Algebraic Geometry

The primitive cohomology lattice of a complete intersection

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Abstract

We describe the primitive cohomology lattice of a smooth even-dimensional complete intersection in projective space. To cite this article: A. Beauville, C. R. Acad. Sci. Paris, Ser. I 347 (2009).

Résumé

La cohomologie primitive d’une intersection complète. Nous décrivons le réseau de cohomologie primitive d’une intersection complète lisse de dimension paire dans l’espace projectif. Pour citer cet article : A. Beauville, C. R. Acad. Sci. Paris, Ser. I 347 (2009).

1. Introduction

Let $X$ be a smooth complete intersection of degree $d$ and even dimension $n$ in projective space. We describe in this note the lattice structure of the primitive cohomology $H^n(X, \mathbb{Z})_0$. Excluding the cubic surface and the intersection of two quadrics, we find

$$H^n(X, \mathbb{Z})_0 = A_{d-1} \oplus pE_8(\pm 1) \oplus qU \quad \text{or} \quad (-d) \oplus p'E_8(\pm 1) \oplus q'U$$

where the numbers $p$, $q$, $p'$, $q'$ and the sign attributed to $E_8$ depend on the multidegree and dimension of $X$ — see Theorem 4 for a precise statement. The proof is an easy consequence of classical facts on unimodular lattices together with the Hirzebruch formula for the Hodge numbers of $X$.

We warn the reader that there are many ways to write an indefinite lattice as an orthogonal sum of indecomposable ones; for instance, when $8|d$, both decompositions above hold. Still it might be useful to have a (semi-) uniform expression for this lattice. Related results, with a different point of view, appear in [3].

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2. Unimodular lattices

As usual we denote by $U$ the rank 2 hyperbolic lattice, and by $\langle d \rangle$ the lattice $\mathbb{Z}e$ with $e^2 = d$. If $L$ is a lattice, $L(-1)$ denotes the $\mathbb{Z}$-module $L$ with the form $x \mapsto -x^2$; if $n$ is a negative number, we put $nL := |n|L(-1)$.

Let $L$ be an odd unimodular lattice. A primitive vector $h \in L$ is said to be characteristic if $h \cdot x \equiv x^2 \pmod{2}$ for all $x \in L$; this is equivalent to saying that the orthogonal lattice $h^\perp$ is even [3, Lemma 3.3].

**Proposition 1.** Let $L$ be a unimodular lattice, of signature $(b^+, b^-)$, with $b^+, b^- \geq 2$; let $h$ be a primitive vector in $L$ of square $d > 0$, such that $h^\perp$ is even. Put $s := b^+ - b^-$, $t = \min(b^+, b^-)$, $u = \min(b^-, b^+ - d)$.

1) If $L$ is even or $8|d$ we have $h^\perp = \langle -d \rangle \oplus \frac{1}{8}E_8 \oplus (t - 1)U$.
2) If $L$ is odd and $d \leq b^+$, we have $h^\perp = A_{d-1} \oplus \frac{1}{8}E_8 \oplus uU$.

**Proof.** A classical result of Wall [6] tells us that $h$ is equivalent under $O(L)$ to any primitive vector $v$ of square $d$, provided $v$ is characteristic if so is $h$. If $L$ is even, we choose a hyperbolic plane $U \subset L$ with a hyperbolic basis $(e, f)$, and we put $v = e + \frac{d}{2}f$; then $v^\perp = \mathbb{Z}(e - \frac{d}{2}f) \oplus U^\perp$, and $U^\perp$ is an indefinite unimodular lattice, hence of the form $pE_8(\pm1) \oplus qU$. Computing $b^+$ and $b^-$ we find the above expressions for $p$ and $q$.

Consider now the case when $L$ is odd. We first observe that since $h$ is characteristic, we have $d = h^2 \equiv s \pmod{8}$ [5, V, Theorem 2]. Let

$$L' := \left( \bigoplus_{l \leq d} \mathbb{Z}e_l \right) \oplus \frac{s - d}{8}E_8 \oplus uU \quad \text{with} \quad e_1^2 = \cdots = e_d^2 = 1.$$  

$L'$ is odd, indefinite and has the same signature as $L$, hence is isometric to $L$. We put $v = e_1 + \cdots + e_d$. The orthogonal of $v$ in $\bigoplus \mathbb{Z}e_l$ is the root lattice $A_{d-1}$. By Wall’s theorem $h^\perp$ is isometric to $v^\perp = A_{d-1} \oplus \frac{s-d}{8}E_8 \oplus uU$.

Suppose moreover that $8$ divides $d$, so that $8|s$. Then $L$ is isomorphic to $\mathbb{Z}e \oplus \mathbb{Z}f \oplus \frac{1}{8}E_8 \oplus (t - 1)U$, with $e^2 = 1$, $f^2 = -1$. Taking $v = (\frac{d}{4} + 1)e + (\frac{d}{4} - 1)f$ gives the result. □

**Remark.** Since the signature of $h^\perp$ is $(b^+ - 1, b^-)$, the condition $d \leq b^+$ is necessary in order that $h^\perp$ contains $A_{d-1}$.

3. Complete intersections

We will check that the hypotheses of the proposition hold for the cohomology of complete intersections; the only non-trivial point is the inequality $d \leq b^+$. We will use the notations of [1]. Let $d = (d_1, \ldots, d_r)$ be a sequence of positive integers. We denote by $V_n(d)$ a smooth complete intersection of multidegree $d$ in $\mathbb{P}^{n+e}$. We put

$$h^{p,q}_n(d) = \dim H^{p,q}(V_{p+q}(d)) \quad \text{and} \quad h^{p,q}_0(d) = h^{p,q}(d) - \delta_{p,q}.$$  

**Lemma 2.** $h^{p+1,q+1}(d) \geq h^{p,q}(d)$.

**Proof.** Following [1] we introduce the formal generating series

$$H(d) = \sum_{p,q \geq 0} h^{p,q}_0(d)y^pz^q \in \mathbb{Z}[y, z];$$

we define a partial order on $\mathbb{Z}[y, z]$ by writing $P \succeq Q$ if $P - Q$ has non-negative coefficients. The assertion of the lemma is equivalent to $H(d) \succeq yzH(d)$. The set $P$ of formal series in $\mathbb{Z}[y, z]$ with this property is stable under addition and multiplication by any $P \geq 0$ in $\mathbb{Z}[y, z]$. The formula
Theorem 4. We assume Lemma 3.

\[ H(d_1, \ldots, d_c) = \sum_{P \subseteq [1,d]} \prod_{i \in P} H(d_i) \]

[1, Corollary 2.4(ii)] shows that it is enough to prove that \( H(d) \) is in \( \mathcal{P} \).

By [1, Corollary 2.4(i)], we have \( H(d) = \frac{1}{2} \) with

\[ P(y, z) = \sum_{i,j \geq 0} \binom{d-1}{i+j+1} y^i z^j \quad \text{and} \quad Q(y, z) = \sum_{i,j \geq 1} \binom{d}{i+j} y^i z^j. \]

Since \( Q \geq yz \), we get \( \frac{1}{2}yz = 1 + \frac{Q(yz)}{1-Q} \geq 0 \), hence \( (1-yz)H \geq 0 \). \( \square \)

Lemma 3. Let \( d = d_1 \cdots d_c \). We have:

a) \( h^{p-p}(d) \geq d \);

b) \( 2h^{p+1-p}(d) + 1 \geq d \), except in the following cases:

- \( d = (2, 2) \);
- \( p = 1, d = (3, 4, (2, 2), (2, 2, 2)) \);
- \( p = 2, d = (2, 2, 2) \).

Proof. We first prove b) in the case \( p = 1 \). Then \( V_2(d) \) is a surface \( S \subset \mathbb{P}^{e+2} \). The canonical bundle \( K_S \) is \( O_S(e) \), with \( e := d_1 + \cdots + d_c - c - 3 \); therefore \( K_S^2 = e^2d \). The cases with \( e \leq 0 \) are excluded, so we assume \( e \geq 1 \). Then the index \( K_S^2 - 8\chi(O_S) \) of the intersection form is negative [4]; if \( e \geq 2 \), we get \( \chi(O_S) > \frac{e}{2} \), hence \( 2h^{2,0}(d) + 1 \geq d \).

If \( e = 1 \), we have \( K_S = O_S(1) \) hence \( p_S = c + 3 \). The possibilities for \( d \) are \( (5, (2, 4), (3, 3) \) and \( (2, 2, 3) \); we have \( 2(c + 3) + 1 \geq d \) in each case.

Since the index is negative, we have \( h^{1,1}(d) > 2h^{2,0}(d) + 1 \); this implies that a) holds (for \( p = 1 \) except perhaps for \( d = (3, 2, 2), (4, (2, 2), (2, 2, 2) \). But the corresponding \( h^{1,1} \) is \( 7, 6, 20, 20, 20 \), which is always \( > d \).

Now assume \( p \geq 2 \). a) follows from the previous case and Lemma 2; similarly it suffices to check b) for the values of \( d \) excluded in the case \( p = 1 \). Using the above formulas we find

\[ h^{3,1}(3) = 1, \quad h^{3,1}(4) = 21, \quad h^{3,1}(2, 3) = 8, \quad h^{3,1}(2, 2, 2) = 27, \quad h^{4,2}(2, 2, 2) = 6, \]

so that \( 2h^{p+1-p}(d) + 1 \geq d \) for \( p \geq 2 \) in the three first cases and for \( p \geq 3 \) in the last one. \( \square \)

Theorem 4. Let \( X \) be a smooth even-dimensional complete intersection in \( \mathbb{P}^{e+c} \), of multidegree \( d = (d_1, \ldots, d_c) \). Let \( d := d_1 \cdots d_c \) be the degree of \( X \), and let \( e \) be the number of integers \( d_i \) which are even.

Let \((b^+, b^-)\) be the signature of the intersection form on \( H^n(X, Z) \); we put

\[ s = b^+ - b^- \quad \text{and} \quad u = \min(b^+ - d, b^-). \]

We assume \( d \neq (2, 2) \) and \( d \neq (3, 2, 2, 2) \) when \( n = 2 \). Then:

- \( H^n(X, Z)_0 = (-d) \oplus \frac{1}{2}E_8 \oplus (t-1)U \) if \( \left( \frac{2}{e} \right) \) is even;

- \( H^n(X, Z)_0 = A_{d-1} \oplus \frac{1}{2}dE_8 \oplus uU \) if \( \left( \frac{2}{e} \right) \) is odd.

For a hypersurface, for instance, we find a lattice of the form \( A_{d-1} \oplus pE_8 \oplus qU \) except if \( d \) is even and \( n \equiv 2 \) (mod. 4).

Proof. We apply Proposition 1 with \( L = H^n(X, Z) \). We take for \( h \) the class of a linear section of codimension \( \frac{d}{2} \), so that \( h^2 = d \).

By [3, Theorem 2.1 and Corollary 2.2], we know that

- \( h \) is primitive;

- \( h^1 \) is even;
• \( L \) is even or odd according to the parity of \( \left( \frac{n^2 + e}{e} \right) \).

To apply the proposition we only need the inequalities \( b^+ \geq d \) and \( b^- \geq 2 \). Note that the statement of the theorem holds trivially for \( d = 2 \), so we may assume \( d \geq 3 \). Let us write \( n = 4k + 2\varepsilon \), with \( \varepsilon \in \{0, 1\} \). By Hodge theory we have

\[
\begin{align*}
   b^+ &= \sum_{p+q=n, \ p \ even} h^{p,q} + \varepsilon, \\
   b^- &= \sum_{p+q=n, \ p \ odd} h^{p,q} - \varepsilon,
\end{align*}
\]

when the inequalities a) and b) of Lemma 3 hold this implies \( b^+ \geq d \) and \( b^- \geq 2 \), so Proposition 1 gives the result.

In the remaining cases \( p = 1, \ d = (3), (2, 3), (2, 2, 2) \) and \( p = 2, \ d = (2, 2, 2) \), the lattice \( L \) is even and we have \( b^+, b^- \geq 2 \), so Proposition 1 still applies.

**Remark.** The two first exceptions mentioned in the theorem are well-known [2, Proposition 5.2]: we have \( H^2(X, \mathbb{Z})_o = E_6 \) for a cubic surface, and \( H^n(X, \mathbb{Z})_o = D_n + 3 \) for an \( n \)-dimensional intersection of two quadrics. For an intersection of 4 quadrics in \( \mathbb{P}^6 \), we have \( d = 16 \), hence by Proposition 1

\[
H^2(X, \mathbb{Z})_o = (-16) \oplus 6E_8(-1) \oplus 15U.
\]

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