The Logarithmic Sobolev Inequality Along The
Ricci Flow: The Case $\lambda_0(g_0) = 0$

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1 Introduction

In [Y1] and [Y2], logarithmic Sobolev inequalities along the Ricci flow in all dimen-
sions $n \geq 2$ were obtained using Perelman’s entropy monotonicity, which lead to Sobolev inequalities and $\kappa$-noncollapsing estimates. In particular, a uniform logarith-
mic Sobolev inequality, a uniform Sobolev inequality and a uniform $\kappa$-noncollapsing
estimate were obtained without any restriction on time, provided that the smallest
eigenvalue $\lambda_0(g_0)$ of the operator $-\Delta + \frac{R}{4}$ for the initial metric is positive. In this
paper, we extend these uniform results to the case $\lambda_0(g_0) = 0$.

Consider a compact manifold $M$ of dimension $n \geq 2$. Let $g = g(t)$ be a smooth
solution of the Ricci flow

$$\frac{\partial g}{\partial t} = -2\operatorname{Ric}$$  \hspace{1cm} (1.1)

on $M \times [0, T)$ for some (finite or infinite) $T > 0$ with a given initial metric $g(0) = g_0$.

**Theorem A** Assume that $\lambda_0(g_0) = 0$. For each $t \in [0, T)$ and each $\sigma > 0$ there holds

$$\int_M u^2 \ln u^2 \, dv \leq \sigma \int_M (|\nabla u|^2 + \frac{R}{4} u^2) \, dv - \frac{n}{2} \ln \sigma + C$$  \hspace{1cm} (1.2)

for all $u \in W^{1,2}(M)$ with $\int_M u^2 \, dv = 1$, where $C$ depends only on $(M, g_0)$.

Note that as in [Y1] a log gradient version of the logarithmic Sobolev inequality
follows as a consequence of (1.2). We omit its statement. Our next result provides a
dependence of the above $C$ on $g_0$ in terms of rudimentary geometric data. To sim-
plifiy the statements, we assume that $|Rm| \leq 1$ for $g_0$, which can always be achieved
by a rescaling. Then we can also assume $T \geq 2\alpha(n)$ for a positive constant $\alpha(n)$ depending only on $n$ such that $|Rm| \leq 2$ on $[0, \alpha(n)]$. (Namely the maximal possible $T$ such that the solution $g = g(t)$ can be extended to a smooth solution of the Ricci flow on $[0, T]$ has this property.)

**Theorem B** There are for each $v_0 > 0$, each $D_0 > 0$, each $\epsilon > 0$ and each integer $l \geq 3$ a positive number $C = C(v_0, D_0, \epsilon, l, n)$ with the following properties. Assume $\lambda_0(g_0) = 0$, $\text{vol}_{g_0}(M) \geq v_0$, $\text{diam}_{g_0} \leq D_0$ and the normalization conditions $|Rm|_{g_0} \leq 1$ and $T \geq 2\alpha(n)$. Then one of the following two cases must occur:

1) $g(\alpha(n))$ lies in the $\epsilon$-neighborhood of a Ricci flat metric on $M$ in the $C^1$ norm, 
2) the logarithmic Sobolev inequality (1.2) holds true for each $t \in [0, T)$, each $\sigma > 0$ and all $u \in W^{1,2}(M)$ with $\int_M u^2 \text{dvol} = 1$, where $C = C(v_0, D_0, \epsilon, l, n)$.

It turns out that we have a better result in dimension $n = 3$. The same holds true in dimension $n = 2$. But Theorem E and Theorem 3.7 in [Y2] provide a stronger result in this dimension.

**Theorem C** Assume that $n = 3$. There is for each $v_0 > 0$ and each $D_0 > 0$ a positive number $C = C(v_0, D_0)$ with the following properties. Assume $\lambda_0(g_0) = 0$, $\text{vol}_{g_0}(M) \geq v_0$, $\text{diam}_{g_0} \leq D_0$ and the normalization conditions $|Rm|_{g_0} \leq 1$ and $T \geq 2\alpha(n)$. Then the logarithmic Sobolev inequality (1.2) holds true for each $t \in [0, T)$, each $\sigma > 0$ and all $u \in W^{1,2}(M)$ with $\int_M u^2 \text{dvol} = 1$, where $C = C(v_0, D_0)$.

As in [Y1], Theorem A, Theorem B and Theorem C lead to Sobolev inequalities along the Ricci flow, which in turn lead to $\kappa$-noncollapsing estimates. We consider only the case $n \geq 3$ although the methods also work for $n = 2$, because the case $n = 2$ is covered by the results in [Y2].

**Theorem D** Assume that $n \geq 3$ and $\lambda_0(g_0) = 0$. Then there holds for each $t \in [0, T)$

$$
\left( \int_M |u|^{\frac{2n}{n-2}} \text{dvol} \right)^{\frac{n-2}{n}} \leq A \int_M (|\nabla u|^2 + \frac{R}{4} u^2) \text{dvol} + B \int_M u^2 \text{dvol}
$$

for all $u \in W^{1,2}(M)$, where $A$ and $B$ depend only on $(M, g_0)$. We have $A = A(v_0, D_0)$ and $B = B(v_0, D_0)$ for given $v_0 > 0$ and $D_0 > 0$, if $n = 3$ and $g_0$ satisfies the conditions in Theorem C. We also have $A = A(v_0, D_0, n)$ and $B = B(v_0, D_0, n)$ for given $v_0 > 0$ and $D_0 > 0$, provided that $g_0$ satisfies the conditions in Theorem B and $g(\alpha(n))$ does not lie in the $\epsilon$-neighborhood of any Ricci flat metric on $M$ in the $C^3$ norm.

**Theorem E** Assume that $n = 3$ and $\lambda_0(g_0) = 0$. Let $L > 0$ and $t \in [0, T)$. Consider the Riemannian manifold $(M, g)$ with $g = g(t)$. Assume $R \leq \frac{1}{L^2}$ on a geodesic ball.
$B(x, r)$ with $0 < r \leq L$. Then there holds
\begin{equation}
\text{vol}(B(x, r)) \geq \left( \frac{1}{2^{n+3}A + 2BL^2} \right)^{\frac{n}{2}} r^n,
\end{equation}
where $A$ and $B$ are from Theorem D.

As in [Y1], the above results extend to various versions of the modified Ricci flows. Moreover, the $\kappa$-noncollapsing estimates ensure that we can obtain smooth blow-up limits at the time infinity under the assumption that $\lambda_0(g_0) = 0$. We omit the statements of those results because they are completely analogous to the corresponding ones in [Y1].

2 The Proofs

Proof of Theorem A Consider a fixed $t_1 \in (0, T)$. Let $u_1$ be a positive eigenfunction for the eigenvalue $\lambda_0(g(t_1))$ associated with the metric $g(t_1)$, such that $\int_M u_1^2 dvol = 1$ with respect to $g(t_1)$. Let $f = f(t)$ be the smooth solution of the equation
\begin{equation}
\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R
\end{equation}
on $[0, t_1]$ with $f(t_1) = -2 \ln u_1$. Note that (2.1) is equivalent to
\begin{equation}
\frac{\partial v}{\partial t} = -\Delta v + Rv,
\end{equation}
where $v = e^{-f}$. So the solution $f(t)$ exists. We also infer $\frac{d}{dt} \int_M v dvol = 0$, and hence $\int_M v dvol = 1$ for all $t \in [0, t_1]$.

We set $u = e^{-\frac{f}{2}}$. By [P, (1.4)] we then have
\begin{equation}
\frac{d}{dt} \int_M (|\nabla u|^2 + \frac{R}{4} u^2) dvol = \frac{1}{4} \frac{d}{dt} \int_M (|\nabla f|^2 + R)e^{-f} dvol \geq \frac{1}{2} \int_M |\text{Ric} + \nabla^2 f|^2 e^{-f} dvol.
\end{equation}

It follows that
\begin{equation}
\lambda_0(g(t_1)) \geq \lambda_0(g_0) + \frac{1}{2} \int_0^{t_1} \int_M |\text{Ric} + \nabla^2 f|^2 e^{-f} dvoldt.
\end{equation}

We choose $t_1 = \min\{\frac{T}{2}, 1\}$. If $\lambda_0(g(t_1)) > 0$, we first apply Theorem A in [Y1] or Theorem A in [Y2] to obtain (1.2) for $g(t)$ on $[0, t_1]$. Then we apply Theorem 4.2 in [Y1] to obtain (1.2) on $[t_1, T)$ with a larger $C$. If $\lambda_0(g(t_1)) = 0$, we deduce from (2.4)
\begin{equation}
\text{Ric} + \nabla^2 f = 0
\end{equation}
on $[0, t_1]$. It follows that $g = g(t)$ is a steady Ricci soliton for all $t$. Hence the logarithmic Sobolev inequality for $g_0$ provided by Theorem 3.3 in [Y1] holds true for all $g(t)$. Actually, [CK, Proposition 5.20] implies that $g_0$ is Ricci flat, hence $g(t) = g_0$ for all $t$.

\[\text{Lemma 2.1} \quad \text{For given } v_0 > 0, D_0 > 0, \epsilon > 0 \text{ and } l \geq 3 \text{ there is a positive constant } \mu_0 = \mu_0(\nu_0, D_0, \epsilon, l, n) \text{ with the following properties. Let } g = g(t) \text{ be a smooth solution of the Ricci flow on } M \times [0, T) \text{ which satisfies the normalization conditions in Theorem B and the conditions } \text{vol}_{g(0)} \geq v_0 \text{ and } \text{diam}_{g(0)} \leq D_0. \text{ Assume } \lambda_0(g_0) = 0. \text{ If } \lambda(g(\alpha(n))) < \mu_0, \text{ then } g(\alpha(n)) \text{ lies in the } \epsilon\text{-neighborhood of a Ricci flat metric with respect to the } C^l\text{-norm.} \]

\[\text{Proof.} \quad \text{Assume that } \mu_0 \text{ does not exist. Then we can find a sequence of manifolds } M_k \text{ of a fixed dimension } n \text{ and a sequence of smooth solutions } g_k = g_k(t) \text{ on } M_k \times [0, T_k] \text{ satisfying the normalization conditions and the conditions } \text{vol}_{g_k(0)} \geq v_0 \text{ and } \text{diam}_{g_k(0)} \leq D_0, \text{ such that } \lambda_0(g_k(0)) = 0, \lambda_0(g_k(\alpha(n))) \to 0, \text{ and } g_k(\alpha(n)) \text{ does not lie in the } \epsilon\text{-neighborhood of any Ricci flat metric with respect to the } C^l\text{-norm. By Gromov-Cheeger-Hamilton compactness theorem } [H], \text{ we can find a subsequence } (M_k, g_k), \text{ which we still denote by } (M_k, g_k), \text{ such that } (M_k, g_k, (\frac{\alpha(n)}{2}, \alpha(n))) \text{ converge smoothly to a limit Ricci flow } (M, g, (\frac{\alpha(n)}{2}, \alpha(n))). \text{ There holds } \lambda_0(g(\alpha(n))) = 0. \text{ By the monotonicity of } \lambda_0 \text{ along the Ricci flow (see } [P] \text{ or the above proof of Theorem A), we have } \lambda_0(g_k(t)) \geq 0 \text{ for all } t \in [0, T_k). \text{ Hence } \lambda_0(g(t)) \geq 0 \text{ for all } t \in [\frac{\alpha(n)}{2}, \alpha(n)]. \text{ Now the argument in the proof of Theorem A implies that } g(\alpha(n)) \text{ is Ricci flat. But } (M_k, g_k(\alpha(n))) \text{ converge smoothly to } (M, g(\alpha(n)), \text{ so } g_k(\alpha(n)) \text{ lies in the } \epsilon\text{-neighborhood of a Ricci flat metric with respect to the } C^l\text{-norm whenever } k \text{ is large enough. This is a contradiction.} \]

\[\text{Proof of Theorem B} \quad \text{Assume that } g(\alpha(n)) \text{ does not lie in the } \epsilon\text{-neighborhood of any Ricci flat metric with respect to the } C^l\text{-norm. Then } \lambda_0(g(\alpha(n))) \geq \mu_0 \text{ by Lemma 2.1. Now we obtain a desired logarithmic Sobolev inequality for } g(t) \text{ on } [0, \alpha(n)] \text{ by Theorem A in } [Y1]. \text{ Alternatively, we can apply the arguments in } [Y3] \text{ for controlling the evolution of the Sobolev constant to bound the Sobolev constant for } g(t) \text{ on } [0, \alpha(n)], \text{ and then apply Theorem 3.3 in } [Y1] \text{ to infer the desired logarithmic Sobolev inequality. Next we apply the bound for the Sobolev constant at } t = \alpha(n), \text{ the bound } \lambda_0 \geq \mu_0 \text{ at } t = \alpha(n) \text{ and the arguments in the proof of Theorem 3.5 in } [Y1] \text{ to deduce a logarithmic Sobolev inequality of the kind } [Y1, (3.11)] \text{ at } t = \alpha(n). \text{ Then we apply Theorem B in } [Y1] \text{ on } [\alpha(n), T) \text{ and combine it with Theorem A in } [Y1]. \text{ Then we arrive at the desired logarithmic Sobolev inequality on } [\alpha(n), T). \]

\[\text{Proof of Theorem C} \quad \text{By } [GIK], \text{ there is for each given flat metric } g \text{ an } \epsilon\text{-neighborhood of } g \text{ with respect to the } C^6\text{-norm, such that the Ricci flow starting at any metric in} \]
the neighborhood converges smoothly to a Ricci flat metric at a fixed exponential rate as \( t \to \infty \). Moreover, the limit Ricci flat metric lies in the \( 2\epsilon \)-neighborhood of \( g \) with respect to the \( C^6 \)-norm. We call such a neighborhood a \textit{Ricci contraction} \( \epsilon \)-neighborhood.

Now for given \( v_0 > 0, D_0 > 0 \) and \( K_0 > 0 \) the moduli space \( \mathcal{M}^0(v_0, D_0, K_0) \) of flat metrics on \( M \) with \( \text{vol} \geq v_0, \text{diam} \leq D_0 \) and \( |Rm| \leq K_0 \) is \( C^\infty \) compact modulo diffeomorphisms by Gromov-Cheeger compactness theorem and the Einstein equation. So there is a uniform \( \epsilon = \epsilon(v_0, D_0, K_0) \) such that each \( g \in \mathcal{M}^0(v_0, D_0, K_0) \) has a Ricci contraction \( \epsilon \)-neighborhood. Moreover, there is a uniform upper bound \( C(v_0, D_0, K_0) \) for the Sobolev constant for \( g \in \mathcal{M}^0(v_0, D_0, K_0) \).

Now consider for given \( v_0 > 0 \) and \( D_0 > 0 \) a smooth solution of the Ricci flow \( g = g(t) \) satisfying the normalization conditions and the conditions \( \text{vol}_{g_0}(M) \geq v_0 \) and \( \text{diam}_{g_0}(M) \leq D_0 \). Let \( \epsilon > 0 \) and \( l = 6 \), where \( \epsilon \) is to be determined. Assume that \( g(\alpha(n)) \) lies in the \( \epsilon \)-neighborhood of a Ricci flat metric \( \bar{g} \) with respect to the \( C^6 \) norm. Since \( n = 3 \), \( \bar{g} \) is flat. There is a positive number \( \epsilon_0 = \epsilon_0(v_0, D_0) \) such that \( \text{vol}_{\bar{g}}(M) \geq \frac{1}{2} v_0, \text{diam}_{\bar{g}}(M) \leq 2D_0 \) and \( |Rm|_{\bar{g}} \leq 3 \), i.e. \( \bar{g} \in \mathcal{M}^0(\frac{1}{2} v_0, 2D_0, 3) \), whenever \( \epsilon \leq \epsilon_0 \). Now we choose \( \epsilon = \min\{\epsilon_0(v_0, D_0), \epsilon(\frac{1}{2} v_0, 2D_0, 4)\} \). Then the maximally extended \( g(t) \) converges smoothly to a flat metric \( g^* \) at exponential rate where the rate depends only on \( v_0 \) and \( D_0 \). We can choose \( \epsilon_0 \) sufficiently small such that \( g^* \in \mathcal{M}^0(\frac{1}{3} v_0, 3D_0, 4) \). We can also make the \( C^6 \) norm of \( g(t) - g^* \) sufficiently small for all \( t \geq \alpha(n) \) such that the Sobolev constant of \( g(t) \) is bounded above by \( 2C(\frac{1}{3} v_0, 3D_0, 4) \) for all \( t \geq \alpha(n) \). A desired logarithmic Sobolev inequality then follows for \( t \in [\alpha(n), T) \).

A desired logarithmic Sobolev inequality for \( g(t) \) on \([0, \alpha(n)]\) follows from Theorem A in [Y1]. It also follows from the arguments for controlling the evolution of the Sobolev constant in [Y3].

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