We prove that static, spherically symmetric, asymptotically flat, regular solutions of the Einstein-Yang-Mills equations are unstable for arbitrary gauge groups. The proof involves the following main steps. First, we show that the frequency spectrum of a class of radial perturbations is determined by a coupled system of radial “Schrödinger equations”. Eigenstates with negative eigenvalues correspond to exponentially growing modes. Using the variational principle for the ground state it is then proven that there always exist unstable modes (at least for “generic” solitons). This conclusion is reached without explicit knowledge of the possible equilibrium solutions.
1 Introduction

In recent years the study of regular and black hole solutions of the Einstein-Yang-Mills (EYM) equations, without and with additional fields (Higgs fields, dilaton fields, etc.) has been actively pursued, and a number of interesting and surprising results have been discovered. Among the regular solutions the most interesting ones are those for which gravity is essential. The first example of this type was found by Bartnik and McKinnon \[1\] for the $SU(2)$-EYM system. For the same model several authors \[2, 3, 4\] discovered later the colored black hole solutions which showed that the classical uniqueness theorem for the Abelian case does not generalize. The existence of both types of solutions, which meanwhile has been established rigorously \[5, 6, 7\], led to a search for corresponding solutions of other related field theories. It turned out, for instance, that the Einstein-Skyrme system has black hole solutions with hair which are at least linearly stable \[8, 9, 10, 11\]. (For a numerical investigation of nonlinear stability, see ref. \[11\].) Several authors studied other models, notably the $SU(2)$-EYM system with a Higgs triplet \[12, 13, 14, 15\], as well as the EYM-dilaton system \[16\], and found in some cases other linearly stable black hole solutions. Interesting black hole solutions have recently been found numerically for the EYM system with a Higgs doublet \[17\], as in the standard electroweak model. These “sphaleron black holes” are expected to be unstable, but this question is not yet fully analyzed. We have shown recently, that the regular solutions are unstable \[18\], but our method cannot be applied directly to the black hole case.

The Bartnik-McKinnon solution and the related black hole solutions were shown to be unstable by some of us \[19, 20, 21, 22\]. On physical grounds one would expect that this remains true for any gauge group, but a mathematical proof of this conjecture presents quite a challenge. We finally succeeded in constructing such a proof for the regular solutions, which will be sketched in the present paper. Similar arguments may perhaps also be used successfully to establish the instability of all black hole solutions, but problems related to the boundary conditions at the horizon have not yet been overcome.

Our strategy is based on the study of the pulsation equations describing linear radial perturbations of the equilibrium solutions. Because the derivation of these equations relies heavily on our previous work \[23, 24\], we first recall some basic facts and equations in the next section. The pulsation equations are then presented in section 3 in a convenient partially decoupled
form. It turns out, that the frequency spectrum of a class of radial perturbations is determined by a coupled system of radial “Schrödinger equations” whose bound states correspond to exponentially growing modes. In section 4 we prove with the variational principle for the ground state that there always exist unstable modes, at least in the generic case (defined in sect. 2). The construction of the trial variations, which are used to establish this, is presumably the main point of the present letter; we succeeded only after a number of failures.

2 Basic facts and equations

Since we consider only spherically symmetric configurations, the metric $g$ of the space-time manifold $M$ can be parametrized as

$$g = -N S^2 dt^2 + N^{-1} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\phi^2),$$

(1)

where the metric functions $N$ and $S$ depend only on the Schwarzschild-like radius $r$ and the time $t$. We use also the usual mass fraction $m(r,t)$, defined by $N =: 1 - 2m/r$. Gauge fields with spherical symmetry have been described in detail in ref. [23]. Let us briefly recall the results, as far as they will be needed below.

We fix a maximal torus $T$ of $G$ with the corresponding integral lattice $I$ (= kernel of the exponential map restricted to the Lie algebra $LT$ of the torus). In addition, we choose a basis $S$ of the root system $R$ of real roots, which defines the fundamental Weyl chamber $K(S) = \{ H \in LT \mid \alpha(H) > 0 \text{ for all } \alpha \in S \}.$

(2)

To a given spherically symmetric configuration, there belongs a canonical element $H_\lambda \in I \cap K(S)$ which characterizes the corresponding principal bundle $P(M, G)$ admitting an $SU(2)$ action (see sect. II of ref. [23]). In [24] we showed that $H_\lambda$ is restricted to a small finite subset of $I \cap K(S)$, if the solution is regular at the origin. In much of our discussion we exclude the possibility that $H_\lambda$ lies on a boundary of the fundamental Weyl chamber and consider only what we call the “generic case”, for which $H_\lambda$ is in the open Weyl chamber $K(S)$.
A spherically symmetric gauge field can be described by a smooth family of linear maps $\Lambda: LSU(2) \rightarrow LG$, depending only on $r$ and $t$ and satisfying the following conditions:

$$\Lambda_1 = [\Lambda_2, \Lambda_3], \quad \Lambda_2 = [\Lambda_3, \Lambda_1], \quad \Lambda_3 = -H_\lambda / 4\pi,$$

(3)

where $\Lambda_k := \Lambda(\tau_k)$ and $2i\tau_k$ are the Pauli matrices. These equations imply that $\Lambda_+ := \Lambda_1 + i\Lambda_2$ lies in the following direct sum of root spaces $L_\alpha$:

$$\Lambda_+ \in \bigoplus_{\alpha \in \Sigma} L_\alpha,$$

(4)

In the generic case $\Sigma$ is a basis of a root system contained in $R$ (see Appendix A of ref. [24]).

For the example of the gauge group $SU(2)$, $H_\lambda$ is an integer multiple of $4\pi \tau_3$: $H_\lambda = 4\pi k \tau_3$ with $k \in \mathbb{Z}$, and the only solutions of (3) are $\Lambda_1 = \Lambda_2 = 0$, $\Lambda_3 = k \tau_3$, or

$$\Lambda_1 = w \tau_1 + \tilde{w} \tau_2, \quad \Lambda_2 = \mp \tilde{w} \tau_1 \pm w \tau_2, \quad \Lambda_3 = \pm \tau_3.$$

(5)

In ref. [23] it is shown that there exists always a (local) gauge such that the gauge potential $A$ takes the form

$$A = \tilde{A} + \hat{A},$$

(6)

with

$$\hat{A} = \Lambda_2 \, d\vartheta + (\Lambda_3 \cos \vartheta - \Lambda_1 \sin \vartheta) \, d\varphi$$

(7)

and

$$\tilde{A} = NS \, \mathcal{A} \, dt + \mathcal{B} \, dr,$$

(8)

where $\mathcal{A}$ and $\mathcal{B}$ commute with $H_\lambda$ (i.e. with $\Lambda_3$). If $H_\lambda$ is generic one knows that its centralizer is the infinitesimal torus $LT$. Hence, $\mathcal{A}$ and $\mathcal{B}$ are $LT$-valued and $\tilde{A}$ is thus Abelian.

The coupled EYM equations, corresponding to the parametrizations (1), (6) – (8) have been derived in [23]. Here, it suffices to write them in a slightly different form for the temporal gauge $\mathcal{A} = 0$.

The Einstein equations reduce to ($\kappa := 8\pi G$)

$$m' = \frac{\kappa}{2} \left\{ NG + p_\theta \right\}, \quad \dot{m} = \frac{\kappa}{2} NH,$$

(9)
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\[
\frac{S'}{S} = \frac{\kappa}{r} G ,
\]  

where

\[
G = \frac{1}{2} \left\{ \left( NS \right)^{-2} |\dot{\Lambda}_+|^2 + |\Lambda'_+ + [\mathcal{B}, \Lambda_+]|^2 \right\} ,
\]

\[
H = \text{Re} \langle \dot{\Lambda}_+, \Lambda'_+ + [\mathcal{B}, \Lambda_+] \rangle ,
\]

\[
p_\theta = \frac{1}{2r^2} \left\{ |\hat{\mathcal{F}}|^2 + |\check{\mathcal{F}}|^2 \right\}
\]

with

\[
\hat{\mathcal{F}} = \frac{r^2}{S} \dot{\mathcal{B}} , \quad \check{\mathcal{F}} = i \frac{1}{2} [\Lambda_+, \Lambda_-] - \Lambda_3 .
\]

Here we have used the following notation: \( \langle \cdot , \cdot \rangle \) denotes a suitably normalized Ad(G)-invariant scalar product on \( LG \), as well as its (hermetian) extension to \( LG_C \), and \( | \cdot | \) is the corresponding norm.

The YM equations decompose into

\[
\frac{2}{NS} \left( \frac{r^2}{S} \dot{\mathcal{B}} \right)' + \left[ \Lambda_+, \Lambda'_- + [\mathcal{B}, \Lambda_-] \right] + \left[ \Lambda_-, \Lambda'_+ + [\mathcal{B}, \Lambda_+] \right] = 0 ,
\]

\[
\frac{1}{S} \left( \frac{1}{NS} \dot{\Lambda}_+ \right)' - \frac{1}{S} \left( NS \{ \Lambda'_+ + [\mathcal{B}, \Lambda_+] \} \right)' - N \left\{ [\mathcal{B}, \Lambda'_+] + [\mathcal{B}, [\mathcal{B}, \Lambda_+]] \right\} + \frac{i}{r^2} [\hat{\mathcal{F}}, \Lambda_+] = 0 ,
\]

\[
2 \left( \frac{r^2}{S} \dot{\mathcal{B}} \right)' + 2 \frac{r^2}{S} [\mathcal{B}, \dot{\mathcal{B}}] + \frac{1}{NS} \left\{ [\Lambda_+, \check{\mathcal{A}}_-] + [\Lambda_-, \check{\mathcal{A}}_+] \right\} = 0 .
\]

The last equation is the Gauss constraint. For the generic case the term proportional to \([\mathcal{B}, \dot{\mathcal{B}}]\) in (17) vanishes.

For static solutions all time derivatives disappear and the basic equations simplify considerably. (For the Bartnik-McKinnon solution \( \Lambda \) is of the form (18) with \( \tilde{w} = 0 \), \( \Lambda_3 = \tau_3 \) and \( \tilde{A} = 0 \) in (18).)
3 Pulsation equations

We consider now a static, regular, asymptotically flat solution of the coupled EYM equations (9), (10), (15) – (17) with a generic $H_\lambda$. Such a solution is of purely magnetic type with vanishing YM charge. This is not yet rigorously proven under satisfactory weak assumptions, but there is strong evidence for this (see [24, 25] for partial results). From now on the symbols $\Lambda_{\pm}$, $N$, $S$, etc. will be used to denote the equilibrium solution. Time-dependent perturbations are denoted by $\delta \Lambda_{\pm}$, $\delta B$, etc..

The details of the reduction of the perturbation equations to a well-adapted form will be described elsewhere. Here, we emphasize only some crucial points and present the final result.

An observation which was already pointed out in [13] for $G = SU(2)$ finds here a natural generalization which simplifies matters considerably: The first of the Einstein equations (9) leads to an expression for $\delta m'$ which can be readily integrated with respect to $r$. An arbitrary function of $t$ which thereby appears for $\delta m$ is then fixed by the second Einstein equation in (9). We find

$$\delta m = \frac{\kappa}{2} N \text{Re} \langle \Lambda'_+ , \delta \Lambda_+ \rangle .$$

(18)

It turns out that a significant decoupling of the perturbation equations is achieved by decomposing $\delta \Lambda_+$ as follows. Let us choose in the space (4) a basis $\{e_\alpha\}$ of the root spaces $L_\alpha$ ($\alpha \in \Sigma$), with respect to which we expand the unperturbed $\Lambda_+$ and its perturbation $\delta \Lambda_+$,

$$\delta \Lambda_+ = \sum_{\alpha \in \Sigma} \delta w^\alpha e_\alpha .$$

(19)

Then we have ($\delta \Lambda_- := c (\delta \Lambda_+)$, $c =$ conjugation in $LG_C$)

$$\delta \Lambda_\pm = \delta X_\pm \pm i \delta Y_\pm$$

(20)

with

$$\delta X_+ = \sum_{\alpha \in \Sigma} \text{Re} \langle \delta w^\alpha \rangle e_\alpha , \quad \delta Y_+ = \sum_{\alpha \in \Sigma} \text{Im} \langle \delta w^\alpha \rangle e_\alpha .$$

(21)

We shall call $\delta X_\pm$, $\delta Y_\pm$ the real and imaginary parts of the perturbations $\delta \Lambda_\pm$. The unperturbed quantity $\Lambda_+$ can be chosen to have only a real part, as was shown in [24].
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With the help of the equilibrium equations one can now bring the perturbation equations after some work into the following standard form: \( \delta X_+ \) decouples,

\[
\delta \ddot{X}_+ + U_{XX} \delta X_+ = 0, \tag{22}
\]

and \( \delta \Phi := (\delta Y_+, \delta B) \) satisfies

\[
T \delta \ddot{\Phi} + U \delta \Phi = 0, \tag{23}
\]

where the operators \( U_{XX} \) and \( U \),

\[
U = \begin{pmatrix} U_{YY} & U_{YB} \\ U_{BY} & U_{BB} \end{pmatrix}, \tag{24}
\]

are given by the expressions

\[
U_{XX} = p_s^2 + \frac{NS^2}{r^2} \text{ad}(i\hat{F}) - \frac{1}{N r^2} \text{ad}(NS\Lambda_+) \text{ad}(NS\Lambda_-) \\
- (p_s\Lambda_+ \frac{\kappa}{r} \left\langle \frac{(NS)'}{NS} + \frac{1}{r} \right\rangle \langle p_s\Lambda_+ , \cdot \rangle \\
+ (p_s\Lambda_+ \frac{\kappa S}{r^3} \langle [\hat{F},\Lambda_+] , \cdot \rangle + [\hat{F},\Lambda_+] \frac{\kappa S}{r^3} \langle p_s\Lambda_+ , \cdot \rangle, \tag{25}
\]

\[
U_{YY} = p_s^2 + \frac{NS^2}{r^2} \text{ad}(i\hat{F}), \tag{26}
\]

\[
U_{BB} = - \text{ad}(NS\Lambda_+) \text{ad}(NS\Lambda_-), \tag{27}
\]

\[
U_{YB} = + p_s \text{ad}(NS\Lambda_+) + \text{ad}(NS p_s\Lambda_+), \tag{28}
\]

\[
U_{BY} = - \text{ad}(NS\Lambda_-) p_s + \text{ad}(NS p_s\Lambda_-) \tag{29}
\]

with

\[
p_s = -iNS \frac{d}{dr}. \tag{30}
\]

The operator \( T \) in (23) acts as multiplication with the diagonal matrix

\[
T = \begin{pmatrix} 1 & 0 \\ 0 & N r^2 \end{pmatrix}. \tag{31}
\]
It is easy to see that $U$ and $T$ are symmetric relative to the following scalar product for $LG_C$-valued functions on $\mathcal{R}_+$:

$$\langle \Psi \mid \Phi \rangle = \int_0^\infty \langle \Psi , \Phi \rangle \frac{dr}{NS} .$$

(32)

(Questions of domains of unbounded operators will be discussed elsewhere.)

For a harmonic time dependence, $\delta \Phi = \xi e^{-i\omega t}$, we obtain from (23)

$$\omega^2 = \frac{\langle \xi \mid U \mid \xi \rangle}{\langle \xi \mid T \mid \xi \rangle} .$$

(33)

Different eigenmodes are orthogonal with respect to the following scalar product defined by $T$:

$$\langle \xi_1 \mid \xi_2 \rangle_T := \langle \xi_1 \mid T \mid \xi_2 \rangle .$$

For the frequency $\omega_0$ of the fundamental mode of eq. (23) we have the minimum principle

$$\omega_0^2 = \min_{\xi} \frac{\langle \xi \mid U \mid \xi \rangle}{\langle \xi \mid T \mid \xi \rangle} ,$$

(34)

which will be used in the next section to show that $\omega_0^2 < 0$.

The perturbation equations (22), (23) do not include the Gauss constraint (17), whose linearization reads

$$\frac{d}{dt} \left\{ p_* \left( \frac{R^2}{S} \delta B \right) - [\Lambda_+ , \delta Y_-] \right\} = 0 .$$

(35)

It will turn out that this is automatically satisfied for physical pulsations (more precisely: in the space orthogonal to pure gauge modes).

4 Instability of all generic EYM solitons

For a given regular solution with $\Lambda_+ = \sum_{\alpha \in \Sigma} w^\alpha e^\alpha$ we construct now a one-parameter family of field configurations $\Lambda_{(\chi)}^+ , B_{(\chi)}$ such that the variational expressions (34) for $\delta \Lambda_\pm = (d\Lambda_{(\chi)}^+/d\chi)_{\chi=0} , \delta B = (dB_{(\chi)}/d\chi)_{\chi=0}$ is negative. This family is chosen of the following form:

$$\Lambda_{(\chi)}^+ = \text{Ad}(\exp(-\chi Z)) \left\{ \Lambda_+ \cos(\chi) + i\Lambda_+(\infty) \sin(\chi) \right\} ,$$

(36)

$$B_{(\chi)} = \chi Z' ,$$

(37)
where \( Z \) is an \( LT \)-valued function of \( r \) with the properties
\[
\lim_{r \to 0, \infty} [Z, \Lambda_+] = i\Lambda_+(\infty), \quad \text{supp} \, Z' \subseteq [1 - \epsilon, 1 + \epsilon] \tag{38}
\]
for an \( \epsilon > 0 \). The existence of such a function can be seen as follows. Let \( \{h_\alpha\}_{\alpha \in \Sigma} \) be the dual basis of \( 2\pi \Sigma \) and put
\[
Z = \sum_{\alpha \in \Sigma} Z_\alpha h_\alpha, \quad Z_\alpha = \begin{cases} w_\alpha(\infty)/w_\alpha(0) & \text{for } r < 1 - \epsilon, \\ 1 & \text{for } r > 1 + \epsilon. \end{cases} \tag{39}
\]
It is easy to verify that both conditions in (38) are satisfied. (It can be shown that \( w_\alpha(0) \neq 0 \) for all \( \alpha \in \Sigma \); see Appendix A of ref. [24].)

We note a few properties of the family (36), (37). The equilibrium solution is clearly obtained for \( \chi = 0 \). Applying a gauge transformation with \( g = \exp(\chi Z) \) we obtain
\[
\Lambda_{(\chi)+} \to \Lambda_+ \cos(\chi) + i\Lambda_+(\infty) \sin(\chi), \quad B_{(\chi)} \to 0. \tag{40}
\]
The first variations of (36) and (37) are
\[
\delta \Lambda_+ = -[Z, \Lambda_+] + i\Lambda_+(\infty), \quad \delta B = Z', \tag{41}
\]
and these satisfy by construction the desired boundary conditions
\[
\lim_{r \to 0, \infty} \delta \Lambda_+ = 0, \quad \lim_{r \to 0, \infty} \delta B = 0. \tag{42}
\]
(\( \delta B \) vanishes even outside \([1 - \epsilon, 1 + \epsilon]\).) Finally, \( \delta \Lambda_+ \) has only an imaginary component
\[
\delta Y_+ = \delta \Lambda_+ = -[Z, \Lambda_+] + i\Lambda_+(\infty) \tag{43}
\]
and thus by (44)
\[
\omega_0^2 \leq \frac{\langle \delta \Phi \mid U \mid \delta \Phi \rangle}{\langle \delta \Phi \mid T \mid \delta \Phi \rangle}. \tag{44}
\]
with \( \delta \Phi = (\delta Y_+, \delta B) \) given by (13) and the second eq. in (41).

It is easy to see that the denominator in this expression is finite if \( \Lambda_+(r) - \Lambda_+(\infty) \) vanishes sufficiently fast for \( r \to \infty \).

The calculation of the numerator in (44) for the operator \( U \) in (24) is somewhat tedious. After several partial integrations, using (42) and the equilibrium equations, one finally arrives at
\[
\langle \delta \Phi \mid U \mid \delta \Phi \rangle = - \int \left\{ \text{NS} |\Lambda'_+|^2 + 2 \frac{S}{r^2} |\hat{F}|^2 \right\} \, dr, \tag{45}
\]
which is finite and strictly negative!

We now return to the linearized Gauss constraint (35) and note that a variation $\delta \Phi$ is orthogonal with respect to $\langle \cdot | \cdot \rangle_T$ to all gauge variations

$$\delta \Phi_{\text{gauge}} = (\hat{i}[\chi, \Lambda_+], \chi'), \quad \chi: \mathcal{R}_+ \to LT,$$  \hspace{1cm} (46)

if and only if the curly bracket in (35) vanishes. This can be shown readily with a partial integration.

This proves the instability of the regular equilibrium solution. It should be emphasized that we were able to draw this conclusion assuming only weak asymptotic conditions for the solitons. Especially, we have used repeatedly that the YM charge vanishes only if $\lim_{r \to \infty} \Lambda(r)$ is a homomorphism from $LSU(2)$ to $LG$ (see eq. (14)).

Acknowledgments

We would like to thank Markus Heusler for discussions and comments and to Michael Volkov for suggestive remarks on stability problems. This work was supported in part by the Swiss National Science Foundation.

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