Redshift/blueshift inside the Schwarzschild black hole

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We consider an observer who moves under the horizon of the Schwarzschild black hole and absorbs a photon. There are two different situations when (i) a photon comes from infinity, (ii) it is emitted by another observer under the horizon (say, by a surface of a collapsing star). We analyze the frequency change for absorption near the event horizon, in an intermediate region and near the singularity and compare the results for both scenarios. Near the singularity, in both scenarios infinite redshift (in the pure radial case) or infinite blueshift are possible, depending on angular momenta of a photon and an observer. The main difference between both scenarios manifests itself, if photons from the star surface are emitted near the horizon and received in the intermediate region or near the singularity by the observer with a nonzero angular momentum.

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I. INTRODUCTION

Sometimes, strong prejudice against careful studies of properties of the inner region of black holes reveals itself because of impossibility to obtain information from there in ”our” world (on the classical level). Meanwhile, black holes is one of the most remarkable and sound predictions of general relativity. They are also found in many other gravitation theories. Thus the presence of a region under the event horizon is quite solid physical result of theory. And, as a part of a physical world, it deserves investigation in spite of some unusual properties of this region (or, by contrary, just due to this fact). The situation in this area of research was recently envisaged in a brief review [1] where it was stressed that because of neglect of the corresponding subject, even some primary notions relevant for this region are described in literature not quite accurately.

As far as the properties of the region inside a black hole is concerned, one of the first questions is the view of surrounding world. What does a falling observer see during his fall? In their popular book, Gurevich and Gliner wrote that when the world line of an observer approaches the singularity, he sees a surrounding world to fade ([2], p. 59). However, they arrive at this conclusion considering the value of the coordinate speed of light - the quantity that does not have direct physical meaning. Quite recently, a similar conclusion was made in [3] where propagation of light from the ”illusory horizon” was considered. This term was coined in [3] to replace a more usual one ”past horizon” in the complete analytically extended Schwarzschild metric. Meanwhile, if, instead of the complete space-time, realistic collapse of a star is considered, its surface follows a time-like trajectory everywhere (including the inner region) and does not approach the would-be past horizon, in contrast to what is supposed in [3] (especially, see page 13 and Fig. 6 there). Therefore, in our opinion, the conclusions derived in [3] about exponential redshift inside the event horizon, hang in the air. And, only radial motion of particles and photons in the inner region was considered in [3].

In the present work, we consider red/blue shift of light under the event horizon in a general case and argue that account for nonzero angular momenta of particles can change a whole picture drastically. In doing so, we concentrate on the properties of a frequency near the singularity. The results are valid both for an eternal black hole or for the realistic collapse of a star outside its surface in the vacuum region. The formulas for the frequency are very simple but, to the best of our knowledge, the corresponding results that follow from
them near the singularity were absent from the literature. This work can be considered as a step towards more general goal - constructing the whole picture seen by an observer inside a black hole. This will have to include transformation of angles under which light comes into view that deserves further separate study. In what follows, we use the geometric system of units in which fundamental constants $G = c = 1$.

II. GENERAL EQUATIONS

A. Metric

Let us consider the black hole metric

$$ds^2 = -dt^2 f + \frac{dr^2}{f} + r^2 d\omega^2,$$

where $d\omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$, $f = f(r)$. The roots of equation $f = 0$ correspond to the horizon. We will discuss the case when there is only one horizon $r_+$ and mainly focus on the Schwarzschild metric. Then, $f = 1 - \frac{r_+}{r}$, $r_+$ being the horizon radius.

We are interested in the region inside the event horizon. As is known, the metric can be described there by its original form (1) but with an important reservation that spacelike and timelike coordinates mutually interchange their roles - see [4] or [5] (page 25). Correspondingly, we redefine $t = y$, $r = -T$, $f = -g$, then the metric (1) takes the form

$$ds^2 = -dT^2 g + dy^2 g + T^2 d\omega^2. \quad (2)$$

Here, all metric coefficients depend on $T$ only. For the Schwarzschild metric,

$$g = \frac{r_+}{(-T)} - 1, \quad -r_+ \leq T \leq 0. \quad (3)$$

The hypersurface $T = \text{const}$ represents a hypercylinder extended in the $y$ direction. It is instructive to write down equations of geodesic motion for massive particles and photons separately.

B. Motion of massive particle

As the metric does not depend on $y$ and $\phi$, the radial momentum $P = mu_y$ and the angular one $\mathcal{L} = mu_\phi$ are conserved, where $u^\mu = \frac{dx^\mu}{d\tau}$ is the four-velocity, $\tau$ being the proper
time. Equations of motion within the plane $\theta = \frac{\pi}{2}$ for a geodesic particle read

$$g \dot{y} = -p,$$  \hspace{1cm} (4)

$$T^2 \dot{\phi} = L,$$  \hspace{1cm} (5)

where $p = \frac{E}{m}$, $L = \frac{L}{m}$, dot denotes derivative with respect to the proper time $\tau$. If a particle having the energy $E > 0$, moves outside, the equation of motion gives $mf \dot{t} = E$ for it. When it enters the inner region, $f = -g < 0$, so for $\dot{y}$ we obtain just (4) with $p = \frac{E}{m}$. However, if a particle inside did not come from the outer region, $p$ can have any sign, $p = \pm |p|$. Under the horizon, $p$ has the meaning of radial momentum and is still conserved. The case $p = 0$ is also possible [6]. From the normalization condition $u^\mu u_\mu = -1$ we have

$$\frac{p^2}{g} + \frac{L^2}{T^2} - \frac{\dot{T}^2}{g} = -1.$$  \hspace{1cm} (6)

Taking into account the forward-in-time condition $\dot{T} > 0$ we obtain

$$\dot{T} = Z,$$  \hspace{1cm} (7)

$$Z = \sqrt{p^2 + g(\frac{L^2}{T^2} + 1)}.$$  \hspace{1cm} (8)

Thus in coordinates $(T, y, \phi)$ we have for the four-velocity $u^\mu = \frac{dx^\mu}{d\tau}$

$$u^\mu = (Z, -\frac{p}{g}, \frac{L}{T^2}),$$  \hspace{1cm} (9)

$$u_\mu = (-\frac{Z}{g}, -p, L).$$  \hspace{1cm} (10)

We omit $u^\theta = 0$.

C. Three-dimensional velocity

It is instructive to define a velocity as a spatial three-dimensional vector $V^{(i)} \ (i = 1, 2)$. Actually, it is two-dimensional for motion within the equatorial plane. To this end, we can define the tetrad (actually, there are only 3 independent vectors since we consider the equatorial plane only but we still use the standard term "tetrads" for convenience):

$$h_{(0)\mu} = \frac{1}{\sqrt{g}}(1, 0, 0),$$  \hspace{1cm} (11)
\[ h_{(1)\mu} = \sqrt{g}(0, 1, 0), \]  
\[ h_{(2)\mu} = |T|(0, 0, 1). \]  

Such a choice of tetrads corresponds to the observer with the four-velocity \( U^\mu = h^\mu_{(0)} \) who remains at rest \( (y = const) \). It is interesting that inside the horizon this is a geodesic observer [6] that was impossible outside, where an observer at rest is accelerated.

Then, the standard definition gives us the tetrad components

\[ V^{(i)} = -\frac{h^{(i)\mu}w^\mu}{h_{(0)\mu}w^\mu}. \]  

It is easy to obtain from (8), (9) and (11) - (13) that

\[ V^{(1)} = -\frac{p}{Z}, \]  
\[ V^{(2)} = \frac{L\sqrt{g}}{|T|Z}, \]  
\[ |\vec{p}| = \frac{V\sqrt{g}}{\sqrt{1-V^2}}. \]

Here \( V = \sqrt{V^{(1)^2} + V^{(2)^2}} \), \( |\vec{p}| = \sqrt{p^2 + \frac{L^2}{T^2}g} \) is the absolute value of the momentum that takes into account both spatial components. Then, (17) has the meaning of the standard relation between the momentum and velocity, redshifted due to the factor \( \sqrt{g} \). One can also define the local energy as \( E_{loc} = -mu_{(0)} \). Then, it follows from (7), (8) and (11) - (13) that \( E_{loc} = \sqrt{m^2 + \frac{|\vec{p}|^2}{g}} \). By substitution of (17), we obtain

\[ E_{loc} = \frac{m}{\sqrt{1-V^2}}. \]

As under the horizon the metric depends on time, the energy is not conserved.

Eqs. (17), (18) can be thought of as counterparts of the formula

\[ E = E_{loc}\sqrt{f} \]  

for the energy \( E \) outside the horizon [7] (see eq. 15 there), now they are valid inside.

It is seen from (17) that in the horizon limit, when \( g \to 0 \), the velocity \( V \to 1 \), so it approaches the speed of light. This applies to each branch of the horizon.
D. Motion of photon

In a similar way, the components \( k_y \equiv -q, k_\phi \equiv l \) of the wave vector \( k^\mu \) are conserved inside the horizon. If a photon came from the outer region, \(|q| = \omega_0\) has the meaning of the frequency emitted by a static observer at infinity. But in general, it can also be emitted already under the horizon.

The normalization condition \( k_\mu k^\mu = 0 \) gives us

\[
-\frac{(k^0)^2}{g} + \frac{l^2}{T^2} + \frac{q^2}{g} = 0,
\]

whence

\[
k^0 = z,
\]

where

\[
z = \sqrt{q^2 + \frac{g}{T^2} l^2}.
\]

Then, the wave vector is equal to

\[
k^\mu = (z, -\frac{q}{g}, \frac{l}{T^2}),
\]

\[
k_\mu = (-\frac{z}{g}, -q, l).
\]

We remind a reader that we consider motion in the equatorial plane only, so all relevant momenta have no \( \theta \) component.

III. FREQUENCY

The frequency measured by an observer with the four-velocity \( u^\mu \) is equal to

\[
\omega = -u_\mu k^\mu.
\]

Taking into account eqs. (10) and (23), we have

\[
\omega = \frac{zZ - pq}{g} - \frac{\mathcal{L} l}{T^2}.
\]

Our main concern is the behavior of the frequency near the horizon and singularity. The relevant quantities enter the general expression (26) in such a way that, as a rule, smooth limiting transitions to the particular cases are impossible. For instance, in the combination \( \frac{\mathcal{L}^2}{T^2} g \) in (8) the result depends strongly on which limit is taken first - \( \mathcal{L} \to 0 \) or \( T \to 0 \), etc. Therefore, we will consider some particular physically interesting situations separately, case by case.
IV. EXACT FORMULAS FOR PARTICULAR CASES

To facilitate reading, we give at first explicitly general exact formulas in particular cases, even in spite of their simplicity. Afterwards, we will analyze them near the horizon and singularity.

A. Radial motion of a photon: \( l = 0, \mathcal{L} \neq 0 \)

It follows from (22) and (26) that

\[
\omega = \frac{|q| (Z - \alpha |p|)}{g},
\]

\[
\alpha = \text{sign}(pq),
\]

(27)

(28)

\( Z \) is given by (8).

B. Radial motion of an observer: \( \mathcal{L} = 0, l \neq 0 \).

Then, we have from (8), (26)

\[
\omega = \sqrt{(p^2 + g)(q^2 + \frac{l^2}{T^2}g) - pq} \frac{-pq}{g}.
\]

(29)

C. Radial motion of an observer and a photon: \( \mathcal{L} = 0, l = 0 \).

Let both \( \mathcal{L} = 0, l = 0 \). Then, it follows from (27) that

\[
\omega = \frac{|q| (\sqrt{p^2 + g} - \alpha |p|)}{g} = \frac{|q|}{\sqrt{p^2 + g + \alpha |p|}}.
\]

(30)

In this case, it is seen from (27), (30) that for both signs of \( \alpha \),

\[
\frac{d\omega}{dg} < 0.
\]

(31)

In the Schwarzschild metric \( \frac{dq}{dr} < 0 \) everywhere under the horizon, so \( \frac{d\omega}{dr} > 0 \). When a particle moves under the horizon towards the singularity \( r = 0 \), \( \omega \) diminishes, so the redshift is increasing in the process of motion.
D. Angular motion of a photon: $q = 0$

The limit $q \to 0$ corresponds to a photon that does not move in the “radial” direction along the leg of a hypercylinder and only circumscribes the half of a full circle in the angular direction (see eq. 69 of [6]). If $q = 0$ it is necessary that $l \neq 0$ to have non-vanishing $k_{\mu}$.

Then, for $\mathcal{L} \neq 0$, it follows from (26) that

$$\omega = \frac{|l|}{T} \left( \sqrt{\frac{\mathcal{L}^2}{T^2} + 1} + \frac{p^2}{g} - \frac{|\mathcal{L}| \text{sgn}(\mathcal{L}l)}{|T|} \right).$$

(32)

For $\mathcal{L} = 0$, we can obtain from (32)

$$\omega = \frac{|l|}{T} \sqrt{\frac{p^2}{g} + 1}. $$

(33)

E. Observer at rest in $y$ direction: $p = 0$

Under the horizon, a geodesic observer can have $p = 0$ and even remain at rest in the corresponding frame (2) if $\mathcal{L} = 0$ as well (see Sec. 2.2. of [6] for details of such a trajectory). This property has no analog outside the horizon since the radial momentum depends on time in the outer region but it is conserved in the inner one. Now, we have from (8), (22), (26)

$$\omega = \sqrt{\frac{c^2}{T^2} + 1} \frac{|q| + \frac{\mathcal{L}^2}{T^2}}{\sqrt{g}} - \frac{\mathcal{L}l}{T^2}.$$ (34)

If $\mathcal{L} = 0$, $l \neq 0$,

$$\omega = \sqrt{\frac{q^2 + c^2}{T^2}} \frac{1}{\sqrt{g}}.$$ (35)

If $\mathcal{L} \neq 0$, $l = 0$,

$$\omega = \sqrt{\frac{c^2}{T^2} + 1} \frac{|q|}{\sqrt{g}}.$$ (36)

If $\mathcal{L} = 0 = l$,

$$\omega = \frac{|q|}{\sqrt{g}}.$$ (37)

If $p = q = 0$, we see from (34) that

$$\omega = \sqrt{\frac{c^2}{T^2} + 1} \frac{|l|}{|T|} - \frac{\mathcal{L}l}{T^2}.$$ (38)
If $p = q = \mathcal{L} = 0$, $l \neq 0$, it follows from (29) that
\[
\omega = \frac{l}{T}.
\] (39)

One can check that all these formulas are mutually consistent with each other. For example, if we put $p = 0$ and $\mathcal{L} = 0$ in (32) or $\mathcal{L} = 0$ in (38) we obtain the same result (39), etc.

Now, on the basis of the obtained formulas, we analyzed behavior of the frequency near the horizon.

V. PHOTON ABSORBED NEAR THE HORIZON

Now, we assume that $g \to 0$.

A. Generic case

If $\alpha = +1$, we have from (26)
\[
\omega(r_+) = \frac{q}{2p} + \frac{(pl - qL)^2}{2pqr_+^2}.
\] (40)

Eq. (40) is valid for generic $\mathcal{L}$, $l$. In the particular case $\mathcal{L} = 0$ it agrees with eq. (10) of [8]. For the pure radial case, $\mathcal{L} = l = 0$,
\[
\omega(r_+) = \frac{q}{2p}.
\] (41)

If a particle crosses the horizon moving from infinity where it was at rest, $p = 1$. Taking also into account that the integral of motion $q$ has the meaning of frequency at infinity, $q = \omega_0$, we have
\[
\frac{\omega(r_+)}{\omega_0} = \frac{1}{2}.
\] (42)

If $\alpha = -1$, eq. (26) gives us
\[
\omega \approx \frac{2|pq|}{g}
\] (43)

independently of $\mathcal{L}$ and $l$, so $\omega \to \infty$ when a photon is absorbed near the horizon. This is a kind of head-on collision. However, such a collision with finite nonzero $p$ and $q$ (say, $p > 0$ and $q < 0$) near the horizon implies that in the complete Schwarzschild space-time a photon
crossed the horizon from the "mirror" universe. The similar effect in the outer region occurs near the white hole horizon \([12]\). Meanwhile, in a more physical situation, if it is emitted from the surface of a collapsing star, the requirement of the finiteness of \(\omega\) entails that \(q\) itself is small (for more details see below).

**B. Angular motion of a photon: \(q = 0, p \neq 0\)**

It follows from (32) that
\[
\omega \approx |l| \frac{|p|}{\sqrt{g} r^+}
\]
(44)
independently of \(|\mathcal{L}|\). Thus \(\omega \to \infty\) in the horizon limit \(g \to 0\). If a massive particle has zero momentum in \(y\) direction, it passes through the bifurcation point \([13]\). The situation when an observer has \(p > 0\), crosses the horizon and meets there a photon with \(q = 0\) is a counterpart of the situation considered in \([9]\), where \(p = 0, q \neq 0\).

**C. Observer at rest in \(y\) direction: \(p = 0, q \neq 0\)**

It follows from (34) that for any \(l\)
\[
\omega \approx \frac{\sqrt{\frac{\mathcal{L}^2}{r^+} + 1} |q|}{\sqrt{g}}
\]
(45)
diverges in the horizon limit \(g \to 0\), so there is an infinite blueshift. Such a high energy collision \([9]\) can be considered as some analogue of the BSW effect \([11]\).

**D. \(q = 0, p = 0\)**

Then, we have from (38) that
\[
\omega(r^+) = \frac{\sqrt{\frac{\mathcal{L}^2}{r^+} + 1} |l|}{r^+} - \frac{\mathcal{L} l}{r^+}
\]
(46)
is finite and nonzero.

We can summarize the results in the table.
Table 1. Behavior of the frequency near the horizon.

| \( pq \)      | \( \omega \)  |
|----------------|---------------|
| \( > 0 \)     | finite        |
| \( < 0 \)     | infinite      |
| \( q = 0, p \neq 0 \) | infinite   |
| \( p = 0, q \neq 0 \) | infinite   |
| \( p = q = 0 \)  | finite        |

VI. BEHAVIOR NEAR THE SINGULARITY

Near the singularity, \( g \to \infty, r \to 0, T \to 0 \).

The result depends strongly on angular momenta of an observer and a photon.

If \( \mathcal{L}l > 0 \), we obtain from (26) that

\[
\omega \approx \frac{l^2}{2 \mathcal{L}} < \infty. \tag{47}
\]

\( \mathcal{L}l < 0 \)

\[
\omega \approx \frac{2 |\mathcal{L}|}{T^2} \to \infty \tag{48}
\]

\( \mathcal{L} = 0, l \neq 0 \)

We have from (29) that

\[
\omega \approx \frac{|l|}{T} \to \infty \tag{49}
\]

Eqs. (47) - (49) are insensitive to \( p \) and \( q \).

\( \mathcal{L} \neq 0, l = 0 \)

From (8), (27) we obtain that

\[
\omega \approx \frac{|q| |\mathcal{L}|}{|T| \sqrt{g}} \tag{50}
\]

In the Schwarzschild case, \( g \approx \frac{r_+}{r^2} \), so

\[
\omega \approx \frac{|q| |\mathcal{L}|}{\sqrt{|T|} \sqrt{r_+}} \tag{51}
\]

diverges when \( T \to 0, g \to \infty \).

\( \mathcal{L} = 0, l = 0 \)
It follows from (30) that

\[ \omega \approx \frac{|q|}{\sqrt{g}} \to 0. \]  

(52)

The results are summarized in the table. Here, the values of \( p \) and \( q \) and their relative sign are irrelevant.

| \( \mathcal{L} l \) | \( \omega \)       |
|-----------------|-----------------|
| \( \mathcal{L} l > 0 \) | finite nonzero |
| \( \mathcal{L} l < 0 \) | infinite blueshift |
| \( \mathcal{L} = 0, l \neq 0 \) | infinite blueshift |
| \( \mathcal{L} \neq 0, l = 0 \) | infinite blueshift |
| \( \mathcal{L} = 0 = l \) | infinite redshift |

Table 2. Behavior of the frequency near the singularity

It is worth noting that although in case \( \mathcal{L} l > 0 \) the frequency is finite and nonzero, it can take any value depending on parameters. In particular, if a photon was emitted with the frequency \( \omega_1 \) and absorbed with the frequency \( \omega_2 \), both limiting cases \( \omega_2 \ll \omega_1 \) and \( \omega_2 \gg \omega_1 \) are possible.

\section{VII. WHAT WILL A FALLING OBSERVER SEE? TYPICAL CASE}

Up to now, we considered the process of absorption of a photon by an observer with given \( \mathcal{L}, p, q \), not specifying a precedent act of emission in which a photon with these characteristics appeared. If a photon enters the region under the horizon from the outside, \( q > 0 \) and the quantity \( q \) is equal to the frequency at infinity \( \omega_0 \). Let us call it scenario (i).

Meanwhile, there is another scenario (ii). It has a two step character. At first, a photon is emitted by observer 1 already under the horizon. Further, it is received by observer 2. In doing so, the angular momentum \( l \) of a photon does not change during the travel between two events. We assume that observer 2 falls from infinity and has \( p_2 > 0 \). The role of observer 1 can be played, say, by a star collapsing surface (then, \( p_1 > 0 \)) or some particle from dust cloud that is able to radiate. We also assume that a photon is sent to meet observer 2, so absorption has a kind of head-on collision: \( p_2 > 0, q < 0 \). Correspondingly, in eq. (28)

\[ \alpha = -1 \]  

(53)

for both observers 1 and 2.
The space-time diagram that depicts both scenario, is presented in Fig. 1. It is nothing else than a standard diagram describing a gravitation collapse. On this diagram, line H corresponds to the horizon.

![Space-time diagram](image)

**FIG. 1.** Space-time diagram describing a star surface (line 1) and a falling observer (line 2).

The most interesting region is the vicinity of the singularity. Meanwhile, the results described above tell us that near the singularity the frequency is insensitive to the sign of \( q \), so the difference between scenarios (i) and (ii) is blurred. And, from Table 2 we see that a quite diverse set of situations becomes possible in each of two scenarios.

While parameter \( q \) becomes irrelevant near the singularity, the crucial role is played by both angular momenta \( \mathcal{L} \) and \( l \). As a result, typically either an unbounded blueshift or finite frequency shift can occur. Only in the exceptional case of radial photon and radial observer, the final outcome is an unbounded redshift in agreement with [2], [3].

Another physically relevant region is the vicinity of the horizon. Let, for simplicity, a signal is emitted by a source that is at rest at infinity. Then, there is a universal relation [42]. Meanwhile, there is no such a relation if a photon is emitted by a star surface. It was shown earlier that, if an emitter crossed the horizon in a point with the value \( V = V_1 \) of the standard Kruskal coordinate and a receiver did it in the point \( V = V_2 \),

\[
\frac{\omega_2}{\omega_1} = \frac{V_1}{V_2} \tag{54}
\]
This was shown for the Schwarzschild metric in [10] and in a more general setting in [8]. In doing so, a photon propagates exactly along the horizon, \( l = 0 = q \) (below, we enlarge these observations mentioned in [8]). Thus, in case (ii) the answer is not as universal as (42).

VIII. SIGNALS EMITTED UNDER THE HORIZON: SPECIAL CASES OF SCENARIO II

In a previous section VII we discussed quite generic situations. Meanwhile, there exist special cases that need an additional care for the analysis in the framework of scenario (ii). In particular, this includes case \( p_1 = 0 \). It does not correspond to a usual gravitational collapse of a star but, for generality, we will consider this ”exotic” example as well. However, we omit from consideration case \( p_1 < 0 \). In the picture of an external black hole it would correspond to a particle coming from the left (”mirror”) universe but such a region is absent in the case of collapse of matter.

Another special case that requires additional attention implies the following configuration. A photon is emitted in point 1 very close to the horizon. If it propagates exactly along the horizon, it will be received in point 2 also on the horizon with some finite redshift [8]. If points 1 or 2 somewhat change slightly, the frequency \( \omega_2 \) also changes slightly. However, it may happen that points 1 and 2 are separated largely, so that \( V_2 \gg V_1 \). Then, according to (54), a redshift can be made as strong as one likes. What is also important, a photon emitted very close to the horizon, in such a situation can be received in some intermediate point or even near the singularity since observer 2 can hit line \( r = 0 \) close to the upper right corner on Fig. 1.

In doing so, new interesting options can appear. Let, for example, a surface of a collapsing star radiate pure radial photons, \( l = 0 \). Assuming that an observer has \( L_2 \neq 0 \) and taking into account Table 2, we come to the conclusion that in the course of his travel, the behavior of frequency \( \omega_2 \) changes radically. There is a strong redshfit near the horizon and in an intermediate region (see below) but, as the singularity is approached, this changes to unbounded blueshift!

Below we describe corresponding behavior of frequency in more detail and derive restrictions on parameters of a photon emitted near the horizon that generalize those in [8].
A. Angular momentum of a photon emitted close to the horizon, $p_1 > 0$

If an observer crosses the horizon and emits a photon in the direction inside the horizon, $\alpha = +1$ and there is nothing special in this process. Instead, if (53) is fulfilled, there are severe restrictions. Indeed, let in point of emission 1 the metric function $g_1$ be very small, formally $g_1 \to 0$. We also assume that $p_1 \neq 0$. For any physically reasonable process with a finite frequency $\omega_1$ it is seen from (26) that in this limit

$$\sqrt{q^2 + \frac{l^2}{r_+^2}}g_1 + |q| \approx \frac{\omega_1}{p_1}g_1.$$  (55)

Bearing in mind that both terms in the left hand side are positive (or, at least, nonnegative) we see that for any finite and nonzero $\omega_1, p_1$, scenario with $l \neq 0$ is impossible since it implies different powers of $g_1$ in both sides of the equation. Eq. (55) is self-consistent, if

$$|q| \approx q_1g_1, \quad l \approx l_1\sqrt{g_1},$$  (56)

where $q_1$ and $l_1$ are some constants. Then, it is seen from (55) that these parameters should satisfy the restriction

$$\sqrt{q_1^2 + \frac{l_1^2}{r_+^2}} + q_1 = \frac{\omega_1}{p_1}.$$  (57)

Thus in the limit under discussion the necessary condition reads $l \to 0, q \to 0$. If, for simplicity, we consider the case $l_1 = 0$, we see from (56), (57) that

$$|q| \approx \frac{\omega_1g_1}{2p_1}.$$  (58)

If a photon with $\alpha = -1$ is emitted just at the moment when an observer crosses the horizon ($g_1 = 0$), $l = 0$ exactly. Then, such a photon is propagates along the horizon (see [8] for details). Meanwhile, another photon (with $\alpha = +1$) can have any $l$ and moves towards a future singularity inside a black hole.

B. Photon received in some intermediate point, $p_1 > 0$

According to explanations above, for a photon with $\alpha = -1$, emitted near the horizon, $l \approx 0$, so our photon moves almost radially. Assuming $l_1 = 0$, we can consider it to be pure radial. Then, we can apply eq. (43), so for a finite $\omega_1 \neq 0$ in the frame comoving with
observer 1 we have eq. (58). Here, the motion of the emitter can be radial or not. The quantity $g_1$ is very small since by assumption point 1 is near the horizon, so $q$ is small as well. Further, this photon is received in point 2 where $g_2 = O(1)$. Let us find its frequency $\omega_2$.

Let $L_2 \neq 0$. From (27) and (58) we have

$$\frac{\omega_2}{\omega_1} \approx \frac{g_1}{2p_1g_2} (Z_2 + \left|p_2\right|). \tag{59}$$

Now, according to (59), $\omega_2 \to 0$ since $g_1 \to 0$. To obtain the case $L_2 = 0$, we can take safely the limit $L_2 \to 0$ in (59), so

$$\frac{\omega_2}{\omega_1} \approx \frac{g_1}{2p_1g_2} (\sqrt{p_2^2 + g_2 + \left|p_2\right|}). \tag{60}$$

C. Angular momentum of a photon emitted close to the horizon, $p_1 = 0$

The situation is different, if $p_1 = 0$. Then, it follows from (34) near the horizon, where $g_1 \to 0$,

$$\sqrt{q^2 + l_1^2 g_1} \approx \left(\omega_1 + \frac{\omega_1 l_1}{r_+^2}\right) \sqrt{g_1}. \tag{61}$$

If $q = O(1)$ is separated from zero, the left hand side remains finite nonzero, so for any finite $\omega_1$, eq. (61) cannot hold. The only way out is to assume that now

$$|q| \approx q_1 \sqrt{g_1}. \tag{62}$$

Then, we obtain the constraint

$$\sqrt{q_1^2 + \frac{l_1^2}{r_+^2}} = \left(\omega_1 + \frac{\omega_1 l_1}{r_+^2}\right) \sqrt{\frac{e_+^2}{r_+^2} + 1}. \tag{63}$$

In doing so, both cases $l = 0$ or $l \neq 0$ separated from zero are allowed. If $l = 0$, the expression simplifies to

$$q_1 = \frac{\omega_1}{\sqrt{\frac{e_+^2}{r_+^2} + 1}}. \tag{64}$$

Thus there are two possible cases. If $p_1 > 0$, an observer, according to (56), can emit a photon near the horizon with a very small $l$ only. If it is emitted just at the moment
when an observer crosses the horizon, \( l = 0 \) exactly. Then, such a photon propagates along the horizon (another photon moves into the inner black hole region). This is the situation described above and in [8].

If \( p_1 = 0 \), such an observer can pass through the horizon in the bifurcation point only [13]. If, additionally, \( l \neq 0 \), a photon cannot remain on the horizon, so both photons emitted move inside. If emission occurs exactly on the horizon, \( g_1 = 0 \) there, so according to (62), \( q = 0 \) exactly.

D. Photon received in some intermediate point, \( p_1 = 0 \)

If \( l = 0 \), it follows from (27) that for \( g_1 \to 0 \)

\[
\frac{\omega_2}{\omega_1} \approx \frac{\sqrt{g_1(Z_2 + p_2)}}{\sqrt{\frac{L^2}{r_+^2} + 1\sqrt{g_2}}} 
\]

(65)

where for simplicity we assumed that \( l = 0 \) exactly.

Comparing (60) and (65) we see that the rate with which \( \omega_2 \) decreases when \( g_1 \to 0 \), is higher (proportional to \( g_1 \)) for \( p_1 > 0 \) than for \( p_1 = 0 \) (proportional to \( \sqrt{g_1} \)).

E. Absorption near the singularity, \( p_1 > 0 \)

As is explained above, for physically reasonable emission near the horizon, \( l = 0 \) or is negligible. Let now \( \mathcal{L}_2 \neq 0 \). Then, eqs. (50), (51), (58) apply,

\[
\frac{\omega_2}{\omega_1} \approx \frac{g_1}{2p_1} \frac{|\mathcal{L}_2|}{|T_2| \sqrt{g_2}}. 
\]

(66)

For the Schwarzschild case,

\[
\frac{\omega_2}{\omega_1} \approx \frac{g_1}{2p_1} \frac{|\mathcal{L}_2|}{\sqrt{|T_2|} \sqrt{r_+}} = \frac{|\mathcal{L}_2| \delta}{2p_1 r_+},
\]

(67)

\( \delta = \frac{g_1 \sqrt{r_+}}{|T_2|} \). Thus we have the play of two small quantities \( g_1 \) and \( T_2 \). If \( \delta \ll 1 \), \( \frac{\omega_2}{\omega_1} \ll 1 \) (strong redshift). When \( \delta \gg 1 \), we have a strong blueshift. When \( \delta = O(1) \), \( \omega_2 \neq 0 \) is finite. Eventually, as the singularity is approached, \( \delta \to \infty \), so blueshift prevails. Thus we see the crucial change of the frequency from a big redshift near the horizon to a strong blueshift.
near the singularity. The transition occurs for

$$|T_2| = r_2 \sim \left( \frac{g_1 |L_2|}{p_1} \right)^2 \frac{1}{r_+}. \quad (68)$$

If $L_2 = 0$, it follows from eq. (37) that

$$\frac{\omega_2}{\omega_1} \approx \frac{g_1}{2 \sqrt{g_2 p_1}} \approx \frac{g_1 \sqrt{|T_2|}}{2 \sqrt{r_+ p_1}} \to 0 \quad (69)$$
due to small $g_1$ and big $g_2$ (small $|T_2|$).

**F. Absorption near the singularity, $p_1 = 0$**

If $l = 0$, $L_2 \neq 0$, eqs. (51) and (64) gives us

$$\frac{\omega_2}{\omega_1} \approx \frac{\sqrt{g_1 |L_2| \sqrt{r_+}}}{\sqrt{|T_2|} \sqrt{L_2^2 + r_+^2}}. \quad (70)$$

This shows that an infinite blueshift occurs in the limit $T_2 \to 0$. If $l = 0$, $L_2 = 0$, eq. (52) is valid. Taking also into account (62), (64) we find

$$\frac{\omega_2}{\omega_1} \approx \frac{\sqrt{g_1}}{\sqrt{g_2} \sqrt{L_2^2 + r_+^2} + 1} \approx \frac{\sqrt{g_1 \sqrt{|T_2|}}}{\sqrt{L_2^2 + r_+^2}}. \quad (71)$$

Eventually, the frequency $\omega_2 \to 0$, when $g_2 \to \infty$.

Now, let $l \neq 0$. Then, if $L_2 = 0$, eq. (49) applies, so independently of $p_2$, there is an infinite blueshift here. If $L_2 \neq 0$, we have eqs. (47) with finite $\omega_2$ or (48) with an infinite blueshift depending on the sign of $L l$.

It is convenient to collect the results in Table 4, where only dependence on coordinate $T_2 \to 0$ is shown.

| $L_2$ | $p_1 > 0$ | $p_1 = 0$ |
|-------|-----------|-----------|
| $l = 0, L_2 \neq 0$ | $\frac{1}{\sqrt{|T_2|}}$ | $\frac{1}{|T_2|}$ |
| $l = 0, L_2 = 0$ | $\sqrt{|T_2|}$ | $\sqrt{|T_2|}$ |
| $l \neq 0, L_2 l > 0$ | finite |
| $l \neq 0, L_2 l < 0$ | $\frac{1}{T_2^2}$ |
| $l \neq 0, L_2 = 0$ | $\frac{1}{T_2^2}$ |

Table 3. Behavior of $\frac{\omega_2}{\omega_1}$ for the cases when a photon is emitted near the horizon and is absorbed near the singularity.
Thus we see that there is no qualitative difference between cases with $p_1 = 0$ and $p_1 \neq 0$, if $l = 0$. If $p_1 > 0$, the case $l \neq 0$ cannot be realized for emission near the horizon (corresponding cells in Table 3 are left empty), as is explained above. Meanwhile, it is quite possible for $p_1 = 0$.

**IX. COMPARISON OF THE TWO MAIN SCENARIOS**

Now, having revealed the main features of scenarios i and ii, it is instructive to compare them. Let us take, for definiteness, the particular case for which $\mathcal{L}_2 \neq 0$ and an observer receives radially propagating signals, $l = 0$. The results of comparison based on previous sections, are collected in Table 4. Here, as before, by ”typical” we mean that points 1 and 2 lie somewhere on the horizon, with $V_1$ and $V_2$ having the same order. Scenario ii is called special, if $\frac{V_2}{V_1} \gg 1$.

| Scenario            | Vicinity of horizon | Intermediate          | Singularity               |
|---------------------|---------------------|-----------------------|----------------------------|
| i, ii (typical)     | Finite              | Finite                | Unbounded blueshift       |
| ii (special)        | unbounded redshift  | unbounded redshift    | Unbounded blueshift       |

Table 4. Properties of $\frac{\omega_2}{\omega_1}$ for different types of scenario with $l = 0$, $\mathcal{L}_2 \neq 0$.

Thus we see that there is no qualitative difference between scenarios i and ii (typical). But there is an essential difference between subcases ii (typical) and ii (special). The combination $\mathcal{L}_2 \neq 0$, $l = 0$ is chosen because the difference under discussion is now especially pronounced. In this sense, this case is especially interesting.

**X. SUMMARY AND CONCLUSIONS**

Thus the results for absorption of light near the singularity are qualitatively different in the pure radial and nonradial cases. And, the change of frequency during a fall of an observer can unfold in different ways for typical and special scenarios. We hope that account for nonradial motion carried out in the present paper will be useful further in investigation of the view of an infalling observer. This should include not only the description of signals from a remote world outside the horizon but also view of a close vicinity of an observer himself.

A separate question is the properties of light absorbed near the event horizon in the
context of the instability issue. We saw that, according to Table 1, there are cases when (almost) infinite blueshift occurs. If $p = 0$, this is in agreement with the results of [9]. If $pq < 0$ with an arbitrary $q < 0$, this corresponds to a photon coming to the inner black hole region from the "mirror" universe that is absent in the realistic collapse. A similar effect occurs near the white hole horizon [12] when a particle emerging from the white hole horizon collides with another one moving towards a black hole one. A question arises, whether this can lead to some new instabilities for the completely extended space-time?

The corresponding conditions of infinite blueshift near the horizon inside require special fine-tuning (almost zero momentum and trajectories near the bifurcation point [13]), so it seems that this constitutes a zero measure set and does not change an overall picture radically. In a sense, the same applies to rotating black holes where the BSW effect needs one of colliding particles to be fine-tuned [11]. For white holes, this is not necessary but their instability is already known (see, e.g. [5] and literature therein). However, this issue remains highly nontrivial and requires further investigations in connection with similar effect near inner horizons.

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[1] A. J. S. Hamilton, Inside astronomically realistic black holes, arXiv:1907.05292.
[2] L. E. Gurevich and E. B. Gliner, General relativity after Einstein. Moscow, 1972 (In Russian).
[3] A. J. S. Hamilton and G. Polhemus, Stereoscopic visualization in curved spacetime: seeing deep inside a black hole, New J. Phys. 12, 123027 (2010), arXiv:1012.4043.
[4] I. D. Novikov, Note on the space-time metric inside the Schwarzschild singular sphere, Sov. Astr. - AJ 5, 423 (1961), (Astron. Zh. 38, 564 (1961)).
[5] V. P. Frolov and I. D. Novikov, Physics of Black Holes (Kluwer Academic, Dordrecht, 1998).
[6] R. Doran, F. S. N. Lobo, P. Crawford, Interior of a Schwarzschild black hole revisited, Found. Phys. 38, 160 (2008), arXiv:gr-qc/0609042.
[7] O. B. Zaslavskii, Acceleration of particles by black holes: Kinematic explanation, Phys. Rev. D 84, 024007 (2011) [arXiv:1104.4802].

[8] A. V. Toporensky, O. B. Zaslavskii, Redshift of a photon emitted along the black hole horizon, Eur. Phys. J. C 77, 179 (2017), [arXiv:1611.09807].

[9] A. V. Toporensky, O. B. Zaslavskii, Zero-momentum trajectories inside a black hole and high energy particle collisions, Journal Cosmol. Astropart. Physics 12 (2019) 063, [arXiv:1808.05254].

[10] K. Kassner, Why ghosts don’t touch: a tale of two adventurers falling one after another into a black hole. Eur. J. Phys. 38, 015605 (2017), [arXiv:1608.07511].

[11] M. Bañados, J. Silk and S.M. West, Kerr Black Holes as Particle Accelerators to Arbitrarily High Energy, Phys. Rev. Lett. 103, 111102 (2009) [arXiv:0909.0169].

[12] A. Grib and Yu. V. Pavlov, Are black holes totally black? Gravitation Cosmol. 21, 13 (2015), [arXiv:1410.5736].

[13] O. B. Zaslavskii, Acceleration of particles near the inner black hole horizon, Phys. Rev. D 85, 024029 (2012) [arXiv:1110.5838].