HIGHER HOMOTOPY STRUCTURE OF GINZBURG ALGEBRAS

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Abstract. We compute the minimal model for Ginzburg algebras associated to acyclic quivers $Q$. In particular, we prove that the Ginzburg algebra is formal and quasi-isomorphic to the preprojective algebra in non-Dynkin type, and in Dynkin type is quasi-isomorphic to a twisted polynomial algebra over the preprojective with a unique higher $A_\infty$-composition. To prove these results, we construct and study an $A_\infty$-category containing the bounded derived category $\mathcal{D}b(Q)$ whose higher compositions encode the triangulated structure of $\mathcal{D}b(Q)$.

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Introduction

Since their introduction in the thesis of Verdier [28], triangulated categories have become central objects of interest in representation theory, topology, geometry, and mathematical physics, among others: Kontsevich’s homological mirror symmetry conjecture phrases the phenomenon of mirror symmetry in terms of equivalences of triangulated categories [18]; the cluster categories of Amiot and Buan, Marsh, Reiten, Reineke, and Todorov are certain triangulated categories encoding the tilting theory of algebras [1, 6]; when applied to a certain algebra constructed from surface triangulations, these cluster categories have applications to Teichmüller theory, and potentially to characteristic classes of surface bundles [8, 24].

The categories above have more structure in common than just being triangulated. They are instances of Calabi-Yau categories in the sense that there bifunctorial isomorphisms

$$\mathcal{F}(X, Y) \cong D\mathcal{F}(Y, X[n])$$

where $D$ denotes $k$-linear dual, and $[1]$ is the suspension functor of $\mathcal{F}$. Here, $n$ is a fixed integer called the CY-dimension of $\mathcal{F}$.

A large source of such categories comes from algebraic geometry. For an $n$-dimensional Calabi-Yau variety $X$, Serre duality implies that

$$\mathcal{D}b(X)(F, G) \cong D\mathcal{D}b(X)(G, F[n])$$

in the derived category of coherent sheaves $\mathcal{D}b(X)$. Hence, $\mathcal{D}b(X)$ is an example of a Calabi-Yau category. This example can be extended by adopting the viewpoint of non-commutative geometry. A theorem of Bondal and Van den Bergh asserts that under mild assumptions, the derived category of a coherent sheaves on a quasi-projective variety is equivalent to the derived category of dg modules over some dg algebra [4].

This suggests two avenues:

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(1) Describe the class of “Calabi-Yau” dg algebras, i.e., those whose derived categories are Calabi-Yau categories.

(2) Compute the triangulated structure of the corresponding Calabi-Yau category.

For the first of these problems, Ginzburg introduced the notion of a Calabi-Yau dg algebra [10], as well as a general construction that associates a 3-Calabi-Yau algebra $\Gamma(Q,W)$ to any quiver with potential $(Q,W)$ in the sense of Derksen, Weyman, and Zelevinsky [7]. These Ginzburg algebras are also used by Amiot to construct cluster categories, and in particular can be used to construct cluster categories associated to surfaces.

The second problem is perhaps best understood through the language of $A_\infty$-categories. A theorem of Keller shows that most triangulated categories arise as the homology of a dg category [15]. More precisely, he shows that for a given triangulated category $\mathcal{T}$, there is a dg category $\mathcal{A}$ and an equivalence

$$\mathcal{T} \cong H_*\mathcal{A}$$

where $\mathcal{T}$ is a graded category with the same objects as $\mathcal{A}$ and morphism spaces

$$\mathcal{T}(X,Y) = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}(X,S^nY).$$

Keller’s result in some sense says that the triangulated structure of $\mathcal{T}$ is the shadow of the richer dg structure of $\mathcal{A}$ which remains after passing to homology. By the now well established theory of homotopy perturbation, the dg structure of $\mathcal{A}$ can be recovered by computing a certain $A_\infty$-structure of $\mathcal{T}$. One should be able to compute this $A_\infty$-structure by using the triangulated structure of $\mathcal{T}$, and this $A_\infty$-structure should further elucidate the “higher” structure of $\mathcal{T}$ contained in $\mathcal{A}$.

Results. In this dissertation, we study the $A_\infty$-structure of the Ginzburg algebras associated to acyclic quivers and in particular compute their minimal models. Our main technical result is the following:

**Theorem** (Theorem 4.8.3). If $Q$ is acyclic, there is a quasi-isomorphism of algebras

$$\Gamma(Q) \to \bigoplus_{n \geq 0} \mathcal{P}_{dg}(Q)(kQ,\tau^{-n}kQ)$$

where $\mathcal{P}_{dg}(Q)$ is the (dg) category of bounded complexes of projective $kQ$-modules and $\tau$ is the Auslander-Reiten translate.

This theorem is proven by constructing a differential graded Galois $\mathbb{Z}$-covering $\tilde{\Gamma}(Q)$ of the algebra $\Gamma(Q)$, together with a quasi-fully-faithful functor $R: \mathcal{C}_{\tilde{\Gamma}(Q)} \to \mathcal{P}_{dg}(Q)$ from the path category of $\tilde{\Gamma}(Q)$. In Section 2 we describe how to recover $\Gamma(Q)$ in terms of the cover $\tilde{\Gamma}(Q)$, providing the quasi-isomorphism of the Theorem.

The derived category $\mathcal{D}^b(Q)$ can be viewed as the zeroth homology of the dg category $\mathcal{P}_{dg}(Q)$, and the total homology category $H_*\mathcal{P}_{dg}(Q)$ is equivalent to $\mathcal{D}^b(Q)^{\mathbb{Z}}$. This determines a minimal $A_\infty$-structure on the category $\mathcal{D}^b(Q)^{\mathbb{Z}}$ by homotopy perturbation. The theorem above then reduces the computation of the minimal model of $\Gamma(Q)$ to the computation of the $A_\infty$-structure of the category $\mathcal{D}^b(Q)^{\mathbb{Z}}$.

**Theorem** (Theorem 3.8.2). If $Q$ is acyclic, the $A_\infty$-structure $(\mu_n)_{n \geq 2}$ has the following properties:

1. the compositions $\mu_n$ are equivariant with respect to the canonical maps $s_X: X \to X[1],$
2. if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a triangle in $\mathcal{D}^b(Q)$ with $X$ indecomposable, then $\mu_3(h,g,f) = s_X$, 
3. $\mu_n = 0$ for $n \neq 2, 3$.

The algebra $\bigoplus_{n \geq 0} \mathcal{D}^b(Q)^{\mathbb{Z}}(kQ,\tau^{-n}kQ)$ has $kQ$ as a subalgebra, and up to a grading shift splits as a direct sum into all indecomposable preprojective objects in $\mathcal{D}^b(Q)$. Combining the above theorems with this observation allows one to compute the minimal model of $\Gamma(Q)$ in terms of the preprojective algebra $\Pi(Q)$. The result depends on if $Q$ is an orientation of a simply-laced Dynkin diagram or not. In the non-Dynkin setting, we recover the following theorem due to Keller [17].

**Theorem** (Corollary 4.8.4). If $Q$ is non-Dynkin and $\mathcal{D}^b(Q)$ is directed, then $\Gamma(Q)$ is formal, and the homology $H_*\Gamma$ is isomorphic to the preprojective algebra $\Pi(Q)$.
In the Dynkin setting, the minimal model is more subtle. Here the underlying associative algebra of the minimal model is a polynomial algebra over \( \Pi(Q) \) twisted by an automorphism (see Section 5.2). This is established by constructing an isomorphism

\[
\Pi(Q)\eta[u] \xrightarrow{\sim} \bigoplus_{n \geq 0} \mathcal{D}^b(Q)^\mathbb{Z}(kQ, \tau^{-n}kQ)
\]

via covering theory. Combining with the \( A_\infty \)-structure theorem for \( \mathcal{D}^b(Q)^\mathbb{Z} \) we obtain the following.

**Theorem** (Theorem 5.3.1). If \( Q \) is Dynkin, then there is an \( A_\infty \)-structure \((\mu_n)_{n \geq 2}\) on the twisted polynomial algebra \( \Pi(Q)^\eta[u] \) making it a minimal model of \( \Gamma(Q) \). Moreover, this \( A_\infty \)-structure is \( u \)-invariant, generates \( \Pi(Q)^\eta[u] \) over \( \Pi(Q) \), and \( \mu_n = 0 \) for \( n \neq 2,3 \).

The higher composition \( \mu_3 \) can be explicitly computed using the triangulated structure of \( \mathcal{D}^b(Q) \) for \( Q \) Dynkin.

**Detailed contents.** We now outline the contents section by section.

In Section 1 we recall preliminary facts about graded, differential graded, and \( A_\infty \)-algebras and categories, and their associated homologies. We introduce the algebra associated to a small category, and show that the associated algebra admits a graded, differential graded, or \( A_\infty \)-structure for such a category; we recall the functorial interpretation of modules over the associated algebra of a category, and compatibility between various homological considerations. Next we discuss minimal models of differential graded algebras and categories, and Kadeishvili’s Theorem, and give an explicit formula for such minimal \( A_\infty \)-structures in terms of planar binary rooted trees using the Homotopy Transfer Theorem. In particular, we give an explicit combinatorial formula for the signs appearing in the tree formula.

In Section 2 we develop the relationship between small \( k \)-categories and associative algebras to accommodate differential graded and \( A_\infty \)-structures. Moreover, we also modify the covering theory of Bongartz and Gabriel [5] to accommodate differential graded and \( A_\infty \)-structures using ideas of Bautista and Liu [3]. In particular, we show compatibility between coverings and homology. Finally, we introduce the notion of a section of a covering to give an interpretation of the algebra of a small category in terms of the algebra of a covering.

In Section 3 we study derived categories considered as the zeroth homology of dg categories. After some preliminaries on graded categories constructed from ordinary categories equipped with automorphism, we show that the derived category of an abelian category is the degree 0 component of an \( A_\infty \)-category generated over degree 0 (cf., Definition 3.2.1). Using this description, we compute the structure of this \( A_\infty \)-category, and prove Theorem 3.8.2.

In Section 4 we introduce Ginzburg algebras. After recalling some preliminary facts about the Auslander-Reiten theory of the derived category \( \mathcal{D}^b(Q) \), we prove Theorem 4.8.3 and Corollary 4.8.4 using Theorem 3.8.2 and the theory developed in Section 2.

In Section 5 we prove Theorem 5.3.1 after developing some theory about twisted polynomial algebras. We describe the Weyl involution of a Dynkin preprojective algebra.

1. Differential graded and \( A_\infty \)-categories

Fix a ground field \( k \), which we will assume to be algebraically closed and characteristic 0. Throughout, all categories are assumed to be \( k \)-categories, all functors are \( k \)-linear, and all unadorned tensor products are over the ground field \( k \). For a category \( \mathcal{C} \) and two objects \( X \) and \( Y \), denote by \( \mathcal{C}(X,Y) \) the space of morphisms from \( X \) to \( Y \) in \( \mathcal{C} \).

1.1. Differential graded categories. We will need to deal with a variety of constructions and sign conventions on graded spaces, and so to fix notation, we recall the theory of graded algebras and modules.

Recall that a graded \( k \)-vector space is a \( k \)-vector space \( V \) with a fixed direct sum decomposition \( V = \bigoplus_{n \in \mathbb{Z}} V_n \). Elements of the subspace \( V_n \) are said to be homogeneous of degree \( n \), and write \( |v| = n \) for \( v \in V_n \).

A \( k \)-linear map \( f : V \to W \) between graded spaces is homogeneous of degree \( n \) if \( f(V_p) \subseteq W_{p+n} \) for every \( p \in \mathbb{Z} \). Graded spaces form a category \( \mathcal{C}^{gr}(k) \) with morphisms

\[
\mathcal{C}^{gr}(k)(V, W) = \bigoplus_{n \in \mathbb{Z}} \mathcal{C}^{gr}_n(k)(V, W)
\]
where $\mathcal{C}^n_k(V,W)$ denotes the space of homogeneous degree $n$ linear maps $f : V \to W$. Note, that $|g \circ f| = |g| + |f|$ for homogeneous linear maps $f$ and $g$.

**Definition 1.1.1.** For reasons that will become clear in the sequel, we call a degree 0 homogeneous graded map a **chain map**.

**Remark 1.1.2.** The additivity of degree under composition implies that graded spaces together with chain maps forms a subcategory $\mathcal{C}_0^{gr}(k)$ of $\mathcal{C}^{gr}(k)$.

Many of the usual constructions from linear algebra have graded analogues. In particular, the tensor product $V \otimes W$ of graded spaces $V$ and $W$ is naturally a graded space with

$$(V \otimes W)_n = \bigoplus_{p+q=n} V_p \otimes W_q.$$

Given maps $f : V \to V'$ and $g : W \to W'$ with $g$ homogeneous, their tensor product $f \otimes g : V \otimes W \to V' \otimes W'$ is defined by

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w)$$

for $w \in W$ and homogeneous $v \in V$. Here we adopt the Koszul sign rule when dealing with graded objects: passing a symbol of degree $p$ across a symbol of degree $q$ results in a sign of $(-1)^{pq}$.

**Definition 1.1.3.** The **shift** of a graded space $V$ is the graded space $V[1]$ with $V[1]_n = V_{n-1}$. In particular, if $V$ is concentrated in degree $n$, the shift $V[1]$ is concentrated in degree $n+1$. Similarly, given a homogeneous linear map $f : V \to W$ of degree $n$, one defines $f[1] : V[1] \to W[1]$ by $f[1] = (-1)^n f$. In this way, shift $[1]$ determines an automorphism of $\mathcal{C}^{gr}(k)$.

The above considerations follow more or less verbatim for modules over a graded algebra. A **graded algebra** is an algebra in the category $\mathcal{C}(k)$, i.e., a graded space $A$ together with a chain map

$$A \otimes A \to A$$

inducing an associative bilinear product. A **graded (right) $A$-module** is a graded space $X$ with a degree 0 homogeneous and associative right action of $A$. The subcategory $\mathcal{C}^{gr}(A)$ of $\mathcal{C}^{gr}(k)$ whose objects are graded $A$-modules and whose morphisms are $A$-linear graded maps is closed under shift and tensor product.

**Definition 1.1.4.** A **graded category** is a category $\mathcal{C}$ enriched over the category $\mathcal{C}(k)$. That is, for any two objects $X$ and $Y$ of $\mathcal{C}$, the space of morphisms

$$\mathcal{C}(X,Y) = \bigoplus_{n \in \mathbb{Z}} \mathcal{C}_n(X,Y)$$

is a graded $k$-vector space, and the composition map

$$\mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$$

is a chain map.

In particular, $|g \circ f| = |g| + |f|$ for homogeneous linear maps $f$ and $g$.

**Examples 1.1.5.**

1. The categories $\mathcal{C}^{gr}(k)$ and $\mathcal{C}^{gr}(A)$ are evidently graded categories.

2. If $A$ is a graded algebra, then the category with one object 0 whose space of endomorphisms is $A$ is a graded category.

3. Given a graded category $\mathcal{C}$, the **opposite category** $\mathcal{C}^{op}$ is the graded category with the same objects as $\mathcal{C}$, morphism spaces $\mathcal{C}^{op}(X,Y) = \mathcal{C}(Y,X)$ and composition given by $g \circ f = (-1)^{|f||g|} f \circ g$.

**Definition 1.1.6.** A **graded functor** is a functor $F : \mathcal{C} \to \mathcal{D}$ so that for any two objects $X$ and $Y$ of $\mathcal{C}$ the induced map $F_{XY} : \mathcal{C}(X,Y) \to \mathcal{D}(FX,FY)$ is a chain map. A **contravariant** graded functor is a graded functor $F : \mathcal{C}^{op} \to \mathcal{D}$.

Recall that a **differential** on a graded space $V$ is a degree $-1$ linear map $d : V \to V$ satisfying $d_V^2 = 0$. A pair $(V,d_V)$ where $d_V$ is a differential on $V$ is a **differential graded** (dg for short) space.

Denote by $\mathcal{C}^{dg}(k)$ the full subcategory of $\mathcal{C}^{gr}(k)$ whose objects are differential graded spaces. Note that we impose no compatibility between the morphisms of $\mathcal{C}^{dg}(k)$ and differentials. Homogeneous degree 0 graded
maps \( f : V \to W \) such that \( d_W \circ f = f \circ d_V \) are called \emph{chain maps}; dg spaces and chain maps form a subcategory \( Z_0 \mathcal{C}^{dg}(k) \) of \( \mathcal{C}^{dg}(k) \). The notation \( Z_0 \mathcal{C}^{dg}(k) \) is meant to be suggestive of the space of cycles in a chain complex (cf., Definition 1.2.1).

\textbf{Remark 1.1.7.} Every graded space \( V \) may be considered as a dg space with differential \( d_V = 0 \). With this differential, \( \mathcal{C}_0^{dg} (k) (V,W) = Z_0 \mathcal{C}^{dg}(k)(V,W) \) so the two uses of the term “chain map” for graded spaces coincide.

Given dg spaces \( V \) and \( W \), the graded tensor product \( V \otimes W \) inherits a differential given by
\[
d_{V \otimes W} = d_V \otimes id_W + id_V \otimes d_W,
\]
and the shift \( V[1] \) is also a dg space with differential \( d_V[1] = d_V[1] = -d_V \).

Again, we can extend the above definitions and constructions for modules over a \emph{differential graded algebra}, i.e., a dg space \( A \) with a chain map
\[
A \otimes A \to A
\]
endowing \( A \) with an associative bilinear product. Explicitly, a dg algebra consists of a graded algebra \( A \) with a differential \( d_A \) satisfying the Leibniz law
\[
d_A(ab) = d_A(a)b + (-1)^{|a|} ad_A(b)
\]
for \( a, b \in A \) with \( a \) homogeneous. A \emph{differential graded (right) \( A \)-module} is a right module \( X \) over the graded algebra \( A \) with a differential \( d_X \) so that
\[
d_X(xa) = d_X(x)a + (-1)^{|x|} x d_A(a)
\]
for \( a \in A \) and homogeneous \( x \in X \).

\textbf{Examples 1.1.8.}  
\begin{enumerate}
\item Any ordinary algebra \( A \) can be viewed as a dg algebra concentrated in degree 0 with \( d_A = 0 \). In this case, dg \( A \)-modules are just chain complexes of ordinary \( A \)-modules.
\item More generally, a dg algebra with \( d_A = 0 \) is a graded algebra.
\end{enumerate}

The subcategory \( \mathcal{C}^{dg}(A) \) of \( \mathcal{C}^{dg}(k) \) whose objects are dg \( A \)-modules and whose morphisms are \( A \)-linear graded maps is closed under tensor products and shifts. Again, there is no imposed compatibility between the morphisms of \( \mathcal{C}^{dg}(A) \) and differentials. Homogeneous degree 0 morphisms commuting with differentials are again called \emph{chain maps}, and the subcategory of \( \mathcal{C}^{dg}(A) \) whose morphisms are chain maps is denoted \( Z_0 \mathcal{C}^{dg}(A) \).

\textbf{Definition 1.1.9.} A \emph{differential graded category} is a category \( \mathcal{A} \) enriched over the category \( Z_0 \mathcal{C}^{dg}(k) \) of dg spaces and chain maps. More explicitly, for any two objects \( X \) and \( Y \) of \( \mathcal{A} \), the set of morphisms
\[
\mathcal{A}(X,Y) = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n(X,Y)
\]
is a graded space and endowed with a differential \( d_{XY} \) of degree \(-1\). Moreover for any three objects \( X, Y, \) and \( Z \) the composition map
\[
\mathcal{A}(Y,Z) \otimes \mathcal{A}(X,Y) \to \mathcal{A}(X,Z)
\]
is a chain map, and so for any two morphisms \( f : X \to Y \) and \( g : Y \to Z \) with \( g \) homogeneous, the Leibniz law
\[
d_{XY}(g \circ f) = d_{YZ}(g) \circ f + (-1)^{|g|} g \circ d_{XY}(f)
\]
is satisfied.

\textbf{Examples 1.1.10.}  
\begin{enumerate}
\item The category \( \mathcal{C}^{dg}(k) \) is a dg category, where \( \mathcal{C}^{dg}(k)(V,W) \) is equipped with the \emph{commutator differential}
\[
d_{VW}(f) = d_W \circ f - (-1)^{|f|} f \circ d_V
\]
where \( d_V \) and \( d_W \) are the differentials of the dg spaces \( V \) and \( W \).
\item For a dg algebra \( A \), \( \mathcal{C}^{dg}(A) \) is a full dg subcategory of \( \mathcal{C}^{dg}(k) \).
\item Any full subcategory of a dg category is evidently a dg category as well. In particular, if \( \mathcal{P}^{dg}(A) \) denotes the full subcategory of \( \mathcal{C}^{dg}(A) \) whose objects are bounded chain complexes of projective modules. Here bounded below means \( X_n = 0 \) for \( n \ll 0 \).
\item The (graded) opposite category of a dg category \( \mathcal{A} \) is a dg category in the evident manner.
\end{enumerate}
**Definition 1.1.11.** A *dg functor* is a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ so that for any two objects $X$ and $Y$ of $\mathcal{A}$ the induced map

$$F_{XY} : \mathcal{A}(X,Y) \rightarrow \mathcal{B}(F(X),F(Y))$$

is a chain map. A *contravariant* dg functor is a dg functor $F : \mathcal{A}^{op} \rightarrow \mathcal{B}.$

1.2. Underlying, homology, and homotopy categories. There are several auxiliary categories which can be attached to a dg category.

**Definition 1.2.1.** The **underlying category** of a dg category $\mathcal{A}$ is the category $Z_0\mathcal{A}$ whose objects are the same as the objects of $\mathcal{A},$ but with morphism spaces $Z_0\mathcal{A}(X,Y)$ given by the 0-cycles of $\mathcal{A}(X,Y).$

Note that a dg functor $F : \mathcal{A} \rightarrow \mathcal{B}$ induces an ordinary functor $Z_0F : Z_0\mathcal{A} \rightarrow Z_0\mathcal{B}$ between underlying categories. The association $F \mapsto Z_0F$ is functorial in the category of small dg categories and dg functors.

**Example 1.2.2.** If $A$ is a dg algebra, then the 0-cycles of the dg space $\mathcal{C}^{dg}(A)(X,Y)$ are the degree 0 maps $f : X \rightarrow Y$ so that $d_{XY}(f) = d_Y \circ f - f \circ d_X = 0,$ i.e., $f \circ d_X = d_Y \circ f.$ Hence the 0-cycles of $\mathcal{C}^{dg}(A)$ are chain maps, justifying the notation from Section ??.

**Definition 1.2.3.** The **homology** of $\mathcal{A}$ is the graded category $H_*\mathcal{A}$ whose objects are the same as the objects of $\mathcal{A},$ but morphism spaces

$$(H_*\mathcal{A})(X,Y) = H_0\mathcal{A}(X,Y),$$

where $H_*\mathcal{A}(X,Y) = \ker d_{XY}/ \text{im } d_{XY}$ is the homology of the dg space $(\mathcal{A}(X,Y),d_{XY}).$

A dg functor $F$ induces a graded functor $H_*F$ in homology, and the association $F \mapsto H_*F$ is functorial for small dg categories.

**Definition 1.2.4.** The **homotopy category** of $\mathcal{A}$ is the degree 0 subcategory $H_0\mathcal{A}$ of the homology $H_*\mathcal{A}.$

Let us look more closely at the homotopy category of a dg category. The collection of boundary subspaces $B_0\mathcal{A}(X,Y)$ form an ideal of the underlying category $Z_0\mathcal{A}$ in the sense that a composition $g \circ f$ lies in $B_0\mathcal{A}$ if either $f$ or $g$ does, and the homotopy category $H_0\mathcal{A}$ is the quotient of $Z_0\mathcal{A}$ by the ideal $B_0\mathcal{A}.$ If $f : X \rightarrow Y$ is a morphism in $Z_0\mathcal{A},$ then there is a degree $-1$ morphism $s : X \rightarrow Y$ with $d_{XY}(s) = f.$

In particular, if $A$ is a dg algebra and $\mathcal{A} = \mathcal{C}^{dg}(A),$ a morphism $f$ lies in $B_0\mathcal{C}^{dg}(A)(X,Y)$ if and only if $f = d_Y \circ s + s \circ d_X,$ and so $B_0\mathcal{C}^{dg}(A)$ is the ideal of null-homotopic chain maps $f : X \rightarrow Y.$ Thus, the homotopy category $H_0\mathcal{C}^{dg}(A)$ is the category whose objects are dg $A$-modules and whose morphisms are given by homotopy classes of chain maps.

Consider now the bounded derived category $\mathcal{D}^b(A)$ of an ordinary algebra $A.$ By definition, it is the localization of $Z_0\mathcal{C}^{dg}(A)$ by the class of quasi-isomorphisms, i.e., morphisms inducing an isomorphism in homology. A theorem of Happel [12] asserts that $\mathcal{D}^b(A)$ is equivalent to $Z_0\mathcal{P}^{dg}(A)$ modulo homotopy, so by the preceding discussion, $\mathcal{D}^b(A)$ is equivalent to the homotopy category $H_0\mathcal{P}^{dg}(A)$.

**Remark 1.2.5.** The preceding description of $\mathcal{D}^b(A)$ can be generalized to model categories. In general for a model category $\mathcal{C},$ the localization $W^{-1}\mathcal{C}$ at the class of weak equivalences is equivalent to the quotient of the full subcategory of $\mathcal{C}$ whose objects are both fibrant and cofibrant with morphisms taken modulo homotopy. The category $Z_0\mathcal{C}^{dg}(A)$ of chain complexes of $A$-modules admits a model structure whose weak equivalences are the quasi-isomorphisms, and whose fibrant and cofibrant objects are complexes of projective modules. For details see e.g., [27].

1.3. $A_\infty$-Algebras and categories. We now recall the theory of $A_\infty$-algebras, which were originally introduced by Stasheff in order to characterize loop spaces among $H$-spaces [26].

**Definition 1.3.1.** An $A_\infty$-**algebra** is a graded space $A$ together with a collection of degree $n-2$ maps $\mu_n : A^{\otimes n} \rightarrow A$ for $n \geq 1$ satisfying the relation

$$0 = \sum_{p+q+r=n \atop p,r \geq 0, \ q \geq 1} (-1)^{p+qr} \mu_{p+1+r} \circ (id^{\otimes p} \otimes \mu_q \otimes id^{\otimes r})$$

for each $n.$ We call the maps $\mu_n$ the $A_\infty$-**structure maps** for the $A_\infty$-algebra $A.$
Let us more explicitly study the relations (1.3.1). The relation for \( n = 1 \) says that the pair \((A, \mu_1)\) is a dg space, and the relation for \( n = 2 \) says that \( \mu_2 : A \otimes A \to A \) is a chain map (where \( \otimes \) is the graded tensor product). For \( n = 3 \), the relation is more subtle: it says that \( \mu_3 \) is associative up to a homotopy provided by \( \mu_3 \). More accurately, it says \( \mu_3 \) is a null-homotopy of the associator \( \mu_2 \circ (\mu_2 \otimes \text{id} - \text{id} \otimes \mu_2) \).

**Examples 1.3.2.**

1. Any graded associative algebra is an \( A_\infty \)-algebra with \( \mu_2 \) given by multiplication, and all other \( \mu_n = 0 \). Conversely, any \( A_\infty \)-algebra with \( \mu_1 = 0 \) determines an associative algebra by forgetting the maps \( \mu_n \) for \( n \geq 3 \).

2. Any differential graded algebra is an \( A_\infty \)-algebra with \( \mu_n = 0 \) for \( n \geq 3 \).

3. Given an associative algebra \( A \) and a Hochschild \( n \)-cocycle \( \lambda \), there is a natural \( A_\infty \)-structure on the polynomial ring \( A[t] \) where \( t \) is an indeterminate of degree \( n - 2 \), such that \( \mu_n \) and \( \mu_2 \) are the only non-trivial structure maps (cf. [16, 23]).

Since an \( A_\infty \)-algebra \( A \) is in particular a dg space, we can form the homology \( H_* A = \ker d_A/ \text{im} d_A \). The structure map \( \mu_2 \) of \( A \) descends to an associative multiplication on \( H_* A \), making it into a graded algebra.

**Definition 1.3.3.** A morphism \( f : A \to B \) of \( A_\infty \)-algebras \((A, \mu_n)\) and \((B, \nu_n)\) is a collection of graded maps \( f_n : A^{\otimes n} \to B \) for \( n \geq 1 \) of degree \( n - 1 \) satisfying the identities

\[
(1.3.2) \quad \sum_{p+q+\tau = n} \sum_{p,r \geq 0, \ q \geq 1} (-1)^{p+q} f_{p+1+r} \circ (\text{id}^{\otimes p} \otimes \mu_q \otimes \text{id}^{\otimes r}) = \sum_{1 \leq d \leq n} (-1)^D \nu_d \circ (f_1 \otimes \cdots \otimes f_d)
\]

for each \( n \) where \( D = (d - 1)(i_1 - 1) + (d - 2)(i_2 - 1) + \cdots + 2(i_{d-2} - 1) + (i_{d-1} - 1) \).

For \( n = 1 \), the relation says that \( f_1 : (A, \mu_1) \to (B, \nu_1) \) is a chain map of the underlying dg spaces. For \( n = 2 \), the relation says that \( f_1 \) is compatible with multiplication up to a homotopy provided by \( f_2 \).

Since \( f_1 \) is a chain map, it induces a homomorphism \((f_1)_* : H_* A \to H_* B \) in homology. The morphism \( f \) is an \( A_\infty \)-quasi-isomorphism if \( f_1 \) is a quasi-isomorphism, i.e., the induced map \((f_1)_* \) is an isomorphism, and \( f \) is strict if \( f_n = 0 \) for \( n \neq 1 \). The identity morphism of an \( A_\infty \)-algebra \( A \) is the strict morphism \( \text{id}_A \) whose \( n = 1 \) component is the identity on \( A \).

The collection of \( A_\infty \)-algebras and morphisms forms a category with composition \( g \circ f \) by setting

\[
(g \circ f)_n = \sum_{1 \leq d \leq n} (-1)^D \nu_d \circ (f_1 \otimes \cdots \otimes f_d)
\]

with \( D \) as in Definition 1.3.3.

**Definition 1.3.4.** Given two morphisms \( f, g : A \to B \) of \( A_\infty \)-algebras, a homotopy from \( f \) to \( g \) is a collection of graded maps \( s_n : A^{\otimes n} \to B \) for \( n \geq 1 \) of degree \( n \) satisfying the identities

\[
(1.3.3) \quad f_n - g_n = \sum_{p+q+r = n} \sum_{p,r \geq 0, \ q \geq 1} (-1)^{p+q} s_{p+1+r} \circ (\text{id}^{\otimes p} \otimes \mu_q \otimes \text{id}^{\otimes r}) + \sum_{p+q = n} (-1)^p \nu_{1+q} \circ (s_p \otimes \text{id}^{\otimes q})
\]

for each \( n \). Two morphisms \( f \) and \( g \) are homotopic (denoted \( f \simeq g \)) if there is a homotopy between them, and a morphism is null-homotopic if it is homotopic to the zero morphism.

Two \( A_\infty \)-algebras are homotopy equivalent if there are morphisms \( f : A \to B \) and \( g : B \to A \) so that \( g \circ f \simeq \text{id}_A \) and \( f \circ g \simeq \text{id}_B \). In this case, the morphism \( g \) is a homotopy inverse of \( f \).

Note that if \( s \) is a homotopy from \( f \) to \( g \), the map \( s_1 \) is a chain homotopy from \( f_1 \) to \( g_1 \) on the underlying dg spaces \((A, \mu_1)\) and \((B, \nu_1)\). In particular, homotopic maps induce the same maps in homology, and homotopy equivalences are \( A_\infty \)-quasi-isomorphisms.

An advantage of studying homotopy theory in the category of \( A_\infty \)-algebras is that \( A_\infty \)-quasi-isomorphisms are invertible up to homotopy; that is, if \( f : A \to B \) is an \( A_\infty \)-quasi-isomorphism, it necessarily admits a homotopy inverse [25].

**1.4. Minimal models.** An \( A_\infty \)-algebra \((A, \mu_n)\) is minimal if \( \mu_1 = 0 \). An \( A_\infty \)-algebra \( B \) is a minimal model for \( A \) if \( B \) is minimal and there is an \( A_\infty \)-quasi-isomorphism \( f : A \to B \). The \( A_\infty \)-algebra \( A \) is formal if it admits a minimal model whose higher multiplications vanish for \( n \geq 3 \).
Remark 1.4.1. From Example 1.3.2, the product $\mu_2$ of a minimal $A_\infty$-algebra is necessarily associative.

If $A$ is any $A_\infty$-algebra, the homology $H_* A$ is minimal; the following theorem of Kadeishvili makes $H_* A$ into a minimal model of $A$ in the case that $A$ is a dg algebra.

Theorem 1.4.2 (Kadeishvili [14]). Let $A$ be a differential graded algebra. Then there is a minimal $A_\infty$-algebra structure $(\mu_n)_{n \geq 2}$ on the homology $H_* A$ with $\mu_2$ equal the usual multiplication map together with an $A_\infty$-quasi-isomorphism $H_* A \to A$.

Moreover, this $A_\infty$-structure on $H_* A$ is unique up to a (non-unique) isomorphism of $A_\infty$-algebras.

In particular, Kadeishvili’s Theorem shows that the minimal model of a dg algebra is unique up to $A_\infty$-isomorphism. By abuse of terminology, we will refer to $H_* A$ with the above $A_\infty$-structure as the minimal model of $A$, and $(\mu_n)_{n \geq 2}$ as the minimal model $A_\infty$-structure of $H_* A$.

Suppose $A$ and $B$ are two dg algebras, and $p : A \to H A$ and $i : H B \to H$ are $A_\infty$-quasi-isomorphisms as per Kadeishvili’s Theorem. If moreover, there is an $A_\infty$-quasi-isomorphism $f : H A \to H B$, then the map $i_1 \circ f_1 \circ p_1 : A \to B$ is a(n ordinary!) quasi-isomorphism of dg algebras. Hence, the quasi-isomorphism type of a dg algebra is determined by the $A_\infty$-quasi-isomorphism type of its minimal model.

Much of the theory of $A_\infty$-structures on algebras can be directly generalized to the categorical setting. These $A_\infty$-categories were first introduced by Fukaya [9].

Definition 1.4.3. An $A_\infty$-category is an $A_\infty$-algebra with multiple objects. More precisely, an $A_\infty$-category $\mathcal{A}$ consists of the data of a class of objects, a graded space $\mathcal{A}(X,Y)$ for every pair of objects $X$ and $Y$ of $\mathcal{A}$, together with $n$-ary compositions

$$\mu_{X_1 \ldots X_n} : \mathcal{A}(X_{n-1}, X_n) \otimes \cdots \otimes \mathcal{A}(X_1, X_2) \otimes \mathcal{A}(X_0, X_1) \to \mathcal{A}(X_0, X_n)$$

satisfying relations analogous to (1.3.1).

Remark 1.4.4. An $A_\infty$-category is in general not a category, since there may not be identity morphisms, and composition may not be associative. However, an $A_\infty$-category in which each $\mu_{X,Y}$ vanishes will be a (strictly associative) graded category (without identity morphisms).

Many of the theorems and techniques available to $A_\infty$-algebras pass with minimal modification to the categorical setting. In particular, every small dg category $\mathcal{A}$ admits its homology category $H_* \mathcal{A}$ as a minimal model, and this $A_\infty$-structure can be explicitly described using the homotopy transfer theorem.

1.5. Homotopy transfer theorem. In order to explicitly construct the minimal $A_\infty$-structure from Kadeishvili’s Theorem we will need the Homotopy Transfer Theorem. To state the Homotopy Transfer Theorem concretely, we will need some elementary facts about planar binary rooted trees.

Definition 1.5.1. A planar binary rooted $n$-tree (PBR $n$-tree for short) is a trivalent planar graph $T$ with $n + 1$ external edges, one of which is distinguished and called the root edge. The other $n$ external edges are called the leaves of $T$. We think of an external edge as being adjacent to only one vertex; all edges adjacent to two vertices are internal edges. Denote the set of PBR $n$-trees by $PBR(n)$.

For any tree $T$ with vertex set $T_0$ and edge set $T_1$, there is a map $\partial : T_1 \to (T_0 \times T_0)/S_2$ from the set of edges to the sending an edge to the unordered pair of adjacent vertices. (Here $S_2$ is the symmetric group on 2 letters acting in the natural way on the product $T_0 \times T_0$.) An orientation of $T$ is a lift of $\partial$ to $T_0 \times T_0$, i.e., a specification of orderings on the sets of vertices incident to the edges of $T$. For an oriented edge $e = \{v,w\}$ of $T$ we write $e : v \to w$.

Equip a rooted tree with the orientation where all edges (internal and external) are oriented towards the root edge. A descendant of a vertex $v$ of $T$ is a vertex $w$ such that there exists an oriented path $w \to \cdots \to v$ in $T$. If the path $w \to v$ consists of a single edge, then $w$ is said to be a child of $v$.

For a PBR tree, every vertex having children has exactly two (one of which may be empty). A planar embedding of a PBR tree gives a well-defined left-to-right ordering of the two children of any vertex $v$. The left child of $v$ is the child $\lambda(v)$ minimal with respect to this ordering, and maximal child is the right child $\rho(v)$. The left subtree is the full planar binary subtree $T_\lambda(v)$ of $T$ whose vertices are the descendants of $\lambda(v)$; it is rooted by with root edge $\lambda(v) \to v$. The right subtree $T_\rho(v)$ is defined similarly.

There is a unique vertex $v_0$ closest to the root of $T$. For brevity, denote $T_\lambda(v_0)$ and $T_\rho(v_0)$ by $T_\lambda$ and $T_\rho$ respectively.
Construction 1.5.2. Define a function
\[ PBR(n + 1) \to \text{GL}_n(\mathbb{Z}) \]
\[ T \mapsto A_T \]
as follows: Choose an embedding \( \sigma : T \to \mathbb{R}^2 \) of \( T \) so that
1. all vertices of \( T \) lie on the cone \( \mathbb{N} \times \mathbb{N} \),
2. all oriented edges \( e : v \to w \) point downwards,
3. for each vertex \( v \) the image of \( T_\lambda(v) \) lies entirely below the image of \( T_\rho(v) \).

The embedding \( \sigma \) determines vertical and horizontal orderings of the internal vertices of \( T \) by pulling back the ordering from \( \mathbb{N} \) under the two projections \( \mathbb{N}^2 \to \mathbb{N} \) onto the \( y \) and \( x \)-axes respectively. The matrix \( A_T \) of an planar binary rooted \((n + 1)\)-tree is the \( n \times n \) matrix with a 1 in the \((i, j)\)-th position for each vertex \( v \) of \( T \) having \( y \)-coordinate \( n + 1 - i \) and \( x \)-coordinate \( j \). All other entries if \( A_T \) are 0.

The matrix \( A_T \) is a permutation matrix corresponding to \((n + 1 - i) \mapsto j\) if there is a vertex with \( y \)-coordinate \( i \) and \( x \)-coordinate \( j \). In particular, \( A_T \) lies in \( \text{GL}_n(\mathbb{Z}) \).

Remark 1.5.3. The condition that \( T_\lambda(v) \) lies below \( T_\rho(v) \) under the embedding \( \sigma \) furnishes a bijection between PBR trees and 231 avoiding permutations. Under this bijection, the sign \( \text{sgn}(T) \) is the sign of the corresponding permutation.

Definition 1.5.4. The sign of a PBR tree \( T \) is the number \( \text{sgn}(T) = \det(A_T) \). The sign \( \text{sgn}(T) = \pm 1 \) as \( A_T \in \text{GL}_n(\mathbb{Z}) \).

Example 1.5.5. If \( T \) is the PBR 5-tree in Figure 1.5, then
\[ A_T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \]
and so \( \text{sgn}(T) = \det(A_T) = +1 \).

Recall that a dg space \( V \) is a homotopy retract of a dg space \( A \) if there is a diagram
\[ \varphi \begin{array}{ccc} A & \xrightarrow{j} & V \\ \downarrow \varphi & & \downarrow \varphi \end{array} \]
such that \( q \varphi = \text{id}_V \) and \( \varphi : \text{id}_A \simeq j \varphi \) is a homotopy equivalence.

Every dg space \( A \) admits its homology \( H_*A \) as a homotopy retract: the map \( j \) is the composition of \( Z_*A \to A \) with a section of \( Z_*A \to H_*A \), and \( q \) is the composition of a retraction of \( Z_*A \to A \) with the
projection $Z_*A \to H_*A$. The homotopy $\varphi$ is the inverse of the differential $d_A$ restricted to the complementary subspace of $Z_*A$ in $A$ (cf. e.g., [19] Lemma 9.4.7).

If $(A, \mu_A, d_A)$ is a dg algebra and $(V, d_V)$ is a homotopy retract, the induced binary product $\mu_V = q \circ \mu_A \circ (j \otimes j)$ need not be associative. However, $V$ does admit the structure of an $A_\infty$-algebra with $\mu_2 = \mu_V$.

Given a PBR $n$-tree $T$, define a map $\mu_T : V^\otimes n \to V$ as follows: place the map $j$ on the leaves of $T$, $q$ on the root, $\varphi$ on each internal edge, and the multiplication $\mu_A$ on each (internal) vertex. The maps go in the direction of the canonical orientation. Note $\mu_T$ has degree $n - 2$, since $\varphi$ has degree 1 and $T$ has $n - 2$ internal edges.

For later use, we give a formal construction of the maps $\mu_T$ for a dg algebra $(A, \mu, d_A)$ by induction on PBR trees. To do so, we define degree $n - 2$ maps $\nu_T : A^\otimes n \to A$ such that $\mu_T = q \circ \nu_T \circ j^\otimes n$ for each PBR tree $T$.

**Construction 1.5.6.** Denote by $Y$ the unique PBR 2-tree, and define $\nu_Y \overset{\text{def}}{=} \mu$ to be the multiplication map, so that $\mu_Y = q \circ \nu_Y \circ j^\otimes 2$. If $T$ is an arbitrary PBR tree, define

$$\nu_T = \mu \circ (\nu'_T \otimes \nu'_T),$$

where $\nu'_T = \varphi \circ \nu_T$ if $T \neq \emptyset$ and is $id_A$ otherwise. (Recall that in our convention, leaves are only adjacent to one edge.)

![Diagram](image)

**Figure 1.5.2.** The map $\mu_T$

**Theorem 1.5.7 (Homotopy Transfer).** The maps

$$\mu_n = \sum_{T \in \text{PBR}_n} \text{sgn}(T)\mu_T$$

endow $V$ with the structure of an $A_\infty$-algebra. Moreover, the maps $q$ and $j$ can be extended to $A_\infty$-quasi-isomorphisms, and $\varphi$ can be extended to an $A_\infty$-homotopy.

**Remark 1.5.8.** The $A_\infty$-extensions of $q$ and $\varphi$ are due to Markl [21]. An explicit formula for the maps $\mu_n$ is due to Merkulov [22].

The construction of the minimal model of Kadeishvili’s Theorem given by the Homotopy Transfer Theorem generalizes quite easily to the categorical setting. Given a small dg category $\mathcal{A}$, one needs to choose homotopy retractions of $\mathcal{A}(X, Y)$ for each pair of objects of $X$ and $Y$. Associated to each PBR $n$-tree $T$, one constructs homogeneous maps

$$\mu_T^{X_0, \ldots, X_n} : \mathcal{A}(X_{n-1}, X_n) \otimes \cdots \otimes \mathcal{A}(X_1, X_2) \otimes \mathcal{A}(X_0, X_1) \to \mathcal{A}(X_0, X_n).$$
analogously to 1.5.6, and the higher compositions
\[ \mu_{n}^{X_{0},...,X_{n}} = \sum_{T \in PBR_{n}} \text{sgn}(T)\mu_{T}^{X_{0},...,X_{n}} \]
define an \( A_{\infty} \)-structure on \( H_{s}\mathcal{A} \).

2. Differential graded covering theory

In this section, we modify the covering theory of Bongartz and Gabriel [5] to accommodate dg structures. To this end, we first recall the relationships between quivers, algebras, and locally finite dimensional categories, and consider their dg structures. The notion of a locally finite dimensional category is not well-adapted to dg considerations, as the naive definition of a locally finite dimensional dg category is not homotopy invariant. To circumvent this issue, we use techniques from Bautista-Liu [3].

2.1. Locally finite dimensional categories. There is a close relationship between small categories and associative algebras which we now recall.

Definition 2.1.1. The associated algebra of a small category \( \mathcal{C} \) is the algebra \( k[\mathcal{C}] \) with underlying vector space \( \bigoplus_{X,Y} \mathcal{C}(X,Y) \) where the direct sum is taken over all pairs of objects \( X, Y \) in \( \mathcal{C} \). The product in \( k[\mathcal{C}] \) is given by composition if the elements are composable, and is 0 otherwise.

If the set of objects of \( \mathcal{C} \) is finite, then the algebra \( k[\mathcal{C}] \) is unital with \( 1 = \sum_{X} id_{X} \); in general the algebra is only locally unital in the sense that \( k[\mathcal{C}]^{2} = k[\mathcal{C}] \).

If \( \mathcal{A} \) is a small graded or dg category, then the associated algebra \( k[\mathcal{A}] \) inherits the structure of a graded or dg algebra. Indeed, \( k[\mathcal{A}]_{n} = \bigoplus_{X,Y} \mathcal{A}_{n}(X,Y) \) where \( \mathcal{A}_{n}(X,Y) \) is the degree \( n \) component of the morphism space \( \mathcal{A}(X,Y) \) and \( d_{k[\mathcal{A}]} = \sum d_{XY} \) where the sum ranges over all pairs of objects \( X, Y \) in \( \mathcal{A} \).

The homology of a small dg category is manifestly small, since it has the same objects as the original dg category. The associated algebra construction is compatible with homology, in the sense that
\[ H_{s}k[\mathcal{A}] = k[H_{s}\mathcal{A}] \]
for a small dg category \( \mathcal{A} \) since homology commutes with direct sum.

A functor \( F : \mathcal{C} \to \mathcal{D} \) between small categories induces an evident (not necessarily unital) algebra homomorphism between the associated algebras whose components are the canonical maps
\[ F_{XY} : \mathcal{C}(X,Y) \to \mathcal{D}(FX,FY) \]
which is a dg (resp. graded) homomorphism if \( f \) is a dg (resp. graded) functor.

Remark 2.1.2. In general an equivalence of categories \( F : \mathcal{C} \to \mathcal{D} \) does not induce an isomorphism \( F_{*} : k[\mathcal{C}] \to k[\mathcal{D}] \), since \( F \) may only be quasi-inverse. However, if \( F \) is an isomorphism of categories, i.e., strictly invertible, then the induced morphism \( F_{*} \) is an isomorphism. Note that for two small categories to be isomorphic, there must be a bijection between object sets.

2.2. Path categories. Recall a quiver is a tuple \( Q = (Q_{0},Q_{1},s,t) \) where \( Q_{0} \) is the vertex set, \( Q_{1} \) is the arrow set, and \( s,t : Q_{1} \to Q_{0} \) are respectively the source and target functions. We will use the notation \( \alpha : i \to j \) to indicate \( \alpha \) is an arrow with \( s(\alpha) = i \) and \( t(\alpha) = j \).

We adopt the concatenation convention when forming paths so that a path in \( Q \) is a sequence \( \alpha_{1}\alpha_{2}\cdots\alpha_{n} \) of arrows with \( t(\alpha_{i}) = s(\alpha_{i+1}) \). Denote the set of paths in \( Q \) of length \( n \) by \( Q_{n} \). A morphism \( f : Q \to Q' \) of quivers is a pair of functions \( f_{n} : Q_{n} \to Q'_{n} \) for \( n = 0,1 \) commuting with the source and target maps; such \( f \) induces for every \( n \) a function \( f_{n} : Q_{n} \to Q'_{n} \) in an evident manner.

Given a quiver \( Q \), the path algebra \( kQ \) is the associative \( k \)-algebra with underlying vector space given by the span of paths in \( Q \), and multiplication given by concatenation. The path category of \( Q \) is the small category \( \mathcal{C}_{Q} \) whose objects are the vertices of \( Q \) and whose morphism spaces are given by \( \mathcal{C}_{Q}(i,j) = e_{j}kQe_{i} \), i.e., the span of paths from \( j \) to \( i \) in \( Q \). Composition in \( \mathcal{C}_{Q} \) is given by multiplication in \( kQ \), i.e.,

\[ \begin{align*}
\mathcal{C}_{Q}(j,k) \otimes \mathcal{C}_{Q}(i,j) & \longrightarrow \mathcal{C}_{Q}(i,k) \\
\text{I} & \text{I} \\
e_{k}kQe_{j} \otimes e_{j}kQe_{i} & \longrightarrow e_{k}kQe_{i}
\end{align*} \]
so \( \alpha \circ \beta = \alpha \beta \).

It is immediate that the algebra \( k[\mathcal{C}_Q] \) associated to the small category \( \mathcal{C}_Q \) is isomorphic to the path algebra \( kQ \).

A quiver morphism \( f : Q \to Q' \) induces both an algebra homomorphism \( f_* : kQ \to kQ' \) and a functor \( F : \mathcal{C}_Q \to \mathcal{C}_{Q'} \) in the evident manner. Both of these associations are functorial, and are compatible with passing to algebras in the sense that \( k[F] = f_* \).

The category \( \mathcal{C}_Q \) is \textit{locally finite dimensional} in the sense that:

1. distinct objects are non-isomorphic,
2. morphism spaces are finite dimensional,
3. and all endomorphism algebras are one dimensional.

In fact all such categories arise as the path category of a quiver modulo an ideal \([5]\).

A \textit{graded} quiver is a quiver whose arrow set is \( \mathbb{Z} \)-graded. The path algebra \( kQ \) of a graded quiver inherits the \( \mathbb{Z} \)-grading, and the path category \( \mathcal{C}_Q \) inherits the structure of a graded category.

\textbf{Definition 2.2.1.} A \textit{differential graded quiver} is a graded quiver \( Q \) together with a degree \(-1\) function \( d : Q_1 \to kQ_2 \) between \( Q_1 \) and the span of the paths of length \( 2 \), such that \( d \) is compatible with the source and target maps in the sense that \( d(\alpha) \in e_{s(\alpha)}kQe_{t(\alpha)} \) for any arrow \( \alpha \).

The differential \( d \) of a dg quiver \( Q \) can be extended by the Leibniz law

\[ d(\alpha \beta) \overset{\text{def}}{=} d(\alpha)\beta + (-1)^{\lvert \alpha \rvert}\alpha d(\beta) \]

to a derivation of the path algebra \( kQ \), making it into a dg algebra. Similarly, the path category \( \mathcal{C}_Q \) of a dg quiver is naturally a dg category.

\textbf{Remark 2.2.2.} The condition that \( d \) take image in \( kQ_2 \subset kQ \) ensures that the differential of \( kQ \) induced by \( d \) has degree \(+1\) with respect to grading by path length.

\textbf{Lemma 2.2.3.} If \( Q \) is a dg quiver, then \( \mathcal{C}_Q \) is a dg category. Moreover, the isomorphism \( k[\mathcal{C}_Q] \cong kQ \) is a dg isomorphism.

\textbf{Proof.} We verify that the differential on \( \mathcal{C}_Q \) satisfies the Leibniz law: Indeed, the composition in \( \mathcal{C}_Q \) is defined so that \( \alpha \circ \beta = \alpha \beta \), and hence

\[ d(\alpha \circ \beta) = d(\alpha \beta) = d(\alpha)\beta + (-1)^{\lvert \alpha \rvert}\alpha d(\beta) = d(\alpha) \circ \beta + (-1)^{\lvert \alpha \rvert}\alpha \circ d(\beta). \]

This also shows that the algebra isomorphism \( k[\mathcal{C}_Q] \xrightarrow{\sim} kQ \) is a dg algebra isomorphism. \( \square \)

\textbf{Remark 2.2.4.} Note that the reversal in the definition of the morphism spaces \( \mathcal{C}_Q(i,j) \) played a crucial role in the proof of the above lemma. Indeed, if we reverse the directions of the morphisms in \( \mathcal{C}_Q \) the differential would not satisfy the Leibniz law.

\textbf{Definition 2.2.5.} A quiver morphism \( f : Q \to Q' \) between dg quivers is a \textit{dg quiver morphism} if it is homogeneous with respect to \( \mathbb{Z} \)-gradings and the diagram

\[
\begin{array}{ccc}
Q_1 & \xrightarrow{f_1} & Q'_1 \\
\downarrow d & & \downarrow d' \\
kQ_2 & \xrightarrow{f_*} & kQ'_2
\end{array}
\]

commutes. Note the restriction of \( f_* \) to the subspace \( kQ_2 \) takes image in \( kQ'_2 \) since \( f_* \) necessarily is homogeneous with respect to path length.

The induced map \( f_* : kQ \to kQ' \) and induced functor \( f_* : \mathcal{C}_Q \to \mathcal{C}_{Q'} \) respect the dg structures of the path algebras and categories when \( f \) is a dg quiver morphism.

\textbf{Lemma 2.2.6.} If \( f : (Q,d) \to (Q',d') \) is a dg quiver morphism, then the induced map \( f_* : kQ \to kQ' \) is a dg homomorphism, and the induced functor \( F_f : \mathcal{C}_Q \to \mathcal{C}_{Q'} \) is a dg functor.
Proof. Since $kQ = k[\mathcal{C}_Q]$ it suffices to show that the induced functor is dg. That is, we want to show that the map induced on morphism spaces

$$\mathcal{C}_Q(i, j) \to \mathcal{C}_Q(f(i), f(j))$$

is a chain map, i.e., $d'(f(w)) = f(d(w))$ for any path $w : j \to i$ in $Q$. We proceed by induction on the length of $w$. If $w = \alpha$ is a path of length 1, i.e., an arrow, then $d'(f(\alpha)) = f(d(\alpha))$ since $d$ is a dg quiver homomorphism. If the length of $w$ is greater than 1, then $w = \alpha u$ for some arrow $\alpha$ and path $u$ with length strictly less than that of $w$. But then

$$d'(f(\alpha u)) = d'(f(\alpha)f(u)) = d'(f(\alpha))f(u) + (-1)^{|\alpha|}f(\alpha)d'(f(u))$$

$$= f(d(\alpha))f(u) + (-1)^{|\alpha|}f(\alpha)f(du) = f(d(\alpha u))$$

where the second and fourth equalities are by the definitions of $d$ and $d'$, and the third is by induction. □

**Definition 2.2.7.** Given an category $\mathcal{A}$, a (right) $\mathcal{A}$-module is a contravariant functor $M : \mathcal{A} \to \mathcal{C}(k)$; a morphism of $\mathcal{C}$-modules is a natural transformation $f : M \to N$. Denote by $\mathcal{C}(\mathcal{A})$ the category of $\mathcal{A}$-modules.

If $\mathcal{A}$ is small and $M$ is a $\mathcal{A}$-module, then $k[M] = \bigoplus_X MX$ is a $k[\mathcal{A}]$-module, where the right $k[\mathcal{A}]$-action is given componentwise by the composition

$$MX \otimes \mathcal{A}(X,Y) \longrightarrow MX \otimes \mathcal{C}(k)(MX,MY) \longrightarrow MY$$

where the first map is $id_M \otimes M_{XY}$ and the second is the evaluation map. This can be extended in a natural way to a functor

$$\mathcal{C}(\mathcal{A}) \to \mathcal{C}(k[\mathcal{A}])$$

which is in fact an equivalence of categories. Hence, we may (and will) view $\mathcal{A}$-modules and $k[\mathcal{A}]$-modules interchangeably.

**Examples 2.2.8.**

1. For any object $X$ of $\mathcal{A}$, the functor $\mathcal{A}(-,X) : \mathcal{A} \to \mathcal{C}(k)$ determines a projective $\mathcal{A}$-module.

2. A quiver $Q$ may be viewed as a locally small category $\mathcal{C}_Q$ whose objects are the vertices of $Q$, and $\mathcal{C}_Q(i,j)$ is the span of the paths from $j$ to $i$ in $Q$. Then representations of $Q$ are by definition $\mathcal{C}_Q$-modules, and $\mathcal{C}_Q(-,i)$ is the canonical projective representation corresponding to the vertex $i$.

**Definition 2.2.9.** If $\mathcal{A}$ is a dg (resp. graded) category, a dg (resp. graded) $\mathcal{A}$-module is a dg (resp. graded) functor $M : \mathcal{A} \to \mathcal{C}^{dg}(k)$ (resp. $M : \mathcal{A} \to \mathcal{C}^{gr}(k)$). Denote by $\mathcal{C}^{dg}(\mathcal{A})$ (resp. $\mathcal{C}^{gr}(\mathcal{A})$) the category of dg (resp. graded) $\mathcal{A}$-modules.

For a dg (resp. graded) $\mathcal{A}$-module $M$ over a small dg (resp. graded) category $\mathcal{A}$, the associated $k[\mathcal{A}]$-module $k[M]$ is naturally dg (resp. graded), and the equivalence $\mathcal{C}(\mathcal{A}) \to \mathcal{C}(k[\mathcal{A}])$ restricts to an equivalence $\mathcal{C}^{dg}(\mathcal{A}) \to \mathcal{C}^{dg}(k[\mathcal{A}])$ of dg (resp. $\mathcal{C}^{gr}(\mathcal{A}) \to \mathcal{C}^{gr}(k[\mathcal{A}])$) of graded module categories.

Homology induces a functor $H_* : \mathcal{C}^{dg}(\mathcal{A}) \to \mathcal{C}^{gr}(H_* \mathcal{A})$ sending a module $M$ to $H_* M$. If $\mathcal{A}$ is small, the homology functor is compatible with $k[-]$ in the sense that

$$\xymatrix{ \mathcal{C}^{dg}(\mathcal{A}) \ar[r]^{H_*} \ar[d] & \mathcal{C}^{gr}(H_* \mathcal{A}) \ar[d] \cr \mathcal{C}^{dg}(k[\mathcal{A}]) \ar[r]^{H_*} & \mathcal{C}^{gr}(H_* k[\mathcal{A}]) }$$

commutes where the vertical maps are the canonical equivalences.

An ideal $\mathcal{I}$ in a category $\mathcal{C}$ is a specification of a subspace $\mathcal{I}(X,Y) \subset \mathcal{C}(X,Y)$ for every pair of objects $X,Y$ of $\mathcal{C}$ such that given any two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, if the composition $g \circ f \in \mathcal{I}(X,Z)$, then either $g \in \mathcal{I}(Y,Z)$ or $f \in \mathcal{I}(X,Y)$. The quotient category $\mathcal{C}/\mathcal{I}$ is the category with the same objects as $\mathcal{C}$, and with morphism spaces

$$(\mathcal{C}/\mathcal{I})(X,Y) = \mathcal{C}(X,Y)/\mathcal{I}(X,Y).$$
The composition is that inherited from $\mathcal{C}$. There is a canonical functor $\pi: \mathcal{C} \to \mathcal{C}/\mathcal{I}$ acting as the identity on objects, and sending a morphism $f: X \to Y$ to the coset $f + \mathcal{I}(X,Y)$.

If $\mathcal{C}$ is small and $\mathcal{I}$ and ideal, then $k[\mathcal{I}] = \bigoplus_{X,Y} \mathcal{I}(X,Y)$ is an ideal of the algebra $k[\mathcal{C}]$. The following lemma is immediate.

Lemma 2.2.10. The induced algebra homomorphism $\pi_* : k[\mathcal{C}] \to k[\mathcal{C}/\mathcal{I}]$ has kernel $k[\mathcal{I}]$ and hence induces an isomorphism

$$k[\mathcal{C}]/k[\mathcal{I}] \xrightarrow{\sim} k[\mathcal{C}/\mathcal{I}].$$

Suppose that $\mathcal{C}$ be a graded category, and $\mathcal{I}$ is a graded ideal in the sense that each $\mathcal{I}(X,Y)$ is generated over $\mathcal{C}_0(X,Y)$ by homogeneous elements. This is equivalent to the property that the inclusion

$$\bigoplus_{n \in \mathbb{Z}} \mathcal{I}(X,Y) \cap \mathcal{C}_n(X,Y) \to \mathcal{I}(X,Y)$$

is an isomorphism. If $\mathcal{C}$ is a dg category, then a dg ideal is a graded ideal where each of the subspaces $\mathcal{I}(X,Y)$ is closed under the differential $d_{XY}$.

Lemma 2.2.11. If $\mathcal{I}$ is a dg ideal of a dg category $\mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ is a dg category. If moreover, $\mathcal{A}$ is small, then $k[\mathcal{I}]$ is a dg ideal of the algebra $k[\mathcal{A}]$ and the isomorphism $k[\mathcal{A}]/k[\mathcal{I}] \xrightarrow{\sim} k[\mathcal{A}/\mathcal{I}]$ is a dg isomorphism.

Proof. We first note that $\mathcal{A}/\mathcal{I}$ is a graded category, with

$$(\mathcal{A}/\mathcal{I})_n(X,Y) = \mathcal{A}_n(X,Y)/(\mathcal{I}(X,Y) \cap \mathcal{A}_n(X,Y))$$

since $\mathcal{I}$ is a graded ideal. If $\mathcal{I}$ is a dg ideal, then the differential $d$ of $\mathcal{A}$ descends to a differential of $\mathcal{A}/\mathcal{I}$; indeed, if $f: X \to Y$ is in $\mathcal{I}(X,Y)$, then $d_{XY}(f)$ is in $\mathcal{I}(X,Y)$ by assumption, so $d$ preserves $\mathcal{I}(X,Y)$-cosets. That the isomorphism $k[\mathcal{A}]/k[\mathcal{I}] \xrightarrow{\sim} k[\mathcal{A}/\mathcal{I}]$ is a dg isomorphism then follows from Lemma 2.2.10.\[\square\]

If $\mathcal{A}$ is a small dg category, its homology $H_*\mathcal{A}$ is also small, so we can form the algebras $k[\mathcal{A}]$ and $k[H_*\mathcal{A}]$. Note that by definition of the homology category, there is a canonical isomorphism $H_*k[\mathcal{A}] \cong [H_*\mathcal{A}]$.

Suppose that $\mathcal{A}$ is a small $A_\infty$-category. Analogous to the situation of ordinary categories, we can form the associated $A_\infty$-algebra $k[\mathcal{A}]= \bigoplus_{X,Y} \mathcal{A}(X,Y)$. The structure maps are given by the higher compositions if defined, and are 0 otherwise. If $\mathcal{A}$ is a small dg category, Kadeishvili’s theorem endows $H_*\mathcal{A}$ with the structure of a (small) minimal $A_\infty$-category $A_\infty$-quasi-isomorphic to $\mathcal{A}$. Hence $k[H_*\mathcal{A}]$ is a minimal model for $k[\mathcal{A}]$, and thus $A_\infty$-isomorphic to $H_*k[\mathcal{A}]$.

2.3. Covering Theory. Following Bongartz-Gabriel [5], a quiver morphism $p: \tilde{Q} \to Q$ is a covering if for every vertex $i \in \tilde{Q}_0$ the induced maps

$$(2.3.1) \quad p_* : s^{-1}(i) \xrightarrow{\sim} s^{-1}(p(i)) \quad \text{and} \quad p_* : t^{-1}(i) \xrightarrow{\sim} t^{-1}(p(i))$$

are bijective.

Given a covering $p: \tilde{Q} \to Q$, the induced functor $p_* : \mathcal{C}_{\tilde{Q}} \to \mathcal{C}_Q$ is a covering of small categories in the sense that the induced maps

$$(2.3.2) \quad \bigoplus_{j' \in p^{-1}(j)} \mathcal{C}_{\tilde{Q}}(i,j') \xrightarrow{\sim} \mathcal{C}_Q(p_*(i),j) \quad \text{and} \quad \bigoplus_{i' \in p^{-1}(i)} \mathcal{C}_{\tilde{Q}}(i',j) \xrightarrow{\sim} \mathcal{C}_Q(i,p_*(j))$$

are isomorphisms.

Definition 2.3.1. A dg quiver morphism $p : (\tilde{Q}, \tilde{d}) \to (Q, d)$ which is also a covering will be called a dg covering. Note that the induced maps (2.3.2) are automatically dg isomorphisms, and the induced functor $p_* : \mathcal{C}_{\tilde{Q}} \to \mathcal{C}_Q$ is a dg functor.
We will mainly be interested in coverings arising from a group action on a quiver \( Q \), i.e., a homomorphism \( G \to \text{Aut}(Q) \) from \( G \) to the group of (orientation preserving) automorphisms of \( Q \). This implies that for each \( n \), \( G \) acts on the set \( Q_n \) of paths of length \( n \) in \( Q \), and this action is compatible with the source and target maps \( s,t : Q_n \to Q_0 \). A \( G \)-quiver is a quiver with a fixed free \( G \)-action. For a dg quiver \( (Q,d) \), the \( G \)-action is said to be differential graded if it acts by dg quiver automorphisms.

If a group \( G \) acts on a quiver \( Q \), it in particular acts on the arrow set \( Q_1 \) and the vertex set \( Q_0 \). The orbit quiver \( Q/G \) as is constructed as follows: The vertex set \( (Q/G)_0 = Q_0/G \) is the set of \( G \)-orbits of the vertices \( Q_0 \), and the arrow set \( (Q/G)_1 = Q_1/G \) is the set of \( G \)-orbits of the arrows \( Q_1 \). The source and target maps \( s,t : (Q/G)_1 \to (Q/G)_0 \) are defined by

\[
s(G\alpha) = Gs(\alpha) \quad \text{and} \quad t(G\alpha) = Gt(\alpha)
\]

which one verifies are independent of the orbit representatives.

Note there is a canonical quiver homomorphism \( \pi : Q \to Q/G \) that sends a vertex/arrow to the corresponding orbit. In fact, \( \pi \) has the following universal property:

**Lemma 2.3.2.** Let \( G \) act on a quiver \( Q \), and let \( f : Q \to Q' \) be a quiver homomorphism having the property that \( f(gi) = f(i) \) and \( f(g\alpha) = f(\alpha) \) for every \( i \in Q_0 \), \( \alpha \in Q_1 \), and \( g \in G \). Then there is a unique quiver homomorphism \( \bar{f} : Q/G \to Q' \) so that \( \pi \circ \bar{f} = f \).

**Proof.** The induced morphism \( \bar{f} \) is given by \( \bar{f}(Gi) = f(i) \) and \( \bar{f}(G\alpha) = f(\alpha) \), which is well-defined by assumption on \( f \). It is clear that \( \bar{f} \) is a quiver morphism, and is the unique such with \( \pi \circ \bar{f} = f \). \( \square \)

Let \( (Q,d) \) be a dg quiver with dg \( G \)-action and define a function \( d_G : (Q/G)_1 \to \mathbf{k}(Q/G)_2 \) by \( d_G(G\alpha) = Gd(\alpha) \). The function \( d_G \) is well-defined since \( G \) acts by dg automorphisms.

**Lemma 2.3.3.** The function \( d_G \) is a differential on \( Q/G \), and the canonical map \( \pi : Q \to Q/G \) is a dg quiver homomorphism.

**Proof.** It is clear that \( d_G^2 = 0 \), since it acts as \( d^2 \) on orbit representatives. The statement that \( \pi \) is a dg quiver homomorphism means that \( \pi \circ d = d_G \circ \pi \) i.e., \( d(\alpha) \) lies in the orbit \( d_G(G\alpha) \), which by definition is the orbit of \( d(\alpha) \). \( \square \)

In order to describe the effect of group actions in the path category, we adopt the covering theory of Bautista Liu [3]. Recall that an action of a group \( G \) on a category \( \mathcal{C} \) is a group homomorphism \( G \to \text{Aut}(\mathcal{C}) \) where \( \text{Aut}(\mathcal{C}) \) is the group of \( \mathbf{k} \)-linear automorphisms of \( \mathcal{C} \). The \( G \)-action is free if the objects \( gX \) and \( X \) are non-isomorphic for any \( g \neq 1 \), and \( X \) indecomposable. For brevity, a category with a fixed free \( G \)-action will be called a \( G \)-category.

The orbit category of a \( G \)-category is the category \( \mathcal{C}^G \) with the same objects as \( \mathcal{C} \), and morphism spaces

\[
\mathcal{C}^G(X,Y) = \bigoplus_{g \in G} (X,gY)
\]

with composition determined by

\[
v \circ u = (gv) \circ u
\]

for \( u : X \to gY \) and \( v : Y \to hZ \) in \( \mathcal{C} \). There is an evident functor \( \pi : \mathcal{C} \to \mathcal{C}^G \) acting as the identity on objects and morphisms.

**Definition 2.3.4** (Bautista Liu [3] Definition 2.3). A functor \( F : \mathcal{C} \to \mathcal{D} \) is said to be \( G \)-stable if there is a collection \( \gamma = \{ \gamma^g : F \circ g \to \tilde{F} \} \) of invertible natural transformations so that

\[
F(ghX) \xrightarrow{\gamma^h_X} FX \\
\downarrow \gamma^g_X \quad \quad \quad \quad \downarrow \gamma^h_X \\
F(hX)
\]

commutes for every object \( X \); the collection \( \gamma \) is called a stabilization of \( F \).
In $\mathcal{C}^G$, there are distinguished morphisms $\sigma_X^g \in \mathcal{C}^G(gX, X)$ whose $g$-component is $id_{gX} : gX \to gX$, and all others are 0. The morphism $\sigma_X^g$ is clearly invertible, with inverse $\sigma^{-1}_X^{-1} = id_X : g^{-1}gX \to X$. The collection of natural transformations $\sigma^g$ satisfy the commutativity property of Definition 2.3.4 and so the functor $\pi : \mathcal{C} \to \mathcal{C}^G$ is $G$-stable.

The following proposition is well-known (cf. [17]).

**Proposition 2.3.5.** The pair $(\pi, \sigma)$ is universal among $G$-stable functors $F : \mathcal{C} \to \mathcal{D}$ with a fixed stabilization. Precisely, if $F : \mathcal{C} \to \mathcal{D}$ is $G$-stable with stabilization $\gamma$, there is a unique functor $\bar{F} : \mathcal{C}^G \to \mathcal{D}$ so that

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow \pi & & \downarrow \bar{F} \\
\mathcal{C}^G & \xrightarrow{\bar{F}} & \mathcal{D}
\end{array}
\]

commutes up to natural isomorphism and $\gamma_X^g = \bar{F}(\sigma_X^g)$ for each $X$ and $g \in G$.

**Proof.** The functor $\bar{F}$ is given by sending a morphism $u : X \to gY$ to the composition $\gamma_X^g \circ Fu$, where $\gamma$ is a $G$-stabilization of $F$. Note that necessarily $\bar{F}(\sigma_X^g) = \gamma_X^g$. □

If $\bar{Q}$ is a $G$-quiver, a covering $p : \bar{Q} \to Q$ is said to be a Galois $G$-cover if the action of $G$ preserves fibres $p^{-1}(i)$ and is fibrewise transitive.

**Lemma 2.3.6.** If $G$ acts freely, then the canonical isomorphism $\pi : Q \to Q/G$ is a Galois $G$-covering.

**Proof.** It is clear from the definitions that both induced maps $p_*$ from (2.3.1) are surjective. Suppose that two arrows $\alpha$ and $\beta$ with source $i$ lie in the same $G$-orbit, i.e., $\alpha = g\beta$ for some $g \in G$. Since $s(\alpha) = s(g\beta) = gs(\beta)$ and $G$ we have $i = gi$ which implies $g = 1$ since $G$ acts freely. Hence $\alpha = \beta$ and so $p_*$ is injective. One proves that the induced map for a fixed target vertex is bijective analogously.

By definition, the fibre $\pi^{-1}(Gi)$ is the $G$-orbit of $i$, hence $G$ preserves fibres and is fibrewise transitive. □

A $G$-action on a quiver $Q$ induces actions on both the path algebra $kQ$ and the path category $\mathcal{C}_Q$ by $g \cdot u = g(u)$ for a path/morphism $u$. If $(Q, d)$ is a dg quiver, the condition of a $G$-action being dg is equivalent to the above actions on $kQ$ and $\mathcal{C}_Q$ commute with differentials.

If $G$ acts freely on $Q$, the induced action on $\mathcal{C}_Q$ is free, and the induced functor $\pi_* : \mathcal{C}_Q \to \mathcal{C}_{Q/G}$ is manifestly $G$-stable with stabilization $\gamma : \pi_* \circ g \xrightarrow{\sim} \pi_*$ given by $\gamma_X^g : Ggi \to Gi$.

**Proposition 2.3.7.** The induced functor $\mathcal{C}_{Q/G} \xrightarrow{\sim} \mathcal{C}_{Q/G}$ is an equivalence (but not necessarily an isomorphism).

**Proof.** It is clear that the functor is surjective on objects, so we only need to show that the maps $\mathcal{C}_Q^G(i, j) \to \mathcal{C}_{Q/G}(Gi, Gj)$ are isomorphisms. Clearly the above map is surjective; an argument analogous to that in the proof of Lemma 2.3.6 shows that it is injective, and hence an equivalence. □

We will want to work with Galois coverings between not necessarily locally finite dimensional categories, and so need a slightly more involved notion of covering than that of Bongartz-Gabriel [5] better suited to our purposes.

**Definition 2.3.8 ([3] Definition 2.8).** A functor $F : \mathcal{C} \to \mathcal{D}$ is a Galois $G$-covering if

1. $F$ is $G$-stable with stabilization $\gamma$ so that the maps

\[
\bigoplus_{g \in G} \mathcal{C}(X, gY) \xrightarrow{\sim} \mathcal{D}(FX, FY) \quad \bigoplus_{g \in G} \mathcal{C}(gX, Y) \xrightarrow{\sim} \mathcal{D}(FX, FY)
\]

\[
(u_g)_{g \in G} \mapsto \sum_{g \in G} \gamma_Y^g \circ F(u_g) \quad (v_g)_{g \in G} \mapsto \sum_{g \in G} F(v_g) \circ (\gamma_X^g)^{-1}
\]

are isomorphisms;
(2) $F$ is essentially surjective;
(3) $FX$ is indecomposable for any indecomposable $X$;
(4) If $X, Y$ are indecomposable with $FX \cong FY$, then there is some $g \in G$ so that $Y = gX$.

The group $G$ is said to be the Galois group the covering.

**Remark 2.3.9.** Suppose that the $G$-action on $\mathcal{C}$ is directed in the sense that for any two indecomposable objects $X$ and $Y$, there is at most one $g$ so that both $\mathcal{C}(gY, X)$ and $\mathcal{C}(X, gY)$ are nontrivial. Then the group element $g \in G$ as per (4) is necessarily unique.

If $\mathcal{C}$ and $\mathcal{D}$ are locally finite dimensional categories, then a Galois covering is simply a covering in the sense of equation (2.3.2) such that $G$ preserves fibres and acts fibrewise transitively. In particular, the covering $\pi_* : \mathcal{C}_Q \to \mathcal{C}_{Q/G}$ is a Galois $G$-covering. By virtue of Proposition 2.3.7, the canonical functor $\mathcal{C}_Q \to \mathcal{C}_{Q/G}$ is a Galois $G$-covering. Moreover, if $Q$ is a dg quiver and $G$ acts by dg automorphisms then it is a dg Galois $G$-covering.

Given small categories $\mathcal{C}$ and $\mathcal{D}$ and a Galois $G$-covering $F : \mathcal{C} \to \mathcal{D}$, an equivalence $\bar{F} : \mathcal{C}^G \to \mathcal{D}$ does not in general induce an isomorphism between the associated algebras $k[\mathcal{C}^G]$ and $k[\mathcal{D}]$. In fact, it induces an isomorphism if and only if the equivalence is strict.

We would like a way of recovering the associated algebra $k[\mathcal{C}]$ from $k[\mathcal{C}^G]$, and in particular, recover $k[\mathcal{C}_{Q/G}]$ from $k[\mathcal{C}_{Q/G}^G]$ for a quiver $Q$. To this end, we introduce the notion of a slice category.

**Definition 2.3.10.** A slice of a Galois $G$-covering $F : \mathcal{C} \to \mathcal{D}$ is an additive section $S$ of the surjective map $F : Ob\mathcal{C} \to Ob\mathcal{D}$ induced by $F$, i.e., a choice of indecomposable object $SY$ in $\mathcal{C}$ with $FSY = Y$ for each indecomposable $Y$ in $\mathcal{D}$. The slice category $\mathcal{C}^S$ is the full subcategory of $\mathcal{C}^G$ generated by the image of $S$.

**Lemma 2.3.11.** Any covering $F$ admits a slice.

**Proof.** Since $F : \mathcal{C} \to \mathcal{D}$ is a covering, the induced map on object sets is surjective. Choose an arbitrary section of this map. Since $\mathcal{C}$ and $\mathcal{C}^G$ have the same objects, this section is a slice of $F$. □

**Remark 2.3.12.** We do not require a slice of $F$ to be a functor itself, but only an assignment on the level of objects. The proof of the following proposition however shows that a fortiori a slice always determines a functor.

**Proposition 2.3.13.** The restriction of the induced functor $\bar{F} : \mathcal{C}^G \to \mathcal{D}$ to a slice category $\mathcal{C}^S$ induces an isomorphism

$$\mathcal{C}^S \xrightarrow{\sim} \mathcal{D}$$

of small categories for any section $S$ of $F$. In particular, the induced algebra homomorphism $k[\mathcal{C}^S] \xrightarrow{\sim} k[\mathcal{D}]$ is an isomorphism.

**Proof.** Since $F$ is a Galois $G$-covering, for every pair of objects $X, Y$ of $\mathcal{D}$, there is by definition an isomorphism

$$\bigoplus_{g \in G} \mathcal{C}(SX, gSY) \xrightarrow{\sim} \mathcal{D}(X, Y),$$

which just so happens to be the canonical map induced by the functor $\bar{F} : \mathcal{C}^G \to \mathcal{D}$. Extend the association $X \mapsto SX$ to a functor $S : \mathcal{D} \to \mathcal{C}^G$ by sending a morphism $u : X \to Y$ to the unique element $S(u)$ of $\mathcal{C}^G(SX, SY)$ mapping to $u$ under $F_{SX, SY}$. By construction, $F \circ S = Id_{\mathcal{D}}$, and the restriction of $S \circ \bar{F} |_{\mathcal{C}^S} = Id_{\mathcal{D}}$, since the objects of $\mathcal{C}^S$ are of the form $SX$ for $X$ in $\mathcal{D}$. □

If $F$ is a dg covering, the slice category $\mathcal{C}^S$ is automatically a dg subcategory of $\mathcal{C}$.

We now want to study the behavior of coverings and slices under quotienting by ideals and passing to homology in the dg setting.

An ideal $\mathcal{I}$ of a $G$-category $\mathcal{C}$ is $G$-stable if $gu \in \mathcal{I}(gX, gY)$ for every $u \in \mathcal{I}(X, Y)$ and $g \in G$. The $G$-action on $\mathcal{C}$ descends to a well-defined $G$-action on the quotient category $\mathcal{C}/\mathcal{I}$ in the obvious manner when $\mathcal{I}$ is $G$-stable.

Let $\mathcal{I}^G(X, Y) = \bigoplus_{g \in G} (X, gY)$, which determines an ideal in the orbit category $\mathcal{C}^G$.

**Lemma 2.3.14.** The categories $\mathcal{C}^G/\mathcal{I}^G$ and $(\mathcal{C}/\mathcal{I})^G$ are naturally equivalent.
Proof. The functor $\mathcal{C}/\mathcal{I} \to \mathcal{C}^{G}/\mathcal{I}^{G}$ is $G$-stable and induces a functor $(\mathcal{C}/\mathcal{I})^{G} \xrightarrow{\sim} \mathcal{C}^{G}/\mathcal{I}^{G}$. The composition $\mathcal{C} \to \mathcal{C}/\mathcal{I} \to (\mathcal{C}/\mathcal{I})^{G}$ is $G$-stable and the induced functor $\mathcal{C}^{G} \to (\mathcal{C}/\mathcal{I})^{G}$ annihilates $\mathcal{I}^{G}$, hence induces a functor $\mathcal{C}^{G}/\mathcal{J}^{G} \to (\mathcal{C}/\mathcal{I})^{G}$ inverse the the other one by universality.

Lemma 2.3.15. Suppose $F : \mathcal{C} \to D$ is a Galois $G$-covering and $\mathcal{I}$ and $\mathcal{J}$ ideals of $\mathcal{C}$ and $\mathcal{D}$ respectively such that $\mathcal{I}$ is $G$-invariant and the isomorphisms (2.3.3) restrict to isomorphisms

$$\bigoplus_{g \in G} \mathcal{I}(X,gY) \xrightarrow{\sim} \mathcal{J}(FX,FY).$$

Then the induced functor $\bar{F} : \mathcal{C}/\mathcal{I} \to \mathcal{D}/\mathcal{J}$ is a Galois $G$-covering.

Proof. The composition $\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{J}$ annihilates $\mathcal{J}$, and so there is an induced functor $\bar{F} : \mathcal{C}/\mathcal{I} \to \mathcal{D}/\mathcal{J}$, which we claim is a Galois covering. Since $\mathcal{I}$ is $G$-stable, there is an induced $G$-action on $\mathcal{C}/\mathcal{I}$, and the maps $\gamma_{Y}^0 + \mathcal{I}(g,Y)$ give a $G$-stabilization of $\bar{F}$.

To show that $\bar{F}$ is Galois, we need to show that for any two objects $X$ and $Y$ of $\mathcal{C}$ the map

$$(2.3.4) \bigoplus_{g \in G} (\mathcal{C}/\mathcal{I})(X,gY) \to (\mathcal{D}/\mathcal{J})(FX,FY)$$

is an isomorphism. The map (2.3.4) fits into a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \bigoplus_{g \in G} \mathcal{I}(X,gY) & \longrightarrow & \bigoplus_{g \in G} \mathcal{C}(X,gY) & \longrightarrow & \bigoplus_{g \in G} (\mathcal{C}/\mathcal{I})(X,gY) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{J}(FX,FY) & \longrightarrow & \mathcal{D}(FX,FY) & \longrightarrow & (\mathcal{D}/\mathcal{J})(FX,FY) & \longrightarrow & 0 \\
\end{array}
\]

where the first and second vertical map are isomorphisms by assumption. Hence (2.3.4) is an isomorphism.

The following corollary is immediate.

Corollary 2.3.16. Given the setup of the previous lemma and a slice $S$ of $F$, then $S$ is also a slice of $\bar{F}$ and the slice category $(\mathcal{C}/\mathcal{I})^S$ is isomorphic to $\mathcal{C}^S/\mathcal{I}^S$ where $\mathcal{I}^S = \mathcal{I}^G \cap \mathcal{C}^S$.

Suppose now that $F : \mathcal{A} \to \mathcal{B}$ is a dg $G$-Galois functor, and the stabilization $\gamma^g$ is given by homogeneous cycles in $\mathcal{B}$, where $G$ acts by dg automorphisms of $\mathcal{A}$. The action of $G$ descends to an action on $Z_*\mathcal{A}$ preserving the ideal $B_*\mathcal{A}$ of boundaries, and hence gives a graded $G$-action on the homology category $H_*\mathcal{A}$.

Lemma 2.3.17. In the situation above, the induced homology functor

$$H_*F : H_*\mathcal{A} \to H_*\mathcal{B}$$

is a Galois $G$-covering.

Proof. First consider the restricted map $\bigoplus_{g \in G} Z_*\mathcal{A}(X,gY) \to Z_*\mathcal{B}(FX,FY)$, which indeed maps into cycles: the $\gamma^g_Y$ are cycles and $F_{X,Y}$ is a dg map. It is injective since $Z_*\mathcal{A}$ is a subcategory of $\mathcal{A}$. If $v : FX \to FY$ is a cycle, let $u = (u_g : X \to gY)$ be the unique sum of morphisms in $\mathcal{A}$ with $v = \sum \gamma^g \circ F(u_g)$. Then

$$0 = dv = \sum (-1)^{|\gamma^g|} \gamma^g \circ F(du_g)$$

and so $du = 0$ by injectivity. Hence, $F$ restricts to a Galois covering of the cycle categories.

For two objects $X$ and $Y$ of $\mathcal{A}$, we claim that the inverse image of $B_*\mathcal{B}(FX,FY)$ is $B_*\mathcal{A}^G(X,Y)$, and so the Lemma will follow from Lemma 2.3.15. Clearly a boundary $X \to gY'$ maps to a boundary, since $\gamma^g_Y$ is a cycle. If $u : X \to gY$ maps to a boundary, then $\gamma^g_Y \circ F(u) = ds$. Let $s'$ be the unique element of $\mathcal{A}(X,gY)$ with $\gamma^g_Y \circ F(s') = s$. Then

$$\gamma^g_Y \circ F(ds') = (-1)^{|\gamma^g_Y|} d(\gamma^g_Y \circ F(s)) = (-1)^{|\gamma^g_Y|} ds$$

and hence $u = \pm ds'$ by injectivity.

Corollary 2.3.18. Given a slice $S$ of $F$, the slice category $\mathcal{A}^S$ is a dg category. Moreover, $S$ defines a slice of $H_*F$, and the slice category $(H_*\mathcal{A})^S$ is isomorphic to $H_*\mathcal{A}$.
3. Derived categories via differential graded categories

3.1. Graded categories via actions. The orbit category \( \mathcal{C}^G \) has additional structure when \( G = \mathbb{Z} \). A category with free \( \mathbb{Z} \)-action is just a category \( \mathcal{T} \) with a distinguished automorphism \( S \) given by the action of \( 1 \in \mathbb{Z} \) on \( \mathcal{T} \). A category \( \mathcal{T} \) with fixed automorphism \( S \) will be called a \( \mathbb{Z} \)-category.

**Examples 3.1.1.**  
(1) Every triangulated category is (equivalent to) a \( \mathbb{Z} \)-category under the translation functor,  
(2) the category of chain complexes in an abelian (or exact) category is a \( \mathbb{Z} \)-category with shift functor given by the shift \([1]\) of chain complexes (see Definition 1.1.3).

The orbit category of a \( \mathbb{Z} \)-category has additional structure: Denote by \( \mathcal{T}_n(X,Y) = \mathcal{T}(X,S^nY) \) so that \( \mathcal{T}_n(X,Y) = \bigoplus_{n \in \mathbb{Z}} \mathcal{T}_n(X,Y) \). Note that composition is given componentwise by maps  
\[
\mathcal{T}_p(Y,Z) \otimes \mathcal{T}_q(X,Y) \to \mathcal{T}_{p+q}(X,Z)
\]
and so \( \mathcal{T}^\mathbb{Z} \) is a graded category. In order to minimize confusion between composition in \( \mathcal{T} \) and composition in \( \mathcal{T}^\mathbb{Z} \), set \( g \cdot f \overset{\text{def}}{=} S^p(g) \circ f \) for \( f \in \mathcal{T}_p(X,Y) \).

If \( F : \mathcal{T} \to \mathcal{D} \) is \( \mathbb{Z} \)-stable, any stabilization \( \gamma \) is uniquely determined by the natural isomorphism \( \gamma^2 : F \circ S \sim F \); conversely, any natural isomorphism \( \gamma : F \circ S \sim F \) determines a \( \mathbb{Z} \)-stabilization of \( F \) by setting  
\[
\gamma_X^n = \gamma_{S^{n-1}X} \circ \gamma_{S^{n-2}X} \circ \cdots \circ \gamma_X.
\]

The canonical \( \mathbb{Z} \)-stabilization \( \sigma \) of \( \pi : \mathcal{T} \to \mathcal{T}^\mathbb{Z} \) is determined by the degree +1 morphisms \( \sigma_X : SX \to X \) of \( \mathcal{T}^\mathbb{Z} \), so \( \sigma : \pi \circ S \sim \pi \) is a degree +1 natural transformation. By slight abuse of terminology, such stabilization will be said to have degree +1.

**Remark 3.1.2.** When the group \( G = \mathbb{Z} \), the “canonical” stabilization \( \sigma \) is no longer canonical. The natural isomorphism \( \bar{\sigma} \) with \( \bar{\sigma}_X = -\sigma_X \) determines a stabilization with \( \bar{\sigma}^n = (-1)^n \sigma^n \). Unless otherwise specified, we will assume that the functor \( \pi : \mathcal{T} \to \mathcal{T}^\mathbb{Z} \) is stabilized by the \( \mathbb{Z} \)-stabilization \( \sigma \) with \( \sigma_X = \text{id}_{SX} \), and refer to \( \bar{\sigma} \) as the anticanonical stabilization.

If \( \mathcal{D} \) is a graded category, and \( F : \mathcal{T} \to \mathcal{D} \) is a \( \mathbb{Z} \)-stable graded functor where we view \( \mathcal{T} \) as a graded category concentrated in degree 0, the induced functor \( F : \mathcal{T}^\mathbb{Z} \to \mathcal{D} \) is graded as well provided some mild assumptions are satisfied.

**Lemma 3.1.3.** If \( F : \mathcal{T} \to \mathcal{D} \) is a \( \mathbb{Z} \)-stable graded functor with degree +1 stabilization \( \gamma : F \circ S \to F \), the induced functor \( F : \mathcal{T}^\mathbb{Z} \to \mathcal{D} \) is a graded functor.

**Proof.** To show that \( F \) is a graded functor, we need to show that the canonical maps  
\[
F_{XY} : \mathcal{T}^\mathbb{Z}(X,Y) \to \mathcal{D}(FX, FY)
\]
are chain (i.e., homogeneous of degree 0) maps. For any \( n \), the restriction of \( F_{XY} \) to \( \mathcal{T}_n^\mathbb{Z}(X,Y) \) is given by the composition \( f \mapsto \gamma^2_X \circ F(f) \). But \( F(f) \) is a degree 0 morphism of \( \mathcal{D} \) by assumption so that \( F_{XY}(f) \) has degree \( n \).

A functor \( F : (\mathcal{T}, S) \to (\mathcal{T}', S') \) between \( \mathbb{Z} \)-categories is a \( \mathbb{Z} \)-functor provided that it strictly commutes with the automorphisms \( S \) and \( S' \). The following is an immediate corollary of the above lemma, which we prove for completeness.

**Corollary 3.1.4.** A \( \mathbb{Z} \)-functor \( F : (\mathcal{T}, S) \to (\mathcal{T}', S') \) induces a graded functor \( F^\mathbb{Z} : \mathcal{T}^\mathbb{Z} \to (\mathcal{T}')^\mathbb{Z} \) so that the diagram  
\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{F} & \mathcal{T}' \\
\pi \downarrow & & \downarrow \pi' \\
\mathcal{T}^\mathbb{Z} & \xrightarrow{F^\mathbb{Z}} & (\mathcal{T}')^\mathbb{Z}
\end{array}
\]
of categories and functors commutes.
Remark 3.2.2. One has for \( f : X \to Y \) in \( \mathcal{T} \), the diagram

\[
\begin{array}{ccc}
(\pi' \circ F)(SX) & \xrightarrow{(\pi' \circ S')(FX)} & (\pi' \circ F)(X) \\
\downarrow (\pi' \circ (Sf)) & & \downarrow (\pi' \circ (S'f)) \\
(\pi' \circ F)(SY) & \xrightarrow{(\pi' \circ S')(FY)} & (\pi' \circ F)(Y)
\end{array}
\]

commutes, since \( S \circ F = S' \circ F \) and \( \sigma' : \pi' \circ S' \xrightarrow{\sim} \pi' \) is a natural transformation. Moreover, the components \( \gamma_X = \pi'(\sigma_{\pi X}') \) have degree 1, so there is an induced functor \( F^Z : \mathcal{T}^Z \to (\mathcal{T}')^Z \) making the diagram commute. \( \square \)

The action of the automorphism \( S \) extends to a graded automorphism \( S^Z \) of \( \mathcal{T}^Z \) with the same action on objects as \( S \), and action on morphisms determined by the maps

\[
S^Z_{X,Y} : \mathcal{T}_n^Z(X,Y) \to \mathcal{T}_n^Z(S^Z X, S^Z Y)
\]

\[
f \mapsto (-1)^n S_{X,S^n Y}(f)
\]

where \( S_{X,S^n Y} : \mathcal{T}(X, S^n Y) \to \mathcal{T}(S X, S^{n+1} Y) \) is the canonical map.

**Lemma 3.1.5.** The application \( S^Z : \mathcal{T}^Z \to \mathcal{T}^Z \) is functorial.

**Proof.** We verify that \( S^Z(g \bullet f) = S^Z(g) \bullet S^Z(f) \). Indeed,

\[
S^Z(g \bullet f) = (-1)^{p+q} S^Z(g \bullet f) = (-1)^{p+q} S^Z(S^p(g) \circ f)
\]

\[
= S^p((-1)^q S^p(g) \circ f) = S^p(S^Z(g)) \circ S^Z(f) = S^Z(g) \bullet S^Z(f)
\]

for \( f : X \to S^n Y \) and \( g : Y \to S^q Z \) in \( \mathcal{T} \). \( \square \)

We now record some lemmas about morphisms in the associated graded category \( \mathcal{T}^Z \).

**Lemma 3.1.6.** The functor \( S^Z : \mathcal{T}^Z \to \mathcal{T}^Z \) is induced by the \( Z \)-functor \( S : \mathcal{T} \to \mathcal{T} \) provided that the functor \( \pi : \mathcal{T} \to \mathcal{T}^Z \) is stabilized by the anticanonical stabilization \( \sigma \).

**Proof.** By universality of the associated graded category, \( S(\sigma_X) = -\sigma_X \). Given a degree \( n \) morphism \( f \), the composition \( \sigma_Y^n \bullet f \) is a degree 0 morphism, and so lies in \( \mathcal{T} \). But then

\[
S(f) = S(\sigma_Y^n) \bullet S(\sigma_Y^n \bullet f) = (-1)^n \sigma_Y^n \bullet S(f) = (-1)^n S(f)
\]

as a degree \( n \) morphism \( X \to Y \) in \( \mathcal{T}^Z \). Hence \( S(f) = S^Z(f) \), completing the proof. \( \square \)

**Lemma 3.1.7.** One has

\[
f \bullet \sigma_X = (-1)^n \sigma_Y \bullet S^Z(f)
\]

for \( f \) homogeneous of degree \( n \).

**Proof.** Notice that \( f \bullet \sigma_X = S f \circ \text{id}_{S^X} = S f \) and \( \sigma_Y \bullet S f = S^n (\text{id}_{S^Y}) \circ S f = S f \), so the lemma holds. \( \square \)

### 3.2. DG structures on associated graded categories

Let \( \mathcal{A} \) be a graded category such that the degree 0 subcategory \( \mathcal{A}_0 \) is a \( Z \)-category with automorphism \( S \).

**Definition 3.2.1.** The category \( \mathcal{A} \) is said to be *generated over degree 0* if there is a degree +1 natural isomorphism \( s : S \xrightarrow{\sim} \text{Id}_0 \) where \( \text{Id}_0 \) is the restriction of the identity functor \( \text{Id} \) of \( \mathcal{A} \) to the degree 0 subcategory \( \mathcal{A}_0 \).

If \( \mathcal{A} \) is a dg category, it is said to be *dg generated over degree 0* if the components \( s_X : SX \to X \) of the natural isomorphism \( s \) are cycles in \( \mathcal{A}(X, SX) \).

**Remark 3.2.2.** Note that if \( s \) is an isomorphism, then

\[
d(s) = d(s \circ s^{-1} \circ s) = d(s) + (-1)^{|s|} s \circ d(s^{-1}) \circ s + d(s)
\]

so that \( d(s) = -(-1)^{|s|} s \circ d(s^{-1}) \circ s \). Hence \( s \) is a cycle if and only if \( s^{-1} \) is a cycle.
Example 3.2.3. If $\mathcal{F}$ is a $\mathbb{Z}$-category, the associated graded category $\mathcal{F}^\mathbb{Z}$ is generated over degree 0.

Generation over degree 0 imposes a great deal of structure on the graded category $\mathcal{A}$. Morally, being generated over degree 0 means that all of the “important” information about $\mathcal{A}$ is contained in its degree 0 subcategory.

Lemma 3.2.4. If $\mathcal{A}$ is generated over degree 0, then the automorphism $S$ of $\mathcal{A}_0$ extends to a graded automorphism $S^\mathbb{Z}$ of $\mathcal{A}$ with
\[ s_Y \circ S^\mathbb{Z}(f) = (-1)^n f \circ s_X \]
for $f : X \to Y$ homogeneous of degree $n$. Moreover, if $\mathcal{A}$ is a dg category dg generated over degree 0, the automorphism $S^\mathbb{Z}$ is a dg automorphism.

Proof. We first remark that any degree 0 morphism $f$ necessarily satisfies $S(f) = s_Y^{-1} \circ f \circ s_X$ by naturality of $s$. For a homogeneous degree $n$ morphism $f : X \to Y$, define $S^\mathbb{Z}(f) = (-1)^n s_Y^{-1} \circ f \circ s_X$. The association $S^\mathbb{Z}$ is clearly functorial, and satisfies the condition of the Lemma.

Suppose now that $\mathcal{A}$ is dg, and $f : X \to Y$ is homogeneous of degree $n$. Then
\[ d_{XY}(S^\mathbb{Z}f) = (-1)^n d_{XY}(s_Y^{-1} \circ f \circ s_X) = (-1)^{n-1}s_Y^{-1} \circ d_{XY}(f) \circ s_X = S^\mathbb{Z}(d_{XY}(f)) \]
since $s_Y^{-1}$ and $s_X$ are cycles and the degree of $s_Y^{-1}$ is $-1$. Hence $S^\mathbb{Z}$ is a dg functor.

Lemma 3.2.5. If $\mathcal{A}$ is a (differential) graded category (dg) generated over degree 0, then the maps
\[ s_* : \mathcal{A}(X,Y) \to \mathcal{A}(X,S^\mathbb{Z}Y)[1] \quad s^* : \mathcal{A}(X,Y) \to \mathcal{A}(S^\mathbb{Z}X,Y)[-1] \]
\[ f \mapsto s_Y^{-1} \circ f \quad f \mapsto (-1)^n f \circ s_X \]
are degree 0 (differential) graded isomorphisms. Moreover, the diagram
\[ \begin{array}{ccc}
\mathcal{A}(X,Y) & \xrightarrow{s_*} & (X,S^\mathbb{Z}Y)[1] \\
\downarrow{s^*} & & \downarrow{s^*[1]} \\
\mathcal{A}(S^\mathbb{Z}X,Y)[-1] & \xrightarrow{s_*[-1]} & \mathcal{A}(S^\mathbb{Z}X,S^\mathbb{Z}Y)
\end{array} \]
of (differential) graded spaces and (differential) graded homomorphisms commutes.

Proof. It is clear that $s_*$ and $s^*$ are graded isomorphisms; let us show that they are compatible with differentials. Indeed, for a homogeneous degree $n$ morphism $f : X \to Y$, one calculates
\[ d_{X,SY}[1](s_*(f)) = -d_{X,SY}(s_Y^{-1} \circ f) = s_Y \circ d_{XY}(f) = s_*d_{XY}(f) \]
\[ d_{S_X,Y}[-1](s^*(f)) = -(-1)^n d_{S_X,Y}(f \circ s_X) = (-1)^{n+1}d_{XY}(f) \circ s_X = s^*(d_{XY}(f)) \]
since $d[\pm 1] = -d$.

Since $S^\mathbb{Z}(f) = (-1)^n s_X \circ f \circ s_Y^{-1}$, in the following diagram
\[ \mathcal{A}_n(X,Y) \xrightarrow{s_*} \mathcal{A}_{n-1}(X,S^\mathbb{Z}Y) \]
\[ \mathcal{A}_{n+1}(S^\mathbb{Z}X,Y) \xrightarrow{s_*} \mathcal{A}_n(S^\mathbb{Z}X,S^\mathbb{Z}Y) \]
the-top right triangle commutes, while the bottom-left triangle anti-commutes. After shifting, the map $s^*[-1] : \mathcal{A}(S^\mathbb{Z}X,Y)[-1] \to \mathcal{A}(S^\mathbb{Z}X,S^\mathbb{Z}Y)$ sends a degree $n+1$ morphism $g : S^\mathbb{Z}X \to Y$ of $\mathcal{F}^\mathbb{Z}$ to $(-1)^n g \circ s_X$ since such $g$ has degree $n$ in the shifted morphism space $\mathcal{A}(S^\mathbb{Z}X,Y)[-1]$. Therefore the diagram commutes after shifting gradings, completing the proof.

Corollary 3.2.6. The morphisms $s_*$ and $s^*$ induce a pair of mutually inverse natural transformations of respective degrees $-1$ and $1$ between $S^\mathbb{Z}$ and the identity functor.
The above corollary implies that the inclusion functor \( \mathcal{A}_0 \to \mathcal{A} \) is \( \mathbb{Z} \)-stable, and so induces a graded functor \( (\mathcal{A}_0)^\mathbb{Z} \to \mathcal{A} \).

**Proposition 3.2.7.** If \( \mathcal{A} \) is generated over degree 0, the induced functor

\[
(\mathcal{A}_0)^\mathbb{Z} \xrightarrow{\sim} \mathcal{A}
\]

is an equivalence of categories.

**Proof.** Recall that the functor \( (\mathcal{A}_0)^\mathbb{Z} \to \mathcal{A} \) is given by the degree 0 maps

\[
(\mathcal{A}_0)^\mathbb{Z}(X,Y) \to \mathcal{A}(X,Y)
\]

\[
f \mapsto (-1)^n s^n_Y \circ f
\]

for \( f : X \to S^nY \) in \( \mathcal{A}_0 \). By repeated application of Lemma 3.2.5, this map is an isomorphism, hence the functor is fully faithful. Since both \( (\mathcal{A}_0)^\mathbb{Z} \) and \( \mathcal{A} \) have the same objects as \( \mathcal{A}_0 \), it is an equivalence. \( \Box \)

One would like that if a dg category \( \mathcal{A} \) is generated over degree 0, then the associated graded category \( (\mathcal{A}_0)^\mathbb{Z} \) is naturally a dg category and the equivalence (3.2.1) is a dg equivalence. Unfortunately, the author does not know of natural conditions on the pair \( (\mathcal{A}_0, S) \) which induce a dg structure on the associated graded category. However, the equivalence (3.2.1) induces a differential on the associated graded category which we now describe.

For such \( \mathcal{A} \), the degree 1 isomorphisms \( s_X : SX \to X \) induce isomorphisms

\[
s^n_X : \mathcal{A}_n(X,Y) \xrightarrow{\sim} \mathcal{A}_0(X,S^nY)[n]
\]

for each \( n \in \mathbb{Z} \). Define a map

\[
\partial_{XY} : \mathcal{A}_0(X,S^nY) \to \mathcal{A}_0(X,S^{n-1}Y)
\]

\[
f \mapsto (-1)^n s_{S^{n-1}Y} \circ d_{X,S^nY}(f)
\]

which is well defined, since \( |\partial(f)| = 1 + |f| - 1 = |f| = 0 \).

**Proposition 3.2.8.** The maps \( \partial_{XY} \) endow \( (\mathcal{A}_0)^\mathbb{Z} \) with the structure of a dg category, and the equivalence \( (\mathcal{A}_0)^\mathbb{Z} \xrightarrow{\sim} \mathcal{A} \) is a dg functor.

**Proof.** We first verify that the maps \( \partial_{XY} \) are differentials. Indeed,

\[
\partial_{XY}^2 = \partial_{XY}(s_{S^{n-1}Y} \circ d_{X,S^nY} f) = s_{S^{n-2}Y} \circ d_{X,S^{n-1}Y}(s_{S^{n-1}Y} \circ d_{X,S^nY} f) = -s_{S^{n-2}Y}^2 d_{X,S^nY}^2 f = 0
\]

since \( s_{S^{n-1}Y} \) is a cycle and \( d^2 = 0 \).

Next we verify the Leibniz law for \( \partial \). Let \( f : X \to S^pY \) and \( g : Y \to S^qZ \) be morphisms in \( \mathcal{A}_0 \), and set \( n = p + q \). Then

\[
\partial_{XZ}(g \cdot f) = (-1)^n s_{S^{n-1}Z} \circ d_{X,S^nZ}(S^p(g) \circ f)
\]

\[
= (-1)^n s_{S^{n-1}Z} \circ d_{Y,S^nZ}(S^p(g)) \circ f + (-1)^n s_{S^{n-1}Z} \circ S^p(g) \circ d_{X,S^nY}(f)
\]

\[
= (-1)^n s_{S^{n-1}Z} \circ S^p(d_{Y,S^nZ}(g)) \circ f + (-1)^n s_{S^{n-1}Z} \circ S^p(g) \circ d_{X,S^nY}(f)
\]

since \( S \) is a dg functor. On the other hand,

\[
\partial_{YZ}(g) \cdot f + (-1)^q g \cdot \partial_{XY}(f)
\]

\[
= (-1)^q s_{S^{n-1}Z} \circ d_{Y,S^nZ}(g) \cdot f + (-1)^n g \cdot (s_{S^{n-1}Y} \circ d_{X,S^nY}(f))
\]

\[
= (-1)^q S^p(s_{S^{n-1}Z} \circ d_{Y,S^nZ}(g)) \circ f + (-1)^n S^p(g) \circ d_{X,S^nY}(f).
\]

The first summand equals \( (-1)^n s_{S^{n-1}Z} \circ d_{S^pY,S^nZ}(S^p(g)) \), and the second summand equals \( (-1)^n s_{S^{n-1}Z} \circ S^p(g) \circ d_{X,S^nY}(f) \) by Lemma 3.2.4. Hence the Leibniz law holds and so \( (\mathcal{A}_0)^\mathbb{Z} \) is a dg category.

To see that the functor \( (\mathcal{A}_0)^\mathbb{Z} \to \mathcal{A} \) is a dg functor, we need to show that the diagram

\[
\begin{array}{ccc}
\mathcal{A}_0(X,S^nY) & \xrightarrow{s^n} & \mathcal{A}_n(X,Y) \\
\partial_{XY} \downarrow & & \downarrow d_{XY} \\
\mathcal{A}_0(X,S^{n-1}Y) & \xrightarrow{s^{n-1}} & \mathcal{A}_{n-1}(X,Y)
\end{array}
\]
commutes. One calculates that
\[(s_n^* - 1 \circ \partial_{XY})(f) = (-1)^n s_n^{-1} \circ s_{S^n - 1} \circ d_{S^n Y}(f) = (-1)^n s_n^* \circ d_{S^n Y}(f) = d_{XY}(s_n^* \circ f) = (d_{XY} \circ s_n^*)(f)\]
so the equivalence is a dg equivalence. This completes the proof of the proposition. \hfill \Box

**Proposition 3.2.9.** For \(A\) as in the previous proposition, the homology \(H_* A\) is generated over degree 0.

**Proof.** The previous proposition implies that there is an isomorphism of chain complexes
\[(A_0 \mathcal{I}^Z(X, S^n Y) \xrightarrow{\sim} A(X, S^n Y)).\]
Inspection at degree 0 gives a commutative diagram
\[
\begin{array}{ccc}
A_1(X, S^n Y) & \longrightarrow & A_0(X, S^n Y) & \longrightarrow & A_{-1}(X, S^n Y) \\
\downarrow & & \downarrow & & \downarrow \\
A_0(X, S^{n+1} Y) & \longrightarrow & A_0(X, S^n Y) & \longrightarrow & A_0(X, S^{n-1} Y)
\end{array}
\]
where the vertical maps are isomorphisms and the rows are chain complexes. Accordingly, \((H_0 A)(X, S^n Y)\) is isomorphic to the homology of the bottom row, which is evidently \(H_n(A_0 \mathcal{I}^Z(X, Y)\). Hence there is a chain of graded equivalences
\[(H_0 A)^Z \xrightarrow{\sim} H_*(A_0 \mathcal{I}^Z) \xrightarrow{\sim} H_* A,\]
proving that \(H_* A\) is generated over degree 0. \hfill \Box

### 3.3. Skeletons.

Recall that a category is **skeletal** if it is small, and all isomorphisms are automorphisms. A **skeleton** of a category \(\mathcal{C}\) is a skeletal full subcategory \(\text{sk}(\mathcal{C})\) such that the inclusion \(\text{sk}(\mathcal{C}) \hookrightarrow \mathcal{C}\) is essentially surjective.

**Remark 3.3.1.** One typically does not require skeletons to be small categories, and instead refers to what we refer to as skeletons as small skeletons. We however are only interested in small skeletons, and so drop the “small” from our definitions.

**Definition 3.3.2.** If \(\mathcal{C}\) is additive, a skeleton \(\text{sk}(\mathcal{C})\) is said to be **additive** if \(X \oplus Y\) is an object of \(\text{sk}(\mathcal{C})\) for \(X, Y\) objects of \(\text{sk}(\mathcal{C})\).

If \(\mathcal{T}\) is a \(\mathbb{Z}\)-category, a skeleton \(\text{sk}(\mathcal{T})\) is said to be **\(\mathbb{Z}\)-stable** if it is closed under the action of the automorphism \(S\).

Following Bautista and Liu [3], a \(\mathbb{Z}\)-category \(T\) is **directed** if for indecomposable objects \(X\) and \(Y\), there is at most one \(n \in \mathbb{Z}\) so that both \(\mathcal{T}(X, S^n Y)\) and \(\mathcal{T}(S^n Y, X)\) are non-zero.

**Lemma 3.3.3.** Suppose \(\mathcal{T}\) is a directed \(\mathbb{Z}\)-category. Then \(\mathcal{T}\) admits a \(\mathbb{Z}\)-stable skeleton \(\text{sk}(\mathcal{T})\).

**Proof.** Choose a skeleton \(\text{sk}(\mathcal{T}^Z)\) of the associated graded category, and let \(\pi^{-1} \text{sk}(\mathcal{T}^Z)\) be the full subcategory of \(\mathcal{T}\) whose objects \(X\) map to objects in \(\text{sk}(\mathcal{T}^Z)\) under the functor \(\pi\). Define \(\text{sk}(\mathcal{T})\) to be the full subcategory of \(\mathcal{T}\) such that \(S^n X\) is in \(\pi^{-1} \text{sk}(\mathcal{T}^Z)\) for some \(n \in \mathbb{Z}\). Note the category \(\text{sk}(\mathcal{T})\) is manifestly \(\mathbb{Z}\)-stable; we claim it is in fact a \(\mathbb{Z}\)-stable skeleton of \(\mathcal{T}\).

Suppose that \(f : X \to Y\) is an isomorphism in \(\text{sk}(\mathcal{T})\). Let \(p, q \in \mathbb{Z}\) so that \(S^p X\) and \(S^q Y\) are in \(\pi^{-1} \text{sk}(\mathcal{T}^Z)\). Then \(\pi S^p X\) and \(\pi S^q Y\) are in \(\text{sk}(\mathcal{T}^Z)\), and
\[
\sigma_q^{-p} \bullet f \circ \sigma_p : \pi S^p X \to \pi S^q Y
\]
is an isomorphism in \(\mathcal{T}^Z\). Hence it is an automorphism, so \(S^p X = S^q Y\). Then \(\mathcal{T}(X, S^q - p Y)\) and \(\mathcal{T}(S^q - p Y, X)\) are both non-zero, as they contain the shift of \(id_{S^q Y} = id_{S^p Y}\). But \(\mathcal{T}(X, Y)\) and \(\mathcal{T}(Y, X)\) are non-zero, containing \(f\) and its inverse respectively. Since \(\mathcal{T}\) is directed, we must have \(q - p = 0\), so \(X = Y\), and therefore \(f\) is an automorphism. \hfill \Box

For many of our applications, the notion of a skeleton is too rigid, so we introduce the following notion.
**Definition 3.3.4.** Let $\gamma$ be a distinguished collection of isomorphisms in $\mathcal{C}$. A $\gamma$-pseudo-skeleton of $\mathcal{C}$ is a small full subcategory $\text{psk}_\gamma(\mathcal{C})$ (or simply $\text{psk}(\mathcal{C})$) of $\mathcal{C}$ such that all isomorphisms are compositions of automorphisms and isomorphisms in $\gamma$.

**Remark 3.3.5.** If $\gamma$ is the collection of automorphisms in $\mathcal{C}$, then a $\gamma$-pseudo-skeleton is a skeleton. If $\gamma$ is the collection of all isomorphisms, then any small category equivalent to $\mathcal{C}$ is a $\gamma$-pseudo-skeleton.

The associated graded category $\mathcal{T}^\mathbb{Z}$ of a $\mathbb{Z}$-category $\mathcal{T}$ (or more generally a graded category generated in degree 1) admits a canonical collection of isomorphisms $\sigma$ provided by the canonical stabilization $\sigma : \pi \circ S \xrightarrow{\sim} \pi$.

**Lemma 3.3.6.** Let $\mathcal{T}$ be a $\mathbb{Z}$-category, with $\mathbb{Z}$-stable skeleton $\text{sk}(\mathcal{T})$. Then

$$\text{psk}(\mathcal{T}^\mathbb{Z}) \overset{\text{def}}{=} \pi(\text{sk}(\mathcal{T}))$$

is a $\gamma$-pseudo-skeleton for $\mathcal{T}^\mathbb{Z}$. In particular, if $\mathcal{T}$ is directed, it admits a pseudo-skeleton.

**Proof.** Suppose that $f : X \to Y$ is an isomorphism in $\mathcal{T}^\mathbb{Z}$, with $X$ and $Y$ in $\text{sk}(\mathcal{T})$. Note that $f = \sigma^n_Y \circ f$ for some $f : X \to S^nY$ in $\mathcal{T}$, namely $f = \sigma^n_Y \circ f$. Since $f$ is invertible in $\mathcal{T}$, $f$ is invertible in $\mathcal{T}$ with inverse $f^{-1} = f^{-1} \circ \sigma^n_Y$, and so is an automorphism, i.e., $S^nY = X$. Hence $f$ is the composition of an automorphism $f$ of $X$ with $\sigma^n_Y$, proving the Lemma.

### 3.4. Minimal model of the derived category

We now want to apply the formalism of $\mathbb{Z}$-categories and dg structures supported on them as developed so far to the study of derived categories.

**Lemma 3.4.1.** If $\mathcal{A}$ is an abelian category, then each of the categories $\mathcal{C}^\text{gr}(\mathcal{A})$, $\mathcal{C}^{dg}(\mathcal{A})$, and $\mathcal{P}^{dg}(\mathcal{A})$ is generated over degree 0.

**Proof.** Note that it suffices to prove the claim for $\mathcal{C}^\text{gr}(\mathcal{A})$, since

$$\mathcal{P}^{dg}(\mathcal{A}) \hookrightarrow \mathcal{C}^{dg}(\mathcal{A}) \hookrightarrow \mathcal{C}^\text{gr}(\mathcal{A})$$

is a chain of full subcategories, each closed under suspension. For a graded space $X$, the identity map of the underlying ungraded space gives a canonical degree 1 isomorphism $s_X : X[1] \to X$ between graded spaces, and $s_Y \circ f[1] = (-1)^n f \circ s_X$ for $f : X \to Y$ a degree $n$ graded map. In particular, if $f$ has degree 0, then $s_Y \circ f[1] = f \circ s_X$, so the isomorphisms $s_X : X[1] \to X$ are natural. Thus $s : [1] \xrightarrow{\sim} 1_{d_0}$ is an equivalence, and so $\mathcal{C}^\text{gr}(\mathcal{A})$ is generated over degree 0. Note that the morphisms $s_X$ are cycles in $\mathcal{C}^{dg}(\mathcal{A})$, and $\mathcal{P}^{dg}(\mathcal{A})$, so these two categories are dg generated over degree 0.

Recall that the bounded derived category of $\mathcal{A}$ is equivalent to $H_0 \mathcal{P}^{dg}(\mathcal{A})$. Combining the above with Proposition 3.2.9 gives the following corollary.

**Corollary 3.4.2.** There is a graded equivalence of categories

$$H_* \mathcal{P}^{dg}(A) \xrightarrow{\sim} \mathcal{D}^{b}(\mathcal{A})^\mathbb{Z}.$$

Hence, the category $\mathcal{P}^{dg}(A)^\mathbb{Z}$ is the homology of a dg category (namely $\mathcal{P}^{dg}(A)$), having the property that the degree 0 subcategory of $\mathcal{P}^{b}(A)^\mathbb{Z}$ is the derived category. Thus it admits via Kadeishvili’s Theorem the structure of a minimal $A_\infty$-category.

**Remark 3.4.3.** Corollary 3.4.2 in particular gives an instance of an “$A_\infty$-envelope” of the triangulated category $\mathcal{D}^{b}(\mathcal{A})$. That is, a minimal $A_\infty$-category whose degree 0 subcategory is equivalent to $\mathcal{D}^{b}(\mathcal{A})$.

A similar construction of an $A_\infty$-envelope follows from a theorem of Keller [15]. In loc. cit., he associates to any algebraic triangulated category $\mathcal{T}$ a (small) dg category $\mathcal{A}'$ with an equivalence $\mathcal{T}^\mathbb{Z} \xrightarrow{\sim} H_* \mathcal{A}'$, and hence an $A_\infty$-envelope for $\mathcal{T}$.

We wish to construct an explicit model of this $A_\infty$-structure by applying the Homotopy Transfer Theorem, but this requires us to replace the category $\mathcal{P}^{dg}(A)$ by a small dg category. In order to get desirable properties for the resulting $A_\infty$-structure, this must be done with pseudo-skeleta.
3.5. **Invariant splittings.** Suppose \( \mathcal{A} \) is a small dg category dg generated over degree 0, and \( \gamma \) a stabilization of the inclusion \( \mathcal{A}_0 \to \mathcal{A} \) by cycles.

**Definition 3.5.1.** A collection of homotopy retractions \( (j_{XY}, q_{XY}, \varphi_{XY}) \) of the morphism spaces \( \mathcal{A}(X,Y) \) is \( \gamma \)-invariant if in the following diagram of homotopy retractions

\[
\begin{array}{cccc}
\mathcal{A}(X,Y) & \xrightarrow{(\gamma_Y)} & \mathcal{A}(X,Y) & \xrightarrow{\gamma_X} \mathcal{A}(S_X,Y) \\
q_{XY} \quad j_{XY} & & q_{XY} \quad j_{XY} & & q_{S_X,Y} \quad j_{S_X,Y} \\
H_* \mathcal{A}(X,Y) & \xrightarrow{[\gamma_Y]^*} & H_* \mathcal{A}(X,Y) & \xrightarrow{[\gamma_X]^*} H_* \mathcal{A}(S_X,Y)
\end{array}
\]

the equalities

\[
\begin{align*}
q_{XY} \circ (\gamma_Y)_* &= [\gamma_Y]^* \circ q_{XY} \\
(j_{XY} \circ [\gamma_Y])_* &= [\gamma_Y]^* \circ j_{XY} \\
(\varphi_{XY} \circ (\gamma_Y)_*) &= (\gamma_Y)^* \circ \varphi_{XY} \\
(\varphi_{S_X,Y} \circ \gamma_X)_* &= [\gamma_X]^* \circ \varphi_{XY}
\end{align*}
\]

hold.

If the degree zero subcategory of \( \mathcal{A} \) is directed, then it admits a \( \sigma \)-pseudo-skeleton for the isomorphisms given by the canonical stabilization \( \sigma \) of the suspension functor by Lemma 3.3.6. In particular, the category \( \mathcal{A}_{\text{psk}}(A) \) for a finite dimensional hereditary algebra \( A \) admits a \( \sigma \)-pseudo-skeleton.

**Lemma 3.5.2.** The dg category \( \mathcal{A}_{\text{psk}}(A) \) admits \( \sigma \)-invariant homotopy retractions.

**Proof.** Since \( \mathcal{A}_0 \) is directed, Lemma 3.3.6 gives a \( \sigma \)-pseudo-skeleton for \( \mathcal{A} \). By construction, the objects of \( \mathcal{A}_{\text{psk}} \) are of the form \( S^nX \) for some \( X \) with \( \pi X \) an object of a fixed skeleton \( \text{sk} \mathcal{A} \). Choose homotopy retractions \( (j_{XY}, q_{XY}, \varphi_{XY}) \) for such \( X \) and \( Y \). For arbitrary \( S^nX \) and \( S^nY \), the equations (3.5.1) uniquely determine \( \sigma \)-invariant homotopy retractions. \( \square \)

3.6. **Suspension invariance.** Let \( (\mu_n)_{n \geq 2} \) be the minimal \( A_\infty \)-structure on the pseudo-skeleton of \( H_* \mathcal{A} \) given by applying the Homotopy Transfer Theorem to the splittings of Lemma 3.5.2.

**Proposition 3.6.1.** If \( \mathcal{A} \) is a small dg category dg generated over degree 0, then the \( A_\infty \)-compositions \( \mu_n \) satisfy

\[
\begin{align*}
[\sigma]_* \circ \mu_n &= \mu_n \circ ([\sigma]_* \otimes \text{id} \otimes (n-1)) \\
\mu_n \circ ([\sigma]_* \otimes \text{id} \otimes (n-1)) &= \mu_n \circ ([\sigma]_* \otimes \text{id} \otimes [\sigma]_* \otimes \text{id} \otimes (n-1)) \\
[\sigma]_* \circ \mu_n &= \mu_n \circ (\text{id} \otimes (n-1)) [\sigma]^*
\end{align*}
\]

for \( 0 < k < n \).

**Proof.** Recall that \( \mu_n \) is given as a sum of maps \( \mu_T \) over PBR \( n \)-trees \( T \), and so it suffices to show that the maps \( \mu_T \) satisfy the formulae of the proposition. The idea of the proof is simple: The map \( \mu_T \) is made by replacing the edges of \( T \) by morphisms which are all \( \sigma \)-equivariant. So, a \( [\sigma] \) term can be freely pushed through the tree from one leaf to another.

To prove the proposition formally, let \( \nu_T \) be the map constructed in Construction 1.5.6 so that \( \mu_T = q \circ \nu_T \circ j_{n} \). Since the splittings \( (j, q, \varphi) \) are \( \sigma \)-invariant, it suffices to show that the \( \nu_T \) satisfy formulae analogous to those of the proposition. We proceed by induction on \( T \).

Note that for the unique PBR 2-tree \( Y \), the map \( \nu_Y \) is just ordinary composition in \( \mathcal{A} \), which manifestly satisfies the above formulae.

For arbitrary \( T \), we are done by induction except for the following cases:

1. If the left subtree of \( T \) is a leaf we still need to show

\[
[\sigma]_* \circ \nu_T = \nu_T \circ ([\sigma]_* \otimes \text{id} \otimes (n-1)),
\]
(2) if $0 < k < n$ and the k-th leaf is on the left subtree of $T$ and the $(k+1)$-st leaf is on the right subtree of $T$, we still need to show
\[
\nu_T \circ \left(id^{\otimes(k-1)} \otimes ([\sigma]^{*} \otimes id) \otimes id^{\otimes(n-k-1)} \right) = \nu_T \circ \left(id^{\otimes(k-1)} \otimes (id \otimes [\sigma]_{*}) \otimes id^{\otimes(n-k-1)} \right),
\]
(3) and if the right subtree of $T$ is a leaf we still need to show
\[
[\sigma]^{*} \circ \nu_T = \nu_T \circ \left(id^{\otimes(n-1)} \otimes [\sigma]^{*} \right).
\]
Let us prove the second case; the other two are similar. Since $\nu_T = \mu \circ (\varphi \circ \mu_{T_{\lambda}}) \circ (\varphi \circ \mu_{T_{\nu}})$, the left-hand side of the equation is
\[
\nu_T \circ \left(id^{\otimes(k-1)}([\sigma]^{*} \otimes id) \otimes id^{\otimes(n-k-1)} \right)
\]
\[
= \mu \circ (\varphi \circ \nu_{T_{\lambda}}) \circ (\varphi \circ \nu_{T_{\nu}}) \circ \left(id^{\otimes(k-1)} \otimes [\sigma]^{*} \otimes id^{\otimes(n-k)} \right)
\]
\[
= \mu \circ \left(\varphi \circ \nu_{T_{\lambda}} \circ (id^{\otimes(k-1)} \otimes [\sigma]^{*}) \otimes (\varphi \circ \nu_{T_{\nu}} \circ id^{\otimes(n-k)} \right)
\]
\[
= \mu \circ \left(\varphi \circ \nu_{T_{\lambda}} \circ (id^{\otimes(k-1)} \otimes ([\sigma]_{*} \circ \varphi \circ \nu_{T_{\nu}} \circ id^{\otimes(n-k)} \right)
\]
by induction. Similarly, the right-hand side is equal to
\[
\mu \circ \left(\varphi \circ \nu_{T_{\lambda}} \circ (id^{\otimes(k-1)} \otimes (\varphi \circ [\sigma]_{*} \circ \nu_{T_{\nu}} \circ id^{\otimes(n-k)} \right),
\]
which equals the left-hand side by $\sigma$-equivariance of the homotopy $\varphi$. This completes the proof of the proposition.

\[\square\]

3.7. Triangles. We now specialize to the case that of psk $\mathcal{P}^{dsk}(A)$ for $A$ a finite dimensional hereditary algebra. Since $\mathcal{P}^{dsk}(A)$ is directed for such an algebra [11], it admits a pseudo-skeleton. Note that $H_{*}\mathcal{P}^{dsk}(A)$ is (weakly) equivalent to $\mathcal{P}^{h}(A)^{Z}$ by Corollary 3.4.2, so $\mu_{3}$ maps triples of morphisms in $\mathcal{P}^{h}(A)$ into $\mathcal{P}^{h}(A)^{Z}$.

**Proposition 3.7.1.** Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a non-split triangle in $\mathcal{P}^{h}(\mathcal{A})$, with $X$ indecomposable. Then up to sign
\[
s_{X} \circ \mu_{3}(h,g,f) = id_{X}
\]
where $s_{X} : X[1] \to X$ is the canonical degree 1 map.

In order to prove the above proposition, we need to introduce the Massey products of a dg category. Suppose $\mathcal{A}$ is a dg category and $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ is a sequence of morphisms in $H_{*}\mathcal{A}$ so that $g \circ f = 0 = h \circ g$. By abuse of notation, denote the representing morphisms of $f$, $g$, and $h$ in $\mathcal{A}$ by the same. Since $g \circ f = 0 = h \circ g$ in homology, we may choose morphisms $u : X \to Z$ and $v : Y \to W$ in $\mathcal{A}$ with $d_{XZ}(u) = g \circ f$ and $d_{YW}(v) = h \circ g$. One verifies the morphism $v \circ f - (-1)^{|h|}h \circ u$ of $\mathcal{A}$ is a cycle, and so represents a morphism in $H_{*}\mathcal{A}$.

**Definition 3.7.2.** The Massey triple product $\langle h, g, f \rangle$ is the subset of $H_{*}\mathcal{A}(X, W)$ of all possible morphisms of the form $v \circ f - (-1)^{|h|}h \circ u$: it is independent of the choice of representative morphisms in $\mathcal{A}$ of $f$, $g$, and $h$. In particular, if either $g \circ f \neq 0$ or $h \circ g \neq 0$ in homology, the Massey product $\langle h, g, f \rangle$ is empty.

**Remark 3.7.3.** Any two elements of $\langle h, g, f \rangle$ differ by an element of the subspace
\[
h \circ H_{|f|+|g|+1}\mathcal{A} + H_{|g|+|h|+1}\mathcal{A} \circ f.
\]
In particular, if $c \in k$ is a scalar and $u : X \to W$, then $cu, u \in \langle h, g, f \rangle$ if and only if either $c = 1$ or $u = 0$.

The following proposition is has been folklore for some time, but the signs have been worked out precisely by [20].
Proposition 3.7.4. Let \((\mu_n)_{n \geq 2}\) be the minimal model \(A_\infty\)-structure on \(H_\bullet \mathcal{A}\). For any composable sequence of morphisms \(X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W\) of \(H_\bullet \mathcal{A}\), one has
\[
\pm \mu_3(h, g, f) \in \langle h, g, f \rangle
\]
provided \(\langle h, g, f \rangle \neq \emptyset\).

Remark 3.7.5. Massey products serve as a sort of coarse approximation to \(A_\infty\)-structures. There are analogous constructions of Massey \(n\)-products, and the natural extension of the above proposition holds for these Massey \(n\)-products. In this sense, the Massey products can be viewed as controlling the possible \(A_\infty\)-structures that arise from the Homotopy Transfer Theorem by choosing different collections of homotopy retracts.

Proof of Proposition 3.7.1. Recall that \(X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]\) is a distinguished triangle in the derived category \(\mathcal{D}^b(A)\) if and only if it is isomorphic to a triangle of the form \(X \xrightarrow{f} Y \xrightarrow{\text{Cone}(f)} Z \xrightarrow{h} X[1]\) where \(\text{Cone}(f)\) is the mapping cone of \(f\). Since Massey products are homotopy invariant, we are reduced to the case \(Z = \text{Cone}(f)\) for \(f\) not null-homotopic. The underlying graded space of \(\text{Cone}(f)\) is \(X[1] \oplus Y\), and in this case \(g\) is the inclusion \(Y \to X[1] \oplus Y\) and \(h\) is the projection \(X[1] \oplus Y \to X[1]\).

Suppose that \(u\) and \(v\) are arbitrary morphisms in \(\mathcal{D}^b(A)\) with \(d(u) = g \circ f\) and \(d(v) = h \circ g\), and assume to the contrary that \(h \circ u - v \circ f\) is null-homotopic. Let \(u_1\) and \(u_2\) denote the linear maps given by composition of \(u\) with projection onto the \(X[1]\) and \(Y\) factors of \(\text{Cone}(f)\) respectively. The condition \(d(u) = g \circ f\) implies that \(u_1\) is a chain map, and \(u_2\) is a homotopy from \(f \circ u_1\). But then \(h \circ u - v \circ f = u_1 - v \circ f\) is homotopic to \(u_1\), since \(d(v) = g \circ f = 0\) implies \(v\) null-homotopic. So if \(h \circ v - u \circ f\) is null-homotopic, \(f \simeq f \circ u_1 \simeq 0\), contradicting \(f \neq 0\) in \(\mathcal{D}^b(A)\). Thus any element of \(\langle h, g, f \rangle\) is non-zero.

We now show that the Massey product \(\langle h, g, f \rangle\) contains the morphism \(s_{X,1}\) when \(Z = \text{Cone}(f)\). Let \(u\) be the composition \(X \to X[1] \to \text{Cone}(f)\), so \(d(u) = g \circ f\), and \(h \circ g = 0 = d(0)\). Thus \(h \circ u = s_{X,1}\) an element of \(\langle h, g, f \rangle\). Since
\[
\mu_3(h, g, f) \in \langle h, g, f \rangle \subset \mathcal{D}^b(A)_{\mathbb{F}}(X, X[1]) = \mathcal{D}^b(\mathcal{A})(X, X)
\]
and \(X\) indecomposable, any other element of \(\langle h, g, f \rangle\) is proportional to \(s_{X,1}\), and hence is \(s_{X,1}\) by Remark 3.7.3. \(\square\)

3.8. Vanishing of higher compositions. Recall that an abelian category \(\mathcal{A}\) is hereditary if subobjects of projective objects are projective, or equivalently, \(\text{Ext}^n_{\mathcal{A}}(X, Y)\) vanishes for \(n > 1\) and any two objects \(X\) and \(Y\). We will say that an abelian category \(\mathcal{A}\) is directed if its (bounded) derived category is directed with respect to the shift functor.

If \(\mathcal{A}\) is hereditary, any indecomposable object of \(\mathcal{D}^b(\mathcal{A})\) is of the form \(X[d]\) for some \(d \in \mathbb{Z}\) and indecomposable object \(X\) of \(\mathcal{A}\) (cf. e.g., [11]). Thus, if \(X[p]\) and \(Y[q]\) are objects of \(\mathcal{D}^b(\mathcal{A})\) with \(X\) and \(Y\) in \(\mathcal{A}\),
\[
\mathcal{D}^b(\mathcal{A})(X[p], Y[q]) = \text{Ext}^n_{\mathcal{A}}(X, Y).
\]
In particular, a hereditary category is directed, since the non-vanishing of both \(\text{Ext}^n_{\mathcal{A}}(X, Y)\) and \(\text{Ext}^n_{\mathcal{A}}(Y, X)\) implies \(n = 0\).

Proposition 3.8.1. If \(\mathcal{A}\) is a hereditary category with enough projectives, then \(\mu_n\) vanishes for \(n > 3\).

Proof. Consider a sequence
\[
X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n
\]
of non-zero composable morphisms in \(\mathcal{D}^b(\mathcal{A})\). By additivity of \(\mu_n\), we may assume that all \(X_k\) are indecomposable. Since \(\mathcal{A}\) is directed, all the \(f_k\) are of non-negative degree, say \(d_k\). Then \(X_0\) is concentrated in degree \(d_0\), \(X_1\) is concentrated in degree \(d_0 + d_1\), and in general \(X_k\) is concentrated in degree \(d_0 + d_1 + \cdots + d_k\).

Thus, \(\mu_n(f_n, \cdots, f_1) : X_0 \to X_n\) is a degree \(d_1 + \cdots + d_n + n - 2\) morphism, and so determines a degree 0 morphism \(X_0 \to X_n[-(d_1 + \cdots + d_n) + n + 2]\). But \(X'_0 = X_0[-d_0]\) and \(X'_n = X_n[-(d_0 + \cdots + d_n)]\) are in the image of the functor \(\mathcal{A} \to \mathcal{D}^b(\mathcal{A})\), and so this morphism represents class in
\[
\mathcal{D}^b(\mathcal{A})(X'_0, X'_n[-n + 2]) = \text{Ext}^{n-2}_{\mathcal{A}}(X'_0, X'_n),
\]

27
which is 0 if \( n > 3 \).

Putting Propositions 3.6.1, 3.7.1, and 3.8.1 together and specializing to the category of representations of an acyclic quiver \( Q \), we get the following:

**Theorem 3.8.2.** If \( Q \) is acyclic, then the minimal \( \mathcal{A}_\infty \)-structure on \( \mathcal{D}^b(kQ) \) satisfies

1. the compositions \( \mu_n \) are \( s \)-equivariant for the maps \( s_X : X[1] \to X \),
2. given a non-split triangle \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \) with \( X \) indecomposable, \( s_X \circ \mu_3(h, g, f) = \pm id_X \),
3. the higher compositions \( \mu_n = 0 \) for \( n \neq 2, 3 \).

4. **Acyclic Ginzburg algebras**

4.1. **Auslander-Reiten Theory.** In this section we recall the Auslander-Reiten theory of the bounded derived category \( \mathcal{D}^b(A) \) of a finite dimensional hereditary algebra \( A \).

4.2. **Almost split sequences.** We recall the fundamental notions of almost split homomorphisms in a module category.

**Definition 4.2.1.** An \( A \)-module homomorphism \( g : X \to Y \) is said to be:

1. right almost split if it is not a split epimorphism and every non-split epimorphism \( h : W \to Y \) factors through \( g \),
2. right minimal if for any endomorphism \( t \in \text{End}_A(X) \) the identity \( got = g \) implies \( t \) is an isomorphism,
3. right minimal almost split if it is both right minimal and right almost split,
4. irreducible if for any factorization \( g = h \circ f \) one has that either \( f \) is a split monomorphism or \( h \) is a split epimorphism.

Dually one can define the notion of left almost split, left minimal, and left minimal almost split homomorphisms. One can show that a right/left minimal almost split morphism is necessarily irreducible.

**Definition 4.2.2.** A short exact sequence

\[
0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0
\]

is almost split if \( f \) is left almost split and \( g \) is right almost split. Equivalently, \( \xi \) is almost split if either \( g \) is right minimal almost split or \( f \) is left minimal almost split.

**Remark 4.2.3.** The condition of \( \xi \) being an almost split sequence is equivalent to either of the following conditions:

1. Every short exact sequence \( u^*(\xi) \) with \( u : U \to Z \) not a split epimorphism splits where \( u^*(\xi) \) is the pullback short exact sequence

\[
0 \to X \xrightarrow{u^*Y} U \to 0
\]

and \( u^*Y \) is the pullback of \( Y \) along \( u \).
2. Every short exact sequence \( v_*\xi(\xi) \) with \( h : X \to V \) not a split monomorphism splits, where \( v_*\xi(\xi) \) is the push-forward short exact sequence.

An almost split sequence is uniquely determined up to isomorphism by the module at which they start (or end). That is given two almost split sequences \( \xi \) and \( \xi' \), and an isomorphism \( h : X \rightarrow X' \), there is a commutative diagram

\[
\begin{array}{ccc}
0 & \to & X & \to & Y & \to & Z & \to & 0 \\
& & h & & \downarrow & & \downarrow & & \\
0 & \to & X' & \to & Y' & \to & Z' & \to & 0 \\
\end{array}
\]

whose vertical maps are isomorphisms.
4.3. **Auslander-Reiten translate.** To describe the existence of almost split sequences, we need the following construction:

**Definition 4.3.1.** Suppose $X$ is a right $A$-module with no non-zero projective summands, and choose a projective presentation $P_1 \xrightarrow{d} P_0 \to X \to 0$ of $X$. The **transpose** $\text{Tr}\, X$ of $X$ is the cokernel of the map

$$\text{Hom}_A(d, A) : \text{Hom}_A(P_0, A) \to \text{Hom}_A(P_1, A).$$

The **Auslander-Reiten translate** of $X$ is $\tau = D \circ \text{Tr}$, where $D$ is the $k$-linear dual.

The transpose is inverse to itself, and so the Auslander-Reiten translate is invertible with inverse $\tau^- = \text{Tr} \circ D$.

**Remark 4.3.2.** The transpose of a module is uniquely defined up to isomorphism, in the sense that different projective presentations give isomorphic modules. It defines an invertible functor

$$\text{mod}\cdot A \to A\text{-mod}$$
on the level of stable categories.

If $A$ is a hereditary, then $\text{Tr}\, X \cong \text{Ext}^1_A(X, A)$ and $\tau X = D \text{Ext}^1_A(X, A)$ since the projective presentation of $X$ can be chosen to be a projective resolution.

The Auslander-Reiten translate and its inverse can be used to construct almost split sequences, by the following fundamental theorem of Auslander and Reiten.

**Theorem 4.3.3** (Auslander-Reiten [2]). If $Z$ is a finitely generated non-projective $A$-module, there is an almost split sequence

$$0 \to \tau Z \to E \to Z \to 0$$

with $E$ finitely generated. Similarly, if $Z$ is a finitely generated non-injective $A$-module, there is an almost split sequence

$$0 \to X \to E' \to \tau^- X \to 0$$

with $E'$ a finitely generated module.

4.4. **Auslander-Reiten quiver.** A homomorphism $f : X \to Y$ is **radical** if any endomorphism of a module $Z$ factoring through $f$ is necessarily non-invertible. The **radical** of $\text{Hom}_A(X, Y)$ is the subspace $\text{rad}(X, Y)$ of radical homomorphisms, and $\text{rad}^2(X, Y)$ is the subspace of $\text{rad}(X, Y)$ whose morphisms are of the form $g \circ f$ for $f \in \text{rad}(X, Z)$ and $g \in \text{rad}(Z, Y)$.

**Remark 4.4.1.** The radical can be used to “count” the number of irreducible morphisms between two modules. More precisely, a homomorphism $f : X \to Y$ between two indecomposable modules $X$ and $Y$ is irreducible if and only if $f \in \text{rad}(X, Y) \setminus \text{rad}^2(X, Y)$.

**Definition 4.4.2.** The **Auslander-Reiten quiver** of the category $\text{mod}\cdot A$ is the quiver constructed as follows:

1. the vertices are isomorphism classes $[X]$ of irreducible $A$-modules $X$,
2. the number of arrows $[X] \to [Y]$ is equal to $\dim_k \text{rad}(X, Y)/\text{rad}^2(X, Y)$.

A module $X$ is **preprojective** if $\tau^n X$ is projective for some $n \geq 0$, and is **preinjective** if $\tau^n X$ is injective for some $n \leq 0$.

4.5. **AR theory in the derived category.** The Auslander-Reiten theory of the module category $\text{mod}\cdot A$ extends naturally to the derived category $D^b(A)$. For a more in depth treatment we refer the reader to [12] and [11].

A non-split triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is **almost split** if either $f$ is left minimal almost split or $g$ is right minimal almost split. An almost split triangle is uniquely determined up to isomorphism by its starting or ending object. The Auslander-Reiten translate $\tau$ of the module category $\text{mod}\cdot A$ extends to $D^b(A)$ by

$$\tau X = X \otimes_A D(A)[-1]$$
where \( D \) denotes \( k \)-linear dual, and for any indecomposable object \( X \) there is an almost split triangle
\[
\tau X \longrightarrow E \longrightarrow X \longrightarrow \tau X[1]
\]
ending at \( X \).

**Remark 4.5.1.** The theory of almost split triangles is more regular than that of almost split sequences. Note that any indecomposable object \( X \) can be completed to an almost split triangle. There is no projectivity/injectivity restriction as there was in the module category setting.

The Auslander-Reiten translate \( \tau \) defines an automorphism of the derived category \( \mathcal{D}^b(A) \) which commutes with the shift functor up to isomorphism. Indeed, \( \tau X \) is determined up to isomorphism by the existence of an almost split triangle
\[
\tau X \rightarrow E \rightarrow X \rightarrow \tau X[1],
\]
which is unique up to isomorphism of triangles. Thus \( \tau \) is an automorphism of \( \mathbb{Z} \)-categories, and so induces a graded automorphism of \( \mathcal{D}^b(Q)\mathbb{Z} \).

4.6. **Happel’s Theorem.** The study of the derived category \( \mathcal{D}^b(Q) \) is intimately related to the combinatorics of the corresponding Auslander-Reiten quiver. We will ultimately want to use various enrichments of the Auslander-Reiten quiver which incorporate the higher degree morphisms of \( \mathcal{D}^b(Q)\mathbb{Z} \) in order to study the Ginzburg algebra of \( Q \). To this end and to fix notations, we recall the mesh category of an acyclic quiver \( Q \).

**Definition 4.6.1.** Let \( Q \) be an acyclic quiver. The **repetitive quiver** of \( Q \) is the bigraded quiver \( Q \times \mathbb{Z} \) with vertex set \( Q_0 \times \mathbb{Z} \) and two families of arrows:

- For each \( \alpha : i \rightarrow j \) in \( Q \) and \( n \in \mathbb{Z} \) there is an arrow \( (\alpha, n) : (i, n) \rightarrow (j, n+1) \) of bidegree \((0,0)\),
- and for each \( \alpha : i \rightarrow j \) in \( Q \) and \( n \in \mathbb{Z} \) there is an arrow \( (\alpha^*, n) : (j, n) \rightarrow (i, n+1) \) of bidegree \((1,0)\).

The first component of the bidegree will be referred to as the weight and we denote the weight of an arrow \( f \) by \( \text{wt}(f) \). The second component of the bidegree at the moment carries no additional information. In the sequel, we will construct honest bigraded quivers obtained from \( Q \times \mathbb{Z} \) by adjoining extra arrows. We denote the total degree of an arrow \( f \) by \( |f| \); at the moment \( |f| = \text{wt}(f) \).

The repetitive quiver has an (orientation preserving) automorphism \( \tau \) defined by
\[
\tau(i, n) = (i, n+1) \quad \text{and} \quad \tau(\alpha, n) = (\alpha, n+1)
\]
and an automorphism \( \sigma \) of the arrow set uniquely determined by the property that \( \sigma(f) : \tau y \rightarrow x \) for an arrow \( f : x \rightarrow y \). The automorphism \( \tau \) is a bigraded automorphism of \( Q \times \mathbb{Z} \), while \( \sigma \) satisfies the identity \( |\sigma f| + |f| = 1 \).

Let \( \mathcal{E}_{Q \times \mathbb{Z}} \) be the ideal of the path category \( \mathcal{E}_{Q \times \mathbb{Z}} \) generated by the **mesh relations**
\[
\rho_x = \sum_{f : y \rightarrow x} (-1)^{|f|} f \circ (\sigma f)
\]
for \( x \in Q_0 \times \mathbb{Z} \). The **mesh category** of \( Q \) is the quotient category
\[
\mathcal{H}_Q \overset{\text{def}}{=} \mathcal{E}_{Q \times \mathbb{Z}}/\mathcal{E}_{Q}.
\]

The following well-known proposition establishes the interest in the mesh category.

**Theorem** (Happel [11]). Let \( \text{ind} \mathcal{D}^b(Q) \) denote the full subcategory of \( \mathcal{D}^b(Q) \) whose objects are given by choosing representatives from each isomorphism class of indecomposable objects. Then there is a fully faithful functor
\[
h : \mathcal{H}_Q \rightarrow \text{ind} \mathcal{D}^b(Q)
\]
such that for every object \( x \), the sequence
\[
h(\tau x) \longrightarrow \bigoplus_{f : y \rightarrow x} h(y) \longrightarrow h(x) \longrightarrow h(\tau x)[1]
\]
is a triangle. Moreover, \( h \) is an equivalence of categories if and only if \( Q \) is Dynkin. In general, the image of \( h \) is the transjective component of \( \mathcal{D}^b(Q) \), i.e., those objects \( X \) such that \( \tau^n X \) is projective for some \( n \in \mathbb{Z} \).
Up to isomorphism, the functor $h$ can be described more explicitly by introducing coordinate functions on the objects of $Q \times \mathbb{Z}$. Denote by $q : Q_0 \times \mathbb{Z} \to Q_0$ and $f : Q_0 \times \mathbb{Z} \to \mathbb{Z}$ the projections onto the first and second coordinates respectively. With this notation, the functor $h : \mathcal{H}_Q \to \mathcal{D}^b(Q)$ sends an object $x$ to $\tau^{f(x)} h_q(x)$.

### 4.7. The derived translation algebra.

We now introduce an algebra which we will later show provides a minimal model for Ginzburg algebras in the acyclic setting.

**Definition 4.7.1.** The derived translation algebra of $Q$ is the algebra

$$U(Q) = \bigoplus_{n \geq 0} \mathcal{D}^b(Q)[Z](kQ, \tau^{-n}kQ)$$

where the product of $f : kQ \to \tau^{-p}kQ$ and $g : kQ \to \tau^{-q}kQ$ is given by

$$f \cdot g = \tau^{-q}(f) \circ g.$$

The derived translation algebra is naturally bigraded: in addition to grading by morphism degree, we define the weight of a morphism $f : kQ \to \tau^{-n}kQ$ to be $n$.

The morphism $kQ = \mathcal{D}^b(Q)(kQ, kQ) \hookrightarrow \mathcal{D}^b(Q)[Z](kQ, kQ) \hookrightarrow U(Q)$ endows the derived translation algebra with the structure of a graded right $kQ$-module. We now want to describe this graded $kQ$-module structure.

**Proposition 4.7.2.** Let $U^{\text{tot}}(Q)$ denote the (singly) graded $kQ$-module given by considering $U(Q)$ as graded by total degree, and set $F = \tau^{-1}[1]$. Then there is a graded isomorphism of $kQ$-modules

$$U^{\text{tot}}(Q) \cong \bigoplus_{n \geq 0} F^n(kQ)$$

where the right-hand side is graded by the usual degree of chain complexes. Moreover, under this isomorphism the submodule $F^n(kQ)$ maps to the weight $n$ component of $U^{\text{tot}}(Q)$.

**Remark 4.7.3.** The functor $F$ plays an important role in the construction of cluster categories associated to hereditary categories [6]. The above proposition can be used to give a direct relationship between the cluster categories of [6] constructed via orbit categories to those of [1] constructed using Ginzburg algebras. We will pursue cluster-theoretic considerations no further in this thesis.

**Proof.** Denote by $P_i$ the indecomposable projective $kQ$-module corresponding to the vertex $i$. Since

$$\mathcal{D}^b(Q)[Z](kQ, \tau^{-n}kQ) = \bigoplus_{i \in Q_0} \mathcal{D}^b(Q)[Z](kQ, \tau^{-n}P_i)$$

it suffices to compute the graded $kQ$-module structure of $\mathcal{D}^b(Q)(kQ, \tau^{-n}P_i)$.

Since $kQ$ is hereditary, there are some integers $n_0$ and $d$ such that $\tau^{-n}P_i$ is quasi-isomorphic to $\tau^{-n_0}P_i[d]$ with $\tau^{-n_0}P_i$ concentrated in degree 0. (If $Q$ is not Dynkin, then $n_0 = n$ and $d = 0$.) Hence $\mathcal{D}^b(Q)(kQ, \tau^{-n_0}P_i)$ is concentrated in degree $d$. Moreover there are graded isomorphisms

$$\mathcal{D}^b(Q)(kQ, \tau^{-n}P_i) = \mathcal{D}^b(Q)(kQ, \tau^{-n_0}P_i[d]) = \mathcal{D}^b(Q)(kQ, \tau^{-n_0}P_i) = \tau^{-n_0}P_i.$$

But the complex $\mathcal{D}^b(Q)(kQ, \tau^{-n_0}P_i)$ lives in degree $d$, so $\mathcal{D}^b(Q)(kQ, \tau^{-n_0}P_i)$ is isomorphic to $\tau^{-n_0}P_i[d] = \tau^{-n}P_i$. Therefore the summand $\mathcal{D}^b(Q)(kQ, \tau^{-n}P_i)$ lies in total degree $d + n$, and so $\mathcal{D}^b(Q)(kQ, \tau^{-n}) = F^n P_i$.

**Remark 4.7.4.** The proof of Proposition 4.7.2 actually shows that $U(Q)$ is a preprojective algebra for $\mathcal{D}^b(Q)$ in the sense that it is a graded algebra containing $kQ$ as a subalgebra, and it splits as a direct sum into all indecomposable preprojective objects in the derived category $\mathcal{D}^b(Q)$. If $Q$ is not Dynkin then every preprojective object of $\mathcal{D}^b(Q)$ is in fact a preprojective module (i.e., concentrated in degree 0).
We now wish to construct a natural $A_\infty$-structure on $U(Q)$. If necessary, we may choose quasi-isomorphisms $\tau^{-n}P_i \leq X_{(n,i)}$ with $X_{(n,i)}$ an object in the pseudo-skeleton $\mathcal{P}^{dg}(Q)$. Such a choice of quasi-isomorphisms induces an isomorphism

\begin{equation}
U(Q) \cong \bigoplus_{n \geq 0} \mathcal{P}^b(Q)^2(X_{(0,i)}, X_{(n,j)})
\end{equation}

which endows $U(Q)$ with the structure of a bigraded $A_\infty$-algebra. Denote by $s_X$ the image of the degree $-1$ map $\sigma_X^{-1} : X[1] \to X$ under the isomorphism (4.7.1). The next proposition is then an immediate corollary of Theorem 3.8.2.

**Proposition 4.7.5.** The $A_\infty$-algebra $U(Q)$ has the following properties:

1. the compositions $\mu_n$ are $s_X$-equivariant,
2. if $f$, $g$, and $h$ are homogeneous of degree $0$ and map to a distinguished triangle under (4.7.1), then $\mu_3(h, g, f) = s_X$ for $f : X \to Y$,
3. $\mu_n = 0$ for $n \neq 2, 3$.

In this section, we will show that the $A_\infty$-algebra $U(Q)$ is a minimal model for the corresponding acyclic Ginzburg algebra. In Section 5, we specialize to the case where $Q$ is Dynkin, where we can explicitly calculate the $A_\infty$-algebra $U(Q)$.

**4.8. Acyclic Ginzburg Algebras.** We now give the construction of the Ginzburg algebra associated to an acyclic quiver $Q$. Let $\hat{Q}$ be the bigraded quiver (i.e., the arrow set $Q_1$ is bigraded) obtained from $Q$ as follows:

1. The arrows of the original quiver have bidegree $(0, 0)$,
2. for each $\alpha : i \to j$ in $Q$ adjoin a reversed arrow $\alpha^* : j \to i$ having bidegree $(1, 0)$,
3. and for each vertex $i \in Q_0$ adjoin a loop $t_i$ at the vertex $i$ of bidegree $(1, 1)$.

The weight and degree of an arrow $\gamma$ are respectively the first and second components of the bidegree of $\gamma$. The total degree is the sum of the weight and degree. Denote the weight of $\gamma$ by $\text{wt}(\gamma)$ and total degree by $|\gamma|$. We may consider $\hat{Q}$ as a graded quiver, with grading given by total degree.

Define a differential $d$ on $\hat{Q}$ by:

1. $d(\alpha) = d(\alpha^*) = 0$ for $\alpha \in Q_1$,
2. and $dt_i = \rho_i$ where

\[ \rho_i = \sum_{\alpha : i \to j \in \hat{Q}} \alpha \alpha^* - \sum_{\beta : j \to i \in \hat{Q}} \beta^* \beta. \]

With the map $d$, and grading by total degree, $\hat{Q}$ is a dg quiver, and so its path algebra is a dg algebra.

**Definition 4.8.1.** The Ginzburg algebra of $Q$ is the dg algebra $\Gamma(Q) = k\hat{Q}$.

**Remarks 4.8.2.** Our grading conventions differ from those most common in the literature. Often one grades the Ginzburg algebra by what is in our language the total degree. Also, we adopt the homological grading convention as opposed to the cohomological one.

1. Classically, a Ginzburg algebra is associated to the date of a quiver with potential $(Q, W)$, i.e., a distinguished element $W$ of $HH_0(kQ)$. When $Q$ is acyclic, $HH_0(kQ) = 0$, so all potentials are trivial.

We will view the Ginzburg algebra as graded by total degree, where each degree homogeneous component is itself graded by weight. That is,

\begin{equation}
\Gamma(Q) = \bigoplus_{n \geq 0} \Gamma_{*, n}(Q)
\end{equation}

where $\Gamma_{*, n}(Q) = \bigoplus_{k \geq 0} \Gamma_{k, n}(Q)$ and $\Gamma_{k, n}(Q)$ is the subspace spanned by the paths in $\hat{Q}$ of weight $k$ and total degree $n$.

The differential of $\Gamma(Q)$ has bidegree $(0, -1)$ so the grading (4.8.1) descends to a grading in homology

\[ H\Gamma(Q) = \bigoplus_{n \geq 0} H_{*, n} \Gamma(Q) \]
where $H_{*,n} \Gamma(Q)$ is the subspace of (total) degree $n$ homology classes. It is itself graded by weight.

Note that with this bigrading,

$$H_{*,0} \Gamma(Q) = kQ/\langle \rho_i : i \in Q_0 \rangle = \Pi(Q)$$

so the homology $H \Gamma(Q)$ contains the preprojective algebra in degree 0 (see Section 5).

The remainder of this section is dedicated to the proof of the following structure theorem about the homology of the Ginzburg algebra:

**Theorem 4.8.3.** For acyclic $Q$, there is a bigraded $k$-algebra isomorphism

$$H \Gamma(Q) \cong U(Q)$$

with the derived translation algebra $U(Q)$.

Modulo the proof of Theorem 4.8.3, we can compute the minimal model of $\Gamma(Q)$ for $Q$ a non-Dynkin quiver. In the process, we recover a theorem of Keller [17].

**Corollary 4.8.4** (See [17] Section 4.2). If $Q$ is not a Dynkin quiver, then the homology $H_{*,} \Gamma(Q)$ is isomorphic to the preprojective algebra $\Pi(Q)$ of $Q$, and $\Gamma(Q)$ is a formal $A_\infty$-algebra.

**Proof.** By Theorem 4.8.3 the homology

$$H_{*} \Gamma(Q) \cong U(Q) = \bigoplus_{n \geq 0} \mathcal{D}^b(Q)Z(kQ, \tau^{-n}kQ).$$

Since $Q$ is not Dynkin, $\tau^{-n}kQ$ is quasi-isomorphic to a complex concentrated in degree 0, so

$$\mathcal{D}^b(Q)Z(kQ, \tau^{-n}kQ) = \mathcal{D}^b(Q)(kQ, \tau^{-n}kQ) \cong \tau^{-n}kQ$$

as $kQ$-modules. Thus, $H_{*} \Gamma(Q)$ is concentrated in degree 0 and hence is isomorphic to $\Pi(Q)$ as a graded algebra.

The $A_\infty$-structure maps $\mu_n$ have degree $n - 2$, and hence are zero for $n \neq 2$, as $H \Gamma$ is concentrated in degree 0.

**Remark 4.8.5.** In terms of the $A_\infty$-structure of $\mathcal{D}^b(Q)Z$, the formality of $\Gamma(Q)$ for $Q$ non-Dynkin is analogous to the fact that for such $Q$ the component of $\mathcal{D}^b(Q)$ containing the preprojective modules is not closed under shift.

Let us turn now to the proof of Theorem 4.8.3. Denote by $\mathcal{G} = \mathcal{C}_Q$ the dg path category of $\hat{Q}$. The idea of the proof is to construct a Galois dg $\mathbb{Z}$-cover $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ and a dg functor $R : \widetilde{\mathcal{G}} \to \mathcal{D}^b_{\mathbb{Z}}(Q)$ inducing a quasi-isomorphism between the algebra $k[\mathcal{G}^S] \cong \Gamma(Q)$ and $U(Q)$, where $S$ is a section of the covering.

Let $\hat{Q} \times \mathbb{Z}$ be the quiver obtained from the repetitive quiver $Q \times \mathbb{Z}$ by adjoining arrows $(t_i, n) : (i, n) \to (i, n + 1)$ of bidegree $(1, 1)$. The automorphism $\tau$ of $Q \times \mathbb{Z}$ extends to $\hat{Q} \times \mathbb{Z}$ by setting $\tau(t_i, n) = (t_i, n + 1)$. There is a unique degree 1 morphism $t_x : x \to \tau x$ for each vertex $x$ of $\hat{Q} \times \mathbb{Z}$. The quiver $\hat{Q} \times \mathbb{Z}$ is a dg quiver with differential determined by

$$df_\gamma = 0$$

$$dt_x = \rho_x = \sum_{\gamma \in Q \times \mathbb{Z}} (-1)^{|\gamma|} \gamma \cdot (\sigma \gamma)$$

for $\gamma : x \to y$ in $Q \times \mathbb{Z}$ and $t_x : x \to \tau x$

and the automorphism $\tau$ is a dg automorphism. Thus the category $\widetilde{\mathcal{G}}$ is dg category, and $\tau$ endows $\widetilde{\mathcal{G}}$ with the structure of a dg $\mathbb{Z}$-category.

There is an evident dg quiver homomorphism $q : \hat{Q} \times \mathbb{Z} \to \hat{Q}$ by given by projection onto the first coordinate. It is a $\mathbb{Z}$-Galois covering, and hence induces a dg Galois $\mathbb{Z}$-covering functor $q : \widetilde{\mathcal{G}} \to \mathcal{G}$. The section $S$ of $q$ given by the objects of the form $(i, 0)$ for $i \in Q_0$ induces an isomorphism

$$\mathcal{G}^S \cong \mathcal{G}.$$
In the path category \( \mathcal{G} \), one has \( \mathcal{G}(x, y) = 0 \) if \( \ell(x) > \ell(y) \), and so
\[
(4.8.2) \quad k[\mathcal{G}^S] = \bigoplus_{x, y \in S, n \geq 0} \mathcal{G}(x, \tau^{-n} y).
\]

The inclusion of quivers \( Q \hookrightarrow \hat{Q} \times \mathbb{Z} \) into degree 0 induces a functor \( \iota : \mathcal{E}_Q \to \mathcal{G} \). For an object \( x \) of \( \mathcal{G} \)

\[
R(x) \overset{\text{def}}{=} \mathcal{G}(-, x) \circ \iota : \mathcal{E}_Q \to \mathcal{E}_{\text{dg}}(k)
\]
is a dg \( \mathcal{E}_Q \)-module. This in turn defines a dg functor
\[
R : \mathcal{G} \to \mathcal{E}_{\text{dg}}(Q)
\]
onto the category of dg \( \mathcal{E}_Q \)-modules. More concretely, for a vertex \( j \in Q_0 \), one has \( R(x)(j) = \mathcal{G}((j, 0), x) \), which is bigraded and equipped with a degree \((0, -1)\) endomorphism \( d \). We think of \( R(x)(j) \) as a chain complex of graded vector spaces, where the internal degree is given by weight.

The following Lemma originally appeared in [13]. We record the proof with slight modification to incorporate dg structures.

**Lemma 4.8.6.** Let \( x \) be an object of \( \mathcal{G} \) and \( \{ f : y \to x \} \) the set of irreducible degree 0 morphisms mapping to \( x \). Then there is a short exact sequence of functors of graded chain complexes
\[
0 \to \bigoplus_{f : y \to x} \mathcal{G}(-, y) \overset{\varphi}{\to} \mathcal{G}(-, x) \overset{\delta}{\to} \mathcal{G}(-, \tau x)[1] \to 0
\]
where \([1]\) denotes shift with respect to the external (i.e. chain) grading.

**Proof.** The map \( \varphi \) is defined on the component \( \mathcal{G}(z, y) \) by \( \varphi(g) = f \circ g \). It is a graded chain map, in the sense that \( d \circ \varphi = (-1)^{|f|} \varphi \circ d \).

Any morphism \( h : z \to x \) can be uniquely decomposed as
\[
(4.8.3) \quad h = t_x \circ s + \sum_{f : y \to x} f \circ g
\]
for \( s : z \to \tau x \) and \( g : z \to y \). Define \( \delta(h) = s \), which is again a graded chain map:
\[
\delta h = t_x \circ ds + \sum_{f : y \to x} f \circ \left( \sigma f \circ s + (-1)^{|f|} dg \right)
\]
and so \( \delta(h) = ds = d\delta(h) \). In other words, \( \delta \circ d = (-1)^{|\delta|} d[1] \circ \delta \).

The uniqueness of the decomposition (4.8.3) implies that the sequence is exact. \( \square \)

Denote by \( \mathcal{G}^- \) the full subcategory of \( \mathcal{G} \) consisting of objects \( x = (i, n) \) with \( n \leq 0 \).

**Proposition 4.8.7.** Let \( x = (i, n) \) be an object of \( \mathcal{G}^- \). The complex \( R(x) \) is a projective resolution of \( \tau^n P_i \), and so in particular the image of the dg functor \( R \) lies in \( \mathcal{P}^{\text{dg}}(Q) \).

**Proof.** We first claim that \( R(x) \) is quasi-isomorphic to the mapping cone of the morphism
\[
R(\tau x) \to \bigoplus_{f : y \to x} R(y)
\]
given by \( \varphi(s) = \sum f : y \to x (-1)^{|\sigma f|} (\sigma f) \circ s \), where the sum is taken over all irreducible degree 0 morphisms \( f : y \to x \). By Lemma 4.8.6 there is a commutative diagram
\[
\begin{array}{c}
R(\tau x) \to R(y) \to \text{Cone}(\varphi) \to R(\tau x)[1] \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
R(\tau x) \to R(y) \to R(x) \to R(\tau x)[1]
\end{array}
\]
in the ordinary derived category \( \mathcal{E}^b(Q) \) where the rows are distinguished triangles. Hence there is a quasi-isomorphism \( \text{Cone}(\varphi) \to R(x) \), and so the claim holds.
Let $L_j(x)$ denote the length of any weight $0$ morphism $(j,0)$ to $x$ in $\mathcal{G}$, and $L(x) = \min \{ L_j(x) : j \in Q_0 \}$. We now proceed by induction on $L(x)$. Note $L(x) = 0$ implies $n = 0$. Thus for any vertex $j \in Q_0$
\[ R(x)(j) = \mathcal{G}((j,0), (i,0)) = C Q(j,i) = P_i(j) \]
and so $R(x) = P_i$ is a projective resolution of $P_i$.

Suppose now that $L(x) > 0$. Since $L(y), L(x) < L(x)$ we may assume by induction that $R(y)$ and $R(\tau x)$ are projective complexes quasi-isomorphic to $\tau^n(x) P_q(y)$ and $\tau^{n+1} P_i$ respectively. But $R(x)$ is quasi-isomorphic Cone($\varphi$) which is an acyclic projective complex. Moreover, $R(x)$ is quasi-isomorphic to the cokernel of
\[ \tau^{n+1} P_i \to \bigoplus_{f:y \to x} \tau^n P_q(y) \]
and so $R(x)$ is a projective resolution of $\tau^n P_i$. \hfill \Box

As a corollary we get that the complexes $R(\tau^n x)$ and $\tau^n R(x)$ are quasi-isomorphic for $x$ in $\mathcal{G}$. Using this fact and the presentation of the sectional algebra from (4.8.2), we get that the functor $R$ induces a dg algebra homomorphism
\[ (4.8.4) \quad k[\mathcal{G}] \to \bigoplus_{x,y \in S} \mathcal{P}^{dg}(Q)(R(x), \tau^n R(y)) \]
where the algebra on the right hand side is quasi-isomorphic to the derived translation algebra $U(Q)$.

We now show that the homomorphism (4.8.4) induces an isomorphism in homology, completing the proof of Theorem 4.8.3.

**Proposition 4.8.8.** The bigraded functor
\[ H_* R : H_* \mathcal{G} \to H_* \mathcal{P}^{dg}(Q) \]
induced in homology is fully faithul.

**Proof.** Let $x, y$ be objects in $\mathcal{G}$ and define $L(x,y)$ to be the length of any degree $0$ morphism $x \to y$ in $\mathcal{G}$. We show that the (weight) graded map
\[ H_n R_{xy} : H_n \mathcal{G}(x,y) \to \mathcal{P}^b(Q) Z_n(kQ)(R(x), R(y)) \]
is an isomorphism for every $n$ by induction on $n$ and $L(x,y)$.

For $n = 0$, one has $H_0 \mathcal{G}(x,y) = \mathcal{H}_Q(x,y)$ and $\mathcal{P}^b(Q) Z_n(kQ)(R(x), R(y)) = \mathcal{P}^b(Q)(R(x), R(y))$. By Proposition 4.8.7, $R(x)$ and $R(y)$ are quasi-isomorphic to $h(x)$ and $h(y)$ respectively, where $h : \mathcal{H}_Q \to \mathcal{P}^b(Q)$ is Happel’s functor (cf. 4.6). If $L(x,y) = 1$, then there is exactly one morphism $x \to y$ in $\mathcal{G}$, and moreover this morphism has weight $0$. Thus the complexes $H_* \mathcal{G}(x,y)$ and $\mathcal{P}^b(Q) Z_n(R(x), R(y))$ are concentrated in degree $0$, and hence isomorphic by Happel’s Theorem.

Suppose now that $L(x,y) > 1$, and consider the degree $0$ irreducible morphisms $f : z \to y$. Note $L(x,\tau y), L(x,z) < L(x,y)$ for every $z$ degree $0$ morphism in $\mathcal{G}$ factors through $\tau y$ or some such $z$. By Lemma 4.8.6 there is a short exact sequence
\[ (4.8.5) \]
of complexes of graded spaces. It induces a long exact sequence in homology and so there is a commutative diagram
\[ (4.8.6) \]
where the direct sums are taken over all degree $0$ irreducible morphisms $f : z \to y$. The first, second, fourth, and fifth vertical maps are isomorphisms by induction, and so by the Five Lemma $H_n \mathcal{G}(x,y) \to \mathcal{P}^b(Q) Z_n(R(x), R(y))$ is an isomorphism. \hfill \Box

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5. The Dynkin case

5.1. The twisted polynomial algebra. We now want to compute the $A_\infty$-structure of $U(Q)$ in the special case where $Q$ is a Dynkin quiver.

Let $K$ be a bialgebra with comultiplication $\Delta$. We will use the Sweedler notation
\[
\Delta(x) = \sum x_{(1)} \otimes x_{(2)}
\]
to denote elements in the image of $\Delta$. Suppose $A$ is a (left) $K$-module algebra, that is $A$ is equipped with an associative product $\mu : A \otimes A \to A$ and a left $K$-action satisfying
\[
x\mu(a, b) = \sum \mu(x_{(1)}a, x_{(2)}b)
\]
for all $a, b \in A$ and $x \in K$. The smash product is the algebra $A \sharp K$ whose underlying $k$-vector space is $A \otimes K$ with multiplication given by
\[
(a \otimes x) \cdot (b \otimes y) = \sum (a \cdot x_{(1)}b) \otimes x_{(2)}y
\]
for $a, b \in A$ and $x, y \in K$. One readily checks that this defines an associative multiplication making $A \sharp K$ into an associative $k$-algebra.

If $x$ is a group-like element of $K$ in the sense that $\Delta(x) = x \otimes x$, then the product (5.1.1) simplifies to $a(xb) \otimes xy$.

**Definition 5.1.1.** The polynomial ring $k[t]$ is naturally a bialgebra having $t$ as a group-like element for the comultiplication. If $A$ is an ordinary algebra and $\varphi$ an algebra automorphism of $A$, then $A$ is naturally a module over the bialgebra $k[t]$, where $t$ acts on $A$ by $\varphi$. We call the smash product $A\sharp k[t]$ the $\varphi$-twisted polynomial algebra and denote it simply by $A_{\varphi}[t]$.

**Lemma 5.1.2.** Let $A$ be a $k$-algebra with automorphism $\varphi$. Let $A[t]$ denote the vector space of polynomials with (left) coefficients in $A$, and let $J$ be the ideal of the tensor algebra $T(A[t])$ generated by the tensors
\[
at^p \otimes b^q - a\varphi^p(b)t^{p+q}
\]
for $a, b \in A$. Then there is an isomorphism
\[
T(A[t])/J \cong A_{\varphi}[t]
\]
of $k$-algebras.

**Proof.** The Lemma is a special case of a more general and intuitively clear statement: If $B$ is a $k$-algebra and $J$ the ideal of $T(B)$ generated by the tensors $x \otimes y - xy$ for $x, y \in B$. The canonical algebra homomorphism $T(B) \to B$ annihilates $J$ and so induces a homomorphism $T(B)/J \to B$, which we claim is an isomorphism.

Let $U_1 \subset U_2 \subset \cdots \subset B$ be the filtration of $B$ given by
\[
U_n = B \oplus B \otimes 2 \oplus \cdots \oplus B \otimes n
\]
and let $J_n = J \cap U_n$ and $K_n = \ker(U_n \to B)$. Note that $J = \bigcup_{n=2}^{\infty} J_n$ and $K = \ker(T(B) \to B) = \bigcup_{n=2}^{\infty} K_n$ so it suffices to prove $K_n = J_n$ for $n \geq 2$.

We proceed by induction. Clearly $K_2 = J_2$. Given $x = x_1 \otimes \cdots \otimes x_n \in U_n$ and $1 \leq i < n$, let
\[
x_i^t = x_1 \otimes \cdots \otimes (x_i x_{i+1}) \otimes \cdots \otimes x_n
\]
so that $x - x^i \in J_n$. If $x \in K_n$, $x^i \in K_{n-1}$ since $x = x - (x - x^i)$ and $x - x^i \in J_n \subset K_n$. Hence $x = (x - x^i) + x^i \in J_n + K_{n-1}$, which by induction is contained in $J_n + J_{n-1} = J_n$.

Suppose that $A = kQ/I$ is a bound path algebra and $\varphi$ is an automorphism $A$ coming from an automorphism of $Q$ such that the induced automorphism of $kQ$ preserves the ideal $I$. We want to describe the twisted polynomial algebra $A_{\varphi}[t]$ as a bound path algebra. Construct a quiver $\Omega$ by adjoining to $Q$ an arrow $u_i : \varphi(i) \to i$ for each vertex $i \in Q_0$. Consider the ideal of $k\Omega$ generated by the elements
\[
\omega_\alpha = u_i \alpha - \varphi(\alpha)u_j
\]
for arrows $\alpha : i \to j$ in $Q$.

**Proposition 5.1.3.** In the situation above the twisted polynomial algebra $A_{\varphi}[t]$ is isomorphic to $k\Omega/I(\omega_\alpha : \alpha \in Q_1)$. 

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Proof. We first prove in the case $A = kQ$, i.e. $I = 0$. Let $\theta : k\Omega \to A_\varphi[t]$ be the $k$-algebra homomorphism determined by $\theta(u_i) = te_i$ and the requirement that the restriction of $\theta$ to $kQ \subset k\Omega$ is the identity. The relator $\omega_\alpha$ maps to
\[
t e_i \alpha - \varphi(\alpha)te_j = ta - ta = 0
\]
under $\theta$. Hence $\omega \in \ker \theta$, and so there is an induced algebra homomorphism $\bar{\theta} : k\Omega/(\omega_\alpha) \to A_\varphi[t]$. We now provide an inverse to $\bar{\theta}$.

Let $J$ be the ideal of $T(A[t])$ as in Lemma 5.1.2 so that $A_\varphi[t] \cong T(A[t])/J$. The map $A[t] \to k\Omega$ evaluating a polynomial at $u$ determines a $k$-algebra homomorphism $\psi : T(A[t]) \to k\Omega$. The homomorphism $\psi$ sends a generator $\alpha t^p \otimes \beta t^q - \alpha \varphi(p)(\beta)t^{p+q}$ of $J$ to
\[
\alpha u^p \beta u^q - \alpha \varphi(p)(\beta)u^{p+q}
\]
which lies in $(\omega)$ by induction on $p$. Hence, $\psi$ descends to a homomorphism $\tilde{\psi} : T(A[t])/J \to k\Omega/(\omega)$, which is readily checked to be inverse to $\bar{\theta}$.

Now suppose that $A = kQ/I$ with $I$ possibly non-zero. We note that $A_\varphi[t] \cong kQ_\varphi[t]/I[t]$ where $I[t]$ is the ideal of $kQ_\varphi[t]$ generated by polynomials with coefficients in $I$. The homomorphism $kQ_\varphi[t] \to k\Omega/(\omega)$ sends the ideal $I[t]$ to $(\omega, I)/(\omega)$, and so there are isomorphisms
\[
A_\varphi[t] \cong \frac{k\Omega/(\omega)}{(\omega, I)/(\omega)} \cong k\Omega/(\omega, I).
\]

\[\Box\]

5.2. Preprojective algebras. Of particular interest to us will be a twisted polynomial algebra with coefficients in the preprojective algebra of a Dynkin quiver $Q$, whose construction we now recall. Denote by $\overline{Q}$ the degree 0 subquiver of $Q$. The preprojective algebra is the bigraded algebra
\[
\Pi(Q) = \overline{kQ}/(\rho_i : i \in Q_0)
\]
where $\rho_i$ are as in Section 4.8. Equivalently, $\Pi(Q) = H_{\rho_i}(\Gamma(Q))$.

Now that with exception of the grading, the quiver $\overline{Q}$ is just the quiver obtained from $Q$ by adjoining reversed arrows $\alpha^* : j \to i$ for each arrow $\alpha : i \to j$ in $Q$. So, the underlying ungraded algebra of $\Pi(Q)$ only depends on the underlying graph of $Q$.

We now restrict to the case where the quiver $Q$ is Dynkin. In this case, $\Pi(Q)$ is equipped with a (non-homogeneous) involution $\eta$ which we now describe.

Let $W$ be the Weyl group of $Q$ and $\Delta$ a set of simple roots for the corresponding root system. The longest word $w_0$ of $W$ gives an automorphism $\eta$ of the simple roots $\Delta$ determined by
\[
w_0 \alpha = -\eta(\alpha)
\]
which extends uniquely to an automorphism of the corresponding Dynkin diagram. This in turn defines an orientation preserving automorphism of the doubled quiver $\overline{Q}$, and hence an algebra automorphism of $k\overline{Q}$. Denote this automorphism by $\bar{\eta}$.

The automorphism $\bar{\eta}$ does not preserve the relators $\rho_i$. In order to remedy this, we need to choose a bipartition of $Q$ i.e., a function $\varpi : Q_0 \to \{\pm 1\}$ with $\varpi(s\alpha) = -\varpi(\alpha)$. Set
\[
\eta(\alpha) = \varpi(s\alpha)^{|\alpha|} \varpi(s\bar{\eta}(\alpha))^{\bar{\eta}(\alpha)}
\]
where $|\alpha|$ denotes total degree. Note that up to sign, the definition of $\eta$ does not depend on the choice of bipartition, since $\varpi' = \pm \varpi$ for any other bipartition $\varpi'$ of $Q$. The isomorphism type of the pair $(k\overline{Q}, \eta)$ is independent of the choice of bipartition.

Lemma 5.2.1. The automorphism $\eta$ sends $\rho_i$ to $\pm \rho_i$. Hence, it descends to an automorphism of $\Pi(Q)$, denoted $\eta$.

Proof. Fix a bipartition $\varpi$. One has for $\alpha : i \to j$ in $\overline{Q}$ the identity
\[
\eta(\alpha)\eta(\alpha^*) = \varpi(i)^{|\alpha|}\varpi(\eta(i))^{|\eta(\alpha)|}\varpi(j)^{|\alpha^*|}\varpi(\eta(j))^{|\eta(\alpha^*)|}\bar{\eta}(\alpha)\bar{\eta}(\alpha^*)
\]
\[
= (-1)^{|\alpha|^+|\eta(\alpha)|^+}\varpi(i)^{|\alpha|+|\eta(\alpha)|}\varpi(\eta(i))^{|\eta(\alpha)|+|\eta(\alpha^*)|^+}\bar{\eta}(\alpha)\bar{\eta}(\alpha^*)
\]
\[
= (-1)^{|\alpha^*|^+|\eta(\alpha)^*|^+}\varpi(i)^{|\alpha|}\varpi(\eta(i))^{|\eta(\alpha)|}\bar{\eta}(\alpha)\bar{\eta}(\alpha^*)
\]
as \( \varpi(s\alpha) = -\varpi(t\alpha) \), and \( |\alpha + \alpha^*| = 1 \).

Since \( \rho_i = \sum (-1)^{i|\alpha|} |\alpha^*| \) where the sum is taken over all arrows \( \alpha : i \to j \) in \( \overline{Q} \), one has

\[
\eta(\rho_i) = \sum_{\alpha : i \to j} (-1)^{i|\alpha| + |\alpha^*| + \eta(\alpha)^*} \varpi(i) \varpi(\eta(i)) \eta(\alpha) \eta(\alpha^*)
\]

\[
= \sum_{\alpha : i \to j} (-1)^{1 + \eta(\alpha)} \varpi(i) \varpi(\eta(i)) \eta(\alpha) \bar{\eta}(\alpha^*)
\]

\[
= -\varpi(i) \varpi(\eta(i)) \sum_{\eta(\alpha) : \eta(i) \to \eta(j)} (-1)^{\eta(\alpha)} = \bar{\eta}(\alpha) \bar{\eta}(\alpha^*)
\]

\[
= -\varpi(i) \varpi(\eta(i)) \rho_{\eta(i)}
\]

completing the proof. \( \square \)

From Proposition 5.1.3 the twisted polynomial algebra \( \Pi_\eta[u] \) is isomorphic to the bound path algebra \( k\Omega_Q/I_Q \) where \( \Omega_Q \) is the quiver obtained from \( \overline{Q} \) by adding arrows \( u_i : i \to \eta(i) \), and \( I_Q = \langle \omega_\alpha, \rho_i : \alpha \in \overline{Q}, i \in Q_0 \rangle \). The involution \( \eta \) naturally extends to \( k\Omega_Q \) by defining \( \eta(u_i) = u_{\eta(i)} \). It satisfies \( \eta(\omega_\alpha) = \omega_{\eta(\alpha)} \) and so descends to an involution of \( k\Omega_Q/I_Q \).

We now extend the grading of \( \overline{Q} \) to \( \Omega_Q \) in the a priori bizarre manner as follows: Define a function \( \tau : Q_0 \to \mathbb{N} \) determined by the condition

\[
\tau^{-N(i)} P_i = P_{\eta(i)[1]}
\]

where \( P_i \) is the indecomposable projective \( kQ \)-module corresponding to the vertex \( i \). The bidegree of the arrow \( u_i \) is defined to be \( (N(i), 1) \).

**Lemma 5.2.2.** The relators \( \omega_\alpha \) are homogeneous with respect to this bigrading, and so the bigrading of \( k\Omega_Q \) descends to \( \Pi_\eta[u] = k\Omega_Q/I_Q \).

**Proof.** We claim that \( N(i) - N(j) = \text{wt}(\alpha) - \text{wt}(\eta(\alpha)) \) for any arrow \( \alpha : i \to j \) in \( \overline{Q} \). The Lemma follows from the claim since \( \text{wt}(\omega u_j) = \text{wt}(\alpha) + N(j) \) and \( \text{wt}(u_i \eta(\alpha)) = N(i) + \text{wt}(\eta(\alpha)) \).

The claim follows from a case-by-case analysis of the weight of \( \alpha \) and \( \eta(\alpha) \). We prove the claim in the case that \( \text{wt}(\alpha) = 1 \) and \( \eta(\alpha) = 0 \); the other three cases are similar. Since \( \alpha^* : j \to i \) and \( \eta(\alpha) : \eta(i) \to \eta(j) \) are arrows in \( Q \), there are irreducible homomorphisms

\[
P_i \to P_j \quad \text{and} \quad P_{\eta(j)} \to P_{\eta(i)}
\]

in \( \text{mod} kQ \). Hence in the Auslander-Reiten quiver of \( \mathcal{D}^b(Q) \) one has

\[
\begin{array}{cccc}
P_i & \cdots & P_j & \cdots & P_{\eta(j)}[1] \\
\downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & \downarrow \\
& \cdots & & \cdots & \\
& & \cdots & & \cdots \\
\end{array}
\]

and so the claim holds. \( \square \)

5.3. **The minimal model in Dynkin type.** Our interest in twisted polynomial algebras comes from the following Theorem:

**Theorem 5.3.1.** Suppose \( Q \) is Dynkin. Then there is a bigraded \( k \)-algebra isomorphism

\[
\Pi_\eta[u] \xrightarrow{\sim} U(Q)
\]

sending \( u_i \) to \( s_i : P_i \to P_i[1] \).

Combining with Theorems 4.8.3 and 3.8.2 we get the following Corollary:
Corollary 5.3.2. If $Q$ is Dynkin, the twisted polynomial algebra $\Pi(Q)_{\eta}[u]$ admits a minimal $A_\infty$-structure so that:

1. The maps $\mu_n$ are $u$-equivariant,
2. $\Pi(Q)_{\eta}[u]$ is generated as an $A_\infty$-algebra by $\Pi$,
3. $\mu_n = 0$ for $n \neq 2, 3$.

Moreover, $\Pi(Q)_{\eta}[u]$ with this $A_\infty$-structure is a minimal model for the Ginzburg algebra $\Gamma(Q)$.

The strategy to prove the Theorem is analogous to that used to prove Theorem 4.8.3: We construct a Galois $\mathbb{Z}$-cover $\tilde{\mathcal{O}}$ of $\mathcal{O} = \mathcal{E}_{Q/\mathbb{Z}}$ of $\Pi_{\eta}[u] = k\Omega Q/I_Q$ and a functor $r : \tilde{\mathcal{O}} \to \mathcal{G}^{h}(Q)^{\mathbb{Z}}$ inducing an isomorphism between $k[\tilde{\mathcal{O}}^S] \cong \Pi(Q)_{\eta}[u]$ and $U(Q)$ for some section $S$ of the covering.

For simplicity fix a Dynkin quiver $Q$, and write $\Omega = \Omega Q$. Let $\tilde{\Omega} = \Omega \times \mathbb{Z}$ be obtained from the repetitive quiver $Q \times \mathbb{Z}$ by adjoining arrows $(u_i, n) : (\eta(i), n - N(\eta(i))) \to (i, n)$ of bidegree $(N(i), 1)$. The automorphism $\eta$ of $\Omega$ lifts to $\tilde{\Omega}$ by setting $\eta(i, n) = (\eta(i), n - N(\eta(i)))$ and $\eta(\gamma, n) = (\eta(\gamma), n - N(s\eta(\gamma)))$ for $\gamma$ an arrow of $\Omega$. Note that the arrow $(u_i, n) : (\eta(i), n) \to (i, n)$ and so for every object $x$ there is a unique degree 1 morphism $u_x : x \to \eta(x)$ in the path category $\mathcal{C}_{\tilde{\Omega}}$.

The automorphism $\tau$ of $Q \times \mathbb{Z}$ extends to $\tilde{\Omega}$ in the evident manner, and commutes with the automorphism $\eta$. This defines an $\eta$-equivariant $\mathbb{Z}$-action on the path category $\mathcal{C}_{\tilde{\Omega}}$. There is an evident covering morphism $q : \tilde{\Omega} \to \Omega$ given by projection onto the first factor which induces an $\eta$-equivariant Galois $\mathbb{Z}$-covering $q : \mathcal{C}_{\tilde{\Omega}} \to \mathcal{C}_\Omega$. The section of $q$ given by the objects $(i, 0)$ for $i \in Q_0$ induces an $\eta$-equivariant isomorphism $\mathcal{C}_{\tilde{\Omega}} \sim \mathcal{C}_\Omega$.

Let $\tilde{\mathcal{J}}$ be the ideal of the category $\mathcal{C}_{\tilde{\Omega}}$ generated by the morphisms $\rho_x = \sum_{f : y \to x \in Q \times \mathbb{Z}} (-1)^{|f|} f \circ (\sigma f) \quad \text{and} \quad \omega_f = u_y \circ f - \eta(f) \circ u_x$ for objects $x$ and weight 0 arrows $f : x \to y$, and set $\tilde{\mathcal{O}} = \mathcal{C}_{\tilde{\Omega}}/\tilde{\mathcal{J}}$.

The ideal $\tilde{\mathcal{J}}$ is preserved by the automorphisms $\tau$ and $\eta$, and the restriction of the covering $q : \mathcal{C}_{\tilde{\Omega}} \to \mathcal{C}_\Omega$ to $\tilde{\mathcal{J}}$ is a covering of the ideal $\mathcal{J}$ of $\mathcal{C}_\Omega$. Thus there is an induced Galois $\mathbb{Z}$-covering $\tilde{\mathcal{O}} \to \mathcal{O}$, and hence an algebra isomorphism $k[\tilde{\mathcal{O}}^S] \sim k[\mathcal{O}]$ induced by the covering $q$.

The inclusion of quivers $Q \rightarrow \tilde{\Omega}$ induces an embedding of path categories $i : \mathcal{C}_Q \to \tilde{\mathcal{O}}$. For an object $x$ of $\tilde{\mathcal{O}}$, there is a contravariant functor

$$r(x) \overset{\text{def}}{=} \tilde{\mathcal{O}}(-, x) \circ i : \mathcal{C}_Q \to \mathcal{C}(k)$$

which we think of as a bigraded $kQ$-module. This in turn defines a functor $r : \tilde{\mathcal{O}} \to \mathcal{G}^{h}(Q)^{\mathbb{Z}}$ $x \mapsto r(x)$ with image in the augmented derived category, since $\tilde{\mathcal{O}}$ has morphisms of non-zero degree. Denote by $\mathcal{C}^-_{\tilde{\Omega}}$ and $\mathcal{O}^-$ the full subcategories of $\mathcal{C}_{\tilde{\Omega}}$ and $\tilde{\mathcal{O}}$ whose objects $x = (i, n)$ have $n \leq 0$. Note that $\eta$ preserves both $\mathcal{C}^-_{\tilde{\Omega}}$ and $\mathcal{O}^-$, but it’s inverse does not.

Lemma 5.3.3. Suppose $x$ is an object of $\mathcal{O}^-$. Composition with $u_x$ induces an isomorphism of graded functors $\mathcal{O}^-(\cdot, x)[1] \sim \mathcal{O}^-(\cdot, \eta(x))$ where shift is taken with respect to degree. Hence, the $kQ$-modules $r(\eta(x))$ and $r(x)[1]$ are isomorphic.
Proof. Composition with \( u_x \) induces a natural transformation
\[
\mathcal{C}_Q(-, x)[1] \to \mathcal{C}_\Omega(-, \eta(x))
\]
which when restricted to \( \mathcal{C}_\Omega^- \) sends the subfunctor \( \tilde{\mathcal{J}}(-, x)[1] \) surjectively onto \( \tilde{\mathcal{J}}(-, \eta(x)) \). Hence it induces a natural transformation
\[
\tilde{\mathcal{O}}^-(\cdot, x)[1] \to \tilde{\mathcal{O}}^-(\cdot, \eta(x)).
\]
One verifies that after restricting to the image of \( \mathcal{C}_Q \) in \( \tilde{\mathcal{O}} \), the natural transformation (5.3.1) is an isomorphism. \( \Box \)

The mesh category \( \mathcal{H}_Q \) is manifestly the subcategory of \( \tilde{\mathcal{O}} \) with degree 0 morphisms, and we denote by \( \mathcal{H}_Q^- \) the corresponding subcategory of \( \tilde{\mathcal{O}}^- \). The endomorphism \( \eta \) of \( \tilde{\mathcal{O}}^- \) preserves \( \mathcal{H}_Q^- \).

**Lemma 5.3.4.** The restriction of \( r \) to \( \mathcal{H}_Q^- \) is naturally equivalent to Happel’s functor \( h \).

**Proof.** It is well-known that the restriction of Happel’s functor \( h \) to the full subcategory of \( \mathcal{H}_Q^- \) consisting of objects \( x = (i, n) \) with \( 0 \geq n > -N(i) \) is naturally equivalent to \( \mathcal{H}_Q(-, x) \circ \iota \) where \( \iota : \mathcal{C}_Q \to \mathcal{H}_Q \) is the embedding induced by \( Q \hookrightarrow Q \times \mathbb{Z} \). For such \( x \), one has
\[
\mathcal{H}_Q(-, x) = \tilde{\mathcal{O}}(-, x)
\]
since the morphisms of the form \( u_x \).

Given arbitrary \( x \) in \( \mathcal{H}_Q^- \), \( \eta(x) = (\eta(i), n - N(\eta(i))) \), and so
\[
h(\eta(x)) = \tau^{-n+N(\eta(i))}P_{\eta(i)} = \tau^{-n}P_i[1] = h(x)[1].
\]
Thus the Proposition holds by the above Lemma and induction. \( \Box \)

**Corollary 5.3.5.** The functors \( \tau \) and \( r \) commute, i.e. \( r(\tau^- x) = \tau^- r(x) \) for \( x \) an object of \( \tilde{\mathcal{O}}^- \).

The functor \( r \) induces a bigraded \( k \)-algebra homomorphism
\[
\mathbb{k}[\tilde{\mathcal{O}}^S] \to \bigoplus_{x,y \in S} \mathcal{D}^b(Q)^\mathbb{Z}(r(x), \tau^{-n}r(y)) = U(Q).
\]

**Proposition 5.3.6.** The functor \( r : \tilde{\mathcal{O}}^- \to \mathcal{D}^b(Q)^\mathbb{Z} \) is fully faithful. Hence, (5.3.2) is an isomorphism.

**Proof.** Recall that the morphisms in \( \mathcal{D}^b(Q)^\mathbb{Z} \) are generated by the morphisms in \( \mathcal{D}^b(Q) \) together with the degree 1 suspension morphisms \( s_X : X \to X[1] \). By Proposition 5.3.4, the induced map \( r_{xy} \) maps onto \( \mathcal{D}^b(Q) \). Moreover, \( r(u_x) = s_{r(x)} \) and so \( r \) is full.

Faithfulness can be demonstrated by noting that every homogeneous morphism \( f : r(x) \to r(y) \) of \( \mathcal{D}^b(Q)^\mathbb{Z} \) can be uniquely written as \( f = g \circ s_X^y \) for some degree 0 morphism \( g : X[n] \to Y \). Hence there is some \( g' \) mapping to \( g \) under Happel’s functor, and \( g' \circ u_x^y \) is the unique morphism of \( \tilde{\mathcal{O}}^- \) mapping to \( f \) under \( r_{xy} \). \( \Box \)

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