Three-dimensional analysis of condensation nanofluid film on an inclined rotating disk by efficient analytical methods

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ABSTRACT
In this study, least square method (LSM) and differential transform method (DTM) applied to solve the three dimensional problem of steady nanofluid deposition on an inclined rotating disk is illustrated. The governing non-linear partial differential equations are reduced to the nonlinear ordinary differential equations system by similarity transform. There is a good agreement between the present analytical and numerical results. Results indicate that increasing nanofluid volume fraction leads to increase in temperature profile and highest temperature obtained for aluminium oxide (Al₂O₃)-water case.

1. Introduction
The removal of a liquid condensate from a cooled, saturated vapor is important in chemical and mechanical engineering processes. Many researchers illustrated this problem at the different conditions. Sparrow and Gregg (1959) studied the removal of the condensate using centrifugal forces on a cooled rotating disc. Sparrow and Gregg (1959) transformed the Navier-Stokes equations into a set of nonlinear ordinary differential equations and numerically integrated for the similarity solution for several finite film thicknesses. The problem is also related to chemical vapor deposition, when a thin fluid film is deposited on a cooled rotating disk (Jensen, Einset, & Fotiadis, 1991).

Enhancement of heat transfer performance in these systems is an essential topic from an energy saving perspective. The low thermal conductivity of conventional heat transfer fluids such as water and oils is a primary limitation in enhancing the performance and the compactness of such systems. Solid typically has a higher thermal conductivity than liquids. For example, copper (Cu) has a thermal conductivity 700 time greater than water and 3000 greater than engine oil. An innovative and new technique to enhance heat transfer is by using solid particles in the base fluid (i.e. nanofluids) in the range of sizes 10–50 nm. Khanafer, Vafai, and Lightstone (2003) firstly conducted a numerical investigation on the heat transfer enhancement by adding nano-particles in a differentially heated enclosure. They found that the suspended nanoparticles substantially increase the heat transfer rate at any given Grashof number. Rashidi, Abelman, and Mehr (2013) considered the analysis of the second law of thermodynamics applied to an electrically conducting incompressible nanofluid fluid flowing over a porous rotating disk. They concluded that using magnetic rotating disk drives has important applications in heat transfer enhancement in renewable energy systems. Ellahi (2013) studied the magnetohydrodynamic (MHD) flow of non-Newtonian nanofluid in a pipe. He observed that the MHD parameter decreases the fluid motion and the velocity profile is larger than that of temperature profile even in the presence of variable viscosities. Free convection heat transfer in a concentric annulus between a cold square and heated elliptic cylinders in presence of magnetic field was investigated by Sheikholeslami, Gorji-Bandpy, and Ganji (2013a). They found that the enhancement in heat transfer increases as Hartmann number increases but it decreases with increase of Rayleigh number. Asymmetric laminar flow and heat transfer of nanofluid between contracting rotating disks was investigated by Hatami, Sheikholeslami, and Ganji (2014). Their results indicated that temperature profile becomes more flat near the middle of two disks with the increase of
injection but an opposite trend is observed with increase of expansion ratio. The problem of laminar nanofluid flow in a semi-porous channel in the presence of transverse magnetic field was investigated analytically by Sheikholeslami, Hatami, and Ganji, (2013b). Their results showed that velocity boundary layer thickness decrease with increase of Reynolds number and it increase as Hartmann number increases. Khan and Pop (2010) published a paper on boundary-layer flow of a nanofluid past a stretching sheet. They indicated that the reduced Nusselt number is a decreasing function of each dimensionless number. Hassan, Tabar, Nemati, Domairry, and Noori (2011) investigated the problem of boundary layer flow of a nanofluid past an inclined rotating disk. They found that the reduced Nusselt number decreases with the increase in Prandtl number for many Brownian motion numbers. Rashidi and Dinarvand (2009) investigated the steady three-dimensional problem of condensation film on inclined rotating disk by analytical methods introduced by Aziz (2006) using Maple software where Bhatti and Zeeshan (2017), Bhatti, Zeeshan, and Ellahi (2016, 2017a), Bhatti, Zeeshan, Ellahi, and Ijaz (2017b), Ezzat (1994, 2001, 2011, 2012), Ezzat and Yousef (2010), Hatami and Ganji (2014a, 2014b), MA. Ezzat, El-Bary, and SM. Ezzat (2013); MA. Ezzat, Abbas, El-Bary, and SM. Ezzat (2014a), M. Ezzat, Sabbah, El-Bary, and S. Ezzat (2014b), Nawaz, Hayat, and Zeeshan (2016), Othman and Ezzat (2001), Zeeshan, Hassan, Ellahi, and Nawaz (2016), Zeeshan and Majeed (2016a, 2016b), and some recent researchers (Devakar, Ramesh, Chouhan, & Raje, 2017; Mahanthesh, Gireesha, & Gorla, 2017; Mirgolbabaee, Ledari, & Ganji, 2017; Saleh, Alali, & Ebaid, 2017; Srinivasacharya & Shafeeurrahman, 2017) used these valuable methods for analyzing the heat transfer and nanofluid flow problems in different geometries.

In this work we shall develop and apply least square method (LSM) and differential transform method (DTM) to solve the problem of three-dimensional analysis of condensation nanofluid film on an inclined rotating disk.

2. Flow analysis and mathematical formulation

Figure 1 (Rashidi & Dinarvand, 2009) shows a disk rotating in its own plane with angular velocity \( \Omega \). The angle between horizontal axis and disk is \( \beta \). A nanofluid film of thickness \( h \) is formed by spraying, with the \( W \) velocity. We assume the disk radius is large compared to the film thickness such that the end effects can be ignored. Vapor shear effects at the interface of vapor and fluid are usually unimportant. The gravitational acceleration, \( g \), acts in the downward direction.

The temperature on the disk is \( T_w \) and the temperature on the film surface is \( T_0 \). Besides, the ambient pressure on the film surface is constant at \( p_0 \) and we can safely say the pressure is a function of \( z \) only.

The nanofluid is a two component mixture with the following assumptions: incompressible; no-chemical reaction; negligible radiative heat transfer; nano-solid-particles and the base fluid are in thermal equilibrium with no slip between them. The thermo physical properties of the nanofluid are given in Table 1.

Neglecting viscous dissipation, the continuity, momentum and energy equations for steady state are given in the Equations (1–5):

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1}
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{\mu_{nf}}{\rho_{nf}} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + g \sin \beta \tag{2}
\]

\[
u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = \frac{\mu_{nf}}{\rho_{nf}} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \tag{3}
\]

\[
u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = \frac{\mu_{nf}}{\rho_{nf}} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - g \cos \beta \frac{p_{nf}}{\rho_{nf}} \tag{4}
\]

\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \frac{k_{nf}}{(\rho C_p)_{nf}} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \tag{5}
\]

In the above equations, \( u, v, \) and \( w \) indicate the velocity components in the \( x, y, \) and \( z \) directions, respectively.
The effective density ($\rho_{nf}$), the effective heat capacity ($\rho C_p)_f$ of the nanofluid and the effective heat capacity ($\rho C_p)_f$ of the nanofluid are defined by Equation (6) (Khanafeder et al., 2003):

$$\rho_{nf} = (1 - \phi) \rho_f + \phi \rho_s, \quad \mu_{nf} = \frac{\mu_f}{(1 - \phi)^{\frac{3}{2}}},$$

(\rho C_p)_{nf} = (1 - \phi) (\rho C_p)_f + \phi (\rho C_p)_s

The effective thermal conductivity of the nanofluid can be approximated by the Maxwell-Garnett (MG) model as given in Equation (7) (Jensen et al., 1991):

$$\frac{k_{nf}}{k_f} = \frac{k_x + 2k_f - 2\phi(k_f - k_x)}{k_x + 2k_f + \phi(k_f - k_x)}$$

Supposing zero slip on the disk and zero shear stress on the film surface, the boundary conditions are represented as Equation (8):

$$u = -\Omega y, \quad v = \Omega x, \quad w = 0, \quad T = T_w \text{ at } z = 0$$

If

$$u_z = 0, \quad v_z = 0, \quad w = -W, \quad T = T_0, \quad p = p_0 \text{ at } z = h$$

Wang introduced the following transform as given in Equation (9) (Jensen et al., 1991):

$$u = -\Omega y \quad g(\eta) + \Omega x \quad f'(\eta) + g \quad k(\eta) \sin \frac{\beta}{\Omega}$$

$$v = \Omega x \quad g(\eta) + \Omega y \quad f'(\eta) + g \quad s(\eta) \sin \frac{\beta}{\Omega}$$

$$w = -2\sqrt{\Omega \nu_{nf}} \quad f(\eta)$$

$$T = (T_0 - T_w) \quad \theta(\eta) + T_w$$

where $\eta$ was introduced as given in Equation (10):

$$\eta = \frac{z}{\sqrt{\nu_{nf}}}$$

When Equation (1) automatically is satisfied, then Equations (2) and (3) can be rewritten as Equations (11–14):

$$f''' - (f')^2 + g^2 + 2ff'' = 0$$

(11)

$$g'' - 2gf' + 2fg' = 0$$

(12)

$$k' - kff' + sg + 2k' = 1 = 0$$

(13)

$$s'' - kg - sf' + 2fs' = 0$$

(14)

If the temperature is a function of the distance $z$ only, then Equation (5) can be rewritten as given in Equation (15):

$$\theta'' + 2\Pr \frac{A_2}{A_1} \frac{A_3}{A_4} \theta' \theta' = 0,$$

$$A_1 = \frac{\rho_{nf}}{\rho_f}, \quad A_2 = \frac{\mu_{nf}}{\mu_f}, \quad A_3 = \frac{(\rho C_p)_nf}{((\rho C_p)_f)_{nf}}, \quad A_4 = \frac{k_{nf}}{k_f}$$

where $Pr = \frac{\nu}{\alpha}$ is the Prandtl number of base fluid.

The boundary conditions for Equations (11–15) areas given in Equation (16):

$$f(0) = 0, \quad f'(0) = 0, \quad f''(\delta) = 0,$$

$$g(0) = 0, \quad g'(0) = 0, \quad k(0) = 0, \quad k'(0) = 0, \quad s(0) = 0, \quad s'(0) = 0,$$

$$\theta(0) = 0, \quad \theta'(0) = 1$$

and $\delta$ is the constant normalized thickness as presented in Equation (17):

$$\delta = h \sqrt{\frac{\Omega}{\nu_{nf}}}$$

which is known through the condensation or spraying velocity as given in Equation (18):

$$f(\delta) = \frac{W}{2\sqrt{\Omega \nu_{nf}}} = \alpha$$

In this case the non-dimensional Nusselt number is obtained using Equation (19):

$$\text{Nu} = \frac{k_{nf}}{k_f} \frac{(\delta)}{(T_0 - T_w)} = A_0 \delta \theta'(0)$$

3. Numerical and analytical applied methods

3.1. Numerical approaches

3.1.1 Fourth-order Runge-Kutta-Fehlberg method

As already mentioned, the current problem type is boundary value problem (BVP) and an appropriate method needs to be selected for this. The available sub-methods in the Maple 15.0 are a combination of the base schemes; trapezoid or midpoint method. There are two major considerations when choosing a method for a problem. The trapezoid method is generally efficient for typical problems, but the midpoint method is capable of handling harmless end-point singularities whereas the trapezoid method cannot. The midpoint method, also known as the fourth-order Runge-Kutta-Fehlberg method, improves the Euler method by adding a midpoint in the step which increases the accuracy by one order. Thus, the midpoint method is used as a suitable numerical technique in this approach. (Aziz, 2006).

3.1.2 Shooting method

Here mathematic10.0 software have been applied to solve BVP by using shooting numerical method. The method will shoot out paths in various directions until a path that has the required boundary value is found. Shooting method is used to compare the results with results obtained via DTM.
### 3.2 Analytical solution

#### 3.2.1 Least square method

LSM is one of the approximation techniques for solving differential equations and this method is also called the weighted residual methods (WRMs). Here a differential operator $D$ is acted on a function $u$ to produce a function $p$ (Hatami & Ganji, 2014a, 2014b) as shown in Equation (20):

$$ D(u(x)) = p(x) $$

(20)

It is considered that $u$ is approximated by a function $\tilde{u}$, which is a linear combination of basic functions chosen from a linearly independent set. That is represented in Equation (21) as,

$$ u \cong \tilde{u} = \sum_{i=1}^{n} c_i \varphi_i $$

(21)

Now, when substituted into the differential operator, $D$, the result of the operations generally isn’t $p(x)$. Hence an error or residual will exist as shown in Equation (22):

$$ R(x) = D(\tilde{u}(x)) - p(x) \neq 0 $$

(22)

The notion in WRMs is to force the residual to zero in some average sense over the domain. That is written as Equation (23):

$$ \int_{X} R(x) W_i(x) = 0 \quad i = 1, 2, \ldots, n $$

(23)

where the number of weight functions $W_i$ is exactly equal to the number of unknown constants $c_i$ in $\tilde{u}$. If the continuous summation of all the squared residuals is minimized, the rationale behind the name can be seen. In other words, a minimum of Equation 24 can be seen.

$$ S = \int_{X} R(x) R(x) dx = \int_{X} R^2(x) dx $$

(24)

In order to achieve a minimum of this scalar function, the derivatives of $S$ with respect to all the unknown parameters must be zero. That is shown in Equation (25),

$$ \frac{\partial S}{\partial c_i} = 2 \int_{X} R(x) \frac{\partial R}{\partial c_i} dx = 0 $$

(25)

Comparing with Equation (23), the weight functions are seen to be as represented in Equation (26)

$$ W_i = 2 \frac{\partial R}{\partial c_i} $$

(26)

However, the “2” coefficient can be dropped, since it cancels out in the equation. Therefore the weight functions for the LSM are just the derivatives of the residual with respect to the unknown constants as given in Equation (27):

$$ W_i = \frac{\partial R}{\partial c_i} $$

(27)

Now, we want to apply this method to the present problem. Because trial functions must satisfy the boundary conditions in Equation (16), so they will be considered as Equation (28),

$$ \begin{align*}
 f(\eta) &= c_1 \eta^2 (\eta - \delta)^3 + c_2 \left( \frac{\eta^3}{6} - \frac{\delta \eta^2}{2} \right) + \ldots \\
 g(\eta) &= 1 + c_1 \eta (\eta - \delta)^2 + c_4 \left( \frac{\eta^2}{2} - \delta \eta \right) + \ldots \\
 k(\eta) &= c_5 \eta (\eta - \delta) + c_6 \left( \frac{\eta^2}{2} - \delta \eta \right) + \ldots \\
 s(\eta) &= c_7 \eta (\eta - \delta)^2 + c_8 \left( \frac{\eta^2}{2} - \delta \eta \right) + \ldots \\
 \theta(\eta) &= \frac{1}{\delta} + c_9 \eta (\eta - \delta) + c_{10} \eta (\eta - \delta)^2 + \ldots 
\end{align*} $$

(28)

It’s necessary to inform that every trial function that satisfy the boundary condition of the problem can be used and its accuracy can be improved by the number of its terms. In this problem, we have five coupled equations (Equations (11–15)) so, five residual functions will appear. Also, in this article we considered two unknown coefficient for each trial function, so 10 unknown coefficients will be seen ($c_1$–$c_{10}$). By substituting the residual functions, $R_1(c_1$–$c_{10})$, $R_2(c_1$–$c_{10})$, $R_3(c_1$–$c_{10})$, $R_4(c_1$–$c_{10})$ and $R_5(c_1$–$c_{10})$, into Equation (25), a set of ten equations will appear and by solving this system of equations, coefficients $c_1$–$c_{10}$ will be determined. For example, using LSM for Cu-water nanofluid with $\phi = 0.04$, Pr = 6.2 and $\delta = 0.5$

Equation (29) will be obtained

$$ \begin{align*}
 f(\eta) &= -0.008817284949\eta^2(\eta - 0.5)^3 - 0.9607426021 \left( \frac{\eta^3}{6} - \frac{\eta^2}{4} \right) \\
 g(\eta) &= 1 + 0.0779439347\eta(\eta - 0.5)^2 + 0.1969431465 \left( \frac{\eta^2}{2} - \frac{\eta}{2} \right) \\
 k(\eta) &= 0.008808870176\eta(\eta - 0.5)^2 - 0.9838801251 \left( \frac{\eta^2}{2} - \frac{\eta}{2} \right) \\
 s(\eta) &= 0.03931075727\eta(\eta - 0.5)^2 + 0.9985540140 \left( \frac{\eta^2}{2} - \frac{\eta}{2} \right) \\
 \theta(\eta) &= 2\eta - 0.1982408923\eta(\eta - 0.5) - 0.2492463454\eta(\eta - 0.5)^2 
\end{align*} $$

(29)

In the same manner for Cu-water nanofluid when $\delta = 1$ it can represented as Equation (30):

$$ \begin{align*}
 f(\eta) &= -0.01914194896\eta^2(\eta - 1)^3 - 0.6564549829 \left( \frac{\eta^3}{6} - \frac{\eta^2}{2} \right) \\
 g(\eta) &= 1 + 0.08162945698\eta(\eta - \delta)^2 + 0.4634992471 \left( \frac{\eta^2}{2} - \eta \right) \\
 k(\eta) &= 0.03926545013\eta(\eta - 1)^2 - 0.8513453189 \left( \frac{\eta^2}{2} - \eta \right) \\
 s(\eta) &= 0.04388591688\eta(\eta - 1)^2 + 0.2695506788 \left( \frac{\eta^2}{2} - \eta \right) \\
 \theta(\eta) &= \eta - 0.4379485922\eta(\eta - 1) - 0.2403810627\eta(\eta - 1)^2 
\end{align*} $$

(30)
Many advantages of LSM compared to other analytical and numerical methods make it more valuable and motivate researchers to use it for solving heat transfer problems. Some of these advantages are listed below (Hatami & Ganji, 2014a):

a. It solves the equations directly and no further simplifications need to be done. For example it solves power nonlinear terms without expanding or using Taylor expansion against differential transformation method (DTM).

b. It does not require any perturbation, linearization or small parameter versus homotopy perturbation method (HPM) and parameter perturbation method (PPM).

c. It is simple and powerful compared to the other numerical methods and reaches the final results faster than numerical procedures. Furthermore while its results are acceptable and have excellent agreement with numerical outcomes, its accuracy can be increased by increasing the statements of the trial functions.

d. It does not require to determine the auxiliary parameter and auxiliary function versus HAM.

### 3.2.2 Differential transform method

Equations (11–15) is transformed to one-dimensional differential transform, using Equation (16) which has resulted in Equations (31–45) Zhou (1986):

\[
f'' \rightarrow (i + 3)(i + 2)(i + 1)F(i + 3)
\]  
(31)

\[
f''' \rightarrow \sum_{r=0}^{i} (i + 2 - r)(i + 1 - r)F(r)F(i + 2 - r)
\]  
(32)

\[
f^4 \rightarrow \sum_{r=0}^{i} (i + 1 - r)(r + 1)F(r + 1)F(i + 1 - r)
\]  
(33)

\[
g'' \rightarrow (i + 2)(i + 1)G(i + 2)
\]  
(34)

\[
k'' \rightarrow (i + 2)(i + 1)K(i + 2)
\]  
(35)

\[
s'' \rightarrow (i + 2)(i + 1)S(i + 2)
\]  
(36)

\[
\theta'' \rightarrow (i + 2)(i + 1)\Theta(i + 2)
\]  
(37)

\[
g^2 \rightarrow \sum_{r=0}^{i} G(r)G(i + r)
\]  
(38)

\[
g' \rightarrow \sum_{r=0}^{i} (i + 1 - r)F(r)G(i + 1 - r)
\]  
(39)

\[
gf \rightarrow \sum_{r=0}^{i} (i + 1 - r)G(r)F(i + 1 - r)
\]  
(40)

\[
gk \rightarrow \sum_{r=0}^{i} (i + 1 - r)F(r)K(i + 1 - r)
\]  
(41)

\[
kf \rightarrow \sum_{r=0}^{i} (i + 1 - r)K(r)F(i + 1 - r)
\]  
(42)

\[
fs' \rightarrow \sum_{r=0}^{i} (i + 1 - r)F(r)S(i + 1 - r)
\]  
(43)

\[
sf' \rightarrow \sum_{r=0}^{i} (i + 1 - r)S(r)F(i + 1 - r)
\]  
(44)

\[
f\theta' \rightarrow \sum_{r=0}^{i} (i + 1 - r)F(r)\Phi(i + 1 - r)
\]  
(45)

Substituting Equations (31–45) into Equations (11–15) according to boundary conditions as presented in Equation (16), we arrive at Equations (46–51):

\[
(i + 3)(i + 2)(i + 1)F(i + 3) = \left\{ -\sum_{r=0}^{i} (i + 1 - r)(r + 1 + F(r + 1)F(i + 1 - r) \right. \\
+ \sum_{r=0}^{i} G(r)G(i + r) \\
+ 2 \sum_{r=0}^{i} (i + 1 - r)i + 2 - rF(i + 2 - r)F(r) \\
i + 2)(i + 1)G(i + 2) = \left\{ -2 \sum_{r=0}^{i} (i + 1 - r)G(r)F(i + 1 - r) \right. \\
+ \sum_{r=0}^{i} S(r)G(i + r) \\
+ 2 \sum_{r=0}^{i} (i + 1 - r)F(r)K(i + 1 - r) + \delta_0(i) \\
(i + 2)(i + 1)\Phi(i + 2) = \left\{ -\sum_{r=0}^{i} G(r)K(i + r) \\
- \sum_{r=0}^{i} (i + 1 - r)S(r)F(i + 1 - r) \\
+ 2 \sum_{r=0}^{i} (i + 1 - r)F(r)S(i + 1 - r) \\
\right\} \\
(i + 2)(i + 1)K(i + 2) = \left\{ -\sum_{r=0}^{i} G(r)K(i + r) \\
- \sum_{r=0}^{i} (i + 1 - r)S(r)F(i + 1 - r) \\
+ 2 \sum_{r=0}^{i} (i + 1 - r)F(r)S(i + 1 - r) \\
\right\} \\
\]

(46)

(47)

(48)

(49)

(50)

(51)
where $F(i)$, $G(i)$, $K(i)$, $S(k)$ and $\Phi(k)$ are the transformation functions of $f(i)$, $g(i)$, $k(i)$, $s(i)$ and $\theta(k)$ respectively and are defined by Equations (52–56):

\[
 f(\eta) = \sum_{i=0}^{\infty} F(i) \eta^i
\]  
(52)

\[
 g(\eta) = \sum_{i=0}^{\infty} G(i) \eta^i
\]  
(53)

\[
 k(\eta) = \sum_{i=0}^{\infty} K(i) \eta^i
\]  
(54)

\[
 s(\eta) = \sum_{i=0}^{\infty} S(i) \eta^i
\]  
(55)

\[
 \theta(\eta) = \sum_{k=0}^{\infty} \Phi(k) \eta^k
\]  
(56)

After using iteration $i = 0, 1, \ldots$, and using Equations (46–51) and Equations (52–56), when $Pr = 1$, $\varphi = 0.4$ and $\delta = 1$, we obtained Equations (57–61):

\[
 f(\eta) = a_1 \eta^2 - \frac{\eta^3}{6} - \frac{\eta^4 a_2}{12} - \frac{1}{60} \eta^5 a_2^2 + \ldots
\]  
(57)

\[
 g(\eta) = 1 + \frac{2 \eta^3 a_1}{3} + \eta a_2 - \frac{\eta^4 a_2}{15} + \frac{1}{12} \eta^4 \left(-2a_1 a_2 + 2\left(-\frac{1}{2} + 2a_1 a_2\right)\right) + \ldots
\]  
(58)

Figure 2. Comparison of LSM and NUM for Cu-water nanofluid with $\phi = 0.04$ and $\delta = 0.5, 0.75$ and 1, respectively.
On which we applied boundary conditions as given in Equation (16) to find $a_i$, $i = 1, 2, \ldots, 5$.

### 4. Results and discussions

The objective of the present study was to apply LSM to obtain an explicit analytic solution of three-dimensional problem of condensation nanofluid film on inclined rotating disk (Figure 1).

The comparison between the obtained results by LSM and DTM with that of numerical results by Maple and Mathematica are shown in Figures 2 and 3. This accuracy gives high confidence validity of this problem and reveals an excellent agreement of engineering accuracy for us. This investigation was completed by depicting the effects of some...
important parameters to evaluate how these parameters influence on this fluid. Effect of normalized thickness on velocity and temperature profiles by numerical and analytical solutions are shown in Figures 2 and 3. Increasing normalized thickness leads to increase in $f$, $f'$ and decrease in $g$, $\theta$. Effect of normalized thickness on $k$ and $s$ are similar to those of $f'$ and $g$, respectively.

Figure 2 shows the effect of nanoparticles volume fraction for Cu-water and the effect of nanoparticles material on temperature profile. The sensitivity of thermal boundary layer thickness to volume fraction of nanoparticles is related to the increased thermal conductivity of the nanofluid. In fact, higher values of thermal conductivity are accompanied by higher values of thermal diffusivity. The high values of thermal diffusivity cause a drop in the temperature gradients and accordingly increase the boundary thickness. Also this figure shows that selecting aluminium oxide (Al$_2$O$_3$) as nanoparticle obtained more enhancement in temperature profile.

5. Conclusions

In this paper, three-dimensional nanofluid flow of condensation film on inclined rotating disk was solved via a sort of analytical methods (least square method (LSM) and differential transform method (DTM)) and numerical methods (the Runge-Kutta method of order 4 and Shooting method). Present analytical methods is a powerful approach for solving nonlinear differential equations such as this problem, also it can be observed that there is a good agreement between the present and numerical results. Results show that increasing nanofluid volume fraction leads to increase in temperature profile and the highest temperature was obtained for Al$_2$O$_3$-water in this case.

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Disclosure statement

We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work. There is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of, the manuscript entitled.

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