Topological Features in Ion Trap Holonomic Computation

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Topological features in quantum computing provide controllability and noise error avoidance in the performance of logical gates. While such an issue is easily dealt with in classical computation, the quantum regime involves intrinsic characteristics which make quantum error avoidance or error correction an important issue for QC. The possible errors, due to the smallness of the system under consideration, include the questionable viability of the approximations considered and the controllability of the experimental setup used to reproduce the desired ideal regimes, as well as thermal noise and spontaneous emission. Two main approaches have been developed in order to overcome these problems. The use of error-correcting algorithms [1], which theoretically enable QC to be carried out even in the presence of quantum noise, and error-avoiding schemes, which produce a computational (decoherence-free [2–4]) subspace of the whole Hilbert space of the system, that is subject to minimal error fluctuations. Such schemes include topological quantum computation [9–11], the use of cooling, adiabaticity or the Zeno effect [4]. Both strategies function with resources additional to those used for ideal encoding and processing quantum algorithms. This extension of resources, properly treated allows the overall control or noise error to be decreased [25]. There is an analogy between the algorithmic strategies and these engineered setups. Loosely speaking, one may regard the error-correcting algorithm as “simulating” a corresponding error-avoiding physical process.

In the literature there has been much interest in topological quantum computation. Ideally, it constitutes within a certain control procedure error-free computation. Although the proposed schemes have hitherto been hard to realize experimentally, the concept is worth investigating. Our particular aim here is to identify some topological features from the geometrical ones [12,13] appearing in the ionic setup. Let us focus on the case of small errors, called here ε. While topological QC has no dependence in ε in all orders, our aim with geometrical QC is to “neutralize” our gates only up to a finite order in ε. As an example, it is easy to visualize in our setup cancellation of the first order in statistical errors in the variables the experimenter is controlling. The parameters Σ of the gates presented here are engineered functions of the experimental parameters in contrast to dynamic QC, where the gate parameters are merely linear functions. In particular, each Σ can be interpreted as the area of a contour lying on a specific surface. This area is resilient to the first order in statistical fluctuations of its border which the experimentalist is traversing [12,15]. The weight factor in the surface integral Σ will further govern the robustness or weakness of the gates.

Let us initially give in an abstract way the main ideas of holonomic quantum computation (HQC) [12–19]. Holonomies are a generalization of the geometrical Berry phases [22] to the case of a multiple, e.g. 2n-fold, degenerate Hamiltonian $H_0$. The quantum information is encoded in an 2n-dimensional, degenerate eigenspace $C$ of $H_0$, with eigenvalue $E_0$, which is usually taken to be the lowest ground energy. The operator $H_0$ is considered to belong to the family $\mathcal{F} = \{ H_\sigma = U(\sigma) H_0 U^\dagger(\sigma) ; \sigma \in \mathcal{M} \}$ of Hamiltonians unitarily $\{U(\sigma) = U^{-1}(\sigma)\}$ equivalent to and therefore iso-spectral with $H_0 = H_{\sigma_0}$ for some $\sigma_0 \in \mathcal{M}$. The $\sigma$’s represent the classical “control” parameters used to manipulate the encoded states $|\psi\rangle \in \mathcal{C}$. Let $C$ be a loop in the control manifold $\mathcal{M}$. When $C$ is slowly traversed, then the evolution is adiabatic and no population is transferred between different energy levels. If $|\psi\rangle_{in} \in \mathcal{C}$ is an initial state in the degenerate eigenspace, at the end of the loop it becomes $|\psi\rangle_{out} = e^{-iT_\mathcal{A}(C)} |\psi\rangle_{in}$, which still belongs in the same degenerate subspace. The first factor is just an overall dynamical phase which is omitted in the following by redefining the energy levels, i.e. by taking $E_0 = 0$. The second contribution is the holonomy $\Gamma_\mathcal{A}(C) \in U(2^n)$ and is a result of the non-trivial topology of the bundle of eigenspaces over $\mathcal{M}$. By introducing the Wilczek-Zee
where $A^{\nu}_{\sigma_i}$ is the $(\nu, \nu)$ matrix element of the $\sigma_i$ component of the connection, one finds $\Gamma_A(C) = \mathbf{P} \exp \int_C A$, where $\mathbf{P}$ denotes path ordering. The set $H(A) := \{ \Gamma_A(C) \; ; \; \forall C \in \mathcal{M} \} \subset U(2^n)$ is known as the holonomy group \cite{23}. In the case where it coincides with the whole unitary group $U(2^n)$ the connection $A$ is called irreducible \cite{13}. The transformations $\Gamma_A(C)$ for suitable $C$’s can be used as logical gates for HQC.

Let us apply these ideas to the ion traps. First of all we need to identify degeneracy between two states. This is provided by the lowest eigenstates of a Jaynes-Cummings Hamiltonian describing two internal ionic levels “dressed” with the vibrational modes of the ion. Manipulating the vibrational modes in an adiabatic cyclic fashion generates a holonomic operator which evolves the degenerate dressed states. These control procedures, which include one and two-mode displacing and squeezing, can be performed by employing lasers which make coherent manipulations between the Fock states of the harmonic modes. Let us look more explicitly at these manipulation and the conditions that need to be satisfied in order to obtain degeneracy.

II. SQUEEZING AND DISPLACING OF VIBRONIC OSCILLATING MODES

In the following, the aim is to produce squeezing and displacing operations on the oscillating mode of a trapped ion. This is produced with lasers which couple the vibrational mode with two internal energy levels of the atom. In particular, two standing-wave lasers are to be used for squeezing. With their appropriate frequencies they will produce coherences between the vibrational states $|n\rangle$ and $|n+1\rangle$ which simulate a squeezing operation. In addition, a traveling wave will be used to insert a displacing term in the overall Hamiltonian. Let us describe briefly the setup first introduced in \cite{5}.

A two-level trapped ion assumed to be located at a common node of two standing-wave laser fields with the frequencies located symmetrically about a carrier frequency $\omega_0$ at which we put an additional traveling-wave field \cite{5}. Using traveling waves for all lasers will produce the same result \cite{6}, with simplifications in the experimental setup, even though it might make it more difficult to meet our theoretical limits \cite{7}. The resulting field $E^{(+)}(\hat{R},t)$ is given by

$$E^{(+)}(\hat{R},t) = E_a \sin(k_a \hat{R})e^{-i(\omega_0-\nu)t-i\phi_a} + E_b \sin(k_b \hat{R})e^{-i(\omega_0+\nu)t-i\phi_b} + E_0 e^{-i\omega_0 t} e^{-i\phi_0}.$$  \hspace{1cm} (2)

The Lamb-Dicke parameter $\eta_j$ is defined by $k_j \hat{R} \equiv \eta_j (a + a^\dagger)$, where $\eta_j = \pi a_0/\lambda_j$ with $a_0$ being the amplitude of the ground state of the trap potential and $\lambda_j$ the optical wavelength. Attention is confined here to the experimentally viable Lamb-Dicke limit $\eta_j \ll 1$ and hence the Hamiltonian can be expanded to the first order in $\eta_j$. Let us define a rotating frame given by the unitary transformation $\rho' = U^\dagger \rho U$ where $U = \exp[-i(\nu a^\dagger a + \omega_0 \sigma_z)/2t]$ and $\rho$ is the density matrix of our system. By omitting the fast oscillating terms by assuming that $\nu$ is much larger than any other parameters characterizing the system, the master equation for $\rho'$ is given by

$$\frac{d\rho'}{dt} = -i[H', \rho'],$$  \hspace{1cm} (3)

where
\[ H' = \sigma_+ (g_\sigma a + g_\sigma a^\dagger) + \frac{\Omega_0}{2} \sigma_+ + \text{h.c.}, \]

where \( a, a^\dagger \) are the annihilation and creation operators for the trap motion, \( \sigma_+ := |e\rangle \langle g|, \sigma_- := |g\rangle \langle e| \) and \( \sigma_z := |e\rangle \langle e| - |g\rangle \langle g| \) are the Pauli matrices describing the two-level transition of frequency \( \omega_0 \). For the above the following conditions were taken; for \( g_j = \eta_j \Omega_j e^{-i\phi_j} / 2 \) with \( \Omega_j \) being the real Rabi frequency of the \( j \)-th laser \( (\phi_0 = 0) \) it is assumed that \( \Omega_0 \ll \Omega_{1,2} \) so that terms of order \( \eta_0 \Omega_0 \) can be neglected.

In this Hamiltonian the excitation of the ion is coupled with two transitions in the vibrational modes. The first, produced by laser 1, is a lowering of the vibrational number from \( n \) to \( n-1 \), while the second, laser 2, raises the vibrational number from \( n \) to \( n + 1 \). Coherent application of these two processes establishes Raman transitions between alternative vibrational states \( (\text{i.e.} \ |n\rangle \text{ and } |n+1\rangle) \), finally giving a nonlinear effect. It is interesting to note that performing these transitions without the intermediate atomic levels would require quadratic terms in the creation and annihilation operators. Going back to a resonance frame by making the inverse transformation \( \tilde{U} = \exp[-i(na^\dagger a + g\sigma_z/2)t] \) one obtains

\[ P \equiv S|\epsilon\rangle D^1(\alpha) \rho D(\alpha)|\epsilon\rangle, \]

satisfying \( dP/dt = -i[H_{JC}, P] \) with

\[ H_{JC} = \nu a^\dagger a + \frac{g}{2} \sigma_z + g (\sigma_+ a e^{-i(g-\nu)t} + a^\dagger \sigma_- e^{i(g-\nu)t}) , \]

and \( \epsilon = r_1 e^{i\theta_1}, r_1 = -1/2 \tan^{-1}(|g_\sigma|/|g_a|), \theta_1 = \phi_a - \phi_b + 2 \nu t, \alpha = -\Omega_0/(2g)e^{i\nu t}, g = |g_a| \cosh r_1 - |g_b| \sinh r_1 \). For \( g = \nu, H_{JC} \) is the Hamiltonian of the Jaynes-Cummings model describing a two-level atom coupled to a harmonic oscillator.

### III. JAYNES-CUMMINGS MODEL AND DEGENERACY

Holonomic quantum computation is based on cyclic adiabatic transformations which allow exchange of populations between states with the same energy. Degeneracy of energy thus has to be introduced in our system. The natural way to achieve this with the JC model is to tune the parameter \( g \) in such a way that two eigenstates of the Hamiltonian become degenerate. In the JC model derived in the foregoing the Hamiltonian has the ground internal state \( |g\rangle \) with energy \( \omega_1 = -g/2 \) and the excited state \( |e\rangle \) with energy \( \omega_2 = g/2 \). The lowest energy eigenstate of \( H_{JC} \) is \( |g, 0\rangle = |g\rangle \otimes |0\rangle \) given as the tensor product of the internal and the vibronic ground states, and it has the energy \( E_0 = \hbar \omega_1 \). The rest of the eigenstates (dressed states) are given, for the resonant case \( \nu = \omega_2 - \omega_1 = g \), by

\[ |n, \pm\rangle = \sqrt{\frac{1}{2}} (|g, n+1\rangle \pm |e, n\rangle) , \]

with the eigenvalues \( E_{n,\pm} = \hbar (\omega_1 + \nu(n+1) \pm \sqrt{n+1}) \), respectively. The resonant condition \( g = \nu \) also creates the desired degenerate condition \( E_{\text{deg}} = E_0 = E_{0-} \) and hence a two-dimensional encoding space is spanned by the orthogonal states

\[ \{ |g, 0\rangle, |0, -\rangle = \sqrt{\frac{1}{2}} (|g, 1\rangle - |e, 0\rangle) \} \equiv \{ |i\} / i = 0, 1 \} , \]

where \( |0\rangle \) and \( |1\rangle \) are the qubit states. None of the rest of the dressed states can have eigenvalues the same as \( E_{\text{deg}} \), which makes the degeneracy strictly twofold. The energy of this subsystem is \( E_{\text{deg}} = \hbar \omega_1 \) and, in contrast to the rest of \( E_{n,\pm} \), it does not depend on the trap frequency \( \nu \). The immediately higher energy state is the \( |1-\rangle \) state with energy \( E_{1-} = \hbar (\omega_1 + (2 - \sqrt{2})\nu) \). The energy difference between it and the degenerate states gives the energy scale with which adiabatic changes have to be compared in order not to have mixing of our encoding states with the rest of the dressed states.

The condition \( g = \nu \) for producing degeneracy has to be compared with the derivation of the Hamiltonian \( H_0 \) where we omitted fast oscillating terms with frequency \( 2\nu \) in relation to \( g = \nu \). A more plausible condition can be obtained if the constructed degeneracy is between the states \( |0, +\rangle \) and \( |1, -\rangle \), where the omitted terms oscillate with frequencies six times as large as the terms depicted in Hamiltonian \( H_0 \). In this case simulations showed that squeezing of the vibrational states can be faithfully produced with maximum amplitude \( r_1 \approx 2 \). Even though the results will
be the same as those presented here, the mathematical derivation is somewhat more complicated, and so attention
will be restricted to the simple case of the degeneracy between \([g, 0]\) and \([0, −]\). In addition, having strong lasers may
influence the internal structure of the atoms due to the Autler-Townes splitting \([17]\). Here we consider the case of
weak binding limit where along the variation of the laser amplitudes the internal atomic levels employed here remain
unaltered.

In the previous section, it was seen that the states of the Hamiltonian \(H\) are the squeezed and displaced states
of \(H_{JC}\). Hence, \(H\) has a degenerate subspace spanned by the rotated basis \(\{|α, ε; i⟩ ≡ D(α)S(ε)|i⟩\ / i = 0, 1\}\). This
allows the degenerate states to be displaced and squeezed at will by varying the phases and amplitudes of the
electric fields of the lasers. Note that the dressing of the internal states with the vibrational modes provides us
with full controllability of the qubit encoded in the two-dimensional degenerate subspace simply by manipulating the
vibrational modes without employing transitions between the internal states.

### IV. ONE-QUBIT HOLONOMIC GATES

At this point we have all the ingredients necessary for performing holonomic quantum computation for one qubit.
The mathematical steps now to be performed are similar to those used to calculate the holonomies for the optical setup \([13]\). With \(H_0\) being the JC Hamiltonian and \(M\) the parametric space spanned by the coordinates \(\{x, y, r_1, θ_1\}\), for
\(α = x + iy\) and \(ε = r_1e^{iθ_1}\), we are able to perform iso-spectral transformations belonging to the family \(\mathcal{F} = \{H_σ = U(σ) H_0 U(σ) / \ σ \in M; U(σ) = D(α)S(ε)\}\) of Hamiltonians unitarily equivalent to and therefore iso-spectral
with \(H_0\), where \(H_0 = H_{σ_0}\) for some \(σ_0 \in M\). With the operator \(\hat{A}_σ = U^{†} \frac{∂}{∂σ} U\) defined for \(U = D(α)S(ε)\), the
holonomic connection \(A_σ\) of our model has the matrix elements \(A_{ij} = \langle i|\hat{A}_σ|j⟩\). Their components are given explicitly
in the following formulas: \(\hat{A}_σ = a_1 \cosh 2r_1 + ae^{−iθ_1} \sinh 2r_1 + \bar{a}_2, \ \bar{A}_σ = −\bar{A}_\bar{σ}, \ \bar{A}_{r_1} = e^{iθ_1} a_1^2 - e^{−iθ_1} a_2^2, \ \bar{A}_{θ_1} = \frac{1}{2}(\cosh 4r_1 − 1)(2a_1^1 a_1 + \frac{1}{2}\sinh 4r_1(e^{iθ_1} a_1^2 + e^{−iθ_1} a_2^2), \text{ eventually yielding}\)

\[
A_x = A_α + A_\bar{α} = \begin{bmatrix}
-iy & -\frac{1}{\sqrt{2}}(\cosh 2r_1 - e^{iθ_1} \sinh 2r_1) \\
\frac{1}{\sqrt{2}}(\cosh 2r_1 - e^{iθ_1} \sinh 2r_1) & iy
\end{bmatrix}
\]

\[
A_y = i(A_α - A_\bar{α}) = \begin{bmatrix}
ix & \frac{1}{\sqrt{2}}(\cosh 2r_1 + e^{iθ_1} \sinh 2r_1) \\
\frac{1}{\sqrt{2}}(\cosh 2r_1 + e^{iθ_1} \sinh 2r_1) & ix
\end{bmatrix}
\]

\[
A_{r_1} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}, \quad A_{θ_1} = \begin{bmatrix}
1 & 0 \\
0 & \frac{3}{2}
\end{bmatrix} \frac{1}{4}(\cosh 4r_1 - 1).
\]

Let us briefly mention that these connection components, \(A_σ\), give rise to two particular field strength components,
\(F_{σ_1, σ_2} = ∂_σ A_σ - ∂_σ A_σ + [A_σ, A_σ],\) analytically given by

\[
F_{r_1x} = -i \begin{bmatrix}
0 & -i \\
i & 0
\end{bmatrix} \sqrt{2}e^{-2r_1}, \quad \text{for } θ_1 = 0, \quad (9)
\]

and

\[
F_{r_1y} = i \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \sqrt{2}e^{-2r_1}, \quad \text{for } θ_1 = π. \quad (10)
\]

Holonomic gates may now be constructed by traversing closed paths in \(M\). In particular, we shall choose two convenient
planes in the four-dimensional control manifold and evaluate the holonomies generated by any loop lying on those
planes. Explicitly, the loop \(C_I \in (x, r_1)_{θ_1 = 0}\) gives \(Γ_A(C_I) = \exp -i\hat{Σ}_I\), with \(Σ_I := \int_{D(C_I)} dxdr_1e^{-2r_1}\). The loop
\(C_{II} \in (y, r_1)_{θ_1 = π}\) gives \(Γ_A(C_{II}) = \exp i\hat{Σ}_{II}\), with area \(Σ_{II} := \int_{D(C_{II})} dydr_1e^{-2r_1}\). In the above, \(D(C_{ρ})\) with
\(ρ = I, II\) is the surface on the relevant sub-manifold \((σ_I, σ_J)\) of \(M\) whose boundary is the path \(C_{ρ}\). These two
unitaries, \(Γ_A(C_I)\) and \(Γ_A(C_{II})\), are sufficient to produce any one qubit gate. The exponential damping feature of the
field strength has been transported into the parameter of the holonomic gates, \(Σ_{ρ}\), making them resilient to control
efforts for large values of the squeezing parameter, \(r_1\). This point will be elaborated in a following section.
An N qubit system can be realized with a chain of N ions. The internal degrees of freedom of each ion are represented by a two-level system, while the external degrees of freedom are described by the collective modes of the crystal. These are the modes which diagonalize the potential (assumed here to be harmonic) and the Coulomb repulsion between the ions. For sufficiently cold crystals the ions oscillate harmonically around their equilibrium position. Each qubit is composed of a two-level system corresponding to the internal two-level transition of the ion, together with a collective mode of the motion. We produce a two-qubit gate by interactions between the collective modes of each qubit. This interaction is realized with a two-mode squeezing or displacing transformation. In the following, these transformations are constructed for two ions.

In the case of a chain of two ions the collective modes are the “center of mass” mode, where the two ions oscillate in phase, and the “stretching” mode, where the two ions oscillate out of phase. Let us define qubit 1 as the combination of the internal levels of ion 1 with the center of mass mode, and qubit 2 as the two levels of ion 2 together with the stretching mode, analogously defined as in eq. (8). Let \( R_i \) be the displacement of ion \( i \) from its equilibrium position and let the oscillations of the ions described by two harmonic oscillators of frequency \( \nu_1, \nu_2 = \sqrt{3} \nu_1 \) for the center of mass and the stretching modes, respectively. The quantized oscillations are described by the operators \( \hat{a}_i, \hat{a}_i^\dagger \), with \( i = 1, 2 \), which are the annihilation and creation operators of the center of mass and the stretching mode, respectively.

Let us consider the following Hamiltonian of two qubits where only the first is coupled with both collective vibronic modes via the appropriately detuned lasers:

\[
H_1 = \nu_1 a_1^\dagger a_1 + \nu_2 a_2^\dagger a_2 + \frac{\omega_1}{2} \sigma_y^1 + \frac{\omega_2}{2} \sigma_y^2 - \frac{d}{\hbar} \left( \sigma_z^1 E^{(+)}(\hat{R}_1, t) + \sigma_z^1 E^{(-)}(\hat{R}_1, t) \right).
\]

(11)

The laser field acts on the first ion only, but couples it to both oscillating modes by the following configuration:

\[
E^{(+)}(\hat{R}_1, t) = E_a \sin(k_a \hat{R}_1) e^{-i(\omega_L - \nu_2)t + i
\phi_a} + E_b \sin(k_b \hat{R}_1) e^{-i(\omega_L + \nu_1)t - i\phi_b},
\]

(12)

with \( k_j \hat{R}_1 = \eta_j^{(1)} (a_1 + a_1^\dagger) + \eta_j^{(2)} (a_2 + a_2^\dagger) \), where index on the frequencies and on the creation and annihilation operators distinguishes between the two collective modes and the index on the Pauli matrices distinguishes between the two atoms. In the Lamb-Dicke limit we are able to expand the trigonometric functions up to the first order in both of the parameters \( \eta^{(1)} \) and \( \eta^{(2)} \). By moving to the rotated frame defined by \( \rho' = \mathcal{U}^\dagger \rho \mathcal{U} \), where \( \mathcal{U} = \exp[-i(\nu_1 a_1^\dagger a_1 + \nu_2 a_2^\dagger a_2 + \frac{\omega_1}{2} \sigma_z^1 + \frac{\omega_2}{2} \sigma_z^2)t] \) we neglect all fast oscillating terms. These are phase factors with rotating frequencies \( \nu_{1,2} \) or \( \nu_1 \pm \nu_2 \). The resulting master equation is given by

\[
\frac{d\rho'}{dt} = -i[H_1', \rho'],
\]

(13)

with

\[
H_1' = g_a^{(2)} \sigma_y^1 a_2 + g_b^{(1)} \sigma_y^2 a_1 + \text{h.c.},
\]

(14)

where \( g_j^{(i)} = \eta_j^{(i)} \Omega_j e^{-i\phi_j}/2 \). Acting on the second ion with the detuning of the lasers exchanged between the two collective frequencies, we obtain the following Hamiltonian

\[
H_2' = g_a^{(1)} \sigma_y^2 a_1 + g_b^{(2)} \sigma_y^1 a_2 + \text{h.c.}.
\]

(15)

If the two interactions described in \( H_1 \) and \( H_2 \) are turned on simultaneously then back to a resonant frame a two-mode squeezed density matrix \( P = M(\zeta) (\rho_1 \otimes \rho_2) M(\zeta) \) is produced satisfying \( dP/dt = -i[H_{rot}, P] \), where the JC Hamiltonian of the two atoms is given by

\[
H_{rot} = \nu_1 a_1^\dagger a_1 + \nu_2 a_2^\dagger a_2 + \frac{g(1)}{2} \sigma_z^1 + \frac{g(2)}{2} \sigma_z^2 + \{ g(1) \sigma_z^1 a_1 + g(2) \sigma_z^2 a_2 + \text{h.c.} \},
\]

(16)

with \( M(\zeta) = \exp(\zeta a_1^\dagger a_2 - \bar{\zeta} a_1 a_2) \), \( \zeta \) being rotated in the interaction frame defined by \( \mathcal{U} \) and given by \( \zeta = r_2 e^{i\phi_2}, r_2 = -\tanh^{-1}(|\eta_1^{(1)}|/|\eta_2^{(1)}|) = -\tanh^{-1}(|\eta_1^{(2)}|/|\eta_2^{(2)}|), g(1) = |\eta_1^{(1)}| \cos r_2 - |\eta_2^{(2)}| \sinh r_2, g(2) = |\eta_2^{(2)}| \cos r_2 - |\eta_1^{(1)}| \sinh r_2 \) and \( \phi_2 = \phi_b + (\nu_1 + \nu_2)t \), where we have taken \( \phi_a = 0 \). In the same way, by interchanging the actions of the two laser fields on the two atoms it is possible to obtain two-mode squeezing and two-mode displacing transformations, the latter being defined by the unitary operator \( N(\zeta) = \exp(\zeta a_1^\dagger a_2 - \bar{\zeta} a_1 a_2) \).
VI. TWO-QUBIT HOLONOMIC GATES

If the ground states of the two atoms $|g\rangle_1$ and $|g\rangle_2$ are kept on the same energy level $\omega_1 = \omega_2^2$, then the two degenerate spaces of the two JC models, created by the conditions $g_{(1)} = \nu_1$ and $g_{(2)} = \nu_2$, are also degenerate between each other. Hence, any adiabatic evolution will allow the mixing of these four degenerate states, but they are kept isolated from the rest state space. The operators $\hat{A}_{r_2} = M^t(\zeta) \frac{\partial}{\partial r} M(\zeta)$ and $\hat{A}_{r_3} = M^t(\zeta) \left( N^t(\xi) \frac{\partial}{\partial r} N(\xi) \right) M(\zeta)$ give rise to the connection components $A^{ij}_\sigma = \langle i | \hat{A}_\sigma | j \rangle$ for $|i\rangle$ and $|j\rangle$ belonging to the tensor product basis of the two qubits $|1\rangle = |00\rangle$, $|2\rangle = |01\rangle$, $|3\rangle = |10\rangle$ and $|4\rangle = |11\rangle$. Hence, we obtain

$$\hat{A}_{r_2} = e^{i\theta_2}a_1^\dagger a_2 - e^{-i\theta_2}a_1a_2, \quad \hat{A}_{r_3} = (e^{i\theta_3}a_1^\dagger a_2 - e^{-i\theta_3}a_1a_2)(\cosh^2 r_2 + \sinh^2 r_2) + e^{i\theta_4}(e^{i\theta_2}a_1^\dagger a_2^\dagger + e^{-i\theta_2}a_2^\dagger a_1^\dagger) \sinh r_2 \cosh r_2 - e^{-i\theta_3}(e^{-i\theta_2}a_1^\dagger a_2 + e^{-i\theta_2}a_2^\dagger a_1) \sinh r_2 \cosh r_2,$$

$$A_{r_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{i\theta_2} & 0 & 0 & 0 \end{bmatrix}, \quad A_{r_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -e^{-i\theta_3} & 0 \\ 0 & e^{i\theta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sqrt{2} (2 \cosh^2 r_2 - 1).$$

The relevant field strength component is given by

$$F_{r_2r_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -e^{-i\theta_3} & 0 \\ 0 & e^{i\theta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sqrt{2} \sinh 2r_2.$$

Two-qubit holonomic gates can be produced by performing the following cyclic evolutions. By means of a loop $C_{III} \in (r_2, r_3)_{\theta_2, \theta_3 = 0}$ we obtain $\Gamma_A(C_{III}) = \exp -i\hat{\sigma}_2^{(12)} \Sigma_{III}$, with $\Sigma_{III} := \int_{D(C_{III})} d^2r_2 d^2r_3 \sinh 2r_2$. For $C_{IV} \in (r_2, r_3)_{\theta_2 = 0, \theta_3 = \pi/2}$ the following holonomy is generated: $\Gamma_A(C_{IV}) = \exp -i\hat{\sigma}_1^{(12)} \Sigma_{IV}$, with the area given by $\Sigma_{IV} := \int_{D(C_{IV})} d^2r_2 d^2r_3 \sinh 2r_2$. In the above we have used

$$\hat{\sigma}_2^{(12)} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \hat{\sigma}_1^{(12)} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. $$

As an example, $\Gamma_A(C_{IV})$ with $\Sigma_{IV} = \pi/4$ gives the following non-trivial two-qubit gate:

$$U = \sqrt{\frac{1}{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix},$$

which together with the holonomies produced by loops $C_1, C_{III}$ compose a universal set of transformations [24].

VII. MEASUREMENT PROCEDURE

At the end of the holonomic procedures it should be possible to distinguish between the states $|g, 0\rangle$ and $|0, -\rangle$ by a measurement. For this purpose, a $\pi/2$ pulse is applied to the ion:

$$U(\phi) = \exp \left[ -i\frac{\pi}{4}(|e\rangle\langle g|e^{-i\phi} + h.c.) \right]. \tag{18}$$

This evolution operator will not affect the state $|g, 0\rangle$, but only $|0, -\rangle$. In the rotated frame we are working in, the phase $\phi$ is shifted to $\phi' = \phi + (\omega_0 - g)$. Hence, the state $|0, -\rangle$ in the laboratory frame becomes $-e^{-i\frac{\pi}{2}g}(|e\rangle, 0)$ when the timing is such that $\phi = -\pi/2$. Direct measurement of the atomic energy levels distinguishes between the two logical states of the qubits.
VIII. CONTROL ERROR RESILIENCE

For the one-qubit gates presented in Section IV the weight factor of the surface integrals $\Sigma_I$ and $\Sigma_{II}$ is $e^{-2r_1}$. Let us introduce errors in the displacing and squeezing parameters. For the rectangular loop of Fig. 2(a) we may take the borders, instead of being positioned at $(x, r_1)$, to be at $(x + \epsilon_1, r_1 + \epsilon_2)$ for small errors $\epsilon_1$ and $\epsilon_2$.

Then the areas $\Sigma_I$ and $\Sigma_{II}$ are varied by

$$\Delta \Sigma = \epsilon_1 \left(1 - e^{-2(r_1 + \epsilon_2)}\right) + xe^{-2r_1} \left(1 - e^{-2\epsilon_2}\right),$$

where its dependence with respect to both errors is given in Fig. 3. For values of $r_1 \approx 2$ the error $\epsilon_2$ is exponentially suppressed in relation to the contribution of $\epsilon_1$, which is linear.

The resilience to the squeezing error is of the following nature. As has been seen, the gate parameter can be viewed as the flux of the field strength $F$ (which can be regarded as a virtual magnetic field). Its amplitude decreases for large $r_1$. Hence, a loop which has one boundary at a large value of $r_1$ will include the same flux for small deformations of its boundary owing to the exponential dumping of the field strength amplitude. This characteristic is conceptually very interesting. A non-trivial bounded geometry, i.e. one whose curvature decreases exponentially from a fixed point, looks like topology from far away. A simple analogy can be drawn with the Aharonov Bohm effect. There, a magnetic field is bounded in a solenoid and a charged particle travels around it. If the particle performs a loop inside the solenoid, it acquires a phase given by the enclosed flux of the magnetic field. This has a one-to-one correspondence with the geometrical phases. But if the particle travels far away from the solenoid, it acquires a fixed amount of phase, varying only by the integer number of circulations the particle makes, which is a topological variable.
FIG. 3. The error in the gate parameter, $\Delta \Sigma$, as a function of the displacing ($\varepsilon_1 = \Delta x$) and squeezing ($\varepsilon_2 = \Delta r_1$) errors. The latter has a much smaller (about two orders of magnitude) influence on $\Delta \Sigma$. The dimensions of the rectangle are taken to be $x = 1$ and $r_1 = 2$.

IX. CONCLUSIONS

In this article, we have constructed an ionic setup where qubits are encoded in the space of degenerate dressed states constructed from internal energy levels of the atoms combined with their vibrational modes. The logical gates are represented by the holonomies acting on the degenerate states. They are produced by adiabatic cyclic evolutions of the oscillating motion of the ions. These holonomies are nonlinear functions of the parameters the experimenter is controlling. This provides a wide range of possibilities for constructing gates resilient to control errors. In particular, we have seen one-qubit gates which afford exponential suppression of errors.

For two qubits the two-mode squeezings and displacings produce holonomies with hyperbolic dependence on the control parameters. It is possible to use two-mode control manipulations of higher order in the creation and annihilation operators than the bilinear forms we have seen with $M(\zeta)$ and $N(\xi)$. Even though their transformations have not been mathematically studied sufficiently to evaluate their holonomies, they appear promising for producing trigonometric [29] or even experimentally decreasing functions of the control parameters, while they are experimentally easy to construct [30].

For quantum computation the threshold of error rate before which error correction can be efficiently applied is of the order of $10^{-4}$. This very low limit is more likely to be achieved by advantageous manipulation strategies rather than by just trying by brute force to improve the controllability of our external parameters. Holonomies provide the advantage that the parameters of our gates are engineered functions of the experimental parameters. Furthermore, they are resilient to statistical errors, thus providing overall an appealing framework for quantum computation.

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