Optimizing shortcut Brownian heat engine

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(Dated: April 19, 2022)

Shortcut to isothermality provides a powerful method to realize quasistatic thermodynamic processes with finite-time manipulation. We implement the shortcut strategy to design and optimize Brownian heat engines, and formulate a rigorous geometric description of the energetics with the thermodynamic length. Along a given closed path in the control parameter space, the optimal control protocol is to manipulate the control parameters with a proper constant velocity to maximize the output power. We also propose the principle to find out the closed path with large maximum power. Our results generalize the previous optimization in the highly underdamped and the overdamped regimes to the general-damped situation, and are applicable for arbitrary finite-time cycles.

Introduction. In the past few decades, the flourishing stochastic thermodynamics has brought great interest in studying the nonequilibrium thermodynamics of small systems featured with fluctuations [1–6]. The microscopic heat engine has been experimentally invented with a single trapped ion or a Brownian particle as the working substance [7, 8]. The optimal precise control of small systems is crucial to designing microscopic machines with high accuracy and low irreversibility of the manipulation. Various methods have been proposed to optimize the control protocol of heat-engine cycles, for example, the optimal control theory to find the optimal configuration of ideal-gas and two-level heat engines [9–11], and the thermodynamic geometry to optimize the control of slow isothermal processes [12–19]. For the Brownian heat engine, the optimal control of the cycle is known in the highly underdamped [20, 21] and the overdamped regimes [22, 23].

Recently, the shortcut to isothermality has been proposed for the Brownian motion model [24, 25]. Such a strategy is feasible to speed up heat-engine cycles [26–28] and the control of biophysical processes [29, 30], and has also been realized in experiments [31–33]. By implementing an auxiliary Hamiltonian, the system is steered to evolve along with the instantaneous equilibrium state of the origin Hamiltonian with finite-time manipulation. In Ref. [34], the shortcut strategy is generalized to include the temperature as a time-dependent control parameter. In Ref. [35], a geometric description of the thermodynamic cost is formulated for the shortcut to isothermality. The irreversible work \( W_{\text{irr}} \) of implementing the auxiliary Hamiltonian is rigorously bounded by the thermodynamic length \( \mathcal{L} \) as

\[
W_{\text{irr}} \geq \mathcal{L}^2 / \tau,
\]

where the equality can be reached by the optimal shortcut control.

In this Letter, we analyze the thermodynamic cost of shortcut processes with time-dependent temperature and work parameter, and give a geometric description of the energetics based on the thermodynamic length. Previous studies of thermodynamic length associated with the corresponding optimization are usually limited to slow-driving regime [12–19, 36–42]. We further optimize the output power of the shortcut Brownian heat engine with a general-damped Brownian particle as the working substance. The maximum power and the efficiency at the maximum power are determined by two geometric quantities of the closed path of the cycle, the area and the thermodynamic length. Therefore, improving the maximum power of the cycle is converted to finding out the closed path with a large ratio between the area and the thermodynamic length. All of our results are beyond the slow-driving regime, and are valid for arbitrary finite-time cycles. We also obtain the maximum power with the given efficiency, namely, the trade-off relation between power and efficiency, for the shortcut Brownian heat engine.

Setup. We consider a heat engine with a single Brownian particle as the working substance. The evolution of the probability distribution \( \rho = \rho(x,p,t) \) for the Brownian particle is governed by the complete Fokker-Planck equation (the Kramer equation) [43]

\[
\frac{\partial \rho}{\partial t} = \mathcal{L}[\rho] + \mathcal{D}[\rho],
\]

where \( \mathcal{L}[\rho] = -\partial_x (\rho \partial_x H) + \partial_p (\rho \partial_p H) \) and \( \mathcal{D}[\rho] = \kappa m \partial_p (\rho \partial_p H + \partial_p \rho / \beta) \) reflect the deterministic and the dissipative evolution with the total Hamiltonian \( H \), the mass \( m \) of the particle, the friction coefficient \( \kappa \), and the inverse temperature \( \beta \) of the environment.

In the shortcut regime, an auxiliary Hamiltonian \( H_a(x,p,t) = \lambda h_{\lambda}(x,p,\lambda) + \beta h_{\beta}(x,p,\lambda,\beta) \) [25, 34] is added to steer the system evolving along with the instantaneous equilibrium states \( \rho = \rho_{\text{eq}} := \exp\{\beta F(\lambda,\beta) - H_{a}(x, p, \lambda)\} \) of the origin Hamiltonian \( H_a(x,p,\lambda) = p^2 / (2m) + U_o(x, \lambda) \), where \( U_o(x, \lambda) \) is the potential with the work parameter \( \lambda \), and the free energy \( F(\lambda,\beta) \) is determined by \( \beta F(\lambda,\beta) = -\ln\{\int \exp[-\beta H_o(x,p,\lambda)] dp dx\} \). The total Hamiltonian is \( H = H_o + H_a \). To construct a heat-engine cycle, the work parameter \( \lambda \) and the inverse temperature \( \beta \) are treated as time-dependent control parameters.
The average input power and the average heat flux are defined as \( \dot{W} := \langle \partial_t H \rangle \) and \( \dot{Q} := \int H \partial_t \rho_{\text{eq}} \, dx \, dp \) [1], where the average is over the instantaneous equilibrium state \( \langle \cdot \rangle := \int \cdot \rho_{\text{eq}} \, dx \, dp \) in the shortcut regime. We divide them into the quasistatic and the irreversible parts

\[
\dot{W} = \dot{W}_o + \dot{W}_{\text{irr}}, \quad (3)
\]
\[
\dot{Q} = \dot{Q}_o + \dot{Q}_{\text{irr}}, \quad (4)
\]
according to the origin Hamiltonian \( H_o \) and the auxiliary Hamiltonian \( H_a \). The quasistatic and the irreversible input powers are \( \dot{W}_o = \dot{\lambda} \langle \partial_t H_o \rangle \) and \( \dot{W}_{\text{irr}} = \langle \partial_t H_a \rangle \). The quasistatic and the irreversible heat fluxes are \( \dot{Q}_o = \int H_o \partial_t \rho_{\text{eq}} \, dx \, dp \) and \( \dot{Q}_{\text{irr}} = \int H_a \partial_t \rho_{\text{eq}} \, dx \, dp \). The first law of thermodynamics is satisfied \( \partial_t \langle H_o \rangle = \dot{W}_o + \dot{Q}_o \) and \( \partial_t \langle H_a \rangle = \dot{W}_{\text{irr}} + \dot{Q}_{\text{irr}} \). For the isotothermal process (\( \beta(t) \equiv \beta \)), the quasistatic work is equal to the free energy change \( W_o = F(\lambda(t), \beta) - F(\lambda(0), \beta) \) [25]. When the inverse temperature \( \beta(t) \) also varies with time, the quasistatic work \( W_o \) relies on the path in the control parameter space, but is independent of the control protocol and the operation time [34]. The irreversible work is obtained as [44]

\[
\dot{Q}_{\text{irr}} = -\int_0^T \left( \dot{\lambda} \frac{\dot{\lambda}}{\dot{\beta}} \right) \mathbf{g}\left( \frac{\dot{\lambda}}{\dot{\beta}} \right) \, dt, \quad (5)
\]
and the irreversible work is

\[
W_{\text{irr}} = \langle H_a \rangle (T^+) - \langle H_a \rangle (T^-) - \dot{Q}_{\text{irr}}, \quad (6)
\]
where \( \langle H_a \rangle (T^+) - \langle H_a \rangle (T^-) = 0 \) since the auxiliary Hamiltonian \( H_a \) is switched off at the beginning and the end of a thermodynamic process, or changes cyclically in a heat-engine cycle. The metric \( \mathbf{g} \) is explicitly

\[
\mathbf{g} = \kappa m \left( \begin{array}{ccc}
\frac{\partial \lambda}{\partial \gamma} & \frac{\partial \lambda}{\partial \beta} & \frac{\partial \lambda}{\partial \gamma} \\
\frac{\partial \lambda}{\partial \gamma} & \frac{\partial \lambda}{\partial \beta} & \frac{\partial \lambda}{\partial \beta} \\
\frac{\partial \lambda}{\partial \gamma} & \frac{\partial \lambda}{\partial \beta} & \frac{\partial \lambda}{\partial \beta}
\end{array} \right), \quad (7)
\]
which is positive semi-definite on the manifold of the work parameter \( \lambda \) and the inverse temperature \( \beta \). Therefore, the irreversible work is always non-negative \( W_{\text{irr}} \geq 0 \). For a given path \( \lambda(s), \beta(s) \) in the control parameter space, the thermodynamic length is defined as

\[
\mathcal{L} := \int g_{\lambda \lambda} \lambda'(s)^2 + 2 g_{\lambda \beta} \lambda'(s) \beta'(s) + g_{\beta \beta} \beta'(s)^2 \, ds, \quad (8)
\]
and the lower bound of the irreversible work of the control on the path is given by the thermodynamic length \( \mathcal{L} \) according to Eq. (1). We emphasize the thermodynamic length defined in the shortcut regime is valid for arbitrary finite time, which leads to the rigorous \( 1/\tau \) scaling of the irreversible work with the assistance of the auxiliary Hamiltonian [35].

For a heat-engine cycle, the control parameters \( \lambda(t) \) and \( \beta(t) \) varies cyclically with time. The output work of a whole cycle is \( -W = Q_+ + Q_- \), where \( Q_+ > 0 \) (\( Q_- < 0 \)) is the heat absorbed from (released to) the environment. By optimizing the control protocol on the given path, the output work is bounded by \( -W \leq -W_o - \mathcal{L}^2/\tau \) with the operation time \( \tau \) of the cycle and the thermodynamic length \( \mathcal{L} \) of the closed path in the control parameter space. The quasistatic work should be negative \( W_o < 0 \) to ensure positive output work. By choosing the operation time \( \tau = \tau_{\text{max}} := 2\mathcal{L}^2/(-W_o) \), the output power \( P := -W/\tau \) reaches the maximum

\[
P_{\text{max}} = \frac{(-W_o)^2}{4\mathcal{L}^2}, \quad (9)
\]
The thermodynamic length \( \mathcal{L} \) guides the optimal control for the shortcut strategy on a given closed path to reach the maximum power \( P_{\text{max}} \). Further optimization of the output power of the finite-time heat-engine cycle is equivalent to finding out the closed path in the control parameter space with a large ratio \(-W_o/\mathcal{L}\).

The efficiency for the shortcut heat engine is defined as \( \eta := -W/Q_+ \). At the maximum power, the efficiency is expressed by the thermodynamic length as

\[
\eta_{\text{EMP}} = \frac{\eta_0}{2 - \eta_0 \mathcal{L}_+ / \mathcal{L}}, \quad (10)
\]
where the efficiency of the quasistatic cycle \( \eta_0 = -W_o/Q_{o,+} \) is upper bounded by the Carnot efficiency \( \eta_C = 1 - T_L/T_H \), and \( \mathcal{L}_+ \) is the thermodynamic length of the path where the heat is absorbed from the environment \( Q > 0 \). The efficiency at the maximum power \( \eta_{\text{EMP}} \) [Eq. (10)] is applicable for arbitrary heat-engine cycles in the shortcut regime, and is determined by the quasistatic efficiency \( \eta_0 \) and the thermodynamic lengths \( \mathcal{L}_+ \) and \( \mathcal{L} \). We also obtain the maximum power \( P_\eta \) with the given efficiency \( \eta \) (the trade-off relation between power and efficiency) as [44]

\[
P_\eta \left( P_{\text{max}} \right) = \frac{4(\eta_0 - \eta)(1 - \eta_0 \mathcal{L}_+/\mathcal{L})}{\eta_0^2(1 - \eta \mathcal{L}_+/\mathcal{L})^2}. \quad (11)
\]
Comparison of the above trade-off relation and the previous results [20, 21, 23, 45] is left in [44]. Equations (9)-(11) are the main result of this work.

Application to power-law potentials. We next realize the shortcut Brownian heat engine with the power-law potentials \( U_o(x, \lambda) = m \lambda^{n+1} x^{2n}/(2n) \). The auxiliary Hamiltonian \( H_a \) is constructed with the auxiliary functions [34]
\[ h_\lambda = \frac{f_n}{4\kappa\lambda m} (p - \kappa mx)^2 + \frac{f_n m}{4\kappa m} \lambda^n x^{2n}, \quad (12) \]
\[ h_\beta = \frac{np^2 + (p - \kappa mx)^2 + m^2 f_n \lambda^{n+1} x^{2n}}{4\beta\kappa mn}, \quad (13) \]

where \( f_n = (n + 1)/n \) is the effective degree of freedom [18].

To gain a geometric interpretation of the quasistatic work \( W_\alpha \), we represent the control parameters \( \lambda \) and \( \beta \) with new variables \( r \) and \( T \) as

\[ \lambda = \lambda_0 e^r, \quad \beta = 1/T, \quad (14) \]

where \( T \) is the temperature of the equilibrium state. Then the quasistatic output work \(-W_\alpha = -(f_n/2) \oint T \, dr\) of the cycle is proportional to the area \( \mathcal{A} = [\oint T \, dr] \) enclosed by the path in the \( T - r \) diagram. The thermodynamic length of the cycle is

\[
\mathcal{L} = \frac{f_n}{2} \oint \frac{(T - dT/dr)^2}{\kappa T} + \kappa_n e^{-f_n r} \frac{T - dT/dr}{\lambda_0^2 T^2 - 1/n (T - dT/dr)^2} dr, \tag{15}
\]

where \( \kappa_n = \kappa m (2n/m)^{1/n} \Gamma (3/(2n)) / \Gamma (1/(2n)) \), and \( \Gamma(\cdot) \) is the gamma function. For the heat-engine cycle on a given closed path, the maximum power Eq. (9) is explicitly

\[ P_{\max} = \frac{f_n^2 \mathcal{A}^2}{16 T_c^2}, \tag{16} \]

where \( \mathcal{A} \) and \( \mathcal{L} \) are the area and the thermodynamic length of the closed path. Each term in Eq. (16) has explicit geometric interpretation.

To form a closed heat-engine cycle, we need to choose two connecting paths between the two isothermal lines \( T = T_L \) and \( T_H \). We consider the harmonic potential \( n = 1 \) and set the bounds of the tunable temperatures to be \( T_L = 0.5 \) and \( T_H = 1 \) in the following numerical calculation. To design the heat-engine cycle with possibly large output power, we need to find the cycle with a large ratio \( \mathcal{A}/\mathcal{L} \). An efficient choice of the connecting paths is to use the geodesic curves according to the metric (15). The explicit expressions of the geodesic equations are left in [44]. Initiated from the points \((r_0, T_L)\) on the low-temperature isothermal line \( T = T_L \), the shortest geodesic curve to reach the high-temperature isothermal line \( T = T_H \) is solved by utilizing the shooting method [46] (green solid curves in Fig. 1). We pick two points \((r_i, T_L)\), \( i = 1, 2 \) with \( r_1 < r_2 \) on the low-temperature isothermal line \( T = T_L \), and choose the connecting paths as the shortest geodesic curves for the two points. Then, the area and the thermodynamic length of the cycle are functions of \( r_1 \) and \( r_2 \), and are represented as \( \mathcal{A}_\text{geo}(r_1, r_2) \) and \( \mathcal{L}_\text{geo}(r_1, r_2) \).

![Figure 1](image)

Figure 1. Cycle diagram of the Brownian heat engine with the harmonic potential \( n = 1 \). The blue and the red horizontal lines give the lower \( (T_L = 0.5) \) and the upper bounds \( (T_H = 1) \) of the tunable temperatures. We compare the shortest geodesic path (green solid curves) and the shortest exponential path (orange dashed curves) to connect one point \((r_0, T_L)\) in the low-temperature isothermal line to the high-temperature isothermal line \( T_H = 1 \). They almost coincide with each other. We also show the optimal connecting path \( \alpha = (n + 1) \) (black dashed curve) and \( \alpha = 1 \) (black dotted curve) in the overdamped and the highly underdamped regimes.

| \( \alpha \) | \( \mathcal{L}_\alpha(r_0) \) |
|---|---|
| 1 | \( \frac{\kappa_n^{1/2} \Gamma (1/2)}{(\lambda_0 e^{\alpha r_0})/n \Gamma (3/(2n)) / \Gamma (1/(2n))} \) |
| \( n + 1 \) | \( \sqrt{T_H - \sqrt{T_L}} \) |

Table I. Expressions of the thermodynamic length of the exponential paths \( \alpha = 1 \) and \( n + 1 \).

We can also choose the connecting paths as the exponential path \( T(r) = T_L \exp(\alpha(r - r_0)) \), \( r_0 \leq r \leq r'_0 := r_0 + \ln(T_H/T_L)/\alpha \). The thermodynamic length of the exponential path is written as \( \mathcal{L}_\exp(\alpha(r_0)) \), and is explicitly carried out for \( \alpha = 1 \) and \( n + 1 \) in Table I. For given \( r_0 \), we can also optimize \( \alpha \) to minimize the thermodynamic length \( \mathcal{L}_\alpha(r_0) \) of the exponential path, and the shortest exponential path (orange dotted curve in Fig. 1) almost coincides with the shortest geodesic path (but not exactly). We remark that the cycles constructed by the exponential paths contain various kinds of cycles in classical thermodynamics, for example, the Carnot cycle \((\alpha_{1,2} = 1)\) and the Stirling cycle \((\alpha_{1,2} = \infty)\). The quasistatic output work \(-W_\alpha = (f_n/2) \mathcal{A}_\exp(r_1, \alpha_1, r_2, \alpha_2)\), the quasistatic efficiency \( \eta_\alpha \), and the thermodynamic length \( \mathcal{L}_\exp(r_1, \alpha_1, r_2, \alpha_2) \) of these cycles are derived analytically in [44]. The maximum power \( P_{\max} \) is then obtained from Eq. (16). To determine the efficiency, we need to divide the whole cycle into the heat absorbed and the heat released paths to calculate \( \mathcal{L}_+ \). With the analytical expression of the quasistatic efficiency \( \eta_\alpha \), the efficiency at the maximum power \( \eta_{\text{EMP}} \) is then obtained from Eq. (10).

In Fig. 2, we present the cyclic control protocol of
Figure 2. Cyclic control protocol of the shortcut heat engine to achieve the maximum power. (a) The cycle diagram with the chosen values \( r_1 = -1 \) and \( r_2 = 0 \). The connecting paths (solid black curves) are solved as the shortest geodesic paths. The whole cycle is divided into four processes (I)–(IV). (b) The control parameters \( r(t) \) and \( T(t) \) as functions of operation time. (c) The implement of the auxiliary Hamiltonian represented by \( \lambda \) and \( \beta \). (d) The input power \( W \) and the heat flux \( Q \) in the cycle. The irreversible heat flux \( Q_{\text{irr}} \) is a constant (green dotted line) in the optimal control to achieve the maximum power.

the shortcut heat engine to achieve the maximum power. Here we choose \( r_1 = -1 \) and \( r_2 = 0 \), and solve the connecting paths as the shortest geodesic paths. The area and the thermodynamic length of the cycle are obtained as \( A_{\text{geo}}(-1, 0) = 0.546 \) and \( L_{\text{geo}}(-1, 0) = 3.59 \), and the operation time of the cycle is \( \tau_{\text{max}} = 47.2 \). The cycle contains four processes, (I) isothermal compression, (II) connecting compression, (III) isothermal expansion, and (IV) connecting expansion. The protocol of changing the control parameters \( r(t) \) and \( T(t) \) is shown in Fig. 2(b).

Figure 2(c) shows the implement of the auxiliary Hamiltonian reflected by \( \lambda \) and \( \beta \). Figure 2(d) plots the input power and the heat flux in a cycle. The heat is absorbed \( Q > 0 \) in processes (II) and (III), and released \( Q < 0 \) in processes (I) and (IV). The irreversible heat flux is a constant \( Q_{\text{irr}} = L^2/\tau_{\text{max}}^2 = 0.00578 \) during the whole cycle.

Figure 3 presents the results of the maximum power \( P_{\text{max}} \) and the efficiency at the maximum power \( \eta_{\text{EMP}} \) of different cycles. In Fig. 3(a), we compare the maximum power \( P_{\text{max}} \) of different heat-engine cycles with exponential paths. The indexes \( \alpha_{1,2} \) are chosen to minimize the thermodynamic length of the exponential paths (orange surface) or fixed as \( \alpha_{1,2} = 1, 2 \) and \( \infty \) (blue, green and red surfaces). Among these choices of the indexes \( \alpha_{1,2} \), the maximum power of the cycle with the shortest exponential paths is the largest, and also coincides with the results by choosing the connecting paths as the shortest geodesic paths (blue dots). Figure 3(b) shows the efficiency at the maximum power \( \eta_{\text{EMP}} \), where \( \alpha_{1,2} \) is chosen to minimize the thermodynamic length of the exponential paths (orange surface). In the highly underdamped regime, the efficiency at the maximum power returns to the Curzon-Ahlborn efficiency [47] \( \eta_{\text{CA}} = 1 - \sqrt{T_H/T_H} \) (green surface). In the overdamped regime, the efficiency at the maximum power agrees with the prediction \( \eta_{\text{EMP}}^{\text{over}} = \eta_0^{\text{over}}/(2 - \eta_0^{\text{over}}/2) \) (red surface) in Ref. [23] where the quasistatic efficiency \( \eta_0^{\text{over}} = \eta_C/\{1 + \eta_C/[f_0(r_2 - r_1)]\} \) includes the heat leakage induced by the kinetic energy.

Discussions. We define a dimensionless quantity

\[
\chi := \frac{\lambda}{\kappa} \left( \frac{T}{m} \right)^{1-1/n}.
\]

The highly underdamped and the overdamped regimes are reflected by \( \chi \gg 1 \) and \( \chi \ll 1 \). In the highly underdamped regime (\( \chi \gg 1 \)), the thermodynamic length [Eq. (15)] is simplified into \( L_{\text{under}} = (f_n/2) \sqrt{(T - dT/dr)^2/(\kappa T)} dr \). The irreversibl work and the thermodynamic length diminish on the exponential path \( T(r) = T_L \exp(r - r_0) \). The connecting processes on these paths, as the adiabatic processes, are free of the irreversible work and does not cost any time (compared to the isothermal processes) in the optimal control. The optimal cycle can be constructed with two isothermal processes connected with the adiabatic processes \( T(r) = T_L \exp(r - r_i), i = 1, 2 \) [20, 21]. The area of the closed path is

\[
A = (T_H - T_L)(r_2 - r_1).
\]

The thermodynamic length of the cycle is

\[
L_{\text{under}} = \frac{f_n}{2} \left( \frac{\sqrt{T_H} + \sqrt{T_L}}{\sqrt{\kappa}} \right)(r_2 - r_1).
\]
We then obtain the maximum power

$$P_{\text{max}} = \frac{\kappa}{4} (\sqrt{TH} - \sqrt{TL})^2. \quad (20)$$

Under the chosen parameters in Fig. 3, the value of the maximum power is $P_{\text{max}} = (3 - 2\sqrt{2})/8 = 0.0214$. The efficiency at the maximum power is the Curzon-Ahlborn efficiency $\eta_{\text{max}} = \eta_{\text{CA}}$. Both the maximum power $P_{\text{max}}$ and the efficiency at the maximum power $\eta_{\text{max}}$ are independent of the shape of the potential $n$, since the dynamics in the highly underdamped regime can be effectively described by similar forms of stochastic differential equation of energy [18, 21].

In the overdamped regime ($\chi \ll 1$), the thermodynamic length [Eq. (15)] is simplified into

$$\mathcal{L}_{\text{over}} = \frac{f_n}{2} \oint \frac{K_n e^{-f_n r}}{\lambda_n T^2} (T - \frac{1}{n+1} \frac{dT}{dr})^2 dr. \quad (21)$$

The connecting paths are set as the exponential path $T(r) = T_L \exp[(n+1)(r - r_0)]$, which correspond to the sudden quench processes with the unchanged position distribution, i.e., $\lambda_n^{n+1}/T = \text{const}$. The optimal cycle can be constructed with two isothermal processes connected with the sudden quench processes $T(r) = T_L \exp[(n+1)(r - r_i)]$, $i = 1, 2$ [23]. The area of the closed path is the same [Eq. (18)], but the thermodynamic length of the cycle is

$$\mathcal{L}_{\text{over}} = \frac{2K_n^{1/2}T_L^{1/(2n)}}{\lambda_n^{1/2}} \left(e^{-f_n r_1/2} - e^{-f_n r_2/2}\right). \quad (22)$$

The maximum power with given $r_1$ and $r_2$ is

$$P_{\text{max}} = \frac{f_n^2 \lambda_n^2 (T_H - T_L)^2 (r_2 - r_1)^2}{64K_n T_L(n - f_n r_1/2 - e^{-f_n r_2/2})^2}. \quad (23)$$

The efficiency at the maximum power is

$$\eta_{\text{EMP}} = \frac{\eta_C}{2 + \{4/[f_n(r_2 - r_1)] - 1\} \eta_C/2}. \quad (24)$$

Both the maximum power (23) and the efficiency at the maximum power (24) depends on the control parameter $r_{1,2}$ and the shape of the potential $n$ in the overdamped regime. These results agree with the results for the harmonic potential $n = 1$ obtained in Ref. [23].

**Conclusion.** We study the thermodynamic length for the finite-time thermodynamic processes boosted by the shortcut strategy. The thermodynamic length defined in the shortcut scheme gives a rigorous and reachable bound of the irreversible work. We further construct and optimize the shortcut Brownian heat engine, and obtain the maximum power, the efficiency at the maximum power, and the maximum power with the given efficiency for this heat engine. The shortcut strategy allows us to optimize the control protocol of the heat-engine cycle in the general-damped situation. The heat-engine cycles are not limited to the finite-time Carnot cycles, which are mostly considered in previous studies [20, 21, 36, 40, 47]. We can choose an arbitrary closed path in the control parameter space, and evaluate the performance of different kinds of finite-time heat-engine cycles, e.g., the Carnot, the Otto, the Stirling cycles, etc. At last, we illustrate that the efficiency at the maximum power approaches the Curzon-Ahlborn efficiency in the highly underdamped regime [21], but deviates from it in the overdamped regime [23].

This work is supported by the National Natural Science Foundation of China (NSFC) under Grants No. 11775001, No. 11534002, No. 11825501, and No. 12147157.

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Supplementary material: Optimizing shortcut Brownian heat engine

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I. WORK AND HEAT IN SHORTCUT THERMODYNAMIC PROCESSES

We rewrite the Kramer equation as

$$\frac{\partial \rho}{\partial t} = {\mathcal L}_o [\rho] + {\mathcal D}_o [\rho] + {\mathcal L}_a [\rho] + {\mathcal D}_a [\rho] ,$$  

where the operators \( {\mathcal L}_o \) and \( {\mathcal D}_o \) corresponding to the deterministic and the dissipative evolution of the origin Hamiltonian \( H_o(x, p, \lambda) = \frac{p^2}{2m} + U_o(x, \lambda) \) are

$$ {\mathcal L}_o [\rho] = - \frac{\partial}{\partial x} \left( \frac{\partial H_o}{\partial p} \rho \right) + \frac{\partial}{\partial p} \left( \frac{\partial H_o}{\partial x} \rho \right), $$

$$ {\mathcal D}_o [\rho] = \kappa m \frac{\partial}{\partial p} \left( \frac{\partial H_o}{\partial p} \rho + \frac{1}{\beta} \frac{\partial \rho}{\partial p} \right), $$

and the operators \( {\mathcal L}_a \) and \( {\mathcal D}_a \) corresponding to the deterministic and the dissipative evolution of the auxiliary Hamiltonian \( H_a \) are

$$ {\mathcal L}_a [\rho] = - \frac{\partial}{\partial x} \left( \frac{\partial H_a}{\partial p} \rho \right) + \frac{\partial}{\partial p} \left( \frac{\partial H_a}{\partial x} \rho \right), $$

$$ {\mathcal D}_a [\rho] = \kappa m \frac{\partial}{\partial p} \left( \frac{\partial H_a}{\partial p} \rho \right). $$

The instantaneous equilibrium state \( \rho_{\text{eq}} := \exp \{ \beta [F(\lambda, \beta) - H_o(x, p, \lambda)] \} \) nullifies the operators \( {\mathcal L}_o [\rho_{\text{eq}}] = 0 \) and \( {\mathcal D}_o [\rho_{\text{eq}}] = 0 \), and its time derivative is

$$ \frac{\partial \rho_{\text{eq}}}{\partial t} = \beta \left[ \frac{\dot{\beta}}{\beta} (F - H_o) + \frac{\dot{\beta}}{\beta} \frac{\partial F}{\partial \beta} + \lambda \left( \frac{\partial F}{\partial \lambda} - \frac{\partial U_o}{\partial \lambda} \right) \right] \rho_{\text{eq}}. $$

Equation (1) gives the principle to construct the auxiliary Hamiltonian [1]

$$ \frac{\partial \rho_{\text{eq}}}{\partial t} = {\mathcal L}_a [\rho_{\text{eq}}] + {\mathcal D}_a [\rho_{\text{eq}}], $$
which is explicitly
\[
\frac{\partial \rho_{\text{eq}}}{\partial t} = \frac{\partial H_a}{\partial x} \frac{\partial \rho_{\text{eq}}}{\partial x} - \frac{\partial H_a}{\partial p} \frac{\partial \rho_{\text{eq}}}{\partial p} + \kappa m \frac{\partial}{\partial p} \left( \frac{\partial H_a}{\partial p} \rho_{\text{eq}} \right).
\] (8)

As done in Ref. [2], both the inverse temperature \( \beta(t) \) and the work parameter \( \lambda(t) \) are treated as the time-dependent control parameters. By setting the auxiliary Hamiltonian as
\[
H_a = \dot{\lambda} h_\lambda(x, p, \lambda) + \dot{\beta} h_\beta(x, p, \lambda, \beta),
\] (9)

Eqs. (6) and (8) lead to the partial differential equations for the auxiliary functions \( h_\lambda(x, p, \lambda) \) and \( h_\beta(x, p, \lambda, \beta) \) as
\[
\frac{\kappa m \partial^2 h_\lambda}{\partial p^2} - \frac{\kappa p \partial h_\lambda}{\partial p} + \frac{\partial U_o}{\partial x} \frac{\partial h_\lambda}{\partial x} - \frac{p \partial h_\lambda}{m \partial x} = \frac{\partial F}{\partial \lambda} - \frac{\partial U_o}{\partial \lambda},
\] (10)
\[
\frac{\kappa m \partial^2 h_\beta}{\partial p^2} - \frac{\kappa p \partial h_\beta}{\partial p} + \frac{\partial U_o}{\partial x} \frac{\partial h_\beta}{\partial x} - \frac{p \partial h_\beta}{m \partial x} = \frac{1}{\beta} (F - H_o) + \frac{\partial F}{\partial \beta}.
\] (11)

The quasi-static work, the quasi-static heat, the irreversible work and the irreversible heat are
\[
W_o = \dot{\lambda} \int \frac{\partial U_o}{\partial \lambda} \rho_{\text{eq}} dx dp,
\] (12)
\[
Q_o = \int H_o \frac{\partial \rho_{\text{eq}}}{\partial t} dx dp,
\] (13)
\[
W_{\text{irr}} = \int \frac{\partial H_o}{\partial t} \rho_{\text{eq}} dx dp,
\] (14)
\[
Q_{\text{irr}} = \int H_o \frac{\partial \rho_{\text{eq}}}{\partial t} dx dp.
\] (15)

Plugging Eq. (8) into Eq. (15), we obtain the explicit expression of the irreversible heat
\[
\frac{d Q_{\text{irr}}}{dt} = \int \left\{ \left( \dot{\lambda} f + \dot{\beta} g \right) \dot{\lambda} \left[ \frac{\partial h_\lambda}{\partial x} \frac{\partial \rho_{\text{eq}}}{\partial x} - \frac{\partial h_\lambda}{\partial p} \frac{\partial \rho_{\text{eq}}}{\partial p} + \kappa m \frac{\partial}{\partial p} \left( \frac{\partial h_\lambda}{\partial p} \rho_{\text{eq}} \right) \right] \right. \\
\left. + \left( \dot{\lambda} f + \dot{\beta} g \right) \dot{\beta} \left[ \frac{\partial h_\beta}{\partial x} \frac{\partial \rho_{\text{eq}}}{\partial x} - \frac{\partial h_\beta}{\partial p} \frac{\partial \rho_{\text{eq}}}{\partial p} + \kappa m \frac{\partial}{\partial p} \left( \frac{\partial h_\beta}{\partial p} \rho_{\text{eq}} \right) \right] \right\} dx dp.
\] (16)

One can verify that only the last term in each line is nonzero in the integral of \( x \) and \( p \), and the result is
\[
\frac{d Q_{\text{irr}}}{dt} = -\kappa m \left( \dot{\lambda} \dot{\beta} \right) \left( \begin{array}{c}
\frac{\partial h_\lambda}{\partial p} \\
\frac{\partial h_\beta}{\partial p}
\end{array} \right) \left( \begin{array}{c}
\frac{\partial h_\lambda}{\partial x} \\
\frac{\partial h_\beta}{\partial x}
\end{array} \right) \left( \begin{array}{c}
\lambda \\
\beta
\end{array} \right).
\] (17)

Then the irreversible work is
\[
\frac{d W_{\text{irr}}}{dt} = \frac{d \langle H_o \rangle}{dt} + \kappa m \left( \dot{\lambda} \dot{\beta} \right) \left( \begin{array}{c}
\frac{\partial h_\lambda}{\partial p} \\
\frac{\partial h_\beta}{\partial p}
\end{array} \right) \left( \begin{array}{c}
\frac{\partial h_\lambda}{\partial x} \\
\frac{\partial h_\beta}{\partial x}
\end{array} \right) \left( \begin{array}{c}
\lambda \\
\beta
\end{array} \right).
\] (18)

Notice that the total differential term \( d \langle H_o \rangle / dt \) does not contribute to the irreversible work since the auxiliary Hamiltonian is switched off at the beginning and the end of the process, or changes cyclically in a heat-engine cycle.
Explicit results for the power-law potentials

For the power-law potential $U_0(x, \lambda) = m\lambda^{n+1}x^{2n}/2n$, the partition function is

$$Z_\lambda(\beta) := \int e^{-\beta H_0} dx dp = 2^{\frac{1}{n} + \frac{1}{2}} \sqrt{\frac{m\pi}{\beta}} \frac{1}{\lambda^{\frac{n+1}{2n}} \Gamma\left(1 + \frac{1}{2n}\right)} \left(\frac{\beta m}{n}\right)^{-\frac{1}{2n}}. \quad (19)$$

The average values associated with the momentum and position are

$$\langle p^2 \rangle = \frac{m}{\beta}, \quad (20)$$
$$\langle x^2 \rangle = \frac{\Gamma\left(\frac{3}{2n}\right)}{\Gamma\left(\frac{1}{2n}\right)} \frac{1}{\lambda} \left(\frac{2n}{\beta \lambda m}\right)^{1/n}, \quad (21)$$
$$\langle x^{2n} \rangle = \frac{\lambda^{-n-1}}{\beta m}. \quad (22)$$

The auxiliary functions $h_\lambda(x, p, \lambda)$ and $h_\beta(x, p, \lambda, \beta)$ are given in the main text, and the average values of them are

$$\langle h_\lambda \rangle = \frac{(n + 1)^2}{4\kappa n^2 \lambda \beta} + \frac{n + 1}{4\lambda^2 \kappa \lambda m} \frac{1}{\frac{1}{2n}} \frac{\Gamma\left(\frac{3}{2n}\right)}{\Gamma\left(\frac{1}{2n}\right)} \left(\frac{2n}{\beta \lambda m}\right)^{1/n}, \quad (23)$$
$$\langle h_\beta \rangle = \frac{(n + 1)^2}{4n^2} \frac{1}{\lambda} \frac{1}{\beta} \frac{1}{\kappa} \frac{\lambda m}{4n \lambda \beta} \frac{1}{\frac{1}{2n}} \frac{\Gamma\left(\frac{3}{2n}\right)}{\Gamma\left(\frac{1}{2n}\right)} \left(\frac{2n}{\beta \lambda m}\right)^{1/n} \frac{\kappa m}{4n \lambda \beta}. \quad (24)$$

The average values of the origin Hamiltonian and the auxiliary Hamiltonian are

$$\langle H_0 \rangle = \frac{n + 1}{2n \beta}, \quad (25)$$
$$\langle H_a \rangle = \left(\frac{n + 1}{4\kappa n^2 \beta}\left(\frac{\lambda}{\lambda} + \frac{\beta}{\beta}\right) + \frac{\kappa m}{4n \lambda \beta} \left(\frac{2n}{\beta \lambda m}\right)^{1/n} \frac{1}{\frac{1}{2n}} \frac{\Gamma\left(\frac{3}{2n}\right)}{\Gamma\left(\frac{1}{2n}\right)} \left(\frac{n + 1}{\lambda} \frac{\lambda + \beta}{\beta}\right) \right). \quad (26)$$

The explicit results of the quasistatic work, the quasistatic heat, the irreversible work and the irreversible heat are

$$\frac{dW_o}{dt} = f_n \frac{\lambda}{2\beta \lambda}, \quad (27)$$
$$\frac{dQ_o}{dt} = -f_n \frac{\lambda}{2\beta} \left(\frac{\lambda}{\lambda} + \frac{\beta}{\beta}\right), \quad (28)$$
$$\frac{dW_{irr}}{dt} = \frac{d\langle H_a \rangle}{dt} + f_n^2 \frac{1}{4 \kappa \beta} \left(\frac{\lambda}{\lambda} + \frac{\beta}{\beta}\right)^2 + \frac{K_n}{\lambda \beta} \left(\frac{\lambda}{\lambda} + \frac{\beta}{(n + 1)\beta}\right)^2, \quad (29)$$
$$\frac{dQ_{irr}}{dt} = -f_n^2 \frac{1}{4 \kappa \beta} \left(\frac{\lambda}{\lambda} + \frac{\beta}{\beta}\right)^2 + \frac{K_n}{\lambda \beta} \left(\frac{\lambda}{\lambda} + \frac{\beta}{(n + 1)\beta}\right)^2, \quad (30)$$

where $f_n = (n + 1)/n$ is the effective degrees of freedom, and

$$K_n = \kappa m \left(\frac{2n}{m}\right)^{\frac{1}{n}} \frac{\Gamma\left(\frac{3}{2n}\right)}{\Gamma\left(\frac{1}{2n}\right)}. \quad (31)$$

For a heat-engine cycle, the irreversible work of a cycle is
\[ W_{irr} = \frac{f_n^2}{4} \int_0^r \left[ \frac{1}{\kappa \beta} \left( \frac{\dot{\lambda}}{\lambda} + \frac{\dot{\beta}}{\beta} \right)^2 + \frac{K_n}{\lambda \beta} \left( \frac{\dot{\lambda}}{\lambda} + \frac{\dot{\beta}}{(n+1)\beta} \right)^2 \right] dt, \]  

(32)

and the thermodynamic length is

\[ \mathcal{L} = \frac{f_n}{2} \int \left\{ \frac{1}{\kappa \beta} \left[ \frac{\lambda'(s)}{\lambda} + \frac{\beta'(s)}{\beta} \right]^2 + \frac{K_n}{\lambda \beta} \left[ \frac{\lambda'(s)}{\lambda} + \frac{\beta'(s)}{(n+1)\beta} \right]^2 \right\} ds, \]  

(33)

where \( \lambda(s) \) and \( \beta(s) \) give the path of the cycle in the control parameter space, and the primes denote their derivatives in respect to arc-length parameter \( s \). By representing the control parameters \( \lambda(t) = \lambda_0 e^r \), \( \beta = 1/T \) with the new parameters \( r \) and \( T \), we obtain the expression (15) in the main text. The elements in the metric are

\[
g_{rr} = \frac{f_n^2}{4} \left( \frac{T}{\kappa} + T^\frac{1}{n} \frac{K_n e^{-f_n r}}{\lambda_0^\frac{1}{n}} \right),
\]

(34)

\[
g_{rT} = -\frac{f_n^2}{4T^2} \left[ \frac{T}{\kappa} + T^\frac{1}{n} \frac{K_n e^{-f_n r}}{(n+1)\lambda_0^\frac{1}{n}} \right],
\]

(35)

\[
g_{TT} = \frac{f_n^2}{4T^2} \left[ \frac{T}{\kappa} + T^\frac{1}{n} \frac{K_n e^{-f_n r}}{(n+1)^2\lambda_0^\frac{1}{n}} \right].
\]

(36)

The Christoffel symbol is determined by the metric as

\[
\Gamma^i_{kl} = \frac{1}{2} g^{in} \left( \frac{\partial g_{nk}}{\partial x^l} + \frac{\partial g_{nl}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^n} \right),
\]

(37)

and the geodesic equation is

\[
\ddot{x}^i + \Gamma^i_{kl} \dot{x}^k \dot{x}^l = 0.
\]

(38)

Here the coordinates \( \dot{x}^i = r \) and \( T \). The geodesic equation is explicitly obtained as

\[
\ddot{r} = \frac{(n+1)^2}{2n^2 \kappa K_n} \left( \frac{\lambda_0 e^r}{T} \right) \left( T - \dot{T} \right) \dot{T} \left( T - \dot{T} \right)^2 + \frac{(n+1)(n+2)}{2n^2} \dot{r}^2 - \frac{(n+2)}{n^2} \frac{\dot{T}^2}{T} - \frac{(n-2)}{2n^2} \ddot{T}^2,
\]

(39)

\[
\ddot{T} = \frac{(n+1)^2}{2n^2 \kappa K_n} \left( \frac{\lambda_0 e^r}{T} \right) \left( T - \dot{T} \right) \dot{T} \left( T - \dot{T} \right)^2 + \frac{(n+1)^2}{n^2} \dot{r}^2 - \frac{2(n+1)}{n^2} \frac{\dot{T}^2}{T} + \frac{(n^2 + 2)}{2n^2} \ddot{T}^2.
\]

(40)

It is difficult to solve the boundary-value problem of the above nonlinear ordinary differential equations, but we can convert it into an initial-value problem. Such a method is known as the shooting method [3]. By solving the geodesic equation with different directions of the initial velocity, we find the shortest geodesic path connecting the point \((r_0, T_L)\) with the high-temperature isothermal line \(T = T_H\).

II. OUTPUT WORK, EFFICIENCY AND THERMODYNAMIC LENGTH FOR CYCLES WITH EXPONENTIAL CONNECTING PATHS

We consider the heat-engine cycle is constructed by two exponential paths \( T(r) = T_L \exp[\alpha_1(r - r_1)], \quad r_1 \leq r \leq r'_1 := r_1 + \ln(T_H/T_L)/\alpha_1 \) (process IV) and \( T(r) = T_L \exp[\alpha_2(r - r_2)], \quad r_2 \leq r \leq r'_2 := r_2 + \ln(T_H/T_L)/\alpha_2 \) (process II) and the isothermal lines \( T = T_L, \quad r_1 \leq r \leq r_2 \) (process I) and \( T = T_H, \quad r'_1 \leq r \leq r'_2 \) (process III). We can
calculate the analytical results of the work and heat. The quasistatic work is proportional to the area of the closed path \( W_o = -(f_n/2)A_{exp}(r_1, \alpha_1, r_2, \alpha_2) < 0 \), and the area is

\[
A_{exp}(r_1, \alpha_1, r_2, \alpha_2) = (r_2 - r_1) (T_H - T_L) + \left( \frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right) [T_H - T_L - T_H \ln (T_H/T_L)].
\]  \hspace{1cm} (41)

The quasistatic heat can be only absorbed in the processes II, III and IV, and the result is

\[
Q_{o,+} = \frac{f_n}{2} \left\{ (r_2 - r_1) T_H - \left( \frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right) T_H \ln \left( \frac{T_H}{T_L} \right) + \max[\frac{\alpha_2 - 1}{\alpha_2} (T_H - T_L), 0] + \max[\frac{-\alpha_1 - 1}{\alpha_1} (T_H - T_L), 0] \right\}.
\]  \hspace{1cm} (42)

The quasistatic efficiency \( \eta_o = W_o/Q_{o,+} \) is explicitly

\[
\eta_o = \frac{(r_2 - r_1) (T_H - T_L) + \left( \frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right) [T_H - T_L - T_H \ln (T_H/T_L)]}{(r_2 - r_1) (T_H) - \left( \frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right) T_H \ln (T_H/T_L) + \max[\frac{\alpha_2 - 1}{\alpha_2} (T_H - T_L), 0] + \max[\frac{-\alpha_1 - 1}{\alpha_1} (T_H - T_L), 0]}.
\]  \hspace{1cm} (43)

For the Carnot cycle \( \alpha_1 = \alpha_2 = 1 \), the quasistatic efficiency is verified to be the Carnot efficiency \( \eta_o = \eta_C := 1-T_L/T_H \).

We next calculate the thermodynamic length. The thermodynamic lengths of the isothermal processes are

\[
\mathcal{L}_1 = \frac{f_n}{2} \int_{r_1}^{r_2} \left[ \frac{T_L}{K} + \frac{K_n T_L^\frac{1}{n}}{\lambda_0^n} e^{-f_n r} dr \right] \hspace{1cm} (44)
\]

\[
= \frac{r_1}{K} \left[ \sinh^{-1} \left( \frac{\lambda_0^n e^{f_n r_1}}{\kappa K_n T_L^\frac{1}{n}} \right) - \sqrt{1 + \frac{\kappa K_n T_L^\frac{1}{n}}{\lambda_0^n} e^{f_n r_1}} \right] \hspace{1cm} (45)
\]

and

\[
\mathcal{L}_{III} = \frac{f_n}{2} \int_{r_1}^{r_2} \left[ \frac{T_H}{K} + \frac{K_n T_H^\frac{1}{n}}{\lambda_0^n} e^{-f_n r} dr \right] \hspace{1cm} (46)
\]

\[
= \frac{r_1}{K} \left[ \sinh^{-1} \left( \frac{\lambda_0^n e^{f_n r_1}}{\kappa K_n T_H^\frac{1}{n}} \right) - \sqrt{1 + \frac{\kappa K_n T_H^\frac{1}{n}}{\lambda_0^n} e^{f_n r_1}} \right]. \hspace{1cm} (47)
\]

The thermodynamic length of process II is

\[
\mathcal{L}_{II} = \frac{f_n}{2} \int_{r_2}^{r_1} \left[ \frac{1}{(1 - \alpha_2)^2} \frac{T_L e^{-\alpha_2 r_2} e^{\alpha_2 r}}{(1 - \alpha_2^2)^2} + \left( 1 - \frac{\alpha_2}{n + 1} \right)^2 \frac{K_n T_L^\frac{1}{n} e^{-\frac{\alpha_2}{n+2} r_2} e^{\left( \frac{\alpha_2}{n} - f_n \right) r}}{\lambda_0^n} \right] dr \hspace{1cm} (48)
\]

\[
= f_n \left[ \left( \frac{1}{\alpha_2} - \frac{1}{n + 1} \right)^2 \frac{K_n T_L^\frac{1}{n} e^{-\frac{\alpha_2}{n+2} r_2} e^{\left( \frac{\alpha_2}{n} - f_n \right) r}}{\lambda_0^n} \sqrt{1 + \Theta_{II} \left[ f_n + (\frac{1}{n} - \frac{1}{2}) \alpha_2 \right] r} + \frac{1}{2} \Gamma \left( \frac{\alpha_2 - f_n}{2 \left( f_n + (1 - \frac{1}{n}) \alpha_2 \right)} \right) \right] \left[ e^{\left( \frac{\alpha_2}{n} - f_n \right) r} \right]_{r=r_2}, \hspace{1cm} (49)
\]

where the constant is
\[ \Theta_{II} = \frac{(1 - \alpha_2)^2}{(1 - \alpha_2/\pi^2)^2} \frac{\lambda_0^n T_L^{1-\frac{\pi}{2}}}{\kappa K_n} e^{-\alpha_2(1-\frac{1}{\pi})r_2}, \]  

and the regularized hyper-geometric function is

\[ \tilde{2F}_1(a; b; c; z) = \frac{2F_1(a, b; c; z)}{\Gamma(c)} = \frac{1}{\Gamma(c)} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k! (c)_k} z^k, \]

with the Pochhammer symbol \((a)_k = \Gamma(a + k)/\Gamma(a)\). The thermodynamic length of process IV is

\[
L_{IV} = \frac{f_n}{2} \int_{r_1}^{r_1'} \sqrt{\frac{(1 - \alpha_1)^2 T_L e^{-\alpha_1 r_1}}{\kappa} e^{\alpha_1 r} + (1 - \frac{\alpha_1}{n+1})^2 K_n T_L^2 e^{-\frac{\alpha_1}{n+1} r_1} e^{(\frac{\alpha_1}{n+1} - f_n) r} dr } \]

\[
= f_n \left[ \sqrt{\frac{1}{\alpha_1} - \frac{1}{n+1}} \frac{K_n T_L^2 e^{-\frac{\alpha_1}{n+1} r_1} e^{\frac{\alpha_1}{n+1} - f_n} r}{\lambda_0^n} \sqrt{1 + \Theta_{IV}(f_n + (1-\frac{1}{n}) \alpha_1) r} \right] \left[ 1 + \frac{1}{2} \frac{\alpha_1}{n+1} - f_n \right] \left( \frac{f_n + (1-\frac{1}{n}) \alpha_1}{2} \right) \left( \frac{2 - \frac{1}{n} \alpha_1 + f_n}{2 (f_n + (1-\frac{1}{n}) \alpha_1)} \right) \right] \bigg|_{r=r_1}^{r_1'},
\]

where the constant is

\[ \Theta_{IV} = \frac{(1 - \alpha_1)^2}{(1 - \alpha_2/\pi^2)^2} \frac{\lambda_0^n T_L^{1-\frac{\pi}{2}}}{\kappa K_n} e^{-\alpha_1(1-\frac{1}{\pi})r_1}. \]

The thermodynamic length of the whole cycle is

\[ L_{\exp}(r_1, \alpha_1, r_2, \alpha_2) = L_I + L_{II} + L_{III} + L_{IV}. \]

We divide the whole cycle into the heat absorbed and the heat released paths with the thermodynamic lengths \(L_+\) and \(L_-\), respectively. To determine \(L_+\), we need to find the path corresponding to \(Q > 0\). We assume that \(Q\) takes the same sign with \(Q_o\), and obtain the thermodynamic length

\[
L_+ = \begin{cases} 
L_{III} & \alpha_1 > 1, \alpha_2 < 1 \\
L_{III} + L_{II} & \alpha_1 > 1, \alpha_2 \geq 1 \\
L_{III} + L_{IV} & \alpha_1 \leq 1, \alpha_2 < 1 \\
L_{III} + L_{II} + L_{IV} & \alpha_1 \leq 1, \alpha_2 \geq 1 
\end{cases}
\]

The analytical results obtained in this section are used to plot Fig. 3 of the main text.

III. MAXIMUM POWER WITH GIVEN EFFICIENCY FOR THE SHORTCUT BROWNIAN HEAT ENGINE

The efficiency of the finite-time cycle is

\[ \eta = \frac{-W}{Q_+} = \frac{-W_o - L^2/\tau}{Q_{o,+} - L_+ L/\tau}. \]

The maximum power and the efficiency at the maximum power are then obtained with the above analytical results of \(A_{\exp}, \eta_o, L_{\exp}\) and \(L_+\).
From Eq. (57), we can also calculate the trade-off relation between the power and the efficiency. The efficiency is rewritten as

\[ \eta = \frac{-W_o - L^2/\tau}{-W_o/\eta_o - L/\tau}. \tag{58} \]

By introducing the function

\[ F = \frac{-W_o}{\tau} - \frac{L^2}{\tau^2} + \zeta \left( \eta - \frac{-W_o - L^2/\tau}{-W_o/\eta_o - L/\tau} \right), \tag{59} \]

we calculate the maximum power with given efficiency through

\[ \frac{\partial F}{\partial \tau} = 0, \quad \frac{\partial F}{\partial \zeta} = 0. \tag{60} \]

The solution is

\[ \tau = \eta_o \frac{(1 - \eta L/\tau)}{\eta_o - \eta} \frac{L^2}{-W_o}, \tag{61} \]

and the maximum power with given efficiency is

\[ \frac{P_{\eta}}{P_{\text{max}}} = \frac{4(\eta_o - \eta)(1 - \eta_o L/\tau)\eta}{\eta_o^2(1 - \eta_o L^2/\tau)^2}. \tag{62} \]

In the highly underdamped regime, the optimal cycle has been obtained in Refs. [4, 5] with \( \eta_o^{\text{under}} = \eta_C \) and \( L^{\text{under}} = \sqrt{T_H/(\sqrt{T_H + \sqrt{T_L})}} \). The maximum power with given efficiency is

\[ \frac{P_{\eta}^{\text{under}}}{P_{\text{max}}} = \frac{4\sqrt{T_L/T_H(\eta_C - \eta)}\eta}{(\eta_C - \eta\eta_{\text{CA}})^2}. \tag{63} \]

Notice that the maximum power with given efficiency in the shortcut regime is different from that of the Curzon-Ahlborn heat engine

\[ \frac{P_{\eta}^{\text{CA}}}{P_{\text{max}}} = \frac{\eta_C - \eta}{(1 - \eta)\eta_{\text{CA}}^2}, \tag{64} \]

but they are quite similar, as shown in Fig. 1. The trade-off relation (64) between the power and the efficiency was obtained in Ref. [6] by considering an endoreversible model with the Newton’s heat-transfer law.

In the overdamped limit, the optimal cycle has been obtained in Ref. [7] with

\[ \eta_o^{\text{over}} = \frac{\eta_C}{1 + \eta_C/[\int_{r_2-r_1}]}, \tag{65} \]

and \( L^{\text{over}} = L^{\text{over}} = 1/2 \). The maximum power with given efficiency is

\[ \frac{P_{\eta}}{P_{\text{max}}} = \frac{4(\eta_o^{\text{over}} - \eta)(1 - \eta_o^{\text{over}}/2)\eta}{(\eta_o^{\text{over}})^2(1 - \eta_o^{\text{over}}/2)^2}. \tag{66} \]

In Fig. 1, we show the maximum power with given efficiency in the highly underdamped and the overdamped regimes by setting \( \eta_C = 1/2 \) and \( \eta_o^{\text{over}} = 1/2 \). We also show the maximum power with given efficiency [Eq. (64)] for the Curzon-Ahlborn heat engine [5], which is quite similar to that of the shortcut heat engine in the highly underdamped regime.

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We set $\eta_C = 1/2$ with $\eta_{CA} = 1 - 1/\sqrt{2}$ for the highly underdamped regime and $\eta_{\text{over}} = 1/2$ for the overdamped regime.

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