Finiteness of Nichols Algebras and Nichols (Braided) Lie Algebras

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Abstract

It is shown that if $\mathcal{B}(V)$ is connected Nichols algebra of diagonal type with $\dim V > 1$, then $\dim(\mathcal{B}(V)) = \infty$ (resp. $\dim(\mathcal{L}(V)) = \infty$) (resp. $\dim(\mathcal{L}^-(V)) = \infty$) if and only if $\Delta(\mathcal{B}(V))$ is an arithmetic root system and the quantum numbers (i.e. the fixed parameters) of generalized Dynkin diagrams of $V$ are of finite order.

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1 Introduction

The question of finite-dimensionality of Nichols algebras dominates an important part of the recent developments in the theory of (pointed) Hopf algebras (see e.g. [AHS08, AS10, An11, He05, He06a, He06b, WZZ15a, WZZ15b, ZWTZ]. The interest in this problem comes from the lifting method by Andruskiewitsch and Schneider to classify finite dimensional (Gelfand-Kirillov) pointed Hopf algebras, which are generalizations of quantized enveloping algebras of semi-simple Lie algebras.
The classification of arithmetic root systems is obtained in [He05] and [He06a]. It is proved in [WZZ15b] that Nichols algebra $B(V)$ is finite-dimensional if and only if Nichols braided Lie algebra $L(V)$ is finite-dimensional. It is shown in [ZWTZ] that $\dim(L^{-}(V)) = \infty$ if $\Delta(B(V))$ is not an arithmetic root system.

The main motivation for the subject of this paper and papers [He05, He06a] is the following question of Andruskiewitsch [A02].

**Question 5.9.** Given a braided vector space $V$ of diagonal type, decide when $B(V)$ is finite dimensional. If so, compute $\dim B(V)$, and give a nice presentation by generators and relations.

Papers [He05] and [He06a] solved the first part of this problem under the following the additional property in [He05, Section 2.2]:

(P) the height of a PBW generator (i.e hard super-letter) of $\mathbb{Z}^n$-degree $e$ is finite if and only if $2 \leq \text{ord}(\chi(e, e)) < \infty$, and in this case it coincides with $\text{ord}(\chi(e, e))$.

In this paper we show that property (P) holds in arithmetic root systems (see Theorem 4.1). Consequently, this paper solves the first part of problem 5.9.

In order to study infinite dimensional Lie algebras V. Kac and R. Moody independently introduced Kac-Moody algebras in 1960s ([Ka85, Wa02]).

In this paper we focus on when a Nichols algebra and a Nichols (braided) Lie algebra have finite dimension. Such finite-dimensional Nichols algebras play a fundamental role in various subjects such as pointed Hopf algebras and logarithmic quantum field theories. There exists a large number of examples of infinite dimensional Lie algebras in Nichols Lie algebras.

The main results of this paper are as follows. If $B(V)$ is connected Nichols algebra of diagonal type with $\dim V > 1$, then $\dim(B(V)) = \infty$ (resp. $\dim(L(V)) = \infty$) if and only if $\Delta(B(V))$ is an arithmetic root system and the quantum numbers (i.e. the fixed parameters) of generalized Dynkin diagrams of $V$ are of finite order.

**Preliminaries**

Throughout this paper braided vector space $V$ is of diagonal type with basis $x_1, x_2, \cdots, x_n$ and braiding $C(x_i \otimes x_j) = p_{ij} x_j \otimes x_i$ without special announcement. Nichols algebra $B(V)$ over braided vector space $V$ has two gradations. A gradation is the length such that $|u| = r$ of length of $u$ for $u = x_{i_1} x_{i_2} \cdots x_{i_r}$. Another gradation is $\mathbb{Z}^n$-graded such that $\deg(x_i) = e_i$ for $1 \leq i \leq n$, where $E = \{e_1, e_2, \cdots, e_n\}$ is a basis of $\mathbb{Z}^n$.

Define linear map $p$ from $B(V) \otimes B(V)$ to $F$ such that $p(u \otimes v) = \chi(\deg(u), \deg(v))$, for any homogeneous element $u, v \in B(V)$. For convenience, $p(u \otimes v)$ is denoted by $p_{uv}$. Let $\tilde{p}_{uv} := p_{uv} p_{vu}$.
Let $A =: \{x_1, x_2, \ldots, x_n\}$ a alphabet; $A^*$ denotes the set of all words in $A$; $A^+ =: A^* \setminus \{1\}$. Define $x_1 < x_2 < \cdots < x_n$ and the order on $A^*$ is the lexicographic orderings. The concepts of words refer [Lo83].

**Definition 1.1.** [Kh99, Def. 1] A word $u$ is called a Lyndon word if $|u| = 1$ or $|u| \geq 2$, and for each representation $u = u_1 u_2$, where $u_1$ and $u_2$ are nonempty words, the inequality $u < u_2 u_1$ holds.

Any word $u \in A^*$ has a unique decomposition into the product of a non-increasing sequence of Lyndon words by [Lo83, Th.5.1.5]. If $u$ is a Lyndon word with $|u| > 1$, then there uniquely exist two Lyndon words $v$ and $w$ such that $u = vw$ and $v$ is shortest (see [Lo83, Pro. 5.1.3] and [He07]) (the composition is called the Shirshov decomposition of $u$).

**Definition 1.2.** We inductively define a linear map $[\ ]$ from $A^+$ to $\mathcal{B}(V)$ as follows:

1. $[u] =: u$ when $u$ is a letter;
2. $[u] =: [w][v] - p_{uw}[v][w]$ when $u$ is a Lyndon word with $|u| > 1$ and $u = vw$ is a Shirshov decomposition;
3. $[u] =: [[[l_1, l_2], l_3] \cdots l_n]$, when $u = l_1 l_2 \cdots l_n$ is a Shirshov decomposition.

$[u]$ is called a nonassociative word for any $u \in A^+$. $[u]$ is called a super-letter if $u$ is a Lyndon word. A multiplication of super-letters is called a super-word.

**Definition 1.3.** [Kh99, Def. 6] A super-letter $[u]$ is said to be hard if it is not a linear combination of super-words with length $|u|$ in greater super-letters than $[u]$.

**Definition 1.4.** [Kh99, Def 7] or [He07, before Th. 10] We said that the height of a super-letter $[u]$ with length $d$ equals a natural number $h$ if $h$ is least with the following properties:

1. $p_{uu}$ is a t-primitive root of unity, and $h = t$;
2. super-word $[u]^h$ is a linear combination of super-words of length $hd$ in greater super-letters than $[u]$.

If the number $h$ with above properties does not exist then we say that the height of $[u]$ is infinite.

Let $\text{ord}(p_{uu})$ denote the order of $p_{uu}$ with respect to multiplication. 

$D(V) = D =: \{[u] \mid [u]$ is a hard super-letter $\}$, we know $h_u > 1$ for $\forall [u] \in D$.

$P(V) = P =: \{[u_1]^{k_1}[u_2]^{k_2} \cdots [u_s]^{k_s} \mid [u_i] \in D, k_i, s \in \mathbb{N}_0; 0 \leq k_i < h_{u_i}; 1 \leq i \leq s; u_s < u_{s-1} < \cdots < u_1\}$

If $[u] \in D$ and $\text{ord}(p_{u,u}) = m > 1$ with $h_u = \infty$, then $[u]$ is called an $m$-infinity element.
Theorem 1.5. [Kh99, Th. 2] or [He05, Th. 1.4.6], \( P \) is a basis of \( \mathfrak{B}(V) \).

Let \( G \) be a nonempty set; \( R \) a nonempty subset of \( G \times G \), and \( \circ : R \rightarrow G \) a map of sets. The pair \((G, \circ)\) is called a groupoid if it satisfies the following conditions.

**GD1** If \((x, y) \in R \) then each of the three elements \( x, y, x \circ y \) is uniquely determined by the other two.

**GD2** If \((x, y), (y, z) \in R \) then \((x \circ y, z), (x, y \circ z) \in R \) and \((x \circ y) \circ z = x \circ (y \circ z) \).

**GD3** If \((x, y), (x \circ y, z) \in R \) then \((y, z), (x, y \circ z) \in R \) and \((x \circ y) \circ z = x \circ (y \circ z) \).

**GD4** If \((y, z), (x, y \circ z) \in R \) then \((x, y), (x \circ y, z) \in R \) and \((x \circ y) \circ z = x \circ (y \circ z) \).

**GD5** For all \( x, y \in G \) there exist unique elements \( e, f, y \in G \) such that \((e, x), (x, f), (y, x) \in R \), \( e \circ x = x \circ f = x \) and \( y \circ x = f \).

**GD6** If \( e \circ e = e, f \circ f = f \) for certain \( e, f \in G \) then there exists \( x \in G \) such that \( e \circ x = x \circ f = x \).

Let \( E \) and \( F \) be bases of \( \mathbb{Z}^n \) and \( \chi \) a bicharacter on \( \mathbb{Z}^n \). Define \( m(e', e'') := \min \{ m \in \mathbb{N}_0 \mid \chi(e', e')^m \chi(e'', e'') = 1 \text{ or } \chi(e', e')^{m+1} = 1, \chi(e', e') \neq 1 \} \) for \( e', e'' \in F \) with \( e' \neq e'' \) and \( m(e', e') := -2 \). \( s_{e', F}(e'') := e'' + m(e', e'')e' \) for \( e', e'' \in F \).

Let \( \widetilde{W} \) denote the groupoid consisting of all pairs \((T, E)\), where \( T \in \text{Aut}(\mathbb{Z}^n) \) and \( E \) is a basis of \( \mathbb{Z}^n \), and the composition \((T_1, E_1) \circ (T_2, E_2)\) is defined (and is then equal to \((T_1 T_2, E_2)\)) if and only if \( T_2(E_2) = E_1 \) (i.e. \( R = \{(T_1, E_1) \times (T_2, E_2) \mid T_2(E_2) = E_1\}) \).

Let \( E \) be a basis of \( \mathbb{Z}^n \) and \( \chi \) a bicharacter on \( \mathbb{Z}^n \). Define \( W_{\chi, E} \) as the smallest subgroupoid of \( \widetilde{W} \) which contains \((id, E)\), and if \((id, F) \in W_{\chi, E}\) for a basis \( F \) of \( \mathbb{Z}^n \) and \( f \in F \), then \((s_{f, F}, F), (id, s_{f, F}(F)) \in W_{\chi, E}\) whenever \( s_{f, F} \) is defined. It is called the Weyl groupoid associated to the pair \((\chi, E)\).

The groupoid \( W_{\chi, E} \) is called full, if \( s_{f, F} \) is well-defined for all bases \( F \) of \( \mathbb{Z}^n \) with \((id, F) \in W_{\chi, E}\) and for all \( f \in F \).

A triple \((\Delta, \chi, E)\) is called an arithmetic root system if \( W_{\chi, E} \) is full and finite with \( \Delta := \cup \{ F \mid (id, F) \in W_{\chi, E}\} \).

\( \Delta^+(\mathfrak{B}(V)) := \{\text{deg}(u) \mid [u] \in D\} \) and \( \Delta(\mathfrak{B}(V)) := \Delta^+(\mathfrak{B}(V)) \cup \Delta^-(\mathfrak{B}(V)) \). Let \( \mathfrak{L}(V) \) denote the braided Lie algebras generated by \( V \) in \( \mathfrak{B}(V) \) under Lie operations \([x, y] = xy - p_{yx}xy\), for any homogeneous elements \( x, y \in \mathfrak{B}(V) \). \( \mathfrak{L}(V), [\ ] \) is called Nichols braided Lie algebra of \( V \). Let \( \mathfrak{L}^-(V) \) denote the Lie algebras generated by \( V \) in \( \mathfrak{B}(V) \) under Lie operations \([x, y]^-- = xy - xy\), for any homogeneous elements \( x, y \in \mathfrak{B}(V) \). \( \mathfrak{L}^-(V), [\ ]^- \) is called Nichols Lie algebra of \( V \).

The dual \( \mathfrak{B}(V)^* \) of Nichols algebra \( \mathfrak{B}(V) \) of rank \( n \) in [He05, Section 1.3]. Let \( y_i \) be a dual basis of \( x_i \). \( \delta(y_i) = g_i^{-1} \otimes y_i, g_i \cdot y_j = p_{ij}^{-1} y_j \) and \( \Delta(y_i) = g_i^{-1} \otimes y_i + y_i \otimes 1 \). There exists a bilinear map \( < \cdot, \cdot > : (\mathfrak{B}(V^*) \# FG) \times \mathfrak{B}(V) \rightarrow \mathfrak{B}(V) \) such that

\( < y_i, uv >= < y_i, u > v + g_i^{-1} u < y_i, v > \) and \( < y_i, < y_j, u > >= < y_i y_j, u > \)

for any \( u, v \in \mathfrak{B}(V) \). Furthermore, for any \( u \in \bigoplus_{i=1}^\infty \mathfrak{B}(V)_{(i)} \), one has that \( u = 0 \) if and only if \( < y_i, u >= 0 \) for any \( 1 \leq i \leq n \).
We have the braided Jacobi identity as follows:

$$[[u, v], w] = [u, [v, w]] + p_{vw}^{-1}[[u, w], v] + (p_{vw} - p_{vw}^{-1})v \cdot [u, w].$$

(1)

$$[u, v \cdot w] = p_{wu}[uv] \cdot w + v \cdot [uw].$$

(2)

**Theorem 1.6.** ([He05 Th. 2.5.3]) If \( \chi \) is a bicharacter on \( \mathbb{Z}^n \) and \((\Delta, \chi, E)\) is an arithmetic root system then for the braided vector space \( V \) of diagonal type with \( \dim V = n \) and with structure constants \( q_{ij} := \chi(e_i, e_j) \) one has \( \Delta(\mathfrak{B}(V)) = \Delta \). Conversely, if \( V \) is a braided vector space of diagonal type such that \( \Delta^+(\mathfrak{B}(V)) \) is finite then \( \Delta(\mathfrak{B}(V)), \chi, E \) is an arithmetic root system, where \( E = \{e_1, e_2, \cdots, e_n\} \) and \( \chi(e_i, e_j) = q_{ij} \) for \( 1 \leq i, j \leq n \).

If \((\Delta, \chi, E)\) is an arithmetic root system and a hyperplane \( H \) with \( 0 \in H \) (i.e. \( \dim H = n - 1 \)), then there exists \( E_H := \{f_1, f_2, \cdots, f_l\} \) such that \((\Delta \cap H, \chi_{\Gamma \times \Gamma}, E_H)\) is an arithmetic root subsystem with \( \Gamma = \mathbb{R}(\Delta \cap H) \cap \mathbb{Z}^n \), written \( \Delta(\chi, f_1, f_2, \cdots, f_l) = (\Delta \cap H, \chi_{\Gamma \times \Gamma}, E_H) \) in short (see [He05 Pro. 2.7.1]).

Throughout, \( \mathbb{Z} := \{x | x \text{ is an integer}\} \), \( \mathbb{R} := \{x | x \text{ is a real number}\} \), \( \mathbb{N}_0 := \{x | x \in \mathbb{Z}, x \geq 0\} \), \( \mathbb{N} := \{x | x \in \mathbb{Z}, x > 0\} \), \( \mathbb{R}_+ := \{x \geq 0 | x \text{ is a real number}\} \). \( F \) denotes the base field, which is an algebraic closed field with characteristic zero. \( F^* = F \setminus \{0\} \). \( S_n \) denotes symmetric group, \( n \in \mathbb{N} \).

## 2 \( m \)-infinity elements

In this section we show that there does not exist any \( m \)-infinity elements in arithmetic root systems.

**Lemma 2.1.** Assume that \( \tilde{f}_1, \tilde{f}_2, \cdots, \tilde{f}_l \) are linearly independent in arithmetic root system \((\Delta, \chi, E)\) with \( \tilde{H} := \mathbb{R}(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \cdots, \tilde{f}_l) \). If

\[
\Delta \cap \tilde{H} \subseteq \mathbb{R}_+(\tilde{f}_1, \tilde{f}_2, \cdots, \tilde{f}_l) \cup -\mathbb{R}_+(\tilde{f}_1, \tilde{f}_2, \cdots, \tilde{f}_l),
\]

(3)

then there exists a hyperplane \( H \) with \( 0 \in H \) and \( E_H = \{f_1, f_2, \cdots, f_l\} \) with \( l = \tilde{l} \), such that \( \Delta \cap H = \Delta \cap \tilde{H} \) and \( (\Delta \cap H, \chi_{\Gamma \times \Gamma}, E_H) \) is an arithmetic root system with \( \Gamma = \mathbb{R}(\Delta \cap H) \cap \mathbb{Z}^n \).

**Proof.** It is clear that there exists \( H_1 \) with \( H_1 \cap \Delta = \emptyset \) such that \( H = \tilde{H} \oplus H_1 \) with \( \dim H = n - 1 \). By [He05 Pro. 2.7.1], there exists \( E_H = \{f_1, f_2, \cdots, f_l\} \) such that \((\Delta \cap H, \chi_{\Gamma \times \Gamma}, E_H)\) is an arithmetic root system with \( \Gamma = \mathbb{R}(\Delta \cap H) \cap \mathbb{Z}^n \) and

\[
E_H \subseteq \Delta \cap H = \Delta \cap \tilde{H} \subseteq \mathbb{R} E_H.
\]

(4)

Considering (3) and (4), we have \( l = \tilde{l} \). \( \square \)
Lemma 2.2. If \((\Delta(\mathfrak{B}(V)), \chi, E)\) is an arithmetic root system with \([v], [w] \in D\) and \(\deg(v) \neq \deg(w)\), then there exists an arithmetic root subsystem \(\Delta(\chi; f_1, f_2)\) of \(\Delta(\mathfrak{B}(V))\) such that \(\deg(v)\) and \(\deg(w)\) are in the subsystem. Furthermore, if \([u] = [[v], [w]]\) is the Shirshov decomposition of \([u] \in D\) then the subsystem \(\Delta(\chi; f_1, f_2)\) is connected and \(\deg(u)\) are in the subsystem.

Proof. We show this by the following steps.

(i) Let \(v_1 = v\) and \(v_2 = w\). If \(\deg(v) - k \deg(w) \notin \mathbb{N} \cdot \Delta(\mathfrak{B}(V))\) for any \(k \in \mathbb{N}\), let \(\tilde{f}_1 := \deg(v_1)\) and \(\tilde{f}_2 := \deg(v_2)\). If there exists a \(k_1 \in \mathbb{N}\) such that \(\deg(v_1) - k_1 \deg(v_2) \notin \mathbb{N} \cdot \Delta(\mathfrak{B}(V))\), then there exist \(v_3 \in \Delta^+(\mathfrak{B}(V))\) and \(l_1 \in \mathbb{Z} \setminus \{0\}\) such that \(\deg(v_1) - k_1 \deg(v_2) = l_1 \deg(v_3)\). Keeping on the step, we obtain \(v_i \in \Delta(\mathfrak{B}(V))\), \(k_i \in \mathbb{N}\) and \(l_i \in \mathbb{Z} \setminus \{0\}\) such that \(\deg(v_i) - k_i \deg(v_{i+1}) = l_i \deg(v_{i+2}) \in \mathbb{N} \cdot \Delta(\mathfrak{B}(V))\) for \(i = 1, 2, \ldots\).

(ii) \(\deg(v_i) \neq \deg(v_j)\) when \(i \neq j\). Indeed, for any \(t\), there exist \(r_t, s_t \in \mathbb{R}\) such that \(\deg(v_i) = r_t \deg(v_1) + s_t \deg(v_2)\) by Part (i). If \(\deg(v_i) = \deg(v_j)\), then \((r_i - r_j) \deg(v_1) = \deg(v_2)(s_j - s_i)\), which contradicts \([\text{He05}, \text{Cor. 2.5.4}]\).

(iii) By Part (ii), the set \(\{v_i \mid i = 1, 2, \ldots\}\) in Part (i) is finite since \(\Delta(\mathfrak{B}(V))\) is finite.

(iv) By Part (i), (ii) and (iii), there exists \(i_0 \in \mathbb{N}\) such that \(\deg(v_i) - k_i \deg(v_{i+1}) = l_i \deg(v_{i+2}) \in \mathbb{N} \cdot \Delta(\mathfrak{B}(V))\) for \(1 \leq i < i_0\) and \(\deg(v_{i_0}) - k \deg(v_{i_0+1}) \notin \mathbb{N} \cdot \Delta(\mathfrak{B}(V))\) for any \(k \in \mathbb{N}\). Let \(\tilde{f}_1 := \deg(v_{i_0})\) and \(\tilde{f}_2 := \deg(v_{i_0+1})\). Let \(\tilde{H} := \mathbb{R}\{\tilde{f}_1, \tilde{f}_2\} = \text{span}\{\tilde{f}_1, \tilde{f}_2\}\).

By \([\text{He05}, \text{Lemma 2.7.2}]\) and Lemma \([27]\), there exists an arithmetic root subsystem \(\Delta(\chi; f_1, f_2)\) such that \(\deg(v)\) and \(\deg(w)\) are in \(\Delta(\chi; f_1, f_2)\).

(v) The subsystem \(\Delta(\chi; f_1, f_2)\) is connected when \([u] = [[v], [w]]\) is the Shirshov decomposition of \([u] \in D\). Otherwise, \(\deg(v), \deg(w), \deg(u) \in \{f_1, f_2\}\), which is a contradiction.

\[\square\]

Proposition 2.3. If \((\Delta, \chi, E)\) is an arithmetic root system, then for every \(\alpha \in \Delta \setminus E\), there exists a connected arithmetic root subsystem \((\Delta \cap H, \chi_{\Gamma \times \Gamma}, E_H)\) such that \(\alpha \in \Delta \cap H\) and \(|E_H| = 2\).

Proof. It follows from Lemma \([22]\). \[\square\]

Theorem 2.4. If \(\mathfrak{B}(V)\) is connected Nichols algebra of diagonal type with \(\dim V > 1\) and \(\Delta(\mathfrak{B}(V))\) is an arithmetic root system, then \(p_{u,u} \neq 1\) for \(\forall [u] \in D\) and there do not exist any \(m\)-infinity elements.

Proof. Set \([u] = [[v], [w]]\) is the Shirshov decomposition of \([u]\). By Lemma \([22]\), \(\deg(u)\) is in a connected subsystem with rank 2. We complete the proof using \([\text{WZZ15}, \text{A.2}]\) and \([\text{He05}, \text{Th. 1.6.1}]\). \[\square\]
3 The classification of Nichols Lie Algebras

In this section we give the classification of Nichols Lie algebras.

Lemma 3.1. Assume that $\mathfrak{B}(V)$ is a Nichols algebra of diagonal type. Let $a := p_i^{-1}$, $b := p_{ij}^{-1}$, $c := p_{ji}^{-1}$ with $i \neq j$. Then

(i) $\tilde{l}_i^m[j] = 0$, when $b = 1, c = 1$.

(ii) $< y_i^k, \tilde{l}_i^m[j] > = \sum_{l=0}^{k-1} (a^l - a^{m-1-l}b) < y_i^{k-1}, \tilde{l}_i^{m-1}[j] > + a^k x_i < y_i^k, \tilde{l}_i^{m-1}[j] > - < y_i^k, \tilde{l}_i^{m-1}[j] > x_i \quad$ for $k \leq m$. \hspace{1cm} (5)

(iii) Assume that $a = 1$. Then $\tilde{l}_i^m[j] \neq 0$, when $b \neq 1$ or $c \neq 1$.

(iv) Assume that $a \neq 1$. Then $\tilde{l}_i^m[j] \neq 0$, when $\text{ord}(a) > m$ with $c \neq 1$ or $b \neq 1$.

(v) $\dim(\mathfrak{L}^-(V)) = \infty$ when there exists $i$ and $j$ with $i \neq j$, $b \neq 1$ or $c \neq 1$ and $\text{ord}(a) = 1$ or $\infty$.

Proof. (i) follows from \cite{WZZ15b} Lemma 4.1 (i)(vi)].

(ii) We show \eqref{5} by induction on $k$ for $k \leq m$. See

$$< y_i, \tilde{l}_i^m[j] > = < y_i, x_i \tilde{l}_i^{m-1}[j] - \tilde{l}_i^{m-1}[j] x_i >$$

$$= \tilde{l}_i^{m-1}[j] - ax_i < y_i, \tilde{l}_i^{m-1}[j] > - < y_i, \tilde{l}_i^{m-1}[j] > x_i - a^{m-1} b \tilde{l}_i^{m-1}[j] >$$

$$= (1 - a^{m-1}b) \tilde{l}_i^{m-1}[j] + ax_i < y_i, \tilde{l}_i^{m-1}[j] > - < y_i, \tilde{l}_i^{m-1}[j] > x_i.$$

Thus equation \eqref{5} holds when $k = 1$. Assume $k > 1$. See

$$< y_i^k, \tilde{l}_i^m[j] >$$

$$= < y_i, \sum_{l=0}^{k-2} (a^l - a^{m-1-l}b) < y_i^{k-1}, \tilde{l}_i^{m-1}[j] > + a^{k-1} x_i < y_i^k, \tilde{l}_i^{m-1}[j] > - < y_i^k, \tilde{l}_i^{m-1}[j] > x_i > \quad \text{by inductive hypothesis}$$

$$= \sum_{l=0}^{k-2} (a^l - a^{m-1-l}b) < y_i^{k-1}, \tilde{l}_i^{m-1}[j] > + a^{k-1} < y_i^{k-1}, \tilde{l}_i^{m-1}[j] >$$

$$+ a^{k-1} x_i a < y_i^k, \tilde{l}_i^{m-1}[j] > - < y_i^k, \tilde{l}_i^{m-1}[j] > x_i - a^{m-k} b < y_i^{k-1}, \tilde{l}_i^{m-1}[j] >$$

$$= \text{the right hand side of \eqref{5}}.$$
Consequently, (5) holds.

Now we show (6) by induction on \( m \). (6) follows from \([\text{WZZ}15b, \text{Lemma 4.1(i)}]\) when \( m = 1 \).

By (5), one obtains

\[
<y^m_i, \tilde{l}^m_i[j]^- > = \sum_{l=0}^{m-1} (a^l - a^{m-1-l}b) < y^{m-1}_i, \tilde{l}^{m-1}_i[j]^- > \\
= (m)a(1-b) < y^{m-1}_i, \tilde{l}^{m-1}_i[j]^- > \\
= \text{the right hand side of (6)}.
\]

Therefore, (6) and (7) hold.

(iii) If \( a = 1 \), then

\[
<y^m_i y_j, \tilde{l}^m_i[j]^- > = (c-1)^m(m)! \text{ and } < y^m_i y_j, \tilde{l}^m_i[j]^- > = (1-b)^m(m)!
\]

by \([\text{WZZ}15b, \text{Lemma 4.1(v)}]\) and (ii).

(iv) It follows from (ii) and \([\text{WZZ}15b, \text{Lemma 4.1 (v)}]\).

(v) If \( a = 1 \), it is clear by (iii). If \( a \neq 1 \), then (v) follows from \([\text{WZZ}15b, \text{Lemma 4.1(iii)}]\) and (iv).

\[\blacksquare\]

**Proposition 3.2.** \((\text{[ZWTZ, Th.10.7(i)]})\) If \( |D| = \infty \), then \( \dim(\mathcal{L}^-(V)) = \infty \).

**Proposition 3.3.** Assume that \( \mathfrak{B}(V) \) is a connected Nichols algebra of diagonal type. Then \( \dim(\mathfrak{B}(V)) = \infty \) and \( \dim(\mathcal{L}^-(V)) < \infty \) if and only if \( \dim V = 1 \) and \( \text{ord}(p_{11}) = 1 \) or \( \infty \).

**Proof.** Sufficiency is clear.

Necessity. Assume \( \dim V > 1 \). By \([\text{ZWTZ, Th.10.7(i)]}\), \( \Delta(\mathfrak{B}(V)) \) is an arithmetic root system. Consequently, there exists \( u \in D \) such that \( \text{ord}(p_{uu}) = 1 \) or \( \infty \). Considering Lemma 3.1 (v) we get a contradiction. \[\blacksquare\]

4 **Finiteness of Nichols Algebras and Nichols (Braided) Lie Algebras**

In this section we obtain an explicit necessary and sufficient condition for a Nichols algebra and a Nichols (braided) Lie algebra to be finite dimensional

**Proposition 4.1.** Assume that \( \mathfrak{B}(V) \) is a connected Nichols algebra of diagonal type. Then \( \dim(\mathfrak{B}(V)) = \infty \) and \( \dim(\mathcal{L}^-(V)) < \infty \) if and only if \( \dim V = 1 \) and \( \text{ord}(p_{11}) = 1 \).

**Proof.** Sufficiency. It is clear. Necessity. By \([\text{WZZ15b, Th. 2.2]}\), \( \dim V = 1 \). Considering \( \dim(\mathfrak{B}(V)) = \infty \) we have \( \text{ord}(p_{11}) = 1 \) or \( \infty \). By \([\text{WZZ15a, Lemma 4.3]}\) \( \text{ord}(p_{11}) = 1 \). \[\blacksquare\]

**Theorem 4.2.** If \( \mathfrak{B}(V) \) is connected Nichols algebra of diagonal type with \( \dim V > 1 \), \( \dim(\mathfrak{B}(V)) = \infty \) if and only if \( \dim(\mathcal{L}^-(V)) = \infty \).
Proof. Sufficiency is clear. Necessity. It follows from Proposition 3.3.

The fixed parameters in [He05, Table A.1, A.2], [He06a, Table B, C] are called quantum numbers of generalized Dynkin diagrams. For example, quantum number of Row 2 in [He05, Table A.1] is $q$; quantum numbers of Row 9 in [He05, Table A.2] are $q$, $r$ and $s$.

**Theorem 4.3.** Assume that $\mathcal{B}(V)$ is connected Nichols algebra of diagonal type with $\dim V > 1$, then the following conditions are equivalent: (i) $\mathcal{B}(V)$ is finite-dimensional; (ii) $\mathcal{L}(V)$ is finite-dimensional; (iii) $\mathcal{L}^-(V)$ is finite-dimensional. (iv) $\Delta(\mathcal{B}(V))$ is an arithmetic root system, $\text{ord}(p_{ii}) \neq \infty$ and $\text{ord}(p_{ij}p_{ji}) \neq \infty$ for all $1 \leq i \neq j \leq n$. (v) $\Delta(\mathcal{B}(V))$ is an arithmetic root system and the quantum numbers (i.e. the fixed parameters) of generalized Dynkin diagrams of $V$ are of finite order.

Proof. (i) $\iff$ (ii). It follows from [WZZ15b, Th. 2.2].

(i) $\iff$ (iii). It follows from Theorem 4.3.

(iv) $\Rightarrow$ (i). By Proposition 4.1, $p_{u,u} \neq 1$ for any $u \in D$. We complete the proof by [He05, Th.1.4.6] or [Kh99, Th. 2] since $\text{ord}(p_{u,u}) < \infty$ for any $u \in D$.

(i) $\Rightarrow$ (iv). Obviously, $\Delta(\mathcal{B}(V))$ is an arithmetic root system and $\text{ord}(p_{ii}) \neq \infty$. Assume $\text{ord}(p_{ij}p_{ji}) = \infty$ for any $i$. By [He05, Lemma 2.2.1], one has $[x_i, x_j] \in D$ or $[x_j, x_i] \in D$. However, the heights of $[x_i, x_j]$ and $[x_j, x_i]$ are infinite. Consequently we get a contradiction.

(iv) $\iff$ (v). It follows from [He05, Table A.1, A.2] and [He06a, Table B, C].

**Corollary 4.4.** Assume that $\mathcal{B}(V)$ is connected Nichols algebra of diagonal type with $\dim V > 1$. Then $\dim(\mathcal{B}(V)) = \infty$ (resp. $\dim(\mathcal{L}(V)) = \infty$) (resp. $\dim(\mathcal{L}^-(V)) = \infty$) if and only if one of the following conditions holds:

(i) $\Delta(\mathcal{B}(V))$ is not an arithmetic root system.

(ii) $\dim V = 2$, $\text{ord}(q) = \infty$ in Row 2-6, Row 11 of [He05, Table A.1].

(iii) $\dim V = 3$, $\text{ord}(q) = \infty$ in Row 1-8, 10 of [He05, Table A.2].

(iv) $\dim V = 3$, $\text{ord}(q) = \infty$ or $\text{ord}(r) = \infty$ or $\text{ord}(s) = \infty$ in Row 9 of [He05, Table A.2].

(v) $\dim V = 4$, $\text{ord}(q) = \infty$ in Row 1-14 of [He06a, Table B].

(vi) $\dim V \geq 5$, $\text{ord}(q) = \infty$ in Row 1-4, 7-10, 16, 20, 22 of [He06a, Table C].

Proof. It follows from Theorem 4.3, [He05, Table A.1, A.2] and [He06a, Table B, C].

## 5 Appendix

The proof of Proposition 3.2.

For completeness, we give the proof of Proposition 3.2.
Let $A := \{x_1, x_2, \ldots, x_n\}$ be an alphabet and a basis of $V = \text{span}(A)$; $A^* := \{u \mid u \text{ is a word}\}$. Let $I$ be a graded ideal of $T(V)$ as algebras such that $\mathfrak{B}(V) = T(V)/I$.

$u$ is called a standard word with respect to $\mathfrak{B}(V)$ if $u$ can not be written as linear combination of strictly greater words in $\mathfrak{B}(V)$.

Let $\text{SW}(\mathfrak{B}(V)) =: \{u \in A^* \mid u \text{ is a standard word with respect to } \mathfrak{B}(V)\}$, written as SW in short;

$L := \{u \in A^* \mid u \text{ is a Lyndon word}\}$. Notice that we view $\text{SW}(\mathfrak{B}(V))$ and $L$ are in $\mathfrak{B}(V)$ often for convenience.

Let $\text{HBLLW}(\mathfrak{B}(V)) =: \{u \in L \mid [u] \text{ is a hard super-letter}\}$,

**Lemma 5.1.** (i) $\text{SW}$ is a basis of $\mathfrak{B}(V)$.

(ii) Any factor of a standard word is a standard word.

(iii) If $u$ is a standard word, then $u = u_1u_2 \cdots u_r$ with $u_1 \geq u_2 \geq \cdots \geq u_r$ and $u_i \in \text{SW} \cap L$ for $1 \leq i \leq r$.

**Proof.** (i) If a word $u \not\in \text{SW}$, then $u = \sum_{v \in \text{SW}, |v| = |u|, v > u} a_v v$. It is clear that $\text{SW}$ is linearly independent in $\mathfrak{B}(V)$. Consequently $\text{SW}$ is a basis of $\mathfrak{B}(V)$.

(ii) It is clear.

(iii) It follows from Part (ii) and [Lo83, Th. 5.1.5].

**Lemma 5.2.** (i) If $l \in L$, then $\lfloor l \rfloor^- = l + \sum_{w > l, |w| = |u|} a_w w$ in $\mathfrak{B}(V)$, where $a_w \in k$.

(ii) If $l \in L$, then $\lfloor l \rfloor^- = a_l l + \sum_{w > l, |w| = |u|} a_w w$ in $\mathfrak{B}(V)$, where $a_l, a_w \in k$ with $a_l \neq 0$.

(iii) $\text{SW} \cap L \subseteq \text{HBLLW}$.

**Proof.** (ii) We show this by induction on $|l|$. It is clear when $|l| = 1$ since $\lfloor l \rfloor = l$. Assume that $l = uv$ is the Shirshow decomposition of $l$. If $u' > u$ and $v' > v$ with $|u'| = |u|$ and $|v'| = |v|$, then $u'v' > uv = l$ and $v'u' > vu > l$.

$\lfloor l \rfloor^- = \lfloor v \rfloor^- u - p_{u,v} \lfloor u \rfloor^- v$

$= (a'_v v + \sum_{v' > v, |v'| = |v|} a'_{v'} v')(a'_u u + \sum_{u' > u, |u'| = |u|} a'_{u'} u')$

$-p_{vu}(a'_u u + \sum_{u' > u, |u'| = |u|} a'_{u'} u')(a'_v v + \sum_{v' > v, |v'| = |v|} a'_{v'} v')$ (by inductive hypothesis)

$= a_l l + \sum_{w > l, |w| = |u|} a_w w$.

(i) The proof is similar to the proof of (ii).
(iii) If \( l \not\in \text{HBLLW} \) with \( l \in L \), then
\[
[l] = \sum_{r=1}^{m} \sum_{u_1 > l, 1 \leq i \leq r, |u_1 u_2 \cdots u_r| = |l|} a'_u [u_1][u_2] \cdots [u_r]
\]
\[
= \sum_{r=1}^{m} \sum_{u_1 > l, 1 \leq i \leq r, |u_1 u_2 \cdots u_r| = |l|} a'_u \prod_{i=1}^{r} (a''_{u_i} u_i + \sum_{w_i > u_i, |w_i| = |u_i|} a''_{w_i} w_i)
\]
\[
= \sum_{w > l, |w| = |l|} a''_w w \quad \text{and} \quad [l] = a'''_l + \sum_{w > l, |w| = |l|} a'''_w w \quad (\text{by (ii)}).
\]

Consequently, \( l = \sum_{w > l, |w| = |w|} a_w w \) and \( l \not\in \text{SW} \cap L \). \( \square \)

**Theorem 5.3.** If \( \mathcal{B}(V) \) is a graded braided Hopf algebras with diagonal braiding, then \( \text{SW}(\mathcal{B}(V)) \cap L = \text{HBLLW}(\mathcal{B}(V)) \). Furthermore, if \( \Delta(\mathcal{B}(V)) \) is not an arithmetic root system, then \( \dim \mathcal{L}^-(\mathcal{B}(V)) = \infty \).

**Proof.** \( \text{SW} \cap L \subseteq \text{HBLLW} \) by Lemma 5.2(iii). If there exists \( u \in \text{HBLLW} \) and \( u \not\in \text{SW} \cap L \), then \( u = \sum_{v_1 \geq v_2 \geq \cdots \geq v_r, v_i \in \text{SW} \cap L, |v_1 v_2 \cdots v_r| = |u|, v_1 v_2 \cdots v_r > u} a_{v_1} v_1 \cdots v_r \) by Lemma 5.1(iii). Consequently, \( [u] = \sum_{v_1 \geq v_2 \geq \cdots \geq v_r, v_i \in \text{SW} \cap L, |v_1 v_2 \cdots v_r| = |u|, v_1 v_2 \cdots v_r > u} a_{v_1} [v_2] \cdots [v_r] \), which contradicts to \( [u], [v_1][v_2] \cdots [v_r] \in P \). \( \square \)

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