1. NRQCD ACTION AND OPERATOR

The forward-backward symmetric evolution of the heavy quark propagator \( G_t \) using the NRQCD action \[ \] is given by

\[
G_{i+1} = \left( 1 - \frac{\delta^2 H}{2} \right) \left( 1 - \frac{H_0}{2n} \right)^n U_4 \left( 1 - \frac{H_0}{2n} \right)^n \left( 1 - \frac{\delta^2 H}{2} \right) G_t, \tag{1}
\]

where \( H_0 = \Delta^2 / 2m_Q^2 \) and \( \delta^2 H \) is defined in \[ \]. The gauge links are tadpole improved by dividing them by \( m_0 = 0.87779 \). The tree-level contribution \( \text{exp}(\Delta^2 / 2m_Q^2) \) of the bare mass \( m_Q^0 \) to the heavy quark propagator is omitted from the calculation. The leading correction proportional to \( \alpha_s \) redefines \( m_Q^0 \): we take this into account perturbatively when matching the lattice theory to the continuum.

The current \( J \), expanded in powers of \( 1/M \), is

\[
J = \eta_0 J^{(0)} + \frac{\eta_1}{2M} J^{(1)} + \frac{\eta_2}{2M} J^{(2)} + \frac{\eta_3}{8M^2} J^{(3)} + \frac{\eta_4}{8M^2} J^{(4)} + \frac{i\eta_5}{4M^2} J^{(5)}, \tag{2}
\]

where we keep all operators of dimension three and four, and only the three dimension five operators that appear at the tree level:

\[
J^{(0)} = \bar{q} \gamma_5 \gamma_4 Q,
J^{(1)} = \bar{q} \left( \vec{\gamma} \cdot \vec{\Delta} \right) \gamma_5 \gamma_4 Q,
J^{(2)} = \left( \bar{q} \cdot \gamma \right) \gamma_5 \gamma_4 Q,
J^{(3)} = \bar{q} \Delta^2 \gamma_5 \gamma_4 Q,
J^{(4)} = \bar{q} \vec{\Sigma} \cdot \vec{B} \gamma_5 \gamma_4 Q,
J^{(5)} = \bar{q} \vec{\alpha} \cdot \vec{E} \gamma_5 \gamma_4 Q. \tag{3}
\]

\( q \) is the four component light spinor and \( Q \) is the nonrelativistic heavy quark spinor whose upper two components are zero.

On transcribing \( J \) on to the lattice, the mixing induced by the hard cutoff has to be included. Dropping dimension five and higher operators, the lattice currents \( J_L \) can be written as

\[
\begin{pmatrix}
J^{(0)}_L \\
J^{(1)}_L \\
J^{(2)}_L
\end{pmatrix} = \begin{pmatrix}
\zeta_0 & a \zeta_1 & a \zeta_2 \\
a^{-1} \zeta_0 & \zeta_1 & \zeta_2 \\
a^{-1} \zeta_2 & \zeta_2 & \zeta_2
\end{pmatrix} \begin{pmatrix}
J^{(0)} \\
J^{(1)} \\
J^{(2)}
\end{pmatrix}. \tag{4}
\]

In the final expression for \( J \), every \( a^{-1} \) appears multiplied by an \( M^{-1} \); all infrared divergences cancel; and the final logarithmic scale dependence is of the form \( \log am_Q^0 \). Apart from these logarithms, all the coefficients are finite for \( 1 \lesssim am_Q^0 \lesssim \infty \). We shall highlight the contribution of the different currents \( J^{(i)} \) to \( f_B \).

2. LATTICE PARAMETERS

The statistical sample consists of 102 \( 16^3 \times 48 \) quenched lattices at \( \beta = 6/g^2 = 6.0 \). Heavy quarks are simulated with \( am_Q^0 = 1.6, 2.0, 2.7 \) with \( n = 2 \); and \( 4.0, 7.0, 10.0 \) with \( n = 1 \). Light quarks with \( \kappa = 0.1369, 0.1375 \) and \( 0.13808 \) (bracketing the strange quark) were simulated using the tadpole improved clover action. Details of
the simulations and fixing of the lattice parameters are given in [1]. Using the data for $m_s^2$ and $m_\rho$ versus $\kappa$ we get $\kappa_l = 0.13917(9)$ and $a^{-1} = 1.92(7)$ GeV. For the strange quark mass we use $\kappa_s = 0.1376(1)$ from $m_K$ to quote the central value, and $\kappa_s = 0.1372(3)$ from $m_K$ to estimate the error. Lastly, a reanalysis of the $B$ spectrum gives $aM_0^b = 2.32(12)$ [3], whereas in [1] we had used $aM_0^b = 2.22(11)$. This revised $aM_0^b$, however, leads to an insignificant change in estimates of decay constants. Estimates of $f_{B_s}$, which only require an interpolation in $\kappa$ and $m_Q^0$, are more reliable than those for $f_B$, which need a large extrapolation in $\kappa$.

3. METHOD OF ANALYSIS

To estimate uncertainty in measuring the mass and amplitude of two point correlators, we extract the decay constants in three ways:

1. Individual local-smeared correlators $\langle J_i S \rangle$ and the smeared-smeared correlators $(SS)$ are fit using the form $A \text{exp} - Et$. From these we get the contribution of each bare operator to the decay constant: $f_1 \sqrt{M} = A_1 / \sqrt{M}$. These $f_1 \sqrt{M}$ are combined using Eq. 3 to obtain the decay constant for each heavy-light combination. Finally, we extrapolate/interpolate in $\kappa$ and $am_Q^0$.

2. We combine the $J_i$ according to Eq. 3 to obtain $\langle JS \rangle$. We fit this and the $\langle SS \rangle$ to exponential forms $A \text{exp} - Et$ and obtain the decay constant $f_i \sqrt{M}$ at each combination of heavy and light quarks. These are then interpolated/extrapolated in $\kappa$ and $am_Q^0$.

3. We make simultaneous fits to each $\langle J_i S \rangle$ and the $\langle SS \rangle$ over the same fit range to obtain the $f_i \sqrt{M}$. We then extrapolate/interpolate these to the physical masses. Finally, we combine these using Eq. 3 to obtain $f_B$ and $f_{B_s}$.

We find that for $am_Q^0 = 1.6, 2.0$ and $2.7$, all three methods of analysis give the same result, whereas for $am_Q^0 = 4.0$ the agreement is marginal. Consequently, we use only the first three values of $am_Q^0$ to obtain our final results.

### Table 1

| $am_Q^0$ | $J^{(0)}$ | $J^{(1)}$ | $J^{(2)}$ | $J^{(3)}$ | $J^{(4)}$ | $J^{(5)}$ |
|---------|----------|----------|----------|----------|----------|----------|
| 1.6     | 164(7)   | -0.025(1)| -0.0036(2)| 0.0036(2)| -0.0036(2)|
| 2.0     | 167(9)   | -0.021(1)| -0.0036(2)| 0.0022(1)| -0.0033(2)|
| 2.7     | 170(11)| -0.016(1)| -0.0022(1)| 0.0011(1)| -0.0018(1)|
| 4.0     | 174(11) | -0.011(1)| -0.0011(1)| 0.0005(0)| -0.0008(1)|

### Table 2

$f_Q \sqrt{M_Q}$ extrapolated to $\kappa_l$ and $\kappa_s$. Results are given for $aq^* = 1$ and $\pi$.

| $am_Q^0$ | $\kappa_l$ | $\kappa_s$ |
|----------|------------|------------|
|          | $\pi/a$    | $\pi/a$    |
| 1.6      | 0.114(6)   | 0.136(4)   | 0.144(4)   |
| 2.0      | 0.121(7)   | 0.144(5)   | 0.153(5)   |
| 2.7      | 0.127(9)   | 0.153(6)   | 0.164(6)   |
| 4.0      | 0.135(9)   | 0.164(6)   | 0.176(6)   |

Estimates of $f_i \sqrt{M}$, multiplied by their tree-level coefficients and extrapolated to $\kappa_l$, are presented in Table 1. We expect that the bare matrix element of $J^{(1)}/2m_Q^0$ should be smaller than that of $J^{(0)}$ by a factor of $O(\alpha/\amQ^0) \sim O(\Lambda_{QCD}/\amQ^0)$, and qualitatively this is borne out by the data. Similarly, the bare matrix elements of $J^{(3)}/8(m_Q^0)^2$, $J^{(4)}/8(m_Q^0)^2$ and $J^{(5)}/4(m_Q^0)^2$ are smaller than that of $J^{(1)}$ by a factor of $O(1/\amQ^0)$.

4. RESULTS

Table 2 presents $f_Q \sqrt{M_Q}$ extrapolated to $\kappa_l$ and $\kappa_s$ for two choices of the lattice scale we believe covers the range relevant to this calculation, $aq^* = 1, \pi$. (The lattice coupling at these scales are $\alpha_V(\pi/a) = 0.1557$ and $\alpha_V(1/a) = 0.2453.$) For our final values we average the two estimates and interpolate to $\amQ^0$:

$$
\begin{align*}
 f_B &= 147(11)(^{+0.9}_{-1.0})(9)(6)\text{MeV} \\
 f_{B_s} &= 175(08)(^{+7.0}_{-1.0})(11)(7)(^{+7.0}_{-0.0})\text{MeV} \\
 \frac{f_{B_s}}{f_B} &= 1.20(4)(^{+0.0}_{-0.7}).
\end{align*}
$$

The error estimates are obtained as follows. To
determine the uncertainty due to setting the lattice scale, we repeat the calculation with $a^{-1} = 1.8$ and 2 GeV. This gives a variation $(+8)_{-12}$ MeV in $f_B$ and $(+7)_{-10}$ in $f_{B_s}$.

To estimate the perturbative errors we consider three issues. (i) The results at $q^* = 1$ and $\pi$ differ by about 5%. (ii) The dimension five operators have been included with tree-level coefficients and their mixing with lower dimensional operators neglected. This, we estimate, introduces an error of $O(\alpha_s/(Ma)^2) \sim 4\%$. (iii) The neglected higher dimension operators are expected to be $O(\Lambda_{QCD}/M)^3 < 1\%$. Altogether we assign a $\sim 7\%$ error coming from these sources.

The remaining discretization errors are $O(\Lambda_{QCD}a)^2$, which we estimate to be 4%. The last error in $f_{B_s}$ and $f_{B_s}/f_B$ is due to the uncertainty in $\kappa_s$. We use the difference in $\kappa_s$ obtained from $K$ and $K^*$ to estimate this.

An alternate approach is to neglect the contribution of the dimension five operators $J^{(3)}$, $J^{(4)}$ and $J^{(5)}$ in the final result, and use them only to estimate errors. The reason is that the neglected mixing with $J^{(0)}$ could introduce $O(\alpha_s/(Ma)^2)$ errors, which can be larger than their $O(\Lambda_{QCD}/M)^2$ contribution. Neglecting the dimension five operators (in which case the calculation is consistent to $O(1/M)$ and $O(\alpha/M)$) would give

$$f_B = 152(11)(+8)_{-12}(10)(6)\text{MeV}$$
$$f_{B_s} = 181(08)(+7)_{-10}(12)(7)(+7)_{-9}\text{MeV}. \quad (6)$$

Finally, we give a breakup of the various contributions. The tree-level estimate is $f_B = 169$ MeV. Of this 195 MeV comes from $J^{(0)}$, $-21$ MeV from $J^{(1)}$, and $-5$ MeV from the $1/M^2$ corrections. The one-loop corrections reduce the estimates by $\sim 13\%$. On the other hand we find that the ratio $f_{B_s}/f_B$ is insensitive to the various corrections.

5. DISCUSSIONS

In Table 3 we compare our results with other calculations. The formalisms used for incorporating heavy quarks are NRQCD, Fermilab and Wilson/clover, and Wilson/clover. Within errors the different approaches give consistent results for the quenched theory. Further improvement of the NRQCD estimates needs the full operator mixing matrix at $O(1/M^2)$, which in turn requires improving the light quark action to $O(a^2)$. A necessary check of our results is to show that they are independent of $a$. We are addressing these issues.

A surprising feature of these data is that the ratio $f_{B_s}/f_B$, which is insensitive to most of the systematic errors, shows a difference between the NRQCD and the other formalisms. We do not understand this. Possible issues to investigate are (i) the chiral extrapolation and (ii) the dependence on the heavy quark mass as this is handled differently in the different approaches.

Acknowledgements

We acknowledge the support of the ACL at LANL and NCSA at Urbana Champagne.

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