Meshless analysis of geometrically nonlinear beams

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Abstract. The meshless method is applied to the response analysis of a geometrically nonlinear beam. The corresponding nonlinear strain operator matrix and tangent rigidity matrix are given. The computational results agree well with Holden's analytic solution. It is found that the meshless method is more efficient for large displacement.

1. Introduction

The geometrical non-linear response of structures is an important subject in engineering. The main characteristic of geometrical non-linear problems is that the fundamental equations are depended on the deformed configuration of structures, so that it is complicated in computation. In the general, the geometrical non-linear problems are divided into two kinds: the large strain and large displacement problems; the small strain and large displacement problems. Many geometrical non-linear problems in engineering practices are the latter. At present, the finite element method, which is a mesh method, is commonly used to compute non-linear problems. However, sometimes the finite element method is ineffective in handling the geometrical non-linear problems due to severe mesh distortions. Meshless methods completely or partly abandon the mesh, and only the information of nodes and the boundary description are needed. So the meshless method is one of promising methods dealing with such problems as the large deformation, crack growth, impact and metal forming etc. [1]. The Element-Free Galerkin method (EFG), which was announced by Belytschko et al. [2-3], is successful used in dynamic modeling of crack growth and 3-D impact analysis. Xia and Wei have analyzed the elastic-plastic response of a plate with a diamond crack by using EFG [4]. Based on Galerkin method, according to function integral transformation, Liu et al. developed Reproducing Kernel Particle Method (RKPM) and applied it to the numerical analysis of the dynamic crack growth and the localization [5], the stress concentration [6], the large deformation problems [7] and the metal forming [8] etc.

The meshless method is one of effective methods of structural analyses. At this moment, although a lot of research has been done in this area, there are still some urgent problems calling for a solution. In this paper, EFG is used to deal with the small strain and large displacement problem. The corresponding non-linear strain operator matrix and tangent rigidity matrix are derivated. The large displacement response of a cantilever beam subjected to the uniform load is analyzed. The computational results of the meshless method are close to Holden's analytic solutions [9]. It is found that

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the computation of the meshless method is efficient for large displacement of the beam because it can analyze the large displacement response of the beam successfully and remesh is not needed. Some useful results and conclusions are obtained.

2. Fundamental theorem

By moving least square approximation (MLS)\cite{10}, we assume the approximate displacement function in domain is in the following form as:

\[ u^h(X) = \sum_{j=1}^{m} p_j(X) a_j(X) = P^T(X)a(X) \]  

(1)

Where, \( X \) is the space coordinates of an arbitrary point, \( P(X) \) is the m-dimension complete polynomial basis. For 2-D problems, \( m = 6 \), the quadratic basis is:

\[ P(X) = (1, x, y, x^2, xy, y^2)^T \]  

(2)

And, \( a_j(X) \) is the coefficients dependent on the coordinates of the point. It can be obtained by minimizing of equation (3) at an arbitrary point \( X \):

\[ J = \sum_{i=1}^{n} w(X - X_i)[P(X_i)a(X) - u_i^*]^2 \]  

(3)

Where, \( u_i^* \) is the displacement characteristic value of Node \( i \), and \( w(X - X_i) \) is the weight function. In this paper, the following weight function is used\cite{2}:

\[ w_i(d_i) = \begin{cases} e^{-(d_i/c)^2} - e^{-(d_{mi}/c)^2} & d_i \leq d_{mi} \\ 1 - e^{-(d_{mi}/c)^2} & d_i > d_{mi} \end{cases} \]  

(4)

Where, \( d_i = \|X - X_i\| \) denotes the distance between \( X \) and \( X_i \); \( c \) is the controlling constant of the relative weights; \( d_{mi} \) the radius of the influence domain of \( X \), and \( n \) is the number of nodes in the influence domain.

By minimizing equation (3), a set of linear algebraic equations are given:

\[ A(X)a(X) = B(X)U^* \]  

(5)

Where,

\[ A(X) = \sum_{i=1}^{n} w_i(X)P(X_i)P^T(X_i) \]

\[ B(X) = [w_1(X)P(X_1), w_2(X)P(X_2), ..., w_n(X)P(X_n)] \]

When the matrix \( A(X) \) is non-singular, we obtain the solution of Equation (5):  

\[ a(X) = A^{-1}(X)B(X)U^* \]  

(6a)
Substituting Equation (6) into Equation (1), the displacement function \( \mathbf{u}^h(\mathbf{X}) \) is written as:

\[
\mathbf{u}^h(\mathbf{X}) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_j(\mathbf{X}) [\mathbf{A}^{-1}(\mathbf{X}) \mathbf{B}(\mathbf{X})]_{ji} \mathbf{u}^*_i = \sum_{i=1}^{n} \phi_i(\mathbf{X}) \mathbf{u}^*_i
\]

(7)

Where, \( \phi_i(\mathbf{X}) = \sum_{j=1}^{m} p_j(\mathbf{X}) [\mathbf{A}^{-1}(\mathbf{X}) \mathbf{B}(\mathbf{X})]_{ji} \) is the shape function of Node \( i \).

On the small strain and large displacement problems, we suppose the constitution of the material is linear and Green strains and Kirchhoff stresses are used. The 2-D strain vector is

\[
\varepsilon_0 = \begin{bmatrix}
\frac{\partial u}{\partial x} + \frac{1}{2} \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right) \\
\frac{\partial v}{\partial y} + \frac{1}{2} \left( \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right)
\end{bmatrix} = \varepsilon_o + \varepsilon_L
\]

(8)

Where, \( \varepsilon_o \) and \( \varepsilon_L \) are the linear strain and the non-linear strain, respectively:

\[
\varepsilon_o = \left( \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^T, \quad \varepsilon_L = \frac{1}{2} \mathbf{L} \cdot \boldsymbol{\theta}
\]

(9)

And,

\[
\mathbf{L} = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & 0 & 0 \\
0 & 0 & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix}, \quad \boldsymbol{\theta} = \begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial x} \\
\frac{\partial v}{\partial y}
\end{bmatrix}
\]

(10)

Then according to the differential relationship of the matrix \( \mathbf{L} \) and the vector \( \boldsymbol{\theta} \), the following equation can be given:

\[
d\varepsilon_L = \frac{1}{2} \mathbf{L} \cdot d\boldsymbol{\theta} + \frac{1}{2} d\mathbf{L} \cdot \boldsymbol{\theta} = \mathbf{L} \cdot d\boldsymbol{\theta}
\]

(11)

For an arbitrary point \( \mathbf{X} \) in the domain \( \Omega \), let the following equations held:

\[
f_{xx} = \frac{\partial u}{\partial x} = \sum_{i=1}^{n} \phi_i \mathbf{u}_i^*, \quad f_{xy} = \frac{\partial v}{\partial x} = \sum_{i=1}^{n} \phi_i \mathbf{v}_i^*
\]
\[ f_{yu} = \frac{\partial u}{\partial y} = \sum_{i=1}^{n} \phi_{i,y} u_i^*, \quad f_{yv} = \frac{\partial v}{\partial y} = \sum_{i=1}^{n} \phi_{i,y} v_i^*, \]

Through differential calculation for vector \( \theta \), we can obtain \( d\theta = G dU^* \), and the element \( G_i \) of \( G \) is presented as:

\[
G_i = \begin{bmatrix}
\phi_{i,x} & 0 \\
0 & \phi_{i,x} \\
\phi_{i,y} & 0 \\
0 & \phi_{i,y}
\end{bmatrix}
\]

Then, Equation (11) may be rewritten as:

\[ d\varepsilon_L = L \cdot G dU^* = B_L dU^* \]

Where, \( B_L \) is called as the non-linear strain operator matrix, its element is \( B_{Li} \):

\[
B_{Li} = \begin{bmatrix}
f_{xy} \cdot \phi_{i,x} & f_{xy} \cdot \phi_{i,x} \\
f_{yv} \cdot \phi_{i,y} & f_{yv} \cdot \phi_{i,y} \\
f_{yu} \cdot \phi_{i,x} + f_{xy} \cdot \phi_{i,y} & f_{yu} \cdot \phi_{i,x} + f_{xy} \cdot \phi_{i,y}
\end{bmatrix}
\]

The strain operator matrix \( B \) can be presented as the sum of the linear strain operator matrix \( B_0 \) and the non-linear strain operator matrix \( B_L \), i.e.:

\[ B = B_0 + B_L \]

Where, the element \( B_{0i} \) of \( B_0 \) is denoted as:

\[
B_{0i} = \begin{bmatrix}
\phi_{i,x} & 0 \\
0 & \phi_{i,y} \\
\phi_{i,x} & \phi_{i,y}
\end{bmatrix}
\]

We define the original configuration as reference configuration and the tangent rigidity matrix \( K_T \) of the geometrical non-linear problems is written as (11):

\[ K_T = K_0 + K_L + K_\sigma \]

Where, \( K_0 \) is the linear rigidity matrix; \( K_L \) is the large displacement matrix; \( K_\sigma \) is the initial stress matrix. They are presented respectively as:

\[ K_0 = \int_{\Omega} B_0^T DB_0 d\Omega + \alpha \int_{\Gamma_c} \Phi^T \Delta \Phi d\Gamma \]

\[ K_L = \int_{\Omega} (B_0^T DB_L + B_L^T DB_L + B_L^T DB_0) d\Omega \]
\[ K_\sigma = \int_\Omega G^T H G d\Omega \]  

(20)

Where, \( \alpha \) is the penalty parameter; Matrix \( \Phi \), \( \Delta \) and \( H \) are denoted, respectively:

\[
\Phi_1 = \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_1 \end{bmatrix}, \quad \Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}, \quad H = \begin{bmatrix} S_{xx} & 0 & S_{xy} & 0 \\ 0 & S_{xx} & 0 & S_{xy} \\ S_{yx} & 0 & S_{yy} & 0 \\ 0 & S_{yx} & 0 & S_{yy} \end{bmatrix}.
\]

3. Computation procedure

The incremental equilibrium equation of the geometrical non-linear problems based on the meshless method is written as:

\[ K_T \Delta U^* = \int_\Omega B^T S d\Omega - P \]

(21)

Where, \( \int_\Omega B^T S d\Omega \) and \( P \) are the internal force and the load vector of nodes, respectively. And \( \left( \int_\Omega B^T S d\Omega - P \right) \) presents the non-equilibrium force vector of the system.

The load acting on the system is divided into \( n \) incremental steps. Newton-Raphson iteration is applied to solve the non-linear problems. For every load incremental step, the following computation procedures is done:

1) From Equation \( K_0 U^* = P \), obtain the first approximation \( U_1^* = U^* \); 
2) Compute \( B = B_0 + B_L(U_1^*) \) and \( S = DBU_1^* \), find the non-equilibrium force vector \( \Delta\phi_1 = \int_\Omega B^T S d\Omega - P \);
3) Compute the tangent rigidity matrix \( K_T = K_0 + K_L(U_1^*) + K_\sigma(S) \);
4) Substitute \( K_T \) into Equation \( K_T \Delta U_1^* = \Delta\phi_1 \), and obtain the increment:
\[ \Delta U_1^* = K_T^{-1} \Delta\phi_1 \]
5) Compute the displacement \( U_2^* = U_1^* + \Delta U_1^* \);
6) Go back Step 2), and repeat the above computational procedures, until \( \Delta\phi_n \) is small enough that it can be neglected.

4. Numerical example analyses

Consider a cantilever beam subjected to the uniform load as shown in Figure 1 (a). The span, height and width of the beam are respectively: \( L = 10cm \), \( h = 1cm \), and \( b = 1cm \). The elastic modulus and Poisson's ratio of the material are \( E = 1.2 \times 10^4 N/cm^2 \) and \( \mu = 0.2 \). According to above computational procedure, we compose the computational program and gain some computational results. These results of the meshless method are compared with Holden's analytic solutions \(^9\).
Two kinds of loads are considered: First, the direction of the load maintains to be vertical, i.e. the load is independent on the deformation of the beam. Second, the direction of the load keeps to be vertical to the axial line of the deformed beam, i.e. the following load.

\[
\kappa = \frac{PL^3}{EI}
\]

**Figure 1** A cantilever beam subjected to uniform load and its nodal arrangements

**Figure 2** Deflection-load curves of the cantilever beam

**Figure 3** Three nodal arrangements
Consider First kind of the loads. Figure 2 gives the load-deflection curves obtained in different computational methods. It is seen that the deflection of the cantilever beam increases with the load, but the relationship curves of the deflection and the load have obvious nonlinear when the influence of the large displacement is included. The computational results of this paper are close to Holden's analytic solutions. But their relative error slightly increases with the increase of the load parameter $k = \frac{PL^3}{EI}$.

When load parameter is equal to 10, the maximum of the relative errors can be found. It is equal to 3.34%.

Figure 3 gives three kinds of the nodal arrangement, these nodal intervals are from sparse to dense. When the nodes are sparse, for example, the interval is 0.5cm×0.5cm (Nodal arrangement 1), the
difference between the deflections computed by the meshless method and Holden's analytic solution is quite large as shown in Figure 4. When nodes are densified, for example, the interval is 0.5cm × 0.25cm (Nodal arrangement 2), the computational results of the two methods are near. However, if the density of nodes is increased to a certain extent, for example, the interval is 0.25cm×0.125cm (Nodal arrangement 3), there isn’t much difference between these computational results, as shown in Figure 4.

From Figure 5, it is seen that the computational results of the deflection doesn’t nearly change when length-width ratio $h/l$ increases.

Consider second kind of the loads, i.e. the following load. In every computational step, the loads are rotated with the same angle as that of the borders of the beam so that the loads keep to be vertical to the top and bottom borders of the deformed beam. The large displacement response of the cantilever beam is computed by the meshless method described in this paper. The deformation of the beam increases with the increase of the load parameter $k$. The computation of the meshless method can be completed successfully when $k=250$ and the beam is rotated over $270^\circ$, as shown in Figure 6.

5. Conclusions
In this paper, a meshless method is applied to analyses of the geometrical non-linear beams. The corresponding non-linear strain operator matrix and tangent rigidity matrix are derivated. The large displacement response of a cantilever beam subjected to the uniform load is investigated. The computational results of the meshless method are compared with Holden's analytic solutions. The computational results of the meshless method are close to Holden's analytic solutions. When the nodal density is in some extent, the computational accuracy of the meshless method increases with the increase of the nodal density. The computation of the meshless method can be completed successfully until the beam is rotated over $270^\circ$.

Because of the independent on the mesh, meshless methods are suitable for some large deformation problems. It can be predicted that meshless methods are promising in the analysis for geometrical non-linear dynamic response of discontinuous solids. Further research in the computational theory and technique is necessary and useful.

References
[1] Zhang X, Song K Z and Lu M W 2003 Chinese J. Comput. Mech. 20 730
[2] Lu Y Y, Belytschko T and Tabbara M 1995 *Comput. Methods Appl. Mech. Engrg.* **126** 131
[3] Belytschko T and Fleming M 1999 *Comput. Struct.* **71** 173
[4] Xia J M and Wei D M 2006 *Int. J. Nonlinear Sci.* **7** 353
[5] Liu W K and Hao S 1999 *Comput. Mater. Sci.* **16** 197
[6] Lee S H, Kim H J and Jun S 2000 *Comput. Mech.* **26** 376
[7] Chen J S, Pan C and Wu C T 1996 *Comput. Methods Appl. Mech. Engrg.* **139** 195
[8] Chen J S and Pan C 1998 *Comput. Mech.* **22** 289
[9] Wang X C 2003 *Finite Element Method* (Beijing: Tsinghua University Press)
[10] Wu Y L 2003 *Methods of Computational Mechanics of Solids* (Beijing: Science Press)
[11] Wang H D and Wu D L 1997 *Finite element method and computational program* (Beijing: Chinese Architecture Industry Press)