Repeatedly Matching Items to Agents Fairly and Efficiently

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ABSTRACT
We consider a novel setting where a set of items are matched to the same set of agents repeatedly over multiple rounds. Each agent gets exactly one item per round, which brings interesting challenges to finding efficient and/or fair repeated matchings. A particular feature of our model is that the value of an agent for an item in some round depends on how often the item has been used by the agent in the past. We present a set of positive and negative results about the efficiency and fairness of repeated matchings.

KEYWORDS
Resource allocation; fair division; envy-freeness; matchings

1 INTRODUCTION
The problem of fairly dividing indivisible items among agents has received enormous attention by the EconCS research community in recent years [1–4, 7, 8]. The standard setting involves a set of items and agents who have values for them. The objective is to compute a fair allocation which gives each item to a single agent. In practice, sometimes, the same set of items must be allocated to the same set of agents repeatedly. Prior work has typically explored various settings where agents’ allocations do not change with time, with few exceptions [6, 7]. More crucially, another feature that distinguishes such scenarios from the standard setting is that the value of an agent for an item changes over time and typically depends on how many times the agent has received the item in the past.

To give an example, consider different research labs that all need access to several expensive research facilities in a university. The access to these facilities must be fairly coordinated/scheduled throughout the year. To be fair among labs and efficient overall, we must take into account the values the labs have for facilities, which typically change over time. For instance, during the first few weeks of access to a facility, the researchers in a lab may need time to learn how to operate it. During that time, the value the lab gets by accessing a facility can be very low, even negative. As the researchers gain more experience, their research output increases, and so does the lab’s value for the facility. Once the researchers have run their intended experiments, the lab’s value for the facility decreases again until the next experiment.

Conceptual contribution. To capture such situations, we introduce a new model of repeated matchings with \( n \) agents who must be matched with exactly one of \( n \) items in each of \( T \) rounds, repeatedly. An important novelty of our model is that valuations are history-dependent: the value an agent has for an item in a round depends on how many times the agent has used the item in previous rounds. We use social welfare to assess the efficiency of repeated matchings. We also use relaxations of envy-freeness as fairness concepts. We adapt the well-known envy-freeness up to one item (EF1) and use it when all valuations are non-negative (i.e., when items are goods). We observe that EF1 is not suitable when valuations can be positive or negative (i.e., when items are mixed), and introduce the notion of swap envy-freeness to assess fairness of repeated matchings for mixed items.

Technical contribution. We prove that the problem of computing a repeated matching with maximum social welfare is NP-hard, even when \( T = 3 \). Our hardness reduction defines instances with non-monotone valuations. The problem becomes solvable in polynomial time when the valuations are monotone. We also consider fair repeated matchings. First, for EF1, we find that under identical valuations, EF1 repeated matchings always exist and can be found tractably. Furthermore, we show that any instance with general valuations and \( T \mod n \in \{0, 1, 2, n − 1\} \) (i.e., including all instances with at most four agents/items) has an EF1 repeated matching, which can be computed efficiently. Unfortunately, EF1 is not compatible with social welfare maximization and even approximating the maximum social welfare over EF1 repeated matchings is NP-hard. This holds even when EF1 solutions and social welfare maximizing solutions can be found in polynomial time separately.

We further propose and study a new fairness notion called swap envy-freeness (swapEF). Here, we find that under identical valuations, swapEF repeated matchings can be found using the same algorithm as used for EF1. Furthermore, we show that swapEF repeated matchings always exist and can be computed efficiently on instances with \( T \mod n \in \{0, 1, 2, n − 2, n − 1\} \) (i.e., including all instances with at most five agents/items). Our hardness results are proved on instances with goods. Our positive results besides those for EF1, apply to instances with mixed items. The complete details can be found in [5].

2 NOTATION AND PRELIMINARIES
Our setting involves a set \( \mathcal{A} \) of \( n \) agents and a set \( \mathcal{G} \) of \( n \) items. We consider instances of the form \( I = (\mathcal{A}, \mathcal{G}, \{v_i\}_{i \in \mathcal{A}}, T) \), where \( T \) denotes the number of rounds and, for each agent \( i \in \mathcal{A}, v_i \) is a function from \( \mathcal{G} \times \{T\} \) to \( \mathbb{R} \), where \( v_i(g, t) \) denotes the valuation of...
agent $i$ for item $g$ when it is matched to the item for the $i$th time. A repeated matching $A = (A^1, \ldots, A^T)$ is simply a collection of matchings, with one matching $A^t$ per each round $t \in [T]$. Furthermore, we denote by $A_i$ the multiset (or bundle) which contains copies of the items to which agent $i \in \mathcal{A}$ is matched in the $T$ rounds.

Hence, defining the bundles $A_i$ for $i \in \mathcal{A}$ given the repeated matching $A$ is trivial. The opposite task is also straightforward. Let $N(B, g)$ be the multiplicity of item $g$ in bundle $B$. Given bundles of items $A_i$ for $i \in \mathcal{A}$ with $|A_i| = T$ (i.e., each agent gets $T$ copies of items) and $\sum_{i \in \mathcal{A}} N(A_i, g) = T$ (i.e., $T$ copies of each item $g$ are allocated), a consistent repeated matching\(^1\) for instance $i$ is obtained as follows. We construct the bipartite multigraph $G = (\mathcal{A}, \mathcal{G}, E)$ so that the set of edges $E$ consists of a (copy of) edge $(i, g)$ for every (copy of) item $g$ such that $g \in A_i$. The graph $G$ is $T$-regular and, thus, by Hall’s matching theorem (see 9), can be decomposed into $T$ matchings of edges $M_1, \ldots, M_T$. These matchings correspond to a repeated matching by interpreting the edge $(i, g)$ in matching $M_t$ as the assignment of item $g$ to agent $i$ in the $t$th round.

With a slight abuse of notation, we use $v_i(B)$ to denote the value agent $i \in \mathcal{A}$ has when she gets the bundle $B$, i.e., $v_i(B) = \sum_{g \in N(B, g)} v_i(g)$. Hence, for a repeated matching $A$, $v_i(A_i)$ is the total value from each item copy agent $i$ receives in all rounds. The social welfare of $A$ is simply the sum of the agents’ values for their bundle, i.e., $SW(A) = \sum_{i \in \mathcal{A}} v_i(A_i)$.

We shall look at specific types of valuations under which we will try to find efficient and/or fair repeated matchings. A well-motivated setting is that of identical valuations where $v_1 = v_2 = \cdots = v_n$. This assumption proves particularly useful in finding fair solutions. Another important class of valuation functions is that of monotone valuations. We extend to repeated matchings envy-freeness of up to one item (EF1) as follows.

**Definition 2.1 (EF1).** A repeated matching $A$ is EF1 if for every pair of agents $i, j \in \mathcal{A}$, there exists an item $g \in \mathcal{G}$ such that $v_i(A_i) \geq v_j(A_i \setminus \{g\})$.

We remark that this repeated matching is not unique. However, this does not affect the values of each agent for her bundle and the bundle of any other agent, which are the same in all different consistent repeated matchings.

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