Bounded Point Evaluations For Rationally Multicyclic Subnormal Operators

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Abstract

Let $S$ be a pure bounded rationally multicyclic subnormal operator on a separable complex Hilbert space $H$ and let $M_z$ be the minimal normal extension of $S$ on $H$. Let $bpe(S)$ be the set of bounded point evaluations on $H$ and let $abpe(S)$ be the set of analytic bounded point evaluations. We show $abpe(S) = bpe(S) \cap \text{Int}(\sigma(S))$. The result affirmatively answers a question asked by J. B. Conway concerning the equality of the interior of the minimal normal extension on a separable complex Hilbert space $z$ and a normal operator $M_z$. For $S, T \in L(H)$, the essential spectrum of $S$ is $\sigma_e(S) = \sigma(S) \cap \mathbb{C}^\times$. For a subset $A \subseteq \mathbb{C}$, we set $\text{Int}(A)$ for its interior, $\hat{A}$ for its closure, and $\chi_A$ for its characteristic function. Let $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$. For $A \subseteq \mathbb{C}$ and $\delta > 0$, we set $B(\lambda, \delta) = \{z : |z - \lambda| < \delta\}$ and $\mathbb{D} = B(0, 1)$. Let $P$ denote the set of polynomials in the complex variable $z$. For a compact subset $K \subseteq \mathbb{C}$, let $\text{Rat}(K)$ be the set of all rational functions with poles off $K$.

1 Introduction

Let $H$ be a separable complex Hilbert space and let $L(H)$ be the space of bounded linear operators on $H$. An operator $T \in L(H)$ is subnormal if there exist a separable complex Hilbert space $K$ containing $H$ and a normal operator $M_z \in L(K)$ such that $M_z|_H = T$. By the spectral theorem of normal operators, we assume that $K = \oplus_{i=1}^\infty L^2(\mu_i)$.

where $\mu_1 \gg \mu_2 \gg \ldots \gg \mu_m$ ($m$ may be $\infty$) are compactly supported finite positive measures on the complex plane $\mathbb{C}$, and $M_z$ is multiplication by $z$ on $K$. For $H = (h_1, \ldots, h_m) \in K$ and $G = (g_1, \ldots, g_m) \in K$, we define

$$\langle H(z), G(z) \rangle = \sum_{i=1}^m h_i(z)g_i(z)\frac{d\mu_i}{d\mu_1}, \quad |H(z)|^2 = \langle H(z), H(z) \rangle.$$  

The inner product of $H$ and $G$ in $K$ is defined by

$$\langle H, G \rangle = \int \langle H(z), G(z) \rangle d\mu_1(z).$$  

$M_z$ is the minimal normal extension if

$$K = \text{clos} \left( \text{span}(M_z^k x : x \in H, k \geq 0) \right).$$

We will always assume that $M_z$ is the minimal normal extension of $S$ and $K$ satisfies (1-1) and (1-4). For details about the functional model above and basic knowledge of subnormal operators, the reader shall consult Chapter II of the book [Conway 1991].

For $T \in L(H)$, we denote by $\sigma(T)$ the spectrum of $T$, $\sigma_e(T)$ the essential spectrum of $T$, $T^*$ its adjoint, $\ker(T)$ its kernel, and $\text{Ran}(T)$ its range. For a subset $A \subseteq \mathbb{C}$, we set $\text{Int}(A)$ for its interior, $\hat{A}$ for its closure, and $\chi_A$ for its characteristic function. Let $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$. For $\lambda \in \mathbb{C}$ and $\delta > 0$, we set $B(\lambda, \delta) = \{z : |z - \lambda| < \delta\}$ and $\mathbb{D} = B(0, 1)$. Let $\mathcal{P}$ denote the set of polynomials in the complex variable $z$. For a compact subset $K \subseteq \mathbb{C}$, let $\text{Rat}(K)$ be the set of all rational functions with poles off $K$.

A subnormal operator $S$ on $H$ is pure if for every non-zero invariant subspace $I$ of $S$ ($SI \subseteq I$), the operator $S|_I$ is not normal. For $F_1, F_2, \ldots, F_N \in H$, let

$$R^2(S|I, F_1, F_2, \ldots, F_N) = \text{clos} \{r_1(S)F_1 + r_2(S)F_2 + \ldots + r_N(S)F_N\}$$  

(1-5)
in \( \mathcal{H} \), where \( r_1, r_2, ..., r_N \in \text{Rat}(\sigma(S)) \) and let

\[
P^2(S|F_1, F_2, ..., F_N) = \text{clos}\{p_1(S)F_1 + p_2(S)F_2 + ... + p_N(S)F_N\}
\]

(1-6)

in \( \mathcal{H} \), where \( p_1, p_2, ..., p_N \in \mathcal{P} \). A subnormal operator \( S \) on \( \mathcal{H} \) is rationally multicyclic \((N\text{–cyclic})\) if there are \( N \) vectors \( F_1, F_2, ..., F_N \in \mathcal{H} \) such that

\[
\mathcal{H} = R^2(S|F_1, F_2, ..., F_N)
\]

and for any \( G_1, ..., G_{N-1} \in \mathcal{H} \),

\[
\mathcal{H} \neq R^2(S|G_1, G_2, ..., G_{N-1}).
\]

\( S \) is multicyclic \((N\text{–cyclic})\) if

\[
\mathcal{H} = P^2(S|F_1, F_2, ..., F_N)
\]

and for any \( G_1, ..., G_{N-1} \in \mathcal{H} \),

\[
\mathcal{H} \neq P^2(S|G_1, G_2, ..., G_{N-1}).
\]

In this case, \( m \leq N \) where \( m \) is as in (1-1).

Let \( \mu \) be a compactly supported finite positive measure on the complex plane \( \mathbb{C} \) and let \( \text{spt}(\mu) \) denote the support of \( \mu \). For a compact subset \( K \) with \( \text{spt}(\mu) \subseteq K \), let \( R^2(K, \mu) \) be the closure of \( \text{Rat}(K) \) in \( L^2(\mu) \). Let \( P^2(\mu) \) denote the closure of \( \mathcal{P} \) in \( L^2(\mu) \).

If \( S \) is rationally cyclic, then \( S \) is unitarily equivalent to multiplication by \( z \) on \( R^2(\sigma(S), \mu_0) \), where \( m = 1 \) and \( F_1 = 1 \). We may write \( R^2(S|F_1) = R^2(\sigma(S), \mu_0) \). If \( S \) is cyclic, then \( S \) is unitarily equivalent to multiplication by \( z \) on \( P^2(\mu_1) \). We may write \( P^2(S|F_1) = P^2(\mu_1) \).

For a rationally \( N\text{–cyclic} \) subnormal operator \( S \) with cyclic vectors \( F_1, F_2, ..., F_N \) and \( \lambda \in \sigma(S) \), we denote the map

\[
E(\lambda) : \sum_{i=1}^{N} r_i(\lambda)F_i \rightarrow \begin{bmatrix} r_1(\lambda) \\ r_2(\lambda) \\ \vdots \\ r_N(\lambda) \end{bmatrix},
\]

(1-7)

where \( r_1, r_2, ..., r_N \in \text{Rat}(\sigma(S)) \). If \( E(\lambda) \) is bounded from \( K \) to \( (\mathbb{C}^N, \| \cdot \|_N) \), where \( \|x\|_N = \sum_{i=1}^{N} |x_i| \) for \( x \in \mathbb{C}^N \), then every component in the right hand side extends to a bounded linear functional on \( \mathcal{H} \) and we will call \( \lambda \) a bounded point evaluation for \( S \). We use \( \text{bpe}(S) \) to denote the set of bounded point evaluations for \( S \). The set \( \text{bpe}(S) \) does not depend on the choices of cyclic vectors \( F_1, F_2, ..., F_N \) (see Corollary 1.1 in Mbekhta et al. (2016)). A point \( \lambda_0 \in \text{int}(\text{bpe}(S)) \) is called an analytic bounded point evaluation for \( S \) if there is a neighborhood \( B(\lambda_0, \delta) \subset \text{bpe}(S) \) of \( \lambda_0 \) such that \( E(\lambda) \) is analytic as a function of \( \lambda \) on \( B(\lambda_0, \delta) \) (equivalently (1-7) is uniformly bounded for \( \lambda \in B(\lambda_0, \delta) \)). We use \( \text{abpe}(S) \) to denote the set of analytic bounded point evaluations for \( S \). The set \( \text{abpe}(S) \) does not depend on the choices of cyclic vectors \( F_1, F_2, ..., F_N \) (also see Remark 3.1 in Mbekhta et al. (2016)). Similarly, for an \( N\text{–cyclic} \) subnormal operator \( S \), we can define \( \text{bpe}(S) \) and \( \text{abpe}(S) \) if we replace \( r_1, r_2, ..., r_N \in \text{Rat}(\sigma(S)) \) in (1-7) by \( p_1, p_2, ..., p_N \in \mathcal{P} \).

Bercovici et al. (1985) show that the Bergman shift has invariant subspaces with the codimension \( N \) property for every \( N \in \{1, 2, ..., \infty\} \). This means that on the Bergman space, the set of all analytic functions \( f \) on the unit disk \( \mathbb{D} \) satisfying

\[
\int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty,
\]

where \( A \) is area measure, there is a closed subspace \( \mathcal{M} \) that is invariant under multiplication by the independent variable \( z \), and such that

\[
dim(\mathcal{M}/(z\mathcal{M})) = N.
\]

Their construction is abstract, and these subspaces are hard to envision. Later, Hedenmalm (1993) gave a concrete construction, using zero sets whose union is not a zero set. Moreover, Aleman et al. (1994) show that there are \( f_1, f_2, ..., f_N \) such that

\[
\mathcal{M} = P^2(S|f_1, ..., f_N),
\]

where \( S \) is multiplication by \( z \) operator on \( \mathcal{M} \), and \( \text{bpe}(S) = \text{abpe}(S) = \mathbb{D} \).

For \( N = 1 \), Thomson (1991) proves a remarkable structural theorem for \( P^2(\mu) \).
Thomson’s Theorem. There is a Borel partition \( \{ \Delta_i \}_{i=0}^{\infty} \) of \( spt \mu \) such that the space \( P^2(\mu|_{\Delta_i}) \) contains no nontrivial characteristic functions and

\[
P^2(\mu) = L^2(\mu|_{\Delta_0}) \oplus \left\{ \bigoplus_{i=1}^{\infty} P^2(\mu|_{\Delta_i}) \right\}.
\]

Furthermore, if \( U_i \) is the open set of analytic bounded point evaluations for \( P^2(\mu|_{\Delta_i}) \) for \( i \geq 1 \), then \( U_i \) is a simply connected region and the closure of \( U_i \) contains \( \Delta_i \).

Conway and Elias (1993) extends some results of Thomson’s Theorem to the space \( R^2(K, \mu) \), while Brennan (2008) expresses \( R^2(K, \mu) \) as a direct sum that includes both Thomson’s theorem and results of Conway and Elias (1993). For a compactly supported complex Borel measure \( \nu \) of \( C \), by estimating analytic capacity of the set \( \{ \lambda : |C\nu(\lambda)| \geq \epsilon \} \), where \( C\nu \) is the Cauchy transform of \( \nu \) (see Section 2 for definition), Brennan (2006, English), Alemán et al. (2009), and Alemán et al. (2010) provide interesting alternative proofs of Thomson’s theorem. Both their proofs rely on X. Tolsa’s deep results on analytic capacity. There are other related research papers for \( N = 1 \) in the history. For example, Brennan (1973), Hruscev (1979, Russian), Brennan and Militzer (2013), and Yang (2014), etc.

Thomson’s Theorem shows in Theorem 4.11 of Thomson (1991) that \( abpe(S) = bpe(S) \) for a cyclic subnormal operator \( S \) (See also Chap VIII Theorem 4.4 in Conway (1991)). The results lead to the next question stated by Conway 7.11 p. 65 of Conway (1991).

Does \( abpe(S) = Int(bpe(S)) \) hold for an arbitrary rationally cyclic subnormal operator \( S \)?

Corollary 5.2 in Conway and Elias (1993) affirmatively answers the question. Our following theorem extends the result to rationally \( N \)-cyclic subnormal operators.

**Theorem 1.** Let \( S \) on \( H \) be a pure subnormal operator and let \( M_z \) on \( K \) (satisfying (1-1) and (1-4)) be its minimal normal extension.

1. If \( S \) is \( N \)-cyclic, then \( abpe(S) = bpe(S) \).
2. If \( S \) is rationally \( N \)-cyclic, then \( abpe(S) = bpe(S) \cap Int(\sigma(S)) \).

**Corollary 1.** Let \( S \) on \( H \) be a pure subnormal operator and let \( M_z \) on \( K \) (satisfying (1-1) and (1-4)) be its minimal normal extension.

1. Suppose \( S \) is \( N \)-cyclic and \( \lambda_0 \in C \). If \( dim(Ker(S - \lambda_0 I)^s) = N \), then \( Ran(S - \lambda_0 I) \) is closed and \( \lambda_0 \in \sigma(S) \cap \sigma_c(S) \).
2. Suppose \( S \) is rationally \( N \)-cyclic and \( \lambda_0 \in Int(\sigma(S)) \). If \( dim(Ker(S - \lambda_0 I)^s) = N \), then \( Ran(S - \lambda_0 I) \) is closed and \( \lambda_0 \in \sigma(S) \cap \sigma_c(S) \).

We prove Theorem 1 and Corollary 1 in section 2.

## 2 The Proofs

Let \( \nu \) be a compactly supported finite measure on \( C \). The Cauchy transform of \( \nu \) is defined by

\[
C\nu(z) = \int \frac{1}{w - z} \, dv(w)
\]

for all \( z \in C \) for which \( \int \frac{dv(w)}{|w - z|^s} < \infty \). A standard application of Fubini’s Theorem shows that \( C\nu \in L^1_{loc}(C) \) for \( 0 < s < 2 \), in particular, it is defined for Area almost all \( z \), and clearly \( C\nu \) is analytic in \( C_{\infty} \setminus spt\nu \), where \( C_{\infty} = C \cup \{ \infty \} \) is the Riemann sphere.

Now suppose that \( \nu \) is a compactly supported finite measure on \( C \) that annihilates the rational functions \( Rat(spt(\nu)) \). Then, for \( r \in Rat(spt(\nu)) \),

\[
\int \frac{r(z) - r(w)}{z - w} \, dv(z) = 0.
\]
Rearranging, we see that
\[ r(w)C\nu(w) = \int \frac{r(z)}{z - w} d\nu(z) \]
for Area almost all \( w \).

Suppose that \( S \) on \( \mathcal{H} \) is a pure rationally \( N \)-cyclic subnormal operator with cyclic vectors \( F_1, F_2, \ldots, F_N \) and let \( M_j \) on \( \mathcal{K} \) (satisfying (1-1) to (1-4)) be its minimal normal extension. Let \( G_i \in \mathcal{K} \) and \( G_i \perp \mathcal{H} \) for \( i = 1, 2, \ldots, N \). Denote
\[ \nu_{ij} = \langle F_i, G_j \rangle \mu_1, \quad (2-1) \]
then \( \nu_{ij} \) annihilates \( \text{Rat}(\sigma(S)) \). We have the following estimation for \( r_i \in \text{Rat}(\sigma(S)) \) and \( \mathcal{B}(\lambda_0, \delta) \subset \text{Int}(\sigma(S)) \).
\[
\int_{\mathcal{B}(\lambda_0, \delta)} \left| \sum_{i=1}^{N} r_i C\nu_{ij} \right| dA \leq \int \left| \sum_{i=1}^{N} r_i F_i \right| d\mu_1 \quad (2-2)
\]
where \( M \) is a constant. Notice that \( C \) refers to the Cauchy transform (not a constant).

For a compact \( K \subset \mathbb{C} \) we define the analytic capacity of \( K \) by
\[ \gamma(K) = \sup \| f'(\infty) \| \]
where the sup is taken over those functions \( f \) analytic in \( \mathbb{C}_\infty \setminus K \) for which \( |f(z)| \leq 1 \) for all \( z \in \mathbb{C}_\infty \setminus K \), and
\[ f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty)). \]
The analytic capacity of a general \( E \subset \mathbb{C} \) is defined to be
\[ \gamma(E) = \sup \{ \gamma(K) : K \subset E, \ K \text{ compact} \}. \]

Good sources for basic information about analytic capacity are [Garnett (1972), Chapter VIII of Gamelin (1969)], Chapter V of Conway (1991), and Tolsa (2014).

A related capacity, \( \gamma_+ \), is defined for \( E \subset \mathbb{C} \) by
\[ \gamma_+(E) = \sup \| \mu \| \]
where now the sup is taken over positive measures \( \mu \) with compact support contained in \( E \) for which \( \| C\mu \|_{L^\infty(\mathbb{C})} \leq 1 \). Since \( C\mu \) is analytic in \( \mathbb{C}_\infty \setminus \text{spt} \mu \) and \( (C\mu)'(\infty) = \| \mu \| \), we have
\[ \gamma_+(E) \leq \gamma(E) \]
for all \( E \subset \mathbb{C} \). Tolsa (2003) proves the astounding result (Tolsa’s Theorem) that \( \gamma_+ \) and \( \gamma \) are actually equivalent. That is, there is an absolute constant \( A_T \) such that
\[ \gamma(E) \leq A_T \gamma_+(E) \]
for all \( E \subset \mathbb{C} \). The following semiadditivity of analytic capacity is a conclusion of Tolsa’s Theorem.
\[
\gamma \left( \bigcup_{i=1}^{m} E_i \right) \leq A_T \sum_{i=1}^{m} \gamma(E_i) \quad (2-3)
\]
where \( E_1, E_2, \ldots, E_m \subset \mathbb{C} \).

We set
\[
\sigma_0(S) = \begin{cases} \mathbb{C}, & \text{if } S \text{ is } N\text{-cyclic} \\ \sigma(S), & \text{if } S \text{ is rationally } N\text{-cyclic} \end{cases} \quad (2-4)
\]

**Theorem 2.** Let \( S \) on \( \mathcal{H} \) be a pure \( N\)-cyclic or rationally \( N\)-cyclic subnormal operator with cyclic vectors \( F_1, F_2, \ldots, F_N \) and let \( M_j \) on \( \mathcal{K} \) (satisfying (1-1) to (1-4)) be its minimal normal extension. Let \( G_j \perp \mathcal{H} \) for \( j = 1, 2, \ldots, N \) and let \( \nu_{ij} \) be as in (2-1). If \( \lambda_0 \in \text{Int}(\sigma_0(S)) \) and \( \nu_{ij} \) satisfy
Let 
exists \ c
for all \ H
Therefore,
we define
Thus, we have
Therefore, we have
Then \ \lambda_0 \ is an analytic bounded point evaluation for S.
Before proving Theorem 2, let us use Theorem 2 to prove Theorem 1 and Corollary 1. For a subnormal operator \ S \ on \ H \ and its minimal normal extension \ M_0 \ on \ \mathcal{K} \ (satisfying (1-1) to (1-4)) with \ \mu_1(\{\lambda_0\}) > 0, \ we define
where \ \rho \ is the residual norm of \ .
Lemma 1. Let \ S \ on \ H \ be a pure \ N-cyclic or rationally \ N-cyclic \ subnormal \ operator \ with \ cyclic \ vectors \ F_1, F_2, ..., F_N \ and \ let \ M_0 \ on \ \mathcal{K} \ (satisfying (1-1) to (1-4)) be its minimal normal extension. Suppose \ \mu_1(\{\lambda_0\}) > 0 \ and \ \lambda_0 \ in \ bpe(S). \ Define

\[ T : F \in \mathcal{H} \rightarrow F^{\lambda_0} = \chi_{(\lambda_0)}^\ast F \in \mathcal{H}_0, \]

then \ T \ is invertible and \ S^{\lambda_0} = TST^{-1}, \ that is, \ S^{\lambda_0} \ is similar to \ S. \ Consequently, \ S^{\lambda_0} \ on \ H_{\lambda_0} \ is a pure \ N-cyclic or rationally \ N-cyclic \ subnormal \ operator \ with \ cyclic \ vectors \ F_1^{\lambda_0}, F_2^{\lambda_0}, ..., F_N^{\lambda_0} \ and \ \lambda_0 \ in \ bpe(S^{\lambda_0}).

Proof: Assume \ S \ is pure and rationally \ N-cyclic (same proof for \ N-cyclic), \ \mu_1(\{\lambda_0\}) > 0, \ and \ \lambda_0 \ in \ bpe(S). \ Then there is a constant \ M_0 > 0 \ such that

\[ |r_k(\lambda_0)|^2 \leq M_0 \left\| \sum_{i=1}^{N} r_i F_i \right\|_{\mathcal{K}_{\lambda_0}}^2 \]

\[ = M_0 \left\| \sum_{i=1}^{N} r_i F_i^{\lambda_0} \right\|_{\mathcal{K}_{\lambda_0}}^2 + M_0 \left\| \sum_{i=1}^{N} r_i(\lambda_0) F_i(\lambda_0) \right\|_{\mathcal{K}_{\lambda_0}}^2 \mu_1(\{\lambda_0\}), \]

where \ r_k \ in \ Rat(\sigma(S)) \ for \ k = 1, 2, ..., N. \ Suppose that \ \lambda_0 \ is not a bounded point evaluation for \ S^{\lambda_0}, \ then there exist \ N \ sequences of rational functions \ \{r_{\lambda_0} \}_{1 \leq i \leq N, 1 \leq n < \infty} \ \subset \ Rat(\sigma(S)) \ such that

\[ \left\| \sum_{i=1}^{N} r_{\lambda_0} F_i^{\lambda_0} \right\|_{\mathcal{K}_{\lambda_0}}^2 \rightarrow 0 \]

and \ |r_{\lambda_0}(\lambda_0)| \rightarrow 1 \ for \ some \ fixed \ i_0. \ Set \ a_n = | \sum_{i=1}^{N} r_{\lambda_0}(\lambda_0) F_i(\lambda_0) |, \ then \ from \ (2-5) \ for \ k = i_0, \ there \ exists \ c_0 > 0 \ such \ that

\[ \lim_{n \rightarrow \infty} a_n \geq c_0. \]

Let

\[ H_n(z) = \frac{\sum_{i=1}^{N} r_{\lambda_0}(z) F_i(z)}{a_n}, \]

then \ \ H_n \ \in \ \mathcal{H}. \ By \ choosing \ a \ subsequence, \ we \ may \ assume \ there \ is \ v \ in \ \mathbb{C}^m \ and \ v \ \neq 0 \ such \ that

\[ \|H_n - \chi_{(\lambda_0)}^\ast v\|_{\mathcal{K}} \rightarrow 0. \]

Therefore, \ \chi_{(\lambda_0)}^\ast v \ \in \ \mathcal{H} \ and \ this \ is \ a \ contradiction \ since \ S \ is \ pure. \ Hence, \ \lambda_0 \ \in \ bpe(S^{\lambda_0}). \ So \ there \ is \ a \ constant \ M_1 > 0 \ such \ that

\[ |r_k(\lambda_0)|^2 \leq M_1 \left\| \sum_{i=1}^{N} r_i F_i^{\lambda_0} \right\|_{\mathcal{K}_{\lambda_0}}^2 \],

where \ r_k \ in \ Rat(\sigma(S)) \ for \ k = 1, 2, ..., N. \ Hence,

\[ \left\| \sum_{i=1}^{N} r_i F_i \right\|_{\mathcal{K}}^2 = \left\| \sum_{i=1}^{N} r_i F_i^{\lambda_0} \right\|_{\mathcal{K}_{\lambda_0}}^2 + \left\| \sum_{i=1}^{N} r_i(\lambda_0) F_i(\lambda_0) \right\|_{\mathcal{K}_{\lambda_0}}^2 \mu_1(\{\lambda_0\}) \leq \left( 1 + M_1 \left\| \sum_{i=1}^{N} |F_i(\lambda_0)| \mu_1(\{\lambda_0\}) \right\|_{\mathcal{K}_{\lambda_0}}^2 \right) \left\| \sum_{i=1}^{N} r_i F_i^{\lambda_0} \right\|_{\mathcal{K}_{\lambda_0}}^2. \]
This implies that $T$ is invertible.

Proof of (2) in Theorem 2. Suppose $\lambda_0 \in \text{bpe}(S) \cap \text{Int}(\sigma(S))$. By Lemma 1 we assume that $\mu_1(\{\lambda_0\}) = 0$. There are $g_1, g_2, \ldots, g_N \in \mathcal{H}$ such that

$$r_j(\lambda_0) = (\sum_{i=1}^N r_i F_i, g_j),$$

where $r_k \in \text{Rat}(\sigma(S))$. Set $G_j = z - \lambda_0 g_j$ for $j = 1, \ldots, N$. Let $\nu_{ij}$ be as in (2-1), then

$$\int \frac{1}{|z - \lambda_0|} d|\nu_{ij}|(z) \leq \|F_i\||g_j| < \infty,$$

and

$$C\nu_{ij}(\lambda_0) = (F_i, g_j) = \delta_{ij}.$$

By Theorem 2, we conclude $\lambda_0 \in \text{bpe}(S)$.

The proof of (1) in Theorem 2 is the same.

Proof of (2) in Corollary 1. From the assumptions of the corollary and Theorem 2, we see $\lambda_0 \in \text{abpe}(S)$. There are $\delta, M > 0$ such that

$$|r_j(\lambda)| \leq M \left(\sum_{i=1}^N r_i F_i\right)_2,$$

for $B(\lambda_0, \delta) \subset \text{Int}(\sigma(S))$, $\lambda \in B(\lambda_0, \delta)$, and $r_j \in \text{Rat}(\sigma(S))$. Using the maximal modulus principle,

$$\sup_{1 \leq j \leq N, \lambda \in B(\lambda_0, \delta)} |r_j(\lambda)| \leq \frac{M}{\delta} \left(\lambda - \lambda_0\right) \left(\sum_{i=1}^N r_i F_i\right)_2.$$

Hence,

$$\int \left(\sum_{i=1}^N r_i F_i\right)_2^2 d\mu_1 \leq \int_{B(\lambda_0, \delta)} \left(\sum_{i=1}^N r_i F_i\right)_2^2 d\mu_1 + \left(\sum_{i=1}^N \|F_i\|^2\right) \sup_{1 \leq j \leq N, \lambda \in B(\lambda_0, \delta)} |r_j(\lambda)|^2.$$

Therefore,

$$\left(\sum_{i=1}^N r_i F_i\right)_2 \leq M_1 \left(\lambda - \lambda_0\right) \left(\sum_{i=1}^N r_i F_i\right)_2,$$

where

$$M_1^2 = \left(1 + M^2 \left(\sum_{j=1}^N \|F_j\|^2\right) / \delta^2\right)^2.$$

So $\text{Ran}(S - \lambda_0)$ is closed. The corollary is proved.

The proof of (1) in Corollary 1 is the same.

Theorem 2 is a generalization of Corollary 2.2 in Aleman et al. (2009) where $N = 1$. There are fundamental differences between $N = 1$, where the existence of analytic bounded point evaluations for $P^*(\mu)$ was first proved in Thomson (1991), and $N > 1$, where analytic bounded point evaluations may not exist (see the example at the end of this section). To prove Theorem 2 we need several lemmas.

The following Lemma is from Lemma B in Aleman et al. (2009).

**Lemma 2.** There are absolute constants $\epsilon_1 > 0$ and $C_1 < \infty$ with the following property. For $R > 0$, let $E \subset \text{clos}(\mathbb{R}^d)$ with $\gamma(E) < R\epsilon_1$. Then

$$|p(0)| \leq \frac{C_1}{R^2} \int_{\text{clos}(\mathbb{R}^d) \setminus E} |p| d\mu / \pi$$

for all $p \in \mathcal{P}$.

**Lemma 3.** Let $\epsilon_1 > 0$ and $C_1 < \infty$ be as in Lemma 2. For $R > 0$, let $E \subset \text{clos}(\mathbb{R}^d)$ with $\gamma(E) < \frac{R}{2}\epsilon_1$. Then

$$|p(\lambda)| \leq \frac{4C_1}{R^2} \int_{\text{clos}(\mathbb{R}^d) \setminus E} |p| d\mu / \pi$$

for all $\lambda \in B(0, \frac{R}{2})$ and $p \in \mathcal{P}$.  

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Proof: For $\lambda \in B(0, \frac{\epsilon}{2})$, let $E_\lambda = \overline{B}(\lambda, \frac{\epsilon}{2}) \cap E - \lambda$. Then $E_\lambda \subset \text{clo} \overline{\text{os}}(\frac{\epsilon}{2}B)\text{ and } \gamma(E_\lambda) \leq \gamma(E) < \frac{\epsilon}{2} \epsilon_1$. From Lemma 2, we have

$$|p(0)| \leq \frac{C_1}{(2\epsilon)^2} \int_{\text{clo} \overline{\text{os}}(\frac{\epsilon}{2}B)\setminus E_\lambda} |p| \frac{dA}{\pi}.$$  

Replacing $p(z)$ by $p(z + \lambda)$, we get

$$|p(\lambda)| \leq \frac{4C_1}{R^2} \int_{\text{clo} \overline{\text{os}}(\frac{\epsilon}{2}B)\setminus E_\lambda} |p(z + \lambda)| \frac{dA(z)}{\pi} = \frac{4C_1}{R^2} \int_{B(\lambda, \frac{\epsilon}{2}) \setminus E} |p| \frac{dA}{\pi} \leq \frac{4C_1}{R^2} \int_{\text{clo} \overline{\text{os}}(B\overline{\text{D}})\setminus E} |p| \frac{dA}{\pi}$$

Let $\nu$ be a compactly supported finite measure on $C$. For $\epsilon > 0$, $C_\epsilon \nu$ is defined by

$$C_\epsilon \nu(z) = \int_{|w - z| > \epsilon} \frac{1}{w - z} d\nu(w),$$

and the maximal Cauchy transform is defined by

$$C_\epsilon \nu(z) = \sup_{\epsilon > 0} |C_\epsilon \nu(z)|.$$  

From Proposition 2.1 of Tolks (2002) and Tolks’s Theorem, we have the following estimation (also see Tolks (2014) Proposition 4.16):

$$\gamma(\{|C_\epsilon \nu| \geq a\}) \leq \frac{C_\gamma}{a} \|\nu\|,$$

where $C_\gamma$ is an absolute positive constant and $a > 0$.

**Lemma 4.** Suppose $\nu$ is a finite compactly supported Borel measure of $C$, $\lambda_0 \in C$, and

$$\int \frac{1}{|z - \lambda|} d|\nu|(z) < \infty.$$  

If $\epsilon_0, a > 0$, then

1. there exist $0 < \delta_a < \frac{1}{4}$ such that

$$2\sqrt{\delta_0} \int \frac{1}{|z - \lambda_0|} d|\nu|(z) + \int_{B(\lambda_0, 2\delta_a)} \frac{1}{|z - \lambda_0|} d|\nu|(z) < \frac{a}{2}$$ (2.7)

and

$$\frac{2C_\gamma}{a} \int_{B(\lambda_0, \sqrt{\delta_0})} \frac{1}{|z - \lambda_0|} d|\nu| < \epsilon_0;$$ (2.8)

2. for $0 < \delta < \delta_a$, there exists $E_\delta \subset \overline{B}(\lambda_0, \delta)$ such that $\gamma(E_\delta) < \epsilon_0 \delta$ and

$$|C \nu(\lambda - C \nu(\lambda_0))| \leq a$$

almost everywhere with respect to the area measure on $B(\lambda_0, \delta) \setminus E_\delta$.

Proof: (2.7) and (2.8) of (1) follow from (2.6). For (2), we fix $0 < \delta < \delta_a$. Let $\nu_0 = \frac{\chi_{B(\lambda_0, \sqrt{\delta})}}{\sqrt{\pi}} \nu$. For $\epsilon < \delta$ and $\lambda \in B(\lambda_0, \delta)$, we get:

$$\overline{B}(\lambda, \epsilon) \subset B(\lambda_0, 2\delta) \subset B(\lambda_0, \sqrt{\delta})$$

$$(\overline{B}(\lambda, \epsilon))^c \cap B(\lambda_0, \sqrt{\delta}) = B(\lambda_0, \sqrt{\delta})^c \cap B(\lambda, \sqrt{\delta} - \delta)^c,$$

and

$$|C_\epsilon \nu(\lambda) - C \nu(\lambda_0)|$$

$$\leq |\lambda - \lambda_0| \int_{|z - \lambda| > \epsilon} \frac{d\nu}{(z - \lambda)(z - \lambda_0)} + \int_{B(\lambda, \epsilon)} \frac{1}{|z - \lambda|} d|\nu|(z)$$

$$\leq \delta \int_{(B(\lambda, \epsilon))^c \cap B(\lambda_0, \sqrt{\delta})^c} \frac{d\nu}{(z - \lambda)(z - \lambda_0)} + \delta \left| \int_{|z - \lambda| > \epsilon} \frac{d\nu}{(z - \lambda)} \right| + \int_{B(\lambda_0, 2\delta)} \frac{1}{|z - \lambda_0|} d|\nu|(z)$$

$$\leq \delta \int_{|z - \lambda| > \sqrt{\delta} - \delta} \frac{1}{|z - \lambda|} d|\nu|(z) + \delta |C_\epsilon \nu_0(\lambda)| + \int_{B(\lambda_0, 2\delta)} \frac{1}{|z - \lambda_0|} d|\nu|(z)$$

$$\leq 2\sqrt{\delta} \int \frac{1}{|z - \lambda_0|} d|\nu|(z) + \delta C_\epsilon \nu_0(\lambda) + \int_{B(\lambda_0, 2\delta)} \frac{1}{|z - \lambda_0|} d|\nu|(z)$$

$$\leq \frac{a}{2} + \delta C_\epsilon \nu_0(\lambda),$$
where the last two steps follow from \( \frac{\delta}{\sqrt{2\delta}} \leq 2\sqrt{3} \) and (2-7). Let
\[
E_\delta = \{ \lambda : C_\nu \nu_\delta(\lambda) \geq \frac{a}{2\delta} \} \cap B(\lambda_0, \delta),
\]
then from (2-9), we get
\[
\{ \lambda : |C_\nu \nu(\lambda) - C_\nu(\lambda_0)| \geq a \} \cap B(\lambda_0, \delta) \subset E_\delta.
\]
From (2-5) and (2-8), we get
\[
\gamma(E_\delta) \leq \frac{2C_\nu \delta}{a} \| \nu_\delta \| < \epsilon_0 \delta.
\]
On \( B(\lambda_0, \delta) \setminus E_\delta \), for \( \epsilon < \delta \), we conclude that
\[
|C_\nu \nu(\lambda) - C_\nu(\lambda_0)| < a.
\]
The lemma follows since
\[
\lim_{\epsilon \to 0} C_\nu \nu(\lambda) = C_\nu(\lambda) \text{ a.e. Area.}
\]

**Remark.** (1) In Aleman et al. (2009) and Aleman et al. (2010), a key step for their alternative proofs of Thomson's theorem is to show that \( |C_\nu(\lambda)| \) is bounded below on \( B(\lambda_0, \delta) \setminus E \), where \( \gamma(E) < \epsilon_0 \delta \) and \( C_\nu(\lambda_0) \neq 0 \). This is directly implied by above lemma. So the lemma provides an alternative proof of the property.

(2) For \( \nu \), \( \lambda_0 \), and \( a > 0 \) in Lemma 4, we have
\[
\text{Area}(\{|C_\nu(\lambda) - C_\nu(\lambda_0)| > a\} \cap B(\lambda_0, \frac{1}{n}))
\leq \frac{1}{a} \int_{B(\lambda_0, \frac{1}{n})} |C_\nu(\lambda) - C_\nu(\lambda_0)| dA(\lambda)
\leq \frac{1}{a} \int_{B(\lambda_0, \frac{1}{n})} |\lambda - \lambda_0| \left| C_\nu(\frac{\nu}{\nu - \lambda_0})(\lambda) \right| dA(\lambda).
\]

Therefore, by Lemma 1 of Browder (1962), we see that \( |C_\nu(\lambda) - C_\nu(\lambda_0)| \leq a \) on a set having full area density at \( \lambda_0 \) whenever \( |\lambda - \lambda_0| \) is sufficiently small. Lemma 4 (2) shows that this inequality holds capacity density which is needed (not area density as Browder considers) in order to apply Lemma 3 in proving our main theorem below.

The following lemma is a simple linear algebra exercise.

**Lemma 5.** Let \( \| \cdot \|_{1,n} \) be the \( t^1 \) norm of \( \mathbb{C}^n \), that is, \( \|x\|_{1,n} = \sum_{i=1}^n |x_i| \). Let \( A = [a_{ij}]_{n \times n} \) be an \( n \times n \) matrix such that \( |a_{ij} - \delta_{ij}| \leq \frac{1}{2n} \). Then
\[
\|xA\|_{1,n} \geq \epsilon_2 \|x\|_{1,n}, \ x \in \mathbb{C}^n,
\]
where \( \epsilon_2 > 0 \) is an absolute constant.

Proof of Theorem 2. Assume \( S \) is rationally \( N \)-cyclic and \( \lambda_0 = 0 \). Let \( A = [a_{ij}]_{N \times N} \) be the inverse of the matrix in condition (2). By replacing \( F_i \) by \( \sum_{k=1}^N a_{ik} F_k \), we may assume that the matrix in the condition (2) is identity.

For \( a = \frac{1}{2N} \), let \( \delta_{ij}^\nu \) be \( \delta_0 \) for \( \nu = \nu_{ij} \) in (1) of Lemma 4. Set
\[
\delta_0 = \min_{1 \leq i, j \leq N} \delta_{ij}^\nu.
\]
From (2) of Lemma 3, for a given \( 0 < \delta < \delta_0 \) with \( B(0, \delta) \subset \text{Int}(\sigma(S)) \) and \( \epsilon_0 = \frac{\epsilon_1}{2AF_N N^7} \), where \( \epsilon_1 \) is in Lemma 4 and \( AF \) is from (2-3), there exists \( E_{ij}^\nu \subset B(0, \delta) \) such that \( \gamma(E_{ij}^\nu) < \epsilon_0 \delta \) and
\[
|C_\nu \nu(\lambda) - \delta_{ij}^\nu| = |C_\nu \nu(\lambda) - C_\nu(\lambda_0)| \leq a
\]
almost everywhere with respect to the area measure on \( B(0, \delta) \setminus E_{ij}^\nu \). Set \( E = \cup_{j=1}^N E_{ij}^\nu \), \( B(\lambda) = |C_\nu \nu(\lambda)|_{N \times N} \), \( \lambda_0 = (\lambda_1, \lambda_2, ..., \lambda_N) \) where \( r_1, r_2, ..., r_N \in \text{Rat}(\sigma(S)) \), and \( R(\lambda) = (b_1(\lambda), b_2(\lambda), ..., b_N(\lambda)) \), then, from Lemma 4 we have
\[
\sum_{j=1}^N |r_j(\lambda)| = \| R(\lambda) B(\lambda) \|_{1,N} \geq \epsilon_2 \| R(\lambda) \|_{1,N} = \epsilon_2 \sum_{j=1}^N |r_j(\lambda)|,
\]
almost everywhere with respect to the area measure on \( B(0, \delta) \setminus E \), where

\[
b_j(\lambda) = \sum_{i=1}^{N} r_i(\lambda)C(\nu_i)(\lambda).\]

From (2-2), we see that

\[
\int_{B(0,\delta)} \sum_{j=1}^{N} |b_j(\lambda)| dA(\lambda) \leq NM\delta \left\| \sum_{i=1}^{N} r_i F_i \right\|.
\]

On the other hand, from (2-3), we see

\[
\gamma(E) \leq AF \sum_{i,j=1}^{N} \gamma(E_{ij}^N) < AFN^2\epsilon_0\delta = \epsilon_1 \frac{\delta}{2}
\]

Therefore, applying Lemma 3 for \( r_1, r_2, \ldots, r_N \in \text{Rat}(\sigma(S)) \) since \( r_1, r_2, \ldots, r_N \) are analytic on \( B(0, \delta) \), we conclude

\[
\sum_{i=1}^{N} |r_i(\eta)| \leq \frac{4C_1}{\pi\epsilon_2 \delta^2} \int_{B(0,\delta) \setminus E} \sum_{i=1}^{N} |r_i| \frac{dA}{\pi}
\]

\[
\leq \frac{4C_1}{\pi\epsilon_2 \delta^2} \int_{B(0,\delta) \setminus E} \sum_{j=1}^{N} |b_j(\lambda)| dA(\lambda)
\]

\[
\leq \frac{4C_1NM}{\pi\epsilon_2 \delta} \left\| \sum_{i=1}^{N} r_i F_i \right\|
\]

for \( \eta \in B(0, \frac{\delta}{4}) \). So \( 0 \) is an analytic bounded point evaluation for \( S \).

If \( S \) is \( N \)-cyclic, we just need to drop the condition that \( B(0, \delta) \subset \text{Int}(\sigma(S)) \) and replace rational functions \( r_1, r_2, \ldots, r_N \in \text{Rat}(\sigma(S)) \) by polynomials \( p_1, p_2, \ldots, p_N \in \mathcal{P} \) in the proof. This completes the proof.

**Example.** A **Swiss cheese** \( K \) can be constructed as

\[
K = \mathbb{D} \setminus \bigcup_{n=1}^{\infty} B(a_n, r_n),
\]

where \( B(a_n, r_n) \subset \mathbb{D}, \, \mathbb{B}(a_i, r_i) \cap \mathbb{B}(a_j, r_j) = \emptyset \) for \( i \neq j \). \( \sum_{n=1}^{\infty} r_n < \infty \), and \( K \) has no interior points.

Let \( \mu \) be the sum of the arc length measures of \( \partial \mathbb{D} \) and all \( \partial B(a_n, r_n) \). Let \( \nu \) be the sum of \( dz \) on \( \partial \mathbb{D} \) and all \( -dz \) on \( \partial B(a_n, r_n) \). For a rational function \( f \) with poles off \( K \), we have

\[
\int fd\nu = 0.
\]

Clearly \( \left( \frac{df}{d\mu} \right) > 0 \), a.e. \( \mu \) and \( \left( \frac{df}{d\mu} \right) \perp R^2(K, \mu) \), so \( S_{\mu} \) (multiplication by \( z \)) on \( R^2(K, \mu) \) is a pure subnormal operator and \( \sigma(S_{\mu}) = \sigma_{\text{e}}(S_{\mu}) = K \). Moreover, there exists a function \( F \in R^2(K, \mu) \) such that \( R^2(K, \mu) = P^2(S_{\mu}[1, F]) \).

**Proof:** Let

\[
H_n(z) = \left( \frac{r_1}{z - a_1} \right)^n \left( \frac{r_2}{z - a_2} \right)^{n-1} \ldots \left( \frac{r_{n-1}}{z - a_{n-1}} \right)^2 \left( \frac{r_n}{z - a_n} \right).
\]

Let \( \{b_n\} \) be a sequence of positive numbers satisfying

\[
\lim_{n \to \infty} \sum_{k=n+1}^{\infty} b_k = 0.
\]

Set \( F = \sum_{n=1}^{\infty} b_n H_n \). By construction, we see that for \( z \in K \), \( |F(z)| \leq \sum_{n=1}^{\infty} b_n < \infty \). Let

\[
p_1 = -\frac{r_1}{b_n(z - a_1)} H_n \sum_{k=1}^{n-1} b_k H_k, \quad p_2 = \frac{r_1}{b_n(z - a_1)} H_n.
\]

Then \( p_1 \) and \( p_2 \) are polynomials and for \( z \in K \),

\[
\left| p_1 + p_2 F - \frac{r_1}{z - a_1} \right| \leq \sum_{k=n+1}^{\infty} \frac{b_k}{b_n} \to 0.
\]
Hence \( \frac{1}{z-a_k} \in P^2(S_\mu|1,F) \). Similarly, one can prove that \( \frac{1}{(z-a_n)^m} \in P^2(S_\mu|1,F) \). Therefore, \( R^2(K,\mu) = P^2(S_\mu|1,F) \), rationally cyclic subnormal operator \( S_\mu \) is 2-cyclic, and \( abpe(S) = \emptyset \).

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