Approximations of strongly continuous families of unbounded operators

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Abstract

The problem of approximating the discrete spectra of families of self-adjoint operators that are merely strongly continuous is addressed. It is well-known that the spectrum need not vary continuously (as a set) under strong perturbations. However, it is shown that under an additional compactness assumption the spectrum does vary continuously, and a family of symmetric finite-dimensional approximations is constructed. An important feature of these approximations is that they are valid for the entire family simultaneously. An application of this result to plasma instabilities is illustrated.

1 Introduction

1.1 Overview

We present a method for obtaining finite-dimensional approximations of the discrete spectrum of families of self-adjoint operators. We study operators that are only required to have a gap in the essential spectrum. This gap is allowed to be a half-line, hence Schrödinger operators are a sub-class of these operators. We are motivated by the following problem:
Problem 1. Consider the family of self-adjoint unbounded operators
\[ \mathcal{M}^\lambda = A + K^\lambda = \begin{bmatrix} -\Delta + 1 & 0 \\ 0 & \Delta - 1 \end{bmatrix} + \begin{bmatrix} K_{\lambda+} & K_{\lambda-} \\ K_{\lambda+} & K_{\lambda-} \end{bmatrix}, \quad \lambda \in [0, 1] \tag{1.1} \]
acting in \( L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d) \), where \( \{K^\lambda\}_{\lambda \in [0, 1]} \) is a bounded, symmetric and strongly continuous family. Is it possible to construct explicit finite-dimensional symmetric approximations of \( \mathcal{M}^\lambda \) whose spectrum in \((-1, 1)\) converges to that of \( \mathcal{M}^\lambda \) for all \( \lambda \) simultaneously?

The problem we have in mind, treated in a separate paper [2], is that of instabilities of the relativistic Vlasov-Maxwell system describing the evolution of collisionless plasmas; it is outlined in section 6 below.

1.2 The main result

Let \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) be a (separable) Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \) and let
\[ A^\lambda = \begin{bmatrix} A^\lambda_+ & 0 \\ 0 & -A^\lambda_- \end{bmatrix} \quad \text{and} \quad K^\lambda = \begin{bmatrix} K^\lambda_+ & K^\lambda_- \\ K^\lambda_+ & K^\lambda_- \end{bmatrix}, \quad \lambda \in [0, 1] \tag{1.2} \]
be two families of operators on \( \mathcal{H} \) depending upon the parameter \( \lambda \in [0, 1] \), where the family \( A^\lambda \) is also assumed to be defined for \( \lambda \in D_0 \) of \([0, 1]\) in the complex plane. They satisfy:

i) **Sectoriality:** The sesquilinear forms \( a^\lambda_\pm \) corresponding to \( A^\lambda_\pm \) are sectorial for \( \lambda \in D_0 \), symmetric for real \( \lambda \), have dense domains \( D(a^\lambda_\pm) \) independent of \( \lambda \in D_0 \), and \( D_0 \ni \lambda \mapsto a^\lambda_\pm[u, v] \) are holomorphic for any \( u, v \in D(a^\lambda_\pm) \). [In the terminology of [4], \( a^\lambda_\pm \) are holomorphic families of type (B).]

ii) **Gap:** \( A^\lambda_+ > 1 \) for every \( \lambda \in [0, 1] \).

iii) **Bounded perturbation:** \( \{K^\lambda\}_{\lambda \in [0, 1]} \subset \mathfrak{B}(\mathcal{H}) \) is a symmetric strongly continuous family.

iv) **Compactness:** There exist symmetric operators \( P_\pm \in \mathfrak{B}(\mathcal{H}_\pm) \) which are relatively compact with respect to the forms \( a^\lambda_\pm \), respectively, satisfying \( K^\lambda = K^\lambda P \) for all \( \lambda \in [0, 1] \) where
\[ P = \begin{bmatrix} P_+ & 0 \\ 0 & P_- \end{bmatrix} . \]

Finally, if the family \( A^\lambda \) does not have a compact resolvent we assume:

v) **Compactification of the resolvent:** There exist nonnegative self-adjoint holomorphic families \( \{W^\lambda\}_{\lambda \in D_0} \) of type (B) with associated forms \( w^\lambda_\pm \) such that \( D(w^\lambda_\pm) \cap D(a^\lambda_\pm) \) are dense for all \( \lambda \in D_0 \) and the inclusion \( (D(w^\lambda_\pm) \cap D(a^\lambda_\pm), \| \cdot \|_{a^\lambda_\pm}) \rightarrow (\mathcal{H}, \| \cdot \|) \) is compact for some \( \lambda \in D_0 \) and all \( \varepsilon > 0 \), where \( a^\lambda_\pm \) is the form associated with
\[ A^\lambda_\pm := A^\lambda + \varepsilon W^\lambda, \quad \lambda \in D_0, \ \varepsilon \geq 0, \tag{1.3} \]
and \( \lambda^* \) is the form associated with
\[
W^\lambda = \begin{bmatrix} W^\lambda & 0 \\ 0 & -W^\lambda \end{bmatrix}, \quad \lambda \in D_0.
\]

Finally, we let \( \{ M^\lambda \}_{\lambda \in [0,1]} \) be a family of (unbounded) operators on \( \mathfrak{H} \) defined as
\[
M^\lambda = A^\lambda + K^\lambda, \quad \lambda \in [0,1]. \tag{1.4}
\]

Our main result is formulated with the general case of continuous spectrum in mind:

**Theorem 2.** Let \( A^\lambda \) be as in (1.3), and define
\[
M^\lambda = A^\lambda + K^\lambda, \quad \lambda \in [0,1]. \tag{1.5}
\]

Let \( \{ e^\lambda_{k,\epsilon} \}_{k \in \mathbb{N}} \subset \mathfrak{H} \) be a complete orthonormal set of eigenfunctions of \( A^\lambda \), let \( G^\lambda_{e,n} : \mathfrak{H} \rightarrow \mathfrak{H} \) be the orthogonal projection operators onto \( \text{span}(e^\lambda_{1,\epsilon}, \ldots, e^\lambda_{n,\epsilon}) \) and let \( \tilde{M}^\lambda_{e,n} \) be the \( n \)-dimensional operator defined as the restriction of \( M^\lambda \) to \( G^\lambda_{e,n}(\mathfrak{H}) \). Fix \( \epsilon^* > 0 \), and define the function
\[
\Sigma : [0,1] \times [0, \epsilon^*] \rightarrow (\text{subsets of } (-1,1), d_H)
\]
\[
\Sigma(\lambda, \epsilon) = (-1,1) \cap \text{sp}(M^\lambda)
\]
and for fixed \( \epsilon > 0 \) the function
\[
\Sigma_* : [0,1] \times \mathbb{N} \rightarrow (\text{subsets of } (-1,1), d_H)
\]
\[
\Sigma_*(\lambda, n) = (-1,1) \cap \text{sp}(\tilde{M}^\lambda_{e,n})
\]
where \( d_H \) is the Hausdorff distance (defined below) and \( \mathbb{N} = \mathbb{N} \cup \{ \infty \} \). Then \( \Sigma \) and \( \Sigma_* \) are continuous.

We recall the definition of the Hausdorff distance between two bounded sets \( X, Y \subset \mathbb{C} \):
\[
d_H(X, Y) = \max \left( \sup_{y \in Y} \inf_{x \in X} |x - y|, \sup_{x \in X} \inf_{y \in Y} |x - y| \right).
\]

As the notation becomes quite cumbersome due to the decomposition \( \mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_- \), we shall first treat the simpler case of semi-bounded operators. So as to avoid repetitions in presentation, we think of the semi-bounded case as the same as before, with \( \mathfrak{H} = \mathfrak{H}_+ \) and the subspace \( \mathfrak{H}_- \) being trivial. For brevity we drop the + subscript. The proof of Theorem 2 is presented in section 3 after the following theorem is proved:

**Theorem 2’.** In the case \( \mathfrak{H} = \mathfrak{H}_+ \) the same conclusion of Theorem 2 holds with \( \text{Ran}(\Sigma) = \text{Ran}(\Sigma_*) = (\text{bounded subsets of } (-\infty,1), d_H) \) defined as
\[
\Sigma(\lambda, \epsilon) = (-\infty,1) \cap \text{sp}(M^\lambda)
\]
and
\[
\Sigma_*(\lambda, n) = (-\infty,1) \cap \text{sp}(\tilde{M}^\lambda_{e,n}).
\]

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1.3 Discussion

One of the main driving forces behind the study of linear operators in the 20th century was the development of quantum mechanics. Particular attention had been given to the characterisation of the spectra of such operators, as it encodes many important physical properties (such as energy levels, for instance). Naturally, operators modelling more complicated phenomena are typically viewed as perturbations of simpler operators. Two of the classic texts on this topic are those written by Kato [4] and Reed and Simon [5]. Both are still widely cited to this day. On the other hand, from a more computational perspective, methods for obtaining finite-dimensional approximations of such operators were developed. We refer to [3] for a recent result, and the references therein.

In this paper we present analytic methods for treating both issues simultaneously. This may be viewed as perturbation theory with two parameters: the continuous parameter \( \lambda \) representing small continuous perturbations, and the discrete parameter \( n \) representing the dimension of the finite-dimensional approximation. One of the important aspects of this theory is that the finite-dimensional approximations apply to the entire family of operators simultaneously. Previously, in [1] a much weaker result of this type was obtained, where the resolvent set of Schrödinger operators with a compact resolvent was shown to be stable under similar perturbations.

There are two substantial difficulties in proving these theorems. If the spectrum of \( \mathcal{A}^\lambda \) were discrete for some \( \lambda \) (and therefore for all \( \lambda \)) we would have a natural way to construct approximations by projecting onto increasing subspaces associated to the eigenvalues of \( \mathcal{M}^\lambda \). However we do not require the spectrum to be discrete, and, indeed, in the type of problems we have in mind it is not. This necessitates the introduction of yet another perturbation parameter, \( \varepsilon \), related to the compactification of the resolvent. The other difficulty is in ensuring that the finite-dimensional approximations approximate the whole family of operators simultaneously. To do this, the compactness assumption (iv) plays a crucial role (see Remark 7 below).

We make several remarks on Theorem 2 and Theorem 2′ and the assumptions (i)-(v):

Remark 3. 1. The compactness requirements (iv) on \( \mathcal{P} \) are motivated by (1.1). If \( \mathcal{A} \) has a compact resolvent (e.g. when acting in \( L^2(\mathbb{T}^d) \oplus L^2(\mathbb{T}^d) \)) where \( \mathbb{T}^d \) is the \( d \)-dimensional torus) we may take \( \mathcal{P} \) to be the identity. Otherwise (e.g. for \( L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d) \)) if the perturbations \( \mathcal{K}^\lambda \) are compactly supported in the sense that

\[
\bigcup_{\lambda \in [0,1], \mu \in \delta} \text{supp}(\mathcal{K}^\lambda u) \subset K
\]  

(1.10)

where \( K = K_+ \times K_- \subset \mathbb{R}^d \times \mathbb{R}^d \) is compact, then we may take \( \mathcal{P}_\pm \) as multiplications by the indicator functions of the sets \( K_\pm \). Indeed, we first note that (1.10) implies that for all \( \lambda \), \( \mathcal{K}^\lambda = \mathcal{P}\mathcal{K}^\lambda \). Then as \( \mathcal{K}^\lambda \) and \( \mathcal{P} \) are symmetric, we deduce that \( \mathcal{K}^\lambda = (\mathcal{K}^\lambda)^* = (\mathcal{K}^\lambda)^* \mathcal{P}^* = \mathcal{K}^\lambda \mathcal{P} \) as required. That \( \mathcal{P} \) is relatively compact with respect to \( -\Delta \) follows from Rellich’s theorem. We also remark that this choice of \( \mathcal{P} \) is in fact the natural inclusion map from \( L^2 \) to \( L^2(K) \).

2. Though the decomposition \( \mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+ \) seems somewhat artificial, under
fairly general assumptions on a holomorphic family $B^\lambda$ it is possible to
construct a unitary holomorphic family $U^\lambda$ such that $B^\lambda = (U^\lambda)^{-1}B^U \lambda$
is a holomorphic family which does have such a decomposition.

3. Care must be taken regarding the spaces we view operators as acting on.
If we view $\mathcal{M}_{\varepsilon,n}^\lambda = \mathcal{G}_{\varepsilon,n}^\lambda : \mathcal{H} \to \mathcal{H}$ then $0$ will always be a spurious
eigenvalue with infinite multiplicity. To remove this unwanted eigenvalue
we must instead consider $\tilde{\mathcal{M}}_{\varepsilon,n}^\lambda : \mathcal{H}_{\varepsilon,n} \to \mathcal{H}_{\varepsilon,n}$ where $\mathcal{H}_{\varepsilon,n}^\lambda = \mathcal{G}_{\varepsilon,n}^\lambda(\mathcal{H})$ is the
$n$-dimensional space corresponding to the eigenprojection $\mathcal{G}_{\varepsilon,n}^\lambda$.

4. The two maps $\Sigma$ and $\Sigma_\varepsilon$ are uniformly continuous by the Heine-Cantor
theorem.

5. In Theorem 2 the Hausdorff distance is well-defined as the spectrum in
$(-\infty, 1)$ is bounded from below due to the uniform boundedness of $\mathcal{K}^\lambda$.

6. Property (ii) implies that there exists $\alpha(\lambda) > 0$ such that $(-\alpha(\lambda) - 1, 1 + \alpha(\lambda))$ is in the resolvent set of $\mathcal{A}^\lambda$. Since the spectrum is continuous in
$\lambda \in [0, 1]$ this implies that there is a uniform constant $\alpha > 0$ such that
$(-\alpha - 1, 1 + \alpha)$ is in the resolvent set of $\mathcal{A}^\lambda$ for all $\lambda \in [0, 1]$.

Let us summarise some of the notation we use throughout this article. For
operators we use upper case calligraphic letter, such as $\mathcal{T}$. As already exhibited
above, the spectrum of $T$ is denoted $\text{sp}(T)$. For the sesquilinear form associated
to an operator we use the same letter in lower case Fraktur font. Hence the
operator $\mathcal{T}$ has the associated form $t$. The space of bounded linear operators on
a Hilbert space $\mathcal{H}$ is denoted $B(\mathcal{H})$. Domains of operators or forms are denoted
by $\mathcal{D}$. The graph norms of an operator $T$ and a form $t$ are denoted $\|\cdot\|_T$ and
$\|\cdot\|_t$, respectively. Strong, strong resolvent and norm resolvent convergence are
denoted by $\overset{s}{\to}$, $\overset{s.r.}{\to}$ and $\overset{n.r.}{\to}$, respectively. For brevity, we denote $\mathbb{N} = \mathbb{N} \cup \{\infty\}$.

This paper is organised as follows. In section 2 we present some results
related to general properties (such as self-adjointness, equivalence of norms,
etc.) of the various operators. In section 3 we construct the finite-dimensional
approximations to our family of operators, which are used in section 4 to prove
Theorem 2. In section 5 these results are extended to families of operators
which are not positive, proving Theorem 2. Finally, in section 6 we give a brief
description of an application of these results, related to plasma instabilities.

2 Preliminary results

We remind the reader that in this section, as well as in section 3 and section 4
we treat the semi-bounded case where $\mathcal{H} = \mathcal{H}_+$ and we drop the $+$ subscript.

Considering the definition (1.4) and the subsequent specifications of the
properties of the various operators and associated forms, we have the following
results.

Lemma 4. For any $\lambda \in [0, 1]$, $\mathcal{M}^\lambda$ is self-adjoint and has the same essential
spectrum and domain as $\mathcal{A}^\lambda$. In particular its spectrum inside $(-\infty, 1)$ is dis-
crete. Furthermore, the associated form $m^\lambda$ has the same domain as $a^\lambda$, which
is independent of $\lambda$.  

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Using the triangle inequality we have
\[ \| \| u \|_\sigma \|_\lambda \leq \| u \|_\lambda \leq C \| u \|_\sigma \]
which proves the uniform equivalence of norms.

**Proof.** Self-adjointness follows from [4 V-Theorem 4.3], due to $A^\lambda$ being self-adjoint for $\lambda \in [0, 1]$, the symmetry assumption (iii) on $K^\lambda$, and the compactness condition (iv). The essential spectrum result follows from Weyl’s theorem as $K^\lambda = K^\lambda P$ is relatively compact with respect to $A^\lambda$ (for any $\lambda$) because $P$ is.

The equality $D(m^\lambda) = D(a^\lambda)$ holds since $K^\lambda$ is bounded for each $\lambda$. The fact that the domains are independent of $\lambda$ is assumed above in (i).

Some clarification is required regarding the domains of the operators $M^\lambda$ and $A^\lambda$ and the associated forms. [Lemma 3] proves that $D(m^\lambda) = D(a^\lambda)$ as a set, and it is easily seen that the associated norms $\| \|_\lambda$ and $\| \|_m$ are equivalent. However, we do not know whether this equivalence is uniform in $\lambda$. Let us show that it is:

**Lemma 5.** The graph norms of $a^\lambda$ and $m^\sigma$ are equivalent uniformly in $\lambda$ and $\sigma$. That is, there is a constant $C \geq 1$ such that for any $\lambda, \sigma \in [0, 1]$ and any $u \in D(a^\lambda)$ we have the norm equivalence:

\[ \frac{1}{C} \| u \|_{\sigma} \leq \| u \|_{\lambda} \leq C \| u \|_{\sigma} \]

where $\sigma, t$ can each be either of $m, n$.

**Proof.** The case where $\sigma, t$ are both $a$ follows from [4 VII-§4.2]. We can extend to when one or both of $\sigma, t$ is $m$ by noting that for $\lambda \in [0, 1]$ and $u \in D(a^\lambda)$ we have

\[ \| u \|_{m^\lambda}^2 = \| u \|^2 + m^\lambda[u] \leq a^\lambda[u] + \left( 1 + \| K^\lambda \|_{B(\mathcal{H})} \right) \| u \|^2 \]

\[ \leq \left( 1 + \sup_{\lambda \in [0, 1]} \| K^\lambda \|_{B(\mathcal{H})} \right) \| u \|_{a^\lambda}^2 \]

where the supremum is finite by the uniform boundedness principle. In an identical fashion, by reversing the roles of $a$ and $m$,

\[ \| u \|_{a^\lambda}^2 \leq \left( 1 + \sup_{\lambda \in [0, 1]} \| K^\lambda \|_{B(\mathcal{H})} \right) \| u \|_{m^\lambda}^2 , \]

which proves the uniform equivalence of norms.

Because of this equivalence we will drop the $\lambda$ and write $D(a)$ for the domain, and for its norm we define $\| \|_a = \| \|_{a^\lambda}$. Next, we turn our attention to the map $\lambda \mapsto M^\lambda$. Intuitively, one would expect $M^\lambda$ to have continuity properties similar to those of $K^\lambda$ and therefore be merely continuous in the strong resolvent sense. In fact, due to the relative compactness assumption on $P$ we have more:

**Proposition 6.** The family $\{M^\lambda\}_{\lambda \in [0, 1]}$ is norm resolvent continuous.

**Proof.** Let $[0, 1] \ni \lambda_n \to \lambda$ as $n \to \infty$. It is sufficient to prove

\[ \| (M^\lambda + i)^{-1} - (M^\lambda + i)^{-1} \|_{B(\mathcal{H})} \to 0 \text{ as } n \to \infty. \]

Using the triangle inequality we have

\[ \| (M^\lambda + i)^{-1} - (M^\lambda + i)^{-1} \|_{B(\mathcal{H})} \leq \| (M^\lambda + i)^{-1} - (A^\lambda + i)^{-1} \|_{B(\mathcal{H})} \]

\[ + \| (A^\lambda + i)^{-1} - (M^\lambda + i)^{-1} \|_{B(\mathcal{H})}. \]
By observing that \( \{A^\sigma + K^\lambda\}_{\sigma \in D_0} \) is also a holomorphic family we deduce that the second term tends to zero as \( n \to \infty \). For the first term we follow the method used to deduce the second Neumann series (see [4 II-(1.13)])

\[
(A^{\lambda_n} + K^{\lambda_n} + i)^{-1} = (A^{\lambda_n} + K^{\lambda_n} + i)^{-1}(1 + (K^{\lambda_n} - K^{\lambda_n})(A^{\lambda_n} + K^{\lambda_n} + i)^{-1})^{-1}
\]  

(2.6)

which is valid whenever \( \|(K^{\lambda_n} - K^{\lambda_n})(A^{\lambda_n} + K^{\lambda_n} + i)^{-1}\|_{\mathfrak{B}(\mathfrak{H})} < 1 \). By the norm resolvent continuity of operator inversion and again using the norm resolvent continuity of the family \( \{A^\sigma + K^\lambda\}_{\sigma \in [0,1]} \), it is sufficient to show that

\[
\|(K^{\lambda_n} - K^{\lambda_n})(A^{\lambda_n} + K^{\lambda_n} + i)^{-1}\|_{\mathfrak{B}(\mathfrak{H})} \to 0 \text{ as } n \to \infty.
\]  

(2.7)

We observe that \( A^{\lambda_n} + K^{\lambda_n} \) is self-adjoint with the same domain as \( A^{\lambda} \) by [4 V, Theorem 4.3], so \( \mathcal{P} \) is also relatively compact with respect to \( A^{\lambda} + K^{\lambda} \). Hence

\[
(K^{\lambda_n} - K^{\lambda_n})(A^{\lambda_n} + K^{\lambda_n} + i)^{-1} = (K^{\lambda_n} - K^{\lambda_n})\mathcal{P}(A^{\lambda_n} + K^{\lambda_n} + i)^{-1}.
\]  

(2.8)

This is a composition of a strongly convergent sequence of operators and the compact operator \( \mathcal{P}(A^{\lambda_n} + K^{\lambda_n} + i)^{-1} \). The compactness converts the strong convergence to norm convergence and proves (2.7).

**Remark 7.** The operator \( \mathcal{P} \) was key to the proof: it is required to obtain convergence \( v_n \to v \) rather than simply \( K^{\lambda_n}v_n \to v \). In general, it is not true that if \( T_n \xrightarrow{s.r.} T_\infty \) and if the closure of the set \( \{T_nu\}_{n \in \mathbb{N}} \) for any \( u \in \mathfrak{H} \) is compact then \( T_n \xrightarrow{s.r.} T_\infty \). Consider for example \( T_nu = \langle e_n, u \rangle e_1 \) where \( \{e_n\}_{n \in \mathbb{N}} \) is an orthonormal basis of \( \mathfrak{H} \). Then \( T_n \xrightarrow{s.r.} 0 = T_\infty \) and the above set is compact. However, consider for instance the sequence \( \|T_ne_n\| = 1 \). Since \( \{e_n\}_{n \in \mathbb{N}} \) form an orthonormal basis, this implies that norm convergence does not hold.

### 3 Constructing approximations

We first treat approximations of operators with discrete spectra, which are naturally defined via a sequence of increasing projection operators. For brevity, we call these approximations \( n \)-approximations (“\( n \)” refers to the dimension of the projection). Our strategy when treating operators with a continuous spectrum is to first “perturb” them by adding a family of unbounded operators (think of adding an unbounded potential to a Laplacian) depending upon a small parameter \( \varepsilon \). For each \( \varepsilon > 0 \) these perturbations are assumed to eliminate any continuous spectrum, so that then we may apply an \( n \)-approximation. We therefore call these \( (\varepsilon, n) \)-approximations. We start with a ‘standard’ result for which we could not find a good reference:

**Lemma 8.** Let \( \mathfrak{H} \) be a Hilbert space and let \( T_n \xrightarrow{s.r.} T \) as \( n \to \infty \) with \( T_n, T \) self-adjoint operators on \( \mathfrak{H} \). Let \( K_n \xrightarrow{s.r.} K \) as \( n \to \infty \) with \( K_n, K \) bounded symmetric operators on \( \mathfrak{H} \). Then \( T_n + K_n \) and \( T + K \) are self-adjoint and \( T_n + K_n \xrightarrow{s.r.} T + K \).

**Proof.** The selfadjointness follows from the Kato-Rellich theorem. For the convergence it is sufficient to prove that \( (T_n + K_n + \alpha i)^{-1} \xrightarrow{s.r.} (T + K + \alpha i)^{-1} \) for some real \( \alpha \neq 0 \). As the \( K_n \) are strongly convergent, by the uniform boundedness
principle they are uniformly bounded in operator norm by some \( M \geq \|K\|_{\mathcal{B}(H)} \).

Letting \( \alpha = 2M \), and using the second Neumann series,

\[
(T_n + K_n + \alpha i)^{-1} = (T_n + \alpha i)^{-1} (1 + K_n (T_n + \alpha i)^{-1})^{-1}
\]

\[
= (T_n + \alpha i)^{-1} \sum_{k=0}^{\infty} (-1)^k (K_n (T_n + \alpha i)^{-1})^k
\]  

(3.1)

is convergent uniformly in \( n \) as \( \|K_n (T_n + \alpha i)^{-1}\|_{\mathcal{B}(H)} \leq \frac{M}{\alpha} = \frac{1}{2} < 1 \). As \( n \to \infty \) each term of the series converges strongly to the corresponding term of the series for \( (T + K + \alpha i)^{-1} \) and as the series converges uniformly in \( n \) we may may swap the order of summation and taking strong limits. \( \square \)

3.1 Operators with discrete spectra

In this paragraph we assume that \( A^\lambda \) has discrete spectrum and compact resolvent for some \( \lambda \) (and, in fact, for all \( \lambda \), as \( A^\lambda \) is a holomorphic family of type (B)). We exploit a property of self-adjoint holomorphic families [4, VII Theorem 3.9 and VII Remark 4.22]: the eigenvalues \( \{\mu^k_\lambda\}_{k \in \mathbb{N}} \) and associated normalised eigenfunctions \( \{e^\lambda_k\}_{k \in \mathbb{N}} \) of \( A^\lambda \) are holomorphic functions of \( \lambda \in [0, 1] \).

An immediate consequence is that the unitary operator defined by

\[
U^\lambda_\sigma : \mathcal{D} \to \mathcal{D}
\]

\[
e^\lambda_k \mapsto e^\lambda_k
\] for any \( k \in \mathbb{N} \) (3.2)

is jointly holomorphic in \( \lambda, \sigma \in [0, 1] \). We now define the \( n \)-truncation operator by

\[
G^\lambda_n : \mathcal{D} \to \mathcal{D}
\]

\[
e^\lambda_k \mapsto \begin{cases} e^\lambda_k & \text{if } k \leq n, \\ 0 & \text{if } k > n. \end{cases}
\] (3.3)

Since the eigenfunctions form a complete orthonormal set we have the convergence \( G^\lambda_n \rightarrow 1 \) as \( n \rightarrow \infty \) for fixed \( \lambda \). Additionally by expressing \( G^\lambda_n = U^\lambda_{\sigma} G^\sigma_n U^\lambda_{\sigma} \) for some fixed \( \sigma \in [0, 1] \) we see that \( G^\lambda_n \) is jointly strongly continuous in \( n \) and \( \lambda \).

We now define the finite-dimensional approximations of \( A^\lambda \) and \( M^\lambda \) by

\[
A^\lambda_n = G^\lambda_n A^\lambda G^\lambda_n \quad \text{and} \quad M^\lambda_n = G^\lambda_n M^\lambda G^\lambda_n,
\] (3.4)

respectively. It is too much to hope for convergence \( M^\lambda_n \overset{n.r.}{\longrightarrow} M^\lambda \) as \( n \rightarrow \infty \), but we can hope for \( M^\lambda_n \overset{\sigma.r.}{\longrightarrow} M^\lambda \). Indeed:

Lemma 9. The family \( \{M^\lambda_n\}_{\lambda \in [0, 1], n \in \mathbb{N}} \) is continuous in the strong resolvent sense, where we use the convention that \( M^\lambda_\infty := M^\lambda \).

Proof. We need to show that if \( \lambda_k \rightarrow \sigma \in [0, 1] \) as \( k \rightarrow \infty \), then

1. if \( m_k \rightarrow m \in \mathbb{N} \), then \( M^\lambda_{m_k} \overset{\sigma.r.}{\longrightarrow} M^\lambda_m \), and

---

1We remind the reader that the definition of a holomorphic family of type (B) is provided in subsection 1.2.
that for any fixed $\lambda$ is compact we deduce that the resolvent of $A$ is compact. Since $A$ is sectorial and that its domain $D$ is the sum of two holomorphic functions of $\lambda$ we see that since $(A + i)^{-1}$ is uniformly bounded the second part converges strongly to zero by the convergence $\mathcal{G}_n^\lambda \overset{\text{s.r.}}{\to} \mathcal{K}$. The latter is true as it is the composition of strong convergences. For the former it is sufficient to show that $(A + i)^{-1} \overset{\text{s.r.}}{\to} (A^\sigma + i)^{-1}$ as $k \to \infty$. Splitting this term as

$$(A_n + i)^{-1} = \mathcal{G}_n^\lambda (A + i)^{-1} + (1 - \mathcal{G}_n^\lambda)(A + i)^{-1},$$

we see that since $(A_n + i)^{-1}$ is uniformly bounded the second part converges strongly to zero by the convergence $\mathcal{G}_n^\lambda \overset{\text{s.r.}}{\to} 1$. For the first part, since $\mathcal{G}_n^\lambda$ is a spectral projection associated with $A_n^\lambda$ we have

$$\mathcal{G}_n^\lambda (A + i)^{-1} = \mathcal{G}_n^\lambda (A + i)^{-1} \mathcal{G}_n^\lambda$$

which converges to $(A^\sigma + i)^{-1}$ by the composition of strong convergences.

\section{Operators with continuous spectra}

We are now ready to turn to the general case of families $\{A^\lambda\}_{\lambda \in [0,1]}$ that may have continuous spectra. Such operators require $(\varepsilon,n)$-approximations. The $\varepsilon$-approximations $A^\lambda_{\varepsilon,n}$ of $A^\lambda$ were defined in (13) and the corresponding approximations $M^\lambda_{\varepsilon,n}$ were defined in (15).

\begin{lemma}
1. For any $\varepsilon > 0$, $\{A^\lambda_{\varepsilon,n}\}_{\lambda \in D_0}$ is a holomorphic family of type (B) with compact resolvent.

2. For any $\lambda \in [0,1], \varepsilon \geq 0$, $A^\lambda_{\varepsilon,n}$ is self-adjoint and we have $A^\lambda_{\varepsilon,n} \geq A^\lambda \geq 1 + \alpha$.
\end{lemma}

\textbf{Proof.} The second claim is obvious since $W^\lambda \geq 0$. For the first we must show $A^\lambda_{\varepsilon,n}$ is sectorial and that its domain $D(a^\lambda_{\varepsilon,n})$ is independent of $\lambda$ and dense in $\mathcal{F}$, and that for any fixed $u \in D(a^\lambda_{\varepsilon,n})$ the function $a^\lambda_{\varepsilon,n}[u]$ is holomorphic in $\lambda \in D_0$. For any $\lambda \in D_0$, $a^\lambda_{\varepsilon,n}$ is the sum of sectorial forms $a^\lambda$ and $\varepsilon w^\lambda$ so by [4] VI.1.6, Theorem 1.3 it is closed and sectorial with domain $D(a) \cap D(w^\lambda)$, which is independent of $\lambda$ since $A^\lambda, W^\lambda$ are holomorphic families of type (B). Furthermore, we assumed that $D(a) \cap D(w^\lambda)$ is dense in $\mathcal{F}$. For any fixed $u \in D(a^\lambda_{\varepsilon,n})$, $a^\lambda_{\varepsilon,n}[u] = a^\lambda[u] + \varepsilon w^\lambda[u]$ is the sum of two holomorphic functions of $\lambda \in D_0$, so $a^\lambda_{\varepsilon,n}[u]$ is also holomorphic in $D_0$. Finally by the assumption that the inclusion from $D(a^\lambda_{\varepsilon,n})$ into $\mathcal{F}$ is compact we deduce that the resolvent of $A^\lambda_{\varepsilon,n}$ is compact.

For each $\varepsilon > 0$ the operator $A^\lambda_{\varepsilon,n}$ has a discrete spectrum, and therefore the $n$-approximations of $A^\lambda_{\varepsilon,n}$ and $M^\lambda_{\varepsilon,n}$ may be defined analogously to (13) via the projection operators

$$\mathcal{G}^\lambda_{\varepsilon,n} : \mathcal{F} \to \mathcal{F}$$

where $\{e^\lambda_{\varepsilon,k}\}_{k \in \mathbb{N}}$ are normalised eigenfunctions of $A^\lambda_{\varepsilon,n}$ as

$$A^\lambda_{\varepsilon,n} = \mathcal{G}^\lambda_{\varepsilon,n} A^\lambda_{\varepsilon,n} \mathcal{G}^\lambda_{\varepsilon,n} \text{ and } M^\lambda_{\varepsilon,n} = \mathcal{G}^\lambda_{\varepsilon,n} M^\lambda_{\varepsilon,n} \mathcal{G}^\lambda_{\varepsilon,n}.$$
We know by \textbf{Lemma 9} that the family \( \{A_{\epsilon n}^\lambda\}_{\lambda \in [0,1], n \in \mathbb{N}} \) is continuous in the strong resolvent sense. In addition, we have:

\textbf{Lemma 11.} The family \( \{A_{\epsilon}^\lambda\}_{\lambda \in [0,1], \epsilon \in (0,\infty)} \) is continuous in the strong resolvent sense.

\textbf{Proof.} Let \( \epsilon_n \to \epsilon \in [0,\infty) \), \( \lambda_n \to \lambda \in [0,1] \) as \( n \to \infty \). We must show that \( A_{\epsilon_n}^{\lambda_n} \xrightarrow{s.r.} A_{\epsilon}^{\lambda} \). The case \( \epsilon > 0 \) is straightforward so we assume that \( \epsilon = 0 \). Without loss of generality we may assume that \( \epsilon_n \downarrow 0 \) and \( \epsilon_n \neq 0 \) for all \( n \). Fix \( f \in \mathcal{D}_1 \), we split

\[
\left\| (A_{\epsilon_n}^\lambda + 1)^{-1} f - (A^\lambda + 1)^{-1} f \right\| \leq \left\| (A_{\epsilon_n}^\lambda + 1)^{-1} f - (A^\lambda + 1)^{-1} f \right\| + \left\| (A^\lambda + 1)^{-1} f - (A^\lambda + 1)^{-1} f \right\|
\]

(3.9)

The second term converges to zero by the holomorphicity of the family \( A^\lambda \). To show the convergence of the first term we will prove that

\[
\left\| (A_{\epsilon_n}^\lambda + 1)^{-1} f - (A^\epsilon + 1)^{-1} f \right\| \to 0
\]

(3.10)

as \( n \to \infty \), uniformly in \( \sigma \in [0,1] \), which is clearly sufficient. The corresponding forms \( a_{\epsilon_n}^\sigma \) are all symmetric and have common domain \( \mathcal{D}(a) \cap \mathcal{D}(w) \subset \mathcal{D}(a) \) and are decreasing in the sense that

\[
a_{\epsilon_n}^\sigma [u] - a_{\epsilon}^\sigma [u] = \epsilon_n w^\sigma [u] \geq 0 \quad \text{for all } u \in \mathcal{D}(a_{\epsilon_n}^\lambda)
\]

(3.11)

Moreover this clearly tends to zero as \( n \to \infty \), and by the holomorphicity of \( w^\lambda \) which implies the relative boundedness of \( w^\sigma \) to \( w^{1/2} \) (say) the convergence is uniform in \( \sigma \in [0,1] \). Moreover, we have proved that for each \( u \in \mathcal{D}(a) \cap \mathcal{D}(w) \) which is a form core for \( a^\sigma \) we have \( a_{\epsilon_n}^\sigma [u] \to a^\sigma [u] \) uniformly in \( \sigma \in [0,1] \). Therefore by the convergence criterion \textbf{[4], VIII Theorem 3.6} on the forms \( a_{\epsilon_n}^{\sigma} \), \( A_{\epsilon_n}^{\sigma} \xrightarrow{\text{a.s.}} A^\sigma \) as \( n \to \infty \), and the convergence is uniform in \( \sigma \in [0,1] \) as all our estimates were and the proof in \textbf{[4]} gives explicit convergence rates. \( \Box \)

\textbf{Corollary 12.} The family \( \{M_{\epsilon}^\lambda\}_{\lambda \in [0,1], \epsilon \in (0,\infty)} \) is continuous in the strong resolvent sense.

\textbf{Proof.} This follows from the stability of strong resolvent continuity with respect to bounded strongly continuous perturbations. \( \Box \)

4 Proof of Theorem 2

4.1 Compactness results

Let \( M^\lambda \) be a family of operators as defined \textbf{(3.3)} and let \( M_{\epsilon}^\lambda \) and \( M_{\epsilon,n}^\lambda \) be the corresponding \( \epsilon \) and \( (\epsilon,n) \)-approximations as defined in \textbf{section 3}. We now show that these approximations are well-behaved, in the following sense:

\textbf{Proposition 13.} Define the set \( \Delta = \mathcal{D}(M_{\epsilon}^\lambda) \times (-\infty,1] \times [0,1] \). Fix \( \epsilon^* > 0 \). Then the set of eigenfunctions

\[
\mathcal{A} = \{(u, \sigma, \lambda, \epsilon) \in \Delta \times [0,\epsilon^*] : \|u\| = 1, M_{\epsilon}^\lambda u = \sigma u\}
\]

(4.1)
is compact. In addition, for any fixed $\varepsilon > 0$ the set of approximated eigenfunctions

$$\mathfrak{A}_\varepsilon = \{(u, \sigma, \lambda, n) \in \Delta \times \mathbb{N} : \|u\| = 1, u = G_{\varepsilon, n}^\lambda u, M_{\varepsilon, n}^\lambda u = \sigma u\} \quad (4.2)$$

is compact.

We will first prove a slightly more general result:

**Lemma 14.** Fix $\varepsilon^* > 0$ and define the set

$$\mathfrak{A}' = \{(u, \sigma, \lambda, \varepsilon, n) \in \Delta \times [0, \varepsilon^*] \times \mathbb{N} : \|u\| = 1, u = G_{\varepsilon^*, n}^\lambda u, M_{\varepsilon^*, n}^\lambda u = \sigma u\}. \quad (4.3)$$

Let $\{(u_k, \sigma_k, \lambda_k, \varepsilon_k, n_k)\}_{k=1}^\infty$ be a sequence in $\mathfrak{A}'$ with $\lambda_k \to \lambda, \sigma_k \to \sigma, \varepsilon_k \to \varepsilon, n_k \to n$ as $k \to \infty$. Then the sequence has a convergent subsequence if $G_{\varepsilon_k, n_k}^\lambda$ has a strong limit as $k \to \infty$.

**Remark 15.** Let $n_k \to \infty$ and $\varepsilon_k \to 0$ both as $k \to \infty$. If we knew that $G_{\varepsilon_k, n_k}^\lambda \to 1$ as $k \to \infty$, then we could in fact relate $n$ to $\varepsilon$ and eliminate one of them.

**Proof.** Each $u_k$ solves the equation

$$G_{\varepsilon_k, n_k}^\lambda A_{\varepsilon_k}^\lambda G_{\varepsilon_k, n_k}^\lambda u_k - \sigma_k u_k + G_{\varepsilon_k, n_k}^\lambda K_{\varepsilon_k, n_k}^\lambda u_k = 0. \quad (4.4)$$

The requirement that $u_k = G_{\varepsilon_k, n_k}^\lambda u_k$ and the fact that $G_{\varepsilon_k, n_k}^\lambda$ commutes with $A_{\varepsilon_k}^\lambda$ means that this is equivalent to

$$A_{\varepsilon_k}^\lambda u_k - \sigma_k u_k + G_{\varepsilon_k, n_k}^\lambda K_{\varepsilon_k, n_k}^\lambda u_k = 0. \quad (4.5)$$

Taking the inner product with $u_k$ we estimate,

$$a^0[u_k] \leq Ca^\lambda_k [u_k] \leq Ca_{\varepsilon_k}^\lambda [u_k] \leq C\sigma_k \|u_k\|^2 + C \sup_{\lambda \in [0, 1]} \|K_{\varepsilon_k, n_k}^\lambda\| \|u_k\|^2 \leq C' \quad (4.6)$$

where $C$ is independent of $k$ comes from the relative form boundedness of the holomorphic family $\{A_{\varepsilon_k}^\lambda\}_{\varepsilon_k \in D_\varepsilon}$ and the supremum is finite by the uniform boundedness principle as $\{K_{\varepsilon_k, n_k}^\lambda\}_{\varepsilon_k \in [0, 1]}$ is strongly continuous. Hence by the relative form compactness of $P$ to $a^0$ we may pass to a subsequence (though we retain the subscript $k$) for which

$$P u_k \to v \in \mathcal{F}. \quad (4.7)$$

Then by rewriting $G_{\varepsilon_k, n_k}^\lambda = K_{\varepsilon_k, n_k} P$ for all $\lambda \in [0, 1]$ we have

$$u_k = -(A_{\varepsilon_k}^\lambda - \sigma_k)^{-1} G_{\varepsilon_k, n_k}^\lambda K_{\varepsilon_k, n_k}^\lambda P u_k \quad (4.8)$$

where the resolvent exists by the assumption that $A_{\varepsilon_k}^\lambda \geq 1 + \alpha$ for all $\lambda \in [0, 1]$. Under the assumption that $G_{\varepsilon_k, n_k}^\lambda$ converges strongly to some bounded operator $\mathcal{G}$ as $k \to \infty$ we then have

$$u_k \to -(A^\lambda - \sigma)^{-1} \mathcal{G} K^\lambda v \quad (4.9)$$

so that $u_k$ is a convergent subsequence. \qed
Lemma 16. The spectrum of the operator $\mathcal{M}_\varepsilon^\lambda$ is bounded below uniformly in $\lambda \in [0, 1]$ and $\varepsilon \in [0, \infty)$.

Proof. It suffices to bound the numerical range. Let $u \in \mathcal{D}(\mathcal{M}_\varepsilon^\lambda)$ with $\|u\| = 1$ then

$$m_\varepsilon^\lambda[u] = a_\varepsilon^\lambda[u] + \langle \mathcal{K}_\varepsilon u, u \rangle \geq a_\varepsilon^\lambda[u] - \sup_{\lambda \in [0, 1]} \|\mathcal{K}_\lambda^\varepsilon\|_{\mathfrak{H}(\lambda)} \geq 1 + \alpha - \sup_{\lambda \in [0, 1]} \|\mathcal{K}_\lambda^\varepsilon\|_{\mathfrak{H}(\lambda)}$$

(4.10)

where the supremum is finite by the uniform boundedness principle. \hfill $\square$

Now we are ready to prove Proposition 13.

Proof of Proposition 13. We first note that we can interpret $\mathfrak{A}$ and $\mathfrak{A}_c$ as subsets of $\mathfrak{A}'$ by

$$\mathfrak{A} \ni (u, \sigma, \lambda, \varepsilon) \mapsto (u, \sigma, \lambda, \varepsilon, \infty) \in \mathfrak{A}'$$

$$\mathfrak{A}_c \ni (u, \sigma, \lambda, n) \mapsto (u, \sigma, \lambda, \varepsilon, n) \in \mathfrak{A}'$$

(4.11)

Let $\{ (u_k, \sigma_k, \lambda_k, \varepsilon_k, n_k) \}_{k=1}^\infty$ be a sequence in $\mathfrak{A}'$. By Lemma 16 the $\sigma_k$ are relatively compact and similarly $\lambda_k \in [0, 1]$, $n_k \in \mathbb{N}$ and $\varepsilon_k \in [0, 1]$ are relatively compact and we may pass to a subsequence (maintaining the index $k$) for which $\lambda_k \to \lambda$, $\sigma_k \to \sigma$, $\varepsilon_k \to \varepsilon$, $n_k \to n$ as $k \to \infty$. Hence Lemma 14 is applicable, and to show a convergent subsequence we must show that $\mathcal{G}_\varepsilon^{\lambda_k}$ has a strong limit as $k \to \infty$. On the one hand if the original sequence was inside $\mathfrak{A}_c \subset \mathfrak{A}'$ then we have $n_k = \infty$ for all $k$. Hence $\mathcal{G}_\varepsilon^{\lambda_k} = 1$ by definition. On the other hand if the original sequence was inside $\mathfrak{A}_c \subset \mathfrak{A}'$ then $\varepsilon_k = \varepsilon > 0$ for all $k$, so that as remarked before $\mathcal{G}_\varepsilon^{\lambda_k}$ is jointly strongly continuous in $\lambda, n$ so that $\mathcal{G}_\varepsilon^{\lambda_k} \to \mathcal{G}_\varepsilon^{\lambda}$ as $k \to \infty$. \hfill $\square$

4.2 Convergence of spectra

We can now use the above compactness results together with the continuity results to prove Theorem 2.

Proof of Theorem 2. We will prove that each of $\Sigma$ and $\Sigma_\varepsilon$ are both upper semi-continuous and lower semi-continuous. The lower semi-continuity of spectra under strong resolvent convergence of self-adjoint operators is standard (e.g. [4] VIII.1.2,Theorem 1.14.). As we have $\mathcal{M}_\varepsilon^\lambda$ continuous in the strong resolvent sense (Lemma 11) we have $\Sigma$ lower semi-continuous. For $\Sigma_\varepsilon$ we must be slightly more careful due to the spurious eigenvalue of $\mathcal{M}_\varepsilon^\lambda$ at $0$ for $n \neq \infty$ (see Remark 3 for further discussion of this eigenvalue, as well as the definition of $\mathcal{M}_\varepsilon^\lambda$ which shall appear below). We instead consider the operator $\tilde{\mathcal{M}}_\varepsilon^\lambda := \mathcal{M}_\varepsilon^\lambda + M(1 - \mathcal{G}_\varepsilon^{-\lambda}) : \mathfrak{H} \to \mathfrak{H}$ where $M > 1$ is some number (note that $\tilde{\mathcal{M}}_\varepsilon^{\lambda, \infty} = \mathcal{M}_\varepsilon^{\lambda, \infty}$). This moves the spurious eigenvalue to $M \not\in (-\infty, 1]$. By Lemma 9 the family $\{ \mathcal{M}_\varepsilon^{\lambda, n} \}_{\lambda \in [0, 1], n \in \mathbb{N}}$ is continuous in the strong resolvent sense, and using the stability of strong resolvent convergence with respect to strongly continuous bounded perturbations $\{ \tilde{\mathcal{M}}_\varepsilon^{\lambda, n} \}_{\lambda \in [0, 1], n \in \mathbb{N}}$ is also continuous in the strong resolvent sense. Moreover, the spectra of $\mathcal{M}_\varepsilon^{\lambda, n}$, $\tilde{\mathcal{M}}_\varepsilon^{\lambda, n}$ and $\tilde{\mathcal{M}}_\varepsilon^{\lambda, n}$ agree in $(-\infty, 1]$ as $M > 1$, which establishes the lower semi-continuity of $\Sigma_\varepsilon$.\hfill $\square$
For the upper semi-continuity we shall use the compactness result [Proposition 13]. As the proof for $\Sigma$ is slightly simpler than that for $\Sigma_\varepsilon$ and otherwise the same we shall leave it to the reader. Let $\lambda_k \to \lambda \in [0,1]$, $n_k \to n$ and $\sigma_k \to \sigma \in (-\infty,1)$ as $k \to \infty$ with $\sigma_k$ an eigenvalue of $\tilde{M}_{\lambda_k}^{\varepsilon,n_k}$. Then it is sufficient to prove that $\sigma$ is an eigenvalue of $\tilde{M}_{\lambda}^{\varepsilon,n}$. Let $u_k$ be the normalised eigenfunctions. Then $\{(u_k,\sigma_k,n_k,\lambda_k)\}_{k=1}^{\infty} \subset \mathfrak{A}_\varepsilon$ is a compact set. Hence we may pass to a subsequence (still indexed with $k$) for which $u_k \to u$.

\[ \tilde{M}_{\lambda_k}^{\varepsilon,n_k} - \sigma_k \xrightarrow{\sigma} \tilde{M}_{\lambda}^{\varepsilon,n} - \sigma \quad \text{as} \quad k \to \infty \]

we see that $u$ is an eigenfunction associated with the eigenvalue $\sigma$. Indeed, if we have some self-adjoint operators $T_k \xrightarrow{\sigma} T$ and elements $z_k \to z$ with $T_k z_k = 0$ then

\[
(T_k + i)z_k = iz_k \quad \iff \quad z_k = i(T_k + i)^{-1}z_k \\
\downarrow \quad \text{as} \quad k \to \infty \quad \text{by} \quad T_k \xrightarrow{\sigma} T \\
z = i(T + i)^{-1}z \quad \iff \quad Tz = 0.
\] (4.12)

\[ \square \]

5 Operators which are not positive: proof of Theorem 2

We define the $\varepsilon$-approximations of $A_{\lambda}^{\pm}$ as before in terms of a pair of holomorphic families $W_{\lambda}^{\pm}$ with the same assumptions. The eigenprojections of $A_{\lambda}^{\pm}$ are then denoted by $G_{\lambda,\varepsilon,n}$ and we define

\[
G_{\lambda,\varepsilon,n} = \begin{bmatrix}
G_{\lambda,\varepsilon,n}^{\pm} & 0 \\
0 & G_{\lambda,\varepsilon,n}^{ \mp}
\end{bmatrix}
\] (5.1)

and

\[
A_{\lambda,\varepsilon,n} = G_{\lambda,\varepsilon,n}^{\pm} A_{\lambda}^{\pm} G_{\lambda,\varepsilon,n}^{\mp} \\
M_{\lambda,\varepsilon,n} = G_{\lambda,\varepsilon,n}^{\pm} M_{\lambda}^{\pm} G_{\lambda,\varepsilon,n}^{\mp}.
\] (5.2)

All the preceding proofs of continuity can be adapted to this case. Indeed, Proposition 6 holds without modification, while Lemma 9 and Lemma 11 can be extended by using the identity

\[
\begin{bmatrix} T_+ & 0 \\ 0 & T_-
\end{bmatrix}^1 \quad \text{and} \quad (T_+ + i)^{-1} = \begin{bmatrix}
(T_+ + i)^{-1} & 0 \\
0 & (T_- + i)^{-1}
\end{bmatrix}
\] (5.3)

and the stability of norm (resp. strong) continuity to symmetric bounded norm (resp. strongly) continuous perturbations. The compactness and spectral continuity results need more modification. Recall that the discrete region of the spectrum is the gap $(-\alpha - 1, 1 + \alpha)$ rather than the half-line $(-\infty, 1 + \alpha)$. We restate this below:
Proposition 17. Fix \( \varepsilon^* > 0 \) and let \( \Delta = \mathfrak{D}(\mathcal{M}_\varepsilon^\lambda) \times [-1, 1] \times [0, 1] \). Then the set of eigenfunctions

\[
\mathfrak{A} = \{(u, \sigma, \lambda, \varepsilon) \in \Delta \times [0, \varepsilon^*] : \|u\| = 1, \mathcal{M}_\varepsilon^\lambda u = \sigma u\}
\]  

is compact. Let \( \varepsilon > 0 \) be fixed then set of approximated eigenfunctions

\[
\mathfrak{A}_\varepsilon = \{(u, \sigma, \lambda, n) \in \Delta \times N : \|u\| = 1, u = G_{\varepsilon, n}^\lambda u, \mathcal{M}_\varepsilon^\lambda u = \sigma u\}
\]  

is compact.

Sketch of proof. To prove the compactness results we use a version of Lemma 14 with \( \sigma \in (-\infty, 1] \) replaced with \( \sigma \in [-1, 1] \) in the definition of \( \mathfrak{A} \). Once we have this the proof is identical to Proposition 13. In the proof of Lemma 14 we need only change (4.6) to the two estimates

\[
a_k^0[u_k^\pm] \leq C a_k^0[u_k^\pm] \leq \sup_{\lambda \in [0, 1]} \|K_{\varepsilon, \lambda}^\pm\| u_k^\pm \leq C'
\]  

obtained by taking the inner product of (4.5) with \( u_k^\pm \) where \( u_k = (u_k^+, u_k^-) \in \tilde{\mathfrak{M}}_+ \times \tilde{\mathfrak{M}}_- \), from which the relative compactness of \( \mathcal{P} u_k \) follows, and lastly note that \( A_{\varepsilon, \lambda} \geq 1 + \alpha \) implies that the resolvent \( (A_{\varepsilon, \lambda}^\pm - \sigma_k)\) exists in (4.8).

With this result the continuity of spectra carries over identically as before, except for the different region of the spectrum considered (previously \((-\infty, 1)\), now \((-1, 1))\). This proves Theorem 2.

6 An application: plasma instabilities

The problem that we have in mind, which we only sketch here briefly and treat separately and at length in the separate paper [2], is that of linear instability of the relativistic Vlasov-Maxwell system. Letting \( f = f(t, x, v) \) be a function measuring the density of electrons that at time \( t \geq 0 \) are located at the point \( x \in \mathbb{R}^d \) and have momentum \( v \in \mathbb{R}^d \) and velocity \( \hat{v} = v/\sqrt{1 + |v|^2} \), the Vlasov equation

\[
\frac{\partial f}{\partial t} + \hat{v} \cdot \nabla_x f + F \cdot \nabla_v f = 0
\]

is a transport equation describing the evolution of the electrons. Here we have taken the mass of the electrons and the speed of light to be 1 for simplicity. The forcing term \( F \) captures the physics of the problem, and in this case it is the Lorentz force

\[
F = -E - \hat{v} \times B
\]

where \( E = E(t, x) \) and \( B = B(t, x) \) are the electric and magnetic fields, respectively. They satisfy Maxwell’s equations

\[
\nabla \cdot E = \rho, \quad \nabla \cdot B = 0, \quad \nabla \times E = \frac{\partial B}{\partial t}, \quad \nabla \times B = j + \frac{\partial E}{\partial t}
\]

where \( \rho = \rho(t, x) = -\int f \, dv \) is the charge density and \( j = j(t, x) = -\int \hat{v} f \, dv \) is the current density (minus signs are due to the electrons being negatively charged). Linearising (6.1) we obtain

\[
\frac{\partial f}{\partial t} + \hat{v} \cdot \nabla_x f + F^0 \cdot \nabla_v f = -F \cdot \nabla_v f^0,
\]

where...
where \( f^0 \) and \( F^0 \) are the equilibrium density and force field, respectively, and \( f \) and \( F \) are their first order perturbations. Substituting into (6.4) the ansatz that all time-dependent quantities behave like \( e^{\lambda t} \) with \( \lambda > 0 \), we get

\[
\lambda f + \hat{v} \cdot \nabla_x f + F^0 \cdot \nabla_v f = -F \cdot \nabla_v f^0.
\] (6.5)

An inversion of this equation leaves us with an expression for \( f \) that depends upon \( F \), and hence upon \( \rho \) and \( j \), and upon \( \lambda \) as a parameter. The final step is to plug the expression for \( f \) into Maxwell’s equations and verify that they hold for some \( \lambda > 0 \). The resulting system is precisely of the form (1.1), and the techniques presented here are necessary to track the eigenvalues of this system, and show that for some \( \lambda > 0 \) there is a non-trivial kernel.

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