Gravitational Waves in the Presence of Viscosity

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We analyze gravitational waves propagating in an isotropic cosmic fluid endowed with a bulk viscosity $\zeta$ and a shear viscosity $\eta$, assuming these coefficients to vary with fluid density $\rho$ as $\rho^{\lambda}$, with $\lambda = 1/2$ favored by experimental evidence. We give the general governing equation for the gravitational waves, and focus thereafter on two examples. The first concerns waves in the very late universe, close to the Big Rip, where the fate of the cosmic fluid is dependent highly on the values of the parameters. Our second example considers the very early universe, the lepton era; the motivation for this choice being that the microscopical bulk viscosity as calculated from statistical mechanics is then at maximum. We find that the gravitational waves on such an underlying medium are damped, having a decay constant equal to the inverse of the conformal Hubble parameter.

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I. INTRODUCTION

In hydromechanics in general the introduction of viscosity coefficients means that one is working to the first order deviations from thermal equilibrium. The usual model is that the viscous stress tensor is taken to be proportional to the symmetric strain tensor $\frac{1}{2}(u_{i,k} + u_{k,i})$. There should accordingly be three viscosity coefficients, but since the case of uniform rotation does not imply any friction forces the number of coefficients is reduced to two. They are the shear viscosity $\eta$ proportional to the strain tensor, and the bulk viscosity $\zeta$ proportional to $\nabla \cdot \mathbf{u}$. The shear viscosity thus relates to fluid velocity gradients, while the bulk viscosity relates only to compression or dilution of the fluid. Usually, $\eta$ is much bigger than $\zeta$.

Also in cosmology the use of viscosity coefficients has been considered. An early extensive treatment is that of Weinberg \cite{1, 2}. A recent review, covering both the early- and the late-time universe, is given by Brevik \textit{et al.} \cite{3} and \cite{4}. There are several research papers in this area; some of them are Refs. \cite{5–16}.

If we specialize to the case of gravitational waves, it turns out that also here the viscosity concept has attracted attention \cite{17–20}. When applying viscosity to cosmology, the following point needs however some attention. Viscosity is basically a macroscopic thermodynamical concept, applicable when when the free paths are much shorter than the wavelengths. The underlying mechanism is statistical mechanics and the Boltzmann equation, and viscosity coefficients are usually calculated via the so-called Chapman-Enskog expansion. This expansion is based on the physical idea that for times much larger than the mean free time, the statistical distribution function develops with time only via the “slowly” varying quantities such as the particle density, the (mean) fluid velocity, and eventually also the temperature \cite{21, 22}.

Now, the wavelengths observed in the present universe lie between 300 and 15,000 km. There is apparently nothing in interstellar space that may provide such a small structure. This point has recently been emphasized by Flauger and Weinberg \cite{23}. In spite of such fundamental issues we will yet apply the viscosity idea in our developments below. The nature of the underlying medium, the dark matter, is after all not well known, and not very much can be said about its local structure. Moreover, the hydrodynamic model enables us to give a coherent formal view of the evolution of the universe, both in its early and in its late epoch when even less is known about the microstructure of the cosmic fluid.

In this paper we consider propagation of gravitational waves in a viscous isotropic fluid. In the presence of bulk and/or shear viscosity there exist massless spin 2 waves, which are influenced by the viscosity, and can increase or decay in amplitude dependent on the parameters. Dissipation, meaning transfer of energy into heat, is present under varying circumstances. In the next section we review the general perturbed Einstein equations which describe gravitational waves. Then, we apply the formalism to the viscous fluid, in general both with a bulk viscosity $\zeta$ and a shear viscosity $\eta$ present. Both of them are modeled to be related to the energy density $\rho$ as $\rho^{1/2}$, as this form appears to be most suitable when comparing with experiments. We also present some numerical estimates. We then go on to

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consider two examples: in Sec. VI we examine gravitational waves near the future Big Rip singularity, distinguishing between cases when amplitudes are growing and when they are decreasing. In Sec. VII we consider a second example from the very early universe, from the lepton epoch, especially from the instant of neutrino decoupling \((10^{10} \text{ K})\), when the bulk viscosity as calculated from statistical mechanics was at maximum. We give the governing equation for gravitational waves on a “slowly varying” geometric background, based upon an experimentally favored bulk viscosity, and find an approximate expression for the decay parameter for the amplitude.

II. BRIEF REVIEW OF GRAVITATIONAL WAVES

Before we consider the gravitational wave in the viscous fluid, we review the propagation of gravitational waves in a general medium. The equation of the gravitational wave is given by the perturbation of the Einstein equation. In the Einstein equation, not only the curvature but the energy-momentum tensor also depends on the metric and therefore the variation of the energy-momentum tensor gives a non-trivial contribution for the propagation of the gravitational wave. We should note that the metric dependence of the energy-momentum tensor highly depends on the model. Even if different models give identical expansion history of the universe, the metric dependence of the energy-momentum tensor and therefore the propagation of the gravitational wave is different in the models (see [24–26] for examples). In general, the perturbed Einstein equation is given by

\[
0 = \frac{1}{2\kappa^2} \left( -\frac{1}{2} \left( \nabla^\mu \nabla_\mu \rho \right) + \frac{1}{2} \left( \nabla^\nu \nabla_\nu \rho \right) \delta g_{\rho \sigma} - \square (0) \delta g_{\mu \nu} - \nabla^\nu \nabla_\nu \left( g^{(0)} \rho \lambda \delta g_{\rho \lambda} \right) - 2 R^{(0)} \lambda \rho \mu \delta g_{\rho \lambda} + R^{(0)} \rho \mu \delta g_{\rho \nu} + R^{(0)} \rho \nu \delta g_{\rho \mu} \right) \\
+ \frac{1}{2} R^{(0)} \delta g_{\mu \nu} + \frac{1}{2} R^{(0)} \left( -\delta g_{\rho \sigma} R^{(0)} \rho \sigma + \nabla^{(0)} \nabla^{(0)} \sigma \delta g_{\rho \sigma} - \square (0) \left( g^{(0)} \rho \sigma \delta g_{\rho \sigma} \right) \right) \right) + \frac{1}{2} \delta T_{\text{matter} \mu \nu}.
\]

(1)

By multiplying \(g^{(0)} \mu \nu\) with (1), we obtain

\[
0 = \frac{1}{2\kappa^2} \left( \nabla^{(0)} \sigma \nabla_{\sigma} \rho \delta g_{\rho \sigma} - \square (0) \left( g^{(0)} \rho \lambda \delta g_{\rho \lambda} \right) + \frac{1}{2} R^{(0)} \left( g^{(0)} \rho \lambda \delta g_{\rho \lambda} \right) - 2 \delta g_{\rho \sigma} R^{(0)} \rho \sigma \right) + \frac{1}{2} \delta T_{\text{matter}}.
\]

(2)

We choose the following gauge condition

\[
0 = \nabla^{(0)} \rho \delta g_{\mu \nu}.
\]

(3)

Then Eq. (1) reduces to

\[
0 = \frac{1}{2\kappa^2} \left( -\frac{1}{2} \left( \square (0) \delta g_{\mu \nu} - \nabla^{(0)} \nabla_{(0)} \left( g^{(0)} \rho \lambda \delta g_{\rho \lambda} \right) - 2 R^{(0)} \lambda \rho \mu \delta g_{\rho \lambda} + R^{(0)} \rho \mu \delta g_{\rho \nu} + R^{(0)} \rho \nu \delta g_{\rho \mu} \right) \\
+ \frac{1}{2} R^{(0)} \delta g_{\mu \nu} + \frac{1}{2} R^{(0)} \left( -\delta g_{\rho \sigma} R^{(0)} \rho \sigma - \square (0) \left( g^{(0)} \rho \sigma \delta g_{\rho \sigma} \right) \right) \right) + \frac{1}{2} \delta T_{\text{matter} \mu \nu},
\]

(4)

and Eq. (2) to

\[
0 = \frac{1}{2\kappa^2} \left( -\square (0) \left( g^{(0)} \rho \lambda \delta g_{\rho \lambda} \right) + \frac{1}{2} R^{(0)} \left( g^{(0)} \rho \lambda \delta g_{\rho \lambda} \right) - 2 \delta g_{\rho \sigma} R^{(0)} \rho \sigma \right) + \frac{1}{2} \delta T_{\text{matter}}.
\]

(5)

By assuming the FRW space-time with flat spatial part [13], we have

\[
\Gamma^i_{ij} = a^2 H \delta_{ij}, \quad \Gamma^i_{jt} = \Gamma^i_{jt} = H \delta^i_j, \quad \Gamma^i_{jk} = \Gamma^i_{jk}, \quad R_{ij} = -\left( \dot{H} + 2H^2 \right) a^2 \delta_{ij}, \quad R_{ijkl} = a^4 H^2 \left( \delta_{ik} \delta_{lj} - \delta_{il} \delta_{kj} \right), \quad R_{tt} = -3 \left( H + H^2 \right), \quad R_{ij} = a^2 \left( \dot{H} + 3H^2 \right) \delta_{ij}, \quad R = 6 \dot{H} + 12H^2, \quad \text{other components} = 0.
\]

(6)

Then \((t, t), (i, j), (t, i)\) components of (1) have the following forms,

\[
0 = \frac{1}{2\kappa^2} \left( \frac{1}{2} \square (0) \delta g_{tt} + \frac{1}{2} \partial_t^2 \left( g^{(0)} \rho \lambda \delta g_{\rho \lambda} \right) + \frac{1}{2} \square (0) \left( g^{(0)} \rho \sigma \delta g_{\rho \sigma} \right) \\
- \frac{1}{2} \left( \dot{H} - H^2 \right) \left( g^{(0)} ij \delta g_{ij} \right) - \frac{3}{2} \left( \dot{H} - H^2 \right) \delta g_{tt} \right) + \frac{1}{2} \delta T_{\text{matter} \, tt}.
\]

(7)
We define the rotation tensor
\[ g_{ij} = \frac{1}{2} \delta g_{ij} + \frac{1}{2} (\partial_i \partial_j - H \delta_{ij} \partial_3) \left( g^{(0)} \rho^\alpha \delta g_{\rho \lambda} \right) - \frac{1}{2} g^{(0)} \partial (g^{(0)} \rho^\sigma \delta g_{\rho \sigma}) + \frac{1}{2} (H + H^2) g^{(0)} \delta g_{tt}, \]
\[ + 2 \left( H + H^2 \right) \delta g_{ij} - \frac{1}{2} g^{(0)} \left( 2 \dot{H} + 2H^2 \right) \left( g^{(0)} \delta g_{kl} \right) + \frac{1}{2} \delta T_{\text{matter}ij}, \]
(8)

or the bulk viscosity by
\[ \theta = \frac{\dot{\rho}}{\rho} \]
We find
\[ \theta \equiv \frac{\dot{\rho}}{\rho} \]
We also define the scalar expansion
\[ \sigma = \rho^\alpha U_{\alpha}, \]
where
\[ \theta = \frac{\dot{\rho}}{\rho} \]
and the bulk viscosity
\[ \eta \equiv \frac{\dot{\rho}}{\rho} \]
and the pressure
\[ p = \rho \frac{\dot{\rho}}{\rho} \]
Hence, we have
\[ p \equiv \rho \dot{\rho} \]
In the coordinates comoving with the viscous fluid the four velocity
(9)
We now consider the energy-momentum tensor
\[ T_{\mu \nu} = \rho U_{\mu} U_{\nu} + (p - \dot{\rho} \theta) h_{\mu \nu} - 2\eta \sigma_{\mu \nu}. \]
(10)

\[ h_{\mu \nu} = g_{\mu \nu} + U_{\mu} U_{\nu}. \]

We define the rotation tensor
\[ \omega_{\mu \nu} \]
and the expansion tensor
\[ \theta_{\mu \nu} \]
as follows,
\[ \omega_{\mu \nu} = \frac{1}{2} \left( U_{\mu ;\alpha} h_{\nu}^\alpha - U_{\nu ;\alpha} h_{\mu}^\alpha \right), \]
\[ \theta_{\mu \nu} = \frac{1}{2} \left( U_{\mu ;\alpha} h_{\nu}^\alpha + U_{\nu ;\alpha} h_{\mu}^\alpha \right), \]
(11)
(12)
We also define the scalar expansion
\[ \theta = \frac{\dot{\rho}}{\rho} \]
by
\[ \theta_{\mu} = \frac{\dot{\rho}}{\rho} \]
In the FRW space-time, whose metric is given by
\[ ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2, \]
(13)
we find \( \theta = 3H \). The shear tensor is given by
\[ \sigma_{\mu \nu} = \theta_{\mu \nu} - \frac{\theta}{3} h_{\mu \nu} = \theta_{\mu \nu} - H h_{\mu \nu}. \]
(14)
We should note that \( \sigma_{\mu \nu} \) = 0 by definition. We may decompose the covariant derivative of \( U_{\mu}, U_{\mu \nu}, \) as follows,
\[ U_{\mu ;\nu} = \omega_{\mu \nu} + \sigma_{\mu \nu} + H h_{\mu \nu} - A_{\mu} U_{\nu}, \]
(15)
where \( A_{\mu} = \dot{U}_{\mu} = U^{\alpha} U_{\mu ;\alpha} \) is the four-acceleration of the fluid.

By assuming the temperature \( T \) is constant, the energy-momentum tensor
\[ T_{\mu \nu} \]
of fluid is given by
\[ T_{\mu \nu} = \rho U_{\mu} U_{\nu} + (p - \dot{\rho} \theta) h_{\mu \nu} - 2\eta \sigma_{\mu \nu}. \]
(16)
Here
\[ p_{\text{eff}} \equiv p - \dot{\rho} \theta = p - 3\dot{H}, \]
(17)
is the effective pressure, lower than \( p \) because of the inequality \( \dot{\zeta} \geq 0 \) which in turn is a consequence of thermodynamics.

IV. GRAVITATIONAL WAVES IN A VISCOUS FLUID

We now consider the energy-momentum tensor \( \delta T_{\mu \nu} \) for viscous fluid in (16). We should note that the energy density \( \rho \) and the pressure \( p \) depend on the metric in general. For simplicity, we assume,
\[ \delta \rho = \rho^\mu \delta g_{\mu \nu}, \quad \delta p = p^\mu \delta g_{\mu \nu}. \]
(18)
The shear viscosity \( \eta \) and the bulk viscosity \( \zeta \) may also depend on the energy density \( \rho \) and the pressure \( p \). In fact, it is often assumed to \( \dot{\zeta} \propto \rho^{\lambda}, \) with a constant. Because we have assumed that the energy density \( \rho \) and the pressure \( p \) should depend on metric as in (18), we may write
\[ \delta \eta = \eta(p) \delta \rho + \eta(p) \delta p = \eta(p) \delta g_{\mu \nu} \equiv \left( \eta(p) \rho^\mu \rho^\nu + \eta(p) p^\mu p^\nu \right) \delta g_{\mu \nu}, \]
\[ \delta \zeta = \zeta^{(p)} \delta \rho + \zeta^{(p)} \delta p = \zeta^{\mu \nu} \delta g_{\mu \nu} = \left( \zeta^{(p)} \rho^{\mu \nu} + \zeta^{(p)} p^{\mu \nu} \right) \delta g_{\mu \nu}. \]  

In the FRW space-time \[ \text{(13)} \] we may assume \( \rho^{\mu \nu}, \ p^{\mu \nu}, \ \eta^{\mu \nu}, \) and \( \zeta^{\mu \nu} \) only depend on the cosmological time \( t \) and do not depend on the spatial coordinates \( (x^i) \). Because \( U^\mu U_\mu = -1 \), the variation of \( U^\mu \) should satisfy the condition

\[ 0 = 2 (\delta U^\mu) + U^\mu U^\nu \delta g_{\mu \nu} = U^\mu (2g_{\mu \nu} \delta U^\nu + \delta g_{\mu \nu} U^\nu), \]

which tells

\[ \delta U^\mu = \frac{1}{2} g^{\mu \rho} (\delta g_{\rho \nu} U^\nu + l_\rho). \]

Here \( l_\rho \) is a vector satisfying the condition \( U^\mu l_\mu = 0 \). The vector \( l_\rho \) will be determined later. Eq. \[ \text{(12)} \] also tells that

\[ \delta \theta_{\mu \nu} = \frac{1}{2} \left( \delta U_{\mu ; \alpha} h^\alpha _\nu - \frac{1}{2} g^{\alpha \lambda} (\nabla_\mu g_{\alpha \lambda} + \nabla_\alpha g_{\mu \lambda} - \nabla_\lambda g_{\mu \alpha}) U_{\nu} h^\alpha _\mu - U_{\nu ; \alpha} U^\alpha h^\mu _\nu \right) \]

\[ + \delta U_{\nu ; \alpha} h^\alpha _\mu - \frac{1}{2} g^{\alpha \lambda} (\nabla_\nu g_{\alpha \lambda} + \nabla_\alpha g_{\nu \lambda} - \nabla_\lambda g_{\nu \alpha}) U_{\mu} h^\alpha _\nu - U_{\nu ; \alpha} U^\alpha g_{\mu \lambda} \delta g_{\nu \lambda} - U_{\nu ; \alpha} \delta U^\alpha U^\mu \right). \]

Therefore we find

\[ \delta \theta = - g^{\mu \rho} \delta g_{\mu \nu} g^{\nu \sigma} \theta_{\rho \sigma} + \delta U_{\mu ; \alpha} h^\alpha _\nu - \frac{1}{2} g^{\alpha \lambda} (\nabla_\mu g_{\alpha \lambda} + \nabla_\alpha g_{\mu \lambda} - \nabla_\lambda g_{\mu \alpha}) U_{\nu} h^\alpha _\mu - \delta U_{\nu ; \alpha} U^\alpha \delta g_{\nu \lambda} \delta g_{\mu \lambda} - U_{\nu} \delta U^\alpha U^\mu \]

and

\[ \delta \sigma_{\mu \nu} = \delta \theta_{\mu \nu} - \frac{1}{3} \delta h_{\mu \nu} - \frac{\theta}{3} (\delta g_{\mu \nu} + \delta U_{\mu ; \lambda} U^\lambda _\nu + \delta U_{\nu ; \lambda} U^\lambda _\mu) \]

\[ = \frac{1}{2} \left( \delta U_{\mu ; \alpha} h^\alpha _\nu - \frac{1}{2} g^{\alpha \lambda} (\nabla_\mu g_{\alpha \lambda} + \nabla_\alpha g_{\mu \lambda} - \nabla_\lambda g_{\mu \alpha}) U_{\nu} h^\alpha _\mu - U_{\nu ; \alpha} U^\alpha \delta g_{\nu \lambda} \delta g_{\mu \lambda} - U_{\nu} \delta U^\alpha U^\mu \right) \]

\[ + \delta U_{\nu ; \alpha} h^\alpha _\mu - \frac{1}{2} g^{\alpha \lambda} (\nabla_\nu g_{\alpha \lambda} + \nabla_\alpha g_{\nu \lambda} - \nabla_\lambda g_{\nu \alpha}) U_{\mu} h^\alpha _\nu - U_{\nu ; \alpha} U^\alpha g_{\mu \lambda} \delta U_{\nu \lambda} - U_{\nu} \delta U^\alpha U^\mu \right). \]

Therefore we find

\[ \delta \theta = - g^{\mu \rho} \delta g_{\mu \nu} g^{\nu \sigma} \theta_{\rho \sigma} + \delta U_{\mu ; \alpha} h^\alpha _\nu - \frac{1}{2} g^{\alpha \lambda} (\nabla_\mu g_{\alpha \lambda} + \nabla_\alpha g_{\mu \lambda} - \nabla_\lambda g_{\mu \alpha}) U_{\nu} h^\alpha _\mu - \delta U_{\nu ; \alpha} U^\alpha \delta g_{\nu \lambda} \delta g_{\mu \lambda} - U_{\nu} \delta U^\alpha U^\mu \]

Then \( \delta T_{\mu \nu} \) has a rather complicated form,

\[ \delta T_{\mu \nu} = \delta \rho U_{\nu} U_\mu + (\rho + p - \zeta \theta) \delta U_{\nu} U_\mu + (\rho + p - \zeta \theta) U_{\mu \delta} U_\nu + (\delta p - \delta \zeta \theta - \zeta \theta \delta U_{\nu} U_\mu) \]

\[ + (p - \zeta \theta) \delta g_{\mu \nu} - 2 \delta \rho \sigma_{\mu \nu} - 2 \eta \delta \sigma_{\mu \nu}. \]

We will now need to simplify the formalism.

A. Case of Bulk Viscosity

First we ignore the shear viscosity by putting \( \eta = 0 \). We also investigate the propagation of the massless spin two mode, where

\[ \delta g_{tt} = \delta g_{\alpha \beta} = 0, \quad \sum_{i=1,2,3} \delta g_{tt} = 0, \quad i = 1, 2, 3. \]

Then because \( \delta g_{tt} = 0 \), we may assume \( \delta U^\mu = 0 \). Furthermore if we assume \( \rho^i, \ p^i \propto \delta^i \), we find \( \delta \rho = \delta p = \delta \zeta = \delta \eta = 0 \). In the FRW space-time \[ \text{(13)} \], we find

\[ \theta_{tt} = \theta_{\alpha \beta} = U_{tt} = U_{tt} = U_{tt} = 0, \quad \theta_{ij} = U_{i ; j} = a^2 H \delta_{ij}, \quad \theta = 3H. \]
Then by using (22), we find
\[ \delta \theta_{tt} = \delta \theta_{ti} = \delta \theta_{ti} = 0, \quad \delta \theta_{ij} = \frac{1}{2} (2 \nabla_i \delta g_{jt} - \nabla_t \delta g_{ij}) = - \frac{1}{2} \partial_i \delta g_{ij}, \quad \delta \theta = 0. \] (28)

Then we obtain
\[ \delta T_{tt} = \delta T_{ti} = \delta T_{ti} = 0, \quad \delta T_{ij} = (p - 3H\zeta) \delta g_{ij}. \] (29)

By using (26) and (29), we see that Eqs. (7) and (9) are trivially satisfied. On the other hand, Eq. (8) has the following form,
\[ 0 = \frac{1}{2\kappa^2} \left( \frac{1}{2} \Box (0) \delta g_{ij} + 2 (H + H^2) \delta g_{ij} \right) + \frac{1}{2} (p - 3H\zeta) \delta g_{ij} \]
\[ = \frac{1}{2\kappa^2} \left( \frac{1}{2} (-\partial_i \delta g_{ij} + a^{-2} \triangle \delta g_{ij}) + \left( 3\dot{H} + 4H^2 \right) \delta g_{ij} \right) + \frac{1}{2} (p - 3H\zeta) \delta g_{ij}, \] (30)

Therefore the massless gravitational wave with spin two exists even if the bulk viscosity is nonzero. This is consistent with Ref. [20].

B. Case of Shear Viscosity

We now consider the case when there exists only a shear viscosity, \( \eta \neq 0 \). First we assume (26) and therefore \( \delta U^\mu = 0 \). Then (24) gives
\[ \delta \sigma_{ti} = 0, \] (31)
\[ \delta \sigma_{i} = \delta \sigma_{it} = \frac{1}{2} \left( \frac{1}{2} g^{\alpha \lambda} (\nabla_i \delta g_{\alpha \lambda} + \nabla_\alpha \delta g_{i \lambda} - \nabla_\lambda \delta g_{i \alpha}) U^\alpha h^\lambda_{ij} \right) = 0, \] (32)
\[ \delta \sigma_{ij} = \frac{1}{2} \left( \frac{1}{2} g^{\alpha \lambda} (\nabla_i \delta g_{\alpha \lambda} + \nabla_\alpha \delta g_{i \lambda} - \nabla_\lambda \delta g_{i \alpha}) U^\alpha h^\lambda_{ij} - \frac{1}{2} g^{\alpha \lambda} (\nabla_j \delta g_{\alpha \lambda} + \nabla_\alpha \delta g_{j \lambda} - \nabla_\lambda \delta g_{j \alpha}) U^\alpha h^\lambda_{ij} \right) \]
\[ - \frac{1}{3} \left( -g^{\eta \zeta} \delta g_{\eta \zeta} g^{\rho \sigma} \rho_{\sigma i} - \frac{1}{2} g^{\alpha \lambda} (\nabla_\eta \delta g_{\alpha \lambda} + \nabla_\lambda \delta g_{\eta \alpha} - \nabla_\alpha \delta g_{\eta \lambda}) U^\lambda h_{ij} \right) h_{ij} - \frac{\theta}{3} \delta g_{ij} \]
\[ = - \frac{1}{2} \partial_i \delta g_{ij} - H \delta g_{ij}. \] (33)

Then Eqs. (7) and (9) are trivially satisfied, again. On the other hand, Eq. (8) has the following form,
\[ 0 = \frac{1}{2\kappa^2} \left( \frac{1}{2} \Box (0) \delta g_{ij} + 2 (H + H^2) \delta g_{ij} \right) + \frac{1}{2} (p - 3H\zeta) \delta g_{ij} - 2\eta \left( -\frac{1}{2} \partial_i \delta g_{ij} - H \delta g_{ij} \right) \]
\[ = \frac{1}{2\kappa^2} \left( \frac{1}{2} (-\partial_i \delta g_{ij} + a^{-2} \triangle \delta g_{ij}) + \left( 3\dot{H} + 4H^2 \right) \delta g_{ij} \right) + \frac{1}{2} (p - 3H\zeta) \delta g_{ij} - 2\eta \left( -\frac{1}{2} \partial_i \delta g_{ij} - H \delta g_{ij} \right), \] (34)

This equation tells that also in this case a massless spin two gravitational wave exists. We should note, however, the presence of the term \( \eta \partial_i \delta g_{ij} \). If \( \eta > 0 \), Eq. (34) tells that the gravitational wave is enhanced. On the other hand, if \( \eta < 0 \), the term \( \eta \partial_i \delta g_{ij} \) expresses a dissipation of the gravitational wave. The absorbed wave energy is transformed into heat.

V. COSMOLOGICAL SPECULATIONS

Because bulk viscosity generates effectively a negative pressure (positive tensile stress), a fluid endowed with bulk viscosity will lead to an accelerated expansion. Although bulk viscosity and shear viscosity have different physical backgrounds (as mentioned the shear viscosity is usually far the greatest among them), it is natural to include them both in the formalism.

For the recently observed gravitational waves [27–32], the distances between the sources and the earth are about a few hundreds Mpc. Because no dissipation or enhancement of the gravitational waves has been observed we find the following constraint on the shear viscosity \( \eta_0 \) in the present universe,
\[ |\kappa^2 \eta_0| \ll (10^3 \text{Mpc})^{-1}, \] (35)
even if the accelerated expansion of the universe is generated by the viscous fluid. Let us express this constraint in physical units: as 1 Mpc=3.086×10^{22} \text{ m} and 1 s=3×10^8 \text{ m} in geometric units, the right hand side of Eq. (33) becomes 10^{-17} \text{ s}^{-1} in physical units. Then, since κ^2 = 1.87 \times 10^{-26} \text{ m/kg}, we can write the constraint as

\[ \eta_0 \ll 5 \times 10^8 \text{ Pa s.} \]  

(36)

Another point we note is that if the dark energy is represented by the viscous fluid, the present bulk viscosity \( \zeta \) satisfies

\[ \frac{1}{\kappa^2 \zeta_0} \sim \frac{1}{H} \sim 10^4 \text{ Mpc}. \]  

(37)

or in physical units

\[ \zeta_0 \sim 5 \times 10^7 \text{ Pa s.} \]  

(38)

It is notable that this value of \( \zeta_0 \) is only one or two orders of magnitude greater than the value derived from comparison with experiments. We return to this point in Sec. VII.

On the other hand, if the inflation in the early universe could be generated by the viscous fluid, we may expect that the shear viscosity could be large. Then the primordial gravitational wave may have been absorbed into the viscous fluid. Then even if the primordial gravitational wave may not be observed in the future experiments, this might be due to the viscous fluid. If the primordial gravitational wave was absorbed into the viscous fluid, the energy of the gravitational wave could be also absorbed into the fluid and the internal energy of the fluid like thermal energy might be increased. Then if the energy density becomes large enough, that is, the energy \( E \) inside the sphere with a radius \( R \) becomes larger than \( \frac{R}{2\eta} \), black hole could have been created. This tells that in the viscous fluid model for the inflation, the primordial gravitational wave may be suppressed but more primordial black hole might be created. In fact, the period of the inflation could be estimated to be \( 10^{-34} \text{ sec.} \sim 10^{15} \text{ eV} = 10^{10} \text{ GeV} \). Here we have used the following natural unit,

\[ 1 \text{ s} = 1.5192674 \times 10^{15} \text{ eV}^{-1}, \quad 1 \text{ m} = 5.0677307 \times 10^6 \text{ eV}^{-1}, \quad 1 \text{ kg} = 5.6095886 \times 10^{35} \text{ eV}. \]  

(39)

This tells that if the primordial gravitational wave is detected, we find

\[ |\eta \kappa^2| \ll 10^{19} \text{ eV} = 10^{10} \text{ GeV}. \]  

(40)

On the other hand, if the scale of the inflation is the GUT scale \( \sim 10^{15} \text{ GeV} \), we find

\[ |\kappa^2 \zeta| \sim H \sim 10^{-19+2 \times 15} = 10^{11} \text{ GeV}. \]  

(41)

By comparing (40) and (41), if \(|\zeta| \sim |\eta| \), Eq. (40) could not be satisfied and therefore the primordial gravitational wave may be absorbed into the viscous fluid. Then if the primordial gravitational wave will be detected by the future observation, we obtain the constraint (40) but if the inflation was generated by the viscous fluid, the primordial gravitational wave might not be detected.

VI. FIRST EXAMPLE: EVOLUTION OF GRAVITATIONAL WAVES NEAR THE BIG RIP SINGULARITY

If the universe is filled with a viscous fluid, a Big Rip singularity may appear. In this section, we investigate the behavior of the gravitational field close to the singularity.

A. Cosmology in the Presence of a Viscous Fluid

Before investigating the propagation and the evolution of the gravitational wave near the Big Rip singularity, we will review the evolution of the viscous fluid by neglecting other components like dark matter and ordinary matter. The energy-momentum tensor in (16) gives the following FRW equations

\[ \frac{3}{\kappa^2} H^2 = \rho, \quad \frac{1}{\kappa^2} \left(3H^2 + 2\dot{H}\right) = p - 3\zeta H; \]  

(42)
which gives the conservation law,

\[ 0 = \dot{\rho} + 3H (\rho + p - 3\zeta H) . \quad (43) \]

We note that the shear viscosity does not contribute to the background evolution. We also assume \( p \propto \rho \) and \( \zeta \propto \rho^\lambda \) with a constant \( \lambda \) as follows,

\[ p = \omega \rho , \quad \zeta = \tilde{\zeta}_0 \rho^\lambda . \quad (44) \]

Here \( \omega \) is the equation of state (EoS) parameter and \( \tilde{\zeta}_0 \) is a constant. (The subscript zero in this section refers to an arbitrary starting point near the singularity, not to the present time as above.) Then by using the first FRW equation in (42), we can rewrite (43) as follows,

\[ 0 = \dot{\rho} + \kappa (\omega + 1) 3\omega \rho^\omega - 3\kappa^2 \tilde{\zeta}_0 \rho^{\lambda + 1} , \]

Eq. (45) can be rewritten as,

\[ w_{\text{eff}} = \omega - 3\omega \tilde{\zeta}_0 , \quad (46) \]

which is nothing but the conservation law for a standard perfect fluid with EoS parameter \( w_{\text{eff}} \). Therefore even if \( \omega > -1 \), in case \( w_{\text{eff}} < -1 \), the fluid becomes effectively phantom and generates a Big Rip singularity, where \( H \) and \( a \) behave as

\[ H = \frac{h_0}{t_s - t} , \quad a = a_0 (t_s - t)^{-h_0} , \quad h_0 \equiv -\frac{2}{3 (w_{\text{eff}} + 1)} > 0 . \quad (48) \]

Hence there occurs a Big Rip singularity at \( t = t_s \).

**B. Propagation Near the Big Rip Singularity**

By using (44), we consider the propagation and evolution of gravitational waves near the singularity in (43). The equation of state (44) is adopted. Because we are considering spatially flat background, we consider a plane wave, where \( \delta g_{ij} \propto e^{i k \cdot x} \). We also assume that \( \zeta \), like \( \eta \), is proportional to \( \rho^\frac{1}{2} \propto H \) and we write

\[ \eta = \frac{\tilde{\eta}_0}{k^2} H . \quad (49) \]

Then Eq. (44) has the following form,

\[
0 \sim \frac{1}{2} \left( \frac{1}{2} \left( -\partial_t^2 \delta g_{ij} - a_0^{-2} (t_s - t)^2 \omega_0 k^2 \delta g_{ij} + \frac{3h_0 + 4h_0^2}{(t_s - t)^2} \delta g_{ij} \right) + \frac{3w_{\text{eff}} h_0^2}{2k^2 (t_s - t)^2} \right) \delta g_{ij} \\
- \frac{2\tilde{\eta}_0 h_0}{k^2 (t_s - t)} \left( \frac{1}{2} \partial_t \delta g_{ij} - \frac{h_0}{t_s - t} \delta g_{ij} \right) \\
\sim \frac{1}{4k^2} \left\{ -\partial_t^2 \delta g_{ij} + \frac{2\tilde{\eta}_0 h_0}{t_s - t} \partial_t \delta g_{ij} + \frac{3h_0 + 4h_0^2 + 6w_{\text{eff}} h_0^2}{(t_s - t)^2} \delta g_{ij} \right\} ,
\]

Because the last equation in (50) is homogeneous, the solution can be obtained by assuming

\[ \delta g_{ij} \propto (t_s - t)^{\alpha} . \quad (51) \]

Here \( \alpha \) is a constant. Then Eq. (50) can be rewritten as an algebraic equation,

\[
0 = \alpha^2 + (2\tilde{\eta}_0 h_0 - 1) \alpha - (3h_0 + 4h_0^2 + 6w_{\text{eff}} h_0^2 + 8\tilde{\eta}_0 h_0^2) \\
= \alpha^2 + (2\tilde{\eta}_0 h_0 - 1) \alpha - (-h_0 - 2h_0^2 + 8\tilde{\eta}_0 h_0^2) .
\]

(52)
Here we have deleted $w_{\text{eff}}$ by using the definition of $h_0$ in \[15\). The solution of \[52\) is given by

$$
\alpha = \alpha_\pm = \frac{1}{2} \left( -2\tilde{\eta}_0 h_0 + 1 \pm \sqrt{(2\tilde{\eta}_0 h_0 - 1)^2 + 4(-h_0 - 2h_0^2 + 8\tilde{\eta}_0 h_0^2)} \right)
$$

$$
= \frac{1}{2} \left( -2\tilde{\eta}_0 h_0 + 1 \pm \sqrt{4h_0^2\tilde{\eta}_0^2 + (-4h_0 + 32h_0^2)\tilde{\eta}_0 + 1 - 4h_0 - 8h_0^2} \right).
$$

(53)

When $\tilde{\eta}_0 = 0$, that is, there is no shear viscosity, $\alpha_\pm$ is always positive because we are assuming $h_0 > 0$ and therefore $\delta g_{ij}$ goes to vanish near the singularity. However in case $-h_0 - 2h_0^2 + 8\tilde{\eta}_0 h_0^2 > 0$, that is,

$$
\tilde{\eta}_0 > \frac{1}{4} + \frac{1}{8h_0},
$$

(54)

$\alpha_-$ becomes negative and therefore $\delta g_{ij}$ will be enhanced near the singularity. We note that $\tilde{\eta}_0$ is positive if Eq. \[54\) is satisfied.

We now consider more details. The determinant $D$ of the quadratic algebraic equation \[52\) with respect to $\alpha$ is given by

$$
D = 4h_0^2\tilde{\eta}_0^2 + (-4h_0 + 32h_0^2)\tilde{\eta}_0 + 1 - 4h_0 - 8h_0^2.
$$

(55)

We note that $D > 0$ when $h_0 < \frac{1}{8}$ or even if $h_0 > \frac{1}{2}$ when

$$
\tilde{\eta}_0 > \frac{1}{2h_0} - 4 + \sqrt{\frac{1}{4h_0^2} - \frac{4}{h_0} + 16 - \frac{1}{4h_0^2} + \frac{1}{h_0} + 2} = \frac{1}{2h_0} - 4 + \sqrt{-\frac{3}{h_0} + 18} \quad \text{or} \quad \tilde{\eta}_0 < \frac{1}{2h_0} - 4 - \sqrt{-\frac{3}{h_0} + 18}.
$$

(56)

On the other hand, $D < 0$ if $h_0 > \frac{1}{2}$ and

$$
\frac{1}{2h_0} - 4 - \sqrt{-\frac{3}{h_0} + 18} < \tilde{\eta}_0 < \frac{1}{2h_0} - 4 + \sqrt{-\frac{3}{h_0} + 18}.
$$

(57)

If $D > 0$, $\alpha_\pm$ are real but if $D < 0$, $\alpha_\pm$ becomes complex and $(\alpha_+)^* = \alpha_-\), therefore the amplitude of the gravitational wave oscillates but if $\tilde{\eta}_0 > \frac{1}{2h_0}$, the amplitude decreases and if $\tilde{\eta}_0 < \frac{1}{2h_0}$, the amplitude increases. We note that the solution of the algebraic equation

$$
\frac{1}{4} + \frac{1}{8h_0} = \frac{1}{2h_0},
$$

(58)

is $h_0 = \frac{3}{2}$. Then when $h_0 > \frac{3}{2}$, $\frac{1}{4} + \frac{1}{8h_0} > \frac{1}{2h_0}$ and when $h_0 < \frac{3}{2}$, $\frac{1}{4} + \frac{1}{8h_0} > \frac{1}{2h_0}$. On the other hand, we find

$$
\frac{1}{2h_0} - 4 - \sqrt{-\frac{3}{h_0} + 18} < \frac{1}{2h_0},
$$

(59)

for positive $h_0$ and the algebraic equation

$$
\frac{1}{2h_0} - 4 + \sqrt{-\frac{3}{h_0} + 18} = \frac{1}{2h_0},
$$

(60)

has the solution $h_0 = \frac{3}{2}$, again. Therefore when $h_0 > \frac{3}{2}$, $\frac{1}{2h_0} - 4 + \sqrt{-\frac{3}{h_0} + 18} > \frac{1}{2h_0}$ and when $\frac{3}{2} > h_0 > \frac{1}{6}$, $\frac{1}{2h_0} - 4 + \sqrt{-\frac{3}{h_0} + 18} < \frac{1}{2h_0}$. We also find that as long as $h_0 > \frac{1}{6}$, $\frac{1}{2h_0} - 4 - \sqrt{-\frac{3}{h_0} + 18} < \frac{1}{4} + \frac{1}{8h_0}$. The solution of the algebraic equation

$$
\frac{1}{2h_0} - 4 + \sqrt{-\frac{3}{h_0} + 18} = \frac{1}{4} + \frac{1}{8h_0},
$$

(61)

is only $h_0 = \frac{3}{2}$. Then when $h_0 \neq \frac{3}{2}$, $\frac{1}{2h_0} - 4 + \sqrt{-\frac{3}{h_0} + 18} < \frac{1}{4} + \frac{1}{8h_0}$.

Then the above results can be summarized as follows,

- Case $0 < h_0 < \frac{1}{6}$;
Therefore if \( \bar{\eta}_0 < \frac{1}{4} + \frac{1}{8h_0} \), the amplitude of the gravitational wave is monotonically decreasing.

- When \( \bar{\eta}_0 > \frac{1}{4} + \frac{1}{8h_0} \), in addition to the mode where the amplitude of the gravitational wave is monotonically decreasing, there appears another mode where the amplitude of the gravitational wave is monotonically increasing.

- Case \( \frac{1}{6} < h_0 < \frac{3}{2} \):
  - When \( \bar{\eta}_0 < \frac{1}{2h_0} - 4 - \sqrt{-\frac{3}{h_0} + 18} \) or \( \frac{1}{2h_0} - 4 + \sqrt{-\frac{3}{h_0} + 18} < \bar{\eta}_0 < \frac{1}{4} + \frac{1}{8h_0} \), the amplitude of the gravitational wave is monotonically decreasing.
  - When \( \frac{1}{2h_0} - 4 - \sqrt{-\frac{3}{h_0} + 18} < \bar{\eta}_0 < \frac{1}{2h_0} - 4 + \sqrt{-\frac{3}{h_0} + 18} \), the amplitude of the gravitational wave is oscillating and decreasing.

- When \( \bar{\eta}_0 > \frac{1}{4} + \frac{1}{8h_0} \), in addition to the mode where the amplitude of the gravitational wave is monotonically decreasing, there appears another mode where the amplitude of the gravitational wave is monotonically increasing.

- Case \( h_0 > \frac{3}{2} \):
  - When \( \bar{\eta}_0 < \frac{1}{2h_0} - 4 - \sqrt{-\frac{3}{h_0} + 18} \) or \( \frac{1}{2h_0} - 4 + \sqrt{-\frac{3}{h_0} + 18} < \bar{\eta}_0 < \frac{1}{8} + \frac{1}{8h_0} \), the amplitude of the gravitational wave is monotonically decreasing.
  - When \( \frac{1}{2h_0} < \bar{\eta}_0 < \frac{1}{2h_0} - 4 + \sqrt{-\frac{3}{h_0} + 18} \), the amplitude is decreasing with oscillation.
  - When \( \bar{\eta}_0 > \frac{1}{4} + \frac{1}{8h_0} \), in addition to the mode where the amplitude of the gravitational wave is monotonically decreasing, there appears another mode where the amplitude of the gravitational wave is monotonically increasing.

We may compare the above result with the case that there is no shear viscosity, \( \eta = 0 \). Then Eq. (53) reduces to

\[
\alpha = \frac{1}{2} \left( 1 \pm \sqrt{1 - 4h_0 - 8h_0^2} \right). \tag{62}
\]

Therefore if \( 1 - 4h_0 - 8h_0^2 > 0 \), that is,

\[
0 < h_0 < \frac{-2 + \sqrt{3}}{8}, \tag{63}
\]

the amplitude of the gravitational wave decreases monotonically. If \( 1 - 4h_0 - 8h_0^2 < 0 \), that is,

\[
h_0 > \frac{-2 + \sqrt{3}}{8}, \tag{64}
\]

the amplitude of the gravitational wave decreases with oscillation. Then we find that there is no case in which the amplitude is enhanced, as in the case of \( \eta \neq 0 \).

VII. SECOND EXAMPLE: GRAVITATIONAL WAVES IN THE LEPTON ERA

As second example we will go to an opposite extreme and apply the above formalism to the early universe, specifically to the end point of the lepton era. The reason for this choice is the following.

The lepton era is characterized by a temperature drop from \( 10^{12} \) K (100 MeV) to \( 10^{10} \) K (1 MeV). In this period the particles present were essentially photons, neutrinos, and electrons, together with their antiparticles. Transfer of momentum took place among the particles because the relativistic and non-relativistic species decreased in temperature following different powers of the scale factor: relativistic particles decreased as \( T \propto a^{-1} \) while the non-relativistic ones decreased as \( T \propto a^{-2} \). The bulk viscosity arose because the photons, electrons and the \((e, \mu, \tau)\) leptons had short mean free paths compared with those of the neutrinos. The maximum momentum transfer, taking place at the instant of neutrino decoupling, \( 10^{10} \) K, marked the maximum of the bulk viscosity.
The theory of viscosities in this very early region was worked out in the extensive study [33], with use of the relativistic Boltzmann equation. Related works are [34] and [35]. A recent detailed treatment, emphasizing the connection with particle physics, is given in [36]. The general behavior of the bulk viscosity in the lepton era is that it is varying very much, by 7-8 orders of magnitude, when regarded as a function of $T$. The reason for this is the influential, but rapidly decaying, $\tau$ and $\mu$ mesons. The maximum value at neutrino decoupling is [36]

$$\zeta \approx 10^{22} \text{ Pa s}. \quad (65)$$

An important point is that this value can be connected with the present value $\zeta_0$ of the bulk viscosity, via the relation

$$\zeta = \zeta_0 \left( \frac{\rho}{\rho_0} \right)^{\lambda}, \quad (66)$$

with $\lambda$ a constant (here again we let subscript zero refer to the present time).

Much work has been done on the phenomenological level, by comparing measured values of $H = H(z)$ with predictions from the FRW equations, in order to determine the value of $\zeta_0$ (for a recent review, see [3]). From our own investigations [13, 14], it turned out that one could restrict $\zeta_0$ to the interval

$$10^4 \text{ Pa s} \leq \zeta_0 \leq 10^6 \text{ Pa s}. \quad (67)$$

It should be noted that this equation is roughly comparable with Eq. (65) above.

Now, combining Eq. (66) with the scaling (66), putting $\lambda = 1/2$, we ended up with the value $\zeta_0 = 10^5 \text{ Pa s}$, thus in the logarithmic middle of the interval (67). This coincidence, which could hardly have been foreseen, gives support to the bulk viscosity numbers above. We here mention also the formula for the energy density as a function of temperature [36],

$$\rho = \frac{\pi^2}{30} g_{\star \rho} T^4, \quad (68)$$

with $g_{\star \rho}$ being the effective degree of freedom for $\rho$. At the instant of neutrino decoupling, $g_{\star \rho} \approx 11$.

Now move on to the shear viscosity $\eta$. From Fig. 8 in [36] we read off

$$\eta \approx 10^{25} \text{ Pa s} \quad (69)$$

at neutrino decoupling. The shear viscosity is thus much greater than the bulk viscosity, as is usual in hydrodynamics. It should be borne in mind that this is a theoretical result, derived from the relativistic Boltzmann equation; there is not an experimental link to the present value $\eta_0$ as was the case for the bulk viscosity. The reason is of course the avoidance of spatial isotropy as soon as shear viscosity is concerned. It might seem natural nevertheless to assume simply that the scaling relation (66) holds approximately also for the shear viscosity,

$$\eta = \eta_0 \left( \frac{\rho}{\rho_0} \right)^{\lambda}. \quad (70)$$

($\eta_0 = \kappa \sqrt{\frac{3}{2\rho_0}}$ in (49) when $\lambda = \frac{1}{2}$.) Then, we can calculate the present shear viscosity $\eta_0$, using (70) and (66), as

$$\eta_0 = \frac{\eta \zeta_0}{\zeta} \approx 10^8 \text{ Pa s} \quad (71)$$

for any constant $\lambda$, assuming $\zeta_0 \approx 10^5 \text{ Pa s}$.

Armed with this formalism, we can now express $\delta T_{\mu \nu}$ in (25) in such a way that the variations of the viscosity coefficients are carried back to the variations of the energy density,

$$\delta \zeta = \frac{\lambda \zeta_0}{\rho_0} \rho^{\lambda-1} \delta \rho, \quad (72)$$

$$\delta \eta = \frac{\lambda \eta_0}{\rho_0^2} \rho^{\lambda-1} \delta \rho. \quad (73)$$

Moreover, for the pressure we can assume, as usual in the radiation epoch,

$$p = \frac{1}{3} \rho. \quad (74)$$
We now consider the gravitational waves. This case is simplified, since \( \delta \rho = \delta p = \delta \zeta = \delta \eta = 0 \). Then by using (84), one gets
\[
0 = \frac{1}{2\kappa^2} \left( \frac{1}{2} (-\partial_i^2 \delta g_{ij} + a^{-2} \Delta \delta g_{ij}) + (3\dot{H} + 4H^2) \delta g_{ij} \right) + \frac{1}{6} (\rho - 9H \zeta) \delta g_{ij} + 2\eta \left( \frac{1}{2} \partial_i \delta g_{ij} + H \delta g_{ij} \right). \tag{75}
\]
As \( \zeta \ll \eta \) at the time of neutrino decoupling, we can omit the \( \zeta \) term and the equation reduces to
\[
0 = \frac{1}{2\kappa^2} \left( \frac{1}{2} (-\partial_i^2 \delta g_{ij} + a^{-2} \Delta \delta g_{ij}) + (3\dot{H} + 4H^2) \delta g_{ij} \right) + \frac{1}{6} (\rho + 12\eta H) \delta g_{ij} + \eta \partial_i \delta g_{ij} = 0 \tag{76}.
\]
We may note that at this instant, in physical units,
\[
\rho c^2 \approx 10^{25} \text{ J/m}^3, \tag{77}
\]
and moreover \( t = 1 \text{s} \) so that \( H = 1/2t = 1/2 \text{s}^{-1} \). Thus the influence from the viscous term \( 12\eta H \) is, perhaps surprisingly, of the same order of magnitude as the energy density term \( \rho c^2 \).

The equation (76) is complicated, since it contains both \( \rho \) and \( H \). We can eliminate \( \rho \) by making use of the FRW equation
\[
3H^2 = \kappa^2 \rho, \tag{78}
\]
and by taking \( \lambda = 1/2 \) we can express the shear viscosity as
\[
\eta = \frac{\eta_0}{\kappa} \sqrt{\frac{3}{\rho_0} H}. \tag{79}
\]
Equation (76) then takes the form
\[
\frac{1}{2} (-\partial_i^2 \delta g_{ij} + a^{-2} \Delta \delta g_{ij}) + 3\dot{H} \delta g_{ij} + \left( 5 + 4\kappa \eta_0 \sqrt{\frac{3}{\rho_0}} \right) H^2 \delta g_{ij} + 2\kappa \eta_0 \sqrt{\frac{3}{\rho_0}} H \partial_i \delta g_{ij} = 0. \tag{80}
\]
Introducing, as usual,
\[
\delta g_{ij} = a^2 h_{ij}, \quad |h_{ij}| \ll 1, \tag{81}
\]
we then have
\[
-\partial_i^2 (a^2 h_{ij}) + \Delta h_{ij} + 6\dot{H} a^2 h_{ij} + 2 \left( 5 + 4\kappa \eta_0 \sqrt{\frac{3}{\rho_0}} \right) H^2 a^2 h_{ij} + 4\kappa \eta_0 \sqrt{\frac{3}{\rho_0}} H \partial_i (a^2 h_{ij}) = 0. \tag{82}
\]
It is advantageous to get information about the numerical magnitudes from viscosities here. Going over to physical units we obtain, inserting \( G = 6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2 \), \( \eta_0 = 10^8 \text{ Pa s} \), \( \rho_0 \approx 10^{-26} \text{ kg/m}^3 \),
\[
4\kappa \eta_0 \sqrt{\frac{3}{\rho_0}} \to \frac{4\sqrt{4\pi G}}{c} \eta_0 \sqrt{\frac{3}{\rho_0 c^2}} \approx 3.0. \tag{83}
\]
This number is remarkably close to one. Thus Eq. (82) can be written as
\[
-\partial_i^2 (a^2 h_{ij}) + \Delta h_{ij} + 6\dot{H} a^2 h_{ij} + 16H^2 a^2 h_{ij} + 3H \partial_i (a^2 h_{ij}) = 0. \tag{84}
\]
This is the governing equation for gravitational waves on a time-dependent background. Recall what is the basis for this equation: it relies upon the connecting formulas (66) and (70) for the viscosities together with the assumption \( \lambda = 1/2 \), and also on the numerical values for \( \zeta_0 \) and \( \eta_0 \). Otherwise, there is no direct link to the neutrino decoupling time in (57), except through the values for \( a \) and \( H \). The equation consequently has a quite a general value; it is applicable to any instant within the radiation dominated epoch.

We may process this equation further by introducing conformal time \( \tau \) via \( dt = ad\tau \), letting a prime henceforth mean derivative with respect to \( \tau \). We also replace \( h_{ij} \) with the quantity \( \mu_{ij} = ah_{ij} \). Then, with \( H = a'/a = aH \) we have
\[
\partial_i^2 (a^2 \mu_{ij}) = \frac{1}{a} \left( \mu'_{ij} + \mathcal{H} \mu'_{ij} - \mathcal{H}^2 \mu_{ij} + \frac{a''}{a} \mu_{ij} \right), \tag{85}
\]
where \( \mathcal{H} = a'/a = aH \) the Hubble parameter.
and by transforming the other terms in (84) similarly we can write the equation as

$$
\mu''_{ij} - 2\mathcal{H}\mu'_{ij} - \left(8\mathcal{H}^2 + \frac{5a''}{a}\right)\mu_{ij} - \Delta\mu_{ij} = 0.
$$

Going over to Fourier space, taking \(\mu_{ij}\) to vary as \(e^{i(\mathbf{k}r - \omega t)}\), we obtain the dispersion relation

$$
\omega^2 = k^2 - 2i\mathcal{H}k - \left(8\mathcal{H}^2 + \frac{5a''}{a}\right) = 0.
$$

We assume that the gravitational wave moves on a slowly varying background (the varying metric). Then, \(k^2\) can be taken to be much larger than \(\mathcal{H}^2\) or \(a''/a\), and the relation reduces to

$$
\omega = k\sqrt{1 - 2i\mathcal{H}/k} \approx k(1 - i\mathcal{H}/k).
$$

This wave is dispersive; it corresponds formally to an electromagnetic wave in a weakly absorbing medium whose complex refractive index is \(n' = 1 + i\mathcal{H}/\omega\). The decay constant with respect to position is thus \(\mathcal{H}^{-1}\), so that the amplitude for a wave with constant \(\omega\) decays as \(e^{-\mathcal{H}r}\).

The following point ought to be noted. It might seem as if memory about viscosity is completely lost in the dispersion relation \((88)\), as only \(\mathcal{H}\) appears. However, \(\mathcal{H}\) comes from the scale factor, calculated from the FRW equations in which the viscosity makes a contribution. Also, recall that \(h_{ij}\) (or \(\mu_{ij}\)) as solved from the governing equation \((84)\) for waves on a slowly varying background relies upon equation \((83)\) which is a viscosity-related approximation.

\[\text{VIII. CONCLUDING REMARKS}\]

It should be borne in mind that basic approach to describing the cosmic fluid is after all the one of statistical mechanics, where the Boltzmann equation and the Chapman-Enskog expansion enables one to calculate the viscosity coefficients \([21, 22]\). A condition for this method to work is that the free mean path is much shorter than the wavelengths. It is not evident that this condition is satisfied for the cosmic fluid. The dominant kind of matter is dark matter, and little is known about its microscopic structure, even at the present time. And if we go to the extreme epochs, either the very early, or the very late, universe, even less is known. In spite of these concerns we have found it reasonable to make use of a viscous model for the cosmic fluid, mainly because of its simplicity and its applicability in almost all areas of fluid mechanics. There might be a microstructure in the dark matter making the viscous approach quite permissible. Notably, the viscous approach has been followed by a large group of investigators, among them the recent Ref. \([20]\).

We derived the governing equation for gravitational waves propagating in a fluid with bulk viscosity \(\zeta\) and shear viscosity \(\eta\) to be as given in \((44)\). In Sec. VI we took the equation of state to be the conventional \(p = w\rho\), and took the bulk and shear viscosities to be given respectively by Eqs. \((44)\) and \((49)\). With the forms for \(H\) and \(a\) given in \((48)\) we could then discuss whether the gravitational wave either increases, or decreases, near the Big Rip singularity. This was our first example. Our second example in Sec. VII was taken from the early universe, specifically the lepton era when the temperature dropped from \(10^{12}\) K to \(10^{10}\) K. The motivation for this choice was that at the end of this era the microscopic bulk viscosity is at maximum. In this case we took only the bulk viscosity into account, in view of the commonly accepted spatial isotropy of the cosmic fluid. We found the governing equation for gravitational waves propagating on the underlying medium, assumed to be “slowly varying”, and found the wave to behave essentially as an absorbing medium in optics where the decay constant was the inverse of the conformal Hubble parameter \(\mathcal{H} = aH\).

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