MIXED RAY TRANSFORM ON SIMPLE 2-DIMENSIONAL RIEMANNIAN MANIFOLDS

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Abstract. We characterize the kernel of the mixed ray transform on simple 2-dimensional Riemannian manifolds, that is, on simple surfaces for tensors of any order.

1. Introduction

We provide a characterization of the kernel of the mixed ray transform on simple 2-dimensional Riemannian manifolds for tensors of any order. The key application pertains to elastic $qS$-wave tomography \cite{3} in weakly anisotropic media.

We let $(M,g)$ be a smooth, compact, connected 2-dimensional Riemannian manifold with smooth boundary $\partial M$. We assume that $(M,g)$ is simple, that is, $\partial M$ is strictly convex with respect to $g$ and $\exp_p:\exp^{-1}(M)\to M$ is a diffeomorphism for every $p\in M$. We let $SM=\{(x,v)\in TM;\|v\|_g=1\}$ be the unit sphere bundle. We use the notation $\nu$ for the outer unit normal vector field to $\partial M$. We write $\partial in(SM)=\{(x,v)\in SM;x\in \partial M,(v,\nu)_g\leq 0\}$ for the vector bundle of inward pointing unit vectors on $\partial M$. For $(x,v)\in SM$, $\gamma_{x,v}(t)$ is the geodesic starting from $x$ in direction $v$, and $\tau(x,v)$ is the time when $\tau_{x,v}$ exits $M$. Since $(M,g)$ is simple $\tau(x,v)<\infty$ for all $(x,v)\in \partial in(SM)$ and the exit time function $\tau$ is smooth in $\partial in(SM)$ \cite[Section 4.1]{15}.

We use the notation $S^kM$, $k\in \mathbb{N}$, for the space of smooth symmetric tensor fields on $M$. We also use the notation $S^kM\times S^\ell M$, $k,\ell \geq 1$ for the space of smooth tensor fields that are symmetric with respect to first $k$ and last $\ell$ variables. The mixed ray transform $L_{k,\ell}$ of a tensor field $f\in S^kM\times S^\ell M$ is given by the formula

\begin{equation}
L_{k,\ell}f(x,v) = \int_0^{\tau(x,v)} f_{i_1,\ldots,i_kj_1,\ldots,j_\ell}(\gamma(t))\dot{\gamma}(t)^{i_1}\cdots\dot{\gamma}(t)^{i_k}\eta(t)^{j_1}\cdots\eta(t)^{j_\ell}dt, (x,v)\in \partial in(SM), \gamma' = \gamma_{x,v},
\end{equation}

where we used the summation convention, while $\eta(t)$ is some unit length vector field on $\gamma$ that is parallel and perpendicular to $\dot{\gamma}(t)$ and depends smoothly on $(x,v)\in \partial in(SM)$. We note that the definition of the mixed ray transform is different in higher dimensions, due to the freedom in the choice of $\eta$ (See \cite[Section 7.2]{15}). We consider the choice of $\eta(t)$ and the mapping properties of $L_{k,\ell}$ in dimension 2.

We define two linear operators the images of which are contained in the kernel of $L_{k,\ell}$. For a $(k\times \ell)$-tensor, $f_{i_1,\ldots,i_kj_1,\ldots,j_\ell}$, we introduce the symmetrization operator as

\begin{equation}
(Sym(i_1,\ldots,i_k)f)_{i_1,\ldots,i_kj_1,\ldots,j_\ell} := \frac{1}{k!}\sum_\sigma f_{i_{\sigma(1)},\ldots,i_{\sigma(k)}j_1,\ldots,j_\ell},
\end{equation}

where $\sigma$ runs over all permutations of $(1,2,\cdots,k)$. This operator symmetrizes $f$ with respect to the first $k$ indices. We define the symmetrization operator $Sym(j_1,\ldots,j_\ell)$, for the last $\ell$ indices analogously.

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We introduce a first operator $\lambda$ the image of which is contained in the kernel of $L_{k,\ell}$. The operator $\lambda : S^{k-1}M \times S^{\ell-1}M \to S^{k}M \times S^{\ell}M$ is defined by

\begin{equation}
(\lambda w)_{i_1, \ldots, i_k, j_1, \ldots, j_\ell} := \text{Sym}(i_1, \ldots, i_k)\text{Sym}(j_1, \ldots, j_\ell)(g_{i_1 j_1}w_{i_2 \ldots, i_k j_2 \ldots j_\ell}).
\end{equation}

Using (2) and (3) it is straightforward to verify that

\begin{equation}
(\lambda w)_{i_1, \ldots, i_k, j_1, \ldots, j_\ell} v^{i_1} \cdots v^{ik}(v^\perp)^{j_1} \cdots (v^\perp)^{j_\ell} = 0, \quad v \in TM,
\end{equation}

where $v^\perp$ is any vector orthogonal to $v$. Therefore (4) implies that

$$\text{Im}(\lambda) \subset \ker(L_{k,\ell}).$$

We use the notation $u_{i_1, \ldots, i_k; \ell}$, for the ($h$) component functions of the covariant derivative $\nabla u$ of the tensor field $u$. We define the second operator, $d'$ say, by the formula,

\begin{equation}
d' : S^{k-1}M \times S^{\ell}M \to S^{k}M \times S^{\ell}M, \quad (d'u)_{i_1, \ldots, i_k, j_1, \ldots, j_\ell} := \text{Sym}(i_1, \ldots, i_k)u_{i_2 \ldots, i_k j_1 \ldots j_\ell}.
\end{equation}

Then the following holds for any $u \in S^{k-1}M \times S^{\ell}M$,

\begin{equation}
\frac{d}{dt} \left( u_{i_1, \ldots, i_k-1, j_1, \ldots, j_\ell}(\gamma(t))\dot{\gamma}(t)^i_1 \cdots \dot{\gamma}(t)^i_k-1 \eta(t)^i_\ell \cdots \eta(t)^i_\ell \right) = (d'u)_{i_1, \ldots, i_k, j_1, \ldots, j_\ell}(\gamma(t))\ddot{\gamma}(t)i_1 \cdots \dot{\gamma}(t)^i_k \eta(t)^i_\ell \cdots \eta(t)^i_\ell.
\end{equation}

If $u|_{\partial M} = 0$, then $L_{k,\ell}(d'u) = 0$ by the fundamental theorem of calculus. Thus

$$\{d'u : u \in S^{k-1}M \times S^{\ell}M, u|_{\partial M} = 0\} \subset \ker(L_{k,\ell}).$$

Our main result shows that the kernel of $L_{k,\ell}$ is spanned by the images of these two linear operators.

**Theorem 1.** Let $(M, g)$ be a simple 2-dimensional Riemannian manifold. Let $f \in S^{k}M \times S^{\ell}M$, $k, \ell \geq 1$. Then

$$L_{k,\ell}f(x, v) = 0, \quad (x, v) \in \partial_n(SM)$$

if and only if

$$f = d'u + \lambda w, \quad u \in S^{k-1}M \times S^{\ell}M, \quad u|_{\partial M} = 0, \quad w \in S^{k-1}M \times S^{\ell-1}M.$$

The key observation needed to prove this theorem is that the mixed ray transform and the geodesic ray transform can be transformed to one another, for arbitrary $k, \ell \geq 1$, if $(M, g)$ is a 2-dimensional simple Riemannian manifold. A similar observation has already been obtained for the transverse ray transform by Sharafutdinov [15, Chapter 5]. The work by Paternain, Salo and Uhlmann [9] proved the s-injectivity of the geodesic ray transform on simple manifolds in dimension 2. In Theorem 1 we characterize the kernel of $L_{k,\ell}$ using their results.

2. Relation with elastic $qS$-wave tomography

We describe a mixed ray transform arising from elastic wave tomography. We follow the presentation in [15, Chapter 7], wherein one can find more details. Let $(x^1, x^2)$ be any curvilinear coordinate system in $\mathbb{R}^2$, where the Euclidean metric is

$$ds^2 = g_{jk} dx^j dx^k.$$

The elastic wave equations

\begin{equation}
\rho \frac{\partial^2 u_j}{\partial t^2} = \sigma_{jk}^{\phantom{k}l} := \sigma_{jk; l}g^{kl}
\end{equation}
describes the waves traveling in a two-dimensional elastic body $M \subset \mathbb{R}^2$. Here $u(x, t) = (u^1, u^2)$ is the displacement vector. The strain tensor is given by

$$
\varepsilon_{jk} = \frac{1}{2} (u_{j;k} + u_{k;j}),
$$

while the stress tensor is

$$
\sigma_{jk} = C_{jklm} \varepsilon^{lm},
$$

where $C(x) = (C_{jklm})$ is the elastic tensor and $\rho(x)$ is the density of mass. Here $\varepsilon^{lm}$ is obtained by raising indices with respect to the metric $g_{jk}$. The elastic tensor has the following symmetry properties

$$
(C_{jklm} = C_{kjlm} = C_{lmjk}).
$$

We assume that the elastic tensor is weakly anisotropic, that is, it can be represented as

$$
C_{jklm} = \lambda g_{jk} g_{lm} + \mu (g_{jl} g_{km} + g_{jm} g_{kl}) + \delta c_{jklm},
$$

where $\lambda$ and $\mu$ are positive functions called the Lamé parameters, and $c = (c_{jklm})$ is an anisotropic perturbation. Here, $\delta$ is a small positive real number. We note here that $c = 0$ corresponds to an isotropic medium.

We construct geometric optics solutions to system (7) using the parameter $\omega = \omega_0 / \delta$,

$$
\begin{align*}
 u_j &= e^{i \omega \iota} \sum_{m=0}^{\infty} \frac{u^{(m)}_{j}}{(i \omega)^{m}}, \\
 \varepsilon_{jk} &= e^{i \omega \iota} \sum_{m=-1}^{\infty} \frac{\varepsilon^{(m)}_{jk}}{(i \omega)^{m}}, \\
 \sigma_{jk} &= e^{i \omega \iota} \sum_{m=-1}^{\infty} \frac{\sigma^{(m)}_{jk}}{(i \omega)^{m}},
\end{align*}
$$

where $\iota(x)$ is a real function.

We substitute the above solutions into equation (7), assume $u^{(-1)} = \varepsilon^{(-2)} = \sigma^{(-2)} = 0$ and equate the terms of the order $-2$ and $-1$ respectively in $\omega$, to obtain

$$
(\lambda + \mu) \langle u^{(0)}, \nabla \iota \rangle_g \nabla \iota + (\mu \| \nabla \iota \|_g^2 - \rho) u^{(0)} = 0.
$$

If we take

$$
\| \nabla \iota \|_g^2 = \frac{\rho}{\mu},
$$

then

$$
\langle u^{(0)}, \nabla \iota \rangle_g = 0.
$$

The solutions $u^{(0)}_j$ represent shear waves ($S$-waves), and the displacement vector $u^{(0)}$ is orthogonal to $\nabla \iota$. We denote $n_s = \rho / \mu$ and $v_s = 1 / n_s$. The characteristics of the eikonal equation (9) are geodesics of the Riemannian metric $n_s^2 ds^2 = n_s^2 g_{jk} dx^j dx^k$.

We choose a geodesic $\gamma$ of metric $n_s^2 ds^2$ and apply the change of variables,

$$
u^{(0)}_j = A_s n_s^{-1} \zeta_j,$$

where

$$A_s = \frac{C}{\sqrt{J \rho v_s}}, \quad J^2 = n_s^2 \det(g_{jk}), \quad C \text{ is a constant}.$$

Then it is shown in [15], Section 7.1.5, that $\zeta$ satisfies the following Rylov's law

$$
\left( \frac{D \zeta}{dt} \right)_j = -i \frac{1}{\rho v_s^6} (\delta_j^q - \hat{\zeta}_j \hat{\zeta}_q) \omega_0 c_{qklm} \hat{\zeta}^k \hat{\zeta}^m \zeta^l,
$$
where \( \frac{D}{\gamma} \) is the covariant derivative along \( \gamma \). We note that \( c_{\gamma \gamma^k \gamma^m} \) is quadratic in \( \gamma \), and symmetric in \( k,m \), so the solution \( \zeta \) of (10) depends only on the symmetrization

\[
 f_{jklm} = -i \frac{1}{4\rho \nu_k} (c_{jklm} + c_{jmlk}).
\]

We assume that for every unit speed geodesic \( \gamma : [a, b] \to M \) (in Riemannian manifold \( (M, n_s^2 ds^2) \)) with endpoints in \( \partial M \), the value \( \zeta(b) \) of a solution to equation (10) is known as \( \zeta(b) = U(\gamma)\zeta(a) \), where \( U(\gamma) \) is the solution operator of (10) and \( \eta(a) \) is the initial value. We formulate an inverse problem.

**Inverse Problem 1.** Determine tensor field \( f \) from \( U(\gamma) \).

We linearize this problem as in [15, Chapter 5]. Take a unit vector \( \xi(t) \perp \dot{\gamma}(t) \), which is also parallel along \( \gamma \). Then \( \xi_1(t) = \xi(t) \) and \( \xi_2(t) = \dot{\gamma}(t) \) form an orthonormal frame along \( \gamma \). In this basis, equation (10) is

\[
 \dot{\zeta}_1 = -i \frac{1}{\rho \nu_\gamma} \omega_0 \hat{c}_{111m} \dot{\gamma}^{[l} \hat{\gamma}^{m]} \zeta_1^l, \quad \dot{\zeta}_2 = 0.
\]

We denote \( F(t) = -i \frac{1}{\rho \nu_\gamma} \omega_0 \hat{c}_{111m}(\gamma(t)) \dot{\gamma}^{[l}(t) \dot{\gamma}^{m]}(t) \). Since (11) is a separable first order ordinary differential equation, its solution is

\[
 \zeta_1(b) = e^{\int_a^b F(t) dt} \zeta_1(a).
\]

We take the first-order Taylor expansion of the right-hand side of the equation above to obtain

\[
 \zeta_1(b) - \zeta_1(a) \sim \int_a^b F(t) \zeta_1^1(a) dt.
\]

Multiplying this equation by \( \eta^1(a) \), we get

\[
 (\zeta_1(b) - \zeta_1(a)) \eta_1^1(a) \sim \int_a^b F(t) \zeta_1^1(a) \eta_1^1(a) dt = \int_a^b \omega_0 f_{111m}(\gamma(t)) \zeta_1^1(a) \zeta_1^1(a) \eta_1^1(t) \dot{\gamma}^m(t) dt.
\]

We denote the vector field \( \eta(t) = \zeta_1^1(a) \xi_1(t), \zeta_2^1(a) = 0 \), and observe that it is parallel along \( \gamma \) and perpendicular to \( \dot{\gamma}(t) \). The right-hand side of (12) then takes the form

\[
 \int_a^b \omega_0 f_{111m}(\gamma(t)) \eta_1^1(t) \eta_1^1(t) \dot{\gamma}^l(t) \dot{\gamma}^m(t) dt,
\]

We arrive at the inverse problem.

**Inverse Problem 2.** Determine the tensor field \( f \) from

\[
 L_{2,2}(f) = \int_a^b f_{jklm}(\gamma(t)) \eta_1^j(t) \eta_1^k(t) \dot{\gamma}^l(t) \dot{\gamma}^m(t) dt
\]

for all \( \gamma \) and \( \eta \perp \gamma \), where \( \eta \) is parallel along \( \gamma \).

**Remark 1.** The tensor field \( f \) possesses the same symmetry properties \( S \) as \( C \). Therefore \( f \in S^2 M \times S^2 M \). Since

\[
 L_{2,2}(f + du + \lambda w) = L_{2,2}(f), \quad \text{for any } u \in S^1 M \times S^2 M, \ w \in S^1 M \times S^1 M,
\]

we can only recover the tensor \( f \) up to the kernel of \( L_{2,2} \). Thus the Inverse Problem \( 2 \) is a special case of Theorem \( 3 \).
3. Context and previous work

We note that if \( \ell = 0 \) in \([11]\), the operator \( L_{k,0} \) is the geodesic ray transform \( I_k \) for a symmetric \( k \)-tensor \( f \). It is well known that \( \text{Sym}(i_1, \ldots, i_k)\nabla u \) is in the kernel of \( I_k \), where \( u \) is a symmetric \( (k-1) \)-tensor with \( u|_{\partial \Omega} = 0 \). If \( I_k f = 0 \) implies \( f = \text{Sym}(i_1, \ldots, i_k)\nabla u \), we say \( I_k \) is s-injective.

When \((M, g)\) is a 2-dimensional simple manifold, Paternain, Salo and Uhlmann \([9]\) proved the s-injectivity of \( I_k \) for arbitrary \( k \). The standard way to prove s-injectivity of \( I_0 \) and \( I_1 \) is to use an energy identity known as the Pestov identity. If \( k \geq 2 \) this identity alone is not sufficient to prove the s-injectivity. The special case \( k = 2 \) was proved earlier \([14]\) using the proof for boundary rigidity \([13]\).

In dimension three or higher, it has been proved that \( I_0 \) is injective \([6, 7]\), and \( I_1 \) is s-injective \([2]\). The s-injectivity of \( I_k \) for \( k \geq 2 \) is still open for simple Riemannian manifolds. Under certain curvature conditions, the s-injectivity of \( I_k \), \( k \geq 2 \) has been proved in \([4, 11, 12, 15]\). Without any curvature condition, it has been proved that \( I_2 \) has a finite-dimensional kernel \([16]\). If \( g \) is in a certain open and dense subset of simple metrics in \( C^r, r \gg 1 \), containing analytic metrics, the s-injectivity is proved by analytic microlocal analysis for \( k = 2 \) \([17]\). Under a different assumption that \( M \) can be foliated by strictly convex hypersurfaces, the s-injectivity has been established for \( m = 0 \) \([20]\), and \( m = 1, 2 \) \([18]\).

The mixed ray transform \((\ell \neq 0, k \neq 0)\) is not studied as extensively as the geodesic ray transform. In dimension two or higher, a result similar to Theorem \([11]\) has been obtained under a restrictive curvature condition \([15]\).

When \( k = 0 \), \( L_{0,\ell} \) is called the transverse ray transform, also denoted by \( J_\ell \). For \( J_\ell \), the situations are quite different for dimension two and higher dimensions. In dimension three or higher, \( J_\ell \) is injective for \( \ell < \dim M \) under certain curvature conditions \([15]\). However, \( J_\ell \) has a nontrivial kernel in dimension 2. This problem is related to polarization tomography, for which some results are given under different conditions \([5, 8, 10]\).

4. Proof of Theorem \([11]\)

Since \((M, g)\) is a 2-dimensional simple Riemannian manifold, there exists a diffeomorphism \( \phi \) from \( M \) onto a closed unit disc \( \mathbb{B} \) of \( \mathbb{R}^2 \). If \( g' \) is the pullback of metric \( g \) under \( \phi^{-1} \) on \( \mathbb{B} \) then \( g' \) is conformally Euclidean, meaning that there exists a change of coordinates after which \( g' = he \), where \( h \) is some positive function and \( e \) is the Euclidean metric; this was shown in \([11]\) Theorem 4] and \([19]\) Proposition 1.3. Therefore there exists global isothermal coordinates \((x_1, x_2)\) on \( M \), so that the metric \( g \) can be written as \( e^{2\alpha(x)}(dx_1^2 + dx_2^2) \) where \( \alpha(x) \) is a smooth real-valued function of \( x \).

The global isothermal coordinate structure makes it possible to define a smooth rotation,

\[
\sigma : TM \to TM, \quad \sigma(v) := (v_2, -v_1),
\]

where \( v = (v_1, v_2) \) in these coordinates. This map satisfies

\[
v \perp \sigma(v) \quad \text{and} \quad \|v\|_g = \|\sigma(v)\|_g.
\]

Moreover, there exists a linear map

\[
(\Phi f)(x, v) := f_{i_1\ldots i_k j_1\ldots j_l}(x) v^{i_1} \ldots v^{i_k} \sigma(v)^{j_1} \ldots \sigma(v)^{j_l}.
\]

Thus each tensor field \( f \in S^k M \times S^l M \) is related to a smooth function on \( SM \) via \([14]\). We note that \( \Phi \) is not one-to-one since \( \Phi(\lambda w) = 0 \) for any \( w \in S^{k-1} M \times S^{l-1} M \), where \( \lambda \) is as in \([3]\). We have the following
Lemma 1. For any $f \in S^k M \times S^\ell M$ it holds that

$$L_{k,\ell} f(x, v) = \int_0^{\tau(x,v)} (\Phi f)(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)) dt, \quad (x, v) \in \partial_M(SM)$$

and

$$L_{k,\ell} : S^k M \times S^\ell M \to C^\infty(\partial_M SM),$$

if we assume that

$$\eta(0) = \sigma(v), \quad (x, v) \in \partial_M(SM).$$

Proof. Let $(x, v) \in \partial_M(SM)$. We define $\eta = \sigma(v)$. Let $P_t(\eta)$ be the parallel transport of $\eta$ from $T_x M$ to $T_{\gamma_{x,v}(t)} M$, $t \in [0, \tau(x,v)]$. By the property of parallel translation, $P_t : T_x M \to T_{\gamma_{x,v}(t)} M$ is an isometry, whence $\|P_t\eta\|_g = 1$ and $\langle P_t\eta, \dot{\gamma}(t) \rangle_g = 0$. Since $M$ is 2-dimensional, the continuity of $P_t\eta$ in $t$ with \([13]\) imply

$$P_t\eta = \sigma(\dot{\gamma}_{x,v}(t)).$$

Because the functions $\Phi f$ and $\tau$ are smooth in $\partial_M(SM)$, the function $L_{k,\ell}(f)$ is smooth in $\partial_M(SM)$ due to \([15]\). \hfill \Box

Let $f \in S^k M \times S^\ell M$. Simplifying the notation, from here on we do not distinguish tensor $f$ from function $\Phi(f)$. We notice first that

$$f(x, v) = (-1)^{\ell-N(j_1, \ldots, j_\ell)} f_{i_1, \ldots, i_k j_1, \ldots, j_\ell}(x) v^{i_1} \cdots v^{i_k} v_1^{\ell-N(j_1, \ldots, j_\ell)} v_2^{N(j_1, \ldots, j_\ell)}, \quad (x, v) \in SM,$$

where $N(j_1, \ldots, j_\ell)$ is the number of 1s in $(j_1, \ldots, j_\ell)$. We let $\delta$ be the map that maps 1s in $(j_1, \ldots, j_\ell)$ to 2s and vice versa. We denote by $\delta(j_1, \ldots, j_\ell)$ the $\ell$-tuple obtained from applying $\delta$ to $(j_1, \ldots, j_\ell)$. Then we define a linear operator

$$A : S^k M \times S^\ell M \to S^k M \times S^\ell M, \quad (Af)_{i_1, \ldots, i_k j_1, \ldots, j_\ell} = (-1)^{\ell-N(j_1, \ldots, j_\ell)} f_{i_1, \ldots, i_k \delta(j_1, \ldots, j_\ell)}.$$

We note that if $\ell = 1$, then $A$ and the Hodge star operator coincide. Formula \([17]\) implies that $A$ is invertible with the following inverse

$$A^{-1} = (-1)^\ell A.$$

We then point out that

$$\quad (Af)_{i_1, \ldots, i_k j_1, \ldots, j_\ell}(x) v^{i_1} \cdots v^{i_k} v^{j_1} \cdots v^{j_\ell} = (\text{Sym} Af)_{i_1, \ldots, i_k j_1, \ldots, j_\ell}(x) v^{i_1} \cdots v^{i_k} v^{j_1} \cdots v^{j_\ell}.$$

The notation Sym$h$ stands for the full symmetrization of the tensor field $h$.

Using equations \([16]\), \([17]\) and \([19]\), we find that

$$L_{k,\ell}(f) = I_{k+\ell}(\text{Sym}(Af)),$$

where $I_{k+\ell}$ is the geodesic ray transform on symmetric tensor field $h \in S^{k+\ell}M$, defined by the formula

$$I_{k+\ell}(h)(x, v) = \int_0^{\tau(x,v)} h_{i_1, \ldots, i_{k+\ell}}(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)) \gamma_{x,v}(t)^{i_1} \cdots \gamma_{x,v}(t)^{i_{k+\ell}} dt, \quad (x, v) \in \partial_M(SM).$$

By \([20]\) and \([9, \text{Theorem 1.1}]\) it holds that for any $h \in S^k M \times S^\ell M,$

$$L_{k,\ell}(h) = 0 \text{ if and only if } \text{Sym} Ah = d^s v, \quad v \in S^{k+\ell-1} M, \quad v|_{\partial M} = 0.$$

In the above, $d^s$ stands for the inner derivative, that is, the symmetrization of the covariant derivative

$$d^s u = \text{Sym}(\nabla u), \quad u \in S^{k+\ell-1} M.$$
If $L_{k,\ell}(f) = 0$ then, with (21) and (24), we can write
\[
 f = (-1)^{\ell} A(\text{Sym}(Af) + (Af - \text{Sym}(Af))) = (-1)^{\ell} A(d^s u) + f + (-1)^{\ell+1} A(\text{Sym}(Af)).
\]
We conclude that the claim of Theorem 1 holds if
\[
 f + (-1)^{\ell+1} A(\text{Sym}(Af)) = \lambda w, \quad A(d^s u - d'u) = \lambda w', \quad d'A = Ad',
\]
for some $w, w' \in S^{k-1}M \times S^{\ell-1}M$ and $u \in S^{k+\ell-1}M$. These equations will be proved in the following subsections.

4.1. **Analysis of operator $A\text{Sym}A$.** In this subsection, we prove the following identity for any $f \in S^kM \times S^\ell M$:

\[
 (23) \quad f + (-1)^{\ell+1} A(\text{Sym}(Af)) = \lambda w \quad \text{for some } w \in S^{k-1}M \times S^{\ell-1}M.
\]

We start with a lemma that characterizes the kernel of $A\text{Sym}A$

**Lemma 2.** For the linear maps $A\text{Sym}A : S^kM \times S^\ell M \to S^kM \times S^\ell M$ and $\lambda : S^{k-1}M \times S^{\ell-1}M \to S^kM \times S^\ell M$ the following holds

\[
 \ker(A\text{Sym}A) = \text{Im}(\lambda).
\]

**Proof.** We use the notation $\otimes_s$ for the symmetric product of tensors. We note that operator $A$ maps a basis element \((\otimes^h dx_1) \otimes_s (\otimes^{k-h} dx^2) \otimes (\otimes^a dx_1) \otimes_s (\otimes^{\ell-a} dx^2), h \in \{0, \ldots, k\}, a \in \{0, \ldots, \ell\}\) of $S^kM \times S^\ell M$ to

\[
 (-1)^{\ell-a} ((\otimes^h dx_1) \otimes_s (\otimes^{k-h} dx^2)) \otimes ((\otimes^{\ell-a} dx_1) \otimes_s (\otimes^a dx^2)).
\]

We also note that the choice of isothermal coordinates implies
\[
 (24) \quad \lambda(a \otimes b) = e^{2\alpha(x)} ((dx_1 \otimes_s a) \otimes (dx_1 \otimes_s b)) + (dx^2 \otimes_s a) \otimes (dx^2 \otimes_s b), \quad a \otimes b \in S^{k-1}M \times S^{\ell-1}M.
\]

Since $A$ is a bijection, it suffices to prove

\[
 (25) \quad \text{Im}(\lambda) = \ker(\text{Sym}A).
\]

We prove first that $\text{Im}(\lambda) \subset \ker(\text{Sym}A)$. In view of the linearity of $\lambda$, it suffices to prove that $\lambda w \in \ker \text{Sym}A$ when

\[
 w = r(x)((\otimes^{h-1} dx_1) \otimes_s (\otimes^{k-h} dx^2)) \otimes ((\otimes^{a-1} dx_1) \otimes_s (\otimes^{\ell-a} dx^2)), \quad h \in \{1, \ldots, k\}, a \in \{1, \ldots, \ell\}.
\]

Then
\[
 (26) \quad e^{-2\alpha(x)} A\lambda w = (-1)^{\ell-a} r(x) \left( ((\otimes^h dx_1) \otimes_s (\otimes^{k-h} dx^2)) \otimes ((\otimes^{\ell-a} dx_1) \otimes_s (\otimes^a dx^2)) \right.
\]
\[
 - \left. ((\otimes^{h-1} dx_1) \otimes_s (\otimes^{k+h+1} dx^2)) \otimes ((\otimes^{\ell-a+1} dx_1) \otimes_s (\otimes^{a-1} dx^2)) \right).
\]

Since $\text{Sym}$ is a linear operator, we have $\text{Sym}A(\lambda w) = 0$. Therefore $\text{Im}(\lambda) \subset \ker(\text{Sym}A)$.

Now we prove that $\ker(\text{Sym}A) \subset \text{Im}(\lambda)$. We assume first that $f = \sum_{m=1}^M u_m$, where

\[
 (27) \quad u_m = r_m(x)((\otimes^h dx_1) \otimes_s (\otimes^{k-h} dx^2)) \otimes ((\otimes^{\ell-a} dx_1) \otimes_s (\otimes^a dx^2)), \quad h + a \leq \min\{k, \ell\}.
\]

Then we can write $f = \sum_{H=0}^{k+\ell} f_H$, where $f_H = 0$, if $H \geq \min\{k, \ell\}$ and otherwise

\[
 f_H = \sum_{h=0}^{H} a_{H,h} f_{H,h}, \quad f_{H,h} := ((\otimes^h dx_1) \otimes_s (\otimes^{k-h} dx^2)) \otimes ((\otimes^{\ell-(H-h)} dx_1) \otimes_s (\otimes^{H-h} dx^2)).
\]
Moreover \( f \in \ker(\text{Sym}A) \) if and only if \( f_{2j} \in \ker(\text{Sym}A) \) for every \( H \in \{1, \ldots, \min\{k, \ell\}\} \). In the following we study the tensor \( f_H \), for a given \( H \in \{1, \ldots, \min\{k, \ell\}\} \).

For \( h \in \{1, \ldots, H\} \) we define \( w_h \in S^{k-1}M \times S^{\ell-1}M \) by formula
\[
 w_h = \left( (\otimes^h dx^1) \otimes_s (\otimes^{k-h} dx^2) \right) \otimes \left( (\otimes^{\ell-(H-h)} dx^1) \otimes_s (\otimes^{H-h} dx^2) \right).
\]

Then (24) yields
\[
 \lambda w_h = e^{2\alpha(x)}(f_{H,h} + f_{H,h-1}).
\]
This implies the recursive formula
\[
 f_{H,h} = \lambda(e^{-2\alpha(x)}w_h) - f_{H,h-1}.
\]
Thus for every \( h \in \{0, \ldots, H\} \) there exists \( w'_h \in S^{k-1}M \times S^{\ell-1}M \) such that
\[
 (28) \quad f_{H,h} = \lambda w'_h + (-1)^h f_{H,0}.
\]
Therefore there exists \( w_H \in S^{k-1}M \times S^{\ell-1}M \) such that
\[
 f_H = \sum_{h=0}^H a_{H,h}f_{H,h} = \lambda w_H + f_{H,0} \sum_{h=0}^H (-1)^h a_{H,h}.
\]
If \( f \in \ker\text{Sym}A \) it holds by the first part of this proof that
\[
 \text{Sym}Af_H = (\text{Sym}Af_{H,0}) \left( \sum_{i=0}^H (-1)^h a_{H,h} \right) = 0.
\]
Since \( \text{Sym}Af_{H,0} \neq 0 \) it follows that \( \sum_{i=0}^H (-1)^h a_{H,h} = 0 \) whence \( f_H = \lambda w_H \). This implies \( f = \lambda w \) for some \( w \in S^{k-1}M \times S^{\ell-1}M \).

If \( f \in \ker\text{Sym}A \) and we cannot write \( f = \sum_{m=1}^M u_m \), where each \( u_m \) satisfies (27), then there exists \( u_m \) that satisfies
\[
 \left( (\otimes^h dx^1) \otimes_s (\otimes^{k-h} dx^2) \right) \otimes \left( (\otimes^{\ell-a} dx^1) \otimes_s (\otimes^a dx^2) \right), \quad \min\{k, \ell\} < h + a \leq \max\{k, \ell\}.
\]
Therefore \( f_H \neq 0 \) for some \( \min\{k, \ell\} < H \leq \max\{k, \ell\} \) and there exist two sub cases. If \( k < H \leq \ell \), then
\[
 f_H = \sum_{h=0}^k a_{H,h}f_{H,h}, \quad f_{H,h} = \left( (\otimes^h dx^1) \otimes_s (\otimes^{k-h} dx^2) \right) \otimes \left( (\otimes^{\ell-(H-h)} dx^1) \otimes_s (\otimes^{H-h} dx^2) \right).
\]
If \( \ell < H \leq k \), then
\[
 f_H = \sum_{h=0}^\ell a_{H,h}f_{H,h}, \quad f_{H,h} = \left( (\otimes^{H-h} dx^1) \otimes_s (\otimes^{k-h} dx^2) \right) \otimes \left( (\otimes^{\ell} dx^1) \otimes_s (\otimes^{H-h} dx^2) \right).
\]
By an analogous recursive argument as before, we find that \( f = \lambda w \), for some \( w \in S^{k-1}M \times S^{\ell-1}M \). This completes the proof. \( \square \)

By the proof of the previous Lemma we can write any \( f \in S^k M \times S^\ell M \) in the form
\[
 f = \lambda w + \sum_{H=0}^{k+\ell} r_H f_{H,0}, \quad r_H \in C^\infty(M),
\]
for some \( w \in S^{k-1}M \times S^{\ell-1}M \). Next, we prove that
\[
 (30) \quad A\text{Sym}Af_{H,0} = (-1)^\ell f_{H,0} + \lambda w, \quad H \in \{1, \ldots, k + \ell\}.
\]
We assume first that \( H \leq \min\{k, \ell\} \). Then
\[
f_{H,0} = (\otimes^k dx^2) \otimes ((\otimes^{\ell-H} dx^1) \otimes_s (\otimes^H dx^2)).
\]
This implies
\[
\text{Sym}Af_{H,0} = (-1)^{\ell} (\otimes^H dx^1 \otimes_s (\otimes^{k+\ell-H} dx^2))
\]
\[
= (-1)^{\ell} \frac{1}{(k+\ell)!} \sum_{h=0}^H A_h (\otimes^h dx^1 \otimes_s (\otimes^{k-h} dx^2)) \otimes (\otimes^{H-h} dx^1 \otimes_s (\otimes^{\ell-H+h} dx^2)),
\]
where \( \sum_{h=0}^H A_h = (k+\ell)! \). Using \( (28) \) we obtain
\[
ASymAf_{H,0} = (-1)^{\ell} \frac{1}{(k+\ell)!} \sum_{h=0}^H (-1)^h A_h f_{H,h} = (-1)^{\ell} \frac{1}{(k+\ell)!} \left( \sum_{h=0}^H A_h \right) f_{H,0} + \lambda w
\]
\[
= (-1)^{\ell} f_{H,0} + \lambda w.
\]
If \( \min\{k, \ell\} < H \leq \max\{k, \ell\} \) it follows by a similar argument that \( ASymAf_{H,0} = (-1)^{\ell} f_{H,0} + \lambda w \).
Therefore, we proved \( (30) \).

Equation \( (23) \) follows from Lemma \( (2) \) and \( (29)-(30) \).

4.2. **Analysis of operator** \( Ad^s \). We note that \( S^{k+\ell} M \subset S^k M \times S^\ell M \). Therefore, we can extend the inner derivative, \( d^s \), to an operator \( d^s : S^{k-1} M \times S^\ell M \to S^k M \times S^\ell M \) and evaluate \( d^s - d' \). In this subsection, we show that for any \( u \in S^{k-1} M \times S^\ell M \) the following equations hold,
\[
A(d^s u - d'u) = \lambda w \quad \text{for some} \ w \in S^{k-1} M \times S^\ell M;
\]
\[
d'A = Ad'.
\]
Since \( Ad^s \) and \( Ad' \) are linear it suffices to prove the claims for
\[
u = r(x) (\otimes^{h-1} dx^1) \otimes_s (\otimes^{k-h} dx^2) \otimes ((\otimes^a dx^1) \otimes_s (\otimes^{\ell-a} dx^2)), \quad r \in C^\infty(M).
\]
By \( (15) \) and \( (17) \) we have
\[
(33) \quad Ad'u = (-1)^{\ell-a} \left( \frac{\partial}{\partial x^1} r(x) - R_1 \right) \left( (\otimes^h dx^1) \otimes_s (\otimes^{k-h} dx^2) \otimes ((\otimes^a dx^1) \otimes_s (\otimes^a dx^2))
\]
\[
+ \left( \frac{\partial}{\partial x^2} r(x) - R_2 \right) \left( (\otimes^{h-1} dx^1) \otimes_s (\otimes^{k-h+1} dx^2) \otimes ((\otimes^a dx^1) \otimes_s (\otimes^a dx^2)) \right),
\]
where \( R_m = \sum_{s=1}^{k+\ell} r_{i_1, \ldots, i_{s-1}, p, i_{s+1}, \ldots, i_{k+\ell}} \Gamma_{p i s}^{m}, \ m \in \{1, 2\} \) and \( r_{i_1, \ldots, i_{s-1}, p, i_{s+1}, \ldots, i_{k+\ell}} \in \{0, r\} \) depending on \( (i_1, \ldots, i_{k+\ell}) \).

We write \( H = h + a \), assume that \( H \leq \min\{k, \ell\} \) and denote \( \tilde{R}_m = \frac{\partial}{\partial x^m} r(x) - R_m \). Then we obtain from \( (17) \) and \( (22) \),
\[
d^s u = \tilde{R}_1 \frac{1}{(k+\ell)!} \sum_{j=0}^H A_j (\otimes^j dx^1) \otimes_s (\otimes^{k-j} dx^2) \otimes ((\otimes^{H-j} dx^1) \otimes_s (\otimes^{\ell+j-H} dx^2))
\]
\[
+ \tilde{R}_2 \frac{1}{(k+\ell)!} \sum_{i=0}^{H-1} B_i (\otimes^i dx^1) \otimes_s (\otimes^{k-i} dx^2) \otimes ((\otimes^{H-i-1} dx^1) \otimes_s (\otimes^{\ell+i-H+1} dx^2)),
\]
Thus \( \lambda v \) and we define

\[
(34)
\]

Finally we prove equation (32). We note that

\[
\text{Ad}^{-1}u = (\log H)^{-1} \sum_{j=0}^{H} (-1)^{\ell-H+j} A_j\left((\otimes^j dx^1) \otimes_s (\otimes^{k-j} dx^2)\right) \otimes \left((\otimes^{\ell+j-H} dx^1) \otimes_s (\otimes^{H-j} dx^2)\right)
\]

We define

\[
g_{H,j} = \left((\otimes^j dx^1) \otimes_s (\otimes^{k-j} dx^2)\right) \otimes \left((\otimes^{\ell+j-H} dx^1) \otimes_s (\otimes^{H-j} dx^2)\right), \quad j \in \{0, \ldots, H\},
\]

and

\[
v_{H,j} = \left((\otimes^j dx^1) \otimes_s (\otimes^{k-j-1} dx^2)\right) \otimes \left((\otimes^{\ell+j-H} dx^1) \otimes_s (\otimes^{H-j-1} dx^2)\right), \quad j \in \{1, \ldots, H\}.
\]

Then (24) implies that \( \lambda v_{H,j} = e^{2\alpha(x)}(g_{H,j} + g_{H,j+1}) \). We obtain

\[
g_{H,j} = \lambda w_{H,j} + (-1)^{H-j} g_{H,H}, \quad \text{for some } w_j \in S^{k-1}M \times S^{\ell-1}M.
\]

Thus

\[
dA' u = (-1)^{\ell-a}\left(\tilde{R}_1 g_{H,H} + \tilde{R}_2 g_{H-1,H-1}\right)
\]

Then (32) holds since the previous equation coincides with (33).


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