Higher Energies in Kähler Geometry I

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Abstract

Let $X \hookrightarrow \mathbb{P}^N$ be a smooth complex projective variety of dimension $n$. Let $\lambda$ be an algebraic one parameter subgroup of $G := SL(N+1, \mathbb{C})$. Let $0 \leq l \leq n+1$. We associate to the coefficients $F_l(\lambda)$ of the normalized weight of $\lambda$ on the $m$th Hilbert point of $X$ new energies $F_{\omega,l}(\varphi)$. The (logarithmic) asymptotics of $F_{\omega,l}(\varphi)$ along the potential deduced from $\lambda$ is the weight $F_l(\lambda)$. $F_{\omega,l}(\varphi)$ reduces to the Aubin energy when $l = 0$ and the K-Energy map of Mabuchi when $l = 1$. When $l \geq 2$ $F_{\omega,l}(\varphi)$ coincides (modulo lower order terms) with the functional $E_{\omega,l-1}(\varphi)$ introduced by X.X. Chen and G.Tian.

§0 The Standard Energies of Kähler Geometry

Recall that Mabuchi’s K-energy map (see [15]) is given by

$$\nu_{\omega}(\varphi) := -\frac{1}{V} \int_0^1 \int_X \varphi_t (\text{Scal}(\varphi_t) - \mu) \omega_t^n dt .$$

$\varphi_t$ denotes a path in $P(X, \omega)$, $\text{Scal}(\varphi_t)$ denotes the scalar curvature of the metric $\omega + \sqrt{-1} \partial \bar{\partial} \varphi_t$ and $\mu$ denotes the average of the scalar curvature. Critical points of the K-Energy are metrics of constant scalar curvature. In [2] Bando and Mabuchi have proved that the K-Energy is bounded from below provided that $(X, \omega)$ admits a Kähler Einstein metric (in this case it is required that $\omega = c_1(K_X^{-1})$). This is noteworthy as the K Energy map is (essentially) the difference of two positive terms as follows.

$$\nu_{\omega}(\varphi) = \frac{1}{V} \int_M \log \left( \frac{\omega^n_{\varphi}}{\omega^n} \right) \omega^n_{\varphi} - \frac{\mu}{n} (I_{\omega}(\varphi) - J_{\omega}(\varphi)) + \frac{1}{V} \int_M h_{\omega} \omega^n_{\varphi}$$

$$I_{\omega}(\varphi) := \frac{1}{V} \int_X \varphi (\omega^n - \omega^n_{\varphi})$$

$$J_{\omega}(\varphi) := \frac{\sqrt{-1}}{V} \int_X \sum_{i=0}^{n-1} \frac{i+1}{n+1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega^n_{\varphi}^{-i-1} .$$

The work of Bando and Mabuchi has been extended to any Kähler class by X.X. Chen and G. Tian.

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Recall that $\nu_\omega$ is proper (see [24]) provided there is an increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $\nu_\omega(\varphi) \geq f(J_\omega(\varphi))$ for all $\varphi \in P(X, \omega)$. This notion is due to Gang Tian. It is known that properness is independent of the Kähler class $\omega$ (see [24]). In [24] Tian has proved (under the assumption $\eta(X) = \{0\}$) that $\nu_\omega$ is proper iff there is a Kähler Einstein metric in the class $\omega$.

In order to detect properness (conjecturally) one restricts attention to the subspace of Bergman metrics inside $P(X, \omega)$ as these are dense in $P(X, \omega)$ (see [22], [21], [26], [4]). The Bergman metrics are induced by the Kodaira embeddings furnished by large multiples of the polarization. More precisely, let $X \hookrightarrow \mathbb{C}P^N$ be the Kodaira embedding defined by a basis $\{S_0, S_1, \ldots, S_N\}$ of $H^0(X, L^\otimes m)$. There is a map $i\{S_j\} : G \to P(X, \omega)$ given as follows

$$i\{S_j\}(\sigma) \equiv \frac{1}{m} \varphi_\sigma = \frac{1}{m} \log \left( \sum_{k=0}^{N} \sum_{l=0}^{N} \sigma_{k,l} S_l \right).$$

$\omega + \sqrt{-1} \partial \bar{\partial} \frac{1}{m} \varphi_\sigma$ are the Bergman metrics. If $E_\omega$ denotes one of the energies we let $E_\omega(\sigma) := E_\omega(\frac{1}{m} \varphi_\sigma)$. The main issue is to let $\sigma = \lambda t$, an algebraic one parameter subgroup, and analyze the small $t$ asymptotics $\lim_{t \to 0} E_\omega(\lambda t)$. It is when we restrict the energies to the subspaces defined by the Bergman metrics that we make contact with Geometric Invariant Theory. The first result in this direction is due to Tian who established the following theorem by exhibiting the K-energy map as the logarithm of a singular metric on a power of the ample divisor on the moduli space of hypersurfaces of degree $d$ in $\mathbb{P}^N$. This is unquestionably the paradigm for all subsequent results in this area of investigation.

**Theorem (G. Tian [23])** Let $Z_f$ be a normal degree $d \geq 2$ hypersurface in $\mathbb{P}^N$, then $Z_f$ is stable if the K-energy is proper, and $Z_f$ is semistable if the K-energy is bounded from below.

There is a related result in higher codimension, which was established independently by S. Zhang and the author. Simon Donaldson has found an outstanding application of this theorem (see [7]).

**Theorem (Zhang [27], Paul [20])** Let $X$ be an $n$-dimensional subvariety of $\mathbb{P}^N$, and let $Z_X := \{R_X = 0\}$ denote the associated hypersurface ($R_X$ is the $X$-resultant). Let $\lambda(t)$ be an algebraic one parameter subgroup of $G$. The weight of the action of $\lambda$ on $R_X$ is denoted by $F_0(\lambda)$. Then the following asymptotic expansion holds as $|t| \to 0$

$$d(n+1)F_0^0(\varphi_{\lambda(t)}) = F_0(\lambda) \log(|t|^2) + O(1).$$

In 1992 Ding and Tian (see [6]) have studied the K-energy asymptotics along the algebraic potentials $\varphi_{\lambda(t)}$ when the limit is an almost Fano variety. They defined the generalized Futaki invariant of a degeneration and proved that the sign of this invariant (which is a rational number) is an obstruction to the existence of a Kähler

\[1^\eta(X) \text{ denotes the Lie algebra of holomorphic vector fields.}\]
Einstein metric in the class $-K_X$. In 2002 (see [8]) Simon Donaldson connected this invariant in an exciting way to Geometric Invariant Theory on the Hilbert scheme. Recently the author and G.Tian have proved the following theorem.

**Theorem** (Paul, Tian [19]) Assume that $(X,L)$ moves in a good family $\mathcal{X}$. Then there is a function $\Psi_X : G \to \mathbb{R}$ such that $-\infty \leq \Psi_X \leq C$ and an asymptotic expansion

$$
(d(n+1))^{\nu_\omega}(\varphi_{\lambda(t)}) - \Psi_X(\lambda(t)) = F_1(\lambda) \log(|t|^2) + O(1) \text{ as } |t| \to 0.
$$

Moreover $\Psi_X(\lambda(t)) = \psi(\lambda) \log(|t|^2) + O(1)$ where $\psi(\lambda) \in \mathbb{Q}_{\geq 0}$ and $\psi(\lambda) \in \mathbb{Q}_+$ if and only if $X^{\lambda(0)}$ (the limit cycle of $X$ under $\lambda$) has a component of multiplicity greater than one. $F_1(\lambda)$ is the generalized Futaki invariant of the degeneration $\lambda$, and $O(1)$ denotes any quantity which is bounded as $|t| \to 0$.

In their study of the Kähler Ricci flow on Kähler Einstein manifolds X.X. Chen and G.Tian introduced a set of new energy functionals $E_{\omega,l}$ ($l = 0, 1, 2, \ldots n$) which monotonically decrease along the flow (under a positivity hypothesis). These have received much attention in the recent literature. It is now known that these energies are bounded from below in the presence of a Kähler Einstein metric. Therefore it is reasonable to attempt to connect these energies to Geometric Invariant Theory in the spirit of the above theorems. Unfortunately this does not seem possible, however in this paper we are able to modify the $E_{\omega,l}$ so that the asymptotic results go through. We call these new energies simply higher energies.

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§1 Statement of Results

Let $X \hookrightarrow \mathbb{P}^N$ be a projective variety. We are concerned with a certain procedure the rough expression of which is the following.

**Step one.** Given $\lambda : \mathbb{C}^* \longrightarrow G$ take the coefficient $F_l(\lambda)$ of $m^{-l}$ in the expansion of the normalized weight of the action of $\lambda$ on the $m$th Hilbert point of $X$.

**Step two.** Choose a virtual bundle $\xi$ such that the determinant of its direct image has weight $F_l(\lambda)$ with respect to the action of $\lambda$.

**Step three.** Define an energy functional $F_{l,\omega}$ as the transgression of the Riemann Roch Hirzebruch integrand with respect to $\xi$. General theory then exhibits this energy as a singular metric on the sheaf obtained in step two.

**Conclusion.** Algebro geometric energy asymptotics of the functional $F_{l,\omega}$.

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2 See [19].
3 See [17] pg. 61.
4 See [8] and [9].
More precisely let $X \xrightarrow{f} S$ be a flat family of subschemes of $\mathbb{P}^N$. Let $L \in \text{Pic}(X)$ be relatively very ample. $Q[L]$ denotes the subring of $K^0(X) \otimes \mathbb{Q}$ generated by $L$. Let $\Delta$ denote the discriminant locus of $f$. That is, $X_z$ is $C^\infty$ for all $z \in S \setminus \Delta$. Our purpose is to study two maps

\begin{align*}
&i) \quad \tau_X : Q[L] \times P_\omega(X_z) \to \mathbb{R} \quad z \in S \setminus \Delta \\
&\quad \quad \tau_X(\xi, \varphi) := \left(\frac{-1}{2\pi}\right)^n \int_X \int_0^1 \frac{\partial}{\partial b} \left( \text{Td}(R_{g_t} + b\frac{\partial g_t}{\partial t} g_t^{-1}) \text{Ch}(F_{H_t}^\xi + b\frac{\partial H_t}{\partial t} H_t^{-1}) \right) \bigg|_{b=0} \quad dt \\
&\text{Where} \\
&\quad g_t = \omega + \frac{-1}{2\pi} \partial \partial \varphi_t, \text{ a family of Kähler metrics on } X, \\
&\quad H_t = e^{-\varphi_t} h, \text{ h a Hermitian metric on } L, \\
&\quad F_{H_t}^\xi, \text{ The curvature of } \xi \text{ with respect to } H_t, \\
&\quad \omega = -\frac{1}{2\pi} \partial \partial \log(h), \text{ the curvature of } h, \\
&\quad R_g := \partial\{\partial(g)g^{-1}\} \in C^\infty(\Lambda^{1,1} \otimes \text{End}(T_X^{0,0})), \text{ the full Kähler curvature tensor.} \\
&\text{and} \\
&ii) \quad \text{Det} : Q[L] \to \text{Pic}(S), \quad \text{Det}(\xi) := \det(R_{f*}(\xi)) \quad .
\end{align*}

For the virtual bundle in step two we make the following choice:

\begin{align*}
E_l(m) := \sum_{\{0 \leq j \leq l\}} \sum_{0 \leq i \leq n+1-j} q_{n+1-j}(m)(-1)^i \binom{n+1-j}{i} L_{m+i} \in Q[L]. \quad (0.1)
\end{align*}

where $0 \leq l \leq n + 1$. The coefficients $q_{n+1-j}(m) \in \mathbb{Q}$ are defined below.

**Theorem 1.** There are invertible sheaves $L_l \ (0 \leq l \leq n+1)$ on $\mathcal{H}ilb_{\mathbb{P}^N}(\mathbb{C})$ such that for any family $X \xrightarrow{f} S$ as above we have

\begin{itemize}
    \item[i)] $\text{Det}(E_l(m)) \cong g^*(L_l)$. Consequently $\text{Det}(E_l(m))$ is independent of $m$.
\end{itemize}

Assume that $S$ is proper, and that $G$ acts on the data, then we can state the following.

\begin{itemize}
    \item[ii)] For any one parameter subgroup $\lambda$ of $G$ and $z \in \mathcal{H}ilb_{\mathbb{P}^N}(\mathbb{C})$
    \end{itemize}

let $w_\lambda(z)$ denote the weight of the $\mathbb{C}^*$ action on $L^\vee |_{z_0}$ where $z_0 = \lambda(0)z$, then $w_\lambda(z) = F_l(\lambda)$.

Assume that GRR is valid for the map $X \xrightarrow{f} S$. Then

\begin{itemize}
    \item[iii)] $c_1(g^* L_l) = \sum_{n+1-l \leq k \leq n+1} f_* \left\{ \frac{C_{l,k}}{k!} Td_{n+1-k}(f) \omega^k \right\}.$
\end{itemize}
Next we introduce our new energy functionals. These simultaneously generalize the Aubin energy, Mabuchi’s K-energy, and (modulo lower order terms) the Chen Tian energy functionals. General theory shows that they are all path independent.

**Definition 1. (Higher Energy Functionals)**

\[ F_{l,\omega}(\varphi) := \tau_X(\mathcal{E}_l(m), \varphi) \quad (0.2) \]

In the statement of the next result we assume that \( X = X_z \) is a generic member in a smooth family \( X \xrightarrow{f} S \).

**Theorem 2.** Let \( \lambda \) be a smooth degeneration of \( X \). Then for all \( 0 \leq l \leq n + 1 \) there is an asymptotic expansion

\[ F_{l,\omega}(\varphi_\lambda(t)) = F_l(\lambda) \log(|t|^2) + O(1) \text{ as } |t| \to 0. \quad (0.3) \]

Where \( O(1) \) denotes any quantity which is bounded as \( |t| \to 0 \). Moreover, we have

\[ F_{0,\omega}(\varphi) = F_0^{\omega}(\varphi) \text{ (Aubin’s Energy)} \]

\[ F_{1,\omega}(\varphi) = \nu_\omega(\varphi) \text{ (Mabuchi’s K-energy)}. \]

**Corollary 1.** If \( F_{l,\omega}(\varphi) \) is bounded from below then for all smooth degenerations \( \lambda \) we have that \( F_l(\lambda) \leq 0 \).

Recall that \( E_{\omega,l} \) is given by the following expression (see [5] for more details).

\[
\frac{dE_{\omega,l}}{dt} := \frac{l + 1}{V} \int_X \Delta_\varphi \left( \frac{\partial \varphi}{\partial t} \right) \text{Ric} (\omega_\varphi)^l \wedge \omega_\varphi^{n-1} - \frac{n-l}{V} \int_X \frac{\partial \varphi}{\partial t} \left( \text{Ric} (\omega_\varphi)^{l+1} - \omega_\varphi^{l+1} \right) \wedge \omega_\varphi^{n-l-1}.
\]

In particular when \( l = 1 \) we have that

\[
\frac{dE_{\omega,1}}{dt} = \frac{2n}{V} \int_M \Delta_\varphi \hat{\varphi}_1 \text{Ric}(\varphi_1) \wedge \omega_\varphi^{n-1} - \frac{n(n-1)}{V} \int_M \hat{\varphi}_1 \left( \text{Ric}(\varphi_1)^2 - \omega_\varphi^2 \right) \wedge \omega_\varphi^{n-2}.
\]

In this paper we limit our study to the first new energy functional \( F_{2,\omega} \). We will take up the further study of this functional in a sequel to this paper. Next we state the following proposition which compares \( E_{1,\omega} \) with \( F_{2,\omega} \).

**Proposition 1.**

\[
\frac{dF_{2,\omega}(\varphi_t)}{dt} = \frac{2n}{V} \int_M \Delta_\varphi \hat{\varphi}_1 \text{Ric}(\varphi_1) \wedge \omega_\varphi^{n-1} - \frac{3n(n-1)}{2V} \int_M \hat{\varphi}_1 \left( \text{Ric}(\varphi_1)^2 - \alpha(g)\omega_\varphi^2 \right) \wedge \omega_\varphi^{n-2} - \frac{1}{V} \int_M \hat{\varphi}_1 \left( ||R(\varphi_1)||^2 - ||\text{Ric}(\varphi_1)||^2 - \beta(g) \right) \omega_\varphi^n - 3\mu \frac{d\nu_\omega}{dt}
\]

\( ||R||^2 \) and \( ||\text{Ric}||^2 \) denote the square norm of the full curvature tensor and Ricci curvature respectively.

\[
\alpha(g) := \frac{1}{V} \int_M \text{Ric}_g^2 \wedge \frac{\omega_\varphi^{n-2}}{n!}, \quad \beta(g) := \frac{1}{V} \int_M \left( ||R||^2 - ||\text{Ric}_g||^2 \right) \frac{\omega_\varphi^n}{n!}
\]
§1 Hilbert Points and Chow Forms

Let \((X, \mathcal{L})\) be a polarized algebraic variety. Assume that \(\mathcal{L}\) is very ample with associated embedding

\[ X \xrightarrow{\varphi_{\mathcal{L}}} \mathbb{P}(H^0(X, \mathcal{L})^*). \]

Fix an isomorphism

\[ \sigma : H^0(X, \mathcal{L})^* \xrightarrow{\cong} \mathbb{C}^{N+1}. \]

In this way we consider \(X\) embedded in \(\mathbb{P}^N\). Let \(m \in \mathbb{Z}\) be a large positive integer. Then there is a surjection

\[ \psi_{X,m} : S^m(\mathbb{C}^{N+1})^* \rightarrow H^0(X, \mathcal{O}(m)) \rightarrow 0. \]

Let \(\chi(m) = \chi(X, \mathcal{O}(m)) = h^0(X, \mathcal{O}(m))\). It is a deep fact (see [25]) that there is an integer \(m(P)\) depending only on the Hilbert polynomial \(P\) such that for all \(m \geq m(P)\), the kernel of \(\psi_{X,m}\)

\[ \text{Ker}(\psi_{X,m}) \in G(\chi(m), S^m(\mathbb{C}^{N+1})^*) \]

completely determines \(X\). In other words, the entire homogeneous (saturated) ideal can be recovered from its \(m\)th graded piece. We have the Plücker embedding

\[ \mathcal{P} : G(\chi(m), S^m(\mathbb{C}^{N+1})^*) \rightarrow \mathbb{P} \left( \bigwedge^{d_m-\chi(m)} S^m(\mathbb{C}^{N+1})^* \right). \]

Next we consider the canonical nonsingular pairing

\[ \bigwedge^{d_m-\chi(m)} S^m(\mathbb{C}^{N+1})^* \otimes \chi(m) \rightarrow \bigwedge S^m(\mathbb{C}^{N+1})^* \rightarrow \text{det}(S^m(\mathbb{C}^{N+1})^*). \]

This induces a natural isomorphism

\[ \mathbb{P} \left( \bigwedge^{d_m-\chi(m)} S^m(\mathbb{C}^{N+1})^* \right) \xrightarrow{\cong} \mathbb{P} \left( \bigwedge^\chi(m) S^m(\mathbb{C}^{N+1})^* \right). \]

Combining this identification with the Plücker embedding we associate to \(\text{Ker}(\psi_{X,m})\) a unique point, called (following Gieseker) the \(m\)th Hilbert Point

\[ \text{Hilb}_m(X) := \iota(\mathcal{P}(\text{Ker}(\psi_{X,m}))) \in \mathbb{P} \left( \bigwedge^\chi(m) S^m(\mathbb{C}^{N+1})^* \right). \]

\(^5\)\(S^m\) denotes the \(m\)th symmetric power operator.

\(^6\)The Grassmannian of \(\chi(m)\) dimensional quotients of \(S^m(\mathbb{C}^{N+1})^*\).
Let \( I = (i_0, i_1, \ldots, i_N) \) be a multiindex with \(|I| := i_0 + i_1 + \cdots + i_N = m, \ i_j \in \mathbb{N} \). Let \( e_0, e_1, \ldots, e_N \) be the standard basis of \( \mathbb{C}^{N+1} \), and \( z_0, z_1, \ldots, z_N \) be the dual basis of linear forms. Consider the monomials \( M_I := e_0^{i_0} e_1^{i_1} \cdots e_N^{i_N} \) and \( M_I^* := z_0^{i_0} z_1^{i_1} \cdots z_N^{i_N} \).

Fix a basis \( \{ f_1, \ldots, f_X(m) \} \) of \( H^0(X, \mathcal{O}(m)) \). Then

\[
\bigwedge^{(m)} \psi_{X,m}(M_{I_1}^* \wedge \cdots \wedge M_{I_{\chi(m)}}^*) = \psi_{X,m}(j_1, \ldots, j_{\chi(m)}) f_1 \wedge \cdots \wedge f_X(m)
\]

\[
\psi_{X,m}(j_1, \ldots, j_{\chi(m)}) \in \mathbb{C}.
\]

Then in homogeneous coordinates we can write

\[
\bigwedge^{(m)} \psi_{X,m} = \sum_{\{I_j, \ldots, I_{\chi(m)}\}} \psi_{X,m}(j_1, \ldots, j_{\chi(m)}) M_{I_1}^* \wedge \cdots \wedge M_{I_{\chi(m)}}^*.
\]

Let \( \lambda : \mathbb{C}^* \rightarrow G \) be an algebraic one parameter subgroup. We may assume that \( \lambda \) has been diagonalized on the standard basis \( \{e_0, e_1, \ldots, e_N\} \). Explicitly, we assume that there are \( r_i \in \mathbb{Z} \) such that

\[
\lambda(t)e_j = t^{r_j}e_j.
\]

Define the weight of \( \lambda \) on the monomial \( M_I \) by

\[
w_{\lambda}(M_I) := r_0i_0 + r_1i_1 + \cdots + r_Ni_N.
\]

**Definition 2.** (Gieseker [12]) The weight of the \( m \)th Hilbert point of \( X \) is the integer

\[
w_{\lambda}(m) := \min \left\{ \{I_1, \ldots, I_{\chi(m)}\} \left( \sum_{1 \leq k \leq \chi(m)} w_{\lambda}(M_{I_{\sigma(k)}}) |\psi_{X,m}(j_1, \ldots, j_{\chi(m)}) \neq 0 \right) \right\}.
\]

Let \( X \subset \mathbb{P}^N \) be an \( n \) dimensional irreducible subvariety of \( \mathbb{P}^N \) with degree \( d \), then the Chow form, or associated hypersurface to \( X \) is defined by

\[
Z_X := \{ L \in \mathbb{G} := \mathbb{G}(N-n-1, \mathbb{C}P^N) : L \cap X \neq \emptyset \}.
\]

It is easy to see that \( Z_X \) is an irreducible hypersurface (of degree \( d \)) in \( \mathbb{G} \). Since the homogeneous coordinate ring of the grassmannian is a UFD, any codimension one subvariety with degree \( d \) is given by the vanishing of a section \( R_X \) of the homogeneous coordinate ring

\[
\{ R_X = 0 \} = Z_X ; \ R_X \in \mathbb{P} H^0(\mathbb{G}, \mathcal{O}(d)).
\]

\( R_X \) is confounded with \( Z_X \). Following [13] we can be more concrete as follows. Let \( M_{n+1,N+1}(\mathbb{C}) \) be the (Zariski open and dense) submanifold of the vector space of \( M_{n+1,N+1}(\mathbb{C}) \) matrices of full rank. We have the canonical projection

\[
p : M_{n+1,N+1}(\mathbb{C}) \rightarrow \mathbb{G}(N-n, \mathbb{C}^{N+1}),
\]

\[\text{See [10] pg. 140 exercise 7.}\]
This map is dominant, so the closure of the preimage
\[ \overline{p^{-1}(Z_X)} \subset M_{n+1,N+1}^0(\mathbb{C}) = M_{n+1,N+1}(\mathbb{C}) \]
is also an irreducible hypersurface of degree \( d \) in \( M_{n+1,N+1}(\mathbb{C}) \). Therefore, there is
a unique\( \text{(symmetric multihomogeneous)} \) polynomial (which will also be denoted by \( R_X \)) such that
\[ Z := \overline{p^{-1}(Z_X)} = \{ R_X(w_{ij}) = 0 \} ; \quad R_X(w_{ij}) \in \mathbb{P}^d[M_{n+1,N+1}(\mathbb{C})]. \]
There is (in principal) an explicit formula for the polynomial \( R_X(w_{ij}) \), which is es-
sentially due to Cayley in his remarkable 1848 note \[1\] on resultants. The modern
formulation of these ideas are due to Grothendieck, Knudsen, and Mumford. See \[14\].

**Theorem (Cayley, Grothendieck, Knudsen, Mumford)**

There is a canonical isomorphism of one dimensional vector spaces
\[ \Delta^{n+1} \det(H^0(X, \mathcal{O}(m))) = \bigotimes_{i=0}^{n+1} \det(H^0(X, \mathcal{O}(m+i)))^{(-1)^{i+1} \binom{n+1}{i}} \cong \mathbb{C} R_X^{(-1)^{n+1}}. \]
\( \Delta \) denotes the first forward difference operator.

It follows from this that the weight is a \( w_\lambda(m) \) is a polynomial in \( m \)
\[ w_\lambda(m) = a_{n+1}(\lambda)m^{n+1} + a_n(\lambda)m^n + O(m^{n-1}). \]

Let \( \chi(m) \) be the Hilbert polynomial of \( X \). Following Gieseker, we consider the ratio
\[ \frac{w_\lambda(m)}{m\chi(m)}. \]

As in Donaldson \[8\] we consider the coefficients of \( m^{-l} \) in the expansion
\[ \frac{w_\lambda(m)}{m\chi(m)} = F_0(\lambda) + F_1(\lambda) \frac{1}{m} + \cdots + F_l(\lambda) \frac{1}{m^l} + \cdots \]

\[ F_l(\lambda) = c_{l,n+1}a_{n+1}(\lambda) + c_{l,n}a_n(\lambda) + c_{l,n-1}a_{n-1}(\lambda) + \cdots + c_{l,n+1-l}a_{n+1-l}(\lambda). \]
The \( c_{l,j} \) are all rational functions of the coefficients of the Hilbert polynomial \( \chi \).

The relationship between Hilbert Points and Chow forms extends to the relative
situation as well. Let \( f : X \rightarrow S \) be a flat morphism of projective varieties.

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\[ ^8 \text{Unique up to scaling.} \]
We assume that $\mathcal{L}$ is a relatively ample line bundle on $\mathfrak{X}$ with respect to the map $f$. The isomorphism $\mathbb{P}(f_*\mathcal{L}^\vee) \cong S \times \mathbb{P}^N$ is equivalent to the existence of an invertible sheaf $\mathcal{A}$ on $S$ such that

$$f_*\mathcal{L} \cong \bigoplus_{N+1} \mathcal{A}.$$ 

Let $\chi$ denote the Hilbert polynomial of the fibers. Let $\text{Hilb}_{\mathbb{P}^N}(\mathbb{C})$ denote the Hilbert scheme. For $m$ large enough there is a map $\varphi_m$ from $S$ into $\text{Hilb}_{\mathbb{P}^N}(\mathbb{C})$

$$\varphi_m : S \to \text{Hilb}_{\mathbb{P}^N}(\mathbb{C}) \hookrightarrow \mathbb{P}^{N(m)} := \mathbb{P}(\bigwedge_{N+1} S^m(\bigoplus_{N+1} \mathbb{C})).$$

Pulling back $\mathcal{O}_{\mathbb{P}^{N(m)}}(1)$ to $S$ via $\varphi_m$ gives the isomorphism

$$\varphi_m^* \mathcal{O}_{\mathbb{P}^{N(m)}}(1) \cong \text{det}(f_*\mathcal{L}^m) \otimes \text{det}(f_*\mathcal{L})^{-\frac{m\chi(m)}{N+1}}.$$ 

Therefore the appropriate generalization of Hilbert points to families is the following invertible sheaf on $S$

$$\text{Hilb}_m(\mathfrak{X} \setminus S) := \text{det}(f_*\mathcal{L}^m) \otimes \text{det}(f_*\mathcal{L})^{-\frac{m\chi(m)}{N+1}}.$$ 

Let $\mathfrak{C}(n, d; \mathbb{P}^N)$ denote the Chow Variety of dimension $n$ and degree $d$ algebraic cycles inside $\mathbb{P}^N$. There is a morphism $\Delta^{n+1}$ from the Hilbert scheme to the Chow variety (see [16] and [9]) which sends a subscheme $\mathcal{I}$ of $\mathbb{P}^N$ with Hilbert polynomial $\chi$ to the Chow form of the top dimensional component of its underlying cycle. We now have the sequence of maps

$$S \xrightarrow{\varphi_m} \text{Hilb}_{\mathbb{P}^N}(\mathbb{C}) \xrightarrow{\Delta^{n+1}} \mathfrak{C}(n, d; \mathbb{P}^N) \xrightarrow{i} \mathbb{P}(H^0(\mathbb{G}, \mathcal{O}(d))).$$

Let $\mathcal{O}(1)$ denote the hyperplane line on $\mathbb{P}(H^0(\mathbb{G}, \mathcal{O}(d)))$. Then we define the Chow form of the map $\mathfrak{X} \xrightarrow{f} S$ to be the invertible sheaf on $S$

$$\text{Chow}(\mathfrak{X} \xrightarrow{f} S) := \varphi_m^* \Delta^{n+1} i^* \mathcal{O}(1).$$

The extension to families of the relationship between Chow forms and Hilbert points can be stated as follows

On the base $S$ there is a canonical isomorphism of invertible sheaves

$$\text{Chow}(\mathfrak{X} \xrightarrow{f} S) \otimes \text{det}(f_*\mathcal{L})^{\frac{d(n+1)}{N+1}} \cong \Delta^{n+1} \text{det}(f_*\mathcal{L}^m).$$

This immediately implies that the following expansion holds (see [14], in particular Theorem 4, for a complete discussion).
**Theorem (Knudsen, Mumford)**
There are invertible sheaves $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_{n+1}$ on $S$ and a canonical and functorial isomorphism:

$$\det(f_*\mathcal{L}^m) \cong \bigotimes_{j=0}^{n+1} \mathcal{M}_j^{(m)}.$$

§2 Higher Sheaves

Let $\omega_{X/S} := K_X \otimes f^*(K_S^{-1})$ denote the relative canonical bundle. In [23] Tian introduced the following invertible sheaf on $S$, which one calls the CM polarization.

$$L_{CM} := \det(f_! 2^{-(n+1)}((n+1)(\omega_{X/S}^{-1} - \omega_{X/S}^{-1})(\mathcal{L} - \mathcal{L}^{-1})^n - \mu(\mathcal{L} - \mathcal{L}^{-1})^{n+1}))^{-1}.$$

We should emphasize that the CM polarization inspired though it is by the G.I.T. approach to moduli, naturally appeared in the setting of analysis and differential geometry. For many reasons it became necessary to have a purely algebraic construction of this sheaf. The following result extends the definition of this sheaf to any flat family of subschemes of some $\mathbb{P}^N$. In particular to the universal family over the Hilbert Scheme. This extension is based on the theory of determinants of Cayley, Grothendieck, Knudsen and Mumford.

**Theorem (Paul, Tian [18])**
We make the following assumptions

i) $X \xrightarrow{f} S$ is a flat, proper, local complete intersection morphism.

ii) The map Pic$(S) \xrightarrow{\cdot 1} H^2(S, \mathbb{Z})$ is injective.

iii) $\mathcal{L}$ is relatively ample on $X$ and $\mathbb{P}(f_*\mathcal{L}) \cong S \times \mathbb{P}^N$.

Then there is a canonical and functorial isomorphism of sheaves on $S$

$$L_{CM} \cong \{\text{Chow}(X) \otimes \det(f_*\mathcal{L})^{\frac{d(n+1)}{n+1}}\}^{n(n+1)+\mu} \otimes \mathcal{M}_n^{-2(n+1)}.$$

As the reader will see, on the right hand side of this isomorphism is the sheaf $L_1$.

Let $f$ be any numerical function, recall that the forward difference of $f$ is defined as follows.

$$\Delta f(m) := f(m+1) - f(m)$$

Inductively we set

$$\Delta^{k+1}f(m) := \Delta^k f(m+1) - \Delta^k f(m).$$

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9See [11].
Let $f_l(m) := m^l$. Then we define polynomials $P_{k,l}(m)$

$$P_{k,l}(m) := \Delta^k f_l(m).$$

It is easy to see that

$$P_{k,l}(m) = \sum_{0 \leq j \leq k} (-1)^{j+1} \binom{k}{j} (m + j)^l.$$

It is not difficult to verify that

$$P_{k,l}(m) = \begin{cases} (-1)^k k!, & \text{if } k = l \\ 0, & \text{if } l < k. \end{cases}$$

In general, $P_{k,k+d}(m)$ is a polynomial in $m$ of degree $d$. Given $0 \leq l \leq n+1$ let $(q_{n+1}(m), q_n(m), q_{n-1}(m), \ldots, q_{n+1-l}(m))$ be the unique solution to the equation

$$\begin{pmatrix} P_{n+1,n+1}(m) & P_{n,n+1}(m) & \cdots & \cdots & P_{n+1,l+1}(m) \\ 0 & P_{n,n}(m) & P_{n-1,n}(m) & \cdots & P_{n+1-l,n+1}(m) \\ 0 & 0 & P_{n-1,n-1}(m) & \cdots & P_{n+1-l,n-1}(m) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & P_{n+1-l,n+1-l}(m) \end{pmatrix} \begin{pmatrix} q_{n+1}(m) \\ q_n(m) \\ q_{n-1}(m) \\ \vdots \\ q_{n+1-l}(m) \end{pmatrix} = \begin{pmatrix} c_{l,n+1} \\ c_{l,n} \\ c_{l,n-1} \\ \vdots \\ c_{l,n-l} \end{pmatrix}.$$

**Definition 3.** (Higher Sheaves) Let $\mathcal{X} \xrightarrow{f} S$ be a flat family of subschemes of $\mathbb{P}^N$. Let $\mathcal{L}$ be ample with respect to $f$. We define sheaves $\mathcal{L}_l$ for all $l = 0, 1, 2, \ldots, n+1$ on $S$ as follows

$$\mathcal{L}_l := \bigotimes_{k=n+1-l}^{n+1} \mathcal{M}_k \prod_{0 \leq j \leq k-1} (-1)^{j+1} \sigma_j(1,2,\ldots,k-1)c_{l,k-j}$$

(0.4)

Where the $\mathcal{M}_k$, $0 \leq k \leq n+1$ are the coefficients in the Cayley, Grothendieck, Knudsen, Mumford expansion.

Observe that we have the following canonical isomorphisms

$$\mathcal{L}_0 \cong \text{Chow}(\mathcal{X} \xrightarrow{f} S) \otimes \det(f_*\mathcal{L})^{-\frac{d(n+1)}{N+1}}$$

$$\mathcal{L}_1 \cong \mathcal{L}_{CM}.$$  

Recall that the Hilbert point of the family $\mathcal{X} \xrightarrow{f} S$ is the invertible sheaf

$$\text{Hilb}_m(\mathcal{X} \setminus S) := \det(f_*\mathcal{L}^m) \otimes \mathcal{A}^{-\frac{mx(m)}{N+1}}$$

$$\mathcal{A} := \det(f_*\mathcal{L}).$$

Then we have the following proposition.
Proposition 2. For all \( l = 0, 1, 2, \ldots, n + 1 \) we have

\[
\bigotimes_{0 \leq p \leq l} \bigotimes_{0 \leq i \leq n + 1 - p} \text{Hilb}_m(\mathfrak{X} \setminus S)^{-1}q_{n+1-p}(m)\binom{n+1-p}{i} \cong \mathcal{L}_i
\]  

(0.5)

Proof

Writing out the left hand side of (0.5) gives

\[
\bigotimes_{0 \leq p \leq l} \bigotimes_{0 \leq i \leq n + 1 - p} \text{det}(f_* \mathcal{L}^{m+i})^{-1}q_{n+1-p}(m)\binom{n+1-p}{i} \otimes \mathcal{A}^\otimes \mathcal{T}^{(-1)^{1+i}(m+i)\chi(m+i)q_{n+1-p}(m)\binom{n+1-p}{i} - 1}.
\]

(0.6)

The exponent of \( \mathcal{A} \) satisfies the following

\[
\sum_{0 \leq p \leq l} \sum_{0 \leq i \leq n + 1 - p} (-1)^{i+1}(m+i)\chi(m+i)q_{n+1-p}(m)\binom{n+1-p}{i} = \begin{cases} 1, & l = 0 \\ 0, & l > 0 \end{cases}.
\]

(0.7)

To see this we first write

\[
(m+i)\chi(m+i) = b_n(m+i)^{n+1} + b_{n-1}(m+i)^n + \cdots + b_j(m+i)^{j+1} + \ldots
\]

Then the left hand side of (0.6) is given by

\[
\sum_{0 \leq p \leq l} \sum_{0 \leq i \leq n + 1 - p} \sum_{0 \leq j \leq n} (-1)^{i+1}q_{n+1-p}(m)b_j(m+i)^{i+1}\binom{n+1-p}{i}.
\]

(0.8)

Recall that we have defined the polynomials \( P_{n+1-p, j+1}(m) \) by the formula

\[
P_{n+1-p, j+1}(m) = \sum_{0 \leq i \leq n + 1 - p} (-1)^{i+1}(m+i)^{i+1}\binom{n+1-p}{i}.
\]

(0.9)

Substituting (0.8) into (0.7), switching the order of summation and appealing to the definition of the \( q_k(m) \) gives

\[
\sum_{n-i \leq j \leq n-j \leq p \leq l} q_{n+1-p}(m)P_{n+1-p, j+1}(m)b_j = \sum_{n-i \leq j \leq n} b_j c_{l, j+1}.
\]

(0.10)

By definition of the \( c_{l,k} \) the right hand side of (0.10) is the coefficient of \( m^{-l} \) in the expansion of

\[
\frac{m\chi(m)}{m\chi(m)} = 1.
\]

From now on we will assume that \( l > 0 \). With this assumption we have

\[
\bigotimes_{0 \leq p \leq l} \bigotimes_{0 \leq i \leq n + 1 - p} \text{Hilb}_{m+i}(\mathfrak{X} \setminus S)^{-1}q_{n+1-p}(m)\binom{n+1-p}{i} \cong \mathcal{L}_i
\]

(0.10)

\[
\bigotimes_{0 \leq p \leq l} \bigotimes_{0 \leq i \leq n + 1 - p} \text{det}(f_* \mathcal{L}^{m+i})^{-1}q_{n+1-p}(m)\binom{n+1-p}{i} \cong \mathcal{L}_i
\]

(0.11)

\[
\bigotimes_{0 \leq p \leq l} \bigotimes_{0 \leq i \leq n + 1 - p} \mathcal{M}_k^{-1}q_{n+1-p}(m)\binom{n+1-p}{i} = \mathcal{M}_k
\]

(0.12)

\[
\bigotimes_{0 \leq p \leq l} \bigotimes_{0 \leq i \leq n + 1 - p} \mathcal{M}_k^{-1}q_{n+1-p}(m)\binom{n+1-p}{i} \cong \mathcal{M}_k
\]

(0.13)
Next we study the exponent of $\mathcal{M}_k$ in (0.12). First we expand the binomial coefficients in of powers of $(m+i)$

$$\binom{m+i}{k} = \frac{1}{k!} \sum_{j=0}^{k-1} (-1)^j \sigma_j(1,2,\ldots,k-1)(m+i)^{k-j}.$$ 

So that we have

$$\sum_{0 \leq i \leq n+1-p} (-1)^i \binom{n+1-p}{i} \binom{m+i}{k} =$$

$$\sum_{0 \leq j \leq k-1} \frac{1}{k!} (-1)^j \sigma_j(1,2,\ldots,k-1) \sum_{0 \leq i \leq n+1-p} (-1)^i \binom{n+1-p}{i} (m+i)^{k-j} =$$

$$\sum_{0 \leq j \leq k-1} \frac{1}{k!} (-1)^{j+1} \sigma_j(1,2,\ldots,k-1) P_{n+1-p,k-j}(m).$$

Therefore,

$$\sum_{0 \leq p \leq l} \sum_{0 \leq i \leq n+1-p} (-1)^i q_{n+1-p}(m) \binom{n+1-p}{i} \binom{m+i}{k} =$$

$$\sum_{0 \leq p \leq l} \sum_{0 \leq j \leq k-1} \frac{1}{k!} (-1)^{j+1} \sigma_j(1,2,\ldots,k-1) P_{n+1-p,k-j}(m) q_{n+1-p}(m) =$$

$$\frac{1}{k!} \sum_{0 \leq j \leq k-1} (-1)^j \sigma_{j+1}(1,2,\ldots,k-1)c_{k-j}.$$

Which completes the proof of the proposition. Recall from the introduction that we have defined $\mathcal{E}_l(m)$ as follows

$$\mathcal{E}_l(m) := \sum_{\{0 \leq i \leq l\}} \sum_{\{0 \leq i \leq n+1-j\}} q_{n+1-j}(m) (-1)^i \binom{n+1-j}{i} L^{m+i} \in \mathbb{Q}[\mathcal{L}].$$

(0.6) says that

$$\bigotimes_{0 \leq p \leq l} \bigotimes_{0 \leq i \leq n+1-p} \mathrm{Hilb}_{m+i}(\mathcal{X} \setminus S)^{(-1)^i q_{n+1-p}(m) (n+1-p)} \cong \mathrm{det}(\mathcal{E}_l(m)) .$$

This completes the proof of part i) of Theorem 1.

Since the base $S$ is closed and $\chi(G) = \{1\}$ ($\chi$ denotes the character group) the action of $G$ on both line bundles must agree, in particular the weights of the respective actions restricted to any one parameter subgroup $\lambda : \mathbb{C}^* \to G$ must agree

$$w_\lambda(\mathcal{L}_i) = w_\lambda \left( \bigotimes_{0 \leq p \leq l} \bigotimes_{0 \leq i \leq n+1-p} \mathrm{Hilb}_{m+i}(\mathcal{X} \setminus S)^{(-1)^i q_{n+1-p}(m) (n+1-p)} \right). \quad (0.14)$$

It is easy to see that the weight on the right hand side of (0.14) is given by $F_l(\lambda)$, which is the claim in part ii). Part iii) of theorem 1 is subsumed in the next section.
§3 Higher Energies

Recall from the introduction the Higher Energy Functionals.

\[ F_{\omega,t}(\varphi) := \left( \frac{\sqrt{-1}}{2 \pi} \right)^n \int_{\mathcal{X}} \int_0^1 \frac{\partial}{\partial b} \left( Td(R_{gt} + b \frac{\partial g_t}{\partial t} g_t^{-1}) Ch(F_{bH_t}^{c_1}) + b \frac{\partial H_t}{\partial t} H_t^{-1} \right) \bigg|_{b=0} dt . \]

The Chern character of \( \mathcal{E}_l(m) \) is calculated according to the rule

\[
\text{Ch}(F_{bH_t}^{c_1}) + b \frac{\partial H_t}{\partial t} H_t^{-1} := 
\sum_{0 \leq j \leq t} \sum_{0 \leq i \leq n+1-j} q_{n+1-j}(m)(-1)^i \binom{n+1-j}{i} \text{Ch}(F_{bH_t}^{m+1}) + b \frac{\partial H_t}{\partial t} H_t^{-1} .
\]

(0.15)

The homogeneous decomposition on the right hand side of (0.15) is in turn given by the expression

\[
\text{Ch}(F_{bH_t}^{m+1}) + b \frac{\partial H_t}{\partial t} H_t^{-1} = \sum_{1 \leq k \leq n+1} \text{Ch}_k(F_{bH_t}^{m+1}) + b \frac{\partial H_t}{\partial t} H_t^{-1} .
\]

(0.16)

We have dropped the degree zero term in (0.16) as it is killed by \( \frac{\partial}{\partial b} \). In the component of type \((k-1, k-1)\) on the right hand side of (0.16) we have defined \( H_t = e^{-(m+i)\varphi_t} h^{(m+i)} \), the induced path of metrics on \( L^{m+1} \). \( F_{bH_t}^{m+1} \) denotes the curvature.

\[
\frac{\sqrt{-1}}{2 \pi} F_{bH_t}^{m+1} = (m+i)(\omega + \frac{\sqrt{-1}}{2 \pi} \partial \bar{\partial} \varphi_t) .
\]

(0.17)

\[
H_t H_t^{-1} = -(m+i)\dot{\varphi}_t .
\]

(0.18)

By definition

\[
\text{Ch}_k(F_{bH_t}^{m+1}) + b \frac{\partial H_t}{\partial t} H_t^{-1} = \frac{1}{k!} \text{Tr} \left( F_{bH_t}^{m+1} + b \frac{\partial H_t}{\partial t} H_t^{-1} \right)^k .
\]

Therefore

\[
\frac{\partial}{\partial b} \text{Ch}_k(F_{bH_t}^{m+1}) + b \frac{\partial H_t}{\partial t} H_t^{-1})|_{b=0} = \frac{(-1)^k}{(k-1)!} (m+i)^k \dot{\varphi}_t \omega_{\varphi_t}^{k-1} .
\]

Putting everything together gives

\[
\frac{\partial}{\partial b} \text{Ch}(F_{bH_t}^{c_1}) + b \frac{\partial H_t}{\partial t} H_t^{-1})|_{b=0} = \sum_{0 \leq j \leq t} \sum_{0 \leq i \leq n+1-j} q_{n+1-j}(m)(-1)^i \binom{n+1-j}{i} \frac{(-1)}{(k-1)!} (m+i)^k \dot{\varphi}_t \omega_{\varphi_t}^{k-1}
\]

\[
= \sum_{0 \leq j \leq t} \sum_{1 \leq k \leq n+1} q_{n+1-j}(m) P_{n+1-j,k}(m) \frac{(-1)}{(k-1)!} \dot{\varphi}_t \omega_{\varphi_t}^{k-1}
\]

\[
= \sum_{0 \leq j \leq t} \sum_{1 \leq k \leq n+1} q_{n+1-j}(m) P_{n+1-j,k}(m) \frac{(-1)}{(k-1)!} \dot{\varphi}_t \omega_{\varphi_t}^{k-1}
\]

\[
= \sum_{0 \leq j \leq t} \sum_{0 \leq k \leq n+1} \frac{(-1)}{(k-1)!} c_{l,k} \dot{\varphi}_t \omega_{\varphi_t}^{k-1} .
\]
Therefore
\[ \frac{\partial}{\partial b} \text{Ch}_k(F_{H_t}^{E_l(m)}) + b \frac{\partial H_t}{\partial t} H_t^{-1} \big|_{b=0} = \frac{(-1)}{(k-1)!} c_{l,k} \omega_{\varphi_t}^{k-1}. \] (0.19)

In the same way one shows that
\[ \text{Ch}(F_{H_t}^{E_l(m)}) = \sum_{\{n+1-l\leq k\leq n\}} \frac{c_{l,k}}{k!} \omega_t^k. \] (0.20)

The top dimensional component of the transgressed GRR integrand is given by
\[
\frac{\partial}{\partial b} \text{Ch}(F_{H_t}^{E_l(m)}) + b \frac{\partial H_t}{\partial t} H_t^{-1} \big|_{b=0} \text{Td}(R_{g_t}) + \frac{\partial}{\partial b} \text{Td}(R_{g_t} + b \frac{\partial g_t}{\partial t} g_t^{-1}) \big|_{b=0} \text{Ch}(F_{H_t}^{E_l(m)}) = \\
\sum_{\{n+1-l\leq k\leq n+1\}} \text{Td}_{n+1-k}(R_{g_t}) \frac{\partial}{\partial b} \text{Ch}_k(F_{H_t}^{E_l(m)}) + b \frac{\partial H_t}{\partial t} H_t^{-1} \big|_{b=0} \\
+ \sum_{\{n+1-l\leq k\leq n\}} \frac{\partial}{\partial b} \text{Td}_{n+1-k}(R_{g_t} + b \frac{\partial g_t}{\partial t} g_t^{-1}) \big|_{b=0} \text{Ch}_k(F_{H_t}^{E_l(m)}).
\]

We summarize what we have in the following lemma.

**Proposition 3.**

\[ F_{\omega, l}(\varphi) = \int_X \int_0^1 \sum_{n+1-l\leq k\leq n} \frac{c_{l,k}}{k!} \frac{\partial}{\partial b} \text{Td}_{n+1-k}(\sqrt{-1} \frac{\omega_t^k}{2\pi} R_{g_t} + b \frac{\partial g_t}{\partial t} g_t^{-1}) \big|_{b=0} \omega_t^k dt \\
- \int_X \int_0^1 \sum_{n+1-l\leq k\leq n+1} \frac{c_{l,k}}{k!} \text{Td}_{n+1-k}(\sqrt{-1} \frac{\omega_t^k}{2\pi} R_{g_t} + b \frac{\partial g_t}{\partial t} g_t^{-1}) \frac{\partial}{\partial t} \omega_t^{k-1} dt.
\]

In particular when \( l = 2 \) we have the following.

\[ \frac{\partial F_{\omega, 2}(\varphi_t)}{\partial t} = \int_X \left( \frac{c_{l,n-1}}{(n-1)!} \frac{\partial}{\partial b} \text{Td}_2(\sqrt{-1} \frac{\omega_t}{2\pi} R_{g_t} + b \frac{\partial g_t}{\partial t} g_t^{-1}) \big|_{b=0} \omega_t^{n-1} - \\
\frac{c_{l,n-1}}{(n-2)!} \text{Td}_2(\sqrt{-1} \frac{\omega_t}{2\pi} R_{g_t} + b \frac{\partial g_t}{\partial t} g_t^{-1}) \frac{\partial}{\partial t} \omega_t^{n-2}\right),
\] (0.21)

In order to compare \( F_{\omega, 2} \) with \( E_{\omega, 1} \) we need to make the right hand side of (0.21) more explicit.

**Proposition 4.**

\[ i) \frac{\partial}{\partial b} \text{Td}_2(\sqrt{-1} \frac{\omega_t}{2\pi} R_{g_t} + b \frac{\partial g_t}{\partial t} g_t^{-1}) \big|_{b=0} \omega_t^{n-1} = \frac{1}{6n} \Delta_{\varphi_t} \varphi_t \text{Scal}(\varphi_t) \omega_t^{n}, \] (0.22)

\[ ii) \text{Td}_2(\sqrt{-1} \frac{\omega_t}{2\pi} R_{g_t} + b \frac{\partial g_t}{\partial t} g_t^{-1}) \frac{\partial}{\partial t} \omega_t^{n-2} = \frac{1}{12n(n-1)} \varphi_t (3 \text{Scal}(\varphi_t)^2 - 4 \|\text{Ric}_t\| + \|R_{g_t}\|^2) .
\] (0.23)
For a vector bundle $\mathcal{E} \to X$ recall that the second Todd class is given by the formula

$$12\text{Td}_2(\mathcal{E}) = \frac{3}{2}c_1(\mathcal{E})^2 - \text{Ch}_2(\mathcal{E}).$$

On a Kähler manifold the Chern-Weil theory implies that for $\mathcal{E} = T^{1,0}_X$ we get

$$12\text{Td}_2\left(\frac{\sqrt{-1}}{2\pi} R_\varphi\right) = \frac{3}{2}\text{Ric}(\omega_\varphi)^2 - \frac{1}{2}\text{Tr}\left(\frac{\sqrt{-1}}{2\pi} R_\varphi\right)^2. \quad (0.24)$$

Therefore we have

$$12\int_X \frac{\partial}{\partial b} \text{Td}_2\left(\frac{\sqrt{-1}}{2\pi} R_\varphi + b(\hat{\varphi}_{ij}g_\varphi^{-1})\right) \bigg|_{b=0} \omega_\varphi^{n-1} =$$

$$\frac{3}{2}\int_X \frac{\partial}{\partial b} \left\{\text{Tr}\left(\frac{\sqrt{-1}}{2\pi} R_\varphi + b(\hat{\varphi}_{ij}g_\varphi^{-1})\right)^2\right\} \bigg|_{b=0} \omega_\varphi^{n-1} +$$

$$-\frac{1}{2}\int_X \frac{\partial}{\partial b} \text{Tr}\left\{\frac{\sqrt{-1}}{2\pi} R_\varphi + b(\hat{\varphi}_{ij}g_\varphi^{-1})\right\} \bigg|_{b=0} \omega_\varphi^{n-1} =$$

$$3\int_X \text{Tr}\left(\frac{\sqrt{-1}}{2\pi} R_\varphi\right) \Delta_\varphi \varphi \omega_\varphi^{n-1} - \int_X \text{Tr}\left(\frac{\sqrt{-1}}{2\pi} R_\varphi(\hat{\varphi}_{ij}g_\varphi^{-1})\right) \omega_\varphi^{n-1} =$$

$$3\int_X \text{Ric}(\omega_\varphi) \Delta_\varphi \varphi \omega_\varphi^{n-1} - \int_X \text{Tr}\left(\frac{\sqrt{-1}}{2\pi} R_\varphi(\hat{\varphi}_{ij}g_\varphi^{-1})\right) \omega_\varphi^{n-1} =$$

$$3\int_X \hat{\varphi} \Delta_\varphi \text{Scal}_\varphi \omega_\varphi^{n-1} - \int_X \text{Tr}\left(\frac{\sqrt{-1}}{2\pi} R_\varphi(\hat{\varphi}_{ij}g_\varphi^{-1})\right) \omega_\varphi^{n-1}.$$

Recall the following fact.

$$\Theta \wedge \omega^{n-k} = \frac{(n-k)!}{n!} \Lambda^k \Theta \omega^n. \quad (0.25)$$

$\Lambda$ denotes contraction against the Kähler form $\omega$ and $\Theta$ denotes a $(k, k)$ form on $X$. An application of (0.25) gives that

$$\int_X n\text{Tr}\left(\frac{\sqrt{-1}}{2\pi} R_\varphi(\hat{\varphi}_{ij}g_\varphi^{-1})\right) \omega_\varphi^{n-1} = \int_X \hat{\varphi} \sum_{1 \leq i, j, k \leq n} \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \left(\Lambda R_g(i, j)g^{ij}\text{det}(g)\right) \frac{\omega^n}{\text{det}(g)} \quad (0.26)$$

On the right hand side of (0.26) we have temporarily replaced $g_\varphi$ by $g$. 

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Next we need some local formulas from Kähler geometry which we collect below.

\[ R_g(i, j) := \overline{\partial}\{\partial(g^{-1})(i, j) = \sum_{k, l} R_{ij}^{kl} \, dz_k \wedge \overline{dz_l} \} \quad \text{(full curvature tensor)} \]

\[ \text{Ric}_g := \text{Tr}(R_g) = \sum_{1 \leq i, l \leq n} \sum_{1 \leq i \leq n} R_{ij}^{il} \, dz_i \wedge \overline{dz_l} \quad \text{(Ricci curvature)} \]

\[ \text{Scal}_\omega := \text{Tr}(\Lambda R_g) = \sum_{1 \leq i, k, l \leq n} g^{kl} R_{ik}^{jl} \quad \text{(scalar curvature)} \]

At the center \( o \) of a normal coordinate system we have that

\[ R_{ij}^{kl}(o) = -\frac{\partial^2 g_{ij}}{\partial z_k \partial \overline{z}_l}(o) = \frac{\partial^2 g_{ij}}{\partial z_k \partial \overline{z}_l}(o) \]

\[ \Delta_g \text{Scal}_\omega(o) = \sum_{1 \leq i, j, k \leq n} \frac{\partial^2 R_{ij}^{kl}}{\partial z_j \partial \overline{z}_j}(o) + \sum_{1 \leq i, k, l \leq n} R_{ij}^{kl}(o) R_{ij}^{kl}(o) . \]

Since \( \text{Ric}_g \) is closed we have the identity

\[ \sum_{1 \leq i \leq n} \frac{\partial^2 R_{ij}^{kl}}{\partial z_p \partial \overline{z}_q} = \sum_{1 \leq i \leq n} \frac{\partial^2 R_{ij}^{kl}}{\partial z_k \partial \overline{z}_l} . \quad (0.27) \]

The commutation \( R_{ijkl} = R_{klji} \) implies that at \( o \) we have

\[ \frac{\partial^2 R_{ij}^{kl}}{\partial z_p \partial \overline{z}_q} = \frac{\partial^2 R_{ij}^{kl}}{\partial z_k \partial \overline{z}_l} + \sum_{1 \leq m \leq n} (R_{km}^{ij} R_{mp}^{kl} - R_{kp}^{jm} R_{mp}^{kl}) . \quad (0.28) \]

Therefore we get

\[ \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \left( \Lambda R_g(i, j) g^{kl} \det(g) \right) = \]

\[ \frac{\partial^2 R_{ij}^{lm}}{\partial z_j \partial \overline{z}_l} + R_{ij}^{lm} R_{mj}^{il} + R_{kl}^{ij} R_{lj}^{mp} - R_{ij}^{mp} R_{lj}^{kl} = \]

\[ \frac{\partial^2 R_{ij}^{kl}}{\partial z_j \partial \overline{z}_l} + R_{im}^{kl} R_{mj}^{ij} - R_{jm}^{ml} R_{ij}^{kl} + R_{ij}^{mp} R_{lj}^{mk} + R_{ij}^{pj} R_{ij}^{qj} - R_{ij}^{mq} R_{ij}^{pq} . \]

Next use the symmetries

\[ \frac{\partial^2 R_{ij}^{kl}}{\partial z_j \partial \overline{z}_l} = \frac{\partial^2 R_{ij}^{kl}}{\partial z_k \partial \overline{z}_l} \]

\[ R_{ij}^{kl} = R_{ij}^{lk} = R_{ij}^{il} = R_{ij}^{kl} . \]
To deduce that
\[ \frac{\partial^2}{\partial z_j \partial z_k} \left( \Lambda R_g(i, j) g^{ik} \det(g) \right) = \]
\[ \frac{\partial^2 R^\pi_{ij}}{\partial z_i \partial z_j} + R^\pi_{im} R^\pi_{mj} - R^\pi_{ij} R^\pi_{mj} + R^\pi_{ij} R^\pi_{mj} - R^\pi_{ij} R^\pi_{mj} = \]
\[ \frac{\partial^2 R^\pi_{ij}}{\partial z_i \partial z_j} + R^\pi_{ij} R^\pi_{ij} = \Delta_g \text{Scal}_\omega(o) . \]

So that we have
\[ \int_X \text{Tr} \left( \sqrt{-1} \frac{1}{2\pi} R_\varphi(\varphi) g^{-1}_\varphi \right) \omega^n = \frac{1}{n} \int_X \varphi \Delta_\varphi \text{Scal}_\omega \omega^n . \]

This establishes i).

By definition we have
\[ \text{Ric}_g = R^\pi_{ij} R^\pi_{ij} d z_k \wedge d z_l \wedge d z_p \wedge d z_q \]  \hspace{1cm} (0.29)
\[ \text{Tr}(R^2_g) = R^\pi_{ij} R^\pi_{ij} d z_k \wedge d z_l \wedge d z_p \wedge d z_q . \]  \hspace{1cm} (0.30)

It is easy to see that
\[ \Lambda^2 \{ d z_k \wedge d z_l \wedge d z_p \wedge d z_q \} = 2 \{ \delta_{kl} \delta_{pq} - \delta_{kp} \delta_{ql} \} . \]  \hspace{1cm} (0.31)

Applying (0.31) to (0.29) and (0.30) respectively gives
\[ \Lambda_g^2 \text{Ric}^2_g = 2(S^2_g - ||\text{Ric}_g||^2) \]
\[ \Lambda_g^2 \text{Tr}(R^2_g) = 2(||\text{Ric}_g||^2 - ||R_g||^2) . \]  \hspace{1cm} (0.32)

Recall the identity
\[ 12 T d_2(\sqrt{-1} R_\varphi) \omega^{n-2}_\varphi = \frac{3}{2} \text{Ric}(\omega_\varphi) \omega^{n-2}_\varphi - \frac{1}{2} \text{Tr}(\sqrt{-1} R_\varphi)^2 \omega^{n-2}_\varphi . \]  \hspace{1cm} (0.33)

An application of (0.25) and (0.32) to the right hand side of (0.33) shows that
\[ 12 T d_2(\sqrt{-1} R_\varphi) \omega^{n-2}_\varphi = \]
\[ \frac{3}{n(n-1)} (S^2_g - ||\text{Ric}_g||^2) - \frac{1}{n(n-1)} (||\text{Ric}_g||^2 - ||R_g||^2) \]
\[ = \frac{1}{n(n-1)} (3 \text{Scal}_\varphi^2 - 4||\text{Ric}_\varphi||^2 + ||R_\varphi||^2) . \]

We leave the computation of the coefficients \( c_{2,j} \) \( (n-1 \leq j \leq n+1) \) to the reader.

This establishes proposition 2.
In this section we exhibit the energy $F_{2,\omega}$ as a singular Hermitian metric on the sheaf $\mathcal{L}_2$. This is a carried out using the method of Tian \textit{mutatis mutandis}. Let $f^{-1}(o) = X_o \subset \mathbb{P}^N$, where $o \in S_\infty := S \setminus \Delta$, where $\Delta$ denotes the discriminant locus of the family. We define for any $z \in S$

$$GX_z := \{(\sigma, y) \in G \times \mathbb{P}^N : y \in \sigma X_z\}.$$ 

Then we have the following diagram, where $p_z$ denotes the evaluation map, i.e. $p_z(\sigma) := \sigma z$.

$$\begin{array}{cccc}
G & \xrightarrow{p_z} & S & \xrightarrow{f} \mathbb{P}(f_*L^\vee) & \cong B \times \mathbb{P}^N & \xrightarrow{p_2} & \mathbb{P}^N \\
\end{array}$$

Given $z \in S \setminus \Delta$ we can consider $T_{X_z}^{1,0}$, the holomorphic tangent bundle of the fiber $X_z$. These fit together holomorphically into a vector bundle $\mathcal{V}$ on $\mathfrak{X} \setminus f^{-1}(\Delta)$. We have the following exact complex (denoted by $(A^\bullet , \partial \cdot)$) over $\mathfrak{X} \setminus f^{-1}(\Delta)$.

$$0 \longrightarrow \mathcal{V} \xrightarrow{\iota_*} T_{\mathfrak{X}}^{1,0}|_{\mathfrak{X} \setminus f^{-1}(\Delta)} \xrightarrow{f_*} f^* T_S^{1,0}|_{\mathfrak{X} \setminus f^{-1}(\Delta)} \longrightarrow 0.$$ 

Let $\omega_X$ and $\omega_S$ denote any Kähler metrics on $X$ and $S$ respectively. The Fubini study form $\omega$ on $\mathbb{P}^N$ induces a Kähler form on each of the smooth fibers of $f$ and so induces a Hermitian metric (which we denote by $h_{\mathcal{V}}$) on $\mathcal{V}$.

More generally let

$$0 \longrightarrow E^0 \xrightarrow{v_0} E^1 \xrightarrow{v_1} \ldots \longrightarrow E^i \xrightarrow{v_i} E^{i+1} \longrightarrow \ldots \longrightarrow E^l \longrightarrow 0$$

be a complex of holomorphic Hermitian vector bundles on a complex manifold $B$. The metric on $E^j$ will be denoted by $h_{E^j}$, the corresponding holomorphic Hermitian connection by $\nabla_{E^j}$. $\nabla_{E^j}^2$ is the curvature. Let $N$ denote the number operator of the complex, i.e. $N$ acts by multiplication by $j$ on $E^j$ ($0 \leq j \leq l$). For $u \geq 0$, let $A_u$ be the Quillen superconnection

$$A_u := \nabla + \sqrt{u} V \ , \ V := v + v^* .$$

Definition 4. \textit{(Bismut, Gillet, Soulé \cite{BGS})}

For $s \in \mathbb{C}$, $\text{Re}(s) > 0$, let $\zeta_{E^\bullet}(s)$ be the collection of forms on $B$ defined by

$$\zeta_{E^\bullet}(s) := -\frac{1}{\Gamma(s)} \int_0^{+\infty} u^{s-1} \text{Tr}_s[N \exp(-A_u^2)]du .$$

$\zeta_{E^\bullet}(s)$ is actually holomorphic on all of $\mathbb{C}$. 

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**Theorem** (Bismut, Gillet, Soulé [3] Theorem 1.15)

On the base $B$ there is a pointwise identity of forms provided that the complex $(E^\bullet, v_\bullet)$ is acyclic.

$$
\sum_{0 \leq j \leq l} (-1)^j \text{Ch}(E^j, h_j) = \partial \bar{\partial} \zeta_{E^\bullet}(0) .
$$ (0.34)

$\zeta_{E^\bullet}(0)^{(p,p)}$ denotes the component of $\zeta_{E^\bullet}(0)$ of degree $(p,p)$. Direct application of (0.34) to the adjunction complex gives forms $\zeta_{A^\bullet}(0)^{(p,p)}$ defined along $X \setminus f^{-1}(\Delta)$ satisfying

$$
\partial \bar{\partial} \zeta_{A^\bullet}(0)^{(p,p)} = 
\text{Ch}(V, \omega)^{(p+1,p+1)} - \text{Ch}(T^{1,0}_X, \omega_X)^{(p+1,p+1)} + f^* \text{Ch}(T^{1,0}_S, \omega_S)^{(p+1,p+1)} .
$$ (0.35)

When $p = 0$ we can describe the function $\zeta_{A^\bullet}(0)^{(0,0)}$ in the following way.

Given $z \in S \setminus \Delta$ we can consider $K_{X_z}$, the canonical bundle of the fiber $X_z$. These fit together holomorphically into a line bundle $K_\infty (\cong \det(V)^{-1})$ on $X \setminus f^{-1}(\Delta)$. On the other hand the relative canonical bundle $K_f$ of the map $f$ is given by

$$
K_f := K_X \otimes f^* K_S^{-1} .
$$

When we restrict $K_f$ to $X \setminus f^{-1}(\Delta)$ we have an isomorphism

$$
K_f \cong K_\infty .
$$

$\iota^* p^*_S \omega_{FS}$ restricts to a Kähler metric on $f^{-1}(z)$ ($z \in S_\infty$) and hence induces a Hermitian metric on the bundle $K_\infty$. We denote its curvature by $R(\iota^* p^*_S (\omega_{FS}))$. The Kähler metrics on $X$ and $S$ induce a metric on the relative canonical bundle $K_f$. We let $R_f$ denote its curvature

$$
R_{X/S} := R(\omega_X) - f^* R(\omega_S) .
$$

In this way we obtain two metrics on the relative canonical bundle over the smooth locus. The curvatures of these metrics are not the same. The relation between them is given in the following “$\partial \bar{\partial}$ lemma along the fibers”.

There is a smooth function $\Psi : X \setminus f^{-1}(\Delta) \to \mathbb{R}$ such that

$$
R_{X/S} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \Psi = R(\iota^* p^*_S (\omega_{FS})) .
$$ (0.36)

Then (up to additive constants) we have the identity

$$
-\zeta_{A^\bullet}(0)^{(0,0)} = \Psi .
$$ (0.37)
Lemma 0.1. There is a continuous metric $\| \|_2$ on the line bundle $L_2$ such that in the sense of currents we have

$$\frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \log(\| \|_2^2) =$$

$$f_* \left\{ \frac{c_{2,n-1}}{(n-1)!} \left( \frac{1}{8} (R_{X/S})^2 p_2^* \omega_{FS}^{n-1} - \frac{1}{12} (\text{Ch}_2(T_{X}^{1,0}, \omega_X) - f^* \text{Ch}_2(T_{S}^{1,0}, \omega_S)p_2^* \omega_{FS}^{n-1}) \right) \right.$$ 

$$- \frac{c_{2,n}}{n!} (R_{X/S}) p_2^* \omega_{FS}^n + \frac{c_{2,n+1}}{(n+1)!} p_2^* \omega_{FS}^{n+1} \right\}.$$  

(0.38)

Next we define a Hermitian metric $h_G$ on the vector bundle $\pi_2^* T_{X_z}^{1,0}$ over the product $G \times X_z$

$$h_G|_{\sigma \times X_z} := (\omega_{FS} + \frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \varphi)|_{X_z}.$$  

(0.39)

Then we have the corresponding Chern-Weil forms

$$\text{Ch}_k(\pi_2^* T_{X_z}^{1,0}, h_G).$$  

(0.40)

Lemma 0.2. Let $\psi$ be a compactly supported form on $G$. Then

$$\int_G F_{2,\omega}(\varphi_\sigma) \overline{\partial} \psi =$$

$$\int_{G \times X_z} \left\{ \frac{c_{2,n-1}}{(n-1)!} \left( \frac{1}{8} c_1(\pi_2^* T_{X_z}^{1,0}, h_G)^2 \pi_2^* \omega_{FS}^{n-1} - \frac{1}{12} \text{Ch}_2(\pi_2^* T_{X_z}^{1,0}, h_G) \pi_2^* \omega_{FS}^{n-1} \right) \right.$$ 

$$- \frac{c_{2,n}}{n!} c_1(\pi_2^* T_{X_z}^{1,0}, h_G) \pi_2^* \omega_{FS}^n + \frac{c_{2,n+1}}{(n+1)!} \pi_2^* \omega_{FS}^{n+1} \right\} \wedge \pi_1^*(\psi).$$  

(0.41)

Next observe that $G X_z$ is biholomorphic to $G \times X_z$. It follows that we have an identity

$$p_{z,2}^* \text{Ch}_k(\mathcal{V}, h_V) = \text{Ch}_k(\pi_2^* T_{X_z}^{1,0}, h_G).$$

It then follows from (0.35) that we have

$$c_1(\pi_2^* T_{X_z}^{1,0}, h_G) = p_{z,2}^* \left( - R_{X/S} + \frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} (0) \right)^{(0,0)}$$

$$\text{Ch}_2(\pi_2^* T_{X_z}^{1,0}, h_G) = p_{z,2}^* \left( \text{Ch}_2(T_{X}^{1,0}, \omega_X) - f^* \text{Ch}_2(T_{S}^{1,0}, \omega_S) + \partial \overline{\partial} (0)^{(1,1)} \right).$$  

(0.42)

This implies the
Corollary 2. The function
\[ \sigma \in G \rightarrow D(\sigma) := F_{2,\omega|X_z}(\sigma) - \log \left( e^{\Psi_S(\sigma z)} \frac{||\sigma z||}{||\sigma||} \right) \] (0.43)
is pluriharmonic. \( \Psi_S : S \setminus \Delta \rightarrow \mathbb{R} \) is given by
\[ \Psi_S(z) := \kappa_1 \int_{X_z} \left( -\frac{1}{\pi} \bar{\zeta}_A(0)^{0,0} R_{X/S} - \frac{1}{4\pi^2} \bar{\zeta}_A(0)^{0,0} \partial \bar{\partial} \bar{\zeta}_A(0)^{0,0} \right) \pi_2^s \omega^{n-1} \]
\[ + \kappa_2 \int_{X_z} \zeta_A(0)^{1,1} \pi_2^s \omega^{n-1} + \kappa_3 \int_{X_z} \frac{\sqrt{1-N}}{2\pi} \partial \bar{\partial} \zeta_A(0)^{0,0} \pi_2^s \omega^n. \]
\[ 8(n-1)!\kappa_1 = c_{2,n-1} - 12(n-1)!\kappa_2 = c_{2,n-1} - n!\kappa_3 = c_{2,n}. \] (0.44)

Standard arguments show that the right hand side of (0.43) is identically zero. This exhibits the new energies as singular metrics on the lines \( L_l \leq l \leq n+1 \), although we have only carried this out in detail when \( l = 2 \). In order to conclude the proof of Theorem 2 let \( \sigma = \lambda(t) \) on the right hand side of (0.43) and let \( t \rightarrow 0 \). Since we have assumed that the limit cycle \( X_z^{\lambda(0)} \) is \( C^\infty \) the term \( \Psi \) is bounded.

Q.E.D.

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