String Amplitudes and Frame-Like Formalism for Higher Spins

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Abstract

We analyze open string vertex operators describing connection gauge fields for spin 3 in Vasiliev’s frame-like formalism and perform their extended BRST analysis. Gauge symmetry transformations, generalized zero torsion constraints relating extra fields to the dynamical frame-like field and relation between dynamical frame-like field and fully symmetric Fronsdal’s field for spin 3 are all realized in terms of BRST constraints on these vertex operators in string theory. Using the construction, we analyze the 3-point correlator for spin 3 field and calculate Chern-Simons type cubic interactions described by 3-derivative Berends-Burgers-Van Dam (BBD) type vertex in the frame-like formalism.

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1. Introduction

Constructing consistent gauge theories of interacting higher spin fields is a long-standing, fascinating and difficult problem (for an incomplete and very subjective list of references see

[1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50], [51], [52], [53], [54], [55], [56], [57], [58], [59], [60], [61], [62], [63], [64], [65], [66], [67], [68])

Despite significant progress in describing the dynamics of higher spin field theories, achieved over recent few decades, our understanding of the general structure of the higher spin interactions is still very far from complete. String theory appears to be a particularly efficient and natural framework to construct and analyze consistent gauge-invariant interactions of higher spins [1], [2], [3], [4], [5], [6], [7], [8], [21], [22], [63], [64], [65], [36], [37], [66], [67], [68].

Within string theory, there are several approaches to this problem. The first approach is based on the observation that excitations with higher spins appear naturally in the massive spectrum of open and closed strings with the masses of the states on the leading Regge trajectory given by $m \sim (\frac{s}{\alpha'})^{\frac{1}{2}}$, so in the tensionless limit $\alpha' \to \infty$ the corresponding operators technically become massless. There are several difficulties within this approach, e.g. it is generally not easy to combine the vertex operators so as to recover the explicit set of the Stueckelberg symmetries of the corresponding states. The known examples of such operators typically mix the excitations with different spin values [68]. In addition, since the tensionless limit is opposite to the low energy one, field theoretic interpretation of the correlation functions of these vertex operators is not easy. This formalism is also hard to extend to the AdS case since the worldsheet correlators of string theory in AdS backgrounds are difficult to analyze beyond the semiclassical limit. Another string theoretic approach to higher spins, based on the formalism of ghost cohomologies, is independent on the tension arguments and in principle allows to circumvent some of the difficulties related to the tensionless limit. This approach is based on new physical (BRST invariant and nontrivial) vertex operators that we analyzed in previous works (see e.g. [60]) that are essentially coupled to the $\beta - \gamma$ system of superconformal ghosts in RNS formalism. This ghost coupling cannot be removed by picture-changing transformation and can be classified in terms of ghost cohomologies [60], [61]. This class of vertex operators is ghost picture-dependent, distinguishing them from standard operators such as a photon or a
graviton, that exist at any picture. In open string sector, there is a subclass of these
operators corresponding to massless higher spin excitations. BRST invariance conditions lead
to Pauli-Fierz on-shell conditions for higher spin fields in Fronsdal’s metric-like formalism
while BRST nontriviality constraints lead to gauge transformations for these operators.
Their worldsheet amplitudes are thus gauge-invariant by construction and describe polynomial
interactions of massless higher spin fields in the low energy effective limit. In
our previous works we calculated some examples of such interactions - cubic interaction of
\( s = 3 - 3 - 4 \), , the disc amplitude of spin 3 operators with the graviton (reproducing the
coupling of spin 3 to gravity through linearized Weyl tensor) and the quartic interaction
of spin 3 and spin 1 gauge fields \([66], [67]\). In practice, however, explicit calculations
involving these operators are in most cases are complicated, as their explicit structure is
generally quite cumbersome. More significantly, due to the picture dependence, in many
physically important cases the options to manipulate with the picture changing are lim-
ited and it is often hard to find the appropriate picture combination of the the higher
spin vertex operator satisfying the correct ghost number balance in correlation functions
to cancel the background charges of the ghosts (e.g. on the sphere all the appropriate
correlators must carry total \( \phi \)-ghost number \(-2\), \( \chi \)-ghost number \(+1\) and \( b - c \) ghost num-
ber \(+3\)). One important example when such complication appears is the cubic interaction
of spin \( s = 3 \) corresponding to cubic amplitude of spin 3 vertex operators in open string
theory. Straightforward calculation of this amplitude using vertex operators for Fronsdal-
type fields requires 4 picture changing transformations which, given cumbersome structure
of the operators, makes the computations practically insurmountable. In this paper we
approach this problem by developing vertex operator formalism for auxiliary (extra) fields
in Vasiliev’s frame-like approach. We construct vertex operators for connection gauge
fields in this formalism. As in the Fronsdal’s case the on-shell conditions on the operators
lead to standard trace and symmetry constraints on the fiber indices of the connection
gauge fields, along with gauge fixing conditions for diffeomorphism symmetries. Gauge
transformations of the connection fields lead, in turn, to shifting the vertex operators by
BRST exact terms that do not affect the correlators that determine the structure of the
interaction terms in the low-energy limit. The generalized zero torsion constraints follow
from ghost cohomology conditions on the vertex operators that will be derived in the next
section.

The rest of the paper is organized as follows. In the Section 2 we review the basic
ideas of the frame-like description of higher spin fields and construct vertex operators for
the dynamical and auxiliary connection gauge fields. In the Section 3 we analyze the 3-point correlation function of these operators for spin 3, limiting ourselves to terms with 3 derivatives. The result is given by the Berends-Burgers-Van Dam type 3-derivative vertex in a certain gauge, modulo total derivative terms. In the Discussion section we outline generalizations of the developed formalism for AdS case and discuss the relation between the vertex operators, constructed in this paper and generators of higher spin algebra in AdS.

2. Frame-like Formalism and Vertex Operators for Connection gauge fields

Frame-like formalism in higher spin field theories, originally proposed by Vasiliev and later developed in a number of works (e.g. see [4], [2], [39], [70], [71], [72], [26], [43], [44]) is a powerful tool to describe gauge-invariant interactions of higher spin fields in various backgrounds including anti de Sitter (AdS) geometry. Unlike the approach used by Frondal that considers higher spin tensor fields as metric-type objects, the frame-like formalism describes the higher spin dynamics in terms of higher spin connection gauge fields that generalize objects such as vielbeins and spin connections in gravity (in standard Cartan-Weyl formulation or Mac Dowell-Mansoury-Stelle-West (MMSW) in case of nonzero cosmological constant). The higher spin connections for a given spin \( s \) are described by collection of two-row gauge fields

\[
\omega^{s-1|t} \equiv \omega_{m}^{a_1...a_{s-1}|b_1...b_t}(x)
\]

\[
0 \leq t \leq s - 1
\]

\[
1 \leq a, b, m \leq d
\]  

traceless in the fiber indices, where \( m \) is the curved \( d \)-dimensional space index while \( a, b \) label the tangent space with \( \omega \) satisfying

\[
\omega_{m}^{(a_1...a_{s-1}|b_1)...b_t} = 0
\]

The gauge transformations for \( \omega \) are given by

\[
\omega_{m}^{a_1...a_{s-1}|b_1...b_t} \rightarrow \omega_{m}^{a_1...a_{s-1}|b_1...b_t} + D_m \rho^{a_1...a_{s-1}|b_1...b_t}
\]

while the diffeomorphism symmetries are

\[
\omega_{m}^{a_1...a_{s-1}|b_1...b_t}(x) \rightarrow \omega_{m}^{a_1...a_{s-1}|b_1...b_t}(x) + \partial_m \epsilon^n(x) \omega_{n}^{a_1...a_{s-1}|b_1...b_t}(x) + \epsilon^n(x) \partial_n \omega_{m}^{a_1...a_{s-1}|b_1...b_t}(x)
\]
The $\omega^{s-1|t}$ gauge fields with $t \geq 0$ are auxiliary fields related to the dynamical field $\omega^{s-1|0}$ by generalized zero torsion constraints:

$$\omega^{a_1 \ldots a_s}_{m^1 \ldots b_t} \sim \partial^{b_1} \ldots \partial^{b_t} \omega^{a_1 \ldots a_s}_{m^1}$$  (5)

skipping pure gauge terms (for convenience of the notations, we set the cosmological constant to 1, anywhere the $AdS$ backgrounds are concerned).

It is also convenient to introduce the $d+1$-dimensional index $A = (a, \hat{d})$ (where $\hat{d}$ labels the extra dimension) and to combine $\omega^{s|t}$ into a single two-row field $\omega^{A_1 \ldots A_{s-1}|B_1 \ldots B_{s-1}}(x)$ identifying $\omega^{s-1|t} = \omega^{a_1 \ldots a_{s-1}|b_1 \ldots b_t \hat{d} \ldots \hat{d}}$

$$\omega^{A_1 \ldots A_{s-1}|B_1 \ldots B_{s-1}} V_{A_{t+1}} \ldots V_{A_{s-1}} = \omega^{A_1 \ldots A_{s-1}|B_1 \ldots B_t}$$  (6)

where $V_A$ is the compensator field satisfying $V_A V^A = 1$. The Fronsdal field $H^{a_1 \ldots a_s}$ is then obtained by symmetrizing $\omega^{(a_1 \ldots a_s)} = e^m(a_s \omega_m^{a_1 \ldots a_{s-1}})$. We now turn to the question of constructing vertex operators for $\omega^{s-1|t}$. The operators for spins greater than 2 constructed in our previous works [66] were in fact limited to the Fronsdal-type objects only. In particular, in RNS superstring theory the operators for $s = 3$ are given by

$$V^{(-3)} = H_{abm}(p) e^{-3\phi} \partial^a X^b \psi^m e^{ipX}$$  (7)

at unintegrated minimal negative picture and

$$V^{(+1)} = K \circ H_{abm}(p) \int dz e^\phi \partial^a X^b \psi^m e^{ipX}$$  (8)

at integrated minimal positive picture +1 where $a, b, m = 0, \ldots d-1$ are Minkowski space-time indices, $X^a(z)$ are space-time coordinates, $\psi^a$ are their worldsheet superpartners, $b, c$ are reparametrizational fermionic ghosts and $\beta, \gamma$ are bosonic superconformal ghosts. The homotopy transformation $K \circ T$ of an integrated operator $T = \oint dz V(z)$ (with $V(z)$ being a primary field of dimension 1) is defined according to

$$K \circ T = T + \frac{(-1)^N}{N!} \oint \frac{dz}{2i\pi} (z-w)^N : K \partial^N W : (z)$$

$$+ \frac{1}{N!} \oint \frac{dz}{2i\pi} \partial^N \partial^N [(z-w)^N K(z)] K \{ Q_{brst}, U \}$$  (9)

where the BRST operator is

$$Q = \oint dz \{ cT - b \partial c - \frac{1}{2} \gamma \psi_m \partial X^m - \frac{1}{4} b \gamma^2 \}$$  (10)
is the BRST operator, $K = -4ce^{2\chi-2\phi}$ is homotopy operator satisfying $\{Q, K\} = 1$. $U$ and $W$ are the operators appearing in the commutator $[Q, V(z)] = \partial U(z) + W(z)$ and the ghost fields are bosonized as usual according to

$$
\begin{align*}
    c &= e^\sigma \\
    b &= e^{-\sigma} \\
    \gamma &= e^{\phi-\chi} \equiv e^{\phi} \eta \\
    \beta &= e^{\chi-\phi} \partial \chi \equiv e^{-\phi} \partial \xi
\end{align*}
$$

The operators (7) and (8) are the elements of negative and positive ghost cohomologies $H_{-3}$ and $H_1$ respectively (see [66] for definitions and review). They are related according to $V^{(+1)} =: Z \Gamma^2 Z \Gamma^2 : V^{(-3)}$ by combination of BRST-invariant transformations by picture-changing operators for $b - c$ and $\beta - \gamma$ systems: $Z =: b \delta(T) :$ and $\Gamma =: \delta(\beta)G :$ ($T$ is the full stress tensor and $G$ is the supercurrent), therefore the on-shell conditions and gauge transformations for $H_{abm}$ at positive and negative pictures are identical. The manifest expression for $V^{(+1)}$ is given by

$$
V_{s=3}(p, w) = \int dz (z - w)^2 U(z) \equiv A_0 + A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 \quad (12)
$$

where

$$
A_0(p, w) = \frac{1}{2} H_{abm}(p) \int dz (z - w)^2 P^{(2)}_{2\phi-2\chi-\sigma} e^\phi \partial X^a \partial X^b \psi^m e^{ip\vec{X}}(z) \quad (13)
$$

and

$$
A_8(w) = H_{abm}(p) \int dz (z - w)^2 \partial cc \partial \xi \xi e^{-\phi} \partial X^a \partial X^b \psi^m \} e^{ip\vec{X}}(z) \quad (14)
$$

have ghost factors proportional to $e^{\phi}$ and $\partial cc \partial \xi \xi e^{-\phi}$ respectively and the rest of the terms
carry ghost factor proportional to \( c\xi \):

\[
A_1(p; w) = -2H_{abm}(p) \oint dz(z - w)^2 c\xi(\bar{\psi}\partial\bar{X})\partial X^a \partial X^b \psi^m e^{ip\vec{X}}(z)
\]

\[
A_2(p; w) = -H_{abm}(p) \oint dz(z - w)^2 c\xi \partial X^a \partial X^b \partial X^m P^{(1)}_{\phi-\chi} e^{ip\vec{X}}(z)
\]

\[
A_3(p; w) = H_{abm}(p) \oint dz(z - w)^2 c\xi \partial X^a \partial X^b \partial^2 X^m e^{ip\vec{X}}(z)
\]

\[
A_4(p; w) = 2H_{abm}(p) \oint dz(z - w)^2 c\xi \partial^a \partial X^b \psi^m e^{ip\vec{X}}(z)
\]

\[
A_5(p; w) = 2H_{abm}(p) \oint dz(z - w)^2 c\xi \partial^a \partial X^b \partial X^m P^{(1)}_{\phi-\chi} e^{ip\vec{X}}(z)
\]

\[
A_6(p; w) = 2iH_{abm}(p) \oint dz(z - w)^2 c\xi (\bar{\psi}\partial\bar{X}) \partial X^a \partial X^b \psi^m e^{ip\vec{X}}(z)
\]

\[
A_7(p; w) = 2iH_{abm}(p) \oint dz(z - w)^2 c\xi (\bar{\psi}\partial\bar{X}) \partial X^a \partial X^b \psi^m e^{ip\vec{X}}(z)
\]

Here \( w \) is an arbitrary point in on the worldsheet; since all the \( w \)-derivatives of \( s = 3 \) operators are BRST-exact in small Hilbert space [66], all the correlation functions involving higher spin operators \( V_{s=3}(p, w) \) are \( w \)-independent and the choice of \( w \) is arbitrary. Conformal dimension \( n \) polynomials \( P^{(n)}_{A\phi+B\chi+C\sigma} \) (where \( A, B, C \) are some numbers) are defined according to

\[
e^{-A\phi(z)-B\chi(z)-C\sigma(z)} \frac{d^n}{dz^n} e^{A\phi(z)+B\chi(z)+C\sigma(z)}
\]

(16)

(where the product is understood in algebraic rather than OPE sense).

As it is straightforward to check, the BRST-invariance constraints on the operators (7) and (8) lead to Pauli-Fierz type conditions

\[
p^2H_{abm} = p^a H_{abm} = \eta^{ab} H_{abm} = 0
\]

(17)

However, in general

\[
\eta^{am} H_{abm} \neq 0
\]

(18)

as the tracelessness in \( a \) and \( m \) or \( b \) and \( m \) indices isn’t required for \( V^{(-3)} \) to be primary field. In what follows below we shall interpret \( H_{abm} \) with the dynamical spin 3 connection form \( \omega^{2|0} \), identifying \( m \) with the manifold index and \( a, b \) with the fiber indices. So the tracelessness condition is generally imposed by BRST invariance constraint on any pair of
fiber indices only (but not on a pair of manifold and fiber indices). The same is actually true also for the vertex operators for frame-like gauge fields of spins higher than 3. Altogether, this corresponds precisely to the double tracelessness constraints for corresponding metric-like Fronsdal’s fields for higher spins (although the zero double trace condition does not of course appear in the case of \( s = 3 \)) As it is clear from the manifest expressions (7), (8) the tensor \( H_{abm} \) is by definition symmetric in indices \( a \) and \( b \) and therefore can be represented as a sum of two Young diagrams. However, only the fully symmetric diagram is the physical state, since the second one (with two rows) can be represented as the BRST commutator in the small Hilbert space:

\[
V^{(-3)} \sim \{Q, W\}
\]

\[
W = H_{abm}(p) c \partial \xi e^{-4\phi + ipX} \partial X^a(\psi^m \partial^2 \psi^b) - 2\psi^m \partial \psi^b \partial \phi + \psi^m \psi^b \left( \frac{5}{13} \partial^2 \phi + \frac{9}{13} (\partial \phi)^2 \right) + a \leftrightarrow b
\]

If \( \Omega_{abm} \) is two-row, the \( V^{(-3)} \) operator is obtained as the commutator of \( W \) with the matter supercurrent term of \( Q \) given by \( \sim \oint \gamma \psi_m \partial X^m \). As \( W \) commutes with \( \oint (-\frac{1}{4} b\gamma^2 - bc \partial c) \) term in \( Q \), \( V^{(-3)} \) is BRST-exact if and only if it commutes the stress energy part of \( Q \) given by \( \oint cT \). This is the case if the integrand of \( W \) is a primary field. It is, however, easy to check that the integrand is primary only when the last term in its expression is present. Since this term is proportional to \( \sim \partial \xi e^{-4\phi + ipX} \partial X^a \psi^m \psi^b \left( \frac{5}{13} \partial^2 \phi + \frac{9}{13} (\partial \phi)^2 \right) \) it is automatically antisymmetric in \( m \) and \( b \) and is absent when multiplied by fully symmetric \( H_{abm} \). In the latter case this term is not a primary since its OPE with \( T \) contains cubic singularities and therefore the commutator of \( Q \) with \( W \) does not give \( V^{(3)} \). Similarly, shifting \( H_{abm} \) by symmetrized derivative \( H_{abm} \rightarrow H_{abm} + p(m \Lambda_{ab}) \) is equivalent to shifting the vertex operator (7) by BRST exact terms given by

\[
V^{(-3)} \rightarrow V^{(-3)} + \{Q, U\}
\]

\[
U = \Lambda_{ab} c \partial \xi e^{-4\phi + ipX} \partial X^a \left( (p\psi) \partial^2 \psi^b - 2(p\psi) \partial \psi^b \partial \phi \right) + (p\psi) \psi^b \left( \frac{5}{13} \partial^2 \phi + \frac{9}{13} (\partial \phi)^2 \right) + \partial X^a \partial X^b \left( (p \partial^2 X) - \partial \phi (p \partial X) \right)
\]

Of course everything described above also applies to the vertex operator (8) at positive picture, with appropriate \( Z, \Gamma \) transformations. This altogether already sends a strong hint to relate (7), (8) to vertex operators for the dynamical frame-like field \( \omega^{2|0} \) describing spin
3. However, to make the relation between string theory and frame-like formalism, we still need the vertex operators for the remaining extra fields $\omega^{2|1}$ and $\omega^{2|2}$. The expressions that we propose are given by

$$V^{2|1}(p) = 2\omega^{ab|c}(p)ce^{-4\phi}(-2\partial\psi^m\psi_c\partial X_a\partial^2 X_b)$$

$$-2\partial\psi^m\partial\psi_c\partial X_a\partial X_b + \psi^m\partial^2\psi_c\partial X_a\partial X_b)e^{ipX}$$

(21)

for $\omega^{2|1}$ and

$$V^{2|2}(p) = -3\omega^{ab|cd}(p)ce^{-5\phi}(\psi^m\partial^2\psi_c\partial^2\psi_d\partial X_a\partial X_b - 2\psi^m\partial\psi_c\partial^3\psi_d\partial X_a\partial^2 X_b$$

$$+ \frac{5}{8}\psi^m\partial\psi_c\partial^2\psi_d\partial X_a \partial X_b + \frac{57}{16}\psi^m\partial\psi_c\partial^2\psi_d\partial^2\partial X_a \partial^2 X_b)e^{ipX}$$

(22)

for $\omega^{2|2}$. We start with analyzing the operator for $\omega^{2|1}$. Straightforward application of $\Gamma$ to this operator gives

$$\Gamma V^{2|1} : (p) = V^{(-3)}(p)$$

$$H^{ab}_m(p) = ip_c\omega^{ab|c}(p)$$

(23)

i.e. the picture-changing of $V^{2|1}$ gives the vertex operator for $\omega^{2|0}$ with the 3-tensor given by the divergence of $\omega^{2|1}$, i.e. for $p_c\omega_{m}^{ab|c}(p) \neq 0$ $V^{2|1}$ is the element of $H_{-3}$. If, however, the divergence vanishes, the cohomology rank changes and $V^{2|1}$ shifts to $H_{-4}$. This is precisely the case we are interested in. Namely, consider the $H_{-4}$ cohomology condition

$$p_c\omega_m^{ab|c}(p) = 0$$

(24)

The general solution of this constraint is

$$\omega_m^{ab|c} = 2p^c\omega_m^{ab} - p^a\omega_m^{bc} - p^b\omega_m^{ac} + p_d\omega_m^{acd;b}$$

(25)

where $\omega_m^{ab}$ is traceless and divergence free in $a$ and $b$ and satisfies the same on-shell constraints as $H^{ab}_m$, while $\omega_m^{acd;b}$ is some three-row field, antisymmetric in $a,c,d$ and symmetric in $a$ and $b$. It is, however, straightforward to check that the operator $V^{2|1}$ with the polarization given by $\omega_m^{ab|c} = p_d\omega_m^{acd;b}$ can be cast as the BRST commutator:

$$p_d\omega_m^{acd;b}(p)V_{ac|b}^m(p) = \{Q, \omega_m^{acd;b}(p)\} = \left\{ \partial\psi^m\psi_c\partial X_a\partial X_b, \right\}$$

(26)
therefore, modulo pure gauge terms the cohomology condition (24) is the zero torsion condition relating the extra field $\omega^{2|1}$ to the dynamical $\omega^{2|0}$ connection. Similarly, constraining $V^{2|2}$ to be the element of $H_{-5}$ cohomology results in the second generalized zero torsion condition

$$\omega_{m}^{ab|cd} = 2p^{d}w^{ab|c} - p^{a}w^{bd|c} - p^{b}w^{ad|c} + 2p^{c}w^{ab|d} - p^{a}w^{bc|d} - p^{b}w^{ac|d}$$

relating $\omega^{2|2}$ to $\omega^{2|1}$ modulo BRST-exact terms $\sim \{Q, W^{2|2}(p)\}$ where

$$W^{2|2}(p) = \omega^{ab;cdf}(p) \int dx^{ip}X_{m} [ (\psi^{m} \partial^{2} \psi_{c} \partial^{3} \psi_{a} \partial X_{b} - 2\psi^{m} \partial \psi_{c} \partial^{3} \psi_{a} \partial X_{b} - 2\psi^{m} \partial \psi_{c} \partial^{3} \psi_{a} \partial X_{b} )$$

$$\times (-5/2 L_{f} \partial^{2} \xi + \partial L_{f} \partial \xi)$$

(28)

where, as previously, $\xi = e^{\chi}$ and

$$L_{f} = e^{-6\phi}(\partial^{2} \psi_{f} - \partial \psi_{f} \partial \phi + 3/25 \psi_{f} ((\partial \phi)^{2} - 4\partial^{2} \phi))$$

(29)

. The gauge transformations for $\omega^{2|1}$ and $\omega^{2|2}$: $\delta \omega_{m}^{ab|c}(p) = p_{m} \Lambda^{ab|c}$ and $\delta \omega_{m}^{ab|cd}(p) = p_{m} \Lambda^{ab|cd}$ with $\Lambda$’s having having the same symmetries in the fiber indices as $\omega$’s shift the operators (21), (22) by terms that are BRST-exact in the small Hilbert space; the explicit expressions for the appropriate BRST commutators are given in the Appendix B. Similarly to the $\omega^{2|0}$ case, for $\omega^{2|1}$ and $\omega^{2|2}$ with the manifold $m$ index antisymmetric with any of the fiber indices $a$ or $b$ the operators (21), (22) become BRST-exact in the small Hilbert space. Given the cohomology (“zero torsion”) conditions (25), (27) ensures that the fully symmetric symmetric $s = 3$ Fronsdal field is related to dynamical field $\omega^{2|0}$ by the gauge transformation removing the two-row diagram. The expressions for the appropriate BRST commutators are given in the Appendix B.

This concludes the construction of the vertex operators for frame-like gauge fields for spin 3. In the next section we shall use this construction to analyze the 3-point open string amplitude for spin 3.

3. 3-Point Amplitude and 3-Derivative Vertex

In this section we use the vertex operator formalism, developed in the previous section, to compute the cubic coupling of massless spin 3 fields. In this paper we limit ourselves to the 3-derivative contributions corresponding to the Berends, Burgers and Van Dam (BBD)
The first step is to choose the ghost pictures of the operators to ensure the correct ghost number balance, i.e. so that the correlator has total $\phi$-ghost number $-2$, $b-c$ ghost number $+3$ and $\chi$-ghost number $+1$. This requires two out of three operators to be taken unintegrated at negative pictures and the third one at positive picture (note that higher spin operators at positive pictures are always integrated). It is convenient to take unintegrated operators at the minimal ghost picture $-3$, i.e. we shall use the $V^{(-3)}$ operator for $\omega^{2,10}$. Then the remaining integrated operator must be taken at picture $+5$, and only the terms proportional to the ghost factor $\sim e^{\chi+4\phi}$ will contribute, while the terms proportional to $\sim dcce^{2\chi+3\phi}$ and to $\sim e^{5\phi}$ will drop out as they don’t satisfy the balance of ghosts. It is therefore appropriate to choose the operator for $\omega^{2,12}$ for the third operator (which minimal positive picture is $+3$) and to apply the picture changing transformation twice to bring it to the picture $+5$. The result is given by

$$\{Q, \xi \{Q, \xi K \circ \omega_{ab|cd}^m(p) \int d\epsilon \phi^m F_{abcd}^{m(12)} \} \} = V_1(p) + V_2(p)$$

(30)

where

$$V_1(p) = \frac{3}{64} \omega_{ab|cd}^{m=12}(p) \int du(u_0 - u)^8 c e^{\chi+4\phi+ipX} R_{ab|cd}^m(u)$$

(31)

and

$$V_2(p) = \frac{1}{9! - 8!} \sum_{n=0}^{7} 2^{n-7} \sum_{\{l,m,p,q\geq 0; l+m+p+q=8-n\}} \sum_{r=0}^{p} \sum_{a=0}^{l} \sum_{b=0}^{q} \sum_{N=0}^{a+b+r+5} (-1)^{a+b+p+q+N}$$

$$\times \omega_{m}^{ab|cd}(p) \int du(u_0 - u)^8 c e^{\chi+4\phi} \partial^{(p-r)} L_{abcd}^{m(N+9)}$$

$$\times \partial^{(m)} P_{2\phi-2\chi|\chi}^{l-a|l} P_{\phi-\chi|\phi-\chi}^{q-b|q} P_{\phi-\chi}^{(5+a+b+r-N)}(u)$$

(32)

Here $u_0$ is an arbitrary point on the boundary (will be fixed later) and

$$F_{abcd}^{m(12)} = (\psi^m \partial^2 \psi_c \partial^3 \psi_d \partial X^a \partial X_b - 2\psi^m \partial \psi_c \partial^4 \psi_d \partial X_a \partial^2 X_b + \frac{5}{8} \psi^m \partial \psi_c \partial^2 \psi_d \partial^3 X_a \partial^2 X_b) e^{ipX}$$

(33)
is dimension $\frac{15}{2}$ primary field (given the on-shell conditions on $\omega$); conformal dimension $N + 9$ fields are defined as the OPE terms in the product of $F_{abcd}^{m(\frac{15}{2})}$ with the matter supercurrent $G = -\frac{1}{2} \psi_n \partial X^n$ on the worldsheet:

$$G_m(z) F_{abcd}^{m(\frac{15}{2})}(w) = \sum_{N=0}^{\infty} (z - w)^{N-1} L_{abcd}^{m(N+9)}(w)$$

or manifestly

$$L_{abcd}^{m(N+9)} = \frac{e^{ipX}}{N!} \left\{ \frac{15}{8} (\partial X_a \partial^3 X_b + \frac{171}{16} \partial^2 X_a \partial^2 X_b)(\partial^{(N+1)} X^m \partial \psi_c \partial^2 \psi_d) \right. - \frac{1}{N + 1} \partial^{(N+2)} X_c \psi^m \partial^2 \psi_d + \frac{2}{(N + 1)(N + 2)} \partial^{(N+3)} X_d \psi^m \partial \psi_c \\
\left. + \psi^m \partial \psi_c \partial^2 \psi_d (-\frac{15}{8(N + 1)} \partial^{N+1} \psi_a \partial^3 X_b - \frac{45}{4(N + 1)(N + 2)(N + 3)} \partial^{N+3} \psi_b \partial X_a \right. \\
\left. - \frac{171}{8(N + 1)(N + 2)} \partial^{N+2} \psi_c \partial^2 X_b) \\
\left. + 3 \partial X_a \partial X_b (\partial^{(N+1)} X^m \partial \psi_c \partial^3 \psi_d - \frac{2}{(N + 1)(N + 2)} \partial^{(N+2)} \psi_b \partial X_a)\right\} \right.$$

The associate ghost polynomials $P_{A_1 \phi + B_1 \chi + C_1 \sigma}^{n|N} A_2 \phi + B_2 \chi + C_2 \sigma$ are conformal dimension $n$ polynomials in derivatives of $\phi, \chi$ and $\sigma$ defined as the terms in the operator product

$$P^{(N)}_{A_1 \phi + B_1 \chi + C_1 \sigma} (z) e^{A_2 \phi + B_2 \chi + C_2 \sigma} (w)$$

$$= \sum_{n=0}^{N} (z - w)^{n-N} P^{n|N}_{A_1 \phi + B_1 \chi + C_1 \sigma} A_2 \phi + B_2 \chi + C_2 \sigma (z) e^{A_2 \phi + B_2 \chi + C_2 \sigma} (w)$$

(see the Appendix A for some of the techniques related to these polynomials). Note that, for example,

$$P^{N|N}_{A_1 \phi + B_1 \chi + C_1 \sigma} A_2 \phi + B_2 \chi + C_2 \sigma \equiv P^{(N)}_{A_1 \phi + B_1 \chi + C_1 \sigma}$$
while
\[ P_{A_1 \phi+B_1 \chi+C_1 \sigma|A_2 \phi+B_2 \chi+C_2 \sigma}^{0|N} = \prod_{k=0}^{n-1} (C_1 C_2 + B_1 B_2 - A_1 A_2 - k) \] (38)

We are now prepared to analyze the 3-point function given by
\[ A(p, k, q) = \omega_n^{s_1 s_2}(p) \omega_p^{t_1 t_2}(k) \omega_m^{a b|c d}(q) \int du (u - u_0)^8 \]
\[ \times < c \partial X_{s_1} \partial X_{s_2} \psi^n e^{ipX(z)} c \partial X_{t_1} \partial X_{t_2} \psi^n e^{ikX(w)} c e^{4\phi+\chi(u)} R_{mab|cd}^m(u) > \] (39)

Using the $SL(2, R)$ symmetry, it is convenient to set $z \to \infty, u_0 = w = 0$ (see [67] where the details related to this choice were discussed). For the notation purposes, however, it is convenient to retain $z$ and $w$ in our notations for a time being. We start with computing the “static” exponential ghost part of the correlator. Simple calculation gives
\[ < c e^{-3\phi(z)} c e^{-3\phi(w)} c e^{4\phi+\chi(u)} > = (z - w)^{-8} (z - u)^{13} (w - u)^{13} \to z^5 (w - u)^{13} \] (40)

where we substituted the $z \to \infty$ limit. Next, consider the $\psi$-part of the correlator. The expression for $R_{mab|cd}^m$ contains two types of terms: those that are quadratic in $\psi$ and those that are quartic $\psi$. Since the remaining two spin 3 operators are linear in $\psi$, only the quadratic terms contribute to the correlator. Note that all the terms quadratic in $\psi$ are also cubic in $\partial X$. So the pattern for the $\psi$-correlators is
\[ < \psi^n(z) \psi^p(w) \partial^{(P_1)} \psi_c \partial^{(P_2)} \psi_d > = P_1! P_2! \left( \frac{\eta^p_c \eta^p_d}{z P_{P_1+1} (w - u)^{P_1+1}} - \frac{\eta^p_c \eta^p_d}{z P_{P_2+1} (w - u)^{P_2+1}} \right) \] (41)

where, according to the manifest expression (35) for $R_{ab|cd}^m$, the numbers $P_1$ and $P_2$ can vary from 0 to $N + 3$ (and $N_{\text{max}} = 8$). Next, consider the $X$-part. As in this paper we limit ourselves to just three-derivative terms, it is sufficient to compute the terms linear in momentum (since the $\omega_2^{l2}$ field already contains 2 derivatives out of 3). According to (31), (32) and (35) the $X$-factor is a combination of the 3-point correlators of the type
\[ \sim (\omega^{2|0}_2)^2 \omega^{2|2} \sim < (\partial X)^2 e^{ipX(z)} (\partial X)^2 e^{ikX(w)} \partial^{(M_1)} X \partial^{(M_2)} X \partial^{(M_3)} X e^{iqX} > \] with different
values of $M_1$, $M_2$ and $M_3$. Straightforward computation gives

$$lim_{z \to \infty} \omega_n^{s_1s_2} (p) \omega_l^{t_1t_2} (k) \omega_m^{abcd} (q)$$

$$ \times < \partial X_{s_1} \partial X_{s_2} e^{ipX} (z) \partial X_{t_1} \partial X_{t_2} e^{ikX} (w) \partial^{(M_1)} X_a \partial^{(M_2)} X_b \partial^{(M_3)} X^m e^{iqX} (u) >$$

$$= M_1! M_2! M_3! \omega_n^{s_1s_2} (p) \omega_l^{t_1t_2} (k) \omega_m^{abcd} (q) \left\{ \frac{2i q t_2 \eta_{s_1 a} \eta_{s_2 b} \eta_{t_1} \eta_{t_2} m}{z^{2+M_1+M_2} (w-u)^2+M_3} + \frac{1}{z^{2+M_1+M_2} (w-u)^2+M_3} \right\}$$

$$+ i q t_2 \eta_{s_1 a} \eta_{s_2 b} \eta_{t_1} \eta_{t_2} \left( \frac{1}{z^{3+M_1} (w-u)1+M_1+M_3} + \frac{1}{z^{3+M_1} (w-u)1+M_1+M_3} \right)$$

$$- \frac{1}{M_3} \eta_{s_1 t_1} \eta_{s_2 a} \eta_{t_2 b} \left( \frac{1}{z^{3+M_1} (w-u)1+M_2+M_3} + \frac{1}{z^{3+M_1} (w-u)1+M_2+M_3} \right)$$

$$+ i p t_2 \eta_{s_1 a} \eta_{s_2 b} \eta_{t_2} \left( \frac{1}{z^{3+M_1} (w-u)1+M_2+M_3} + \frac{1}{z^{3+M_1} (w-u)1+M_2+M_3} \right)$$

$$+ i p \eta_{s_1 t_1} \eta_{s_2 a} \eta_{t_2} \left( \frac{1}{M_2} \right)$$

$$= \frac{1}{M_3} \eta_{s_1 t_1} \eta_{s_2 a} \eta_{t_2 b} \left( \frac{1}{z^{3+M_1} (w-u)1+M_2+M_3} + \frac{1}{z^{3+M_1} (w-u)1+M_2+M_3} \right)$$

$$+ i p \eta_{s_1 t_1} \eta_{s_2 a} \eta_{t_2} \left( \frac{1}{M_2} \right)$$

(42)

Comparing this with the explicit expression (30)-(35) for the $\omega^{2|2}$ vertex operator, it is easy to notice that, while the static ghost factor (40) is proportional to $\sim z^5$, the $z$-asymptotics of the $\psi$-correlator (41) is $\sim \frac{1}{z} + O(\frac{1}{z^2})$ and the asymptotics for the $X$-correlator is $\sim \frac{1}{z} + O(\frac{1}{z^2})$. This means that only the terms proportional to $\sim z^0$ contribute to the interaction vertex. Terms proportional to negative powers of $z$ disappear in the limit $z \to \infty$ and correspond to pure gauge contributions. There are no terms proportional to positive powers of $z$ (their presence would be a signal of problems with the gauge invariance). Moreover the $z$-asymptotics further simplifies the analysis of the ghost polynomials in the expressions (30)-(32) for the $\omega^{2|2}$ operator; namely, all the polynomials have to couple to the $ce^{-3\phi}$ ghost exponent of the $\omega^{2|0}$ operator sitting at $w$ as any couplings of these polynomials with the operator sitting at $z$ produce contributions vanishing in the $z \to \infty$ limit.

Combining (30), (32), (41) and (42) we arrive to the following expression for the main
matter building block for the matter part of the correlator:

\[
\lim_{z \to \infty} \omega_n^{s_1 s_2}(p) \omega_p^{t_1 t_2}(k) \omega_m^{a b c d}(q) \\
\times < \psi^n \partial X_{s_1} \partial X_{s_2} e^{i p X(z)} \psi^p \partial X_{t_1} \partial X_{t_2} e^{i k X(w)} L_{abcd}^{m(N)}(u; q) > \]

\[
= z^{-5}(w - u)^{-9 - N} A_N(p, k, q)
\]

\[
A_N(k, p, q) = \omega_n^{s_1 s_2}(p) \omega_p^{t_1 t_2}(k) \omega_m^{a b c d}(q) \times \{ \eta^{n m} \eta_{p d} (-72(N + 5)\eta^{s_1 a} \eta^{s_2 b} t_1 c q ^{t_2} + 144 - \frac{45}{4} N(N + 1)) \eta^{s_1 t_1} \eta^{s_2 a} \eta^{t_2 b} k ^{c} + Symm(m, a, b) \}
\]

Using the manifest expression (35) for the \( V_{2|2} \) vertex operator in terms of \( L_{9+N} \) and their derivatives, it is now straightforward to calculate the cubic coupling. First of all, it is immediately clear that only the \( V_2 \)-part of \( V_{2|2} \) contributes to the overall correlator. No terms from \( V_1 \) contribute since, as it was pointed out above, all the ghost polynomials entering \( V_{2|2} \) must be completely absorbed by the ghost exponent \( \sim ce^{-3\phi} \) located at \( w \) (no couplings to the exponent at \( z \) are allowed as they would result in contributions vanishing at \( z \to \infty \)). At the same time all the terms in \( V_1 \) carry the factors of \( P_{2|2}^{(n)}(n = 11, 12, 13) \) which cannot be absorbed by \( ce^{-3\phi} \) (i.e. their OPE’s with \( ce^{-3\phi} \) are less singular than \( (z - w)^{-n} \)). Indeed, since \( e^{2\phi - 2\chi b(z)}ce^{-3\phi}(w) \sim (z - w)^{5}e^{-\phi - 2\chi}(w) + O(z - w)^{6} \), clearly for \( n \geq 5 \)

\[
\partial^{(n)}(e^{2\phi - 2\chi b}(z)) \equiv ce^{-3\phi}(w) \equiv: P_{2\phi - 2\chi - \sigma}^{(n)}e^{2\phi - 2\chi b} : ce^{-3\phi}(w)
\]

\[
\sim \frac{n!}{(n - 5)!} : P_{2\phi - 2\chi - \sigma}^{(n-5)}e^{-\phi - 2\chi} : (w) + O(z - w)
\]

implying that

\[
P_{2\phi - 2\chi - \sigma}^{(n)}(z)ce^{-3\phi}(w) \sim O\left(\frac{1}{z^5}\right)
\]

i.e. no complete contractions for \( n \geq 6 \). Next, combining (42) with the expression (35) for \( V_2(q) \) we obtain the following result for the overall correlator:

\[
\int \frac{1}{m!}(p - r)!r!(N - r)!(5 + l + b + r - N)!(N - l - b - r - 2)!(N + 8)! \times \alpha_{3;0;1}^{2;0;1}(q - b|q) \alpha_{2;1;0}^{-3;0;1}(s|s) \alpha_{3;1;0}^{1;0;1}(n|8) A_N(p, k, q)
\]

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where we used the fact that the only nonzero contributions from the summation over $a$ are the terms with $a = l$ for which $P_{\chi|\phi-\chi}^{l-a|l} = P_{\chi|\phi-\chi}^{0|l} = (-1)^l l!$; while for $a \neq l$ $P_{\chi|\phi-\chi}^{l-a|l}$ are the polynomials in $\chi$ of dimension $l - a$ which aren’t contractible with $ce^{-3\phi}$. The numbers $\alpha_{A_3; B_3; C_3}^{A_1; B_1; C_1|A_2; B_2; C_2}(n|N)$ appearing in (46) are the coefficients in front of the leading order terms in the operator products

$$P_{A_1|\phi+B_1|\chi+C_1|\sigma}^{A_3; B_3; C_3}(z)e^{A_3+\phi+B_3+\chi+C_3|\sigma}(w) \sim \frac{\alpha_{A_1; B_1; C_1|A_2; B_2; C_2}(n|N)}{(z-w)^n} e^{A_3+\phi+B_3+\chi+C_3|\sigma}(w)$$

(47)

The calculation of these coefficients is explained and the values are given in the Appendix A (see (58), (60)). Finally, substituting for $A_N(p, k, q)$ and evaluating the series (46) we obtain the following answer for the cubic coupling:

$$A(p, k, q) = \frac{691072283467i}{720} \omega^{s_1s_2}(p)\omega^{t_1t_2}(k)\omega^{ab|cd}(q) \times \left\{ \eta^{nm}_{pd} \left( \eta^{s_1a}_{p} \eta^{s_2b}_{q} \eta^{t_1c}_{t} \eta^{t_2d}_{t} \right) + \frac{4}{3} \eta^{t_1a}_{t} \eta^{s_1b}_{s} \eta^{t_2c}_{t} k^{s_2} + \frac{1}{12} \eta^{s_1a}_{s} \eta^{s_2a}_{t} \eta^{t_2b}_{t} k^{c} - \eta^{t_1a}_{s} \eta^{s_2a}_{t} \eta^{t_2b}_{t} p^{b} \right\} + \text{Symm}(m, a, b)$$

(48)

This concludes the calculation of the 3-derivative part of the cubic vertex. Inclusion of the appropriate Chan-Paton’s indices is straightforward and leads to vertices of the type considered in [21], [73].

4. Conclusion and Discussion

In this paper we performed the analysis of open string vertex operators describing generalized connection gauge fields in Vasiliev’s frame-like formalism for higher spin fields. We have shown that generalized zero curvature conditions relating auxiliary connections $\omega^{s-1|t}$ to the dynamical $\omega^{s|0}$ fields are realized (up to BRST-exact terms) through ghost cohomology conditions on vertex operators that ensure that the fields with higher values of $t$ belong to cohomologies of higher orders. We have also given precise BRST arguments relating $\omega^{2|0}$ to symmetric Fronsdal field for spin 3, presenting BRST commutator for non-symmetric spin 3 diagram (an important point which has been somewhat obscure before). We also demonstrated how the 3-derivative cubic vertex of spin 3 fields appears from string-theoretic 3-point amplitude. Computed in this work. Obvious directions for future research include the computation of the 5-derivative vertex in the flat space (which
technically appears to be significantly more tedious than the 3-derivative one) and to generalize the construction proposed in this work to frame-like gauge fields with spins greater than 3. We hope to present these results soon in our future papers. The cubic vertex computed in this work is the one for the flat space and an important next step would be to generalize it to AdS. For that, one has to generalize the computation, analyzing of the 3-point function of operators for frame-like spin 3 fields in the sigma-model background studied in [74]. That is, one has to perturb the flat background with the vertex operators for spin 2 vielbeins and connections in AdS space constructed in [74], [75]. These operators carry negative cosmological constant and the vacuum solution of the low-energy equations of motion is described by AdS geometry. To calculate the cubic coupling of spin 3 frame-like fields in the AdS space one has to consider their disc amplitude with insertions of closed string operators for spin 2 connections. As the insertions carry the dependence on the cosmological constant parameter, important question to explore is the relation of this amplitude with to AdS deformations of flat vertices considered by Vasiliev by methods of vertex complex analysis [26]. It is particularly interesting to clarify how the insertions of the closed string operators give rise to terms with lower the number of derivatives, as it was observed in [26] for the AdS deformations of vertices in Minkowski space. Another issue is to explore the relevance of the zero momentum parts of the vertex operators for frame-like fields to space-time symmetry generators and higher spin algebra in AdS space. Typically, physical vertex operators in string theory are related to generators of global space-time symmetries in the zero momentum limit. For example, a photon operator at \( p = 0 \) is the generator of translations. The similar question can be asked about the vertex operators of frame-like gauge fields for higher spins at zero momentum. While in general these operators at \( p = 0 \) do not generate global symmetries for RNS string theory in flat space, it is possible that they realize the symmetries of the sigma model perturbed by operators for spin 2 connections and vielbeins, provided that AdS vacuum constraints are imposed on spin 2. So far we have been able to show this for spin 3 only [76] and this conjecture needs to be generalized for higher spins. If the vertex operators for the frame-like gauge fields are indeed related to the symmetries of the sigma-model, their operator algebras may provide nice realizations of higher spin algebras in various AdS backgrounds.

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Appendix A. Associate Ghost Polynomials and $\alpha_{A_3;B_3;C_3}^{A_1;B_1;C_1|A_2;B_2;C_2}(n|N)$-coefficients

In this Appendix section we explain some of the techniques to calculate the $\alpha$-coefficients that appear in the series (46) for the spin 3 cubic coupling. As was explained above, the associate ghost polynomials $P_{A_1;B_1;C_1|A_2;B_2;C_2}(n|N)$ are defined as coefficients in front of the leading order $n$ terms in the OPE

$$P_{A_1;B_1;C_1|A_2;B_2;C_2}^{(N)}(z)e^{A_2\phi+B_2\chi+C_2\sigma}(w)$$

where the conformal dimension $N$ polynomials $P_{A_1;B_1;C_1|A_2;B_2;C_2}^{(N)}$ are defined according to (16). Then the $\alpha$-coefficients $\alpha_{A_3;B_3;C_3}^{A_1;B_1;C_1|A_2;B_2;C_2}(n|N)$ are defined as coefficients in front of the leading order $n$ terms in the OPE

$$P_{A_1;B_1;C_1|A_2;B_2;C_2}^{(n)}(z)e^{A_2\phi+B_2\chi+C_2\sigma}(w) \sim$$

$$\alpha_{A_1;B_1;C_1|A_2;B_2;C_2}(n|N)e^{A_3\phi+B_3\chi+C_3\sigma}(w)$$

(if the actual leading order of a given OPE is less than $n$, the appropriate coefficient is zero) Although the manifest form of the associate ghost polynomials (AGP) is generally complicated, there is an algorithm significantly simplifying the computations of both AGP and related $\alpha$-coefficients. The algorithm is based on comparison of two similar operator products. Below we shall explain the algorithm and present the results for $\alpha_{3;1;0|1;1;0}(q-b|q)$ (where $0 \leq b \leq q \leq 8$) and $\alpha_{2;2;0|1;0;1}(n|8)$ entering the series (46). Consider the operator product

$$e^{A_1\phi+B_1\chi+C_1\sigma}(z)e^{A_2\phi+B_2\chi+C_2\sigma}(w) = \sum_{m=0}^{\infty} \frac{(z-w)^{-A_1A_2+B_1B_2+C_1C_2+m}}{m!} \times :e^{(A_1+A_2)\phi+(B_1+B_2)\chi+(C_1+C_2)\sigma}(z)P_{A_1;B_1;C_1}^{(m)}(w) :$$

around the point $w$. Differentiating $N$ times over $z$ we get

$$\partial^{(N)}e^{A_1\phi+B_1\chi+C_1\sigma}(z)e^{A_2\phi+B_2\chi+C_2\sigma}(w)$$

$$= \sum_{m=0}^{\infty} \prod_{l=0}^{N-1} (-A_1A_2+B_1B_2+C_1C_2+m-l)(z-w)^{-A_1A_2+B_1B_2+C_1C_2+m-N} \frac{1}{m!} \times :e^{(A_1+A_2)\phi+(B_1+B_2)\chi+(C_1+C_2)\sigma}(z)P_{A_1;B_1;C_1}^{(m)}(w) :$$
On the other hand this product by definition coincides with the OPE:

\[
\begin{align*}
P_{A_1 \phi + B_1 \chi + C_1 \sigma}^{(N)} & : \ e^{A_1 \phi + B_1 \chi + C_1 \sigma} : (z) e^{A_2 \phi + B_2 \chi + C_2 \sigma} (w) \\
& = \sum_{j,k=0}^{\infty} \sum_{n=0}^{N} \frac{(z - w)^{-A_1 A_2 + B_1 B_2 + C_1 C_2 + n - N + j + k}}{j! k!} \\
& \times : \partial^{(j)} P_{A_1 \phi + B_1 \chi + C_1 \sigma}^{n|N} \times P_{A_1 \phi + B_1 \chi + C_1 \sigma}^{(k)} e^{(A_1 + A_2) \phi + (B_1 + B_2) \chi + (C_1 + C_2) \sigma} : (w)
\end{align*}
\]

(the derivative of the associate polynomials appear since in the definition (49) the polynomials are located at \( z \) while the OPE (53) is around \( w \)). This gives the characteristic equation on \( P_{A_1 \phi + B_1 \chi + C_1 \sigma}^{n|N} : A_2 \phi + B_2 \chi + C_2 \sigma : \):

\[
\begin{align*}
\sum_{j,k=0}^{\infty} \sum_{n=0}^{N} \frac{(z - w)^{-A_1 A_2 + B_1 B_2 + C_1 C_2 + n - N + j + k}}{j! k!} \\
& \times : \partial^{(j)} P_{A_1 \phi + B_1 \chi + C_1 \sigma}^{n|N} P_{A_1 \phi + B_1 \chi + C_1 \sigma}^{(k)} e^{(A_1 + A_2) \phi + (B_1 + B_2) \chi + (C_1 + C_2) \sigma} : \\
& = \sum_{m=0}^{\infty} \prod_{l=0}^{N-1} (-A_1 A_2 + B_1 B_2 + C_1 C_2 + m - l) \\
& \times \frac{(z - w)^{-A_1 A_2 + B_1 B_2 + C_1 C_2 + m - N}}{m!} P_{A_1 \phi + B_1 \chi + C_1 \sigma}^{(m)}
\end{align*}
\]

Matching the coefficients in front of each power of \( (z - w) \) (starting from the most singular term) then gives recurrence relations on \( P_{A_1 \phi + B_1 \chi + C_1 \sigma}^{n|N} : A_2 \phi + B_2 \chi + C_2 \sigma : \) expressing them in terms of conformal dimension \( n \) combinations of \( P_{A_1 \phi + B_1 \chi + C_1 \sigma}^{(l)} \) and their derivatives (with \( 1 \leq l \leq n \)). The coefficients \( \alpha_{A_1;B_3;C_3}^{A_1;B_1;C_1;A_2;B_2;C_2} (n|N) \) are then obtained by replacing each of \( \partial^{(k)} P^{(l)} \) according to

\[
P_{A_1 \phi + B_1 \chi + C_1 \sigma}^{(l)} \rightarrow (-1)^k \prod_{i=0}^{k-1} \prod_{j=0}^{l-1} (l + i)(-A_1 A_3 + B_1 B_3 + C_1 C_3 - j)
\]

in each of these combinations since

\[
P_{A_1 \phi + B_1 \chi + C_1 \sigma}^{(l)} (z) e^{A_3 \phi + B_3 \chi + C_3 \sigma} (w) \\
\sim (z - w)^{-l} \prod_{j=0}^{l-1} (-A_1 A_3 + B_1 B_3 + C_1 C_3 - j) e^{A_3 \phi + B_3 \chi + C_3 \sigma} (w) + O(z - w)^{1-l}
\]

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Applied to $P^{n|8}_{2\phi-2\chi-\sigma|\chi}$ this procedure gives the recurrence relations

$$
P^{n|8}_{2\phi-2\chi-\sigma|\chi} = 9! P^{(n)}_{2\phi-2\chi-\sigma|\chi} - \sum_{k=1}^{n-1} \sum_{l=0}^{k} \frac{\partial^{(l)} P^{k|8}_{2\phi-2\chi-\sigma|\chi} P^{(n-l-k)}_{2\phi-2\chi-\sigma}}{l!(k-l)!}$$

(57)

and the corresponding $\alpha$-coefficients $\alpha_{2,-2,-1|0,1,0}^{3,0,1}(n|8)$ are given by

$$
\begin{align*}
\alpha_{2,-2,-1|0,1,0}^{3,0,1}(0|8) &= 9! \\
\alpha_{2,-2,-1|0,1,0}^{3,0,1}(1|8) &= -8! \times 40 \\
\alpha_{2,-2,-1|0,1,0}^{3,0,1}(2|8) &= 8! \times 150 \\
\alpha_{2,-2,-1|0,1,0}^{3,0,1}(3|8) &= -8! \times 300 \\
\alpha_{2,-2,-1|0,1,0}^{3,0,1}(4|8) &= 8! \times 275 \\
\alpha_{2,-2,-1|0,1,0}^{3,0,1}(5|8) &= -8! \times 94 \\
\alpha_{2,-2,-1|0,1,0}^{3,0,1}(6|8) &= 0 \\
\alpha_{2,-2,-1|0,1,0}^{3,0,1}(7|8) &= 0 \\
\alpha_{2,-2,-1|0,1,0}^{3,0,1}(8|8) &= 0.
\end{align*}
$$

(58)

Similarly, when applied to $P^{q-b|q}_{3\phi+\chi|\phi-\chi}$ ($0 \leq b \leq q \leq 8$) this procedure gives the recurrence relations

$$
P^{q-b|q}_{3\phi+\chi|\phi-\chi} = \frac{(q+3)!}{3!} P^{(q)}_{3\phi+\chi} (\delta_{1;q} + \delta_{2;q} + \delta_{3;q}) - \sum_{k=1}^{q-1} \sum_{l=0}^{k} \frac{\partial^{(l)} P^{k|8}_{3\phi+\chi} P^{(n-l-k)}_{3\phi+\chi}}{l!(k-l)!}$$

(59)
and the corresponding $\alpha$-coefficients are $\alpha_{3,-1,0|1,-1,0}^{-3,0,1}(q-b|q)(0 \leq b \leq q \leq 8)$ are

$$\alpha_{3,-1,0|1,-1,0}^{-3,0,1}(0|q) = \frac{(q+3)!}{6}$$

$$(\alpha_{3,-1,0|1,-1,0}^{-3,0,1}(1|q) = -\frac{3}{2}(q+2)!q \sum_{p=1}^{8} \delta_{p;2q}$$

$$\alpha_{3,-1,0|1,-1,0}^{-3,0,1}(2|q) = (-6(q+3)! + 12(q+2)!q + 72(q+1)!) \sum_{p=2}^{8} \delta_{p;2q}$$

$$\alpha_{3,-1,0|1,-1,0}^{-3,0,1}(3|q) = (28(q+3)! - 42(q+2)!q - 504q!) \sum_{p=3}^{8} \delta_{p;2q}$$

$$\alpha_{3,-1,0|1,-1,0}^{-3,0,1}(4|q) = (84(q+3)! + 126(q+2)!q - 1512(q+1)!) \sum_{p=4}^{8} \delta_{p;2q}$$

$$\alpha_{3,-1,0|1,-1,0}^{-3,0,1}(5|q) = (-639(q+3)! - 339(q+2)!q + 4896(q+1)! - 7560q!) \sum_{p=5}^{8} \delta_{p;2q}$$

$$\alpha_{3,-1,0|1,-1,0}^{-3,0,1}(6|q) = (1352(q+3)! + 600(q+2)!q - 6984(q+1)! + 10080q!) \sum_{p=6}^{8} \delta_{p;2q}$$

$$\alpha_{3,-1,0|1,-1,0}^{-3,0,1}(7|q) = (1158(q+3)! - 3918(q+2)!q + 7992(q+1)! - 52920q!) \sum_{p=7}^{8} \delta_{p;2q}$$

$$\alpha_{3,-1,0|1,-1,0}^{-3,0,1}(8|8) = 9!$$

This fully determines the $\alpha$-coefficients in the series (46).

**Appendix B. BRST Relations and Gauge Transformations for $\omega^{2|1}$ and $\omega^{2|2}$.**

In this Appendix section we present explicit expressions for BRST commutators leading to the gauge transformations for the frame-like fields and relating frame-like and Fronsdal fields for spin 3. The gauge transformation for the $\omega^{2|1}$ field:

$$\omega_{m}^{ab|c}(p) \rightarrow \omega_{m}^{ab|c}(p) + p_{m} \Lambda_{m}^{ab|c}(p)$$  \hspace{1cm} (61)

leads to shifting the $V^{2|1}$ vertex operator (21) by BRST-exact terms: $V^{2|1}(p) \rightarrow V^{2|1}(p) +$
\{Q, W^{2|1}_1(p)\} \text{ where, up to overall numerical factor,}

\begin{equation}
W^{2|1}_1(p) \sim \Lambda^{ab|c}(p) \int dz ce^{-5\phi + ipX} ((p\partial\psi)(\psi_c \partial^2 X_b - 2\partial\psi_c \partial X_b) + (p\psi)\partial^2 \psi_c \partial X_b) \\
\times (\frac{2}{5} \partial L_a \partial \xi - L_a \partial^2 \xi)
\end{equation}

where

\begin{equation}
L_a = \partial^2 \psi_a - 2\partial\psi_a \partial\phi + \frac{1}{13} \psi_a (5\partial^2 \phi + 9(\partial\phi)^2)
\end{equation}

and \(\Lambda\) has the same symmetry in the fiber indices as \(\omega^{2|1}\). This operator is BRST-exact if \(\omega\) is transverse in the \(a, b\) fiber indices (which, in turn, is the invariance condition). Next, if \(\omega^{a|b|c}_m(p)\) is antisymmetric in \(m\) and \(a\) (so that the corresponding \(\omega^{2|0}\) is the two-row field), \(V^{2|1}\) is again the BRST commutator in the small Hilbert space:

\begin{equation}
V^{2|1}(p) = \{Q, W^{2|1}_2(p)\}
\end{equation}

with

\begin{equation}
W^{2|1}_2(p) \sim \omega^{a|b|c}_m(p) \int dz ce^{-5\phi + ipX} (\psi_c \partial^2 X_b - \partial\psi_c \partial X_b) \\
\times (\frac{2}{5} \partial \psi [m \partial L_a] \partial \xi - \partial \psi [m \partial L_a] \partial^2 \xi) + \partial^2 \psi_c \partial X_b (\frac{2}{5} \psi [m \partial L_a] \partial \xi - \psi [m \partial L_a] \partial^2 \xi)
\end{equation}

Next, we analyze \(\omega^{2|2}\) and its vertex operator (22). The gauge transformation for the \(\omega^{2|2}\) field:

\begin{equation}
\omega^{a|b|c|d}_m(p) \rightarrow \omega^{a|b|c|d}_m(p) + p_m \Lambda^{a|b|c|d}(p)
\end{equation}

leads to shifting the \(V^{2|2}(p)\) vertex operator (21) by BRST-exact terms: \(V^{2|2}(p) \rightarrow V^{2|2}(p) + \{Q, W^{2|2}_1(p)\}\) with

\begin{equation}
W^{2|2}_1(p) \sim \Lambda^{a|b|c|d}(p) \int dz ce^{-6\phi + ipX} \\
\times \{(\frac{1}{4} (p_n \partial N^n) \partial \xi - (p_n N^n) \partial^2 \xi) (\partial^2 \psi_c \partial^3 \psi_d \partial X^a \partial X_b - 2\partial\psi_c \partial^3 \psi_d \partial X^a \partial X_b) \\
+ \frac{5}{8} \partial\psi_c \partial^2 \psi_d \partial X_a \partial^3 X_b + \frac{57}{16} \partial\psi_c \partial^2 \psi_d \partial^2 X_a \partial^2 X_b)\}
\end{equation}

where

\begin{equation}
N_n = \partial^3 X_n - \frac{3}{2} \partial^2 X_n - \frac{1}{3} \partial X_n ((\partial\phi)^2 - \frac{17}{6} \partial^2 \phi)
\end{equation}
As before, this operator is BRST-exact if $\omega$ is transverse in the $a, b$ fiber indices. Finally, if $\omega_{m}^{ab|cd}(p)$ is antisymmetric in $m$ and $a$ or $b$ (so that the corresponding $\omega^{2|0}$ is the two-row field), $V^{2|2}$ is again the BRST commutator in the small Hilbert space:

$$V^{2|2}(p) = \{Q, W^{2|2}_{2}(p)\}$$

with

$$W^{2|2}_{2}(p) \sim \omega_{m}^{ab|cd}(p) \int dz ce^{-6\phi+i\phi X} \left\{ \left( \frac{1}{4}N^{m}_{a} \partial \xi - (N^{m})_{a} \partial^{2} \xi \right) \right.$$

$$\times (\partial^{2}\psi_{c} \partial^{3}\psi_{d} \partial X^{a} \partial X_{b} - 2\partial \psi_{c} \partial^{3}\psi_{d} \partial X_{a} \partial^{2} X_{b} - (a \leftrightarrow m) \right\}$$

$$+ \frac{5}{8} \partial \psi_{c} \partial^{2}\psi_{d} \partial X_{a} \partial^{3} X_{b} + \frac{57}{16} \partial \psi_{c} \partial^{2}\psi_{d} \partial^{2} X_{a} \partial^{2} X_{b} - (a \leftrightarrow m) \right\}$$

(70)
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