Improper interval edge colorings of graphs

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A $k$-improper edge coloring of a graph $G$ is a mapping $\alpha : E(G) \rightarrow \mathbb{N}$ such that at most $k$ edges of $G$ with a common endpoint have the same color. An improper edge coloring of a graph $G$ is called an improper interval edge coloring if the colors of the edges incident to each vertex of $G$ form an integral interval. In this paper we introduce and investigate a new notion, the interval coloring impropriety (or just impropriety) of a graph $G$ defined as the smallest $k$ such that $G$ has a $k$-improper interval edge coloring; we denote the smallest such $k$ by $\mu_{int}(G)$. We prove upper bounds on $\mu_{int}(G)$ for general graphs $G$ and for particular families such as bipartite, complete multipartite and outerplanar graphs; we also determine $\mu_{int}(G)$ exactly for $G$ belonging to some particular classes of graphs. Furthermore, we provide several families of graphs with large impropriety; in particular, we prove that for each positive integer $k$, there exists a graph $G$ with $\mu_{int}(G) = k$. Finally, for graphs with at least two vertices we prove a new upper bound on the number of colors used in an improper interval edge coloring.

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1. Introduction

A proper edge coloring of a graph $G$ is called an interval $t$-coloring if exactly $t$ colors appear on the edges of $G$ and the colors of the edges incident to every vertex $v$ of $G$ form an interval of integers. This notion was introduced by Asratian and Kamalian [3] (available in English as [4]), motivated by the problem of constructing timetables without “gaps” for teachers and classes. Generally, it is an NP-complete problem to determine whether a bipartite graph has an interval coloring [25]. However some classes of graphs have been proved to admit interval colorings; it is known, for example, that trees, regular and complete bipartite graphs [3,15,20], bipartite graphs with maximum degree at most three [15], doubly convex bipartite graphs [2,21], grids [11], and outerplanar bipartite graphs [12] have interval colorings. Additionally, all $(2, b)$-biregular graphs [15,16,22] and $(3, 6)$-biregular graphs [6] admit interval colorings, where an $(a, b)$-biregular graph is a bipartite graph where the vertices in one part all have degree $a$ and the vertices in the other part all have degree $b$. By contrast, it is unknown whether all $(3, 4)$-biregular graphs have interval colorings, despite efforts by various authors (see e.g. [6]).

Improper (or defective) colorings were first considered independently by Andrews and Jacobson [1], Harary and Jones [17], and Cowen et al. [7]. This coloring model is a well-known generalization of ordinary graph coloring with applications in various scheduling and assignment problems, see e.g. the recent survey [26], or [7].

Motivated by scheduling and assignment problems with compactness requirements, but where a certain degree of conflict is acceptable, we consider improper interval edge colorings in this paper. An improper edge coloring of a graph is
called an improper interval (edge) coloring if the colors on the edges incident to every vertex of the graph form a set of consecutive integers. This edge coloring model seems to have been first considered by Hudak et al. [18], although their investigation has a different focus than ours.

Note that unlike the case for interval colorings, every graph trivially has an improper interval edge coloring. An improper interval coloring is \( k \)-improper if at most \( k \) edges with a common endpoint have the same color. We define \( \mu_{int}(G) \) to be the smallest \( k \) such that \( G \) has a \( k \)-improper interval edge coloring. We call the parameter \( \mu_{int}(G) \) the interval coloring impropriety (or just impropriety) of \( G \).

Improper interval edge colorings have immediate applications in scheduling problems, where an optimal schedule without waiting periods or idle times is desirable, but a certain level of conflict is allowed. For a bipartite graph \( G \), representing a scheduling problem, the parameter \( \mu_{int}(G) \) has a natural interpretation as the minimum degree of conflict necessary in a schedule with no waiting periods. Moreover, in view of the fact that not every graph has an interval coloring, the parameter \( \mu_{int}(G) \) may be viewed as a natural measure of how far from being interval colorable a graph is.

Trivially, if \( G \) has an interval coloring, then \( \mu_{int}(G) = 1 \). Thus, determining \( \mu_{int}(G) \) for a given graph \( G \) is an intractable problem. Moreover, given the relatively few positive results on graphs admitting interval colorings, the parameter \( \mu_{int} \) appears to be a quite difficult graph invariant.

The purpose of this paper is to initiate the investigation of the impropriety of graphs. We provide several families of graphs with large impropriety; in particular, we prove that for each positive integer \( k \), there is a graph \( G \) with \( \mu_{int}(G) = k \). A related interesting open question is to determine, for any \( k \), the least integer \( \Delta \) such that there is a graph with maximum degree \( \Delta \) satisfying \( \mu_{int}(G) = k \); this question is open even for the case \( k = 3 \) (see Problem 4.5).

We prove general upper bounds on \( \mu_{int}(G) \) and determine \( \mu_{int}(G) \) exactly for some families of graphs \( G \); in particular we prove that

- \( \mu_{int}(G) \leq 2 \) if \( \Delta(G) \leq 5 \), and
  \[
  \mu_{int}(G) \leq \min \left\{ 2 \left\lceil \frac{\Delta(G)}{\delta(G)} \right\rceil, \frac{\Delta(G)}{2} \right\},
  \]
  for any graph \( G \) with \( \Delta(G) \geq 6 \), where \( \Delta(G) \) and \( \delta(G) \) denote the maximum and minimum degree of a graph \( G \), respectively;
- \( \mu_{int}(G) \leq \left\lceil \frac{\Delta(G)}{4} \right\rceil \) if \( G \) is a bipartite graph and has no vertices of degree three, and
  \[
  \mu_{int}(G) \leq \min \left\{ \frac{\Delta(G)}{\delta(G)}, \left\lceil \frac{\Delta(G)}{3} \right\rceil \right\},
  \]
  for any bipartite graph \( G \);
- \( \mu_{int}(G) \leq \left\lceil \frac{\Delta(G)}{\delta(G)} \right\rceil \) if \( G \) is a complete \( r \)-partite graph.

Furthermore, we conjecture that outerplanar graphs have impropriety at most 2 and we prove this conjecture for graphs with maximum degree at most 8. Finally, we consider the number of colors in an improper interval edge coloring and obtain a new upper bound on the number of colors used in such a coloring.

2. Preliminaries

The degree of a vertex \( v \) of a graph \( G \) is denoted by \( d_G(v) \). \( \Delta(G) \) and \( \delta(G) \) denote the maximum and minimum degrees of \( G \), respectively. For two positive integers \( a \) and \( b \) with \( a \leq b \), we denote by \( [a, b] \) the interval of integers \( \{a, \ldots, b\} \).

We shall need a classic result from factor theory. A 2-factor of a multigraph \( G \) (where loops are allowed) is a 2-regular spanning subgraph of \( G \).

**Theorem 2.1** (Petersen’s Theorem). Let \( G \) be a 2-regular multigraph (where loops are allowed). Then \( G \) has a decomposition into edge-disjoint 2-factors.

If \( \alpha \) is an edge coloring of \( G \) and \( v \in V(G) \), then \( S_G(v, \alpha) \) (or \( S(v, \alpha) \)) denotes the set of colors appearing on edges incident to \( v \); the smallest and largest colors of the spectrum \( S(v, \alpha) \) are denoted by \( \underline{S}(v, \alpha) \) and \( \overline{S}(v, \alpha) \), respectively.

The chromatic index \( \chi'(G) \) of a graph \( G \) is the minimum number \( t \) for which there exists a proper \( t \)-edge coloring of \( G \).

**Theorem 2.2** (Vizing’s Theorem). For any graph \( G \), \( \chi'(G) = \Delta(G) \) or \( \chi'(G) = \Delta(G) + 1 \).

A graph \( G \) is said to be Class 1 if \( \chi'(G) = \Delta(G) \), and Class 2 if \( \chi'(G) = \Delta(G) + 1 \). The next result gives a sufficient condition for a graph to be Class 1 (see, for example, [10]).

**Theorem 2.3**. If \( G \) is a graph where no two vertices of maximum degree are adjacent, then \( G \) is Class 1.

Every bipartite graph is Class 1, as the following well-known proposition, known as König’s edge coloring theorem, states.
Theorem 2.4 (König’s Edge Coloring Theorem). If G is bipartite, then \( \chi'(G) = \Delta(G) \).

We shall also need some preliminary results on interval edge colorings. The following result was proved by Hansen [15].

Theorem 2.5. If G is a bipartite graph with maximum degree \( \Delta(G) \leq 3 \), then G has an interval coloring.

Finally, we need the notion of a projective plane.

Definition 2.6. A finite projective plane \( \pi(n) \) of order \( n \) (\( n \geq 2 \)) has \( n^2 + n + 1 \) points and \( n^2 + n + 1 \) lines, and satisfies the following properties:

- **P1** any two points determine a line;
- **P2** any two lines determine a point;
- **P3** every point is incident to \( n + 1 \) lines;
- **P4** every line is incident to \( n + 1 \) points.

3. Improper interval edge colorings of some non-interval-colorable graphs

In this section we determine the impropriety of some well-known families of graphs that in general do not admit interval colorings; in particular we describe constructions of bipartite graphs with arbitrarily large impropriety.

3.1. The impropriety of some non-interval-colorable graphs

Regular Class 1 graphs are trivially interval colorable, while no Class 2 graphs are [3,5,14]; however, all regular graphs have small impropriety.

Proposition 3.1. If G is a regular graph, then

\[
\mu_{\text{int}}(G) = \begin{cases} 
1, & \text{if } G \text{ is Class 1}, \\
2, & \text{if } G \text{ is Class 2}.
\end{cases}
\]

Proof. Let G be a regular graph. It is well-known that G is interval colorable if and only if G is Class 1. Hence, it suffices to prove that \( \mu_{\text{int}}(G) \leq 2 \); we shall give an explicit 2-improper interval coloring of G.

Suppose first that the vertex degrees of G are even, say \( d_G(v) = 2k \) for every vertex \( v \in V(G) \). By Petersen’s theorem G has a decomposition into 2-factors \( F_1, \ldots, F_k \). By coloring all edges of \( F_i \) by color \( i \), \( i = 1, \ldots, k \), we obtain a 2-improper interval coloring of G.

Suppose now that \( d_G(v) = 2k - 1 \) for all \( v \in V(G) \). By taking two copies \( G_1 \) and \( G_2 \) of G and adding an edge between corresponding vertices of \( G_1 \) and \( G_2 \), we obtain a 2k-regular supergraph \( H \). By the preceding paragraph, \( H \) has a 2-improper interval coloring. Moreover, every vertex of \( H \) is incident to two edges of every color. Thus, by taking the restriction of this coloring to \( G_1 \), it follows that \( \mu_{\text{int}}(G) \leq 2 \). \( \square \)

Note that Proposition 3.1 implies that for cycles \( C_n \) (\( n \geq 3 \)) and complete graphs \( K_n \) it holds that

\[
\mu_{\text{int}}(C_n) = \mu_{\text{int}}(K_n) = \begin{cases} 
1, & \text{if } n \text{ is even}, \\
2, & \text{if } n \text{ is odd}.
\end{cases}
\]

Next, we consider generalizations of two families of bipartite graphs with no interval colorings introduced by Giaro et al. [13]. For any \( a, b, c \in \mathbb{N} \), define the graph \( S_{a,b,c} \) as follows:

\[
V(S_{a,b,c}) = \{u_0, u_1, u_2, u_3, v_1, v_2, v_3\} \cup \{x_1, \ldots, x_a, y_1, \ldots, y_b, z_1, \ldots, z_c\}
\]

and

\[
E(S_{a,b,c}) = \{u_0u_1, u_1u_2, u_2u_3, v_1v_2, v_2v_3, u_3u_1\} \cup \{u_0x_i, u_1x_i : 1 \leq i \leq a\}
\]

\[
\cup \{u_0y_j, u_2y_j : 1 \leq j \leq b\} \cup \{u_0z_k, u_3z_k : 1 \leq k \leq c\}.
\]

Fig. 1 shows the graph \( S_{7,7,7} \).

Next, we define a family of graphs \( M_{a,b,c} \) (\( a, b, c \in \mathbb{N} \)). We set

\[
V(M_{a,b,c}) = \{u_0, u_1, u_2, u_3\} \cup \{x_1, \ldots, x_a, y_1, \ldots, y_b, z_1, \ldots, z_c\}
\]

and

\[
E(M_{a,b,c}) = \{u_0u_1, u_1u_2, u_2u_3 : 1 \leq i \leq a\} \cup \{u_0y_j, u_2y_j, u_3y_j : 1 \leq j \leq b\}
\]

\[
\cup \{u_0z_k, u_3z_k, u_1z_k : 1 \leq k \leq c\}.
\]

Fig. 2 shows the graph \( M_{5,5,5} \).

Clearly, \( S_{a,b,c} \) and \( M_{a,b,c} \) are connected bipartite graphs. Giaro et al. [13] showed that the graphs \( S_k = S_{k,k,k} \) and \( M_l = M_{l,l,l} \) do not admit interval colorings if \( k \geq 7 \), and \( l \geq 5 \), respectively; in fact they proved that graphs from these
families can have arbitrarily large so-called deficiency; the deficiency of a graph $G$ is the minimum number of pendant edges whose addition to $G$ yields a graph with an interval coloring. Thus, the deficiency of a graph is another measure of how far from being interval colorable a graph is.

Here we shall prove that all graphs in the families $\{S_{a,b,c}\}$ and $\{M_{a,b,c}\}$, despite having large deficiency, satisfy that $\mu_{\text{int}}(S_{a,b,c}) \leq 2$ and $\mu_{\text{int}}(M_{a,b,c}) \leq 2$, respectively.

**Theorem 3.2.** For any $a, b, c \in \mathbb{N}$, $\mu_{\text{int}}(S_{a,b,c}) \leq 2$ and $\mu_{\text{int}}(M_{a,b,c}) \leq 2$.

**Proof.** Without loss of generality, we may assume that $a \leq b \leq c$.

We first construct an edge coloring $\alpha$ of the graph $S_{a,b,c}$. We define this coloring as follows:

1. for $1 \leq i \leq a$, let $\alpha(u_0x_i) = \alpha(u_1x_i) = i$;
2. for $1 \leq j \leq b$, let $\alpha(u_0y_j) = \alpha(u_2y_j) = j$;
(3) for $1 \leq k \leq c$, let $\alpha(u_0z_k) = \alpha(u_2z_k) = a + k$;
(4) $\alpha(u_1v_1) = \alpha(u_1v_2) = 1$, $\alpha(u_2v_2) = b + 1$, $\alpha(v_2u_3) = b + 2$, $\alpha(u_3v_3) = a$, $\alpha(v_3u_1) = a + 1$.

It is straightforward that $\alpha$ is a 2-improper interval coloring of $S_{a,b,c}$.

Next we define an edge coloring $\beta$ of the graph $M_{a,b,c}$ as follows:

(1') for $1 \leq i \leq a$, let $\beta(u_0x_i) = i$;
(2') for $1 \leq i \leq a$, let $\beta(u_1x_i) = \beta(u_2x_i) = i + 1$;
(3') for $1 \leq j \leq b$, let $\beta(u_0y_j) = j$;
(4') for $1 \leq j \leq b$, let $\beta(u_2y_j) = \beta(u_3y_j) = j + 1$;
(5') for $1 \leq k \leq c$, let $\beta(u_0z_k) = a + k$;
(6') for $1 \leq k \leq c$, let $\beta(u_2z_k) = \beta(u_1z_k) = a + k + 1$.

It is easy to verify that $\beta$ is a 2-improper interval coloring of $M_{a,b,c}$. We conclude that $\mu_{int}(S_{a,b,c}) \leq 2$ and $\mu_{int}(M_{a,b,c}) \leq 2$. □

Lastly, let us consider two elementary classes of graphs that have been proved not to always admit interval colorings. The join of two disjoint graphs $G$ and $H$ is the graph obtained from $G$ and $H$ by adding all possible edges between $V(G)$ and $V(H)$. Recall that a wheel graph $W_n$ on $n$ vertices ($n \geq 4$) is defined as the join of $C_{n-1}$ and $K_1$. It is well-known that only few wheels are interval colorable, but they all have small impropriety (which in fact is implicit in [18]).

**Proposition 3.3.** If $W_n$ is a wheel graph on $n$ vertices, then

$$
\mu_{int}(W_n) = \begin{cases} 
1, & \text{if } n = 4, 7, 10, \\
2, & \text{otherwise.}
\end{cases}
$$

**Proof.** Let $W_n$ be a wheel graph. In [5,14], it was shown that $W_n$ has an interval coloring if and only if $n = 4, 7$ or 10. Hence, it suffices to prove that $\mu_{int}(W_n) \leq 2$; this follows from a result in [18]: in fact the improper interval $(n-1)$-coloring of $W_n$ described in the proof of Theorem 2.8 in [18] is a 2-improper interval coloring of $W_n$. □

In [8], the authors considered the problem of constructing interval edge colorings of so-called generalized $\theta$-graphs; a generalized $\theta$-graph, denoted by $\theta_m$, is a graph consisting of two vertices $u$ and $v$ together with $m$ internally-disjoint $(u, v)$-paths, where $2 \leq m < \infty$. These graphs also have small impropriety.

**Proposition 3.4.** For any $m \geq 2$,

$$
\mu_{int}(\theta_m) = \begin{cases} 
1, & \text{if } \theta_m \text{ is not an Eulerian graph with an odd number of edges}, \\
2, & \text{otherwise.}
\end{cases}
$$

**Proof.** In [8], it was proved that $\theta_m$ has an interval coloring if and only if it is not an Eulerian graph with an odd number of edges. Hence, it suffices to prove that $\mu_{int}(\theta_m) \leq 2$; for $i = 1, \ldots, m$, we color all edges of the $i$th path between $u$ and $v$ by color $i$. Thus, trivially $\mu_{int}(\theta_m) \leq 2$. □

### 3.2. Graphs with large impropriety

In this section we describe several families of graphs with large impropriety. We begin our considerations with constructions based on subdivisions.

Let $G$ be a graph and $V(G) = \{v_1, \ldots, v_n\}$. Define graphs $S(G)$ and $\widehat{G}$ as follows:

$$
V(S(G)) = \{v_1, \ldots, v_n\} \cup \{w_{ij} : v_i v_j \in E(G)\},
$$
$$
E(S(G)) = \{v_i w_{ij}, v_j w_{ij} : v_i v_j \in E(G)\},
$$
$$
V(\widehat{G}) = V(S(G)) \cup \{u\}, u \notin V(S(G)), E(\widehat{G}) = E(S(G)) \cup \{uw_{ij} : v_i v_j \in E(G)\}.
$$

In other words, $S(G)$ is the graph obtained by subdividing every edge of $G$, and $\widehat{G}$ is the graph obtained from $S(G)$ by connecting every inserted vertex to a new vertex $u$. Note that $S(G)$ and $G$ are bipartite graphs.

**Theorem 3.5.** If $G$ is a connected graph and

$$
|E(G)| > k \left(1 + \max_{P \in P} \sum_{v \in V(P)} (d_{\widehat{G}}(v) - 1)\right),
$$

where $P$ is the set of all shortest paths in $S(G)$ connecting vertices $w_{ij}$, then $\mu_{int}(\widehat{G}) > k$.  

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Suppose, to the contrary, that $\widehat{G}$ has a $k$-improper interval $t$-coloring $\alpha$; then $t \geq \frac{|E(G)|}{k}$, because there is a vertex $u$ in $\widehat{G}$ that is adjacent to all vertices in $V(S(G)) \setminus V(G)$.

Consider the vertex $u$, and let $w$ and $w'$ be two vertices adjacent to $u$ satisfying that $\alpha(uw) = S(u, \alpha) = s$ and $\alpha(uw') = S(u, \alpha) \geq s + \frac{|E(G)|}{k} - 1$. Since $\widehat{G} - u$ is connected, there is a shortest path $P(w, w')$ in $\widehat{G} - u$ joining $w$ with $w'$, where

$$P(w, w') = x_1e_1x_2 \ldots x_ie_{i+1} \ldots x_je_jx_{j+1}$$

and $x_1 = w, x_{r+1} = w'$. 

Note that

$$\alpha(x_ix_{i+1}) \leq s + \sum_{j=1}^{r} (d_G(x_j) - 1) \text{ for } 1 \leq i \leq r$$

and

$$\alpha(x_{r+1}u) = \alpha(u'x) \leq s + \sum_{j=1}^{r+1} (d_G(x_j) - 1).$$

Hence

$$s + \frac{|E(G)|}{k} - 1 \leq S(u, \alpha) = \alpha(uw') \leq s + \sum_{j=1}^{r+1} (d_G(x_j) - 1) \leq s + \max_{p \in P} \sum_{v \in V(P)} (d_G(v) - 1)$$

and thus

$$|E(G)| \leq k \left( 1 + \max_{p \in P} \sum_{v \in V(P)} (d_G(v) - 1) \right),$$

which is a contradiction. □

**Theorem 3.5** can be used for obtaining infinite families of graphs with arbitrarily large impropriety; indeed, we have the following.

**Corollary 3.6.** If $n^2 - n > 2k(2n + 3)$, then $\mu_{\text{int}}(\widehat{K}_n) > k$.

**Corollary 3.7.** If $mn > k(m + n + 5)$, then $\mu_{\text{int}}(\widehat{K}_{m,n}) > k$.

Our next construction uses techniques first described in [24] and generalizes the family of so-called Hertz graphs first described in [13].

Let $T$ be a tree and let $P$ be the set of all paths in $T$. We set

$$F(T) = \{ v \in V(T) : d_T(v) = 1 \},$$

and define $M(T)$ as follows:

$$M(T) = \max_{p \in P} \{|E(P)| + |\{uw \in E(T) : u \in V(P), w \not\in V(P)\}|\}.$$

Thus $F(T)$ is the set of all leaves in $T$, and $M(T)$ is the maximum number of edges with at least one endpoint in a single path in $T$.

Now let us define the graph $\overline{\widehat{T}}$ as follows:

$$V(\overline{\widehat{T}}) = V(T) \cup \{ u \}, \ u \not\in V(T), \ E(\overline{\widehat{T}}) = E(T) \cup \{ uv : v \in F(T) \}.$$ 

Clearly, $\overline{\widehat{T}}$ is a connected graph with $\Delta(\overline{\widehat{T}}) = |F(T)|$. Moreover, if $T$ is a tree in which the distance between any two leaves is even, then $\overline{\widehat{T}}$ is a connected bipartite graph.

**Theorem 3.8.** If $T$ is a tree and $|F(T)| > k(M(T) + 2)$, then $\mu_{\text{int}}(\overline{\widehat{T}}) > k$.

**Proof.** Suppose, to the contrary, that $\overline{\widehat{T}}$ has a $k$-improper interval $t$-coloring $\alpha$ for some $t \geq \frac{|F(T)|}{k}$.

Consider the vertex $u$. Let $v$ and $v'$ be two vertices adjacent to $u$ such that $\alpha(uv) = S(u, \alpha) = s$ and $\alpha(uv') = S(u, \alpha) \geq s + \frac{|F(T)|}{k} - 1$. Since $\overline{T} - u$ is a tree, there is a unique path $P(v, v')$ in $\overline{T} - u$ joining $v$ with $v'$, where

$$P(v, v') = x_1e_1x_2 \ldots x_ie_{i+1} \ldots x_je_{r+1}$$

and $x_1 = v, x_{r+1} = v'. $

Note that

$$\alpha(x_ix_{i+1}) \leq s + \sum_{j=1}^{r} (d_T(x_j) - 1) \text{ for } 1 \leq i \leq r.$$
From this, we have
\[\alpha(x_i, x_{i+1}) = \alpha(x_i, v') \leq s + 1 + \sum_{j=1}^{r}(d_j(x_j) - 1) \leq s + M(T).\]

Hence
\[s + \frac{|F(T)|}{k} - 1 \leq S(u, \alpha) = \alpha(uv') \leq s + 1 + M(T),\]
and thus \(|F(T)| \leq k (M(T) + 2)\), which is a contradiction. \(\square\)

**Corollary 3.9.** If \(T\) is a tree in which the distance between any two leaves is even and \(|F(T)| > k (M(T) + 2)\), then the bipartite graph \(\tilde{T}\) has no \(k\)-improper interval coloring.

Recall that the deficiency of a graph \(G\) is the minimum number of pendant edges whose addition to \(G\) yields a graph with an interval coloring. As mentioned above, our constructions by trees generalize the so-called Hertz’s graphs \(H_{p,q}\), first described in [13]. Hertz’s graphs are known to have a high deficiency, so let us specifically consider the impropriety of such graphs.

In [13] the Hertz’s graph \(H_{p,q}(p, q \geq 2)\) was defined as follows:
\[V(H_{p,q}) = \{a, b_1, b_2, \ldots, b_p, d\} \cup \{c_{ij}^{(i)} : 1 \leq i \leq p, 1 \leq j \leq q\}\]
\[E(H_{p,q}) = E_1 \cup E_2 \cup E_3,\]
where
\[E_1 = \{ab_i : 1 \leq i \leq p\}, E_2 = \{b_i c_{ij}^{(i)} : 1 \leq i \leq p, 1 \leq j \leq q\}\]
\[E_3 = \{c_{ij}^{(i)}d : 1 \leq i \leq p, 1 \leq j \leq q\}.\]

The graph \(H_{p,q}\) is bipartite with maximum degree \(\Delta(H_{p,q}) = pq\) and \(|V(H_{p,q})| = pq + p + 2\). We are now able to prove the following result; our main result of this section.

**Theorem 3.10.** For any \(k \in \mathbb{N}\), there exists a bipartite graph \(G\) such that \(\mu_{\text{int}}(G) = k\).

**Proof.** For a given \(k\), choose \(p\) so that \(p \geq 2k^2 - 1\). Let us consider the tree \(T = H_{p,k} - d\). Note that \(M(T) = p + 2k\), by e.g. taking a path between two vertices \(b_i\) and \(b_j\), and \(|F(T)| = pk\). Moreover, since the graph \(H_{p,k}\) is isomorphic to \(T\), by Theorem 3.8, we obtain that \(\mu_{\text{int}}(H_{p,k}) > k - 1\). On the other hand, let us define an edge coloring \(\alpha\) of \(H_{p,k}\) as follows:

1. For \(1 \leq i \leq p\), let \(\alpha(ab_i) = i + 1\);
2. For \(1 \leq i \leq p\) and \(1 \leq j \leq k\), let \(\alpha(b_i c_j^{(i)}) = \alpha(c_j^{(i)}d) = i\).

It is easy to verify that \(\alpha\) is a \(k\)-improper interval coloring of \(H_{p,k}\); thus \(\mu_{\text{int}}(H_{p,k}) \leq k\). \(\square\)

In the last part of this section we use finite projective planes for constructing bipartite graphs with large impropriety. This family of graphs was first described in [24]. Let \(\pi(n)\) be a finite projective plane of order \(n \geq 2\), \(\{1, 2, \ldots, n^2 + n + 1\}\) be the set of points and \(L = \{l_1, l_2, \ldots, l_{n^2+n+1}\}\) the set of lines of \(\pi(n)\). Let \(A_i = \{k \in l_i : 1 \leq k \leq n^2 + n + 1\}\) for every \(1 \leq i \leq n^2 + n + 1\); then \(|A_i| = n + 1\) for every \(i\), and \(A_i \neq A_j\) if \(i \neq j\). For a sequence of \(n^2 + n + 1\) integers \(r_1, r_2, \ldots, r_{n^2+n+1} \in \mathbb{N}\) \((r_1 \geq \cdots \geq r_{n^2+n+1} \geq 1)\), we define the
Theorem 3.11. If $\sum_{i=2}^{n+1} r_i - (k-1)\sum_{i=1}^{n+1} r_i > 2k(n+1)$, then $\mu_{\text{int}}(\text{Erd}(r_1, \ldots, r_{n^2+n+1})) > k$. 

Proof. Suppose, to the contrary, that the graph $G = \text{Erd}(r_1, \ldots, r_{n^2+n+1})$ has a $k$-improper interval $t$-coloring $\alpha$ for some $t \geq \frac{n^2+n+1}{k}$. 

Consider the vertex $u$ of $G$, and let $v_{p}^{(l_0)}$ and $v_{q}^{(l_0)}$ be two vertices adjacent to $u$ such that $\alpha(v_{p}^{(l_0)}) = \alpha(v_{q}^{(l_0)}) = S(u, \alpha) = s$ and 

$$\alpha(v_{p}^{(l_0)}) = \alpha(v_{q}^{(l_0)}) = S(u, \alpha) = s + \frac{n^2+n+1}{k} - 1.$$ 

If $l_0 = l_0'$, then, by the construction of $G$ there exists $k_0$ such that $k_0v_{p}^{(l_0)}$, $k_0v_{q}^{(l_0)} \in E(G)$. If, on the other hand $l_0 \neq l_0'$, then $l_0 \cap l_0' = \emptyset$ (by property P2); so again, by the construction of $G$, there exists $k_0$ such that $k_0v_{p}^{(l_0)}$, $k_0v_{q}^{(l_0)} \in E(G)$.

Now, we have $d(v_{p}^{(l_0)}) = d(v_{q}^{(l_0)}) = n+2$ and

$$\alpha(k_0v_{p}^{(l_0)}) \leq s + d(v_{p}^{(l_0)}) - 1 = s + n + 1,$$

and thus

$$\alpha(k_0v_{q}^{(l_0)}) \leq s + n + 1 + d(k_0) - 1 \leq s + n + \sum_{i=1}^{n+1} r_i.$$ 

This implies that

$$\sum_{i=1}^{n+1} r_i \leq \frac{n^2+n+1}{k} - 1 \leq \alpha(v_{q}^{(l_0)}) \leq s + n + \sum_{i=1}^{n+1} r_i + d(v_{q}^{(l_0)}) - 1 = s + 2n + 1 + \sum_{i=1}^{n+1} r_i,$$

Hence,

$$\sum_{i=n+2}^{n^2+n+1} r_i \leq (k-1)\sum_{i=1}^{n+1} r_i \leq 2k(n+1),$$

which is a contradiction. 

Using Theorem 3.11 we can generate infinite families of graphs with large impropriety. For example, if $r_1 = r_2 = \cdots = r_{n^2+n+1} = r$, where $r$ is some constant, then $\mu_{\text{int}}(\text{Erd}(r_1, \ldots, r_{n^2+n+1})) > k$ if $n^2r - (k-1)r(n+1) > 2k(n+1)$.

4. Upper bounds on the impropriety of graphs

In this section, we give general upper bounds on $\mu_{\text{int}}(G)$ for several different families of graphs.

There is a prominent line of research on interval colorings of bipartite graphs; we begin this section by considering improper interval colorings of bipartite graphs.

4.1. Bipartite graphs

As mentioned above, Hansen [15] proved that if $G$ is bipartite and satisfies that $\Delta(G) \leq 3$, then $G$ has an interval coloring, while the question of interval colorability for bipartite graphs of maximum degree 4 is open. However, using Hansen’s result and König’s edge coloring Theorem 2.4, we deduce the following upper bound.
Theorem 4.1. If $G$ is bipartite, then

(i) $\mu_{\text{int}}(G) \leq \left\lceil \frac{\Delta(G)}{\delta(G)} \right\rceil$.

(ii) $\mu_{\text{int}}(G) \leq \left\lceil \frac{\Delta(G)}{\delta(G) - \frac{1}{2}} \right\rceil$.

Proof. Let $G$ be a bipartite graph. To prove (i), we construct a new bipartite graph $H$ from $G$ by proceeding in the following way: for every vertex $v$ of degree at least $\delta(G) + 1$, we split $v$ into as many vertices of degree $\delta(G)$ as possible, and one vertex of degree less than $\delta(G)$. Since the graph $H$ has maximum degree $\delta(G)$, by König’s edge coloring theorem, it has a proper $\delta(G)$-edge coloring. Let $\varphi_H$ be the coloring of $G$ induced by this coloring of $H$. Since each vertex of $G$ is split into at most $\left\lceil \frac{\Delta(G)}{\delta(G)} \right\rceil$ vertices, the coloring $\varphi_G$ is a $\left\lceil \frac{\Delta(G)}{\delta(G)} \right\rceil$-improper interval coloring of $G$ using $\delta(G)$ colors.

Part (ii) can be proved similarly to part (i), except that we apply Theorem 2.5 to the graph obtained from $G$ by splitting every vertex of $G$ into vertices of degree at most three. □

We note that part (i) of the preceding theorem is sharp for regular bipartite graphs, while we believe that part (ii) can probably be improved.

If $G$ is bipartite, and, in addition, has no vertices of degree 3, then we have the following:

Proposition 4.2. If $G$ is bipartite and has no vertices of degree three, then $\mu_{\text{int}}(G) \leq \left\lceil \frac{\Delta(G)}{4} \right\rceil$.

Proof. We proceed as in the preceding proof. From the bipartite graph $G$, we construct a graph $G'$ by splitting every vertex of degree at most five into as many vertices of degree four as possible, and one vertex of degree at most three. From $G'$, we construct a graph $G''$ with even vertex degrees by taking two copies of the graph $G'$ and joining any two corresponding vertices of degree three or one by an edge. Finally, we construct a 4-regular multigraph $H$ by adding a loop at every vertex of degree two. Now, by Petersen’s Theorem 2.1, $H$ has a decomposition into two 2-factors $F_1$ and $F_2$. In $G''$, the subgraph $F_i$ corresponds to a collection of even cycles, $i = 1, 2$. By coloring the edges of every cycle in $G''$ corresponding to a cycle of $F_1$ alternately by colors 1, 2; and the edges of every cycle corresponding to a cycle of $F_2$ alternately by colors 3, 4, we obtain an interval edge coloring $\varphi$ of $G''$, where every vertex of degree 2 has colors 1 and 2, or 3 and 4, on its incident edges.

Since there are no vertices of degree three in $G$, and each vertex of $G$ is split into at most $\left\lceil \frac{\Delta(G)}{4} \right\rceil$ different vertices in $G'$, the coloring $\varphi$ corresponds to a $\left\lceil \frac{\Delta(G)}{4} \right\rceil$-improper interval edge coloring of $G$. □

For bipartite graphs with small vertex degrees we deduce some consequences of the above results.

Corollary 4.3. If $G$ is bipartite and $\Delta(G) \leq 6$, then $\mu_{\text{int}}(G) \leq 2$.

Corollary 4.4. If $G$ is bipartite, Eulerian and $\Delta(G) \leq 8$, then $\mu_{\text{int}}(G) \leq 2$.

In general, for $k \geq 2$, it would be interesting to determine or bound the smallest integer $f_{\text{bip}}(k)$ for which there exists a graph $G$ with maximum degree $f_{\text{bip}}(k)$ satisfying $\mu_{\text{int}}(G) = k$. Even the case $k = 2$ of this problem is open. It is known, however, that $4 \leq f_{\text{bip}}(2) \leq 11$, see e.g. [24]. Moreover, by the results of Hertz graphs, $f_{\text{bip}}(3) \leq 51$, and by the above corollary $f_{\text{bip}}(3) \geq 7$.

4.2. General graphs

Let us now deduce some upper bounds on the impropriety of general graphs.

As for bipartite graphs, we define $f(k)$ as the smallest integer such that there exists a graph $G$ with maximum degree $f(k)$ and $\mu_{\text{int}}(G) = k$. In terms of maximum degree, the smallest graphs with impropriety 2 are odd cycles; thus $f(2) = 2$.

We believe that the following question is of particular interest:

Problem 4.5. Determine $f(3)$, that is, determine the smallest integer $\Delta$, such that there is a graph $G$ with maximum degree $\Delta$ satisfying $\mu_{\text{int}}(G) = 3$.

The following result shows that $f(3) > 5$ in Problem 4.5.

Theorem 4.6. If $G$ is a graph with $\Delta(G) \leq 5$, then $\mu_{\text{int}}(G) \leq 2$.

Proof. If $G$ has maximum degree 2, then trivially $\mu_{\text{int}}(G) \leq 2$. Let us now consider the case when $G$ satisfies $3 \leq \Delta(G) \leq 4$; again, we shall use Petersen’s 2-factor theorem. From $G$ we form a new graph $G'$ by taking two copies of $G$ and adding an edge between any two corresponding vertices of odd degree. From $G'$ we form a new 4-regular graph $H$ by adding a loop at every vertex of degree 2 in $G'$. By Petersen’s theorem, $H$ has a decomposition into two 2-factors $F_1$ and $F_2$. In $G'$, $F_1$ corresponds to a collection of cycles, $i = 1, 2$. By coloring edges of all cycles of $F_i$ by color $i$, we obtain a 2-improper
interval coloring $\varphi$ of $G'$, and the result now follows by coloring $G$ according to the restriction of $\varphi$ to one of the copies of $G$ in $G'$.

Let us now consider the case when $\Delta(G) = 5$. Let $G_5$ be the subgraph of $G$ induced by the vertices of degree 5 in $G$. Let $M$ be a maximum matching in $G_5$. Since $M$ is maximum, the graph $H = G - M$ either has maximum degree 4 or no two vertices of degree 5 in $H$ are adjacent. It follows that $H$ has a proper 5-edge coloring; in the former case by Vizing's theorem, and in the latter case $H$ is Class 1 by Theorem 2.3. If $H$ has maximum degree 5, then we set $H' = H - M'$, where $M'$ is a matching covering all vertices with degree 5 in $H$. If $H$ has maximum degree 4, then we set $H' = H$ and $M' = \emptyset$.

Now, by the argument in the preceding paragraph, $H'$ has a 2-improper interval coloring $\alpha$ with colors 1,2 such that for any vertex $v$ with degree 4 or 3, $S(v, \alpha) = \{1, 2\}$, and for any vertex $v$ with degree 2 or 1, $S(v, \alpha) = \{1\}$ or $S(v, \alpha) = \{2\}$.

Let us define a new edge coloring $\beta$ of $G - M'$ by coloring the edges of $M$ as follows: for every $e \in E(G - M')$, let

$$\beta(e) = \begin{cases} \alpha(e), & \text{if } e \in E(H'), \\ 3, & \text{if } e \in M. \end{cases}$$

Since $M$ is a maximum matching in $G_5$, the coloring $\beta$ is a 2-improper interval 3-coloring of $G - M'$. From $\beta$ we define an edge coloring $\gamma$ of $G$ as follows: for every $uv \in E(G)$, let

$$\gamma(uv) = \begin{cases} \beta(uv), & \text{if } uv \in E(G - M'), \\ 3, & \text{if } uv \in M', \text{ and } S(u, \beta) = S(u, \beta) = 2, \\ 0, & \text{otherwise}. \end{cases}$$

If there is an edge $e_0$ such that $\gamma(e_0) = 0$, then we define an edge coloring $\gamma'$ of $G$ as follows: $\gamma'(e) = \gamma(e) + 1$ for every $e \in E(G)$. It is straightforward that if this holds, then $\gamma'$ is a 2-improper interval 4-coloring of $G$; otherwise $\gamma$ is a 2-improper interval 3-coloring of $G$. Thus, $\mu_{int}(G) \leq 2$. $\square$

We note that the upper bound in Theorem 4.6 is in fact sharp, since any regular Class 2 graph is not interval colorable.

It also seems that graphs $G$ whose vertex degrees are sufficiently concentrated satisfy $\mu_{int}(G) \leq 2$; for instance, as pointed out above (Proposition 3.1), any regular graph $G$ satisfies that $\mu_{int}(G) \leq 2$. We strengthen this observation slightly as follows.

**Proposition 4.7.** If $G$ is a graph with $\Delta(G) - \delta(G) \leq 1$, then $\mu_{int}(G) \leq 2$.

**Proof.** Let $G$ be a graph satisfying $\Delta(G) - \delta(G) \leq 1$, and let $G_\Delta$ be the subgraph of $G$ induced by the vertices of maximum degree in $G$. By the preceding proposition, we may assume that $\Delta(G) > 5$. Let $M$ be a maximum matching in $G_\Delta$. Since $M$ is maximum, the graph $H = G - M$ either has maximum degree $\Delta(G) - 1$ or no two vertices of degree $\Delta(G)$ in $H$ are adjacent. Then $H$ has a proper $\Delta(G)$-edge coloring $\varphi$; in the former case by Vizing’s theorem, and in the latter case $H$ is Class 1 by Theorem 2.3.

Let $M_1$, be the set of edges with color 1 under $\varphi$, $M_\Delta$ be the edges of color $\Delta(G)$ under $\varphi$, and consider the edge-induced subgraph $H[M_1 \cup M_\Delta]$. Since $\Delta(H) \geq \Delta(G) - 1$, this graph is a spanning subgraph, and, furthermore, every component of this graph is an even cycle or a path. Let $D$ be an orientation of $H[M_1 \cup M_\Delta]$ where every vertex has indegree at most 1 and outdegree at most 1; such an orientation exists since every component of $H[M_1 \cup M_\Delta]$ is a cycle or a path.

We define a new proper edge coloring of $H$ from $\varphi$ by recoloring some of the edges in $H[M_1 \cup M_\Delta]$ in the following way: for every arc $(a, b)$ of $D$, if there is a color $c \in \{2, \ldots, \Delta(G) - 1\}$ which does not appear on an edge incident to $b$ under $\varphi$, then we recolor the edge $ab$ with color $c$; if there is no such color $c$, then we retain the color of the edge $ab$. Denote the obtained coloring by $\varphi'$. Finally, we extend the coloring $\varphi'$ to a coloring $\alpha$ of $G$ by coloring every edge of $M$ by color $\Delta(G)$.

Let us prove that $\alpha$ is a 2-improper interval edge coloring of $G$. The color $\Delta(G)$ appears at most twice at a vertex of $G$, and if two edges of $H[M_1 \cup M_\Delta]$, both of which are incident to a common vertex $u$, are recolored by the same color $j \in \{2, \Delta(G) - 1\}$, then $j$ does not appear on any edge incident to $u$ under $\varphi$. Hence, every color appears at most twice at any vertex of $G$.

Suppose now that the colors on the edges incident to some vertex $v$ of $G$ under $\varphi$ do not form an interval. Since $\alpha$ uses $\Delta(G)$ colors, this means that there is some color $j \in \{2, \ldots, \Delta(G) - 1\}$ that does not appear on an edge incident to $v$ under $\varphi$. Moreover, since $\alpha$ is obtained from $\varphi$ by recoloring only edges of color 1 or $\Delta(G)$, $j$ does not appear at $v$ under $\varphi$. Now, since $\varphi$ is a proper $\Delta(G)$-edge coloring of $H$ and $\Delta(H) \geq \Delta(G) - 1$, we must have $d_H(v) = \Delta(G) - 1$; and so $v$ is incident to an edge colored $j$ under $\varphi$, for every $i \in \{1, \ldots, j - 1, j + 1, \ldots, \Delta(G)\}$; in particular, $v$ has degree 2 in $H[M_1 \cup M_\Delta]$, and therefore one of the edges in $H[M_1 \cup M_\Delta]$ would have been recolored $j$ in the process of constructing $\varphi'$. This is a contradiction, and so it follows that $\alpha$ is a 2-improper interval edge coloring of $G$. $\square$

Using the preceding proposition, we can prove the following, by splitting vertices.

**Proposition 4.8.** If $G$ is a graph, then $\mu_{int}(G) \leq 2 \left\lceil \frac{\Delta(G)}{\delta(G)} \right\rceil$.
Proof. We proceed as in the proof of Theorem 4.1: from $G$ we form a new graph $G'$ by splitting every vertex of degree at least $\delta(G) + 1$ into as many vertices of degree exactly $\delta(G)$ as possible, and one vertex of degree at most $\delta(G)$. Let $H$ be a $\delta(G)$-regular supergraph of $G'$. By Proposition 4.7, $H$ has a 2-improper interval edge coloring. This coloring induces a $2\left\lceil \frac{\Delta(G)}{2} \right\rceil$-improper interval coloring of $G$. □

Both preceding propositions can in general not be improved since regular Class 2 graphs are not interval colorable. Finally, we have the following general upper bound.

**Theorem 4.9.** If $G$ is a graph with $\Delta(G) \geq 6$, then $\mu_{\text{int}}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil$.

**Proof.** Since any regular graph has improriety at most 2, it suffices to consider the case when $\delta(G) < \Delta(G)$. Furthermore, without loss of generality, we assume that $G$ is connected.

Let $H$ be the graph obtained by taking two copies of $G$ and adding an edge between any two corresponding vertices of odd degree. We shall consider $G$ as a subgraph of $H$.

Since all vertex degrees in $H$ are even, it has an Eulerian circuit $T$. By coloring all edges of $T$ by 1 and 2 alternately along $T$, we obtain an improper interval coloring $\varphi$ of $H$. Let $\alpha$ be the improper interval coloring of $G$ induced by $\varphi$. If every vertex of $G$ is incident to at most $\left\lceil \frac{\Delta(G)}{2} \right\rceil$ edges with the same color, then the desired result follows, so assume that this does not hold. Then there is a vertex $v$ which is incident to exactly $\left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$ edges with the same color under $\alpha$, say 1. Indeed, since the edges of $T$ are colored alternately by colors 1 and 2, $v$ must be the first vertex of the Eulerian circuit $T$ in $H$. Moreover, $v$ must be a vertex of maximum degree in $G$.

Let us first consider the case when $\Delta(G)$ is odd, that is, $\Delta(G) = \Delta(H) - 1$. Let $T_1$ be a shortest subtrail of $T$ from $v$ to a vertex $u$ of degree at most $\Delta(G) - 1$ in $G$, which starts by an edge colored 1. We define a new coloring $\varphi'$ of $H$ from $\varphi$ by recoloring the edges on $T_1$ in the following way:

$$
\varphi'(e) = \begin{cases} 
1, & \text{if } \varphi(e) = 2 \text{ and } e \in E(T_1), \\
2, & \text{if } \varphi(e) = 1 \text{ and } e \in E(T_1), \\
\varphi(e), & \text{if } e \notin E(T_1).
\end{cases}
$$

Since $d_c(u) \leq \Delta(G) - 1$, and all vertices of $H$ except $v$ and $u$ are incident with exactly many edges of color $i$ under $\varphi'$ as under $\varphi$, $i = 1, 2$, it follows that the restriction of $\varphi'$ to $G$ is a $\left\lceil \frac{\Delta(G)}{2} \right\rceil$-improper interval coloring of $G$.

Let us now consider the case when $\Delta(G)$ is even, i.e. $\Delta(H) = \Delta(G)$. Let $T_1$ be a shortest subtrail of $T$ from $v$ to a vertex $u$ of degree at most $\Delta(G) - 1$ in $G$, which starts by an edge colored 1, and suppose $e_1$ is the last edge of $T_1$. We consider the following different cases.

(a) If $e_1 \notin E(G)$ or $d_c(x) \leq \Delta(G) - 2$, then we define a new coloring $\varphi'$ from $\varphi$ by recoloring all edges of $T_1$ by setting $\varphi'(e) = 1$ if $\varphi(e) = 2$, $\varphi'(e) = 2$ if $\varphi(e) = 1$, and retaining the color of every other edge of $H$. The coloring $\alpha'$ of $G$ induced by $\varphi'$ is a $\left\lceil \frac{\Delta(G)}{2} \right\rceil$-improper interval coloring of $G$.

(b) If $e_1 \in E(G)$, $d_c(x) = \Delta(G) - 1$, $\varphi(e_1) = 1(2)$, and $x$ is incident to an edge in $E(H) \setminus E(G)$ of color 2(1) under $\varphi$, then we proceed as in (a).

(c) If $e_1 \in E(G)$, $d_c(x) = \Delta(G) - 1$, $\varphi(e_1) = 1(2)$, and $x$ is incident to an edge $e_2 \neq e_1$ in $H$ of color 1(2) under $\varphi$ that is not in $G$, then we proceed as follows: let $T_2$ be the subtrail of $T$ beginning with $v$ whose last edge is $e_2$. By proceeding as in (a) and switching colors on $T_2$, and taking the restriction of the obtained coloring to $G$, we obtain a $\left\lceil \frac{\Delta(G)}{2} \right\rceil$-improper interval coloring of $G$. □

4.3. Outerplanar graphs

In this section we consider outerplanar graphs. Recall that a graph is outerplanar if it has an embedding in the plane for which all vertices belong to the outer face. We do not know of any outerplanar graph $G$ with $\mu_{\text{int}}(G) \geq 3$; in fact, we believe that there is no such graph.

**Conjecture 4.10.** For any outerplanar graph $G$, $\mu_{\text{int}}(G) \leq 2$.

Since there are examples of outerplanar graphs with no interval edge coloring, the upper bound in Conjecture 4.10 would be sharp if true. Next, we shall prove that this conjecture holds for graphs with maximum degree at most eight.

**Proposition 4.11.** If $G$ is an outerplanar graph and $\Delta(G) \leq 8$, then $\mu_{\text{int}}(G) \leq 2$.

For the proof of this result, we shall use the well-known fact that an outerplanar graph is Class 1 unless it is an odd cycle [9].

**Proof.** By rotating the colors of edge colorings of different blocks of a graph, it is evident that a graph $G$ has a $k$-improper interval coloring if every block of $G$ has a $k$-improper interval coloring; thus it suffices to consider the case when $G$ is 2-connected.
Consequently, assume that \( G \) is 2-connected; then it has a Hamiltonian cycle \( C \). The graph \( G - E(C) \) has maximum degree at most 6. If \( |V(C)| \) is even, then we define a proper edge coloring \( \varphi \) of \( C \) by coloring its edges alternately by colors 2 and 3. If \( |V(C)| \) is odd, then we define \( \varphi \) in the following way: it is easy to see that every 2-connected outerplanar graph has a vertex \( v \) of degree 2; we color the edges of \( C \) alternately by colors 2 and 3, and beginning and ending with color 2 at the edges incident to \( v \).

Now, consider the graph \( H = G - E(C) \). Since \( H \) is an outerplanar graph (or consisting of several outerplanar components), it has a proper edge coloring \( \alpha \) using at most 6 colors 1, ..., 6. From \( \alpha \), we define a new edge coloring \( \alpha' \) by recoloring any edges of colors 5 and 6 by colors 1 and 4, respectively. It is straightforward to verify that the colorings \( \varphi \) and \( \alpha' \) taken together form a 2-improper interval coloring of \( G \). □

Using the same vertex splitting technique as several times before, we deduce the following corollary. Note that if \( G \) is outerplanar and \( v \in V(G) \), then given integers \( k \) and \( l \) such that \( k + l = \deg(v) \), it is always possible to split the vertex \( v \) into two new vertices \( v' \) and \( v'' \) of degrees \( k \) and \( l \), respectively, so that the resulting graph is outerplanar (or a union of vertex-disjoint outerplanar graphs). We state this observation as a lemma.

**Lemma 4.12.** If \( G \) is outerplanar, \( v \) is a vertex of \( G \) and \( k \) and \( l \) are positive integers satisfying \( \deg(v) = k + l \), then we can split the vertex \( v \) into two new vertices of degrees \( k \) and \( l \), respectively, in such a way that the resulting graph is outerplanar (or a union of vertex-disjoint outerplanar graphs).

**Corollary 4.13.** If \( G \) is an outerplanar graph, then \( \mu_{\text{int}}(G) \leq \left\lceil \frac{\Delta(G)}{4} \right\rceil + 1 \).

**Proof.** By Proposition 4.11, we may assume that \( \Delta(G) \geq 9 \). As in the preceding proof, it suffices to consider the case when \( G \) is 2-connected. Let \( C \) be a Hamiltonian cycle of \( G \); we color \( C \) as in the proof of Proposition 4.11. The result now follows by splitting all vertices of \( G - E(C) \) into as many vertices of degree 4 as possible, and possibly one additional vertex of degree at most 3; by repeatedly applying Lemma 4.12, this can be done so that the resulting graph \( J \) is outerplanar (or a union of disjoint outerplanar graphs).

Now, since \( \Delta(J) = 4 \), \( J \) has a proper 4-edge coloring. This proper edge coloring, together with the coloring of \( C \) is the required improper interval edge coloring of \( G \). □

### 4.4. Complete multipartite graphs

In this section we prove an upper bound for the imprropriety of complete multipartite graphs. A graph \( G \) is called complete \( r \)-partite \((r \geq 2)\) if its vertices can be partitioned into \( r \) nonempty independent sets \( V_1, \ldots, V_r \) such that each vertex in \( V_i \) is adjacent to all the other vertices in \( V_j \) for \( 1 \leq i < j \leq r \). Let \( K_{n_1, n_2, \ldots, n_r} \) denote a complete \( r \)-partite graph with independent sets \( V_1, V_2, \ldots, V_r \) of sizes \( n_1, n_2, \ldots, n_r \).

**Theorem 4.14.** For any \( n_1, n_2, \ldots, n_r \in \mathbb{N} \),

\[
\mu_{\text{int}}(K_{n_1, n_2, \ldots, n_r}) \leq \left\lceil \frac{r}{2} \right\rceil.
\]

Before proving Theorem 4.14, let us briefly describe the proof idea. We partition the vertex set of a complete \( r \)-partite graph \( G \) into two sets \( A \) and \( B \), where \( A \) contains \( \left\lceil \frac{r}{2} \right\rceil \) independent sets from \( G \), and the second set \( B \) contains the remaining \( \left\lfloor \frac{r}{2} \right\rfloor \) independent sets from \( G \). Now, the subgraph induced by two independent sets from \( A \) (or \( B \)) is a complete bipartite graph, and therefore has an interval coloring. We properly color the edges of every such complete bipartite subgraph \( K_{a,b} \) of \( G \) using colors 1, \( a + b - 1 \) so that the coloring is interval. Then every color appears at most \( \left\lceil \frac{r}{2} \right\rceil - 1 \) times at any vertex in the resulting coloring. Finally we properly color the bipartite graph induced by the edges with one endpoint in \( A \) and one endpoint in \( B \). Hence, the proof idea is simple but we have to choose each partial coloring of a subgraph of \( G \) carefully to ensure that the resulting edge coloring indeed is a \( \left\lceil \frac{r}{2} \right\rceil \)-improper interval coloring. The details are given in the following proof:

**Proof of Theorem 4.14.** Without loss of generality, we may assume that \( n_1 \geq n_2 \geq \cdots \geq n_r \). We partition the independent sets \( V_1, \ldots, V_r \) into two groups: \( X_1, X_2, \ldots, X_{\left\lfloor \frac{r}{2} \right\rfloor} \) (first \( \left\lceil \frac{r}{2} \right\rceil \) independent sets) of sizes \( n_1, n_2, \ldots, n_{n_1} \) and \( Y_1, Y_2, \ldots, Y_{\left\lfloor \frac{r}{2} \right\rfloor} \) of sizes \( n_{n_1+1}^{(1)}, n_{n_1+2}^{(1)}, \ldots, n_r \) (remaining \( \left\lfloor \frac{r}{2} \right\rceil \) independent sets). Let \( X = X_1 \cup X_2 \cup \cdots \cup X_{\left\lfloor \frac{r}{2} \right\rceil} \) and \( Y = Y_1 \cup Y_2 \cup \cdots \cup Y_{\left\lfloor \frac{r}{2} \right\rceil} \). We also label the vertices of \( X \) and \( Y \) as follows:

\[
X = \left\{ x_1, x_2, \ldots, x_{n_1}, x_{n_1+1}, \ldots, x_{n_1+n_2}, \ldots, x_{n_1+n_2+\cdots+n_{\left\lfloor \frac{r}{2} \right\rceil}} \right\}
\]

and

\[
Y = \left\{ y_1, y_2, \ldots, y_{n_1}^{(1)}, y_{n_1+1}^{(1)}, \ldots, y_{n_1+n_2}^{(1)}, \ldots, y_{n_1+n_2+\cdots+n_{\left\lfloor \frac{r}{2} \right\rceil}}^{(1)} \right\}.
\]
Let $s_i = \sum_{j=i}^i n_j$ ($1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$) and $t_i = \sum_{j=i}^{\left\lceil \frac{n}{2} \right\rceil + 1} n_j$ ($1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$).

Now let us consider the subgraphs $H$ and $H'$ of $K_{n_1,n_2,...,n_r}$ induced by the vertices of $X$ and $Y$, respectively. We first define an edge coloring $\alpha$ of $H \cup H'$.

1. For $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1$, $1 \leq j \leq s_i$ and $1 \leq k \leq n_{i+1}$, let
   \[ \alpha(xy_{i+k}) = j + k - 1. \]

2. For $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1$, $1 \leq j \leq t_i$ and $1 \leq k \leq n_{i+1}$, let
   \[ \alpha(y_jy_{i+k}) = j + k - 1. \]

By the definition of $\alpha$, we have
(a) for $1 \leq k \leq n_1$,
   \[ S_H(x_k, \alpha) = [k, k + n_2 - 1]. \]
(b) for $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 2$ and $1 \leq k \leq n_{i+1}$,
   \[ S_H(x_{i+k}, \alpha) = [k, s_i + k - 1] \cup [s_i + k, s_{i+1} + k - 1] = [k, s_{i+1} + k - 1]. \]
(c) for $1 \leq k \leq n_{i+1}$,
   \[ S_H(x_{i_{\frac{j}{2}} - 1} + k, \alpha) = [k, s_i + k - 1 + 1]. \]
(d) for $1 \leq k \leq n_{i+1}$,
   \[ S_H(y_{k}, \alpha) = [k, k + n_{i+2} - 1]. \]
(e) for $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 2$ and $1 \leq k \leq n_{i+1}$,
   \[ S_H(y_{i+k}, \alpha) = [k, t_i + k - 1] \cup [t_i + k, t_{i+1} + k - 1] = [k, t_{i+1} + k - 1]. \]
(f) for $1 \leq k \leq n_r$,
   \[ S_H(y_{i_{\frac{j}{2}} - 1} + k, \alpha) = [k, t_i + k - 1]. \]

Note that for every vertex $v$ of $H \cup H'$, each color can occur at most $\left\lceil \frac{n}{2} \right\rceil - 1$ times at $v$ under the coloring $\alpha$. Hence, $\alpha$ is an $(\left\lceil \frac{n}{2} \right\rceil - 1)$-improper interval coloring of $H \cup H'$.

Next, we define an edge coloring $\beta$ of $K_{n_1,n_2,...,n_r} - E(H \cup H')$ as follows: for $1 \leq i \leq s_i$, and $1 \leq j \leq t_i$, let
   \[ \beta(xy) = i + j - 1. \]

Now we are able to define an edge coloring $\gamma$ of $K_{n_1,n_2,...,n_r}$ by taking the colorings $\alpha$ and $\beta$ together; that is, for any $e \in E(K_{n_1,n_2,...,n_r})$, we set
   \[ \gamma(e) = \begin{cases} \alpha(e), & \text{if } e \in E(H \cup H'), \\ \beta(e), & \text{otherwise.} \end{cases} \]

By the definition of $\gamma$, we have
(a') for $1 \leq k \leq n_1$,
   \[ S(x_k, \gamma) = [k, k + n_2 - 1] \cup [k, k + t_{i_{\frac{n}{2}}} - 1]. \]
(b') for $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 2$ and $1 \leq k \leq n_{i+1}$,
   \[ S(x_{i+k}, \gamma) = [k, s_{i+1} + k - 1] \cup [s_i + k, k + t_{i_{\frac{n}{2}}} - 1]. \]
(c') for $1 \leq k \leq n_{i+1}$,
   \[ S(x_{i_{\frac{n}{2}} - 1} + k, \gamma) = [k, s_i - 1 + k - 1] \cup [s_i - 1 + k + k + t_{i_{\frac{n}{2}}} - 1]. \]

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(d') for $1 \leq k \leq n_1 \frac{k}{2} + 1$, 
$$S(y_k, \gamma) = \left[ k, k + n_1 \frac{k}{2} + 1 \right] \cup \left[ k, k + s_1 \frac{k}{2} - 1 \right].$$

(e') for $1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor - 2$ and $1 \leq k \leq n_1 \frac{k}{2} + 1$, 
$$S(y_{i+k}, \gamma) = \left[ k, t_i + 1 + k - 1 \right] \cup \left[ k, t_i + k + s_1 \frac{k}{2} - 1 \right].$$

(f') for $1 \leq k \leq n_t$, 
$$S\left(y_{i+1}, t_i \right) = \left[ k, t_i + 1 + k - 1 \right] \cup \left[ k, t_i + k + s_1 \frac{k}{2} - 1 \right].$$

It is not difficult to see that $\gamma$ is an $\left\lfloor \frac{k}{2} \right\rfloor$-improper interval coloring of $K_{n_1, n_2, \ldots, n_t}$; thus $\mu_{\text{int}} \left(K_{n_1, n_2, \ldots, n_t}\right) \leq \left\lfloor \frac{k}{2} \right\rfloor$. □

**Corollary 4.15.** If $G$ is a complete 3-partite or 4-partite graph, then $\mu_{\text{int}}(G) \leq 2$.

In fact, we believe that a more general result is true:

**Conjecture 4.16.** If $G$ is a complete $r$-partite graph, then $\mu_{\text{int}}(G) \leq 2$.

Since there are complete multipartite graphs of Class 2, Conjecture 4.16, if true, would be best possible.

4.5. Cartesian products of graphs

In this section we consider the impropriety of Cartesian products of graphs. The Cartesian product $G \square H$ of two graphs $G$ and $H$ is defined by setting

$$V(G \square H) = V(G) \times V(H),$$

and

$$E(G \square H) = \{(u_1, v_1)(u_2, v_2) : (u_1 = u_2 \land v_1 = v_2) \lor (v_1 = v_2 \land u_1 = u_2) \in E(G)\}.$$

**Proposition 4.17.** For any graphs $G$ and $H$,

$$\mu_{\text{int}}(G \square H) \leq \max \{\mu_{\text{int}}(G), \mu_{\text{int}}(H)\}.$$

**Proof.** In the proof of this proposition we follow the idea from [12] (Theorem 2.4). Let $\alpha$ be a $k_1$-improper interval $t_1$-coloring of $G$ and $\beta$ be a $k_2$-improper interval $t_2$-coloring of $H$, where $k_1 = \mu_{\text{int}}(G)$ and $k_2 = \mu_{\text{int}}(H)$.

We define an edge coloring $\gamma$ of $G \square H$ as follows: for every $(u_1, v_1)(u_2, v_2) \in E(G \square H)$, let

$$\gamma((u_1, v_1)(u_2, v_2)) = \begin{cases} \alpha(u_1u_2) + S(v_1, \beta) - 1, & \text{if } v_1 = v_2 \land u_1u_2 \in E(G), \\ \beta(v_1v_2) + \overline{S}(u_1, \alpha), & \text{if } u_1 = u_2 \land v_1v_2 \in E(H). \end{cases}$$

By the definition of $\gamma$, for every vertex $(u, v) \in V(G \square H)$, we have

$$S((u, v), \gamma) = \left[ S(u, \alpha) + S(v, \beta) - 1, \overline{S}(u, \alpha) + S(v, \beta) - 1 \right] \cup$$

$$\cup \left[ \overline{S}(u, \alpha) + S(v, \beta) - 1, \overline{S}(u, \alpha) + \overline{S}(v, \beta) \right].$$

Since for every vertex $(u, v)$ of $G \square H$, each color can occur at most $\max\{k_1, k_2\}$ times at $(u, v)$ under the coloring $\gamma$, this implies that $\gamma$ is a max-$k_1, k_2$-improper interval $(t_1 + t_2)$-coloring of $G \square H$. Thus, $\mu_{\text{int}}(G \square H) \leq \max \{\mu_{\text{int}}(G), \mu_{\text{int}}(H)\}$. □

Clearly, this upper bound on the impropriety in Proposition 4.17 is sharp for all Cartesian products of graphs when the factors are interval colorable. Let us note that there are graphs $G$ and $H$ such that $\mu_{\text{int}}(G \square H) < \max \{\mu_{\text{int}}(G), \mu_{\text{int}}(H)\}$. For example, if $G$ and $H$ are both isomorphic to the Petersen graph, then, by Proposition 3.1, $\mu_{\text{int}}(G) = \mu_{\text{int}}(H) = 2$, but $\mu_{\text{int}}(G \square H) = 1$, since $G$ and $H$ contain perfect matchings [23]. On the other hand, if we consider the Cartesian product of two odd cycles $C_{2m+1} \square C_{2n+1}$, then again, by Proposition 3.1, $\mu_{\text{int}}(C_{2m+1}) = \mu_{\text{int}}(C_{2n+1}) = 2$, but in this case $\mu_{\text{int}}(C_{2m+1} \square C_{2n+1}) = 2$, since $C_{2m+1} \square C_{2n+1}$ is Class 2. So, the upper bound on the impropriety in Proposition 4.17 is also sharp for all Cartesian products of regular graphs when the factors and the Cartesian product of factors are Class 2.

5. The number of colors in an improper interval coloring

Following [18], we denote by $\mathcal{I}(G)$ the maximum number of colors used in an improper interval edge coloring of $G$. In [18], Hudak et al. proved upper bounds on the parameter $\mathcal{I}(G)$ in terms of the number of vertices, the maximum degree and the diameter of graphs. They also obtained upper bounds for specific families of graphs like wheels, prisms and complete graphs. In particular, they proved the following two results.
Theorem 5.1 (\cite{18}). For each connected triangle-free graph $G$ on $n$ vertices, $\hat{t}(G) \leq n - 1$. Moreover, the upper bound is sharp.

Theorem 5.2 (\cite{18}). For each connected graph $G$ on $n$ vertices, $\hat{t}(G) \leq 2n - 1$.

Here we slightly improve the general upper bound from the last theorem.

Proposition 5.3. For each connected graph $G$ on $n$ vertices ($n \geq 2$), $\hat{t}(G) \leq 2n - 3$.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $\alpha$ be an improper interval $\hat{t}(G)$-coloring of $G$. Define an auxiliary graph $H$ as follows:

$$V(H) = U \cup W,$$

where

$$U = \{u_1, u_2, \ldots, u_n\}, \quad W = \{w_1, w_2, \ldots, w_n\}$$

and

$$E(H) = \left\{u_i w_j : v_i v_j \in E(G), 1 \leq i \leq n, 1 \leq j \leq n\right\} \cup \{u_i w_j : 1 \leq i \leq n\}.$$

Clearly, $H$ is a bipartite graph with $|V(H)| = 2n$.

Define an edge coloring $\beta$ of $H$ as follows:

1. For every edge $v_i v_j \in E(G)$, let $\beta(u_i w_j) = \beta(u_i w_i) = \alpha(v_i v_j) + 1$,

2. For $i = 1, 2, \ldots, n$, let $\beta(u_i w_j) = S(v_i, \alpha) + 2$.

It is easy to see that $\beta$ is an edge coloring of the graph $H$ with colors $2, 3, \ldots, \hat{t}(G) + 2$ and $S(u_i, \beta) = S(u_i, \beta)$ for $i = 1, 2, \ldots, n$. We construct an improper interval $\hat{t}(G) + 2$-coloring of the graph $H$ by picking an edge $u_i w_j$ with $S(u_i, \beta) = S(u_i, \beta) = 2$ and recoloring it with color 1. The obtained coloring is an improper interval $\hat{t}(G) + 2$-coloring of $H$. Since $H$ is a connected bipartite graph, by Theorem 5.1, we have

$$\hat{t}(G) + 2 \leq |V(H)| - 1 = 2n - 1,$$

thus $\hat{t}(G) \leq 2n - 3$. \hfill \square

We note that the upper bound in the preceding proposition is sharp for $K_2$.

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