Edge of the wedge theorem for tempered ultra-hyperfunctions

E. Brüning\textsuperscript{a*} and S. Nagamachi\textsuperscript{b}

\textsuperscript{a}School of Mathematical Sciences, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, South Africa; \textsuperscript{b}Faculty of Engineering, The University of Tokushima, Tokushima 770-8506, Japan

Communicated by S. Krantz

(Received 23 October 2012; final version received 2 March 2013)

Tempered ultra-hyperfunctions do not have the same type of localization properties as Schwartz distributions or Sato hyperfunctions; but the localization properties seem to play an important role in the proofs of the various versions of the edge of the wedge theorem. Thus, for tempered ultra-hyperfunctions, one finds a global form of this result in the literature, but no local version. In this paper we propose and prove a formulation of the edge of the wedge theorem for tempered ultrahyperfunctions, both in global and local form. We explain our strategy first for the one variable case. We argue that in view of the cohomological definition of hyperfunctions and ultra-hyperfunctions, the global form of the edge of the wedge theorem is not surprising at all.

Keywords: tempered ultra-hyperfunctions; edge of the wedge theorem; local and global form of eowt

AMS Subject Classifications: 46F15; 32A45; 32A70

1. Introduction

Nowadays, i.e. about 50 years after its discovery, there are many versions of the ‘edge of the wedge theorem’ which originated through the challenges of relativistic quantum field theory and the theory of dispersion relations for scattering amplitudes (see [1,2]). In quantum field theory, the important fact that the Wightman functions are holomorphic in the region of all totally space-like points is shown by a simple application of the edge of the wedge theorem.

The statements in these various versions of the edge of the wedge theorem assert the extendability of holomorphic functions defined in wedges in complex space $\mathbb{C}^n$ with edge in real space $\mathbb{R}^n$, under certain conditions.

Recall the original version of Bogoliubov (see [2]):

**Theorem 1.1** [Bogoliubov] Let $C \subset \mathbb{R}^n$ be an open proper cone with vertex at the origin and denote by $C_r = C \cap B(0, r)$ the intersection of $C$ with the open ball of radius $r > 0$ centered at the origin; for an open nonempty set $E \subset \mathbb{R}^n$ introduce the wedges

\*Corresponding author. Email: bruninge@ukzn.ac.za

© 2013 Taylor & Francis
$W^\pm = E \pm iC_r$ with common edge $E$. If now $F_1$ is a holomorphic function on $W^+$ and $F_2$ is holomorphic on $W^-$ and if $F_1$ and $F_2$ have the same boundary values on $E$,

$$\lim_{y \to 0, y \in C_r} F_1(x + i y) = \lim_{y \to 0, y \in C_r} F_2(x - i y), \quad x \in E,$$

then $F_1$ and $F_2$ can be extended holomorphically to a complex neighborhood $\Omega$ of $W^+ \cup E \cup W^-$. Naturally, in (1) it is important to specify in which sense the boundary values are considered. In Bogoliubov’s version, these boundary values are taken in the sense of Schwartz distributions. Note that for $n = 1$ and when the boundary values are taken in the sense of continuous functions, this result is easily proven by using Morera’s theorem.

Over the last 50 years, this result has been extended in several directions:

- the type of generalized functions for the boundary values in (1), e.g. Schwartz distributions, Sato hyperfunctions, Fourier hyperfunctions, and ultradistributions;
- the number $m$ of wedges, $m > 2$;
- the ‘topological nature’ of the edge $E$, e.g. an open nonempty set or a maximal real submanifold.

We comment here on the case of Fourier hyperfunctions. A Fourier hyperfunction $f$ has two realizations. One is as a dual element of the test-function space $\mathcal{O}(D^n)$ and the other is as a formal sum

$$f(x) = \sum_{j=1}^m F_j(z)$$

where $F_j(z)$ is holomorphic in a wedge $W_j = D^n + i \Gamma_j$, that is, an element of the relative Čech cohomology group $H^n(\mathcal{O}, \mathcal{O}'; \mathcal{O})$ which is isomorphic to $H^n_{\text{dR}}(Q^n; \mathcal{O})$ for a suitable relative covering $(\mathcal{O}, \mathcal{O}')$ of the pair $(Q^n, Q^n \setminus D^n)$, where $\mathcal{O}$ is the sheaf of slowly increasing holomorphic functions on $Q^n = D^n + i \mathbb{R}^n$ and $D^n = \mathbb{R}^n \cup S^{n-1}_\infty$ is the radial compactification of $\mathbb{R}^n$ (see [3,4]). In the case of hyperfunctions and Fourier hyperfunctions, the edge of the wedge theorem tells us when the above sum is zero. Note that $H^n(\mathcal{O}, \mathcal{O}'; \mathcal{O})$ can be expressed as the following quotient space (see [5])

$$\oplus_{j=1}^m \hat{\mathcal{O}}(W_j)/[\oplus_{j<k} \hat{\mathcal{O}}(W_j + W_k)].$$

This denominator appears in the general edge of the wedge theorem for Fourier hyperfunctions (see [6]).

**Remark 1** Relation (2) ‘contains’ the most important versions of the EOW: If $\sum_{j=1}^m F_j(z) = f(x) = 0$, then there exist functions $H_{jk} \in \hat{\mathcal{O}}(W_j + W_k)$ for $j < k$ such that

$$F_j(z) = \sum_{k=1}^m H_{jk}(z), \quad j = 1, \ldots, m,$$

where we put

$$H_{jk}(z) = -H_{kj}(z) \text{ for } j > k \text{ and } H_{jj}(z) = 0.$$ 

The above statement is Martineau’s version of the EOW theorem.[7–9]
Note that conversely, for the functions $F_j$ defined by (3) with functions $H_{jk}$ satisfying relation (4), the sum
\[ \sum_{j=1}^{m} F_j(z) = \sum_{j=1}^{m} \sum_{k=1}^{m} H_{jk}(z) \]
is reduced to zero and defines the zero Fourier hyperfunction.

When $m = 2$, we have Epstein’s version of the EOW theorem,[10] i.e. if $F_j \in \hat{O}(W_j)(j = 1, 2)$ define the same Fourier hyperfunction, then $F_j \in \hat{O}(W_j)(j = 1, 2)$ are analytically continued to $H_{12} \in \hat{O}(W_1 + W_2)$. (Epstein’s original version is formulated in terms of Schwartz distributions as boundary values).

If $\Gamma_2 = -\Gamma_1$ and if the boundary values are taken in the sense of Schwartz distributions, then we have Bogoliubov’s version of the EOW theorem.

In our recent investigations of relativistic quantum field theory with a fundamental length (see [11–13], in particular Theorem 5.11 of [13]) we need a version of the edge of the wedge theorem for tempered ultra-hyperfunctions and it is this version which is treated in this article.

2. Preliminaries

2.1. Global and local versions of the EOW Theorem

In these preliminary considerations, we put the global and the local forms of the edge of the wedge theorems for hyperfunctions and ultra-hyperfunctions into the perspective of cohomology theory.

Let us recall the global form of this theorem for hyperfunctions (for simplicity and our intended application to quantum field theory, we consider in this paper only Bogoliubov’s version for $\Gamma = V_+$):

Let $V_+ \subset \mathbb{R}^4$ denote the forward light-cone; suppose that $F_1$ is an analytic function in $T(V_+) = \mathbb{R}^4 + iV_+$ and $F_2$ an analytic function in $T(-V_+)$. Then the two functions $F_i$ ($i = 1, 2$) define hyperfunctions $f_i$ on $\mathbb{R}^4$. If $f_1 = f_2$, then $F_i$ are analytically continued to an entire function $F$.

The local form of the EOW theorem for hyperfunctions can be formulated as follows. Let $U$ be an open set in $\mathbb{R}^n$ and $V$ an open set in $\mathbb{C}^n$ such that $U = V \cap \mathbb{R}^n$. Then we have the canonical restriction map
\[ H^n_{\mathbb{C}^n}(\mathbb{C}^n, \mathcal{O}) \to H^n_{\mathbb{R}^n}(V, \mathcal{O}) = H^{n-1}(V \setminus U, \mathcal{O}), \] where $\mathcal{O}$ is the sheaf of holomorphic functions on $\mathbb{C}^n$. $H^n_{\mathbb{R}^n}(V, \mathcal{O})$ is independent of the complex neighborhood $V$ of $U$ by the excision theorem, and the presheaf $\{ U \to H^n_{\mathbb{R}^n}(V, \mathcal{O}) \}$ is the sheaf of hyperfunctions on $\mathbb{R}^n$ which is often denoted by $B$. The local form of the EOW theorem now reads (we use the notation from above):

If $f_1 = f_2$ in $U$ (or the restrictions $f_j|_U$ ($j = 1, 2$) coincide or the support of $f_1 - f_2$ is contained in the complement of $U$), then $F_j$ are analytically continued to each other through $U$.

There are two ways to treat the EOW theorem; the functional method (see [14]) and the cohomological method (see [5,15]). The functional method uses the notion of the analytic wave front set $WF_a(f)$ of hyperfunctions $f$ on $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and the decomposition of
The colomological method uses the notion of the flabby sheaf $C$ of microfunctions on $\mathbb{R}^n \times S^{n-1}$ and the exact sequence

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \pi_* C \rightarrow 0,$$

where $\mathcal{A}$ is the sheaf of real analytic functions on $\mathbb{R}^n$ and $\pi_* C$ is the direct image of $C$ under the projection $\pi : \mathbb{R}^n \times S^{n-1} \rightarrow \mathbb{R}^n$, i.e., the sheaf on $\mathbb{R}^n$ defined by the correspondence

$$\mathbb{R}^n \ni U \rightarrow C(\pi^{-1}(U)).$$

Finally we comment on the difficulties for the EOW for tempered ultra-hyperfunctions. An element of $T(T(\mathbb{R}^n))'$ is a tempered ultra-hyperfunction (see Definition 3.1 for the topology of $T(T(\mathbb{R}^n))$). For an open set $W$ in $\mathbb{C}^n$, $O_0(W)$ is the space of all tempered (polynomially increasing) holomorphic functions in $W$ (see Definition 3.5 and [16]).

Any element of $T(T(\mathbb{R}^n))'$ belongs to some $T(T(K))'$, $K = [-k, k]^n$ for suitable $k > 0$ and

$$T(T(K))' = H^n(\mathcal{M}, \mathcal{M}'; O_0),$$

where $\mathcal{M} = \mathcal{M}' \cup \{\mathbb{C}^n\}$ and $\mathcal{M}' = \{T(E_j); j = 1, \ldots, n\}$, $E_j = \{y \in \mathbb{R}^n; |y_j| > k\}$ are relative covering of $\mathbb{C}^n$, $\mathbb{C}^n \setminus T(K)$ (see [16,17]). Hyperfunctions are localized in a relatively open set $U$ of the closed set $\mathbb{R}^n$ in $\mathbb{C}^n$ by the formula (5). On the other hand, tempered ultra-hyperfunctions may be localized in a relatively open set $U$ of the closed set $T(K)$ of $\mathbb{C}^n$, but not in a relatively open set $U$ of the closed set $\mathbb{R}^n$ in $\mathbb{C}^n$. This is the reason why tempered ultra-hyperfunctions have no (standard) localization property in $\mathbb{R}^n$. In [11,12,18] this property has been successfully applied to axiomatic quantum field theory in order to formulate such a theory with a fundamental length.

The global form of the edge of the wedge theorem for tempered ultra-hyperfunctions reads: Let $\Gamma = \{y \in \mathbb{R}^n; y_j > k, j = 1, \ldots, n\}$. If $F_1$ is a polynomially increasing holomorphic function in $T(\Gamma)$ and $F_2$ a polynomially increasing holomorphic function in $T(-\Gamma)$, then these functions $F_i$ ($i = 1, 2$) define tempered ultra-hyperfunctions $f_i$ and if $f_1 = f_2$, then $F_i$ are analytically continued to a polynomial $F$ (see [16]).

**Remark 2** It is interesting to note that even though $T(\Gamma)$ and $T(-\Gamma)$ are separated by a gap of size $2\sqrt{n}k$ the functions $F_1$ and $F_2$ are analytically continued to each other!

At first sight this seems to be quite surprising. However, from the point of view of the cohomological definition of ultra-hyperfunctions, this result is not so surprising, since $H^n(\mathcal{M}, \mathcal{M}'; O_0)$ has the following representation

$$O_0(\mathbb{C}^n \# T(K)) \bigg/ \sum_{j=1}^n O_0(W_j),$$

where

$$\mathbb{C}^n \# T(K) = T(E_1) \cap \cdots \cap T(E_n),$$

$$W_j = T(E_1) \cap \cdots \cap \overline{T(E_j)} \cap \cdots \cap T(E_n)$$

and the denominator of this representation then shows this result. We explain this in more detail in the next subsection for the one-dimensional case. The cohomological treatment of ultra-hyperfunctions is given in [19,20] and the above representation was presented in [16].
The global form of the EOW theorem for tempered ultra-hyperfunctions has been shown in [21] and some preliminary version in [22,23].

In the functional method [14], Hörmander used a kernel \( K(z) \) defined by
\[
K(z) = (2\pi)^{-n} \int e^{i(z,\xi)}/I(\xi)d\xi,
\]
\[
I(\xi) = \int_{|\omega|=1} e^{-i(\omega,\xi)}d\omega
\]
(7)
to prove the edge of the wedge theorem. For the proof of the local form of the EOW theorem for tempered ultra-hyperfunctions we also use the functional method with some modification \( K_r(z) = r^{-n}K(z/r) \) of this kernel \( K(z) \) for \( r > 0 \).

The local form of the edge of the wedge theorem for hyperfunction has the following formulation (\( F_j \) and \( f_j \) are related as in the above results): If \( f_1 = f_2 \) in an open set \( O \), then \( F_1 \) and \( F_2 \) are analytically continued to each other through \( O \).

Since tempered ultra-hyperfunctions have no localization property, it is not easy to formulate a local form of the edge of the wedge theorem. In this paper, we suggest a formulation of the local version of the edge of the wedge theorem by using the notion of a carrier.

Remark 3 For the one-dimensional case, the Cauchy-Hilbert transformation (10) gives the isomorphism of the space \( O(L)^\prime \) of analytic functionals with carriers in \( L \) onto the relative cohomology group of covering (11) of the pair \( (\mathbb{C}, \mathbb{C}\setminus L) \). But for the multidimensional case, we need the additional assumption to have the expression of the space \( O(L)^\prime \) of analytic functionals with carriers in a compact set \( L \subset \mathbb{C}^n \) as a collection of holomorphic functions. In fact, the isomorphism of the space \( O(L)^\prime \) of analytic functionals with carriers in a compact set \( L \subset \mathbb{C}^n \) onto the relative cohomology group \( H^n_L(\mathbb{C}^n, O) \) is proven under the condition a) \( H^p(L, O) = 0 \) for \( p = 1, 2, \ldots \) (Martineau-Harvey theorem, see [24]), and the isomorphism of \( H^n_L(\mathbb{C}^n, O) \) onto the cohomology group \( H^n_\mathcal{W}(\mathcal{W}, \mathcal{W}^\prime; O) \) of a covering \( (\mathcal{W}, \mathcal{W}^\prime) \) of the pair \( (\mathbb{C}^n, \mathbb{C}^n \setminus L) \) is proven under the condition b) \( H^p(W_{\lambda_0} \cap \ldots \cap W_{\lambda_m}) = 0 \) for \( p \geq 1 \) and \( W_{\lambda_0}, \ldots, W_{\lambda_m} \in \mathcal{W} \) (Leray theorem, see [24]). In the case of hyperfunctions, a compact set \( L \subset \mathbb{R}^n \) satisfies condition (a) and there exists a relative covering \( (\mathcal{W}, \mathcal{W}^\prime) \) of the pair \( (\mathbb{C}^n, \mathbb{C}^n \setminus L) \) which satisfies condition (b) (see [24]). But in the case of analytic functional \( O(L)^\prime \), conditions (a) and (b) may not necessarily be satisfied for some compact set \( L \subset \mathbb{C}^n \) and any relative covering \( (\mathcal{W}, \mathcal{W}^\prime) \) of the pair \( (\mathbb{C}^n, \mathbb{C}^n \setminus L) \). Therefore, in this paper, we employ the functional method.

The main result in this regard is Corollary 4.3 which has an intimate connection to axiomatic quantum field theory with a fundamental length (see [13]). This corollary says that if \( f_1 - f_2 \in T(L)^\prime \) for some \( \ell \)-neighborhood
\[
L = \{w \in \mathbb{C}^4; \exists x \in V \ |\text{Re } w - x| + |\text{Im } w|_1 < \ell, \}
\]
of the light-cone \( V, \ell > 0 \), then \( F_i (i = 1, 2) \) are analytically continued to each other through a set
\[
\{x \in \mathbb{R}^4; \text{dist}(x,V) > (\sqrt{2} + 1)\ell. \}
\]

2.2. The one-dimensional case

In order to explain the basic idea of our strategy of proof for the local version of the EOW theorem for tempered ultrahyperfunctions, we illustrate it here for the technically
much simpler case of one dimension. And we prepare this with explaining the proof for hyperfunctions in one variable.

The space of hyperfunctions of one variable is the quotient space

\[ B(\mathbb{R}) = \mathcal{O}(\mathbb{C}\setminus \mathbb{R})/\mathcal{O}(\mathbb{C}). \]

Let \( F_1 \) (resp. \( F_2 \)) be a holomorphic function in the upper (resp. lower) half plane. Then the pair of functions \((F_1, F_2) \in \mathcal{O}(\mathbb{C}\setminus \mathbb{R})\) defines an element \( f (= F_1 - F_2) \) of \( B(\mathbb{R}) \). If \( f = 0 \) \((F_1 = F_2)\), then \((F_1, F_2) \in \mathcal{O}(\mathbb{C})\). This shows that \( F_1, F_2 \) coincide with an element \( F \) of \( \mathcal{O}(\mathbb{C}) \). Thus the EOW theorem automatically follows from the cohomological definition of hyperfunctions.

The local version of the EOW theorem for hyperfunctions is also a direct consequence of the cohomological definition of hyperfunctions on an open set \( U \) of \( \mathbb{R} \):

\[ B(U) = \mathcal{O}(V \setminus U)/\mathcal{O}(V), \]

where \( V \) is a complex neighborhood of \( U \) such that \( U = V \cap \mathbb{R} \).

Now consider the case of tempered ultra-hyperfunctions. Denote \( K = [-k, k] \) for \( k > 0 \) and \( T(K) = \{ z \in \mathbb{C}; |\text{Im} \ z| \leq k \} \). Note that formula (6) shows that any tempered ultra-hyperfunction \( f \in T(T(\mathbb{R}))' \) can be expressed as an element of the space

\[ \mathcal{O}_0(\mathbb{C}\setminus T(K))/\mathcal{O}_0(\mathbb{C}) \]

for some \( k > 0 \). Let \( F_1 \) (resp. \( F_2 \)) be a polynomially increasing holomorphic function in \( \{ z \in \mathbb{C}; |\text{Im} \ z| > k \} \) (resp. \( \{ z \in \mathbb{C}; -|\text{Im} \ z| > k \} \)). Then the pair of functions \((F_1, F_2) \in \mathcal{O}_0(\mathbb{C}\setminus T(K))\) defines an element \( f \) of \( \mathcal{O}_0(\mathbb{C}\setminus T(K))/\mathcal{O}_0(\mathbb{C}) \). If \( f = 0 \), then \((F_1, F_2) \in \mathcal{O}_0(\mathbb{C})\). This shows that \( F_1, F_2 \) coincide with an element \( F \) of \( \mathcal{O}_0(\mathbb{C}) \). Thus the global form of the EOW theorem for tempered ultra-hyperfunctions automatically follows from the cohomological definition of tempered ultra-hyperfunctions.

Since the notion of localization for ultra-hyperfunctions is not available in the above sense, there is no literature about the local version of the EOW theorem for ultra-hyperfunctions. The notion of localization for generalized functions has an intimate connection with the notion of support. However, ultra-hyperfunctions have no supports in general, but they are a special kind of analytic functionals and have carriers.

**Definition 2.1** Let \( L \) be a compact set in \( \mathbb{C} \). \( L \) is called a **carrier of an analytic functional** \( f \) (a continuous linear functional on the space \( \mathcal{O}(\mathbb{C}) \) of entire functions), if \( f \) satisfies

\[ |f(\phi)| \leq C_V \sup_{z \in V} |\phi(z)| \]

for any open neighborhood \( V \) of \( L \).

Let \( K = [a, b] \subset \mathbb{R} \) and \( H^1_K(\mathbb{C}, \mathcal{O}) \) be the space of hyperfunctions with supports in \( K \), which is isomorphic to the space of analytic functionals with carriers in \( K \):

\[ H^1_K(\mathbb{C}, \mathcal{O}) \cong \mathcal{O}(\mathbb{C}\setminus K)/\mathcal{O}(\mathbb{C}). \]

Every \( F \in \mathcal{O}(\mathbb{C}\setminus K) \) defines a functional on the space of functions \( \phi \) which are holomorphic in a complex neighborhood \( V \) of \( K \) by the formula

\[ \phi \rightarrow f(\phi) = -\int_C F(z)\phi(z)dz, \] (8)
where $C$ is a closed path that encircles $K$ once in the positive direction.

Let $E^t_\xi(z) = (4\pi t)^{-1/2}e^{-(z-\xi)^2/4t}$ for $\xi \in \mathbb{R}\setminus K$. Clearly $E^t_\xi$ is holomorphic in a complex neighborhood $V$ of $K$ and, as $t \to 0$, $E^t_\xi \to \delta(x-\xi)$ and one has the estimate

$$|f(E^t_\xi)| \leq M_C \sup_{x+iy \in C} (4\pi t)^{-1/2}e^{(y^2-(x-\xi)^2)/4t} \to 0$$

as $t \to 0^+$, since $C$ can be chosen arbitrary close to $K$.

Let $L = [a, b] + i[-\ell, \ell]$ and $f$ be an analytic functional with carrier $L$. Then the Cauchy-Hilbert transformation $F$ of $f$ is defined by

$$F(z) = (2\pi i)^{-1} f_x(1/(w-z)),$$

it is an element of

$$\mathcal{O}_0(\mathbb{C}\setminus L)/\mathcal{O}_0(\mathbb{C}),$$

i.e. the space of tempered ultra-hyperfunctions with carriers in $L$, and it reproduces the functional $f$ by formula (8) with a closed path $C$ that encircles $L$, i.e.

$$-\int_C F(z)\phi(z)dz = f_x(1/(w-z))$$

For a tempered ultra-hyperfunction $f$ with carrier $L$, we have (9) if $\xi \in \mathbb{R}\setminus [a-\ell, b+\ell]$. This means that $\xi$ is considered to be outside of $[a, b]$ only if $\xi \in \mathbb{R}\setminus [a-\ell, b+\ell]$!

**Remark 4** Heuristically we read this fact as follows:

An ultra-hyperfunction $f$ becomes ‘aware’ of $\xi$ being outside of $[a, b]$ only if $\xi \in \mathbb{R}\setminus [a-\ell, b+\ell]$, while a hyperfunction can be ‘aware’ of $\xi$ being outside of $[a, b]$ if $\xi \in \mathbb{R}\setminus [a, b]$.

This difference can also be understood through cohomological considerations. Let $U$ be an open set in $\mathbb{R}$ such that $[a, b] \cap U = \emptyset$ and $U = V \cap \mathbb{R}$ for an open set $V$ in $\mathbb{C}$. In the case of hyperfunctions, we have the restriction

$$H^1_K(\mathbb{C}, \mathcal{O}) \to H^1_U(V, \mathcal{O}) \cong \mathcal{O}(V\setminus K)/\mathcal{O}(V) = \mathcal{O}(V)/\mathcal{O}(V) = 0.$$

But for the case of ultra-hyperfunctions, we have

$$\mathcal{O}_0(\mathbb{C}\setminus L)/\mathcal{O}_0(\mathbb{C}) \to \mathcal{O}_0(V\setminus L)/\mathcal{O}_0(V),$$

and there exists an open set $V$ in $\mathbb{C}$ satisfying $[a, b] \cap U = \emptyset$ and $U = V \cap \mathbb{R}$ but $V\setminus L \neq V$ and $\mathcal{O}_0(V\setminus L)/\mathcal{O}_0(V) \neq 0$.

Now we study the EOW theorem by using Hörmander’s kernel $K(z)$. First, recall that Dirac’s $\delta$ function can be expressed as follows:

$$\delta(x) = (2\pi)^{-1} \lim_{\epsilon \to 0} \int_{-\infty}^\infty e^{i\xi x} d\xi$$

$$= (2\pi)^{-1} \lim_{\epsilon \to 0} \int_{-\infty}^\infty \frac{e^{i\xi(x+i\epsilon)}}{e^\xi + e^{-\xi}} d\xi + (2\pi)^{-1} \lim_{\epsilon \to 0} \int_{-\infty}^\infty \frac{e^{i\xi(x-i\epsilon)}}{e^\xi + e^{-\xi}} d\xi$$

$$= (2\pi)^{-1} \lim_{\epsilon \to 0} \int_{-\infty}^\infty \frac{e^{i\xi(x+i\epsilon)}}{e^\xi + e^{-\xi}} d\xi + (2\pi)^{-1} \lim_{\epsilon \to 0} \int_{-\infty}^\infty \frac{e^{i\xi(x-i\epsilon+i)}}{e^\xi + e^{-\xi}} d\xi$$

$$= (2\pi)^{-1} \sum_{\omega = \pm 1} \int_{-\infty}^\infty \frac{e^{i\xi(x+i(1-\epsilon)\omega)}}{e^\xi + e^{-\xi}} d\xi.$$
Now introduce the function $K(z)$ of (7) for $n = 1$:

$$K(z) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\xi z} \frac{1}{2} \text{sech} \xi d\xi = \frac{1}{4} \text{sech} (\pi z/2)$$

for $\text{sech} \xi = \frac{2}{e^{\xi} + e^{-\xi}}$. Then the above representation of the Dirac’s delta function can be rewritten as

$$\delta(x) = \lim_{\epsilon \to +0} \frac{1}{\epsilon} \left[ K(x + i\epsilon - i) + K(x - i\epsilon + i) \right]$$

$$= \lim_{\epsilon \to +0} (1/4)[\text{sech} (x + i\epsilon - i)/2 + \text{sech} (x - i\epsilon + i)/2]$$

$$= \lim_{\epsilon \to +0} (1/4)i[\text{cosech} (x + i\epsilon)/2 - \text{cosech} (x - i\epsilon)/2], \quad (12)$$

where $\text{cosech} \xi = \frac{2}{e^{\xi} + e^{-\xi}}$. Since the difference between $(1/4)i\text{cosech} \pi z/2$ and $-1/(2\pi iz)$ is a holomorphic function in $\{z \in \mathbb{C}; |\text{Im} z| < 1\}$, formula (12) is equivalent to the famous formula

$$\delta(x) = -(2\pi i)^{-1} \lim_{\epsilon \to +0} [1/(x + i\epsilon) - 1/(x - i\epsilon)].$$

The singular points of $\text{sech} \xi$ are $\xi = i(1 + 2n)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$ and those of $K(z)$ are $z = i(1 + 2n), n = 0, \pm 1, \pm 2, \ldots$. Therefore we have, for $\phi \in T(T(\mathbb{R}))$ and $0 < R \leq 1$

$$\phi(0) = \lim_{\epsilon \to +0} \int \left[ K(x + i\epsilon - i) + K(x - i\epsilon + i) \right] \phi(x) dx$$

$$= \lim_{\epsilon \to +0} \sum_{\omega = \pm 1} \int K(x - i\omega - i\epsilon + i\omega) \phi(x) dx$$

$$= \lim_{\epsilon \to +0} \sum_{\omega = \pm 1} \int K(x - i\omega R - i\epsilon + i\omega) \phi(x - i\omega) dx$$

$$= \sum_{\omega = \pm 1} \int K(x - i\omega R + i\omega) \phi(x - i\omega) dx$$

Let $K_r(z) = r^{-1}K(z/r)$. Then, we can reformulate the above relation as

$$\delta(x) = \lim_{\epsilon \to +0} \left[ K_r(x + i\epsilon R - i\epsilon R) + K_r(x - i\epsilon R + i\epsilon R) \right]$$

and for $0 < R \leq r$

$$\sum_{\omega = \pm 1} \int_{-\infty}^{\infty} K_r(x + i(r - R)\omega) \phi(x - i\omega R) dx = \phi(0)$$

and

$$\sum_{\omega = \pm 1} \int_{-\infty}^{\infty} K_r(x - t + i(r - R)\omega) \phi(x - i\omega R) dx$$

$$= \sum_{\omega = \pm 1} \int_{-\infty}^{\infty} K_r(x + i(r - R)\omega) \phi(x + t - i\omega R) dx = \phi(t). \quad (13)$$
Let $\Gamma_1 = \{y \in \mathbb{R}; y > \ell\}$ and $\Gamma_2 = \{y \in \mathbb{R}; -y > \ell\}$. Given $F_j \in \mathcal{O}_0(T(\Gamma_j))$ denote by $u_j$ the tempered ultra-hyperfunction defined by

$$u_j(\phi) = \int_{-\infty}^{\infty} F_j(x + iy)\phi(x + iy)dx$$

for $y \in \Gamma_j$. (The definitions of $u_j$ do not depend on the choice of the $y \in \Gamma_j$ because the paths of integration can be shifted by Cauchy’s integral theorem.) Choose $r > \ell$ and define

$$U_j(z) = u_j \ast K_r(z) = \int_{-\infty}^{\infty} F_j(\xi + i\eta)K_r(z - \xi - i\eta)d\xi$$

for $\eta \in \Gamma_j$. Then $U_j(z)$ is analytic in $V_j$,

$$V_1 = \cup_{\eta \in \Gamma_1} \{z \in \mathbb{C}; |\text{Im} (z - i\eta)| < r\} = \{z \in \mathbb{C}; \text{Im} z > \ell - r\},$$
$$V_2 = \cup_{\eta \in \Gamma_2} \{z \in \mathbb{C}; |\text{Im} (z - i\eta)| < r\} = \{z \in \mathbb{C}; \text{Im} z < -\ell + r\}.$$

Note that (13) implies

$$\sum_{\omega = \pm 1} \int_{-\infty}^{\infty} U_j(x + i(r - R)\omega)\phi(x - iR\omega)dx$$
$$= \sum_{\omega = \pm 1} \int_{-\infty}^{\infty} u_j \ast K_r(x + i(r - R)\omega)\phi(x - iR\omega)dx$$
$$= \sum_{\omega = \pm 1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_j(\xi + i\eta)K_r(x - \xi - i\eta + i(r - R)\omega)\phi(x - iR\omega)d\xi d\eta$$
$$= \int_{-\infty}^{\infty} F_j(\xi + i\eta)\phi(\xi + i\eta)d\xi = u_j(\phi).$$

(14)

If $u_1 = u_2$, then $U_1(z) = U_2(z)$ in $V_1 \cap V_2 = \{z \in \mathbb{C}; |\text{Im} z| < r - \ell\}$, and the two functions $U_1, U_2$ are continued to a function $U$ which is analytic in $V_1 \cup V_2 = \mathbb{C}$.

Now introduce the function

$$H(z) = \sum_{\omega = \pm 1} U(z + ir\omega);$$

clearly $H$ is an entire function, and we have for $y \in \Gamma_1, \phi \in \mathcal{T}(T(\mathbb{R}))$, and $0 < \sigma \leq r - \ell$

$$\int_{-\infty}^{\infty} H(x + iy)\phi(x + iy)dx = \sum_{\omega = \pm 1} \int_{-\infty}^{\infty} U(x + iy + ir\omega)\phi(x + iy)dx$$
$$= \sum_{\omega = \pm 1} \int_{-\infty}^{\infty} U(x + i(r - [\ell + \sigma])\omega)\phi(x - i(\ell + \sigma)\omega)dx$$
$$= \sum_{\omega = \pm 1} \int_{-\infty}^{\infty} U_1(x + i(r - [\ell + \sigma])\omega)\phi(x - i(\ell + \sigma)\omega)dx$$
$$= u_1(\phi) = \int_{-\infty}^{\infty} F_1(x + iy)\phi(x + iy)dx$$
where we used the fact that $U(z) = U_1(z)$ for $\text{Im } z > \ell - r$ and relation (14), hence
\[
\int_{-\infty}^{\infty} H(x + iy)\phi(x + iy)dx = \int_{-\infty}^{\infty} F_1(x + iy)\phi(x + iy)dx
\]
for $\phi \in \mathcal{T}(T(\mathbb{R}))$. This shows that $H(z) = F_1(z)$ in $T(\Gamma_1)$, and in the same way we get $H_1(z) = F_2(z)$ in $T(\Gamma_2)$. This proves the global form of the EOW theorem for tempered ultra-hyperfunctions of one variable.

Next we discuss the local version. To this end, assume that the carrier of $u_1 - u_2$ is contained in $L = [a, b] + i[-\ell, \ell]$, instead of $u_1 = u_2$. This will lead to the local form of the EOW theorem.

Introduce the function $U_{12}$ by
\[
U_{12}(z) = (u_1 - u_2) \ast K_r(z).
\]
Since the singular points of $K_r(z)$ are $z = i(1 + 2n)r, n = 0, \pm 1, \pm 2, \ldots, K_r(z - w)$, for $\text{Re } z \not\in [a, b]$, is holomorphic in a neighborhood of $L$ and $U_{12}(z)$ is holomorphic in $Z = \{z \in \mathbb{C}; \text{Re } z \not\in [a, b]\}$. Since $u_1 = u_2 + u_1 - u_2$, $U_1(z) = U_2(z) + U_{12}(z)$.

$U_1(z)$ is holomorphic in $V_1$ and $U_2(z) + U_{12}(z)$ is holomorphic in $Z \cap V_2$. Therefore, in the same way as in the global case, $U_1(z)$ can be analytically continued to a function $U_1(\xi)$ which is holomorphic in $Z \cup V_1$ and $F_1(z)$ can be analytically continued to $F_1(\xi) = \sum_{\omega = \pm 1} U_1'(z + i\omega) \xi$ which is holomorphic in $Z \cup T(\Gamma_1)$. In the same way, $F_2(\xi)$ can be analytically continued to $F_2(\xi)$ which is holomorphic in $Z \cup T(\Gamma_2)$. In order to show $H_1(\xi) = H_2(\xi)$ in $Z$, we introduce a path $C = C_1 + \ldots + C_5$ consisting of line segments $C_1 = (-\infty, a - 2\ell]$, $C_2 = [a - 2\ell, a + i2\ell]$, $C_3 = [a + i2\ell, b + i2\ell]$, $C_4 = [b + i2\ell, b + 2\ell]$ and $C_5 = [b + 2\ell, \infty)$ (see Figure 1). Let $E_1^r(z) = (4\pi t)^{-1/2} e^{-((\xi - z)^2 + t) / 4t}$ and $\xi \in \mathbb{R}$ such that $\xi < a - 2\ell$ or $\xi > b + 2\ell$. Then, we have
\[
u_1(E_1^r(z)) = \int_{-\infty}^{\infty} F_1(x + i2\ell)E_1^r(x + i2\ell)dx
\]
\[
= \int_{-\infty}^{\infty} H_1(x + i2\ell)E_1^r(x + i2\ell)dx = \int_C H_1(z)E_1^r(z)dz \to H_1(\xi)
\]
as $t \to 0+$. Here we used the fact that if $|\xi - \text{Re } z| > |\text{Im } z|$, $|E_1^r(z)| = (4\pi t)^{-1} e^{-((\xi - \text{Re } z)^2 + |\text{Im } z|^2) / 4t} \to 0$ as $t \to 0+$ and if $z \in \mathbb{R}$, $E_1^r(z) \to \delta(z - \xi)$, i.e. $E_1^r(z) \to 0$ for $z \in C_2 + C_3 + C_4$, and if $\xi \in C_1$ (resp. $\xi \in C_5$) then $E_1^r(z) \to \delta(z - \xi)$ and $E_1^r(z) \to 0$ for

![Figure 1](image.png)

Figure 1. The path of integration $C = C_1 + C_2 + C_3 + C_4 + C_5$. 
z \in C_5 \text{ (resp. } z \in C_1 \text{). If we choose a curve } C' = C'_1 + \ldots + C'_5, \text{ where } C'_1 = (-\infty, a - 2\ell], C'_2 = [a - 2\ell, a - i2\ell], C'_3 = [a - i2\ell, b - i2\ell], C'_4 = [b - i2\ell, b + \ell], C'_5 = [b + 2\ell, \infty), \text{ then we have}

\begin{equation}
u_2(E'_k) = \int_{-\infty}^{\infty} H_2(x - i2\ell)E'_k(x - i2\ell)dx = \int_{C'} H_2(z)E'_k(z)dz \to H_2(\xi)
\end{equation}
as \( t \to 0^+ \). Moreover, we find

\begin{equation}
|\nu_1(E'_k) - \nu_2(E'_k)| \leq M_V \sup_{z \in V} |E'_k(z)| \to 0
\end{equation}
as \( t \to 0^+ \) for some neighborhood \( V \) of \( L \). Thus, \( H_1(\xi) = H_2(\xi) \) follows and consequently there exists a function \( H \) which is holomorphic in \( \mathbb{C}\setminus L \) and which is the common extension of \( F_1 \) and \( F_2 \). Therefore, \( F_1 \) and \( F_2 \) have the common extension \( H \in \mathcal{O}_0(\mathbb{C}\setminus L) \), and \( H \) is an element of \( \mathcal{O}_0(\mathbb{C}\setminus L)/\mathcal{O}_0(\mathbb{C}) \) of (11).

It is interesting to note that the cohomology group \( H^1_L(\mathbb{C}, \mathcal{O}_0) \) of tempered ultra-hyperfunctions with carriers in \( L \) appears again here in the argument of EOW theorem using the kernel \( K_r(z) \).

### 3. Global edge of the wedge theorem

In this section, we prove the global version of EOW theorem in higher dimension using the functional method. This will help in understanding the proof of the local version.

First we recall the test-function space \( T(T(\mathbb{R}^n)) \) of tempered ultra-hyperfunction. Let \( T(A) = \mathbb{R}^n + iA \) for \( A \subset \mathbb{R}^n \). Let \( K \subset \mathbb{R}^n \) be a convex compact set and \( T_b(T(K)) \) the set of continuous functions on \( T(K) \) which are holomorphic in the interior of \( T(K) \) and satisfy

\begin{equation}
\|f\|_{T(K), j} = \sup\{|z^p f(z)|; z \in T(K), |p| \leq j \} < \infty, \quad j = 1, 2, \ldots.
\end{equation}

There is a natural restriction mapping \( P_{K, L} : T_b(T(K)) \to T_b(T(L)) \) for \( K \supseteq L \).

**Definition 3.1** \( T(T(\mathbb{R}^n)) \) is the projective limit

\begin{equation}
T(T(\mathbb{R}^n)) = \lim_{\leftarrow} T_b(T(K)), \quad K \uparrow \mathbb{R}^n
\end{equation}
of the projective system \( (T_b(T(K)), P_{K, L}) \), where \( K \) runs through the convex compact sets in \( \mathbb{R}^n \).

It is known that the kernel \( K(z) \) of (7) is a rapidly decreasing holomorphic function in

\begin{equation}
X = \{z \in \mathbb{C}^n; |\text{Im } z|^2 < 1 + |\text{Re } z|^2\};
\end{equation}

more precisely the following lemma holds which is Lemma 3.1 of [6].

**Lemma 3.2** \( K(z) \) is a holomorphic function in \( X \) and we have, for some \( c > 0 \), \( K(z) = O(e^{-c|z|}) \) as \( z \to \infty \) in \( \{z \in \mathbb{C}^n; |\text{Im } z| < |\text{Re } z|/2\} \).
This allows to prove:

**Lemma 3.3** For any $0 < R \leq 1$ and $\phi \in T(T(R^n))$ one has

\[
\phi(t) = \int_{|\omega|=1} d\omega \int_{\mathbb{R}^n} K(x - iR\omega + i\omega - t)\phi(x - iR\omega)dx.
\]

**Proof** Consider the Fourier transform $\hat{\phi}$ of $\phi$ and note that it can be represented as

\[
\hat{\phi}(\xi) = \int_{\mathbb{R}^n} \phi(x)e^{i(x,\xi)}dx = \int_{\mathbb{R}^n} \phi(x - iR\omega)e^{i(x,\xi)}dx.
\]

Thus, we get

\[
\int_{|\omega|=1} d\omega \int_{\mathbb{R}^n} K(x - iR\omega + i\omega - t)\phi(x - iR\omega)dx
= (2\pi)^{-n} \int_{|\omega|=1} d\omega \int_{\mathbb{R}^n} d\xi \phi(x - i\omega)\int_{\mathbb{R}^n} d\xi e^{i(x - iR\omega + i\omega - t,\xi)} I(\xi)
= (2\pi)^{-n} \int_{|\omega|=1} d\omega \int_{\mathbb{R}^n} d\xi \phi(x - i\omega)e^{i(x - iR\omega,\xi)}e^{i(\omega - t,\xi)}I(\xi)
= (2\pi)^{-n} \int_{|\omega|=1} d\omega \int_{\mathbb{R}^n} d\xi \phi(x - i\omega)e^{i(\omega - t,\xi)}I(\xi)
= (2\pi)^{-n} \int_{|\omega|=1} d\omega \int_{\mathbb{R}^n} d\xi \phi(x - i\omega)e^{i(\omega - t,\xi)}I(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} d\xi \hat{\phi}(\xi)e^{-i(\xi,\xi)} = \phi(t).
\]

**Corollary 3.4** Let $K_r(z) = r^{-n}K(z/r)$. Then $K_r(z)$ is a rapidly decreasing holomorphic function in $\{z \in \mathbb{C}^n; |\text{Im } z/r|^2 < 1 + |\text{Re } z/r|^2\} = \{z \in \mathbb{C}^n; |\text{Im } z|^2 < r^2 + |\text{Re } z|^2\}$, and for $0 < R \leq r$, the identity

\[
\phi(t) = \int_{|\omega|=1} d\omega \int_{\mathbb{R}^n} K_r(x - iR\omega + i\omega - t)\phi(x - iR\omega)dx
\]
holds for any $\phi \in T(T(R^n))$.

**Proof** A straightforward calculation yields

\[
\int_{|\omega|=1} d\omega \int_{\mathbb{R}^n} K_r(x - iR\omega + i\omega - t)\phi(x - iR\omega)dx
= r^{-n} \int_{|\omega|=1} d\omega \int_{\mathbb{R}^n} K_1((x - iR\omega + i\omega - t)/r)\phi(x - iR\omega)dx
= \int_{|\omega|=1} d\omega \int_{\mathbb{R}^n} K_1(y - iR\omega/r + i\omega - t/r)\phi(y - iR\omega/r)dy
= \phi(r(t/r)) = \phi(t).
\]

Thus, we can remove the restriction $R \leq 1$. 

**Definition 3.5** Let $O$ be an open set in $\mathbb{R}^n$. Denote by $\mathcal{O}_0(T(O))$ the set of those functions $F(z)$ which are holomorphic in $T(O)$ and which satisfy the following condition: For any compact set $K \subset O$, there exists a natural number $j > 0$ such that
\[
\sup_{z \in T(K)} |F(z)|(1 + |z|)^{-j} < \infty.
\]

Now we can state the global form of the edge of the wedge theorem for tempered ultra-hyperfunctions.

**Theorem 3.6** Let $V_+ = \{ y \in \mathbb{R}^n; y_1 > \sqrt{\sum_{j=2}^n y_j^2} \}$ be the forward light-cone in $\mathbb{R}^n$, $e = (1, 0, \ldots, 0) \in \mathbb{R}^n$ and $\Gamma = \ell e + V_+$. Let $F_1(z) \in \mathcal{O}_0(T(\Gamma))$, $F_2(z) \in \mathcal{O}_0(T(-\Gamma))$ and $C_j^\eta = \{ z_j \in \mathbb{C}; z_j = x + ir, -\infty < x < \infty \}$. Define
\[
u_1(\phi) = \int_{\Pi_j} C_j^{\eta} F_1(z)\phi(z)dz,
\]
\[
u_2(\phi) = \int_{\Pi_j} C_j^{-\eta} F_2(z)\phi(z)dz.
\]
for $\phi \in T(\mathbb{R}^n)$ and $\eta = (\eta_1, \ldots, \eta_n) \in \Gamma$. If $\nu_1 = \nu_2$, then $F_1(z)$ and $F_2(z)$ can be continued analytically to each other and define a tempered entire function $H(z) \in \mathcal{O}_0(T(\mathbb{R}^n))$.

**Proof** Note first that $\nu_1(\phi)$ (resp. $\nu_2(\phi)$) does not depend on the path $C_j^{\eta_j}$ (resp. $C_j^{-\eta_j}$). This can be proved by applying Cauchy’s integral theorem $n$ times. Let $r > \ell$. Observe that the function $U_1$ defined by
\[
U_1(z) = \nu_1 * K_r(z) = \int_{\mathbb{R}^n} F_1(\xi + i\eta)K_r(z - \xi - i\eta)d\xi,
\]
for $\eta \in \Gamma$, is analytic in
\[
V_1 = \bigcup_{\eta \in \Gamma} \{ z \in \mathbb{C}^n; |\text{Im } (z - i\eta)| < r \} = \{ z \in \mathbb{C}^n; \text{dist } (\text{Im } z, \Gamma) < r \}
\]
\[
\sup \{ z \in \mathbb{C}^n; \text{dist } (\text{Im } z, \ell e) < r \} \sup \{ z \in \mathbb{C}^n; |\text{Im } z| < r - \ell \}
\]
and similarly $U_2(z) = \nu_2 * K_r(z)$ is analytic in
\[
V_2 = \bigcup_{\eta \in -\Gamma} \{ z \in \mathbb{C}^n; |\text{Im } (z - i\eta)| < r \} = \{ z \in \mathbb{C}^n; \text{dist } (\text{Im } z, -\Gamma) < r \}
\]
\[
\sup \{ z \in \mathbb{C}^n; \text{dist } (\text{Im } z, -\ell e) < r \} \sup \{ z \in \mathbb{C}^n; |\text{Im } z| < r - \ell \}.
\]
Note that $U_j(z) \in \mathcal{O}_0(V_j)$. In fact, since $K_r(z - i\eta)$ is rapidly decreasing in $|\text{Im } (z - i\eta)| < r$, we can estimate as follows:
\[
\left| \int_{\mathbb{R}^n} F_j(\xi + i\eta)K_r(z - \xi - i\eta)d\xi \right| \leq C \int_{\mathbb{R}^n} (1 + |\xi + i\eta|)^k|K_r(x + iy - \xi - i\eta)|d\xi
\]
\[
= C \int_{\mathbb{R}^n} (1 + |\xi + i\eta - x|)^k|K_r(iy - \xi - i\eta)|d\xi
\]
\[
\leq C(1 + |x|)^k \int_{\mathbb{R}^n} (1 + |\xi + i\eta|)^k|K_r(iy - \xi - i\eta)|d\xi
\]
\[
\leq C'(1 + |x|)^k \leq C'(1 + |z|)^k
\]
for $|\text{Im} \,(z-i\eta)| < r$, where we used the fact that $|K_r(iy-\xi-i\eta)|$ is decreasing exponentially as $|\xi| \to \infty$, which follows from Lemma 3.2. Since the compact set $K$ in $V_j$ is covered by $\cup_{\eta \in I_j} T\{y \in \mathbb{R}^n; \ |y-\eta| < r\}$ for some finite set $I_j$ in $\Gamma$ (for $j=1$ or $-\Gamma$ for $j=2$), we have $U_j(z) \in O_0(V_j)$. Corollary 3.4 implies

$$\int_{|\omega|=1} d\omega \int dx U_j(x + i(r-R)\omega) \phi(x - iR\omega) = \int_{|\omega|=1} d\omega \int dx u_j * K_r(x + i(r-R)\omega) \phi(x - iR\omega)$$

$$= \int_{|\omega|=1} d\omega \int dx \int d\xi F_j(\xi + i\eta) K_r(x + i(r-R)\omega - \xi - i\eta) \phi(x - iR\omega)$$

$$= \int d\xi \int_{|\omega|=1} d\omega \int dx F_j(\xi + i\eta) K_r(x + i(r-R)\omega - \xi - i\eta) \phi(x - iR\omega)$$

$$= \int d\xi F_j(\xi + i\eta) \phi(\xi + i\eta) = u_j(\phi).$$

The relation $u_1(\phi) = u_2(\phi)$ implies $U_1(z) = U_2(z)$ in $V_1 \cap V_2 \supset \{z \in \mathbb{C}^n; \ |\text{Im} \, z| < r - \ell \}$. Then $U_1(z)$ and $U_2(z)$ are continued to an analytic function $U(z)$ in $V_1 \cup V_2$. Moreover, $U(z)$ is analytically continued to the convex envelope of $V_1 \cup V_2$ by Bochner’s theorem on tubular domains (see [25–27]). Since the convex envelope of $V_1 \cup V_2$ is the entire space $\mathbb{C}^n$, $U(z)$ is analytically continued to an entire function, and the maximum principle shows that $U(z)$ belongs to $O_0(T(\mathbb{R}^n))$. In fact, let $K = \prod_{l=1}^n [-k_l, k_l]$ for $k_l > 0$ be such that $\mathbb{R}^n + i(k_1) \times \prod_{l=2}^n [-k_l, k_l] \subset V_1$ and $\mathbb{R}^n + i\{-k_1\} \times \prod_{l=2}^n [-k_l, k_l] \subset V_2$. Since $U_j(z) \in O_0(V_j)$ for $j=1, 2$, there exist $R, r_l \in \mathbb{N}$ such that

$$|W_j(z)| = \left| \prod_{l=1}^n (z_l - iR)^{-r_l} U_j(z) \right| \leq M$$

for some $M > 0$ in $V_j \cap T(K) \cap \mathbb{C}^n$ and $W_j(z) \to 0$ as $V_j \cap T(K) \cap \mathbb{C}^n \ni z \to \infty$. Let

$$W(z) = \prod_{l=1}^n (z_l - iR)^{-r_l} U(z).$$

In order to show that $U(z) \in O_0(T(\mathbb{R}^n))$, it suffices to show that $|W(z)| \leq M$ in $T(K) \cap \mathbb{C}^n$. Suppose there exists $\zeta \in T(K) \cap \mathbb{C}^n$ such that $|W(\zeta_1, \zeta_2, \ldots, \zeta_n)| > M$. Consider the holomorphic function $W(z_1, \zeta_2, \ldots, \zeta_n)$ of the variable $z_1 \in D = \mathbb{R} + i[-k_1, k_1]$. The maximum is attained at a point in the boundary $\partial D = \mathbb{R} + i\{-k_1, k_1\}$ of $D$. Since $(\zeta_2, \ldots, \zeta_n) \in \mathbb{R}^{n-1} + i\prod_{l=2}^n [-k_l, k_l]$ and $\mathbb{R}^n + i\{-k_1, k_1\} \times \prod_{l=2}^n [-k_l, k_l] \subset V_j \cup V_2$, we have $W(z_1, \zeta_2, \ldots, \zeta_n) \leq M$ in $D$ which contradicts the inequality $|W(\zeta_1, \zeta_2, \ldots, \zeta_n)| > M$.

The function

$$H(z) = \int_{|\omega|=1} d\omega U(z + i\omega),$$

for $|\text{Im} \,(z-i\eta)| < r$, where we used the fact that $|K_r(iy-\xi-i\eta)|$ is decreasing exponentially as $|\xi| \to \infty$, which follows from Lemma 3.2. Since the compact set $K$ in $V_j$ is covered by $\cup_{\eta \in I_j} T\{y \in \mathbb{R}^n; \ |y-\eta| < r\}$ for some finite set $I_j$ in $\Gamma$ (for $j=1$ or $-\Gamma$ for $j=2$), we have $U_j(z) \in O_0(V_j)$. Corollary 3.4 implies

$$\int_{|\omega|=1} d\omega \int dx U_j(x + i(r-R)\omega) \phi(x - iR\omega) = \int_{|\omega|=1} d\omega \int dx u_j * K_r(x + i(r-R)\omega) \phi(x - iR\omega)$$

$$= \int_{|\omega|=1} d\omega \int dx \int d\xi F_j(\xi + i\eta) K_r(x + i(r-R)\omega - \xi - i\eta) \phi(x - iR\omega)$$

$$= \int d\xi \int_{|\omega|=1} d\omega \int dx F_j(\xi + i\eta) K_r(x + i(r-R)\omega - \xi - i\eta) \phi(x - iR\omega)$$

$$= \int d\xi F_j(\xi + i\eta) \phi(\xi + i\eta) = u_j(\phi).$$
is an entire function which satisfies
\[\int_{\prod_j C^{\eta_j}} H(z) \phi(z) dz = \int_{\prod_j C^{\eta_j}} dz \int_{|\omega|=1} d\omega U(z + i\omega) \phi(z)\]
\[= \int_{|\omega|=1} d\omega \int_{\mathbb{R}^n} dx U(x + i\eta + i\omega) \phi(x + i\eta)\]
\[= \int_{|\omega|=1} d\omega \int_{\mathbb{R}^n} dx U_1(x + i(r - \ell - \sigma)\omega) \phi(x - i(\ell + \sigma)\omega)\]
\[= u_1(\phi) = \int_{\prod_j C^{\eta_j}} F_1(z) \phi(z) dz\]
for \(0 < \sigma \leq r - \ell\), where we used the fact that \(U(z) = U_1(z)\) in \(\{z \in \mathbb{C}^n; |\text{Im}\ z| < r - \ell\}\) and Relation (15). Thus we have
\[\int_{\prod_j C^{\eta_j}} H(z) \phi(z) dz = \int_{\prod_j C^{\eta_j}} F_1(z) \phi(z) dz\]
for \(\phi \in \mathcal{T}(T(\mathbb{R}^n))\). This shows that \(H(z) = F_1(z)\) in \(T(\Gamma)\), and in the same way we get \(H(z) = F_2(z)\) in \(T(-\Gamma)\). Thus, \(F_1(z)\) and \(F_2(z)\) are analytically continued to a tempered entire function \(H(z)\).

4. Local edge of the wedge theorem

We begin by proving a regularization result for tempered ultra-hyperfunctions using the kernel \(K_r\).

**Proposition 4.1** Let \(L\) be an open set in \(\{w \in \mathbb{C}^n; |\text{Im}\ w| < r\}\) and \(u \in \mathcal{T}(L)'\). Then \(U(z) = K_r \ast u(z)\) is holomorphic in
\[Z = \{z \in \mathbb{C}^n; |\text{Im}\ (z - w)|^2 < r^2 + |\text{Re}\ (z - w)|^2, \forall w \in L\}.
Furthermore, introduce the function
\[g_r(x) = \inf \left\{ \sqrt{r^2 + |x - \text{Re}\ w|^2} - |\text{Im}\ w|; w \in L \right\}.\]
(16)

Then the following inclusion is valid.
\[Z \supset \{z = x + iy \in \mathbb{C}^n; |y| < g_r(x), x \in \mathbb{R}^n\}.\]
(17)

**Proof** Since \(K_r(z)\) is a rapidly decreasing holomorphic function in \(\{z \in \mathbb{C}^n; |\text{Im}\ z|^2 < r^2 + |\text{Re}\ z|^2\}\), if \(z \in Z\), \(K_r(z - w)\) is a rapidly decreasing holomorphic function of \(w\) in a neighborhood of \(L\), and consequently, \(U(z)\) is holomorphic in \(Z\). The inclusion (17) can be shown as follows:
We use the same notation as in Theorem 3.6. Then
\[ Z = \left\{ z \in \mathbb{C}^n \mid |\text{Im}(z - w)| < \sqrt{r^2 + |\text{Re}(z - w)|^2}, \forall w \in L \right\} \]
\[ \supset \left\{ z \in \mathbb{C}^n \mid |z| + |\text{Im} w| < \sqrt{r^2 + |\text{Re}(z - w)|^2}, \forall w \in L \right\} \]
\[ \supset \left\{ z = x + iy \in \mathbb{C}^n \mid |y| < g_r(x), x \in \mathbb{R}^n \right\}. \]

\[ \square \]

**Theorem 4.2** Let \( F_i(z) \) (\( i = 1, 2 \)) be holomorphic functions and \( u_i \) be the tempered ultra-hyperfunctions defined by \( F_i(z) \) as in Theorem 3.6. Let \( L \) be an open set in \( \{ w \in \mathbb{C}^n \mid |\text{Im} w| < \ell \} \) such that the set \( O = \{ x \in \mathbb{R}^n ; g_r(x) > r \} \) contains an open set \( Q \) such that \( \text{dist}(\partial O, Q) > 2\ell \). for \( r > \ell/(\sqrt{2} - 1) \) and \( g_r(x) \) of (16). Assume that \( u_1 - u_2 \in \mathcal{T}(L)' \). Then \( F_i(z) \) are analytically continued to \( O \) and coincide there.

**Proof** We use the same notation as in Theorem 3.6. Then \( U_1(z) = u_1 * K_r(z) \) is holomorphic in
\[ V_1 = \{ z \in \mathbb{C}^n \mid \text{dist}(\text{Im} z, \Gamma) < r \} \]
and \( U_2(z) = u_2 * K_r(z) \) is holomorphic in
\[ V_2 = \{ z \in \mathbb{C}^n \mid \text{dist}(\text{Im} z, -\Gamma) < r \}. \]

Since \( u_1 - u_2 \in \mathcal{T}(L)' \), it follows from Proposition 4.1 that \( U_{12}(z) = (u_1 - u_2) * K_r(z) \) is holomorphic in
\[ Z \supset \{ z = x + iy \in \mathbb{C}^n \mid |y| < g_r(x), x \in \mathbb{R}^n \}. \]

Since \( u_1 * K_r(z) = u_2 * K_r(z) + (u_1 - u_2) * K_r(z) \), \( U_1(z) \) is holomorphic in
\[ V_1 \cup (V_2 \cap \{ z = x + iy \in \mathbb{C}^n \mid |y| < g_r(x), x \in \mathbb{R}^n \}) \supset \{ x \in \mathbb{R}^n ; g_r(x) > r + \delta \} \]
\[ \times i (\{ y \in \mathbb{R}^n ; \text{dist}(y, \Gamma) < r \} \cup (\{ y \in \mathbb{R}^n ; \text{dist}(y, -\Gamma) < r \} \cap B_{r+\delta})) \]
\[ = \{ x \in \mathbb{R}^n ; g_r(x) > r + \delta \} \times i (\{ y \in \mathbb{R}^n ; \text{dist}(y, \Gamma) < r \} \cup B_{r+\delta}) \]

for any \( r + \delta < \sqrt{2r} + \ell \), where \( B_r = \{ y \in \mathbb{R}^n ; |y| < r \} \) (see Figure 2). Note that if \( r > \ell/(\sqrt{2} - 1) \) then there exists \( \delta > 0 \) such that \( r + \delta < \sqrt{2r} + \ell \).

Denote
\[ W_{r,\delta} = \{ y \in \mathbb{R}^n ; \text{dist}(y, \Gamma) < r \} \cup B_{r+\delta}. \]

If
\[ r + \delta > \sqrt{(r/\sqrt{2})^2 + (r/\sqrt{2} - \ell)^2} \]
then \( \cap_{|\omega|=1}(W_{r,\delta} + r\omega) \) strictly contains the set \( \Gamma \), and if \( \delta > 0 \) then \( \cap_{|\omega|=1}(W_{r,\delta} + r\omega) \) contains a connected set \( \bar{\Gamma}_{\delta} \) containing \( \Gamma \) and an open ball \( B_\delta \) (see Figure 3).

As before, consider
\[ H_1(z) = \int_{|\omega|=1} d\omega U_1(z + ir\omega). \]

\( H_1(z) \) is holomorphic in the set
\[ \cap_{|\omega|=1}(V_1 \cup (V_2 \cap \{ z = x + iy \in \mathbb{C}^n ; |y| < g_r(x), x \in \mathbb{R}^n \})) + ir\omega] \]
which contains the sets $T(\Gamma)$ and

$$\{x \in \mathbb{R}^n; g_r(x) > r + \delta\} \times i \cap_{|\omega|=1} (W_{r+\delta} + r\omega) \supset \{x \in \mathbb{R}^n; g_r(x) > r + \delta\} \times i\Gamma_{\delta}.$$ 

This implies

$$\int_{\prod_j C^{\eta_j}} H_1(z) \phi(z) dz = \int_{\prod_j C^{\eta_j}} dz \int_{|\omega|=1} d\omega U_1(z + i r \omega) \phi(z)$$

$$= \int_{|\omega|=1} d\omega \int_{\mathbb{R}^n} dy U_1(y + i\eta + i r \omega) \phi(y + i\eta)$$

$$= \int_{|\omega|=1} d\omega \int_{\mathbb{R}^n} dy U_1(y + i(r - [\ell + \sigma])\omega) \phi(y - i[\ell + \sigma] \omega)$$

$$= u_1(\phi) = \int_{\prod_j C^{\eta_j}} F_1(z) \phi(z) dz.$$
Figure 3. The connected set $\tilde{\Gamma}_\delta$ containing $\Gamma$ and $B_\delta$.

This shows

$$\int_{\prod_j C^{n_j}} H_1(z)\phi(z)dz = \int_{\prod_j C^{n_j}} F_1(z)\phi(z)dz$$

for $\phi \in T(T(\mathbb{R}^n))$ and therefore $H_1(z) = F_1(z)$ in $T(\Gamma)$ and $F_1(z)$ is analytically continued to

$$\{x \in \mathbb{R}^n; g_r(x) > r + \delta\} + i\tilde{\Gamma}_\delta.$$ 

In the same way, we get $H_2(z) = F_2(z)$ in $T(-\Gamma)$ and $F_2(z)$ is analytically continued to

$$\{x \in \mathbb{R}^n; g_r(x) > r + \delta\} - i\tilde{\Gamma}_\delta.$$ 

In order to show $H_1(z) = H_2(z)$, we choose a surface $S_1$ contained in the domain of the holomorphy of $H_1(z)$ such that

$$S_1 = \{z = x + iy \in \mathbb{C}^n; y_1 = f_1(x), y_j = 0, 2 \leq j \leq n, x \in \mathbb{R}^n\},$$
where $0 \leq f_1(x) \leq \ell + \delta$ is a continuous function satisfying $f_1(x) = 0$ for $x \in O = \{x \in \mathbb{R}^n; g_r(x) > r\}$ and $f_1(x) = \ell + \delta$ for

$$x \in \left\{x \in \mathbb{R}^n; g_r(x) < \sqrt{(r/\sqrt{2})^2 + (r/\sqrt{2} - r)^2}\right\}.$$ 

For $\xi \in O \subset O$, define $E_\xi^t(z) = (4\pi t)^{-n/2} e^{-|\xi - z|^2/4t}$. Note that the open ball with center $\xi$ and radius $2\ell$ is contained in $O$, $B_{2\ell}(\xi) \subset O \subset S_1$. Then, for $\xi \in O$,

$$u_1(E_\xi^t) = \int_{S_1} H_1(z) E_\xi^t(z) dz = \int_{B_{2\ell}(\xi)} H_1(z) E_\xi^t(z) dz + \int_{S_1 \setminus B_{2\ell}(\xi)} H_1(z) E_\xi^t(z) dz$$

and

$$\int_{B_{2\ell}(\xi)} H_1(z) E_\xi^t(z) dz = \int_{B_{2\ell}(\xi)} H_1(z)(4\pi t)^{-n/2} e^{-|\xi - z|^2/4t} dx \to H_1(\xi)$$
as $t \to 0+$. Since

$$\int_{S_1 \setminus B_{2\ell}(\xi)} H_1(z) E_\xi^t(z) dz = \int_{S_1 \setminus B_{2\ell}(\xi)} H_1(z)(4\pi t)^{-n/2} e^{-|\xi - z|^2/4t} dz$$

and for $z \in S_1 \setminus B_{2\ell}(\xi)$,

$$|e^{-|\xi - z|^2/4t}| = e^{-|\xi - x|^2/4t} e^{y_1^2/4t} |e^{-2i(\xi - x) y_1/4t}| \leq e^{-|\xi - x|^2/4t} e^{(\ell + \delta)^2/4t},$$

we get

$$\int_{S_1 \setminus B_{2\ell}(\xi)} H_1(z) E_\xi^t(z) dz \to 0$$
as $t \to 0+$, where we used the relation $(\xi - x)^2 \geq 4\ell^2$ and consequently, $(\xi - x)^2/2 \geq (\ell + \delta)^2$ for $0 < \delta < \ell/3$. It follows

$$\lim_{t \to 0} u_1(E_\xi^t) = H_1(\xi).$$

In the same way, we get

$$\lim_{t \to 0} u_2(E_\xi^t) = H_2(\xi).$$

Let $x \in O = \{x \in \mathbb{R}^n; g_r(x) > r\}$. Then

$$\sqrt{r^2 + |x - \text{Re } w|^2 - |\text{Im } w|} > r$$

for any $w \in L$, that is, $|x - \text{Re } w| > 0$. Let $\xi \in Q$. Then $|\xi - \text{Re } w| > 2\ell$ for any $w \in L$ and

$$\sup_{w \in L} (1 + |w|)^j (4\pi t)^{-n/2} \exp[-(\xi - \text{Re } w)^2 + (\text{Im } w)^2]/4t \leq \sup_{w \in L} (1 + |w|)^j (4\pi t)^{-n/2} \exp[-(\xi - \text{Re } w)^2/2 + \ell^2]/4t \to 0$$
as $t \to 0+$. $u_1 - u_2 \in T(L)'$ implies

$$0 = \lim_{t \to 0} (u_1 - u_2)(E_\xi^t) = \lim_{t \to 0} u_1(E_\xi^t) - \lim_{t \to 0} u_2(E_\xi^t) = H_1(\xi) - H_2(\xi).$$
Thus, \(H_1(z)\) and \(H_2(z)\) are analytically continued to each other.

The following corollary helps to understand Theorem 5.11 of [13].

**Corollary 4.3** Let \(V_+ = \{ y \in \mathbb{R}^4; y_0 > \sqrt{\sum_{j=1}^{3} y_j^2} \}\) be the forward light-cone in \(\mathbb{R}^4\), \(e = (1, 0, 0, 0) \in \mathbb{R}^4\) and \(\Gamma = \ell e + V_+.\) Let \(F_1(z) \in \mathcal{O}_0(T(\Gamma))\), \(F_2(z) \in \mathcal{O}_0(T(-\Gamma))\) and \(C_j' = \{ z_j \in \mathbb{C}; z_j = x + ir, -\infty < x < \infty \}\). Define

\[
\begin{align*}
    u_1(\phi) &= \int_{\Pi_j c_j^{-\alpha_j}} F_1(z) \phi(z) dz, \\
    u_2(\phi) &= \int_{\Pi_j c_j^{-\alpha_j}} F_2(z) \phi(z) dz
\end{align*}
\]

for \(\phi \in \mathcal{T}(T(\mathbb{R}^4))\) and \(w = (\eta_0, \ldots, \eta_3) \in \Gamma\). Let

\[
    L = \{ w \in \mathbb{C}^4; \exists x \in V |\text{Re } w - x| < |\text{Im } w|_1 < \ell \},
\]

where \(V\) is the light-cone and \(|y| = |y^0| + |y|\). Assume that \(u_1 - u_2 \in \mathcal{T}(L)^\prime\). Then \(F_i(z)\) \((i = 1, 2)\) are analytically continued to the set

\[
    \left\{ x \in \mathbb{R}^4; \text{dist}(x, V) > (\sqrt{2} + 1)\ell \right\}
\]

and coincide there.

**Proof** The corollary follows from Theorem 4.2 and the following lemma.

**Lemma 4.4** Let \(L\) be the set defined in the above corollary and \(g_r(x)\) the function defined by (16) with \(r > \ell/(\sqrt{2} - 1)\). Then the set \(O = \{ x \in \mathbb{R}^4; g_r(x) > r \}\) contains a set

\[
    \left\{ x \in \mathbb{R}^4; \text{dist}(x, V) > (\sqrt{2} + 1)\ell \right\}.
\]

The open set

\[
    Q = \{ x \in \mathbb{R}^4; \text{dist}(x, V) > (\sqrt{2} + 3)\ell \}
\]

is contained in \(O\) and \(\text{dist}(\partial O, Q) > 2\ell\).

**Proof** We can find \(\xi \in \bar{V}\) such that \(\text{dist}(x, V) = |x - \xi|\). Since \(|\text{Re } w - \xi| + |\text{Im } w|_1 < \ell\) for \(w \in L\), we have for \(x \in \mathbb{R}^4 \setminus L\)

\[
    g_r(x) = \inf \left\{ \sqrt{r^2 + |x - \text{Re } w|^2} - |\text{Im } w|; w \in L \right\}
\]

\[
    \geq \inf \left\{ \sqrt{r^2 + |x - \text{Re } w|^2} - \ell + |\text{Re } w - \xi|; \text{Re } w \in [\xi, x] \cap L \right\}
\]

\[
    \geq \inf \left\{ \sqrt{r^2 + (|x - \xi| - |\text{Re } w - \xi|)^2} - \ell + |\text{Re } w - \xi|; \text{Re } w \in [\xi, x] \cap L \right\}
\]

\[
    \geq \sqrt{r^2 + |x - \xi|^2} - \ell = \sqrt{r^2 + \text{dist}(x, V)^2} - \ell
\]

where \([\xi, x]\) is the line segment connecting \(\xi\) and \(x\). The inequality \(g_r(x) > r\) follows from

\[
    \sqrt{r^2 + \text{dist}(x, V)^2} > r + \ell \iff \text{dist}(x, V)^2 > \ell(2r + \ell).
\]
If \( r = \ell / (\sqrt{2} - 1) \) then \( \ell (2r + \ell) = (\sqrt{2} + 1)^2 \ell^2 \). Therefore, if \( \text{dist}(x, V) > (\sqrt{2} + 1)\ell \), there exists \( r > \ell / (\sqrt{2} - 1) \) such that \( \text{dist}(x, V)^2 > \ell (2r + \ell) \). The second statement is obviously true.

References

[1] Bogoliubov NN, Shirkov DV. Introduction to the theory of quantized fields. New York: Wiley-Interscience; 1959.
[2] Bogolubov NN, Logunov AA, Oksak AI, Todorov IT. General principles of quantum field theory. Vol. 10, Mathematical Physics and Applied Mathematics. Dordrecht: Kluwer Academic; 1990.
[3] Kawai T. On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients. J. Fac. Sci. Univ. Tokyo, Sect. I.A. 1970;17:467–517.
[4] Brüning E, Nagamachi S. Hyperfunction quantum field theory: basic structural results. J. Math. Phys. 1989;30:2340–2359.
[5] Kaneko A. Introduction to hyperfunctions, mathematics and its applicatons (Japanese Series). Dordrecht: Kluwer Academic; 1988.
[6] Nagamachi S, Nishimura T. Edge of the wedge theorem for Fourier hyperfunctions. Funkcialaj Ekvacioj. 1993;36:499–517.
[7] Martineau A. Distributions et valeurs au bord des fonctions holomorphes. In: Proceedings of International Summer Course on the Theory of Distributions, Lisbon, 1964. p. 195–326.
[8] Martineau A. Théorème sur le prolongement analytique du type ‘Edge of the Wedge Theorem’, Séminaire Bourbaki, 20-ième année, No. 340, 1967/68.
[9] Martineau A. Le ‘edge of the wedge theorem’ en théorie des hyperfonctions de Sato. In: Proceedings of International Conference on Functional Analysis, Tokyo, 1969; Tokyo: University Tokyo Press; 1970. p. 95–106.
[10] Epstein H. Generalization of the ‘edge of the wedge’ theorem. J. Math. Phys. 1960;1:524–531.
[11] Brüning E, Nagamachi S. Relativistic quantum field theory with a fundamental length. J. Math. Phys. 2004;45:2199–2231.
[12] Brüning E, Nagamachi S. Solutions of a linearized model of Heisenberg’s fundamental equation II. J. Math. Phys. 2008;49:052304–1–052304-22.
[13] Nagamachi S, Brüning E. Frame independence of the fundamental length in relativistic quantum field theory. J. Math. Phys. 2010;51:022305-1–022305-18.
[14] Hörmander L. The analysis of linear partial differential operators I. Vol. 256, Grundlehren der mathematischen Wissenschaften. Berlin: Springer-Verlag; 1983.
[15] Morimoto M. Edge of the wedge theorem and hyperfunction. In: Hyperfunctions and pseudo-differential equations. Proc. Conf., Katata. 1971 Lecture Notes in Math., Vol. 287. Berlin: Springer; 1973. p. 41–81.
[16] Morimoto M. Convolutors for ultrahyperfunctions. In: International Symposium on Mathematical Problems in Theoretical Physics, Vol. 39 of Lecture Notes in Phys. Berlin: Springer; 1975. p. 49–54.
[17] Hasumi M. Note on the \( n \)-dimensional tempered ultra-distributions. Tohoku Math. J. 1961;13:94–104.
[18] Nagamachi S, Brüning E. Solutions of a linearized model of Heisenberg’s fundamental equation I. 2008;arXiv:0804.1663 [math-ph].
[19] Morimoto M. Sur les ultradistributions cohomologiques. Ann. Inst. Fourier. 1969;19:129–153.
[20] Morimoto M. La décomposition de singularités d’ultradistributions cohomologiques. Proc. Japan Acad. 1972;48:129–153.
[21] Suwa M. Distributions of exponential growth with support in a proper cone. Publ. RIMS, Kyoto Univ. 2004;40:565–603.
[22] Franco DH. The edge of the wedge theorem for tempered ultrahyperfunctions. 2006;arXiv:math/0609751.
[23] Franco DH. The edge of the wedge theorem for tempered ultrahyperfunctions II, A generalized version. 2007;arXiv:0708.0252.
[24] Morimoto M. An introduction to Sato’s hyperfunctions. Translations of Mathematical Monographs Vol. 129. American Mathematical Society; Providence, Rhode Island, 1993.
[25] Vladimirov VS. methods of the theory of functions of many complex variables. Cambridge: The M.I.T Press; 1964.
[26] Hörmander L. An introduction to complex analysis in several variables. Princeton: D. van Nostrand Comp.; 1966. Vol. 256.
[27] Hounie J. A proof of Bochner’s tube theorem. Proc. Am. Math. Soc. 2009;137:4203–4207.