Quantum–classical crossover close to quantum critical point

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We analyze the quantum-classical crossover in the vicinity of the continuous quantum critical point (QCP) of a Boson system. The analysis is based on the Keldysh approach for the description of the non-equilibrium quantum dynamics. The critical behavior close to QCP has three different regimes (modes): adiabatic quantum mode (AQM), dissipative classical mode (CCDM) and dissipative quantum critical mode (QCDM). Crossover among these regimes is possible: it is shown that the experimentally observed changing of the critical exponents close to QCP is the dynamical effect accompanying the crossover from CCDM where the thermal fluctuations dominate to QCDM where the quantum fluctuations determine the critical behavior. In this case the effective dimension of the $d$-dimensional system continuously changes from $D_{eff} = d$ to $D_{eff} = d + 2$ while the universality class of the system does not change.

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Recently there has been considerable interest in the experimental and theoretical studies of the quantum phase transition dynamics. This interest is quite natural as quantum phase transitions are essentially dynamic [1]. In this case the time acts as an additional space dimension [2–9]. However, usually one considers only a simplified case assuming that in the vicinity of the critical point it is possible to distinguish two regimes: in one of them the energy of thermal fluctuations exceeds the energy of quantum fluctuations, $k_B T \gg \hbar \omega_T$ ($\omega_T$ is the quantity reciprocal to the relaxation time of the system, $\tau_T$), the critical mode being described by classical dynamics; in the other one the energy of thermal fluctuations becomes less than the energy of quantum fluctuations, $k_B T \ll \hbar \omega_T$, the system being described by quantum mechanics [1]. This description is not complete since it does not take into account the effect of dissipation in the quantum fluctuation regime, though it is well known that dissipation drastically change the critical properties [10–13]. It is clear that the system turns from the mode of dissipative dynamics of thermal fluctuations into the adiabatic mode of purely quantum fluctuations, then there should exist some intermediate dissipative mode of quantum fluctuations. The crossover between these critical modes has not been theoretically studied so far. It will be shown below that within a unified approach based on the Keldysh technique of non-equilibrium dynamics description, the crossover among all three critical modes in the vicinity of the quantum critical point will be described. The special attention will be devoted to the to the experimentally observable situation [14] when the crossover from the classical to the quantum criticality takes place with the corresponding continuous change of the critical indexes. Below we will show, that in this case the system universality class does not change. The modification of the critical indexes is the result of the change of the effective dimension of the dynamic system, which occurs at the transition from the thermal fluctuations to the quantum fluctuations.

To describe quantum critical dynamics theoretically, it is most convenient to use the Keldysh technique initially formulated for quantum systems. Let the system of our interest be the Boson system, whose state is described with the scalar field of the one-component order parameter $\phi$, and the potential energy is determined by the functional $U(\phi)$, e.g. $U \propto \phi^4$. Let us assume that $\hbar = 1$ and $k_B = 1$. In the static, to say more correctly, in the stationary, not quantum case the physics of the system is determined by the partition function:

$$Z = N \int \mathcal{D}\phi \exp \left[-S(\phi)\right],$$

where $\int \mathcal{D}\phi$ denotes the functional $\phi$-field integration, $S$ is the action that in the general form is as follows:

$$S(\phi) = \frac{1}{T} \int dk \left( \phi^4 G^{-1} \phi + U(\phi) \right),$$

where $T$ is temperature of the system, $U = v\phi^4$, $\Delta$ is the governing parameter, that tends to zero at the critical point.

There are different methods for the description of the transition from equilibrium statics to non-equilibrium dynamics. Note, that all of them result in doubling the fields describing the system, suggest the interaction of the system with the thermostat and are essentially equivalent to each other. As it has been mentioned above, we are going to use the Keldysh technique, which seems most convenient. In this case the role of the partition function is played by the functional path integral that after Wick rotation has the form [15]:

$$Z = N \int \mathcal{D}\phi^c \mathcal{D}\phi^q \exp \left[-S(\phi^c, \phi^q)\right],$$

$$S(\phi^c, \phi^q) = \int d\omega dk \left( \phi^c G^{-1} \phi^c + U(\phi^c + \phi^q) - U(\phi^c - \phi^q) \right), \quad \phi = \{\phi^q, \phi^c\},$$
where $\phi^q$ and $\phi^d$ are pair of fields called “quantum” and “classical” respectively. In the case of the Boson system the matrix of the inverse correlation functions is the following [15]:

$$
\hat{G}^{-1} = \begin{bmatrix}
0 & \omega^2 + \varepsilon_k + i\Gamma \omega \\
\omega^2 + \varepsilon_k - i\Gamma \omega & 2\Gamma \omega \coth(\omega/T)
\end{bmatrix}, \quad (1)
$$

where $\Gamma$ is the kinetic coefficient, and the function $f(\omega, T) = \coth(\omega/T)$ is the function of the density of states of the ideal Boson gas. The advancing, retard and Keldysh parts of both the correlation functions matrix and the inverse matrix are connected by the relation known as the fluctuation-dissipation theory (FDT):

$$
[\hat{G}^{-1}]^K = 2 \coth(\omega/T) \text{Im} \left([\hat{G}^{-1}]^A\right).
$$

The expressions given above allow us to describe the critical dynamics of the system theoretically in the vicinity of the critical point. They are general and allow the system to be described both within the classical, $T \gg \omega$, and the quantum, $\omega \gg T$, limits.

Let us consider the first region, where thermal fluctuations dominate, $\omega \ll T$. Note that the plane $\omega = 0$ is entirely located in this region. The critical dynamics of the system is determined by the Keldysh element of the matrix of Green functions, $[\hat{G}^{-1}]^K = 2\Gamma \omega \coth(\omega/T)$. Within $T \gg \omega$ this function tends to $\lim_{T \gg \omega} [\hat{G}^{-1}]^K \approx 2\Gamma T$.

Note that in this case the effect of the thermostat on the system (the action of the statistical ensemble on its own element) corresponds to the influence of the external “white” noise. The fluctuations with the smallest wave vectors and energies ($k \to 0$, $\omega \to 0$) are considered to be relevant (significant) in the vicinity of the critical point, hence only the terms with the lowest degrees $k$ and $\omega$ are retained in the expressions. As a result, in the fluctuation field the system is described by the standard classical non-equilibrium dynamics:

$$
\hat{G}^{-1} = \begin{bmatrix}
0 & \varepsilon_k + i\Gamma \omega \\
\varepsilon_k - i\Gamma \omega & 2\Gamma T
\end{bmatrix},
$$

satisfying the standard form of FDT:

$$
[\hat{G}^{-1}]^K = (T/\omega) \text{Im} \left([\hat{G}^{-1}]^A\right).
$$

The dispersion relation in this case is: $\omega \propto k^2$, whence it follows that the dynamic critical exponent of the theory (scaling dimension) in the first approximation will be: $z = 2$. The dimension of the system is: $D = d + z = d + 2$, but due to the presence of “white” noise the effective dimension of the system is: $D_{\text{eff}} = D - 2 = d + 1$. Naturally, it results in the coincidence of the critical dimensions of the dynamic and static theories, the critical behavior of the system being described by the classical critical dynamics of the $d$-dimensional system. Let us refer to this mode as the mode of the classical critical dynamics (CCDM) ($T \gg \omega, \Gamma T \gg |\Delta|$).

Now let us consider the case when the quantum fluctuations dominate, $\omega \gg T$. In this case [17] has the form:

$$
\hat{G}^{-1} \approx \begin{bmatrix}
0 & \varepsilon_k + i\Gamma \omega \\
\varepsilon_k - i\Gamma \omega & 2\Gamma T
\end{bmatrix},
$$

so the system “does not know” that it has got temperature. In spite of the absence of thermal fluctuation in the quantum case FDT still exists and has the following form:

$$
[\hat{G}^{-1}]^K = 2 \text{sign}(\omega) \text{Im} \left([\hat{G}^{-1}]^A\right),
$$

and the action of the statistic ensemble on the system does not depend on the temperature. Note, that close to the phase transition ($\Delta \approx 0$), when $\varepsilon_k \to 0$, we get $G^K(\omega) = 2/|\Gamma| |\omega|$. This is the so called 1/f-noise (Flicker noise), whose intensity does not depend on the temperature. The latter significantly changes the critical properties of the system. As in the case of classical critical dynamics the dimension in this case is $D = d + 2$. However, the 1/f noise in contrast to the “white”-noise, does not decrease the effective dimension [17], therefore the effective dimension of the dissipative quantum system is greater by 2 than its static dimension, $D_{\text{eff}} = d + 2$. The disagreement of the static and dynamic theories is accounted for by the fact that in the quantum case there is no statistic limit, and the only correct results are those of the dynamic theory. This dynamic mode can be referred to as the mode of the quantum critical dynamics (QCDM) ($T \ll \omega, \Gamma \omega \gg |\Delta|$).

With $\omega \gg T$ also the case is possible when the coherence time of the system appears much shorter than the inverse frequency of quantum fluctuations, $\Gamma \omega \ll |\Delta|$, the dynamics of the system changes into an adiabatic mode, in which the dissipation can be neglected, $\Gamma \to 0$, thus:

$$
\hat{G}^{-1} \approx \begin{bmatrix}
0 & \omega^2 + \varepsilon_k \\
\omega^2 + \varepsilon_k & 0
\end{bmatrix}, \quad (2)
$$

the dispersion relation takes the form $\omega \propto k$, and as a result, $z = 1$. In this region the critical behavior is described as the critical behavior of the static system with the dimension $D_{\text{eff}} = d + 1$. It is easy to see that this is the region in which the Matsubara formalism works. Also one can see that in this case the critical behavior of a three-dimension system has the simple description within the mean field theory (Ginzburg–Landau theory), since the effective dimension is equal to the critical dimension, $D_{\text{eff}} = d_+ = 4$. This regime can be referred to the adiabatic quantum mechanical mode (AQM) ($T \ll \omega, \Gamma \omega \ll |\Delta|$).

The schematic picture of the different critical regions is shown in Fig. 4: the surfaces indicate the regions of the crossover between the critical modes. Let us consider separately the region of “crossing” of all modes, which is in the vicinity of the plane $\omega = T$. Here the thermal and quantum fluctuations are equal, thus this area is the region of crossover between classical and quantum dynamic
The crossover from CCDM to QCDM can be observed experimentally \[14\]. For example, the experimental dependence of critical index $\beta$ on the temperature is shown in Fig. 2 \[14\]. At the relatively high temperatures, $T/\omega \gg 1$, this exponent corresponds to the value characteristic for the three-dimensional classic system. However, at small temperatures, $T/\omega \ll 1$, it becomes equivalent to the corresponding critical exponent of mean field theory, since the effective dimension of the system becomes greater than the critical dimension: $D_{\text{eff}} = d + z \geq d^\ast$.

At first sight this dependence of the critical exponents on the temperature seems strange, because the critical exponents are dependent on the system universality class, which can not continuously change with temperature. However, in the critical dynamics the critical exponents do not only depend on the universality class, but also on the nature of the critical fluctuations. Therefore, the reason of this continuous change of the critical exponents is the crossover from the thermal fluctuations mode to the quantum fluctuations mode with the corresponding increase of the effective dimension, while the system's universality class does not change.

One can show that this crossover is driven by the temperature dependence of the density of states. The point is that all diagrams giving the main contribution in the renormalization theory, contain the loop consisting of one retarded (or advanced) Green function, and one Keldysh Green function \[17\]. In the long-wavelength limit these contributions have the form: \[
\sim \int 2\Gamma \omega f(\omega, T) / \omega^3 d\omega^{1+2\nu/2},
\] where $\Gamma$ is dimensionless. Therefore the integral in \[3\] logarithmically diverges when $\omega f(\omega, T) \sim \omega^{2-d/2}$. There is no longer any difference from the usual renormalization procedure. If the function $\omega f(\omega)$ would be the power function, $\omega f(\omega) = \omega^\lambda$, then the problem is reduced to the usual problem of the investigation of the critical behavior of $d + 2\lambda$-dimensional system.

Unfortunately the function $\omega f(\omega, T) = \omega \coth(\omega/T)$ is the complicated non power one. This fact seemingly does usless our usual reasoning for deriving of the temperature dependence of the critical exponents in analytical form. However, the slope tangent to this function continuously changes in the bounded limits: from 0 at $\omega/T \ll 1$ to 1 at $\omega/T \gg 1$ \[17\]. Since in the experiment $\omega/T$ is controlled by the temperature and the characteristic energy of the quantum fluctuations, $\omega_0$, then for every value $\omega_0/T$ the function of density of states can be approximated by the power function $\omega f(\omega, T) \approx \omega^\lambda(\omega_0/T)$, where $\Lambda(\omega) = x\theta \ln(x^d/\omega) / \partial_\omega$ ($0 \leq \lambda \leq 1$), and $\omega_0$ is the characteristic frequency of the system, which depends the quantum fluctuation energy. Thus, the exponent $\Lambda$ is the only function of temperature $\Lambda(\omega_0/T) = (\coth(\omega_0/T) - x \cosh^2(\omega_0/T)) \tanh(\omega_0/T)$.

The above approximation allows us to use the conventional renormalization procedure for the calculation of the critical indexes. If at the temperature $T'$ the function of density of states can be approximated by some power function $\omega f(\omega, T') \sim \omega^{\Lambda(\omega)}$, then the critical behavior of the system is identical to the classical (non-quantum) critical behavior of the $(d + 2\Lambda)$-dimensional system.

In order to estimate the temperature dependence of the critical exponent $\beta$ we can use the well known relations for the critical exponents: $\alpha + 2\beta + \gamma = 2$, $d \cdot \nu = 2 - \alpha$, $\gamma = \nu \cdot (2 - \eta)$. These exponents characterize the heat capacity, $C_v \sim |\Delta|^{-\alpha}$, susceptibility, $\chi \sim |\Delta|^{-\gamma}$, magnetization, $\langle \phi \rangle \sim |\Delta|^{\nu}$, correlation radius, $r_c \sim |\Delta|^{\nu}$, and Green function, $G(\nu) \sim r^{-d+2-\eta}$ ($\eta$ is the anomalous dimension index). According to above,
in the case of the crossover from CCDM to QCDM, when \( d \to d' = d + 2\Lambda \), relations for the critical exponents are valid for efficient exponents: \( \nu \to \nu'(\Lambda) \), \( \eta \to \eta'(\Lambda) \), \( \beta \to \beta'(\Lambda) \), \( \gamma \to \gamma'(\Lambda) \), \( \alpha \to \alpha'(\Lambda) \).

First dependence can be derived from the well known renormalization group equations (in the one-loop approximation) [18]:

\[
\frac{d \ln |\Delta|}{d \xi} = 2 - \frac{\nu'}{8\pi^2}, \quad \frac{d \ln \nu}{d \xi} = \frac{\nu'}{8\pi^2} - \epsilon,
\]

where \( \xi \) is the regularization parameter, \( \epsilon = (4-d')/2 = (4-d)/2 - \Lambda = \nu - \Lambda \). From the fixing condition for \( \nu \), \( d/\xi = 0 \), and definition \( \nu' \equiv (d \ln |\Delta|/d\xi)^{-1} \) one can get: \( \nu' = 1/(2 - \epsilon' + 3) = 1/(\nu^{-1} + (2 - \nu^{-1})\Lambda) \). In the three-dimension case \( \nu \to 1 \), and \( \nu \approx 0.642 \) [18]. Because \( \eta \approx 0 \) for all dimensions, then we have \( \gamma' \approx 2\nu' \), and \( (d+2\Lambda)\nu' = 2(\beta' + \nu') \). As a result \( \beta' = (d/2 + \Lambda - 1)\nu' \). Using these formulas and the dependence \( \Lambda(\omega_0/T) \) we can calculate functions \( \beta'(T) \), \( \nu'(T) \), \( \gamma'(T) \), and \( \alpha'(T) \) (Fig. 3). One can see that \( \beta'(T) \) dependence is in good qualitative agreement with experimental data [14].

From above one can see that the critical behavior in the vicinity of the quantum critical point is multi-critical. The functional technique of theoretical description of non-equilibrium dynamics allows us to describe the entire spectrum of critical modes in the vicinity of quantum phase transition within a single formalism. In particular, it describes the crossover between CCDM and QCDM, and the unusual temperature dependence of the system critical exponents. Note that in this case the system universality class does not change. The continuous change of the critical exponents is the dynamical effect, which is caused by the crossover from the thermal fluctuation mode to the quantum fluctuation mode.

![Figure 3: Theoretical dependence of critical exponent \( \beta' \) on the \( T/\omega_0 \) ratio for the \( \phi^4 \)-model.](Image)

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