Exact $N=2$ Landau-Ginzburg Flows

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We find exactly solvable $N=2$-supersymmetric flows whose infrared fixed points are the $N=2$ minimal models. The exact $S$-matrices and the Casimir energy (a $c$-function) are determined along the entire renormalization group trajectory. The $c$-functions run from $c=3$ (asymptotically) to the $N=2$ minimal-model values, leading us to interpret these theories as the Landau-Ginzburg models with superpotential $X^{k+2}$. The calculation of the elliptic genus is consistent with this interpretation. We also find an integrable model in this hierarchy with spontaneously-broken supersymmetry and superpotential $X$, and a series of integrable models with $(0,2)$ supersymmetry. The flows exhibit interesting behavior in the UV, including a relation to the $N=2$ super sine-Gordon model. We speculate about the relation between the kinetic term and the cigar target-space metric.

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1. Introduction

The Landau-Ginzburg description of $N=2$ theories \cite{1,2} starts with a $N=2$ superspace action written in terms of a chiral superfield (or, more generally, several chiral superfields) as

$$ S = \int d^2z d^4\Theta \, K(X, \overline{X}) + \int d^2z d^2\Theta \, W(X) + h.c., \quad (1.1) $$

where, for example, $K = \overline{X}X$ gives the standard flat-space kinetic terms. This lagrangian has IR difficulties and appears to be intractable. We do know, however, that the superpotential $W$ is not renormalized (other than by wavefunction renormalization) along the renormalization group flow of this theory into the infrared. This is the $N=2$ nonrenormalization theorem. By dimension counting, changing the kinetic term $K$ amounts to an irrelevant perturbation so an IR fixed point should be completely determined by the superpotential $W$. IR fixed points of the RG flow are $N=2$ superconformal theories, so the LG approach allows us to describe many such theories simply in terms of a superpotential. This is why the LG description of $N=2$ theories is so useful in spite of the fact that directly analyzing the theory along the flow is impossible at the present time.

A wide variety of $N=2$ theories can be described in this way, including the $N=2$ minimal models \cite{2} and, via orbifolds, superstring vacua \cite{3}. The power of the LG description also is applicable to perturbed $N=2$ superconformal theories \cite{4-8}.

Given the success of the LG description it is natural to want to understand at a deeper level the flow of (1.1) into its IR fixed point. In this paper we present integrable scattering theories which flow into the $N=2$ minimal models. We argue that these theories exactly describe the LG models along their entire renormalization group trajectories. These scattering theories are described in terms of exact $S$-matrices for the scattering of the massless excitations which survive in the IR limit. Because this is an exact and completely non-perturbative description of these quantum field theories, it is a great challenge to connect these results with a direct analysis in terms of the RG flow of the action (1.1). The hope is that these exact results could lead to new insight into $N=2$ LG theories and quantum field theory in general. For example, from an exact $S$-matrix one can compute exact form factors \cite{9}, which can then be used to extract correlators.

Because of the difficulty in obtaining a direct connection between our exact scattering theories and the RG flow of a LG action into the $N=2$ minimal models, we will provide some (highly non-trivial) checks. By analyzing the thermodynamics of the scattering theories we will exactly determine the Casimir energy (a $c$-function \cite{10,11}) along the entire RG
trajectory in terms of a solution of coupled (TBA) integral equations. We verify in sect. 2 that this $c$-function does flow from $c = 3$ (asymptotically) in the UV to the minimal model value of the central charge in the IR, as would be expected for the LG theory.

Because our $N=2$ scattering theories have individually-conserved left and right fermion numbers $F_L$ and $F_R$, a second check on our scattering matrices is to evaluate their elliptic genera \[ T_{\tau} e^{i\alpha F_L} (-1)^{F_R} q^H \tau^{FR}. \] (1.2) The fact that this quantity only receives contributions from states with $H_R=0$ is nicely exhibited in our scattering theories. The insertion of $(-1)^{F_R}$ causes the right-moving particles to decouple; only left movers contribute to (1.2) all along the RG flow, all the way from the far UV to the far IR. We see that (1.2), as computed in our scattering theories, is constant along the RG flow. This behavior is required for (1.2) because it is an index. We derive the leading term (in the thermodynamic limit $q \to 1$) from our scattering theories and verify that it agrees with this limit of the exact expression, which was recently computed using the $N=2$ LG field theory in [14] and verified in terms of the known $N=2$ minimal model characters in [15].

In sect. 4 we discuss two related $S$-matrices. The first describes the scattering of Goldstinos resulting from spontaneously-broken $N=2$ supersymmetry. We argue for a LG potential which describes this case and which fits in nicely into our overall picture. This result also indicates that the potential $K$ is not the simple one $XX$. The second model has $(0,2)$ supersymmetry and might be of interest for string theory model-building.

We then attempt to understand a little more about these LG theories, especially their ultraviolet fixed points, whose properties are somewhat confusing. In sect. 5 and 6 we calculate power-series expansions of of the $c$-function and the ground-state energy in an external background field around the UV fixed point; we find that the results are the same as those of the $N=2$ super-sine-Gordon model except for some alternating signs. In sect. 7 we speculate about comparing the UV limit of our exact results with the LG theory (1.1) whose kinetic term $K$ corresponds to a cigar metric.

2. Exact results for $N=2$ Landau-Ginzburg flows

We study integrable RG flows whose IR limit is a non-trivial conformal field theory. The basic idea is that the theories can be described by massless excitations for which we can find an exact $S$-matrix. In particular, we have a set of left movers and a set of right
movers and $S$-matrix elements describing left-left, right-right, and left-right interactions. Because of the left-right interaction the massless theory is not conformal. In the IR limit of the RG flow, the left-right $S$-matrix becomes trivial, the left and right movers decouple from each other, and we obtain the IR fixed point conformal theory. The left-left and right-right $S$-matrices are independent of the RG scale and encode a description of the IR fixed point. Such massless scattering theories arise, for example, in the continuum limit of the XXX spin chain \cite{16,17} and have recently been proposed for flows from the $p^{th}$ to $(p-1)^{th}$ $N=0$ minimal models \cite{18,19}, flows from the $SU(2)$ principal chiral model with a WZW term into the $SU(2)_1$ CFT \cite{20}, deformations of the O(3) sigma model with topological term $\theta = \pi$ \cite{20,21} and for the Kondo problem \cite{22}.

The theories which we will be considering here are the $N=2$ minimal models described by the superpotential \cite{2}

$$W = g X^{k+2},$$

(2.1)

where $g$ is a coupling constant. We do not have a direct argument that such flows should be integrable. However, it was shown in \cite{14} that, even in the off-critical LG theory, there is a full $N=2$ superconformal algebra acting on the pure left-moving states, i.e. those with $H_R=0$, and likewise for the pure right moving states. Since this is an infinite-dimensional symmetry algebra, it is plausible that it leads to the conserved currents required for integrability. We will assume integrability and find the simplest massless scattering theories consistent with the symmetries and the various requirements which exact $S$-matrices must satisfy. The final results provide highly nontrivial checks on our assumptions.

All excitations are all massless, with $H = |P|$. The left movers form representations of the left-moving $N=2$ supersymmetry algebra and are annihilated by the right-moving generators. In particular, the simplest representation is a doublet $(u_L(\theta), d_L(\theta))$ under the action of the generators $Q^\pm_L$:

$$Q^-_L |u_L(\theta)\rangle = \sqrt{2Me^{-\theta/2}} |d_L(\theta)\rangle, \quad Q^+_L |d_L(\theta)\rangle = \sqrt{2Me^{-\theta/2}} |u_L(\theta)\rangle.$$

These excitations are eigenstates of $H_L = H - P = \{Q^+_L, Q^-_L\}$ with eigenvalue $2Me^{-\theta}$, where $M$ is a scale parameter, and have left-moving fermion number $F_L$ given by $(f, f-1)$ for some $f$. Likewise, the right-moving excitations form doublets under $Q^\pm_R$, are eigenstates of $H_R = H + P$ with eigenvalue $2Me^\theta$, and are annihilated by the left-moving generators. These are eigenstates of the right fermion number $F_R$.
The simplest scattering theory consists of a single left \((u_L, d_L)\) doublet and a single right doublet. The fermion numbers of the doublets are \((\frac{1}{2}, -\frac{1}{2})\) and the \(d\) excitations are the anti-particles of the \(u\) excitations. Consider first the left-left \(S\)-matrix \(S_{LL}\) for scattering of the \(u_L(\theta)\) and \(d_L(\theta)\) among themselves. Demanding that this \(S\)-matrix commutes with the supersymmetry generators along with the requirements of crossing and unitarity, and the stipulation that there be no extra bound states completely determines the matrix \(S_{LL}\). In fact, the \(S\)-matrix is formally the same as that obtained in [6] for massive \(N=2\) theories with a spontaneously broken \(Z_2\) symmetry. Denoting \(u_L(\theta_1)\) by \(u_1\), \(S_{LL}\) is

\[
\begin{pmatrix}
  u_2 u_1 & d_2 u_1 & u_2 d_1 & d_2 d_1 \\
  u_1 u_2 & 0 & 0 & 0 \\
  u_1 d_2 & 0 & -iZ(\theta) \tanh \frac{\theta}{2} & Z(\theta) \frac{1}{\cosh \frac{\theta}{2}} \\
  d_1 u_2 & 0 & Z(\theta) \frac{1}{\cosh \frac{\theta}{2}} & -iZ(\theta) \tanh \frac{\theta}{2} \\
  d_1 d_2 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(2.2)

where \(\theta = \theta_1 - \theta_2\) and

\[
Z(\theta) = \exp \left( \frac{i}{4} \int_{-\infty}^{\infty} d\omega \frac{\sin \omega \theta}{\omega \cosh^2 \frac{\pi \omega}{2}} \right).
\]

The \(S\)-matrix \(S_{RR}\) for right-right scattering is also given by (2.2).

Without a left-right interaction, these \(S\)-matrices would describe an IR fixed point CFT. In order to describe a massless but not conformal flow, we now consider coupling the above left and right scattering theories with some nontrivial left-right \(S\)-matrix. Consider \(u_L(\theta_1)u_R(\theta_2)\) scattering. By fermion-number conservation the final state must be \(A(\theta)u_R(\theta_2)u_L(\theta_1)\) for some function \(A(\theta)\). Since the \(S\)-matrix must commute with the generators \(Q^\pm_L\) and \(Q^\pm_R\), we see that the \(S\)-matrix for left-right scattering must be diagonal with all elements equal to the same function \(A(\theta)\). The simplest nontrivial choice for this function which satisfies the left-right \(S\)-matrix requirements [18] is

\[
S_{LR}(\theta) = \tanh \left( \frac{\theta}{2} - i \frac{\pi}{4} \right).
\]

(2.3)

This \(S\)-matrix does indeed becomes trivial in the UV and IR fixed point limits.

The above is the simplest possible scattering theory for a massless but not conformal \(N=2\) supersymmetric flow. We conjecture that it describes the LG flow associated with \(k=1\) case of [2.7], i.e. the simplest, massless \(N=2\) LG flow.

As a highly nontrivial check on the LG conjecture, we calculate the partition function on a torus with euclidean time \(\beta\) and length \(L\) from this scattering theory. This is done
by using the thermodynamic Bethe ansatz $[23]$. We take the fermion boundary conditions to be periodic in the $L$ direction and we insert the operator $e^{i\alpha_L F_L} e^{i\alpha_R F_R}$ to give a field with fermion numbers $(q_L, q_R)$ the twisted boundary conditions $-e^{i(\alpha_L q_L + \alpha_R q_R)}$ in the $\beta$ direction. In the thermodynamic limit $L \to \infty$, we derive exact integral equations for the corresponding free energy. We give the final result without elaboration because the techniques involved were discussed at length in $[6]$; it is

$$c(\alpha_L, \alpha_R; M\beta) = \frac{6\beta}{\pi L} \log \text{Tr} e^{i\alpha_L F_L} e^{i\alpha_R F_R} e^{-\beta H}$$

$$= \frac{3}{\pi^2} \sum_a \int d\theta \nu_a(\theta) \log(1 + \lambda_a e^{-\epsilon_a(\theta)}),$$

(2.4)

where $\epsilon_a(\theta)$ are obtained by solving the coupled integral equations

$$\epsilon_a(\theta) = \nu_a(\theta) - \sum_b l_{ab} \int \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} \log(1 + \lambda_a e^{-\epsilon_a(\theta')}).$$

(2.5)

The index $a$ as well as the $\nu_a(\theta)$, the $\lambda_a$, and $l_{ab}$ in these equations are conveniently encoded in the diagram:

The index $a$ runs over each node in this diagram. The $\nu_a(\theta)$ are given by zero if node $a$ is open, $\frac{1}{2} M\beta e^{-\theta}$ for the node with a $L$ and $\frac{1}{2} M\beta e^{\theta}$ for the node with a $R$ in it. The $\lambda_a$ are one except for the four outside nodes, for which the $\lambda_a$ are given by the phases indicated in the diagram. Finally, $l_{ab} = 1$ if the nodes $a$ and $b$ are connected by a line in the diagram and zero otherwise. This result follows almost immediately from the results of $[18]$ and $[1]$; the coupling between the $L$ and $R$ nodes follows from the $S$-matrix (2.3), while the extra massless nodes arise from “diagonalizing” the $S$-matrix (2.2).

The above scattering theory and the resulting integral equations are to be associated with the $k=1$ case of (2.1). The $S$-matrices associated with the higher $k$ cases are more complicated and we will not write them out explicitly here. They follow from the results of $[19]$, where analogous scattering theories describing the flows between the $N=0$ minimal models are discussed. The spectrum in the $N=0$ case consists of massless solitons
interpolating between adjacent wells of a $\phi^{2(k+1)}$-type potential with $k+1$ degenerate wells. In the $N=2$ case each particle merely becomes a $u, d$ doublet; the $LL$ and $RR$ scattering is then the $N=0$ $S$-matrix tensored with $^{(2.2)}$, while $S_{LR}$ is the same in the $N=0$ and $N=2$ cases $^1$. The TBA integral equations for higher $k$ become an obvious generalization of the above ones: they are given by the same expressions $^{(2.4)}$ and $^{(2.5)}$ but with the diagram generalized to

![Diagram](https://example.com/diagram)

where there are $k - 1$ open nodes between the $L$ one and the $R$ one.

The reason for our normalization and notation in the expression $^{(2.4)}$ is that, by changing our interpretation about which cycle on the above torus is length and which one is time, the free energy per unit length is proportional to the ground-state (Casimir) energy of the quantum field theory on a circle of radius $\beta$. This is, in turn, proportional to the central charge at the fixed points $^{[24]}$. The function $c(\alpha_L, \alpha_R; M\beta)$ can be interpreted as a $c$-function for the RG flow of the theory in the sector with fermion boundary conditions twisted by $\alpha_L$ and $\alpha_R$. $M\beta$ is the RG parameter which runs from zero in the far UV to infinity in the far IR. In these limits the function $c(\alpha_L, \alpha_R; \beta M)$ coincides with the minimum value of $c - 12(h + \bar{h})$ in the fixed point theory in the sector with the twisted fermion boundary conditions.

This allows a very convincing check that our scattering theories describe the LG flow of $^{(2.1)}$ because $^{(2.4)}$ gives the correct central charge in both the IR and the UV limits. Because our scattering theory is defined in terms of the massless excitations associated with the IR limit of the RG flow, we first verify that the IR limit of the TBA equations $^{(2.5)}$ do indeed correctly give the central charge of the $N=2$ minimal models associated with the IR fixed points of $^{(2.1)}$. In the IR $\beta M \to \infty$ limit the integral equations $^{(2.5)}$ effectively break up into a piece corresponding to the left movers described by the diagram

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$^1$ If one replaces the $S$-matrix $^{(2.2)}$ with the $SU(2)$ doublet $S$-matrix of $^{[20]}$, this gives the $S$-matrix for the flows into $SU(2)_k$ WZW models.
and an analogous piece for the right movers. We can evaluate the corresponding leading
collection to (2.4) exactly in terms of Rogers dilogarithm functions using the standard
trick [23]. We omit the details, as they closely follow calculations discussed in our previous
papers. The final result is

$$c(\alpha_L, \alpha_R, \beta_M \to \infty) = \frac{3k}{k+2} \left(1 - \frac{\alpha_L^2}{\pi^2} - \frac{\alpha_R^2}{\pi^2}\right).$$

(2.6)

When $\alpha_L=\alpha_R=0$, i.e. in the NS sector, the minimum value of $h+\bar{h}$ is zero, corresponding to
the NS vacuum. The result (2.4) then gives the correct value $c = 3k/(k+2)$ for the central
charge of the $N=2$ minimal model associated with the IR fixed point of (2.1). The value of
(2.6) for nonzero $\alpha_L$ and $\alpha_R$ is exactly as expected for the sector with fermion boundary
conditions twisted by $\alpha_L$ and $\alpha_R$: by spectral flow of the NS vacuum, the minimum value of
$h+\bar{h}$ in the sector with twisted fermion boundary conditions is $c((\alpha_L/2\pi)^2 + (\alpha_R/2\pi)^2)/6,$
in agreement with (2.6). This agreement is a check that the conserved charges $F_L$ and
$F_R$ associated with our scattering theory do, indeed, correspond to left and right fermion
number in the IR theory.

More support for the LG interpretation comes from finding the dimension of the
irrelevant operator which dominates the final stage of the RG flow into the
$N=2$ minimal model fixed point. This can be obtained from (2.5) either numerically or by using the
“periodicity” argument discussed in [18]. For $k \neq 1$ the dimension is $2 + (4/k + 2)$. This shows that for our scattering theories the kinetic term in (1.1) flows as $K = K_{IR} + \lambda(XX)^2 + \cdots$, with $\lambda \to 0$ in the IR limit. For $k=1$, the dimension is 3, which
corresponds to adding $(XX)^3$. Since the flow reaches the IR fixed point by a pure kinetic-term perturbation, this is a good hint that it is purely LG kinetic-term all the way as conjectured.

We now turn to the UV limit. The $M\beta \to 0$ limit of (2.3) can, again, be evaluated in
terms of dilogarithm functions. In particular, it is seen for $\alpha_L=\alpha_R=0$, that $c(M\beta \to 0)=3,$
for all $k$. This is precisely what we would expect for the UV limit of the LG theory: the
superpotential (2.1) is driven to zero by wavefunction renormalization and we are left
with the central charge of a free superfield. We should, however, expect subtleties. By considering the Witten index or the elliptic genus to be discussed in the next section, it is seen that the effect of the superpotential can’t completely go away, even in the far UV limit. A manifestation of this is that \( c=3 \) is only reached asymptotically. The TBA equations (2.5) give

\[
c(M\beta \to 0) = 3 - \frac{3\pi^2(k + 2)}{2(-\log M\beta + \text{const.})^2} + \ldots,
\]

where the \( \ldots \) includes higher \( 1/\log(M\beta) \) terms as well as powers of \( M\beta \). It is possible to derive this result from (2.3) along the lines of the discussion in [21], but the calculation is subtle and we will not present it here. We verified this result by obtaining the numerical solution of equations (2.5) for small \( M\beta \); the results fit (2.7) well. The \( 1/\log(M\beta) \) terms in (2.7) reveal that \( c=3 \) is only approached asymptotically. For example, the derivative of the \( c \)-function blows up at \( M\beta=0 \) so in this region the theory can not be thought of as a CFT with a small perturbation. We indeed see that the superpotential doesn’t really go away in the UV limit. This is the statement that the UV fixed point is an “infinite distance” from the IR fixed point in whose neighborhood the \( S \)-matrix is defined, an unavoidable consequence of the fact that the two fixed points have different Witten indices [25,26]. In terms of a LG theory (1.1), one might also expect such log terms in the UV, coming from unbounded fluctuations of the (near) constant modes of the bosonic component of the superfield \( X \). In sect. 6 we will discuss these \( 1/\log \) terms in the context of a particular choice of the kinetic term \( K \).

We also mention that the UV limit of (2.3) exhibits some interesting features when \( \alpha_L \) and \( \alpha_R \) are turned on. Because the UV region cannot be described by a \( c=3 \) CFT plus a small perturbation, we shouldn’t be surprised to find that (2.3) differs in the UV from the result \( 3(1 - \frac{1}{2}(\alpha_L/\pi)^2 - \frac{1}{2}(\alpha_R/\pi)^2) \) of a \( c=3 \) CFT with boundary conditions twisted by \( \alpha_L \) and \( \alpha_R \). In particular, as will be discussed in the following section, at \( \alpha_R = \pi \) we are computing the elliptic genus and we know there that (2.3) can’t be of this form. Especially interesting behavior occurs when \( \alpha_L = \pm \alpha_R \equiv \alpha \). For example, for \( k=1 \) (2.3) yields \( c(\alpha; M\beta \to 0) = \)

\[
3 - 5\frac{\alpha^2}{\pi^2} + \frac{f_1(\alpha)}{\log^2 M\beta} + \ldots \quad |\alpha| \leq \frac{\pi}{2}
\]

\[
(5 - 3\frac{|\alpha|}{\pi})(1 - \frac{|\alpha|}{\pi})f_2(\alpha)(M\beta)^{4(2|\alpha| - \pi)/3\pi} + \ldots \quad \frac{\pi}{2} \leq |\alpha| \leq \pi
\]
It seems that at $\alpha = \pi/2$ the log correction turns into a power series one with continuously varying exponent; the behavior is continuous in $\alpha$ but not analytic. This reveals an interesting transition in $\alpha$, possibly due to a level crossing associated with a state which would be eliminated when $\alpha_L \neq \pm \alpha_R$ by the twisted boundary conditions.

3. The Elliptic Genus

The elliptic genus $[12,13]$ of a supersymmetric theory having separately-conserved left and right fermion numbers $F_L$ and $F_R$ is given by (1.2). (If, as in heterotic theories, there is no $F_L$, the $e^{i\alpha L F_L}$ term should, of course, be omitted). By standard arguments $[25]$, only those states with $H_R = 0$ contribute to (1.2) and, thus, the elliptic genus is independent of $\bar{q}$. The elliptic genus is an “index” — it is invariant under continuous deformations of the theory.

These features of the elliptic genus are nicely exhibited in our scattering theory. In particular, consider the TBA equations (2.5) with $\alpha_R = \pi$. Because these equations were obtained by taking the thermodynamic limit where the length $L \to \infty$, they yield the $q = e^{-2\pi \beta/L} \to 1$ limit of (1.2). The fact that the elliptic genus is a holomorphic function of $q$, receiving contributions only from $H_R = 0$ states, is reflected in the fact that the insertion of $(-1)^{F_R}$ causes the right-moving excitations to decouple. To see this from our integral equations (2.4) notice that a solution of (2.5) is given by

$$e^{-\epsilon_R(\theta)} = 0, \quad \epsilon_{\pm F_R}(\theta) = 0,$$

(3.1)

where $\epsilon_R$ is the $\epsilon_a$ for the node labeled by an $R$ and $\epsilon_{\pm F_R}$ are the $\epsilon_a$ for the two nodes labeled by $e^{\pm i\alpha_R}$ in the diagram. Therefore $\log(1 + e^{-\epsilon_R(\theta)}) = 0$ for all $\theta$ and thus the $R$ node as well as the two end nodes to the right of it are effectively cut off of the diagram; we are left with the diagram appearing before (2.6). The remaining system of integral equations is independent of the RG flow parameter $M\beta$, as is to be expected since we are computing an index. This is easily seen by noting that a rescaling of $M\beta$ can now be absorbed into a shift of $\theta$ in (2.4) and (2.5). The value of (2.4) along the entire RG flow is thus given by the IR expression (2.6) with $\alpha_R = \pi$; so our value for the thermodynamic limit of (1.2) is

$$Tr e^{i\alpha L F_L}(-1)^{F_R}e^{-\beta H} = \exp\left(\frac{\pi k L}{4\beta(k + 2)}(1 - \frac{\alpha_R^2}{\pi^2})\right)$$

(3.2)
along the entire renormalization-group trajectory. This agrees with the $q \to 1$ limit of the exact $N=2$ minimal models result, recently computed exactly using a free field theory associated with the LG description in [14] and verified in terms of the known minimal model characters in [15]. Note that when $\alpha_L = \pm \pi$, (1.2) becomes equal to the Witten index and the expression (3.2) properly becomes independent of $\beta$ and $L$.

We also note that, as discussed in [27], the thermodynamic limit of $\text{Tr} F(-1)^F e^{-\beta H}$ is a “pseudo-topological” index which is invariant under pure “D-term” variations of the theory. This index is especially useful for analyzing massive theories, where the elliptic genus cannot be defined. In the present context it is simply follows from the elliptic genus: taking the derivative of (3.2) with respect to $\alpha_L$ and setting $\alpha_L = \pm \pi$ gives $\text{Tr} F_L (-1)^F R e^{-\beta H} = \mp \frac{L}{2\pi} \left( k/(k+2) \right)$. Adding or subtracting the same expression with $L$ and $R$ interchanged gives $\text{Tr} F(-1)^F e^{-\beta H}$ for the two choices of $F = F_L \pm F_R$. The result agrees with the minimal model result [27].

4. Related models: spontaneously-broken and $(0,2)$ supersymmetry

We briefly discuss two types of related models. The first model, that of Goldstinos resulting from spontaneously-broken $N=2$ supersymmetry, in fact fits in nicely with the above LG flows and gives a great deal of independent support for the picture we have described.

4.1. $N=2$ Goldstinos and the superpotential $W = X$

If supersymmetry is spontaneously broken, one expects a massless fermionic excitation for each broken generator. A simple and elegant example of this is in the flow from the tricritical Ising model to the Ising model. The $N=1$ supersymmetry of the tricritical Ising model is spontaneously broken; in the IR limit the resulting Goldstinos become the massless free Majorana fermion of the Ising model [1]. In the Landau-Ginzburg picture, this is described by a supersymmetric $\phi^4$ Lagrangian; the effective Goldstino action is given by integrating out the boson. These interactions are irrelevant so at the end of the flow we are left with the free massless fermion, but in the midst of the flow the fermions interact. This flow is integrable, and as shown in [18], one can find the $S$-matrix for the left and right fermion: $S_{LR}$ is given by (2.3), while $S_{LL}$ and $S_{RR}$ must be 1 because the infrared limit is a free theory.
A virtually identical situation happens with \( N=2 \) supersymmetry. The only difference is that because there are two supersymmetries, we will have two left-moving particles with fermion number \( F_L = \pm 1 \) and likewise two right movers with \( F_R = \pm 1 \). These can be thought of as a massless Dirac fermion. In the IR limit, the fermion is free, so \( S_{LL} = S_{RR} = 1 \). For the same reasons as in the \( N=1 \) case [18], \( S_{LR} \) is given by (2.3) for all scattering processes, independent of the charge. The resulting TBA is given by the diagram

\[
\begin{array}{c}
L \\
\downarrow \\
R \\
\downarrow \\
R \\
\downarrow \\
L
\end{array}
\]

In the IR, this TBA system gives \( c=1 \), as required. In the UV limit, we have \( c=3 \) and log-type corrections given by (2.7) with \( k = -1 \). Note that this diagram fits in nicely with the \( D_{k+4} \) Dynkin diagrams obtained in the previous section: it is the \( k = -1 \) member of this series. In sect. 6 we will also see how this model can be related to \( N=2 \) sine-Gordon at a coupling corresponding to \( k = -1 \).

This allows an intriguing Landau-Ginzburg interpretation of this model. Plugging \( k = -1 \) into (2.1) gives a superpotential of \( W = X \). In this case the Witten index is zero, so it is possible for supersymmetry to be spontaneously broken. Such a superpotential is trivial if the kinetic term is \( XX \), but the scattering indicates that this theory is certainly interacting. This indicates that our kinetic term is not the simple one, an issue we return to in sect. 7.

4.2. \( (0,2) \) supersymmetry

Models with two right-moving supersymmetries and no left-moving supersymmetries are of interest to string phenomenologists. Motivated by this, we briefly mention that we can couple our massless \( N=2 \) right-moving excitations to \( N=0 \) left moving excitations to obtain a theory with non-trivial RG flow into a conformal theory with left movers a \( N=0 \) minimal model and right movers a \( N=2 \) minimal model. This naturally associates a specific \( N=0 \) model with each \( N=2 \) model.

Above, we saw that the \( N=2 \) \( S \)-matrices were given by the \( N=0 \) ones [18,19] with an extra \( N=2 \) piece tensored to \( S_{LL} \) and \( S_{RR} \). The \( N=2 \) right-moving massless excitations
can be coupled to left-moving $N=0$ massless excitations merely by tensoring the $N=2$ piece only for $S_{RR}$, i.e.

$$S_{RR} = S_{RR}^{N=0} \otimes S_{u,d}^{N=2}, \quad S_{LL} = S_{LL}^{N=0}, \quad S_{LR} = S_{LR}^{N=0},$$

(4.1)

where $S_{u,d}^{N=2}$ is that displayed in (2.2). This yields a TBA system described by the equations (2.4) and (2.5) with the diagram corresponding to removing the two open nodes on the far left (the $e^{\pm i\alpha_L}$ nodes) of the $\hat{D}_{k+4}$ diagram obtained in sect. 2. (Removing just one of the nodes gives a (1,2) theory which flows into the left movers of a $N=1$ minimal model combined with the right movers of a $N=2$ model). The $c$-function $c(\alpha_R = 0, M\beta)$ flows from $(2k+3)/(k+3)$ in the UV to the average of $c_L = 1 - (6/(k+2)(k+3))$ and $c_R = 3k/(k+2)$ in the IR corresponding to the fact that the IR theory is a $N=0$ minimal model for left movers combined with a $N=2$ minimal model for right movers. Note also that inserting $(-1)^{F_R}$ (by setting $\alpha_R = \pi$ in the integral equations) removes the $R$ node and its outside $\pm F_R$ nodes from the diagram, leaving just the left-moving $N=0$ massless scattering theory along the entire flow.

It remains to be seen the extent to which these theories make sense and if they can be used in the construction of (0,2) string vacua.

5. Adding a background field

Further information about the physics encoded in our exact $S$-matrices can be extracted by studying the response of the theories to a constant external background field. The two conserved charges $F_L$ and $F_R$ can be coupled to two independent background fields $A_L$ and $A_R$, modifying the hamiltonian to be $H = H_0 + A_L F_L + A_R F_R$. The exact $S$-matrices can be used to directly calculate the contribution of these background fields to the energy density. Because the background fields have dimension of mass, their strength controls the position of our theory on its renormalization group trajectory. In this section we will focus on the IR and UV limits, corresponding to small and large background field strengths, respectively. More detailed information about the theories away from their UV and IR fixed points will be discussed in the next section.

First we consider the flow with $k=1$. The effect of background fields with, say, $A_L$ and $A_R$ positive is to introduce $d_L$ and $d_R$-type particles (of charge $-\frac{1}{2}$) into the ground state. Left movers fill all levels with rapidity greater than some value $B_L$, while the right
movers fill all with rapidity less than $-B_R$. Using the earlier work \cite{19,21} we can write the answer down: the “dressed” particle energies solve

$$\epsilon_L(\theta) = \frac{1}{2} A_L - \frac{M}{2} e^{-\theta} + \int_{B_L}^{\infty} d\theta' \phi_{LL}(\theta - \theta')\epsilon_L(\theta') + \int_{-\infty}^{-B_R} d\theta' \phi_{LR}(\theta - \theta')\epsilon_R(\theta'), \quad (5.1)$$

and likewise for $\epsilon_R(\theta)$, where

$$\phi_{ab}(\theta) = -\frac{i}{2\pi} \frac{\partial \ln S_{ab}(\theta)}{\partial \theta}.$$ 

$\phi_{LL}$ follows from the $dd$ scattering in (2.2), while $\phi_{LR}$ follows from (2.3):

$$\phi_{LL}(\theta) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\cos \omega \theta}{4 \cosh^2 \frac{\pi \omega}{2}} \quad \phi_{LR}(\theta) = \frac{1}{\cosh \theta}.$$ 

The ground-state energy density is then given by

$$E(A_L, A_R) = -\int_{B_L}^{\infty} \frac{d\theta}{4\pi} M e^{-\theta} \epsilon_L(\theta) - \int_{-\infty}^{-B_R} \frac{d\theta}{4\pi} M e^{\theta} \epsilon_R(\theta). \quad (5.2)$$

The $B$ are determined by the boundary condition $\epsilon_L(B_L) = \epsilon_R(B_R) = 0$.

Finding the equations for arbitrary $k$ requires a little more effort: one must include zero-mass, zero-charge “pseudoparticles” to account for the fact that we can have different kinds of solitons in the vacuum. After simplification, the answer is given by (5.1)–(5.2) where $\phi_{LL}$ and $\phi_{LR}$ are now

$$\phi_{LL}(\theta) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \cos \omega \theta \left[ \frac{1}{4 \cosh^2 \frac{\pi \omega}{2}} + \frac{\sinh \frac{(k-1)\pi \omega}{2}}{2 \sinh \frac{k\pi \omega}{2} \cosh \frac{\pi \omega}{2}} \right]$$

$$\phi_{LR}(\theta) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \cos \omega \theta \frac{\sinh \frac{\pi \omega}{2}}{2 \sinh \frac{k\pi \omega}{2} \cosh \frac{\pi \omega}{2}}.$$ 

These equations can be solved in the $M/A \to 0$ (UV) and $M/A \to \infty$ (IR) limits. In the IR limit, the last term in (5.1) is negligible and the equations for $\epsilon_L$ and $\epsilon_R$ decouple. In the UV limit, one follows the method of \cite{21}, yielding

$$E_{IR}(A_L, A_R) = \frac{k}{k+2} \frac{A_L^2 + A_R^2}{4\pi}$$

$$E_{UV}(A_L, A_R) = \left( 1 - \frac{2k}{(k+2)(k+4)} \right) \frac{A_L^2 + A_R^2}{4\pi} + \frac{8}{k+4} \frac{A_L A_R}{4\pi}. \quad (5.3)$$

The IR limit is in perfect agreement with the general result of \cite{28} that the coefficient of $A_L^2 + A_R^2$ at a critical point must be $c/12\pi$. The UV limit is not that of a $c=3$ CFT, but this is not surprising given the subtleties of the UV limit mentioned at the end of sect. 2. We discuss this further in sect. 6.
6. The UV limit and the $N=2$ sine-Gordon model

The UV limit of our LG flow theories is related to that of a massive integrable theory, the $N=2$ super-sine-Gordon model. The $S$-matrix of $N=2$ sine-Gordon is a tensor product of the $N=0$ sine-Gordon $S$-matrix with the basic $N=2$ $S$-matrix \[ (2.2) \] \[ (25) \]. The value of the sG coupling constant which is related to our $k$-th $N=2$ LG flow theory corresponds to taking the $N=0$ sine-Gordon $S$-matrix at $\beta^2_{N=0} = 8\pi(k+2)/(k+3)$. The $N=2$ sine-Gordon TBA system for integer $k \geq 0$ is described by the diagram \[ (7) \]

where there are $k+4$ open nodes in all (including the ones at the ends). As before $\nu_a = 0$ for the open nodes, while for the node $\otimes$, it is $M \cosh \theta$. The phases correspond to turning on $e^{i(\alpha_F F + \alpha_T T)}$ in the partition function, where $F$ is the conserved fermion number charge and $T$ is the conserved topological charge counting solitons minus antisolitons. Comparing this diagram to the one obtained for our LG flow theories reveals an obvious similarity: they are both $\tilde{D}_{k+4}$ Dynkin diagrams. We also see obvious differences corresponding to the fact that the LG flow theory has massless excitations and a non-trivial IR fixed point at $c_{IR} = 3k/(k+2)$ whereas the $N=2$ sine-Gordon theory is massive and must flow to $c_{IR} = 0$. Also, the conserved charges in the LG flow theory are left and right fermion number whereas the conserved charges of $N=2$ sine-Gordon are the total fermion number $F$ and the topological charge $T$. We will see that the two theories, while very different in the IR, are deeply related in the UV.

First we note that both theories have $c=3$ in the UV. In fact, the $N=2$ sine-Gordon theory also has precisely the same following term in \[ (2.7) \] (so it too only asymptotically approaches its UV fixed point). The fact that this second term agrees for the two theories is a consequence of the fact that they both have the same diagram. The $c$-functions for the two models of course are not the same for all $M\beta$; numerically, we find that the leading difference between the two fits nicely to the power $(M\beta)^{4/(k+2)}$. The similarity between the two theories in the UV limit remains if we turn on $\alpha_L=\alpha_R=\alpha$ in the LG flow theory and $\alpha_F=(k+2)\alpha_T=\alpha$ in the $N=2$ sG theory, as can be seen by comparing their diagrams.
We can make this connection even stronger by comparing the two theories in the presence of a background field. As discussed in the previous section we can compute the ground-state energy of the LG theory when the conserved charges $F_L$ and $F_R$ are coupled to constant background fields $A_L$ and $A_R$. Likewise, we can consider the $N=2$ sG theory when the conserved charges $F$ and $T$ are coupled to background fields $A_F$ and $A_T$. We will, in particular, compare the LG flow theory with $A_L=A_R=A$ to the $N=2$ sG theory with $A_F=(k+2)A_T=A$. Our result will be the following relation between the two theories. For the LG theory the energy density is of the form

$$\mathcal{E}(A) = b_0 A^2 + \sum_{n=1}^{\infty} (-1)^n b_n \left( \frac{M}{A} \right)^{4n/(k+4)},$$

while for the $N=2$ sG theory the energy has exactly the same expansion with the same $b_n$, only without the $(-1)^n$. This result indicates that the models have the same UV fixed point (as $M \to 0$), and that the perturbing operators are related in a simple manner. This relation is analogous to perturbation by $\pm \Phi_{1,3}$ in the $N=0$ minimal models: with one sign the perturbation flows to the next minimal model and has a massless spectrum, while the other gives a massive field theory. A power series around the common UV fixed point has the same behavior, with alternating signs in one case but not the other. Another analogous example is the $CP^1$ sigma model where the theories with $\Theta = \pi$ and $\Theta = 0$ are so related [21].

We start with the $N=2$ sine-Gordon model. For our range of the sG coupling the spectrum consists of four solitons with fermion number $F$ and topological charge $T$ given by $(F,T)= (\pm \frac{1}{2}, \pm 1)$, for the four different sign choices. As discussed above, our result requires coupling the background fields to the two conserved charges as $A_F=(k+2)A_T=A$. It so happens that this ratio of the background fields is special. For this ratio, with $A > 0$, only the solitons with $(F,T)= (-\frac{1}{2}, -1)$ appear in the vacuum and the relevant equations follow immediately from their $S$-matrix element — there is no need for pseudoparticles. This is seen as follows. Suppose we had $A_F > 0$ and $A_T=0$. The ground state then fills with the solitons with $(F,T)= (-\frac{1}{2},1)$ and $(-\frac{1}{2}, -1)$. Now start to increase $A_T$. At some special value of $A_T$ it will no longer be energetically favorable to have the $(-\frac{1}{2},1)$ particles in the vacuum. For this ratio of $A_T$ to $A_F$ there will only be $(-\frac{1}{2}, -1)$ states in the vacuum, and from their single $S$-matrix element we immediately obtain

$$\epsilon(\theta) = \frac{1}{2} A_F + A_T - M \cosh \theta + \int_{-B}^{B} d\theta' \Phi(\theta - \theta') \epsilon(\theta'),$$

(6.2)
where
\[ \Phi(\theta) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \cos \omega \theta \left[ \frac{1}{4 \cosh^2 \frac{\pi \omega}{2}} + \frac{\sinh \left( \frac{(k+1)\pi \omega}{2} \right)}{2 \sinh \left( \frac{(k+2)\pi \omega}{2} \right) \cosh \frac{\pi \omega}{2}} \right], \tag{6.3} \]

the first term resulting from the \(N=2\) part of the \(S\)-matrix and the second from the sine-Gordon part. The energy density is here
\[ \mathcal{E}(A) = -\frac{m^2}{2\pi} \int_{-B}^{B} d\theta \cosh \theta \epsilon(\theta) \tag{6.4} \]
with \(B\) determined by the boundary condition \(\epsilon(\pm B) = 0\). In the UV limit, it is straightforward to show that \[21\]
\[ \mathcal{E}(A \to \infty) = -\frac{q^2 A^2}{2\pi} \frac{1}{1 - \Phi(0)} \tag{6.5} \]
where \(\Phi\) is the Fourier transform of \(\Phi\). Thus for our case
\[ \mathcal{E}(A \to \infty) = -\frac{4(k+2)}{k+4} \frac{\left( \frac{1}{2} A_F + A_T \right)^2}{2\pi} \tag{6.6} \]

To prove that this special ratio is \(A_F = (k+2)A_T\), we consider two other choices of the background fields by setting either \(A_T\) or \(A_F\) to zero. For \(A_F > 0\) and \(A_T=0\), the vacuum fills with both \((F, T)= (-\frac{1}{2}, 1)\) and \((-\frac{1}{2}, -1)\) particles and so relation (5.2) with kernel (5.3) doesn’t hold. Since these particles do not scatter diagonally (their \(S\)-matrix involves the sG \(S\)-matrix), we need pseudoparticles just as in the TBA. Because the equations are a generalization of (5.2) and are linear in the \(\epsilon_a(\theta)\), we can simplify the expressions and remove the pseudoparticles. Similar considerations can be applied to the case of \(A_T > 0\) and \(A_F=0\). For these two cases we end up with the same equation (6.2), with \(A_T\) or \(A_F\) appropriately zero, and with the kernel \(\Phi\) replaced by, respectively
\[ \Phi_F(\theta) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \cos \omega \theta \left[ \frac{1}{4 \cosh^2 \frac{\pi \omega}{2}} + \frac{\cosh \left( \frac{(k+3)\pi \omega}{2} \right)}{4 \cosh \left( \frac{(k+1)\pi \omega}{2} \right) \cosh^2 \frac{\pi \omega}{2}} \right], \tag{6.7} \]
\[ \Phi_T(\theta) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \cos \omega \theta \left[ 1 - \frac{\cosh \left( \frac{(k+3)\pi \omega}{2} \right)}{2 \sinh \left( \frac{(k+2)\pi \omega}{2} \right) \cosh \frac{\pi \omega}{2}} \right]. \]

Using \(\Phi_F\) and \(\Phi_T\) in (6.3) we obtain
\[ \mathcal{E}(A_F \to \infty, A_T = 0) = -\frac{A_F^2}{2\pi}, \quad \mathcal{E}(A_F = 0, A_T \to \infty) = -2(k+2) \frac{A_T^2}{2\pi}. \tag{6.8} \]
These results are in agreement with the general results of [28]; the first gives $c=3$ and the second also shows that the $\beta_{N=2}$ in the superpotential $W = \cos \frac{1}{2} \beta_{N=2} X$ is related to the parameter $k$ in the $S$-matrix by $\beta_{N=2}^2 = 8\pi(k + 2)$, a relation which was obtained in [29] by quantum-group symmetry and which can be viewed as a reflection of $N=2$ nonrenormalization. The discussion in [28] also requires that

$$\mathcal{E}(A_F \to \infty, A_T \to \infty) = -\frac{A_F^2}{2\pi} - 2(k + 2) \frac{A_T^2}{2\pi}$$

because the two currents are independent (this also follows from analyticity because symmetry rules out a cross term $A_F A_T$). Comparing this to equation (6.6) obtained above using the kernel (6.3), we see that kernel (6.3) is valid when $A_F = (k + 2)A_T$.

We now compare (6.2), with $A_F=(k+2)A_T=A$ and with kernel (6.3), to the LG flow equations (5.1) with $A_L=A_R=A$. First, in the $A \to \infty$ UV limit they both give

$$\mathcal{E}(A \to \infty) = -\frac{k+4}{k+2} \frac{A^2}{2\pi}.$$ 

For the $N=2$ sG theory, as discussed above, this result is expected from the considerations of [28] for coupling to our peculiar combination of fermion number and topological charge. For the LG flow theory, we are coupling to $F_L + F_R$ which might be interpreted as the total fermion number $F$ (at least in the IR or on pure left-moving or pure right-moving states). This result looks odd in light of the discussion in [28]. Once again, we are seeing that the UV limit of the LG flow theory is subtle. Perhaps the analogy with $N=2$ sG where some topological charge is mixed in with the fermion number will be useful for better understanding the UV limit of the LG flow theory. We also note that, if we continue $A$ to imaginary $A = i\alpha/\beta$, we make contact with the LG theory with $\alpha_L=\alpha_R=\alpha$ and the $N=2$ sG theory with $\alpha_F=(k+2)\alpha_T=\alpha$. These two UV limits coincide here as in the TBA, but we do not see the peculiar transition (2.8) which the TBA Casimir energy exhibits. This suggests that this particular aspect of the UV limit is due to the presence of a level crossing which occurs in the finite-size TBA but not in the infinite-volume background-field calculation.

The more detailed result (5.1) is obtained by analyzing our two sets of background field energy equations using a generalized Weiner-Hopf technique [30]. Since this has been described in detail for several very similar models [31,21,13], we do not present the full calculation here. Instead, we will explain how to extract the relevant information from the kernels. The technique relies on the usual Weiner-Hopf trick of dividing the Fourier
transforms of the kernels into a product of two pieces, the first of which has no poles or zeroes in the lower half plane and the second none in the upper half plane. Writing $1 - \tilde{\Phi}(\omega) = 1/K_+(\omega)K_-(\omega)$, expressions of the form

$$\oint f(\omega)\frac{g(\omega)e^{2i\omega B}}{(\omega-i)^2}d\omega = \frac{g(\omega)}{\tilde{K}_+ - \tilde{K}_-}$$

occur regularly in the analysis; the contour covers the upper half plane. The function $f(\omega)$ is different depending on where we are in the analysis, but it is analytic in the upper half plane. For our case of $N=2$ sG the kernel (6.3) gives

$$1 - \tilde{\Phi} = \frac{\sinh\left(\frac{(k+4)\pi\omega}{2}\right)}{4\sinh\left(\frac{(k+2)\pi\omega}{2}\right)\cosh^2\frac{\pi\omega}{2}}.$$  (6.10)

Notice that the double pole in $\Phi$ at $\omega = i$ becomes a double zero in $K_+$ which cancels the explicit double pole appearing in (6.9). Ordinarily, such a pole results in a bulk term proportional to $M^2$; this is a nice check, because such terms do not appear in supersymmetric theories. The poles in the contour are the zeros of (6.10), which are at $\omega = 2ni/(k + 4)$. Thus (6.9) can be written as a series in $\exp(-4B/(k + 4))$. In particular, the boundary condition results in an equation

$$\frac{M}{A}e^B = \text{const} + \sum_n f_ng_ne^{-4nB/(k+4)}$$

where $f_n$ and $g_n$ are the residues of $f(\omega)/(\omega-i)^2$ and $g(\omega)$, respectively. The $f_n$ themselves also obey an equation of this form, so for large $A/M$, we can write $e^B$ and $f_n$ each as a series in $(A/M)^{-4/(k+4)}$. The energy is also given by a term like (6.9), so it too must be a series in $(A/M)^{-4/(k+4)}$.

For the $A_T=0$ or $A_F=0$ cases, the kernels (6.7) result in power-series expansions with different exponents. This indicates that there is not just one perturbing operator in the model: the different background fields isolate different operators.

This gives the result (6.1) for $N=2$ sine-Gordon. For the LG flow, we must first rewrite the equations (5.1) in Weiner-Hopf form. The general result is that

$$qA \rightarrow q(1 + \frac{\tilde{\phi}_{LR}(0)}{1 - \tilde{\phi}_{LL}(0)})A$$

$$\frac{1}{K_+(\omega)K_-(\omega)} = 1 - \tilde{\phi}_{LL} - \frac{\tilde{\phi}_{LR}^2}{1 - \tilde{\phi}_{LL}}$$

$$g(\omega) = \frac{K_+(\omega)}{K_-(\omega)}\frac{\tilde{\phi}_{LR}}{1 - \tilde{\phi}_{LL}}.$$
The $K_+$ and $K_-$ obtained from the LG flow kernels discussed in sect. 5 are exactly the same as the ones obtained above for $N=2$ sG. The extra piece in the above expression for $g(\omega)$ is $\sinh \pi \omega / \sinh (\pi \omega (k + 2)/2)$ here. It results in no additional poles in the contour because of the zeros in $g$; its only effect is to change $g_n$ to $(-1)^n g_n$. Since the $f_n$ above are not changed, we then find that the series for the LG flow is exactly the same as in $N=2$ sine-Gordon, except that the signs of every other term are different. This is what we set out to prove, and shows that these two models are mysteriously deeply related.

The goldstino $S$-matrix discussed in sect. 4 can be related to the sine-Gordon model at $k = -1$, giving further evidence that the flow in this case has a superpotential $W = X$. At $k = -1$, the $N=0$ part of the $N=2$ sG $S$-matrix becomes trivial, leaving the only non-trivial scattering in the $N=2$ labels. The TBA diagram is given by

and the log term is that of (2.7) with $k = -1$, as it is for the Goldstinos. The $N=2$ sG background-field calculation is covered by the previous calculation with $k = -1$, while for the Goldstinos it is given by (5.1)–(5.2) with

$$\phi_{LL}(\theta) = 0 \quad \phi_{LR} = \frac{1}{\cosh \theta}.$$  

It is simple to verify that the two expansions are of the form (5.4) with an extra “bulk” term for the Goldstinos because the pole at $\omega = i$ is not cancelled; this is allowed because the supersymmetry is spontaneously broken.

We note that the $k=0$ case of the LG flow (corresponding to $W = X^2$) is massive with a trivial fixed point. It seems to be not just related but actually identical to the $N=2$ sine-Gordon theory at $k=0$. We don’t know why this is so, but this may be helpful in understanding the relation between the two theories for all $k$.

7. Questions and conclusions

We have seen that the quantitative results from our “LG flow” scattering theory match a variety of quantitative and qualitative expectations for the LG theory (1.1) with the superpotential (2.7). One might hope to be able to do better by connecting the
results to some more detailed aspects of the LG theory. For example, in the UV limit we might be able to make contact with a perturbative analysis of the action (1.1) with some particular kinetic term $K$ and with the superpotential (2.1) small. As discussed in the introduction, this is hard; there are difficulties in regulating and analyzing the theory with the superpotential (2.1). The basic problem is that the bosonic component of $X$ appears directly in the action and, with the standard standard $K = X \overline{X}$ kinetic term, the boson fluctuates too much. We can handle derivatives and exponentials of the boson, but the logarithms in the boson-boson correlation functions make it difficult to work with the boson itself.

We saw that the Goldstino case $k = -1$ is described by the superpotential $X$. If the kinetic term were $XX$, even if only in the UV, this model would be trivial all along the flow. Since our $S$-matrix is certainly that of an interacting theory, we are motivated to try a different kinetic term. We can hope that this will also make the boson better behaved. We consider taking the boson to live on some sigma model with metric $G_{xx} = \partial_x \partial_{\overline{x}} K$. In order to have this theory give the right elliptic genus the sigma model should be topologically equivalent to the plane. A choice of metric which looks promising in the UV is the cigar

$$G_{xx} = (al^{\frac{b}{\pi}} + bx)^{-1}$$ (7.1)

where $a$ and $b$ are constants and $l$ is the RG length scale, say $l = M \beta$, so $l \to 0$ in the UV. The kinetic term corresponding to this metric is a dilogarithm function. Unlike the black hole of [32], we do not have a dilaton so our metric has the above nontrivial RG flow; it is a solution of the flow equation discussed in [21] when the superpotential is turned off. Treating the superpotential as a small perturbation (a “tachyon condensate”), there will be order $g^4$ corrections to the RG flow of this metric. The superpotential (2.1) scales to zero in the UV limit provided $b(k + 2)^2 < 2$.

It remains to be seen if one can develop a sensible perturbation theory using the cigar metric with the superpotential (2.1). One way of finding the appropriate kinetic terms for these theories may be to study classical integrable equations, along the lines of [33]. We simply note that it appears possible for this theory to reproduce one aspect of our calculation, the asymptotic $1/\log^2 M \beta$ behavior (2.7). As in [34] these terms come from the fields which are (nearly) constant in the $\beta$ cycle of the torus. For $\alpha_L$ and $\alpha_R$ zero, the fermions are antiperiodic in this cycle so we can neglect them. The contribution of the boson constant modes to the ground state energy reduces to a quantum mechanics problem
in the constant mode coordinates $x_0$ and $\overline{x}_0$. It seems plausible that this gives the behavior \((2.7)\), since in the $N=0$ sausage sigma model the identical terms arise \([21]\). The metric \((7.1)\) is also given qualitative support by some recent results \([35]\), where similar metrics are considered with the superpotential \((2.1)\) in a Landau-Ginzburg description of some black-hole-motivated sigma models, along the lines of \([36]\). This opens up the intriguing possibility that our exact LG flow scattering theories are related to the world-sheet physics of 2D black holes.

Understanding the ultraviolet limit of these theories better will almost certainly shed more light on the situation. Indeed, we have a number of unanswered questions here: Can we see the connection to $N=2$ sine-Gordon theory directly from the Lagrangians? Why in this correspondence is topological charge mixed in with fermion number? Why is the $k=0$ model identical in both these cases?

Our exact massless scattering theories provide quantitative information which, as we have seen, agree with results and expectations for the $N=2$ LG flows. On the other hand, the program of describing RG flows to nontrivial IR CFTs in terms of integrable scattering theories of massless excitations is very new and not yet completely understood. For example, it is not known in detail how in the IR the decoupled left and right massless scattering theories are “equivalent” to the usual descriptions of the IR CFTs. For example, the left and right scattering theories in the IR limit are totally decoupled, but we know that the left and right CFTs are not. While we know much about these massless excitations (e.g. their exact $S$-matrices and the free energy), one wonders what these excitations “really are”. Are they real particles, or are they just a way of encoding exact information like the free energy? We hope that these $N=2$ examples will shed some light on these general issues. In particular, perhaps the $N=2$ superconformal algebra obtained in \([14]\) acting on purely left-moving or right-moving states will allow for a better understanding as to what these massless excitations really are.

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