DYNAMICS FOR A NON-AUTONOMOUS REACTION DIFFUSION MODEL WITH THE FRACTIONAL DIFFUSION

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Abstract. In this paper, we study the dynamics of a non-autonomous reaction diffusion model with the fractional diffusion on the whole space. We firstly prove the existence of a \((L^2, L^2)\) pullback \(D\)–attractor of this model. Then we show that the pullback \(D\)–attractor attract the \(D\) class (especially all \(L^2\)-bounded set) in \(L^{2+\delta}\)–norm for any \(\delta \in [0, \infty)\). Moreover, the solution of the model is shown to be continuous in \(H^s\) with respect to initial data under a slightly stronger condition on external forcing term. As an application, we prove that the \((L^2, L^2)\) pullback \(D\)–attractor indeed attract the class of \(D\) in \(H^s\)–norm, and thus the existence of a \((L^2, H^s)\) pullback \(D\)–attractor is obtained.

1. Introduction. Reaction diffusion equations, arising from heat diffusion and other numerous areas of applied sciences, have been extensively studied. The dynamics of these equations are also quite well understood, see e.g. \([13, 16, 17, 32, 34, 40]\) and references therein. It is known that the heat equation is a special reaction diffusion equation. If \(-\Delta\) in the heat equation is replaced by a nonlocal operator \((-\Delta)^s\) \((s \in (0, 1))\), then we have a fractional diffusion equation. This class of equations are arising in many contexts. For instance, the critical quasi-geostrophic equation appearing in hydrodynamics (see e.g., \([12, 26]\)), fractional porous medium models (see e.g., \([7, 11, 31]\)), the context of stable processes (see e.g., \([8, 23]\)) and many others. These fractional diffusion equations have been studied a lot in recent years, such as the existence of global solutions and the global regularity of smooth solutions.

However, the understanding of the dynamics of fractional diffusion equations is rather limited, especially the theory of attractors of these equations is far less
developed. The main goal of this paper is to study the existence and properties of pullback attractors of the following non-autonomous fractional diffusion equation:

\[
\begin{aligned}
\frac{\partial u}{\partial t} + (-\Delta)^s u + \lambda u + f(u) &= g(x, t) \quad \text{in } \mathbb{R}^N \times (\tau, \infty), \\
\quad u(x, \tau) &= u_\tau, \quad \text{for } x \in \mathbb{R}^N,
\end{aligned}
\]

where \( \lambda > 0 \), \( 0 < s < 1 \), \( \tau \in \mathbb{R} \) is the initial time, and where \( u_\tau \in L^2(\mathbb{R}^N) \), \( g \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^N)) \).

The function \( f \in C^1(\mathbb{R}, \mathbb{R}) \) satisfies the following conditions:

\[
\exists \ C_1, C_2, k_1, k_2 > 0, \ s.t. \quad \lambda > k_1 > 0 \text{ and } \ C_1|s|^p - k_1|s|^2 \leq f(s)s \leq C_2|s|^p + k_2|s|^2, \forall \ s \in \mathbb{R},
\]

and

\[
\exists \ l \leq \lambda \ s.t. \ f'(s) \geq -l, \ \forall \ s \in \mathbb{R}.
\]

The nonlocal operator \((-\Delta)^s\) is defined in terms of the Fourier transform

\[
(-\Delta)^s u(\xi) = (2\pi|\xi|)^{2s} \hat{u}(\xi), \quad \text{for a.e. } \xi \in \mathbb{R}^N,
\]

and we use the notation \( \Lambda = (-\Delta)^{1/2} \).

The nonlocal operator of the form \((-\Delta)^s\), which is usually called the fractional Laplacian, has attracted considerable attentions in recent years. This is because the interests in both mathematics and practical applications, including harmonic analysis, stochastic processes of Lévy type, potential theory, continuum mechanics, statistical physics, finance and so on. See e.g., \([8–10, 20, 23, 24, 30, 37–39]\) where further references can be found.

Different from the standard Laplacian \(-\Delta\) which is defined by pointwise differentiation, the fractional Laplacian \((-\Delta)^s\) is defined by a global integration with respect to a singular kernel (see e.g., \([22]\)). The difference between these two types of operators is best seen in the theory of diffusion. The operator \(-\Delta\) is often adopted to model the Brownian motion, whose feature is that the space scale of the propagation of the probability distribution is proportional to \( t^{1/2} \). When the space scale of the propagation of a diffusion is proportional to another power of time, \((-\Delta)^s\) is more proper to model the diffusion. This type of diffusion is often known as anomalous diffusion (see e.g., \([6,18,25]\)). The fractional Laplacian operators are also the infinitesimal generators of stable Lévy processes (see e.g., \([3]\)).

Define

\[
F(s) = \int_0^s f(\tau)d\tau.
\]

Then by \([4]\), there exist positive constants \( \tilde{C}_i, \tilde{k}_i \) for \( i = 1, 2 \) such that

\[
\tilde{C}_1|s|^p - \tilde{k}_1|s|^2 \leq F(s) \leq \tilde{C}_2|s|^p + \tilde{k}_2|s|^2. \quad (6)
\]

The first result of this paper provides the existence of pullback \( \mathcal{D}_\mu^-\)-attractor in \( L^2 \), see Section 3. In order to do this, we need the positive constants \( \lambda, C_1, C_2, k_1, k_1, \tilde{k}_2 \) satisfying the following conditions:

\[
\lambda - k_1 - \frac{2\tilde{C}_1\tilde{k}_2}{\tilde{C}_2} > 0, \quad (7)
\]

and

\[
\lambda + 1 - 2\tilde{k}_1 > 0. \quad (8)
\]
It is obvious that if $\lambda$ is large enough, then (7) and (8) are always true.

**Theorem 1.1.** Let $\theta = \min \{2, \lambda - k_1, \frac{\lambda - k_1 - 2\mu}{\mu k_2 - \lambda + 1}\} > 0$, $\mu \in (0, \theta)$ and $\mathcal{D}_\mu$ be the class of all families $\{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2(\mathbb{R}^N))$ of nonempty subsets of $L^2(\mathbb{R}^N)$ such that

$$
\lim_{\tau \to -\infty} e^{\sigma \tau} \sup \{\|u\|_{L^2(\mathbb{R}^N)} : u \in D(\tau)\} = 0.
$$

Assume that the conditions (2)-(8) hold, and let

$$
\int_{-\infty}^{t} e^{\sigma s} \|g(s)\|_{L^2(\mathbb{R}^N)} ds < +\infty, \text{ for all } t \in \mathbb{R}.
$$

Then the process $U(t, \tau)$ associated with equation (1) possesses a unique minimal $(L^2(\mathbb{R}^N), L^2(\mathbb{R}^N))$ pullback $\mathcal{D}_\mu$-attractor $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$.

In the proof of Theorem 1.1, we firstly provide the existence of $(L^2(\mathbb{R}^N), H^s(\mathbb{R}^N) \cap L^p(\mathbb{R}^N))$ pullback $\mathcal{D}_\mu$-absorbing sets; then establish a “Tail estimate” (see Lemma 3.8), which, together with Rellich-Kondrachov’s theorem, implies that the process $U$ generated by the equation (1) is pullback $\mathcal{D}_\mu$-limit-set compact in $L^2$. We emphasize that the pullback $\mathcal{D}_\mu$-limit-set compactness of $U$ cannot be deduced simply by the fact that $U$ has a $(L^2(\mathbb{R}^N), H^s(\mathbb{R}^N))$ pullback $\mathcal{D}_\mu$-absorbing set, since the Sobolev embedding $H^s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ is only continuous but not compact.

After the existence of the pullback attractor is obtained, we turn to study properties of the attractor. Note that the higher-order attraction of attractor is significant to understand the dynamics of a system. The second result of this paper is concerned with the $L^{2+\delta}(\mathbb{R}^N)$-attraction ($\delta > 0$) of the pullback attractor.

**Theorem 1.2.** Let $\theta$ be a positive constant given in Theorem 1.1, $\mu \in (0, \theta)$. Assume that the conditions (2)-(9) hold. Then for any $\delta \in [0, \infty)$, the minimal $(L^2, L^2)$ pullback attractor $\mathcal{A} = \{A(t) : A(t) \in \mathcal{P}(L^2(\mathbb{R}^N)) : t \in \mathbb{R}\}$ obtained in Theorem 1.1 pullback attracts $\mathcal{D}_\mu$-class in $L^{2+\delta}$-norm, i.e. for any $D \in \mathcal{D}_\mu$, $t \in \mathbb{R}$,

$$
\lim_{\tau \to -\infty} \text{dist}_{L^{2+\delta}}(U(t, \tau)D(\tau), A(t)) = 0.
$$

Here $\text{dist}_{L^{2+\delta}}$ is defined as follows:

$$
\text{dist}_{L^{2+\delta}}(E, F) := \sup_{u_1 \in E, u_2 \in F} \inf_{u \in F} \|u_1 - u_2\|_{L^{2+\delta}(\mathbb{R}^N)}, \text{ for any } E, F \subset L^2(\mathbb{R}^N).
$$

Note that $g$ only satisfies the $L^2$-integrability (3) and (9), and $f$ grows as the polynomial of degree $p - 1$. The solutions of (1) at most belong to $W^{2s, 2p - 2}(\mathbb{R}^N)$. On the other hand, the embedding $W^{2s, 2p - 2}(\mathbb{R}^N) \hookrightarrow L^{2+\delta}(\mathbb{R}^N)$ is not true when both the spatial dimension $N$ and $\delta$ are large enough. Therefore, for any $\delta \in (0, \infty)$, the $L^{2+\delta}(\mathbb{R}^N)$-norm pullback $\mathcal{D}_\mu$-attraction obtained in Theorem 1.2 is not trivial.

The last result of this paper investigates the dynamics of equation (1) in $H^s(\mathbb{R}^N)$.

It states that for any $\delta \in [0, \infty)$, the $(L^2(\mathbb{R}^N), L^2(\mathbb{R}^N))$ pullback $\mathcal{D}_\mu$-attractor obtained in Theorem 1.1 attracts the $\mathcal{D}_\mu$-class (especially for any $L^2(\mathbb{R}^N)$-bounded set) in $H^s(\mathbb{R}^N)$-norm.

**Theorem 1.3.** Let $\theta$ be a positive constant defined in Theorem 1.1, $\mu \in (0, \theta)$. Suppose that (5)-(8), (10) hold. Furthermore, we assume that $f$ satisfies an additional condition

$$
f \in C^1(\mathbb{R}) \text{ and there exists a positive } C_f \text{ such that for any } s \in \mathbb{R},
$$

$$
|f'(s)| \leq C_f(1 + |s|^{p-2}).
$$

(11)
Then the minimal \((L^2, L^2)\) pullback \(\mathcal{A}_\mu\)-attractor \(\mathcal{A} = \{A(t) : t \in \mathbb{R}\}\) obtained in Theorem 1.2 pullback attracts the \(\mathcal{A}_\mu\)-class in the topology of \(H^s(\mathbb{R}^N)\), i.e., for any \(\mathcal{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{A}_\mu\),

\[
\lim_{\tau \to -\infty} \text{dist}_{H^s(\mathbb{R}^N)}(U(t, \tau)D(\tau), A(t)) = 0 \quad \text{for any } t \in \mathbb{R}.
\] (12)

The condition (11), which is natural and has been used in many contexts (see e.g. [28], [32], [34] and references therein), is adapted to prove the continuity of solutions of (1) in \(H^s(\mathbb{R}^N)\), see Lemma 5.2.

By Theorem 1.3, we find that the \((L^2(\mathbb{R}^N), L^2(\mathbb{R}^N))\) pullback \(\mathcal{A}_\mu\)-attractor is in fact a \((L^2(\mathbb{R}^N), H^s(\mathbb{R}^N))\) pullback \(\mathcal{A}_\mu\)-attractor, see Remark 5. In order to prove the existence of the \((L^2(\mathbb{R}^N), H^s(\mathbb{R}^N))\) pullback \(\mathcal{A}_\mu\)-attractor, generally we need to show that the process \(U\) associated with the equation \((I)\) not only has a \((L^2(\mathbb{R}^N), H^s(\mathbb{R}^N))\) pullback \(\mathcal{A}_\mu\)-absorbing set but also is pullback \(\mathcal{A}_\mu\)-asymptotically compact in the Sobolev space \(H^s(\mathbb{R}^N)\).

The existence of a \((L^2(\mathbb{R}^N), H^s(\mathbb{R}^N))\) pullback \(\mathcal{A}_\mu\)-absorbing set is proved in Theorem 1.1. However, it is more difficult to show that \(U\) is pullback \(\mathcal{A}_\mu\)-asymptotically compact in \(H^s(\mathbb{R}^N)\). The reasons are listed as follows. Firstly, since \(q(t)\) only satisfies \(L^2\) integrability (w.r.t. \(t\)), we cannot differentiate the equation (w.r.t. \(t\)) to get a nice a priori estimate of \(\partial_t u\). Secondly, a sufficient condition, which is called the “Tail estimate” in \(H^s(\mathbb{R}^N)\), is difficult to establish because of the nonlocal operator \((-\Delta)^s\). Thirdly, for a non-autonomous reaction-diffusion equation defined on a bounded domain \(\Omega\), Lukaszewicz [28] proved the existence of a unique minimal pullback attractor in \(H^s_0(\Omega)\) by the Kuratowski measure of non-compactness of a bounded set together with a new Gronwall lemma. However, because our model is a nonlocal equation and defined on the whole space \(\mathbb{R}^N\). The method presented in [28] is inapplicable to prove that \(U\) is pullback \(\mathcal{A}_\mu\)-asymptotically compact in \(H^s(\mathbb{R}^N)\). Based on the above arguments, new methods are needed.

This paper is organized as follows. We begin with some preliminary results which will be used throughout this paper. Section 3 is devoted to the study of the existence of the \((L^2(\mathbb{R}^N), L^2(\mathbb{R}^N))\) pullback \(\mathcal{A}_\mu\)-attractor. In Section 4, first some higher-order integrability for the difference of solutions near the initial time (Lemma 4.1) is proved by using a bootstrap argument, and then Theorem 1.2 is proved. In Section 5, we first show the continuity of solutions in \(H^s(\mathbb{R}^N)\) with respect to the initial data (Lemma 5.1), and then we complete the proof of Theorem 1.3. Finally, as a corollary, the existence of a \((L^2(\mathbb{R}^N), H^s(\mathbb{R}^N))\) pullback \(\mathcal{A}_\mu\)-attractor (Remark 5) is obtained.

2. Preliminaries.

2.1. Abstract results on pullback attractors. In this section, we recall some abstract results on pullback attractors, which were originally introduced by Crâuel and Flandoli [19] and generalized by Caraballo, Lukaszewicz and Real [14], see e.g., [15], [19], [27] for more details. Firstly, we recall the definition of a process.

**Definition 2.1** (Kloeden and Rasmussen [27]). Let \((X, d)\) be a complete metric space with the metric \(d(\cdot, \cdot)\). A process is a continuous mapping \(U(t, \tau) : X \to X\) for \(t, \tau \in \mathbb{R}\) with \(t \geq \tau\), which satisfies the initial value and evolution properties:

(i) \(U(t, t)x = x\) for all \(t \in \mathbb{R}\) and \(x \in X\),

(ii) \(U(t, \tau)x = U(t, r)U(r, \tau)x\) for all \(\tau \leq r \leq t\) and \(x \in X\).
A process is also called a two-parameter semigroup on $X$ in contrast with the one parameter semigroup of an autonomous semi-dynamical system since it depends on both the initial time $t$ and the actual time $t$ rather than just the elapsed time $t - \tau$.

Suppose that $\mathcal{D}$ is a nonempty class of parameterized sets $\mathcal{D} = \{ D(t) \in \mathcal{P}(X) : t \in \mathbb{R} \}$, where $\mathcal{P}(X)$ denotes the family of all nonempty subsets of $X$.

**Definition 2.2** (Carvalho et al. [15, 27]). A process on $X$ is said to be pullback $\mathcal{D}$-asymptotically compact if for any $t \in \mathbb{R}$, any $D \in \mathcal{D}$ and any sequence $\{ \tau_n \} \subset (-\infty, t]$ and $\{ x_n \} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D(\tau_n)$ for all $n$, the sequence $\{ u(t, \tau_n)x_n \}$ is relatively compact in $X$.

**Definition 2.3** (Lukaszewicz [29]). Let $B$ be a nonempty bounded set in $X$. The Kuratowski measure of noncompactness is defined by

$$\alpha(B) = \inf \{ \delta : B \text{ admits a finite cover by sets of diameter } \leq \delta \}.$$ 

**Definition 2.4** (Lukaszewicz [29]). A process $U(t, \tau)$ on $X$ is said to be a pullback $\mathcal{D}$-limit-set compact if for any $D \in \mathcal{D}$,

$$\lim_{s \to -\infty} \alpha \left( \bigcup_{\tau \leq s} U(t, \tau)D(\tau) \right) = 0.$$ 

**Definition 2.5** (Anguiano et al. [2, 15, 27]). Let $X, Y$ be two Banach spaces. We say that $\mathcal{B} = \{ B(t) : t \in \mathbb{R} \} \subset \mathcal{P}(Y)$ is $(X, Y)$ pullback $\mathcal{D}$-absorbing for the process $U$ on $X$ if for any $t \in \mathbb{R}$ and any $D \in \mathcal{D} \subset \mathcal{P}(X)$, there exists a $\tau_0(t, D) \leq t$ such that $U(t, \tau)D(\tau) \subset B(t)$ for any $\tau \leq \tau_0(t, D)$.

**Definition 2.6** (Anguiano et al. [2, 15, 27]). Let $X, Y$ be two Banach spaces. The family $\mathcal{A} = \{ A(t) : A(t) \in \mathcal{P}(Y), t \in \mathbb{R} \}$ is called to be a $(X, Y)$ pullback $\mathcal{D}$-attractor for the process $U(\cdot, \cdot)$ if

(i) $A(t)$ is closed in $X$ and compact in $Y$ for all $t \in \mathbb{R}$;

(ii) $\mathcal{A}$ is pullback $\mathcal{D}$-attracting in the topology of $Y$, i.e.

$$\lim_{\tau \to -\infty} \text{dist}_Y(U(t, \tau)D(\tau), A(t)) = 0,$$

for all $D \in \mathcal{D} \subset \mathcal{P}(X)$ and all $t \in \mathbb{R}$,

where $\text{dist}_Y$ is the usual Hausdorff semidistance in $Y$: $\text{dist}_Y(A, B) := \sup_{a \in A} \inf_{b \in B} \| a - b \|_Y$;

(iii) $\mathcal{A}$ is invariant, i.e.

$$U(t, \tau)A(\tau) = A(t) \text{ for any } -\infty < \tau \leq t < \infty.$$ 

We have the following result on the existence of a minimal pullback $\mathcal{D}$-attractor.

**Theorem 2.7** (Lukaszewicz [29]). Let $X$ be a Banach space. Consider a continuous process $U : \mathbb{R}^2 \times X \to X$, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\mathcal{B} = \{ B(t) : t \in \mathbb{R} \} \subset \mathcal{P}(X)$ which is $(X, X)$ pullback $\mathcal{D}$-absorbing for $U$, and assume also that $U(t, \tau)$ is pullback $\mathcal{D}$-limit-set compact in $X$. Then the family $\mathcal{A} = \{ A(t) : t \in \mathbb{R} \}$ defined by

$$A(t) = \omega(\mathcal{B}, t) = \bigcap_{s \leq t \tau \leq s} \bigcup_{t < \tau \leq s} U(t, \tau)B(\tau)$$

is a minimal $(X, X)$ pullback $\mathcal{D}$-attractor for $U$. The family $\mathcal{A}$ is minimal in the sense that if $\mathcal{C} = \{ C(t) : t \in \mathbb{R} \} \subset \mathcal{P}(X)$ is a family of closed sets such that for any $D = \{ D(t) : t \in \mathbb{R} \} \in \mathcal{D}$,

$$\lim_{\tau \to -\infty} \text{dist}_X(U(t, \tau)D(\tau), C(t)) = 0,$$

then $A(t) \subset C(t)$. 

Let $U(t, \tau)$ be a process on a Banach space $X$. Denote by $\mathcal{K}$ the collection of all complete trajectories of $U(t, \tau)$, that is

$$\mathcal{K} := \{ \hat{\mathcal{u}} = \{u(t) : t \in \mathbb{R}\} : U(t, \tau)u(\tau) = u(t), \text{ for any } -\infty < \tau \leq t < \infty \}.$$ 

We also recall the following result of the structure of pullback $\mathcal{P}$–attractors (see [15, 27] for more details), which means that each pullback $\mathcal{P}$–attractor contains at least one complete trajectory.

**Lemma 2.8** (Carvalho et al. [15, 27]). Let $U(t, \tau)$ be a process on a Banach space $X$, let $\mathcal{A} = \{A(t), t \in \mathbb{R}\} \subseteq \mathcal{D}$ be the pullback $\mathcal{P}$–attractor of $U(t, \tau)$, then for any $t \in \mathbb{R}$

$$A(t) = \bigcup_{\hat{\mathcal{u}} \in \mathcal{K} \cap \mathcal{P}} u(t),$$

and consequently, there exists at least one complete trajectory $\hat{v}$ of $U(t, \tau)$ that satisfies $\hat{v} \in \mathcal{P}$.

2.2. **Fractional Sobolev spaces.** The main aim of this subsection is to prove the following preliminary result:

Let $s \in (0, 1)$. If $u \in L^\infty(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$, then $|u|^{p-2}u \in L^\infty(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$ for any $p \geq 2$.

At the beginning, we provide the definition of fractional Sobolev spaces and state some preliminary lemmas, we refer readers to [4, 20, 35] where details can be found. Let $S$ be the usual Schwartz class on $\mathbb{R}^N$ and $S'$ be the space of tempered distributions. The Fourier transform $\hat{f}$ of a $L^1$-function $f$ is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} f(x)e^{-2\pi i x \cdot \xi}dx.$$

For $f \in S'$, the Fourier transform of $f$ is defined by

$$(\hat{f}, g) = (f, \hat{g}) \text{ for any } g \in S.$$ 

The fractional Laplacian $(-\Delta)^s$ with $s \in \mathbb{R}$ is defined through its Fourier transform, namely

$$(\hat{-\Delta})^s f(\xi) = (2\pi |\xi|)^{2s} \hat{f}(\xi).$$

Recall that for $s \in \mathbb{R}$, the inhomegenous Sobolev space is defined by

$$H^s(\mathbb{R}^N) := \{ f \in S' : (1 + |\xi|^2)^{s/2} |\hat{f}(\xi)| \in L^2(\mathbb{R}^N) \}. \quad (13)$$

**Definition 2.9.** Let $\Omega$ be a smooth open set in $\mathbb{R}^N$, and $q \in [1, \infty)$,

(i) For any $s \in (0, \infty) \cap \mathbb{Z}$, the Sobolev space $W^{s,q}(\Omega)$ is studied in many contexts. (see e.g., [1])

(ii) For any $s \in (0, \infty) \cap \mathbb{Z}$, i.e., $s$ is not an integer, let $s = k + \sigma$, where $k = \max \{ k \leq s : k \in \mathbb{Z} \}$ and $\sigma \in (0, 1)$, we define the fractional Sobolev space $W^{s,q}(\Omega)$ as follows (see e.g., [20]):

$$W^{s,q}(\Omega) := \left\{ u, D^k u \in L^q(\Omega) : \frac{|D^k u(x) - D^k u(y)|}{|x-y|^{N+\sigma}} \in L^q(\Omega \times \Omega) \right\}, \quad (14)$$

endowed with the norm

$$\|u\|_{W^{s,q}(\Omega)}$$

$$= \left( \int_\Omega |u(x)|^q dx + \int_\Omega |D^k u(x)|^q dx + \int_\Omega \int_\Omega \frac{|D^k u(x) - D^k u(y)|^q}{|x-y|^{N+\sigma q}} dx dy \right)^{\frac{1}{q}}. \quad (15)$$
Note that for any $s \in (0, \infty)$, \[13\] is equivalent to \[14\] for the case $q = 2$ and $\Omega = \mathbb{R}^N$. In fact, we have the following result:

**Lemma 2.10** (Di Nezza et al. \[20\]). Let $s \in (0, 1)$. Then the fractional Sobolev space $H^s(\mathbb{R}^N)$ defined in \[13\] coincides with $H^s(\mathbb{R}^N)$ defined in \[14\]. In particular, there exists a constant $c(N,s)$, such that, for any $u \in H^s(\mathbb{R}^N)$

$$
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy = 2c(N,s)^{-1} \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi.
$$

Thus we use the notation $H^s(\mathbb{R}^N)$ to denote $W^{s,q}(\mathbb{R}^N)$ for the case $q = 2$. The following result states that any function in the fractional Sobolev space $W^{s,q}(\mathbb{R}^N)$ can be approximated by a sequence of smooth functions with compact support. However, $C_0^\infty(\Omega)$ is not dense in $W^{s,q}(\Omega)$ when $\Omega \subset \mathbb{R}^N$.

**Lemma 2.11** (Di Nezza et al. \[20\]). For any $s \in (0, \infty)$ and any $q \in [1, \infty)$, the space $C_0^\infty(\mathbb{R}^N)$ of smooth functions with compact support is dense in $W^{s,q}(\mathbb{R}^N)$.

The following result is a Sobolev-type inequality involving the fractional norm $\|\cdot\|_{W^{s,q}}$.

**Lemma 2.12** (Di Nezza et al. \[20\]). Let $s \in (0, \infty)$ and $q \in [1, \infty)$ be such that $sq < N$. Let $\Omega \subset \mathbb{R}^N$ be a bounded subset with a smooth boundary. Then there exists a positive constant $C = C(N,q,s,\Omega)$ such that, for any $u \in W^{s,q}(\Omega)$,

$$
\|u\|_{L^r(\Omega)} \leq C\|u\|_{W^{s,q}(\Omega)},
$$

for any $r \in [1,q^*]$, where $q^* = \frac{Nq}{N - sq}$. That is, the space $W^{s,q}(\Omega)$ is continuously embedded in $L^r(\Omega)$ for any $r \in [1,q^*]$. If $\Omega = \mathbb{R}^N$, then $W^{s,q}(\mathbb{R}^N)$ is continuously embedded in $L^r(\mathbb{R}^N)$ for any $r \in [q,q^*]$. If $sq > N$, then $W^{s,q}(\Omega)$ is continuously embedded in $C^{0,\alpha}(\Omega)$, $\alpha = \frac{N}{q} - s$, where $\Omega$ is a subset with a smooth boundary or the whole space $\mathbb{R}^N$.

We also need the following compact embedding, see e.g. \[20\].

**Lemma 2.13** (Rellich-Kondrachov). Let $s \in (0, \infty)$ and $q \in [1, \infty)$ be such that $sq < N$. Let $r \in [1,q^*)$, $\Omega \subset \mathbb{R}^N$ be a bounded subset with a smooth boundary. Suppose that the sequence $\{u_n : n \in \mathbb{N}\}$ is bounded in $W^{s,q}(\Omega)$, then $\{u_n\}$ is pre-compact in $L^r(\Omega)$.

Let $u$ be a measurable function on $\mathbb{R}^N$, the positive and negative parts of $u$ is defined by $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$. Clearly $u = u^+ + u^-$ and $|u| = u^+ - u^-$. 

**Lemma 2.14.** Let $s \in (0, 1)$, $u \in H^s(\mathbb{R}^N)$, then $u^+, u^-, |u| \in H^s(\mathbb{R}^N)$.

**Proof.** Note that $|u^+(x) - u^+(y)| \leq |u(x) - u(y)|$ for any $x, y \in \mathbb{R}^N$. This, together with \[14\] and \[15\], implies that $u^+ \in H^s(\mathbb{R}^N)$ and

$$
\|u^+\|_{H^s(\mathbb{R}^N)} \leq \|u\|_{H^s(\mathbb{R}^N)}.
$$

Moreover, by $u^+ = u^+ - u - u^-$ and $|u| = u^+ - u^-$ we deduce that $u^-, |u| \in H^s(\mathbb{R}^N)$ and

$$
\|u^-\|_{H^s(\mathbb{R}^N)} \leq \|u\|_{H^s(\mathbb{R}^N)}, \quad \|u^+\|_{H^s(\mathbb{R}^N)} \leq \|u\|_{H^s(\mathbb{R}^N)}.
$$

\[\square\]
The following result states that the product of two functions in a fractional Sobolev space is still in this Sobolev space.

**Lemma 2.15** (Bahouri et al. [4]). For any positive real number \( s \) and any \( q \in [1, \infty] \), if \( u, v \in L^\infty(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N) \), then the product \( uv \in L^\infty(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N) \) and a constant \( c = c(q) \) exists such that

\[
\|uv\|_{W^{s,q}(\mathbb{R}^N)} \leq c\left(\|u\|_{L^\infty(\mathbb{R}^N)}\|v\|_{W^{s,q}(\mathbb{R}^N)} + \|u\|_{W^{s,q}(\mathbb{R}^N)}\|v\|_{L^\infty(\mathbb{R}^N)}\right). \tag{16}
\]

We usually consider the action of smooth functions on the space \( W^{s,q}(\mathbb{R}^N) \). More precisely, if \( f \) is a smooth function vanishing at 0, and \( u \) is a function of \( W^{s,q}(\mathbb{R}^N) \), then \( f \circ u \) also belongs to \( W^{s,q}(\mathbb{R}^N) \). This result reads as follows.

**Lemma 2.16** (Bahouri et al. [4]). Let \( f : \mathbb{R} \to \mathbb{R} \) be a smooth function vanishing at 0, a positive real number and \( q \in [1, \infty] \). If \( u \in L^\infty(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N) \), then so does \( f \circ u \) and we have

\[
\|f \circ u\|_{W^{s,q}(\mathbb{R}^N)} \leq C(s, q', \|u\|_{L^\infty(\mathbb{R}^N)})\|u\|_{W^{s,q}(\mathbb{R}^N)}.
\]

Then we are ready to prove the result which is formulated at the beginning of this subsection.

**Lemma 2.17.** Let \( s \in (0, 1) \), \( p \geq 2 \) be arbitrary. Assume that \( u \in L^\infty(\mathbb{R}^N) \cap H^s(\mathbb{R}^N) \), then we have \( |u|^{p-2}u \in L^\infty(\mathbb{R}^N) \cap H^s(\mathbb{R}^N) \).

**Proof.** By Lemma 2.14 we have \( u^+ \), \( |u| \in L^\infty(\mathbb{R}^N) \cap H^s(\mathbb{R}^N) \). For any \( r \in \mathbb{R} \), define \( f(r) = r^{p-1} \), it is obvious that \( f(0) = 0 \) and \( f' \in C(\mathbb{R}) \). Since both \( u^+ \) and \( |u| \) belong to \( L^\infty(\mathbb{R}^N) \), we deduce that there exists a constant \( M = M(f, \|u\|_{L^\infty(\mathbb{R}^N)}) \in [0, \infty) \) such that \( |f'(u(x))| \leq M \) for any \( x \in \mathbb{R}^N \). Thus, by Lemma 2.16 we know that both \( (u^+)^{p-1} \) and \( |u|^{p-1} \) belong to \( L^\infty(\mathbb{R}^N) \cap H^s(\mathbb{R}^N) \) and satisfying corresponding norm estimates. Note that \( |u|^{p-1} = |u|^{p-2}u^+ - |u|^{p-2}u^- = (u^+)^{p-1} - |u|^{p-2}u^- \), thus we deduce that \( |u|^{p-2}u^- \in L^\infty(\mathbb{R}^N) \cap H^s(\mathbb{R}^N) \). Finally, by the equality \( |u|^{p-2}u = |u|^{p-1} + 2|u|^{p-2}u^- \), we obtain that \( |u|^{p-2}u \in L^\infty(\mathbb{R}^N) \cap H^s(\mathbb{R}^N) \) for any \( p \geq 2 \) and the proof is completed.

2.3. **Useful inequalities.** The following generalized Gronwall’s inequality, which is derived by Lukaszewicz [28], is crucial to prove that the process \( U \) associated with \( \{\} \) has pullback \( \mathcal{P}_{\mu} \)-absorbing sets in \( H^s(\mathbb{R}^N) \).

**Lemma 2.18** (Lukaszewicz [28]). Suppose that

\[ y'(s) + \lambda y(s) \leq h(s) \]

for some \( \lambda > 0 \), \( \tau \in \mathbb{R} \) and for \( s > \tau \), where the function \( y, y', h \) are assumed to be locally integrable with \( y \), \( h \) nonnegative on the interval \( t < s < t + r \) for some \( t \geq \tau \). Then

\[ y(t + r) \leq e^{-\lambda(t+r)}\frac{2}{\tau} \int_t^{t+r} y(s)ds + e^{-\lambda(t+r)}\int_t^{t+r} e^{\lambda h(s)}ds. \]

We also need the following classical Strook-Varopoulos inequality.

**Lemma 2.19** (Karch [25]). For \( s \in [0, 1] \), \( k \geq 2 \) and any \( \varphi \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \), then by Lemma 2.17 \( |\varphi|^{k-2}\varphi \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) and the following inequality holds:

\[ \int_{\mathbb{R}^N} \Lambda^2 \varphi |\varphi|^{k-2}\varphi dx \geq \frac{4(k-1)}{k^2} \int_{\mathbb{R}^N} |\Lambda^s(|\varphi|^{\frac{k}{2}})|^2 dx. \]
3. Existence of a \((L^2(\mathbb{R}^N), L^2(\mathbb{R}^N))\) pullback \(\mathcal{D}_\mu\)-attractor.

3.1. The well-posedness. In this subsection, motivated by the results of Dlotko et al. in [21], we establish some elementary results of the equation (1), including the existence and uniqueness of both the strong solution and the weak solution, and a \(L^\infty\) apriori estimate for the weak solution.

**Definition 3.1** (Strong solutions). A function \(u = u(x, t)\) defined in \(\mathbb{R}^N \times [\tau, T]\) is called a strong solution of (1) if
\[
 u \in L^2(\tau, T; H^{2s}(\mathbb{R}^N)) \cap C(\tau, T; H^s(\mathbb{R}^N)) \cap L^\infty(\tau, T; L^p(\mathbb{R}^N)),
\]
\[
 u' \in L^2(\tau, T; L^2(\mathbb{R}^N)),
\]
and the equation (1) holds almost everywhere in \(\mathbb{R}^N \times [\tau, T]\).

The following result about the existence and uniqueness of the strong solution has been essentially proved by Dlotko et al. [21].

**Lemma 3.2.** Let \(g\) satisfy (3), \(f\) satisfy (4), (5) and the initial data \(u_\tau \in H^s(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)\), then for any \(T > \tau\), there exists a unique strong solution \(u\) to (1) satisfying
\[
 u \in L^2(\tau, T; H^{2s}(\mathbb{R}^N)) \cap C(\tau, T; H^s(\mathbb{R}^N)) \cap L^\infty(\tau, T; L^p(\mathbb{R}^N)),
\]
\[
 u' \in L^2(\tau, T; L^2(\mathbb{R}^N)),
\]
and the equation (1) holds in \(L^2(\mathbb{R}^N)\) for almost all \(t \in (\tau, T)\).

**Definition 3.3** (Weak solutions). The function \(u = u(\cdot; \tau, u_\tau)\) is called a weak solution to the Cauchy problem (1) if for any \(T > \tau\), it satisfies
\[
 u(t) \in C([\tau, T]; L^2(\mathbb{R}^N)) \cap L^2(\tau, T; H^s(\mathbb{R}^N)) \cap L^p(\tau, T; L^p(\mathbb{R}^N)),
\]
\(u\) also satisfies the initial condition \(u(\tau) = u_\tau\) and the mapping \(u_\tau \to u(t; \tau, u_\tau)\) is continuous in \(L^2(\mathbb{R}^N)\). In addition, for all \(v \in H^s(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)\) and for almost every \(t \in (\tau, T)\), the following identity holds:
\[
 (\partial_t u(t), v)_{L^2(\mathbb{R}^N)} + (\Lambda^s u(t), \Lambda^s v)_{L^2(\mathbb{R}^N)} + (\lambda u(t) + f(u(t)), v)_{L^2(\mathbb{R}^N)} = (g(t), v)_{L^2(\mathbb{R}^N)},
\]
(19)

The following result states the existence and uniqueness of the weak solution of the problem (1) – (2), the proof is based on the approximation.

**Lemma 3.4.** Let \(T > \tau\). Assume that (2) – (5) hold. Then there exists a unique weak solution \(u\) to the equation (1).

**Proof.** Since the embedding \(H^s(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)\) is dense, it follows that there exists a sequence \(\{u_\tau^{(m)}\}\) satisfying
\[
 \{u_\tau^{(m)}\} \subset H^s(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \text{ and } u_\tau^{(m)} \to u_\tau \text{ in } L^2(\mathbb{R}^N).
\]
By Lemma 3.2 for each \(u_\tau = u_\tau^{(m)}\), the equation (1) exists a unique strong solution \(u^{(m)}(t) = U(t, \tau)u_\tau^{(m)}\). It is obvious that every strong solution of (1) is also a weak solution, thus \(u^{(m)}(t) \in C([\tau, T]; L^2(\mathbb{R}^N)) \cap L^2(\tau, T; H^s(\mathbb{R}^N)) \cap L^p(\tau, T; L^p(\mathbb{R}^N))\), and
\[
 \partial_t u^{(m)} \in L^2((\tau, T); L^2(\mathbb{R}^N)),
\]
(20)
and for any \(v \in H^s(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)\) and almost every \(t \in (\tau, T)\),
\[
(\partial_t u^{(m)}(t), v)_{L^2(\mathbb{R}^N)} + (\Lambda^s u^{(m)}(t), \Lambda^s v)_{L^2(\mathbb{R}^N)} + (\lambda u^{(m)}(t) + f(u^{(m)}(t)), v)_{L^2(\mathbb{R}^N)} = (g(t), v)_{L^2(\mathbb{R}^N)}.
\]
(21)

By \([20]\), we have
\[
\|u^{(m)}(t)\|^2_{L^2(\mathbb{R}^N)} - \|u_\tau^{(m)}\|^2_{L^2(\mathbb{R}^N)} = 2 \int_\tau^t (\partial_t u^{(m)}(s), u^{(m)}(s))_{L^2(\mathbb{R}^N)} ds.
\]

Taking \(v = u^{(m)}(t)\) in (21) and integrating on \((\tau, t)\), we obtain that for any \(t \in [\tau, T]\),
\[
\frac{1}{2} \|u^{(m)}(t)\|^2_{L^2(\mathbb{R}^N)} + \int_\tau^t \|\Lambda^s u^{(m)}(s)\|^2_{L^2(\mathbb{R}^N)} ds + \lambda \int_\tau^t \|u^{(m)}(s)\|^2_{L^2(\mathbb{R}^N)} ds + \int_\tau^t (f(u^{(m)}(s)), u^{(m)}(s))_{L^2(\mathbb{R}^N)} ds
\]
\[
= \frac{1}{2} \|u_\tau^{(m)}\|^2_{L^2(\mathbb{R}^N)} + \int_\tau^t (g(s), u^{(m)}(s))_{L^2(\mathbb{R}^N)} ds.
\]

This, together with \([4]\) and Cauchy’s inequality, yields that
\[
\|u^{(m)}(t)\|^2_{L^2(\mathbb{R}^N)} + c_{\Lambda, k_1} \int_\tau^t \|u^{(m)}(s)\|^2_{H^s(\mathbb{R}^N)} ds + 2 \int_\tau^t \|u^{(m)}(s)\|^p_{L^p(\mathbb{R}^N)} ds
\]
\[
\leq \|u_\tau^{(m)}\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{\lambda - k_1} \int_\tau^t \|g(s)\|^2_{L^2(\mathbb{R}^N)} ds,
\]
where \(c_{\Lambda, k_1} = \min\{2, \lambda - k_1\}\). Therefore \(\{u_\tau^{(m)}\}\) is bounded in \(L^\infty(\tau, T; L^2(\mathbb{R}^N)) \cap L^2(\tau, T; H^s(\mathbb{R}^N)) \cap L^p(\tau, T; L^p(\mathbb{R}^N))\). Consequently there exists an element \(u(t, x)\) and a subsequence which is still denoted by \(\{U(t, \tau)u_\tau^{(m)}\}\) without lose of generality, such that
\[
u^{(m)} \rightarrow u \quad \text{weakly}^* \quad \text{in} \quad L^\infty(\tau, T; L^2(\mathbb{R}^N)),
\]
\[
u^{(m)} \rightarrow u \quad \text{weakly} \quad \text{in} \quad L^2(\tau, T; H^s(\mathbb{R}^N)) \cap L^p(\tau, T; L^p(\mathbb{R}^N)),
\]
\[
f(u^{(m)}) \rightarrow \Phi \quad \text{weakly} \quad \text{in} \quad L^p(\tau, T; L^p(\mathbb{R}^N)).
\]

Let \(w = u^{(m)} - u^{(n)}\), it is obvious that \(w\) is the solution of the following equation
\[
\begin{cases}
\partial_t w + \Lambda^s w + \lambda w + f(u^{(m)}) - f(u^{(n)}) = 0, & \text{in} \ \mathbb{R}^N \times (\tau, T), \\
u(\tau) = u_\tau^{(m)} - u_\tau^{(n)}, & \text{for} \ x \in \mathbb{R}^N.
\end{cases}
\]
(22)

Multiplying (22) by \(w\) and integrating on \(\mathbb{R}^N \times (\tau, T)\), we obtain
\[
\|u^{(m)}(t) - u^{(n)}(t)\|^2_{L^2(\mathbb{R}^N)} \leq e^{2(t-\tau)} \|u_\tau^{(m)} - u_\tau^{(n)}\|^2_{L^2(\mathbb{R}^N)},
\]
which yields that \(\{u^{(m)}\}\) is a Cauchy sequence in \(C([\tau, T]; L^2(\mathbb{R}^N))\). Consequently, by the uniqueness of the limit, it follows that
\[
u^{(m)} \rightarrow u \quad \text{in} \quad C([\tau, T]; L^2(\mathbb{R}^N)).
\]

Hence, extracting a subsequence if necessary, we deduce that \(f(u^{(m)}) \rightarrow f(u)\) a.e. in \(\mathbb{R}^N \times (\tau, T)\). This, together with the fact that \(f(u^{(m)})\) is uniformly bounded in \(L^p(\tau, T; L^p(\mathbb{R}^N))\), yields that \(f(u^{(m)}) \rightarrow f(u)\) weakly in \(L^p(\tau, T; L^p(\mathbb{R}^N))\). Therefore, by the uniqueness of the weak limit, \(\Phi = f(u)\). Let \(m \rightarrow \infty\) in (21),
we finally obtain that \( u \) is the unique weak solution of the problem \((1)\) and the proof is completed. \(\qed\)

By Lemma 3.4, we can define a continuous process \(\{ U(t, \tau); -\infty < \tau \leq t < \infty \} \) in \( L^2(\mathbb{R}^N) \) by

\[
U(t, \tau) u_\tau = u(t) := u(t; \tau, u_\tau) \quad \text{for all } t \geq \tau,
\]

where \( u(t) \) is the weak solution to the problem \((1)\) with \( u(\tau) = u_\tau \).

We will frequently use the technique of approximation in the rest of this paper. Thus it is necessary to deduce the following \( L^\infty \)-type result about the weak solution of \((1)\).

**Lemma 3.5.** Let \( T > \tau \). Assume that \((2)\)-\((5)\) hold. If \( u_\tau \in L^\infty(\mathbb{R}^N) \) and \( g \in L^\infty(\tau, T; L^\infty(\mathbb{R}^N)) \), then there exists a constant \( M \), which depends on \( \| u_\tau \|_{L^\infty(\mathbb{R}^N)} \), \( \| g \|_{L^\infty(\tau, T; L^\infty(\mathbb{R}^N))} \), \( \lambda \), \( C_i \), and \( k_i \) (\( i = 1, 2 \)), such that the weak solution \( u = U(t, \tau) u_\tau \) of \((1)\) satisfies \( \| u \|_{L^\infty(\tau, T; L^\infty(\mathbb{R}^N))} \leq M \).

**Proof.** Let \( M_1 = \max \{ \| u_\tau \|_{L^\infty(\mathbb{R}^N)}, \| g \|_{L^\infty(\tau, T; L^\infty(\mathbb{R}^N))} \} \). We deduce from \((1)\) that there exists a constant \( s_0 > 0 \), which depends on \( M_1 \), \( \lambda \), \( C_i \) and \( k_i \) (\( i = 1, 2 \)), such that

\[
f(s) + \lambda s \geq M_1 \quad \text{for any } s \geq s_0 \quad \text{and } f(s) + \lambda s \leq -M_1 \quad \text{for any } s \leq -s_0.
\]

Set \( M = \max \{ M_1, s_0 \} + 1 \). It is clear that \( (u - M)_+ (:= \max\{u - M, 0\}) \in C([\tau, T]; L^2(\mathbb{R}^N)) \cap L^2(\tau, T; H^s(\mathbb{R}^N)) \cap L^p(\tau, T; L^p(\mathbb{R}^N)) \), where \( u \) is the weak solution of \((1)\). We claim that \( \| u \|_{L^\infty(\tau, T; L^\infty(\mathbb{R}^N))} \leq M \). To do this, first let \( \varphi = (T - t)(u - M)_+ \). It is obvious that \( \varphi \in C([\tau, T]; L^2(\mathbb{R}^N)) \cap L^2(\tau, T; H^s(\mathbb{R}^N)) \cap L^p(\tau, T; L^p(\mathbb{R}^N)) \); moreover, \( \varphi \in L^1(\tau, T; L^1(\mathbb{R}^N)) \) and

\[
\int_\tau^T \int_{\mathbb{R}^N} M \varphi \, dx \, dt < +\infty.
\]

Then multiplying \((1)\) by \( \varphi \) and integrating on \((\tau, T) \times \mathbb{R}^N\), we obtain

\[
\int_\tau^T \int_{\mathbb{R}^N} \frac{\partial u}{\partial t} \varphi + \Lambda^{2s} u \varphi + \lambda u \varphi + f(u) \varphi \, dx \, dt = \int_\tau^T \int_{\mathbb{R}^N} g \varphi \, dx \, dt.
\]

Adding both sides of the above equality by \(-\int_\tau^T \int_{\mathbb{R}^N} M \) yields

\[
\int_\tau^T \int_{\mathbb{R}^N} \frac{\partial u}{\partial t} \varphi + \Lambda^{2s} u \varphi + (f(u) + \lambda u - M_1) \varphi \, dx \, dt = \int_\tau^T \int_{\mathbb{R}^N} (g - M_1) \varphi \, dx \, dt. \quad (24)
\]

Now we estimate integrals of \((24)\) one by one. Firstly, note that for any convex function \( \phi \), we have the point-wise inequality \( \Lambda^{2s} u \phi'(u) \geq \Lambda^{2s} \phi(u) \), see e.g. [22]. Let \( \phi(u) = (u - M)_+ \). Thus \( \phi \) is convex and the following inequality holds:

\[
\int_\tau^T \int_{\mathbb{R}^N} \Lambda^{2s} u \phi \, dx \, dt = \int_\tau^T \int_{\mathbb{R}^N} (T - t) \Lambda^{2s} u (u - M)_+ \, dx \, dt \\
\geq \int_\tau^T (T - t) \int_{\mathbb{R}^N} \left( \Lambda^{2s} (u - M)_+ \right) (u - M)_+ \, dx \, dt \\
= \int_\tau^T (T - t) \int_{\mathbb{R}^N} |\Lambda^s (u - M)_+|^2 \, dx \, dt \\
\geq 0.
\]

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Secondly, we use the definition of $M$ to obtain
\[
\int_{\tau}^{T} \int_{\mathbb{R}^N} (f(u) + \lambda u - M_1) \varphi dx dt \\
= \int_{\tau}^{T} \int_{\mathbb{R}^N} (f(u) + \lambda u - M_1)(T - t)(u - M)_+ dx dt \geq 0,
\]
and
\[
\int_{\tau}^{T} \int_{\mathbb{R}^N} (g - M_1) \varphi dx dt \\
= \int_{\tau}^{T} \int_{\mathbb{R}^N} (g - M_1)(T - t)(u - M)_+ dx dt \leq 0.
\]

Thirdly, integrating by parts yields
\[
\int_{\tau}^{T} \int_{\mathbb{R}^N} \frac{\partial u}{\partial t}(T - t)(u - M)_+ dx dt = \frac{1}{2} \int_{\tau}^{T} (T - t) \frac{d}{dt} \int_{\mathbb{R}^N} (u - M)_+^2 dx dt \\
= \frac{1}{2} \int_{\tau}^{T} \int_{\mathbb{R}^N} (u - M)_+^2 dx dt.
\]
Hence, substituting (25)–(28) into (24), we deduce that
\[
\frac{1}{2} \int_{\tau}^{T} \int_{\mathbb{R}^N} (u - M)_+^2 dx dt \leq 0.
\]

This implies that $u \leq M$ for almost every $(x, t) \in \mathbb{R}^N \times [\tau, T]$. On the other hand, taking $\varphi = (t - T)(u + M)_-$, $t \in [\tau, T]$, and repeating the above argument, we obtain that $u \geq -M$ for almost every $(x, t) \in \mathbb{R}^N \times (\tau, T)$. Therefore $\|u\|_{L^\infty(\tau, T; L^\infty(\mathbb{R}^N))} \leq M$ and the proof is completed. \hfill \Box

3.2. Existence of a $(L^2, L^2)$ pullback $\mathcal{D}_\mu$-attractor. In this subsection we prove the existence of $(L^2(\mathbb{R}^N), L^2(\mathbb{R}^N))$ pullback $\mathcal{D}_\mu$-attractor. Firstly, by using Lemma 2.18 we prove the existence of $(L^2(\mathbb{R}^N), H^s(\mathbb{R}^N) \cap L^p(\mathbb{R}^N))$ pullback $\mathcal{D}_\mu$-absorbing sets. Then following the idea presented in [40], where autonomous reaction diffusion equations are studied, we establish in addition a more important technical lemma called “Tail estimate”. This, together with Rellich-Kondrachov’s theorem, implies that the process $U(t, \tau)$ defined in (23) is pullback $\mathcal{D}_\mu$-limit-set compact.

Assume that (4) and (7) hold. Then define
\[
\theta = \min \left\{ 2, \lambda - k_1, \frac{C_1}{C_2}, \frac{\lambda - k_1 - \frac{3k_1k_2}{C_2}}{\lambda + 1} \right\}.
\]
It follows that $\theta > 0$. Now let $\mu \in (0, \theta)$ and let $g(t)$ satisfy the following assumption
\[
\int_{-\infty}^{t} e^{\mu s} \|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds < +\infty, \text{ for all } t \in \mathbb{R}.
\]

**Definition 3.6.** For any $\mu \in (0, \theta)$, we denote by $\mathcal{D}_\mu$ the class of all families of nonempty subsets $\mathcal{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2(\mathbb{R}^N))$ such that
\[
\lim_{\tau \to -\infty} e^{\mu \tau} \sup \{\|u\|_{L^2(\mathbb{R}^N)} : u \in D(\tau)\} = 0.
\]

The following result reveals that the process $U(t, \tau)$ defined in (23) possesses a family of $(L^2(\mathbb{R}^N), H^s(\mathbb{R}^N) \cap L^p(\mathbb{R}^N))$ pullback $\mathcal{D}_\mu$-absorbing sets.
Lemma 3.7. Assume that (2)–(8) and (30) hold. Then the process \( U(t, \tau) \) corresponding to (1) possesses a family \( B \) of \( (L^2(\mathbb{R}^N), H^s(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)) \) pullback \( \mathcal{D}_\mu \)-absorbing sets.

Proof. Taking \( u \) as a test function in (19) and using (4), we obtain that for almost every \( t \in (\tau, \infty) \),

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2(\mathbb{R}^N)}^2 + \|\Lambda^s u(t)\|_{L^2(\mathbb{R}^N)}^2 + (\lambda - k_1) \|u(t)\|_{L^2(\mathbb{R}^N)}^2 + C_1 \|u(t)\|_{L^p(\mathbb{R}^N)}^p \\
\leq \frac{\lambda - k_1}{2} \|u(t)\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2(\lambda - k_1)} \|g(t)\|_{L^2(\mathbb{R}^N)}^2,
\]

and this implies that for a.e. \( t \in (\tau, \infty) \),

\[
\frac{d}{dt} \|u(t)\|_{L^2(\mathbb{R}^N)}^2 + 2 \|\Lambda^s u(t)\|_{L^2(\mathbb{R}^N)}^2 + (\lambda - k_1) \|u(t)\|_{L^2(\mathbb{R}^N)}^2 \\
+ 2C_1 \|u(t)\|_{L^p(\mathbb{R}^N)}^p \\
\leq \frac{1}{\lambda - k_1} \|g(t)\|_{L^2(\mathbb{R}^N)}^2.
\]

Multiplying (32) by \( e^{\mu t} \), where \( \mu \) comes from (30), it follows that

\[
\frac{d}{dt} (e^{\mu t} \|u(t)\|_{L^2(\mathbb{R}^N)}^2) + (\lambda - k_1 - \mu) e^{\mu t} \|u(t)\|_{L^2(\mathbb{R}^N)}^2 \\
\leq \frac{e^{\mu t}}{\lambda - k_1} \|g(t)\|_{L^2(\mathbb{R}^N)}^2.
\]

Integrating the above inequality with respect to \( t \) on the interval \((\tau, t)\) and using \( \mu < \lambda - k_1 \), we have in particular

\[
\|u(t)\|_{L^2(\mathbb{R}^N)}^2 \leq e^{-\mu (t-\tau)} \|u(\tau)\|_{L^2(\mathbb{R}^N)}^2 + \frac{e^{-\mu t}}{\lambda - k_1} \int_{\tau}^{t} e^{\mu s} \|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds.
\]

On the other hand, taking \( \partial_t u \) as a test function in (19) and using (3), we deduce that

\[
\|u_t(t)\|_{L^2(\mathbb{R}^N)} + \frac{1}{2} \frac{d}{dt} \|\Lambda^s u(t)\|_{L^2(\mathbb{R}^N)}^2 + \lambda \frac{d}{dt} \|u(t)\|_{L^2(\mathbb{R}^N)}^2 \\
+ \frac{d}{dt} \int_{\mathbb{R}^N} F(u) dx \\
\leq \int_{\mathbb{R}^N} g(t) u_t dx \\
\leq \frac{1}{2} \|g(t)\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2} \|u_t(t)\|_{L^2(\mathbb{R}^N)}^2,
\]

hence

\[
\frac{d}{dt} \left( \lambda \|u\|_{L^2(\mathbb{R}^N)}^2 + \|\Lambda^s u(t)\|_{L^2(\mathbb{R}^N)}^2 + 2 \int_{\mathbb{R}^N} F(u) dx \right) \leq \|g(t)\|_{L^2(\mathbb{R}^N)}^2.
\]

Combining (32) and (34), it yields that

\[
\frac{d}{dt} \left( (\lambda + 1) \|u(t)\|_{L^2(\mathbb{R}^N)}^2 + \|\Lambda^s u(t)\|_{L^2(\mathbb{R}^N)}^2 + 2 \int_{\mathbb{R}^N} F(u) dx \right) + 2 \|\Lambda^s u(t)\|_{L^2(\mathbb{R}^N)}^2 \\
+ (\lambda - k_1) \|u(t)\|_{L^2(\mathbb{R}^N)}^2 + 2C_1 \|u(t)\|_{L^p(\mathbb{R}^N)}^p \\
\leq (\frac{1}{\lambda - k_1} + 1) \|g(t)\|_{L^2(\mathbb{R}^N)}^2,
\]
This, together with the fact that
\[
\begin{align*}
(\lambda - k_1)\|u(t)\|_{L^2(\R^N)}^2 + 2C_1\|u(t)\|_{L^p(\R^N)}^p \\
= (\lambda - k_1)\|u(t)\|_{L^2(\R^N)}^2 + \frac{2C_1}{C_2}\|u(t)\|_{L^p(\R^N)}^p \\
= (\lambda - k_1)\|u(t)\|_{L^2(\R^N)}^2 + \frac{2C_1}{C_2}\|u(t)\|_{L^p(\R^N)}^p \\
+ \frac{2C_1}{C_2}\|\tilde{k}_2\|L^2(\R^N) - \frac{2C_1}{C_2}\|u(t)\|_{L^2(\R^N)}^2 \\
= (\lambda - k_1 - \frac{2C_1\tilde{k}_2}{C_2})\|u(t)\|_{L^2(\R^N)}^2 + \frac{2C_1}{C_2}\|u(t)\|_{L^2(\R^N)}^2 + \frac{2C_1}{C_2}\int_{\R^N} F(u)dx, \\
\end{align*}
\]
implies that
\[
\begin{align*}
\frac{d}{dt} \left( (\lambda + 1)\|u(t)\|_{L^2(\R^N)}^2 + \|\Lambda^*u(t)\|_{L^2(\R^N)}^2 + 2\int_{\R^N} F(u)dx \right) \\
+ 2\|\Lambda^*u(t)\|_{L^2(\R^N)}^2 + \left( \lambda - k_1 - \frac{2C_1\tilde{k}_2}{C_2} \right)\|u(t)\|_{L^2(\R^N)}^2 + \frac{2C_1}{C_2}\int_{\R^N} F(u)dx
\leq (\frac{1}{\lambda - k_1} + 1)\|u(t)\|_{L^2(\R^N)}^2.
\end{align*}
\]
Let
\[
H(u) = (\lambda + 1)\|u(t)\|_{L^2(\R^N)}^2 + \|\Lambda^*u(t)\|_{L^2(\R^N)}^2 + 2\int_{\R^N} F(u)dx.
\]
It follows from (6) and (8) that
\[
H(u) \geq (\lambda + 1 - 2\tilde{k}_1)\|u(t)\|_{L^2(\R^N)}^2 + \|\Lambda^*u(t)\|_{L^2(\R^N)}^2 + 2C_1\|u(t)\|_{L^p(\R^N)}^p
\geq 0.
\]
Note that \(\mu \in (0, \theta)\) and
\[
\theta = \min \left\{ 2, \lambda - k_1, \frac{C_1}{C_2}, \frac{\lambda - k_1 - \frac{2C_1\tilde{k}_2}{C_2}}{\lambda + 1} \right\}.
\]
Then, by (36) and (37), we have that
\[
\begin{align*}
\frac{d}{dt} H(u) + \mu H(u) \\
\leq \frac{d}{dt} H(u) + \theta H(u) \\
\leq \frac{d}{dt} \left( (\lambda + 1)\|u(t)\|_{L^2(\R^N)}^2 + \|\Lambda^*u(t)\|_{L^2(\R^N)}^2 + 2\int_{\R^N} F(u)dx \right) \\
+ \frac{\lambda - k_1 - \frac{2C_1\tilde{k}_2}{C_2}}{\lambda + 1} (\lambda + 1)\|u(t)\|_{L^2(\R^N)}^2
\end{align*}
\]
Consequently, by Lemma 2.18, we have

\[ H(u(t + r)) \leq e^{-\frac{1}{2} \mu r \frac{2}{\tau}} \int_{t}^{t+\tau} H(u(s))ds + \left( \frac{1}{\lambda - k_1} + 1 \right) \|g(t)\|_{L^2(\mathbb{R}^N)}^2. \]  

(38)

Consequently, by Lemma 2.18 we have

\[ H(u(t + r)) \leq e^{-\frac{1}{2} \mu r \frac{2}{\tau}} \int_{t}^{t+\tau} H(u(s))ds + (\frac{1}{\lambda - k_1} + 1)e^{-\mu(t+r)} \int_{t}^{t+r} e^{\mu s}\|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds. \]  

(39)

Now it suffices to estimate the first term on the right side of the inequality (39), and we establish this as follows. Combining (32) and (35), we see that

\[ \frac{d}{dt}\|u(t)\|_{L^2(\mathbb{R}^N)}^2 + \theta H(u(t)) \leq \|u(t)\|_{L^2(\mathbb{R}^N)}^2 + 2\|\Lambda^s u(t)\|_{L^2(\mathbb{R}^N)}^2 + (\lambda - k_1)\|u(t)\|_{L^2(\mathbb{R}^N)}^2 + 2C_1\|u(t)\|_{L^p(\mathbb{R}^N)}^p \leq \frac{1}{\lambda - k_1} \|g(t)\|_{L^2(\mathbb{R}^N)}^2. \]

Integrating the above inequality with respect to time on \((t, t + \frac{\tau}{2})\) and observing that

\[ \int_{t}^{t+\frac{\tau}{2}} \|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds \leq e^{-\mu t} \int_{-\infty}^{t+\frac{\tau}{2}} e^{\mu s}\|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds, \]

we deduce that

\[ \theta \int_{t}^{t+\frac{\tau}{2}} H(u(s))ds \leq \|u(t)\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{\lambda - k_1} \int_{t}^{t+\frac{\tau}{2}} \|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds \]

\[ \leq \|u(t)\|_{L^2(\mathbb{R}^N)}^2 + \frac{e^{-\mu t}}{\lambda - k_1} \int_{-\infty}^{t+\frac{\tau}{2}} e^{\mu s}\|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds \]

\[ \leq e^{-\mu(t-\tau)}\|u(t)\|_{L^2(\mathbb{R}^N)}^2 + \frac{e^{-\mu t}}{\lambda - k_1} \int_{-\infty}^{t} e^{\mu s}\|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds \]

\[ + \frac{e^{-\mu t}}{\lambda - k_1} \int_{-\infty}^{t+\frac{\tau}{2}} e^{\mu s}\|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds \]

\[ \leq e^{-\mu(t-\tau)}\|u(t)\|_{L^2(\mathbb{R}^N)}^2 + 2e^{-\mu t}\|\Lambda^s u(t)\|_{L^2(\mathbb{R}^N)}^2 + \frac{4e^{-\mu(t+\frac{\tau}{2})}}{\theta \tau (\lambda - k_1)} \int_{-\infty}^{t+r} e^{\mu s}\|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds \]

(40)

where (33) is used to prove the third inequality. Thus the estimate of the first term on the right hand side of (39) is obtained.

Substituting (40) into (39), it follows that there exists a constant \(\tau_0 = \tau_0(t, \mathcal{D}) < t\) such that

\[ H(u(t + r)) \leq e^{-\frac{1}{2} \mu r \frac{2}{\tau}} e^{-\mu(t-\tau)}\|u(t)\|_{L^2(\mathbb{R}^N)}^2 \]

\[ + \left( \frac{1}{\lambda - k_1} + 1 \right) e^{-\mu(t+r)} + 4e^{-\mu(t+\frac{\tau}{2})} \frac{\theta \tau (\lambda - k_1)}{\lambda - k_1} \int_{-\infty}^{t+r} e^{\mu s}\|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds \]

\[ \leq C_{\mu, \theta, r, \lambda, k_1} e^{-\mu t} \left( 1 + \int_{-\infty}^{t+r} e^{\mu s}\|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds \right), \]
for any $u(\tau) \in D(\tau)$ and $\tau < \tau_0$. Let $t = t' - 1$, $r = 1$. Then the above inequality implies that

$$H(u(t')) \leq C_{\mu,\theta,\lambda,k_1} e^{-\mu t'} \left(1 + \int_{-\infty}^{t'} e^{\mu s} \|g(s)\|_{L^2(B(\tau))}^2 ds\right).$$

Substituting (37) into the above inequality, we obtain that for any $\tau < \tau_0$ and $u(\tau) \in D(\tau)$,

$$\|u(t)\|_{L^2(B(\tau))}^2 + \|A^* u(t)\|_{L^2(B(\tau))}^2 + \|u(t)\|_{L^p(B(\tau))}^p \leq C_{\mu,\theta,\lambda,k_1} e^{-\mu t} \left(1 + \int_{-\infty}^{t} e^{\mu s} \|g(s)\|_{L^2(B(\tau))}^2 ds\right).$$

We denote by $B = \{B(t) : t \in \mathbb{R}\}$ the closed ball in phase space $H^s(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ with center 0 and radius $(R(t))^\frac{2}{p}$, where

$$R(t) = C_{\mu,\theta,\lambda,k_1} e^{-\mu t} \left(1 + \int_{-\infty}^{t} e^{\mu s} \|g(s)\|_{L^2(B(t))}^2 ds\right) < +\infty \text{ for all } t \in \mathbb{R}.$$

Therefore $B = \{B(t) : t \in \mathbb{R}\}$ is a class of pullback $\mathcal{D}_\mu$-absorbing sets in $H^s(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ and the proof is completed.

Since the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ is only continuous but not compact, the following “Tail estimate” is necessary. This, together with Rellich-Kondrachov’s theorem, implies that the process $U$ defined in (23) is pullback $\mathcal{D}_\mu$-limit-set compact.

**Lemma 3.8** (Tail estimate). Assume that (2), (7), and (30) hold. Let $\varepsilon > 0$, $t \in \mathbb{R}$ and let $D$ denote the set $\{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\mu$, where $\mathcal{D}_\mu$ is given in Definition 3.6. Then there exist positive constants $k = k(\int_{-\infty}^{0} e^{\mu s} \|g(s)\|_{L^2(B(\tau))}^2 ds, \lambda, k_1, \mu, t)$ and $\tau_0 = \tau_0(\varepsilon, \mu, t, D)$ such that

$$\int_{(B_k)^c} [U(t, \tau) u_\tau(x)]^2 dx < \varepsilon \text{ for all } \tau < \tau_0 \text{ and } u_\tau \in D(\tau),$$

where $B_k = \{x \in \mathbb{R}^N : |x| < k\}$, $(B_k)^c$ is the complement of $B_k$ in $\mathbb{R}^N$ and $U(t, \tau) u_\tau$ is the weak solution of (4).

**Proof.** Let $\eta(x) \in C^\infty(\mathbb{R}^N)$, $0 \leq \eta \leq 1$, $|\eta'| \leq 3$ and

$$\eta(x) = \begin{cases} 1, & |x| \geq 1, \\ 0, & |x| \leq \frac{1}{2}, \end{cases} \quad (42)$$

We define $\eta_k(x) = \eta(\frac{x}{k})$, it is obvious that $1 - \eta_k \in C^\infty_0(\mathbb{R}^N)$. Let $t$ be an arbitrary fixed time and $u(t) \in H^s(\mathbb{R}^N)$, we deduce that $\eta_k u(t) = u(t) - (1 - \eta_k) u(t) \in H^s(\mathbb{R}^N)$ by Lemma 2.15. Taking $\eta_k u$ as a test function in (19) and using (4), we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} u^2(t) \eta_k dx + (\Lambda^{2s} u, \eta_k)_{H^{-s}(\mathbb{R}^N), H^s(\mathbb{R}^N)} + (\lambda - k_1) \int_{\mathbb{R}^N} u^2(t) \eta_k dx \leq \frac{\lambda - k_1}{2} \int_{\mathbb{R}^N} u^2(t) \eta_k dx + \frac{1}{2(\lambda - k_1)} \|g(t)\|_{L^2(\mathbb{R}^N)}^2.$$ 

This implies that

$$\frac{d}{dt} \int_{\mathbb{R}^N} u^2(t) \eta_k dx + (\lambda - k_1) \int_{\mathbb{R}^N} u^2(t) \eta_k dx \leq -(\Lambda^{2s} u, \eta_k)_{H^{-s}(\mathbb{R}^N), H^s(\mathbb{R}^N)} + \frac{1}{\lambda - k_1} \|g(t)\|_{L^2(\mathbb{R}^N)}^2, \quad (43)$$
It has been proved that (see e.g. (4.7)–(4.8) in [18, p.484] )

\[(\Lambda^{2s} u, u\eta_{k})_{H^{-s}(\mathbb{R}^N), H^{s}(\mathbb{R}^N)} \geq \frac{1}{2} \int_{\mathbb{R}^N} u^2 \Lambda^{2s} \eta_{k} dx, \tag{44}\]

and that

\[|\Lambda^{2s} \eta_{k}| \leq \frac{C}{k^{2s}}, \tag{45}\]

for some constant \(C\) depending only on \(N, s, \eta\). Combining (43), (44) and (45), it yields that

\[\frac{d}{dt} \int_{\mathbb{R}^N} u^2(t) \eta_{k} dx + (\lambda - k_1) \int_{\mathbb{R}^N} u^2(t) \eta_{k} dx \leq \frac{C}{k^{2s}} \int_{\mathbb{R}^N} u^2(t) \eta_{k} dx + \frac{1}{\lambda - k_1} \|g(t) \eta_{k}\|_{L^2(\mathbb{R}^N)}^2. \]

Multiplying by \(e^{(\lambda-k_1)t}\), it follows that, for each pair \(\tau, t\) with \(-\infty < \tau < t < \infty\),

\[\frac{d}{dt} \left( e^{(\lambda-k_1)t} \int_{\mathbb{R}^N} u^2(t) \eta_{k} dx \right) \leq \frac{C}{k^{2s}} e^{(\lambda-k_1)t} \|u(t)\|_{L^2(\mathbb{R}^N)}^2 + \frac{e^{(\lambda-k_1)t}}{\lambda - k_1} \|g(t) \eta_{k}\|_{L^2(\mathbb{R}^N)}^2. \]

Integrating the above inequality with respect to time on \((\tau, t)\), we have

\[\int_{\mathbb{R}^N} u^2(t) \eta_{k} dx \leq e^{-(\lambda-k_1)(t-\tau)} \int_{\mathbb{R}^N} u^2(\tau) \eta_{k} dx + \frac{C e^{-(\lambda-k_1)t}}{k^{2s}} \int_{\tau}^{t} e^{(\lambda-k_1)r} \|u(r)\|_{L^2(\mathbb{R}^N)}^2 dr + \frac{e^{-(\lambda-k_1)t}}{\lambda - k_1} \int_{\tau}^{t} e^{(\lambda-k_1)r} \|g(r) \eta_{k}\|_{L^2(\mathbb{R}^N)}^2 dr \tag{46}\]

\[\leq e^{-(\lambda-k_1)(t-\tau)} \|u(\tau)\|_{L^2(\mathbb{R}^N)}^2 + \frac{C e^{-(\lambda-k_1)t}}{k^{2s}} \int_{\tau}^{t} e^{(\lambda-k_1)r} \|u(r)\|_{L^2(\mathbb{R}^N)}^2 dr + \frac{e^{-(\lambda-k_1)t}}{\lambda - k_1} \int_{\tau}^{t} e^{(\lambda-k_1)r} \|g(r) \eta_{k}\|_{L^2(\mathbb{R}^N)}^2 dr \]

\[: = I_1(\tau) + I_2(\tau) + I_3(\tau). \]

Now we estimate \(I_1(\tau), I_2(\tau), I_3(\tau)\) one by one. Firstly, since \(t - \tau > 0\) and \(\mu \in (0, \theta)\), where \(\theta\) is defined in [29], we have

\[e^{-(\lambda-k_1)(t-\tau)} < e^{-\mu(t-\tau)}. \tag{47}\]

Thus

\[I_1(\tau) = e^{-(\lambda-k_1)(t-\tau)} \|u(\tau)\|_{L^2(\mathbb{R}^N)}^2 < e^{-\mu(t-\tau)} \|u(\tau)\|_{L^2(\mathbb{R}^N)}^2.\]

By Definition [3.6] for the fixed \(\varepsilon\) and \(t\), we easily obtain that there exists \(\tau_0 = \tau_0(\varepsilon, \mu, t, \mathcal{D}) < t\) such that \(I_1(\tau) < \frac{\varepsilon}{2}\) for all \(\tau < \tau_0\) and \(u(\tau) \in D(\tau)\).

For \(I_3(\tau)\), using [47] we obtain

\[I_3(\tau) = \frac{e^{-(\lambda-k_1)t}}{\lambda - k_1} \int_{\tau}^{t} e^{(\lambda-k_1)r} \|g(r) \eta_{k}\|_{L^2(\mathbb{R}^N)}^2 dr \leq \frac{e^{-\mu t}}{\lambda - k_1} \int_{\tau}^{t} e^{\mu r} \|g(r) \eta_{k}\|_{L^2(\mathbb{R}^N)}^2 dr \leq \frac{e^{-\mu t}}{\lambda - k_1} \int_{-\infty}^{t} e^{\mu r} \|g(r) \eta_{k}\|_{L^2(\mathbb{R}^N)}^2 dr.\]
Then it follows from (30) and (42) that there exists a constant \( k = k(\varepsilon, \lambda, k_1, \mu, t, \int_0^t e^{\mu s}\|g(s)\|^2_{L^2(\mathbb{R}^N)}ds > 0 \) such that \( I_3(\tau) < \frac{\varepsilon}{2} \).

Finally, using (33), we have
\[
I_2(\tau) \leq \frac{Ce^{-\lambda k_1 t}}{k^{2s}} \int_{t}^{\lambda k_1 t} e^{\lambda k_1 r} \left( e^{-\mu (r-t)} \|u(\tau)\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{\lambda - k_1} \int_{-\infty}^{t} e^{\mu s} \|g(s)\|^2_{L^2(\mathbb{R}^N)} ds \right) dr
\]
\[
\leq \frac{Ce^{-\lambda k_1 t}}{k^{2s}(\lambda - k_1 - \mu)} \left( e^{\mu t} \|u(\tau)\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{\lambda - k_1} \int_{-\infty}^{t} e^{\mu s} \|g(s)\|^2_{L^2(\mathbb{R}^N)} ds \right).
\]

Therefore there exists \( k = k(\varepsilon, \lambda, k_1, \mu, t, \int_0^t e^{\mu s}\|g(s)\|^2_{L^2(\mathbb{R}^N)}ds > 0 \) such that \( I_2(\tau) < \frac{\varepsilon}{2} \) and the result is proved.

**Proof of Theorem 1.1.** By Theorem 2.7, Lemmas 3.4 and 3.7, it is sufficient to prove that \( U(t, \tau) \) is pullback \( \mathcal{D}_\mu \)-limit-set compact in \( L^2(\mathbb{R}^N) \). That is, for any \( \varepsilon > 0 \) and any \( \mathcal{D} \in \mathcal{D}_\mu \), there exists \( \tau_{\varepsilon, \mathcal{D}} < t \) such that
\[
\alpha \left( \bigcup_{\tau \leq \tau_{\varepsilon, \mathcal{D}}} U(t, \tau) D(\tau) \right) \leq \varepsilon.
\]

Let \( B_k = \{ x \in \mathbb{R}^N : |x| < k \} \). We denote the characteristic function of \( B_k \) by \( \chi_{B_k} \). For any \( \varepsilon > 0 \) and any \( \mathcal{D} \in \mathcal{D}_\mu \), by applying Lemma 3.8 and (33), there exist \( k = k(\varepsilon, \lambda, k_1, \mu, t, \int_0^t e^{\mu s}\|g(s)\|^2_{L^2(\mathbb{R}^N)}ds \) and \( \tau_0 = \tau_0(\varepsilon, \mu, t, \mathcal{D}) \) such that
\[
\int_{|x| > k} |U(t, \tau) u_\tau(x)|^2 dx < \frac{\varepsilon}{2} \text{ for all } \tau < \tau_0 \text{ and all } u_\tau \in D(\tau),
\]
which implies that
\[
\alpha \left( \bigcup_{\tau \leq \tau_0} \bigcup_{u_\tau \in D(\tau)} U(t, \tau) u_\tau (1 - \chi_{B_k}) \right) \leq \frac{\varepsilon}{2}.
\]

On the other hand, for any \( \mathcal{D} \in \mathcal{D}_\mu \), Lemma 3.7 implies that \( \bigcup_{\tau \leq \tau_0} U(t, \tau) D(\tau) \subset B(t) \), that is, \( \bigcup_{\tau \leq \tau_0} U(t, \tau) D(\tau) \) is bounded in \( H^s(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \), where \( B(t) \) (\( t \in \mathbb{R} \)) is given in the proof of Lemma 3.7. Therefore \( \bigcup_{\tau \leq \tau_0} \bigcup_{u_\tau \in D(\tau)} U(t, \tau) u_\tau (x) \chi_{B_k}(x) \) is also bounded in \( H^s(B_k) \cap L^p(B_k) \). Then by Rellich-Kondrachov’s theorem we have
\[
\alpha \left( \bigcup_{\tau \leq \tau_0} \bigcup_{u_\tau \in D(\tau)} U(t, \tau) D(\tau) \right) < \frac{\varepsilon}{2}.
\]

Therefore
\[
\alpha \left( \bigcup_{\tau \leq \tau_0} U(t, \tau) D(\tau) \right) \leq \alpha \left( \bigcup_{\tau \leq \tau_0} U(t, \tau) D(\tau) \chi_{B_k} \right) + \alpha \left( \bigcup_{\tau \leq \tau_0} U(t, \tau) D(\tau)(1 - \chi_{B_k}) \right)
\]
\[
\leq \frac{\varepsilon}{2}.
\]
\[ = \alpha \left( \bigcup_{\tau \leq t_0} \bigcup_{u_r \in D(\tau)} U(t, \tau) u_\tau(1 - \chi_{B_k}) \right) + \alpha \left( \bigcup_{\tau \leq t_0} \bigcup_{u_r \in D(\tau)} U(t, \tau) u_\tau(1 - \chi_{B_k}) \right) \]
\[ \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}. \]

Setting \( \tau, \sigma = t_0 \), we have \( U(t, \tau) \) is pullback \( \mathcal{D}_\mu \)-limit set compact in \( L^2(\mathbb{R}^N) \), as required. The result is proved. \( \square \)

**Remark 1.** The approach presented in the proof of Theorem 1.1 is rather classical. We point out that the pre-compactness of \( U(t, \tau) \) can be also obtained by the energy method. We refer the interested reader to J. M. Ball [5] for more details on the energy method.

4. **The pullback attraction in \( L^{2+\delta}(\mathbb{R}^N) \)-norm.** In this section, we study the properties of the pullback attractor obtained in Theorem 1.1. In fact, motivated by the results in \( 13, 33 \), we prove that for any \( \delta > 0 \), the \( (L^2, L^2) \) pullback \( \mathcal{D}_\mu \)-attractor obtained in Theorem 1.1 indeed attract the \( \mathcal{D} \) by the results in \( 13, 33 \). We prove that for any \( \delta > 0 \), the \( (L^2, L^2) \) pullback \( \mathcal{D}_\mu \)-attractor obtained in Theorem 1.1 indeed attract the \( \mathcal{D}_\mu \)-class in \( L^{2+\delta}(\mathbb{R}^N) \)-norm. At the beginning, we establish an a priori estimate, which is called the higher-order integrability, for the difference of solutions near the initial time.

4.1. **Higher-order integrability near the initial time.**

**Lemma 4.1.** Let \( u^i \) be the weak solution of the problem (1)–(5) corresponding to the initial data \( u^i_0(i = 1, 2) \). We denote by \( w \) the difference of \( u^i \), that is, \( w(t) = u^1(t) - u^2(t) = U(t, \tau) u^1_\tau - U(t, \tau) u^2_\tau \). Then for any \( T > \tau \) and any \( k = 1, 2, \ldots \), there exists a positive constant \( M_k = M(T - \tau, N, l, k, \lambda, s) \|w(t)\|_{L^2(\mathbb{R}^N)}^k \), such that
\[ (t - \tau)^{\frac{N}{2 - \sigma}} \| (t - \tau)^b w(t) \|_{L^2(\mathbb{R}^N)}^k \leq M_k \text{ for all } t \in [\tau, T], \quad (A_k) \]
and
\[ \int_\tau^T \left( \int_{\mathbb{R}^N} |(t - \tau)^{b_k} w(t)|^2 \frac{dt}{(\mathbb{R}^N)^k} \right)^{\frac{N}{N - 2s}} dt \leq M_k, \quad (B_k) \]
where
\[ b_1 = 1 + \frac{1}{2}, \quad b_2 = b_1 + 1, \quad b_{k+1} = b_k + \frac{1 + \frac{N}{2s}}{2(\frac{N}{2s} - 1)} k, \quad \text{for } k = 2, 3, \ldots . \quad (48) \]

**Remark 2.** This result states that for any \( k \in \mathbb{Z}^+ \), the difference of two solutions is decay in \( L^{2(\frac{N}{2s})^k}(\mathbb{R}^N) \)-norm near the initial time.

**Proof.** **Step 1.** Approximate solutions.

Since the embedding \( L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \) and \( L^\infty(\tau, T; L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)) \hookrightarrow L^2(\tau, T; L^2(\mathbb{R}^N)) \) are both dense, it follows that there exist sequences \( \{u^i_{\tau m}\}_{m=1}^\infty(i = 1, 2) \) and \( \{g_m\}_{m=1}^\infty \) satisfying
\[ u^i_{\tau m} \in L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N), \]
\[ g_m \in L^\infty(\tau, T; L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)), \]
and
\[ u^i_{\tau m} \to u^i_\tau \quad \text{in } L^2(\mathbb{R}^N) \text{ as } m \to \infty, \quad i = 1, 2, \]
\[ g_m \to g \quad \text{in } L^2(\tau, T; L^2(\mathbb{R}^N)) \text{ as } m \to \infty . \quad (49) \]
By Lemmas 3.4 and 3.5 if both \( u_\tau = u_{\tau m} \) and \( g = g_m \) hold, then the equation (1) exists a unique weak solution \( u^m_m(t) = U(t, \tau)u_{\tau m} \) with the properties

\[ u^m_m \in L^\infty(\tau, T; L^\infty(\mathbb{R}^N)), \]

and for \( i = 1, 2, \)

\[ u^i_m \rightarrow u^i \text{ in } L^2(\tau, T; L^2(\mathbb{R}^N)) \text{ as } m \rightarrow \infty, \]

where \( u^i \) is the weak solution of the problem [1–5] corresponding to the initial data \( u^i_\tau \) and forcing term \( g(t) \).

**Step 2.** \( A_k \) and \( B_k \) are both true for approximate solutions.

We prove that \( A_k \) and \( B_k \) are both true for \( w_m \) for all \( m \in \mathbb{Z}^+ \). Let \( w_m(t) = u^1_m(t) - u^2_m(t) \). We deduce from (50) and Lemma 2.17 that for any \( \theta \geq 2 \), the element \( (w_m(t))^{\theta-2}w_m(t) \) belongs to \( L^\infty(\mathbb{R}^N) \cap H^\infty(\mathbb{R}^N) \) for almost every \( t \in (\tau, T) \).

It is obvious that \( w_m(t) \) is the solution of the following equation

\[
\begin{cases}
\partial_t w_m + \Lambda^2 w_m + \lambda w_m + f(u^1_m) - f(u^2_m) = 0, & \text{in } \mathbb{R}^N \times (\tau, T), \\
|w_m(\tau) = u^1_\tau - u^2_\tau, & \text{for } x \in \mathbb{R}^N.
\end{cases}
\]

We prove the result by induction on \( k \).

**Case 1.** \( k = 1 \). Firstly, multiplying (52) by \( w \) and integrating over \( \mathbb{R}^N \), using (5) and definition of \( H^\infty(\mathbb{R}^N) \) in Section 2.3, we obtain that

\[
\frac{1}{2} \frac{d}{dt} \| w_m(t) \|_{L^2(\mathbb{R}^N)}^2 + c_\lambda \| w_m(t) \|_{H^\infty(\mathbb{R}^N)}^2 \leq l \| w_m(t) \|_{L^2(\mathbb{R}^N)}^2, \quad \text{for a.e. } t \in (\tau, T),
\]

where \( c_\lambda = \min\{1, \lambda\} > 0 \). Integrating the above inequality on \( (\tau, t) \), it follows that

\[
\| w_m(t) \|_{L^2(\mathbb{R}^N)}^2 \leq e^{2(l-t)} \| w_m(\tau) \|_{L^2(\mathbb{R}^N)}^2, \quad \forall t \in (\tau, T),
\]

and

\[
c_\lambda \int_{\tau}^{T} \| w_m(t) \|_{H^\infty(\mathbb{R}^N)}^2 dt \leq \frac{1}{2} \| w_m(\tau) \|_{L^2(\mathbb{R}^N)}^2 + l \int_{\tau}^{T} \| w_m(r) \|_{L^2(\mathbb{R}^N)}^2 dr
\]

\[
\leq \frac{1}{2} \| w_m(\tau) \|_{L^2(\mathbb{R}^N)}^2 + l \int_{\tau}^{T} e^{2(l-t)} \| w_m(\tau) \|_{L^2(\mathbb{R}^N)}^2 dr
\]

\[
= \frac{e^{2(T-\tau)}}{2} \| w_m(\tau) \|_{L^2(\mathbb{R}^N)}^2.
\]

By Lemma 2.12 we have

\[
\int_{\tau}^{T} \| w_m(r) \|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}^2 dr \leq c_\lambda^2 \| w_m(\tau) \|_{L^2(\mathbb{R}^N)}^2,
\]

which implies that

\[
\int_{\tau}^{T} \| (r - \tau)^{b_1} w_m(r) \|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}^2 dr = \int_{\tau}^{T} (r - \tau)^{2b_1} \| w_m(r) \|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}^2 dr
\]

\[
\leq (T - \tau)^{2b_1} \int_{\tau}^{T} \| w_m(r) \|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}^2 dr
\]

\[
\leq (T - \tau)^{2b_1} c_\lambda^2 \| w_m(\tau) \|_{L^2(\mathbb{R}^N)}^2.
\]

Secondly, we multiply (52) by \( \| w_m \|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}^{-2} \cdot w_m \) and then integrate on \( \mathbb{R}^N \). By Lemma 2.19 and (5), it follows that for a.e. \( t \in (\tau, T) \) (here we let \( c_i (i = 1, 2, 3, 4) \)
Multiplying (54) by \((t - \tau)^b_1 \frac{2N}{\nu_2}\), and combining with the fact that
\[
\frac{d}{dt} \| (t - \tau)^b_1 w_m(t) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}^2 = \frac{d}{dt} \left( (t - \tau)^b_1 \frac{2N}{\nu_2} \| w_m(t) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}^2 \right)
\]
\[
= \left( t - \tau \right)^b_1 \frac{2N}{\nu_2} \frac{d}{dt} \| w_m(t) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}
\]
\[
+ b_1 \left( \frac{2N}{N - 2s} \right) (t - \tau)^b_1 \frac{2N}{\nu_2} - 1 \| w_m(t) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}^2,
\]
we deduce that
\[
\frac{d}{dt} \| (t - \tau)^b_1 w_m(t) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}^2 + c_1 \| (t - \tau)^b_1 w_m(t) \|_{\frac{2N}{\nu_2} H^s(\mathbb{R}^N)}^2 
\leq c_2 \| (t - \tau)^b_1 w_m(t) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}^2 + c_3 (t - \tau)^b_1 \frac{2N}{\nu_2} - 1 \| w_m(t) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}^2
\]
\[
\leq c_4 (1 + (t - \tau)^{-1}) \| (t - \tau)^b_1 w_m(t) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}^2.
\]
Multiplying by \(t - \tau\), for a.e. \(t \in (\tau, T)\), we see that
\[
(t - \tau) \frac{d}{dt} \| (t - \tau)^b_1 w_m(t) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}^2 \leq c_4 (T - \tau + 1) \| (t - \tau)^b_1 w_m(t) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}^2.
\]
Hence for a.e. \(t \in (\tau, T)\),
\[
(t - \tau) \frac{d}{dt} \| (t - \tau)^b_1 w_m(t) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}^2 \leq c_4 (T - \tau + 1) \| (t - \tau)^b_1 w_m(t) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}^2.
\]
Integrating the above inequality on \([\tau, t]\), we obtain that for any \(t \in (\tau, T)\),
\[
\int_{\tau}^{t} \left( r - \tau \right) \frac{d}{dr} \| (r - \tau)^b_1 w_m(r) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}^2 dr
\]
\[
= (t - \tau) \| (t - \tau)^b_1 w_m(t) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}^2 - \int_{\tau}^{t} \| (r - \tau)^b_1 w_m(r) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}^2 dr
\]
\[
\leq c(T - \tau + 1) \int_{\tau}^{t} \| (r - \tau)^b_1 w_m(r) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}^2 dr.
\]
Substituting (53) into (56), it follows that for any \(t \in (\tau, T)\),
\[
(t - \tau) \| (t - \tau)^b_1 w_m(t) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}^2 \leq c(T - \tau + 2) \int_{\tau}^{t} \| (r - \tau)^b_1 w_m(r) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)}^2 dr
\]
\[
\leq c(T - \tau + 2)(T - \tau)^{2b_1} c_{s,N}^2 c_{2L^2(T - \tau)} w_m(t) \|_{L^2(\mathbb{R}^N)}^2,
\]
which implies that for any \(t \in (\tau, T)\),
\[
(t - \tau) \frac{2N}{\nu_2} \| (t - \tau)^b_1 w_m(t) \|_{\frac{2N}{\nu_2} L^{\frac{2N}{\nu_2}}(\mathbb{R}^N)} \leq C_{T - \tau, N, s, l, b_1} \| w_m(t) \|_{L^2(\mathbb{R}^N)}^2.
\]
Let
\[
M'_{1m} = C_{T - \tau, N, s, l, b_1} \| w_m(t) \|_{L^2(\mathbb{R}^N)}^2.
\]
Then for any $t \in [\tau, T]$,
\[
(t - \tau) \frac{2N}{N - 2s} \left\| (t - \tau)^{b_1} w_m(t) \right\|_{L^\frac{2N}{N - 2s}(\mathbb{R}^N)} \leq M'_1 m,
\]
thus (A1) is proved. In order to prove (B1), we multiply (55) by $t - \tau \frac{2N}{N - 2s}$, and thus for a.e. $t \in (\tau, T)$
\[
(t - \tau) \frac{2N}{N - 2s} \frac{d}{dt} \left\| (t - \tau)^{b_1} w_m(t) \right\|_{L^\frac{2N}{N - 2s}(\mathbb{R}^N)} + t - \tau \frac{2N}{N - 2s} c_1 \left\| (t - \tau)^{b_1} w_m(t) \right\|_{L^\frac{2N}{N - 2s}(\mathbb{R}^N)}^2 \leq c_4 \left( (t - \tau) \frac{2N}{N - 2s} + (t - \tau) \frac{2N}{N - 2s} - 1 \right) \left\| (t - \tau)^{b_1} w_m(t) \right\|_{L^\frac{2N}{N - 2s}(\mathbb{R}^N)}^2 + c(T - \tau, N) M'_1 m \leq c(T - \tau, N, s) M'_1 m,
\]
where we use (57) to prove the last inequality. Integrating the above inequality on $[\tau, T]$, it follows that
\[
\int_\tau^T c_1 \left\| (t - \tau)^{b_1 + 1} w_m(t) \right\|_{L^\frac{2N}{N - 2s}(\mathbb{R}^N)}^2 \, dt \leq \int_\tau^T \frac{2N}{N - 2s} (t - \tau) \frac{2N}{N - 2s} - 1 \left\| (t - \tau)^{b_1} w_m(t) \right\|_{L^\frac{2N}{N - 2s}(\mathbb{R}^N)} \, dt + c(T - \tau, N) M'_1 m \leq cM'_1 m \int_\tau^T \frac{2N}{N - 2s} - 1 \frac{N}{N - 2s} \, dt + c(T - \tau, N, s) M'_1 m \leq c(T - \tau, N, s) M'_1 m,
\]
where (57) is used again in the proof of the second inequality. Hence it follows from the Sobolev inequality that
\[
\int_\tau^T \left( \int_{\mathbb{R}^N} \left\| (t - \tau)^{b_2} w_m(t) \right\|^{2l} \frac{N}{N - 2s} \, dx \right)^{\frac{N - 2s}{N}} \, dt \leq c_{N, s} \int_\tau^T \left\| (t - \tau)^{b_1 + 1} w_m(t) \right\|_{L^\frac{2N}{N - 2s}(\mathbb{R}^N)}^2 \, dt \leq c_{N, \lambda, s} C_{T - \tau, N, s} M'_1 m.
\]
Take $M'_1 m$ to be $\max\{M'_1 m, c_{N, \lambda, s} C_{T - \tau, N, s} M'_1 m\}$. Then the proof of (A1) and (B1) is completed by (57) and (58).

**Case 2.** $k > 1$. Assume that (A$k$) and (B$k$) are both true. In the following we show that (A$k+1$) and (B$k+1$) are also true.

We multiply (52) by $|w_m|^{2\frac{N}{N - 2s} - k - 1} w_m$ and integrate on $\mathbb{R}^N$. Then applying Lemma 2.19 and (5), we obtain that for a.e. $t \in (\tau, T)$
\[
\frac{d}{dt} \left\| w_m(t) \right\|_{L^\frac{2N}{N - 2s}^{k+1}(\mathbb{R}^N)} + c_1 \left\| w_m(t) \right\|_{L^\frac{2N}{N - 2s}^{k+1}(\mathbb{R}^N)} \leq c_2 t \left\| w_m(t) \right\|_{L^\frac{2N}{N - 2s}^{k+1}(\mathbb{R}^N)}.
\]

(59)
Multiplying (59) by \((t - \tau)^{2(N-\alpha)} \tau^{k+k+1}\), and using the fact that
\[
\frac{d}{dt} \left\| (t - \tau)^{k+k+1} w_m(t) \right\|^{2(N-\alpha)}_{L^{2(N-\alpha)}} (R^N) = \frac{d}{dt} \left( (t - \tau)^{2(N-\alpha)} \tau^{k+k+1} \right) \left\| w_m(t) \right\|^{2(N-\alpha)}_{L^{2(N-\alpha)}} (R^N) = (t - \tau)^{2(N-\alpha)} \tau^{k+k+1} \frac{d}{dt} \left\| w_m(t) \right\|^{2(N-\alpha)}_{L^{2(N-\alpha)}} (R^N) + 2 \left( \frac{N}{N-2\alpha} \right)^{k+k+1} (t - \tau)^{2(N-\alpha)} \tau^{k+k+1} \left\| w_m(t) \right\|^{2(N-\alpha)}_{L^{2(N-\alpha)}} (R^N),
\]
we deduce that
\[
\frac{d}{dt} \left\| (t - \tau)^{k+k+1} w_m(t) \right\|^{2(N-\alpha)}_{L^{2(N-\alpha)}} (R^N) + c_1 \left\| (t - \tau)^{k+k+1} w_m(t) \right\|^{2(N-\alpha)}_{H^1(R^N)} 
\leq c_2 \left\| (t - \tau)^{k+k+1} w_m(t) \right\|^{2(N-\alpha)}_{L^{2(N-\alpha)}} (R^N) + c_3 (t - \tau)^{2(N-\alpha)} \tau^{k+k+1} \left\| w_m(t) \right\|^{2(N-\alpha)}_{L^{2(N-\alpha)}} (R^N) \leq c_1, N, k, \alpha, (1 + (t - \tau)^{-1}) \left\| (t - \tau)^{k+k+1} w_m(t) \right\|^{2(N-\alpha)}_{L^{2(N-\alpha)}} (R^N), \tag{60}
\]
Multiplying (60) by \(t - \tau\), it follows that for a.e. \(t \in (\tau, T)\)
\[
(t - \tau) \frac{d}{dt} \left\| (t - \tau)^{k+k+1} w_m(t) \right\|^{2(N-\alpha)}_{L^{2(N-\alpha)}} (R^N) \leq c_T, T, l, N, k, \lambda, s, \left\| (t - \tau)^{k+k+1} w_m(t) \right\|^{2(N-\alpha)}_{L^{2(N-\alpha)}} (R^N), \tag{61}
\]
which yields that for a.e. \(t \in (\tau, T)\)
\[
(t - \tau) \frac{d}{dt} \left\| (t - \tau)^{k+k+1} w_m(t) \right\|^{2(N-\alpha)}_{L^{2(N-\alpha)}} (R^N) \leq c_T, T, l, N, k, \lambda, s, \left\| (t - \tau)^{k+k+1} w_m(t) \right\|^{2(N-\alpha)}_{L^{2(N-\alpha)}} (R^N), \tag{61}
\]
Integrating (61) on \([\tau, t]\) and using \((B_k)\), we deduce that for all \(t \in [\tau, T]\)
\[
(t - \tau) \left\| (t - \tau)^{k+k+1} w_m(t) \right\|^{2(N-\alpha)}_{L^{2(N-\alpha)}} (R^N) \leq c_T, T, l, N, k, \lambda, s, + 1 \int_{\tau}^{T} \left\| (t - \tau)^{k+k+1} w_m(t) \right\|^{2(N-\alpha)}_{L^{2(N-\alpha)}} (R^N) dt \leq (c_T, T, l, N, k, \lambda, s, + 1) M_{km}.
\]
This implies that for all \(t \in [\tau, T]\)
\[
(t - \tau)^{\frac{N}{2\alpha}} \left\| (t - \tau)^{k+k+1} w_m(t) \right\|^{2(N-\alpha)}_{L^{2(N-\alpha)}} (R^N) \leq (c_T, T, l, N, k, \lambda, s, + 1) M_{km} \frac{N}{2\alpha}, \tag{62}
\]
consequently we have proved \((A_{k+1})\).
In order to prove \( (B_{k+1}) \), multiplying (60) by \( (t - \tau)^{1 + \frac{N}{2N - 2} - \frac{N}{2}} \), it follows that
\[
(t - \tau)^{1 + \frac{N}{2N - 2} - \frac{N}{2}} \frac{d}{dt} \left\| (t - \tau)^{b_{k+1}} w_m(t) \right\|_{L^2(\mathbb{R}^N)}^{2(\frac{N}{2N - 2})^{k+1} + 1} + c_1 \left\| (t - \tau)^{b_{k+1}} w_m(t) \right\| \left( \frac{N}{2N - 2} \right)^{k+1} \right\|_{H^s(\mathbb{R}^N)}^2 \]
\[
\leq c_{l, N, \lambda, s} (T - \tau + 1) (t - \tau)^{\frac{N}{2N - 2}} \| (t - \tau)^{b_{k+1}} w_m(t) \|_{L^2(\mathbb{R}^N)^{k+1}}^{2(\frac{N}{2N - 2})^{k+1}}.
\] (63)

Substituting (62) into (63), we have that a.e. \( t \in [\tau, T] \),
\[
(t - \tau)^{1 + \frac{N}{2N - 2} - \frac{N}{2}} \frac{d}{dt} \left\| (t - \tau)^{b_{k+1}} w_m(t) \right\|_{L^2(\mathbb{R}^N)}^{2(\frac{N}{2N - 2})^{k+1} + 1} + c_1 \left\| (t - \tau)^{b_{k+1}} w_m(t) \right\| \left( \frac{N}{2N - 2} \right)^{k+1} \right\|_{H^s(\mathbb{R}^N)}^2 \]
\[
\leq c_{T - \tau, l, N, \lambda, s} M_\text{km}^{\frac{N}{2N - 2}}.
\]

Integrating the above inequality on \([\tau, T]\) and using (62) again, we have that
\[
\int_\tau^T \left\| (t - \tau)^{b_{k+2}} w_m(t) \right\|_{L^2(\mathbb{R}^N)^{k+1}}^{2(\frac{N}{2N - 2})^{k+1}} \, dt \leq c_{T - \tau, l, N, \lambda, s} M_\text{km}^{\frac{N}{2N - 2}}.
\]

Then applying the Sobolev embedding once more, we obtain that
\[
\int_\tau^T \left( \int_{\mathbb{R}^N} \left\| (t - \tau)^{b_{k+2}} w_m(t) \right\|_{L^2(\mathbb{R}^N)^{k+1}}^{2(\frac{N}{2N - 2})^{k+2}} \right) \frac{N - 2}{N} \, dx \, dt \leq c_{T - \tau, l, N, \lambda, s} M_\text{km}^{\frac{N}{2N - 2}}.
\] (64)

Therefore by setting
\[
M_{(k+1)m} = \max\{ (c_{T - \tau, l, N, \lambda, s} + 1) M_\text{km} \}^{\frac{N}{2N - 2}}, c_{T - \tau, l, N, \lambda, s} M_\text{km}^{\frac{N}{2N - 2}} \text{,}
\]
(62) and (64) imply immediately that \( (A_{k+1}) \) and \( (B_{k+1}) \) hold respectively.

Therefore, combining Case 1 and Case 2, we obtain that \( A_k \) and \( B_k \) are both true for approximate solutions. That is, for any \( T > \tau \) and any \( k = 1, 2, \cdots \), there exists a positive constant \( M_\text{km} = M(T - \tau, N, l, k, \lambda, s) \left\| w_m(\tau) \right\|_{L^2(\mathbb{R}^N)^k} \), such that
\[
(t - \tau)^{\frac{N}{2N - 2}} \left\| (t - \tau)^{b_k} w_m(t) \right\|_{L^2(\mathbb{R}^N)^{k+1}}^{2(\frac{N}{2N - 2})^{k+1}} \leq M_\text{km} \text{ for all } t \in [\tau, T], \quad (A_{km})
\]
and
\[
\int_\tau^T \left( \int_{\mathbb{R}^N} \left\| (t - \tau)^{b_{k+1}} w_m(t) \right\|_{L^2(\mathbb{R}^N)^{k+1}}^{2(\frac{N}{2N - 2})^{k+2}} \right) \frac{N - 2}{N} \, dx \, dt \leq M_\text{km}, \quad (B_{km})
\]
where \( b_k \) are given in (48).

**Step 3.** Estimates for weak solutions.

By (49), we have
\[
\left\| w_m(\tau) \right\|_{L^2(\mathbb{R}^N)} \leq \left\| w(\tau) \right\|_{L^2(\mathbb{R}^N)}^2 + 1.
\] (65)

Furthermore, it follows from (51) that there exists a subsequence \( \{w_m_j \} \) and still denoted by \( \{w_m \} \) without lose of generality, such that
\[
w_m(t) \to w(t) \text{ a.e. in } (\tau, T) \times \mathbb{R}^N \text{ as } m \to \infty.
\] (66)
Therefore, by passing to the limit for \((A_{km})\) and \((B_{km})\), and then using Fatou’\(\) lemma, we obtain that
\[
(t-\tau)^{N \frac{\alpha}{2}} \left\| (t-\tau)^{b_k} w(t) \right\|^2 \leq M_k \text{ for all } t \in [\tau, T],
\]
and
\[
\int_{\tau}^{T} \left( \int_{\mathbb{R}^N} |(t-\tau)^{b_k+1} w(t)|^2 \frac{N}{\alpha} \right)^{\frac{N-2}{N}} dt \leq M_k.
\]
Hence the proof of Lemma 4.1 is completed. \(\square\)

**Remark 3.** Note that the \(L^\infty\)-estimate of \(w(t)\) can be also obtained. In fact, the monotone increasing sequence \(\{b_k\}\) convergent to \(b \in \mathbb{R}\). On the other hand, there exists a constant \(c = c(T - \tau, N, l, \lambda, s)\) such that
\[
\lim_{k \to \infty} M_k \left( \frac{N}{\alpha} \right)^b \leq c \|w(t)\|_{L^2(\mathbb{R}^N)}.
\]
Therefore, we deduce from \((A_k)\) that \(\|w(t)\|_{L^\infty(\mathbb{R}^N)} \leq c(t - \tau, N, l, \lambda, s)\|w(t)\|_{L^2(\mathbb{R}^N)}\) for every \(t \in (\tau, T)\).

### 4.2. The \((L^2(\mathbb{R}^N), L^{2+\delta}(\mathbb{R}^N))\) pullback attraction

In the following subsection, the attraction of the \((L^2, L^2)\) pullback \(\mathcal{P}_\mu\)-attractor can be proved to be \(L^{2+\delta}\text{-norm}\) for any \(\delta \in [0, \infty)\). Note that generally we do not know whether the pullback \(\mathcal{P}_\mu\)-attractor \(\mathcal{A}\) or \(U(t, \tau)D(\tau)\) belongs to \(L^{2+\delta}(\mathbb{R}^N)\) or not. However, because of the estimates about decay rate of solutions in \(L^2(\mathbb{R}^N)\)-norm given by Lemma 4.1, the difference of any two elements which derived from \(D(\tau)\) and \(A(\tau)\) respectively will belongs to \(L^{2+\delta}(\mathbb{R}^N)\).

**Proof of Theorem 1.2.** For any fixed \(\delta \in [0, \infty)\), there exists a unique positive integer \(k_0 \in \mathbb{N} \cap (\log \frac{N}{\alpha} - \frac{2+\delta}{2}, \log \frac{N}{\alpha} - \frac{2+\delta}{2} + 1)\) such that
\[
2\left( \frac{N}{N - 2s} \right)^{k_0} > 2 + \delta.
\]
By interpolation, there exists a constant \(\gamma \in (0, 1)\) such that for any \(u_1, u_2 \in L^2(\mathbb{R}^N)^{k_0}\),
\[
\|u_1 - u_2\|_{L^{2+\delta}(\mathbb{R}^N)} \leq \|u_1 - u_2\|_{L^2(\mathbb{R}^N)}^{\gamma} \|u_1 - u_2\|_{L^2(\mathbb{R}^N)}^{1-\gamma}. \tag{67}
\]

Using the definition of pullback attractors, we deduce that for each \(t \in \mathbb{R}\) the section \(A(t)\) is compact in \(L^2(\mathbb{R}^N)\), and so is bounded in \(L^2(\mathbb{R}^N)\). We denote by \(B(t)\) the 1-neighborhood of \(A(t)\) under the \(L^2(\mathbb{R}^N)\)-norm, then \(B(t)\) is bounded in \(L^2(\mathbb{R}^N)\). For any fixed \(t < T\), applying Lemma 4.1 to the case of \(\tau = t - 1\) and \(k = k_0\), it follows that there exists a positive constant \(M_{k_0}\), which depends only on \(k_0, N, l, c_{\lambda,s,N}\) and \(\|B(t-1)\|_{L^2(\mathbb{R}^N)}\), such that
\[
\|U(t, t-1)u_1^i - U(t, t-1)u_2^i\|_{L^2(\mathbb{R}^N)^{k_0}} \leq M_{k_0}, \quad \forall u_1^i, u_2^i \in B(t-1). \tag{68}
\]

For the above \(t, M_{k_0}\), and for any \(\varepsilon > 0\), by the definition of pullback attractors again, we deduce that for any \(\mathcal{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{P}_\mu\), there is a time \(\tau_0(t < t - 1)\) depending only on \(t, \varepsilon > 0\) and \(\mathcal{D}\), such that
\[
U(t-1, \tau)D(\tau) \subset B(t-1) \text{ for all } \tau \leq \tau_0, \tag{69}
\]
and
\[
\text{dist}_{L^2}(U(t, \tau)D(\tau), A(\tau)) \leq \varepsilon^{\frac{1}{b}} \cdot M_{k_0}^{\frac{s-1}{b}} \text{ for all } \tau \leq \tau_0. \tag{70}
\]
Then it is a immediate consequence of (67)–(70) that for each \( \tau \leq \tau_0 \),
\[
\text{dist}_{L^{2+p}}(U(t, \tau)D(\tau), A(t))
\leq \text{dist}_{L^2}^5(U(t, \tau)D(\tau), A(t))\left( \sup_{u_1 \in U(t, \tau)} \| u_1 - u_2 \|_{L^2}^{1-\gamma} N^{\frac{n \gamma}{N-2}} \right) + \text{growth-}
\leq \text{dist}_{L^2}^5(U(t, \tau)D(\tau), A(t))\left( \sup_{u_2 \in A(t)} \| u_1 - u_2 \|_{L^2}^{1-\gamma} N^{\frac{n \gamma}{N-2}} \right)
\leq \text{dist}_{L^2}(U(t, \tau)D(\tau), A(t))\left( \sup_{u_1 \in U(t, \tau)B(t-1)} \| u_1 - u_2 \|_{L^2}^{1-\gamma} N^{\frac{n \gamma}{N-2}} \right)
\leq \text{dist}_{L^2}(U(t, \tau)D(\tau), A(t))\left( \sup_{i=1,2} \| u_1 - u_2 \|_{L^2}^{1-\gamma} N^{\frac{n \gamma}{N-2}} \right)
\leq \left( \varepsilon \cdot M_{k_0}^{-\gamma} \right)^\gamma M_{k_0}^{1-\gamma} = \varepsilon.
\]
By the arbitration of \( \varepsilon \), the proof is completed.

5. Dynamics in \( H^s(\mathbb{R}^N) \). In this section, we study the dynamics of the equation \( (1) \) in \( H^s(\mathbb{R}^N) \). Firstly, without any restriction on spatial dimension \( N \) and growth-index \( p \) of nonlinear term \( f \), we show that the solution of \( (1) \) is continuous in \( H^s(\mathbb{R}^N) \) with respect to the initial data (Lemma 5.2). Then we prove that for any \( \delta > 0 \), the pullback \( \mathcal{D}_\mu \)-attractor obtained in Theorem 1.1 attract the \( \mathcal{D}_\mu \)-class in the norm of \( H^s(\mathbb{R}^N) \) (Theorem 1.3). As a corollary, we obtain the existence of a \( (L^2(\mathbb{R}^N), H^s(\mathbb{R}^N)) \) pullback \( \mathcal{D}_\mu \)-attractor (Remark 5).

5.1. A priori estimate.

Lemma 5.1. Let \( u(t) = U(t, \tau)u_\tau \ (t \geq \tau) \) be the weak solution of the problem \( (1) \). Then \( u \) satisfies following estimates: for any \( T > \tau \),
\[
\int_{\mathbb{R}^N}|u(s)|^pdx \leq M \text{ for all } s \in [\frac{\tau+T}{2}, T],
\]
and
\[
\int_{\frac{\tau+T}{2}}^T \int_{\mathbb{R}^N}|u(s)|^{2p-2}dxds \leq M,
\]
where the constant \( M \) depends on \( T-\tau, p, C_1, k_1, \lambda, \| u_\tau \|_{L^2(\mathbb{R}^N)}, \| g \|_{L^2((\tau,T) \times \mathbb{R}^N)} \).

Proof. First of all, multiplying \( (1) \) by \( u \) and integrating with respect to \( x \) on \( \mathbb{R}^N \), it follows that for a.e. \( t \in (\tau, T) \),
\[
\frac{1}{2} \frac{d}{dt} \| u(t) \|_{L^2(\mathbb{R}^N)}^2 + \| \Lambda^s u(t) \|_{L^2(\mathbb{R}^N)}^2 + \lambda \| u(t) \|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} f(u)udx
\]
\[
= \int_{\mathbb{R}^N} g(t)udx.
\]
This, together with \( (4) \) and Cauchy’s inequality, implies that for any \( t \in [\tau, T] \),
\[
\frac{1}{2} \frac{d}{dt} \| u(t) \|_{L^2(\mathbb{R}^N)}^2 + \| \Lambda^s u(t) \|_{L^2(\mathbb{R}^N)}^2 + (\lambda - k_1) \| u(t) \|_{L^2(\mathbb{R}^N)}^2 + C_1 \| u(t) \|_{L^p(\mathbb{R}^N)}^p
\]
\[
\leq \frac{\lambda - k_1}{2} \| u(t) \|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2(\lambda - k_1)} \| g(t) \|_{L^2(\mathbb{R}^N)}^2.
\]
Integrating with respect to time on \((\tau, t)\), thus we obtain that for any \(t \in [\tau, T]\),
\[
\|u(t)\|_{L^2(\mathbb{R}^N)}^2 + 2C_1 \int_{\tau}^{t} \|u(s)\|_{L^p(\mathbb{R}^N)}^p ds + \int_{\tau}^{t} \sum_{\lambda} \|f(s)\|_{L^2(\mathbb{R}^N)}^2 ds.
\]
(73)

Claim. there exists a fixed time \(t_0 \in [\tau, \frac{\tau + T}{2}]\) such that \(u(t_0) \in L^p(\mathbb{R}^N)\) and
\[
\|u(t_0)\|_{L^p(\mathbb{R}^N)}^p \leq \frac{1}{C_1(T - \tau)} \left( \|u(\tau)\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{\lambda - k_1} \int_{\tau}^{\tau + \frac{T}{2}} \|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds \right).
\]
(74)

We prove the Claim by contradiction. If such \(t_0\) does not exist, then
\[
\|u(t)\|_{L^p(\mathbb{R}^N)}^p > \frac{1}{C_1(T - \tau)} \left( \|u(\tau)\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{\lambda - k_1} \int_{\tau}^{\tau + \frac{T}{2}} \|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds \right)
\]
for every \(t \in [\tau, \frac{\tau + T}{2}]\).

This yields that
\[
\|u(\frac{\tau + T}{2})\|_{L^2(\mathbb{R}^N)}^2 + 2C_1 \int_{\tau}^{\tau + \frac{T}{2}} \|u(s)\|_{L^p(\mathbb{R}^N)}^p ds > \|u(\frac{\tau + T}{2})\|_{L^2(\mathbb{R}^N)}^2
\]
\[
+ 2C_1 \frac{T - \tau}{2} \left( \|u(\tau)\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{\lambda - k_1} \int_{\tau}^{\tau + \frac{T}{2}} \|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds \right)
\]
\[
\geq \|u(\tau)\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{\lambda - k_1} \int_{\tau}^{\tau + \frac{T}{2}} \|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds,
\]
which contradicts the inequality (73). Thus the Claim is proved.

Now multiplying (1) by \(|u|^{p-2}u\) and integrating with respect to \(x\) on \(\mathbb{R}^N\), it follows that for a.e. \(t \in (\tau, T)\),
\[
\frac{1}{p} \int_{\mathbb{R}^N} |u(t)|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \Lambda^2 u|u|^{p-2} u dx + \lambda \|u(t)\|_{L^p(\mathbb{R}^N)}^p + \int_{\mathbb{R}^N} f(u)|u|^{p-2} u dx
\]
\[
= \int_{\mathbb{R}^N} g(t)|u|^{p-2} u dx.
\]

Using (18), (4) and Cauchy’s inequality, we deduce that for a.e. \(t \in (\tau, T)\),
\[
\frac{1}{p} \frac{d}{dt} \|u(t)\|_{L^p(\mathbb{R}^N)}^p + (\lambda - k_1) \|u(t)\|_{L^p(\mathbb{R}^N)}^p + C_1 \int_{\mathbb{R}^N} |u|^{2p-2} dx
\]
\[
\leq \frac{1}{2C_1} \|g(t)\|_{L^{2p}(\mathbb{R}^N)}^2 + \frac{C_1}{2} \|u(t)\|_{L^{2p}(\mathbb{R}^N)}^{2p-2}.
\]
(75)

For any \(t \in [\tau + \frac{T}{2}, T]\), integrating (75) with respect to time on \((t_0, t)\), we deduce that
\[
\|u(t)\|_{L^p(\mathbb{R}^N)}^p + \frac{C_1 p}{2} \int_{t_0}^{t} \|u(s)\|_{L^2(\mathbb{R}^N)}^{2p-2} ds
\]
\[
\frac{1}{C(T - \tau)} \left( \|u(\tau)\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{\lambda - k_1} \int_\tau^\tau \|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds \right) + \frac{1}{C_1} \int_\tau^\tau \|g(s)\|_{L^2(\mathbb{R}^N)}^2 ds,
\]

which implies (71) - (72) immediately. \(\square\)

5.2. Continuity in \(H^s(\mathbb{R}^N)\) with respect to the initial data. At the beginning, we recall some previous results on the continuity of reaction-diffusion equations in \(H^1\). Robinson \(\cite{32}\) proved the continuity of solutions in \(H_0^1(\Omega)\) only for the case of \(N \leq 2\) and the case of \(N = 3\) with a additional growth restriction \((p \leq 4)\) on the nonlinear term \(f\). In 2008 Trujillo and Wang \(\cite{36}\) reported a positive answer for other situations, i.e. any spatial dimension \(N\) and any growth condition on \(f\).

The proof of Trujillo and Wang \(\cite{36}\) essentially depends on differentiating the equation with respect to \(t\) in order to get a nice apriori estimate for \(\partial_t u\). However, the method of Trujillo and Wang \(\cite{36}\) is inapplicable to our non-autonomous nonlinear term \(f\) because \(g(t)\) only satisfies some \(L^2\) integrability \((3), (9)\). We cannot differentiate \((1)\) with respect to \(t\) to get nice apriori estimate for \(\partial_t u\).

In this subsection, based on Lemma 4.1, we provide a method to prove the continuity of the solution of \((1)\) in \(H^s(\mathbb{R}^N)\). To do this, we assume further that the nonlinear term \(f\) satisfies

\[f \in C^1(\mathbb{R}) \text{ and there exists a positive } C_f \text{ such that for any } s \in \mathbb{R}, \]

\[|f'(s)| \leq C_f (1 + |s|^{p-2}).\]  

\[(78)\]

**Lemma 5.2.** Assume that \((3) - (5), (78)\) hold. Then for any \(\tau \in \mathbb{R}\) and any \(t > \tau\), if \(\{u_{n\tau}\} \subset L^2(\mathbb{R}^N)\) satisfying \(u_{n\tau} \to u_\tau\) in \(L^2(\mathbb{R}^N)\) as \(n \to \infty\), then

\[U(t, \tau)u_{n\tau} \to U(t, \tau)u_\tau \text{ in } H^s(\mathbb{R}^N) \text{ as } n \to \infty,\]  

\[(79)\]

where \(U(t, \tau)u_{n\tau}\) and \(U(t, \tau)u_\tau\) are the weak solutions corresponding to initial data \(u_{n\tau}\) and \(u_\tau\) respectively. More precisely, the following estimate holds:

\[\|U(t, \tau)u_{n\tau} - U(t, \tau)u_\tau\|_{H^s(\mathbb{R}^N)} \leq L_1 \|u_{n\tau} - u_\tau\|_{L^2(\mathbb{R}^N)} + L_2 \|u_{n\tau} - u_\tau\|_{L^{2\vartheta}(\mathbb{R}^N)},\]  

\[(80)\]

where the constant \(\vartheta \in (0, 1)\) is the exponent of the interpolation \(\|\cdot\|_{L^{2\vartheta}(\mathbb{R}^N)} \leq \|\cdot\|_{L^{2\vartheta}(\mathbb{R}^N)}^{1/\vartheta} \|\cdot\|_{L^{2\vartheta}(\mathbb{R}^N)}^{1-\vartheta}\) with some \(\alpha \in \mathbb{N}\) satisfying \(2(\frac{N}{N-2})^\alpha > 2p - 2\); the constant \(L_1\) depends only on \(t - \tau, N, p, l, \vartheta, \lambda, s\), and the constant \(L_2\) depends only on \(l, \vartheta, \lambda, s, \|u_{n\tau}\|_{L^2(\mathbb{R}^N)}, \|u_\tau\|_{L^2(\mathbb{R}^N)}, \|g\|_{L^2((\tau, T) \times \mathbb{R}^N)}, l - \tau, N, C_1, k_1, p\).
Proof. Let \( u_n(t) = U(t, \tau)u_{n\tau} \) and \( u(t) = U(t, \tau)u_\tau \) be the solutions to the problem \([1],[2]\) with initial data \( u_{n\tau} \) and \( u_\tau \) respectively. Define \( \varrho_n(t) = U(t, \tau)u_{n\tau} - U(t, \tau)u_\tau \). Then \( \varrho_n(t) \) is a solution of the following equation

\[
\begin{align*}
\partial_t \varrho_n + \Lambda^{2s} \varrho_n + \lambda \varrho_n + f(u_n(t)) - f(u(t)) &= 0, \\
\varrho_n(\tau) &= u_{n\tau} - u_\tau.
\end{align*}
\] (81)

Multiplying the equation (81) by \( \partial_t \varrho_n \) and integrating on \( \mathbb{R}^N \), it follows that

\[
2\|\partial_t \varrho_n(t)\|^2_{L^2(\mathbb{R}^N)} + C_\lambda \frac{d}{dt}\|\varrho_n(t)\|^2_{H^s(\mathbb{R}^N)}
= 2 \int_{\mathbb{R}^N} (f(u(t)) - f(u_n(t))) \partial_t \varrho_n(t) \, dx
\leq 2 \int_{\mathbb{R}^N} |f'(u_n(t)) - f(u(t))| \varrho_n(t) \, dx
\leq \|\partial_t \varrho_n(t)\|^2_{L^2(\mathbb{R}^N)} + C_f \int_{\mathbb{R}^N} (1 + |u_n(t)|^p - 2 + |u(t)|^p - 2)^2 \|\varrho_n(t)\|^2_{L^2(\mathbb{R}^N)} \, dx
\leq \|\partial_t \varrho_n(t)\|^2_{L^2(\mathbb{R}^N)} + \|u_n(t)\|^2_{L^{2p-2}(\mathbb{R}^N)} + \|u(t)\|^2_{L^{2p-2}(\mathbb{R}^N)} \|\varrho_n(t)\|^2_{L^{2p-2}(\mathbb{R}^N)},
\]

where \( C_\lambda = \min\{1, \lambda\} \) and \( \rho(t) \in (0, 1) \). Hence

\[
\frac{d}{dt}\|\varrho_n(t)\|^2_{H^s(\mathbb{R}^N)}
\leq c\|\varrho_n(t)\|^2_{L^2(\mathbb{R}^N)} + c(\|u_n(t)\|^2_{L^{2p-2}(\mathbb{R}^N)} + \|u(t)\|^2_{L^{2p-2}(\mathbb{R}^N)}) \|\varrho_n(t)\|^2_{L^{2p-2}(\mathbb{R}^N)},
\] (82)

where \( c = c(C_\lambda, C_f) \).

Since the result is obvious when \( p = 2 \), we suppose that \( p > 2 \) in the rest of the proof. Let \( \alpha \) be the unique integer in \( \mathbb{N} \cap (\log \frac{N}{N-2\alpha} (p-1), 1 + \log \frac{N}{N-2\alpha} (p-1)) \) such that

\[
2\left(\frac{N}{N-2\alpha}\right)^\alpha > 2p - 2.
\]

By interpolation, there exists a unique \( \vartheta \in (0, 1) \), \( \vartheta = \vartheta(p, \alpha) \) such that

\[
\|\varrho_n(t)\|_{L^{2p-2}(\mathbb{R}^N)} \leq \|\varrho_n(t)\|^{\vartheta}_{L^{2p-2}(\mathbb{R}^N)} \cdot \|\varrho_n(t)\|^{1-\vartheta}_{L^{2p-2}(\mathbb{R}^N)}.
\] (83)

For the above \( \alpha \), we may as well choose \( b_\alpha \) as in (48) and set \( r \) as follows:

\[
\frac{N - 2\vartheta}{N - 2\alpha} \cdot \frac{2 - 2\vartheta}{2(N - 2\alpha)} = (2 - 2\vartheta)b_\alpha > 0.
\]

By applying Lemma 4.1 to the case of \( u_{n\tau}, u_\tau, \tau, t, b_\alpha \), it follows that there exists a constant \( M_\alpha = M(t - \tau, N, l, \alpha, \vartheta, \lambda, s, \|u_{n\tau}\|_{L^2(\mathbb{R}^N)}, \|u_\tau\|_{L^2(\mathbb{R}^N)}) \) such that

\[
\begin{align*}
(s - \tau)^\vartheta \|\varrho_n(t)\|^2_{L^{2p-2}(\mathbb{R}^N)}
&= \left( \frac{N - 2\alpha}{N - 2\vartheta} \right)^\alpha \|\varrho_n(t)\|^2_{L^{2p-2}(\mathbb{R}^N)} \left( \frac{N - 2\alpha}{N - 2\vartheta} \right)^\alpha \|\varrho_n(t)\|^2_{L^{2p-2}(\mathbb{R}^N)} \left( \frac{N - 2\alpha}{N - 2\vartheta} \right)^\alpha \|\varrho_n(t)\|^2_{L^{2p-2}(\mathbb{R}^N)}
\leq M_\alpha,
\end{align*}
\] (84)

where \( s \in [\tau, t] \).
Let $t > \tau$ and $s \in [\tau + \frac{t-\tau}{2}, t]$. We multiply (82) by $(s - (\tau + \frac{t-\tau}{2}))^{r+1}$, then by \((83)\) and \((84)\), we have for a.e. $s \in [\tau + \frac{t-\tau}{2}, t]$,
\[
(s - (\tau + \frac{t-\tau}{2}))^{r+1} \frac{d}{ds} \|w_n(s)\|_{H^r(\mathbb{R}^N)}^2 \\
\leq c(s - \tau)^{r+1} \|w_n(s)\|^2_{L^2(\mathbb{R}^N)} \\
+ c(\|u_n(s)\|_{L^p(\mathbb{R}^N)}^{2p-4} + \|u(s)\|_{L^p(\mathbb{R}^N)}^{2p-4})(s - \tau)^{r+1} \|w_n(s)\|^2_{L^{2p-2}(\mathbb{R}^N)} \\
\leq c(s - \tau)^{r+1} \|w_n(s)\|^2_{L^2(\mathbb{R}^N)} \\
+ c(\|u_n(s)\|_{L^p(\mathbb{R}^N)}^{2p-4} + \|u(s)\|_{L^p(\mathbb{R}^N)}^{2p-4})(s - \tau)\|w_n(s)\|^{2\theta}_{L^{2}(\mathbb{R}^N)} M_\alpha.
\]
We integrate the above inequality with respect to $s$ over the interval $[\tau + \frac{t-\tau}{2}, t]$. Therefore
\[
\frac{t-\tau}{2} \|w_n(t)\|^2_{H^r(\mathbb{R}^N)} \\
\leq (r + 1)(\frac{t-\tau}{2})^r \int_{\tau + \frac{t-\tau}{2}}^t \|w_n(s)\|^2_{H^r(\mathbb{R}^N)} ds \\
+ (t - \tau) M_\alpha \int_{\tau + \frac{t-\tau}{2}}^t (\|u_n(s)\|_{L^p(\mathbb{R}^N)}^{2p-4} + \|u(s)\|_{L^p(\mathbb{R}^N)}^{2p-4}) \|w_n(s)\|^{2\theta}_{L^{2}(\mathbb{R}^N)} ds \\
\leq (r + 1)(\frac{t-\tau}{2})^r \int_{\tau + \frac{t-\tau}{2}}^t \|w_n(s)\|^2_{H^r(\mathbb{R}^N)} ds \\
+ (t - \tau) M_\alpha \left( \int_{\tau + \frac{t-\tau}{2}}^t \|u_n(s)\|^{2p-2}_{L^{p}(\mathbb{R}^N)} + \|u(s)\|^{2p-2}_{L^{p}(\mathbb{R}^N)} ds \right)^{\frac{p-2}{p}} \tag{85}
\]
where the third inequality is obtained by Lemma 5.1 and the constant $M$ comes from Lemma 5.1. Now we are in a position to estimate $(\frac{t-\tau}{2})^{r+1} \|w_n(t)\|^2_{H^r(\mathbb{R}^N)}$. Multiplying (81) by $w_n$, it implies that for any $t \geq \tau$,
\[
\|w_n(t)\|^2_{L^2(\mathbb{R}^N)} \leq e^{2(t-\tau)} \|w_n(\tau)\|^2_{L^2(\mathbb{R}^N)} \tag{86}
\]
and
\[
\int_\tau^t \|w_n(s)\|^2_{H^r(\mathbb{R}^N)} ds \leq e^{2(t-\tau)} \|w_n(\tau)\|^2_{L^2(\mathbb{R}^N)}. \tag{87}
\]
Consequently
\[
(r + 1)(\frac{t-\tau}{2})^r \int_{\tau + \frac{t-\tau}{2}}^t \|w_n(s)\|^2_{H^r(\mathbb{R}^N)} ds \\
\leq (r + 1)(\frac{t-\tau}{2})^r e^{2(t-\tau)} \|w_n(\tau)\|^2_{L^2(\mathbb{R}^N)} \\
+ (t - \tau) M_\alpha \|w_n(t)\|^2_{L^2(\mathbb{R}^N)} ds \tag{87}
\]
Proof of Theorem 1.3. Under the \(L^N\)-norm, the \((L^N, L^N)\) pullback attraction. In this subsection, we prove that the \((L^2, L^2)\) pullback \(\mathcal{A}_\mu\)-attractor \(\mathcal{A}\) obtained in Theorems 1.1 attract the class of \(\mathcal{D}\) in \(H^s(\mathbb{R}^N)\)-norm. The proof is based on Lemma 5.2.

**Proof of Theorem 1.3.** Let \(t \in \mathbb{R}\). We denote by \(B(t)\) the 1-neighborhood of \(A(t)\) under the \(L^2\)-norm, thus we have that \(B(t)\) is bounded in \(L^2\). Applying Lemma 5.2 (especially the estimate (80)) to the case of \(\tau = t - 1\) and \(u, v \in B(t - 1)\), we know that there exist two constants \(L_1, L_2\) satisfying

\[
\|U(t, t - 1)u - U(t, t - 1)v\|_{L^2(\mathbb{R}^N)}^2 \leq L_1\|u - v\|_{L^2(\mathbb{R}^N)}^2 + L_2\|u - v\|_{L^2(\mathbb{R}^N)}^{2\alpha}, \quad \forall \ u, v \in B(t - 1).
\]

Since \(\mathcal{A}\) is the \((L^2, L^2)\) pullback \(\mathcal{D}_\mu\)-attractor, for any \(\varepsilon > 0\) and any \(D = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\mu\), there is a time \(\tau_1(t < t - 1)\) depending only on \(t, \varepsilon\) and \(D\), such that

\[
U(t - 1, \tau)D(\tau) \subset B(t - 1) \quad \text{for all } \tau \leq \tau_1,
\]

and

\[
\text{dist}_{L^2(\mathbb{R}^N)}(U(t - 1, \tau)D(\tau), A(t - 1)) \leq \varepsilon \quad \text{for all } \tau \leq \tau_1.
\]

This, together with (91)-(92), implies that for \(\tau \leq \tau_1\),

\[
\text{dist}_{H^s(\mathbb{R}^N)}^2(U(t, \tau)D(\tau), A(t)) = \text{dist}_{H^s(\mathbb{R}^N)}^2(U(t, t - 1)U(t - 1, \tau)D(\tau), U(t, t - 1)A(t - 1)) \leq L_1\text{dist}_{L^2(\mathbb{R}^N)}^2(U(t - 1, \tau)D(\tau), A(t - 1)) + L_2\text{dist}_{L^2(\mathbb{R}^N)}^2(U(t - 1, \tau)D(\tau), A(t - 1)) \leq L_1\varepsilon^2 + L_2\varepsilon^{2\alpha}.
\]

Therefore the result follows from the arbitration of \(\varepsilon\) and \(D\).
Remark 5. By Lemma 2.8, Lemma 5.2 and Theorem 1.3, we see that $A(t) - y_t$ is compact in $H^s(\mathbb{R}^N)$, where $y_t$ be an arbitrary point in $A(t)$. Furthermore, it follows from Lemma 3.7 that $A(t)$ is also a $(L^2(\mathbb{R}^N), H^s(\mathbb{R}^N))$ pullback $\mathcal{D}_\mu$-absorbing set. Hence it is easy to verify that the $(L^2(\mathbb{R}^N), L^2(\mathbb{R}^N))$ pullback $\mathcal{D}_\mu$-attractor $\mathcal{A}$ is also a $(L^2(\mathbb{R}^N), H^s(\mathbb{R}^N))$ pullback $\mathcal{D}_\mu$-attractor.

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