HIERARCHIES OF SIMPLICIAL COMPLEXES VIA THE
BGG-CORRESPONDENCE

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Abstract. Via the BGG-correspondence a simplicial complex $\Delta$ on $[n]$ is transformed into a complex of coherent sheaves $\mathcal{L}(\Delta)$ on the projective space $\mathbb{P}^{n-1}$. In general we compute the support of each of its cohomology sheaves.

When the Alexander dual $\Delta^*$ is Cohen-Macaulay there is only one such non-zero cohomology sheaf. We investigate when this sheaf can be an $a$th syzygy sheaf in a locally free resolution and show that this corresponds exactly to the case of $\Delta^*$ being $a + 1$-Cohen-Macaulay as defined by K.Baclawski [4].

By putting further conditions on the sheaves we get nice subclasses of $a + 1$-Cohen-Macaulay simplicial complexes whose $f$-vector depends only on $a$ and the invariants $n, d,$ and $c$. When $a = 0$ these are the bi-Cohen-Macaulay simplicial complexes, when $a = 1$ and $d = 2c$ cyclic polytopes are examples, and when $a = c$ we get Alexander duals of the Steiner systems $S(c, d, n)$.

We also show that $\Delta^*$ is Gorenstein* iff the associated coherent sheaf of $\Delta$ is an ideal sheaf.

Introduction

A simplicial complex $\Delta$ on $\{1, \ldots, n\}$ corresponds to a monomial ideal in the exterior algebra. Via the BGG-correspondence such an ideal transforms to a complex of coherent sheaves $\mathcal{L}(\Delta)$ on the projective space $\mathbb{P}^{n-1}$. In a recent paper, [12], J.E.Vatne and the author studied when the BGG-correspondence applied to a simplicial complex $\Delta$ gives rise to a complex $\mathcal{L}(\Delta)$ with at most one non-zero cohomology sheaf $\mathcal{S}(\Delta)$. We showed that this happens exactly when the Alexander dual $\Delta^*$ is a Cohen-Macaulay simplicial complex. We then singled out a particularly nice class of coherent sheaves, the locally Cohen-Macaulay sheaves and asked when $\mathcal{S}(\Delta)$ belonged to this class. This turned out to happen when both $\Delta$ and $\Delta^*$ are Cohen-Macaulay.

In this paper we study this further in several directions. Firstly we develop in Section 3 criteria for telling which cohomology sheaves of $\mathcal{L}(\Delta)$ are non-zero and if so, what their support is. We also describe the multigraded Hilbert functions of these cohomology sheaves. This is related to the cellular complexes of [6].

Secondly, we again single out nice classes of coherent sheaves and ask when a simplicial complex $\Delta$ is such that $\mathcal{S}(\Delta)$ is in such a class. This time
we consider the class of coherent sheaves $\mathcal{S}_{a}$ which can occur as $a$'th syzygy sheaf in a locally free resolution of a coherent sheaf. Thus when $a$ is 1 we have the torsion free sheaves, when $a$ is 2 the reflexive sheaves and when $a$ is $n - 1$ the vector bundles. In Section 4 we consider the class $\mathcal{C}\mathcal{L}_{a}$ of simplicial complexes $\Delta$ such that $S(\Delta)$ is in $\mathcal{S}_{a}$, giving us a "hierarchy" $\mathcal{C}\mathcal{L} = \mathcal{C}\mathcal{L}_{0} \supset \mathcal{C}\mathcal{L}_{1} \supset \cdots$ of simplicial complexes. Note that $\mathcal{C}\mathcal{L}$ is the class of Alexander duals of Cohen-Macaulay simplicial complexes. There turns out to be nice, compact criteria for membership in the classes $\mathcal{C}\mathcal{L}_{a}$ and when taking restrictions and links of a $\Delta$ in $\mathcal{C}\mathcal{L}_{a}$ these are predictably in $\mathcal{C}\mathcal{L}_{a}$ and $\mathcal{C}\mathcal{L}_{a-1}$ respectively. (Via Alexander duals this generalizes the fact that the link of a Cohen-Macaulay simplicial complex is Cohen-Macaulay.)

The three most important invariants of a simplicial complex $\Delta$ is $n$, its dimension $d - 1$ and $c$, the largest integer such that all $c$-sets are in $\Delta$. Via certain nice additional properties which one may put on sheaves in $\mathcal{S}_{a}$ we further single out a particular nice subclass of $\mathcal{C}\mathcal{L}_{a}$. When $a$ is 0 this subclass turns out to be the class studied in [12], the bi-Cohen-Macaulay simplicial complexes. When $a$ is $c$, the maximal interesting value for $a$, it turns out that we get exactly the Steiner systems $S(c, d, n)$. In general for each $a$, the simplicial complexes of this nice subclass of $\mathcal{C}\mathcal{L}_{a}$ will have $f$-vectors depending only on $c, d$ and $n$.

Now taking Alexander duals of the classes $\mathcal{C}\mathcal{L}_{a}$ we get in Section 5 a "hierarchy" of Cohen-Macaulay simplicial complexes $\mathcal{C}\mathcal{M} = \mathcal{C}\mathcal{M}_{0} \supset \mathcal{C}\mathcal{M}_{1} \supset \cdots$. The simplicial complexes in $\mathcal{C}\mathcal{M}_{a}$ turn out to be exactly the $a + 1$-Cohen-Macaulay simplicial complexes as defined by K. Baclawski [4]. (Thus the simplicial complexes in $\mathcal{C}\mathcal{L}_{a}$ are the Alexander duals of the $a + 1$-Cohen-Macaulay simplicial complexes.) After translating most of the results for the classes $\mathcal{C}\mathcal{L}_{a}$ and its subclasses to the classes $\mathcal{C}\mathcal{M}_{a}$ and its subclasses, we consider the subclass of $\mathcal{C}\mathcal{M}$ consisting of Gorenstein* simplicial complexes $\Delta$. This class corresponds exactly to the subclass of $\mathcal{C}\mathcal{M}_{1}$ with $\tilde{H}_{d-1}(\Delta) = k$ and is the class of Cohen-Macaulay simplicial complexes $\Delta$ such that $S(\Delta^*)$ is a torsion free rank one coherent sheaf. Moreover we show that this sheaf may be naturally identified with the associated coherent sheaf of the ideal defining the Stanley-Reisner ring of $\Delta^*$.

Having described the contents of Sections 3, 4, and 5 we inform that Section 1 contains preliminaries and techniques for dealing with simplicial complexes. Section 2 describes the BGG-correspondence and the classes of coherent sheaves we single out for study, namely the classes of sheaves which can be $a$'th syzygy sheaves in a locally free resolution. In the last section, Section 6, we pose some problems for further study.

1. Simplicial complexes.

Denote by $[n]$ the set of integers $\{1, \ldots, n\}$. A simplicial complex $\Delta$ on $[n]$ is a family of subsets of $[n]$ such that if $Y \subseteq X \subseteq [n]$ and $X$ is in $\Delta$ then $Y$ is in $\Delta$. 
1.1. **Notions.** We recall some notions for simplicial complexes. An element in \( \Delta \) is called a **face** of \( \Delta \), and a maximal face is a **facet**. If \( d \) is the maximal cardinality of a face of \( \Delta \), the **dimension** of \( \Delta \) is \( d - 1 \). If \( c \) is the largest integer such that all \( c \)-sets in \([n]\) are in \( \Delta \), we call \( c - 1 \) the **frame dimension** of \( \Delta \). By convention the empty simplex \( \emptyset \) has \( c = -1 \) (while \( \{\emptyset\} \) has \( c = 0 \)).

The **Alexander dual** simplicial complex \( \Delta^* \) is the simplicial complex on \([n]\) consisting of all \( F \) in \([n]\) such that the complement \( F^c = [n] \setminus F \) is not in \( \Delta \). Let \( d^* - 1 \) and \( c^* - 1 \) be the dimension and frame dimension of \( \Delta^* \). It is easily seen that

\[
  n = d + c^* + 1, \quad n = d^* + c + 1.
\]

Now we introduce the following notation. Let \( R \cup S \cup T \) be a partition of \([n]\) into three disjoint subsets, and let

\[
  \Delta_{R,S,T} = \{ F \subseteq R \mid F \cup S \in \Delta \}
\]

which is a simplicial complex on \( R \). Note that \( \Delta_{S,\emptyset} \) is the link \( \text{lk}_\Delta S \), and that \( \Delta_{R,\emptyset} \) is the restriction \( \Delta_R \) to \( R \). Also note that \( \Delta_{R,S,T} \) is \( \emptyset \) iff \( S \) is not a face of \( \Delta \).

**Lemma 1.1.**

\[
  (\Delta_{R,S,T})^* = (\Delta^*)_{R,S}. \quad (\Delta_{R,S,T})^* = (\Delta^*)_{R,S}.
\]

**Proof.** Let \( F \) be a subset of \( R \). That \( F \) is in \( (\Delta_{R,S,T})^* \) means that \( R \setminus F \) is not in \( \Delta_{R,S,T} \) or \( (R \setminus F) \cup S \) not in \( \Delta \).

That \( F \) is in \( (\Delta^*)_{R,S} \) means that \( F \cup T \) is in \( \Delta^* \) or \([n]\setminus(F \cup T) \) not in \( \Delta \). But the latter is \((R \setminus F) \cup S \). \( \square \)

1.2. **Homology.** Let \( V \) be a vector space over a field \( k \) with basis \( e_1, \ldots, e_n \) and \( E = E(V) \) the exterior algebra \( \oplus_0^\infty V \wedge^i V \). Let \( W = V^* \) be the dual space with dual basis \( x_1, \ldots, x_n \). Consider the monomials \( e_{i_1} \cdots e_{i_r} \) in \( E \) such that \( \{i_1, \ldots, i_r\} \) is not in \( \Delta \). They form a basis for an ideal \( J_\Delta \) in \( E \).

Dualizing the inclusion \( J_\Delta \subseteq E \) we get an exact sequence of \( E \)-modules

\[
  0 \to C_\Delta \to E(W) \to (J_\Delta)^* \to 0.
\]

where \( C_\Delta \) is the kernel. A basis for \( C_\Delta \) consist of all monomials \( x_{i_1} \cdots x_{i_r} \) such that \( \{i_1, \ldots, i_r\} \) are in \( \Delta \). Left multiplication with \( u = e_1 + e_2 + \cdots + e_n \) gives a differential \( d \) on \( C_\Delta \) and the reduced homology of \( \Delta \) is defined by \( (W \) has degree \(-1)\)

\[
  \tilde{H}_p(\Delta) = H^{-p-1}(C_\Delta, d).
\]

Note that via the isomorphism of \( E \)-modules between \( E(V) \) and \( E(W)(-n) \), the ideal \( J_\Delta \) gets identified with the submodule \( C_{\Delta^*}(-n) \) of \( E(W)(-n) \). We therefore get an exact sequence

\[
  0 \to C_\Delta \to E(W) \to (C_{\Delta^*})^*(n) \to 0.
\]

**Lemma 1.2.** \( \tilde{H}_{n-3-p}(\Delta^*) \cong \tilde{H}_p(\Delta)^* \).
Proof. Since the differential on \( E(W) \) is acyclic we get from (2)
\[
H^{-p-1}(C_\Delta) \cong H^{n-2-p}((C_\Delta^*)^*) = H^{p+2-n}(C_\Delta^*)^*.
\]

\[ \square \]

1.3. Reducing. Now if \([n]\) is partitioned as \( R \cup \{x\} \) we get a short exact sequence of \( E \)-modules
\[
0 \to C_{\Delta_R} \to C_\Delta \to C_{lk_\Delta(x)}(1) \to 0.
\]
The following is a basic kind of general deduction from this type of exact sequence.

**Proposition 1.3.** Let \( R \cup S \cup T \) be a partition of \([n]\) and suppose \( \tilde{H}_p(\Delta_R^{ST}) \) is non-zero.

a) (Reducing \( S \).) Given \( S' \subseteq S \). Then there are \( R' \supseteq R \) and \( T' \supseteq T \) such that \( \tilde{H}_p'(\Delta_R^{ST}) \) is non-zero, where \( p' - p \) is the cardinality of \( R' \setminus R \).

b) (Reducing \( T \).) Given \( T' \subseteq T \). Then there are \( R' \supseteq R \) and \( S' \supseteq S \) such that \( \tilde{H}_p'(\Delta_R^{ST'}) \) is non-zero.

c) (Reducing \( R \).) Given \( R' \subseteq R \). Then there are \( S' \supseteq S \) and \( T' \supseteq T \) such that \( \tilde{H}_p'(\Delta_R^{S'T}) \) is non-zero where \( p - p' \) is the cardinality of \( S' \setminus S \).

**Proof.** All of these just follow from the exact sequence
\[
0 \to C_{\Delta_R^{ST \cup \{x\}}} \to C_{\Delta_R^{S \cup \{x\}}} \to C_{\Delta_R^{S \cup \{x\}, T}}(1) \to 0
\]
where \( R \cup S \cup T \cup \{x\} \) is a partition of \([n]\), by running long exact cohomology sequences. \[ \square \]

**Remark 1.4.** This proposition generalizes Corollary 4.4 of [1].

**Notation 1.5.** We shall usually write \( \Delta_\cdot \) for \( \Delta_{\cdot \setminus R} \) (in which case \( R = S \cup T \)). Often we shall also drop the \( T \) and write this simply as \( \Delta_\cdot \). In general when \( Y \) is a set we shall use the lower case letter \( y \) to denote the cardinality of \( Y \). For instance given \( \Delta_\cdot \) then \( r \) and \( s \) will be the cardinalities of \( R \) and \( S \) respectively.

**Corollary 1.6.** Suppose \( \tilde{H}_p(\Delta_R^{ST}) \) is non-zero. Then \( c - 1 \leq p + s \leq d - 1 \) and the following holds.

a) If \( p + s = c - 1 \) and \( S' \subseteq S \) then \( \tilde{H}_p'(\Delta_{\cdot \cup \{x\}}) \) is non-zero, where \( p' + s' = c - 1 \).

b) If \( p + s = d - 1 \) and \( T' \subseteq T \) then \( \tilde{H}_p(\Delta_{\cdot \cup \{y\}}) \) is non-zero.

c) If \( p \geq 0 \) then \( \tilde{H}_{p-1}(lk_\Delta S') \) is non-zero for some \( S' \) strictly containing \( S \).

d) \( \tilde{H}_{p'}(\Delta_R) \) is non-zero for some \( R \) and \( p' \geq p \).

e) \( \tilde{H}_p(\Delta) \) is non-zero for some \( \Delta \) containing \( S \).
Proof. The frame dimension of $\Delta_{ST}^{R}$ is $\geq c - s - 1$. If therefore $p < c - s - 1$ then $\tilde{\tilde{H}}(\Delta_{ST}^{R})$ is zero. Similarly the dimension of $\mathrm{lk}_S$ is $\leq d - s - 1$ and so if $p \geq d - s$ then $\tilde{\tilde{H}}(\Delta_{ST}^{R})$ is zero.

a) Reduce $S$ to $S'$ and get $\tilde{\tilde{H}}(\Delta_{ST}^{R'})$ non-zero. First note we must have $p' + s' \geq c - 1$. Also we have $p' - p = s + t - s' - t'$. Since $t' \geq t$ we must have $T = T'$ and $p' + s' = c - 1$. Part b) is shown similarly as a).

c) We have $\tilde{\tilde{H}}(\Delta_{ST}^{R})$ non-zero. Now reduce $R$. Sooner or later $S$ increases to $\tilde{S} = S \cup \{x\}$ such that

\[(3)\quad \tilde{\tilde{H}}_{p-1}(\Delta_{\tilde{S},T}^{R}) \neq 0.\]

This is so since otherwise we would get $\tilde{\tilde{H}}(\Delta_{st}^{R})$ non-zero which is impossible. Now reducing $T$ in (3) to $\emptyset$ we get $\tilde{\tilde{H}}_{p-1}(\Delta_{S,T}^{R})$ non-zero.

d) and e) follow by reducing $S$, respectively $T$, to the empty set. \(\square\)

For later we need the following.

Lemma 1.7. If $\Delta$ is not the $n - 1$-simplex or the empty simplex, then $H_{c-1}(\mathrm{lk}_S\Delta)$ is non-zero for some $S$.

Proof. Let $R$ be a $c + 1$-set which is not a face of $\Delta$. Then clearly $H_{c-1}(\Delta_R)$ is non-zero. Now reduce $T(=R^c)$ to $\emptyset$, and get $H_{c-1}(\mathrm{lk}_S\Delta)$ non-zero. \(\square\)

1.4. CLeray and Cohen-Macaulay simplicial complexes. For an integer $e$, a simplicial complex $\Delta$ is called $e$-Leray, [15] p.12], if $H_p(\Delta_R) = 0$ for all $p \geq e$ and subsets $R$ of $[n]$. Motivated by this we make the following definition.

Definition 1.8. A simplicial complex $\Delta$ is CLeray if $\tilde{\tilde{H}}(\Delta_R) = 0$ for all $p \geq c$, that is for all $p$ greater than the frame dimension of $\Delta$, and subsets $R$ of $[n]$. (Note that we do not speak of a complex being say, 3Leray or 2Leray. One can speak of a complex being CLeray or 3-Leray.)

Note that when $c = 0$ this gives that $\Delta$ is a simplex on the vertices it contains.

Recall that $\Delta$ is Cohen-Macaulay if $\tilde{\tilde{H}}(\mathrm{lk}_S\Delta) = 0$ when $p + s \leq d - 2$. The class of CLeray and Cohen-Macaulay simplicial complexes are now seen to be Alexander dual.

Proposition 1.9. $\Delta$ is CLeray iff $\Delta^*$ is Cohen-Macaulay.

Proof. By Lemmata 1.1 and 1.2

\[\tilde{\tilde{H}}(\Delta_R) = \tilde{\tilde{H}}_{p-3-p}(\Delta_{\emptyset}^{R,p,c}) = \tilde{\tilde{H}}_{p-3-p}(\Delta_{\emptyset}^{R,c}) = \tilde{\tilde{H}}_{p-3-p}(\mathrm{lk}_S\Delta_{p,c}).\]

The condition that $p \geq c$ is equivalent to $n - 3 - p \leq d^* - 2$ and so we get the statement. \(\square\)

Another description of $\Delta$ being CLeray is the following.

Proposition 1.10. $\Delta$ is CLeray iff $\tilde{\tilde{H}}(\mathrm{lk}_S\Delta) = 0$ for all $S$ in $\Delta$. 


Proof. Suppose $\Delta$ is not CLeray. Then $\tilde{H}_p(\Delta_R)$ is non-zero for some $R$ and $p \geq c$. By Corollary 1.6 e), we get $\tilde{H}_p(\text{lk}_S \Delta)$ non-zero for some $S$ and so $\tilde{H}_c(\text{lk}_S \Delta')$ non-zero for some $S'$ in $\Delta$ by 1.6 c).

Conversely, if $\tilde{H}_c(\text{lk}_S \Delta)$ is non-zero, then by Corollary 1.6 d), $\tilde{H}_p(\Delta_R)$ is non-zero for some $p \geq c$ and so $\Delta$ is not CLeray. □

Corollary 1.11. $\Delta$ is Cohen-Macaulay iff $\tilde{H}_p(\Delta - R)$ is zero for $p + r = d - 2$.

Proof. Using Lemma 1.2 this follows from the above. □

2. The BGG-correspondence and a hierarchy of coherent sheaves

Recall that $W = V^*$ is the dual space of $V$ and let $S = S(W)$ be the symmetric algebra on $W$. If $M = \oplus_{i \in \mathbb{Z}} M_i$ is a graded module over $E$ we can form the complex

$$L(M) : \cdots \to S(i) \otimes_k M_i \xrightarrow{d^i} S(i + 1) \otimes_k M_{i+1} \to \cdots$$

with differential given by

$$d^i(s \otimes m) = \sum_{\alpha=1}^{n} sx_\alpha \otimes e_\alpha m.$$

Sheafifying this we get a complex of coherent sheaves

$$\tilde{L}(M) : \cdots \to \mathcal{O}_{\mathbb{P}^{n-1}}(i) \otimes_k M_i \xrightarrow{d^i} \mathcal{O}_{\mathbb{P}^{n-1}}(i + 1) \otimes_k M_{i+1} \to \cdots$$

on the projective space $\mathbb{P}^{n-1}$. This, in short, is the BGG-correspondence between graded modules over the exterior algebra and complexes of coherent sheaves on $\mathbb{P}^{n-1}$, originally from [3]. Our main reference for this will be [10].

Let us give some properties of this correspondence.

2.1. Restriction to linear subspaces. Let $V'$ be a subspace of $V$ so $W' = V'^*$ is a quotient space of $W$. Via $E(V') \to E(V)$, the module $M$ may be restricted to a module $\text{res} \ M$ over $E(V')$. Also via $S(W) \to S(W')$ we may form the quotient complex $L(M) \otimes_S S(W')$. These are related by

$$L'(\text{res} M) = L(M) \otimes_S S(W')$$

where $L'$ is the corresponding functor for $E(V')$-modules.

2.2. Duals of complexes. Now we consider $\wedge^n W$ to be a module over $E$ in degree $-n$.

Let $M' = \text{Hom}_k(M, \wedge^n W)$. Then we have canonically

$$L(M') = \text{Hom}_S(L(M), S(-n) \otimes_k \wedge^n W)[n].$$

Sheafifying this, note that the canonical sheaf $\omega_{\mathbb{P}^{n-1}}$ naturally identifies with the sheafification of $S(-n) \otimes_k \wedge^n W$, we get

$$(4) \quad \tilde{L}(M') = \text{Hom}_{\mathcal{O}_{\mathbb{P}^{n-1}}}(\tilde{L}(M), \omega_{\mathbb{P}^{n-1}})[n].$$
In particular if the only nonzero cohomology of $\tilde{L}(M)$ is $F = H^{-a}\tilde{L}(M)$, a coherent sheaf, we see that

$$\text{Ext}_{\mathcal{O}_{\mathbb{P}^{n-1}}}^p(F, \omega_{\mathbb{P}^{n-1}}) = H^{a+r-n}(M^\vee).$$

2.3. Calculating Tor’s. For a graded module $M$ over $E(V)$ and $v$ an element of $V$, let $H^p(M, v)$ be the cohomology of the complex

$$M_{p-1} \xrightarrow{v} M_p \xrightarrow{v} M_{p+1}.$$

Letting $k(v)$ be $\mathcal{O}_{\mathbb{P}^{n-1}}/I_v$ where $I_v$ is the ideal sheaf of the point $v$ in $\mathbb{P}^{n-1}$ we have, due to [11],

$$H^p(M, v) = H^p(\tilde{L}(M) \otimes_{\mathcal{O}_{\mathbb{P}^{n-1}}} k(v)).$$

In particular, if $F = H^0\tilde{L}(M)$ is the only non-zero cohomology

$$H^{-p}(M, v) = \text{Tor}_{\mathcal{O}_{\mathbb{P}^{n-1}}}^p(F, k(v)).$$

This is what will be needed about the BGG-correspondence. Now we turn to describing some classes of coherent sheaves.

2.4. A hierarchy of coherent sheaves.

Definition 2.1. A coherent sheaf $F$ on $\mathbb{P}^{n-1}$ is an \textit{a}'th syzygy sheaf if there is a locally free resolution of some coherent sheaf $M$

$$M \leftarrow \mathcal{E}_0 \leftarrow \ldots \leftarrow \mathcal{E}_{i-1} \leftarrow d_i \mathcal{E}_i \leftarrow \ldots$$

such that $F$ is the kernel of $d_{i-1}$.

Now let $\text{Sh} = \text{Sh}^0$ be the category of all coherent sheaves on $\mathbb{P}^{n-1}$. For $a \geq 1$ let $\text{Sh}_a$ be the full subcategory of $\text{Sh}$ consisting of the sheaves which can occur as an \textit{a}'th syzygy sheaves. There is then a filtration into “nicer and nicer” classes of coherent sheaves

$$\text{Sh}_0 \supseteq \text{Sh}_1 \supseteq \ldots \supseteq \text{Sh}_{n-1}.$$

Note that $\text{Sh}_1$ are the torsion free sheaves, $\text{Sh}_2$ are the reflexive sheaves, i.e. sheaves $F$ such that the natural map $F \rightarrow F^\vee$ is an isomorphism (here $F^\vee$ is $\text{Hom}_{\mathcal{O}_{\mathbb{P}^{n-1}}} (F, \omega_{\mathbb{P}^{n-1}})$), and $\text{Sh}_{n-1}$ are the vector bundles.

Proposition 2.2. $F$ is in $\text{Sh}_a$ if and only if the codimension of the support of $\text{Ext}_{\mathcal{O}_{\mathbb{P}^{n-1}}}^i(F, \omega_{\mathbb{P}^{n-1}})$ is greater or equal to $i + a$ for all $i > 0$.

Proof. If $F$ is in $\text{Sh}_a$, then with the notation in (6), $\text{Ext}_{\mathcal{O}_{\mathbb{P}^{n-1}}}^i(F, \omega_{\mathbb{P}^{n-1}})$ is equal to $\text{Ext}_{\mathcal{O}_{\mathbb{P}^{n-1}}}^{i+\alpha}(M, \omega_{\mathbb{P}^{n-1}})$ for $i > 0$ and the statement follows by [8, 20.9].

For later use note the following.

Fact. If $F$ is a coherent sheaf with support in codimension $r$, then the sheaf $\text{Ext}_{\mathcal{O}_{\mathbb{P}^{n-1}}}^i(F, \omega_{\mathbb{P}^{n-1}})$ has codimension $r$ when $i = r$ and is zero when $0 < i < r$. This follows for instance from [8, 18.4] and its proof.
Now assume $\mathcal{E}xt^i(\mathcal{F}, \omega_{\mathbb{P}^{n-1}})$ has codimension greater or equal to $i + a$ for $i \geq 1$. If $a \geq 1$ it follows from the above fact that $\mathcal{F}$ cannot have a torsion subsheaf, and so $\mathcal{F}$ is in $\mathcal{S}h_1$.

Now let $a \geq 2$. Since $\mathcal{F}$ is in $\mathcal{S}h_1$, there is an exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}^\vee \to \mathcal{P} \to 0$$

where $\mathcal{P}$ is the cokernel. Let $1 + r$ be the codimension of $\mathcal{P}$. Since any torsion free sheaf is locally free outside a codimension 2 subset, $1 + r \geq 2$. Now we get a long exact sequence

$$\mathcal{E}xt^i(\mathcal{F}^\vee, \omega_{\mathbb{P}^{n-1}}) \to \mathcal{E}xt^i(\mathcal{F}, \omega_{\mathbb{P}^{n-1}}) \to \mathcal{E}xt^{i+1}(\mathcal{P}, \omega_{\mathbb{P}^{n-1}})$$

$$\to \mathcal{E}xt^{i+1}(\mathcal{F}^\vee, \omega_{\mathbb{P}^{n-1}}).$$

Hence if $\mathcal{P}$ is non-zero, that is $\mathcal{F}$ is not in $\mathcal{S}h_2$, then $\mathcal{E}xt^i(\mathcal{F}, \omega_{\mathbb{P}^{n-1}})$ has codimension $1 + r$, which is against our assumption. Hence $\mathcal{P}$ is zero and so $\mathcal{F}$ is in $\mathcal{S}h_2$.

Now we look at the situation when $a \geq 3$. We know then that $\mathcal{F}$ is reflexive. Let

$$\mathcal{F} \leftarrow \mathcal{E}_0 \leftarrow \cdots \leftarrow \mathcal{E}_m$$

be a locally free resolution of $\mathcal{F}$. Dualizing we get

$$\mathcal{F}^\vee \to \mathcal{E}_0^\vee \to \cdots \to \mathcal{E}_m^\vee$$

with cohomology $\mathcal{E}xt^i(\mathcal{F}, \omega_{\mathbb{P}^{n-1}})$ at $\mathcal{E}_i^\vee$. Now we claim that $\mathcal{E}xt^i(\mathcal{F}^\vee, \omega_{\mathbb{P}^{n-1}})$ is zero for $0 < i < a - 1$. More generally, if $\mathcal{K}_r$ is the kernel of $\mathcal{E}_r^\vee \to \mathcal{E}_{r+1}^\vee$, then $\mathcal{E}xt^i(\mathcal{K}_r, \omega_{\mathbb{P}^{n-1}})$ is zero for $0 < i < r + a - 1$. This follows easily by breaking the complex $\mathcal{E}$ into short exact sequences and using the fact noted above. If we take a locally free resolution $\mathcal{D}$ of $\mathcal{F}^\vee$ and dualize this we get a complex

$$\mathcal{F}^\vee \to \mathcal{D}_{a-1}^\vee \to \cdots \to \mathcal{D}_0^\vee \to \mathcal{D}_1^\vee \to \cdots$$

which is exact at all $\mathcal{D}_i^\vee$ when $i > 0$. Hence $\mathcal{F}$ which is equal to $\mathcal{F}^\vee$ is in $\mathcal{S}h_a$.

To obtain even nicer classes of coherent sheaves, we can consider the subcategory $\mathcal{S}h^+_{a}$ of $\mathcal{S}h_a$ consisting of $\mathcal{F}$ such that there exist $r > 0$ with $\mathcal{E}xt^i(\mathcal{F}, \omega_{\mathbb{P}^{n-1}})$ zero except possibly when $i$ is 0 or $r$.

3. **Simplicial complexes via the BGG-correspondence**

Given a simplicial complex $\Delta$ over $[n]$, recall the module $C_\Delta$ over $E(V)$. Using the BGG-correspondence, we may form the complex $\tilde{L}(C_\Delta)$ of coherent sheaves on $\mathbb{P}^{n-1}$. Of interest then is the cohomology sheaves of $\tilde{L}(C_\Delta)$. The following basic result was established in [12].

**Theorem 3.1.** a) $\tilde{L}(C_\Delta)$ has at most one non-zero cohomology group $\mathcal{F}$ if and only if $\Delta$ is CLeray. In this case $\mathcal{F} = H^{-c}\tilde{L}(C_\Delta)$.

b) The cohomology sheaf $\mathcal{F}$ is locally Cohen-Macaulay sheaf if and only if $\Delta$ is both CLeray and Cohen-Macaulay.
Notation 3.2. We write \( L(\Delta) \) for \( L(C_\Delta) \) and \( S^a(\Delta) \) and \( S^a(\Delta) \) for the \( a \)'th cohomology groups of \( L(\Delta) \) and \( \tilde{L}(\Delta) \) respectively. In case \( \Delta \) is CLeray we simply write \( S(\Delta) \) for \( S^c(\Delta) \). If \( \Delta \) is both CLeray and Cohen-Macaulay we call \( \Delta \) bi-Cohen-Macaulay.

We shall now establish a more general result, namely determine for any \( \Delta \) which cohomology sheaves of \( \tilde{L}(\Delta) \) are non-zero. We first establish some other results.

Lemma 3.3. The associated primes of each cohomology group \( H^{-p}L(\Delta) \) are of the form \((x_i)_{i \in R}\) where \( R \) is a subset of \([n]\).

Proof. \( H^{-p}L(\Delta) \) is graded by \( \mathbb{Z}^{n+1} \). By [8, Exc. 35] an associated prime is also graded by \( \mathbb{Z}^{n+1} \). But the only such prime ideals are of the above form. \( \square \)

3.1. Local ranks. The following Proposition and Corollary is due to T. Ekedahl. Let \( v = \sum \lambda_i e_i \) be a linear form in \( E(V) \). It can also be considered as a point in \( \mathbb{P}^{n-1} \). We call the set \( R = \{i \mid \lambda_i \neq 0\} \) the support of \( v \).

Proposition 3.4. Let \( v = \sum \lambda_i e_i \) be a point in \( \mathbb{P}^{n-1} \) with support \( R \). Then
\[
H^{-p}(\tilde{L}(\Delta) \otimes_{\mathcal{O}_{\mathbb{P}^{n-1}}} k(v)) = \bigoplus_{S \subseteq R} \tilde{H}_{p-s}(\Delta_S^{S,T}).
\]

Proof. First note the following general fact. If \( v = \sum \lambda_i e_i \) where all \( \lambda_i \) are non-zero, let \( u = \sum_{i=1}^{n} e_i \). Then
\[
H^{-p}(C_\Delta, v) = H^{-p}(C_\Delta, u) = \tilde{H}_{p-1}(\Delta).
\]
This is because the map \( C_\Delta \to C_\Delta \) defined by \( x_i \mapsto \lambda_i x_i \) gives an isomorphism between the complexes \( (C_\Delta, v) \) and \( (C_\Delta, u) \).

Now by Paragraph 2.3
\[
H^{-p}(\tilde{L}(\Delta) \otimes_{\mathcal{O}_{\mathbb{P}^{n-1}}} k(v)) = H^{-p}(C_\Delta, v).
\]
Let \( V_R \) be the subspace of \( V \) spanned by the \( e_i \) where \( i \) is in \( R \). Restricting \( C_\Delta \) via \( E(V_R) \to E(V) \) we get
\[
\text{res} C_\Delta = \bigoplus_{S \subseteq R} C_{\Delta_S^{S,T}}(s).
\]
Therefore
\[
H^{-p}(C_\Delta, v) = \bigoplus_{S \subseteq R} H^{-p+s}(C_{\Delta_S^{S,T}, v}) = \bigoplus_{S \subseteq R} \tilde{H}_{p-s-1}(\Delta_{R}^{S,T}).
\]
\( \square \)

Corollary 3.5. a) The rank of \( H^{-p}L(\Delta) \) is the dimension of \( \tilde{H}_{p-1}(\Delta) \).

b) Let \( \Delta \) be CLeray and \( v \) a point in \( \mathbb{P}^{n-1} \) with support \( R \). Then
\[
\text{Tor}_p^{\mathcal{O}_{\mathbb{P}^{n-1}}}(S(\Delta), k(v)) = \bigoplus_{S \subseteq R} \tilde{H}_{p+c-1-s}(\Delta_{R}^{S,T}).
\]

Proof. Immediate. \( \square \)
Example 3.6. Let the dimension of $\Delta$ be 1, so $\Delta$ is a graph and suppose $c = 1$. Then $\Delta$ is CLeray iff $\Delta$ is a forest, $\Delta$ is Cohen-Macaulay iff $\Delta$ is connected and $\Delta$ is bi-Cohen-Macaulay iff $\Delta$ is a tree.

We can compute the rank of $S(\Delta)$ at a point $v$ with support $R$ as follows. (We use lower case letters to denote the dimension of a cohomology group.)

$$\text{rank } S(\Delta)_v = \tilde{h}_0(\Delta_R) + \sum_{x \in R^c} \tilde{h}_{-1}(\Delta_R^{\{x\}}).$$

Note that $\tilde{h}_{-1}(\Delta_R^{\{x\}})$ is non-zero iff $\Delta_R^{\{x\}}$ is $\{\emptyset\}$ and this holds iff $R \cap \text{lk}_\Delta \{x\}$ is $\emptyset$. Thus

$$\text{rank } S(\Delta)_v = \tilde{h}_0(\Delta_R) + \# \{ x \mid R \cap \text{lk}_\Delta \{x\} = \emptyset \}.$$ 

3.2. Cohomology modules. Now we shall investigate the cohomology modules of $L(\Delta)$. Note that the complex $L(\Delta)$ is simply the cellular complex, see [6], obtained by associating to vertex $i$ in $\Delta$ the monomial variable $x_i$. For $a$ in $\mathbb{N}^n$ let $\Delta_{\leq a}$ be the subcomplex of $\Delta$ on the vertices $i$ for which $a_i > 0$, and call the set of such $i$ the support of $a$. The following observation is part of the proof of [6, Prop.1.2] and gives the multigraded Hilbert functions of the cohomology modules.

Proposition 3.7. a) $H^{-p}L(\Delta)_a$ is isomorphic to $\tilde{H}_{p-1}(\Delta_{\leq a})$.

b) Let $m$ be a monomial and $b = a + \deg_Z m$. There is then a commutative diagram

$$\begin{array}{ccc}
\tilde{H}_{p-1}(\Delta_{\leq a}) & \longrightarrow & \tilde{H}_{p-1}(\Delta_{\leq b}) \\
\downarrow & & \downarrow \\
H^{-p}L(\Delta)_a & \xrightarrow{m} & H^{-p}L(\Delta)_b
\end{array}$$

Proof. The graded part $L(\Delta)_a$ identifies naturally with the complex $(C_{\Delta_{\leq a}}, d)$ as follows. Let $M$ be the monomial corresponding to $a$. Now $L^{-p}(\Delta) = S(-p) \otimes_k (C_{\Delta})_{-p} = \oplus_I Su_I$ where $u_I = 1 \otimes_k x_I$. There is a natural map

$$(C_{\Delta_{\leq a}})_p \rightarrow L^{-p}(\Delta)_a$$

given by

$$x_I \mapsto M/x_I \cdot u_I.$$ 

We verify easily that this is an isomorphism and commutes with the differentials in $C_{\Delta_{\leq a}}$ and $L(\Delta)_a$. This gives a) and b). □

Corollary 3.8. $H^{-p}L(\Delta)$ is zero iff $\tilde{H}_{p-1}(\Delta_R)$ is zero for all $R \subseteq [n]$. 

Now we look closer at the cohomology modules of $L(\Delta)$. For a subset $R$ of $[n]$ let $P(R)$ be the homogeneous prime ideal in the polynomial ring $S$ defining the linear subspace $P^{n-1}$ spanned by the $e_i$ where $i$ is in $R$. That is $P(R) = (x_i)_{i \notin R}$. 


The last statement, let $M$ be the $i$th graded $\mathbb{Z}$-module of the resolution $L$, we find that $\rho_{\triangledown}(\Delta)$ is zero. The first statement follow from $a)$ and $c)$ of Theorem 3.9. To prove $\rho_{\triangledown}(\Delta)$ is non-zero. Hence the length $\rho_{\triangledown}(\Delta)$ is zero since $\rho_{\triangledown}(\Delta)$ is non-zero. Let $\rho_{\triangledown}(\Delta)$ be in $\mathbb{N}$ have 1 in position $i$ if $i$ is not in $R$ and 0 otherwise. Since $H^{-p}L(\Delta)_{\delta(R)}$ is isomorphic to $H^{p-1}(\Delta)$ there is a sequence

$$0 \to (S/P(R))(-r)^{\rho} \to H^{-p}L(\Delta) \to Q \to 0$$

where $\rho$ is the $k$-dimension of $\tilde{H}_{p-1}(\Delta)$ and $Q$ is the cokernel. Now localization we find that $Q_{(P(R))}$ is zero since $\Pi_{R}x_i \cdot 0$ is zero.

Corollary 3.10. a) The sheaf $H^{-p}L(\Delta)$ has no embedded linear subspaces and is supported on a reduced union of coordinate linear subspaces of $\mathbb{P}^{n-1}$. b) For $p \geq 1$, $H^{-p}L(\Delta)$ is the graded global sections of the sheaf $H^{-p}L(\Delta)$.

Proof. The first statement follow from $a)$ and $c)$ of Theorem 3.9. To prove the last statement, let $M = H^{-p}L(\Delta)$. By b) of Theorem 3.9 the local cohomology groups $H^0_M(M)$ and $H^1_M(M)$ are zero, [8, Thm. A4.3]. That $\oplus_{k \in \mathbb{Z}} \Gamma(\mathbb{P}^{n-1}, \widetilde{M}(k))$ is equal to $M$ now follows by [8, Thm. A4.1]. 

4. Hierarchies of CLEary complexes

We consider the various hierarchies of coherent sheaves $S_{\Delta}$ and subclasses of them and investigate simplicial complexes $\Delta$ such that $S(\Delta)$ is in such a class.
4.1. \(a+1\)-CLeray simplicial complexes. Let \(\mathcal{CL}_a\) be the class of CLeray simplicial complexes \(\Delta\) such that \(S(\Delta)\) is in \(\mathfrak{Sh}_a\). Note that if \(\Delta\) is the \(c-1\)-skeleton of the \(n-1\)-simplex then \(S(\Delta)\) is the vector bundle \(\Omega_{p_{n-1}}^c\) and so \(\Delta\) is in \(\mathcal{CL}_a\) for all \(a\). We shall however make the convention that the \(c-1\)-skeleton of the \(n-1\)-simplex is in \(\mathcal{CL}_a\) iff \(c \geq a - 1\). Thus for instance \(\mathcal{CL}_a\) contains the 1-simpleton but not the 0-simpleton, \(\emptyset\), or \(\emptyset\). The simplicial complexes in \(\mathcal{CL}_a\) are now called \(a+1\)-CLeray. The following gives a criterion for \(\Delta\) to be \(a+1\)-CLeray.

**Theorem 4.1.** \(\Delta\) is \(a+1\)-CLeray iff \(c \geq a - 1\) and 
\[
\tilde{H}_{c-a}(lk_\Delta S) = 0, \quad \text{for } s \geq a.
\]
In particular, if \(\Delta\) is not the \(a-2\)-skeleton of the \(n-1\)-simplex then \(c \geq a\).

**Proof.** a) That \(\Delta\) is in \(\mathcal{CL}_a\) means that \(\mathcal{E}xt^i(S(\Delta), \omega_{p_{n-1}})\) has codimension \(\geq i + a\) for all \(i \geq 1\). By \([5]\) this sheaf is the cohomology sheaf 
\[
H^{c+i-n}(C_\Delta)^{\vee},
\]
Now by \([2]\) there is an exact sequence 
\[
0 \to C_{\Delta^*} \to E(W) \to (C_\Delta)^{\vee} \to 0
\]
and so we get \(\mathcal{E}xt^i(S(\Delta), \omega_{p_{n-1}})\) equal to \(H^{c+i-n+1}(\Delta^*)\). That this is a sheaf with support in codimension \(\geq i + a\) means that 
\[
\tilde{H}_{n-i-2-c}(\Delta^*)_{-s} = 0
\]
when \(s < i + a\). By Lemma \([1, 1]\) \((\Delta^*)_{-s} = (lk_\Delta S)^*\). And so by Lemma \([1, 2]\) \(7\) is the same as 
\[
\tilde{H}_{c+i-s-1}(lk_\Delta S) = 0
\]
for \(s < i + a\). Now put \(s = i + a - 1 - p\) where \(p \geq 0\). Then this becomes 
\[
\tilde{H}_{c-a+p}(lk_\Delta S) = 0
\]
for \(s \geq a - p\). By Corollary \([1, 6]\) \(c\) if \(c \geq a\) then \(\tilde{H}_{c-a}(lk_\Delta S)\) zero for \(s \geq a\) implies \(9\), so this latter is the condition that \(\Delta\) is in \(a+1\)-CLeray when \(c \geq a\). When \(c < a\), then again by Corollary \([1, 6] c\), \(9\) is implied by \(\tilde{H}_{-1}(lk_\Delta S)\) zero for \(s \geq c + 1\) and this means that \(\Delta\) has no faces of dimension \(c\). Hence it is the \(c-1\)-skeleton of the \(n-1\)-simplex and by our convention we have \(c = a - 1\). \(\Box\)

**Corollary 4.2.** a) \(\Delta\) is \(a+1\)-CLeray iff \(\tilde{H}_p(\Delta^S_R)\) is zero for \(p + s \geq c\) and \(p \geq c - a\).

b) If \(\Delta\) is \(a+1\)-CLeray, then \(\Delta_R\) is \(a+1\)-CLeray.

**Proof.** a) Suppose \(\tilde{H}_p(\Delta^S_R)\) is nonzero where \(p + s \geq c\) and \(p \geq c - a\) where \(c \geq a\). By Corollary \([1, 6] c\) \(\tilde{H}_p(lk_\Delta S)\) is nonzero for some \(S'\) containing \(S\) and
by Corollary 1.6(c) we get $\tilde{H}_{c-a}(lk_\Delta S''')$ is nonzero where $s'' - s' \geq p - (c-a)$. But then

$$s'' \geq p + s' - (c-a) \geq c - (c-a) = a$$

and this is against assumption.

b) This is clear since the frame dimension of $\Delta_R$ is greater or equal to that of $\Delta$. □

As stated after Definition 1.8 when $\Delta$ is 1-CLeray with $c = 0$, then $\Delta$ is a simplex on its vertices. In view of Theorem 4.1 it is of interest to investigate the $a + 1$-CLeray $\Delta$'s with minimal interesting frame dimension which is $a - 1$ (when the frame dimension is $a - 2$ it is the $a - 2$-skeleton of the $n-1$-simplex).

**Proposition 4.3.** $\Delta$ is $c + 1$-CLeray iff any two distinct facets intersect in a subset of cardinality less than $c$. (If $c = 0$ this means that there is only one facet.)

**Proof.** Suppose $\Delta$ is $c + 1$-CLeray. Let $F$ and $G$ be two distinct facets such that the cardinality of $S = F \cap G$ is maximal. Now we claim that $lk_\Delta S$ is disconnected. Suppose not and let $F = F_1 \cup S$ and $G = G_1 \cup S$ where $F_1 \cap G_1$ is empty. Then there would be a path from some vertex $f_1$ in $F_1$ to some vertex $g_1$ in $G_1$ in $lk_\Delta S$. Say the path starts with \{f_1, x\} where $x$ is not a vertex in $F_1$. Then $H = S \cup \{f_1, x\}$ is a face of $\Delta$, $H$ is not contained in $F$ and $F \cap H$ has cardinality larger than $S$ which goes against our assumptions. Hence $H_0(lk_\Delta S)$ is non-zero. By Theorem 1.1(a) we get that $s < c$.

In the other direction, given that distinct facets always intersect in cardinality $< c$, we see that $lk_\Delta S$ when $s \geq c$ is always a simplex. Hence $H_0(lk_\Delta S)$ is zero and so $\Delta$ is $c + 1$-CLeray. □

By Corollary 4.6 if $p + s \leq c - 2$ then $\tilde{H}_p(lk_\Delta S)$ is zero and Corollary 4.2 gives conditions on homology when $p + s \geq c$. It is therefore of interest to investigate what happens when $p + s = c - 1$. The following is motivating for the classes studied in Subsection 4.3.

**Proposition 4.4.** $(p + s = c - 1)$

a) If $\tilde{H}_{c-s-1}(lk_\Delta S)$ is zero and $S'$ contains $S$ then $\tilde{H}_{c-s'-1}(lk_\Delta S')$ is zero.

b) Suppose $\Delta$ is $a + 1$-CLeray. Then $H_{c-a}(lk_\Delta S)$ is nonzero for any face $S$ with $s = a - 1$.

**Proof.** a) follows form Corollary 1.6(a) by contraposition.

b) Let $S'$ be a facet containing $S$ where $s$ is $a - 1$. Then $\tilde{H}_{-1}(\Delta_{S', S''})$ is non-zero. Reduce $S'$ to $S$ and get $\tilde{H}_p(\Delta_{R, S''})$ non-zero and so $p + s \geq c - 1$. Now reduce $T$ to $\emptyset$ and get $\tilde{H}_p(lk_\Delta S)$ non-zero where $\hat{S} \supseteq S$. Since $\Delta$ is in $C\Delta_0$, $\tilde{H}_p(lk_\Delta S)$ is zero when $p + \hat{s} \geq c$ and $p \geq c - a$. Hence either $p + \hat{s} \leq c - 1$ or $p \leq c - a - 1$. The latter is impossible since $p + a - 1$ which
is $p + s$ is $\geq c - 1$ and the former gives $s = s = a - 1$ and so $\tilde{H}_p(\text{lk}_\Delta S)$ is non-zero.

We now give a somewhat more conceptual description of what it means for a complex to be $a + 1$-CLeray. First a lemma.

**Lemma 4.5.** If $\Delta$ is 2-CLeray, the frame dimension of $lk_\Delta \{x\}$ is one less than that of $\Delta$.

**Proof.** Since $\Delta$ is 2-CLeray, $c \geq 0$. If $c = 0$ this holds, so assume $c \geq 1$. We have $\tilde{H}_p(\text{lk}_\Delta \{x\} \cup S)$ zero for $p + s + 1 \geq c$ and $p \geq c - 1$, which reduces simply to the condition $p \geq c - 1$. Thus by Lemma 4.5 the frame dimension of $\text{lk}_\Delta \{x\}$ is $\leq c - 2$ and must be equal to $c - 2$. $\square$

**Theorem 4.6.** $\Delta$ is a $a + 1$-CLeray iff every link $\text{lk}_\Delta S$ with $s = a$ is CLeray with frame dimension $c - a - 1$.

**Proof.** We may assume $c \geq a \geq 1$. If $\Delta$ is $a + 1$-CLeray then

$$\tilde{H}_{c-a}(\text{lk}_\Delta (S \cup T) = 0, \text{ when } s = a \text{ and } t \geq 0.$$ 

Since by Lemma 4.5 the frame dimension of $\text{lk}_\Delta S$ is $c - a - 1$, we get that $\text{lk}_\Delta S$ is CLeray. Conversely, if each $\text{lk}_\Delta S$ is CLeray of frame dimension $c - a - 1$ then (4.1) holds and so $\Delta$ is $a + 1$-CLeray. $\square$

4.2. The classes $\mathcal{C}L^+_a$. Let $\mathcal{C}L^+_a$ be the class of CLeray simplicial complexes $\Delta$ such that $S(\Delta)$ is in $\mathcal{SH}^+_a$. The following gives a criterion for $\Delta$ to be in $\mathcal{C}L^+_a$.

**Theorem 4.7.** $\Delta$ is in $\mathcal{C}L^+_a$ iff $\tilde{H}_p(\text{lk}_\Delta S)$ is zero for all $p$ and $s$ in the range $c \leq p + s \leq d - 2$

and also when $p, s$ are in the range $p + s = d - 1, p \geq c - a$.

In this case every facet has dimension $c - 1$ or $d - 1$.

**Proof.** For $S(\Delta)$ to be in $\mathcal{C}L^+_a$, there is some $r$ such that $\text{Ext}^i(S(\Delta), \omega_{P_{n-1}})$ is zero except possibly when $i$ is 0 or $r$, and in the latter case it has codimension $\geq r + a$. By the argument of Theorem 4.1 this means that

(10) $H_{c+i-s-1}(\text{lk}_\Delta S) = 0$ for all $s$ when $i \neq 0, r$

$H_{c+r-s-1}(\text{lk}_\Delta S) = 0$ for $s < a + r$.

Letting $p = c + i - s - 1$ this gives $H_p(\text{lk}_\Delta S)$ zero when $p + s \geq c$ and $p + s \neq c + r - 1$ or when $p + s = c + r - 1$ and $p \geq c - a$. But let $S$ be a facet. Then $\text{lk}_\Delta S$ is $\{\emptyset\}$ so $\tilde{H}_{-1}(\text{lk}_\Delta S)$ is non-zero. Hence if $s \geq c + 1$, letting $s = c + i$, we see from (10) that we must have $i = r$ and so $s = c + r$. So all facets $S$ with $s \geq c + 1$ must have $s = c + r$ which is then equal to $d$. $\square$
Corollary 4.8. a) \( \Delta \) is in \( \mathcal{CL}_a^\dagger \) iff \( \tilde{H}_p(\Delta^S_R) \) is zero when \( p, s \) (and \( r \)) are in the range
\[
c \leq p + s \leq d - 2
\]
and also when \( p \) and \( s \) are in the range
\[
p + s = d - 1, \quad p \geq c - a.
\]
b) If \( \Delta \) is in \( \mathcal{CL}_a^\dagger \) then \( \Delta^\ast \) is in \( \mathcal{CL}_a^\dagger \).

Proof. The proofs are analogous to those of Corollary 4.2.

Theorem 4.9. \( \Delta \) is in \( \mathcal{CL}_a^\dagger \) iff every link \( \text{lk}_S \Delta \) with \( s = a \) is in \( \mathcal{CL}_0^\dagger \) with dimension \( d - a - 1 \) and frame dimension \( c - a - 1 \) or is the \( c - a - 1 \) skeleton of the simplex on \( [n] \setminus S \).

Proof. We can assume \( c \geq a \geq 1 \). If now \( \Delta \) is in \( \mathcal{CL}_a^\dagger \), the facets have dimension \( c - 1 \) or \( d - 1 \). In view of Lemma 1.7 the statement follows immediately from Theorem 4.6.

Remark 4.10. The if direction is not true unless we assume the links to be of dimension \( d - a - 1 \) or \( c - a - 1 \). If one simply assume the links are of frame dimension \( c - a - 1 \) and are in \( \mathcal{CL}_a^\dagger \), then a counterexample is given by starting with the disjoint union of a 2-simplex and a 3-simplex and then adding all line segments between pairs of vertices of them.

4.3. The classes \( \mathcal{CL}_a^\circ \). By Proposition 4.4 the nicest behaviour one can expect for the homology groups \( \tilde{H}_p(\text{lk}_S \Delta) \) when \( p + s = c - 1 \) and \( \Delta \) is in \( \mathcal{CL}_a^\dagger \) is
\[
\tilde{H}_{c-a-1}(\text{lk}_S \Delta) = 0, \quad \text{when } s = a.
\]
We now let \( \mathcal{CL}_a^\circ \) be the complexes \( \Delta \) such that \( \Delta \) is in \( \mathcal{CL}_a^\dagger \) and fulfills the condition (11).

(Note that when \( \Delta \) is in \( \mathcal{CL}_{a+1} \) then by Proposition 4.4 \( \tilde{H}_{c-a-1}(\text{lk}_S \Delta) \) is nonzero for \( s = a \), so if \( \Delta \) is in \( \mathcal{CL}_a \) and fulfills (11) then it is not in \( \mathcal{CL}_{a+1} \).)

Two special cases are interesting to take note of.

Example 4.11. If \( \Delta \) in \( \mathcal{CL}_0^\circ \) has \( c = a \), the lowest interesting value for \( c \), then the condition (11) says that \( H_{-1}(\text{lk}_S \Delta) \) is zero for \( s = c = a \). Hence by Theorem 4.7 all facets of \( \Delta \) have dimension \( d - 1 \), and by Proposition 4.3 any two facets intersect in a face of dimension \( \leq c - 2 \). Hence we get precisely the Steiner systems \( S(c, d, n) \). In particular the \( f \)-vector only depends on \( c, d, \) and \( n \).

Example 4.12. When \( \Delta \) is in \( \mathcal{CL}_0^\circ \) the condition (11) says \( \tilde{H}_{c-1}(\Delta) \) is zero and so \( S(\Delta) \) is a torsion sheaf. Then \( \mathcal{E}xt^i(S(\Delta), \omega_{\mathbb{P}^{n-1}}) \) is nonzero only for \( i = d - c \) and so \( S(\Delta) \) is a locally Cohen-Macaulay sheaf. By [12] such \( \Delta \) are bi-Cohen-Macaulay, i.e. both \( \Delta \) and \( \Delta^* \) are Cohen-Macaulay and we
showed that the $f$-vector of such $\Delta$ again only depends on $n, d,$ and $c$. In fact we showed that the $f$-polynomial $f_\Delta(t) = \sum_i f_{i-1} t^i$ is

$$f_\Delta(t) = (1 + t)^{d-c}(1 + (n - d + c)t + \cdots + \left(\frac{n - d + c}{c}\right) t^c).$$

By the following theorem and its corollary this generalizes.

**Theorem 4.13.** $\Delta$ is in $\mathcal{CL}^0_a$ iff each $\mathrm{lk}_\Delta S$ with $s = a$ is bi-Cohen-Macaulay with invariants $n - a$, $d - a$, and $c - a$.

**Proof.** Suppose $\Delta$ is in $\mathcal{CL}^0_a$. The condition \[(\ref{eqn:condition})\] together with Proposition 4.3(a) gives $H_{d-1}(\mathrm{lk}_\Delta S)$ zero when $s = c$. Hence by Proposition 4.3 all facets of $\Delta$ have dimension $d - 1$. This shows that $\mathrm{lk}_\Delta S$ has invariants $n - a, d - a,$ and $c - a$. It is now immediate that $\mathrm{lk}_\Delta S$ fulfills the condition to be in $\mathcal{CL}^0_a$ and this means that $\mathrm{lk}_\Delta S$ is bi-Cohen-Macaulay. In the converse direction it is immediate to see that $\Delta$ fulfills the condition to be in $\mathcal{CL}^0_a$. \[\square\]

**Corollary 4.14.** Suppose $\Delta$ in $\mathcal{CL}^0_a$ has invariants $n, d,$ and $c \geq a$. Then $f_\Delta(t)$ is a polynomial $f_a(n,d,c;t)$ depending only on $a, n, d,$ and $c,$ and is given inductively as follows.

1) $f'_a(n,d,c;t)/n = f_{a-1}(n-1,d-1,c-1;t)$

2) $f_0(n,d,c;t) = (1 + t)^{d-c}(1 + (n - d + c)t + \cdots + \left(\frac{n - d + c}{c}\right) t^c).$

**Proof.** We may assume $a \geq 1$ and assume first that $c \geq 2$. By induction the $f$-polynomial of $\mathrm{lk}_\Delta \{x\}$ is

$$g(t) = f_{a-1}(n-1,d-1,c-1;t)$$

and is independent of $x$. Now if $F$ is any $i$-dimensional face of $\Delta$, then $F \setminus \{x\}$ is an $i - 1$-dimensional face of $\mathrm{lk}_\Delta \{x\}$ for each $x$ in $F$. Therefore $ng_{i-1}$ counts all pairs $(F,x)$ where $x$ is in $F$. But this is also equal to to $f_i(i+1)$ and so

$$ng_{i-1} = f_i(i+1)$$

$$ng(t) = f'_\Delta(t).$$

In case $c = 1$ then $a = 1$ and $\Delta$ is a disjoint union of $n/d$ simplexes of dimension $d - 1$ and so

$$f_\Delta(t) = (n/d)(1 + t)^{d-1} - n/d$$

and so 1) also holds in this case since

$$f_0(n-1,d-1,0;t) = (1 + t)^{d-1},$$

\[\square\]

**Remark 4.15.** The $\Delta$ in $\mathcal{CL}^0_a$ are $a$-$(n,d,\lambda)$ block designs (see [5, Chap.14]), where $\lambda$ is the number of facets of $\mathrm{lk}_\Delta S$ for $S$ any face of cardinality $a$. One may see that $\lambda$ is $\left(\frac{n - d + c - a}{c - a}\right)$.
The following generalizes the fact that when $\Delta$ is in $\mathcal{CL}_0$ then $S(\Delta)$ is a locally Cohen-Macaulay sheaf of codimension $d-c$ and so $\mathcal{E}xt^{d-c}(S(\Delta), \omega_{P^{n-1}})$ is a locally Cohen-Macaulay sheaf.

**Proposition 4.16.** If $\Delta$ is in $\mathcal{CL}_a$, then the only non-vanishing higher $\mathcal{E}xt$-sheaf, $\mathcal{E}xt^{d-c}(S(\Delta), \omega_{P^{n-1}})$, is a locally Cohen-Macaulay sheaf.

**Proof.** For short write $E$ for $\mathcal{E}xt^{d-c}(S(\Delta), \omega_{P^{n-1}})$. The sequence $0 \to C_{\Delta^*} \to E(W) \to (C_{\Delta})^\vee \to 0$ gives an exact sequence $0 \to \tilde{L}(\Delta^*) \to \tilde{L}(E(W)) \to \tilde{L}((C_{\Delta})^\vee) \to 0$.

Hence by (4) and (5), $\tilde{L}(\Delta^*)$ only has possibly non-zero cohomology sheaves $E$ in degree $c + (d - c) - n + 1$ which is $-c^*$, and $H^{-d^*} \tilde{L}(\Delta^*)$ in degree $-d^*$. The condition (11) for $\Delta$, when translated to $\Delta^*$, see (13) and the paranthetical remark after, gives that all generators of $H^{-d^*}L(\Delta^*)$ have degree $\geq n - a + 1$. Since they are of characteristic type (see the proof of Theorem 3.9) a free resolution has length $\leq a - 1$.

Hence $E$ has a locally free resolution of length $\leq d^* - c^* + a$ which is equal to $d - c + a$. By the Auslander-Buchsbaum theorem [3], Thm. 19.9, $E$ then has local depth $\geq (n - 1) - (d - c + a)$. But $\Delta$ being in $\mathcal{CL}_a$, this sheaf has local dimension $\leq (n - 1) - (d - c + a)$. Since local dimension is greater or equal to local depth we must have equalities everywhere and so $E$ is locally Cohen-Macaulay. $\Box$

5. **Hierarchies of Cohen-Macaulay simplicial complexes**

The Alexander duals of CLeray simplicial complexes are the Cohen-Macaulay simplicial complexes. Therefore by taking Alexander duals of the various hierarchies of CLeray complexes, we get hierarchies of Cohen-Macaulay simplicial complexes. This section will contain the Alexander dual versions of most of the statements in Section 4. But we shall also give a description of what it means for a simplicial complex $\Delta$ to be Gorenstein* in terms of the associated coherent sheaf $S(\Delta^*)$. We prove that $\Delta$ is Gorenstein* iff $S(\Delta^*)$ is an ideal sheaf, i.e. a subsheaf of $\mathcal{O}_{P^{n-1}}$. In fact it turns out to be the associated sheaf of the ideal defining the Stanley-Reisner ring of $\Delta^*$.

5.1. **a+1-Cohen-Macaulay simplicial complexes.** Let $\mathcal{CM}_a$ be the class of Cohen-Macaulay simplicial complexes which are Alexander duals of the simplicial complexes in the class $\mathcal{CL}_a$. We shall show that this is exactly the class of $a+1$-Cohen-Macaulay simplicial complexes as defined by Baclawski [4].

**Theorem 5.1.** $\Delta$ is in $\mathcal{CM}_a$ iff $d \leq n - a$ and $\tilde{H}_p(\Delta_{-R})$ is zero when $p + r = d + a - 2$ and $p \leq d - 2$. In particular, if $\Delta$ is not the $n - a - 1$-skeleton of the $n - 1$-simplex, then $d \leq n - a - 1$. 

Proof. By Theorem 4.1 $\Delta^*$ is in $\mathcal{L}_a$ means that $d \leq n - a$ and

\[(12) \quad H_{c^* - a}((\Delta^*)^S_S) = 0 \quad \text{for } s \geq a.\]

Now $(\Delta^*)^S_S$ is equal to $(\Delta_S)^*$ and so by Lemma 1.2 (12) is equivalent to

\[\tilde{H}_{(n-s)+a-c^*+3}(\Delta_S) = 0 \quad \text{for } s \geq a.\]

Since $n - 1 - c^* = d$ we get the statement.

\[\square\]

Corollary 5.2. a) $\Delta$ is in $\mathcal{C}_a$ iff $\tilde{H}_p(\Delta^S_R)$ is zero when $p + s \leq d - 2$ and $p + r \leq d + a - 2$.

b) If $\Delta$ is in $\mathcal{C}_a$, then $\text{lk}_a \Delta S$ is also in $\mathcal{C}_a$.

Proof. This is just the Alexander dual versions of Corollary 4.2 using Lemma 1.2.

\[\square\]

Corollary 5.3. Suppose $\Delta$ has dimension one, i.e. $\Delta$ is a graph. Then $\Delta$ is in $\mathcal{C}_a$ if and only if $\Delta$ contains at least $a + 2$ vertices and is $a + 1$-connected.

Proof. $\Delta$ is in $\mathcal{C}_a$ iff $\tilde{H}_{r-1}(\Delta_S)$ is zero for $r = a + 1$ and $\tilde{H}_0(\Delta_S)$ is zero for $r = a$. This translates precisely to the above.

The following describes the objects of $\mathcal{C}_a$ with upper extremal values for $d$. Recall that a subset $F$ of $[n]$ is a missing face of $\Delta$ if $F$ is not in $\Delta$.

Now if $\Delta$ is in $\mathcal{C}_a$ and not the $n - a - 1$-skeleton of the $n - 1$-simplex then $d \leq n - a - 1$ or equivalently $a \leq n - d - 1$.

Proposition 5.4. $\Delta$ is in $\mathcal{C}_{n-d-1}$ iff the cardinality of $F \cup G$ is $\geq d + 2$ for all distinct missing faces $F$ and $G$.

Proof. $\Delta$ is in $\mathcal{C}_{n-d-1}$ iff $\Delta^*$ is $c^* + 1$-CLeray. If $F$ and $G$ are two distinct missing faces then the complements $F^c$ and $G^c$ are faces of $\Delta^*$ and so the cardinality of $F^c \cap G^c$ is $\leq c^* - 1$. But then the cardinality of $F \cup G$ is $\geq n - c^* + 1$ and this is $d + 2$.

The following shows that $\Delta$ is in $\mathcal{C}_a$ iff $\Delta$ is $a + 1$-Cohen-Macaulay ($a + 1$-CM) as defined by Baclawski.

Theorem 5.5. $\Delta$ is in $\mathcal{C}_a$ iff every restriction $\Delta_{-R}$ with $r = a$ is Cohen-Macaulay of the same dimension as $\Delta$.

Proof. This is just the Alexander dual version of Theorem 4.6.

\[\square\]

From [1] we have the following means of constructing $a + 1$-CM simplicial complexes.

Theorem 5.6 ([1]). If $\Delta$ and $\Delta'$ are $a + 1$-CM simplicial complexes, then the join $\Delta \ast \Delta'$ is also $a + 1$-CM.

In particular, the $\Delta'$ consisting of $m \geq a + 1$ vertices is $a + 1$-CM and so the $m$-point suspension of $\Delta$ will be $a + 1$-CM.

Remark 5.7. If $F_*$ is a free resolution of the Stanley-Reisner ring of $\Delta$, it follows from Theorem 5.4 and Hochsters description of $\text{Tor}_i^S(k[\Delta], k)$, see [17]
II.4.8], that $\Delta$ is $(a + 1)$-CM iff it is Cohen-Macaulay and $F_i = S(-i - d)^{e_i}$ for $n - d \geq i > n - d - a$.

5.2. The classes $\mathcal{CM}_a^\dagger$. Let $\mathcal{CM}_a^\dagger$ be the class of simplicial complexes which are Alexander duals of the simplicial complexes in $\mathcal{CL}_a^\dagger$.

**Theorem 5.8.** $\Delta$ is in $\mathcal{CM}_a^\dagger$ iff $\mathcal{H}_p(\Delta_{-R})$ is zero for all $r$ when $p$ is in the range $c \leq p \leq d - 2$ and also when $p = c - 1$ and $r < d + a - c$.

*Proof.* This is the Alexander dual version of Theorem

**Corollary 5.9.** a) $\Delta$ is in $\mathcal{CM}_a^\dagger$ iff $\mathcal{H}_p(\Delta_{-R})$ is zero for $p, s$ (and $r$) in the range $c \leq p + s \leq d - 2$ and also when $p + s = c - 1$ and $s + r < d + a - c$.

b) If $\Delta$ is in $\mathcal{CM}_a^\dagger$ then $lk_{\Delta} S$ is in $\mathcal{CM}_a^\dagger$.

*Proof.* This is the Alexander dual version of Corollary

**Theorem 5.10.** $\Delta$ is in $\mathcal{CM}_a^\dagger$ iff each restriction $\Delta_{-R}$ with $r = a$ is either in $\mathcal{CM}_a^\dagger$ with the same dimension and frame dimension as $\Delta$ or is the $d - 1$-skeleton of the simplex on $[n]\setminus R$.

*Proof.* This is the Alexander dual version of Theorem

5.3. The classes $\mathcal{CM}_a^\circ$. We let $\mathcal{CM}_a^\circ$ be the class of simplicial complexes which are Alexander duals of the simplicial complexes in $\mathcal{CL}_a^\circ$. If $\Delta$ is in $\mathcal{CM}_a^\circ$ so $\Delta^*$ is in $\mathcal{CL}_a^\circ$ the condition (1) is equivalent to the condition

$$\mathcal{H}_{d-1}(\Delta_{-R}) = 0 \quad \text{when } r = a.$$ (13)

(It then follows by Proposition 4.4 that $\mathcal{H}_{d-1}(\Delta_{-R}) = 0$ for $r \geq a$.) Thus $\Delta$ is in $\mathcal{CM}_a^\circ$ iff it is in $\mathcal{CM}_a^\dagger$ and fulfills (13).

**Example 5.11.** If $\Delta$ is a cyclic polytope of odd dimension, then $\Delta$ is in $\mathcal{CM}_1^\circ$. To see this, note that by Alexander duality for Gorenstein* simplicial complexes [17, p.66]

$$\mathcal{H}_p(\Delta_{-R}) = \mathcal{H}_{d-2-p}(\Delta_R)^*.$$ (14)

Now we can apply Theorem 5.8. For a cyclic polytope of odd dimension, $d = 2c$ and we see that if $c \leq p \leq d - 2$ then $0 \leq d - 2 - p \leq c - 2$ and so (14) is zero. Also $p = c - 1$ gives $d - 2 - p = c - 1$ and $r \leq d + a - c - 1$ is the same as $r \leq c$. Thus (14) is zero and so $\Delta$ is in $\mathcal{CM}_1^\dagger$. That (13) holds is immediate from (14) and so $\Delta$ is in $\mathcal{CM}_1^\circ$.

**Theorem 5.12.** $\Delta$ is in $\mathcal{CM}_a^\circ$ iff every restriction $\Delta_{-R}$ where $r = a$, is bi-Cohen-Macaulay of the same dimension and frame dimension as $\Delta$.

*Proof.* This is the Alexander dual version of Theorem

**Corollary 5.13.** If $\Delta$ is in $\mathcal{CM}_a^\circ$ and $r \leq a$, then $\Delta_{-R}$ is in $\mathcal{CM}_{a-r}^\circ$. 

5.4. Gorenstein* simplicial complexes. We shall now describe where Gorenstein* complexes fit in our scheme. Recall that $\Delta$ is Gorenstein* if $\tilde{H}_p(\operatorname{lk}_x \Delta) = k$ when $p + s = d - 1$ and $S$ a face of $\Delta$, and zero otherwise. The following theorem is certainly well known (except the last statement). It follows for instance from [1 4.7].

**Theorem 5.14.** $\Delta$ is Gorenstein* iff $\Delta$ is 2-CM and $\tilde{H}_{d - 1}(\Delta) = k$, i.e. iff $S(\Delta^*)$ is a torsion free rank one sheaf.

**Proof.** Suppose $\Delta$ is Gorenstein*. By Theorem 5.1 we must show that $\tilde{H}_p(\Delta_{-R})$ is zero when $p + r = d - 1$ and $p \leq d - 2$. By the Alexander duality theorem for Gorenstein* simplicial complexes [17, p.66], $\tilde{H}_p(\Delta_{-R})$ is equal to $\tilde{H}_{d - 2 - p}(\Delta_R)^*$. Hence we need to show that $\tilde{H}_{d - 2 - p}(\Delta_R)^*$ is zero when $r = p + 1$. But this is true simply because the homology of any simplicial complex on $n$ elements always vanishes in degrees $\geq n - 1$.

Now suppose $\Delta$ is in $\mathbb{C}M_1$ with $\tilde{H}_{d - 1}(\Delta)$ equal to $k$. By Corollary 5.2 a) then $\tilde{H}_p(\operatorname{lk}_x \Delta)$ is zero for $p + s \leq d - 2$. We must therefore show in addition that $\tilde{H}_p(\operatorname{lk}_x \Delta)$ is $k$ when $p + s = d - 1$ and $S$ a face of $\Delta$. Since $\operatorname{lk}_x \{x\}$ is in $\mathbb{C}M_1$ it is enough by induction to show that $\tilde{H}_{d - 2}(\operatorname{lk}_x \{x\})$ is $k$ when $\{x\}$ is a face of $\Delta$. But there is a sequence

$$\tilde{H}_{d - 1}(\Delta_{-\{x\}}) \to \tilde{H}_{d - 1}(\Delta) \to \tilde{H}_{d - 2}(\operatorname{lk}_x \{x\}) \to \tilde{H}_{d - 2}(\Delta_{-\{x\}}).$$

Since $\Delta$ is in $\mathbb{C}M_1$, $\tilde{H}_{d - 2}(\Delta_{-\{x\}})$ is zero. Since $\operatorname{lk}_x \{x\}$ is in $\mathbb{C}M_1$, applying Proposition 4.3 b) we get that $\tilde{H}_{d - 2}(\operatorname{lk}_x \{x\})$ is non-zero. And so we must have $\tilde{H}_{d - 2}(\operatorname{lk}_x \{x\})$ equal to $k$. \hfill $\Box$

**Proposition 5.15.** When $\Delta$ is Gorenstein* there is a natural identification of $S(\Delta^*)$ with the ideal defining the Stanley-Reisner ring of the simplicial complex $\Delta^*$. (And so this ideal defines a subscheme of codimension $c + 1$ in $\mathbb{P}^{n - 1}$.)

**Proof.** The paranathetical remark is because the codimension of the subscheme defined by the ideal of the Stanley-Reisner ring of $\Delta^*$ is $n - d^*$ which is $c + 1$.

First we shall suppose $c \geq 1$. Then $\tilde{H}_{d - 1}(\Delta_{-\{x\}})$ by Alexander duality is equal to $\tilde{H}_{-1}(\Delta_{\{x\}})$ which is zero. Hence $\Delta$ fulfills condition [13] and by the last remark in Subsection 5.3 $H^{-d} \tilde{L}(\Delta)$ is equal to $\mathcal{O}_{\mathbb{P}^{n - 1}}(-n)$.

Now by [5] and the sequence

$$0 \to C_\Delta \to E(W) \to (C_{\Delta^*})^\vee \to 0$$

we get

$$\mathcal{H}om(S(\Delta^*), \omega_{\mathbb{P}^{n - 1}}) = H^{-d} \tilde{L}(\Delta) = \mathcal{O}_{\mathbb{P}^{n - 1}}(-n) = \omega_{\mathbb{P}^{n - 1}}.$$

Since $S(\Delta^*)$ is torsion free

$$S(\Delta^*) \hookrightarrow S(\Delta^*)^{\vee\vee} = \mathcal{H}om(\omega_{\mathbb{P}^{n - 1}}, \omega_{\mathbb{P}^{n - 1}}) = \mathcal{O}_{\mathbb{P}^{n - 1}}.$$
Taking graded global sections
\[ S(\Delta^*) = \oplus_{m \in \mathbb{Z}} \Gamma(\mathbb{P}^{n-1}, S(\Delta^*)(m)) \hookrightarrow \oplus_{m \in \mathbb{Z}} \Gamma(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(m)) = S \]
we get \( S(\Delta^*) \) as an ideal in \( S \).

If \( c = 0 \), let \( R \) consist of the \( x \) such that \( \{ x \} \) is a face in \( \Delta \). Let \( r \) be the cardinality of \( R \). In this case \( H^{-d}L(\Delta) \) is \( \mathcal{O}_{\mathbb{P}^{n-1}}(-r) \). Hence \( S(\Delta^*)^{\wedge} \) is \( \mathcal{O}_{\mathbb{P}^{n-1}}(r-n) \). By the inclusion \( \mathcal{O}_{\mathbb{P}^{n-1}}(r-n) \xrightarrow{\Pi_{ig \in R^*}} \mathcal{O}_{\mathbb{P}^{n-1}} \)
we also get \( S(\Delta^*) \) as an ideal sheaf in \( \mathcal{O}_{\mathbb{P}^{n-1}} \) and taking graded global sections we get \( S(\Delta^*) \) as an ideal in \( S \).

Now we want to identify this ideal as the ideal of the Stanley-Reisner ring associated to \( \Delta^* \). By the sequence
\[ 0 \to C_{\Delta^*} \to E(W) \to (C_{\Delta})^\vee \to 0 \]
we get (using (4)) a sequence of complexes
\[ 0 \to L(\Delta^*) \to L(E(W)) \to \text{Hom}_S(L(\Delta), S(-n))[n] \to 0. \]
Now the latter complex is
\[ S(d-n)^{f_{d-1}} \leftarrow S(d-1-n)^{f_{d-2}} \leftarrow \cdots \leftarrow S(-n) \]
where the cohomological degree of the first term is \( d-n \). Taking the long exact cohomology sequence of (15) we get that the only cohomology of (16) is in cohomological degree \( d-n \) and is
\[ H^{d-n+1}L(\Delta^*) = H^{-c^*}L(\Delta^*) = S(\Delta^*). \]
Hence (16) is a resolution of the ideal \( S(\Delta^*) \) in \( S \). Since it is a multi-graded resolution the generators of \( S(\Delta^*) \) in \( S \) will have multidegrees the multidegrees of the indices \( J \) when writing
\[ S(d-n)^{f_{d-1}} = \oplus J S(d-n)u_J. \]
But by the BGG-correspondence we recognise the \( J \)'s as the multidegrees of the facets of \( \Delta \). But this means exactly that \( S(\Delta^*) \) is the ideal of the Stanley-Reisner ring of \( \Delta^* \).

**Remark 5.16.** By the result of [9] the Stanley-Reisner ideal \( I_{\Delta^*} \) in \( S \) of a simplicial complex \( \Delta^* \) has a linear resolution iff \( \Delta \) is Cohen-Macaulay. The same is valid for the the analog \( J_{\Delta^*} \) of the Stanley-Reisner ideal in the exterior algebra \( E \). But this is the same as saying that the ideal \( J_{\Delta^*} \) is a Koszul module over \( E \) (see [10]). The algebras \( E \) and \( S \) are Koszul duals. Via the functors relating (complexes of) modules over them, \( J_{\Delta^*} \) then transfers to a Koszul module over \( S \). The above Proposition 5.15 then shows that this transformed module is an ideal in \( S \) exactly when \( \Delta \) is Gorenstein*.

Summing up, \( \Delta \) is Cohen-Macaulay iff \( J_{\Delta^*} \) in \( E \) is a Koszul ideal (which transfers to a Koszul module), and \( \Delta \) is Gorenstein* iff \( J_{\Delta^*} \) in \( E \) is a Koszul ideal transforming to a Koszul ideal in \( S \).
Remark 5.17. If $\Delta$ is in $\CM_a$ then the link $lk_{\Delta}S$ is also in $\CM_a$. Hence if $S$ is of dimension $d-2$, then $lk_{\Delta}S$ consists of a set of vertices of, being in $\CM_a$, cardinality $\geq a+1$.

The subclass $G_a$ of $\CM_a$ such that this cardinality is always the minimum possible, namely $a+1$, might be a reasonable generalization of Gorenstein* complexes, since $G_1$ would be exactly this class. If $\Delta$ and $\Delta'$ are in $G_a$ then the join $\Delta \ast \Delta'$ is also in $G_a$, in particular the $a+1$-point suspension of $\Delta$ is in $G_a$.

In contrast to the case for Gorenstein* simplicial complexes there does however not seem to be any formula for $\tilde{H}_{d-1}(\Delta)$ for $\Delta$ in $G_a$ for instance in terms of $n$ and $d$.

6. Problems

We pose the following two problems.

**Problem 1.** What are the possible $f$-vectors (or $h$-vectors) of the simplicial complexes in the classes $\CL_a$ and $\CM_a$.

This is likely to be a very difficult problem since any answer also would include an answer to what the $h$-vectors of Gorenstein* simplicial complexes are. However, any conjecture about this would be highly interesting since it would contain as a subconjecture what the $h$-vectors of Gorenstein* simplicial complexes are.

**Problem 2.** Construct simplicial complexes in the classes $\CL_a^\circ$ and $\CM_a^\circ$ for various parameters of $n, d, c$, and $a$.

As has been pointed out this has been done in a number of particular cases. When $a = 0$ we have the bi-Cohen-Maculay simplicial complexes constructed in [12]. When $a = 1$ and $d = 2c$ we have the cyclic polytopes in $\CM_1^\circ$, and when $a = c$, many Steiner systems $S(c, d, n)$ have been constructed, which give simplicial complexes in $\CL_a^\circ$.

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