Commutants and Reflexivity of Multiplication tuples on Vector-valued Reproducing Kernel Hilbert Spaces

Sameer Chavan, Shubhankar Podder, and Shailesh Trivedi

Abstract. Motivated by the theory of weighted shifts on directed trees and its multivariable counterpart, we address the question of identifying commutant and reflexivity of the multiplication $d$-tuple $\mathcal{M}_z$ on a reproducing kernel Hilbert space $\mathcal{H}$ of $E$-valued holomorphic functions on $\Omega$, where $E$ is a separable Hilbert space and $\Omega$ is a bounded domain in $\mathbb{C}^d$ admitting bounded approximation by polynomials. In case $E$ is a finite dimensional cyclic subspace for $\mathcal{M}_z$, under some natural conditions on the $B(E)$-valued kernel associated with $\mathcal{H}$, the commutant of $\mathcal{M}_z$ is shown to be the algebra $H^\infty_{B(E)}(\Omega)$ of bounded holomorphic $B(E)$-valued functions on $\Omega$, provided $\mathcal{M}_z$ satisfies the matrix-valued von Neumann’s inequality. This generalizes a classical result of Shields and Wallen (the case of $\dim E = 1$ and $d = 1$). As an application, we determine the commutant of a Bergman shift on a leafless, locally finite, rooted directed tree $\mathcal{T}$ of finite branching index. As the second main result of this paper, we show that a multiplication $d$-tuple $\mathcal{M}_z$ on $\mathcal{H}$ satisfying the von Neumann’s inequality is reflexive. This provides several new classes of examples as well as recovers special cases of various known results in one and several variables. We also exhibit a family of tri-diagonal $B(C^2)$-valued kernels for which the associated multiplication operators $\mathcal{M}_z$ are non-hyponormal reflexive operators with commutants equal to $H^\infty_{B(C^2)}(\mathbb{D})$.

1. Introduction

This paper is motivated by some recent developments pertaining to the function theory of weighted shifts on rooted directed trees and its multivariable counterpart (refer to [32, 17, 36, 13, 15, 16]). In particular, it is centered on the investigation of two topics from classical function-theoretic operator theory, namely commutants and reflexivity of multiplication tuples on reproducing kernel Hilbert spaces of vector-valued holomorphic functions (refer to [50, 44, 20] for a comprehensive account on commutants and reflexivity of unilateral weighted shifts and multiplication operators on reproducing kernel Hilbert spaces of scalar-valued holomorphic functions; refer to [42, 20, 23] for a masterful exposition on reflexivity of algebras of commuting operators). Via a construction of Shimorin [53], any bounded linear left-invertible weighted shift on a rooted directed tree can be modeled as the operator of multiplication by the coordinate function on a reproducing kernel Hilbert space of $E$-valued holomorphic functions [17], where $E$ is a separable Hilbert space.

2010 Mathematics Subject Classification. Primary 46E22, 47A13, Secondary 46E40, 47B37.

Key words and phrases. operator-valued reproducing kernel, multiplication tuple, commutant, reflexivity, weighted shift, directed trees.

The research of the third author was supported by the National Post-doctoral Fellowship (Ref. No. PDF/2016/001681), SERB.
On the other hand, a classical result of Shields and Wallen \([51]\) Theorem 2 identifies the commutant of a contractive multiplication operator on a reproducing kernel Hilbert space of scalar-valued holomorphic functions on the open unit disc with the algebra of bounded holomorphic functions (see \([48]\) Theorem 1, \([19]\) Chapter II, Theorem 5.4, \([54]\) Chapter VI, Corollary 3.7 for its variants). This provides essential motivation for vector-valued analog of the aforementioned theorem of Shields and Wallen (see \([48]\) Theorem 3 for a vector-valued version of \([48]\) Theorem 1). Essential ingredients in the proof of the main result of Section 3 (Theorem \([5.1]\)) include an adaptation of the technique from \([51]\) to the present situation, matrix-valued version of von Neumann’s inequality \([39]\), and the role of the simultaneous boundedness of reproducing kernel and its inverse along the diagonal (cf. \([21]\) Theorem 5.2). Theorem \([5.1]\) is applicable to the so-called Bergman shifts on locally finite, rooted directed trees of finite branching index (see Proposition \([5.1]\)). It is worth noting that the simultaneous growth of associated Bergman kernel and its inverse along the diagonal is at most of polynomial order (the reader is referred to \([25]\) and \([35]\), where asymptotic behavior of scalar-valued kernels has been studied).

The second main result of this paper (Theorem \([1.1]\)) ensures reflexivity of multiplication tuple \(\mathcal{M}_x\) on a reproducing kernel Hilbert space \(\mathcal{H}\) of vector-valued holomorphic functions with essentially the only assumption that \(\mathcal{M}_x\) satisfies the von Neumann’s inequality. This provides several new classes of examples and recovers special cases of various known results in one and several variables, see \([47]\) Theorem 3, \([50]\) Section 10, Proposition 37, \([10]\) Theorem 15, \([12]\), \([33]\) Theorem 5.2, \([46]\) Theorem 4.2, \([37]\) Section 0, Theorem 4, \([4]\) Theorem A (cf. \([18]\) Proposition 4.4, \([43]\) Theorem 3, \([37]\) Theorem 3, \([54]\) Theorem 3.1, \([26]\) Corollary 3.7, \([7]\) Theorem 1.2, \([27]\) Theorem 2.11, \([30]\) Corollary 7). It is worth noting that the techniques employed in the proofs of Theorems \([5.1]\) and \([1.1]\) have some common features (e.g. bounded approximation by polynomials in the sense of \([51]\) and \([37]\)). We conclude the paper by exhibiting a two-parameter family of tri-diagonal matrix-valued kernels to which Theorems \([5.1]\) and \([1.1]\) are applicable.

We set below the notations used throughout this text. For a set \(X\) and integer \(d\), \(\text{card}(X)\) denotes the cardinality of \(X\) and \(X^d\) stands for the \(d\)-fold Cartesian product of \(X\). The symbol \(\mathbb{N}\) stands for the set of nonnegative integers, while \(\mathbb{C}\) denotes the field of complex numbers. For \(\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d\), we use \(|\alpha| := \prod_{j=1}^d \alpha_j!\) and \(\alpha! := \prod_{j=1}^d \alpha_j!\). For \(w = (w_1, \ldots, w_d) \in \mathbb{C}^d\) and \(\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d\), the complex conjugate \(\overline{w} \in \mathbb{C}^d\) of \(w\) is given by \((\overline{w}_1, \ldots, \overline{w}_d)\), while \(w^{\alpha} \in \mathbb{C}^d\) centered at the origin, while the open unit ball in \(\mathbb{C}^d\) centered at the origin is denoted by \(\mathbb{B}^d\). In case \(d = 1\), we prefer the notation \(\mathbb{D}\) in place of \(\mathbb{D}^1\) or \(\mathbb{B}^1\). Let \(\mathcal{H}\) be a complex Hilbert space. If \(F\) is a subset of \(\mathcal{H}\), the closure of \(F\) is denoted by \(\overline{F}\), while the closed linear span of \(F\) is denoted by \(\overline{\{ x : x \in F \}}\). In case \(F\) is single-ton \(\{ x \}\), then \(\overline{\{ x \}}\) is denoted by the simpler notation \([x]\). If \(\mathcal{M}\) is a finite dimensional subspace of \(\mathcal{H}\), then \(\text{dim} \mathcal{M}\) denotes the vector space dimension of \(\mathcal{M}\). For a closed subspace \(\mathcal{M}\) of \(\mathcal{H}\), the orthogonal projection of \(\mathcal{H}\) onto \(\mathcal{M}\) is denoted by \(P_{\mathcal{M}}\). For a positive integer \(d\), the orthogonal direct sum of \(d\) copies of \(\mathcal{H}\) is denoted by \(\mathcal{H}^{(d)}\). Let \(B(\mathcal{H})\) denote the unital Banach algebra of bounded linear operators on \(\mathcal{H}\). The multiplicative identity \(I\) of \(B(\mathcal{H})\) is sometimes denoted by \(I_{\mathcal{H}}\). For a subspace \(\mathcal{M}\) of \(B(\mathcal{H})\), \(\mathcal{M}_{\text{WFOT}}\) denotes the closure of \(\mathcal{M}\) in the weak operator topology in \(B(\mathcal{H})\). For clarity, the norm \(\| \cdot \|\) on a normed linear space \(X\) is occasionally denoted by \(\| \cdot \|_X\). Sometimes, this is denoted by the pair \((X, \| \cdot \|_X)\). For a subset \(\Omega\) of \(\mathbb{C}^d\) and a normed linear space \(X\), the sup norm of a function \(\Phi : \Omega \to X\) is given by \(\| \Phi \|_{\infty, \Omega} := \sup_{w \in \Omega} \| \Phi(w) \|_X\). If \(T \in B(\mathcal{H})\), then \(\ker(T)\)
denotes the kernel of $T$. $T(H)$ denotes the range of $T$, $T^*$ denotes the Hilbert space adjoint of $T$, while $T^{(d)} \in B(H^{(d)})$ denotes for the orthogonal direct sum of $d$ copies of $T$. Given $x, y \in H$, by the rank one operator $x \otimes y$, we understand the bounded linear operator $x \otimes y(h) = \langle h, y \rangle x$, $h \in H$.

An operator $T \in B(H)$ is left-invertible if $T^*T$ is invertible in $B(H)$. The Cauchy dual of a left-invertible operator $T \in B(H)$ is given by $T^* := T(T^*T)^{-1}$. We say that $T \in B(H)$ is analytic if $\cap_{n \in \mathbb{N}} T^n(H) = \{0\}$. An operator $T \in B(H)$ is irreducible if it does not admit a proper reducing subspace. By a commuting $d$-tuple $T = (T_1, \ldots, T_d)$ in $B(H)$, we mean a collection of commuting operators $T_1, \ldots, T_d$ in $B(H)$. The notations $\sigma(T)$, $\sigma_H(T)$ and $\sigma_p(T)$ are reserved for the Taylor spectrum, Hartee spectrum and joint point spectrum of a commuting $d$-tuple $T$ respectively. The Hilbert adjoint of the commuting $d$-tuple $T = (T_1, \ldots, T_d)$ is the $d$-tuple $T^* = (T_1^*, \ldots, T_d^*)$, and the joint kernel $\cap_{j=1}^d \ker(T_j)$ of $T$ is denoted by $\ker(T)$. Further, for $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$, by $T - \lambda$, we understand the commuting $d$-tuple $(T_1 - \lambda_1I_H, \ldots, T_d - \lambda_dI_H)$. A commuting $d$-tuple $T = (T_1, \ldots, T_d)$ is said to be a contraction (resp. a joint contraction) if $T_j^*T_j \leq I$ for every $j = 1, \ldots, d$ (resp. $\sum_{j=1}^d T_j^*T_j \leq I$). The commutant $\mathcal{J}$ of a subset $\mathcal{J}$ of $B(H)$ is given by

$$\mathcal{J} := \{T \in B(H) : ST = TS \text{ for all } S \in \mathcal{J}\}.$$  

For a commuting $d$-tuple $T = (T_1, \ldots, T_d)$ in $B(H)$, we use the simpler notation $\{T\}$ for $\mathcal{J}$, where $\mathcal{J} = \{T_1, \ldots, T_d\}$. Note that $\mathcal{J}$ is a unital closed subalgebra of $B(H)$. If $C \in B(H)$, then Lat $C$ denotes the set of all closed linear subspaces of $H$ that are invariant under $C$. Let $\mathcal{W}$ be a subalgebra of $B(H)$ containing the identity operator $I_H$, and let Lat $\mathcal{W}$ be the set of all closed linear subspaces of $H$ that are invariant under every operator $W \in \mathcal{W}$. The set

$$\text{AlgLat } \mathcal{W} = \{C \in B(H) : \text{Lat } \mathcal{W} \subseteq \text{Lat } C\}$$

turns out to be a WOT-closed subalgebra of $B(H)$, which contains $\mathcal{W}$. We say that $\mathcal{W}$ is reflexive if $\mathcal{W} = \text{AlgLat } \mathcal{W}$. For a commuting $d$-tuple $T = (T_1, \ldots, T_d)$ in $B(H)$, let $\mathcal{W}_T$ stand for the WOT-closed subalgebra of $B(H)$ generated by $T_1, \ldots, T_d$ and the identity operator $I_H$:

$$\mathcal{W}_T = \{p(T) : p \in \mathbb{C}[z_1, \ldots, z_d]^{\text{WOT}}\},$$

where $\mathbb{C}[z_1, \ldots, z_d]$ denotes the vector space of complex polynomials in $z_1, \ldots, z_d$ and $p(T)$ is given by the polynomial functional calculus of $T$. A commuting $d$-tuple $T$ is reflexive if $\mathcal{W}_T$ is reflexive.

Here is the outline of the paper. In Section 2, we collect essential facts pertaining to the operator-valued kernels and associated reproducing kernel Hilbert spaces. Further, we formally introduce the notion of functional Hilbert space and discuss some properties of associated multiplication tuple. Sections 3 and 4 are devoted to main results of this paper (and their immediate consequences) on commutants and reflexivity of multiplications tuples on functional Hilbert spaces respectively. In the final section, we discuss applications of the main results to the theory of weighted shifts on rooted directed trees. Among various applications, we derive the curious fact that the commutant of a Bergman shift on a leafless, locally finite rooted directed tree $\mathcal{I}$ of finite branching index is abelian if and only if $\mathcal{I}$ is graph isomorphic to the rooted directed tree without any branching vertex.

2. Operator-valued Reproducing Kernels

Before we introduce the so-called functional Hilbert spaces, we briefly recall from [1, 40] some definitions and facts pertaining to Hilbert spaces associated with operator-valued kernels. Let $E$ be a Hilbert space and let $X$ be any set. A
weak $B(E)$-valued kernel on $X$ is a function $\kappa : X \times X \to B(E)$ such that, for any finite set $\{\lambda_1, \ldots, \lambda_n\} \subseteq X$ and any vectors $v_1, \ldots, v_n \in E$, we have
\[
\sum_{i,j=1}^{n} \langle \kappa(\lambda_i, \lambda_j)v_i, v_j \rangle_E \geq 0.
\]
If, in addition, $\kappa(\lambda, \lambda) \neq 0$ for any $\lambda \in X$, then $\kappa$ is referred to as a $B(E)$-valued kernel on $X$.

With any $B(E)$-valued kernel $\kappa : X \times X \to B(E)$, one can associate a Hilbert space $\mathcal{H}$ of $E$-valued functions on $X$ such that for every $\lambda \in X$,

(C1) the evaluation at $\lambda$ is a continuous linear function from $\mathcal{H}$ to $E$,

(C2) $\{f(\lambda) : f \in \mathcal{H}\} \neq \{0\}$

(see [1] Theorem 2.60, [30] Theorem 6.12). In this case, for any $g \in E$ and $\lambda \in X$, we have
\[
\kappa(\cdot, \lambda)g \in \mathcal{H},
\]
\[
(f, \kappa(\cdot, \lambda)g_F) = \langle f(\lambda), g_E \rangle, \quad f \in \mathcal{H}
\]
(refer to [1] Remark 2.65 for details). Conversely, any Hilbert space $\mathcal{H}$ of $E$-valued functions on a set $X$ satisfying conditions (C1) and (C2) can be shown to be a reproducing kernel Hilbert space associated with a weak $B(E)$-valued kernel $\kappa_F$ on $X$ (see [1] Theorem 2.60). However, if $\mathcal{H}$ contains $E$, then one can ensure that any weak $B(E)$-valued kernel is indeed a $B(E)$-valued kernel. Indeed, if $\kappa(\lambda, \lambda)g = 0$ for some $\lambda \in \Omega$ and $g \in E$, then
\[
\|\kappa(\cdot, \lambda)g\|^2 = \langle \kappa(\cdot, \lambda)g, \kappa(\cdot, \lambda)g \rangle_F = \langle \kappa(\cdot, \lambda)g, g \rangle_E,
\]
which implies that $\kappa(\cdot, \lambda)g = 0$, and hence by [1] applied to the constant function $f = g$, we get $g = 0$.

A bounded open connected subset $\Omega$ of $\mathbb{C}^d$ is said to be an admissible domain if it has the following property: For any bounded holomorphic function $\phi : \Omega \to \mathbb{C}$, there exists a sequence $\{p_n\}_{n=1}^{\infty}$ of polynomials such that

- for some $M > 0$, $\|p_n\|_{\infty, \Omega} \leq M\|\phi\|_{\infty, \Omega}$ for every integer $n \geq 1$,
- $p_n(w)$ converges to $\phi(w)$ as $n \to \infty$ for every $w \in \Omega$.

It is well-known that if a bounded domain $\Omega$ has polynomially convex closure in $\mathbb{C}^d$, then $\Omega$ is admissible provided it is star-shaped or strictly pseudoconvex with $C^2$ boundary (see, for instance, [37] Proof of Theorem 4, [7] Lemma 2.2]). In what follows, we also need the notion of vector-valued holomorphic function $f : \Omega \to Z$, where $\Omega$ is a domain in $\mathbb{C}^d$ and $Z$ is a normed linear space. Recall that $f$ is holomorphic if $\phi \circ f$ is holomorphic for every bounded linear functional $\phi$ on $Z$.

Although the following is not standard, we find it convenient for our purpose.

**Definition 2.1.** A functional Hilbert space is the quadruple $\mathcal{H}, \kappa_F, \Omega, E$, where $\Omega$ is an admissible domain in $\mathbb{C}^d$, $E$ is a separable Hilbert space and $\mathcal{H}$ is a reproducing kernel Hilbert space of holomorphic functions $f : \Omega \to E$ associated with the $B(E)$-valued kernel $\kappa_F : \Omega \times \Omega \to B(E)$ satisfying the following:

- (z-invariance) for any $f \in \mathcal{H}$, the function $z_jf : \Omega \to E$ given by
  \[
  (z_jf)(w) = w_jf(w), \quad w = (w_1, \ldots, w_d) \in \Omega
  \]
  belongs to $\mathcal{H}$ for every $j = 1, \ldots, d$,
- (Density of polynomials) the space of $E$-valued polynomials in $z_1, \ldots, z_d$ forms a dense subspace of $\mathcal{H}$: $\mathcal{H} : \{z^\alpha g : \alpha \in \mathbb{N}^d, g \in E\} = \mathcal{H}$.

**Remark 2.2.** Suppose that $\Omega$ contains the origin $0$. Then the condition
\[
\kappa(\lambda, 0) = I_E, \quad \lambda \in \Omega,
\]
ist the origin $0$. Then the condition
\[
\kappa(\lambda, 0) = I_E, \quad \lambda \in \Omega,
\]
together with the reproducing property implies that \( \mathcal{H} \) contains the space \( \mathcal{E} \) of all \( E \)-valued constant functions. This fact combined with the \( z \)-invariance of \( \mathcal{H} \) implies that indeed \( \mathcal{H} \) contains the subspace \( \mathcal{P} \) of all \( E \)-valued polynomials in \( z_1, \ldots, z_d \). Further, the condition (1) allows to rephrase the normalization condition as

\[
(f - f(0), g)_{\mathcal{H}} = 0, \quad g \in E, \ f \in \mathcal{H}.
\]

Let \( (\mathcal{H}, \kappa_{\mathcal{H}}, \Omega, E) \) be a functional Hilbert space. A function \( \Phi : \Omega \to B(E) \) is said to be a multiplier of \( \mathcal{H} \) if \( \Phi \) is holomorphic and

\[
\Phi f \in \mathcal{H} \quad \text{whenever} \quad f \in \mathcal{H},
\]

where \( (\Phi f)(w) = \Phi(w)f(w) \) for \( w \in \Omega \). Any multiplier \( \Phi \) induces the bounded linear operator \( \mathcal{M}_\Phi : \mathcal{H} \to \mathcal{H} \) given by

\[
\mathcal{M}_\Phi f = \Phi f, \quad f \in \mathcal{H}.
\]

Indeed, in view of the closed graph theorem, this is immediate from

\[
\langle \Phi f, \kappa_{\mathcal{H}}(\cdot, w)h \rangle_{\mathcal{H}} = \langle \Phi(w)f(w), h \rangle_E, \quad f \in \mathcal{H}, \ w \in \Omega, \ h \in E.
\]

By the multiplier norm of \( \Phi \), we understand the operator norm of \( \mathcal{M}_\Phi \). We say that \( \Phi : \Omega \to B(E) \) is bounded if \( \|\Phi\|_{\infty, \Omega} < \infty \). In this paper, we will be interested in the algebra

\[
H^\infty_{R(E)}(\Omega) := \left\{ \Phi : \Omega \to B(E) \mid \Phi \text{ is a bounded holomorphic function} \right\}.
\]

It can be easily deduced from Weierstrass convergence theorem [45, Chapter I, Theorem 1.9] that \( H^\infty_{R(E)}(\Omega) \) is a Banach algebra endowed with the sup norm \( \|\cdot\|_{\infty, \Omega} \).

In case \( E = \mathbb{C} \), we use the simpler and standard notation \( H^\infty(\Omega) \) for \( H^\infty_{R(E)}(\Omega) \).

**Remark 2.3.** By the definition of the functional Hilbert space, \( z_j I_E : \Omega \to B(E) \) given by

\[
(z_j I_E)(w) = w_j I_E, \quad w = (w_1, \ldots, w_d) \in \Omega,
\]

is a multiplier of \( \mathcal{H} \) for \( j = 1, \ldots, d \). In particular, \( \mathcal{M}_{z_j I_E} \) defines a bounded linear operator on \( \mathcal{H} \). We find it convenient to denote \( \mathcal{M}_{z_j I_E} \) by \( \mathcal{M}_{z_j} \). Note that the \( d \)-tuple \( \mathcal{M}_z = (\mathcal{M}_{z_1}, \ldots, \mathcal{M}_{z_d}) \) is a commuting \( d \)-tuple in \( B(\mathcal{H}) \).

We collect below several elementary properties of functional Hilbert spaces and associated multiplication operators.

**Proposition 2.4.** Let \( (\mathcal{H}, \kappa_{\mathcal{H}}, \Omega, E) \) be a functional Hilbert space and let \( \mathcal{M}_z \) denote the commuting \( d \)-tuple of multiplication operators \( \mathcal{M}_{z_1}, \ldots, \mathcal{M}_{z_d} \) in \( B(\mathcal{H}) \). Then, we have

(i) \( \big\{ \kappa_{\mathcal{H}}(\cdot, w)g : w \in \Omega, g \in E \big\} = \mathcal{H} \),
(ii) \( \kappa_{\mathcal{H}}(\cdot, w)(g + h) = \kappa_{\mathcal{H}}(\cdot, w)g + \kappa_{\mathcal{H}}(\cdot, w)h \) for every \( w \in \Omega \) and \( g, h \in E \),
(iii) for any linearly independent subset \( G \) of \( E \), \( \{ \kappa_{\mathcal{H}}(\cdot, w)g : g \in G \} \) is linearly independent in \( \mathcal{H} \) for every \( w \in \Omega \),
(iv) \( \kappa_{\mathcal{H}}(\cdot, w)E \subseteq \ker(\mathcal{M}_z^* - \overline{w}) \) for every \( w \in \Omega \).

In addition, if \( E \) is finite dimensional, then for any \( w \in \Omega \), we have

(v) \( \kappa_{\mathcal{H}}(w, w) \in B(E) \) is invertible such that \( \|\kappa_{\mathcal{H}}(w, w)^{-1}\|\kappa_{\mathcal{H}}(w, w)\| \geq 1 \),
(vi) \( \|\kappa_{\mathcal{H}}(w, w)^{-1}\|\kappa_{\mathcal{H}}(w, w)\| = 1 \) if and only if \( \kappa_{\mathcal{H}}(w, w) = \mu(w)I_E \) for some scalar \( \mu(w) > 0 \),
(vii) \( \ker(\mathcal{M}_z^* - \overline{w}) = \kappa_{\mathcal{H}}(\cdot, w)E \),
(viii) \( \dim \kappa_{\mathcal{H}}(\cdot, w)E = \dim E \).
Proof. Let \( w \in \Omega \). The part (i) follows from the fact that any \( f \in \mathcal{H} \) orthogonal to \( \kappa_{\mathcal{H}}(\cdot, w)E \) satisfies \( f(w) = 0 \) in view of (1). Part (ii) follows from (1) and additivity of the inner-product. To see (iii), in view of (ii), we may suppose that \( \kappa_{\mathcal{H}}(\cdot, w)g = 0 \) for some \( g \in E \). Thus \( \kappa_{\mathcal{H}}(w, w)g = 0 \), and hence by the injectivity of \( \kappa_{\mathcal{H}}(w, w) \) (see the discussion following (1)), \( g = 0 \). This completes the verification of (iii).

Assume that \( E \) is finite dimensional, and let \( w \in \Omega \). Since \( \kappa_{\mathcal{H}}(w, w) \) is injective and \( E \) is finite dimensional, \( \kappa_{\mathcal{H}}(w, w) \in B(E) \) is invertible. The remaining part in (v) is now obvious. To see (vi), note that by (2), \( \kappa_{\mathcal{H}}(w, w) \) is a positive operator. By the spectral theorem, \( \kappa_{\mathcal{H}}(w, w) \) is unitarily equivalent to a diagonal matrix with positive diagonal entries, say, \( \mu_j(w) \), \( j = 1, \ldots, \dim E \). Further,

\[
\|\kappa_{\mathcal{H}}(w, w)\| = \max_{1 \leq j \leq \dim E} \mu_j(w), \quad \|\kappa_{\mathcal{H}}(w, w)^{-1}\| = \left( \min_{1 \leq j \leq \dim E} \mu_j(w) \right)^{-1}.
\]

It follows that \( \|\kappa_{\mathcal{H}}(w, w)^{-1}\|\|\kappa_{\mathcal{H}}(w, w)\| = 1 \) if and only if

\[
\max_{1 \leq j \leq \dim E} \mu_j(w) = \min_{1 \leq j \leq \dim E} \mu_j(w),
\]

which is possible if and only if \( \mu_1(w) = \cdots = \mu_{\dim E}(w) \). In this case, \( \kappa_{\mathcal{H}}(w, w) \) must be a scalar multiple of \( I_E \). This completes the verification of (vi).

The facts (iv), (vii) and (viii) may be deduced from (1) and the density of \( \mathcal{H} \)-valued polynomials in \( z_1, \ldots, z_d \). Indeed, these parts have been noted implicitly in the proof of [13, Corollaries 4.1.11 and 4.2.11]. \( \square \)

Remark 2.5. Note that \( \kappa_{\mathcal{H}}(\cdot, w)g \) is precisely the value of the adjoint of the evaluation map \( E_w: \mathcal{H} \to E \) evaluated at \( g \) as discussed in [20, 21].

Corollary 2.6. Let \((\mathcal{H}, \kappa_{\mathcal{H}}, \Omega, E)\) be a functional Hilbert space and let \( \mathcal{M}_z \) denote the commuting \( d \)-tuple of multiplication operators \( \mathcal{M}_z = \mathcal{M}_{z_1}, \ldots, \mathcal{M}_{z_d} \) in \( B(\mathcal{H}) \). Then the Harte spectrum \( \sigma_H(\mathcal{M}_z) \) of \( \mathcal{M}_z \) contains the closure of \( \Omega \).

Proof. By Proposition 2.4 iv), \( \{ w \in \mathbb{C}^d : w \in \Omega \} \) is contained in the joint point spectrum \( \sigma_{p}(\mathcal{M}_z) \) of \( \mathcal{M}_z \). However, by the general theory [22],

\[
\sigma_{p}(\mathcal{M}_z) \subseteq \sigma_H(\mathcal{M}_z) = \{ w \in \mathbb{C}^d : w \in \sigma_H(\mathcal{M}_z) \}.
\]

It follows that \( \Omega \subseteq \sigma_H(\mathcal{M}_z) \). The desired conclusion now follows from the fact that the Harte spectrum is closed. \( \square \)

Remark 2.7. Note that the Harte spectrum \( \sigma_H(\mathcal{M}_z) \) of \( \mathcal{M}_z \) is dominating for the algebra \( H^\infty(\Omega) \) in the following sense:

\[
\|f\|_{\infty, \Omega} = \|f\|_{\infty, \sigma_H(\mathcal{M}_z) \cap \Omega}, \quad f \in H^\infty(\Omega).
\]

The condition that a spectral system (e.g. Harte spectrum, essential Taylor spectrum, essential Harte spectrum) is dominating for the algebra of bounded holomorphic functions appears in a variety of results on the invariant subspaces or reflexivity of tuples (see [26, Corollary 3.7], [42, Theorem 10.2.2], [29, Theorem 4.2]).

We find it convenient to introduce the following terminologies, which resemble with that of von Neumann \( d \)-tuple (refer to [19, 23]).

Definition 2.8. Let \((\mathcal{H}, \kappa_{\mathcal{H}}, \Omega, E)\) be a functional Hilbert space and let \( \mathcal{M}_z \) denote the commuting \( d \)-tuple of multiplication operators \( \mathcal{M}_z = \mathcal{M}_{z_1}, \ldots, \mathcal{M}_{z_d} \) in \( B(\mathcal{H}) \). We say that \( \mathcal{M}_z \) satisfies von Neumann’s inequality if there exists a constant \( K > 0 \) such that

\[
\|p(\mathcal{M}_z)\|_{B(\mathcal{H})} \leq K \|p\|_{\infty, \Omega}, \quad p \in \mathbb{C}[z_1, \ldots, z_d].
\]
We say that $\mathcal{M}_z$ satisfies the matrix-valued von Neumann’s inequality if there exists a constant $K > 0$ such that

$$\| (p_{i,j}(\mathcal{M}_z))_{1 \leq i,j \leq m}\|_{B(\mathcal{H}^\infty)} \leq K \| (p_{i,j})_{1 \leq i,j \leq m}\|_{\infty,\Omega}, \quad p_{i,j} \in \mathbb{C}[z_1, \ldots, z_d], \ m \in \mathbb{N}.$$  

**Remark 2.9.** Note that the multiplication tuple $\mathcal{M}_z$ satisfies von Neumann’s inequality (resp. matrix-valued von Neumann’s inequality) if the polynomial functional calculus $\Phi(p) = p(\mathcal{M}_z)$ ($p \in \mathbb{C}[z_1, \ldots, z_d]$) is bounded (resp. completely bounded) in the sense of $[39]$ and $[41]$.

We record the following known fact for ready reference (see $[39]$ Corollary 7.7] or $[6]$ Theorem 1.2.2] and the remark following it).

**Lemma 2.10.** Let $\Omega$ be a bounded domain in $\mathbb{C}^d$ and let $T$ be a commuting $d$-tuple in $B(\mathcal{H})$. Suppose there exists a commuting $d$-tuple $N$ of normal operators in $B(K)$ for some Hilbert space $K$ containing $\mathcal{H}$ such that

$$\sigma(N) \subseteq \bar{\Omega} \text{ and } p(T) = P_K p(N)|_H, \quad p \in \mathbb{C}[z_1, \ldots, z_d].$$

Then $T$ satisfies

$$\| (p_{i,j}(T))_{1 \leq i,j \leq m}\|_{B(\mathcal{H}^\infty)} \leq \| (p_{i,j})_{1 \leq i,j \leq m}\|_{\infty,\Omega}, \quad p_{i,j} \in \mathbb{C}[z_1, \ldots, z_d], \ m \in \mathbb{N}.$$  

**List 2.11.** We list here some known cases $[54]$ Chapter I, Theorems 4.1], $[5]$ Pg 88], $[31]$ Theorem 1.1] $[38]$ Pg 987], $[8]$ Proposition 2] in which the pair $(\Omega, T)$ satisfies the hypothesis of Lemma 2.10:

- $\Omega = \mathbb{D}$, $T$ is any contraction,
- $\Omega = \mathbb{D}^2$, $T$ is a contractive 2-tuple,
- $\Omega = \mathbb{D}^d$, $T$ is a contractive $d$-variable weighted shift with positive weights,
- $\Omega = \mathbb{D}^d$, $T$ is a joint contractive $d$-tuple,
- $\Omega = \mathbb{R}^d$, $T$ is a commuting $d$-tuple such that

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{\alpha \in \mathbb{N}^d, |\alpha| = j} \frac{j!}{\alpha!} T_\alpha^* T_\alpha \geq 0, \quad k = 1, \ldots, d.$$  

Under some natural assumptions on the $B(E)$-valued kernels $\kappa_{\mathcal{H}}$ (see $[4]$ and $[3]$), the commutants of multiplication tuples $T = \mathcal{M}_z$ in $B(\mathcal{H})$, falling in any one of the classes mentioned in List 2.11, can be identified with the algebra $H^\infty_{\mathcal{H}(\Omega)}$ of $B(E)$-valued bounded holomorphic functions on $\Omega$ (see Corollary $[4,3]$). We will also show that $\mathcal{M}_z$ is reflexive in all the above cases (see Corollary $[4,3]$).

### 3. Commutants

The first main result of this paper identifies commutants of multiplication tuples $\mathcal{M}_z$ on certain functional Hilbert spaces. A special case of this result (under the additional assumption that the joint kernel of $\mathcal{M}_z^* - \lambda I$ is 1-dimensional for every $\lambda \in \Omega$) has been essentially obtained in $[21]$ Theorem 5.2].

**Theorem 3.1.** Let $(\mathcal{H}, \kappa_{\mathcal{H}}, \Omega, E)$ be a functional Hilbert space with finite dimensional $E$ and let $\mathcal{M}_z$ denote the commuting $d$-tuple of multiplication operators $\mathcal{M}_{z_1}, \ldots, \mathcal{M}_{z_d}$ in $B(\mathcal{H})$. Suppose that the reproducing kernel $\kappa_{\mathcal{H}}$ satisfies the following conditions:

$$\langle f(\cdot), g \rangle_E h \in \mathcal{H} \text{ for every } g, h \in E \text{ and } f \in \mathcal{H}, \quad (4)$$

$$\sup_{w \in \Omega} \| \kappa_{\mathcal{H}}(w, w) \|_{B(E)} \| \kappa_{\mathcal{H}}(w, w)^{-1} \|_{B(E)} < \infty \quad (5)$$
(see Proposition 2.4(v)). Then the commutant \( \mathcal{M}_z \)' of \( \mathcal{M}_z \) is isometrically isomorphic to the Banach algebra

\[
\mathcal{R} := \{ \Phi \in H^\infty_{\mathbb{N}^d}(\Omega) : \mathcal{M}_\Phi \in \{ \mathcal{M}_z \}' \}
\]

endowed with the multiplier norm \( \| \cdot \|_{B(\mathcal{R})} \). In addition, if the multiplication \( d \)-tuple \( \mathcal{M}_z \) satisfies the matrix-valued von Neumann’s inequality, then \( \{ \mathcal{M}_z \}' \) is isometrically isomorphic to the Banach algebra \( H^\infty_{\mathbb{N}^d}(\Omega) \), \( \| \cdot \|_{B(\mathcal{R})} \).

Remark 3.2. The conditions (1) and (4) are natural in the following sense:

- Any \( E \)-valued polynomial \( f \) satisfies (1).
- In case \( \dim E = 1 \), \( \kappa_{\mathcal{R}} \) satisfies (1) as well as (4).

Proof. Let \( T \in B(\mathcal{H}) \) be in the commutant of \( \mathcal{M}_z \) and let \( w = (w_1, \ldots, w_d) \) be in \( \Omega \). By Proposition 2.4(vii), for any \( g \in E \) and \( j = 1, \ldots, d \),

\[
(T_{z_j} - \bar{w}_j)T^* \kappa_{\mathcal{R}}(\cdot, w)g = T^*(T_{z_j} - \bar{w}_j)\kappa_{\mathcal{R}}(\cdot, w)g = 0.
\]

Thus \( T^* \) maps \( \kappa_{\mathcal{R}}(\cdot, w)E \) into itself. Consequently, there exists an operator \( \Phi(w) \) in \( B(E) \) such that

\[
T^* \kappa_{\mathcal{R}}(\cdot, w)g = \kappa_{\mathcal{R}}(\cdot, w)\Phi(w)^* g, \quad g \in E
\]

(see, for instance, [1] Pg 32]). Note that for any \( h \in \mathcal{H} \) and \( g \in E \),

\[
\langle (Th)(w), g \rangle_E = \langle h, T^* \kappa_{\mathcal{R}}(\cdot, w)g \rangle_{\mathcal{H}} \quad \equiv \quad \langle h, \kappa_{\mathcal{R}}(\cdot, w)\Phi(w)^* g \rangle_{\mathcal{H}} \equiv \langle \Phi(w)h(w), g \rangle_E.
\]

This shows that \( T = \mathcal{M}_\Phi \) for a map \( \Phi : \Omega \to B(E) \). Further, since \( Tg \) is holomorphic for every \( g \in E \), so is \( \Phi \). We claim that

\[
\sup_{w \in \Omega} \| \kappa_{\mathcal{R}}(w, w) \|_{B(\mathcal{E})} \leq \| \mathcal{M}_\Phi \|.
\]

Indeed, for any \( w \in \Omega \) and unit vectors \( g, \bar{g} \in E \),

\[
\| \langle \Phi(w)\kappa_{\mathcal{R}}(w, w)g, \bar{g} \rangle_E \| \leq \| \langle M_\Phi \kappa_{\mathcal{R}}(w, w)g, \kappa_{\mathcal{R}}(w, w)\bar{g} \rangle \| \leq \| M_\Phi \| \| \kappa_{\mathcal{R}}(w, w)g \|_{\mathcal{H}} \| \kappa_{\mathcal{R}}(w, w)\bar{g} \|_{\mathcal{R}} \leq \| M_\Phi \| \| \kappa_{\mathcal{R}}(w, w) \|_{B(\mathcal{E})}.
\]

Taking supremum over all unit vectors \( g, \bar{g} \in E \), the claim stands verified. Combining (1) with the assumption (4), we obtain for any \( w \in \Omega \),

\[
\| \Phi(w) \|_{B(\mathcal{E})} \leq \| \Phi(w)\kappa_{\mathcal{R}}(w, w) \|_{B(\mathcal{E})} \| \kappa_{\mathcal{R}}(w, w)^{-1} \|_{B(\mathcal{E})} \leq \| \mathcal{M}_\Phi \| \sup_{w \in \Omega} \| \kappa_{\mathcal{R}}(w, w) \|_{B(\mathcal{E})} \| \kappa_{\mathcal{R}}(w, w)^{-1} \|_{B(\mathcal{E})},
\]

and hence \( T = \mathcal{M}_\Phi \) for \( \Phi \) in the algebra \( H^\infty_{\mathbb{N}^d}(\Omega) \). Clearly, \( (\mathcal{R}, \| \cdot \|_{B(\mathcal{R})}) \) is a Banach algebra. It follows that the mapping \( \mathcal{F} : \{ \mathcal{M}_z \}' \to (\mathcal{R}, \| \cdot \|_{B(\mathcal{R})}) \) given by \( \mathcal{F}(\mathcal{M}_\Phi) = \Phi \) is an isometric isomorphism.

To see the remaining part, assume that \( \mathcal{M}_z \) satisfies the matrix-valued von Neumann’s inequality. It suffices to check that for every bounded holomorphic function \( \Phi : \Omega \to B(E) \) is a multiplier of \( \mathcal{H} \). To see that, let \( \Phi : \Omega \to B(E) \) be a bounded holomorphic function. Let \( m := \dim E \) and \( \mathcal{B} := \{ b_j : j = 1, \ldots, m \} \) be an orthonormal basis of \( E \). For \( w \in \Omega \), let \( (\phi_{i,j}(w))_{1 \leq i, j \leq m} \) be the matrix representation of \( \Phi(w) \) with respect to the basis \( \mathcal{B} \) of \( E \). Fix \( i, j = 1, \ldots, m \). Since \( \Phi \) is bounded holomorphic, so is \( \phi_{i,j} \). By assumption, \( \Omega \) is an admissible domain in \( \mathbb{C}^d \), and hence there exists a sequence \( \{ p_{i,j}^{(n)} \}_{n=1}^\infty \subseteq \mathbb{C}[z_1, \ldots, z_d] \) such that

\[
(P1) \quad \| p_{i,j}^{(n)} \|_{\infty, \Omega} \leq M \| \phi_{i,j} \|_{\infty, \Omega} \quad \text{for some constant } M > 0,
\]
(P2) \( p_{i,j}^{(n)}(w) \) converges to \( \phi_{i,j}(w) \) as \( n \to \infty \) for every \( w \in \Omega \).

Let \( M_{n}(\mathbb{C}) \) denote the Banach algebra of \( m \times m \) matrices of complex entries endowed with the operator norm, and recall the fact that

\[
\| (a_{i,j}) \|_{M_{n}(\mathbb{C})} = \max_{1 \leq i,j \leq m} |a_{i,j}|, \quad (a_{i,j}) \in M_{n}(\mathbb{C}).
\]

(9)

Since \( \mathcal{M}_{z} \) satisfies the matrix-valued von Neumann’s inequality, for some constant \( K > 0 \), we obtain

\[
\| (p_{i,j}^{(n)}(\mathcal{M}_{z})) \|_{B(\mathcal{H}^{(m)})} \leq K \sup_{z \in \Omega} \| (p_{i,j}^{(n)}(z)) \|_{M_{n}(\mathbb{C})}
\]

\[
\leq K' \max_{1 \leq i,j \leq m} \| p_{i,j}^{(n)} \|_{\infty, \Omega}
\]

(10)

\[
\leq K' M \max_{1 \leq i,j \leq m} \| \phi_{i,j} \|_{\infty, \Omega},
\]

where \( K' = m K \). Thus for any \( F \in \mathcal{H}^{(m)} \), \( \{ (p_{i,j}^{(n)}(\mathcal{M}_{z})) \}_{n=1}^{\infty} \) is a bounded sequence in \( \mathcal{H}^{(m)} \). By [52] Theorem 3.6.11, \( \{ (p_{i,j}^{(n)}(\mathcal{M}_{z})) \}_{n=1}^{\infty} \) admits a weakly convergent subsequence. For simplicity, we assume that \( \{ (p_{i,j}^{(n)}(\mathcal{M}_{z})) \}_{n=1}^{\infty} \) itself converges weakly to, say, \( \tilde{F} \in \mathcal{H}^{(m)} \). That is,

\[
\lim_{n \to \infty} \langle (p_{i,j}^{(n)}(\mathcal{M}_{z})) F, H \rangle = \langle \tilde{F}, H \rangle \quad \text{for all } H \in \mathcal{H}^{(m)}.
\]

(11)

Let \( F = \bigoplus_{i=1}^{m} f_{i} \in \mathcal{H}^{(m)} \) and write \( \tilde{F} = \bigoplus_{i=1}^{m} \tilde{f}_{i} \). Fix an integer \( k = 1, \ldots, m \), \( g \in E \), \( w \in \Omega \), and set \( H := \bigoplus_{j=1}^{m} h_{j} \), where

\[
h_{j} = \begin{cases} \kappa_{\mathcal{H}}(\cdot, w)g & \text{if } j = k, \\ 0 & \text{otherwise}. \end{cases}
\]

Note that

\[
\langle (p_{i,j}^{(n)}(\mathcal{M}_{z})) F, H \rangle_{\mathcal{H}^{(m)}} = \langle \bigoplus_{i=1}^{m} (\sum_{j=1}^{m} p_{i,j}^{(n)}(\mathcal{M}_{z}) f_{j}) \bigoplus_{j=1}^{m} h_{j}, G \rangle_{\mathcal{H}^{(m)}}
\]

\[
= \sum_{i=1}^{m} \langle p_{k,i}^{(n)}(\mathcal{M}_{z}) f_{i}, \kappa_{\mathcal{H}}(\cdot, w)g \rangle_{\mathcal{H}^{(m)}}
\]

\[
= \sum_{i=1}^{m} p_{k,i}^{(n)}(w) \langle f_{i}(w), g \rangle_{E}
\]

Since \( p_{i,j}^{(n)}(w) \) converges to \( \phi_{i,j}(w) \) (see (P2)), after letting \( n \to \infty \) on both sides, by [10], we obtain

\[
\langle \tilde{F}, H \rangle_{\mathcal{H}^{(m)}} = \sum_{i=1}^{m} \phi_{k,i}(w) \langle f_{i}(w), g \rangle_{E}
\]

However, \( \langle \tilde{F}, H \rangle_{\mathcal{H}^{(m)}} = \langle \tilde{f}_{k}(w), g \rangle_{E} \), so that

\[
\langle \tilde{f}_{k}(w), g \rangle_{E} = \sum_{i=1}^{m} \phi_{k,i}(w) \langle f_{i}(w), g \rangle_{E} \quad \text{for every } g \in E.
\]

Consequently,

\[
\tilde{f}_{k}(w) = \sum_{i=1}^{m} \phi_{k,i}(w) f_{i}(w), \quad w \in \Omega, \quad k = 1, \ldots, m.
\]

(11)
The preceding discussion shows that for every \( F = \oplus_{i=1}^{m} f_{i} \in \mathcal{H}^{(m)} \), we obtain \( \tilde{F} = \oplus_{i=1}^{m} f_{i} \in \mathcal{H}^{(m)} \) governed by (11). We apply this association to \( F^{(k)} = \oplus_{i=1}^{m} f_{i}^{(k)}, k = 1, \ldots, m \), given by
\[
\tilde{f}_{i}^{(k)}(w) := \langle f(w), g_{i} \rangle_{E} g_{k}, \quad w \in \Omega, \ i = 1, \ldots, m, \tag{12}
\]
where \( f \in \mathcal{H} \). By assumption (11), \( f_{i}^{(k)} \in \mathcal{H} \) for all \( i = 1, \ldots, m \). Hence the above association yields \( \tilde{F}^{(k)} = \oplus_{i=1}^{m} f_{i}^{(k)} \in \mathcal{H}^{(m)}, k = 1, \ldots, m \). It follows from (11) that
\[
\tilde{f}_{j}^{(k)}(w) = \sum_{i=1}^{m} \phi_{j,i}(w)f_{i}^{(k)}(w), \quad w \in \Omega, \ j = 1, \ldots, m,
\]
and hence for any \( w \in \Omega \),
\[
\sum_{k=1}^{m} \tilde{f}_{k}^{(k)}(w) = \sum_{k=1}^{m} \sum_{i=1}^{m} \phi_{k,i}(w)f_{i}^{(k)}(w)
= \sum_{k=1}^{m} \sum_{i=1}^{m} \phi_{k,i}(w)\langle f(w), g_{i} \rangle_{E} g_{k} = (\mathcal{M}_{k}f)(w).
\]
Since \( \tilde{f}_{k}^{(k)} \in \mathcal{H} \) for \( k = 1, \ldots, m \), \( \mathcal{M}_{k}f \in \mathcal{H} \), and hence \( \Phi \) is a multiplier of \( \mathcal{H} \). Trivially, \( \mathcal{M}_{k} \) commutes with \( \mathcal{M}_{z} \).

Remark 3.3. It is evident from the proof that the matrix-valued von Neumann’s inequality is required only for the choice \( m = \dim E \). Further, it has been pointed out by the anonymous referee that one can renorm \( \mathcal{H} \) (assuming (11)), so that it is a RKHS in this equivalent norm with kernel of the form \( \kappa(z, w)I_{E} \), where \( \kappa \) is a scalar-valued kernel. This can be achieved by endowing \( \mathcal{H}_{j} = \{(f(\cdot), e_{j}) : f \in \mathcal{H}\} \) with the norm, which makes \( f \mapsto (f(\cdot), e_{j}) \) a quotient map, where \( \{e_{1}, \ldots, e_{\dim E}\} \) is an orthonormal basis. Note that (11) allows us to identify, up to similarity, the \( \mathbb{C}[z_{1}, \ldots, z_{d}] \)-Hilbert modules \( \mathcal{H} \) and \( \oplus_{j=1}^{\dim E} \mathcal{H}_{j} \). In particular, Theorem 3.1 can be recovered from its scalar-valued counterpart. The advantage gained in this process is that the assumption (5) can be relaxed.

In the remaining part of this section, we discuss several applications of Theorem 3.1. The first of which is a direct consequence of Theorem 5.1 and Lemma 2.10.

Corollary 3.4. Let \( (\mathcal{H}, \kappa_{\mathcal{H}}, \Omega, E) \) be a functional Hilbert space with finite dimensional \( E \) and let \( \mathcal{M}_{z} \) denote the commuting \( d \)-tuple of multiplication operators \( \mathcal{M}_{z} \). Suppose that the reproducing kernel \( \kappa_{\mathcal{H}} \) satisfies (4) and (5). If the pair \( (\Omega, \mathcal{M}_{z}) \) falls in the List 2.11 then the commutant \( \{\mathcal{M}_{z}\}' \) of \( \mathcal{M}_{z} \) is equal to the algebra \( H_{B(\mathcal{H})}^{\infty}(\Omega) \).

Remark 3.5. Assume that \( \dim E = 1 \). Then, by Remark 3.2. \( \kappa_{\mathcal{H}} \) satisfies (4) and (5). Thus the above corollary is applicable to any pair \( (\Omega, T := \mathcal{M}_{z}) \) falling in the List 2.11.

The following answers when the commutant of the multiplication tuple \( \mathcal{M}_{z} \) is abelian (cf. 5.4 Theorem 4.12, [50] Section 4, Corollary 2, [55] Theorem 7).

Corollary 3.6. Under the assumptions of Theorem 3.1 the following statements are equivalent:
(i) The commutant \( \{\mathcal{M}_{z}\}' \) of \( \mathcal{M}_{z} \) is abelian.
(ii) \( \mathcal{M}_{z} \) is irreducible.
(iii) \( \dim E = 1 \).

Proof. In the proof, we need the following properties of multipliers:
(a) \( \mathcal{M}_{\Psi} \mathcal{M}_{\Phi} = \mathcal{M}_{\Phi} \mathcal{M}_{\Psi} \) if and only if \( \Phi \Psi = \Psi \Phi \).
Suppose that (i) holds. If $S, T \in \{\mathcal{M}_z\}'$ then by the preceding theorem, $S = \mathcal{M}_\Phi$ and $T = \mathcal{M}_\Psi$ for some bounded holomorphic $B(E)$-valued functions $\Phi, \Psi$ on $\Omega$. But then by (a), we must have $\Phi \Psi = \Psi \Phi$. Clearly, in case $E > 1$, there are constant (and hence bounded and holomorphic) functions $\Phi, \Psi$ which do not commute. Thus (i) holds if and only if $\dim E = 1$. This proves the equivalence of (i) and (iii).

Suppose that $\dim E = 1$. Let $P$ be an orthogonal projection belonging to $\{\mathcal{M}_z\}'$. By Theorem 5.1, $P = \mathcal{M}_\Phi$ for some bounded holomorphic $\Phi : \Omega \to B(E)$. By (b), $\Phi^2 = \Phi$, and hence either $\Phi = 0$ or $\Phi = I_E$. This proves that (iii) $\Rightarrow$ (ii).

Suppose that $\dim E \geq 2$. Consider the constant $B(E)$-valued rank one orthogonal projection $\Phi$. By (b), $\mathcal{M}_\Phi$ is an orthogonal projection. Further, by (a), $\mathcal{M}_\Phi$ belongs to $\{\mathcal{M}_z\}'$. This shows that $\mathcal{M}_z$ is reducible, and hence (ii) $\Rightarrow$ (iii).

**Remark 3.7.** Unlike the case of $\dim E = 1$ (see [51, Theorem 2]), the commutant of $\mathcal{M}_z$ differs from the WOT-closed algebra $\mathcal{W}_z$ generated by $\mathcal{M}_z$ and the identity operator $I_{\mathcal{H}}$ on $\mathcal{H}$. Indeed, if $\dim E > 1$, then $\{\mathcal{M}_z\}'$ is non-abelian, whereas $\mathcal{W}_z$ is easily seen to be abelian.

The following identifies the commutant of an orthogonal direct sum of finitely many copies of a contractive multiplication operator on a reproducing kernel Hilbert space of scalar-valued holomorphic functions.

**Corollary 3.8.** Let $(\mathcal{H}, \kappa, \mathbb{D}, \mathbb{C})$ be a functional Hilbert space and let $m$ be a positive integer. If the operator $\mathcal{M}_z$ of multiplication by $z$ is contractive, then the commutant $(\mathcal{M}_z(m))'$ of $\mathcal{M}_z(m) \in B(\mathcal{H}^{(m)})$ is equal to the algebra $H^{\infty}_{\mathbb{D}[\mathbb{C}^m]}(\mathbb{D})$.

**Proof.** Suppose that $\mathcal{M}_z$ is contractive. Let $f = \sum_{j=1}^m f_j \in \mathcal{H}^{(m)}$ and let $K(z, w) = \kappa(\mathcal{H}(z, w) I_{\mathbb{C}^m})$. Then, for any $x = (x_1, \ldots, x_m) \in \mathbb{C}^m$,

$$\langle f, K(\cdot, w)x \rangle = \sum_{j=1}^m \langle f_j, k(\cdot, w)x_j \rangle = \sum_{j=1}^m f_j(w) x_j = \langle f(w), x \rangle.$$ 

In particular, $\mathcal{H}^{(m)}$ is a reproducing kernel Hilbert space associated with the $B(\mathbb{C}^m)$-valued kernel $K$ (see [40, Pg 99-100]). This also shows that

$$\langle f(\cdot), x \rangle y = \sum_{j=1}^m x_j f_j(\cdot)y \in \mathcal{H}^{(m)}$$ 

for every $x, y \in \mathbb{C}^m$ and $f \in \mathcal{H}^{(m)}$, which is precisely the condition 4. Since $K$ trivially satisfies the boundedness condition 5, the desired conclusion is immediate from Theorem 3.1.

4. Reflexivity

The main result of this section shows that the multiplication tuple $\mathcal{M}_z$ on any functional Hilbert space $\mathcal{H}$ satisfying von Neumann’s inequality is reflexive. Our proof is inspired by the technique usually employed either to compute commutant or to establish reflexivity of the multiplication tuple $\mathcal{M}_z$ on a reproducing kernel Hilbert space of scalar-valued holomorphic functions (cf. [51, Proof of Lemma 5], [50, Section 10, Proposition 37], [19, Chapter VII, Lemma 8.2], and [37, Section 0, Theorem 4]). The novelty of this refined technique is that it ensures reflexivity of $\mathcal{M}_z$ under the mild assumption that it satisfies von Neumann’s inequality.

**Theorem 4.1.** Let $(\mathcal{H}, \kappa, \mathcal{H}(\cdot), \mathbb{C})$ be a functional Hilbert space and let $\mathcal{M}_z$ denote the commuting $d$-tuple of multiplication operators $\mathcal{M}_{z_1}, \ldots, \mathcal{M}_{z_d}$ in $B(\mathcal{H})$. Suppose that $\mathcal{M}_z$ satisfies von Neumann’s inequality. Then $\mathcal{M}_z$ is reflexive.
Proof. Clearly, \( W_{\mathcal{H}_z} \subseteq \text{AlgLat } W_{\mathcal{H}_z} \). To see the reverse inclusion, let \( A \) belong to \( \text{AlgLat } W_{\mathcal{H}_z} \), and note that \( \text{Lat } W_{\mathcal{H}_z} \subseteq \text{Lat } A^* \). By Proposition 2.4(iv), \[ M_{w,g} := \{ ak_{\mathcal{H}}(\cdot, w)g : a \in \mathbb{C} \} \in \text{Lat } W_{\mathcal{H}_z}, \quad w \in \Omega, \; g \in E. \]

Thus there exists a scalar \( \phi_g(w) \) such that
\[ A^* k_{\mathcal{H}}(\cdot, w)g = \phi_g(w) k_{\mathcal{H}}(\cdot, w)g, \quad w \in \Omega, \; g \in E. \] (13)

Let \( \{ g_n \}_{n \in \Lambda} \) be an orthonormal basis of \( E \). We contend that
\[ \phi_{g_j} = \phi_{g_k}, \quad j, k \in \Lambda. \] (14)

For \( w \in \Omega \) and \( j, k \in \Lambda \), by two applications of Proposition 2.4(ii), we have
\[ \phi_{g_j}(w) k_{\mathcal{H}}(\cdot, w)g_j + \phi_{g_k}(w) k_{\mathcal{H}}(\cdot, w)g_k = A^* k_{\mathcal{H}}(\cdot, w)g_j + A^* k_{\mathcal{H}}(\cdot, w)g_k = A^* k_{\mathcal{H}}(\cdot, w)(g_j + g_k) \quad \text{and hence} \]
\[ \phi_{g_j}(w) k_{\mathcal{H}}(\cdot, w)g_j + \phi_{g_k}(w) k_{\mathcal{H}}(\cdot, w)g_k = \phi_{g_j + g_k}(w)(k_{\mathcal{H}}(\cdot, w)g_j + k_{\mathcal{H}}(\cdot, w)g_k). \] (15)

However, by Proposition 2.4(iii), \( \{ k_{\mathcal{H}}(\cdot, w)g_j \}_{j \in \Lambda} \) forms a linearly independent subset of \( \mathcal{H} \), and hence we conclude that
\[ \phi_{g_j}(w) = \phi_{g_j + g_k}(w) = \phi_{g_k}(w). \]

This yields (14). Let \( \phi : \Omega \to \mathbb{C} \) be a function such that \( \phi_{g_j} = \phi \) for all \( j \in \Lambda \). It is now immediate from (13) that
\[ A^* k_{\mathcal{H}}(\cdot, w)g_j = \phi(w)k_{\mathcal{H}}(\cdot, w)g_j, \quad w \in \Omega, \; j \in \Lambda. \] (15)

This implies that
\[ \| \phi(w) k_{\mathcal{H}}(\cdot, w)g_j \| = \| A^* k_{\mathcal{H}}(\cdot, w)g_j \| \leq \| A^* \| \| k_{\mathcal{H}}(\cdot, w)g_j \|, \quad w \in \Omega, \; j \in \Lambda. \]

Since \( k_{\mathcal{H}}(\cdot, w)g_j \neq 0 \) (see Proposition 2.4(iii)), the above estimate shows that \( \| \phi \|_{\infty, \Omega} \leq \| A^* \| \), and hence \( \phi \) is bounded. Further, for any \( f \in \mathcal{H} \) and \( w \in \Omega \),
\[ (Af)(w) = \sum_{j \in \Lambda} (Af)(w)g_j g_j = \sum_{j \in \Lambda} \langle f, k_{\mathcal{H}}(\cdot, w)g_j \rangle g_j = \phi(w) \sum_{j \in \Lambda} \langle f, k_{\mathcal{H}}(\cdot, w)g_j \rangle g_j = \phi(w)f(w). \] (16)

Since \( Af \in \mathcal{H} \), \( \phi f \in \mathcal{H} \) for every \( f \in \mathcal{H} \). This shows that \( \phi g_j \in \mathcal{H} \) for every \( j \in \Lambda \). However, \( \phi(w) = \langle \phi(w)g_j, g_j \rangle_E \), \( j \in \Lambda \), and hence \( \phi \) is holomorphic. Since \( \Omega \) is admissible, there exists a sequence of polynomials \( \{ p_n \}_{n \in \mathbb{N}} \subseteq \mathbb{C}[z_1, \ldots, z_d] \) such that for some positive constant \( M \),
\[ \| p_n \|_{\infty, \Omega} \leq M \| \phi \|_{\infty, \Omega}, \quad \lim_{n \to \infty} p_n(w) = \phi(w), \quad w \in \Omega. \] (17)

Note that for \( f \in \mathcal{H} \), \( w \in \Omega \) and \( g \in E \),
\[ \lim_{n \to \infty} \langle p_n(\mathcal{M}_z)f, k_{\mathcal{H}}(\cdot, w)g \rangle_{\mathcal{H}} = \lim_{n \to \infty} \langle p_n(w)f(w), g \rangle_E = \langle \phi(w)f(w), g \rangle_E = \langle (Af)(w), g \rangle_E = \langle Af, k_{\mathcal{H}}(\cdot, w)g \rangle_{\mathcal{H}}. \] (18)

By von Neumann’s inequality and (17), \( \{ p_n(\mathcal{M}_z) \}_{n \in \mathbb{N}} \) is a bounded sequence. This combined with (18) and Proposition 2.4(i) shows that \( \{ p_n(\mathcal{M}_z) \}_{n \in \mathbb{N}} \) converges to
A in WOT. This shows that $A \in \mathcal{W}_M$, and hence $\text{AlgLat} \mathcal{W}_M \subseteq \mathcal{W}_M$. This completes the proof of the theorem. \hfill \square

**Remark 4.2.** We note the following:

1. It is evident from the proof of Theorem 4.1 that the assumption of the density of $E$-valued analytic polynomials is inessential (cf. Remark 2.2).
2. Theorem 4.1 is applicable to the multiplication tuple $M$, acting on a functional Hilbert space, which is a $\Gamma$-contraction in the sense of [2].
3. As pointed out by the anonymous referee, the part of Theorem 4.1 till (16) can also be deduced from [9] Corollary 2.2. However, after applying it to the algebra $\mathcal{F}$ of all diagonal operators on $E$, one may conclude that every operator in $\text{AlgLat}(\mathcal{W}_M)$ is of the form $M_\phi$ for a $\mathcal{F}$-valued multiplier $\phi$. Since dim $\mathcal{F}$ could be bigger than 1, it is not clear to the authors how to deduce that the multiplier $\phi$ is indeed scalar-valued.

We discuss below several consequences of Theorem 4.1.

**Corollary 4.3.** Let $(\mathcal{H}, \kappa, \Omega, E)$ be a functional Hilbert space and let $M_z$ denote the commuting $d$-tuple of multiplication operators $M_z, \ldots, M_{zd}$ in $B(\mathcal{H})$. If the pair $(\Omega, M_z)$ falls in the List [27], then $M_z$ is reflexive.

The next corollary shows that certain joint subnormal multiplication tuples acting on functional Hilbert spaces are reflexive. In particular, it recovers special cases of [38] Theorem 3, [11] Theorem 2.4, [28] Theorem 3.8, [24] Theorem 5 and [30] Theorem 1. Recall that a commuting $d$-tuple $S = (S_1, \ldots, S_d)$ in $B(\mathcal{H})$ is joint subnormal if there exist a Hilbert space $K \supseteq \mathcal{H}$ and a commuting $d$-tuple $N$ of normal operators $N_1, \ldots, N_d$ in $B(\mathcal{K})$ such that $S_j = N_j|_\mathcal{H}$ for $j = 1, \ldots, d$.

**Corollary 4.4.** Let $(\mathcal{H}, \kappa, \Omega, E)$ be a functional Hilbert space and let $M_z$ denote the commuting $d$-tuple of multiplication operators $M_z, \ldots, M_{zd}$ in $B(\mathcal{H})$. If $M_z$ is a joint subnormal $d$-tuple with normal extension $N$ such that $\sigma(N) \subseteq \mathcal{T}$, then $M_z$ is reflexive.

**Proof.** By the spectral theorem for normal tuples [11], $M_z$ satisfies von Neumann’s inequality. Now apply Theorem 4.1. \hfill \square

A celebrated result of Brown and Chevreau [12] states that any contraction with isometric $H^\infty$-functional calculus is reflexive (see [23] Chapter III, Theorem 11.3 for exact statement). The following provides a sufficient condition for reflexivity of polynomially bounded operators (cf. [18] Proposition 4.4, [4] Theorem A). Recall that $T \in B(\mathcal{H})$ is polynomially bounded if there exists a constant $M > 0$ such that $\|p(T)\|_{B(\mathcal{H})} \leq M \|p\|_{\infty, \mathbb{D}}$ for every $p \in \mathbb{C}[z]$.

**Corollary 4.5.** Any left-invertible, analytic polynomially bounded $T$ in $B(\mathcal{H})$ is reflexive provided the spectral radius of the Cauchy dual operator $T^*$ is at most 1.

**Proof.** This is immediate from Theorem 4.1 and Shimorin’s analytic model for left-invertible analytic operators [53], where the assumption that $r(T^*) \leq 1$ ensures that the $E$-valued functions in the model space $\mathcal{H}$ of $T$ are holomorphic in the open unit disc $\mathbb{D}$. \hfill \square

The last corollary is applicable to any analytic operator $T$ in $B(\mathcal{H})$ satisfying the following inequality:

$$\|Tx + y\|^2 \leq 2(\|x\|^2 + \|Ty\|^2), \quad x, y \in \mathcal{H}.$$ 

In fact, an examination of the proof of [53] Theorem 3.6 shows that the Cauchy dual $T^*$ of $T$ exists and satisfies $I - 2T^*T^* + T^*2T^2 \leq 0$. It can be concluded from [46] Lemma 1 that $T$ is a contraction and the spectral radius of $T^*$ is at most 1.
5. Applications to weighted shifts on rooted directed trees

The reader is referred to [32] for all the relevant definitions pertaining to the rooted directed trees and associated weighted shifts. Let \( \mathcal{T} = (V, \mathcal{E}) \) be a leafless, rooted directed tree and let

\[
V_\prec := \{ v \in V : \text{card(Chi}(v)) \geq 2 \}
\]

denote the set of branching vertices. The **branching index** \( k_\mathcal{T} \) of \( \mathcal{T} \) is defined as

\[
k_\mathcal{T} := \begin{cases} 
1 + \sup \{ d_w : w \in V_\prec \} & \text{if } V_\prec \text{ is non-empty}, \\
0 & \text{otherwise},
\end{cases}
\]

where \( d_w \) is the unique non-negative integer such that \( w \in \text{Chi}(d_w) \text{(root)} \) (see [32] Corollary 2.1.5)). We refer to \( d_w \) as the **depth** of \( w \) in \( \mathcal{T} \). Let \( S_\lambda \) be a weighted shift on a rooted directed tree \( \mathcal{T} \). Then \( E := \text{ker}(S_\lambda) \) is finite dimensional if and only if \( \mathcal{T} \) is locally finite with finite branching index (see [17] Proposition 2.1).

Let \( \mathcal{T} = (V, \mathcal{E}) \) be a leafless, locally finite rooted directed tree. For an integer \( a \geq 2 \), the **Bergman shift** \( B_a \) is the weighted shift on \( \mathcal{T} \) with weights given by

\[
\lambda_{u,a} = \frac{1}{\sqrt{\text{card(Chi}(v))}} \sqrt{\frac{d_v + 1}{d_v + a}}, \quad u \in \text{Chi}(v), \ v \in V,
\]

where \( d_v \) is the depth of \( v \) in \( \mathcal{T} \). Needless to say, the shift \( B_2 \) is the Bergman shift if \( \mathcal{T} \) is the rooted directed tree without any branching vertex. By [15] Proposition 5.1.8, \( B_a \) is unitarily equivalent to the operator \( \mathcal{M}_{e_a} \) of multiplication by the coordinate function \( z \) on \( \mathcal{H}_a \), where \( \mathcal{H}_a \) is the reproducing kernel Hilbert space associated with the reproducing kernel \( \kappa_{\mathcal{H}_a} : \mathbb{D} \times \mathbb{D} \rightarrow B(\mathcal{E}) \) given by

\[
\kappa_{\mathcal{H}_a}(z, w) = \! \sum_{n=0}^{\infty} \! \binom{n+a-1}{n} z^n P_{\text{root}} \! + \! \sum_{v \in V_\prec} \! \sum_{n=0}^{\infty} \frac{(d_v + n + a)!}{(d_v + a)!} \frac{(d_v + n + 1)!}{(d_v + n)!} z^n P_{\text{root} \text{(Chi}(v)) \text{root}[r_v]}, \quad z, w \in \mathbb{D}.
\]

Here \( E = \text{ker}(B_a) \) and \( \Gamma_v : \text{Chi}(v) \rightarrow \mathbb{C} \) is given by \( \Gamma_v = \sum_{u \in \text{Chi}(v)} \lambda_{u,v} e_u \). Clearly, \( \kappa_{\mathcal{H}_a}(\lambda, 0) = I_E \) for any \( \lambda \in \mathbb{D} \). Further, it can be easily seen that

\[
\|z^n g\|^2 = \frac{(d_v + a - 1)!}{(d_v + n + a - 1)!} \|g\|^2, \quad g \in l^2(\text{Chi}(d_v) \text{(root)}).
\]

**Proposition 5.1.** Let \( \mathcal{T} = (V, \mathcal{E}) \) be a leafless, locally finite rooted directed tree and let \( B_a \) be the Bergman shift on \( \mathcal{T} \). If \( \mathcal{T} \) has finite branching index, then the commutant \( \{ B_a \}^* \) of \( B_a \) is isometrically isomorphic to \( \mathcal{H}_a^\infty(\mathbb{D}), \| \cdot \|_{B(\mathcal{H})} \), where the finite dimensional Hilbert space \( E \) equals the kernel of \( B_a \).

**Proof.** Suppose that \( \mathcal{T} \) has finite branching index. Note that

\[
\sup_{v \in V} \sum_{u \in \text{Chi}(v)} \lambda_{u,a}^2 \leq \sup_{v \in V} \frac{d_v + 1}{d_v + a} = 1,
\]

and hence by [32] Proposition 3.1.8, \( B_a \) is a contraction. Hence, in view of Theorem 6.1 and the discussion prior to the statement of Proposition 5.1, it suffices to check that the reproducing kernel \( \kappa_{\mathcal{H}_a} \), as given by (20), satisfies (1) and (5). To see (1), let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H} \), where \( \{a_n\}_{n \in \mathbb{N}} \subseteq E \). Let \( \{g_1, \ldots, g_{\dim E}\} \) be an orthonormal basis of \( E \) such that for any \( i = 1, \ldots, \dim E \),

\[
g_i \in l^2(\text{Chi}(k_i) \text{(root)}) \text{ for some } k_i \in \mathbb{N}.
\]
It is easy to see using (21) that \((z^n g_j, z^n g_i) = 0\) if \(i \neq j\). Also, since \(\{z^n E\}_{n \in \mathbb{N}}\) are mutually orthogonal (Lemma 5.2.7), we have
\[
\|f\|^2 = \sum_{n=0}^{\infty} \|z^n a_n\|^2 = \sum_{n=0}^{\dim E} \|\sum_{j=1}^{\dim E} (a_n, g_j)_E z^n g_j\|^2
\]
\[
= \sum_{j=1}^{\dim E} \sum_{n=0}^{\infty} |(a_n, g_j)_E|^2 \|z^n g_j\|^2. \tag{23}
\]
Note further that
\[
\langle f(w), g_j \rangle_E g_k = \sum_{n=0}^{\infty} (a_n, g_j)_E w^n g_k, \quad w \in \mathbb{D},
\]
and hence by the mutual orthogonality of \(\{z^n E\}_{n \in \mathbb{N}}\), we obtain
\[
\|\langle f(\cdot), g_j \rangle_E g_k\|^2 = \sum_{n=0}^{\infty} |(a_n, g_j)_E|^2 \|z^n g_k\|^2 = \sum_{n=0}^{\infty} |(a_n, g_j)_E|^2 \|z^n g_j\|^2 \\|z^n g_k\|^2. \tag{24}
\]
To complete the verification of (4), in view of (23), it suffices to check that the sequence \(\{\|z^n g_k\|^2/\|z^n g_j\|^2\}_{n \in \mathbb{N}}\) is bounded. Indeed, this sequence is convergent in view of (21) and (24).

To see (5), fix \(w \in \mathbb{D}\), and note that \(\kappa_{\mathcal{K}}(w, w)\) is a positive diagonal operator with respect to the orthonormal bases of \([e_{\alpha\nu}]\) and \(l^2(\text{Chi}(v)) \oplus [\Gamma_v]\), \(v \in V_{\mathcal{F}}\). Moreover, the diagonal entries of \(\kappa_{\mathcal{K}}(w, w)\) are given by
\[
\sum_{n=0}^{\infty} \binom{n+a-1}{n} |w|^{2n}, \quad \sum_{n=0}^{\infty} \binom{d_v + n + a}{d_v} (d_v + 1)! \binom{d_v + n}{d_v} (d_v + n + 1)! |w|^{2n} \quad v \in V_{\mathcal{F}}.
\]
Consider the bi-sequence \(\{a_{m,n}\}_{m,n \in \mathbb{N}}\) given by
\[
a_{m,n} = \frac{(m+n+a)!(m+1)!}{(m+a)!(m+n+1)!}, \quad m, n \in \mathbb{N}. \tag{24}
\]
Since \(a \geq 2\), \(\{a_{m,n}\}_{m \in \mathbb{N}}\) is decreasing for every \(n \in \mathbb{N}\). Further, since
\[
\binom{n+a-1}{n} \geq a_{0,n}, \quad n \in \mathbb{N},
\]
it follows that the minimum \(\mu_{\min}(w)\) and maximum \(\mu_{\max}(w)\) of eigenvalues of \(\kappa_{\mathcal{K}}(w, w)\) are given respectively by
\[
\mu_{\min}(w) = \sum_{m=0}^{\infty} a_{m,n} |w|^{2n}, \quad m_0 := \max \{d_v : v \in V_{\mathcal{F}}\},
\]
\[
\mu_{\max}(w) = \sum_{n=0}^{\infty} \binom{n+a-1}{n} |w|^{2n},
\]
where \(m_0\) is finite since \(\mathcal{F}\) has finite branching index. Thus (5) is equivalent to
\[
\sup_{w \in \mathbb{D}} \frac{\mu_{\max}(w)}{\mu_{\min}(w)} < \infty. \tag{25}
\]
To see (24), note that
\[
\lim_{n \to \infty} \frac{\binom{n+a-1}{n}}{a_{m_0,n}} = \frac{(m_0 + a)!}{(m_0 + 1)!(a - 1)!}.
\]
It follows now from (24) that there exists a positive integer \(n_0\) such that
\[
\binom{n+a-1}{n} \leq (m_0 + a)! a_{m_0,n}, \quad n \geq n_0.
\]
Consequently,
\[
\frac{\mu_{\text{max}}(\omega)}{\mu_{\text{min}}(\omega)} = \frac{\sum_{n=0}^{\infty} \binom{n+a-1}{n} |\omega|^{2n}}{\sum_{n=0}^{\infty} a_{m_0,n} |\omega|^{2n}} \leq \frac{\sum_{n=0}^{\infty} \binom{n+a-1}{n} |\omega|^{2n}}{\sum_{n=0}^{\infty} a_{m_0,n} |\omega|^{2n}} \leq \sum_{n=0}^{\infty} \binom{n+a-1}{n} |\omega|^{2n} + \frac{\sum_{n=0}^{\infty} (m_0 + a)! a_{m_0,n} |\omega|^{2n}}{\sum_{n=0}^{\infty} a_{m_0,n} |\omega|^{2n}} 
\]
and hence we obtain the conclusion in (25). 

\[\square\]

**Remark 5.2.** An examination of the proof shows that there exists a real polynomial \( p \) such that
\[
\| \kappa_{\mathscr{H}_a}(\omega, \omega) \|_{B(E)} \| \kappa_{\mathscr{H}_a}(\omega, \omega)^{-1} \|_{B(E)} \leq p(|\omega|^2), \quad \omega \in \mathbb{D}. \tag{26}
\]

The following corollary is immediate from Corollary 3.6, Proposition 5.1 and Proposition 3.5.1.

**Corollary 5.3.** Let \( \mathcal{T} = (V, E) \) be a leafless, locally finite rooted directed tree and let \( \mathcal{R}_a \) be the Bergman shift on \( \mathcal{T} \). If \( \mathcal{T} \) has finite branching index then the commutant \( \{ \mathcal{R}_a \} \) of \( \mathcal{R}_a \) is abelian if and only if \( \mathcal{T} \) is graph isomorphic to the rooted directed tree without any branching vertex.

It would be of independent interest to characterize \( B(E) \)-valued reproducing kernels \( \kappa \) which satisfy (20) for a polynomial \( p \). In the context of Bergman shifts on \( \mathcal{T} \), this problem seems to be closely related to the notion of finite branching index of \( \mathcal{T} \). One may further ask for a multivariable counter-part of Proposition 5.1. We believe that similar arguments can be used to obtain a counter-part of Proposition 5.1 for multivariable analogs \( S_{\lambda_a} \) of Bergman shifts \( \mathcal{R}_a \) (refer to [15] Chapter 5). Further, it may be concluded from Corollary 1.3 and Theorem 5.2.6 and Example 5.3.5 that the \( d \)-tuple \( S_{\lambda_a} \) is reflexive for any integer \( a \geq d \). In order to avoid book-keeping, we skip these verifications. Here we discuss one family of weighted multishift to which Theorem 4.1 is applicable. The following can be seen as a 2-variable counterpart of [14] Theorem 10 (the reader is referred to [15] for the definitions of directed Cartesian product of directed trees and associated multishifts).

**Proposition 5.4.** Let \( \mathcal{T} = (V, E) \) be the directed Cartesian product of locally finite, rooted directed trees \( \mathcal{T}_1, \mathcal{T}_2 \) and let \( S_{\lambda} = (S_{\lambda_1}, S_{\lambda_2}) \) be a multishift on \( \mathcal{T} \) consisting of left-invertible operators \( S_1 \) and \( S_2 \). Let \( E \) be the joint kernel of \( S_{\lambda} \). Assume that \( S_{\lambda} \) satisfies
\[
E \subseteq \ker(S_{\lambda_1}^* S_{\lambda_2}^\alpha) \cap \ker(S_{\lambda_2}^* S_{\lambda_1}^\alpha) \quad \text{for all } (\alpha_1, \alpha_2) \in \mathbb{N}^2, \tag{27}
\]
and that the 2-tuple \( (S_{\lambda_1}^*, S_{\lambda_2}^*) \) consists of commuting Cauchy dual operators of spectral radii at most 1. If \( S_{\lambda} \) is contractive, then it is reflexive.

**Proof.** By [15] Theorem 4.2.4, there exist a reproducing kernel Hilbert space \( \mathcal{H} \) of \( E \)-valued holomorphic functions defined on the unit bidisc centered at the origin and a unitary \( U : l^2(V) \to \mathcal{H} \) such that \( U S_j = \mathcal{M}_{z_j} U \) for \( j = 1, 2 \). The desired conclusion now follows from Theorem 4.1, List 2.11 and Lemma 2.10. \( \square \)

**Remark 5.5.** In case any one of \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) is without any branching vertex, (27) always holds (see [15] Corollary 4.2.9).
5.1. A two-parameter family of tridiagonal $B(E)$-valued kernels. We conclude the paper by exhibiting a two parameter family of a tridiagonal $B(E)$-valued kernels, which satisfy the assumptions of Theorems 3.1 and 4.1. Consider the rooted directed tree $\mathcal{T} = (V, E)$ as discussed in [32, Section 6.2] (see also [17, Example 3]). Recall that the set $V$ of vertices of $\mathcal{T}$ is given by

$$V := \{ (0,0) \} \cup \{(1,i), (2,i) : i \geq 1 \}$$

with root $= (0,0)$, and the edges are governed by $\text{Chi}(0,0) = \{(1,1), (2,1)\}$ and $\text{Chi}(1,i) = \{(1,i+1)\}$, $\text{Chi}(2,i) = \{(2,i+1)\}$, $i \geq 1$.

For positive numbers $s$ and $t$ with $t \neq 1$, consider the weight system $\lambda_{s,t}$ given by

$$\begin{align*}
\lambda_{(1,1)} &= s = \lambda_{(2,1)}, \\
\lambda_{(1,2)} &= 1 = \lambda_{(2,3)}, \\
\lambda_{(2,2)} &= t = \lambda_{(1,3)}, \\
\lambda_{(j,i)} &= 1, \quad j = 1,2, \quad i \geq 4.
\end{align*}$$

(28)

Let $S_{\lambda_{s,t}}$ be the weighted shift with weight system $\lambda_{s,t}$ and let $E := \ker(S_{\lambda_{s,t}}^*)$. Then, as noted in [17, Proposition 4.1], $\lambda_{s,t}$ is unitarily equivalent to the multiplication operator $M$, on the reproducing kernel Hilbert space $H$ of $E$-valued holomorphic functions on the unit disc $\mathbb{D}$. The reproducing kernel $\kappa_{\mathcal{H}} : \mathbb{D} \times \mathbb{D} \rightarrow B(E)$ of $\mathcal{H}$ is given by

$$\kappa_{\mathcal{H}}(z, w) = I_E + a_0(x \otimes y \otimes z^2 + y \otimes x z^2 w) + \sum_{k=1}^{\infty} \left( a_k x \otimes x + a_{k+1} y \otimes y \right) z^2 w^k, \quad z, w \in \mathbb{D},$$

(29)

where $x = e_{(0,0)}$, $y = s(e_{(1,1)} - e_{(2,1)})$ are orthogonal basis vectors for $E$, and

$$a_k := \begin{cases} 
\frac{1}{24} (1 - t^{-2}) & \text{if } k = 0, \\
\frac{1}{24} & \text{if } k = 1, \\
\frac{1}{24} (1 + t^{-2}) & \text{if } k = 2, \\
\frac{1}{24t} & \text{if } k \geq 3.
\end{cases}$$

Clearly, $\kappa_{\mathcal{H}}$ satisfies the normalization condition [3]. One may argue as in [17, Example 4] to deduce that

$$B := \{ x \} \cup \left\{ a_k z^k p(z) \right\}_{k \in \mathbb{N}} \cup \left\{ b_k z^k q(z) \right\}_{k \in \mathbb{N}}$$

(30)

forms an orthonormal basis of $\mathcal{H}$, where

$$a_0 = a_1 = 1, \quad a_k = \frac{1}{4}, \quad k \geq 2, \quad b_0 = 1, \quad b_k = \frac{1}{4}, \quad k \geq 1,$$

(31)

$$p(z) = \frac{1}{24} (xz + y), \quad q(z) = \frac{1}{24} (xz - y).$$

(32)

**Lemma 5.6.** The $B(E)$-valued kernel $\kappa_{\mathcal{H}}$, as given by (29), satisfies the conditions [1] and [5] of Theorem [37].

**Proof.** Let $f \in \mathcal{H}$. Since $B$, as given by (30), forms an orthonormal basis for $\mathcal{H}$, there exist $c, c_k, d_k \in \mathbb{C}$, $k \geq 0$, such that

$$f(z) = cx + \sum_{k=0}^{\infty} c_k a_k z^k p(z) + \sum_{k=0}^{\infty} d_k b_k z^k q(z), \quad z \in \mathbb{D}.$$

It follows that $\|f\|_{\mathcal{H}}^2 = |c|^2 + \sum_{k=0}^{\infty} |c_k|^2 + \sum_{k=0}^{\infty} |d_k|^2$. That is, $\{c_k\}_{k \geq 0}$, $\{d_k\}_{k \geq 0}$ are in $l^2(\mathbb{N})$. For $g, h \in E$, define $f_{g,h}(w) = \langle f(w), g \rangle_E h$, $w \in \mathbb{D}$. Since $\{x, y\}$ is
an orthogonal basis of \( E \), in order to show that \( f_{g,h} \in \mathcal{H} \), it suffices to check that 
\( f_{x,y}, f_{y,x}, f_{x,x}, f_{y,y} \in \mathcal{H} \). It is easy to see using (32) that 
\[
f_{x,y}(w) = cy + \frac{1}{2\bar{g}} \sum_{k=0}^{\infty} (a_k c_k + b_k d_k) w^{k+1} y, \quad w \in \mathbb{D}.
\]
Since \( \{z^n y\}_{n \in \mathbb{N}} \) is orthogonal and for \( n \in \mathbb{N} \),
\[
\|z^n y\|^2 = s^2 \|S_{\lambda,1}^n (c_{1,1} - c_{2,1})\|^2_{\mathcal{F}(V)} = s^2 \|S_{\lambda,1}^n (c_{1,1})\|^2_{\mathcal{F}(V)} + \|S_{\lambda,1}^n (c_{2,1})\|^2_{\mathcal{F}(V)} \leq \max\{2s^2, 2a t^2, s^2(1 + t^2)\},
\]
it follows from \( \{c_k\}_{k \in \mathbb{N}}, \{d_k\}_{k \in \mathbb{N}} \in \mathcal{F}(\mathbb{N}) \) and (31) that \( f_{x,y} \in \mathcal{H} \). Along the similar lines, one can check that \( f_{y,x}, f_{x,x}, f_{y,y} \in \mathcal{H} \). This yields (4).

To see (5), note that \( \|y\| = s\sqrt{2} \). Thus, for \( w \in \mathbb{D} \), by (30), we have
\[
\kappa_{\mathcal{F},w} y = k_1(w, w) x + \alpha_0 s \sqrt{2} w^2 \frac{y}{\|y\|},
\]
\[
k_1(w, w) = 1 + \sum_{k=1}^{\infty} \alpha_k |w|^{2k}.
\]
Similarly, for \( w \in \mathbb{D} \), we have
\[
\kappa_{\mathcal{F},w} y \frac{y}{\|y\|} = \alpha_0 s \sqrt{2} w^2 \frac{x + k_2(w, w)}{\|y\|},
\]
\[
k_2(w, w) = 1 + \sum_{k=1}^{\infty} \alpha_k + 2a |w|^{2k}.
\]
The matrix representation, say \( A(w) \), of the positive operator \( \kappa_{\mathcal{F},w} \) with respect to the basis \( \{x, \frac{y}{\|y\|}\} \) is given by
\[
A(w) = \begin{bmatrix} k_1(w, w) & aw^2 \frac{y}{\|y\|} \\ aw^2 \frac{y}{\|y\|} & k_2(w, w) \end{bmatrix},
\]
where \( a := \alpha_0 s \sqrt{2} \). The eigenvalues \( x_+(w) \) and \( x_-(w) \) of \( A(w) \) are given by
\[
x_{\pm}(w) = \frac{1}{2} \left( k_1(w, w) + k_2(w, w) \pm \sqrt{(k_1(w, w) - k_2(w, w))^2 + 4a^2|w|^6} \right).
\]
Clearly, \( x_+(w) \geq x_-(w) \) for all \( w \in \mathbb{D} \). It follows that for any \( w \in \mathbb{D} \),
\[
\frac{x_+(w)}{x_-(w)} = \frac{k_1(w, w) + k_2(w, w) + \sqrt{(k_1(w, w) - k_2(w, w))^2 + 4a^2|w|^6}}{k_1(w, w) + k_2(w, w) - \sqrt{(k_1(w, w) - k_2(w, w))^2 + 4a^2|w|^6}} \leq \frac{k_1(w, w) + k_2(w, w) + |k_1(w, w) - k_2(w, w)| + 2a|w|^3}{k_1(w, w) + k_2(w, w) - |k_1(w, w) - k_2(w, w)| - 2a|w|^3} \leq \max\left\{ \frac{k_1(w, w) + a}{k_2(w, w) - a}, \frac{k_2(w, w) + a}{k_1(w, w) - a} \right\},
\]
which, in view of (33) and (34), is easily seen to be of polynomial order as a function of \( |w|^2 \). This completes the verification of (5). \( \Box \)

Assume that \( s \in (0, 1/\sqrt{2}) \) and \( t \in (0, 1) \). By Proposition 3.1.8, \( S_{\lambda,1} \) is a contraction. Combining Lemma 5.3 with Theorem 5.1, we conclude that the commutant of \( S_{\lambda,1} \) is isometrically isomorphic to \( (H_{\mathcal{F}(\mathbb{R})}^\infty (\mathbb{D}), \|\cdot\|_{\mathcal{F}(\mathcal{H})}) \), where \( E = \ker(S_{\lambda,1}^*) \) is the 2-dimensional space spanned by \( x \) and \( y \). Further, by Theorem 4.1, \( S_{\lambda,1} \) is reflexive. It is worth mentioning that \( S_{\lambda,1} \) is never hypnormal, that is, \( S_{\lambda,1}^*, S_{\lambda,1} - S_{\lambda,1}, S_{\lambda,1}^* \neq 0 \) (see [32, Theorem 5.1.2]).
Acknowledgment. A part of this paper was written while the second author visited the Department of Mathematics and Statistics, IIT Kanpur. He expresses his gratitude to the faculty and the administration of this unit for their warm hospitality. The authors appreciate the suggestions of Md. Ramiz Reza and Deepak Kumar Pradhan pertaining to the definition of the functional Hilbert space improving the earlier presentation. The authors would like to thank Jan Stochel for his continual support and encouragement. Finally, the authors are grateful to the anonymous referee for several important remarks (see Remark 3.3 and Remark 4.2). In particular, it has been pointed out by the referee that Theorem 3.1 can be recovered from its scalar-valued counterpart (the case of dim $E = 1$).

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