+1} dv_{s+k+1} d\tau \leq \leq C_{d,\alpha} (s+k-1) T R^d \theta^{(d-1)/2}
\]

\[
\int_0^T \int_{B_{2R}^d} \int_{S_{d-1}} 1(\tau, v_{s+k+1}, \omega_{k+1}) \in B_{V,II}^+ d\omega_{k+1} dv_{s+k+1} d\tau \leq \leq C_{d,\alpha} (s+k-1) T R^d \theta^{(d-1)/2}
\]
To conclude, we let $B^+ = B_1^+ \cup B_II^+ \cup B_{II}^+ \cup B_{IV}^+ \cup B_{III}^+ \cup B_{V}^+ \cup B_{V}^+ \cup B_{V}^+$; then we have
\[
\int_0^T \int_{B_{2R}} \int_{\mathbb{S}^d_{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in B^+} \, d\omega_{k+1} \, dv_{s+k+1} \, d\tau \leq C_d (s + k) T R^d \left[ \alpha + \frac{y}{\eta T} + C_{d, \alpha} \left( \frac{\eta}{R} \right)^{d-1} + C_{d, \alpha} \theta^{(d-1)/2} \right]
\]

Then again, by assumption, we have $\sin \theta > c_d y^{-1}\varepsilon$; as long as $c_d$ is chosen sufficiently large, we always have
\[
Z_{s,s+k+1}[Z_s, t + \tau; t_1 + \tau, \ldots, t_k + \tau, 0; v_{s+1}, \ldots, v_{s+k}, v_{s+k+1};
\omega_1, \ldots, \omega_{k+1}; t_1, \ldots, t_k, i_k + 1] \in K_{s+k+1} \cap U_{s+k+1}^0
\]
whenever $(\tau, v_{s+k+1}, \omega_{k+1}) \in A^+ \setminus B^+$. \qed

9. THE BOLTZMANN HIERARCHY

We will say that a sequence of continuous symmetric functions $\{f_{s}^{(s)}(t, Z_s)\}_{s \in \mathbb{N}^*}$ with $Z_s \in \mathbb{R}^{2ds}$, satisfies the Boltzmann hierarchy if the following equation holds for each $s$ in the sense of distributions:
\[
\left( \frac{\partial}{\partial t} + V_s \cdot \nabla X_s \right) f_{\infty}^{(s)}(t, Z_s) = \ell^{-1} C_{s+1,0} f_{\infty}^{(s+1)}(t, Z_s) \quad (140)
\]
The collision operators $C_{s,s+1}^0$ are defined as follows:
\[
C_{s+1}^0 = \sum_{i=1}^s C_{i,s+1}^0 \quad (141)
\]
\[
C_{i,s+1}^0 = C_{i,s+1}^{0,+} - C_{i,s+1}^{0,-} \quad (142)
\]
\[
C_{i,s+1}^{0,+} f_{\infty}^{(s+1)}(t, Z_s) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^d_{d-1}} [\omega \cdot (v_{s+1} - v_i)]_+ \times
\]
\[
\times f_{\infty}^{(s+1)}(t, x_1, v_{1}, \ldots, x_i, v_i^*, \ldots, x_s, v_s, x_i, v_{s+1}) \, d\omega \, dv_{s+1} \quad (143)
\]
\[
C_{i,s+1}^{0,-} f_{\infty}^{(s+1)}(t, Z_s) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^d_{d-1}} [\omega \cdot (v_{s+1} - v_i)]_- \times
\]
\[
\times f_{\infty}^{(s+1)}(t, x_1, v_{1}, \ldots, x_i, v_i, v_{s+1}, \ldots, x_s, v_s, x_i, v_{s+1}) \, d\omega \, dv_{s+1} \quad (144)
\]
where
\[
\begin{align*}
v_i^* &= v_i + \omega \cdot (v_j - v_i) \\
v_j^* &= v_j - \omega \cdot (v_j - v_i)
\end{align*} \quad (145)
\]
We also define the free transport operators $T_s^0(t)$, which act on functions $f_{\infty}^{(s)} : \mathbb{R}^{2ds} \rightarrow \mathbb{R}$ as follows:
\[
\left( T_s^0(t) f_{\infty}^{(s)} \right)(X_s, V_s) = f_{\infty}^{(s)}(X_s - V_s t, V_s) \quad (146)
\]
Just as for the BBGKY hierarchy, the Boltzmann hierarchy admits a formal Duhamel series expressing the solution in terms of the data,

\[
f^{(s)}_\infty(t) = \sum_{k=0}^{\infty} \ell^{-k} \times \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} T^0_s(t - t_1)C^0_{s+1} \cdots T^0_{s+k}(t_k) f^{(s+k)}_\infty(0) dt_k \cdots dt_1
\]

The convergence of this series (for small data) follows from the well-posedness theorem which is proven in the following section.

**Remark.** If \( f_t(x, v) \) is a sufficiently smooth solution of the Boltzmann equation then the sequence \( \{ f_t^{\otimes s} \}_{s \in \mathbb{N}} \) is a solution of the Boltzmann hierarchy.

We will now construct pseudo-trajectories for the Boltzmann hierarchy, directly analogous to those we have constructed for the BBGKY hierarchy. Given \( Z_s \in \mathbb{R}^{2ds} \), along with times \( 0 \leq t_k \leq \cdots \leq t_1 \leq t \), velocities \( v_{s+1}, \ldots, v_{s+k} \), impact parameters \( \omega_1, \ldots, \omega_k \), and indices \( i_1, \ldots, i_k \), we will define

\[
Z^0_{s,s+k}[Z_s, t; t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k] = (X'_{s+k}, V'_{s+k}) \in \mathbb{R}^{2d(s+k)}
\]

We assume \( i_1 \in \{1, \ldots, s\}, i_2 \in \{1, \ldots, s, s+1\}, \ldots, i_j \in \{1, 2, \ldots, s+j-1\} \).

To begin the induction, for \( Z_s = (X_s, V_s) \in \mathbb{R}^{2ds} \) and \( t > 0 \) we define

\[
Z^0_{s,s}[Z_s, t] = (X_s - V_s t, V_s)
\]

More generally, if the symbol

\[
Z^0_{s,s+k}[Z_s, t; t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k] = (X'_{s+k}, V'_{s+k}) \in \mathbb{R}^{2d(s+k)}
\]

is defined, then for \( \tau > 0 \) we define

\[
Z^0_{s,s+k}[Z_s, t + \tau; t_1 + \tau, \ldots, t_k + \tau; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k] = (X'_{s+k} - V'_{s+k} \tau, V'_{s+k})
\]

Similarly, if the symbol

\[
Z^0_{s,s+k}[Z_s, t; t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k] = (X'_{s+k}, V'_{s+k}) \in \mathbb{R}^{2d(s+k)}
\]

is defined (including the possibility \( k = 0 \)) then for any given velocity \( v_{s+k+1} \in \mathbb{R}^d \), any index \( i_{k+1} \in \{1, \ldots, s, s+1, \ldots, s+k\} \), and any choice of
impact parameter $\omega_{k+1} \in \mathbb{S}^{d-1}$, if $\omega_{k+1} \cdot (v_{s+k+1} - v'_{ik+1}) \leq 0$ we define

$$Z_0^{0, s+k+1} [Z_s; t; t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; v_{s+k+1};$$

$$\omega_1, \ldots, \omega_k, \omega_{k+1}; i_1, \ldots, i_k, i_{k+1}] = (x'_1, v'_1, \ldots, x'_{ik+2}, v'_{ik+1}, \ldots, x'_s, v'_s, x'_{ik+1}, v_{s+k+1})$$

(153)

whereas if $\omega_{k+1} \cdot (v_{s+k+1} - v'_{ik+1}) > 0$ then we define

$$Z_0^{0, s+k+1} [Z_s; t; t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; v_{s+k+1};$$

$$\omega_1, \ldots, \omega_k, \omega_{k+1}; i_1, \ldots, i_k, i_{k+1}] = (x'_1, v'_1, \ldots, x'_{ik+2}, v'_{ik+1} + \omega_{k+1} \omega_{k+1} \cdot (v_{s+k+1} - v'_{ik+1}),$$

$$\ldots, x'_s, v'_s, x'_{ik+1}, v_{s+k+1} - \omega_{k+1} \omega_{k+1} \cdot (v_{s+k+1} - v'_{ik+1}))$$

(154)

Now we construct the collision kernel $b_0^{0, s+k} [Z_s; t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k]$. First we define

$$b_0^{0, s} [Z_s; t] = 1$$

(155)

If we have defined

$$b_0^{0, s+k} [Z_s; t; t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k]$$

(156)

then for any $\tau > 0$ we define

$$b_0^{0, s+k} [Z_s; t + \tau; t_1 + \tau, \ldots, t_k + \tau; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k] =$$

$$b_0^{0, s+k} [Z_s; t; t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k]$$

(157)

and we also define

$$b_0^{0, s+k+1} [Z_s; t; t_1, \ldots, t_k, 0; v_{s+1}, \ldots, v_{s+k}; v_{s+k+1};$$

$$\omega_1, \ldots, \omega_k, \omega_{k+1}; i_1, \ldots, i_k, i_{k+1}] =$$

$$\omega_{k+1} \cdot (v_{s+k+1} - v'_{ik+1}) \times$$

$$b_0^{0, s+s+k+1} [Z_s; t; t_1, \ldots, t_k, v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k]$$

(158)

Then the formal Duhamel series (147) becomes

$$f_s^{(s)}(t, Z_s) = \sum_{k=0}^{\infty} e^{-k \times}$$

$$\times \sum_{i_1=1}^{s} \ldots \sum_{i_k=1}^{s+k-1} \int_0^t \ldots \int_0^{t_{k-1}} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}}^{k} \frac{\prod_{m=1}^{k} d\omega_m d v_{s+m} d t_m}{k} \times$$

$$\times \left( b_0^{0, s+k+1} [Z_s; t; t_1, \ldots, t_k, v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k] \right)$$

(159)
10. SMALL SOLUTIONS OF THE BOLTZMANN HIERARCHY

We will prove a global well-posedness result for the Boltzmann hierarchy with small data \( F_\infty(0) = \{ f^{(s)}_\infty(0) \}_{s \in \mathbb{N}} \) in vacuum. The proof is based on a fixed point iteration and a dispersive estimate. \[\text{[2, 17]}\] If, in addition to the hypotheses of the theorem, we have \( f^{(s)}_\infty(0) = f^{(s)}_0 \) for some smooth function \( f_0(x, v) \), then it is well-known that the Boltzmann equation has a unique non-negative smooth solution \( f_t \) \[\text{[6, 8]}\], and \( \{ f^{(s)}_t \}_{s \in \mathbb{N}} \) solves the Boltzmann hierarchy. Then the uniqueness part of the following theorem implies that \( F_\infty(t) = \{ f^{(s)}_t \}_{s \in \mathbb{N}} \), i.e., the Boltzmann hierarchy propagates chaoticity.

**Theorem 10.1.** (Illner & Pulvirenti 1986) Suppose \( F_\infty(0) = \{ f^{(s)}_\infty(0) \}_{s \in \mathbb{N}} \) is a sequence of functions such that each \( f^{(s)}_\infty(0) : \mathbb{R}^{2d} \rightarrow \mathbb{R} \) is continuous and symmetric, and for some \( \beta_0 > 0, \mu_0 \in \mathbb{R} \),

\[
\sup_{s \in \mathbb{N}} \sup_{Z_s \in \mathbb{R}^{2d}} \left| f^{(s)}_\infty(0, Z_s) e^{\beta_0 [E_s(Z_s) + I_s(Z_s)]} e^{\mu_0 s} \right| \leq 1
\]

Then if \( d \geq 3 \) and \( \ell^{-1} e^{-\mu_0} \beta_0^{-\frac{d+1}{2}} \) is sufficiently small (depending only on \( d \)), then there exists a unique sequence \( F_\infty(t) = \{ f^{(s)}_\infty(t) \}_{s \in \mathbb{N}} \), with each \( f^{(s)}_\infty(t, Z_s) : [0, \infty) \times \mathbb{R}^{2d} \rightarrow \mathbb{R} \) continuous and symmetric, such that

\[
\sup_{t \geq 0} \sup_{s \in \mathbb{N}} \sup_{Z_s \in \mathbb{R}^{2d}} \left| f^{(s)}_\infty(t, Z_s) e^{\frac{1}{2} \beta_0 [E_s(Z_s) + I_s((X_s - V_s, t, V_s))]} e^{(\mu_0 - 1)s} \right| \leq 2
\]

and for each \( s \in \mathbb{N} \) there holds

\[
\left( \frac{\partial}{\partial t} + V_s \cdot \nabla X_s \right) f^{(s)}_\infty(t, Z_s) = \ell^{-1} C^{0}_{s+1} f^{(s+1)}(t, Z_s)
\]

in the sense of distributions.

**Proof.** Recall the free evolution \( (T^0_s)(t) f^{(s)}_\infty(Z_s) = f^{(s)}_\infty(X_s - V_s t, V_s) \), where \( Z_s \in \mathbb{R}^{2d} \). Subject to the estimates stated in the theorem, and the continuity of \( f^{(s)}_\infty(t, Z_s) \), the weak form of the Boltzmann hierarchy is equivalent to the following mild form:

\[
f^{(s)}_\infty(t) = T^0_s(t) f^{(s)}_\infty(0) + \ell^{-1} \int_0^t T^0_s(t - \tau) C^{0}_{s+1} f^{(s+1)}(\tau) \, d\tau
\]

At this point it is convenient to change the coordinates. Let us define \( G_\infty(t) = \{ g^{(s)}_\infty(t) \}_{s \geq 1} \) by \( g^{(s)}_\infty(t) = T^0_s(-t) f^{(s)}_\infty(t) \), and write

\[
V^0_{s+1}(\tau) = T^0_s(-\tau) C^{0}_{s+1} T^0_{s+1}(\tau)
\]

Then we have

\[
g^{(s)}_\infty(t) = g^{(s)}_\infty(0) + \ell^{-1} \int_0^t V^0_{s+1}(\tau) g^{(s+1)}_\infty(\tau) \, d\tau
\]
We record an explicit formula for the action of the operator $V_{s+1}^0(\tau)$:

$$V_{s+1}^0(\tau) = V_{s+1}^{0,+}(\tau) - V_{s+1}^{0,-}(\tau)$$

(166)

$$\left( V_{s+1}^{0,+}(\tau) g_{\infty}^{(s+1)}(t) \right) (Z_s) = \sum_{i=1}^{s} \int_{\mathbb{R}^d} \int_{S^{d-1}} d\omega dv_{s+1} \left[ \omega \cdot (v_{s+1} - v_i) \right]_+ \times$$

$$\times g_{\infty}^{(s+1)}(t, x_1, v_1, \ldots, x_i - (v_i^* - v_i) \tau, v_i^*, \ldots, \ldots, x_s, v_s, x_i - (v_i^* - v_i) \tau, v_i^*)$$

(167)

$$\left( V_{s+1}^{0,-}(\tau) g_{\infty}^{(s+1)}(t) \right) (Z_s) = \sum_{i=1}^{s} \int_{\mathbb{R}^d} \int_{S^{d-1}} d\omega dv_{s+1} \left[ \omega \cdot (v_{s+1} - v_i) \right]_- \times$$

$$\times g_{\infty}^{(s+1)}(t, x_1, v_1, \ldots, x_i - (v_i^* - v_i) \tau, v_i^*, \ldots, \ldots, x_s, v_s, x_i - (v_i^* - v_i) \tau, v_i^*)$$

(168)

We will prove pointwise bounds for the operators $V_{s+1}^{0,\pm}(\tau)$. If $0 < \beta' < \beta$, $\mu' < \mu$, $t, \tau \geq 0$, then we have:

$$\left| \left( e^{\mu' s} e^{\beta'(E_s(Z_s) + I_s(Z_s))} V_{s+1}^{0,+}(\tau) g_{\infty}^{(s+1)}(t) \right) (Z_s) \right| \leq$$

$$\leq \sum_{i=1}^{s} \int_{\mathbb{R}^d} \int_{S^{d-1}} d\omega dv_{s+1} |v_{s+1} - v_i| e^{-(\beta' - \beta) E_s(Z_s)} e^{-(\mu - \mu') s} \times$$

$$\times e^{-\frac{1}{2} \beta |v_{s+1}|^2} e^{\frac{1}{2} \beta (|x_1|^2 - |x_i - (v_i^* - v_i)|^2) \tau^2 - |x_i - (v_i'^* - v_i)|^2} e^{\mu} \times$$

$$\times e^{\mu(s+1)} e^{\frac{1}{2} \beta \sum_{i=1}^{s+1} |v_i|^2} e^{\frac{1}{2} \beta (|x_1|^2 + \cdots + |x_i - (v_i^* - v_i)|^2 + \cdots + |x_{s+1} + s| - (v_i^* - v_i)|^2) \tau^2} \times$$

$$\times \left| g_{\infty}^{(s+1)}(t, x_1, v_1, \ldots, x_i - (v_i^* - v_i) \tau, v_i^*, \ldots, \ldots, x_s, v_s, x_i - (v_i^* - v_i) \tau, v_i^*) \right|$$

$$\leq \sum_{i=1}^{s} \int_{\mathbb{R}^d} \int_{S^{d-1}} d\omega dv_{s+1} |v_{s+1} - v_i| e^{-(\beta' - \beta) E_s(Z_s)} e^{-(\mu - \mu') s} \times$$

$$\times e^{-\frac{1}{2} \beta |v_{s+1}|^2} e^{\frac{1}{2} \beta (|x_1|^2 - |x_i - (v_i^* - v_i)|^2) \tau^2 - |x_i - (v_i'^* - v_i)|^2} e^{\mu} \times$$

$$\times \left| e^{\mu(s+1)} e^{\beta(E_{s+1}(Z_{s+1}) + I_{s+1}(Z_{s+1}))} g_{\infty}^{(s+1)}(t, Z_{s+1}') \right|_{L^2_{s+1}}$$
and similarly
\[
\left| (e^{\mu s} e^{\beta'(E_s(Z_s)+I_s(Z_s))} V_{s+1}^0(\tau) g_s^{(s+1)}(t) ) (Z_s) \right| \leq \\
\sum_{i=1}^s \int_{\mathbb{R}^d} \int_{\mathbb{S}^d-1} d\omega dv_{s+1} |v_{s+1} - v_i| e^{-(\beta - \beta') E_s(Z_s)} e^{-(\mu - \mu') s} \times \\
e^{-\frac{1}{2} \beta |v_{s+1}|^2} e^{-\frac{1}{2} \beta |x_i - (v_{s+1} - v_i)\tau|^2} e^{-\mu} \times \\
e^{\mu(s+1)} e^{\beta E_s(Z_s)} e^{-\mu} \times \\
\left| g_s^{(s+1)} (t, x_1, v_1, \ldots, x_i, v_i, \ldots, x_s, v_s, x_{s+1} - (v_{s+1} - v_i)\tau, v_{s+1}) \right|
\]

Therefore we obtain a bound on the full operator \( V_{s+1}^0(\tau) \),
\[
\left| (e^{\mu s} e^{\beta'(E_s(Z_s)+I_s(Z_s))} V_{s+1}^0(\tau) g_s^{(s+1)}(t) ) (Z_s) \right| \leq \\
2 \sum_{i=1}^s \int_{\mathbb{R}^d} \int_{\mathbb{S}^d-1} d\omega dv_{s+1} |v_{s+1} - v_i| e^{-(\beta - \beta') E_s(Z_s)} e^{-(\mu - \mu') s} \times \\
e^{-\frac{1}{2} \beta |v_{s+1}|^2} e^{-\frac{1}{2} \beta |x_i - (v_{s+1} - v_i)\tau|^2} e^{-\mu} \times \\
\left| e^{\mu(s+1)} e^{\beta E_s(Z_s)+I_s(Z_s')} g_s^{(s+1)} (t, Z_{s+1}) \right|_{L_{s+1}^\infty} \leq \sum_{i=1}^s \int_{\mathbb{R}^d} \int_{\mathbb{S}^d-1} d\omega dv_{s+1} |v_{s+1} - v_i| e^{-(\beta - \beta') E_s(Z_s)} e^{-(\mu - \mu') s} \times \\
e^{-\frac{1}{2} \beta |v_{s+1}|^2} e^{-\frac{1}{2} \beta |x_i - (v_{s+1} - v_i)\tau|^2} e^{-\mu} \times \\
\left| e^{\mu(s+1)} e^{\beta E_s(Z_s)+I_s(Z_s')} g_s^{(s+1)} (t, Z_{s+1}) \right|_{L_{s+1}^\infty}
\]

We use the following dispersive inequality [2]:
\[
\| \zeta(x - \nu t, v) \|_{L^\infty_x L^1_v} \leq |t|^{-d} \| \zeta(x, v) \|_{L^1_x L^\infty_v}
\]

which implies the pointwise bound
\[
\left| (e^{\mu s} e^{\beta'(E_s(Z_s)+I_s(Z_s))} V_{s+1}^0(\tau) g_s^{(s+1)}(t) ) (Z_s) \right| \leq \\
C_d e^{-\beta^2} (1 + \tau)^{-d} \left( s^{\frac{1}{2}} E_s(Z_s) \right)^{\frac{1}{2}} e^{-(\beta - \beta') E_s(Z_s)} e^{-(\mu - \mu') s} \times \\
\left| e^{\mu(s+1)} e^{\beta E_s(Z_s)+I_s(Z_s')} g_s^{(s+1)} (t, Z_{s+1}) \right|_{L_{s+1}^\infty}
\]

The following identity follows from elementary manipulation:
\[
|x_i|^2 + |x_i - (v_{s+1} - v_i)|^2 - |x_i - (v^s_{s+1} - v_i)|^2 - |x_i - (v^s_{s+1} - v_i)|^2 = 0 \quad (169)
\]
and therefore also implies

\[
\left\| \left( e^{\mu} s e^{\beta(t)}(E_s(Z_s) + I_s(Z_s)) V_{s+1}^0(\tau) g_\infty^{(s+1)}(t) \right) (Z_s) \right\|_{L_\infty^{Z_s}} \leq \\
\leq C_d e^{-\mu - \frac{d}{2}} (1 + \tau)^{-d} \left( \frac{1}{\sqrt{\beta - \beta^t} \cdot \sqrt{\mu - \mu^t} + \beta^{-\frac{1}{2}}} \right) \times \\
\times \left\| e^{\mu(s+1)} e^{\beta(t)}(E_{s+1}(Z_{s+1}')) + I_{s+1}(Z_{s+1}')) g_\infty^{(s+1)}(t, Z_{s+1}')) \right\|_{L_\infty^{Z_{s+1}+1}}
\]

(173)

Fix a sequence of positive numbers \( r_0, r_1, r_2, \ldots \) such that \( 0 < r_{k+1} < r_k \) and \( \sum_{k=0}^\infty r_k = 1 \). We define continuous decreasing functions \( \beta(t), \mu(t) \), for \( t \geq 0 \):

\[
\beta(t) = \beta_0 \cdot \left[ 1 - \frac{1}{2} \sum_{0 \leq k < n} r_k - \frac{1}{2} r_n (t - n) \right] \quad \forall \quad t \in [n, n + 1) \quad (174)
\]

\[
\mu(t) = \mu_0 - \sum_{0 \leq k < n} r_k - r_n (t - n) \quad \forall \quad t \in [n, n + 1) \quad (175)
\]

Using the pointwise bound (170), we obtain

\[
\left| \left( e^{\mu(t)} s e^{\beta(t)}(E_s(Z_s) + I_s(Z_s)) \right) \left( V_{s+1}^0(\tau) g_\infty^{(s+1)}(\tau) \right) (Z_s) d\tau \right| \leq \\\n\leq C_d e^{-(\mu_0 - 1)} \left( \frac{\beta_0}{2} \right)^{-\frac{3}{2}} \left( s^2 E_s(Z_s) \right)^{-\frac{1}{2}} + \left( \frac{\beta_0}{2} \right)^{-\frac{1}{2}} \times \\\n\times \int_0^t (1 + \tau)^{-d} e^{-(\beta(t) - \beta(t))} E_s(Z_s) e^{-(\mu(t) - \mu(t))} s d\tau \times \\\n\times \left\| e^{\mu(t') (s+1)} e^{\beta(t')} (E_{s+1}(Z_{s+1}')) + I_{s+1}(Z_{s+1}')) g_\infty^{(s+1)}(t', Z_{s+1}')) \right\|_{L_\infty^{Z_{s+1}+1}}
\]

(176)

Then by a straightforward computation we have

\[
\int_0^t (1 + \tau)^{-d} e^{-(\beta(t) - \beta(t))} E_s(Z_s) e^{-(\mu(t) - \mu(t))} s d\tau \leq \sum_{k=0}^\infty r_k^{-1} \frac{(1 + k)^{-d}}{s + \frac{\beta_0}{2} E_s(Z_s)}
\]

(177)

Observe that if \( d \geq 3 \) then we may choose \( r_k \) such that \( r_k \sim k^{-d+\frac{1}{2}} \) as \( k \to \infty \), and \( \sum_{k=0}^\infty r_k = 1 \); then, we will also have \( \sum_{k=0}^\infty r_k^{-1} (1 + k)^{-d} < \infty \).
Then we may define the operator $V$ depend on time. Then the Boltzmann hierarchy may be written as

$$\|V\| \leq C_\delta \varepsilon \to X \to X$$

Since $V$ where $N$ we may view the data

$$d$$

chaotic data, we can use the solvability of the Boltzmann equation near vacuum (see [8]), combined with the local well-posedness of the Boltzmann

Hence for $d \geq 3$ there holds

$$\left\| e^{\mu(t)} e^{\beta(t)(E_s(Z_s)+I_s(Z_s))} \right\|_{L_t^\infty L_{Z_s}^\infty} \leq$$

$$\leq C_d \varepsilon^d \beta_0^{d+1} \times$$

$$\times \left\| e^{\mu(t)(s+1)} e^{\beta(t)(E_{s+1}(Z_{s+1})+I_{s+1}(Z_{s+1}))} \right\|_{L_t^\infty L_{Z_{s+1}}^\infty} \ (178)$$

The Boltzmann hierarchy can be written in the following vector form:

$$G_\infty(t) = G_\infty(0) + \ell^{-1} \int_0^t V^0(\tau) G_\infty(\tau) \ d\tau \ (179)$$

where $V^0(\tau) G_\infty(t) = \left\{ V_{s+1}(\tau) g_{\infty}(s+1)(t) \right\}_{s \in \mathbb{N}}$. We work in the Banach space $(X, \| \cdot \|)$ of sequences $G_\infty(t) = \left\{ g_{\infty}(s)(t) \right\}_{s \in \mathbb{N}}$ with each function $g_{\infty}(s)(t) : [0, \infty) \times \mathbb{R}^{2d_s} \to \mathbb{R}$ continuous and symmetric, and with norm

$$\|G_\infty\| = \sup_{t \geq 0} \sup_{s \in \mathbb{N}} \left\| e^{\mu(t)} e^{\beta(t)(E_s(Z_s)+I_s(Z_s))} \right\|_{L_t^\infty L_{Z_s}^\infty} \ (180)$$

Then we may define the operator $\mathcal{V} : \mathcal{X} \to \mathcal{X}$,

$$(\mathcal{V} G_\infty)(t) = \ell^{-1} \int_0^t V^0(\tau) G_\infty(\tau) \ d\tau \ (181)$$

We may view the data $G_\infty(0)$ as an element of $\mathcal{X}$ which simply does not depend on time. Then the Boltzmann hierarchy may be written as

$$G_\infty = G_\infty(0) + \mathcal{V} G_\infty \ (182)$$

Since $\|\mathcal{V}\|_\text{op} \leq C_d \varepsilon^{-d \mu} \beta_0^{d+1} \to X \to X$ and as soon as $\ell^{-d \mu} \beta_0^{d+1} \to X \to X$ is sufficiently small we can invert this equation to give

$$G_\infty = (I - \mathcal{V})^{-1} G_\infty(0) = \sum_{j=0}^{\infty} \mathcal{V}^j G_\infty(0) \ (183)$$

which is the unique solution of the Boltzmann hierarchy. \hfill $\Box$ \hfill $\Box$

**Remark.** We cannot apply the above argument, as written, in the case $d = 2$; this is due to the failure of integrability at large times. However, this is a technical restriction since Theorem 6.1 gives us *a priori* bounds for the BBGKY hierarchy, independent of $N$, for all $d \geq 2$. Indeed, a slightly different argument from the one above actually implies that Theorem 10.1 holds when $d = 2$ (see [17]); note that the only difference in their proof was that while they could not show that $\sum_j \|\mathcal{V}\|_\text{op}^j < \infty$, they could at least prove that $\sum_j \|\mathcal{V}^j G_\infty(0)\| < \infty$, under the same assumptions. Alternatively, for chaotic data, we can use the solvability of the Boltzmann equation near vacuum (see [8]), combined with the local well-posedness of the Boltzmann
hierarchy; this line of reasoning would still be completely sufficient to reach the conclusions of Theorem 12.1 in the case $d=2$.

To conclude this section, we quote a couple of local-in-time well-posedness results for the Boltzmann hierarchy. The proofs are well-known and similar to the proof presented above.

**Theorem 10.2.** Suppose $F_\infty(0) = \{f^{(s)}_\infty(0)\}_{s \in \mathbb{N}}$ is a sequence of functions such that each $f^{(s)}_\infty(0) : \mathbb{R}^{2ds} \to \mathbb{R}$ is continuous and symmetric, and for some $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$,

$$\sup_{s \in \mathbb{N}} \sup_{Z_s \in \mathbb{R}^{2ds}} \left| f^{(s)}_\infty(0, Z_s) \right| e^{\beta_0 [E_s(Z_s) + I_s(Z_s)]} e^{\mu_0 s} \leq 1$$  \hspace{1cm} (184)

Then there is a constant $C_d > 0$, depending only on $d$, such that if $T_L < C_d e^{\mu_0 \beta_0^d/2}$, then there exists a unique sequence $F_\infty(t) = \{f^{(s)}_\infty(t)\}_{s \in \mathbb{N}}$, with each $f^{(s)}_\infty(t, Z_s) : [0, T_L] \times \mathbb{R}^{2ds} \to \mathbb{R}$ continuous and symmetric, such that

$$\sup_{0 \leq t \leq T_L} \sup_{s \in \mathbb{N}} \sup_{Z_s \in \mathbb{R}^{2ds}} \left| f^{(s)}_\infty(t, Z_s) \right| e^{\beta_0 [E_s(Z_s) + I_s((X_s - V_s, t))] + (\mu_0 - 1)s} \leq 2$$  \hspace{1cm} (185)

and for each $s \in \mathbb{N}$ there holds

$$\left( \frac{\partial}{\partial t} + V_s \cdot \nabla X_s \right) f^{(s)}_\infty(t, Z_s) = \ell^{-1} C^{0}_{s+1} f^{(s+1)}_\infty(t, Z_s)$$  \hspace{1cm} (186)

in the sense of distributions, for $0 \leq t \leq T_L$.

**Theorem 10.3.** Suppose $F_\infty(0) = \{f^{(s)}_\infty(0)\}_{s \in \mathbb{N}}$ is a sequence of functions such that each $f^{(s)}_\infty(0) : \mathbb{R}^{2ds} \to \mathbb{R}$ is continuous and symmetric, and for some $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$,

$$\sup_{s \in \mathbb{N}} \sup_{Z_s \in \mathbb{R}^{2ds}} \left| f^{(s)}_\infty(0, Z_s) \right| e^{\beta_0 E_s(Z_s)} e^{\mu_0 s} \leq 1$$  \hspace{1cm} (187)

Then there is a constant $C_d > 0$, depending only on $d$, such that if $T_L < C_d e^{\mu_0 \beta_0^d/2}$, then there exists a unique sequence $F_\infty(t) = \{f^{(s)}_\infty(t)\}_{s \in \mathbb{N}}$, with each $f^{(s)}_\infty(t, Z_s) : [0, T_L] \times \mathbb{R}^{2ds} \to \mathbb{R}$ continuous and symmetric, such that

$$\sup_{0 \leq t \leq T_L} \sup_{s \in \mathbb{N}} \sup_{Z_s \in \mathbb{R}^{2ds}} \left| f^{(s)}_\infty(t, Z_s) \right| e^{\beta_0 E_s(Z_s)} e^{(\mu_0 - 1)s} \leq 2$$  \hspace{1cm} (188)

and for each $s \in \mathbb{N}$ there holds

$$\left( \frac{\partial}{\partial t} + V_s \cdot \nabla X_s \right) f^{(s)}_\infty(t, Z_s) = \ell^{-1} C^{0}_{s+1} f^{(s+1)}_\infty(t, Z_s)$$  \hspace{1cm} (189)

in the sense of distributions, for $0 \leq t \leq T_L$. 


11. Construction of the initial data

We introduce the \( N \)-particle density \( f_N \)
\[
f_N(0, Z_N) = Z_N^{-1} 1_{Z_N \in \mathcal{D}_N} f_0^\otimes N(Z_N)
\]  
where \( Z_N \) is the partition function,
\[
Z_N = \int_{\mathbb{R}^{2dN}} 1_{Z_N \in \mathcal{D}_N} f_0^\otimes N(Z_N) dZ_N
\]  
We also use the notation \( Z_s \) for \( 1 \leq s \leq N \) (note carefully the implicit dependence on \( \varepsilon \)),
\[
Z_s = \int_{\mathbb{R}^{2ds}} 1_{Z_s \in \mathcal{D}_s} f_0^\otimes s(Z_s) dZ_s
\]

The proofs in this section are almost identical to those in the literature; we include them for the sake of completeness.

**Lemma 11.1.** For \( 1 \leq s < N \), and any probability density \( f_0(x, v) \) on \( \mathbb{R}^{2d} \) with \( f_0 \in L_x^\infty L_v^1 \), in the Boltzmann-Grad scaling \( N\varepsilon^{d-1} = \ell^{-1} \) there holds
\[
Z_{s+1} \geq Z_s \left( 1 - \ell^{-1} |B_1^d| \| f_0 \|_{L_x^\infty L_v^1} \varepsilon \right)
\]  
where \( B_1^d \) is the unit ball in \( \mathbb{R}^d \) and \( Z_s \) is given by (192).

**Proof.** For \( 1 \leq s < N \), we have
\[
Z_{s+1} = \int_{\mathbb{R}^{2d(s+1)}} 1_{Z_{s+1} \in \mathcal{D}_{s+1}} f_0^\otimes (s+1)(Z_{s+1}) dZ_{s+1}
\]
\[
= \int_{\mathbb{R}^{2d(s+1)}} 1_{Z_s \in \mathcal{D}_s} \left( \prod_{i=1}^{s} 1_{|x_i - x_{s+1}| > \varepsilon} \right) f_0^\otimes (s+1)(Z_{s+1}) dZ_{s+1}
\]
\[
= \int_{\mathbb{R}^{2ds}} 1_{Z_s \in \mathcal{D}_s} \left[ \int_{\mathbb{R}^{2d}} f_0(z_{s+1}) \left( \prod_{i=1}^{s} 1_{|x_i - x_{s+1}| > \varepsilon} \right) dz_{s+1} \right] f_0^\otimes s(Z_s) dZ_s
\]
We bound the quantity in brackets from below, uniformly in \( Z_s \).
\[
\int_{\mathbb{R}^{2d}} f_0(z_{s+1}) \left( \prod_{i=1}^{s} 1_{|x_i - x_{s+1}| > \varepsilon} \right) dz_{s+1}
\]
\[
\geq \int_{\mathbb{R}^{2d}} f_0(z_{s+1}) \left( 1 - \sum_{i=1}^{s} 1_{|x_i - x_{s+1}| \leq \varepsilon} \right) dz_{s+1}
\]
\[
\geq 1 - s\varepsilon^d |B_1^d| \| f_0 \|_{L_x^\infty L_v^1}
\]
\[
\geq 1 - N\varepsilon^{d-1} |B_1^d| \| f_0 \|_{L_x^\infty L_v^1} \varepsilon
\]
\[
= 1 - \ell^{-1} |B_1^d| \| f_0 \|_{L_x^\infty L_v^1} \varepsilon
\]
We have used the Boltzmann-Grad scaling \( N\varepsilon^{d-1} = \ell^{-1} \) in the last step. Finally we are able to conclude, for \( 1 \leq s < N \),
\[
Z_{s+1} \geq Z_s \left( 1 - \ell^{-1} |B_1^d| \| f_0 \|_{L_x^\infty L_v^1} \varepsilon \right)
\]  
(194)
as claimed.

Lemma 11.2. For $1 \leq s < N$, and any probability density $f_0(x,v)$ on $\mathbb{R}^{2d}$ with $f_0 \in L^\infty_x L^1_v$, in the Boltzmann-Grad scaling $N \varepsilon^{-d-1} = \ell^{-1}$ there holds

$$1 \leq Z^{-1}_N Z_{N-s} \leq \left(1 - \ell^{-1} |B_1^d| \|f_0\|_{L^\infty_x L^1_v} \varepsilon\right)^{-s} \quad (195)$$

where $B_1^d$ is the unit ball in $\mathbb{R}^d$ and $Z_s$ is given by (192).

Proof. For the first inequality, we note that clearly $Z_N \leq Z_s Z_{N-s}$, then use the fact that $Z_s \leq 1$. The second inequality follows directly from Lemma 11.1 by induction on $s$. □ □

Lemma 11.3. For $1 \leq s \leq N$, and any probability density $f_0(x,v)$ on $\mathbb{R}^{2d}$ with $f_0 \in L^\infty_x L^1_v$, in the Boltzmann-Grad scaling $N \varepsilon^{-d-1} = \ell^{-1}$ there holds

$$f^{(s)}_N(0, Z_s) \leq 1_{Z_s \in \mathcal{D}_s} f^{\otimes s}_0(Z_s) \left(1 - \ell^{-1} |B_1^d| \|f_0\|_{L^\infty_x L^1_v} \varepsilon\right)^{-s} \quad (196)$$

where $B_1^d$ is the unit ball in $\mathbb{R}^d$ and $f^{(s)}_N(0)$ is the marginal of the data $f_N(0)$ given by (191).

Proof. We proceed by computation.

$$f^{(s)}_N(0, Z_s) = \int_{\mathbb{R}^{2d(N-s)}} Z^{-1}_N Z_{N-s} 1_{Z_s \in \mathcal{D}_s} f^{\otimes N}_0(0, Z_N) dZ_{(s+1):N}$$

$$\leq \int_{\mathbb{R}^{2d(N-s)}} Z^{-1}_N Z_{N-s} 1_{Z_s \in \mathcal{D}_s} f^{\otimes N}_0(0, Z_N) dZ_{(s+1):N}$$

$$= Z^{-1}_N Z_{N-s} 1_{Z_s \in \mathcal{D}_s} f^{\otimes s}_0(Z_s)$$

Then the result follows from Lemma 11.2. □ □

Lemma 11.4. For $1 \leq s \leq N$, and any probability density $f_0(x,v)$ on $\mathbb{R}^{2d}$ with $f_0 \in L^\infty_x L^1_v$, in the Boltzmann-Grad scaling $N \varepsilon^{-d-1} = \ell^{-1}$ there holds

$$f^{(s)}_N(0, Z_s) \geq 1_{Z_s \in \mathcal{D}_s} f^{\otimes s}_0(Z_s) \left(1 - (s + 1) \ell^{-1} |B_1^d| \|f_0\|_{L^\infty_x L^1_v} \varepsilon\right) \quad (197)$$

where $B_1^d$ is the unit ball in $\mathbb{R}^d$ and $f^{(s)}_N(0)$ is the marginal of the data $f_N(0)$ given by (191).
Proof. We proceed by computation.

\[
\begin{align*}
\hat{f}_N^{(s)}(0, Z_s) &= \int_{\mathbb{R}^{2d(N-s)}} Z_N^{-1} 1_{Z_N \in \mathcal{D}_N} f_0^{\otimes N}(Z_N) dZ_{(s+1):N} \\
&= \int_{\mathbb{R}^{2d(N-s)}} Z_N^{-1} 1_{Z_s \in \mathcal{D}_s} 1_{Z_{(s+1):N} \in \mathcal{D}_{N-s}} \times \\
&\quad \times \left( \prod_{1 \leq i \leq s} \prod_{s < j \leq N} 1_{|x_i - x_j| > \varepsilon} \right) f_0^{\otimes N}(Z_N) dZ_{(s+1):N} \\
&= Z_N^{-1} 1_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) \int_{\mathbb{R}^{2d(N-s)}} 1_{Z_{(s+1):N} \in \mathcal{D}_{N-s}} \times \\
&\quad \times \left( \prod_{1 \leq i \leq s} \prod_{s < j \leq N} 1_{|x_i - x_j| > \varepsilon} \right) f_0^{\otimes (N-s)}(Z_{(s+1):N}) dZ_{(s+1):N}
\end{align*}
\]

Now observe that

\[
\prod_{1 \leq i \leq s} \prod_{s < j \leq N} 1_{|x_i - x_j| > \varepsilon} \geq 1 - \sum_{1 \leq i \leq s} \sum_{s < j \leq N} 1_{|x_i - x_j| \leq \varepsilon} \quad (198)
\]

Then again, for \(1 \leq i \leq s, s < j \leq N\), we have

\[
\int_{\mathbb{R}^{2d(N-s)}} 1_{Z_{(s+1):N} \in \mathcal{D}_{N-s}} f_0^{\otimes (N-s)}(Z_{(s+1):N}) dZ_{(s+1):N} \leq Z_{N-s-1} \varepsilon |B_1^d| \|f_0\|_{L_1^\infty L_1^\infty} \quad (199)
\]

Therefore,

\[
\begin{align*}
\hat{f}_N^{(s)}(0, Z_s) &\geq Z_N^{-1} 1_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) \times \\
&\quad \times \left[ Z_{N-s} - s(N-s) Z_{N-s-1} \varepsilon d|B_1^d| \|f_0\|_{L_1^\infty L_1^\infty} \right] \quad (200)
\end{align*}
\]

We use Lemma 11.1, Lemma 11.2, and the Boltzmann-Grad scaling \(N \varepsilon^{d-1} = \ell^{-1}\) to conclude

\[
\hat{f}_N^{(s)}(0, Z_s) \geq 1_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) \left( 1 - (s+1) \ell^{-1} \varepsilon \right) \|f_0\|_{L_1^\infty L_1^\infty} \quad (201)
\]

□ □ □

Corollary 11.5. For any probability density \(f_0(x, v) > 0\) on \(\mathbb{R}^{2d}\) with \(f_0 \in L_1^\infty L_1^1\), in the Boltzmann-Grad scaling \(N \varepsilon^{d-1} = \ell^{-1}\), if \(N\) is sufficiently large, then simultaneously for all \(1 \leq s \leq N\) there holds

\[
\left\| 1_{Z_s \in \mathcal{D}_s} \left( \frac{\hat{f}_N^{(s)}(0, Z_s)}{f_0^{\otimes s}(Z_s)} - 1 \right) \right\|_{L_1^\infty L_1^s} \leq \left[ \left( 1 - \varepsilon^{-1} |B_1^d| \|f_0\|_{L_1^\infty L_1^1} \right)^{(s+1)} - 1 \right] \quad (202)
\]

where \(f_N^{(s)}(0)\) is the marginal of the data \(f_N(0)\) given by (140).
Theorem 12.1. Suppose in order to prove uniform convergence on a set of “good” phase points.

Result for the BBGKY hierarchy. We will use the stability result from Section 2, subject to the Boltzmann-Grad scaling

Further suppose \( \ell \) chaotic (see Section 2), then

Density \( f \) functions such that each \( \mathbb{F}(i) \)

Local-in-time propagation of chaos, and with each function \( f \) (ii) the Boltzmann hierarchy has a unique continuous symmetric solution

\( \beta > 0 \), \( \mu_0 \), \( \in \mathbb{R} \). Then for any \( \mu' < \mu \) we have for all sufficiently

\( N \) in the Boltzmann-Grad scaling \( N_{\varepsilon d-1} = \ell^{-1} \) the estimate

Suppose \( f_N(t) = \left\{ f_N^{(s)}(t) \right\} \) is a solution of the BBGKY hierarchy, subject to the Boltzmann-Grad scaling \( N_{\varepsilon d-1} = \ell^{-1} \), and with each function \( f_N^{(s)} : [0, \infty) \times \mathbb{R}^{2d_s} \rightarrow \mathbb{R} \) symmetric under particle interchange. Further suppose \( f_\infty(0) = \left\{ f_\infty^{(s)}(0) \right\} \) is a sequence of functions such that each \( f_\infty^{(s)}(0) : \mathbb{R}^{2d_s} \rightarrow \mathbb{R} \) is continuous and symmetric. Assume that for some \( \beta_0 > 0 \), \( \mu_0 \), \( \in \mathbb{R} \),

Then there is a constant \( C_d > 0 \), depending only on \( d \), such that if \( T_L < C_d e^{\mu_0 \beta_0} \), then all of the following are true:

(i) \( f_N(t) \) satisfies the bound

(ii) the Boltzmann hierarchy has a unique continuous symmetric solution \( f_\infty(t), t \in [0, T_L] \), satisfying the bound

(iii) if \( f_\infty^{(s)}(0) = f_0^{\otimes s} \) \( \forall s \), \( \in \mathbb{N} \) for some Lipschitz-continuous probability density \( f_0(x,v) \), and likewise \( \left\{ \left\{ f_N^{(s)}(0) \right\} \right\} \) is nonuniformly \( f_0 \)-chaotic (see Section 2), then \( f_\infty^{(s)}(t) = f_t^{\otimes s} \) \( \forall s \), \( \in \mathbb{N} \) for \( t \in [0, T_L] \) where \( f_t \)
solves Boltzmann’s equation, and \( \{ f_N^{(s)}(t) \}_{1 \leq s \leq N} \) is nonuniformly \( f_t \)-chaotic for \( t \in [0, T_L] \).

**Proof.** The local well-posedness of the Boltzmann hierarchy, and the bounds \[\text{(207)(208)}\], are direct consequences of Theorem 5.1 and Theorem 10.3.

We introduce a smooth cut-off function \( \chi : [0, \infty) \to \mathbb{R} \), decreasing, with \( 0 \leq \chi \leq 1 \), \( \chi(z) = 1 \) for \( 0 \leq z \leq 1 \), \( \| \chi' \|_{\infty} \leq 2 \), and \( \chi(z) = 0 \) for \( z \geq 2 \). Given parameters \( R > 0 \) and \( n \in \mathbb{N} \), we define

\[
f_{N,n,R}^{(s)}(0, Z_s) = f_N^{(s)}(0, Z_s)1_{1 \leq s \leq n} \chi \left( \frac{1}{R^2} E_s(Z_s) \right)
\]

and let \( F_{N,n,R}(0) = \{ f_{N,n,R}^{(s)}(0) \}_{1 \leq s \leq N} \). We let \( F_{N,n,R}(t) \) be the solution of the BBGKY hierarchy \[\text{(36)}\] with initial data \( F_{N,n,R}(0) \). Similarly, given initial data \( F_{\infty}(0) = \{ f_{\infty}^{(s)}(0) \}_{s \in \mathbb{N}} \), define

\[
f_{\infty,n,R}^{(s)}(0, Z_s) = f_{\infty}^{(s)}(0, Z_s)1_{1 \leq s \leq n} \chi \left( \frac{1}{R^2} E_s(Z_s) \right)
\]

and let \( F_{\infty,n,R}(0) = \{ f_{\infty,n,R}^{(s)}(0) \}_{s \in \mathbb{N}} \). We let \( F_{\infty,n,R}(t) \) be the solution of the Boltzmann hierarchy with data \( F_{\infty,n,R}(0) \). Using Theorem 5.1 and Theorem 10.3, and the linearity of the BBGKY and Boltzmann hierarchies, and dividing \( C_d \) by \( e \cdot 2^{d+1} \) in the statement of the theorem, we immediately obtain the following estimates:

\[
\sup_{1 \leq s \leq N} \sup_{t \in [0, T_L]} \sup_{Z_s \in \mathcal{D}_s} \left| \left( f_{N,n,R}^{(s)} - f_{\infty,n,R}^{(s)} \right)(t, Z_s) \right| e^{\frac{1}{2} \beta_0 E_s(Z_s) e^{(\mu_0 - 2)s}} \leq e^{-\frac{1}{2} \beta_0 R^2} + e^{-n} \tag{211}
\]

\[
\sup_{s \in \mathbb{N}} \sup_{t \in [0, T_L]} \sup_{Z_s \in \mathbb{R}^{2ds}} \left| \left( f_{\infty}^{(s)} - f_{\infty,n,R}^{(s)} \right)(t, Z_s) \right| e^{\frac{1}{2} \beta_0 E_s(Z_s) e^{(\mu_0 - 2)s}} \leq 2 \left( e^{-\frac{1}{2} \beta_0 R^2} + e^{-n} \right) \tag{212}
\]

The remainder of the proof consists of comparing the two functions \( f_{N,n,R}(t) \) and \( f_{\infty,n,R}(t) \).

We have the following Duhamel series:

\[
f_{N,n,R}^{(s)}(t, Z_s) = \sum_{k=0}^{n-s} a_{N,k,s} \times \]

\[
\times \sum_{i_1=1}^{s} \cdots \sum_{i_k=1}^{s+k-1} \int_0^t \cdots \int_0^t \int_{\mathbb{R}^{dk}} \int_{(\mathbb{S}^{d-1})^k} \left( \prod_{m=1}^k d\omega_m dv_{s+m} dt_m \right) \times \]

\[
\times \left( b_{s,s+k} [ f_{N,n,R}^{(s+k)}(0, Z_{s,s+k}) ] \right) \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] \tag{213}
\]
where
\[ a_{N,k,s} = \frac{(N-s)!}{(N-s-k)!} \epsilon^{k(d-1)} \]

It is not hard to show that all terms appearing in the finite series \([213,214]\) are finite for all \( t \geq 0 \). Note that the expression \([213]\) is meaningful as a measurable function if the data is integrable and compactly supported (see [17] for a detailed proof of this fact), whereas the expression \([214]\) makes sense due to the continuity of the data \( F_{\infty,n,R}(0) \).

Let us now define a new function, \( \tilde{f}_{N,n,R}^{(s)}(t) \), which is closely related to \( f_{N,n,R}^{(s)}(t) \).

\[ \tilde{f}_{N,n,R}^{(s)}(t, Z_s) = \sum_{k=0}^{n-s} \ell^{-k} \times \]
\[ \times \sum_{i_1=1}^{s} \cdots \sum_{i_k=1}^{s+k-1} \int_0^t \cdots \int_0^{t_{k-1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \prod_{m=1}^{k} d\omega_m dv_{s+m} dt_m \right) \times \]
\[ \times \left( b_{s,s+k} [.] \right) F_{\infty,n,R}(0, Z_{s,s+k} [.]) \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] \]

Note that \( \left| a_{N,k,s} - \ell^{-k} \right| \leq \left[ 1 - \left( 1 - \frac{n}{N} \right)^n \right] \ell^{-k} \) for \( 0 \leq k \leq n - s \); therefore,

\[ \left| \tilde{f}_{N,n,R}^{(s)}(t, Z_s) - f_{N,n,R}^{(s)}(t, Z_s) \right| \leq \left[ 1 - \left( 1 - \frac{n}{N} \right)^n \right] \times \]
\[ \sum_{k=0}^{n-s} \sum_{i_1=1}^{s} \cdots \sum_{i_k=1}^{s+k-1} \ell^{-k} \int_0^t \cdots \int_0^{t_{k-1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \prod_{m=1}^{k} d\omega_m dv_{s+m} dt_m \right) \times \]
\[ \times \left( b_{s,s+k} [.] \right) \left| F_{N,n,R}(0, Z_{s,s+k} [.]) \right| \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] \]

To estimate the series in \([217]\), we recall that \( f_{N,n,R}^{(s+k)}(0) \) is absolutely bounded by \( e^{-\mu_0(s+k)} \) and is supported in the set \( E_{s+k}(Z_{s+k}) \leq 2R^2 \). Hence, due to energy conservation, all the iterated integrals appearing in \([217]\) range over compact sets and we can evaluate the maximum possible contributions explicitly. Note that this is a significant over-estimate since we are not using the exponential decay of \( f_{N,n,R}^{(s+k)}(0) \) at large energies; nevertheless, this crude
estimate will suffice for the proof. We obtain

\[ |f^{(s)}_{N,n,R}(t, Z_s) - f^{(s)}_{N,n,R}(t, Z_s)| \leq \left[ 1 - \left( 1 - \frac{n}{N}\right)^n \right] e^{-\mu_0s} \exp \left[ C_d \ell^{-1} n R^{d+1} e^{-\mu_0 t} \right] \tag{218} \]

Observe that the right-hand side of (218) tends to zero as \( N \to \infty \) when \( n, R, Z_s, t \) are all held fixed.

Let us now fix \( Z_s \in \mathcal{K}_s \cap U^0, t \in [0, T_L], \) with \( E_s(Z_s) \leq 2R^2. \) Let us pick parameters \( \eta, \vartheta, \alpha, y > 0 \) such that \( \vartheta > \eta \) and \( \sin \theta > c_d y^{-1} \varepsilon, \) where \( c_d \) is as in the statement of Proposition 8.3. Let us define

\[ \mathcal{A}_{n,R} = \sum_{k=0}^{n} C_d^k \ell^k R^{k(d+1)} n^k e^{-\mu_0 k R^k} \tag{219} \]

where the constant \( C_d \) is to be chosen in the next step. Then, by repeated application of Proposition 8.3 we can construct sets \( \{ \mathcal{B}_k \}_{k=0}^{n-s}, \) dependent on \( Z_s, t, \) with

\[ \mathcal{B}_k \subset \left( [0, \infty) \times \mathbb{R}^d \times \mathbb{S}^{d-1} \times \mathbb{N} \right)^k \tag{220} \]

such that

\[ \sum_{k=0}^{n-s} \sum_{s+k-1}^{s+1} \sum_{i_k=1}^{s+k-1} \prod_{k=1}^{n} \int_0^t \int_0^{t_{k-1}} \int_{\mathcal{B}_k^d} \int_{\mathcal{S}^{d-1}} 1_{\mathcal{B}_k} \prod_{m=1}^{k} d\omega_m dv_{s+m} dt_m \times \]

\[ \times \left( |b_{s,s+k}[\cdot]| f^{(s+k)}_{N,n,R}(0, Z_{s,s+k}[\cdot]) \right) \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^{k} \right] \leq e^{-\mu_0 s} n^2 \mathcal{A}_{n,R} \left[ \alpha + \frac{y}{\eta T_L} + C_{d, \alpha} \left( \frac{\eta}{R} \right)^{d-1} + \vartheta (d-1)/2 \right] \tag{221} \]

\[ \sum_{k=0}^{n-s} \sum_{i_k=1}^{s+k-1} \prod_{k=1}^{n} \int_0^t \int_0^{t_{k-1}} \int_{\mathcal{B}_k^d} \int_{\mathcal{S}^{d-1}} 1_{\mathcal{B}_k} \prod_{m=1}^{k} d\omega_m dv_{s+m} dt_m \times \]

\[ \times \left( |b_{0,s+k}[\cdot]| f^{(s+k)}_{N,n,R}(0, Z_{0,s+k}[\cdot]) \right) \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^{k} \right] \leq e^{-\mu_0 s} n^2 \mathcal{A}_{n,R} \left[ \alpha + \frac{y}{\eta T_L} + C_{d, \alpha} \left( \frac{\eta}{R} \right)^{d-1} + \vartheta (d-1)/2 \right] \tag{222} \]

and such that whenever

\[ \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^{k} \]

\[ \in \left( [0, T_L] \times B_{2R}^d \times \mathbb{S}^{d-1} \times \mathbb{N} \right)^k \setminus \{ 0 \leq t_k \leq \cdots \leq t_1 \leq t \} \tag{223} \]

there holds

\[ \left| (Z_{s,s+k}[:]) - Z_{0,s,s+k}[:]) \right|_{\infty} \leq k \varepsilon \tag{224} \]
\[ b_{s,s+k} \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] = b_{s,s+k}^0 \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] \] (225)

\[ Z_{s,s+k} \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] \in K_{s+k} \cap U_{s+k}^0 \] (226)

Here \( |Z_j|_\infty = \sup_{i=1,...,j} \max(|x_i|, |v_i|) \).

**Remark.** The sets \( B_k \) collect all integration points for which the Duhamel series (214) and (216) fail to agree. At the remaining points, the pseudo-trajectories \( Z_{s,s+k} \) and \( Z_{0,s,s+k} \) are identical, up to \( O(\varepsilon) \) perturbations of the particles’ spatial positions. These perturbations are harmless because the Boltzmann hierarchy propagates smoothness forwards in time.

As long as we are away from \( B_k \), we can use the triangle inequality:

\[ \left| \left( f^{(s+k)}(0, Z_{s,s+k}^0 [\cdot]) - f^{(s+k)}(0, Z_{s,s+k} [\cdot]) \right) \right| \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] \]

\[ \leq \left| \left( f^{(s+k)}(0, Z_{s,s+k}^0 [\cdot]) - f^{(s+k)}(0, Z_{s,s+k} [\cdot]) \right) \right| \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] \]

\[ + \left| \left( f^{(s+k)}(0, Z_{s,s+k}^0 [\cdot]) - f^{(s+k)}(0, Z_{s,s+k} [\cdot]) \right) \right| \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] \] (227)

We can easily control the first term using the regularity assumption on \( f^{(s+k)}(0) \) combined with the stability estimate (224). On the other hand, due to (226), in order to control the second term, we only need to estimate \( |f^{(s+k)}(0) - f^{(s+k)}(0)| \) on \( K_{s+k} \cap U_{s+k}^0 \).

**Remark.** Carefully observe that it is entirely possible that \( Z_{0,s,s+k}^0 [\cdot] \notin K_{s+k} \cap U_{s+k}^0 \), even away from \( B_k \). This is because in the construction of \( B_k \), we never ruled out events wherein two particles only “barely” miss each other under the backwards flow.
Now we easily obtain
\[
\sup_{0 \leq t \leq T_L} \left| \left( f_N^{(s)} - f_\infty^{(s)} \right)(t, Z_s) \right| 1_{Z_s \in K_j \cap U_0^q} 1_{E_s(Z_s) \leq 2R^2}
\]
\[
\leq 3e^{-(\mu_0 - 2)s} \left( e^{-\frac{1}{2} \beta_0 R^2} + e^{-n} \right) +
\]
\[
+ \left[ 1 - \left( 1 - \frac{n}{N} \right)^n \right] e^{-\mu_0 s} C_d e^{\ell-1} n R^{d+1} e^{-\mu_0 T_L} +
\]
\[
+ 2e^{-\mu_0 s} n^2 A_{n, R} \left[ \alpha + \frac{y}{\eta T_L} + C_{d, \alpha} \left( \frac{n}{R} \right)^{d-1} + C_{d, \alpha} \theta(d-1)/2 \right] +
\]
\[
+ C_{d, n}^{\frac{3}{2}} e^{\ell-1} n R^{d+1} e^{-\mu_0 T_L} +
\]
\[
+ C_d e^{\ell-1} n R^{d+1} e^{-\mu_0 T_L} \sup_{1 \leq j \leq n} \left\{ \left| \nabla Z_j f_\infty^{(j)}(0, Z_j) \right| \right\}_2 1_{E_j(Z_j) \leq 2R^2} +
\]
\[
+ C_d e^{\ell-1} n R^{d+1} e^{-\mu_0 T_L} \sup_{1 \leq j \leq n} \left\{ \left| \left( f_N^{(j)} - f_\infty^{(j)} \right)(0, Z_j) \right| \right\}_2 1_{Z_j \in K_j \cap U_0^q} 1_{E_j(Z_j) \leq 2R^2}
\]
\[
(228)
\]
where \( \left| \nabla Z_j f^{(s)} \right|^2 = \sum_{i=1}^{s} \left( \left| \nabla x_i f^{(s)} \right|^2 + \left| \nabla y_i f^{(s)} \right|^2 \right) \). According to the definition of nonuniform \( f_0 \)-chaoticity, we may let \( \eta = \varepsilon^\kappa \) for some fixed \( \kappa \in (0, 1) \). We will then let \( y = \varepsilon^{(1+\kappa)/2} \) and \( \theta \sim \varepsilon^{(1-\kappa)/4} \); in particular, the constraint \( \sin \theta \geq c_d y_0^{-1} \varepsilon \) is satisfied. Now let \( N \to \infty \) and \( \varepsilon \to 0 \) simultaneously in the Boltzmann-Grad scaling, \( N \varepsilon^{d-1} = \ell^{-1} \), and use the fact that \( f_\infty^{(j)}(0) = f_0^{(j)} \) and that \( \left\{ \left\{ f_N^{(j)}(0) \right\}_1, 1 \leq j \leq n, N \in \mathbb{N} \right\} \) is nonuniformly \( f_0 \)-chaotic.

\[
\limsup_{N \to \infty} \sup_{0 \leq t \leq T_L} \sup_{Z_s \in \mathbb{R}^{2d}} \left| \left( f_N^{(s)} - f_\infty^{(s)} \right)(t, Z_s) \right| 1_{Z_s \in K_j \cap U_0^q} 1_{E_s(Z_s) \leq 2R^2}
\]
\[
\leq 3e^{-(\mu_0 - 2)s} \left( e^{-\frac{1}{2} \beta_0 R^2} + e^{-n} \right) + 2e^{-\mu_0 s} n^2 A_{n, R^2} \alpha
\]
\[
(229)
\]
Since \( \alpha > 0 \) is arbitrary we have

\[
\limsup_{N \to \infty} \sup_{0 \leq t \leq T_L} \sup_{Z_s \in \mathbb{R}^{2d}} \left| \left( f_N^{(s)} - f_\infty^{(s)} \right)(t, Z_s) \right| 1_{Z_s \in K_j \cap U_0^q} 1_{E_s(Z_s) \leq 2R^2}
\]
\[
\leq 3e^{-(\mu_0 - 2)s} \left( e^{-\frac{1}{2} \beta_0 R^2} + e^{-n} \right)
\]
\[
(230)
\]
Since \( n \) is arbitrary, the second term on the right-hand side can be thrown away. On the other hand, the left-hand side only increases as \( R \) increases, so we can throw away the first term on the right-hand side as well. Since the Boltzmann hierarchy propagates chaoticity, we have \( f_\infty^{(s)}(t) = f_t^{(s)} \) for
We conclude that \( \{ \{ f(s)(t, Z_s) \}_{1 \leq s \leq N} \}_{N \in \mathbb{N}} \) is nonuniformly \( f_t \)-chaotic for \( t \in [0, T_L] \).

\[ \limsup_{N \to \infty} \sup_{0 \leq t \leq T_L} \left| \sum_{s \in \mathbb{R}^{2d}} \epsilon_i \right| f_N(t, Z_s) - f_t^{\otimes s}(Z_s) \right| 1_{Z_s \in \mathcal{K}_t} \leq 2R^2 = 0 \] (231)

Remark. We can deduce part \( (i) \) of Theorem 2.1 directly from Theorem 12.1 by splitting the time interval \([0, T]\) into smaller intervals \([0, T_L], [T_L, 2T_L], \ldots\) for some sufficiently small time \( T_L \).

Appendix A. Proof of Part \( (ii) \) of Theorem 2.1

The proof consists of three parts. The first part is the introduction of an unsymmetric Boltzmann-Enskog hierarchy; we show that this auxiliary hierarchy propagates partial factorization. The second part is to show that a certain class of pseudo-trajectories for the BBGKY dynamics coincide (with high probability) with the corresponding pseudo-trajectories for the unsymmetric Boltzmann-Enskog hierarchy. The third part is to add up all the sources of error pointwise, as in Section 12. We outline the proof of the first step, provide full technical estimates for the second step, and skip the third step (which is tedious yet straightforward). We remark that a much more general version of the same result (accounting for correlations of any finite number of particles) is currently under investigation.\(^\text{13}\) (The proof in the case of general \( m \) is significantly more difficult than the two-particle case and therefore deserves a separate treatment.) This result and the proof were largely inspired by the techniques of M. Pulvirenti and S. Simonella.\(^\text{28}\)

A.1. An Unsymmetric Boltzmann-Enskog Hierarchy. We are going to construct an infinite hierarchy of equations which tracks correlations between the first \( m-1 \) labeled particles while ignoring all correlations between the remaining particles. Clearly, such a hierarchy cannot preserve symmetry between all particles. Nevertheless, we will be able to prove a partial factorization property which will be the key to part \( (ii) \) of Theorem 2.1. The factorization property we will prove for the resulting hierarchy is that if \( s \geq m \geq 2 \) then

\[ g^{(s)}(t) = g^{(m-1)}(t) \otimes g^{(s-m+1)}(t) \] (232)

if such factorization holds at the initial time; here \( g(t) \) is the solution to a Boltzmann-Enskog type equation.

\(^{13}\)To appear, JMP Vol 58 Issue 12 – the result and proof presented in this Appendix is special to two particles \((m - 1 = 2)\).
Let us introduce the unsymmetric \( s \)-particle phase space, where \( m \geq 2 \) is fixed and \( s \geq m - 1 \):

\[
\tilde{D}_s = \left\{ Z_s = (X_s, V_s) \in \mathbb{R}^{ds} \times \mathbb{R}^{ds} \mid \forall 1 \leq i < j \leq m - 1, |x_i - x_j| > \varepsilon \right\}
\]  

(233)

Observe that in the definition of \( \tilde{D}_s \), we only enforce an exclusion condition between the first \( m - 1 \) particles. We do not have exclusion for any pair of particles for which at least one particle index is greater than \( m - 1 \). We define the collision operators,

\[
\tilde{C}_{s+1} = \sum_{i=1}^{s} \left( \tilde{C}_{i,s+1}^+ - \tilde{C}_{i,s+1}^- \right)
\]

(234)

where

\[
\tilde{C}_{i,s+1}^+ g^{(s+1)}(t, Z_s) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \left[ \omega \cdot (v_{s+1} - v_i) \right]_+ \times \nonumber \\
\times g^{(s+1)}(t, x_1, v_1, \ldots, x_i, v_i^*, \ldots, x_s, v_s, x_i + \varepsilon \omega, v_{s+1}) \, d\omega dv_{s+1}
\]

(235)

\[
\tilde{C}_{i,s+1}^- g^{(s+1)}(t, Z_s) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \left[ \omega \cdot (v_{s+1} - v_i) \right]_- \times \nonumber \\
\times g^{(s+1)}(t, x_1, v_1, \ldots, x_i, v_i, \ldots, x_s, v_s, x_i + \varepsilon \omega, v_{s+1}) \, d\omega dv_{s+1}
\]

(236)

and

\[
v_i^* = v_i + \omega \cdot (v_{s+1} - v_i)
\]

\[
v_{s+1}^* = v_{s+1} - \omega \cdot (v_{s+1} - v_i)
\]

(237)

The function \( g^{(s)}(t, Z_s) \) is defined for \( 0 \leq t < T \) and \( Z_s \in \tilde{D}_s, s \geq m - 1 \), as the solution to the following hierarchy of equations:

\[
(\partial_t + V_s \cdot \nabla x_s) g^{(s)}(t) = \ell^{-1} \tilde{C}_{s+1} g^{(s+1)}(t) \quad \text{if } s \geq m - 1
\]

(238)

with boundary condition

\[
g^{(s)}(t, Z_s^*) = g^{(s)}(t, Z_s) \quad \text{a.e. } (t, Z_s) \in [0, T) \times \partial \tilde{D}_s
\]

(239)

and initial conditions \( g^{(s)}(0, Z_s) \) defined for \( s \geq m - 1 \) and \( Z_s \in \tilde{D}_s \). We also introduce the function \( g_\varepsilon(t, x, v) \) \((t \geq 0, x, v \in \mathbb{R}^d)\) which is defined to be the solution to the equation

\[
(\partial_t + v \cdot \nabla x) g_\varepsilon(t) = \ell^{-1} \tilde{C}_2 (g_\varepsilon(t) \otimes g_\varepsilon(t))
\]

(240)

with prescribed initial data \( g_\varepsilon(0) \).

We now introduce a mild form for (238)(239). For any \( s \geq m - 1 \), let \( \tilde{T}_s(t) \) denote the strongly continuous semigroup on \( L^2(\tilde{D}_s) \) with generator \(-V_s \cdot \nabla x_s \) and specular reflection boundary conditions along \( \partial \tilde{D}_s \). The operators \( \tilde{T}_s(t) \) extend to other functional spaces by standard density arguments. Then, under sufficiently strong regularity conditions, the hierarchy
The sequence \( \{g_{\varepsilon}^{(s)}(t)\} \) is equivalent to the following hierarchy written in mild form:

\[
g_{\varepsilon}^{(s)}(t) = \hat{T}_s(t)g_{\varepsilon}^{(s)}(0) + \ell^{-1} \int_0^t \hat{T}_s(t-\tau)\hat{C}_{s+1}g_{\varepsilon}^{(s+1)}(\tau)d\tau \quad (s \geq m-1)
\]

Following Lanford’s fixed point argument, we are able to prove existence and uniqueness of solutions to (241) on a short time interval. However, under the conditions of Lanford’s proof, the distributional form (238-239) and the mild form (241) are equivalent, so we are free to work with either formulation for our computations. Note that, in a similar fashion, we can define mild solutions for (240), and solutions can be constructed on a short time interval by a fixed point argument.

We will state a well-posedness theorem for (238-239) so that we can refer to the result later. The proof follows Lanford’s fixed point argument so we omit it.

**Proposition A.1.** Fix an integer \( m \geq 2 \). Let \( \{g_{\varepsilon}^{(s)}(0)\}_{s \geq m-1} \) be a sequence of functions, with each \( g_{\varepsilon}^{(s)}(0) \) defined for \( Z_s \in \hat{D}_s \). Furthermore, suppose that there exists \( \beta_0 > 0 \) and \( \mu_0 \in \mathbb{R} \) such that

\[
\sup_{s \geq m-1} \sup_{Z_s \in \hat{D}_s} e^{\mu_0 s} e^{\beta_0 E_s(Z_s)} \left| g_{\varepsilon}^{(s)}(0, Z_s) \right| \leq 1
\]

Then there exists a constant \( C_d > 0 \) such that the following is true: If \( T_L < C_d \ell e^{\mu_0 \beta_0^{m+1}} \) then there exists a unique sequence of functions \( \{g_{\varepsilon}^{(s)}(t)\}_{s \geq m-1} \) defined for \( t \in [0, T_L] \) such that (i), (ii), and (iii) below all hold.

(i) For any bounded open set \( \mathcal{O} \subset [0, T_L] \times \hat{D}_s \), we have \( (\partial_t + V_s \cdot \nabla X_s) g_{\varepsilon}^{(s)} \in L^1(\mathcal{O}) \).

(ii) We have the bound:

\[
\sup_{s \geq m-1} \sup_{t \in [0, T_L]} \sup_{Z_s \in \hat{D}_s} e^{\mu_0 (s-1)} e^{\beta_0 E_s(Z_s)} \left| g_{\varepsilon}^{(s)}(t, Z_s) \right| \leq 2
\]

(iii) The sequence \( \{g_{\varepsilon}^{(s)}(t)\}_{s \geq m-1} \) solves (238-239) in the sense of distributions on \( [0, T_L] \) with initial data \( \{g_{\varepsilon}^{(s)}(0)\}_{s \geq m-1} \); note that the equation is well-defined thanks to (i) and (ii).

We now turn to the main result of this section:

**Proposition A.2.** Fix an integer \( m \geq 2 \). Let \( \{g_{\varepsilon}^{(s)}(t)\}_{s \geq m-1} \) be a sequence of functions, with each \( g_{\varepsilon}^{(s)}(t, Z_s) \) defined for \( (t, Z_s) \in [0, T) \times \hat{D}_s \). Let \( g_{\varepsilon}(t, x, v) \) be defined for \( t \in [0, T) \) and \( x, v \in \mathbb{R}^d \). Further suppose that there exists \( \beta_T > 0 \) and \( \mu_T \in \mathbb{R} \) such that

\[
\sup_{s \geq m-1} \sup_{t \in [0, T]} \sup_{Z_s \in \hat{D}_s} e^{\mu_T s} e^{\beta_T E_s(Z_s)} \left| g_{\varepsilon}^{(s)}(t, Z_s) \right| \leq 1
\]
and
\[ \sup_{t \in [0,T]} \sup_{x,v \in \mathbb{R}^d} e^{\mu t} e^{\frac{1}{2}\beta t|v|^2} |g_\varepsilon(t,x,v)| \leq 1 \] (245)
and that \((\partial_t + V_s \cdot \nabla_x) g_\varepsilon^{(s)} \in L^1(\mathcal{O})\) for any bounded open set \(\mathcal{O} \subset [0,T] \times \mathcal{D}_s\). Then, if \(\{g_\varepsilon^{(s)}(t)\}_{s \geq m-1}\) solve (248), \(g_\varepsilon(t)\) solves (240), and
\[ g_\varepsilon^{(s)}(0) = g_\varepsilon^{(m-1)}(0) \otimes g_\varepsilon(0)^{\otimes (s-m+1)} \] (246)
for all \(s \geq m\), then for \(t \in [0,T]\) there holds
\[ g_\varepsilon^{(s)}(t) = g_\varepsilon^{(m-1)}(t) \otimes g_\varepsilon(t)^{\otimes (s-m+1)} \] (247)
for all \(s \geq m\).

**Proof.** We proceed by constructing a solution of the unsymmetric Boltzmann-Enskog hierarchy (238-239) with the desired property; then, the conclusion follows by uniqueness. Let \(T_L < C_d e^{\mu T} \beta_{\varepsilon}^{\frac{d+1}{2}}\), where \(C_d\) is the constant appearing in Proposition A.1.

Recall that \(g_\varepsilon(t)\) is the solution to (230), with initial data \(g_\varepsilon(0)\). Let us now define \(u_\varepsilon^{(m-1)}(t)\) to be the solution of the following equation, for \(0 \leq t \leq T_L:\)
\[ (\partial_t + V_{m-1} \cdot \nabla_{x_{m-1}}) u_\varepsilon^{(m-1)}(t) = \ell^{-1} \tilde{C}_m \left( u_\varepsilon^{(m-1)}(t) \otimes g_\varepsilon(t) \right) \] (248)
with boundary condition \(u_\varepsilon^{(m-1)}(t, Z_{m-1}) = u_\varepsilon^{(m-1)}(t, Z_{m-1})\) along \([0,T_L] \times \partial \mathcal{D}_m\), and initial data \(g_\varepsilon^{(m-1)}(0)\). The existence and uniqueness for (248) on a time interval of size \(T_L\) follows from a modified version of Lanford’s fixed point argument; moreover, the solution obeys the following bound:
\[ \sup_{t \in [0,T_L]} \sup_{Z_{m-1} \in \mathcal{D}_m} e^{2(\mu t - 1)} e^{\frac{1}{2} \beta t E_{m-1}(Z_{m-1})} |u_\varepsilon^{(m-1)}(t, Z_{m-1})| \leq 2 \] (249)
Having defined \(u_\varepsilon^{(m-1)}(t)\), let us define, for \(s \geq m\),
\[ u_\varepsilon^{(s)}(t) = u_\varepsilon^{(m-1)}(t) \otimes g_\varepsilon(t)^{\otimes (s-m+1)} \] (250)
Now it is straightforward to verify that the sequence \(\{u_\varepsilon^{(s)}(t)\}_{s \geq m-1}\) satisfies (238-239) for \(t \in [0,T_L]\); by uniqueness, we conclude that \(g_\varepsilon^{(s)}(t) = u_\varepsilon^{(s)}(t)\) for all \(s \geq m-1\) and \(t \in [0,T_L]\).

We can iterate the same argument on the time intervals \([T_L, 2T_L], [2T_L, 3T_L], \ldots\), until we have covered the full time interval \([0,T]\). \(\Box\)

**A.2. Series Solution for the Unsymmetric Boltzmann-Enskog Hierarchy.** We will develop a series expansion and corresponding pseudo-trajectories for the unsymmetric Boltzmann-Enskog hierarchy (238-239). The main differences between the unsymmetric Boltzmann-Enskog hierarchy and the BBGKY hierarchy are twofold: first, the former is an infinite
hierarchy, whereas the latter is finite; and second, the former tracks correlations between \( m - 1 \) particles, whereas the latter tracks correlations between all particles. Since the two hierarchies are so similar, the developments in this section will be almost identical to those of Section 7. Nevertheless, there are a few subtle differences which are important in our proof, so in the interest of completeness we repeat the construction in this case.

The main point we wish to emphasize is that there is a new dynamics, given by a measurable measure-preserving map \( \tilde{\psi}_s^t : \tilde{\mathcal{D}}_s \to \tilde{\mathcal{D}}_s \), with the property that

\[
\left( \tilde{T}_s(t) g^{(s)} \right)(Z_s) = g^{(s)} \left( \tilde{\psi}_s^{-t} Z_s \right)
\]

where \( g^{(s)}(Z_s) \) is an arbitrary measurable function with finite integral, and \( \tilde{T}_s \) is the transport operator appearing in the mild form (241) of the unsymmetric Boltzmann-Enskog hierarchy. The dynamics \( \tilde{\psi}_s^t \) forces collisions between the first \( m - 1 \) labeled particles, whereas any pair of particles with \((i, j)\) with \( 1 \leq i < j \leq s \) may pass through each other without colliding. We will need to use \( \tilde{\psi}_s^t \) in place of \( \psi_s^t \) in the construction of pseudo-trajectories for the unsymmetric Boltzmann-Enskog hierarchy.

Similar to the BBGKY hierarchy, we can write down an iterated Duhamel series for the unsymmetric Boltzmann-Enskog hierarchy (241), like so:

\[
g_s^{(s)}(t) = \sum_{k=0}^{\infty} \ell^{-k} \times \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \tilde{T}_s(t - t_1) \tilde{C}_{s+1} \cdots \tilde{T}_{s+k}(t_k) g^{(s+k)}(0) dt_k \cdots dt_1
\]

(if \( s \geq m - 1 \))

(252)

Notice that the collision operators \( \tilde{C}_{s+1} \) have replaced the collision operators \( C_{s+1} \) which appear in the Duhamel series for the BBGKY hierarchy, and the transport operators \( \tilde{T}_s \) have replaced \( T_s \). The collision operators \( \tilde{C}_{s+1} \) do not enforce any exclusion condition, as can be seen from (234-236); this fact will have to be reflected in the construction of pseudo-trajectories.

Fix an integer \( m \geq 2 \) and let \( s \geq m - 1 \). We will be defining the symbols

\[
\tilde{Z}_{s,s+k} \left[ Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right]
\]

where \( Z_s \in \tilde{\mathcal{D}}_s \), \( 0 \leq t_k < \cdots < t_2 < t_1 < t \), \( i_1 \in \{1, 2, \ldots, s\} \), \( i_2 \in \{1, 2, \ldots, s + 1\} \), \ldots, \( i_k \in \{1, 2, \ldots, s + k - 1\} \), \( v_{s+j} \in \mathbb{R}^d \), and \( \omega_j \in S^{d-1} \). Given \( Z_s \in \mathcal{D}_s \) and \( t > 0 \) we define

\[
\tilde{Z}_{s,s} [Z_s, t] = \tilde{\psi}_s^{-t} Z_s
\]

and \( \tilde{Z}_{s,s} [Z_s, 0] = Z_s \). If the symbol

\[
\tilde{Z}_{s,s+k} \left[ Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] \in \tilde{\mathcal{D}}_{s+k}
\]

(255)
is defined, then for all $\tau > 0$ we define
\[
\tilde{Z}_{s,s+k} \left[ Z_s, t + \tau; \{ t_j + \tau, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] = \\
= \tilde{\gamma}_{s+k} \tilde{Z}_{s,s+k} \left[ Z_s, t; \{ t_j, v_{s+k}, \omega_j, i_j \}_{j=1}^k \right] 
\] (256)

Now suppose that the symbol
\[
\tilde{Z}_{s,s+k} \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] = (X'_{s+k}, V'_{s+k}) \in \tilde{D}_{s+k} 
\] (257)
is defined, $t_{k+1} = 0$, $v_{s+k+1} \in \mathbb{R}^d$, $\omega_{k+1} \in S^{d-1}$, and $i_{k+1} \in \{1, 2, \ldots, s + k\}$. Further suppose that $\omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) \leq 0$. Then we define
\[
\tilde{Z}_{s,s+k+1} \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] = (X'_{s+k}, V'_{s+k}, x'_{i_{k+1}} + \varepsilon \omega_{k+1}, v_{s+k+1}) 
\] (258)

Similarly, suppose that the symbol
\[
\tilde{Z}_{s,s+k} \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] = (X'_{s+k}, V'_{s+k}) \in \tilde{D}_{s+k} 
\] (259)
is defined, $t_{k+1} = 0$, $v_{s+k+1} \in \mathbb{R}^d$, $\omega_{k+1} \in S^{d-1}$, and $i_{k+1} \in \{1, 2, \ldots, s + k\}$. Further suppose that $\omega \cdot (v_{s+k+1} - v'_{i_{k+1}}) > 0$. Then we define
\[
\tilde{Z}_{s,s+k+1} \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] = \\
= (x'_1, v'_1, \ldots, x'_{i_{k+1}}, v'_{i_{k+1}} + \omega_{k+1} \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}), \ldots, x'_s, v'_s, \\
x'_{i_{k+1}} + \varepsilon \omega_{k+1}, v_{s+k+1} - \omega_{k+1} \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}})) 
\] (260)

Now we define the iterated collision kernel, again using induction. If $Z_s \in \tilde{D}_s$ and $t \geq 0$ we define
\[
\tilde{b}_{s,s} [Z_s,t] = 1 
\] (261)
If the symbol
\[
\tilde{b}_{s,s+k} \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] 
\] (262)
is defined and $\tau > 0$ then we define
\[
\tilde{b}_{s,s+k} \left[ Z_s, t + \tau; \{ t_j + \tau, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] = \\
= \tilde{b}_{s,s+k} \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] 
\] (263)
If the symbol
\[
\tilde{b}_{s,s+k} \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] 
\] (264)
is defined, and $t_{k+1} = 0, v_{s+k+1} \in \mathbb{R}^d, \omega_{k+1} \in S^{d-1}$, and $i_{k+1} \in \{1, 2, \ldots, s+k\}$ then we define
\[
\tilde{b}_{s,s+k+1} \left[ Z_s; t; \{t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^{k+1} \right] = \\
= \tilde{b}_{s,s+k} \left[ Z_s; t; \{t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^{k} \right] \times \omega_{k+1} \cdot (v_{s+k+1} - v_{k+1}^t)
\]
(265)

We now have the following identity which holds pointwise for $Z_s \in \mathcal{D}_s$ and $t \geq 0$:
\[
g_{\varepsilon}^{(s)}(t, Z_s) = \sum_{k=0}^{\infty} \varepsilon^{-k} \times \times \sum_{i_1=1}^{s} \cdots \sum_{i_k=1}^{s+k-1} \int_0^t \cdots \int_0^{t_{k-1}} \int_{\mathbb{R}^d} \int_{(S^{d-1})^k} \left( \prod_{m=1}^{k} d\omega_m dv_{s+m} dt_m \right) \times (266)
\]
\[
\times \left( \tilde{b}_{s,s+k[1]} g_{\varepsilon}^{(s+k)} \left( 0, \tilde{Z}_{s,s+k[1]} \right) \right) \left[ Z_s; t; \{t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^{k} \right]
\]

A.3. Stability of pseudo-trajectories. This subsection is concerned purely with the pseudo-trajectories generated by the BBGKY hierarchy. We are going to show that, if $Z_s$ is such that all but the first two particles have free trajectories under the backwards particle flow\footnote{including the possibilities that the first two particles collide or “pass through” each other, or miss entirely}, then adding a particle preserves this property with high probability. This result is important because it allows us to compare pseudo-trajectories for the BBGKY hierarchy with those of the unsymmetric Boltzmann-Enskog hierarchy. Then it is straightforward to conclude that partial factorization is propagated by the BBGKY hierarchy, because by Proposition A.2 partial factorization is propagated by the unsymmetric Boltzmann-Enskog hierarchy.

Remark. We fix $m = 3$ in order to justify the result $f_N^{(s)}(t) \approx f_N^{(2)}(t) \otimes f_i^{(s-2)}$ on the set $\mathcal{G}_s \cap \mathcal{U}_s^q(\varepsilon)$, introduced in Section 2 under the assumption that the entire sequence $\{F_N(0)\}_N$ is 2-nonuniformly $f_0$-chaotic.

We will require the following sets:
\[
\mathcal{G}_s = \left\{ \begin{array}{c} Z_s = (X_s, V_s) \in \mathcal{D}_s \quad \forall \tau > 0, \forall 3 \leq i \leq s, \\
\left( \psi_{s-\tau}^t Z_s \right)_i = (x_i - v_i \tau, v_i) \\
\text{and, } \forall \tau > 0, \forall 1 \leq i \leq 2, \forall 3 \leq j \leq s, \\
|(x_i - x_j) - (v_i - v_j)\tau| > \varepsilon \end{array} \right\}
\]
(267)
\[
\mathcal{V}_s^\eta = \left\{ \begin{array}{c} (Z_u, Z'_s) \in \mathcal{D}_s \times \mathcal{D}_s \quad \inf_{1 \leq i \neq j \leq s} |v_i - v'_j| > \eta \\
\text{and} \\
\inf_{1 \leq i \leq s: |v_i - v'_i| \neq 0} |v_i - v'_i| > \eta \end{array} \right\}
\]
(268)
whenever

However, it will not be enough to delete times for which particles at a particle addition times for which particles are too concentrated in space.

and points generated in this way does not concentrate in space. For recollisions arising from the collisional dynamics, we gather together

Proof.

Claim (i) for all \( \tau \geq 0 \) we have

\[
Z_{s,s+k} [Z_s,t+\tau;\tau,t_1+\tau,\ldots,t_k+\tau,\ldots,\omega_k;\omega_1,\ldots,\omega_k;i_1,\ldots,i_k] \\
\in G_{s+k} \cap U^n_{s+k}
\]  

(271)

(i) for any \( i_{k+1} \in \{1,2,\ldots,s+k \} \), and for any \( \alpha,y > 0 \) and \( \theta \in (0,\frac{\pi}{2}) \) such that \( \sin \theta > c_d y^{-1} \varepsilon \), there exists a measurable set \( \mathcal{B} \subset [0,\infty) \times \mathbb{R}^d \times S^{d-1} \), which may depend on \( Z_s, t \), and \( \{t_j,v_{s+j},\omega_j,i_j\}_{j=1}^k \), such that

\[
\forall T > 0,
\int_0^T \int_{B^d_{2R}} \int_{S^{d-1}} 1_{(\tau,v_{s+k+1},\omega_{k+1})} \in \mathcal{B} d\omega_{k+1} d\tau \\
\leq C_{d,s,k} T R^d \left[ \alpha + \frac{y}{\eta T} + C_{d,\alpha} \left( \frac{\eta}{R} \right)^{d-1} + C_{d,\alpha} \theta^{(d-1)/2} \right]
\]

(272)

and

\[
Z_{s,s+k+1} [Z_s,t+\tau;\tau,t_1+\tau,\ldots,t_k+\tau,\ldots,\omega_k;\omega_1,\ldots,\omega_k;i_1,\ldots,i_k,i_{k+1}] \\
\in G_{s+k+1} \cap \hat{U}^n_{s+k+1}
\]  

(273)

\( \text{whenever } (\tau,v_{s+k+1},\omega_{k+1}) \in (0,\infty) \times \mathbb{R}^d \times S^{d-1} \) \( \setminus \mathcal{B} \).

Proof. Claim (i) is trivial so we turn to claim (ii). The first step is to delete particle addition times for which particles are too concentrated in space. However, it will not be enough to delete times for which particles at a single time-slice are nearby. Instead, in order to eventually control all possible recollisions arising from the collisional dynamics, we gather together all line segments generated by the flow prior to time \( \tau \) and project them via free flight to land on the \( \tau \) time-slice. Then we ask that the entire set of phase points generated in this way does not concentrate in space. For \( t' \in \mathbb{R} \) and \( Z^0_s = (X^0_s, V^0_s) \in \mathbb{R}^d x \mathbb{R}^d \) we define

\[
\hat{Z}^0_s (t') = (X^0_s + V^0_s t', V^0_s)
\]

(274)
We can easily estimate the measure of

Also, we define

If \( Z_s^0, Z_s^1 \in \mathbb{R}^{2d} \), then we define

\[
d_X(Z_s^0, Z_s^1) = \min \left( \inf_{1 \leq i \neq j \leq s} |x_i^0 - x_j^0|, \inf_{1 \leq i \leq s} \inf_{(x_i^0, v_i^0) \neq (x_i^1, v_i^1)} |x_i^0 - x_i^1| \right)
\]

(275)

\[B_I = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in [0, \infty) \times \mathbb{R}^d \times S^{d-1} \text{ such that } \tau = 0 \text{ or } \exists t', t'' \geq 0 : d_X(Z_{s+k}^\prime(\tau; t'), Z_{s+k}^\prime(\tau; t'')) \leq y \right\}
\]

(276)

We can easily estimate the measure of \( B_I \) due to the condition \( Z_{s+k}^\prime \in \mathcal{G}_{s+k} \); it suffices to consider (at most) two possible line segments for each of the first two particles (corresponding to whether the particles are allowed to collide, or pass through each other, or miss entirely), and for each \( 3 \leq i \leq s + k \), the unique backwards line segment available to one of the \( i \)th particle. Distinct line segments are compared pairwise to find collisions or near-collisions. Since \( Z_{s+k}^\prime \in \hat{U}_{s+k}^\eta \), any two line segments can only be within a distance \( y \) (along a fixed time slice \( \tau \)) for a time \( \Delta \tau \) of order \( \eta^{-1} \) (this is where we explicitly use the integral in the creation time \( \tau \)). We have

\[
\int_0^T \int_{B_{2R}^d} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in B_I} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_{d,s,k} R^d \eta^{-1} y
\]

(277)

At this point it is useful to distinguish between pre-collisional and post-collisional configurations for the added particle. Therefore we introduce two sets,

\[
A^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in [0, \infty) \times \mathbb{R}^d \times S^{d-1} \text{ such that } \omega_{k+1} : (v_{s+k+1} - v'_{i_{k+1}} (\tau; 0)) > 0 \right\}
\]

(278)

\[
A^- = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in [0, \infty) \times \mathbb{R}^d \times S^{d-1} \text{ such that } \omega_{k+1} : (v_{s+k+1} - v'_{i_{k+1}} (\tau; 0)) \leq 0 \right\}
\]

(279)

We also delete collisions which are close to grazing:

\[
B_{II} = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in [0, \infty) \times \mathbb{R}^d \times S^{d-1} \text{ such that } \omega_{k+1} : (v_{s+k+1} - v'_{i_{k+1}} (\tau; 0)) \leq (\sin \alpha) |v_{s+k+1} - v'_{i_{k+1}} (\tau; 0)| \right\}
\]

(280)

We have

\[
\int_0^T \int_{B_{2R}^d} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in B_{II}} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_{d} T R^d \alpha
\]

(281)

The pre-collisional configurations and post-collisional configurations are dealt with separately.
Pre-collisional configurations. We must guarantee that the \((s+k+1)\)-particle state is in \(\mathcal{U}^\partial_{s+k+1}\) at the time of particle creation. Let us define

\[
\mathcal{B}^-_{III} = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^- \text{ such that } \begin{array}{l}
\exists \tilde{i} \in \{1, 2, \ldots, s + k\}, t' \geq 0 : |v_{s+k+1} - v'_{\tilde{i}}(\tau; t')| \leq \eta \end{array} \right\} \tag{282}
\]

If \((\tau, v_{s+k+1}, \omega_{k+1}) \notin \mathcal{B}^-_{III}\), and the \((s+k+1)\) particle’s backwards trajectory is free (which follows from the next step), we can be sure that the \((s+k+1)\)-particle state is in \(\mathcal{U}^\partial_{s+k+1}\). We have

\[
\int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}^-_{III}} d\omega_{k+1}d v_{s+k+1} d\tau \leq C_{d,s,k} T \eta^d \tag{283}
\]

Finally we need to make sure that the backwards trajectory of the added particle is free (accounting for all possible histories of particles 1 and 2). Let us define

\[
\mathcal{B}^-_{IV} = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^- \text{ such that } \begin{array}{l}
\exists \tilde{i} \in \{1, 2, \ldots, s + k\}, t' \geq 0 : \\
|\frac{(x'_{\tilde{i}_{k+1}}(\tau; 0) + \varepsilon \omega - x'_{\tilde{i}}(\tau; t')) \cdot (v_{s+k+1} - v'_{\tilde{i}}(\tau; t'))}{|v_{s+k+1} - v'_{\tilde{i}}(\tau; t')|} | \geq \cos \theta \\
\end{array} \right\} \tag{284}
\]

We have

\[
\int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}^-_{IV}} d\omega_{k+1}d v_{s+k+1} d\tau \leq C_{d,s,k} T \eta^d \tag{285}
\]

To conclude, we let \(\mathcal{B}^- = \mathcal{B}_I \cup \mathcal{B}_{III} \cup \mathcal{B}^-_{IV} \); then we have

\[
\int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}^-} d\omega_{k+1}d v_{s+k+1} d\tau \leq C_{d,s,k} T \eta^d \tag{286}
\]

Then again, by assumption, \(\sin \theta > c_d \eta^d \varepsilon\); by choosing \(c_d\) sufficiently large we may guarantee that

\[
Z_{s,s+k+1} \left[ Z_s, t + \tau; t_1 + \tau, \ldots, t_k + \tau, 0; v_{s+1}, \ldots, v_{s+k}, v_{s+k+1}; \omega_1, \ldots, \omega_k, \omega_{k+1}; i_1, \ldots, i_k, i_k+1 \right] \in \mathcal{G}_{s+k+1} \cap \mathcal{U}^{\partial}_s \tag{287}
\]

whenever \((\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^- \setminus \mathcal{B}^-\).

Post-collisional configurations. The post-collisional case is very similar to the pre-collisional case; the only difference is that we must account
Using Lemma 8.1 we have

\[ v^s_{s+k+1} = v_{s+k+1} - \omega_{k+1}\omega_{k+1} \cdot \left( v_{s+k+1} - v^I_{i_{k+1}}(\tau;0) \right) \]
\[ v^v_{i_{k+1}} = v^I_{i_{k+1}}(\tau;0) + \omega_{k+1}\omega_{k+1} \cdot \left( v_{s+k+1} - v^I_{i_{k+1}}(\tau;0) \right) \]

(288)

for the collisional change of variables. We define

\[ B_{II} = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^+ \setminus \mathcal{B}_{II} \text{ such that} \right. \]
\[ \exists i \in \{1, 2, \ldots, s + k\}, \ t' \geq 0 : \left| v^s_{s+k+1} - v^I_t(\tau; t') \right| \leq \eta \right\} \]
(289)

\[ B_{IV}^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^+ \setminus \mathcal{B}_{II} \text{ such that} \right. \]
\[ \exists i \in \{1, 2, \ldots, s + k\} \setminus \{i_{k+1}\}, \ t' \geq 0 : \left| v^s_{i_{k+1}} - v^I_t(\tau; t') \right| \leq \eta \right\} \]
(290)

\[ B_v^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^+ \text{ such that} \left| v_{s+k+1} - v^I_t(\tau;0) \right| \leq \eta \right\} \]
(291)

We remove particle creations with velocities being too close to some other particle’s velocity:

\[
\int_0^T \int_{B_{2R}} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in B_{II}} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_{d,s,k} C_{d,\alpha} T R^d \eta^{d-1}
\]
(292)

\[
\int_0^T \int_{B_{2R}} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in B_{IV}^+} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_{d,s,k} C_{d,\alpha} T R^d \eta^{d-1}
\]
(293)

\[
\int_0^T \int_{B_{2R}} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in B_v^+} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_d T \eta^d
\]
(294)

The last estimate will remove possible recollisions; define the sets

\[ B_{VI}^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^+ \setminus \mathcal{B}_{II} \text{ such that} \right. \]
\[ \exists i \in \{1, 2, \ldots, s + k\} \setminus \{i_{k+1}\}, \ t' \geq 0 : \left| v^s_{i_{k+1}}(\tau;0) + \varepsilon \omega - x^I_t(\tau; t') \cdot (v^s_{s+k+1} - v^I_t(\tau; t')) \right| \geq \cos \theta \right\}
(295)

\[ B_{VII}^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^+ \setminus \mathcal{B}_{II} \text{ such that} \right. \]
\[ \exists i \in \{1, 2, \ldots, s + k\} \setminus \{i_{k+1}\}, \ t' \geq 0 : \left| v^s_{i_{k+1}}(\tau;0) + \varepsilon \omega - x^I_t(\tau; t') \cdot (v^s_{s+k+1} - v^I_t(\tau; t')) \right| \geq \cos \theta \right\}
(296)
Using Lemma 8.1 and Lemma 8.2, we have
\[
\int_0^T \int_{B^d_{2R}} \int_{S^{d-1}} 1_{(\tau,v_{s+k+1},\omega_{k+1})} \in B_{VI}^d \ d\omega_{k+1} dv_{s+k+1} d\tau \leq C_{d,s,k} C_{d,\alpha} T R^d \theta^{(d-1)/2}
\]
(297)
\[
\int_0^T \int_{B^d_{2R}} \int_{S^{d-1}} 1_{(\tau,v_{s+k+1},\omega_{k+1})} \in B_{VII}^d \ d\omega_{k+1} dv_{s+k+1} d\tau \leq C_{d,s,k} C_{d,\alpha} T R^d \theta^{(d-1)/2}
\]
(298)

To conclude, we let \( B^+ = B_I \cup B_{II}^+ \cup B_{IV}^+ \cup B_{VI}^+ \cup B_{V}^+ \cup B_{VII}^+ \); then we have
\[
\int_0^T \int_{B^d_{2R}} \int_{S^{d-1}} 1_{(\tau,v_{s+k+1},\omega_{k+1})} \in B^+ \ d\omega_{k+1} dv_{s+k+1} d\tau \leq C_{d,s,k} T R^d \left[ \alpha + \frac{y}{\eta T} + C_{d,\alpha} \left( \frac{\eta}{R} \right)^{d-1} + C_{d,\alpha} \theta^{(d-1)/2} \right]
\]
(299)

Then again, by assumption, \( \sin \theta > c_d y^{-1} \varepsilon \); by choosing \( c_d \) sufficiently large we may guarantee that
\[
Z_{s,s+k+1} [Z_s, t + \tau; t_1 + \tau, \ldots, t_k + \tau, 0; v_{s+1}, \ldots, v_{s+k}, v_{s+k+1}; \omega_1, \ldots, \omega_k, \omega_{k+1}; i_1, \ldots, i_k, i_{k+1}] \in G_{s+k+1} \cap \hat{U}^n_{s+k+1}
\]
whenever \( (\tau, v_{s+k+1}, \omega_{k+1}) \in A^+ \setminus B^+ \).

**Appendix B. Acknowledgements**

This paper is an expanded version of the author’s dissertation at New York University. I would like to thank my PhD advisor, Nader Masmoudi, for valuable guidance and immeasurable patience. I would also like to thank Laure Saint-Raymond, for reading multiple drafts of the paper and providing feedback which has improved the presentation significantly. I would also like to thank Clément Mouhot, Pierre Germain, and Pierre-Emmanuel Jabin, for delightful discussions and many insights into the mathematical universe. Finally, I would like to thank the anonymous reviewers for pointing out a number of necessary corrections and clarifications, which have improved the overall quality of the manuscript.

**References**

[1] R. K. Alexander, *The infinite hard-sphere system*, Ph.D. Thesis, 1975.
[2] C. Bardos and P. Degond, *Global existence for the Vlasov-Poisson equation in 3 space variables with small initial data*, Annales de l’institut Henri Poincaré (C) Analyse non linéaire 2 (1985), no. 2, 101–118.
[3] T. Bodineau, I. Gallagher, and L. Saint-Raymond, *From hard spheres dynamics to the Stokes-Fourier equations: an L² analysis of the Boltzmann-Grad limit*, arXiv:1511.03057 (2015).
[4] ________, *The Brownian motion as the limit of a deterministic system of hard spheres*, Invent. math. (2015).
[5] T. Bodineau, I. Gallagher, L. Saint-Raymond, and S. Simonella, *One-sided convergence in the Boltzmann-Grad limit*, arXiv:1612.03722 (2016), available at [1612.03722](https://arxiv.org/abs/1612.03722).

[6] L. Boudin and L. Desvillettes, *On the singularities of the global small solutions of the full Boltzmann equation*, Monatshefte für Mathematik 131 (2000), no. 2, 91–108.

[7] C. Cercignani, V. I. Gerasimenko, and D. Ya. Petrina, *Many-particle dynamics and kinetic equations*, Kluwer Academic Publishers, 1997.

[8] C. Cercignani, R. Illner, and M. Pulvirenti, *The mathematical theory of dilute gases*, Springer Verlag, 1994.

[9] Carlo Cercignani, *On the boltzmann equation for rigid spheres*, Transport Theory and Statistical Physics 2 (1972), no. 3, 211–225, available at [https://doi.org/10.1080/00411457208232538](https://doi.org/10.1080/00411457208232538).

[10] R. L. Dobrushin, *Vlasov equations*, Func. anal. and appl. 13 (1979), no. 2.

[11] I. Gallagher, L. Saint-Raymond, and B. Texier, *From Newton to Boltzmann: Hard spheres and short-range potentials*, Zurich Lec. Adv. Math. (2014).

[12] G. Gallavotti, *Divergences and the approach to equilibrium in the Lorentz and the wind-tree models*, Phys. Rev. 185 (1969), no. 1, 308–322.

[13] V. I. Gerasimenko, *Approaches to derivation of the Boltzmann equation with hard sphere collisions*, arXiv:1308.1789 (2013).

[14] H. Grad, *On the kinetic theory of rarefied gases*, Comm. Pure and Appl. Math. 2 (1949), 331–407.

[15] E. Hewitt and L. J. Savage, *Symmetric measures on Cartesian products*, Trans. Amer. Math. Soc. 80 (1955), 470–501.

[16] R. Illner, *On the number of collisions in a hard sphere particle system in all space*, Transport Theory and Stat. Phys. 18 (1989), no. 1, 71–86.

[17] R. Illner and M. Pulvirenti, *Global validity of the Boltzmann equation for a two-dimensional rare gas in vacuum*, Comm. Math. Phys. 105 (1986), no. 2, 189–203.

[18] Pierre-Emmanuel Jabin and Maxime Hauray, *Particles approximations of Vlasov equations with singular forces: Propagation of chaos.*, hal-00609453 (2011).

[19] M. Kac, *Foundations of kinetic theory*, Proceedings of the third Berkeley symposium on mathematical statistics and probability, volume 3: Contributions to astronomy and physics, 1956, pp. 171–197.

[20] F. King, *BGK hierarchy for positive potentials*, Ph.D. Thesis, 1975.

[21] O. E. Lanford, *Time evolution of large classical systems*, Dynamical systems, theory and applications, 1975, pp. 1–111.

[22] J. L. Lebowitz and H. Spohn, *Steady state self-diffusion at low density*, J. Stat. Phys. 29 (1982), no. 1, 39–55.

[23] H. P. McKean, *An exponential formula for solving Boltzmann’s equation for a Maxwellian gas*, J. Comb. Theory 2 (1967), no. 3, 358–382.

[24] Pierre-Emmanuel Jabin and Maxime Hauray, *Particles approximations of Vlasov equations with singular forces: Propagation of chaos.*, hal-00609453 (2011).

[25] M. Kac, *Foundations of kinetic theory*, Proceedings of the third Berkeley symposium on mathematical statistics and probability, volume 3: Contributions to astronomy and physics, 1956, pp. 171–197.

[26] F. King, *BBGKY hierarchy for positive potentials*, Ph.D. Thesis, 1975.

[27] O. E. Lanford, *Time evolution of large classical systems*, Dynamical systems, theory and applications, 1975, pp. 1–111.

[28] J. L. Lebowitz and H. Spohn, *Steady state self-diffusion at low density*, J. Stat. Phys. 29 (1982), no. 1, 39–55.

[29] H. P. McKean, *An exponential formula for solving Boltzmann’s equation for a Maxwellian gas*, J. Comb. Theory 2 (1967), no. 3, 358–382.

[30] Pierre-Emmanuel Jabin and Maxime Hauray, *Particles approximations of Vlasov equations with singular forces: Propagation of chaos.*, hal-00609453 (2011).

[31] M. Kac, *Foundations of kinetic theory*, Proceedings of the third Berkeley symposium on mathematical statistics and probability, volume 3: Contributions to astronomy and physics, 1956, pp. 171–197.

[32] F. King, *BBGKY hierarchy for positive potentials*, Ph.D. Thesis, 1975.
[30] ______, *Large scale dynamics of interacting particles*, Theoretical and Mathematical Physics, Springer-Verlag Berlin Heidelberg, 1991.

[31] H. van Beijeren, O. E. Lanford, J. L. Lebowitz, and H. Spohn, *Equilibrium time correlation functions in the low-density limit*, Journal of Statistical Physics **22** (1980), no. 2, 237–257.