Non-existence of Bose-Einstein condensation in Bose-Hubbard model in dimensions 1 and 2

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Abstract

We apply the Bogoliubov inequality to the Bose-Hubbard model to rule out the possibility of Bose-Einstein condensation. The result holds in one and two dimensions, for any filling at any nonzero temperature. This result can be considered as complementary to analogous, classical result known for interacting bosons in continuum.

Keywords: Statistical mechanics; Bogolyubov inequality; Bose-Hubbard model; Mermin-Wagner theorem

1 Introduction

Many-body boson systems were among first quantum systems, where the problem of phase transitions has been noticed. In a system of non-interacting bosons it was investigated in the very beginning of quantum theory [1], [2], [3], [4]. For an excellent review of non-interacting boson systems see [5].

In systems of interacting bosons, the problem turned out much more difficult. Significant progress has been made in late 40’s of XX-th century, when the existence of phase transition in 3d systems has been (non-rigorously) shown by Bogolyubov [6]. Further contributions into development of the theory have been made, among others, by Penrose, Onsager, [7], [8], Feynman [9], Lee, Huang, Yang [10], [11]. (For exhaustive reviews see [12], [13]).
The Bogolyubov approach was heuristic one and it cannot be regarded as a rigorous proof of
the Bose-Einstein condensation. It has been based on profound understanding of physics of the
problem, however, it lacked a rigorous justification, which has been made only half century later
(but only in certain set of physical parameters) \([14]\). (For another approach, assuming however
fulfilling of certain boundary conditions, see \([15]\).

The situation in lower dimensions, i.e. 1 and 2, turned out to be more tractable. In the sixties of
XX-th century there appeared convincing explanation that in systems of continuous bosons there is
no condensation at positive temperatures \([16]\). This paper was based on deep physical insight, but
it also cannot be considered as a proof – even the Hamiltonian didn’t appear in the paper! About
ten years later, Bouziane and Martin in \([17]\) analysed the problem rigorously and their results can
be considered as a proof of the lack of Bose-Einstein condensation in dimensions one and two.

Technical tool, being the cornerstone of the proof by Bouziane and Martin, was (certain version
of) the \textit{Bogolyubov inequality}.

It was used in a whole series of papers, where non-existence of Long Range Order (LRO) has
been proven in systems possessing the continuous symmetry group. Boson system also belong to
this class. More precisely, celebrated Mermin-Wagner theorem \([18]\) tells that in lattice spin systems
fulfilling the following assumptions:

1. they possess continuous symmetry group;
2. dimension of the lattice is 1 or 2;
3. the interactions exhibits sufficiently fast spatial fall-off,

there can be no LRO at positive temperatures.

This theorem – half-century-old at present – is an one of few general beautiful and powerful
results, concerning the (im)possibility of ordering in lattice systems. It was generalized and extended
in numerous directions (for some of these achievements, see \([20]\)).

The Mermin-Wagner theorem has been proven with the use of Bogolyubov inequality \([19]\). For
(finite) spin or fermion systems this is a \textit{matrix} inequality, whose proof is tricky but elementary. For
bosonic systems, however, one encounters technical complications due to the fact that the bosonic
Hilbert space is infinite dimensional even for finite lattices. One is forced to develop suitable \textit{operator}
variant of Bogolyubov inequality. It was the case of the paper \([17]\), where it has been proven certain
version of Bogolyubov inequality, whose application resulted in the proof of lack of Bose-Einstein
condensation in continuous systems in dimensions one and two. Another operator versions of the
Bogolyubov inequality have also been proven in papers \([21]\) and \([22]\), where they have been applied
to prove lack of ordering in quantum rotor models under aforementioned conditions.

Besides the fact that the lack of Bose-Einstein condensation has been proven in continuous
systems in \(d = 1\) and \(d = 2\) more than 40 years ago, this is not the case of \textit{lattice} models of interacting
bosons. Canonical example of such lattice models is the \textit{Bose-Hubbard} one. The fermionic Hubbard
model is widely known from early sixties of last century \([23]\). Its bosonic version was much less
famous up to eighties, where it became popular after the paper \([24]\). In this paper basic features of
these model have been examined. A breakthrough in an interest to the model dates about ten years
ago, when it was applied to description of trapped atomic gases and Bose-Einstein condensation
therein \([25]\). Although there is a broad ‘folk knowledge’ that there is no Bose-Einstein condensation
at positive temperatures in dimensions one and two, we couldn’t find – perhaps surprisingly – a proof of this fact. This opportunity motivated us to fill this gap.

Two problems appear here. One is the formal calculation, based on the Bogolyubov inequality. We define the order parameter to be the average of zero-momentum annihilation operator (like in [17] and other papers). By a suitable choice of operators one shows that this order parameter is zero in dimensions one and two at positive temperatures.

The second aspect is justification of these calculations; the ordinary (finite dimensional) Bogolyubov inequality is not sufficient here. It turned out that only in limited range one can use the technique developed in the paper [17]. Instead, we consider certain sequence of finite-dimensional approximations together with taking the infinite-dimensional limits on the level of thermal averages and showing that these limits exist.

The organization of the paper is as follows. In the Sec. 2 we present a ‘setup’, i.e. the notation, formalism and definition of the Bose-Hubbard model. This Section contains also some preparatory theorems, necessary for further considerations. In the Sec. 3 we elaborate some general aspects of Bogolyubov inequality necessary in further considerations. The Sec. 4 presents the choice of operators in the Bogolyubov inequality and calculation of necessary commutators. The Sec. 5 describes passing to the thermodynamic limit to prove the lack of the Bose-Einstein condensation in dimensions one and two at positive temperatures. The Sec. 6 is devoted to summary, conclusions and some open (as far as we know) problems.

2 Setup for the Bose-Hubbard model: definitions and preparatory theorems

In this section we introduce the model and prove self-adjointness of its Hamiltonian. Let $K$ be a separable Hilbert space and $(e_0, e_1, \ldots)$ an orthonormal basis; $e_n$ will also be denoted by $|n\rangle$. Let $c^\dagger, c$ be the standard creation and anihilation operators on $K$ with respect to the orthonormal basis $(e_n = |n\rangle)$:

$$c^\dagger |n\rangle := \sqrt{n + 1}|n + 1\rangle, \quad n \geq 0; \quad c|0\rangle := 0, \quad c|n\rangle := \sqrt{n}|n - 1\rangle, \quad n > 0 \quad (1)$$

and extended by linearity to the space of finite linear combinations of basis vectors. Clearly on this space they satisfy $[c, c^\dagger] = 1$ and for the number operator $\hat{n} := c^\dagger c$ we have $\hat{n}|n\rangle = n|n\rangle$.

For a finite set $\Lambda$ we define

$$\mathcal{H}_\Lambda := \bigotimes_{x \in \Lambda} H_x, \quad \text{where} \quad H_x := K. \quad (2)$$

Vectors of the induced orthonormal basis in $\mathcal{H}_\Lambda$ will be abbreviated as:

$$|n_1, n_2, \ldots, n_{|\Lambda|}\rangle := |n_1\rangle \otimes |n_2\rangle \otimes \cdots \otimes |n_{|\Lambda|}\rangle$$

Let $\mathcal{D}$ be the space of finite linear combinations of elements of basis in $\mathcal{H}_\Lambda$:

$$\mathcal{D} := \text{span}\{|n_1, \ldots, n_{|\Lambda|}\rangle : n_1, \ldots, n_{|\Lambda|} \in \mathbb{N} \cup \{0\}\}, \quad (3)$$

clearly $\mathcal{D}$ is dense in $\mathcal{H}_\Lambda$. 

3
Let us define several linear operators acting on $\mathcal{D}$. For $x \in \Lambda$ by $c_x^\dagger$, $c_x$, $\hat{n}_x$ we denote linear operators acting as $c^\dagger$, $c$, $\hat{n}$ on $x$-th "slot" and as 1 on remaining "slots" i.e. 

$$
\begin{align*}
  c_x^\dagger|n_1, \ldots, n_x, \ldots, n_{|\Lambda|}⟩ &= \sqrt{n_x + 1}|n_1, \ldots, n_x + 1, \ldots, n_{|\Lambda|}⟩ \\
  c_x|n_1, \ldots, n_x, \ldots, n_{|\Lambda|}⟩ &= \sqrt{n_x}|n_1, \ldots, n_x - 1, \ldots, n_{|\Lambda|}⟩ \\
  \hat{n}_x|n_1, \ldots, n_x, \ldots, n_{|\Lambda|}⟩ &= n_x|n_1, \ldots, n_x, \ldots, n_{|\Lambda|}⟩.
\end{align*}
$$

(4)

Clearly $\hat{n}_x = c_x^\dagger c_x$. It is straightforward to check that these operators, as operators on $\mathcal{D}$, satisfy 

$$
\begin{align*}
  [c_x, c_y^\dagger] &= \delta_{xy}, & [n_x, c_y] &= -\delta_{xy}c_y, \\
  [n_x, c_y^\dagger] &= \delta_{xy}c_y^\dagger, & [n_x, c_y^\dagger c_z] &= (\delta_{xy} - \delta_{xz})c_y^\dagger c_z
\end{align*}
$$

(5)

**Remark 1** It is known that $c_x$ and $c_x^\dagger$ are closable and $c_x^\dagger = (c_x)^*|\mathcal{D}$. In what follows for a linear operator $A$ acting on $\mathcal{D}$ and such that $\mathcal{D} \subset D(A^*)$ we will often use notation $A^\dagger := (A^*)|_{\mathcal{D}}$.

The total number operator $\hat{N}_\Lambda$ and the operator $\hat{N}_{2,\Lambda}$ are defined as

$$
\hat{N}_\Lambda := \sum_{x \in \Lambda} \hat{n}_x, \quad \hat{N}_{2,\Lambda} := \sum_{x \in \Lambda} \hat{n}_x^2
$$

(6)

so that

$$
\begin{align*}
  \hat{N}_\Lambda|n_1, \ldots, n_{|\Lambda|}⟩ &= (n_1 + \cdots + n_{|\Lambda|})|n_1, \ldots, n_{|\Lambda|}⟩ \\
  \hat{N}_{2,\Lambda}|n_1, \ldots, n_{|\Lambda|}⟩ &= (n_1^2 + \cdots + n_{|\Lambda|}^2)|n_1, \ldots, n_{|\Lambda|}⟩
\end{align*}
$$

(7)

If it doesn’t lead to any confusion we will omit subscript $\Lambda$ and denote these operators by $\hat{N}$ and $\hat{N}_2$. For $m, M \in \mathbb{N} \cup \{0\}$ let

$$
\begin{align*}
  \mathcal{D}_m &= \text{span}\{|n_1, \ldots, n_{|\Lambda|}⟩ : n_1 + \cdots + n_{|\Lambda|} = m\} \\
  \mathcal{D}_M &= \text{span}\{|n_1, \ldots, n_{|\Lambda|}⟩ : n_1 + \cdots + n_{|\Lambda|} \leq M\} = \bigoplus_{m=0}^M \mathcal{D}_m
\end{align*}
$$

(8)

i.e. $\mathcal{D}_m$ is the eigenspace of $\hat{N}$ with eigenvalue $m$. Using this notation we have $\mathcal{H}_\Lambda = \bigoplus_{m=0}^\infty \mathcal{D}_m$ (orthogonal direct sum). Moreover

$$
M < L \Rightarrow \mathcal{D}_M \subset \mathcal{D}_L \quad \text{and} \quad \mathcal{D} = \bigcup_{M \in \mathbb{N}} \mathcal{D}_M
$$

(9)

It is routine to check that operators $\hat{N}$ and $\hat{N}_2$ are essentially self-adjoint; their self-adjoint closures will be denoted by $\overline{\hat{N}}$ and $\overline{\hat{N}}_2$, respectively. Note the type of font, we will use similar notation for other operators as well. In general, the closure of an operator $A$ will be denoted by $\overline{A}$ as well as complex conjugation of a number $z$ by $\overline{z}$.

We define two more operators on $\mathcal{D}$:

$$
\begin{align*}
  T_\Lambda &:= \sum_{x, y \in \Lambda} t_{xy}c_x^\dagger c_y \sum_{x \neq y} t_{xy}c_x^\dagger c_y + \sum_x t_{xx}\hat{n}_x =: T' + T'' \quad t_{xy} = \overline{t_{yx}} \in \mathbb{C} \\
  L_\Lambda &:= \sum_x (c_x^\dagger + c_x).
\end{align*}
$$

(10)

(11)
Let us note that operators $T', T'', \hat{N}_2$ and $L$ are symmetric on $\mathcal{D}$ and
\begin{equation}
T'D_m \subset D_m, \ T''D_m \subset D_m, \ \hat{N}_2D_m \subset D_m, \ LD_m \subset D_{m-1} \oplus D_{m+1}
\end{equation}
(12)

Finally, for $u > 0, \mu, \lambda \in \mathbb{R}$, let us consider the grand canonical ensemble hamiltonian:
\[ H_\Lambda(u) := u \hat{N}_2 + T_\Lambda + \mu \hat{N}_\Lambda + \lambda L_\Lambda \]  
(13)

(again we will omit subscript $\Lambda$ in $T_\Lambda$ and $L_\Lambda$). Of course $H_\Lambda$ depends of $\lambda$ and $\mu$ as well, but we will use dependence of $u$ explicitly, so we underline it in notation.

The operator $T_\Lambda$ possess physical interpretation as the lattice hopping term, and $u\hat{N}_2$ is the on-site interaction term. We have also the $L$ term, which is responsible for breaking of the $U(1)$ symmetry. Realize that in the canonical ensemble the average of the operator $\sum_x c_x^\dagger c_x$ (as well as $\sum_x c_x c_x^\dagger$) is equal to zero. However, in the grand canonical ensemble both averages can be non-zero. One takes (suitably scaled) averages of the operator $\sum_x c_x^\dagger c_x$ (or $\sum_x c_x c_x^\dagger$) as the ‘order parameter’, i.e.
condensate density.

Let us also fix two numbers $M, M_d$ satisfying:
\begin{equation}
\max \left\{ \sum_{y \in \Lambda} |t_{xy}|, \ x \in \Lambda \right\} \leq M, \ \max\{|t_{xx}| : x \in \Lambda\} \leq M_d
\end{equation}
(14)

**Remark 2** This is not a restriction on a single system, but later on, in termodynamic limit, we shall assume that $M$ and $M_d$ fulfilling (14) can be chosen independently of $\Lambda$. Clearly we may assume $M_d \leq M$.

At this moment we don’t impose any other restrictions on the model; they will appear later on. The rest of this section is devoted to the proof of self-adjointness of the Hamiltonian (13).

**Theorem 3** Let $u > 0$ and $\mu, \lambda \in \mathbb{R}$. Then
\begin{enumerate}
  \item The operator $H(u)$ defined by (13) is essentially self-adjoint and its closure $H(u)$ is equal to $H(u) = u\hat{N}_2 + T + \mu \hat{N} + \lambda L$ (in particular this equality means that $D(H(u)) = D(\hat{N}_2)$).
  \item $H(u)$ is bounded from below with lower bound $\gamma(u)$ satisfying
  \begin{equation}
  \gamma(u) \geq -K \frac{|\Lambda|(M/2 + 2|\lambda|) - (M_d + |\mu|)}{1 - \frac{|\Lambda|}{uK}(|\Lambda|(\sqrt{3}/2M + 2|\lambda|) + M_d + |\mu|)}
  \end{equation}
  for any $K \in \mathbb{N}$ such that $K > \frac{|\Lambda|}{uK}(|\Lambda|(\sqrt{3}/2M + 2|\lambda|) + M_d + |\mu|)$
  \item The operator $\exp(-\beta H(u))$ is trace class for every $\beta > 0$
\end{enumerate}

The idea of the proof is to show that the hamiltonian (13) is a “small” perturbation of the operator $u\hat{N}_2$ and then to use some general results of perturbation theory – they are summarized in Prop. 7. After the proof of this proposition, which essentially consists of pointing to relevant results from literature, we prove – in Prop. 11 – estimates that enable us to apply Prop. 7 to Bose-Hubbard Model.
2.1 Some results about unbounded perturbations.

Here we deal with unbounded perturbation and for completeness of exposition we recall the following

Definition 4 ([26], Ch. 5, 4.1) Let \( T \) and \( A \) be operators acting on a Hilbert space. The operator \( A \) is \( T \)-bounded iff \( D(T) \subset D(A) \) and there exist \( a, b \in \mathbb{R}_+ \cup \{0\} \) such that for every \( \psi \in D(T) \):

\[
||A\psi|| \leq a||\psi|| + b||T\psi||.
\] (16)

In this situation we will also say that \( A \) is \( T \)-bounded with constants \((a,b)\).

A is \( T \)-bounded with the relative bound \( 0 \) if for every \( \epsilon > 0 \) there exist \( a \) such that \( A \) is \( T \)-bounded with constants \((a,\epsilon)\).

Remark 5 There is an equivalent definition ([26], Ch. 5, 4.2) in which inequality (16) is replaced by

\[
||A\psi||^2 \leq \tilde{a}^2||\psi||^2 + \tilde{b}^2||T\psi||^2.
\] (17)

Indeed, if (17) is satisfied then also (16) holds with \( a = \tilde{a} \) and \( b = \tilde{b} \). In the opposite direction, if (16) holds then taking \( \tilde{a} := a\sqrt{1 + 1/\epsilon} \) and \( \tilde{b} := b\sqrt{1 + \epsilon} \) for any \( \epsilon > 0 \) we obtain (17).

Remark 6 From the very definition it follows that relative boundedness is transitive: if \( A \) is \( T \)-bounded with constants \((a_1,b_1)\) and \( T \) is \( S \)-bounded with constants \((a_2,b_2)\) then \( A \) is \( S \)-bounded with constants \((a_1 + b_1 a_2, b_1 b_2)\).

Now we can formulate the proposition being the main tool of our analysis.

Proposition 7 Let \( T \) be a self-adjoint operator acting on a Hilbert space \( \mathcal{H} \). Let \( D \subset \mathcal{H} \) be a dense linear space and \( A : D \to \mathcal{H} \) be a symmetric operator. Assume that

i) \( T \) is essentially self-adjoint on \( D \);

ii) The operator \( A \) is \( T|_D \)-bounded with constants \((a,b)\) and \( b < 1 \).

iii) \( T \) is bounded from below with the lower bound \( \gamma \);

iv) \( e^{-\beta T} \) is a trace class operator for every \( \beta > 0 \).

Then

1. The operator \( T|_D + A \) is essentially self-adjoint and \( \mathcal{S} := (T|_D + A) = T + A \) is self-adjoint on \( D(T) \).

2. The self-adjoint operator \( S \) is bounded from below with the lower bound \( \gamma_S \) and

\[
\gamma_S \geq \gamma - \max \left\{ \frac{a}{1 - b}, a + b|\gamma| \right\}
\] (18)

3. The operator \( e^{-tS} \) is a trace class operator for any \( t > 0 \).

Remark 8 If a pair \((T, A)\) fulfils assumptions i) – iv) with constants \((a, b, \gamma)\) then for \( \nu > 0 \) and \( 0 \leq \mu \leq \nu \) the pair \((\nu T, \mu A)\) fulfils i) – iv) with constants \((\mu a, \mu b, \nu \gamma)\).
Proof of Prop. 7

1) This statement is just the Thm 4.4, Ch.5 of [26] (Kato-Rellich Theorem) applied to operators $T|\mathcal{D}$ and $\gamma$; only assumptions i) and ii) are used.

2) From the proof of mentioned theorem, it follows that $\mathcal{A}$ is $T$-bounded with (the same) constants $(a,b)$. The statement (2) is the Thm 4.11, Ch.5 of [26] applied to the pair $(T,\mathcal{A})$;

3) Let us notice that, by the remark $\S$ it is enough to prove the third claim for $t=1$. Therefore we shall prove that $e^{-S}$ is a trace class operator. To this end we are going to use the following

**Lemma 9** ([28], Thm. 4) Let $A, B$ be self-adjoint operators and $\mathcal{D} \subset D(A) \cap D(B)$ be a dense linear space. Assume that $\text{Tr}(e^{-A}) < \infty$, $B$ is bounded from below and $S := (A+B)|\mathcal{D}$ is self-adjoint. Then $\text{Tr}(e^{-S}) \leq \text{Tr}(e^{-A}e^{-B})$ and therefore $e^{-S}$ is trace class. $\square$

Let us take $a$ satisfying $b < a < 1$ and consider operators $\alpha T$ and $A$. These operators satisfy assumptions of the theorem with the same $\mathcal{D}$ and constants $\gamma_1 := \alpha \gamma$, $a_1 := a$ and $b_1 := \frac{b}{\alpha} < 1$. Therefore we know, by 1) and 2), that the operator $S_1 := \alpha T + \mathcal{A}$ is self-adjoint (on $D(T)$) and $\mathcal{D}$ is a core for $S_1$. It is also bounded from below with the bound

$$\gamma S_1 \geq \alpha \gamma - \max \left\{ \frac{a\alpha}{\alpha - b}, a + b|\gamma| \right\}$$

Therefore we can write the operator $S$ as a sum of two self-adjoint, bounded from below operators:

$$S = T + \mathcal{A} = (1-a)T + (\alpha T + \mathcal{A}), \tag{19}$$

where $\text{exp}(-(1-a)T)$ is trace class (by iii)) and $\mathcal{D}$ is a core for $S$. Now by Lemma $\S$ the operator $\text{exp}(-S)$ is trace class. $\square$

**Corollary 10** Let $T$ and $A$ satisfy assumptions i) – iii) of Thm. [27] and let $B$ be $T$-bounded with constants $(\tilde{a}, \tilde{b})$. Then, for any $\beta > 0$, the operator $B \exp(-\beta(T + \mathcal{A}))$ is bounded and

$$||B \exp(-\beta(T + \mathcal{A}))|| \leq (\tilde{a} + \frac{\tilde{b}}{1-b}) ||\exp[-\beta(T + \mathcal{A})]| | + \frac{\tilde{b}}{1-b} ||(T + \mathcal{A}) \exp[-\beta(T + \mathcal{A})]| | \tag{20}$$

**Proof:** Let us denote $S := T + \mathcal{A}$. By the functional calculus of self-adjoint operators (see e.g. [27], XII.2.7(c)) we know that $\exp(-\beta S) : \mathcal{H} \rightarrow D(S) = D(T) \subset D(B)$, therefore $B \exp(-\beta S)$ is defined on the whole space $\mathcal{H}$ and for $\varphi \in \mathcal{H}$:

$$||B \exp(-\beta S) \varphi|| \leq \tilde{a} ||\exp(-\beta S) \varphi|| + \tilde{b} ||T \exp[-\beta(T + \mathcal{A})] \varphi|| \tag{21}$$

For $\psi \in D(T) = D(S) \subset D(A)$ we have:

$$||T \psi|| \leq ||S \psi|| + ||\mathcal{A} \psi|| \leq ||S \psi|| + a ||\psi|| + b ||T \psi||,$$

and, since $b < 1$,

$$||T \psi|| \leq \frac{a}{1-b} ||\psi|| + \frac{1}{1-b} ||S \psi||$$

Using this estimate in (21) for $\psi := \exp(-\beta S) \varphi$ we obtain (20). $\square$
2.2 Self-adjointness of Bose-Hubbard Hamiltonian

To prove Thm. 3 we use Prop. 7 with \( T := u\tilde{N}_2 \) and \( A := T + \mu \tilde{N} + \lambda L \). It is clear that \( \tilde{N}_2 \geq 0 \) and it is easy to check that \( \exp(-\beta\tilde{N}_2) \) is trace class for every \( \beta > 0 \), so the same is true for \( \exp(-\beta u\tilde{N}_2) \). Thus to prove the theorem we need to show that the operator \( T + \mu \tilde{N} + \lambda L \) is \( u\tilde{N}_2 \)-bounded with some constants \((a,b), b < 1 \). In fact we are proving in Prop. 11 that each of operators \( T', T'', \tilde{N} \) and \( L \) is \( \tilde{N}_2 \)-bounded with relative bound 0. Notice that this is enough, since it is straightforward to see that then, for any \( \lambda \) and \( \mu \), the operator \( T + \mu \tilde{N} + \lambda L \) is \( \tilde{N}_2 \)-bounded with relative bound 0; moreover from the very definition, it follows that if some operator \( A \) is \( \tilde{N}_2 \)-bounded with relative bound 0, it is also \( u\tilde{N}_2 \)-bounded with relative bound 0 for any \( u \in \mathbb{R} \).

By considerations above, the proof of the following proposition will complete the proof of 1) and 3) of Thm. 3.

**Proposition 11** For any \( K \in \mathbb{N} \) and any \( \psi \in \mathcal{D} \) we have the following estimates:

\[
\|T'\psi\|^2 \leq \frac{1}{4}M^2|\Lambda|^2(K+1)^2\|\psi\|^2 + \frac{3}{4}(K+1)^2\|\tilde{N}_2\psi\|^2
\]

(22)

\[
\|\tilde{N}\psi\|^2 \leq K^2\|\psi\|^2 + \frac{|\Lambda|^2}{(K+1)^2}\|\tilde{N}_2\psi\|^2
\]

(23)

\[
\|T''\psi\|^2 \leq M^2\|\psi\|^2 + \frac{M^2|\Lambda|^2}{(K+1)^2}\|\tilde{N}_2\psi\|^2
\]

(24)

\[
\|L\psi\| \leq 2|\Lambda|(K+1)\|\psi\| + \frac{2|\Lambda|^2}{K+1}\|\tilde{N}_2\psi\|
\]

(25)

**Proof:** Notice that, as mentioned above, these inequalities imply that operators \( T', T'', \tilde{N} \) and \( L \) are \( \tilde{N}_2 \)-bounded with relative bound 0. Let us start with the following

**Lemma 12** For any \( \varphi \in \mathcal{D}_m \) we have:

\[
\|T'\varphi\| \leq M\frac{m+1}{2}|\Lambda|\|\varphi\|
\]

(26)

\[
\|\tilde{N}_2\varphi\| \geq \frac{m^2}{|\Lambda|}\|\varphi\|
\]

(27)

**Proof:** The first estimate we are going to prove is:

\[
\|c_x^\dagger c_y\varphi\| \leq \frac{m+1}{2}\|\varphi\| \quad \text{for} \quad x \neq y \quad \text{and} \quad \varphi \in \mathcal{D}_m
\]

(28)

Observe that each subspace \( \mathcal{D}_m \) is an orthogonal sum \( \mathcal{D}_m = \bigoplus_{k=0}^m h_k \), where \( h_k := \text{span}\{n_1, \ldots, n_{|\Lambda|}\} \in \mathcal{D}_m : n_x + n_y = k \}. We have \( c_x^\dagger c_y h_k \subset h_k \) and each \( h_k \) is isomorphic to \( h'_k \otimes h''_k \), where

\[
h'_k := \text{span}\{m_{l-k} : l = 0, \ldots, k\}, \quad h''_k := \text{span}\{n_{l+\ldots+n_{|\Lambda|-2}} : n_1 + \ldots + n_{|\Lambda|-2} = m-k\}
\]

and \( c_x^\dagger c_y(x'_k \otimes x''_k) = A_k(x'_k) \otimes x''_k \) where

\[
A_k[l, k-l] = \sqrt{k-l} \sqrt{l+1} \sqrt{l+1+l-k-1} \cdot l = 0, \ldots, k
\]
So in the basis \( f_l := |l, k - l\rangle, l = 0, \ldots, k \) the matrix of \( A_k \) is

\[
A := \begin{pmatrix}
0 & 0 & \ldots & \ldots & 0 \\
0 & a_0 & 0 & \ldots & 0 \\
0 & 0 & a_1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \ldots & a_{k-1}
\end{pmatrix}, \quad a_l := \sqrt{l + 1} \sqrt{k - l}
\]

then \( ||A||^2 = \max\{|a_l|^2 : l = 0, \ldots, k - 1\} \); this immediately follows from \( ||A||^2 = ||A^* A|| \). Therefore

\[
||A_k||^2 = \max\{(l + 1)(k - l), l = 0, \ldots, k - 1\} = \begin{cases} 
\frac{(k+1)^2}{4} & k - \text{odd} \\
\frac{k(k+2)}{4} & k - \text{even} 
\end{cases}
\]

and

\[
||A_k||^2 = \max\{||A_k||^2, k = 0, \ldots, m\} \leq \left( \frac{m+1}{2} \right)^2
\]
as in (28). Now, for \( \varphi \in \mathcal{D}_m \) we compute

\[
||T' \varphi|| = ||\sum_{x \neq y} t_{xy} c_k^x c_y \varphi|| \leq \sum_{x \neq y} |t_{xy}| ||c_k^x c_y \varphi|| \leq \frac{m+1}{2} ||\varphi|| \sum_{x \neq y} |t_{xy}| \leq |\Lambda| \frac{m+1}{2} ||\varphi||
\]
as claimed in (29).

It remains to prove (27). This immediately follows from the fact that the minimal value of \( x_1^2 + x_2^2 + \cdots + x_l^2 \) on the hyperplane \( x_1 + x_2 + \cdots + x_l = m \) is equal to \( l \left( \frac{m}{2} \right)^2 \). □

Now we can prove the proposition. Let us fix \( K \in \mathbb{N} \). For any \( \psi \in \mathcal{D}_l \), because of (30) we may assume, that \( \psi \in \mathcal{D}_{K+L} = \bigoplus_{m=0}^{K+L} \mathcal{D}_m \) for some integer \( L > 1 \) and write \( \psi = \sum_{m=0}^{K+L} \varphi_m, \varphi_m \in \mathcal{D}_m \) (orthogonal sum). By (12) and the estimates in the lemma we have:

\[
||T' \psi||^2 = \sum_{m=0}^{K+L} ||T' \varphi_m||^2 \leq \frac{1}{4} |\Lambda|^2 M^2 \sum_{m=0}^{K+L} (m+1)^2 ||\varphi_m||^2 \tag{29}
\]

and

\[
\sum_{m=0}^{K+L} (m+1)^2 ||\varphi_m||^2 = ||\psi||^2 + \sum_{m=0}^{K} (m^2 + 2m) ||\varphi_m||^2 = \]

\[
= ||\psi||^2 + \sum_{m=0}^{K} (m^2 + 2m) ||\varphi_m||^2 + \sum_{m=K+1}^{K+L} (m^2 + 2m) ||\varphi_m||^2
\]

\[
\leq ||\psi||^2 + (K^2 + 2K) ||\psi||^2 + \sum_{m=K+1}^{K+L} (m^2 + 2m) \frac{|\Lambda|^2}{m^2} ||\tilde{N}_2(\varphi_m)||^2 \leq \]

\[
(\bar{K} + 1)^2 ||\psi||^2 + |\Lambda|^2 \left( \frac{1}{(K+1)^2} + \frac{2}{(K+1)^3} \right) \sum_{m=K+1}^{K+L} ||\tilde{N}_2(\varphi_m)||^2 \leq \]

\[
(\bar{K} + 1)^2 ||\psi||^2 + |\Lambda|^2 \left( \frac{3|\Lambda|^2}{(K+1)^2} ||\tilde{N}_2 \psi||^2 \right)
\]

(30)
Thus the estimate \( (22) \) follows. In the similar way:

\[
||\hat{N}\psi||^2 = \sum_{m=0}^{K+L} m^2 ||\phi_m||^2 = \sum_{m=0}^{K} m^2 ||\phi_m||^2 + \sum_{m=K+1}^{K+L} m^2 ||\phi_m||^2 \leq
\]

\[
\leq K^2||\psi||^2 + \frac{||A||^2}{(K+1)^2} \sum_{m=K+1}^{K+L} m^2 \leq K^2||\psi||^2 + \frac{||A||^2}{(K+1)^2} ||\hat{N}\psi||^2
\]

This is the estimate \( (23) \). Finally, we compute:

\[
|\psi \cdot T''\psi| = \left| \left(\psi \sum_{x \in \Lambda} t_{xx} \hat{n}_x \psi \right) \right| \leq \left| \sum_{x \in \Lambda} |t_{xx}| |\psi \cdot \hat{n}_x \psi| \right| \leq \max \{|t_{xx}| : x \in \Lambda \} \sum_{x \in \Lambda} (\psi \cdot \hat{n}_x \psi) = M_d(\psi \cdot \hat{N} \psi),
\]

then, for \( \phi \in \mathcal{D}_m \) we get

\[
|\phi \cdot T''\phi_m| \leq M_d(\phi \cdot \hat{N} \phi_m) = M_d m ||\phi_m||^2
\]

therefore

\[
||T''|\mathcal{D}_m|| \leq M_d m, \quad \text{and}
\]

\[
||T''\phi||^2 = \sum_{m=0}^{K+L} ||T''\phi_m||^2 \leq M_d^2 \sum_{m=0}^{K+L} m^2 ||\phi_m||^2 = M_d^2 ||\hat{N}\psi||^2 \leq M_d^2 \left( K^2 ||\psi||^2 + \frac{||A||^2}{(K+1)^2} ||\hat{N}\psi||^2 \right),
\]

where in the last step we use second claim of the proposition, and this is \( (24) \).

Finally to prove the last inequality consider the lemma

**Lemma 13** For every \( \psi \in \mathcal{D} \) the following inequalities hold

\[
||c_x \psi|| \leq ||\hat{N}^{1/2} \psi||, \quad ||c_x^\dagger \psi|| \leq ||(\hat{N} + 1)^{1/2} \psi||
\]

and, consequently, operators \( c_x^\dagger (\hat{N} + 1)^{-1/2} \) and \( c_x (\hat{N} + \rho)^{-1/2} \) extend to bounded operators with the norm \( \leq 1 \) for any \( \rho > 0 \).

**Proof:** This immediately follows from definitions of \( c_x, c_x^\dagger \) and \( \hat{N} \). \( \square \)

This way we obtain

\[
||\sum_x c_x^\dagger (\hat{N} + 1)^{-1/2}|| \leq |A|, \quad ||L(\hat{N} + 1)^{-1/2}|| \leq 2|\lambda|,
\]

and

\[
||L\psi|| = ||L(\hat{N} + 1)^{-1/2}(\hat{N} + 1)^{1/2}\psi|| \leq 2|\lambda||(\hat{N} + 1)^{1/2}\psi|| \leq 2|\lambda||(||\hat{N}\psi|| + ||\psi||)
\]

and \( (25) \) follows now from \( (23) \). The proposition is proven as well as points 1) and 3) of Thm. \( 3 \).

It remains to prove the inequality \( (14) \) in 2). By Prop. \( 11 \)

\[
||(T + \mu \hat{N} + \lambda L)\psi|| \leq a(u)||\psi|| + b(u)||u\hat{N}\psi||, \quad \psi \in \mathcal{D},
\]

where

\[
a(u) := (K + 1) \left( |\lambda|(|M/2 + 1|) - (M_d + |\mu|) \right)
\]
\[ b(u) := \frac{|A|}{u(K+1)} \left( |A|\left(\frac{\sqrt{3}}{2} M + 2|\lambda|\right) + M_d + |\mu| \right). \]

Remembering that \( u\tilde{N}_2 \geq \gamma = 0 \), we have \( \gamma(u) \geq -\frac{a(u)}{1 - b(u)} \) by (18), and the inequality (15) follows.

By the proof of Thm. 3 we have:

**Corollary 14** For any \( \rho \in \mathbb{R} \) an operator \( H(u) + \rho I \) is self-adjoint, bounded from below and \( \exp(-\beta(H(u) + \rho I)) \) is trace class.

\[ \Box \]

### 3 Bogolyubov inequality for systems of bosons

The fundamental technical tool in proving the absence of ordering is the **Bogolyubov inequality**. Working with bosons, one needs a kind of *infinite dimensional* version of that inequality involving unbounded operators. One approach is to obtain directly such an inequality for a given hamiltonian and appropriate class of operators – this way for Bose gas in continuum was used in [17]. In our approach we proceed in different manner. Namely, we start with *finite dimensional* Bogolyubov inequality for finite dimensional “approximations” of relevant operators and show by limiting procedure that finite dimensional averages converge to “true” averages for the model.

Let \( V \) be a *finite dimensional* Hilbert space and \( H = H^* \) a self-adjoint operator on \( V \). For an operator \( B \) on \( V \) let

\[ \langle B \rangle := \frac{\text{Tr}_V(Be^{\beta H})}{\text{Tr}_V(e^{\beta H})}, \beta > 0. \]

Let \( A, C \) be linear operators on \( V \). The Bogolyubov inequality [19] reads:

\[ \frac{\beta}{2} \langle A^*A + AA^* \rangle_{V} (\langle [C,H], C^* \rangle_{V}) \geq |\langle [C,A] \rangle_{V}|^2. \] (35)

Assume now, that \( V \) is a subspace of a Hilbert space \( \mathcal{H} = V \oplus V^\perp \) and let \( P_V \) be the orthogonal projection on \( V \). If \( A \) is an operator on \( V \) let \( \tilde{A} := AP_V \) be its extension (by 0 on \( V^\perp \)) on \( \mathcal{H} \). It is easy to see that for linear operators \( A, A_1, \ldots, A_k, H \) on \( V \) we have

\[ \tilde{A}^* = (\tilde{A})^*, \quad \text{Tr}_V(Ae^{\beta H}) = \text{Tr}(\tilde{A}e^{\beta \tilde{H}}), \quad P(A_1,\ldots,A_k)^\perp = P(\tilde{A}_1,\ldots,\tilde{A}_k), \]

for any polynomial \( P(x_1,\ldots,x_k) \) of noncommuting variables. Thus the inequality (35) can be written as

\[ \frac{\beta}{2} \langle \tilde{A}^* \tilde{A} + \tilde{A} \tilde{A}^* \rangle_{V} (\langle [\tilde{C},\tilde{H}], \tilde{C}^* \rangle_{V}) \geq |\langle [\tilde{C},\tilde{A}] \rangle_{V}|^2, \] (36)

where, for operators \( B : \mathcal{H} \to \mathcal{H} \) satisfying \( B = BP_V = P_VB \), we denote

\[ \langle B \rangle_{V} := \frac{\text{Tr}(Be^{\beta \tilde{H}})}{\text{Tr}(P_Ve^{\beta \tilde{H}})}. \]

Finally, and this is the situation we deal with, let us assume that \( \mathcal{D} \subset \mathcal{H} \) is a dense linear space with \( V \subset \mathcal{D} \) and let \( H, A, C : \mathcal{D} \to \mathcal{H} \) be linear operators; assume moreover that \( H \) is symmetric and \( \mathcal{D} \subset D(A^*) \cap D(C^*) \). Then restricting operators to \( V \) and then extending to \( \mathcal{H} \) we obtain the following inequality:

\[ \frac{\beta}{2} \langle (AV)^*AV + AV(AV)^* \rangle_{V} (\langle [CV,H_V], (CV)^* \rangle_{V}) \geq |\langle [CV,AV] \rangle_{V}|^2, \] (37)
where \( A_V := P_V A P_V \), etc, and the average \( \langle B \rangle_V \) for operators satisfying \( B = B_V \) is defined this time as

\[
\langle B \rangle_V := \frac{\text{Tr}(B e^{-\beta H_V})}{\text{Tr}(P_V e^{-\beta H_V})}
\]  

(38)

Using notation just introduced we can formulate:

**Proposition 15** Let \( D \subset H \) be a dense linear space and \( H, A, C : D \to D \) be linear operators; assume moreover that \( H \) is symmetric and \( D \subset D(A^*) \cap D(C^*) \). Let \( V, W \) be finite dimensional subspaces satisfying \( W \subset V \subset D \).

If \( P_V C \subset CP_V \) then

\[
[C_V, A_V] = [C, A]_V
\]

(39)

[[C_V, H_V], (C_V)^*] = [[C, H], C^\dagger]_V

(40)

If, moreover \( P_V A \subset AP_W \) then additionally:

\[
A_V(A_V^*) + (A_V)^*A_V = (AA^\dagger + A^\dagger A)_V - (P_V - P_W)A^\dagger A(P_V - P_W)
\]

(41)

and

\[
\frac{\beta}{2} \langle (AA^\dagger + A^\dagger A)_V \rangle_{V} \langle [[C, H], C^\dagger]_V \rangle_V \geq |\langle [C, A]_V \rangle_V|^2,
\]

(42)

**Proof:** It is straightforward to verify that for any operators \( E, F : D \to D \):

\[
[E_V, F_V] = [E, F]_V + P_V E(P_V F - FP_V)P_V + P_V F(E P_V - P_V E)P_V.
\]

(43)

If an operator \( P \) satisfies \( PP_V = P \) then \( P(P_V E - EP_V)P_V = 0 \). In particular if either \( P_V E \subset EP_V \) or \( P_V F \subset FP_V \) then \( \langle 43 \rangle \) reduces to \( [E_V, F_V] = [E, F]_V \). So \( \langle 39 \rangle \) follows from \( \langle 13 \rangle \) by assumptions on \( C \).

To prove \( \langle 40 \rangle \) notice that our assumptions on \( C \) imply:

\[
(C_V)^* = (C^\dagger)_V = C^\dagger P_V \supset P_V C^\dagger,
\]

(44)

and for any operator \( E : D \to D \):

\[
[[C_V, E_V], (C^\dagger)_V] = P_V C P_V E P_V C^\dagger P_V - P_V E P_V C P_V C^\dagger P_V +
\]

\[
- C^\dagger P_V C P_V E P_V + C^\dagger P_V E P_V C P_V =
\]

\[
= P_V C E C^\dagger P_V - P_V E C C^\dagger P_V - P_V C^\dagger C E P_V + P_V C^\dagger E C P_V =
\]

\[
= P_V [[C, E], C^\dagger] P_V,
\]

(45)

In particular for \( E = H \) we obtain \( \langle 40 \rangle \).

To prove \( \langle 41 \rangle \) let us notice that

\[
(A_V)^* = (A^\dagger)_V = A^\dagger P_V \supset P_W A^\dagger,
\]

Using these relations, let us compute

\[
A_V(A_V^*) = P_V A P_V A^\dagger P_V = P_V A P_V A^\dagger P_V P_V = P_V A P_V P_W A^\dagger P_V = P_V A P_W A^\dagger P_V = P_V A A^\dagger P_V
\]

(46)

\[
(A_V^*)^* A_V = P_V A^\dagger P_V A P_V = P_V A^\dagger A P_W P_V = P_V A^\dagger A P_W
\]

(47)
and
\[ A_V(A_V)^* + (A_V)^*A_V = P_VAA_V^*P_V + P_VA_V^*AP_W = P_V(AA_V^* + A^*A)V - P_VA_V^*A(P_V - P_W), \]
\[ = P_V(AA_V^* + A^*A)V - (P_V - P_W)A_V^*A(P_V - P_W) \]
The last equality follows from (15):
\[ P_WA_V^*A(P_V - P_W) ⊂ A_V^*P_VA(P_V - P_W) ⊂ A_V^*AP_W(P_V - P_W) = 0 \]
It remains to prove (42), in fact it follows from inequality (37). The RHS of (37) reads:
\[ |\langle [C, A_V]\rangle_V|^2 = |\langle [C, A]\rangle_V|^2, \]
and, using an abbreviation \( q_V := P_V - P_W \), the LHS (without \( \frac{\beta}{2} \)):
\[ \langle (A_V)^*A_V + A_V(A_V)^*\rangle_V\langle [C_V, H_V], (C_V)^*\rangle_V = \]
\[ = \langle (AA_V^* + A^*A)V\langle [C, H], C^*V \rangle_V - \langle q_VA_V^*Aq_V\rangle_V\langle [C, H_V], (C_V)^*\rangle_V \]
It is known that \( \langle [C_V, H_V], (C_V)^*\rangle_V ≥ 0 \), (see e.g [34]) and \( \langle q_VA_V^*Aq_V\rangle_V ≥ 0 \) (as a thermal average of positive operator) therefore inequality (37) gives
\[ \frac{\beta}{2} \langle (AA_V^* + A^*A)V\langle [C, H], C^*V \rangle_V \geq |\langle [C, A]\rangle_V|^2. \]

We are going to use sequence of inequalities (37) for \( V := D_M \). In this situation we will write \( P_M \) for the orthogonal projection on \( D_M \), and for an operator \( B : D(B) \rightarrow \mathcal{H} \) with \( D \subset D(B) \subset \mathcal{H} \) we will write
\[ B_M := P_MB_P_M, \quad (B_M)_M := \frac{\text{Tr}(B_Me^{-\beta H_M})}{\text{Tr}(P_Me^{-\beta H_M})}. \]
For operators of interest we will show convergence of these finite dimensional approximations to their “true” thermal averages.

For a bounded operator \( A \) its thermal average \( \langle A \rangle \) is defined as
\[ \langle A \rangle := \frac{\text{Tr}(A_{\exp(-\beta H)})}{\text{Tr}(\exp(-\beta H))}, \]
and for \( H \equiv H(u) \) it is well defined and finite by Thm. 3. Let us note that thermal averages don’t change under replacements \( H \rightarrow H + \rho I \) for any \( \rho \in \mathbb{R} \). In the following, we have to consider more general case of some unbounded observables \( A \). For them we have to show that the formula (47) is meaningful. More precisely we have:

**Proposition 16** Let operators \( T \) and \( A \) satisfy assumptions i) – iv) of Prop. 7 and let \( B \) be a \( T \)-bounded operator. Then for any \( \beta > 0 \) the operator \( B_{\exp(-\beta(T+\bar{A}))} \) is trace class.

**Proof:** Let \( S := T + \bar{A} \). By Cor. 10 the operator \( B_{\exp(-\beta S)} \) is bounded. Since \( B_{\exp(-\beta S)} = (B_{\exp(-\beta T)})_{\exp(-\beta S)} \) and, by Prop. 4 the operator \( \exp(-\beta S) \) is trace class for every \( \beta > 0 \), the result is clear. □

Now we consider the problem of convergence. Our aim is to prove that for \( B : D → \mathcal{H} \), possibly satisfying some additional conditions, \( \lim_{M \rightarrow \infty} \langle B_M \rangle_M = \langle B \rangle \). We will prove this by replacing \( \exp(-\beta H_M) \) in (46) by a different operator, say \( \exp(-\beta H_M) \), without changing the value of \( \langle B_M \rangle_M \) and the new sequence of operators will converge in \( L^1(\mathcal{H}) \) i.e. the Banach space of trace class operators on \( \mathcal{H} \).
Proposition 17 Let operators $T, A$, a linear space $D$ and numbers $(a, b)$ satisfy assumptions of Prop. [A] Let $(P_n)_{n \in \mathbb{N}}$ be an increasing sequence of finite dimensional orthogonal projections satisfying

$$\text{strong} - \lim_{n \to \infty} P_n = I \quad , \quad P_n(\mathcal{H}) \subset D \quad , \quad D \subset \bigcup_{n \in \mathbb{N}} P_n(\mathcal{H})$$

(48)

For $\alpha$ satisfying $1 > \alpha > b$ and $\rho$ such that $\alpha T + \overline{A} + \rho I \geq 0$ let us define, cf. [17], a sequence of self-adjoint operators $S_n^\alpha$ on $D(S_n^\alpha) := D(T)$:

$$S_n^\alpha := (1 - \alpha)T + P_n(\alpha T + \overline{A} + \rho I)P_n = (1 - \alpha)T + P_n(\alpha T|_D + A + \rho I)P_n.$$

Then for any $t > 0$

$$\lim_{n \to \infty} \exp[-tS_n^\alpha] = \exp[-t(T + \overline{A} + \rho I)] \quad \text{in } L^1(\mathcal{H})$$

Proof: We will use the following

Lemma 18 [29, lemma p. 271] Let $(H_n)_{n \in \mathbb{N}}$ be a sequence of self-adjoint operators and $H_-, H$ be self-adjoint operators on a Hilbert space $\mathcal{H}$. Assume that

1) $H_- \leq H_n \forall n \in \mathbb{N}$
2) $\exp(-\beta H_-) \in L^1(\mathcal{H})$ for any $\beta > 0$
3) $\lim_{n \to \infty} H_n = H$ in a strong-generalized sense.

Then

1) $\exp(-\beta H) \in L^1(\mathcal{H})$ for any $\beta > 0$
2) $\lim_{n \to \infty} \exp(-\beta H_n) = \exp(-\beta H)$ in $L^1(\mathcal{H})$. \hfill $\square$

Let us recall that the sequence of self-adjoint operators $T_n$ converges to a self-adjoint operator $T$ in a strong generalized sense iff for every $z \in \mathbb{C} \setminus \mathbb{R}$ the sequence of resolvents $(T_n - zI)^{-1}$ converges strongly to $(T - zI)^{-1}$. The useful criterion for strong convergence of resolvents is:

Lemma 19 [29, Ch. VIII, Cor. 1.6] Let $(T_n)$ be a sequence of self-adjoint operators and $T$ a self-adjoint operator. Assume $D$ is a core for $T$ and $T_n \psi \to T \psi$ for every $\psi \in D$. Then for any $z \in \mathbb{C} \setminus \mathbb{R}$ the sequence of resolvents $(T_n - zI)^{-1}$ converges strongly to $(T - zI)^{-1}$. \hfill $\square$

Now, to prove our statement we use Lemma [18]. We put $H_- = (1 - \alpha)T$, $H_n := S_n^\alpha$, and $H := T + \overline{A} + \rho I$. Clearly $\exp(-\beta(1 - \alpha)T)$ is trace class for any $\beta > 0$ and $S_n^\alpha \geq (1 - \alpha)T$ due to the definition of $\rho$. It remains to verify that $S_n^\alpha$ converges to $T + \overline{A} + \rho I$ in a strong generalized sense. We use Lemma [19]. For any $\psi \in D$, $P_n \psi = \psi$ for sufficiently large $n$, therefore

$$\lim_{n \to \infty} S_n^\alpha \psi = \lim_{n \to \infty} [(1 - \alpha)T + P_n(\alpha T|_D + A + \rho I)P_n] \psi =$$

$$= (1 - \alpha)T \psi + \alpha \lim_{n \to \infty} P_n T \psi + \lim_{n \to \infty} P_n A \psi + \rho \psi = T \psi + A \psi + \rho \psi =$$

$$= (T + \overline{A} + \rho I) \psi,$$

Since $D$ is a core for $T + \overline{A} + \rho I$ the result follows. \hfill $\square$

The next theorem is essential for convergence of finite dimensional approximations of thermal averages. In its proof we will use the following lemma:

Lemma 20 Let $A$ be $T$-bounded with constants $(a, b)$. Then for every contraction $P$ (i.e. $\|P\| \leq 1$) satisfying $PT \subset TP$, operators $AP$ and $PAP$ are $T$-bounded with (the same) constants $(a, b)$. 

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\textbf{Proof}: It is straightforward to check that if \( P \) is any contraction the operator \( PA \) is \( T \)-bounded with (the same) constants \((a, b)\). So it is enough to prove that if \( PT \subset TP \) then \( AP \) is \( T \)-bounded with constants \((a, b)\). By assumptions we have \( PD(T) \subset D(T) \subset D(A) \). Therefore for \( \psi \in D(T) \):

\[
TP\psi = PT\psi \text{ and }
||AP\psi|| \leq a||P\psi|| + b||TP\psi|| \leq a||\psi|| + b||PT\psi||
\]

\( \Box \)

\textbf{Theorem 21} Let us keep notation and assumptions of Prop. \[47\]. In particular, \( A \) is \( T\)-bounded with constants \((a, b)\). We add two more assumptions:

1) \( b < \frac{1}{3} \) and
2) \( P_nT \subset TP_n \) for every \( n \in \mathbb{N} \).

If \( B \) and \( B^* \) are \( T \)-bounded, then

\[
\lim_{n \to \infty} \text{Tr}[P_nBP_n \exp(-\beta S_n^\alpha)] = \text{Tr}[B \exp(-\beta(T + \overline{A} + \rho I))]
\]

\textbf{Proof:} Let \( S := T + \overline{A} + \rho I \). The operator \( B \exp(-\beta S) \) is trace class by Cor. \[16\]. Below we prove that operators \( B \exp(-\beta S_n^\alpha), n \in \mathbb{N} \) are bounded for any \( \beta > 0 \) (therefore also trace class) and give uniform (in \( n \)) bound for their norm. Assuming this has been done, we can write:

\[
P_nBP_n \exp(-\beta S_n^\alpha) - B \exp(-\beta S) = [P_nBP_n \exp(-\beta S_n^\alpha) - B \exp(-\beta S_n^\alpha)] + B \exp(-\beta S_n^\alpha) - B \exp(-\beta S) = [*] + [**]
\]

The first term can be written as:

\[
[*] = -(1 - P_n)BP_n \exp(-\beta S_n^\alpha) - B(1 - P_n) \exp(-\beta S_n^\alpha)
\]

Note that \( D(T(1 - P_n)) = D(T) \) and \( T = (1 - P_n)T(1 - P_n) + P_nTP_n \), therefore

\[
S_n^\alpha = P_n(T + \overline{A} + \rho I)P_n + (1 - \alpha)(1 - P_n)T(1 - P_n) \quad \text{and}
S_n^\alpha P_n \supset P_nS_n^\alpha, \quad \exp(-\beta S_n^\alpha)P_n = P_n \exp(-\beta S_n^\alpha)
\]

To shorten notation let us denote \( Q_n := 1 - P_n \). By the second equality in \[50\]:

\[
\text{Tr}[Q_nBP_n \exp(-\beta S_n^\alpha)] = \text{Tr}[Q_nBP_n \exp(-\beta S_n^\alpha)P_n] = 0
\]

and, again by \[51\],

\[
Q_n \exp(-\beta S_n^\alpha) = Q_n \exp(-\beta S_n^\alpha)Q_n = Q_n \exp[-\beta(1 - \alpha)T] Q_n
\]

So we have:

\[
|\text{Tr}[*]| = |\text{Tr}[B Q_n \exp(-\beta S_n^\alpha)]| = \\
|\text{Tr}[B Q_n \exp\left(-\frac{\beta}{2}(1 - \alpha)T\right) Q_n \exp\left(-\frac{\beta}{2}(1 - \alpha)T\right) Q_n]| = \\
\leq ||Q_nBQ_n \exp\left(-\frac{\beta}{2}(1 - \alpha)T\right)|| \text{Tr}[Q_n \exp\left(-\frac{\beta}{2}(1 - \alpha)T\right) Q_n] = \\
= ||Q_nBQ_n \exp\left(-\frac{\beta}{2}(1 - \alpha)T\right)|| \text{Tr}[Q_n \exp\left(-\frac{\beta}{2}(1 - \alpha)T\right) Q_n]
\]

Since the operator \( \exp[-\beta(1 - \alpha)T] \) is trace class:

\[
\lim_{n \to \infty} \text{Tr}[Q_n \exp[-\beta(1 - \alpha)T] Q_n] = 0.
\]
and we obtain \( \lim_{n \to \infty} \text{Tr}(\{s\}) = 0 \) provided that

\[
\sup_{n \in \mathbb{N}} \| (1 - P_n)B(1 - P_n) \exp \left[ -\frac{\beta}{2}(1 - \alpha)T \right] \| < \infty. \tag{52}
\]

Since \( (1 - P_n)T \subset T(1 - P_n) \), by the lemma \( 20 \) the operator \( (1 - P_n)B(1 - P_n) \) is \( T \)-bounded with some constants \( (a_0, b_0) \) independent of \( n \) therefore for \( \psi \in D(T) \):

\[
\| (1 - P_n)B(1 - P_n) \exp \left[ -\frac{\beta}{2}(1 - \alpha)T \right] \psi \| \leq a_0 \| \exp \left[ -\frac{\beta}{2}(1 - \alpha)T \right] \psi \| + b_0 \| T \exp \left[ -\frac{\beta}{2}(1 - \alpha)T \right] \psi \|
\]

So the inequality \( 52 \) is true and \( \lim_{n \to \infty} \text{Tr}(\{s\}) = 0 \).

Let us now consider \([**]\):

\[
B \exp(\beta S_n^\alpha) - B \exp(-\beta S) = B \exp(\frac{\beta}{2}S_n^\alpha) - B \exp(\frac{\beta}{2}S) + B \exp(\frac{\beta}{2}S_n^\alpha - \frac{\beta}{2}S) \exp(\frac{\beta}{2}S).
\]

The first term:

\[
\left| \text{Tr} \left[ B \exp(\frac{\beta}{2}S_n^\alpha) \left( \exp(-\frac{\beta}{2}S_n^\alpha) - \exp(-\frac{\beta}{2}S) \right) \right] \right| \leq \| B \exp(\frac{\beta}{2}S_n^\alpha) \| \text{Tr} \left| \exp(\frac{\beta}{2}S_n^\alpha - \frac{\beta}{2}S) \right|
\]

The second term:

\[
\text{Tr} \left[ B \left( \exp(-\frac{\beta}{2}S_n^\alpha) - \exp(-\frac{\beta}{2}S) \right) \exp(-\frac{\beta}{2}S) \right] = \text{Tr} \left[ \left( \exp(-\frac{\beta}{2}S)B \right) \left( \exp(-\frac{\beta}{2}S_n^\alpha) - \exp(-\frac{\beta}{2}S) \right) \right]
\]

Note that, by assumptions, \( B \) is closable and

\[
\exp(-\frac{\beta}{2}S)B \subset \exp(-\frac{\beta}{2}S)T \Rightarrow (\exp(-\frac{\beta}{2}S))^*B^{**} = (B^*\exp(-\frac{\beta}{2}S))^*
\]

We have assumed that \( B^* \) is \( T \)-bounded, so by Cor. \( 11 \) the operator \( B^* \exp(-\frac{\beta}{2}S) \) is bounded, as well as \( \exp(-\frac{\beta}{2}S)B \). By the previous theorem, \( \lim_{n \to \infty} \exp(-\frac{\beta}{2}S_n^\alpha) = \exp(-\frac{\beta}{2}S) \) in \( L^1(H) \) therefore the second term goes to 0. For the 0 limit of the first term it is sufficient to obtain the uniform (in \( n \)) bound for \( \| B \exp(-\frac{\beta}{2}S_n^\alpha) \| \) (note that we have assumed in the beginning of the proof that \( B \exp(-\frac{\beta}{2}S_n^\alpha) \) is bounded). Now we prove this fact.

Recall the definition of \( S_n^\alpha \):

\[
S_n^\alpha = (1 - \alpha)T + P_n(\alpha T + A + \rho I)P_n = (1 - \alpha)T + P_n(\alpha T|_D + A + \rho I)P_n,
\]

Let \( T_1 := (1 - \alpha)T \) and \( A_1 := \alpha T|_D + A + \rho I \), then \( D \subset D(A_1) \) and, for \( \psi \in D(T) \):

\[
\| A_1 \psi \| \leq \alpha \| T \psi \| + \| A \psi \| + \| \rho \| \| \psi \| \leq (a + |\rho|) \| T \psi \| + (a + |\rho|) \| \psi \| + \frac{a + b}{1 - \alpha} \| T_1 \psi \|
\]

but taking \( b < \alpha < \frac{1}{3} \) we get \( \frac{a + b}{1 - \alpha} < 1 \), therefore \( T_1 \) and \( A_1 \) satisfy assumptions \( i) - iii) \) of Prop. \( 4 \) with constants \( a_0 := (a + |\rho|) \) and \( b_0 := \frac{a + b}{1 - \alpha} < 1 \). By the lemma \( 20 \) \( P_nA_1P_n \) are also \( T_1 \) bounded with the same constants, and by the corollary \( 10 \) we get uniform bound on \( \| B \exp(-\frac{\beta}{2}S_n^\alpha) \| \). The proof is completed.

Notice that due to \( 50 \): \( P_n \exp(-\beta S_n^\alpha)P_n = P_n \exp(-\beta P_nSP_n)P_n \) therefore

\[
\text{Tr} \left[ P_nBP_n \exp(-\beta S_n^\alpha) \right] = \text{Tr} \left[ P_nBP_n \exp(-\beta P_nSP_n) \right]
\]

and we obtain, for \( B \) and \( S \) as in Thm. \( 21 \)

\[
\lim_{n \to \infty} \text{Tr} \left[ P_nBP_n \exp(-\beta P_nSP_n) \right] = \text{Tr} \left[ B \exp(-\beta S) \right] \tag{53}
\]
4 Bose-Hubbard system and Bogolyubov inequality

So far, we didn’t make any assumptions concerning the set of sites \( \Lambda \) (besides its finiteness). Now we specify \( \Lambda \) to be the cubic lattice

\[
\Lambda := \{0, 1, \ldots, N - 1\}^d \subset \mathbb{Z}^d \subset \mathbb{R}^d, \quad d \in \mathbb{N}
\]  

We will work with ‘momentum’ variables from the first Brillouin zone:

\[
\hat{\Lambda} := \{k \in \mathbb{R}^d : k_i = \frac{2\pi}{N}n_i, n_i \in \{0, \ldots, (N - 1)\}\}
\]  

Then \( |\hat{\Lambda}| = |\Lambda| = N^d \). At some moment we will assume that our interaction is “translationally invariant” and to express this condition we will think of the lattice \( \Lambda \) as of embedded into torus (or simply as of direct product cyclic group \( \mathbb{Z}_N^d \)) and we will add elements of \( \Lambda \) as elements of \( \mathbb{Z}_N^d \):

\[
x + y := ((x_1 + y_1) \mod N, \ldots, (x_d + y_d) \mod N)
\]  

The corresponding subtraction operation will be denoted by \( x - y \). Note, for \( k \in \hat{\Lambda}, x, y \in \Lambda \), equalities:

\[
\exp(ik \cdot (x - y)) = \exp(ik \cdot (x - y)) , \quad \sum_k e^{ikx} = |\Lambda| \delta_x 0
\]  

For \( k \in \hat{\Lambda} \) let us define:

\[
c(k) := \frac{1}{\sqrt{|\Lambda|}} \sum_x e^{ikx} c_x
\]

Then \( c^\dagger(k) = \frac{1}{\sqrt{|\Lambda|}} \sum_x e^{-ikx} c^\dagger_x \) and

\[
c_x = \frac{1}{\sqrt{|\Lambda|}} \sum_k e^{-ikx} c(k), \quad c^\dagger_x = \frac{1}{\sqrt{|\Lambda|}} \sum_k e^{ikx} c^\dagger(k).
\]

Moreover, the following easy equality will be used later on:

\[
\sum_x c^\dagger_x c_x = \sum_k c^\dagger(k)c(k).
\]

Partially motivated by the paper [31] (where absence of superconducting order in the fermionic Hubbard model has been proved) we define the operators \( C \) and \( A \) (as operators on \( \mathcal{D} \)) as follows:

\[
A \equiv A(k) := c^\dagger(k) = \frac{1}{\sqrt{|\Lambda|}} \sum_x e^{-ikx} c^\dagger_x, \quad C \equiv C(k) := \frac{1}{\sqrt{|\Lambda|}} \sum_x e^{ikx} n_x.
\]

We will need various commutators, recall that all of them have the meaning as operators on \( \mathcal{D} \).

Using relations (5) by straightforward (although a little bit lengthy) computations we obtain:

\[
[A^\dagger, A] = 1
\]

\[
[C, A] = \frac{1}{|\Lambda|} \sum_x c^\dagger_x = \frac{1}{\sqrt{|\Lambda|}} c^\dagger(0)
\]

\[
[C, \hat{N}] = [C, \hat{N}_2] = 0,
\]

\[
[C, L] = \frac{1}{\sqrt{|\Lambda|}} \sum_x e^{ikx}(c^\dagger_x - c_x) , \quad [C, L^\dagger] = -\frac{1}{|\Lambda|} \sum_x (c^\dagger_x + c_x) = -\frac{L}{|\Lambda|}
\]
\[ [C, T] = \frac{1}{\sqrt{|\mathcal{A}|}} \sum_{x,z} t_{xz} (e^{ikx} - e^{ikz}) \xi_x^\dagger \xi_z \]

\[ [[C, T], C^\dagger] = \frac{2}{|\mathcal{A}|} \sum_{x,y} t_{xy} (1 - \cos(\mathbf{k} \cdot (\mathbf{x} - \mathbf{y}))) \xi_x^\dagger \xi_y \quad (66) \]

\[ [[C, H], C^\dagger] = [[C, T], C^\dagger] + \lambda[[C, L], C^\dagger] = \]

\[ = -\frac{2}{|\mathcal{A}|} \sum_{x,y} t_{xy} (1 - \cos(\mathbf{k} \cdot (\mathbf{x} - \mathbf{y}))) \xi_x^\dagger \xi_y - \frac{\lambda}{|\mathcal{A}|} L \quad (67) \]

We are going to use inequality (12) for a subspace \( V := \mathcal{D}_M \). Recall that \( P_M \) denotes the orthogonal projection on \( \mathcal{D}_M \) and let \( q_M \) be the orthogonal projection on \( \mathcal{D}_M \); there are obvious relations

\[ q_M + P_{M-1} = P_M, \quad q_M P_M = q_M = P_M q_M \quad (68) \]

It is straightforward to verify that operators \( A, C \) defined in (61) satisfy assumptions of Prop. 15 with \( W := P_{M-1}(\mathcal{H}) \) and \( V := P_M(\mathcal{H}) \) i.e.

\[ P_M C \subset C P_M, \quad (C_M)^* = (C_M^\dagger) = C^\dagger P_M \supset P_M C^\dagger \]

\[ P_M A \subset A P_{M-1}, \quad (A_M)^* = (A_M^\dagger) = A^\dagger P_M \supset P_{M-1} A^\dagger, \]

therefore, using this proposition and notation introduced in (66) we obtain:

\[ [C_M, A_M] = [C, A] \quad , \quad [C_M, H_M], (C_M)^* = [[C, H], C^\dagger]_M \]

\[ A_M (A_M)^* + (A_M)^* A_M = (A A^\dagger + A^\dagger A)_M - q_M A^\dagger A q_M, \]

and

\[ \frac{\beta}{2} \langle (A A^\dagger + A^\dagger A)_M \rangle_M [[C, H], C^\dagger]_M \geq \langle [[C, A]_M \rangle_M^2 \quad (69) \]

Let us now, compute more explicitly terms appearing in (69).

By (63):

\[ ||[C, A]_M||^2 = \frac{1}{|\mathcal{A}|} ||(c_M^\dagger(0))_M||^2 =: m_M. \quad (70) \]

The quantity \( m_M \) is the “finite dimensional approximation” of order parameter and its estimate is main goal of further considerations.

Since \( 0 \leq \langle [[C, H], (C_M)^*]_M \rangle_M = \langle [[C, H], C^\dagger]_M \rangle_M \) we can write using (67)

\[ \langle [[C, H], C^\dagger]_M \rangle_M = \langle [[[C, H], C^\dagger]_M \rangle_M \]

\[ = \left| \frac{2}{|\mathcal{A}|} \sum_{x,y} t_{xy} (1 - \cos(\mathbf{k} \cdot (\mathbf{x} - \mathbf{y}))) \langle c_x^\dagger c_y \rangle_M - \frac{\lambda}{|\mathcal{A}|} \langle L_M \rangle_M \right| \leq \]

\[ \leq \frac{2}{|\mathcal{A}|} \left| \sum_{x,y} t_{xy} (1 - \cos(\mathbf{k} \cdot (\mathbf{x} - \mathbf{y}))) \langle c_x^\dagger c_y \rangle_M \right| + \frac{|\lambda|}{|\mathcal{A}|} \langle L_M \rangle_M | \quad (71) \]

The second term in above inequality is estimated in the following

\[ \textbf{Lemma 22} \]

\[ ||\langle L_M \rangle_M|| \leq \langle N_M \rangle_M + |\mathcal{A}| \]

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Remark. 2). Using translational invariance together with (59) we compute

Later on, in thermodynamic limit, we will assume that

Our translational invariance condition means:

\[ \forall x, y, z \in \Lambda : \quad t_{x,x+z} = t_{y,y+z} =: \tilde{t}_z \]  \tag{72} \]

Let \( M_2 \) satisfies:

\[ \sum_z |\tilde{t}_z|^2 \leq M_2 \]  \tag{73} \]

Later on, in thermodynamic limit, we will assume that \( M_2 \) can be chosen independently of \( \Lambda \) (cf. Remark 2). Using translational invariance together with (59) we compute

\[ \sum_{x,z} \tilde{t}_z (1 - \cos(k \cdot z)) c^+_x c_{x+z} = \sum_{x,z} \tilde{t}_z (1 - \cos(k \cdot z)) \sum_x \frac{1}{|\Lambda|} \sum_j e^{i(j-1)x} e^{-i(z \cdot c^+_x)(j \cdot c(l))} = \sum_{x,l} \tilde{t}_z (1 - \cos(k \cdot z)) e^{-i(z \cdot c^+_x)(l \cdot c(l))} \]

and the first term of the inequality (71) can be estimated as

\[ |\sum_{x,y} t_{xy} (1 - \cos(k \cdot (x - y))) (c^+_x c_M) M| \leq \sum_{x,l} \tilde{t}_z (1 - \cos(k \cdot z)) e^{-i(z \cdot c^+_x)(l \cdot c(l))} M \leq \sum_{x,l} |\tilde{t}_z (1 - \cos(k \cdot z))|(c^+_x c_M) M| \leq \frac{1}{2} |k|^2 \langle \hat{\mathcal{N}}_M \rangle_M | \sum_{z} |\tilde{t}_z|^2 |z|^2 \leq \frac{1}{2} M_2 |k|^2 \langle \hat{\mathcal{N}}_M \rangle_M , \]

where elementary inequalities \( 1 - \cos x \leq \frac{1}{2} x^2, \ (k \cdot z)^2 \leq |k|^2 |z|^2 \) together with the equality \( |\langle c^+_x c_M \rangle_M| = |\langle c^+_x c \rangle_M| \) were used.

This way we obtain the estimate:

\[ \langle [C, H], C^M \rangle_M \leq |\lambda| + \frac{\langle \hat{\mathcal{N}}_M \rangle_M |\lambda|}{|\Lambda|} (M_2 |k|^2 + |\lambda|). \]

Notice that by (62):

\[ \langle (AA^\dagger + A^\dagger A) \rangle_M = \langle (2A A^\dagger + 1) \rangle_M = \langle (2c^+_x (k)(k)) \rangle_M + 1 \]  \tag{74} \]

Now, by (70, 71) we can write the Begolyubov inequality (69) as:

\[ m_\Lambda = \frac{|\langle c^+_0 \rangle_M \rangle_M|^2}{|\Lambda|} \leq \frac{\beta}{2} \left( \frac{\langle 2c^+_x (k)(k) \rangle_M \rangle_M + 1}{|\lambda| + \frac{\langle \hat{\mathcal{N}}_M \rangle_M |\lambda|}{|\Lambda|} (M_2 |k|^2 + |\lambda|) \right) \]
Let us introduce a “finite dimensional approximation” of density
\[ \rho_M := \frac{\langle \hat{N}_M \rangle_M}{|\Lambda|}. \] (75)

Now the Begolyubov inequality reads:
\[ \frac{m_M}{|\lambda| + \rho_M(M^2|k|^2 + |\lambda|)} \leq \frac{\beta}{2} \left( \langle (2c^\dagger(k)c(k))_M \rangle_M + 1 \right). \]

Summing over \( k \in \hat{\Lambda} \) using (60) and dividing by \(|\Lambda|\) we get:
\[ m_M \sum_k \frac{1}{\rho_M M^2|k|^2 + |\lambda| (\rho_M + 1)} \leq \beta (\rho_M + \frac{1}{2}) \] (76)

Now we shall show that \( m_M \) and \( \rho_M \) have limits as \( M \to \infty \):

**Proposition 23**

\[ \lim_{M \to \infty} \langle \hat{N}_M \rangle_M = \langle \hat{N} \rangle, \quad \lim_{M \to \infty} \langle (c^\dagger(0))_M \rangle_M = \langle (c(0))^* \rangle \]

**Proof:** We are going to use Thm. 21 with the following identifications: \( T := u\hat{N} \) (then \( T|_D = u\hat{N} \)), \( A := T + \mu \hat{N} + \lambda L \) and our sequence of projections is \( P_M \). By Prop. 11 the operator \( A \) is \( u\hat{N}^2 \)-bounded with relative bound 0, so clearly we can choose \( b < \frac{1}{3} \) and we have a self-adjoint operator \( S := H(u) + \rho I \).

We know that \( \hat{N} \) is \( u\hat{N}^2 \)-bounded, so (compare the proof of Prop. 4) \( \hat{N} = \hat{N}^* \) is \( u\hat{N}^2 \)-bounded, so it can be put as operator \( B \) in Thm. 21.

By the Lemma 13 \( c(0) \) and \( c^\dagger(0) \) are \( u\hat{N}^2 \) bounded. Clearly \( c(0) = (c^\dagger(0))^*|_D \) and \( c^\dagger(0) = (c(0))^*|_D \) so both of them are closable and their closures are \( u\hat{N}^2 \)-bounded, so each of them can be used as \( B \) in Thm. 21. So, as in (63):
\[ \lim_{M \to \infty} \text{Tr} \left[ P_M \hat{N} P_M \exp(-\beta P_M (H(u) + \rho I)P_M) \right] = \text{Tr} \left[ \hat{N} \exp(-\beta (H(u) + \rho I)) \right] \]
\[ \lim_{M \to \infty} \text{Tr} \left[ P_M c^\dagger(0) P_M \exp(-\beta P_M (H(u) + \rho I)P_M) \right] = \text{Tr} \left[ (c(0))^* \exp(-\beta (H(u) + \rho I)) \right] \]
\[ \lim_{M \to \infty} \text{Tr} \left[ P_M \exp(-\beta P_M (H(u) + \rho I)P_M) \right] = \text{Tr} \left[ \exp(-\beta (H(u) + \rho I)) \right] \]

Since thermal average is invariant with respect to shift of Hamiltonian by a constant, the proposition follows. \( \square \)

Let us denote
\[ m_\Lambda := \lim_{M \to \infty} m_M = \frac{|\langle (c(0))^* \rangle|}{|\Lambda|} \quad \text{and} \quad \rho_\Lambda := \lim_{M \to \infty} \rho_M = \frac{\langle \hat{N} \rangle}{|\Lambda|}. \]

Clearly \( m_\Lambda \geq 0 \) and \( \rho_\Lambda > 0 \). Now, passing to the limit \( M \to \infty \) in (76) and multiplying by \( \rho_\Lambda \) we obtain:
\[ m_\Lambda \sum_k \frac{1}{M^2|k|^2 + |\lambda|(1 + 1/\rho_\Lambda)} \leq \beta \rho_\Lambda (\rho_\Lambda + \frac{1}{2}) \] (77)
5 Thermodynamic limit

At this moment it is useful to state clearly assuptions we made before passing to the limit. We will consider only Hamiltonians defined by [13] which satisfy the following conditions:

1. The lattice $\Lambda$ is a cubic lattice as in (54);
2. Translational invariance of $T$ as in (72): $t_{x,x+z} =: \tilde{t}_z$;
3. There exists $M_2$ independent of $\Lambda$ such that $\sum_{z \in \Lambda} |\tilde{t}_z|^2 \leq M_2$;
4. There exists $M_d$ such that $|\tilde{t}_0| \leq M_d$; (These three conditions imply that $M$ defined in (14) can be chosen independently of $\Lambda$);
5. There exist constants $\rho_1 > 0$ and $\rho_2$ independent of $\Lambda$ and $\lambda$ such that

$$0 < \rho_1 \leq \rho_\Lambda \leq \rho_2,$$

(78)

Using equality $|\Lambda| = N^d$, (78) and (77) we have

$$m_\Lambda \left( \frac{2\pi}{N} \right)^d \sum_k M_2 |k|^2 + |\lambda|(1 + 1/\rho_1) \leq m_\Lambda \left( \frac{2\pi}{N} \right)^d \sum_k \frac{1}{M_2 |k|^2 + |\lambda|(1 + 1/\rho_\Lambda)} \leq (2\pi)^d \beta \rho_\Lambda (\rho_\Lambda + \frac{1}{2}) \leq (2\pi)^d \beta \rho_2 (\rho_2 + \frac{1}{2})$$

(79)

It is easy to see, that when $N \to \infty$ the quantity $\left( \frac{2\pi}{N} \right)^d \sum_k \frac{1}{M_2 |k|^2 + |\lambda|(1 + 1/\rho_1)}$ converges to the integral

$$\int_{[0,2\pi]^d} \frac{d^dk}{M_2 |k|^2 + |\lambda|(1 + 1/\rho_1 + 1)}$$

This integral for $d \geq 3$ is finite for any $\lambda \in \mathbb{R}$ and

for $d = 1$:

$$\int_{[0,2\pi]} \frac{dx}{\frac{M_2}{\sqrt{\lambda}} x^2 + |\lambda|(1/\rho + 1)} = \frac{\arctan \left( \frac{2\pi \sqrt{M_2}}{\sqrt{\lambda}} \right)}{\sqrt{M_2 \alpha}}, \alpha := |\lambda|(1 + 1/\rho)$$

for $d = 2$:

$$\int_{[0,2\pi]^2} \frac{dxdy}{\frac{M_2}{\sqrt{\lambda}} (x^2 + y^2) + \alpha} \geq \int_0^{\pi/2} d\varphi \int_0^{2\pi} d\rho \int_0^{\frac{\pi}{2}} dr \frac{rdr}{\frac{M_2}{\sqrt{\lambda}} + \alpha} = \frac{\pi}{4M_2} \log \left( 1 + \frac{4M_2 \pi^2}{\alpha} \right)$$

and both integrals are divergent to $\infty$ as $\lambda \to 0$, therefore we finally get “no condensation” result:

**Proposition 24** Let $d = 1$ or $d = 2$. For any $\epsilon > 0$ there exist $\delta_\epsilon > 0$ and $N_\epsilon$ such that:

$$|\lambda| < \delta_\epsilon \text{ and } N \geq N_\epsilon \Rightarrow |m(\lambda, \Lambda)| \leq \epsilon$$

□

6 Summary

We have rigorously shown that for one- and two-dimensional Bose–Hubbard model, with translationally invariant interaction and periodic boundary conditions, with arbitrary hoppings, which fall-off sufficiently fast with a distance, for any filling and any positive temperature, there is no Bose–Einstein condensation. We allowed arbitrary occupation of every site. For models where the
occupation is bounded by some constant, we have also absence of Bose–Einstein condensation, as it is rather easy corollary from considerations in the paper (32).

In the area of bosonic Hubbard model, there is still quite a few problems, waiting for rigorous treatment (the situation is similar for fermionic version). Even refraining ourselves to two dimensions and positive temperature, one encounters open problems; one of the most important ones is proving existence of the the Kosterlitz–Thouless type transition [33]. It has been rigorously established for some classical models (XY, sine-Gordon, Coulomb gas ones) [34]. Despite extensive numeric and non-rigorous treatment, the proofs for quantum models, including 2d Bose-Hubbard one, are – as far as we know – still lacking.

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