NILPOTENT ELEMENTS IN THE COHOMOLOGY OF THE CLASSIFYING SPACE OF A CONNECTED LIE GROUP

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ABSTRACT. We give an example of a compact connected Lie group of the lowest rank such that the mod 2 cohomology ring of its classifying space has a nonzero nilpotent element.

1. INTRODUCTION

Let $p$ be a prime number and $G$ a compact Lie group. In [Qui71], Quillen defined a homomorphism,

$$q_G : H^*(BG;\mathbb{Z}/p) \to \lim_{\mathcal{A} \in \mathcal{A}} H^*(BA;\mathbb{Z}/p),$$

where $\mathcal{A}$ is the category of elementary abelian $p$-subgroups of $G$ and proved that $q_G$ is an $F$-isomorphism, that is, each element in the kernel of $q_G$ is nilpotent and for each

$$y \in \lim_{\mathcal{A} \in \mathcal{A}} H^*(BA;\mathbb{Z}/p),$$

there is a positive integer $n$ such that $y^{p^n}$ belongs to the image of $q_G$. For $p = 2$, $H^*(BA;\mathbb{Z}/2)$ is a polynomial ring. Hence, a nonzero element in the image of $q_G$ is not nilpotent. So, the nilradical of $H^*(BG;\mathbb{Z}/2)$ is precisely the kernel of $q_G$. Thus, the above homomorphism $q_G$ is injective if and only if the mod 2 cohomology ring $H^*(BG;\mathbb{Z}/2)$ has no nonzero nilpotent element. In [KY93], Kono and Yagita showed that $q_G$ is not injective for $p = 2$, $G = \text{Spin}(11), E_7$ by showing the existence of a nonzero nilpotent element in $H^*(BG;\mathbb{Z}/2)$. For an odd prime number $p$, Adams conjectured that $q_G$ is injective for all compact connected Lie groups. Adams’ conjecture remains an open problem.

On the other hand, for a compact connected Lie group $G$, a maximal torus $T$ exists. Let $W$ be the Weyl group $N(T)/T$. We denote by $H^*(BT;\mathbb{Z})^W$ the ring of invariants of $W$. We denote by Tor the torsion part of $H^*(BG;\mathbb{Z})$. Then, the inclusion map of $T$ induces a homomorphism,

$$\iota_T^* : H^*(BG;\mathbb{Z})/\text{Tor} \to H^*(BT;\mathbb{Z})^W.$$

Borel showed that $\iota_T^*$ is injective. In [Fes81], Feshbach gave a criterion for $\iota_T^*$ to be surjective, hence an isomorphism. In particular, after localized at $p$, $\iota_T^*$ is surjective if and only if the $E_\infty$-term of the mod $p$ Bockstein spectral sequence of $BG$,

$$H^*(BG;\mathbb{Z})/\text{Tor} \otimes \mathbb{Z}/p,$$

has no nonzero nilpotent element. For $p = 2$, Feshbach showed that for $G = \text{Spin}(12)$, the $E_\infty$-term of the mod 2 Bockstein spectral sequence of $BG$ has a

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nonzero nilpotent element. As for spin groups, Spin$(n)$, Benson and Wood [BW95] computed the ring of invariants of the Weyl group and they showed that $i^*_T$ is not surjective if and only if $n \geq 11$ and $n \equiv 3, 4, 5 \mod 8$. However, as in the case of Adams’ conjecture, for an odd prime number $p$, no example of a compact connected Lie group $G$ such that the $E_\infty$-term of the mod $p$ Bockstein spectral sequence of $BG$ has a nonzero nilpotent element is known.

So, nonzero nilpotent elements in the cohomology of the classifying spaces of compact connected Lie groups are exciting subjects for study. However, no example of a compact connected Lie group $G$ such that $H^*(BG; \mathbb{Z}/2)$ has a nonzero nilpotent element is known except for spin groups and the exceptional Lie group $E_7$. The purpose of this paper is to give a more straightforward example to shed some light on the existence of nonzero nilpotent elements in the mod 2 cohomology of the classifying space of a connected Lie group.

First, we define a compact connected Lie group $G$. Let us consider the three fold product $SU(2)^3$ of the special unitary groups $SU(2)$. Its center is an elementary abelian 2-group $(\mathbb{Z}/2)^3$. Let $\Gamma$ be the kernel of the group homomorphism $\varphi: (\mathbb{Z}/2)^3 \to \mathbb{Z}/2$ defined by $\varphi(a_1, a_2, a_3) = a_1a_2a_3$. We define $G$ to be $SU(2)^3/\Gamma$.

Next, we state our results, saying that $G = SU(2)^3/\Gamma$ satisfies the required conditions. Since $SU(2)^3/(\mathbb{Z}/2)^3 = SO(3)^3$, we have the following fiber sequence:

$$B\mathbb{Z}/2 \to BG \to BSO(3)^3.$$  

Let $\pi_i: BSO(3)^3 \to BSO(3)$ be the projection onto the $i^{\text{th}}$ factor. The mod 2 cohomology ring of $BSO(3)$ is given by

$$H^*(BSO(3); \mathbb{Z}/2) = \mathbb{Z}/2[w_2, w_3],$$

where $w_i$ is the universal $i^{\text{th}}$ Stiefel-Whitney class for $i = 2, 3$.

Let $w'_k = \pi^*(\pi_1(w_k))$ and $w''_k = \pi^*(\pi_2(w_k))$. Let $w_{16}(\rho)$ be the Stiefel-Whitney class $w_{16}(\rho)$ of a real representation $\rho: G \to O(16)$. We will give the definition of $\rho$ in Section 2. Let $f_5, f_9, g_4, g_7, g_8$ be polynomials defined by

\[
\begin{align*}
f_5 &= w'_2w'_3 + w''_2w'_3, \\
f_9 &= w''_2w'_2 + w''_3w'_3, \\
g_4 &= w'_2w'_2, \\
g_7 &= w'_2w'_2(w'_2 + w''_2), \\
g_8 &= w'_3w'_3(w'_2 + w''_2),
\end{align*}
\]

respectively. Then, our results are stated as follows:

**Theorem 1.1.** The mod 2 cohomology ring of $BG$ is

$$\mathbb{Z}/2[w'_2, w''_2, w'_3, w''_3, w_{16}]/(f_5, f_9)$$

and its nilradical is generated by $g_7, g_8$.

**Theorem 1.2.** The $E_\infty$-term of the mod 2 Bockstein spectral sequence of $BG$ is

$$\mathbb{Z}/2[w''_2, w''_3, w_{16}] \otimes \Delta(g_4, g_8),$$

where $\Delta(g_4, g_8)$ is the vector space over $\mathbb{Z}/2$ spanned by 1, $g_4$, $g_8$ and $g_4g_8$. Its nilradical is generated by $g_8$. 
The computations involved in these theorems are similar to those of Quillen in [Qui71b] and Kono in [Kon86]. We have no claim for novelty in this respect.

The rank of $SU(2)^3/\Gamma$ is 3. If the rank of a compact connected Lie group is lower than 3, then it is homotopy equivalent to one of $T$, $SU(2)$, $T^2$, $T \times SU(2)$, $SU(2) \times SU(2)$, $SU(3)$, $G_2$ or their quotient groups by their central subgroups. For such a compact connected Lie group, the mod 2 cohomology ring of its classifying space is a polynomial ring so that it has no nonzero nilpotent element. Thus $SU(2)^3/\Gamma$ is a lowest rank Lie group such that the mod 2 cohomology of its classifying space has a nilpotent element.

We hope our results shed some light on Adams’ conjecture since, contrary to spin groups, we have an odd prime analog of the group $SU(2)^3/\Gamma$. Let $\Gamma_2$ be the kernel of the determinant homomorphism $det: (S^1)^3 \rightarrow S^1$. Consider the quotient group $U(p^3)/\Gamma_2$. It is the odd prime counterpart as the group $U(2)^3/\Gamma_2$ is the central extension of the group $SU(2)^3/\Gamma$ by $S^1$. But that is another story and we wish to deal with the group $U(p^3)/\Gamma_2$ in another paper.

In what follows, we assume that $G$ is the compact connected Lie group $SU(2)^3/\Gamma$. We also denote the mod 2 cohomology ring of $X$ by $H^*(X)$ rather than $H^*(X; Z/2)$.

This paper is organized as follows: In Section 2, we compute the Leray-Serre spectral sequence associated with the fiber sequence

$$BZ/2 \to BG \to BSO(3)^3$$

to describe the mod 2 cohomology ring $H^*(BG)$ and prove Theorem 1.1. In Section 3 we define and compute the $Q_0$-cohomology of $H^*(BG)$ to complete the proof of Theorem 1.2.

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2. THE MOD 2 COHOMOLOGY RING

In this section, we compute the mod 2 cohomology ring of $BG$ by the Leray-Serre spectral sequence associated with the fiber sequence

$$BZ/2 \to BG \to BSO(3)^3.$$

First, we recall the mod 2 cohomology rings of $BSO(3)$ and $BSO(3)^3$. As stated in Section 1, the mod 2 cohomology ring is given by

$$H^*(BSO(3); Z) = Z/2[w_2, w_3].$$

Let $Q_i$ be the Milnor operation

$$Q_i: H^k(X) \to H^{k+2i+1-1}(X)$$

defined inductively by

$$Q_0 = Sq^1, \quad Q_{i+1} = Sq^{2i+1} Q_i + Q_i Sq^{2i+1}$$

for $i \geq 0$. The Wu formula yields

$$Q_0(w_2) = w_3, \quad Q_1(w_2) = w_2w_3, \quad Q_2(w_2) = w_3^2 + w_3^3.$$
Let us define elements $v_2$, $v_3$ by

$$v_2 = w'_2 + w''_2 + w'''_2,$$
$$v_3 = w'_3 + w''_3 + w'''_3,$$

and ideals $I_1$, $I_2$ by

$$I_1 = (v_2, v_3),$$
$$I_2 = (v_2, v_3, Q_1(v_2)).$$

Again, by abuse of notation, let

$$f_5 = w'_3 w'_3 + w''_3 w'_3,$$
$$f_9 = w'_3 w'_3 + w''_3 w'_3 \in H^*(BSO(3)^3).$$

Then, by direct calculations, we have

$$Q_0 v_2 = v_3, \quad Q_1 v_2 \equiv f_5 \pmod{I_1}, \quad Q_2 v_2 \equiv f_9 \pmod{I_2}.$$

Now, we compute the Leray-Serre spectral sequence. The $E_2$-term is given by

$$E_2^{p,q} = H^p(BSO(3)^3) \otimes H^q(B\mathbb{Z}/2),$$

so that

$$E_2 = \mathbb{Z}/2[w'_2, w''_2, v_2, w'_3, w''_3, v_3, u_1],$$

where $u_1$ is the generator of $H^1(B\mathbb{Z}/2) \cong \mathbb{Z}/2$. A possible first nontrivial differential is $d_2$. Let $\iota_i: SU(2) \to SU(2)^3$ be the inclusion map to the $i$th factor,

$$\iota_1(g) = (g, 1, 1), \quad \iota_2(g) = (1, g, 1), \quad \iota_3(g) = (1, 1, g).$$

Then, they induce the following commutative diagram.

$$\begin{array}{ccc}
B\mathbb{Z}/2 & \longrightarrow & B\mathbb{Z}/2 \\
\downarrow & & \downarrow \\
BSU(2) & \longrightarrow & BG \\
\downarrow & & \downarrow \\
BSO(3) & \longrightarrow & BSO(3)^3.
\end{array}$$

Since the differential $d_2$ in the Leray-Serre spectral sequence associated with the left column homotopy fibration is

$$d_2(u_1) = w_2,$$

we have

$$d_2(u_1) = v_2$$

in the Leray-Serre spectral sequence for the right column homotopy fibration.

To compute the higher differentials, we consider the following diagram. Let $K(\mathbb{Z}/2, 2)$ be the Eilenberg-MacLane space. Let

$$k: BSO(3)^3 \rightarrow K(\mathbb{Z}/2, 2)$$
be a map representing the cohomology class $v_2 \in H^2(\text{BSO}(3)^3)$ such that

$$k^*(u_2) = v_2$$

where $u_2$ is the generator of $H^2(K(\mathbb{Z}/2, 2)) \cong \mathbb{Z}/2$. Putting the path space fibration over $K(\mathbb{Z}/2, 2)$ in the right column, we have the following commutative diagram.

$$
\begin{array}{c}
\text{BZ}/2 \\
\downarrow \\
\text{BG} \\
\downarrow \\
\text{BSO}(3)^3 \\
\end{array}
\quad
\begin{array}{c}
\rightarrow \\
\downarrow \\
\text{PK(}\mathbb{Z}/2, 2) \\
\downarrow \\
\text{K(}\mathbb{Z}/2, 2) \\
\end{array}
$$

The mod 2 cohomology rings and the Leray-Serre spectral sequence for the path space fibration are known. We refer the reader to Serre’s classical paper [Ser53]. Its $E_2$-term is

$$E_2 = \mathbb{Z}/2[u_2, Sq^1 u_2, Sq^2 Sq^1 u_2, \ldots] \otimes \mathbb{Z}/2[u_1]$$

and nontrivial differentials are given by

$$d_{2n+1}(u_1^{2n}) = Sq^{2n-1} \cdots Sq^1 u_2$$

for $n \geq 0$.

**Lemma 2.1.** For $x \in H^2(X)$ and $k \geq 1$, we have

$$Q_k(x) = Sq^{2k} \cdots Sq^0(x).$$

**Proof.** We prove this lemma by induction on $k$. Suppose $k = 1$. By the unstable condition, we have $Sq^1(x) = x^2$. By the Cartan formula, we have $Sq^1(x^2) = 0$. Hence, we have

$$Q_1(x) = Sq^2 Q_0(x) + Q_0 Sq^2(x) = Sq^2 Q_0(x).$$

For $k \geq 2$, by the definition of $Q_{k+1}$ and the unstable condition, we have

$$Q_k(x) = Sq^{2k} Q_{k-1}(x) + Q_{k-1} Sq^{2k}(x) = Sq^{2k} Q_{k-1}(x). \quad \square$$

From $d_2(u_1) = v_2$ and the action of $Q_0$, $Q_1$, $Q_2$ on $v_2$, by Lemma 2.1 and the Leray-Serre spectral sequence for the above path space fibration, we have

$$d_3(u_1^2) = v_3,$$
$$d_5(u_1^4) = f_5,$$
$$d_9(u_1^8) = f_9.$$
It is easy to see that
\[
\begin{align*}
E_3 &= \mathbb{Z}/2[w'_2, w''_2, w'_3, w''_3, v_3, u^2_1], \\
E_4 &= \mathbb{Z}/2[w'_2, w''_2, w'_3, w''_3, u^4_1], \\
E_6 &= \mathbb{Z}/2[w'_2, w''_2, w'_3, w''_3, u^8_1]/(f_5),
\end{align*}
\]
In \(\mathbb{Z}/2[w'_2, w''_2, w'_3, w''_3]\), we consider the sequence \(f_5, f_9\). It is a regular sequence since their greatest common divisor is 1. Therefore, we have
\[
E_{10} = \mathbb{Z}/2[w'_2, w''_2, w'_3, w''_3, u^{16}_1]/(f_5, f_9).
\]

To prove that the spectral sequence collapses at the \(E_{10}\)-term, we consider the Stiefel-Whitney class of a real representation
\[
\rho: G \to O(16)
\]
defined as follow. On the one hand, since \(\mathbb{C}\) is isomorphic to \(\mathbb{R}^2\) as a vector space over \(\mathbb{R}\), \(\mathbb{C}^2\) is isomorphic to \(\mathbb{R}^4\). Then, the tautological representation of \(SU(2)\) on \(\mathbb{C}^2\) induces the inclusion map
\[
j: SU(2) \to SO(4).
\]
On the other hand, we have an isomorphism
\[
SO(4) = SU(2) \times_{\mathbb{Z}/2} SU(2).
\]
Since
\[
G = SU(2) \times_{\mathbb{Z}/2} (SU(2) \times_{\mathbb{Z}/2} SU(2)) = SU(2) \times_{\mathbb{Z}/2} SO(4),
\]
we may regard \(G\) as a subgroup of
\[
SO(4) \times_{\mathbb{Z}/2} SO(4)
\]
with the inclusion map induced by
\[
j \times 1: SU(2) \times SO(4) \to SO(4) \times SO(4).
\]
Let
\[
\varphi: SO(4) \times SO(4) \to O(16)
\]
be the real representation given by
\[
(g_1, g_2)m = g_1 m g_2^{-1}
\]
where \((g_1, g_2) \in SO(4) \times SO(4)\) and \(m\) is a 4 \(\times\) 4 matrix with real coefficients. Then, \(\varphi\) induced a 16-dimensional real representation.
\[
\varphi': SO(4) \times_{\mathbb{Z}/2} SO(4) \to O(16).
\]
We define the representation \(\rho\) as the restriction of \(\varphi'\) to \(G\).

**Proposition 2.2.** The Stiefel-Whitney class \(w_{16}(\rho)\) of the real representation \(\rho\) is indecomposable in \(H^*(BG)\). It is represented by \(u^1_{16}\) in the Leray-Serre spectral sequence associated with the fiber sequence
\[
B\mathbb{Z}/2 \to BG \to BSO(3)^3.
\]
Proof. Let $\iota: \mathbb{Z}/2 \to G$ be the inclusion map of the center of $G$. We may regard $\mathbb{Z}/2$ as a subgroup of the center of $SU(2)^3$. Thus, the inclusion map $\iota$ factors through the projection $SU(2)^3 \to G$.

The restriction of $\rho$ to the center of $G$ is $16\lambda$, where $\lambda$ is the nontrivial 1-dimensional real representation of $\mathbb{Z}/2$. So, the Stiefel-Whitney class $w_{16}(\rho \circ \iota)$ is nonzero. If $d_r(u_{16}^2) \neq 0$ for some $r$, up to degree $\leq 16$, $H^*(BG)$ is generated by $w'_2, w''_2, w'_3, w''_3$. However, since $\iota$ factors through $SU(2)^3$, and since $BSU(2)^3$ is 3-connected, the induced homomorphism sends $w'_2, w''_2, w'_3, w''_3$ to zero. So, $w_{16}(\rho \circ \iota)$ is zero. It is a contradiction. Therefore, $u_{16}^2$ is a permanent cycle in the Leray-Serre spectral sequence and it is represented by $w_{16}(\rho)$. □

By Proposition 2.2, the spectral sequence collapses at the $E_{10}$-term, that is, $E_{\infty} = E_{10}$ and we obtain the first half of Theorem 1.1.

**Proposition 2.3.** We have

$$H^*(BG) = \mathbb{Z}/2[w'_2, w''_2, w'_3, w''_3, u_{16}]/(f_5, f_9),$$

where $u_{16}$ is the Stiefel-Whitney class $w_{16}(\rho)$.

To prove the second half of Theorem 1.1, let us define a ring homomorphism

$$\eta: \mathbb{Z}/2[w'_2, w''_2, w'_3, w''_3, u_{16}] \to \mathbb{Z}/2[w'_2, w''_2, u, u_{16}]$$

by $\eta(w'_2) = w'_2$, $\eta(w''_2) = w''_2$, $\eta(w'_3) = w'_2u$, $\eta(w''_3) = w''_2u$, $\eta(u_{16}) = u_{16}$. It induces the following ring homomorphism

$$\eta': H^*(BG) \to \mathbb{Z}/2[w'_2, w''_2, u, u_{16}]/(w^3w'_2w''_2(w'_2 + w''_2)).$$

Let $R_0 = \mathbb{Z}/2[w'_2, w''_2, w'_3, w''_3, u_{16}]$.

From Proposition 2.2, using the fact that $f_5, f_9$ is a regular sequence in $R_0$, the Poincaré series of $H^*(BG)$ is given by

$$PS(H^*(BG), t) = \frac{(1 - t^5)(1 - t^9)}{(1 - t^2)^2(1 - t^3)^2(1 - t^{16})}.$$

On the other hand, it is also easy to see that the image of $\eta'$ is spanned by monomials

$$u^kw'^m_w''^nw''_{16},$$

where $k$ ranges over all non-negative integers, for $\ell = 0, 1, 2$, $(m, n)$ satisfies the condition $m + n \geq \ell$, and for $\ell \geq 3$, $(m, n)$ satisfies one of the following conditions: $m \geq \ell$, $n = 0$ or $m = 1$, $n \geq \ell - 1$ or $m = 0$, $n \geq \ell$. Thus, the Poincaré series $PS(\text{Im} \eta', t)$ is

$$\frac{1}{1 - t^{16}} \left( \frac{1}{(1 - t^2)^2} - 1 \right) + t^2 \left( \frac{1}{(1 - t^2)^2} - 1 - 2t^2 \right) + \sum_{\ell=3}^{\infty} \frac{3t^3\ell}{1 - t^2}.$$

Then, we have

$$PS(H^*(BG), t) = PS(\text{Im} \eta', t).$$

Thus, $\eta'$ is injective. In view of this injective homomorphism $\eta'$, it is easy to see that elements $g_7, g_8$ corresponding to $uw'_2w''_2(w'_2 + w''_2)$, $u^2w'_2w''_2(w'_2 + w''_2)$, respectively, are nilpotent. So we obtain the following second half of Theorem 1.1.

**Proposition 2.4.** The nilradical of $H^*(BG)$ is the ideal generated by two elements $g_7$ and $g_8$. 
3. The mod 2 Bockstein spectral sequence

For each \( i \geq 0 \), we have \( Q_iQ_i = 0 \). Hence, for a graded vector space \( M \) over \( \mathbb{Z}/2 \) with \( Q_i \)-action, we may define \( Q_i \)-cohomology \( H^*(M, Q_i) \) by

\[
\text{Ker} Q_i / \text{Im} Q_i.
\]

In particular, the \( E_2 \)-term of the mod 2 Bockstein spectral sequence of \( BG \) is the \( Q_0 \)-cohomology \( H^*(H^*(BG), Q_0) \). In this section, to show that the mod 2 Bockstein spectral sequence of \( BG \) collapses at the \( E_2 \)-term, we compute the \( Q_0 \)-cohomology of the mod 2 cohomology of \( BG \), i.e.

\[
H^*(H^*(BG), Q_0) = \text{Ker} Q_0 / \text{Im} Q_0.
\]

First, we recall the action of \( Q_0 \) on \( H^*(BG) \). The action of \( Q_0 \) on \( w_2, w_2'', w_3', w_3'' \) is clear from that on \( H^*(BSO(3)) \). We need to determine the action of \( Q_0 \) on \( w_{16} \).

**Proposition 3.1.** In \( H^*(BG) \), we have \( Q_0(w_{16}) = 0 \).

**Proof.** The generator \( w_{16} \) is defined as the Stiefel-Whitney class \( w_{16}(\rho) \) of the 16-dimensional real representation \( \rho : G \to O(16) \). Hence, \( w_{17}(\rho) = 0 \). Since \( BG \) is simply-connected, we also have \( w_1(\rho) = 0 \). By the Wu formula, we have \( \text{Sq}^1 w_{16}(\rho) = w_{17}(\rho) + w_1(\rho)w_{16}(\rho) \). Therefore, we have the desired result. \( \square \)

Let

\[
R_0 = \mathbb{Z}/2[w_2, w_2'', w_3', w_3'', u_{16}].
\]

We consider the action of \( Q_0 \) on \( w_2, w_2'', w_3', w_3'', u_{16} \) in \( R_0 \). It is given by

\[
Q_0(w_2') = w_2', \quad Q_0(w_2'') = w_2'', \quad Q_0(w_3') = 0, \quad Q_0(w_3'') = 0, \quad Q_0(u_{16}) = 0.
\]

Let

\[
R_1 = R_0/(f_5), \quad R_2 = R_0/(f_5, f_9).
\]

It is clear that \( R_2 = H^*(BG) \) and \( H^*(H^*(BG), Q_0) = H^*(R_2, Q_0) \). We will prove the following Proposition 3.2 at the end of this section.

**Proposition 3.2.** We have

\[
H^*(R_2, Q_0) = \mathbb{Z}/2[w_2^2, w_2''^2, u_{16}] \otimes \Delta(g_4, g_8).
\]

The \( E_1 \)-term of the mod 2 Bockstein spectral sequence of \( BG \) is the mod 2 cohomology ring of \( BG \) and \( d_1 \) is \( Q_0 \). Since, by Proposition 3.2, the \( E_2 \)-term has no nonzero odd degree element, the spectral sequence collapses at the \( E_2 \)-term. It is also clear that \( g_4^2 = w_2^2 w_2''^2 \neq 0 \), \( g_8^2 = 0 \) from Theorem 1.1. Hence, we obtain Theorem 1.2.

Now, we complete the proof of Theorem 1.2 by proving Proposition 3.2.

**Proof of Proposition 3.2.** We start with \( H^*(R_0, Q_0) \). It is clear that

\[
H^*(R_0, Q_0) = \mathbb{Z}/2[w_2^2, w_2''^2, u_{16}].
\]

We denote by \( (-) \times a \) the multiplication by \( a \). Consider a short exact sequence

\[
0 \to R_0 \xrightarrow{(-) \times f_5} R_0 \to R_1 \to 0.
\]

Since \( Q_0 \) is a derivation and \( Q_0 f_5 = 0 \), \( Q_0 \) commutes with \( (-) \times f_5 \). Hence, this short exact sequence induces a long exact sequence in \( Q_0 \)-cohomology:

\[
\cdots \to H^i(R_0, Q_0) \to H^i(R_1, Q_0) \xrightarrow{\delta_i} H^{i-4}(R_0, Q_0) \to \cdots
\]
Since $H^{\text{odd}}(R_0, Q_0) = 0$, this long exact sequence splits into short exact sequences:

$$0 \to H^{2i}(R_0; Q_0) \to H^{2i}(R_1, Q_0) \xrightarrow{\delta} H^{2i-4}(R_0, Q_0) \to 0$$

and $H^{\text{odd}}(R_1, Q_0) = 0$. Since $Q_0 g_4 = f_5$ in $R_0$, $g_4$ is nonzero in $R_1$ and $\delta(g_4) = 1$. Therefore, we have

$$H^*(R_1, Q_0) = \mathbb{Z}/2[w^2_2, w'^2_2, u_{16}] \otimes \Delta(g_4).$$

Next, let us consider a short exact sequence

$$0 \to R_1 \xrightarrow{(-) \times f_9} R_1 \to R_2 \to 0.$$  

Again, since $Q_0 f_9 = 0$ and $Q_0$ is a derivation, it induces a long exact sequence in $Q_0$-cohomology. As above, since $H^{\text{odd}}(R_1, Q_0) = \{0\}$, we have short exact sequences

$$0 \to H^{2i}(R_1; Q_0) \to H^{2i}(R_2, Q_0) \xrightarrow{\delta} H^{2i-8}(R_1, Q_0) \to 0$$

and $H^{\text{odd}}(R_2, Q_0) = \{0\}$. Since $Q_0 g_8 = f_9$, we obtain the desired result

$$H^*(R_2, Q_0) = \mathbb{Z}/2[w^2_2, w'^2_2, u_{16}] \otimes \Delta(g_4, g_8).$$

□

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