Towards a more realistic modelling of the uncertainty on identified mode shapes due to measurement noise

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Abstract. Many damage identification methods use the information from mode shapes. In order to test the robustness of these methods, it is a common practice to introduce uncertainty on the mode shapes in the form of independent noise at each measured location. In doing so, the potential spatial correlation in the mode shapes uncertainty is not taken into account. A better approach consists in adding uncorrelated noise on the time domain responses at each sensor before doing the identification. The spatial correlation resulting from the identification can then be evaluated using the covariance matrices of the identified mode shapes. In this study, we apply this approach to the numerical example of a simply supported beam. Modal identification is performed using stochastic subspace based algorithms developed in the toolbox MACEC. The covariance matrices of the mode shapes shows that there is a strong spatial correlation in the mode shapes uncertainty. This result shows that adding independent noise directly on the mode shapes is not a very realistic approach to assess the impact of noise on damage identification methods. The approach used to characterize noise uncertainty on modeshapes identification is totally general and can be applied to any mode, structure or sensing technology.

1. Introduction
Dynamic signatures of structures are widely used for various purposes such as finite element model updating, vibration control, structural design or structural health monitoring (SHM). At low frequencies, the dynamic signature is a combination of a few modeshapes corresponding to eigen frequencies with modal damping coefficients. These modal properties can be extracted from the dynamic response using modal identification techniques and are often used as input for model updating [1] or damage identification methods [2]. Traditionally, when testing the robustness of such types of methods, numerical models are used in order to compute the modeshapes which will be used as input measurements, and noise is added directly on the modeshapes in the form of independent noise at each measured location [3, 4]. In [5, 6], the authors consider noise added directly on the time-domain sensor responses, which seems to be more realistic, but the methods developed are not based on modal properties. The aim of this paper is therefore to study the effect of noise added directly on the time-domain sensor responses on modal properties identified using the stochastic subspace identification method [11] and to compare it with the traditional approach consisting in adding spatially uncorrelated noise directly on the modeshapes. In order to do this, we consider a simply supported beam equipped with 11 strain sensors. Five thousand
samples of the dynamic response of the beam excited by a band-limited white noise signal are computed and the modal properties are extracted for each sample. This data is used to compute the covariance matrix of the identified modeshapes from which information on spatial correlation between the sensors is presented. Although the structure studied is very simple, the approach presented in this paper is very general and can be applied to any kind of structure, sensing technology and modeshape.

2. Description of the case study
We consider a simply supported beam of 1m length and a square (0.1m x 0.1m) cross section made of concrete (Figure 1). The beam is modeled with 100 Euler-Bernoulli beam elements using the Structural Dynamics Toolbox under Matlab [9] and a force with a band-limited white noise [0 4000 Hz] exciting the first four modes (193 Hz, 773 Hz, 1740 Hz and 3094 Hz) is applied. The time-domain response is computed using an in-house code based on the modal properties computed from the finite element beam model. A network of 11 equally spaced strain sensors with a gauge length of 0.01m is placed on the beam and the measurements are performed over a period of 10 sec with a sampling rate of 8000Hz resulting in a set of 80000 measurement points for each sensor. Noise is added on the sensors responses under the following form:

\[ y_i(t) = y_i^0(t) + \beta_1 \lambda_{\text{max}} x_{|\tau \in [0,10]|}(y_i^0(t^*)) \]

where \( y_i(t) \) and \( y_i^0(t) \) are the noisy and non-noisy responses of sensor \( i \) (\( i = 1, \cdots, 11 \)) at time \( t \) respectively. \( \lambda \) is the random parameter, with its continuous distribution \( f(\lambda) \) following a Gaussian distribution with zero mean and unitary standard deviation. The level of noise \( \beta_1 \) is fixed at 5%, which lead to the signal-to-noise ratios (SNR) given in Table 1. Modal identification is performed using the covariance based stochastic subspace identification method (SSI-cov) [7] implemented in the Macec Toolbox [8] under Matlab, using output-only measurements.

![Figure 1: Simply supported concrete beam equipped with 11 strain sensors](image)

Table 1: Signal-to-noise ratios.

| Sensor | 1   | 2   | 3   | 4   | 5    | 6    | 7    | 8    | 9    | 10   | 11   |
|--------|-----|-----|-----|-----|------|------|------|------|------|------|------|
| SNR    | 24.95 | 24.86 | 26.52 | 26.55 | 22.79 | 26.74 | 26.78 | 26.12 | 24.04 | 23.7672 | 23.24 |
The modal identification is performed five thousand times with different realizations of the input excitation signal and the output measurement noise on the sensors. Only the first mode shape is studied in the present paper and each realization is noted \( \Phi_k, (k = 1, \cdots, 5000) \). The reference mode shape \( \Phi^0 \) is the mean of the identified mode shapes:

\[
\Phi^0 = \frac{1}{5000} \sum_{k=1}^{5000} \Phi_k,
\]

with \( \Phi_k = [\phi_{k1} \cdots \phi_{k11}]^T \). An alternative very simple manner to introduce noise on mode shapes which has been adopted in several studies is to add spatially uncorrelated noise directly on the mode shapes:

\[
\phi_{ki} = \phi^0_i + \beta_2 \lambda \phi^0_i,
\]

where \( \phi_{ki} \) and \( \phi^0_i \) are respectively the noisy and reference component \( i (i = 1, \cdots, 11) \) of the \( k^{th} \) sample of the identified mode shape (mode 1 in this study), \( \beta_2 \) is the level of noise and \( \lambda \) follows the \( f(\lambda) \) distribution (normal distribution with zero mean and unitary standard deviation).

Because the effect on mode shapes of a noise introduced following Equations (1) or (2) is very different, we can not choose \( \beta_1 = \beta_2 \) if we wish to have noise on mode shapes that are of the same order of magnitude. There is no way to link \( \beta_1 \) with \( \beta_2 \), and we have therefore arbitrarily fixed the value of \( \beta_2 \) using the mean value of noise on all sensors. This idea lead to \( \beta_2 = 0, 858\% \). For clarity, we will refer to the solutions obtained with Equations (1) and (3) as schemes \( S_1 \) and \( S_2 \) respectively.

3. Study of noise based on covariance matrices

3.1. Computation of covariances

The SSI-cov method implemented in \textit{Macec} normalizes the modes with respect to the largest component which leads to a zero variance for that component. In order to avoid that, the modes are renormalized using their norm: \( \tilde{\Phi}_k = \Phi_k / \| \Phi_k \| \). Let \( \Sigma \) be the classical covariance of the mode shapes:

\[
cov(\tilde{\Phi}_1 \cdots \tilde{\Phi}_{5000}) = \Sigma = \begin{bmatrix}
    s_1^2 & \cdots & s_1 s_{11} \\
    \vdots & \ddots & \vdots \\
    s_{11} s_1 & \cdots & s_{11}^2
\end{bmatrix}
\]

We have chosen a noise model which is proportional to the amplitude at each sensor, which induces a spatial shaping of the noise. In order to get rid of it, we will study the normalized covariance \( \tilde{\Sigma} \) given by:

\[
\tilde{\Sigma} = \begin{bmatrix}
    \hat{s}_1^2 & \cdots & \hat{s}_1 \hat{s}_{11} \\
    \vdots & \ddots & \vdots \\
    \hat{s}_{11} \hat{s}_1 & \cdots & \hat{s}_{11}^2
\end{bmatrix},
\]

where

\[
\hat{s}_i \hat{s}_j = \frac{s_i s_j}{\phi^0_i \phi^0_j}
\]

In addition, we have checked that the variability in the modes was mainly due to the noise added on the sensors and not to the identification process itself. In order to do that, we have performed five thousand identifications without noise added on the sensors (with different realizations of the input signal) and computed the covariance matrix. It was found to be several orders of magnitude lower than the covariance matrix computed with the noise added on the sensors.
3.2. Analysis of the covariances

Before analyzing the covariance matrices, it is necessary to check their convergence with respect to the number of samples used. In order to do that, we have considered each column of $\Sigma$ and computed the evolution of the Normalized Modal Difference (NMD) as a function of the number of samples [10]:

$$NMD = \sqrt{1 - \frac{MAC}{MAC}},$$

(7)

where the $MAC$ is computed between the column of the covariance matrix computed using $n$ samples and the column of the covariance matrix computed using all (5000) samples. The NMD should be close to zero if two vectors are identical. Figure 2 shows the NMD for the first column of the covariance matrix $\hat{\Sigma}$. The NMD is below 0.5% when using 3000 samples or more, showing that the covariance matrix has converged sufficiently for 5000 samples. The same conclusion can be drawn for the other columns.

![Figure 2: Evolution of NMD with the number of samples (first column of $\hat{\Sigma}$).](image)
Because noise is added independently on each component of the mode shape with scheme S_2, the covariance matrix of \( \tilde{\Phi}_i \) will converge to a diagonal matrix when the number of samples increases. This characteristic can be seen with 5000 samples. On the other hand, with the same number of samples, the covariance matrix for scheme S_1 is clearly not diagonal, showing that the noises on each component of the mode shapes are not independent. Plotting the columns of the covariance matrix (or the rows because this matrix is diagonal) can therefore give some information on the spatial correlation between noise at sensor \( i \) and noise at sensor \( j \). Figure 3 shows some columns of the covariance matrices of schemes S_1 and S_2. Scheme S_2 presents columns close to pulses because noise is added independently on each component. Noise at sensor \( i \) has therefore no influence on the other sensors \( i \neq j \). On the other hand, with scheme S_1, noise at sensor \( i \) has definitely influence on noise at the other sensors. Note that correlation can exist between very distant sensors such as sensors 1 and 11 (Figure 3 (a)).

Performing a singular value decomposition of the covariance matrix, it is possible to represent the variability in the form of uncorrelated variables. Because \( \hat{\Sigma} \) is symmetric, the singular value decomposition is equivalent to an eigenvalue analysis:

\[
\hat{\Sigma} = [U][S][U]^T
\]  \hspace{1cm} (8)

If each noisy normalized mode shape can be decomposed in its noisy and non-noisy parts as \( \tilde{\Phi}_k = \Phi^0 + \delta\tilde{\Phi}_k \), the noise of sample \( k \) can be rebuilt by combining the eigen vectors:

\[
\delta\tilde{\Phi}_k = [U]\alpha_k^T
\]  \hspace{1cm} (9)

which describes the noise with independent random variables. In our application, the \( \alpha_k \) probability distributions have been found to be approximatively normal, and their covariance matrix...
is diagonal as expected.

In Figure 4, we compare the singular values obtained with schemes $S_1$ and $S_2$:

![Comparison of singular values](image)

**Figure 4**: Comparison of the singular values for schemes $S_1$ and $S_2$.

Clearly, the two schemes show different energy distribution. Scheme $S_1$ has an exponentially decreasing energy distribution while scheme $S_2$ shows an almost constant energy distribution. Comparing the eigen vectors $u_i$ of $[U]$ allows to understand how the energy is distributed (Figure 5):

![Eigen vectors comparison](image)
The eigen vectors of scheme $S_1$ are waves of different length (due to the spatial correlation) and different energy. Because the energy content is decreasing, the noise can be approximated by considering a combination of a few independent variables containing the major part of the energy. For scheme $S_2$, the eigen vectors are pulses with equal energy, so that the noise cannot be approximated by a reduced set of independent variables.

4. Conclusion
In this work, we have studied the impact of noise added directly on time-domain sensor responses on the identified modeshapes using an output-only covariance based stochastic subspace identification method (SSI-cov). Using Monte-Carlo simulations to compute the covariance matrix of the first modeshape of a simply supported beam equipped with eleven strain sensors, we have demonstrated that the modal identification introduces some spatial correlation in the noise between the sensors, so that adding spatially uncorrelated noise on the modeshapes directly is not correct. Although it is possible to represent this noise in an independent variables space using the singular value decomposition, the singular vectors and values used to reconstruct the noise are problem dependent and it does not seem possible to draw general conclusions which could lead to the possibility to model the noise directly on the modeshapes using some kind of a priori spatial correlation model. The best solution seems to be to add noise directly on the time domain responses before performing the identification, although it is more time consuming. The approach proposed to characterize the uncertainty on identified modeshapes was presented for a simple structure and only for its first modeshape, but can be applied to any mode of any structure, as well as any sensing technology. The next step might be therefore to use the technique with more complex numerical cases, and experimentally.

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