ORDER COMPLEXES OF COSET POSETS OF FINITE GROUPS ARE NOT CONTRACTIBLE

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ABSTRACT. We show that the order complex of the poset of all cosets of all proper subgroups of a finite group $G$ is never $\mathbb{F}_2$-acyclic and therefore never contractible. This settles a question of K. S. Brown.

1. Introduction

We settle a question asked by K. S. Brown in [9]. For a group $G$, $\mathcal{C}(G)$ will denote the poset of all cosets of all proper subgroups of $G$, ordered by inclusion. For a poset $P$, $\Delta P$ will denote the order complex of $P$. Other terms used but not defined in this introduction are defined in Section 2.

**Theorem 1.1.** If $G$ is a finite group, then $\Delta \mathcal{C}(G)$ is not $\mathbb{F}_2$-acyclic, and therefore is not contractible.

With some explicitly stated exceptions, the groups, partially ordered sets and simplicial complexes considered herein are assumed to be finite. We assume some familiarity with topological combinatorics (see for example [4, 44]), along with the rudiments of algebraic topology (see for example [20, 28]) and group theory (see for example [2, 14]).

1.1. History and motivation. The topology of $\Delta \mathcal{C}(G)$ was studied by Brown in [9]. More general coset complexes were studied from a somewhat different point of view by Abels and Holz in [1]. However, from our perspective (and that of Brown), the story begins with the work of P. Hall, who in [19] introduced generalized Möbius inversion in order to enumerate generating sequences. Hall considered the probability $P_G(k)$ that a $k$-tuple $(g_1, \ldots, g_k)$ of elements of a group $G$, chosen uniformly with replacement, includes a generating set for $G$. He showed that

$$P_G(k) = \sum_{H \leq G} \mu(H, G) [G : H]^{-k},$$

where $\mu$ is the Möbius function on the subgroup lattice of $G$. (We mention that Weisner introduced generalized Möbius inversion independently in [45]. See [42, Chapter 3] for a comprehensive discussion of this theory.)
Bouc observed that $-P_G(-1)$ is the reduced Euler characteristic $\tilde{\chi}(\Delta C(G))$. Indeed, Hall showed in [19] that if $\hat{P}$ is obtained from $P$ by adding a minimum element $\hat{0}$ and a maximum element $\hat{1}$, then

$$\tilde{\chi}(\Delta P) = \mu_{\hat{P}}(0, \hat{1}).$$

A straightforward computation shows that

$$\mu_{\tilde{C}(G)}(0, \hat{1}) = -P_G(-1).$$

This led to Brown's work, in which he obtained divisibility results for $P_G(-1)$ using group actions on $\Delta C(G)$.

Brown found no group $G$ for which $P_G(-1) = 0$. As the reduced Euler characteristic of a contractible complex is zero, the question of contractibility arises naturally. Previous progress on this question involved showing that $P_G(-1) \neq 0$. Gaschütz showed in [17, Satz 2] that $P_G(-1) \neq 0$ when $G$ is solvable. (Brown refined this result by calculating the homotopy type of $\Delta C(G)$ for a solvable group $G$ in [9, Proposition 11].) Patassini proved $P_G(-1) \neq 0$ for many almost simple groups $G$ in [30, 31]. He obtained further results for some groups with minimal normal subgroups that are products of alternating groups in [34]. The question of whether $P_G(-1)$ is nonzero for all (finite) $G$ remains open.

Abels and Holz consider in [1] a more general class of posets. Let $G$ be a (possibly infinite) group and $\mathcal{H}$ be a collection of proper subgroups of $G$ that is closed under intersection. Abels and Holz study the order complex of the poset $\mathcal{C}_\mathcal{H}(G)$ of all cosets of all subgroups in $\mathcal{H}$. In their Theorem 2.4, they describe relations between connectivity properties of $\Delta \mathcal{C}_\mathcal{H}(G)$ and the structure of $G$. Our Theorem 1.1 says that $\Delta \mathcal{C}_\mathcal{H}(G)$ is not infinitely connected when $G$ is finite and $\mathcal{H}$ contains all proper subgroups of $G$. In contrast, Ramras in [36, Remark 2.4] noticed that $\Delta C(G)$ is contractible when $G$ is not finitely generated.

Some other papers on the topology of $\Delta C(G)$ are [13, 47, 48].

1.2. A brief description of our proof. Our proof has three main ingredients, namely, a "join theorem" of Brown, the Classification of Finite Simple Groups, and P. A. Smith Theory.

Brown showed that, given a group $G$ and normal subgroup $N$, there is a subposet $\mathcal{C}(G, N)$ of $\mathcal{C}(G)$ such that $\Delta \mathcal{C}(G)$ is homotopy equivalent to the join $\Delta \mathcal{C}(G, N) \ast \Delta \mathcal{C}(G/N)$. This result allows us to use induction on $|G|$. We complete the proof by showing that $\Delta \mathcal{C}(G, N)$ is not $\mathbb{F}_2$-acyclic when $N$ is a minimal normal subgroup of $G$. Such a subgroup is a direct product of pairwise isomorphic simple groups.

In order to show $\Delta \mathcal{C}(G, N)$ is not $\mathbb{F}_2$-acyclic, we use Smith Theory and the Classification. For each possible minimal normal subgroup $N$, we describe a group $E$ such that $E$ acts on $\mathcal{C}(G, N)$ with no fixed point. Using results of
Smith and Oliver, we choose $E$ so as to preclude a fixed-point-free action on an $\mathbb{F}_2$-acyclic complex.

1.3. Further comments.

1.3.1. Theorem 1.1 stands in clear contrast to other results on order complexes of posets naturally associated to finite groups. Consider the poset $L(G)$ of nontrivial proper subgroups of $G$, ordered by inclusion. The complex $\Delta L(G)$ is contractible for many groups, including all those with nontrivial Frattini subgroup.

Next, let $p$ be a prime. Consider the subposet $S_p(G)$ of $L(G)$ consisting of all $p$-subgroups and the subposet $A_p(G)$ of $S_p(G)$ consisting of all elementary abelian $p$-subgroups. The order complexes $\Delta S_p(G)$ and $\Delta A_p(G)$ were first studied, respectively, by Brown in [6, 7] and Quillen in [35]. These two complexes are homotopy equivalent. They are contractible when $G$ has a nontrivial normal $p$-subgroup. The converse of this last statement is a well-known conjecture of Quillen, see [35, Conjecture 2.9].

1.3.2. The identity $-P_G(-1) = \tilde{\chi}(\Delta L(G))$ can be considered to be an example of the phenomenon of combinatorial reciprocity. Often some objects of interest are counted by evaluating an appropriate function at positive integers. Combinatorial reciprocity occurs when evaluation of the same function at negative integers counts some closely related objects. Combinatorial reciprocity is discussed in [3, 40, 41, 42]. At this point we know of no interesting interpretation of $P_G(-n)$ for integers $n > 1$. More generally, one can evaluate $P_G$ at any complex number $s$. The study of $P_G(s)$ as a complex function was initiated by Boston in [5] and Mann in [27], and was continued by various authors. See for example [10, 11, 12, 13, 30, 32, 33, 38].

1.3.3. In Lemma 3.17 (3) below, we note that if $L$ is a finite simple group of Lie type or a sporadic simple group, then there exists some odd prime $p$ such that $L = \langle P, R \rangle$ whenever $P$ is a Sylow $p$-subgroup of $L$ and $R$ is a Sylow 2-subgroup of $L$. This property need not hold when $L$ is an alternating group.

However, one can ask whether for each $n$ there exist primes $p = p(n)$ and $r = r(n)$ such that $\langle P, R \rangle = A_n$ whenever $P$ is a Sylow $p$-subgroup and $R$ is a Sylow $r$-subgroup of $A_n$. This interesting question remains open. It is related to a question raised by Dolfi, Guralnick, Herzog and Praeger in [15, Section 6]. These authors ask whether for each $n$ there exist conjugacy classes $C, D$ in $A_n$ consisting of elements of prime-power order, such that $\langle c, d \rangle = A_n$ for all $(c, d) \in C \times D$. A positive answer to their question immediately implies a positive answer to ours. We will address related questions in a forthcoming paper.
1.3.4. To our knowledge, the first use of Smith Theory in combinatorics appears in work of Kahn, Saks and Sturtevant. In the paper [23], these authors use the work of Smith and Oliver mentioned above to obtain a striking result about computational complexity.

1.4. Contents of the paper. In Section 2 we introduce some basic facts and definitions. In Section 3 we reduce the proof of Theorem 1.1 to some claims about nonabelian finite simple groups. In the remaining sections, we use the Classification to prove these claims. In Sections 4 and 5 we prove the required result for alternating groups. The group $A_7$ requires more care than the other alternating groups. In Section 6 we handle sporadic groups and groups of Lie type.

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2. Preliminaries

Here we introduce some basic definitions and facts. A reader who is familiar with topological combinatorics and group theory can skip this section safely, and refer to it as necessary.

2.1. Groups and cosets. As is standard, we write $K^g$ for $g^{-1}Kg$ whenever $K \subseteq G$ and $g \in G$, and write $x^g$ for $g^{-1}xg$. Similarly, we write $K^\alpha$ and $x^\alpha$ for the images of $K$ and $x$ under an automorphism $\alpha$ of $G$.

When referring to a coset, we mean a right coset. This causes no loss of generality, as every coset of every subgroup of $G$ is a right coset of some subgroup. Indeed, $xH = H^{x^{-1}}x$. It is not hard to see that every coset is a right coset of a unique subgroup.

2.2. Simplicial complexes. An abstract simplicial complex is a collection $\Delta$ of sets (called faces) such that if $S \in \Delta$ and $T \subseteq S$ then $T \in \Delta$. We make no distinction between an abstract simplicial complex and its geometric realization.
Let $P$ be a (finite) poset. The order complex $\Delta P$ is the simplicial complex whose $k$-dimensional faces are the chains of length $k$ (size $k+1$) in $P$.

If $\Delta$ and $\Gamma$ are simplicial complexes on disjoint vertex sets, the join $\Delta \ast \Gamma$ is the complex whose faces are all sets $S \cup T$ such that $S \in \Delta$ and $T \in \Gamma$.

Associated to a simplicial complex $\Delta$ and a ring $R$ are the reduced simplicial homology groups $\tilde{H}_i(\Delta; R)$, as described (for example) in [28]. A complex $\Delta$ is called $R$-acyclic if $\tilde{H}_i(\Delta; R) = 0$ for every integer $i$. Every contractible complex is $R$-acyclic for all $R$. Every nonempty $R$-acyclic complex has at least one nonempty face. Indeed, $\tilde{H}_{-1}(\{\emptyset\}; R) \cong R$, hence the complex $\{\emptyset\}$ is not acyclic over any ring $R$. The simplicial complexes that we consider all contain the empty face $\emptyset$.

3. Proof of Theorem 1.1

Here we prove Theorem 1.1, although we defer the proofs of some key lemmas on simple groups to later sections. Let us first collect some main ingredients in the proof.

3.1. Brown’s Join Theorem for $\Delta C(G)$. Given a normal subgroup $N$ of $G$, we define the relative coset poset to be

$$C(G, N) := \{Hx \in C(G) : HN = G\}.$$  

The next result, due to Brown, is key to our proof.

**Theorem 3.1 (Brown’s Join Theorem [9, Proposition 10]).** If $G$ is a group and $N$ is a normal subgroup of $G$, then $\Delta C(G)$ is homotopy equivalent to $\Delta C(G/N) \ast \Delta C(G, N)$.

**Corollary 3.2.** Let $p$ be a prime. Let $N$ be a normal subgroup of $G$. The complex $\Delta C(G)$ is $\mathbb{F}_p$-acyclic if and only if at least one of $\Delta C(G, N)$ and $\Delta C(G/N)$ is $\mathbb{F}_p$-acyclic.

**Proof.** The result follows immediately from Theorem 3.1 and the Künneth Formula for joins (see for example [4, (9.12)]). □

We see now that Theorem 1.1 follows quickly from the next result.

**Theorem 3.3.** If $N$ is a minimal normal subgroup of $G$, then $\Delta C(G, N)$ is not $\mathbb{F}_2$-acyclic.

**Proof (of Theorem 1.1, assuming Theorem 3.3).** We proceed by induction on the order of $G$. If $|G| = 1$, then $\Delta C(G) = \{\emptyset\}$. Now assume $|G| > 1$, and let $N$ be a minimal normal subgroup of $G$. The complex $\Delta C(G/N)$ is not $\mathbb{F}_2$-acyclic by inductive hypothesis and $\Delta C(G, N)$ is not $\mathbb{F}_2$-acyclic by Theorem 3.3. Theorem 1.1 now follows from Corollary 3.2. □
It remains to prove Theorem 3.3. In the rest of Section 3, we show how to reduce the proof to certain claims about finite simple groups.

3.2. Group actions and Smith theory. In order to prove Theorem 3.3 we use Smith Theory.

Given a group $E$ acting by automorphisms (order preserving bijections) on a poset $Q$, we write $Q^E$ for the fixed point set

$$Q^E := \{ q \in Q : q^g = q \text{ for all } g \in E \}.$$ 

The action of $E$ on $Q$ induces a simplicial action of $E$ on $\Delta Q$.

Work of Smith in [39] and of Oliver in [29] shows that, given a prime $p$, certain groups cannot act without fixed points on $\mathbb{F}_p$-acyclic complexes. (A clear summary of this work appears in [29, Section 1].) Applying their results to actions on order complexes, we obtain immediately the next result.

**Theorem 3.4** (Smith [39], Oliver [29]). Let $p$ and $r$ be primes. Let $Q$ be a poset such that $\Delta Q$ is $\mathbb{F}_p$-acyclic. Let $E$ be a group admitting a normal series $P \trianglelefteq H \trianglelefteq E$ such that

1. $P$ is a $p$-group,
2. $H/P$ is cyclic, and
3. $E/H$ is an $r$-group.

If $E$ acts on $Q$ by automorphisms, then $Q^E \neq \emptyset$.

**Remark 3.5.** It is not necessary that the primes $p, r$ in Theorem 3.4 be distinct.

**Remark 3.6.** If $E \cong P \times K$, where $P$ is a $p$-group and $K$ is either cyclic or an $r$-group, then $E$ satisfies conditions (1)-(3) of Theorem 3.4. The same holds for $E \cong (P \times K) \rtimes R$, where $P$ is a $p$-group, $R$ is an $r$-group, and $K$ is cyclic.

We will apply Theorem 3.4 to $C(G,N)$. There are two actions on $C(G)$ that we wish to consider. The first action is that of $G \times G$ by left and right translation. That is,

$$Hx \cdot (g,h) = g^{-1}Hxh = H^g g^{-1}xh \text{ for } (g,h) \in G \times G. \quad (3.1)$$

The second action is by $\text{Aut}(G)$, where

$$\alpha \in \text{Aut}(G). \quad (Hx)\alpha = H^\alpha x^\alpha \quad (3.2)$$

The component-wise action of $\text{Aut}(G)$ on $G \times G$ gives rise to the semidirect product $A := (G \times G) \rtimes \text{Aut}(G)$. The actions described in (3.1) and (3.2) combine to form a well-defined action of $A$ on $C(G)$, with $(g,h,p)$ mapping $Hx$ to

$$(Hx \cdot (g,h))\alpha = (g^{-1}Hxh)^\alpha = (Hx)\alpha \cdot (g^\alpha, h^\alpha).$$
Remark 3.7. If $|G| > 1$, then the action of $A$ has a nontrivial kernel $N$. The quotient $A/N$ is called the holomorph of $G$. The kernel of this action will be of no concern to us.

In all but one of our arguments, we will use subgroups of $(G \times G) \rtimes \text{Aut}(G)$ that are contained in $G \times G$. These subgroups will be of the form $P \times K$ with $P, K \leq G$. When we mention an action of such a subgroup on $\mathcal{C}(G)$, we always mean that $P$ acts by left translation and $K$ acts by right translation.

Suppose $N \triangleleft G$. If $HN = G$ then $HgN = G$ for all $g \in G$. It follows that $P \times K$ acts on $\mathcal{C}(G,N)$. Note that $P \times 1$ fixes $Hx$ if and only if $P \leq H$ and $1 \times K$ fixes $Hx$ if and only if $Kx^{-1} \leq H$. The next result follows.

Lemma 3.8. A subgroup $P \times K$ of $G \times G$ fixes $Hx \in \mathcal{C}(G)$ if and only if $\langle P, Kx^{-1} \rangle \leq H$.

3.3. Minimal normal subgroups. Along with Smith theory, we use the Classification of Finite Simple Groups to prove Theorem 3.3. Suppose $G$ is nontrivial, and $N$ is a minimal normal subgroup of $G$. There exist some positive integer $t$ and some simple group $L$ such that $N$ is isomorphic with the direct product of $t$ copies of $L$. (See for example [14, Theorem 4.3A(iii)].) In this situation, we abuse notation by writing $N = L^t$ and representing an element of $N$ as a $t$-tuple of elements of $L$.

The case where the simple group $L$ is cyclic of prime order was already handled by Brown.

Lemma 3.9. If $G$ has an abelian minimal normal subgroup $N$, then $\Delta \mathcal{C}(G,N)$ is not $\mathbb{F}_2$-acyclic.

Proof. As noted in [9, Proposition 9], the poset $\mathcal{C}(G,N)$ is an antichain of size divisible by $|N|$. Therefore, $\Delta \mathcal{C}(G,N)$ is not connected if it contains a nonempty face. It follows that one of $\tilde{H}_{-1}(\Delta \mathcal{C}(G,N); \mathbb{F}_2)$ or $\tilde{H}_0(\Delta \mathcal{C}(G,N); \mathbb{F}_2)$ is nontrivial. □

We turn now to the case where $N = L^t$ with $L$ nonabelian simple. A subgroup $K \leq L$ can be embedded in $N$ diagonally, as follows.

Definition 3.10. Given $N = L^t$ and $K \leq L$, we define

$$K^\text{diag} := \{(k, \ldots, k) : k \in K\} \leq N.$$ 

The next definition is key for finding useful group actions on $\Delta \mathcal{C}(G,N)$.

Definition 3.11. Let $G$ be a group, let $H \leq G$ and let $p$ be a prime. We say that $H$ universally $p$-generates $G$ if $\langle H, P \rangle = G$ whenever $P$ is a Sylow $p$-subgroup of $G$. 
**Remark 3.12.** Let \( p, r \) be primes. A Sylow \( r \)-subgroup of \( G \) universally \( p \)-generates \( G \) if and only if every maximal subgroup of \( G \) has index divisible by at least one of \( p \) and \( r \).

The importance of Definition 3.11 is apparent from the following lemma.

**Lemma 3.13.** Let \( G \) be a group and let \( p \) be a prime. Let \( N \trianglelefteq G \). If \( K \leq N \) universally \( p \)-generates \( N \) and \( P \) is any Sylow \( p \)-subgroup of \( N \), then \( C(G, N)^{P \times K} = \emptyset \).

**Proof.** Assume for contradiction that \( Hx \in C(G, N)^{P \times K} \). By Lemma 3.8 \( H \) contains both \( P \) and \( K^{x^{-1}} \). As \( K \) universally \( p \)-generates \( N \), so does \( K^{x^{-1}} \). Therefore, \( H \) contains \( N \). This is impossible, as \( H < G \) and \( HN = G \). \( \square \)

**Lemma 3.14.** Let \( L \) be a simple group, let \( p \) be a prime, and let \( t \) be a positive integer. If a proper subgroup \( K < L \) universally \( p \)-generates \( L \), then \( K^{\text{diag}} \) universally \( p \)-generates \( N := L^t \).

**Proof.** Let \( P \) be a Sylow \( p \)-subgroup of \( N \). By assumption, \( N = \prod_{i=1}^t L_i \) with each \( L_i \cong L \). It is not hard to see that \( P = \prod_{i=1}^t (P \cap L_i) \). Moreover, \( P \cap L_i \) is a Sylow \( p \)-subgroup of \( L_i \) for each \( i \in [t] \). The standard projection of \( \langle P, K^{\text{diag}} \rangle \) onto \( L_i \) thus contains both \( P \cap L_i \) and \( K \) and is therefore all of \( L_i \). It follows now from (the conjugacy part of) Sylow’s Theorem that \( \langle P, K^{\text{diag}} \rangle \) contains every Sylow \( p \)-subgroup of \( L_i \).

As \( K < L \) universally \( p \)-generates \( L \), \( P \cap L_i \) is nontrivial. As \( L \) is simple, it follows that the Sylow \( p \)-subgroups of \( L_i \) together generate \( L_i \). Hence, \( L_i \leq \langle P, K^{\text{diag}} \rangle \) for each \( i \in [t] \). \( \square \)

**Corollary 3.15.** Let \( L \) be a simple group, let \( p \) be a prime, and let \( t \) be a positive integer. Let \( G \) be a group with normal subgroup \( N = L^t \), and let \( P \) be a Sylow \( p \)-subgroup of \( N \). If a proper subgroup \( K < L \) universally \( p \)-generates \( L \), then \( C(G, N)^{P \times K^{\text{diag}}} = \emptyset \).

**Proof.** This follows immediately from Lemmas 3.13 and 3.14. \( \square \)

**Corollary 3.16.** Let \( N, G, L, p \) and \( K \) be as in Corollary 3.15. If \( K \) is either cyclic or of prime-power order, then \( \Delta C(G, N) \) is not \( \mathbb{F}_p \)-acyclic.

**Proof.** This follows directly from Corollary 3.15 and Theorem 3.4. \( \square \)

Our strategy is now clear. We go through the list of nonabelian simple groups, as provided by the Classification. For each such group \( L \), we look for some \( K < L \) that universally 2-generates \( L \) and is either cyclic or of prime-power order. This strategy fails only when \( L = A_7 \), in which case we use a slight extension of Corollary 3.16.

Every nonabelian finite simple group is, up to isomorphism, an alternating group \( A_n \) with \( n \geq 5 \), a group of Lie type, or one of twenty six sporadic
groups. Note that the small alternating groups $A_5 \cong PSL_2(5)$, $A_6 \cong PSL_2(9)$ and $A_8 \cong PSL_4(2)$ are all isomorphic with simple groups of Lie type. (See for example [13, Theorem 2.2.10].)

**Lemma 3.17.** If $L$ is simple and $L \ncong A_7$, then there is some $K < L$ that is either cyclic or of prime-power order, and that universally 2-generates $L$.

Indeed, the following claims hold.

1. If $L = A_n$ with $n \geq 9$ odd and $h \in L$ is an $n$-cycle, then $\langle h \rangle$ universally 2-generates $L$.
2. If $L = A_n$ with $n \geq 10$ even and $h \in L$ is an $(n-1)$-cycle, then $\langle h \rangle$ universally 2-generates $L$.
3. If $L$ is a sporadic simple group or a simple group of Lie type, then there is some odd prime $p$ such that a Sylow $p$-subgroup of $L$ universally 2-generates $L$.

We will prove Claims (1) and (2) in Section 4, and Claim (3) in Section 6. We examine $A_7$ in Section 5 where we prove the following result.

**Lemma 3.18.** If $G$ has a minimal normal subgroup $N \cong A_7$, then $\Delta C(G,N)$ is not $\mathbb{F}_2$-acyclic.

Theorem 3.3 (and so Theorem 1.1) follows from Lemma 3.9, Corollary 3.16, Lemma 3.17 and Lemma 3.18.

### 4. Alternating Groups of High Degree

Here we prove Claims (1) and (2) of Lemma 3.17.

Our proof involves the standard division of subgroups of $S_n$ into three classes. Such a subgroup $H$ is **transitive** if for each $i, j \in [n]$ there is some $x \in H$ such that $ix = j$, and **intransitive** otherwise. A transitive subgroup $H$ is **imprimitive** if there is some partition $\pi = \{\pi_1, \ldots, \pi_\ell\}$ of $[n]$ into subsets, such that $1 < \ell < n$, and such that for each $x \in H$ and each $i \in [\ell]$, there exists some $j \in [\ell]$ with $\pi_i x = \pi_j$. In this case, we say that $H$ stabilizes $\pi$. A subgroup $H$ is **primitive** if $H$ is transitive but not imprimitive. So, each subgroup of $S_n$ is intransitive, imprimitive or primitive. We begin with a classical result of Jordan.

**Theorem 4.1** (Jordan [22]. See also [14, Example 3.3.1]). If $n \geq 9$, then every primitive subgroup of $S_n$ containing an element with exactly $n - 4$ fixed points contains $A_n$.

When $n \geq 4$, every Sylow 2-subgroup of $A_n$ contains an element with exactly $n - 4$ fixed points, namely, the product of two disjoint transpositions.

**Corollary 4.2.** If $n \geq 9$, then no primitive proper subgroup of $A_n$ contains a Sylow 2-subgroup of $A_n$. 
Suppose that the transitive subgroup \( H \leq S_n \) stabilizes the partition \( \pi = \{\pi_1, \ldots, \pi_\ell\} \) of \([n]\), with \( 1 < \ell < n \). The transitivity of \( H \) forces \( |\pi_i| = |\pi_j| \) for all \( i, j \in [\ell] \). Each \( \pi_i \) has size \( d = n/\ell \). The full stabilizer \( G_\pi \) of \( \pi \) in \( S_n \) (which contains \( H \)) is isomorphic with the wreath product \( S_d \wr S_\ell \), and thus has order \( d!\ell! \). Now \( G_\pi \not\leq A_n \), as \( G_\pi \) contains a transposition. It follows that \( G_\pi \cap A_n \) contains a Sylow 2-subgroup of \( A_n \) if and only if \( n!d!\ell! \) is odd.

**Lemma 4.3.** If \( n \) is odd, then no imprimitive subgroup of \( A_n \) contains a Sylow 2-subgroup of \( A_n \).

**Proof.** By the preceding discussion, it suffices to show that \( n!d!\ell! \) is even whenever \( d \) is a nontrivial proper divisor of \( n \) and \( \ell = n/d \). Straightforward manipulations yield
\[
\frac{n!}{d!\ell!} = \prod_{j=1}^{\ell} \left( \frac{jd-1}{d-1} \right).
\]
It suffices to show any term in the product on the right is even. We calculate
\[
\frac{2d-1}{d} \left( \frac{2d-2}{d-1} \right) = \frac{2d-1}{d} \cdot 2 \left( \frac{2d-3}{d-1} \right).
\]
Since the divisor \( d \) of \( n \) is odd, the result follows. \( \square \)

**Lemma 4.4.** If \( n \) is even, then no imprimitive subgroup of \( A_n \) contains an \((n-1)\)-cycle.

**Proof.** Assume for contradiction that the \((n-1)\)-cycle \( h \in A_n \) stabilizes the partition \( \pi = \{\pi_1, \ldots, \pi_\ell\} \) with \( 1 < \ell < n \). Without loss of generality, the unique fixed point \( j \) of \( h \) lies in \( \pi_1 \). Now \( \pi_1 h = \pi_1 \). As \( \langle h \rangle \) is transitive on \([n] \setminus \{j\}\) and \( |\pi_1| > 1 \), we obtain the contradiction \( [n] = \pi_1 \). \( \square \)

No intransitive subgroup of \( A_n \) contains an \( n \)-cycle. Thus Claim (1) follows from Corollary 4.2 and Lemma 4.3. To prove Claim (2), it remains to show that when \( n \geq 10 \) is even, no intransitive subgroup of \( A_n \) contains both a Sylow 2-subgroup of \( A_n \) and an \((n-1)\)-cycle. It suffices to show a Sylow 2-subgroup \( P \) of \( A_n \) contains a fixed-point-free element. Depending on \( n \mod 4 \), \( P \) contains a conjugate of either
\[
(1, 2, 3, 4)(5, 6) \cdots (n-1, n) \quad \text{or} \quad (1, 2)(3, 4) \cdots (n-1, n).
\]

5. The alternating group \( A_7 \)

Here we prove Lemma 3.18. The conclusion of Claim (1) of Lemma 3.17 does not hold for \( A_7 \). Indeed, \( A_7 \) has proper primitive subgroups that contain both a 7-cycle and a Sylow 2-subgroup of \( A_7 \). (Such subgroups are isomorphic to \( PGL_3(2) \), and are embedded in \( A_7 \) through actions on points and lines in the Fano plane.) As a
result, we cannot apply Corollary 3.15 or Corollary 3.16. However, we can still apply Smith Theory (as in Theorem 3.4) when $G$ has a minimal normal subgroup $N = A_7^t$.

We begin by collecting some information on $A_7$. All these facts are straightforward to confirm, or can be verified with [16] or [46].

**Lemma 5.1.** The group $A_7$ has the following properties.

1. There exist conjugacy classes $K_1$ and $K_2$ of subgroups of $A_7$ satisfying the following conditions.
   - If $K \in K_1 \cup K_2$, then $K \cong PGL_3(2)$, and so $[A_7 : K] = 15$.
   - A proper subgroup $K$ of $A_7$ contains both a Sylow 2-subgroup of $A_7$ and a 7-cycle if and only if $K \in K_1 \cup K_2$.
   - If $K \in K_1 \cup K_2$, $P$ is a Sylow 2-subgroup of $K$, and $R$ is a Sylow 7-subgroup of $K$, then $\langle P^r : r \in R \rangle = K$.

2. There is an involution $\phi \in \text{Aut}(A_7)$ satisfying the following conditions.
   - The automorphism $\phi$ normalizes both a Sylow 2-subgroup and a Sylow 7-subgroup of $A_7$.
   - If $K \in K_1$ then $K^\phi \in K_2$, and if $K \in K_2$ then $K^\phi \in K_1$.

Indeed, the automorphism arising from conjugation by $(1, 2)(3, 4)(5, 6) \in S_7$ has the desired properties.

The next result follows directly from Lemma 5.1 and the universal property of direct products.

**Lemma 5.2.** Let $N$ be the direct product $\prod_{i=1}^t L_i$ with each $L_i \cong A_7$. There is an involution $\rho \in \text{Aut}(N)$ such that the following claims hold for each $i \in [t]$.

1. The automorphism $\rho$ normalizes $L_i$.
2. The automorphism $\rho$ normalizes both a Sylow 2-subgroup $P_i$ of $L_i$ and a Sylow 7-subgroup $\langle h_i \rangle$ of $L_i$.
3. If $H \leq L_i$ is normalized by $\rho$ and contains both $P_i$ and $\langle h_i \rangle$, then $H = L_i$.

Moreover, $\rho$ normalizes both $P := P_1P_2 \cdots P_t$ and $K := \langle (h_1, h_2, \ldots, h_t) \rangle$.

**Remark 5.3.** The automorphism $\rho$ may be realized concretely by embedding $N \cong A_7^t$ in $S_{7^t}$, setting $x := (1, 2)(3, 4)(5, 6) \in S_7$, and conjugating by the element $(x, x, \ldots, x) \in S_{7^t}$.

The next lemma is a special case of a theorem of P. Jin in [21]. It also can be proved directly using standard facts from the cohomology of groups (see [8, Chapter IV], particularly Corollary IV.6.8 therein).

We write $\text{Inn}(M)$ for the inner automorphism group of a group $M$ and $\text{Out}(M)$ for the outer automorphism group $\text{Aut}(M)/\text{Inn}(M)$.
Lemma 5.4 (See [21, Corollary C]). Let $N \leq G$ with $Z(N) = 1$ and let $\rho \in \text{Aut}(N)$. If the coset $\text{Inn}(N)\rho$ is in $Z(\text{Out}(N))$, then there exists some $\theta \in \text{Aut}(G)$ such that

1. $|\theta| = |\rho|,$
2. $\theta$ normalizes $N,$ and
3. the restriction $\theta_N$ of $\theta$ to $N$ is $\rho.$

Applying Lemma 5.4 to the involution $\rho$ described in Lemma 5.2, we obtain the following corollary.

Corollary 5.5. Suppose that the group $G$ has a normal subgroup $N = \prod_{i=1}^{t} L_i$ with each $L_i \cong A_7.$ If $\rho \in \text{Aut}(N)$ is as in Lemma 5.2, then there exists an involution $\theta \in \text{Aut}(G)$ such that $\theta$ normalizes $N$ and $\theta_N = \rho.$

Proof. Note first that $\text{Aut}(A_7) \cong S_7$ (see for example [14, Section 8.2]). It follows that $\text{Aut}(N) \cong S_7 \wr S_t$ (this is [14, Exercise 4.3.9]). As $Z(N) = 1,$ it follows in turn that $\text{Inn}(N) \cong N$ and $\text{Out}(N) \cong Z_2 \wr S_t.$ Therefore $\text{Out}(N)$ has a central element $z$ of order 2.

Let $\varphi \in \text{Aut}(N).$ The coset $\text{Inn}(N)\varphi$ is equal to $z$ if and only if the conditions

(a) $\varphi$ normalizes $L_i,$ and
(b) the restriction of $\varphi$ to $L_i$ is not in $\text{Inn}(L_i)$

are satisfied for each $i \in [t].$ The automorphism $\rho$ meets conditions (a) and (b). The conclusion now follows from Lemma 5.4. \qed

We are ready to complete the proof of Lemma 3.18. Suppose that the group $G$ has a minimal normal subgroup $N = \prod_{i=1}^{t} L_i$ with each $L_i \cong A_7.$ The automorphism $\theta$ obtained in Corollary 5.5 normalizes both groups $P,K$ described in Lemma 5.2. Using the componentwise action of $\theta$ on $P \times K,$ we obtain the semidirect product

$$E := (P \times K) \rtimes \langle \theta \rangle \leq (G \times G) \rtimes \text{Aut}(G).$$

The group $E$ acts on $C(G)$ as described in Section 3.2. Since $\theta$ normalizes $N,$ this action restricts to $C(G,N).$ Since $|\theta| = 2,$ the group $E$ meets the conditions of Theorem 3.4 (as discussed in Remark 3.6). It thus suffices to show that $C(G,N)^E = \emptyset.$

Assume for contradiction that $Hx \in C(G,N)^E.$ Then $H$ contains $\langle P, K^{x^{-1}} \rangle$ and is normalized by $\theta.$ In particular, for each $i \in [t],$ the intersection $H \cap L_i$ is normalized by $\theta$ and contains $P_i.$ Moreover, the projection of $H \cap N$ to $L_i$ (which contains $H \cap L_i$ as a normal subgroup) is normalized by $\theta$ and contains $\langle h_i \rangle.$ By Lemma 5.2 (3), this projection is $L_i.$ As $A_7$ is simple and $H \cap L_i$ is nontrivial, it follows that $H \cap L_i = L_i.$ Therefore $N \leq H.$ This is impossible, as $H < G$ and $HN = G.$
6. Simple groups of Lie type and sporadic groups

Here we prove Claim (3) of Lemma 3.17. We refer the reader to [18, Chapter 2] for an introduction to the finite simple groups of Lie type, with original references. Each such group is determined by an irreducible crystallographic root system $\Sigma$, a (possibly trivial) automorphism $\sigma$ of the Dynkin diagram of $\Sigma$ and a finite field $\mathbb{F}_q$ of order $q$. If $\sigma$ has order $d$, we say that the type of the associated simple group is $d\Sigma(q)$, suppressing $d$ when $d = 1$. Much of what we need has already been proved by Damian and Lucchini in [10, Section 4]. We summarize their results as follows.

**Theorem 6.1** (Damian and Lucchini [10]). If $L$ is a finite simple group of Lie type or a sporadic simple group, then one of the following conditions holds.

(a) There is some cyclic subgroup $C \leq L$ of prime order that universally 2-generates $L$.
(b) The group $L$ is of Lie type $B_n(q)$ ($n \geq 3$), $D_n(q)$ ($n \geq 4$), or $G_2(q)$, and $q$ is odd.
(c) The group $L$ is of Lie type $A_5(2)$, $C_3(2)$, $D_4(2)$ or $^2A_3(2)$.
(d) The group $L$ is the McLaughlin sporadic group $McL$.

In fact, Damian and Lucchini give further restrictions on $L$ when condition (b) of Theorem 6.1 holds, but we will not need these.

As every Sylow $p$-subgroup of a group $G$ contains a conjugate of every element of order $p$ in $G$, it remains to examine the groups listed in cases (b), (c) and (d) of Theorem 6.1.

Suppose $L$ is a simple group of type $d\Sigma(q)$ and $q$ is a power of the prime $p$. We say that $L$ has characteristic $p$ and call a subgroup $M \leq L$ parabolic if $M$ contains the normalizer of some Sylow $p$-subgroup of $L$. The following result is attributed to Tits by Seitz in [37, (1.6)].

**Lemma 6.2** (Tits). Let $L$ be a simple group of Lie type in characteristic $p$, and let $P$ be a Sylow $p$-subgroup of $L$. Every maximal subgroup of $L$ containing $P$ is parabolic.

The groups of types $A_5(2)$, $C_3(2)$, $D_4(2)$ and $^2A_3(2)$ are all classical groups. The orders of parabolic subgroups of classical groups are known, and can be found in [25]. It is straightforward to confirm that the index of each parabolic subgroup of each of the four given groups is divisible by three. The same holds for the index of each maximal subgroup of odd index in $McL$, as can be confirmed by consulting [46]. We obtain the following result.

**Lemma 6.3.** If the simple group $L$ is listed in case (c) or case (d) of Theorem 6.1, then $L$ is universally 2-generated by any of its Sylow 3-subgroups.

It remains to handle case (b). Key to the work of Damian and Lucchini in [10] is a result of Liebeck and Saxl in [26], in which the authors describe all
primitive permutation groups of odd degree. (See also the paper [24] of Kantor, in particular Lemma 2.3 therein.) Such a description necessarily includes a list of all pairs \((M, L)\) such that \(L\) is a finite simple group and \(M\) is a maximal subgroup of odd index in \(L\). Consulting this list, we obtain the following result.

**Lemma 6.4** (Liebeck and Saxl [26]; see also [24, Lemma 2.3]). If \(L\) is a simple group of Lie type in odd characteristic and some parabolic subgroup of \(L\) contains a Sylow 2-subgroup of \(L\), then the type of \(L\) is one of \(A_n(q)\) or \(E_6(q)\).

Consulting [18, Theorem 2.2.10], we see that, assuming the lower bounds on \(n\) listed in case (b), there exists no isomorphism between a group of type \(B_n(q), D_n(q),\) or \(G_2(q)\) with \(q\) odd and a group of type \(A_n(q)\) or \(E_6(q)\). Thus Lemmas 6.2 and 6.4 together complete our proof of Claim (3).

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