Violation of the “information–disturbance relationship” in finite-time quantum measurements

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Abstract The effect of measurement attributes (quantum level of precision, finite duration) on the classical and quantum correlations is analyzed for a pair of qubits immersed in a common reservoir. We show that the quantum discord is enhanced as the precision of the measuring instrument is increased, and both the classical correlation and the quantum discord experience noticeable changes during finite-time measurements performed on a neighboring partition of the entangled system. The implication of these results on the “information–disturbance relationship” is examined, with critical analysis of the delicate roles played by quantum non-locality and non-Markovian dynamics in the violation of this relationship, which appears surprisingly for a range of measurement attributes. This work highlights that the fundamental limits of quantum mechanical measurements can be altered by exchanges of non-classical correlations such as the quantum discord with external sources, which has relevance to cryptographic technology.

Keywords Quantum correlations · Quantum measurements · Non-Markovian dynamics · Information–disturbance relationship · Quantum cryptography

1 Introduction

The role of measurements in quantum correlation and decoherence processes presents a challenging area of investigation of quantum systems [1–8]. Some progress has been made in the understanding of the links between quantum measurements and...
decoherence processes, which results in the breakdown of phases in the superposed states [9]. In the “decoherence scheme,” a quantum system that survives while in contact with an environment is guided to its pointer states [10]. The view that an open quantum system is equivalent to a system that is continuously measured by its environment has been examined using various approaches in other works [11,12]. Some issues still remain in relation to the links between quantum correlations and non-locality. This stems from difficulties in formulating a rigorous definition for non-locality, partly due to the non-objectivity–non-locality issue. In general, quantum theory presents a well-defined platform in which to investigate quantum measurements, with no apparent conflict if the notion that wave functions do collapse is excluded in favor of quantum state reduction [13].

Quantum measurements have been used in the formulation of several measures of quantum correlations, including the quantum discord [14–16], an entity that is known to possess more generalized properties than other well-established measures such as the Wootters concurrence [17]. The quantum discord is useful in differentiating processes that are based on locally accessible correlations from those that incorporate generically non-classical features [18–22]. The quantum discord of a specific system is obtained by performing a set of positive-operator-valued measurements (POVM) in a neighboring partition. At zero quantum discord, a measurement procedure allows external observers to obtain all information about a bipartite system without disturbing it.

Recently, the trade-off between information gained due to quantum measurement and the disturbance on the observed quantity has been examined in several works [23–27]. One study [24] showed that an informative measurement affects at least one state of the system, and the quantity of disturbance on the state is lower bounded by the amount of information that can distinguish the input and corresponding output states. The trade-off between the magnitude of information obtained via quantum measurements and disturbance on the evolution of the system could be reinterpreted using the Heisenberg uncertainty principle [25–27]. Heisenberg [28] first raised the idea that any attempt to measure the position of a particle with higher precision will result in a greater disturbance as quantified by the mean square deviations of the momentum measurements. In a recent experimental study [29] involving weak measurements, Heisenberg’s “measurement–disturbance relationship” was noted to be violated. In another work [30] involving “entanglement-enhanced measurement,” the spinning of electrons in atoms was observed without any disturbance to the atomic cloud.

This study is aimed at examining the attributes of the measurement process and imperfections that distort the quantum correlations present in entangled systems. Following the approach in an earlier work [31], we focus on two key attributes: a) the measurement duration and b) the quantum-level precision for a model system of a qubit pair immersed in a common reservoir. In order to keep the numerical analysis tractable, we adopt the Feynman’s path integral framework [32,33] to interpret quantum measurements and, in particular, employ a variant of this formalism based on the restricted path integral formalism [8,34,35]. Within the restricted path framework, the continuous measurement of a quantity with a given result is monitored by constraints imposed on the Feynman’s path integral. Accordingly, an anti-Hermitian term is added to the Hamiltonian that describes the dynamics of the measured system. In
the presence of non-Hermitian terms introduced during highly precise measurements, the measured system may evolve via one or more complex routes, and the final readout becomes ill-defined. The key feature in this work is thus the perusal of the idea that state reduction can be associated with imperfect measurements, which departs from conventional treatments.

This paper is organized as follows. In Sect. 2 we describe the restricted path integral approach for energy measurements incorporating the non-ideal attributes of the measuring device. Description of the quantum discord measures and details of the qubit pair system under study are provided in Sect. 3, including an analysis of the influence of critical parameters in the violation of a Bell inequality associated with the quantum state of the qubit pair. In Sect. 4, the effect of the measurement precision and finite time duration on the quantum correlation measure is evaluated and numerical results are presented. The information–measurement precision trade-off relations are examined in Sect. 5, and the implication of results obtained in Sect. 4 on the “information–disturbance relationship” is discussed. In Sect. 6 non-Markovianity as quantified by the fidelity difference is used to analyze the flow of information during quantum measurements, and the issues of quantum non-locality and non-Markovian dynamics during violation of the “information–disturbance relationship” are examined. The conclusion is provided in Sect. 7.

2 The restricted path integral approach for energy measurements

We recall the two key elements in Feynman’s path integral formalism [32,33]: The first involves the superposition principle which yields the transition amplitude for a given quantum process, and in the absence of measurements. Under this scheme, the probability amplitude of the transition from the initial to the final state of the system is obtained via summation of the amplitudes of all possible paths which could also interfere with each other. The second feature in Feynman’s formalism involves the weight attached to each individual path that is included during the summation procedure. This weight provides a measure of contribution of each constituent path.

The restricted path integral is derived [8,34–37] from the Feynman’s path integral through the incorporation of a weight functional within the integrand that represents the summation of all tracks linking the origin and destination point. We recall that the Feynman’s propagator, $K_{[E]}(q', \tau; q, 0)$, in the phase-space representation at time $\tau$ is given by [32,33]

$$K(q', \tau; q, 0) = \int \mathcal{D}[p] \mathcal{D}[q] e^{\frac{i}{\hbar} \int_0^\tau [p \dot{q} - H_0(q, p)]dt}$$

where $H_0$ is the Hamiltonian of the closed (unmeasured) quantum system and $[p]$ and $[q]$ are the paths in the momentum and configuration spaces, respectively. In Mensky’s formalism, the output of a measured quantum system is expressed in terms of constrained paths associated with a weight functional $w_{[E]}$ [8]. This functional may assume a Gaussian form, with a damping magnitude that is proportional to the squared difference of the observed value along the paths and the actual measurement result.
system subjected to measurement therefore evolves via a propagator which modifies Eq. (1) according to [37,38]

\[
K_{[E]}(q', \tau; q, 0) = \int d[q]d[p]e^{\frac{i}{\hbar} \int_0^\tau [p\dot{q} - \tilde{H}_0(q, p)]dt} w_{[E]}
\] (2)

This relation highlights the dependence of a selected measurement output such as \( E \) for a measuring instrument that incurs an error \( E_r \) during a measurement duration, \( \tau \).

The sensitivity or error during measurements of the energy levels of a two-level system is known to influence interlevel transitions [38–40]. In a recent work [31], singularities known as exceptional points [41] are shown to appear at the branch point of eigenfunctions at a critical measurement precision \( E_{cr} \). The significance of Mensky’s formalism lies in the inclusion of attributes of the measuring device that may influence the dynamics of the quantum system under observation. This has obvious implications for the evaluation of the quantum discord in quantum systems, as will be shown later in this work. The use of the Gaussian measure, \( w_{[E]} = \exp\left\{ -\frac{\langle (H_0 - E)^2 \rangle}{\Delta E^2} \right\} \), enables the effect of the measurement to be incorporated via the effective Hamiltonian [38,39] for a two-level system

\[
\tilde{H}_{\text{eff}} = \tilde{H}_0 - i\frac{\hbar}{\tau E_r^2} (\tilde{H}_0 - E)^2
\] (3)

where \( \langle \cdots \rangle \) denotes the time average for the duration \( \tau \) during which measurement was performed. As noted earlier, \( E \) [see Eq. (2)] is the selected measurement output after a time \( \tau \) and \( E_r \) is the error made during the measurement of the energy, \( E \). It is evident that maximization of the product \( \tau E_r \) ensures minimal disturbance associated with the measurement process. This product term is linked to the uncertainty principle, so that a lower limit \( \tau E_r \) would ensure maximal disturbance on the monitored system. A large error \( E_r \) and duration \( \tau \) appear as key attributes of a weak measurement. The finite duration \( \tau \) yields a degree of uncertainty in energy of the observed quantum system. We therefore consider a weak non-Hermitian term, so that the system under observation evolves as \( i\hbar \frac{d}{dt} |\psi(t)\rangle = \tilde{H}_{\text{eff}} |\psi(t)\rangle \). By expanding the state of the system within the unperturbed basis states \( |n\rangle \) of the unmeasured system with Hamiltonian \( \tilde{H}_0 \) as \( |\psi(t)\rangle = \sum_n C_n(t) |n\rangle \), the coefficients \( C_n(t) \) can be determined using the Schrödinger equation based on the Hamiltonian in Eq. (3).

The Hamiltonian \( \tilde{H}_0 \) of the unmeasured qubit with energies \( E_1 \) (\( E_2 \)) at state \( |0\rangle \) (\( |1\rangle \)) is of the form

\[
\tilde{H}_0 = -\hbar \left( \frac{\Delta \omega}{2} \sigma_z + V(t) \sigma_x \right)
\] (4)

where the Pauli matrices \( \sigma_x = |0\rangle \langle 1| + |1\rangle \langle 0| \), \( \sigma_z = |1\rangle \langle 1| - |0\rangle \langle 0| \), \( \Delta \omega = 2(E_1 + E_2) \), and the potential \( V(t) \) which induces transitions between the two levels. The perturbation potential terms are taken to be \( V_{00} = V_{11} = 0 \) and \( V_{01} = V_{10}^* = V_0 e^{i\omega(t-t_0)} \) with \( V_0 \) as a real number. The state of the measured system, \( |\psi(t)\rangle \), evolves as [31]
\[ |\psi(t)\rangle = e^{-i(E_1-i\lambda_1/4)t} C_1(t) \left| 0 \right\rangle + e^{-i(E_2-i\lambda_2/4)t} C_2(t) \left| 1 \right\rangle \] (5)

where \( \lambda_1 = \frac{(E_1-E)^2}{2rE_r^2} \) and \( \lambda_2 = \frac{(E_2-E)^2}{2rE_r^2} \) for a renormalized \( E_r \).

The coefficients \( C_1(t) \), \( C_2(t) \) in Eq. (5) are obtained using [31]

\[
\begin{bmatrix}
C_1(t) \
C_2(t)
\end{bmatrix} =
\begin{bmatrix}
\cos \kappa t - i\alpha_1 & -i\alpha_2 \\
-i\alpha_2 & \cos \kappa t + i\alpha_1
\end{bmatrix}
\begin{bmatrix}
C_1(0) \
C_2(0)
\end{bmatrix},
\] (6)

where \( \alpha_1 = \cos \theta \sin \kappa t, \alpha_2 = \sin \theta \sin \kappa t, \cos \theta = \frac{q}{\kappa}, \kappa = \sqrt{q^2 + V_0^2}, q = \frac{1}{2}(\omega - \Delta E + i\Omega/2), \Delta E = (E_2 - E_1), \) and \( \Omega = \lambda_2 - \lambda_1 \). The qubit states of the monitored system therefore incorporate non-Hermitian terms, which are functions of the measurement attributes

\[ |\chi_s(t)\rangle = e^{-\lambda_s t/4} (\cos \kappa t - i \cos \theta \sin \kappa t) \left| 0 \right\rangle \\
-ie^{-\lambda_s t/4} \sin \theta \sin \kappa t \left| 1 \right\rangle \]

\[ |\chi_a(t)\rangle = e^{-\lambda_a t/4} (\cos \kappa t + i \cos \theta \sin \kappa t) \left| 1 \right\rangle \\
-ie^{-\lambda_a t/4} \sin \theta \sin \kappa t \left| 0 \right\rangle,
\] (7)

where \( \lambda_s = \frac{\Delta E^2}{2rE_r^2} \). For measurement procedures that introduce very large errors, \( E_r \to \infty, \lambda_1 = \lambda_2 = \lambda_s = \cos \theta = 0 \), and the qubit oscillates coherently between the two levels with the Rabi frequency \( 2\kappa = 2V_0 \) as is well known in the unmeasured system.

For a system in which the initial state at \( t = 0 \) is \( \left| 1 \right\rangle \) and the final state at time \( t \) is either \( \left| 1 \right\rangle \) or \( \left| 0 \right\rangle \), the probability \( P_{11} \) (\( P_{10} \)) of the system to be in the state \( \left| 1 \right\rangle \) (\( \left| 0 \right\rangle \)) depends on the relation between \( V_0 \) and \( \lambda_s \). At the resonance frequencies, \( \omega = \Delta E \), the Rabi frequency \( 2\kappa_0 = (4V_0^2 - (\lambda_s/2)^2)^{1/2} \), and \( \cos \theta = -i\lambda_s/4\kappa_0 \). There exist two tunneling regimes with \( V_0 > \frac{\lambda_s}{4} \) (\( V_0 < \frac{\lambda_s}{4} \)) applicable to the coherent (incoherent) cases. For the coherent tunneling regime, we obtain [31]

\[ P_{11} = e^{-\lambda_s t/2} \left[ \cos \kappa_0 t - \frac{\lambda_s}{4\kappa_0} \sin \kappa_0 t \right]^2 \] (8)

\[ P_{10} = e^{-\lambda_s t/2} \left| \frac{V_0^2}{2\kappa_0^2} \sin^2 \kappa_0 t, \right. \] (9)

where \( \lambda_s = \frac{(E_2-E_1)^2}{2rE_r^2} \). The total probabilities \( P_{11} + P_{10} \leq 1 \), and the loss of normalization depends on the measurement precision, \( E_r \) as expected. For the system undergoing incoherent tunneling, we replace \( \sin[x] \) (\( \cos[x] \)) by \( \sinh[x] \) (\( \cosh[x] \)). The dynamics at the exceptional point occurs at \( \kappa_0 = 0, V_0 = \frac{\lambda_s}{4}, \) and both regimes merge to a point in topological space. The two-level system can be seen as a non-ideal dissipative quantum system due to its coupling to a multitude of decay states associated with the measurement process.
3 Classical correlation and quantum discord

Following the formulation of quantum discord in Refs. [14–16], we express the quantum mutual information of a composite state \( \rho \) of two subsystems \( A \) and \( B \) as

\[
\mathcal{I}(\rho) = S(\rho_A) + S(\rho_B) - S(\rho)
\]

for a density operator in \( \mathcal{H}_A \otimes \mathcal{H}_B \). \( \rho_A \) (\( \rho_B \)) is the reduced density matrix associated with \( A \) \( (B) \), and \( S(\rho) \) \((i = A, B)\) denotes the well-known von Neumann entropy of the density operator \( \rho \), where

\[
S(\rho) = -\text{tr}(\rho \log \rho)
\]

The mutual information can also be written in terms of quantum conditional entropy

\[
S(\rho|\rho_A) = S(\rho) - S(\rho_B)
\]

\( I(\rho) = S(\rho_B) - S(\rho|\rho_A) \) (10)

A series of one-dimensional orthogonal projectors \( \{\Pi_k\} \) induced in \( \mathcal{H}_A \) results in different outcomes of the measurement in \( \mathcal{H}_B \) via the post-measurement conditional state

\[
\rho_B|k = \frac{1}{p_k} (\Pi_k \otimes I_B) \rho (\Pi_k \otimes I_B)
\]

where the probability \( p_k = \text{tr}[\rho (\Pi_k \otimes I_B)] \) and \( \{\Pi_k\} \) denote the one-dimensional projector indexed by the outcome \( k \). From the cumulative effect of the mutually exclusive measurements on \( A \), we obtain a conditional entropy of the subsystem \( B \) based on \( \rho_B|k \)

\[
S(\rho|\{\Pi_k\}) = \sum_k p_k S(\rho_B|k)
\]

which is used to obtain the measurement-induced mutual information \( \mathcal{I}(\rho|\{\Pi_k\}) = S(\rho_B) - S(\rho|\{\Pi_k\}) \). The classical correlation measure based on optimal measurements made on \( A \) is obtained as [14–16]

\[
C_A(\rho) = \sup_{\{\Pi_k\}} \mathcal{I}(\rho|\{\Pi_k\})
\]

The difference in \( \mathcal{I}(\rho) \) and \( C_A(\rho) \) yields the non-symmetric quantum discord \( D_A(\rho) = \mathcal{I}(\rho) - C_A(\rho) \). The discord \( D_B(\rho) \) associated with measurements on subsystem \( B \) can be evaluated likewise, and in general, \( D_A(\rho) \neq D_B(\rho) \). Measurements made on a neighboring partition hold the key to determining the classical correlation measure between the subsystems.

3.1 A qubit pair immersed in a common reservoir

The joint evolution of a pair of two-level qubit subsystems \( A, B \) undergoing decoherence in a common reservoir is determined by a completely positive, trace-preserving map expressed in the operator-sum form [42–44]
\[ \varepsilon (\rho_{AB}) = \sum_{i,j} \Gamma_i(A) \Gamma_j(B) \rho_{AB} \Gamma_i^\dagger(B) \Gamma_j^\dagger(A), \]  

(14)

where \( \Gamma_i(A) \) (\( \Gamma_i(B) \)) is the Kraus operator associated with the decoherence process at \( A \) (\( B \)). For the phase flip channel in which there is loss of quantum information with conservation of energy, the Kraus operators are given in the basis \( \{|0\}, |1\rangle \) for both subsystems, \( k = A, B \) as \([19,42]\) \( \Gamma_0(A) = \text{diag}(\sqrt{1-p/2}, \sqrt{1-p/2}) \otimes \mathbf{1}_B, \Gamma_1(A) = \text{diag}(\sqrt{p/2}, -\sqrt{p/2}) \otimes \mathbf{1}_B, \Gamma_0(B) = \mathbf{1}_A \otimes \text{diag}(\sqrt{1-p/2}, \sqrt{1-p/2}) \) and \( \Gamma_1(B) = \mathbf{1}_A \otimes \text{diag}(\sqrt{p/2}, -\sqrt{p/2}) \). The parameter \( p = 1 - \exp(-\gamma t) \), where \( \gamma \) denotes the phase damping rate.

To simplify the numerical analysis, we consider a joint state of the pair of two-level qubit subsystems \( A, B \) in an initial X-type state with maximally mixed marginals \( (\rho_{AB}) = I_{A(B)}/2, S(\rho_A(t)) = S(\rho_B(t)) = 1 \). The density matrix appears in the form \( \rho(0) = \frac{1}{4} \left[ 1 + \sum_{i=1,3} c_i \sigma_A^i \otimes \sigma_B^i \right] \), where \( I \) is the identity operator associated with the qubit pair, \( \sigma_A^i \), and \( \sigma_B^i \), and \( \sigma_A^j \) (\( j = A, B, i = 1, 2, 3 \)) are the Pauli operators of each qubit. \( c_i (0 \leq |c_i| \leq 1) \) are real numbers, with the Werner states sharing a common \( |c_1| = |c_2| = |c_3| = c \) and \( c = 1 \) for the Bell basis states. We assume that the usual unit trace and positivity conditions of the density operator \( \rho \) are satisfied. For the class of states where \( |c_1| = |c_2| = c, |c_3| = c_3 \), the evolution of the joint system is described by the matrix

\[ \rho_{A,B}(t) = \frac{1}{4} \begin{pmatrix} 1 + c_3 & 0 & 0 & 0 \\ 0 & 1 - c_3 & 2c e^{\mu t} & 0 \\ 0 & 2c e^{\mu t} & 1 - c_3 & 0 \\ 0 & 0 & 0 & 1 + c_3 \end{pmatrix}, \]  

(15)

where \( \mu = [-2\gamma - i(\Delta \omega_A - \Delta \omega_B)] t \), and as noted earlier, \( \gamma \) is the phase damping rate. To simplify the analysis, we have considered the same damping rate for the two-qubit subsystems. \( \Delta \omega_i, i = A,B \) denotes the difference in energy levels of each qubit subsystem, and we consider equivalent energy levels in the qubit pair. The mutual information of state \( \rho_{A,B} \) in Eq. (15) is evaluated using \( I(\rho_A : \rho_B) = 2 + \sum_{i=1}^4 \lambda_i \log \lambda_i \) where the eigenvalues \( \lambda_i \) of \( \rho_{A,B} \) are \( \lambda_{1,2} = \frac{1}{4}(1 + c_3), \lambda_3 = \frac{1}{4}(1 - c_3 + 2c e^{-2\gamma t}), \) and \( \lambda_4 = \frac{1}{4}(1 - c_3 - 2c e^{-2\gamma t}) \).

3.2 Influence of \( c_3 \) and \( c \) in the violation of a Bell inequality

The violation of the CHSH-Bell inequality function \( B \) quantifies quantum non-local correlations that cannot be created by classical means \([45,46]\). The CHSH inequality Bell function \( B \) is \(|B| \leq 2 \), where \( B = M(a, b) - M(a, b') + M(a', b) + M(a', b') \), where \( M(a, b) \) is the correlated results (±1) arising from the measurement of two qubits in directions \( a \) and \( b \). The CHSH-Bell inequality is violated when \( B \) exceeds 2, and the correlations are considered inaccessible by any classical means of information transfer, while for values less than 2, the local hidden-variable theory satisfies the CHSH-Bell inequality.
We first investigate the influence of parameters \( c_1 = c_2 = c \) and \( c_3 \) in a possible violation of the CHSH-Bell inequality. These results will be compared with the effect of \( c \) and \( c_3 \) on classical and quantum correlations in the next section. For the density matrix in Eq. (15), \( B \) based on correlation averages is obtained using the following relations [47]:

\[
B(t, c, c_3) = \text{Max} \{ B_1(t, c, c_3), B_2(t, c, c_3) \} \\
B_1(t, c, c_3) = 2\sqrt{e^{-4gt}c^2 + c_3^2} \\
B_2(t, c, c_3) = 2\sqrt{2ce^{-2gt}}
\]  

In general, the interplay of several parameters \((c, c_3, t, g)\) makes it a complex problem to examine the non-locality of the two-qubit density matrix, \( \rho_{A,B} \). To simplify the approach, we note that the eigenvalues \( \lambda_i \) of \( \rho_{A,B} \) in Eq. (15) need to satisfy the positivity criteria of assuming only nonnegative values. To this end, \( \lambda_4 = \frac{1}{4}(1 - c_3 - 2ce^{-2\gamma t}) \) is most susceptible to violating this criteria when \( c_3^m = 1 - 2ce^{-2\gamma t} \).

Figure 1a, b shows values of \( B(t, c, c_3) \) as a function of \( c, t \), at two damping rates \( g \), with \( c_3 = c_3^m \). The results indicate that with increasing \( c \), the system is likely to violate the Bell inequality during the initial period of measurement. There are subtle differences in the system non-locality arising from use of low and high \( g \) as can be inferred from Eq. (16). In Fig. 1c where \( c_3 \) is not constrained, the system best exhibits classical features at low \( c \approx 0.1 \) and \( c_3 \approx c \). There is a gradual shift toward possible violation of the CHSH-Bell inequality as \( c \) is increased, and when \( \lambda_4 \) becomes negative.

4 Measurement precision and quantum correlations

The classical correlation measure is obtained through all possible local measurements on one of the subsystems, say \( A \). For the ideal case of a local measurement that is instantaneous, we utilize a set of orthogonal projectors \( \{ \Pi_k = |\theta_k\rangle \langle \theta_k|, k = ||, \perp \} \), which are defined in terms of the orthogonal states

\[
|\theta||\rangle = \cos \theta |0\rangle + e^{i\phi} \sin \theta |1\rangle, \\
|\theta\perp\rangle = e^{-i\phi} \sin \theta |0\rangle - \cos \theta |1\rangle,
\]

where \( 0 \leq \theta \leq \pi/2 \) and \( 0 \leq \phi \leq 2\pi \). In a recent work, Galve et al. [48] showed that orthogonal measurements are sufficient to evaluate the quantum discord pertaining to rank 2 states of two-qubit systems, but provide tight upper bounds for higher rank (3 and 4) states.

Using Eq. (7) as a basis, we modify the projection operators in Eq. (17) using generalized projectors that incorporate measurement attributes

\[
|\theta||\rangle_p = R(\theta) |0\rangle + e^{i\phi} S(\theta) |1\rangle, \\
|\theta\perp\rangle_p = e^{-i\phi} S(\theta) |0\rangle - R(\theta) |1\rangle,
\]

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The terms $R(\theta) = e^{-\lambda_r t/4} \left( \cos \theta - i \frac{\lambda_r t}{4\theta} \sin \theta \right)$ and $S(\theta) = e^{-\lambda_r t/4} \sqrt{\frac{\theta^2 + (\lambda_r t/4)^2}{\theta}}$ sin $\theta$ with $\lambda_r = \frac{\Delta E^2}{2\tau E_f^2}$, $\Delta E$ being the energy difference between the $|1\rangle$ and $|0\rangle$ states of the qubit. In the limit of $\lambda_r \rightarrow 0$, Eq. (18) reverts back to the orthogonal set in Eq. (17). It is implicit that the orthogonal measurement projections in Eq. (18) may be in a state of evolution during the measurement process.

The generalized measurements as specified by the constituent maps in Eq. (18) can be projected as follows:

$$\frac{1}{2} \left| \theta || \theta \rangle \right|_p \langle \theta || \theta \rangle_p + \frac{1}{2} \left| \theta \perp \right|_p \langle \theta \perp \right|_p, \quad (19)$$

$$= \begin{pmatrix}
| R(\theta) |^2 + | S(\theta) |^2 & 0 \\
0 & | R(\theta) |^2 + | S(\theta) |^2
\end{pmatrix}, \quad \tau = \lambda_r = 0$$

The incorporation of measurement attributes ($\tau$, $p$) therefore leads to non-preservation of the trace of the density matrix in Eq. (19). This can be attributed to loss of the particle from the system due to the observational mapping process. The dependence of the
projected states given in Eq. (18) on the measurement attributes, \( \tau \) and \( \lambda_r \), results in the dependence of the classical correlation on these same attributes that we examine next.

The reduced density matrices, \( \rho_B^{(k)} \), of the neighboring subsystem \( B \) in accordance with the two projective measurements \( (p_\parallel = p_\perp = 1/2) \) in subsystem \( A \) are obtained as

\[
\rho_B^\parallel = \left( \begin{array}{cc}
\frac{1}{2} (1 - c_3 e^{-p t/2} [\varphi_1 - \varphi_2]) & \frac{1}{2} (1 + c_3 e^{-p t/2} [\varphi_1 - \varphi_2]) \\
\frac{1}{2} e^{-2g t} \varphi_1 \varphi_2 & \frac{1}{2} (1 + c_3 e^{-p t/2} [\varphi_1 - \varphi_2])
\end{array} \right),
\]

\[
\rho_B^\perp = \left( \begin{array}{cc}
\frac{1}{2} (1 + c_3 e^{-p t/2} [\varphi_1 - \varphi_2]) & -\frac{1}{2} e^{-2g t} \varphi_1 \varphi_2 \\
-\frac{1}{2} e^{-2g t} \varphi_1 \varphi_2 & \frac{1}{2} (1 - c_3 e^{-p t/2} [\varphi_1 - \varphi_2])
\end{array} \right).
\]

where \( \varphi_1 = \cos^2 \theta - \left(\frac{p t}{4\theta}\right)^2 \sin^2 \theta - \frac{\xi^2}{\theta^2} \sin^2 \theta, \varphi_2 = \frac{\xi^2}{\theta^2} \sin^2 \theta \) and \( \xi = \sqrt{\theta^2 + (p t/4)^2} \).

\( g = \gamma \tau \), the dimensionless time, \( \tau \) is obtained via division with the measurement duration, \( \tau = 2\pi/V_0 \) and the dimensionless precision parameter, \( p = \lambda_t \tau \). The eigenvalues of the two reduced density matrices, \( \rho_B^{(k)} \), of the neighboring subsystem \( B \) are obtained as

\[
\xi_{1,2}^{(k)} = \frac{1}{2} (1 \pm \Theta),
\]

\[
\Theta^2 = c_3^2 e^{-p t} [\varphi_1 - \varphi_2]^2 + 4 c^2 e^{-4g t} e^{-p t} \varphi_1 \varphi_2
\]

Using Eq. (22), we obtain \( S(\rho_B^\parallel) = S(\rho_B^\perp) = -\text{tr}(\rho \log \rho) = -\frac{1-\Theta}{2} \log_2 \left[ \frac{1-\Theta}{2} \right] - \frac{1+\Theta}{2} \log_2 \left[ \frac{1+\Theta}{2} \right] \). The classical correlation given in Eq. (13) is evaluated using

\[
C(\rho) = 1 - \min_{\theta,\phi} R(\Theta),
\]

Further evaluation of \( C(\rho) \) is simplified by the elimination of the parameter \( \phi \) due to the choice of similar parameters, \( |c_1| = |c_2| = c \). The maximal value of \( \Theta \) is dependent on \( c_3, c, \gamma \), the precision parameter \( p \), and the measurement duration \( \tau \). It is obvious from Eq. (22) that the influence of the phase damping rate becomes pronounced at \( c > c_3 \).

The quantum discord is evaluated using

\[
D(\rho) = 2 + \sum_{k=1}^{4} \lambda_k \log_2 \lambda_k - C(\rho),
\]

Figures 2 and 3 highlight changes in the classical correlation \( C \) and quantum discord \( D \) as a function of \( t \) \((0 < t < \tau)\), based on numerical evaluation of Eqs. (22), (23), and (24), and the eigenvalues \( \lambda_i \) of \( \rho_{AB} \) in Eq. (15). \( C \) and \( D \) undergo noticeable changes due to finite-time measurements on a neighboring partition at nonzero \( p \) for input parameters \( c_3 > c \). Figures 2b and 3b show that the quantum discord \( D \) is enhanced, with a corresponding decrease in \( C \) as \( p \) is increased. The enhancement in the quantum discord \( D \) is pronounced at a higher ratio \( \frac{c_3}{c} \). However, at \( c > c_3 \), the
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Fig. 2 a Classical correlation $\mathcal{C}$ as a function of $t$ ($0 < t < \tau$) for various values of the measurement precision $p$. $\tau$, the measurement time duration, is set at 1 and $g = \gamma \tau = 0.6$, where $\gamma$ is the phase damping rate. The real numbers $c_1 = c_2 = c = 0.2$ and $c_3 = 0.6$. The curves from top to bottom correspond to the unitless measurement precision, $p = 0$, 0.05, 0.2, 0.5 and b quantum discord $D$ as a function of $t$ ($0 < t < \tau$) for various values of the measurement precision $p = 0.5, 0.2, 0.05, 0$ (top to bottom). All other parameters are the same as in a

Fig. 3 a Classical correlation $\mathcal{C}$ as a function of $t$ ($0 < t < \tau$) for various values of the measurement precision $p$. $\tau$, the measurement time duration, is set at 1 and $g = \gamma \tau = 0.6$. The real numbers $c_1 = c_2 = c = 0.1$ and $c_3 = 0.7$. The curves from top to bottom correspond to the unitless measurement precision, $p = 0$, 0.05, 0.2, 0.5 and b quantum discord $D$ as a function of $t$ for various values of the measurement precision $p = 0.5, 0.2, 0.05, 0$ (top to bottom). All other parameters are the same as in a

Numerically evaluated classical correlation was noted to be almost insignificant, $\mathcal{C} \approx 0.01$, and $D$ was seen to be independent of $p$. These results indicate a trend toward more non-classical behavior at increased $p$ for the case when $c_3 > c$; however, it is not immediately clear why a low $\mathcal{C}$ is obtained at $c_3 < c$. Figures 4 and 5 illustrate the changes in $\mathcal{C}$ and $D$ due to the dephasing rate $g$ for various values of the measurement precision $p$. At $c < c_3$, the classical correlation $\mathcal{C}$ remains independent of $g$, a trend that appears only beyond a critical $g$ at $c > c_3$. This has also been noted in earlier works [19,20] for the specific case, $p = 0$. Imperfect measurements carried out on the subsystem $A$ therefore enhance the non-classical correlations of the adjacent subsystem $B$ at $c_3 > c$.

It is not immediately clear as to the influence of quantum measurements in the region where $c < c_3$, even though it appears that there is less impact of measurements. Further investigations are needed to confirm the role of the CHSH-Bell inequality, $B$ in relation to $c$, $c_3$ parameters. For instance in Fig. 1, it appears that states that exhibit greater classical features (at low $c$, $c_3 > c$) are more likely to have reduced $\mathcal{C}$ and increased $D$ with measurements carried out in an adjacent subsystem. On the other hand, states that are close to CHSH-Bell inequality violation ($c > c_3$) seem less influenced by imprecise measurement procedures.
5 Information–measurement precision trade-off

The results in Sect. 4 highlight the effect of disturbance on the classical and non-classical correlations of the adjacent reduced density matrix, $\rho_{B}^{(k)}$. This disturbance is quantified by the measurement precision $p$ and finite time duration $\tau$ associated with the imperfect projective measurements [Eq. (18)] on subsystem $A$. In this section, we examine the implications of these results on the “information–disturbance relationship” on subsystem $B$ as a result of imperfect measurements on subsystem $A$. We note the two sources of uncertainty ($p$, $\tau$) which give rise to a probabilistic distribution of the quantity being measured.

Two important measures will be used to examine the trade-off between information gained due to quantum measurement and the disturbance caused during observation: fidelity and trace distance. The fidelity, $F$ [50], which quantifies the distance between two states appears as

$$F[\rho_1, \rho_2] = \left\{ \text{Tr} \left[ \sqrt{\rho_1 \rho_2 \rho_1^*} \right] \right\}^2,$$

(25)
and is bounded by $0 \leq F[\rho_1, \rho_2] \leq 1$. The measurement disturbance on the dynamics of a system can be quantified using [24]

\[ D_i = 1 - F[\rho_1, \rho_2] \tag{26} \]

Based on the reduced density matrices $\rho_B(t, p)$ [Eq. (20)], the disturbance $D_i$ can be evaluated as a function of $t$ and $p$.

To quantify the information gained from the system, we define the uncertainty $H(\nu)$ based on the parameter $\nu$ where

\[ \nu = \frac{1}{2} - \frac{1}{2} T_d(\rho_1, \rho_2) \tag{27} \]

The trace distance, $T_d$, between density matrices, $\rho_1, \rho_2$, is given by half of the trace norm of the difference of the matrices as $T_d(\rho_1, \rho_2) = \frac{1}{2} \text{Tr}(|\rho_1 - \rho_2|)$ and $H(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$. In order to analyze the influence of the precision $p$ and $t$, we consider $\rho_1 = \rho_B(t = 0, p = 0)$ and $\rho_2 = \rho_B(t, p)$. We next investigate the information–disturbance trade-off relation [24]

\[ 1 - F[\rho_1, \rho_2] \geq 1 - H(\nu(p, t)) \tag{28} \]

where the mutual information $(1 - H(\nu(p, t)))$ is evaluated based on $\nu$ [Eq. (27)] for given values of the measurement attributes, $p, t$. Equation (28) specifies that the disturbance between two states $\rho_1, \rho_2$ has a lower bound, quantified by the gain in information due to measurements. An alternative interpretation of Eq. (28) was also provided [24] through the existence of a lower limit to the sum of the disturbance $1 - F$ and the uncertainty $H(\nu(p, t))$ that is based on verification of the difference in the two states, $\rho_1, \rho_2$. The difference between disturbance $D_i (\times 100 \%)$ and gain in information $(1 - H(\nu(p, t))) (\times 100 \%)$ as a function of $t$ and precision $p$ is shown in Fig. 6a, b. We note that for the input parameters ($c_1 = c_2 = c = 0.4, c_3 = 0.1$), there exists a range of $p$ and $t$ for which the “information–disturbance relationship” [Eq. (28)] formulated in Ref.[24] is violated.

Comparing the results in Fig. 6a, b with those in Figs. 2 and 3, one notes that the appearance of increased quantum discord is invariably linked to the non-violation of the inequality in Eq. (28) when $c_3 > c$. The difference between the disturbance $D_i$ [Eq. (26)] and the mutual information $(1 - H(\nu(p, t)))$ yields a measure of the quantum discord. This difference is accentuated at increasing $p$, a trend that is also observed in the enhancement of the quantum discord $D$ at higher $p$. One can expect a zero discord when the lower bound in Eq. (28) is reached, at which point the disturbance on the system equates the amount of information that can be retrieved. Another important observation relates to the case when $c > c_3$, where we earlier noted the existence of a small $C \approx 0.01$ and $D$ that was immune to changes in $p$. Interestingly, we note that a violation of Eq. (28) occurs when $c > c_3$ and for a range of $p, t$ as illustrated in Fig. 6b.

One possible explanation for the noted violation may lie in the presence of other unseen neighboring subsystems, which gives rise to a net deficit in quantum discord, as
Fig. 6  

(a) Contour plots showing the trade-off between the measurement disturbance and information gain. Difference between disturbance $D_i \times 100\%$ and gain in information $(1 - H(v(p, t))) \times 100\%$ as a function of $t$ and precision $p$, $\rho_1 = \rho^0(t = 0, p = 0)$ and $\rho_2 = \rho^0(t, p)$ [Eq. (20)]. $c_1 = c_2 = c = 0.1, c_3 = 0.7, g = \gamma \tau = 0.5$ and $\theta = 90$.  

(b) Description and parameters are the same as in (a) with the exception of $c_1 = c_2 = c = 0.4, c_3 = 0.1$. A range of $p$ and $t$ for which the difference between disturbance and gain in information is negative becomes noticeable.

far as the two known subsystems $A, B$ are concerned. This results in greater retrieval of information than the actual disturbance on the system, with bearings very similar to the Maxwell’s demon model [49]. In the latter model, positive entropy production during measurement arises from work performed by other agents, ensuring that the second law of thermodynamics remains intact.

The results in Fig. 6a, b can partly be interpreted on the basis of earlier obtained results in Fig. 1 a–c, where we noted that at higher $c > c_3$, there is trend toward violation of the CHSH-Bell inequality. The violation of Heisenberg’s “measurement–disturbance relationship” as evidenced by the nature of the $c, c_3$ input parameters may have its origins in non-local quantum states which are also influenced by these same parameters. To this end, investigations involving rigorous mathematical formulations [51] of the underlying abstract Hilbert space are needed to provide greater insight into the links between the information–disturbance trade-off relation, quantum discord, and quantum non-locality based on the CHSH-Bell inequality function $B$ [Eq. (16)]. The results obtained here may be useful in the interpretation of experimental results [29] showing similar violation of the “measurement–disturbance relationship.”

6 Non-Markovianity during quantum measurements

To better understand the flow of information during quantum measurements, we consider the appearance of non-Markovianity, in relation to the attributes, $p, t$. Quantum systems undergoing Markovian dynamics observe a completely positive, trace-preserving dynamical map $\Lambda(t), \rho(0) \rightarrow \rho(t) = \Lambda(t)\rho(0)$, which constitutes the one-parameter semigroup obeying the composition law [52–54], $\Lambda(t_1)\Lambda(t_2) = \Lambda(t_1 + t_2), t_1, t_2 \geq 0$. Accordingly, the fidelity function $F[\rho(t), \rho(t + \tau)]$ involving the initial state $\rho(t)$ and the evolved state $\rho(t + \tau)$ at a later time $t + \tau$, under Markovian dynamics, satisfies the inequality [53]
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Fig. 7 Fidelity difference $\Delta(t, \tau)$ as a function of $t$ and $\theta$ for the density matrices corresponding to $\rho_1 = \rho_B^\parallel(t = 0, p = 0)$ and $\rho_2 = \rho_B^\parallel(t, p)$ [Eq. (20)]. We set $\tau = 0.1, c_1 = c_2 = c = 0.1, c_3 = 0.8, g = \gamma \tau = 0.1$. Values of $\rho$ are indicated at the top of each figure. Negative values indicating non-Markovianity are shaded green, increase with precision $p$, and are dominant around $25 < \theta < 42^\circ$ (Color figure online)

\[
F[\rho(t), \rho(t + \tau)] \equiv F[\Lambda(t)\rho(0), \Lambda(t)\rho(\tau)] \\
\Rightarrow F[\rho(t), \rho(t + \tau)] \geq F[\rho(0), \rho(\tau)].
\] (29)

Any violation of this inequality is a signature of non-Markovian dynamics which can be observed via the fidelity difference function

\[
\Delta(t, \tau) = \frac{F[\rho(t), \rho(t + \tau)] - F[\rho(0), \rho(\tau)]}{F[\rho(0), \rho(\tau)]},
\] (30)

Negative values of $\Delta(t, \tau)$ serve as sufficient but not necessary condition of non-Markovianity. Using Eq. (30), we have evaluated the fidelity difference $G(t, \tau)$ as a function of $t$ and $\theta$ for the density matrices corresponding to $\rho_1 = \rho_B^\parallel(t = 0, p = 0)$ and $\rho_2 = \rho_B^\parallel(t, p)$ [Eq. (20)], as illustrated in Figs. 7 and 8. The figures highlight important differences between systems where $c_3 > c$ and those with $c > c_3$. In regions midway with $25 < \theta < 42$, there is enhancement of non-Markovianity with precision $p$ when $c_3 > c$. In systems where $c_3 < c$, the non-Markovian regions are located at the peripheral regions, $\theta \approx 0, 90$. We note that at $c_3 > c$ the optimized angle $\theta$ used in the evaluation of the classical correlation in Eq. (23) is about 90, decreasing gradually with increase $p$. At $c_3 < c$, the optimized angle $\theta < 30$. These results highlight the role of $c, c_3$ parameters in determining the links between non-Markovian dynamics and optimization processes associated with the classical correlation measure.
Fig. 8 Fidelity difference \( \Delta(t, \tau) \) as a function of \( t \) and \( \theta \) for the density matrices corresponding to \( \rho_1 = \rho_B(t = 0, p = 0) \) and \( \rho_2 = \rho_B(t, p) \) [Eq. (20)]. We set \( \tau = 0.1, c_1 = c_2 = c = 0.4, c_3 = 0.1, g = \gamma \tau = 0.1 \). Values of \( p \) are indicated at the top of each figure. Negative values indicating non-Markovianity are shaded green and are located at the peripheral regions, \( \theta \approx 0, 90 \) (Color figure online)

The findings in Figs. 7 and 8 may provide a speculative basis to examine links between violation of the “measurement–disturbance relationship” in Eq. (28) and non-Markovianity. Is it possible that a pathway for violation of this relationship is attained when gain in information exceeds disturbance between two states via non-Markovian processes? This challenging question needs experimental verification using more generalized systems, as numerical results related to just two subsystems have been provided here.

7 Conclusion

In conclusion, we have presented results of the influence of non-ideal attributes such as the measurement precision and finite measurement time duration on the classical correlation and quantum discord for a qubit pair immersed in a common environment. The results show that the quantum discord is enhanced as the precision of the measuring instrument is increased for a range of parameters, and both the classical correlation and the quantum discord undergo noticeable changes during the duration when measurements are performed on a neighboring partition. We also conclude that increased quantum discord within two subsystems is invariably linked to the non-violation of the inequality associated with the “information–disturbance relationship.” We note that a zero discord corresponds to the lower bound in this inequality, consistent with the
point at which the disturbance on the system equates the amount of information that can be retrieved. A violation of this inequality indicates a deficit in quantum discord, with the possibility that other undetected agents may be responsible in giving rise to a greater retrieval of information than the actual disturbance on the system. Overall, the results obtained in this work indicate that the fundamental limits of quantum mechanical measurements may be altered by exchanges involving non-classical correlations such as the quantum discord with external sources.

The violation of the “information–disturbance relationship” may have links with quantum non-locality and non-Markovian quantum dynamics of states that do not necessarily evolve via a completely positive, trace-preserving dynamical maps. This study identifies (though not conclusively) that a possible pathway for violation of this relationship may occur when gain in information exceeds disturbances between two states via non-Markovian processes. Further scrutiny of the intricate links between these entities requires mathematically rigorous approaches [51] and experimental verifications; however, the results obtained in this study have wider implications for exploiting the subtleties of the uncertainty principle in multipartite systems. Cryptographic technologies specify that eavesdroppers can be detected as a result of disturbances caused by their measuring activities. The results obtained here indicate that eavesdroppers can remain undetected in some instances (when positivity of the density matrix of the observed system is violated). These ideas may be further extended to the development of sensitive quantum probes of fragile material systems, involving single atoms and molecules, and including living tissue matter.

Lastly, this study shows that the joint examination of several entities (non-locality, non-Markovianity, negative quantum discord) is needed in investigations involving quantum measurements of optics and nanostructure systems [55,56]. The possibility that an analogous “information–disturbance relationship” may be violated in light-harvesting systems [57–62] is an area for future investigation of systems that display exceptionally high efficiencies of energy transfer processes.

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