PWB-method and Wiener criterion for boundary regularity under generalized Orlicz growth

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Abstract
Perron’s method and Wiener’s criterion, the Dirichlet problem associated with the Laplacian equation was entirely solved. Since then, these ideas have been used for more general equations. So, in this paper, we extend these methods to study the regularity of boundary points of bounded domain concerning the Dirichlet problem associated with quasilinear equations under Musielak-Orlicz growth.

Keywords Generalized Orlicz-Sobolev spaces · Generalized $\Phi$-functions · $G(\cdot)$-capacity · $G(\cdot)$-potential · Perron method · Wiener criterion

Mathematics Subject Classification 31B25 · 32U20 · 35J25

1 Introduction
In this paper, we are concerned with the regularity of boundary points of a bounded domain $\Omega$ of $\mathbb{R}^n$ for the Dirichlet problem associated with $G(\cdot)$-Laplace operator defined by:

$$-\Delta_{G(\cdot)}(u) := -\text{div} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u,$$

where $g(\cdot)$ is the density of a generalized Orlicz function $G(\cdot)$ that have been previously used in [3–5, 13–15]. This equation covers for example, the p-Laplace equation $G(x, t) = t^p$, the variable exponent case $G(x, t) = t^{p(x)}$ and its perturbation...
$G(x, t) = t^{p(x)} \log(e + t)$, the double phase case $G(x, t) = t^p + a(x)t^q$, and the Orlicz case $G(x, t) = G(t)$. More examples and details can be found in [13, 21, 27].

Historically, Riemann proposed in 1851 the Dirichlet principle, which states that a harmonic function always exists in the interior of a domain with boundary conditions given by a continuous function. However, Lebesgue produced in 1912 an example of the bounded domain on which the Dirichlet problem was not always solvable. To overcome this problem, there is a method based on the works of Perron, Wiener and Brelot, well known as Perron’s method or PWB method [29] based on the search for the largest subharmonic function with limit values lower than desired values. The advantage of this method is that one can construct reasonable solutions for arbitrary boundary data. After that, in 1924, Wiener introduced in [32] the harmonic capacity to give his famous criterion of the regularity of a boundary point, which allows us to solve the Dirichlet problem for the Laplace equation completely. Since then, Perron’s method and Wiener’s criterion have attracted the attention of the researchers for applying these ideas to study the Dirichlet problem in the more general equations.

For $f \in W^{1,G(\cdot)}(\Omega)$, Harjulehto, Hästö, and Klén proved in [15] the existence of the solution to the Dirichlet-Sobolev problem

\[
\begin{cases}
-\Delta_{G(\cdot)}(u) = 0 & \text{in } \Omega \\
u - f \in W^{1,G(\cdot)}_0(\Omega),
\end{cases}
\]

where $W^{1,G(\cdot)}(\Omega)$ and $W^{1,G(\cdot)}_0(\Omega)$ are the generalized Orlicz-Sobolev spaces, also called Musielak-Orlicz-Sobolev spaces (see Sect. 2). So, the first question concerns the regularity of the Sobolev boundary point $x_0 \in \partial \Omega$, i.e.

$$\lim_{x \to x_0} u(x) = f(x_0),$$

for any $f \in W^{1,G(\cdot)}(\Omega) \cap C(\overline{\Omega})$.

In the p-Laplace equation, $G(x, t) = t^p$, if $\Omega$ satisfies the exterior sphere condition (see Sect. 3) then $\Omega$ is a Sobolev $p$-regular domain. By the work of Harjulehto and Hästö in the locally fat set [14], we generalize this result in our situation. As a consequence, we solve the Dirichlet problem for a domain with a smooth boundary and construct the Poisson modification of our functions. Therefore, by the ideas of Granlund, Lindqvist, and Martio [12], we can apply Perron’s method to the $G(\cdot)$-Laplace equation. More correctly, the regularity of the boundary points is defined in connection with the solution of the generalized Dirichlet problem (see [18, 29]), not only for the Dirichlet-Sobolev solution. Precisely, we say a boundary point $x_0 \in \partial \Omega$ is $G(\cdot)$-regular if

\[
\lim_{x \to x_0} H_f(x) = f(x_0),
\]

for each $f \in C(\partial \Omega)$ where $H_f$ is the Perron solution with boundary data $f$ (see Sect. 5).

In the linear case, the celebrated Wiener criterion gives the condition for the regularity boundary points. This criterion has been generalized in the $p$-Laplacian equation.
Indeed, the sufficient part was proved by Maz’ya in [24]. The necessary part has been proved by Kilpelainen and Maly in [19]. Next, Trudinger and Wang [31] gave a new method based on Poisson modification and Harnack inequality. Mikkonen has treated the weighted situation in [26]. Björn has developed the proof of this criterion in the method based on Poisson modification and Harnack inequality. Mikkonen has treated

Note that, a generalized $\Phi'$-function, denoted by $G$, is $G$-regular if and only if for some $\rho > 0$,

$$
\int_0^\rho g^{-1} \left( \frac{\text{cap}_{G(t)}(B(x_0, t) \cap \Omega, B(x_0, 2t))}{t^{n-1}} \right) \, dt = \infty.
$$

**2 Preliminary**

**Definition 2.1** A function $G : \Omega \times [0, \infty) \to [0, \infty]$ is called a generalized $\Phi$-function, denoted by $G(\cdot) \in \Phi(\Omega)$, if the following conditions hold

- For each $t \in [0, \infty)$, the function $G(\cdot, t)$ is measurable.
- For a.e $x \in \Omega$, the function $G(x, \cdot)$ is an $\Phi$-function, i.e.
  
  1. $G(x, 0) = \lim_{t \to 0^+} G(x, t) = 0$ and $\lim_{t \to \infty} G(x, t) = \infty$;
  2. $G(x, \cdot)$ is increasing, left-continuous and convex.

Note that, a generalized $\Phi$-function $G(\cdot)$ can be represented as

$$
G(x, t) = \int_0^t g(x, s) \, ds,
$$

where $g(x, \cdot)$ is the right-hand derivative of $G(x, \cdot)$. Furthermore, for each $x \in \Omega$, the function $g(x, \cdot)$ is right-continuous and nondecreasing. So, we have the following inequality

$$
g(x, a)b \leq g(x, a)a + g(x, b)b, \quad \text{for } x \in \Omega \text{ and } a, b \geq 0. \quad (2.1)
$$

We denote $G^+_B(t) := \sup_B G(x, t)$, $G^-_B(t) := \inf_B G(x, t)$. We say that $G(\cdot)$ satisfies $(SC)$: If there exist two constants $g_0$, $g^0 > 1$ such that,

$$
1 < g_0 \leq \frac{tg(x, t)}{G(x, t)} \leq g^0.
$$
(A₀) : If there exists a constant c₀ > 1 such that,
\[ \frac{1}{c₀} \leq G(x, 1) \leq c₀, \text{ a.e } x \in \Omega. \]

(A₁) : If there exists C > 0 such that, for every x, y ∈ BR ⊂ Ω with R ≤ 1, we have
\[ G_B(x, t) \leq CG_B(y, t), \text{ when } G_B(t) \in \left[ 1, \frac{1}{R^n} \right]. \]

(A₁ₙ) : If there exists C > 0 such that, for every x, y ∈ BR ⊂ Ω with R ≤ 1, we have
\[ G_B(x, t) \leq CG_B(y, t), \text{ when } t \in \left[ 1, \frac{1}{R} \right]. \]

The following lemma gives a more flexible characterization of (A₁ₙ) [13].

Lemma 2.1 Let Ω ⊂ \mathbb{R}ⁿ be convex, G(·) ∈ Φ(Ω) and 0 < r ≤ s. Then G(·) satisfies (A₁ₙ) if, and only if, there exists C > 0 such that, for every x, y ∈ BR ⊂ Ω with R ≤ 1, we have
\[ G_B(x, t) \leq CG_B(y, t) \text{ when } t \in \left[ r, \frac{s}{R} \right]. \]

Under the structure condition (SC), we have the following inequalities
\[ σ^0G(x, t) \leq G(x, σt) \leq σ^0G(x, t), \text{ for } x \in Ω, \ t ≥ 0 \text{ and } σ ≥ 1. \quad (2.2) \]
\[ σ^0G(x, t) \leq G(x, σt) \leq σ^0G(x, t), \text{ for } x \in Ω, \ t ≥ 0 \text{ and } σ ≤ 1. \quad (2.3) \]

We define \( G^*(·) \) the conjugate \( Φ \)-function of \( G(·) \), by
\[ G^*(x, s) := \sup_{t ≥ 0} (st - G(x, t)), \text{ for } x \in Ω \text{ and } s ≥ 0. \]

Note that \( G^*(·) \) is also a generalized \( Φ \)-function and can be represented as
\[ G^*(x, t) = \int_0^t g^{-1}(x, s) \, ds, \]
with \( g^{-1}(x, s) := \sup \{ t ≥ 0 : g(x, t) ≤ s \} \). Furthermore, if \( G(·) \) satisfies (SC), then \( G^*(·) \) satisfies also (SC), as follows
\[ \frac{g^0}{g^0 - 1} \leq \frac{tg^{-1}(x, t)}{G^*(x, t)} \leq \frac{g_0}{g_0 - 1}. \quad (2.4) \]
The functions $G(\cdot)$ and $G^*(\cdot)$ satisfies the following Young inequality

$$st \leq G(x, t) + G^*(x, s), \text{ for } x \in \Omega \text{ and } s, t \geq 0.$$ 

Further, we have the equality if $s = g(x, t)$ or $t = g^{-1}(x, s)$. So, if $G(\cdot)$ satisfies $(SC)$, we have the following inequality

$$G^*(x, g(x, t)) \leq (g^0 - 1) G(x, t), \quad \forall x \in \Omega, t \geq 0.$$  \tag{2.5}

**Definition 2.2** We define the generalized Orlicz space, also called Musielak-Orlicz space, by

$$L^{G(\cdot)}(\Omega) := \{ u \in L^0(\Omega) : \lim_{\lambda \to 0} \rho_{G(\cdot)}(\lambda |u|) = 0 \},$$

where $\rho_{G(\cdot)}(t) = \int_{\Omega} G(x, t) \, dx$ equipped with the norm:

$$\| u \|_{G(\cdot)} = \inf \{ \lambda > 0 : \int_{\Omega} G \left( x, \frac{|u(x)|}{\lambda} \right) \, dx \leq 1 \}.$$ 

If $G(\cdot)$ satisfies $(SC)$, then

$$L^{G(\cdot)}(\Omega) = \{ u \in L^0(\Omega) : \rho_{G(\cdot)}(|u|) < \infty \}.$$

**Definition 2.3** We define the generalized Orlicz-Sobolev space by

$$W^{1,G(\cdot)}(\Omega) := \{ u \in L^{G(\cdot)}(\Omega) : |\nabla u| \in L^{G(\cdot)}(\Omega), \text{ in the distribution sense} \},$$

equipped with the norm

$$\| u \|_{1,G(\cdot)} = \| u \|_{G(\cdot)} + \| \nabla u \|_{G(\cdot)}.$$ 

**Definition 2.4** $W^{1,G(\cdot)}_0(\Omega)$ is the closure of $C^\infty_0(\Omega) \cap W^{1,G(\cdot)}(\Omega)$ in $W^{1,G(\cdot)}(\Omega)$.

Note that, in such spaces we have the following Poincaré inequality in modular form (see Proposition 6.2.10 in [13]).

**Theorem 2.1** Let $G(\cdot) \in \Phi(\Omega)$ satisfy $(A_0)$ and $(A_1)$. There exists a constant $C > 0$ such that

$$\int_{\Omega} G(\cdot, \frac{|u|}{diam(\Omega)}) \, dx \leq C \left( \int_{\Omega} G(\cdot, |\nabla u|) \, dx + |\{ \nabla u \neq 0 \} \cap \Omega| \right),$$

for every $u \in W^{1,1}_0(\Omega)$ with $\rho_{G(\cdot)}(|\nabla u|) \leq 1$. 

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Definition 2.5 Let $G(·) \in \Phi(\Omega)$ and $K \subset \Omega$ be a compact set. The relative $G(·)$-capacity of $K$ with respect to $\Omega$ is

$$cap_{G(·)}(K; \Omega) = \inf_{u \in S_{G(·)}(K; \Omega)} \int_{\Omega} G(x, |\nabla u|) \, dx,$$

where $S_{G(·)}(K; \Omega) = \{u \in W^{1,G(·)}_0(\Omega) : u \geq 1 \text{ in a neighborhood of } K\}$.

Proposition 2.1 Let $G(·) \in \Phi(\Omega)$.

(i) $cap_{G(·)}(\emptyset; \Omega) = 0$.

(ii) If $K, K'$ are compact sets and $\Omega'$ is an open set such that $K \subset K' \subset \Omega' \subset \Omega$, then

$$cap_{G(·)}(K; \Omega) \leq cap_{G(·)}(K'; \Omega').$$

(iii) If moreover $G(·)$ satisfy $(A_0)$, $(A_1)$ and, $K \subset B(x_0, r), 0 < r \leq s \leq 2r$, then

$$cap_{G(·)}(K; B(x_0, 2s)) \leq cap_{G(·)}(K; B(x_0, 2r)) \leq C \left( cap_{G(·)}(K; B(x_0, 2s)) + s^n \right).$$

Proof For i) and ii), we can see [2].

iii) Since the first inequality is trivial, it suffices to verify the second inequality in the extremal case $s = 2r$. Let $\eta \in C_0^\infty(B(x, 2r))$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B(x, r)$ and $|\nabla \eta| \leq \frac{C}{r}$. If $u \in S_{G(·)}(K; B(x, 4r))$, then $\eta u \in S_{G(·)}(K; B(x, 2r))$, so by Theorem 2.1, we have

$$cap_{G(·)}(K; B(x_0, 2r)) \leq \int_{B(x_0, 2r)} G(x, |\nabla \eta u|) \, dx$$

$$\leq C \left( \int_{B(x_0, 2r)} G(x, \eta |\nabla u|) \, dx + \int_{B(x_0, 2r)} G(x, u |\nabla \eta|) \, dx \right)$$

$$\leq C \left( \int_{B(x_0, 4r)} G(x, |\nabla u|) \, dx + \int_{B(x_0, 4r)} G(x, \frac{u}{r}) \, dx \right)$$

$$\leq C \left( \int_{B(x_0, 4r)} G(x, |\nabla u|) \, dx + r^n \right).$$

Taking the infimum over all such functions $u$, we obtain

$$cap_{G(·)}(K; B(x_0, 2r)) \leq C \left( cap_{G(·)}(K; B(x_0, 4r)) + r^n \right).$$

This concludes the proof. □
3 \textbf{G}(\cdot)-Laplace equation}

Let \( G(\cdot) \in \Phi(\Omega) \), we consider the following \( G(\cdot) \)-Laplace equation

\[ - \text{div} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u = 0 \tag{3.1} \]

\textbf{Definition 3.1} A function \( h \in W^{1,G(\cdot)}(\Omega) \) is \( G(\cdot) \)-harmonic in \( \Omega \) if it is continuous and \( G(\cdot) \)-solution to equation (3.1) in \( \Omega \) i.e

\[ \int_{\Omega} \frac{g(x, |\nabla h|)}{|\nabla h|} \nabla h \cdot \nabla \varphi \, dx = 0, \]

whenever \( \varphi \in W^{1,G(\cdot)}_{0}(\Omega) \).

\textbf{Definition 3.2} A function \( u \in W^{1,G(\cdot)}(\Omega) \) is a \( G(\cdot) \)-supersolution (resp, \( G(\cdot) \)-subsolution) to equation (3.1) in \( \Omega \) if

\[ \int_{\Omega} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \varphi \, dx \geq 0 \text{ (resp,} \leq 0), \]

whenever \( \varphi \in W^{1,G(\cdot)}_{0}(\Omega) \) and nonnegative.

Given \( v_0 \in W^{1,G(\cdot)}(\Omega) \) and \( \psi: \Omega \to [-\infty, \infty] \) be any function. Construct the upper obstacle set:

\[ K_{\psi,v_0}(\Omega) = \{ u \in W^{1,G(\cdot)}(\Omega) \ u \leq \psi, \ \text{a.e in} \ \Omega \ \text{and} \ u - v_0 \in W^{1,G(\cdot)}_{0}(\Omega) \}. \]

We say that \( u \) is a solution of the obstacle problem in \( K_{\psi,v_0}(\Omega) \) if

\[ \int_{\Omega} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot (\nabla v - \nabla u) \, dx \geq 0, \]

whenever \( v \in K_{\psi,v_0}(\Omega) \).

\textbf{Theorem 3.1} Let \( G(\cdot) \in \Phi(\Omega) \cap C^{1}(\mathbb{R}^{+}) \) satisfy (SC), (A\(_{0}\)) and (A\(_{1}\)). If \( K_{\psi,v_0}(\Omega) \) is nonempty, then there exists a solution of the obstacle problem in \( K_{\psi,v_0}(\Omega) \).

\textbf{Proof} We define a mapping \( T: K_{\psi,v_0}(\Omega) \to (W^{1,G(\cdot)}(\Omega))^* \) by

\[ \langle T(u), v \rangle := \int_{\Omega} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v \, dx. \]

We apply the Kinderlehrer-Stampacchia inequality, see Corollary III .1.8, page 87 in [20], which asserts that if \( K_{\psi,v_0}(\Omega) \) is a closed convex subset and \( T \) is monotone, coercive and weakly continuous on \( K_{\psi,v_0}(\Omega) \), then there exists an element \( u \) in \( K_{\psi,v_0}(\Omega) \) such that
\[\langle T(u), v - u \rangle \geq 0, \text{ whenever } v \in K_{\psi,v_0}(\Omega).\]

**Step 1.** We prove that \(K_{\psi,v_0}(\Omega)\) is closed and convex. The convexity of \(K_{\psi,v_0}(\Omega)\) is immediate. To show that \(K_{\psi,v_0}(\Omega)\) is a closed, let \((v_i)_{i \in \mathbb{N}} \subset K_{\psi,v_0}(\Omega)\) be a convergent sequence and \(v \in W^{1,G(\cdot)}(\Omega)\) such that \(\|v_i - v\|_{1,G(\cdot)} \to 0\). So \((v_i - v_0) - (v - v_0)\|_{1,G(\cdot)} \to 0\). Since \(v_i - v_0 \in W^{1,G(\cdot)}_0(\Omega)\), then \(v - v_0 \in W^{1,G(\cdot)}_0(\Omega)\). As \(v_i \to v \in L^{G(\cdot)}(\Omega)\), by the lemma 2.1 in [5], there exists a subsequence \((v_{i_j})\) such that \(v_{i_j} \to v\) a.e. in \(\Omega\). By \(v_i \leq \psi\) a.e. in \(\Omega\), we obtain \(v \leq \psi\) a.e. in \(\Omega\). Therefore \(v \in K_{\psi,v_0}(\Omega)\).

**Step 2.** We prove that \(T\) is monotone, coercive and weakly continuous on \(K_{\psi,v_0}(\Omega)\). We start with the coercivity of \(T\). As for a.e. \(x \in \Omega\), we have \(G(x,t) \leq g(x,t)t\), then

\[
\langle T(u), u \rangle = \int_{\Omega} g(x, |\nabla u|) \nabla u \cdot \nabla u \, dx \\
= \int_{\Omega} g(x, |\nabla u|) |\nabla u| \, dx \\
\geq \int_{\Omega} G(x, |\nabla u|) \, dx.
\]

Since, for any \(a, b \in \mathbb{R}^n\) we have \(G(x, |a|) \geq CG(x, |a-b|) - G(x, |b|)\), then

\[
\langle T(u), u \rangle \geq C \int_{\Omega} G(x, |\nabla u - \nabla v_0|) \, dx - \int_{\Omega} G(x, |\nabla v_0|) \, dx.
\]

By Proposition 2.1 in [25], we have

\[
\langle T(u), u \rangle \geq C \min\{\|\nabla u - \nabla v_0\|_{G(\cdot)}^0, \|\nabla u - \nabla v_0\|_{G(\cdot)}^{g_0}\} - \int_{\Omega} G(x, |\nabla v_0|) \, dx.
\]

Using Poincaré inequality, see Theorem 6.2.8 in [13], we obtain

\[
\langle T(u), u \rangle \geq C \min\{\|u - v_0\|_{1,G(\cdot)}^0, \|u - v_0\|_{1,G(\cdot)}^{g_0}\} - \int_{\Omega} G(x, |\nabla v_0|) \, dx.
\]

As \(g^0 \geq g_0 > 1\), then

\[
\frac{\langle T(u), u \rangle}{\|u\|_{1,G(\cdot)}} \xrightarrow{\|u\|_{1,G(\cdot)} \to \infty} \infty.
\]

Hence \(T\) is coercive.

Next, we show that \(T\) is weakly continuous. In fact, let \(u_i \in K_{\psi,v_0}(\Omega)\) be a convergent sequence to an element \(u \in K_{\psi,v_0}(\Omega)\) in \(W^{1,G(\cdot)}(\Omega)\). By the lemma 2.1 in [5], there exists a subsequence \(u_i\) such that \(u_i \to u\) and \(\nabla u_i \to \nabla u\) a.e. in \(\Omega\). Since the mapping \(t \mapsto g(x,t)\) is continuous for a.e. \(x\), then for any \(v \in W^{1,G(\cdot)}(\Omega)\), we have

\[
g(x, |\nabla u_i|) \nabla u_i \cdot \nabla v \to g(x, |\nabla u|) \nabla u \cdot \nabla v \text{ a.e. in } \Omega.
\]
Moreover, if \( E \subset \Omega \) is any measurable subset, then by Hölder inequality and inequality (2.5), we have

\[
\left| \int_E \frac{g(x, |\nabla u_i|)}{|\nabla u_i|} \nabla u_i \cdot \nabla v \, dx \right| \leq \int_E g(x, |\nabla u_i|) |\nabla v| \, dx \leq C \parallel g(x, |\nabla u_i|) \parallel_{G^{(\cdot)}} \parallel \nabla v \parallel_{G^{(\cdot)}} \\
\leq C \parallel \nabla u_i \parallel_{G^{(\cdot)}} \parallel \nabla v \parallel_{G^{(\cdot)}} \\
\leq C \parallel v \parallel_{L^1(\Omega)}.
\]

So, applying Vitali convergence theorem, we obtain

\[
\int_{\Omega} \frac{g(x, |\nabla u_i|)}{|\nabla u_i|} \nabla u_i \cdot \nabla v \, dx \to \int_{\Omega} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v \, dx.
\]

Then \( T \) is weakly continuous. Moreover, by Corollary 8.3 in [15] we have \( T \) is monotone. Therefore, there exists an element \( u \in K_{\psi,v_0}(\Omega) \) such that

\[
\int_{\Omega} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot (\nabla v - \nabla u) \, dx \geq 0,
\]

whenever \( v \in K_{\psi,v_0}(\Omega) \).

**Theorem 3.2** Let \( G(\cdot) \in \Phi(\Omega) \cap C^1(\mathbb{R}^+) \) satisfy \((SC)\), \((A_0)\) and \((A_1)\). Then for every \( v_0 \in W^{1,G(\cdot)}(\Omega) \), there exists \( u \in W^{1,G(\cdot)}(\Omega) \) a \( G(\cdot) \)-solution to equation (3.1) in \( \Omega \), such that \( u - v_0 \in W^{1,G(\cdot)}(\Omega) \).

If \( G(\cdot) \) is strictly convex, the \( G(\cdot) \)-solution is unique, and if \((A_1,n)\) hold, then it is continuous.

**Proof** Let \( G(\cdot) \in \Phi(\Omega) \) satisfy \((SC)\) and \( v_0 \in W^{1,G(\cdot)}(\Omega) \). Then \( K_{\infty,v_0}(\Omega) \neq \emptyset \), so by Theorem 3.1 there exists a solution \( u \) of the obstacle problem in \( K_{\infty,v_0}(\Omega) \).

Let \( \varphi \in W^{1,G(\cdot)}_0(\Omega) \) then \( u - \varphi, u + \varphi \in K_{\infty,v_0}(\Omega) \). Hence

\[
\int_{\Omega} \frac{g(x, |u|)}{|u|} \nabla u \cdot \nabla \varphi \, dx \geq 0
\]

and

\[
- \int_{\Omega} \frac{g(x, |u|)}{|u|} \nabla u \cdot \nabla \varphi \, dx \geq 0.
\]

Consequently

\[
\int_{\Omega} \frac{g(x, |u|)}{|u|} \nabla u \cdot \nabla \varphi \, dx = 0,
\]

whenever \( \varphi \in W^{1,G(\cdot)}_0(\Omega) \). Then \( u \) is a \( G(\cdot) \)-solution to equation (3.1) in \( \Omega \) such that \( u - v_0 \in W^{1,G(\cdot)}_0(\Omega) \).
If moreover, \( G(\cdot) \) is strictly convex, then by the comparison weak principle Lemma 4.3 in [5] the \( G(\cdot) \)-solution is unique, and from Theorem 1.3 in [16] it is locally bounded. If \( G(\cdot) \) satisfies \( (A_{1,n}) \), then by Corollary 4.1 in [4] the \( G(\cdot) \)-solution is locally Hölder continuous. This concludes the proof. \( \square \)

4 Sobolev \( G(\cdot) \)-regular boundary points and exterior sphere condition

**Definition 4.1** Let \( G(\cdot) \in \Phi(\Omega) \) strictly convex and satisfy \((SC)\), \((A_0)\), \((A_1)\) and \((A_{1,n})\). A boundary point \( x_0 \) of a bounded open set \( \Omega \) is said to be Sobolev \( G(\cdot) \)-regular if, for each function \( v_0 \in W^{1,G(\cdot)}(\Omega) \cap C(\Omega) \), the \( G(\cdot) \)-harmonic function \( h \) in \( \Omega \) with \( h - v_0 \in W^{1,G(\cdot)}_0(\Omega) \) satisfies

\[
\lim_{x \to x_0} h(x) = v_0(x_0).
\]

Furthermore, we say that a bounded open set \( \Omega \) is Sobolev \( G(\cdot) \)-regular if each \( x_0 \in \partial \Omega \) is Sobolev \( G(\cdot) \)-regular.

In [14], Harjulehto and Händö gave the following sufficient condition for the Sobolev \( G(\cdot) \)-regular point.

**Theorem 4.1** Let \( x_0 \in \partial \Omega \). Let \( G(\cdot) \in \Phi(\mathbb{R}^n) \) be strictly convex and satisfy \((SC)\), \((A_0)\), \((A_1)\), and \((A_{1,n})\). If there exist \( C \in (0, 1) \) and \( R > 0 \) such that

\[
\cap_{G(\cdot)}(B(x_0, r) \setminus \Omega); B(x_0, 2r)) \geq C \cap_{G(\cdot)}(B(x_0, r); B(x_0, 2r)) \quad \text{for all} \quad 0 < r < R,
\]

then \( x_0 \) is a Sobolev \( G(\cdot) \)-regular point.

**Definition 4.2** We say that a boundary point \( x_0 \) of a bounded open set \( \Omega \) satisfies the exterior sphere condition, if there is a ball \( B(y_0, \rho) \) such that \( B(y_0, \rho) \cap \Omega = \{x_0\} \).

Furthermore, if all boundary points of \( \Omega \) satisfy exterior sphere condition, then \( \Omega \) satisfies exterior sphere condition.

**Lemma 4.1** Let \( G(\cdot) \in \Phi(\sigma B) \) with \( \sigma > 1 \) satisfy \((SC)\). Then there exits a positive constant \( C = C(n, g_0^0, g_0, \sigma) \) such that

\[
\frac{1}{C} |B|G_{\sigma B}^{-}\left(\frac{1}{r}\right) \leq \cap_{G(\cdot)}(B; \sigma B) \leq C |B|G_{\sigma B}^{+}\left(\frac{1}{r}\right).
\]

**Proof** Let \( u \in W^{1,G(\cdot)}_0(\sigma B) \) be such that \( 0 \leq u \leq 1, u = 1 \) in \( B \) and \( |\nabla u| \leq \frac{C}{r} \). Then by the condition \((SC)\), we have

\[
\cap_{G(\cdot)}(B; \sigma B) \leq \int_{\sigma B} G(x, |\nabla u|) \, dx \leq \int_{\sigma B} G_{\sigma B}^+\left(\frac{C}{r}\right) \, dx \leq C |B|G_{\sigma B}^+\left(\frac{1}{r}\right).
\]
For the opposite inequality by Jensen-type inequality in [13] and the definition of I-capacity that

\[
\int_{\sigma B} G(x, |\nabla u|) \, dx \geq \int_{\sigma B} G_{\sigma B}^{-}(|\nabla u|) \, dx
\]

\[
= |\sigma B| \int_{\sigma B} G_{\sigma B}^{-}(|\nabla u|) \, dx
\]

\[
\geq C|\sigma B| G_{\sigma B}^{-} \left( \int_{\sigma B} |\nabla u| \, dx \right)
\]

\[
\geq C|\sigma B| G_{\sigma B}^{-} \left( \frac{\text{cap}_1(B; \sigma B)}{|\sigma B|} \right).
\]

Since by Example 2.12 in [17] we have \( \text{cap}_1(B; \sigma B) = C r^{n-1} \), then by the condition (SC), we get

\[
\int_{\sigma B} G(x, |\nabla u|) \, dx \geq C|\sigma B| G_{\sigma B}^{-} \left( \frac{1}{r} \right).
\]

This concludes the proof. \( \square \)

**Theorem 4.2** Let \( G(\cdot) \in \Phi(\Omega) \) strictly convex and satisfy (SC), (A0), (A1) and (A1,\( n \)). If \( \Omega \) satisfies the exterior sphere condition, then \( \Omega \) is Sobolev \( G(\cdot) \)-regular.

**Proof** Let \( \Omega \) satisfy the exterior sphere condition. Then for every \( x_0 \in \partial \Omega \), there exists a ball \( B(y_0, r) \) such that \( B(y_0, r) \cap \overline{\Omega} = \{x_0\} \). So we have \( B(x_0, 3r) \setminus \Omega \) contains \( B(y_0, r) \). Then, by Proposition 2.1 and Lemma 4.1, we have

\[
\text{cap}_{G(\cdot)}(B(x_0, 3r) \setminus \Omega; B(x_0, 6r)) \geq C \text{cap}_{G(\cdot)}(B(y_0, r); B(x_0, 6r))
\]

\[
\geq C \text{cap}_{G(\cdot)}(B(y_0, r); B(y_0, 8r))
\]

\[
\geq C|B(y_0, r)| G_{B(y_0, 8r)}^{-} \left( \frac{1}{r} \right)
\]

\[
\geq C|B(y_0, 3r)| G_{B(y_0, 8r)}^{-} \left( \frac{1}{r} \right).
\]

By the condition (A1,\( n \)) there exists a constant \( C > 0 \) such that

\[
G_{B(y_0, 8r)}^{+} \left( \frac{1}{r} \right) \leq CG_{B(y_0, 8r)}^{-} \left( \frac{1}{r} \right).
\]

Using again Lemma 4.1 we obtain

\[
\text{cap}_{G(\cdot)}(B(x_0, 3r) \setminus \Omega; B(x_0, 6r)) \geq C|B(x_0, 3r)| G_{B(y_0, 8r)}^{+} \left( \frac{1}{r} \right)
\]

\[
\geq C|B(x_0, 3r)| G_{B(x_0, 6r)}^{+} \left( \frac{1}{r} \right)
\]

\[
\geq C \text{cap}_{G(\cdot)}(B(x_0, 3r); B(x_0, 6r)).
\]

for \( r \) small enough, so by Theorem 4.1 we have \( \Omega \) is Sobolev \( G(\cdot) \)-regular. \( \square \)
Corollary 4.1  Let \( G(\cdot) \in \Phi(\Omega) \) strictly convex and satisfy (SC), (A0), (A1) and (A1,\(n\)). All balls are Sobolev \( G(\cdot) \)-regular.

Consequently, every open set can be exhausted by Sobolev \( G(\cdot) \)-regular open sets as a consequence of this corollary.

5 The Perron-Wiener-Brelot method

5.1 Upper and lower Perron \( G(\cdot) \)-solution

Let \( G(\cdot) \in \Phi(\Omega) \). A function \( u : \Omega \to \mathbb{R} \cup \{\infty\} \) is called \( G(\cdot) \)-superharmonic in \( \Omega \) if

(i) \( u \) is lower semicontinuous,
(ii) \( u \neq \infty \) in \( \Omega \),
(iii) for each domain \( D \subset \subset \Omega \) the comparison principle holds: if \( h \in C(D) \) is \( G(\cdot) \)-harmonic in \( D \) and \( u \geq h \) on \( \partial D \), then \( u \geq h \) in \( D \).

A function \( v : \Omega \to \mathbb{R} \cup \{-\infty\} \) is called \( G(\cdot) \)-subharmonic in \( \Omega \) if

(i) \( u \) is upper semicontinuous,
(ii) \( u \neq -\infty \) in \( \Omega \),
(iii) for each domain \( D \subset \subset \Omega \) the comparison principle holds: if \( h \in C(D) \) is \( G(\cdot) \)-harmonic in \( D \) and \( u \leq h \) on \( \partial D \), then \( u \leq h \) in \( D \).

For \( f : \partial \Omega \to [-\infty, \infty] \) a function, we define as in classical potential theory [18] two classes of functions:

- The upper class \( U_f \) consists of all functions \( v : \Omega \to (-\infty, \infty] \) such that
  (i) \( v \) is \( G(\cdot) \)-superharmonic in \( \Omega \),
  (ii) \( v \) is bounded below,
  (iii) \( \liminf_{x \to \xi} v(x) \geq f(\xi) \) when \( \xi \in \partial \Omega \).

- The lower class \( L_f \) consists of all functions \( u : \Omega \to [-\infty, \infty) \) such that
  (i) \( u \) is \( G(\cdot) \)-subharmonic in \( \Omega \),
  (ii) \( u \) is bounded above,
  (iii) \( \limsup_{x \to \xi} u(x) \leq f(\xi) \) when \( \xi \in \partial \Omega \).

We define at each point in \( \Omega \)

\[
\text{the upper Perron } G(\cdot)\text{-solution } \overline{H}_f(x) = \inf_{v \in U_f} v(x)
\]

and

\[
\text{the lower Perron } G(\cdot)\text{-solution } \underline{H}_f(x) = \sup_{v \in L_f} v(x).
\]

If \( U_f = \emptyset \) (or \( L_f = \emptyset \)), then we have \( \overline{H}_f = \infty \) (and \( \underline{H}_f = -\infty \) respectively). The following lemma gives simple properties for Perron \( G(\cdot) \)-solutions.
Lemma 5.1 Let \( f : \partial \Omega \to [-\infty, \infty] \) be a function, we have the following properties

1. \( H_f = -\overline{H}_{-f} \)
2. \( \overline{H}_f \leq \overline{H}_{f} \)
3. if \( f \leq g \), then \( \overline{H}_f \leq \overline{H}_{g} \)
4. for \( \lambda \in \mathbb{R} \), we have \( \overline{H}_{f + \lambda} = \overline{H}_f + \lambda \) and \( \overline{H}_{\lambda f} = \lambda \overline{H}_f \).

For (3) and (4), a similar statement is true if \( \overline{H}_f \) is replace by \( \underline{H}_f \).

5.2 The Poisson modification

Generally, the Harnack Convergence theorem and the comparison principle are needed to construct the Poisson modification. From [11], we have

Theorem 5.1 (Harnack Convergence theorem) Let \( G(\cdot) \in \Phi(\Omega) \) satisfy (SC). If \((u_i)\) is a sequence of \( G(\cdot)\)-harmonic functions such that

\[ 0 \leq u_1 \leq u_2 \leq \ldots, \ u = \lim_{i \to \infty} u_i, \ \text{pointwise in } \Omega, \]

then either \( u = \infty \) or \( u \) is a \( G(\cdot)\)-harmonic in \( \Omega \).

Lemma 5.2 (Comparison principle) Let \( G(\cdot) \in \Phi(\Omega) \) satisfy (SC). Suppose that \( u \) is a \( G(\cdot)\)-subharmonic and \( v \) is a \( G(\cdot)\)-superharmonic in \( \Omega \) such that

\[ \lim_{x \to y} \sup u(x) \leq \lim_{x \to y} \inf v(x), \]

for all \( y \in \partial \Omega \). If the left and right-hand sides are neither \( \infty \) nor \(-\infty \) at the same time, then

\[ u \leq v \ \text{in } \Omega. \]

Let \( G(\cdot) \in \Phi(\Omega) \) strictly convex and satisfy (SC), \((A_0),(A_1)\) and \((A_{1,n})\). Given a Sobolev \( G(\cdot)\)-regular subdomain \( D \subset \subset \Omega \) (see Corollary 4.1) and \( v \) is a \( G(\cdot)\)-superharmonic function in \( \Omega \). Since \( v \) is lower semicontinuous in \( \Omega \), there exists a sequence \( v_i \in C^\infty(\Omega) \) such that

\[ v_1 \leq v_2 \leq \ldots \leq v \text{ and } \lim_{i \to \infty} v_i(x) = v(x) \text{ at each } x \in \Omega. \]

Let \( h_i \) be the \( G(\cdot)\)-harmonic function in \( D \) such that \( h_i - v_i \in W^{1,G(\cdot)}_0(D) \). Applying the Sobolev \( G(\cdot)\)-regularity of \( D \) and the comparison principle, we get

\[ h_1 \leq h_2 \leq \ldots \leq v \text{ in } D. \]

By the Harnack convergence theorem, the function \( h = \lim_{i \to \infty} h_i \) is \( G(\cdot)\)-harmonic. We define the Poisson modification \( P(v, D) \) as follows

\[ P(v, D) = \begin{cases} h & \text{in } D \\ v & \text{in } \Omega \setminus D. \end{cases} \]
Theorem 5.2 Let $G(\cdot) \in \Phi(\Omega)$ strictly convex and satisfy $(SC)$, $(A_0)$, $(A_1)$ and $(A_{1,n})$. Let $D \subset \Omega$ be a $G(\cdot)$-regular subdomain and $v$ is a $G(\cdot)$-superharmonic function in $\Omega$. Then the Poisson modification $P(v, D)$ is $G(\cdot)$-superharmonic in $\Omega$, $G(\cdot)$-harmonic in $D$ and $P(v, D) \leq v$.

Proof By the construction of the Poisson modification, we have $P(v, D)$ is a $G(\cdot)$-harmonic function in $D$, and $h \leq v$ in $D$, so

$$P(v, D) \leq v \text{ in } \Omega.$$ 

We show that $P(v, D)$ is lower semicontinuous. Let $\xi \in \partial D$

$$\liminf_{x \to \xi} P(v, D) = \liminf_{x \to \xi} v(x) \geq v(\xi) = P(v, D)(\xi)$$

and

$$\liminf_{x \to \xi} P(v, D)(x) = \liminf_{x \to \xi} h(x) \geq \liminf_{x \to \xi} h_1(x) = v_1(\xi).$$

So,

$$\liminf_{x \to \xi} P(v, D)(x) \geq v(\xi) = P(v, D)(\xi).$$

Next, we prove $P(v, D)$ satisfies the comparison principle. Indeed, let $G \subset \subset \Omega$ is a domain and $H \in C(G)$ is $G(\cdot)$-harmonic function in $G$ with $H|_{\partial G} \leq P(v, D)|_{\partial G}$.

We have $P(v, D) \leq v$ in $\Omega$, then $H|_{\partial G} \leq v|_{\partial G}$. As $v$ is $G(\cdot)$-superharmonic function, then $H \leq v$ in $G$. Hence,

$$H \leq P(v, D) \text{ in } G \setminus D.$$ 

Let $\xi \in \partial (G \cap D)$, we have

$$H(\xi) \leq v(\xi) \leq \liminf_{x \to \xi} h(x).$$

So

$$\liminf_{x \to \xi} H(x) \leq \liminf_{x \to \xi} h(x).$$

Then

$$H \leq h = P(v, D) \text{ in } D \cap G.$$
Hence

\[ H \leq P(v, D) \text{ in } G. \]

Therefore \( P(v, D) \) is \( G(\cdot) \)-superharmonic function in \( \Omega \).

\[ \square \]

### 5.3 \( G(\cdot) \)-resolutivity

**Definition 5.1** Let \( G(\cdot) \in \Phi(\Omega) \) strictly convex and satisfy \((SC)\), \((A_0)\), \((A_1)\) and \((A_{1,n})\). We say that a function \( f : \Omega \rightarrow [-\infty, \infty] \) is \( G(\cdot) \)-resolutive if the upper and the lower Perron \( G(\cdot) \)-solution \( \overline{H}_f \) and \( \underline{H}_f \) coincide and are \( G(\cdot) \)-harmonic in \( \Omega \).

**Definition 5.2** A family \( U \) of functions is downward directed if for each \( u, v \in U \), there is \( s \in U \) with \( s \leq \min(u, v) \).

The following Lemma is fundamental in PWB method [17]. First recall that the lower semicontinuous regularization \( u^* \) of any function \( u : \Omega \rightarrow [-\infty, \infty] \) is defined by

\[ u^*(x) := \lim_{r \to 0} \inf_{\Omega \cap B(x, r)} u. \]

**Lemma 5.3** (Choquet’s topological lemma) Suppose \( E \subset \mathbb{R}^N \) and that \( U = \{ u_\gamma, \gamma \in I \} \) is a family of functions \( u_\gamma : E \rightarrow [-\infty, \infty] \). Let \( u = \inf U \). If \( U \) is downward directed, then there is a decreasing sequence of functions \( v_j \in U \) with limit \( v \) such that the lower semicontinuous regularizations \( u^* \) and \( v^* \) coincide.

**Theorem 5.3** Let \( G(\cdot) \in \Phi(\Omega) \) strictly convex and satisfy \((SC)\), \((A_0)\), \((A_1)\) and \((A_{1,n})\). Then one of the following alternatives is true

1. \( \overline{H}_f \) is \( G(\cdot) \)-harmonic in \( \Omega \),
2. \( \overline{H}_f \equiv -\infty \),
3. \( \overline{H}_f \equiv \infty \).

A similar statement is true for \( \underline{H}_f \).

**Proof** If the upper class \( U_f \) is empty, then \( \overline{H}_f = \infty \).

Suppose that the upper class \( U_f \) is not empty, then \( U_f \) is downward directed. So, by Choquet’s topological lemma, there exists a decreasing sequence of functions \( u_i \in U_f \) convergent to a function \( u \) such that \( u^* = \overline{H}_f \) in \( \Omega \).

Let \( D \subset \subset \Omega \) be a Sobolev \( G(\cdot) \)-regular and consider the Poisson modification \( P(u_i, D) \). Using Theorem 5.2, we have \( P(u_i, D) \in U_f \). Then, by the Harnack convergence theorem, \( \lim_{i \to \infty} P(u_i, D) \) is either \( G(\cdot) \)-harmonic or identically \( -\infty \) in \( D \). As \( \overline{H}_f \leq P(u_i, D) \leq u_i \) and \( u^* = \overline{H}_f \), then \( \overline{H}_f = \lim_{i \to \infty} P(u_i, B) \) in \( D \). Therefore \( \overline{H}_f \) is either \( G(\cdot) \)-harmonic or identically \( -\infty \) in \( \Omega \).

**Theorem 5.4** (Wiener theorem) Let \( G(\cdot) \in \Phi(\Omega) \) strictly convex and satisfy \((SC)\), \((A_0)\), \((A_1)\) and \((A_{1,n})\). Suppose that \( f : \partial \Omega \rightarrow \mathbb{R} \) is continuous. Then \( f \) is \( G(\cdot) \)-resolutive in \( \Omega \), i.e \( \overline{H}_f = \underline{H}_f := H_f \).
Proof Let \( f : \partial \Omega \to \mathbb{R} \) be a continuous function. By the Tietze extension theorem, we can assume \( f \in C(\mathbb{R}^n) \). Then, for all \( \epsilon > 0 \) there exists \( \varphi_i \in C^\infty(\mathbb{R}^n) \) such that
\[
\varphi_i(\xi) - \epsilon < f(\xi) < \varphi_i(\xi) + \epsilon \quad \text{when} \quad \xi \in \partial \Omega.
\]
Thus,
\[
H_{\varphi_i} - \epsilon \leq H_{\varphi_i} - \epsilon \leq H_f \leq \overline{H}_f \leq \overline{H}_{\varphi_i} + \epsilon \leq \overline{H}_{\varphi_i} + \epsilon.
\]
So, if \( H_{\varphi_i} = \overline{H}_{\varphi_i} \), then \( H_f = \overline{H}_f \). Hence, it suffices to prove the result for \( \varphi_i \).

Let \( H_i \) be a \( G(\cdot) \)-harmonic in \( \Omega \) such that \( H_i - \varphi_i \in W_0^{1,G(\cdot)}(\Omega) \). Let \( v_i \) denote the \( G(\cdot) \)-solution to obstacle problem with \( \varphi_i \) acting as obstacle and also boundary data. So \( v_i \in U_f \). Choose Sobolev \( G(\cdot) \)-regular domains \( D_j \subset \subset \Omega \) such that \( \Omega = \bigcup_{j \geq 1} D_j \) and \( D_1 \subset D_2 \subset \ldots \). Construct the sequence of Poisson modification
\[
P_{i,j} = P(v_i, D_j).
\]
Then \( \{P_{i,j}\} \) is non-increasing, \( P_{i,j} \in U_f \) and \( P_{i,j} - \varphi_i \). Then \( P_{i,j} - \varphi_i = P_{i,j} - v_i + v_i - \varphi_i \in W_0^{1,G(\cdot)}(\Omega) \). Let \( P_i = \lim_{j \to \infty} P_{i,j} \). As \( \overline{H}_{\varphi_i} \leq P_{i,j} \), then by the Harnack convergence theorem, \( P_i \) is \( G(\cdot) \)-harmonic in \( \Omega \) and \( P_i - \varphi_i \in W_0^{1,G(\cdot)}(\Omega) \). So, \( P_i = H_i \) in \( \Omega \). Hence \( \overline{H}_{\varphi_i} \leq P_i = H_i \). By a similar proof, we have \( H_i \leq \overline{H}_{\varphi_i} \).

Then
\[
H_i \leq H_{\varphi_i} \leq \overline{H}_{\varphi_i} \leq H_i.
\]
Hence
\[
H_{\varphi_i} = \overline{H}_{\varphi_i}.
\]
This concludes the proof. \( \square \)

As a consequence of the previous theorem, the Perron \( G(\cdot) \)-solution coincides with the \( G(\cdot) \)-solution of Dirichlet-Sobolev with boundary \( f \).

Corollary 5.1 Let \( G(\cdot) \in \Phi(\Omega) \) strictly convex and satisfy (SC), (A0), (A1) and (A1,n). If \( f \in W_0^{1,G(\cdot)}(\Omega) \cap C(\overline{\Omega}) \). Then \( \overline{H}_f \) is the unique \( G(\cdot) \)-harmonic function such that \( \overline{H}_f - f \in W_0^{1,G(\cdot)}(\Omega) \).

6 \( G(\cdot) \)-potential

Definition 6.1 Let \( G(\cdot) \in \Phi(\Omega) \) strictly convex and satisfy (SC), (A0), (A1) and (A1,n). Let \( K \subset B \) be compact and \( \psi \in C_0^\infty(B) \) be such that \( \psi = 1 \) on \( K \). We define the \( G(\cdot) \)-potential for \( K \) with respect to \( B \) as follows
\[
\mathcal{R}_{G(\cdot)}(K, B) := \begin{cases} 
2. \text{in } B \setminus K, \\
1 \text{ in } K.
\end{cases}
\]
where $h$ is the unique $G(\cdot)$-harmonic function in $B \setminus K$ such that $h - \psi \in W^{1, G(\cdot)}_0 (B \setminus K)$.

**Remark 6.1** The definition of $\mathcal{R}_{G(\cdot)} (K, B)$ is independent of the particular choice of $\psi$. Indeed, if $\tilde{\psi}$ is another such that $\tilde{h}$ is the unique $G(\cdot)$-harmonic function in $B \setminus K$ such that $\tilde{h} - \tilde{\psi} \in W^{1, G(\cdot)}_0 (B \setminus K)$, then $h - \tilde{h} \in W^{1, G(\cdot)}_0 (B \setminus K)$ and by the uniqueness we have $h = \tilde{h}$ in $W^{1, G(\cdot)}_0 (B \setminus K)$.

### 6.1 $G(\cdot)$-potential and $G(\cdot)$-capacity

Using the same method, as in [10], we have the following lemma.

**Lemma 6.1** Let $G(\cdot) \in \Phi (B) \cap C^1 (\mathbb{R}^+)$ satisfy (SC), $(A_0)$ and $(A_1)$. If $u = \mathcal{R}_{G(\cdot)} (K, B)$ the $G(\cdot)$-potential for $K$ with respect to $B$. Then $u$ is a $G(\cdot)$-supersolution in $B$.

**Proof** Let $G(\cdot) \in \Phi (B) \cap C^1 (\mathbb{R}^+)$. Following [15], $u$ is a $G(\cdot)$-supersolution in $B$ is equivalent to

$$
\int_B G(x, |\nabla u|) \, dx \leq \int_B G(x, |\nabla (u + \varphi)|) \, dx,
$$

for every nonnegative function $\varphi$ in $W^{1, G(\cdot)}_0 (B)$. Note that, it suffices to prove the previous inequality to $u + \varphi \leq 1$, in $B$. Indeed, for any $\psi \in W^{1, G(\cdot)}_0 (B)$ nonnegative we have the function $\varphi = \min \{ \psi, 1 - u \}$ satisfy $\varphi \in W^{1, G(\cdot)}_0 (B)$, $u + \varphi \leq 1$ in $B$ and

$$
\int_B G(x, |\nabla u|) \, dx \leq \int_B G(x, |\nabla (u + \varphi)|) \, dx \\
= \int_B G(x, |\nabla \min \{ u + \psi, 1 \}|) \, dx \\
\leq \int_B G(x, |\nabla (u + \psi)|) \, dx.
$$

Since $u = 1$ in $K$, then the inequality $u + \varphi \leq 1$ implies that $\varphi = 0$ on $K$. Hence, by Lemma 6.1.10 and Theorem 6.4.7 in [15], we have $\varphi \in W^{1, G(\cdot)}_0 (B \setminus K)$. As $u$ is a $G(\cdot)$-harmonic function in $B \setminus K$, then

$$
\int_B G(x, |\nabla u|) \, dx = \int_{B \setminus K} G(x, |\nabla u|) \, dx \\
\leq \int_{B \setminus K} G(x, |\nabla (u + \varphi)|) \, dx \leq \int_B G(x, |\nabla (u + \varphi)|) \, dx.
$$

Therefore $u$ is a $G(\cdot)$-supersolution in $B$. \qed

Using the Riesz representation theorem, see Theorem 2.14 in [30], we have the following theorem.
Lemma 6.2 Let $G(\cdot) \in \Phi(\Omega)$. For every $G(\cdot)$-supersolution $u$ in $\Omega$, there is a Radon measure $\mu[u] \in \left( W_0^{1,G(\cdot)}(\Omega) \right)^*$ such that

$$
\int_{\Omega} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} \varphi \, d\mu[u],
$$

whenever $\varphi \in W_0^{1,G(\cdot)}(\Omega)$.

Proof Since $u$ is $G(\cdot)$-supersolution, then for all nonnegative $\eta \in W_0^{1,G(\cdot)}(4B)$ we have

$$
\int_{\Omega} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \eta \, dx \geq 0.
$$

Using the Riesz representation theorem, see Theorem 2.14 in [30], we conclude that there exists a Radon measure $\mu[u] \in \left( W_0^{1,G(\cdot)}(\Omega) \right)^*$ such that

$$
\int_{\Omega} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} \varphi \, d\mu[u],
$$

whenever $\varphi \in W_0^{1,G(\cdot)}(\Omega)$. In abbreviated notation, we write $-\Delta_{G(\cdot)}(u) = \mu[u]$. $\square$

Theorem 6.1 Let $G(\cdot) \in \Phi(B) \cap C^1(\mathbb{R}^+)$ satisfy (SC), (A0) and (A1) and $K$ be a compact subset of $B$. If $u = \mathcal{R}_{G(\cdot)}(K, B)$ is the $G(\cdot)$-potential for $K$ with respect to $B$ and $\mu[u]$ its associated Radon measure in $\left( W_0^{1,G(\cdot)}(\Omega) \right)^*$, then there exists a constant $C > 0$ such that

$$
\frac{1}{C} \text{Cap}_{G(\cdot)}(K; B) \leq \mu[u](K) \leq C \text{Cap}_{G(\cdot)}(K; B).
$$

Proof Let $u$ the $G(\cdot)$-potential for $K$ with respect to $B$ and $\mu[u]$ its associated Radon measure in $\left( W_0^{1,G(\cdot)}(\Omega) \right)^*$. As $u$ is $G(\cdot)$-harmonic in $B \setminus K$, then the support of the measure $\mu[u]$ is contained in $K$. Hence

$$
\mu[u](K) = \mu[u](B) = \int_B u \, d\mu[u] = \int_B \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla u \, dx. \quad (6.1)
$$

On the one hand, as $u \in S_{G(\cdot)}(K; \Omega)$ then

$$
cap_{G(\cdot)}(K; B) \leq \int_B g(x, |\nabla u|) \, dx \leq C \int_B \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla u \, dx \leq C \mu[u](K).
$$

On the other hand, let $\varphi \in S_{G(\cdot)}(K; \Omega)$ and we consider $\psi := \max\{\varphi, u\}$. So, the nonnegative function $\psi - u \in W_0^{1,G(\cdot)}(B)$. By Lemma 6.1, we have $u$ is a $G(\cdot)$-supersolution, hence

$\square$ Springer
\[ \int_B \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla (\varphi - u) \, dx \geq \int_B \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla (\psi - u) \, dx \geq 0. \]

Then
\[ \int_B \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \psi \, dx \leq \int_B \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \varphi \, dx. \]

Using the inequality (2.1), we get
\[ \int_B G(x, |\nabla u|) \, dx \leq C \int_B g(x, |\nabla u|) |\nabla \varphi| \, dx \leq \frac{1}{2} \int_B G(x, |\nabla u|) \, dx + C \int_B G(x, |\nabla \varphi|) \, dx. \]

Hence
\[ \int_B G(x, |\nabla u|) \, dx \leq C \int_B G(x, |\nabla \varphi|) \, dx. \]

By the equality (6.1), we have
\[ \mu[u](K) \leq C \int_B G(x, |\nabla u|) \, dx \leq C \int_B G(x, |\nabla \varphi|) \, dx. \]

Taking the infimum of the functions \( \varphi \in S_{G(\cdot)}(K; B) \), we obtain
\[ \mu[u](K) \leq C \text{cap}_{G(\cdot)}(K; B). \]

This concludes the proof. \( \Box \)

### 6.2 Estimation of \( G(\cdot) \)-potential

In [4], we proved the following Caccioppoli type estimate of supersolutions to equation (3.1).

**Lemma 6.3** Let \( G(\cdot) \in \Phi(2B) \) satisfy (SC). Let \( u \) be a nonpositive \( G(\cdot) \)-supersolution of (3.1) in a ball \( 2B \), \( \eta \in C_0^\infty(2B) \) with \( 0 \leq \eta \leq 1 \) and \( |\nabla \eta| \leq \frac{1}{r} \). Then, there exits a constant \( C \) such that
\[ \int_{2B} G(x, |\nabla u|) \eta^0 \, dx \leq C \int_{2B} G^+ \left( \frac{-u}{r} \right) \, dx. \]

**Lemma 6.4** Let \( G(\cdot) \in \Phi(B(x_0, 2r_0)) \) satisfy (SC), (A0) and (A1,n). If \( u \) is a nonnegative bounded \( G(\cdot) \)-supersolution in \( B(x_0, 2r_0) \), then for some constant \( C > 0 \), we have
\[ Cr g^{-1} \left( x_0, \frac{\mu[u](B(x_0, r))}{r^{n-1}} \right) \leq \text{ess inf}_{B(x_0, r)} u + r, \]
with \( r \in (0, r_0] \) and \( \mu[u] \) is the associated Radon measure to \( u \) in \( \left( W_{0,1}^1 \right)^* \).

**Proof** We set \( B = B(x_0, r) \), \( b = \inf_B u \) and, \( v = \min\{u, b\} + r \). Choose \( \omega = v \eta^g \) such that \( \eta \in C_0^\infty(2B) \) with \( 0 \leq \eta \leq 1 \), \( \eta = 1 \) in \( \overline{B} \) and \( |\nabla \eta| \leq \frac{C}{r} \), we have

\[
(b + r)\mu[u](B) \leq \int_{2B} \omega \, d\mu[u] = \int_{2B} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \omega \, dx \\
\leq \int_{2B} \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v \right) \eta^g \, dx + C \int_{2B} \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \eta \right) \eta^{g-1} v \, dx.
\]

By the condition \((SC)\), we have

\[
I_1 := \int_{2B} \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v \right) \eta^g \, dx \\
\leq g^0 \int_{2B} G(x, |\nabla v|) \eta^g \, dx
\]

and

\[
I_2 := \int_{2B} \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \eta \right) \eta^{g-1} v \, dx \\
\leq \int_{2B} g(x, |\nabla v|) |\nabla \eta| \eta^{g-1} v \, dx.
\]

As \( v \leq b + r \) and \( |\nabla \eta| < \frac{C}{r} \), we have

\[
I_2 \leq C \frac{b + r}{r} \int_{2B} g(x, |\nabla v|) \eta^{g-1} \, dx.
\]

Using inequality (2.1) for \( a' = |\nabla v| \) and \( b' = \frac{b + r}{\eta r} \), and the condition \((SC)\), we get

\[
I_2 \leq C \left( \int_{2B} G(x, |\nabla v|) \eta^g \, dx + \int_{2B} G \left( x, \frac{b + r}{r} \right) \, dx \right).
\]

Collecting the previous estimations of \( I_1 \) and \( I_2 \), we obtain

\[
(b + r)\mu[u](B) \leq C \left( \int_{2B} G(x, |\nabla v|) \eta^g \, dx + \int_{2B} G \left( x, \frac{b + r}{r} \right) \, dx \right).
\]
By Lemma 6.3, we have
\[
\int_B G(x, |\nabla (v - (b + r))|)\eta \, dx \leq C \int_{2B} G^+ \left( \frac{b + r - v}{r} \right) \, dx.
\]
Hence
\[
(b + r)\mu[u](B) \leq C \int_{2B} G^+ \left( \frac{b + r}{r} \right) \, dx.
\]
Since \( L^G(B) \subset L^{g_0}(B) \) (see [13]), we have
\[
1 \leq \frac{b + r}{r} \leq \|u\|_{\infty, B(x_0, 2r_0)} + r.
\]
Then, by Lemma 2.1, there exists a constant \( C > 0 \) such that
\[
G^+ \left( \frac{b + r}{r} \right) \leq CG \left( x_0, \frac{b + r}{r} \right).
\]
Hence
\[
(b + r)\mu[u](B) \leq Cr^n G \left( x_0, \frac{b + r}{r} \right).
\]
So, by the condition (SC), we have
\[
\mu[u](B) \leq Cr^{-n} g \left( x_0, \frac{b + r}{r} \right).
\]
From inequalities 2.4, 2.2 and 2.3, we have
\[
Cr^{-1} \left( x_0, \frac{\mu[u](B)}{r^{-n-1}} \right) \leq \inf_B u + r.
\]
This concludes the proof. \( \square \)

By a similar proof as in [8], we have the following lemma.

**Lemma 6.5** Let \( G(\cdot) \in \Phi(\Omega) \) satisfy (SC). Let \( f \in W^{1,G(\cdot)}(\Omega) \) and \( v \) be a \( G(\cdot) \)-supersolution in \( \Omega \) such that \( f - v \in W^{1,G(\cdot)}_0(\Omega) \). Then the solution of the obstacle problem with the obstacle \( v \) and the boundary data \( f \) is a \( G(\cdot) \)-solution in \( \Omega \).

**Theorem 6.2** Let \( x_0 \in \partial \Omega \). Let \( G(\cdot) \in \Phi(\mathbb{R}^n) \cap C^1(\mathbb{R}^+) \) be strictly convex and satisfy (SC), (A0), (A1), and (A1, n). Fix \( r > 0 \), and let \( u := R_{G(\cdot)}(\overline{B}(x_0, r) \setminus \Omega, B(x_0, 4r)) \)
be the $G(\cdot)$-potential for $\overline{B(x_0, r)} \setminus \Omega$ with respect to $B(x_0, 4r)$. Then for $0 < \rho \leq r$ and $x \in B(x_0, \rho)$, we have

$$1 - u(x) \leq \exp \left( -C \int_\rho^r g^{-1} \left( x_0, \frac{\text{cap}_{G(\cdot)}(\overline{B(x_0, t)} \cap \Omega^C \setminus \cap \Omega^C; B(x_0, 2t))}{rt^{n-1}} \right) dt + Cr \right).$$

**Proof** Let $x_0 \in \partial \Omega$, $r > 0$, and $B_j = B(x_0, r_j)$ where $r_j = 4^{1-j}r$, $j = 0, 1, 2, \ldots$. Let $u$ be the $G(\cdot)$-potential for $\overline{B_1} \cap \Omega^C$ with respect to $B_0$. By Lemma 6.4, we have

$$m_1 := \inf_{\frac{1}{2}B_0} u \geq \frac{C r_0}{2} g^{-1} \left( x_0, \frac{\mu[u](\frac{1}{2}B_0)}{\frac{r_0}{2}^{n-1}} \right) - \frac{r_0}{2} \geq C r_0 g^{-1} \left( x_0, \frac{\mu[u](\overline{B_1} \cap \Omega^C)}{r_0^{n-1}} \right) - \frac{r_0}{2}$$

Using Theorem 6.1, we get

$$m_1 \geq C r_0 g^{-1} \left( x_0, \frac{\text{cap}_{G(\cdot)}(\overline{B_1} \cap \Omega^C; B_0)}{r_0^{n-1}} \right) - \frac{r_0}{2}. \quad (6.2)$$

As $1 + t \leq e^t$, then

$$1 - m_1 \leq 1 - C r_0 g^{-1} \left( x_0, \frac{\text{cap}_{G(\cdot)}(\overline{B_1} \cap \Omega^C; B_0)}{r_0^{n-1}} \right) + \frac{r_0}{2} \leq \exp \left( -C r_0 g^{-1} \left( x_0, \frac{\text{cap}_{G(\cdot)}(\overline{B_1} \cap \Omega^C; B_0)}{r_0^{n-1}} \right) + \frac{r_0}{2} \right). \quad (6.3)$$

Next, let $D_1 = B_1 \setminus (\overline{B_2} \cap \Omega^C)$ and let $f_1 \in W^{1,G(\cdot)}_0(B_0)$ such that $f_1 = m_1$ on $\partial B_1$ and $f_1 = 1$ on $\overline{B_2}$. Let $u_1$ be the solution of the obstacle problem in $D_1$ with the upper obstacle $u$ and the boundary values $f_1$ extended to $\overline{B_2} \cap \Omega^C$ by the constant 1. Then $\frac{u_1 - m_1}{1 - m_1}$ is the $G(\cdot)$-potential for $\overline{B_2} \cap \Omega^C$ with respect to $B_1$. So, by inequality (6.2), we have

$$\inf_{\frac{1}{2}B_1} \frac{u_1 - m_1}{1 - m_1} \geq C r_1 g^{-1} \left( x_0, \frac{\text{cap}_{G(\cdot)}(\overline{B_2} \cap \Omega^C; B_1)}{r_1^{n-1}} \right) - \frac{r_1}{2}.$$ 

Hence

$$m_2 := \inf_{\frac{1}{2}B_1} u_1 \geq C r_1 (1 - m_1) g^{-1} \left( x_0, \frac{\text{cap}_{G(\cdot)}(\overline{B_2} \cap \Omega^C; B_1)}{r_1^{n-1}} \right) - \frac{r_1}{2} (1 - m_1) + m_1.$$
Consequently

\[ 1 - m_2 \leq -Cr_1(1 - m_1)g^{-1}\left(x_0, \frac{\text{cap}_{G(\cdot)}(\overline{B}_2 \cap \Omega^C; B_1)}{r_1^{n-1}}\right) + (1 + \frac{r_1}{2})(1 - m_1) \]

\[ \leq (1 - m_1) \left(1 - Cr_1(1 - m_1)g^{-1}\left(x_0, \frac{\text{cap}_{G(\cdot)}(\overline{B}_2 \cap \Omega^C; B_1)}{r_1^{n-1}}\right) + \frac{r_1}{2}\right). \]

Then

\[ 1 - m_2 \leq (1 - m_1) \exp\left(-Cr_1g^{-1}\left(x_0, \frac{\text{cap}_{G(\cdot)}(\overline{B}_2 \cap \Omega^C; B_1)}{r_1^{n-1}}\right) + \frac{r_1}{2}\right). \]

A similar method, let \( D_j = B_j \setminus (\overline{B}_{j+1} \cap \Omega^C) \) and let \( f_j \in W_0^{1,G(\cdot)}(B_{j-1}) \) such that \( f_j = m_j \) on \( \partial B_j \) and \( f_j = 1 \) on \( \overline{B}_{j+1} \).

\[
 f_j = \begin{cases} 
 m_j & \text{on } \partial B_j \\
 1 & \text{on } \overline{B}_{j+1}.
\end{cases}
\]

Let \( u_j \) be the solution of the obstacle problem in \( D_j \) with the upper obstacle \( u_{j-1} \) and the boundary values \( f_j \) extended to \( \overline{B}_{j+1} \cap \Omega^C \) by the constant 1. Then we have

\[ 1 - m_{j+1} \leq (1 - m_j) \exp\left(-Cr_jg^{-1}\left(x_0, \frac{\text{cap}_{G(\cdot)}(\overline{B}_{j+1} \cap \Omega^C; B_j)}{r_j^{n-1}}\right) + \frac{r_j}{2}\right). \]

with \( m_{j+1} := \inf_{\overline{B}_j} u_j \). Iterating this inequality and using inequality (6.3), we get for \( k = 1, 2, ... \),

\[ 1 - m_{k+1} \leq \exp\left(-C \sum_{j=0}^{k} r_jg^{-1}\left(x_0, \frac{\text{cap}_{G(\cdot)}(\overline{B}_{j+1} \cap \Omega^C; B_j)}{r_j^{n-1}}\right) + \sum_{j=0}^{k} \frac{r_j}{2}\right). \]

As \( u \geq u_1 \) and \( u_j \geq u_{j+1} \) in \( B_{j+1}; \ j = 1, 2, ..., \), then

\[ 1 - u \leq \exp\left(-C \sum_{j=0}^{k} r_jg^{-1}\left(x_0, \frac{\text{cap}_{G(\cdot)}(\overline{B}_{j+1} \cap \Omega^C; B_j)}{r_j^{n-1}}\right) + \sum_{j=0}^{k} \frac{r_j}{2}\right) \text{ on } 1/2 \overline{B}_k. \]

(6.4)
Fix $\rho > 0$ so that $\rho \leq r$ and choose an integer $k$ so that $r_{k+3} < \rho \leq r_{k+2}$, we have

$$
\sum_{j=0}^{k} r_j g^{-1} \left( x_0, \frac{\text{cap}_{G()}(\overline{B}_{j+1} \cap \Omega^C; B_j)}{t_{j+1}} \right)
\geq C \sum_{j=0}^{k} \int_{r_j}^{r_{j+1}} g^{-1} \left( x_0, \frac{\text{cap}_{G()}(\overline{B}_{j+1} \cap \Omega^C; B_j)}{t_{j+1}} \right) dt.
$$

Or using $r_{j+2} \leq t \leq r_{j+1}$ and Proposition 2.1, we get

$$
\text{cap}_{G()}(\overline{B}(x_0, t) \cap \Omega^C; B(x_0, 2t)) \leq C \left( \text{cap}_{G()}(\overline{B}(x_0, t) \cap \Omega^C; B(x_0, 4t)) + t^n \right)
\leq C \left( \text{cap}_{G()}(\overline{B}(x_0, t) \cap \Omega^C; B(x_0, 8t)) + t^n \right)
\leq C \left( \text{cap}_{G()}(\overline{B}_{j+1} \cap \Omega^C; B_j) + t^n \right).
$$

Then, we have

$$
\int_{r_{j+2}}^{r_{j+1}} g^{-1} \left( x_0, \frac{\text{cap}_{G()}(\overline{B}_{j+1} \cap \Omega^C; B_j)}{t_{j+1}} \right) dt
= \int_{r_{j+2}}^{r_{j+1}} g^{-1} \left( x_0, \frac{\text{cap}_{G()}(\overline{B}_{j+1} \cap \Omega^C; B_j) + t^{n-1}}{t_{j+1}} \right) dt
\geq C \int_{r_{j+2}}^{r_{j+1}} g^{-1} \left( x_0, \frac{\text{cap}_{G()}(\overline{B}_{j+1} \cap \Omega^C; B_j) + t^{n-1}}{t_{j+1}} \right) dt
\geq C \int_{r_{j+2}}^{r_{j+1}} g^{-1} \left( x_0, \frac{\text{cap}_{G()}(\overline{B}(x_0, t) \cap \Omega^C; B(x_0, 2t))}{t_{j+1}} \right) dt
\geq C \int_{r_{j+2}}^{r_{j+1}} g^{-1} \left( x_0, \frac{\text{cap}_{G()}(\overline{B}(x_0, t) \cap \Omega^C; B(x_0, 2t))}{t_{j+1}} \right) dt.
$$

Hence, by the condition $(A_0)$, we obtain

$$
\sum_{j=0}^{k} r_j g^{-1} \left( x_0, \frac{\text{cap}_{G()}(\overline{B}_{j+1} \cap \Omega^C; B_j)}{t_{j+1}} \right)
\geq C \int_{r}^{r} g^{-1} \left( x_0, \frac{\text{cap}_{G()}(\overline{B}(x_0, t) \cap \Omega^C; B(x_0, 2t))}{t_{j+1}} \right) dt - C r.
$$

Then, for $x \in B(x_0, \rho)$, we get

$$
1 - u(x) \leq \exp \left( -C \int_{r}^{r} g^{-1} \left( x_0, \frac{\text{cap}_{G()}(\overline{B}(x_0, t) \cap \Omega^C; B(x_0, 2t))}{t_{j+1}} \right) dt + C r \right).
$$

This concludes the proof. \qed
Theorem 6.3 Let $x_0 \in \partial \Omega$. Let $G(\cdot) \in \Phi(\mathbb{R}^n) \cap C^1(\mathbb{R}^+) \text{ be strictly convex and satisfy (SC), (A_0), (A_1), and (A_{1,n}). Fix } r > 0 \text{ and let } u \text{ be the } G(\cdot)-\text{potential for } \overline{B}(x_0, r) \setminus \Omega \text{ with respect to } B(x_0, 4r). \text{ Then}

\[
\liminf_{x \to x_0} u(x) \leq C \left( \int_0^{4r} g^{-1} \left( x_0, \frac{\text{cap}_{G(\cdot)}(\overline{B}(x_0, t) \cap \Omega^C; B(x_0, 2t))}{t^{n-1}} \right) \, dt + r \right).
\]

**Proof** Let $u$ be the $G(\cdot)$-potential for $\overline{B}(x_0, r) \setminus \Omega$ with respect to $B(x_0, 4r)$. Then by the Wolff potential upper estimate Theorem 5.12 in [6] and Theorem 4.4 in [14], we have

\[
\liminf_{\rho \to 0} \inf_{B(x_0, \rho)} u(x) \leq C \left( r + \inf_{B(x_0, r)} u + \int_0^{2r} g^{-1} \left( x_0, \frac{\mu(u)(B(x_0, t))}{t^{n-1}} \right) \, dt \right).
\]

Next, let $0 < t \leq 2r$, $B = B(x_0, r)$ and $\mu_t$ be the restriction of $\mu(u)$ to $B(x_0, t)$, and let $u_t \in W_0^{1, G(\cdot)}(4B)$ be the $G(\cdot)$-supersolution in $4B$ associated with $\mu_t$. So, for all nonnegative $\eta \in W_0^{1, G(\cdot)}(4B)$, we have

\[
\int_{4B} \frac{g(x, |\nabla u_t|)}{|\nabla u_t|} \nabla u_t \cdot \nabla \eta \, dx = \int_{4B} \eta \, d\mu_t[u]
\]

\[
\leq \int_{4B} \eta \, d\mu[u] = \int_{4B} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \eta \, dx.
\]

As $(u_t - u)_+ \in W_0^{1, G(\cdot)}(4B)$, then by the weak comparison principle (see Lemma 3.8 in [6]), we have

\[0 \leq u_t \leq u \leq 1 \text{ in } 4B.\]

Let $\varphi \in S_{G(\cdot)}(\overline{B}(x_0, t) \setminus \Omega; 4B)$. Then, by Young inequality, we have

\[
\mu(B(x_0, t)) \leq \int_{4B} \varphi \, d\mu_t[u] = \int_{4B} \frac{g(x, |\nabla u_t|)}{|\nabla u_t|} \nabla u_t \cdot \nabla \varphi \, dx \leq C \left( \int_{4B} G(x, |\nabla u_t|) \, dx + \int_{4B} G(x, |\nabla \varphi|) \, dx \right).
\]

Similarly in the proof of Theorem 6.1, we have

\[
\int_{4B} G(x, |\nabla u_t|) \, dx \leq C \int_{4B} G(x, |\nabla \varphi|) \, dx.
\]

Hence

\[
\mu(B(x_0, t)) \leq C \int_{4B} G(x, |\nabla \varphi|) \, dx.
\]

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Taking infimum over all such \( \varphi \) and using Proposition 2.1, we obtain

\[
\mu(B(x_0, t)) \leq \text{cap}_{G(\cdot)}(\overline{B}(x_0, t) \cap \Omega^C; 4B) \leq \text{cap}_{G(\cdot)}(\overline{B}(x_0, t) \cap \Omega^C; B(x_0, 2t)).
\]

Let \( \lambda = \inf_{B} u \) and \( B(y, \frac{r}{4}) \subset B \cap \Omega^C \), so by the condition \((SC)\), we get

\[
r^{n-1} g \left( x_0, \frac{\lambda}{r} \right) \leq C \int_{B(y, \frac{r}{4})} G \left( x_0, \frac{1}{r} \right) dx.
\]

As \( 1 \leq \frac{u + r}{r} \leq \frac{2}{r} \), then, by Lemma 2.1, we have

\[
\int_{4B} G \left( x_0, \frac{u + r}{r} \right) dx \leq \int_{4B} G \left( x, \frac{u + r}{r} \right) dx.
\]

Then, using the Poincaré inequality in modular form and the condition \((A_0)\), we obtain

\[
r^{n-1} g \left( x_0, \frac{\lambda}{r} \right) \leq C \left( \int_{4B} G(x, \frac{u}{r}) dx + \int_{4B} G(x, 1) dx \right)
\]

\[
\leq C \left( \int_{4B} G(x, |\nabla u|) dx + r^n + G(x_0, 1) |4B| \right)
\]

\[
\leq C \left( \int_{4B} G(x, |\nabla u|) dx + r^n \right).
\]

From Lemma 6.2, if we choose \( \varphi = u \), we obtain

\[
\int_{4B} G(x, |\nabla u|) dx \leq C \int_{4B} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla u dx = C \int_{4B} u d\mu[u] \leq C \mu[u](4B).
\]

Then

\[
r^{n-1} g \left( x_0, \frac{\lambda}{r} \right) \leq C \left( \mu[u](4B) + r^n \right).
\]

From Theorem 6.1, we have

\[
r^{n-1} g(x_0, \frac{\lambda}{r}) \leq C \left( \text{cap}_{G(\cdot)}(\overline{B}(x_0, r) \setminus \Omega; 4B) + r^n \right).
\]
Using inequalities 2.4, 2.2 and 2.3, we get
\[ \lambda \leq C \left( r g^{-1} \left( x_0, \frac{\text{cap}_{G(\cdot)}(\overline{B}(x_0, r) \setminus \Omega; 4B)}{r^{n-1}} \right) + r^2 \right). \]

Therefore
\[ \inf_{2B} u \leq C \left( \int_{r}^{2r} g^{-1} \left( x_0, \frac{\text{cap}_{G(\cdot)}(\overline{B}(x_0, t) \cap \Omega^c; B(x_0, 2t))}{t^{n-1}} \right) dt + r^2 \right). \]

Hence
\[ \liminf_{x \to x_0} u(x) \leq C \left( \int_{0}^{4r} g^{-1} \left( x_0, \frac{\text{cap}_{G(\cdot)}(\overline{B}(x_0, t) \cap \Omega^c; B(x_0, 2t))}{t^{n-1}} \right) dt + r \right). \]

This concludes the proof. \(\Box\)

## 7 Wiener Criterion

First of all, the notion of the regularity of boundary points is defined in connection with Perron \(G(\cdot)\)-solutions.

**Definition 7.1** Let \(G(\cdot) \in \Phi(\Omega)\). A boundary point \(x_0\) of an open set \(\Omega\) is called \(G(\cdot)\)-regular if
\[ \lim_{x \to x_0} \overline{H}_f(x) = f(x_0), \]
for each continuous \(f : \partial\Omega \to \mathbb{R}\).

The following lemma shows that \(G(\cdot)\)-regularity is a local property.

**Lemma 7.1** Let \(G(\cdot) \in \Phi(\Omega)\). A boundary \(x_0\) of \(\Omega\) is \(G(\cdot)\)-regular if and only if
\[ \lim_{x \to x_0} \overline{H}_f(x) = f(x_0), \]
for each bounded \(f : \partial\Omega \to \mathbb{R}\), continuous at \(x_0\).

**Proof** Let \(x_0 \in \partial\Omega\) be \(G(\cdot)\)-regular and fix \(\epsilon > 0\). Let \(U\) be an neighborhood of \(x_0\) such that \(|f - f(x_0)| < \epsilon\) on \(U \cap \partial\Omega\). Then, choose a continuous function \(g : \partial\Omega \to [f(x_0) + \epsilon, \sup |f| + \epsilon]\) such that \(g(x_0) = f(x_0) + \epsilon\) and \(g = \sup |f| + \epsilon\) on \(\partial\Omega \setminus U\). Now \(g \geq f\) on \(\partial\Omega\) and hence we have
\[ \limsup_{x \to x_0} \overline{H}_f(x) \leq \lim_{x \to x_0} \overline{H}_g(x) = g(x_0) = f(x_0) + \epsilon. \]
Similarly, we have
\[
\liminf_{x \to x_0} \overline{H} f(x) \geq f(x_0) - \epsilon.
\]
Thus we conclude
\[
\lim_{x \to x_0} \overline{H} f(x) = f(x_0).
\]
and the lemma is proved. □

**Lemma 7.2** Let \( G(\cdot) \in \Phi(\Omega) \). Assume that \( f : \partial\Omega \to \mathbb{R} \) is \( G(\cdot) \)-resolutive. Let \( \Omega' \subset \Omega \) be open and define \( \tilde{f} : \partial\Omega' \to \mathbb{R} \) by
\[
\tilde{f}(x) = \begin{cases} 
  f(x) & \text{if } x \in \partial\Omega \cap \partial\Omega' \\
  H f(x) & \text{if } x \in \Omega \cap \partial\Omega'.
\end{cases}
\]
Then \( \tilde{f} \) is \( G(\cdot) \)-resolutive with respect to \( \Omega' \) and the Perron \( G(\cdot) \)-solution for \( \tilde{f} \) in \( \Omega' \) is \( \overline{H} f|_{\Omega'} \).

**Proof** Let \( f : \partial\Omega \to \mathbb{R} \) be a \( G(\cdot) \)-resolutive, \( \Omega' \subset \Omega \) and \( u \in U_f \). As \( u \) is lower semicontinuous, then for each \( y \in \Omega' \)
\[
\lim_{y \to x} u(y) \geq \tilde{f}(x) \quad \text{for all } x \in \partial\Omega'.
\]
Hence \( u \in U_{\tilde{f}} \) for \( \tilde{f} \) in \( \Omega' \). So taking infimum over all \( u \), we have
\[
\overline{H} f \leq \overline{H} f|_{\Omega'}.
\]
Applying the same argument to \( -f \), we obtain
\[
\underline{H} f \leq \overline{H} f \leq \overline{H} f = -H_{-f} \leq -\overline{H} f \leq \underline{H} f \quad \text{in } \Omega'.
\]
This concludes the proof. □

**Theorem 7.1** Let \( G(\cdot) \in \Phi(\mathbb{R}^n) \cap C^1(\mathbb{R}^+ \) be strictly convex and satisfy \( SC \), \( (A_0) \), \( (A_1) \), and \( (A_{1,n}) \). The point \( x_0 \in \partial\Omega \) is \( G(\cdot) \)-regular if and only if for some \( \rho > 0 \),
\[
\int_0^\rho g^{-1} \left( x_0, \frac{\text{cap}_{G(\cdot)}(B(x_0, t) \cap \Omega \setminus B(x_0, 2t))}{t^{n-1}} \right) \, dt = \infty. \tag{7.1}
\]

**Proof** Let \( f \in C(\partial\Omega) \) and \( \epsilon > 0 \) be arbitrary. There exists \( r > 0 \) such that
\[
\sup_{\partial\Omega \cap B(x_0, 2r)} |f - f(x_0)| \leq \epsilon.
\]
Let $u$ be the $G(\cdot)$-potential for $B(x_0, r) \setminus \Omega$ with respect to $B(x_0, 4r)$ and $\tilde{f}$ be as in Lemma 7.2 with $\Omega' := \Omega \cap 4B$. So, we put $B = B(x_0, r), m = \sup_{\partial \Omega \cap 2B} (f - f(x_0))$ and $M = \sup_{\partial \Omega} (f - f(x_0))$. Then, we have

$$\tilde{f} - f(x_0) \leq m + M(1 - u) \text{ on } \partial \Omega'.$$

Using Lemma 5.1 and Lemma 7.2, we get

$$H_f - f(x_0) = H_f \bigg|_{\Omega'} - f(x_0) \leq H_{f - f(x_0)} \bigg|_{\Omega'} \leq H_{m + M(1 - u)} = m + M(1 - u) \text{ on } \Omega'.$$

Hence, from Theorem 6.2, we have

$$\sup_{\Omega \cap B(x_0, \rho)} (H_f - f(x_0)) \leq \sup_{\partial \Omega \cap 2B} (f - f(x_0)) + \sup_{\partial \Omega} (f - f(x_0)) \exp \left(-C \int_\rho^r g^{-1} \left(x_0, \frac{\text{cap}_{G(\cdot)} (\overline{B}(x_0, t) \cap \Omega^c \cap B(x_0, 2t))}{t^{n-1}} \right) dt + Cr \right).$$

So, by the condition (7.1) for all sufficiently small $0 < \rho \leq r$, we get

$$\sup_{\Omega \cap B(x_0, \rho)} (H_f - f(x_0)) \leq 2\varepsilon.$$

Then $H_f$ is continuous at $x_0$ and as $f \in C(\partial \Omega)$ was arbitrary, which implies that $x_0$ is $G(\cdot)$-regular.

For the converse, by Theorem 6.3, we have

$$\liminf_{x \to x_0} u(x) \leq C \left(\int_0^{2r} g^{-1} \left(x_0, \frac{\text{cap}_{G(\cdot)} (\overline{B}(x_0, t) \cap \Omega^c \cap B(x_0, 2t))}{t^{n-1}} \right) dt + r \right).$$

By the condition (7.1), we can find $r > 0$ sufficiently small so that

$$\liminf_{x \to x_0} u(x) < 1.$$

As $u$ is solution of the Sobolev-Dirichlet problem in $4B \setminus (\overline{B} \cap \Omega^c)$ with the continuous boundary data 1 on $K$ and 0 on $\partial(4B)$, then $x_0$ is not $G(\cdot)$-regular. □

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