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On inertia and ratio type bounds for the $k$-independence number of a graph and their relationship

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ABSTRACT

For $k \geq 1$, the $k$-independence number $\alpha_k$ of a graph is the maximum number of vertices that are mutually at distance greater than $k$. The well-known inertia and ratio bounds for the (1-)independence number $\alpha(=\alpha_1)$ of a graph, due to Cvetković and Hoffman, respectively, were generalized recently for every value of $k$. We show that, for graphs with enough regularity, the polynomials involved in such generalizations are closely related and give exact values for $\alpha_k$, showing a new relationship between the inertia and ratio type bounds. Additionally, we investigate the existence and properties of the extremal case of sets of vertices that are mutually at maximum distance for walk-regular graphs. Finally, we obtain new sharp inertia and ratio type bounds for partially walk-regular graphs by using the predistance polynomials.

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1. Introduction

Given a graph $G$ with diameter $D$ and an integer $k \in [1, D-1] = \{1, \ldots, D-1\}$, the $k$-independence number $\alpha_k = \alpha_k(G)$ of $G$ is the maximum number of vertices that are at distance greater than $k$ from each other. This is a natural generalization of the well-known independence number $\alpha = \alpha_1$. In fact, the $k$-independence number of $G$ corresponds to the (standard) independence number of the power graph $G^k$, which has the same vertex set as $G$ and two vertices $u$ and $v$ are adjacent in $G^k$ when they are at distance at most $k$ in $G$. However, even the simplest algebraic or combinatorial parameters of $G^k$ cannot be deduced easily from the similar parameters of $G$. For instance, neither the spectrum (Das and Guo [17], and Abiad, Coutinho, Fiol, Nogueira, Zeijlemaker [3, Section 2]), nor the average degree (Devos, McDonald, Scheide [18]), nor the rainbow connection number (Basavaraju, Chandran, Rajendraprasad, and Ramaswamy [6]) can be, in general, derived directly from the original graph. This provides the initial motivation for this work.

The study of the $k$-independence number of a graph has received a considerable amount of attention. Kong and Zhao [37] showed that, for every $k \geq 2$, determining $\alpha_k(G)$ is an NP-complete problem for general graphs. They also

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proven that this problem remains NP-complete for regular bipartite graphs when \( k \geq 2 \) [38]. Duckworth and Zito extended a simple heuristic-based algorithm to approximate the independence number of connected random \( d \)-regular graphs to the 2-independence number [19]. Hota, Pal, and Pal [33] provided an efficient algorithm for finding a maximum weight \( k \)-independent set on trapezoid graphs. The \( k \)-independence number has also been studied in other contexts. For instance, Atkinson and Frieze [5] studied \( \alpha_k \) in relation to random graphs. The mentioned complexity results on \( \alpha_k \) provide motivation for finding tight bounds. Fibry and Haviland [26] showed lower and upper bound for \( \alpha_k(G) \) in a connected graph on \( n \) vertices as a function of \( n \) and \( k \). Beis, Duckworth, and Zito [7] provided upper bounds for \( \alpha_k(G) \) in random \( r \)-regular graphs for each of fixed integers \( k \geq 2 \) and \( r \geq 3 \). O. Shi, and Taoqiu [43] showed tight upper bounds for the \( k \)-independence number in an \( n \)-vertex regular graph for each positive integer \( k \geq 2 \) and \( r \geq 3 \) and with given minimum and maximum degrees. The case of \( k = 2 \) has also received some attention: Jou, Lin, and Lin [34] presented a tight upper bound for the 2-independence number of a tree. Recently, Li and Wu [39] gave bounds for the \( k \)-independence number of a graph \( G \) in terms of its order and vertex-connectivity. The \( k \)-independence number is also directly related to the study of distance-\( j \) ovoids in incidence geometry. This study started with generalized polygons by Thas [44], who investigated the existence of distance-2 ovoids in generalized quadrangles and distance-3 ovoids in generalized hexagons (which are simply known as ovoids). The existence of distance-\( j \) ovoids is related to the existence of particular perfect codes, see Cameron, Thas, and Payne [14], the separability of particular groups in Cameron and Kazanian [13], and various other topics.

A useful idea to find upper bounds for \( \alpha_k(G) \) is to consider the (adjacency) spectrum of the graph \( G \). The first eigenvalue bounds for the \( k \)-independence number, an inertia-type and a ratio-type bounds, were shown by Abiad, Cioabă, and Tait [1]. Such inertia-type and ratio-type bounds were improved by Abiad, Coutinho, and Fiol [2] by using more general polynomials that achieve equality for the corresponding bounds. The polynomials for the ratio type bound were optimized by Fiol [24] in the case of partially walk-regular graphs. Analogously, Abiad, Coutinho, Fiol, Nogueira, and Zeijlemaker [3] formulated mixed integer linear programs (MILP) that optimize the choice of polynomials for the inertia-type bound. Abiad, Elphick, and Wocjan [45] proved that the inertia-type bound for the \( k \)-independence number also applies to the quantum \( k \)-independence number. The fact that this quantum parameter is not known to be computable justifies the use of optimization methods to find the exact value of the bounds. A common feature of all the aforementioned spectral bounds for \( \alpha_k \) is that they involve various families of polynomials. Examples of this are the well-known predistance polynomials (which have been successfully applied for the characterization of distance-regular graphs, for instance, in the well-known Spectral Excess Theorem by Fiol and Garriga [25]), or the sign and minor polynomials, see Fiol [24].

Despite the fact that both bounds provide proof for the EKR theorem, not much is known about the relationship between both bounds besides the fact that both use some type of interlacing (see information on this technique in Haemers [30]). The results in [4,20] can all be viewed as attempts to provide unifications of such spectral bounds. In this work, we show that for graphs with enough regularity, the sign and the minor polynomials are sometimes closely related and provide exact values for \( \alpha_k \). This provides a relationship between the inertia-type and ratio-type bounds for \( \alpha_k \). To do so, we introduce the notion of \( k \)-Cvetković–Hoffman graphs to study when the two classes of polynomials that upper bound \( \alpha_k \) are linearly related and when they both provide tight bounds.

This paper is structured as follows. In Section 3, we focus on investigating \( k \)-Cvetković–Hoffman graphs for the extreme cases: \( k = 1 \) for regular graphs, and \( k = d - 1 \) for walk-regular graphs (where \( d \) is the number of distinct eigenvalues minus one). All tight 1-Cvetković–Hoffman graphs holding the bounds with equality have quantum independence number \( \alpha_k = \alpha \), thus we extend the list of graphs for which the quantum independence number is known. In the latter case, we consider the maximum cardinality of a set of vertices that are mutually at maximum distance, that is \( D = D \), where \( D \) is the diameter of \( G \). In particular, in Section 4, we study the existence and properties of \( d \)-cliques (also called \( d \)-spreads in the literature), that is, sets of vertices that are mutually at maximum distance \( d \), when \( G \) is a walk-regular graph. Our results provide a way to unify the inertia-type and ratio-type bounds for the \( k \)-independence number. In particular, for \( k \)-partially walk-regular graphs, we show that both bounds for \( \alpha_k \) can be seen as a linear combination of eigenvalue multiplicities. Note that, so far, a relationship between both bounds has only been found with \( k = 1 \) for graphs with enough regularity (see Haemers and Higman [31], where they established it for graphs with exactly three distinct eigenvalues, thus showing a characterization of tight 1-Cvetković–Hoffman graphs that are strongly regular). In Sections 2.2 and 5, we prove that such a relationship can also be shown for \( k \)-partially walk-regular graphs. Additionally, in Section 5, we use the predistance polynomials to provide new tight inertia and ratio bounds for \( k \)-partially walk-regular graphs.

2. Preliminaries

Let \( G = (V, E) \) be a graph with \( n = |V| \) vertices, \( m = |E| \) edges, and adjacency matrix \( A \) with spectrum 
\[
\{\theta_0^m, \theta_1^m, \ldots, \theta_d^m\},
\]
where the different eigenvalues are, in decreasing order, \( \theta_0 > \theta_1 > \cdots > \theta_d \) and the superscripts stand for their multiplicities. Since \( G \) is assumed to be connected, we have that \( m_0 = 1 \). When the eigenvalues are presented with possible repetitions, we shall indicate them by 
\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.
\]

Let us now recall some known concepts. A graph \( G \) is called walk-regular if the number of closed walks of any length from a vertex to itself does not depend on the choice of the vertex, a concept introduced by Godsil and McKay in [29].
A graph $G$ is called \textit{k-partially walk-regular} for some integer $k \geq 0$, if the number of closed walks of a given length $l \leq k$, rooted at a vertex $v$, only depends on $l$. In other words, if $G$ is a $k$-partially walk-regular graph, for any polynomial $p \in \mathbb{R}_k[x]$ (where $\mathbb{R}_k[x]$ denotes the polynomials on $x$ of degree at most $k$ with real coefficients) the diagonal of $p(A)$ is constant with entries

$$(p(A))_{uv} = \frac{1}{n} \text{tr} p(A) = \frac{1}{n} \sum_{i=1}^{n} p(\lambda_i) \quad \text{for all } u, v \in V.$$ 

Thus, every (simple) graph is $k$-partially walk-regular for $k = 0, 1$ and every regular graph is 2-partially walk-regular.

A graph $G$ is called \textit{distance-regular} if, for any two vertices $u$ and $v$ at distance $l$, the number of vertices at distance $i$ from $u$ and at distance $j$ from $v$, denoted by $p_{ij}^l$, depends only on $l, i, j$. A graph $G$ is called \textit{k-partially distance-regular} if it is distance-regular up to distance $k$. Note that every $k$-partially distance-regular is $2k$-partially walk-regular. Moreover, $G$ is $k$-partially walk-regular for any $k$ if and only if $G$ is walk-regular. For example, it is well known that every distance-regular graph is walk-regular (but the converse does not hold). A $d$-regular graph $G$ on $n$ vertices is \textit{strongly regular} if every pair of adjacent (respectively, non-adjacent) vertices has $a$ (respectively, $c$) common adjacent vertices. Then, the parameters of $G$ are indicated by $(n, d, a, c)$. So, if connected, $G$ is distance-regular with diameter two.

The first well-known spectral bound (\textit{inertia bound}) for the independence number $\alpha = \alpha_1$ of $G$ is due to Cvetković [15]:

$$\alpha \leq \min \{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}. \quad (1)$$

When $G$ is regular, another well-known bound (\textit{ratio bound}) for the independence number of $G$ is due to Hoffman (unpublished):

$$\alpha \leq \frac{n}{1 - \frac{\lambda_1}{\lambda_n}}. \quad (2)$$

For the sake of comparison, in Table 1, we show the values of the independence number, the (floor of the) Hoffman bound (2), and the Cvetković bound (1) of some known regular graphs. Note that, in all these examples, the ratio bound is either equal to or smaller than the inertia bound, and this seems to be the case in general. However, there are graphs where the contrary happens, such as the Clebsh graph and the Higman–Sims graph. These graphs can be found in Table 2, where all the (non-trivial) triangle-free strongly regular graphs are considered. The bounds of the Clebsh and Higman–Sims graphs are highlighted in bold. Note that, in both cases, their independence numbers coincide with their degree (the maximum independent sets are the neighborhoods of each vertex). In fact, this appears to be a very special property in strongly regular graphs. We checked it for strongly regular graphs with at most 250 vertices, and it only holds for the above two graphs and the pentagon (with parameters $(n, d, a, c) = (5, 2, 0, 1)$).

In addition to the Clebsh graph and Higman–Sims graph, the following strongly regular graphs from Brouwer’s database [8] have smaller inertia bound than ratio bound:

(a) The McLaughlin Family (see Brouwer and vanLint [11]): (112, 30, 2, 10), (162, 56, 10, 24), (243, 110, 37, 60), (275, 112, 30, 56), and (276, 140, 58, 84).

(b) $O^-(6, q)$, that is, the graphs whose vertices are points on an elliptic nondegenerate quadric in $PG(5, q)$, with adjacency defined by orthogonality: (27, 10, 1, 5), (112, 30, 2, 10), (325, 68, 3, 17), and (756, 130, 4, 26).

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2.2. Partially walk-regular graphs

Corollary 2.2. Let $G$ be a partially walk-regular graph with diameter $D$ and spectrum $(\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_d^{m_d})$. Let $h(x)$ be the Heaviside function, that is, $h(x) = 1$ if $x \geq 0$, and $h(x) = 0$ otherwise. Then (3) and (4) can be stated as follows in terms of the multiplicities.

(i) Let $p = s \in \mathbb{R}[x]$ be a polynomial satisfying $\lambda(s) = \min_{i \in [1,d]} \{s(\theta_i)\} = -1$ and $\text{tr} \, s(A) = 0$. Then,
\[
\alpha_k \leq \sum_{i=0}^d m_i h(s(\theta_i)).
\]
(ii) Let \( p = f \in \mathbb{R}_k[x] \) be a polynomial satisfying \( \lambda(f) = \min_{i \in [1,d]} \{ f(\theta_i) \} = 0 \) and \( f(\theta_0) = 1 \). Then,

\[
\alpha_k \leq \sum_{i=0}^{d} m_i f(\theta_i). \tag{8}
\]

Since both results concern the same parameter, one would expect that a ‘good’ polynomial \( s \) provides a ‘good’ polynomial \( f \), and vice versa.

**Remark 2.3.** The following relations between the polynomials \( s \) and \( f \) hold.

(i) If \( s \in \mathbb{R}_k[x] \) satisfies \( \min_{i \in [1,d]} \{ s(\theta_i) \} = -1 \) and \( \text{tr} s(A) = 0 \), then \( f(x) = \frac{1+s(x)}{1+s(\theta_0)} \) satisfies \( \min_{i \in [1,d]} \{ f(\theta_i) \} = 0 \) and \( f(\theta_0) = 1 \). Then, from (8), we get

\[
\alpha_k \leq \frac{n}{1+s(\theta_0)}. \tag{9}
\]

(ii) If \( f \in \mathbb{R}_k[x] \) satisfies \( \min_{i \in [1,d]} \{ f(\theta_i) \} = 0 \) and \( f(\theta_0) = 1 \), then \( s(x) = \frac{n}{\text{tr} f(A)} f(x) - 1 \) has \( \text{tr} s(A) = 0 \) and \( \min_{i \in [1,d]} \{ s(\theta_i) \} = -1 \). Then, from (7), we get

\[
\alpha_k \leq \sum_{i=0}^{d} m_i h (f(\theta_i) - \frac{1}{n} \text{tr} f(A)). \tag{10}
\]

### 2.2.1. Optimizing the upper bounds

To optimize the upper bounds (3) and (4), the following polynomials were introduced by Abiad, Coutinho, Fiol, Nogueira, and Zeijlemaker [3] and Fiol [24], respectively.

**Bound (7):** If \( s \in \mathbb{R}_k[x] \) is a polynomial satisfying \( \text{tr} s(A) = 0 \), the best result is obtained by the so-called sign polynomial \( s_k \), obtained as follows. Let \( b = (b_0, \ldots, b_d) \in [0,1]^{d+1} \) and \( m = (m_0, \ldots, m_d) \) (the vector of multiplicities). For a given \( k < D \leq d \), the sign polynomial \( s_k(x) = a_k x^k + \cdots + a_0 \) is the one with coefficients being the solution of the following MILP (mixed-integer linear programming) problem with variables \( a_1, \ldots, a_k \) and \( b_0, \ldots, b_d \):

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=0}^{d} m_i b_i \\
\text{subject to} & \quad \sum_{i=0}^{d} m_i s_k(\theta_i) = 0 \\
& \quad \sum_{i=0}^{d} a_i \theta_j^i - Mb_j + \epsilon \leq 0, \quad j = 0, \ldots, d \quad (\ast) \\
& \quad b \in [0,1]^{d+1}
\end{align*}
\tag{11}
\]

Here \( M \) is set to be a large integer and \( \epsilon > 0 \) a small number. The idea of the formulation is that each \( b_j = 1 \) represents an index \( j \) so that \( s_k(\theta_j) \geq \omega(p) = 0 \). In fact, condition (\ast) gives that \( s_k(\theta_j) \geq 0 \) implies \( b_j = 1 \). So, upon minimizing the number of such indices \( j \), we are optimizing \( s_k(x) \) and the corresponding bound \( \alpha_k \leq \sum_{i=0}^{d} m_i b_i \). See Abiad, Coutinho, Fiol, Nogueira, and Zeijlemaker [3] for more details.

**Bound (8):** If \( f \in \mathbb{R}_k[x] \) is a polynomial satisfying \( \lambda(f) = 0 \) and \( f(\theta_0) = 1 \), the best result is obtained with the so-called minor polynomial \( f_k \) that minimizes \( \sum_{i=0}^{d} m_i f_k(\theta_i) \). This polynomial \( f_k \) is defined by \( f_k(\theta_0) = x_0 = 1 \) and \( f_k(\theta_i) = x_i \) for \( i = 1, \ldots, d \), where the vector \( (x_1, x_2, \ldots, x_d) \) is a solution of the following linear programming (LP) problem:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=0}^{d} m_i x_i \\
\text{subject to} & \quad f[\theta_0, \ldots, \theta_m] = 0, \quad m = k + 1, \ldots, d \\
& \quad x_i \geq 0, \quad i = 1, \ldots, d
\end{align*}
\tag{12}
\]

Here, \( f[\theta_0, \ldots, \theta_m] \) denote the \( m \)th divided differences of Newton interpolation, recursively defined by \( f[\theta_1, \ldots, \theta_j] = \frac{f[\theta_1, \ldots, \theta_j] - f[\theta_1, \ldots, \theta_j-1]}{\theta_j - \theta_{j-1}} \), where \( j > i \), starting with \( f[\theta_i] = p(\theta_i) = x_i \) for \( 0 \leq i \leq d \). Note that by equating these values to zero, we guarantee that \( f_k \in \mathbb{R}_k[x] \).

Some known examples, properties, and approximations of the minor polynomials (‘MP’) are the following. For more details, see Fiol [24].

**MP0.** For every \( k = 0, 1, \ldots, d \), every \( k \)-minor polynomial \( f_k \) has degree \( k \) with its zeros in the interval \( [\theta_d, \theta_0] \subset \mathbb{R} \). Moreover, \( f_k \) can always be chosen to have its \( k \) zeros in the mesh \( \{ \theta_1, \ldots, \theta_d \} \) (see [24, Prop. 3.2]). In the remainder of this section, we always choose \( f_k \) such that it satisfies this condition.

**MP1.** \( f_0(x) = 1 \) and \( f_1(x) = \frac{x - \theta_0}{\theta_d - \theta_0} \). The degree one polynomial gives the bound \( \alpha_1 \leq n \frac{\theta_d - \theta_0}{\theta_d - \theta_0} \), which corresponds to the Hoffman bound (2).
\textbf{MP2.} \(f_2(x) = \frac{(x-\theta_1)(x-\theta_{i+1})}{(\theta_0-\theta_1)(\theta_0-\theta_{i+1})},\) where \(\theta_i\) is the smallest eigenvalue greater than \(-1\). This gives the following bound (see Abiad, Fiol, and Coutinho [2]):
\[
\alpha_2 \leq n \frac{\theta_0 + \theta_{i+1}}{(\theta_0 - \theta_1)(\theta_0 - \theta_{i+1})}.
\]

\textbf{MP3.} \(f_3(x) = \frac{(x-\theta_1)(x-\theta_{i+1})}{(\theta_0-\theta_1)(\theta_0-\theta_{i+1})},\) where, as before, \(\theta_i\) is the smallest eigenvalue such that \(\theta_i > -1\) (in fact, the only possible zeros of \(f_3\) are \(\theta_d\) and the consecutive pair \(\theta_i, \theta_{i+1}\) for some \(i \in [1, d - 1]\)). Then the polynomial \(f_3f_2\) gives the following bound (see Fiol [24]):
\[
\alpha_3 \leq n \frac{\Delta - \theta_0 + \theta_d - \theta_0(\theta_i + \theta_{i+1} + \theta_d)}{(\theta_0 - \theta_1)(\theta_0 - \theta_{i+1})(\theta_0 - \theta_d)},
\]
where \(\Delta = u(x^2) = W(x^2) = (A^3)_{uu}\) for any \(u \in V\), (recall that here we are assuming that \(G\) is 3-partially walk-regular).

\textbf{MP4.} \(f_5(x) = \frac{1}{g(\theta_0)}(g(x) - \lambda(g)),\) where \(g(x) = x^3 + bx^2 + cx\) is the polynomial with coefficients \(b = -(\theta_1 + \theta_{i+1} + \theta_d)\) and \(c = \theta_0\theta_d + \theta_i\theta_{i+1} + \theta_0\theta_{i+1}\), and \(\theta_i\) is the smallest eigenvalue such that
\[
\theta_i \geq -\frac{\theta_0^2 + \theta_0\theta_d - W(x^3)}{\theta_0(1 + \theta_d)},
\]
where \(W(x^3) = \max_{v \in V}(|(A^3)_{uu}|)\) (see Kavi, Newman, and Sajna [36]). In fact, this exact formula for \(f_5\) gives the same expression as Eq. (14) for the bound of \(\alpha_3\). This means that both polynomials, \(f_3f_2\) and \(f_5\), have zeros \(\theta_i, \theta_{i+1},\) and \(\theta_d\) where, in each case, \(\theta_i\) is the smallest eigenvalue satisfying the condition in \textbf{MP3} or \textbf{MP4}, respectively. To compare both conditions, note that
\[
-\frac{\theta_0^2 + \theta_0\theta_d - W(x^3)}{\theta_0(1 + \theta_d)} \leq -1 \iff W(x^3) \leq \theta_0(\theta_0 - 1).
\]

Since the condition on \(W(x^3)\) holds for any (regular) graph, we infer that the condition in \textbf{MP3} for the first zero of \(f_3f_2\), that is \(\theta_i > -1\), implies condition (15) in \textbf{MP4} for the first zero of \(f_5\). In other words, the condition in \textbf{MP3} is weaker than the one in \textbf{MP4} (in principle), but in most cases the resulting \(\theta_i\) could be the same, and so the approximation in \textbf{MP3} could be exact. In this case, \(f_5 = f_3f_2\).

\textbf{MP5.} \(f_{d-1}\) takes only one non-zero value at the mesh \([\theta_1, \ldots, \theta_d]\), which, because of the condition \(\lambda(f_{d-1}) = 0\), must be located at some eigenvalue \(\theta_i\) with odd index \(i\). In fact, experimental evidence seems to suggest that such eigenvalue is either \(\theta_i\) or one of \(\theta_{i-1}, \theta_{d}\) (depending on the parity of \(i\)). This gives the bound
\[
\alpha_{d-1} \leq \min_{i \text{ odd}} \left\{1 + \frac{m_i}{\pi_0}\right\},
\]
where \(\pi_i = \prod_{j \neq i} |\theta_i - \theta_j|\) (see Fiol [24]).

\textbf{MP6.} \(f_0(x) = \frac{1}{n}H(x),\) where \(H\) is the Hoffman polynomial [32] characterizing regularity by \(H(A) = J\), the all-1 matrix.

\textbf{3. k-Cvetković-Hoffman graphs}

From the above definitions, it seems interesting to know when, for a given value of \(k\), the \(k\)-sign polynomial and the \(k\)-minor polynomial are linearly related according to Remark 2.3. Moreover, if both polynomials give the same (inertia and ratio) bounds on the \(k\)-independence number, we can consider such polynomials to be ‘essentially’ the same. A graph satisfying this property will be called a \(k\)-Cvetković-Hoffman graph (\(k\)-CH graph for short). If the equal bounds are tight as well, we call it a tight \(k\)-CH graph. These definitions are motivated by the following results concerning the cases \(k = 1\) and \(k = d - 1\).

We begin with two simple cases.

\textbf{Lemma 3.1.}

\textbf{(i)} For any graph, the 1-sign polynomial and the 1-minor polynomial are linearly related.

\textbf{(ii)} Every regular bipartite graph with an even number \(d + 1\) of different eigenvalues is a tight 1-CH graph.

\textbf{Proof.} Note that (1) implies that the 1-sign polynomial of \(G\) is \(s_1 = \pm x\), whereas the 1-minor polynomial is a linear function, as shown in \textbf{MP1}. This proves (i).

Under the hypothesis of (ii), the spectrum of \(G\) is symmetric around 0 and it has no zeros. Hence, if \(G\) has \(n\) vertices, both polynomials \(s_1\) and \(f_1\) give the exact independence number \(\alpha = \frac{n}{2}\). \(\square\)

A more interesting result is the following characterization of tight 1-CH strongly regular graphs, due to Haemers and Higman in [31, Theorem 2.6].
Theorem 3.2 ([31]). Let $G$ be a strongly regular graph with maximum independent set $U \subset V$. Then both the inertia and ratio bounds are tight if and only if the graph induced by $U = V \setminus U$ is strongly regular.

The other extreme case is when $k = d - 1$. Then, we deal with maximally independent sets of vertices. For the case of triangle-free strongly regular graphs, we have the following result.

Lemma 3.3. Let $G(n)$ denote a triangle-free strongly regular. If $G(n)$ exists with feasible parameters

\[
(n^4 + 5n^3 + 6n^2 - n - 1, n^2(n + 2), 0, n^2) \quad \text{for } n = 1, 2, \ldots,
\]

then it is a tight 1-CH graph.

Proof. Since the spectrum of a strongly regular graph is related to its parameters (see, for instance, Godsil [28]), the proof is a simple computation. □

In fact, the only known graphs of this family are $G(1) = P$, the Petersen graph with parameters $(5, 2, 0, 1)$, and $G(2) = M_{22}$, the Mesner graph with parameters $(77, 16, 0, 4)$. Brouwer [9] told us that this family is mentioned in Brouwer and van Maldeghem [10, Section 8.5.8] as subconstituents of the ‘Krein graphs without triangles’, which are only known for $r = 1$, the Clebsch graph, and for $r = 2$, the Higman–Sims graph. For $r = 3$, no example exists and nothing is known for $r > 3$. According to Brouwer, it could be possible that $G(3)$ can be embedded in a $(324, 57, 0, 12)$ strongly regular graph as a second subconstituent, and then it would follow that such a graph does not exist.

Another infinite family of tight 1-CH is given by the Kneser graphs. Given integers $n$ and $k \leq n$, the Kneser graph $K(n, k)$ has as vertices the $\binom{n}{k}$-subsets of the set $[1, n]$, and two vertices are adjacent if and only if their corresponding subsets are disjoint. For instance, $K(n, 2)$ is the complement of the triangular graph $T_n = L(K_n)$ for all $n \geq 3$, where $L(G)$ represents the line graph of $G$. The Kneser graph $K(n, k)$ has order $\binom{n}{k}$, eigenvalues $\mu_j = (-1)^{j} \binom{n-k-j}{k-j}$ for $j = 0, \ldots, k$, and multiplicities $m_0 = 1$ and $m_j = \binom{n}{j} - \binom{n}{j-1}$ for $j > 0$. (Notice that, with this notation, the eigenvalues $\mu_0, \mu_1, \ldots, \mu_k$ are not in decreasing order. In particular, the above least eigenvalue $\theta_0$ is now $\mu_1$.) If $n = 2k$, the Kneser graph $K(n, 2k)$ is disconnected (constituted by different copies of $K_2$), and we omit this trivial case in the following result.

Corollary 3.4. The Kneser graph $K(n, k)$, with $n > 2k$, is a tight 1-CH graph.

Proof. From the above values of the eigenvalues and multiplicities, we find that both the inertia and ratio bounds (with the respective sign and minor polynomials) give $\binom{n-1}{k-1}$. Under the hypothesis and by the Erdős–Ko–Rado Theorem [21] (see Katona [35] for a simple proof), this is known to be the exact value of the independence number $\alpha$ of $K(n, k)$. In fact, note that a maximum set of independent vertices can be formed by considering all $k$-subset containing some fixed digit in $[1, n]$. □

Other examples of tight 1-CH graphs are the Taylor 2-graphs for $U_3(q)$ with $q \in \{3, 5, 7, 9\}$.

In the next section, we prove, among other facts, the following results for graphs with larger diameter:

(i) The Odd graph $O_\ell$, with even degree $\ell$, is a $(d-1)$-CH graph. Moreover, the Odd graph $O_\ell$ is a tight $(d-1)$-CH graph for every $\ell \in \{2, 3, 4, 6, 7, 8, 10, 12, 14, 16\}$ (see Table 5).

(ii) The antipodal distance-regular graphs with odd diameter are tight $(d-1)$-CH graphs.

4. Maximally independent sets

In this section, we focus on the extreme case when $G$ is $(d-1)$-partially walk-regular (that is, walk-regular), and we search for the maximum cardinality of a set $U$ of vertices that are mutually at distance $d$ (that is, $k = d - 1$). In the literature, such a set is known as a $d$-spread or $d$-clique. Thus, the $(d-1)$-independence number $\alpha_{d-1}$ coincides with the so-called the $d$-clique number or $d$-spread number $\alpha_d$. In this context, Dalfó, Fiol, and Garriga [16] proved the following result.

Theorem 4.1 ([16]). Let $G$ be a walk-regular graph on $n$ vertices with spectrum $\{\theta_0^{m_0}, \ldots, \theta_d^{m_d}\}$ and spectrally maximum diameter $D = d$ (that is, the number of distinct eigenvalues minus one). Let $U \subset V$ be a $d$-clique with $r$ vertices. Then the set of projected points $E_i U = \{E_i e_u : u \in U\}$, where $E_i$ is the $i$th idempotent representing the projection on the $\theta_i$-eigenspace $E_i \cong \mathbb{R}^{m_i}$, are the vertices of an $(r-1)$-simplex (that is, an $(r-1)$-dimensional polytope which is the convex hull of its $r$ vertices) in $E_i$, with barycenter $c_i = \frac{1}{r} \sum E_i e_u$ at distance from $S$ to the origin, radius $R$ (distance from $c_i$ to the origin), and edge length $L$ satisfying

\begin{align}
S &= \sqrt{n \left( \frac{m_i + (-1)^{(r-1)} \mu_0}{\mu_1} \right)^2}, \\
R &= \sqrt{\frac{r-1}{n} \left( \frac{m_i + (-1)^{(r-1)} \mu_0}{\mu_1} \right)^2}. \tag{17}
\end{align}
\[ L = \sqrt{\frac{2}{n} \left( m_i - (-1)^{\pi_0} \frac{\pi_0}{\pi_i} \right)}, \]  

where \( \pi_i = \prod_{j \neq i} |\theta_i - \theta_j| \) for \( i = 0, \ldots, d \).

Since \( R, L \geq 0 \) and the maximum number of points mutually at a given distance equals \( m_i + 1 \) in an \( m_i \)-dimensional space, we have the following consequence.

**Corollary 4.2.** Let \( G \) be a walk-regular graph as above. Let \( U \subset V \) be a d-clique with \( r \) vertices. Then the eigenvalue multiplicities and the \((d - 1)\)-independence number satisfy the following results.

(i) If \( i \) is even, then \( m_i \geq \frac{R_0}{\pi_i} \). Besides, if \( m_i \neq \frac{R_0}{\pi_i} (R \neq 0) \), then \( \alpha_{d-1} \leq 1 + m_i \).

(ii) If \( i \) is odd, then \( m_i \geq (r - 1)\frac{R_0}{\pi_i} \) and \( \alpha_{d-1} \leq 1 + m_i \frac{R_0}{\pi_i} \).

Moreover, equality for the multiplicity in (i) is attained if and only if the simplex with vertices \( E_i U \) collapses into a point \((L = R = 0)\), while equality for the multiplicity in (ii) is attained if the corresponding simplex is centered at the origin \((S = 0)\).

Using our polynomials \( s(x) \) and \( f(x) \), we can obtain an alternative proof of the above upper bounds. First, notice that if \( D < d \), then \( \alpha_{d-1} = 1 \), so we may assume that \( G \) has spectrally maximum diameter \( D = d \).

**Theorem 4.3.** Let \( G \) be a walk-regular graph with spectrum \( \{\theta_0, \theta_1, \ldots, \theta_d\} \). Let \( \pi_i = \prod_{j \neq i} |\theta_i - \theta_j| \), for \( i = 0, \ldots, d \). Then the following holds.

(i) For every \( i = 1, \ldots, [d/2] \) such that \( m_{2i} \neq \frac{R_0}{\pi_{2i}} \),

\[ \alpha_{d-1} \leq m_{2i} \quad \text{(inertia bound).} \]  

(ii) For every \( i = 1, \ldots, [d/2] \),

\[ \alpha_{d-1} \leq 1 + m_{2i-1} \quad \text{(inertia bound).} \]  

and

\[ \alpha_{d-1} \leq 1 + m_{2i-1} \frac{\pi_{2i-1}}{\pi_0} \quad \text{(ratio bound).} \]  

**Proof.** (i) Let \( \tilde{s} \in \mathbb{R}_{d-1}[x] \) be the monic polynomial with zeros at \( \theta_j \) for \( j \neq 0, 2i \) for some \( i = 1, \ldots, [d/2] \), that is, \( \tilde{s}(x) = \prod_{j \neq 0, 2i} (x - \theta_j) \). Then \( \tilde{s}(\theta_0) > 0, \tilde{s}(\theta_{2i}) < 0 \) and

\[
\text{tr} \, \tilde{s}(A) = \sum_{j=0}^{d} m_j \tilde{s}(\theta_j) = \tilde{s}(\theta_0) + m_{2i} \tilde{s}(\theta_{2i}) = m_0 \frac{\pi_0}{\theta_0 - \theta_{2i}} + m_{2i} \frac{\pi_{2i}}{\theta_{2i} - \theta_0} \\
= (\theta_0 - \theta_{2i})^{-1} \left[ \pi_0 - m_0 \pi_{2i} \right].
\]

Now, we claim that \( \text{tr} \, \tilde{s}(A) \leq 0 \). By contradiction, if \( \text{tr} \, \tilde{s}(A) > 0 \), there is a constant \( \epsilon > 0 \) such that the polynomial \( s'(x) = s - \epsilon \) has \( \text{tr} \, s'(A) = 0 \), and takes only one positive value at \( \theta_0 \). Then, Corollary 2.2(i) would imply that \( \alpha_{d-1} \leq 1 \), contradicting that the diameter is \( d \). (Notice that the condition \( \lambda(s') = -1 \) is not necessary to apply to the corollary, although we could multiply \( s' \) by a constant, if desired.) Moreover, from (23) and the hypothesis on \( m_{2i} \), we know that \( \text{tr} \, \tilde{s}(A) \neq 0 \). Hence, \( \text{tr} \, \tilde{s}(A) < 0 \) and there exists a constant \( \sigma > 0 \) such that the polynomial \( s(x) = \tilde{s}(x) + \sigma \) satisfies \( s(x) = 0 \), and it takes the only negative value at \( \theta_{2i} \). See Fig. 1(a) for an example with \( d = 4 \). Thus, the result follows from Corollary 2.2(i) by using the polynomial \( -s(x) \).

To prove (ii), consider first the polynomial \( \tilde{s}(x) = \prod_{j \neq 0, 2i-1} (x - \theta_j) \), now satisfying \( \text{tr} \, \tilde{s}(A) > 0 \) since \( \tilde{s}(\theta_0), \tilde{s}(\theta_{2i-1}) \neq 0 \). Then, there exists a constant \( \tau > 0 \) such that the polynomial \( s(x) = \tilde{s}(x) - \tau \) has \( \text{tr} \, s(A) = 0 \) and possible positive values only at \( \theta_0 \) and \( \theta_{2i-1} \), as illustrated in Fig. 1(b). Thus, Corollary 2.2(i) gives (21). To prove (20), we consider the polynomial \( f(x) = \frac{s(x)}{\pi_0} \), which satisfies the conditions in Corollary 2.2(ii), that is, \( \lambda(f) = 0 \) and \( f(\theta_0) = 1 \). Then, the result comes from \( \alpha_{d-1} \leq \text{tr} \, f(A) \).

Of course, in (i) we could derive a ratio bound by applying Corollary 2.2(ii) with the polynomial \( f(x) = \frac{\tilde{s}(x) + \tilde{s}(\theta_0)}{\tilde{s}(\theta_0) + \tilde{s}(\theta_{2i})} \), but the result is not good, since \( \lambda(f) > 0 \) (instead of \( \lambda(f) = 0 \), as it should be when we use the optimal minor polynomials.) In fact, the minor polynomial \( f_{d-1} \) only takes non-zero values at \( \theta_0 \) and \( \theta_{2i} \) for some \( i \), odd, as in case (ii).

Note that in (ii), one of the bounds (21) or (22) is better than the other depending on whether \( \pi_{2i-1} > \pi_0 \) or \( \pi_{2i-1} < \pi_0 \). However, it appears that the second bound is always either equal to or better than the first one. Although sometimes \( \frac{\pi_{2i-1}}{\pi_0} > 1 \), there always seem to be \( i, j \) such that \( m_{2i-1} \geq \frac{\pi_{2i-1}}{\pi_0} m_{2i-1} \). For instance, we checked all small graphs up to seven vertices, all Sage’s named graphs and many well-known graph families, and this is always the case. Moreover, the bounds coincide when \( \pi_{2i-1} = \pi_0 \). As we will see later, this happens for the Odd graphs \( G_{\ell} \) with even \( \ell = d + 1 \) and \( 2i - 1 = d \). In fact, the exact value of \( \frac{\pi_{2i-1}}{\pi_0} \) can be easily computed for some families of graphs. The following list gives some examples.
Corollary 4.4. Let G be a walk-regular graph with spectrum \(\{\theta_0, \theta_1, \ldots, \theta_d\}\) and diameter \(D = d\).

(i) If \(d\) is even, then \(\alpha_{d-1} \leq m_d\).

(ii) If \(d\) is odd, then \(\alpha_{d-1} \leq 1 + m_d \cdot \min(1, \frac{\pi_2}{\pi_0})\).

Next, we will show that the bounds of Corollary 4.4 are tight for some Odd graphs. First, recall that, for every integer \(\ell \geq 2\), the Odd graph \(O_\ell\) can be defined as the Kneser graph \(K(2\ell - 1, \ell - 1)\). In other words, \(O_\ell\) has vertices corresponding to the \((\ell - 1)\)-subsets of a \((2\ell - 1)\)-set, and the adjacencies are defined by void intersection. Note that \(O_3\) is the Petersen graph \(P\). In general, \(O_\ell\) is an \(\ell\)-regular graph of order \(n = (2\ell - 1)^{\ell - 1}\), diameter \(D = \ell - 1\), and its eigenvalues and multiplicities are \(\mu_i = (-1)^i(\ell - i)\) and \(m(\mu_i) = m_i = \frac{(2\ell - 1)^{\ell - i} - (\ell - i)^{\ell - i}}{\ell - i}\) for \(i = 0, 1, \ldots, \ell - 1\), so that \(d = D = \ell - 1\). As for general Kneser graphs, these eigenvalues \(\mu_0, \mu_1, \ldots, \mu_d\) are again not in decreasing order, and the least eigenvalue is \(\mu_1\). The following result was shown by Fiol [24] and Abiad, Coutinho, Fiol, Nogueira, and Zeijlemaker [3].

Proposition 4.5 ([3,24]). For \(i = 0, \ldots, \ell - 1\), let \(\mu_i\) and \(m_i\) be the eigenvalues and multiplicities of the Odd graph \(O_\ell = O_{d+1}\). Then,

\[
\alpha_{d-1}(O_{d+1}) \leq \begin{cases} 
1 + \frac{\pi_2}{\pi_0} & \text{for even } d, \\
1 + \frac{\pi_2}{\pi_0} + \frac{\pi_2}{\pi_0} & \text{for odd } d.
\end{cases}
\]

(24)

Now we provide an alternative proof of Proposition 4.5 using our polynomials. Moreover, we also show that, for \(\ell\) even, the Odd graphs \(O_\ell\) are \((d - 1)\)-CH graphs (that is, the inertia-type and ratio-type bounds for \(\alpha_{d-1}(O_\ell)\) coincide and are obtained by sign and minor polynomials that are linearly related), as stated in CH1 at the end of the previous section.

Proposition 4.6. For \(i = 0, \ldots, \ell - 1\), let \(\mu_i\) and \(m_i\) be the eigenvalues and multiplicities of the Odd graph \(O_\ell = O_{d+1}\).

(i) The \((d - 1)\)-independence number of \(O_{d+1}\) satisfies

\[
\alpha_{d-1}(O_{d+1}) \leq 1 + m_2 \frac{\pi_2}{\pi_0} = \begin{cases} 
2d + \frac{d - 2}{2d + 1} & \text{for even } d, \\
2d + \frac{d - 2}{2d + 1} & \text{for odd } d.
\end{cases}
\]

(25)

(ii) If \(\ell\) is even, \(O_\ell\) is a \((d - 1)\)-CH graph. Moreover, in this case,

\[
\frac{m_1}{m_2} = \frac{\pi_2}{\pi_0}.
\]

(26)

Proof. (i) This is because the minor polynomial \(f_{d-1} = f_{\ell-2}\) of \(O_{d+1} = O_\ell\) can be taken as \(f_{d-1}(x) = \prod_{\mu \neq 0.2(\mu - \mu_i)}\), which gives

\[
\text{tr}_{f_{d-1}}(A) = 1 + m_2 f_{d-1}(\mu_2) = 1 + m_2 \frac{\pi_2}{\pi_0} = \begin{cases} 
2d + \frac{d - 2}{2d + 1} & \text{for even } d, \\
2d + \frac{d - 2}{2d + 1} & \text{for odd } d.
\end{cases}
\]
In Table 4, we show their values at the mesh \( \theta \) of the minor polynomial \( f_d(x) \) and sign polynomial \( s_d(x) \) of the Odd graphs \( O_5 \) and \( O_6 \).

\[
\begin{array}{cccccccc}
\hline
k & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{tr } f_d(A) \\
\hline
1 & 1 & 0 & 2/9 & 5/9 & 7/9 & 1 & 56 \\
2 & 5/14 & 0 & 0 & 5/14 & 1 & 13.5 \\
3 & 0 & 0 & 0 & 0 & 5/18 & 1 & 8.5 \\
\hline
\end{array}
\]

The values \( x_i = f_k(\theta_i) \) at the different eigenvalues \( \theta_d < \theta_{d-1} < \cdots < \theta_0 \) of the Odd graphs \( O_5 \) and \( O_6 \) are shown in Table 3.

If the bounds are tight, there are some interesting examples where the inertial and ratio bound coincide (and, as commented, the involved polynomials are linearly related according to Remark 2.3.). For instance, in the Odd graph \( O_6 \), the minor polynomial for \( k = 4 \) is

\[
f_d(x) = \frac{1}{15}(x^4 - 2x^2 - 13x^2 + 14x + 24),
\]

whereas the corresponding sign polynomial is

\[
s_d(x) = \frac{1}{12}(x^4 - 2x^2 - 13x^2 + 14x + 12).
\]

In Table 4, we show their values at the mesh \( \theta_5, \theta_4, \ldots, \theta_0 \), together with the value of the traces of the matrices when evaluated at \( A \). Then, the ratio bound for \( \alpha_4 \) is 11, which coincides with the inertia bound \( m_0 + m_5 = 11 \).

(ii) For every Odd graph \( O_e \) with even \( e \) (that is, odd \( d \)), we can also take the minor polynomial \( f_{d-1} \) whose only non-zeros are at \( \theta_0 = \ell \) and \( \theta_d = -\ell + 1 \), giving again the bound \( \alpha_{d-1} \leq 2d + 1 \). In fact, this is the same bound obtained by the sign polynomial that, following Remark 2.3.(ii), can be written as \( s_{d-1}(x) = \text{tr } f_d(A) - 1 \). Hence, the minor and sign polynomials are linearly related. Finally, (26) comes from equating the bounds in (24) and (25) when \( \ell \) is even.

The values \( x_i = f_k(\theta_i) \) at the different eigenvalues \( \theta_d < \theta_{d-1} < \cdots < \theta_0 \) of the Odd graphs \( O_5 \) and \( O_6 \) are shown in Table 3.

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In Abiad, Coutinho, Fiol, Nogueira, and Zeilinger [3], it was shown that the bounds of Proposition 4.5 are tight for \( \ell \in \{4, 6, 7, 8, 10, 12, 14\} \), and this also holds for \( \ell = 2 \) (\( K_4 \)) and \( \ell = 3 \) (the Petersen graph). However, this is not the case when \( \ell \in \{5, 9, 11\} \). The known exact values of \( \alpha_{d-1} \) and the corresponding upper bounds are shown in Table 5. The exact values for odd \( \ell \) were found by computer search. However, when \( \ell \) is even, the exact value of \( \alpha_{d-1} \) can be proved theoretically through its relation with symmetric designs.

Let \( \ell \geq 4 \) be even. Then the bound in Proposition 4.5 is tight, that is, \( \alpha_{\ell-2}(O_e) = \ell - 1 \), if and only if the vertices of a maximum \((\ell-2)\)-independent set constitute a \( 2 - (2\ell - 1, \ell - 1, \frac{1}{2} \ell - 1) \) symmetric design (see Hall [41] for its definition). In terms of intersecting set systems, this is equivalent to finding the largest system of \((\ell - 1)\)-subsets in a \((2\ell - 1)\)-set such that the intersection of any two sets has size \( \ell/2 - 1 \). Such designs are known to exist for \( \ell = 4, 6, \ldots, 16 \) (see Stinton [42]), which correspond to the entries in Table 5. Moreover, there exists a Hadamard matrix of order \( 4m \) if and only if there exists a symmetric \( 2-(4m - 1, 2m - 1, m - 1) \)-design, for \( m > 1 \) (see Stinton [42, Th. 4.5] again). It is known that Hadamard matrices of order \( 4m \) exist whenever \( 4m = 2^n \). Then, the bound in Proposition 4.5 is also tight for every \( \ell \) a power of 2. It would be interesting to know other exact values in order to show a more general result for other values of \( \ell \).

Next, we extend a result by Dalfó, Fiol, and Garriga [16] by showing that every antipodal distance-regular graph with odd diameter is a tight \((d - 1)\)-CH graph (see CHZ).

**Proposition 4.7.** Let \( G \) be a walk-regular graph with spectrum \( \{\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\} \), diameter \( D = d \), and \((d - 1)\)-independence number \( \alpha_{d-1} = r \). Then, for \( i = 1, \ldots, [d/2] \), the multiplicities satisfy the bounds

\[
m_{2i} \geq \frac{\pi_0}{\pi_{2i}} \quad \text{and} \quad m_{2i-1} \geq (r - 1) \frac{\pi_0}{\pi_{2i-1}}.
\]

Moreover, if the mean number of vertices at distance \( d \) from every vertex equals \( r - 1 \), equalities hold in (27) for every \( i = 1, \ldots, [d/2] \) if and only if \( G \) is an \( r \)-antipodal distance-regular graph (that is, a distance-regular graph in which the sets of antipodal vertices have all cardinality \( r \)).
Table 5
The known exact values of $\alpha_{d-1}$ and the upper bounds for the Odd graphs $O_{d+1}$.

| Graph       | $\alpha_{d-1}$ | Bound                          |
|-------------|----------------|--------------------------------|
| $O_2 (K_3)$ | $\alpha_0 = 3$ | $m_0 + m_1 = 3$                |
| $O_3$ (Petersen) | $\alpha_1 = 4$ | $m_1 = 4$                      |
| $O_4$       | $\alpha_2 = 7$ | $m_0 + m_1 = 7$                |
| $O_5$       | $\alpha_3 = 7$ | $m_1 = 8$                      |
| $O_6$       | $\alpha_4 = 11$| $m_0 + m_1 = 11$               |
| $O_7$       | $\alpha_5 = 12$| $m_1 = 12$                     |
| $O_8$       | $\alpha_6 = 15$| $m_0 + m_1 = 15$               |
| $O_9$       | $\alpha_7 = 15$| $m_1 = 16$                     |
| $O_{10}$    | $\alpha_8 = 19$| $m_0 + m_1 = 19$               |
| $O_{11}$    | $\alpha_9 = 19$| $m_1 = 20$                     |
| $O_{12}$    | $\alpha_{10} = 23$| $m_0 + m_1 = 23$             |
| $O_{14}$    | $\alpha_{12} = 27$| $m_0 + m_1 = 27$             |
| $O_{16}$    | $\alpha_{14} = 31$| $m_0 + m_1 = 31$             |

Proof. Note that in the proof of Theorem 4.3, we already showed that $m_{2i} \geq \frac{\pi_0}{\pi_{2i}}$. Finally, $m_{2i-1} \geq (r - 1 - \frac{\pi_0}{\pi_{2i-1}})$ is a consequence of (22).

For the case of equality, let us first show that $G$ is distance-regular. For this, we can use the spectral excess theorem by Fiol and Garriga [25], which states that a regular graph is distance-regular if and only if the spectral excess $p_d(\theta_0)$ (see Eq. (29)) equals the average excess $\overline{\theta}_d$ (the mean of the number of vertices at distance $d$ from each vertex, in our case $r - 1$). Note that $G$ has order

$$n = \sum_{i=0}^{d} m_i = \sum_{i \text{ even}} \frac{\pi_0}{\pi_i} + (r - 1) \sum_{i \text{ odd}} \frac{\pi_0}{\pi_i} = r \sum_{i \text{ odd}} \frac{\pi_0}{\pi_i} = \frac{r}{2} \sum_{i=0}^{d} \frac{\pi_0}{\pi_i},$$

(28)

where we used that $\sum_{i \text{ even}} \frac{\pi_0}{\pi_i} = \sum_{i \text{ odd}} \frac{\pi_0}{\pi_i}$ (see Fiol [23]). Thus, $\sum_{i=0}^{d} \frac{\pi_0}{\pi_i} = \frac{2n}{r}$. Combined with the expressions for the multiplicities, this gives

$$\sum_{i=0}^{d} \frac{\pi_i^2}{m_i \pi_i^2} = \sum_{i \text{ even}} \frac{\pi_0}{\pi_i} + \sum_{i \text{ odd}} \frac{\pi_0}{(r - 1)\pi_i} = \sum_{i \text{ even}} \frac{\pi_0}{\pi_i} \left(1 + \frac{1}{r - 1}\right) = \frac{r}{2(r - 1)} \sum_{i=0}^{d} \frac{\pi_0}{\pi_i} = \frac{n}{r - 1}.$$

Consequently, the spectral excess of $G$ is

$$p_d(\theta_0) = n \left(\sum_{i=0}^{d} \frac{\pi_i^2}{m_i \pi_i^2}\right)^{-1} = r - 1 = \overline{\theta}_d,$$

(29)

and the spectral excess theorem implies that $G$ is distance-regular. Finally, we use a result of Fiol [22], stating that a distance-regular graph is $r$-antipodal if and only if the multiplicities are given by the above expressions. \(\square\)

In fact, to conclude that $G$ is an $r$-antipodal distance-regular graph, some of the above conditions can be relaxed, namely:

- Since $G$ is assumed to be walk-regular, (27) holds. Hence, to have equalities, we only need to require that the order $n$ is given by (28).
- Alternatively, if we assume equalities in (27), to infer that $G$ is an $r$-antipodal distance-regular graph, we only need to assume that $G$ is regular (as well as the condition $\overline{\theta}_d = r - 1$).

5. New bounds for $\alpha_k$ using the predistance polynomials

Let $G$ be a graph with spectrum as above. Then we can define the scalar product

$$\langle p, q \rangle_G = \frac{1}{n} \text{tr} p(A) = \frac{1}{n} \sum_{i=0}^{d} m_i pq(\theta_i)$$

for $p, q \in \mathbb{R}_k[x]$. The predistance polynomials $p_i$, for $i = 0, 1, \ldots, d$, which were introduced by Fiol and Garriga [25] and were used to prove the well-known Spectral Excess Theorem, are a sequence of orthogonal polynomials with respect to

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the above scalar product, that is,
\[(p_i, p_j)_C = 0, \text{ for } i \neq j,\]
normalized in such a way that \(p_i(\theta_0) = \|p_i\|_C\) (see Cámara, Fàbrega, Fiol, and Garriga [12] for some applications of these polynomials).

When \(G\) is distance-regular, the predistance polynomials become the distance polynomials that, applied to \(A\), give the corresponding distance matrices. In other words, \(p_i(A) = A_i\) for \(i = 0, \ldots, d\). Thus, in this case, the \(k\)-power graph \(G^k\) has adjacency matrix \(A^k = q_k(A) = \sum_{i=0}^{d} p_i(A)\), hence the (not necessarily distinct) eigenvalues of \(G^k\) are
\[q_k(\theta_0), q_k(\theta_1), \ldots, q_k(\theta_d),\]
repeated \(m_0, m_1, \ldots, m_d\) times respectively.

Since it is known that \(q_k(\theta_0) \geq q_k(\theta_1)\) for \(i = 1, \ldots, d\) [12, Coro. 2.4], we use the bounds (1)–(2) on \(\alpha_k\) to extend a result by Abiad, Coutinho, Fiol, Nogueira, and Zeijlemaker [3, Corollary 2.3].

**Proposition 5.1.**

(i) Let \(G\) be a \(k\)-partially walk-regular graph on \(n\) vertices, with eigenvalues \(\lambda_1 \geq \cdots \geq \lambda_n\) and predistance polynomials \(p_0, p_1, \ldots, p_d\). Let \(q_k = \sum_{i=0}^{d} p_i\). Then,
\[
\alpha_k \leq \min\{|i : q_k(\lambda_i) \geq 0\}, |i : q_k(\lambda_i) \leq 0\}.
\]

(ii) Moreover, if \(\lambda(\theta_0) > \theta_1 > \cdots > \theta_d\) then,
\[
\alpha_k \leq \frac{n}{1 - \frac{q_k(\theta_0)}{q_k(\theta_1)}}
\]
where \(\lambda(q_k) = \min_{i \in [1, d]}\{q_k(\theta_i)\}\).

**Proof.** Using the same reasoning as in Dalfó, Fiol, and Garriga [16, Proposition 2.1], we conclude that \(G\) is \(k\)-partially walk-regular if and only if the matrices \(p_i(A)\), for \(i = 1, \ldots, k\), have zero diagonals. Hence, this also holds for the matrix \(q_k(A)\). By considering a suitable \(\alpha_k \times \alpha_k\) principal zero submatrix of \(A\) and using interlacing, we then prove (i). Similarly, by taking the appropriate \(2 \times 2\) quotient matrix, interlacing yields (ii). \(\square\)

Next, we study when the bound from **Proposition 5.1** is tight. Table 6 compares **Proposition 5.1** to several known upper bounds on the 2-independence number. We limit ourselves to 2-walk-regular graphs that are not distance-regular, since otherwise, one can simply use [3, Corollary 2.3]. Note that for several graphs, we obtain a better bound than in Abiad, Coutinho, and Fiol [2] and again [3], for example, for the Gray and Hoffman graphs. Moreover, if \(k = 2\), an infinite family for which the bound of **Proposition 5.1** is tight can be found among the circulant graphs. For any positive integer \(n\) and set \(S \subseteq \{1, \ldots, \lfloor n/2 \rfloor\}\), the circulant graph \(C_n^S\) is the undirected Cayley graph on \(\mathbb{Z}_n\) with generating set \(S \cup (-S)\). The graph \(a_{\{1, \ldots, m\}}\) is connected if and only if \(\gcd(n, s_1, \ldots, s_m) = 1\). In particular, for all \(i > j\) such that \(\gcd(i, j) = 1\), \(C_{\{i,j\}}\) is a connected 3-regular graph. If \(j\) is odd, these graphs are isomorphic to the Möbius ladder graphs, and for even \(j\) they are prism graphs, also known as circular ladder graphs. In both cases, **Proposition 5.1** gives a tight bound on the 2-independence number whenever \(4 \mid i\). Moreover, this bound is also tight for noncircular prism graphs if their order is not a multiple of eight. Note that these graphs are all \(2\)-partially walk-regular, but not \(2\)-partially distance-regular.

In the extremal case \(k = d = 1\), we obtain the following result.

**Corollary 5.2.** Let \(G\) be a walk-regular graph on \(n\) vertices, with distinct eigenvalues \(\theta_0 > \cdots > \theta_d\) and predistance polynomial \(p_d\). Let \(A(p_d) = \max_{i \in [1, d]} p_d(\theta_i)\). Then,
\[
\alpha_{d-1} \leq \frac{n(1 + A(p_d))}{n + A(p_d) - p_d(\theta_0)}.
\]

**Proof.** Notice that, since the Hoffman polynomial is \(H = p_0 + \cdots + p_d\) and \(p_0 = 1\), we have \(q_{d-1}(x) = H(x) - p_d(x) - 1\). But \(H(\theta_0) = n\) and \(H(\theta_i) = 0\) for \(i = 1, \ldots, d\). Then, \(q_{d-1}(\theta_0) = n - p_d(\theta_0) - 1\) and \(\lambda(q_{d-1}) = -A(p_d) - 1\). Then (31) gives the result. \(\square\)

When \(G\) is an \(r\)-antipodal distance-regular graph, the bound in (32) is tight, since \(A(p_d) = -p_d(\theta_0) = -r + 1\). So, we get \(\alpha_{d-1} = r\).

**Remark 5.3.** Note that for the regular case, all the bounds for \(\alpha_k\) directly yield bounds for the distance chromatic number \(\chi_k\), see again Abiad, Coutinho, Fiol, Nogueira, and Zeijlemaker [3, Section 3] for details.
Table 6
Comparison between $\alpha_2$ and several of its upper bounds.

| Graph's name          | $\vartheta_2$ | $\alpha_2$ | $\alpha_2$ | $\alpha_2$ | $\alpha_2$ |
|-----------------------|---------------|------------|------------|------------|------------|
| Balaban 10-cage       | 17            | 17         | 19         | 19         | 19         | 17         |
| Frucht               | 3             | 3          | 3          | 3          | 3          | 3          |
| Meredith             | 14            | 10         | 10         | 10         | 14         | 10         |
| Moebius–Kantor       | 4             | 4          | 6          | 4          | 4          | 4          |
| Bidikakis cube        | 3             | 2          | 4          | 3          | 3          | 2          |
| Gray                  | 14            | 11         | 19         | 19         | 13         | 11         |
| Nauru                 | 6             | 5          | 8          | 8          | 6          | 6          |
| Blanusa First Snark  | 4             | 4          | 4          | 4          | 4          | 4          |
| Blanusa Second Snark | 4             | 4          | 4          | 4          | 4          | 4          |
| Brinkmann             | 4             | 3          | 6          | 6          | 3          | 3          |
| Harborth              | 12            | 9          | 13         | 13         | 11         | 10         |
| Harries               | 17            | 17         | 18         | 18         | 18         | 17         |
| Bucky Ball            | 16            | 12         | 16         | 16         | 15         | 12         |
| Harries-Wong          | 17            | 17         | 18         | 18         | 18         | 17         |
| Robertson             | 3             | 3          | 5          | 5          | 3          | 3          |
| Hoffman               | 3             | 2          | 5          | 4          | 2          | 2          |
| Holt                  | 6             | 3          | 7          | 7          | 4          | 3          |
| Szekeres Snark        | 12            | 10         | 13         | 13         | 13         | 9          |
| Tietze                | 3             | 3          | 4          | 3          | 3          | 3          |
| Double star snark     | 7             | 7          | 9          | 9          | 7          | 6          |
| Durer                 | 3             | 2          | 3          | 3          | 3          | 2          |
| Klein 3-regular       | 13            | 13         | 19         | 18         | 14         | 12         |
| Truncated Tetrahedron | 3             | 3          | 4          | 4          | 3          | 3          |
| Dyck                  | 8             | 8          | 8          | 8          | 8          | 8          |
| Tutte                 | 11            | 10         | 13         | 13         | 11         | 10         |
| F26A                  | 6             | 6          | 7          | 7          | 6          | 6          |
| Watkins Snark         | 14            | 9          | 13         | 13         | 13         | 9          |
| Flower Snark          | 5             | 5          | 7          | 7          | 5          | 5          |
| Markstroem            | 6             | 6          | 7          | 7          | 6          | 6          |
| Folkman               | 4             | 3          | 5          | 4          | 3          | 3          |
| McGee                 | 6             | 5          | 7          | 6          | 6          | 5          |
| Franklin              | 3             | 2          | 4          | 3          | 3          | 2          |

Data availability
No data was used for the research described in the article.

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