Thomas–Fermi model for a bulk self-gravitating stellar object in two dimensions

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Abstract

In this article we have solved a hypothetical problem related to the stability and gross properties of two-dimensional self-gravitating stellar objects using the Thomas–Fermi model. The formalism presented here is an extension of the standard three-dimensional problem discussed in the book on statistical physics, Part I by Landau and Lifshitz. Further, the formalism presented in this article may be considered a class problem for post-graduate-level students of physics or may be assigned as a part of their dissertation project.

Keywords: Thomas–Fermi model, main sequence star, Lane–Emden equation, Chandrasekhar equation

1. Introduction

The study of gross properties of bulk self-gravitating objects using the Thomas–Fermi model has been discussed in a very lucid manner in the textbook on statistical physics by Landau and Lifshitz [1]. In this book the model has also been extended for the bulk system with ultra-relativistic electrons as one of the constituents. Analogous to the conventional white dwarf model, these electrons in both non-relativistic and ultra-relativistic cases provide degeneracy pressure to make the bulk system stable against gravitational collapse. The results are therefore an alternative to the Lane—Emden equation or Chandrasekhar equation [2, 3]. The mathematical formalism along with the numerical estimates of various parameters, e.g., mass, radius, etc., for white dwarf stars may be taught as standard astrophysical problems in the MSc level classes for the students of astrophysics and cosmology. The standard version of the Thomas–Fermi model has also been taught in the MSc-level atomic physics and statistical mechanics general classes.

More than two decades ago, Bhaduri et al [4] developed a formalism for the Thomas–Fermi model for a two-dimensional atom. The problem can also be given to advanced level
MSc students in the quantum mechanics classes. The numerical evaluation of various quantities associated with the two-dimensional atoms are also found to be useful for the students to learn numerical techniques and computer programming along with the physics of the problem. The work of Bhaduri et al is an extension of the standard Thomas–Fermi model for heavy atoms into a two-dimensional scenario. A two-dimensional version of the Thomas–Fermi model has also been used to study the stability and some of the gross properties of the two-dimensional star cluster [5]. In our opinion this is the first attempt to apply the Thomas–Fermi model to a two-dimensional gravitating object. However, to the best of our knowledge the two-dimensional generalization of the Thomas–Fermi model to study gross properties of bulk self-gravitating objects, e.g., white dwarfs, has not been reported earlier. This problem can also be treated as a standard MSc level class problem for the advanced level students of astrophysics and cosmology. In this article we shall therefore develop a formalism for a two-dimensional version of the Thomas–Fermi model to investigate some of the gross properties of two-dimensional hypothetical white dwarf stars. The work is essentially an extension of the standard three-dimensional problem which is discussed in the statistical physics book by Landau and Lifshitz [1]. The motivation of this work is to study Newtonian gravity in two-dimensions. An analogous Coulomb problem with logarithmic-type potential has been investigated in an extensive manner. However, the identical problem for gravitating objects has not been thoroughly studied (except in [5]). One can use this two-dimensional gravitational picture as a model calculation to study the stability of a giant molecular cloud during star formation and also in galaxy formation.

The article is arranged in the following manner. In the next section we have developed the basic mathematical formalism for a two-dimensional hypothetical white dwarf star. In section 3, we have investigated the gross properties of white dwarf stars in two-dimensions. In section 4, the stability of two-dimensional white dwarfs with ultra-relativistic electrons as one of the constituents has been studied. Finally in the last section we have given the conclusion of this work.

2. Basic formalism

In this section we start with the conventional form of Poisson’s equation, given by [5] (see also [6])

\[ \nabla^2 \phi = 2\pi G \rho \]

where \( \phi \) is the gravitational potential, \( \rho \) is the surface density of matter and for the two-dimensional scenario \( \nabla^2 \) has to be expressed in its two-dimensional form. Let us assume that the bulk object in two-dimensional geometry is a hypothetical white dwarf star for which the inward gravitational pressure is balanced by the outward degeneracy pressure of the two-dimensional electron gas. The mass of the object is coming from the heavy nuclei distributed in two-dimensional bounded geometry.

Now the Fermi energy for a two-dimensional electron gas is given by

\[ \mu_e = \frac{p_{Fe}^2}{2m_e} \]

where \( p_{Fe} \) and, \( m_e \) are, respectively, the Fermi momentum and the mass of the electrons. Further, the number of electrons per unit surface area is given by
n_e = \frac{1}{2\pi\hbar^2}p_e^2. \quad (3)

Hence the electron Fermi energy is

\mu_e = \frac{\pi\hbar^2}{m_e}n_e. \quad (4)

Let \( m_p \) be the baryon mass per electron [1]; then it can very easily be shown that

\mu_e = \frac{\pi\hbar^2}{m_e m_p} \rho \quad (5)

where \( \rho \) is the mass density (mass per unit area) of the matter. Then following [1], the Thomas–Fermi condition is given by

\mu_e + m_p \phi = \text{constant.} \quad (6)

Since \( \phi \) is a function of radial coordinate \( \vec{r} \), the chemical potential \( \mu_e \) and the matter density \( \rho \) also depend on the radial coordinate \( \vec{r} \). Then replacing the gravitational potential \( \phi \) with the electron Fermi energy \( \mu_e \) from equation (6), the Poisson’s equation (equation (1)) can be written as

\nabla^2 \mu_e = -2\pi G m_p \rho. \quad (7)

Assuming circular symmetry, i.e., independent of the \( \theta \) coordinate, replacing \( \rho(r) \) by \( \mu_e(r) \) from equation (5) and expressing \( \nabla^2 \) in two-dimensional polar form, we have from equation (7)

\frac{d^2 \mu_e}{dr^2} + \frac{1}{r} \frac{d \mu_e}{dr} = -\frac{2Gm_e m_p^2}{\hbar^2} \mu_e \quad (8)

Substituting \( x = \frac{r\lambda}{\hbar} \), where \( x \) may be called the scaled radial coordinate with the constant

\lambda = \frac{2Gm_e m_p^2}{\hbar^2} \quad (9)

we can write down the Poisson’s equation in the following form:

\frac{d^2 \mu_e}{dx^2} + \frac{1}{x} \frac{d \mu_e}{dx} + \mu_e = 0. \quad (10)

It is quite obvious that in this case the solution is given by [7]

\mu_e(x) = AJ_0(x) \quad (11)

where \( J_0(x) \) is the ordinary Bessel function of order zero. The surface of the two-dimensional bulk object is obtained from the first zero \( x_s \)(say) of the Bessel function \( J_0(x) \), i.e.,

\mu_e(x_s) = AJ_0(x_s) = 0. \quad (12)

Hence the radius of the object is given by \( R = x_s/\lambda \). Further, at the centre, i.e., for \( x = 0 \),

\mu_0(0) = AJ_0(0) = A = \text{constant}. \quad (13)

Therefore the solution can also be written in the form

\mu_e(x) = \mu_e(0)J_0(x). \quad (14)
Hence the central density is given by
\[ \rho(0) = \frac{\lambda}{2\pi G m_\rho} \mu(0). \] (15)

Therefore we can also write
\[ \rho(x) = \rho(0) J_0(x) \] (16)
gives the variation of matter density with \( r \).

3. Mass–radius relation

With circular symmetry, the Poisson’s equation in the polar coordinate can also be written in the following form:
\[ \frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) = 2\pi G \rho \] (17)

Let us now integrate this differential equation with respect to \( r \) from \( r = 0 \), which is the centre to \( r = R \), the surface of the bulk two-dimensional stellar object. Then we have
\[ GM = R \left. \frac{d\phi}{dr} \right|_{r=R} = x_s \left. \frac{d\phi}{dx} \right|_{x=x_s} \] (18)

where
\[ M = \int_0^R 2\pi \rho r dr \] (19)
is the mass of the object. Now expressing \( \phi \) in terms of \( \mu_\xi \) and using the solution for \( \mu_\xi \) (equation (14)), we finally get
\[ GM = \frac{x_s \mu(0)}{m_\rho} J_1(x_s) \] (20)

where we have used the standard relation \( dJ_0(x)/dx = -J_1(x) \) [7]. Therefore this transcendental equation (equation (20)) can be solved numerically to get the mass-radius relation for the two-dimensional stellar objects in the Thomas–Fermi model in the non-relativistic scenario. In figure 1, we have shown graphically the form of the mass–radius relation for such objects. For the sake of illustration we have chosen the central density \( \rho(0) \) in such a manner that the maximum mass of the object is \( \sim 1.4 M_\odot \). The average density can also be obtained from the definition
\[ \bar{\rho} = M \frac{1}{x_s R^2} \] (21)

which can be expressed in terms of central density, and is given by
\[ \bar{\rho} = \frac{2\rho(0)}{x_s} J_1(x_s) \] (22)

4. Ultra-relativistic scenario

In this section, following [1] we have considered a two-dimensional hypothetical white dwarf star composed of massive ions and degenerate ultra-relativistic electron gas in a two-
dimensional geometrical configuration. The stability of the object is governed by electron degeneracy pressure. In the ultra-relativistic scenario the energy of an electron is given by the usual expression

\[ \epsilon = cp \] (23)

where \( c \) is the velocity of light and \( p \) is the electron momentum. The Fermi energy for this degenerate electron gas is then given by

\[ \mu_e = cp_F \] (24)

where \( P_F \) is the electron Fermi momentum. Hence the number density (the surface value) for the ultra-relativistic electron gas can be written in the following form:

\[ n_e = \frac{p_F^2}{2\pi\hbar^2}. \] (25)

Therefore the electron Fermi energy

\[ \mu_e = c\hbar \left( \frac{2\pi n_e}{m} \right)^{\frac{1}{2}} = c\hbar \left( \frac{2\pi\rho}{m_p} \right)^{\frac{1}{2}} \] (26)

where \( n_e \approx \rho m_p \), and \( \rho \) is the mass density for such ultra-relativistic matter.

Then following equation (7), we have the Thomas–Fermi equation in two-dimensions in the ultra-relativistic electron gas scenario

\[ \frac{d^2 \mu_e}{dx^2} + \frac{1}{x} \frac{d \mu_e}{dx} + \mu_e^2 = 0 \] (27)
where $x$ is the scaled radial coordinate and the constant

$$\lambda = \frac{Gm_p^2}{(\hbar c)^2}$$

Unfortunately, the non-linear differential equation given by equation (27) cannot be solved analytically. To obtain its numerical solution we have used the standard four-point Runge–Kutta numerical technique and a code is written in FORTRAN 77 to solve equation (27) using the initial conditions (i) $\mu(0) = 2\pi\hbar\rho(0)m_p$, at $x = 0$ (which indicates the origin of the two-dimensional white dwarf), which is the maximum value of $\mu$; then as a consequence (ii) $d\mu/dx = 0$, the other initial condition. The surface of this bulk two-dimensional object is then obtained from the boundary condition $\mu(x_s) = 0$, where $x_s$ is the scaled radius parameter.

From the numerical solution of equation (27) with the initial conditions (i) and (ii), the actual value of the radius can be obtained. As a cross check, we have used MATHEMATICA in the LINUX platform and got almost the same result. In our formalism, instead of solving for the gravitational potential in two dimensions, we have solved numerically for the electron chemical potential with the initial conditions (i) and (ii). The initial condition (i) depends on the central density $\rho(0)$, which we have supplied by hand as input. In the non-relativistic case, by a trial and error method we have fixed the value $\rho(0) \approx 10^9$ g cm$^{-3}$ to get $M/M_\odot \sim 1.4$. Whereas in the ultra-relativistic electron gas scenario, we have started from $\rho(0) = 10^6$ g cm$^{-3}$ and gone up to $10^{12}$ g cm$^{-3}$. Since we are solving numerically for $\mu$ and the central density is supplied by hand, the singularity at the origin ($x = 0$) does not appear in our analysis. Singularity at the origin also does not appear when Lane–Emden equation (Newtonian picture) and TOV equation (general relativistic scenario) are solved numerically by supplying the central density by hand. In this case the mass-radius relation can also be derived from equation (27) with $\phi$ replaced by $\mu$ from equation (6), the Thomas–Fermi
condition, and is given by

\[ GM = -\frac{x_s}{m_p} \left. \frac{d\rho}{dx} \right|_{x=x_s} \]  

(29)

Unlike the non-relativistic scenario, in this case the derivative term at the surface has to be obtained from the numerical solution of the Thomas–Fermi equation (equation (27)). In figure 2 we have shown the graphical form of the mass–radius relation for the ultra-relativistic case. The qualitative nature of figures 1 and 2 are completely different. Unlike the non-relativistic situation, where the mass–radius relation is obtained only for a particular central density, in this case each point on the mass–radius curve corresponds to a particular central density. This is obvious from the initial condition (i), where \( \rho(0) \) is the given central density. Since the central density has a fixed value in the non-relativistic scenario, the mass of the object increases with the increase of its radius. To make these two figures more understandable, in figures 3 and 4 we have plotted the variations of mass and radius, respectively, for the object against the central density in the ultra-relativistic scenario. In figure 3 we have plotted the mass of the object expressed in terms of solar mass, with the central density of the object expressed in terms of normal nuclear density. In figure 4 we have shown the variation of the radius of the object with central density. From the study of density profiles of the object in the ultra-relativistic scenario, we have noticed that for the low value of the central density, the numerical solution of equation (27) shows that the matter density vanishes or becomes negligibly small for quite a large radius value, whereas for very high central density, the matter density vanishes very quickly. Therefore in this model the stellar objects with very low central density are large in size and since the matter density is low enough, the mass is also quite small. On the other hand for the large values of central density, the objects are small in size, i.e., they are quite compact in size but massive enough. This
study explains the nature of variations of mass and radius with central density for the ultra-relativistic case. Further, in the ultra-relativistic scenario the average density of matter inside the bulk two-dimensional stellar object is given by

\[ \bar{\rho} = \rho(0) \frac{d\mu}{dx} \bigg|_{x=x_s} \]  

which, exactly like the non-relativistic case, also depends on the central density and the radius of the object. However, unlike the non-relativistic situation, here the average density depends on the surface value of the gradient of electron Fermi energy, which has to be obtained numerically from the solution of equation (27).

\[ \bar{\rho} = \rho(0) \frac{d\mu}{dx} \bigg|_{x=x_s} \]  

5. Conclusion

In conclusion we would like to comment that although the formalism developed here is for a hypothetical stellar object, which is basically an extension of the standard three-dimensional problem discussed in the book by Landau & Lifshitz, we strongly believe that it may be considered an interesting post-graduate level problem for physics students, including its numerical part. This problem can also be treated as part of a dissertation project for post-graduate physics students. The study of the Thomas–Fermi model in two-dimensions for self-gravitating objects may be used for model calculations of star formation and galaxy formation from a giant gaseous cloud. Finally, we believe that the problem solved here has some academic interest.
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References

[1] Landau L D and Lifshitz E M 1998 Statistical Physics: I (Oxford: Butterworth/Heinemann) p 320
[2] Shapiro S L and Teukolsky S A 1983 Black Holes, White Dwarfs and Neutron Stars (New York: Wiley) p 61, 188
[3] Huang H Q and Yu K N 1998 Stellar Astrophysics (Berlin: Springer) 337
[4] Bhaduri R K, das Gupta S and Lee S J 1990 Am. J. Phys 58 983
[5] Sanudo and Pacheco A F 2006 Rev. Mex. Fis. E E53 82
[6] Engineer S, Srinivasan K and Padmanabhan T 1999 Astrophys. J. 512 1
[7] Abramowitz M and Stegun I A (ed) 1972 Handbook of Mathematical Functions (New York: Dover) p 355