1. Introduction

By a **kinematical Lie algebra** in dimension D, we mean a real \(\frac{1}{2}(D + 1)(D + 2)\)-dimensional Lie algebra with generators \(R_{ab} = -R_{ba}\), with \(1 \leq a, b \leq D\), spanning a Lie subalgebra \(\tau \cong so(D)\); that is,

\[
[R_{ab}, R_{cd}] = \delta_{bc}R_{ad} - \delta_{ac}R_{bd} - \delta_{bd}R_{ac} + \delta_{ad}R_{bc},
\]

and 2D + 1 generators \(B_a, P_a\) and \(H\) which transform according to the vector, vector and scalar representations of \(so(D)\), respectively – namely,

\[
[R_{ab}, B_c] = \delta_{bc}B_a - \delta_{ac}B_b, \quad [R_{ab}, P_c] = \delta_{bc}P_a - \delta_{ac}P_b \quad \text{and} \quad [R_{ab}, H] = 0.
\]

The rest of the brackets between \(B_a, P_a\) and \(H\) are only subject to the Jacobi identity: in particular, they must be \(\tau\)-equivariant. The kinematical Lie algebra where those additional Lie brackets vanish is called the **static** kinematical Lie algebra and shall be denoted \(s\). Every other kinematical Lie algebra will be, by definition, a deformation of \(s\). A partial converse, which is an easy consequence of the Hochschild–Serre factorisation theorem [1], is that for \(D \geq 3\) every deformation of \(s\) is kinematical. This fails for \(D = 2\) because \(so(2)\) is not semisimple and there are many more deformations of \(s\) in \(D = 2\) than the ones which concern us in this paper.

Up to isomorphism, there is only one kinematical Lie algebra in \(D = 0\): it is one-dimensional and hence abelian. For \(D = 1\), \(\tau = 0\) and hence any three-dimensional Lie algebra is kinematical. The classification is therefore the same as the celebrated Bianchi classification of three-dimensional real Lie algebras [2]. Chronologically, the next classification was for \(D = 3\) by Bacry and Nuyts [3], following up from earlier work of Bacry and Lévy-Leblond [4]. A deformation theory approach to that (and related) classifications is given in [5] based on earlier work [6]. This same approach has been used in [7] to classify kinematical Lie algebras for \(D \geq 4\) and in [8] for \(D = 2\). The purpose of this brief note is to summarise the results of the papers [5, 8, 7], which contain the details of the necessary calculations.

Recall that a Lie algebra \(\mathfrak{t}\) is said to be **metric** if it admits an ad-invariant (also called **associative**) inner product; that is, a non-degenerate symmetric bilinear form \(\langle - , - \rangle : \mathfrak{t} \times \mathfrak{t} \to \mathbb{R}\) which satisfies

\[
\langle [x, y], z \rangle = \langle x, [y, z] \rangle \quad \forall x, y, z \in \mathfrak{t}.
\]

It follows from Cartan’s semisimplicity criterion that semisimple Lie algebras are metric relative to the Killing form, but there are non-semisimple metric Lie algebras, where \(\langle - , - \rangle\) is an additional piece of data. For example, any inner product on an abelian Lie algebra is invariant. We also indicate in our results which of the Lie algebras in our classifications are metric.

**Notation.** We use the perhaps non-standard notation for Lie algebras described in Table 1. A caret adorning a symbol means a nontrivial central extension, e.g., \(\hat{g}\) is the Bargmann algebra, et cetera.

| Notation | Name            | Notation | Name            |
|----------|-----------------|----------|-----------------|
| \(a\)    | abelian         | \(p\)    | Poincaré        |
| \(s\)    | static          | \(so\)   | orthogonal      |
| \(n_+\)  | (euclidean) Newton | \(c\)   | orthogonal + dilatation |
| \(n_-\)  | (lorentzian) Newton | \(g\)  | galilean        |
| \(e\)    | euclidean       | \(c\)    | Carroll         |

2. \(D = 1\): Bianchi revisited

Apart from the abelian Lie algebra (Bianchi I) and the simple three-dimensional Lie algebras (Bianchi VIII and IX), all other three-dimensional Lie algebras have the structure of an abelian two-dimensional Lie algebra extended by an outer derivation. Letting \(B\) and \(P\) denote the generators of the abelian Lie algebra and \(H\) the outer derivation, we arrive at the following nonzero brackets:

\[
[H, B] = aB + cP \quad \text{and} \quad [H, P] = bB + dP,
\]
which can be brought to a normal form. Table 2 lists the different isomorphism classes and relates them to the Bianchi classification. It should be mentioned that the parameter $\gamma$ in Bianchi VI is not the traditional parameter, but a “compactification” to the interval.

| Bianchi I | Nonzero Lie brackets | Comments | Metric? |
|-----------|----------------------|----------|---------|
| II        | $[H, B] = P$          | $g (\geq s)$ | ✓       |
| III       | $[H, B] = P$          |           |         |
| IV        | $[H, B] = B + P$      |           |         |
| V         | $[H, B] = B$          |           |         |
| VI0       | $[H, B] = -B$         | $n_\pm (\geq p)$ |         |
| VIγ       | $[H, B] = \gamma B$   | $0 \neq \gamma \in (-1, 1)$ |         |
| VII0      | $[H, B] = P$          | $n_\pm (\geq \epsilon)$ |         |
| VIIγ      | $[H, B] = \gamma B$   |           |         |
| VII       | $[H, B] = P$          |           |         |
| IX        | $[H, B] = P$          |           |         |

In this dimension we already see many of the types of kinematical Lie algebras which exist for generic $D$. There are some isomorphisms which are low-dimensional accidents, such as between the Carroll and galilean algebras, between the Newton and euclidean/Poincaré algebras and also between the de Sitter/hyperbolic and anti de Sitter algebras. Of the Lie algebras in Table 2, only the abelian (I) and simple (VII, IX) cases are metric.

3. $D = 2$

This and the next dimension have a richer set of kinematical Lie algebras than for generic $D$. In the case of $D = 2$ it has to do with the $so(2)$-invariant symplectic structure on the vector representation. In this case the rotational algebra is one-dimensional and hence abelian and equation (2) takes the simpler form

$$[R, B_a] = \epsilon_{ab} B_b \quad \text{and} \quad [R, P_a] = \epsilon_{ab} P_b,$$

with $\epsilon_{ab}$ the Levi-Civita symbol normalised to $\epsilon_{12} = +1$. We may diagonalise the action of $R$ by complexifying. To this end we introduce $B = B_1 + iB_2$ and $P = P_1 + iP_2$ and extend the Lie brackets complex-linearly, so that now

$$[R, B] = -iB \quad \text{and} \quad [R, P] = -iP.$$

We also have $B = B_1 - iB_2$ and $P = P_1 - iP_2$, which satisfy

$$[R, B] = iB \quad \text{and} \quad [R, P] = iP.$$

The complex span of $R, H, B, P, \bar{B}, \bar{P}$ subject to the brackets (5) (and their complex conjugates) defines a complex Lie algebra $s_c$. This complex Lie algebra has a conjugation (that is, a complex-antilinear involutive automorphism) denoted by $*$ and defined by $H^* = H, R^* = R, B^* = \bar{B}$ and $P^* = \bar{P}$. We see that the real Lie subalgebra of $s_c$ consisting of real elements (i.e., those $X \in s_c$ such that $X^* = X$) is the static kinematical Lie algebra $s$. The same holds for any other kinematical Lie algebra in $D = 2$: its complexification admits the above conjugation. We find it convenient in the summary given in Table 3 to use the complex form of the Lie algebra.

The kinematical Lie algebras below the horizontal line in Table 3 are unique to $D = 2$ and owe their existence to the invariant symplectic structure $\epsilon_{ab}$ or, equivalently, to the complex structure which in the complex version of the algebra is simply multiplication by $i$. We see that in $D = 2$, the Carroll, euclidean and Poincaré algebras are metric, as well as two of the kinematical Lie algebras which are unique to this dimension.

4. $D = 3$

This is the other classical case. It is convenient here and in the case of $D > 3$ as well, to follow the abbreviated notation introduced in [4], by which we let $B$ and $P$ stand for the $D$-dimensional vectors of generators $B_\alpha$ and $P_\alpha$, and write the Lie brackets without indices, with the understanding that the indices in the LHS also appear on the RHS, albeit possibly contracted with the rotationally invariant tensors: the Kronecker $\delta$ and the Levi-Civita $\epsilon$.

All kinematical Lie algebras share the Lie brackets in equations (1) and (2), which in abbreviated notation are written as

$$[R, R] = R \quad [R, B] = B \quad [R, H] = 0 \quad \text{and} \quad [R, P] = P.$$

The kinematical Lie algebras in Table 4 which lie below the line are unique to $D = 3$: indeed, they owe their existence to the vector product in $\mathbb{R}^3$, which is invariant under rotations. In $D = 3$ the metric Lie algebras are the simple Lie algebras and in addition four of the Lie algebras which are unique to this dimension.

5. $D \geq 4$

We use the same abbreviated notation as in the case $D = 3$ and, as in that case, we list only those Lie brackets which do not involve the rotational generators. The only metric Lie algebras for $D > 3$ are the simple Lie algebras.
Table 3. Kinematical Lie algebras in $D = 2$ (complex form)

| Nonzero Lie brackets | Comments | Metric? |
|----------------------|----------|---------|
| $[H, B] = P$         |          | $s$     |
| $[H, B] = B$         |          | $\partial$ |
| $[H, P] = -P$        |          | $n_-$   |
| $[H, P] = \lambda + i\theta$ |          | $\lambda \in (-1, 1), \theta \in \mathbb{R}$ |
| $[H, B] = B + P$     |          | $c$     |
| $[H, P] = 2(H - i\lambda)$ |          | $so(3, 1)$ |
| $[H, P] = -2H$       |          | $c$     |
| $[H, P] = -i\theta$  |          | $p$     |
| $[H, B] = B$         |          | $\partial$ |
| $[H, P] = B$         |          | $n_+$   |
| $[H, P] = -2i\theta$ |          | $so(4)$ |
| $[H, P] = 2i\theta$  |          | $so(2, 2)$ |
| $[B, B] = iH$        |          | $\partial$ |
| $[P, P] = i(H + R)$  |          | $\partial$ |

Table 4. Kinematical Lie algebras in $D = 3$

| Nonzero Lie brackets | Comments | Metric? |
|----------------------|----------|---------|
| $[H, B] = -P$        |          | $s$     |
| $[H, B] = -B$        |          | $\partial$ |
| $[H, P] = -B$        |          | $n_-$   |
| $[H, P] = -B$        |          | $n_+$   |
| $[H, B] = B$         |          | $\gamma \in (-1, 1)$ |
| $[H, B] = B + P$     |          | $\alpha > 0$ |
| $[H, P] = \alpha P - B$ |          |          |
| $[H, P] = B + P$     |          | $c$     |
| $[B, P] = H$         |          |          |
| $[B, B] = R$         |          |          |
| $[B, B] = -R$        |          | $p$     |
| $[B, P] = H - R$     |          | $so(4, 1)$ |
| $[B, B] = R$         |          | $so(5)$ |
| $[B, B] = -R$        |          | $so(3, 2)$ |
| $[B, P] = B - R$     |          | $\partial$ |
| $[B, P] = R - B$     |          | $\partial$ |
| $[B, B] = B$         |          |          |
| $[B, B] = P$         |          |          |

6. One-dimensional extensions of kinematical Lie algebras ($D \geq 3$)

For $D \geq 3$, the static kinematical Lie algebra $s$ with nonzero brackets given by (1) and (2) admits a one-dimensional central extension $\hat{s}$, with additional bracket

$$[B_\alpha, P_\beta] = \delta_{\alpha\beta}Z \quad \text{(or in abbreviated form } [B, P] = Z.)$$

In [5], based on the earlier work [6], we classified the deformations of $\hat{s}$ for $D = 3$ and in [7] also for $D \geq 4$. I am not aware of any results for $D = 2$, perhaps due to the fact that $\dim H^2(s; \mathbb{R}) = 5$. One of the central generators makes nonzero the $[R, H]$ bracket, which would perhaps disqualify it as “kinematical”, since we would like to retain the identification of $R$ as a rotation. In that case, it is the relative cohomology $H^2(s; \mathbb{R}, r)$, with $r$ the one-dimensional subalgebra spanned by $R$, which one has to calculate. One now finds $\dim H^2(s, r; \mathbb{R}) = 4$. In addition, one finds that $\dim H^2(s, r, r) = 11$ and $\dim H^2(s, r, r, r) = 29$. We have not yet fully analysed the integrability of these infinitesimal deformations.

Table 6 lists the deformations of the universal central extension of the static kinematical Lie algebra for $D \geq 3$. The table is divided into three by horizontal lines. The top third consists of nontrivial central extensions of kinematical...
Table 5. Kinematical Lie algebras in $D \geq 4$

| Nonzero Lie brackets | Comments | Metric? |
|----------------------|----------|---------|
| $[H, B] = P$         | $\mathfrak{g}$                        |
| $[H, B] = -B$        | $[H, P] = P$                         | $n_-$   |
| $[H, B] = P$         | $[H, P] = -B$                       | $n_+$   |
| $[H, B] = \gamma B$  | $[H, P] = P$                        | $\gamma \in (-1, 1)$ |
| $[H, B] = \alpha B + P$ | $[H, P] = \alpha P - B$         | $\alpha > 0$ |
| $[H, B] = B + P$     | $[H, P] = P$                        |         |

| $[B, P] = H$          | $\mathfrak{c}$                       |
| $[B, P] = H + R$      | $[B, P] = H$                        | $so(D + 1, 1)$ | $\checkmark$ |
| $[B, P] = -R$         | $[B, P] = -R$                       | $so(D + 2)$ | $\checkmark$ |

Table 6. Deformations of $\hat{a}$ in $D \geq 3$

| Nonzero Lie brackets | Comments | Metric? |
|----------------------|----------|---------|
| $[B, P] = Z$         | $\hat{a}$                         |
| $[H, B] = B$         | $[H, P] = -P$                       | $\hat{n}_-$ |
| $[B, P] = Z$         | $[H, B] = -P$                       | $\hat{n}_+$ |
| $[B, P] = H$         | $[H, B] = -P$                       | $\hat{g}$ |
| $[B, P] = H + R$     | $[H, B] = B$                        | $so(D + 1, 1) \oplus R$ | $\checkmark$ |
| $[B, P] = -R$        | $[H, B] = -B$                       | $so(D + 2) \oplus R$ | $\checkmark$ |
| $[B, P] = H$         | $[H, B] = -P$                       | $R \oplus R$ |
| $[H, B] = \gamma B$  | $[H, P] = P$                        | $\gamma \in (-1, 1)$ |
| $[H, B] = \alpha B + P$ | $[H, P] = \alpha P + B$            | $\alpha > 0$ |
| $[H, B] = Z$         | $[Z, B] = P$                        | $co(D + 1) \times \mathbb{R}^{D+1}$ |
| $[H, B] = -P$        | $[H, P] = P$                        | $co(D, 1) \times \mathbb{R}^{D,1}$ |

Lie algebras, the middle third of trivial central extensions of kinematical Lie algebras, and the bottom third of non-central extensions of kinematical Lie algebras. The metric Lie algebras here are only the trivial extensions of the simple kinematical Lie algebras.

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