On Strong Observational Refinement and Forward Simulation

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Abstract

Hyperproperties are correctness conditions for labelled transition systems that are more expressive than traditional trace properties, with particular relevance to security. Recently, Attiya and Enea studied a notion of strong observational refinement that preserves all hyperproperties. They analyse the correspondence between forward simulation and strong observational refinement in a setting with finite traces only. We study this correspondence in a setting with both finite and infinite traces. In particular, we show that forward simulation does not preserve hyperliveness properties in this setting. We extend the forward simulation proof obligation with a progress condition, and prove that this progressive forward simulation does imply strong observational refinement.

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1 Introduction

Hyperproperties [2] form a large class of properties over sets of sets of traces, characterising, in particular, security properties such as generalised non-interference that are not expressible over a single trace. Like trace properties, which can be characterised by a conjunction of a safety and a liveness property, every hyperproperty can be characterised as the conjunction of a hypersafety and hyperliveness property.

Recently, Attiya and Enea proposed strong observational refinement, a correctness condition that preserves all hyperproperties, even in the presence of an adversarial scheduler. An object $O_1$ strongly observationally refines an object $O_2$ if the executions of any program $P$ using $O_1$ as scheduled by some admissible deterministic scheduler cannot be observationally distinguished from those of $P$ using $O_2$ under another deterministic scheduler. They showed that strong observational refinement preserves all hyperproperties. Furthermore, they prove that forward simulation [6] implies strong observational refinement. Forward simulation alone is sound but not complete for ordinary refinement, and in general both backward and forward simulation are required. Forward simulation is furthermore known to not preserve liveness.
int * current_val initially 0

int fetch_and_add(int k):
F1. do
F2. n = LL(&current_val)
F3. while (!SC(&current_val, n + k))
F4. return n

Figure 1 A fetch-and-add implementation with a nonterminating schedule when LL and SC are implemented using the algorithm of [4].

properties, which motivates our study of forward simulation and observational refinement in the context of infinite traces and hyperliveness.

As a result we show – by example – that forward simulation does not preserve hyperliveness. Furthermore, forward simulation alone cannot guarantee strong observational refinement when requiring admissibility of schedulers, i.e., when schedulers are required to continually schedule enabled actions. To address these limitations, we employ a version of forward simulation extended with a progress condition, thereby guaranteeing strong observational refinement and preservation of hyperliveness.

2 Motivating Example

We start by giving an example of an abstract atomic object \( O_2 \) and a non-atomic implementation \( O_1 \) such that there is a forward simulation from \( O_1 \) to \( O_2 \), but hyperliveness properties are not preserved for all schedules.

As the atomic abstract object \( O_2 \) we choose a fetch-and-add object with just one operation, \( \text{fetch\_and\_add}(\text{int } k) \), which adds the value integer \( k \) to a shared integer variable and returns the value of that variable before the addition. Let \( P \) be a program with two threads \( t_1 \) and \( t_2 \), each of which executes one \( \text{fetch\_and\_add} \) operation and assigns the return value to a local variable of the thread. Clearly, for any scheduler \( S \), the variable assignment of both threads will eventually occur. This “eventually” property can be expressed as a hyperproperty.

Now, consider the fetch-and-add implementation presented in Figure 1. This implementation uses the load-linked/store-conditional (LL/SC) instruction pair. The \( \text{LL}(\text{ptr}) \) operation loads the value at the location pointed to by the pointer \( \text{ptr} \). The \( \text{SC}(\text{ptr},v) \) conditionally stores the value \( v \) at the location pointed to by \( \text{ptr} \) if the location has not been modified by another \( \text{SC} \) since the executing thread’s most recent \( \text{LL}(\text{ptr}) \) operation. If the update actually occurs, \( \text{SC} \) returns \text{true}, otherwise the location is not modified and \( \text{SC} \) returns \text{false}. In the first case, we say that the \( \text{SC} \) succeeds. Otherwise, we say that it fails.

Critically, we stipulate that the LL and SC operations are implemented using the algorithm of [4]. This algorithm has the following property. If thread \( t_1 \) executes an LL operation, and then thread \( t_2 \) executes an LL operation before \( t_1 \) has executed its subsequent SC operation, then that SC is guaranteed to fail. This happens even though there is no intervening modification of the location.

Now, let \( O_1 \) be a labeled transition systems (LTS) representing a multithreaded version of this fetch_and_add implementation, using the specified LL/SC algorithm. Consider

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1. There are several ways to represent a multithreaded program or object as an LTS, e.g., [7, 5].
furthermore the program $P$ (above) running against the object $O_1$. A scheduler can continually alternate the LL at line $F2$ of $t_1$ and that of $t_2$, such that neither the `fetch_and_add` operation ever completes. Therefore, unlike when using the $O_2$ object, the variable assignments of $P$ will never occur, so the $O_1$ system does not satisfy the hyperproperty for all schedulers.

There is, however, a forward simulation from $O_1$ to $O_2$. The underlying LL/SC implementation can be proven correct by means of forward simulation, as can the `fetch_and_add` implementation. Therefore, standard forward simulation is insufficient to show that all hyperproperties are preserved.

3 Forward Simulation and Strong Observational Refinement

Next, we give a formalization of the above discussed concepts. In the definitions we closely follow Attiya and Enea’s notation $\mathcal{I}$. Objects as well as programs using objects are described by labelled transition systems. An LTS $A = (Q, \Sigma, s_0, \delta)$ consists of a (possibly infinite) set of states $Q$, an alphabet $\Sigma$, an initial state $s_0$ and a transition relation $\delta \subseteq Q \times \Sigma \times Q$. We write $s \xrightarrow{\delta} A s'$ for $(s, a, s') \in \delta$ and extend this to sequences of actions, i.e., $s \xrightarrow{\delta_1 \cdots \delta_n} A s'$ if there exists $s_0, \ldots, s_n$ such that $s_0 = s$, $s_{n+1} = s'$ and $s_i \xrightarrow{\delta_i} A s_{i+1}$, $0 \leq i \leq n$. We in particular have $s \xrightarrow{\varepsilon} s$, $\varepsilon$ being the empty sequence. An action $a$ is said to be enabled in a state $s$ if there is a transition $s \xrightarrow{\delta} A s'$ for some $s'$.

An execution of an LTS $A$ is a finite or infinite sequence $s_0 \cdot a_1 \cdot s_1 \cdot a_2 \cdot \ldots$ that alternates states and actions, and ends with a state if finite. In this, $s_0$ is the initial state of $A$, $s_{i-1} \xrightarrow{\delta} A s_i$ must hold for all steps of the sequence. A trace is the sequence of actions of such an execution. The set of traces is denoted $T(A)$. This set contains finite traces $\sigma = a_1 \cdot a_2 \cdot \ldots \cdot a_n$ with length $\#\sigma = n \geq 0$, including the empty sequence $\varepsilon$, and infinite traces, typically denoted by $\tau$. The prefix relation on traces is denoted $\sigma \sqsubseteq \tau$ (where $\tau$ may be finite or infinite), $\sigma \sqsubset \tau$ denotes a proper prefix. $\tau|n$ is the prefix of length $n$ of an infinite trace $\tau$, $\tau|\infty$ is the next action after $\tau|n$. We assume that every LTS has a special `idle` action, that is enabled whenever no other action is enabled, and that does not change the state. This implies that any finite trace $\sigma$ can be extended to an infinite trace $\tau$ with $\sigma \sqsubseteq \tau$.

Terminating executions are modelled by infinite sequences composed of a finite sequence of “proper” actions, followed by an infinite sequence of `idle` actions. Nonterminating executions contain no `idle` actions.

An LTS $A$ is deterministic if for any finite trace $\sigma \in \Sigma^*$ there is a single state $s'$ with $s_0 \xrightarrow{\delta} A s'$. Hence, we can define the state $\text{state}(\sigma)$ to be reached after $\sigma$ as this $s'$. For some alphabet $\Gamma \subseteq \Sigma$ and (finite or infinite) trace $\tau$, we define the projection $\tau|\Gamma$ to be the maximal subsequence of $\tau$ containing $\Gamma$-actions only.

Both programs and objects will be given as deterministic LTSs. A program $P$ executes actions out of its own alphabet $\Sigma_P$ plus `call` actions out of a set $C$ and `return` actions of $R$. We let $\Gamma_P$ be $\Sigma_P \cup C \cup R$. Objects $O$ implement the operations being called (either atomically or non-atomically), and thus are LTSs over the alphabet $C \cup R$ plus some alphabet of internal actions $\Sigma_O$ (e.g., the actions corresponding to LL and SC of the `fetch_and_add` implementation are internal actions). Program $P$ and object $O$ synchronize via call and return actions. Formally, this is defined as the usual product of LTSs, denoted $P \times O$.

Adversaries in this setting are modelled by schedulers. A scheduler drives the execution of $P \times O$ in a particular direction. For a deterministic LTS over alphabet $\Sigma$, a scheduler is
On Strong Observational Refinement and Forward Simulation

a function \( S : \Sigma^* \rightarrow 2^\Sigma \). This function prescribes the actions that can be taken in a next step. A (finite or infinite) trace \( \tau \) is consistent with a scheduler \( S \) if \( \tau[n] \in S(\tau \upharpoonright n) \) for every proper prefix \( \tau \upharpoonright n \subseteq \tau \). We write \( T(A, S) \) for the set of traces of \( A \) consistent with \( S \). A scheduler is admitted by an LTS \( A \) if for all finite traces \( \sigma = a_1 \ldots a_n \) of \( A \) consistent with \( S \), the scheduler satisfies (i) \( S(\sigma) \) is non-empty and (ii) all actions in \( S(\sigma) \) are enabled in state(\( \sigma \)).

Besides being admissible, schedulers for programs and objects (LTSs of the form \( P \times O \)) also have to be deterministic: they must resolve the nondeterminism on the actions of the object. A scheduler \( S \) of an LTS \( P \times O \) is deterministic if (i) \( S(\tau) \subseteq \Sigma_P \) (i.e., it can choose several program actions) or (ii) \( |S(\tau)| = 1 \) (i.e., if \( S \) chooses an action of \( O \), then exactly 1).

Now we are ready to define strong observational refinement.

\textbf{Definition 1.} An object \( O_1 \) strongly observationally refines the object \( O_2 \), written \( O_1 \leq_S O_2 \), iff for every deterministic scheduler \( S_1 \) admitted by \( P \times O_1 \) there exists a deterministic scheduler \( S_2 \) admitted by \( P \times O_2 \) such that \( T(P \times O_1, S_1)|\Sigma_P = T(P \times O_2, S_2)|\Sigma_P \) for all programs \( P \) over alphabet \( \Gamma_P = \Sigma_P \cup C \cup R \).

Note that the definition of strong observational refinement here considers both finite and infinite traces, which is necessary for preservation of all hyperproperties.

\textbf{Definition 2.} Let \( A_i = (Q_i, \Sigma_i, s_{i0}, \delta_i) \), \( i = 1, 2 \), be two LTSs and \( \Gamma \) an alphabet.

A relation \( F \subseteq Q_1 \times Q_2 \) is a \( \Gamma \)-forward simulation from \( A_1 \) to \( A_2 \) iff \( (s_0^i, s_0^j) \in F \) and for all \( (s_1, s_2) \in F \) if \( s_1 \xrightarrow{\alpha} s_1' \) then there exist \( \alpha \in \Sigma_2^* \) and \( s_2' \in Q_2 \) such that \( \alpha|\Gamma = \alpha|\Gamma \), \( s_2 \xrightarrow{\alpha} s_2' \) and \( (s_1', s_2') \in F \).

Note that \( \alpha|\Gamma \) in the above definition may in particular be \( \varepsilon \). This is then called a stuttering step: the abstract object \( O_2 \) matches the internal action \( a \) of object \( O_1 \) by an empty step. One of the main theorems of \( \Pi \) now states the following property.

\( O_1 \leq_S O_2 \) if and only if there exists a \( (C \cup R) \)-forward simulation from \( O_1 \) to \( O_2 \).

As we have exemplified in Section 2, we might however have objects which have executions with an infinite number of internal actions. For the above given \texttt{fetch and add} implementation this might occur if there are two threads concurrently trying to add to some variable. In that case the loops (of both threads) might not terminate. Now, assume \( P \) to be the program sketched above consisting of these threads calling \texttt{fetch and add} and then assigning return values to local variables. A deterministic admissible scheduler \( S_1 \) for \( P \) and \( O_1 \) can drive \( P \times O_1 \)'s execution along the infinite trace of LL and SC operations, thereby making the calls to \texttt{fetch and add} never return. On the other hand, any scheduler for the \( O_2 \) system must eventually execute call and return actions for both \texttt{fetch and add} operations, and subsequently execute the writes to the program variables. This is because an admissible scheduler must schedule enabled actions until no more are available, from which point only \texttt{idle} actions can be scheduled. The projection of such a completed trace to program actions is not in the set \( T(P \times O_1, S_1)|\Sigma_P \). This is true, even in a setting with only consider finite traces.

Hence, the existence of a \( (C \cup R) \)-forward simulation does not imply strong observational refinement, contradicting Lemma 5.2 of \( \Pi \).
4 Progressive Forward Simulation implies Strong Observational Refinement

The above example already indicates where a possible repair of this lemma could start. The forward simulation has to guarantee some sort of progress, so that the scheduler $S_2$ is always able to schedule some action without producing a trace not present in $P \times O_1$ under $S_1$. This guarantee can be made if we disallow infinite stuttering.

Definition 3 (Progressive Forward Simulation). Let $A_i = (Q_i, \Sigma_i, s_i^0, \delta_i), i = 1, 2$, be two LTSs and $\Gamma$ an alphabet. A relation $F \subseteq Q_1 \times Q_2$ together with a well-founded order $\ll \subseteq Q_1 \times Q_1$ is called a progressive $\Gamma$-forward simulation from $A_1$ to $A_2$ iff

\[
(s_1^0, s_2^0) \in F, \text{ and }
\]

for all $(s_1, s_2) \in F$, if $s_1 \xrightarrow{a} s_1'$ and $a \in \Sigma_1$, then there exist $\alpha \in \Sigma_2^*$ and $s_2' \in Q_2$ such that $a \mid \Gamma = \alpha \mid \Gamma$, $s_2 \xrightarrow{\alpha} s_2'$ and $(s_1', s_2') \in F$. When $\alpha = \varepsilon$ then $s_1' \ll s_1$ is required.

The definition requires that the concrete state decreases in the well-founded order when the abstract sequence $\alpha$ in the forward simulation is empty and $s_2 = s_2'$ (stuttering). Thinking in terms of the usual commuting diagrams in a forward simulation this is when the diagram formed by the states $s_1, s_1', s_2$ is triangular. Progressiveness prohibits an infinite sequence of concrete internal steps that map to the empty abstract sequence. For the above given object $O_1$ with the \texttt{fetch\_and\_add} implementation no such well-founded ordering on states satisfying the progress condition can be given.

The use of a forward simulation with well-founded ordering has already been used in the context of non-atomic refinement [3]. With this change in place, the desired implication now holds.

Theorem 4. If there exists a progressive $(C \cup R)$-forward simulation from $O_1$ to $O_2$, then $O_1 \leq_S O_2$.

The proof can be found in the appendix.

5 Conclusion

In this paper, we have reported on our findings that forward simulation does not imply strong observational refinement in a setting with infinite traces. We have furthermore proposed a notion of progressive forward simulation implying strong observational refinement. In future work, we will investigate whether strong observational refinement implies progressive forward simulation, thereby hopefully re-establishing an equivalence.

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On Strong Observational Refinement and Forward Simulation

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Figure 2 Constructing \( f(\tau_1) \in T(P \times O_2) \) from \( \tau_1 \in T(P \times O_1, S_1) \) with \( a_3, a_4 \in \Sigma_1 \setminus (C \cup R), \alpha, a_1, a_2 \in (\Sigma_2 \setminus (C \cup R))^* \) and \( cs_2 \preceq cs_1 \).

A Proof of Theorem 4

Given \( O_1 \) and \( O_2 \) for which a progressive forward simulations exists, and an arbitrary program \( P \) together with a scheduler \( S_1 \) for traces over \( P \times O_1 \) our proof has to construct a scheduler \( S_2 \) such that \( T(P \times O_1, S_1)[\Sigma_P] = T(P \times O_2, S_2)[\Sigma_P] \). The construction is in two steps: First a function \( f \) is constructed that maps traces \( \tau_1 \in T(P \times O_1, S_1) \) to traces \( f(\tau_1) \in T(P \times O_2) \). This function has to be carefully defined to then allow the definition of a scheduler \( S_2 \) that schedules exactly all the \( f(\tau_1) \). Progressiveness is key to ensure that for an infinite trace \( \tau_1 \) the trace \( f(\tau_1) \) is infinite as well, which allows to schedule actions for any prefix.

The construction of \( f \) shown in Fig. 2 first has to fix a unique sequence of abstract actions in \( f(\tau_1) \) that correspond to a single step of \( \tau_1 \). To do this a mapping \( m \) is defined. For two states \( cs \in Q_1 \) and \( as \in Q_2 \) with \( F(cs, as) \) and an action \( a \in \Sigma_1 \), \( m \) returns a fixed sequence \( \alpha \in \Sigma_2 \) such that \( F(cs', as) \) holds again for the (unique) states with \( cs \xrightarrow{a} cs' \) and \( as \xrightarrow{a} as' \). The existence of \( \alpha \) is guaranteed by the main proof obligation for a forward simulation. To be useful for constructing traces over \( P \times O_2 \) when a step of a trace over \( P \times O_1 \) is given, we extend the definition to allow a program action \( a \in \Sigma_p \) as well. In this case \( m \) just returns the one element sequence of \( a \). Intuitively, in addition to the commuting diagrams of the forward simulation this defines commuting diagrams that map program steps one-to-one. Formally

\[
m : Q_1 \times (\Sigma_1 \cup \Sigma_p) \times Q_2 \rightarrow (\Sigma_2 \cup \Sigma_p)^*
\]
is defined to return \( m(cs, a, as) := a \) when \( a \in \Sigma_p \), and to return the fixed sequence \( \alpha \) as described above when \( a \in \Sigma_1 \).

It is then possible to define partial functions \( f_0, f_1, \ldots \) (viewed as sets of pairs) with \( \text{dom}(f_n) = \{ \sigma_1 \in T(P \times O_1, S_1) : \#\sigma_1 \leq n \}, \text{cod}(f_n) \subseteq T(P \times O_2) \), such that \( f_0 \subseteq f_1 \subseteq \ldots \) inductively as follows:

\[
f_0 = \{ (\varepsilon, \varepsilon) \}
\]

\[
f_{n+1} = f_n \cup \{ (\sigma_1 \cdot a, f(\sigma_1) \cdot \alpha) \mid \sigma_1 \cdot a \in T(P \times O_1, S_1), \#\sigma_1 = n, \alpha = m(\text{state}(\sigma_1).\text{obj}, a, \text{state}(f(\sigma_1)).\text{obj}) \}
\]

The inductive definition maps the new action \( a \in S_1(\sigma_1) \) to the corresponding sequence \( \alpha \) that is chosen by \( m \). In the definition \( (ps, cs).\text{obj} := cs \) when the final state of \( \sigma_1 \) is \( \text{state}(\sigma_1) = (ps, cs) \). Analogously \( (ps, as).\text{obj} = as \).

The states \( (ps, cs) = \text{state}(\sigma_1) \) and \( (ps', as) = \text{state}(f_n(\sigma_1)) \) reached at the end of two corresponding traces always satisfy \( ps = ps' \) and \( F(cs, as) \). The use of \( m \) in the construction guarantees that all the \( f_n \) are prefix-monotone: if \( f_n \) is defined on \( \sigma \) and \( \sigma' \subseteq \sigma \) then \( f_n(\sigma') \subseteq f_n(\sigma) \).
Now, define \( f := \bigcup_n f_n \). Function \( f \) is obviously prefix-monotone as well. Intuitively, it maps each finite trace of \( T(P \times O_1, S_1) \) to a corresponding abstract trace, where \( m \) is used in each commuting diagram to choose the abstract action sequence.

If \( \tau_1 \) is an infinite trace from \( T(P \times O_1, S_1) \), and \( \sigma_n := f(\tau_1|n) \), then \( \sigma_0 \subseteq \sigma_1 \subseteq \sigma_2 \subseteq \ldots \). The length of \( \sigma_n \) always eventually increases again, otherwise the concrete trace would execute infinitely many stutter steps, which is ruled out by the forward simulation being progressive. Therefore the \( \sigma_n \) converge to an infinite sequence \( \tau_2 \in T(P \times O_2) \) and we can extend the definition of \( f \) to have \( f(\tau_1) := \tau_2 \).

We will now define a scheduler \( S_2 \), that will schedule exactly those traces in \( \sigma_2 \in T(P \times O_2) \) where \( \sigma_2 \) is a prefix of some \( f(\tau_1) \), where \( \tau_1 \) is an infinite trace in \( T(P \times O_1, S_1) \). Before we can do this properly, a number of lemmas is needed.

**Lemma 5.** \( f(\sigma_1)|\Gamma_P = \sigma_1|\Gamma_P \) for all \( \sigma_1 \in T(P \times O_1, S_1) \).

**Proof.** This should be obvious from the construction, since the forward simulation guarantees that \( m(cs, a, as)|\Gamma_P = a|\Gamma_P \) for all \( a \in \Sigma_\rho \) is mapped by identity.

**Lemma 6.** For two finite traces \( \sigma_1, \sigma_1' \in T(P \times O_1, S_1) \): if \( f(\sigma_1) \) and \( f(\sigma_1') \) have the same program actions in \( \Gamma_P \), then \( \sigma_1 \) is a prefix of \( \sigma_1' \) or vice versa, and the longer one just adds internal actions of \( O_1 \).

**Proof.** Lemma 5 implies \( \sigma_1|\Gamma_P = \sigma_1'|\Gamma_P \). If the lemma were wrong, then there would be a maximal common prefix \( \sigma_0 \) and two actions \( a \neq a' \) such that \( \sigma_0 \cdot a \subseteq \sigma_1 \) and \( \sigma_0 \cdot a' \subseteq \sigma_1' \). The case where both \( a \) and \( a' \) are external actions is impossible, otherwise the external actions in \( \sigma_1 \) and \( \sigma_1' \) would not be the same. If however one of them is internal, then \( S_1(\sigma_0) \) is a one-element set, and both \( a \) and \( a' \) must be in the set, contradicting \( a \neq a' \).

**Lemma 7.** For all finite prefixes \( \sigma_2 \) of \( f(\tau_1) \), there is a unique \( n \), such that \( f(\tau_1|n) \subseteq \sigma_2 \subseteq f(\tau_1|n) \cdot \alpha \), where \( \alpha := m(\text{state}(\tau_1|n).\text{obj}, \tau[n], \text{state}(\tau_1|n).\text{obj}) \neq \epsilon \).

Intuitively, each element of \( f(\tau_1) \) is added by a uniquely defined commuting diagram.

**Proof.** First, note that \( f(\tau_1|n+1) = f(\tau_1|n) \cdot \alpha \). Since the lengths of \( f(\tau_1|n) \) are increasing with \( n \) to infinity (and \( f(\tau_1|n) = f(\varepsilon) = \varepsilon \) \( n \) is the biggest index where the length of \( f(\tau_1|n) \) is still less or equal to \( \#\sigma \).

**Lemma 8.** Assume \( \tau_1, \tau_1' \in T(P \times O_1, S_1) \). if \( \sigma_2 \) is a prefix of both \( f(\tau_1) \) and \( f(\tau_1') \), then there is \( m \) such that \( \tau_1|m = \tau_1'|m \) and \( \sigma_2 \subseteq f(\tau_1|m) \).

The lemma says, that a common prefix of two traces in the image of \( f \) is possible only as the result of a common prefix in the domain of \( f \).

**Proof.** Since \( \sigma_2 \subseteq f(\tau_1) \) and each step from \( f(\tau_1|n) \) to \( f(\tau_1|n+1) \) adds at most one program action, a minimal index \( n \) can be found such that \( \sigma_2 \) has the same program actions as \( f(\tau_1|n) \), while \( f(\tau_1|n-1) \) has fewer when \( n \neq 0 \). Similarly, a minimal index \( n' \) can be found such that \( \sigma_2|\Gamma_P = f(\tau_1'|n') \). By Lemma 5 above, it follows that \( \tau_1|n \) is a prefix of \( \tau_1'|n' \) or vice versa, with only internal \( O_1 \)-actions added to the longer one. If both are equal, then \( n \) and \( n' \) and \( m \) can be set to be \( n \). However, when the two are not equal, the longer one, say \( \tau_1'|n' \), ends with an internal \( O_1 \)-action. But then, since this action is mapped to a sequence of internal \( O_2 \)-actions \( f(\tau_1'|n'-1) \) also has the same program actions than \( \sigma_2 \), contradicting the minimality of \( n' \).
Equipped with these lemmas, it is now possible to define the scheduler \( S_2 \) and to prove it is well-defined.

**Definition 9.** We define \( S_2(\sigma_2) \) for any finite prefix \( \sigma_2 \) of any \( f(\tau_1) \), where \( \tau_1 \in T(P \times O_1, S_1) \). The definition uses Lemma 7 to find a unique index \( n \), such that \( f(\tau_{1|n}) \subseteq \sigma_2 \subseteq f(\tau_{1|n+1}) \), where \( n = m(\text{state}(\tau_{1|n}).\text{obj}, \tau_{1|n}),\text{state}(f(\tau_{1|n})).\text{obj}) \neq \epsilon \). Since \( \sigma_2 \) is a proper prefix, there is an event \( a \), such that \( \sigma_2 \cdot a \subseteq f(\tau_{1|n}) \cdot \alpha \), and \( a \) is an element of \( \alpha \).

If \( a \) is an external action in \( \Gamma_P \) then \( a \) must be equal to \( \tau[n] \) (\( \alpha \) contains either \( \tau[n] \) if it is an external action, or no external action at all). In this case we set \( S_2(\sigma_2) := S_1(\tau_{1|n}) \). Note that \( a \) is enabled and in \( S_1(\tau_{1|n}) \) in this case. Otherwise, when \( a \notin \Gamma_P \), we set \( S_2(\sigma_2) := \{a\} \).

**Theorem 10.** \( S_2 \) is well-defined.

**Proof.** Assume that \( \sigma_2 \) is a prefix of two traces \( f(\tau_1) \) and \( f(\tau_1') \). We prove that this never leads to two different definitions of \( S_2(\sigma_2) \). First, Lemma 7 gives an index \( m \) with \( \tau_1|n = \tau_{1'|m} \) and \( \sigma_2 \subseteq f(\tau_{1|m}) \). If \( \sigma_2 \) is a proper prefix of \( f(\tau_{1|m}) \), then the \( n \) used in the construction of \( S_2 \) must satisfy \( n + 1 \geq m \), and the prefix \( f(\tau_{1|m+1}) = f(\tau_{1|n}) \cdot \alpha \) on which the definition of \( S_2 \) is based, is the same for both traces. The remaining case is \( m = n + 1 \) and \( \sigma_2 \subseteq f(\tau_{1|m+1}) \).

In this case the next elements \( \tau_{1|n+1} \) and \( \tau_{1'|n+1} \) in the two traces \( \tau_1 \) and \( \tau_1' \) could be different. If one of them is internal (i.e. not in \( \Gamma_P \)), then this is not possible, since then \( S_1(\tau_{1|m+1}) \) is a one-element set that contains both of them. However, it is possible that \( \tau[n] \) and \( \tau'[n + 1] \) are two different program events \( a \neq a' \), both in \( \Gamma_P \), but in \( S_1(\tau_{1|m}) \). However, in this case \( S_2(f(\tau_{1|m})) \) is defined in both cases to be \( S_1(\tau_{1|m+1}) \).

**Theorem 12.** \( T(P \times O_2, S_2) = \{\sigma_2 : \sigma_2 \subseteq f(\tau_1) : \tau_1 \in T(P \times O_1, S_1)\} \).
Proof. The proof is by contraction. If the theorem does not hold, then there is a trace $\sigma_2 \sqsubseteq T(P \times O_2, S_2)$ of minimal length and some action $a_2$, such that $a_2 \in S_2(\sigma_2)$ is not equivalent to the existence of some $\tau'_1 \in T(PxO_1, S_1)$ such that $\sigma_2 \cdot a_2 \sqsubseteq f(\tau'_1)$. However, this equivalence is asserted by the Lemma 11. ◀

Theorem 13. $T(P \times O_1, S_1)|\Gamma_P = T(P \times O_2, S_2)|\Gamma_P$, so $T(P \times O_1, S_1)|\Sigma_P = T(P \times O_2, S_2)|\Sigma_P$ as well.

Proof. This is a simple consequence of Theorem 12. If $\tau_1 \in T(P \times O_1, S_1)$ then $f(\tau_1)$ is in $T(P \times O_2, S_2)$ and has the same program actions in $\Gamma_P$, and every $\tau_2 \in T(P \times O_2, S_2)$ is some $f(\tau_1)$ such that $\tau_1 \in T(P \times O_1, S_1)$ with the same program actions. ◀