Finite-size scaling analysis of localization transitions in the disordered two-dimensional Bose-Hubbard model within the fluctuation operator expansion method

Andreas Geißler\textsuperscript{1,2,}\textsuperscript{*}
\textsuperscript{1} ISIS, University of Strasbourg and CNRS, 67000 Strasbourg, France
\textsuperscript{2} School of Physics \& Astronomy, University of Nottingham, NG7 2RD Nottingham, UK
(Dated: November 23, 2020)

The disordered Bose-Hubbard model in two dimensions at non-integer filling admits a superfluid to Bose-glass transition at weak disorder. Far less understood are the properties of this system at strong disorder and energy density far from the ground state. In this work we put the Bose-glass transition of the ground state in relation to a finite energy localization transition, the mobility edge of its quasiparticle spectrum, which is a critical energy separating extended from localized excitations. We use the fluctuation operator expansion, which also considers effects of many-body entanglement. The level spacing statistics of the quasiparticle excitations, the fractal dimension and decay of the corresponding wavefunctions are consistent with a many-body mobility edge, while the finite-size scaling of the lowest gaps yields a correction to the mean-field prediction of the superfluid to Bose-glass transition. In its vicinity we further discuss spectral properties of the ground state in terms of the dynamic structure factor and the spectral function which also shows distinct behavior above and below the mobility edge.

PACS numbers: 67.85.De, 03.75.Lm, 05.30.Jp, 63.20.Pw
Keywords: Many-body localization, Bose glass, Bose-Hubbard model, two dimensions

I. INTRODUCTION

The inclusion of local disorder in the Bose-Hubbard model is able to induce a low temperature superfluid to insulator transition at arbitrary filling. The resulting Bose glass (BG) phase is distinct from the Mott phase at integer filling but has a vanishing gap similar to the superfluid (SF) \cite{1-5}. Numerous works have given numerical evidence \cite{6-10} and analytical results \cite{3, 4, 11-16} showing its existence. In addition the BG phase has been probed experimentally in one \cite{17} and three dimensional \cite{18} cold atom setups with an optical lattice as well as for bosonic quasiparticles in a doped quantum magnet \cite{19}. In two dimensions the scaling properties at criticality have been studied extensively in the Bose-Hubbard model \cite{20} and its hard-core boson limit \cite{21, 23} using a wide range of advanced numerical tools, with results comparing mostly quite well with earlier analytical predictions \cite{22, 24, 25}.

In a recent work we have studied the related phenomenon of localized excitations finding that disorder induces mobility-edges for all values of the local interaction in the full excitation spectrum of a disordered Bose-Hubbard model \cite{25}. Such a system is a candidate for the study of so-called many-body localization (MBL) \cite{20, 29}, a topic which has received increased interest in real years with exciting connections to the fields of topological states \cite{30, 31} or quantum computing \cite{32, 33} to name a few \cite{34, 35}. One of its most renown features is its incompatibility with the eigenstate thermalization hypothesis resulting from an extensive number of local integrals of motion \cite{36, 38}. A complete demonstration of MBL in principle requires complete knowledge of the spectrum, limiting exact diagonalization based analyses to small system sizes \cite{39, 40}. Numerous perturbative arguments \cite{26, 27, 41, 42} and increasing numerical evidence \cite{28, 29, 43} have supported its existence in two dimensions, involving a mobility edge (ME) separating mobile from localized states in the spectrum. Due to its unconstrained local basis, bosonic lattice systems have turned out to be especially hard for numerical simulations, limiting most works to small scale one-dimensional \cite{39, 44} and two-dimensional systems with a constrained local basis \cite{40}, although strong arguments have been put forward in favour of an MBL transition in a disordered continuum system of ultracold bosonic particles in two spatial dimensions \cite{45, 46}, even as a function of temperature consistent with a ME. Nevertheless, despite a rigorous proof for certain one-dimensional spin-chains \cite{47, 48}, recent numerical works have challenged the possibility of a thermal phase transition for two dimensional systems \cite{49, 51} and even argued for the absence of a proper localization-delocalization transition in the thermodynamic limit for a one-dimensional spin-chain \cite{52} sparking some counter arguments in \cite{53}. Also, it has been argued recently that the necessary length- and timescales that have to be reached to uniquely identify MBL-type localization are currently out of reach both experimentally and theoretically \cite{54}. Nevertheless, some experimental realizations have already shown strong signs of localization in cold atom setups, where a disorder potential can be imprinted onto the optical lattice in one \cite{55, 56} and two dimensions \cite{57, 58}.

\textsuperscript{*} andreas.geissler87@gmail.com
In this work we analyze the quantum phases of the disordered two-dimensional Bose-Hubbard model (BHM) in order to determine the critical scaling of its mean-field ground state SF to BG transition. While we have already discussed the critical scaling at the ME in [23], we further focus on the low energy excitations to put MBL and quantum glass phenomenology in a direct relation [59, 60]. In second quantization the grand canonical BHM with disorder using $\hbar = 1$ can be written as

$$\hat{H} = \sum_{\ell} \left( \mu_{\ell} \hat{b}_{\ell} \hat{b}_{\ell}^\dagger + \frac{U}{2} \hat{b}_{\ell} \hat{b}_{\ell}^\dagger \hat{b}_{\ell} \hat{b}_{\ell}^\dagger \right) - t \sum_{\langle \ell, \ell' \rangle} (\hat{b}_{\ell} \hat{b}_{\ell'} + \text{h.c.})$$ \hspace{1cm} (1)

with $\mu_{\ell} = -\mu + \epsilon_{\ell}$ given by the local potential $\mu$ and the random potential $\epsilon_{\ell}$, while $U$ and $t$ are the Bose-Hubbard interaction and tunneling rate, respectively. We always choose $\mu$ such that the mean occupation number $n = \langle n_{\ell} \rangle_{d} = 0.5$ where $\langle \cdot \rangle_{d}$ is the disorder average and $n_{\ell} = \langle \hat{b}_{\ell}^\dagger \hat{b}_{\ell} \rangle$. For $\epsilon_{\ell}$ we assume a Gaussian distribution $P(\epsilon_{\ell}) = (2\pi W^2)^{-1/2} \exp\left(-\frac{\epsilon_{\ell}^2}{2W^2}\right)$ as has been realized in recent experiments [52–55] with $W$ its standard deviation. This describes a homogeneous system insofar as $\langle \mu_{\ell} \rangle_{d} = -\mu$. We furthermore consider a simple $L \times L$ square lattice with spacing $a$ and periodic boundary conditions.

In previous works the ground state of (1) has been investigated for the hard-core limit $U \rightarrow \infty$ [21–24] analyzing the SF to BG transition. Regarding the regime of moderate interaction strength, it has been shown that due to disorder there is no direct SF to Mott insulator phase transition [8–10] at unit filling, which in the ground state instead happens via an intermediate BG phase. For non-integer filling (or small $U$) only the latter is directly accessible on the mean-field level.

Here, we evaluate mean-field and quasiparticle spectral properties of the disordered BHM [1] in terms of the fluctuation operator expansion (FOE) method [23–25] a beyond mean-field quasiparticle expansion method. For all disorder strengths we find a critical point in the ground state of moderate interaction strength, it has been shown that due to disorder there is no direct SF to Mott insulator phase transition [20–24] an analyzing the SF to BG transition. Regarding the regime of moderate interaction strength, it has been shown that due to disorder there is no direct SF to Mott insulator phase transition [8–10] at unit filling, which in the ground state instead happens via an intermediate BG phase. For non-integer filling (or small $U$) only the latter is directly accessible on the mean-field level. Considering the fractal dimension of an inhomogeneous Gutzwiller-type mean-field representation of the ground state wave-function [4, 64, 65] we find finite-size scaling exponents that match surprisingly well with earlier (analytical) predictions [3, 4, 11] in contrast to results from more advanced numerical simulations [23, 24]. As the FOE method gives access to the complete spectrum of quasiparticle (QP)s, we use it to discuss spectral properties of experimental interest by considering the beyond mean-field QP ground state. We note that all QP excitations tend to resemble approximate local integrals of motion (LIOM) for sufficiently strong disorder.

The remainder of this work is structured in four main sections and a summary. First we determine the mean-field ground state of the disordered BHM in order to characterize the SF to BG transition in terms of the Edwards-Anderson parameter and the fractal dimension in Sec. [11]. In particular we determine the finite-size scaling collapse for the fractal dimension of the mean-field (MF) ground state. Next, we go through a detailed discussion of the full quasiparticle spectrum in Sec. [111]. To do so we first discuss the energy level statistics and localization properties of the fluctuation wave-functions in order to discern localized and non-local states separated by a ME. By considering a simple finite-size scaling ansatz we further establish a relation between the lowest excited states and the SF to BG transition in the ground state. In the following Sec. [11V] we then focus on spectral properties of the FOE’s quasiparticle ground state in the vicinity of the phase transition which nicely reflect the phenomenology discussed in the previous sections. For a proper understanding, in the last section we detail the FOE method giving access to the discussed quasiparticle spectrum beyond the weak-coupling ansatz of the Bogoliubov method. In Sec. [11V B] we also discuss a numerical test of its applicability for the disordered BHM [1] we find finite-size scaling exponents that match surprisingly well with earlier (analytical) predictions [3, 4, 11] in contrast to results from more advanced numerical simulations [23, 24].

II. MEAN-FIELD CRITICAL POINT

We start by characterizing the ground state properties of (1), specifically in relation to the aforementioned occurrence of a Bose-glass phase [20–24] in the low temperature regime. Here, we consider a simple Gutzwiller MF product ansatz of the form $|\psi_{\text{MF}}\rangle = \prod_{\ell} |\psi_{0\ell}\rangle_{\ell}$ where each $|\psi_{0\ell}\rangle$ is given in terms of a linear combination over the local Fock-basis truncated at some fixed number $N$. Their, in general, complex amplitudes can either be found via a minimization of the energy or a self-consistent procedure (see Sec. [11VA]). On this mean-field level we focus on two observables to characterize the occurrence of a transition point in the ground state phase for an increasing disorder potential, where a SF to BG transition is expected. We note that the superfluid fraction is expected to vanish at this transition while the condensate fraction is not. Thus, one has to consider another witness of the transition, as only the latter is directly accessible on the mean-field level. Therefore we define an Edwards-Anderson-type order parameter

$$q_{\text{EA}} = \frac{1}{L^2} \sum_{\ell} \left( \langle n_{\ell} \rangle_{d} - \langle n_{\ell} \rangle_{d} \langle n_{\ell} \rangle_{d} \right) \hspace{1cm} (2)$$

with $n_{\ell} = \langle \hat{n}_{\ell} \rangle$ the expectation value of the local boson number density. By construction it is always zero in a homogeneous state and only non-zero if the correlations between the density and the disorder are extensive [60].

A. System

B. Overview
Furthermore, we consider the fractal dimension $D_\phi$ of the condensate wave-function $\phi_\ell = \langle \hat{b}_\ell \rangle$, for which we use the definition [69, 70]

$$D_\phi = \left( \log_L \left( \frac{\sum_\ell L^2 |\phi_\ell|^2}{\max_\ell |\phi_\ell|^2} \right) \right)_d. \quad (3)$$

We evaluate both characteristics over a range of parameters $U/t \in [1, 25]$ and $W/t \in [0, 15]$, and for the linear system sizes $L \in \mathcal{L} = \{10, 20, 24, 32, 40\}$ while averaging over $N_r = 60$ disorder realizations each time.

As an example we show $q_{EA}$ and $D_\phi$ for $U/t = 20$ in Fig. 1. While $q_{EA}$ in panel (a) is almost independent from the considered system sizes, it also barely exhibits any extremal behavior except for the soft kink at $W/t \approx 5$ visible in the numerical derivative $\Delta q_{EA}/\Delta W$ [inset Fig. 1(a)]. Still, a nonzero value of $q_{EA}$ indicates the occurrence a glassy ground state for increasing disorder. The fractal dimension $D_\phi$ in panel (b), on the other hand, features a much more pronounced drop in the vicinity of the critical point. In or-

FIG. 1. Characterization of the MF critical point for the 2D BHM with disorder. The Edwards-Anderson parameter $q_{EA}$ (a) and the fractal dimension $D_\phi$ (b) are shown together with their respective numerical derivatives in the insets. Both are given as a function of the disorder $W$ for fixed interaction $U/t = 20$ and various system sizes (see legend).

FIG. 2. Finite-size scaling and critical points for the MF ground state of the disordered 2D BHM. (a) shows the shift of $D_0$ at the inflection point as function of $L$ for $\alpha = 0.44$. Grey colors correspond to $L$ as in Fig. 1(b) and lines are best fits to Eq. 5 used to determine $D_0$ in the limit $L \to \infty$. Corresponding best infinite system size predictions for $D_\phi$ (left pointing triangles) and $W_c(U)$ (right pointing triangles) as determined by the crossing point in the partial collapse [inset (c)] is shown in (b) for fixed values of $\alpha$ and $U/t = 20$. The adjusted parameter of convergence of these fits is given in the inset of (b). (c) shows the scaling collapse of $D_\phi$ as a function of the rescaled disorder (unscaled in the inset) for $W_c/t = 2.9$ and $U/t = 20$. (d) depicts the MF critical disorder $W_c$ determined via the collapse of $D_\phi$ according to Eq. 4 and a corresponding collapse for $U/t = 3$ in the inset.

A. Finite-size scaling

Such a finite-size shift indicates a critical point with a scaling that is typically of the form [24]

$$D_{\phi,L,U}(W) - D_c = L^{-\alpha} \tilde{D}_U \left( |W - W_c(U)| L^{1/\nu} \right), \quad (4)$$

with the critical fractal dimension $D_c$, the critical disorder $W_c(U)$, a universal function $\tilde{D}_U$ with parameter $U$, as well as the critical exponents $\alpha$ and $\nu$. For the scaling collapse of the inflection points $D_{0,L,U}(L)$ onto the inflection point of the scaling function $D_U(W_0)$, where $W_0 = |W_0(L) - W_c| L^{1/\nu}$ is the rescaled disorder, we thus expect

$$D_{0,L,U}(L) = \frac{\tilde{D}_U(W_0)}{L^\alpha} + D_c. \quad (5)$$

As this expression has three unknown parameters, compared to the five system sizes $L \in \mathcal{L} = \{10, 20, 24, 32, 40\}$ considered for each value of $U$, we first determine the best
fit parameters \( \tilde{D}_U(\tilde{W}_0) \) and \( D_c \) for fixed values of \( \alpha \) and \( U/t \in \{15, 20, 25\} \) to obtain the functional relation \( D_c(\alpha) \) shown in Fig. 2(b), while exemplary fits for \( \alpha = 0.44 \) are shown in Fig. 2(a). By definition \( D_c \) is limited from above so the collapse of the inflection points gives a lower bound \( \alpha > 0.4 \) [see Fig. 2(b)]. As the finite-size scaling Eq. 4 is independent of the scaling exponent \( \nu \) at the critical point \( W_c(U) \), we can further determine \( W_c(U) \) if we scale only the fractal dimension according to \((D_\phi - D_c)L^\alpha\) to obtain the crossing point of all system sizes, as shown in the inset of Fig. 2(c). This way we get the best candidates for the critical point \((W_c(U), D_c)\) as a function of \( \alpha \), exemplarily depicted in Fig. 2(b) for \( U/t = 20 \). To quantify the goodness of these fits we consider the adjusted coefficient of determination \( R^2 \) given in the inset of Fig. 2(b) with errorbars representing the standard deviation when sampling over \( U/t \in \{15, 20, 25\} \) and six distinct subsets of 10 disorder realizations each. The value of \( R^2 \) for these fits is almost constantly at its optimum for the considered range of \( \alpha \).

For the full collapse we only have to consider \( \alpha \) and \( \nu \) in order to minimize the mean relative variance as a measure for the goodness of the collapse:

\[
\chi_{D_\phi} = \sum_U \frac{\chi_{D_\phi}(U)}{N_U} = \sum_U \sum_{L' > L} \sum_{\bar{W}} \frac{(D_{\phi;L,U}(\bar{W}) - D_{\phi;L',U}(\bar{W}))^2}{\sigma D_{\phi;L,U}(\bar{W}) + \sigma D_{\phi;L',U}(\bar{W})} \cdot \frac{1}{C_{D_\phi}}
\]

Here, \( \sigma D_{\phi;L,U}(\bar{W}) \) are the standard errors of the mean determined from the disorder sampling while the normalization constant \( C_{D_\phi} \) is given by the total number of terms, \( C_{D_\phi} = N_U \sum_{L' > L} \sum_{\bar{W}} 1 \) with \( N_U \) the number of considered interaction values. For an ideal collapse this measure should be on the order of 1. In order to estimate the error of the obtained scaling exponents this finite-size scaling procedure is repeated for 6 independent subsets of 10 disorder realizations each, while the interaction sum takes into account all considered values \( U/t \in \{1, 3, 5, 10, 15, 20, 25\} \). The free parameters of this collapse are \( \nu \) and \( \alpha \), the latter of which implicitly determines \( D_c(\alpha) \) via the scaling of the inflection points [see Figs. 2(a, b)] as well as \( W_c(U) \) via the unique crossing point of the rescaled fractal dimension [as in inset Fig. 2(c)].

B. Results

An exemplary collapse for \( U/t = 20 \) is given in Fig. 2(c) which has the individual relative variance \( \chi_{D_\phi}(U = 20t) = 0.44(15) \). In combination the mean relative variance Eq. 6 for all interaction values together is \( \chi_{D_\phi} = 2.8(5) \). It is greater then one primarily due to substantial finite-size corrections far from the critical point at weak interaction resulting in \( \chi_{D_\phi}(U = 3t) = 11(2) \) [inset of Fig. 2(d)]. For all best collapses taken together we find the scaling exponents

\[
\alpha = 0.44(2), \quad \nu = 2.0(2)
\]

and a critical fractal dimension \( D_\phi^c/t = 1.97(3) \) indistinguishable from its upper limit. The corresponding critical line \( W_c(U) \) is depicted in Fig. 2(d).

To summarize, for weak interaction \( U/t \lesssim 10 \) the critical disorder strength is close to zero. At strong interaction values \( U/t \gtrsim 20 \), on the other hand, we find a ground state transition point that is consistent with previous predictions of a superfluid to Bose-glass transition also at half-filling but in the hard-core boson limit \( U/t \to \infty \) with box-disorder \( \epsilon_s \in [-W,W] \) for the local potential [21–24]. Additionally, considering earlier results for this system [3, 4, 11] and the nonzero \( g_{\text{E}}A \) for \( W > W_c(U) \) we associate this critical line with a SF to BG transition. Notably, the MF scaling exponents we find match some early Monte-Carlo predictions surprisingly well [21].

III. Characterization of the QP Spectrum

In this section we extend our discussion beyond the ground state by considering and characterizing general QP fluctuations obtained within the FOE method discussed in Sec. V. On the one hand, we analyze the distribution of the QP energy levels and their gap statistics. On the other hand, we specifically discuss the exponential localization of the FOE wave-functions associated with the excitations. Both aspects can be summarized in terms of two simple and fundamentally different measures related to localization. These are (i) the QP energy level spacing ratio \( r \in [0, 1] \) and (ii) the multi-fractality \( D \in [0, 1] \) for the second moment of the QP fluctuation wave functions. They reveal and allow for an independent characterization of the many-body ME as discussed in detail in [25].

a. Superfluid vs. Bose-glass gap scaling Before we discuss the QP spectrum we first take into account the lowest QP excitations only, in order to discuss their relation to the ground state. To do so we consider the \( n \) lowest QP excitations \( \omega_n \) with \( \gamma \leq n \). We note that for the local basis truncation \( N \to \infty \) (see Sec. VA) in the presence of a MF condensate follows \( \omega_n \to 0 \) in which case this mode actually has to be represented by
the momentum-like operator $P$, as discussed in Sec. [\ref{sec:V.C}]
Thus the lowest relevant average $n$-gaps are given by $\Delta \omega_n = \langle \omega_n - \omega_0 \rangle_d$. In a superfluid it is well known that the lowest energy excitations are Goldstone modes following a linear dispersion relation. Irrespective of the spatial dimension the smallest possible lattice momenta on an isotropic lattice have $|k_{\text{min}}| = \pi/La, \sqrt{2}\pi/La, \ldots$ implying $\Delta \omega_n \propto 1/L$ for sufficiently small $\gamma$. In contrast, for strong disorder excitations are expected to be increasingly uncorrelated such that the average level spacing becomes inversely proportional to the total number of levels. Therefore, we expect $\Delta \omega_n \propto 1/L^2$ for sufficiently strong disorder as the number of QP modes within the FOE is proportional to the number $L^2$ of lattice sites. As we are only interested in the scaling with $L$ it is numerically beneficial to consider the average of the 8 lowest gaps $\Delta \overline{\omega} \equiv \frac{1}{8} \sum_{n=1}^{8} \Delta \omega_n$ corresponding to the longest wavelength modes $|k| \in \{ \pi/La, \sqrt{2}\pi/La \}$ of the superfluid. For this average we assume the following generic scaling,
\begin{equation}
\Delta \overline{\omega} = \frac{\omega_l}{L^p} + \omega_{\text{off}} \label{eq:scaling}
\end{equation}
Here, the first term represents the system size scaling with some power $p$ and an effective local single-site gap $\omega_l$ while $\omega_{\text{off}}$ is an offset energy. These parameters are determined via fitting. We perform this scaling for $U/t = 20$, $W/t \in [0, 40]$ and $N \in \{3, 4, 5\}$ corresponding to light grey, dark grey and black in Fig. \ref{fig:scaling}(a). Exemplary data for $W/t \in [0, 10, 20]$ is shown in Fig. \ref{fig:scaling}(a) with errors of the mean from the disorder sampling. The obtained values for $\omega_{\text{off}}$, $p$ and $\omega_l$ are given in Figs. \ref{fig:scaling}(b, c), while the corresponding adjusted coefficient of determination $R^2$ is shown in Fig. \ref{fig:scaling}(d). All fits are nearly exact with an adjusted parameter of convergence $R^2 \approx 1$.
Regarding $p$, a truncation $N > 3$ is sufficient to determine this scaling exponent in the vicinity of the SF to BG transition for the considered system sizes (see Fig. \ref{fig:scaling}, although one has to be careful for disorder $W/t > 20$ [23]. Just as expected we find $p = 1$ for sufficiently weak disorder consistent with a SF phase while the exponent increases approximately linearly beyond 1 above a critical disorder $W_c, \Delta \omega_l$. Linear fits (dashed lines) in Fig. \ref{fig:scaling}(b) cross $p = 1$ at $W_c, \Delta \omega_l = 5.98(8), 7.5(2), 7.7(7)$ corresponding to $N = 3, 4, 5$, respectively, and thus well above the MF result. Regarding the effective single site gap $\omega_l \approx U$ for sufficiently weak disorder, as one would expect in the single site limit. Notably, in the opposite limit at strong disorder $W > 20t$ ($W > U$) we find best fits with $p > 2$ and $\omega_l \gg U, W$ which also have the lowest fit quality [see Figs. \ref{fig:scaling}(b), (c) and (d)]. As such a runaway effective local gap seems unphysical, we also assume a fixed value $p = 2$ as discussed earlier for $W \geq 20t$. We then find fits of nearly identical quality [dashed lines in Fig. \ref{fig:scaling}(d)] but with much lower effective local gaps $\omega_l$ [dashed lines in Fig. \ref{fig:scaling}(c)].
In conclusion we find a finite-size scaling of lowest excitations consistent with a dissolving spectrum of Goldstone modes for increasing disorder, as expected for a SF to BG transition. Furthermore, the behavior of the scaling exponent $p$ remains unclear at strong disorder $W > 20t$ ($W > U$) where in earlier works we have shown the need for even greater truncation $N > 5$ to obtain converged lowest energy QP excitations [23].

b. Level spacing statistics. Next, we focus on the QP spectrum beyond the low energy regime. To characterize an MBL(-like) transition the gap ratio $r = r_\gamma$ is the most prevalent measure, which in terms of the QP energy gaps $\Delta \omega_\gamma = \omega_{\gamma + 1} - \omega_\gamma$ we define as
\begin{equation}
r_\gamma \equiv \left( \frac{\min[\Delta \omega_\gamma-1, \Delta \omega_\gamma]}{\max[\Delta \omega_\gamma-1, \Delta \omega_\gamma]} \right)_d. \label{eq:ratio}
\end{equation}
The statistical properties of $r$ and the rescaled level spacing $s \equiv \Delta \omega_\gamma / \overline{\Delta \omega}$, where $\overline{\Delta \omega}$ is the mean level spacing, are well known from random matrix theory [39, 71]. In
while the Poissonian has the simple form \( P(s) = \frac{s}{2} \exp\left(-\frac{s}{2}\right) \) with \( P(r) = 1/(1 + r^2) \). In Fig. 4 we show distributions obtained for the QP spectra at \( U/t = 20 \) and \( W/t = 5 \) in the vicinity of the low energy ME at about \( \omega_r/t \approx 3.5 \). While the spectra at low \( \omega_r \) reproduce the GOE prediction, the distributions approach P behavior for increased energies [see Fig. 4(a, b)], consistent with crossing a ME somewhere in between. If, on the other hand, we increase the system size while keeping the energy window fixed to \( \omega_r/t = 5 \pm 0.5 \) [see Fig. 4(c, d)] we again find that the distributions interpolate from near GOE to P-like behavior. This finite-size scaling behavior is consistent with QP states that are on the localized side of the ME.

Random matrix theory furthermore predicts the expectation value of \( r \) within each ensemble to \( r_G \approx 0.5307 \) and \( r_p = 2m_2 - 1 \approx 0.3863 \) for the GOE and P statistics, respectively [11]. In Fig. 5 we show \( r \) as a function of the QP energies \( \omega_r \). For sufficiently low energies most \( r_r \approx r_G \), as expected for non-localized states. Outliers towards extremely small values result from a systematic finite-size effect. For not too strong disorder, such as \( W/U = 0.25 \) in this case, the low energy part of the QP spectrum is only weakly disturbed, as visible by the nearly plane wave character of the wave function in the first inset of Fig. 5. Thus one finds clusters of near-degenerate QP excitations in the spectrum for which the lattice momentum \( k \) still is a good approximate quantum number. The number of states in each cluster is related to the underlying 90° rotational and reflection symmetries of the corresponding disorder-free excitation bands, as can be seen in a clustering of the fractal dimension \( D = D(\gamma) \) of the QP fluctuation wave functions \( \psi(\gamma) \) (see Fig. 5, blue dots) which we discuss in the following.

c. Decay of fluctuations While the discussed level statistics are fully consistent with a ME in the disordered BHM we now consider the localization properties of the fluctuation wave function \( \psi(\gamma) \) directly. Analogous to the fractal dimension we analyze the typical radial wave-function amplitude \( A(r) \) oriented at its center-of-mass \( r_0 \) for each level and disorder realization. The most relevant notion of distance is given by the minimal number of links between two sites. Thus, we define the norm \( |\cdot| \) of a lattice vector \( r \) via its spatial components \( x \) and \( y \) as

\[
|r| = \sum_{i=x,y} |r_i| \tag{11}
\]

while we consider the center-of-mass \( r_0 = \left( \sum_\ell r_\ell |\psi(\gamma)_\ell|^2 / \sum_\ell |\psi(\gamma)_\ell|^2 \right) \) with \(|\cdot|\) denoting a rounding
Note the deviations from a circular shape of the contour lines for a typical state $\langle \log_{10} |v_\ell|^2 \rangle_d$, depicted in the inset of Fig. 6 (a) for $\omega/t = 5.5$, justifying our distance definition. At this moderate disorder the fluctuation wave functions become strongly localized above some QP energy $\omega$, associated with the mobility edge as visible by the exponential decay of $\mathcal{A}$ and $\mathcal{I}$ for large disorder. Especially the behavior of $\mathcal{I}$ at high energies implies that the majority of QP state is constrained to the sites close to some central site, while the examples given in Fig. 6 show that of these sites usually only a few actually contribute. To quantify the decay of the QP wave-function we consider the following ansatz for the tail of $\mathcal{A}(r)$ fitted up to $r \leq L/2$:

$$\mathcal{A}(r) \approx \exp \left( -\frac{r}{\lambda} + \xi \right). \quad (14)$$

Its parameters are an irrelevant offset $\xi$ related to the onset of the tail and the decay length $\lambda$. We always find $1/\lambda > 0$ (with an adjusted parameter of convergence that mostly is $\hat{R}^2 > 0.99$) for any excitation of sufficiently high energy – that is, above the ME – in the thus localized part of the spectrum. We find this behavior for any local interaction $U$ and disorder $W$ which is also strongly convergent for sufficiently large $L > 10$ and $N \geq 3$. Thus, we can consider the inverse of the decay length as an order parameter, as $\lambda$ diverges at the transition from localized to extended states. Indeed, starting at high energies [see Fig. 6 (c)] or strong disorder [for sufficiently low energy, see Fig. 6 (d)] and lowering either the energy or the disorder strength, $1/\lambda$ eventually tends to zero within the resolvable states (limited by $N$ and $L$). This is nicely captured by linear fits for small $1/\lambda$ to

$$\frac{1}{\lambda} = a(\omega - \omega_0). \quad (15)$$

Here, $a$ is the slope with proper units and $\omega_0$ is the zero. Both terms are determined by fits for different $L$ [see Fig. 6 (c) and inset]. The scaling of $\omega_0$ in turn follows a simple relation of the form

$$\omega_0 = \omega_c + \frac{\bar{\omega}}{L^{1/\nu}}, \quad (16)$$

with the critical energy $\omega_c$ corresponding to the ME, the rescaled energy $\bar{\omega}$ and the finite-size scaling exponent $1/\nu = 0.91(4)$ determined in our previous works 24. We note however that determining $\lambda$ for $\mathcal{A}(r)$ is problematic at small system sizes $L < 10$ for the least localized excitations which have a substantial inner region such that the onset of the decay is shifted outwards [see Fig. 6 (a)]. Then $\lambda$ may be overestimated for small $L$ and the least localized low energy excitations. This is visible in Fig. 6 (c) where the deviation in $1/\lambda$ for $L = 10$ and $L > 10$ increases with $1/\lambda \to 0$. When fitting (16) to determine $\omega_c$ we therefore distinguish two cases, one with $L = 10$ included [solid line in the inset of Fig. 6 (c)]

Fig. 6. Decay behavior of $|v_\ell^{(\gamma)}|^2$ for $U/t = 20$ in (1) with $L = 32$ and $N = 3$ if not specified otherwise. For $W/t = 7$ examples of $\mathcal{A}(r)$ and $\mathcal{I}(r)$ are shown in (a) and (b). All distributions are averaged over bins of 16 levels closest in energy to $\omega/t \in [0.75, 5.75]$ increasing in steps of 0.5 from light gray to black (top to bottom at large distances). Inset in (a) shows a corresponding typical state $\langle \log_{10} |v_\ell^{(\gamma)}|^2 \rangle_d$ at $\omega/t = 5.5$. Dashed lines in (a) are exponential fits of eq. (14) and corresponding inverse decay lengths $1/\lambda$ are given in (c) for $W/t = 10$ and $W/t \in [1, 15]$ in (d). Data sets in (c) for $L \in [10, 20, 24, 32, 40]$ are accompanied by linear fits (solid lines) and circles have $L = 32$ and $N = 4$ (see legend). Zeros of these fits are given in the inset with solid (dashed) linear fit lines for all $L (L > 10)$. $\omega_c$ in (c) and lines in (d) represent the ME determined in 23 (see legend), while circles (crosses) are derived from the decay length $\lambda$ ($\bar{\lambda}$) for all sizes ($L > 10$).

to the nearest site. Using these we define

$$\tilde{\mathcal{A}}_c(r) = \sum_{\ell |r_{\ell} - r| = r} \exp \left[ \langle \log |v_\ell^{(\gamma)}|^2 \rangle_d \right], \quad (12)$$
$$\tilde{\mathcal{I}}_c(r) = \sum_{r > r} \tilde{\mathcal{A}}(r') \quad (13)$$

giving the surface integral of the typical wave-function $\tilde{\mathcal{A}}_c(r)$ and its radial integral $\tilde{\mathcal{I}}_c(r)$. For convenience we scale either by its respective maximum: $\tilde{\mathcal{A}}_c(r) = \tilde{\mathcal{A}}_c(r)/\max_r (\tilde{\mathcal{A}}_c(r))$ and the latter by the full sum:

$$\tilde{\mathcal{I}}_c(r) = \tilde{\mathcal{I}}_c(r)/\tilde{\mathcal{I}}_c(0).$$

For $U/t = 20, W/t = 7$ and $L = 32$ a few examples of both are shown in Fig. 6 (a) and (b), respectively.

In the following, energies are binned over consecutive energy levels so we discard the level index from here on and instead consider the mean energies of the bins. The behavior of either one is consistent with our findings so far. Note the deviations from a circular shape of the
and labeled $\lambda$ in (d)] and the other with $L = 10$ excluded [dashed line in the inset of Fig. 3(c) and labeled $\lambda$ in (d)].

Considering the drop in fit quality of (14) for $L = 10$ due to the shift of the onset of decay and the weak decay when approaching the ME from the localized side we find a ME that closely overlaps with our earlier predictions, which relied on the finite-size scaling of the gap ratio and the fractal dimension of the excitation states [22]. For the sake of completeness we therefore finish this section with a brief discussion of the fractal dimension of the QP wave-functions to show how the various observables related to the ME compare.

d. Fractal dimension of fluctuations

Analogous to the scaling of q-moments $R_q = \sum_{|\gamma|} |\psi_\gamma|^q$ of many-body eigenstates where $n$ labels the partial amplitudes of a given many-body basis [69, 70, 72], our analysis is based on the local amplitudes of the wave function $|\psi_i^{(\gamma)}|^2 = \sum_{\ell > 0} |\psi_{\ell, i}^{(\gamma)}|^2$ (and $q = 2$):

$$D \equiv D^{(2)}_\ell = -\log L^2 \left[ \frac{\sum_{\ell} |\psi_{\ell, i}^{(\gamma)}|^4}{\sum_{\ell} |\psi_{\ell, i}^{(\gamma)}|^2} \right].$$  (17)

In contrast to many-body eigenstates the fluctuation wave function preserves real-space information in its amplitudes, so $D^{(2)}_\ell$ characterizes the spatial extension of QP fluctuations in relation to the system size (see insets in Fig. 5). As shown in our previous work $r$ and $D$ can be used to determine the ME of the QP spectrum [see Fig. 5(c, d)] via the critical values $r_c(W)$ and $D_c(W)$ [22]. As we have already seen for the level statistics, where high energy excitations have P statistics corresponding to localized states, also $D$ quickly tends to 0 above the ME where the QP states are centered at arbitrary sites and only involve a few of the nearest sites [see Figs. 5 and 6(a, b)]. We note that this behavior is very typical of LIOMs implying that the QP modes can be considered their lowest order approximation via the definition $I^{(0)}_\gamma = \beta^\dagger_\gamma \beta_\gamma$. In case of the existence of actual LIOMs, corrections to this lowest order can be determined by the far neglected Hamiltonian terms $H^{(3)}$ and $H^{(4)}$, analogous to a weak coupling expansion [73]. But as the FOE is effectively a strong coupling expansion, already the lowest order goes beyond a single particle description. In particular using the FOE method the QP ground state as well as its excitations can be highly entangled, as we have already shown [22].

In summary, we have shown that the localization of the QP states is well characterized by the level spacing statistics, the decay length of the fluctuation wave-functions and their related fractal dimension. From these we obtain matching predictions of the ME. Furthermore, we have shown that the finite-size scaling of the lowest gaps is consistent with a SF to BG transition of the ground state. In the next section we will complete this picture by discussing various spectral functions of the quasiparticle ground state $|\psi_{\text{QP}}\rangle$.

### IV. SPECTRAL FUNCTIONS

As shown in Sec. [12], one can derive a simple implicit definition for a corrected QP ground state $|\psi_{\text{QP}}\rangle$ by requiring the condition $\beta_\gamma |\psi_{\text{QP}}\rangle = 0$ for all QP modes $\gamma$ (and $\mathcal{P}|\psi_{\text{QP}}\rangle = 0$). Using this definition it is straightforward to determine the single particle spectral functions. Here, we focus on the spectral function and normalized dynamic structure factor for the QP ground state of the disordered BHM [1] up to second order in the FOE, respectively given by

$$A(k, \omega) = \Theta(\omega) A^{(2)}_k(k, \omega) - \Theta(-\omega) A^{(2)}_s(k, \omega)$$

$$= \Theta(\omega) \langle \psi_{\text{QP}} | \tilde{b}_k \delta(\tilde{H}^{(2)} - E_0 - \omega) | \psi_{\text{QP}} \rangle_d - \Theta(-\omega) \langle \psi_{\text{QP}} | \tilde{b}^\dagger_k \delta(\tilde{H}^{(2)} - E_0 + \omega) | \psi_{\text{QP}} \rangle_d,$$

$$= \Theta(\omega) \langle \psi_{\text{QP}} | n_k \delta(\tilde{H}^{(2)} - E_0 - \omega) n_k | \psi_{\text{QP}} \rangle_d / N_p \text{.}$$  (19)

with the total number of particles $N_p$ and the label $d$ signifying the disorder average, as well as their static counter parts, the momentum distribution $n(k) = -\int_0^\infty A^{(2)}_k(k, \omega) d\omega$ and the static structure factor $S(k) = \int_0^\infty S(k, \omega)$. These are given in terms of Fourier transforms of the local creation (annihilation) and number operators,

$$\tilde{b}_k = \frac{1}{\sqrt{L^2}} \sum_\ell e^{-ik \cdot r_\ell} \tilde{b}_\ell,$$

$$\tilde{b}^\dagger_k = \frac{1}{\sqrt{L^2}} \sum_\ell e^{ik \cdot r_\ell} \tilde{b}^\dagger_\ell,$$

$$\tilde{n}_k = \sum_\ell e^{ik \cdot r_\ell} \tilde{n}_\ell.$$  (22)

Furthermore, due to the completeness of each eigenbasis $\{|\ell, i\rangle\}$ of the local MF Hamiltonians [22] any local operator $\hat{O}^{(\ell)}$ has an exact representation within this basis, in terms of the local Gutzwiller operators:

$$\hat{O}^{(\ell)} = \sum_{i,j \geq 0} \ell \langle i | \hat{O}^{(\ell)} | j \rangle \ell \delta_{\ell, \ell'} | j \rangle$$

$$= \sum_{i,j \geq 0} \langle i | \delta_{\ell \ell'} | j \rangle | j \rangle$$

$$\equiv \sum_{i \geq 0} \left( O_{00}^{(i)} \sigma^{(i)}_\ell + O_{01}^{(i)} \sigma^{(i)}_\ell \right) + O_{00}^{(i)} \mathbb{1}_N$$

$$+ \sum_{i,j \geq 0} \left( O_{ij}^{(i)} - \delta_{i,j} O_{00}^{(i)} \right) \sigma^{(i)}_\ell \sigma^{(i)}_\ell.$$  (23)

Using the inverse of (26) and (37) we can then use the implicit definition for the QP ground state to compute the spectral functions. We note that, while (36) is a non-linear representation, (26) and (37) are linear. Thus, due to the implicit definition of the QP ground state, only terms of even order in the Gutzwiller operators matter.
for (18) and (19). As we will see in Sec. V B the average number \( \kappa \) of local Gutzwiller excitations in the QP ground state is on the order of a few percent, so we neglect the fourth order terms which would only contribute \( O(\kappa^2) \).

For \( U/t = 20 \) we consider a system with 1600 sites \( (L = 40) \) at weak \( (W/t = 1) \) and moderate \( (W/t = 5) \) disorder averaged over \( N_r = 10 \) disorder realizations using a truncation of \( N = 4 \) to discuss signatures of localization in the static properties of the ground state as well as in the spectrum of its FOE excitations. Firstly, Fig. 7 depicts the weak disorder case for which the momentum distribution (panel \( a \)) has a very pronounced peak at \( \mathbf{k} = 0 \) corresponding to the condensate fraction while the static structure factor (panel \( c \)) displays only weak fluctuations due to the disorder but otherwise follows the behavior of the homogeneous case as well. The spectral function (panel \( b \), inset panel \( a \)) and the dynamic structure factor (panel \( d \)) on the other hand already present strong signatures of localized fluctuations, especially at large excitation energies in the first gapped band where the spectral weights are spread over all lattice momenta. Conversely, the ungapped (Goldstone) band is well resolved as the ME [dashed line in Fig. 7(b, d)] is identical to its upper band edge. Especially the low-energy states are almost exactly the low-momentum eigenstates following the linear dispersion of a SF.

In contrast, as visible for the spectral function (panel \( b \), inset panel \( a \)) and dynamic structure factor (panel \( d \)) given in Fig. 8 at an enhanced disorder of \( W/t = 5 \) the entire spectrum of QP states above the ME (dashed lines in panels \( b \) and \( d \)) becomes smeared-out over all lattice momenta, while both lowest bands are merging due to the disorder driven local energy fluctuations. Only for QP energies below the ME one can still find a prevailing linear dispersion of low-momentum QP states (inset panel \( a \)). Regarding the QP ground state itself, the increased disorder results in a further decreased zero-momentum peak in \( n(\mathbf{k}) \) (panel \( a \)). Also, there is an almost complete loss of non-trivial non-local density correlations, visible in the nearly flat static structure factor (panel \( c \)) close to the BG phase which indicates a nearly uncorrelated distribution of particles. The value of the flat background \( s_0 = 0.54(3) \) corresponds to the only non-trivial (local) correlations via \( c(\mathbf{r}) = N_p L^{-2} \sum_{\mathbf{k}} \exp(i \mathbf{kr}) L^{-2} S(\mathbf{k}) \approx \langle \hat{n}_\mathbf{r} \rangle_d (\hat{n}_{\mathbf{r}})_d + \delta_{\mathbf{r},0} s_0 \) where \( S(\mathbf{k}) \approx s_0 + \delta_{\mathbf{k},0} N_p \). Here, \( c(d) = \sum_{\mathbf{q}} \langle \mathbf{Q} | \hat{n}_\mathbf{q} \hat{n}_d | \mathbf{Q} \rangle_d / L^2 \) is the lattice and disorder average of the density correlations where \( d = \mathbf{r}_\ell - \mathbf{r}_{\ell'} \).
and $q = (r_x + r_y)/2$. While an uncorrelated placement of particles would imply $P$ correlations with $c_p(0) = n^2 + n = 3/4$, the local correlations $c(0) = 0.519(14)$ at this disorder are sub-Poissonian due to the repulsive local interactions $U$. This value increases towards the Poissonian value above the critical disorder of the SF to BG transition. Altogether, this discussion of spectral functions nicely reflects our predictions of the ME and is consistent with a superfluid ground state dissolving in favor of a Bose glass phase for increasing disorder.

V. FLUCTUATION OPERATOR EXPANSION

We now discuss the fluctuation operator expansion (FOE) with a main focus on its application to systems with broken translational invariance such as $\Gamma$, in order to investigate its properties beyond the previous discussion of the MF ground state. Given any such MF state the FOE constitutes a systematic expansion of all beyond first-order fluctuation operator terms of $\Gamma$ neglected on the MF level – commonly only non-local terms – in terms of a quadratic map onto local complete sets of generators of MF excitations, the Gutzwiller operators. Within the approximation of a negligible small density of local Gutzwiller fluctuations these operators are quasi-bosonic and their second-order contribution to the FOE representation constitutes a quadratic map that is exact in the limit $N \to \infty$ with $N$ the truncation of the local Gutzwiller eigenbases. To consider a sufficient convergence of these bases we set $N_b = 3N$. It is convenient to introduce the local Gutzwiller raising and lowering operators as well as their compound terms for all $i > 0$:

$$\sigma_{\ell}^{(i)} \equiv |i\rangle_{\ell} \langle 0|, \quad 1_N - \sum_{i>0} \sigma_{\ell}^{(i)} \sigma_{\ell}^{(i)} = |0\rangle_{\ell} \langle 0|, \quad (26)$$

$$\sigma_{\ell}^{(i)} \equiv |0\rangle_{\ell} \langle i|, \quad \text{and} \quad \sigma_{\ell}^{(i)} \sigma_{\ell}^{(j)} = |i\rangle_{\ell} \langle j|. \quad (27)$$

Using these operators one obtains the formally exact representation $\hat{H} = \sum_{\ell} \hat{H}_{\ell}^{(\ell)} + \hat{H}^{(2)} + \hat{H}^{(3)} + \hat{H}^{(4)}$, where each term $\hat{H}^{(n)}$ refers to a different order in the Gutzwiller operators. We note that the self-consistency condition guarantees the absence of first order terms. While the second order term $\hat{H}^{(2)}$ describes the (linearized) QP fluctuations of the approximate product ground state $|\psi_{MF}\rangle = \prod_{\ell} |0\rangle_{\ell}$, the higher order terms introduce interactions among them. The main approximation of this work is to assume that quasiparticle interaction terms $\eta > 2$ can be neglected, meaning we will consider the beyond mean-field Hamiltonian $\hat{H}^{(2)} \equiv \sum_{\ell} \hat{H}_{\ell}^{(\ell)} + \hat{H}^{(2)}$. This can be justified in the vicinity of the ground state that can implicitly be defined as the state not containing any QP excitations (discussed in Secs. V B and V D), resulting in very good predictions both in Mott-type and superfluid phases. Furthermore, in the localized phase at strong disorder the eigenstates of this approximate Hamiltonian tend to become a representation of approximate LIOMs, as discussed in Sec. III. These also have the property that their spectra are (nearly) unaffected by one another, resulting in the absence of level repulsion in the localized regime and causing the well-known Poisson statistics of the level spacings also discussed in Sec. III.

A. Gutzwiller operator representation

The FOE is a quasiparticle method based on an expansion of a second quantized Hamiltonian such as $\Gamma$ in terms of the eigenstates $|i\rangle_{\ell}$ of its local mean-field Hamiltonians (given a truncation $N_b$ of the local bosonic number states)

$$\hat{H}_{MF}^{(\ell)} = \hat{H}_{\ell} - t \sum_{\langle \ell' | \ell, \ell' \rangle} \left( \hat{b}_{\ell}^{\dagger} \phi_{\ell'} + \text{h.c.}. \right). \quad (24)$$

These are defined in terms of the fluctuation operators $\hat{b}_{\ell} \equiv \hat{b}_{\ell} - \phi_{\ell}$ and the fields $\phi_{\ell} \equiv \langle \ell | 0 \rangle \hat{b}_{\ell} | 0 \rangle_{\ell}$ which are to be determined self-consistently. Drawing from variational concepts the FOE allows for a systematic improvement over standard Bogoliubov theory by considering in principle general local fluctuations, giving access to the complete QP spectrum of

$$\hat{H} = \sum_{\ell} \hat{H}_{MF}^{(\ell)} - t \sum_{\langle \ell, \ell' \rangle} \left( \hat{b}_{\ell}^{\dagger} \hat{b}_{\ell'} - \phi_{\ell}^{\dagger} \phi_{\ell'} + \text{h.c.}. \right). \quad (25)$$

Due to the completeness of each local eigenbasis $\{|i\rangle_{\ell}\}$, the FOE representation $\hat{b}_{\ell} = \sum_{i,j< N} \epsilon(i) \hat{b}_{\ell} |j\rangle_{\ell} \langle i|_{\ell} \langle j|$ constitutes a quasiparticle interaction term.

B. Quasi-Bosonic commutation relations

Before we can attempt to diagonalize $\hat{H}^{(2)}$ we first have to bring it into a standard Nambu-type form, which is straightforward for regular bosons. To do so in our case we have to consider the actual commutation relations that characterize the Gutzwiller operators $\sigma_{\ell}^{(i)}$ and $\sigma_{\ell}^{(j)}$. One can easily show that they obey quasi-bosonic commutation relations of the form

$$\left[ \sigma_{\ell}^{(j)}, \sigma_{\ell}^{(i)} \right] = \delta_{i,j} \delta_{\ell,\ell'} - \delta_{\ell,\ell'} \hat{R}^{(i,j)}(\ell), \quad (28)$$

$$\left[ \sigma_{\ell}^{(i)} \sigma_{\ell}^{(j)} \right] = \left[ \sigma_{\ell}^{(i)} \sigma_{\ell}^{(j)} \right] = 0. \quad (29)$$

Here, we introduce the residual operator $\hat{R}^{(i,j)}(\ell)$ quantifying the deviation from bosonic behavior. Its form is
written in terms of the vectors \( \sigma = (\sigma_1^{(1)}, \ldots, \sigma_L^{(N-1)})^T \) and \( \sigma^T = (\sigma_1^{(1)^T}, \ldots, \sigma_L^{(N-1)^T})^T \). Using these and the approximation \( \langle \hat{R}^{(i,j)}(\ell) \rangle \to 0 \) in the commutation relation \( [\hat{R}, \sigma] \) one can bring the Hamiltonian into a Nambu-type form, so

\[
\hat{H}^{(2)} \approx \hat{H}_{\text{QP}}^{(2)} = \frac{1}{2} \left( \sigma^T + \sigma \right) \hat{H}_{\text{QP}}^{(2)} \left( \sigma + \sigma^T \right) - \frac{1}{2} \text{Tr}(h),
\]

with \( \hat{H}_{\text{QP}}^{(2)} = \left( \frac{h}{\Delta} \right) \). (32)

As we have to get half of the normal ordered pairs \( \sigma^T \sigma \) into anti-normal order, we obtain the scalar term \( \text{Tr}(h) \) along the way. Within this approximation the introduced Hamiltonian matrix \( \hat{H}_{\text{QP}}^{(2)} \) has a size of \( 2(N-1)L^2 \times 2(N-1)L^2 \). Its individual entries are given in terms of \( \langle \hat{b}_i \hat{b}_j \rangle \) matrix elements, each within the local Gutzwiller bases, so the explicit matrix entries are given by

\[
h_{(\ell,\ell'),(j,\ell')} = -t_{\ell,\ell'} F_{(\ell,\ell')}^{(j)} + \delta_{\ell,\ell'} \delta_{j,\ell} E_i^{(\ell)},
\]

\[\Delta_{(\ell,\ell'),(j,\ell')} = -t_{\ell,\ell'} F_{(\ell,\ell')}^{(j)} + \delta_{\ell,\ell'} \delta_{j,\ell} E_i^{(\ell)},\]

Both expressions are given in terms of the tunneling matrix, whose matrix elements \( t_{\ell,\ell'} = t \forall \{\ell, \ell'\} \{\ell, \ell'\} \) are nonzero for all neighboring sites, and the excitation energies \( E_i^{(\ell)} \) of the ith Gutzwiller excited state at each site \( \ell \). The remaining terms are the matrix elements of the non-local products of local operators

\[
F_{(\ell,\ell')}^{(j)} = B_{(\ell,\ell')}^{(j)} + B_{(\ell,\ell')}^{(j)} + B_{(\ell,\ell')}^{(j)} + B_{(\ell,\ell')}^{(j)},
\]

\[\text{with } B_{(\ell,\ell')}^{(j)} = \delta_{\ell,\ell'} \delta_{j,\ell} \phi_i E_i^{(\ell)},\]

where \( \phi_i \) are the previously defined self-consistent mean-field values associated with the local annihilation operator.

D. Diagonalization of \( \hat{H}_{\text{QP}}^{(2)} \)

In order to preserve the bosonic structure of the operators, the diagonalization of \( [\hat{R}, \sigma] \) has to be performed on the symplectic space, namely by diagonalizing \( \Sigma \hat{H}_{\text{QP}}^{(2)} \), where \( \Sigma = \begin{pmatrix} (N-1)L^2 & 0 \\ 0 & -1 \end{pmatrix} \). This yields the representation of \( \hat{H}_{\text{QP}}^{(2)} \) in terms of generalized Bogoliubov-type QP mode operators

\[
\beta_i \equiv \chi^{(\gamma)^T} \Sigma \left( \begin{array}{c} \sigma \\ \sigma^T \end{array} \right) \equiv \mu^{(\gamma)^T} \sigma + \nu^{(\gamma)^T} \sigma^T,
\]

\[\beta_i^T \equiv -\chi^{(\gamma)^T} \Sigma \left( \begin{array}{c} \sigma \\ \sigma^T \end{array} \right) \equiv \nu^{(\gamma)^T} \sigma + \mu^{(\gamma)^T} \sigma^T.
\]
These are given by the eigenvectors of the eigenvalue equations $\Sigma \mathcal{H}_{\text{QP}}^2 \mathbf{x}^{(\gamma)} = \omega^{(\gamma)} \mathbf{x}^{(\gamma)}$ and $\Sigma \mathcal{H}_{\text{QP}}^2 \mathbf{y}^{(\gamma)} = -\omega^{(\gamma)} \mathbf{y}^{(\gamma)}$. Thus all QP frequencies $\omega^{\gamma}$ appear in pairs and those with a nonzero imaginary part represent unstable modes \[83\]. By requiring the normalization condition $|\mathbf{u}^{(\gamma)}|^2 - |\mathbf{v}^{(\gamma)}|^2 = 1$ in analogy to regular Bogoliubov theory, we preserve the (approximate) bosonic commutation relations \[28\] and \[29\], so $[\beta_{\gamma}, \beta^{\dagger}_{\gamma'}] = \delta_{\gamma, \gamma'}$. We note that $\mathbf{u}^{(\gamma)}$ and $\mathbf{v}^{(\gamma)}$ can be interpreted as dual wave functions analogous to particle and hole fluctuations.

In the presence of a condensate one encounters a degenerate two-dimensional subspace constituted by an identification of the energy pair which becomes numerically exact only for $N \to \infty$. In the case of an exact degeneracy the eigenvalue equation becomes $\Sigma \mathcal{H}_{\text{QP}}^2 \mathbf{p} = 0$ and can be solved by an eigenvector of the form $\mathbf{p} = (\mathbf{u}^{(0)}, -\mathbf{u}^{(0)})^T$. In order to complete the representation of this two-dimensional subspace one has to introduce a second vector $\mathbf{q}$ within this subspace, which is best defined implicitly via $\Sigma \mathcal{H}_{\text{QP}}^2 \mathbf{q} = -i\mathbf{p} / \bar{m}$, where $\bar{m}$ is a mass-like scalar. Therefore, we obtain two different operators taking the places of the Bogoliubov-like operators \[36\] and \[37\] for the doubly degenerate mode (these are discussed in further detail in \[80\]):

\[
\mathcal{P} \equiv \mathbf{p}^\dagger \Sigma \begin{pmatrix} \sigma^1 \\ \sigma^\dagger \end{pmatrix} = \left( \begin{array}{c} \mathbf{u}^{(0)} \\ -\mathbf{u}^{(0)*} \end{array} \right) \begin{pmatrix} \sigma^1 \\ \sigma^\dagger \end{pmatrix},
\]

\[
\mathcal{Q} \equiv -i \mathbf{q}^\dagger \Sigma \begin{pmatrix} \sigma^1 \\ \sigma^\dagger \end{pmatrix} = -i \begin{pmatrix} \mathbf{v}^{(0)} \\ \mathbf{v}^{(0)*} \end{pmatrix} \begin{pmatrix} \sigma^1 \\ \sigma^\dagger \end{pmatrix}.
\]

We note that $\mathcal{P}$ is a momentum-like operator that can be considered as the generator of translations in the global phase of the condensate mode \[84\], so it represents the free motion of the complex phase factor of the condensate.

As a result of the (approximately) exact commutation relations of the QP mode operators, the second order Hamiltonian $\mathcal{H}_{\text{QP}}^2$ generally has the form

\[
\mathcal{H}_{\text{QP}}^2 \approx \sum_{\gamma} \omega^{\gamma} \beta^\dagger_{\gamma} \beta^{\gamma} + \frac{\mathcal{P}^2}{2\bar{m}} + \frac{1}{2} \left( \sum_{\gamma} \omega^{\gamma} - \text{Tr}(h) \right).
\]

This representation is given in terms of the generalized Bogoliubov creation (annihilation) operators $\beta^\dagger_{\gamma}$ ($\beta^{\gamma}$) where the notation $\sum_{\gamma}$ represents the fact that the $\gamma = 0$ term in the sum is to be replaced by $\mathcal{P}$ whenever a condensate is present for $N \to \infty$. Otherwise, for small $N$, the lowest mode remains gapped such that the $\mathcal{P}$ term can be replaced by $\omega_0 \beta^\dagger_0 \beta_0$. As all $\omega_0 > 0$ the form \[40\] implies that the quasiparticle ground state is characterized by $\langle \psi_{\text{QP}} | \beta^\dagger_{\gamma} \beta^{\gamma} | \psi_{\text{QP}} \rangle = 0$ ($\langle \psi_{\text{QP}} | \mathcal{P}^2 | \psi_{\text{QP}} \rangle = 0$), so we can use $\beta^{\gamma} | \psi_{\text{QP}} \rangle = 0$ ($\mathcal{P} | \psi_{\text{QP}} \rangle = 0$) for all $\gamma$ as its implicit definition. This allows for the \textit{a posteriori} check of the central FOE approximation discussed in Sec. \[V\].

Regarding the spectral properties discussed in Sec. \[IV\] consideration of $\mathcal{P}$ and $\mathcal{Q}$ only yields a sub-leading [even self-canceling for $A(k, \omega)$ correction at $k = 0$ and $\omega = 0$ in the thermodynamic limit \[62\]], so we may neglect both for our purposes. By expressing $\mathcal{H}_{\text{QP}}^2$ with $\beta^\dagger_{\gamma}$ and $\beta^{\gamma}$ in normal order we find a further scalar contribution proportional to $\sum_{\gamma} \omega^{\gamma}$. Note that both scalar terms generate a shift of the total energy. While both contributions $\sum_{\gamma} \omega^{\gamma}$ and $\text{Tr}(h)$ would diverge individually without truncation ($N \to \infty$), even in a finite system, in combination they only yield a finite correction of the quasiparticle ground-state energy. They effectively lower the energy of $| \psi_{\text{MF}} \rangle$ in relation to the energy of the MF state $| \psi_{\text{MF}} \rangle$ as a result of an average down shift of the QP mode energies in relation to the energies of the Gutzwiller excitations.

This concludes the diagonalization of the disordered BHM up to second order in the Gutzwiller operators. We note that the obtained representation is reminiscent of the emergent LIOMs predicted within the MBL phase \[36\] \[38\] \[73\] \[85\] \[86\], albeit on the lowest order of approximation where all coupling terms are disregarded. This parallel is expected to hold especially for strongly localized QP states, where we consider the FOE to yield a representation in terms of approximate LIOMs for which the corresponding QP states are expected to be strongly localized. Indeed they are, as we show in Sec. \[III\] where we characterize the corresponding spectrum via its energies and the spatial localization of the QP eigenstates, further reinforcing this argument.

VI. SUMMARY

In this work we have explored the properties of the two-dimensional BHM with disorder, both in the ground state and in its FQ quasiparticle spectrum, in order to obtain some insight on the relation between the well-known BG ground state phase at moderate disorder and the more elusive (many-body) localization phenomena in the excitations at strong disorder. Regarding the BG phase, we find that a surprisingly simple fractal dimension analysis of the mean-field ansatz already suffices to reveal a critical disorder strength above which we observe a finite Edwards-Anderson parameter, implying the onset of the BG phase. Furthermore, we show that FOE gives corrections to this result by considering the finite-size scaling of the lowest energy gaps.

Regarding the excitations of this corrected ground state we find QP level spacing statistics that are consistent with a (many-body) ME and an exponential decay for the fluctuation wave-functions in the localized part of the spectrum. An analysis of the spectral function and dynamic structure factor yields a weak broadening of the spectrum of entangled QP excitations below the ME while above it they become smeared out over the lattice momenta. In addition, the static structure factor becomes flat at the onset of the BG indicating the
transition to a phase with vanishing non-local density correlations.

Finally, we argue that the FOE method yields a very good approximation of the ground state and its excitations, due to the observed very low fraction of local fluctuations, the interaction of which is neglected when deriving the FOE spectrum. As this holds throughout the whole range of considered disorder and local interaction values, we expect the method to be ideally suited to evaluate the dynamics of typical experimental quenching protocols, for example in order to determine the evolution of the entanglement entropy after a sudden quench of the disorder potential.

ACKNOWLEDGMENTS

The author would like to thank L. Rademaker for insightful discussions and especially G. Pupillo for his extensive support and many comments. Support by the Leopoldina Fellowship Programme of the German National Academy of Sciences Leopoldina grant no. LPDS 2018-14, the ANR ERA-NET QuantERA - Projet RonTe (ANR-18-QUAN-0005-01) and the High Performance Computing center of the University of Strasbourg, providing access to computing resources and scientific support, is gratefully acknowledged. Part of the computing resources were funded by the Equipex Equi@Meso project (Programme Investissements d’Avenir) and the CPER Alsacalcul/Big Data. G.P. is further supported by USIAS in Strasbourg and the Institut Universitaire de France (IUF).

[1] J. A. Hertz, L. Fleishman, and P. W. Anderson, Physical Review Letters 43, 942 (1979)
[2] A. Gold, Zeitschrift für Physik B Condensed Matter 52, 1 (1983)
[3] D. S. Fisher and M. P. A. Fisher, Physical Review Letters 61, 1847 (1988)
[4] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Physical Review B 40, 546 (1989)
[5] I. Bloch, J. Dalibard, and W. Zwerger, Reviews of Modern Physics 80, 885 (2008)
[6] R. T. Scalettar, G. G. Batrouni, and G. T. Zimanyi, Physical Review Letters 66, 3144 (1991)
[7] U. Bissbort and W. Hofstetter, EPL (Europhysics Letters) 86, 50007 (2009)
[8] L. Pollet, N. V. Prokof’ev, B. V. Svistunov, and M. Troyer, Physical Review Letters 103, 140402 (2009)
[9] V. Gurarie, L. Pollet, N. V. Prokof’ev, B. V. Svistunov, and M. Troyer, Physical Review B 80, 214519 (2009)
[10] S. G. Soyler, M. Kiselev, N. V. Prokof’ev, and B. V. Svistunov, Physical Review Letters 107, 185301 (2011)
[11] I. F. Herbut, Physical Review Letters 79, 3502 (1997)
[12] I. F. Herbut, Physical Review B 57, 13729 (1998)
[13] G. M. Falco, T. Nattermann, and V. L. Pokrovsky, Physical Review B 80, 104515 (2009)
[14] G. M. Falco, T. Nattermann, and V. L. Pokrovsky, EPL (Europhysics Letters) 85, 30002 (2009)
[15] Z. Ristivojevic, A. Petkovic, P. LeDoussal, and T. Giamarchi, Physical Review B 90, 125144 (2014)
[16] B. Wang and Y. Jiang, The European Physical Journal D 70, 257 (2016)
[17] L. Fallani, J. E. Lye, V. Guarrera, C. Fort, and M. Inguscio, Physical Review Letters 98, 130404 (2007)
[18] C. Meldgin, U. Ray, P. Russ, D. Chen, D. M. Ceperley, and B. DeMarco, Nature Physics 12, 646 (2016)
[19] R. Yu, L. Yin, N. S. Sullivan, J. S. Xia, C. Huan, A. Paduan-Filho, N. F. Oliveira Jr, S. Haas, A. Steppke, C. F. Miclea, F. Weickert, R. Movshovich, E.-D. Mun, B. L. Scott, V. S. Zapf, and T. Rosclde, Nature 489, 379 (2012)
[20] J. Kisker and H. Rieger, Physical Review B 55, R11981 (1997)
[21] M. Makivić, N. Trivedi, and S. Ullah, Physical Review Letters 71, 2307 (1993)
[22] S. Zhang, N. Kawashima, J. Carlson, and J. E. Gubernatis, Physical Review Letters 74, 1500 (1995)
[23] A. Priyadarshree, S. Chandrasekharan, J.-W. Lee, and H. U. Baranger, Physical Review Letters 97, 115703 (2006)
[24] J. P. Álvarez Zúñiga, D. J. Luitz, G. Lemeni, and N. Laforencie, Physical Review Letters 114, 155301 (2015)
[25] A. Geißler and G. Pupillo, arXiv:1909.09247
[26] B. L. Altshuler, Y. Gefen, A. Kamenev, and L. S. Levitov, Physical Review Letters 78, 2803 (1997)
[27] D. Basko, I. Aleiner, and B. Altshuler, Annals of Physics 321, 1126 (2006)
[28] V. Oganesyan and D. A. Huse, Physical Review B 75, 155111 (2007)
[29] A. Pal and D. A. Huse, Physical Review B 82, 174411 (2010)
[30] D. A. Huse, R. Nandkishore, V. Oganesyan, A. Pal, and S. L. Sondhi, Physical Review B 88, 014206 (2013)
[31] B. Bauer and C. Nayak, Journal of Statistical Mechanics: Theory and Experiment 2013, P09005 (2013), arXiv:1306.5753
[32] J. Smith, A. Lee, P. Richerme, B. Neyenhuis, P. W. Hess, P. Hauke, M. Heyl, D. A. Huse, and C. Monroe, Nature Physics 12, 907 (2016)
[33] K. S. C. Decker, D. M. Kennes, J. Eisert, and C. Karrasch, arXiv:1902.02259
[34] R. Nandkishore and D. A. Huse, Annual Review of Condensed Matter Physics 6, 15 (2015)
[35] D. A. Abanin and Z. Papić, Annalen der Physik 529, 1700169 (2017)
[36] M. Serbyn, Z. Papić, and D. A. Abanin, Physical Review Letters 111, 127201 (2013)
[37] D. A. Huse, R. Nandkishore, and V. Oganesyan, Physical Review B 90, 174202 (2014)
