The modeling of multiphoton ionization by path integral approach

A A Biryukov¹ and Ya V Degtyareva¹

¹Samara National Research University, Moskovskoye shosse 34, Samara, Russia, 443086
e-mail: biryukov_1@mail.ru, degt-yana@yandex.ru

Abstract. The probability of quantum transitions of a molecule between its states under the action of an electromagnetic field is represented as an integral over trajectories of a real alternating functional. A method is proposed for computing the integral using recurrence relations. The method is used to describe the ionization of hydrogen atom.

1. Introduction
Nonlinear processes are actively being studied of interaction of microsystems with laser radiation of various configurations and varying degrees of intensity (excitation and dissociation of molecules, ionization of atoms under the action of laser radiation, etc.). It’s relevant to investigate the probabilities of transitions in multilevel quantum systems outside the perturbation theory and approximations which impose limits on the structure of laser radiation, both in intensity and in the form of pulses.

For this purpose, G. V. Ryazanov [1] proposed to represent the probabilities of quantum transitions by path integrals of an alternating-sign functional.

The theory of representing the probabilities of quantum transitions by integrals along trajectories from the real alternating functional in the energy representation is developed in the articles [2],[3].

However, the calculation of probabilities by using this method is too difficult because of the large dimensions of the integrals in the expressions. Hence, the actual task is to find an efficient method for calculating the proposed integrals.

2. The quantum system model
Consider a quantum system, interacting with an external electromagnetic field. We can restrict a valence electron in an atom or a molecule(or a molecule containing two or more atoms as a system). A quantum system has got a certain discrete energy spectrum. The energy values and the wave functions conforming them are determined by the specific structure of the system under investigation. We describe a system with Hamiltonian $\hat{H}_{\text{sys}}$, the energy level and the state of the system being determined by the equation

$$\hat{H}_{\text{sys}}|n\rangle = E_n|n\rangle, \quad \langle n'|n\rangle = \delta_{nn'}, \quad \sum_n |n\rangle\langle n| = 1, \quad n = 1, 2, ...$$ (1)
where $E_n$ is the energy value in quantum state $|n\rangle$, $\hat{H}_{syst}$ is the system Hamiltonian that does not explicitly depend on time $t$. Note: taking Eq. (1) into account we can represent the Hamiltonian of the system in the form

$$\hat{H}_{syst} = \sum_n E_n |n\rangle \langle n|.$$  

(2)

The electromagnetic field effect on a quantum system is characterized by operator $\hat{V}_{inf}(t)$:

$$\hat{V}_{inf}(\tau) = qx E_0 V(\tau),$$  

(3)

where $q$ is the particle charge, $x$ is the particle coordinate, $E_0$ is the electromagnetic field amplitude, $V(\tau)$ is the scalar function of the time $\tau$-dependence field. It is convenient to represent Eq. (3) in the form

$$\hat{V}_{inf}(\tau) = \sum_{n',n} \hbar \Omega_{n'n}^{R} E_0 V(\tau) |n\rangle \langle n|,$$  

(4)

where

$$\Omega_{n'n}^{R} = \frac{qx_{n'n} E_0}{\hbar},$$  

(5)

is Rabi frequency; $x_{n'n} = |\langle n'|\hat{x}|n\rangle|$ is the absolute value of the matrix element of the coordinate operator.

The total Hamiltonian of the system has the form:

$$\hat{H} = \hat{H}_{syst} + \hat{V}_{inf}(t).$$  

(6)

We will describe the state of the system at each instant of time by statistical operator $\hat{\rho}(t)$ governed by Dirac equation

$$\dot{\hat{\rho}}(t) = \hat{U}_D(t) \hat{\rho}(0) \hat{U}_D^+(t),$$  

(7)

where $\hat{\rho}(0)$ is the statistical operator at the initial time, the evolution operator being represented as

$$\hat{U}_D(t,t_0) = T \exp\left[-i \frac{t}{\hbar} \int_{t_0}^{t} \hat{V}_D(\tau) d\tau\right],$$  

(8)

where

$$\hat{V}_D(\tau) = \exp\left[i \frac{\tau}{\hbar} \hat{H}_{syst}\tau\right] \hat{V}_{inf}(\tau) \exp\left[-i \frac{\tau}{\hbar} \hat{H}_{syst}\tau\right].$$  

(9)

Using the expressions (4), (5), the interaction operator (9) is represented in a form that is convenient for the evolution operator, concretizing:

$$\hat{V}_D(\tau) = \sum_{n',n} X_{n'n'} |n\rangle \langle n|,$$  

(10)

where

$$X_{n'n'}(\tau) = \hbar \Omega_{n'n}^{R} V(\tau) \exp[i \omega_{n'n} \tau],$$  

(11)

$\omega_{n'n}$ is the quantum transition frequency of the system between stationary states with energies $E_{n'}$ and $E_n$:

$$\omega_{n'n} = \frac{E_{n'} - E_n}{\hbar}.$$  

(12)

We use the energy representation of the evolution equation for the statistical density matrix (7):

$$\rho_{n'm'}(t) = \sum_{n_0,m_0} \langle n'|\hat{U}_D(t)|n_0\rangle \rho_{n_0,m_0} \langle m_0|\hat{U}_D^+(t)|m_f\rangle,$$  

(13)
where

\[ \rho_{nf,mf}(t) = \langle nf|\hat{\rho}(t)|mf \rangle, \quad \rho_{n0,m0} = \langle n0|\hat{\rho}(0)|m0 \rangle. \tag{14} \]

Vectors \( |n\rangle, |m\rangle \) are eigenvectors of system Hamiltonian \( H_{\text{sys}} \).

Using the group properties of operator \( \hat{U}_D \) and completeness of state vectors \( |n_k\rangle \) we can present the evolution operator kernel \( \langle nf|\hat{U}_D(t,0)|n0\rangle \) as the product of elementary kernels:

\[ \langle nf|\hat{U}_D(t,0)|n0\rangle = \sum_{n_1,..,n_K=1}^{N} \prod_{k=1}^{K+1} \langle n_k|\hat{U}_D(t_k,t_{k-1})|n_{k-1}\rangle, \tag{15} \]

where \( t_{K+1} = t, \ n_{K+1} = n_f, \ t_0 = 0 \) and

\[ \langle n_k|\hat{U}_D(t_k,t_{k-1})|n_{k-1}\rangle = \langle n_k|\exp\left[-\frac{i}{\hbar} \int_{t_{k-1}}^{t_k} \hat{V}_D(\tau) d\tau\right]|n_{k-1}\rangle, \tag{16} \]

with \( t_k > t_{k-1} \). It is proved in paper [1] that for small time intervals \((t_k - t_{k-1} \rightarrow 0)\) the evolution operator kernels \( \langle n_k|\hat{U}_D(t_k,t_{k-1})|n_{k-1}\rangle \) have the form

\[ \langle n_k|\hat{U}_D(t_k,t_{k-1})|n_{k-1}\rangle = \int_{0}^{1} \exp[-i\Delta S[n_k,t_k;n_{k-1},t_{k-1};\xi_{k-1}]]d\xi_{k-1}, \tag{17} \]

where

\[ \Delta S[n_k,t_k;n_{k-1},t_{k-1};\xi_{k-1}] = 2\pi(n_k - n_{k-1})\xi_{k-1} + 2\Omega^R_{n_k,n_{k-1}}V(t_k)(\cos(2\pi(n_k - n_{k-1})\xi_{k-1} + \omega_{n_k,n_{k-1}}t_k)(t_k - t_{k-1}). \tag{18} \]

is the dimensionless action (in terms of \( \hbar \)) in the energy representation.

Taking into account Eq. (17), the total transition amplitude (15) takes the form

\[ \langle nf|\hat{U}_D(t,0)|n0\rangle \sum_{n_1,..,n_K=1}^{N} \int_{0}^{1} \int_{0}^{1} \exp[-iS[n_f,n_K,\xi_K;..;n_k,n_{k-1},\xi_{k-1};..;n_1,n_0,\xi_0]]d\xi_0..d\xi_K \tag{19} \]

where action \( S \) is a functional on the continuum of trajectories defined in the discrete space of variables \( n_k \), the dimensions of which are determined by the number of quantum levels of the system under study, and the continuous variable bounded by \([0,1]\) space of real variables \( \xi_k \):

\[ S[n_f,n_K,\xi_K;..;n_k,n_{k-1},\xi_{k-1};..;n_1,n_0,\xi_0] = \sum_{k=1}^{K+1} \Delta S[n_k,n_{k-1},\xi_{k-1}] \tag{20} \]

with \( t_{K+1} = t, \ n_{K+1} = n_f, \ t_0 = 0 \).

Taking into account Eq. (19), we write the evolution equation for density matrix (13) in the form:

\[ \rho_{mf,mf}(t) = \sum_{n_0,..,n_K} \sum_{m_0,..,m_K} \int_{0}^{1} \int_{0}^{1} d\xi_0..d\xi_K d\xi_0..d\xi_K \rho_{n_0,m_0}(0) \times \]

\[ \times \exp[-i\{S[n_f,n_K,\xi_K;..;n_k,n_{k-1},\xi_{k-1};..;n_1,n_0,\xi_0]|S[m_f,m_K,\xi_K;..;m_k,m_{k-1},\xi_{k-1};..;m_1,m_0,\xi_0]|0\}]. \tag{21} \]

where functional \( S[m_f,m_K,\xi_K;..;m_1,m_0,\xi_0] \) has the same structure as well as \( S[n_f,n_K,\xi_K;..;n_1,n_0,\xi_0] \) with replace \( n_i, \xi_i \) on \( m_i, \zeta_i \).
The quantum transition probability from a pure quantum state $\hat{\rho}(0) = |n_i\rangle\langle n_i|$ with

$$\rho_{n_0, m_0}(0) = \delta(n_0 - n_i)\delta(m_0 - n_i)$$

at initial time $t = 0$ to the final pure quantum state $|n_f\rangle$ at time $t$ according to expression (2) has the form

$$P(n_f, t|n_i, 0) = \sum_{n_1, \ldots, n_K} \sum_{m_1, \ldots, m_K} \int_0^1 \frac{d\xi_0 \cdots d\xi_K d\xi_0 \cdots d\xi_K}{0} \times \exp[-i(S[n_f, \ldots, n_i; \xi_K, \ldots, \xi_0] - S[n_f, \ldots, m_i, \ldots, n_i; \xi_K, \ldots, \xi_0])]. \quad (22)$$

In article [2] it is proved that the quantum transition probability (2) can be represented as a functional integral with a real integrand:

$$P(n_f, t_f|n_i, 0) =$$

$$= A_n \sum_{n_1, \ldots, n_K} \sum_{m_1, \ldots, m_K} \int_0^1 \frac{1}{0} \cos[\sum_{k=1}^{K+1} \Delta S(n_k, n_k-1; m_k, m_k-1; \xi_k-1)] d\xi_0 d\xi_K d\xi_0 d\xi_K, \quad (23)$$

where $A_n$ is a constant that preserves the normalization condition $\sum_{n_f} P(n_f, t_f|n_i, 0) = 1$,

$$\Delta S(n_k, n_k-1; m_k, m_k-1; \xi_k-1) = \Delta S(n_k, n_k-1, t_k-1; \xi_k-1) - \Delta S(m_k, m_k-1, t_k-1; \xi_k-1), \quad (24)$$

actions $\Delta S(n_k, n_k-1, t_k-1; \xi_k-1)$, $\Delta S(m_k, m_k-1, t_k-1; \xi_k-1)$ take form (18).

Formulas (18) and (23) – (24) definitely represents the quantum transition probability. However, analytical calculation of high-dimensional sums and integrals in formula (23) for real models is impossible. Because of this the calculation of transition probabilities by formula (23) may be implemented numerical simulations only.

### 3. Algorithm for calculating the functional integral

The quantum transitions probability calculation by numerical methods using formula (23) leads to correct results in agreement with the experiment. However, calculations can be carried out only for small time intervals $t$. For large intervals $t$ the requirements for computer resources and computational errors rise sharply. Therefore, for numerical calculations of probabilities in accordance with formula (23) a recurrence relation method is proposed that will reduce the counting time for large $t$ and improve the accuracy of calculations.

To construct an algorithm for the numerical calculation of the probability of a quantum transition we introduce a probability function represented by the expression

$$\tilde{P}_{\cos}(n_{K+1}, m_{K+1}, t_{K+1}; n_i, t_0) = \tilde{R}_{K} \cdots \tilde{R}_0 \{A^{-1} \cos[\sum_{k=1}^{K+1} \Delta S(n_k, n_k-1; m_k, m_k-1; \xi_k-1)]\}, \quad (25)$$

where operators $\tilde{R}_k$ take form:

$$\tilde{R}_k = \sum_{n_k=1}^{N} \sum_{m_k=1}^{N} \int_0^1 \frac{d\xi_k}{0} \int_0^1 \frac{d\xi_k}{0}, \quad k = 1, 2, \ldots, \quad (26)$$
The IV International Conference on Information Technology and Nanotechnology
IOP Publishing
IOP Conf. Series: Journal of Physics: Conf. Series 1096 (2019) 012145
doi:10.1088/1742-6596/1096/1/012145

The transition probability between states \(|n_{in}\rangle\) and \(|n_{K+1}\rangle\) is obtained from function (25) with \(m_{K+1} = n_{K+1}\):

\[
P(n_{K+1}, t_{K+1}; n_{in}, 0) = \tilde{P}_{\cos}(n_{K+1}, m_{K+1}, t_{K+1}; n_{in}, 0)\delta_{n_{K+1}m_{K+1}},
\]

where \(\delta_{n_{K+1}m_{K+1}}\) is Kronecker symbol.

In formula (25) we expand the cosine of the sum by the recurrence formula, that is, we represent it in the form (for simplicity, we omit the normalization factor):

\[
\begin{align*}
\tilde{P}_{\cos}(n_{K+1}, m_{K+1}, t_{K+1}; n_{in}, 0) &= \hat{R}_K \hat{R}_0 \times \\
&\times \{ \cos[\Delta S(n_{K+1}, n_K, \xi_K, m_{K+1}, m_K, \zeta_K)] \cos[\sum_{k=1}^{K} \Delta S(n_k, n_{k-1}, \xi_k, m_k, m_{k-1}, \zeta_k)] - \\
&- \sin[\Delta S(n_{K+1}, n_K, \xi_K, m_{K+1}, m_K, \zeta_K)] \sin[\sum_{k=1}^{K} \Delta S(n_k, n_{k-1}, \xi_k, m_k, m_{k-1}, \zeta_k)] \}.
\end{align*}
\]

Taking into account the properties of operators \(\hat{R}_0, \ldots, \hat{R}_K\) equation (28) may be represented in the form

\[
\begin{align*}
\tilde{P}_{\cos}(n_{K+1}, m_{K+1}, t_{K+1}; n_{in}, 0) &= \\
&= \hat{R}_K \cos[\Delta S(n_K, n_{K-1}, \xi_{K-1}, m_{K-1}, \zeta_{K-1})] \hat{R}_0 \cos[\sum_{k=1}^{K} \Delta S(n_k, n_{k-1}, \xi_k, m_k, m_{k-1}, \zeta_k)] - \\
&- \hat{R}_K \sin[\Delta S(n_K, n_{K-1}, \xi_{K-1}, m_{K-1}, \zeta_{K-1})] \hat{R}_0 \sin[\sum_{k=1}^{K} \Delta S(n_k, n_{k-1}, \xi_k, m_k, m_{k-1}, \zeta_k)].
\end{align*}
\]

We introduce the auxiliary probability functions in Eq. (29)

\[
\begin{align*}
\tilde{P}_{\cos}(n_K, m_K, t_K; n_{in}, t_0) &= \hat{R}_{K-1} \ldots \hat{R}_0 \cos[\Delta S(n_K, n_{K-1}, \xi_{K-1}, m_K, m_{K-1}, \zeta_{K-1})], \\
\tilde{P}_{\sin}(n_K, m_K, t_K; n_{in}, t_0) &= \hat{R}_{K-1} \ldots \hat{R}_0 \sin[\Delta S(n_K, n_{K-1}, \xi_{K-1}, m_K, m_{K-1}, \zeta_{K-1})].
\end{align*}
\]

Then (29) takes the form:

\[
\begin{align*}
\tilde{P}_{\cos}(n_{K+1}, m_{K+1}, t_{K+1}; n_{in}, t_0) &= \\
&= \hat{R}_K \{ \cos[\Delta S(n_{K+1}, n_K, \xi_K, m_{K+1}, m_K, \zeta_K)] \tilde{P}_{\cos}(n_K, m_K, t_K; n_{in}, t_0) \} - \\
&- \hat{R}_K \{ \sin[\Delta S(n_{K+1}, n_K, \xi_K, m_{K+1}, m_K, \zeta_K)] \tilde{P}_{\sin}(n_K, m_K, t_K; n_{in}, t_0) \}
\end{align*}
\]

Formula (32) is a recurrence relation between \(\tilde{P}_{\cos}(n_{K+1}, n_{K+1}, t_{K+1}; n_{in}, t_0)\) and functions defined on a space of variables of lesser dimension \(\tilde{P}_{\cos}(n_K, m_K, t_K; n_{in}, t_0), \tilde{P}_{\sin}(n_K, m_K, t_K; n_{in}, t_0)\). For calculating the functions \(\tilde{P}_{\cos}(n_k, m_k, t_k; n_0, t_0), \tilde{P}_{\sin}(n_k, m_k, t_k; n_0, t_0)\) on step \(k\) we use similar recurrence relations:

\[
\begin{align*}
\tilde{P}_{\cos}(n_k, m_k, t_k; n_0, t_0) &= \\
&= \hat{R}_{k-1} \{ \cos[\Delta S(n_k, n_{k-1}, \xi_{k-1}, m_k, m_{k-1}, \zeta_{k-1})] \tilde{P}_{\cos}(n_{k-1}, m_{k-1}, t_{k-1}; n_{in}, t_0) \} - \\
&- \hat{R}_{k-1} \{ \sin[\Delta S(n_k, n_{k-1}, \xi_{k-1}, m_k, m_{k-1}, \zeta_{k-1})] \tilde{P}_{\sin}(n_{k-1}, m_{k-1}, t_{k-1}; n_{in}, t_0) \}
\end{align*}
\]
To calculate the probability functions according to the proposed scheme, it is necessary to specify the initial conditions. At initial time \( t = 0 \) the system is in the state \( |n_{in}\rangle \), therefore

\[
\overline{P}_{\text{cos}}(n_{in}, m_{in}, 0; n_{in}, 0) = 1, \quad \overline{P}_{\text{sin}}(n_{in}, m_{in}, 0; n_{in}, 0) = 0.
\]

Further calculations are carried out according to the scheme indicated.

Calculating the values of \( \overline{P}_{\text{cos}}(n_K, m_K, t_K; n_{in}, 0) \) and \( \overline{P}_{\text{sin}}(n_K, m_K, t_K; n_{in}, 0) \) and normalization constant \( A \), we find the probability of a quantum transition from state \( |n_{in}\rangle \) to state \( |n_K\rangle \):

\[
P(n_K, t_K; n_{in}, 0) = \overline{P}_{\text{cos}}(n_K, m_K, t_K; n_{in}, 0) \delta_{n_{in}+1} m_{K+1}.
\]

4. The modeling of multiphoton ionization hydrogen atom

By means of the proposed numerical method we will describe the non-linear process: quantum transitions in the hydrogen atom under the laser radiation action.

Let us consider the hydrogen atom with a Hamiltonian:

\[
\hat{H} = -\frac{\hbar^2}{2m} \overleftarrow{\nabla}^2 - \frac{e^2}{r},
\]

where \( m \) - the reduced mass of the electron, \(-\frac{e^2}{r}\) - potential energy of interaction of an electron with a nucleus, \( e \) - absolute value of electron charge. The energy of an electron \( E_n \) and its state functions \( \Psi_{nlm}(r, \theta, \phi) \) are defined by equation:

\[
\hat{H} \Psi_{nlm}(r, \theta, \phi) = E_n \Psi_{nlm}(r, \theta, \phi),
\]
where

\[ E_n = -\frac{e^4 m}{2\hbar^2} n^2, \quad n = 1, 2, \ldots \]  

(39)

The electron energy has negative values. The energy spectrum is degenerate. The energy level corresponding to ionization of the atom is \( E_{\text{ion}} = 0 \). It is determined from condition \( \lim_{n \to \infty} E_n = E_{\text{ion}} \).

For numerical calculations, it is convenient to start the counting of the electron energy from the first energy level:

\[ E_1 = -\frac{e^4 m}{2\hbar^2} \]  

(40)

The energy of an electron \( E'_n \) will be determined by expression:

\[ E'_n = E_n - E_1 \]  

(41)

The ionization energy of a hydrogen atom is determined by the expression:

\[ E'_{\text{ion}} = |E_1| = \frac{e^4 m}{2\hbar^2} = 13.55\text{eV}. \]  

(42)

The frequencies of quantum transitions are determined by expression:

\[ \omega_{n'n} = \frac{1}{\hbar} (E'_{n'} - E_n). \]  

(43)

We put the atom in an alternating electromagnetic field. The vector of field strength changes according to the law:

\[ \overrightarrow{E}(t) = \overrightarrow{E}_0 \cos(\omega_{21}t). \]  

(44)

The Rabi frequency for quantum transitions is given by:

\[ \Omega_{n'n} = \frac{1}{\hbar} e |r_{n'n}||\overrightarrow{E}_0|. \]  

(45)

where

\[ r_{n'n} = \int_0^\infty R_{n'l+1}(r)r^3 R_{n'l}(r)dr \]  

(46)

\( R_{n'l}(r) \) - radial functions of quantum states.

Under the influence of an electromagnetic field, an electron in an atom passes between energy levels.

Consider the atom of level 4:

\[ E'_1 = 0, E'_2 = 10.15\text{eV}, E'_3 = 12.0\text{eV}, E'_4 = E'_{\text{ion}} = 13.55\text{eV}. \]  

(47)

Let the electrons at the initial instant of time be at the first level with energy \( E'_1 = 0 \). Using the proposed method, we calculate the probabilities of the electron transition to the levels \( E'_2 \), \( E'_{\text{ion}} \) as a function of time and magnitude of the amplitude of the strength \( E \) of the alternating electromagnetic field. The results of numerical calculations are presented on Fig.1 and Fig.2.

We interpret the mathematical modeling performed on the basis of the photon model of electromagnetic radiation. A single-mode electromagnetic wave of frequency \( \omega_{21} \) is a flux of photons. Each photon has an energy \( \hbar \omega_{21} = E_2 - E_1 = 10.15\text{eV} \). The density of the photon flux is proportional to the amplitude of the field strength \( E \). The interaction of one photon with an electron of the atom excites it to the level \( E_2 = 10.15\text{eV} \). Graphs of the probability for this one-photon process are presented in Fig.1 as functions of the time and amplitude of the field.
Figure 1. The transition probability of the electron of the hydrogen atom from the first level $E'_1 = 0$ to the second level $E'_2 = 10.15 eV$ as a function of the time of interaction with the field and the amplitude of the field strength $E = 1 eV, 3 eV, 5 eV$.

Figure 2. The probabilities of electron transition to the ionization level $E'_{ion} = 13.55 eV$ as a function of the time $t$ of interaction with the field and the amplitude of the field intensity $E = 1 eV, 3 eV, 5 eV$.

One photon cannot increase the electron energy to the ionization level of the atom $E_{ion} = 13.55 eV$.

When interacting with an electron simultaneously two, three, $n$ photons, their total energy $n \hbar \omega_{21} (n = 2, 3, ..)$ is sufficient to excite the electron to the level of ionization. Graphs of the probability of multiphoton processes are presented in Fig.2, as functions of the time and amplitude of the field strength. Analysis of the graphs shows that the probability of multiphoton ionization of an atom increases with increasing amplitude $E$ of the electromagnetic field strength.

From the analysis of the graphs on Fig.1 it follows that as the amplitude of the electromagnetic field increases, the probability of one-photon excitation of the electron to the level $E'_2$ decreases. This is due to the fact that multiphoton processes of electron excitation appear.

The proposed method of digital simulation of multiphoton ionization of atoms and molecules is in agreement with experimental and alternative theoretical methods of research [5]. The probability of multiphoton ionization of an atom depends on the quantum state of the electron before ionization and after ionization, on the frequency and amplitude of the intensity of the electromagnetic wave interacting with the atom.

A general regularity in all experiments of multiphoton ionization is a nonlinear increase in the ionization probability with increasing amplitude of the electromagnetic field intensity. This
pattern is observed in the proposed model for describing multiphoton ionization, as can be seen from the analysis of the graphs in Fig.2. A detailed description of empirical regularities requires more detailed exposition.

5. Conclusions
The obtained results show that the constructed method of modeling quantum transitions successfully describes multiphoton processes. Numerical methods made it possible, in agreement with experiment, to describe multiphoton ionization of an atom and two-photon oscillations of Rabi [6]. The main advantage of the method proposed consists in a relatively low resource intensity, which allows it to be used for solving more complex problems.

6. References
[1] Ryazanov G V 1958 JETP 35(1) 121
[2] Biryukov A A and Shleenkov M A 2012 Teoreticheskaia fizika 13 8
[3] Biryukov A A and Shleenkov M A 2015 Samara State Technical Univ. 19 221
[4] Landau L.D. and Lifshitz E.M. 1991 Quantum Mechanics: Non-Relativistic Theory. Oxford; Boston: Butterworth - Heinemann
[5] Delone N.B. 1975 Usp. Phys. Sci 115 361
[6] Biryukov A A, Degtyarova Ya v and Shleenkov M A 2018 SPIE Proc 10717 107170S