NONSTANDARD LAGRANGIAN SUBMANIFOLDS IN $\mathbb{C}P^n$

RIVER CHIANG

Abstract. We use Hamiltonian actions to construct nonstandard (as opposed to $\mathbb{R}P^n$ and $T^n$) Lagrangian submanifolds of $\mathbb{C}P^n$. First of all, a quotient of $\mathbb{R}P^3$ by the dihedral group $D_3$ is a Lagrangian submanifold of $\mathbb{C}P^3$. Secondly, $SU(n)/\mathbb{Z}_n$ are Lagrangian submanifolds of $\mathbb{C}P^{n^2-1}$.

1. Introduction

Let $(M, \omega)$ be a symplectic manifold. A submanifold $L \subset M$ is Lagrangian if $\dim L = \frac{1}{2} \dim M$ and $\omega$ vanishes on $T(L)$.

Using Floer homology, Paul Biran has obtained information on the topology of Lagrangian submanifolds of some closed manifolds. One of Paul Biran’s theorems concerns the Lagrangian submanifolds of $\mathbb{C}P^n$ [B-1, B-2]:

Theorem 1.1. Let $L$ be a Lagrangian submanifold of $\mathbb{C}P^n$ such that $H_1(L; \mathbb{Z})$ is 2-torsion (that is, $2H_1(L; \mathbb{Z}) = 0$). Then

1. $H^*(L; \mathbb{Z}_2) \cong H^*(\mathbb{R}P^n; \mathbb{Z}_2)$ as graded vector spaces.
2. If $n$ is even, the isomorphism in (1) is an isomorphism of graded algebras.

Seidel [Se] also has a theorem with regard to the same subject:

Theorem 1.2. Let $L$ be a Lagrangian submanifold of $\mathbb{C}P^n$. Then

1. $H^1(L; \mathbb{Z}_{2n+2}) \neq 0$.
2. $H^1(L; \mathbb{Z}_{2n+2}) \not\cong (\mathbb{Z}_2)^g$ for any $g \geq 2$.
3. If $H^1(L; \mathbb{Z}_{2n+2}) \cong \mathbb{Z}_2$, then $H^i(L; \mathbb{Z}_2) \cong \mathbb{Z}_2$ for all $i = 0, \ldots, n$.

To test the limits of these topological restrictions on Lagrangian submanifolds of $\mathbb{C}P^n$, we construct new examples using Hamiltonian actions. The constructions in this paper rely on the following well-known fact:

Proposition 1.3. Let a compact Lie group $G$ act on a symplectic manifold $(M, \omega)$ in a Hamiltonian fashion with a moment map $\Phi : M \rightarrow \mathfrak{g}^*$. Assume $\alpha \in \mathfrak{g}^*$ is a fixed point of the coadjoint action. Then for any $m \in \Phi^{-1}(\alpha)$, the orbit $G \cdot m$ is an isotropic submanifold of $M$.

Proof. An orbit is smooth. We only need to show that it is isotropic. Let $\xi$ and $\eta$ be any vector fields in the Lie algebra $\mathfrak{g}$ of $G$ and let $\xi_M$ and $\eta_M$ be their induced vector fields on $M$.

$$\omega(m)(\xi_M(m), \eta_M(m)) = d\langle \Phi(m), \xi \rangle(\eta_M(m))$$
$$= \langle d\Phi(\eta_M), \xi \rangle$$
$$= \langle \eta_{\mathfrak{g}^*}(\Phi(m)), \xi \rangle$$
$$= \frac{d}{dt} \bigg|_0 \exp t \eta \cdot \alpha, \xi \bigg)$$
$$= \langle 0, \xi \rangle$$

1
Therefore \( \omega(m) = 0 \) for all \( m \in \Phi^{-1}(\alpha) \). In particular, the orbit \( G \cdot m \) is a smooth isotropic submanifold of \( M \). \( \square \)

In particular, any orbit of an abelian action is isotropic as well as any orbit in the zero level set of a nonabelian action.

**Example 1.4.** Consider the action of the torus \( T^n \) on \( \mathbb{CP}^n \) given by

\[
(\lambda_1, \ldots, \lambda_n) \cdot [z_0, z_1, \ldots, z_n] = [z_0, \lambda_1^{-1} z_1, \ldots, \lambda_n^{-1} z_n].
\]

This action is Hamiltonian and its moment map is

\[
\Phi([z_0, z_1, \ldots, z_n]) = \frac{1}{2} \left( \frac{|z_1|^2}{\|z\|^2}, \ldots, \frac{|z_n|^2}{\|z\|^2} \right),
\]

where \( \|z\|^2 = \sum_{i=0}^{n} |z_i|^2 \). The free orbit \( T^n \) is a Lagrangian submanifold of \( \mathbb{CP}^n \).

Our first example is a Lagrangian submanifold of \( \mathbb{CP}^3 \). Its topology has an interesting deviation from Theorems 1.1 and 1.2.

**Theorem 1.5.** There exists a Lagrangian submanifold of \( \mathbb{CP}^3 \), which is a quotient of \( \mathbb{RP}^3 \) by the dihedral group \( D_3 \).

We discuss the topology of this submanifold in Section 3.

We use different actions to construct a second family of examples. It provides a generalization of the standard Lagrangian submanifold \( \mathbb{RP}^3 \) of \( \mathbb{CP}^3 \).

**Theorem 1.6.** \( \text{SU}(n)/\mathbb{Z}_n \) is a Lagrangian submanifold of \( \mathbb{CP}^{n^2-1} \).

### 2. First example

View \( \mathbb{C}^4 \) as the space of homogeneous polynomials of degree 3 in two variables. We can define the irreducible SU(2) representation \( \rho: \text{SU}(2) \to \text{GL}(\mathbb{C}^4) \) by

\[
\rho(A)(p)(z) = p(zA)
\]

where \( p \in \mathbb{C}^4 \), \( A \in \text{SU}(2) \) and \( z = (x, y) \in \mathbb{C}^2 \). The moment map for this representation as a map from \( \mathbb{C}^4 \) into \( \mathbb{R} \times \mathbb{C} \cong \text{su}(2)^* \) is given by

\[
(u_0, u_1, u_2, u_3) \mapsto \frac{3}{2} |u_0|^2 + \frac{1}{2} |u_1|^2 - \frac{1}{2} |u_2|^2 - \frac{3}{2} |u_3|^2, u_0 \overline{u}_1 + u_1 \overline{u}_2 + u_2 \overline{u}_3.
\]

So the zero level set of the moment map must satisfy

\[
\begin{align*}
3|u_0|^2 + |u_1|^2 &= |u_2|^2 + 3|u_3|^2, \\
u_0 \overline{u}_1 + u_1 \overline{u}_2 + u_2 \overline{u}_3 &= 0.
\end{align*}
\]

This SU(2) action commutes with the diagonal \( S^1 \) action on \( \mathbb{C}^4 \), and therefore descends to an action on \( \mathbb{CP}^3 \).

Clearly the point \([1, 0, 0, 1]\) is in the zero level set. We know that

\[
(x, y) \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = (\alpha x - \beta y, \beta x + \alpha y),
\]

where \(|\alpha|^2 + |\beta|^2 = 1\). To compute the stabilizer, we set \( \lambda(x^3 + y^3) = (\alpha x - \beta y)^3 + (\beta x + \alpha y)^3 \) for any possible \( \lambda \) in \( S^1 \). Direct computation shows that \( \lambda = \pm 1 \) when \( \beta = 0 \) and \( \lambda = \pm i \) when \( \alpha = 0 \). So the stabilizer is generated by

\[
(2.1) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-6} \end{pmatrix}, \alpha^6 = 1 \right\}
\]
Note this SU(2) action is not effective on CP\(^3\). It induces an effective SO(3) action and the stabilizer for \([1, 0, 0, 1]\) is the semidirect product \(\mathbb{Z}_2 \ltimes \mathbb{Z}_3\), namely, the dihedral group \(D_3\).

Therefore, we have a Lagrangian \(L = SO(3)/D_3 \cong \mathbb{R}P^3/D_3\) in CP\(^3\).

In fact, this is the only orbit in the zero level set. According to the Guillemin-Marle-Sternberg local normal form theorem, a neighborhood of the orbit of \([1, 0, 0, 1]\) is equivariantly symplectomorphic to a neighborhood of the zero section in \(Y = SO(3) \times_{D_3} \mathbb{R}^3\). It is easy to check that there exists one single orbit in the zero level set in \(Y\). Since the zero level set is connected and since the orbit is closed, it follows that the zero level set in \(CP^3\) consists of one orbit only.

3. Topology for the first example

Let \(\Gamma\) denote the group generated by the elements in (2.1). Then the fundamental group of \(L\) is \(\pi_1(L) = \Gamma\) since \(L = \mathbb{R}P^1/D_3 = S^1/\Gamma\).

Let \(b\) denote \((0, i)\) and \(a\) denote \((\frac{1}{2}, 0, 0)\), \(a^3 = 1\). Then \(b^2 = -1\) and \(ba = a^b b\). The commutator subgroup \(A = \{ x^{-1}y^{-1}xy \mid x, y \in \pi_1(L) \}\) is \(\{ 1, a^2, a^4 \}\). By Hurewicz Theorem, the first homology group \(H_1(L; \mathbb{Z})\) is \(\{ 1, b, b^2, b^3 \} \cong \mathbb{Z}_4\).

Using the Universal Coefficient Theorem and Poincaré duality, we can calculate that

\[
\begin{align*}
H_0(L; \mathbb{Z}) &= \mathbb{Z}, & H^0(L; \mathbb{Z}) &= \mathbb{Z}, & H^0(L; \mathbb{Z}_2) &= \mathbb{Z}_2, \\
H_1(L; \mathbb{Z}) &= \mathbb{Z}_4, & H^1(L; \mathbb{Z}) &= 0, & H^1(L; \mathbb{Z}_2) &= \mathbb{Z}_2, \\
H_2(L; \mathbb{Z}) &= 0, & H^2(L; \mathbb{Z}) &= \mathbb{Z}_4, & H^2(L; \mathbb{Z}_2) &= \mathbb{Z}_2, \\
H_3(L; \mathbb{Z}) &= \mathbb{Z}, & H^3(L; \mathbb{Z}) &= \mathbb{Z}, & H^3(L; \mathbb{Z}_2) &= \mathbb{Z}_2.
\end{align*}
\]

Also, \(H^1(L; \mathbb{Z}_2) = \mathbb{Z}_4\).

The Lagrangian \(L\) we found does not satisfy the assumption of 2-torsion as in Theorem [12] and Theorem [123], but \(H^*(L; \mathbb{Z}_2)\) is isomorphic to \(H^*(\mathbb{R}P^3; \mathbb{Z}_2)\) as graded vector spaces.

4. Second family of examples

Consider \(\mathbb{C}^n\) as the set of \(n\) by \(n\) matrices. SU\((n)\) acts naturally by left multiplication. This action is Hamiltonian and its moment map is given by

\[\Phi(Z) = \frac{i}{2} ZZ^* + \frac{1}{2i} I\]

where \(I\) is the identity matrix.

This SU\((n)\) action commutes with the circle action of multiplication by \(\lambda I\). Therefore, it descends to an action on CP\(^{n^2-1}\) where the center \(\mathbb{Z}_n\) of SU\((n)\) acts trivially. Therefore, a free orbit in the zero level set of \(\mathbb{C}^n\) descends to an orbit in the zero level set of CP\(^{n^2-1}\) with stabilizer \(\mathbb{Z}_n\). Since the dimension of SU\((n)\) is \(n^2 - 1\), by Proposition [3] SU\((n)/\mathbb{Z}_n\) is a Lagrangian submanifold of CP\(^{n^2-1}\).

For \(n = 2\), the above-mentioned orbit SU\((2)/\mathbb{Z}_2\) is the standard Lagrangian \(\mathbb{R}P^3\) in CP\(^3\). Indeed it is the projective space of the real 4 dimensional space \(V\) spanned.

\[\text{I would like to thank Michèle Audin for showing me a beautiful alternative construction of these examples in terms of real forms of Lie groups [A].}\]
by the following matrices:

\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}, \quad \begin{pmatrix}
i & 0 \\
0 & i
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & i \\
-i & 0
\end{pmatrix}.
\]

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