Homological finiteness properties of wreath products

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Abstract

We study the homological finiteness properties $FP_m$ of wreath products $\Gamma = H \wr_X G$. We show that when $H$ has infinite abelianization $\Gamma$ has type $FP_m$ if and only if both $H$ and $W$ have type $FP_m$, all stabilizers of the diagonal action of $G$ on $X^i$ are of type $FP_{m-i}$, and $G \setminus X^i$ is finite for all $1 \leq i \leq m$. If $\chi$ is a non-trivial discrete character of $\Gamma$ such that $\chi(H) = 0$ we establish a criterion when $[\chi] \in \Sigma^m(\Gamma, \mathbb{Z})$. If furthermore $H$ is torsion-free we find a criterion for $\Gamma$ to be Bredon-$FP_m$ with respect to the class of finite subgroups of $\Gamma$.

1 Introduction

In this paper we consider the homological finiteness properties $FP_m$ of a group $G$. By definition $G$ is of type $FP_m$ if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ has a projective resolution with all projectives finitely generated in dimensions up to $m$. If $G$ is

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finitely presented and of type $F \mathcal{P}_m$ for some $m \geq 2$ then $G$ is of homotopical type $F_m$ i.e. there is a $K(G, 1)$ with finite $m$-skeleton.

In general it is hard to determine the homological type $F \mathcal{P}_m$ of a group $G$. Even in the case of metabelian groups $G$ there is no complete classification of the ones of type $F \mathcal{P}_m$ for $m > 2$ though there is an open conjecture, the $F \mathcal{P}_m$-Conjecture, that relates the homological type $F \mathcal{P}_m$ with the Bieri-Renz-Strebel invariant $\Sigma^1(G)$ [4]. Both properties $FP_2$ and $F_2$ (i.e. finite presentability) coincide for a metabelian group $G$ and the $FP_2$-Conjecture holds [7]. For a general group $G$ the properties $FP_2$ and $F_2$ do not coincide [2]. Though there are some sufficient conditions, see [11], there is no complete classification of finite presentability even in the class of nilpotent-by-abelian groups.

In [13] the classification of finitely presented wreath products $\Gamma = H \wr_X G$ was established. It was shown that $\Gamma$ is finitely presented if and only if both $H$ and $G$ are finitely presented, $G$ acts on $X$ and on $X^2$ with finitely many orbits and the stabilizers of $G$ on $X^2$ are finitely generated. We generalize this result by showing in Lemma [32] a criterion when $G$ is of type $FP_2$. If $H$ has infinite abelianization we prove a criterion for $\Gamma$ to be of type $FP_m$ for $m \geq 3$. The sufficiency of the conditions of the criterion do not require that $H$ has infinite abelianization, see Proposition [24] but our proof of the necessity of the conditions uses significantly the fact that $H$ has infinite abelianization.

Our main results are the following theorems. The second is a homotopy version of the first one. The proof of Theorem A is homological and Theorem B is an easy corollary of Theorem A and the fact that for $m \geq 2$ a group is of type $F_m$ if and only if it is of type $FP_m$ and is finitely presented.

**Theorem A** Let $\Gamma = H \wr_X G$ be a wreath product, where $X \neq \emptyset$ and $H$ has infinite abelianization. Then the following are equivalent:

1. $\Gamma$ is of type $FP_m$;
2. $H$ is of type $FP_m$, $G$ is of type $FP_m$, $G$ acts on $X^i$ with finitely many orbits and stabilizers of type $FP_{m-1}$ for all $1 \leq i \leq m$.

**Theorem B** Let $\Gamma = H \wr_X G$ be a wreath product, where $X \neq \emptyset$ and $H$ has infinite abelianization. Then the following are equivalent:

1. $\Gamma$ is of type $F_m$;
2. $H$ is of type $F_m$, $G$ is of type $F_m$, $G$ acts on $X^i$ with finitely many orbits and stabilizers of type $F_{m-1}$ for all $1 \leq i \leq m$.

As a corollary we obtained some results about the Bredon-$FP_m$ type with respect to the class of finite subgroups, denoted $FP_m$. This requires that $H$ is torsion-free in order to control the conjugacy classes of finite subgroups in $\Gamma = H \wr_X G$. In general a group is of type $FP_m$ if it has finitely many conjugacy classes of finite subgroups and the centralizer of every finitely generated subgroup is of type $FP_m$ [14]. The homotopical counterpart of the property $FP_\infty$ was studied earlier in [15].
Theorem C Let $\Gamma = H \wr X G$ be a wreath product, where $X \neq \emptyset$ and $H$ be torsion-free, with infinite abelianization. Then $\Gamma$ has type $FP_m$ if and only if the following conditions hold:
1. $G$ has type $FP_m$;
2. $H$ has type $FP_m$;
3. for every finite subgroup $K$ of $G$ and every $1 \leq i \leq m$ the centralizer $C_G(K)$ acts on $(K \setminus X)^i$ with finitely many orbits and stabilizers of type $FP_{m-i}$.

Finally a monoid version of Theorem A is shown in Theorem 53 for the monoid $\Gamma^{\chi} = \{ g \in \Gamma | \chi(g) \geq 0 \}$, where $\chi : \Gamma \to \mathbb{R}$ is a discrete non-trivial character of $\Gamma$ such that $\chi(H) = 0$. This describes the discrete points of the Bieri-Renz-Strebel invariant $\Sigma^m(\Gamma, \mathbb{Z})$ in terms of the invariant $\Sigma^m(G, \mathbb{Z})$ and the action of $G^{\chi}$ on $X^i$ for $i \leq m$.

For a finitely generated group $G$ the Bieri-Renz-Strebel invariants were first defined for metabelian groups for $m = 1$ in [7] and later on were generalized for every $m$ in [5]. In general the homological invariant $\Sigma^m(G, \mathbb{Z})$ is an open subset of the unit sphere $S(G)$ and $\Sigma^m(G, \mathbb{Z})$ determines which subgroups of $H$ above the commutator are of homological type $FP_m$ [5]. The homological and homotopical $\Sigma^m$-invariants of a group are quite difficult to calculate but they are known for right angle Artin groups [17], the R. Thompson group $F$ [6], metabelian groups of finite Prufer rank [10].

2 Preliminaries on the homological type $FP_m$

We recall that an $R$-module $M$ is of type $FP_m$ if $M$ has a projective resolution with all projectives finitely generated up to dimension $m$. If not otherwise stated $\otimes$ is the tensor product over $\mathbb{Z}$ and the modules and groups actions considered are left ones.

Lemma 1. [3, Prop. 1.4] Let $0 \to M_1 \to M \to M_2 \to 0$ be a short exact sequence of $R$-modules.

a) if $M$ and $M_2$ are of type $FP_s$ and $s \geq 1$ then $M_1$ is of type $FP_{s-1}$;
b) if $M_1$ and $M_2$ are of type $FP_s$ then $M$ is of type $FP_s$;
c) if $M_1$ is of type $FP_{s-1}$ and $M$ is of type $FP_s$ then $M_2$ is of type $FP_s$.

The following results follow easily from Lemma 1. For completeness we include proofs.

Lemma 2. Let

$$\cdots \to Q_i \xrightarrow{d_i} Q_{i-1} \to \cdots \to Q_0 \xrightarrow{d_0} V \to 0$$

be an exact complex of $R$-modules with $Q_i$ of type $FP_{m-i}$ for all $i \leq m$. Then $V$ has type $FP_m$.

Proof. Apply Lemma 1 (c) for the short exact sequences

$$0 \to Im(d_{i+1}) = Ker(d_i) \to Q_i \to Im(d_i) \to 0$$

for $0 \leq i \leq m-1$ to prove by inverse induction on $i$ that $Im(d_i)$ is $FP_{m-i}$.
Lemma 3. Let $0 \leq k \leq m$ be integers and let
\[
\cdots \to Q_i \overset{d_i}{\to} Q_{i-1} \to \cdots \to Q_0 \overset{d_0}{\to} V \to 0
\]
be an exact complex of $R$-modules with $Q_i$ of type $FP_{m-i}$ for all $i < k$. Suppose further that $V$ has type $FP_m$. Then $Im(d_k)$ has type $FP_{m-k}$.

Proof. Consider the exact complex
\[
0 \to Im(d_k) \to Q_{k-1} \overset{d_{k-1}}{\to} Q_{k-2} \to \cdots \to Q_0 \overset{d_0}{\to} V \to 0
\]
and apply Lemma 1(a) for the short exact sequences associated to the above complex. \qed

Lemma 4. Let
\[
\cdots \to Q_i \overset{d_i}{\to} Q_{i-1} \to \cdots \to Q_0 \overset{d_0}{\to} V \to 0
\]
be a complex of $R$-modules with $Q_i$ of type $FP_{m-i}$ for all $0 \leq i \leq m$ and $H_i(Q)$ of type $FP_{m-i-1}$ for $0 \leq i \leq m-1$. Then $V$ has type $FP_m$.

Proof. Apply Lemma 1 for the short exact sequences
\[
0 \to Ker(d_i) \to Q_i \to Im(d_i) \to 0
\]
and
\[
0 \to Im(d_{i+1}) \to Ker(d_i) \to H_i(Q) \to 0
\]
to prove by inverse induction on $i$ that $Im(d_i)$ is $FP_{m-i}$ for $0 \leq i \leq m$. \qed

The following lemma appeared as a footnote in [1] and was later on explained with more details in [8, Prop. 4.1].

Lemma 5. A retract of a group of type $FP_m$ is a group of type $FP_m$ i.e. if the split extension $\Gamma = M \rtimes G$ is of type $FP_m$ then $G$ is of type $FP_m$.

Let $\Gamma = M \rtimes G$ be a group. Then $ZM$ is a left $\Gamma$-module, where $G$ acts via conjugation on $M$ and $M$ acts on $ZM$ via left multiplication. Note that the augmentation ideal $Aug(ZM)$ is a $\Gamma$-submodule of $ZM$.

Lemma 6. Let $\Gamma = M \rtimes G$ be a group. Then $\Gamma$ is $FP_m$ if and only if $G$ is $FP_m$ and $Aug(ZM)$ is $FP_{m-1}$ as $\Gamma$-module.

Proof. By Lemma [1] applied for the short exact sequence $0 \to Aug(\Gamma G) \to \Gamma G \to Z \to 0$ the group $\Gamma$ is of type $FP_m$ if and only if $Aug(\Gamma G)$ is $FP_{m-1}$ as $\Gamma$-module. Consider the short exact sequence of $\Gamma$-modules
\[
0 \to Z \Gamma Aug(ZG) \overset{\alpha}{\to} Aug(\Gamma G) \to Aug(ZM) \to 0,
\]
where $\alpha$ is the inclusion map.

If $\Gamma$ is of type $FP_m$ by Lemma [5] $G$ is of type $FP_m$, so $Aug(ZG)$ is $FP_{m-1}$ as $ZG$-module and the induced $\Gamma$-module $Z \Gamma Aug(ZG)$ is $FP_{m-1}$. Then by Lemma 1 c) applied for (7) $Aug(ZM)$ is $FP_{m-1}$ as $\Gamma$-module.

If $G$ is of type $FP_m$ and $Aug(ZM)$ is $FP_{m-1}$ as $\Gamma$-module then by Lemma [1] b) applied for (7) $Aug(\Gamma G)$ is $FP_{m-1}$ as $\Gamma$-module. \qed
Lemma 8. Let $S$ be a subring of $R$ such that $R$ is flat as right $S$-module and $M$ be a left $S$-module. Futhermore assume that the inclusion map $S \rightarrow R$ of right $S$-modules splits. Then $M$ is of type $FP_m$ as $S$-module if and only if $R \otimes_S M$ is $FP_m$ as $R$-module.

Proof. 1. Suppose that $M$ is of type $FP_m$ as $S$-module. Let $\mathcal{F}$ be a projective resolution of $M$ with projectives finitely generated in dimensions $\leq m$. Since $R \otimes_S -$ is an exact functor $R \otimes_S \mathcal{F}$ is a projective resolution of the $R$-module $R \otimes_S M$ with projectives finitely generated in dimensions $\leq m$, so $R \otimes_S M$ is $FP_m$ as $R$-module.

2. Suppose that $R \otimes_S M$ is $FP_m$ as $R$-module. We will prove by induction on $m$ that $M$ is $FP_m$ as $S$-module.

Suppose first that $m = 0$. Let $X = \{a_i = \sum_j r_{i,j} \otimes m_{i,j}\}$ be a finite generating set of $R \otimes_S M$ as $R$-module, where $r_{i,j} \in R, m_{i,j} \in M$. Let $M_0$ be the $S$-submodule of $M$ generated by the finite set $Y = \{m_{i,j}\}_{i,j}$. Since $R$ is flat as $S$-module there is a short exact sequence $0 \rightarrow R \otimes_S M_0 \rightarrow R \otimes_S M \rightarrow R \otimes_S (M/M_0) \rightarrow 0$ and by the choice of $Y$ the map $\alpha$ is surjective, so $\alpha$ is an isomorphism. Then $R \otimes_S (M/M_0) = 0$ has a direct summand $S \otimes_S (M/M_0) = M/M_0$, so $M_0 = M$.

Suppose now that $M$ is $FP_{m-1}$ as $S$-module. Let

\[ \mathcal{F} : \ldots \rightarrow F_i \xrightarrow{d_i} F_{i-1} \rightarrow \ldots \rightarrow F_0 \rightarrow M \rightarrow 0 \]

be a projective resolution of $S$-modules with $F_i$ finitely generated for $i \leq m - 1$. Consider the exact complex induced by $\mathcal{F}$

\[ R : 0 \rightarrow A = \ker(d_{m-1}) \rightarrow F_{m-1} \xrightarrow{d_{m-1}} F_{m-2} \rightarrow \ldots \rightarrow F_0 \rightarrow M \rightarrow 0. \]

Since $R$ is flat as right $S$-module

\[ R \otimes_S R : 0 \rightarrow R \otimes_S A \rightarrow R \otimes_S F_{m-1} \xrightarrow{id_R \otimes d_{m-1}} R \otimes_S F_{m-2} \rightarrow \ldots \rightarrow R \otimes_S F_0 \rightarrow R \otimes_S M \rightarrow 0 \]

is an exact complex and $R \otimes_S F_i$ is a finitely generated projective $R$-module for $i \leq m - 1$. By Lemma 8 since $R \otimes_S M$ is $FP_m$ as $R$-module we have that $R \otimes_S A$ is finitely generated as $R$-module. Then by the case $m = 0$ we deduce that $A$ is finitely generated as $S$-module, so $M$ is $FP_m$ as $S$-module.

Corollary 9. Let $L$ be a subgroup of a group $G$ and $A$ be a (left) $\mathbb{Z}L$-module. Then $ZG \otimes_{\mathbb{Z}L} A$ is $FP_m$ as $ZG$-module if and only if $A$ is $FP_m$ as $\mathbb{Z}L$-module.

3 Exterior and tensor powers of semi-induced modules

Let $G$ be a group and $S$ be a free $\mathbb{Z}$-module with basis $X \neq \emptyset$. If $S$ is a $ZG$-module via an action of $G$ on the set $X$, we call $S$ an induced module. If $S$ is a $ZG$-module via an action of $G$ on the set $X \cup -X$ we say that $S$ is a semi-induced
module, in this case we write $S = Z_t X$ with index $t$ standing for twisted action of $G$.

Let $G$ be a group acting on a set $X$, say $G \times X \to X$ sends $(g, x)$ to $g \ast x$ and for every $G$-orbit $V$ of $G$ in $X$ we fix a homomorphism of groups $\chi_V : G \to \{ \pm 1 \}$. Then we have a semi-induced module $Z_t X$ with

$$gx = \chi_{G \ast x}(g \ast x) \in X \cup -X.$$ (10)

The converse holds too, any semi-induced module is obtained from an induced $ZG$-module $ZX$ in this way. For a semi-induced $ZG$-module defined by $X \cup -X$ we call the $G$-action on $X$, satisfying (10), and denoted by $*$, the associated action of $G$ on $X$.

**Lemma 11.** $Z_t X$ is of type $FP_k$ as $ZG$-module if and only if $ZX$ is of type $FP_h$ as $ZG$-module, where the action of $G$ on $X$ is the associated one.

**Proof.** Note first that $Z_t X$ is $FP_0$ as $ZG$-module (i.e. finitely generated) if and only if $ZX$ is $FP_0$ as $ZG$-module (i.e. finitely generated). And this happens precisely when $G \setminus X$ is finite.

Assume from now on that $G \setminus X$ is finite. For every orbit $G \ast x$ of $G$ on $X$ consider $\text{Ker}(\chi_{G \ast x})$ and note that $[G : \text{Ker}(\chi_{G \ast x})] \leq 2$. Let $G_0$ be the intersection of $\text{Ker}(\chi_{G \ast x})$ over all possible $G$-orbits $G \ast x$ in $X$, so $G_0$ has finite index in $G$. Note that for a $ZG$-module $M$ we have that $M$ is of type $FP_m$ as $ZG$-module if and only if $M$ is of type $FP_m$ as $ZG_0$-module. In particular $Z_t X$ is of type $FP_k$ as $ZG_0$-module if and only if $Z_t X$ is $FP_k$ as $ZG$-module. Similarly $ZX$ is of type $FP_h$ as $ZG_0$-module if and only if $ZX$ is $FP_k$ as $ZG$-module. Finally $Z_t X = ZX$ as $ZG_0$-modules since for $g \in G_0$ and $x \in X$ we have $gx = g \ast x$.

**Proposition 12.** Let $M = Z_t X$ be a semi-induced $ZG$-module. Then the following are equivalent:

1. $\wedge^i M$ is of type $FP_{m-i}$ for all $1 \leq i \leq m$;
2. $\otimes^i M$ is of type $FP_{m-i}$ for all $1 \leq i \leq m$;
3. all stabilizers of the diagonal action of $G$ on $X^i$ (via the associated action of $G$ on $X$) are of type $FP_{m-i}$ and $\wedge^i M$ is finitely generated as $ZG$-module for all $1 \leq i \leq m$;
4. all stabilizers of $i$ element subsets of $X$ (via the associated action of $G$ on $X$) are of type $FP_{m-i}$ and $\wedge^i M$ is finitely generated as $ZG$-module for all $1 \leq i \leq m$;
5. all stabilizers of the diagonal action of $G$ on $X^i$ (via the associated action of $G$ on $X$) are of type $FP_{m-i}$ and $G \setminus X^i$ is finite for all $1 \leq i \leq m$.

**Proof.** Observe that $\wedge^i M = Z_t Y_i$, where $Y_i$ is the set of $i$-element subsets of $X$ and $\otimes^i M = Z_t Z_i$, where $Z_i = X^i$. Then by Lemma 11 $\wedge^i M$ is $FP_k$ as $ZG$-module if and only if $Z Y_i$ is $FP_k$ as $ZG$-module. Note that by Corollary 9 $Z Y_i$ is $FP_{m-i}$ as $ZG$-module for $1 \leq i \leq m$ if and only if item 4 holds (note that for the finite generation part we used again Lemma 11). Thus 1. and 4. are equivalent.
Similarly by Lemma 11 \( \otimes^i M \) is \( FP_k \) as \( ZG \)-module if and only if \( ZZ_i \) is \( FP_k \) as \( ZG \)-module. Note that by Corollary 9 \( ZZ_i \) is \( FP_{m-i} \) as \( ZG \)-module for \( 1 \leq i \leq m \) if and only if item 3 holds (note that for the finite generation part we used again Lemma 11). Thus 2. and 3. are equivalent.

Note that all stabilizers of the diagonal action of \( G \) on \( X^i \) are of type \( FP_{m-i} \) for all \( 1 \leq i \leq m \) if and only if all stabilizers of \( i \) element subsets of \( X \) are of type \( FP_{m-i} \) for all \( 1 \leq i \leq m \). Thus 3. implies 4.

Now we show that 4. implies 3. If \( \otimes^i M \) is finitely generated as \( ZG \)-module for \( i \leq m \) we can finish as in the above paragraph. Note that since \( \Lambda^i M \) is finitely generated as \( ZG \)-module we have that \( G \) acts on the set of \( i \) element subsets of \( X \) with finitely many orbits for all \( i \leq m \). Then \( G \) acts on \( X^i \) with finitely many orbits for all \( 1 \leq i \leq m \), so \( \otimes^i M \) is finitely generated as \( ZG \)-module for all \( 1 \leq i \leq m \).

Finally the equivalence between 3. and 5. is obvious. \( \square \)

**Lemma 13.** Let \( X \) be a (left) \( G \)-set such that \( G \) acts with finitely many orbits on \( X^i \) and with stabilizers of type \( FP_{m-i} \) for all \( 1 \leq i \leq m \).

1. Let \( i_1, i_2, \ldots, i_s \) be non-negative integers such that \( i_1 + \ldots + i_s = r \leq m \) and \( X^{i_1, i_2, \ldots, i_s} \) be the set of \( s \)-tuples \( (X_1, \ldots, X_s) \) of pairwise disjoint subsets \( X_1, \ldots, X_s \) of \( X \) such that \( X_j \) has cardinality \( i_j \) for all \( 1 \leq j \leq s \). Then \( ZX^{i_1, i_2, \ldots, i_s} \) is of type \( FP_{m-r} \) as \( ZG \)-module.

2. Let \( k \leq m \) and

\[ X^{(k)} = \{(x_1, \ldots, x_k) | x_i \in X, x_i \neq x_j \text{ for } i \neq j \} \]

and \( \tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) \) be a \( k \)-tuple of positive integers such that \( \tilde{\alpha}_1 \leq \tilde{\alpha}_2 \leq \ldots \leq \tilde{\alpha}_k \).

Let

\[ V_{\tilde{\alpha}} = \mathbb{Z}(X^{(k)}) / \sim \]

be defined by the equivalence relation \( \sim \) in \( X^{(k)} \) generated by

\[ (x_1, \ldots, x_i, x_{i+1}, \ldots, x_k) \sim (-1)^{\tilde{\alpha}_i - \tilde{\alpha}_{i+1}} (x_1, \ldots, x_{i+1}, x_i, \ldots, x_k) \]

whenever \( \tilde{\alpha}_i = \tilde{\alpha}_{i+1}, 1 \leq i < k-1 \). Then \( V_{\tilde{\alpha}} \) is of type \( FP_{m-k} \).

**Proof.** 1. Let \( B \) be the stabilizer in \( G \) of the \( s \)-tuple \( (X_1, \ldots, X_s) \) i.e. \( B \) is the set of \( g \in G \) such that \( gX_i = X_i \) for all \( 1 \leq i \leq s \). Then since all \( X_i \) are finite \( B \) is a subgroup of finite index in the group \( D \), where \( D \) is the stabilizer in \( G \) of the set \( \cup_{1 \leq i \leq s} X_i \). Note that \( | \cup_{1 \leq i \leq s} X_i | = i_1 + \ldots + i_s = r \). Then by Proposition 12 i.e. item 5 implies item 4. \( D \) is of type \( FP_{m-r} \), hence \( B \) is of type \( FP_{m-r} \). Note that \( G \setminus X^r \) finite implies that \( G \setminus X^{i_1, i_2, \ldots, i_s} \) is finite. Then by Lemma 9 \( \mathbb{Z}X^{i_1, i_2, \ldots, i_s} \) is of type \( FP_{m-r} \) as \( ZG \)-module.

2. Let \( i_1 + \ldots + i_s = k, \tilde{\alpha}_1 = \ldots = \tilde{\alpha}_s < \tilde{\alpha}_{i_1+1} = \ldots = \tilde{\alpha}_{i_1+i_2} < \ldots < \tilde{\alpha}_{i_1+i_2+\ldots+i_{s-1}+1} = \ldots = \tilde{\alpha}_k \). We claim that \( V_{\tilde{\alpha}} \simeq \mathbb{Z}X^{i_1, i_2, \ldots, i_s} \) as \( ZG \)-modules. Indeed for

\[ Y = \{(x_1, \ldots, x_k) | x_i \in X, x_1 < \ldots < x_i, x_{i+1} < \ldots < x_{i_1+i_2}, \ldots, x_{i_1+i_2+\ldots+i_{s-1}+1} < \ldots < x_k \} \]
where we have fixed a linear order \( \leq \) on \( X \), we have \( V_\alpha = \mathbb{Z}_d Y \) is a semi-induced \( \mathbb{Z}G \)-module. By Lemma 14, \( V_\alpha \) is \( FP_{m-k} \) as \( \mathbb{Z}G \)-module if and only if \( ZY \) is \( FP_{m-k} \) as \( \mathbb{Z}G \)-module. Note that \( ZY = ZX^{(i_1,i_2,\ldots,i_s)} \), so by part 1 applied for \( r = k \) we have that \( ZY \) is \( FP_{m-k} \) as \( \mathbb{Z}G \)-module. \( \square \)

4 Type \( FP_m \) for split extensions of induced modules

Let \( \Gamma = M \rtimes G \) be a group, where \( M \) is abelian. Note we do not impose restrictions on \( G \). We consider \( M \) as a left \( \mathbb{Z}G \)-module, where \( G \) acts by conjugation.

**Proposition 14.** Let \( G \) be a group of type \( FP_m \), \( M = \oplus_i ZG \otimes_{ZH_i} \mathbb{Z} \) a \( \mathbb{Z}G \)-module and \( \wedge^i M \) be of type \( FP_{m-i} \) as \( \mathbb{Z}G \)-module via the diagonal action for all \( 1 \leq i \leq m \). Then \( \Gamma = M \rtimes G \) is of type \( FP_m \).

**Proof.** Let \( X \) be the disjoint union \( \cup_i G/H_i \), so \( X \) is a basis of \( M \) as a free \( \mathbb{Z} \)-module. Since \( M \) is a torsion-free abelian group, there is a Koszul complex \( [18] \)

\[
\cdots \to ZM \otimes_{\mathbb{Z}} (\wedge^k M) \to ZM \otimes_{\mathbb{Z}} (\wedge^{k-1} M) \to \cdots \to ZM \otimes_{\mathbb{Z}} M \to ZM \otimes_{\mathbb{Z}} \mathbb{Z} \to 0
\]

(15)

with differential given by

\[
d_k(m_1 \wedge \ldots \wedge m_k) = \sum_{1 \leq i \leq k} (-1)^i \epsilon(m_i) \otimes m_1 \wedge \ldots \wedge m_{i-1} \wedge m_{i+1} \wedge \ldots \wedge m_k,
\]

where \( \epsilon(m_i) = m_i - 1 \in ZM \) and \( m_1, \ldots, m_k \in X \). Thus (15) gives an exact complex

\[
Q : \cdots \to Q_{k-1} = ZM \otimes_{\mathbb{Z}} (\wedge^k M) \to Q_{k-2} = ZM \otimes_{\mathbb{Z}} (\wedge^{k-1} M) \to \cdots
\]

\[
	o Q_0 = ZM \otimes_{\mathbb{Z}} M \to Q_{-1} = \text{Aug}(ZM) \to 0.
\]

(16)

Note that for a left \( \mathbb{Z}G \)-module \( V \) there is an isomorphism of abelian groups

\[
\mathbb{Z}G \otimes_{\mathbb{Z}G} V \simeq ZM \otimes_{\mathbb{Z}} V
\]

and thus the action of \( \Gamma \) on the induced module \( \mathbb{Z}G \otimes_{\mathbb{Z}G} V \) gives an action of \( \Gamma \) on \( ZM \otimes_{\mathbb{Z}} V \). We apply this for \( V = \wedge^h M \) and since \( \wedge^h M \) is of type \( FP_{m-k} \) as \( \mathbb{Z}G \)-module and \( \mathbb{Z}G \otimes_{\mathbb{Z}G} \) is an exact functor

\[
\mathbb{Z}G \otimes_{\mathbb{Z}G} (\wedge^h M) \simeq ZM \otimes_{\mathbb{Z}} (\wedge^h M)
\]

is of type \( FP_{m-k} \) as \( \mathbb{Z}G \)-module. Note that since \( X \) is a \( G \)-invariant set the differentials in (14) are homomorphisms of \( \mathbb{Z}G \)-modules, hence are homomorphisms of \( \mathbb{Z}G \)-modules. Then (14) is an exact complex with \( Q_i \) of homological type \( FP_{m-i-1} \) as \( \mathbb{Z}G \)-module for \( 0 \leq i \leq m - 1 \). Then by Lemma 2 applied for the complex (14) \( \text{Aug}(ZM) \) is of type \( FP_{m-1} \) as \( \mathbb{Z}G \)-module, hence by Lemma 6 \( \Gamma \) is of type \( FP_m \). \( \square \)
Lemma 17. Suppose that $\Gamma = M \rtimes G$ is a group of type $FP_m$, $M = \oplus_i \mathbb{Z}G \otimes_{\mathbb{Z}H_i} \mathbb{Z}$ a $\mathbb{Z}G$-module and $\wedge^i M$ is of type $FP_{m-i-1}$ as $\mathbb{Z}G$-module (via the diagonal action) for all $1 \leq i \leq m-1$. Then $M$ is of type $FP_{m-1}$ as $\mathbb{Z}G$-module.

Proof. By Lemma 6, $\Omega = Aug(ZM)$ is of type $FP_{m-1}$ as $\mathbb{Z}G$-module. Note that $M \simeq \Omega/\Omega^2 \simeq \mathbb{Z} \otimes_{\mathbb{Z}M} Aug(ZM)$. Let

$$\mathcal{F} : \ldots \to F_i \to F_{i-1} \to \ldots \to F_0 \to \Omega \to 0$$

be a free resolution of $\Omega$ as $\mathbb{Z}G$-module with $F_i$ finitely generated for $i \leq m-1$. Let $Q$ be the complex $\mathbb{Z} \otimes_{\mathbb{Z}M} \mathcal{F}$ i.e.

$$Q : \ldots \to Q_i \to Q_{i-1} \to \ldots \to Q_0 \to M \to 0 \quad (18)$$

is a complex of free $\mathbb{Z}G$-modules with $Q_i$ finitely generated for $i \leq m-1$ and its homology groups are

$$H_i(Q) \simeq Tor_i^{ZM}(Z, \Omega) \text{ for } i \geq 1.$$ 

Note that the long exact sequence in $Tor$ applied for the short exact sequence $\Omega \to ZM \to Z$ gives

$$Tor_i^{ZM}(Z, \Omega) \simeq Tor_{i+1}^{ZM}(Z, Z) = H_{i+1}(M, Z) \simeq \wedge^{i+1} M \text{ for } i \geq 1.$$ 

The last isomorphism is [9, Ch. V, Thm. 6.4]. Then $H_i(Q)$ is of type $FP_{m-i-1}$ as $\mathbb{Z}G$-module for all $i \leq m-2$. Then by Lemma 4 applied for the complex $Q$ $M$ is of type $FP_{m-1}$ as $\mathbb{Z}G$-module.

Proposition 19. Let $\Gamma = M \rtimes G$ be a group of type $FP_m$ and $M = \oplus_i \mathbb{Z}G \otimes_{\mathbb{Z}H_i} \mathbb{Z}$ a $\mathbb{Z}G$-module. Then $\wedge^i M$ is of type $FP_{m-i}$ as $\mathbb{Z}G$-module via the diagonal $G$-action for all $1 \leq i \leq m$.

Proof. We induct on $m$ and assume that the proposition holds for smaller values of $m$, in particular $\wedge^i M$ is $FP_{m-i-1}$ as $\mathbb{Z}G$-module. By Lemma 17, $M$ is of type $FP_{m-1}$ as $\mathbb{Z}G$-module, thus the proposition holds for $i = 1$. We proceed by induction on $i$ i.e. assume that we have proved that $\wedge^i M$ is of type $FP_{m-j}$ as $\mathbb{Z}G$-module via the diagonal $G$-action for all $1 \leq j \leq i - 1$ and will show that $\wedge^i M$ is of type $FP_{m-i-1}$ as $\mathbb{Z}G$-module.

Consider the Koszul complex

$$\cdots \to ZM \otimes_{\mathbb{Z}} (\wedge^k M) \xrightarrow{d_k} ZM \otimes_{\mathbb{Z}} (\wedge^{k-1} M) \to \cdots \to ZM \otimes_{\mathbb{Z}} M \xrightarrow{d_1} ZM \to Z \to 0 \quad (20)$$

and its modified version

$$Q : \cdots \to Q_{k-1} = ZM \otimes_{\mathbb{Z}} (\wedge^k M) \xrightarrow{d_k} Q_{k-2} = ZM \otimes_{\mathbb{Z}} (\wedge^{k-1} M) \to \cdots \to Q_0 = ZM \otimes_{\mathbb{Z}} M \xrightarrow{d_1} Q_{-1} = Aug(ZM) \to 0 \quad (21)$$

Since $\wedge^k M$ is $FP_{m-k}$ as $\mathbb{Z}G$-module for $k \leq i - 1$ the induced module $Q_{k-1} = ZM \otimes_{\mathbb{Z}} (\wedge^k M)$ is $FP_{m-k}$ as $\mathbb{Z}G$-module for $k \leq i - 1$. Note that by Lemma 6
$\text{Aug}(\mathbb{Z}M)$ is $FP_{m-1}$ as $\mathbb{Z}\Gamma$-module. Then by Lemma 3 applied for the complex $\alpha$, $\text{Im}(d_i)$ is of type $FP_{m-1-(i-1)}$ as $\mathbb{Z}\Gamma$-module. Note that

$$\text{Im}(d_i) \simeq (\mathbb{Z}M \otimes (\wedge^i \Gamma))/\text{Ker}(d_i) \simeq (\mathbb{Z}M \otimes (\wedge^i \Gamma))/\text{Im}(d_{i+1}).$$

Denote $(\mathbb{Z}M \otimes (\wedge^i \Gamma))/\text{Im}(d_{i+1})$ by $V$ i.e. $V$ has type $FP_{m-i}$ as $\mathbb{Z}\Gamma$-module. Let $W$ be $\mathbb{Z} \otimes_{\mathbb{Z}M} V$. Since $\mathbb{Z} \otimes_{\mathbb{Z}M}$ is right exact there is an exact sequence

$$\mathbb{Z} \otimes_{\mathbb{Z}M} \text{Im}(d_{i+1}) \to \mathbb{Z} \otimes_{\mathbb{Z}M} (\mathbb{Z}M \otimes (\wedge^i \Gamma)) \to W \to 0.$$

By the definition of $d_{i+1}$ we have $\beta = 0$, furthermore the module in the middle is isomorphic to $\wedge^i \mathbb{M}$, so

$$W \simeq \wedge^i \mathbb{M}.$$

If $W$ is of type $FP_{m-i}$ as $\mathbb{ZG}$-module the inductive step is completed.

Let $\mathcal{F}: \ldots \to F_j \to F_{j-1} \to \ldots \to F_0 = V \to 0$ be a free resolution of $V$ as $\mathbb{Z}\Gamma$-module with $F_j$ finitely generated for $j \leq m - i$. Let $\mathcal{R}$ be the complex $\mathbb{Z} \otimes_{\mathbb{Z}M} \mathcal{F}$ i.e.

$$\mathcal{R}: \ldots \to R_j \to R_{j-1} \to \ldots \to R_0 = W \to 0$$

is a complex of free $\mathbb{ZG}$-modules with $R_j$ finitely generated for $j \leq m - i$ and its homology groups are

$$H_j(\mathcal{R}) \simeq \text{Tor}^\mathbb{Z}_{j+1}(\mathbb{Z}, V) \text{ for } j \geq 1.$$

Note that for the $\mathbb{Z}M$-module $V = \text{Im}(d_i)$ the Koszul complex (20) gives a free resolution

$$C: \ldots \to \mathbb{Z}M \otimes_{\mathbb{Z}} (\wedge^k \mathbb{M}) \xrightarrow{d_k} \mathbb{Z}M \otimes_{\mathbb{Z}} (\wedge^{k-1} \mathbb{M}) \to \ldots$$

$$\mathbb{Z} \otimes_{\mathbb{Z}M} \mathcal{C} : \ldots \to \wedge^{i+2} \mathbb{M} \to \wedge^{i+1} \mathbb{M} \to \wedge^i \mathbb{M} \to \mathbb{Z} \otimes_{\mathbb{Z}M} \text{Im}(d_i) = W \to 0$$

and by the definition of the differentials $d_k$ the complex

has all zero differentials except possibly $\alpha$. Thus

$$H_j(\mathcal{R}) \simeq \text{Tor}^\mathbb{Z}_{j+1}(\mathbb{Z}, V) \simeq H_j(\mathbb{Z} \otimes_{\mathbb{Z}M} \mathcal{C}) \simeq \wedge^{j+1} \mathbb{M} \text{ for } j \geq 1.$$  

By Lemma 3 applied for the complex $\mathcal{R}$, for $W$ to be of type $FP_{m-i}$ as $\mathbb{ZG}$-module it is sufficient that

$$H_j(\mathcal{R}) \text{ is of type } FP_{m-i-j-1} \text{ as } \mathbb{ZG}\text{-module for } 1 \leq j \leq m - i - 1.$$  

Thus we need for $k = i + j$ that

$$\wedge^k \mathbb{M} \text{ is of type } FP_{m-k-1} \text{ as } \mathbb{ZG}\text{-module for } k \leq m - 1$$

but this follows from the fact that the proposition holds for $m - 1$. \hfill $\square$

**Theorem 24.** Let $\Gamma = M \rtimes G$ be a group and $M = \bigoplus_i \mathbb{Z}G \otimes_{\mathbb{Z}H_i} \mathbb{Z}$ a $\mathbb{ZG}$-module. Then $\Gamma$ is of type $FP_m$ if and only if $G$ is of type $FP_m$ and $\wedge^i \mathbb{M}$ is of type $FP_{m-i}$ as $\mathbb{ZG}$-module via the diagonal action for all $1 \leq i \leq m$.

**Proof.** Follows directly from Lemma 5, Proposition 4 and Proposition 19. \hfill $\square$
5 Homological type $FP_m$ for wreath products

5.1 Preliminaries on tensor products of complexes

Let $A_i$ be a non-negative complex of free $\mathbb{Z}$-modules i.e. all non-zero modules are in dimension $\geq 0$. Consider the tensor product $A_1 \otimes \ldots \otimes A_s$ with differential given by

$$\partial(a_1 \otimes \ldots \otimes a_s) = \sum_{1 \leq j \leq s} (-1)^{\deg(a_1) + \ldots + \deg(a_j)} a_1 \otimes \ldots \otimes \partial(a_j) \otimes \ldots \otimes a_s. \quad (25)$$

For the transposition $\pi = (i, i+1) \in S_s$ and each $a_i$ an element of one of the free modules in $A_i$ we define

$$\pi(a_1 \otimes \ldots \otimes a_s) = a_1 \otimes \ldots \otimes a_{i+1} \otimes a_i \otimes \ldots \otimes a_s := (-1)^{\deg(a_i) \deg(a_{i+1})}(a_1 \otimes \ldots \otimes a_s). \quad (26)$$

Since the symmetric group $S_s$ is generated by $\{(i, i+1)\}_{1 \leq i \leq s-1}$ we get an action of $S_s$ on the tensor product $A_1 \otimes \ldots \otimes A_s$. As the action of the symmetric group $S_s$ commutes with the differential from (25), this induces an action of $S_s$ on the homology groups of $A_1 \otimes \ldots \otimes A_s$.

5.2 Wreath products: proofs of Theorem A and Theorem B

Proposition 27. Let $\Gamma = H \wr X G$ be a wreath product such that both $H$ and $G$ are of type $FP_m$, $G$ acts (diagonally) on $X^i$ with finitely many orbits and stabilizers of type $FP_{m-i}$ for $1 \leq i \leq m$. Then $\Gamma$ has type $FP_m$.

Proof. Let

$$\mathcal{H} : \ldots \to ZH \otimes ZY_1 \xrightarrow{\partial_1} ZH \otimes ZY_{i-1} \to \ldots \to ZH \otimes ZY_1 \xrightarrow{\partial_1} ZH \to Z \to 0$$

be a free resolution of the trivial $ZH$-module $Z$ with all $Y_i$ finite for $i \leq m$.

Consider $H_x$ an isomorphic copy of the group $H$ and let $H_x$ be the complex obtained from $\mathcal{H}$ by substituting $H$ with $H_x$. We write $\partial_x$ for the differential of $\mathcal{H}_x$ and if we want to stress its degree, say $i$, we write $\partial_{i,x}$. Let $M$ be the normal closure of $H$ in $\Gamma$. Thus $M$ is the subgroup of the direct product $\prod_{x \in X} H_x$ that contains the elements with all but finitely many trivial coordinates.

We can take the tensor product over $Z$ of the deleted complexes $\mathcal{H}_x^{del}$ for $x \in X$ and obtain the deleted complex $\mathcal{F}^{del}$. This makes sense for infinite $X$ as the direct limit of the tensor products of any finite number of the complexes $\mathcal{H}_x$. Note that we have fixed a linear order $\leq$ on $X$ and the tensor product $\mathcal{H}_x^{del} \otimes \ldots \otimes \mathcal{H}_x^{del}$ is made for $x_1 < \ldots < x_t$ and $t \geq 1$. By Künneth formula we get a free resolution of the trivial $ZM$-module $Z$.

$$\mathcal{F} : \ldots \to ZM \otimes W_1 \xrightarrow{\partial_1} ZM \otimes W_{i-1} \to \ldots \to ZM \otimes W_1 \xrightarrow{\partial_1} ZM \to Z \to 0,$$
where $W_i$ is a free abelian group with a basis $Z_i$, $Z_i$ is the disjoint union

$$Z_i = \bigcup(Y_1^{i_1}) \times (Y_2^{i_2}) \times \ldots \times (Y_s^{i_s}) \times X^{(i_1,i_2,\ldots,i_s)},$$

over all possible $s \geq 1$ such that $i_1 + 2i_2 + \ldots + si_s = i$, $i_j \geq 0$ and $X^{(i_1,i_2,\ldots,i_s)}$ is the set of $s$-tuples $(X_1,\ldots,X_s)$ of pairwise disjoint subsets $X_1,\ldots,X_s$ of $X$ such that $X_j$ has cardinality $i_j$ for all $1 \leq j \leq s$. We write the elements of $Z_i$ as formal products $y_1 \ldots y_j x_1 \ldots x_j$, which indicates that in the tensor product $y_i \in Y = \bigcup_{i \geq 1} Y_i$ is taken from the complex $H_{x_i}$ and $x_1 < \ldots < x_j$ are elements of $X$, recall we have fixed some linear order $\leq$ on $X$. Indeed the element $y_1 \ldots y_j x_1 \ldots x_j$ corresponds to the element $B_1 \times \ldots \times B_s \times (X_1,\ldots,X_s)$ of $Z_i$, where $X_r$ is the set of those elements $x_i$ of $(x_1,\ldots,x_j)$ for which $y_i \in Y_r$. We write $X_r = \{x_{i_1},\ldots,x_{i_r}\}$ where $x_{i_1} < \ldots < x_{i_r}$ and set $\beta_r = (y_{i_1},\ldots,y_{i_r}) \in Y^{i_r}$.

Sometimes it will be convenient to use general products $y_1 \ldots y_j x_1 \ldots x_j$, where $x_1,\ldots,x_j$ are pairwise different elements of $X$ but we do not assume that $x_1 < \ldots < x_k$. We follow (26) and define for all $1 \leq i \leq j - 1$

$$y_1 \ldots y_j x_1 \ldots x_j = (-1)^{\deg(y_1)\deg(y_{i+1})}y_1 \ldots y_{i+1}y_i \ldots y_j x_1 \ldots x_{i+1}x_i \ldots x_j.$$

This completes the definition of the general products $y_1 \ldots y_j x_1 \ldots x_j$ and shows that they belong to $Z_i \cup -Z_i$.

As $X$ is a (left) $G$-set we get an action of $G$ on the general products given by

$$g(y_1 \ldots y_j x_1 \ldots x_j) = y_1 \ldots y_j (gx_1) \ldots (gx_j).$$

(28)

Thus $W_i$ is a semi-induced $\mathbb{Z}G$-module i.e. $W_i = \mathbb{Z}_iZ_i$. Using the other notation for the elements of $Z_i$ we have for $B_1 \times \ldots \times B_s \times (X_1,\ldots,X_s) \in Z_i$ and $g \in G$ that

$$g(B_1 \times \ldots \times B_s \times (X_1,\ldots,X_s)) = \pm (gB_1 \times \ldots \times gB_s \times (gX_1,\ldots,gX_s))$$

where $gB_k$ is obtained from $B_k \in Y^{i_k}_k$ by some permutation (possible the trivial one) of the $i_k$ coordinates. Then by going down to a subgroup of finite index in $G$ we can assume that

$$g(B_1 \times \ldots \times B_s \times (X_1,\ldots,X_s)) = \pm (B_1 \times \ldots \times B_s \times (gX_1,\ldots,gX_s)).$$

(29)

We assume from now on (until the end of this section) that (29) holds. Then the free $\mathbb{Z}$-module with basis $(Y_1^{i_1}) \times (Y_2^{i_2}) \times \ldots \times (Y_s^{i_s}) \times X^{(i_1,i_2,\ldots,i_s)}$ is a semi-induced $\mathbb{Z}G$-module with an associated $G$-action on $(Y_1^{i_1}) \times (Y_2^{i_2}) \times \ldots \times (Y_s^{i_s}) \times X^{(i_1,i_2,\ldots,i_s)}$ induced by the $G$ action on $X^{(i_1,i_2,\ldots,i_s)}$. Since $G$ acts (diagonally) on $X^i$ with finitely many orbits and stabilizers of type $FP_{m-i}$ for $1 \leq i \leq m$ by Lemma 13 part (1) and Lemma 11 we get that the free $\mathbb{Z}$-submodule of $(W_i)$ with basis $(Y_1^{i_1}) \times (Y_2^{i_2}) \times \ldots \times (Y_s^{i_s}) \times X^{(i_1,i_2,\ldots,i_s)}$ is a $\mathbb{Z}G$-module of type $FP_{m-i}$. Since $W_i = \mathbb{Z}_iZ_i$ is a finite direct sum of such modules we deduce that $W_i$ is of type $FP_{m-i}$ as $\mathbb{Z}G$-module for $1 \leq i \leq m$. Then $\mathbb{Z}G \otimes_{\mathbb{Z}G} W_i \simeq \mathbb{Z}M \otimes W_i$ is of type $FP_{m-i}$ as $\mathbb{Z}G$-module.
Since the action of the symmetric group in (26) commutes with the differential (26) for a general product $y_1 \ldots y_j x_1 \ldots x_j$ the differential of $F$ is

$$d_s(y_1 \ldots y_j x_1 \ldots x_j) = \left( \sum_{\deg(y_i) \geq 2} (-1)^{\deg(y_1) + \ldots + \deg(y_i)} y_1 \ldots \partial_{x_i}(y_i) \ldots y_j x_1 \ldots x_j \right) +$$

$$\left( \sum_{\deg(y_i) = 1} (-1)^{\deg(y_1) + \ldots + \deg(y_i)} y_1 \ldots \partial_{x_i}(y_i) \ldots y_j x_1 \ldots \tilde{x}_i \ldots x_j \right)$$

(30)

where $s = \sum_{1 \leq i \leq j} \deg(y_i) \geq 1$. Note that by (28) the above differential commutes with the $G$-action, hence commutes with the $\Gamma$-action. Then by Lemma 2 applied for the complex (obtained from $F$)

$$\tilde{F} : \ldots \to \mathbb{Z}M \otimes W_1 \overset{d_1}{\to} \mathbb{Z}M \otimes W_{i-1} \to \ldots \to \mathbb{Z}M \otimes W_1 \overset{d_1}{\to} \text{Aug}(\mathbb{Z}M) \to 0,$$

we deduce that $\text{Aug}(\mathbb{Z}M)$ is of type $FP_{m-1}$ as $\mathbb{Z}\Gamma$-module, so by Lemma [9] $\Gamma$ is of type $FP_{m}$.

**Lemma 32.** Let $\Gamma = H \wr_X G$ be a group with $H \neq 1, X \neq \emptyset$. Then $\Gamma$ is of type $FP_2$ if and only if $H$ is of type $FP_2$ and $G$ is of type $FP_2$ and $G$ acts on $X$ with finitely many orbits and with stabilizers of type $FP_{2-i}$ for $1 \leq i \leq 2$.

**Proof.** One of the directions is a particular case of the previous theorem. For the other direction assume that $\Gamma$ is of type $FP_2$. By the classification of the finitely generated wreath products $\Gamma$ from [13] we know that $G$ acts on $X$ with finitely many orbits. Note that it follows from [13, Prop. 2.10] that if either $G \setminus X^2$ is infinite or some $x \in X$ has infinitely generated stabilizer, then for every finitely presented group that maps onto $\Gamma$, the kernel maps onto a non-abelian free group. This contradicts the fact that by [9, p. 198]

a group $T$ is $FP_2$ if and only if there is a finitely presented group $T_0$

such that $T \simeq T_0/N$ and $N$ is perfect, (33)

applied for $T = \Gamma$ (note that a perfect group $N$ cannot map onto a non-trivial free group).

By Lemma 5 a retract of a group of type $FP_2$ is $FP_2$, hence $G$ is $FP_2$.

Finally assume that $H$ is not $FP_2$. Using (9.2) we see that a group $C$ is of type $FP_2$ if and only if for every sequence of epimorphisms $C_i \to C_{i+1}$ whose direct limit is $C$ and all $C_i$ are finitely generated, for sufficiently large $i$ the kernel of the canonical map $C_i \to C$ is perfect. Since $H$ is not $FP_2$ there is a sequence of epimorphisms $H_i \to H_{i+1}$ whose direct limit is $H$, all $H_i$ are finitely generated and for every $i$ the kernel $K_i$ of the canonical map $H_i \to H$ is not perfect. Then $H_i^{(X)} \rtimes G$ is a surjective direct system of finitely generated groups with direct limit $\Gamma$ and the kernel of $H_i^{(X)} \rtimes G \to \Gamma$ is $K_i^{(X)}$, which is therefore non-perfect, so $\Gamma$ fails to be $FP_2$, a contradiction. Here $H_i^{(X)}$ denotes the subgroup of $H_i^X$ of elements of finite support i.e. all but finitely many coordinates are trivial. 

\[ \Box \]
Lemma 3 applied for the complex (31) $W_\Gamma$ be the canonical projection. By the definition of $Y$ and will prove that we can assume that $\Gamma$ is of type $FP_{m-1}$ and $G$ acts on $X^1$ with finitely many orbits and with stabilizers of type $FP_{m-i}$ for $1 \leq i \leq m$. Then $H$ is of type $FP_m$.

Proof. By the main result of [13] the case $m = 1$ holds. The case $m = 2$ is a particular case of Lemma [32]. So we may assume that $m \geq 3$.

Let $Z_i$ and $Y_i$ be as in the proof of Proposition 27. Since $H$ is of type $FP_{m-1}$ we can assume that $Y_1, Y_2, \ldots, Y_{m-1}$ are all finite and will prove that $Y_m$ can be chosen finite. The description of $Z_m$ and $X^{(0, \ldots, 0, 1)} = X$ imply that

$$ZZ_m = T_{1,m} \oplus T_{2,m},$$

where $T_{1,m} = Z[Y_m \times X]$ and $T_{2,m} = Z[Z_m \setminus (Y_m \times X)]$ i.e. $T_{1,m} = ZY_m \otimes ZX$. Observe that since $Y_1, \ldots, Y_{m-1}$ are all finite $T_{2,m}$ is finitely generated as $ZG$-module. Note that by the proof of Proposition 27, $ZM \otimes ZZ_i$ is $FP_{m-i}$ as $Z\Gamma$-module for $1 \leq i \leq m - 1$. Since $\Gamma$ is of type $FP_m$ by Lemma 6 and Lemma 3 applied for the complex (31) $Im(d_m)$ is finitely generated as $Z\Gamma$-module. Consider the splitting of the domain $ZM \otimes ZZ_m$ of the differential $d_m$ as

$$ZM \otimes ZZ_m = W_{1,m} \oplus W_{2,m},$$

where $W_{1,m} = ZM \otimes T_{1,m}$ and $W_{2,m} = ZM \otimes T_{2,m}$. Since $W_{2,m}$ is finitely generated as $Z\Gamma$-module the condition that $Im(d_m)$ is finitely generated is equivalent with $d_m(W_{1,m})/d_m(W_{1,m}) \cap d_m(W_{2,m})$ is finitely generated as $Z\Gamma$-module.

Let $w_1 \in W_{1,m}, w_2 \in W_{2,m}$ be such that $d_m(w_1) = d_m(w_2)$. Then $w_1 - w_2 \in Ker(d_m) = Im(d_{m+1})$. Let

$$p : W_{1,m} \oplus W_{2,m} \to W_{1,m}$$

be the canonical projection. By the definition of $W_{1,m}$ and [5,2] applied for $s = m + 1$ we get that

$$w_1 = p(w_1 - w_2) \in Im(d_{m+1})$$

$$pd_{m+1}((ZM \otimes Z[Y_1 \times Y_m \times X^{(1,0,\ldots,0,1)})] \oplus (ZM \otimes Z[Y_{m+1} \times X])].$$

Then since $d_{m+1}(ZM \otimes Z[Y_{m+1} \times X]) \subseteq W_{1,m}$ we have $pd_{m+1}(ZM \otimes Z[Y_{m+1} \times X]) = d_{m+1}(ZM \otimes Z[Y_{m+1} \times X])$ and so

$$d_m(w_1) \subseteq d_m pd_{m+1}(ZM \otimes Z[Y_1 \times Y_m \times X^{(1,0,\ldots,0,1)})] + d_m pd_{m+1}(ZM \otimes Z[Y_{m+1} \times X]) =$$

$$d_m pd_{m+1}(ZM \otimes Z[Y_1 \times Y_m \times X^{(1,0,\ldots,0,1)})] + d_m d_{m+1}(ZM \otimes Z[Y_{m+1} \times X]) =$$

$$d_m pd_{m+1}(ZM \otimes Z[Y_1 \times Y_m \times X^{(1,0,\ldots,0,1)})]) =$$

(35)

Recall that $\partial_{j,x}$ denotes the differential of $\mathcal{H}_x$ in dimension $j$ and when we do not want to stress the dimension we omit the index $j$ and use $\partial_x$. Note that for $y_1 \in Y_1, y_m \in Y_m, x_1, x_m \in X, x_1 \neq x_m$

$$d_m pd_{m+1}(y_1 y_m x_1 x_m) = d_m p(-\partial_{x_1}(y_1)y_m x_m + (-1)^m y_1 \partial_{x_m}(y_m)x_1 x_m) =$$
\[ d_m(-\partial_x(y_1 \cdot y_m \cdot x_m)) = (-1)^{m+1} \partial_x(y_1) \partial_{x_m}(y_m) x_m. \quad (36) \]

Decompose \( M = S \times H \), where \( S = \prod_{i \neq x} H_i \). Thus in (36) we have \( \partial_{x_m}(y_m) \in \text{Im}(\partial_{m,x_m}), \partial_x(y_1) \in \text{Aug}(\text{Aug}S_{x_m}) \). Then by (35) and (36)

\[
\begin{align*}
&\quad d_m(w_1) \in \bigoplus_{y_m \in Y_1, y_1 \in Y_1, x_1, x_m \in X, x_1 \neq x_m} \text{Aug}(\text{Aug}S_{x_m}) \text{Aug}(\text{Aug}S_{x_m}) \text{Aug}(\text{Aug}S_{x_m}) \\
&\quad \text{Im}(\partial_{m,x_m}) x_m = \\
&\quad \bigoplus_{x \in X} \text{Aug}(\text{Aug}S_{x_m}) \text{Aug}(\text{Aug}S_{x_m}) \text{Aug}(\text{Aug}S_{x_m}) \text{Aug}(\text{Aug}S_{x_m}) \text{Aug}(\text{Aug}S_{x_m}) \text{Aug}(\text{Aug}S_{x_m}) \text{Aug}(\text{Aug}S_{x_m}) \\
&\quad \bigoplus_{x \in X} Z M \text{Im}(\partial_{m,x_m}) x_m = \\
&\quad \bigoplus_{x \in X} Z H_{x_m} \text{Im}(\partial_{m,x_m}) x = \bigoplus_{x \in X} Z S_x \text{Im}(\partial_{m,x_m}) x. \quad (37)
\end{align*}
\]

Furthermore for every \( j \in J \) there is \( w_1 \in W_1, w_2 \in W_2, m \) with \( d_m(w_1) = d_m(w_2) \) and \( d_m(w_1) = j \). Thus by (37) and (38)

\[ K := d_m(W_1, m) / J \cong \bigoplus_{x \in X} \text{Im}(\partial_{m,x_m}) x \subseteq \bigoplus_{x \in X} Z H_{x_m} x. \]

Thus the condition that \( \text{Im}(d_m) \) is finitely generated as \( Z \Gamma \)-module is equivalent with \( K \) is finitely generated as \( Z \Gamma \)-module. Take a finite set \( D \) of generators of \( K \) such that \( D \subseteq \bigcup_{x \in X} \text{Im}(\partial_{m,x_m}) x \). Write \( d \in D \) as \( a_x x \) and define \( A_x = \bigcup_{x \in X} \text{Im}(\partial_{m,x_m}) x \) as the set of all possible \( a_x \). Thus the set of the images of the finite set \( \bigcup_{x \in X} A_x \) in \( \text{Im}(\partial_{m,x_m}) \), where we send canonically every \( H_x \) to \( H \) by just forgetting the index \( x \), generates \( \text{Im}(\partial_{m,x_m}) \) as \( Z \Gamma \)-module. Thus \( \text{Im}(\partial_{m,x_m}) \) is finitely generated as \( Z \Gamma \)-module, so \( Y_m \) can be chosen finite.

\[ \square \]

**Theorem 39.** Let \( m \geq 1 \) and \( \Gamma = H \wr_X G \) be of type \( FP_m \), where \( X \neq \emptyset \) and \( H \) has infinite abelianization. Then \( H \) is of type \( FP_m \) and \( G \) acts on \( X^i \) with finitely many orbits and stabilizers of type \( FP_{m-1} \) for all \( 1 \leq i \leq m \).

**Proof.** We induct on \( m \). The case \( m = 1 \) is trivial as the property \( FP_1 \) is equivalent with finite generation and then [13] Prop. 2.1 applies.

Since \( H \) has infinite abelianization \( Z \) is a retract of \( H \), so \( \Gamma_0 = Z \wr_X G \) is retract of \( \Gamma \). By Lemma 3 since \( \Gamma \) is of type \( FP_m \) the group \( \Gamma_0 \) is \( FP_m \) too. Then by Proposition 12 and Theorem 24 \( G \) acts on \( X^i \) with finitely many orbits and stabilizers of type \( FP_{m-1} \) for all \( 1 \leq i \leq m \). Then Proposition 34 completes the inductive step.

\[ \square \]

**Theorem 40.** Let \( \Gamma = H \wr_X G \) be a wreath product, where \( X \neq \emptyset \) and \( H \) has infinite abelianization. Then the following are equivalent:

1. \( \Gamma \) is of type \( FP_m \);
2. \( H \) is of type \( FP_m \), \( G \) is of type \( FP_m \), \( G \) acts on \( X^i \) with finitely many orbits and stabilizers of type \( FP_{m-1} \) for all \( 1 \leq i \leq m \).

**Proof.** Follows from Lemma 3 Proposition 27 and Theorem 39

The following is a homotopy version of Theorem 40.
Theorem 41. Let $\Gamma = H \wr X G$ be a wreath product, where $X \neq \emptyset$ and $H$ has infinite abelianization. Then the following are equivalent:

1. $\Gamma$ is of type $F^m$;
2. $H$ is of type $F^m$, $G$ is of type $F^m$, $G$ acts on $X^i$ with finitely many orbits and stabilizers of type $FP_{m-i}$ for all $1 \leq i \leq m$.

Remark. Observe that in the condition 2 we have a mixture of both properties $F^m$ and $FP^m$ and not just the homotopical property $F^m$.

Proof. In the case $m \leq 2$ this is the main result of [13]. So we can assume that $m > 2$. In general a group is of type $F^m$ if and only if it is finitely presented (i.e. is $F^2$) and is of type $FP^m$. Then the result follows from Theorem 40 and the fact that the case $m = 2$ of Theorem 41 holds.

6 An example

Let $F$ be the Richard Thompson group with infinite presentation

$\langle x_0, x_1, x_2 \ldots | x_j^{x_i} = x_{j+1} \text{ for } 0 \leq i < j \rangle$.

Consider its realization as piecewise linear transformations on the interval $[0, 1]$, see [10]. Let $X = (0, 1) \cap \mathbb{Z}^{\frac{1}{2}}$. Then $F$ acts with finitely many orbits on $X^i$ i.e. the $F$ orbit of $x = (x_1, \ldots, x_i) \in X^i$ contains $y = (y_1, \ldots, y_i)$ if and only if there is a permutation $\sigma \in S_i$ such that $x_{\sigma(1)} \leq x_{\sigma(2)} \leq \ldots \leq x_{\sigma(i)}$ and $y_{\sigma(1)} \leq y_{\sigma(2)} \leq \ldots \leq y_{\sigma(i)}$ and $x_{\sigma(k)} = x_{\sigma(k+1)}$ if and only if $y_{\sigma(k)} = y_{\sigma(k+1)}$. Furthermore the stabilizer in $F$ of the point $x = (x_1, \ldots, x_i) \in X^i$ is $F^{j+1}$, where $j$ is the number of different coordinates of $x$. By [12] $F$ is of type $FP_{\infty}$, so all stabilizers are of type $FP_{\infty}$. The following result is a direct corollary of Theorem 40 and Theorem 41.

Corollary 42. Let $X$ and $F$ be as above and $\Gamma = H \wr X F$ be a wreath product. Then $H$ has type $FP^m$ (resp. $F^m$) if and only if $\Gamma$ is of type $FP^m$ (resp. $F^m$).

Proof. Follows directly from Proposition 27 and Proposition 43.

7 Bredon type $FP^m$ for wreath products

7.1 Preliminaries on Bredon homology

The homotopical finiteness Bredon properties $F^\infty$ were considered in [15]. Here we need the following homological version of Luck’s result for $FP^m$.

Theorem 43. [17] A group $\Gamma$ is of type $FP^m$ if and only if $\Gamma$ has finitely many conjugacy classes of finite subgroups and for every finite subgroup $K$ of $\Gamma$ the centralizer $C_{\Gamma}(K)$ is of type $FP^m$. 
7.2 Bredon type $FP_m$ for wreath products: a proof of Theorem C

Theorem 44. Let $\Gamma = H \wr_X G$ be a wreath product such that $H$ is torsion-free. Then every finite subgroup of $\Gamma$ is conjugate to a subgroup of $G$.

Proof. In this proof we consider $X$ as a right $G$-set (if $X$ is equipped with a left $G$-action it becomes a right $G$-action by $xg = g^{-1}x$). Denote by $\pi : \Gamma \to G$ the natural homomorphism. Let $M$ be the normal closure of $H$ in $\Gamma$. Elements $\gamma$ of $\Gamma$ may uniquely be written as $\gamma = mg$ for $m \in M$ and $g \in G$; we have $\pi(\gamma) = g$, and write $\gamma @ x := mx$, where $mx$ is the $x$-coordinate of $m \in M \subseteq H^X$. The assertion of the theorem is that for every finite subgroup $K \leq \Gamma$ there exists $\delta \in \Gamma$ such that $(k^s)@x = 1$ for all $x \in X$, $k \in K$.

Consider first two points $x, y \in X$ in the same orbit under $\pi(K)$. Then there exists a unique $h_{x,y} \in H$ such that $k@x = h_{x,y}$ for all $k \in K$ with $x^{\pi(k)} = y$.

Indeed, consider two such $k, k' \in K$. Then $\pi(k'k^{-1})$ fixes $x$, and has finite order $s$ because $k'k^{-1}$ belongs to $K$; so $1 = (k'k^{-1})^s @ x = ((k'k^{-1}) @ x)^s$ implies $1 = (k'k^{-1}) @ x = (k' @ x)(k @ x)^{-1}$. Note the cocycle identity $h_{x,y}h_{y,z} = h_{x,z}$.

In each orbit $\Omega$ of $\pi(K)$ on $X$, choose a representative $x_\Omega$, and define $\delta \in H^X$ by $\delta_x = h_{x,x_\Omega}$ whenever $x$ lies in the orbit $\Omega$. Note that since $K$ is finite there are only finitely many $x, y \in X$ such that $h_{x,y} \neq 1$. Therefore $\delta$ is a finitely supported function on $X$ i.e. $\delta \in M$.

Now consider $k \in K$, and write $y = x^{\pi(k)}$. Note that $x_\Omega = y_\Omega$. Then

$$(k^s)@x = (\delta^{-1}@x) (k@x) (\delta @ y) = h_{x,x_\Omega}^{-1} h_{x,y} h_{y,y_\Omega} h_{x,x_\Omega} = 1.$$ \hfill \Box

Theorem 45. Let $\Gamma = H \wr X G$ be a wreath product, $X \neq \emptyset$ and $H$ be torsion-free, with infinite abelianization. Then $\Gamma$ has type $FP_m$ if and only if the following conditions hold:

1. $G$ has type $FP_m$;
2. $H$ has type $FP_m$;
3. for every finite subgroup $K$ of $G$ and every $1 \leq i \leq m$ the centralizer $C_G(K)$ acts on $(K \setminus X)^i$ with finitely many orbits and stabilizers of type $FP_{m-i}$.

Proof. By Theorem 44, we have to understand the centralizers and the conjugacy classes of finite subgroups in $\Gamma$. By Theorem 44, every finite subgroup $K$ of $\Gamma$ is conjugated to a finite subgroup $K_0$ of $G$. Thus $\Gamma$ has finitely many conjugacy classes of finite subgroups if and only if $G$ has finitely many conjugacy classes of finite subgroups. This completes the proof for $m = 0$.

Furthermore for a finite subgroup $K$ of $G$ we have that

$$C_\Gamma(K) = C_M(K) \rtimes C_G(K) = H \wr_{K \setminus X} C_G(K),$$

where $M$ is the normal closure of $H$ in $\Gamma$ and we consider $X$ as a left $G$-set. Then by Theorem 44, the centralizer $C_\Gamma(K)$ is of type $FP_m$ if and only if $C_G(K)$ is $FP_m$ and conditions 2 and 3 hold. Then Theorem 43 completes the proof. \hfill \Box
8 \( \Sigma \)-invariants

8.1 Some basic properties

Following [5] for a non-zero real character \( \chi \) of \( G \)
\[
\Sigma^m(G, \mathbb{Z}) = \{ [\chi] \mid \text{\( \chi \) is \( \mathbb{Z}G \)-module} \},
\]
where \( G_\chi = \{ g \in G \mid \chi(g) \geq 0 \} \) and \( [\chi] = \mathbb{R}_{>0} \chi \).

**Lemma 46.** Let \( m \geq 1 \) be an integer, \( \Gamma = M \rtimes G \) be a finitely generated group and \( [\chi] \in \Sigma^m(\Gamma, \mathbb{Z}) \) such that \( \chi(M) = 0 \). Then \( [\chi_0] \in \Sigma^m(G, \mathbb{Z}) \), where \( \chi_0 \) is the restriction of \( \chi \) to \( G \).

**Proof.** This is a monoid version of Lemma 5 and Aberg’s idea (i.e. the footnote from [1]) can be easily modified in this context. Indeed by [3] we have that \( [\chi] \in \Sigma^m(\Gamma, \mathbb{Z}) \) is equivalent with
\[
\text{Tor}^Z(\mathbb{Z}, \prod \Gamma_\chi) = 0
\]
for all \( 1 \leq i \leq m - 1 \). Note that by the functoriality of \( \text{Tor}^S(\mathbb{Z}, \prod S) \) on \( S \) applied for the monoids \( S \in \{ G_\chi \cap \Gamma \} \) and the fact that \( \Gamma_\chi = M \rtimes G_{\chi_0} \) we get that \( \text{Tor}^Z(\mathbb{Z}, \prod \Gamma_\chi) = 0 \) for all \( 1 \leq i \leq m - 1 \). Thus \( [\chi_0] \in \Sigma^m(G, \mathbb{Z}) \).

**Lemma 47.** Let \( m \geq 1 \) be an integer, \( \Gamma = M \rtimes G \) be a finitely generated group and \( \chi : \Gamma \to \mathbb{R} \) be a non-zero homomorphism such that \( \chi(M) = 0 \). Denote by \( \chi_0 \) the restriction of \( \chi \) to \( G \). Then \( [\chi] \in \Sigma^m(\Gamma, \mathbb{Z}) \) if and only if \( [\chi_0] \in \Sigma^m(G, \mathbb{Z}) \) and \( \text{Aug}(\mathbb{Z}M) \) is \( \mathbb{F} \mathbb{P} \_m \) as \( \Sigma \Gamma_\chi \)-module.

**Proof.** In the proof of Lemma 5 substitute \( \Gamma \) with \( \Gamma_\chi \) and substitute \( G \) with \( G_{\chi_0} \).

**Lemma 48.** Let \( k \geq 0 \) be an integer, \( \chi : G \to \mathbb{R} \) be a non-zero homomorphism, \( L \) a subgroup of \( G \) and \( X = G/L = \{ gL \}_{g \in G} \). Denote by \( \chi_0 \) the restriction of \( \chi \) on \( L \). Then
1) if \( \chi \) is discrete we have that \( \mathbb{Z}G_\chi \) is a (right or left) free \( \mathbb{Z}L_{\chi_0} \)-module with a basis of cardinality \( |G : L| \);
2) in general \( \mathbb{Z}G_\chi \) is a flat (right or left) \( \mathbb{Z}L_{\chi_0} \)-module and the inclusion map of \( \mathbb{Z}L_{\chi_0} \)-modules \( \mathbb{Z}L_{\chi_0} \to \mathbb{Z}G_\chi \) splits;
3) \( \mathbb{Z}X \) is \( \mathbb{F} \mathbb{P} \) as \( \mathbb{Z}G_\chi \)-module (i.e. \( [\chi] \in \Sigma^k(G, \mathbb{Z}X) \)) if and only if \( \chi_0 \neq 0 \) and \( \mathbb{Z} \) is \( \mathbb{F} \mathbb{P} \) as \( \mathbb{Z}L_{\chi_0} \)-module (i.e. \( [\chi_0] \in \Sigma^k(L, \mathbb{Z}) \)).

**Proof.** Consider \( T \) a left transversal of \( L \) in \( G \) (i.e. \( G \) is the disjoint union \( tL \) over \( t \in T \)) such that \( 1 \in T \) and \( \chi(t) \geq 0 \) for all \( t \in T \). Then
\[
\mathbb{Z}G_\chi = \oplus_{t \in T} tL_{\chi_0 \geq \chi(t)}, \tag{49}
\]
where \( L_{\chi_0 \geq r} = \{ g \in L \mid \chi_0(g) \geq r \} \). Since \( 1 \in T \) the inclusion map of (right) \( \mathbb{Z}L_{\chi_0} \)-modules \( \mathbb{Z}L_{\chi_0} \to \mathbb{Z}G_\chi \) splits.
1) We consider right modules here. The case of left modules is similar. Since \( \chi_0 \) is discrete \( \mathbb{Z}L_{\chi_0} \simeq \mathbb{Z}L_{\chi_0} \) as \( \mathbb{Z}L_{\chi_0} \)-modules, so
\[
\mathbb{Z}G_{\chi} = \bigoplus_{t \in T} t \mathbb{Z}L_{\chi_0} \simeq \bigoplus_{t \in T} t \mathbb{Z}L_{\chi_0}
\]
i.e. \( \mathbb{Z}G_{\chi} \) is a free (right) \( \mathbb{Z}L_{\chi_0} \)-module.

2) Note that \( \mathbb{Z}L_{\chi_0} \simeq - \chi(t) \) is the direct limit of \( \mathbb{Z}L_{\chi_0} \geq s \), where \( s \in \text{Im}(\chi_0) = \chi(L) \) and \( s \geq - \chi(t) \). Since any such \( \mathbb{Z}L_{\chi_0} \geq s \) as \( \mathbb{Z}L_{\chi_0} \)-module we get that \( \mathbb{Z}L_{\chi_0} \) is flat as a (right) \( \mathbb{Z}L_{\chi_0} \)-module.

3) If \( \chi_0 = 0 \) then \( L \subseteq \text{Ker}(\chi) \subseteq G \). If \( \mathbb{Z}X \) is finitely generated as \( \mathbb{Z}G_{\chi} \)-module, \( \mathbb{Z}(G/\text{Ker}(\chi)) = \text{Im}(\chi) \) is finitely generated as \( \mathbb{Z}G_{\chi} \)-module, a contradiction. So from now on we can assume that \( \chi_0 \neq 0 \).

Note that \( X \) is one \( G \)-orbit with generator \( e = L \), the monoid \( G_{\chi} \) acts by restriction on \( X, X = G_{\chi} e \) and the centralizer of \( e \) in \( G_{\chi} \) is \( G_{\chi} \cap L = L_{\chi_0} \).

Then we have an isomorphism of \( \mathbb{Z}G_{\chi} \)-modules
\[
\mathbb{Z}G_{\chi} \otimes_{\mathbb{Z}L_{\chi_0}} \mathbb{Z} \simeq \mathbb{Z}G \otimes_{\mathbb{Z}L} \mathbb{Z} = \mathbb{Z}X.
\] (50)

By Lemma [S] and case 2) \( \mathbb{Z}G_{\chi} \otimes_{\mathbb{Z}L_{\chi_0}} \mathbb{Z} = FP_m \) as \( \mathbb{Z}G_{\chi} \)-module if and only if \( \mathbb{Z}X \) is \( FP_m \) as \( \mathbb{Z}L_{\chi_0} \)-module. Then [50] completes the proof of 3).

8.2 Semi-induced modules

Lemma 51. Let \( \chi : G \to \mathbb{R} \) be a non-zero discrete homomorphism. Let \( Z_\chi X \) be a semi-induced \( \mathbb{Z}G \)-module such that \( G \cdot X \) is finite via the associated action of \( G \) on \( X \). Then \( Z_\chi X \) is of type \( FP_k \) as \( \mathbb{Z}G_{\chi} \)-module if and only if \( \mathbb{Z}X \) is of type \( FP_k \) as \( \mathbb{Z}G_{\chi} \)-module.

Proof. For every orbit \( G \cdot x \) of \( G \) on \( X \) consider \( \text{Ker}(\chi_{G \cdot x}) \), so \( [G : \text{Ker}(\chi_{G \cdot x})] \leq 2 \), where \( \chi_{G \cdot x} \) is the character defined at the beginning of Section 4. Let \( G_0 \) be the intersection of \( \text{Ker}(\chi_{G \cdot x}) \) over all possible \( G \)-orbits of \( X \). Then \( G_0 \) has finite index in \( G \), so by Lemma [S] \( \mathbb{Z}G_{\chi} \) is a finitely generated free \( \mathbb{Z}(G_0)_{\chi_0} \)-module. Thus a \( \mathbb{Z}G_{\chi} \)-module \( V \) is of type \( FP_k \) if and only if \( V \) is of type \( FP_k \) as \( \mathbb{Z}(G_0)_{\chi_0} \)-module (this is an obvious translations of a proof of [9]). We apply this for \( V = Z_\chi X \) and for \( V = \mathbb{Z}X \). Finally \( Z_\chi X = \mathbb{Z}X \) as \( \mathbb{Z}(G_0)_{\chi_0} \)-module. □

Proposition 52. Let \( M = Z_\chi X \) be a semi-induced \( \mathbb{Z}G \)-module (in particular \( X \neq \emptyset \)) and \( \chi : G \to \mathbb{R} \) a non-zero discrete homomorphism. Then the following are equivalent:

1. \( \wedge^i M \) is of type \( FP_{m-i} \) as \( \mathbb{Z}G_{\chi} \)-module for all \( 1 \leq i \leq m \);
2. \( \otimes^i M \) is of type \( FP_{m-i} \) as \( \mathbb{Z}G_{\chi} \)-module for all \( 1 \leq i \leq m \);
3. \( \otimes^i M \) is finitely generated as \( \mathbb{Z}G_{\chi} \)-module via the diagonal action and for representatives \( \{ H_1, \ldots, H_{m^i} \} \) of all \( G \)-orbits of stabilizers of the diagonal action of \( G \) on \( X \) (via the associated \( G \)-action on \( X \)) the restriction \( \chi_j \) of \( \chi \) on \( H_j \) is non-zero and \( [\chi_j] \in \Sigma(m^{-1}(H_j, \mathbb{Z})) \) for all \( 1 \leq i \leq m \);
4. \( \wedge^i M \) is finitely generated as \( \mathbb{Z}G_{\chi} \)-module via the diagonal action and for representatives \( \{ H_1, \ldots, H_{m^i} \} \) of all \( G \)-orbits of stabilizers of the action of \( G \) on
the \( i \)-element subsets of \( X \) (via the associated \( G \)-action on \( X \)) the restriction \( \chi_j \) of \( \chi \) on \( H_j \) is non-zero and \( [\chi_j] \in \Sigma^{m-i}(H_j, \mathbb{Z}) \) for all \( 1 \leq i \leq m \);

5. \( G \setminus X^i \) is finite and for representatives \( \{H_1, \ldots, H_{s_3}\} \) of all \( G \)-orbits of stabilizers of the diagonal action of \( G \) on \( X^i \) (via the associated \( G \)-action on \( X \)) the restriction \( \chi_j \) of \( \chi \) on \( H_j \) is non-zero and \( [\chi_j] \in \Sigma^{m-i}(H_j, \mathbb{Z}) \) for all \( 1 \leq i \leq m \).

**Proof.** In the proof of Proposition 12 substitute \( G \) with \( G_\chi \), apply Lemma 48 and use Lemma 51 instead of Lemma 11. Where \( G \setminus Y \) finite appears should be substituted with there is a finite subset \( Y_0 \) of \( Y \) such that \( G_\chi Y_0 = Y \).

**Theorem 53.** Let \( \Gamma = H \wr_{X} G \) be a wreath product of type \( FP_m \), where \( X \neq \emptyset \) and \( H \) has infinite abelianization. Let \( \chi: \Gamma \to \mathbb{R} \) be a non-zero discrete character such that \( \chi(H) = 0 \). Then the following are equivalent:

1. \( [\chi] \in \Sigma^m(\Gamma, \mathbb{Z}) \);
2. \( [\chi|_G] \in \Sigma^m(G, \mathbb{Z}) \) and condition 5 from Proposition 52 holds.

**Proof.** The proof is the same as the proof of Theorem 40 substituting \( G \) with \( G_\chi \) with the only difference that in the proofs of the results leading to the proof of Theorem 40 where \( G \setminus Y \) finite appears should be substituted with there is a finite subset \( Y_0 \) of \( Y \) such that \( G_\chi Y_0 = Y \).

The fact that 2. implies 1. is a monoid version of Proposition 27. Note that in the proof of Proposition 27 we substitute \( G \) with a subgroup of finite index. This step requires justification in its \( \Sigma \)-version i.e. if \( G_0 \) is a subgroup of finite index in \( G \) and \( \chi_0 \) is the restriction of \( \chi \) to \( \Gamma_0 = H \wr_{X} G_0 \) then \( [\chi] \in \Sigma^m(\Gamma, \mathbb{Z}) \) if and only if \( [\chi_0] \in \Sigma^m(\Gamma_0, \mathbb{Z}) \). Indeed by Lemma 18 part 1 since \( \chi \) is discrete \( \mathbb{Z}\Gamma_\chi \) is a finitely generated free \( \mathbb{Z}(\Gamma_0)_{\chi_0} \)-module, so the trivial \( \mathbb{Z}\Gamma_\chi \)-module \( \mathbb{Z} \) is \( FP_m \) if and only if the trivial \( \mathbb{Z}(\Gamma_0)_{\chi_0} \)-module \( \mathbb{Z} \) is \( FP_m \).

The fact that item 1 implies item 2 follows from Lemma 46 applied for the retracts \( G \) and \( \mathbb{Z} \wr_{X} G \) of \( \Gamma \). Indeed we get that \( [\chi|_G] \in \Sigma^m(G, \mathbb{Z}) \) and \( [\chi|_{\Gamma_1}] \in \Sigma^m(\Gamma_1, \mathbb{Z}) \), where \( \Gamma_1 = \mathbb{Z} \wr_{X} G \). By the monoid version of Theorem 24 the result holds for \( \Gamma_1 \) so condition 5 from Proposition 52 holds.

Note that in Proposition 52 we assume that the character \( \chi \) is discrete too. Thus the proof of the result uses substantionally that \( \chi \) is discrete but it is not clear whether for non-discrete characters the result holds or not.

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