Existence of Chaos for a Singularly Perturbed NLS Equation

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Abstract. The work [1] is generalized to the singularly perturbed nonlinear Schrödinger (NLS) equation of which the regularly perturbed NLS studied in [1] is a mollification. Specifically, the existence of Smale horseshoes and Bernoulli shift dynamics is established in a neighborhood of a symmetric pair of Silnikov homoclinic orbits under certain generic conditions, and the existence of the symmetric pair of Silnikov homoclinic orbits has been proved in [2].

The main difficulty in the current horseshoe construction is introduced by the singular perturbation $\varepsilon \partial_{xx}$ which turns the unperturbed reversible system into an irreversible system. It turns out that the equivariant smooth linearization can still be achieved, and the Conley-Moser conditions can still be realized.

1. Introduction

Consider the singularly perturbed nonlinear Schrödinger (NLS) equation,

$$
 iq_t = q_{\zeta\zeta} + 2|q|^2 - \omega^2 q + i\varepsilon(q_{\zeta\zeta} - \alpha q + \beta),
$$

(1.1)

where $q = q(t, \zeta)$ is a complex-valued function of the two real variables $t$ and $\zeta$, $t$ represents time, and $\zeta$ represents space. $q(t, \zeta)$ is subject to periodic boundary condition of period $2\pi$, and even constraint, i.e.

$$
 q(t, \zeta + 2\pi) = q(t, \zeta), \quad q(t, -\zeta) = q(t, \zeta).
$$

$\alpha > 0$ and $\beta > 0$ are constants, and $\varepsilon > 0$ is the perturbation parameter. For simplicity of presentation, we restrict $\omega$ by $\omega \in (1/2, 1)$. In [2], the following theorem on the existence of Silnikov homoclinic orbits was proved.

**Theorem 1.1.** There exists a $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists a codimension 1 surface in the external parameter space $(\alpha, \beta, \omega) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ where $\omega \in (1/2, 1)/S$, $S$ is a finite subset, and $\omega \beta < \alpha$. For any $(\alpha, \beta, \omega)$ on the codimension 1 surface, the singularly perturbed nonlinear Schrödinger equation (1.1) possesses a symmetric pair of Silnikov homoclinic orbits asymptotic to a saddle $Q_\varepsilon$. The codimension 1 surface has the approximate representation given by $\alpha = 1/\kappa(\omega)$, where $\kappa(\omega)$ is plotted in Figure 1.

Notice that if $q(t, \zeta)$ is a homoclinic orbit, then $q(t, \zeta + \pi)$ is another homoclinic orbit. Thus $q(t, \zeta)$ and $q(t, \zeta + \pi)$ form a symmetric pair of homoclinic orbits. Based

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upon the above theorem, we will construct Smale horseshoes in a neighborhood of the symmetric pair of homoclinic orbits. The construction is a generalization of that in [1] where the singular perturbation $\epsilon \partial^2_x$ is mollified into a bounded Fourier multiplier. The main difficulty in the current horseshoe construction is introduced by the singular perturbation $\epsilon \partial^2_x$ which turns the unperturbed reversible system into an irreversible system. Specifically, denote by $F^t_\epsilon$ the evolution operator of the singularly perturbed nonlinear Schrödinger equation \( \text{(1.1)} \). When $\epsilon = 0$, $F^t_0$ is a group. When $\epsilon > 0$, $F^t_\epsilon$ is only a semigroup. It turns out that the equivariant smooth linearization can still be achieved, and the Conley-Moser conditions can still be realized. Of course, one has to replace the inverse of the evolution operator $F^t_\epsilon$ by preimage. The article is organized as follows: In section 2, we present equivariant smooth linearization. In section 3, we present the Poincaré map and its representation. In section 4, the fixed points of the Poincaré map is studied. In section 5, we present the existence of chaos. Finally, in section 6, numerical evidence for the generic conditions is presented.

2. Equivariant Smooth Linearization

The symmetric pair of Silnikov homoclinic orbits is asymptotic to the saddle $Q_\epsilon = \sqrt{I}e^{i\theta}$, where
\begin{equation}
I = \omega^2 - \epsilon \frac{1}{2\omega} \sqrt{\beta^2 - \alpha^2\omega^2 + \cdots}, \quad \cos \theta = \frac{\alpha \sqrt{I}}{\beta}, \quad \theta \in (0, \pi/2).
\end{equation}
Its eigenvalues are
\begin{equation}
\lambda^\pm_n = -\epsilon \alpha + n^2 \pm 2 \sqrt{(n^2/2 + \omega^2 - I)(3I - \omega^2 - n^2/2)},
\end{equation}
where $n = 0, 1, 2, \cdots$, $\omega \in (1/2, 1)$, and $I$ is given in \( \text{(2.1)} \). The crucial points to notice are: (1). only $\lambda^+_1$ and $\lambda^+_1$ have positive real parts, $\text{Re}\{\lambda^+_1\} < \text{Re}\{\lambda^+_1\}$; (2).
all the other eigenvalues have negative real parts among which the absolute value of $\text{Re}\{\lambda^+_n\} = \text{Re}\{\lambda^+_0\}$ is the smallest; (3). $|\text{Re}\{\lambda^+_2\}| < \text{Re}\{\lambda^+_0\}$. Actually, items (2) and (3) are the main characteristics of Silnikov homoclinic orbits.

**Lemma 2.1.** For any fixed $\epsilon \in (0, \epsilon_0)$, let $E_\epsilon$ be the codimension 1 surface in the external parameter space, on which the symmetric pair of Silnikov homoclinic orbits are supported (cf: Theorem 1.1). For almost every $(\alpha, \beta, \omega) \in E_\epsilon$, the eigenvalues $\lambda^\pm_n(\epsilon)\in\Lambda$ satisfy the nonresonance condition of Siegel type: There exists a natural number $s$ such that for any integer $n \geq 2$,

$$\left|\Lambda_n - \sum_{j=1}^r \Lambda_{l_j}\right| \geq 1/r^s,$$

for all $r = 2, 3, \ldots, n$ and all $l_1, l_2, \ldots, l_r \in \mathbb{Z}$, where $\Lambda_n = \lambda^+_n$ for $n \geq 0$, and $\Lambda_n = \lambda^-_{n-1}$ for $n < 0$.

Proof. The same proof as in [1] can be carried through here. QED

Thus, in a neighborhood of $Q_\epsilon$, the singularly perturbed NLS (1.1) is analytically equivalent to its linearization at $Q_\epsilon$. [3]. In terms of eigenvector basis, (1.1) can be rewritten as

$$\begin{align*}
\dot{x} &= -ax - by + N_x(Q), \\
\dot{y} &= bx - ay + N_y(Q), \\
\dot{z}_1 &= \gamma_1 z_1 + N_{z_1}(Q), \\
\dot{z}_2 &= \gamma_2 z_2 + N_{z_2}(Q), \\
\dot{Q} &= LQ + N_Q(Q);
\end{align*}$$

(2.3)

where $a = -\text{Re}\{\lambda^+_2\}$, $b = \text{Im}\{\lambda^+_2\}$, $\gamma_1 = \lambda^+_0$, $\gamma_2 = \lambda^+_1$; $N$'s vanish identically in a neighborhood of $Q = 0$, $Q = (x, y, z_1, z_2, Q)$, $Q$ is associated with the rest of eigenvalues, $L$ is given as

$$LQ = -iQ_{\zeta\zeta} - 2i|Q|^2 - \omega^2)Q + Q^2 \bar{Q} + e[-\alpha Q + Q_{\zeta\zeta}],$$

and $Q_\epsilon$ is given in (2.3). The following theorem on well-posedness is standard [4]. Let $F^t$ $(0 \leq t < \infty)$ be the evolution operator of the singularly perturbed NLS (2.3), and $H^s$ be the Sobolev space.

**Theorem 2.2.** For any $s \geq 1$, and any $\bar{Q}_0 \in H^{s+2}$, $F^t(\bar{Q}_0) \in C^0([0, \infty); H^{s+2}) \cap C^1([0, \infty); H^s)$. For any fixed $t \in [0, \infty)$, $F^t$ is a $C^2$ map in $H^s$.

3. The Poincaré Map and Its Representation

Denote by $h_k$ $(k = 1, 2)$ the symmetric pair of Silnikov homoclinic orbits. The symmetry $\sigma$ of half spatial period shifting has the new representation in terms of the new coordinates

$$\sigma \circ (x, y, z_1, z_2, Q) = (x, y, z_1, -z_2, \sigma \circ Q).$$

(3.1)

We have the following facts about the homoclinic orbits:

1. The homoclinic orbits are classical solutions,
2. As $t \to -\infty$, the homoclinic orbits are tangent to the positive $z_1$-axis at $Q = 0$.

The same proof as in [1] works here for item (2). Since $\sigma$ is the smallest attracting rate, we assume that
As $t \to +\infty$, the homoclinic orbits are tangent to the $(x, y)$-plane at $\vec{Q} = 0$.

The Poincaré section is defined as in [1].

**Definition 3.1.** The Poincaré section $\Sigma_0$ is defined by the constraints:

\[
\begin{align*}
    y &= 0, \quad \eta \exp\{-2\pi a/b\} < x < \eta; \\
    0 < z_1 < \eta, \quad -\eta < z_2 < \eta, \quad \|Q\| < \eta;
\end{align*}
\]

where $\eta$ is a small parameter.

The horseshoes are going to be constructed on this Poincaré section. The auxiliary Poincaré section is defined differently from that in [1].

**Definition 3.2.** The Poincaré section $\Sigma_1$ is defined by the constraints:

\[
\begin{align*}
    z_1 &= \eta, \quad -\eta < z_2 < \eta, \\
    \sqrt{x^2 + y^2} < \eta, \quad \|Q\| < \eta.
\end{align*}
\]

The Poincaré map is defined as follows.

**Definition 3.3.** The Poincaré map $P$ is defined as:

\[
P : U \subset \Sigma_0 \mapsto \Sigma_0, \quad P = P_0^1 \circ P_1^0,
\]

where

\[
P_0^1 : U_0 \subset \Sigma_0 \mapsto \Sigma_1, \quad \forall \vec{Q} \in U_0, \quad P_0^1(\vec{Q}) = F^{t_0}(\vec{Q}) \in \Sigma_1,
\]

and $t_0 = t_0(\vec{Q}) > 0$ is the smallest time $t$ such that $F^t(\vec{Q}) \in \Sigma_1$, and

\[
P_1^0 : U_1 \subset \Sigma_1 \mapsto \Sigma_0(= \Sigma_0 \cup \partial \Sigma_0), \quad \forall \vec{Q} \in U_1, \quad P_1^0(\vec{Q}) = F^{t_1}(\vec{Q}) \in \Sigma_0,
\]

and $t_1 = t_1(\vec{Q}) > 0$ is the smallest time $t$ such that $F^t(\vec{Q}) \in \Sigma_0$.

The map $P_0^1$ has the explicit representation: Let $\vec{Q}_0$ and $\vec{Q}_1$ be the coordinates on $\Sigma_0$ and $\Sigma_1$ respectively, $y_0 = 0$, and $z_1^0 = \eta$, then $t_0 = \frac{1}{\gamma_1} \ln \frac{\eta}{z_1^0}$, and

\[
\begin{align*}
    x^1 &= \left(\frac{z_1^0}{\eta}\right)^{1/\gamma_1} x^0 \cos \left[\frac{b}{\gamma_1} \ln \frac{\eta}{z_1^0}\right], \\
    y^1 &= \left(\frac{z_1^0}{\eta}\right)^{1/\gamma_1} x^0 \sin \left[\frac{b}{\gamma_1} \ln \frac{\eta}{z_1^0}\right], \\
    z_2^1 &= \left(\frac{\eta}{z_2^0}\right)^{\gamma_2} z_2^0, \\
    Q^1 &= e^{\lambda R} \vec{Q}_0.
\end{align*}
\]

Let $\vec{Q}_0^*$ and $\vec{Q}_1^*$ be the intersection points of the homoclinic orbit $h_1$ with $\Sigma_0$ and $\Sigma_1$ respectively. The discussion with respect to the other homoclinic orbit $h_2$ is the same. In a small neighborhood of $\vec{Q}_1^*$, the map $P_0^1$ has an approximate representation. By virtue of the fact that the homoclinic orbit $h_1$ is a classical solution, the Well-Posedness Theorem 2.2 implies that $F^t(\vec{Q}_1^*)$ is $C^1$ in $t$. Thus for $\vec{Q}_1$ in a small neighborhood of $\vec{Q}_1^*$,

\[
P_1^0(\vec{Q}_1^*) = P_0^1(\vec{Q}_1^*) + \mathcal{O}(\vec{Q}_1 - \vec{Q}_1^*) + \mathcal{O}(\|\vec{Q}_1 - \vec{Q}_1^*\|^2),
\]

(3.2)
EXISTENCE OF CHAOS FOR A SINGULARLY PERTURBED NLS EQUATION

\[ L(\tilde{Q}^1 - \tilde{Q}^1_\ast) = \partial_t F_t^1(\tilde{Q}^1_\ast) \circ (\tilde{Q}^1 - \tilde{Q}^1_\ast) + \partial_t F_t^1(\tilde{Q}^1_\ast) \circ \partial_{\tilde{Q}^1} F_t^1(\tilde{Q}^1_\ast) \circ (\tilde{Q}^1 - \tilde{Q}^1_\ast), \]

and \( t_1 = t_1(\tilde{Q}^1) \) is defined by the constraint that the \( y \)-coordinate of \( F_t^1(\tilde{Q}^1) \) vanishes,

\[ F_t^1(\tilde{Q}^1) = 0. \]

Thus

\[ \partial_t F_t^1(\tilde{Q}^1_\ast) + \partial_t F_t^1(\tilde{Q}^1_\ast) \partial_{\tilde{Q}^1} F_t^1(\tilde{Q}^1_\ast) = 0, \]

i.e.

\[ (3.3) \]

\[ \frac{\partial t_1}{\partial \tilde{Q}^1}(\tilde{Q}^1_\ast) = -\frac{1}{\partial_t F_t^1(\tilde{Q}^1_\ast)} \partial_{\tilde{Q}^1} F_t^1(\tilde{Q}^1_\ast). \]

Let \( \tilde{x}^0_\ast = \tilde{Q}^0 - \tilde{Q}^0_\ast \), and \( \tilde{x}^1_\ast = \tilde{Q}^1 - \tilde{Q}^1_\ast \), then \( P^1_\ast \) has the approximate representation

\[ (3.4) \]

\[ \begin{pmatrix} \hat{x}^0 \\ \hat{y}^0 \\ \hat{z}^0 \\ \hat{Q}^0 \end{pmatrix} = C \begin{pmatrix} \tilde{x}^1 \\ \tilde{y}^1 \\ \tilde{z}^1 \\ \tilde{Q}^1 \end{pmatrix} + \Xi, \]

where

\[ \Xi \sim O\left((\tilde{x}^1)^2 + (\tilde{y}^1)^2 + (\tilde{z}^1)^2 + \|\tilde{Q}^1\|^2\right), \]

\[ C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & C_{14} \\ c_{21} & c_{22} & c_{23} & C_{24} \\ c_{31} & c_{32} & c_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{pmatrix}, \]

in which \( c_{jl} \) \((j, l = 1, 2, 3)\) are real constants, \( C_{j4} \) \((j = 1, 2, 3, 4)\) and \( C_{ul} \) \((l = 1, 2, 3)\) are linear operators.

4. The Fixed Points of the Poincaré Map \( P \)

As \( t_0 \to +\infty \), to the leading order, the fixed points of \( P \) satisfy

\[ (4.1) \]

\[ \begin{pmatrix} \hat{x}^0 \\ 0 \\ 0 \\ \hat{Q}^0 \end{pmatrix} = C \begin{pmatrix} x^0_0 \cos bt_0 \\ x^0_0 \sin bt_0 \\ \hat{x}^1 \\ \hat{Q}^0 \end{pmatrix}, \]

where

\[ \hat{z}_2 = e^{at_0} \hat{z}_2, \quad \hat{x}^0 = e^{at_0} \hat{x}^0, \quad \hat{Q}^0 = e^{at_0} \hat{Q}^0. \]
Explicitly, the second and the third equations in (4.1) are:

\[ x_0^* \left[ c_{21} \cos bt_0 + c_{22} \sin bt_0 \right] + c_{23} \hat{z}_2^{1} = 0, \]  
(4.2)

\[ x_2^* \left[ c_{31} \cos bt_0 + c_{32} \sin bt_0 \right] + c_{33} \hat{z}_2^{1} = 0. \]

**Lemma 4.1.** \( c_{23} \) and \( c_{33} \) do not vanish simultaneously.

Proof. Notice that \( W^u(\mathcal{Q}) \) is two-dimensional, and intersects \( \Sigma_0 \) (or its extension to \( -\eta < z_1 < \eta \)) into a one-dimensional curve with tangent vector

\[ v = C \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \]

Notice also that for any \( \vec{Q} \in h_1 \),

\[ \dim \{ T_{\vec{Q}}W^u(\mathcal{Q}) \cap T_{\vec{Q}}W^s(\mathcal{Q}) \} = 1, \]

where \( T_{\vec{Q}} \) denotes the tangent space at \( \vec{Q} \). If \( c_{23} \) and \( c_{33} \) vanish simultaneously, then \( v \in T_{\vec{Q}}W^s(\mathcal{Q}) \) which implies that

\[ \dim \{ T_{\vec{Q}}W^u(\mathcal{Q}) \cap T_{\vec{Q}}W^s(\mathcal{Q}) \} = 2 \]

which contradicts (4.3). The lemma is proved. QED

Let

\[ \Delta_1 = c_{21}c_{33} - c_{31}c_{23}, \quad \Delta_2 = c_{22}c_{33} - c_{32}c_{23}. \]

We assume that

- (A2). \( \Delta_1 \) and \( \Delta_2 \) do not vanish simultaneously.

Then (4.2) has infinitely many solutions:

\[ t^{(l)}_0 = \frac{1}{b} [l\pi - \varphi_1], \quad l \in \mathbb{Z}; \]

where

\[ \varphi_1 = \arctan \{ \Delta_1 / \Delta_2 \}. \]

Without loss of generality, we assume \( c_{23} \neq 0 \). Then, solving Eqs. (4.2), we have

\[ z_2^{(1,l)} = -x_0^*[c_{23}]^{-1} \{ c_{21} \cos bt_0^{(l)} + c_{22} \sin bt_0^{(l)} \}. \]

Solving (4.3), we have

\[ \hat{x}^{(0,l)} = x_0^* \left[ c_{11} \cos bt_0^{(l)} + c_{12} \sin bt_0^{(l)} \right] + c_{13} \hat{z}_2^{(1,l)}; \]

\[ \hat{Q}^{(0,l)} = x_0^* \left[ C_{41} \cos bt_0^{(l)} + C_{42} \sin bt_0^{(l)} \right] + C_{43} \hat{z}_2^{(1,l)}. \]

Finally, by the implicit function theorem, there exist infinitely many fixed points of \( P \), which have the approximate expressions given above [1]. Specifically, we have
Theorem 4.2. The Poincaré map $P$ has infinitely many fixed points labeled by $l$ ($l \geq l_0$):

$$t_0 = t_{0,l}, \; \hat{z}^0 = \hat{z}^0_l, \; \hat{Q}^0 = \hat{Q}^0_l, \; \hat{z}^1 = \hat{z}^1_{2,l},$$

where as $l \to +\infty$,

$$t_{0,l} = \frac{1}{b} [l \pi - \varphi_1] + o(1),$$

$$\hat{z}^0_l = \hat{z}^{(0,l)} + o(1),$$

$$\hat{Q}^0_l = \hat{Q}^{(0,l)} + o(1),$$

$$\hat{z}^1_{2,l} = \hat{z}^{(1,l)} + o(1),$$

in which $\hat{z}^{(0,l)}$, $\hat{Q}^{(0,l)}$ and $\hat{z}^{(1,l)}$ are given in \([4,7], [4,3], [4,4]\).

5. Existence of Chaos

One can construct Smale horseshoes in the neighborhoods of the fixed points of $P$.

Definition 5.1. For sufficiently large natural number $l$, we define slab $S_l$ in $\Sigma_0$ as follows:

$$S_l = \left\{ \begin{array}{c} \hat{Q} \in \Sigma_0 \mid \eta \exp\{-\gamma_1(t_{0,2(t+1)} - \frac{\pi}{2b})\} \\ \hat{z}_1^0(\hat{Q}) \leq \eta \exp\{-\gamma_1(t_{0,2l} - \frac{\pi}{2b})\}, \\ \hat{z}_2^0(\hat{Q}) \leq \eta \exp\{-\frac{1}{2} a \cdot t_{0,2l}\}, \\ \hat{z}_2^1(P_0^1(\hat{Q})) \leq \eta \exp\{-\frac{1}{2} a \cdot t_{0,2l}\}, \\ \|\hat{Q}_{1}(P_0^1(\hat{Q}))\| \leq \eta \exp\{-\frac{1}{2} a \cdot t_{0,2l}\} \end{array} \right\},$$

where the notations $\hat{z}_1^0(\hat{Q})$, $\hat{z}_2^1(P_0^1(\hat{Q}))$, etc. denote the $x^0$ coordinate of the point $\hat{Q}$, the $\hat{z}_1^1$ coordinate of the point $P_0^1(\hat{Q})$, etc.

$S_l$ is defined so that it includes two fixed points $p_i^+\;\text{and}\;p_i^-$ of $P$ (Theorem 4.2), $S_l, P_0^1(S_l)$, and $L P_0^1(S_l)$ are illustrated in Figure 4, where $L$ is defined in \([4,2]\).

$\{e_{x^0}, e_{\hat{z}_1^0}, e_{\hat{z}_2^0}, e_{\hat{Q}^0}\}$ denotes the unit vectors along $(\hat{x}^0, \hat{z}_1^0, \hat{z}_2^0, \hat{Q}^0)$-directions in $\Sigma_0$, $\{e_{\hat{z}_1^1}, e_{\hat{y}_1^1}, e_{\hat{z}_2^1}, e_{\hat{Q}_1^1}\}$ denotes the unit vectors along $(\hat{x}_1^1, \hat{y}_1^1, \hat{z}_2^1, \hat{Q}_1^1)$-directions in $\Sigma_1$, and under the linear map $L$, $\{e_{\hat{z}_1^1}, e_{\hat{y}_1^1}, e_{\hat{z}_2^1}, e_{\hat{Q}_1^1}\}$ are mapped into $\{e_{\hat{z}_1^1}, e_{\hat{y}_1^1}, e_{\hat{z}_2^1}, e_{\hat{Q}_1^1}\}$. Let $e_\theta$ be the unit angular vector at $P_0^1(p_i^\pm)$ of the polar coordinate frame on the $(\hat{x}_1^1, \hat{y}_1^1)$-plane. Let $E_\theta = L e_\theta$, and we assume that

- (A3). $\text{Span}\{e_{\hat{x}^0}, e_{\hat{Q}^0}, E_\theta, e_{\hat{z}_2^1}\} = \Sigma_0$.

Let $S_{l,\sigma} = \sigma \circ S_l$ where the symmetry $\sigma$ is defined in \([8,1]\). We need to define a larger slab $\hat{S}_l$ such that $S_l \cup S_{l,\sigma} \subset \hat{S}_l$. 


Figure 2. An illustration of $S_l$, $P^1_0(S_l)$, and $LP^1_0(S_l)$.

Figure 3. (a) shows one of the homoclinic orbits, and (b) shows the blow-up of the neighborhood of the saddle $Q_+$.

Definition 5.2. The larger slab $\hat{S}_l$ is defined as

$$\hat{S}_l = \left\{ \vec{Q} \in \Sigma_0 \mid \eta \exp\{-\gamma_1(t_{0,2(t+1)} - \frac{\pi}{2b})\} \leq z_1^0(\vec{Q}) \leq \eta \exp\{-\gamma_1(t_{0,2t} - \frac{\pi}{2b})\},
\|\vec{Q}\| \leq \eta \exp\{-\frac{1}{2}a t_{0,2t}\} \right\}.$$
where $z^1_{2^*}$ is the $z^1_2$-coordinate of $\tilde{Q}^1_1$.

**Definition 5.3.** In the coordinate system $\{\tilde{x}^0, \tilde{z}^0_1, \tilde{z}^0_2, \tilde{Q}^0\}$, the stable boundary of $\hat{S}_1$, denoted by $\partial_s \hat{S}_1$, is defined to be the boundary of $\hat{S}_1$ along $(\tilde{x}^0, \tilde{Q}^0)$-directions, and the unstable boundary of $\hat{S}_1$, denoted by $\partial_u \hat{S}_1$, is defined to be the boundary of $\hat{S}_1$ along $(\tilde{z}^0_1, \tilde{z}^0_2)$-directions. A stable slice $V$ in $\hat{S}_1$ is a subset of $\hat{S}_1$, defined as the region swept out through homeomorphically moving and deforming $\partial_s \hat{S}_1$ in such a way that the part

$$\partial_s \hat{S}_1 \cap \partial_u \hat{S}_1$$

of $\partial_s \hat{S}_1$ only moves and deforms inside $\partial_u \hat{S}_1$. The new boundary obtained through such moving and deforming of $\partial_s \hat{S}_1$ is called the stable boundary of $V$, which is denoted by $\partial_s V$. The rest of the boundary of $V$ is called its unstable boundary, which is denoted by $\partial_u V$. An unstable slice of $\hat{S}_1$, denoted by $H$, is defined similarly.

As shown in [1], under the assumption (A3), when $l$ is sufficiently large, $P(\hat{S}_i)$ and $P(\hat{S}_i)$ intersect $\hat{S}_1$ into four disjoint stable slices $\{V_1, V_2\}$ and $\{V_{-1}, V_{-2}\}$ in $\hat{S}_i$. $V_j$'s ($j = 1, 2, -1, -2$) do not intersect $\partial_s \hat{S}_1$; moreover,

$$\partial_s V_i \subset P(\partial_s \hat{S}_1), (i = 1, 2); \partial_s V_i \subset P(\partial_s \hat{S}_1), (i = -1, -2).$$

Let

$$H_j = P^{-1}(V_j), \quad (j = 1, 2, -1, -2),$$

where and for the rest of this article, $P^{-1}$ denotes preimage of $P$. Then $H_j$ ($j = 1, 2, -1, -2$) are unstable slices. More importantly, the Conley-Moser conditions are satisfied as shown in [1]. Specifically, Conley-Moser conditions are:

**Conley-Moser condition (i):**

$$\begin{align*}
V_j &= P(H_j),
\partial_s V_j &= P(\partial_s H_j), \quad (j = 1, 2, -1, -2)
\partial_s V_j &= P(\partial_s H_j).
\end{align*}$$

**Conley-Moser condition (ii):** There exists a constant $0 < \nu < 1$, such that for any stable slice $V \subset V_j$ ($j = 1, 2, -1, -2$), the diameter decay relation

$$d(\tilde{V}) \leq \nu d(V)$$

holds, where $d(\cdot)$ denotes the diameter [1], and $\tilde{V} = P(V \cap H_k), \quad (k = 1, 2, -1, -2)$; for any unstable slice $H \subset H_j$ ($j = 1, 2, -1, -2$), the diameter decay relation

$$d(\tilde{H}) \leq \nu d(H)$$

holds, where $\tilde{H} = P^{-1}(H \cap V_k), \quad (k = 1, 2, -1, -2)$.

The Conley-Moser conditions are sufficient conditions for establishing the topological conjugacy between the Poincare map $P$ restricted to a Cantor set in $\Sigma_0$, and the shift automorphism on symbols.

Let $W$ be a set which consists of elements of the doubly infinite sequence form:

$$a = (\cdots a_{-2} a_{-1} a_0, a_1 a_2 \cdots),$$

where $a_k \in \{1, 2, -1, -2\}; \quad k \in Z$. We introduce a topology in $W$ by taking as neighborhood basis of

$$a^* = (\cdots a_{-2}^* a_{-1}^* a_0^*, a_1^* a_2^* \cdots),$$
the set
\[ W_j = \left\{ a \in \mathcal{W} \mid a_k = a_k^* \ (|k| < j) \right\} \]
for \( j = 1, 2, \cdots \). This makes \( \mathcal{W} \) a topological space. The shift automorphism \( \chi \) is defined on \( \mathcal{W} \) by
\[ \chi : \mathcal{W} \rightarrow \mathcal{W}, \quad \forall a \in \mathcal{W}, \ \chi(a) = b, \text{ where } b_k = a_{k+1}. \]
The shift automorphism \( \chi \) exhibits sensitive dependence on initial conditions, which is a hallmark of chaos.

Let
\[ a = (\cdots a_{-2}a_{-1}a_0, a_1a_2 \cdots), \]
be any element of \( \mathcal{W} \). Define inductively for \( k \geq 2 \) the stable slices
\[ V_{a_0a_{-1}} = P(H_{a_{-1}}) \cap H_{a_0}, \]
\[ V_{a_0a_{-1}\cdots a_{-k}} = P(V_{a_{-1}\cdots a_{-k}}) \cap H_{a_0}. \]
By Conley-Moser condition (ii),
\[ d(V_{a_0a_{-1}\cdots a_{-k}}) \leq \nu_1 d(V_{a_0a_{-1}\cdots a_{-(k-1)}}) \leq \cdots \leq \nu_{k-1} d(V_{a_0a_{-1}}). \]
Then,
\[ V(a) = \bigcap_{k=1}^{\infty} V_{a_0a_{-1}\cdots a_{-k}} \]
defines a 2 dimensional continuous surface in \( \Sigma_0 \); moreover,
\[ (5.3) \quad \partial V(a) \subset \partial_u S_{\lambda}. \]
Similarly, define inductively for \( k \geq 1 \) the unstable slices
\[ H_{a_0a_1} = P^{-1}(H_{a_1} \cap V_{a_0}), \]
\[ H_{a_0a_1\cdots a_k} = P^{-1}(H_{a_1\cdots a_k} \cap V_{a_0}). \]
By Conley-Moser condition (ii),
\[ d(H_{a_0a_1\cdots a_k}) \leq \nu_2 d(H_{a_0a_1\cdots a_{k-1}}) \leq \cdots \leq \nu_k d(H_{a_0}). \]
Then,
\[ H(a) = \bigcap_{k=0}^{\infty} H_{a_0a_1\cdots a_k} \]
defines a codimension 2 continuous surface in \( \Sigma_0 \); moreover,
\[ (5.4) \quad \partial H(a) \subset \partial_s \tilde{S}_{\lambda}. \]
By \[ (5.3, 5.4) \] and dimension count,
\[ V(a) \cap H(a) \neq \emptyset \]
consists of points. Let
\[ p \in V(a) \cap H(a) \]
be any point in the intersection set. Now we define the mapping
\[ \phi : \mathcal{W} \mapsto \hat{S}_t, \]
\[ \phi(a) = p. \]
By the above construction,
\[ P(p) = \phi(\chi(a)). \]
That is,
\[ P \circ \phi = \phi \circ \chi. \]
Let
\[ \Lambda \equiv \phi(\mathcal{W}), \]
then \( \Lambda \) is a compact Cantor subset of \( \hat{S}_t \), and invariant under the Poincare map \( P \). Moreover, with the topology inherited from \( \hat{S}_t \) for \( \Lambda \), \( \phi \) is a homeomorphism from \( \mathcal{W} \) to \( \Lambda \). Thus we have the theorem.

**Theorem 5.4 (Horseshoe Theorem).** Under the generic assumptions (A1)-(A3) for the perturbed nonlinear Schrödinger system [1,4], there exists a compact Cantor subset \( \Lambda \) of \( \hat{S}_t \), \( \Lambda \) consists of points, and is invariant under \( P \). \( P \) restricted to \( \Lambda \), is topologically conjugate to the shift automorphism \( \chi \) on four symbols \( 1, 2, -1, -2 \). That is, there exists a homeomorphism
\[ \phi : \mathcal{W} \mapsto \Lambda, \]
such that the following diagram commutes:
\[
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\phi} & \Lambda \\
\chi \downarrow & & \downarrow P \\
\mathcal{W} & \xrightarrow{\phi} & \Lambda
\end{array}
\]

6. Numerical Evidence for the Generic Assumptions

6.1. Generic Assumption (A1). Figure 3 shows a numerical result of Mark Winograd. It indicates that the homoclinic orbits are indeed tangent to the \((x, y)\)-plane as \( t \to +\infty \). Specifically, Figure 3 (a) shows one of the homoclinic orbits, and Figure 3 (b) shows the blow-up of the neighborhood of the saddle \( Q_\epsilon \).

6.2. Generic Assumptions (A2) and (A3). Numerical simulation of the generic assumptions (A2) and (A3) is planned for a future work.

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