Grassmannians $\text{Gr}(N-1, N+1)$, closed differential $N-1$-forms and $N$-dimensional integrable systems*

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Abstract
Integrable flows on the Grassmannians $\text{Gr}(N-1, N+1)$ are defined by the requirement of closedness of the differential $N-1$-forms $\Omega_{N-1}$ naturally associated with $\text{Gr}(N-1, N+1)$. Gauge-invariant parts of these flows, given by the systems of the $N-1$ quasi-linear differential equations, describe coisotropic deformations of $(N-1)$-dimensional linear subspaces. For the class of solutions which are Laurent polynomials in one variable these systems coincide with $N$-dimensional integrable systems such as the Liouville equation ($N=2$), dispersionless Kadomtsev–Petviashvili equation ($N=3$), dispersionless Toda equation ($N=3$), Plebanski second heavenly equation ($N=4$) and others. Gauge-invariant part of the forms $\Omega_{N-1}$ provides us with the compact form of the corresponding hierarchies. Dual quasi-linear systems associated with the projectively dual Grassmannians $\text{Gr}(2, N+1)$ are defined via the requirement of the closedness of the dual forms $\Omega^*_{N-1}$. It is shown that at $N=3$ the self-dual quasi-linear system, which is associated with the harmonic (closed and co-closed) form $\Omega_2$, coincides with the Maxwell equations for orthogonal electric and magnetic fields.

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1. Introduction

Multidimensional integrable systems which are equivalent to the commutativity of multidimensional vector fields

$$D_\alpha = \sum_{k=0}^{N} a_{\alpha k} (x; \lambda) \frac{\partial}{\partial x_k}, \quad \alpha = 1, 2,$$

$^*$ In memory of S V Manakov.
where $a_{jk}(x; \lambda)$ are rational functions of the spectral parameter $\lambda$ have been invented by Zakharov and Shabat in the seminal paper [1]. Concrete examples of such systems and an extension of this scheme to the case when operators $D_\alpha$ contain also the derivatives with respect to the spectral parameter $\lambda$ ($\frac{\partial}{\partial \lambda} = \frac{\partial}{\partial \alpha}$) have been considered in [2–12]. This class of multidimensional integrable systems includes second heavenly equation [2], hyper-Kähler hierarchy [3, 4], dispersionless Kadomtsev–Petviashvili (dKP) equation [5–7], Manakov–Santini system [8, 9], Boyer–Finley (dispersionless Toda) equation [10, 11], Dunaski equation [12] and their generalizations. In recent years, various properties of such integrable systems (wave breaking, reductions, etc) have been intensively studied [13–21]. In particular, in the paper [16] it was observed that in addition to the equations for the variables $a_{jk}(x)$ the equations for the Jacobian $J$ of the solutions $\Psi$ of the systems $D_\alpha \Psi = 0$, $\alpha = 1, 2$ play an important role in the whole method. For instance, it allows us to obtain a compact form of the corresponding hierarchies.

In this paper, we show that all these integrable systems arise naturally in the families of the Grassmannians $\text{Gr}(N - 1, N + 1)$ parameterized by $N + 1$ variables $x_0, x_1, \ldots, x_N$. Namely, they are equivalent to the closedness conditions for differential $N - 1$-forms

$$\Omega_{N-1} = \omega_0 \wedge \ldots \wedge \omega_{N-2},$$

where 1-forms $\omega_\beta = \sum_{\alpha=0}^{N} \alpha \delta x_\alpha$, $\beta = 0, \ldots, N - 2$ are built of $N - 1$ vectors $P^\omega = (P_{\alpha}^0, P_1^\beta, \ldots, P_N^\beta)$ defining $(N - 1)$-dimensional linear subspaces. In terms of Plücker coordinates $\pi_{i_0, i_1, \ldots, i_{N-2}} = \det(P_{i_k}^\beta, k=0, \ldots, N-2, i_k = 0, 1, \ldots, N$, one has the system of linear equations

$$\left[\frac{\partial \pi_{i_0, i_1, \ldots, i_{N-2}}}{\partial x_{i_{N-1}}}\right] = 0,$$

where the bracket $[ \cdots ]$ means antisymmetrization over all indices. This system is equivalent to the system of $N + 1$ quasi-linear equations

$$\frac{\partial J}{\partial x_{N-1}} + \sum_{l=0}^{N-2} \frac{\partial (Ja_{1m})}{\partial x_m} = 0, \quad \frac{\partial J}{\partial x_N} + \sum_{l=0}^{N-2} \frac{\partial (Ja_{2m})}{\partial x_m} = 0,$$

$$\frac{\partial a_{ik}}{\partial x_k} - \frac{\partial a_{2k}}{\partial x_{i-1}} + \sum_{l=0}^{N-2} \left( a_{2l} \frac{\partial a_{1k}}{\partial x_l} - a_{1l} \frac{\partial a_{2k}}{\partial x_l} \right) = 0, \quad k = 0, 1, \ldots, N - 2,$$

for $J = \pi_{01 \ldots N-1}$ and $2(N - 1)$-dimensional affine coordinates

$$a_{1k} = (-1)^k J^{-1} \pi_{0 \ldots k-1 k+1 \ldots N-1}, \quad a_{2k} = (-1)^k J^{-1} \pi_{0 \ldots k-1 k+1 \ldots N-2},$$

where $k = 0, \ldots, N - 2$. The subsystem (5) coincides with the system describing the coisotropic deformations of the $(N - 1)$-dimensional linear space defined by the equations

$$p_{N-1} + \sum_{k=0}^{N-2} a_{1k} p_k = 0, \quad p_N + \sum_{k=0}^{N-2} a_{2k} p_k = 0.$$

The system (3) provides us with the linearization of the nonlinear equation (5) in terms of the variables $J, Ja_{1m}, Ja_{2m}$, where $J$ is a solution of equations (4).

For the class of solutions polynomial in the variable $x_0 (= \lambda)$, the system (5) becomes the system of the integrable $N$-dimensional equations of the type considered in the papers [2–12, 19, 20]. Hierarchies of systems arising in such a way can be represented in the compact form

$$(J^{-1} \Omega_{N-1})_+ = (J^{-1} d\Psi_0 \wedge d\Psi_1 \wedge \cdots \wedge d\Psi_{N-2})_+ = 0,$$
where $\Psi_0, \Psi_1, \ldots, \Psi_{N-2}$ are Darboux-type coordinates for the closed $N - 1$-form $\Omega_{N-1}$, which are the Laurent series in $x_0 = \lambda$ and $(\cdots)_-$ denotes the projection on the negative part of Laurent series. Simple generating formulae for the variables $\Psi_k$ are also presented in [20].

Duality between the Grassmannians $\text{Gr}(N - 1, N + 1)$ and $\text{Gr}(2, N + 1)$ suggests to consider the system dual to the system (4), (5). It is equivalent to the closedness condition for the $2$-form $\Omega_2^* = \bullet \Omega_{N-1}$, where $\bullet$ denotes the Hodge star (or duality) operation or to the co-closedness of the form $\Omega_{N-1}$ and it is of the form

$$
\frac{\partial J^*}{\partial x_k} + \frac{1}{2} \sum_{m=0}^{N-1} \frac{\partial (J^* a_{km}^*)}{\partial x_m} = 0, \quad k = 2, \ldots, N,
$$

(7)

$$
\frac{\partial a_{jl}^*}{\partial x_k} - \frac{1}{2} \sum_{m=0}^{N-1} \left( a_{km}^* \frac{\partial a_{jl}^*}{\partial x_m} - a_{jm}^* \frac{\partial a_{kl}^*}{\partial x_m} \right) = 0, \quad l = 0, 1; \quad j, k = 2, \ldots, N,
$$

(8)

where $J^* = \pi_{01}^*, a_{0j}^* = J^{*-1} \pi_{1j}^*, a_{kj}^* = -J^{*-1} \pi_{0j}^*$ and $\pi_{kl}^*$ are Plücker coordinates dual to $\pi_{01}, \ldots, \pi_{N1}$. At $N = 3$, the system (8) is quite similar to the original system (5). For $N \geq 4$, the system (8) is the semi-decoupled system of $\frac{(N-2)(N-1)}{2}$ four-dimensional subsystems.

Self-dual systems are associated with the harmonic forms $\Omega_{N-1}$, i.e. the forms which are closed and co-closed. The self-dual system in $\text{Gr}(N - 1, N + 1)$ is the systems of $(N - 1)^2$ equations for $(2(N - 1))$-dependent variables $\alpha_{1k}, \alpha_{2k}, k = 0, 1, \ldots, N - 2$. At $N = 3$, such self-dual system coincides in form with the system of sourceless Maxwell equations with vanishing second invariant (orthogonal electric and magnetic fields) that provides us with a wide class of their solutions.

The paper is organized as follows. Basic facts on the Grassmannians are collected in section 2. Closed differential ($N - 1$) forms and associated systems of quasi-linear differential equations are considered in section 3. Solutions of these systems which are Laurent polynomials in one variable and their relation to $N$-dimensional integrable equations are studied in section 4. The compact form of these equations and corresponding hierarchies are discussed in section 5. Dual and self-dual quasi-linear systems are considered in section 6.

2. Grassmannians $\text{Gr}(N - 1, N + 1)$

Grassmannian manifold $\text{Gr}(m, n)$ is by definition the parameter space for the totality of $m$-dimensional linear subspaces $V_m$ in the $n$-dimensional space $V_n$ (see e.g. [22–24]). Equivalently,

$$
\text{Gr}(m, n) = \{m\text{-frames in } V_n\} / \text{GL}(m),
$$

(9)

where an $m$-frame means an $m$-tuple $\{p^0, p^1, \ldots, p^{m-1}\}$ of linearly independent vectors with the coordinates $p^i_j$ ($i = 0, 1, \ldots, n - 1$) in a given basis $\{e^0, e^1, \ldots, e^{n-1}\}$ in $V_n$. These coordinates can be arranged in the $n \times m$ matrix $F$ with the elements $p^i_j$. The dimension of $\text{Gr}(m, n)$ is equal to $m(n - m)$.

Grassmanian $\text{Gr}(m, n)$ can be viewed as an algebraic submanifold of the $\left( \begin{array}{c} n-1 \end{array} \right)$-dimensional space $\wedge^m V_n$ via the correspondence of a $m$-frame $\{p^0, p^1, \ldots, p^{m-1}\}$ with the exterior product $p^0 \wedge p^1 \wedge \ldots \wedge p^{m-1} \in \wedge^m V_n$ (the canonical projective embedding). One has

$$
p^0 \wedge p^1 \wedge \ldots \wedge p^{m-1} = \sum_{0 \leq i_0 < \cdots < i_m \leq n-1} \pi_{i_0, i_1, \ldots, i_m} e^{i_0} \wedge e^{i_1} \wedge \cdots \wedge e^{i_{m-1}},
$$

(10)
where \( \pi_{i_0i_1...i_{m-1}} = \det(p^k_{i_j})_{i_j=0,...,m-1} \). These coefficients \( \pi_{i_0i_1...i_{m-1}} \) are called the Plücker coordinates and satisfy the Plücker’s relations

\[
\sum_{j=0}^{m} (-1)^j \pi_{i_0...i_{m-2}j} \pi_{j...j} = 0, \tag{11}
\]

where indices \( i_j \) and \( j \) range over all possible values and the notation \( \tilde{j} \) means the deletion of this number. The Plücker coordinates completely define embedding of \( \text{Gr}(m, n) \) into \( \wedge^m \mathcal{V}_n \).

Plücker coordinates also define the subspaces \( \mathcal{V}_n \) in \( \mathcal{V}_n \). Namely, a point in \( \mathcal{V}_n \) with the coordinates \((y_0, y_1, \ldots, y_{n-1})\) lies in \( \mathcal{V}_m \) iff (see e.g. [22]); they obey the system of equations

\[
\sum_{j=0}^{m} (-1)^j \pi_{i_0...i_{m-2}j} y_j = 0. \tag{12}
\]

In virtue of the Plücker’s relations there are \( n - m \) linearly independent equations among them. As a consequence, the system (11) is equivalent to the system of \( n - m \) equations

\[
y_\gamma + \sum_{k=0}^{m-1} a_{\gamma k} y_k = 0, \quad \gamma = m, m + 1, \ldots, n - 1, \tag{13}
\]

where \( a_{\gamma k} \) are independent affine coordinates in \( \text{Gr}(m, n) \).

The Grassmannian \( \text{Gr}(m, n) \) is projectively equivalent to the Grassmannian \( \text{Gr}(n - m, n) \) [22]. The latter has coordinates \( \pi^*_{i_0i_1...i_{n-1}} \) defined by the relation

\[
epsilon_{i_0i_1...i_{n-1}} \pi^*_{i_0i_1...i_{n-1}} = \pi_{i_0i_1...i_{n-1}},
\]

where \( \epsilon_{i_0i_1...i_{n-1}} \) is the totally antisymmetric tensor in \( N + 1 \) dimensions. The coordinates \( \pi^*_{i_0i_1...i_{n-1}} \) obey equations (11), and the system dual to (12) is

\[
\sum_{j=0}^{n} \pi^*_{i_0...i_{m-1}j} y_j^* = 0. \tag{14}
\]

This system of equations is equivalent to the following one:

\[
y_{n-\gamma} + \sum_{k=0}^{n-m-1} a^*_{\gamma k} y_k^* = 0, \quad \gamma = n - m, \ldots, n, \tag{15}
\]

where \( a^*_{\gamma k} \) are independent affine coordinates in \( \text{Gr}(n - m, n) \).

In this paper, we will consider Grassmannians \( \text{Gr}(N - 1, N + 1) \) for arbitrary \( N \). They are the \( 2(N - 1) \)-dimensional sets of \((N - 1)\)-dimensional linear subspaces in the \((N + 1)\)-dimensional space defined by co-dimension two constraint

\[
y_{N-1} + \sum_{k=0}^{N-2} a_{1k} y_k = 0, \quad y_{N} + \sum_{k=0}^{N-2} a_{2k} y_k = 0 \tag{16}
\]

with \( a_{1k} = (-1)^k J^{-1} \epsilon_{i_0...i_{k-1}j+1...N-1} \), \( a_{2k} = (-1)^k J^{-1} \epsilon_{i_0...i_{k-1}j+1...N} \), \( k = 0, \ldots, N - 2 \), \( J = \pi_{01...N-2} \). Affine coordinates \( a_{1k} \) and \( a_{2k} \) are completely defined by the set of \((N - 1)\)-independent vectors in \( \mathcal{V}_{N+1} \).

In particular, at \( N = 3 \) for the Grassmannian \( \text{Gr}(2, 4) \), equations (13) are of the form

\[
y_2 + a_{11} y_1 + a_{10} y_0 = 0, \quad y_3 + a_{21} y_1 + a_{20} y_0 = 0. \tag{17}
\]

Plücker coordinates obey the single equation

\[
\pi_{01} \pi_{23} - \pi_{02} \pi_{13} + \pi_{03} \pi_{12} = 0 \tag{18}
\]

and

\[
\pi_{01} = \det \begin{pmatrix} p^0_0 & p^1_0 \\ p^0_1 & p^1_1 \end{pmatrix}, \quad \pi_{02} = \det \begin{pmatrix} p^0_0 & p^1_0 \\ p^0_2 & p^1_2 \end{pmatrix}, \quad \pi_{03} = \det \begin{pmatrix} p^0_0 & p^1_0 \\ p^0_3 & p^1_3 \end{pmatrix}.
\]
\[ \pi_{12} = \det \left( \begin{array}{cc} p_0^0 & p_1^1 \\ p_0^1 & p_1^0 \end{array} \right), \quad \pi_{13} = \det \left( \begin{array}{cc} p_0^0 & p_1^1 \\ p_0^3 & p_1^3 \end{array} \right), \quad \pi_{23} = \det \left( \begin{array}{cc} p_0^0 & p_1^1 \\ p_0^3 & p_1^3 \end{array} \right). \]

where \( p^0 = (p_0^0, p_0^1, p_0^2, p_0^3) \) and \( p^1 = (p_0^1, p_1^1, p_1^2, p_1^3) \) are two vectors in \( V_4 \) defining the plane (16). Due to relations (16) one has

\[ J = \pi_{01}, \quad \pi_{02} = -Ja_{11}, \quad \pi_{03} = -Ja_{21}, \quad \pi_{12} = Ja_{10}, \quad \pi_{13} = Ja_{20}, \quad \pi_{23} = J(a_{21}a_{10} - a_{12}a_{20}). \]

The Grassmannian dual to \( \text{Gr}(N-1, N+1) \) is \( \text{Gr}(2, N+1) \). It has the same dimension 2\((N-1)\) as the former one, but is the set of two-dimensional linear subspaces in the \( (N+1) \)-dimensional space \( V_{N+1} \) defined by the constraints

\[ \gamma y_0^+ + a_{\gamma 0}^+ y_1^+ + a_{\gamma 0}^- y_0^- = 0, \quad \gamma = 2, \ldots, N. \]  

At \( N = 3 \), equations (17), (18) and their dual equations (21) have the same form.

3. Closed differential \( N - 1 \)-forms and systems of quasi-linear differential equations

Now we introduce an infinite family of Grassmannians \( \text{Gr}(N-1, N+1) \) parametrized by \( N+1 \) variables \( x_0, x_1, \ldots, x_N \). Thus, all quantities considered in the previous section become functions of these variables. A geometrical realization of such family is provided, for example, by the \( (2N+1) \)-dimensional cotangent bundle with the local coordinates \( x_0, x_1, \ldots, x_N \) in the base manifold and coordinates \( p_0, p_1, \ldots, p_N \) in cotangent space \( T^*_N \) and the Grassmannian \( \text{Gr}(N-1, N+1) \) in \( T^*_N \) attached to each point of the base space.

For each vector \( p^0 \) defining the linear \( (N-1) \)-dimensional subspace in \( V_{N+1} \) there is a naturally associated differential 1-form \( \omega_0 = \sum_{i=0}^N p_i^0 dx_i \). The projective embedding of the Grassmannian \( \text{Gr}(N-1, N+1) \) (formula (10)) suggests to introduce differential \( N-1 \)-form

\[ \Omega_{N-1} = \omega_0 \wedge \ldots \wedge \omega_{N-2} \]

associated with the member of the family of Grassmannians corresponding to the point \( (x_0, x_1, \ldots, x_N) \). One has

\[ \Omega_{N-1} = \sum_{0 \leq i_0 < \ldots < i_{N-2} \leq N} \pi_{i_0 i_1 \ldots i_{N-2}}(x) dx_{i_0} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_{N-2}}, \]

where all indices \( i_k \) take values \( 0, 1, \ldots, N \). The homogeneous coordinates \( \pi_{i_0 i_1 \ldots i_{N-2}} \) are defined up to the common factor \( \rho \) depending on \( x \). So, the form (23) is defined up to the scaling transformations \( \Omega_{N-1} \rightarrow \rho(x) \Omega_{N-1} \) with the arbitrary function \( \rho(x) \) which can be viewed as the gauge transformations.

To select a special family of Grassmannians \( \text{Gr}(N-1, N+1) \), we require that this form is closed. Thus, we require that the Plücker coordinates obey the equations

\[ \frac{\partial \pi_{i_0 i_1 \ldots i_{N-2}}}{\partial x_{i_N}} = 0, \]

where indices \( i_k \) take values \( 0, 1, \ldots, N \) and the bracket \( [ \ldots ] \) means antisymmetrization over all these indices. This system of \( 2\binom{N+1}{N-1} = N + 1 \) differential equations together with the algebraic Plücker relations (11) forms a full system of equations which characterizes the family of Grassmannians \( \text{Gr}(N-1, N+1) \) for which the form \( \Omega_{N-1} \) is closed.

The system (24) is equivalent to the system

\[ \frac{\partial J}{\partial x_{N-1}} + \sum_{m=0}^{N-2} \frac{\partial (Ja_{1m})}{\partial x_m} = 0, \quad \frac{\partial J}{\partial x_N} + \sum_{m=0}^{N-2} \frac{\partial (Ja_{2m})}{\partial x_m} = 0, \]

where \( p_0 = (p_0^0, p_0^1, p_0^2, p_0^3) \) and \( p_1 = (p_0^1, p_1^1, p_1^2, p_1^3) \) are vectors in \( V_4 \) defining the plane (16). Due to relations (16) one has

\[ J = \pi_{01}, \quad \pi_{02} = -Ja_{11}, \quad \pi_{03} = -Ja_{21}, \quad \pi_{12} = Ja_{10}, \quad \pi_{13} = Ja_{20}, \quad \pi_{23} = J(a_{21}a_{10} - a_{12}a_{20}). \]
The subsystem (26), invariant under the gauge transformations mentioned above, is the gauge invariant form of the system (24). Two equations (25) can be viewed as the equations for the gauge variable \( J \) which transforms as \( J \rightarrow \rho J \) under the gauge transformations. It is a straightforward check that equations (25) are compatible due to the subsystem (26). We note that the systems (25) and (26) can be rewritten in the form

\[
D_1 \ln J + \sum_{m=0}^{N-2} \frac{\partial a_{1m}}{\partial x_m} = 0, \quad D_2 \ln J + \sum_{m=0}^{N-2} \frac{\partial a_{2m}}{\partial x_m} = 0,
\]

(27)

\[
D_2 a_{1k} - D_1 a_{2k} = 0, \quad k = 0, \ldots, N - 2,
\]

(28)

where \( D_1 \) and \( D_2 \) are the vector fields

\[
D_1 = \frac{\partial}{\partial x_N} + \sum_{m=0}^{N-2} a_{1m} \frac{\partial}{\partial x_m}, \quad D_2 = \frac{\partial}{\partial x_N} + \sum_{m=0}^{N-2} a_{2m} \frac{\partial}{\partial x_m}.
\]

(29)

So the subsystem (26) is equivalent to the commutativity \([D_1, D_2] = 0\) of the vector fields \( D_1 \) and \( D_2 \).

The system (26) of nonlinear equations admits an obvious linearization. Indeed, for a solution \( a_{1k}, a_{2k} \) of this system one can find \( J \) solving linear equations (25). Then for variables \( J, Ja_{1k}, Ja_{2k} \) one has a linear system which coincides with the system (24).

The system (26) has another geometrical meaning. Let us denote by \( \mathcal{I} \) the family of ideals of polynomials of the variables \( y_0, y_1, \ldots, y_N \) generated by linear variety (16) and require that \( \mathcal{I} \) is the Poisson subalgebra, i.e.

\[
\{ \mathcal{I}, \mathcal{J} \} < \mathcal{I},
\]

(30)

where \( \{,\} \) is the standard Poisson bracket with \( x_0, x_1, \ldots, x_N; y_0, y_1, \ldots, y_N \) being the canonical Darboux coordinates. Condition (30) defines the so-called coisotropic deformations of the linear algebraic variety defined by equations (12) [25, 26]. Such deformations are described exactly by equations (26). Thus, the coisotropic deformations of the ideal \( \mathcal{I} \) represent the necessary and sufficient gauge-invariant conditions for the closedness of the \((N-1)\)-forms (22).

At \( N = 2 \), we have a two-dimensional family of straight lines in the three-dimensional space, 1-form \( \Omega_1 = \sum_{k=0}^{2} \rho_k dx_k \) and the conditions of its closedness are given by equations (\( J = p_0 \)):

\[
\frac{\partial p_0}{\partial x_0} - \frac{\partial p_1}{\partial x_0} = 0, \quad \frac{\partial p_0}{\partial x_2} - \frac{\partial p_1}{\partial x_2} = 0, \quad \frac{\partial p_1}{\partial x_0} - \frac{\partial p_2}{\partial x_0} = 0,
\]

or

\[
\frac{\partial J}{\partial x_1} + \frac{\partial (Ja_{10})}{\partial x_0} = 0, \quad \frac{\partial J}{\partial x_2} + \frac{\partial (Ja_{20})}{\partial x_0} = 0,
\]

\[
\frac{\partial a_{10}}{\partial x_1} - \frac{\partial a_{10}}{\partial x_0} = 0, \quad \frac{\partial a_{20}}{\partial x_2} - \frac{\partial a_{20}}{\partial x_0} = 0, \quad \frac{\partial a_{10}}{\partial x_0} - \frac{\partial a_{20}}{\partial x_0} = 0,
\]

(31)

where \( J = p_0, a_{10} = -J^{-1} p_1, a_{20} = -J^{-1} p_2 \).

At \( N = 3 \), one has the family of planes (17), 2-form \( \Omega_2 \) given by

\[
\Omega_2 = J(dx_0 \wedge dx_1 - a_{11}dx_0 \wedge dx_2 - a_{23}dx_0 \wedge dx_3 + a_{10}dx_1 \wedge dx_2
\]

\[
+ a_{20}dx_1 \wedge dx_3 - (a_{11}a_{20} - a_{10}a_{21}) dx_2 \wedge dx_3)
\]

(32)
and the equations

\[ \frac{\partial J}{\partial x_2} + \sum_{m=0}^{1} \frac{\partial (Ja_{1m})}{\partial x_m} = 0, \quad \frac{\partial J}{\partial x_3} + \sum_{m=0}^{1} \frac{\partial (Ja_{2m})}{\partial x_m} = 0, \tag{33} \]

\[ \frac{\partial a_{1k}}{\partial x_2} - \frac{\partial a_{2k}}{\partial x_2} + \sum_{l=0}^{1} \left( a_{2l} \frac{\partial a_{1k}}{\partial x_l} - a_{1l} \frac{\partial a_{2k}}{\partial x_l} \right) = 0, \quad k = 0, 1; \tag{34} \]

where \( J, a_{1k}, a_{2k} \) are defined by relations (19) and (20).

Finally, we note that at arbitrary \( N \) the \( N-1 \)-form \( \Omega_{N-1} \) has a factorized form

\[ \Omega_{N-1} = J\Omega_{N-1}, \tag{35} \]

where \( \Omega_{N-1} \) is the gauge invariant \( N-1 \)-form depending on the affine coordinates \( a_{1k}, a_{2k} \) only.

4. Solutions rational in one variable and \( N \)-dimensional integrable systems

The system (26) of \( N-1 \) equations for \( 2(N-1) \)-dependent variables is always the undetermined system. In order to make it determined one should impose constraints. The simplest constraint is to choose \( a_{2k} \) to be certain functions of \( a_{1k} \), i.e. \( a_{2k} = \varphi_k(a_{10}, a_{11}, \ldots, a_{N-2}), \) \( k = 0, 1, \ldots, N-2 \). In this case, the system (26) becomes the system of (\( N+1 \))-dimensional hydrodynamic-type equations

\[ \frac{\partial a_{1k}}{\partial x_N} + \sum_{m=0}^{N-2} A_{km} \frac{\partial a_{1m}}{\partial x_{N-1}} + \sum_{m=0}^{N-2} B_{km} \frac{\partial a_{1m}}{\partial x_l} = 0, \quad k = 0, 1, \ldots, N-2, \tag{36} \]

where

\[ A_{km} = -\frac{\partial \varphi_k}{\partial a_{1m}}, \quad B_{km} = \delta_{km} \varphi_l - a_{1l} \frac{\partial \varphi_k}{\partial a_{1m}}, \quad k, m, l = 0, 1, \ldots, N-2. \tag{37} \]

At \( N = 2 \), it is effectively the (1+1)-dimensional Hopf–Burgers-type equation

\[ \Lambda \frac{\partial a_{10}}{\partial t} + B \frac{\partial a_{10}}{\partial x_0} = 0, \tag{38} \]

where \( \Lambda = 1 - \frac{a_{10}}{a_{11}}, B = \varphi_0 - a_{10} \frac{\partial \varphi_1}{\partial a_{10}}, \) and \( t = \frac{1}{\lambda} (x_2 - x_1) \). Different types of reductions of the system (26) are associated with the restriction to a special subclass of its solutions, for example, to solutions which are Laurent polynomials in one variable, say \( x_0 \). Considering, for example, solutions of the systems (26) of the form

\[ a_{1k} = \sum_{n=-m_1}^{m_1} a_{k0} (x_1, \ldots, x_N) x_0^n, \quad a_{2k} = \sum_{n=-m_2}^{m_2} a_{k0} (x_1, \ldots, x_N) x_0^n, \tag{39} \]

where \( k = 0, 1, \ldots, N-2, \) with appropriate \( n_1, n_1, n_2, m_2, \) one obtains the systems of \( N \)-dimensional equations for the functions \( a_{10}, a_{11}, \) together with the corresponding constraints between them. A complete classification of determined systems which can be obtained in such a way is beyond the scope of this paper. Here we will present several examples to demonstrate that one can obtain number of known integrable systems within this approach. In the simplest case \( N = 2 \) with the ansatz

\[ a_{10} = u_0 + \lambda u_1, \quad a_{20} = v_0 + \lambda v_1 + \lambda^2 v_2, \tag{40} \]

where \( \lambda = x_0, x = (x_1, x_2, x_3) \) the system (31) assumes the form

\[ u_{0x_2} - v_{0x_3} + u_{1v_0} - u_{0v_1} = 0, \quad u_{1x_2} - v_{1x_3} - 2u_{0v_2} = 0, \quad v_{2x_1} + u_{1v_2} = 0. \tag{41} \]
This system implies that \( u_1 = -\varphi_1 \), where \( \varphi = \ln v_2 \). Under the constraint \( v_1 = 0 \) the above system becomes

\[
\varphi_{x_1,x_2} + 2u_0 e^\varphi = 0, \quad u_{0x_1} - \tilde{v}_1 e^{-\varphi} = 0,
\]

where \( \tilde{v} = 2v_1 e^\varphi \). Under the further constraint \( u_0 = -\frac{1}{2}, \tilde{v} = 2 \) one obtains the Liouville equation

\[
\varphi_{x_1,x_2} = e^\varphi,
\]

for which

\[
a_{10} = -\frac{1}{2} - \lambda \varphi_{x_1}, \quad a_{20} = e^{-\varphi} + \lambda^2 e^\varphi.
\]

Choosing

\[
a_{10} = -\frac{1}{2} - \lambda \varphi_{x_1}, \quad a_{20} = e^{-\varphi} + \lambda^2 \varphi_{x_2} + \lambda^3 e^{2\varphi},
\]

one obtains the higher Liouville equation

\[
\varphi_{x_1,x_2} - \varphi_{x_1} \varphi_{x_2} - \frac{1}{2} \lambda e^{2\varphi} = 0
\]

which in terms of the variable \( \Psi = \exp(-\frac{1}{2} \varphi) \) is of the form

\[
\left( \frac{\Psi_{x_1,x_2}}{\Psi} \right)_{x_2} - \frac{3}{4} \Psi^{-4} = 0.
\]

Solutions of the Liouville and higher Liouville equations describe two-parameter deformations of the straight line defined by the linear system

\[
p_1 + a_{10} p_0 = 0, \quad p_2 + a_{20} p_0 = 0
\]

in the three-dimensional space with \( a_{10}, a_{20} \) given above. In geometrical terms, solutions of these equations describe special classes of congruences of the lines in the three-dimensional space. In a different context, the interrelation between congruences of lines and system of equations of hydrodynamic type has been studied in [27, 28].

In the case \( N = 3 \), the first nontrivial choice is

\[
a_{10} = u_0(x), \quad a_{11} = u_1(x) + \lambda, \quad a_{20} = v_0(x) + \lambda v_1(x), \quad a_{21} = v_2(x) + \lambda v_3(x) + \lambda^2.
\]

Substituting this ansatz into two equations (34) and denoting \( x = x_1, y = x_2, t = x_3 \), one obtains the system of equations

\[
u_0 t - v_0 y + v_2 u_0 - u_1 v_0 = u_0 v_1 = 0,
\]

\[
u_1 t - v_2 y + v_0 + v_2 u_1 - u_1 v_2 = u_0 u_1 = 0,
\]

\[
u_1 - 2u_0 - v_2 x - v_3 y + v_3 u_1 = u_1 v_3 x = 0,
\]

\[
u_1 y + v_0 x + u_1 v_1 x - v_3 u_0 x = 0,
\]

\[
u_1 x - v_3 x = 0,
\]

\[
u_0 x - u_1 x = 0.
\]

Equations (53) and (54) imply that \( v_1 = u_0, v_3 = u_1 \) and equations (51) and (52) become

\[
u_0 + u_1 y + v_2 x = 0, \quad u_0 y + v_0 x = 0.
\]
The last equation (55) implies the existence of the function $u$ such that $u_0 = -u_1$, $v_0 = v_1$.

As a result from the first equation (55) one obtains $u_1 = v_1$, $v_2 = u - v_1$, where $v$ is some function. Substituting these expressions into equations (49) and (50), one obtains the system

$$u_{xt} + u_{xy} + u_x^2 + (u - v_1)u_{xx} + v_1u_{xy} = 0,$$

$$v_{xt} + v_{yy} + v_x v_y + (u - v_1)v_{xx} = 0.$$  

It is the Manakov–Santini system found in [8, 9]. Under the constraint $v = 0$, it reduces into the dKP equation

$$u_{xt} + u_{xy} + (u m_1) = 0,$$

while at $u = 0$ it is the equation

$$v_{xt} + v_{yy} + v_x v_y - v_x v_{xx} = 0$$

considered in [29–31]. Thus, the Manakov–Santini system is the gauge-invariant form of the closedness condition for the 2-form

$$\Omega_2 = J(d\lambda \wedge dx + (\phi_1 + \phi_2 + \lambda v_1 + \lambda^2)dx \wedge dy - (u - v_1 + \lambda v_1 + \lambda^2)dx \wedge dy$$

$$+ (u_1 - \lambda u_2)dx \wedge dt - (u_1 u - u_1 v_1 + u_2 v_1 + \lambda u_1)dy \wedge dt)$$

(60)

associated with the family of Grassmannians Gr(2,4). The function $J$ is a solution of the system

$$J_t - u_2 J_x + (\lambda + v_1)J_x + v_1 J = 0,$$

$$J_t + (u_1 - \lambda u_2)J_x + (u - v_1 + \lambda v_1 + \lambda^2)J_x + (-v_{xy} + \lambda v_{xx})J = 0.$$  

(62)

The second example of the three-dimensional integrable system corresponds to the choice (again $\lambda = x_0$, $x = x_1$, $y = x_2$, $t = x_3$)

$$a_{10} = -\lambda \left( \phi_1 - \phi_2 \frac{m_1}{m_2} \right), \quad a_{11} = -\lambda - \frac{m_1}{m_2},$$

$$a_{20} = -e^{\phi} \frac{m_1}{m_2}, \quad a_{21} = \frac{1}{\lambda} \frac{e^{\phi}}{m_2},$$

(63)

where $\phi$ and $m$ are the functions of $x$, $y$, $t$. The system (34) takes the form

$$(e^{\phi})_{xx} - m_2 \phi_{xy} + m_1 \phi_{xz} = 0,$$

$$e^{\phi} m_{xx} - m_2 m_{xy} + m_1 m_{xz} = 0.$$  

(65)

It is the generalization of the dispersionless two-dimensional Toda lattice (2DTL) equation given in [21]. Under the reduction $m = x$ it is the dispersionless 2DTL equation

$$(e^{\phi})_{xx} - \phi_{xy} = 0$$

(66)

and at $\phi = 0$ it is the equation

$$m_{xx} - m_2 m_{xy} + m_1 m_{xz} = 0$$

(67)

considered in [29, 31, 30].

The corresponding 2-form is given by

$$\Omega_2 = J(d\lambda \wedge dx + (\lambda - \frac{m_2}{m_1})dx \wedge dy + \frac{1}{\lambda} e^{\phi} dx \wedge dt + \lambda \left( \phi_1 - \phi_2 \frac{m_1}{m_2} \right) dx \wedge dy$$

$$+ e^{\phi} \frac{m_1}{m_2} dx \wedge dt - e^{\phi} \frac{m_1}{m_2} (\lambda \phi_1 + \phi_2) dy \wedge dt)$$

(68)
and $J$ is a solution of the system

$$J_y + \lambda \left( \phi_y - \phi_x \frac{m_x}{m_y} \right) J_k - \left( \frac{\lambda + m_x}{m_y} \right) J_x + \left( \phi_y - \phi_x \frac{m_y}{m_x} - \frac{m_y}{m_x} \right) J_z = 0,$$

$$J_k - \frac{\epsilon^\phi \phi_x}{m_x} J_x - \frac{1}{\lambda} \frac{\epsilon^\phi}{m_x} J_k - \frac{1}{\lambda} \left( \frac{\epsilon^\phi}{m_x} \right) J_z = 0. \quad (69)$$

Now let us consider the case $N = 4$ and choose the functions $a_{1k}$ and $a_{2k}$, $k = 0, 1, 2$, as

$$a_{10} = u_0, \quad a_{11} = u_1 + \lambda, \quad a_{12} = u_2,$$

$$a_{20} = v_0, \quad a_{21} = v_1, \quad a_{22} = v_2 + \lambda, \quad (70)$$

where $u_k, v_k, k = 0, 1, 2$ are the functions of the variables $x = x_1, y = x_2, z = x_3, t = x_4$ and $\lambda = x_0$. Substituting this ansatz into the system (26) with $N = 4$, one obtains the equations

$$u_{0t} = v_{0x} + v_{1u_0} + v_{2u_0} = u_1 v_{0u} - u_2 v_{0y} = 0, \quad (71)$$

$$u_{1t} = v_{1z} + v_{0u_2} + v_{2u_1} - u_1 v_{1u} - u_2 v_{1y} = 0, \quad (72)$$

$$u_{2t} = v_{2z} - u_0 + v_{1u_2} + v_{2u_2} - u_1 v_{2u} - u_2 v_{2y} = 0, \quad (73)$$

$$u_{iy} - v_{ux} = 0, \quad k = 0, 1, 2. \quad (74)$$

The last equations imply that

$$u_0 = u_1, \quad v_0 = v_1, \quad u_1 = u_2, \quad v_1 = v_2, \quad u_2 = w_x, \quad v_2 = w_y, \quad (75)$$

where $w, u, v$ are the functions obeying the equations

$$u_{yt} = u_{yz} + v_y u_{yx} + (w_y - v_x) u_{xy} - w_x u_{yy} = 0, \quad (76)$$

$$v_{yt} = v_{yz} + u_y + v_y v_{xx} + (w_y - v_x) v_{xy} - w_x v_{yy} = 0, \quad (77)$$

$$w_{yt} - u_{yz} - u_x + v_x w_{xx} + (w_x - v_y) w_{xy} - w_x w_{yy} = 0. \quad (78)$$

The 3-form $\Omega_3$ is the sum of ten terms

$$\Omega_3 = J (dx \land dy \land dz - w_x d\lambda \land dx \land dz - (\lambda + w_y) d\lambda \land dx \land dt + (\lambda + v_x) d\lambda \land dy \land dz + \cdots) \quad (79)$$

and equations for $J$ are

$$J_x + u_x J_x + (\lambda + v_x) J_x + u_y J_y + (v_x + w_y) J_x = 0, \quad (80)$$

$$J_y + u_y J_y + v_y J_x + (\lambda + v_x) J_y + (v_x + w_y) J_y = 0. \quad (81)$$

The system (76)–(78) admits several reductions. An obvious one is $u = 0$ for which it is the system of two equations (77) and (78) with $u = 0$. Less trivial reduction corresponds to the constraint $v_y = w_x = 0$. It implies that $v = \Theta_y$ and $w = -\Theta_x$ and the system (76)–(78) takes the form

$$u_{yt} - u_{yz} + \Theta_{xy} u_{xx} - 2 \Theta_{xy} u_{xy} + \Theta_{xx} u_{yy} = 0, \quad (82)$$

$$u = \Theta_y - \Theta_x + \Theta_x^2 - \Theta_{xx} \Theta_{yy}. \quad (83)$$

It is the Dunajski system proposed in [12]. Under the additional constraint $u = 0$ one has the Plebanski second heavenly equation [2]

$$\Theta_{xy} - \Theta_{xx} + \Theta_{yy}^2 - \Theta_{xx} \Theta_{yy} = 0. \quad (84)$$

Integrable systems for any $N$ can be constructed in a similar way. An interesting problem of finding the systems which describe solutions of the system (26) which have poles in one variable, similar to the Calogero–Moser system for the rational solutions of the Korteweg–de Vries equation [32], will be discussed in a separate paper.
5. Compact form of the hierarchies

Considering the functions \( a_{1k} \) and \( a_{2k} \) which are higher order polynomials in \( x_0 = \lambda \), one obtains the higher order Manakov–Santini and Dynajski equations. (For simplicity, we will not consider here the two-component dispersionless 2DTL case which requires Laurent polynomials; however, it can also be considered in the similar framework, see [21, 18].) The forms \( \Omega_{N-1} \) provide us also with compact forms of these hierarchies. These differential forms have rank \( N-1 \). Hence, the condition \( d\Omega_{N-1} = 0 \) implies the existence of \( N-1 \) variables \( \Psi^0, \Psi^1, \ldots, \Psi^{N-2} \) such that (see e.g. [33, 34])

\[
\Omega_{N-1} = d\Psi^0 \wedge d\Psi^1 \wedge \cdots d\Psi^{N-2}.
\]

This means that the components of the vectors \( \rho^k, k = 0, 1, \ldots, N-2 \), which define \((N-1)\) dimensional linear subspaces in the Grassmannian \( \text{Gr}(N-1, N+1) \) can be taken as the derivatives \( \rho^k_m = \frac{\partial \Psi^k}{\partial x_m}, k = 0, 1, \ldots, N-2; m = 0, 1, \ldots, N \), and equations (16) take the form

\[
D_1\Psi^k = 0, \quad D_2\Psi^k = 0, \quad k = 0, 1, \ldots, N-2,
\]

where the operators \( D_1 \) and \( D_2 \) are given by (29).

Since \( \Omega_{N-1} \) is a complicated function in \( x_0 = \lambda \), the variables \( \Psi^k \) are, in general, certain Laurent series in \( \lambda \). An important property of this form is that the \( N-1 \)-form \( \Omega_{N-1} \) defined in (35) is a polynomial function in \( \lambda \) for polynomials \( a_{1k} \) and \( a_{2k} \). Hence,

\[
(J^{-1}d\Psi^0 \wedge d\Psi^1 \wedge \cdots d\Psi^{N-2})_\lambda = 0,
\]

where \((\cdots)_\lambda\) denotes the projection on the part of \((\cdots)\) with negative powers in \( \lambda \) and

\[ J = \pi_{01\ldots N-2} = \det(\partial\Psi^m)_{m,l=0,\ldots,N-2}. \]

This is the compact form of the hierarchies of multidimensional integrable systems considered in [20].

The generating relation (87) implies Lax–Sato equations which define the evolution of the series \( \Psi^1, \ldots, \Psi^{N-2} \) with the coefficients depending on \( x_m \) (\( 1 \leq m \leq N-2 \)) with respect to the times \( t_1 = x_{N-1}, t_2 = x_N \) (and also higher times corresponding to commuting flows of the hierarchy). Indeed, the coefficients of the form (85) can be written as \( \pi_{01t_1\ldots t_{N-2}} = \det(\partial_t\Psi^m)_{m,l=0,\ldots,N-2} \) (compare with relation (10)). The functions \( \Psi_k \) satisfy identically the relations corresponding to formula (12):

\[
\sum_{l=0}^{N-1} (-1)^l \pi_{l_0\ldots l_{k-1}l_{k+1}\ldots l_N} \partial_t l_k \Psi_k = 0.
\]

An important step is to observe that due to the generating relation (87) the affine coefficients of linear relations (88) are analytic, so we obtain nontrivial relations

\[
\sum_{l=0}^{m} (-1)^l (J^{-1} \pi_{l_0\ldots l_{k-1}l_{k+1}\ldots l_m}) \partial_t l_k \Psi_k = 0.
\]

Taking the basic relations corresponding to constraints (16), we obtain Lax–Sato equations

\[
\partial_{N-1}\Psi^q + \sum_{k=0}^{N-2} a_{1k} \partial_k \Psi^q = 0,
\]

\[
\partial_N \Psi^q + \sum_{k=0}^{N-2} a_{2k} \partial_k \Psi^q = 0
\]
with
\[ a_{1k} = (-1)^k (J^{-1} \pi_{0,k-1+k+1, \ldots, N-2N-1})_+, \]
\[ a_{2k} = (-1)^k (J^{-1} \pi_{0,k-1+k+1, \ldots, N-2N})_+, \]
where \( k = 0, \ldots, N-2 \). Under reasonable assumptions, the coefficients \( a_{1k} \) and \( a_{2k} \) contain only a finite number of coefficients of the series \( \Psi^0 \), and equations (90) and (91) define the evolution of these series with respect to times \( t_1 = x_{N-1}, t_2 = x_N \) (similar equations can be written for the hierarchy of commuting flows). The compatibility conditions for equations (90) and (91) considered as linear equations for \( \Psi^0 \) are of the form (26), and they define \( N \)-dimensional integrable systems for the coefficients of polynomials \( a_{1k}, a_{2k} \). They can also be obtained directly from the condition of the closedness of the form \( \Omega \).

The general multidimensional integrable hierarchy arising from the generating relation of the form (87) was considered in [20]. Here we give a brief outline of this construction, emphasizing the connections with the construction developed in this work. First, following [20], we impose the generating relation (87) on the formal series of the variable \( \lambda \):
\[ \Psi^0 = \lambda + \sum_{n=1}^{\infty} \Psi^0_n (t^1, \ldots, t^N) \lambda^{-n}, \]
(92)
\[ \Psi^k = \sum_{n=0}^{\infty} \Psi^k_n (\Psi^0)^n + \sum_{n=0}^{\infty} \Psi^k_n (t^1, \ldots, t^N) (\Psi^0)^{-n}, \]
(93)
where \( 1 \leq k \leq N-2 \), depending on \( k \) infinite sequences of independent variables \( t^k = (t^k_0, \ldots, t^k_n, \ldots) \), \( t^k_0 = x_k, \lambda = x_0 \). Some motivation of introduction of the series of this type and higher times of the hierarchy is given in [3] in the simpler case of hyper-Kähler hierarchies (vector fields do not contain a derivative over the spectral variable). The setting of the hierarchy with an infinite number of higher times corresponds to \( \text{Gr}(N-1, \infty) \).

Slightly modifying notations in formulae (90) and (91), we obtain an infinite set of Lax–Sato equations of the hierarchy in the form
\[ \partial_{t_k} \Psi^l + (-1)^{N-1} \sum_{k=0}^{N-2} d_{nk} \partial_{\tilde{t}_k} \Psi^l = 0, \]
(94)
where \( l = 1, \ldots, N-2, n = 1, \ldots, \infty, \partial_{\tilde{t}_k} = \frac{\partial}{\partial t^k} \),
\[ d_{nk} = (-1)^k \left( J^{-1} \left[ D(\Psi^0, \Psi^1, \ldots, \Psi^{N-1}, \Psi^N) \right] \right)_+, \]
and the fraction denotes a Jacobian matrix, and \( \tilde{t}_k \) is the absent element. Using the series (92) and (93) and estimating the projections, it is possible to simplify these expressions and get rid of the derivatives over higher times
\[ (-1)^{N-1} d_{nk} = - (-1)^k (-1)^l \left( J^{-1} (\Psi^0)^l \left[ D(\Psi^0, \ldots, \Psi^l, \ldots, \Psi^{N-2}) \right] \right)_+. \]
The corresponding Lax–Sato equations are
\[ \partial_{t_k} \Psi^l = \sum_{k=0}^{N-2} (-1)^k (-1)^l \left( J^{-1} (\Psi^0)^l \left[ D(\Psi^0, \ldots, \Psi^l, \ldots, \Psi^{N-2}) \right] \right)_+ \partial_{\tilde{t}_k} \Psi^l. \]
These equations have an evolutionary form and define the dynamics of the series \( \Psi^l \) with the coefficients depending on \( x_1, \ldots, x_{N-2} \) with respect to the higher times. The evolutionary
form of Lax–Sato equations is connected with the special choice of the form of the series (92) and (93); it is a serious argument in favor of this choice.

Using the Jacobian matrix

\[(\text{Jac}_0) = \begin{pmatrix} D(\Psi^0, \ldots, \Psi^{N-2}) \\ D(x_0, \ldots, x_{N-2}) \end{pmatrix}, \quad \det(\text{Jac}_0) = J,\]

we obtain the Lax–Sato equation in the form introduced in [20],

\[\partial_t^k \Psi = \sum_{i=0}^{N-2} ((\text{Jac}_0)^{-1})_i {\cal H}(\Psi^0)^n_i \partial_i \Psi, \quad 1 \leq k \leq N-2, \]

where \(1 \leq n < \infty, \Psi = (\Psi^0, \ldots, \Psi^{N-2}).\) It was proved in [20] that a complete set of Lax–Sato equations (95) is equivalent to the generating relation (87) and that Lax–Sato flows are compatible. First flows of the hierarchy read

\[\partial_t^k \Psi = \left( \lambda \partial_k - \sum_{p=1}^{N-2} (\partial_p u_p) \partial_p - (\partial_q u_0) \partial_k \right) \Psi, \quad 1 \leq k \leq N-2, \]

where \(u_0 = \Psi^0, u_k = \Psi^k, 1 \leq k \leq N-2.\) A compatibility condition for any pair of linear equations (e.g., with \(\partial_t^k \) and \(\partial_t^q, k \neq q\)) implies a closed nonlinear \(N\)-dimensional system of PDEs for the set of functions \(u_k, u_0,\) which can be written in the form

\[\partial_t^k \partial_q \hat{u} - \partial_t^q \partial_k \hat{u} + [\partial_q \hat{u}, \partial_k u] = (\partial_k u_0) \partial_q - (\partial_q u_0) \partial_k,\]

\[\partial_t^k \partial_0 u_0 - \partial_t^q \partial_0 u_0 + (\partial_q \hat{u}) \partial_0 u_0 - (\partial_q \hat{u}) \partial_q u_0 = 0, \]

where \(\hat{u}\) is a vector field, \(\hat{u} = \sum_{\ell=1}^{N-2} u_\ell \partial_\ell.\) For \(N = 4,\) this system corresponds to the system (76)–(78) and, under a volume-preservation reduction \(J = 1,\) to the Dunajski system (82), (83). The case \(N = 3\) corresponds to the hierarchy connected with the system (56), (57) (the Manakov–Santini hierarchy, see [20]).

### 6. Dual and self-dual quasi-linear systems

For the dual Grassmannian \(\text{Gr}(2, N+1),\) one defines the differential 2-form

\[\Omega^*_2 = \omega_{N-1} \wedge \omega_N, \]

where 1-forms \(\omega_\gamma = \sum_{i=0}^{N} \omega^*_\gamma d x_i.\) So,

\[\Omega^*_2 = \sum_{l_0, l_1} \pi^{*}_{l_0 l_1} d x_{l_0} \wedge d x_{l_1}, \]

This 2-form and \(N-1\)-form \(\Omega_{N-1}\) are connected by the Hodge star (duality) operation \(\star:\)

\[\Omega^*_2 = \star \Omega_{N-1}, \]

where by definition (see e.g. [34])

\[\star (d x_{l_0} \wedge d x_{l_1} \wedge \ldots \wedge d x_{l_m}) = \frac{1}{(n-m-1)!} \sum_{l_0, l_1, \ldots, l_m} \epsilon_{l_0, l_1, \ldots, l_m} d x_{l_0} \wedge \ldots \wedge d x_{l_m} \wedge \ldots \wedge d x_{l_m}. \]

Operator \(\delta\) dual to the exterior differential \(d\) is defined as

\[\delta \Omega = (-1)^{m+p+m+1} \star d \star \Omega, \]

(102)
where $m$ is the dimension of the space and $p$ is an order of the form $\Omega$. In particular, $\delta \delta^* = \dd \dd$ and the operator $\Delta = \dd \delta + \delta \dd$ is self-dual. The differential form $\Omega$ obeying the condition $\delta \Omega = 0$ (or $d \Omega^* = 0$) is called co-closed. The form which is closed and co-closed, i.e. $\Delta \Omega = 0$ is referred to as the harmonic form (see e.g. [34]).

The condition of closedness of the form $\Omega^2_\Omega$ is given by the system of equations

\[
\left[ \frac{\partial \pi^*_{\ell_0 \ell_1}}{\partial x_{\ell_0}} \right] = 0,
\]

where indices take all values $0, 1, \ldots, N$. In virtue of (21) this system is equivalent to

\[
\frac{\partial J^*}{\partial x_\gamma} + \frac{\partial (J^* a^*_{\gamma 0})}{\partial x_0} + \frac{\partial (J^* a^*_{\gamma 1})}{\partial x_1} = 0, \quad \gamma = 2, \ldots, N,
\]

\[
\frac{\partial a^*_{ji}}{\partial x_k} - \frac{a^*_{ji}}{\partial x_j} + \sum_{m=0}^{1} \left( a^*_{km} \frac{\partial a^*_{ji}}{\partial x_m} - a^*_{jm} \frac{\partial a^*_{ji}}{\partial x_m} \right) = 0, \quad l = 0, 1; \quad j, k = 2, \ldots, N.
\]

At $N=3$, the system (104), (105) consists of four equations which in the variables $J^*$ and $a^*_j$ have the form similar to the original system (33), (34). In the original variables, due to the relation $\pi^*_\alpha = \sum_{l=0}^{3} e_{ilm} \pi_{lm}$, i.e.

$\pi^*_{01} = 2\pi_{23} = -2J(a_{21}a_{10} - a_{11}a_{20}), \quad \pi^*_{02} = -2\pi_{13} = -2Ja_{20}, \quad \pi^*_{03} = 2\pi_{12} = 2Ja_{10},$

$\pi^*_{12} = 2\pi_{03} = -2Ja_{21}, \quad \pi^*_{13} = -2\pi_{02} = 2Ja_{11}, \quad \pi^*_{23} = 2\pi_{01} = 2J$

it is of the form

\[
\frac{\partial J}{\partial x_1} - \frac{\partial (Ja_{21})}{\partial x_3} + \frac{\partial (Ja_{11})}{\partial x_2} = 0,
\]

\[
\frac{\partial J}{\partial x_0} - \frac{\partial (Ja_{20})}{\partial x_3} + \frac{\partial (Ja_{10})}{\partial x_2} = 0,
\]

\[
\frac{\partial (Ja_{11})}{\partial x_0} - \frac{\partial (Ja_{10})}{\partial x_1} + \frac{\partial}{\partial x_3} [J(a_{21}a_{10} - a_{11}a_{20})] = 0,
\]

\[
\frac{\partial (Ja_{21})}{\partial x_0} - \frac{\partial (Ja_{20})}{\partial x_1} - \frac{\partial}{\partial x_3} [J(a_{21}a_{10} - a_{11}a_{20})] = 0.
\]

Due to equations (107) and (108), equations (109) and (110) are equivalent to the following:

\[
\frac{\partial a_{10}}{\partial x_1} - \frac{\partial a_{11}}{\partial x_0} + a_{10} \frac{\partial a_{11}}{\partial x_2} + a_{20} \frac{\partial a_{11}}{\partial x_3} - a_{11} \frac{\partial a_{10}}{\partial x_2} - a_{21} \frac{\partial a_{10}}{\partial x_3} = 0,
\]

\[
\frac{\partial a_{20}}{\partial x_1} - \frac{\partial a_{21}}{\partial x_0} + a_{10} \frac{\partial a_{21}}{\partial x_2} + a_{20} \frac{\partial a_{21}}{\partial x_3} - a_{11} \frac{\partial a_{20}}{\partial x_2} - a_{21} \frac{\partial a_{20}}{\partial x_3} = 0.
\]

This dual system is quite similar to the original system (34).

If one requires that the form $\Omega^2_\Omega$ is a harmonic one, then one has a system of four equations (34), (111), (112) for four dependent variables $a_{10}, a_{11}, a_{20}$ and $a_{21}$. This system has a simple physical meaning. Indeed, the conditions of closedness and co-closedness of the form $\Omega^2_\Omega$, i.e. equations (3) and (103) are equivalent to

\[
\frac{\partial \pi^*_{ik}}{\partial x_i} + \frac{\partial \pi^*_{il}}{\partial x_j} + \frac{\partial \pi^*_{li}}{\partial x_k} = 0, \quad \sum_{k=0}^{3} \frac{\partial \pi^*_{ik}}{\partial x_k} = 0, \quad i, k, l = 0, 1, 2, 3.
\]
This system can be viewed as the second and first pairs of the sourceless Maxwell equations for the electromagnetic field tensor $\pi_{\Delta i}$, i.e. $\pi_{i0} = E_0$, $\pi_{\alpha 0} = -\sum_{\gamma=1}^{3} \epsilon_{\alpha \beta \gamma} H_\gamma$, $\alpha, \beta = 1, 2, 3$, where $\epsilon_{\alpha \beta \gamma}$ is totally antisymmetric three-dimensional tensor. Plücker’s relation (18) in terms of this is of the form

$$\sum_{\alpha=1}^{3} E_\alpha H_\alpha = 0. \quad (114)$$

Thus, for the harmonic 2-form $\Omega_2$ the equations for the Plücker coordinates coincide with the sourceless Maxwell equations with perpendicular electric and magnetic fields. This observation provides us with the variety of solutions of these equations (see e.g. [35]). Given such a solution one obtains solution of the system (34), (111) and (112) via

$$a_{10} = -\frac{H_3}{E_1}, \quad a_{11} = -\frac{E_2}{E_1}, \quad a_{20} = \frac{H_2}{E_1}, \quad a_{21} = -\frac{E_3}{E_1} \quad (115)$$

and $J = E_1$.

It is a simple check that under the Manakov–Santini ansatz (48) the system (34), (111), (112) has only trivial solution. Rational solutions of this system will be studied elsewhere.

At $N = 4$, the dual system (104), (105) is the system of nine equations

$$\frac{\partial J^*}{\partial x_2} + \frac{\partial (J^* a_{20}^*)}{\partial x_0} + \frac{\partial (J^* a_{21}^*)}{\partial x_0} = 0, \quad (116)$$

$$\frac{\partial J^*}{\partial x_3} + \frac{\partial (J^* a_{30}^*)}{\partial x_0} + \frac{\partial (J^* a_{31}^*)}{\partial x_0} = 0, \quad (117)$$

$$\frac{\partial J^*}{\partial x_4} + \frac{\partial (J^* a_{40}^*)}{\partial x_0} + \frac{\partial (J^* a_{41}^*)}{\partial x_0} = 0, \quad (118)$$

$$\frac{\partial a_{20}^*}{\partial x_5} - \frac{\partial a_{20}^*}{\partial x_2} + a_{20}^* \frac{\partial a_{20}^*}{\partial x_0} - a_{20}^* \frac{\partial a_{20}^*}{\partial x_4} + a_{20}^* \frac{\partial a_{20}^*}{\partial x_2} - a_{20}^* \frac{\partial a_{20}^*}{\partial x_4} = 0, \quad (119)$$

$$\frac{\partial a_{21}^*}{\partial x_5} - \frac{\partial a_{21}^*}{\partial x_3} + a_{21}^* \frac{\partial a_{21}^*}{\partial x_0} - a_{21}^* \frac{\partial a_{21}^*}{\partial x_3} + a_{21}^* \frac{\partial a_{21}^*}{\partial x_3} - a_{21}^* \frac{\partial a_{21}^*}{\partial x_3} = 0. \quad (120)$$

$$\frac{\partial a_{20}^*}{\partial x_4} - \frac{\partial a_{20}^*}{\partial x_2} + a_{20}^* \frac{\partial a_{20}^*}{\partial x_0} - a_{20}^* \frac{\partial a_{20}^*}{\partial x_4} + a_{20}^* \frac{\partial a_{20}^*}{\partial x_2} - a_{20}^* \frac{\partial a_{20}^*}{\partial x_4} = 0. \quad (121)$$

$$\frac{\partial a_{21}^*}{\partial x_4} - \frac{\partial a_{21}^*}{\partial x_3} + a_{21}^* \frac{\partial a_{21}^*}{\partial x_0} - a_{21}^* \frac{\partial a_{21}^*}{\partial x_4} + a_{21}^* \frac{\partial a_{21}^*}{\partial x_3} - a_{21}^* \frac{\partial a_{21}^*}{\partial x_4} = 0. \quad (122)$$

$$\frac{\partial a_{30}^*}{\partial x_4} - \frac{\partial a_{30}^*}{\partial x_3} + a_{30}^* \frac{\partial a_{30}^*}{\partial x_0} - a_{30}^* \frac{\partial a_{30}^*}{\partial x_4} + a_{30}^* \frac{\partial a_{30}^*}{\partial x_3} - a_{30}^* \frac{\partial a_{30}^*}{\partial x_4} = 0. \quad (123)$$

$$\frac{\partial a_{31}^*}{\partial x_4} - \frac{\partial a_{31}^*}{\partial x_3} + a_{31}^* \frac{\partial a_{31}^*}{\partial x_0} - a_{31}^* \frac{\partial a_{31}^*}{\partial x_4} + a_{31}^* \frac{\partial a_{31}^*}{\partial x_3} - a_{31}^* \frac{\partial a_{31}^*}{\partial x_4} = 0. \quad (124)$$

It is quite different from the original system (25, 26) at $N = 4$. The equations (116), (117), (119) and (120) form a closed subsystem for the dependent variables $J^*$, $a_{20}^*$, $a_{21}^*$, $a_{30}^*$, $a_{31}^*$. The same is valid for the groups of variables $J^*$, $a_{20}^*$, $a_{21}^*$, $a_{40}^*$, $a_{41}^*$ and $J^*$, $a_{30}^*$, $a_{31}^*$, $a_{40}^*$, $a_{41}^*$. So, the whole system (116)–(124) is decomposed into three independent subsystems. Each of these subsystems coincides with the system (33) and (34) for Gr(2, 4). Such a decomposition remains valid also for the rational solutions discussed in section 4. So, the simplest rational solutions of
the system (119)–(124) can be constructed as the common solutions of three independent Manakov–Santini or two-component dispersionless 2DTL systems. The subsystem (119)–(124), which is the gauge invariant form of the system (103), describes the coisotropic deformations of the family of the planes defined by (21). It contains twice as many equations as the subsystem (26) at $N = 4$.

Similar situation takes place for the dual systems for Gr $(N − 1, N + 1)$ at $N \geq 5$.

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References

[1] Zakharov V E and Shabat A B 1979 Integration of nonlinear equations of mathematical physics by the method of inverse scattering problem: II Funkts. Anal. Prilozh. 13 13
[2] Plebański J F 1975 Some solutions of complex Einstein equations J. Math. Phys. 16 2395–402
[3] Takasaki K 1989 An infinite number of hidden variables in hyper-Kähler metrics J. Math. Phys. 30 1515–21
[4] Kodama Y and Gibbons J 1990 Integrability of the dispersionless KP hierarchy Proc. 4th Workshop on Nonlinear and Turbulent Processes in Physics (Singapore: World Scientific) pp 160–80
[5] Zakharov V E 1994 Dispersionless limit of integrable systems in 2+1 dimensions Singular Limits of Dispersive Waves ed N M Ercolani et al (New York: Plenum)
[6] Krichever I M 1994 The $t$-function of the universal Witten hierarchy, matrix models and topological field theories Commun. Pure Appl. Math. 47 437–75
[7] Manakov S V and Santini P M 2004 The Cauchy problem on the plane for the dispersionless Kadomtsev–Petviashvili equation JETP Lett. 83 462–6
[8] Manakov S V and Santini P M 2007 A hierarchy of integrable PDEs in 2+1 dimensions associated with 2-dimensional vector fields Theor. Math. Phys. 152 1004–11
[9] Takasaki K and Takebe T 1991 SDiff(2) Toda equation—hierarchy, Tau function, and symmetries Lett. Math. Phys. 23 205–14
[10] Takasaki K and Takebe T 1995 Integrable hierarchies and dispersionless limit Rev. Math. Phys. 7 743–808
[11] Manakov S V and Santini P M 2008 On the solutions of the dKP equation: nonlinear Riemann Hilbert problem, longtime behaviour, implicit solutions and wave breaking J. Phys. A: Math. Theor. 41 055204
[12] Manakov S V and Santini P M 2009 The dispersionless 2D Toda equation: dressing, Cauchy problem, longtime behavior, implicit solutions and wave breaking J. Phys. A: Math. Theor. 42 04013
[13] Manakov S V and Santini P M 2011 Solvable vector nonlinear Riemann problems, exact implicit solutions of dispersionless PDEs and wave breaking J. Phys. A: Math. Theor. 44 345203
[14] Bogdanov L V 2010 On a class of reductions of the Manakov–Santini hierarchy connected with the interpolating system J. Phys. A: Math. Theor. 43 115206
[15] Bogdanov L V 2011 Interpolating differential reductions of multidimensional integrable hierarchies Theor. Math. Phys. 167 705–13
[16] Bogdanov L V 2012 Dunajski–Tod equation and reductions of the generalized dispersionless 2DTL hierarchy arXiv:1204.3780
[17] Bogdanov L V, Dryuma V S and Manakov S V 2007 Dunajski generalization of the second heavenly equation: dressing method and the hierarchy J. Phys. A: Math. Theor. 40 14383–93
[18] Bogdanov L V 2009 A class of multidimensional integrable hierarchies and their reductions Theor. Math. Phys. 160 887–93
[19] Bogdanov L V 2010 Non-Hamiltonian generalizations of the dispersionless 2DTL hierarchy J. Phys. A: Math. Theor. 43 434008
[20] Hodge W V D and Pedoe D 1994 Methods of Algebraic Geometry vol 1,2 (Cambridge: Cambridge University Press)
[23] Sato M and Sato Y 1982 Soliton equations as dynamical systems in infinite dimensional Grassmann manifold
Lect. Notes Num. Appl. Anal. 5 259–71
[24] Takasaki K 1989 Geometry of universal Grassmann manifold from algebraic point of view Rev. Math.
Phys. 1 1–46
[25] Konopelchenko B and Magri F 2007 Coisotropic deformations of associative algebras and dispersionless
integrable hierarchies Commun. Math. Phys. 274 627–58
[26] Konopelchenko B and Ortenzi G 2009 Coisotropic deformations of algebraic varieties and integrable systems
J. Phys. A: Math. Theor. 42 415207
[27] Agafonov S I and Ferapontov E V 1996 Systems of conservation laws form the point of view of the projective
theory of congruences Izv. Ross. Akad. Nauk. Ser. Mat. 60 3–30
[28] Ferapontov E V 2001 Transformations of quasilinear systems originating from the projective theory of
congruences Geometry of Solitons (CRM Proceedings and Lecture Notes vol 29) (Providence, RI: American
Mathematical Society) 179–90
[29] Martínez Alonso L and Shabat A B 2002 Energy-dependent potentials revisited: a universal hierarchy of
hydrodynamic type Phys. Lett. A 300 58–64
[30] Pavlov M V 2003 Integrable hydrodynamic chains J. Math. Phys. 44 4134–56
[31] Martínez Alonso L and Shabat A B 2004 Hydrodynamic reductions and solutions of a universal hierarchy Theor.
Math. Phys. 140 1073–85
[32] Airault H, McKean H P and Moser J 1977 Rational and elliptic solutions of the KdV equation and related
many-body problems Commun Pure Appl. Math. 30 95–198
[33] Sternberg S 1964 Lectures on Differential Geometry (Englewood Cliffs, NJ: Prentice-Hall)
[34] de Rham G 1955 Variétes Differentiables. Formes, Courants, Formes Harmoniques (Paris: Hermann)
[35] Landau L D and Lifshits E M 1967 Course of Theoretical Physics: Field Theory vol 2 (Moscow: Nauka)