1. Introduction

The Galilei group is the spacetime symmetry of non-relativistic systems. It is the low velocity limit of the Poincaré group, which is the true spacetime symmetry of relativistic systems. We are all very familiar with the success of relativistic field theories in describing the interactions of fundamental particles. What is the point then of studying Galilei invariant field theories?

First of all, some Galilean field theories can be understood as non-relativistic limits of relativistic field theories. As such, they offer a simpler setting for studying generic field-theoretic concepts, such as renormalization and renormalization group, anomalies, and solitons. Second, Galilean field theories are a second-quantized description of quantum mechanics, and are therefore useful for addressing many body problems in non-relativistic (condensed matter) systems. A second-quantized description often gives insight into an otherwise intractable problem. An example of this is the Aharonov-Bohm scattering problem. The exact solution is well known and well behaved, but perturbation theory runs into problems. Second quantization shows that the source of the discrepancy is a conformal anomaly arising from the perturbative divergence of the $1/r^2$ problem, which must be canceled by the inclusion of a contact interaction of critical strength.

Finally, the most surprising application of Galilei invariant field theories is to relativistic strings. The spacetime symmetry of the $D$-dimensional relativistic string in light-cone gauge is the transverse $D - 2$-dimensional Galilei group. Only when $D = 26$ does one regain $D$-dimensional Poincaré invariance. The light-cone string can be understood as a composite object, whose constituent "bits" bind to form a closed chain. The dynamics of these string-bits are governed by a Galilei invariant
Supersymmetric extensions of the Galilei group were first suggested by Puzalowski in 3+1 dimensions. The minimal superalgebra is called $S_1 G$, and introduces a single supercharge $Q$, with $Q^2 = M$. There is also an extended superalgebra, called $S_2 G$, that introduces another supercharge $R$, with $R^2 = H$. Puzalowski constructed field theoretic representations of the minimal superalgebra $S_1 G$, which can be understood as the non-relativistic limit of an $\mathcal{N} = 1$ Super-Poincaré algebra. We are more interested in the extended superalgebra $S_2 G$, which can be understood either as the non-relativistic limit of an $\mathcal{N} = 2$ Super-Poincaré algebra, or as the light-cone subalgebra of a one higher dimension $\mathcal{N} = 1$ Super-Poincaré algebra. The first point of view is relevant to low-energy physics, and the second point of view is relevant to superstring-bits.

Section 2 reviews the two-dimensional Super-Galilei algebras. Section 3 reviews the construction of Super-Galilei invariant field theories, and in section 4 we derive Non-relativistic Super-Chern-Simons theory as a special case. In section 5 we construct Super-Galilei matrix field theories, and discuss their implication on a composite formulation of superstring theory. Section 6 is devoted to a discussion of the results and future directions.

2. The $d = 2$ Super-Galilei Algebra

To start, let us review the Galilei algebra in $d = 2$ space dimensions. The generators consist of the momentum vector $P_i$, the boost vector $K_i$, the angular momentum pseudo-scalar $J$, the Hamiltonian $H$ and the number operator $N$. The non-trivial part of the algebra is given by:

$$[P_i, K_j] = i\delta_{ij} mN, \quad [P_i, J] = -i\epsilon_{ij} P_j, \quad [H, K_i] = i P_i, \quad [K_i, J] = -i\epsilon_{ij} K_j. \quad (2.1)$$

One usually thinks of this algebra as the non-relativistic (N-R) limit ($c \to \infty$) of the $D = 3$ Poincaré algebra:

$$[P^\rho, M^{\mu\nu}] = i \left( \eta^{\nu\rho} P^\mu - \eta^{\mu\rho} P^\nu \right) \quad (2.2)$$

In the N-R limit the Poincaré generators become:

$$P^0 \to mN + H \quad M^{0i} \to K^i \quad M^{12} \to J, \quad (2.3)$$

which in turn satisfy the Galilei algebra (2.1). Alternatively, eq. (2.1) can be interpreted as the light-cone subalgebra of the $D = 4$ Poincaré algebra, in which

$$mN = P^0 + P^3, \quad H = (P^0 - P^3)/2, \quad K^i = M^{0i} + M^{3i}, \quad J = M^{12}. \quad (2.4)$$

Lower case $d$ will denote space dimensions only, and upper case $D$ will denote spacetime dimensions.
Minimal supersymmetric extension of the Galilei group is achieved by introducing a complex supercharge $Q$ satisfying:

$$[Q, J] = Q/2, \quad \{Q, Q^\dagger\} = mN.$$  \hfill (2.5)

The resulting superalgebra is called $S_1G$. It can be understood as the non-relativistic limit of the $D = 3, \mathcal{N} = 1$ Super-Poincaré algebra:

$$[Q_a, M^\mu] = \frac{1}{2} \gamma^\mu_{ab} Q_b, \quad \{Q_a, Q_b\} = (\gamma^\mu \gamma^0)_{ab} P_\mu, \hfill (2.6)$$

where $Q_a$ is a real two-component spinor, and $\gamma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. The non-relativistic limit gives $(Q_1 + iQ_2)/\sqrt{2} \to Q$.

We extend the superalgebra to $S_2G$ by introducing another supercharge $R$, satisfying:

$$[R, J] = -R/2, \quad [R, K^-] = iQ, \quad \{R, Q^\dagger\} = -P^\dagger/2, \quad \{R, R^\dagger\} = H/2,$$  \hfill (2.7)

with all other (anti-)commutators vanishing.\footnote{We use the notation $V^\pm = V_1 \pm iV_2$.} The extended superalgebra can arise as the non-relativistic limit of the $D = 3, \mathcal{N} = 2$ Super-Poincaré algebra, with a non-trivial central charge. The relevant part of the Super-Poincaré algebra takes the form:

$$[Q_a, M^\mu] = \frac{1}{2} \gamma^\mu_{ab} Q_b, \quad \{Q_a, Q_b^\dagger\} = \gamma^\mu_{ab} P_\mu - \delta_{ab} T,$$  \hfill (2.8)

where $Q_a$ is a complex two-component spinor. In the basis in which $\gamma^0 = \sigma_3$ the non-relativistic limit gives $Q_1/\sqrt{2} \to Q$ and $Q_2/\sqrt{2} \to R$, provided that the N-R limit of the central charge is $T \to -mN$. Alternatively, $S_2G$ can be interpreted as the light-cone subalgebra of the $D = 4, \mathcal{N} = 1$ Super-Poincaré algebra. The relevant part of the superalgebra is given by

$$[Q_a, M^\mu] = \frac{1}{2} \sigma^\mu_{ab} Q_b, \quad \{Q_a, Q_b^\dagger\} = \sigma^\mu_{ab} P_\mu,$$  \hfill (2.9)

where $Q_a$ is a complex two-component Weyl spinor, and $\sigma^{\mu\nu} = i(\sigma^\mu \sigma^\nu - \sigma^\nu \sigma^\mu)/2$. The Galilean supercharges are then given by $Q = \pm Q_1$ and $R = \mp Q_2/\sqrt{2}$.

### 3. $S_2G$-Invariant Field Theory

Introduce a complex field $\phi(x)$, and a complex one-component Grassmann field $\psi(x)$, satisfying canonical commutation and anti-commutation relations respectively:

$$[\phi(x), \phi^\dagger(y)] = \{\psi(x), \psi^\dagger(y)\} = \delta(x - y),$$  \hfill (3.1)

The free field representation of the supercharges is given by:

$$Q = -i\sqrt{m} \int dx \psi^\dagger(x) \phi(x), \quad R_0 = \frac{1}{2\sqrt{m}} \int dx \psi^\dagger(x) \theta^+ \phi(x),$$  \hfill (3.2)
resulting in the free Hamiltonian

\[ H_0 = 2 \{ R_0, R_0^\dagger \} = \frac{1}{2m} \int dx \left[ |\nabla \phi(x)|^2 + |\nabla \psi(x)|^2 \right]. \]  \hspace{1cm} (3.3)

Since the Hamiltonian of any \( S_2G \)-invariant field theory is the square of a supercharge \( R \), we can construct interacting \( S_2G \)-invariant field theories by adding an interaction term to \( R_0 \), so that \( R = R_0 + R' \). If we restrict to quartic operators, the \( S_2G \) algebra implies that \( R' \) must be of the form

\[ R' = \int dx \, dy \, W^+(y - x) \psi^\dagger(x) \rho(y) \phi(x), \]  \hspace{1cm} (3.4)

where \( \rho = \phi^\dagger \phi + \psi^\dagger \psi \), and the superpotential \( W^+ \) is given by

\[ W^+(x) = (\partial^1 + i\partial^2) \mathcal{F}(|x|) = \partial^+ \mathcal{F}(|x|), \]  \hspace{1cm} (3.5)

where \( \mathcal{F} \) is complex in general. The total supercharge can now be written concisely as

\[ R = \frac{1}{2\sqrt{m}} \int dx \psi^\dagger(x) D^+ \phi(x), \]  \hspace{1cm} (3.6)

where

\[ D^+ = \partial^+ - i \int dy W^+(y - x) \rho(y). \]  \hspace{1cm} (3.7)

The Hamiltonian is obtained by squaring \( R \), and is given by

\[ H = \frac{1}{2m} \int dx \left[ |D^+ \phi(x)|^2 + |D^+ \psi(x)|^2 \right] \\
+ \frac{1}{m} \int dx \, dy \, \nabla y \times W(y - x) \left[ \psi^\dagger(x) \phi^\dagger(y) \psi(y) \phi(x) - \psi^\dagger(x) \rho(y) \psi(x) \right]. \]  \hspace{1cm} (3.8)

Note that for

\[ W(x) = \nabla \ln |x| \]  \hspace{1cm} (3.9)

we get \( \nabla \times W(x) = -2\pi \delta(x) \), and the Hamiltonian reduces to the self-dual form

\[ H_{SD} = \frac{1}{2m} \int dx \left[ |D^+ \phi(x)|^2 + |D^+ \psi(x)|^2 \right]. \]  \hspace{1cm} (3.10)

Alternatively, the Hamiltonian can be expressed in terms of the vector

\[ D = \nabla - i \int dy W(y - x) \rho(y) \]  \hspace{1cm} (3.11)

as

\[ H = \frac{1}{2m} \int dx \left[ |D \phi|^2 + |D \psi|^2 \right] + \frac{1}{2m} \int dx \, dy \, \nabla y \times W(y - x) \\
\times \left[ : (|\phi(x)|^2 - |\psi(x)|^2) \rho(y) : + 2\psi^\dagger(x) \phi^\dagger(y) \psi(y) \phi(x) \right]. \]  \hspace{1cm} (3.12)
Note that this Hamiltonian possesses a local symmetry given by

\[
W \rightarrow W + \nabla f
\]

\[
\phi, \psi(x) \rightarrow \phi, \psi(x) \exp \left[ i \int dy f(y - x) \rho(y) \right],
\]

which suggests identifying \( \mathcal{D} \) with a covariant derivative in a background abelian gauge field given by

\[
A(x) = \int dy W(y - x) \rho(y).
\]

The above symmetry is then simply the gauge invariance

\[
A \rightarrow A + \nabla \Lambda,
\]

with \( \Lambda = \int dy f(y - x) \rho(y) \), and the Hamiltonian in (3.8) or (3.12) is interpreted as minimal coupling to the gauge field plus non-minimal couplings. We will see that for a specific choice (up to the local symmetry in (3.13)) of the superpotential \( W \) this theory reduces to a Super-Galilean gauge theory.

4. N-R Super-Chern-Simons Theory

Gauge theories in odd spacetime dimensions with the so called Chern-Simons kinetic term are topological, and therefore invariant under all coordinate transformations, including both the Poincaré group and the Galilei group. When coupled to non-relativistic matter only the Galilei group survives, and one ends up with a non-relativistic (Galilei invariant) gauge theory.

The action for 2+1-dimensional Chern-Simons theory coupled to non-relativistic matter is given by

\[
S_{CS} = \int d^3 x \left[ \kappa \partial_t A \times A - \kappa A^0 B + \phi \left( i D_t + \frac{D^2}{2m} \right) \phi + \psi \left( i D_t + \frac{D^2}{2m} \right) \psi 
- \frac{e}{2m} B |\psi|^2 + \lambda_1 |\phi|^4 + \lambda_2 |\phi|^2 |\psi|^2 \right].
\]

The Pauli interaction has been included explicitly, as well as the two possible non-minimal quartic interactions with coefficients \( \lambda_1, \lambda_2 \). Higher power terms have been suppressed.

In addition to Galilei invariance, this action also possesses an SO(2, 1) conformal symmetry, which is however broken quantum mechanically. For the following values of the coupling constants

\[
\lambda_1 = -\frac{1}{2m\kappa}, \quad \lambda_2 = 3\lambda_1
\]

the theory is supersymmetric (under \( S_2G \)) and the conformal anomaly vanishes. This is N-R Super-Chern-Simons theory. The Hamiltonian of this theory is easily derived from the action (4.1) at the critical point (4.2),

\[
H_{SCS} = \frac{1}{2m} \int dx \left[ |D\phi|^2 + |D\psi|^2 + \frac{1}{\kappa} |\phi|^4 + \frac{2}{\kappa} |\phi|^2 |\psi|^2 \right].
\]
The gauge field is completely determined by Gauss’ law, and is given in Coulomb gauge by
\[ A(x) = -\frac{1}{\kappa} \int dy \left[ \nabla_y \times \ln |y - x| \right] \rho(y) . \] (4.4)
Consequently we see that our Hamiltonian (3.12) reduces to \( H_{\text{SCS}} \) when \( W(x) = -(1/\kappa) \nabla \times \ln |x| \). As a bonus we see that since this is precisely the superpotential for which \( H \) was self-dual, we have
\[ H_{\text{SCS}} = H_{\text{SD}} , \] (4.5)
in agreement with Leblanc et. al.\[11\] All this suggests an interesting connection among \( d = 2 \) Galilean supersymmetry, self-duality and \( SO(2,1) \) conformal symmetry, which may or may not go beyond this Chern-Simons example.

5. Matrix Fields
A composite formulation of superstring theory was mentioned as one of the motivations for studying Super-Galilei invariant field theories. The main idea of superstring-bit models is to replace the free string by a chain of particles with dynamics given by nearest-neighbor interactions,
\[ h = \sum_{k=1}^{N} \left[ \frac{p_{k}^2}{2m} + V(x_{k+1} - x_{k}) \right] . \] (5.1)
Second-quantization of this system requires the use of matrix-valued fields. We are led to a Super-Galilei invariant \( N_c \times N_c \) matrix field theory, whose
- \( N_c \rightarrow \infty \) limit yields the nearest-neighbor interactions, which become the free string in the continuum limit.
- \( 1/N_c \) expansion produces the chain splitting and joining processes which become the string interactions in the continuum limit.
The fields transform in the adjoint representation of a global \( U(N_c) \) ”color” group, and satisfy canonical (anti-)commutation relations,
\[ [\phi(x)_{\alpha}^{\beta}, \phi^\dagger(y)_{\gamma}^{\delta}] = \{ \psi(x)_{\alpha}^{\beta}, \psi^\dagger(y)_{\gamma}^{\delta} \} = \delta(x - y) \delta_{\alpha}^{\delta} \delta_{\gamma}^{\beta} . \] (5.2)

5.1. The Supercharges
The generators of \( S_2G \) are singlets, and are given by traces of products of matrix fields. The one-body (free) operators can be obtained from their single-component field-theoretic counterparts (5.2) by elevating the fields to matrices and tracing, e.g.
\[ R_0 = \frac{1}{2\sqrt{m}} \int dx \text{Tr} \left[ \psi^\dagger(x) \partial^+ \phi(x) \right] . \] (5.3)
For the two body operators \((R', H)\) this procedure is ambiguous, since different (non-cyclic) matrix orderings yield different traces. In addition, one is not guaranteed that a particular ordering will satisfy the superalgebra. Consider the two possibilities for ordering the matrix fields in \(R'\):

\[
R'_1 = \int dx dy \, W^+(y - x) : \mathrm{Tr} \left[ \psi^\dagger(x) \rho(y) \phi(x) \right] :
\]

\[
R'_2 = \int dx dy \, W^+(y - x) : \mathrm{Tr} \left[ \psi^\dagger(x) \phi(x) \rho(y) \right] :
\]

(5.4)

These have the same (normal) ordering of operators, but different color correlations. The corresponding supercharges are given by \(R_1 = R_0 + R'_1\) and \(R_2 = R_0 + R'_2\), neither of which satisfy the superalgebra, since:

\[
\{R_1, Q^\dagger\} \neq -P^+/2 \quad \text{for all } W^+(x) \neq 0 \quad \text{and} \quad \{R_2, R_2\} \neq 0.
\]

(5.5)

However, the second anti-commutator involves the factor

\[
W^+(x - z)W^+(z - y) + W^+(x - y)W^+(y - z) + W^+(z - x)W^+(x - y),
\]

(5.6)

which vanishes for \(W^+(x) = \alpha \, \partial^+ \ln |x|\), or in vector notation:

\[
W(x) = \alpha \, \nabla \ln |x| + \alpha_2 \nabla \times \ln |x| \quad \text{where} \quad \alpha = \alpha_1 - i \alpha_2.
\]

(5.7)

With this superpotential, \(R_2\) satisfies the \(\mathcal{S}_2 \mathcal{G}\) algebra, and the field theory defined by \(H = 2 \{R_2, R'_2\}\) is \(\mathcal{S}_2 \mathcal{G}\)-invariant.

### 5.2. Large \(N_c\) and Closed Chains

The Fock space contains states transforming under various irreducible representations of \(U(N_c)\), but we are mostly interested in the singlet closed-chain states:

\[
|\Psi\rangle = \prod_{k=1}^N (dx_k d\theta_k) \mathrm{Tr} \left[ \Phi^\dagger(x_1, \theta_1) \cdots \Phi^\dagger(x_N, \theta_N) \right] |0\rangle \Psi(x_1, \theta_1, \cdots, x_N, \theta_N),
\]

(5.8)

where \(\Phi^\dagger(x, \theta) = \phi^\dagger(x) + \psi^\dagger(x) \theta\), and \(\Psi\) is a many-body wavefunction. The action of two-body operators on this state will give rise to terms with two traces, corresponding to splitting the chain into two chains. Operators with color-correlated annihilation operators will also produce single trace terms, preserving the integrity of the chain. These will be enhanced by a factor of \(N_c\) relative to the chain-splitting terms. To see how this works in a simple setting consider a matrix field \(a(x)\). The kinds of two-body operators we are dealing with are:

\[
\Omega_1 = \frac{1}{N_c} \int dx dy V(y - x) \mathrm{Tr} \left[ a^\dagger(x) a^\dagger(y) a(y) a(x) \right]
\]

\[
\Omega_2 = \frac{1}{N_c} \int dx dy V(y - x) : \mathrm{Tr} \left[ a^\dagger(x) a(y) a^\dagger(y) a(x) \right] :
\]

(5.9)

In \(\Omega_1\) the annihilation operators are correlated, and in \(\Omega_2\) they are not. Acting on a single \(N\)-particle chain \(|\psi\rangle = \mathrm{Tr} \left[ a^\dagger(x_1) \cdots a^\dagger(x_N) \right] |0\rangle\), gives after one contraction
\[ \Omega_1 |\psi\rangle = \frac{1}{N_c} \int dy \sum_k V(y - x_k) \times \]
\[ \times \text{Tr} \left[ a_1^\dagger(x_1) a_1^\dagger(y) a(y) a_1^\dagger(x_{k+1}) \cdots a_1^\dagger(x_N) a_1^\dagger(x_1) \cdots a_1^\dagger(x_{k-1}) \right] |0\rangle \]
\[ \Omega_2 |\psi\rangle = \frac{1}{N_c} \int dy \sum_k V(y - x_k) \times \]
\[ \times \text{Tr} \left[ a_1^\dagger(x_k) : a(y) a_1^\dagger(y) : a_1^\dagger(x_{k+1}) \cdots a_1^\dagger(x_N) a_1^\dagger(x_1) \cdots a_1^\dagger(x_{k-1}) \right] |0\rangle . \quad (5.10) \]

The last contraction of \( a(y) \) with one of the creation operators to its right will almost always break the trace into two traces. The one exception is when \( a(y) \) contracts with \( a_1^\dagger(x_{k+1}) \) in the action of \( \Omega_1 \), giving the original trace with a factor of \( N_c \) coming from the trace of the identity. In the limit \( N_c \to \infty \) we therefore get

\[ \Omega_1 |\psi\rangle = \sum_{k=1}^N V(x_{k+1} - x_k) |\psi\rangle , \quad \Omega_2 |\psi\rangle = 0 , \quad (5.11) \]

exhibiting nearest-neighbor interactions for correlated annihilation operators.

The supercharge \( R_2 \) (and thus \( H \)) will not give rise to nearest-neighbor interactions, since its annihilation matrices are uncorrelated. On the other hand \( R_1 \) will, but it failed to satisfy the superalgebra. Both properties are required to build supersymmetric chains. It turns out that the combination\(^c\)

\[ R' = \int W^+(y-x) : \text{Tr} \left[ \left( [\phi_1^\dagger(y), \phi(y)] + \{\psi_1^\dagger(y), \psi(y)\} \right) \left[ \psi_1^\dagger(x), \phi(x) \right] \right] : , \quad (5.12) \]

does precisely this. Again we find that the superalgebra requires the superpotential (5.7). Note that this superpotential is the same as the one that gave N-R Super-Chern-Simons theory in the previous section, up to a local symmetry transformation (3.13). This seems to imply that the Super-Galilean matrix field theory is related to non-abelian N-R Super-Chern-Simons theory. This issue is under current investigation.\(^d\)

### 5.3. Chain Quantum Mechanics

In the limit \( N_c \to \infty \) the supercharge \( R = R_0 + R' \) preserves the integrity of the chain, and gives rise to a first-quantized Hamiltonian given by

\[ h = \frac{1}{2m} \sum_{k=1}^N \left\{ -\nabla_k^2 + 2iW_{k,k+1} \cdot (\nabla_k - \nabla_{k+1}) + 2i\partial_{\theta_k} W_{k,k+1}^+ \right. \]
\[ - i \left[ \partial_{\theta_{k+1}} W_{k,k+1}^- - \partial_{\theta_k} W_{k,k+1}^+ \right] (\theta_{k+1} - \theta_k) \left( \frac{\partial}{\partial \theta_{k+1}} - \frac{\partial}{\partial \theta_k} \right) \]
\[ + 2 |W_{k,k+1}|^2 - W_{k-1,k}^- W_{k,k+1}^- - W_{k,k+1}^+ W_{k-1,k}^- \left\} , \quad (5.13) \]

\(^c\)The (anti-)commutators apply only to matrix ordering.
where the shorthand notation \( W_{k,k+1} \equiv W(x_k - x_{k+1}) \) was used. We wish to determine whether chains really form as bound states in this theory. Since \( \alpha \) is a dimensionless (complex) parameter, this system is classically scale invariant, and it seems unlikely that any finite energy bound state should form unless an anomaly appears. We will show that there is no anomaly, and scale invariance is exact. This has the unfortunate effect of ruling out this theory as a superstring-bit model.

Take a closer look at the two-body sector, namely what would be a single link in the chain. The two-body wavefunction has four components:

\[
  u_1(x_1, x_2) + (\theta_1 + \theta_2)u_2(x_1, x_2) + (\theta_1 - \theta_2)u_3(x_1, x_2) + \theta_1 \theta_2 u_4(x_1, x_2), \quad (5.14)
\]

and the relative coordinate space Hamiltonian is given by

\[
  h_2 = -\frac{\nabla^2}{m} + \frac{1}{m^2} \left[ 2i\alpha_1 \mathbf{x} \cdot \nabla - 2i\alpha_2 \mathbf{x} \times \nabla + |\alpha|^2 \right] + \frac{2\pi i}{m} (\alpha_1 \pm i\alpha_2) \delta^{(2)}(x). \quad (5.15)
\]

The upper and lower signs in the coefficient of the \( \delta \)-function hold for \( u_1, u_2 \) and \( u_3, u_4 \), respectively. \( \alpha_1 \) can be eliminated by a phase redefinition

\[
  u_n(x) = x^{-i\alpha_1} \tilde{u}_n(x), \quad (5.16)
\]

corresponding to a "gauge" transformation \( (3.13) \), with \( f(x) = \alpha_1 \ln |x| \). This leaves precisely the two-body Hamiltonian arising from the self-dual NR Matter-Chern-Simons theory, which is known to be scale invariant to all orders in perturbation theory. There can thus be no finite energy bound state. It has in fact been shown that the bound state energy vanishes in this case. A very simple argument can be made utilizing supersymmetry. Since \( h = \{ r, r^\dagger \} \), the Hamiltonian is a positive definite operator, which forbids a negative energy bound state. A positive energy bound state is likewise forbidden since the potential vanishes at infinity. Therefore the bound state energy must vanish.

6. Discussion

Since the Galilei group (and its extension \( S_2G \)) can be understood both as a NR limit and as a light-cone subgroup of Poincaré groups, it is relevant in the description of both non-relativistic (condensed matter) systems, and relativistic systems in an infinite momentum (light-cone) frame. We have stressed the latter point of view, as it applies to reformulating string theory in terms of constituents.

Quartic 2 + 1-dimensional \( S_2G \)-invariant field theories are defined by a vector superpotential given by

\[
  W(x) = \nabla f(|x|) + \nabla \times g(|x|), \quad (6.1)
\]

\( ^d \) The above argument is known to fail for certain singular superpotentials, which develop negative energy bound states. It seems unlikely however that this happens here, since the Hamiltonian \( (5.13) \) with \( \alpha_1 = 0 \) is equivalent to the Aharonov-Bohm problem, for which the exact solution is scale invariant.
where \( f(|x|) \) makes no contribution to the dynamics. N-R Super-Chern-Simons theory emerges as a special case when \( g(|x|) = \ln |x| \). If the fields are matrices, \( S_2G \)-invariance requires

\[
W(x) = \alpha_1 \nabla \ln |x| + \alpha_2 \nabla \times \ln |x| , 
\]

(6.2)
suggesting an interpretation in terms of non-abelian N-R Super-Chern-Simons theory. This in turn implies an exact conformal symmetry, precluding a finite energy bound state. If the particles do not bind into chains, the continuum limit cannot be a string theory. It is still possible that the chains are zero-energy bound states, but then it isn’t clear how the string tension will arise. To achieve finite energy bound chains in an \( S_2G \)-invariant matrix theory, it may be necessary to beyond quartic terms in the supercharge.

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