The Integral Transform of N.I. Akhiezer

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Abstract. We study the integral transform which appeared in a different form in Akhiezer’s textbook “Lectures on Integral Transforms”.

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1. The Akhiezer Integral Transforms: a formal definition.

In the present paper we consider the one-parametric family of pairs $\Phi_\omega, \Psi_\omega$ of linear integral operators. The parameter $\omega$ which enumerates the family can be an arbitrary positive number and is fixed in the course of our consideration. Formally the operators $\Phi_\omega, \Psi_\omega$ are defined as convolution operators according the formulas

\[
(\Phi_\omega x)(t) = \int_\mathbb{R} \Phi_\omega(t - \tau)x(\tau)d\tau, \quad t \in \mathbb{R},
\]

\[
(\Psi_\omega x)(t) = \int_\mathbb{R} \Psi_\omega(t - \tau)x(\tau)d\tau, \quad t \in \mathbb{R}.
\]

In (1.1), $x(\tau)$ is $2 \times 1$ vector-column,

\[
x(\tau) = \begin{bmatrix} x_1(\tau) \\ x_2(\tau) \end{bmatrix},
\]

which entries $x_1(\tau), x_2(\tau)$ are measurable functions, and $\Phi_\omega, \Psi_\omega$ are $2 \times 2$ matrices,

\[
\Phi_\omega(t) = \begin{bmatrix} C_\omega(t) & S_\omega(t) \\ S_\omega(t) & C_\omega(t) \end{bmatrix},
\]

\[
\Psi_\omega(t) = \begin{bmatrix} C_\omega(t) & -S_\omega(t) \\ -S_\omega(t) & C_\omega(t) \end{bmatrix}.
\]
where
\[ C_\omega(t) = \frac{\omega}{\pi} \cdot \frac{1}{\cosh \omega t}, \quad S_\omega(t) = \frac{\omega}{\pi} \cdot \frac{1}{\sinh \omega t}. \quad (1.4) \]

Here and in what follows, \( \sinh, \cosh, \tanh, \text{sech} \) are hyperbolic functions.

For \( z \in \mathbb{C} \)
\[ \sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}, \quad \tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}, \quad \text{sech} z = \frac{2}{e^z + e^{-z}}. \]

The operators \( \Phi_\omega, \Psi_\omega \) are naturally decomposed into blocks:
\[
\Phi_\omega = \begin{bmatrix} C_\omega & S_\omega \\ S_\omega & C_\omega \end{bmatrix}, \quad \Psi_\omega = \begin{bmatrix} C_\omega & -S_\omega \\ -S_\omega & C_\omega \end{bmatrix}, \quad (1.5)
\]

where \( C_\omega \) and \( S_\omega \) are convolution operators:
\[
(C_\omega x)(t) = \int_{\mathbb{R}} C_\omega(t - \tau)x(\tau)d\tau, \quad (1.6a)
\]
\[
(S_\omega x)(t) = \int_{\mathbb{R}} S_\omega(t - \tau)x(\tau)d\tau. \quad (1.6b)
\]

In (1.6), \( x \) is a \( \mathbb{C} \)-valued function.

The function \( C_\omega(\xi) \) is continuous and positive on \( \mathbb{R} \). It decays exponentially as \( |\xi| \to \infty \):
\[
\frac{\omega}{\pi} e^{-\omega|\xi|} \leq C(\xi) < \frac{2\omega}{\pi} e^{-\omega|\xi|}, \quad \forall \xi \in \mathbb{R}. \quad (1.7)
\]

Since
\[ |\tau| - |t| \leq |t - \tau| \leq |\tau| + |t|, \]
the convolution kernel \( C(t - \tau) \) admits the estimate
\[
\frac{\omega}{\pi} e^{-\omega|t|} e^{-\omega|\tau|} \leq C(t - \tau) < \frac{2\omega}{\pi} e^{\omega|t|} e^{-\omega|\tau|}, \quad \forall t \in \mathbb{R}, \quad \forall \tau \in \mathbb{R}. \quad (1.8)
\]

**Definition 1.1.** The set \( L^1_\omega \) as the set of all complex valued functions \( x(t) \) which are measurable, defined almost everywhere with respect to the Lebesgue measure on \( \mathbb{R} \) and satisfy the condition
\[
\int_{\mathbb{R}} |x(\xi)| e^{-\omega|\xi|} d\xi < \infty. \quad (1.9)
\]

The set \( L^1_\omega + L^1_\omega \) is the set of all \( 2 \times 1 \) columns \( x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \) such that \( x_1(t) \in L^1_\omega \) and \( x_2(t) \in L^1_\omega \).

**Lemma 1.2.** Let \( x(\tau) \) be a \( \mathbb{C} \)-valued function which belongs to the space \( L^1_\omega \). Then the integral in the right hand side of (1.6a) exists\(^1\) for every \( t \in \mathbb{R} \).

We define the function \((C_\omega x)(t)\) by means of the equality (1.6a).

\(^1\)That is the value of this integral is a finite complex number for every \( t \in \mathbb{R} \).
Remark 1.3. For $x(\tau) \in L^1_\omega$, the function $(C_\omega x)(t)$ is a continuous function well defined on the whole $\mathbb{R}$. Nevertheless the function $(C_\omega x)(t)$ may not belong to the space $L^1_\omega$. The operator $C_\omega$ does not map the space $L^1_\omega$ into itself. (In other words, the operator $C_\omega$ considered as an operator in $L^1_\omega$ is unbounded.)

The situation with the integral in the right hand side of (1.6b) is more complicated. The function $S_\omega(\xi)$ also decays exponentially as $|\xi| \to \infty$:

$$|S(\xi)| < \frac{2\omega}{\pi(1 - e^{-2\omega|\xi|})} e^{-\omega|\xi|}, \quad \forall \xi \in \mathbb{R}. \quad (1.10)$$

However the function $S_\omega$ has the singularity at the point $\xi = 0$:

$$S_\omega(\xi) = \frac{1}{\pi\xi} + r(\xi), \quad (1.11)$$

where $r(\xi)$ is a function which is continuous and bounded for $\xi \in \mathbb{R}$. Thus the convolution kernel $S_\omega(t - \tau)$ has a non-integrable singularity on the diagonal $t = \tau$:

$$\int_{(t-\varepsilon,t+\varepsilon)} |S_\omega(t - \tau)| d\tau = \infty, \quad \forall t \in \mathbb{R}, \forall \varepsilon > 0.$$  

Therefore the integral in the right hand side of (1.6b) may not exist as a Lebesgue integral. Given a function $x(\tau)$, the equality

$$\int_\mathbb{R} |S_\omega(t - \tau)x(\tau)| d\tau = \infty \quad (1.12)$$

holds at every point $t \in \mathbb{R}$ which is a Lebesgue point of the function $x$ and $x(t) \neq 0$. Nevertheless, under the condition (1.9) we can attach a meaning to the integral $\int S_\omega(t - \tau)x(\tau)d\tau$ for almost every $t \in \mathbb{R}$.

**Lemma 1.4.** Let $x(\tau)$ be a $\mathbb{C}$-valued function which belongs to the space $L^1_\omega$. Then the principal value integral

$$\text{p.v.} \int_\mathbb{R} S_\omega(t - \tau)x(\tau)d\tau \overset{\text{def}}{=} \lim_{\varepsilon \to +0} \int_\mathbb{R} \setminus (t-\varepsilon,t+\varepsilon) S_\omega(t - \tau)x(\tau)d\tau \quad (1.13)$$

exists for almost every $t \in \mathbb{R}$.

We define the function $(S_\omega x)(t)$ by means of the equality (1.6b), where the integral in the right hand side of (1.6b) is interpreted as a principal value integral.

Under the condition (1.9), the integral $\int S_\omega(t - \tau)x(\tau)d\tau$ exists as a Lebesgue integral for every $\varepsilon > 0$:

$$\int_\mathbb{R} \setminus (t-\varepsilon,t+\varepsilon) |S_\omega(t - \tau)x(\tau)| d\tau < \infty, \quad \forall t \in \mathbb{R}, \forall \varepsilon > 0.$$
This follows from the estimate (1.10). The assertion that the limit in (1.13) exists for almost every $t \in \mathbb{R}$ will be proved in section 4 using the Hilbert transform theory.

**Remark 1.5.** Under the assumption (1.9), the function

$$y(t) = \text{p.v.} \int_{\mathbb{R}} S_\omega(t - \tau)x(\tau)d\tau,$$

which is defined for almost every $t$, is not necessary locally summable. It may happen that $\int_{[a,b]} |y(t)|dt = \infty$ for every finite interval $[a, b], -\infty < a < b < \infty$.

Let us define the transforms $\Phi_\omega$ and $\Psi_\omega$ formally.

**Definition 1.6.** For $x(\tau) = \begin{bmatrix} x_1(\tau) \\ x_2(\tau) \end{bmatrix} \in L_1^\omega + L_1^\omega$, we put

$$\begin{align*}
(\Phi_\omega x)(t) &= y(t), \\
(\Psi_\omega x)(t) &= z(t),
\end{align*}$$

(1.14)

where $y(t)$ and $z(t)$ are $2 \times 1$ columns:

$$\begin{align*}
y(t) &= \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \\
z(t) &= \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix},
\end{align*}$$

(1.15)

with the entries

$$\begin{align*}
y_1(t) &= (C_\omega x_1)(t) + (S_\omega x_2)(t), \\
y_2(t) &= (S_\omega x_1)(t) + (C_\omega x_2)(t), \\
z_1(t) &= (C_\omega x_1)(t) - (S_\omega x_2)(t), \\
z_2(t) &= -(S_\omega x_1)(t) + (C_\omega x_2)(t).
\end{align*}$$

(1.16a, 1.16b)

The operators $C_\omega$ and $S_\omega$ are the same that appeared in Lemmas 1.2 and 1.4 respectively.

According to Lemmas 1.2 and 1.4, the values $y(t)$ and $z(t)$ are well defined for almost every $t \in \mathbb{R}$.

The integral transforms (1.1)-(1.3) are said to be the Akhiezer integral transforms.

In another form, these transforms appear in [1, Chapter 15]. (See Problems 3 and 4 to Chapter 15.) The matrix nature of the Akhiezer transforms was camouflaged there.

In what follows, we consider the Akhiezer transform in various functional spaces. We show that the operators $\Phi_\omega$ and $\Psi_\omega$ are mutually inverse in spaces of functions growing slower than $e^{\omega|t|}$.

2. The operators $C_\omega$ and $S_\omega$ in $L^2$.

The Fourier transform machinery is an adequate tool for study convolution operators.

1. Studing the operators $C_\omega$ and $S_\omega$ by means of the Fourier transform technique, we deal with the spaces $L^1$ and $L^2$. Both these spaces consist of measurable functions defined almost everywhere on the real axis $\mathbb{R}$ with
respect to the Lebesgue measure. The spaces are equipped by the standard linear operations and the standard norms. If $u \in L^1$, then

$$\|u\|_{L^1} = \int_\mathbb{R} |u(t)| dt.$$  \hfill (2.1)

The space $L^1$ consists of all $u$ such that $\|u\|_{L^1} < \infty$. If $u \in L^2$, then

$$\|u\|_{L^2} = \left\{ \int_\mathbb{R} |u(\xi)|^2 d\xi \right\}^{1/2}.$$  \hfill (2.2)

The space $L^2$ consists of all $u$ such that $\|u\|_{L^2} < \infty$. This space is equipped by inner product $\langle \cdot, \cdot \rangle_{L^2}$. If $u' \in L^2$, $u'' \in L^2$, then

$$\langle u', u'' \rangle_{L^2} = \int_\mathbb{R} u'(t) \overline{u''(t)} dt.$$  \hfill (2.3)

2. The Fourier-Plancherel operator $\mathcal{F}$:

$$\mathcal{F}u = \hat{u}$$  \hfill (2.4)

where

$$\hat{u}(\lambda) = \int_\mathbb{R} u(t) e^{it\lambda} dt,$$  \hfill (2.5)

maps the space $L^2$ onto itself isometrically:

$$\|\hat{u}\|_{L^2}^2 = 2\pi \|u\|_{L^2}^2, \quad \forall u \in L^2.$$  \hfill (2.6)

The inverse operator $\mathcal{F}^{-1}$ is of the form

$$\mathcal{F}^{-1}v = \check{v},$$  \hfill (2.7)

where

$$\check{v}(t) = \frac{1}{2\pi} \int_\mathbb{R} v(\lambda) e^{-it\lambda} d\lambda.$$  \hfill (2.8)

Lemma 2.1. Let $f \in L^2$ and $k \in L^1$. Then

1. The integral

$$g(t) = \int_\mathbb{R} k(t - \tau) f(\tau) d\tau$$  \hfill (2.9)

exists as a Lebesgue integral (i.e. $\int_\mathbb{R} |k(t - \tau)g(\tau)| d\tau < \infty$) for almost every $t \in \mathbb{R}$.

2. The function $g$ belongs to $L^2$, and the inequality

$$\|g\|_{L^2} \leq \|k\|_{L^1}\|f\|_{L^2}$$  \hfill (2.10)

holds.
3. The Fourier-Plancherel transforms \( \hat{f} \) and \( \hat{g} \) are related by the equality
\[
\hat{g}(\lambda) = \hat{k}(\lambda) \cdot \hat{f}(\lambda), \quad \text{for a.e. } \lambda \in \mathbb{R},
\]
where
\[
\hat{k}(\lambda) = \int_{\mathbb{R}} k(t)e^{it\lambda} dt, \quad \forall \lambda \in \mathbb{R}.
\]
(2.11)

This lemma can be found in [2], Theorem 65 there. See also [3], Theorem 3.9.4.

Let us calculate the Fourier transforms of the functions \( C_\omega \) and \( S_\omega \). The function \( C_\omega \) belongs to \( L^1 \). So its Fourier transform
\[
\hat{C}_\omega(\lambda) = \int_{\mathbb{R}} C_\omega(t)e^{it\lambda} dt
\]
is well defined for every \( \lambda \in \mathbb{R} \).

**Lemma 2.2.** The Fourier transforms \( \hat{C}_\omega(\lambda) \) of the function \( C_\omega(t) \) is:
\[
\hat{C}_\omega(\lambda) = \text{sech} \frac{\pi \lambda}{2\omega}, \quad \forall \lambda \in \mathbb{R}.
\]
(2.14)

The formula (2.14) can be found in [2], where it appears as (7.1.6).

4. The function \( S_\omega(t) \) does not belong to \( L^1 \). This function has non-integrable singularity at the point \( t = 0 \). Therefore the integral
\[
\int_{\mathbb{R}} S_\omega(t)e^{2\pi it\lambda} dt
\]
does not exist as a Lebesgue integral. However
\[
\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} |S_\omega(t)| dt < \infty, \forall \varepsilon > 0.
\]
So the integral
\[
\hat{S}_{\omega,\varepsilon}(\lambda) = \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} S_\omega(t)e^{it\lambda} dt
\]
exists as a Lebesgue integral for every \( \varepsilon > 0 \). We define the Fourier transform \( \hat{S}_\omega(\lambda) \) as a principal value integral:
\[
\hat{S}_\omega(\lambda) = \lim_{\varepsilon \to +0} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} S_\omega(t)e^{it\lambda} dt.
\]
(2.16)

**Lemma 2.3.** The limit in (2.16) exists for every \( \lambda \in \mathbb{R} \). The Fourier transforms \( \hat{S}_\omega(\lambda) \) of the function \( S_\omega(t) \) is:
\[
\hat{S}_\omega(\lambda) = i \cdot \tanh \frac{\pi \lambda}{2\omega}, \quad \forall \lambda \in \mathbb{R}.
\]
(2.17)

The difference
\[
\varrho_\omega(\lambda, \varepsilon) = \hat{S}_\omega(\lambda) - \hat{S}_{\omega,\varepsilon}(\lambda), \quad \forall \lambda \in \mathbb{R}, \varepsilon > 0.
\]
(2.18)
satisfies the conditions
\[
\lim_{\varepsilon \to +0} \varrho_\omega(\lambda, \varepsilon) = 0, \quad \forall \lambda \in \mathbb{R},
\]
(2.19)
and

\[ \sup_{\lambda \in \mathbb{R}, 0 < \varepsilon \leq \frac{\pi}{4}} |Q_\omega(\lambda, \varepsilon)| < \infty. \]  \tag{2.20} 

The formula (2.17) can be found in [2], where it appears as (7.2.3).

5. In Section 1 we already have defined the functions \( C_\omega x \) and \( S_\omega x \) for \( x \) from the space \( L^1_\omega \). The space \( L^2 \) is contained in \( L^1_\omega \). If \( x \in L^2 \), then

\[ \int_{\mathbb{R}} |x(t)| e^{-\omega|t|} dt \leq \left\{ \int_{\mathbb{R}} |x(t)|^2 dt \right\}^{1/2} \left\{ \int_{\mathbb{R}} e^{-2\omega|t|} dt \right\}^{1/2} < \infty. \]  \tag{2.21} 

According to Lemmas 1.2 and 1.4, if \( f \in L^1_\omega \), then the function \( (C_\omega f)(t) \) is defined for every \( t \in \mathbb{R} \) and the function \( (S_\omega f)(t) \) is defined for almost every \( t \in \mathbb{R} \). However for \( f \in L^2(\mathbb{R}) \), we can obtain much more accurate results.

Lemma 2.4. Let \( f \in L^2 \) and \( g = C_\omega f \), i.e.

\[ g(t) = \int_{\mathbb{R}} C_\omega(t - \tau)f(\tau)d\tau. \]  \tag{2.22} 

Then \( g \in L^2 \), and the Fourier-Plancherel transforms \( \hat{f}, \hat{g} \) of functions \( f \) and \( g \) are related by the equality

\[ \hat{g}(\lambda) = \hat{C}_\omega(\lambda) \cdot \hat{f}(\lambda), \quad \text{a.e. on } \mathbb{R}, \]  \tag{2.23} 

where \( \hat{C}_\omega(\lambda) \) is determined by the equality (2.14).

Proof. Lemma 2.4 is a direct consequence of Lemma 2.1. \( \square \)

Lemma 2.5. Let \( f \in L^2 \) and \( g = S_\omega f \), i.e.

\[ g(t) = \operatorname{p.v.} \int_{\mathbb{R}} S_\omega(t - \tau)f(\tau)d\tau. \]  \tag{2.24} 

Then \( g \in L^2 \), and the Fourier-Plancherel transforms \( \hat{f}, \hat{g} \) of functions \( f \) and \( g \) are related by the equality

\[ \hat{g}(\lambda) = \hat{S}_\omega(\lambda) \cdot \hat{f}(\lambda), \quad \text{a.e. on } \mathbb{R}, \]  \tag{2.25} 

where \( \hat{S}_\omega(\lambda) \) is determined by the equality (2.17).

Proof. Since \( S_\omega \notin L^1 \), Lemma 2.5 does not follow from Lemma 2.1 directly. Let

\[ S_{\omega, \varepsilon}(t) = \begin{cases} S_\omega(t), & \text{if } t \in \mathbb{R} \setminus (-\varepsilon, \varepsilon), \\ 0, & \text{if } t \in (-\varepsilon, \varepsilon). \end{cases} \]  \tag{2.26} 

The function \( S_{\omega, \varepsilon} \) belongs to \( L^1 \) for every \( \varepsilon > 0 \). Let

\[ g_{\varepsilon}(t) = \int_{\mathbb{R}} S_{\omega, \varepsilon}(t - \tau)f(\tau)d\tau. \]  \tag{2.27}
Applying Lemma 2.1 to $k = S_{\omega, \varepsilon}$, we conclude that $g_\varepsilon \in L^2$ for every $\varepsilon > 0$ and that the Fourier-Plancherel transforms $\hat{g}_\varepsilon, \hat{f}$ of the functions $g_\varepsilon, f$ are related by the equality

$$\hat{g}_\varepsilon(\lambda) = \hat{S}_{\omega, \varepsilon}(\lambda) \cdot \hat{f}(\lambda),$$

where $\hat{S}_{\omega, \varepsilon}(\lambda)$ is defined by (2.15). According to Lemma 2.3

$$\hat{g}_\varepsilon(\lambda) = \hat{S}_\omega(\lambda) \cdot \hat{f}(\lambda) - \hat{h}_\varepsilon(\lambda),$$

where

$$\hat{h}_\varepsilon(\lambda) = \hat{Q}_\omega(\lambda, \varepsilon) \hat{f}(\lambda),$$

and the family $\{Q_\omega(\lambda, \varepsilon)\}_{0<\varepsilon<\infty}$ satisfies the conditions (2.19) and (2.20).

From (2.19), (2.20), (2.30) and the Lebesgue Dominated Convergence Theorem it follows that

$$\lim_{\varepsilon \to +0} \int_{\mathbb{R}} |\hat{h}_\varepsilon(\lambda)|^2 d\lambda = 0.$$

In other words,

$$\|\hat{g}_\varepsilon(\lambda) - \hat{g}\|_{L^2} = 0,$$

where

$$\hat{g} \overset{\text{def}}{=} \hat{S}_\omega(\lambda) \cdot \hat{f}(\lambda).$$

From (2.31) it follows that $\|g_\varepsilon - \hat{g}\|_{L^2} \to 0$ as $\varepsilon \to +0$, i.e.

$$\|S_{\omega, \varepsilon}f - \hat{g}\|_{L^2} \to 0 \quad \text{as} \quad \varepsilon \to +0,$$

where $\hat{g} = \mathcal{F}^{-1}\hat{g} \in L^2$. From the other side, $(S_{\omega, \varepsilon}f)(t) \to g(t)$ for a.e. $t \in \mathbb{R}$ by Lemma 1.4. Hence $g = \hat{g}$, and $\hat{g} = \hat{S}_\omega \hat{f}$. □

6. The equality

$$|\hat{C}_\omega(\lambda)|^2 + |\hat{S}_\omega(\lambda)|^2 = 1, \quad \forall \lambda \in \mathbb{R},$$

plays a crucial role in this paper. This equation is a direct consequence of the explicite expressions (2.14) and (2.17) for $\hat{C}_\omega$ and $\hat{S}_\omega$ and the identity

$$(\cosh \zeta)^2 - (\sinh \zeta)^2 = 1, \quad \forall \zeta \in \mathbb{C}.$$  

(2.35)

**Lemma 2.6.** The operators $C_\omega$ and $S_\omega$ are contractive in the space $L^2(\mathbb{R})$. Moreover the equality

$$\|C_\omega f\|_{L^2}^2 + \|S_\omega f\|_{L^2}^2 = \|f\|_{L^2}^2, \quad \forall f \in L^2,$$

holds.

**Proof.** Let $g_c = C_\omega f, g_s = S_\omega f$ and let $\hat{f}, \hat{g}_c, \hat{g}_s$ be the Fourier-Plancherel transforms of the functions $f, g_c, g_s$. According to Lemmas 2.4 and 2.5 the equalities

$$\hat{g}_c(\lambda) = \hat{C}_\omega(\lambda) \hat{f}(\lambda), \quad \hat{g}_s(\lambda) = \hat{S}_\omega(\lambda) \hat{f}(\lambda), \quad \text{for a.e.} \ \lambda \in \mathbb{R}$$

hold. From (2.34) it follows that

$$|\hat{g}_c(\lambda)|^2 + |\hat{g}_s(\lambda)|^2 = |\hat{f}(\lambda)|^2, \quad \text{for a.e.} \ \lambda \in \mathbb{R}.$$
Integrating with respect to $\lambda$, we obtain the equality $\|\hat{g}_c\|_{L^2}^2 + \|\hat{g}_s\|_{L^2}^2 = \|\hat{f}\|_{L^2}^2$. In view of (2.6), the last equality is equivalent to the equality (2.36). □

3. The Akhiezer operators $\Phi_\omega$ and $\Psi_\omega$ in $L^2 \oplus L^2$.

**Definition 3.1.** The space $L^2 \oplus L^2$ is the set of all $2 \times 1$ columns $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $x_1(t) \in L^2$ and $x_2(t) \in L^2$. The set $L^2 \oplus L^2$ is equipped by the natural linear operations and by the inner product $\langle \cdot , \cdot \rangle_{L^2 \oplus L^2}$.

If $x'(t) = \begin{bmatrix} x'_1(t) \\ x'_2(t) \end{bmatrix}$ and $x''(t) = \begin{bmatrix} x''_1(t) \\ x''_2(t) \end{bmatrix}$ belong to $L^2 \oplus L^2$, then

$$\langle x', x'' \rangle_{L^2 \oplus L^2} \stackrel{\text{def}}{=} \langle x'_1, x''_1 \rangle_{L^2} + \langle x'_2, x''_2 \rangle_{L^2}.$$  \hspace{1cm} (3.1)

The inner product (3.1) generates the norm

$$\|x\|_{L^2 \oplus L^2} = \sqrt{\|x_1\|^2_{L^2} + \|x_2\|^2_{L^2}} \quad \text{for} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in L^2 \oplus L^2. \hspace{1cm} (3.2)$$

Since $L^2 \subset L^1_\omega$, also $L^2 \oplus L^2 \subset L^1_\omega + L^1_\omega$. Thus if $x \in L^2 \oplus L^2$, then the values $y(t) = (\Phi_\omega x)(t)$ and $z(t) = (\Psi_\omega x)(t)$ are defined by (1.16) for almost every $t \in \mathbb{R}$. Using Lemmas 2.4 and 2.5, we conclude from (1.16) that the operators $\Phi_\omega$ and $\Psi_\omega$ are bounded operators in the space $L^2 \oplus L^2$. In particular, the values $y(t)$ and $z(t)$ belong to $L^2 \oplus L^2$.

**Theorem 3.2.** Each of the operators $\Phi_\omega$ and $\Psi_\omega$ is an isometric operator in the space $L^2 \oplus L^2$: \hspace{1cm}

$$\|\Phi_\omega x\|_{L^2 \oplus L^2} = \|x\|_{L^2 \oplus L^2}, \quad \|\Psi_\omega x\|_{L^2 \oplus L^2} = \|x\|_{L^2 \oplus L^2}, \quad \forall x \in L^2 \oplus L^2. \hspace{1cm} (3.3)$$

**Theorem 3.3.** The operators $\Phi_\omega$ and $\Psi_\omega$ are mutually inverse in the space $L^2 \oplus L^2$: \hspace{1cm}

$$\Psi_\omega \Phi_\omega x = x, \quad \forall x \in L^2 \oplus L^2, \hspace{1cm} (3.4a)$$

$$\Phi_\omega \Psi_\omega x = x, \quad \forall x \in L^2 \oplus L^2. \hspace{1cm} (3.4b)$$

**Proofs of Theorem 3.2.** Let us associate the $2 \times 2$ matrix functions $\widehat{\Phi}_\omega(\lambda)$ and $\widehat{\Psi}_\omega(\lambda)$ with the operators $\Phi_\omega$ and $\Psi_\omega$:

$$\widehat{\Phi}_\omega(\lambda) = \begin{bmatrix} \widehat{C}_\omega(\lambda) & \widehat{S}_\omega(\lambda) \\ \widehat{S}_\omega(\lambda) & \widehat{C}_\omega(\lambda) \end{bmatrix}, \quad \lambda \in \mathbb{R}, \hspace{1cm} (3.5a)$$

$$\widehat{\Psi}_\omega(\lambda) = \begin{bmatrix} \widehat{C}_\omega(\lambda) & -\widehat{S}_\omega(\lambda) \\ -\widehat{S}_\omega(\lambda) & \widehat{C}_\omega(\lambda) \end{bmatrix}, \quad \lambda \in \mathbb{R}, \hspace{1cm} (3.5b)$$

\(^2\text{See} \text{[221].}\)
where $\widehat{C}_\omega(\lambda)$ and $\widehat{S}_\omega(\lambda)$ are the same that in (2.14) and (2.17). Let

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in L^2 \oplus L^2, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \Phi_\omega x,$$

and let $\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$, $\hat{y} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix}$, where $\hat{x}_1, \hat{x}_2, \hat{y}_1, \hat{y}_2$ are the Fourier-Plancherel transforms of the functions $x_1, x_2, y_1, y_2$ respectively. According to the equality (1.16) and to Lemmas 2.4 and 2.5, the equality

$$\hat{y}(\lambda) = \Phi_\omega(\lambda)\hat{x}(\lambda)$$

(3.6)

holds for almost every $\lambda \in \mathbb{R}$.

From the equality (2.34) it follows that the matrix $\Phi_\omega(\lambda)$ is unitary for each $\lambda \in \mathbb{R}$:

$$(\Phi_\omega(\lambda))^*\Phi_\omega(\lambda) = I, \quad \forall \lambda \in \mathbb{R},$$

(3.7)

where $I$ is $2 \times 2$ identity matrix. From (3.6) and (3.7) it follows that

$$(\hat{y}(\lambda))^*\hat{y}(\lambda) = (\hat{x}(\lambda))^*\hat{x}(\lambda), \quad \text{for a.e. } \lambda \in \mathbb{R},$$

i.e.

$$|\hat{y}_1(\lambda)|^2 + |\hat{y}_2(\lambda)|^2 = |\hat{x}_1(\lambda)|^2 + |\hat{x}_2(\lambda)|^2,$$

for a.e. $\lambda \in \mathbb{R}$.

Integrating with respect to $\lambda$ over $\mathbb{R}$ and using the Parseval identity (2.6), we conclude that

$$\|y_1\|^2_{L^2} + \|y_2\|^2_{L^2} = \|x_1\|^2_{L^2} + \|x_2\|^2_{L^2},$$

that is $\|\Phi_\omega x\|_{L^2 \oplus L^2} = \|x\|_{L^2 \oplus L^2}$. The equality $\|\Psi_\omega x\|_{L^2 \oplus L^2} = \|x\|_{L^2 \oplus L^2}$ can be obtained analogously.

**Proof of Theorem 3.3.** Let

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in L^2 \oplus L^2, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \Phi_\omega x, \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \Psi_\omega y.$$

Let $\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$, $\hat{y} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix}$, $\hat{z} = \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix}$, where $\hat{x}_1, \hat{x}_2, \hat{y}_1, \hat{y}_2, \hat{z}_1, \hat{z}_2$ are the Fourier-Plancherel transforms of the functions $x_1, x_2, y_1, y_2, z_1, z_2$ respectively. We already proved the equality (3.6). In the same way the equality

$$\hat{z}(\lambda) = \Phi_\omega(\lambda)\hat{x}(\lambda), \quad \text{for a.e. } \lambda \in \mathbb{R},$$

(3.8)

can be established. From (3.6) and (3.8) it follows that

$$\hat{z}(\lambda) = \Phi_\omega(\lambda)\Phi_\omega(\lambda)\hat{x}(\lambda), \quad \text{for a.e. } \lambda \in \mathbb{R},$$

(3.9)

From the equality (2.34) it follows that the matrices $\Phi_\omega(\lambda)$ and $\Psi_\omega(\lambda)$ are mutually inverse:

$$\Phi_\omega(\lambda)\Psi_\omega(\lambda) = I, \quad \forall \lambda \in \mathbb{R},$$

(3.10)

where $I$ is $2 \times 2$ identity matrix. From (3.9) and (3.10) we conclude that

$$\hat{z}(\lambda) = \hat{x}(\lambda), \quad \text{for a.e. } \lambda \in \mathbb{R},$$

Finally $z = x$. 

The equality (3.4a) is proved. The equality (3.4b) can be proved in the same way.

□

4. The Hilbert transform

Definition 4.1. Let $u(\tau)$ be a complex-valued function which is defined for almost every $\tau \in \mathbb{R}$. We assume that the function $u$ satisfies the condition

$$\int_{\mathbb{R}} \frac{|u(\tau)|}{1 + |\tau|} d\tau < \infty. \quad (4.1)$$

Then the integral

$$H_\varepsilon u(t) = \frac{1}{\pi} \int_{\mathbb{R} \setminus (t-\varepsilon, t+\varepsilon)} \frac{u(\tau)}{t-\tau} d\tau \quad (4.2)$$

exists for every $t \in \mathbb{R}$ and $\varepsilon > 0$. For each $\varepsilon > 0$, the function $H_\varepsilon u(t)$ is a continuous function of $t$ for $t \in \mathbb{R}$. The function $Hu(t)$ is defined for those $t \in \mathbb{R}$ for which the value $H_\varepsilon u(t)$ tends to a finite limit as $\varepsilon \to +0$:

$$Hu(t) \overset{\text{def}}{=} \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{\mathbb{R} \setminus (t-\varepsilon, t+\varepsilon)} \frac{u(\tau)}{t-\tau} d\tau. \quad (4.3)$$

The function $Hu$ is said to be the Hilbert transform of the function $u$.

Theorem. (A.I.Plessner.) Let $u(\tau)$ be a function which is defined for almost every $\tau \in \mathbb{R}$. If the function $u(\tau)$ satisfies the condition (4.1), then its Hilbert transform $Hu(t)$ exists for almost every $t \in \mathbb{R}$.

Proof of this Plessner’s Theorem can be found in [2], Theorem 100 there.

If $u$ is a function from $L^2$, then $u$ satisfies the condition (4.1). By Plessner’s theorem, the Hilbert transform $v(t) = (Hu)(t)$ exists for almost every $t \in \mathbb{R}$.

Theorem. (E.C. Titchmarsh.) Let $u$ be a function from $L^2$. Then:

1. Its Hilbert transform $v = Hu$ also belongs to $L^2$, and the equality

$$\|v\|_{L^2} = \|u\|_{L^2} \quad (4.4)$$

holds.
2. The equality

$$(Hv)(t) = -u(t) \quad (4.5)$$

holds for almost every $t \in \mathbb{R}$.

This theorem means that the Hilbert transform, considered as an operator in $L^2$, is an unitary operator which satisfies the equality

$$H^2 = -I, \quad (4.6)$$

where $I$ is the identity operator in $L^2$. 
Proof of Lemma 1.4. We use the decomposition (1.11) of the kernel $S(t - \tau)$ into the sum of the Hilbert kernel $\frac{1}{\pi(t - \tau)}$ and the ‘regular’ kernel $r(t - \tau)$. Let $(a, b) \subset \mathbb{R}$ be an arbitrary finite interval of the real axis. We split the function $f(\tau)$ into the sum of two summands.

\[ f(\tau) = g(\tau) + h(\tau), \quad (4.7) \]

where

\[ g(\tau) = \begin{cases} f(\tau), & \text{if } \tau \in (a, b), \\ 0, & \text{if } \tau \in \mathbb{R} \setminus (a, b). \end{cases} \quad (4.8) \]

So

\[ h(\tau) = 0, \quad \text{if } \tau \in (a, b). \quad (4.9) \]

According to (1.11) and (4.7), the equality

\[ \int_{\mathbb{R} \setminus (t-\varepsilon, t+\varepsilon)} S_\omega(t - \tau)f(\tau) d\tau = I_{1,\varepsilon}(t) + I_{2,\varepsilon}(t) + I_{3,\varepsilon}(t) \quad (4.10) \]

holds, where

\[ I_{1,\varepsilon}(t) = \int_{\mathbb{R} \setminus (t-\varepsilon, t+\varepsilon)} \frac{1}{\pi(t - \tau)}g(\tau)d\tau, \quad (4.11) \]

\[ I_{2,\varepsilon}(t) = \int_{\mathbb{R} \setminus (t-\varepsilon, t+\varepsilon)} r(t - \tau)g(\tau)d\tau, \quad (4.12) \]

\[ I_{3,\varepsilon}(t) = \int_{\mathbb{R} \setminus (t-\varepsilon, t+\varepsilon)} S_\omega(t - \tau)h(\tau)d\tau. \quad (4.13) \]

The function $g$ satisfies the condition (1.1). According Plessner’s Theorem, \( \lim_{\varepsilon \to +0} I_{1,\varepsilon}(t) \) exists for almost every $t \in \mathbb{R}$. Since the function $g$ is finitely supported and the kernel $r(t - \tau)$ is continuous, \( \lim_{\varepsilon \to +0} I_{2,\varepsilon}(t) \) exists for every $t \in \mathbb{R}$. Since the function $h(\tau)$ vanishes for $\tau \in (a, b)$ and \( \int_{\mathbb{R}} \frac{|h(\tau)|}{\cosh \omega \tau} d\tau < \infty \),

\( \lim_{\varepsilon \to +0} I_{3,\varepsilon}(t) \) exists for every $t \in (a, b)$. In view of (4.10), the limit in (4.13) exists for almost every $t \in (a, b)$. Since $(a, b)$ is an arbitrary finite interval, the limit in (1.13) exists for almost every $t \in \mathbb{R}$. 

\[ \square \]

5. The operators $C_\omega$ and $S_\omega$ in $L^2_\sigma$.

In this section we consider the operators $C_\omega$ and $S_\omega$ acting in spaces of functions growing slower than $e^{\omega|t|}$ as $t \to \pm \infty$. 

Lemma 5.2. Assume that the means of the formula (2.22) are bounded in the space $L^2_{\sigma}$, and we proved that the operators $C_{\omega}$ in Section 2. Moreover, we show that if $f \in L^2_{\sigma}$, then $F_a = \omega < \sigma < \omega$ or $\sigma = 0$.

In Section 1 we already have defined the functions $C_{\omega}x$ and $S_{\omega}x$ for $x$ from the space $L^1_{\omega}$. For $\sigma < \omega$, the space $L^2_{\sigma}$ is contained in $L^1_{\omega}$. If $f \in L^2_{\sigma}$, then

$$\int_{\mathbb{R}} |x(t)| e^{-\omega|t|} dt \leq \left\{ \int_{\mathbb{R}} |x(t)|^2 e^{-2\sigma|t|} dt \right\}^{1/2} \left\{ \int_{\mathbb{R}} e^{-2(\omega-\sigma)|t|} dt \right\}^{1/2} < \infty. \tag{5.2}$$

According to Lemmas 1.2 and 1.4, if $f \in L^1_{\omega}$, then the function $(C_{\omega}f)(t)$ is defined for every $t \in \mathbb{R}$ and the function $(S_{\omega}f)(t)$ is defined for almost every $t \in \mathbb{R}$.

In Section 2 we obtained that if $f \in L^2$, then $C_{\omega}f \in L^2$ and $S_{\omega}f \in L^2$. Moreover, we proved that the operators $C_{\omega}$ and $S_{\omega}$ are contractive in $L^2$: see Corollary 2.6. In this section we show that if $0 < \sigma < \omega$ and $f \in L^2_{\sigma}$, than $C_{\omega}f \in L^2_{\sigma}$ and $S_{\omega}f \in L^2_{\sigma}$. Moreover, we show that the operators $C_{\omega}$ and $S_{\omega}$ are bounded in the space $L^2_{\sigma}$.

Lemma 5.2. Assume that $0 \leq \sigma < \omega$. Let $f \in L^2_{\sigma}$, and $g$ is related to $f$ by means of the formula (2.22), i.e. $g = C_{\omega}f$. Then $g \in L^2_{\sigma}$, and

$$\|g\|_{L^2_{\sigma}} \leq \frac{M_{c}}{1 - \sigma/\omega} \|f\|_{L^2_{\sigma}}, \tag{5.3}$$

where $M_{c} < \infty$ is a value which does not depend on $\omega$ and $\sigma$.

Proof. Let

$$u(\tau) = f(\tau)e^{-\sigma|\tau|}, \quad v(t) = g(t)e^{-\sigma|t|}. \tag{5.4}$$

Since $f \in L^2_{\sigma}$, $u \in L^2$. The equality (2.22) can be rewritten as

$$v(t) = \int_{\mathbb{R}} e^{-\sigma|t|+\sigma|\tau|} C_{\omega}(t-\tau) u(\tau) d\tau. \tag{5.5}$$

Let us estimate the kernel

$$K_{c}(t, \tau) = e^{-\sigma|t|+\sigma|\tau|} C_{\omega}(t-\tau). \tag{5.6}$$

For $\sigma \geq 0$ the inequality

$$| -\sigma|t| + \sigma|\tau| | \leq \sigma|t-\tau|, \quad \forall t \in \mathbb{R}, \tau \in \mathbb{R}, \tag{5.7}$$

holds. Hence

$$e^{-\sigma|t|+\sigma|\tau|} \leq e^{\sigma|t-\tau|}, \quad \forall t \in \mathbb{R}, \tau \in \mathbb{R}. \tag{5.8}$$
From this inequality and from the expression (1.4) for $C_\omega$ we conclude that

$$|K_c(t, \tau) \leq k_{c, \omega}(t - \tau), \quad \forall t \in \mathbb{R}, \tau \in \mathbb{R},$$

(5.9)

where

$$k_{c, \omega}(\xi) = \frac{\omega e^{\sigma |\xi|}}{\pi \cosh \omega \xi}, \quad \xi \in \mathbb{R}.$$  

(5.10)

For $0 \leq \sigma < \omega$, the function $k_{c, \omega}$ belongs to $L^1$ and

$$\|k_{c, \omega}\|_{L^1} < \frac{4}{\pi} \int_0^\infty \frac{\cosh a \xi}{\cosh \xi} d\xi,$$

(5.11)

where

$$a = \frac{\sigma}{\omega}.$$  

(5.12)

The integral in (5.11) can be calculated explicitly:

$$\int_0^\infty \frac{\cosh a \xi}{\cosh \xi} d\xi = \frac{\pi}{2 \cos \frac{\pi}{2} a},$$

(5.13)

Thus

$$\|k_{c, \omega}\|_{L^1} < \frac{2}{\sin \frac{\pi}{2} (1 - a)}.$$  

(5.14)

Since $\sin \frac{\pi}{2} \eta \geq \eta$ for $0 \leq \eta \leq 1$, the inequality (5.14) implies the inequality

$$\|k_{c, \omega}\|_{L^1} < \frac{2}{1 - \sigma/\omega}.$$  

(5.15)

From (5.6) and (5.9) we obtain the inequality $|v(t)| \leq w(t), \forall t \in \mathbb{R}$, and

$$\|v\|_{L^2} \leq \|w\|_{L^2},$$

(5.16)

where

$$w(t) = \int_\mathbb{R} k_{c, \omega}(t - \tau)|u(\tau)|d\tau.$$  

(5.17)

According to Lemma 2.1, $w \in L^2$, and the inequality

$$\|w\|_{L^2} \leq \|k_{c, \omega}\|_{L^1} \cdot \|u\|_{L^2}$$

(5.18)

holds. The inequality

$$\|v\|_{L^2} \leq \frac{2}{1 - \sigma/\omega} \|u\|_{L^2}$$

is a consequence of the equalities (5.16), (5.18) and (5.15). According to (5.4), $\|u\|_{L^2} = \|f\|_{L^2_\sigma}$, $\|v\|_{L^2} = \|g\|_{L^2_\sigma}$. So the inequality (5.3) holds with $M_c = 2$. $\square$

**Lemma 5.3.** Assume that $0 \leq \sigma < \omega$. Let $f \in L^2_\sigma$, and $g$ is related to $f$ by means of the formula (2.24), i.e. $g = S_\omega f$. Then $g \in L^2_\sigma$, and

$$\|g\|_{L^2_\sigma} \leq \frac{M_s}{(1 - \sigma/\omega)^2} \|f\|_{L^2_\sigma},$$

(5.19)

where $M_s < \infty$ is a value which does not depend on $\omega$ and $\sigma$. 
Proof. Let \( u(\tau), v(t) \) be defined according to (5.4). Since \( f \in L^2_\sigma, u \in L^2 \). The equality (2.24) can be rewritten as

\[
v(t) = \text{p.v.} \int_{\mathbb{R}} e^{-|t| + \sigma |\tau|} S_\omega(t - \tau)u(\tau) d\tau.
\]

(5.20)

We present \( v(t) \) as

\[
v(t) = v_1(t) + v_2(t),
\]

(5.21)

where

\[
v_1(t) = \text{p.v.} \int_{\mathbb{R}} S_\omega(t - \tau)u(\tau) d\tau,
\]

(5.22)

\[
v_2(t) = \int_{\mathbb{R}} (e^{-|t| + \sigma |\tau|} - 1) S_\omega(t - \tau)u(\tau) d\tau,
\]

(5.23)

Let us estimate the kernel

\[
K_s(t, \tau) = (e^{-|t| + \sigma |\tau|} - 1) S_\omega(t - \tau).
\]

(5.24)

From the inequalities \(|e^\xi - 1| \leq |\xi|e^{|\xi|}\), from (5.7) and from the expression (1.4) for \( S_\omega \) we conclude that

\[
|K_s(t, \tau)| \leq k_{\sigma, \omega}^s(s(t - \tau)),
\]

(5.25)

where

\[
k_{\sigma, \omega}^s(\xi) = \frac{\omega}{\pi} \cdot \frac{\sigma |\xi| e^{\sigma |\xi|}}{\sinh \omega |\xi|}.
\]

(5.26)

The function \( k_{\sigma, \omega}^s \) belongs to \( L^1 \), and

\[
\|k_{\sigma, \omega}^s\|_{L^1} < \frac{4a}{\pi} \int_0^\infty \frac{\xi \cosh a\xi}{\sinh \xi} d\xi,
\]

(5.27)

where \( a \) is the same that in (5.12). The integral in (5.27) can be calculated explicitly:

\[
\int_0^\infty \frac{\xi \cosh a\xi}{\sinh \xi} d\xi = \frac{\pi^2}{4 \sin^2 \frac{\pi}{2}(1 - a)}.
\]

(5.28)

Thus the inequality

\[
\|k_{\sigma, \omega}^s\|_{L^1} < \frac{\pi}{(1 - \sigma/\omega)^2}
\]

(5.29)

holds. From (5.23), (5.24) and (5.25) it follows that

\[
\|v_2\|_{L^2} \leq \|w\|_{L^2},
\]

(5.30)

where

\[
w(t) = \int_{\mathbb{R}} k_{\sigma, \omega}^s(t - \tau)|u(\tau)| d\tau.
\]

(5.31)

According to Lemma 2.1, \( w \in L^2 \), and the inequality

\[
\|w\|_{L^2} \leq \|k_{\sigma, \omega}^s\|_{L^1} \cdot \|u\|_{L^2}
\]

(5.32)
holds. From (5.29), (5.32) and (5.30) we conclude that
\[ \|v_2\|_{L^2} \leq \frac{\pi}{(1 - \sigma/\omega)^2}\|u\|_{L^2}. \] (5.33)

According to Lemma 2.6 the inequality
\[ \|v_1\|_{L^2} \leq \|u\|_{L^2} \] (5.34)
holds. From (5.21), (5.34) and (5.33) we derive the inequality
\[ \|v\|_{L^2} \leq \frac{M_s(1 - \sigma/\omega)^2}{(1 - \sigma/\omega)^2}\|u\|_{L^2} \] (5.35)
with \( M_s = \pi + 1 \).

6. The Akhiezer operators \( \Phi_\omega \) and \( \Psi_\omega \) in \( L^2_\sigma \oplus L^2_\sigma \).

Definition 6.1. The space \( L^2_\sigma \oplus L^2_\sigma \) is the set of all \( 2 \times 1 \) columns \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) such that \( x_1(t) \in L^2_\sigma \) and \( x_2(t) \in L^2_\sigma \), where \( L^2_\sigma \) was defined in Definition 5.1.

The set \( L^2_\sigma \oplus L^2_\sigma \) is equipped by the natural linear operations and by the norm
\[ \|x\|_{L^2_\sigma \oplus L^2_\sigma} = \sqrt{\|x_1\|_{L^2_\sigma}^2 + \|x_2\|_{L^2_\sigma}^2}. \] (6.1)

Since \( L^2_\sigma \subset L^1_\sigma \), also \( L^2_\sigma \oplus L^2_\sigma \subset L^1_\sigma \oplus L^1_\sigma \). Thus if \( x \in L^2_\sigma \oplus L^2_\sigma \), then the values \( y(t) = (\Phi_\omega x)(t) \) and \( z(t) = (\Psi_\omega x)(t) \) are defined by (1.16) for almost every \( t \in \mathbb{R} \). From Lemmas 5.2 and 5.3 we derive

Lemma 6.2. We assume that \( 0 \leq \sigma < \omega \). Let \( x \in L^2_\sigma \oplus L^2_\sigma \) and let \( y = \Phi_\omega x \), \( z = \Psi_\omega x \) be defined by (1.16). Then \( y \in L^2_\sigma \oplus L^2_\sigma \), \( z \in L^2_\sigma \oplus L^2_\sigma \), and the estimates hold
\[ \|\Phi_\omega x\|_{L^2_\sigma \oplus L^2_\sigma} \leq \frac{M}{1 - \sigma/\omega}\|x\|_{L^2_\sigma \oplus L^2_\sigma}, \] (6.2)
\[ \|\Psi_\omega x\|_{L^2_\sigma \oplus L^2_\sigma} \leq \frac{M}{1 - \sigma/\omega}\|x\|_{L^2_\sigma \oplus L^2_\sigma}, \] (6.3)
where \( M < \infty \) is a value which does not depend on \( \sigma, \omega, x \).

The following theorem is a main result of this paper.

Theorem 6.3. We assume that \( 0 \leq \sigma < \omega \). Then for every \( x \in L^2_\sigma \oplus L^2_\sigma \) the equalities
\[ \Psi_\omega \Phi_\omega x = x, \quad \Phi_\omega \Psi_\omega x = x \] (6.4)
hold.

Proof. From Lemma 6.2 it follows that the operators \( \Psi_\omega \Phi_\omega \) and \( \Phi_\omega \Psi_\omega \) are bounded linear operators in the space \( L^2_\sigma \oplus L^2_\sigma \). The set \( L^2 \oplus L^2 \) is a dense subset of the space \( L^2_\sigma \oplus L^2_\sigma \). By Theorem 3.3, the equalities (6.4) holds for every \( x \in L^2 \oplus L^2 \). By continuity, the equalities (6.4) can be extended from \( L^2 \oplus L^2 \) to \( L^2_\sigma \oplus L^2_\sigma \). \( \square \)

\(^3\)See (5.2).
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