GEOMETRY OF DIVISION RINGS

IGOR V. NIKOLAEV

Abstract. We prove an analog of Belyi’s theorem for the algebraic surfaces. Namely, any non-singular algebraic surface can be defined over a number field if and only if it covers the complex projective plane with ramification at three knotted two-dimensional spheres.

1. Introduction

Belyi’s theorem says that any non-singular algebraic curve over a number field is a covering of the complex projective line \( P^1(\mathbb{C}) \) ramified at the three points \( \{0, 1, \infty\} \) [Belyi 1979] [1, Theorem 4]. The aim of our note is an analog of Belyi’s theorem for the algebraic surfaces based on an approach of [6]. Namely, we associate to the countable division ring an avatar, see definition 1.1. If the ring is commutative (non-commutative, resp.), then its avatar is an algebraic curve (an algebraic surface, resp.) defined over the field \( \mathbb{C} \). For example, an avatar of the ring of rationals (rational quaternions, resp.) is the complex projective line \( P^1(\mathbb{C}) \) (complex projective plane \( P^2(\mathbb{C}) \), resp.) Belyi’s theorem follows from an embedding of rationals into the field of algebraic numbers [6, Section 4] and its analog for surfaces from an embedding of rational quaternions into a quaternion algebra, see Section 4 in below. An extension of Belyi’s theorem to complex surfaces was studied independently in [González-Diez 2008] [3].

Recall that an analogy between the number fields and complex algebraic curves is well known [Eisenbud & Harris 1999] [2, p. 83]. The Grothendieck’s theory of schemes cannot explain this relation [Manin 2006] [4, Section 2.2]. The problem can be solved in terms of the \( C^* \)-algebras [6]. To explain the solution, let \( R \) be a (discrete) associative ring, let \( M_2(R) \) be the matrix ring over \( R \) and let

\[ \rho : M_2(R) \rightarrow \mathcal{B}(\mathcal{H}) \]  

be a self-adjoint representation of \( M_2(R) \) by the bounded linear operators on a Hilbert space \( \mathcal{H} \). Taking the norm-closure of \( \rho(M_2(R)) \) in the strong operator topology, one gets a \( C^* \)-algebra \( \mathcal{A}_R \). Likewise, let \( B(V, \mathcal{L}, \sigma) \) be the twisted homogeneous coordinate ring of a complex projective variety \( V \), where \( \mathcal{L} \) is an invertible sheaf and \( \sigma \) is an automorphism of \( V \) [Stafford & van den Bergh 2001] [8, p. 173]. Recall that the Serre \( C^* \)-algebra, \( \mathcal{A}_V \), is the norm closure of a self-adjoint representation of the ring \( B(V, \mathcal{L}, \sigma) \) in \( \mathcal{B}(\mathcal{H}) \) [5, Section 5.3.1]. Finally, let \( \mathcal{K} \) be the \( C^* \)-algebra of all compact operators on the Hilbert space \( \mathcal{H} \). We refer the reader to [6] for the motivation and examples illustrating the following definition.

2010 Mathematics Subject Classification. Primary 12E15, 32J15; Secondary 46L85.

Key words and phrases. quaternion algebra, algebraic surface, Belyi’s theorem.
Definition 1.1. The complex projective variety $V$ is called an avatar of the ring $R$, if there exists a $C^*$-algebra homomorphism:

$$\mathcal{A}_V \to \mathcal{A}_R \otimes \mathcal{K}.$$ (1.2)

Theorem 1.2. ([6]) Let $\overline{\mathbb{Q}}$ be the algebraic closure of the field $\mathbb{Q}$. Then:

(i) $P^1(\mathbb{C})$ is an avatar of the field $\mathbb{Q}$;

(ii) the avatar of a field $K \subset \overline{\mathbb{Q}}$ is a non-singular algebraic curve $C(\overline{\mathbb{Q}})$;

(iii) the field extension $\mathbb{Q} \subset K$ defines a covering $C(\overline{\mathbb{Q}}) \to P^1(\mathbb{C})$ ramified at the points $\{0, 1, \infty\}$.

Remark 1.3. Belyi’s theorem follows from item (iii) of theorem 1.2. Roughly speaking, it can be shown that each non-singular algebraic curve $C(\overline{\mathbb{Q}})$ is the avatar of a field $K \subset \overline{\mathbb{Q}}$. We refer the reader to [6, Section 4] for the details.

To formalize our results, we use the following notation. Denote by $\left(\frac{a}{b} K\right)$ a quaternion algebra, i.e the algebra over a field $F$, such that $\{1, i, j, ij\}$ is a basis for the algebra and

$$i^2 = a, \quad j^2 = b, \quad ji = -ij$$ (1.3)

for some $a, b \in F^\times$ [Voight 2021] [9, Section 2.2]. The quaternion algebra with $a = b = -1$ will be written as $\mathbb{H}(F)$. The algebra $\mathbb{H}(\mathbb{R})$ corresponds to the Hamilton quaternions and the algebra $\mathbb{H}(\mathbb{Q})$ corresponds to the rational quaternions. Our main result is a generalization of theorem 1.2 to the division rings $\left(\frac{a}{b} K\right)$, where $K \subset \overline{\mathbb{Q}}$.

Theorem 1.4. Let $\left(\frac{a}{b} K\right)$ be a division ring, such that $K \subset \overline{\mathbb{Q}}$. Then:

(i) $P^2(\mathbb{C})$ is an avatar of the division ring $\mathbb{H}(\mathbb{Q})$;

(ii) the avatar of a division ring $\left(\frac{a}{b} K\right)$ is a non-singular algebraic surface $S(\overline{\mathbb{Q}})$;

(iii) the field extension $\mathbb{Q} \subset K$ defines a covering $S(\overline{\mathbb{Q}}) \to P^2(\mathbb{C})$ ramified at three knotted two-dimensional spheres $P^1(\mathbb{C}) \cup P^1(\mathbb{C}) \cup P^1(\mathbb{C})$.

Remark 1.5. A relation between algebraic surfaces and division rings follows from [5, Section 7.2]. Indeed, each non-singular algebraic surface is a smooth 4-dimensional manifold and the arithmetic topology relates such manifolds to the cyclic division algebras [5, Theorem 7.2.1].

Remark 1.6. The knotting type of $P^1(\mathbb{C}) \cup P^1(\mathbb{C}) \cup P^1(\mathbb{C})$ depends on the arithmetic of the field $K$ and extends the Grothendieck’s theory of dessin d’enfant to the case of algebraic surfaces.

The paper is organized as follows. A brief review of the preliminary facts is given in Section 2. Theorem 1.4 is proved in Section 3. An analog of Belyi’s theorem for the algebraic surfaces is proved in Section 4.
2. Preliminaries

This section is a brief review of the quaternion and Serre $C^*$-algebras. We refer the reader to [Voight 2021] [9, Section 2.2] and [5, Section 5.3.1] for a detailed account.

2.1. Quaternion algebras.

**Definition 2.1.** The algebra $(\frac{a,b}{F})$ over a field $F$ is called a quaternion algebra if there exists $i, j \in (\frac{a,b}{F})$ such that $\{1, i, j, ij\}$ is a basis for $(\frac{a,b}{F})$ and

$$i^2 = a, \quad j^2 = b, \quad ji = -ij$$

for some $a, b \in F^\times$.

**Example 2.2.** If $F \cong \mathbb{R}$ and $a = b = -1$, then the quaternion algebra $(\frac{-1}{\mathbb{R}})$ consists of the Hamilton quaternions $\mathbb{H}(\mathbb{R})$; hence the notation. If $F \cong \mathbb{Q}$, then the quaternion algebra $(\frac{-1}{\mathbb{Q}})$ consists of the rational quaternions $\mathbb{H}(\mathbb{Q})$.

A $*$-involution on $(\frac{a,b}{F})$ is defined by the formula $(1, i, j, k) \mapsto (1, -i, -j, -k)$.

The norm $N(u) = uu^*$ of an element $u = x_0 + xi + yj + zk \in (\frac{a,b}{F})$ is a quadratic form:

$$N(x_0 + xi + yj + zk) = x_0^2 - ax^2 - by^2 + abz^2. \quad (2.2)$$

Since $N(1) = 1$ and $N(uv) = N(u)N(v)$, one concludes that the $(\frac{a,b}{F})$ is a division algebra if and only if quadratic form $(2.2)$ vanishes only at the zero element $u = 0$. Thus $(2.2)$ must be a positive form for all $x_0, x, y, z \in F$.

It is easy to see, that the form $(2.2)$ admits non-trivial zeroes if and only if there are such zeroes for the ternary quadratic form:

$$Q(x, y, z) = -ax^2 - by^2 + abz^2. \quad (2.3)$$

The substitution $a' = \frac{1}{a}, \quad b' = \frac{1}{a}$ maps the zeroes of $(2.3)$ to the $F$-points of a conic surface given by the equation:

$$z^2 = ax^2 + by^2. \quad (2.4)$$

The following classification of the quaternion algebras is well known.

**Theorem 2.3.** ([9, Theorem 5.1.1]) The formula

$$\left(\frac{a,b}{F}\right) \mapsto Q(x, y, z) \quad (2.5)$$

maps isomorphic quaternion algebras to the similar ternary quadratic forms. Equivalently, the quaternion algebras are classified by the isomorphism classes of the conic surfaces $(2.4)$. 

2.2. Serre C∗-algebras. Let V be an n-dimensional complex projective variety endowed with an automorphism \( \sigma : V \to V \) and denote by \( B(V, \mathcal{L}, \sigma) \) its twisted homogeneous coordinate ring [Stafford & van den Bergh 2001] [8]. Let \( R \) be a commutative graded ring, such that \( V = \text{Proj}(R) \). Denote by \( R[t, t^{-1}; \sigma] \) the ring of skew Laurent polynomials defined by the commutation relation \( b^* t = tb \) for all \( b \in R \), where \( b^* \) is the image of \( (a, b) \) under automorphism \( \sigma \). It is known, that \( R[t, t^{-1}; \sigma] \cong B(V, \mathcal{L}, \sigma) \).

Let \( H \) be a Hilbert space and \( \mathcal{B}(H) \) the algebra of all bounded linear operators on \( H \). For a ring of skew Laurent polynomials \( R[t, t^{-1}; \sigma] \), consider a homomorphism:

\[
\rho : R[t, t^{-1}; \sigma] \to \mathcal{B}(H).
\] (2.6)

Recall that \( \mathcal{B}(H) \) is endowed with a \(*\)-involution; the involution comes from the scalar product on the Hilbert space \( H \). We shall call representation (2.6) \(*\)-coherent, if (i) \( \rho(t) \) and \( \rho(t^{-1}) \) are unitary operators, such that \( \rho^*(t) = \rho(t^{-1}) \) and (ii) for all \( b \in R \) it holds \( (\rho^*(b))^\sigma = \rho^*(b^*) \), where \( \sigma(\rho) \) is an automorphism of \( \rho(R) \) induced by \( \sigma \). Whenever \( B = R[t, t^{-1}; \sigma] \) admits a \(*\)-coherent representation, \( \rho(B) \) is a \(*\)-algebra. The norm closure of \( \rho(B) \) is a \( C^* \)-algebra denoted by \( \mathcal{A}_V \). We refer to \( \mathcal{A}_V \) as the Serre \( C^* \)-algebra of the complex projective variety \( V \).

3. Proof

3.1. Part I. Let us prove item (i) of theorem 1.4. Denote by \( \mathcal{O} \) the ring of integers of the quaternion algebra \( \left( \frac{a, b}{\mathcal{Q}} \right) \). Consider a polynomial ring:

\[
\mathfrak{R} = \mathbb{Z}[x, y, z]/[\mathcal{Q}],
\] (3.1)

where \( [\mathcal{Q}] \) is an ideal generated by the quadratic form \( Q(x, y, z) \) given by formula (2.3). The proof of item (i) is based on the following lemma.

Lemma 3.1. The matrix rings \( M_2(\mathcal{O}) \) and \( M_2(\mathfrak{R}) \) are isomorphic.

Proof. Roughly speaking, lemma 3.1 follows from the classification of the quaternion algebras given by Theorem 2.3. Namely, the quaternion algebras are classified by the conic surfaces defined by (2.4) or, equivalently, by their coordinate rings \( \mathfrak{R} \). The same is true for the corresponding matrix rings. Let us pass do a detailed argument.

\[
\begin{array}{ccc}
\left( \frac{a, b}{\mathcal{Q}} \right) & \text{isomorphism} & \left( \frac{a', b'}{\mathcal{Q}} \right) \\
F & & F \\
\mathcal{Q}(x, y, z) & \text{similarity} & \mathcal{Q}'(x, y, z)
\end{array}
\] (i) One can recast theorem 2.3 as a commutative diagram in Figure 1. We wish to upgrade the map \( F \) to a ring isomorphism. The simplest non-commutative ring
attached to the ternary quadratic form $Q(x, y, z)$ is the matrix ring $M_2(\mathcal{R})$ over the ring $\mathcal{R}$ defined by (3.1). On the other hand, the quaternion algebra $\left(\frac{a}{\mathcal{R}}, \frac{b}{\mathcal{R}}\right)$ is simple and, therefore, cannot be isomorphic to $M_2(\mathcal{R})$. However, the ring of integers $\mathcal{O}$ of the algebra $\left(\frac{a}{\mathcal{R}}, \frac{b}{\mathcal{R}}\right)$ admits non-trivial two-sided ideals. Here again, the ring $\mathcal{O}$ cannot be isomorphic to $M_2(\mathcal{R})$, since $\mathcal{O}$ is a domain while $M_2(\mathcal{R})$ admits the zero divisors, e.g. the projections. Thus we must consider the matrix ring $M_2(\mathcal{O})$ as a candidate for the required ring isomorphism. Let us show that $M_2(\mathcal{O}) \cong M_2(\mathcal{R})$ whenever $Q(x, y, z) = F\left(\frac{a}{\mathcal{R}}, \frac{b}{\mathcal{R}}\right)$.

(ii) Indeed, it follows from [Voight 2021] [9, Corollary 5.5.2] that the similar quadratic forms $Q(x, y, z)$ and $Q'(x, y, z)$ correspond to the isomorphic conic surfaces (2.4) and, therefore, to the isomorphic rings $\mathcal{R}$ and $\mathcal{R}'$. Since $\mathcal{O} \subset \left(\frac{a}{\mathcal{R}}, \frac{b}{\mathcal{R}}\right)$, we conclude that an isomorphism between $\left(\frac{a}{\mathcal{R}}, \frac{b}{\mathcal{R}}\right)$ and $\left(\frac{a'}{\mathcal{R}}, \frac{b'}{\mathcal{R}}\right)$ induces an isomorphism between $\mathcal{O}$ and $\mathcal{O}'$. In other words, the diagram in Figure 2 must be commutative.

(iii) The tensor product with the matrix ring $M_2$ in Figure 2 gives us $\mathcal{O} \otimes M_2 \cong M_2(\mathcal{O})$ and $\mathcal{R} \otimes M_2 \cong M_2(\mathcal{R})$. The functor $F$ extends to the tensor product and one gets a commutative diagram in Figure 3.

(iv) It remains to show that the map $F$ defines a ring isomorphism $M_2(\mathcal{O}) \cong M_2(\mathcal{R})$. Indeed, let $x_0 \in M_2(\mathcal{O})$. The left multiplication $y \mapsto x_0 y$ (addition $y \mapsto x_0 + y$, resp.) defines a morphism $\phi_{x_0}^{\text{mult}}$ ($\phi_{x_0}^{\text{add}}$, resp.) of the ring $M_2(\mathcal{O})$. Since the map $F$ preserves morphisms, we conclude that $F(\phi_{x_0}^{\text{mult}})$ ($F(\phi_{x_0}^{\text{add}})$, resp.) is a morphism of the ring $M_2(\mathcal{R})$. Namely, the morphism $F(\phi_{x_0}^{\text{mult}})$ ($F(\phi_{x_0}^{\text{add}})$, resp.) acts by the formula $F(y) \mapsto F(x_0)F(y)$ ($F(\phi_{x_0}^{\text{add}})$, resp.) Thus one gets $F(x_0 y) = F(x_0)F(y)$ and $F(x_0 + y) = F(x_0) + F(y)$ for all $x_0, y \in M_2(\mathcal{O})$. In other words, the map $F$ defines an an isomorphism between the rings $M_2(\mathcal{O})$ and $M_2(\mathcal{R})$.

Lemma 3.1 is proved.

Lemma 3.2. Conic surface (2.4) is an avatar of the quaternion algebra $\left(\frac{a}{\mathcal{R}}, \frac{b}{\mathcal{R}}\right)$. 
Proof. (i) According to definition 1.1, we must consider a self-adjoint representation \( \rho \) of the rings \( M_2(\mathcal{R}) \cong M_2(\mathcal{O}) \) by the bounded linear operators on a Hilbert space \( \mathcal{H} \). We take the norm-closure of \( \rho \) in the strong operator topology. Lemma 3.1 implies the following isomorphism of the \( C^* \)-algebras:

\[
\rho(M_2(\mathcal{R})) \otimes \mathcal{K} \cong \rho(M_2(\mathcal{O})) \otimes \mathcal{K}.
\]  

(ii) On the other hand, from the definition of the Serre \( C^* \)-algebra \( \mathcal{A}_V \) one gets the following isomorphisms:

\[
\begin{cases}
\rho(M_2(\mathcal{R})) \cong \mathcal{A}_V(\mathbb{Q}) \\
\rho(M_2(\mathcal{R})) \otimes \mathcal{K} \cong \mathcal{A}_V(\mathbb{C}),
\end{cases}
\]  

where \( V(\mathbb{Q}) \) (\( V(\mathbb{C}) \), resp.) is the conic surface (2.4) over the field of rational numbers \( \mathbb{Q} \) (complex numbers \( \mathbb{C} \), resp.) Thus the LHS of (3.2) is the Serre \( C^* \)-algebra of the conic surface (2.4).

(iii) It is immediate that the \( \rho(M_2(\mathcal{O})) \) at the RHS of (3.2) is the \( C^* \)-algebra \( \mathcal{A}_R \) of the ring \( R \cong \left( \frac{a,b}{\mathbb{Q}} \right) \).

(iv) Using (ii) and (iii), we can write (3.2) in the form:

\[
\mathcal{A}_V(\mathbb{C}) \cong \mathcal{A}_R \otimes \mathcal{K}.
\]  

(v) It remains to compare (3.4) and the definition 1.1, where the connecting map in (1.2) is an isomorphism between the \( C^* \)-algebras. We conclude that the conic surface (2.4) is an avatar of the quaternion algebra \( \left( \frac{a,b}{\mathbb{Q}} \right) \).

Lemma 3.2 is proved. \( \square \)

Lemma 3.3. The (2.4) is a rational surface over the field \( k \cong \mathbb{Q}(\sqrt{-1}, \sqrt{-a}, \sqrt{-b}) \). In particular, the complex points of (2.4) define a simply connected 4-dimensional manifold.
Proof. (i) Let us show that the conic (2.4) is a rational surface over the field \( \mathbb{Q}(\sqrt{-1}, \sqrt{-a}, \sqrt{-b}) \). Indeed, the reader can verify that a parametrization \((u, v) \mapsto (x, y, z)\) of (2.4) is given by the formulas:

\[
\begin{align*}
    x &= \frac{u^2 - v^2}{\sqrt{a}} \\
    y &= \frac{2uv}{\sqrt{b}} \\
    z &= \sqrt{-1}(u^2 + v^2)
\end{align*}
\]  

(3.5)

We conclude that the conic (2.4) is a rational surface \( P^2(k) \) defined over the field \( k \cong \mathbb{Q}(\sqrt{-1}, \sqrt{-a}, \sqrt{-b}) \).

(ii) It is well known that the rational complex projective variety is simply connected as a manifold. By item (i) surface (2.4) is rational and therefore the underlying 4-dimensional manifold is simply connected.

Lemma 3.3 is proved.

Remark 3.4. Notice that \( k \neq \mathbb{Q} \). For otherwise the ternary quadratic form (2.3) admits (infinitely many) non-trivial zeroes and \( \left( \begin{smallmatrix} a & b \\ b & c \end{smallmatrix} \right) \) is no longer a division ring, see Section 2.1.

Corollary 3.5. \( P^2(\mathbb{C}) \) is an avatar of the division ring \( \mathbb{H}(\mathbb{Q}) \).

Proof. Our proof is based on the result [Piergallini 1995] [7] which says that for each smooth 4-dimensional manifold \( M^4 \) there exists a transverse immersion \( X \rightarrow S^4 \) of a 2-dimensional surface \( X \) into the 4-dimensional sphere \( S^4 \), such that \( M^4 \) is the 4-fold PL cover of \( S^4 \) branched at the points of \( X \). We pass to a detailed argument.

(i) Recall that if \( J \) is the complex conjugation, then \( P^2(\mathbb{C})/J \cong S^4 \). Using Piergallini’s Theorem, we conclude that \( J \) acts on the conic surface (2.4) so that it becomes a branched cover of \( P^2(\mathbb{C}) \). In view of lemma 3.3, the conic surface is rational over the field \( k \cong \mathbb{Q}(\sqrt{-1}, \sqrt{-a}, \sqrt{-b}) \). Thus there exists a regular map:

\[
P^2(k) \rightarrow \mathbb{P}^2(k_0),
\]  

(3.6)

where \( k_0 \subset k \) is the minimal non-trivial subfield of \( k \).

(ii) But the minimal non-trivial subfield of \( k \cong \mathbb{Q}(\sqrt{-1}, \sqrt{-a}, \sqrt{-b}) \) independent of the parameters \( a \) and \( b \) coincides with the field \( k_0 \cong \mathbb{Q}(\sqrt{-1}) \). Clearly, the \( k_0 \) corresponds to the case \( a = b = -1 \). All other possible combinations \( a = \pm 1, \ b = \pm 1 \) must be excluded since the ternary quadratic form (2.3) must be positive-definite.

(iii) It remains to notice that the quaternion algebra with \( a = b = -1 \) corresponds to the rational quaternions \( \mathbb{H}(\mathbb{Q}) \).

Corollary 3.5 is proved.

Item (i) of theorem 1.4 follows from corollary 3.5.
3.2. **Part II.** Let us prove item (ii) of theorem 1.4. We proceed with construction of an algebraic surface \( S(\overline{Q}) \) from the quaternion algebra \( \left( \frac{a,b}{Q} \right) \) with \( K \subset \overline{Q} \). From Part I if \( K \cong Q \), then \( S(\overline{Q}) \) is given by the equation \( Q(x,y,z) = 0 \), where
\[
Q(x,y,z) = -ax^2 - by^2 + abz^2. \tag{3.7}
\]

(i) Let \( K \subset \overline{Q} \) be a number field and let \( a, b \in K \). Denote by \( p \in \mathbb{Z}[u] \) and \( q \in \mathbb{Z}[w] \) the minimal polynomials of \( a \) and \( b \), respectively. We set \( a = h, \ b = w \) and we write (3.7) in the form:
\[
F(x,y,z,u,w) = -ux^2 - wy^2 + uwz^2 \in \mathbb{Z}[x,y,z,u,w]. \tag{3.8}
\]

(ii) Solving the equation \( F(x,y,z,u,w) = 0 \), one gets:
\[
u = \frac{wy^2}{wz^2 - x^2}, \quad w = \frac{ux^2}{uz^2 - y^2}. \tag{3.9}
\]

(iii) The required algebraic surface \( S(\overline{Q}) \) is defined as an intersection of two hyper-surfaces given the equations:
\[
\begin{align*}
   p \left( \frac{wy^2}{wz^2 - x^2} \right) &= 0, \\
   q \left( \frac{ux^2}{uz^2 - y^2} \right) &= 0. \tag{3.10}
\end{align*}
\]

(iv) The reader can verify that the surface (3.10) is defined over the field \( \overline{Q} \) and coincides with the conic surface (3.7) when \( K \cong Q \).

Item (ii) of theorem 1.4 is proved.

3.3. **Part III.** Let us prove item (iii) of theorem 1.4. For the sake of clarity, we consider the case \( K \cong Q \) first, and then the general case \( K \subset \overline{Q} \).

3.3.1. **Case \( K \cong Q \).** (i) Lemma 3.2 says that conic surface (2.4) is an avatar of the quaternion algebra \( \left( \frac{a,b}{Q} \right) \). On the other hand, it is known that (2.4) is a rational surface \( P^2(k) \) over the number field \( k \cong Q(\sqrt{-1}, \sqrt{-a}, \sqrt{-b}) \), see lemma 3.3.

(ii) Let \( k_1 \cong Q(\sqrt{-1}), k_2 \cong Q(\sqrt{-a}) \) and \( k_3 \cong Q(\sqrt{-b}) \). Consider a regular map \( P^2(k_i) \to P^2(Q) \) between the rational surfaces \( P^2(k_i) \) and \( P^2(Q) \) shown at the lower level in Figure 4.

(iii) Using [Piergallini 1995] [7] (see proof of corollary 3.5), we conclude that each \( P^2(k_i) \to P^2(Q) \) is a covering map ramified over an embedded 2-dimensional surface \( X \). To determine the genus of \( X \), recall that the group of deck transformations of the covering \( P^2(k_i) \to P^2(Q) \) is isomorphic to the Galois group \( Gal (k_i/Q) \cong \mathbb{Z}/2\mathbb{Z} \) of the field \( k_i \). In particular, the ramification set \( X \) is fixed by the deck transformations and, therefore, corresponds to the field \( Q \) fixed by the group \( Gal (k_i/Q) \). But the avatar of \( Q \) is a projective line \( P^1(C) \cong X \), see item (i) of theorem 1.2. We conclude that the surface \( X \) has genus zero.
(iv) Since \( k \cong Q(\sqrt{-1}, \sqrt{-a}, \sqrt{-b}) \), one gets for each \( i = 1, 2, 3 \) a regular map \( P^2(k) \rightarrow P^2(k_i) \) as shown at the upper level of diagram in Figure 4. Composing these maps with the maps \( P^2(k_i) \rightarrow P^2(Q) \), one concludes the algebraic surface \( P^2(k) \) is a covering of \( P^2(C) \) ramified over three knotted two-dimensional spheres \( P^1(C) \cup P^1(C) \cup P^1(C) \) embedded in \( P^2(C) \).

Remark 3.6. The knotting type of \( P^1(C) \cup P^1(C) \cup P^1(C) \) depends on the arithmetic of the fields \( k_i \) and extends the Grothendieck’s theory of \textit{dessin d’enfant} to the case of algebraic surfaces.

3.3.2. Case \( K \subset Q \). (i) Let \( S(Q) \) is an avatar of the quaternion algebra \( \left( \frac{a,b}{K} \right) \) constructed in Part II. It follows from formulas (3.8)-(3.10) that there exists a regular map:

\[
f : S(Q) \rightarrow P^2(k).
\] (3.11)

(ii) Let \( f^{-1}(P^1(C) \cup P^1(C) \cup P^1(C)) \) be the pre-image of the three knotted two-dimensional spheres embedded in \( P^2(C) \) under the map (3.11). It is not hard to see, that such a pre-image consists again of three spheres \( P^1(C) \cup P^1(C) \cup P^1(C) \) but knotted differently when compared to the case of the surface \( P^2(k) \).

(iii) Since our surface is an avatar of the quaternion algebra \( \left( \frac{a,b}{K} \right) \), it is defined over \( Q \), see item (ii) of theorem 1.4. This argument finishes the proof of item (iii) of theorem 1.4.

Theorem 1.4 is proved.

4. Belyi’s theorem for algebraic surfaces

The aim of this section is an analog of Belyi’s theorem for algebraic surfaces. Our approach is geometric and follows from theorem 1.4. We refer the reader to [González-Diez 2008] [3] for an analytic treatment to this problem.

Theorem 4.1. A non-singular algebraic surface is defined over a number field \( K \) if and only if it is a covering of the complex projective plane \( P^2(C) \) ramified at three knotted two-dimensional spheres \( P^1(C) \cup P^1(C) \cup P^1(C) \).
Proof. Our proof is based on item (iii) of theorem 1.4 and the following lemma.

**Lemma 4.2.** For each algebraic surface $S(\mathbb{Q})$ there exists a quaternion algebra $\left(\frac{a,b}{K}\right)$ such that the avatar of $\left(\frac{a,b}{K}\right)$ is isomorphic to $S(\mathbb{Q})$.

**Proof.** (i) In view of [Piergallini 1995] [7] (see proof of corollary 3.5), the algebraic surface $S(\mathbb{Q})$ is a covering of the projective plane $P^2(\mathbb{C})$ ramified over a knotted 2-dimensional surface $X \subset P^2(\mathbb{C})$. In particular, there exists a regular map $\phi : S(\mathbb{Q}) \to P^2(\mathbb{C})$.

(ii) Recall that the $P^2(\mathbb{C})$ is an avatar of the division ring $\left(\frac{-1,-1}{\mathbb{Q}}\right)$, see item (i) of theorem 1.4. Moreover, the field inclusion $\mathbb{Q} \subset \overline{\mathbb{Q}}$ gives rise to a regular map $\psi_0 : S_0(\overline{\mathbb{Q}}) \to P^2(\mathbb{C})$, where $S_0(\overline{\mathbb{Q}})$ is an avatar of the quaternion algebra $\left(\frac{-1,-1}{\overline{\mathbb{Q}}}\right)$.

(iii) One gets a regular map $\psi : S(\mathbb{Q}) \to S_0(\overline{\mathbb{Q}})$ by closing arrows of the commutative diagram in Figure 5. Notice that $\psi$ is a finite covering of the surface $S_0(\overline{\mathbb{Q}})$, since $\phi$ and $\psi_0$ are mappings of finite degree.

(iv) On the other hand, it follows from equations (3.8)-(3.10) that each finite covering $S(\mathbb{Q})$ of the avatar $S_0(\overline{\mathbb{Q}})$ of the algebra $\left(\frac{-1,-1}{\mathbb{Q}}\right)$ must be avatar of an algebra $\left(\frac{a,b}{K}\right)$ for some $a,b \in K$. Thus there exists a quaternion algebra $\left(\frac{a,b}{K}\right)$, such that algebraic surface $S(\mathbb{Q})$ is the avatar of $\left(\frac{a,b}{K}\right)$.

Lemma 4.2 is proved. $\square$

Returning to the proof of theorem 4.1, one combines lemma 4.2 with the conclusion of item (iii) of theorem 1.4. This argument finishes the proof of theorem 4.1. $\square$
References

1. G. V. Belyi, *Galois extensions of a maximal cyclotomic field*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), 267-276.
2. D. Eisenbud and J. Harris, *The Geometry of Schemes*, GTM **197**, Springer, 1999.
3. G. González-Diez, *Belyi’s theorem for complex surfaces*, Amer. J. Math. **130** (2008), 59-74.
4. Yu. I. Manin, *The notion of dimension in geometry and algebra*, Bull. Amer. Math. Soc. **43** (2006), 139-161.
5. I. V. Nikolaev, *Noncommutative Geometry*, De Gruyter Studies in Math. **66**, Second Edition, Berlin, 2022.
6. I. V. Nikolaev, *Geometry of integers revisited*, J. Fixed Point Theory Appl. (to appear)
7. R. Piergallini, *Four-manifolds as 4-fold branched covers of $S^4$*, Topology **34** (1995), 497-508.
8. J. T. Stafford and M. van den Bergh, *Noncommutative curves and noncommutative surfaces*, Bull. Amer. Math. Soc. **38** (2001), 171-216.
9. J. Voight, *Quaternion Algebras*, GTM **288**, Springer, 2021.

¹ Department of Mathematics and Computer Science, St. John’s University, 8000 Utopia Parkway, New York, NY 11439, United States.

Email address: igor.v.nikolaev@gmail.com