SCAFFOLD: Stochastic Controlled Averaging for On-Device Federated Learning

Sai Praneeth Karimireddy
EPFL, Lausanne

Satyen Kale
Google Research, New York

Mehryar Mohri
Google Research and Courant Institute, New York

Sashank J. Reddi
Google Research, New York

Sebastian U. Stich
EPFL, Lausanne

Ananda Theertha Suresh
Google Research, New York

Abstract

Federated learning is a key scenario in modern large-scale machine learning. In that scenario, the training data remains distributed over a large number of clients, which may be phones, other mobile devices, or network sensors and a centralized model is learned without ever transmitting client data over the network. The standard optimization algorithm used in this scenario is Federated Averaging (FedAvg). However, when client data is heterogeneous, which is typical in applications, FedAvg does not admit a favorable convergence guarantee. This is because local updates on clients can drift apart, which also explains the slow convergence and hard-to-tune nature of FedAvg in practice. This paper presents a new Stochastic Controlled Averaging algorithm (SCAFFOLD) which uses control variates to reduce the drift between different clients. We prove that the algorithm requires significantly fewer rounds of communication and benefits from favorable convergence guarantees.

1. Introduction

A learning scenario playing a key role in modern large-scale applications is that of federated learning. Unlike standard settings where models are trained using large datasets stored in a central server (Dean et al., 2012; Iandola et al., 2016; Goyal et al., 2017), in federated learning, the training data remains distributed over a large number of clients, which may be phones, other mobile devices, network sensors, or alternative local information sources (Konečný et al., 2016b,a; McMahan et al., 2017b; Mohri et al., 2019). A centralized model is thus learned without ever transmitting client data over the network, thereby ensuring a basic level of privacy and limiting network communication.

*. Work done during internship at Google Research, New York.
This centralized model benefits from all client data and can often result in a beneficial performance, as reported in several tasks, including next word prediction (Hard et al., 2018; Yang et al., 2018), emoji prediction (Ramaswamy et al., 2019), decoder models (Chen et al., 2019b), vocabulary estimation (Chen et al., 2019a), low latency vehicle-to-vehicle communication (Samarakoon et al., 2018), and predictive models in health (Brisimi et al., 2018). Nevertheless, federated learning raises several types of issues and has been the topic of multiple research efforts studying the learning and generalization properties in that scenario (Mohri et al., 2019), systems, networking and communication bottleneck problems due to frequent exchanges between the central server and the clients, with unreliable or relatively slow network connections (McMahan et al., 2017a), and many others.

This paper deals with the key question of the optimization task in federated learning and specifically that of designing an efficient optimization solution with convergence guarantees. The optimization task in federated learning has been the topic of multiple research work. That includes the design of more efficient communication strategies (Konečný et al., 2016b,a; Suresh et al., 2017; Stich et al., 2018; Karimireddy et al., 2019; Basu et al., 2019), the study of lower bounds for parallel stochastic optimization with a dependency graph (Woodworth et al., 2018b), devising efficient distributed optimization methods benefiting from differential privacy guarantees (Agarwal et al., 2018), stochastic optimization solutions for the agnostic formulation (Mohri et al., 2019), and incorporating cryptographic techniques (Bonawitz et al., 2017), see (Li et al., 2019a) for an in-depth survey of recent work in federated learning.

Training in federated learning typically involves alternating rounds of communication and local updates. At each such round, a subset of clients are sampled and each sampled client receives the shared global model. Clients then perform local updates to this model, which involve only their local training data. Then, at the end of the round, the clients sampled send their updates to the server, which aggregates the updates to form the new global model. There are three key aspects which differentiate federated learning from parallel or distributed training: (1) the data, and thus the loss function, on the different clients may be very heterogeneous and this far from being representative of the joint data; (2) only a small subset of the devices selected by a central server participate in each round; (3) the server never keeps track of any individual client information and only uses aggregates to ensure privacy.

The standard optimization algorithm for federated learning is Federated Averaging (FedAvg) (McMahan et al., 2017b). For this algorithm, the subset of clients participating in the current round receive the global parameters $\boldsymbol{x}$. Each client $i$ performs a fixed (say $K$) steps of SGD using its local data and outputs the update $\Delta \boldsymbol{y}_i$. The updates are then aggregated to update the global parameters. However, FedAvg does not benefit from a favorable convergence guarantee and can be quite slow when client data is heterogeneous, which is typical in applications. Empirically, FedAvg is known to be sensitive to its hyperparameters and tends to diverge if not chosen carefully. This, along with its slow convergence, can make it hard to use out of the box (Li et al., 2019a, Sec 2.3). This is due to the key problem of drifting of client updates, which we now briefly discuss.

We distinguish between the server optimum $\boldsymbol{x}^*$, parameters which work well for the combined data, and client $i$’s optimum $\{\boldsymbol{x}_i^*\}$, parameters which work well on the client data for client $i$. Since client data is heterogeneous, the server optimum $\boldsymbol{x}^*$ is usually
quite different from the client optima \( \{x^*_i\} \). Suppose we run FedAvg starting close to the server optimum \( x^* \). Each client \( i \) updates its local model towards \( x^*_i \) since \( x^* \) is not the client optimum. This drift away from the true server optimum suggests that, to ensure convergence, FedAvg requires a carefully decreasing step-size sequence.

The FedProx algorithm by Li et al. (2019a) seeks to minimize the local drift by imposing additional regularization on each client. While this can slightly reduce the effect, it does not eliminate it. This argument can be formalized to prove that FedAvg (and FedProx) are necessarily significantly slower than standard SGD, even without any stochasticity and with all clients participating at every round. This is because standard SGD ensures that the local clients are always in sync through frequent communication.

The main idea behind the design of our Stochastic Controlled Averaging algorithm is to use control variates to reduce client drift and ensure that the client updates are aligned with each other. Each client \( i \) is assigned a control variate \( c_i \) and the global control variate is defined to be their uniform average \( c = \frac{1}{N} \sum_{i=1}^{N} c_i \). The control variate \( c_i \) represents the direction of the local update we expect to see from client \( i \) and \( c \) to be the aggregate direction in which the server updates. Given access to \( c \) and \( c_i \), the client can perform the following correction to its local update \( \Delta_i \) to better align itself with the server update

\[
\Delta'_i = \Delta_i + c - c_i.
\]

Assume that \( c_i \) is set to be equal to \( \Delta_i \). Then, the corrected update \( \Delta'_i = \frac{1}{N} \sum_i \Delta_i \) on every client \( i \) is exactly the server update, thereby removing all drift. While this fixes the issue, we are left with a chicken-and-egg problem: we need to know the client update direction \( \Delta_i \) in order to set \( c_i \) and we need \( c_i \) in order to compute \( \Delta_i \). We break the cycle by using only an estimate for \( \Delta_i \) in order to set \( c_i \). After performing the actual update, this estimate \( c_i \) can be further refined. This leads to our new optimization algorithm SCAFFOLD, which we describe more formally and analyze in detail in the next sections.

**Related work.** For identical clients, FedAvg coincides with parallel SGD analyzed by Zinkevich et al. (2010) who proved identical asymptotic convergence. Stich (2018) and, more recently Stich and Karimireddy (2019) and Patel and Dieuleveut (2019), gave a sharper analysis of the same method, under the name of local SGD, also for identical functions. However, there still remains a gap between their upper bounds and the lower bound of Woodworth et al. (2018a).

The analysis of FedAvg for heterogeneous clients is more delicate since it faces the local client drift issue discussed earlier. Several analyses bound this drift by using a very small step-size and assuming that the local updates admit bounded magnitude (Wang et al., 2019; Yu et al., 2019; Li et al., 2019b). Some other analyses view the drift as a second source of stochastic noise and provide guarantees asymptotically worse than standard SGD (Khaled et al., 2019). Similarly, Li et al. (2019a) prove convergence under an assumption which effectively implies that the client optima are \( \epsilon \)-close, and therefore that the drift is negligible. Finally, Zhao et al. (2018) propose global sharing of the clients’ data. While this does address the drifting issue, sending client data defeats the framework and the main purpose of federated learning.

The use of control variates is a classical technique to reduce variance in Monte Carlo sampling methods (cf. Glasserman (2013)). In optimization, they were used for finite-sum minimization by SVRG (Johnson and Zhang, 2013; Zhang et al., 2013) and then in SAGA.
(Defazio et al., 2014) to simplify the linearly convergent method SAG (Schmidt et al., 2017). Numerous variations and extensions of the technique are studied in (Hanzely and Richtárik, 2019). In a very similar vein, control variates were used to obtain linearly converging decentralized algorithms under the guise of ‘gradient-tracking’ in (Shi et al., 2015; Nedich et al., 2016) and for gradient compression as ‘compressed-differences’ in (Mishchenko et al., 2019). Our method can be viewed as seeking to remove the ‘variance’ in the gradients across the clients, though there still remains additional stochasticity.

The problem of drifting we described is a common phenomenon in distributed optimization. In fact, classic techniques such as ADMM mitigate this drift, though they are not applicable in federated learning. For well structured convex problems, CoCoA uses the dual variable as the control variates, enabling flexible distributed methods (Smith et al., 2016). DANE by Shamir et al. (2014) obtain a closely related primal only algorithm, which was later accelerated by Reddi et al. (2016). Stochastic Controlled Averaging can be viewed as an improved version of DANE where, instead of solving a proximal sub-problem at every iteration, a fixed number of (stochastic) gradient steps are taken.

The rest of the paper is organized as follows. In Section 2, we describe the optimization problem we consider, describe the assumptions adopted about client functions and specify the notation used. In Section 3, we describe our Stochastic Controlled Averaging algorithm in the simpler case where there is no sampling of clients. The convergence analysis of the algorithm is presented in Section 4. In Section 5, we discuss the more general setup of our algorithm relevant to federated learning where, at each round, a subset of clients is sampled. The convergence analysis is presented in Section 6.

2. Problem setup

2.1 Optimization problem

The problem we consider is that of minimizing a sum of stochastic functions, with only access to stochastic samples:

\[
    f(x^*) = \min_{x \in \mathbb{R}^d} \left\{ f(x) := \frac{1}{N} \sum_{i=1}^{N} \{ f_i(x) := \mathbb{E}_{\zeta}[f_i(x; \zeta)] \} \right\}.
\]

The functions \( \{ f_i \} \) are present on separate clients which can intermittently communicate amongst themselves. Our results also extend to the case when functions are weighted with respect to the number of samples \( m_i \).

2.2 Assumptions

We will adopt the following standard assumptions:

(A1) Each function \( f_i \) is \( \beta \)-smooth and for any \( x \) and \( y \) satisfies

\[
    \|\nabla f_i(x) - \nabla f_i(y)\|^2 \leq \beta \langle \nabla f_i(x) - \nabla f_i(y), x - y \rangle.
\]
In particular, using \( y = x^* \) and the convexity of \( f \) this implies that

\[
\frac{1}{N} \sum_{i=1}^{N} \| \nabla f_i(x) - \nabla f_i(x^*) \|^2 \leq \frac{\beta}{N} \sum_{i=1}^{N} \langle \nabla f_i(x) - \nabla f_i(x^*), x - x^* \rangle 
\]

\[
\leq \beta (f(x) - f(x^*)). 
\] (1)

Further, by the Cauchy-Schwarz inequality, the smoothness of \( f_i \) implies that the gradient of \( f_i \) is Lipschitz and for \( x \) and \( y \) gives

\[
\| \nabla f_i(x) - \nabla f_i(y) \| \leq \beta \| x - y \|. 
\] (2)

(A2) Each function \( f_i \) is \( \mu \)-strongly convex and for any \( x \) and \( y \) satisfies

\[
\langle \nabla f_i(x), y - x \rangle \leq -\left( f_i(x) - f_i(y) + \frac{\mu}{2} \| x - y \|^2 \right). 
\] (3)

(A3) We are given an independent unbiased stochastic gradient \( g_i(x) = \nabla f_i(x; \zeta) \) with \( \mathbb{E}_{\zeta}[g_i(x)] = \nabla f_i(x) \) and bounded variance

\[
\mathbb{E}_{\zeta}[\| g_i(x) - \nabla f_i(x) \|^2] \leq \sigma^2. 
\] (4)

Note that we do not make any assumptions regarding the similarity between functions \( \{f_i\} \).

2.3 Notation

Here, we summarize the notation used throughout the paper:

- \( \| \cdot \| \) denotes the euclidean norm, and \( [M] = \{1, \ldots, M\} \).
- we have \( N \) clients, \( R \) rounds of communication, and \( K \) (local) client update steps between two communication rounds.
- \( x^r \) represents the model parameters of the server after round \( r \in [R] \) of communication.
- \( y_{i,k}^r \) for \( i \in [N] \) and \( k \in [K] \) represents the model of client \( i \) after performing \( k \) local update steps in round \( r \).
- \( c^r \) and \( c_i^r \) represent the control variates at the server and client \( i \) respectively computed after round \( r \). We always maintain the invariant that \( c^r = \frac{1}{N} \sum_{i=1}^{N} c_i^r \).

3. Stochastic Controlled Averaging algorithm – without sampling of clients

Our algorithm (with client sampling) is presented in Algorithm 1. As a warm-up, we will study the case when all the \( N \) clients participates every round. We will use the following notation: \( \{y_i\} \) represent the client models, \( x \) is the aggregate server model, and \( c_i \) and \( c \) are the client and server control variates. Client \( i \) in round \( r \) performs the following updates
Algorithm 1 SCAFFOLD: Stochastic Controlled Averaging for federated learning

1: **server input** initial parameters $x$, control variate $c$, and global step-size $\eta_g$
2: **client input** for each $i$ local control variate $c_i$, and local step-size $\eta_l$
3: **for** each communication round $r = 1, \ldots, R$ **do**
4:   select a subset of clients $S \subseteq \{1, \ldots, N\}$
5:   communicate $(x, c)$ to all clients $i \in S$
6: **on each client** $i \in S$ **do**
7:   initialize local parameters $y_i \leftarrow x$
8:   **for** each local step $k = 1, \ldots, K$ **do**
9:      compute a stochastic gradient $g_i(y_i)$ of $f_i$
10:     $y_i \leftarrow y_i - \eta_l (g_i(y_i) - c_i + c)$ \Comment{local updates with correction}
11: **end for**
12: $c_i^+ \leftarrow (i)\ g_i(x)$, or (ii) $c - c_i + \frac{1}{K\eta_l}(x - y_i)$ \Comment{compute new control variate}
13: communicate $(\Delta y_i, \Delta c_i) \leftarrow (y_i - x, c_i^+ - c_i)$
14: $c_i \leftarrow c_i^+$ \Comment{update control variate}
15: **end client**
16: $\Delta x \leftarrow \frac{1}{|S|}\sum_{i \in S} \Delta y_i$, and $\Delta c \leftarrow \frac{1}{|S|}\sum_{i \in S} \Delta c_i$ \Comment{aggregate client outputs}
17: $x \leftarrow x + \eta_g \Delta x$ and $c \leftarrow c + \frac{|S|}{N}\Delta c$ \Comment{update parameters and control}
18: **end for**

- Starting from the shared global parameters $x_{i,r}^0 = x^{r-1}$, we update the local parameters for $k \in [K]$
  $$y_{i,k}^r = y_{i,k-1}^r - \eta_l v_{i,k}^r,$$  \Comment{local updates with correction}
  where $v_{i,k}^r := g_i(y_{i,k-1}^r) - c_i^{r-1} + c^{r-1}$ \Comment{(5)}

- Update the control iterates using any of the following options:
  $$c_i^r = \begin{cases} \text{Option I.} & g_i(x^{r-1}) , \\ \text{Option II.} & c^{r-1} - c_i^{r-1} + \frac{1}{K\eta_l}(x^{r-1} - x_{i,K}^r) . \end{cases} \Comment{(6)}$$

- Compute the new global parameters and global control variate
  $$x^r = x^{r-1} + \frac{\eta_g}{N}\sum_{i=1}^N (y_{i,K}^r - x^{r-1}) \quad \text{and} \quad c^r = \frac{1}{N}\sum_{i=1}^N c_i^r . \Comment{(7)}$$

Note that if we remove the correction $(c^{r-1} - c_i^{r-1})$ in (5) or equivalently always set $c_i = 0$ in (6), we recover the standard FedAvg algorithm. As we discussed previously, the main issue with FedAvg is that the updates of the clients $i$ and $j$ may be very different from each other leading to ‘drift’. The correction $(c^{r-1} - c_i^{r-1})$ is introduced to exactly reduce this drift. For example, suppose that we can set the control variate every step $k$ to be $c_i^{r-1} = g_i(y_{i,k-1}^r)$ in (5), then the update becomes identical for all clients
  $$y_{i,k}^r = y_{i,k-1}^r - \frac{1}{N}\sum_{i=1}^N \eta_l g_i(y_{i,k-1}^r) .$$
Unfortunately, we cannot set \( c_i = g_i(y_{i,k-1}^r) \) since computing the corresponding global control iterate \( \frac{1}{N} \sum_{i=1}^{N} c_i \) requires all the clients communicating with each other every step. We will instead use the easily computable \( c_i = g_i(x_{r-1}) \) with \( \frac{1}{N} \sum_{i=1}^{N} c_i \) for the whole round \( r \) (option I in (6)). Since the gradient of \( f_i \) is Lipschitz, we can hope that \( g_i(x_{r-1}) \approx g_i(y_{i,k-1}^r) \) as long as our local updates are not too large and \( x_{r-1} \approx y_{i,k} \). This idea of control iterates is inspired by (and is similar to) those used for variance reduction in SVRG (Johnson and Zhang, 2013) and SAGA (Defazio et al., 2014).

There are other choices of the control variate \( c_i \) which are correlated with \( g_i(y_{i,k-1}^r) \) and hence also suffice. One could use an update similar to that of SARAH (Nguyen et al., 2017) and continuously perform local updates to \( c_i \) instead of keeping it fixed. In another approach, option II in (6) which is known as gradient-tracking in decentralized algorithms (Shi et al., 2015; Nedich et al., 2016) uses

\[
c_{i}^r = c_{i}^{r-1} - c_{i}^{r-1} + \frac{1}{K} (x_{r-1} - x_{i,K}^r) = \frac{1}{K} \sum_{k=1}^{K} g_i(y_{i,k-1}^r).
\]

By using an average of many stochastic gradients, option II has lower variance at the cost of slightly higher bias. This option results in a method similar to a very recent independent work of Anonymous (2019). However, in general i) their algorithm and proof can not be extended to support client sampling, and ii) they do not use a global step-size and hence their rates have a worse dependence on the number of clients \( N \).

The final output of the algorithm is a weighted average for some positive weights \( \{w_r\} \) for \( r \in \{1, \ldots, R+1\} \)

\[
\bar{x}^R = \frac{1}{\sum_r w_r} \sum_r w_r x_{r-1}^r. \tag{8}
\]

### 4. Convergence analysis – without sampling

We show the following rate of convergence for strongly-convex functions. Similar extensions can be derived for the general convex, and non-convex settings.

**Theorem 1** Suppose that each of the functions \( f_i \) satisfies assumptions A1–A3. Then, there exist weights \( \{w_r\} \) and local step-sizes \( \eta_i \leq \frac{1}{8\beta g_i K} \) such that for any \( \eta_g \geq 1 \) the output (8) generated using (5)–(7) for any \( R \geq \frac{8\beta}{\mu} \) satisfies \(^1\):

\[
\mathbb{E}[f(\bar{x}^R)] - f(x^*) \leq \hat{O}\left(\frac{\sigma^2}{\mu R} \left( \frac{1}{N} + \frac{1}{\eta_g^2} \right) + \mu \|x^0 - x^*\|^2 \exp\left(-\frac{\mu R}{8\beta}\right)\right).
\]

Thus by setting \( \eta_g = \sqrt{N} \), we get a communication complexity of \( \hat{O}\left(\frac{\beta}{\mu} + \frac{\sigma^2}{\mu^2 NK}\right) \). When the variance is large (which is usually the case) or if the required accuracy is small, the rate of convergence is typically dominated by \( \sigma^2 / (\mu \epsilon) \). The result above shows that increasing the number of local steps \( K \) as well as number of clients \( N \) can decrease the number of communication rounds required.

\(^1\) We use \( O(\cdot) \) to suppress constant factors and \( \tilde{O}(\cdot) \) to hide logarithmic terms.
Theorem 1 improves upon the best-know upper bounds even when all functions are identical by a factor $N$—in comparison, Stich and Karimireddy (2019) show a communication complexity of $\tilde{O}(\frac{N^\beta}{\mu} + \frac{\sigma^2}{\mu k N \epsilon})$ for identical functions. When $\sigma = 0$ our communication complexity becomes $\tilde{O}(\frac{2}{\mu})$ and nearly matches the lower bound of $\tilde{\Omega}(\sqrt{\frac{\beta}{\mu}})$ for distinct functions by Arjevani and Shamir (2015). The improved square-root dependence on condition number can be achieved via acceleration (Nesterov, 2018), a direction which we do not explore here.

Finally, note that we are free to choose an $\eta_g$ which is even larger than $\sqrt{N}$ while retaining the same rate. However, the local step-size $\eta_l$ also correspondingly becomes smaller. In the limit when $\eta_g \rightarrow \infty$, $\eta_l = 0$ and we recover SGD with a large batch size of $NK$. Thus, we fail to show a strict improvement due to picking a small $\eta_l$. This is not surprising since the lower-bound by Arjevani and Shamir (2015) rules out the possibility of improvement over SGD for general convex functions. In fact, even when all functions are identical (in which case the lower bound of Arjevani and Shamir (2015) does not apply), showing a strict advantage of taking local steps remains an open question.

4.1 Usefulness of control variates

Let us examine how our correction using control variates might help us. We first show that by using the control variates, the server update direction does not change. We drop the indices for the round $r$ and local step $k$ whenever obvious from context.

(P1) The local update (5) of SCAFFOLD aggregated across clients is similar to that of FedAvg

$$\frac{1}{N} \sum_{i=1}^{N} (y_i - \eta v_i) = \frac{1}{N} \sum_{i=1}^{N} (y_i - \eta (g_i(y_i) - c_i + c)) = \frac{1}{N} \sum_{i=1}^{N} (y_i - \eta g_i(y_i)).$$

While the aggregate update direction remains unchanged, the drift across the clients is reduced by our use of control variates. For simplicity, in this section we only examine (option I) choice of $c_i = g_i(x)$ in (6) and delay the general case for Section 6.

(P2) The drift from the starting point $x$ due to the local updates of SCAFFOLD is bounded as follows for any $a > 0$ and $\eta_l \leq \frac{1}{\beta}$

$$\frac{1}{N} \sum_{i=1}^{N} \|(y_i - \eta v_i) - x\|^2 \leq \frac{(1+a)}{N} \sum_{i=1}^{N} \|y_i - x\|^2 + \eta_l^2 (1+1/a)\beta (f(x) - f(x^*)) + 5\eta_l^2 \sigma^2.$$

The proof of (P2) can be found in Appendix D.1. Suppose we start from the optimum point and $x = x^*$, and further $\sigma^2 = 0$. Then we can set $a \rightarrow 0$ in (P2) to get

$$\frac{1}{N} \sum_{i=1}^{N} \|(y_i - \eta v_i) - x\|^2 \leq \frac{1}{N} \sum_{i=1}^{N} \|y_i - x\|^2 \Rightarrow \|(y_i - \eta v_i) - x\|^2 = 0.$$

Thus, SCAFFOLD overcomes the problem of client drift (at least close to the optimum). Further, the above argument proves that, unlike for FedAvg, the optimum $x^*$ is a fixed point of SCAFFOLD (up to the noise $\sigma^2$ in the stochastic gradients).
4.2 Proof summary

We can show the following progress for our algorithm between two communication rounds.

**Lemma 1 (one round progress)** Suppose our updates satisfy (P1) and assumptions A1–A3. For any step-size satisfying \( \eta \leq \frac{1}{8gK\eta_g} \) and effective step-size \( \eta := K\eta_g\eta_i \),

\[
\mathbb{E}\|x^r - x^*\|^2 \leq (1 - \frac{\mu\eta}{2}) \mathbb{E}\|x^{r-1} - x^*\|^2 + \frac{g^2\sigma^2}{K} + \eta\mathbb{E}[f(x^{r-1})] - f(x^*) + 3\beta\eta\delta_r,
\]

where \( \delta_r \) is the drift caused by the local updates on the clients

\[
\delta_r := \frac{1}{KN} \sum_{k=1}^{K} \sum_{i=1}^{N} \mathbb{E}_r[\|y_{i,k}^r - x^{r-1}\|^2].
\]

Here, \( \mathbb{E}_r[\cdot] \) is the expectation over all randomness in round \( r \), and conditioned on \( x^{r-1} \). The lemma above is valid for any algorithm which satisfies (P1) and is hence also valid for FedAvg. The difference between the two algorithms is bound on the drift-term \( \delta_r \) which becomes smaller as we approach the optimum, and does not depend on the heterogeneity across the functions.

**Lemma 2 (bounded drift)** Suppose our updates satisfy (P2) and assumptions A1–A3. For any step-size satisfying \( \eta \leq \frac{1}{8gK\eta_g} \), then we can bound the drift as

\[
3\beta\delta_r \leq \frac{1}{2\eta_g^2}(\mathbb{E}[f(x^{r-1})]) - f(x^*) + \frac{2g^2\sigma^2}{K\eta_g^2}.
\]

Note that (P2) was shown only for (option I) choice of \( c_i = g_i(x) \) in (6). Option II and other variations are analyzed in Section 6.

**Proof of Theorem 1** Combining Lemmas 1 and 2 gives the following recurrence

\[
\mathbb{E}\|x^r - x^*\|^2 \leq (1 - \frac{\mu\eta}{2}) \mathbb{E}\|x^{r-1} - x^*\|^2 + \frac{g^2\sigma^2}{K} + \eta\mathbb{E}[f(x^{r-1})] - f(x^*) + 3\beta\eta\delta_r.
\]

Rearranging the equation above gives the following one step progress

\[
w_r(\mathbb{E}[f(x^{r-1})] - f(x^*)) \leq \frac{w_r}{\eta} (1 - \mu\eta) d_{r-1} - \frac{w_r}{\eta} d_r + c\eta w_r,
\]

with the notation that \( d_r = \mathbb{E}\|x^r - x^*\|^2 \), \( \eta = \eta/2 \), and \( c = \frac{2g^2}{K} \left( \frac{1}{N} + \frac{2}{\eta_g^2} \right) \). Now, Lemma 5 is applicable with \( \eta_{\text{max}} = 1/(16\beta) \). Using the step-size \( \eta \) and weights \( \{w_r\} \) as defined in Lemma 5, the following holds for all \( R \geq \frac{1}{2\eta_{\text{max}}} = \frac{8\beta}{\mu} \):

\[
\frac{1}{(\sum_{r=1}^{R+1} w_r)} \sum_{r=1}^{R+1} w_r(\mathbb{E}[f(x^{r-1})] - f(x^*)) \leq \tilde{O}\left( \frac{\sigma^2}{\mu KR} \left( \frac{1}{N} + \frac{1}{\eta_g^2} \right) + \mu\|x^0 - x^*\|^2 \exp\left( -\frac{\mu R}{8\beta} \right) \right).
\]

Using convexity of \( f \) completes the proof of the theorem.

Note that the local step-size we need to take to get optimal rates is bounded as \( \eta_i \leq \frac{1}{8gK}\). The reason why we need to scale by \( K \) is because if the functions \( f_i \) are completely unrelated to each other, then taking multiple local-steps may not really help the optimization of the average function. In practice, larger step-sizes can be used since the functions \( f_i \) are typically more closely related to each other.
5. Stochastic Controlled Averaging algorithm – with sampling

Control variates can also be used even when only a small subset of devices (say $S \ll N$) participate each round. We will describe the algorithm using notation here which is convenient for the proofs: $\{y_i\}$ represent the client models, $x$ is the aggregate server model, and $c_i$ and $c$ are the client and server control variates. For an equivalent description which is easier to implement, we refer to Algorithm 1. The server maintains a global control variate $c$ as before and each client maintains its own control variate $c_i$. In round $r$, a subset of clients $S^r$ of size $S$ are sampled uniformly from $\{1, \ldots, N\}$. Suppose that every client performs the following updates

- Starting from the shared global parameters $y^r_{i,r} = x^{r-1}$, we update the local parameters for $k \in [K]$

  \[ y^r_{i,k} = y^r_{i,k-1} - \eta_k v^r_{i,k}, \quad \text{where} \quad v^r_{i,k} := g_i(y^r_{i,k-1}) - c_i^{r-1} + c^{r-1} \]

- Update the control iterates using (option II):

  \[ \tilde{c}_i^r = c_i^{r-1} - c_i^{r-1} + \frac{1}{K\eta_k} (x^{r-1} - x^r_{i,K}) = \frac{1}{K} \sum_{k=1}^K g_i(y^r_{i,k-1}) \]

  We update the local control variates only for clients $i \in S^r$

  \[ c_i^r = \begin{cases} 
  \tilde{c}_i^r & \text{if } i \in S^r \\
  c_i^{r-1} & \text{otherwise.} 
\end{cases} \]

- Compute the new global parameters and global control variate using only updates from the clients $i \in S^r$:

  \[ x^r = x^{r-1} + \frac{\eta_k}{S} \sum_{i \in S^r} (y^r_{i,K} - x^{r-1}) \quad \text{and} \quad c^r = \frac{1}{N} \sum_{i=1}^N c_i^r = \frac{1}{N} \left( \sum_{i \in S^r} c_i^r + \sum_{j \notin S^r} c_j^{r-1} \right). \]

Note that the clients are agnostic to the sampling and their updates are identical to when all clients are participating. Also note that the control variate choice (10) corresponds to (option II) in step 12 of Algorithm 1. Further, the updates of the clients $i \notin S^r$ is forgotten and is defined only to make the proofs easier. While actually implementing the method, only clients $i \in S^r$ participate and the rest remain inactive (see Algorithm 1).

After running for $R$ rounds of communication, the final output of the algorithm is, as before, a weighted average for some positive weights $\{w_r\}$ for $r \in \{1, \ldots, R + 1\}$

\[ \bar{x}^R = \frac{1}{\sum_r w_r} \sum_r w_r x^r \]

To get some intuition about the new method, examine what happens when the number of machines ($N$) is large and $S = 1$. Let $K = 1$, making the local and global iterates are the
same i.e. \( y_i = x \). Also suppose that \( \sigma^2 = 0 \) implying \( g_i(x^r) = \nabla f_i(x^r) \). Then the update in round \( r \) can be written for a randomly chosen \( i \) to be

\[
x^r = x^{r-1} - \eta(\nabla f_i(x^{r-1}) + c^{r-1} - c_i^{r-1}) \text{ where we update } c_i^r = \nabla f_i(x^{r-1}).
\]

In this setting, SCAFFOLD turns out to be equivalent to SAGA (Defazio et al., 2014). In the other extreme, if all the data is on a single machine \((N = 1)\) and \( R = T \), the method reduces to the standard SGD updates as expected. Thus, SCAFFOLD captures a wide range of algorithms and their corresponding rates as special cases, while simultaneously generalizing to many new useful settings of the parameters \( N, K, K \) and \( \sigma^2 \).

6. Convergence analysis – with sampling of clients

We prove the following rate of convergence for strongly convex functions. One can extend our technique to derive rates for general convex functions and non-convex functions.

**Theorem 2** Suppose that each of the functions \( f_i \) satisfies assumptions A1–A3. Then, there exist weights \( \{w_i\} \) and local step-sizes \( \eta \leq \min\left(1, \frac{\beta k}{\mu L}, \frac{1}{\mu KN} \right) \) such that for any \( \eta \geq 0 \) the output (13) of Algorithm 1 using option (ii) in step 12 satisfies for all \( R \geq \max\left(\frac{81\beta}{\mu}, \frac{15N}{S}\right) \):

\[
\mathbb{E}[f(x^R)] - f(x^*) \leq \tilde{O}\left(\frac{\sigma^2}{\mu KR} \left(\frac{1}{S} + \frac{1}{\eta^2}\right) + \mu d_0 \exp\left(-\min\left(\frac{S}{15N}, \frac{2}{81\beta}\right) R\right)\right),
\]

where \( d_0 := (\|x^0 - x^*\|^2 + \frac{S}{N} \sum_{i=1}^{N} \|c_i^0 - \nabla f_i(x^*)\|^2) \).

Setting \( \eta \geq \frac{1}{\sqrt{S}} \) gives a communication complexity of \( \tilde{O}(\frac{N}{S} + \frac{\beta}{\mu} + \frac{\sigma^2}{\mu KN}) \). If all clients participate and \( S = N \), we recover the communication complexity of \( \tilde{O}(\frac{\beta}{\mu} + \frac{\sigma^2}{\mu KN}) \) given by Theorem 1. This proves Theorem 1 even when option (II) is used for the update of the control variate. If \( S < N \), then the additional \( N/S \) term is necessary since we would need to communicate with every device at least once. Also note that when \( \sigma^2 = 0 \), Theorem 2 recovers the linear rate \( \tilde{O}(\frac{N}{S} + \frac{\beta}{\mu}) \) which matches that of SAGA. In fact, when \( K > 1 \) we obtain an interesting generalization of SAGA with additional local steps.

Instead of counting each round of communication as a single unit of cost, we can count the total amount of communication received from clients. This is an important metric since it represents the amount of work done by the clients and also captures the cost to privacy. With this metric the communication cost with all devices participating is \( \tilde{O}(\frac{N}{S} + \frac{\beta}{\mu} + \frac{\sigma^2}{\mu KN}) \). In contrast, with sampling the algorithm has a cost \( \tilde{O}(N + \frac{S\beta}{\mu} + \frac{\sigma^2}{\mu KN}) \). Since typically \( S \ll N \), this represents a significant reduction.

6.1 Overcoming sampling with control variates

The main question is what variant of properties (P1) and (P2) still hold when we sample a small number of clients. Consider the accumulated local updates of the clients \( i \in S \) starting from local models \( \{y_i\} \):

(P3) It is easy to see that \( \mathbb{E}_S[c - c_i] = 0 \) where the expectation is over the sampling \( S \)

\[
\mathbb{E}_S\left[\frac{1}{N}\sum_{i \in S}(y_i - \eta_i v_i)\right] = \mathbb{E}_S\left[\frac{1}{S}\sum_{i \in S}(y_i - \eta_i(g_i(y_i) - c_i + c))\right] = \frac{1}{N}\sum_i (y_i - \eta_i g_i(y_i)).
\]
Thus, in expectation over the sampling of $\mathcal{S}$, our update matches that of the usual FedAvg. We now examine what we can say about the drift of the local client parameters. The challenge with bounding the drift with client sampling is that even with syncing every step (i.e. $K = 1$) SGD may drift at the optimum due to the variance across the clients. Thus, the use of control variates is critical here. We now analyze the general case with any choice of control variate.

(P4) The drift from the starting point $x$ due to the local update of SCAFFOLD in local step $k$ or rounds $r$ is bounded as below for any $a > 0$ and $\eta_l \leq \frac{1}{\beta}$

$$
\frac{1}{S} \mathbb{E}_r \left[ \sum_{i \in \mathcal{S}} \| (y_i - \eta_l v_i) - x \|^2 \right] \leq \frac{1}{S} \mathbb{E}_r \left[ \sum_{i \in \mathcal{S}} \| y_i - x \|^2 \right] + \eta_l^2 \sigma^2 + 6\eta_l^2(1+1/a)\beta(f(x) - f(x^*)) + 3\eta_l^2(1+\frac{1}{a})\frac{1}{N} \sum_{i=1}^{N} \| c_i - \nabla f_i(x^*) \|^2.
$$

The proof of (P4) can be found in Appendix E.1. By comparing (P4) with (P2) we see that the final term which depends on the control iterates $c_i$ to be extra. Thus any choice of $c_i$ which ‘learns’ $\nabla f_i(x^*)$ (up to the noise in the stochastic gradients) as the algorithm progresses would suffice. E.g. suppose we start at the optimum $x = x^*$. Then, by setting $c_i = g_i(x)$ and assuming $\sigma^2 = 0$, we see once again that using $a \to 0$ in (P4) proves that there is no drift at the optimum

$$
\frac{1}{S} \mathbb{E}_r \left[ \sum_{i \in \mathcal{S}} \| (y_i - \eta_l v_i) - x \|^2 \right] \leq \frac{1}{S} \mathbb{E}_r \left[ \sum_{i \in \mathcal{S}} \| y_i - x \|^2 \right] \Rightarrow \|(y_j - \eta_l v_j) - x\|^2 = 0 \ \forall j \in [N].
$$

The challenge here is then to bound the term $\| c_i - \nabla f_i(x^*) \|^2$ as the algorithm progresses.

6.2 Proof summary

Just like in the full sampling case, we can prove the following progress between two communication rounds. For notational convenience, assume that $x_i^r := x^0$ for all $r < 0$ and $i \in [N]$.

Lemma 3 (one round progress) Suppose our updates satisfy (P1) and assumptions A1–A3. Then the following holds for any step-size satisfying $\eta_l \leq \min\left(\frac{1}{8\beta K\eta_g}, \frac{S\eta}{16\mu N K \eta_g}\right)$, effective step-size $\tilde{\eta} := K\eta_g \eta_l$, and control variates updated using (10),

$$
\mathbb{E} \left[ \| x^r - x^* \|^2 + \frac{9N\tilde{\eta}^2}{S} C_r \right] \leq (1 - \frac{\tilde{\eta}^2}{2}) \left( \mathbb{E} \| x^{r-1} - x^* \|^2 + \frac{9N\tilde{\eta}^2}{S} C_{r-1} \right)
$$

$$
+ \frac{\tilde{\eta}^2 \sigma^2}{KS} - \frac{2\tilde{\eta}}{25\hat{\eta}_r} \mathbb{E} [f(x^{r-1})] - f(x^*) + 3\beta \hat{\eta}_r - \frac{2\tilde{\eta}^2}{3} C_{r-1},
$$

where $C_r$ is the error in our control variate defined as $C_r := \frac{1}{N} \sum_{j=1}^{N} \| \mathbb{E} [c_i^j] - \nabla f_i(x^*) \|^2$. 

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and $\delta_r$ is the drift caused by the local updates on the clients

$$
\delta_r := \frac{1}{KN} \sum_{k=1}^K \sum_{i=1}^N \mathbb{E}\left\|y_{i,k}^r - x_r^{r-1}\right\|^2.
$$

In addition to keeping track of distance from the optimum as we did in Lemma 1, we also need to keep track how far our control variate is from its value at the optimum using $C_r$. This is because there is some ‘lag’ in our updates of the control variates $c_i$ since only a small subset of them are updated each round. We can also bound the drift term $\delta_r$.

**Lemma 4 (bounded drift)** Suppose our step-sizes satisfy $\eta_l \leq \frac{1}{81\beta K \eta_g}$ and $f_i$ satisfies assumptions $A1–A3$. Then, for any global $\eta_g \geq 1$ we can bound the drift as

$$
3\beta \tilde{\eta} \delta_r - \frac{2\tilde{\eta}^2}{3} C_{r-1} \leq \frac{\tilde{\eta}}{2\eta_g} (\mathbb{E}[f(x_r^{r-1})] - f(x^*)) + \frac{\tilde{\eta}^2}{K \eta_g} \sigma^2.
$$

Here again, the optimal step-size should not scale as $1/K$ but should be much larger depending on the similarity between the functions. There is an additional bound on the step-size depending on the number of clients sampled $S$. However, typically $\mu$ is very small making $1/\mu N$ reasonably large. Hence the condition that $\tilde{\eta} \leq S/(\mu N)$ can be safely ignored while setting the learning rate in practice.

**Proof of Theorem 2** Combining Lemmas 3 and 4 and rearranging the terms we can show that for any weights $\{w_r\}$

$$
w_r (\mathbb{E}[f(x_r^{r-1})] - f(x^*)) \leq \frac{w_r}{\eta} (1 - \mu \eta) d_{r-1} - \frac{w_r}{\eta} d_r + c \eta w_r,
$$

where $d_r = \mathbb{E}[\|x_r - x^*\|^2 + \frac{13N\tilde{\eta}^2}{S} C_r]$, $\eta = \tilde{\eta}/2$, and $c = \frac{2a^2}{S} \left(\frac{1}{N} + \frac{1}{\eta_g}\right)$ for any $\eta \leq \frac{1}{2} \min\left(\frac{S}{100aN}, \frac{1}{\eta_g}\right)$. The rest proceeds exactly as in the proof of Theorem 1.

7. Conclusion

We observe that FedAvg may experience ‘drift’ due to the updates of heterogeneous local clients, leading to slow convergence and necessitating careful learning rate scheduling. We instead propose SCAFFOLD, a new method which uses control variates to overcome this issue and prove that it has excellent theoretical properties. We believe that the increased stability of SCAFFOLD to heterogeneity of the clients even with sampling would make it easy to tune in practice. This, along with the ease of implementation of SCAFFOLD, we believe facilitates easy adoption.

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Appendix A. Algorithm without sampling

Here, we outline our algorithm when all devices participate every round.

Algorithm 2 Stochastic Controlled Averaging (without sampling)

```
1: input initial parameters $x$, control variate $c$, global and local step-sizes $\eta_g, \eta_l$
2: initialize for each client $i$ control variates $c_i$
3: for each communication round $r = 1, \ldots, R$ do
4: communicate to all clients $(x, c)$
5: on each client $i \in \{1, \ldots, N\}$ do
6: initialize local parameters $y_i \leftarrow x$
7: for each local step $k \in \{1, \ldots, K\}$ do
8: compute a stochastic gradient $g_i(y_i)$ of $f_i$
9: $y_i \leftarrow y_i - \eta_l (g_i(y_i) - c_i + c)$ \> local updates with correction
10: end for
11: $c_i^+ \leftarrow (i) g_i(x)$, or (ii) $c - c_i + \frac{1}{\eta_l} (y_i^k - x)$ \> compute new control variate
12: communicate $(\Delta y_i, \Delta c_i) \leftarrow (y_i - x, c_i^+ - c_i)$
13: $c_i \leftarrow c_i^+$ \> update control variate
14: end client
15: $\Delta x \leftarrow \frac{1}{N} \sum_{i=1}^{N} \Delta y_i$ and $\Delta c \leftarrow \frac{1}{N} \sum_{i=1}^{N} \Delta c_i$ \> aggregate client outputs
16: $x \leftarrow x + \eta_g \Delta x$ and $c \leftarrow c + \Delta c$ \> update parameters and control
17: end for
```

Appendix B. Some technical lemmas

In this section we cover some technical lemmas which are useful for computations later on.

The lemma below is useful to unroll recursions and derive convergence rates.

**Lemma 5 (convergence rate)** For every non-negative sequence $\{d_{r-1}\}_{r \geq 1}$ and any parameters $\mu > 0$, $\eta_{\text{max}} \in (0, 1/\mu]$, $c \geq 0$, $R \geq \frac{1}{2\eta_{\text{max}} \mu}$, there exists a constant step-size $\eta \leq \eta_{\text{max}}$ and weights $w_r := (1 - \mu \eta)^{1-r}$ such that for $W_R := \sum_{r=1}^{R+1} w_r$,

$$
\Psi_R := \frac{1}{W_R} \sum_{r=1}^{R+1} \left( \frac{w_r}{\eta} (1 - \mu \eta) d_{r-1} - \frac{w_r}{\eta} d_r + c \eta w_r \right) = \tilde{O} \left( \mu d_0 \exp(-\mu \eta_{\text{max}} R) + \frac{c}{\mu R} \right).
$$

**Proof** By substituting the value of $w_r$, we observe that we end up with a telescoping sum and estimate

$$
\Psi_R = \frac{1}{\eta W_R} \sum_{r=1}^{R+1} \left( w_{r-1} d_r - w_r d_{r-1} + \frac{c \eta}{W_R} w_r \right) \leq \frac{d_0}{\eta W_R} + c \eta.
$$

When $R \geq \frac{1}{2\eta \mu}$, $(1 - \mu \eta)^R \leq \exp(-\mu \eta R) \leq \frac{2}{3}$. For such an $R$, we can lower bound $\eta W_R$ using

$$
\eta W_R = \eta (1 - \mu \eta)^{-R} \sum_{r=0}^{R} (1 - \mu \eta)^r = \eta (1 - \mu \eta)^{-R} \frac{1 - (1 - \mu \eta)^R}{\mu \eta} \geq (1 - \mu \eta)^{-R} \frac{1}{3\mu}.
$$
This proves that for all \( R \geq \frac{1}{2\mu R} \),

\[
\Psi_R \leq 3\mu d_0 (1 - \mu \eta)^R + c\eta \leq 3\mu d_0 \exp(-\mu \eta R) + c\eta.
\]

The lemma now follows by carefully tuning \( \eta \). Consider the following two cases depending on the magnitude of \( R \) and \( \eta_{\text{max}} \):

- Suppose \( \frac{1}{2\mu R} \leq \eta_{\text{max}} \leq \frac{\log(\max(1, \mu^2 Rd_0/c))}{\mu R} \). Then we can choose \( \eta = \eta_{\text{max}} \),

\[
\Psi_R \leq 3\mu d_0 \exp[-\mu \eta_{\text{max}} R] + c\eta_{\text{max}} \leq 3\mu d_0 \exp[-\mu \eta_{\text{max}} R] + \tilde{O} \left( \frac{c}{\mu R} \right).
\]

- Instead if \( \eta_{\text{max}} > \frac{\log(\max(1, \mu^2 Rd_0/c))}{\mu R} \), we pick \( \eta = \frac{\log(\max(1, \mu^2 Rd_0/c))}{\mu R} \) to claim that

\[
\Psi_R \leq 3\mu d_0 \exp[-\log(\max(1, \mu^2 Rd_0/c))] + \tilde{O} \left( \frac{c}{\mu R} \right) \leq \tilde{O} \left( \frac{c}{\mu R} \right).
\]

Next, we state a relaxed triangle inequality true for the squared \( \ell_2 \) norm.

**Lemma 6 (relaxed triangle inequality)** Let \( \{v_1, \ldots, v_\tau\} \) be \( \tau \) vectors in \( \mathbb{R}^d \). Then the following are true:

1. \( \|v_i + v_j\|^2 \leq (1 + a)\|v_i\|^2 + (1 + \frac{1}{a})\|v_j\|^2 \) for any \( a > 0 \), and
2. \( \|\sum_{i=1}^\tau v_i\|^2 \leq \tau \sum_{i=1}^\tau \|v_i\|^2 \).

**Proof** The proof of the first statement for any \( a > 0 \) follows from the identity:

\[
\|v_i + v_j\|^2 = (1 + a)\|v_i\|^2 + (1 + \frac{1}{a})\|v_j\|^2 - \|\sqrt{a}v_i + \frac{1}{\sqrt{a}}v_j\|^2.
\]

For the second inequality, we use the convexity of \( x \to \|x\|^2 \) and Jensen’s inequality

\[
\left\| \frac{1}{\tau} \sum_{i=1}^\tau v_i \right\|^2 \leq \frac{1}{\tau} \sum_{i=1}^\tau \|v_i\|^2.
\]

Next we state an elementary lemma about expectations of norms of random vectors.

**Lemma 7 (separating mean and variance)** Let \( \{\Xi_1, \ldots, \Xi_\tau\} \) be \( \tau \) random variables in \( \mathbb{R}^d \) not necessarily independent such that their mean is \( \mathbb{E}[\Xi_i] = \xi_i \) and variance is bounded as \( \mathbb{E}[\|\Xi_i - \xi_i\|^2] \leq \sigma^2 \). Then, the following holds

\[
\mathbb{E}[\|\sum_{i=1}^\tau \Xi_i\|^2] \leq \|\sum_{i=1}^\tau \xi_i\|^2 + \tau^2 \sigma^2.
\]

If further, the variables \( \{\Xi_i - \xi_i\} \) form a martingale difference sequence we can improve the bound to

\[
\mathbb{E}[\|\sum_{i=1}^\tau \Xi_i\|^2] \leq \|\sum_{i=1}^\tau \xi_i\|^2 + \tau \sigma^2.
\]
Proof} For any random variable \( X \), \( \mathbb{E}[X^2] = (\mathbb{E}[X] - \mathbb{E}[X])^2 + (\mathbb{E}[X])^2 \) implying
\[
\mathbb{E}[\| \sum_{i=1}^{\tau} \Xi_i \|^2] = \| \sum_{i=1}^{\tau} \xi_i \|^2 + \mathbb{E}[\| \sum_{i=1}^{\tau} \Xi_i - \xi_i \|^2].
\]
Expanding the above expression using relaxed triangle inequality (Lemma 6) proves the first claim:
\[
\mathbb{E}[\| \sum_{i=1}^{\tau} \Xi_i - \xi_i \|^2] \leq \tau \sum_{i=1}^{\tau} \mathbb{E}[\| \Xi_i - \xi_i \|^2] \leq \tau^2 \sigma^2.
\]
For the second statement, we use the tighter expansion:
\[
\mathbb{E}[\| \sum_{i=1}^{\tau} \Xi_i - \xi_i \|^2] = \sum_{i,j} \mathbb{E}[\| \Xi_i - \xi_i \| \cdot \| \Xi_j - \xi_j \|] = \sum_{i} \mathbb{E}[\| \Xi_i - \xi_i \|^2] \leq \tau \sigma^2.
\]
The cross terms in the above expression have zero mean since \( \{ \Xi_i - \xi_i \} \) form a martingale difference sequence.

Appendix C. Properties of convex functions

We now study two lemmas which hold for any smooth and strongly-convex functions. The first is a generalization of the standard strong convexity inequality (3), but can handle gradients computed at slightly perturbed points.

**Lemma 8 (perturbed strong convexity)** The following holds for any \( \beta \)-smooth and \( \mu \)-strongly convex function \( h \), and any \( x, y, z \) in the domain of \( h \):
\[
\langle \nabla h(x), z - y \rangle \geq h(z) - h(y) + \frac{\mu}{4} \| y - z \|^2 - \beta \| z - x \|^2.
\]

**Proof** Given any \( x, y, \) and \( z \), we get the following two inequalities using smoothness and strong convexity of \( h \):
\[
\langle \nabla h(x), z - x \rangle \geq h(z) - h(x) - \frac{\beta}{2} \| z - x \|^2,
\]
\[
\langle \nabla h(x), x - y \rangle \geq h(x) - h(y) + \frac{\mu}{2} \| y - x \|^2.
\]
Further, applying the relaxed triangle inequality gives
\[
\frac{\mu}{2} \| y - x \|^2 \geq \frac{\mu}{4} \| y - z \|^2 - \frac{\mu}{2} \| x - z \|^2.
\]
Combining all the inequalities together we have
\[
\langle \nabla h(x), z - y \rangle \geq h(z) - h(y) + \frac{\mu}{4} \| y - z \|^2 - \frac{\beta + \mu}{2} \| z - x \|^2.
\]
The lemma follows since \( \beta \geq \mu \).

Here, we see that a gradient step is a contractive operator.
Lemma 9 (contractive mapping) For any $\beta$-smooth and $\mu$-strongly convex function $h$, points $x, y$ in the domain of $h$, and step-size $\eta \leq \frac{1}{\beta}$, the following is true
\[ \|x - \eta \nabla h(x) - y + \eta \nabla h(y)\| \leq (1 - \mu \eta) \|x - y\|^2. \]

Proof
\[
\|x - \eta \nabla h(x) - y + \eta \nabla h(y)\|^2 = \|x - y\|^2 + \eta^2 \|\nabla h(x) - \nabla h(y)\|^2 - 2\eta \langle \nabla h(x) - \nabla h(y), x - y \rangle
\]
\[
\leq \|x - y\|^2 + \eta^2 \|\nabla h(x) - \nabla h(y)\|^2 - 2\eta \langle \nabla h(x) - \nabla h(y), x - y \rangle
\]
\[
\leq \|x - y\|^2 + (\eta^2 - 2\eta) \langle \nabla h(x) - \nabla h(y), x - y \rangle.
\]
Recall our bound on the step-size $\eta \leq \frac{1}{\beta}$ which implies that $(\eta^2 - 2\eta) \leq -\eta$. By the $\mu$-strong convexity of $h$,
\[
\langle \nabla h(x) - \nabla h(y), x - y \rangle \geq \mu \|x - y\|^2.
\]

Appendix D. Convergence of SCAFFOLD (without sampling)

D.1 Properties of SCAFFOLD (Proof of (P2))

Recall that (P2) claims
\[
\frac{1}{N} \sum_{i=1}^{N} \| (y_i - \eta_t v_i) - x \|^2 \leq \frac{1}{N} \sum_{i=1}^{N} \| y_i - x \|^2 + \eta_t^2 (1 + 1/a) \beta (f(x) - f(x^*)) + 5\eta_t^2 \sigma^2.
\]

Using the definition of $v_i$ and the fact that $c_i = g_i(x)$ (an independent stochastic gradient computed at $x$), we can expand as
\[
\frac{1}{N} \sum_{i=1}^{N} \| (y_i - \eta_t v_i) - x \|^2 = \frac{1}{N} \sum_{i=1}^{N} \| (y_i - \eta_t g_i(y_i) + \eta_t g_i(x) - x - \eta_t g(x)) \|^2.
\]

Taking expectations on both sides and separating the mean and variance using Lemma 7 gives
\[
\frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \| (y_i - \eta_t v_i) - x \|^2 \leq \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \| y_i - \eta_t \nabla f_i(y_i) + \eta_t g_i(x) - x - \eta_t g(x) \|^2 + \sigma^2
\]
\[
\leq \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \| y_i - \eta_t \nabla f_i(y_i) + \eta_t \nabla f_i(x) - x - \eta_t \nabla f(x) \|^2 + 5\sigma^2
\]
\[
\leq \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \| y_i - \eta_t \nabla f_i(y_i) + \eta_t \nabla f_i(x) - x \|^2
\]
\[
+ (1 + 1/a) \eta_t^2 \| \nabla f(x) \|^2 + 5\sigma^2.
\]
The third step used the relaxed triangle inequality (Lemma 6) and holds for any $a > 0$. Now, we use the smoothness and convexity of $f_i$ and Lemma 9 to claim that

$$\mathbb{E}\|y_i - \eta_l \nabla f_i(y_i) + \eta_l \nabla f_i(x) - x\|^2 \leq \mathbb{E}\|y_i - x\|^2,$$

and the smoothness of $f$ to bound

$$\|\nabla f(x)\|^2 \leq \beta(f(x) - f(x^*)) .$$

Putting the above inequalities together yields the proof.

D.2 Progress between communication rounds (Proof of Lemma 1)

**Lemma 1 (one round progress)** Suppose our updates satisfy (P1) and assumptions A1–A3. For any step-size satisfying $\eta_l \leq \frac{1}{2\beta K_\eta}$ and effective step-size $\tilde{\eta}_l := K_\eta \eta_l$,

$$\mathbb{E}\|x^r - x^*\|^2 \leq (1 - \frac{\mu^2}{2}) \mathbb{E}\|x^{r-1} - x^*\|^2 + \frac{\tilde{\eta}_l^2 \sigma^2}{KN} - \tilde{\eta}_l (\mathbb{E}[f(x^{r-1})] - f(x^*)) + 3\beta \tilde{\eta} \delta_r ,$$

where $\delta_r$ is the drift caused by the local updates on the clients

$$\delta_r := \frac{1}{KN} \sum_{k=1}^K \sum_{i=1}^N \mathbb{E} \left[ \|y^r_{i,k} - x^{r-1}\|^2 \right] .$$

**Proof** We start with (P1) which states that

$$\frac{1}{N} \sum_{i=1}^N (y_i - \eta_l v_i) = \frac{1}{N} \sum_{i=1}^N (y_i - \eta_l (g_i(y_i) - c_i) = \frac{1}{N} \sum_{i=1}^N (y_i - \eta_l g_i(y_i)) .$$

This implies that the server update can be written as

$$\Delta x = - \frac{K_\eta \eta_l}{KN} \sum_{k,i} g_i(y_{i,k-1}) = - \frac{\tilde{\eta}_l}{KN} \sum_{k,i} g_i(y_{i,k-1}) ,$$

where we dropped the superscript $r$ everywhere, set $\tilde{\eta}_l = K_\eta \eta_l$, and the sum over $k$ runs from 1 through $K$ and over $i$ from 1 through $N$. We can then expand

$$\mathbb{E}\|x + \Delta x - x^*\|^2 = \mathbb{E}\|x - x^*\|^2 - \frac{2\tilde{\eta}_l}{KN} \mathbb{E} \sum_{k,i} \langle \nabla f_i(y_{i,k-1}), x - x^* \rangle + \frac{\tilde{\eta}_l^2}{N^2 K^2} \mathbb{E} \sum_{k,i} g_i(y_{i,k-1})^2$$

$$\leq \frac{2\tilde{\eta}_l}{KN} \mathbb{E} \sum_{k,i} \langle \nabla f_i(y_{i,k-1}), x^* - x \rangle + \frac{\tilde{\eta}_l^2}{N^2 K^2} \mathbb{E} \sum_{k,i} \langle \nabla f_i(y_{i,k-1}), x^* - x \rangle$$

$$+ \mathbb{E}\|x - x^*\|^2 + \frac{\tilde{\eta}_l^2 \sigma^2}{KN} .$$
We use Lemma 7 to separate the variance and the mean of the stochastic gradients. Expanding the second term $T_2$ using the relaxed triangle inequality with $a = 1$,

$$T_2 = \tilde{\eta}^2 \left\| \frac{1}{KN} \sum_{k,i} (\nabla f_i(y_{i,k-1}) - \nabla f_i(x)) + \nabla f(x) \right\|^2 \leq 2\tilde{\eta}^2 \|\nabla f(x)\|^2 + 2\tilde{\eta}^2 \left\| \frac{1}{KN} \sum_{k,i} \nabla f_i(y_{i,k-1}) - \nabla f_i(x) \right\|^2 \leq 2\tilde{\eta}^2 \|\nabla f(x)\|^2 + \frac{2\eta^2}{KN} \sum_{k,i} \|\nabla f_i(y_{i,k-1}) - \nabla f_i(x)\|^2 \leq 2\tilde{\eta}^2 \|\nabla f(x)\|^2 + \frac{2\beta^2 \tilde{\eta}^2}{KN} \sum_{k,i} \|y_{i,k-1} - x\|^2 \leq 2\beta \tilde{\eta}^2 (f(x) - f(x^*)) + 2\beta^2 \eta^2 \delta .$$

The inequality before the last used that the gradient of $f_i$ is $\beta$-Lipschitz, and the last inequality follows from the smoothness of $f$. We next turn our attention to the first term $T_1$. We can directly apply Lemma 8 with $h = f_i$, $x = y_{i,k-1}$, $y = x^*$, and $z = x$ to get

$$T_1 = \frac{2\tilde{\eta}}{KN} \sum_{k,i} (\nabla f_i(y_{i,k-1}), x^* - x) \leq \frac{2\tilde{\eta}}{KN} \sum_{k,i} \left( f_i(x^*) - f_i(x) + \beta \|y_{i,k-1} - x\|^2 \frac{\mu}{4} \|x - x^*\|^2 \right) = -2\tilde{\eta} \left( f(x) - f(x^*) + \frac{\mu}{4} \|x - x^*\|^2 \right) + 2\beta \tilde{\eta} \delta .$$

Combining the bounds for $T_1$ and $T_2$,

$$\mathbb{E}\|x + \Delta x - x^*\|^2 \leq \mathbb{E}\|x - x^*\|^2 + \frac{\tilde{\eta}^2 \sigma^2}{KN} - 2\tilde{\eta} \left( f(x) - f(x^*) + \frac{\mu}{4} \|x - x^*\|^2 \right) + 2\beta \tilde{\eta} \delta + 2\beta^2 \tilde{\eta}^2 (f(x) - f(x^*)) + 2\beta^2 \tilde{\eta}^2 \delta \leq (1 - \frac{\mu}{2}) \|x - x^*\|^2 + \frac{\tilde{\eta}^2 \sigma^2}{KN} + (2\beta \tilde{\eta}^2 - 2\tilde{\eta}) (f(x) - f(x^*)) + (2\beta \tilde{\eta} + 2\beta^2 \tilde{\eta}^2) \delta .$$

The claim now follows from observing that $\tilde{\eta} = K \eta_\beta \eta \leq \frac{1}{2\beta}$ implies that $2\beta \tilde{\eta}^2 \leq \tilde{\eta}$. □

D.3 Bounding the drift (Proof of Lemma 2)

**Lemma 2 (bounded drift)** Suppose our updates satisfy (P2) and assumptions A1–A3. For any step-size satisfying $\eta_i \leq \frac{1}{8\beta K \eta_\beta}$, then we can bound the drift as

$$3\beta \delta_r \leq \frac{1}{2\eta_\beta^2} (\mathbb{E}[f(x^{r-1})] - f(x^*)) + \frac{2\tilde{\eta} \sigma^2}{K \eta_\beta^2} .$$
Proof First note that if \( K = 1 \), \( \delta_r = 0 \) since \( y_{i,0} = x \) for all \( i \in [N] \). Assuming \( K > 1 \), recalling property (P2) which bounds the increase in the drift at the local step \( k \) with \( a = 1/(K-1) \)

\[
\frac{1}{N} \sum_{i=1}^{N} \| y_{i,k} - x \|^2 \leq \frac{(1 + \frac{1}{K-1})}{N} \sum_{i=1}^{N} \| y_{i,k-1} - x \|^2 + K \eta_k^2 \beta (f(x) - f(x^*)) + 5 \eta_k^2 \sigma^2.
\]

Since we know that \( y_{i,0} = x \), we just have to unroll the above recursion over \( k \) to get

\[
\frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \| y_{i,k} - x \|^2 \leq (K \eta_k^2 (f(x) - f(x^*)) + 5 \eta_k^2 \sigma^2)(\sum_{j=0}^{k-1} (1 + \frac{1}{K-1})^j) \\
\leq (K \eta_k^2 (f(x) - f(x^*)) + 5 \eta_k^2 \sigma^2)(K - 1)((1 + \frac{1}{K-1})^K - 1) \\
\leq 7 \beta^2 K^2 \eta_k^2 (f(x) - f(x^*)) + 35 \beta K \eta_k^2 \sigma^2.
\]

Again averaging over \( k \) and multiplying by \( 3 \beta \) gives

\[
3 \beta \delta_r \leq 21 \beta^2 K^2 \eta_k^2 (f(x) - f(x^*)) + 15 \beta \eta_k^2 K \sigma^2.
\]

Note that our upper bound on the step-size \( \eta_l \leq \frac{1}{8 \bar{\eta} \eta_g K} \) implies that \( 3 \beta^2 K^2 \eta_g^2 \eta_l^2 \leq \frac{1}{2} \) and that \( 15 \beta \eta K \eta_g \leq 2 \). This completes the the proof of the lemma statement

\[
3 \beta \delta_r \leq \frac{1}{\eta_g^2} (f(x^{r-1})) - f(x^*) + \frac{2 \eta \sigma^2}{K \eta_g^2}
\]

where we additionally used \( \bar{\eta} = K \eta_l \eta_g \).

Appendix E. Convergence of SCAFFOLD (with sampling)

E.1 Properties of SCAFFOLD updates

Proof of (P4). Recall that (P4) claims for any control variates \( \{c_i\} \), the local update of SCAFFOLD in local step \( k \) or rounds \( r \) the following holds for any \( a > 0 \)

\[
\frac{1}{S} \mathbb{E}_r \left[ \sum_{i \in S} \| (y_i - \eta_l v_i) - x \|^2 \right] \leq \frac{(1 + a)}{S} \mathbb{E}_r \left[ \sum_{i \in S} \| y_i - x \|^2 \right] + \eta_l^2 \sigma^2 \\
+ 3 \eta_l^2 (1 + 1/a) \beta (f(x) - f(x^*)) + \eta_l^2 (1 + \frac{1}{a}) \frac{6}{N} \sum_{j=1}^{N} \| c_i - \nabla f_i(x^*) \|^2.
\]
Starting from the definition of the update (9) and then applying the relaxed triangle inequality, we can expand

$$
\frac{1}{S} \mathbb{E}_r \left[ \sum_{i \in S} \| (y_i - \eta v_i) - x \|^2 \right] = \frac{1}{S} \mathbb{E}_r \left[ \sum_{i \in S} \| y_i - \eta g_i(y_i) + \eta c - \eta c_i - x \|^2 \right] 
\leq \frac{1}{S} \mathbb{E}_r \left[ \sum_{i \in S} \| y_i - \eta \nabla f_i(y_i) + \eta c - \eta c_i - x \|^2 \right] + \eta^2 \sigma^2 
\leq \frac{(1 + a)}{S} \mathbb{E}_r \left[ \sum_{i \in S} \| y_i - \eta \nabla f_i(y_i) + \eta \nabla f_i(x) - x \|^2 \right]
\leq \frac{(1 + \frac{a}{\beta}) \eta^2}{S} \mathbb{E}_r \left[ \frac{1}{S} \sum_{i \in S} \| c - c_i + \nabla f_i(x) \|^2 \right] + \eta^2 \sigma^2.
$$

The final step follows from the relaxed triangle inequality (Lemma 6). Applying the contractive mapping Lemma 9 for $\eta_i \leq 1/\beta$ shows

$$
\mathcal{T}_3 = \frac{1}{S} \sum_{i \in S} \| y_i - \eta \nabla f_i(y_i) + \eta \nabla f_i(x) - x \|^2 \leq \| y_i - x \|^2.
$$

Once again using our relaxed triangle inequality to expand the other term $\mathcal{T}_4$, we get

$$
\mathcal{T}_4 = \mathbb{E}_r \left[ \frac{1}{S} \sum_{i \in S} \| c - c_i + \nabla f_i(x) \|^2 \right]
= \frac{1}{N} \sum_{j=1}^N \| c - c_i + \nabla f_i(x) \|^2
= \frac{1}{N} \sum_{j=1}^N \| c - c_i + \nabla f_i(x^*) + \nabla f_i(x) - \nabla f_i(x^*) \|^2
\leq 3 ||c||^2 + \frac{3}{N} \sum_{j=1}^N ||c_i - \nabla f_i(x^*)||^2 + \frac{3}{N} \sum_{j=1}^N ||\nabla f_i(x) - \nabla f_i(x^*)||^2
\leq \frac{6}{N} \sum_{j=1}^N ||c_i - \nabla f_i(x^*)||^2 + \frac{3}{N} \sum_{j=1}^N ||\nabla f_i(x) - \nabla f_i(x^*)||^2
\leq \frac{6}{N} \sum_{j=1}^N ||c_i - \nabla f_i(x^*)||^2 + 3\beta(f(x) - f(x^*)).
$$

The last step used the smoothness of $f_i$. Combining the bounds on $\mathcal{T}_3$ and $\mathcal{T}_4$ completes the proof.

**Variance of server update.** We next observe how the variance of the server update can be bounded.
Lemma 10  For updates (9)—(12), we can bound the variance of the server update in any round $r$ and any $\bar{\eta} := \eta \eta y K \geq 0$ as follows

\[
\mathbb{E}[\| \Delta x^r \|^2] \leq 4 \beta \bar{\eta}^2 (\mathbb{E}[f(x^{r-1})] - f(x^*)) + 8 \bar{\eta}^2 C_{r-1} + 4 \bar{\eta}^2 \beta^2 \delta_r + \frac{12 \bar{\eta}^2 \sigma^2}{K S}.
\]

**Proof** Recall that the server update in round $r$ can be written as follows (dropping the superscript $r$ everywhere)

\[
\mathbb{E}[\| \Delta x \|^2] = \mathbb{E}\left[ \left\| -\frac{\bar{\eta}}{K S} \sum_{k,i \in S} v_{i,k} \right\|^2 \right] = \mathbb{E}\left[ \left\| \frac{\bar{\eta}}{K S} \sum_{k,i \in S} (g_i(y_{i,k-1}) + c - c_i) \right\|^2 \right],
\]

which can then be expanded as

\[
\mathbb{E}[\| \Delta x \|^2] \leq \mathbb{E}\left[ \left\| \frac{\bar{\eta}}{K S} \sum_{k,i \in S} (g_i(y_{i,k-1}) + c - c_i) \right\|^2 \right]
\]

\[
\leq 4 \mathbb{E}\left[ \left\| \frac{\bar{\eta}}{K S} \sum_{k,i \in S} g_i(y_{i,k-1}) - \nabla f_i(x) \right\|^2 + 4 \bar{\eta}^2 \mathbb{E}\| c \|^2 \right] + 4 \mathbb{E}\left[ \left\| \frac{\bar{\eta}}{K S} \sum_{k,i \in S} \nabla f_i(x^*) - c_i \right\|^2 \right]
\]

\[
+ 4 \mathbb{E}\left[ \left\| \frac{\bar{\eta}}{K S} \sum_{k,i \in S} \nabla f_i(x) - \nabla f_i(x^*) \right\|^2 \right]
\]

\[
\leq 4 \mathbb{E}\left[ \left\| \frac{\bar{\eta}}{K S} \sum_{k,i \in S} g_i(y_{i,k-1}) - \nabla f_i(x) \right\|^2 + 4 \bar{\eta}^2 \mathbb{E}\| c \|^2 \right] + 4 \mathbb{E}\left[ \left\| \frac{\bar{\eta}}{S} \sum_{i \in S} \nabla f_i(x^*) \right\|^2 \right]
\]

\[
+ 4 \beta \bar{\eta}^2 (\mathbb{E}[f(x)] - f(x^*)) + \frac{12 \bar{\eta}^2 \sigma^2}{K S}.
\]

The inequality before the last used the smoothness of $\{f_i\}$. The last inequality which separates the mean and the variance is an application of Lemma 7: the variance of $(\frac{1}{K S} \sum_{k,i \in S} g_i(y_{i,k-1}))$ is bounded by $\sigma^2/K S$. Similarly, $c_j$ as defined in (10) for any $j \in [N]$ has variance smaller than $\sigma^2/K$ and hence the variance of $(\frac{1}{S} \sum_{i \in S} c_i)$ is smaller than $\sigma^2/K S$. Then, using $c = \frac{1}{N} \sum_i c_i$ and Jensen’s inequality, we can simplify the expression as

\[
\mathbb{E}[\| \Delta x \|^2] \leq \frac{4 \bar{\eta}^2}{KN} \sum_{k,i} \mathbb{E}[\| \nabla f_i(y_{i,k-1}) - \nabla f_i(x) \|^2 + 4 \bar{\eta}^2 \mathbb{E}[c_i]^2 + \frac{4 \bar{\eta}^2}{N} \sum_i \| \nabla f_i(x^*) - \mathbb{E}[c_i] \|^2
\]

\[
+ 4 \beta \bar{\eta}^2 (\mathbb{E}[f(x)] - f(x^*)) + \frac{12 \bar{\eta}^2 \sigma^2}{K S}
\]

\[
\leq \frac{4 \bar{\eta}^2}{KN} \sum_{k,i} \mathbb{E}[\| \nabla f_i(y_{i,k-1}) - \nabla f_i(x) \|^2 + \frac{8 \bar{\eta}^2}{N} \sum_i \| \nabla f_i(x^*) - \mathbb{E}[c_i] \|^2
\]

\[
+ 4 \beta \bar{\eta}^2 (\mathbb{E}[f(x)] - f(x^*)) + \frac{12 \bar{\eta}^2 \sigma^2}{K S}.
\]
Since the gradient of \( f_i \) is \( \beta \)-Lipschitz, \( T_5 \leq \frac{\beta^2 K^2}{N} \sum_{k,i} \mathbb{E} \| y_{i,k-1} - x \|^2 = 4\eta^2 \beta^2 \delta. \) The definition of the error in the control variate \( C_{r-1} := \frac{1}{N} \sum_{j=1}^{N} \mathbb{E} \| [c_i] - \nabla f_i(x^*) \|^2 \) completes the proof.

**Change in control error.** We have previously related the variance of the server update to the error in the control variate updates. We now examine how this control error grows each round.

**Lemma 11** For updates (9)—(12) with the control update (10), the following holds true for any \( \tilde{\eta} := \eta \eta_0 K \in [0,1/\beta] \):

\[
C_r \leq (1 - \frac{S}{N}) C_{r-1} + \frac{S}{N} (2\beta(\mathbb{E}[f(x^{r-1})] - f(x^*)) + 2\beta^2 \delta_r).
\]

**Proof** Recall that after round \( r \), the control update rule (10) implies that \( c_i^r \) is set as per

\[
c_i^r = \begin{cases} 
  c_i^{r-1} & \text{if } i \notin S^r \text{ i.e. with probability } (1 - \frac{S}{N}), \\
  \frac{1}{K} \sum_{k=1}^{K} g_i(y_{i,k-1}^{r}) & \text{with probability } \frac{S}{N}.
\end{cases}
\]

Taking expectations on both sides yields

\[
\mathbb{E}[c_i^r] = (1 - \frac{S}{N}) \mathbb{E}[c_i^{r-1}] + \frac{S}{N} \sum_{k=1}^{K} \mathbb{E}[\nabla f_i(y_{i,k-1}^r)], \ \forall \ i \in [N].
\]

Plugging the above expression in the definition of \( C_r \) we get

\[
C_r = \frac{1}{N} \sum_{i=1}^{N} \|\mathbb{E}[c_i^r] - \nabla f_i(x^*)\|^2 \\
= \frac{1}{N} \sum_{i=1}^{N} \| (1 - \frac{S}{N}) (\mathbb{E}[c_i^{r-1}] - \nabla f_i(x^*)) + \frac{S}{N} (\frac{1}{K} \sum_{k=1}^{K} \mathbb{E}[\nabla f_i(y_{i,k-1}^r)]) - \nabla f_i(x^*) \|^2 \\
\leq (1 - \frac{S}{N}) C_{r-1} + \frac{S}{N} \sum_{k=1}^{K} \mathbb{E}[\nabla f_i(y_{i,k-1}^r)] - \nabla f_i(x^*)\|^2.
\]

The final step is an application of Jensen’s inequality. We can then further simplify using the relaxed triangle inequality as

\[
\mathbb{E}_r[C_r] \leq \left(1 - \frac{S}{N}\right) C_{r-1} + \frac{S}{N^2 K} \sum_{i,k} \mathbb{E}[\nabla f_i(y_{i,k-1}^r) - \nabla f_i(x^*)]^2 \\
\leq \left(1 - \frac{S}{N}\right) C_{r-1} + \frac{2S}{N^2} \sum_{i} \mathbb{E}[\nabla f_i(x^{r-1}) - \nabla f_i(x^*)]^2 + \frac{2S}{N^2 K} \sum_{i,k} \mathbb{E}[\nabla f_i(y_{i,k-1}^r) - \nabla f_i(x^{r-1})]^2 \\
\leq \left(1 - \frac{S}{N}\right) C_{r-1} + \frac{2S}{N^2} \sum_{i} \mathbb{E}[\nabla f_i(x^{r-1}) - \nabla f_i(x^*)]^2 + \frac{2S}{N^2 K} \beta^2 \sum_{i,k} \mathbb{E}[y_{i,k-1}^r - x^{r-1}]^2 \\
\leq \left(1 - \frac{S}{N}\right) C_{r-1} + \frac{S}{N} (2\beta(\mathbb{E}[f(x^{r-1})] - f(x^*)) + \beta^2 \delta_r).
\]

The last two inequalities follow from smoothness of \{\( f_i \)\} and the definition \( \delta_r = \frac{1}{N} \beta^2 \sum_{i,k} \mathbb{E}[y_{i,k-1}^r - x^{r-1}]^2. \)
E.2 Progress between communication rounds (Proof of Lemma 3)

Lemma 3 (one round progress) Suppose our updates satisfy (P1) and assumptions A1–A3. Then the following holds for any step-size satisfying \( \eta \leq \min\left(\frac{1}{8\eta_\text{eff}}, \frac{S}{10\mu N K \eta_\delta}\right) \), effective step-size \( \tilde{\eta} := K \eta \eta_\delta \), and control variates updated using (10),

\[
\mathbb{E}\left[\|x^r - x^*\|^2 + \frac{9N\tilde{\eta}^2}{S} C_r\right] \leq \left(1 - \frac{\tilde{\eta}^2}{2}\right) \left(\mathbb{E}\|x^{r-1} - x^*\|^2 + \frac{9N\tilde{\eta}^2}{S} C_{r-1}\right) + \frac{\tilde{\eta}^2 \sigma^2}{K S} - \frac{2\tilde{\eta}}{S} \mathbb{E}[f(x^{r-1}) - f(x^*)] + 3\beta \tilde{\eta} \delta_r - \frac{9\tilde{\eta}^2}{4} C_{r-1},
\]

where \( C_r \) is the error in our control variate defined as

\[
C_r := \frac{1}{N} \sum_{j=1}^{N} \|\mathbb{E}[c_j^r] - \nabla f_i(x^*)\|^2,
\]

and \( \delta_r \) is the drift caused by the local updates on the clients

\[
\delta_r := \frac{1}{K N} \sum_{k=1}^{K} \sum_{i=1}^{N} \mathbb{E}\|y_{i,k}^r - x^{r-1}\|^2.
\]

Proof We will follow nearly exactly the same steps as in the proof of Lemma 1, except taking care that we do not have samples from all the machines. Starting from observation (P3), the server update can be written as

\[
\Delta x = -\frac{\tilde{\eta}}{K S} \sum_{k,i \in S} (g_i(y_{i,k-1}) + c - c_i), \quad \text{and} \quad \mathbb{E}_S[\Delta x] = -\frac{\tilde{\eta}}{K N} \sum_{k,i} g_i(y_{i,k-1}).
\]

We can then apply Lemma 10 to bound the second moment of the server update as

\[
\mathbb{E}_r \|x + \Delta x - x^*\|^2 = \mathbb{E}_r \|x - x^*\|^2 - \frac{2\tilde{\eta}}{K S} \mathbb{E}_r \sum_{k,i \in S} \langle \nabla f_i(y_{i,k-1}), x - x^* \rangle + \mathbb{E}_r \|\Delta x\|^2
\]

\[
\leq \frac{2\tilde{\eta}}{K S} \mathbb{E}_r \sum_{k,i \in S} \langle \nabla f_i(y_{i,k-1}), x^* - x \rangle + \mathbb{E}_r \|x - x^*\|^2 - 6\tilde{\eta} \delta_r - \frac{9\tilde{\eta}^2}{4} C_{r-1} + 3\beta \tilde{\eta} \delta_r - \frac{9\tilde{\eta}^2}{4} C_{r-1},
\]

The term \( \mathcal{T}_6 \) can be bounded by using Lemma 8 with \( h = f_i, x = y_{i,k-1}, y = x^* \), and \( z = x \) to get

\[
\mathbb{E}[\mathcal{T}_6] = \frac{2\tilde{\eta}}{K S} \mathbb{E}_r \sum_{k,i \in S} \langle \nabla f_i(y_{i,k-1}), x^* - x \rangle
\]

\[
\leq \frac{2\tilde{\eta}}{K S} \mathbb{E}_r \sum_{k,i \in S} \left( f_i(x^*) - f_i(x) + \beta \|y_{i,k-1} - x\|^2 - \frac{\mu}{4} \|x - x^*\|^2 \right)
\]

\[
= -2\tilde{\eta} \mathbb{E} \left( f(x) - f(x^*) + \frac{\mu}{4} \|x - x^*\|^2 \right) + 2 \beta \tilde{\eta} \delta.
\]
Plugging $T_0$ back, we can further simplify the expression to get

$$
\mathbb{E}\|x + \Delta x - x^*\|^2 \leq \mathbb{E}\|x - x^*\|^2 - 2\tilde{\eta}\left(f(x) - f(x^*) + \frac{\mu}{4}\|x - x^*\|^2\right) + 2\beta\tilde{\eta}\delta
$$

$$
+ \frac{12\tilde{\eta}^2\sigma^2}{KS} + 4\beta\tilde{\eta}^2(\mathbb{E}[f(x^r-1)] - f(x^*)) + 8\tilde{\eta}^2C_{r-1} + 4\tilde{\eta}^2\delta
$$

$$
= (1 - \frac{\mu\tilde{\eta}}{2})\|x - x^*\|^2 + (4\beta\tilde{\eta}^2 - 2\tilde{\eta})(f(x) - f(x^*))
$$

$$
+ \frac{12\tilde{\eta}^2\sigma^2}{KS} + (2\beta\tilde{\eta} + 4\beta^2\tilde{\eta}^2)\delta + 8\tilde{\eta}^2C_{r-1}.
$$

The final term which depends on $C_{r-1}$ is new and was not present in the proof of the case without sampling. We can use Lemma 11 to bound this term. Multiplying by $9\tilde{\eta}^2 \frac{N}{S}$,

$$
9\tilde{\eta}^2 \frac{N}{S}C_r \leq (1 - \frac{\mu\tilde{\eta}}{2})9\tilde{\eta}^2 \frac{N}{S}C_{r-1} + 9(\frac{9\tilde{\eta}N}{2S} - 1)\tilde{\eta}^2C_{r-1} + 9\tilde{\eta}^2(2\beta(\mathbb{E}[f(x^r-1)] - f(x^*)) + 2\beta^2\delta)
$$

Together, this proves that

$$
\mathbb{E}\|x + \Delta x - x^*\|^2 + \frac{9\tilde{\eta}^2NC_r}{S} \leq (1 - \frac{\mu\tilde{\eta}}{2})\left(\mathbb{E}\|x - x^*\|^2 + \frac{9\tilde{\eta}^2NC_{r-1}}{S}\right) + (22\beta\tilde{\eta}^2 - 2\tilde{\eta})(f(x) - f(x^*))
$$

$$
+ \frac{12\tilde{\eta}^2\sigma^2}{KS} + (2\beta\tilde{\eta} + 22\beta^2\tilde{\eta}^2)\delta + \frac{9\mu\tilde{\eta}N}{2S} - 1)\tilde{\eta}^2C_{r-1}
$$

Finally, the lemma follows from noting that $\tilde{\eta} \leq \frac{1}{8\mu\beta}$ implies $22\beta^2\tilde{\eta}^2 \leq \frac{24}{25}\beta$ and $\tilde{\eta} \leq \frac{S}{10\mu N}$ implies $\frac{9\mu\tilde{\eta}N}{2S} \leq \frac{1}{3}$.

**E.3 Bounding the drift (Proof of Lemma 4)**

**Lemma 4 (bounded drift)** Suppose our step-sizes satisfy $\eta_i \leq \frac{1}{8\mu \beta K\eta_g}$ and $f_i$ satisfies assumptions A1–A3. Then, for any global $\eta_g \geq 1$ we can bound the drift as

$$
3\beta\tilde{\eta}\delta_r - \frac{2\delta^2}{3}C_{r-1} \leq \frac{\tilde{\eta}}{25\eta_g}(\mathbb{E}[f(x^r-1)] - f(x^*)) + \frac{\bar{\eta}^2}{K\eta_g}\sigma^2.
$$

**Proof** First, observe that if $K = 1$, $\delta_r = 0$ since $y_{i,0} = x$ for all $i \in [N]$ and that $C_{r-1}$ and the right hand side are both positive. Thus the lemma is trivially true if $K = 1$. For $K > 1$, we build a recursive bound of the drift starting from (P4) with $a = 1/(K - 1)$:

$$
\frac{1}{N} \sum_{i} \mathbb{E}\|y_{i,k} - x\|^2 \leq \frac{1 + \frac{1}{K-1}}{N} \sum_{i} \mathbb{E}\|y_{i,k-1} - x\|^2 + \eta_i^2\sigma^2
$$

$$
+ 6\eta_i^2K\beta(f(x) - f(x^*)) + \frac{6K\eta_i^2}{N} \sum_{i} \mathbb{E}\|c_i - \nabla f_i(x^*)\|^2.
$$
Recall that with the choice of $c_i$ in (10), the variance of $c_i$ is less than $\frac{\sigma^2}{K}$. Separating its mean and variance gives

$$
\frac{1}{N} \sum_i \mathbb{E}\|y_{i,k} - x\|^2 \leq \left(1 + \frac{1}{K-1}\right) \frac{1}{N} \sum_i \mathbb{E}\|y_{i,k-1} - x\|^2 + \tilde{\eta}^2 \sigma^2 + 6\eta^2 K \beta (f(x) - f(x^*)) + \frac{6K\eta^2}{N} \sum_i \|\mathbb{E}[c_i] - \nabla f_i(x^*)\|^2 \tag{14}
$$

Unrolling the recursion (14), we get the following for any $k \in \{1, \ldots, K\}$

$$
\frac{1}{N} \sum_i \mathbb{E}\|y_{i,k} - x\|^2 \leq \left(6K^2\beta^2 \eta^2 f(x) - f(x^*)) + 6K\eta^2 C_{r-1} + 7\beta \eta^2 \sigma^2\right) \left(\sum_{j=0}^{k-1} (1 + \frac{1}{K-1})^j\right)
$$

$$
\leq (6K^2\beta^2 \eta^2 f(x) - f(x^*)) + 6K\eta^2 C_{r-1} + 7\beta \eta^2 \sigma^2)(K-1)((1 + \frac{1}{K-1})^K - 1)
$$

$$
\leq (6K^2\beta^2 \eta^2 f(x) - f(x^*)) + 6K\eta^2 C_{r-1} + 7\beta \eta^2 \sigma^2) 3K
$$

$$
\leq 18K^2\beta^2 \eta^2 f(x) - f(x^*)) + 18K^2\eta^2 C_{r-1} + 21K\beta \eta^2 \sigma^2.
$$

The inequality $(K-1)((1 + \frac{1}{K-1})^K - 1) \leq 3K$ can be verified for $K = 2, 3$ manually. For $K \geq 4$,

$$(K-1)((1 + \frac{1}{K-1})^K - 1) < K(\exp(\frac{K}{K-1}) - 1) \leq K(\exp(\frac{4}{3}) - 1) < 3K.
$$

Again averaging over $k$ and multiplying by $3\beta$ yields

$$
3\beta \delta_r \leq 54K^2\beta^2 \eta^2 f(x) - f(x^*)) + 54K^2\beta \eta^2 C_{r-1} + 63\beta K^2 \eta^2 \sigma^2
$$

$$
= \frac{1}{\tilde{g}} \left(54\beta \tilde{\eta}^2 (f(x) - f(x^*)) + 54\beta \tilde{\eta}^2 C_{r-1} + 63\beta \tilde{\eta}^2 \sigma^2 \right)
$$

$$
\leq \frac{1}{\tilde{g}} \left(\frac{1}{\tilde{g}} (f(x) - f(x^*)) + \frac{2}{\tilde{g}} \tilde{\eta} C_{r-1} + \tilde{\eta}^2 \sigma^2 \right).
$$

The equality follows from the definition $\tilde{\eta} = K \eta \eta_9$, and the final inequality uses the bound that $\tilde{\eta} \leq \frac{1}{81\beta}$. ■