OCCUPATION KERNEL HILBERT SPACES AND THE SPECTRAL ANALYSIS OF NONLOCAL OPERATORS

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Abstract. This manuscript introduces a space of functions, termed occupation kernel Hilbert space (OKHS), that operate on collections of signals rather than real or complex functions. To support this new definition, an explicit class of OKHSs is given through the consideration of a reproducing kernel Hilbert space (RKHS). This space enables the definition of nonlocal operators, such as fractional order Liouville operators, as well as spectral decomposition methods for corresponding fractional order dynamical systems. In this manuscript, a fractional order DMD routine is presented, and the details of the finite rank representations are given. Significantly, despite the added theoretical content through the OKHS formulation, the resultant computations only differ slightly from that of occupation kernel DMD methods for integer order systems posed over RKHSs.

1. Introduction. Despite the proliferation of numerical methods for and applications of fractional order and nonlocal dynamical systems over the past twenty years (cf. [50, 12, 41, 31]), there are several long standing problems in the modeling of nonlocal dynamical systems and system identification techniques for nonlinear fractional order systems (cf. [11, 45, 13]). Principle among these problems is the effective representation of data for modeling in nonlocal nonlinear systems. This manuscript enables several novel data driven approaches to modeling nonlocal dynamical systems, including a nonlocal variant of Dynamic Mode Decomposition (DMD), by addressing the problem of data representation for nonlinear time-fractional order dynamical systems. In particular, this work builds on work involving occupation kernels, which embeds signal information into a function within a reproducing kernel Hilbert space (RKHS) and thus positions signal data as the fundamental unit of information of a dynamical system (cf. [55, 52, 51]), by extending the idea beyond merely functions inside of a space, but by generating Hilbert spaces of functions on collections of signals based on the occupation kernels themselves.

Fractional order dynamical systems have been applied broadly in nearly every scientific discipline. For example, applications include two-phase flows [66, 67], turbulence modeling [59], and stochastic systems (e.g. [14]). Other research has exploited the memory capabilities of fractional order operators in the context of materials, where classically they were used to model visco-elastic materials [33] and more recently biological tissues [32], and fractional order methods have also been employed in biological applications for guidance in sensor placement [62]. Extensions of control theory have been realized through fractional order PID controllers [44], which have seen applications to problems in aerospace and control of flexible systems [39, 58].

Despite the wide application of fractional order systems, given a real world system, the development of a corresponding fractional order model is nontrivial. In the absence of first principles through which a model may be realized, many applications

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*This research was supported by the Air Force Office of Scientific Research (AFOSR) under contract numbers FA9550-20-1-0127, and the National Science Foundation (NSF) under award 2027976. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the sponsoring agencies.

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of fractional order dynamical systems leverage fractional operators in an unmotivated way. This manuscript provides a new Hilbert space setting for the modeling of nonlinear fractional order systems, which will provide a data driven motivation for the implementation of fractional order operators and models.

System identification and learning for fractional order systems has been largely constrained to linear fractional order systems. For example, [45] developed a system identification routine through the Laplace transform, while [20] gives a learning method for a collection linear fractional order PDEs. A notable exception is fPiNNs (cf. [35] and [41]), which uses physics informed neural network to provide a model of a fractional order PDE, and also that of the authors which leveraged a regression approach with occupation kernels [30]. The methods herein differ from that of fPiNNs in that they are operator based (rather than neural network based) approaches to system identification and modeling, and constitute a contribution in a new direction for both data driven modeling for fractional order system and for DMD analysis techniques as a whole.

One of the hurtles that have prevented the development of system identification for nonlocal nonlinear systems is a matter of data representation, which is resolved in the sequel by the invocation of occupation kernels. This builds on the work in [30] in that some of the occupation kernels take the same form as in [30]. The objective of the work of [30] was to obtain an approximation of the dynamics themselves. The principle difference is the development of nonlocal operators and a new Hilbert space framework, which allows for spectral decompositions of these operators for expressing a model for the system.

Section 4 introduces occupation kernel Hilbert spaces (OKHSs). The construction presented here is for the particular case of the Caputo fractional derivative, but is immediately generalizable to a wide range of time fractional operators where initial value problems may be resolved using Voltera integral equations. The development of OKHSs enables the generalization of a key operator used in the study of nonlinear dynamical systems, namely the Liouville operator. Some Liouville operators are continuous generators of semigroups of Koopman operators [9, 10, 15, 16, 17, 18], which are the pivotal tools in the study of nonlinear dynamical systems of integer order.

1.1. Overview of Dynamic Mode Decomposition Methods. DMD emerged as an effective data-driven method of learning dynamical systems from trajectory data with no prior knowledge. The DMD method aims to analyze finite dimensional nonlinear dynamical systems as operators over infinite dimensional Hilbert spaces, where the tools of traditional linear systems may be employed to make predictions about nonlinear systems from captured trajectory data (or snapshots). Though based on early work by Koopman and Von Neumann in the 1930s, DMD more recently came to prominence as a method of identifying underlying governing principles of nonlinear fluid flows (cf. [5, 7, 26, 36, 37, 64, 65]), which compared favorably with principle orthogonal decomposition (POD) analyses.

Underlying classical DMD methods are Koopman operators, which are operators over function spaces that represent discrete time dynamics [7, 26]. That is, given a discrete dynamical system, \( x_{i+1} = F(x_i) \), and a Hilbert space of functions, \( H \), the corresponding Koopman operator, \( \mathcal{K}_F : \mathcal{D}(\mathcal{K}_F) \to H \) is given as \( \mathcal{K}_F g = g \circ F \) for all \( g \in \mathcal{D}(\mathcal{K}_F) \subset H \), where \( \mathcal{D}(\mathcal{K}_F) \) is a given subset of \( H \) corresponding to the domain of \( \mathcal{K}_F \). For a collection of snapshots of a dynamical system, \( x_1, x_2, \ldots, x_m \), the dynamics are then represented through observables pulled from \( H \), as \( g(x_1), g(x_2), \ldots, g(x_m) \), where \( g(x_{i+1}) = g(F(x_i)) = \mathcal{K}_F g(x_i) \) [26, 64]. When \( H \) is a reproducing kernel Hilbert
space (RKHS) over \( \mathbb{R}^n \) with kernel function \( K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) (the definition of which is given in the Technical Background below), then this action on the observables may be expressed as

\[
\langle g, K(\cdot, x_{i+1}) \rangle_H = g(x_{i+1}) = \langle K_F g, K(\cdot, x_i) \rangle_H = \langle g, K_F^* K(\cdot, x_i) \rangle_H.
\]

Hence, discrete time dynamics may be captured through the adjoint of the Koopman operator acting on the kernel functions of a RKHS.\(^1\) This connection between Koopman operators and RKHSs is the core observation of kernel-based extended DMD presented in [65, 24]. The kernel-based extended DMD algorithm then expands the kernel function as a column of features, collects a sequence of input-output data, and then reduces the system through a singular value decomposition. The DMD procedure is completed upon determining the eigenvectors and eigenvalues of the reduced system, and then lifting up these eigenvalues and eigenfunctions to the Koopman operator to establish what are called Koopman modes for the dynamical system [65].

So described is the DMD procedure for discrete time dynamical systems. Practitioners analyze continuous time dynamics by first fixing a time-step and then examining a discrete time proxy for the continuous time system [26]. The intuition employed in this approach is that by taking small time steps, an adequate approximation of the Liouville operator may be determined by using a Koopman operator.

The limiting perspective using Koopman operators has several major theoretical drawbacks; the most significant and fundamental is its restriction to dynamical systems that admit a discretization. The symbol (or discrete dynamics) of a Koopman operator must be defined over all of the state space for the Koopman operator to be well defined. The consideration of the dynamics \( \dot{x} = 1 + x^2 \) yields a discretization of \( x_{m+1} = \tan(\Delta t + \arctan(x_m)) \), where it can be seen that the selection of \( x_m = \tan(\pi/2 - \Delta t) \) gives an undefined value for \( x_{m+1} \). Hence, many polynomial systems do not fall under the purview of Koopman based techniques, and the particular requirement that guarantees the existence of a discretization of a dynamical system is forward completeness (cf. [25, Chapter 11]) which is usually established by demonstrating that a dynamical system is globally Lipschitz continuous (cf. [6] and [23, Theorem 3.2]).

To address the above limitation of the Koopman perspective, [51] introduced a combination of Liouville operators and occupation kernels to determine a finite rank representation of Liouville operators for spectral analysis. This allows for the analysis of dynamical systems that do not admit discretizations, and as such, this allows for the analysis of Liouville operators that were not also Koopman generators. Moreover, the introduction of occupation kernels and scaled Liouville operators in [51] over the Bargmann-Fock space yielded a compact operator for a wide range of dynamics, and scaled Liouville operators allows for norm convergence of finite rank approximations used in DMD procedures. The main thrust of this manuscript is to provide a framework for the transportation of these tools to (time) fractional order dynamical systems, where data driven modeling techniques using DMD procedures could not be previously applied.

2. Motivation for the present approach. Suppose that for \( 0 < q < 1 \) the Caputo fractional derivative is given as\(^2\) \( D^q x(t) = C_{1-q} \int_0^t (t - \tau)^{-q} \dot{x}(\tau) \, d\tau \), and let

\[^1\]In the context of function theoretic operator theory, the function \( F \), which defines the discrete time dynamics, is called the symbol of the Koopman operator. This terminology is common among many other such operators over RKHSs (cf. [49, 48, 56])

\[^2\]Here \( C_q : = 1/\Gamma(q) \) is used to avoid confusion with the occupation kernels.
$D^q_t x(t) = f(x(t))$ be a nonlinear fractional order system with the initial condition $x(0) = x_0 \in \mathbb{R}^n$. The Riemann-Liouville fractional integral is given as $J^q x(t) = C_q \int^t_0 (t-\tau)^{q-1} x(\tau) \, d\tau$, and the initial value problem may be resolved as $x(t) = x_0 + \int^t_0 D^q_t x(\tau) \, d\tau$ (cf. [12, 19]). Hence, $\dot{x}(t) = \frac{d}{dt} J^q f(x(t)) = D^{1-q} f(x(t))$, where $D^q$ is the Riemann-Liouville fractional derivative operator (cf. [12, 19, 40]).

Exploiting this core idea, generalizations of Liouville operators and occupation kernels may be defined.

Specifically, two variants on the Liouville operator are explored in this manuscript; the Liouville operator of order $q$ given as $A_{f,q} g := \nabla g(\cdot) D^{1-q} f(\cdot)$, and the fractional Liouville operator of order $q$ given as $A_{f,q} g = A_{f,q} g(\cdot)(T_\gamma) = \frac{1}{\Gamma(q)} \int^T_0 (T_\gamma - t)^{-q} \nabla g(\gamma(t)) \theta^{1-q} f(\gamma(t)) \, dt$ where $g$ is a function on a collection of signals in a Hilbert space that will be defined in Section 4. If the fractional Liouville operator was poised over a RKHS consisting of functions over $\mathbb{R}^n$, then the image of $A_{f,q} g$ would be a function of $\mathbb{R}^n$, which would mean that for any $x \in \mathbb{R}^n$, the quantity $A_{f,q} g(x)$ would be well defined. However, as a state carries no “history,” the term $D^{1-q} f(x)$ is not well defined. This difficulty prevents the straightforward generalization of Liouville operators for integer order dynamical systems to that of fractional order dynamical systems. Section 4 of this manuscript provides a resolution to this problem through the construction of a Hilbert space consisting of functions over signals. Thus, in the sequel, the domain of $A_{f,q} g$ is a collection of signals, and for a given continuously differentiable signal $\theta : [0,T] \to \mathbb{R}^n$, the quantity $D^{1-q} f(\theta(t))$ is well defined. The astute reader will ask how a gradient of a function, $g$, over a collection of signal is defined and this will be quantified more precisely in Section 4.

To accommodate the nonlocal requirements of the fractional Liouville operator, Section 4 constructs a Hilbert space from a RKHS, $H_{\text{RKHS}}$. Specifically, for any function, $g$, in a continuously differentiable RKHS over $\mathbb{R}^n$ a mapping over collection of signals arises naturally via $\phi_g[\theta](t) := g(\theta(t))$. Hence, $\phi_g$ maps the signal $\theta$ to the signal $g(\theta(\cdot))$. In combination with the canonical identification, $g \mapsto \phi_g$, and the inner product of the RKHS, $\langle \cdot, \cdot \rangle_{H_{\text{RKHS}}}$, an inner product on the vector space $H_{\text{OKHS}} := \{ \phi_g : g \in H_{\text{RKHS}} \}$ may be established, and the resultant Hilbert space will be called an occupation kernel Hilbert space (OKHS). The name for this Hilbert space arises from occupation kernels, which will play a key role in the analysis of nonlinear dynamical systems of fractional order, just as they have for integer order systems (cf. [55]). For a given signal, $\theta : [0,T] \to \mathbb{R}^n$, the occupation kernel of order $q > 0$ is given as $\Gamma_{\theta,q}(x) := C_q \int^T_0 (T - \tau)^{q-1} K(x, \theta(\tau)) \, d\tau$. Occupation kernels represent integration after composition of the signal with a function in a RKHS, $\langle g, \Gamma_{\theta,q} \rangle = C_q \int^T_0 (T - \tau)^{q-1} g(\theta(\tau)) \, d\tau$.

The introduction of OKHSSs and associated Liouville operators directly addresses the problem of data integration. That is, OKHSSs directly incorporate trajectory data in a kernel function contained in the Hilbert space. The definition of the Hilbert space, which consists of functions on signals, allows for the definition of nonlocal Liouville operators. In turn, these nonlocal Liouville operators allow for the development of DMD procedures on nonlinear fractional order systems.

3. Prerequisites.

3.1. Reproducing Kernel Hilbert Spaces. A (real-valued) reproducing kernel Hilbert space (RKHS) is a Hilbert space, $H$, of real valued functions over a set $X$ such that for all $x \in X$ the evaluation functional $E_x g := g(x)$ is bounded (cf. [42, 60, 63, 1, 3, 47, 54]). As such, the Riesz representation theorem guarantees, for
all \( x \in X \), the existence of a function \( k_x \in H \) such that \( \langle g, k_x \rangle_H = g(x) \), where \( \langle \cdot, \cdot \rangle_H \) is the inner product for \( H \). The function \( k_x \) is called the reproducing kernel function at \( x \), and the function \( K(x, y) = \langle k_y, k_x \rangle_H \) is called the kernel function corresponding to \( H \).

To establish a connection between RKHSs and nonlinear dynamical systems the following operator was introduced, which was inspired by the study of occupation measures and related concepts (cf. [27, 2, 34, 21, 57, 22, 4, 5, 8, 28, 29, 36, 38, 61]).

**Definition 3.1.** Let \( \dot{x} = f(x) \) be a dynamical system with the dynamics, \( f : \mathbb{R}^n \to \mathbb{R}^n \), Lipschitz continuous, and suppose that \( H \) is a RKHS over a set \( X \), where \( X \subset \mathbb{R}^n \) is compact. The Liouville operator with symbol \( f \), \( A_f : D(A_f) \to H \), is given as \( A_f g := \nabla_x g \cdot f \), where \( D(A_f) := \{ g \in H : \nabla_x g \cdot f \in H \} \).

As a differential operator, \( A_f \) is not expected to be a bounded over many RKHSs. However, as differentiation is a closed operator over RKHSs consisting of continuously differentiable functions (cf. [60, 46]), it can be similarly established that \( A_f \) is closed under the similar circumstances (cf. [55]). Thus, \( A_f \) is a closed operator for RKHSs consisting of continuously differentiable functions. Consequently, the adjoints of densely defined Liouville operators are themselves densely defined (cf. [43]).

Associated with Liouville operators in particular are a special class of functions within the domain of the Liouville operators’ adjoints. The following definition and proposition were given in [55], and these generalize the framework of [27] on occupation measures.

**Definition 3.2.** Let \( X \subset \mathbb{R}^n \) be compact, \( H \) be a RKHS of continuous functions over \( X \), and \( \gamma : [0, T] \to X \) be a continuous signal. The functional \( g \mapsto \int_0^T g(\gamma(\tau))d\tau \) is bounded, and may be represented as \( \int_0^T g(\gamma(\tau))d\tau = \langle g, \Gamma_\gamma \rangle_H \), for some \( \Gamma_\gamma \in H \) by the Riesz representation theorem. The function \( \Gamma_\gamma \) is called the occupation kernel corresponding to \( \gamma \) in \( H \).

**Proposition 3.3.** Let \( H \) be a RKHS of continuously differentiable functions over a compact set \( X \), and suppose that \( f : \mathbb{R}^n \to \mathbb{R}^n \) is Lipschitz continuous. If \( \gamma : [0, T] \to X \) is a trajectory that satisfies \( \dot{\gamma} = f(\gamma) \), then \( \Gamma_\gamma \in D(A_\gamma^\dagger) \), and \( A_\gamma^\dagger \Gamma_\gamma = K(\cdot, \gamma(T)) - K(\cdot, \gamma(0)) \).

Proposition 3.3 thus integrates nonlinear dynamical systems with RKHSs through the Liouville operator. In particular, the action of the adjoint of the Liouville operator acting on an occupation kernel expressed as a difference of kernels, enables continuous time DMD analyses that do not require an a priori discretization of the dynamical system [51].

### 3.2. Time Fractional Dynamical Systems

Time fractional derivatives, such as the Riemann-Liouville fractional derivative and the Caputo fractional derivative, are defined with respect to two operators: integer order derivatives and the Riemann-Liouville fractional integral given as \( J^{\alpha} \gamma(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \gamma(\tau) d\tau \). The Riemann-Liouville fractional derivative of order \( 0 < q < 1 \) of a function \( \gamma \) is then given as \( D_q^\alpha \gamma(t) := \frac{d}{d(t)} J^{1-q} \gamma(t) \), while the Caputo fractional derivative of order \( 0 < q < 1 \) is given as \( D_q^\alpha \gamma(t) := J^{1-q} \gamma'(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-\tau)^{\alpha-1} \gamma'(\tau) d\tau \). Initial value problems for the Caputo fractional derivative may be expressed as

\[
D_q^\alpha x = f(x) \quad \text{with} \quad x(0) = x_0,
\]

and their solution can be written in terms of a Volterra operator as \( x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau) d\tau \). Riemann-Liouville based initial value problems require initial
values with respect to fractional derivatives, which are difficult to express for a physical system [12]. Hence, the present manuscript will focus on dynamical systems arising from the Caputo fractional derivative.

4. Occupation Kernel Hilbert Spaces. To provide for a nonlocal Hilbert space framework, the objective of Section 4 is to develop further the concept of occupation kernels. At this point, occupation kernels have been viewed as a part of a RKHS rather than as a generator of a Hilbert space on their own. In what follows, this will remain true to some extent, where many OKHSs will arise from RKHSs. However, the central objects of study will be bounded functionals that come from trajectories. The focus on trajectories rather than points in \( \mathbb{R}^n \) allows for the treatment of operators on nonlocal information. Section 4 formalizes the necessary Hilbert space framework, which will enable the development of DMD analyses of (time) fractional order dynamical systems.

**Definition 4.1.** Let \( q > 0 \). For fixed \( d, e \in \mathbb{N} \) with \( e < d \), let \( X = \cup_{T>0} C^d([0,T], \mathbb{R}^n) \) and \( Y = \cup_{T>0} C^e([0,T], \mathbb{R}) \). The OKHS of order \( q \), \( H \), over \( X \) is a Hilbert space that consists of functions \( g : X \to Y \) such that for each \( x \in X \) and \( g(x) : [0,T] \to \mathbb{R} \) in \( Y \) the functional \( g \mapsto J_g \) is bounded, with \( g(x) \in Y \) indicating the mapping of \( g \) from \( X \) to \( Y \) by \( g \in H \).

For each such functional, the Riesz representation theorem yields a function, \( \Gamma_{\gamma,q} \in H \), such that \( \langle g, \Gamma_{\gamma,q} \rangle_H = \frac{1}{\Gamma(q)} \int_0^T (T - t)^{q-1} g(t) dt \), this function is called the occupation kernel corresponding to \( H \).

4.1. OKHSs from RKHSs. The existence of OKHSs follows immediately from that of RKHSs.

**Definition 4.2.** Let \( X = \cup_{T>0} C^d([0,T], \mathbb{R}^n) \) and \( Y = \cup_{T>0} C^e([0,T], \mathbb{R}^m) \) and \( Y \) be a vector space of infinitely differentiable functions \( g : \mathbb{R}^n \to \mathbb{R}^m \). For each \( g \in Y \) let \( \phi_g : X \to Y \) be the mapping that takes \( \gamma \in X \) to \( \phi_g(\gamma) := g(\gamma(t)) \in Y \). Let \( \mathcal{W} \) be the vector space of maps \( \varphi : X \to Y \), under the standard operations. The operator, \( \mathcal{T} : \mathcal{Y} \to \mathcal{W} \) given by \( \mathcal{T} g := \phi_g \), is called the canonical mapping of the scalar valued function \( g \) to the signal valued function \( \phi_g \).

Note, since \( g \) is infinitely differentiable, \( \phi_g(\gamma) \) is at least as smooth as \( \gamma \). Hence \( \phi_g(\gamma) \in Y \).

**Theorem 4.3.** Let \( H_{RKHS} \) be an RKHS of infinitely differentiable scalar-valued functions over \( \mathbb{R}^n \) with kernel function \( K \). The space, \( \mathcal{T} H_{RKHS} := \{ Tg : g \in H_{RKHS} \} \) is an OKHS of any order \( q \geq 0 \) where the inner product is taken to be \( \langle \phi, \psi \rangle_{OKHS} = \langle \mathcal{T}^{-1} \phi, \mathcal{T}^{-1} \psi \rangle_{RKHS} \).

**Proof.** From the discussion before the theorem statement, \( \phi_g \) is well defined as a map from \( X \) to \( Y \) for any \( g \in H_{RKHS} \). To demonstrate that \( H_{OKHS} := \mathcal{T} H_{RKHS} \) is an occupation kernel Hilbert space, it must be demonstrated that the space is complete with respect to the norm induced by the inner product and that for any \( q > 0 \), the collection of functionals in Definition 4.1 are bounded.

Note that \( \mathcal{T} \) is linear, and hence \( H_{OKHS} \) is a vector space. That \( \mathcal{T} \) is injective follows from the consideration of constant signals. If \( g_1, g_2 \in H_{RKHS} \) are distinct functions, then there is a point \( x \in \mathbb{R}^n \) such that \( g_1(x) \neq g_2(x) \). Considering \( \theta_x(t) \equiv x \) for \( t \in [0,1] \) which resides in \( X \) given in Definition 4.1, it follows that \( \phi_{g_1}(\theta_x)(t) = g_1(\theta_x(t)) = g_1(x) \neq g_2(x) = \phi_{g_2}(\theta_x)(t) \) for all \( t \).
Once injectivity is established, the inner product on \( H_{OKHS} \), given as

\[
\langle \phi, \psi \rangle_{OKHS} := \langle T^{-1} \phi, T^{-1} \psi \rangle_{RKHS},
\]

is well defined. Moreover, \( T \) is automatically continuous with respect to the induced norm on \( H_{OKHS} \).

To demonstrate completeness, suppose that \( \{ \phi_m \}_{m=1}^{\infty} \subset TH_{RKHS} \) is Cauchy with respect to the norm induced by the inner product on \( H_{OKHS} \). For each \( m \), there is a function \( g_m \in H_{RKHS} \) such that \( \phi_m = Tg_m \). Given any \( \epsilon > 0 \), there is an \( M \) such that for all \( m, m' > M \), the following holds,

\[
\| \phi_m - \phi_{m'} \|_{OKHS}^2 = \sqrt{\langle \phi_m - \phi_{m'}, \phi_m - \phi_{m'} \rangle_{OKHS}} \leq \sqrt{\langle g_m - g_{m'}, g_m - g_{m'} \rangle_{RKHS}} = \| g_m - g_{m'} \|_{RKHS}^2.
\]

Hence, \( g_m \) is a Cauchy sequence in \( H_{RKHS} \), and there is a function \( g \in H_{RKHS} \) such that \( g_m \to g \). The continuity of the canonical identification yields \( \phi_{g_m} \to \phi_g \), and thus the limit of \( \phi_m \) resides in \( TH_{RKHS} \). Therefore, \( TH_{RKHS} \) is complete and \( TH_{RKHS} = H_{OKHS} \).

To demonstrate the boundedness of the functional \( \phi \mapsto C_q \int_0^T (T - \tau)^{q-1} \phi(\gamma(\tau))d\tau \) for each \( \gamma \in X \) and \( q > 0 \), let \( g = T^{-1} \phi \). It follows that

\[
\begin{align*}
\left| C_q \int_0^T (T - \tau)^{q-1} \phi(\gamma(\tau))d\tau \right| &\leq \| g \|_{RKHS} \cdot C_q \int_0^T (T - \tau)^{q-1} \sqrt{K(\gamma(\tau), \gamma(\tau))}d\tau \\
&= \| \phi \|_{OKHS} \cdot C_q \int_0^T (T - \tau)^{q-1} \sqrt{K(\gamma(\tau), \gamma(\tau))}d\tau.
\end{align*}
\]

As \( K \) is a continuous function on \( \mathbb{R}^n \) and \( \gamma([0, T]) \) is compact in \( \mathbb{R}^n \), it follows that the last integral is bounded for any \( q > 0 \). Note that for \( 0 < q < 1 \), the quantity \( (T - \tau)^{q-1} \) has an integrable singularity at \( \tau = T \).

**Remark 1.** The operator \( T : H_{RKHS} \to H_{OKHS} \) is an isometric isomorphism from the base reproducing kernel Hilbert space to the constructed occupation kernel Hilbert space. In fact,

\[
\| T \| = \sup_{g \neq 0} \left\{ \frac{\| Tg \|}{\| g \|} : g \in H_{RKHS} \right\} = \sup_{g \neq 0} \left\{ \| g \| : g \in H_{RKHS} \right\} = 1.
\]

**Remark 2.** In general, if \( g \) is arbitrary signal valued function, then \( g(\gamma) \) and \( \gamma \) are not necessarily defined over the same interval. However, if \( H \) is an OKHS arising from an RKHS under the canonical mapping, then given \( a \in H \) and \( \gamma : [0, T] \to \mathbb{R}^n \) it follows that \( g(\gamma) \) is also defined over \([0, T]\).

A formula for \( \Gamma_{\gamma, g} \) for an OKHS derived from an RKHS is obtainable in terms of the reproducing kernel.

**Lemma 4.4.** Let \( H_{RKHS} \) be a reproducing kernel Hilbert space of infinitely differentiable functions then for a fixed \( q > 0 \) and \( \gamma : [0, T] \to \mathbb{R}^n \) continuous the functional

\[
g \mapsto C_q \int_0^T (T - t)^{q-1} g(\gamma(t))dt
\]
is bounded and thus there exists a function $K_{\gamma,q} \in H_{RKHS}$ such that

$$\langle g, K_{\gamma,q} \rangle_{RKHS} = C_q \int_0^T (T-t)^{q-1} g(\gamma(t))dt.$$  

Moreover, that function is given by

$$K_{\gamma,q}(x) = C_q \int_0^T (T-t)^{q-1} K(x, \gamma(t))dt,$$

where $K$ is the reproducing kernel for $H_{RKHS}$.

Proof. Similar to the above, let $\gamma : [0, T] \to \mathbb{R}^n$ be continuous. If $g_1, g_2 \in H_{RKHS}$ then

$$|g_1(\gamma(t)) - g_2(\gamma(t))| \leq \|g_1 - g_2\|_{RKHS} \|K_{\gamma(t)}\|_{RKHS}$$

for all $t \in [0, T]$ where $\|K_{\gamma(t)}\|_{RKHS} = \sqrt{K(\gamma(t), \gamma(t))}$. Since $\gamma$ is a continuous function on a compact set, it follows that $K(\gamma(t), \gamma(t))$ is also bounded on $[0, T]^2$ since $K$ is at least continuous in each variable. Therefore, $\|K_{\gamma(t)}\|$ is bounded as a function of $t$ and the integral is bounded as well. Hence, for each $\gamma$ and $q > 0$ there exists a function, denoted $K_{\gamma,q} \in H_{RKHS}$, such that

$$\langle g, K_{\gamma,q} \rangle_{RKHS} = C_q \int_0^T (T-t)^{q-1} g(\gamma(t))dt.$$  

Note that

$$K_{\gamma,q}(x) = \langle K_{\gamma,q}, K_x \rangle_{RKHS}$$

$$= \langle K_x, K_{\gamma,q} \rangle_{RKHS}$$

$$= C_q \int_0^T (T-t)^{q-1} K(\gamma(t), x)dt$$

$$= C_q \int_0^T (T-t)^{q-1} K(x, \gamma(t))dt.$$  

THEOREM 4.5. Let $H_{RKHS}$ be a reproducing kernel Hilbert space of infinitely differentiable functions over a compact set and $H_{OKHS}$ be the OKHS obtained from $H_{RKHS}$ under the mapping $T$ above. 

Proof. Let $\phi_\gamma \in H_{OKHS} = T(H_{RKHS})$ be an arbitrary element in $H_{OKHS}$. By Lemma 4.4,

$$\langle \phi_\gamma, \Gamma_{\gamma,q} \rangle_{OKHS} = C_q \int_0^T (T-t)^{q-1} g(\gamma(t))dt = \langle g, K_{\gamma,q} \rangle_{RKHS}.$$  

Since $T$ is an isometric isomorphism, $T^{-1}$ is also an isometric isomorphism. Hence, 

$$\langle \phi_\gamma, \Gamma_{\gamma,q} \rangle_{OKHS} = \langle T(\phi_\gamma), T^{-1}(\Gamma_{\gamma,q}) \rangle_{RKHS} = \langle g, T^{-1}(\Gamma_{\gamma,q}) \rangle_{RKHS}.$$  

Since $\phi_\gamma$ was arbitrary, it follows that $g$ is also arbitrary. Moreover, $\langle g, T^{-1}(\Gamma_{\gamma,q}) \rangle_{RKHS} = \langle g, K_{\gamma,q} \rangle_{RKHS}$. This is enough to show that $\Gamma_{\gamma,q} = T(K_{\gamma,q})$.  

8
5. Nonlocal Liouville Operators. The objective of this manuscript is to generalize existing methods for integer order dynamical systems for system identification and DMD analysis to that of fractional order dynamical systems of Caputo type. This section introduces two fractional order generalizations of the Liouville operator, which will be employed in the sequel for DMD analysis.

**Definition 5.1.** Let \( \varphi \in H_{\text{OKHS}}^1 \) where \( H_{\text{OKHS}}^1 \) is an OKHS of order 1 arising from an RKHS, \( H_{\text{RKHS}} \), under the canonical mapping. Since \( \varphi \in H_{\text{OKHS}}^1 \), \( \varphi = \phi_\varphi = T(g) \) where \( T \) is the canonical mapping and \( g : \mathbb{R}^n \to \mathbb{R} \) is a member of \( H_{\text{RKHS}} \). Given \( \gamma : [0, T] \to \mathbb{R}^n \), and noting \( \nabla g : \mathbb{R}^n \to \mathbb{R}^n \), define

\[
\nabla \varphi(\gamma) = \nabla g(\gamma(\cdot))
\]

as a signal from \( [0, T] \) to \( \mathbb{R}^n \). Moreover, given an \( f : \mathbb{R}^n \to \mathbb{R}^n \) and any \( \gamma : [0, T] \to \mathbb{R}^n \) such that \( f \circ \gamma : [0, T] \to \mathbb{R}^n \) is differentiable, define

\[
D^{1-q}f(\gamma) = [D^{1-q}(f \circ \gamma_1(\cdot)), \ldots, D^{1-q}(f \circ \gamma_n(\cdot))]^T
\]

as a signal from \( [0, T] \) to \( \mathbb{R}^n \). Here, \( (f \circ \gamma)_i, i = 1, \ldots, n \), are the components of \( f \circ \gamma \), and interpret \( D^{1-q} \) on the right hand side as the one dimensional Caputo fractional derivative. Finally, using the standard dot product, define

\[
(\nabla \varphi \cdot D^{1-q}f)(\gamma) = \nabla g(\gamma(\cdot)) \cdot D^{1-q}(f(\gamma(\cdot)))
\]

as a signal from \( [0, T] \) to \( \mathbb{R} \). Similarly, if \( \varphi \in H_{\text{OKHS}}^3 \) then define

\[
t \mapsto C_q \int_0^t (t - \tau)^{-q} \nabla \varphi(\gamma(\tau)) \cdot D^{1-q}f(\gamma(\tau))d\tau
\]

as a signal from \( [0, T] \) to \( \mathbb{R} \).

Let \( f : \mathbb{R}^n \to \mathbb{R}^n \), \( D^q_x(t) = f(x(t)) \) be a dynamical system of Caputo type with \( 0 < q < 1 \), and recall that

\[
\dot{x}(t) = D^{1-q}f(x(t)).
\]

**Definition 5.2.** Let \( \mathcal{D}(A_{f,q}) := \{ \varphi \in H_{\text{OKHS}} : \nabla \varphi \cdot D^{1-q}f \in H_{\text{OKHS}} \} \). The Liouville operator, \( A_{f,q} : \mathcal{D}(A_{f,q}) \to H_{\text{OKHS}} \), corresponding to the dynamical system in (3.1) over an \( H_{\text{OKHS}} \) arising from an RKHS is defined as \( A_{f,q}[\varphi](t) = \nabla g(\gamma(t))D^{1-q}f(\gamma(t)) \).

Alternatively, let \( \mathcal{D}(A_{f,q}) := \{ \varphi \in H_{\text{OKHS}}^3 : J^{1-q} (\nabla \varphi \cdot D^{1-q}f) \in H_{\text{OKHS}}^3 \} \). The fractional Liouville operator, \( A_{f,q} : \mathcal{D}(A_{f,q}) \to H_{\text{OKHS}}^3 \), corresponding to (3.1) over \( H_{\text{OKHS}}^3 \) is given as \( A_{f,q}[\varphi](t) = \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{-q} \nabla \varphi(\gamma(\tau)) \cdot D^{1-q}f(\gamma(\tau))d\tau \).

The clearest way to distinguish between the two operators is through the examination of the performance of their eigenfunctions on a trajectory, \( \gamma \), satisfying (3.1). Specifically, suppose that \( \phi_\varphi \in H_{\text{OKHS}} \) is an eigenfunction of \( A_{f,q} \) with eigenvalue \( \lambda \), then

\[
\frac{d}{dt} \phi_\varphi(t) = \nabla g(\gamma(t))\dot{\gamma}(t) = \nabla g(\gamma(t))D^{1-q}f(\gamma(t)) = A_{f,q}\phi_\varphi(t) = \lambda \phi_\varphi(t),
\]

hence \( \phi_\varphi(t) = \phi_\varphi(0)e^{\lambda t} \). This relation on the eigenfunctions of the Liouville operator agrees with that expressed for continuous time integer order systems in [51].
In contrast, suppose that \( \phi_h \in \tilde{H}_{OKHS} \) is an eigenfunction for \( A_{f,q} \) with eigenvalue \( \lambda \), then

\[
D^q \phi_h[\gamma](t) = C_q \int_0^t (t - \tau)^{-q} \nabla h(\gamma(\tau)) \dot{\gamma}(\tau) d\tau
= C_q \int_0^t (t - \tau)^{-q} \nabla h(\gamma(\tau)) D^{1-q} f(\gamma(\tau)) d\tau = A_{f,q} \phi_h[\gamma](t) = \lambda \phi_h[\gamma](t).
\]

Hence, \( \phi_h[\gamma](t) = \phi_h[\gamma](0) E_q(\lambda^t) \), where \( E_q \) is the Mittag-Leffler function of order \( q \), given as \( E_q(t) := \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(qm+1)} \).

These two different perspectives on the Liouville operator each has their own advantages. Specifically, the decompositions of these operators will leverage either exponentials or Mittag-Leffler functions. In the case of the Liouville operator, \( A_{f,q} \), the decomposition for the dynamics resulting from this operator will be a linear combination of exponential functions, which not only align with classical DMD analyses (cf. \([51]\)), but are easier to compute. The decomposition of the dynamics given through the fractional Liouville operator, \( A_{f,q} \), will be a linear combination of Mittag-Leffler functions, which are likely to yield better predictions for fractional order dynamical systems, given the intimate relationship between fractional derivatives and Mittag-Leffler functions.

Interestingly, the best choice of RKHS for the foundation of an OKHS depends on the selection of the approach to the generalization of Liouville operators. In particular, the exponential dot product kernel’s native space (frequently referred to as the Bargmann-Fock space), aligns well with the first type of Liouville operator, \( A_{f,q} \), since its kernel functions are of the form \( K(x,y) = \exp(x^T y) \). However, for the fractional Liouville operator, \( \hat{A}_{f,q} \), a multi-variable generalization of the Mittag-Leffler RKHS of the slitted plane (cf. \([54]\)) would be more appropriate, where \( K(s,t) = E_q(s^{\alpha} t^{\alpha}) \).

Similar to the case of the integer order Liouville operator, each Liouville operator interacts nicely with occupation kernels from their respective spaces. In particular,

\[
\langle A_{f,q} \phi, \Gamma \rangle_{H_{OKHS}} = g(\gamma(T)) - g(\gamma(0)) = \phi_g[\gamma](T) - \phi_g[\gamma](0),
\]

and

\[
\langle A_{f,q} h, \tilde{\Gamma} \rangle_{\tilde{H}_{OKHS}} = \phi_h[\gamma](T) - \phi_h[\gamma](0),
\]

where \( \Gamma \) is the occupation kernel corresponding to \( H_{OKHS} \) and \( \tilde{\Gamma} \) is the occupation kernel corresponding to \( \tilde{H}_{OKHS} \). At this point, it is important to note that each OKHS has “bounded point evaluations” through constant trajectories, \( \alpha_{x_0} : [0, 1] \to \mathbb{R}^n \) where \( \alpha_{x_0}(t) = x_0 \). Point evaluation is then expressed as \( \langle \phi_g, \Gamma_{\alpha_{x_0}} \rangle_{H_{OKHS}} = \int_0^1 \phi_g[\alpha_{x_0}](t) dt = \int_0^1 g(x_0) dt = g(x_0) = \phi_g[\alpha_{x_0}](0) \), and a similar result is feasible for \( \tilde{H}_{OKHS} \) at the cost of a constant multiple \( C_q \). Combining this result with the inner product relations yields

\[
A_{f,q} \Gamma = \Gamma_{\alpha_{\gamma}(T)} - \Gamma_{\alpha_{\gamma}(0)} \text{ and } A_{f,q} \tilde{\Gamma} = C_q \left( \tilde{\Gamma}_{\alpha_{\gamma}(T)} - \tilde{\Gamma}_{\alpha_{\gamma}(0)} \right).
\]

These relations will be pivotal in the development of a DMD method for fractional order systems.
6. Finite Rank Representations of Liouville Operators. 

For \( q > 0 \), the objective of this section is to leverage observed trajectories, \( \mathcal{M} = \{ \gamma_i : [0, T_i] \to \mathbb{R}^n \}_{i=1}^M \), that satisfy \( D^2_x \gamma_i = f(\gamma_i) \) for each \( i = 1, \ldots, M \) and an unknown \( f : \mathbb{R}^n \to \mathbb{R}^n \) to derive a model, \( G : [0, T] \to \mathbb{R}^n \), for the system for which a given initial value \( x(0) \in \mathbb{R}^n \) the trajectory \( x : [0, T] \to \mathbb{R}^n \) satisfying \( D^2_x x = f(x) \) can be estimated as \( x(t) \approx G(t) \). To obtain this model, a finite rank representation of the operators \( A_{f,q} \) and \( A_{f,q}^* \) over a given RKHS, \( H \), will be determined through the associated occupation kernels and the relations established in Section 5. The finite rank representation of the operator will then be leveraged as a proxy for the actual densely defined operator, where a spectral decomposition will be determined and the eigenfunctions will form the foundation of the data driven model. In practice, the eigenfunctions will be determined as a linear combination of the associated occupation kernels.

6.1. Finite Rank Representation of the Liouville Operator, \( A_{f,q} \). Let \( H \) be an OKHS determined through a RKHS, \( \tilde{H} \), of continuously differentiable functions over \( \mathbb{R}^n \). Let \( \beta \) be the ordered basis of occupation kernels corresponding to \( \mathcal{M} \) given as \( \beta := \{ \Gamma_{\gamma,q} \}_{\gamma,q=1}^M \). The vector space \( \text{span} \beta \) is a finite dimensional subspace of \( H \) and hence, closed. Let \( P_{\beta} \) denote the projection operator from \( H \) onto \( \text{span} \beta \). As demonstrated in Section 5, each occupation kernel corresponding to \( \gamma_i \) in the domain of the adjoint of the operator \( A_{f,q} \). Hence, the operator \( P_{\beta} A_{f,q}^* P_{\beta} \) is well defined over \( H \), and this operator is of finite rank. Note that the matrix, \( [P_{\beta} A_{f,q}^* P_{\beta}]_\beta \), defined in terms of the ordered basis \( \beta \) may be expressed simply as \( [P_{\beta} A_{f,q}^* P_{\beta}]_\beta \), since the domain of definition for the matrix is \( \text{span} \beta \) and \( P_{\beta} h = h \) for all \( h \in \text{span} \beta \).

For a function \( h \in H \), the projection of \( H \) onto \( \text{span} \beta \) may be written as \( P_{\beta} h = \sum_{i=1}^M w_i \Gamma_{\gamma_i,q} \), where \( w = (w_1, \ldots, w_M)^T \in \mathbb{R}^M \) is obtained by solving the matrix equation

\[
\begin{pmatrix}
\langle \Gamma_{\gamma_1,q}, \Gamma_{\gamma_1,q} \rangle_H & \cdots & \langle \Gamma_{\gamma_M,q}, \Gamma_{\gamma_1,q} \rangle_H \\
\vdots & \ddots & \vdots \\
\langle \Gamma_{\gamma_1,q}, \Gamma_{\gamma_M,q} \rangle_H & \cdots & \langle \Gamma_{\gamma_M,q}, \Gamma_{\gamma_M,q} \rangle_H \\
\end{pmatrix}
\begin{pmatrix}
w_1 \\
\vdots \\
w_M \\
\end{pmatrix}
= \begin{pmatrix}
\langle h, \Gamma_{\gamma_1,q} \rangle_H \\
\vdots \\
\langle h, \Gamma_{\gamma_M,q} \rangle_H \\
\end{pmatrix}.
\]

Consequently, the finite rank representation of \( A_{f,q}^* \) may be determined as

\[
[P_{\beta} A_{f,q}^* P_{\beta}]_\beta = \left(\begin{array}{ccc}
\langle \Gamma_{\gamma_1,q}, \Gamma_{\gamma_1,q} \rangle_H & \cdots & \langle \Gamma_{\gamma_M,q}, \Gamma_{\gamma_1,q} \rangle_H \\
\vdots & \ddots & \vdots \\
\langle \Gamma_{\gamma_1,q}, \Gamma_{\gamma_M,q} \rangle_H & \cdots & \langle \Gamma_{\gamma_M,q}, \Gamma_{\gamma_M,q} \rangle_H \\
\end{array}\right)^{-1}
\times
\begin{pmatrix}
\langle A_{f,q}^* \Gamma_{\gamma_1,q}, \Gamma_{\gamma_1,q} \rangle_H & \cdots & \langle A_{f,q}^* \Gamma_{\gamma_1,q}, \Gamma_{\gamma_M,q} \rangle_H \\
\vdots & \ddots & \vdots \\
\langle A_{f,q}^* \Gamma_{\gamma_M,q}, \Gamma_{\gamma_1,q} \rangle_H & \cdots & \langle A_{f,q}^* \Gamma_{\gamma_M,q}, \Gamma_{\gamma_M,q} \rangle_H \\
\end{pmatrix}.
\]

In (6.2), the entries of the matrix may be computed using 5.3 and the canonical identification \( \phi_q \) developed in Section 4. To wit, noting that \( q = 1 \) for the Liouville operator and occupation kernel relation:

\[
\langle A_{f,q}^* \Gamma_{\gamma_i,q}, \Gamma_{\gamma_j,q} \rangle_H = \langle \Gamma_{\alpha_{\gamma_i}(T),1} - \Gamma_{\alpha_{\gamma_i}(0),1}, \Gamma_{\gamma_j,q} \rangle_H \\
= \int_0^T \tilde{K}(\gamma_j(\tau), \gamma_i(T)) - \tilde{K}(\gamma_j(\tau), \gamma_i(0)) d\tau.
\]
Similarly, for the fractional Liouville operator, \( A_{f,q} \), the finite rank representation is given as

\[
[P_{\beta} A_{f,q}]_\beta = \left( \langle \Gamma_{\gamma_1,q}, \Gamma_{\gamma_1,q} \rangle_H \cdots \langle \Gamma_{\gamma_M,q}, \Gamma_{\gamma_M,q} \rangle_H \right)^{-1} \times \\
\left( \langle A_{f,q}^* \Gamma_{\gamma_1,q}, \Gamma_{\gamma_1,q} \rangle_H \cdots \langle A_{f,q}^* \Gamma_{\gamma_M,q}, \Gamma_{\gamma_M,q} \rangle_H \right),
\]

where each entry can be computed as

\[
\langle A_{f,q}^* \Gamma_{\gamma_i,q}, \Gamma_{\gamma_j,q} \rangle_H = \langle \Gamma_{\gamma_i(q(T)),q} - \Gamma_{\gamma_i,0,q}, \Gamma_{\gamma_j,q} \rangle_H \\
= C_q \int_0^T (T - \tau)^{q-1} \tilde{K}(\gamma_j(\tau), \gamma_i(T)) - \tilde{K}(\gamma_j(\tau), \gamma_i(0)) d\tau.
\]

The entries of the Gram matrix in (6.2) and (6.3) can be expressed as

\[
\langle \Gamma_{\gamma_i,q}, \Gamma_{\gamma_j,q} \rangle_H = \langle \Gamma_{\gamma_i(T)},q - \Gamma_{\gamma_i,0}, q, \Gamma_{\gamma_j,q} \rangle_H \\
= (C_q)^2 \int_0^T (T - \tau)^{q-1}(T - t)^{q-1} \tilde{K}(\gamma_j(\tau), \gamma_i(t)) d\tau dt,
\]

with \( q = 1 \) for the Liouville operator and \( q \) matching the order of the system for the fractional Liouville operator.

7. Dynamic Mode Decompositions with Liouville Modes and Fractional Liouville Modes. In Dynamic Mode Decomposition (DMD) methods for the data driven analysis of dynamical systems, the identity function (also called the full state observable), \( g_{id} : \mathbb{R}^n \to \mathbb{R}^n \), given as \( g_{id}(x) = x \) is individually decomposed with respect to an eigenbasis in a RKHS corresponding to finite rank representations of Koopman and Liouville operators similar to those determined in Section 6. The result is a linear combination of scalar valued eigenfunctions multiplied by a collection of vectors obtained through the projection of the component of \( g_{id} \) onto the eigenbasis. These vectors are called the dynamic modes of the system. When connected with particular operators, they are also called Koopman or Liouville modes. Subsequently, a model for the dynamical system is determined by exploiting certain features of the eigenfunctions, which ultimately replace the eigenfunctions with exponential functions.

This section discusses the two different models that are determined through a choice of using the Liouville operator or the fractional Liouville operator over a OKHS. In place of the identity function, whose image is in \( \mathbb{R}^n \), the DMD method presented here leveraged the signal valued analogue, \( \phi_{g_{id}} \), and exploits identities (5.1) and (6.3).

From the data driven perspective, the Liouville operator and Fractional Liouville operator for a particular collection of trajectories is not directly accessible. This motivates the use of finite rank representations such as those given in Section 6. An eigenvector, \( v = (v_1, \ldots, v_M)^T \in \mathbb{C}^M \), with eigenvalue \( \lambda \in \mathbb{C} \) obtained from (6.2) or (6.3) corresponds to a function in the OKHS as \( \varphi = \frac{1}{\sqrt{M}} \sum_{i=1}^M v_i \Gamma_{\gamma_i,q} \) with \( q = 1 \) for (6.2) and \( q \) as the fractional order of the system in (6.3), which is an eigenfunction for...
the corresponding finite rank operator. To derive a model for the dynamical system, the eigenfunctions obtained from the finite rank operators will be used in place of proper eigenfunctions of the Liouville and fractional Liouville operators.

Specifically, suppose that the eigenfunctions \( \{ \hat{\phi}_i \}_{i=1}^M \) corresponding to the eigenvalues \( \{ \lambda_i \}_{i=1}^M \) diagonalize the finite rank operator \( P_{\beta} A_{f,q} P_{\beta} \). The full state observable for the OKHS, \( \phi_{g,d} \), may then be decomposed as

\[
\phi_{g,d}(t) = \sum_{i=1}^{M} \xi_i \hat{\phi}_i(t) \approx \sum_{i=1}^{M} \xi_i \hat{\phi}_i(0)e^{\lambda_i t}.
\]

Hence, a model for the dynamical system has been derived from the observed trajectories. However, since the dynamics are governed by the Caputo fractional derivative, a model obtained from the finite rank representation of the fractional Liouville operator may be more suitable. In this case, suppose that \( \{ \hat{\phi}_i \}_{i=1}^M \) are eigenfunctions of the finite rank operator \( A_{f,q} \) with eigenvalues \( \{ \lambda_i \}_{i=1}^M \). Following the same procedure as above and using (5.2), the resultant model for the dynamical system is given as

\[
x(t) = \phi_{g,d}(x(t)) \approx \sum_{i=1}^{M} \hat{\xi}_i \hat{\phi}_i(x(t)) \approx \sum_{i=1}^{M} \hat{\xi}_i \hat{\phi}_i(x(0))E_q(\lambda_i t^\gamma),
\]

with \( \xi_i \) being the \( i \)-th fractional Liouville mode. Heuristically, the latter model is expected to match the state more closely, since \( t \mapsto E_q(\lambda_i t^\gamma) \) is the solution to a fractional order linear system. Additionally, if the finite sums above are replaced with a series obtained from a collection of eigenfunctions that diagonalize the original operators, then the expressions are expected to result in equalities.

**8. Discussion.** It should be noted that while the theoretical developments given above are for a scalar valued function, these methods extend naturally to vector valued quantities by treating each dimension separately and then later stacking the dynamic modes. This is equivalent to using vector valued kernels that are diagonal.

Despite the theoretical developments in the manuscript, the actual implementation of the DMD method does not differ greatly from standard occupation kernel DMD. As seen in Section 6, the computations ultimately are performed on the underlying RKHS rather than directly on the OKHS. Hence, the computation of Gram matrices only need a slight adjustment to accommodate the fractional integrals. In fact, these computations are exactly in line with [30], and follow from Newton-Cotes and Gaussian quadrature methods.

Finally, the fractional order framework can also be posed over signal valued RKHSs [53], which gives another approach to handling nonlocal data interpretation. OKHSs differ in that the definition of OKHSs are independent of that of RKHSs, where the RKHSs are used in this manuscript as a particular realization of OKHSs. Moreover, while signal valued spaces correspond to trajectories of a fixed length (and may artificially adjust the lengths of trajectories using indicator functions), OKHSs allow for arbitrarily long trajectories natively.
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