ON LONG TIME BEHAVIOR OF MOORE-GIBSON-THOMPSON EQUATION WITH MOLECULAR RELAXATION

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ABSTRACT. A third order in time nonlinear equation is considered. This particular model is motivated by High Frequency Ultra Sound (HFU) technology which accounts for thermal and molecular relaxation. The resulting equations give rise to a quasilinear-like evolution with a potentially degenerate damping [23]. The purpose of this paper is twofold: (1) to provide a brief review of recent results in the area of long time behavior of solutions to of MGT equation, (2) to provide recent results pertaining to decay of energy associated with the model accounting for molecular relaxation which is locally distributed.

1. Introduction.

1.1. Physical model. The MGT equation is one of the equations of nonlinear acoustics describing acoustic wave propagation in gases and liquids. The behavior of acoustic waves depends strongly on the medium property related to dispersion, dissipation and nonlinear effects. It arises from modeling high frequency ultrasound (HFU) waves [21, 23, 38]. The derivation of the equation, based on continuum and fluid mechanics, takes into account viscosity and heat conductivity as well as effect of the radiation of heat on the propagation of sound. The original derivation dates back to [46]. The corresponding system consists of equations of motion (conservation of momentum) for the variable \( v \) which denotes fluid particle viscosity (Navier-Stokes type), continuity equation (conservation of mass) relating density of the

2010 Mathematics Subject Classification. 35L25, 35L75, 35L90, 35L05, 93D15, 93D20, 37C75.

Key words and phrases. MGT equation, thermal and molecular relaxation, third order in time equations, viscoelastic-memory dependent dynamics, global solutions for small data.

Research of Irena Lasiecka has been partially supported by NSF Grant DMS-1108871 and AFOSR Grant FA9550-12-1-0354. Research of Valéria Neves Domingos Cavalcanti has been partially supported by the CNPq Grant 304895/2003-2. Research of Arthur Henrique Caixeta has been partially supported by CAPES.

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medium to fluid/gas velocity and conservation of energy (entropy equation, first law of thermodynamics). The system closes with the so called state equation describing the relation between the pressure, density and the entropy [20, 39, 45]. What is specific to MGT equations is the fact that heat conductivity is described by Maxwell-Cattaneo law, rather than a standard Fourier’s law. Maxwell-Cattaneo law accounts for finite speed of propagation of the heat transfer, thus eliminates infinite speed paradox associated with Fourier’s laws. This phenomenon is known as the paradox of heat conduction [14, 34]. Introducing the Maxwell Cattaneo flux law, with a small relaxation parameter
\[ \tau > 0 \] (intrinsic relaxation time of the heat flux [12]), the heat flow propagates through the medium over time as a thermal wave. This phenomenon is also referred to as second sound [14, 20]. The value of this parameter (relatively small) depends on physical parameters of the fluid or gas. Some specific examples are given in [39].

1.2. PDE model. The resulting model, written for a variable \( u \) denoting the pressure, is a third order in time scalar equation with a “small” parameter \( \tau \) denoting thermal relaxation. The relaxation parameter accounts for finite speed of thermo-acoustic waves, addressing the paradox of infinite speed of propagation occurring in modeling acoustic waves [24]. See [11, 21, 26, 42] for more information. In addition, one may account for molecular relaxation entering the state equation, which then leads to an addition of memory term with a relaxation kernel \( g(s) \). The model under the consideration is the following
\[
\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t + \int_0^t g(t-s) \Delta u(s) \, ds = \frac{d^2}{dt^2} \left[ \frac{1}{c^2} \left( 1 + \frac{B}{2A} \right) u^2 \right],
\]
(1)
with either Dirichlet or Neumann/Robin boundary conditions prescribed on the boundary of a smooth domain defined in \( \mathbb{R}^3 \). Here \( b > 0 \) denotes a parameter of diffusivity, \( c^2 \) the speed of sound and \( \alpha \) represents friction. Positive parameters \( B, A \) represent nonlinear interactions. See [23, 39] for physical interpretation.

It is often convenient to write an abstract version of this problem as:
\[
\tau u_{ttt} + \alpha u_{tt} + c^2 \mathcal{A} u + b \mathcal{A} u_t - \int_0^t g(t-s) \mathcal{A} u(s) \, ds = k \frac{d^2}{dt^2} (u^2),
\]
(2)
where \( \mathcal{A} \) is a positive self-adjoint operator on a Hilbert space \( H \), on which we use the notation \( || \cdot || \) and \( (\cdot, \cdot) \) for norm and inner product. The state space then becomes
\[
\mathcal{H} \triangleq D(\mathcal{A}^{1/2}) \times D(\mathcal{A}^{1/2}) \times H
\]
with the associated energy functional:
\[
E_0(t) \triangleq ||u_{tt}||^2 + ||\mathcal{A}^{1/2} u_t||^2 + ||\mathcal{A}^{1/2} u||^2,
\]
(3)
In addition to the basic space \( \mathcal{H} \) we shall also consider “higher energy” spaces
\[
\mathcal{H}_1 \triangleq D(\mathcal{A}) \times D(\mathcal{A}^{1/2}) \times H
\]
\[
\mathcal{H}_2 \triangleq D(\mathcal{A}) \times D(\mathcal{A}) \times D(\mathcal{A}^{1/2})
\]
It has been shown [24, 33] that as long as \( b > 0 \), the linear part of equation (2) with \( g = 0 \) generates a strongly continuous group on each \( \mathcal{H}_i, i = 0, 1, 2 \). The wellposedness -both local and global-of a nonlinear (quasilinear) model (1) without a memory has been established in [25]. See also a comprehensive survey article [23]. Wellposedness of solutions accounting for memory terms has been considered in [30, 31] in the linear case and in [27] in the nonlinear case.
The aim of this paper is to study the interaction between several dissipative mechanisms (frictional, diffusive and viscoelastic-memory) and their effects on long time behavior of the corresponding solutions. Of particular interest is the analysis of essential support of various damping mechanisms and their role in dissipating the energy. This latter topic is presented in the last section.

1.3. Notation. The following notation will be used throughout the paper.

- \[ ||u|| \equiv ||u||_{H}, \ (u, v) \equiv (u, v)_{H}. \]
- \[ ||u||_{L^p(J_T; L^q(\Omega))} \equiv ||u||_{L^p(J_T; L^q(\Omega))}, \ J_T = (0, T) \]
- \[ g \circ u \equiv \int_0^t g(t-s)||u(t)-u(s)||^2 \, ds \]
- \[ g \ast u(t) \equiv \int_0^t g(t-s)u(s) \, ds \]
- \[ g \circ u(t) \equiv \int_0^t g(t-s)(u(s) - u(t)) \, ds \]
- \[ G(t) \equiv \int_0^t g(s) \, ds \]
- \[ \gamma = \alpha - \frac{\tau^2}{b} \]
- \[ B_X(r) \] denotes a ball in a Banach space \( X \) with a radius \( r > 0 \).

2. Wellposedness and stability of solutions at various energetic levels. In this section we shall provide a brief overview of wellposedness results, both local and global, pertinent to various configurations of MGT equations. Both linear and nonlinear models with and without memory will be discussed.

2.1. Linear models. As a consequence of the presence of the parameter \( \tau \), the original “parabolic” like model \( (\tau = 0) \) becomes hyperbolic \( (\tau > 0) \) - see [24, 25, 33] with revealing spectral analysis carried out in [33]. Thus, the issues of wellposedness, stabilization and long time behavior becomes more subtle due to the presence of infinitely many unstable eigenvalues. Roughly speaking, with \( \tau > 0 \), \( (1) \) becomes ill-posed when diffusion parameter \( b = 0 \) [17]; on the other hand with \( b > 0 \) and \( g = 0 \) the linearization of the system generates a strongly continuous group in the variables \( U \equiv (u, u_t, u_{tt}) \) on the space \( H^1_0(\Omega) \times H^1_0(\Omega) \times L^2(\Omega) \), e.g in the case of Dirichlet boundary conditions.

The following is a standing assumption imposed on the relaxation kernel.

Assumption 2.1. \( g \in C^1(R), g' \leq 0, g(0) > 0, G(\infty) < c^2 \)

Theorem 2.2. Consider the linear problem by taking \( k = 0 \) in \( (2) \).

- \( g = 0 \). The system described by \( (2) \), in the variable \( u, u_t, u_{tt} \), generates a group \( e^{At} \) on \( \mathcal{H} \). Moreover, with a suitably selected inner product the corresponding group is contractive, as long as \( \gamma \equiv \alpha - \frac{\tau^2}{b} > 0 \). Similar result of generation holds on other spaces \( \mathcal{H}_i, \ i = 1, 2 \).

- Let Assumption 2.1 be in force. Then \( (2) \) is well posed in \( \mathcal{H} \). This is to say that for all initial data \( U(0) = (u_0, u_1, u_2) \in \mathcal{H}, \) there exists unique solution \( U(t) = (u(t), u_t(t), u_{tt}(t)) \in C([0, \infty); \mathcal{H}) \) which is continuously dependent (in the norm of \( \mathcal{H} \)) on the initial data. The same result is valid on \( \mathcal{H}_i, \ i = 1, 2 \), provided \( G(\infty) < b \).

The first part of the theorem has been proved in [24]. The second part in [30].

2.2. Nonlinear models. Here we have quasilinear and degenerate behavior which leads to major subtleties in the treatment of wellposedness. First of all, the theory requires small data for the construction of potential well which is sufficiently regular.
This, in turn, commands introduction of more regular spaces and energies. In order to describe the results we introduce a higher level energy

\[ E_2(t) \equiv \|Au(t)\|^2 + \|Au_t(t)\|^2 + \|A^{1/2}u_{tt}\|^2 \]

and

\[ \mathcal{H}_2 \equiv \mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A^{1/2}) \]

**Theorem 2.3.**
- Let \( g = 0 \) and \( k > 0 \) in (2). Assuming that \( \|U(0)\|_{\mathcal{H}_1} \leq \rho, \rho > 0 \) and \( \rho \) is sufficiently small, there exists unique solution \( U(t) = (u(t), u_t(t), u_{tt}(t)) \) such that \( \|U(t)\|_{\mathcal{H}_1} \leq C(\rho) \) for all times \( t > 0 \).
- **Memory case with \( g \) subject to assumption 2.1.** Let \( \gamma > 0 \). In addition assume that \( G(\infty) < b \). Problem is locally well posed in \( C(0,T;\mathcal{H}_2) \). If, in addition \( g' + \omega g \leq 0 \) with some \( \omega > 0 \), then the said solution is global. This means that the size of initial data can be made uniform with respect to the running time \( t > 0 \).
- **Memory revisited.** Assume critical value of \( \gamma = 0 \). The result as above holds for the model where viscoelasticity kernel contains the velocity \( u_t + \frac{c^2}{b}u \)

The first part of the theorem is proved in [25] while the second part in [27].

### 2.3. MGT with nonlinear control.

Since the nonlinear (quasilinear) theory requires small initial data assumption, there has been an effort made in order to eliminate such assumption by introducing control mechanism. The crux of the matter is to keep solutions small enough so that degeneracy will not affect solvability. The above aspect brings forward another idea. The idea is of controlling degeneracy. By this we mean that a feedback control will prevent from leaving solvability well, rather than just a smallness of solutions. Here we shall explore the concept of feedback control counteracting ill-posedness of the problem.

In [6] the following model with nonlinear control feedback was considered.

\[ \tau u_{ttt} + \alpha u_{tt} + c^2 Au + bAu_t + \beta u_t^3 = 2ku_t^2 + p(u) \quad (4) \]

where the parameter \( \beta \geq 0 \) and \( p(u) \) denotes an active force. The term \( \beta u_t^3 \) represents the nonlinear control. With this model the following energy functional is associated

\[ \mathcal{E}(t) = E_0(t) + \frac{\beta}{4} \int_\Omega u_t^2 - \frac{2k}{3} \int_\Omega u_t^3 dx \]

In this case, we assume that \( \gamma > 0 \), the operator \( A \) is strictly positive with control of the norm \( \|u\|_H \leq C_0\|A^{1/2}u\|_H \), \( u \in \mathcal{D}(A^{1/2}) \), and \( p \) is a real function satisfying

**Assumption 2.4.**
- \( p \in C^1(\mathbb{R}) \);
- Its derivative satisfies

\[ -\delta \leq p'(s) \leq m, \quad s \in \mathbb{R}, \]

for \( 0 \leq m \leq \min\left\{ \frac{c^2}{4C^2_0}, \frac{c^2\gamma}{16\tau\theta C^2_0} \right\} \) and \( 0 < \delta < \min\left\{ m, \frac{b\gamma}{4\tau\theta C^2_0} \right\} \).

where \( \theta = \frac{1}{2}(\frac{\alpha}{\tau} + \frac{c^2}{b\tau}) \).

Under the Assumption 2.4 it was proved in [6] we that (4) with initial data of arbitrary size in \( \mathcal{H} \) is locally and globally well-posed with respect to the topology induced by \( E_0(t) \).

The energy \( \mathcal{E}(t) \) may be negative, however when \( \beta > 0 \) it is bounded from above and below by a continuous function depending on the energy \( E_0(t) \). In fact, the following result is available in [6].
Theorem 2.5. (1) Suppose $b > 0$, $\gamma := \alpha - \frac{r^2}{b} > 0$, $\beta \geq 0$. For any initial condition $U_0 = (u_0, u_1, u_2) \in \mathcal{H}$, there exists a time $T = T(U(0)) > 0$ such that the problem (4) is well-posed on $(0, T)$, i.e., there exists a unique mild solution $U \in C^0([0, T]; \mathcal{H})$ solving (4) and such that
\[
\|(u(t), u_t(t), u_{tt}(t))\|_{\mathcal{H}}^2 = \left\{ \|A^{1/2}u(t)\|^2 + \|A^{1/2}u_t(t)\|^2 + \|u_{tt}(t)\|^2 \right\}
\]
is finite for all $t \in [0, T]$. The same result is valid when replacing $\mathcal{H}$ by $\mathcal{H}_2$;
(2) If $\beta > 0$, the constant responsible for the nonlinear feedback control of long-time behavior for solutions corresponding to initial data of arbitrary size, is positive, then we can take $T = +\infty$;
(3) If we assume, in addition, that $\beta > \beta_0$ where $\beta_0 = \frac{4\tau k^2 \theta C_0^2}{b\gamma}$, then the dynamical system is gradient. In particular, the energy $E(t)$ is decreasing.

Even in the absence of a gradient property, the controlled dynamical system generated by (4) enjoys an absorbing property.

Theorem 2.6. Let $\beta > 0$. With respect to the global (in time) solutions obtained via Theorem 2.5, the system $(\mathcal{H}, S(t))$ generated by (4) in the energy space $\mathcal{H}$ is ultimately dissipative, i.e., there exists a positive real number $R > 0$, possessing the property: for any bounded set $B \subset \mathcal{H}$, there exists a time $t_0 = t_0(B)$ such that $\|S(t)U\|_{\mathcal{H}} \leq R$ for all $U \in B$ and $t \geq t_0$. The radius $R$ does not depend on the size of $B$.

This result is critical for the study of attractors in the non-gradient situation.

Theorem 2.7. Assume Theorem 2.6 holds true. Then the semiflow $S(t)$ generated by the problem (4) is asymptotically smooth and, consequently, possesses a compact global attractor $A$.

Theorem 2.8. Let $\beta > 0$. Then the attractor in reference above
- has a finite fractal dimension;
- enjoys partial smoothness
  \[
  \|u_{tt}\|_{H^1(\Omega)} + \|A(c^2u + bu_t)\| \leq C_k.
  \]
- If $p \in C^\infty$ then the trajectories on the attractor are infinitely many times differentiable (in time).

Theorems 2.5-2.8 have been proved in [6].

2.4. Comments. We shall make several comments which provide a road map to the proofs of the results formulated above.

1. As already mentioned one of the main obstacles in studying the system under consideration is the fact that system is not dissipative and hence does not posses the usual a priori bounds. It is the mixture of second order and third order dynamics that prevents the usual dissipative relations. The theories known in this area [25] and references therein, deal with sufficiently small and regular solutions, in which case the super linear terms have a natural “home” to reside. Instead, weak (finite energy) solutions, and without smallness assumptions imposed on the initial data do not fit a natural framework for consideration. In order to obtain global existence of finite energy solutions suitable damping is introduced with a positive damping parameter $\beta$. For parameter $\beta > 0$ the said solutions are captured by an absorbing set, hence are
bounded uniformly in time. The above property does not follow from energy relation and needs to be established by resorting to barrier’s method. As a consequence global solutions of finite energy can be established - as stated in Theorem 2.5.

2. Another difficulty, though related, is the lack (when $\beta > 0$ is not required to be sufficiently large) of gradient structure and of related Lyapunov function. Thus an existence of global attractor depends on a priori existence of the absorbing set (the property which guarantees an existence of universal set trapping inside all the solutions - regardless the location of initial conditions). In order to prove ultimate dissipativity (absorbing set) we shall resort to the “barrier method” - a device used in the theory of quasilinear systems.

3. The issue of asymptotic compactness. We note that the nonlinear control term is at the critical level of nonlinearity when dimension of $\Omega$ equals to 3. In order to establish the needed compactness property, a new compensated compactness criterion is used that is based on a certain functional formulation which allows to exhibit strong sequential convergence of sequences that are only bounded - but subject to certain structural property [9].

4. Finally smoothness and finite dimensionality of attractor is shown by resorting to a new method introduced in [16], which then leads to the establishment of quasistability property [10].

In closing this section, we wish to emphasize that the present paper is, to our best knowledge, the first treatment of global attractors and related long time behavior in the context of third order equations. Potential further developments may evolve in the following directions

- Consider more general structures of static damping where $g(s)$ is an arbitrary monotone with $g(s)s > s^4$ for $s$ large. Consider a larger class of functions $p(s)$ which conform with typical dissipativity conditions at infinity. The methods presented in this manuscript would require only minor modifications.
- Asymptotic analysis for the model with respect to the parameter $\tau$. When $\tau \to 0$ we obtain a well studied strongly damped wave equation where the results are known to be optimal [8] and references therein. It would be nice to reconstruct the limit attractor and prove uppersemicontinuity of the family of attractors.

3. Long time behavior-revisited. Long time behavior of MGT nonlinear models is subtle. It depends on several parameters. We recall the important stability parameter which is assumed to be non-negative

$$\gamma \equiv \alpha - c^2 \beta^{-1} \geq 0 \quad (5)$$

It is known that when $\gamma > 0$, then the corresponding linear group with $g = 0$ is exponentially stable on a full scale of spaces $H_i$ [24, 33]. However, when $\gamma = 0$ the semigroup is conservative; no decay of the energy occurs, despite the presence of diffusivity ($b > 0$) and of frictional “damping” ($\alpha > 0$). The spectrum of the linear operator associated with the dynamics $U(t)$ consists of a continuous spectrum converging to the point $-c^2/b$ and two branches of eigenvalues with real parts negative for $\gamma > 0$ converging asymptotically to $-\gamma$. The “optimal” damping parameter $\alpha$ corresponds to the value $\alpha_0 = \frac{3\tau c^2}{b}$ and provides the optimal location of the essential spectrum with respect to dynamic stability. For “stable” values of $\gamma > 0$, it has been shown in [25] that the nonlinear equation (2) admits global
solutions for sufficiently small and smooth initial data. We also note that in the parabolic case \( \tau = 0 \), global solvability of the nonlinear equation has been proved in [36] by using maximal parabolic regularity. In what follows we shall present precise formulations of stability results pertinent to each model considered. We shall begin with models which do not account for memory effects.

3.1. Models without the memory. Here we shall consider quasilinear and degenerate model without accounting for memory effects. The Theorem stated below has been proved in [25]. See also survey article [23].

**Theorem 3.1.** With reference to quasilinear equation (2) with \( g = 0 \) the following properties are valid [25]:

- (a) **Local (in time) well-posedness:** Let \( T > 0 \) be arbitrary and \( \gamma \geq 0 \). There exists \( C(T) > 0 \) such that if the initial data \( U(0) = (u_0, u_1, u_2) \) satisfy \( E_1(0) \leq C(T) \), then there exists a unique mild solution \( U(t) = (u, u_1, u_t) \) and such that \( E_1(t) < \infty \) for all \( t \in [0, T] \). The said solution is unique and depends continuously (with respect to the topology generated by \( E_1 \)) on the initial data. An alternative formulation of local existence may be given as follows: for a given \( U(0) \in H_1 \) there exists \( T(U(0)) > 0 \) such that a mild solution exists on \((0, T(U(0)))\).

- (b) **Global well-posedness:** Assume, in addition, that \( \gamma > 0 \). Then there exists \( C > 0 \), \( C_1 > 0 \) such that solutions corresponding to initial data \((u_0, u_1, u_2)\) with \( E_1(0) \leq C \) exist for all \( t > 0 \) and satisfy \( E_1(t) \leq C_1 \) for all \( t > 0 \).

- (c) **Exponential decay rates:** Assuming \( \gamma > 0 \), with initial data \((u_0, u_1, u_2)\) satisfying \( E_1(0) \leq C \) for some \( C > 0 \). Then there exist constants \( C_1, \omega > 0 \) such that solutions satisfy \( E_1(t) \leq C_1 e^{-\omega t} \) for all \( t > 0 \).

3.2. Models with relaxed memory. Motivated by modeling aspects of nonlinear acoustics where memory effects are known to play an important role in modeling molecular relaxation [22, 32, 45], leads us to consider viscoelasticity in the model. In fact, for high frequency waves in liquids and gases an additional relaxation process may be present. This is exhibited by the so called molecular relaxation where the pressure depends on density at all prior times, thus it depends on the history of the process [39]. This leads to an integral term in the equation with a kernel - say \( g(s) \) associated with a particular relaxation mechanism excitations of molecular degrees of freedom, impurity effects etc. The corresponding kernels are of exponential type - say \( g(t) = mc_0^2 e^{-\omega t} \) where relaxation parameter \( m = \frac{c_0^2 - c_1^2}{c_1} \) with \( c_1 \) denoting the sound velocity in the high frequency limit [39]. The relaxation time \( \omega^{-1} \) depends on the nature of the process - e.g. for sea water typical values are \( 10^{-3}\text{sec} \). Concrete values for relaxation parameters depend on the medium - some examples are given in Chapter 1 in [39]. Dependence on molecular relaxation can also be introducing time derivative of the pressure tensor in the state equation with a suitable time relaxation parameter [32].

We then begin with linear model and consider

\[
\tau u_{ttt} + \alpha u_{tt} + \epsilon^2 Au + bAu_t - \int_0^t g(t - s)Aw(s)ds = 0, \tag{6}
\]

where the variable \( w \) is a linear combination of \( u \) and \( u_t \). This is to say

\[
w = au_t + \lambda u. \tag{7}
\]
While in the equation of the second order, memory term represents itself by the variable corresponding to potential energy, in the case of the third order equation the dynamics is richer and one may have dependence on both variables $u$ and $u_t$. We shall analyze the situation from the point of view of effectiveness of the memory term on the stabilization, and more generally of long time behavior of the resulting dynamics. In what follows we shall refer to Memory Type I when in (7) $a = 0, \lambda > 0$, Type II when $\lambda = 0, a > 0$ and Type III when $a > 0, \lambda > 0$.

It has been shown in [31] that in the subcritical case $\gamma > 0$ the addition of memory damping (under certain conditions) leads to uniform stability of the overall dynamics. The conditions involved are the ones limiting the size of $G(\infty)$ with respect to the parameters of equation such as $c^2$ and $b$. In view of this of particular interest is the “critical” value of $\gamma = 0$, within the context of both linear (6) and the nonlinear equation (2). In this case the original linearized dynamics associated with (2) is unstable. Thus, one expects that memory relaxation provides some stabilizing effect. These are known to produce dissipative effects. In fact, it is well known by now that the convolution type of memory, when added to the second order conservative systems, provides a stabilizing effect [40, 41, 43]. The strength of this stabilizing effect depends on the properties of convolution kernel. Detailed analysis with optimal estimates is given in [40]. Starting with memory term depending on the pressure only $w = u$, it has been recently discovered that even in the case when relaxation function $g(t)$ is exponentially decaying, the overall model is not exponentially stable in the critical case $\gamma = 0$. This result has been proved in [13] by analyzing roots of transfer function corresponding to fourth order equation. However, in this case one still has strong stability-in fact polynomial decays. Thus, there is a dissipative stabilizing effect, however not as strong as in the case of second order dynamics. The said result will be described below. Defining the corresponding energy

$$F(t) = \|A^{\frac{1}{2}}u(t)\|^2 + \|A^{\frac{1}{2}}\partial_t u(t)\|^2 + \|\partial_t u(t)\|^2 - \int_0^t g'(s)\|A^{\frac{1}{2}}u(t) - A^{\frac{1}{2}}u(t-s)\|^2 ds,$$

the following theorem from [31] holds.

**Theorem 3.2.** In the subcritical case $\gamma > 0$ the energy decays exponentially, that is, there exist constants $C \geq 1$ and $\omega > 0$ such that

$$F(t) \leq CF(0)e^{-2\omega t}. \tag{8}$$

In the critical case, we have instead

**Theorem 3.3.** With reference to the model (6), $\gamma = 0$. with $w = u$ and $g'(t) + \omega g \leq 0$, $g'' > 0$ one obtains

- The exponential decay of the energy $F(t)$ takes place only if $A$ is bounded.
- However, one can show that

$$F(t) \leq \frac{c}{t+1}E_1(0), \ E_1(0) = \|Au\|^2 + \|Au_t\|^2 + \|A^{1/2}u_{tt}\|^2$$

The first part of Theorem has been proved in [31] while the negative result of the second part is proved in [13].

Motivated by the lack of exponential stability in the critical case with memory affecting the pressure only $u$, one asks a question what about more general form of the memory involving also $u_t$. In fact, in this case there is a positive result of exponential stabilizability which reads as follows.
Theorem 3.4. Consider (2), $\gamma = 0$, with $w = u_t + \frac{c^2}{b} u$. Under the same exponential characteristics of the kernel one obtains that $F(t)$ is exponentially stable. This is to say, there exists $\omega > 0$ such that

$$F_1(t) \leq C e^{-\omega t} F(0).$$

Remark 1. An interesting question is the following: what about stability of the system in critical regime by accounting for velocity memory depending only on $u_t$. Calculations reveal that this causes the critical parameter to become subcritical, however memory dissipation provides at the same time anti-damping effect. The system can be written like the original equation with subcritical $\gamma > 0$, however the memory term with $u$ only comes with a “wrong sign”. This suggest the following dichotomy: memory in $u_t$ shift the essential spectrum in the right direction, however it also affects point spectrum by moving eigenvalues to wrong parts of complex plane. This phenomena explains why a suitable calibration of the damping as in Theorem 3.3 leads to the desired results.

The case of general relaxation kernel—not only of exponential type has been also considered [30]. The standing assumption is the following

Assumption 3.5. $g \in C^1(R)$, monotone decreasing and such that $g'(t) + H(g) \leq 0$ for some convex, continuous, increasing function $H : R^+ \to R^+$ and $G(\infty) < c^2$.

Here are the main findings regarding this model. We begin by introducing the energy function which consists of two parts: acoustic and memory energy. $E(t) \equiv E_0(t) + E_m(t)$ where

- $E_0(t) = ||A^{1/2} u(t) + \frac{c^2}{b} A^{1/2} u(t)||^2 + \tau || u_t(t) + \frac{c^2}{b} u(t)||^2$
- $E_m(t) = g \circ A^{1/2} (u_t + \frac{c^2}{b} u)$
- $E(t) = E_0(t) + E_m(t) + ||A^{1/2} u||^2$

Theorem 3.6. Let’s assume that the relaxation kernel satisfies Assumption 3.5.

Then

- When $\gamma > 0$, the energy of (6) with $w = u$ satisfies:
  $$E(t) \leq S(t), t > 1$$
  where $S(t)$ satisfies the following ODE:
  $$S_t + \tilde{H}(S) = 0, S(0) = E(0)$$

- When $\gamma = 0$, $w = u_t + \frac{c^2}{b} u$, $G(\infty) < b$, the energy decays according to the same ODE law.

where $\tilde{H}(s) = CH(cs)$ for some intrinsic constants $c, C > 0$.

Remark 2. Note that $E(t)$ -on the dynamics is topologically equivalent to $H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$.

The proof of the above theorem can be found in [27] and [30]. It is interesting to note that the viscoelastic damping acting on the pressure $u$ is not sufficient to produce exponential decays (counterexample given in [13]). However, when viscoelasticity involves a combination of the pressure and its velocity $u, u_t$ then the decay is indeed exponential -in the case of exponential decay kernel [31] and more generally uniform reconstructing decays of the relaxation kernel [30].
We shall analyze next a similar problem within the context of the third order quasilinear system. This study has been undertaken in [27].

Below we present the main result on the existence of global solutions with the corresponding decay rates which hold under additional restrictions imposed on the relaxation kernel.

**Assumption 3.7.** $g'(s) + \omega_0 g(s) \leq 0$ for some positive constant $\omega_0 > 0$.

**Theorem 3.8.** [Global wellposedness] Consider equation given by (2). Under the Assumptions 3.5 and 3.7, there exist a constant $r > 0$ such that if the $U(0) = (u_0, u_1, u_2) \in B_{H^2}(r)$, then there exists unique global solution solving (2) and such that

$$U = (u, u_t, u_{tt}) \in C([0, \infty); H^2)$$

Moreover, such solution is exponentially decaying in the topology of both $H$ and $H^2$. This is to say $|U(t)|_H \leq C(r) e^{-\omega t}$ for $t > 0$, $U(0) \in B_H(r)$ and some $\omega > 0$. The same holds with $H$ replaced by $H^2$.

**Remark 3.** It may be interesting to analyze what happens when $\lambda \neq c_b^2$ where $w = u_t + \lambda u$. In order to gain some insight it is instructive to write

$$g * u_t = g(0)u(t) - g(t)u(0) + g' * u$$

(9)

Based on the formula above it is clear that in the critical case, memory of Type 2 moves the critical value $\gamma = 0$ to the noncritical value $\gamma_{new} = g(0)\tau b^{-1} > 0$. Thus, adding the Type 2 memory puts the problem in the noncritical regime. However, based on the formula (9), memory of Type 2 introduces $g' * Au$, which gives a perturbation with a wrong sign. This explains why Type 2 memory alone has no chances for stabilizing. However, adding Type 1 memory compensates for the convolution term resulting from Type 2 memory and leads to the stabilization. In fact, this “heuristics” has been confirmed quantitatively in [30].

Another interesting question to ask is what happens when both frictional (represented by $\alpha$) and viscoelastic damping is localized to a subset of the domain. More specifically, how the presence of viscoelasticity can shift the critical parameter $\gamma$.

The next section is devoted to this issue.

4. **Localized and degenerate memory damping.** Let us now consider the linear version of MGT equation with a localized damping

$$\tau u_{ttt} + \alpha(\cdot) u_{tt} - c^2 \Delta u - b \Delta u_t + \int_0^t g(t-s) div (a(\cdot) \nabla u(s)) \, ds = 0,$$

(10)

and initial data in the phase space $H$. The coefficients $a \in C^1(\overline{\Omega})$ and $\alpha \in L^\infty(\Omega)$ are both non-negative but they can be degenerate. Thus, the damping mechanism considered in previous sections is no longer effective. Our goal is to investigate this new situation and to determine suitable support of each damping in order to obtain the decay of the overall energy. In order to focus on the main difficulties associated with the degeneracy of the dissipation, we shall focus on the linear model only. However, nonlinear the results for versions of the same problem can also be obtained by incorporating the analysis of the present section with the previously developed nonlinear theory.
4.1. **Main result.** The following assumption describing quantitatively the degeneracy of the dissipation is introduced.

**Assumption 4.1.** Let \( \delta > \|a\|_\infty = M \) and assume that we have \( \alpha(x) > 0 \) almost everywhere in \( \Omega \) and

\[
a(x) + \alpha(x) \geq \delta \quad \text{a.e. in } \Omega;
\]

- The memory kernel \( g \in C^2(\mathbb{R}^+) \) satisfies
  - (i) \( g \) is convex and \( g(t) \geq 0 \);
  - (ii) \( \int_0^\infty g(t)dt < \frac{c_2}{\|a\|_\infty} \);
  - (iii) There exist \( c_1, c_2 > 0 \) such that \( g'(t) \leq -c_1 g(t) \) and \( g''(t) \leq c_2 g(t) \) for all \( t \geq 0 \);
- The parameters in the equation satisfy \( \frac{c_2 b}{\tau} < \delta - M \).

The first assumption describes the balance between viscoelastic and frictional damping. Degeneracy of one type of damping must be compensated by active support of the second type of dissipation. The second assumption is the usual one on memory kernels, which gives well-posedness and exponential characteristics for the decay rates. The third assumption is related to the dissipativeness of the system, analogue to the condition \( \gamma > 0 \) mentioned in the previous sections.

We introduce the usual energy function corresponding to memory models.

\[
E_a \sim \|u_t\|^2 + \|\nabla u_t\|^2 + \|\nabla u\|^2 - \int_\Omega a(g' \circ \nabla u)
\]

With the above functional one has exponential decay of \( H \) energy of the system.

**Theorem 4.2.** Under the Assumptions 4.1 we have

\[
E_a(t) \leq Ce^{-\omega t}E_a(0) = Ce^{-\omega t}\|U\|_H
\]

for some positive constants \( C, \omega > 0 \)

**Remark 4.** When the coefficients \( a(x) \) and \( \alpha(x) \) are constant, the above theorem reconstructs previous results obtained in subcritical regime. However, in the case of variable coefficients, it allows to trade off geometrically frictional damping against molecular damping. While it is known that viscoelastic damping cannot stabilize in the critical case, what is possible instead that frictional damping which is almost critical locally can be compensated by viscoelastic damping. This is an useful observation which is rigorously proved.

4.2. **Proof of Theorem 4.2.** Under the assumptions made we can prove the following results

**Lemma 4.3.** The weak solutions for (10) satisfy the energy identity

\[
\frac{dE_a}{dt}(t) + J_\theta(t) = 0,
\]
where $c^2/b < \theta < (\delta - M)/\tau$, $E_a \sim \|u_{tt}\|^2 + \|\nabla u_t\|^2 + \|\nabla u\|^2 - \int_{\Omega} a \,(g' \circ \nabla u)$ and

$$0 \leq J_\theta(t) := \int_{\Omega} (\alpha(x) - \tau \theta) \, |u_{tt}(t)|^2 \, dx + (\theta b - c^2) \|\nabla u_t(t)\|^2$$
$$+ \frac{1}{2} \int_{\Omega} a(x) \, \|g'' - \theta g'\| \, \nabla u(t) \, dx$$
$$+ \frac{1}{2} \int_{\Omega} a(x) \, |\theta g(t) - g'(t)| \, |\nabla u(t)|^2 \, dx.$$

Proof. We multiply the equation by $u_{tt}$ and $\theta u_t$ and collect the terms with derivatives to obtain the desired identity. The above procedure is carried out on sufficiently smooth solutions (corresponding to smooth initial data), and then extended by density to finite energy solutions.

Lemma 4.4. Let

$$R(t) := \tau(u_{tt}(t), u(t)) - \frac{\tau}{2} \|u_{tt}(t)\|^2 + (\alpha(\cdot)u_t(t), u(t)) + \frac{b}{2} \|\nabla u(t)\|^2$$

and

$$P(t) := \|u_{tt}\|^2 + \|\nabla u_t\|^2 + \|\nabla u\|^2 - \int_{\Omega} a \,(g' \circ \nabla u)$$

then $|R(t)| \leq M_2 P(t)$, $t > 0$ for some positive constant $M_2 > 0$.

Proof. The repeated use of the Cauchy-Schwarz, Poincaré and Young inequalities gives us the stated estimate. We also note that the convolution term in the definition of $P(t)$ is negative, making $- \int_{\Omega} a \,(g' \circ \nabla u)$ a positive term.

Define $F := R + \mathbf{N} \cdot E_a$, where $\mathbf{N}$ is a positive constant to be determined later.

Lemma 4.5. For $N$ sufficiently large, there exists a constant $C_N = C(N) > 0$ satisfying

$$\frac{dF}{dt}(t) \leq -C_N P(t).$$

Proof. Multiplying the original equation by $\upsilon_t$, integrating by parts and collecting the derivatives we obtain

$$\frac{d}{dt} \left\{ \tau(u_{tt}(t), u(t)) - \frac{\tau}{2} \|u_{tt}(t)\|^2 + (\alpha(\cdot)u_t(t), u(t)) + \frac{b}{2} \|\nabla u(t)\|^2 \right\}$$
$$- \int_{\Omega} \alpha(x) \|u_t(t)\|^2 \, dx + \int_{\Omega} k(t,x) \|\nabla u_t(t)\|^2 \, dx$$
$$+ \int_{\Omega} a(x) \nabla u(t) \cdot (g \circ \nabla u(t)) \, dx = 0. \quad (12)$$

Thus, for $L = c^2 - \|a\|_\infty \int_{\Omega} g(t) \, dt$,

$$\frac{dR}{dt}(t) = \int_{\Omega} \alpha(x) \|u_t(t)\|^2 \, dx - \int_{\Omega} k(t,x) \|\nabla u_t(t)\|^2 \, dx$$
$$- \int_{\Omega} a(x) \nabla u(t) \cdot (g \circ \nabla u(t)) \, dx$$
$$\leq \lambda \|a\|_\infty \|\nabla u_t(t)\|^2 - \int_{\Omega} k(t,x) \|\nabla u_t(t)\|^2 \, dx$$
$$+ \int_{\Omega} a(x) \nabla u(t) \cdot (g \circ \nabla u(t)) \, dx$$
Theorem 4.6. There exist positive constants $C$ and $\bar{C}$ such that

$$E_a(t) \leq C E_a(0) e^{-\bar{C} t}.$$  

Proof. By the Lemma 4.4, $|R(t)| \leq M_2 P(t)$. There exists $m_1 > 0$ such that, $P(t) \leq \frac{1}{m_1} E_a(t)$. Hence

$$-\frac{M_2}{m_1} E_a(t) \leq R(t) \leq \frac{M_2}{m_1} E_a(t).$$

Adding $NE_a(t)$ in the above inequality, we obtain that $F(t)$ is equivalent to $E_a$, as long as the constant $N > \frac{M_2}{m_1}$ satisfies Lemma 4.5.

Applying Lemma 4.5 we obtain

$$\frac{dF}{dt}(t) \leq -CP(t) \leq -CM_1 E_a(t) \leq -\bar{C} F(t).$$
Thus,
\[ F(t) \leq F(0)e^{-\bar{C}t} \]
Since \( F \) is topologically equivalent to \( E_a \), we conclude
\[ E_a(t) \leq CE_a(0)e^{-\bar{C}t}. \]

**Remark 5.** One could consider nonlinear version of the model in (10) under the assumptions introduced in 4.1. By combining methods of the present section with nonlinear analysis introduced earlier, one could establish global existence of nonlinear solutions subject to the smallness hypothesis imposed on the initial data. The details are somewhat technical, but the road map has been already established.

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Received August 2016; revised September 2016.

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