HAMILTONIAN MECHANICS IN 1+2 DIMENSIONS

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Abstract

A complete list of all transitive symplectic manifolds of the Poincaré and Galilei group in 1+2 dimensions is given.

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1 Introduction

Recently we have given a complete analysis of the projective unitary irreducible representations of the Poincaré and Galilei groups in 1+2 dimensions [1], [2]. In the context of constant interest in physics in 1+2 dimensions, we think it useful to provide the same analysis in the framework of Hamiltonian mechanics. Namely, we intend to provide a complete list of all the transitive symplectic actions for the Poincaré and Galilei groups in 1+2 dimensions. The method used is, essentially, the orbit method of Kostant, Souriau and Kirillov [3-6]. However, if we want the complete list of these symplectic actions (not only up to a covering like in the usual formulation of the orbit method) we need a generalization of this method appearing in [7].

In Section 2 we present the method following [7]. In Section 3 and 4 we apply the method for the Poincaré and the Galilei groups respectively. The first case is rather standard and offers no surprises. The second case is on the contrary a very good "laboratory" for practically all the symplectic techniques described in full generality in Section 2.

2 Transitive Symplectic Actions of Lie Groups

A. Basic Definitions

Let \((M_i, \Omega_i)\) \(i = 1, 2\) be symplectic manifolds. A diffeomorphism \(\phi : M_1 \rightarrow M_2\) is called **symplectic** if:

\[ \phi^* \Omega_2 = \Omega_1. \]  

(2.1)

Let \((M, \Omega)\) be a symplectic manifold and \(G\) a Lie group (not necessarily connected) acting on \(M\): \(G \ni g \mapsto \phi_g \in \text{Diff}(M)\).

This action is called **symplectic** and \((M, \Omega)\) is called a **\(G\)-symplectic manifold** if:

\[ (\phi_g)^* \Omega = \Omega, \ \forall g \in G. \]  

(2.2)

If \((M_i, \Omega_i)\) \(i = 1, 2\) are two \(G\)-symplectic manifolds, they are called **\(G\)-symplectomorphic** if there exists a symplectic map \(\phi : M_1 \rightarrow M_2\) which is also \(G\)-equivariant.

Physically, a \(G\)-symplectic manifold can be considered as the phase space of a given Hamiltonian system for which \(G\) is the covariance group. Two such Hamiltonian systems are identical from the physical point of view if the corresponding \(G\)-symplectic manifolds are \(G\)-symplectomorphic. One can also argue that elementary systems are described by transitive \(G\)-symplectic manifolds. The orbit method describes transitive \(G\)-symplectic manifolds up to \(G\)-symplectomorphisms.

B. The Basic Theorem

We denote by \(\text{Lie}(G) \equiv T_e(G)\) the Lie algebra of \(G\), by \(G \ni g \mapsto A_d \in \text{End}(\text{Lie}(G))\) the adjoint action and by \(G \ni g \mapsto A_d^* \in \text{End}((\text{Lie}(G))^*)\) the coadjoint action of \(G\). A
2-cocycle for $\text{Lie}(G)$ is a bilinear antisymmetric map: $\sigma : \text{Lie}(G) \times \text{Lie}(G) \to \mathbb{R}$ verifying the cocycle identity [6]:

$$\sigma([X_1, X_2], X_3) + \text{cyclic permutations} = 0 \quad \forall X_i \in \text{Lie}(G) \ (i = 1, 2, 3). \quad (2.3)$$

We denote by $Z^2(\text{Lie}(G), \mathbb{R})$ the linear space of all 2-cocycles. If $\sigma \in Z^2(\text{Lie}(G), \mathbb{R})$ we define:

$$h_\sigma \equiv \{X \in \text{Lie}(G) | \sigma(X, Y) = 0, \forall Y \in \text{Lie}(G)\} \quad (2.4)$$

and one finds out that $h_\sigma \subset \text{Lie}(G)$ is a Lie subalgebra. We denote by $H_\sigma \subset G$ the connected Lie subgroup immersed in $G$ and associated to the Lie subalgebra $h_\sigma$.

There is a natural action of $G$ on $Z^2(\text{Lie}(G), \mathbb{R})$, namely the coadjoint action given by:

$$(Ad_g^*)\sigma(X_1, X_2) = \sigma(Ad_{g^{-1}}(X_1), Ad_g(X_2)). \quad (2.5)$$

We denote by $G_\sigma \subset G$ the stability subgroup of $\sigma$ with respect to this action. $H_\sigma$ is a normal subgroup of $G_\sigma$. An $Ad^*$-orbit $\mathcal{O}$ in $Z^2(\text{Lie}(G), \mathbb{R})$, is called regular if for some $\sigma \in \mathcal{O}$ (then for all $\sigma \in \mathcal{O}$), the subgroup $H_\sigma$ is closed.

If $K$ is a Lie group and $N \subset K$ is a normal subgroup we denote by $K/N$ the factor Lie group. Two subgroups $Q, Q' \subset K/N$ are called conjugated if there exists $k_0 \in K$ such that:

$$Q' = \{k_0kk_0^{-1}N | kN \in Q\}. \quad (2.6)$$

Now we can formulate the main theorem [7]:

**Theorem** Take one representative $\sigma$ from every regular orbit in $Z^2(\text{Lie}(G), \mathbb{R})$. Let $\mathcal{H}_\sigma$ be the set of discrete subgroups of $G_\sigma/H_\sigma$ and $\mathcal{C}_\sigma$ the set of conjugacy classes in $\mathcal{H}_\sigma$. Let $\bar{H} \in \mathcal{H}_\sigma$ be a representative of a given conjugacy class $[\bar{H}] \in \mathcal{C}_\sigma$ and:

$$H \equiv \{h \in G_\sigma | hH_\sigma \in [\bar{H}]\}. \quad (2.7)$$

Then $H \subset G$ is a closed subgroup and $G/H$ is a $G$-symplectic manifold with the symplectic form $\Omega^\sigma$ uniquely determined by:

$$\sigma = (\pi^*\Omega^\sigma)_e. \quad (2.8)$$

(here $\pi : G \to G/H$ is the canonical submersion.)

Every $G$-symplectic manifold of $G$ is $G$-symplectomorphic with a manifold of the form $(G/H, \Omega^\sigma)$ described above. Moreover, to different couples $(\sigma, [\bar{H}]) \neq (\sigma', [\bar{H}'])$ correspond $G$-symplectic manifolds which are not $G$-symplectomorphic.

This theorem is quite general and affords a complete classification in a very constructive way. Loosely speaking, the regular orbits classify the $G$-symplectic manifolds, up to a covering; the various coverings are classified by some classes of discrete subgroups in $G_\sigma/H_\sigma$.

In applications, we will first determine for every regular orbit, the maximal symplectic manifold corresponding to $H = H_\sigma$ in the construction above. Then the various symplectic manifolds covered by this maximal manifold will be determined by factorizing $(G/H_\sigma, \Omega^\sigma)$ to the (symplectic) action of some suitable chosen discrete subgroup of $G$.
C. A Particular Case

A 2-coboundary is an element of $Z^2(\text{Lie}(G), \mathbb{R})$ of the form:

$$\sigma(X_1, X_2) = - < \eta, [X_1, X_2] >, \ \forall X_1, X_2 \in \text{Lie}(G). \quad (2.9)$$

Here $<,>$ is the duality form between $\text{Lie}(G)$ and $(\text{Lie}(G))^*$ and $\eta \in (\text{Lie}(G))^*$ is arbitrary. The linear space of all 2-coboundaries is denoted by $B^2(\text{Lie}(G), \mathbb{R})$. We will also need the second cohomology group with real coefficients

$$H^2(\text{Lie}(G), \mathbb{R}) \equiv Z^2(\text{Lie}(G), \mathbb{R})/B^2(\text{Lie}(G), \mathbb{R}).$$

Finally, an 1-cocycle for $\text{Lie}(G)$ is any element $\eta \in (\text{Lie}(G))^*$ verifying:

$$< \eta, [X_1, X_2] > = 0, \ \forall X_1, X_2 \in \text{Lie}(G). \quad (2.10)$$

We denote by $Z^1(\text{Lie}(G), \mathbb{R})$ the linear space of all 1-cocycles and by $H^1(\text{Lie}(G), \mathbb{R}) \equiv Z^1(\text{Lie}(G), \mathbb{R})$ the first cohomology group with real coefficients.

For every $X \in \text{Lie}(G)$, let $X_\mathcal{O}$ be the associated vector field on $\mathcal{O}$:

$$X_\mathcal{O} \equiv \frac{d}{ds} \text{Ad} e^{sX} |_{s=0}. \quad (2.11)$$

Then we have:

**Corollary:** Let $\mathcal{O} \subset (\text{Lie}(G))^*$ a coadjoint orbit. Then $\mathcal{O}$ becomes a symplectic manifold with respect to the Kostant-Souriau-Kirillov symplectic form $\Omega^{KSK}_\eta$ which is uniquely determined by:

$$\Omega^{KSK}_\eta(X_\mathcal{O}, Y_\mathcal{O}) = - < \eta, [X, Y] >, \quad (2.12)$$

$(\forall \eta \in \mathcal{O}, \forall X, Y \in \text{Lie}(G)).$

Two different coadjoint orbits are not $G$-symplectomorphic.

Suppose that $H^i(\text{Lie}(G), \mathbb{R}) = 0$ ($i = 1, 2$). If the stability subgroup $G_\eta$ ($\eta \in \mathcal{O}$ arbitrary) is connected, then every transitive $G$-symplectic manifold is $G$-symplectomorphic with a coadjoint orbit. If $G_\eta$ is not connected, then the coadjoint orbits are the maximal symplectic manifolds and the various symplectic manifolds covered by $\mathcal{O}$ are classified by the conjugacy classes in $G_\eta/(G_\eta)^0$.

In applications one can use the factorization method outlined at the end of Subsection 2B. Let us remark that if $H^i(\text{Lie}(G), \mathbb{R}) \neq 0$ for $i = 1$ or $i = 2$ then the list of all transitive $G$-symplectic manifolds is not exhausted by the construction outlined above and based on coadjoint orbits.
D. Extended Coadjoint Orbits

We close this section with another interesting construction. We have seen in Subsection 2B that if $H^i(Lie(G), \mathbb{R}) = 0$ ($i = 1, 2$) then one can conveniently describe transitive $G$-symplectic manifolds as coadjoint orbits of $G$ (or their factorization). One may wonder if something analogous works in the general case. The answer is positive and the construction works as follows [4]-[5].

Let $c$ be a 2-cocycle for the Lie group $G$, i.e. a map $c : G \times G \to \mathbb{R}$ verifying:

$$c(g_1, g_2) + c(g_1g_2, g_3) = c(g_2, g_3) + c(g_1, g_2g_3) \quad (\forall g_1, g_2, g_3 \in G),$$  (2.13)

$$c(e, g) = c(g, e) = 0 \quad (\forall g \in G).$$  (2.14)

We construct the central extension $G^c$ of $G$ which is set-theoretically $G^c = G \times \mathbb{R}$ with the composition law:

$$(g; \zeta) \cdot (g'; \zeta') = (gg'; \zeta + \zeta' + c(g, g')).$$  (2.15)

Then if $c$ is smooth, $G^c$ is also a Lie group. We identify $Lie(G^c) \simeq Lie(G) + \mathbb{R}$ and $(Lie(G^c))^* \simeq (Lie(G))^* + \mathbb{R}$ in a natural way.

Then one can show that the coadjoint action of $G^c$ has the form:

$$Ad^c_{g^c}(\eta; \rho) = (Ad^*_{g}(\eta) + \rho \alpha_{g^{-1}}; \rho).$$  (2.16)

Here $\alpha_g \in (Lie(G))^*$ is given by:

$$< \alpha_g, X > = \left. \frac{d}{ds} \left[ c \left( g, e^{sX} \right) - c \left( e^{sAd_g(X)}g, g \right) \right] \right|_{s=0}.$$  (2.17)

One notices that the orbits of the action (2.16) are of the form $(O; \rho)$ where $O$ are orbits in $(Lie(G))^*$ relative to a modified coadjoint action. In particular, we consider the case $\rho = 1$ and obtain the modified coadjoint action:

$$Ad^c_{g^c}(\eta) \equiv Ad^*_{g}(\eta) + \alpha_{g^{-1}}.$$  (2.18)

Because $Ad^*_{g^c}$ is modified only by an $\eta$- independent translation, it is clear that $Ad^c_{g^c}$ will remain a symplectic transformation with respect to $\Omega^{KSK}$. It follows that in this way we obtain new coadjoint orbits as transitive $G$-symplectic manifolds.

One can prove that two construction of this type, based on the 2-cocycles $c_1$ and $c_2$ respectively, give the same result (up to a $G$-symplectomorphism) iff $c_1$ and $c_2$ are cohomologous, i.e:

$$c_1(g, g') - c_2(g, g') = d(g) - d(gg') + d(g')$$  (2.19)

for some smooth $d : G \to \mathbb{R}$.

In conclusion on can obtain new $G$-symplectic manifolds (beside the usual coadjoint orbits), by classifying all 2-cocycles of $G$, up to the equivalence relation (2.19), selecting a representative from every cohomology class and working with the central extension $G^c$.

There is no guarantee that we will obtain all the transitive $G$-symplectic manifolds in this way although this happens, for instance, for the Galilei group in 1+3 dimensions. In fact the Galilei group in 1+2 dimensions provide an example for which we do not obtain all the transitive $G$-symplectic manifolds in this way. The list can be however completed by some simple tricks performed on the coadjoint orbits.
3 Transitive Symplectic Actions for the Poincaré Group in 1+2 Dimensions

We denote by $M$ the 1+2-dimensional Minkowski space i.e. $\mathbb{R}^3$ with coordinates $(x^0, x^1, x^2)$ and with the Minkowski bilinear form:

$$\{x, y\} \equiv x^0 y^0 - x^1 y^1 - x^2 y^2. \quad (3.1)$$

We will also need the Minkowski norm: $\|x\| \equiv \{x, x\}$

The Lorentz group is:

$$L \equiv \{\Lambda \in \text{End}(M) | \{Lx, Ly\} = \{x, y\}, \forall x, y \in M\}$$

considered as group with respect to operator multiplication.

The proper orthochronous Lorentz group is:

$$L^\uparrow_+ \subset L: \quad L^\uparrow_+ \equiv \{\Lambda \in L | \det(L) = 1, \ L_{00} > 0\}$$

The proper orthochronous Poincaré group is a semi-direct product: set theoretically $P^\uparrow_+$ is formed from couples $(L, a)$ with $L \in L^\uparrow_+$ and $a \in M$ and the composition law is:

$$(L, a) \cdot (L', a') = (LL', a + La'). \quad (3.2)$$

It is well known that: $H^i(\text{Lie}(P^\uparrow_+), \mathbb{R}) = 0 \ (i = 1, 2)$ (see e.g. [6]) so we can apply the corollary from Subsection 2C.

One can identify $(\text{Lie}(P^\uparrow_+))^* \simeq \wedge^2 M + M$ (see [6]). One can naturally extend to this space the action of $P^\uparrow_+$, the Minkowski bilinear form $\{,\}$ and the Minkowski norm $\| \cdot \|$. The coadjoint action is then given by:

$$\text{Ad}_{L,a}^*(\Gamma, P) = (L \Gamma + a \wedge LP, LP). \quad (3.3)$$

One can easily compute the coadjoint orbits of $P^\uparrow_+$. If $e_0, e_1, e_2$ is the canonical base in $M$, then they are:

(a) $M^\uparrow_{m,s} \equiv \{(\Gamma, P) | \|P\|^2 = m^2, \ sign(P_0) = \epsilon, \ \Gamma \wedge P = \epsilon e_0 \wedge e_1 \wedge e_2\} \ (m \in \mathbb{R}_+, \ s \in \mathbb{R}, \ \epsilon = \pm)$

(b) $M^\uparrow_s \equiv \{(\Gamma, P) | \|P\|^2 = 0, \ sign(P_0) = \epsilon, \ \Gamma \wedge P = \epsilon e_0 \wedge e_1 \wedge e_2\} \ (s \in \mathbb{R}, \ \epsilon = \pm)$

(c) $M^\uparrow_{m,s} \equiv \{(\Gamma, P) | \|P\|^2 = -m^2, \ \Gamma \wedge P = \epsilon e_0 \wedge e_1 \wedge e_2\} \ (m \in \mathbb{R}_+, \ s \in \mathbb{R})$

(d1) $\tilde{M}^\uparrow_{m} \equiv \{(\Gamma, 0) | \|\Gamma\|^2 = m^2, \ sign((\ast \Gamma)_{00}) = \epsilon\} \ (m \in \mathbb{R}_+, \ \epsilon = \pm)$

(d2) $\tilde{M}^\uparrow_{m} \equiv \{(\Gamma, 0) | \|\Gamma\|^2 = 0, \ sign((\ast \Gamma)_{00}) = \epsilon\} \ (\epsilon = \pm)$

(d3) $\tilde{M}^\uparrow_{m} \equiv \{(\Gamma, 0) | \|\Gamma\|^2 = -m^2\} \ (m \in \mathbb{R}_+)$

(here $\ast$ is the Hodge operator.)

Computing the stability subgroups for a given reference point from every orbit we obtain only connected Lie subgroups. Applying the corollary from Subsection 2C it follows that (a)-(d) is the complete list of the transitive $P^\uparrow_+$ symplectic manifolds.
Remark 1: A different realization of $M_{m,s}^\epsilon$ also appeared in [8]. It is interesting to establish the connection between these two realizations. The idea is to identify $\wedge^2 M \ni \Gamma \leftrightarrow J = *\Gamma \in M$. (3.4)

Then the action (3.3) becomes:

$$Ad_{L,a}^\ast(J, P) = (LJ + a \times LP, LP)$$ (3.5)

where, for any $a, b \in M$ we define $a \times b \in M$ according to:

$$(a \times b)^\rho \equiv \varepsilon^{\rho\mu\nu} a_\mu b_\nu.$$ (3.6)

Then the manifolds (a)-(d) above become subsets of points $(J, P) \in M \times M$:

(a) $M_{m,s}^\epsilon \equiv \{(J, P) ||P||^2 = m^2, \ sign(P_0) = \epsilon, \ {J, P} = \epsilon ms\} \ (m \in \mathbb{R}_+, \ s \in \mathbb{R}, \ \epsilon = \pm)$

(b) $M_s^\epsilon \equiv \{(J, P) ||P||^2 = 0, \ sign(P_0) = \epsilon, \ {J, P} = \epsilon s\} \ (s \in \mathbb{R}, \ \epsilon = \pm)$

(c) $M_{m,s} \equiv \{(J, P) ||P||^2 = -m^2, \ {J, P} = ms\} \ (m \in \mathbb{R}_+, \ s \in \mathbb{R})$

(d1) $\tilde{M}_m^\epsilon \equiv \{(J, 0)||J||^2 = m^2, \ sign(J_{00}) = \epsilon\} \ (m \in \mathbb{R}_+, \ \epsilon = \pm)$

(d2) $M^\epsilon \equiv \{(J, 0)||J||^2 = 0, \ sign(J_{00}) = \epsilon\} \ (\epsilon = \pm)$

(d3) $M_m \equiv \{(J, 0)||J||^2 = -m^2\} \ (m \in \mathbb{R}_+)$

Remark 2: One can investigate now the notion of localisability for the systems described above following the lines of [9]. It is not hard to establish that only the system corresponding to $M_{m,0}^\epsilon$ can be localisable, namely on the Euclidean space $\mathbb{R}^2$.

Remark 3: In all cases the identity map is a bona fide momentum map.

Remark 4: If we compare the actions above with the list of projective unitary irreducible representations of the same group [1] it is apparent that there are representations which do not have a classical analogue.

4 The Transitive Symplectic Actions for the Galilei Group in 1+2 Dimensions

A. Notations

We define directly the proper orthochronous Galilei group in $1 + 2$ dimensions $G^\uparrow_{1+2}$ as the group of $4 \times 4$ real matrices of the form:

$$(R, v, \tau, a) \equiv \begin{pmatrix} R & v & a \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{pmatrix}$$ (4.1)

where $\tau \in \mathbb{R}$, $R \in SO(2)$ is a $2 \times 2$ real orthogonal matrix and the vectors $v, a \in \mathbb{R}^2$ are considered as column matrices.
As for any matrix group, we identify the Lie algebra $\text{Lie}(G_+^1)$ with the linear space of $4 \times 4$ real matrices of the form:

$$(\alpha, u, t, x) \equiv \begin{pmatrix} \alpha A & u & x \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.2}$$

Here $u, x \in \mathbb{R}^2$, $t, \alpha \in \mathbb{R}$, $A \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and the exponential map is the usual matrix exponential. One can easily obtains the Lie bracket as:

$$[[\alpha_1, u_1, t_1, x_1], (\alpha_2, u_2, t_2, x_2)] = (0, A(\alpha_1 u_2 - \alpha_2 u_1), 0, A(\alpha_1 x_2 - \alpha_2 x_1) + t_2 u_1 - t_1 u_2). \tag{4.3}$$

We have established [2] that $H^2(\text{Lie}(G_+^1), \mathbb{R}) \neq 0$ (in fact it is a three-dimensional real space) so we will have to apply directly the theorem from Subsection 2B.

First we choose a convenient representation for an arbitrary element from $Z^2(\text{Lie}(G_+^1), \mathbb{R})$. From [2] it follows that a generic element is of the form: $[m, F, S, G, P]$ ($m, F, S \in \mathbb{R}, G, P \in \mathbb{R}^2$) given by the following formula:

$$[m, F, S, G, P] = m \xi_0 + F \xi_1 + S \xi_2 + [G, P] \tag{4.4}$$

where $\xi_0, \xi_1, \xi_2$ are non-trivial cocycles (i.e. they are not coboundaries) and they have the following expressions:

$$\xi_0((\alpha_1, u_1, t_1, x_1), (\alpha_2, u_2, t_2, x_2)) = x_1 \cdot u_2 - x_2 \cdot u_1 \tag{4.5}$$

$$\xi_1((\alpha_1, u_1, t_1, x_1), (\alpha_2, u_2, t_2, x_2)) = <u_1, u_2> \tag{4.6}$$

$$\xi_2((\alpha_1, u_1, t_1, x_1), (\alpha_2, u_2, t_2, x_2)) = \alpha_1 t_2 - \alpha_2 t_1 \tag{4.7}$$

and $[G, P]$ is a coboundary of the form:

$$[G, P]((\alpha_1, u_1, t_1, x_1), (\alpha_2, u_2, t_2, x_2)) =$$

$$< P, \alpha_1 x_2 - \alpha_2 x_1 > + P \cdot (t_2 u_1 - t_1 u_2) - < G, \alpha_1 u_2 - \alpha_2 u_1 >. \tag{4.8}$$

We have denoted the usual scalar product in $\mathbb{R}^2$ by $x \cdot y$ and $<, >$ is the symplectic form on $\mathbb{R}^2$:

$$< x, y > = x \cdot Ay. \tag{4.9}$$

**B. Coadjoint Orbits in $Z^2(\text{Lie}(G_+^1), \mathbb{R})$**

We need the corresponding coadjoint action. First, we compute from (4.1) and (4.2) the adjoint action:

$$Ad_{R, \nu, \tau, a}(\alpha, u, t, x) = (R, \nu, \tau, a)(\alpha, u, t, x)(R, \nu, \tau, a)^{-1} =$$

$$(\alpha, Ru - \alpha Av, t, Rx + tv + \alpha(\tau v - a) - \tau Ru). \tag{4.10}$$
Then, applying (2.4) we get the desired coadjoint action:

\[ Ad^*_{R, v, \tau, a}[m, F, S, G, P] = [m, F, S, R(G + ma) - \tau R(P + mv) - FAv, RP + mv]. \] (4.11)

It is clear that the structure of the coadjoint orbits will depend on \( F \). In particular we have two cases \( F = 0 \) and \( F \neq 0 \).

(I) \( F = 0 \)
In this case the coadjoint orbits are:

(a) \( O^1_{m, S} \equiv \{ [m, 0, S, G, P] | G, P \in \mathbb{R}^2 \} \) \( m \in \mathbb{R}^*, S \in \mathbb{R} \)

(b) \( O^2_{s, k, \lambda} \equiv \{ [0, 0, S, G, P] | P^2 = k^2, G \wedge P = \lambda e_1 \wedge e_2 \} \) \( k \in \mathbb{R}^+, S, \lambda \in \mathbb{R} \)

(c) \( O^3_{S, k} \equiv \{ [0, 0, S, 0] | G^2 = k^2 \} \) \( k \in \mathbb{R}^+ \cup \{0\} \)

(II) \( F \in \mathbb{R}^* \)

(a) \( O^4_{m, s} \equiv \{ [m, F, S, G, P] | G, P \in \mathbb{R}^2 \} \) \( m \in \mathbb{R}^*, S \in \mathbb{R} \)

(b) \( O^5_{m, s} \equiv \{ [0, F, S, G, 0] | G \in \mathbb{R}^2, P^2 = k^2 \} \) \( k \in \mathbb{R}^+ \cup \{0\}, S \in \mathbb{R} \)

Above we have denoted with \( e_1 \) and \( e_2 \) the natural basis in \( \mathbb{R}^2 \).

C. Computation of the Transitive Symplectic Actions

As indicated in the statement of the theorem from Subsection 2B, we need to provide a list of subgroups \( H \subset G^+ \) such that \( G^+/H \) is a symplectic manifold with the symplectic form given by (2.8). Of course, this is a very implicit way to exhibit the symplectic transitive actions of \( G^+ \).

As suggested in Subsection 3B, we divide the study in two cases. One has to take some reference point \( \sigma \) on every orbit \( O^i (i = 1, ..., 5) \) and thereafter to compute \( G^\sigma \) and \( H^\sigma \) and the discrete subgroups of \( G^\sigma/H^\sigma \). The computations are elementary and we provide only the final results. We point out that in all the cases the action of \( G^\sigma \) on \( G^\sigma/H^\sigma \) is trivial, so \( C^\sigma = H^\sigma \) (in the notations of the theorem from 2B).

(I) \( F = 0 \)
(a) \( \sigma = [m, 0, S, 0, 0] \)
One finds two subcases:

(a1) \( S = 0 \)

\[ H^\sigma = \{(R, 0, \tau, 0) | R \in SO(2), \tau \in \mathbb{R}\} \]

(a2) \( S \neq 0 \)

\[ H^\sigma = \{(1, 0, 0, 0)\} \]

In both cases we have

\[ G^\sigma = \{(R, 0, \tau, 0) | R \in SO(2), \tau \in \mathbb{R}\} \]

So, in the case (a1), \( G^\sigma/H^\sigma \) is trivial and in the case (a2) \( G^\sigma/H^\sigma \simeq G^\sigma \simeq SO(2) \times \mathbb{R} \).

(b) \( \sigma = [0, 0, S, \lambda e_1, k e_2] \)
One finds:

\[ H^\sigma = \{(1, v, 0, a) | v_2 = 0, a_1 = 0\} \]
\[ G^\sigma = \{(1, v, 0, a) | v, a \in \mathbb{R}^2\} \]
\[ G_\sigma/H_\sigma \simeq \{(1,v,0,a)|v_1 = 0, \ a_2 = 0\} \simeq \mathbb{R} \times \mathbb{R} \]

(c) \( \sigma = [0,0,S,ke_2,0] \)
Again we have two subcases:
(c1) \( k \in \mathbb{R}_+ \)

\[ H_\sigma = \left\{ (1,v,\tau,a)|v_1 = -\frac{S}{k}, \ \tau, v_2 \in \mathbb{R}, \ a \in \mathbb{R}^2 \right\} \]
\[ G_\sigma = \{(1,v,\tau,a)|\tau \in \mathbb{R}, \ v, a \in \mathbb{R}^2\} \]
\[ G_\sigma/H_\sigma \simeq \{(1,v,0,0)|v_2 = 0\} \simeq \mathbb{R} \]

(c2) \( k = 0 \)
If \( S \neq 0 \), we have:
\[ H_\sigma = \{(1,v,0,a)|v, a \in \mathbb{R}^2\} \]
\[ G_\sigma = \mathcal{G}_+^1 \]
\[ G_\sigma/H_\sigma \simeq \{(R,0,\tau,0)|R \in SO(2), \ \tau \in \mathbb{R}\} \simeq SO(2) \times \mathbb{R}. \]
If \( S = 0 \), we have \( G_\sigma = H_\sigma = \mathcal{G}_+^1 \) so this case is trivial.

(II) \( F \in \mathbb{R}^* \)
(a) \( \sigma = [m,F,S,0,0] \)
We obtain the same subgroups \( G_\sigma \) and \( H_\sigma \) as in the case \( F = 0 \).
(b) \( \sigma = [0,F,S,0,ke_2] \)
We have two subcases:
(b1) \( k \in \mathbb{R}_+ \)

\[ H_\sigma = \left\{ \left(1,ve_1,\frac{Fv}{k},\left(\frac{SFv}{k^2} + \frac{Fv^2}{2k^2}\right)e_1 + ae_2\right)|v, a \in \mathbb{R}\right\} \]
\[ G_\sigma = \left\{ \left(1,ve_1,\frac{Fv}{k},a\right)|v \in \mathbb{R}, \ a \in \mathbb{R}^2\right\} \]
\[ G_\sigma/H_\sigma = \{(1,0,0,be_1)|b \in \mathbb{R}\} \simeq \mathbb{R} \]

(b2) \( k = 0 \)
There are two possibilities:
(b21) \( S \neq 0 \)

\[ H_\sigma = \{(1,0,0,a)|a \in \mathbb{R}^2\} \]

(b21) \( S = 0 \)
\[ H_\sigma = \{(R,0,\tau,a)|R \in SO(2), \ \tau \in \mathbb{R}, \ a \in \mathbb{R}^2\} \]
Regardless of the value of \( S \) we have:
\[ G_\sigma = \{(R,0,\tau,a)|R \in SO(2), \ \tau \in \mathbb{R}, \ a \in \mathbb{R}^2\} \]
So we have for the first possibility

\[ G_\sigma / H_\sigma = \{(R, 0, \tau, 0) | R \in SO(2), \tau \in \mathbb{R} \} \cong SO(2) \times \mathbb{R} \]

and for the second possibility the factor group is trivial.

As regards the discrete subgroups of \( G_\sigma / H_\sigma \) we have only three non-trivial possibilities: \( \mathbb{R}, \mathbb{R} \times \mathbb{R} \) and \( SO(2) \times \mathbb{R} \) as it is apparent from the list above. It is well known that the discrete subgroups are in these cases \( \bar{H}_\gamma \equiv \gamma \mathbb{Z}, H_{\gamma_1, \gamma_2} \equiv \gamma_1 \mathbb{Z} \times \gamma_2 \mathbb{Z} \) and \( H_{r, \gamma} \equiv Z_r \times \gamma \mathbb{Z} \) respectively. Here \( \gamma, \gamma_1, \gamma_2 \in \mathbb{R}_+ \) \( \cup \{0\} \) and \( Z_r \in SO(2) \) is the cyclic group of order \( r \in \mathbb{N}^*: Z_r \equiv \{R(2\pi k/r)|k = 0, ..., r-1\} \) (as usual \( R(\phi) = e^{i\phi} \) is the rotation of angle \( \phi \)).

Combining the results obtained above, we can formulate the main result:

**Proposition 1:** Every transitive \( G^+_\gamma \)- symplectic manifold is \( G^+_\gamma \)- symplectomorphic with one of the form \( (G^+_\gamma / H, \Omega^\gamma) \) where \( H \) and \( \sigma \) can be:

1. \( H^1 = \{(R, 0, \tau, 0) | R \in SO(2), \tau \in \mathbb{R} \} \)
2. \( \sigma^1 = [m, F, S, 0, 0] \; \; m, S \in \mathbb{R}^+, \; F \in \mathbb{R} \)
3. \( H^2 = \{(R(2\pi k/r), 0, n\gamma, 0) | k = 0, ..., r-1, n \in \mathbb{Z} \} \)
4. \( \sigma^2 = [m, F, 0, 0, 0] \; \; m \in \mathbb{R}^+, \; F \in \mathbb{R}, \; r \in \mathbb{N}^+, \; \gamma \in \mathbb{R}_+ \cup \{0\} \)
5. \( H^3 = \{(1, ne_1 + \gamma_1 ne_2, 0, \gamma_2 me_1 + ae_2) | v, a \in \mathbb{R}, \; m, n \in \mathbb{Z} \} \)
6. \( \sigma^3 = [0, 0, S, \lambda e_1, ke_2] \; \; k \in \mathbb{R}_+, \; \lambda, S \in \mathbb{R}, \; \gamma_1, \gamma_2 \in \mathbb{R}_+ \cup \{0\} \)
7. \( H^4 = \left\{ \left(1, ve_1, \frac{Fv}{k}, \left(\frac{SFv}{k^2} + \frac{Fv^2}{2k^2} + \gamma n\right)e_1 + ae_2 \right) | v, a \in \mathbb{R}, \; n \in \mathbb{Z} \right\} \)
8. \( \sigma^4 = [0, F, S, 0, ke_2] \; \; k \in \mathbb{R}_+, \; S \in \mathbb{R}, \; F \in \mathbb{R}^*, \; \gamma \in \mathbb{R}_+ \cup \{0\} \)
9. \( H^5 = \left\{ \left(1, \left(\gamma - \frac{S\tau}{k}\right)e_1 + ve_2, \tau, ae_2 \right) | \tau, v \in \mathbb{R}, \; a \in \mathbb{R}^2, \; n \in \mathbb{Z} \right\} \)
10. \( \sigma^5 = [0, 0, S, ke_2, 0] \; \; k \in \mathbb{R}_+, \; S \in \mathbb{R}, \; \gamma \in \mathbb{R}_+ \cup \{0\} \)
11. \( H^6 = \{(R(2\pi k/r), v, n\gamma, a) | k = 0, ..., r-1, \; v, a \in \mathbb{R}^2, \; n \in \mathbb{Z} \} \)
12. \( \sigma^6 = [0, 0, S, 0, 0] \; \; S \in \mathbb{R}^*, \; r \in \mathbb{N}^*, \; \gamma \in \mathbb{R}_+ \cup \{0\} \)
13. \( H^7 = \{(R(2\pi k/r), 0, n\gamma, a) | k = 0, ..., r-1, \; a \in \mathbb{R}^2, \; n \in \mathbb{Z} \} \)
14. \( \sigma^7 = [0, F, S, 0, 0] \; \; F, S \in \mathbb{R}^*, \; r \in \mathbb{N}^*, \; \gamma \in \mathbb{R}_+ \cup \{0\} \)
(8) \[ H^8 = \{(R, 0, \tau, a)| R \in SO(2), \tau \in \mathbb{R}, a \in \mathbb{R}^2\} \]
\[ \sigma^8 = [0, F, 0, 0, 0] \quad F \in \mathbb{R}^* \]

For distinct couples \((H, \Omega) \neq (H', \Omega')\) the corresponding manifolds are not \(G_+^1\)-symplectomorphic.

**D. Central Extensions of \(G_+^1\)**

In principle, the analysis of the transitive symplectic manifolds for \(G_+^1\) was completed above. However, it is interesting to exhibit such manifolds as coadjoint orbits. For this we need \(H^2(G_+^1, \mathbb{R})\). We compute this group taking advantage of the knowledge of \(H^2(\tilde{G}_+^1, \mathbb{R})\) which was determined in [2]. For the definitions of \(\tilde{G}_+^1\) and of the covering map \(\delta : \tilde{G}_+^1 \to G_+^1\) see also [2].

Let \(c \in Z^2(\tilde{G}_+^1, \mathbb{R})\) be arbitrary. We define \(\tilde{c} : \tilde{G}_+^1 \times \tilde{G}_+^1 \to \mathbb{R}\) by:
\[ \tilde{c}(\tilde{g}, \tilde{g}') \equiv c(\delta(\tilde{g}), \delta(\tilde{g}')). \]

Then it is elementary to show that \(\tilde{c} \in Z^2(\tilde{G}_+^1, \mathbb{R})\). Applying the result obtained in [2] it follows that \(\tilde{c}\) is cohomologous with \(mc_0 + Fc_1 + Sc_2\) where:
\[ \tilde{c}_0(\tilde{g}, \tilde{g}') = \frac{1}{2}[a \cdot R(x)v' - v \cdot R(x)a' + \tau'v \cdot R(x)v']. \]  \hfill (4.12)
\[ \tilde{c}_1(\tilde{g}, \tilde{g}') = \frac{1}{2} < v, R(x)v' > \]  \hfill (4.13)
\[ \tilde{c}_2(\tilde{g}, \tilde{g}') \equiv \tau x'. \]  \hfill (4.14)

Here \(\tilde{g} = (x, v, \tau, a), \tilde{g}' = (x', v', \tau', a').\) Explicitely we have:
\[ c(\delta(\tilde{g}), \delta(\tilde{g}')) = mc_0(\tilde{g}, \tilde{g}') + Fc_1(\tilde{g}, \tilde{g}') + Sc_2(\tilde{g}, \tilde{g}') + \tilde{d}(\tilde{g}) - \tilde{d}(\tilde{g}\tilde{g}') + \tilde{d}(\tilde{g}') \]  \hfill (4.15)

with \(\tilde{d} : \tilde{G}_+^1 \to \mathbb{R}\) a smooth function. By redefining \(\tilde{d} \to \tilde{d}'\) where:
\[ \tilde{d}'(x, v, \tau, a) = \tilde{d}(x, v, \tau, a) - \frac{x}{2\pi} \tilde{d}(2\pi, 0, 0, 0) \]

we still have (4.15) but the new \(\tilde{d}\) also verifies
\[ \tilde{d}(2\pi, 0, 0, 0) = 0. \]

If we make \(x \to x + 2\pi\) in (4.15) we get more, namely that the function \(\tilde{d}\) is periodic in \(x\) with period \(2\pi\). Finally, if we make \(x' \to x' + 2\pi\) in (4.15) we get \(S = 0\). Then it follows from (4.15) that:
\[ c(g, g') = mc_0(g, g') + Fc_1(g, g') + d(g) - d(gg') + d(g') \]
where
\[ c_0(g, g') \equiv \frac{1}{2}(a \cdot Rv' - v \cdot Ra' + \tau'v \cdot Rv'). \] (4.16)
\[ c_1(g, g') \equiv \frac{1}{2} < v, Rv' > \] (4.17)
with \( g = (R, v, \tau, a), g' = (R', v', \tau', a') \) and \( d \) is uniquely determined by \( d \circ \delta = \tilde{d} \).

**Proposition 2:** \( H^2(G_+^\dagger, \mathbb{R}) \) is a two dimensional real space. Explicitly, every 2-cocycle of \( G_+^\dagger \) is cohomologous with one of the form \( mc_0 + Fc_1 \).

According to Subsection 2D we must consider the central extension \((G_+^\dagger)c\) where \( c = mc_0 + Fc_1 \).

It is to be expected that we will obtain only the symplectic manifolds corresponding to \( S = 0 \) from the list included in the statement of Proposition 1.

**E. Coadjoint Orbits of \((G_+^\dagger)c\)**

We must first compute the function \( \alpha_g \) according to (2.17). An elementary computation gives:
\[ \alpha_{R,v,\tau,a}(\alpha, u, t, x) = \\
m \left( R^{-1}a \cdot u - R^{-1}v \cdot x - \alpha < a, v > - \frac{1}{2}tv^2 \right) + F \left( \frac{1}{2}\alpha v^2 + < R^{-1}v, u > \right). \] (4.18)

To compute the extended action (2.16) we identify \((\text{Lie}(G_+^\dagger))^* \simeq \mathbb{R} + \mathbb{R}^2 + \mathbb{R} + \mathbb{R}^2\) using the duality form
\[ < (\beta, G, E, P), (\alpha, u, t, x) > = -\beta \alpha - G \cdot u - Et + P \cdot x. \] (4.19)

Then (2.18) gives for the modified coadjoint action:
\[ Ad_{R,v,\tau,a}^c(\beta, G, E, P) = \\
(\beta + < RG + ma, v > - < RP, a > - \frac{1}{2}Fv^2, RG + ma - \tau(RP + mv) - FAv, \\
E + RP \cdot v + \frac{1}{2}mv^2, RP + mv). \] (4.20)

It is elementary to compute the orbits of this action. They are:
(a) \( O_{m,s,F,E}^{1} \equiv \{(\beta, G, E, P) \mid |P \wedge G - (FE - m\beta)e_1 \wedge e_2| = ms, \ E - \frac{F^2}{2m} = E\} \)
\( m \in \mathbb{R}^+, \ s \in \mathbb{R}^+ \cup \{0\}, \ F, E \in \mathbb{R} \)
(b) \( O_{k,\lambda,F}^{2} \equiv \{(\beta, G, E, P) \mid P \wedge G = (FE - m\beta)e_1 \wedge e_2, P^2 = k^2\} \)
\( m = 0, \ k \in \mathbb{R}^+, \ \lambda, F \in \mathbb{R} \)
(c) \( O_{E,s,F}^{3} \equiv \{(\beta, G, E, 0) \mid 2F\beta + G^2 = s\} \)
\( m = 0, \ s, E, F \in \mathbb{R} \)

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Now it is easy to match these coadjoint orbits with symplectic manifolds appearing in the statement of Proposition 1 and corresponding to $S = 0$. We will get maximal manifolds as it is easy to anticipate. Namely we have:
- $O^1_{m,s,E}$ is for any $s$, $E$ the maximal manifold of case (2)
- $O^2_{k,\lambda,F}$ is the maximal manifold of case (3) and respectively (4) (both for $S = 0$).
- $O^3_{E,s,F}$ is for any $E, s$ the maximal manifold of case (5)(for $S = 0$) and respectively (8).

F. Maximal Symplectic Manifolds for $S \neq 0$

- To obtain case (1) we give another realization of case (2). Namely $M = \mathbb{R}^2 \times \mathbb{R}^2$ with coordinates $(q, p)$, the symplectic form:

$$\Omega = \sum_{i=1}^{2} dq_i \wedge dp_i + \frac{2F}{m^2} dp_1 \wedge dp_2$$

and the action of $G^+_+$:

$$\phi_{R,v,\tau,a}(q, p) = \left( Rq + a - \frac{\tau}{m}(Rp + mv), Rp + mv \right).$$

One can see in [7] the corresponding 1+3-dimensional case. Taking a suggestion from this case we build case (1) as follows. $M = \mathbb{R}^2 \times \mathbb{R}^2 \times S^1 \times \mathbb{R}$ with coordinates $(q, p, \nu, l)$, the symplectic form:

$$\Omega = \sum_{i=1}^{2} dq_i \wedge dp_i + \frac{2F}{m^2} dp_1 \wedge dp_2 + \frac{S}{m} d\phi \wedge dl$$

(4.23)

where $\nu = (\cos(\varphi), \sin(\varphi))$, and the action

$$\phi_{R,v,\tau,a}(q, p, \nu, l) = \left( Rq + a - \frac{\tau}{m}(Rp + mv), Rp + mv, R\nu + l + m\tau \right).$$

(4.24)

- In the cases (3)-(5) for $S \neq 0$, the trick is to modify a little the extended action (4.20), namely:

$$Ad^S_{R,v,\tau,a}(\beta, G, E, P) =$$

$$\left( \beta + \langle R G, v \rangle - \langle R P, a \rangle - \frac{1}{2}Fv^2 - S\tau, R G - \tau R P - FAv, E + R P \cdot v, R P \right).$$

(4.25)

and to keep the KSK-symplectic form unchanged.

- Case (6) can be realized by: $M = \mathbb{R} \times S^1$ with coordinates $(T, \nu)$, the symplectic form

$$\Omega = Sd\varphi \wedge dT.$$

(4.26)

and the action

$$\phi_{R,v,\tau,a}(T, \nu) = (T + \tau, R\nu).$$

(4.27)
Finaly case (7) is given by $M = \mathbb{R} \times \mathbb{R}^2 \times S^1$ with coordinates $(T, V, \nu)$, the symplectic form

$$\Omega = F dV_1 \wedge dV_2 + S d\phi \wedge dT + k(-\sin(\phi)dV_1 + \cos(\phi)dV_2) \wedge dT$$

(4.28)

and the action

$$\phi_{R,v,\tau,a}(T, V, \nu) = (T + \tau, R V + v, R \nu).$$

(4.29)

Of course in (4.26) and (4.28) $\nu = (\cos(\phi), \sin(\phi))$ as in case (1).

G. Factorized Symplectic Manifolds

In all cases except (1) and (8) we have, beside the maximal manifolds exhibited above, some families of factorized manifolds. We indicate briefly the results in these remaining cases.

(2) If $\gamma = 0$ and $r = 2, 3, ...$ then we modify (4.24) as follows;

$$\phi_{R,v,\tau,a}(q, p, \nu, l) = (R q + a - \frac{\tau}{m}(R p + m v), R p + m v, R^r \nu, l + m \tau).$$

(4.30)

To obtain the cases with $\gamma \in \mathbb{R}_+$ one factorizes the previous cases to the following action of $\mathbb{Z}$:

$$n \cdot (q, p, \nu, l) = (q, p, \nu, l + n \gamma m).$$

(4.31)

(3) One factorizes the maximal manifolds to an action of $\mathbb{Z} \times \mathbb{Z}$ namely:

$$(n, m) \cdot (\beta, G, E, P) = (\beta + m k \gamma_2, G, E + n k \gamma_1, P).$$

(4.32)

(4)-(5) The maximal manifold is factorized to the following action of $\mathbb{Z}$:

$$n \cdot (\beta, G, E, P) = (\beta + n k \gamma, G, E, P).$$

(4.33)

(6) If $\gamma = 0$ and $r = 2, 3, ...$ we proceed as at (2) above, namely we modify (4.27) to:

$$\phi_{R,v,\tau,a}(T, \nu) = (T + \tau, R^r \nu).$$

(4.34)

The case $\gamma \in \mathbb{R}_+$ is obtained factorizing the previous cases to the following action of $\mathbb{Z}$:

$$n \cdot (T, \nu) = (T + n \gamma, \nu).$$

(4.35)

(7) Similarly with (6): the action on the variable $V$ is not changed.

Remark 5: The notion of localisability can be investigated as in [9]. It is manifest from the first part of Subsection F that cases (1) and (2) (i.e. non-zero mass systems) are localisable on $\mathbb{R}^2$. One also finds out that the cases (5)-(7) are localisable on $S^1$.

Remark 6: The existence of a momentum map is more subtle than in the case of the Poincaré group. Namely for $S \neq 0$ such a map does not exists and for $S = 0$ a momentum map for the maximal manifolds is the identity map in the coadjoint representation from Subsection 4E. If $S = 0$ but we are dealing with a factorized manifold, again the momentum map does not exists.
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