SECOND ORDER ESTIMATE ON TRANSITION LAYERS

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Abstract. In this paper we establish a uniform $C^{2,\theta}$ estimate for level sets of stable solutions to the singularly perturbed Allen-Cahn equation in dimensions $n \leq 10$ (which is optimal). The proof combines two ingredients: one is the infinite dimensional reduction method which enables us to reduce the $C^{2,\theta}$ estimate for these level sets to a corresponding one on solutions of Toda system; the other one uses a small regularity theorem on stable solutions of Toda system to establish various decay estimates on these solutions, which gives a lower bound on distances between different sheets of solutions to Toda system or level sets of solutions to Allen-Cahn equation.

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1. Introduction

1.1. Main result. In this paper, continuing the study in [19], we establish a second order estimate on level sets of stable solutions to the singularly perturbed Allen-Cahn equation

\[ \varepsilon \Delta u_\varepsilon = \frac{1}{\varepsilon} W'(u_\varepsilon), \quad |u_\varepsilon| < 1 \quad \text{in} \quad B_1(0) \subset \mathbb{R}^n. \]

Here \( W(u) \) is a general double well potential, that is, \( W \in C^3([-1, 1]) \) satisfying

- \( W > 0 \) in \((-1, 1)\) and \( W(\pm 1) = 0; \)
- \( W'(\pm 1) = 0 \) and \( W''(-1) = W''(1) = 1; \) (Note a slight notation difference here with other literatures.)
- there exists only one critical point of \( W \) in \((-1, 1)\), which we assume to be 0.

A typical model is given by \( W(u) = (1 - u^2)^2 / 8. \)

Under these assumptions on \( W \), it is known that there exists a unique solution to the following one dimensional problem

\[ g''(t) = W'(g(t)), \quad g(0) = 0 \quad \text{and} \quad \lim_{t \to \pm \infty} g(t) = \pm 1. \]

A solution of (1.1) is stable if for any \( \eta \in C^\infty_0(B_1(0)), \)

\[ \int_{B_1(0)} \left[ \varepsilon^2 |\nabla \eta|^2 + W''(u_\varepsilon)\eta^2 \right] \geq 0. \]

By Sternberg-Zumbrun [14], the stability condition is equivalent to

\[ \int_{B_1(0)} |\nabla \eta|^2 \varepsilon |\nabla u_\varepsilon|^2 \geq \int_{B_1(0)} \eta^2 |B(u_\varepsilon)|^2 \varepsilon |\nabla u_\varepsilon|^2, \quad \forall \eta \in C^\infty_0(B_1(0)). \]

Here

\[ |B(u_\varepsilon)|^2 = \begin{cases} \frac{|\nabla^2 u_\varepsilon|^2 - |\nabla |\nabla u_\varepsilon||^2}{|\nabla u_\varepsilon|^2}, & \text{if } |\nabla u_\varepsilon| \neq 0 \\ 0, & \text{otherwise.} \end{cases} \]

It is known that if \( |\nabla u_\varepsilon(x)| \neq 0 \), it holds that

\[ |B(u_\varepsilon)(x)|^2 = |A_x(x)|^2 + |\nabla_T \log |\nabla u_\varepsilon(x)||^2, \]

where \( A_x(x) \) is the local area of the level set of \( u_\varepsilon \) at \( x \) and \( \nabla_T \) is the tangential gradient.
where $A_\varepsilon(x)$ is the second fundamental form of the level set $\{u_\varepsilon = u_\varepsilon(x)\}$ and $\nabla_T$ denotes the tangential derivative along the level set $\{u_\varepsilon = u_\varepsilon(x)\}$.

The main result of this paper is

**Theorem 1.1.** For any $\theta \in (0,1)$, $0 < b_1 \leq b_2 < 1$ and $\Lambda > 0$, there exist two constants $C = C(\theta, b_1, b_2, \Lambda)$ and $\varepsilon_* = \varepsilon(\theta, b_1, b_2, \Lambda)$ so that the following holds. Suppose $u_\varepsilon$ is a stable solution of (1.1) in $B_1(0) \subset \mathbb{R}^n$ satisfying

$$\left|\nabla u_\varepsilon\right| \neq 0 \quad \text{and} \quad |B(u_\varepsilon)| \leq \Lambda, \quad \text{in } \{|u_\varepsilon| \leq 1 - b_2\} \cap B_1(0).$$

If $n \leq 10$ and $\varepsilon \leq \varepsilon_*$, then for any $t \in [-1 + b_1, 1 - b_1]$, $\{u_\varepsilon = t\}$ are smooth hypersurfaces and the $C^\alpha$ norm of their second fundamental forms are bounded by $C$. Moreover,

$$|H(u_\varepsilon)| \leq C\varepsilon (\log |\log \varepsilon|)^2,$$

where $H(u_\varepsilon)$ denotes the mean curvature of $\{u_\varepsilon = t\}$.

Several corollaries follow from this theorem.

**Corollary 1.2.** For any $\theta \in (0,1)$, $b \in (0,1)$ and $Q > 0$, there exist two constants $\varepsilon_1$ and $C_1$ so that the following holds. Suppose that $u_\varepsilon$ is a sequence of stable solution of (1.1) in $C_1 := B_1^{n-1}(0) \times (-1,1) \subset \mathbb{R}^n$ with $\varepsilon \to 0$, satisfying

(H1) there exists a sequence of $t_\varepsilon \in (-1 + b, 1 - b)$ such that $\{u_\varepsilon = t_\varepsilon\}$ consists of $Q$ connected components

$$\Gamma_{\alpha, \varepsilon} = \{x_n = f_{\alpha, \varepsilon}(x'), \ x' := (x_1, \ldots, x_{n-1}) \in B_1^{n-1}\}, \quad \alpha = 1, \ldots, Q,$$

where $-1/2 < f_{1, \varepsilon} < f_{2, \varepsilon} < \cdots < f_{Q, \varepsilon} < 1/2$;

(H2) for each $\alpha$, $\nabla f_{\alpha, \varepsilon}$ are uniformly continuous in $B_1^{n-1}(0)$.

If $n \leq 10$, then the same conclusion of Theorem 1.1 holds for all $u_\varepsilon$ if $\varepsilon \leq \varepsilon_1$, with $C$ replaced by $C_1$.

**Corollary 1.3.** For any $\theta \in (0,1)$ and $b \in (0,1)$, there exist three constants $\varepsilon_2$, $\delta_2$ and $C_2$ so that the following holds. Suppose that $u_\varepsilon$ is a sequence of stable solutions of (1.1) in $B_1(0) \subset \mathbb{R}^n$, satisfying for any $x \in \{u_\varepsilon = 0\} \cap B_{1-\varepsilon}(0)$,

$$\sup_{y \in B_\varepsilon(x)} |u_\varepsilon(y) - g \left(\frac{y \cdot e - t}{\varepsilon}\right)| \leq \delta_2,$$

where $e$ is a unit vector and $t$ is a constant, both depending on $x$. If Stable Bernstein Conjecture is true in dimension $n$, then the same conclusion of Corollary 1.2 holds with $\varepsilon_1, C_1$ replaced by $\varepsilon_2, C_2$.

Note that Stable Bernstein Conjecture is expected to be true for $n \leq 7$ and it has been verified for $n = 3$, see do Carmo and Peng [7], Fischer-Colbrie and Schoen [9] and Pogorelov [13]. For regularity theory of stable minimal surfaces in higher dimensions we refer to Wickramasekera [20].

Some remarks are in order.

**Remark 1.4.**

- The $n = 2$ case is essentially contained in our paper [19]. Recently Chodosh and Mantoulidis established this second order regularity result for the $n = 3$ case, which was used in their analysis of Allen-Cahn approximation to minimal surfaces in three dimensional manifold, see [3]. The relations between the number of ends and Morse index is discussed in Mantoulidis [12].
The assumption (1.8) says \( u_\varepsilon \) is close to the one dimensional solution \( g \) at \( O(\varepsilon) \) scales. This is guaranteed by assumptions (H1) and (H2), see [19] as well as Section 11.

The dimension bound \( n \leq 10 \) is sharp. If \( n \geq 11 \), there exists a smooth, radially symmetric, stable solution to the Liouville equation (i.e. two component Toda system)

\[
\Delta f = e^{-f}, \quad \text{in } \mathbb{R}^{n-1}.
\]

Agudelo-del Pino-Wei [1] constructed a family of solutions \( u_\varepsilon \) of (1.1) in \( \mathbb{R}^n \), with its nodal set \( \{ u_\varepsilon = 0 \} \) given by the graph \( \{ x_n = \pm f_\varepsilon(x') \} \), \( x' \in \mathbb{R}^{n-1} \), where

\[
f_\varepsilon(x) \approx \varepsilon f \left( \varepsilon^{-\frac{1}{2}} x \right) + \varepsilon |\log \varepsilon|.
\]

Clearly we have

\[
|\nabla^2 f_\varepsilon(0)| \approx |\nabla^2 f(0)|
\]

while

\[
\lim_{\varepsilon \to 0} |\nabla^2 f_\varepsilon(x)| = 0, \quad \forall x \neq 0.
\]

Hence \( \nabla^2 f_\varepsilon \) is not uniformly continuous.

The stability condition is also necessary for this second order regularity. Without the stability condition it is not true even for \( n = 2 \). Counterexamples are provided by the multiple end solutions of (1.1) in \( \mathbb{R}^2 \), constructed by Del Pino-Kowalczyk-Pacard-Wei [5]. By utilizing solutions of Toda system

\[
f_\alpha^n = e^{-(f_\alpha-f_{\alpha-1})} - e^{-f_{\alpha+1}-f_\alpha}, \quad \text{on } \mathbb{R}, \quad 1 \leq \alpha \leq Q,
\]

they constructed a family of solutions \( u_\varepsilon \) of (1.1) in \( \mathbb{R}^2 \), with its nodal set \( \{ u_\varepsilon = 0 \} \) given by the graph of

\[
f_{\alpha,\varepsilon}(x) \approx \varepsilon f_\alpha \left( \varepsilon^{-\frac{1}{2}} x \right) + \alpha \varepsilon |\log \varepsilon|.
\]

As in the previous case, \( \nabla^2 f_\varepsilon \) is not uniformly continuous.

Although this second order regularity does not hold any more for \( n \geq 11 \). A partial regularity result may still hold. For example, under assumptions of Corollary 1.2, there should exist a closed set of Hausdorff dimension at most \( n - 10 \) such that in any compact set outside this singular set, uniform second order regularity of level sets for stable solutions of (1.11) still hold.

It seems that the second order regularity problem is quite different in nature from the first order regularity problem, i.e. uniform \( C^{1,\theta} \) estimates on level sets. See Caffarelli-Cordova [2] and Tonegawa-Wickramasekera [16]. For example, it can be checked that the above counterexamples ([1] [5]) to second order regularity still enjoy a uniform \( C^{1,\theta} \) estimate.

We do not touch any aspect on higher order regularity (e.g. \( C^{k,\theta} \) regularity for \( k \geq 3 \)) of level sets. It will be interesting to obtain such a result even for the multiplicity one case.

### 1.2. Outline of proof.

The proof of Theorem 1.1 consists of the following three steps.

**Step 1. Infinite dimensional Lyapunov-Schmidt reduction.** Sections 2-7 are devoted to this reduction procedure. It is almost the same with the one in [19], but here various simplifications and improvements will be given.

The main difference is that in [19], it is either assumed that there are only finitely many connected components of transition layers (as in Corollary 1.2) or the distance between
different connected components of transition layers has a lower bound in the form $c|\log \varepsilon|$ (see [19, Section 17]), but now both assumptions are removed and we only need the assumption that the distance between different connected components of transition layers to be $\gg \varepsilon$ (see Lemma 2.1 below) as a starting point. Moreover, now we can show that all estimates in this step hold uniformly with respect to the number of connected components of transition layers. Hence there is no assumption on the number of connected components of transition layers in Theorem 1.1 and Corollary 1.3.

The reduction method proceeds as follows. First from the assumptions in Theorem 1.1 (or Corollary 1.2 or 1.3), it follows that the solution $u_\varepsilon$ is close to the one dimensional profile at $O(\varepsilon)$ scales, see Section 2. Therefore the solution has the form

$$u_\varepsilon = \sum_\alpha g_{\alpha,\varepsilon} + \phi_\varepsilon,$$

where $g_{\alpha,\varepsilon}$ is the one dimensional solution in composition with the distance function to $\Gamma_{\alpha,\varepsilon}$, a connected component of $\{u_\varepsilon = 0\}$, and $\phi_\varepsilon$ is a small error between our solution $u_\varepsilon$ and the approximate solution $\sum_\alpha g_{\alpha,\varepsilon}$.

Writing $u_\varepsilon$ in this way, the single equation for $u_\varepsilon$, (1.1), is almost decoupled into two equations: one is the equation for the level set $\{u_\varepsilon = 0\}$ and the other one is an equation for $\phi_\varepsilon$. Such a decoupling is possible by choosing an optimal approximation in (1.9), which then implies that $\phi_\varepsilon$ lies in the subspace orthogonal to the kernel space at $\sum_\alpha g_{\alpha,\varepsilon}$, see Proposition 4.1 for a precise statement. To this end, it is necessary to take a small perturbation in the normal direction of each $\Gamma_{\alpha,\varepsilon}$ so that $g_{\alpha,\varepsilon}$ is the optimal approximation to $u_\varepsilon$ in the normal direction. Here it is convenient to introduce Fermi coordinates with these $\Gamma_{\alpha,\varepsilon}$ and rewrite everything in these coordinates, see Section 3-4.

Since $\Gamma_{\alpha,\varepsilon}$ are far from each other and they are almost parallel, the interaction pattern between different $g_{\alpha,\varepsilon}$, which represents the interaction between different components of $\{u_\varepsilon = 0\}$, can be determined by using asymptotic expansions of the one dimensional profile at infinity. This gives the equation for $\Gamma_{\alpha,\varepsilon}$,

$$H_{\alpha,\varepsilon} = \frac{2A_{[-1]}^2}{\varepsilon} e^{-\frac{|d_{\alpha-1,\varepsilon}|}{\varepsilon}} - \frac{2A_{[-1]}^2\varepsilon}{\varepsilon} - \frac{|d_{\alpha+1,\varepsilon}|}{\varepsilon} + \text{higher order terms},$$

where $H_{\alpha,\varepsilon}$ is the mean curvature of $\Gamma_{\alpha,\varepsilon}$, $|d_{\alpha-1,\varepsilon}|$ and $|d_{\alpha+1,\varepsilon}|$ are distances to $\Gamma_{\alpha-1,\varepsilon}$ and $\Gamma_{\alpha+1,\varepsilon}$ respectively, see Section 5 for a precise statement.

Higher order terms in (1.10) involve some terms containing $\phi_\varepsilon$. In order to get a good reduced problem, a precise estimate on $\phi_\varepsilon$ is needed. This is established in Section 6 and Section 7. Since $\phi_\varepsilon$ is known to be a small perturbation, it satisfies an almost linearized equation. (This is the reduction procedure, i.e. we partially linearize (1.1) in the $\phi_\varepsilon$ component.) To estimate $\phi_\varepsilon$, we need to consider two separate cases: the inner problem near $\{u_\varepsilon = 0\}$, and the outer one which is concerned with the part far away from $\{u_\varepsilon = 0\}$. It is important here that these two parts are still almost decoupled, which is guaranteed by the fast decay of the one dimensional profile at infinity.

**Step 2. Reduction of the stability condition.** Now the $C^{2,\theta}$ estimate is reduced to a corresponding one on (1.10). It turns out that this depends in an essential way on lower bounds on $|d_{\alpha-1,\varepsilon}|$ and $|d_{\alpha+1,\varepsilon}|$, as observed in [19]. To get these lower bounds, we use the stability condition (1.4). In Section 8, we show that if $u_\varepsilon$ is a stable solution, then solutions to the reduced problem (1.10) satisfies an almost stability condition. This is
achieved by choosing test functions in (1.4) to be
\[ \sum_{\alpha} \eta_{\alpha} g'_{\alpha, \varepsilon}, \]
where \( \eta_{\alpha} \in C_0^\infty(\Gamma_{\alpha, \varepsilon}) \). In other words, we consider variations along directions tangential to \( \{ u_\varepsilon = 0 \} \). This choice of test functions in the stability condition is similar to the one used in [1] and [3, Appendix D]. Then by a careful analysis of contributions from tangential parts, normal parts, cross terms and the interaction between different components, we get a stability condition on solutions to (1.10), see Proposition 8.1.

Step 3. Decay estimates. Finally, a small regularity theorem on stable solutions of (1.10) will be employed to give decay estimates on \( e^{-|d_{\alpha, \varepsilon}|/\varepsilon} \) in the interior, which then leads to a \( C^{2,\theta} \) estimate on (1.10).

This small regularity result has been established by the first author in [17, 18] in the setting of stable solutions for the Liouville equation and it can be generalized to Toda system (1.10). Here the dimension restriction \( n \leq 10 \) appears, due to the fact that this small regularity theorem requires an \( L^1 \) smallness assumption on \( e^{-|d_{\alpha, \varepsilon}|/\varepsilon} \) as the starting point. This \( L^1 \) smallness condition holds unconditionally only in \( n \leq 10 \), which can be proved by an \( L^p \) estimate of Farina [8].

A reduction procedure is still needed in order to apply this small regularity theorem to (1.10). In this paper, two of such approaches are employed. The first one is extrinsic and uses the graph representation (with respect to a fixed hyperplane) of \( \Gamma_{\alpha, \varepsilon} \). This works well when they are very close, which implies that different \( \Gamma_{\alpha, \varepsilon} \) are almost parallel to each other. This then allows us to represent distances between them by differences of functions, and replace the minimal surface operator in (1.10) by the standard Laplacian operator.

The second reduction method is intrinsic and uses the Jacobi field construction introduced in Chodosh and Mantoulidis [3]. Here we fix an \( \Gamma_{\alpha, \varepsilon} \) and view other components as graphs of functions defined on this component. Combined with some elliptic estimates on these functions, this approach gives a stronger distance lower bound. This then implies that the exponential nonlinearity in (1.10) is dominated by mean curvature terms. Using this we can construct positive Jacobi fields as in [3].

In this paper, a constant is called universal if it depends only on the dimension \( n \), the double well potential \( W \) and the constants \( b_1, b_2, \Lambda \) in Theorem 1.1. If \( A \leq CB \) for a universal constant \( C \), then we denote it by \( A \lesssim B \) or \( A = O(B) \). If the constant \( C \) depends on a parameter \( K \), it is written as \( A = O_K(B) \).

By letting \( u(x) := u_\varepsilon(\varepsilon x) \), we obtain the unscaled Allen-Cahn equation
\[ \Delta u = W'(u). \]

2. Preliminary analysis

In the following we will only be concerned with one level set of \( u_\varepsilon \), \( \{ u_\varepsilon = 0 \} \). It will be clear that our proof goes through without any change when 0 is replaced by any other \( t \in [-1 + b_1, 1 - b_1] \), and all of the following estimates are uniform in \( t \in [-1 + b_1, 1 - b_1] \).

By standard elliptic regularity theory, \( u_\varepsilon \in C^{4,\theta}_{\text{loc}}(B_1(0)) \). Concerning the regularity of \( \{ u_\varepsilon = 0 \} \), we first prove that different components of it are at least \( O(\varepsilon) \) apart. In the following a connected component of \( \{ u_\varepsilon = 0 \} \) is denoted by \( \Gamma_{\alpha, \varepsilon} \), where \( \alpha \) is the index. The following lemma also shows that the cardinality of the index set is always finite for fixed \( \varepsilon \), although it could go to infinity as \( \varepsilon \to 0 \).
Lemma 2.1. For any \( \alpha \) and \( x_\varepsilon \in \Gamma_{\alpha,\varepsilon} \cap B_{7/8}(0) \), as \( \varepsilon \to 0 \), \( \tilde{u}_\varepsilon(x) := u_\varepsilon(x_\varepsilon + \varepsilon x) \) converges to a one dimensional solution in \( C^{2}_{\text{loc}}(\mathbb{R}^n) \). In particular,

\[
\varepsilon^{-1} \text{dist}(x_\varepsilon, \{u_\varepsilon = 0\} \setminus \Gamma_{\alpha,\varepsilon}) \to +\infty \quad \text{uniformly.}
\]

Proof. In \( B_{\varepsilon^{-1}/8}(0) \), \( \tilde{u}_\varepsilon(x) \) satisfies the Allen-Cahn equation (1.11). By standard elliptic regularity theory, \( \tilde{u}_\varepsilon(x) \) is uniformly bounded in \( C^{4}_{\text{loc}}(\mathbb{R}^n) \). By Arzela-Ascoli theorem, as \( \varepsilon \to 0 \), it converges to a limit function \( \tilde{u}_\infty \in C^{2}_{\text{loc}}(\mathbb{R}^n) \). Clearly \( \tilde{u}_\infty \) is a stable solution of (1.11) in \( \mathbb{R}^n \).

Since \( \tilde{u}_\varepsilon(0) = 0 \), \( \tilde{u}_\infty(0) = 0 \). By our assumption on the double well potential \( W \), the constant solution 0 is not stable. Hence \( \tilde{u}_\infty \) is a non-constant solution. As a consequence, by unique continuation principle, the critical set \( \{\nabla \tilde{u}_\infty = 0\} \) has zero Lebesgue measure.

By (1.6),

\[
|B(\tilde{u}_\varepsilon)| \leq \Lambda \varepsilon, \quad \text{in } \{|\tilde{u}_\varepsilon| \leq 1 - b\} \cap B_{\varepsilon^{-1}/7}(0).
\]

By the convergence of \( \tilde{u}_\varepsilon \), we can pass this inequality to the limit in any compact set outside \( \{\nabla \tilde{u}_\infty = 0\} \), which leads to \( |B(\tilde{u}_\infty)| \equiv 0 \) in \( \{|\tilde{u}_\infty| \leq 1 - b\} \). Hence by (1.5) and Sard theorem, almost all level sets \( \{\tilde{u}_\infty = t\}, \; t \in [-1 + b, 1 - b], \) are hyperplanes. Then it is directly verified that \( \tilde{u}_\infty \) is one dimensional.

Let \( \tilde{\Gamma}_{\alpha,\varepsilon} := \varepsilon^{-1}(\Gamma_{\alpha,\varepsilon} - x_\varepsilon) \). By the convergence of \( \tilde{u}_{\alpha,\varepsilon} \), in any compact set of \( \mathbb{R}^n \), \( \tilde{\Gamma}_{\alpha,\varepsilon} \) converge to \( \{\tilde{u}_\infty = 0\} \) in the Hausdorff distance. Since \( \{\tilde{u}_\infty = 0\} \) is a single hyperplane, we get

\[
\text{dist}\left(0, \{\tilde{u}_\varepsilon = 0\} \setminus \tilde{\Gamma}_{\alpha,\varepsilon}\right) \to +\infty,
\]

where the convergence rate depends only on \( \varepsilon \). This gives (2.1). \( \square \)

The above proof implies that the Implicit Function Theorem can be applied to \( u_\varepsilon \) at \( O(\varepsilon) \) scales, which gives the \( C^4 \) regularity of \( \{u_\varepsilon = 0\} \). Of course it is not known whether there exists a uniform bound independent of \( \varepsilon \).

The following lemma can be proved by combining the curvature bound (1.6) with the fact that different connected components of \( \{u_\varepsilon = u_\varepsilon(0)\} \) are disjoint. (This fact has been used a lot in minimal surface theory, see for instance [4].)

Lemma 2.2. There exist two universal constants \( \sigma \in (0, 1/16) \) and \( C(\sigma, \Lambda) \) so that the following holds. For any \( x_\varepsilon \in \{u_\varepsilon \leq 1 - b\} \cap B_{7/8}(0) \), in a suitable coordinate system, \( \{u_\varepsilon = u_\varepsilon(x_\varepsilon)\} \cap B_\sigma(x_\varepsilon) \) is a family of graphs \( \cup_{\alpha} \{x_n = f_\alpha(x')\} \), where \( f_\alpha \in C^1(B_{2\varepsilon^{-1}}(x'_\varepsilon)) \) satisfying \( \|f_\alpha\|_{C^{1,1}(B_{2\varepsilon^{-1}}(x'_\varepsilon))} \leq C(\sigma, \Lambda) \).

3. Fermi coordinates

3.1. Definition. For simplicity of presentation, we now work in the stretched version and do not write the dependence on \( \varepsilon \) explicitly.

By denoting \( R = \varepsilon^{-1} \), \( u(x) = u_\varepsilon(\varepsilon x) \) satisfies the Allen-Cahn equation (1.11) in \( B_R(0) \). Its nodal set \( \{u = 0\} \) consists of finitely many connected components, \( \Gamma_\alpha \).

By our assumption, for each \( \alpha \), the second fundamental form \( A_\alpha \) of \( \Gamma_\alpha \) satisfies

\[
|A_\alpha(y)| \leq \Lambda \varepsilon, \quad \forall y \in \Gamma_\alpha \cap B_R(0).
\]

We will assume \( \Lambda \) is sufficiently small, perhaps after restricting to a small ball and then rescaling its radius to 1.

Let \( y \) be a local coordinates of \( \Gamma_\alpha \). The Fermi coordinate is defined as \( (y, z) \mapsto x \), where \( x = y + zN_\alpha(y) \). Here \( N_\alpha(y) \) is a unit normal vector to \( \Gamma_\alpha \), \( z \) is the signed distance to \( \Gamma_\alpha \).

By (3.1), Fermi coordinates are well defined and smooth in \( B_R(0) \).
By Lemma 2.2 (recall that we have assumed $\Lambda \ll 1$), after a rotation,
\begin{equation}
\Gamma_\alpha \cap B_R(0) = \{x_n = f_\alpha(x')\}, \quad \forall \alpha.
\end{equation}
Therefore a canonical way to choose local coordinates of $\Gamma_\alpha$ is by letting $y = x'$ for each $\alpha$. Then the induced metric on $\Gamma_\alpha$ is
\[ g_{\alpha,ij}(y) = \delta_{ij} + \frac{\partial f_\alpha}{\partial y_i}(y) \frac{\partial f_\alpha}{\partial y_j}(y). \]

By Lemma 2.2 and (3.1), we get a universal constant $C$ such that
\begin{equation}
|\nabla f_\alpha| \leq C \quad \text{and} \quad |\nabla^2 f_\alpha| \leq C\varepsilon, \quad \text{in } B_R^n(0).
\end{equation}

Sometimes the signed distance to $\Gamma_\alpha$ is also denoted by $d_\alpha$. Since $\Gamma_\alpha \cap \Gamma_\beta = \emptyset$ for any $\alpha \neq \beta$, we can choose the sign so that $\{d_\alpha > 0\} \cap \{d_\beta > 0\} \neq \emptyset$ for any $\alpha \neq \beta$.

For any $z \in (-R, R)$, let $\Gamma_{\alpha,z} := \{\text{dist}(x, \Gamma_\alpha) = z\}$. Hence $\Gamma_{\alpha,0}$ is just $\Gamma_\alpha$. Define the vector field
\[ X_i := \frac{\partial}{\partial y^i} + z \frac{\partial N_\alpha}{\partial y^i} = \sum_{j=1}^{n-1} (\delta_{ij} - z A_{\alpha,ij}) \frac{\partial}{\partial y^j}, \quad 1 \leq i \leq n - 1. \]

The tangent space of $\Gamma_{\alpha,z}$ is spanned by $X_i$. The Euclidean metric restricted to $\Gamma_{\alpha,z}$ is denoted by $g_{\alpha,ij}(y,z)dy^i \otimes dy^j$, where
\begin{equation}
g_{\alpha,ij}(y,z) = \begin{array}{ll}
X_i(y,z) \cdot X_j(y,z) & (\text{for } i, j = 1, \ldots, n) \\
n_{\alpha,ij}(y,z) & (\text{for } i = 1, \ldots, n; j = 1, \ldots, n-1)
\end{array}
\end{equation}
\begin{equation}
= g_{\alpha,ij}(y,0) - 2z \sum_{k=1}^{n-1} A_{\alpha,ik}(y,0)g_{jk}(y,0) + z^2 \sum_{k,l=1}^n g_{\alpha,kl}(y,0)A_{\alpha,ik}(y,0)A_{\alpha,jl}(y,0).
\end{equation}

The second fundamental form of $\Gamma_{\alpha,z}$ has the form
\begin{equation}
A_\alpha(y,z) = [I - z A_\alpha(y,0)]^{-1} A_\alpha(y,0).
\end{equation}

3.2. Some notations. In the remaining part of this paper the following notations will be employed.

- Given a point on $\Gamma_\alpha$ with local coordinates $(y,0)$ in the Fermi coordinates, denote $D_\alpha(y) := \min_{\beta \neq \alpha} |d_\beta(y,0)|$.
- For any $x \in B_R(0)$ and $r \in (0, R - |x|)$, denote
\[ A(r;x) := \max_{y \in \Gamma_\alpha \cap B_r(x)} e^{-D_\alpha(y)}. \]
- The covariant derivative on $\Gamma_{\alpha,z}$ with respect to the induced metric is denoted by $\nabla_{\alpha,z}$.
- The area form on $\Gamma_{\alpha,z}$ with respect to the induced metric is denoted by $dA_{\alpha,z} = \lambda_\alpha(y,z)dy$, where $\lambda_\alpha(y,z) = \sqrt{\det [g_{\alpha,ij}(y,z)]}$.
- We use $B^\alpha_r(y)$ to denote the open ball on $\Gamma_\alpha$ with center $y$ and radius $r$, which is measured with respect to intrinsic distance.
- For $\lambda \in \mathbb{R}$, let
\[ M_\alpha^\lambda := \{|d_\alpha| < |d_{\alpha-1}| + \lambda \quad \text{and} \quad |d_\alpha| < |d_{\alpha+1}| + \lambda\}. \]
- In the Fermi coordinates with respect to $\Gamma_\alpha$, there exist two continuous functions $\rho_{\alpha}^+(y)$ such that
\[ M_\alpha^0 = \{(y,z) : \rho_{\alpha}^+(y) < z < \rho_{\alpha}^+(y)\}. \]
3.3. Deviation in \( z \). In this subsection we collect several estimates on the deviation of various terms in \( z \), when \( z \neq 0 \). Recall that \( \varepsilon \) is the upper bound on curvatures of level sets of \( u \), see (3.1).

By (3.1), \( |A_\alpha(y,0)| \lesssim \varepsilon \). Thus by (3.5), for \( |z| < R \), \( |A_\alpha(y,z)| \lesssim \varepsilon \). As in [19], we also have

**Lemma 3.1.** For any \( y \in \Gamma_\alpha \cap B_{R-1}(0) \),

\[
|\nabla_{\alpha,0} A_\alpha(y,0)| + |\nabla_{\alpha,0}^2 A_\alpha(y,0)| \lesssim \varepsilon.
\]

By (3.5), we have

\[
|A_\alpha(y,z) - A_\alpha(y,0)| \lesssim |z||A_\alpha(y,0)|^2 \lesssim \varepsilon^2 |z|.
\]

Similarly, by (3.4), the deviation of metric tensors is

\[
|g_{\alpha,ij}(y,z) - g_{\alpha,ij}(y,0)| \lesssim \varepsilon |z|,
\]

\[
|g_{ij}(y,z) - g_{ij}(y,0)| \lesssim \varepsilon |z|.
\]

As a consequence, the deviation of mean curvature is

\[
|H_\alpha(y,z) - H_\alpha(y,0)| \lesssim \varepsilon^2 |z|.
\]

By (3.3) and (3.6), for any \( |z| < R \),

\[
\sum_{i,j=1}^{n-1} ([\nabla_{\alpha,z} g_{\alpha,ij}(y,z)] + [\nabla_{\alpha,z} g_{\alpha,ij}^2(y,z)] + [\nabla_{\alpha,z} g_{\alpha,ij}(y,z)] + [\nabla_{\alpha,z}^2 g_{\alpha,ij}^2(y,z)]) \lesssim \varepsilon.
\]

The Laplacian operator in Fermi coordinates has the form

\[
\Delta_{\mathbb{R}^n} = \Delta_{\alpha,z} - H_\alpha(y,z) \partial_z + \partial_{zz},
\]

where \( \Delta_{\alpha,z} \) is the Beltrami-Laplace operator on \( \Gamma_{\alpha,z} \), that is,

\[
\Delta_{\alpha,z} = \sum_{i,j=1}^{n-1} \frac{1}{\sqrt{\det(g_{\alpha,ij}(y,z))}} \frac{\partial}{\partial y_j} \left( \sqrt{\det(g_{\alpha,ij}(y,z))} g_{ij}^a(y,z) \frac{\partial}{\partial y_i} \right)
\]

\[
= \sum_{i,j=1}^{n-1} g_{ij}^a(y,z) \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{n-1} b_i^a(y,z) \frac{\partial}{\partial y_i}
\]

with

\[
b_i^a(y,z) = \frac{1}{2} \sum_{j=1}^{n-1} g_{ij}^a(y,z) \frac{\partial}{\partial y_j} \log \det(g_{\alpha,ij}(y,z)).
\]

By (3.8) and (3.10), we get

**Lemma 3.2.** For any function \( \varphi \in C^2(\Gamma_\alpha) \) and \( |z| < R \),

\[
|\Delta_{\alpha,z} \varphi - \Delta_{\alpha,0} \varphi| \lesssim \varepsilon |z| \left( |\nabla_{\alpha,0}^2 \varphi| + |\nabla_{\alpha,0} \varphi| \right).
\]

Finally we recall a commutator estimate from [19].

**Lemma 3.3.** For any \( \varphi \in C^3(\Gamma_\alpha) \) and \( |z| < R \),

\[
\left| \frac{\partial}{\partial y_i} \Delta_{\alpha,z} \varphi - \Delta_{\alpha,z} \frac{\partial \varphi}{\partial y_i} \right| \lesssim \varepsilon \left( |\nabla_{\alpha,0}^2 \varphi| + |\nabla_{\alpha,0} \varphi| \right).
\]
3.4. Comparison of distance functions. Given a point $X$, let $\Pi_\alpha(X)$ be the nearest point on $\Gamma_\alpha$ to $X$. The following lemma is taken from [19].

**Lemma 3.4.** For any $K > 0$, there exists a constant $C(K)$ so that the following holds. For any $X \in B_{7R/8}(0)$ and $\alpha \neq \beta$, if $|d_\alpha(X)| \leq K|\log \varepsilon|$ and $|d_\beta(X)| \leq K|\log \varepsilon|$ at the same time, then we have

$$
\text{dist}_{\Gamma_\beta} (\Pi_\beta \circ \Pi_\alpha(X), \Pi_\beta(X)) \leq C(K) \varepsilon^{1/2} |\log \varepsilon|^{3/2},
$$

$$
|d_\beta (\Pi_\alpha(X)) + d_\alpha (\Pi_\beta(X))| \leq C(K) \varepsilon^{1/2} |\log \varepsilon|^{3/2},
$$

$$
|d_\alpha(X) - d_\beta(X) + d_\beta (\Pi_\alpha(X))| \leq C(K) \varepsilon^{1/2} |\log \varepsilon|^{3/2},
$$

$$
|d_\alpha(X) - d_\beta(X) - d_\beta (\Pi_\beta(X))| \leq C(K) \varepsilon^{1/2} |\log \varepsilon|^{3/2},
$$

$$
1 - \nabla d_\alpha(X) \cdot \nabla d_\beta(X) \leq C(K) \varepsilon^{1/2} |\log \varepsilon|^{3/2}.
$$

The following lemma is an easy consequence of Lemma 2.2.

**Lemma 3.5.** For any $\alpha \neq \beta$, both $\Pi_\beta|_{\Gamma_\alpha}$ and its inverse are Lipschitz continuous with their Lipschitz constants bounded by a universal constant $C$.

Finally, the following fact will be used a lot in this paper.

**Lemma 3.6.** For any $y \in \Gamma_\alpha$,

$$
\sum_{\beta \neq \alpha} e^{-|d_\beta(y,0)|} \lesssim e^{-D_\alpha(y)}.
$$

**Proof.** By Lemma 2.1 and Lemma 3.4, there exists a constant $C \gg 1$ such that

$$
|d_\beta(y,0)| \geq D_\alpha(y) + C (|\beta - \alpha| - 1).
$$

Summing in $\beta$ we conclude the proof.

\[ \square \]

4. An approximate solution

4.1. Optimal approximation. Fix a function $\zeta \in C_0^\infty(-2,2)$ with $\zeta \equiv 1$ in $(-1,1)$, $|\zeta'| + |\zeta''| \leq 16$. Let

$$
\bar{g}(t) = \zeta(4|\log \varepsilon|t) g(t) + [1 - \zeta(4|\log \varepsilon|t)] \text{sgn}(t), \quad t \in (-\infty, +\infty).
$$

In particular, $\bar{g} \equiv 1$ in $(8|\log \varepsilon|, +\infty)$ and $\bar{g} \equiv -1$ in $(-\infty, -8|\log \varepsilon|)$.

$\bar{g}$ is an approximate solution to the one dimensional Allen-Cahn equation, that is,

$$
\bar{g}'' = W'(\bar{g}) + \bar{\xi},
$$

where $\text{spt}(\bar{\xi}) \subset \{4|\log \varepsilon| < |t| < 8|\log \varepsilon|\}$, and $|\bar{\xi}| + |\bar{\xi}'| + |\bar{\xi}''| \lesssim \varepsilon^3$.

We also have (see Appendix A for the definition of $\sigma_0$)

$$
\int_{-\infty}^{+\infty} \bar{g}'(t)^2 dt = \sigma_0 + O(\varepsilon^3).
$$

Suppose $u$ has the same sign as $(-1)^\alpha d_\alpha$ near $\Gamma_\alpha$. Given a function $h_\alpha \in C^2(\Gamma_\alpha)$, let

$$
g_\alpha(y, z; h_\alpha) := \bar{g}((-1)^\alpha (z - h_\alpha(y))),
$$

where $(y, z)$ is the Fermi coordinates with respect to $\Gamma_\alpha$. 

Given a sequence of functions \((h_\alpha) =: h\), define the function \(g(y, z; h)\) in the following way: for each \(\alpha\),
\[
g(y, z; h) := g_\alpha + \sum_{\beta < \alpha} \left[ g_\beta - (-1)\beta \right] + \sum_{\beta > \alpha} \left[ g_\beta + (-1)\beta \right] \quad \text{in } \mathcal{M}_\alpha^0.
\]

By the definition of \(\bar{g}\) and Lemma 2.1, there are only finitely many terms in the above sum.

For simplicity of notation, denote
\[
g'(y, z; h_\alpha) = \bar{g}'((-1)^\alpha (z - h_\alpha(y))), \quad g''(y, z; h_\alpha) = \bar{g}''((-1)^\alpha (z - h_\alpha(y))), \quad \cdots.
\]

**Proposition 4.1.** There exists \(h = (h_\alpha)\) with \(|h_\alpha| \ll 1\) for each \(\alpha\), such that for any \(\alpha\) and \(y \in \Gamma_\alpha \cap B_{7R/8}(0)\),
\[
(4.3) \quad \int_{-\infty}^{+\infty} [u(y, z) - g(y, z; h)] g'_\alpha(y, z; h_\alpha) dz = 0,
\]
where \((y, z)\) denotes the Fermi coordinates with respect to \(\Gamma_\alpha\).

**Proof.** Denote
\[
F(h) := \left( \int_{-\infty}^{+\infty} [u(y, z) - g(y, z; h)] g'_\alpha(y, z; h_\alpha) dz \right),
\]
which is viewed as a map from the Banach space \(\mathcal{X} := \bigoplus_\alpha C^0(\Gamma_\alpha \cap B_{7R/8}(0))\) to itself.

Clearly \(F\) is a \(C^1\) map. Furthermore, \((DF(h))_\alpha\), the \(\alpha\)-component of \(DF(h)\), equals
\[
(1)^\alpha \xi_\alpha(y) \int_{-\infty}^{+\infty} \left[ g'_\alpha(y, z; h_\alpha)^2 - (u(y, z) - g(y, z; h)) g''(y, z; h_\alpha) \right] dz
\]
\[\quad + \sum_{\beta \neq \alpha} (1)^\beta \int_{-\infty}^{+\infty} \xi_\beta(\Pi_\beta(y, z)) g'_\beta(y, z; h_\alpha) g'(y, z; h_\beta) \nabla d_\beta \cdot \nabla d_\alpha(y, z) dz.
\]

By Lemma 2.1, there exists a \(\tau > 0\) such that if for all \(\alpha\), \(|h_\alpha| \ll 1\), \(\mathcal{X} \leq C\). By Lemma 2.1, for all \(\varepsilon\) small enough, \(|F(0)|_{\mathcal{X}} \ll 1\). The existence of \(h\) then follows from the inverse function theorem.

**Remark 4.2.** The proof shows that for each \(\alpha\), \(|h_\alpha| \ll 1\), \(\mathcal{X} \leq C\). By differentiating (4.3), we can show that \(|h_\alpha| \ll 1\) for each \(\alpha\).

Denote \(g_\alpha(y, z) := g_\alpha(y, z; h_\alpha)\) and \(g_\star(y, z) := g(y, z; h)\), where \(h\) is given in the previous proposition. As before we denote
\[
g'_\alpha(y, z) = g'_\alpha(y, z; h_\alpha), \quad g''(y, z) = g''(y, z; h_\alpha), \quad \cdots.
\]

Let \(\phi := u - g_\star\) be the error between the solution \(u\) and the approximate solution \(g_\star\).

In the Fermi coordinates with respect to \(\Gamma_\alpha\), \(\phi\) satisfies the following equation
\[
\Delta_{\alpha, z}\phi - H_\alpha(y, z) \partial_z \phi + \partial_{zz} \phi
\]
\[(4.4) \quad W'(g_\ast + \phi) - \sum_\beta W'(g_\beta) + (-1)^{\alpha} g'_\alpha \left[ H_\alpha(y, z) + \Delta_{\alpha, z} h_\alpha(y) \right] - g''_\alpha |\nabla_{\alpha, z} h_\alpha|^2 \\
+ \sum_{\beta \neq \alpha} \left[ (-1)^{\beta} g'_\beta R_{\beta, 1} - g''_\beta R_{\beta, 2} \right] - \sum \xi_\beta, \]

where for each \(\beta\), in the Fermi coordinates with respect to \(\Gamma_\beta\),
\[\xi_\beta(y, z) = \hat{\xi} \left( (-1)^{\beta} (z - h_\beta(y)) \right),\]
\[R_{\beta, 1}(y, z) := H_{\beta}(y, z) + \Delta_{\beta, z} h_\beta(y),\]
\[R_{\beta, 2}(y, z) := |\nabla_{\beta, z} h_\beta(y)|^2.\]

4.2. Interaction terms. In this subsection we collect several estimates on the interaction term \(I := W'(g_\ast) - \sum_\beta W'(g_\beta)\) between different components.

**Lemma 4.3.** In \(M^3_\alpha\),
\[(4.5) \quad I = \left[ W''(g_\alpha) - 1 \right] \left[ g_{\alpha-1} - (-1)^{\alpha-1} \right] + \left[ W''(g_\alpha) - 1 \right] \left[ g_{\alpha+1} - (-1)^{\alpha+1} \right] + O \left( e^{-2d_{\alpha-1}} + e^{2d_{\alpha+1}} \right) + O \left( e^{-d_{\alpha-2} - |d_{\alpha}|} + e^{d_{\alpha+2} - |d_{\alpha}|} \right).\]

The following upper bound on the interaction term will be used a lot in the below.

**Lemma 4.4.** For any \((y, z) \in M^3_\alpha\),
\[|I(y, z)| \lesssim e^{-D_\alpha(y)} + \varepsilon^2.\]

The Lipschitz norm of interaction terms can also be estimated in a similar way.

**Lemma 4.5.** For any \((y, z) \in M^3_\alpha\),
\[\|I\|_{\text{Lip}(B_1(y, z))} \lesssim \max_{B_1^2(y)} e^{-D_\alpha} + \varepsilon^2.\]

4.3. Controls on \(h\) using \(\phi\). The choice of optimal approximation in Subsection 4.1 has the advantage that \(h\) is controlled by \(\phi\). This will allow us to iterate various elliptic estimates in Section 6 below.

**Lemma 4.6.** For each \(\alpha\) and \(y \in \Gamma_\alpha\), we have
\[(4.6) \quad \|h_\alpha\|_{C^{2, \theta}(B_1^2(y))} \lesssim \|\phi\|_{C^{2, \theta}(B_1(y, 0))} + \max_{B_1^2(y)} e^{-D_\alpha},\]
\[\|\nabla_{\alpha, 0} h_\alpha\|_{C^{1, \theta}(B_1^2(y))} \lesssim \|\nabla_{\alpha, 0} \phi\|_{C^{1, \theta}(B_1(y, 0))} + \varepsilon^{1/6} \max_{B_1^2(y)} e^{-D_\alpha}\]
\[+ \left( \max_{\beta : |d_{\beta}(y, 0)| \leq 8 |\log \varepsilon|} \|\nabla_{\beta, 0} h_\beta\|_{C^{1, \theta}(B_2^2(\Pi_{\beta}(y, 0)))} \right) \left( \max_{B_1^2(y)} e^{-D_\alpha} \right).\]

**Proof.** Fix an \(\alpha\). In the Fermi coordinates with respect to \(\Gamma_\alpha\), because \(u(y, 0) = 0\),
\[\phi(y, 0) = -\tilde{g} \left( (-1)^{\alpha+1} h_\alpha(y) \right) - \sum_{\beta < \alpha} \left[ \tilde{g} \left( (-1)^{\beta} (d_{\beta}(y, 0) - h_{\beta}(\Pi_{\beta}(y, 0))) \right) - (-1)^{\beta} \right] \]
\[(4.8) \quad - \sum_{\beta > \alpha} \left[ \tilde{g} \left( (-1)^{\beta} (d_{\beta}(y, 0) - h_{\beta}(\Pi_{\beta}(y, 0))) \right) + (-1)^{\beta} \right].\]
Note that for $\beta \neq \alpha$, $|h_\beta(\Pi_\beta(y,0))| \ll 1$. Then using Lemma 3.6, we get
\begin{equation}
|h_\alpha(y)| \lesssim |\phi(y,0)| + \sum_{\beta \neq \alpha} e^{-|d_\beta(y,0)|} \lesssim |\phi(y,0)| + e^{-D_\alpha(y)}.
\end{equation}

Differentiating (4.8), we get
\[
\nabla_{\alpha,0}\phi(y,0) = (-1)^{\alpha} g' ((-1)^{\alpha+1} h_\alpha(y)) \nabla_{\alpha,0} h_\alpha(y) - \sum_{\beta \neq \alpha} (-1)^{\beta} g'_\beta(y,0) \nabla_{\alpha,0} (d_\beta - h_\beta \circ \Pi_\beta)(y,0),
\]
and
\[
\nabla^2_{\alpha,0}\phi(y,0) = (-1)^{\alpha} g' ((-1)^{\alpha+1} h_\alpha(y)) \nabla^2_{\alpha,0} h_\alpha(y) - g'' ((-1)^{\alpha} h_\alpha(y)) \nabla_{\alpha,0} h_\alpha(y) \otimes \nabla_{\alpha,0} h_\alpha(y) - \sum_{\beta \neq \alpha} (-1)^{\beta} g''_\beta(y,0) \nabla_{\alpha,0} (d_\beta - h_\beta \circ \Pi_\beta)(y,0) \otimes \nabla_{\alpha,0} (d_\beta - h_\beta \circ \Pi_\beta)(y,0).
\]

By Lemma 3.4, if $g'_\beta(y,0) \neq 0$, we get
\[
|\nabla_{\alpha,0} d_\beta| = \sqrt{1 - \nabla d_\beta \cdot \nabla d_\alpha} \lesssim \varepsilon^{1/6}.
\]
Thus
\[
|\nabla_{\alpha,0} h_\alpha(y)| \lesssim |\nabla_{\alpha,0} \phi(y,0)| + \left( \varepsilon^{1/6} + |\nabla_{\beta,0} h_\beta(\Pi_\beta(y,0))| \right) \left( e^{d_{\alpha+1}(y,0)} + e^{-d_{\alpha-1}(y,0)} \right).
\]
A similar calculation leads to an upper bound on $|\nabla^2_{\alpha,0} h_\alpha(y)|$.

Finally, the Hölder estimate in (4.7) follows by combining the above representation formula, Lemma 4.5 and the bound
\[
|\nabla^2_{\alpha,0} d_\beta| \leq |\nabla^2 d_\beta| \lesssim \varepsilon.
\]
Here it is useful to note that $\nabla^2 d_\beta$ is the second fundamental form of $\Gamma_{\beta,z}$, as well as the fact that $\Pi_\beta(B^1_\alpha(y)) \subset B^3_2(\Pi_\beta(y,0))$ if $|d_\beta(y,0)| \leq 8|\log \varepsilon|$.

\section{5. A Toda system}

In the Fermi coordinates with respect to $\Gamma_\alpha$, multiplying (4.4) by $g'_\alpha$ and integrating in $z$ leads to
\begin{equation}
\int_{-\infty}^{+\infty} \left[ g'_\alpha \Delta_{\alpha,z} \phi - H_\alpha(y,z) g'_\alpha \partial_z \phi + g'_\alpha \partial_{zz} \phi \right]
= \int_{-\infty}^{+\infty} \left[ W'(g_\alpha + \phi) - \sum_{\beta} W'(g_\beta) \right] g'_\alpha + (-1)^{\alpha} \int_{-\infty}^{+\infty} [H_\alpha(y,z) + \Delta_{\alpha,z} h_\alpha(y)] g'_\alpha(z)^2
- \int_{-\infty}^{+\infty} g''_\alpha g'_\alpha \left| \nabla_{\alpha,z} h_\alpha \right|^2 + \sum_{\beta \neq \alpha} \int_{-\infty}^{+\infty} \left[ (-1)^{\beta} g'_\alpha g''_{\beta,1} R_{\beta,1} - g'_\alpha g''_{\beta,2} \right] - \sum_{\beta} \int_{-\infty}^{+\infty} \xi_{\beta} g'_\alpha.
\end{equation}

From this equation we deduce that
\begin{equation}
H_\alpha(y,0) + \Delta_{\alpha,0} h_\alpha(y) = \frac{2A^2(\alpha-1)}{\sigma_0} e^{-d_{\alpha-1}(y,0)} - \frac{2A^2(\alpha)}{\sigma_0} e^{d_{\alpha+1}(y,0)} + E^0_\alpha(y),
\end{equation}
where $E^0_\alpha$ is a higher order term. (See Appendix A for the definition of $A_1$ and $A_{-1}$.) More precisely, we have
Lemma 5.1. For any \( x \in B_{5R/6}(0) \) and \( r \in (0, R/7) \),
\[
\max_{\alpha} \left\| E^0_\alpha \right\|_{C^0(\Gamma_\alpha \cap B_r(x))} \lesssim \varepsilon^2 + \varepsilon^{\frac{2}{3}} A (r + 10|\log \varepsilon|; x) + A (r + 10|\log \varepsilon|; x)\frac{2}{3}
\]
\( (5.3) \)

\[+ \max_{\alpha} \left\| H_\alpha + \Delta_\alpha,0h_\alpha \right\|_{C^0(\Gamma_\alpha \cap B_{r+10}|\log \varepsilon|; x))} + \| \phi \|^2_{C^{2,\theta}(B_{r+10}|\log \varepsilon|; x))}.\]

The proof is given in Appendix B.

Since all terms in the right hand side of (5.3) are small quantities, a direct consequence of this lemma is

Corollary 5.2. There exists a universal constant \( C > 0 \) such that for any \( x \in B_{5R/6}(0) \) and \( r \in (0, R/7) \),
\[
\max_{\alpha} \left\| H_\alpha + \Delta_\alpha,0h_\alpha \right\|_{C^0(\Gamma_\alpha \cap B_r(x))}
\]
\( (5.4) \)

\[\leq \frac{1}{4} \left( \max_{\alpha} \left\| H_\alpha + \Delta_\alpha,0h_\alpha \right\|_{C^0(\Gamma_\alpha \cap B_{r+10}|\log \varepsilon|; x))} + \| \phi \|^2_{C^{2,\theta}(B_{r+10}|\log \varepsilon|; x))}\right)
\]
\[+ C\varepsilon^2 + CA (r + 10|\log \varepsilon|; x).\]

6. Estimates on \( \phi \)

In this section we prove the following \( C^{2,\theta} \) estimate on \( \phi \).

Proposition 6.1. For any \( x \in B_{5R/6}(0) \) and \( r \in (0, R/7) \),
\[
\max_{\alpha} \left\| H_\alpha + \Delta_\alpha,0h_\alpha \right\|_{C^0(\Gamma_\alpha \cap B_r(x))} + \| \phi \|^2_{C^{2,\theta}(B_r(x))} \lesssim \varepsilon^2 + A (r + 50|\log \varepsilon|^2; x). \]

The first order Hölder estimates of \( \phi \) will be established in Subsection 6.1 and Subsection 6.2. The second order Hölder estimate will be proved in Subsection 6.3.

To prove the first order Hölder estimate on \( \phi \), fix a large constant \( L > 0 \), for each \( \alpha \) define
\[
\Omega_\alpha^1 := \{ -L < d_\alpha < L \} \cap \mathcal{M}_\alpha^0, \quad \Omega_\alpha^2 := \{ d_\alpha > L/2 \} \cap \mathcal{M}_\alpha^0,
\]
and
\[
\Omega_\alpha^3 := \{ -L/2 \leq d_\alpha \leq L/2 \} \cap \mathcal{M}_\alpha^0.
\]

We will estimate the \( C^{1,\theta} \) norm of \( \phi \) in \( \Omega_\alpha^1 \cap B_r(x) \) and \( \Omega_\alpha^2 \cap B_r(x) \) separately. Roughly speaking, in \( \Omega_\alpha^1 \), \( \phi \) satisfies
\[-\Delta \phi + W''(g_\alpha)\phi = \text{interaction terms + parallel component + errors}.\]

Together with the orthogonal condition (4.3) we get a control on \( \phi \), which is possible by the decay estimate of the operator \(-\Delta + W''(g)\) in the class of functions satisfying the orthogonal condition (4.3), see for example [6]. In \( \Omega_\alpha^2 \), \( \phi \) satisfies
\[-\Delta \phi + \phi = \text{interaction terms + errors}.\]

Hence a control on \( \phi \) is possible by using the decay estimate of the coercive operator \(-\Delta + 1\).
6.1. $C^{1,\theta}$ estimate in $\Omega^2_\alpha$. We start with the easy case. In $\Omega^2_\alpha$, the equation for $\phi$ can be written in the following way.

**Lemma 6.2.** For any $\alpha$, in $\Omega^2_\alpha$,
\[
\Delta_{\alpha,z}\phi(y, z) - H_\alpha(y, z)\partial_z\phi(y, z) + \partial_{zz}\phi(y, z) = [1 + o(1)]\phi(y, z) + E^2_\alpha(y, z),
\]
where
\[
\|E^2_\alpha\|_{L^\infty(\Omega^2_\alpha \cap B_r(x))} \leq C\varepsilon^2 + CA (r + 10\log \varepsilon; x) + C\|\phi\|_{C^2,\theta(B_r + 10\log \varepsilon)(x)}^2 + Ce^{-L}\max_{\alpha}\|H_\alpha + \Delta_{\alpha,0}h_\alpha\|_{L^\infty(\Gamma_\alpha \cap B_{r + 10\log \varepsilon})(x)}.
\]

**Proof.** The $L^\infty$ estimate on $E^2_\alpha$ is a consequence of the following estimates on those terms in (4.4).
- First we have $W'(g_\alpha + \phi) - W'(g_\alpha) = [W''(g_\alpha) + O(\phi)]\phi = [1 + o(1)]\phi$.
- By Lemma 4.4, $W'(g_\alpha) - \sum_\beta W'(g_\beta) = O(e^{-D\phi}) + O(\varepsilon^2)$.
- By (3.7)-(3.12), we get
\[
g'_\alpha [H_\alpha(y, z) + \Delta_{\alpha,z}h_\alpha]
= g'_\alpha [H_\alpha(y, 0) + \Delta_{\alpha,0}h_\alpha(y)] + g'_\alpha [H_\alpha(y, z) - H_\alpha(y, 0)] + g'_\alpha [\Delta_{\alpha,z}h_\alpha - \Delta_{\alpha,0}h_\alpha]
= O(e^{-L}[H_\alpha(y, 0) + \Delta_{\alpha,0}h_\alpha(y)]) + O(\varepsilon^2) + O(\|
\n
Concerning estimates on the last two terms involving $h_\alpha$, we use Lemma 4.6.
- Similarly, estimates on $g''_\alpha|\nabla_{\alpha,z}h_\alpha|^2$ follow from Lemma 4.6.
- Those two terms involving $g''_\beta R_{\beta,1}$ and $g''_\beta R_{\beta,2}$ can be estimated as in the above two cases, but now in Fermi coordinates with respect to $\Gamma_\beta$. Note that we need only to consider those $\beta$ satisfying $\Gamma_\beta \cap B_{r + 8|\log \varepsilon|(x)} \neq \emptyset$, because otherwise $g''_\beta = 0$ in $B_r(x)$. To put all estimates of $\beta \neq \alpha$ together, we use Lemma 3.6.
- Finally, by definition of $\xi_\beta$, $\sum_\beta \xi_\beta = O(\varepsilon^3|\log \varepsilon|) = O(\varepsilon^2)$.

By standard interior elliptic estimates on the coercive operator $-\Delta + 1$, we deduce that, for any $\alpha$,
\[
\|\phi\|_{C^{1,\theta}(\Omega^2_\alpha \cap B_r(x))} \leq Ce^{-cL} \left(\|\phi\|_{C^{1,\theta}(\Omega^2_\alpha \cap B_{r + 10\log \varepsilon})(x)) + \|\phi\|_{C^{1,\theta}(\Omega^2_\alpha \cap B_{r + 10\log \varepsilon})(x)}\right)
+ C\varepsilon^2 + CA (r + 10\log \varepsilon; x).
\]

(6.2)

6.2. $C^{1,\theta}$ estimate in $\Omega^1_\alpha$. In $\Omega^1_\alpha$, the equation for $\phi$ can be written in the following way.

**Lemma 6.3.** In $\Omega^1_\alpha$,
\[
\Delta_{\alpha,0}\phi + \partial_{zz}\phi = W''(g_\alpha)\phi + (-1)^\alpha g'_\alpha [H_\alpha(y, 0) + \Delta_{\alpha,0}h_\alpha] + E^1_\alpha,
\]
where for some constant $C(L)$,
\[
\|E^1_\alpha\|_{L^\infty(\Omega^1_\alpha \cap B_r(x))} \leq C(L)\varepsilon^2 + CA (r + 10\log \varepsilon; x).\]

**Proof.** The proof is similar to the one for Lemma 6.2, in particular,
- we use Cauchy inequality and (3.12) to bound $\Delta_{\alpha,z}\phi - \Delta_{\alpha,0}\phi$ (here it is usefull to note that $|z| < 2L$ in $\Omega^1_\alpha$);
- we use Cauchy inequality and the fact that $|H_\alpha(y, z)| \lesssim \varepsilon$ to bound $H_\alpha(y, z)\partial_z\phi$;
- we use Lemma 4.4 to bound interaction terms;
- we use Lemma 4.6 to bound those terms involving $h_\alpha$. 

by the exponential decay of \( \bar{g}' \) at infinity and Lemma 3.6, \( \sum_{\beta \neq \alpha} g'_\beta R_{\beta,1} \) and \( \sum_{\beta \neq \alpha} g'_\beta R_{\beta,2} \) are bounded by \( e^{-D_\alpha(y)} \) in \( \Omega^2_\alpha \). (Although there are constants \( e^{CL} \) appearing when we bound \( g'_\beta \) by \( O(e^{-D_\alpha(y)}) \), they can be incorporated because \( |R_{\beta,1}| + |R_{\beta,2}| \ll 1 \) while \( L \), although large, is a fixed constant.) □

Take a function \( \xi \in C^\infty_c(-2L, 2L) \) satisfying \( \xi \equiv 1 \) in \( (-L, L) \), \( |\xi'| \lesssim L^{-1} \) and \( |\xi''| \lesssim L^{-2} \). Let \( \phi_{\alpha}(y, z) := \phi(y, z)\xi(z) - c_{\alpha}(y)g'_{\alpha}(y, z) \), where

\[
(6.3) \quad c_{\alpha}(y) = \frac{\int_{-\infty}^{+\infty} \phi(y, z) (\xi(z) - 1) g'_{\alpha}(y, z) \, dz}{\int_{-\infty}^{+\infty} g'_{\alpha}(y, z)^2 \, dz}.
\]

Hence by (4.3) we still have the orthogonal condition

\[
(6.4) \quad \int_{-\infty}^{+\infty} \phi_{\alpha}(y, z) g'_{\alpha}(y, z) \, dz = 0, \quad \forall y \in \Gamma_{\alpha}.
\]

We have the following estimates on \( c_{\alpha} \).

**Lemma 6.4.** For any \( y \in \Gamma_{\alpha} \),

\[
|c_{\alpha}(y)| \lesssim e^{-L} \max_{L < |z| < 8|\log \epsilon|} |\phi(y, z)|,
\]

\[
|\nabla_{\alpha,0} c_{\alpha}(y)| \lesssim e^{-L} \max_{L < |z| < 8|\log \epsilon|} (|\phi(y, z)| + |\nabla_{\alpha,z} \phi(y, z)|),
\]

\[
|\nabla_{\alpha,0}^2 c_{\alpha}(y)| \lesssim e^{-L} \max_{L < |z| < 8|\log \epsilon|} \left( |\phi(y, z)| + |\nabla_{\alpha,z} \phi(y, z)| + |\nabla_{\alpha,z}^2 \phi(y, z)| \right).
\]

**Proof.** By (6.3) and the definition of \( \bar{g} \) and \( \xi \),

\[
|c_{\alpha}(y)| \lesssim \left( \max_{L < |z| < 8|\log \epsilon|} |\phi(y, z)| \right) \int_{L}^{+\infty} e^{-z} \, dz \lesssim e^{-L} \max_{L < |z| < 8|\log \epsilon|} |\phi(y, z)|.
\]

Differentiating (6.3) gives

\[
\nabla_{\alpha,0} c_{\alpha}(y) \left( \int_{-\infty}^{+\infty} g'_{\alpha}(y, z)^2 \, dz \right) + c_{\alpha}(y) \left( \nabla_{\alpha,0} \int_{-\infty}^{+\infty} g'_{\alpha}(y, z)^2 \, dz \right)
\]

\[
= \int_{-\infty}^{+\infty} \nabla_{\alpha,0} \phi(y, z) (\xi(z) - 1) g'_{\alpha}(y, z) \, dz
\]

\[
- \, (-1)^\alpha \nabla_{\alpha,0} h_{\alpha}(y) \int_{-\infty}^{+\infty} \phi(y, z) (\xi(z) - 1) g''_{\alpha}(y, z) \, dz.
\]

The second estimate follows as above. The third one can be proved in the same way. □

The factor \( e^{-L} \) reveals the fact that behavior of \( \phi \) in \( \Omega^2_\alpha \) has little effect on the behavior of \( \phi \) in \( \Omega^1_\alpha \), that is, these two parts are almost decoupled.

In the Fermi coordinates with respect to \( \Gamma_{\alpha} \), the equation satisfied by \( \phi_{\alpha} \) reads as

\[
(6.5) \quad \Delta_{\alpha,0} \phi_{\alpha} + \partial_{zz} \phi_{\alpha} = W''(g_{\alpha}) \phi_{\alpha} + p_{\alpha}(y)g'_{\alpha} + F_{\alpha},
\]

where

\[
p_{\alpha}(y) = (-1)^\alpha [H_{\alpha}(y, 0) + \Delta_{\alpha,0} h_{\alpha}(y)] - \Delta_{\alpha,0} c_{\alpha}(y)
\]

and

\[
F_{\alpha}(y, z) = 2(-1)^\alpha \nabla_{\alpha,0} c_{\alpha}(y) \cdot \nabla_{\alpha,0} h_{\alpha}(y) g''_{\alpha}(y, z)
\]
Combining this expression with Lemma 6.3, Lemma 6.4 and the definition of $\xi$, we obtain

**Lemma 6.5.** For any $x \in B_{5R/6}(0)$ and $r \in (0, R/7)$,

$$\|F_\alpha\|_{L^\infty(\Omega_{\alpha}^1 \cap B_r(x))} \leq C (L) \varepsilon^2 + C(L)A (r + 10 \log \varepsilon; x) + C(L)\|\phi\|_{C^{2,\theta}(B_{r+10} \log \varepsilon; x)}^2$$

$$+ Ce^{-L} \max_\alpha \|H_\alpha + \Delta_\alpha \delta \phi\|_{L^\infty(\Gamma_{\alpha} \cap B_{r+10} \log \varepsilon; x)} + \|\phi\|_{C^{2,\theta}(B_{r+10} \log \varepsilon; x)}.$$

By (6.5) and the orthogonal condition (6.4), applying standard estimates on the linearized operator $-\Delta + W''(g)$ (see for example [6, Proposition 4.1]) leads to

$$\|\phi\|_{C^{1,\theta}(B_r(x))} \leq Ce^{-cL} \|\phi\|_{C^{1,\theta}(B_{r+10} \log \varepsilon; x)} + Ce^{-L} \max_\alpha \|H_\alpha + \Delta_\alpha \delta \phi\|_{L^\infty(\Gamma_{\alpha} \cap B_{r+10} \log \varepsilon; x)} + \|\phi\|_{C^{2,\theta}(B_{r+10} \log \varepsilon; x)}.$$

Coming back to $\phi$, by the fact that $\|\phi\|_{C^{2,\theta}(B_{r+10} \log \varepsilon; x)} \ll 1$ and the estimates on $c_\alpha$ in Lemma 6.4, we get

$$\|\phi\|_{C^{1,\theta}(\Omega_{\alpha}^1 \cap B_r(x))} \leq Ce^{-cL} + C A (r + 10 \log \varepsilon; x)$$

$$+ Ce^{-L} \max_\alpha \|H_\alpha + \Delta_\alpha \delta \phi\|_{L^\infty(\Gamma_{\alpha} \cap B_{r+10} \log \varepsilon; x)} + \|\phi\|_{C^{2,\theta}(B_{r+10} \log \varepsilon; x)}.$$

Combining (6.2) and (6.6), by choosing $L$ large enough and denoting $\sigma(L) := Ce^{-cL}$, we obtain

$$\|\phi\|_{C^{1,\theta}(B_r(x))} \leq C (L) \varepsilon^2 + C(L)A (r + 10 \log \varepsilon; x)$$

$$+ \sigma(L) \left( \max_\alpha \|H_\alpha + \Delta_\alpha \delta \phi\|_{L^\infty(\Gamma_{\alpha} \cap B_{r+10} \log \varepsilon; x)} + \|\phi\|_{C^{2,\theta}(B_{r+10} \log \varepsilon; x)} \right).$$

**6.3. Second order Hölder estimates on $\phi$.** First we have the following Hölder bounds on the right hand side of (4.4).

**Lemma 6.6.** For any $x \in B_{5R/6}(0)$ and $r \in (0, R/7)$,

$$\|\Delta \phi - W''(g) \phi\|_{C^\theta(B_r(x))} \leq \varepsilon^2 + A (r + 10 \log \varepsilon; x) + \|\phi\|_{C^{2,\theta}(B_{r+10} \log \varepsilon; x)}^2$$

$$+ A (r + 10 \log \varepsilon; x) \left( \max_\alpha \|H_\alpha + \Delta_\alpha \delta \phi\|_{C^\theta(\Gamma_{\alpha} \cap B_{r+10} \log \varepsilon; x)} + \|\phi\|_{C^{2,\theta}(B_{r+10} \log \varepsilon; x)} \right).$$

The proof is given in Appendix C.

By (6.7) and Schauder estimates, we get

$$\|\phi\|_{C^{2,\theta}(B_r(x))} \leq \sigma(L) \left( \max_\alpha \|H_\alpha + \Delta_\alpha \delta \phi\|_{C^\theta(\Gamma_{\alpha} \cap B_{r+10} \log \varepsilon; x)} + \|\phi\|_{C^{2,\theta}(B_{r+10} \log \varepsilon; x)} \right)$$

$$+ C(L) \varepsilon^2 + C(L)A (r + 10 \log \varepsilon; x).$$
Combining this estimate with Corollary 5.2, we get
\[
\max_{\alpha} \|H_\alpha + \Delta_{\alpha,0}h_\alpha\|_{C^\theta(\Gamma_{\alpha}\cap B_r(x))} + \|\phi\|_{C^{2,\theta}(B_r(x))} \\
\leq \frac{1}{2} \left( \max_{\alpha} \|H_\alpha + \Delta_{\alpha,0}h_\alpha\|_{C^\theta(\Gamma_{\alpha}\cap B_{r+10}\log \varepsilon_1(x))} + \|\phi\|_{C^{2,\theta}(B_{r+10}\log \varepsilon_1(x))} \right) \\
+ C(L)\varepsilon^2 + C(L)A(r + 10|\log \varepsilon|; x).
\]

An iteration of this inequality from \( r + 50|\log \varepsilon|^2 \) to \( r \) leads to (6.1). The proof of Proposition 6.1 is thus complete.

7. Improved estimates on horizontal derivatives

In this section we prove an improvement on the \( C^{1,\theta} \) estimates of horizontal derivatives of \( \phi, \phi_i := \partial\phi/\partial y_i, \ 1 \leq i \leq n - 1 \).

**Proposition 7.1.** For any \( x \in B_{5R/6}(0) \) and \( r \in (0, R/7) \),
\[
\|\phi_i\|_{C^{1,\theta}(B_r(x))} \lesssim \varepsilon^2 + A \left( r + 60|\log \varepsilon|^2; x \right)^{3/2} + \varepsilon^{1/6}A \left( r + 60|\log \varepsilon|^2; x \right).
\]

Combining this with Lemma 4.6, we obtain

**Corollary 7.2.** For any \( x \in B_{5R/6}(0) \) and \( r \in (0, R/7) \),
\[
\max_{\alpha} \|\nabla h_\alpha\|_{C^{1,\theta}(\Gamma_{\alpha}\cap B_r(x))} \lesssim \varepsilon^2 + A \left( r + 60|\log \varepsilon|^2; x \right)^{3/2} + \varepsilon^{1/6}A \left( r + 60|\log \varepsilon|^2; x \right).
\]

To prove Proposition 7.1, as in Section 6 we still estimate \( \phi_i \) in \( \Omega^1_{\alpha} \) and \( \Omega^2_{\alpha} \) separately. To this end, we first rewrite (4.4) as
\[
(7.1) \quad \Delta_{\alpha,z}\phi + \partial_{zz}\phi = W''(g_\alpha)\phi + (-1)\alpha g'_\alpha [H_\alpha(y,0) + \Delta_{\alpha,0}h_\alpha(y)] + \mathcal{I} + E_\alpha,
\]
where
\[
E_\alpha = H_\alpha(y,z)\partial_z\phi + \left[ W'(g_\alpha + \phi) - W'(g_\alpha) - W''(g_\alpha)\phi \right] \\
+ (-1)\alpha g'_\alpha [H_\alpha(y,z) - H_\alpha(y,0) + \Delta_{\alpha,z}h_\alpha(y) - \Delta_{\alpha,0}h_\alpha(y)] \\
- g'_\alpha \nabla_{\alpha,z}h_\alpha^2 + \sum_{\beta \neq \alpha} \left[ (-1)^{2\beta/3}g_{\beta,1} - g_{\beta,2}^2 \right] - \sum_\beta \xi_\beta.
\]

The following \( L^\infty \) bound on \( E_\alpha \) follows from (6.1) and the calculation in Appendix C.

**Lemma 7.3.** For any \( x \in B_{5R/6}(0) \) and \( r \in (0, R/7) \),
\[
\|E_\alpha\|_{L^\infty(\mathcal{M}^0_{\alpha}(\cap B_r(x)) \lesssim \varepsilon^2 + A \left( r + 50|\log \varepsilon|^2; x \right)^{3/2}.
\]

Differentiating (7.1) in \( y_i \), we obtain an equation for \( \phi_i := \phi_{y_i} \), which in Fermi coordinates with respect to \( \Gamma_\alpha \) reads as
\[
(7.2) \quad \Delta_{\alpha,z}\phi_i + \partial_{zz}\phi_i = W''(g_\alpha)\phi_i - (-1)\alpha g'_\alpha [H_{\alpha,i}(y,0) + \Delta_{\alpha,0}h_{\alpha,i}(y)] + \partial_{y_i}\mathcal{I} + \partial_i E_\alpha + E_i,
\]
where \( H_{\alpha,i}(y,0) := \partial_{y_i}H_\alpha(y,0), h_{\alpha,i}(y) := \partial_{y_i}h_\alpha \) and the remainder term
\[
E_i = \left( \Delta_{\alpha,z}\phi_i - \partial_{y_i}\Delta_{\alpha,z}\phi \right) + \left[ W''(g_\alpha) - W''(g_\alpha) \right] \phi_i \\
+ W'''(g_\alpha)\phi \left( \sum_{\beta \neq \alpha} (-1)^{2\beta/3}g_{\beta} \partial_{y_i}h_{\alpha,i} - \sum_{j=1}^{n-1} (h_{\beta,j} \circ \Pi_{\beta}) \frac{\partial \Pi_{\beta}^j}{\partial y_i} \right) \\
+ \sum_{\beta \neq \alpha} (-1)^{2\beta/3}g_{\beta} \partial_{y_i}h_{\alpha,i} \left( \sum_{j=1}^{n-1} (h_{\beta,j} \circ \Pi_{\beta}) \frac{\partial \Pi_{\beta}^j}{\partial y_i} \right)
\]
We have the following $L^\infty$ bound on $E_i$.

**Lemma 7.4.** For any $x \in B_{5R/6}(0)$ and $r \in (0, R/7)$,
\[
\|E_i\|_{L^\infty(M_0^0 \cap B_r(x))} \lesssim \varepsilon^2 + A (r + 60 \log \varepsilon^2; x)^{3/2} + \varepsilon^{1/3} A (r + 60 \log \varepsilon^2; x).
\]

**Proof.** We estimate the five terms one by one.

1. By Lemma 3.3, for $(y, z) \in M_0^0 \cap B_r(x)$,
\[
|I| \lesssim \varepsilon (|\nabla^2_{\alpha,0} \phi(y, z)| + |\nabla_{\alpha,0} \varphi(y, z)|) \lesssim \varepsilon^2 + \|\phi\|_{\mathcal{C}^2, \sigma(B_r(x))}^2.
\]

2. For $(y, z) \in M_0^0 \cap B_r(x)$, by Taylor expansion and Lemma 3.6 we get
\[
\left| W'(g_\ast + \phi) - W'(g_\alpha) \right| \lesssim |\phi| + \sum_{\beta \neq \alpha} g'_\beta 
\]
\[
\lesssim \|\phi\|_{L^\infty(B_r(x))} + \max_{\Gamma_\alpha \cap B_r(x)} e^{-D_\alpha} + \varepsilon^2.
\]

Hence
\[
\|II\|_{L^\infty(M_0^0 \cap B_r(x))} \lesssim \|\phi\|_{\mathcal{C}^2, \sigma(B_r(x))} \left(\|\phi\|_{\mathcal{C}^2, \sigma(B_r(x))} + \max_{\Gamma_\alpha \cap B_r(x)} e^{-D_\alpha} + \varepsilon^2\right)
\]
\[
\lesssim \|\phi\|_{C^2, \sigma(B_r(x))} + A(r; x)^{1/2} \|\phi\|_{\mathcal{C}^2, \sigma(B_r(x))} + \varepsilon^2.
\]

3. For $\beta \neq \alpha$, if $g'_\beta \neq 0$, by Lemma 3.4,
\[
(7.3) \quad \left| \frac{\partial d_\beta}{\partial y_i} \right| \lesssim \varepsilon^{1/6}.
\]

By the Cauchy inequality, Lemma 3.6 and Lemma 4.6, we obtain
\[
\|III\|_{L^\infty(M_0^0 \cap B_r(x))} \lesssim \|\phi\|_{C^1, \sigma(B_r+8\log \varepsilon; x)}^2 + A(r + 8 \log \varepsilon; x)^2 + \varepsilon^{1/6} \|\phi\|_{C^1, \sigma(B_r+8\log \varepsilon; x)}.
\]

4. By Lemma 3.3 and Lemma 4.6,
\[
\|IV\|_{L^\infty(M_0^0 \cap B_r(x))} \lesssim \varepsilon^2 + \|\phi\|_{C^2, \sigma(B_r(x))}^2 + A(r; x)^2.
\]

5. By the Cauchy inequality and Lemma 4.6,
\[
\|V\|_{L^\infty(M_0^0 \cap B_r(x))} \lesssim \|\phi\|_{C^2, \sigma(B_r(x))}^2 + \max_{\Gamma_\alpha \cap B_r(x)} e^{-2D_\alpha} + \|H_\alpha + \Delta_{\alpha,0} h_\alpha\|_{L^\infty(\Gamma_\alpha \cap B_r(x))}^2.
\]

Putting these estimates together and applying (6.1) we conclude the proof. 

Finally, the order of $\partial_{y_i} \mathcal{I}$ is increased by one due to the appearance of one more term involving horizontal derivatives of $\phi$.

**Lemma 7.5.** For any $x \in B_{6R/7}(0)$ and $r \in (0, R/8)$,
\[
\|\partial_{y_i} \mathcal{I}\|_{L^\infty(B_r(x))} \lesssim \varepsilon^2 + A (r + 60 \log \varepsilon^2; x)^2 + \varepsilon^{1/6} A (r + 60 \log \varepsilon^2; x).
\]
Proof. We have
\[
\partial_y I = \sum_{\beta} (-1)^{\beta+1} [W''(g_*) - W''(g_{\beta})] g'_\beta \left( \frac{\partial d_\beta}{\partial y_i} - \sum_{j=1}^{n-1} h_{\beta,j} (\Pi_{\beta}(y,z)) \frac{\partial \Pi_{\beta}^j}{\partial y_i}(y,z) \right).
\]

Let us first give an estimate on \([W''(g_*) - W''(g_{\beta})]\) \(g'_\beta\) in \(M_0^0\). There are two cases.

- If \(\beta = \alpha\), we have
  \[
  \left| W''(g_*) - W''(g_{\alpha}) \right| g'_\alpha \lesssim g'_\alpha \sum_{\beta \neq \alpha} (1 - g_{\beta}^2) \lesssim e^{-D\alpha},
  \]
  where the last inequality follows the same reasoning in the proof of Lemma 4.4.

- If \(\beta \neq \alpha\), still as in the proof of Lemma 4.4,
  \[
  (7.4) \quad \left[ W''(g_*) - W''(g_{\beta}) \right] g'_\beta = \left[ W''(g_{\alpha}) - W''(1) + O \left( \sum_{\beta \neq \alpha} (1 - g_{\beta}^2) \right) \right] g'_\beta
  = O (\varepsilon^2) + O \left( e^{-|d_{\beta}(y,0)|} \right).
  \]

Hence by Lemma 3.6, we have
\[
(7.5) \quad \sum_{\beta} (-1)^{\beta+1} \left[ W''(g_*) - W''(g_{\beta}) \right] g'_\beta = O (\varepsilon^2) + O \left( e^{-D\alpha} \right).
\]

Using Lemma 4.6 to bound \(h_{\beta,j}\), we see if \(g'_\beta \neq 0\), (7.3) holds and
\[
(7.6) \quad \left| \sum_{j=1}^{n-1} h_{\beta,j} (\Pi_{\beta}(y,z)) \frac{\partial \Pi_{\beta}^j}{\partial y_i}(y,z) \right| \lesssim A(r;x) + \|\phi\|_{C^1,\theta(B_r(x))}.
\]

Combining (7.5), (7.3) and (7.6), using the Cauchy inequality and applying (6.1) we conclude the proof. \(\square\)

Differentiating (4.3) we obtain for any \(\alpha\) and \(y \in \Gamma_\alpha \cap B_r(x)\),
\[
(7.7) \quad \int_{-\infty}^{+\infty} \phi_i g_{\alpha,i}' \, dz = h_{\alpha,i}(y) \int_{-\infty}^{+\infty} \phi g_{\alpha}' \, dz = O \left( \|\phi\|_{C^1(B_r; \log \varepsilon(x))}^2 + \max_{\Gamma_\alpha \cap B_r(x)} e^{-2D\alpha} \right).
\]

In view of Lemma 7.3, Lemma 7.4 and Lemma 7.5, combining (7.2) and the almost orthogonal condition (7.7), proceeding as in Section 6 we get Proposition 7.1. Note that although here we only have an \(L^\infty\) estimate on \(E_{\alpha}\) instead of \(\partial_y E_{\alpha}\), we can still use the \(W^{2,p}\) estimates (for a sufficiently large \(p\)) of the linear elliptic operator \(-\Delta + 1\) (in \(\Omega_\alpha^0\), see [6, Proposition 4.1]) and \(-\Delta + W''(g_{\alpha})\) (in \(\Omega_\alpha^0\)) to get the \(C^{1,\theta}\) bound on \(\phi_i\).

Now (5.2) can be rewritten in the following way.

**Corollary 7.6.** For any \(x \in B_{6R/7}(0)\) and \(r < R/8\), in \(B_r(x)\) it holds that
\[
H_{\alpha}(y,0) + \Delta_{\alpha,0} h_{\alpha}(y) = \frac{2A^2}{\sigma_0} \varepsilon^{-d_{\alpha}-1(y,0)} - \frac{2A^2}{\sigma_0} e^{d_{\alpha+1}(y,0)} + O (\varepsilon^2)
+ O \left( A(r + 60|\log \varepsilon|^2; x)^{3/2} \right) + O \left( \varepsilon^{1/6} A(r + 60|\log \varepsilon|^2; x) \right).
\]
8. Reduction of the stability condition

In this section we show that if \( u \) is a stable solution of the Allen-Cahn equation, then solutions to the Toda system (5.2) constructed in Section 5 satisfies an almost stable condition.

Given a point \( x \in B_{5R/6}(0) \) and \( r \in (0, R/7) \), and finitely many functions \( \eta_\alpha \in C_0^\infty (\Gamma_\alpha \cap B_r(x)) \), using Fermi coordinates with respect to \( \Gamma_\alpha \) we define

\[
\varphi_\alpha(y, z) := \eta_\alpha(y) g_\alpha(y, z).
\]

In the following we will view \( \eta_\alpha \) as a function defined in \( B_{r+8|\log \varepsilon|(x)} \) by identifying it with \( \eta_\alpha \circ \Pi_\alpha \).

Let \( \varphi := \sum \varphi_\alpha \). By definition \( \varphi \in C_0^\infty (B_{r+8|\log \varepsilon|(x)}) \). The stability condition for \( u \) says that

\[
\int_{B_{r+8|\log \varepsilon|(x)}} \left[ |\nabla \varphi|^2 + W''(u)\varphi^2 \right] \geq 0.
\]

The purpose of this section is to rewrite this inequality as a stability condition for the Toda system (5.2).

**Proposition 8.1.** If \( \eta_\alpha \) are given as above, then we have

\[
\sum_\alpha \int_{\Gamma_\alpha} |\nabla_{\alpha,0} \eta_\alpha|^2 dA_{\alpha,0} + Q(\eta)
\geq \sum_\alpha \frac{2A^2(-1)^{\alpha-1}}{\sigma_0} \int_{\Gamma_\alpha} e^{-d_{\alpha-1}(y)} \left[ \eta_\alpha(y) - \eta_{\alpha-1}(\Pi_{\alpha-1}(y, 0)) \right]^2 dA_{\alpha,0},
\]

where

\[
|Q(\eta)| \lesssim \left[ \varepsilon^{\frac{1}{4}} + A \left( r + 60|\log \varepsilon|^2; x \right)^{\frac{1}{2}} \right] \left( \sum_\alpha \int_{\Gamma_\alpha} |\nabla_{\alpha,0} \eta_\alpha|^2 dA_{\alpha,0} \right)
+ \left[ \varepsilon^2 + A \left( r + 60|\log \varepsilon|^2; x \right)^{\frac{3}{2}} + \varepsilon A \left( r + 60|\log \varepsilon|^2; x \right) \right] \left( \sum_\alpha \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0} \right).
\]

Since

\[
\int_{B_{r+8|\log \varepsilon|(x)}} \left[ |\nabla \varphi|^2 + W''(u)\varphi^2 \right] = \sum_\alpha \int_{B_{r+8|\log \varepsilon|(x)}} \left[ |\nabla \varphi_\alpha|^2 + W''(u)\varphi_\alpha^2 \right]
+ \sum_{\alpha \neq \beta} \int_{B_{r+8|\log \varepsilon|(x)}} \left[ \nabla \varphi_\alpha \cdot \nabla \varphi_\beta + W''(u)\varphi_\alpha \varphi_\beta \right],
\]

we first consider the first integrals and estimate the tangential part \( \nabla_{\alpha,z} \varphi_\alpha(y, z) \) in Subsection 8.1, then the normal part \( \partial_\gamma \varphi_\alpha \) in Subsection 8.2, where an interaction term appears and it is studied in Subsection 8.3, and finally in Subsection 8.4 estimates on cross terms are given. Proposition 8.1 follows by putting these estimates together.

### 8.1. The tangential part

In this subsection we prove

**Lemma 8.2.** The horizontal part has the expansion

\[
\int_{B_{r+8|\log \varepsilon|(x)}} |\nabla_{\alpha,z} \varphi_\alpha(y, z)|^2 = [\sigma_0 + O(\varepsilon + A(r; x))] \int_{\Gamma_\alpha} |\nabla_{\alpha,0} \eta_\alpha|^2 dA_{\alpha,0} + Q_\alpha(\eta),
\]

where
where
\[ |Q_\alpha(\eta_\alpha)| \lesssim \left[ \varepsilon^2 + A \left( r + 60 \log |\varepsilon|^2 ; x \right) \right] \frac{1}{2} + \varepsilon^\frac{1}{2} A \left( r + 60 \log |\varepsilon|^2 ; x \right) \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0} \]

Proof. A direct differentiation shows that in Fermi coordinates with respect to \( \Gamma_\alpha \),
\[ \nabla_{\alpha,z} \varphi_\alpha(y, z) = g'_\alpha(y, z) \nabla_{\alpha,z} \eta_\alpha(y) + (-1)^{\alpha+1} \eta_\alpha(y) g''_\alpha(y, z) \nabla_{\alpha,z} h_\alpha(y). \]
Hence
\[
\int_{B_{r+\delta|\log \varepsilon|}(x)} |\nabla_{\alpha,z} \varphi_\alpha(y, z)|^2
\]
\[
= \left[ \int_{-\infty}^{+\infty} \int_{\Gamma_\alpha} |\nabla_{\alpha,z} \eta_\alpha|^2 |g'_\alpha|^2 \lambda_\alpha dy dz \right] + \left[ \int_{-\infty}^{+\infty} \int_{\Gamma_\alpha} \eta_\alpha^2 |\nabla_{\alpha,z} h_\alpha|^2 |g''_\alpha|^2 \lambda_\alpha dy dz \right]
\]

These three integrals are estimated in the following way.

1. By (3.8), we have
\[ |\nabla_{\alpha,z} \eta_\alpha|^2 = [1 + O(\varepsilon|z|)] |\nabla_{\alpha,0} \eta_\alpha|^2 \]
and
\[ \lambda_\alpha(y, z) = \lambda_\alpha(y, 0) + O(\varepsilon|z|). \]

Hence by the exponential decay of \( g'_\alpha \) at infinity, we get
\[
I = \int_{\Gamma_\alpha} |\nabla_{\alpha,0} \eta_\alpha|^2 \left( \int_{-\infty}^{+\infty} |g'_\alpha|^2 dy \right) dA_{\alpha,0}
\]

2. By (6.1) and Lemma 4.6,
\[ II \lesssim [\varepsilon^2 + A(r + 60 \log \varepsilon|z|^2)] \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}. \]

3. Integrating by parts on \( \Gamma_{\alpha,z} \) leads to
\[
III = (-1)^{\alpha} \left[ \int_{-\infty}^{+\infty} \int_{\Gamma_{\alpha,z}} \eta_\alpha^2 \Delta_{\alpha,z} h_\alpha g'_\alpha g''_\alpha \lambda_\alpha dy dz \right]
\]
\[ + \int_{-\infty}^{+\infty} \int_{\Gamma_{\alpha,z}} \eta_\alpha^2 |\nabla_{\alpha,z} h_\alpha|^2 \left( |g'_\alpha|^2 + g''_\alpha g'_\alpha \right) \lambda_\alpha dy dz. \]

By Corollary 7.2 we get
\[ |III| \lesssim [\varepsilon^2 + A(r + 60 \log \varepsilon|z|^2)] \frac{1}{2} + \varepsilon^\frac{1}{2} A(r + 60 \log \varepsilon|z|^2) \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}. \]

Putting all of these together we finish the proof. \( \Box \)
8.2. The normal part. As before we have
\[ \partial_z \varphi_\alpha(y, z) = \eta_\alpha(y) g''_\alpha(y, z). \]
Integrating by parts in \( z \) we get
\[
\begin{align*}
\int_{ \Gamma_\alpha } \int_{ -\infty }^{ +\infty } \eta_\alpha(y)^2 |g''_\alpha(y, z)|^2 \lambda_\alpha(y, z) dz dy \\
= - \int_{ \Gamma_\alpha } \eta_\alpha(y)^2 \left[ \int_{ -\infty }^{ +\infty } W''(g_\alpha(y, z)) |g'_\alpha(y, z)|^2 \lambda_\alpha(y, z) dz \right] dy \\
+ \int_{ \Gamma_\alpha } \eta_\alpha(y)^2 \left[ \frac{1}{2} \int_{ -\infty }^{ +\infty } |g'_\alpha(y, z)|^2 \partial_z \lambda_\alpha(y, z) dz - \int_{ -\infty }^{ +\infty } g'_\alpha(y, z) \xi'_\alpha(y, z) \lambda_\alpha(y, z) dz \right] dy.
\end{align*}
\]
By (3.4) and the definition of \( \lambda_\alpha \) we have
\[ |\partial_z \lambda_\alpha(y, z)| \lesssim |A_\alpha(y)|^2 \lesssim \varepsilon^2. \]
Using this together with estimates on \( \xi_\alpha \) we get
\[ |II| \lesssim \varepsilon^2 \int_{ \Gamma_\alpha } \eta_\alpha^2 dA_{\alpha,0}. \]
It remains to rewrite the integral
\[
\int_{ \Gamma_\alpha } \eta_\alpha(y)^2 \int_{ -\infty }^{ +\infty } [W''(u(y, z)) - W''(g_\alpha(y, z))] |g'_\alpha(y, z)|^2 \lambda_\alpha(y, z) dz dy,
\]
which will be the goal of the next subsection.

8.3. The interaction part. Multiplying (4.4) by \( \eta_\alpha g''_\alpha \lambda_\alpha \) and then integrating in \( y \) and \( z \) gives
\[ I - II + III = IV + V - VI + VII - VIII, \]
where
\[
\begin{align*}
I &:= \int_{ \Gamma_\alpha } \eta_\alpha^2 \left[ \int_{ -\infty }^{ +\infty } \Delta_{\alpha,z} \psi g''_\alpha \lambda_\alpha dz \right] dy, \\
II &:= \int_{ \Gamma_\alpha } \eta_\alpha^2 \left[ \int_{ -\infty }^{ +\infty } H_\alpha(y, z) \psi z g''_\alpha \lambda_\alpha dz \right] dy, \\
III &:= \int_{ \Gamma_\alpha } \eta_\alpha^2 \left[ \int_{ -\infty }^{ +\infty } \phi_{zz} g''_\alpha \lambda_\alpha dz \right] dy, \\
IV &:= \int_{ \Gamma_\alpha } \eta_\alpha^2 \left[ \int_{ -\infty }^{ +\infty } \left( W'(u) - \sum_\beta W'(g_\beta) \right) g''_\alpha \lambda_\alpha dz \right] dy, \\
V &:= (-1)^\alpha \int_{ \Gamma_\alpha } \eta_\alpha^2 \left[ \int_{ -\infty }^{ +\infty } (H_\alpha(y, z) + \Delta_{\alpha,z} h_\alpha(y)) g''_\alpha \lambda_\alpha dz \right] dy, \\
VI &:= \int_{ \Gamma_\alpha } \eta_\alpha^2 \left[ \int_{ -\infty }^{ +\infty } |g''_\alpha|^2 \lambda_\alpha dz \right] dy, \\
VII &:= \sum_{ \beta \neq \alpha } \int_{ \Gamma_\alpha } \eta_\alpha^2 \left[ \int_{ -\infty }^{ +\infty } (-1)^\beta g'_\beta R_{\beta,1} - g''_\beta R_{\beta,2} g''_\alpha \lambda_\alpha dz \right] dy,
\end{align*}
\]
We need to estimate each of them.

1. By Proposition 7.1,

\[ |I| \lesssim \left[ \varepsilon^2 + A \left( r + 60 |\log \varepsilon|^2; x \right) \right]^2 + \varepsilon^2 A \left( r + 60 |\log \varepsilon|^2; x \right) \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}. \]

2. Because \( H_\alpha = O(\varepsilon) \), by (6.1),

\[ |II| \lesssim \varepsilon \| \phi \|_{C^2, \theta(B_{r+8}|\log \varepsilon|)(x)} \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0} \]

\[ \lesssim \left[ \varepsilon^2 + A \left( r + 60 |\log \varepsilon|^2; x \right) \right] \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}. \]

3. Integrating by parts in \( z \) gives

\[ III = - \int_{\Gamma_\alpha} \eta_\alpha^2 \left[ \int_{-\infty}^{+\infty} \phi_z g''_\alpha \lambda_\alpha d\zeta \right] d\eta - \int_{\Gamma_\alpha} \eta_\alpha^2 \left[ \int_{-\infty}^{+\infty} \phi_z g''_\alpha \partial_z \lambda_\alpha d\zeta \right] d\eta \]

\[ - \int_{\Gamma_\alpha} \eta_\alpha^2 \left[ \int_{-\infty}^{+\infty} \phi_z \xi'_\alpha \lambda_\alpha d\zeta \right] d\eta - \int_{\Gamma_\alpha} \eta_\alpha^2 \left[ \int_{-\infty}^{+\infty} \phi_z g''_\alpha \partial_z \lambda_\alpha d\zeta \right] d\eta. \]

Because \( \xi_\alpha = O(\varepsilon^2) \), the length \( |\{ z : \xi'_\alpha(z) \neq 0 \}| \lesssim |\log \varepsilon| \) and \( \partial_z \lambda_\alpha = O(\varepsilon) \) (see (3.4) and the definition of \( \lambda_\alpha \)), using (6.1) and reasoning as in the previous case we obtain

\[ (8.2) \quad III = - \int_{\Gamma_\alpha} \eta_\alpha^2 \left[ \int_{-\infty}^{+\infty} W''(g_\alpha) g'_\alpha \phi_z \lambda_\alpha d\zeta \right] d\eta + O \left( \varepsilon^2 + A \left( r + 60 |\log \varepsilon|^2; x \right)^2 \right) \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}. \]

4. Integrating by parts in \( z \) leads to

\[ IV = - \int_{\Gamma_\alpha} \eta_\alpha^2 \left[ \int_{-\infty}^{+\infty} W''(u) g'_\alpha \phi_z \lambda_\alpha d\zeta \right] d\eta \]

\[ - \int_{\Gamma_\alpha} \eta_\alpha^2 \left[ \int_{-\infty}^{+\infty} \left[ W''(u) - W''(g_\alpha) \right] |g'_\alpha|^2 \lambda_\alpha d\zeta \right] d\eta \]

\[ - \sum_{\beta \neq \alpha} \int_{\Gamma_\alpha} \eta_\alpha^2 \left[ \int_{-\infty}^{+\infty} \left[ W''(u) - W''(g_\beta) \right] g'_\alpha g'_\beta \left( \frac{\partial \lambda_\beta}{\partial z} - \frac{\partial}{\partial z} (h_\beta \circ \Pi_\beta) \right) \lambda_\alpha d\zeta \right] d\eta \]

\[ - \int_{\Gamma_\alpha} \eta_\alpha^2 \left[ \int_{-\infty}^{+\infty} \left( W'(u) - \sum_{\beta} W'(g_\beta) \right) g'_\alpha \partial_z \lambda_\alpha d\zeta \right] d\eta. \]

The first integral cancel with III (see (8.2)) up to a higher order term. The second integral is the one we want to rewrite in Subsection 8.2.
First let us estimate the term $X$. By Taylor expansion we have

$$W'(u) - \sum_{\beta} W'(g_{\beta}) = \mathcal{I} + O(\phi)$$

Then by (6.1), Lemma 4.4 and the fact that $\partial_x \lambda_\alpha = O(\epsilon)$, we get

$$|X| \lesssim \left[ \epsilon^2 + A(r + 60|\log \epsilon|^2; x) \right] \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}.$$  

(8.3)

It remains to rewrite the integral $IX_{\beta}$. First replace $W''(u)$ by $W''(g_\ast)$. This introduces an error bounded by

$$O\left( \|\phi\|_{C^{2,\beta}(B_{r+8|\log \epsilon|}(x))} \right) \int_{\Gamma_\alpha} \eta_\alpha^2 \int_{-\infty}^{+\infty} g'_\alpha g'_\beta \lambda_\alpha dzdy$$

$$\lesssim \left[ \epsilon^2 + A(r + 60|\log \epsilon|^2; x) \right] \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}.$$  

(8.4)

Next, if $g'_\alpha \neq 0$ and $g'_\beta \neq 0$ at the same time, by Lemma 3.4,

$$\frac{\partial d_\beta}{\partial z} = 1 + O\left( \epsilon^{1/3} \right).$$

(8.5)

$$d_\beta(y, z) = d_\beta(y, 0) \pm z + O\left( \epsilon^{1/3} \right).$$

(8.6)

Replace $\frac{\partial d_\beta}{\partial z}$ by 1 and throw away the term involving $h_\beta$. This introduces another error controlled by

$$\left[ \epsilon^\frac{1}{3} + A(r + 60|\log \epsilon|^2; x) \right] \int_{\Gamma_\alpha} \eta_\alpha^2 \int_{-\infty}^{+\infty} |W''(g_\ast) - W''(g_{\beta})| g'_\alpha g'_\beta \lambda_\alpha dzdy$$

(8.7)

$$\lesssim \left[ \epsilon^\frac{1}{3} + A(r + 60|\log \epsilon|^2; x) \right] \int_{\Gamma_\alpha} \eta_\alpha^2 \int_{-\infty}^{+\infty} g'_\alpha g'_\beta \sum_{\gamma} g'_\gamma dzdy$$

$$\lesssim \left[ \epsilon^\frac{1}{3} A(r + 60|\log \epsilon|^2; x) + A(r + 60|\log \epsilon|^2; x)^2 \right] \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}.$$  

Finally, in order to determine

$$XI_{\beta} := \int_{-\infty}^{+\infty} [W''(g_\ast) - W''(g_{\beta})] g'_\alpha g'_\beta \lambda_\alpha dz,$$

by (8.6) we can assume for each $\beta \neq \alpha$,

$$g_\beta(y, z) = \bar{g}\left( (-1)^{\beta}(d_\beta(y, 0) \pm z) \right).$$

We can also replace $\lambda_\alpha(y, z)$ by $\lambda_\alpha(y, 0)$. These two procedures lead to a third error, which can be estimated as in (8.7).

Then proceeding as in Case (8) in Appendix B and applying Lemma A.1 we get

$$XI_{\beta} = -2A_{(-1)^{\beta}e^{-|d_\beta(y, 0)|}} + O\left( e^{-\frac{1}{3}|d_\beta(y, 0)|} \right)$$

$$+ O\left( \epsilon^\frac{1}{3} |\log \epsilon|e^{-|d_\beta(y, 0)|} \right) + O\left( A(r + 60|\log \epsilon|^2; x)|d_\beta(y, 0)|e^{-|d_\beta(y, 0)|} \right).$$

If $|\beta - \alpha| \geq 2$ and $|d_\beta(y, 0)| \leq 2|\log \epsilon|$, by Lemma 3.4 we have

$$|d_\beta(y, 0)| \geq 2 \min_{\alpha} \min_{\Gamma_\alpha \cap B_{r}(x)} D_\alpha + C|\beta - \alpha| - C.$$
In particular, for any $|\beta - \alpha| \geq 2$,

(8.9) $|XI_\beta| \lesssim e^{-C|\beta - \alpha|} A(r; x)^2$.

Summing (8.8) in $\beta$ and applying Lemma 3.6, (8.5), (8.7) and (8.9) we get

\[
\sum_{\beta \neq \alpha} IX_\beta = -2 \int_{\Gamma_\alpha} \eta_\alpha(y) \left[ A_{(1)}^{2} e^{-d_{\alpha} - 1}(y, 0) + A_{(1)}^{2} e^{d_{\alpha} + 1}(y, 0) \right] dA_{\alpha, 0} \nonumber \\
+ O \left( \epsilon^2 + \epsilon^3 A (r + 60|\log \epsilon|^2; x) + A (r + 60|\log \epsilon|^2; x)^{\frac{3}{2}} \right) \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha, 0}.
\]

Combining (8.4) and (8.9) we get

\[
IV = - \int_{\Gamma_\alpha} \eta_\alpha^2 \int_{-\infty}^{+\infty} W''(u) g_\alpha' \phi z \lambda_\alpha dz dy \nonumber \\
- \int_{\Gamma_\alpha} \eta_\alpha^2 \int_{-\infty}^{+\infty} [W''(u) - W''(g_\alpha)] |g_\alpha'|^2 \lambda_\alpha dz dy \nonumber \\
- 2 \int_{\Gamma_\alpha} \eta_\alpha^2 \left[ A_{(1)}^{2} e^{-d_{\alpha} - 1} + A_{(1)}^{2} e^{d_{\alpha} + 1} \right] dA_{\alpha, 0} \nonumber \\
+ O \left( \epsilon^2 + \epsilon^3 A (r + 60|\log \epsilon|^2; x) + A (r + 60|\log \epsilon|^2; x)^{\frac{3}{2}} \right) \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha, 0}.
\]

(5) Integrating by parts in $z$ leads to

\[
V = \frac{(-1)^{\alpha+1}}{2} \int_{\Gamma_\alpha} \eta_\alpha^2 \left[ \int_{-\infty}^{+\infty} \frac{\partial}{\partial z} (H_\alpha(y, z) + \Delta_{\alpha, z} h_\alpha(y)) |g_\alpha'|^2 \lambda_\alpha dz \right] dy \nonumber \\
+ \frac{(-1)^{\alpha+1}}{2} \int_{\Gamma_\alpha} \eta_\alpha^2 \left[ \int_{-\infty}^{+\infty} (H_\alpha(y, z) + \Delta_{\alpha, z} h_\alpha(y)) |g_\alpha'|^2 \partial_z \lambda_\alpha dz \right] dy.
\]

By (3.5),

\[
\frac{\partial}{\partial z} H_\alpha(y, z) = O(\epsilon^2).
\]

By (3.4), (3.11), (6.1) and Lemma 4.6,

\[
\left| \frac{\partial}{\partial z} \Delta_{\alpha, z} h_\alpha(y) \right| \lesssim \epsilon \left( |\nabla_{\alpha, 0} h_\alpha(y)| + |\nabla_{\alpha, 0} h_\alpha(y)| \right) \lesssim \epsilon^2 + A (r + 60|\log \epsilon|^2; x)^2.
\]

Finally, because $\partial_z \lambda_\alpha = O(\epsilon)$, we have

\[
\int_{-\infty}^{+\infty} (H_\alpha(y, z) + \Delta_{\alpha, z} h_\alpha(y)) |g_\alpha'|^2 \partial_z \lambda_\alpha dz 
= \int_{-\infty}^{+\infty} (H_\alpha(y, 0) + \Delta_{\alpha, 0} h_\alpha(y) + O(\epsilon^2|z|) + O(\epsilon|z|)) |g_\alpha'|^2 \partial_z \lambda_\alpha dz 
= O \left( \epsilon^2 + A (r + 60|\log \epsilon|^2; x)^2 \right). \quad (By \ (5.1))
\]

Combining these three estimates we see

\[
|V| \lesssim \left[ \epsilon^2 + A (r + 60|\log \epsilon|^2; x)^2 \right] \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha, 0}.
\]
(6) By Lemma 4.6 and (6.1), we get
\[ |VI| \lesssim \left[ \varepsilon^2 + A (r + 60 \log \varepsilon^2; x)^{\frac{3}{2}} \right] \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}. \]

(7) Following the proof of (B.8), we get a similar estimate on
\[ \int_{-\infty}^{+\infty} \left[ (-1)^{r} g_\beta^r R_{\beta,1} - g_\beta^r R_{\beta,2} \right] g_\alpha^r \lambda_\alpha dz, \]
which then gives
\[ |VII| \lesssim \left[ \varepsilon^2 + A (r + 60 \log \varepsilon^2; x)^{\frac{3}{2}} \right] \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}. \]

(8) Finally, by the definition of \( \xi_\beta \) and (3.6), and because \( \{ g_\alpha^r (y, \cdot) \neq 0 \} \) has length at most \( 16 |\log \varepsilon| \), we obtain
\[ |VIII| \lesssim \varepsilon^3 |\log \varepsilon| \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0} \lesssim \varepsilon^2 \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}. \]

Combining all of these estimates together, we obtain
\[
\begin{align*}
\int_{\Gamma_\alpha} \eta_\alpha^2 \int_{-\infty}^{+\infty} [W''(u) - W''(g_\alpha)] [g_\alpha^r]^2 \lambda_\alpha dzdy &= \int_{\Gamma_\alpha} \eta_\alpha^2 \int_{-\infty}^{+\infty} [W''(u) - W''(g_\alpha)] g_\alpha^r \phi_2 \lambda_\alpha dzdy \\
&- 2 \int_{\Gamma_\alpha} \eta_\alpha^2 \left[ A_{(-1)^{\alpha - 1}}^2 e^{-d_{\alpha - 1}} + A_{(-1)^{\alpha}}^2 e^{d_{\alpha + 1}} \right] dA_{\alpha,0} \\
&+ \ O \left( \varepsilon^2 + \varepsilon^{\frac{1}{4}} A (r + 60 |\log \varepsilon^2; x) + A (r + 60 |\log \varepsilon^2; x)^{\frac{3}{2}} \right) \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}.
\end{align*}
\]

The first integral in the right hand side of this equation is estimated in the following way. As in (8.3) and Lemma 4.4,
\[ |W''(u) - W''(g_\alpha)| \lesssim |\phi| + \sum_{\beta \neq \alpha} (1 - g_\beta^2). \]

Then arguing as in the proof of (B.7) and using (6.1) to estimate \( \phi \), we see this integral is also bounded by
\[ \int_{\Gamma_\alpha} \eta_\alpha^2 \int_{-\infty}^{+\infty} [W''(u) - W''(g_\alpha)] [g_\alpha^r]^2 \lambda_\alpha dzdy \]
\[ \lesssim \varepsilon^3 |\log \varepsilon| \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}. \]

Therefore we arrive at the following form
\[
\begin{align*}
\int_{\Gamma_\alpha} \eta_\alpha^2 \int_{-\infty}^{+\infty} [W''(u) - W''(g_\alpha)] [g_\alpha^r]^2 \lambda_\alpha dzdy &= -2 \int_{\Gamma_\alpha} \eta_\alpha g_\alpha^2 \left[ A_{(-1)^{\alpha - 1}}^2 e^{-d_{\alpha - 1}} + A_{(-1)^{\alpha}}^2 e^{d_{\alpha + 1}} \right] dA_{\alpha,0} \\
&+ \ O \left( \varepsilon^2 + \varepsilon^{\frac{1}{4}} A (r + 60 |\log \varepsilon^2; x) + A (r + 60 |\log \varepsilon^2; x)^{\frac{3}{2}} \right) \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0}.
\end{align*}
\]

This completes the reduction of the vertical part.
8.4. Cross terms. In this section we estimate the integral of cross terms,
\[ \sum_{\alpha \neq \beta} \int_{B_{r+8|\log \varepsilon|}(x)} \left[ \nabla \varphi_\alpha \cdot \nabla \varphi_\beta + W''(u) \varphi_\alpha \varphi_\beta \right]. \]

**Lemma 8.3.** For any \( \alpha \neq \beta \), we have
\[
\left| \int_{B_{r+8|\log \varepsilon|}(x)} \nabla_{\alpha,z} \varphi_\alpha \cdot \nabla_{\alpha,z} \varphi_\beta \right| \lesssim A(r;x)^{1/2} \left[ \int_{\Gamma_\alpha} |\nabla_{\alpha,0} \eta_\alpha|^2 dA_{\alpha,0} + \int_{\Gamma_\beta} |\nabla_{\beta,0} \eta_\beta|^2 dA_{\beta,0} \right] + \left( \varepsilon^2 + \varepsilon^{3/2} A(r;x) + A(r;x)^{3/2} \right) \left[ \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0} + \int_{\Gamma_\beta} \eta_\beta^2 dA_{\beta,0} \right].
\]

**Proof.** In Fermi coordinates with respect to \( \Gamma_\alpha \), write
\[
\nabla_{\alpha,z} \varphi_\alpha \cdot \nabla_{\alpha,z} \varphi_\beta = \left[ g'_\alpha \nabla_{\alpha,z} \eta_\alpha - (-1)^{\alpha} \eta_\alpha g''_\alpha \nabla_{\alpha,z} h_\alpha \right] \times \left[ g'_\beta \nabla_{\alpha,z} \eta_\beta + (-1)^{\beta} \eta_\beta g''_\beta \left( \nabla_{\alpha,z} d_\beta - \nabla_{\beta,0} h_\beta \circ D_{\alpha,z} \Pi_\beta \right) \right]
\]

In the following we assume \( \beta < \alpha \). \( \Gamma_\alpha \) and \( \Gamma_\beta \) divide \( B_{r+8|\log \varepsilon|}(x) \) into three domains, \( \Omega_{0\alpha\beta}^0 \) the one between them, \( \Omega_{0\alpha\beta}^+ \) the one above \( \Gamma_\alpha \) and \( \Omega_{0\alpha\beta}^- \) the one below \( \Gamma_\beta \).

**Case 1.** In \( \Omega_{0\alpha\beta}^0 \), we have
\[
g'_\alpha(y,z)g'_\beta(y,z) + g'_\alpha(y,z)g''_\beta(y,z) + g'_\beta(y,z)g''_\alpha(y,z) + g'_\alpha(y,z)g''_\beta(y,z) \lesssim e^{-d_\beta(y,0)}.
\]
Using Lemma 4.6 and (6.1) to estimate terms involving \( h \), using Lemma 3.4 to estimate \( \nabla_{\alpha,z} d_\beta \) (note that if \( g'_\alpha(y,z)g''_\beta(y,z) \neq 0 \), then \( d_\beta(y,0) \leq 16|\log \varepsilon| \) ), we get
\[
\left| \nabla_{\alpha,z} \varphi_\alpha(y,z) \cdot \nabla_{\alpha,z} \varphi_\beta(y,z) \right| \lesssim e^{-d_\beta(y,0)} |\nabla_{\alpha,z} \eta_\alpha| |\nabla_{\alpha,z} \eta_\beta|
\]
\[
\lesssim \left[ \varepsilon^{1/2} A \left( r + 60 |\log \varepsilon|^2; x \right) + A \left( r + 60 |\log \varepsilon|^2; x \right)^2 \right] e^{-d_\beta(y,0)} \eta_\alpha \eta_\beta
\]
\[
+ \left[ \varepsilon^{3/2} A \left( r + 60 |\log \varepsilon|^2; x \right) \right] e^{-d_\beta(y,0)} \eta_\alpha |\nabla_{\alpha,z} \eta_\beta|
\]
\[
+ \left[ \varepsilon^{3/2} A \left( r + 60 |\log \varepsilon|^2; x \right) \right] e^{-d_\beta(y,0)} \eta_\beta |\nabla_{\alpha,z} \eta_\alpha|.
\]

**Subcase 1.1.** Here we show how to estimate the integral of the first term in the right hand side of (8.13). First by Lemma 3.5 we can replace \( \nabla_{\alpha,z} \eta_\alpha \) by \( \nabla_{\alpha,0} \eta_\alpha \). Then by Lemma 3.4 and Cauchy inequality we obtain
\[
\int_{\Omega_{0\alpha\beta}^0} e^{-d_\beta(y,0)} |\nabla_{\alpha,0} \eta_\alpha| |\nabla_{\alpha,z} \eta_\beta| \lesssim \int_{\Omega_{0\alpha\beta}^0} e^{-d_\beta(y,0)} |\nabla_{\alpha,0} \eta_\alpha|^2 + \int_{\Omega_{0\alpha\beta}^0} e^{-d_\beta(y,0)} |\nabla_{\alpha,z} \eta_\beta|^2.
\]
Since \( \Omega_{0\alpha\beta}^0 \subset \{(y,z) : |z| < 2d_\beta(y,0)\} \), the first integral is controlled by
\[
\int_{\Gamma_\alpha} \left( \int_{-2d_\beta(y,0)}^0 e^{-d_\beta(y,0)} dz \right) |\nabla_{\alpha,0} \eta_\alpha|^2 dA_{\alpha,0} \lesssim \left( \max_{\Gamma_\alpha \cap B_r(x)} d_\beta e^{-d_\beta} \right) \int_{\Gamma_\alpha} |\nabla_{\alpha,0} \eta_\alpha|^2 dA_{\alpha,0}.
\]
The second one can be estimated in the same way.
Subcase 1.2. To estimate the integral of $e^{1/6}e^{-d_3(y,0)}\eta_0\eta_\beta$, the above method needs a revision. Here we note that the domain of integration can be restricted to $\{|z| < 8|\log \varepsilon|\} \cap \{d_\beta < 8|\log \varepsilon|\}$, because otherwise $g_\alpha'$ or $g_\beta' = 0$. Hence we have

\[
\varepsilon^{\frac{1}{6}} \int_{\Omega_{\alpha,0} \cap \{z < 8|\log \varepsilon|\} \cap \{d_\beta(y,z) < 8|\log \varepsilon|\}} e^{-d_3(y,0)} \eta_0 \eta_\beta 
\lesssim \varepsilon^{\frac{1}{6}} \int_{\Omega_{\alpha,0} \cap \{z < 8|\log \varepsilon|\}} e^{-d_3(y,0)} \eta^2_\alpha + \varepsilon^{\frac{1}{6}} \int_{\Omega_{\alpha,0} \cap \{d_\beta(y,z) < 8|\log \varepsilon|\}} e^{-d_3(y,0)} \eta^2_\beta.
\]

The first integral is rewritten as

\[
\varepsilon^{\frac{1}{6}} \int_{\Gamma_\alpha} \left( \int_{6|\log \varepsilon|} e^{-d_3(y,0)} dy \right) \eta^2_\alpha dA_{\alpha,0} \lesssim \varepsilon^{\frac{1}{6}} |\log \varepsilon| \left( \max_{y \in \Gamma_\alpha \cap B_r(x)} e^{-d_3(y,0)} \right) \int_{\Gamma_\alpha} \eta^2_\alpha dA_{\alpha,0}
\lesssim \varepsilon^{\frac{1}{6}} A(r; x) \int_{\Gamma_\alpha} \eta^2_\alpha dA_{\alpha,0}.
\]

The second integral in (8.14) and integrals involving $\eta_\alpha |\nabla_{\alpha,z} \eta_\beta|$ as well as $\eta_\beta |\nabla_{\alpha,z} \eta_\alpha|$ can be estimated in a similar way.

Case 2. In $\Omega_{\alpha,0}^+$, we have

\[
\int_{\Omega_{\alpha,0}^+ \cap \{z < 8|\log \varepsilon|\}} e^{-d_3(y,0) - z|\nabla_{\alpha,0} \eta_\alpha||\nabla_{\beta,z} \eta_\beta|}
\lesssim \left( \int_{\Omega_{\alpha,0}^+ \cap \{z < 8|\log \varepsilon|\}} e^{-2d_3(y,0) - z|\nabla_{\alpha,0} \eta_\alpha|^2} \right)^\frac{1}{2} \left( \int_{\Omega_{\alpha,0}^+ \cap \{z < 8|\log \varepsilon|\}} e^{-z|\nabla_{\beta,z} \eta_\beta|^2} \right)^\frac{1}{2}
\]

\[
\lesssim \left( \int_{\Gamma_\alpha} \int_{0} e^{-2d_3(y,0) - z|\nabla_{\alpha,0} \eta_\alpha(y)|^2} dy dA_{\alpha,0} \right)^\frac{1}{2} \left( \int_{\Gamma_\beta} \int_{0} e^{-z|\nabla_{\beta,z} \eta_\beta|^2} dz dA_{\beta,0} \right)^\frac{1}{2}
\]

Other terms and integrals in $\Omega_{\alpha,0}^-$ can be estimated in the same way and we conclude the proof.

Lemma 8.4. For any $\alpha \neq \beta$, we have

\[
\int_{B_{r+8|\log \varepsilon|}(x)} \partial_x \varphi_\alpha \partial_x \varphi_\beta = -\int_{\Gamma_\alpha} \int_{-\infty}^{+\infty} \eta_\alpha \eta_\beta W''(g_\beta) g'_\alpha g'_\beta \lambda_\alpha dy + \mathcal{Q}_{\alpha,\beta}(\eta),
\]

where

\[
|\mathcal{Q}_{\alpha,\beta}(\eta)| \lesssim \left[ \varepsilon^{\frac{1}{6}} A(r; x) + A(r; x)^{\frac{3}{2}} \right] \left[ \int_{\Gamma_\alpha} \eta^2_\alpha dA_{\alpha,0} + \int_{\Gamma_\beta} \eta^3_\beta dA_{\beta,0} \right].
\]
+ \varepsilon \frac{1}{7} A(r; x) \int_{\Gamma_{\beta,0}} |\nabla_{\beta,0}\eta_\beta|^2 dA_{\beta,0}.

Proof. We have

\begin{equation}
(8.15) \quad \partial_2 \varphi_\alpha \partial_2 \varphi_\beta = \eta_\alpha \eta_\beta g_\alpha'' g_\beta'' \left[ \frac{\partial d_\beta}{\partial z} - (-1)^\beta \nabla h_\beta \cdot \frac{\partial \Pi_\beta}{\partial z} \right] + \eta_\alpha g_\alpha'' g_\beta' \left( \nabla_{\beta,0} \eta_\beta \cdot \frac{\partial \Pi_\beta}{\partial z} \right).
\end{equation}

By Lemma 3.4, if \( g_\alpha'' g_\beta'' \not= 0 \) or \( g_\alpha' g_\beta'' \not= 0 \), then

\begin{equation}
(8.16) \quad \left| \frac{\partial d_\beta}{\partial z} - 1 \right| + \left| \frac{\partial \Pi_\beta}{\partial z} \right| \lesssim \varepsilon^{1/6},
\end{equation}

We can proceed as in the proof of Lemma 8.3 to estimate the integral of

\[ \eta_\alpha g_\alpha'' \eta_\beta g_\beta'' \left( \frac{\partial d_\beta}{\partial z} - 1 - (-1)^\beta \nabla h_\beta \cdot \frac{\partial \Pi_\beta}{\partial z} \right) + \eta_\alpha g_\alpha'' g_\beta' \left( \nabla_{\beta,0} \eta_\beta \cdot \frac{\partial \Pi_\beta}{\partial z} \right). \]

It remains to determine the integral

\[ \int_{B_{r+8|\log \varepsilon|}(x)} \eta_\alpha \eta_\beta g_\alpha'' g_\beta''. \]

Write this in Fermi coordinates with respect to \( \Gamma_\alpha \). Integrating by parts in \( z \) leads to

\[ \int_{\Gamma_\alpha} \int_{-\infty}^{+\infty} \eta_\alpha \eta_\beta g_\alpha'' g_\beta'' \lambda_\alpha dz dy \]

\[ = - \int_{\Gamma_\alpha} \int_{-\infty}^{+\infty} \eta_\alpha \eta_\beta W''(g_\beta) g_\alpha' g_\beta'' \lambda_\alpha - \int_{\Gamma_\alpha} \int_{-\infty}^{+\infty} \eta_\alpha \left( \nabla_{\beta,0} \eta_\beta \cdot \frac{\partial \Pi_\beta}{\partial z} \right) g_\alpha g_\beta'' \lambda_\alpha \]

\[ - \int_{\Gamma_\alpha} \int_{-\infty}^{+\infty} \eta_\alpha \eta_\beta g_\alpha'' \xi_\beta' \lambda_\alpha - \int_{\Gamma_\alpha} \int_{-\infty}^{+\infty} \eta_\alpha \eta_\beta g_\alpha' g_\beta'' \partial_2 \lambda_\alpha dz \cdot dA. \]

When \( g_\alpha'' g_\beta'' \not= 0 \), by Lemma 3.4,

\[ \left| \frac{\partial}{\partial z} \eta_\beta \right| \leq \varepsilon \frac{1}{7} |\nabla_{\beta,0} \eta_\beta|. \]

Hence as in the proof of Lemma 8.3 (here it is useful to observe that \( g_\alpha' g_\beta'' = 0 \) outside the \( 8|\log \varepsilon| \) neighborhood of \( \Gamma_\alpha \cup \Gamma_\beta \)), we get

\begin{equation}
(8.17) \quad |I| \lesssim \varepsilon \frac{1}{7} A(r; x) \left[ \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0} + \int_{\Gamma_\beta} |\nabla_{\beta,0} \eta_\beta|^2 dA_{\beta,0} \right].
\end{equation}

By the definition of \( \xi_\beta \), we also have

\begin{equation}
(8.18) \quad |II| \lesssim \varepsilon^2 \left[ \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0} + \int_{\Gamma_\beta} \eta_\beta^2 dA_{\beta,0} \right].
\end{equation}

Because \( \partial_2 \lambda_\alpha = O(\varepsilon) \), we get

\begin{equation}
(8.19) \quad |III| \lesssim \varepsilon \frac{1}{7} A(r; x) \left[ \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0} + \int_{\Gamma_\beta} \eta_\beta^2 dA_{\beta,0} \right].
\end{equation}

The conclusion follows by combining (8.17)-(8.19). \( \Box \)
Lemma 8.5. We have
\[
\sum_{\beta \neq \alpha} \int_{B_{r+8}|\log \varepsilon|} \eta_\alpha \eta_\beta \left[ W''(u) - W''(g_\beta) \right] g_\alpha' g_\beta'
\]
\[
= -4 \sum_{\beta = \alpha, \alpha+1} A_{(-1)\alpha-1}^2 \int_{\Gamma_\alpha} \eta_\alpha(y) \eta_\beta (\Pi_\beta(y,0)) e^{-|d_\beta(y,0)|} dA_{\alpha,0}
\]
\[
+ O \left( \varepsilon^{\frac{1}{2}} A(r; x) + A (r + 60 |\log \varepsilon|^2; x) \right) \sum_{\beta \neq \alpha} \int_{\Gamma_\beta} |\nabla_{\beta,0} \eta_\beta|^2 dA_{\beta,0}
\]
\[
+ O \left( \varepsilon^2 + \varepsilon^{\frac{3}{2}} A(r; x) + A (r + 60 |\log \varepsilon|^2; x) \right) \sum_{\beta} \int_{\Gamma_\beta} \eta_\beta^2 dA_{\beta,0}.
\]

Proof. The proof is divided into three steps.

**Step 1.** First by Taylor expansion and (6.1), proceeding as in the proof of Lemma 8.3 we get
\[
\left| \int_{B_{r+8}|\log \varepsilon|} \left[ W''(u) - W''(g_\alpha) \right] \varphi_\alpha \varphi_\beta \right|
\lesssim \|\varphi\|_{L^\infty(B_{r+8}|\log \varepsilon|)} \int_{B_{r+8}|\log \varepsilon|} \eta_\alpha \eta_\beta g_\alpha' g_\beta'
\lesssim \left[ \varepsilon^2 + A (r + 60 |\log \varepsilon|^2; x) \right] \left[ \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0} + \int_{\Gamma_\beta} \eta_\beta^2 dA_{\beta,0} \right].
\]

**Step 2.** Next by Lemma 3.6 we have
\[
\int_{\Gamma_\alpha} \eta_\alpha(y) \left( \int_{-\infty}^{+\infty} |\eta_\beta (\Pi_\beta(y,0)) - \eta_\beta (\Pi_\beta(y,z))| \left[ W''(g_\alpha) - W''(g_\beta) \right] g_\beta' g_\alpha' \lambda_\alpha dz \right) dy
\lesssim \int_{0}^{1} \int_{\Gamma_\alpha} \eta_\alpha(y) \left( \int_{-\infty}^{+\infty} \left| \frac{d}{dt} \eta_\beta ((1 - t)\Pi_\beta(y,0) + t\Pi_\beta(y,z)) g_\beta' g_\alpha' \lambda_\alpha dz \right) dy
\lesssim \varepsilon^{\frac{1}{2}} \int_{0}^{1} \int_{\Gamma_\alpha} \eta_\alpha(y) \left( \int_{-\infty}^{+\infty} |\nabla_{\beta,0} \eta_\beta ((1 - t)\Pi_\beta(y,0) + t\Pi_\beta(y,z)) g_\beta' g_\alpha' \lambda_\alpha dz \right) dy
\lesssim \varepsilon^{\frac{1}{2}} A(r; x) \left[ \int_{\Gamma_\alpha} \eta_\alpha^2 dA_{\alpha,0} + \int_{\Gamma_\beta} |\nabla_{\beta,0} \eta_\beta|^2 dA_{\beta,0} \right].
\]

**Step 3.** Finally, by Lemma A.1 (and Lemma 3.4 to estimate errors coming from comparing distances) we get for $\beta = \alpha + 1$ or $\alpha - 1$,
\[
\int_{-\infty}^{+\infty} \left[ W''(g_\alpha(y,z)) - W''(g_\beta(y,z)) \right] g_\beta(y,z) g_\alpha' (y,z) dz
\]
\[
= -2A_{(-1)\alpha-\beta}^2 e^{-|d_\beta(y,0)|} + O \left( e^{-\frac{4}{3}|d_\beta(y,0)|} \right).
\]

and for $|\beta - \alpha| \geq 2$, we use the estimate
\[
\left| \int_{-\infty}^{+\infty} \left[ W''(g_\alpha(y,z)) - W''(g_\beta(y,z)) \right] g_\beta(y,z) g_\alpha' (y,z) dz \right|
\lesssim D_\alpha(e^{-|d_\beta(y,0)|})
\[
\leq \varepsilon^2 + \left( \max_{\alpha} \max_{y \in \Gamma_\alpha \cap B_r(x)} D_\alpha \right) A(r; x)^2
\]
\[
\leq \varepsilon^2 + A(r; x)^{3/2},
\]
where we have used the fact that if \(|d_\beta(y, 0)| \leq 3|\log \varepsilon|\), then by Lemma 3.4 we have
\[
|d_\beta(y, 0)| = |d_{\beta-1}(y, 0)| + |\Pi_\beta(y, 0) - \Pi_{\beta-1}(y, 0)| + o(1).
\]

Combining these three steps we finish the proof. \qed

9. A decay estimate

Recall the definition of \(A(r; x)\) in Section 3. In this section we establish the following decay estimate.

**Proposition 9.1.** There exist two universal constants \(M \gg K \gg 1\) such that for any \(r \in [2R/3, 5R/6]\), if
\[
\kappa := A(r; 0) \geq M \varepsilon^2 |\log \varepsilon|,
\]
then we have
\[
A(r - KR_\ast; 0) \leq \frac{1}{2} A(r; 0),
\]
where
\[
R_\ast := \max \left\{ \kappa^{-\frac{1}{2}}, 200|\log \varepsilon|^2 \right\}.
\]

9.1. Reduction to a decay estimate for Toda system. In this subsection we reduce the proof of Proposition 9.1 to a decay estimate for Toda system.

By (5.2) and Corollary 7.2, for any \(\beta\) and \(y \in \Gamma_\beta \cap B_{r-R}\),
\[
H_\beta(y, 0) = \frac{2A^2_{\beta}(0)}{\sigma_0} e^{-d_{\beta-1}(y, 0)} - \frac{2A^2_{\beta-1}(y, 0)}{\sigma_0} e^{d_{\beta+1}(y, 0)} + O \left( \kappa^{7/6} \right) + O \left( \varepsilon^2 \right).
\]

Take an arbitrary index \(\alpha\) and \(x \in \Gamma_\alpha \cap B_{r-KR_\ast}\). To prove Proposition 9.1, it suffices to show that
\[
e^{-D_\alpha(x)} \leq \kappa^2.
\]

After a rotation and a translation, assume \(x = 0\). In the finite cylinder \(C_{R/7}(0) := B_{R/7}^{-1}(0) \times (-R/7, R/7)\), \(\Gamma_\alpha\) is represented by the graph \(\{x_n = f_\alpha(y)\}\), where \(y \in B_{R/7}^{-1}(0)\). Without loss of generality assume it holds that
\[
f_\alpha(0) = 0, \quad \nabla f_\alpha(0) = 0.
\]

In the following, we also assume
\[
|d_\alpha(0)| \geq d_{\alpha+1}(0), \quad \text{and} \quad d_{\alpha+1}(0) \leq 2|\log \varepsilon|.
\]

Then by Lemma 2.2, we get a function \(f_{\alpha+1}\) such that
\[
\Gamma_{\alpha+1} \cap C_{R/7}(0) = \{x_n = f_{\alpha+1}(x')\}.
\]

Moreover, we have the Lipschitz bound
\[
\|\nabla f_\alpha\|_{L^\infty(B_{R/7}^{-1}(0))} + \|\nabla f_{\alpha+1}\|_{L^\infty(B_{R/7}^{-1}(0))} \leq C.
\]

Curvature bounds on \(\Gamma_\alpha\) and \(\Gamma_{\alpha+1}\) can be transformed into
\[
\|\nabla^2 f_\alpha\|_{L^\infty(B_{R/7}^{-1}(0))} + \|\nabla^2 f_{\alpha+1}\|_{L^\infty(B_{R/7}^{-1}(0))} \lesssim \varepsilon.
\]
By (9.4) and (9.7), for any \( y \in B_{K^{-1/2}}(0) \),
\[
|\nabla f_\alpha(y)| \lesssim \varepsilon |y| \lesssim K \varepsilon K^{-1/2} \lesssim KM^{-1/2} \log \varepsilon^{-1/2}.
\]

Concerning \( f_{\alpha+1} \) we have the following estimates. In the following a positive constant \( \delta < \min\{1/48,(1 - \theta)/32\} \) will be fixed.

**Lemma 9.2.** For \( y \in B_{K^{-1/2}}(0) \), we have
\[
|\nabla f_{\alpha+1}(y)| = O_K \left( M^{-\delta} \log |\varepsilon|^{-\delta} \right).
\]

**Proof.** By (9.5) and (9.6) we have
\[
\max_{B_{K^{-1/2}(0)}} (f_{\alpha+1} - f_\alpha) \lesssim K\kappa^{-1/2} \lesssim KM^{-1/2} \log \varepsilon^{-1/2} \varepsilon^{-1}.
\]

As in Lemma 3.4, combining this bound with an interpolation argument we get
\[
\max_{B_{K^{-1/2}(0)}} |\nabla f_{\alpha+1} - \nabla f_\alpha| \lesssim C(K)M^{-\delta} \log |\varepsilon|^{-\delta}.
\]

Combining this estimate with (9.8) we get (9.9). \( \square \)

The following lemma shows that \( d_{\alpha+1} \) is well approximated by vertical distances. The proof uses the fact that under assumptions (9.4) and (9.5), \( \Gamma_\alpha \) and \( \Gamma_{\alpha+1} \) are almost parallel and horizontal.

**Lemma 9.3.** For \( y \in B_{K^{-1/2}}(0) \), if \( e^{-|d_{\alpha+1}(y)|} \geq \varepsilon^2 \), then
\[
e^{-|d_{\alpha+1}(y)|} = e^{-(f_{\alpha+1}(y) - f_\alpha(y))} + O_K \left( M^{-1} \kappa \right).
\]

**Proof.** Assume the nearest point on \( \Gamma_{\alpha+1} \) to \( (y,f_\alpha(y)) \) is \( (y_*,f_{\alpha+1}(y_*)) \). Because
\[
d_\alpha(y_*,f_{\alpha+1}(y_*)) \leq |d_{\alpha+1}(y,f_\alpha(y))| \leq 2|\log \varepsilon|,
\]

using Lemma 3.4 we deduce that
\[
|\nabla f_{\alpha+1}(y_*) - \nabla f_\alpha(y_*)| \lesssim \varepsilon^{1/3}.
\]

Combining this estimate with (9.8) and noting the fact that \( M^{-1/2} \log \varepsilon^{-1/2} \gg \varepsilon^{1/3} \), we get
\[
|\nabla f_{\alpha+1}(y_*)| \lesssim KM^{-1/2} \log |\varepsilon|^{-1/2}.
\]

By (9.6), we have
\[
|f_{\alpha+1}(y) - f_\alpha(y)| \lesssim |d_{\alpha+1}(y)| \lesssim |\log \varepsilon|.
\]

Hence we also have
\[
|y - y_*| \lesssim |\log \varepsilon|.
\]

Then by (9.7), we get
\[
\text{dist} \left( (y,f_{\alpha+1}(y)), T_{y_*,f_{\alpha+1}(y_*)} \Gamma_{\alpha+1} \right) \lesssim \varepsilon |\log \varepsilon|^2,
\]

where \( T_{y_*,f_{\alpha+1}(y_*)} \Gamma_{\alpha+1} \) denotes the tangent hyperplane of \( \Gamma_{\alpha+1} \) at \( (y_*,f_{\alpha+1}(y_*)) \).

Let \( \vartheta \) be the angle between \( N_{\alpha+1}(y_*) \) (normal vector of \( \Gamma_{\alpha+1} \) at \( (y_*,f_{\alpha+1}(y_*)) \) and the direction \( e_{n+1} = (0, \cdots, 0, 1) \). By (9.11), we get
\[
|\vartheta| \lesssim KM^{-1/2} |\log \varepsilon|^{-1/2}.
\]

From this we deduce that
\[
f_{\alpha+1}(y) - f_\alpha(y) \leq |d_{\alpha+1}(y)| \left[ 1 + C (\sin \vartheta)^2 \right] + \varepsilon |\log \varepsilon|^2
\]
Then by Taylor expansion and the fact that \( e^{d_{\alpha+1}(y)} \leq \kappa \), we obtain (9.10).

By this lemma, now the Toda system (9.2) is rewritten as, for any \( y \in B^{n-1}_{K(n-1)/2}(0) \),
\[
\begin{align*}
\text{(9.14)} & \quad \text{div} \left( \frac{\nabla f_{\alpha}(y)}{\sqrt{1+|\nabla f_{\alpha}(y)|^2}} \right) \geq -\frac{2A^2}{\sigma_0} e^{-[f_{\alpha+1}(y)-f_{\alpha}(y)]} + O_K(M^{-1}\kappa), \\
\text{div} \left( \frac{\nabla f_{\alpha+1}(y)}{\sqrt{1+|\nabla f_{\alpha+1}(y)|^2}} \right) & \leq \frac{2A^2}{\sigma_0} e^{-[f_{\alpha+1}(y)-f_{\alpha}(y)]} + O_K(M^{-1}\kappa), 
\end{align*}
\]
Taking the difference we obtain the equation for \( f_{\alpha+1} - f_{\alpha} \),
\[
\text{(9.15)} \quad \text{div} [A_{\alpha} \nabla (f_{\alpha+1} - f_{\alpha})] \leq \frac{4A^2}{\sigma_0} e^{-[f_{\alpha+1}(y)-f_{\alpha}(y)]} + O_K(M^{-1}\kappa).
\]
Here \( A_{\alpha} \) is the symmetric matrix with entries defined by
\[
\text{(9.16)} \quad \int_0^1 \left[ \frac{\delta_{ij}}{\sqrt{1+|\nabla f_{\alpha}(t)|^2}} - \frac{\partial_i f_{\alpha}^t \partial_j f_{\alpha}^t}{(1+|\nabla f_{\alpha}(t)|^2)^{3/2}} \right] dt, \quad 1 \leq i, j \leq n-1,
\]
where \( f_{\alpha}^t := (1-t)f_{\alpha} + tf_{\alpha+1} \) and \( \delta_{ij} \) denotes Kronecker delta.

In view of (9.8) and (9.9), we have
\[
\text{(9.17)} \quad |A_{\alpha}(y) - Id| \lesssim M^{-\delta}, \quad \forall y \in B^{n-1}_{K(n-1)/2}(0).
\]
Define the function in \( B^{n-1}_{K(n-1)/2}(0) \),
\[
\tilde{v}_{\alpha}(y) := f_{\alpha+1}(\kappa^{-1/2}y) - f_{\alpha}(\kappa^{-1/2}y) - |\log \kappa|.
\]
It satisfies in \( B^{n-1}_{K(n-1)/2}(0) \),
\[
\text{(9.18)} \quad \text{div} \left( \tilde{A}_{\alpha} \nabla v_{\alpha} \right) \leq \frac{4A^2}{\sigma_0} e^{-v_{\alpha}} + O_K(M^{-1}).
\]
Here \( \tilde{A}_{\alpha}(y) := A_{\alpha}(\kappa^{-1/2}y) \), which still satisfies (9.17).

9.2. Completion of the proof of Proposition 9.1. First we show that \( v_{\alpha} \) is almost stable.

**Lemma 9.4.** For any \( \tilde{\eta}_0 \) and \( \tilde{\eta}_{\alpha+1} \in C_{c}^{\infty}(B^{n-1}_{K(n-1)/2}(0)) \),
\[
\begin{align*}
\sum_{\beta=\alpha,\alpha+1} \int_{B^{n-1}_{K(n-1)/2}(0)} |\nabla \tilde{\eta}_\beta|^2 dy & + O_K(M^{-\delta}) \sum_{\beta=\alpha,\alpha+1} \int_{B^{n-1}_{K(n-1)/2}(0)} \tilde{\eta}_\beta^2 dy \\
& \geq \frac{2A^2}{\sigma_0} \int_{B^{n-1}_{K(n-1)/2}(0)} e^{-v_{\alpha}} [\tilde{\eta}_{\alpha+1} - \tilde{\eta}_{\alpha}]^2 dy.
\end{align*}
\]

**Proof.** For \( y \in B^{n-1}_{K(n-1)/2}(0) \), let \( \eta_{\alpha}(y) := \tilde{\eta}_0(\kappa^{1/2}y) \) and \( \eta_{\alpha+1} \) be defined similarly. We will view them as functions on \( \Gamma_{\alpha} \) (respectively \( \Gamma_{\alpha+1} \)), by identifying \( y \) with \( (y, f_{\alpha}(y)) \) etc.

Then Proposition 8.1 says
\[
\begin{align*}
\sum_{\beta} \int_{B^{n-1}_{K(n-1)/2}(0)} |\nabla \eta_{\beta}|^2 \left[ 1 + O(\|\nabla f_{\beta}|^2) \right] dy & + Q(\eta_{\alpha}, \eta_{\alpha+1}) \\
& \geq \sum_{\beta} \frac{2A^2(-1)^{\beta-1}}{\sigma_0} \int_{B^{n-1}_{K(n-1)/2}(0)} e^{-d_{\alpha+1}(y,0)} \left[ \eta_{\alpha}(y) - \eta_{\alpha+1}(\Pi_{\alpha+1}(y,0))^2 \right] \left[ 1 + O(\|\nabla f_{\beta}|^2) \right] dy,
\end{align*}
\]
where

\[
|Q(\eta)| \lesssim \left[ \varepsilon^\frac{1}{2} + \kappa \right] \left( \sum_\beta \int_{B_{K^{-1/2}}(n)} |\nabla \eta_\beta(y)|^2 \left[ 1 + O \left( |\nabla f_\beta(y)|^2 \right) \right] dy \right) + \left( \varepsilon^2 + \kappa^\frac{1}{2} + \varepsilon \kappa \right) \left( \sum_\beta \int_{B_{K^{-1/2}}(n)} \eta_\beta(y)^2 \left[ 1 + O \left( |\nabla f_\beta(y)|^2 \right) \right] dy \right) \\
\lesssim \kappa^{1/8} \left( \sum_\beta \int_{B_{K^{-1/2}}(n)} |\nabla \eta_\beta(y)|^2 dy \right) + \frac{\kappa}{M} \left( \sum_\beta \int_{B_{K^{-1/2}}(n)} \eta_\beta(y)^2 dy \right).
\]

Substituting Lemma 9.3 into this inequality and taking a rescaling, we obtain

\[
\left[ 1 + O \left( M^{-\delta} \right) \right] \sum_{\beta=\alpha,\alpha+1} \int_{B_{K^{-1/2}}(n)} |\nabla \tilde{\eta}_\beta|^2 dy + O \left( M^{-\delta} \right) \sum_{\beta=\alpha,\alpha+1} \int_{B_{K^{-1/2}}(n)} \tilde{\eta}_\beta^2 dy \geq \frac{2A_0^2}{\sigma_0^{1/2}} \int_{B_{K^{-1/2}}(n)} e^{-\nu_\alpha} [\tilde{\eta}_{\alpha+1} - \tilde{\eta}_\alpha]^2 dy.
\]

Moving the coefficient \( 1 + O \left( M^{-\delta} \right) \) before the first group of integrals to other groups, we conclude the proof. \( \Box \)

This almost stability condition implies an \( L^1 \) estimate.

**Lemma 9.5.** For any \( \sigma > 0 \), if \( n \leq 10 \) and we have chosen \( M \gg K \gg 1 \) (depending only on \( n \) and \( \sigma \)), then

\[
\int_{B_{K^{-1/2}}(n)} e^{-\nu_\alpha} \leq \sigma.
\]

**Proof.** Let \( V_\alpha := e^{-\nu_\alpha} \). Direct calculation using (9.18) gives

\[
- \text{div} \left( \tilde{A}_\alpha \nabla V_\alpha \right) \leq \frac{4A_0^2}{\sigma_0} V_\alpha^2 - \left[ 1 + O \left( M^{-\delta} \right) \right] V_{\alpha-1}^2 |\nabla V_\alpha|^2 + CM^{-\delta} V_\alpha.
\]

Following Farina [8], for any \( \eta \in C_0^{\infty}(B_{K^{-1/2}}(n)) \) and \( q > 0 \), multiplying (9.20) by \( V_\alpha^{2q-1} \eta^2 \) and integrating by parts, we get

\[
\left[ 2q + O_K \left( M^{-\delta} \right) \right] \int V_\alpha^{2q-2} |\nabla V_\alpha|^2 \eta^2 - \frac{1}{2q} \int V_\alpha^{2q} \Delta \eta^2 \leq \frac{4A_0^2}{\sigma_0} \int V_\alpha^{2q+1} \eta^2 + O_K \left( M^{-\delta} \right) \int V_\alpha^{2q} \eta^2.
\]

On the other hand, for any \( \eta \in C_0^{\infty}(B_{K^{-1/2}}(n)) \), substituting \( \eta_\alpha = \eta \) and \( \eta_{\alpha+1} = -\eta \) into Lemma 9.4 gives

\[
\int_{B_{K^{-1/2}}(n)} |\nabla \eta|^2 dy + C(K) M^{-\delta} \int_{B_{K^{-1/2}}(n)} \eta^2 dy \geq \frac{4A_0^2}{\sigma_0} \int_{B_{K^{-1/2}}(n)} V_\alpha \eta^2 dy.
\]

Taking \( V_\alpha^{2q} \eta \) as test functions in (9.22) leads to

\[
\frac{4A_0^2}{\sigma_0} \int V_\alpha^{2q+1} \eta^2 \leq q^2 \int V_\alpha^{2q-2} |\nabla V_\alpha|^2 \eta^2 + C \int V_\alpha^{2q} (|\Delta \eta|^2 + |\nabla \eta|^2) + C(K) M^{-\delta} \int V_\alpha^{2q} \eta^2.
\]
Combining (9.21) and (9.23) we get, if
\[ 2q > q^2 \left( 1 + C(K)M^{-\delta} \right), \]
which is true if \( M \) is large enough and \( q < 15/8 \), then
\[
\int_{V} V_{\alpha}^{2q-2} |\nabla V_{\alpha}|^2 + \int V_{\alpha}^{2q+1} \eta^2 \leq C(q) \int V_{\alpha}^{2q} \left( |\Delta \eta|^2 + |\nabla \eta|^2 + C(K)M^{-\delta} \eta^2 \right). \tag{9.24}
\]
Still as in Farina [8], replace \( \eta \) by \( \eta^m \) for some \( m \gg 1 \) (depending only on \( q \)), where \( \eta \) is a standard cut-off function. Applying the Hölder inequality to (9.24) gives, for any \( q < 15/8 \),
\[
\int_{\mathcal{B}_{r-1}^n(0)} V_{\alpha}^{2q-2} |\nabla V_{\alpha}|^2 + V_{\alpha}^{2q+1} \eta^2 \leq C(q)K^{n-1-2(2q+1)} + C(q,K)M^{-\delta}K^{n-1}. \tag{9.25}
\]
If \( n \leq 10 \) and \( q > 7/4 \), then
\[ n - 1 - 2(2q + 1) < 0. \]
First choose a \( K \) so large that \( C(q)K^{n-1-2(2q+1)} < \sigma/2 \), then take an \( M \) so large that \( C(q,K)M^{-\delta}K^{n-1} < \sigma/2 \), we get (9.19) by the Hölder inequality. \( \square \)

Now we improve this \( L^1 \) estimate to an \( L^\infty \) estimate. To this end, we need the following decay estimate.

**Lemma 9.6.** There exist two universal constants \( \sigma_* \) and \( \tau_* \in (0,1) \) so that the following holds. For any \( y \in B_{1}^{n-1}(0) \) and \( r \in (0,1) \), suppose
\[
r^{3-n} \int_{B_{r}^{n-1}(y)} V_{\alpha} \leq \sigma_*, \tag{9.26}
\]
then
\[
(\tau_* r)^{3-n} \int_{B_{\tau_* r}^{n-1}(y)} V_{\alpha} \leq \frac{1}{2} r^{3-n} \int_{B_{r}^{n-1}(y)} V_{\alpha}. \tag{9.27}
\]

The proof follows the method introduced by the first author in [17, 18] with minor modifications. In that proof what we need are:

(i) the elliptic inequality (9.18);
(ii) an integral estimate
\[
\int_{B_{r/2}^{n-1}(y)} V_{\alpha}^2 \lesssim \left[ r^{-2} + C(K)M^{-\delta} \right] \int_{B_{r}(y)} V_{\alpha}. \tag{9.28}
\]
The estimate (9.28) follows by taking \( q = 1/2 \) in (9.24), which gives, for any \( \eta \in C_0^\infty(B_{2}^{n-1}(0)) \),
\[
\int_{B_{2}^{n-1}(0)} V_{\alpha}^2 \eta^2 \lesssim \int_{B_{2}^{n-1}(0)} V_{\alpha} \left( |\eta| |\Delta \eta| + |\nabla \eta|^2 + M^{-\delta} \eta^2 \right), \tag{9.29}
\]
and then choosing \( \eta \) to be a standard cut-off function with \( \eta \equiv 1 \) in \( B_{r/2}^{n-1}(y) \), \( \eta \equiv 0 \) outside \( B_{2}^{n-1}(y) \), \( |\nabla^2 \eta| + |\nabla \eta|^2 \lesssim r^{-2} \).

Using this lemma we get
Lemma 9.7. If
\[ \int_{B_2^{n-1}(0)} V_\alpha \leq \sigma_*, \]
then
\[ \max_{B_1^{n-1}(0)} V_\alpha \leq \frac{1}{2}. \]

Proof. For any \( y \in B_1^{n-1}(0) \),
\[ \int_{B_1^{n-1}(0)} V_\alpha \leq \int_{B_2^{n-1}(0)} V_\alpha \leq \sigma_* . \]

Then by the previous lemma, we get
\[ (9.32) \int_{B_{\kappa k+1}(y)} V_\alpha \leq \frac{1}{2} \int_{B_{\kappa k}(y)} V_\alpha , \quad \forall k \geq 1. \]

In other words, for any \( y \in B_1^{n-1}(0) \) and \( r \in (0, 1) \),
\[ (9.33) \int_{B_r^{n-1}(y)} V_\alpha \leq C \sigma_* r^{n-3+\frac{\log 2}{\log \tau_*}}. \]

Combining standard elliptic estimates in Morrey spaces and (9.30) we get (9.31). \( \square \)

In view of Lemma 9.5, Lemma 9.7 is applicable to \( V_\alpha \) for all \( \kappa \) small. Rescaling (9.31) back we get (9.3). The proof of Proposition 9.1 is thus complete.

10. Distance bound

In this section we give a lower bound on \( D_\alpha \).

Proposition 10.1. There exists a universal constant \( C \) such that
\[ A(2R/3; 0) \leq C \varepsilon^2 (\log |\log \varepsilon|)^2 . \]

Remark 10.2. This is a small improvement of [3, Proposition 3.2], where they established a bound in the form
\[ A(2R/3; 0) = o \left( \varepsilon^2 |\log \varepsilon| \right) . \]

We do not know if there exists a universal constant \( C \) such that
\[ A(2R/3; 0) \leq C \varepsilon^2 . \]

10.1. Non-optimal lower bounds. Before proving this proposition, we first provide three non-optimal lower bounds.

Lemma 10.3. For any \( \theta \in (0, 1) \), for any \( \alpha \) and \( y \in \Gamma_\alpha \cap B_{19R/24}(0) \),
\[ D_\alpha(y) \geq (1 + \theta) |\log \varepsilon|. \]

Proof. Assume by the contrary \( A(19R/24; 0) \geq \varepsilon^{1+\theta} \). Then by the monotone dependence of \( A(r; 0) \) on \( r \), we have
\[ A(r; 0) \geq A(19R/24; 0) \geq \varepsilon^{1+\theta} , \quad \forall r \in [19R/24, 5R/6]. \]

Now Proposition 9.1 is applicable, which says
\[ A \left( r - KR^{\frac{1+\theta}{\theta}}; 0 \right) \leq \frac{1}{2} A(r; 0) . \]
Here we have used the estimate on the constant in Proposition 9.1, \( R_* \leq R \frac{1 + \theta}{2} \).

An iteration of this decay estimate from \( r = 5R/6 \) to \( 19R/24 \) leads to a contradiction
\[
A \left( 19R/24; 0 \right) \leq 2^{-cK^{-1}R \frac{1 + \theta}{2}} A \left( 5R/6; 0 \right) \leq \varepsilon^2.
\]
In the last inequality we have used \( A(5R/6; 0) \leq 1 \), which is a consequence of Lemma 2.1. \( \square \)

**Lemma 10.4.** There exists a universal constant \( C \) such that for any \( \alpha \) and \( y \in \Gamma \alpha \cap B_{3R/4}(0) \),
\[
D_\alpha(y) \geq 2 - \log |\log \varepsilon|.
\]

**Proof.** By Lemma 10.3, we can assume
\[
A(19R/24; 0) \leq \varepsilon^{1 + \theta},
\]
where \( \theta \) is very close to 1 (to be determined below).

Now assume by the contrary \( A(3R/4; 0) \geq \varepsilon^2 |\log \varepsilon|^2 \). Then by the monotone dependence of \( A(r; 0) \) on \( r \), we have
\[
A(r; 0) \geq A(3R/4; 0) \geq \varepsilon^2 |\log \varepsilon|^2, \quad \forall r \in [3R/4, 19R/24].
\]

Now Proposition 9.1 is applicable, which says
\[
A \left( r - K \frac{R}{\log R}; 0 \right) \leq \frac{1}{2} A(r; 0).
\]
Here we have used the estimate on the constant in Proposition 9.1, \( R_* \leq R/\log R \).

An iteration of this decay estimate from \( r = 19R/24 \) to \( 3R/4 \) leads to a contradiction, i.e.
\[
A \left( 3R/4; 0 \right) \leq 2^{-cK^{-1}R \frac{1 + \theta}{2}} A \left( 5R/6; 0 \right) \leq C \varepsilon^{1 + \theta \log 2 / K} \leq \varepsilon^2,
\]
provided \( 1 + \theta + \frac{\log 2}{K} > 2 \), i.e. \( \theta \) has been chosen to be very close to 1. \( \square \)

**Lemma 10.5.** There exists a universal constant \( C \) such that for any \( \alpha \) and \( y \in \Gamma \alpha \cap B_{17R/24}(0) \),
\[
D_\alpha(y) \geq 2 - \log |\log \varepsilon| - C.
\]

**Proof.** By Lemma 10.3, we can assume
\[
A(3R/4; 0) \leq cK \varepsilon^2 |\log \varepsilon|^2.
\]

Now assume by the contrary \( A(17R/24; 0) \geq M \varepsilon^2 |\log \varepsilon| \) for some \( M > 0 \) large. Then by the monotone dependence of \( A(r; 0) \) on \( r \), we have
\[
A(r; 0) \geq A(17R/24; 0) \geq M \varepsilon^2 |\log \varepsilon|^2, \quad \forall r \in [17R/24, 3R/4].
\]

Now Proposition 9.1 is applicable, which says
\[
A \left( r - K \sqrt{M/A} \log R; 0 \right) \leq \frac{1}{2} A(r; 0).
\]
Here we have used the estimate on the constant in Proposition 9.1, \( R_* \leq R/\sqrt{M \log R} \).

An iteration of this decay estimate from \( r = 3R/4 \) to \( 17R/24 \) leads to a contradiction, i.e.
\[
A \left( 17R/24; 0 \right) \leq 2^{-cK^{-1}} \sqrt{M \log R} A \left( 3R/4; 0 \right) \leq 2^{-2cK^{-1}} \sqrt{M \log R} \varepsilon^2 |\log \varepsilon|^2 \leq \varepsilon^2 |\log \varepsilon|.
\]
The last inequality follows from the estimate
\[
2^{-cK^{-1}} \sqrt{M \log \varepsilon} |\log \varepsilon| = 2^{-cK^{-1}} \sqrt{M (\log \varepsilon)^2 + (\log 2)(\log |\log \varepsilon|)} \leq 1,
\]
which is true if $\varepsilon$ is small enough. \qed

10.2. Proof of Proposition 10.1. By Lemma 10.5, now we can assume
\begin{equation}
A(17R/24; 0) \lesssim \varepsilon^2 |\log \varepsilon|.
\end{equation}
Hence by (5.4), for any $\alpha$
\begin{equation}
\|H_\alpha\|_{\dot{L}^\infty(B_{17R/24}(0))} \lesssim \varepsilon^2 |\log \varepsilon|.
\end{equation}

Denote $\rho := |d\alpha + 1|$. By [3, Eqn. (2.41), Lemma 2.9 and Appendix A] and (9.2), $\rho$ satisfies
\begin{equation}
L\rho(y) + |A_\alpha(y)|^2 \rho(y) + N(\rho) \leq \frac{8A_1^2}{\sigma_0} e^{-\rho(y)} + O(\varepsilon^2).
\end{equation}
Here $L$ is the linear uniformly elliptic operator
\begin{equation}
L\varphi := a(y)^{-1} \text{div}_\alpha \left[ a(y) \nu_\alpha(y) \cdot \nu_{\alpha + 1}(y, \rho(y)) \nabla_{\alpha + 1} \varphi \right],
\end{equation}
where
\begin{equation}
a(y) := \frac{\lambda_\alpha(y, 0)}{\lambda_{\alpha + 1}(y, \rho(y))}.
\end{equation}
The nonlinear error term $N(\rho)$ satisfies
\begin{equation}
|N(\rho)(y)| \lesssim \varepsilon^3 |\rho(y)|^2 + \varepsilon |\nabla_\alpha \rho(y)|^2.
\end{equation}

**Lemma 10.6.** For any $y \in \Gamma_\alpha \cap B_{11R/16}(0)$, if $\rho(y) \leq 2 |\log \varepsilon|$, then
\begin{equation}
|\nabla_\alpha \rho(y)| \lesssim \varepsilon |\rho(y)|.
\end{equation}

**Proof.** Fix an $\alpha$ and a point $y_* \in \Gamma_\alpha \cap B_{11R/16}(0)$. By our assumption and Lemma 10.5,
\begin{equation}
2 |\log \varepsilon| - \log |\log \varepsilon| - C \leq \rho(y_*) \leq 2 |\log \varepsilon|.
\end{equation}

Choose a coordinate system such that $\Gamma_\alpha \cap B_{R/48}(0)$ and $\Gamma_{\alpha + 1} \cap B_{R/48}(0)$ are represented by graphs of functions $f_\alpha$ and $f_{\alpha + 1}$, and $\rho(y_*)$ is attained at $(y_*, f_{\alpha + 1}(y_*))$. Therefore we have
\begin{equation}
\rho(y_*) = f_{\alpha + 1}(y_*) - f_\alpha(y_*), \quad \rho(y) \leq f_{\alpha + 1}(y) - f_\alpha(y),
\end{equation}
and consequently
\begin{equation}
\nabla \rho(y_*) = \nabla f_{\alpha + 1}(y_*) - \nabla f_\alpha(y_*).
\end{equation}

Define
\begin{equation}
\tilde{\rho}(\tilde{y}) := \rho(y_*)^{-1} \left[ f_{\alpha + 1}(y_* + \varepsilon^{-1} \tilde{y}) - f_\alpha(y_* + \varepsilon^{-1} \tilde{y}) \right], \quad \tilde{y} \in B_{1/48}^{n-1}(0).
\end{equation}

Taking difference of (10.4), by (10.7) we obtain
\begin{equation}
\left\| \text{div} \left( \tilde{A}_\alpha \nabla \tilde{\rho} \right) \right\|_{\dot{L}^\infty(B_{1/48}^{n-1}(0))} \leq C.
\end{equation}

Here $\tilde{A}_\alpha, \varepsilon^1(\tilde{y}) = A_\alpha(y_* + \varepsilon^{-1} \tilde{y})$ and $A_\alpha$ is defined as in (9.16).

Note that $\tilde{\rho} > 0$ in $B_{1/48}^{n-1}(0)$. On the other hand, by our assumption and Lemma 10.5, $\tilde{\rho}(0) = 1$. Then by Moser’s Harnack inequality for inhomogeneous equations (see [10, Theorem 8.17 and 8.18]), there exists a $\sigma > 0$ such that
\begin{equation}
1/2 \leq \tilde{\rho} \leq 2, \quad \text{in } B_{\sigma}^{n-1}(0).
\end{equation}

Using standard elliptic estimates we get a universal constant $C$ such that $\|\tilde{\rho}\|_{C^{1,1/2}(B_{\sigma}^{n-1}(0))} \leq C$. In particular, $|\nabla \tilde{\rho}(0)| \leq C$. Rescaling back and using (10.8) we conclude the proof. \qed
The proof, in particular, (10.10) implies that

**Corollary 10.7.** If \( \rho(y) \leq 2|\log \varepsilon| \), then

\[
\sup_{B_{sR} \cap \Gamma} \rho \leq 4|\log \varepsilon|.
\]

Substituting (10.11) and Lemma 10.6 into (10.6) we obtain

**Corollary 10.8.** If \( \rho(y) \leq 2|\log \varepsilon| \), then

\[
\sup_{B_{sR} \cap \Gamma} \rho \leq 4|\log \varepsilon|.
\]

Next we give a decay estimate with a weaker assumption than Proposition 9.1.

**Proposition 10.9.** There exist two universal constants \( M \gg K \gg 1 \) such that for any \( r \in [2R/3, 5R/6] \), if

\[
\kappa := A(r; 0) \geq M \varepsilon^2,
\]

then we have

\[
A(r - KR; 0) \leq \frac{1}{2} A(r; 0),
\]

where

\[
R_* := \max \left\{ \kappa^{-\frac{2}{3}}, 200|\log \varepsilon|^2 \right\}.
\]

**Proof.** Fix an \( \alpha \) and a point \( x_* \in \Gamma_{\alpha} \cap B_{r-KR} \). Assume

\[
e^{-\rho(x_*)} \geq \frac{\kappa}{2} \geq \frac{M}{2} \varepsilon^2.
\]

Let

\[
\tilde{\Gamma} := \kappa^{\frac{2}{3}} (\Gamma_{\alpha} - x_*), \quad \tilde{\rho}(\tilde{y}) := \rho(\kappa^{-\frac{2}{3}} \tilde{y} + \log \kappa), \quad \forall \tilde{y} \in \tilde{\Gamma} \cap B_K(0).
\]

By (9.1), we have

\[
|A_{\tilde{\Gamma}}| \leq \frac{C \varepsilon}{\kappa^{1/2}} \leq \frac{C}{M}.
\]

Hence \( \tilde{\Gamma} \) is very close to a hyperplane in \( B_K(0) \).

A direct rescaling of (10.5) leads to

\[
\tilde{L}\tilde{\rho}(\tilde{y}) + O \left( \kappa^{-1} \varepsilon^3 |\log \varepsilon|^2 \right) \leq \frac{8A_{\Gamma}^2}{\sigma_0} e^{-\tilde{\rho}(\tilde{y})} + O \left( \frac{\varepsilon^2}{\kappa} \right).
\]

Here \( \tilde{L} \) is the rescaling of \( L \), but we rewrite it as

\[
L = a(y)^{-1} \text{div} \left( \tilde{A}_\alpha(y) \nabla_{\alpha} \right),
\]

where \( |a(y) - 1| \ll 1 \) and

\[
\| \tilde{A}_\alpha - Id \|_{L^\infty(\Gamma_{\alpha} \cap B_{K\kappa^{-1/2}}(0))} \ll 1,
\]

by a derivation similar to the one of (9.17). (Note that under assumption (10.14), we have (10.13), which then enables us to apply Lemma 3.4 to estimate \( \nu_{\alpha} \cdot \nu_{\alpha+1} \) etc.)

As in (9.22), for any \( \eta \in C_0^\infty(\tilde{\Gamma} \cap B_K(0)) \), we still have

\[
\int_{\Gamma \cap B_K(0)} |\nabla \eta|^2 + C(K) M^{-\delta} \int_{\Gamma \cap B_K(0)} \eta^2 \geq \frac{4A_{\Gamma}^2}{\sigma_0} \int_{\Gamma \cap B_K(0)} e^{-\tilde{\rho}} \eta^2 dy.
\]

Then proceeding as in Subsection 9.2 we conclude the proof. □
As in Subsection 10.1, we use this decay estimate to prove Proposition 10.1.

**Proof of Proposition 10.1.** By Lemma 10.5, we can assume
\begin{equation}
A(17R/24; 0) \lesssim \varepsilon^2 |\log \varepsilon|.
\end{equation}

Now assume by the contrary there exists a large constant \(M\) such that
\begin{equation}
A(2R/3; 0) \geq M\varepsilon^2 (\log |\log \varepsilon|)^2.
\end{equation}

Then by the monotone dependence of \(A(r; 0)\) on \(r\), we have
\[\kappa := A(r; 0) \geq A(2R/3; 0) \geq M\varepsilon^2 (\log |\log \varepsilon|)^2, \quad \forall r \in [2R/3, 17R/24].\]

Now Proposition 10.9 is applicable, which says
\[A(r - K\sqrt{M \log |\log \varepsilon|}; 0) \leq \frac{1}{2} A(r; 0).\]

Here we have used the estimate on the constant in Proposition 10.9, \(R_* \leq \frac{R}{\sqrt{\varepsilon}}\).

An iteration of this decay estimate from \(r = 17R/24\) to \(2R/3\) leads to a contradiction, i.e.
\[A(2R/3; 0) \leq 2^{-cK^{-1}\sqrt{M \log |\log \varepsilon|}} A(17R/24; 0) \leq C 2^{-cK^{-1}\sqrt{M \log |\log \varepsilon|^2 \log \varepsilon}} \lesssim \frac{(\log |\log \varepsilon|)^2}{2} \varepsilon^2.\]

The last inequality is true provided \(M\) is large enough. This is a contradiction with (10.19) and the proof is complete. \(\square\)

11. **Proof of main results**

In this section we prove Theorem 1.1 and its two corollaries, Corollary 1.2 and 1.3.

**Proof of Theorem 1.1.** Substituting Proposition 10.1 into (6.1), we get
\begin{equation}
\|\phi\|_{C^{2, \theta}(B_{2R/3}(0))} + \max_{\alpha} \|H_{\alpha} + \Delta_{\alpha, 0} h_{\alpha}\|_{C^{\theta}(\Gamma_{\alpha} \cap B_{2R/3}(0))} \lesssim \varepsilon^{1+\theta}.\end{equation}

By Lemma 4.6, for any \(\alpha\),
\[\|H_{\alpha}\|_{C^{\theta}(\Gamma_{\alpha} \cap B_{R/2}(0))} \lesssim \|\phi\|_{C^{2, \theta}(B_{2R/3}(0))} + \|H_{\alpha} + \Delta_{\alpha, 0} h_{\alpha}\|_{C^{\theta}(\Gamma_{\alpha} \cap B_{2R/3}(0))} + A(2R/3; 0) \lesssim \varepsilon^{1+\theta}.\]

After rescaling back to \(u_\varepsilon\), this says for any connected component of \(\{u_\varepsilon = 0\}\), say \(\Gamma_{\alpha, \varepsilon}\), its mean curvature satisfies
\begin{equation}
\|H_{\alpha, \varepsilon}\|_{L^\infty(\Gamma_{\alpha, \varepsilon} \cap B_{1/2}(0))} \lesssim \varepsilon (\log |\log \varepsilon|)^2, \quad \|H_{\alpha, \varepsilon}\|_{C^{\theta}(\Gamma_{\alpha, \varepsilon} \cap B_{1/2}(0))} \leq C.
\end{equation}

Because \(\Gamma_{\alpha, \varepsilon} \cap B_R(0)\) is a Lipschitz graph in some direction (see Lemma 2.2), by standard estimates on the minimal surface equations (see for example [10, Chapter 16] or [11, Appendix C]) we obtain a uniform bound on the \(C^{\theta}\) norm of its second fundamental form \(A_{\alpha, \varepsilon}\).

As mentioned at the beginning of Section 2, all of these estimates hold uniformly for \(t \in [-1 + b_1, 1 - b_1]\). This completes the proof of Theorem 1.1. \(\square\)

Next we show how Corollary 1.2 and 1.3 follow from Theorem 1.1.
Proof of Corollary 1.2. This follows the same reasoning in [19, Section 7]. Here we include the proof for completeness.

First as in Lemma 2.1 or [19, Lemma 7.1], we deduce (1.8) from (H1) and (H2). Thus for all $\varepsilon$ small, $\nabla u_\varepsilon \neq 0$ in $\{|u| < 1 - 2\}$ and hence $|B(u_\varepsilon)|$ is well defined. In order to apply Theorem 1.1, it suffices to establish a uniform bound on $|B(u_\varepsilon)|$ as in (1.6).

Now assume by the contrary, as $\varepsilon \to 0$,

$$
\lim_{\varepsilon \to 0} \max_{x \in \{|u_\varepsilon| < 1 - 2\} \cap \mathcal{C}_{\varepsilon/3}} |B(u_\varepsilon)(x)| = +\infty.
$$

Let $x_\varepsilon \in \mathcal{C}_1 \cap \{|u_\varepsilon| \leq 1 - 2\}$ attain the following maxima (we denote $x = (x', x_n)$)

$$
\max_{\mathcal{C}_1 \cap \{|u_\varepsilon| \leq 1 - 2\}} (1 - |x'|) |B(u_\varepsilon)(x)|.
$$

By (H1), $x_\varepsilon \in \{|x_n| \leq 1/2\}$.

Denote

$$
L_\varepsilon := |B(u_\varepsilon)(x_\varepsilon)|, \quad r_\varepsilon := \left(\frac{3}{2} - |x'_\varepsilon|\right)/2.
$$

Then by definition

$$
L_\varepsilon r_\varepsilon \geq \frac{1}{3} \sup_{\mathcal{C}_{\varepsilon/3} \cap \{|u_\varepsilon| \leq 1 - 2\}} |B(u_\varepsilon)(x)| \to +\infty.
$$

In particular, $L_\varepsilon \to +\infty$. On the other hand, by (1.8), we get

$$
L_\varepsilon = o\left(\frac{1}{\varepsilon}\right).
$$

By the choice of $r_\varepsilon$ at (11.5), we have (here $\mathcal{C}_{r_\varepsilon}(x'_\varepsilon) := B_{r_\varepsilon}^{n-1}(x'_\varepsilon) \times (-1, 1)$)

$$
\max_{x \in \mathcal{C}_{r_\varepsilon}(x'_\varepsilon) \cap \{|u_\varepsilon| \leq 1 - 2\}} |B(u_\varepsilon)(x)| \leq 2L_\varepsilon.
$$

Let $\kappa := L_\varepsilon$ and define $u_\kappa(x) := u_\varepsilon(x + L_\varepsilon^{-1}x)$. Then $u_\kappa$ satisfies (1.11) with parameter $\kappa$ in $B_{L_\varepsilon r_\varepsilon}(0)$. By (11.7), $\kappa \to 0$ as $\varepsilon \to 0$. For any $t \in [-1+b, 1-b]$, the level set $\{u_\kappa = t\}$ consists of $Q$ Lipschitz graphs

$$
\{x_n = f_{\beta, \kappa}(x') := L_\varepsilon \left[f_{\beta, \varepsilon}(x'_\varepsilon + L_\varepsilon^{-1}x') - f_{\alpha, \varepsilon}(x'_\varepsilon)\right]\},
$$

where $\alpha$ is chosen so that $x_\varepsilon$ lies in the connected component of $\{|u_\varepsilon| \leq 1 - 2\}$ containing $\Gamma_{\alpha, \varepsilon}$.

By (11.8), we also have

$$
|B(u_\kappa)| \leq 2, \quad \text{for } x \in \mathcal{C}_1 \cap \{|u_\kappa| \leq 1 - 2\}.
$$

Now Theorem 1.1 is applicable to $u_\kappa$. Hence $f_{\alpha, \kappa}$ are uniformly bounded in $C_{\text{loc}}^{2, \theta}(\mathbb{R}^{n-1})$. After passing to a subsequence, it converges to a limit $f_\infty$, which by (1.7) is an entire solution of the minimal surface equation. Since the rescaling (11.9) preserves the Lipschitz constants, $f_\infty$ is global Lipschitz. Then by Moser’s Liouville theorem on minimal surface equations (see [11, Theorem 17.5]), $f_\infty$ is an affine function. In particular,

$$
\nabla^2 f_\infty \equiv 0.
$$

On the other hand, by the construction we have $|B(u_\kappa)(0)| = 1$. If $n = 2$, as in the proof of [19, Theorem 3.6], we get

$$
|B(u_\kappa)(0)|^2 \lesssim \kappa^\theta,
$$
a contradiction with (11.10). If \( n \geq 3 \), we have
\[
1 = |B(u_\kappa)(0)|^2 = |\nabla^2 f_{\alpha,\kappa}(0)|^2 + O(\kappa^\theta).
\]
(The only difference here with the \( n = 2 \) case is that now the Hessian of the distance function to \( \Gamma_{\alpha,\kappa} \) does not converge to 0, but its leading order term is exactly \( \nabla^2 f_{\alpha,\kappa}(0) \), see (3.7).) This gives
\[
\lim_{\kappa \to 0} |\nabla^2 f_{\alpha,\kappa}(0)|^2 = 1,
\]
a contradiction with (11.10). This contradiction implies that the assumption (11.3) cannot hold and the proof is thus complete.

Proof of Corollary 1.3. If \( \delta_2 \) is sufficiently small in (1.8), by unique continuation principle \( \nabla u_\varepsilon \neq 0 \) in \( \{|u_\varepsilon| \leq 1 - b\} \) and hence \( |B(u_\varepsilon)| \) is well defined. As in the proof of Corollary 1.2, the proof is reduced to a uniform bound on \( |B(u_\varepsilon)| \).

Assume by the contrary, we perform a similar blow up analysis as in the proof of Corollary 1.2. This gives another sequence of solutions \( u_\kappa \) defined in an expanding domain. Moreover, \( u_\kappa \) satisfies all of the assumptions in Theorem 1.1. Hence the connected component of \( \{u_\kappa = 0\} \) passing through 0 is a minimal hypersurface in \( \mathbb{R}^n \), denoted by \( \Sigma \). Its second fundamental form satisfies \( |A_{\Sigma}| \leq 3 \) and \( |A_{\Sigma}(0)| = 1 \) (as in the proof of Corollary 1.3).

We claim that \( \Sigma \) is stable. This then leads to a contradiction if Stable Bernstein conjecture is true, which states that \( \Sigma \) must be a hyperplane and hence \( A_{\Sigma} \equiv 0 \). The stability of \( \Sigma \) follows from the general analysis in [3]: first if there are at least two interfaces of \( u_\kappa \) both converging to \( \Sigma \), we can construct a positive Jacobi field on \( \Sigma \) as in [3, Theorem 4.1], which implies the stability of \( \Sigma \); secondly, if there is only one such an interface, then there exist \( \sigma > 0 \) and \( C > 0 \) such that
\[
\int_{B_\sigma(0)} \left[ \frac{\kappa}{2} |\nabla u_\kappa|^2 + \frac{1}{\kappa} W(u_\kappa) \right] \leq C.
\]
Because \( u_\kappa \) is stable, the stability of \( \Sigma \) then follows by applying the main result in [15].

Appendix A. Some facts about the one dimensional solution

In this appendix we recall some facts about one dimensional solution of (1.11), see [19] for more details.

It is known that the following identity holds for \( g \),
\[
g'(t) = \sqrt{2W(g(t))} > 0, \quad \forall t \in \mathbb{R}. \tag{A.1}
\]
Moreover, as \( t \to \pm \infty \), \( g(t) \) converges exponentially to \( \pm 1 \) and the following quantity is well defined
\[
\sigma_0 := \int_{-\infty}^{+\infty} \left[ \frac{1}{2} g'(t)^2 + W(g(t)) \right] dt \in (0, +\infty).
\]

In fact, as \( t \to \pm \infty \), the following expansions hold. There exists a positive constant \( A_1 \) such that for all \( t > 0 \) large,
\[
g(t) = 1 - A_1 e^{-t} + O(e^{-2t}), \quad g'(t) = A_1 e^{-t} + O(e^{-2t}), \quad g''(t) = -A_1 e^{-t} + O(e^{-2t}),
\]
and a similar expansion holds as $t \to -\infty$ with $A_1$ replaced by another positive constant $A_{-1}$.

The following result describes the interaction between two one dimensional profiles.

**Lemma A.1.** For all $T > 0$ large, we have the following expansion:

$$\int_{-\infty}^{+\infty} \left[ W''(g(t)) - 1 \right] [g(-t - T) + 1] g'(t) dt = -2A_{-1}^2 e^{-T} + O \left( e^{-\frac{4}{3}T} \right).$$

$$\int_{-\infty}^{+\infty} \left[ W''(g(t)) - 1 \right] [g(T - t) - 1] g'(t) dt = 2A_1^2 e^{-T} + O \left( e^{-\frac{4}{3}T} \right).$$

Next we discuss the spectrum of the linearized operator at $g$,

$$\mathcal{L} = -\frac{d^2}{dt^2} + W''(g(t)).$$

By a direct differentiation we see $g'(t)$ is an eigenfunction of $\mathcal{L}$ corresponding to eigenvalue 0. By (A.1), 0 is the lowest eigenvalue. In other words, $g$ is stable.

Concerning the second eigenvalue, we have

**Theorem A.2.** There exists a constant $\mu > 0$ such that for any $\varphi \in H^1(\mathbb{R})$ satisfying

(A.2) \[ \int_{-\infty}^{+\infty} \varphi(t) g'(t) dt = 0, \]

we have

$$\int_{-\infty}^{+\infty} [\varphi'(t)^2 + W''(g(t)) \varphi(t)^2] dt \geq \mu \int_{-\infty}^{+\infty} \varphi(t)^2 dt.$$

This can be proved via a contradiction argument.

**APPENDIX B. PROOF OF LEMMA 5.1**

The proof of Lemma 5.1 is similar to the one given in [19, Appendix B]. However, since the setting is a little different (see Step 1 in Subsection 1.2), for reader’s convenience, we will include a complete proof.

Before proving Lemma 5.1, we first derive the exponential nonlinearity in Toda system (5.2).

**Lemma B.1.** For any $y \in \Gamma_\alpha$,

$$\int_{-\infty}^{+\infty} \mathcal{I}(y, z) g'(y, z) dz = (-1)^\alpha \left[ 2A_{-1}^2 e^{-d_{\alpha-1}(y,0)} - 2A_1^2 e^{d_{\alpha+1}(y,0)} \right] + A_\alpha(y),$$

where

$$\|A_\alpha\|_{C^0(B_1(y))} \lesssim e^2 + e^{1/3} \max_{B_1^0(y)} e^{-D_\alpha} + \max_{B_1^0(y)} e^{-\frac{4}{3}D_\alpha} + \max_{\beta:|d_\beta(y,0)| \leq 8|\log \epsilon|} \max_{B_1^0(\Pi_\beta(y,0))} e^{-2D_\beta} + \max_{|z| < 8|\log \epsilon|} \|\phi^\beta\|_{C^{2,\theta}(B_1(y, z))}^2.$$}

**Proof.** To determine the integral $\int_{-\infty}^{+\infty} \mathcal{I} g'$, consider for each $\beta$, the integral on $(-\infty, +\infty) \cap \mathcal{M}_{\beta}^0$, which we assume to be an interval $(\rho^-_\beta(y), \rho^+_\beta(y))$.

**Step 1.** If $\beta \neq \alpha$, by Lemma 4.4, in $(\rho^-_\beta(y), \rho^+_\beta(y))$,

$$|\mathcal{I}| \lesssim e^{-(|d_\beta| + |d_{\beta-1}|)} + e^{-(|d_\beta| + |d_{\beta+1}|)} + e^2.$$
We only consider the case \( \beta > \alpha \) and estimate

\[
\int_{\rho_{\alpha}^\alpha(y)}^{\rho_{\beta}^\beta(y)} e^{-(|d_\beta| + |d_{\beta-1}|)} g'_\alpha.
\]

If \(|z|, |d_\beta| \) and \(|d_{\beta-1}| \) are all smaller than \(8 \log \varepsilon\) at the same time, by Lemma 3.4,

(B.1) \[ d_\beta(y, z) = z + d_\beta(y, 0) + O \left( \varepsilon^{1/3} \right), \]

(B.2) \[ d_{\beta-1}(y, z) = z + d_{\beta-1}(y, 0) + O \left( \varepsilon^{1/3} \right). \]

Note that since \( \beta > \alpha \), by our convention on the sign of \( d_\beta \), we have \( z > 0 \) and \( d_\beta(y, 0) < d_{\beta-1}(y, 0) \leq 0 \).

By (B.1) and (B.2) we get

\[
\int_{\rho_{\beta}^\alpha(y)}^{\rho_{\beta}^\beta(y)} e^{-(|d_\beta| + |d_{\beta-1}|)} g'_\alpha \lesssim \int_{\rho_{\beta}^\alpha(y)}^{\rho_{\beta}^\alpha(y)} e^{-\left(|z| + |z + d_{\beta-1}(y, 0)| + |z + d_\beta(y, 0)|\right)}
\]

\[
\lesssim \int_{\rho_{\beta}^\beta(y)}^{\rho_{\beta}^\beta(y)} e^{-\left(z + d_{\beta-1}(y, 0) - d_\beta(y, 0)\right)} + \int_{\rho_{\beta}^\beta(y)}^{\rho_{\beta}^\beta(y)} e^{-\left(3z + d_{\beta-1}(y, 0) + d_\beta(y, 0)\right)}
\]

\[
\lesssim e^{-\left(d_{\beta-1}(y, 0) - d_\beta(y, 0)\right)} e^{-\left(d_{\beta-1}(y, 0) - 2d_\beta(y, 0)\right)}.
\]

By definition,

\[-d_\beta(y, \rho_{\beta}^{-\beta}(y)) = d_{\beta-1}(y, \rho_{\beta}^{-\beta}(y)).\]

Thus by (B.1) and (B.2),

\[
\rho_{\beta}^{-\beta}(y) = \frac{d_{\beta-1}(y, 0) + d_\beta(y, 0)}{2} + O \left( \varepsilon^{1/3} \right).
\]

Substituting this into the above estimate gives

\[
\int_{\rho_{\beta}^\alpha(y)}^{\rho_{\beta}^\beta(y)} e^{-(|d_\beta| + |d_{\beta-1}|)} g'_\alpha \lesssim e^{-\frac{1}{2} \left(d_{\beta-1}(y, 0) - 3d_\beta(y, 0)\right)} + e^{-\left(d_{\beta-1}(y, 0) - 2d_\beta(y, 0)\right)}.
\]

If \( \beta = \alpha + 1 \), because \( d_{\beta-1}(y, 0) = 0 \), the right hand side is bounded by \( O \left( e^{\frac{2}{3} d_{\alpha+1}(y, 0)} \right) \).

If \( \beta \geq \alpha + 2 \), the right hand side is bounded by \( O \left( e^{d_{\alpha+2}(y, 0)} \right) \).

**Step 2.** It remains to consider the integration in \((\rho_{\alpha}^{-\alpha}(y), \rho_{\alpha}^{\alpha}(y))\). In this case we use Lemma 4.3, which gives

(B.3) \[
\int_{\rho_{\alpha}^\alpha(y)} \int g'_\alpha
\]

\[
= \int_{\rho_{\alpha}^\alpha(y)} \left[ W''(g_\alpha) - 1 \right] [g_{\alpha-1} - (-1)^{\alpha-1}] g'_\alpha + \int_{\rho_{\alpha}^\alpha(y)} \left[ W''(g_\alpha) - 1 \right] [g_{\alpha+1} + (-1)^\alpha] g'_\alpha
\]

\[
+ \int_{\rho_{\alpha}^\alpha(y)} \left[ O \left( e^{-2d_{\alpha-1}} + e^{2d_{\alpha+1}} \right) + O \left( e^{-d_{\alpha-2} - |z|} + e^{d_{\alpha+2} - |z|} \right) \right] g'_\alpha.
\]

Because \( g'_\alpha \lesssim e^{-|z|} \) and

\[
e^{-2d_{\alpha-1}} \lesssim e^{-2d_{\alpha-1}(y, 0) - 2z} + \varepsilon^2,
\]
we get
\[
\int_{\rho_0^+}^{\rho_\alpha^+} e^{-2d_{\alpha-1} y} g_\alpha' \lesssim \varepsilon^2 + e^{-2d_{\alpha-1}(y,0)} \left[ \int_{\rho_\alpha}^{\rho_\alpha^+} e^{-z} dz + \int_{\rho_0^+}^{\rho_\alpha^+} e^{-3z} dz \right]
\]
\[
\lesssim \varepsilon^2 + e^{-2d_{\alpha-1}(y,0) - \rho_\alpha^+} \lesssim \varepsilon^2 + e^{-\frac{3}{2}d_{\alpha-1}(y,0)}.
\]

Similarly, we have
\[
\int_{\rho_\alpha^-}^{\rho_\alpha^+} e^{2d_{\alpha+1} y} g_\alpha' \lesssim \varepsilon^2 + e^{\frac{3}{2}d_{\alpha+1}(y,0)},
\]
\[
\int_{\rho_\alpha^-}^{\rho_\alpha^+} O \left( e^{-d_{\alpha-2} - \mid z \mid} + e^{d_{\alpha+2} - \mid z \mid} \right) g_\alpha' \lesssim e^{-d_{\alpha-2}} + e^{d_{\alpha+2}}.
\]

To determine the first integral in the right hand side of (B.3), arguing as in Step 1, if both $g_\alpha$ and $g_{\alpha-1} - (-1)^{\alpha-1}$ are nonzero, then
\[
g_{\alpha-1}(y, z) = \tilde{g} \left( (-1)^{\alpha-1} \left( z + d_{\alpha-1}(y,0) + h_{\alpha-1}(\Pi_{\alpha-1}(y, z)) + O \left( \varepsilon^{1/3} \right) \right) \right).
\]

Therefore
\[
\int_{\rho_\alpha^-}^{\rho_\alpha^+} \left[ W''(g_\alpha) - 1 \right] \left( g_{\alpha-1} - (-1)^{\alpha-1} \right) g_\alpha'
\]
\[
= \int_{\rho_\alpha^-}^{\rho_\alpha^+} \left[ W''(\tilde{g} \left( (-1)^{\alpha}(z - h_\alpha(y)) \right)) - 1 \right] \tilde{g}' \left( (-1)^{\alpha}(z - h_\alpha(y)) \right)
\]
\[
\times \left[ \tilde{g} \left( (-1)^{\alpha-1} \left( z + d_{\alpha-1}(y,0) + h_{\alpha-1}(\Pi_{\alpha-1}(y, z)) + O \left( \varepsilon^{1/3} \right) \right) \right) \right] \left( (-1)^{\alpha-1} \right) dz
\]
\[
= \int_{-\infty}^{+\infty} \left[ W''(\tilde{g} \left( (-1)^{\alpha}(z - h_\alpha(y)) \right)) - 1 \right] \tilde{g}' \left( (-1)^{\alpha}(z - h_\alpha(y)) \right)
\]
\[
\times \left[ \tilde{g} \left( (-1)^{\alpha-1} \left( z + d_{\alpha-1}(y,0) + h_{\alpha-1}(\Pi_{\alpha-1}(y, z)) + O \left( \varepsilon^{1/3} \right) \right) \right) \right] \left( (-1)^{\alpha-1} \right) dz
\]
\[
+ O \left( e^{-\frac{3}{2}d_{\alpha-1}(y,0)} \right)
\]
\[
= (-1)^{\alpha} 2 A_{\alpha-1}^2 e^{-d_{\alpha-1}(y,0)} + O \left( |h_\alpha(y)| + |h_{\alpha-1}(\Pi_{\alpha-1}(y, z))| + \varepsilon^{1/3} \right) e^{-d_{\alpha-1}(y,0)}
\]
\[
+ O \left( e^{-\frac{3}{2}d_{\alpha-1}(y,0)} \right).
\]

Step 3. What we have proven says
\[
\left| A_\alpha(y) \right| \lesssim \varepsilon^2 + \left( |h_\alpha(y)| + |h_{\alpha-1}(\Pi_{\alpha-1}(y, z))| + \varepsilon^{1/3} \right) e^{-d_{\alpha-1}(y,0)}
\]
\[
+ \left( |h_\alpha(y)| + |h_{\alpha+1}(\Pi_{\alpha+1}(y, z))| + \varepsilon^{1/3} \right) e^{d_{\alpha+1}(y,0)}
\]
\[
+ e^{-\frac{3}{2}d_{\alpha-1}(y,0)} + e^{\frac{3}{2}d_{\alpha+1}(y,0)} + e^{-d_{\alpha-2}(y,0)} + e^{d_{\alpha+2}(y,0)}.
\]

By taking derivatives of $\int_{-\infty}^{+\infty} I g_\alpha'$ in $y$, and then using Lemma 4.4 and Lemma 4.6, we see the $C^q(B_1^0(y))$ norm (in fact, the Lipschitz norm) of $A_\alpha$ is bounded by
\[
\left( \varepsilon^2 + \max_{B_1^0(y)} e^{-D_\alpha} \right) \left( \sum_{|\beta| < 8 \log \varepsilon} \| h_\beta \|_{C^2,\theta(B_1^0(\Pi_{\theta}(y)))} \right)
\]
\begin{align*}
\lesssim \varepsilon^2 + \max_{\beta' |\beta'(y_0)| \leq 8} \max_{B_1(y_0)} e^{-2D_\beta} + \max_{|z| < 8} \max_{\log \varepsilon} \frac{\|\phi\|^2_{C^{2,\#}(B_1(y,z))}}{B_1(y,z)}.
\end{align*}

Similarly, by Lemma 3.4, the $C^\theta(B_1(y))$ norm (in fact, the Lipschitz norm) of $2A_{(1-\varepsilon)} e^{-d_{a-1}(y,0)} - 2A_{(1-\varepsilon)} e^{d_{a+1}(y,0)}$ is controlled by $\varepsilon^{1/3} \max_{B_1(y)} e^{-D_\alpha}$. This gives the estimate on $A_\alpha$. \hfill \Box

Now let us prove Lemma 5.1. Differentiating (7.7) twice leads to

\begin{align}
(B.4) \quad \int_{-\infty}^{+\infty} \left[ \frac{\partial \phi}{\partial y} g'_{\alpha} + (-1)^{\alpha-1} \phi g''_{\alpha} \frac{\partial h_{\alpha}}{\partial y} \right] = 0
\end{align}

and

\begin{align}
(B.5) \quad & \int_{-\infty}^{+\infty} \left[ \frac{\partial^2 \phi}{\partial y^2} g'_{\alpha} + (-1)^{\alpha-1} \frac{\partial \phi}{\partial y} g''_{\alpha} \frac{\partial h_{\alpha}}{\partial y} + (-1)^{\alpha-1} \frac{\partial \phi}{\partial y} g''_{\alpha} \frac{\partial h_{\alpha}}{\partial y} \right] \\
& + \int_{-\infty}^{+\infty} \left[ (-1)^{\alpha-1} \phi g''_{\alpha} \frac{\partial^2 h_{\alpha}}{\partial y^2} + \phi g''_{\alpha} \frac{\partial h_{\alpha}}{\partial y} \partial h_{\alpha} \right] = 0.
\end{align}

Therefore

\begin{align}
(B.6) \quad \int_{-\infty}^{+\infty} \Delta_{\alpha,0} \phi(y,z) g'_{\alpha} = (-1)^{\alpha} \Delta_{\alpha,0} h_{\alpha} \int_{-\infty}^{+\infty} \phi g''_{\alpha} - |\nabla_{\alpha,0} h_{\alpha}|^2 \int_{-\infty}^{+\infty} \phi g''_{\alpha} \\
& + 2(-1)^{\alpha-1} \int_{-\infty}^{+\infty} g'_{\alpha}(y,0) \frac{\partial \phi}{\partial y} \frac{\partial h_{\alpha}}{\partial y} g''_{\alpha}.
\end{align}

Substituting (B.6) into (5.1), we obtain

\begin{align*}
& \int_{-\infty}^{+\infty} (\Delta_{\alpha,\phi} - \Delta_{\alpha,0}) g'_{\alpha} + (-1)^{\alpha} \left( \int_{-\infty}^{+\infty} \phi g''_{\alpha} \right) \Delta_{\alpha,0} h_{\alpha} - \left( \int_{-\infty}^{+\infty} \phi g''_{\alpha} \right) |\nabla_{\alpha,0} h_{\alpha}(y)|^2 \\
& + 2(-1)^{\alpha-1} \int_{-\infty}^{+\infty} g''_{\alpha} \frac{\partial \phi}{\partial y} \frac{\partial h_{\alpha}}{\partial y} - \int_{-\infty}^{+\infty} H_{\alpha}(y,z) g'_{\alpha} \phi_z + \int_{-\infty}^{+\infty} \xi''_{\alpha} \phi \\
& = \int_{-\infty}^{+\infty} \left[ W''(g_{\alpha}) - W''(g_{\alpha}) \right] g'_{\alpha} + \int_{-\infty}^{+\infty} R(\phi) g'_{\alpha} \\
& + \int_{-\infty}^{+\infty} I g'_{\alpha} + (-1)^{\alpha} \left( \int_{-\infty}^{+\infty} |g'_{\alpha}|^2 \right) [H_{\alpha}(y,0) + \Delta_{\alpha,0} h_{\alpha}(y)] \\
& + (-1)^{\alpha} \int_{-\infty}^{+\infty} |g'_{\alpha}|^2 [H_{\alpha}(y,z) - H_{\alpha}(y,0)] + (-1)^{\alpha} \int_{-\infty}^{+\infty} |g'_{\alpha}|^2 [\Delta_{\alpha,0} h_{\alpha}(y) - \Delta_{\alpha,0} h_{\alpha}(y)] \\
& + \frac{1}{2} \left( \int_{-\infty}^{+\infty} |g'_{\alpha}|^2 \frac{\partial g_{\alpha}}{\partial z \phi_{\alpha}} \phi_{\alpha} \frac{\partial h_{\alpha}}{\partial y} \frac{\partial h_{\alpha}}{\partial y} \right) \\
& + \sum_{\beta \neq \alpha} (-1)^{\beta} \int_{-\infty}^{+\infty} g''_{\alpha} g'_{\beta} \lambda_{\beta,1} - \sum_{\beta \neq \alpha} \int_{-\infty}^{+\infty} g''_{\alpha} g'_{\beta} \lambda_{\beta,2} - \sum_{\beta \neq \alpha} \int_{-\infty}^{+\infty} g''_{\alpha} \xi_{\beta}.
\end{align*}

We estimate the Hölder norm of these terms one by one.

1. By (3.12), we have

\begin{align*}
\left\| \int_{-\infty}^{+\infty} (\Delta_{\alpha,\phi} - \Delta_{\alpha,0}) g'_{\alpha} \right\|_{C^0(B_1(y))} \lesssim \varepsilon \max_{|z| < 8} \|\phi\|_{C^2,\#(B_1(y,z))} \\
\lesssim \varepsilon^2 + \max_{|z| < 8} \|\phi\|_{C^2,\#(B_1(y,z))}.
\end{align*}
(2) By the exponential decay of \( g' \) and Lemma 4.6, we have
\[
\left\| \left( \int_{-\infty}^{+\infty} \phi g''_\alpha \right) \Delta_{\alpha,0} h_\alpha \right\|_{C^{\theta}(B^*_1(y))} \lesssim \|h_\alpha\|_{C^{2,\theta}(B^*_1(y))} \max_{|z|<8} \log \varepsilon \|\phi\|_{C^{\theta}(B_1(y,z))} \\
\lesssim \|h_\alpha\|_{C^{2,\theta}(B^*_1(y))}^2 + \max_{|z|<8} \log \varepsilon \|\phi\|_{C^{\theta}(B_1(y,z))}^2 \\
\lesssim \max_{B^*_1(y)} e^{-2D_\alpha} + \max_{|z|<8} \log \varepsilon \|\phi\|_{C^{2,\theta}(B_1(y,z))}^2.
\]

(3) By the exponential decay of \( g' \) and Lemma 4.6, we have
\[
\left\| \int_{-\infty}^{+\infty} g''_\alpha g_{ij}(y,0) \frac{\partial \phi}{\partial y_i} \frac{\partial h_\alpha}{\partial y_j} \right\|_{C^{\theta}(B^*_1(y))} \lesssim \|h_\alpha\|_{C^{1,\theta}(B^*_1(y))} \max_{|z|<8} \log \varepsilon \|\phi\|_{C^{1,\theta}(B_1(y,z))} \\
\lesssim \|h_\alpha\|_{C^{1,\theta}(B^*_1(y))}^2 + \max_{|z|<8} \log \varepsilon \|\phi\|_{C^{1,\theta}(B_1(y,z))}^2 \\
\lesssim \max_{B^*_1(y)} e^{-2D_\alpha} + \max_{|z|<8} \log \varepsilon \|\phi\|_{C^{2,\theta}(B_1(y,z))}^2.
\]

(4) By the exponential decay of \( g' \) and Lemma 4.6, we have
\[
\left\| \left( \int_{-\infty}^{+\infty} \partial^2 \phi \right) |\nabla_{\alpha,0} h_\alpha(y)|^2 \right\|_{C^{\theta}(B^*_1(y))} \lesssim \|h_\alpha\|_{C^{2,\theta}(B^*_1(y))} \max_{|z|<8} \log \varepsilon \|\phi\|_{C^{\theta}(B_1(y,z))} \\
\lesssim \|h_\alpha\|_{C^{2,\theta}(B^*_1(y))}^2 + \max_{|z|<8} \log \varepsilon \|\phi\|_{C^{2,\theta}(B_1(y,z))}^2 \\
\lesssim \max_{B^*_1(y)} e^{-2D_\alpha} + \max_{|z|<8} \log \varepsilon \|\phi\|_{C^{2,\theta}(B_1(y,z))}^2.
\]

(5) By (3.6) and the exponential decay of \( g' \), we have
\[
\left\| \int_{-\infty}^{+\infty} H_\alpha(y,z) g'_\alpha \phi_2 \right\|_{C^{\theta}(B^*_1(y))} \lesssim \varepsilon \max_{|z|<8} \log \varepsilon \|\phi\|_{C^{2,\theta}(B_1(y,z))} \\
\lesssim \varepsilon^2 + \max_{|z|<8} \log \varepsilon \|\phi\|_{C^{2,\theta}(B_1(y,z))}.
\]

(6) The \( C^{\theta}(B^*_1(y)) \) norm of \( \int_{-\infty}^{+\infty} [W''(g_\alpha) - W''(g_\alpha)] g'_\alpha \phi \) is bounded by
\[
\left( \max_{|z|<8} \log \varepsilon \|\phi\|_{C^{\theta}(B_1(y,z))} \right) \left( \max_{B^*_1(y)} \int_{-\infty}^{+\infty} \left( |g''_{\alpha-1} - 1| + |g''_{\alpha+1} - 1| \right) g'_\alpha \right) \\
\lesssim \left( \max_{|z|<8} \log \varepsilon \|\phi\|_{C^{2,\theta}(B_1(y,z))} \right) \left( \max_{B^*_1(y)} D_\alpha e^{-\frac{3}{2}D_\alpha} \right) \\
\lesssim \max_{|z|<8} \log \varepsilon \|\phi\|_{C^{2,\theta}(B_1(y,z))} + \max_{B^*_1(y)} e^{-\frac{3}{2}D_\alpha}.
\]

(7) The \( C^{\theta}(B^*_1(y)) \) norm of \( \int_{-\infty}^{+\infty} R(\phi) g'_\alpha \) is bounded by \( \max_{|z|<8} \log \varepsilon \|\phi\|_{C^{2,\theta}(B_1(y,z))} \).

(8) By the definition of \( \bar{g} \) (see Subsection 4.1), the \( C^{\theta}(B^*_1(y)) \) norm of \( \int_{-\infty}^{+\infty} |g'_\alpha|^2 - \sigma_0 \) is bounded by \( O(\varepsilon^2) \).

(9) By (3.6) and (3.9), the \( C^{\theta}(B^*_1(y)) \) norm of \( \int_{-\infty}^{+\infty} |g'_\alpha|^2 [H_\alpha(y,z) - H_\alpha(y,0)] \) is bounded by \( O(\varepsilon^2) \).

(10) By (3.12) and Lemma 4.6, the \( C^{\theta}(B^*_1(y)) \) norm of \( \int_{-\infty}^{+\infty} |g'_\alpha|^2 [\Delta_{\alpha,0} h_\alpha(y) - \Delta_{\alpha,z} h_\alpha(y)] \) is bounded by
\[
\varepsilon \|h_\alpha\|_{C^{2,\theta}(B^*_1(y))} \lesssim \varepsilon^2 + \|h_\alpha\|_{C^{2,\theta}(B^*_1(y))}^2 \lesssim \varepsilon^2 + \max_{B^*_1(y)} e^{-2D_\alpha} + \|\phi\|_{C^{2,\theta}(B_1(y,z))}^2.
\]
(11) By (3.4), the $C^θ(B^α_1(y))$ norm of $\left( \int_{-∞}^{+∞} |g''_α|^2 \frac{∂^2 g^i_j}{∂y^i∂z^j} \right) \frac{∂h_{αn}}{∂y^i} \frac{∂h_{αr}}{∂y^j}$ is bounded by

$$\varepsilon \|h_{α}\|^2_{C^{1,θ}(B^α_1(y))} \lesssim \max_{B^α_1(y)} e^{-2D_α} + \|\phi\|^2_{C^{2,θ}(B_1(y,0))}.$$ 

(12) For $β \neq α$, if $|d_β(y,0)| > 8|\log ε|$, the $C^θ(B^α_1(y))$ norm of $\int_{-∞}^{+∞} g'_α g'_β R_{β,1}$ is bounded by $O(ε^2)$.

If $|d_β(y,0)| \leq 8|\log ε|$, first note that in Fermi coordinates with respect to $Γ_β$, we have the decomposition

$$g'_α g'_β R_{β,1} = \underbrace{g'_α g'_β [H_β(y,0) + Δ_β,0 h_β(y)]}_{I} + \underbrace{g'_α g'_β [H_β(y,z) - H_β(y,0)]}_{II} + \underbrace{g'_α g'_β [Δ_β,z h_β(y) - Δ_β,0 h_β(y)]}_{III}.$$ 

These three terms are estimated in the following way. First we have

$$\|I\|_{C^θ(B_1(y,z))} \lesssim e^{-|d_α(y,z)| - |z|} \|H_β + Δ_β,0 h_β\|_{C^θ(B^α_1(y))}.$$ 

By (3.9), we get

$$\|II\|_{C^θ(B_1(y,z))} \lesssim ε^2 |z| e^{-|d_α(y,z)| - |z|},$$ 

and by (3.12), we get

$$\|III\|_{C^θ(B^α_1(y))} \lesssim ε^2 |z| e^{-|d_α(y,z)| - |z|} \|H_β\|_{C^{2,θ}(B^α_1(Π_β(y,0)))}.$$ 

Putting these estimates together and coming back to Fermi coordinates with respect to $Γ_α$, applying Lemma 3.4 to change distances to be measured with respect to $Γ_α$, we see the $C^θ(B^α_1(y))$ norm of $\int_{-∞}^{+∞} g'_α g'_β R_{β,1}$ is controlled by

$$\varepsilon^2 + |d_β(y,0)| e^{-|d_β(y,0)|} \|H_β + Δ_β,0 h_β\|_{C^θ(B^α_1(Π_β(y,0)))} + ε|d_β(y,0)| e^{-|d_β(y,0)|} \|H_β + Δ_β,0 h_β\|_{C^θ(B^α_1(y,0))}.$$ 

Summing in $β \neq α$ and applying Lemma 3.6 we obtain

$$\sum_{β \neq α} \varepsilon^2 + \max_{B^α_1(y)} \varepsilon^2 D_α + \max_{β \neq α; |d_β(y,0)| \leq 8|\log ε|} \|H_β + Δ_β,0 h_β\|^2_{C^θ(B^α_1(Π_β(y,0)))}.$$ 

(13) By the same reasoning as in the previous case, for $β \neq α$, the $C^θ(B^α_1(y))$ norm of $\sum_{β \neq α} \int_{-∞}^{+∞} g'_α g'_β R_{β,2}$ is controlled by

$$\varepsilon^2 + \max_{|z| < 8|\log ε|} \|\phi\|^2_{C^{2,θ}(B_1(y,z))} + \sum_{β \neq α; |d_β(y,0)| \leq 8|\log ε|} |d_β(y,0)| e^{-|d_β(y,0)| + 2D_β(Π_β(y,0))}.$$ 

$$\lesssim \varepsilon^2 + \max_{β; |d_β(y,0)| \leq 8|\log ε|} \max_{B^α_1(Π_β(y,0))} \varepsilon^{-\frac{ξ}{2} D_β} + \max_{|z| < 8|\log ε|} \|\phi\|^2_{C^{2,θ}(B_1(y,z))}.$$ 

(14) By the definition of $ξ$, the $C^θ(B^α_1(y))$ norm of $\sum_β (-1)^β - 1 \int_{-∞}^{+∞} g'_α ξ_β$ is bounded by $O(ε^2)$.

Combining all of these estimates we get (5.3).
APPENDIX C. PROOF OF LEMMA 6.6

We estimate the Hölder norm of the right hand side of (4.4) term by term. Since they reproduce similar patterns on each $\mathcal{M}_0^\alpha$, it is sufficient to consider one of such domains.

1. Because

$$ R(\phi) = W'(g_\ast + \phi) - W'(g_\ast) - W''(g_\ast)\phi, $$

we get

$$ \| R(\phi) \|_{C^\alpha(B_r(x))} \lesssim \| \phi \|_{C^\alpha(B_r(x))}^2. $$

2. By Lemma 4.5, we have

$$ \left\| W'(g_\ast) - \sum_\beta W'(g_{\beta}) \right\|_{C^\alpha(M_0^\alpha \cap B_r(x))} \lesssim \varepsilon^2 + A(r; x). $$

3. Take the decomposition

$$ g'_\alpha [H_\alpha(y, z) + \Delta z h_\alpha(y)] = g'_\alpha [H_\alpha(y, 0) + \Delta_0 h_\alpha(y)] $$

$$ + g'_\alpha [H_\alpha(y, z) - H_\alpha(y, 0)] + g'_\alpha [\Delta_{\alpha, z} h_\alpha(y) - \Delta_{\alpha, 0} h_\alpha(y)]. $$

First we have

$$ \| g'_\alpha (y, z) [H_\alpha(y, z) - H_\alpha(y, 0)] \|_{C^\alpha(M_0^\alpha \cap B_r(x))} $$

$$ \lesssim \max_{(y, z) \in M_0^\alpha \cap B_r(x)} e^{-|z|} |H_\alpha(y, z) - H_\alpha(y, 0)| $$

$$ + \max_{(y, z) \in M_0^\alpha \cap B_r(x)} |z||e^{-|z|}|A_\alpha(y, 0)||\nabla_{\alpha, 0} A_\alpha(y, 0)| $$

$$ \lesssim \varepsilon^2. $$

Next because

$$ \Delta_{\alpha, z} h_\alpha(y) - \Delta_{\alpha, 0} h_\alpha(y) = \sum_{i, j=1}^{n-1} \left[ g_{ij}'(y, z) - g_{ij}'(y, 0) \right] \frac{\partial^2 h_\alpha}{\partial y_i \partial y_j} + \sum_{i=1}^{n-1} \left[ b_i'(y, z) - b_i'(y, 0) \right] \frac{\partial h_\alpha}{\partial y_i}, $$

we get

$$ \| g'_\alpha [\Delta_{\alpha, 0} h_\alpha(y) - \Delta_{\alpha, z} h_\alpha(y)] \|_{C^\alpha(M_0^\alpha \cap B_r(x))} $$

$$ \lesssim \max_{(y, z) \in M_0^\alpha \cap B_r(x)} e^{-|z|} \left[ |g_{ij}'(y, z) - g_{ij}'(y, 0)||\nabla_{\alpha, 0} h_\alpha(y)| + |b_i'(y, z) - b_i'(y, 0)||\nabla_{\alpha, 0} h_\alpha(y)| \right] $$

$$ + \max_{(y, z) \in M_0^\alpha \cap B_r(x)} e^{-|z|} \left( |\nabla_{\alpha, 0}^2 h_\alpha(y)| + |\nabla_{\alpha, 0}^2 h_\alpha(y)| \right) $$

$$ \times \left[ \left| g_{ij}'(\tilde{y}, \tilde{z}) - g_{ij}'(\tilde{y}, 0) \right|_{C^\alpha(B_1(y, z))} + \left| b_i'(\tilde{y}, \tilde{z}) - b_i'(\tilde{y}, 0) \right|_{C^\alpha(B_1(y, z))} \right] $$

$$ + \| h_\alpha \|_{C^2, 0(\Gamma_\alpha \cap B_r(x))} \left( \max_{(y, z) \in M_0^\alpha \cap B_r(x)} e^{-|z|} \left[ |g_{ij}'(y, z) - g_{ij}'(y, 0)| + |b_i'(y, z) - b_i'(y, 0)| \right] \right) $$

$$ \lesssim \varepsilon \| h_\alpha \|_{C^2, 0(\Gamma_\alpha \cap B_r(x))} $$

$$ \lesssim \varepsilon^2 + \| h_\alpha \|_{C^2, 0(\Gamma_\alpha \cap B_r(x))}^2 $$

(by Cauchy inequality)

$$ \lesssim \varepsilon^2 + \| \phi \|_{C^2, 0(\Gamma_\alpha \cap B_r(x))}^2 + \max_{\Gamma_\alpha \cap B_r(x)} e^{-2D_\alpha}. $$

(by Lemma 4.6)
(4) By Lemma 4.6, we have
\[
\|g'_\beta \nabla_{\alpha,z} h_\alpha\|_2 \lesssim C_\theta (M^{1\theta}_\alpha \cap B_r(x))
\]
\[
\lesssim |\nabla_{\alpha,0} h_\alpha|^2_{L^\infty(G_\alpha \cap B_r(x))} + |\nabla_{\alpha,0} h_\alpha|^2_{L^\infty(G_\alpha \cap B_r(x))}
\lesssim \|\phi\|_{2^\theta(B_{r+9\log \varepsilon}(x))}^2 + \max_{\alpha} \max_{\Gamma_\alpha \cap B_{r+9\log \varepsilon}(x)} e^{-2D_\alpha}.
\]

(5) As in the previous case, we first estimate the Hölder norm of \(R_{\beta,1}\) in Fermi coordinates with respect to \(\Gamma_\beta\) for each \(\beta \neq \alpha\). Coming back to Fermi coordinates with respect to \(\Gamma_\alpha\) and noting that if \(g'_\beta \neq 0\), then \(|d_\beta(y, z)| < 8|\log \varepsilon|\), we obtain
\[
\|g'_\beta R_{\beta,1}\|_{C^\theta(M^{1\theta}_\alpha \cap B_r(x))} \lesssim \max_{(y, z) \in M^{1\theta}_\alpha \cap B_r(x)} e^{-|d_\beta(y, z)|}
\times \left(\varepsilon^2 + \|\phi\|_{2^\theta(B_{r+9\log \varepsilon}(x))}^2 + \max_{\alpha} \max_{\Gamma_\alpha \cap B_{r+9\log \varepsilon}(x)} e^{-2D_\alpha}\right)
\]
\[
+ \max_{(y, z) \in M^{1\theta}_\alpha \cap B_r(x), |d_\beta(y, z)| \leq 8|\log \varepsilon|} e^{-|d_\beta(y, z)|} \|H_\beta + \Delta_\beta h_\beta\|_{C^\theta(B_{r}^\theta(\Pi_\beta(y, 0)))}.
\]
Then using Lemma 3.6 and summing in \(\beta\), we get
\[
\sum_{\beta \neq \alpha} \|g'_\beta R_{\beta,1}\|_{C^\theta(M^{1\theta}_\alpha \cap B_r(x))} \lesssim \varepsilon^2 + \|\phi\|_{2^\theta(B_{r+9\log \varepsilon}(x))}^2 + \max_{\alpha} \max_{\Gamma_\alpha \cap B_{r+9\log \varepsilon}(x)} e^{-2D_\alpha}
\]
\[
+ \left(\max_{\alpha} \max_{\Gamma_\alpha \cap B_{r+9\log \varepsilon}(x)} e^{-\frac{D_\alpha}{2}}\right) \left(\max_{\alpha} \|H_\alpha + \Delta_\alpha h_\alpha\|_{C^\theta(B_{r}^\theta(\Pi_\beta(y, 0)))}\right).
\]

(6) Similar to the previous case, we have
\[
\sum_{\beta \neq \alpha} \|g'_\beta R_{\beta,2}\|_{C^\theta(M^{1\theta}_\alpha \cap B_r(x))} \lesssim \varepsilon^2 + \|\phi\|_{2^\theta(B_{r+9\log \varepsilon}(x))}^2 + \max_{\alpha} \max_{\Gamma_\alpha \cap B_{r+9\log \varepsilon}(x)} e^{-2D_\alpha}
\]
\[
+ \left(\max_{\alpha} \max_{\Gamma_\alpha \cap B_{r+9\log \varepsilon}(x)} e^{-\frac{D_\alpha}{2}}\right) \left(\max_{\alpha} \|H_\alpha + \Delta_\alpha h_\alpha\|_{C^\theta(B_{r}^\theta(\Pi_\beta(y, 0)))}\right).
\]

(7) For any \(\beta, \xi_\beta(y, z) \neq 0\) only if \(|d_\beta(y, z)| \leq 8|\log \varepsilon|\). Hence by Lemma 2.1,
\[
\|\sum_{\beta} \xi_\beta\|_{C^\theta(M^{1\theta}_\alpha \cap B_r(x))} \lesssim \varepsilon^3 |\log \varepsilon| \lesssim \varepsilon^2.
\]
Putting these estimates together we finish the proof of Lemma 6.6.
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