Asymptotic solvers for second-order differential equation systems with multiple frequencies

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Abstract In this paper, an asymptotic expansion is constructed to solve second-order differential equation systems with highly oscillatory forcing terms involving multiple frequencies. An asymptotic expansion is derived in inverse of powers of the oscillatory parameter and its truncation results in a very effective method of discretizing the differential equation system in question. Numerical experiments illustrate the effectiveness of the asymptotic method in contrast to the standard Runge–Kutta method.

Keywords Highly oscillatory problems · Second-order differential equations · Modulated Fourier expansions · Multiple frequencies · Numerical analysis

Mathematics Subject Classification 65L05 · 65D30 · 42B20 · 42A10
1 Introduction

The behaviour of signals comprising several non-commensurate frequencies is very important in the design and analysis of electronic circuits. Non-linearities in circuits can result in such signals giving rise to distortion and resulting in degradation of performance. Hence, the accurate and efficient simulation of circuit behaviour in the presence of such signals is essential. It is to address this issue that the current paper is directed. As an example of nonlinear circuits, the Van der Pol oscillator shall be considered. It has numerous applications in science and engineering, for example, from describing the action potentials of biological neurons [8,12] to the modelling of resonant tunneling diode circuits [15]. A coupled Van der Pol-Duffing system shall also be considered. Such coupled systems have applications in secure communications [9,13]. In addition, these coupled systems can be realised using analog circuitry [2].

Consider a second-order differential equation system of the form

\[ y''(t) + f(y(t))y'(t) + g(y(t)) = F_\omega(t), \quad t \geq 0, \]  

(1.1)

where \( f : \mathbb{C}^d \rightarrow \mathbb{C}^d \), \( g : \mathbb{C}^d \rightarrow \mathbb{C}^d \) are two analytical functions,

\[
\begin{pmatrix}
  f_{11}(y) & f_{12}(y) & \cdots & f_{1d}(y) \\
  f_{21}(y) & f_{22}(y) & \cdots & f_{2d}(y) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{d1}(y) & f_{d2}(y) & \cdots & f_{dd}(y)
\end{pmatrix}
\]  

(1.2)

\[
\begin{pmatrix}
  g_1(y) \\
  g_2(y) \\
  \vdots \\
  g_d(y)
\end{pmatrix}
\]  

(1.3)

where every entry \( f_{jv}(y) : \mathbb{C}^d \rightarrow \mathbb{R} \) and \( g_j(y) : \mathbb{C}^d \rightarrow \mathbb{R} \) is an analytic scalar function for \( j, v = 1, 2, \ldots, d \). The initial conditions are \( y(0) = y_0 \in \mathbb{C}^d \) and \( y'(0) = y'_0 \in \mathbb{C}^d \), and the forcing term \( F_\omega(t) \) is

\[
F_\omega(t) = \sum_{m=1}^{M} a_m(t)e^{i\omega_m t},
\]  

(1.4)

in which \( a_1, \ldots, a_M \in \mathbb{R}_+ \rightarrow \mathbb{C}^d \) are analytic functions. Note that we have assumed that there is a finite set of frequencies \( \omega_1, \ldots, \omega_M \in \mathbb{R} \setminus \{0\} \) in the forcing term. At least some of these frequencies are large which results in a highly oscillatory solution and one which is very expensive to obtain with classical discretization methods. Furthermore, assume that the functions \( f(y(t)) \) and \( g(y(t)) \) are analytic to ensure the existence and uniqueness of the solution \( y(t) \).

When \( d = 1, \omega_{2m-1} = m\omega, \omega_{2m} = -m\omega, m = 0, 1, \ldots, \lfloor \frac{M}{2} \rfloor, \omega \gg 1 \), the above multiple frequency case reduces to a single frequency case.
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\[ y''(t) + f(y(t))y'(t) + g(y(t)) = \sum_{k=-\infty}^{\infty} b_k(t)e^{ik\omega t}, \quad t \geq 0, \quad (1.5) \]

which has been already analysed in Condon et al. [6].

Differential equations of this type abound in a wide variety of different contexts, cf. for example [2,12,13] and [15]. We note that there are diverse origins of highly oscillatory systems of differential equations originating in a variety of applications. Some, e.g. those originating in the linear Schrödinger equation, are outside the scope of our discussion.

Modulated Fourier expansion for oscillatory ODEs has provided a natural framework for highly oscillatory ODEs. This type expansion has been analysed as an essential tool in Geometric Numerical Integration [3,4,11,14].

In current context, modulated Fourier expansions are used to solve the case in which the linear part of the equation possesses the highly oscillatory parameter [11]. Compared to the modulated Fourier expansions in [5,6], the asymptotic ansatz for single frequency is not suitable for the multi-frequency setting (1.1) in which a new expansion, using index sequences, must be constructed. Further complications originate in the transition from a scalar to a vector and matrix form. However, modulated Fourier expansion should be carried out in a different manner, incorporating all the frequencies [7] using the iterated index sets. In contrast to [7], we require that \( p_{0,m} \equiv 0 \) and \( p_{1,m} \equiv 0 \) for \( m \neq 0 \), since otherwise differentiation with respect to \( t \) would produce a positive power of \( \omega \) which does not appear in the original equation (1.1).

It is important to emphasise a critical difference between the methods of this paper and traditional numerical algorithms. In particular, asymptotic expansions in general are not subject to classical notions of convergence [1]. Having said so, it is legitimate to ask under which circumstances our asymptotic expansions converge in classical terms. This is a highly nontrivial issue which will be addressed in future publications.

2 Construction of the asymptotic expansion

We start with a set \( U_0 = \{1, 2, \ldots, M\} \) and \( \omega_j = \kappa_j \omega, \quad j = 1, 2, \ldots, M \), where \( \omega \) serves as the oscillatory parameter. Therefore, the original equation (1.1) may be written as

\[ y''(t) + f(y(t))y'(t) + g(y(t)) = \sum_{m=1}^{M} a_m(t)e^{i\kappa_m t \omega t} = \sum_{m \in U_0} a_m(t)e^{i\kappa_m t \omega t}, \quad t \geq 0. \]

(2.1)

Our basic ansatz is that the solution \( y(t) \) admits an expansion in inverse powers of the oscillatory parameter \( \omega \),

\[ y(t) \sim \sum_{r=0}^{\infty} \frac{1}{\omega^r} \sum_{m \in U_r} p_{r,m}(t)e^{i\sigma_m t \omega t}, \quad (2.2) \]
The sets \( \mathcal{U}_r \), the scalars \( \sigma_m \) and the functions \( p_{r,m}(t) \) will be described in the sequel. The important point to note as this stage is that the functions \( p_{r,m}(t) \) are independent of \( \omega \) and can be derived recursively: \( p_{r,0}(t) \) by solving a non-oscillatory ODE and \( p_{r,m}(t) \) for \( m \neq 0 \) by recursion.

It is very important, however, to impose \( p_{0,m} \equiv 0 \) and \( p_{1,m} \equiv 0 \) for \( m \neq 0 \) so that differentiation does not result in the presence of a positive power of \( \omega \) in the resultant equations for \( p_{r,m}(t) \) involved in the asymptotic method.

Therefore, the proposed solution \( y(t) \) in inverse powers of the oscillatory parameter \( \omega \) is,

\[
y(t) \sim p_{0,0}(t) + \frac{1}{\omega} p_{1,0}(t) + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{m \in \mathcal{U}_r} p_{r,m}(t) e^{i \sigma_m \omega t}, \tag{2.3}
\]

As just stated it will be assumed in the ansatz that \( p_{1,0} \) is the only non-zero \( p_{1,m} \) in \( \mathcal{U}_1 \). So \( \mathcal{U}_1 = \{0\} \).

Following the approach in Condon et al. [7], the expression for \( y(t) \) is substituted into the second-order differential equation. The first-order derivative of \( y(t) \) is

\[
y' \sim p_{0,0}' + \frac{1}{\omega} p_{1,0}' + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{m \in \mathcal{U}_r} \left( p_{r,m}' e^{i \sigma_m \omega t} + i \sigma_m \omega p_{r,m} e^{i \sigma_m \omega t} \right)
\]

\[
= p_{0,0}' + \frac{1}{\omega} p_{1,0}' + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{m \in \mathcal{U}_r} p_{r,m}' e^{i \sigma_m \omega t} + \sum_{r=2}^{\infty} \frac{1}{\omega^r - 1} \sum_{m \in \mathcal{U}_r} i \sigma_m p_{r,m} e^{i \sigma_m \omega t}
\]

\[
= p_{0,0}' + \frac{1}{\omega} p_{1,0}' + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{m \in \mathcal{U}_r} p_{r,m}' e^{i \sigma_m \omega t} + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m \in \mathcal{U}_{r+1}} i \sigma_m p_{r+1,m} e^{i \sigma_m \omega t},
\]

\[
= p_{0,0}' + \frac{1}{\omega} \left[ p_{1,0}' + \sum_{m \in \mathcal{U}_2} i \sigma_m p_{2,m} e^{i \sigma_m \omega t} \right]
\]

\[
+ \sum_{r=2}^{\infty} \frac{1}{\omega^r} \left[ \sum_{m \in \mathcal{U}_r} p_{r,m}' e^{i \sigma_m \omega t} + \sum_{m \in \mathcal{U}_{r+1}} i \sigma_m p_{r+1,m} e^{i \sigma_m \omega t} \right]. \tag{2.4}
\]

The second-order derivative of \( y(t) \) is

\[
y''(t) \sim p_{0,0}'' + \frac{1}{\omega} \left[ p_{1,0}'' + \sum_{m \in \mathcal{U}_2} i \sigma_m \left( p_{2,m}' + \left( i \sigma_m \omega p_{2,m} \right) e^{i \sigma_m \omega t} \right) \right] + \sum_{r=2}^{\infty} \frac{1}{\omega^r}
\]

\[
\times \left[ \sum_{m \in \mathcal{U}_r} \left( p_{r,m}'' + i \sigma_m \omega p_{r,m}' \right) e^{i \sigma_m \omega t} + \sum_{m \in \mathcal{U}_{r+1}} i \sigma_m \left( p_{r+1,m}' + i \sigma_m \omega p_{r+1,m} \right) e^{i \sigma_m \omega t} \right],
\]
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\[
\begin{align*}
= p_{0,0}'' + \sum_{m \in \mathcal{U}_2} (i \sigma_m)^2 p_{2,m} e^{i \sigma_{m,0} t} + \frac{1}{\omega} \left[ p_{1,0}'' + 2 \sum_{m \in \mathcal{U}_2} (i \sigma_m) p_{2,m} e^{i \sigma_{m,0} t} ight] \\
+ \sum_{m \in \mathcal{U}_3} (i \sigma_m)^2 p_{3,m} e^{i \sigma_{m,0} t} \right] + \sum_{r=2}^{\infty} \frac{1}{\omega r} \left[ \sum_{m \in \mathcal{U}_r} p_{r,m} e^{i \sigma_{m,0} t} \right] \\
+ 2 \sum_{m \in \mathcal{U}_{r+1}} (i \sigma_m) p_{r+1,m} e^{i \sigma_{m,0} t} + \sum_{m \in \mathcal{U}_{r+2}} (i \sigma_m)^2 p_{r+2,m} e^{i \sigma_{m,0} t} \right].
\end{align*}
\]

(2.5)

The matrix function \( f(y(t))_{d \times d} \) defined in (1.2) is analytic and its Taylor expansion about \( p_{0,0}(t) \) may be determined. \( f_{j,v}^{(n)}(p_{0,0})[\eta_1, \ldots, \eta_n] : \mathbb{C}^d \times \mathbb{C}^d \times \cdots \times \mathbb{C}^d \rightarrow \mathbb{C} \) is the \( n \)th derivative operator which is linear in each of \( \eta_k \)s such that

\[
f_{j,v}(y_0 + t \epsilon) = f_{j,v}(y_0) + \sum_{n=1}^{\infty} \frac{t^n}{n!} f_{j,v}^{(n)}(y_0)[\epsilon, \ldots, \epsilon]
\]

for sufficiently small \(|t| > 0 \) and \( j, v = 1, 2, \ldots, d \). Hence,

\[
f_{j,v}(y) = f_{j,v} \left( p_{0,0}(t) + \frac{1}{\omega} p_{1,0}(t) + \sum_{r=2}^{\infty} \frac{1}{\omega r} \sum_{m \in \mathcal{U}_r} p_{r,m}(t) e^{i \sigma_{m,0} t} \right)
\]

\[
= f_{j,v}(p_{0,0}) + \sum_{n=1}^{\infty} \frac{1}{n!} f_{j,v}^{(n)}(p_{0,0}) \left[ \sum_{\ell_1=1}^{\infty} \frac{1}{\omega_{\ell_1}} \sum_{k_1 \in \mathcal{U}_{\ell_1}} \ldots \sum_{k_n \in \mathcal{U}_{\ell_n}} \ldots \sum_{k_n \in \mathcal{U}_{\ell_n}} \right] \]

\[
= f_{j,v}(p_{0,0}) + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\ell_1=1}^{\infty} \ldots \sum_{\ell_n=1}^{\infty} \frac{1}{\omega_{\ell_1} + \cdots + \omega_{\ell_n}} \sum_{k_1 \in \mathcal{U}_{\ell_1}} \ldots \sum_{k_n \in \mathcal{U}_{\ell_n}} \ldots \sum_{k_n \in \mathcal{U}_{\ell_n}} f_{j,v}^{(n)}(p_{0,0}) \left[ p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n} \right] e^{i(\sigma_{k_1} + \cdots + \sigma_{k_n}) t}
\]

\[
= f_{j,v}(p_{0,0}) + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{r=n}^{\infty} \frac{1}{\omega r} \sum_{\ell \in \mathcal{U}_{\ell, r}} \ldots \sum_{k_n \in \mathcal{U}_{\ell_n}} f_{j,v}^{(n)}(p_{0,0}) \left[ p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n} \right] e^{i(\sigma_{k_1} + \cdots + \sigma_{k_n}) t}
\]

\[
= f_{j,v}(p_{0,0}) + \sum_{r=1}^{\infty} \frac{1}{\omega r} \sum_{n=1}^{r} \frac{1}{n!} \sum_{\ell \in \mathcal{U}_{\ell, r}} \ldots \sum_{k_n \in \mathcal{U}_{\ell_n}} f_{j,v}^{(n)}(p_{0,0}) \left[ p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n} \right] e^{i(\sigma_{k_1} + \cdots + \sigma_{k_n}) t}
\]
where

\[ \mathbb{I}^o_{n,r} = \left\{ \ell = (\ell_1, \ldots, \ell_n)^T \in \mathbb{N}^n : \sum_{j=1}^n \ell_j = r \right\}, \quad 1 \leq n \leq r. \]

To avoid redundancy, set

\[ \mathbb{I}_{n,r} = \left\{ \ell = (\ell_1, \ldots, \ell_n)^T \in \mathbb{N}^n : \sum_{j=1}^n \ell_j = r, \quad \ell_1 \leq \ell_2 \leq \cdots \leq \ell_n \right\}, \]

and the symbol \( \theta_\ell \) stands for the multiplicity of \( \ell \). This is the number of terms in \( \mathbb{I}_{n,r}^o \) that can be brought to it by permutation. It follows that

\[
f_{jv}(y(t)) = f_{jv}(p_{0,0}) + \sum_{r=1}^\infty \frac{1}{\omega^r} \sum_{n=1}^r \frac{1}{n!} \sum_{\ell \in \mathbb{I}_{n,r}} \theta_\ell \sum_{k_1 \in \mathcal{U}_{\ell_1}} \cdots \sum_{k_n \in \mathcal{U}_{\ell_n}} \left[ p_{\ell_1,k_1} \cdots p_{\ell_n,k_n} \right] e^{i(\sigma_{k_1} + \cdots + \sigma_{k_n})\omega t}. \tag{2.6}
\]

Similar expansions can be applied to the vector function

\[
g(y(t)) = g(p_{0,0}) + \sum_{r=1}^\infty \frac{1}{\omega^r} \sum_{n=1}^r \frac{1}{n!} \sum_{\ell \in \mathbb{I}_{n,r}} \theta_\ell \sum_{k_1 \in \mathcal{U}_{\ell_1}} \cdots \sum_{k_n \in \mathcal{U}_{\ell_n}} \left[ p_{\ell_1,k_1} \cdots p_{\ell_n,k_n} \right] e^{i(\sigma_{k_1} + \cdots + \sigma_{k_n})\omega t}. \tag{2.7}
\]

Incorporating these expansions into the second differential equation, we have

\[
p''_{0,0} + \sum_{m \in \mathcal{U}_2} (i\sigma_m)^2 p_{2,m} e^{i\sigma_{m\omega t}} + \frac{1}{\omega} \left[ p''_{1,0} + 2 \sum_{m \in \mathcal{U}_2} (i\sigma_m) p'_{2,m} e^{i\sigma_{m\omega t}} \right] + \sum_{m \in \mathcal{U}_3} (i\sigma_m)^2 p_{3,m} e^{i\sigma_{m\omega t}} + \sum_{r=2}^\infty \frac{1}{\omega^r} \left[ \sum_{m \in \mathcal{U}_r} p''_{r,m} e^{i\sigma_{m\omega t}} \right] + 2 \sum_{m \in \mathcal{U}_{r+1}} (i\sigma_m) p'_{r+1,m} e^{i\sigma_{m\omega t}} + \sum_{m \in \mathcal{U}_{r+2}} (i\sigma_m)^2 p_{r+2,m} e^{i\sigma_{m\omega t}}
\]

\[
+ f(y(t)) \left( p'_{0,0} + \frac{1}{\omega} \left( p'_{1,0} + \sum_{m \in \mathcal{U}_2} i\sigma_m p_{2,m} e^{i\sigma_{m\omega t}} \right) \right) + \sum_{r=2}^\infty \frac{1}{\omega^r} \left( \sum_{m \in \mathcal{U}_r} p'_{r,m} e^{i\sigma_{m\omega t}} + \sum_{m \in \mathcal{U}_{r+1}} i\sigma_m p_{r+1,m} e^{i\sigma_{m\omega t}} \right) d \times 1
\]
\[ + g(p_{0,0}) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\ell \in I_n} \sum_{k_1 \in \mathcal{U}_\ell} \cdots \sum_{k_n \in \mathcal{U}_{t_n}} \left( p_{0,0} \right) \right] e^{i(\sigma_{k_1} + \cdots + \sigma_{k_n})_{\text{out}}} = \sum_{m \in \mathcal{U}} a_m(t)e^{i\kappa_m_{\text{out}}}. \] (2.8)

### 3 Construction of the asymptotic expansion

In this section, we derive the constituent terms of the asymptotic expansion. Explicit expressions shall be given for the first \( r \) values in the expansion to aid in the understanding of the general expression for \( r \geq 0 \). The expression for \( y(t) \) exhibits two distinct hierarchies of scales—amplitudes \( \omega^{-r} \) for \( r \geq 1 \) and for each \( r \), frequencies \( e^{i\sigma_{k,1} \cdots \sigma_{k,n}_{\text{out}}} \). Construction involves separating amplitudes and then separating frequencies. However, before deriving the expansion, the procedure for incorporation of the initial conditions shall be addressed.

#### 3.1 The initial conditions

The initial conditions for the second-order differential equations are

\[ y(0) = p_{0,0}(0) = y_0, \quad y'(0) = p'_{0,0}(0) = y'_0. \] (3.1)

The ansatz must satisfy the same initial conditions

\[ p_{0,0}(0) + \frac{1}{\omega} p_{1,0}(0) + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{m \in \mathcal{U}} p_{r,m}(0) = y_0; \] (3.2)

\[ p'_{0,0}(0) + \frac{1}{\omega} \left[ p'_{1,0}(0) + \sum_{m \in \mathcal{U}_2} i\sigma_m p_{2,m}(0) \right] + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \left[ \sum_{m \in \mathcal{U}_r} p'_{r,m}(0) + \sum_{m \in \mathcal{U}_{r+1}} i\sigma_m p_{r+1,m}(0) \right] = y'_0. \] (3.3)

To this end, we set

\[ p_0(0) = y_0, \quad p'_{0}(0) = y'_0; \]

\[ p_{1,0}(0) = 0, \quad p'_{1,0}(0) = -\sum_{m \in \mathcal{U}_2} i\sigma_m p_{2,m}(0); \]

\[ p_{2,0}(0) = -\sum_{m \in \mathcal{U}_2 \setminus \{0\}} p_{2,m}(0), \quad p'_{2,0}(0) = -\sum_{m \in \mathcal{U}_2 \setminus \{0\}} p'_{2,m}(0) - \sum_{m \in \mathcal{U}_3} i\sigma_m p_{3,m}(0); \]

\[ \vdots. \]
\[ p_{\ell,0}(0) = - \sum_{m \in \mathcal{U}_1 \setminus \{0\}} p_{\ell,m}(0), \quad p_{\ell,0}'(0) = - \sum_{m \in \mathcal{U}_1 \setminus \{0\}} p_{\ell,m}'(0) - \sum_{m \in \mathcal{U}_{t+1}} i \sigma_m p_{\ell+1,m}(0); \]

\[ \vdots. \]

3.2 The zeroth term \( r = 0 \)

We extract the \( O(1) \) terms from Eq. (2.8),

\[ p''_{0,0} + \sum_{m \in \mathcal{U}_2} (i \sigma_m)^2 p_{2,m} e^{i \sigma_{m0}} + f(p_{0,0}) p'_{0,0} + g(p_{0,0}) = \sum_{m \in \mathcal{U}_0} a_m(t) e^{i \kappa_{m0}}, \]

where

\[ f(p_{0,0}) = \begin{pmatrix} f_{11}(p_{0,0}) & f_{12}(p_{0,0}) & \cdots & f_{1d}(p_{0,0}) \\ f_{21}(p_{0,0}) & f_{22}(p_{0,0}) & \cdots & f_{2d}(p_{0,0}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{d1}(p_{0,0}) & f_{d2}(p_{0,0}) & \cdots & f_{dd}(p_{0,0}) \end{pmatrix}_{d \times d}, \quad (3.4) \]

\[ g(p_{0,0}) = \begin{pmatrix} g_1(p_{0,0}) \\ g_2(p_{0,0}) \\ \vdots \\ g_d(p_{0,0}) \end{pmatrix}_{d \times 1}. \quad (3.5) \]

Equating the \( e^{i \sigma_{m0}} \) terms results in for \( m = 0 \)

\[ p''_{0,0}(t) + f(p_{0,0}) p'_{0,0}(t) + g(p_{0,0}) = 0, \quad t \geq 0, \quad p_{0,0}(0) = y_0, \quad p'_{0,0}(0) = y'_0. \quad (3.6) \]

By setting \( \mathcal{U}_2 = \mathcal{U}_0 \cup \{0\}, \quad \sigma_m = \kappa_m, \quad m = 1, \ldots, M; \quad \sigma_0 = \kappa_0 = 0, \quad m \in \mathcal{U}_2, \)

the following relation is obtained for \( m \neq 0 \)

\[ p_{2,m}(t) = \frac{a_m(t)}{(i \kappa_m)^2}, \quad m \neq 0. \quad (3.7) \]

3.3 The \( r = 1 \) terms

When \( r = 1 \), note that \( n = 1 \) and \( \| \|_{1,1} = 1 \) and \( \theta_\ell = 1 \), Then

\[ \sum_{n=1}^{r} \frac{1}{n!} \sum_{\ell \in \mathcal{U}_n} \theta_\ell \sum_{k_1 \in \mathcal{U}_{t_1}} \cdots \sum_{k_n \in \mathcal{U}_{t_n}} f_{j,v}^{(n)}(p_{0,0}) [p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n}] e^{i (\sigma_{k_1} + \cdots + \sigma_{k_n}) ot} \]

\[ = \sum_{m \in \mathcal{U}_1} f_{j,v}^{(1)}(p_{0,0}) [p_{1,m}] e^{i \sigma_{m0}}. \]
Based on the assumption that, \( p_{1,m} = 0, m \neq 0 \) and that \( \mathcal{U}_1 = \{0\} \), the above term is reduced to \( f^{(1)}_{jv}(p_{0,0})[p_{1,0}] \).

We now consider the \( O(\frac{1}{m^2}) \) terms

\[
p''_{1,0} + 2 \sum_{m \in \mathcal{U}_2} (i\sigma_m)p'_{2,m}e^{i\sigma_{m_{tot}}} + \sum_{m \in \mathcal{U}_3} (i\sigma_m)^2 p_{3,m}e^{i\sigma_{m_{tot}}} + f(p_{0,0})p'_{1,0} + f(p_{0,0}) \sum_{m \in \mathcal{U}_2} (i\sigma_m)p_{2,m}e^{i\sigma_{m_{tot}}} + f^{(1)}(p_{0,0})[p_{1,0}]p'_{0,0} + g^{(1)}(p_{0,0})[p_{1,0}] = 0,
\]

where

\[
f^{(1)}(p_{0,0})[p_{1,0}] = \begin{pmatrix} f^{(1)}_{11}(p_{0,0})[p_{1,0}] & f^{(1)}_{12}(p_{0,0})[p_{1,0}] & \cdots & f^{(1)}_{1d}(p_{0,0})[p_{1,0}] \\ f^{(1)}_{21}(p_{0,0})[p_{1,0}] & f^{(1)}_{22}(p_{0,0})[p_{1,0}] & \cdots & f^{(1)}_{2d}(p_{0,0})[p_{1,0}] \\ \vdots & \vdots & \ddots & \vdots \\ f^{(1)}_{d1}(p_{0,0})[p_{1,0}] & f^{(1)}_{d2}(p_{0,0})[p_{1,0}] & \cdots & f^{(1)}_{dd}(p_{0,0})[p_{1,0}] \end{pmatrix}_{d \times d}, \tag{3.8}
\]

\[
g^{(1)}(p_{0,0})[p_{1,0}] = \begin{pmatrix} g^{(1)}_{11}(p_{0,0})[p_{1,0}] \\ g^{(1)}_{21}(p_{0,0})[p_{1,0}] \\ \vdots \\ g^{(1)}_{d1}(p_{0,0})[p_{1,0}] \end{pmatrix}_{d \times 1}, \tag{3.9}
\]

in which

\[
f^{(1)}_{jv}(p_{0,0})[p_{1,0}] = \nabla (f_{jv}(p_{0,0}))_{1 \times d} \times [p_{1,0}]_{d \times 1}, \tag{3.10}
\]

\[
g^{(1)}_{j}(p_{0,0})[p_{1,0}] = \nabla (g_{j}(p_{0,0}))_{1 \times d} \times [p_{1,0}]_{d \times 1}. \tag{3.11}
\]

If \( m = 0 \), the unperturbed equation and the associated initial conditions are obtained

\[
p''_{1,0} + f(p_{0,0})p'_{1,0} + f^{(1)}(p_{0,0})[p_{1,0}]p'_{0,0} + g^{(1)}(p_{0,0})[p_{1,0}] = 0, \tag{3.12}
\]

\[
p^{(1)}_{1,0}(0) = 0, \quad p^{(1)}_{1,0}(0) = -\sum_{m \in \mathcal{U}_2} (i\kappa_m)p^{(1)}_{2,m}(0). \tag{3.13}
\]

If \( m \neq 0 \), we extract the terms with \( e^{i\sigma_{m_{tot}}} \). Set \( \mathcal{U}_3 = \mathcal{U}_0 \cup \{0\} = \{0, 1, 2, \ldots, M\} = \mathcal{U}_2 \). It follows that

\[
\sigma_0 = 0, \quad \sigma_m = \kappa_m, \quad m \in \mathcal{U}_3 = \mathcal{U}_0 \cup \{0\}
\]

and

\[
p^{(1)}_{3,m} = \frac{-1}{i\kappa_m} [2p'_{2,m} + f(p_{0,0})p_{2,m}], \quad m \neq 0. \tag{3.14}
\]
When $r = 2$, 

$$
\begin{align*}
&\left[ \sum_{m \in \mathcal{M}_2} p''_{2,m} e^{i\sigma_m \omega t} + 2 \sum_{m \in \mathcal{M}_3} (i\sigma_m) p'_{3,m} e^{i\sigma_m \omega t} + \sum_{m \in \mathcal{M}_4} (i\sigma_m)^2 p_{4,m} e^{i\sigma_m \omega t} \right] \\
&+ f(p_{0,0}) \left( \sum_{m \in \mathcal{M}_2} p'_{2,m} e^{i\sigma_m \omega t} + \sum_{m \in \mathcal{M}_3} i\sigma_m p_{3,m} e^{i\sigma_m \omega t} \right) \\
&+ f^{(1)}(p_{0,0}) [p_{1,0}] \left( p'_{1,0} + \sum_{m \in \mathcal{M}_2} i\sigma_m p_{2,m} e^{i\sigma_m \omega t} \right) \\
&+ \sum_{n=1}^{2} \frac{1}{n!} \sum_{\ell \in \mathcal{L}_n} \theta_{\ell} \sum_{k_1 \in \mathcal{U}_{\ell_1}} \cdots \sum_{k_n \in \mathcal{U}_{\ell_n}} f^{(n)}(p_{0,0})[p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n}] e^{i(\sigma_{k_1} + \cdots + \sigma_{k_n}) \omega t} p'_{0,0} \\
&+ \sum_{n=1}^{2} \frac{1}{n!} \sum_{\ell \in \mathcal{L}_n} \theta_{\ell} \sum_{k_1 \in \mathcal{U}_{\ell_1}} \cdots \sum_{k_n \in \mathcal{U}_{\ell_n}} g^{(n)}(p_{0,0})[p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n}] e^{i(\sigma_{k_1} + \cdots + \sigma_{k_n}) \omega t} = 0.
\end{align*}
$$

where

$$
\begin{align*}
&\sum_{n=1}^{2} \frac{1}{n!} \sum_{\ell \in \mathcal{L}_n} \theta_{\ell} \sum_{k_1 \in \mathcal{U}_{\ell_1}} \cdots \sum_{k_n \in \mathcal{U}_{\ell_n}} f^{(n)}(p_{0,0})[p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n}] e^{i(\sigma_{k_1} + \cdots + \sigma_{k_n}) \omega t} \\
&= \sum_{m \in \mathcal{M}_2} f^{(1)}(p_{0,0})[p_{2,m}] e^{i\sigma_m \omega t} + \frac{1}{2} \sum_{m_1 \in \mathcal{M}_1, m_2 \in \mathcal{M}_1} f^{(2)}(p_{0,0})[p_{1,m_1}, p_{1,m_2}] e^{i(\sigma_{m_1} + \sigma_{m_2}) \omega t} \\
&= \sum_{m \in \mathcal{M}_2} f^{(1)}(p_{0,0})[p_{2,m}] e^{i\sigma_m \omega t} + \frac{1}{2} f^{(2)}(p_{0,0})[p_{1,0}, p_{1,0}],
\end{align*}
$$

and

$$
\begin{align*}
&\sum_{n=1}^{2} \frac{1}{n!} \sum_{\ell \in \mathcal{L}_n} \theta_{\ell} \sum_{k_1 \in \mathcal{U}_{\ell_1}} \cdots \sum_{k_n \in \mathcal{U}_{\ell_n}} g^{(n)}(p_{0,0})[p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n}] e^{i(\sigma_{k_1} + \cdots + \sigma_{k_n}) \omega t} \\
&= \sum_{m \in \mathcal{M}_2} g^{(1)}(p_{0,0})[p_{2,m}] e^{i\sigma_m \omega t} + \frac{1}{2} g^{(2)}(p_{0,0})[p_{1,0}, p_{1,0}],
\end{align*}
$$

in which

$$
f^{(1)}(p_{0,0})[p_{2,m}] = \begin{pmatrix} f^{(1)}_{11}(p_{0,0})[p_{2,m}] & f^{(1)}_{12}(p_{0,0})[p_{2,m}] & \cdots & f^{(1)}_{1d}(p_{0,0})[p_{2,m}] \\
 f^{(1)}_{21}(p_{0,0})[p_{2,m}] & f^{(1)}_{22}(p_{0,0})[p_{2,m}] & \cdots & f^{(1)}_{2d}(p_{0,0})[p_{2,m}] \\
 \vdots & \vdots & \ddots & \vdots \\
 f^{(1)}_{d1}(p_{0,0})[p_{2,m}] & f^{(1)}_{d2}(p_{0,0})[p_{2,m}] & \cdots & f^{(1)}_{dd}(p_{0,0})[p_{2,m}] \end{pmatrix}_{d \times d}, \quad (3.15)
$$
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\[ f^{(2)}(p_{0,0})[p_{1,0}, p_{1,0}] = \left( f^{(2)}_{11}(p_{0,0})[p_{1,0}, p_{1,0}] f^{(2)}_{12}(p_{0,0})[p_{1,0}, p_{1,0}] \cdots f^{(2)}_{1d}(p_{0,0})[p_{1,0}, p_{1,0}] \right) \]

\[ \vdots \]

\[ f^{(2)}_{d1}(p_{0,0})[p_{1,0}, p_{1,0}] f^{(2)}_{d2}(p_{0,0})[p_{1,0}, p_{1,0}] \cdots f^{(2)}_{dd}(p_{0,0})[p_{1,0}, p_{1,0}] \right)_{d \times d} \]

\[ g^{(1)}(p_{0,0})[p_{2,m}] = \left( g^{(1)}_{1}(p_{0,0})[p_{2,m}] \right) \]

\[ g^{(2)}(p_{0,0})[p_{1,0}, p_{1,0}] = \left( g^{(2)}_{1}(p_{0,0})[p_{1,0}, p_{1,0}] \right) \]

\[ g^{(2)}_{d}(p_{0,0})[p_{1,0}, p_{1,0}] \right)_{d \times 1} \]

and

\[ f^{(1)}_{jv}(p_{0,0})[p_{2,m}] = \nabla(f^{(1)}_{jv}(p_{0,0}))_{1 \times d} \times [p_{2,m}]_{d \times 1} \]

\[ f^{(2)}_{jv}(p_{0,0})[p_{1,0}, p_{1,0}] = [p_{1,0}]_{1 \times d} \left( \frac{\partial^2 f^{(1)}_{jv}(p_{0,0})}{\partial y_{m_1} \partial y_{m_2}} \right) [p_{1,0}]_{d \times 1} \]

\[ g^{(1)}_{j}(p_{0,0})[p_{2,m}] = \nabla(g^{(1)}_{j}(p_{0,0}))_{1 \times d} \times [p_{2,m}]_{d \times 1} \]

\[ g^{(2)}_{j}(p_{0,0})[p_{1,0}, p_{1,0}] = [p_{1,0}]_{1 \times d} \left( \frac{\partial^2 g^{(1)}_{j}(p_{0,0})}{\partial y_{m_1} \partial y_{m_2}} \right) [p_{1,0}]_{d \times 1} \]

with \( p_{1,m}(t) \equiv 0 \) for \( m \neq 0 \).

If \( m = 0 \),

\[ p''_{2,0} + f(p_{0,0})p'_{2,0} + f^{(1)}(p_{0,0})[p_{1,0}][p'_{1,0}] + \frac{1}{2} f^{(2)}(p_{0,0})[p_{1,0}, p_{1,0}]p'_{0,0} + g^{(1)}(p_{0,0})[p_{2,0}] \]

\[ + \frac{1}{2} g^{(2)}(p_{0,0})[p_{1,0}, p_{1,0}] = 0; \]

\[ p_{2,0}(0) = - \sum_{m \in \mathcal{U}_2 \setminus \{0\}} p_{2,m}(0); \quad p'_{2,0}(0) = - \sum_{m \in \mathcal{U}_2 \setminus \{0\}} p'_{2,m}(0) - \sum_{m \in \mathcal{U}_3} i \kappa_m p_{3,m}(0). \]

\[ (3.23) \]
Set $\mathcal{U}_4 = \mathcal{U}_0 \cup \{0\} = \{0, 1, 2, \ldots, M\}$. This means that $\mathcal{U}_2 = \mathcal{U}_3 = \mathcal{U}_4$ and $\sigma_m = \kappa_m, m \in \mathcal{U}_4$. If $m \neq 0$,

$\begin{align*}
    p_{4,m} &= -\frac{1}{(ik_m)^2} \left[ p''_{2,m} + 2(i\kappa_m)p'_{3,m} + f(p_{0,0}) \left( p'_{2,m} + i\kappa_m p_{3,m} \right) \right] \\
    &+ f^{(1)}(p_{0,0})[p_{1,0}i\kappa_m p_{2,m} + f^{(1)}(p_{0,0})[p_{2,m}]p'_{0,0} + g^{(1)}(p_{0,0})[p_{2,m}]], \quad m \neq 0.
\end{align*}$

3.5 $r = 3$

When $r = 3$, the $O(\frac{1}{\omega^3})$ terms are collected

$$
\sum_{m \in \mathcal{U}_3} p''_{3,m} e^{i\sigma_{m\text{out}}} + 2 \sum_{m \in \mathcal{U}_4} (i\sigma_m) p'_{4,m} e^{i\sigma_{m\text{out}}} + \sum_{m \in \mathcal{U}_5} (i\sigma_m)^2 p_{5,m} e^{i\sigma_{m\text{out}}} \\
+ f(p_{0,0}) \left( \sum_{m \in \mathcal{U}_3} p'_{3,m} e^{i\sigma_{m\text{out}}} + \sum_{m \in \mathcal{U}_4} i\sigma_m p_{4,m} e^{i\sigma_{m\text{out}}} \right) \\
+ f_1(p_{0,0})[p_{1,0}] \left( \sum_{m \in \mathcal{U}_2} p'_{2,m} e^{i\sigma_{m\text{out}}} + \sum_{m \in \mathcal{U}_3} i\sigma_m p_{3,m} e^{i\sigma_{m\text{out}}} \right) \\
+ \left( \sum_{m \in \mathcal{U}_2} f^{(1)}(p_{0,0})[p_{2,m}] e^{i\sigma_{m\text{out}}} + \frac{1}{2} f^{(2)}(p_{0,0})[p_{1,0}, p_{1,0}] \right) \\
\times \left( p'_{1,0} + \sum_{m \in \mathcal{U}_2} i\sigma_m p_{2,m} e^{i\sigma_{m\text{out}}} \right) \\
+ \left[ \sum_{n=1}^{3} \frac{1}{n!} \sum_{\ell_1, \ldots, \ell_n} \sum_{k_1, \ldots, k_n} f^{(n)}(p_{0,0}) \left[ p_{\ell_1, k_1}, \ldots, p_{\ell_n, k_n} \right] e^{i(\sigma_{k_1} + \cdots + \sigma_{k_n})\text{out}} \right] p_{0,0} \\
+ \sum_{n=1}^{3} \frac{1}{n!} \sum_{\ell_1, \ldots, \ell_n} \sum_{k_1, \ldots, k_n} g^{(n)}(p_{0,0}) \left[ p_{\ell_1, k_1}, \ldots, p_{\ell_n, k_n} \right] e^{i(\sigma_{k_1} + \cdots + \sigma_{k_n})\text{out}} = 0.
$$

Since $I_{1,3} = \{3\}$, $\theta_3 = 1$, $I_{2,3} = \{(1, 2), (2, 1)\}$, $\theta(1,2) = 2$, $I_{3,3} = \{(1, 1, 1)\}$, $\theta(1,1,1) = 1$ and $p_{1,m} = 0$ for $m \neq 0$, we get

$$
\sum_{n=1}^{3} \frac{1}{n!} \sum_{\ell_1, \ldots, \ell_n} \sum_{k_1, \ldots, k_n} f^{(n)}(p_{0,0}) \left[ p_{\ell_1, k_1}, \ldots, p_{\ell_n, k_n} \right] e^{i(\sigma_{k_1} + \cdots + \sigma_{k_n})\text{out}} \\
= \sum_{m \in \mathcal{U}_3} f^{(1)}(p_{0,0})[p_{3,m}] e^{i\sigma_{m\text{out}}} + \sum_{m \in \mathcal{U}_2} f^{(2)}(p_{0,0})[p_{1,0}, p_{2,m}] e^{i\sigma_{m\text{out}}} \\
+ \frac{1}{6} f^{(3)}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}].
$$
Therefore, the complete equation for \( r = 3 \) becomes

\[
\sum_{m \in \mathcal{U}_3} p''_{3,m} e^{i \sigma_{m\text{rot}} t} + 2 \sum_{m \in \mathcal{U}_4} (i \sigma_m) p'_{4,m} e^{i \sigma_{m\text{rot}} t} + \sum_{m \in \mathcal{U}_5} (i \sigma_m)^2 p_{5,m} e^{i \sigma_{m\text{rot}} t} + f(p_{0,0}) \left( \sum_{m \in \mathcal{U}_3} p'_{3,m} e^{i \sigma_{m\text{rot}} t} + \sum_{m \in \mathcal{U}_4} i \sigma_m p'_{4,m} e^{i \sigma_{m\text{rot}} t} \right) + f^{(1)}(p_{0,0})[p_{1,0}] \left( \sum_{m \in \mathcal{U}_3} p_{3,m} e^{i \sigma_{m\text{rot}} t} + \sum_{m \in \mathcal{U}_5} i \sigma_m p_{5,m} e^{i \sigma_{m\text{rot}} t} \right) + f^{(1)}(p_{0,0})[p_{1,0}] \left( \sum_{m \in \mathcal{U}_3} f^{(1)}(p_{0,0})[p_{3,m}] e^{i \sigma_{m\text{rot}} t} + \sum_{m \in \mathcal{U}_5} f^{(2)}(p_{0,0})[p_{1,0}, p_{1,0}] e^{i \sigma_{m\text{rot}} t} \right) + \frac{1}{2} f^{(2)}(p_{0,0})[p_{1,0}, p_{1,0}] \sum_{m \in \mathcal{U}_2} i \sigma_m p_{2,m} e^{i \sigma_{m\text{rot}} t} + \sum_{m_1 \in \mathcal{U}_2} \sum_{m_2 \in \mathcal{U}_2} f^{(1)}(p_{0,0})[p_{2,m_2}](i \sigma_{m_1}) p_{2,m_1} e^{i(\sigma_{m_1} + \sigma_{m_2}) t} + \frac{1}{6} f^{(3)}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] \sum_{m \in \mathcal{U}_3} f^{(1)}(p_{0,0})[p_{3,m}] e^{i \sigma_{m\text{rot}} t} + \sum_{m \in \mathcal{U}_5} f^{(2)}(p_{0,0})[p_{1,0}, p_{1,0}, p_{2,m}] e^{i \sigma_{m\text{rot}} t} + \frac{1}{6} f^{(3)}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] = 0,
\]

where the terms \( f^{(1)}(p_{0,0})[p_{3,m}] \), \( f^{(2)}(p_{0,0})[p_{1,0}, p_{2,m}] \), \( g^{(1)}(p_{0,0}) \) and \( g^{(2)}(p_{0,0})[p_{1,0}, p_{2,m}] \) have the same form as the case \( r = 2 \).

\[
f^{(3)}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] = \begin{pmatrix}
    f^{(3)}_{11}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] & f^{(3)}_{12}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] & \cdots & f^{(3)}_{1d}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] \\
    f^{(3)}_{21}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] & f^{(3)}_{22}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] & \cdots & f^{(3)}_{2d}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] \\
    \vdots & \vdots & \ddots & \vdots \\
    f^{(3)}_{d1}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] & f^{(3)}_{d2}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] & \cdots & f^{(3)}_{dd}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}]
\end{pmatrix}_{d \times d},
\]

\[
g^{(3)}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] = \begin{pmatrix}
    g^{(3)}_{1}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] \\
    g^{(3)}_{2}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] \\
    \vdots \\
    g^{(3)}_{d}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}]
\end{pmatrix}_{d \times 1}.
\]

(3.24)
in which

\[
    f_j^{(3)}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] = \sum_{j_1=1}^{d} \sum_{j_2=1}^{d} \sum_{j_3=1}^{d} \frac{\partial^3 f_{j_0,0}(p_{0,0})}{\partial y_{j_1} \partial y_{j_2} \partial y_{j_3}} (p_{1,0})_{j_1}(p_{1,0})_{j_2}(p_{1,0})_{j_3},
\]

\[3.25\]

\[
    g_j^{(3)}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] = \sum_{j_1=1}^{d} \sum_{j_2=1}^{d} \sum_{j_3=1}^{d} \frac{\partial^3 g_{j_0,0}(p_{0,0})}{\partial y_{j_1} \partial y_{j_2} \partial y_{j_3}} (p_{1,0})_{j_1}(p_{1,0})_{j_2}(p_{1,0})_{j_3}
\]

\[3.26\]

and \((p_{j,k})_s\) denotes the \(s\)th element of the vector \(p_{j,k}\). We now consider the construction of the set \(U_5\). \(U_5\) must include terms with \(\kappa_i + \kappa_j\) for \(i, j = 1, 2, \ldots, M\). Therefore, let

\[
    U_5 = U_4 \cup \{(m_1, m_2) : 1 \leq m_1 \leq m_2 \leq M\}.
\]

\[3.27\]

Now for \(i \neq j\), \(\kappa_i + \kappa_j\) may result in a term in \(U_4\). However, it may also result in terms not in \(U_4\). These are the terms included in \(U_5 \setminus U_4\). For \(0 \leq \ell_1 \leq \ell_2 \leq M\), define the multiplicity \(\rho_{\ell_1,\ell_2}^m\) as the number of cases when \(\kappa_{\pi(\ell_1)} + \kappa_{\pi(\ell_2)} = \kappa_m\), where \(\pi(\ell)\) is a permutation of \(\ell\). Let \(\rho_{\ell_1,\ell_2}^{m_1,m_2}\) be the number of permutations such that \(\kappa_{\pi(\ell_1)} + \kappa_{\pi(\ell_2)} = \kappa_{m_1} + \kappa_{m_2}\) for \(0 \leq \ell_1 \leq \ell_2 \leq M\) and \(1 \leq m_1 \leq m_2 \leq M\).

If \(m = 0\), the terms with respect to \(\sigma_0 = 0\) obey

\[
    p_{3,0}'' + f(p_{0,0})p_{3,0}' + f^{(1)}(p_{0,0})[p_{1,0}]p_{2,0} + f^{(1)}(p_{0,0})[p_{2,0}]p_{1,0}' + \sum_{\ell_1 \in \ell_2} \sum_{\ell_2 \in \ell_2, \kappa_{\ell_1} + \kappa_{\ell_2} = 0} \rho_0^{0,0}(\kappa_{\ell_1}) f^{(1)}(p_{0,0})[p_{2,0}]p_{2,0} \ell_1' + \left[ f^{(1)}(p_{0,0})[p_{3,0}] + f^{(2)}(p_{0,0})[p_{1,0}, p_{2,0}] + \frac{1}{6} f^{(3)}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] \right] p_{3,0}'' + g^{(1)}(p_{0,0})[p_{3,0}] + g^{(2)}(p_{0,0})[p_{1,0}, p_{2,0}] + \frac{1}{6} g^{(3)}(p_{0,0})[p_{1,0}, p_{1,0}, p_{1,0}] = 0,
\]

\[3.28\]

with

\[
    p_{3,0}(0) = - \sum_{m \in \ell_3 \setminus \{0\}} p_{3,m}(0),
\]

\[
    p_{3,0}'(0) = - \sum_{m \in \ell_3 \setminus \{0\}} p_{3,m}'(0) - \sum_{m \in \ell_4} i\kappa_m p_{4,m}(0).
\]

\[3.29\]

Then match all the terms in \(U_4 \setminus \{0\} \subset U_5\). This yields the recurrence

\[
    -(i\kappa_m)^2 p_{5,m}'' = p_{3,m}'' + 2(i\kappa_m) p_{4,m}' + f(p_{0,0}) \left( p_{3,m} + i\kappa_m p_{4,m} \right) + f^{(1)}(p_{0,0})[p_{1,0}] \left( p_{2,m} + i\kappa_m p_{3,m} \right) + f^{(1)}(p_{0,0})[p_{2,m}] p_{1,0}'
\]

\[\]
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\[ + \frac{1}{2} J^{(2)}(p_{0,0})[p_{1,0}, p_{1,0}] i \kappa m p_{2,2,0} + \sum_{\kappa \ell_1 \kappa \ell_2 = \kappa m} \rho^{m}_{\ell_1, \ell_2} (i \kappa \ell_1) f^{(1)}(p_{0,0})[p_{2,\ell_2}] p_{2,\ell_1} \]

\[ + \left( f^{(1)}(p_{0,0})[p_{3,m}] + f^{(2)}(p_{0,0})[p_{1,0}, p_{2,2,1}] \right) p'_{0,0} \]

\[ + g^{(1)}(p_{0,0})[p_{3,m}] + g^{(2)}(p_{0,0})[p_{1,0}, p_{2,2,1}] \].

Finally, we match the terms in \( U_5 \setminus U_4 \). Define the pairs \((\ell_1, \ell_2)\) satisfying \( \ell_1 \leq \ell_2 \) and \( \kappa \ell_1 + \kappa \ell_2 \neq \sigma_j \) for \( j = 0, 1, \ldots, M \). It follows that

\[ (i (\kappa m_1 + \kappa m_2))^2 p_{5,m} = - \sum_{\kappa \ell_1 + \kappa \ell_2 = \kappa m_1 + \kappa m_2} \rho^{m_1, m_2}_{\ell_1, \ell_2} (i \kappa \ell_1) f^{(1)}(p_{0,0})[p_{2,\ell_2}] p_{2,\ell_1} \]

These are all terms with respect to \( r = 3 \).

3.6 The general case \( r \geq 1 \)

The terms in \( U_{r+1} \) are composed of \( \kappa j_1 + \kappa j_2 + \cdots + \kappa j_q \), \( q \leq r \) and \( j_1 \leq j_2 \leq \cdots \leq j_q \). Set \( \rho^{m_1, \ldots, m_q}_{\ell_1, \ldots, \ell_p} \) to be the number of distinct \( p \)-tuples \((\ell_1, \ldots, \ell_p)\), where each \( \ell_i \) and \( m_i \) lie in \( \{0, 1, 2, \ldots, M\} \), \( m_1 \leq m_2 \leq \cdots m_q \), such that

\[ \sum_{i=1}^{p} \kappa \ell_i = \sum_{i=1}^{q} \kappa m_i. \]

Extract all the terms at the level \( r \)

\[ \left[ \sum_{m \in U_r} p_{r,m} e^{i \sigma m_{\text{cot}}} + 2 \sum_{m \in U_{r+1}} (i \sigma m) p_{r+1,m} e^{i \sigma m_{\text{cot}}} + \sum_{m \in U_{r+2}} (i \sigma m)^2 p_{r+2,m} e^{i \sigma m_{\text{cot}}} \right] \]

\[ + f(p_{0,0}) \left( \sum_{m \in U_r} p'_{r,m} e^{i \sigma m_{\text{cot}}} + \sum_{m \in U_{r+1}} i \sigma m p_{r+1,m} e^{i \sigma m_{\text{cot}}} \right) \]

\[ + \sum_{n=1}^{r} \frac{1}{n!} \sum_{\ell \in U_n} \theta \ell \sum_{k_1 \in U_{k_1}} \cdots \sum_{k_n \in U_{k_n}} f^{(n)}(p_{0,0}) \left[ p_{\ell_1, k_1, \ldots, k_n} e^{i (\sigma k_1 + \cdots + \sigma k_n)_{\text{cot}}} \right] p'_{0,0} \]

\[ + \left[ \sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\ell \in U_n} \theta \ell \sum_{k_1 \in U_{k_1}} \cdots \sum_{k_n \in U_{k_n}} f^{(n)}(p_{0,0}) \left[ p_{\ell_1, k_1, \ldots, k_n} e^{i (\sigma k_1 + \cdots + \sigma k_n)_{\text{cot}}} \right] \right] \]

\[ \times \left( p'_{1,0} + \sum_{m \in U_2} i \sigma m p_{2,m} e^{i \sigma m_{\text{cot}}} \right) \]

\[ + \sum_{r=1}^{r-1} \frac{1}{n!} \sum_{\ell \in U_n} \theta \ell \sum_{k_1 \in U_{k_1}} \cdots \sum_{k_n \in U_{k_n}} f^{(n)}(p_{0,0}) \left[ p_{\ell_1, k_1, \ldots, k_n} e^{i (\sigma k_1 + \cdots + \sigma k_n)_{\text{cot}}} \right] \]
where

\[
f^{(n)}(p_{0,0})[p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n}] = \begin{pmatrix}
  f^{(n)}_{11}(p_{0,0})[p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n}] \\
  f^{(n)}_{21}(p_{0,0})[p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n}] \\
  \vdots \\
  f^{(n)}_{dd}(p_{0,0})[p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n}]
\end{pmatrix}_{d \times d}
\]

and

\[
f^{(n)}_{j_1}(p_{0,0})[p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n}] = \sum_{j_1=1}^{d} \cdots \sum_{j_n=1}^{d} \frac{\partial^n f_{j_1}(p_{0,0})}{\partial y_{j_1} \cdots \partial y_{j_n}}(p_{\ell_1,k_1})j_1 \cdots (p_{\ell_n,k_n})j_n, \tag{3.32}
\]

\[
s^{(n)}_{j}(p_{0,0})[p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n}] = \sum_{j_1=1}^{d} \cdots \sum_{j_n=1}^{d} \frac{\partial^n g_{j}(p_{0,0})}{\partial y_{j_1} \cdots \partial y_{j_n}}(p_{\ell_1,k_1})j_1 \cdots (p_{\ell_n,k_n})j_n. \tag{3.33}
\]

Considering the form of the frequency terms \(e^{i\sigma_m ot}\) in the above extracted equation, there are two kinds of frequency form that appear:

\[
e^{i(\sigma_1 + \cdots + \sigma_n) ot} = e^{i\eta ot}, \quad e^{i(\sigma_q + \sigma_1 + \cdots + \sigma_n) ot} = e^{i\eta ot}, \tag{3.34}
\]

where \(r_1 \geq 2\) is fixed: their difference is in the presence of the \(\sigma_q\) term. Now let

\[
\eta = \sum_{j=1}^{\xi} k_{m_j},
\]

for \(\xi \in \{1, 2, \ldots, r\} \) or \(r - 1\) and \(0 \leq m_1 \leq m_2 \leq \cdots \leq m_\xi \leq M\). If \(\eta \in U_{r+1}\), there exists \(\sigma_m = \eta\) for \(m \in U_{r+1}\). Otherwise, add to \(U_{r+1}\) the ordered \(\xi\)-tuple \((m_1, m_2, \ldots, m_\xi)\) to generate the set \(U_{r+2}\).
We impose the same natural partial ordering on $U_r$ as in Condon et al. [7]: first the singletons in lexicographic ordering, second the pairs in lexicographic ordering, then the triplets etc. This defines a relation $m_1 \preceq m_2$ for all $m_1, m_2 \in U_r$,

$$\mathcal{W}_{r,m}^n = \left\{ (\ell, k) : k_i \in U_{\ell_i}, \ell \in \mathbb{I}_{n,r}, \sum_{i=1}^{n} \sigma_{k_i} = \sigma_m, k_1 \leq \cdots \leq k_n \right\}$$  \hspace{1cm} (3.35)

for all $m \in U_r$ and $n \in \{1, 2, \ldots, r\}$. For the second frequency form, for fixed $r_1 \geq 2$ and $q \in U_{r_1}$, define the relation

$$\mathcal{W}_{r,m-q}^n = \left\{ (r_1, q, \ell, k) : k_i \in U_{\ell_i}, \ell \in \mathbb{I}_{n,r}, m \in U_r, \sum_{i=1}^{n} \sigma_{k_i} = \sigma_m - \sigma_q, k_1 \leq \cdots \leq k_n, q \in U_{r_1} \right\}.$$  \hspace{1cm} (3.36)

The corresponding number of distinct $p$-tuples $(k_1, \ldots, k_p)$ is defined as $\rho_{k}^{m-q}$ such that $k_1 \leq k_2 \leq \cdots \leq k_p$,

$$\sigma_{k_1} + \cdots + \sigma_{k_p} = \sigma_m - \sigma_q$$  \hspace{1cm} (3.37)

for the fixed $q \in U_{r_1}, r_1 \geq 2$.

Firstly, we obtain the terms in $U_{r+2}$ which lie in the set $U_r$, i.e. the case $m \in U_r \subset U_{r+2}$. Then we get

\begin{align*}
(i\sigma_m)^2 p_{r+2,m} &= -p''_{r,m} - 2(i\sigma_m)p'_{r+1,m} - f(p_{0,0}) \left( p'_{r,m} + i\sigma_m p_{r+1,m} \right) \\
&- \sum_{n=1}^{r} \frac{1}{n!} \sum_{\ell \in \mathbb{I}_{n,r}} \sum_{(t,k) \in \mathcal{W}_{r,m}^n} \rho_k^m f^{(n)}(p_{0,0}) \left[ p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n} \right] p'_{0,0} \\
&- \sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\ell \in \mathbb{I}_{n,r-1}} \sum_{(t,k) \in \mathcal{W}_{r,m}^n} \rho_k^m f^{(n)}(p_{0,0}) \left[ p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n} \right] p'_{1,0} \\
&- \sum_{q \in U_{r_1}} \sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\ell \in \mathbb{I}_{n,r-1}} \sum_{(t,k) \in \mathcal{W}_{r,m-q}^n} \rho_k^{m-q} f^{(n)}(p_{0,0}) \left[ p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n} \right] i\sigma_q p_{2,q} \\
&- \sum_{r_1=2}^{r} \sum_{n=1}^{r-r_1} \frac{1}{n!} \sum_{\ell \in \mathbb{I}_{n,r-r_1}} \sum_{(t,k) \in \mathcal{W}_{r,m-q}^n} \rho_k^{m-q} f^{(n)}(p_{0,0}) \left[ p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n} \right] \\
&\times \left( \sum_{q \in U_{r_1}} p'_{r_1,q} + \sum_{q \in U_{r_1+1}} i\sigma_q p_{r_1+1,q} \right) \\
&- \sum_{n=1}^{r} \frac{1}{n!} \sum_{\ell \in \mathbb{I}_{n,r}} \sum_{(t,k) \in \mathcal{W}_{r,m}^n} \rho_k^m f^{(n)}(p_{0,0}) \left[ p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n} \right].  \hspace{1cm} (3.38)
\end{align*}
When \( m = 0 \) in the above equation, \( \sigma_0 = 0 \). Therefore, the term \( p_{r,0} \) can be computed as

\[
\begin{align*}
&\sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\ell \in n,r} \theta_\ell \sum_{(t,k) \in W_{r,0}^n} \rho_k^{0} f^{(n)}(p_{0,0}) \left[ \sum_{\ell,1, \ldots, p_{\ell_n,k_n}} \right] p_{0,0}' + \\
&\sum_{r=1}^{r-1} \frac{1}{n!} \sum_{\ell \in n,r} \theta_\ell \sum_{(t,k) \in W_{r,0}^n} \rho_k^{0} f^{(n)}(p_{0,0}) \left[ \sum_{\ell,1, \ldots, p_{\ell_n,k_n}} \right] p_{1,0}' + \\
&\sum_{r=1}^{r-1} \left[ \sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\ell \in n,r} \theta_\ell \sum_{(t,k) \in W_{r,0}^n} \rho_k^{0} f^{(n)}(p_{0,0}) \left[ \sum_{\ell,1, \ldots, p_{\ell_n,k_n}} \right] \right] \\
&\times \left( \sum_{q \in U_{r+1}} p_{r+1,q} + \sum_{q \in U_{r+1}} i \sigma_q p_{r+1,q} \right) \\
&\sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\ell \in n,r} \theta_\ell \sum_{(t,k) \in W_{r,0}^n} \rho_k^{0} f^{(n)}(p_{0,0}) \left[ \sum_{\ell,1, \ldots, p_{\ell_n,k_n}} \right] 0 = 0.
\end{align*}
\]

The corresponding initial conditions are

\[
\begin{align*}
p_{r,0}(0) &= - \sum_{m \in U_r \setminus \{0\}} p_{r,m}(0), \\
p_{r,0}'(0) &= - \sum_{m \in U_r \setminus \{0\}} p_{r,m}'(0) - \sum_{m \in U_r \setminus \{0\}} i \sigma_m p_{r,m+1}(0).
\end{align*}
\]

Secondly, consider \( m \in U_{r+1} \setminus U_r \) which belong to \( U_{r+2} \),

\[
(i \sigma_m)^2 p_{r+2,m} = -2(i \sigma_m) p_{r+1,m}' - f(p_{0,0}) (i \sigma_m) p_{r+1,m}'
\]

\[
- \sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\ell \in n,r} \theta_\ell \sum_{(t,k) \in W_{r+1,m}^n} \rho_k^m f^{(n)}(p_{0,0}) \left[ \sum_{\ell,1, \ldots, p_{\ell_n,k_n}} \right] p_{0,0}' + \\
- \sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\ell \in n,r} \theta_\ell \sum_{(t,k) \in W_{r+1,m}^n} \rho_k^m f^{(n)}(p_{0,0}) \left[ \sum_{\ell,1, \ldots, p_{\ell_n,k_n}} \right] p_{1,0}' + \\
- \sum_{r=1}^{r-1} \left[ \sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\ell \in n,r} \theta_\ell \sum_{(t,k) \in W_{r+1,m}^n} \rho_k^{m-q} f^{(n)}(p_{0,0}) \left[ \sum_{\ell,1, \ldots, p_{\ell_n,k_n}} \right] i \sigma_q p_{2,0}' + \\
- \sum_{r=1}^{r-1} \left[ \sum_{n=1}^{r-1} \frac{1}{n!} \sum_{\ell \in n,r} \theta_\ell \sum_{(t,k) \in W_{r+1,m}^n} \rho_k^{m-q} f^{(n)}(p_{0,0}) \left[ \sum_{\ell,1, \ldots, p_{\ell_n,k_n}} \right] \right]
\end{align*}
\]
From this equation, the term $p_{r+2,m}, m \in \mathcal{U}_{r+1} \setminus \mathcal{U}_r \subseteq \mathcal{U}_{r+2}$ is derived.

Finally, extract the terms $p_{r+2,m}, m \in \mathcal{U}_{r+2} \setminus \mathcal{U}_{r+1}$.

$$(i\sigma_m)^2 p_{r+2,m} = -p_{0,0}^r \left( \sum_{n=1}^{r-1} \sum_{k \in \mathcal{A}_r} \frac{1}{n!} \sum_{\ell \in \mathcal{A}_{n-1}} \theta_\ell \rho_k^m f^{(n)}(p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n}) \right)$$

$$- \sum_{n=1}^{r-1} \sum_{k \in \mathcal{A}_r} \frac{1}{n!} \sum_{\ell \in \mathcal{A}_{n-1}} \theta_\ell \rho_k^m f^{(n)}(p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n})\rho_k^{m-q} f^{(n)}(p_{0,0}) \left( p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n} \right) i\sigma_q p_{2,q}$$

$$- \sum_{n=1}^{r-1} \sum_{k \in \mathcal{A}_r} \frac{1}{n!} \sum_{\ell \in \mathcal{A}_{n-1}} \theta_\ell \rho_k^m g_n(p_{0,0}) \left( p_{\ell_1,k_1}, \ldots, p_{\ell_n,k_n} \right).$$

We can compute the term $p_{r+2,m}, m \in \mathcal{U}_{r+2} \setminus \mathcal{U}_{r+1}$ directly from this equation.

4 Numerical experiments

In this section, we present some examples that illustrate the construction and properties of the expansion given in Sect. 3. In all cases, we will compare the approximation given by the first few terms of the asymptotic-numerical solver with the exact solution (which will be computed numerically with standard Maple routines up to prescribed accuracy). We use the notation

$$e_s = \left| y(t) - \sum_{r=0}^{s} \frac{\psi_n(t)}{\omega^r} \right|, \quad s \geq 0$$

for the error.

If the nonlinear functions $f$ and $g$ are analytic, the Taylor expansion can be used to approximate them. In particular, when they are polynomials, the Taylor expansions
are simpler and more manageable in the practical case for the Van der Pol and Duffing equations.

4.1 Example 1

We consider first the scalar Van der Pol oscillator. Its governing equation is

\[ y''(t) - \mu (1 - y^2)y'(t) + \varepsilon y(t) = F_\omega(t), \quad y(0) = 1, \quad y'(0) = 0, \]

where

\[ F_\omega(t) = \sum_{m \in \mathcal{U}_0} a_m(t)e^{i\kappa_m \omega t}, \]

where \( \mathcal{U}_0 = \{1, 2\}, \kappa_1 = 1, \kappa_2 = \sqrt{2}, \mu = 0.744313, \varepsilon = 0.983299, a_1(t) = 1 \) and \( a_2(t) = t \) which are the same as in [2].

Herewith the solution by ode45 for Example 1 with \( \omega = 500 \) in Fig. 1:

We will work out the first four terms of the asymptotic expansion (\( r = 0, 1, 2, 3 \))

\[
y(t) \sim p_{0,0}(t) + \frac{1}{\omega} p_{1,0}(t) + \frac{1}{\omega^2} \left[ p_{2,0}(t) + \sum_{m \in \mathcal{U}_2 \setminus \{0\}} p_{2,m}(t)e^{i\sigma_m \omega t} \right] \\
+ \frac{1}{\omega^3} \left[ p_{3,0}(t) + \sum_{m \in \mathcal{U}_3 \setminus \{0\}} p_{3,m}(t)e^{i\sigma_m \omega t} \right].
\]

![Fig. 1](image) The real part of the solution by ode45 for \( \omega = 500 \)
Taking \( r = 0 \), we obtain the two equations

\[
p''_{0,0} + \mu (p^2_0 - 1)p'_{0,0} + \varepsilon p_{0,0}, \quad p_0(0) = 1, \quad p'_0(0) = 0.
\]

and

\[
p_{2,m}(t) = \frac{a_m(t)}{(ik_m)^2}, \quad m = 1, 2,
\]

where

\[
p_{2,1}(t) = \frac{1}{(ik_1)^2}, \quad p_{2,2}(t) = \frac{t}{(ik_2)^2};
\]

When \( r = 1 \), the resulting equations are

\[
p''_{1,0} + \mu (p^2_{0,0} - 1)p'_{1,0} + 2\mu p'_{0,0}p_{0,0}p_{1,0} + \varepsilon p_{1,0} = 0, \quad p_{1,0}(0) = 0, \quad p'_{1,0}(0) = \frac{-1}{ik_1}.
\]

and

\[
p_{3,m}(t) = \frac{-1}{ik_m} \left[ 2p'_{2,m} + \mu (p^2_{0,0} - 1)p_{2,m} \right], \quad m = 1, 2,
\]

Thus

\[
p_{3,1}(t) = \frac{-\mu (p^2_{0,0} - 1)}{(ik_1)^3}, \quad p_{3,2}(t) = \frac{-1}{(ik_2)^3} \left( 2 + \mu (p^2_{0,0} - 1) \right);
\]

With \( r = 2 \), the term \( p''_{2,0} \) satisfies the differential equation

\[
p''_{2,0} + \mu (p^2_{0,0} - 1)p'_{2,0} + 2\mu p_{0,0}p'_{1,0}p_{1,0} + 2\mu p'_{0,0}p_{0,0}p_{2,0} + \mu p'_{0,0}p_{0,0}p_{1,0} + \varepsilon p_{2,0} = 0,
\]

\[
p_{2,0}(0) = \frac{-1}{(ik_1)^2}, \quad p'_{2,0}(0) = \frac{1}{(ik_2)^2}.
\]

In addition, \( p_{4,m} \) for \( m \neq 0 \)

\[
p_{4,1}(t) = \frac{-1}{(ik_1)^4} \left[ -2\mu p_{0,0}p'_{0,0} - \mu^2 (p^2_{0,0} - 1)^2 + 2\mu (ik_1)p_{0,0}p_{1,0} + \varepsilon \right],
\]

\[
p_{4,2}(t) = \frac{-1}{(ik_2)^4} \left[ -3\mu (p^2_{0,0} - 1) - 2\mu p_{0,0}p'_{0,0} - \mu^2 t (p^2_{0,0} - 1)^2 + 2\mu (ik_2)p_{0,0}p_{1,0} + \varepsilon t \right];
\]
When $r = 3$, the term $p_{3,0}$ obeys

$$
p''_{3,0} + \mu (p_{0,0}^2 - 1)p'_{3,0} + 2\mu p_{0,0}p_{1,0}p'_{2,0} + 2\mu p_{1,0}p_{0,0}p_{2,0} + \mu p'_{1,0}p_{1,0}p_{0,0} + 2\mu p_{0,0}(p_{0,0}p_{3,0} + p_{1,0}p_{2,0}) + \epsilon p_{3,0} = 0 \quad (4.1)
$$

subject to the initial conditions

$$p_{3,0}(0) = \frac{2}{(i\kappa_2)^3}, \quad p'_{3,0}(0) = \frac{\epsilon}{(i\kappa_1)^3}.$$  

Figures 2 and 3 show the asymptotic error with $r = 0, 1, 2, 3$ for $\omega = 500$ (the left) and $\omega = 5,000$ (the right), respectively compared with the ode45 in Matlab with the very high tolerance $abstol = reltol = 1e^{-28}$, respectively. As evident from these figures, the asymptotic error decreases for increasing $r$. Furthermore, the accuracy of the asymptotic method increases greatly for the same number of $r$ levels for higher values of $\omega$. This feature makes the method most suitable for simulation of modern electronic systems where ever-rising frequencies are present.

Focussing on the CPU time, some impressive results are obtained when the asymptotic method is compared to the Runge–Kutta method. While the asymptotic method takes about 5 s to compute the solution for $\omega = 500$ and $\omega = 5,000$, the rkf45 method takes about 207 s for $\omega = 500$. Moreover, this increases to 2,432 s for $\omega = 5,000$. Again, this marks the significant value of the method for computing results involving very high frequencies.

4.2 Example 2

To illustrate the use of the asymptotic method for a second-order differential equation system, a coupled Van der Pol-Duffing oscillator is chosen from [2]

$$
\ddot{x} - \mu_1 (1 - x^2)\dot{x} + \epsilon x = \alpha_1 \dot{y} + \alpha_2 y + i(t) \\
\ddot{y} + \mu_2 \dot{y} + \gamma y + c_0 y^3 = \alpha_3 \dot{x} + \alpha_4 x \quad (4.2)
$$

The forcing term $i(t)$ is composed of two non-commensurate frequencies.

$$i(t) = \sum_{m=1}^{M} a_m(t)e^{i\omega_m t},$$

In equation (4.2), $\mu_1 = 0.744313, \mu_2 = 0.668083, \alpha_1 = 0.235191, \alpha_2 = 0, \alpha_3 = 0.981204, \alpha_4 = 0, \gamma = -1.6, c_0 = 0.222375, \epsilon = 0.983299, \kappa_1 = \sqrt{2}, \kappa_2 = 2$.

$$a_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad a_2(t) = \begin{pmatrix} \sin(t) \\ 0 \end{pmatrix}$$
and the initial conditions are \( x(0) = 1, x'(0) = 0, y(0) = 0, y'(0) = 0 \). Let \( \mathbf{p}_{\ell,k} = \begin{pmatrix} (\mathbf{p}_{\ell,k})_1 \\ (\mathbf{p}_{\ell,k})_2 \end{pmatrix} \), where \((\mathbf{p}_{\ell,k})_j\) is the \(j\)-th elements of the vector \( \mathbf{p}_{\ell,k} \). The corresponding representation functions in this equation are

\[
\begin{align*}
\mathbf{f}(\mathbf{y}) &= \begin{pmatrix} -\mu_1 (1 - x^2) & -\alpha_1 \\ -\alpha_3 & \mu_2 \end{pmatrix}, \\
\mathbf{f}^{(1)}(\mathbf{y}) &= \begin{pmatrix} (2\mu_1 x, 0) \\ (0, 0) \end{pmatrix}, \\
\mathbf{f}^{(2)}(\mathbf{y})[\mathbf{p}_{\ell_1,k_1}, \mathbf{p}_{\ell_2,k_2}] &= \begin{pmatrix} 2\mu_1 (\mathbf{p}_{\ell_1,k_1})_1 (\mathbf{p}_{\ell_2,k_2})_1 \\ 0 \end{pmatrix}, \\
\mathbf{g}(\mathbf{y}) &= \begin{pmatrix} \epsilon x \\ y y + c_0 y^3 \end{pmatrix}, \\
\mathbf{g}^{(1)}(\mathbf{y}) &= \begin{pmatrix} (\epsilon, 0) \\ (0, y + 3c_0 y^2) \end{pmatrix}, \\
\mathbf{g}^{(2)}(\mathbf{y})[\mathbf{p}_{\ell_1,k_1}, \mathbf{p}_{\ell_2,k_2}] &= \begin{pmatrix} 0 \\ 6c_0 y (\mathbf{p}_{\ell_1,k_1})_2 (\mathbf{p}_{\ell_2,k_2})_2 \end{pmatrix}.
\end{align*}
\]

Fig. 2 The top row: the real and imaginary parts of \( e_0 \) (left) and \( e_1 \) (right). The second row: the real and imaginary parts of \( e_2 \) (the left) and \( e_3 \) (right) with \( \omega = 500 \).
Fig. 3 The top row: the real and imaginary parts of $e_0$ (left) and $e_1$ (right). The second row: the real and imaginary parts of $e_2$ (left) and $e_3$ (right) with $\omega = 5,000$

$$g^{(3)}(y)[p_{\ell_1,k_1}, p_{\ell_2,k_2}, p_{\ell_3,k_3},] = \begin{pmatrix} 0 \\ 6c_0(p_{\ell_1,k_1})_2(p_{\ell_2,k_2})_2(p_{\ell_3,k_3})_2 \end{pmatrix}$$

The remaining terms are zero.

Assume that the the unknown functions $x(t)$ and $y(t)$ are

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \sim p_{0,0} + \frac{1}{\omega} p_{1,0} + \frac{1}{\omega^2} \left[ p_{2,0} + p_{2,1}e^{ik_1\omega t} + p_{2,2}e^{ik_2\omega t} \right]$$

$$+ \frac{1}{\omega^2} \left[ p_{3,0} + p_{3,1}e^{ik_1\omega t} + p_{3,2}e^{ik_2\omega t} \right].$$

The initial conditions are

$$p_{0,0}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad p'_{0,0}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
Fig. 4 The top row: the real (left) and imaginary (right) parts of $e_0$. The middle row: the real (left) and imaginary (right) parts of $e_1$. The third row: the real (left) and imaginary (right) parts of $e_2$. The fourth row: the real (left) and imaginary (right) parts of $e_3$ for $x(t)$ with $\omega = 100$. 

\[ e_0 \approx 0.01, \quad e_1 \approx 5 \times 10^{-7}, \quad e_2 \approx 1 \times 10^{-8}, \quad e_3 \approx 5 \times 10^{-9} \]
Fig. 5 The top row: the real (left) and imaginary (right) parts of $e_0$. The middle row: the real (left) and imaginary (right) parts of $e_1$. The third row: the real (left) and imaginary (right) parts of $e_2$. The fourth row: the real (left) and imaginary (right) parts of $e_3$ for $y(t)$ with $\omega = 100$
Fig. 6 The top row: the real (left) and imaginary (right) parts of $e_0$. The middle row: the real (left) and imaginary (right) parts of $e_1$. The third row: the real (left) and imaginary (right) parts of $e_2$. The fourth row: the real (left) and imaginary (right) parts of $e_3$ for $x(t)$ with $\omega = 1,000$. 

\[ \text{Asymptotic solvers for second-order differential equation systems} \]
Fig. 7 The top row: the real (left) and imaginary (right) parts of $\varepsilon_0$. The middle row: the real (left) and imaginary (right) parts of $\varepsilon_1$. The third row: the real (left) and imaginary (right) parts of $\varepsilon_2$. The fourth row: the real (left) and imaginary (right) parts of $\varepsilon_3$ for $y(t)$ with $\omega = 1,000$
Asymptotic solvers for second-order differential equation systems

\[ p_{1,0}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad p'_{1,0}(0) = -\left[(i\kappa_1)p_{2,1}(0) + (i\kappa_2)p_{2,2}(0)\right], \quad (4.5) \]

\[ p_{r,0}(0) = -\left[p_{r,1}(0) + p_{r,2}(0)\right], \quad (4.6) \]

\[ p'_{r,0}(0) = -\left[p'_{r,1}(0) + p'_{r,2}(0)\right] - \left[(i\kappa_1)p_{r+1,1}(0) + (i\kappa_2)p_{r+1,2}(0)\right], \quad r \geq 2. \quad (4.7) \]

The base equation is

\[
\begin{pmatrix} (p_{0,0})'' \\ (p_{0,0})''_2 \end{pmatrix} + \begin{pmatrix} \mu_1((p_{0,0})^2 - 1) - \alpha_1 \\ -\alpha_3 \end{pmatrix} \begin{pmatrix} (p_{0,0})_1' \\ (p_{0,0})_2' \end{pmatrix} + \begin{pmatrix} \epsilon(p_{0,0})_1 \\ \gamma(p_{0,0})_2 + c_0(p_{0,0})^3_2 \end{pmatrix} = 0
\]  
with

\[ p_{0,0}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad p'_{0,0}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (4.9) \]

Moreover, we have

\[ p_{2,1}(t) = \frac{1}{(i\kappa_1)^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad p_{2,2}(t) = \frac{1}{(i\kappa_2)^2} \begin{pmatrix} \sin(t) \\ 0 \end{pmatrix}. \quad (4.10) \]

For \( r = 1 \), the equation for \( p_{1,0} \) is

\[
\begin{pmatrix} (p_{1,0})'' \\ (p_{1,0})''_2 \end{pmatrix} + \begin{pmatrix} \mu_1((p_{1,0})^2 - 1)(p_{1,0})'_1 - \alpha_1(p_{1,0})'_2 \\ -\alpha_3(p_{1,0})'_1 + \mu_2(p_{1,0})'_2 \end{pmatrix} + \begin{pmatrix} 2\mu_1(p_{0,0})_1(p_{0,0})_1'(p_{0,0})'_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \epsilon(p_{1,0})_1 \\ \gamma + 3c_0(p_{0,0})^2_2(p_{1,0})_2 \end{pmatrix} = 0
\]  
with the initial conditions

\[ p_{1,0}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad p'_{1,0}(0) = \frac{-1}{i\kappa_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

and

\[ p_{3,1}(t) = \frac{-1}{(i\kappa_1)^3} \begin{pmatrix} \mu_1((p_{0,0})^2 - 1) \\ -\alpha_3 \end{pmatrix}, \quad (4.12) \]

\[ p_{3,2}(t) = \frac{-1}{(i\kappa_2)^3} \begin{pmatrix} \mu_1((p_{0,0})^2 - 1) \sin(t) + 2\cos(t) \\ -\alpha_3 \sin(t) \end{pmatrix}. \quad (4.13) \]
The \( r = 2 \) terms are

\[
\left( \frac{(p_{2,0})''}{(p_{2,0})_2} \right) + \left( \frac{\mu_1((p_{0,0})^2_1 - 1)(p_{2,0})'_1 - \alpha_1(p_{2,0})'_2}{(p_{2,0})'_2} \right) \\
+ \left( 2\mu_1(p_{0,0})_1(p_{1,0})'_1 \right) + \left( 0 \right) + \left( 2\mu_1(p_{0,0})_1(p_{2,0})'_1 \right) + \left( \epsilon(p_{2,0})_1 \right) \\
+ \left( \mu_1(p_{1,0})_1(p_{0,0})'_1 \right) + \left( \gamma + 3c_0(p_{0,0})^2_2 \right) \\
+ \left( 3c_0(p_{0,0})_2(p_{1,0})^2_2 \right) = 0
\]

(4.14)

with the initial conditions

\[
p_{2,0}(0) = -\frac{1}{(i\kappa_1)^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad p'_{2,0}(0) = \left( \frac{1}{(i\kappa_2)^2} \frac{i\kappa_2}{-\alpha_3} \right).
\]

and

\[
p_{4,1}(t) = -\frac{1}{(i\kappa_1)^4} \left[ \left( -4\mu_1(p_{0,0})_1(p_{0,0})'_1 \right) + \left( \mu_1((p_{0,0})^2_1 - 1)^2 + \alpha_1\alpha_3 + \mu_1((p_{0,0})^2_1 - 1) - \mu_2\alpha_3 \right) \right] \\
+ (i\kappa_1) \left( 2\mu_1(p_{0,0})_1(p_{1,0})'_1 \right) + \left( 2\mu_1(p_{0,0})_1(p_{2,0})'_1 \right) + \left( \epsilon(p_{2,0})_1 \right),
\]

(4.15)

\[
p_{4,2}(t) = -\frac{1}{(i\kappa_2)^4} \left[ \left( -\sin(t) \right) + \left( \sin(t) \right) \right] \\
-2 \left( \mu_1 \cos(t) \left( (p_{0,0})^2_1 - 1 \right) + 2\mu_1 \sin(t)(p_{0,0})_1(p_{0,0})'_1 - 2 \sin(t) \right) \\
- \left( \mu_1 \left( (p_{0,0})^2_1 - 1 \right) - \alpha_1 \right) \left( -\mu_1 \left( (p_{0,0})^2_1 - 1 \right) \right) \\
+ (i\kappa_2) \left( 2\mu_1(p_{0,0})_1(p_{1,0})'_1 \right) + \left( 0 \right) \right],
\]

(4.16)

Similarly, the term \( p_{3,0} \) satisfies

\[
\left( \frac{(p_{3,0})''}{(p_{3,0})_2} \right) + \left( \frac{\mu_1((p_{0,0})^2_1 - 1) - \alpha_1}{\mu_2} \right) \left( \frac{(p_{3,0})'_1}{(p_{3,0})'_2} \right) \\
+ \left( 2\mu_1(p_{0,0})_1(p_{1,0})'_1 \right) + \left( 0 \right) + \left( 2\mu_1(p_{0,0})_1(p_{2,0})'_1 \right) + \left( \epsilon(p_{3,0})_1 \right) \\
+ \left( \mu_1(p_{1,0})_1(p_{0,0})'_1 \right) + \left( 2\mu_1(p_{0,0})_1(p_{3,0})'_1 \right) + \left( 2\mu_1(p_{2,0})_1(p_{1,0})'_1 \right) + \left( \gamma + 3c_0(p_{0,0})^2_2 \right) \\
+ \left( 6c_0(p_{0,0})_2(p_{3,0})_2 \right) = 0.
\]

(4.17)
with initial conditions
\[ p_{3,0}(0) = \left( \frac{2}{(i\kappa_2)^3} \frac{\alpha_3}{(i\kappa_3)^3} \right), \quad p'_{3,0}(0) = \left( \frac{\epsilon - \alpha_1 \alpha_3}{(i\kappa_1)^3} \frac{1}{2\alpha_3} \left( \frac{\mu_2}{(i\kappa_3)^3} + \frac{\mu_2}{(i\kappa_1)^3} \right) \right). \]

For \( r = 0, 1, 2, 3 \) in the expansion, Fig. 4 illustrates the real and imaginary part of the errors for the first variable \( x(t) \) with \( \omega = 100 \), compared with the Maple routine of Runge–Kutta method. Figure 5 illustrates the results for \( y(t) \).

In Figs. 6 and 7 we have displayed the real and imaginary parts of the errors when the oscillatory parameter is \( \omega = 1,000 \).

It can be seen that the error of the asymptotic method reduces greatly with increasing \( \omega \). In terms of the CPU time, the asymptotic method takes 9.5 s, compared to the Runge–Kutta method which takes 375 s for \( \omega = 500 \) and 5,145 s for \( \omega = 5,000 \).

References

1. Bleistein, N., Handlesman, R.: Asymptotic Expansions of Integrals. Dover, New York (1975)
2. Chedjou, J.C., Fotsin, H.B., Woafo, P., Domngang, S.: Analog simulation of the dynamics of a Van der Pol oscillator coupled to a Duffing oscillator. IEEE Trans. Circ. Syst. I: Fundam. Theory Appl. 48(6), 748–757 (2001)
3. D. Cohen. Analysis and numerical treatment of highly oscillatory differential equations. PhD thesis, University of Geneva (2004)
4. Cohen, D., Hairer, E., Lubich, C.: Modulated Fourier expansions of highly oscillatory differential equations. Found. Comput. Math. 3, 327–345 (2003)
5. Condon, M., Deaño, A., Iserles, A.: On systems of differential equations with extrinsic oscillation. Discret. Cont. Dyn. Syst. 28, 1345–1367 (2010)
6. Condon, M., Deaño, A., Iserles, A.: On second order differential equations with highly oscillatory forcing terms. Proc. Royal Soc. A. 466, 1809–1828 (2010)
7. Condon, M., Deaño, A., Gao, J., Iserles, A.: Asymptotic solvers for ordinary differential equations with multiple frequencies. NA2011/11
8. Fitzhugh, R.: Impulses and physiological states in theoretical models of nerve membranes. Biophys. J. 1, 445–466 (1961)
9. Fodjoung, G.J., Fotsin, H.B., Woafo, P.: Synchronizing modified van der Pol-Duffing oscillators with offset terms using observer design: application to secure communications. Phys. Scr. 75, 638–644 (2007)
10. Hairer, E., Norsett, S., Wanner, G.: Solving Ordinary Differential Equations I: Nonstiff Problems. Springer, Berlin (1993)
11. Hairer, E., Norsett, S., Wanner, G.: Geometric Numerical Integration, 2nd edn. Springer, Berlin (2006)
12. Nagumo, J., Arimoto, S., Yoshizawa, S.: An active pulse transmission line simulating nerve axon. Proc. IRE 50, 2061–2070 (1962)
13. Njah, A.N., Vincent, U.E.: Chaos synchronization between single and double wells Duffing-Van der Pol oscillators using active control. Chaos Solitons Fractals 37, 1356–1361 (2008)
14. Sanz-Serna, J.M.: Modulated Fourier expansions and heterogeneous multiscale methods.IMA J. Numer. Anal. 29, 595–605 (2009)
15. Slight, T.J., et al.: A Lienard oscillator resonant tunnelling diode-laser diode hybrid integrated circuit: model and experiment. IEEE J. Quantum Electron. 44(12), 1158–1163 (2008)