COLLISION-AVOIDING IN THE SINGULAR CUCKER-SMALE MODEL WITH NONLINEAR VELOCITY COUPLINGS

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Abstract. Collision avoidance is an interesting feature of the Cucker-Smale (CS) model of flocking that has been studied in many works, e.g. [2, 1, 4, 6, 7, 20, 21, 22]. In particular, in the case of singular interactions between agents, as is the case of the CS model with communication weights of the type \( \psi(s) = s^{-\alpha} \) for \( \alpha \geq 1 \), it is important for showing global well-posedness of the underlying particle dynamics. In [4], a proof of the non-collision property for singular interactions is given in the case of the linear CS model, i.e. when the velocity coupling between agents \( i, j \) is \( v_j - v_i \). This paper can be seen as an extension of the analysis in [4]. We show that particles avoid collisions even when the linear coupling in the CS system has been substituted with the nonlinear term \( \Gamma(\cdot) \) introduced in [12] (typical examples being \( \Gamma(v) = |v|^{2(\gamma - 1)} \) for \( \gamma \in (\frac{1}{2}, \frac{3}{2}) \)), and prove that no collisions can happen in finite time when \( \alpha \geq 1 \).

1. Introduction. A Cucker-Smale (CS) type of model deals with an interacting system of \( N \) autonomous, self-driven particles (agents). The main model postulate is that the agents adjust their velocities by taking a weighted average of their relative velocities to all other agents. If we let \( (x_i, v_i) \in \mathbb{R}^d \times \mathbb{R}^d \) be the phase space position of the \( i \)th particle for \( 1 \leq i \leq N \), and \( d \geq 1 \) be the physical dimension, the dynamics of the particle motion is governed by the system:

\[
\begin{align*}
\frac{d}{dt} x_i(t) &= v_i(t), & i = 1, \ldots, N, & t > 0, \\
\frac{d}{dt} v_i(t) &= \frac{1}{N} \sum_j \psi(|x_i - x_j|)(v_j - v_i),
\end{align*}
\]

given some initial data \( (x_i(0), v_i(0)) = (x_{i0}, v_{i0}), \ i = 1, \ldots, N \). Throughout this paper, the symbol \( \sum \) is used as an abbreviation of \( \sum_{1 \leq i \leq N} \). The function \( \psi(r) \), \( r \geq 0 \), quantifies the interaction between two agents and it is to be referred as the communication weight of the interaction. It is positive, nonincreasing, and vanishes as \( r \to \infty \). We observe that in our model, \( \psi(\cdot) \) depends on the metric distance

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between two agents. The main question that arises in the study of (1) is whether the system emerges to a flock, i.e., all the particle velocities align asymptotically in time and the agents stay connected forever. The prototype example in the CS model was $\psi(r) = (1 + r^2)^{-\beta}$, for $\beta \geq 0$. In [9, 10, 15] it was shown that flocking is guaranteed if $\beta \leq \frac{1}{2}$, and if $\beta > \frac{1}{2}$ then the system might converge to a flock only under certain conditions on the initial positions and velocities. The phase transition that happens when $\beta = \frac{1}{2}$ is typical of the system (1) and supports the more general result that when weight $\psi(\cdot)$ has a non integrable tail (i.e. $\int_{-\infty}^{\infty} \psi(s) ds = \infty$) then flocking occurs regardless of the initial configuration of agents.

After its introduction in [9, 10] (based on an earlier idea from [25]), research on the CS model took several different routes. The original flocking results were simplified and improved in [15, 16]. The CS system was studied in the presence of Rayleigh friction forces in [13], as well as other repulsion/alignment/turning forces [1]. The effect of a flock leader in emergent behavior was considered in [23]. The model was also studied with extra random noise terms in [8, 14, 24]. S. Motsch and E. Tadmor proposed a model that resolves some of the drawbacks of the CS system by normalizing the communication weights in [19] and established flocking conditions. The CS system was studied with delay terms in [5, 11].

An interesting variation to system (1) was proposed in [12] and describes the particle system where the linear coupling term $v_j - v_i$ is substituted by a nonlinear vector $\Gamma(v_j - v_i) : \mathbb{R}^d \rightarrow \mathbb{R}^d$, i.e.

\[
\begin{align*}
\frac{d}{dt} x_i(t) &= v_i(t), \quad i = 1, \ldots, N, \quad t > 0, \\
\frac{d}{dt} v_i(t) &= \frac{1}{N} \sum_j \psi(|x_i - x_j|) \Gamma(v_j - v_i).
\end{align*}
\] (2)

The justification given in [12] of this nonlinear version of (1) lies in the fact that there seems to be no underlying physical principle that requires most alignment models to be linear, other than a modeling convenience. It is therefore of paramount importance to know that model (1) is robust under small variations in all parameters, including the velocity couplings. We refer to system (2) from now on as NL CS (nonlinear CS). The continuous coupling vector $\Gamma(v_j - v_i)$ that appears in (2) has the following properties:

- (A1) (skew symmetry) $\Gamma(-v) = -\Gamma(v)$ for $v \in \mathbb{R}^d$.
- (A2) (coercivity) There exists some $C_1 > 0$ and $\gamma \in (\frac{1}{2}, \frac{3}{2})$ such that $\langle \Gamma(v), v \rangle \geq C_1 |v|^{2\gamma}$. Here by $\langle \cdot, \cdot \rangle$ we denote the inner product in $\mathbb{R}^d$ and with $|\cdot|$ its induced norm.

System (2) exhibits similar phase transition properties as (1). For a communication weight with a non integrable tail and under assumptions (A1) – (A2), it can be shown that flocking occurs for $\gamma \in (\frac{1}{2}, \frac{3}{2})$ with a rate that depends explicitly on the value of $\gamma$. In more detail, when $\gamma \in (\frac{1}{2}, 1)$ we have flocking that occurs in finite time $T^* < \infty$, algebraically fast. If $\gamma \in (1, \frac{3}{2})$ we have emergence of a flock in infinite time $T^* = \infty$, with algebraic decay rate. Finally, the case $\gamma = 1$ reduces to the linear case with $T^* = \infty$, and flocking that happens at an exponential decay rate.

A question that requires special investigation is the presence of collisions between agents. The original CS systems (1) and (2) with the weights we just mentioned does not exclude the possibility of collisions. For obvious reasons, the modeling of alignment in animal flocks, aerial vehicles, unmanned drones etc. should in many
cases incorporate a mechanism for avoiding collisions. The way to design systems like that is by introducing an interaction that becomes singular when two particles collide. This interaction might have the form of an extra repulsion forcing term, like in [2, 1, 6, 7], or it might simply be a communication weight that is singular at the origin, e.g. [4, 20, 21, 22]. In this work our focus shifts to the latter scenario.

The purpose of this article is to study the problem of collision avoidance for the NL CS model (2). We prove that for all the cases where flocking is possible, we have the absence of collisions in finite time for any interaction of the type \( \psi(s) = s^{-\alpha} \), with \( \alpha \geq 1 \). This result is in complete agreement with the linear case treated in [4]. Our approach shows that the methodology used in [4] is not just specific to the linear case but can be easily adopted to a non linear scenario. It also serves as a further indication of the robustness of the choice of linear couplings in the classical CS model.

Furthermore, we derive uniform estimates for the general case of a weight with expanded singularity \( \psi(s) = (s - \delta)^{-\alpha} \), \( \delta \geq 0 \). We use distance functions of the type \( L^\beta(t) = \frac{1}{N(N-1)} \sum_{i \neq j} |(x_i(t) - x_j(t)) - \delta|^{-\beta} \) for some appropriately chosen \( \beta = \beta(\alpha, \gamma) > 0 \), and prove that \( L^\beta(t) \leq O(T), \forall t \in [0, T], \) when \( \alpha \geq 2\gamma \). This gives an estimate for the minimum interparticle distance like \( \inf_{i \neq j} |x_i(t) - x_j(t)| \geq O((TN^2)^{-\frac{\delta}{2}}) \), which is enough to conclude the well-posedness of the dynamics for a fixed number \( N \) of agents (given that \( \Gamma(\cdot) \) is also Lipschitz). Unfortunately, this estimate is useless as \( N \to \infty \) and leaves the question of passage to the mean field equation (for singular communication weights) still open, see e.g. [17, 18].

The rest of this paper is structured as follows. In Section 2 we briefly review the theory for problem (2) and the flocking result presented in [12]. In the end of Section 2 we present the main result of this paper and give its proof in Section 3. Finally, in Section 4 we give and prove the uniform estimates in the case of a communication weight \( \psi(s) = (s - \delta)^{-\alpha} \), for \( \alpha \geq 2\gamma \) and \( \delta \geq 0 \).

2. Preliminaries and main result. In what follows, we denote with \( x(t), v(t) \) the position and velocity of the whole \( N \) particle system, i.e. \( x(t) := (x_1, \ldots, x_N) \) and \( v(t) := (v_1, \ldots, v_N) \). We also denote with \( (x(t), v(t)) \) a solution to the CS system if \( x(t), v(t) \) solve the NL system at time \( t \). In the same spirit, the notation \( (x_0, v_0) := (x_{10}, \ldots, x_{N0}, v_{10}, \ldots, v_{N0}) \) represents the vector of initial data. We now give a formal definition of flocking for a particle system \( (x(t), v(t)) \).

**Definition 2.1** (Asymptotic flocking). A given particle system \( (x(t), v(t)) \) is said to converge to a flock, iff the following two conditions hold,

\[
\sup_{t>0} \sup_{i,j} |x_i(t) - x_j(t)| < \infty, \quad \lim_{t \to \infty} \sup_{i,j} |v_i(t) - v_j(t)| = 0. \tag{3}
\]

We need to keep in mind that the definition we just gave is independent of the configuration of initial velocities and positions \( (x_0, v_0) \). This definition corresponds to the so called unconditional flocking scenario. If flocking holds for a certain class of initial configurations then we speak of conditional flocking.

Now that we stated the definition of flocking, we may proceed with the invariants of particle dynamics for system (2). For this, we define the first three moments of particle motion,

\[
m_0(t) := \sum_i 1, \quad m_1(t) = \sum_i v_i, \quad m_2(t) := \sum_i |v_i|^2. \tag{4}
\]
The following lemma shows how these moments propagate in time.

**Lemma 2.2** (propagation of moments (see [12])). Assume that the conditions (A1)-(A2) hold. Suppose also that \((x(t), v(t))\) is a solution to the NL CS system. Then, the three velocity moments satisfy

\[
\frac{d}{dt} m_0(t) = \frac{d}{dt} m_1(t) = 0, \quad \frac{d}{dt} m_2(t) \leq - \frac{C_1}{N} \sum_{i,j} \psi(|x_i - x_j|) |v_i - v_j|^{2\gamma}. \tag{5}
\]

*Proof.* We give the short proof for completion. The first equation is trivial since \(m_0(t) = N\). The equation for \(m_1(t)\) follows from the symmetry of \(\psi(\cdot)\) and (A1),

\[
m_1(t) = \sum_i \dot{v}_i(t) = \frac{1}{N} \sum_{i,j} \psi(|x_i - x_j|) \Gamma(v_j - v_i) = \frac{1}{N} \sum_{i,j} \psi(|x_i - x_j|) \Gamma(v_i - v_j) \tag{A1} = - \frac{1}{N} \sum_{i,j} \psi(|x_i - x_j|) \Gamma(v_j - v_i) = 0,
\]

where \(\sum_{i,j} = \sum_i \sum_j\). For the second moment \(m_2(t)\) we have

\[
m_2(t) = \frac{d}{dt} \sum_i |v_i(t)|^2 = 2 \sum_i \langle \dot{v}_i, v_i \rangle = \frac{2}{N} \sum_{i,j} \psi(|x_i - x_j|) \langle \Gamma(v_j - v_i), v_i \rangle \tag{A1} = - \frac{2}{N} \sum_{i,j} \psi(|x_i - x_j|) \langle \Gamma(v_j - v_i), v_j \rangle \\
= - \frac{1}{N} \sum_{i,j} \psi(|x_i - x_j|) \langle \Gamma(v_j - v_i), v_j - v_i \rangle \tag{A2} \leq - \frac{C_1}{N} \sum_{i,j} \psi(|x_i - x_j|) |v_i - v_j|^{2\gamma}.
\]

\[\square\]

A direct consequence of the invariance of the first moment is that the bulk velocity \(v_c(t) := \frac{1}{N} \sum_i v_i\) remains constant in time, i.e., \(v_c(t) = v_c(0)\). For the mean position vector \(x_c(t) := \frac{1}{N} \sum_i x_i\), we easily see that \(x_c(t) = v_c(0)t + x_c(0)\). Based on these observations we can define the standard deviation of positions and velocities for a group of \(N\) agents by

\[
\sigma_x(t) := \sqrt{\frac{1}{N} \sum_i |x_i(t) - x_c(t)|^2}, \quad \sigma_v(t) := \sqrt{\frac{1}{N} \sum_i |v_i(t) - v_c(t)|^2},
\]

and use them to study the flocking behavior in (2). Indeed, the flocking conditions in definition (3) are equivalent to showing \(\sup_{t \geq 0} \sigma_x(t) < \infty\) and \(\lim_{t \to \infty} \sigma_v(t) = 0\). The main flocking result for the NL CS system was given in [12].

**Proposition 1** (see [12]). We assume that conditions (A1)-(A2) hold and that \((x(t), v(t))\) is a smooth solution to the NL CS system (2) with the following constraint on initial configurations

\[
\sigma_v^{3-2\gamma}(0) \leq C_2 (3 - 2\gamma) \int_{\sigma_x(0)}^{\infty} \psi(2\sqrt{Ns}) \, ds, \tag{6}
\]

for some \(C_2 = C_2(\gamma, C_1, N) > 0\). Then, for \(\gamma \in \left(\frac{1}{2}, \frac{3}{4}\right)\) the particle system emerges to a flock.
The proof of Proposition 1 is actually pretty straightforward and a brief sketch of it is possible in few lines. First, one easily shows that the inequality for the dissipation of $\sigma_v(t)$ is

$$\frac{d}{dt}\sigma_v(t) \leq -C_2(2\sqrt{N}\sigma_x(t))\sigma_v^{2\gamma-1}(t). \quad (7)$$

Inequality (7) would be enough to ensure the convergence $\sigma_v(t) \to 0$, as long as we had a uniform bound on $\sigma_x(t)$ (some $\sigma_x^\infty < \infty$ such that $\sup_{t>0} \sigma_x(t) \leq \sigma_x^\infty$). For this, we may use the Lyapunov functionals

$$E^\pm(t) := \frac{\sigma_v^{3-2\gamma}(t)}{3-2\gamma} \pm C_2 \int_0^{\sigma_x(t)} \psi(2\sqrt{N}s) \, ds,$$

and show (with the help of (7)) that $E^\pm(t)$ are dissipative ($E^\pm(t) \leq E^\pm(0)$ for $t \geq 0$). Finally, using condition (6) and the dissipation of $E^\pm(t)$ it can be shown that $\sup_{t>0} \sigma_x(t) \leq \sigma_x^\infty < \infty$ which concludes with the proof.

**Remark 1.** It might appear that the flocking condition (6) in Proposition 1 is an unnecessary restriction but it is consistent with the phase transition character in the classical CS model (1). This condition is satisfied trivially in the case of long range interactions with $\int_0^\infty \psi(s) \, ds = \infty$, giving unconditional flocking when the communication weight $\psi(\cdot)$ has a heavy tail. When the interaction between agents has a short range, the emergence of a flock is only conditionally possible.

We note that in the statement of Proposition 1, flocking is proven for smooth solutions to system (2). The well-posedness of system (2) is of course a separate problem and everything depends on the regularity of interaction $\psi(\cdot)$ and coupling $\Gamma(\cdot)$. Naturally, local well-posedness can be proven for more singular interactions between agents and velocity couplings (see e.g. [3]). For global results, showing that particles avoid collisions implies that the singular weigh $\psi(\cdot)$ is effectively smooth, and thus creates no issues in terms of well-posedness. On the other hand, a non Lipschitz velocity coupling might create problems with uniqueness and we may end up with an infinite number of solutions. The Lipschitz condition on $\Gamma(\cdot)$ for $\gamma \in [1, \frac{3}{2})$ is a natural one, since the prototype example is $\Gamma(v) = \gamma|v|^{2(\gamma-1)}$, so this assumption is made in our main result. When $\gamma \in (\frac{1}{2}, 1)$ we don’t have uniqueness in general, as the Lipschitz property fails in $\Gamma(v) = v|v|^{2(\gamma-1)}$. Hence, our result gives a definite answer to the existence of smooth solutions for $\gamma \in [1, \frac{3}{2})$ when $\psi(s) = s^{-\alpha}$ and $\alpha \geq 1$.

The existence of collisions is not the end of story when it comes to the question of well-posedness. The low singularity of $\psi(\cdot)$ when $0 < \alpha < 1$ is good enough to give existence of piecewise weak solutions for $\gamma = 1$ (see [21]). Later, it was shown by the same author in [22] that when $0 < \alpha < \frac{1}{2}$ we have existence and uniqueness of global solutions in the linear CS system. It remains an open problem to see if and how these results can be reproduced for the NL CS system.

We now give the main result that we prove in the next section.

**Theorem 2.3.** Consider the CS system (2) with $\gamma \in (\frac{1}{2}, \frac{3}{2})$ and initial data $(x_0, v_0)$ that satisfy

$$x_{i0} \neq x_{j0} \quad \text{for} \quad i \neq j.$$  

We consider the communication weight $\psi(s) = s^{-\alpha}$, with $\alpha \geq 1$. Furthermore, if $\gamma \in [1, \frac{3}{2})$ we assume that $\Gamma(\cdot)$ is Lipschitz continuous. Then for any solution of the NL CS system the particle trajectories remain non-collisional for $t > 0$. 

The following easy lemma will prove helpful.

**Lemma 2.4.** Given \( p > 0 \), then for any \( q > 0 \) there exists a constant \( C_{pq} := C(p, q) > 0 \) such that

\[
|a^{-p} - b^{-p}| \geq C_{pq}|a^q - b^q| \quad \text{for} \quad 0 < a, b < 1.
\]

**Proof.** This is an exercise in calculus. For \( a = b \) it holds trivially. If \( a \neq b \), we set \( x = a^q, y = b^q \) and we consider the function \( f(x, y) = \frac{y^{p/q} - x^{p/q}}{x - y} \) on the triangle \( 0 < y < x < 1 \). We can show that the function has a positive lower bound \( C_{pq} > 0 \) for any pair \( p, q > 0 \).

3. Collision-avoiding for singular interactions.

**Proof.** The idea of the proof follows closely the steps in [4] in its first part. We assume that at some finite time \( t_C > 0 \) the first collision between a group of particle happens. Then, based on this assumption and estimates that we derive for the dynamics of the group of particles that collide, we reach a contradiction. We denote the group of particles that collide at time \( t_C \) with \( C \), and their number by \( |C| \) i.e.

\[
|x_i(t) - x_j(t)| \to 0 \quad \text{as} \quad t \searrow t_C \quad \text{for} \quad (i, j) \in C^2 := C \times C
\]

\[
|x_i(t) - x_j(t)| \geq \delta > 0 \quad \text{for some} \quad \delta > 0, \quad (i, j) \notin C^2, \quad t \in [0, t_C].
\]

We define the position and velocity fluctuation for the particles in the collisional group by

\[
\|x\|_C(t) := \sqrt{\sum_{(i,j) \in C^2} |x_i(t) - x_j(t)|^2} \quad \text{and} \quad \|v\|_C(t) := \sqrt{\sum_{(i,j) \in C^2} |v_i(t) - v_j(t)|^2}.
\]

Here \( \sum_{(i,j) \in C^2} \) is the sum over all pairs \((i, j)\) where both indices are members of group \( C \). According to the definition we just gave, we have that \( \|x\|_C(t) \to 0 \) as \( t \searrow t_C \). We also have the following uniform bounds for \( \|x\|_C(t) \) and \( \|v\|_C(t) \) as a result of the particle dynamics. There exist \( M > 0 \) and \( R = R(t_C) > 0 \), such that for all \( t \in [0, t_C] \) we have

\[
\|v\|_C(t) \leq M := \sqrt{2}|C| \sup_i |x_{i_0}|, \quad \|x\|_C(t) \leq R := \sqrt{2}|C| (\sup_i |x_{i_0}| + \sup_i |v_0| t_C).
\]

It is easy to show using the definition of \( \|x\|_C(t) \) that

\[
\left| \frac{d}{dt} \|x\|_C(t) \right| \leq \|v\|_C(t). \tag{8}
\]

Our plan is to show a sharp inequality for the dissipation of \( \|v\|_C(t) \) in the spirit of [4]. In more detail, we show that:

- If \( \frac{1}{2} < \gamma < 1 \),

\[
\frac{d}{dt} \|v\|^2_C(t) \leq -2c_0\psi(\|x\|_C(t))\|v\|^2_C(t) + 2c_1\|x\|_C(t)\|v\|_C(t) + 2c_2\|v\|_C(t). \tag{9}
\]

- If \( 1 \leq \gamma < \frac{3}{2} \),

\[
\frac{d}{dt} \|v\|^2_C(t) \leq -2c_0\psi(\|x\|_C(t))\|v\|^2_C(t) + 2c_1\|x\|_C(t)\|v\|_C(t) + 2c_2\|v\|^2_C(t). \tag{10}
\]
For the derivation of (9)-(10) we compute the time evolution of \( \|v\|_C(t) \), i.e.

\[
\frac{d}{dt} \|v\|_C^2 = 2 \sum_{(i,j) \in C^2} \left( v_i - v_j, \frac{1}{N} \sum_k \psi(|x_k - x_i|) \Gamma(v_k - v_i) - \frac{1}{N} \sum_k \psi(|x_k - x_j|) \Gamma(v_k - v_j) \right) 
\]

\[
= \frac{2}{N} \left( \sum_{(i,j) \in C^2, k \in C} + \sum_{(i,j) \in C^2, k \notin C} \right) (\psi(|x_k - x_i|) \langle v_i - v_j, \Gamma(v_k - v_i) \rangle) 
- \psi(|x_k - x_j|) \langle v_i - v_j, \Gamma(v_k - v_j) \rangle) =: J_1 + J_2.
\]

The computation for the first term \( J_1 \) gives

\[
J_1 = \frac{2}{N} \sum_{(i,j) \in C^2, k \in C} \psi(|x_k - x_i|) \langle v_i - v_j, \Gamma(v_k - v_i) \rangle - \psi(|x_k - x_j|) \langle v_i - v_j, \Gamma(v_k - v_j) \rangle) 
\]

\[
\overset{(i.o)}{=} \frac{4}{N} \sum_{(i,j) \in C^2, k \in C} \psi(|x_k - x_i|) \langle v_i - v_j, \Gamma(v_k - v_i) \rangle 
\]

\[
\overset{(i.o)}{=} \frac{2}{N} \sum_{(i,j) \in C^2, k \in C} \psi(|x_k - x_i|) \langle v_i - v_j, \Gamma(v_k - v_i) \rangle 
\]

\[
\overset{(A1)}{=} \frac{2}{N} \sum_{(i,j) \in C^2, k \in C} \psi(|x_k - x_i|) \langle v_i - v_k, \Gamma(v_k - v_i) \rangle 
\]

\[
= - \frac{2}{N} \sum_{(i,j) \in C^2, k \in C} \psi(|x_k - x_i|) \langle v_k - v_i, \Gamma(v_k - v_i) \rangle 
\]

\[
\overset{(A2)}{\leq} - \frac{2C_1|C|}{N} \sum_{(i,j) \in C^2} \psi(|x_i - x_j|) |v_i - v_j|^{2\gamma}.
\]

Then, using the definition of \( \|x\|_C, \|v\|_C \) and the monotonicity of \( \psi(\cdot) \)

\[
J_1 \leq -2c_0 \psi(\|x\|_C) \|v\|_C^{2\gamma}
\]

for \( c_0 = \frac{C_1|C|}{N} \).

For \( J_2 \) we have

\[
J_2 = \frac{2}{N} \sum_{(i,j) \in C^2, k \notin C} \psi(|x_k - x_i|) \langle v_i - v_j, \Gamma(v_k - v_i) \rangle 
\]

\[
= \frac{2}{N} \sum_{(i,j) \in C^2, k \notin C} \psi(|x_k - x_j|) \langle v_i - v_j, \Gamma(v_k - v_j) \rangle 
\]

\[
= \frac{2}{N} \sum_{(i,j) \in C^2, k \notin C} \psi(|x_k - x_i|) - \psi(|x_k - x_j|) \langle v_i - v_j, \Gamma(v_k - v_j) \rangle 
\]

\[
= \frac{2}{N} \sum_{(i,j) \in C^2, k \notin C} \left( \psi(|x_k - x_i|) \langle v_i - v_j, \Gamma(v_k - v_i) \rangle - \psi(|x_k - x_j|) \langle v_i - v_j, \Gamma(v_k - v_j) \rangle \right).
\]
\[ + \frac{2}{N} \sum_{k \in C^2} \sum_{(i,j) \in C^2} \psi(|x_k - x_i|)(v_i - v_j, \Gamma(v_k - v_i) - \Gamma(v_k - v_j)) := J_{21} + J_{22}. \]

The first term \( J_{21} \) is bounded by

\[ J_{21} \leq \frac{2}{N} \Gamma_M L_{\delta} \sum_{(i,j) \in C^2} |x_i - x_j| |v_i - v_j| \leq 2c_1 \|x\|_C \|v\|_C, \quad c_1 = \frac{N - |C|}{N} \Gamma_M L_{\delta}, \]

where \( L_{\delta} \) is the Lipschitz constant of \( \psi(\cdot) \) on the interval \((\delta, \infty)\) and \( \Gamma_M := \max_{\{c : \|v\| \leq \|M\|\}} |\Gamma(v)|. \) Similarly, since \( |x_k - x_i| > \delta \), it follows that \( \psi(|x_k - x_i|) < \psi(\delta) \) and we have the following bounds for the second term \( J_{22} \):

If \( \frac{1}{2} < \gamma < 1 \),

\[ J_{22} \leq \frac{4}{N} \psi(\delta) \Gamma_M \sum_{(i,j) \in C^2} |v_i - v_j| \leq 2c_2 \|v\|_C, \quad c_2 = \frac{2[C/(N - |C|)]}{N} \psi(\delta) \Gamma_M, \]

and if \( 1 \leq \gamma < \frac{3}{2} \) (using the Lipschitz property of \( \Gamma(\cdot) \), \( |\Gamma(v) - \Gamma(w)| \leq L_{\Gamma}|v - w| \))

\[ J_{22} \leq \frac{2}{N} \psi(\delta) L_{\Gamma} \sum_{(i,j) \in C^2} |v_i - v_j|^2 \leq 2c_2 \|v\|_C^2, \quad c_2 = \frac{N - |C|}{N} \psi(\delta) L_{\Gamma}. \]

The derivation of estimates (9)-(10) is complete. We mention a couple of differences with the linear case \( \Gamma(v) = v \). First, the term \( J_1 \) introduces the nonlinearity which makes it impossible to use a differential Gronwall lemma given the additional terms. Also, in contrast to the linear case where \( J_{22} \leq 0 \), here \( J_{22} \) is an extra term to be handled.

We keep in mind that for the singular weights \( \psi(s) = s^{-\alpha} \) we are considering with \( \alpha \geq 1 \), their primitive \( \Psi(s) = \int^s \psi(t)dt \) is also singular at 0. The following bound on the increase of \( \Psi(\|x\|_C(\cdot)) \) on the interval \((s, t)\) is useful.

\[ |\Psi(\|x\|_C(t))| = \left| \int_s^t \frac{d}{dt} \Psi(\|x\|_C(\tau)) \, d\tau + \Psi(\|x\|_C(s)) \right| \]

\[ = \left| \int_s^t \Psi'(\|x\|_C(\tau)) \frac{d}{d\tau} \|x\|_C(\tau) \, d\tau + \Psi(\|x\|_C(s)) \right| \]

\[ \leq \int_s^t \psi(\|x\|_C(\tau)) \|v\|_C(\tau) \, d\tau + |\Psi(\|x\|_C(s))|. \quad (11) \]

If we can show that \( \Psi(\|x\|_C(t_c)) < \infty \), the singularity of \( \Psi(\cdot) \) at 0 implies that \( \|x\|_C(t_c) \neq 0 \) which is a contradiction to our initial hypothesis. In our study, we consider the cases \( \gamma \in \left(\frac{1}{2}, 1\right] \) and \( \gamma \in (1, \frac{3}{2}) \) separately. The first is rather trivial, but the latter requires a bit of analysis.

- Case \( \gamma \in \left(\frac{1}{2}, 1\right] \):

The case of \( \frac{1}{2} < \gamma \leq 1 \) is pretty straightforward. From estimate (9) we get directly that

\[ \int_s^{t_c} \psi(\|x\|_C(\tau)) \|v\|_C^{2\gamma - 1}(\tau) \, d\tau < \infty. \]

We have that \( 0 \leq 2\gamma - 1 \leq 1 \) which, combined with fact that \( \|v\|_C(t) \leq M \), yields

\[ \int_s^{t_c} \psi(\|x\|_C(\tau)) \|v\|_C(\tau) \, d\tau \leq M^{2 - 2\gamma} \int_s^{t_c} \psi(\|x\|_C(\tau)) \|v\|_C^{2\gamma - 1}(\tau) \, d\tau < \infty. \]
In view of (11) we have $\Psi(||x||_{C}(t_C)) < \infty$.

- Case $\gamma \in (1, \frac{3}{2})$:
  This case is more elaborate. We know that $||v||_{C}(t)$ can only vanish at $t_C$, otherwise because of (10) it would be 0 on some interval $(s, t_C)$ and $t_C$ cannot be the time of the first collision. Thus, we have
  \[
  \frac{d}{dt}||v||_{C}(t) \leq -c_0\psi(||x||_{C})||v||_{C}^{2\gamma-1}(t) + c_1||x||_{C}(t) + c_2||v||_{C}(t). \quad (12)
  \]

Regarding (12), we observe that it is a nonlinear, nonhomogeneous differential inequality, with the nonlinear term being responsible for the dissipation of $||v||_{C}$. Despite its phenomenal simplicity, and unlike its linear version, there is no differential Gronwall lemma known for (12). On the other hand, all the integral versions of linear and nonlinear generalized Gronwall inequalities have a major caveat. To our best knowledge, they require positivity in the nonlinear term, and thus they are not helpful in that we cannot use them to take advantage of the dissipation the nonlinear term provides for large $||v||_{C}$. However, we may reach a contradiction by assuming a collision and doing a bit of qualitative analysis in (12), without the need to derive a Gronwall estimate for (12). The idea is actually pretty simple: For the three terms that appear in the rhs of (12) we study what happens when each of them is the dominant as $t \nearrow t_C$. For this, we consider the following three cases

(C1) $\psi(||x||_{C}(t))||v||_{C}^{2\gamma-1}(t) < ||x||_{C}(t)$,  
(C2) $\psi(||x||_{C}(t))||v||_{C}^{2\gamma-1}(t) < ||v||_{C}(t)$

and

(C3) $\frac{d}{dt}||v||_{C}(t) \leq -\psi(||x||_{C}(t))||v||_{C}^{2\gamma-1}(t)$.

Notice that for now we make the assumption that constants $c_0 = c_1 = c_2 = 1$. Later on we keep close track of all the constants involved.

We begin by checking what happens when each of (C1)-(C3) holds on some interval $(t_0, t_C)$. When (C1) holds on an interval $(t_0, t_C)$, we show that $||v||_{C}$ is practically so small that a collision cannot happen in finite time. Indeed, we have

\[
||v||_{C}(t) < \left( \frac{||x||_{C}(t)}{\psi(||x||_{C}(t))} \right)^{\frac{1}{2-\gamma}} = ||x||_{C}^{\frac{\alpha+1}{2-\gamma}}(t) \quad \text{for} \quad t \geq t_0.
\]

By (8), we have $\frac{d}{dt}||x||_{C}(t) > -||v||_{C}(t) - \frac{4}{\delta}||x||_{C}(t) > -||x||_{C}^{\frac{\alpha+1}{2-\gamma}}(t)$. We solve this differential inequality by integrating from $t_0$ to $t$ to get

\[
||x||_{C}(t) > \left( ||x||_{C}(t_0) - \lambda(t - t_0) \right)^{\frac{1}{\delta}}, \quad (13)
\]

where $\lambda = 1 - \frac{\alpha+1}{2-\gamma} = \frac{2(\gamma-1)-\alpha}{2-\gamma-1} < 0$. Now, setting $t = t_C$ in (13) leads to an obvious contradiction since $||x||_{C}(t) > 0$ for all $t \geq t_0$. It is useful for our proof to compute the change of $\Psi(||x||_{C}(\tau))$ on the interval $(t_0, t_C)$.

\[
|\Psi(||x||_{C}(t_C))| - |\Psi(||x||_{C}(t_0))| \leq \int_{t_0}^{t_C} |\psi(||x||_{C}(\tau))||v||_{C}(\tau)| d\tau < \int_{t_0}^{t_C} ||x||_{C}^{\mu}(\tau) d\tau, \quad (14)
\]

where $\mu = -\alpha + \frac{\alpha+1}{2-\gamma} = \frac{-2\alpha(\gamma-1)+1}{2-\gamma-1}$. If $\mu \geq 0$, it follows trivially that $|\Psi(||x||_{C}(t_C))| - |\Psi(||x||_{C}(t_0))| \leq R^\mu(t_C - t_0)$. If on the other hand $\mu < 0$, then we have using (13)
in (14)

\[
|\Psi(\|x\|_C(t_C))| - |\Psi(\|x\|_C(t_0))| < \int_{t_0}^{t_C} (\|x\|_C^\lambda(t_0) - \lambda(t_0)^\nu) \, dt
\]

\[
= \frac{1}{\lambda(\nu + 1)} \left(\|x\|_C^{\lambda(\nu + 1)}(t_0) - \lambda(t_0)^{\nu + 1}\right)_{t_0}^{t_C}
\]

\[
= - \frac{1}{\lambda(\nu + 1)} \left(\|x\|_C^{\lambda(t_0)}(t_0) - \lambda(t_0)^{\nu + 1} - \|x\|_C^{\lambda(t_0) + 1}(t_0)\right) < \infty,
\]

where \( \nu = \frac{\mu}{\lambda} = -\frac{2\alpha(\gamma - 1) + 1}{2(\gamma - 1)} > 0. \) The last inequality implies that \( |\Psi(\|x\|_C(t_C))| - |\Psi(\|x\|_C(t_0))| \leq C_\nu(t_C - t_0), \) for some constant \( C_\nu := C(\nu) > 0, \) due to the basic inequality \( |a^\nu - b^\nu| \leq C_p|a - b|, \) for \( 0 < a, b < 1, p \geq 1. \)

Similarly, if (C2) holds on some interval \((t_0, t_C)\) we have

\[
\|v\|_C(t) < \left(\frac{1}{\psi(\|x\|_C(t))}\right)^{\frac{1}{2-2\gamma}} = \|x\|_C^{\frac{2-2\gamma}{\gamma}}(t) \quad \text{for} \quad t \geq t_0.
\]

By (8), we have \( \frac{d}{dt}\|x\|_C(t) > -\|x\|_C^{\frac{2-2\gamma}{\gamma}}(t). \) The solution is once again given by (13), only this time \( \lambda = \frac{2(\gamma - 1) - \alpha}{2\gamma - 2} < 0. \) Setting \( t = t_C \) in (13) we get a contradiction like before. Also, the change of \( \Psi(\|x\|_C(t)) \) over \((t_0, t_C)\) is bounded like in (14), with \( \mu = -\alpha + \frac{2\alpha}{2\gamma - 2}. \) Now \( \mu > 0 \) and we get the estimate \( |\Psi(\|x\|_C(t_C))| - |\Psi(\|x\|_C(t_0))| \leq R^\mu(t_C - t_0). \) Overall, we have shown that \( |\Psi(\|x\|_C(t))| \) is Lipschitz in time when (C1) or (C2) holds, with a constant \( C_\mu = R^\mu \) for \( \mu \geq 0, \) or \( C_\mu = \frac{C_\nu}{\nu + 1} \) for \( \mu < 0. \)

Finally, we check what happens if (C3) holds on interval \((t_0, t_C). \) In this case, we show that although \( \|v\|_C(t) \) is not necessarily “small”, it dissipates so quickly to 0 that the collision cannot happen in finite time. Using a Gronwall inequality for (C3) we have

\[
\|v\|_C(t) \leq \left(\|v\|_C^{2-2\gamma}(t_0) + 2(\gamma - 1) \int_{t_0}^t \psi(\|x\|_C(\tau)) \, d\tau\right)^{1/(2-2\gamma)} \quad \text{for} \quad t \geq t_0.
\]

Then substituting eq. (16) in (11) we have

\[
|\Psi(\|x\|_C(t_C))| \leq \int_{t_0}^{t_C} \psi(\|x\|_C(t)) \left(\|v\|_C^{2-2\gamma}(t_0) + 2(\gamma - 1) \int_{t_0}^t \psi(\|x\|_C(\tau)) \, d\tau\right)^{\frac{1}{2-2\gamma}} \, dt
\]

\[
+ |\Psi(\|x\|_C(t_0))| = - \frac{1}{3 - 2\gamma} \left(\|v\|_C^{2-2\gamma}(t_0) + 2(\gamma - 1) \int_{t_0}^{t_C} \psi(\|x\|_C(\tau)) \, d\tau\right)^{\frac{2-2\gamma}{3 - 2\gamma}} \bigg|_{t = t_0}^{t C}
\]

\[
+ |\Psi(\|x\|_C(t_0))| \leq \frac{1}{3 - 2\gamma} \|v\|_C^{3-2\gamma}(t_0) + |\Psi(\|x\|_C(t_0))| < \infty.
\]

We may get a similar estimate for (C3) by integrating inequality \( \frac{1}{(3 - 2\gamma)} \frac{d}{dt}\|v\|_C^{3-2\gamma} \leq -\psi(\|x\|_C)\|v\|_C \) (which we get if we multiply (C3) by \( \|v\|_C^{3-2\gamma}. \) Hence,

\[
|\Psi(\|x\|_C(t))| - |\Psi(\|x\|_C(s))| \leq - \frac{1}{(3 - 2\gamma)} (\|v\|_C^{3-2\gamma}(t) - \|v\|_C^{3-2\gamma}(s)).
\]

We now proceed to the last part of the proof. We have investigated what happens when there is an interval \((t_0, t_C)\) over which one of the terms in the rhs of (12) is dominant over the others. Of course, this is by no means the only possible scenario.
In reality, we have to exclude the possibility of infinite “oscillations” between cases (C1)-(C3) right before the collision. Therefore, we divide the interval \((t_0, t_C)\) into two regions depending on the dominant terms in (12), i.e.

\[
\frac{1}{2} c_0 \psi(\|x\|_C(t)) \|v\|_C^{2\gamma-1}(t) < c_1 \|x\|_C(t), \quad t \in I_1,
\]

\[
\frac{d}{dt} \|v\|_C(t) < -\frac{1}{2} c_0 \psi(\|x\|_C(t)) \|v\|_C^{2\gamma-1}(t) + c_2 \|v\|_C(t), \quad t \in I_2,
\]

where

\[
I_1 = (t_0, t_1) \cup \ldots \cup (t_{2n}, t_{2n+1}) \cup \ldots, \quad I_2 = (t_1, t_2) \cup \ldots \cup (t_{2n+1}, t_{2n+2}) \cup \ldots.
\]

We start with region \(I_1\) which is practically case (C1) that we studied earlier. When \(t \in I_1\), we have \(\|v\|_C(t) < c_4 \|x\|_C^{\frac{4+\gamma}{2}}(t)\), \(c_3 = \left(\frac{2}{c_0}\right)^{1/(2\gamma-1)}\). We have already proved the Lipschitz property of \(\Psi(\|x\|_C(\cdot))\) on any interval in \(I_1\). Region \(I_2\) is a “hybrid” of cases (C2) and (C3), and we want to know how \(\Psi(\|x\|_C(\cdot))\) changes on some interval in \(I_2\). First we multiply eq. (19) by \(\|v\|_C^{2\gamma-2}(t)\) to get

\[
\frac{1}{3 - 2\gamma} \frac{d}{dt} \|v\|_C^{3-2\gamma}(t) < -\frac{1}{2} c_0 \psi(\|x\|_C(t)) \|v\|_C(t) + c_2 \|v\|_C^{3-2\gamma}(t).
\]

Now using the definition of \(\psi(\cdot)\) and integrating on the interval \((t_{2k+1}, t_{2k+2})\) we get an expression which is the equivalent to (17) for ineq. (19)

\[
|\Psi(\|x\|_C(t_{2k+2}))| - |\Psi(\|x\|_C(t_{2k+1}))| \leq -\frac{2}{(3 - 2\gamma)c_0} (\|v\|_C^{3-2\gamma}(t_{2k+2)})
\]

\[
- \|v\|_C^{3-2\gamma}(t_{2k+1})) + \frac{2c_2}{c_0} \int_{t_{2k+1}}^{t_{2k+2}} \|v\|_C^{3-2\gamma}(\tau) d\tau.
\]

(21)

The second rhs term in (21) is bounded by \(c_4(t_{2k+2} - t_{2k+1})\), with \(c_4 = \frac{2c_2}{c_0} M^{3-2\gamma}\).

Summing the second term from \(k = 1\) to \(n\) we have

\[
\sum_{k=0}^{n} \frac{2c_2}{c_0} \int_{t_{2k+1}}^{t_{2k+2}} \|v\|_C^{3-2\gamma}(\tau) d\tau \leq c_4 m(I_2),
\]

where \(m(I_2)\) is the Lebesgue measure of \(I_2\). For the first term, we have from ineq. (20) after we integrate from \(t_{2k+1}\) to \(t_{2k+2}\) that \(\|v\|_C^{3-2\gamma}(t_{2k+2}) - \|v\|_C^{3-2\gamma}(t_{2k+1}) \leq (3 - 2\gamma)c_2 M^{3-2\gamma}(t_{2k+2} - t_{2k+1})\) and

\[
\sum_{k=0}^{n} (\|v\|_C^{3-2\gamma}(t_{2k+2}) - \|v\|_C^{3-2\gamma}(t_{2k+1})) \leq (3 - 2\gamma)c_2 M^{3-2\gamma} m(I_2), \quad \forall n \geq 0.
\]

Of course, this bound is not enough since we need a bound of this term from below!

The trick lies in the fact that \(\|v\|_C(t)\) does not change drastically on \(I_1\). We have shown that (see ineq. (13)) \(|x|^\lambda(t_{2k+1}) - |x|^\lambda(t_{2k})| \leq -\lambda c_3(\tau_{2k+1} - t_{2k})\), where \(\lambda = 1 - \frac{2\gamma + 1}{2\gamma - 1} < 0\). Using the fact that \(\|v\|_C(t_k) = c_3 |x|^\frac{\lambda}{2\gamma - 1}(t_k)\) and Lemma 2.4 (with \(p = \frac{\lambda}{2\gamma - 1}\), and \(q = 3 - 2\gamma\)), we get

\[
\|v\|_C^{3-2\gamma}(t_{2k+1}) - \|v\|_C^{3-2\gamma}(t_{2k}) < c_5 (t_{2k+1} - t_{2k}), \quad \text{for} \quad c_5 = \frac{-\lambda c_3^{\frac{\lambda(2\gamma - 1)}{2\gamma - 1} + 1}}{C_{pq}}.
\]

Hence, taking the sum we get

\[
\sum_{k=0}^{n} (\|v\|_C^{3-2\gamma}(t_{2k+1}) - \|v\|_C^{3-2\gamma}(t_{2k})) < c_5 m(I_1), \quad \forall n \geq 0.
\]

(22)
Finally, using (22) we have \( \forall n \geq 0 \)
\[
- \sum_{k=0}^{n} (\|v\|_{C}^{3-2\gamma}(t_{2k+2}) - \|v\|_{C}^{3-2\gamma}(t_{2k+1})) \leq c_{5}m(I_{1}) + \|v\|_{C}^{3-2\gamma}(t_{0}) - \|v\|_{C}^{3-2\gamma}(t_{2n+2}).
\]

We now decompose \( |\Psi(||x||C(t_{n}))| \) in the following manner
\[
|\Psi(||x||C(t_{2n+2}))| = \sum_{k=0}^{n} (|\Psi(||x||C(t_{2k+2}))| - |\Psi(||x||C(t_{2k+1}))|)
+ \sum_{k=0}^{n} (|\Psi(||x||C(t_{2k+1}))| - |\Psi(||x||C(t_{2k}))| + |\Psi(||x||C(t_{0}))|).
\]

We have done all the work required to bound the two sums, and we can show that \( |\Psi(||x||C(t_{2n+2}))| < \infty \). We mention that a decomposition could also be performed for \( |\Psi(||x||C(t_{2n+1}))| \) with terms that are treated in similar manner. We have shown how we can bound the first term of this decomposition when we treated the terms that appear in (21). Indeed,
\[
\sum_{k=0}^{n} (|\Psi(||x||C(t_{2k+2}))| - |\Psi(||x||C(t_{2k+1}))|) \leq c_{4}m(I_{2})
+ 2c_{5}m(I_{1}) + 2\|v\|_{C}^{3-2\gamma}(t_{0}) - 2\|v\|_{C}^{3-2\gamma}(t_{2n+2})
\]
\[
(3 - 2\gamma)c_{0}.
\]

The second term of the decomposition can be easily bounded due to the Lipschitz property that we showed for (C1), so
\[
\sum_{k=0}^{n} (|\Psi(||x||C(t_{2k+1}))| - |\Psi(||x||C(t_{2k}))|) \leq c_{3}C_{\mu} \sum_{k=0}^{n} (t_{2k+1} - t_{2k})
\leq c_{3}C_{\mu}m(I_{1}) < \infty \quad \forall n \geq 0.
\]

Putting those two sums together, we have
\[
|\Psi(||x||C(t_{C}))| \leq \limsup |\Psi(||x||C(t_{n}))| < c_{3}C_{\mu}m(I_{1}) + c_{4}m(I_{2})
+ 2c_{5}m(I_{1}) + 2\|v\|_{C}^{3-2\gamma}(t_{0})
\]
\[
(3 - 2\gamma)c_{0} + |\Psi(||x||C(t_{0}))| < \infty,
\]
which contradicts our hypothesis of a collision at time \( t_{C} \).

\[\square\]

**Remark 2.** We should point out that the result we just proved, for the values of \( \alpha \) that give non-collisional particle dynamics, is sharp. For \( 0 < \alpha < 1 \) collisions are possible for any \( \gamma \in \left(\frac{1}{2}, \frac{3}{2}\right) \), and there is no dependence between the critical value of \( \alpha \) (between collisional and non-collisional dynamics) and \( \gamma \). This is not a surprising statement, since the main ingredient in our proof was the non integrability of \( \psi(\cdot) \) at \( 0 \). We may easily show that collisions exist when \( 0 < \alpha < 1 \) by solving the simplest possible system of two particles that move on a line (1-d motion).

4. **Uniform estimates on the particle distance for the communication weight** \( \psi_{\delta}(s) = (s - \delta)^{-\alpha} \), with \( \alpha \geq 2\gamma \). In this section, we give estimates for the minimum inter-particle distance in the case of weights of the type \( \psi_{\delta}(s) = (s - \delta)^{-\alpha} \)
for some fixed $\delta \geq 0$. We introduce the distance function $\mathcal{L}^\beta(t)$ for the particle system $(x_i(t), v_i(t))$, with $|x_i - x_j| > \delta$ for $1 \leq i \neq j \leq N$.

$$
\mathcal{L}^\beta(t) := \frac{1}{N(N-1)} \sum_{i \neq j} (|x_i(t) - x_j(t)| - \delta)^{-\beta} \quad \text{with} \quad \beta > 0.
$$

The symbol $\sum_{i \neq j}$ is short for the sum over all pairs $i, j$ for which $i \neq j$. For the special case $\beta = 0$ we define $\mathcal{L}^0(t) := \frac{1}{N(N-1)} \sum_{i \neq j} \log(|x_i(t) - x_j(t)| - \delta)$.

This function is chosen so that if $\mathcal{L}^\beta(t) < \infty$ on some interval $[0, T]$, then it follows that particles do not collide and that $|x_i(t) - x_j(t)| > \delta$ for $i \neq j$ on $[0, T]$, provided that $|x_{i0} - x_{j0}| > \delta$ for $i \neq j$. We will in fact show that if $\mathcal{L}^\beta(0) < \infty$ and given any $T > 0$, we have that $\mathcal{L}^\beta(t) < O(T)$ for all $t \in [0, T]$. Of course, the choice of $\beta$-distance we use depends directly on $\alpha$ and $\gamma$. In the spirit of [4], we introduce the maximal collisionless life-span of a solution with initial datum $x_0$, i.e.

$$
T_0 := \sup \{ s \geq 0 : \forall (x(t), v(t)) \text{ to } (2), \text{ there are no collisions on } [0, s] \}.
$$

We then prove

**Theorem 4.1.** Suppose that $\alpha \geq 2\gamma$ and that the CS system has initial data $(x_{i0}, v_{i0})$ satisfying $|x_{i0} - x_{j0}| > \delta$ for $1 \leq i \neq j \leq N$. Then, for any global smooth solution $(x(t), v(t))$ to the NL CS particle system we have $T_0 = \infty$. Moreover, we have the following estimates for $t \in [0, T_0]$:

(i) For $\alpha = 2\gamma$ we have

$$
\mathcal{L}^0(t) + \frac{1}{2C_1\gamma(N-1)} \sum_i |v_i(t)|^2 \leq \frac{2\gamma - 1}{2\gamma} t + \mathcal{L}^0(0) + \frac{1}{2C_1\gamma(N-1)} \sum_i |v_{i0}|^2.
$$

(ii) For $\alpha > 2\gamma$ we choose $\beta = \frac{\alpha}{2\gamma} - 1$ and have

$$
\mathcal{L}^\beta(t) + \frac{\beta}{2C_1\gamma(N-1)} \sum_i |v_i(t)|^2 \leq \frac{(2\gamma - 1)\beta}{2\gamma} t + \mathcal{L}^\beta(0) + \frac{\beta}{2C_1\gamma(N-1)} \sum_i |v_{i0}|^2.
$$

**Proof.** First, observe that for $\beta > 0$ we have

$$
\frac{d}{dt} \mathcal{L}^\beta(t) = -\frac{\beta}{N(N-1)} \sum_{i \neq j} (|x_i - x_j| - \delta)^{-\beta-1} \left\langle \frac{x_i - x_j}{|x_i - x_j|}, v_i - v_j \right\rangle
$$

and

$$
\frac{d}{dt} \mathcal{L}^0(t) = \frac{1}{N(N-1)} \sum_{i \neq j} (|x_i - x_j| - \delta)^{-1} \left\langle \frac{x_i - x_j}{|x_i - x_j|}, v_i - v_j \right\rangle.
$$

(i) If $\alpha = 2\gamma$, then we choose the $\beta$-distance with $\beta = \alpha/[2\gamma - 1] = 0$. We then have

$$
\frac{d}{dt} \mathcal{L}^0(t) = \frac{1}{N(N-1)} \sum_{i \neq j} (|x_i - x_j| - \delta)^{-1} \left\langle \frac{x_i - x_j}{|x_i - x_j|}, v_i - v_j \right\rangle
$$

$$
\leq \frac{1}{N(N-1)} \left( \frac{2\gamma - 1}{2\gamma} \sum_{i \neq j} 1 + \frac{1}{2\gamma} \sum_{i \neq j} (|x_i - x_j| - \delta)^{-\alpha} |v_i - v_j|^{2\gamma} \right).
$$
at which $\Psi$ becomes singular at 0 is sufficiently fast. We introduce a $\beta$ related to this $\psi$ of the type presented above. For this, we assume that the communication weight is not necessarily which improves the derived estimates in [4].

Integrating from 0 to $t$ we get our estimate.

(ii) If $\alpha > 2\gamma$, we choose $\beta = \frac{\alpha}{2\gamma} - 1$ once again. We similarly have

$$\frac{d}{dt} \mathcal{L}^\beta(t) \leq \frac{(2\gamma - 1)\beta}{2\gamma} + \frac{\beta}{2\gamma N(N - 1)} \sum_{i \neq j} |v_i(t)|^2 |v_i - v_j|^{2\gamma},$$

that yields

$$\frac{d}{dt} \left( \mathcal{L}^\beta(t) + \frac{\beta}{2C_1 \gamma (N - 1)} \sum_i |v_i(t)|^2 \right) \leq \frac{(2\gamma - 1)\beta}{2\gamma},$$

which gives our estimate.

\[ \square \]

**Remark 3.** We note that the estimates (23)-(24) we just gave generalize the ones in [4], as they are valid for any $\gamma > \frac{1}{2}$. Another interesting observation is that there is no need to use Gronwall’s lemma for their derivation. As a result, the minimum inter-particle distance estimate has a growth in time that is $O(t)$ instead of $O(e^{Ct})$ which improves the derived estimates in [4].

We may now give a slightly more general version of the uniform estimates presented above. For this, we assume that the communication weight is not necessarily of the type $\psi(s) = s^{-\alpha}$, but has a primitive $\Psi(\cdot)$ that is singular at 0, and the rate at which $\Psi$ becomes singular at 0 is sufficiently fast. We introduce a $\beta$-distance related to this $\Psi(\cdot)$ by

$$\mathcal{L}^\beta(t) := \frac{1}{N(N - 1)} \sum_{i \neq j} |\Psi(|x_i(t) - x_j(t)|)|^\beta \quad \text{for} \quad \beta > 0.$$ 

Similarly, we define $\mathcal{L}^0(t) := \frac{1}{N(N - 1)} \sum_{i \neq j} \log |\Psi(|x_i(t) - x_j(t)|)|$. Then, with a computation based on elementary techniques like in the previous result, we show

**Theorem 4.2.** Consider system (2) with $\gamma > \frac{1}{2}$ and initial data $(x_0, v_0)$ that satisfy $x_{i0} \neq x_{j0}$ for $i \neq j$.

We also assume that the communication weight $\psi(\cdot)$ has a primitive $\Psi(s)$ that is singular at $s = 0$ and satisfies the inequality

$$\Psi'(s) \leq C|\Psi(s)|^{(1-\beta)(2\gamma/(2\gamma-1)} \quad \text{for some} \quad C > 0 \quad \text{and} \quad 1 > \beta \geq 0. \quad (25)$$

We have that any solution to (2) remains non-collisional for $t > 0$. Moreover, we have for $\beta > 0$

$$\mathcal{L}^\beta(t) + \frac{\beta}{2C_1 \gamma (N - 1)} \sum_i |v_i(t)|^2 \leq C\frac{(2\gamma - 1)\beta}{2\gamma} t + \mathcal{L}^\beta(0) + \frac{\beta}{2C_1 \gamma (N - 1)} \sum_i |v_{i0}|^2, \quad (26)$$
and for $\beta = 0$
\[ \mathcal{L}_0(t) + \frac{1}{2C_1\gamma(N-1)} \sum_i |v_i(t)|^2 \leq C\frac{2\gamma - 1}{2\gamma} t + \mathcal{L}_0(0) + \frac{1}{2C_1\gamma(N-1)} \sum_i |v_{i0}|^2. \]

(27)

**Proof.** First, let us calculate the time evolution of $\mathcal{L}_\beta(t)$ for $\beta > 0$
\[
\frac{d}{dt} \mathcal{L}_\beta(t) = \beta \frac{\beta}{N(N-1)} \sum_{i\neq j} \Psi'(|x_i - x_j|) |\Psi(|x_i - x_j|)|^{\beta - 1} \frac{\Psi(|x_i - x_j|)}{|\Psi(|x_i - x_j|)|} \left( \frac{x_i - x_j}{|x_i - x_j|}, v_i - v_j \right)
\]
\[
\leq \beta \frac{(2\gamma - 1)}{2\gamma N(N-1)} \sum_{i\neq j} \Psi'(|x_i - x_j|) |\Psi(|x_i - x_j|)|^{(\beta - 1)/2\gamma(2\gamma - 1)}
\]
\[
+ \beta \frac{2\gamma}{2\gamma N(N-1)} \sum_{i\neq j} \Psi'(|x_i - x_j|) |v_i - v_j|^{2\gamma}. \]

Once again we made use of Young’s inequality for $a = |v_i - v_j|$, $b = |\Psi(|x_i - x_j|)|^{\beta - 1}$ and $p = 2\gamma$, $q = \frac{2\gamma}{2\gamma - 1}$.

Now using condition (25) and the second moment estimate in (5) we get
\[
\frac{d}{dt} \left( \mathcal{L}_\beta(t) + \frac{\beta}{2C_1\gamma(N-1)} \sum_i |v_i(t)|^2 \right) \leq C\frac{\beta(2\gamma - 1)}{2\gamma}. \]

For $\beta = 0$, we get
\[
\frac{d}{dt} \left( \mathcal{L}_0(t) + \frac{1}{2C_1\gamma(N-1)} \sum_i |v_i(t)|^2 \right) \leq C\frac{2\gamma - 1}{2\gamma}. \]

\[\square\]

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