On the Connection between Generalized Hypergeometric Functions and Dilogarithms

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Abstract

Several integrals involving powers and ordinary hypergeometric functions are rederived by means of a generalized hypergeometric function of two variables (Appell’s function) recovering some well-known expressions as particular cases. Simple connections between dilogarithms and a kind of Appell’s function are shown. A relationship is generalized to polylogarithms.
Hypergeometric functions play an important role in mathematical physics since they are related to a wide class of special functions appearing in a large variety of fields. In particular, it is well-known a long time ago that integrals emerging from loop calculations in Feynman diagrams can be written in terms of hypergeometric functions \(1\). More recently, generalized hypergeometric functions of one or several variables have been used in the evaluation of scalar one-loop Feynman integrals \(2\) or multiloop ones \(3\).

In this work we firstly rederive \(1\) an integral expression involving two ordinary Gauss’ functions yielding a generalized hypergeometric function of two variables (Appell’s function). Several formulae appearing in standard tables (\(e.g.\) Gradshteyn and Ryzhik \(5\)) of utility for the evaluation of Feynman loop integrals are obtained as particular cases. Moreover, we have shown a simple relationship between a kind of Appell’s function and dilogarithms \(6\), contributing to enlarge the knowledge on the connection between them. In the appendices at the end of the paper we present a brief survey on the generalized Gauss’ functions establishing the notation employed and revising some of their properties needed in this work.

**expression 1**

\[
\int_0^1 du \, u^\gamma(1-u)^\rho-1 \, _2F_1[\sigma, \eta; \gamma; zu] \, _2F_1[\alpha, \beta; \rho; k(1-u)] =
\]

\[
= \frac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma+\rho)} \, F_3[\alpha, \beta, \gamma; \eta; \sigma; \rho; k, z]\]

provided that \(Re(\gamma) > 0, \, Re(\rho) > 0, \, |arg(1-k)| < \pi, \, |arg(1-z)| < \pi.\)

**Proof.** We will first show that Eq. (1) holds in the domain of convergence of the series. Expanding one of the two \(2F_1\) functions as a power series leads to:

\[
\sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\rho)_n} \frac{k^n}{n!} \int_0^1 du \, u^{\gamma-1}(1-u)^{\rho+n-1} \, _2F_1[\sigma, \eta; \gamma; zu] \quad |k| < 1, \, |z| < 1
\]

where we have interchanged the order of summation and integration on account of the dominated convergence theorem of Lebesgue, provided that \(Re(\gamma) > 0, \, Re(\rho) > 0.\) Now, performing the integration over \(u\) one gets from (A.3):

\[
\Gamma(\gamma) \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\rho)_n} \frac{k^n}{n!} \frac{\Gamma(\rho+n)}{\Gamma(\gamma+\rho+n)} \, _3F_2[\gamma, \sigma, \eta; \gamma+\rho+n, \gamma; z] =
\]

\[
= \frac{\Gamma(\gamma)}{\Gamma(\gamma+\rho)} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\rho)_n} \frac{k^n}{n!} \frac{\Gamma(\rho+n)}{\Gamma(\gamma+\rho+n)} \, _2F_1[\sigma, \eta; \gamma+\rho+n; z]
\]

where a cancellation between two parameters in the \(3F_2\) function occurred.

\(^1\)An exhaustive set of integrals involving generalized Gauss functions containing ours as a particular case can be found in \(4\).
Finally, using that: \( \Gamma(p + n) = \Gamma(p)\Gamma(n) \), \( \Gamma(\gamma + \rho + n) = \Gamma(\gamma + \rho)\Gamma(n) \), one arrives at

\[
\frac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma + \rho)} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma + \rho)_n} n! \, _2F_1[\sigma, \eta; \gamma + \rho + n; z]
\]

which leads to Eq. (1) at once in virtue of (B.2). Moreover, the integral (1) furnishes a single-valued function of two variables beyond the domain of convergence of the series by imposing the cuts: \( |arg(1 - k)| < \pi, |arg(1 - z)| < \pi \).

**expression 1.1**

Setting \( \rho = \delta - \gamma, \alpha = \delta - \sigma \) and \( \beta = \delta - \eta \) in Eq. (1) the formula 7.512.7 of reference [3] is recovered:

\[
\int_0^1 du \, u^{\gamma-1} (1 - u)^{\delta-\gamma-1} \, _2F_1[\sigma, \eta; \gamma; zu] \, _2F_1[\delta - \sigma, \delta - \eta; \delta - \gamma; k(1 - u)] = \frac{\Gamma(\gamma)\Gamma(\delta - \gamma)}{\Gamma(\delta)} \, _2F_1[\sigma, \eta; \delta; k + z - k\gamma]
\]

provided that \( Re(\delta) > Re(\gamma) > 0, |arg(1 - k)| < \pi, |arg(1 - z)| < \pi \).

This can be directly obtained from Eq. (1) taking further into account the property (B.6) which here implies:

\[
_3F_3[\delta - \sigma, \sigma, \delta - \eta, \eta; \delta; k, z] = (1 - k)^{\sigma + \eta - \delta} \, _2F_1[\sigma, \eta; \delta; k + z - k\gamma]
\]

**expression 1.2**

With the aid of (A.4) the left hand side of Eq. (1) can be written as:

\[
\int_0^1 du \, u^{\gamma-1} (1 - u)^{\rho-1} (1 - k(1 - u))^{-\alpha} \, _2F_1[\sigma, \eta; \gamma; zu] \, _2F_1[\alpha, \rho - \beta; \rho; \frac{k(1 - u)}{k - 1 - ku}]
\]

Now, let us assume that \( z \) and \( k \) are related through \( k = z/(z - 1) \). Then

\[
(1 - z)^{\alpha} \int_0^1 du \, u^{\gamma-1} (1 - u)^{\rho-1} (1 - zu)^{-\alpha} \, _2F_1[\sigma, \eta; \gamma; zu] \, _2F_1[\alpha, \rho - \beta; \rho; \frac{z(1 - u)}{1 - zu}]
\]

\[
= \frac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma + \rho)} \, _3F_3[\alpha, \sigma, \beta, \gamma + \rho; z/(z - 1), z]
\]

Next, let us suppose further that \( \beta = \gamma + \rho - \eta \). Then taking into account consecutively the properties (B.5) and (B.4) the right hand side of the last expression becomes:

\[
\frac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma + \rho)} (1 - z)^{\alpha} \, _2F_1[\eta; \alpha; \gamma + \rho; z] = \frac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma + \rho)} (1 - z)^{\alpha} \, _2F_1[\sigma + \alpha, \eta; \gamma + \rho; z]
\]

\[\text{Except the exponent of } (1 - k) \text{ which in our notation would read: } 2\sigma - \delta. \text{ Clearly this is an error since the result should be invariant under the interchange of } \sigma \text{ and } \eta, \text{ as the I.h.s. certainly is. The original source [3] is equally wrong.} \]
Hence one recovers the formula 7.512.8 of ref. [3]:

$$\int_0^1 du \ u^{\gamma-1}(1-u)^{\rho-1}(1-zu)^{-\alpha} \ 2F_1[\sigma, \eta; \gamma; zu] \ 2F_1[\alpha, \eta - \gamma; \rho; \frac{z(1-u)}{(1-zu)}] =$$

$$= \frac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma + \rho)} \ 2F_1[\sigma + \alpha, \eta; \gamma + \rho; z]$$

(3)

provided that $Re(\gamma) > 0$, $Re(\rho) > 0$, $|arg(1-z)| < \pi$.

Let us now go back again to Eq. (1) and consider $\eta = \gamma$ as a new special case. Then two parameters of a hypergeometric function in the integrand cancel, i.e. $2F_1[\sigma, \gamma; \gamma; zu] = 1F_0[\sigma; zu] = (1-zu)^{-\sigma}$, yielding:

expression 2

$$\int_0^1 du \ u^{\gamma-1}(1-u)^{\rho-1}(1-zu)^{-\sigma} \ 2F_1[\alpha, \beta; \gamma; ku] =$$

$$= \frac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma + \rho)} \ (1-z)^{-\sigma}F_3[\alpha, \sigma, \beta, \rho; \gamma + \rho; k, z/(z-1)]$$

(4)

provided that $Re(\gamma) > 0$, $Re(\rho) > 0$, $|arg(1-k)| < \pi$, $|arg(1-z)| < \pi$.

Proof. It follows immediately as a particular case of the expression 1 by means of the change of the integration variable: $u \rightarrow 1-u$ and interchanging the $\gamma$ and $\rho$ parameters.

An alternative (direct) proof is achieved with the aid of the integral representation of the $F_3$ Appell’s function. Starting from (B.3) and making the consecutive changes of the integration variables: $v \rightarrow 1-v$ and $u \rightarrow uv$ it follows that

$$\frac{\Gamma(\beta)\Gamma(\rho)\Gamma(\gamma - \beta)}{\Gamma(\gamma + \rho)} \ F_3[\alpha, \sigma, \beta, \rho; \gamma + \rho; k, z/(z-1)] =$$

$$(1-z)^{\beta-1} \int_0^1 dv \ du \ v^{\gamma-1}u^{\rho-1}(1-v)^{\rho-1}(1-u)^{\gamma-1}(1-zv)^{-\sigma}(1-kvu)^{-\alpha}$$

Hence the expression 2 is immediately obtained by expressing $2F_1[\alpha, \beta; \gamma; kv]$ in its Euler’s integral representation (A.2).

expression 2.1

Setting $k = 1$ in expression 2 reproduces the result 7.512.9 of ref. [3]:

$$\int_0^1 du \ u^{\gamma-1}(1-u)^{\rho-1}(1-zu)^{-\sigma} \ 2F_1[\alpha, \beta; \gamma; u] =$$

$$= \frac{\Gamma(\gamma)\Gamma(\rho)(\gamma + \rho - \alpha - \beta)}{\Gamma(\gamma + \rho - \alpha)\Gamma(\gamma + \rho - \beta)}(1-z)^{-\sigma} \ 3F_2[\rho, \sigma, \gamma+\rho-\alpha-\beta; \gamma+\rho-\alpha, \gamma+\rho-\beta; z/(z-1)]$$

provided additionally that: $Re(\gamma + \rho - \alpha - \beta) > 0$. 

4
This can be easily shown by rewriting the power expansion of \( F_3 \) following (B.2) in terms of \( 2F_1[\alpha, \beta; \gamma + \rho + n; 1] \) supposed the convergence of the series, and using the Gauss’ summation relation:

\[
2F_1[a, b; c; 1] = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(a)\Gamma(c - b)}; \quad Re(c - a - b) > 0
\]

**Particular values of the parameters of \( F_3 \)**

Let us now take the particular values of the parameters: \( \alpha = \beta = \sigma = \eta = \gamma = 1 \) and \( \rho = 2 \) in the \( F_3[\alpha, \sigma, \beta, \eta; \gamma + \rho; x, y] \) Appell’s function. From our expression 1, it is easy to see that making \( y = 1 \) ones gets

\[
x \cdot F_3[1, 1, 1, 1; 3; x, y = 1] = 2x \cdot 3F_2[1, 1, 1; 2, 2; x] = 2 \cdot Li_2(x)
\]

where Eq. (A.8) has been taken into account. In fact, one can get the same result by expanding \( F_3[1, 1, 1, 1; 3; x, 1] \) in terms of \( 2F_1[1, 1; 3 + m; 1] \) and using the Gauss’ summation relation. (See also appendix B for a generalization to polylogarithms.) The restriction \(|x| < 1\) can be dropped on account of analytic continuation, extending the domain of analyticity over the complex \( x \)-plane cut from 1 to \( \infty \) along the real axis. It is obvious from symmetry, that an equivalent expression for \( y \) must be satisfied. In fact, Eq. (5) can be viewed as a particular case of a more general relationship between this Appell’s series and dilogarithms:

**expression 3**

\[
\frac{1}{2} \cdot xy \cdot F_3[1, 1, 1, 1; 3; x, y] = L_i_2(x) + L_i_2(y) - L_i_2(x + y - xy)
\]

\(|arg(1 - x)| < \pi, |arg(1 - y)| < \pi.

**Proof.** This formula can be again derived from expression 1 by calculating directly the integral in terms of dilogarithms. Instead, we will prove it by differentiating both sides with respect to \( x \) and \( y \) consecutively. Expanding the Appell’s function as a double series, the result of differentiating the l.h.s. reads:

\[
\frac{1}{2} F_3[1, 1; 2, 1; 3; x, y]
\]

Now, invoking the property (B.6):

\[
F_3[\alpha, \gamma - \alpha, \beta, \gamma - \beta; x, y] = (1 - y)^{\alpha + \beta - \gamma} \cdot 2F_1[\alpha, \beta; \gamma; x + y - xy]
\]

which is valid in a suitable small open polydisc centered at the origin, we conclude that (7) can be rewritten as

\[
\frac{1}{2} \cdot 2F_1[1, 2; 3; x + y - xy]
\]
Next, differentiating twice the r.h.s of Eq. (6) one gets:
\[
-\frac{1}{x+y-xy}[1+\frac{1}{x+y-xy}\ln(1-(x+y-xy))] = \frac{1}{2} \text{ _2F_1}[1,2;3;x+y-xy]
\]
the last step coming from (A.5). Then both sides in Eq. (6) would differ in \(f(x)+g(y)\):
\[
\frac{1}{2}xy \text{ _3F_2}[1,1,1,1;3;x,y] = \text{Li}_2(x) + \text{Li}_2(y) - \text{Li}_2(x+y-xy) + f(x) + g(y)
\]
where \(f(x)\) and \(g(y)\) are functions to be determined by taking particular values of the variables. Setting \(x = 0\) and \(y = 0\) it is easy to see that \(f(x) = g(y)\equiv 0\).

Now, by analytic continuation we dispense with the restriction on the small polydisc, extending its validity to a suitable domain of \(C^2\): in order to get a single-valued function, with a well-defined branch for each dilogarithm in Eq. (6), we assume further that \(|\arg(1-x)| < \pi, |\arg(1-y)| < \pi\).  

**expression 3.1**

\[
x^2 \text{ _2F_3}[1,1,1,1;3;x,-x] = \text{Li}_2(x^2) \tag{8}
\]
for \(x\) real.

**Proof.** It follows directly from Eq. (6) using the relation \(\text{Li}_2(x) + \text{Li}_2(-x) = \frac{1}{2} \text{Li}_2(x^2)\).

**expression 3.2**

\[
x^2 \text{ _3F_3}[1,1,1,1;3;x,x] = 4 \text{Li}_2\left(\frac{1}{2-x}\right) + 2\ln^2(2-x) - \frac{\pi^2}{3} \tag{9}
\]
for \(x\) real and less than unity.

**Proof.** It follows directly from Eq. (6) using the relation: \(\text{Li}_2(2x-x^2) = 2\text{Li}_2(x) - 2\text{Li}_2(1/(2-x)) + \pi^2/6 - \ln^2(2-x)\).

**expression 3.3**

\[
\lim_{y\to0} \frac{xy}{2} \text{ _2F_3}[1,1,1,1;3;x,y] = y \left[1 + \frac{1-x}{x}\ln(1-x)\right] \tag{10}
\]

**Proof.** It follows directly from Eq. (6) using the relation: \(\text{ _2F_1}[1,1;3;x] = (1-x)^{-1} \text{ _2F_1}[1,2;3;x/(x-1)]\) and (A.5). If besides \(x\to0\), the limit \(xy/2\) is quickly recovered.

The set of expressions 3 provide new connections (not shown in literature to our knowledge) between dilogarithms and a certain \(\text{ _3F_2}\) Appell’s function.

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3Observe that then each function of one complex variable obtained from (6) by fixing the other variable is analytic in the corresponding subset of \(C^2\). Then the function of two variables \(\frac{1}{2}xy\text{ _3F_3}\) is analytic according to the theorem of Hartogs-Osgood
Appendices

A

Generalized Gauss’ Functions of one variable

Hypergeometric functions can be introduced at first as series within a certain domain of convergence \[^8\] \[^9\] \[^10\]. We write, using the abbreviate notation:

\[
pFq[{a \choose p}; {b \choose q}; z] = \sum_{n=0}^{\infty} \frac{(a)_n \ldots (a_p)_n}{(b)_n \ldots (b_q)_n} \frac{z^n}{n!}
\] (A.1)

where \((a)_n = \Gamma(a+n)/\Gamma(a)\) stands for the Pochhammer symbol. We suppose that none of the denominator parameters is a negative integer or zero. This series converges for all values of \(z\), real or complex, when \(p \leq q\), and for \(|z| < 1\) when \(p = q + 1\). In the latter case, it also converges (absolutely) on the circle \(|z| = 1\) if \(\text{Re} (\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i) > 0\). If \(p > q + 1\), the series never converges, except either when \(z = 0\) or when the series terminates, that is when one at least of the \(a\) parameters is zero or a negative integer.

Hypergeometric series admit in general an integral representation of the Euler’s type \[^10\] \[^11\] \[^12\]\ which permits the corresponding analytic continuation in the complex \(z\)-plane beyond the unit disc.

For the ordinary hypergeometric series, we have:

\[
\pFq{2}{1}{a}{b}{c}{z} = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_0^1 du \frac{u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a}}{u}
\] (A.2)

with \(\text{Re}(c) > \text{Re}(b) > 0\). In order to get a single-valued analytic function in the whole complex \(z\)-plane we will follow the customary convention of assuming a cut along the real axis from 1 to \(\infty\).

For the generalized hypergeometric function of one variable the integral representation of the Euler’s type is:

\[
pFq{p}{q}{a_{p-1}}{b_1}{b_q}{z} = \frac{\Gamma(b_1)}{\Gamma(a_1) \Gamma(b_1-a_1)} \int_0^1 du \frac{u^{a_1-1}(1-u)^{b_1-a_1-1} F_{q-1}[{a_{p-1} \choose p-1}; {b_q \choose q}; zu]}{u}
\] (A.3)

under the constraints: \(p \leq q + 1\), \(\text{Re}(b_i) > \text{Re}(a_i) > 0\) and none of \(b_i\), \(i = 1 \ldots q\), is zero or a negative integer, giving the analytic continuation in the whole complex \(z\)-plane, cut along the positive axis from 1 to \(\infty\) again.

A well-known transformation between Gauss’ hypergeometric functions of one variable, needed in the main text is: \[^12\]

\[
\pFq{2}{1}{a}{b}{c}{z} = (1-z)^{-a} \pFq{2}{1}{c-a,b}{c-a-b}{c}{z/(z-1)}
\]

\[
= (1-z)^{-a} \pFq{2}{1}{a-c,b}{a-c-b}{a-c}{z/(z-1)}
\] (A.4)

An interesting relation between an ordinary Gauss’ function and an elementary function not usually shown in specialized tables is:

\[
z \pFq{2}{1}{1,2}{0}{1,2}{z} = -2 \left[ 1 + \frac{1}{z} \ln (1-z) \right]
\] (A.5)

which can be proved by expanding both sides as power series.
The dilogarithm and its relation to the $3F_2$ function

The dilogarithmic function is defined as:

$$Li_2(z) = -\int_0^1 du \frac{\ln(1 - zu)}{u} \quad (A.6)$$

for values of $z$ real or complex. If $|z| < 1$, the dilogarithm may be expanded as the power series:

$$Li_2(z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)^2} \quad (A.7)$$

corresponding to the principal value. We can also write:

$$Li_2(z) = z \int_0^1 du \ _2F_1[1, 1; 2; zu] = z \ _3F_2[1, 1, 1; 2, 2; z] \quad (A.8)$$

The derivative of the dilogarithm is

$$\frac{d}{dz} Li_2(z) = -\frac{\ln(1 - z)}{z} \quad (A.9)$$
**B**

*Generalized Gauss’ Functions of two variables: Appell’s functions*

In this paper we are involved in particular with the $F_3$ Appell’s function, so we write its series expansion:

$$F_3[a, a', b, b'; c, x, y] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n x^m y^n}{(c)_{m+n} m! n!}$$

(B.1)

which exists for all real or complex values of $a, a', b, b'$, and $c$ except $c$ a negative integer. With regard to its convergence, the $F_3$ series is absolutely convergent when both $|x| < 1$ and $|y| < 1$. Then there is no problem with internal rearrangements of the series.

The $F_3$ function can be rewritten in terms of ordinary Gauss’ functions:

$$F_3[a, a', b, b'; c, x, y] = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m x^m}{(c)_m m!} \, {}_2F_1[a', b; c + m; y]$$

(B.2)

where we have made use of the relation: $(c)_{m+n} = (c)_{m} (c + m)_{n}$.

Moreover, the $F_3$ function admits the following integral representation:

$$F_3[a, a', b, b'; c, x, y] = \frac{\Gamma(c)}{\Gamma(b) \Gamma(b') \Gamma(c - b - b')} \frac{1}{1 - u - v}$$

(B.3)

$$= \int \int du \, dv \, \frac{u^{b-1}v^{b'-1}(1-u-v)^{-a} (1-yv)^{-a'}}{(1-xu)^{-a}}$$

where the integral is taken over the triangular region $0 \leq u$, $0 \leq v$, $u + v \leq 1$, under the conditions: $\Re(b) > 0$, $\Re(b') > 0$, $\Re(c - b - b') > 0$. This expression furnishes a single-valued analytic function in the domain defined by the Cartesian product of the complex planes of $x$ and $y$ with the restrictions $|\arg(1-x)| < \pi$, $|\arg(1-y)| < \pi$. Hence, the order of integration may be reversed according to Fubini’s theorem.

Some properties and relations between Appell’s functions needed in this paper are given below:

$$F_1[a; b, b'; c, x] = 2F_1[a, b + b'; c; x] = 2F_1[b + b', a; c; x]$$

(B.4)

$$F_3[a, c-a, b, b'; c, x, (y-1)] = F_3[b', b, c-a, a; c; y/(y-1), x] = (1-y)^b F_1[a; b, b'; c, x, y]$$

(B.5)

$$F_3[a, c-a, b, c-b; c, x, y] = F_3[c-a, a, c-b, b; c, y, x] = (1-y)^{a+b-c} 2F_1[a, b; c, x+y-xy]$$

(B.6)
The polylogarithm and its relation to the generalized Campé de Fériet function $F_B^{(2)}$

The polylogarithm $Li_q(z)$ is defined as a series as

$$Li_q(z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)^q} \quad (q > 1) \quad (B.7)$$

which can be expressed according to

$$Li_q(z) = z \int_0^1 du q F_{q-1}[\{1\}_q; \{2\}_q; zu] = z q^2 F_q[\{1\}_q; \{2\}_q; z]$$

A generalized Campé de Fériet function $F_B^{(2)}$, of particular interest for us, is defined as

$$F_B^{(2)}[\{b\}_r, \{b'\}_s; \{d\}_t; c; x, y] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(b_1)_m \ldots (b_r)_m (b'_1)_n \ldots (b'_s)_n}{(d_1)_m \ldots (d_t)_m (c)_{m+n}} \frac{x^m y^n}{m! n!} \quad (B.8)$$

Thus the following equality is satisfied

$$x F_B^{(2)}[\{1\}_q, \{1\}_2; \{2\}_q; 3; x, y = 1] = 2 Li_q(x) \quad (B.9)$$

which is the generalization of Eq. (5).

\[\textsuperscript{[4]}\text{Campé de Fériet functions are special cases of generalized Lauricella functions of two variables.}\]
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