Overlooked Contribution to the Hall Effect in Ferromagnetic Metals

J. E. Hirsch

Department of Physics, University of California, San Diego
La Jolla, CA 92039-0319
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It is pointed out that in ferromagnetic metals a contribution to the Hall voltage arises when a non-zero spin current exists, which is generally the case in the presence of a charge current. This contribution is independent of any scattering effects and exists down to zero temperature. The sign of the resulting Hall coefficient may be either equal or opposite to the one of the ordinary Hall coefficient depending on the band filling. This effect seems to have been left out in previous analyses of the Hall effect in ferromagnetic metals.

The Hall coefficient of ferromagnetic metals is found to be larger than that of non-magnetic metals and to exhibit a strong dependence on the magnetic field \( B \). It is found that the Hall resistivity

\[
\rho_H = E_y / j_x
\]

with \( E_y \) the transverse electric field and \( j_x \) the longitudinal current density, can be fitted empirically by the formula

\[
\rho_H = R_o B + 4\pi R_s M \equiv \rho_H^o + \rho_{sH}^H
\]

with \( B \) the applied magnetic field and \( M \) the magnetization per unit volume. \( R_o \) is the “ordinary” Hall coefficient and \( R_s \) the “anomalous” Hall coefficient. Various explanations for the origin of \( R_s \) have been proposed, all of them involving scattering processes of the conduction electrons together with the spin-orbit interaction.

In some models the carriers are assumed to be magnetic and the scattering centers non-magnetic, while in others the situation is reversed.

We should point out at the outset that the contribution to the anomalous Hall effect considered in this paper is not proportional to the magnetization as given by Eq. (2) but rather to the magnetization current. While in some cases the two quantities will be proportional, this is not necessarily so, and one can even have situations where a magnetization current exists in the absence of net magnetization, as discussed later. We will nevertheless use the definition Eq. (2) for \( R_s \) whenever possible for consistency with earlier work.

In the theory of Karplus and Luttinger, the anomalous Hall effect was explained as arising from interband matrix elements of the applied electric potential in the presence of spin-orbit coupling in a perfectly periodic lattice. This theory was criticized by Smit, who showed that in fact within the Karplus-Luttinger treatment a periodic spin-orbit interaction will not give rise to an anomalous Hall voltage when all terms are properly taken into account. Instead, according to Smit, the effect arises from skew scattering by perturbations that break the periodicity and lead to finite resistivity, e.g. impurities and phonons. Later, it was proposed that in addition to skew scattering a ‘side-jump’ occurs when a magnetic carrier scatters off an impurity, which will also give a contribution to the anomalous Hall effect. While there has not been general agreement on whether skew scattering or side jump are dominant in various cases, and on whether impurity scattering or phonon scattering dominates, there seems to be a consensus that the anomalous Hall effect only arises from scattering by potentials that break the lattice periodicity. Furthermore it is also generally assumed that the effect is proportional to the magnetization of the system as given by Eq. (2).

For the case of ferromagnetic transition metals it is generally accepted that the electrons that give rise to magnetism are itinerant. Here we adopt this point of view. The purpose of this paper is to point out that a contribution to \( R_s \) necessarily arises simply from the fact that in general a spin current exists in ferromagnetic metals when a charge current exists. We show that quite generally a spin current circulating in a solid will give rise to a transverse Hall field. This effect is independent of any scattering processes, and seems to have been omitted in previous discussions of the origin of \( R_s \) in ferromagnetic metals. It gives a contribution to the anomalous Hall coefficient of the same order of magnitude as the ordinary Hall coefficient.

As is well known, for the ordinary Hall effect there is a simple classical explanation. It arises from the Lorentz force that acts on a moving charge \( q \)

\[
\mathbf{F} = q \mathbf{v} \times \mathbf{B}
\]

and will appear whenever a charge current circulates in a metal in the presence of a magnetic field perpendicular to the current direction. The force Eq. (3) is balanced in steady state by a compensating electric field

\[
E_y = \frac{v}{c} B
\]
in direction perpendicular to the current and the magnetic field. Dividing by the current density

\[ j_x = n q v \]  \hspace{1cm} (5)

yields the Hall resistivity

\[ \rho_H = \frac{E_y}{j_x} = \frac{1}{n q c} B \]  \hspace{1cm} (6)

where \( n \) is the number of electrons in the band if the band is almost empty, and \( q = -e \), with \( e \) the magnitude of the electron charge. If the band is almost full, Eq. (6) applies with \( q = +e \) and \( n \) the number of holes. Of course, for a real metal a quantitative evaluation of the Hall effect is considerably more complicated than what is described above. Nevertheless, these simple considerations capture the essence of the effect.

One may ask whether there is not similarly a simple classical argument that will predict a transverse Hall field in the presence of a spin current. We show here that such an effect is indeed expected. The Hall field will arise whenever a spin current circulates, with or without charge current and with or without net magnetization, in the presence of a perfectly periodic potential, even at zero temperature, just as the ordinary Hall effect.

Consider an infinite line of equally spaced magnetic moments \( \mathbf{m} \) pointing along the \( z \) direction, moving with velocity \( \mathbf{v} \) along the \( x \) direction, as shown in Figure 1. An electric field results in the laboratory frame, which is identical to that generated by an infinite line of stationary electric dipoles pointing in the \((-y)\) direction, given by

\[ \mathbf{p} = \gamma \frac{\mathbf{v}}{c} \times \mathbf{m} \]  \hspace{1cm} (7)

with \( \gamma = (1 - v^2/c^2)^{-1/2} \). This is seen as follows: the magnetic field of a magnetic dipole in its rest frame is

\[ \mathbf{B} = \frac{3\mathbf{n}(\mathbf{m} \cdot \mathbf{n}) - \mathbf{m}}{r^3} \]  \hspace{1cm} (8)

with \( \mathbf{n} \) a unit vector from the position of \( \mathbf{m} \) to the point where \( \mathbf{B} \) is observed, and \( r \) the distance from \( \mathbf{m} \) to the observation point. From a Lorentz transformation we find that there is an electric field in the laboratory frame, given by:

\[ \mathbf{E}_\perp = -\frac{\gamma}{c} \mathbf{v} \times \mathbf{B} \]  \hspace{1cm} (9)

Here, \((-\mathbf{v})\) is the velocity of the lab frame with respect to the rest frame of the moments and \( \perp \) indicates directions perpendicular to the velocity. By symmetry there is neither electric nor magnetic fields in the direction of \( \mathbf{v} \). Eqs. (8) and (9) yield

\[ \mathbf{E}_\perp = \frac{\gamma}{c} \frac{-3(\mathbf{v} \times \mathbf{n})\mathbf{m} \cdot \mathbf{n} + \mathbf{v} \times \mathbf{m}}{r^3} \]  \hspace{1cm} (10)

On the other hand the electric field from the dipole Eq. (7) is

\[ \mathbf{E}_\perp = \gamma \frac{3n_\perp(\mathbf{v} \times \mathbf{m}) \mathbf{n} - \mathbf{v} \times \mathbf{m}}{r^3} \]  \hspace{1cm} (11)

with \( n_\perp \) the projection of \( \mathbf{n} \) in the plane perpendicular to \( \mathbf{v} \). Although the expressions Eq. (10) and (11) are different for a single magnetic dipole, when integrated over the infinite line they yield the same answer.

Hence we may think of the moving magnetic moments as stationary electric dipoles, of magnitude given by Eq. (7). In the absence of external potentials there will be no net transverse force on these dipoles, hence no spontaneous Hall effect. However, as we show next, the periodic lattice potential will exert a transverse force on these dipoles that tends to deflect them; in steady state, the transverse force is balanced by accumulation of charge on the edges under open circuit conditions as in the ordinary Hall effect.

Consider an array of charges \( Q \) in a simple cubic lattice of spacing \( a \), as shown in Fig. 2. The electric potential at point \((x, y, z)\) is given by

\[ V(x, y, z) = Q \times \sum_{n_1, n_2, n_3} \frac{1}{[(x - an_1)^2 + (y - an_2)^2 + (z - an_3)^2]^{1/2}} \]  \hspace{1cm} (12)

The force in the \( y \) direction on an electric dipole of magnitude \( p \) pointing in that direction is

\[ F_y(x, y, z) = \frac{\partial^2 V}{\partial y^2} = p Q \times \sum_{n_1, n_2, n_3} \frac{2(y - an_2)^2 - (x - an_1)^2 - (z - an_3)^2}{[(x - an_1)^2 + (y - an_2)^2 + (z - an_3)^2]^{5/2}} \]  \hspace{1cm} (13)

The sign of the force depends on the location of the dipole. In particular, it is negative on (010) planes that go through lattice points and positive on (010) planes midway between lattice points, as indicated in Figure 2.

Assume that the magnetic moments propagate in rectilinear motion along the \( x \) direction with constant velocity. It is then appropriate to average the force Eq. (13) over different \( x \)-positions in the unit cell, yielding

\[ F_y(y, z) = \frac{1}{a} \int_0^a dx F_y(x, y, z) = \frac{p Q}{a} \times \sum_{n_1, n_2, n_3} \frac{[2(y - an_2)^2 - (z - an_3)^2]}{(G_1 - G_2)} \]  \hspace{1cm} (14)

with

\[ G_1 = F_1(\omega_2) - F_1(\omega_1) \]  \hspace{1cm} (15a)

\[ G_2 = F_2(\omega_2) - F_2(\omega_1) \]  \hspace{1cm} (15b)

\[ \omega_1 = -an_1 \]  \hspace{1cm} (15c)

\[ \omega_2 = a(\frac{1}{2} - n_1) \]  \hspace{1cm} (15d)
\[ F_1(\omega) = \frac{2\omega(2\omega^2 + 3r^2)}{3r^2(w^2 + r^2)^{3/2}} \]  
\[ F_2(\omega) = \frac{2\omega^3}{3r^2(w^2 + r^2)^{3/2}} \]  
\[ r^2 = (y - an_2)^2 + (z - an_3)^2 \]  
\[ \langle \bar{F}_y \rangle = \frac{1}{2} \left[ \bar{F}_y(r\cos\theta, r\sin\theta) + \bar{F}_y(r\cos(\pi/2 - \theta), r\sin(\pi/2 - \theta)) \right] \]

Eq. (14), with \( p \) given by eq. (7), is the transverse force acting on a magnetic moment that propagates with uniform velocity along a straight path parallel to one of the principal axes of the cubic lattice. We find the remarkable result
\[ < l_y > \equiv \frac{1}{2} \left[ F_y(r\cos\theta, r\sin\theta) + F_y(r\cos(\pi/2 - \theta), r\sin(\pi/2 - \theta)) \right] = \frac{pQ}{a^3} \]

\[ F_y = \frac{2\pi Q}{3} \frac{m|v|}{qc} \]  
\[ M = \frac{\nu m}{a^3} \]  
\[ E_y = \frac{2\pi |v|}{3} c M. \]  
\[ j_x = -n_\uparrow e v \]  
\[ R_s = -\frac{1}{6n_\uparrow e c} = \frac{R_\phi}{6} \]

Consider now a general situation in a solid. The transverse field from the effect discussed here is obtained by averaging Eq. (19) over all charge carriers. The particle number current for electrons of spin \( \sigma = \uparrow, \downarrow \) is:

\[ j_\sigma = \sum_\nu \int \frac{d^3k}{(2\pi)^3} \frac{g(\varepsilon^\nu_\sigma(k))}{h} v^\nu_\sigma(k) \]  

where \( g \) is the electron distribution function, \( \nu \) labels the bands, and the velocity is given by

\[ v^\nu_\sigma(k) = \frac{1}{\hbar} \frac{d\varepsilon^\nu_\sigma(k)}{dk} \]  

with \( \varepsilon^\nu_\sigma(k) \) the band energy. The charge current is given by

\[ j_{ch} = (-e)(j_\uparrow + j_\downarrow) \]  

and the spin current, or more precisely magnetic moment current, by

\[ j_{spin} = -\mu_B(j_\uparrow - j_\downarrow) \]  

with \( \mu_B \) the Bohr magneton. The Hall field originating from the effect under discussion here is then

\[ E_y = \frac{2\pi \gamma}{3} c \frac{j_{spin}}{j_{ch}} \]  

so that the contribution to the Hall resistivity due to this effect is

\[ \rho_H = \frac{2\pi \gamma}{3} \frac{j_{spin}}{j_{ch}} \frac{\hbar}{e} \]  

and the anomalous Hall coefficient is given by

\[ R_s = \frac{1}{6} \frac{\gamma j_{spin}}{j_{ch} M} \]  

with the magnetization \( M \) given by

\[ M = -\mu_B \sum_\nu \int \frac{d^3k}{(2\pi)^3} \left[ f(\varepsilon^\nu_\uparrow(k)) - f(\varepsilon^\nu_\downarrow(k)) \right] \]  

with \( f \) the Fermi function.

Within semiclassical transport theory and the relaxation time approximation we have for the current in the presence of an applied longitudinal electric field \( E \):

\[ j_\sigma = (-e) \sum_\nu \tau_\nu(k) \left[ \frac{d}{dk} \frac{d^2\varepsilon^\nu_\sigma(k)}{d^2k} \right] v^\nu_\sigma(k) v^\nu_\sigma(k) E \]

with \( \tau \) the collision time. Assuming \( \tau \) depends on momentum only through the band energy and temperatures much smaller than the Fermi energy Eq. (30) can be rewritten as

\[ j_\sigma = (-e) \sum_\nu \tau_\nu(\varepsilon_F) \left[ \int \frac{d^3k}{(2\pi)^3} \frac{1}{\hbar^2} \frac{d^2\varepsilon^\nu_\sigma(k)}{dk^2} \varepsilon(\varepsilon(k)) \right] E \]
Consider in particular the case of a nearly empty band with isotropic Fermi surface. We have
\[
\frac{1}{\hbar^2} \frac{d^2 \varepsilon_{\sigma}}{dk^2 \ell} = \frac{1}{m_{\sigma}} \quad (32a)
\]
and for small $\Delta$ we have
\[j_{\sigma} = \frac{\tau_{\sigma} n_{\sigma}}{m_{\sigma}} E \quad (32b)\]
so that
\[M = -\mu_B (n_\uparrow - n_\downarrow) \quad (32c)\]
so that
\[R_s = -\gamma \frac{n_\uparrow \tau_\uparrow - n_\downarrow \tau_\downarrow}{6ec \left( \frac{n_\uparrow \tau_\uparrow}{m_\uparrow} + \frac{n_\downarrow \tau_\downarrow}{m_\downarrow} \right)} (n_\uparrow - n_\downarrow) \quad (33)\]
For the case where $m_\uparrow = m_\downarrow$ and $\tau_\uparrow = \tau_\downarrow$ the anomalous Hall coefficient reduces to Eq. (21). However because majority and minority spin Fermi surfaces will be different, the relaxation times could be very different. Furthermore, when there is spin polarization the effective masses of spin up and down electrons could be very different due to interaction effects. If the effective masses for up and down electrons are very different Eq. (33) leads to a Hall coefficient $R_s$ that can be substantially larger than the ordinary one.

Even without assuming different effective masses for up and down electrons the Hall resistivity from this effect will in general be different than Eq. (21). Consider again a single band, and a Stoner-like description of magnetism. In the magnetic state the band energies are shifted by the exchange splitting $\Delta$:
\[
\varepsilon_\uparrow(k) = \varepsilon(k) - \frac{\Delta}{2} \quad (34a)
\]
\[
\varepsilon_\downarrow(k) = \varepsilon(k) + \frac{\Delta}{2} \quad (34b)
\]
the magnetization is given by eq. (29) (for a single band) and for small $\Delta$ we have
\[M = -\mu_B g(\varepsilon_F) \Delta \quad (35)\]
with $g(\varepsilon_F)$ the density of states at the Fermi energy. The spin current Eq. (25) is
\[j_{\text{spin}} = e\tau(\varepsilon_F) \mu_B \Delta \left[ \int \frac{d^3k}{(2\pi)^3} \frac{1}{\hbar^2} \frac{d^2 \varepsilon(k)}{dk^2} \left( -\frac{df}{d\varepsilon} \right) \right] E \quad (36)\]
or
\[j_{\text{spin}} = -e\tau(\varepsilon_F) M \left( \frac{1}{\hbar^2} \frac{d^2 \varepsilon(k)}{dk^2} \right)_{F.S.} E \quad (37)\]
where $M$ is the magnetization. The quantity in brackets is the effective mass tensor
\[M^{-1}(k) = \frac{1}{\hbar^2} \frac{d^2 \varepsilon(k)}{dk^2} \quad (38)\]
and it is averaged over the Fermi surface. In contrast the charge current is given by
\[j_{\text{charge}} = e^2 \tau(\varepsilon_F) \left[ \int \frac{d^3k}{(2\pi)^3} \frac{1}{\hbar^2} \frac{d^2 \varepsilon(k)}{dk^2} f(\varepsilon(k)) \right] E \quad (39)\]
so that it involves the effective mass tensor integrated over the occupied (or empty) states:
\[\langle M^{-1} \rangle_{\text{occ}} = \frac{\int \frac{d^3k}{(2\pi)^3} \frac{1}{\hbar^2} \frac{d^2 \varepsilon(k)}{dk^2} f(\varepsilon(k))}{\int \frac{d^3k}{(2\pi)^3} f(\varepsilon(k))} \quad (40)\]
Eq. (27) then yields
\[\rho_H = -\gamma \frac{\langle M^{-1} \rangle_{F.S.}}{6ec \langle M^{-1} \rangle_{\text{occ}}^{\uparrow \text{occ}}} 4\pi M \quad (41)\]
where the number of carriers $n$ is given by the denominator of Eq. (40). We have assumed that the applied electric field is along a principal axis of the effective mass tensor labeled by $i$.

Eq. (41) shows that this contribution to the Hall voltage is proportional to the magnetization, as Eq. (2) indicates. The magnitude depends on the detail of the band structure through the averages of the effective mass tensors given in Eq. (41). Although for an almost empty (or almost full) band the two averages in Eq. (41) will be very similar, in other cases they could be very different.

It is important to emphasize that the anomalous Hall effect discussed here originates in the spin current and not in the magnetization. Consider for example the situations depicted in Figs. 3(a) and (b). In both cases the carriers in the spin-up band, that carries the current, are electrons, since it is less than 1/2-full, so that the ordinary Hall coefficient $R_s < 0$. However, in (a) the spin-down band is empty while in (b) it is full. Hence, in (a) the current carriers have magnetic moment parallel to the overall magnetization and the effect discussed in this paper predicts an anomalous Hall coefficient of the same sign as the ordinary one, $R_s < 0$. Instead in (b), the current carriers have magnetic moment opposite to the overall magnetization, hence the anomalous Hall coefficient $R_s > 0$, opposite in sign to the ordinary one.

Consider then the Hall voltages for a single band that undergoes full spin polarization. For simplicity we assume the band is symmetric around its center, as occurs for example for a tight binding band in a bipartite lattice structure. At onset of spin polarization the ordinary and anomalous Hall coefficient will have the same sign, negative for $n < 1$ (less than half-filled band) and positive for $n > 1$. For full spin polarization instead, the ordinary and anomalous Hall coefficients as function of the total band filling $n = n_\uparrow + n_\downarrow$ (i.e. the band filling of the unpolarized band) have the behavior indicated in Figure
4. In particular, they have opposite sign for band fillings between one quarter and three quarters. Note also that $R_s$ is discontinuous at half filling because the spin current switches sign at that point.

In the presence of several bands the analysis will become more complicated, but it is clear that a detailed analysis can provide useful information on the band structure of the metal. It should also be kept in mind that contributions to the ordinary Hall coefficient will arise both from current carriers in bands involved in the magnetism as well as those in non-magnetic bands. Temperature dependence of the effect discussed here will arise both from the temperature dependence of the exchange splitting, the relaxation time $\tau(k)$, and possibly the effective masses. It is clear that the temperature dependence of spin and charge currents could be very different, as the relaxation time will involve different phonons for the majority and minority spin bands. Thus the Hall voltage Eq. (26) may exhibit strong temperature dependence, as observed experimentally. It is possible that the effect discussed here is an important contribution to the anomalous Hall effect observed in ferromagnetic metals.

In summary, we have shown here that a Hall field will arise when a spin current flows in a solid. This effect should contribute to the anomalous Hall effect of ferromagnetic metals because flow of a charge current in the presence of nonzero magnetization will generally (although not always) be accompanied by flow of a spin current. Although in the simplest model this contribution to the Hall effect is of similar magnitude to that of the ordinary effect, in general the contribution will be different and could be substantially larger. The detailed temperature dependence in various cases should be investigated.

One could also have a situation where a spin current flows in the absence of magnetization. In a hypothetical half-metallic antiferromagnet [10], flow of charge current is accompanied by flow of spin current in the absence of magnetization, and a transverse Hall field will result. In the model of spin-split metals discussed in Ref. 17, a spin current flows in the absence of both magnetization and charge current, and a transverse Hall field should also exist. Other examples are discussed in Ref. 18, in connection with manganites, and in Ref. 19 for superfluid $^3$He. Finally, a pure spin current is also predicted to occur in the geometry of the ‘spin Hall effect’ discussed in Ref. 20, where opposite edges of a conducting sample are connected through a transverse conducting strip.

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FIG. 1. Line of magnetic moments pointing in the $z$ direction moving along the $x$ direction with velocity $v$. The electric field generated in the laboratory frame is the same as that obtained from a line of equally spaced stationary electric dipole moments pointing along the $(-y)$ direction, of magnitude given by Eq. (7).

FIG. 2. Simple cubic lattice of charges $Q$. Magnetic moments pointing in the $z$ direction are propagating in the $x$ direction and are equivalent to dipoles $p$ shown. The direction of the force on the dipoles due to the periodic lattice of charges $Q$ depends on the position of the dipoles and is indicated by the bold arrows. The length of these arrows indicates qualitatively the relative magnitudes for the three different positions shown.

FIG. 3. Examples where the anomalous and ordinary Hall coefficients have the same sign (a) and opposite sign (b). In both cases $R_o < 0$. The dashed lines indicate the positions of the Fermi level. In (a), the current carriers have magnetic moment parallel, in (b) antiparallel, to the total magnetization.

FIG. 4. Ordinary (full line) and anomalous (dashed line) Hall coefficients versus total band filling $n = n_\uparrow + n_\downarrow$ for a fully polarized single band symmetric around its center (schematic). For unpolarized band filling less than one quarter ($n < 1/2$) and more than three quarters ($n > 3/2$) the ordinary and anomalous Hall coefficients have the same sign. The long-dash short-dash line indicates the ordinary Hall coefficient at onset of spin polarization.
Figure 2
Figure 3

(a) $R_s < 0$

(b) $R_s > 0$
Figure 4