SU(3)-SKEIN ALGEBRAS AND WEBS ON SURFACES

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Abstract. The SU₃-skein algebra of a surface $F$ is spanned by isotopy classes of certain framed graphs in $F \times I$ called 3-webs subject to the skein relations encapsulating relations between $U_q(sl(3))$-representations. These skein algebras are quantizations of the $SL(3, \mathbb{C})$-character varieties of surfaces. It is expected that their theory parallels that of the Kauffman bracket skein algebras. We make the first step towards developing that theory by proving that the reduced SU₃-skein algebra of any surface of finite type is finitely generated.

We achieve that result by developing a theory of canonical forms of webs in surfaces. Specifically, we show that for any ideal triangulation of $F$ every 3-web without trivial components, bigons, or 1- or 2-gons parallel to $\partial F$ can be uniquely decomposed into unions of pyramid formations of hexagons and disjoint arcs in the faces of the triangulation with possible additional “crossbars” connecting their edges along the ideal triangulation. We show that such canonical position is unique up to “crossbar moves”. That leads us to the associated system of coordinates for webs in triangulated surfaces (counting intersections of the web with the edges of the triangulation and their rotation numbers inside of the faces of the triangulation) which determine a reduced web uniquely.

1. Introduction

The SU($n$)-skein module of a 3-manifold $M$ is a $q$-deformation of the $SL(3)$-character variety of $\pi_1(M)$ introduced in [S2]. For the cylinder over a surface, $F \times I$, the skein module is an algebra, $S_n(F)$, quantizing the $SL(n)$-character variety of $F$, [S1]. For $n = 2$ this is the Kauffman bracket skein algebra, which is one of the central subjects of quantum topology, and has been a subject of a very active research over the recent years – see for example [BW, BFK, FrGe, FKL, Ma, Le2, Le3, Mu, PS1, PS2, Th, Tu] and references within. That research connected skein algebras to quantum invariants of 3-manifolds, (quantum) hyperbolic geometry, (quantum) cluster algebras, (derived algebraic geometry) of moduli spaces, and quantum field theory.

2010 Mathematics Subject Classification. 57M25, 57M27.
Key words and phrases. skein algebra, web, character variety.
In this paper, we make the first step towards a development of an analogous theory for $SU(3)$-skein algebras. The main difficulty in it is that, unlike for $n = 2$, the skein definition of $SU(n)$-skein algebras goes beyond arcs and links and involves also a special class of $n$-valent graphs called $n$-webs. These graphs are considered up to skein relations encapsulating relations appearing in the $U_q(sl(n))$-representation theory, which for $n = 3$ were considered first in [Ku] (cf. also [SW, CKM]).

Since the combinatorics associated with isotopy of webs in surfaces is much richer than for embedded one-manifolds, the properties of $SU(n)$-skein algebras are much harder to prove for $n > 2$ than for $n = 2$.

From now we will work with $SU(3)$-skein algebras only, refer to 3-webs as “webs”, and allow for web endpoints in $\partial F$. To be specific we will be working mostly with the reduced version, $\mathcal{R}S(F, M)$, of this algebra obtained by killing all webs with boundary parallel 1-, 2- and 3-gons.

The main results of this paper are:

- Theorem 7 stating that the reduced $SU(3)$-skein algebras of all marked surfaces of finite type are finitely generated. (An analogous result for the Kauffman bracket skein relations without marked points is due to Bullock, [Bu] (cf. also [AF]) and with marked points in [PS2].)
- Theorem 22 which for marked surfaces $(F, M)$ with an ideal triangulation provides an explicit basis of $\mathcal{R}S(F, M)$ of “canonical webs” and provides a coordinate system for those webs.

Let us elaborate on the latter result. We show that each reduced web (i.e. a non-elliptic web without boundary parallel 1-, 2- and 3-gons) admits a canonical position through “tidying up”, Theorem 20, in which it is composed of a disjoint union of certain hexagonal patterns (which we call pyramids) and some simple arcs in each face of the triangulation. These triangle pieces may have additional “cross-bars” connecting their edges along the edges of ideal triangulation. We show that such canonical position is unique up to “crossbar moves”. That leads us to an associated system of coordinates for webs in triangulated surfaces (counting intersections of the web with the edges of the triangulation and their rotation numbers inside of the faces of the triangulation) which determine a reduced web uniquely (Theorem 22). Daniel Douglas and Zhe Sun informed us that they also have a set of coordinates for webs (which appear to be similar but different than ours), [DZ].

Bonahon-Wang and others constructed embeddings of Kauffman bracket skein algebras for non-closed surfaces $F$ into quantum Teichmüller
space of Chekhov–Fock, [BW, CF, Le2, Le3, Mu]. These constructions are based on the coordinate system for unoriented multi-curves $m$ in $F$ counting geometric intersections numbers of $m$ with the edges of an ideal triangulation of $F$. The quantum Teichmüller space is a quantum torus whose free variables correspond to the edges of the ideal triangulation. We believe that an analogous embeddings of $SU(3)$-skein algebras into quantum tori exist and that the our above coordinate systems for 3-webs is an essential step towards its construction.

Furthermore, we believe that this subject is intimately connected to Fock-Goncharov theory, [FoGo], in similar vain to Allegretti-Kim work for $SU(2)$, [AK]. Since webs are functions on the $SL(3, \mathbb{C})$-character variety of $F$, our canonical webs form a canonical basis for functions on this variety. We believe that this basis is closely related to the Fock-Goncharov canonical bases for the function spaces on their moduli spaces of framed and decorated ($A$- and $X$-) local systems for $SL(3, \mathbb{R})$.

2. $SU_3$–SKEIN ALGEBRAS

A marked surface is an oriented surface $F$ together with a finite subset $M$ of $\partial F$ of marked points ($\partial F$ may be empty). In Sec. 5, we will alternatively consider $\hat{F} = F - M$ instead with a hyperbolic metric on it so that $M$ is a set of points at infinity and the boundary arcs of $F$ connecting them are infinite geodesics.

Let $I = [0, 1]$. A web in $(F, M) \times I$ consists of

- 1-valent external vertices in fibers $\{m\} \times I$ over (some of) the marked points $m \in M$
- 3-valent internal vertices which are either sources or sinks.
- oriented edges connecting the (internal and external) vertices
- oriented loops
- A framing, which can be formally defined by
  - a choice of a properly embedded oriented compact surface $S$ having the web as a spine, called framing, such that the projection on the first factor $p : F \times I \to F$ restricts to a local homeomorphism $p|_S : S \to F$.
  - $S \cap \partial F \times I$ is a collection of horizontal arcs, each of which containing a unique point of $M$.

A web may be empty, $\emptyset$, or disconnected. If it does not contain any internal vertices then it is a union of ribbons and annuli. Webs in $(F, M) \times I$ are considered up to isotopy which is a homotopy within the space of all such webs.

We say that a boundary component of $F$ is marked if it contains a point of $M$. Connected components of marked boundary components
of $F$ with points of $M$ removed are the boundary intervals of $(F, M)$. We orient them arbitrarily (for example, with the orientation induced from $F$) and we consider that choice of orientation as a part of the structure of a marked surface $(F, M)$.

It is convenient to represent webs by their 1-dimensional diagrams in $F$. A web diagram in $(F, M)$ is an oriented graph composed of

- 1-valent (external) vertices in boundary intervals of $(F, M)$ (i.e. away from the marked points)
- 3-valent (internal) vertices, sinks or sources, in the interior of $F$
- crossings, i.e. 4-valent vertices with the overpass marked, cf. Figure 1
- oriented edges connecting the above vertices
- oriented loops.

Figure 1. Types of internal vertices in 3-web diagrams

Each web diagram represents a web in $(M, F) \times I$ obtained by pushing the diagram endpoints (external vertices) in every marked boundary interval starting with $m \in M$ into $\{m\} \times I \subset F \times I$ so that the points closer to $m$ go to lower levels. (Since marked boundary components are oriented, each marked boundary interval has a starting point.) This operation in the context of framed links was discussed in [PS2]. Keeping the web endpoints away from $M$ rather than at $M$ makes the theory easier.

Conversely, every web in $(M, F) \times I$ is represented by a diagram, which is unique up to the Reidemeister-type moves of Fig. 2

Figure 2. Reidemeister-type moves for 3-web diagrams. Orientations are not marked and all are possible.

One of the results of this paper is a system of coordinates for webs in $(F, M)$, cf. Sec. 12.

Let $R$ be a commutative ring with a specified invertible element $q^{1/3}$. We use standard exponential notation to denote the powers of this element, eg. $q$ is its cube. The $SU(3)$-skein algebra $S(F, M)$ of
\((F,M)\) is the free \(R\)-module with the basis of all webs in \((F,M) \times I\) quotiented by the following skein relations:

\[
(1) \quad \begin{array}{c}
\begin{array}{c}
\text{\textcircle{}}
\end{array}
\end{array} - (q^2 + 1 + q^{-2})\emptyset, \quad \begin{array}{c}
\begin{array}{c}
\text{\textcircled{}}
\end{array}
\end{array} + (q + q^{-1})\rightarrow
\end{array}
\]

\[
(2) \quad \begin{array}{c}
\begin{array}{c}
\text{\textcircled{-}}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\text{\textcircled{-}}
\end{array}
\end{array}
\end{array}
\]

\[
(3) \quad \begin{array}{c}
\begin{array}{c}
\text{\textcircled{\textcircled{-}}} - q^\frac{1}{2} - q^{-\frac{2}{3}}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\text{\textcircled{\textcircled{-}}} - q^{-\frac{1}{2}} - q^{\frac{2}{3}}
\end{array}
\end{array}
\end{array}
\]

All these diagrams have the standard, horizontal framing. (Our \(q\) is \(q^{1/2}\) in \([Ku]\).

Let \(W_1 \cdot W_2\) denote a web obtained by stacking the web \(W_1\) on top of \(W_2\) in \(F \times I\). That operation extends to a product on \(\mathcal{S}(F,M)\) making it into an associative \(R\)-algebra with the identity \(\emptyset\). (That explains the term “algebra” in its name.)

As alluded earlier, it is easier to operate on web diagrams than webs. In the diagrammatic approach, the stacking of \(W_1\) on top of \(W_2\) requires that the endpoints of \(W_1\) lie after the points of \(W_2\) in the respective boundary intervals of \((F,M)\). (Here again we use the fact that boundary intervals are oriented. Kauffman bracket skein algebras for surfaces with marked points were considered in \([BW, Le2, Le3, Mu, PS2]\). Our approach follows that last paper.)

Note that the above skein relations allow for resolving any crossing as a linear combination of the vertexless and the I-resolutions, Fig. 3.

\[
\text{Figure 3. The vertexless resolution and the I-resolution of a crossing.}
\]

**Definition 1.** We say that two web diagrams are isotopic if they represent webs isotopic in \((F,M) \times I\). We say that two crossingless web diagrams are planarly isotopic if they are isotopic in the space of crossingless web diagrams in \((F,M)\).

Note that neither isotopy nor planar isotopy allows for endpoints passing each other in \(\partial F\). Isotopy is more flexible than planar isotopy, because it allows for a flip of two parallel loop components, (made up of two Reidemeister II moves), cf. Fig. 4. We call it a flip move. In fact,
any two isotopic crossingless web diagrams differ by a planar isotopy and flip moves.

A crossingless web diagram without trivial components (contractible loops) nor internal 2- or 4-gons is called non-elliptic. (“Internal” refers to regions in the interior of $F$.)

The results and methods of [SW] imply:

**Theorem 2.** Non-elliptic web diagrams in $(F,M)$ up to isotopy form a basis of $S(F,M)$.

We have seen in [PS2, Example 8] that the Kauffman bracket skein algebra is not finitely generated in general, even in the case of $(D^2,M)$ for $M \neq \emptyset$. For the same reason, the $SU_3$-skein algebra of $(D^2,M)$ is not finitely generated either. To achieve a finite generation of the $SU(3)$-skein algebras we will consider their reduced version, analogous to that in [Le2, Le3, Mu, PS2].

3. **Reduced $SU_3$-skein algebras**

Let $a$ be a fixed invertible element of $R$ and let $D(F,M)$ be the $R$-submodule of $S(F,M)$ spanned by the boundary skein relations of Fig. 4.

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=1cm]{figure4} \\
\includegraphics[width=1cm]{figure4}
\end{array} = 0,
\begin{array}{c}
\includegraphics[width=1cm]{figure4} \\
\includegraphics[width=1cm]{figure4}
\end{array} = a \cdot \begin{array}{c}
\includegraphics[width=1cm]{figure4} \\
\includegraphics[width=1cm]{figure4}
\end{array},
\begin{array}{c}
\includegraphics[width=1cm]{figure4} \\
\includegraphics[width=1cm]{figure4}
\end{array} = a^{-1} \cdot \begin{array}{c}
\includegraphics[width=1cm]{figure4} \\
\includegraphics[width=1cm]{figure4}
\end{array}
\end{align*}
\]

**Figure 5.** Additional skein relations of reduced skein algebras. The blue horizontal line denotes $\partial F$. The orientations of the arc and of the trigon are arbitrary.

**Lemma 3.** $D(F,M)$ is a 2-sided ideal in $S(F,M)$.

**Proof.** For any web diagrams $W_1, W_2$ in $(F,M)$, the endpoints of $W_2$ in $W_1 \cdot W_2$ and in $W_2 \cdot W_1$ are either after or before the endpoints of $W_1$ in each boundary interval. Consequently, they do no affect the above skein relations. \qed
\( \mathcal{RS}(F,M) = \mathcal{S}(F,M)/\mathcal{D}(F,M) \) is the reduced \( SU(3) \)-skein algebra of \((F,M)\) which is the main subject of this paper.

A non-elliptic web diagram is reduced if it does not contain any path of \( \leq 3 \) edges parallel to a boundary interval, cf. Figure 5. (A path of edges of \( W \) is parallel to a boundary interval if together with it bounds a 2-disk whose interior is disjoint from \( W \).)

These web diagrams considered up to isotopy span \( \mathcal{RS}(F,M) \). However, for \( M \neq \emptyset \), the relation of Figure 7 shows a linear dependence between them.

For that reason we say that two reduced crossingless web diagrams are equivalent if they are related by isotopy of webs (of Definition 1) and by a permutation of parallel arc components. By the method of confluence, cf. \[SW\], we prove:

**Proposition 4.** \( \mathcal{RS}(F,M) \) has a basis of consisting of one representative per each equivalence class of reduced crossingless web diagrams in \((F,M)\).

In order to make such a basis explicit, we need to choose a convention about parallel loops and arcs similar to the conventions governing car traffic.

For the sake of precision, all pictures in this paper show \( F \) oriented counterclockwise. We call a pair of parallel loops or arcs a
British highway if they are oriented inconsistently with the boundary of the annulus or of the rectangle they bound. Note that such parallel arcs and loops resemble the left-side traffic of the countries of the former British empire. As for the sake of safe and efficient traffic, one needs to mandate either the left-side and the right-side rule, so we do in order to fix a basis of $\mathcal{RS}(F, M)$. Specifically, as a consequence Proposition 4 we obtain:

**Corollary 5.** $\mathcal{RS}(F, M)$ has a basis of reduced webs in $(F, M)$ (up to planar isotopy) without British highways.

Let us discuss generating sets for skein algebras now.

A **triad** is a connected web in $(F, M) \times I$ with a single 3-valent vertex (source or sink).

We say that an arc diagram in $(F, M)$ is descending if transversing it according to its orientation one passes each crossing through an overpass first. Similarly, a triad is descending if there is an ordering of its three edges such that the above property holds when transversing first the 1st edge, then the 2nd, and finally the 3rd one. Finally, a knot diagram $D$ is descending if there is a point in $D$ such that transversing $D$ according to its orientation, starting from that point, one passes each crossing through an overpass first.

A knot, arc, or triad in $(F, M) \times I$ is descending if it has a descending diagram.

**Theorem 6** (Proof in Sec. 4). (1) For every marked surface $(F, M)$, the algebra $\mathcal{RS}(F, M)$ is generated by descending knots, arcs, and triads.

(2) $\mathcal{RS}(D^2, M)$ is generated by crossingless arc and triad diagrams with endpoints at distinct boundary intervals.

Since each crossingless arc or triad diagram in $(D^2, M)$ is determined by its orientation and the boundary intervals containing its endpoints, the cardinality of the above generating set for $F = D^2$ is $2\binom{|M|}{2} + 2\binom{|M|}{3}$, where $\binom{k}{l} = 0$ for $k < l$. However, for $F \neq D^2$ the above generating set is infinite.

The following much stronger and harder to prove result is one of the two main results of this paper. It is a consequence of its graded version, Theorem 11. Following the standard definition, we say that $(F, M)$ is of finite type if $F$ is obtained by removing a finite set of internal points from a compact surface.

**Theorem 7** (Proof in Sec. 6). $\mathcal{RS}(F, M)$ is a finitely generated $R$-algebra for every marked surface $(F, M)$ of finite type.
4. Proof of Theorem 6

A size of a web diagram is the number of its vertices plus twice the number of crossings. Size defines a filtration of $RS(F, M)$ (which is not an algebra filtration).

A union of web diagrams stacked one on top of another, is called a stack. Specifically, it is a web diagram of the form $W_1 \cup \ldots \cup W_n$, such that $W_i$ lies on top of $W_{i+1}$ for $i = 1, \ldots, n - 1$. It is like a product of webs, except that the endpoints of webs in a stack do not have to be arranged according to the rule of the multiplication of webs. We have

$$W = q^{-2/3} \cdot \quad + q^{1/3} \cdot \quad = q^{2/3} \cdot \quad,$$

and

$$W = q^{-1/3} \cdot \quad + q^{2/3} \cdot \quad = q^{-1/3} \cdot \quad,$$

where the blue horizontal line denotes $\partial F$ as usual. Hence, a stack $W_1 \cup \ldots \cup W_n$ equals to $W_1 \cdot \ldots \cdot W_n$ times a power of $q$ and $a$ in $RS(F, M)$.

Theorem 6(1) follows by induction on size from the lemma below.

Lemma 8. Every web diagram $W$ in $(F, M)$ of size $s$ equals in $RS(F, M)$ to a power of $q$ times a stack of descending knot, arc, and triad diagrams of total size at most $s$ plus a linear combination of webs of size less than $s$.

Proof. Let $W$ be a web in $(F, M)$ of size $s$. By (3), up to a multiplicative factor of power of $q$, one can replace all internal edges of $W$ by crossings of arbitrary sign, plus linear combinations of webs of smaller size. Since we can control the signs of these crossings, the above operations can transform $W$ into a stack of webs diagrams without internal edges (i.e. composed of diagrams of knots, arcs, and triads) represented by a diagram of size $s$, plus a linear combination of webs of smaller size. Furthermore, these diagrams of knots, arcs, and triads can be assumed in descending position.

Proof of part (2) of Theorem 6 follows from the fact that the only descending knots, arcs and triads in $(D^2, M)$ are trivial (i.e. crossing-less). For the triad that follows from the fact that any crossing between two of its edges can be undone by a twist. The possible side effect – twisting of the framing of the edges can be eliminated by a multiplication by a power of $q$. The fact that the endpoints of arc and triad diagrams can be assumed to lie in different boundary intervals follows from the relations of the reduced skein algebras of Fig. 5.
5. Alternative Approach to Marked Surfaces, Ideal Triangulations

Since $RS(F,M)$ is not affected by the removal of unmarked boundary components from $F$, we can assume that they are never there (and think of them as “punctures”).

Furthermore, by abuse of notation, from now on we will identify $(F,M)$ with $\hat{F} = F - M$. That should not lead to confusion since one can recover $(F,M)$ from $F$. (Note that $\hat{F}$ is a surface whose boundary consists of open arcs only.)

Furthermore, from now on we will not consider webs in $(F,M) \times I$ directly but rather their representations by web diagrams in $(F,M)$ or in $\hat{F}$ only. (This does not lead to any issues since web diagrams cannot touch points of $M$ anyway.) For that reason we will call web diagrams webs in $\hat{F}$ for simplicity.

Let us denote the genus of $F$ by $g$ and the number of boundary components and ends of $F$ by $k$.

**Lemma 9.** If $4 - 4g - 2k - |M| < 0$ then $\hat{F}$ can be given a hyperbolic metric such that

- points $M$ are at infinity and all boundary intervals are infinite geodesics.
- all ends of $\hat{F}$, other than points of $M$ or punctures.

**Proof.** Let $\bar{F}$ be two copies of $\hat{F}$ glued along their corresponding boundary intervals. The Euler characteristic of $\bar{F}$ is $4 - 4g - 2k - |M| < 0$ and, hence, it admits a complete hyperbolic structure. The boundary intervals of $F$ are arcs in $\bar{F}$ connecting its ends and so they can be isotoped to geodesics. □

Although $\hat{F}$ is not necessarily hyperbolic, we will call its ends other than the points of $M$ punctures. An infinite arc in $\hat{F}$ is an embedding of $(-\infty, \infty)$ into $\hat{F}$ with each of its ends either in $M$ or one of the punctures of $\hat{F}$. (In particular, every boundary arc of $(F,M)$ is an infinite arc in $\hat{F}$.)

An ideal triangulation of $\hat{F}$ is a disjoint collection of infinite arcs $\gamma_1, \ldots, \gamma_N$ in $\hat{F}$ which includes all boundary intervals of $\hat{F}$ and which decomposes $\hat{F}$ into ideal triangles.

By Lemma 9

**Corollary 10.** For every non-closed marked surface $(F,M)$, other than the open disk, the closed disk with at most two marked points, or an
annulus \((J \times S^1, \emptyset)\), where \(J = (0, 1)\) or \([0, 1)\) or \([0, 1]\), \(\hat{F}\) has an ideal triangulation.

**Figure 9.** An ideal triangulation \(\Gamma = \{\gamma_0, \gamma_1, \gamma_2\}\) (in green) of a torus with one boundary component.

The proof of finite generation of the reduced Kauffman bracket skein algebras of punctured surfaces in [AF, PS2] uses the fact that the reduced Kauffman bracket skein algebra has a basis of multi-curves, each of which has a canonical position with respect to any ideal triangulation of \(\hat{F}\). Specifically, each multi-curve in \((F, M)\) can be placed so that it consists of mutually disjoint simple (unoriented) arcs in each face of the ideal triangulation. Such canonical position is unique. As alluded earlier, we prove an analogous statement for web diagrams in Sec. 11.

### 6. The Graded Skein Algebra

In this section we will consider gradings on \(RS(\hat{F})\) induced by ideal triangulations \(\Gamma = \{\gamma_1, ..., \gamma_N\}\) of \(\hat{F}\). A weight \(w(W)\) of a web \(W\) in \(\hat{F}\) with respect to \(\Gamma\) (as above) is the minimal geometric intersection number of \(W\) with \(\Gamma\). We do not allow for the interior vertices and crossings of \(W\) to lie in the edges \(\gamma_1, ..., \gamma_N\) (although we do allow isotopies moving vertices and crossings through \(\gamma\)'s).

Let \(F_\ell\) be the subspace of \(RS(\hat{F})\) spanned by webs of weight \(\leq \ell\). (Similar filtrations were considered in [CM, Mn, Le1, AF, PS2].) Note that \(\{F_\ell\}_{\ell \geq 0}\) is an algebra filtration of \(RS(\hat{F})\) and, therefore, it defines the graded reduced skein algebra, \(GRS(\hat{F})\). (Clearly, it depends on \(\Gamma\) but we suppress that in the notation.) It is easy to see that \(GRS(\hat{F})\) has the same basis of Proposition 4 (or Corollary 5) as \(RS(\hat{F})\) has.

**Theorem 11** (Proof in Sec. 15). \(GRS(\hat{F})\) is finitely generated as an \(R\)-algebra for every finite ideal triangulation of a marked surface \(\hat{F}\).

**Proof of Theorem 7.** For a closed surface \(F\) there is a natural epimorphism \(RS(F', \emptyset) \to RS(F, \emptyset)\) where \(F'\) is \(F\) with some points removed. Therefore, for the sake of proof of Theorem 7 it is enough to assume that \(F\) is not closed.
By a further removal of points from \( \hat{F} \) if necessary, we can ensure that \( \hat{F} \) has a finite ideal triangulation (cf. Corollary \[10\]). Since any element of the associated graded algebra \( GA \) of an algebra \( A \) lifts to an element of \( A \) and any generating set of \( GA \) lifts to a generating set of \( A \), Theorem \[11\] immediately implies Theorem \[7\].

The proof of Theorem \[11\] requires several ingredients discussed in the remainder of the paper.

7. Minimizing geometric intersection numbers

From now on, when speaking of a web \( W \) in \( \hat{F} \) in presence of infinite arcs, we always assume that \( W \) is in general position with respect to them.

Proposition 12 (Proof in Sec. \[8\]). For any infinite arc \( \gamma \) in \( \hat{F} \), and any non-elliptic web \( W \) in \( \hat{F} \), any planarly isotopic image \( W' \) of \( W \) with a minimal geometric intersection number with \( \gamma \) can be obtained from \( W \) by cap, vertex, crossbar moves (pictured below), the flip move (cf. Fig. \[4\]) and an isotopy away from \( \gamma \). Note that cap, vertex, crossbar moves reduce the intersection number of \( W \) with \( \gamma \) and the crossbar move does not change it. Consequently, the above moves monotonically reduce the intersection number of \( W \) with \( \gamma \).

We call cap and vertex moves intersection reduction moves.

\[
\begin{array}{c}
\text{cap move} \\
\text{vertex move} \\
\text{crossbar move}
\end{array}
\]

Figure 10. Intersection reduction moves (cap move, vertex move) and a crossbar move. The arc \( \gamma \) of the triangulation is vertical green.

Corollary 13. Every crossingless web in \( \hat{F} \) can be isotoped though cap, vertex, and crossbar moves so that it minimizes its geometric intersection number with each arc of any disjoint collection of infinite arcs of \( \hat{F} \) simultaneously.

Proof. It is an immediate consequence of the fact that (a) none of the moves of Proposition \[12\] increases the intersection numbers with other infinite arcs and (b) the flip move is independent of others (i.e. it commutes with others) and does not change the intersection numbers with other infinite arcs. Therefore, it is unnecessary for achieving a minimal intersection number. □
A web as in Corollary 13 is said to be in a minimal position with respect to a collection of infinite arcs.

**Proposition 14.** A minimal position of any non-elliptic web with respect to any disjoint in $\hat{F}$ collection of infinite arcs is unique up to crossbar moves and flip moves (and planar isotopy within the complement of these arcs).

**Proof.** The proof is by induction on the number $n$ of infinite arcs $\gamma_1, \ldots, \gamma_n$. For $n = 1$ it follows from Proposition 12 and the fact that only the crossbar and flip moves can be applied to a diagram in a minimal position. For the inductive step, suppose that the statement holds for $n$ and suppose $W$ and $W'$ are two isotopic reduced webs, both in minimal position with respect to infinite arcs $\gamma_1, \ldots, \gamma_{n+1}$. Since $W, W'$ are minimal with respect to $\gamma_{n+1}$, they can be made isotopic in $\hat{F} - \gamma_{n+1}$ by flips and crossbar moves. Now the statement follows from the inductive assumption. $\square$

8. **Surface graphs and their face indices. Proof of Proposition 12**

A surface graph $G$ in $F$ is a compact graph properly embedded in $F$. Zero-valent vertices are not allowed and the monovalent vertices of $G$ must coincide with $G \cap \partial F$. We allow for loop components in $G$ though.

Closures (in $F$) of connected components of $F - G$ are called faces of $G$.

Our proof of Proposition 12 relies on the notion of index of a face. Specifically, the index of a face $f$ of $G$ is

$$\text{(6)} \quad \text{ind}(f) = \chi(f) - \frac{\text{int}(f)}{2} - \text{ext}(f) + \sum_v \frac{1}{\text{val}(v)},$$

where

- $\chi(f)$ is the Euler characteristic of $f$
- $\text{int}(f)$ and $\text{ext}(f)$ are the numbers of interior and exterior edges of $f$. (An exterior edge is a segment of $\partial F$ which may or may not be an edge of $G$.)
- $\text{val}(v)$ is the number of faces a vertex $v$ of $G$ belongs to.

If an interval appears twice as an edge of a face $f$ then it is counted twice in $\text{int}(f)$. Similarly, if a vertex appears many times in $\partial f$, $\text{val}(v)$ counts that face with a multiplicity. (Note that components of $\partial f$ which are internal loops of $G$ or coincide with components of $\partial F$, do not contribute to that sum. Ends of $F$ do not contribute to it either.)
Lemma 15.\
(7) \[ \chi(F) = \sum \ind(f) \]
where the sum is over all faces \( f \) of \( G \).

We will need it only for surface graphs \( G \) in compact \( F \) with contractible faces. In this case this equality follows from the fact that \( G \) defines a cell decomposition of \( F \), and since the Euler characteristics of every face is 1, \( \sum \ind(f) \) adds up to the number of 2-cells minus the number of edges (1-cells) plus the number of vertices (0-cells).

From now on we will assume that \( G \) is a web in \( F \) (with some possible endpoints in \( \partial F \)). In particular, all internal vertices of \( G \) are 3-valent.

![Figure 11. Positive index faces. Blue line denotes \( \partial F \).](image)

Figure 11 shows all possible positive index faces for surface graphs. From left to right, the exterior bigon has index \( \frac{1}{2} \), the exterior trigon has index \( \frac{1}{3} \), the exterior quadrigon has index \( \frac{1}{6} \), the disk has index 1, the monogon has index \( \frac{1}{2} \), the bigon has index \( \frac{2}{3} \), and the quadrigon has index \( \frac{1}{3} \).

The proof of Proposition 12 is based on:

**Lemma 16.** Any non-elliptic web \( W \) in \( \hat{F} \) in general position with respect to disjoint, parallel infinite arcs \( \gamma_1, \gamma_2 \) bounding a bigon \( B \) can be isotoped through the intersection reduction moves (through \( \gamma_1 \) and \( \gamma_2 \)) and crossbar moves so that it intersects \( B \) in a collection of disjoint, parallel arcs connecting \( \gamma_1 \) with \( \gamma_2 \), depicted by vertical arcs in figure below.

![Diagram](image)

**Proof by contradiction:** Suppose that the statement is false. Among webs \( W \) contradicting the statement consider a web \( W \) with the minimal number of vertices in \( B \). Let \( W' \) be \( W \cap B \) with all simple arcs connecting \( \gamma_1 \) and \( \gamma_2 \) (pictured as vertical segments in the picture above) removed. Formula (7) implies that the sum of indices of faces of \( W' \) in \( B \) is 1, the Euler characteristic of \( B \). Let \( f_1, f_2 \) be the faces of \( W' \)
containing $m_1$ and $m_2$. By discussion above (cf. Figure 11) $f_1, f_2$ have indices $\leq 1/2$. Furthermore, none of them can be $1/2$ since that would imply that $f_1$ or $f_2$ is a bigon and, hence, $W'$ contains a simple arc – contradicting the definition of $W'$.

Hence, $W'$ must contain another face of positive index in $B$. Since $W'$ is non-elliptic, that face must be an external bigon, trigon, or quadrigon. That face can be eliminated by either an intersection reduction move (cap or vertex move) or a crossbar move (away from $m_1, m_2$) contradicting the minimality of vertices of $W \cap B$. □

Proof of Proposition 12: By applying a sequence of cap, vertex, and crossbar moves to $W$ we can assume that no further reduction of $|W \cap \gamma|$ through those moves is possible. The proof continues by contradiction: Assume that an arc $\gamma'$ is isotopic to $\gamma$ rel its endpoints with

\[ |\gamma' \cap W| < |\gamma \cap W|. \]  

By applying an arbitrarily small isotopy to $\gamma$ and $\gamma'$ if necessary, one can assume that $\gamma$ and $\gamma'$ are in general position and that $|\gamma \cap \gamma'|$ is finite. Let us assume for $\gamma, \gamma'$ as above that $|\gamma \cap \gamma'|$ is as small as possible. By a theorem of Epstein, there is a bigon $B$ bounded by a subarc $\alpha$ of $\gamma$ and a subarc $\alpha'$ of $\gamma'$.

Finally, let as assume that for $\gamma, \gamma', W$ and $B$ as above, the number of vertices of $W$ in $B$ is as small as possible. Given these assumptions, Lemma 16 (applied to $\hat{F}$ with the endpoints of $\alpha, \alpha'$ removed) implies that $W$ intersects $B$ in a collection of simple, parallel, arcs connecting $\alpha$ with $\alpha'$.

If $\gamma \cap \gamma' = \emptyset$ then $\alpha = \beta, \alpha' = \beta'$ and the above statement contradicts (8). If $\gamma \cap \gamma' \neq \emptyset$ then pushing $\gamma$ through $B$ reduces the number of intersections with $\gamma'$ – contradicting the assumption about the minimality of $|\gamma \cap \gamma'|$. □

9. Webs in bigons

Let $B$ be the (ideal) bigon $D^2 - \{m_1, m_2\}$ with its boundary intervals (the connected components of $\partial D^2 - \{m_1, m_2\}$) denoted by $\partial_+ B, \partial_- B$, oriented from $m_1$ to $m_2$, and so that $\partial_+ B$ lies on the left and $\partial_- B$ on the right assuming (as usual) the counterclockwise orientation of $B$.

A crossbar web of index $n$ in $B$ consists of $n$ vertical lines, which are simple, mutually disjoint arcs connecting $\partial_+ B$ and $\partial_- B$ and a certain number of horizontal intervals ("crossbars") connecting some of the adjacent vertical lines, as in Figure 12. (Together they form a graph,
which one orients according to web orientation rules.) In particular, there are $2^n$ index $n$ crossbar webs without crossbars, called trivial.

The signs of intersections of the external edges of a crossbar web $W$ with $\partial_+ B$ and with $\partial_- B$ form the plus-signature $(p_1, \ldots, p_n) \in \{\pm\}^n$ and the minus-signature $(q_1, \ldots, q_n) \in \{\pm\}^n$, of $W$, respectively, cf. Figure 12.

A braid of index $n$ in $B$ consists of $n$ oriented braided strands connecting $\partial_+ B$ with $\partial_- B$, which may have opposing orientations. We say that braid is proper if no two strings of it with coinciding orientations cross. (Note that the notions of “braid” and “proper” are non-standard and that we do not identify webs related by Markov moves.)

A proper braid in $B$ is minimal if no two strands in it cross more than once.

**Lemma 17.** A proper braid in $B$ is minimal iff no two strands in it cross twice consecutively (forming a bigon).

**Proof.** The implication $\Rightarrow$ is obvious. Proof of $\Leftarrow$: Let $\beta$ be a braid in which two strands cross (at least) twice. The arcs connecting these two crossings form a bigon, which $\beta$ may intersect. There may be a number of bigons of this form cut out by $\beta$. Let $B$ be a minimal one — that is one which does not contain a smaller one inside. Since $\beta$ is proper, no strand of $\beta$ may intersect both sides of $B$. Therefore, either

- $\beta$ does not intersect the interior of $B$, implying that $B$ is formed by two strands in it crossing twice consecutively, or
- $\beta$ intersects $B$ in an arc with both endpoints on one side of $B$. That double intersection defines a smaller bigon in $B$ — contradicting the minimality of $B$.

$\Box$

Note that the I-resolution of crossings (cf. Fig 3) in a proper braid $\beta$ yields a crossbar web. We denote it by $H_\beta$. Furthermore, every crossbar web is obtained that way. (Note however that (a) isotopic braids (i.e. related by Markov moves) may define different crossbar webs and (b) $H_\beta$ is insensitive to crossing changes in $\beta$.) $H_\beta$, for $\beta$ minimal, is called a minimal crossbar web.
For the sake of arguments in the remainder of this paper it is useful to consider the notion of weakly reduced webs by which you mean non-elliptic webs without exterior bigons and triangles (the first two diagrams in Fig. 6), but which may have triple edge paths parallel to $\partial F$. Clearly, every minimal crossbar web in $B$ is weakly reduced. Furthermore, we have:

**Lemma 18.** (1) Every weakly reduced web $W$ is a minimal crossbar web.

(2) A minimal $\beta$ such that $W = H_{\beta}$ is unique up to crossing sign changes.

(3) Every weakly reduced web is determined by its signature uniquely (in the set of crossbar webs).

*Proof.* (1) By Lemma 16, every web in a bigon can be trivialized by intersection reduction and crossbar moves. Since a weakly reduced web does not allow for intersection reduction, it must be a crossbar web, $W = H_{\beta}$ for some proper $\beta$. If $\beta$ is not minimal then it contains two strings with consecutive crossings by Lemma 17. These consecutive crossings yield crossbars which together with vertical lines connecting them form a 4-gon in $W$, contradicting the non-ellipticity of $W$.

(2) and (3) follow from the fact that (a) the signature of $\beta$ coincides with that of $W = H_{\beta}$ and (b) this signature determines a minimal braid uniquely. □

10. **Canonical webs in triangles**

A disk with three marked points in its boundary is called a **triangle**. An arbitrarily chosen side of a triangle $T$ will be drawn horizontally and called the **base**. The degree $d \in \mathbb{Z}_{>0}$ pyramid $H_d$ in $T$ is a web consisting of:

- $d$ horizontal lines parallel to the base, which we number from 1 (the shortest) to $d$ (the longest)
- $i$ vertical intervals connecting the $i$-th line with the $(i+1)$-th line, for $i = 1, 2, \ldots, d$, so that all vertical lines are in a staggered pattern. Here, the $(d+1)$-th line is the base (which is not the part of $H_d$). These vertical intervals split the horizontal lines into horizontal intervals.
- the horizontal and vertical intervals are oriented according to the web orientation rules, so that all endpoints of $H_d$ point outwards,

c.f. Figure 13.
Figure 13. Top: A pyramid of degree 1 and a pyramid of degree 2 with two different choices of the base. Bottom: A pyramid of degree 3. Note the $2\pi/3$ rotational symmetry of the honeycomb form of the pyramid.

Note that $H_d$ does not depend on the choice of base of $T$, cf. Figure 13. By reversing all orientations of $H_d$ we obtain $H_{-d}$. Therefore, $H_d$ is defined for all $d \in \mathbb{Z}$ with $H_0 = \emptyset$.

We say that a web in $T$ is canonical if it consists of $H_d$, for some $d \in \mathbb{Z}$, and of a number of simple oriented arcs, mutually disjoint from each other and from $H_d$, each connecting different sides of the ideal triangle, cf. Figure 14. These are corner arcs.

Figure 14. An example of a canonical web in a triangle.

11. JOY-SPARKING AND CANONICAL WEBS

Let $\hat{F}$ be a marked surface with an ideal triangulation $\Gamma = \{\gamma_1, ..., \gamma_N\}$. Choose arbitrary orientations of $\gamma_1, ..., \gamma_N$ and their disjoint tabular neighborhoods $\mathcal{N}(\gamma_1), ..., \mathcal{N}(\gamma_N)$.

Recall from Sec. 5 that all boundary arcs are included among the edges of $\Gamma$; we call them external. Each external edge $\gamma_i$ necessarily coincides with one of the two boundary intervals of $\mathcal{N}(\gamma_i)$.

We call $\mathcal{N}(\gamma_1), ..., \mathcal{N}(\gamma_N)$ a padded ideal triangulation of $\hat{F}$ and $\mathcal{N}(\gamma_1), ..., \mathcal{N}(\gamma_N)$ are called its bigons. Each of them is an ideal bigon. (In the case of $\hat{F} = S^1 \times (0, 1)$, it is a bigon with both of its sides identified.)
The connected components of $F \setminus \bigcup_{i=1}^{N} \mathcal{N}(\gamma_i)$ are its faces. By the definition of an ideal triangulation, each of them is an ideal triangle.

The orientations of $\gamma_1, ..., \gamma_N$ define the oriented boundary intervals $\partial_+ \mathcal{N}(\gamma_i), \partial_- \mathcal{N}(\gamma_i)$ of the bigons $\mathcal{N}(\gamma_i)$ for $i = 1, ..., N$, as in Sec. 9.

From now on we consider non-elliptic webs only, unless specifically stated otherwise. Taking advantage of Corollary 13, we will assume that these webs are in the minimal position with respect to infinite arcs $\partial_\pm \mathcal{N}(\gamma_i), ..., \partial_\pm \mathcal{N}(\gamma_N)$ and simply say that they are in a minimal position.

By Proposition 14, such position is unique up to crossbar and flip moves. We call crossbar moves pushing crossbars into bigons tidying up operations, cf. Figure 15.

**Figure 15.** Tidying up operation.

Consider a non-elliptic web $W$ in minimal position and apply to it tidying up operations. Inspired by Marie Kondo, we say that such web sparks joy if no further tidying up operations are possible. As a consequence of Proposition 14 we have:

**Corollary 19.** The result of putting a non-elliptic web $W$ (considered up to planar isotopy) in a joy-sparking position (with respect to $\mathcal{N}(\gamma_1), ..., \mathcal{N}(\gamma_N)$) is unique up to flip moves and crossbar passes (cf. Fig. 16).

**Figure 16.** A crossbar pass. Orientations not specified.

In order to characterize joy-sparking webs, it is best to narrow focus to the weakly reduced webs introduced in Sec. 9. We say that a weakly reduced web $W$ in $\hat{F}$ is in a canonical position with respect to an ideal triangulation $\Gamma$ if $W \cap \mathcal{T}$ is canonical for every face $T$ of the triangulation and $W \cap \mathcal{N}(\gamma_i)$ is a minimal crossbar web for every $i$.

**Theorem 20.** A weakly reduced web is in a minimal position and sparks joy iff it is in a canonical position.
Proof of $\Leftarrow$: A web in a canonical position is in a minimal position and sparks joy, since it does not allow for any tidying up operations.

Proof of $\Rightarrow$: Since $W$ is in a minimal position with respect to $\partial_+ \mathcal{N}(\gamma_i)$, Lemma 16 implies that $W \cap \mathcal{N}(\gamma_i)$ is a crossbar web. It is minimal because $W$ is weakly reduced. Therefore, it is enough to prove that $W$ is in canonical position in every triangle $T$ of the triangulation. By Lemma 16, the only connected components of $W$ which do not touch all sides of $T$ are corner arcs. Let $W'$ be $W$ stripped of all these corner arcs.

Let us assume that $W' \neq \emptyset$ now since otherwise we are done. Consider the faces $f_0, f_1, f_2$ of $W'$ in $T$, as in Sec. 8 containing the vertices $v_0, v_1$ and $v_2$ of $T$, respectively. By the above discussion, $f_i$ is bounded by a path of $n_i$ edges of $W'$ for $n_i \geq 2$. Since $f_i$ has one external edge (with $v_i$ inside of it),

$$\text{ind}(f_i) = 1 - n_i/2 - 1 + 2 \cdot \frac{1}{2} + (n_i - 1) \frac{1}{3} = \frac{1}{3} - \frac{n_i - 2}{6}.$$ 

Since $W'$ is non-elliptic web without 1-, 2- and 3-gons parallel to the three sides of $T$, the faces $f_0, f_1, f_2$ are the only faces with potentially positive index. Since the sum of the face indices is $\chi(T) = 1$, we see that $\text{ind}(f_i) = \frac{1}{3}$ and $n_i = 2$ for $i = 0, 1, 2$. That implies that all faces other than $f_0, f_1, f_2$ have index 0. In particular, all internal faces of $W'$ are hexagonal and all external faces other than $f_0, f_1, f_2$ are bounded by a 4-gon, as in Fig. 13. Consequently, $W'$ is a pyramid.  

12. Web coordinates

The intersection coordinates of a web $W$ in a canonical position with respect to a triangulation $\Gamma$ are given by the numbers of + and − signs of the intersections of $W$ with the bigons of the triangulation. Specifically, the intersection coordinates of $W$ are

- the numbers, $e_{+,i}(W)$, of positive endpoints of $W$ in $\partial_+ \mathcal{N}(\gamma_i)$ or, equivalently, in $\partial_- \mathcal{N}(\gamma_i)$. (Because $W \cap B$ is a crossbar web, these numbers of positive endpoints coincide, cf. Sec. 9.)
- the numbers, $e_{-,i}(W)$, of negative endpoints of $W$ in $\partial_+ \mathcal{N}(\gamma_i)$ or, equivalently, in $\partial_- \mathcal{N}(\gamma_i)$.

The coordinates $e_{\pm,i}(W)$ are indexed by $i = 1, \ldots, N$, where $N$ is the number of edges in $\Gamma$.

Note that these coordinates alone do not determine the isotopy class of a canonical web in a triangle, cf. Fig. 17. For that reason we need the rotation number of a web $W$ in canonical position in a face $T$, ...
defined as
\[ r_T = \sum_{\alpha} \varepsilon(\alpha), \]
where the sum is over all corner arcs of \( W \cap T \) and \( \varepsilon(\alpha) = 1 \) if \( \alpha \) is oriented clockwise and \( \varepsilon(\alpha) = -1 \) otherwise.

The two intersection coordinates per each edge \( \gamma_i \), and the rotation number per each face of the triangulation are simply called the coordinates of a web in a canonical position.

According to Corollary 19, the joy-sparking position of a weakly reduced web is unique up to flip moves and up to crossbar passes. Since these operations do not affect the coordinates, we say that the coordinates of any weakly reduced web \( W \) are those of \( W \) put into a canonical position. By from Sec. 3 that two weakly reduced webs are equivalent if they are related by isotopy and by permutations of parallel arc components. Since these operations preserve coordinates, we obtain:

**Corollary 21.** The above coordinates are well defined on weakly reduced webs considered up to equivalence.

These coordinates do not determine weakly reduced webs in when \( \partial \hat{F} \neq \emptyset \), because a boundary bigon may contain an arbitrary minimal crossbar web and those are not determined by intersection coordinates. However, somewhat surprisingly we have:

**Theorem 22** (Proof in Sec. 13).

Each reduced web in \( \hat{F} \) is determined up to equivalence by its coordinates.

For a web \( W \) in a canonical position, the degree of the pyramid in \( T \cap W \) is simply called the degree of \( W \) in \( T \). The proof of Theorem 22 will use the following observation: If \( T \) is bounded by \( \gamma_{i_0}, \gamma_{i_1}, \) and \( \gamma_{i_2} \), with the orientations induced by that of \( T \), then the degree of \( W \)
in $T$ is
\begin{equation}
\frac{1}{3} \sum_{k=0}^{2} e_{+\gamma_{ik}}(W) - e_{-\gamma_{ik}}(W).
\end{equation}

If $\gamma_{ik}$ is oriented in the opposite way to $\partial T$ then $e_{+\gamma_{ik}}(W)$ and $e_{-\gamma_{ik}}(W)$ need to be interchanged in the above formula.

13. Proof of Theorem 22

The coordinates of $W$ determine the degree of $W$ in every triangle of an ideal triangulation by (9). Furthermore:

**Lemma 23.** The coordinates of a web in canonical position determine the multiplicities of all six types of corner arcs in each triangle of the triangulation.

**Proof.** Let us assume as before that a face $T$ of triangulation is bounded by $\gamma_{i0}, \gamma_{i1},$ and $\gamma_{i2}$, with the orientations induced by that of $T$. (The proof for other orientations is analogous.) Let $e_{\pm, j} = e_{\pm, \gamma_{ij}}(W) - d$ for $i = 0, 1, 2$, for simplicity, where $d$ is the degree of the pyramid in $T$ given by (9). Then $e_{\pm,0}, e_{\pm,1}, e_{\pm,2}$ count the ends of corner arcs in $W \cap T$. Specifically, denote the numbers of corner arcs by $n_{i,j}$, for $i \neq j$, as in Figure 18.

![Figure 18. Corner arcs in a face of an ideal triangulation.](image)

Then
\begin{equation}
\begin{pmatrix}
  e_{+,0} \\
  e_{-,0} \\
  e_{+,1} \\
  e_{-,1} \\
  e_{+,2} \\
  e_{-,2} \\
  r
\end{pmatrix}
  =
  \begin{pmatrix}
    0 & 1 & 0 & 0 & 1 & 0 \\
    1 & 0 & 0 & 0 & 0 & 1 \\
    1 & 0 & 0 & 1 & 0 & 0 \\
    0 & 1 & 1 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 1 \\
    0 & 0 & 0 & 1 & 1 & 0 \\
    1 & -1 & 1 & -1 & 1 & -1
  \end{pmatrix}
  \cdot
  \begin{pmatrix}
    n_{01} \\
    n_{10} \\
    n_{12} \\
    n_{21} \\
    n_{20} \\
    n_{02}
  \end{pmatrix}
\end{equation}
The above $7 \times 6$-matrix has zero nullity and, hence, $e_{\pm,0}, e_{\pm,1}, e_{\pm,2}$ and the rotation number $r$ determine the numbers $n_{ij}$. □

**Proof of Theorem 22.** Let reduced webs $W_1, W_2$ in $\hat{F}$ have coinciding coordinates. Without loss of generality we can assume that they are in canonical position with respect to $\Gamma$. By [9] and Lemma [23], $W_1$ and $W_2$ coincide in each face of the triangulation up to a permutation of parallel corner arcs in different directions. Although the statement of the theorem is of topologically-combinatorial nature, we find it easiest to prove by applying some of the machinery of the $SU(3)$-skein algebras developed in this paper. Specifically, we are going to use the following trick: Let $a = 1$ in the coefficient ring $R$. Then by Proposition 4 and Figure 7 two reduced webs in $\hat{F}$ are equivalent iff they coincide in $GRS(\hat{F})$. Therefore, we will complete the proof by showing that $W_1 = W_2$ in $GRS(\hat{F})$.

For the sake of that proof it will be convenient to expand the class of webs under consideration. We say that a web $W$ is in an almost canonical position with respect to a padded ideal triangulation $\mathcal{N}(\gamma_1), \ldots, \mathcal{N}(\gamma_N)$ if it is in a canonical position in every face of the triangulation and it is a crossbar web in $\mathcal{N}(\gamma_i)$ for every $i$. However, we do not assume that $W$ is reduced and, hence, $W$ may contain 4-gons, either internal or external. Every internal 4-gon involves two parallel crossbars either inside of one bigon $\mathcal{N}(\gamma_i)$ or two different ones, cf. Figure 19.

However, by (2), the two parallel crossbars in any internal 4-gon can be removed without affecting the value of the web in $GRS(\hat{F})$, cf. Figure 19. That follows immediately from the fact that removing these crossbars corresponds to one of the resolutions of (2), while the second one reduces the intersection number with $\Gamma$ and, hence, vanishes in the graded skein algebra.

![Figure 19. 4-gon eliminations. (Equalities in $GRS(\hat{F})$)](image)

Although not essential for us here, one can easily prove that almost canonical webs have no 2-gons (neither internal nor external), nor trivial components (contractible loops), nor external trigons. In other words, almost canonical webs satisfy all conditions of being reduced, except possibly having quadrigons.
Let $W'$ be obtained now from $W_1$ by transposing two arcs in a face $T$ of the triangulation and by adding crossbars in the bigons on both sides of these arcs, as in Figure 20.

**Figure 20.** A transposition of arcs of opposite orientation.

Then $W'$ is in an almost canonical position and equals to $W_1$ in $\text{GRS}(\hat{F})$. Let us repeat this process, until we obtain a web which coincides with $W_2$ in all faces of the triangulation. This web is in an almost canonical position and it equals to $W_1$ in $\text{GRS}(\hat{F})$. Let $W''$ be the web obtained by the elimination of all 4-gons in the bigons. That elimination maintains its almost canonical position and does not affect its value in $\text{GRS}(\hat{F})$. Since $W''$ and $W_2$ coincide in all faces of the triangulation, the crossbar webs $W'' \cap B$ and $W_2 \cap B$ have coinciding signatures for internal every bigon $B$ of the triangulation. That is also the case for the external bigons since $W''$ and $W_2$ are trivial in these. Since the crossbar webs $W'' \cap B$ and $W_2 \cap B$ are minimal, they coincide by Lemma 18(3). Consequently, $W'' = W_2$. □

14. THE COORDINATE MONOID

The coordinates of the canonical webs with respect to a triangulation $\Gamma$ form a subset $C$ of $\mathbb{Z}^{\Gamma+\Gamma+\mathcal{F}(\Gamma)}_{\geq 0}$, where $\mathcal{F}(\Gamma)$ denotes the set of faces of $\Gamma$.

The profile of a canonical web $W$ is $\varepsilon_W : \mathcal{F}(\Gamma) \to \{\pm 1\}$, where $\varepsilon_W(T)$ is the sign of the degree of the pyramid of $W$ in an ideal triangle $T$ of the ideal triangulation. Denote the set of coordinates of all canonical webs of profile $\varepsilon$ by $S_\varepsilon \subset \mathbb{Z}^{\Gamma+\Gamma+\mathcal{F}(\Gamma)}_{\geq 0}$.

**Proposition 24.** $S_\varepsilon$ is a finitely generated additive submonoid of $\mathbb{Z}^{\Gamma+\Gamma+\mathcal{F}(\Gamma)}_{\geq 0}$ for every $\varepsilon$.

**Proof.** Consider first the 7 coordinates $(n_{01}, n_{10}, n_{12}, n_{21}, n_{20}, n_{02}, d) \in \mathbb{Z}^7_{\geq 0}$ for each face of $\Gamma$ for webs in $S_\varepsilon$, where $\varepsilon(T) \cdot d$ is the degree of the pyramid in $T$. Combined over all faces of $\Gamma$, they form an alternative set of web coordinates with values in $\mathbb{Z}^{7\mathcal{F}(\Gamma)}_{\geq 0}$.

If $T$ is bounded by $\gamma_{i_0}, \gamma_{i_1}, \gamma_{i_2}$, oriented according to the orientation of $T$ then these coordinates determine the intersection coordinates by
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\[
\begin{pmatrix}
 e_{+\gamma_0}(W) \\
e_{-\gamma_0}(W) \\
e_{+\gamma_1}(W) \\
e_{-\gamma_1}(W) \\
e_{+\gamma_2}(W) \\
e_{-\gamma_2}(W) \\
r
\end{pmatrix} =
\begin{pmatrix}
 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 1 & -1 & 1 & -1 & 1 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
 n_{01} \\
n_{10} \\
n_{12} \\
n_{21} \\
n_{20} \\
n_{02} \\
d
\end{pmatrix},
\]
assuming the degree of the pyramid in \( T \) is non-negative. For the negative degree one needs to interchange ones and zeros in the first six entries of the last column. The above numbers in \( \mathbb{Z}^{7|\mathcal{F}(\Gamma)|}_{\geq 0} \) are realized by a web if for every internal edge \( \gamma \) of the triangulation, the numbers \( e_{\pm \gamma}(W) \) coming from the two adjacent ideal triangles coincide. Such \( 7|\mathcal{F}(\Gamma)| \)-tuples form an additive submonoid of \( \mathbb{Z}^{7|\mathcal{F}(\Gamma)|}_{\geq 0} \). This monoid is finitely generated by Gordan’s Lemma. \( \mathcal{S}_\epsilon \) is the image of this monoid under the linear map given by the matrix above and, hence, a finitely generated (additive) monoid as well.

The empty web, which has all its coordinates zero is the unit of \( \mathcal{S}_\epsilon \).

15. PROOF OF THEOREM 11

Note that the degree \( d > 0 \) pyramid in a triangle \( T \) is obtained by \( I \)-resolutions of all crossings in the stack of triads, \( ST_d \), in Figure 21. (Negative degree pyramid is obtained from stacking sink triads.) Let

\[ W \]
be a non-elliptic web in \( \hat{F} \) containing \( T \) as a face of its triangulation. If \( W \cap T = ST_d \) then the geometric intersection number of \( W \) with \( \partial T \) is \( |ST_d \cap \partial T| = 3d \). Note that on the other hand, any web \( W' \) obtained by a vertexless resolution (as in Figure 3) of one of the crossings of \( ST_d \) contains a vertex with two edges connecting it the same side of
Corollary 25. Let $W$ be a web in $\hat{F}$ with $ST_d$ in $T$ and $W'$ the same web with the pyramid $P_d$ in $T$ instead. Then $W$ and $W'$ equal in $GRS(\hat{F})$ up to multiplication by a power of $q$.

Let webs $W_1, W_2$ be reduced webs in canonical positions, of the same profile $\varepsilon$. Let us stack them up so that:

- in each face $T$ of the ideal triangulation:
  - (a) the corner arcs of $W_1$ and of $W_2$ are disjoint,
  - (b) the pyramids of $W_1$ and of $W_2$ intersect like in the Figure 22. (Hence, if they are of degrees $d_1$ and $d_2$ then they intersect $d_1 \cdot d_2$ times.)
- in each bigon, the vertical lines of $W_1$ and of $W_2$ are disjoint, the crossbars (horizontal lines) are disjoint and any vertical line of one intersects a horizontal line of the other at most once.

![Figure 22. Stacking pyramids](image)

By (4) and (5), $W_1 \cup W_2$ (as above) equals to $W_1 \cdot W_2$ in $GRS(\hat{F})$ up to multiplication by a power of $q$.

Let $W_{12}$ be obtained by I-resolutions of all crossings of $W_1 \cup W_2$. Since $H_d$ can be obtained by I-resolutions of crossings in a horizontal stack of $d$ triads, the I-resolution of all vertices of the web in Fig. 22 yields a pyramid of degree $d_1 + d_2$, cf. Fig. 23.

![Figure 23. Horizontal triad stacks](image)

Hence, $W_{12}$ is non-elliptic and in canonical position in each triangle $T$ of the triangulation.
In principle, $W_{12}$ may contain however 4-gons in bigons, cf. [24].

![Figure 24. Left: Two crossbar webs stacked in a bigon. Middle: The result of I-resolving all crossings. Right: The result of 4-gon elimination.](image)

Each such 4-gon in a bigon $B$ resolves into two horizontal arcs plus two vertical ones, cf. [2]. However, the horizontal resolution leads to a web with a lower intersection number with $\partial_\pm B$. Let $\overline{W_{12}}$ be obtained from $W_{12}$ by resolving all 4-gons in bigons vertically. Then by the above discussion,

$$W_1 \cdot W_2 = q^c \cdot \overline{W_{12}}$$

in $\mathcal{GRS}(\hat{F})$ for some $c \in \frac{1}{3}\mathbb{Z}$. Observe also that the coordinates of $\overline{W_{12}}$ are the sum of those of $W_1$ and $W_2$:

$$\left(e_-(W_{12}), e_+(W_{12}), r(W_{12})\right) = \left(e_+(W_1), e_-(W_1), r(W_1)\right) + \left(e_+(W_2), e_-(W_2), r(W_2)\right) \in \mathbb{Z}_{\geq 0}^{\Gamma + \Gamma + \mathcal{F}(\Gamma)}.$$

Now we are ready for:

**Proof of Theorem 11**: By Proposition 24, $S_\varepsilon$ is generated by a finite set $G_\varepsilon \subset S_\varepsilon$ for every $\varepsilon : \mathcal{F}(\Gamma) \to \{\pm 1\}$. Consider the union $G = \bigcup_\varepsilon G_\varepsilon$ of these generating sets for all $\varepsilon$’s. We identify these elements with the reduced webs in $\hat{F}$ they correspond to through Theorem 22. We claim that they generate $\mathcal{GRS}(\hat{F})$. To this end we will show that every reduced web $W$ is a polynomial in those in $G$ by contradiction: Let $W$ be a web of smallest weight for which it is not the case. By Theorem 20, $W$ can be put into canonical position. Let $\varepsilon$ be the profile of $W$. The coordinates of $W$ are either one of those in $G_\varepsilon$, implying that $W \in G$, or they decompose

$$\left(e_+(W), e_-(W), r(W)\right) = \left(e_+(W_1), e_-(W_1), r(W_1)\right) + \left(e_+(W_2), e_-(W_2), r(W_2)\right),$$

for some non-empty webs $W_1, W_2$, which by our assumption are polynomial expressions in elements of $G$. By (11) and (12), $W$ equals $q^c \cdot W_1 \cdot W_2$ in $\mathcal{GRS}(\hat{F})$, for some $c \in \frac{1}{3}\mathbb{Z}$. Since $W_1, W_2$ are of lower weight than $W$, that contradicts the assumption of $W$ being a lowest weight web which is not a polynomial expression in the webs in $G$. **
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