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A tour through $N = 2$ strings.

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Abstract

I give an overview of open, closed and heterotic $N = 2$ strings. At the tree level I derive the effective field theories of all the strings, and discuss the group theory of the $N = 2$ open string and the interaction between its open and closed sectors. The two-dimensional effective field theory of the open $N = 2$ string is a sigma model, while the four-dimensional theory gives self-dual Yang-Mills (SDYM) in a self-dual gravity (SDG) background. The theory can have any gauge group, unlike the usual Chan-Paton ansatz. The four-dimensional closed string gives SDG, and the heterotic string is related to SDYM. At one loop $N = 2$ string loop amplitudes and partition functions have incurable infra-red divergences, and show puzzling disagreements on the dimension of spacetime when compared to their effective field theories. I show that the known closed-string three-point amplitude can be written directly in terms of a Schwinger parameter, so explicitly exhibiting the inconsistency. I finally discuss the possibility that the puzzles posed by the loop amplitudes could be solved if the $N = 2$ theories were Lorentz invariant and supersymmetric, and I speculate on possible modifications of the string calculations.

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1. The basics of $N = 2$ strings

1.1. Introduction

I would like to thank the organizers of this conference for giving me the opportunity to talk about $N = 2$ strings in Rome. It would be a pleasure to give any talk here, but it is particularly appropriate to talk about $N = 2$ strings, since they were first studied in the famous papers by Ademollo et al., and so have 11 Italian (and 2 foreign) fathers. Now, although $N = 2$ strings naturally live in four dimensions, it is a four-dimensional spacetime with signature $(2, 2)$ or $(4, 0)$. Thus, unlike the subjects of the first two talks here, $N = 2$ strings can not be directly relevant to nature. So, before jumping into the physics, I would like to give you some reasons for talking to you about them, aside from the aptness of the location.

First, there are interesting mathematical aspects to these theories. As we shall see, the field theories describing these strings are self-dual gravity (SDG) for the closed string and self-dual Yang-Mills (SDYM) for the open and heterotic strings. SDYM in Euclidean space is of course related to instantons, which have a whole host of interesting physical and mathematical properties. In addition, the dimensional reductions of SDYM to two dimensions gives rise to many integrable systems, and it has even been conjectured that all two-dimensional integrable systems are reductions of SDYM. $N = 2$ strings thus may give us some notion of what a quantum theory of instantons should be.

What is more interesting from my point of view, and perhaps also from the point of view of the fathers of the string, are the “stringy” reasons for examining these theories. Although $N = 2$ strings are unrealistic, studying them may provide us with better insight into string theory in general because, in some ways, they are remarkably simple theories. An example of this will be our ability to directly compare the loop amplitudes of the $N = 2$ strings to those of their effective field theories. Since $N = 2$ strings do not have Regge trajectories, and contain only massless particles, they give rise to field theories containing only a finite number of fields. In this way they are similar to the $c = 1$ theories, but are even more simple since, being Lorentz invariant, they can not contain any discrete states. Also, $N = 2$ amplitudes are local and many of them vanish, possibly indicating some kind of topological structure of the theories. It will turn out that, despite their simplicity, our understanding of $N = 2$ theories is still immature with many puzzles remaining. I would like to view this as a challenge, rather than a problem, and I hope that other people will be encouraged to enter into the subject.

The original work that I will present in this talk consists of the reconsideration of the $N = 2$ open string in $2\mathbb{C}$ dimensions. However, instead of concentrating on this particular case, I shall rather give a general tour through various $N = 2$ strings, telling you their status and pointing out their problems. This means that I shall be discussing the work of many other people, with additional commentary of my own in various places, and I shall of course be biased to those aspects of the theories that
are particularly interesting to me. For example, I shall not discuss non-critical $N = 2$ strings at all, nor the $N = 2$ strings with background charges\(^8\). Also I shall not go into too much detail on aspects of any particular theory; you can find more details in the various original references.

The outline of my talk is the follows: I first give a basic review of $N = 2$ strings. Then I calculate various tree-level string amplitudes, to find the effective field theory actions describing the strings. I then present some one-loop amplitudes to illustrate the problems intrinsic in the amplitudes, and in comparing the amplitudes to those of the effective field theory. I conclude with a discussion of the status of the $N = 2$ strings, and of possible solutions to their various problems.

1.2. $N = 2$ supergravity in two dimensions

At this stage, I should explain to those who do not know that the “$N$” of $N = 2$ refers to the number of local world-sheet supersymmetries of the string. Thus the $N = 0$ string is the bosonic string, which is basically a theory of matter coupled to 2-dimensional world-sheet gravity, and the $N = 1$ string is a theory of supermatter coupled to world-sheet $N = 1$ supergravity. (In heterotic strings, the numbers of left- and right-handed supersymmetries are different.) $N = 2$ supergravity was first considered by Brink and Schwarz\(^9\). The $N = 2$ supergravity multiplet consists of a vielbein, a complex gravitino and a $U(1)$ gauge field denoted, respectively:

$$e^{a}_{\alpha}, \begin{pmatrix} \chi^{(1)}_{\alpha} \\ \chi^{*-1}_{\alpha} \end{pmatrix} \quad \text{and} \quad A_{\alpha}. \tag{1}$$

Here the numbers in parenthesis show the $U(1)$ charges of the gravitini. As is usual in string theory, all the supergravity fields can be locally gauged away using the gauge symmetries of the theory, and there is no action for them: The vielbein is removed by general coordinate invariance, local Lorentz invariance and local Weyl transformations; the gravitini by the complex supersymmetry and complex super-Weyl transformations, and the two components of the $U(1)$ gauge field by vector and chiral $U(1)$ gauge symmetries on the world sheet\(^*\). These $U(1)$ symmetries have no counterpart in the $N = 0$ and $N = 1$ theories, and their existence leads to many of the special features of $N = 2$ strings.

To get a string theory, one must couple the supergravity to some ($N = 2$) matter. The simplest such matter is an $N = 2$ chiral superfield $X^i$, with $i$ some internal (space-time!) index. The component fields are seen in the $\theta$ expansion of these superfields given, somewhat schematically, by:

$$X^i \sim x^i + \theta^{(-1)} \psi^{(1)}_i \quad \text{and} \quad \bar{X}^{\bar{i}} \sim \bar{x}^{\bar{i}} + \theta^{*(-1)} \psi^{*(1)}_{\bar{i}}. \tag{2}$$

\(^*\)The existence of the chiral invariance was pointed out by Fradkin and Tseytlin in ref. 10.
The string world-sheet action is:

\[
S = \int d^2z \sqrt{g} \left( \frac{1}{2} g^{\alpha\beta} \partial_\alpha x^i \partial_\beta \bar{x}^i + i \bar{\psi}^j \gamma^\alpha \psi_i + A_\alpha \bar{\psi}^j \gamma^\alpha \psi_i + (\partial_\alpha \bar{x}^i + \bar{\psi}^j \chi) \chi^\beta \gamma^{\alpha\beta} \psi_i + \text{c.c.} \right),
\]

where \( \mathcal{D} \) denotes a gravitationally covariant derivative of the spinor, containing a spin-connection piece. Note that, although the scalar field \( x^i \) is complex, it does not have a \( U(1) \) charge and does not couple to the gauge field \( A_\alpha \). On the other hand, the spinors \( \psi^i \) are charged, and I have explicitly shown their minimal coupling in the action.

1.3. The critical dimension of \( N = 2 \) strings

The first question to ask in any string theory is what is the critical dimension of the theory? Ademollo et al. found that in \( N = 2 \) strings \( D = 2 \), where \( D \) is the number of chiral superfields \( X^i \). The modern derivation of this result goes as follows:\(^{10}\) gauge fixing the \( N = 2 \) supergravity algebra gives rise to independent left and right-handed \( N = 2 \) constraint algebras:

\[
\begin{pmatrix}
T \\
G & G^* \\
J 
\end{pmatrix}_L \quad \text{and} \quad \begin{pmatrix}
\bar{T} \\
\bar{G} & \bar{G}^* \\
\bar{J}
\end{pmatrix}_R.
\]

In both the left and right sectors one has the usual \( (b,c) \) ghosts, complex \( (\beta,\gamma) \) supersymmetry ghosts and a \( (b',c') \) system of \( U(1) \) ghosts with conformal spins \( (1,0) \). Canceling the conformal anomaly means

\[
c = -26 + 2 \cdot 11 - 2 + 2D \cdot (1 + \frac{1}{2}) = 0,
\]

resulting in \( D = 2 \). An even easier calculation is to cancel the axial-vector \( U(1) \) anomaly between the \( \psi \)'s and the \( (\beta,\gamma) \) ghosts:

\[
c = 0 = -2 + D.
\]

This gives the same result, since all the anomalies of the theory fall into a single supermultiplet.

Since we have 2 complex scalars \( x^i \), it would now appear to be obvious that the theory should live in a \( 2\mathbb{C} \)-spacetime; in terms of real dimensions this is a \( 4\mathbb{R} \)-dimensional spacetime that must have a signature \( (2,2) \) or \( (4,0) \). However, here the history of the subject is a little peculiar. In their original work, Ademollo et al. defined a real superfield \( Y = X + \bar{X} \), instead of working with the chiral superfield \( X \). This superfield contains the same information as \( X \), except that the imaginary part of the zero-mode \( x \) appears only as a derivative. Based on this, they decided
not to excite positions and momenta in these “imaginary” coordinates. This means that they dimensionally reduced the theory by brute force to \(2\mathbb{R}\) dimensions. (Note that the anomaly cancellation still works in the truncated theory, since only the zero-modes of the fields have been changed, and the theory still contains 4 real bosonic and fermionic fields.) When Fradkin and Tseytlin rederived the critical dimension of the theory they considered the imaginary coordinates to be physical\(^{10}\), and this was later taken by D’Adda and Lizzi to imply a Lorentz-invariant \(4\mathbb{R}\)-dimensional spacetime\(^{11}\). Since then the theory has been firmly ensconced in \(2\mathbb{C}\) dimensions.

1.4. The spectrum of \(N = 2\) strings

To see the spectrum of the theory one needs to calculate the theory on the cylinder. This means that we must discuss the boundary conditions of the fields. As usual, since they are world-sheet tensor fields, the \(x^i\) fields and the anticommuting ghosts are periodic (although other possibilities were considered in ref. 2). However, a new feature of the \(N = 2\) string is that the fermions \(\psi^i\) and the commuting ghosts are charged under the \(U(1)\) symmetry. Since one can have a constant gauge field on the cylinder (more precisely one can have non-trivial Wilson lines around the cylinder), one can continuously change the boundary conditions of the fermions from NS to R by turning on this gauge field. Because of this, one usually considers just the purely NS part of the theory, and argues that all other sectors are equivalent to it by the \(N = 2\) spectral flow\(^{12}\). I shall generally do this throughout the talk, although I shall discuss other possibilities in the conclusion.

Now naively, since we are in \(D = 2\), one would expect that one could go to a lightcone gauge with \(D - 2 = 0\) transverse dimensions! This suggests that \(N = 2\) theories should have no oscillator excitations. The light-cone argument also suggests that the mass squared of this state is \((2 - D)/24 = 0\). The decoupling of the massive states of the string was indeed seen by Ademollo et al., and the same result has been confirmed by a BRST analysis\(^{13}\). But the simplest way to find the spectrum is to calculate the partition function of the theory: Aside from zero modes, the four \(x\) oscillators are canceled by the \((b, c)\) and \((b', c')\) ghosts, while, for any boundary conditions, the two complex charged \(\psi^i\)'s are canceled by the charged \((\beta, \gamma)\) ghost system. This leaves only massless states in the spectrum. Summarizing:

\[a)\] \(N = 2\) strings live in \(2\mathbb{C}\) dimensions.

\[b)\] All their oscillator excitations vanish.

\[c)\] In each sector of the theory, only a massless “scalar” state propagates.

and, thus:

\[d)\] \(N = 2\) string amplitudes must satisfy duality with no infinite sums over massive states!
\textit{e}) \textit{N = 2 string field theory is an ordinary field theory with a finite number of particles.}

1.5. The Lorentz symmetry of the \textit{N = 2} strings

We have now seen where the \textit{N = 2} strings live and have calculated their spectrum. One remaining question that we should ask is what is the “Lorentz” invariance of these theories. Note that in the gauge-fixed theory, where one turns off the supergravity fields $A_\alpha$ and $\chi_\alpha$ in the string action of eq. (3), there is no difference between matter fields and their complex conjugates in the action. Based on this, D’Adda and Lizzi argued that the theory has an $SO(2,2)$ Lorentz invariance\textsuperscript{11} (choosing the “Minkowski” rather than the Euclidean signature). However, the $\psi^i$'s and $\bar{\psi}^i$'s couple differently to the gauge field $A_\alpha$, since they have opposite charges. This difference is then fed to the $x^i$ and $\bar{x}^i$ fields by the gravitini $\chi_\alpha$. Another way of looking at this is that the constraint multiplets of eq. (4) distinguish the different fields. In the works of Ooguri and Vafa that sparked the renewed interest in \textit{N = 2} strings\textsuperscript{5}, space-time is thus considered to be intrinsically complex (actually Kähler), with a $U(1,1)$ symmetry group.

Recently, however, Siegel has argued\textsuperscript{14} that the \textit{N = 2} string is the same as the “\textit{N = 4}” string, which was also discovered by Ademollo et al.\textsuperscript{2}. This string has an $SU(2)$ local symmetry, is hard to quantize covariantly, and apparently has $D = -2!$ If Siegel is correct, the manifest $U(1,1)$ invariance of the \textit{N = 2} string should be extendible to the full $SO(2,2)$ Lorentz invariance. This is known to be the case in the tree-level amplitudes of refs. 5–7, as I will discuss. A bigger Lorentz symmetry would be important, since the $U(1,1)$ spaces have no little group, so one cannot define the “spins” of particles, while the $SO(2,2)$ Poincaré group can have representations of continuous spin. Using the larger Lorentz group, Siegel has also argued that the Ramond sectors of the \textit{N = 2} strings should describe fermions\textsuperscript{14}, and that the effective field theories of the \textit{N = 2} strings should be maximally spacetime supersymmetric\textsuperscript{15}. Such results necessitate a modification of the string amplitudes. I shall return to these issues in the discussion section at the end of my talk.

2. Tree amplitudes and effective actions

2.1. \textit{N = 2} strings in two real dimensions

We have seen that \textit{N = 2} strings do not have any oscillator excitations, and so can be described by effective field theories with finite numbers of fields. The most straightforward approach to finding these field theories is simply to calculate the tree-level amplitudes of the string theory, and to equate them to those of the field theory. Here I shall follow an approximately historical approach, and first consider the strings in two real dimensions, à la Ademollo \textit{et al.} It may seem a bit strange to the post-Polyakov physicist that the $2R$-dimensional theory makes any sense at all. However, we have already argued that (local) anomalies are still canceled in this
truncated theory, and tree amplitudes are consistent even in dimensions which do not give anomaly cancellation. We would expect this truncation to make troubles at some stage, but since the rules for writing down $N = 2$ strings are so ill-defined, it is even possible that these theories are consistent as is. It turns out that these truncated theories illustrate many of the general properties of $N = 2$ strings, and have interesting effective field theories in their own right.

I now turn to the case of the open string in $2_{RR}$ dimensions\(^2\). As I suggested above, the spectrum of this theory should be a single massless scalar. However, as is usual in open strings, Ademollo et al. added $(SU(2))$ Chan-Paton factors\(^16\) to the string, ending up with a theory of three scalars in the adjoint of $SU(2)$. Since the $2_{RR}$-dimensional theory is the truncation of the $2_{CC}$ theory, which I shall consider in detail later, I shall simply present its amplitudes without derivation: The first nonvanishing amplitude in the theory is the 4-point function, since all odd-point functions vanish by the G-parity of the theory. It is given by

$$A_4 = g^2 \frac{u}{2} \text{Tr} (\lambda_1 \lambda_2 \lambda_3 \lambda_4) + \text{perms} \tag{7}$$

where $u$ is the usual Mandelstam variable, and the $\lambda$'s are the Chan-Paton factors. Note that this amplitude would vanish without the Chan-Paton factors, since it would be proportional to $s + t + u = 0$. Also, recall that in the scattering of two massless particles in $2_{RR}$-dimensions one has the kinematic relation

$$stu = 0 \tag{8}$$

One can get the equation of motion corresponding to eq. (7) from an $SU(2)$ sigma model\(^1\), with action

$$S = \int d^2x \text{Tr} \left( g^{-1} \partial_i g \ g^{-1} \partial^i g \right) \tag{9}$$

and equation of motion

$$\partial_i \left( g^{-1} \partial^i g \right) = 0 \tag{10}$$

Ademollo et al. argued that this equivalence should be true to all orders, using uniqueness arguments based on Regge behaviour and Adler zeroes, which I do not claim to follow.

We can already see several intriguing features of this theory, which will generalize to all $N = 2$ theories. Unlike other string theories, and about any other field theories, the amplitude of eq. (7) is a local function of the momenta. That this had to happen is pretty much forced on us by the opposing requirements of the duality of the amplitudes and the fact that the theory contains only massless particles. This should therefore be true for all nonvanishing amplitudes. Ademollo et al. argued that all $4N + 2$ point functions vanish. (Thus the next possibly nonvanishing amplitude

\(^1\)Because the sign of the amplitude has not been fixed, the correct theory may be the coset theory $SU(2)_{CC}/SU(2)^6$.\n
6
is the eight-point function—not surprisingly, it has never been calculated.) It will turn out that there are many vanishing amplitudes in $N = 2$ string calculations, for reasons that are not completely clear. Finally, as one might hope for a theory based on strings, the effective field theory has a nice geometrical nature.

I shall now briefly turn to the case of closed and heterotic strings in $2\mathbb{R}$ dimensions. These were both studied in a paper by Green in 1987\(^1\). Using the rules for obtaining closed-string amplitudes from open ones\(^2\), he found that the closed $2\mathbb{R}$-dimensional $N = 2$ string has a vanishing four-point function, and is presumably free. On the other hand, the heterotic string has a four-point amplitude given by

$$A_4^{\text{het}} = A_4^{\text{open}} + \alpha' t \delta_{12} \delta_{34} ,$$

which can be interpreted as a sigma-model amplitude with a correction from a graph with an internal “graviton”.

2.2. Closed $N = 2$ strings in $2\mathbb{C}$-dimensions

Continuing on the historical path, I now turn to a discussion of the closed $N = 2$ string in $2\mathbb{C}$ dimensions, following Ooguri and Vafa\(^5\). I will consider this case in somewhat more detail, since I will take over many of the conventions and results for the open case. The world-sheet action of the theory, with all the supergravity fields gauge fixed, can be written as

$$S = \int \frac{d^2z}{\pi} d^2\theta d^2\bar{\theta} K_0(X^i, \bar{X}^\bar{i}) ,$$

where $X^i$ is the $N = 2$ chiral superfield. Here $i$ runs from 0 to 1, corresponding to a real (2,2)-dimensional spacetime, and $K_0$ is the flat Kähler potential $\eta_{ij}X^i \bar{X}^\bar{j}$. The only state of the theory is a single massless scalar, corresponding to a perturbation around the flat Kähler potential. The superspace-vertex operator for emitting this scalar with (complex) momentum $k$ is

$$V_c = \frac{\kappa}{\kappa} e^{i(k \cdot \bar{X} + \bar{k} \cdot X)} . \tag{13}$$

The first amplitude that one can calculate is the three-point function. Fixing the super-Möbius transformations, this is given by

$$A_{ccc} = \left< V_{c|\theta=0}(0) \cdot \int d^2\theta d^2\bar{\theta} V_c(1) \cdot V_{c|\theta=0}(\infty) \right> \tag{14}$$

$$= \kappa c_{12}^2 ,$$

where

$$c_{12} \equiv (k_1 \cdot \bar{k}_2 - \bar{k}_1 \cdot k_2) \tag{15}$$

is the extra invariant product of the momenta (other than the usual dot product which has a plus sign) that exists in $2\mathbb{C}$ dimensions, when the Lorentz group $SO(2,2)$
is reduced to $U(1, 1)$. Note that $c_{ij}$ is antisymmetric with respect to its two indices, and is additive in the sense that $c_{i,j} + c_{i,k} = c_{i,j+k}$. Using momentum conservation, one sees that $A_{ccc}$ is totally symmetric, as it should be. It is important to note that in $(2, 2)$ dimensions, unlike the familiar $(3,1)$-dimensional case, there is sufficient phase space to describe one massless particle splitting into two others. The three-point function of the $2_{CC}$-dimensional closed string therefore implies a truly nontrivial $S$-matrix element, and not just some unphysical vertex.

The four-point function can be calculated similarly, and results in an apparently standard string amplitude:

$$A_{cccc} \sim \int d^2z \left( V_c|_{\theta=0}(0) \cdot \int d^2\theta d^2\bar{\theta} V_c(z) \cdot \int d^2\theta d^2\bar{\theta} V_c(1) \cdot V_c|_{\theta=0}(\infty) \right)$$

$$= \frac{\kappa^2}{\pi} F^2 \frac{\Gamma(1-s/2) \Gamma(1-t/2) \Gamma(1-u/2)}{\Gamma(s/2) \Gamma(t/2) \Gamma(u/2)} .$$

(16)

However, Ooguri and Vafa noted that, like the $2_{RR}$-dimensional kinematic identity $stu = 0$, in the scattering of massless particles in $(2, 2)$ dimensions there is again a relation:

$$F \equiv 1 - \frac{c_{12}c_{34}}{su} - \frac{c_{23}c_{41}}{tu} = 0 .$$

(17)

This means that the four-point function of eq. (16) vanishes on shell. Ooguri and Vafa proposed that this vanishing comes from some kind of topological nature of the theory, but this is still somewhat of a mystery. They also conjectured that all higher-point functions in the $2_{CC}$-dimensional case vanish; this is, as yet, unchecked. (As I argued before, duality arguments just show that the amplitudes should be local.) It is simple to see that if the string is reduced to $2_{RR}$ dimensions, as in the work of Green\textsuperscript{17}, the theory becomes trivial: In $2_{RR}$ dimensions $c_{ij}$ vanishes identically, so the three-point function now vanishes. Also, while in $2_{RR}$ dimensions $F \to 1$, the four-point function still vanishes because of the identity $stu = 0$.

Returning to $2_{CC}$ dimensions, one sees that the local three-point function and vanishing four-point function can be obtained from the action\textsuperscript{5}

$$\mathcal{L}_c = \int d^4x \left( \frac{1}{2} \partial^i \phi \bar{\partial}_i \phi + \frac{2\kappa}{3} \phi \partial \bar{\partial} \phi \wedge \partial \bar{\partial} \phi + O(\phi^5) \right) ,$$

(18)

where the $O(\phi^5)$ term needs to be determined from the five-point function. If all higher-order terms are absent the equation of motion of this action has a name—the Plebanski equation\textsuperscript{19}—and has a nice geometrical meaning. It is therefore very reasonable that the action is exact. (Note that this conjecture and the conjecture that higher-point amplitudes vanish are logically independent.) The Plebanski equation is

$$\partial^i \bar{\partial}_i \phi - 2\kappa \partial \bar{\partial} \phi \wedge \partial \bar{\partial} \phi = 0 .$$

(19)

Its geometrical meaning is seen by considering $\phi$ to be a perturbation of a Kähler potential around the flat $2_{CC}$ space, resulting in a spacetime metric

$$g_{\bar{i}j} = \partial_i \bar{\partial}_j \left( x_k x^k + 4\kappa \phi \right) .$$

(20)
The Plebanski equation of motion then becomes simply
\[ \det g_{ij} = -1. \]  
(21)

Now since in a Kähler space the Ricci tensor \( R_{ij} \) is given by
\[ R_{ij} = \partial_i \partial_j \log \det g, \]  
(22)

the Plebanski equation is the condition for the Ricci flatness of the \( 2\mathbb{C} \)-dimensional Kähler space. An important theorem of Atiyah, Hitchin and Singer\(^{20}\), which is quite nontrivial in the forward direction, is that a \( 4\mathbb{R} \)-dimensional Riemann space is self-dual \textit{iff} it is Kähler and Ricci flat. Thus the equation of motion of the \( N = 2 \) closed string can be written elegantly as
\[ R = \tilde{R}, \]  
(23)

and the \( N = 2 \) string describes self-dual gravity. It is intriguing that although the entire formulation of the string has been carried out in a \( 2\mathbb{C} \)-dimensional Kähler space, the final equation of motion can be written in a completely \( (2, 2) \) Lorentz-invariant way. I shall discuss this further at the end of the talk.

2.3. Purely open \( N = 2 \) strings in \( 2\mathbb{C} \)-dimensions

Since, despite apparently being a scalar theory, the closed \( N = 2 \) string turns out to describe self-dual gravity, it is reasonable to expect that the open and heterotic strings describe self-dual Yang-Mills. In the heterotic string\(^{6}\), which was studied by Ooguri and Vafa soon after the closed case, this is basically true, but there are some complications. Since the geometrical structure of the heterotic string is not that well understood, I shall leave the historical path and turn rather to the open string, returning to make a few comments on the heterotic string later.

I shall thus now turn to my work; the reconsideration of the open string in \( 2\mathbb{C} \) dimensions\(^{7,8}\). The open string sweeps out a world sheet that is a super-Riemann surface, but now with boundaries. For example, in the case of the super upper-half plane which gives the tree-level amplitudes of the theory, the boundary is given by \( z = \bar{z} \equiv \sigma, \ \theta = \bar{\theta} \equiv \theta \). The action of the string is the same as that of the closed string in eq. (12). To calculate with it, one also needs the boundary conditions for the fields, which are given by \( \partial x = \bar{\partial} x \big|_{z = \bar{z}} \) and \( \psi_R = \psi_L \equiv \psi \big|_{z = \bar{z}} \).

As in the closed string, the spectrum of the open string is a single massless scalar. However, as is usually the case in open strings, we want to append “Chan-Paton” group theory factors to the string amplitudes, and so end up with a multiplet of scalars \( \varphi^a \). The superspace vertex operator to emit these scalars is the same as that of the closed scalars:
\[ V_a = g e^{i(k \cdot \bar{X} + \bar{k} \cdot X)}, \]  
(24)

\(^{7}\)There were previous attempts to calculate open string amplitudes in \( 2\mathbb{C} \) dimensions in refs. 11 and 21. However in the first paper it was not realised that the three-point amplitude is non-zero, and in the second that the four-point amplitude is zero. The nature of the theory was therefore misunderstood.
but it has a different interpretation, since it is inserted on the boundary of the super-Riemann surface. After integrating out the fermionic coordinates one obtains

\[ V_o^{\text{int}} = \int d^2 \theta V_o = \frac{g}{2} (i k \cdot \partial_\sigma \bar{x} - i \bar{k} \cdot \partial_\sigma x - 4k \cdot \bar{\psi} \bar{k} \cdot \psi) e^{i(k \cdot \bar{x} + \bar{k} \cdot x)}. \]  

(The strange factors of 2 are because I am using conventions appropriate to the closed string.)

The open-string three-point amplitude is then given by

\[ A_{oo\theta} = \left\langle V_o |_{\theta=0}(0) \cdot \int d^2 \theta V_o(1) \cdot V_o |_{\theta=0}(\infty) \right\rangle = gc_{12} \times (-i f_{abc}), \] 

where I have inserted the “group-theory” factor \(-if_{abc}\) by hand. Without this factor the amplitude would be totally antisymmetric with respect to the three scalars, and so vanish. In the Chan-Paton ansatz one would have \(f_{abc} = \text{Tr} (\Lambda^a [\Lambda^b, \Lambda^c])\), but since we have not yet established the principles for constructing \(N = 2\) strings, I shall here consider the most general possible ansatz consistent with principles to be given later. Thus, at this stage, \(f_{abc}\) is a general unspecified totally antisymmetric tensor. (The notation may be a little suggestive, however.) Note that, as usual\(^{18}\), the open amplitude is the square root of the closed-string amplitude of eq. (14). It is therefore the same as that of the heterotic string, and can be derived from the same field-theory action\(^6\):

\[ \mathcal{L}_3 = \int dx \left( \frac{1}{2} \partial_i \varphi^a \partial_i \varphi^a - i \frac{g}{3} f_{abc} \varphi^a \partial^i \varphi^b \partial_i \varphi^c \right) + O(\varphi^4). \]  

The four-point amplitude of the string is given by

\[ A_{oooo} \sim \int_0^1 dx \left\langle V_o |_{\theta=0}(0) \cdot \int d^2 \theta V_o(x) \cdot \int d^2 \theta V_o(1) \cdot V_o |_{\theta=0}(\infty) \right\rangle \]

\[ = g^2 \frac{4F}{\Gamma(1 - 2s) \Gamma(1 - 2t) \Gamma(2u)} \text{,} \] 

and, as in the closed string, vanishes because of the \(F\) factor. Because this amplitude vanishes, the usual unitarity constraints\(^{22}\) on the group-theory factors do not apply. However, one can still find a constraint on them by calculating the four-point function of the field theory action of eq. (27) (supplemented by a quartic term \(\mathcal{L}_{4o}\)), and demanding that it vanishes in agreement with the string result of eq. (28). Using some kinematical identities, the field-theory result can be written as

\[ A_{ooooFT} = -g^2 \left\{ \frac{C_{12}C_{34}}{s} \left( f_{abx} f_{xcd} - f_{acx} f_{xbd} - f_{bcx} f_{xda} \right) + uf_{bcx} f_{xda} + tf_{cax} f_{xbd} \right\} - V_4. \]
Since $V_{40}$ must be a local vertex, the factor in the parentheses must vanish for the amplitude to be zero. This factor is simply the Jacobi identity, so we see that the $f^{abc}$'s must be the structure constants of some group, as one might have been expected. The group is unspecified, and can be any semisimple group times a product of $U(1)$'s. This is unlike the case of the bosonic and $N=1$ open strings, where classically the group can only be $SO(N)$, $USp(N)$ or $U(N)^{22\S}$. We thus have an ansatz for the group theory that is more general than (but also even more ugly and ad hoc than) that of Chan and Paton$^{16}$. The resulting quartic interaction determined by eq. (29) is now:\footnote{Of course, in the case of the superstring, we know that the cancellation of anomalies uniquely picks out the group $SO(32)^{23}$.}

$$\mathcal{L}_{40} = \int d^4x \left( -\frac{g^2}{6} f^{abc}_x f^{x \alpha \beta} \partial^a \varphi^b \partial^c \varphi \right) . \tag{30}$$

The action of eqs. (27) and (30) does not appear to have any clear meaning. However, its resulting equation of motion can be written in a nice compact way: Defining the anti-hermitian matrix $\varphi$, the equation of motion becomes

$$\bar{\partial}_i \left( e^{-2i\varphi} \partial^j e^{2i\varphi} \right) = 0 , \tag{31}$$

which is known as Yang's equation$^{24}$. As expected, since we again have an equation with a name, it has a nice meaning and is indeed related to self-dual Yang-Mills (SDYM). This can be seen by noting that in a Kähler space the SDYM equation $F = \tilde{F}$ breaks into three pieces. Defining the holomorphic $(2,0)$ form $\omega$ and the Kähler form $k$, one first has

$$F_{ij} = 0 \ (F \wedge \bar{\omega} = 0) \iff A_i \equiv e^{-i\varphi} \partial_i e^{i\varphi}$$

$$F_{i\bar{j}} = 0 \ (F \wedge \omega = 0) \iff \bar{A}_i \equiv e^{i\varphi} \bar{\partial}_i e^{-i\varphi} . \tag{32}$$

(Here, choosing the $\varphi$'s in $A_i$ and $\bar{A}_i$ to be the same means that we have fixed the gauge-invariance of the theory.) The third self-duality equation is then

$$F^2_i = 0 \ (F \wedge k = 0) \iff \left[ \mathcal{D}_i, \mathcal{D}_j \right] = 0$$

$$\iff g^{ij} \bar{\partial}_j \left( e^{-2i\varphi} \partial_i e^{2i\varphi} \right) = 0 . \tag{33}$$

Thus the purely open string equation, eq. (31), describes SDYM in a flat $(2,2)$-dimensional spacetime. The action giving Yang’s equation can be written order by order in $\varphi$, as we have started to do in eqs. (28) and (30). Since the equation is a generalization of the Wess-Zumino-Witten equation in two dimensions, it may not

\footnote{Since the heterotic string also has a vanishing four-point amplitude, this result is again in agreement with that of the pure Yang-Mills sector of 6. However, the heterotic string also gets contribution from graphs with an intermediate graviton, modifying $\mathcal{L}_{40}$. In an open theory such graphs have the topology of an annulus, and are not part of the classical theory.}

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be surprising that one can also write a more geometrical action in terms of coset elements in a five-dimensional space\textsuperscript{25}. Summarizing, once the dust has settled, the equations of motion of the string can again be written in a completely Lorentz-invariant way:

\[ F = \tilde{F} ! \]  

\[ (34) \]

2.4. Coupled open and closed $N = 2$ strings in 2\textsuperscript{3}-dimensions

We have seen that, as we expected, the open string describes SDYM. However, we know that all strings should contain gravity, so there should also be a gravitational sector of the theory. In the heterotic string the gravitational sector indeed exists and, as in the 2\textsuperscript{3} case, it changes the equations of motion of the string from being exactly the self-duality of the Yang-Mills. In open strings, gravity exists because open strings can always join together to form closed ones. Open theories thus always have closed sectors, and one should consider interactions between the sectors. Actually, not many calculations of this type have been carried out, at least in the Polyakov formalism, so such calculations have several relatively unfamiliar aspects to them. For example, you should bear in mind that in open string theories the tree-level equations of motion are the same as those of the closed string. However, the tree level of the open sector comes from the disk, or upper-half plane (UHP), at “genus 1/2”, so the very notion of the classical limit of the theory is not so well defined. For example, the closed and open-string couplings are related by $\kappa \sim \sqrt{\bar{h}g^2}$.

I shall not go into the details of the mixed open-closed amplitude calculations\textsuperscript{7}, but shall just give the results. First, all amplitudes with only one open string and an arbitrary number of closed ones vanish, because of the twist symmetry of the theory. This is fortunate, since the open sector has a group theory index and such amplitudes imply a breaking of the group. The only mixed three-point amplitude therefore involves the scattering of two open strings with one closed one on the UHP. It is given by

\[ A_{ooc} \sim \int_{-\infty}^{\infty} dx \left.V_o|_{\theta=0}(x) \cdot \int d^2\theta d^2\bar{\theta} V_c(z = i) \cdot V_o|_{\theta=0}(\infty)\right) = \kappa \delta^{ab} c_{12}^2. \]  

(35)

(Note that this amplitude involves an integration over the position of one of the vertices, even though it is a three-point function!) Since the amplitude gives the coupling of the gravitational sector to the quadratic term of the open scalars, I have chosen to append the group-theory factor $\delta^{ab}$, as in the kinetic term. This vertex is similar to that of the gravitational self-interaction of eq. (14), showing some kind of universality in the couplings of the various fields to gravity. The existence of the
vertex means that one has to add the interaction
\[ L_{ooc} = \int d^4x \left( 2\kappa \phi \bar{\partial} \varphi^a \wedge \partial \bar{\varphi}^a \right) \] (36)
to our open-closed action.

The first interesting four-point function involves the scattering of three open strings with one closed one on the UHP. It is given by
\[ A_{oooc} = \frac{i}{2} \kappa g f^{abc} F \frac{\Gamma(s) \Gamma(t) \Gamma(u)}{\Gamma(-s) \Gamma(-t) \Gamma(-u)} \left(c_{12} t + c_{23} s\right), \] (37)
and it again vanishes on shell because of the \( F \) factor. Calculating the same amplitude in the field theory, one sees that one has to add yet another term to the action:
\[ L_{ooc} = \int d^4x \left( -\frac{4}{3} i\kappa f^{abc} \bar{\partial} \varphi^a \wedge \varphi^b \partial \varphi^c \right) . \] (38)

In principle, one should also consider the amplitude with two closed and two open scalars. Since it is really messy to calculate, I shall assume that it vanishes because of an \( F \) factor, like all the other four-point amplitudes. This vanishing is found from the field theory without having to add any new terms to the Lagrangian.

There is one final type of amplitude that I have not yet discussed, because it has a different structure and interpretation than all the other amplitudes. This is the scattering of three closed strings on the UHP and on the projective plane \( \mathbb{R}P^2 \), which should be combined with the UHP graph, since it is also of Euler number 1. Unlike graphs with open vertices, whose tree-level amplitudes are defined on the UHP (since \( \mathbb{R}P^2 \) has no boundaries, it does not appear in open amplitudes), the closed three-point tree-level amplitude of the theory comes from the sphere. These genus 1/2 graphs thus give a type of quantum corrections to the theory. Their amplitudes are difficult to calculate, but one can see that they both have the form:
\[ A_{ccc}' \propto \frac{\kappa^3}{g^2} c_{12}^4, \] (39)
with some finite coefficients. Since the amplitudes contains no open-string vertices, their group-theory factors are not fixed and one can consider them with arbitrary coefficients. (In the Chan-Paton scheme the overall factor would be proportional to \( N - 2 \) for the group \( SO(N) \), etc.) The interpretation of eq. (39) is unclear and, in the following, I will generally choose the coefficients to make the contribution vanish.

At this stage, we have enough of the action to see what the full equations of motion of the theory should be. The open-sector equation is:
\[ g_{ij}(\phi) \partial^j \left( e^{-2ig\phi} \partial^i e^{2ig\phi} \right) = 0 \iff \ F \wedge k = 0, \] (40)
where \( k(\phi) \) and \( g(\phi) \) are now the full Kähler form and metric, defined in terms of \( \phi \) as in eq. (20). This gives the self-duality of the Yang-Mills field strength in the
background of the curved Kähler space of the closed sector. The closed equation of motion is now modified to
\[ \det g_{ij} = -1 - \frac{2\kappa^2}{g^2} \text{Tr} \left( F_{ij}^2 \right), \]  
(41)

or
\[ \partial \bar{\partial} K \wedge \partial \bar{\partial} K = 2\omega \wedge \bar{\omega} - \frac{4\kappa^2}{g^2} \text{Tr} \left( F \wedge F \right). \]  
(42)

Since the Ricci tensor is defined in terms of derivatives of \( \det g \) (eq. (22)), one sees that the Ricci-flatness condition is modified by a source term from the open sector, in the same way that the Einstein tensor gets a contribution from the matter stress tensor in usual gravity. Recall that, since there is relation \( \kappa \sim \sqrt{\bar{h}} g^2 \), the source term is some kind of quantum mechanical correction to the equation of motion. (If we had included the quantum amplitude of eq. (39), there would also have been a gravitational source term on the right of eqs. (41) and (42).) The spacetime is thus no longer Ricci flat, and is no longer self-dual. I do not know of any Lorentz-invariant way of writing this gravitational equation with the source.

2.5. Heterotic \( N = 2 \) strings in \( 2\mathbb{C} \) (?) dimensions

For completeness, I will now briefly discuss the heterotic \( N = 2 \) strings\(^6\). The major problem with this string is that the left-hand side of the string lives in \((2, 2)\) dimensions, while the right-hand side lives in either \((25, 1)\) or \((9, 1)\) dimensions. One therefore needs to get rid of one of the LHS time coordinates. This is done, essentially, by compactifying the theory to either \((2, 1)\) or \((1, 1)\) dimensions, somewhat messing up the geometrical interpretation of the theories. As in the open case, the pure gauge sector of the theory is described by Yang’s equation for SDYM, now reduced to three (or two) dimensions\(^6\). Somewhat surprisingly, the resulting Yang-Mills particles are actually tachyonic scalars! In addition, the theory also contains massless vector-like particles in the gravitational sector, whose couplings are rather poorly understood. As in the \((1,1)\)-dimensional case, the intermediate vector particles also induce an \( O(\alpha') \) modification to the equation of motion of the scalars, so the gauge sector of the theory is not simply SDYM. Since our understanding of these theories is so confused, I shall not say anything more about them in this talk.

3. Loops

3.1. The closed string partition function

Thus far, aside from the heterotic string, we have had a nice interpretation of all of the amplitudes in the theory. However once we continue to loop amplitudes we are going to see all kinds of problems. I will give the results for several amplitudes here, and discuss various possible ways around the problems in the conclusion of the talk.

The first loop calculation done in an \( N = 2 \) string—the partition function of the closed string—was carried out by Mathur and Mukhi\(^{27}\). Unlike other strings,
this calculation includes an integration over the possible Wilson lines on the torus, which is equivalent to an integration over the boundary conditions of the world-sheet fermions. However, since each field in the theory is accompanied by a ghost field of the same charge but different spin, and spin is irrelevant on the torus, this means that the partition function is independent of the boundary conditions. After doing a careful integration over all the zero modes in the theory, one obtains

$$Z_{\text{string}} = \frac{1}{4\pi} \int_{\mathcal{M}} \frac{d\tau d\bar{\tau}}{\tau_2^{D/2}},$$

where $\mathcal{M}$ is the usual “keyhole” of the moduli space of the torus. Recall that in the $2\mathbb{R}$-dimensional version of the theory one still has all four scalars and only the $x$ zero-mode integration is changed, so this result is valid both in $2\mathbb{R}$ and $2\mathbb{C}$ dimensions. The partition function is only modular invariant in the $2\mathbb{C}$ case, giving the first evidence that this is where the theory should actually be defined.

However, the string partition is not that which one would expect of a (2,2)-dimensional scalar, such as that of our Plebanski lagrangian of eq. (18)! That would be given by

$$Z_{\text{part}} = \sum \frac{1}{2} \hbar \omega$$

$$= \frac{1}{2} \text{Tr} \log(p^2 + m^2)$$

$$= \frac{1}{2} \frac{1}{(4\pi)^{D/2}} \int_0^\infty \frac{ds}{s^{1+D/2}} e^{-sm^2},$$

where $s = \pi \alpha' \tau_2$ is the Schwinger parameter. As usual, the range of integration in the string case is different from that of the particle. What is really peculiar is that the string result agrees with that of a massless particle in $2\mathbb{R}$ and not $2\mathbb{C}$ dimensions. Technically, this is because the zero modes of the extra $U(1)$ ghosts of the string contribute an extra factor of $\tau_2$, but this does not explain the physical discrepancy between eqs. (43) and (44).

The discrepancy also gives rise to another problem in $N = 2$ loop amplitudes: Instead of considering the string in a (2,2)-dimensional Minkowski space, one can compactify one of the complex dimensions to a complex torus. This was done by Ooguri and Vafa, using the results of Dixon et al. The partition function then becomes

$$Z_{\text{torus}} \sim \int_{\mathcal{M}} \frac{d\tau d\bar{\tau}}{\tau_2} \cdot \sum q^{p_L^2/2} \bar{q}^{p_R^2/2}$$

$$\tau_2 \to \infty \int_0^\infty \frac{d\tau_2}{\tau_2} \left(1 + e^{-2\pi M^2 \tau_2} + \cdots\right),$$

and so develops a IR divergence as $\tau_2 \to \infty$. Such divergences will turn out to plague the interpretation of $N = 2$ loop amplitudes.
3.2. The open string partition function

The open-string partition function shows many of the same problems: In an open theory the partition function is found by calculating the path integral on all the genus zero graphs: the torus, the Klein bottle, the Möbius band and the annulus. Defining the proper time $t$, the total partition function should be given by

$$Z = \frac{1}{2} Z_{\text{torus}} + \frac{1}{4\pi} \int_0^\infty \frac{dt}{t^2} \left( \frac{1}{2} + \frac{1}{2} c_{\text{annulus}} - \frac{1}{2} c_{\text{Möbius}} \right),$$

(46)

where the coefficients $c$ are group-theory factors. (I have put $c_{\text{Klein}} = 1$, since I want the sum of the contributions of the torus and Klein bottle to give the one scalar of the closed sector. The factors of $1/2$ are due to the nonorientability of the theory, and should be dropped if nonorientable graphs are discarded.) In our general group-theory ansatz, we do not have any a priori knowledge of the coefficients. In order to get the spectrum right, one needs the relation

$$c_{\text{annulus}} + c_{\text{Möbius}} = 2 \dim G,$$

(47)

which is a new, not very obvious, constraint on the ansatz. In the Chan-Paton case this constraint is natural, since $c_{\text{annulus}} = N^2$ and $c_{\text{Möbius}} = \pm N$ for the groups $SO(N)$ and $USp(N)$.

One can isolate the divergences of the partition function by doing appropriate “modular-like” transformations on eq. (46)\textsuperscript{29}. The result is

$$Z = \frac{1}{2} Z_{\text{torus}} + \frac{1}{16\pi} \left[ \int_0^1 \frac{dq}{q} \left( 2 + \frac{1}{2} c_{\text{annulus}} \right) + \int_{-1}^0 \frac{dq}{q} 2c_{\text{Möbius}} \right].$$

(48)

This has an IR divergence at $q = 0$, which in the Chan-Paton case can be regulated only for the (very uninteresting) group $SO(2)$. This is the analogue of the groups $SO(32)$ for the superstring\textsuperscript{30} and $SO(8192)$ for the bosonic string\textsuperscript{29}. However, since there are no known anomalies for the bosonic and $N = 2$ strings, the argument for these special groups is not very compelling in these cases.

3.3. One-loop three-point functions

So far we have seen problems in the calculation of partition functions. One could argue that these are not very relevant physically, although this would not be the case for free energy calculations, but actual scattering amplitude calculations also show peculiar behaviour. This was first seen by Bonini, Gava and Iengo\textsuperscript{31} in the one-loop three-point scattering of the closed string. They found the result

$$A_3^{(1)} = 2 \frac{(2\pi)^6 g^3 c_{12}^6}{32\pi^6} \sum_{n,m} \frac{\tau_2^3}{|n + m\tau|^6},$$

(49)

where $c_{12}$ is the usual kinematic factor of eq. (15). This amplitude presents several surprising features: Although it is a one-loop amplitude, it is completely local! (This
is also the case for the similar genus $1/2$ term of the open-closed theory in eq. (39), but that is more expected since it is not a true quantum term.) Also, the terms in the sum with $m = 0$ again give a nasty IR divergence. Note that this divergence can not be cured in the usual way: Normally, one would calculate the interference terms between this graph and the tree-level graph in the first-order correction to the three-point cross section. The IR divergence would then be expected to be canceled by the singular contribution of the tree-level four-point function. However, in this theory the four-point function vanishes identically, so this cancellation can not work (even if one knew how to define a cross-section in a spacetime with two times)!

A good thing about this amplitude is that, because of the simplicity of the integrand, the integration over modular space can (almost) be carried out explicitly\textsuperscript{32}, using the techniques of Dixon et al.\textsuperscript{28}. These allow one to convert the sum over $m$ and $n$ in the amplitude to a sum over different copies of the moduli space, giving:

\begin{equation}
A_3^{(1)} = 2 (2\pi)^6 g^3 \epsilon^6_{12} \times \int_0^\infty d\tau_2 \frac{\tau_2}{\bar{t}^7}.
\end{equation}

The “almost” in carrying out the calculation is that we are still stuck with the IR divergence. Nevertheless, it is very interesting that the string amplitude can be rewritten as an integral with a field-theory Schwinger parameter, rather than as an integral over some complicated region in moduli space. This means that the amplitude can be directly compared to the corresponding amplitude of the effective field theory, in the same way as can the free-energy of string theories\textsuperscript{26}. It is a reasonable conjecture that, except for partition functions, string theory amplitudes can always be rewritten in this way. This means that common statements about “strings naturally providing a UV cut-off”, and so being intrinsically different from field theories are not true, and string theories can really be studied in terms of their effective field theories.

Unfortunately, when we attempt to compare this particular amplitude to that of its field theory we find the same problem that arose in the partition function calculations. The amplitude calculated from the Plebanski action of eq. (18) is

\begin{equation}
A_3^{(1)}_{FT} = \frac{1}{(4\pi)^{D/2}} g^3 \epsilon^6_{12} \times \int_0^\infty ds \frac{s^{2-D/2}}{7!}.
\end{equation}

This almost agrees with the string result (even up to the $7!$, if not up to $2$’s and $\pi$’s!) except that, one again, the field theory wants to live in $2_{RR}$ dimensions!

4. Conclusions

4.1. Tree level theory

Having told you all that I know about the amplitudes of the $N = 2$ theories, I would now like to stand back and to summarize the status of the strings. At the classical level, they are elegant and well understood. They are naturally defined in $2_{CC}$ dimensions, although one can consider them truncated to $2_{RR}$ dimensions. This
implies that they live in a spacetime with signature \((2,2)\), and are theories with two times! Their spectrum contains only massless scalar particles, so their space-time string field theories are simply field theories of the usual sort. These theories all turn out to have nice geometrical meanings: In \(2\mathbb{R}\) dimensions, the closed string is trivial\(^{17}\), while the open string describes a sigma model theory, either on a group manifold\(^2\) \(G\), or on the coset space\(^6\) \(G_{\mathbb{C}}/G\). The heterotic string gives a modified sigma model\(^{17}\).

In \(2\mathbb{C}\) dimensions, the closed theory describes self-dual gravity (SDG) in a \((2,2)\)-dimensional spacetime, with the scalar of the string describing the Kähler potential of the spacetime\(^5\). The open string describes self-dual Yang-Mills (SDYM) propagating in the Kähler background of the closed sector\(^7\), and its scalars parameterize the gauge field as described by Yang\(^{24}\). The \(N = 2\) open theory can have any gauge group, since the usual unitarity constraints\(^{22}\) on the Chan-Paton factors no longer apply, but the insertion of the group theory into the string is then very \textit{ad hoc}. The heterotic string is less understood. Its gauge sector also describes SDYM, with a gauge group coming from the compactification of an internal 24-dimensional space, but the theory must be compactified to \((2,1)\) dimensions\(^6\). This means that the scalars becomes tachyonic! In addition, the geometry of the gravity sector of the heterotic string is poorly understood, and it in turn induces interactions modifying the SDYM gauge structure.

The amplitudes of \(N = 2\) strings are very unusual since, although the strings only have massless particles, they must still have consistent dual amplitudes. This means that all the amplitudes of the theory—even the loops!—are local in momenta. In \(2\mathbb{R}\) dimensions, the (open and heterotic) theories have nontrivial local four-point amplitudes; the next possibly nonvanishing amplitude is the eight-point function, which has never been calculated. In \(2\mathbb{C}\) dimensions, all the theories have nonvanishing three-point functions, which imply a “physical” decay process in the \((2,2)\)-dimensional spacetime. All the four-point functions that have been calculated vanish, and it is reasonable to conjecture that the higher-point functions also vanish. This conjecture is related to the folk-theorem that there is no classical scattering in SDYM\(^{33}\), and it has been suggested that one may be able to prove it using the much-studied twistor formalism\(^{34}\) for instantons. However, although there is a light-cone field theory calculation to support it\(^{35}\), the conjecture remains unproven. The vanishing of these amplitudes suggests that there is some kind of topological structure to the \(N = 2\) string theories, but it is not of the usual form.

### 4.2. Loops and their problems

In contrast to the situation at the tree level, \(N = 2\) loops are a bit of a mess! They present two problems: First, they have nasty infra-red divergences. These are not intrinsically stringy, but can also be seen in the effective field theory. They may thus not seem to be so surprising, since one is dealing with massless particles, but the divergences cannot be cured in the usual ways. In addition, when one tries to compare the partition functions of the strings to those of their effective field theories, the field-theory loop integrations must be carried out in \(2\mathbb{R}\) dimensions to obtain the
$2c$-dimensional result of the strings$^{27}$! A very nice feature of the $N = 2$ string is that, because of the simplicity of its spectrum, one can take a (closed three-point) one-loop amplitude$^{31}$, and rewrite it so that it is explicitly an integration over a Schwinger parameter instead of an integration over moduli space$^{32}$. I conjecture that this should be possible for all amplitudes in all string theories, except for vacuum amplitudes. However, this particular $N = 2$ string amplitude also shows both the IR divergence and the incompatibility of dimension with that of its field theory.

### 4.3. Lorentz invariance and the other spin structures

An important issue that may have a bearing on the problems of the loop amplitudes, is “what is the Lorentz invariance of the $N = 2$ strings in (2, 2) dimensions?”. As we have seen, the $N = 2$ theories are naturally defined in a Kähler space, and so have a $U(1, 1)$ invariance group. Since the $U(1, 1)$ Poincaré group has no little group, this means that one can not define the “spin” of the particles of the various sectors of the theories. However, despite the Kähler nature of the theory (seen, for example, in the ubiquitous $c_{12}$ factors in the amplitudes), the equations of motion of the closed and open $N = 2$ strings can be written simply as $R = \tilde{R}$ and $F = \tilde{F}$, respectively, and have a manifest $SO(2, 2)$ Lorentz invariance! The reason for this is unclear. While these equations of motion are Lorentz invariant, it is well known that one can not write a Lorentz invariant action for them without including new anti-self-dual fields$^{36}$, unless one uses nonlinear Lagrange multiplier fields$^{37}$. This makes it difficult to calculate quantum corrections in the field theory in a Lorentz invariant way. There has been a recent attempt to write a Lorentz-invariant action for SDYM in harmonic space$^{38}$, which seems very reasonable since the harmonic approach is deeply related to twistors. However, one can show that this action does not give a correct description of the quantum SDYM theory$^{39}$.

I should now like to amplify a bit more on the propositions of Siegel concerning the spectrum and invariances of the $N = 2$ strings$^{14, 15}$. These are rather controversial, and I shall not try to pronounce a final verdict on them. He first argues that the $N = 2$ strings are the same as the “$N = 4$” strings of Ademollo et al.$^2$, which have an $SU(2)$ world sheet symmetry$^{14}$. This would mean that the $x$ oscillators of the theory can be taken to be $SO(2, 2)$ vectors, while the $\psi$’s would be $SO(2, 2)$ spinors. The constraint system of the $N = 4$ theories is complicated to quantize, but the equivalence immediately implies that $N = 2$ strings must be completely Lorentz invariant. This also has consequences for the non-NS sectors of the $N = 2$ string, which I have generally ignored in the talk. These sectors are normally regarded as duplicates of the NS sector$^{2, 5}$, but if the theory is Lorentz invariant the $R$ sectors of the $N = 2$ strings must describe fermions$^{14}$. Since no amplitudes of non-NS sectors have ever been calculated, there is as yet no stringy evidence for or against this proposition. The NS tree amplitudes are unaffected by the existence of other sectors, and it is hard to draw conclusions from the loop amplitudes, since they are so confusing.
Siegel has also argued that the $N = 2$ strings should actually be maximally spacetime supersymmetric\textsuperscript{15}! This conjecture is based on the fact that (without getting into nonlinear Lagrange multipliers) one can write a Lorentz-invariant superspace action for SDYM and SDG only if they have $N = 4$ and $N = 8$ spacetime supersymmetries, respectively. I feel that the arguments for this conjecture can be evaded, since there are supersymmetric theories which do not have full-superspace actions\textsuperscript{40}, and the ordinary type II string does not have a usual Lorentz-invariant action, since it has a self-dual field with no anti-dual partner. If this proposition is nevertheless true, it has very far-reaching consequences for the $N = 2$ theories: First, all the other fields of the extended supermultiplet must show up in the theories. For example, the other sectors of the open string must give rise to four spinors and six scalars, in addition to the aforementioned anti-dual fields. These numbers have to be included by hand, and appear to be somewhat ad hoc. In the case of the heterotic string, the gauge groups must also be far smaller than those derived by Green\textsuperscript{17} and Ooguri and Vafa\textsuperscript{6}. In the open string, it should not be possible for the open and closed sectors to interact, since the two sectors have different $N = 4$ and $N = 8$ supersymmetries. This contradicts to the various mixed amplitudes that I have presented here\textsuperscript{7}. Somehow forcing these amplitudes to vanish would have the good feature of removing the non-Lorentz invariant “1/2-loop” corrections to the SDG equations in eqs. (41) and (42), but it also means that the SDYM of the open string could live only in flat space!

The true loop amplitudes of the various strings would also have to be changed: supersymmetry means that the partition functions of all the strings must vanish, with cancellations between the bosonic and fermionic sectors of the theories. This contradicts the standard result of ref. 27, in which all sectors contribute with the same phase. In addition, the IR-divergent non-Lorentz-invariant one loop closed-string amplitude of eq. (50)\textsuperscript{31}, which gave us so much trouble, should also vanish. In fact, the proposition of maximal supersymmetry solves all the loop problems of the $N = 2$ theories rather dramatically: All loops must vanish identically!

4.4. Modifying string amplitudes?

All these results follow from calculations in the (supposed) effective field theories of the $N = 2$ strings. If they are to be relevant to the strings themselves, one has to implement them at the string level. I would like to conclude my talk by considering the possibility that the $N = 2$ string amplitudes that I have talked about at such length should be modified, and if so, how.

The mixed open-closed amplitudes can only be made to vanish by declaring that they should not exist. Since I inserted group-theory factors by hand into these amplitudes, this is possible to do. It is, however, somewhat unsatisfying. If one does not wish to do this, the gravity sector of the $N = 2$ open string could have at most an $N = 4$ supersymmetry, as in the usual superstring, and not the maximal $N = 8$ supersymmetry.

In the case of the true one-loop amplitudes of eqs. (43) and (50), one has more
freedom to modify the string. These amplitudes involve the adding together of differ-
ent spin structures, all of which give the same contribution. Normally, one would add
them with fixed phases, but the only convincing argument for this is that the final
result should be modular invariant and factorizable. In the case of \( N = 2 \) strings all
the sectors are independently modular invariant, and unitarity and factorizability are
not understood in a spacetime with two times. It thus may be reasonable to make
the total amplitudes vanish. In fact, since the spacetime constraints on \( N = 2 \) strings
are so unclear, one could be even more extreme and drop the possibility that the
theories need be modular invariant: The \( 2\mathbb{R} \)-dimensional truncations of the theories
could then be consistent. While this may seem to be a remarkably ugly thing to do,
the possibility can not be ruled out by space-time arguments.

Finally, I should also point out that all the calculations that I have presented here
are in the zero-instanton sector of the theory. The existence of the \( U(1) \) gauge field of
the string means that one should sum over world-sheet configurations with different
instanton number, possibly weighted with a \( \theta \)-term: \( \theta \int d^2 z F \). The necessity of such
calculations has been mentioned by various groups but they have never been carried
out. One such calculation that we can do very simply is the partition function on the
torus: Since the contribution of the charged \( \psi \) fields is canceled by that of the \( (\beta, \gamma) \)
ghosts, the partition function is independent of the instanton number, giving:

\[
Z \to Z_0 \times \sum_n e^{in\theta} = 2\pi \delta(\theta) Z_0 .
\]

This is a rather odd result (although it explains the strong-CP problem!), and it
may need to be modified—possibly to zero—as suggested above. The contribution of
the sectors with nonvanishing instanton number to scattering amplitudes should be
further investigated.

As a final summing up: our investigations of the \( N = 2 \) strings have led to many
unexpected results. The theories are still not understood deeply, and we still need to
find the basic rules for constructing and calculating with these theories. It should be
interesting!
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