An answer to an open problem on seminormed fuzzy integral

Michał Boczek, Marek Kaluszka∗

Institute of Mathematics, Lodz University of Technology, 90-924 Lodz, Poland

Abstract

We give an answer to Problem 9.3 stated by by Mesiar and Stupňanová [7]. We show that the class of semicopulas solving this problem contains any associative semicopula $S$ such that for each $a \in [0,1]$ the function $x \mapsto S(a, x)$ is continuous and increasing on a countable number of intervals.

Keywords: Generalized Sugeno integral; Seminorm; Capacity; Monotone measure, Semicopula; Fuzzy integrals.

1 Introduction

Let $(X, \mathcal{A})$ be a measurable space, where $\mathcal{A}$ is a $\sigma$-algebra of subsets of a non-empty set $X$, and let $\mathcal{S}$ be the family of all measurable spaces. The class of all $\mathcal{A}$-measurable functions $f: X \to [0, 1]$ is denoted by $\mathcal{F}(X, \mathcal{A})$. A capacity on $\mathcal{A}$ is a non-decreasing set function $\mu: \mathcal{A} \to [0, 1]$ with $\mu(\emptyset) = 0$ and $\mu(X) = 1$. We denote by $\mathcal{M}(X, \mathcal{A})$ the class of all capacities on $\mathcal{A}$.

Suppose $S: [0, 1]^2 \to [0, 1]$ is a non-decreasing function in both coordinates with the neutral element equal to 1, called a semicopula or a t-seminorm (see [1, 2]). It is clear that $S(x, y) \leq x \wedge y$ and $S(x, 0) = 0 = S(0, x)$ for all $x, y \in [0, 1]$. We denote the class of all semicopulas by $\mathcal{S}$. Typical examples of semicopulas include: $M(a, b) = a \wedge b$, $\Pi(a, b) = ab$, $S(x, y) = xy(x \vee y)$ and $S_L(a, b) = (a + b - 1) \vee 0$; $S_L$ is called the Łukasiewicz t-norm [5]. Hereafter, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

The generalized Sugeno integral is defined by

$$I_S(\mu, f) := \sup_{t \in [0, 1]} S\left(t, \mu(\{f \geq t\})\right),$$

∗Corresponding author. E-mail adress: kaluszka@p.lodz.pl; tel.: +48 42 6313859; fax.: +48 42 6363114.
where \( \{ f \geq t \} = \{ x \in X : f(x) \geq t \} \), \((X, \mathcal{A}) \in \mathcal{S}\) and \((\mu, f) \in \mathcal{M}(X, \mathcal{A}) \times \mathcal{F}(X, \mathcal{A})\). In the literature the functional \( I_S \) is also called seminormed fuzzy integral \([3, 6, 8]\). Replacing semicopula \(S\) with \(M\), we get the Sugeno integral \([10]\). Moreover, if \(S = \Pi\), then \(I_\Pi\) is called the Shilkret integral \([9]\).

Below we present Problem 9.3 from \([7]\), which was posed by Hutník during the conference FSTA 2014 The Twelfth International Conference on Fuzzy Set Theory and Applications held from January 26 to January 31, 2014 in Liptovský Ján, Slovakia.

**Problem 9.3** To characterize a class of semicopulas \(S\) for which the property

\[
\left( \forall a \in [0,1] \right) \quad I_S(\mu, S(a, f)) = S(\mu, I_S(\mu, f))
\]

holds for all \((X, \mathcal{A}) \in \mathcal{S}\) and all \((\mu, f) \in \mathcal{M}(X, \mathcal{A}) \times \mathcal{F}(X, \mathcal{A})\).

Hutník et al. \([4, 7]\) conjecture that the class of semicopulas solving Problem 9.3 will contain only the (semi)copulas \(M\) and \(\Pi\). We show that this conjecture is false, that is, the property \([11]\) is satisfied by any associative semicopula with continuous selections which satisfy some mild conditions.

## 2 Main results

Let \(\mathfrak{S}_0\) denote the set of all semicopulas \(S\) which fulfill the following two conditions:

(C1) \(S\) is associative, i.e. \(S(S(x, y), z) = S(x, S(y, z))\) for all \(x, y, z \in [0, 1]\),

(C2) \([0, 1] \ni x \mapsto S(a, x)\) is continuous for each \(a \in (0, 1)\).

The class \(\mathfrak{S}_0\) is non-empty, e.g., \(M, \Pi, S_L \in \mathfrak{S}_0\). We prove that the property \([11]\) implies that \(S \in \mathfrak{S}_0\).

**Theorem 2.1.** If the equality \([11]\) holds for all \((X, \mathcal{A}) \in \mathcal{S}\) and all \((\mu, f) \in \mathcal{M}(X, \mathcal{A}) \times \mathcal{F}(X, \mathcal{A})\), then \(S \in \mathfrak{S}_0\).

**Proof.** The equality \([11]\) is obvious for \(a \in \{0, 1\}\), so we assume that \(a \in (0, 1)\). First, we show that the property \([11]\) implies that \(S\) is an associative semicopula. Indeed, put \(f = b \mathbb{1}_A\) in \([11]\), where \(b \in [0, 1]\) and \(A \in \mathcal{A}\). Then \([11]\) takes the form

\[
\sup_{t \in [0,1]} S \left( t, \mu(A \cap \{S(a, b) \geq t \}) \right) = S \left( a, \sup_{t \in [0,1]} S(t, \mu(A \cap \{b \geq t \})) \right).
\]

Clearly, \(\{S(a, b) \geq t \} = X\) and \(\{b \geq t \} = X\) for \(t \in [0, S(a, b)]\) and \(t \in [0, b]\), respectively, and both the sets are empty otherwise. Hence

\[
\sup_{t \in [0, S(a, b)]} S(t, \mu(A)) = S \left( a, \sup_{t \in [0, b]} S(t, \mu(A)) \right).
\]
Let $X$ denote by $L = P$ since the following conditions are satisfied:

(C2a) $x \mapsto S(a, x)$ is right-continuous for each $a \in (0, 1)$,

(C2b) $x \mapsto S(a, x)$ is left-continuous for each $a \in (0, 1)$.

Denote by $L$ and $P$ the left-hand side and the right-hand side of equation (1), respectively. Let $X = [0, 1]$. Putting $\mu(A) = 0$ for $A \neq X$, we get

\[
\begin{align*}
L &= \sup_{t \in [0, \inf_x S(a, f(x))]} S(t, 1) = \inf_{x \in [0, 1]} S(a, f(x)), \\
P &= S(a, \sup_{x \in [0, \inf_x f(x)]} S(z, 1)) = S(a, \inf_{x \in [0, 1]} f(x)).
\end{align*}
\]

Let $b_n \searrow b$ for a fixed $b \in [0, 1)$ and $f(x) = b_n \mathbb{1}_{\left(\frac{1}{n+1}, \frac{1}{n}\right)}(x)$ for $x \in (0, 1)$ with $f(0) = f(1) = 1$, where $\mathbb{1}_A$ denotes the indicator of $A$. Hereafter, $a_n \searrow a$ means that $\lim_{n \to \infty} a_n = a$ and $a_n > a_{n+1}$ for all $n$. Since $L = P$, $P = S(a, b)$ and

\[
L = \inf_{x \in [0, 1]} S(a, f(x)) = \lim_{n \to \infty} S(a, b_n),
\]

the condition (C2a) holds. Now we show that (C2b) is fulfilled. Set $\mu(A) = 1$ for all $A \neq \emptyset$. Obviously,

\[
L = \sup_{t \in [0, \sup_x S(a, f(x))]} S(t, 1) = \sup_{x \in [0, 1]} S(a, f(x)), \quad P = S(a, \sup_{x \in [0, 1]} f(x)).
\]

Let $b_n \nearrow b$ for some $b \in (0, 1)$, $f(x) = b_n \mathbb{1}_{\left(\frac{1}{n+1}, \frac{1}{n}\right)}(x)$ for $x \in (0, 1)$ and $f(0) = f(1) = 0$. Since $L = P$, $P = S(a, b)$ and $L = \sup_{x \in [0, 1]} S(a, f(x)) = \lim_{n \to \infty} S(a, b_n)$, we obtain the condition (C2b).

Next, we show that under an additional assumption on $S$, the condition $S \in \mathfrak{S}_0$ is necessary and sufficient for (1) to hold.

**Theorem 2.2.** Suppose that for each $a \in (0, 1)$ function $x \mapsto S(a, x)$ is increasing on some countable number of intervals. Then the equality (1) holds true for all $(X, A) \in \mathcal{S}$ and all $(\mu, f) \in \mathcal{M}_{(X, A)} \times \mathcal{F}_{(X, A)}$ if and only if $S \in \mathfrak{S}_0$.

To proof this result we need the following lemma.

**Lemma 2.1.** Suppose $g, h: [0, 1] \to [0, 1]$ and $g$ is non-decreasing.
(a) If \( g \) is right-continuous, then \( g\left( \inf_{x \in [0,1]} h(x) \right) = \inf_{x \in [0,1]} g(h(x)) \).

(b) If \( g \) is left-continuous, then \( g\left( \sup_{x \in [0,1]} h(x) \right) = \sup_{x \in [0,1]} g(h(x)) \).

Proof. (a) Observe that \( \inf_{x \in [0,1]} g(h(x)) \geq g\left( \inf_{x \in [0,1]} h(x) \right) \). Let \( (x_n) \subset [0,1] \) be such that \( h(x_n) \searrow \inf_{x \in [0,1]} h(x) \). From the right-continuity of \( g \), we have

\[
g\left( \inf_{x \in [0,1]} h(x) \right) = g\left( \lim_{n \to \infty} h(x_n) \right) = \lim_{n \to \infty} g(h(x_n)) \geq \inf_{x \in [0,1]} g(h(x)) \,.
\]

Hence \( g\left( \inf_{x \in [0,1]} h(x) \right) = \inf_{x \in [0,1]} g(h(x)) \).

(b) Clearly, \( \sup_{x \in [0,1]} g(h(x)) \leq g\left( \sup_{x \in [0,1]} h(x) \right) \). Let \( h(x_n) \nearrow \sup_{x \in [0,1]} h(x) \). Due to the left-continuity of \( g \),

\[
g\left( \sup_{x \in [0,1]} h(x) \right) = g\left( \lim_{n \to \infty} h(x_n) \right) = \lim_{n \to \infty} g(h(x_n)) \leq \sup_{x \in [0,1]} g(h(x)) ,
\]

thus \( g\left( \sup_{x \in [0,1]} h(x) \right) = \sup_{x \in [0,1]} g(h(x)) \).

\( \square \)

**Proof of Theorem 2.2** By (C2) and Lemma 2.1(a) we get

\[
P = S\left( a, \sup_{z \in [0,1]} S(z, \mu(\{ f \geq z \})) \right) = \sup_{z \in [0,1]} S\left( a, S(z, \mu(\{ f \geq z \})) \right).
\]

Applying (C1) we obtain

\[
\sup_{z \in [0,1]} S\left( a, S(z, \mu(\{ f \geq z \})) \right) = \sup_{z \in [0,1]} S\left( S(a, z), \mu(\{ f \geq z \}) \right).
\]

In consequence,

\[
P = \sup_{z \in [0,1]} S\left( S(a, z), \mu(\{ f \geq z \}) \right). \quad (3)
\]

We prove first that \( L = P \) if \( x \mapsto S(a, x) \) is continuous and increasing in one interval, that is,

\[
S(a, x) = \begin{cases} 0, & \text{if } x \in [0, z_0], \\ g(x), & \text{if } x \in [z_0, z_1], \\ a, & \text{if } x \in [z_1, 1], \end{cases}
\]

where \( 0 \leq z_0 < z_1 \leq 1 \) and \( g \colon [z_0, z_1] \to [0, a] \) is an increasing and continuous function with \( g(z_0) = 0 \) and \( g(z_1) = a \). Both \( z_0 \) and \( z_1 \) may depend on \( a \). From (3), we get

\[
P = \sup_{z \in [0, z_0]} S\left( 0, \mu(\{ f \geq z \}) \right) \lor \sup_{z \in [z_0, z_1]} S\left( S(a, z), \mu(\{ f \geq z \}) \right) \lor \sup_{z \in [z_0, z_1]} S\left( a, \mu(\{ f \geq z \}) \right)
\]

\[
= \sup_{z \in [z_0, z_1]} S\left( S(a, z), \mu(\{ S(a, f) \geq S(a, z) \}) \right) \lor \sup_{z \in [z_0, z_1]} S\left( a, \mu(\{ f \geq z \}) \right).
\]
since \( S(0, y) = 0 \) for all \( y \). Observe that
\[
\sup_{z \in [z_1, 1]} S\left(a, \mu\left(\{ f \geq z \}\right)\right) = S\left(a, \mu\left(\{ f \geq z_1 \}\right)\right) = S\left(a, \mu\left(\{ S(a, f) \geq S(a, z_1) \}\right)\right)
\]
\[
= S\left(S(a, z_1), \mu\left(\{ S(a, f) \geq S(a, z_1) \}\right)\right),
\]
as \( S(a, z_1) = a \). From \( S(a, [z_0, z_1]) = [0, a] \) we conclude that
\[
P = \sup_{z \in [z_0, z_1]} S\left(S(a, z), \mu\left(\{ S(a, f) \geq S(a, z) \}\right)\right)
\]
\[
= \sup_{t \in [0, a]} S\left(t, \mu\left(\{ S(a, f) \geq t \}\right)\right) = L.
\]
Now, we generalize the above case to a countable number of intervals, i.e.
\[
S(a, x) = \begin{cases} 
  t_k, & \text{if } x \in [z_{2k-1}, z_{2k}], \\
  g_k(x), & \text{if } x \in [z_{2k}, z_{2k+1}], \\
  t_{k+1}, & \text{if } x \in [z_{2k+1}, z_{2k+2}],
\end{cases}
\]
where \( 0 = t_0 \leq t_1 \leq \ldots \leq a \) with \( \bigcup_{k \geq 0} [t_k, t_{k+1}] = [0, a], \) \( 0 = z_{-1} \leq z_0 < z_1 \leq \ldots \leq 1 \) and \( g_k: [z_{2k}, z_{2k+1}] \to [t_k, t_{k+1}] \) is an increasing function with \( g_k(z_{2k}) = t_k \) and \( g_k(z_{2k+1}) = t_{k+1} \) for each \( k \geq 0 \). By (3)
\[
P = \sup_k \left[ \sup_{z \in [z_{2k}, z_{2k+1}]} S\left(S(a, z), \mu\left(\{ S(a, f) \geq S(a, z) \}\right)\right) \right].
\]
Since
\[
\sup_{z \in [z_{2k}, z_{2k+1}]} S\left(S(a, z), \mu\left(\{ S(a, f) \geq S(a, z) \}\right)\right) = \sup_{t \in [t_k, t_{k+1}]} S\left(t, \mu\left(\{ S(a, f) \geq t \}\right)\right)
\]
we get
\[
P = \sup_{t \in [0, a]} S\left(t, \mu\left(\{ S(a, f) \geq t \}\right)\right) = L.
\]
The proof is complete. \( \square \)

Now we show that if capacity \( \mu \) in Problem 9.3 is continuous from below, then the equality in (1) may also hold for discontinuous semicopulas \( S \). Recall that \( \mu \) is said to be continuous from below if \( \lim_{n \to \infty} \mu(A_n) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right) \) for \( A_n \subset A_{n+1} \).

Denote by \( \mathcal{S} \) the class of all associative semicopulas \( S \) such that for any \( a \in (0, 1) \), the function \( x \mapsto S(a, x) \) is increasing and left-continuous and has only isolated discontinuity points, i.e. for any discontinuity point \( z_k(a) \) there exists an interval \( (z_k(a), z_k(a) + \varepsilon_k(a)) \) with \( \varepsilon_k(a) > 0 \), on which function \( x \mapsto S(a, x) \) is continuous.
Theorem 2.3. If $S \in \mathcal{S}_1$, then the equality \(1\) holds true for all $(X, \mathcal{A}) \in \mathcal{S}$, all $f \in \mathcal{F}_{(X, \mathcal{A})}$ and all continuous from below capacities $\mu \in \mathcal{M}_{(X, \mathcal{A})}$.

Proof. First, we assume that $x \mapsto S(a, x)$ has only one point of discontinuity, say $z = z(a)$, for each $a$. Set $t_1 = S(a, z)$ and $t_2 = \lim_{x \searrow z} S(a, x)$. Clearly, $[0, a] = S(a, [0, 1]) \cup B$, where $B = (t_1, t_2]$. Hence

$$L = \sup_{t \in [0, a]} S\left(t, \mu(\{S(a, f) \geq t\})\right)$$
$$= \sup_{t \in S(a, [0, 1])} S\left(t, \mu(\{S(a, f) \geq t\})\right) \vee \sup_{t \in B} S\left(t, \mu(\{S(a, f) \geq t\})\right).$$

(4)

Observe that for all $t \in B$

$$\{x: S(a, f(x)) \geq t\} = \{x: S(a, f(x)) \geq t_2\}.$$

Therefore

$$\sup_{t \in B} S\left(t, \mu(\{S(a, f) \geq t\})\right) = \sup_{t \in B} S\left(t, \mu(\{S(a, f) \geq t_2\})\right)$$
$$= S\left(t_2, \mu(\{S(a, f) \geq t_2\})\right).$$

(5)

Combining (4) and (5) yields

$$L = \sup_{t \in S(a, [0, 1])} S\left(t, \mu(\{S(a, f) \geq t\})\right) \vee S\left(t_2, \mu(\{S(a, f) \geq t_2\})\right).$$

(6)

Furthermore

$$\sup_{t \in S(a, [0, 1])} S\left(t, \mu(\{S(a, f) \geq t\})\right) = \sup_{z \in [0, 1]} S\left(S(a, z), \mu(\{S(a, f) \geq S(a, z)\})\right).$$

(7)

From the associativity of $S$ and Lemma 2.1(b), we get

$$\sup_{z \in [0, 1]} S\left(S(a, z), \mu(\{S(a, f) \geq S(a, z)\})\right) = S\left(a, \sup_{z \in [0, 1]} S(z, \mu(\{f \geq z\}))\right) = P.$$ 

(8)

Summing up, from (6) and (8) we obtain

$$L = P \vee S\left(t_2, \mu(\{S(a, f) \geq t_2\})\right).$$

It remains to prove that $S\left(t_2, \mu(\{S(a, f) \geq t_2\})\right) \leq P$. Since $z < 1$, there exists a sequence $(u_n)$ such that $u_n \searrow t_2$ and $u_n = S(a, z_n)$ for a sequence $(z_n)$ such that $z_n > z$ for all $n$. Observe that

$$\lim_{n \to \infty} S\left(u_n, \mu(\{S(a, f) \geq u_n\})\right) \geq \lim_{n \to \infty} S\left(t_2, \mu(\{S(a, f) \geq u_n\})\right)$$
$$= \lim_{n \to \infty} S\left(t_2, \mu(\{S(a, f) \geq u_n\})\right),$$


as \( u_n > t_2 \) and the sequence \( (\mu(\{S(a, f) \geq u_n\}))_n \) is non-decreasing. From the left-continuity of \( x \mapsto S(a, x) \) and the continuity from below of \( \mu \), we get

\[
\liminf_{n \to \infty} S\left(u_n, \mu(\{S(a, f) \geq u_n\})\right) \geq S\left(t_2, \mu(\{S(a, f) \geq t_2\})\right).
\]

(9)

Since \( u_n \in S(a, [0,1]) \), we have

\[
\liminf_{n \to \infty} S\left(u_n, \mu(\{S(a, f) \geq u_n\})\right) \leq \sup_{t \in S(a, [0,1])} S\left(t, \mu(\{S(a, f) \geq t\})\right).
\]

(10)

Combining (9) and (10) with (7) and (8) gives

\[
S\left(t_2, \mu(\{S(a, f) \geq t_2\})\right) \leq \sup_{t \in S(a, [0,1])} S\left(t, \mu(\{S(a, f) \geq t\})\right) = \text{P}.
\]

Thus, we have shown that if there exists only one point of discontinuity, then \( L = \text{P} \). The proof for the case of many isolated points proceeds in much the same way as above. \( \Box \)

**References**

[1] B. Bassan, F. Spizzichino, Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes, Journal of Multivariate Analysis 93 (2005) 313–339.

[2] F. Durante, C. Sempi, Semicopulas, Kybernetika 41 (2005) 315-328.

[3] F. Suárez García, P. Gil Álvarez, Two families of fuzzy integrals, Fuzzy Sets and Systems 18 (1986) 67-81.

[4] J. Borzová-Molnárová, L. Halčinová, O. Hutník, The smallest semicopula-based universal integrals I: properties and characterizations, Fuzzy Sets and Systems (2014), http://dx.doi.org/10.1016/j.fss.2014.09.023

[5] E.P. Klement, R. Mesiar, E. Pap, Triangular norms, Kluwer Academic Publishers, Dordrecht, 2000.

[6] E.P. Klement, R. Mesiar, E. Pap, A universal integral as common frame for Choquet and Sugeno integral, IEEE Transactions Fuzzy Sets and Systems 18 (2010) 178-187.

[7] R. Mesiar, A. Stupňanová, Open problems from the 12th International Conference on Fuzzy Set Theory and Its Applications, Fuzzy Sets and Systems 261 (2015) 112-123.

[8] Y. Ouyang, R. Mesiar, On the Chebyshev type inequality for seminormed fuzzy integral, Applied Mathematics Letters 22 (2009) 1810-1815.

[9] N. Shilkret, Maxitive measure and integration, Indagationes Mathematicae 33 (1971) 109-116.
[10] M. Sugeno, Theory of Fuzzy Integrals and its Applications, Ph.D. Dissertation, Tokyo Institute of Technology, 1974.