Liouville-Type Theorems for Steady Flows of Degenerate Power Law Fluids in the Plane

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Abstract. We extend the Liouville-theorems of Gilbarg and Weinberger and of Koch, Nadirashvili, Seregin and Sverák valid for the stationary variant of the classical Navier–Stokes equations in 2D to the degenerate power law fluid model.

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1. Introduction

To begin with we look at a velocity field $u: \mathbb{R}^2 \to \mathbb{R}^2$ and a pressure function $\pi: \mathbb{R}^2 \to \mathbb{R}$ satisfying the stationary equations of Navier–Stokes
\begin{equation}
\begin{aligned}
-\Delta u + u^k \partial_k u + \nabla \pi &= 0, \\
\text{div } u &= 0 \quad \text{on } \mathbb{R}^2,
\end{aligned}
\end{equation}
which correspond to the flow of an incompressible Newtonian fluid with constant viscosity (w.l.o.g. equal to 1). Here we study entire solutions, and a natural question is the search for suitable conditions which force $u$ (and thereby $\pi$) to be constant. We recall two prominent examples of such Liouville-type results for the Navier–Stokes equation (1.1): if $u$ is a finite energy solution, i.e. if we have
\begin{equation}
\int_{\mathbb{R}^2} |\nabla u|^2 \, dx < \infty, \tag{1.2}
\end{equation}
then Gilbarg and Weinberger [13] proved $u = \text{const}$ making extensive use of the fact that the vorticity function $\omega := \partial_2 u_1 - \partial_1 u_2$ satisfies a nice elliptic equation. Recently, Koch et al. [14] discussed the instationary variant of (1.1) and, as a byproduct of their investigations, they showed that in the stationary case (1.2) can be replaced by
\begin{equation}
\sup_{x \in \mathbb{R}^2} |u(x)| < \infty \tag{1.3}
\end{equation}
implying the constancy of the vector field $u$. In connection with the Navier–Stokes equation we like to remark that according to [19] the hypothesis
\begin{equation}
\int_{\mathbb{R}^2} |u|^t \, dx < \infty \quad \text{for some } t > 1
\end{equation}
(replacing (1.1) or (1.3)) implies the vanishing of $u$, whereas in [6] it is observed that $u = \text{const}$ is still true if the growth of $|u(x)|$ as $|x| \to \infty$ is not too strong.
In [3, 5, 19] the situation for generalized Newtonian fluids being either of shear thickening or shear thinning type is studied. For this case Eq. (1.1) has to be replaced by

\[
- \text{div} \left[ DH(\varepsilon(u)) \right] + u^k \partial_k u + \nabla \pi = 0, \quad \text{div } u = 0 \quad \text{on } \mathbb{R}^2
\]  

(1.4)

with a strictly convex potential \( H \) of class \( C^2 \) acting on symmetric \((2 \times 2)\)-matrices (\( \varepsilon(u) \) denoting the symmetric gradient of the velocity field \( u \)) and being of the form

\[
H(\varepsilon) = h(|\varepsilon|)
\]

(1.5)

for a function \( h: [0, \infty) \rightarrow [0, \infty) \) for which \( \mu(t) := h'(t) / t \) either decreases or increases. Note that according to (1.5) we have \( DH(\varepsilon) = \mu(|\varepsilon|) \varepsilon \), thus \( \mu \) plays the role of a shear dependent viscosity. For further physical and mathematical explanations we refer to the monographs [8, 9, 15, 16] or [4].

The most severe restriction concerns the existence and the behaviour of \( D^2 H(0) \), which in particular means that we require

\[
D^2 H(0)(\varepsilon, \varepsilon) \geq \lambda |\varepsilon|^2
\]

(1.6)

for some positive constant \( \lambda \). Assuming (1.6) it is shown: suppose that \( u \in C^1(\mathbb{R}^2, \mathbb{R}^2) \) is an entire weak solution of (1.4), i.e. it holds \( \text{div } u = 0 \) together with

\[
0 = \int_{\mathbb{R}^2} DH(\varepsilon(u)) : \varepsilon(\varphi) \, dx + \int_{\mathbb{R}^2} u^k \partial_k u^i \varphi^i \, dx
\]

(1.7)

for all \( \varphi \in C^\infty_0(\mathbb{R}^2, \mathbb{R}^2) \) such that \( \text{div } \varphi = 0 \). Then we have \( u \equiv \text{const} \), if either (1.3) holds or if we replace (1.2) through the appropriate hypothesis

\[
\int_{\mathbb{R}^2} h(|\nabla u|) \, dx < \infty.
\]

(1.8)

Clearly these results apply to non-degenerate \( p \)-fluids for which \( h(t) = (1 + t^2)^{p/2} \) (modulo physical constants) with exponent \( p \in (1, \infty) \) but not to the degenerate power law model, i.e. to the potential \( H \) with function \( h(t) = t^p \).

In the present paper we are going to investigate the degenerate \( p \)-case, i.e. from now on we assume that \( H \) is given by

\[
H(\varepsilon) = |\varepsilon|^p
\]

for some \( 1 < p < \infty \) and that \( u \in C^1(\mathbb{R}^2, \mathbb{R}^2) \) with \( \text{div } u = 0 \) solves Eq. (1.7). Then our results are as follows:

**Theorem 1.1.** Suppose that \( 1 < p \leq 2 \).

i) If \( u \) belongs to the space \( L^\infty(\mathbb{R}^2, \mathbb{R}^2) \), i.e. if condition (1.3) holds, then \( u \) is a constant vector.

ii) If \( p < 2 \), if

\[
0 < \alpha < \frac{2 - p}{6 + p}
\]

(1.9)

and if we have

\[
\limsup_{|x| \to \infty} |u(x)||x|^{-\alpha} < \infty,
\]

(1.10)

then the conclusion of i) holds.
Remark 1.1. For the choice $p = 2$ we reproduce the contribution of Koch, Nadirashvili, Seregin and Sverák [14], for $1 < p < 2$ condition (1.10) allows even a certain growth of $|u(x)|$ as $|x| \to \infty$. In Theorem 1.5 we will discuss in more detail the admissible a priori growth rates of $u$ in the case $p = 2$.

The next two theorems extend the Liouville result of Gilbarg and Weinberger [13] to exponents $p$ not necessarily equal to 2.

**Theorem 1.2.** Let $6/5 < p \leq 2$ and assume that
\[ \int_{\mathbb{R}^2} |\nabla u|^p \, dx < \infty, \]
which means that (1.8) is satisfied. Then $u$ has to be constant.

**Theorem 1.3.** Theorem 1.2 remains valid for exponents $p \in [2, 3]$.

Theorem 1.4 is the counterpart to Theorem 1.1, ii) for $p > 2$ involving formally the same exponent $(p - 2)/(p + 6)$.

**Theorem 1.4.** Let $p > 2$ and let $u_\infty \in \mathbb{R}^2$ denote a vector such that

i) in case $2 < p < 6$
\[ \sup_{|x| \geq R} |u(x) - u_\infty| |x|^{\frac{p-2}{p+6}} \to 0 \quad \text{as } R \to \infty; \]  

ii) in case $p = 6$:
\[ \limsup_{|x| \to \infty} |u(x) - u_\infty| |x|^{\frac{1}{2}} < \infty; \]  

iii) in case $p > 6$:
\[ \sup_{|x| \geq R} |u(x) - u_\infty| |x|^{\frac{1}{2}} \to 0 \quad \text{as } R \to \infty. \]  

Then $u \equiv u_\infty$ follows.

Remark 1.2. It remains an open question, if in case $p > 2$ bounded solutions are constant without imposing a decay condition.

An inspection of the proofs of Theorem 1.1–1.4 will show:

**Corollary 1.1.** Let $p \in (1, \infty)$ and suppose that $u: \mathbb{R}^2 \to \mathbb{R}^2$ is a solution of the $p$-Stokes system in the plane, i.e. a solution of (1.7) with $H(\varepsilon) = |\varepsilon|^p$, where now the convective term is neglected. Then $u$ is a constant vector if either $u \in L^\infty(\mathbb{R}^2, \mathbb{R}^2)$ or if $u$ is of finite energy, i.e. $\int_{\mathbb{R}^2} |\nabla u|^p \, dx < \infty$.

**Remark 1.3.** Clearly Corollary 1.1 can be generalized in the sense that for $1 < p < 2$ a certain growth of $u$ can be included which might be even stronger in comparison to the formulation given in (1.9) and (1.10). We leave the details to the reader.

We finish this introduction with an extension of the Liouville results obtained in [14] and [6] for the case of the classical Navier–Stokes equation.

**Theorem 1.5.** Suppose that $u: \mathbb{R}^2 \to \mathbb{R}^2$ is a solution of (1.1) such that
\[ \limsup_{|x| \to \infty} |u(x)||x|^{-\alpha} < \infty \]  

for some $\alpha < 1/3$. Then the constancy of $u$ follows.

**Remark 1.4.** It would be interesting to know the optimal bound for the number $\alpha$ occurring in (1.14).
Our paper is organized as follows: in Sect. 2 we give estimates for the energy \( \int_{B_r(x_0)} |\nabla u|^p \, dx, 1 < p < \infty \), on disks in terms of the radius under various hypotheses imposed on \( u \). Section 3 is devoted to the case \( 1 < p < 2 \), i.e. we will present the proofs of Theorem 1.1 and of Theorem 1.2 by combining the results of Sect. 2 with estimates for the “second derivatives” due to Wolf [18].

Since these estimates are not available for \( p > 2 \), we have to find alternatives leading to Theorem 1.3 and to Theorem 1.4. This is done in Sect. 4.

In Sect. 5 we give a proof of Theorem 1.5. Moreover, we collect some technical tools in an Appendix.

2. Estimates for the \( p \)-Energy on Disks

In this section we describe the growth of the energy \( \int_{B_r(x_0)} |\nabla u|^p \, dx \) of weak solutions \( u \) to (1.7) in terms of the radius of the disk under various conditions concerning the growth of \( u \).

Lemma 2.1. Let \( u \in C^1(\mathbb{R}^2, \mathbb{R}^2) \), \( \text{div} \ u = 0 \), denote a solution of (1.7) for the choice \( H(\varepsilon) = |\varepsilon|^p \) with exponent \( p \in (1, \infty) \).

i) Then, for any real number \( \beta < 1 \), it holds

\[
\int_{B_r(x_0)} |\nabla u|^p \, dx \leq c \left[ r^{-p} \int_{B_{2r}(x_0)} |u|^p \, dx + r^{-1+\beta} \int_{B_{2r}(x_0)} |u|^2 \, dx \right. \\
+ r^{-1} \int_{B_{2r}(x_0)} |u|^3 \, dx + r^{-1-\beta} \int_{B_{2r}(x_0)} |u|^4 \, dx \right]
\]

for all disks \( B_{2r}(x_0) \). Here, the positive constant \( c \) is independent of \( x_0, r \) and \( u \).

ii) If \( u \) is bounded, then it follows by choosing \( \beta = 0 \)

\[
\int_{B_r(x_0)} |\nabla u|^p \, dx \leq c \left( \|u\|_{L^\infty(\mathbb{R}^2)} \right) \left[ r^{-p} \int_{B_{2r}(x_0)} |u|^p \, dx + r^{-1} \int_{B_{2r}(x_0)} |u|^2 \, dx \right]
\]

again for all disks. In particular it holds

\[
\int_{B_R(0)} |\nabla u|^p \, dx \leq c \left( \|u\|_{L^\infty(\mathbb{R}^2)} \right) R
\]

for radii \( R \geq 1 \).

If \( u_\infty \in \mathbb{R}^2 \) is some fixed vector, then (2.2) is also valid for the function \( \tilde{u} := u - u_\infty \) in place of \( u \).

iii) Suppose that

\[
\limsup_{|x| \to \infty} |u(x)| |x|^{-\gamma} < \infty
\]

for some number \( \gamma \) such that

\[
\gamma \in \begin{cases} 
[0, 1), & \text{if } 1 < p \leq 2, \\
[-1/2, 0), & \text{if } p > 2.
\end{cases}
\]

Then it holds for any \( R \geq 1 \)

\[
\int_{B_R(0)} |\nabla u|^p \, dx \leq c R^{1+3\gamma}.
\]
Proof of Lemma 2.1. Ad i) & ii).
Consider \( \eta \in C_0^\infty (B_2(x_0)) \) such that \( 0 \leq \eta \leq 1, \eta \equiv 1 \) on \( B_1(x_0) \) and \( |\nabla \eta| \leq c/r \). In Eq. (1.7) we let \( \varphi = \eta^2 u - w \), where the field \( w \) is defined on \( B_{2r}(x_0) \), vanishing on \( \partial B_{2r}(x_0) \) with the properties
\[
\text{div } w = \text{div}(\eta^2 u) = \nabla \eta^2 \cdot u \quad \text{on } B_{2r}(x_0),
\]
\[
\|\nabla w\|_{L^2(B_{2r}(x_0))} \leq c\|\nabla \eta^2 \cdot u\|_{L^2(B_{2r}(x_0))}.
\]
Note that (2.6) holds with the same field \( w \) both for the choice \( q = 2 \) and for the choice \( q = p \) (cf. Lemma A.1). The integer \( l \) will be determined later. We have
\[
\int_{B_{2r}(x_0)} DH(\varepsilon(u)) : \varepsilon(u) \eta^{2l} \, dx = -\int_{B_{2r}(x_0)} DH(\varepsilon(u)) : (\nabla \eta^{2l} \otimes u) \, dx
\]
\[
+ \int_{B_{2r}(x_0)} DH(\varepsilon(u)) : \varepsilon(w) \, dx
\]
\[
- \int_{B_{2r}(x_0)} u^k \partial_k u \cdot \eta^{2l} \, dx + \int_{B_{2r}(x_0)} u^k \partial_k u \cdot w \, dx
\]
\[
=: T_1 + T_2 + T_3 + T_4. \tag{2.7}
\]
Young’s inequality yields for any \( \delta > 0 \)
\[
|T_1| \leq c \int_{B_{2r}(x_0)} |\varepsilon(u)|^{p-1} \eta^{2l-1} |\nabla \eta||u| \, dx
\]
\[
\leq \delta \int_{B_{2r}(x_0)} |\varepsilon(u)|^p \eta^{2l-1} \frac{r^{2l-1}}{r^p} \, dx + c(\delta) \int_{B_{2r}(x_0)} |\nabla \eta|^p |u|^p \, dx
\]
\[
\leq \delta \int_{B_{2r}(x_0)} \eta^{2l} |\varepsilon(u)|^p \, dx + c(\delta)r^{-p} \int_{B_{2r}(x_0)} |u|^p \, dx,
\]
provided that we choose \( l \) so large that \((2l - 1)p/(p - 1) \geq 2l \). For small enough \( \delta \) the bound for \(|T_1|\) in combination with (2.7) yields
\[
\int_{B_{2r}(x_0)} |\varepsilon(u)|^p \eta^{2l} \, dx \leq c \left[ r^{-p} \int_{B_{2r}(x_0)} |u|^p \, dx + |T_2| + |T_3| + |T_4| \right]. \tag{2.8}
\]
Next we use (2.6) for \( q = p \) and obtain by Young’s inequality
\[
|T_2| \leq \delta \int_{B_{2r}(x_0)} |\varepsilon(u)|^p \, dx + c(\delta) \int_{B_{2r}(x_0)} |\varepsilon(w)|^p \, dx
\]
\[
\leq \delta \int_{B_{2r}(x_0)} |\varepsilon(u)|^p \, dx + c(\delta)r^{-p} \int_{B_{2r}(x_0)} |u|^p \, dx,
\]
thus by (2.8)
\[
\int_{B_r(x_0)} |\varepsilon(u)|^p \, dx \leq \delta \int_{B_{2r}(x_0)} |\varepsilon(u)|^p \, dx + c(\delta)r^{-p} \int_{B_{2r}(x_0)} |u|^p \, dx + |T_3| + |T_4|. \tag{2.9}
\]
Finally we observe using an integration by parts
\[
|T_3| = \frac{1}{2} \int_{B_{2r}(x_0)} u^k |u|^2 \partial_k \eta^{2l} \, dx \leq cr^{-1} \int_{B_{2r}(x_0)} |u|^3 \, dx \tag{2.10}
\]
and
\[ T_4 = - \int_{B_{2r}(x_0)} u^i u^k \partial_k w^i \, dx, \]
thus
\[ |T_4| \leq \left[ \int_{B_{2r}(x_0)} |u|^4 \, dx \right]^{\frac{1}{2}} \left[ \int_{B_{2r}(x_0)} |\nabla w|^2 \, dx \right]^{\frac{1}{2}}, \]
and the use of (2.6) now with the choice \( q = 2 \) shows
\[ |T_4| \leq \left[ \int_{B_{2r}(x_0)} |u|^4 \, dx \right]^{\frac{1}{2}} \left[ r^{-2} \int_{B_{2r}(x_0)} |u|^2 \, dx \right]^{\frac{1}{2}} \]
\[ = \left[ r^{-1-\beta} \int_{B_{2r}(x_0)} |u|^4 \, dx \right]^{\frac{1}{2}} \left[ r^{-1+\beta} \int_{B_{2r}(x_0)} |u|^2 \, dx \right]^{\frac{1}{2}} \]
\[ \leq c r^{-1+\beta} \int_{B_{2r}(x_0)} |u|^2 \, dx + c r^{-1-\beta} \int_{B_{2r}(x_0)} |u|^4 \, dx. \] (2.11)

Combining (2.9) with (2.10) and (2.11) and using Lemma A.4 it follows
\[ \int_{B_r(x_0)} |\varepsilon(u)|^p \, dx \leq c \left[ r^{-p} \int_{B_{2r}(x_0)} |u|^p \, dx + r^{-1+\beta} \int_{B_{2r}(x_0)} |u|^2 \, dx \right. \]
\[ + r^{-1} \int_{B_{2r}(x_0)} |u|^3 \, dx + r^{-1-\beta} \int_{B_{2r}(x_0)} |u|^4 \, dx \].

Applying Korn’s inequality in \( W^1_p(B_{2r}(x_0), \mathbb{R}^2) \) (cf. Lemma A.2) we arrive at (2.1). From (2.1) the claims (2.2) and (2.3) immediately follow.

For the second statement of ii) we observe that \( \tilde{u} = u - u_\infty \) solves Eq. (1.7) with the additional term \( \int u^k \partial_k \tilde{u} \cdot \varphi \, dx \) and the choice \( \varphi = \eta^2 \tilde{u} - \tilde{w} \) (with an obvious meaning of \( \tilde{w} \)) leads to (2.2) for \( \tilde{u} \) with the help of elementary identities like
\[ u^k_{\infty} \int_{B_{2r}(x_0)} \partial_k \tilde{u}^i \eta^2 \tilde{u}^i \, dx = -\frac{1}{2} u^k_{\infty} \int_{B_{2r}(x_0)} |\tilde{u}|^2 \partial_k \eta^2 \, dx. \]

Ad iii).

Suppose that we have
\[ \limsup_{|x| \to \infty} |u(x)| |x|^{-\gamma} < \infty \] (2.12)
with \( \gamma \) satisfying (2.4).

Case 1: \( \gamma \in [0, 1) \) and \( 1 < p \leq 2 \). In this case (2.12) implies the growth condition
\[ \sup_{B_R(0)} |u| \leq c R^{\gamma} \]
for all \( R \geq 1 \). (2.13)

Quoting inequality (2.1) choosing \( x_0 = 0, r = R \geq 1 \) and \( \beta = \gamma \), (2.13) gives
\[ \int_{B_R(0)} |\nabla u|^p \, dx \leq c [R^{2-p+\gamma \gamma} + R^{1+3\gamma}], \]
and since \( 2 - p + p \gamma \leq 1 + 3 \gamma \), we get (2.5).
Case 2: $\gamma \in [-1/2, 0)$ and $p > 2$. From (2.12) we deduce the boundedness of $u$ together with
\[ \sup_{R \leq |x| \leq 2R} |u| \leq R^\gamma \]  
for $R$ sufficiently large. We return to the beginning of the proof and replace $\varphi$ through the modified test-function (with $\eta$ as before and with $w^* \in \mathcal{W}^{1,q}_0(T_R(0), \mathbb{R}^2)$ given according to Lemma A.1– again we will make use both of the choice $q = 2$ and of the choice $q = p$ in this Lemma)
\[ \varphi^* = \begin{cases} u & \text{on } B_R(0), \\ \eta^2 u - w^* & \text{on } T_R(0), \end{cases} \]
where we always set
\[ T_R(x_0) := B_{2R}(x_0) - B_R(x_0). \]
We have
\[ \text{div } w^* = \text{div}(\eta^2 u) = \nabla \eta^2 \cdot u \quad \text{on } T_R(0), \]
\[ ||\nabla w^*||_{L^q(T_R(0))} \leq c||\nabla \eta^2 \cdot u||_{L^q(T_R(0))}. \]
Note that $\int_{T_R(0)} \text{div}(\eta^2 \cdot u) \, dx = 0$. We then obtain a version of (2.7) with $x_0 = 0$, $w$ being replaced by $w^*$ and where in $T_2$ and $T_4$ the integration is performed over the annulus $T_R(0)$. In place of (2.9) we get after specifying $c(\delta)$
\[ \int_{B_R(0)} |\varepsilon(u)|^p \, dx \leq \delta \int_{T_R(0)} |\varepsilon(u)|^p \, dx + c\delta^{1-p}R^{1+p} \int_{T_R(0)} |u|^p \, dx + [T_3] + [T_4]. \]  
(2.15)
For $T_3$ it holds (compare (2.10))
\[ |T_3| \leq cR^{-1} \int_{T_R(0)} |u|^3 \, dx \]
and for $T_4$ we just observe
\[ |T_4| \leq cR^{-1} \left[ \int_{T_R(0)} |u|^4 \, dx \right]^{1/2} \left[ \int_{T_R(0)} |u|^2 \, dx \right]^{1/2}. \]
Thus (2.15) implies (recalling (2.14))
\[ \int_{B_R(0)} |\varepsilon(u)|^p \, dx \leq \delta \int_{T_R(0)} |\varepsilon(u)|^p \, dx + c[\delta^{1-p}R^{2-p+\gamma} + R^{1+3\gamma}]. \]  
(2.16)
Since $u$ is bounded, we can apply (2.3) to the first term on the r.h.s. of (2.16), hence
\[ \int_{B_R(0)} |\varepsilon(u)|^p \, dx \leq c[\delta R + \delta^{1-p}R^{2-p+\gamma} + cR^{1+3\gamma}]. \]  
(2.17)
Suppose now that we have for some $n = 0, 1, 2$
\[ \int_{B_R(0)} |\varepsilon(u)|^p \, dx \leq cR^{1+n\gamma}, \]  
(2.18) which by (2.3) in fact is true in the case $n = 0$. Then, instead of (2.17), we have using assumption (2.18)
\[ \int_{B_R(0)} |\varepsilon(u)|^p \, dx \leq c[\delta R^{1+n\gamma} + \delta^{1-p}R^{2-p+\gamma} + cR^{1+3\gamma}]. \]  
(2.19)
We choose \( \delta = R^\gamma \) in (2.19):
\[
\int_{B_R(0)} |\varepsilon(u)|^p \, dx \leq c \left[ R^{1+(n+1)\gamma} + R^{1-\gamma p} R^{2-p+\gamma} + R^{1+3\gamma} \right] \\
\leq c R^{1+(n+1)\gamma},
\]
provided that we have \((n+1) \leq 3\) (which clearly is true since we suppose \( n \leq 2 \)—recall \( \gamma \leq 0 \) in the case under consideration) and if we have in addition
\[
\gamma + 2 - p \leq 1 + (n+1)\gamma \iff 1 - p \leq \gamma n.
\]
(2.21)

Note that for \( \gamma \in \left(-\frac{1}{2}, 0\right) \) and \( p \geq 2 \) (2.21) holds true up to the choice \( n = 2 \) and as the final result we obtain
\[
\int_{B_R(0)} |\varepsilon(u)|^p \, dx \leq c R^{1+3\gamma}.
\]
(2.22)

Applying the version of Korn’s inequality stated in Lemma A.2, \( \text{iii} \), to (2.22) we obtain
\[
\int_{B_R(0)} |\nabla u|^p \, dx \leq c \left[ R^{1+3\gamma} + R^{-p+2+p\gamma} \right]
\]
and thereby (2.5) which completes the proof of Lemma 2.1. \( \square \)

From Lemma 2.1 we immediately obtain

**Corollary 2.1.** Suppose that \( p > 2 \) and that
\[
\limsup_{|x| \to \infty} |u(x)||x|^{-\gamma} < \infty
\]
holds for some number \( \gamma < -1/3 \). Then \( u \) must be identically zero.

**Proof of Corollary 2.1.** W.l.o.g. we may assume \( \gamma \in (-1/2, -1/3) \) since otherwise we replace the (negative) exponent \( \gamma \) through \(-1/2\). But then (2.5) yields the claim by passing to the limit \( R \to \infty \). \( \square \)

### 3. The Case \( 1 < p < 2 \)

During this section we always assume that \( u \in C^1(\mathbb{R}^2, \mathbb{R}^2) \) is a solenoidal field satisfying (1.7) for the choice \( H(\varepsilon) = |\varepsilon|^p \) with exponent \( p \in (1, 2) \). Note that on account of Corollary I in the paper \([18]\) of Wolf weak solutions of (1.7) from the space \( W^{1}_{p,\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \) are of class \( C^1 \) if we require \( p > 3/2 \).

The proofs of Theorem 1.1 and Theorem 1.2 make extensive use of the following preliminary result, where we let

\[
V(\varepsilon) := \begin{cases} 
|\varepsilon|^{\frac{p-2}{2}} & \text{if } \varepsilon \neq 0, \\
0 & \text{if } \varepsilon = 0.
\end{cases}
\]

**Lemma 3.1.** The velocity field \( u \) is an element of the space \( W^{1}_{p,\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \) and for any disk \( B_r(x_0) \) it holds (recall \( T_r(x_0) = B_{2r}(x_0) - B_r(x_0) \))
\[
\int_{B_r(x_0)} V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \, dx \leq c \left[ r^{-2} \int_{T_r(x_0)} |\nabla u|^p \, dx + r^{-1} \int_{T_r(x_0)} |u| |\nabla u|^2 \, dx \right],
\]
(3.1)
where \( c \) denotes a finite constant independent of \( u, r \) and \( x_0 \).
Proof of Lemma 3.1. The existence of the second order weak derivatives in $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ has been established by Naumann [17] in Theorem 2 of his paper. Actually Naumann considers slow flows, i.e. the convective term is neglected, but his arguments cover the case of volume forces $f \in L^p_{\text{loc}}$, and since $u$ is a $C^1$-function, we just put $f := -u^k \partial_k u$.

For proving estimate (3.1) we benefit from the basic inequality (3.24) in Wolf’s paper [18]: let $\eta \in C_0^\infty(B_{2r}(x_0))$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_r(x_0)$ and $|\nabla \eta| \leq cr^{-1}$, $l = 1, 2$. Choosing

$$S_{ij} = \frac{\partial H}{\partial e_{ij}}, \quad \lambda = 0, \quad \xi = \eta, \quad \tilde{f} := -u^k \partial_k u$$

and using the symbol $\pi$ for the pressure we obtain from (3.24) in [18] (replacing $r$ by $2r$)

$$c(p) \int_{B_{2r}(x_0)} V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \eta^2 \, dx \leq \sum_{i=1}^6 I_i \quad (3.2)$$

with $I_i$ defined exactly as in the above reference and for a constant $c(p) > 0$. We have ($c$ denoting positive constants with values varying from line to line but being independent of $x_0$ and $r$)

$$|I_1| \leq c \int_{T_r(x_0)} |\varepsilon(u)|^{p-1} |\nabla u| [[|\nabla \eta|^2 + |\nabla^2 \eta|] \, dx$$

$$\leq cr^{-2} \int_{T_r(x_0)} |\nabla u|^p \, dx \quad (3.3)$$

and by Young’s inequality (using also the estimate $|\nabla^2 u| \leq c|\nabla \varepsilon(u)|$ and recalling the definition of $V$)

$$|I_2| \leq c \int_{B_{2r}(x_0)} |\varepsilon(u)|^{p-1} |\nabla^2 u| \eta |\nabla \eta| \, dx$$

$$\leq c \int_{B_{2r}(x_0)} V(\varepsilon(u)) |\nabla \varepsilon(u)| \eta |\varepsilon(u)|^{\frac{5}{2}} |\nabla \eta| \, dx$$

$$\leq \delta \int_{B_{2r}(x_0)} V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \eta^2 \, dx + c(\delta) \int_{T_r(x_0)} |\varepsilon(u)|^p |\nabla \eta|^2 \, dx.$$ 

Choosing $\delta$ small enough and quoting (3.3) we deduce from (3.2)

$$\int_{B_{2r}(x_0)} V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \eta^2 \, dx \leq c \left[ r^{-2} \int_{T_r(x_0)} |\nabla u|^p \, dx + |I_3 + I_4| + |I_5| + |I_6| \right]. \quad (3.4)$$

Next we rewrite the quantity $|I_3 + I_4|$ in the following form:

$$|I_3 + I_4| = \left| \int_{B_{2r}(x_0)} \pi \partial_k(\partial_i \eta^2 \partial_k u^i) \, dx \right| = \left| \int_{B_{2r}(x_0)} \pi \, \text{div} \, \varphi \, dx \right|,$$

where $\varphi^k := \partial_i \eta^2 \partial_k u^i$. From (1.4) it follows that

$$\int_{B_{2r}(x_0)} \pi \, \text{div} \, \varphi \, dx = \int_{B_{2r}(x_0)} DH(\varepsilon(u)) : \varepsilon(\varphi) \, dx + \int_{B_{2r}(x_0)} u^k \partial_k u \cdot \varphi \, dx,$$
hence
\[
|I_3 + I_4| \leq c \left[ \int_{B_2(x_0)} |\varepsilon(u)|^{p-1} |\nabla \eta|^2 |\nabla^2 u| \, dx + \int_{B_2(x_0)} |\varepsilon(u)|^{p-1} |\nabla^2 \eta|^2 |\nabla u| \, dx \right] \\
+ \int_{B_2(x_0)} u^k \partial_k u^i \partial_l \eta^2 \partial_i u^l \, dx \\
=: c(J_1 + J_2 + J_3).
\]

\(J_1\) is handled in the same way as \(I_2\), \(J_2\) corresponds to \(I_1\), thus we get from (3.4)
\[
\int_{B_r(x_0)} V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \, dx \leq c \left[ r^{-2} \int_{T_r(x_0)} |\nabla u|^p \, dx + |J_5| + |J_6| + J_3 \right].
\]

(3.5)

We estimate \(I_5\):
\[
|I_5| = \left| \int_{B_2(x_0)} u^k \partial_k u^i \partial_l \eta^2 \partial_i u^l \, dx \right| \leq r^{-1} \int_{T_r(x_0)} |u||\nabla u|^2 \, dx.
\]

For \(I_6\) it holds:
\[
|I_6| = \left| \int_{B_2(x_0)} u^k \partial_k u^i \partial_l u^i \eta^2 \, dx \right| = \left| \int_{B_2(x_0)} \partial_l (u^k \partial_k u^i \eta^2) \partial_i u^l \, dx \right| \\
= \left| \int_{B_2(x_0)} \partial_l u^k \partial_k u^i \partial_l u^i \eta^2 \, dx \right| + \left| \int_{B_2(x_0)} u^k \partial_l \partial_k u^i \eta^2 \partial_l u^i \, dx \right| + \left| \int_{B_2(x_0)} u^k \partial_k u^i \partial_l u^i \partial_l \eta^2 \, dx \right| \\
=: |K_1 + K_2 + K_3|.
\]

Since we are in the 2 D-case, we have \(K_1 = 0\). For \(K_2\) we observe
\[
|K_2| = \left| \int_{B_2(x_0)} \frac{1}{2} u^k \partial_k |\nabla u|^2 \eta^2 \, dx \right| = \left| \int_{T_r(x_0)} \frac{1}{2} u^k |\nabla u|^2 \partial_k \eta^2 \, dx \right| \\
\leq cr^{-1} \int_{T_r(x_0)} |u||\nabla u|^2 \, dx,
\]
and clearly the same bound holds for \(K_3\). With (3.5) we therefore arrive at
\[
\int_{B_r(x_0)} V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \, dx \leq c \left[ r^{-2} \int_{T_r(x_0)} |\nabla u|^p \, dx + R^{-1} \int_{T_r(x_0)} |u||\nabla u|^2 \, dx + J_3 \right].
\]

(3.6)

By the definition of \(J_3\) we finally have
\[
J_3 \leq cr^{-1} \int_{T_r(x_0)} |u||\nabla u|^2 \, dx,
\]
and our claim (3.1) follows from (3.6).

With the help of Lemma 3.1 we now give the

**Proof of Theorem 1.1.** Suppose that \(1 < p < 2\) and that we have (1.9) together with (1.10) (the case \(p = 2\) together with bounded field \(u\) follows by the same arguments setting \(\alpha = 0\)).
From Lemma 2.1, \( iii \), it follows with the choice \( x_0 = 0 \) on account of \( \alpha < \frac{1}{3} \)

\[
\lim_{R \to \infty} R^{-2} \int_{B_R(0)} |\nabla u|^p \, dx = 0.
\]  
(3.7)

Thus (3.1) will imply

\[
V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 = 0 \quad \text{a.e. on } \mathbb{R}^2
\]
(3.8)
as soon as we can show that the remaining integral on the r.h.s. of (3.1) can be estimated in a suitable way.

Obviously it is also sufficient to discuss the integral of \( |u||\nabla u|^2 \) with \( T_r(x_0) \) replaced by \( \Delta_r(x_0) := B_{3r/2}(x_0) - B_r(x_0) \). In fact, inequality (3.1) remains true with \( \Delta_r(x_0) \) as domain of integration on the r.h.s., which follows by appropriate choice of \( \eta \).

In order to estimate the integral \( \int_{\Delta_r(x_0)} |u||\nabla u|^2 \, dx \) we choose a new cut-off function \( \eta \in C_0^\infty(B_{2r}(x_0)) \) such that \( 0 \leq \eta \leq 1, \eta \equiv 1 \) on \( \Delta_r(x_0) \) and \( |\nabla \eta| \leq c/r \). Moreover, we note that (1.10) implies with a positive constant

\[
|u(x)| \leq c(1 + |x|^{2\alpha})^2 =: h(x).
\]

Using this bound we obtain after an integration by parts

\[
r^{-1} \int_{\Delta_r(x_0)} |u||\nabla u|^2 \, dx \leq cr^{-1} \int_{B_{2r}(x_0)} h\eta^2 \partial_k u^i \partial_k u^i \, dx
\]

\[
= -cr^{-1} \int_{B_{2r}(x_0)} h u^i \partial_k u^i \partial_k \eta^2 \, dx
\]

\[
- cr^{-1} \int_{B_{2r}(x_0)} h u^i \partial_k u^i \partial_k \eta^2 \, dx
\]

\[
- cr^{-1} \int_{B_{2r}(x_0)} \partial_k h u^i \partial_k u^i \eta^2 \, dx
\]

\[
\leq cr^{-1} \int_{B_{2r}(x_0)} (1 + |x|^{2\alpha}) |\nabla \varepsilon(u)| \, dx
\]

\[
+ cr^{-2} \int_{B_{2r}(x_0)} (1 + |x|^{2\alpha}) |\nabla u| \, dx + cr^{-1}|T|,
\]

where

\[
T := \int_{B_{2r}(x_0)} \partial_k h u^i \partial_k u^i \eta^2 \, dx.
\]
On the set \( \{ \varepsilon(u) = 0 \} \) we clearly have \( \nabla \varepsilon(u) = 0 \), if \( \varepsilon(u) \neq 0 \), then we use the definition of \( V(\varepsilon) \) and obtain from Young’s inequality

\[
\begin{align*}
 r^{-1} \int_{\Delta_r(x_0)} |u| |\nabla u|^2 \, dx &\leq cr^{-1} \int_{B_2r(x_0)} (1 + |x|)^{2\alpha} V(\varepsilon(u)) |\nabla \varepsilon(u)| |\varepsilon(u)|^{1-\frac{p}{2}} \, dx \\
 &+ cr^{-2} \int_{B_2r(x_0)} (1 + |x|)^{2\alpha} |\nabla u| \, dx + cr^{-1} |T|
\end{align*}
\]

\[
\leq \delta \int_{B_2r(x_0)} V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \, dx
+ c(\delta) r^{-2} \int_{B_2r(x_0)} (1 + |x|)^{4\alpha} |\varepsilon(u)|^{2-p} \, dx
+ cr^{-2} \int_{B_2r(x_0)} (1 + |x|)^{2\alpha} |\nabla u| \, dx + cr^{-1} |T|. \tag{3.9}
\]

Let us look at the quantity \( T \): it holds

\[
T = \int_{B_2r(x_0)} \partial_k h \frac{1}{2} \partial_k |u|^2 \eta^2 \, dx
= - \int_{B_2r(x_0)} \partial_k \partial_k h \frac{1}{2} |u|^2 \eta^2 \, dx
- \int_{B_2r(x_0)} \partial_k h \frac{1}{2} |u|^2 \partial_k \eta^2 \, dx,
\]

hence (recalling the bound for \( |u| \) and the definition of \( h \))

\[
|T| \leq c \left[ \int_{B_2r(x_0)} (1 + |x|)^{3\alpha-2} \, dx + r^{-1} \int_{B_2r(x_0)} (1 + |x|)^{3\alpha-1} \, dx \right].
\]

It is worth remarking that the quantity \( \int_{B_2r(x_0)} h u^i \partial_k u^j \partial_k \eta \, dx \) could have been estimated in a similar way. We insert (3.9) combined with the estimate for \( |T| \) into the r.h.s. of (3.1) (in the version for the annulus \( \Delta_r(x_0) \) in place of \( T_r(x_0) \)) with the result

\[
\begin{align*}
\int_{B_2r(x_0)} V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \, dx &\leq \delta \int_{B_2r(x_0)} V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \, dx + c(\delta) r^{-2} \int_{B_2r(x_0)} |\nabla u|^p \, dx \\
&+ r^{-2} \int_{B_2r(x_0)} (1 + |x|)^{4\alpha} |\nabla u|^{2-p} \, dx + r^{-2} \int_{B_2r(x_0)} (1 + |x|)^{2\alpha} |\nabla u| \, dx
+ r^{-1} \int_{B_2r(x_0)} (1 + |x|)^{3\alpha-2} \, dx + r^{-2} \int_{B_2r(x_0)} (1 + |x|)^{3\alpha-1} \, dx \]. \tag{3.10}
\]
Note that (3.10) holds for all $\delta > 0$ and any disk $B_{2r}(x_0)$. Then Lemma A.4 applied to (3.10) yields for all disks

$$\int_{B_r(x_0)} V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \, dx \leq c \left[ r^{-2} \int_{B_{2r}(x_0)} |\nabla u|^p \, dx + r^{-2} \int_{B_{2r}(x_0)} (1 + |x|)^{4\alpha} |\nabla u| \, dx \right]$$

$$+ r^{-2} \int_{B_{2r}(x_0)} (1 + |x|)^{2\alpha} |\nabla u| \, dx + r^{-1} \int_{B_{2r}(x_0)} (1 + |x|)^{3\alpha-2} \, dx + r^{-2} \int_{B_{2r}(x_0)} (1 + |x|)^{3\alpha-1} \, dx \right]. \quad (3.11)$$

At this point we make the particular choice $x_0 = 0$. We obtain for $r = R$ sufficiently large

$$\int_{B_R(0)} V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \, dx \leq c \left[ R^{-2} \int_{B_{2R}(0)} |\nabla u|^p \, dx + R^{-2+4\alpha} \int_{B_{2R}(0)} |\nabla u|^{2-p} \, dx + R^{-2+2\alpha} \int_{B_{2R}(0)} |\nabla u| \, dx \right]$$

$$+ R^{-1} \int_{B_{2R}(0)} (1 + |x|)^{3\alpha-2} \, dx + R^{-2} \int_{B_{2R}(0)} (1 + |x|)^{3\alpha-1} \, dx \right]. \quad (3.12)$$

The first integral on the r.h.s. of (3.12) is already discussed in (3.7). For the second one we observe with the help of (2.5):

$$R^{-2+4\alpha} \int_{B_{2R}(0)} |\nabla u|^{2-p} \, dx \leq c R^{-2+4\alpha} \left[ \int_{B_{2R}(0)} |\nabla u|^p \, dx \right]^{\frac{2-p}{p}} R^{2p-2}$$

$$= c R^{-2+4\alpha} R^{(1+3\alpha)\frac{2-p}{p}} R^{2p-2}$$

$$= c R^{\frac{2-p}{p}} R^{\alpha \frac{p+6}{p}} \to 0 \quad \text{as } R \to \infty,$$

where we used the fact that (1.9) is equivalent to

$$\frac{p-2}{p} + \alpha \frac{p+6}{p} < 0.$$ 

Next we note that (1.9) gives by elementary calculations

$$\alpha < \frac{1}{2p+3}, \quad (3.13)$$

which shows

$$R^{-2+2\alpha} \int_{B_{2R}(0)} |\nabla u| \, dx \leq c R^{-2+2\alpha} \left[ \int_{B_{2R}(0)} |\nabla u|^p \, dx \right]^{\frac{1}{p}} R^{2(1-\frac{1}{p})}$$

$$\leq c R^{-2+2\alpha+\frac{1+3\alpha}{p}+2-\frac{2}{p}}$$

$$= c R^{-\frac{1}{p}+\alpha \frac{2p+3}{p}} \to 0 \quad \text{as } R \to \infty.$$
Finally we discuss the last two integrals on the r.h.s. of (3.12): we have
\[
\begin{align*}
R^{-1} \int_{B_{2R}(0)} (1 + |x|)^{3\alpha - 2} \, dx &= 2\pi R^{-1} \int_0^{2R} (1 + t)^{3\alpha - 2} \, dt \\
&\leq 2\pi R^{-1} \int_0^{2R} (1 + t)^{3\alpha - 1} \, dt \\
&= \frac{2\pi}{3\alpha} R^{-1} [(1 + 2R)^{3\alpha - 1} - 1] \to 0
\end{align*}
\]
as \( R \to \infty \) on account of \( \alpha < 1/3 \). Moreover,
\[
\begin{align*}
R^{-2} \int_{B_{2R}(0)} (1 + |x|)^{3\alpha - 1} \, dx &\leq c R^{-2} R^{3\alpha - 1} \to 0
\end{align*}
as \( R \to \infty \), and with (3.12) we have shown
\[
\int_{\mathbb{R}^2} V(\varepsilon(u)) |\nabla \varepsilon(u)|^2 \, dx = 0,
\]
which implies (3.8).

On the set \( |\varepsilon(u) = 0| \) we once more observe \( \nabla \varepsilon(u) = 0 \), hence \( \nabla^2 u = 0 \) by recalling the inequality \( |\nabla^2 u| \leq c |\nabla \varepsilon(u)| \) a.e. On the set \( |\varepsilon(u) \neq 0| \) we deduce \( \nabla \varepsilon(u) = 0 \) from (3.8). Thus \( \nabla^2 u = 0 \) on \( \mathbb{R}^2 \), which means that \( u \) is affine. However, since we assume the growth condition (1.10), the constancy of \( u \) is established, which completes the Proof of Theorem 1.1. \( \square \)

The proof of Theorem 1.2 additionally needs the following auxiliary results:

**Lemma 3.2.** If \( u \) is as in Lemma 3.1, then \( v := |\varepsilon(u)|^{p/2} \) belongs to the space \( W^{1,2}_{2,\text{loc}}(\mathbb{R}^2) \) and
\[
\int_{\Omega} |\nabla v|^2 \, dx \leq c \int_{\Omega} V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \, dx
\]
for any domain \( \Omega \subseteq \mathbb{R}^2 \).

**Proof of Lemma 3.2.** Let \( v_\delta := (\delta + |\varepsilon(u)|)^{p/2} \), \( \delta > 0 \). From \( u \in W^2_{p,\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \) it easily follows that \( v_\delta \in W^1_{2,\text{loc}}(\mathbb{R}^2) \) together with
\[
|\nabla v_\delta|^2 \begin{cases} 
\leq c V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 & \text{on the set } |\varepsilon(u) \neq 0|, \\
= 0 & \text{on the set } |\varepsilon(u) = 0|,
\end{cases}
\]
so that the sequence \( \{v_\delta\} \) is locally uniformly bounded in \( W^1_{2,\text{loc}}(\mathbb{R}^2) \), thus
\[
v_\delta \rightharpoonup \tilde{v} \quad \text{in } W^1_{2,\text{loc}}(\mathbb{R}^2).
\]
Clearly \( \tilde{v} = v \), and the desired estimate for \( \int_{\Omega} |\nabla v|^2 \, dx \) follows from (3.14) and lower semicontinuity. \( \square \)

**Lemma 3.3.** Suppose that \( v \in C^1(\mathbb{R}^2) \) satisfies \( \int_{\mathbb{R}^2} |\nabla v|^p \, dx < \infty \) for some \( p \in (1,2) \). Then it holds
\[
\limsup_{R \to \infty} R^{-2} \int_{B_R(0)} |v| \, dx < \infty,
\]
in particular we deduce for any \( \beta > 2 \)
\[
\lim_{R \to \infty} R^{-\beta} \int_{B_R(0)} |v| \, dx = 0.
\]
Proof of Lemma 3.3. W.l.o.g. let $x_0 = 0$ and fix some real number $\gamma > 0$. Introducing polar coordinates $r, \theta$ we define

$$f(r, \theta) = |v(r \cos(\theta), r \sin(\theta))| + \gamma.$$

The following calculations are essentially due to Gilbarg and Weinberger (see [13], proof of Lemma 2.1). We have by Hölder’s inequality

$$\frac{d}{dr} \left[ \int_0^{2\pi} f(r, \theta)^p \, d\theta \right]^{\frac{1}{p}} \leq \int_0^{2\pi} f(r, \theta)^{p-1} |f_r(r, \theta)| \, d\theta \leq \int_0^{2\pi} f(r, \theta)^p \, d\theta \leq \left[ \int_0^{2\pi} |f_r(r, \theta)|^p \, d\theta \right]^{\frac{1}{p}},$$

where we use the symbol $f_r$ for the partial derivative of $f$ with respect to the variable $r$. Thus, for any $\gamma > 0$ we have shown (recall that $f$ is depending on the parameter $\gamma$)

$$\frac{d}{dr} \left[ \int_0^{2\pi} f(r, \theta)^p \, d\theta \right] \leq \left[ \int_0^{2\pi} |f_r(r, \theta)|^p \, d\theta \right]^{\frac{1}{p}}. \quad (3.15)$$

Now let

$$\varphi(t) := \left[ \int_0^{2\pi} f(t, \theta)^p \, d\theta \right]^{\frac{1}{p}}.$$

From (3.15) we get for any $R > 1$:

$$\varphi(R) - \varphi(1) \leq \int_1^{R} \left[ \int_0^{2\pi} |f_r(r, \theta)|^p \, d\theta \right]^{\frac{1}{p}} \, dr$$

$$= \int_1^{R} \left[ \int_0^{2\pi} |f_r(r, \theta)|^p \, d\theta \right]^{\frac{1}{p}} \frac{1}{r^{\frac{1}{p}} - \frac{1}{p}} \, dr$$

$$\leq \left[ \int_1^{R} \int_0^{2\pi} |f_r(r, \theta)|^p \, d\theta \right]^{\frac{1}{p}} \left[ \int_1^{R} r^{\frac{1}{p}} \, dr \right]^{\frac{1}{p} - 1} \left[ \int_1^{R} r^{-\frac{1}{p}} \, dr \right]^{1 - \frac{1}{p}},$$

where we have used Hölder’s inequality once more. This shows (recall $p < 2$)

$$\varphi(R) \leq \varphi(1) + c(p) \left[ \int_1^{R} \int_0^{2\pi} |f_r(r, \theta)|^p r \, d\theta \, dr \right]^{\frac{1}{p}}$$

and since

$$|f_r(r, \theta)| \leq |\nabla v| (r e^{i\theta}) ,$$

we deduce

$$\varphi(R) \leq \varphi(1) + c(p) \left[ \int_{B_R(0) - B_1(0)} |\nabla v|^p \, dx \right]^{\frac{1}{p}}. \quad (3.16)$$
In (3.16) we pass to the limit $\gamma \to 0$ and the finiteness of the energy then yields the inequality

$$\sup_{R \geq 1} \int_{0}^{2\pi} |v(R \cos(\theta), R \sin(\theta))|^p \, d\theta < \infty. \quad (3.17)$$

Hence, for any $R > 1$ we obtain from (3.17)

$$\int_{B_R(0)} |v|^p \, dx = \int_{0}^{R} \int_{0}^{2\pi} |v(r \cos(\theta), r \sin(\theta))|^p r \, d\theta \, dr$$

$$\leq c + \int_{1}^{R} \int_{0}^{2\pi} |v(r \cos(\theta), r \sin(\theta))|^p r \, d\theta \, dr \leq c(1 + R^2),$$

which proves Lemma 3.3.

Proof of Theorem 1.2. Now our assumption on $u$ is

$$\int_{\mathbb{R}^2} |\nabla u|^p \, dx < \infty, \quad (3.18)$$

and in view of this hypothesis and by quoting Lemma 3.1 we have to discuss the quantity

$$r^{-1} \int_{T_{r}(x_0)} |u| |\nabla u|^2 \, dx$$

in order to verify (3.8) for the situation at hand. Let

$$A := \int_{T_{r}(x_0)} u \, dx.$$ 

Clearly it holds

$$r^{-1} \int_{T_{r}(x_0)} |u| |\nabla u|^2 \, dx \leq cr^{-1} \int_{T_{r}(x_0)} |u - A| |\nabla u|^2 \, dx + cr^{-1} |A| \int_{T_{r}(x_0)} |\nabla u|^2 \, dx. \quad (3.19)$$

In (3.19) we apply Hölder’s and Young’s inequality and get for any $\delta > 0$

$$r^{-1} \int_{T_{r}(x_0)} |u| |\nabla u|^2 \, dx \leq c \left[ \int_{T_{r}(x_0)} \left( \frac{|u - A|}{r} \right)^{\frac{p}{p-1}} \, dx \right]^{\frac{p-1}{p}} \left[ \int_{T_{r}(x_0)} |\nabla u|^{2p} \, dx \right]^{\frac{1}{p}} \quad + \delta \int_{T_{r}(x_0)} |\nabla u|^{2p} \, dx + c(\delta)^{\frac{2}{p-1}} \left[ r^{-3} \int_{T_{r}(x_0)} |u| \, dx \right]^{\frac{p}{p-1}}. \quad (3.20)$$

To the first integral on the r.h.s. of (3.20) we apply the Sobolev–Poincaré inequality: let $p^* := \frac{2p}{(2 + p')}$, $p' := \frac{p}{(p - 1)}$, so that $p'$ is the Sobolev exponent of $p^*$.

Let us first consider the case $p \geq 4/3$ for which $p^* \leq p$. Then we have

$$\left[ \int_{T_{r}(x_0)} |u - A|^{p'} \, dx \right]^{\frac{1}{p'}} \leq c \left[ \int_{T_{r}(x_0)} |\nabla u|^{p^*} \, dx \right]^{\frac{1}{p^*}},$$
and by Hölder’s inequality
\[
\left[ \int_{T_r(x_0)} |u - A|^p \, dx \right]^{\frac{1}{p'}} \leq c \left[ \int_{T_r(x_0)} |\nabla u|^p \, dx \right]^{\frac{1}{p'}} \left[ \int_{T_r(x_0)} |\nabla u|^{2p} \, dx \right]^{\frac{1}{p}} = cr^{2(1 - \frac{4}{p'})} = \frac{1}{p} = c r^{3 - \frac{4}{p}}.
\]

We therefore obtain
\[
\begin{align*}
& r^{-1} \int_{T_r(x_0)} |u||\nabla u|^2 \, dx \leq c r^{2 - \frac{4}{p}} \left[ \int_{T_r(x_0)} |\nabla u|^p \, dx \right]^{\frac{1}{p}} \left[ \int_{T_r(x_0)} |\nabla u|^{2p} \, dx \right]^{\frac{1}{p'}} \\
& + \delta \int_{T_r(x_0)} |\nabla u|^{2p} \, dx + c(\delta)r^2 \left[ r^{-3} \int_{T_r(x_0)} |u| \, dx \right]^{\frac{p}{p'-1}}. \tag{3.21}
\end{align*}
\]

Let \( \gamma := 2 - 4/p \) and assume w.l.o.g. that \( p < 2 \), hence \( \gamma < 0 \). Using our assumption \((3.18)\) in \((3.21)\), we find
\[
\begin{align*}
& r^{-1} \int_{T_r(x_0)} |u||\nabla u|^2 \, dx \leq \delta \int_{T_r(x_0)} |\nabla u|^{2p} \, dx + c r^\gamma \left[ r^{-3} \int_{T_r(x_0)} |u| \, dx \right]^{\frac{p}{p'-1}} \\
& + c(\delta)r^2 \left[ r^{-3} \int_{T_r(x_0)} |u| \, dx \right]^{\frac{p}{p'-1}},
\end{align*}
\]
and another application of Young’s inequality shows
\[
\begin{align*}
& r^{-1} \int_{T_r(x_0)} |u||\nabla u|^2 \, dx \leq 2\delta \int_{T_r(x_0)} |\nabla u|^{2p} \, dx + c r^\gamma \left[ r^{-3} \int_{T_r(x_0)} |u| \, dx \right]^{\frac{p}{p'-1}} + r^2 \left[ r^{-3} \int_{T_r(x_0)} |u| \, dx \right]^{\frac{p}{p'-1}}. \tag{3.22}
\end{align*}
\]

Next we discuss the quantity \( \int_{B_2r(x_0)} |\nabla u|^{2p} \, dx \): by Korn’s inequality Lemma A.2, \( ii \), we have
\[
\begin{align*}
& \int_{B_2r(x_0)} |\nabla u|^{2p} \, dx \leq c \left[ \int_{B_2r(x_0)} |\varepsilon(u)|^{2p} \, dx \right]^{\frac{1}{p'}} \left[ \int_{B_2r(x_0)} |u|^{2p} \, dx \right]^{\frac{1}{p}}. \tag{3.23}
\end{align*}
\]

Since \( u \) is a function of class \( C^1(\mathbb{R}^2, \mathbb{R}^2) \) and thereby an element of the space \( W^{1}_{2p, \text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \) we can apply the \( L^{2p} \)-variant of Korn’s inequality to get \((3.23)\). Let \( B := \int_{B_2r(x_0)} u \, dx \) and \( q := 4p/(2 + 2p) \), i.e. \( 2p \) is the Sobolev exponent of \( q \). We therefore get from the Sobolev–Poincaré inequality
\[ \|u\|_{L^p(B_{2r}(x_0))} \leq c \left[ \|u - B\|_{L^p(B_{2r}(x_0))} + |B| r^{\frac{1}{p}} \right] \leq c \left[ \|\nabla u\|_{L^p(B_{2r}(x_0))} + |B| r^{\frac{1}{p}} \right] \]

\[ \leq c \left[ \int_{B_{2r}(x_0)} |\nabla u|^p \, dx \int r^{\frac{1}{p}} r^{2(\frac{1}{2} - \frac{1}{p})} \right], \]

hence (quoting (3.18))

\[ r^{-2p} \int_{B_{2r}(x_0)} |u|^{2p} \, dx \leq c [r^{-2} + |B|^{2p} r^{2-2p}]. \quad (3.24) \]

By Lemma 3.2 the function \( v := |\varepsilon(u)|^{p/2} \) is in the local space \( W^{1,1}_{2,\text{loc}}(\mathbb{R}^2) \), and from Lemma A.3 we obtain

\[ \int_{B_{2r}(x_0)} |\varepsilon(u)|^{2p} \, dx \leq c \left[ \int_{B_{2r}(x_0)} |\varepsilon(u)|^p \, dx \int_{B_{2r}(x_0)} |\nabla v|^2 \, dx + r^{-2} \left[ \int_{B_{2r}(x_0)} |\varepsilon(u)|^p \, dx \right]^2 \right], \]

thus by (3.18) and the estimate for \( \int_{B_{2r}(x_0)} |\nabla v|^2 \, dx \) stated in Lemma 3.2 we find

\[ \int_{B_{2r}(x_0)} |\varepsilon(u)|^{2p} \, dx \leq c \left[ \int_{B_{2r}(x_0)} V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \, dx + r^{-2} \right]. \quad (3.25) \]

Inserting (3.23)–(3.25) into (3.22) we get

\[ r^{-1} \int_{T_r(x_0)} |u| |\nabla u|^2 \, dx \leq 2 \delta \int_{B_{2r}(x_0)} V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \, dx \]

\[ + c(\delta) \left[ r^{-2} + |B|^{2p} r^{2-2p} + r^{-3} + r^2 \left[ \int_{B_{2r}(x_0)} |u|^p \, dx \right]^{2p} \right]. \quad (3.26) \]

Next we return to (3.1) estimating the second term on the r.h.s. through (3.26) with the result (replacing \( \delta \) by \( \delta/2 \))

\[ \int_{B_r(x_0)} V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \, dx \leq \delta \int_{B_{2r}(x_0)} V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \, dx \]

\[ + c(\delta) \left[ r^{-2} + r^2 \left[ \int_{B_{2r}(x_0)} |u|^p \, dx \right]^{2p} \right] \]

\[ \quad + r^2 \left[ \int_{B_{2r}(x_0)} |u|^p \, dx \right]^{\frac{2p}{p-1}}. \]

Applying the \( \delta \)-Lemma A.4 we arrive at (after choosing \( r = R \geq 1 \) and \( x_0 = 0 \))

\[ \int_{B_R(0)} V(\varepsilon(u))^2 |\nabla \varepsilon(u)|^2 \, dx \leq c \left[ R^{-2} + R^7 \left[ \int_{B_{2R}(0)} |u|^2 \, dx \right] R^{\frac{p-1}{p-1}} + R^{\frac{p-3}{p-1}} \int_{B_{2R}(0)} |u|^2 \, dx \right] \]

\[ + \left[ R^{2} \left[ \int_{B_{2R}(0)} |u|^2 \, dx \right] \right]^{\frac{p}{p-1}}. \quad (3.27) \]
By Lemma 3.3 it follows that the r.h.s. of (3.27) vanishes as \( R \to \infty \), thus we obtain (3.8) and, as outlined at the end of the proof of Theorem 1.1, \( u \) has to be an affine function. But then (3.18) yields the constancy of \( u \), which proves Theorem 1.2 in the case \( p \geq 4/3 \).

If \( 6/5 < p < 4/3 \) we return to (3.21) and estimate the r.h.s. of the inequality stated in (3.20) in a different way: observing that by the choice of \( p \)
\[
p < p^* = \frac{2p}{3p - 2} < 2p,
\]
we can apply the interpolation inequality
\[
\|\nabla u\|_{p^*} \leq \|\nabla u\|_p \|\nabla u\|_{2p}^{1 - \alpha},
\]
where all norms are calculated over \( T_r(x_0) \) and where
\[
\frac{1}{p^*} = \frac{\alpha}{p} + \frac{1 - \alpha}{2p}, \quad \text{hence} \quad \alpha = \frac{2p}{p^*} - 1.
\]
This gives using (3.18)
\[
\left[ \int_{T_r(x_0)} \left| \frac{u - A}{r} \right|^{p^*} \frac{r}{p} \, dx \right]^{\frac{p-1}{p^*}} \left[ \int_{T_r(x_0)} |\nabla u|^{2p} \, dx \right]^\frac{1}{2} \leq cr^{-1} \|\nabla u\|_{p^*} \|\nabla u\|_{2p}^2 \leq cr^{-1} \|\nabla u\|_p^\alpha \|\nabla u\|_{2p}^{2+1-\alpha} \leq cr^{-1} \left[ \int_{T_r(x_0)} |\nabla u|^{2p} \, dx \right]^{\frac{3-\alpha}{2p}}.
\]
With elementary calculations one obtains
\[
\frac{3 - \alpha}{2p} = \frac{6 - 3p}{2p}
\]
and we find that
\[
\frac{3 - \alpha}{2p} < 1
\]
is true under our hypothesis \( p > 6/5 \). This gives us the flexibility to apply Young’s inequality with the result
\[
r^{-1} \left[ \int_{T_r(x_0)} |\nabla u|^{2p} \, dx \right]^{\frac{3-\alpha}{2p}} \leq c \left[ r^{-\kappa} + \int_{T_r(x_0)} |\nabla u|^{2p} \, dx \right]
\]
with a suitable positive exponent \( \kappa \). Using this estimate in (3.20) the proof can be finished as before. \( \square \)

4. The Case \( p > 2 \)

We start with an appropriate variant of Lemma 3.1 which is more difficult to establish since now we can no longer benefit from the higher weak differentiability results of Naumann [17] and Wolf [18].

Lemma 4.1. Let \( u \in C^1(\mathbb{R}^2, \mathbb{R}^2) \) denote a solenoidal field satisfying (1.7) with \( H(\varepsilon) = |\varepsilon|^p \) for some exponent \( p > 2 \). Moreover, let
\[
W := W(\varepsilon(u)) := |\varepsilon(u)|^{\frac{2-2}{2}} \varepsilon(u).
\]
Then it holds:

i) \( W \) is in the space \( W_{2,\text{loc}}^1(\mathbb{R}^2, \mathbb{R}^{2 \times 2}) \).
ii) There exists a finite constant $c$ independent of $u$ such that for any $\delta > 0$ and for each $q > 2$

$$
\int_{B_r(x_0)} |\nabla W|^2 \, dx \leq \delta \int_{B_{2r}(x_0)} |\nabla W|^2 \, dx + c \left[ \delta^{-1} r^{-2} \int_{T_r(x_0)} |\nabla u|^p \, dx \right] + r^{-1} \left[ \int_{T_r(x_0)} |u|^q \, dx \right]^{1-\frac{2}{q}} \left[ \int_{T_r(x_0)} |\nabla u|^q \, dx \right]^{\frac{2}{q}} \tag{4.1}
$$

for any disk $B_r(x_0)$.

Proof. We use the difference quotient technique and let

$$
\Delta_h^\alpha v(x) := \frac{1}{h} (v(x + he_\alpha) - v(x))
$$

for functions $v$, parameters $h \neq 0$ and a coordinate direction $e_\alpha$, $\alpha = 1, 2$. If $\varphi \in C^1_0(\mathbb{R}^2, \mathbb{R}^2)$ satisfies $\operatorname{div} \varphi = 0$, then we have the Eq. (1.7) together with the identity

$$
0 = \int_{\mathbb{R}^2} DH(\varepsilon(u))(x + he_\alpha) : \varepsilon(\varphi)(x) \, dx + \int_{\mathbb{R}^2} (u^k \partial_k u)(x + he_\alpha) \varphi(x) \, dx,
$$

hence after subtracting the equations and after dividing by $h$

$$
\int_{\mathbb{R}^2} \Delta_h^\alpha (DH(\varepsilon(u))) : \varepsilon(\varphi) \, dx + \int_{\mathbb{R}^2} \Delta_h^\alpha (u^k \partial_k u) \cdot \varphi \, dx = 0, \tag{4.2}
$$

and (4.2) clearly extends to solenoidal fields from $W^1_{p,\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ with compact support. Alternatively—taking into account the pressure function $\pi$ in the weak form of (1.4)—we can replace (4.2) by

$$
0 = \int_{\mathbb{R}^2} \Delta_h^\alpha (DH(\varepsilon(u))) : \varepsilon(\varphi) \, dx + \int_{\mathbb{R}^2} \Delta_h^\alpha (u^k \partial_k u) \cdot \varphi \, dx - \int_{\mathbb{R}^2} \Delta_h^\alpha \pi \, \operatorname{div} \varphi \, dx
$$

$$
=: T_1 + T_2 + T_3 \tag{4.3}
$$

valid for all $\varphi \in W^1_{p,\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ with compact support. In (4.3) we choose $\varphi := \varphi_\alpha := \eta^2 \Delta_h^\alpha u$ with $\alpha = 1, 2$ being fixed (no summation convention w.r.t. $\alpha$) and with $\eta \in C^2_0(B_{2r}(x_0))$, $0 < \eta \leq 1$, $\eta = 1$ on $B_r(x_0)$, $|\nabla \eta| \leq cr^{-1}$. We discuss the quantities $T_i$ from (4.3) related to our choice of $\varphi$: it holds

$$
T_1 = \int_{B_{2r}(x_0)} \Delta_h^\alpha (DH(\varepsilon(u))) : \varepsilon(\Delta_h^\alpha u) \eta^2 \, dx
$$

$$
+ \int_{B_{2r}(x_0)} \Delta_h^\alpha (DH(\varepsilon(u))) : (\nabla \eta^2 \otimes \Delta_h^\alpha u) \, dx
$$

$$
=: U_1 + U_2,
$$

and for $U_1$ we observe

$$
\Delta_h^\alpha (|\varepsilon(u)|^{p-2} \varepsilon(u))(x) : \varepsilon(\Delta_h^\alpha u)(x) = \frac{1}{h} \left[ |\varepsilon(u)|^{p-2} (x + he_\alpha) \varepsilon(u)(x + he_\alpha) - |\varepsilon(u)|^{p-2}(x) \varepsilon(u)(x) \right] : \varepsilon(\Delta_h^\alpha u)(x)
$$

$$
\geq c \left[ |\varepsilon(u)|^{p-2} (x + he_\alpha) + |\varepsilon(u)|^{p-2}(x) \right] \Delta_h^\alpha \varepsilon(u)(x) : \Delta_h^\alpha \varepsilon(u)(x),
$$
where the last inequality can be easily deduced from Lemma A.5, ii). At the same time, Lemma A.5, i), implies
\[
\frac{1}{|A|} |\varepsilon(u)|^p - 2(x + he_\alpha)\varepsilon(u)(x + he_\alpha) - |\varepsilon(u)|^p - 2(x)\varepsilon(u)(x) - e^2 \leq c\left[|\varepsilon(u)|^2(x + he_\alpha) + |\varepsilon(u)|^2(x)\right]^{\frac{p-2}{2}} \frac{1}{|A|} |\varepsilon(u)(x + he_\alpha) - \varepsilon(u)(x)|,
\]
thus using Young’s inequality
\[
|U_2| \leq c \int_{B_2(x_0)} \left[|\varepsilon(u)(x + he_\alpha) + |\varepsilon(u)(x)|^p - 2(x)\Delta_h^2\varepsilon(u)\Delta_h^2 u|\nabla \eta|^2 \right] dx
\]
\[
\leq \delta \int_{B_2(x_0)} \left[|\varepsilon(u)(x + he_\alpha) + |\varepsilon(u)(x)|^p - 2(x)\Delta_h^2\varepsilon(u) : \Delta_h^2 \varepsilon(u) \right] dx + c\delta^{-1} \int_{B_2(x_0)} \left[|\varepsilon(u)(x + he_\alpha) + |\varepsilon(u)(x)|^p - 2(x)\nabla \eta|^2 \Delta_h^2 u \cdot \Delta_h^2 u \right] dx
\]
for any \(\delta > 0\). Combining these estimates, returning to (4.3) and choosing \(\delta\) small enough we find
\[
\int_{B_2(x_0)} \left[|\varepsilon(u)|^p - 2(x + he_\alpha) + |\varepsilon(u)|^p - 2(x)\right] \eta^2 \Delta_h^2 \varepsilon(u) : \Delta_h^2 \varepsilon(u) \right] dx
\]
\[
\leq c \int_{T_r(x_0)} \left[|\varepsilon(u)|^p - 2(x + he_\alpha) + |\varepsilon(u)|^p - 2(x)\right] |\nabla \eta|^2 \Delta_h^2 u \cdot \Delta_h^2 u \right] dx + |T_2| + |T_3|.
\]
Next we look at the pressure term \(T_3\): we have
\[
\text{div}(\eta^2 \Delta_h^2 u) = \nabla \eta^2 \cdot \Delta_h^2 u =: f_h^\alpha
\]
where the function \(f_h^\alpha\) is compactly supported in \(T_r(x_0)\). Moreover, we have by the definition of \(f_h^\alpha\) and the properties of \(\eta\)
\[
\int_{T_r(x_0)} f_h^\alpha dx = \int_{T_r(x_0)} \text{div}(\eta^2 \cdot \Delta_h^2 u) dx
\]
\[
= - \int_{\partial T_r(x_0)} \Delta_h^2 u(x) \cdot \frac{x - x_0}{r} d\mathcal{H}^1(x)
\]
\[
= - \int_{B_r(x_0)} \text{div}(\Delta_h^2 u) dx = 0,
\]
where \(\mathcal{H}^1\) denotes the one-dimensional Hausdorff-measure. According to Lemma A.1 we find \(\psi_h^\alpha \in W_2^1(T_r(x_0), \mathbb{R}^2)\) satisfying \(\text{div} \psi_h^\alpha = f_h^\alpha\) on \(T_r(x_0)\) and sharing the usual estimates on the annulus \(T_r(x_0)\). We get
\[
|T_3| = \int_{T_r(x_0)} \Delta_h^2 \pi \text{div}(\eta^2 \Delta_h^2 u) dx = \int_{T_r(x_0)} \Delta_h^2 \pi f_h^\alpha dx
\]
\[
= \int_{T_r(x_0)} \Delta_h^2 \pi \text{div} \psi_h^\alpha dx
\]
and if we use (4.3) with $\psi_h^\alpha$ as test function it follows

$$
|T_3| = \left| \int_{T_r(x_0)} \Delta_h^\alpha (DH(\varepsilon(u))) : \varepsilon(\psi_h^\alpha) \, dx + \int_{T_r(x_0)} \Delta_h^\alpha (u^k \partial_k u) \cdot \psi_h^\alpha \, dx \right|
= \left| S_1 + S_2 \right|
$$

For $S_1$ we first observe (compare the discussion of $U_2$)

$$
|S_1| \leq c \int_{T_r(x_0)} |\varepsilon(u)|^p (|\varepsilon(\psi_h^\alpha)|^p \varepsilon(u) + |\varepsilon(u)|^p) \varepsilon(u) \varepsilon(\psi_h^\alpha) \, dx
\leq c \int_{T_r(x_0)} (|\varepsilon(u)|(x + h e_\alpha) + |\varepsilon(u)|(x))^p |\Delta_h^\alpha \varepsilon(u)| \varepsilon(\psi_h^\alpha) \, dx
$$

and then use Young’s inequality to get for any $\delta > 0$

$$
|S_1| \leq \delta \int_{T_r(x_0)} (|\varepsilon(u)|(x + h e_\alpha) + |\varepsilon(u)|(x))^p |\Delta_h^\alpha \varepsilon(u)| \, dx
+ c\delta^{-1} \int_{T_r(x_0)} (|\varepsilon(u)|(x + h e_\alpha) + |\varepsilon(u)|(x))^p |\varepsilon(\psi_h^\alpha)|^2 \, dx.
$$

According to [8], Theorem 3.2, p. 130, the support of $\psi_h^\alpha$ is compact in $T_r(x_0)$ and by quoting Lemma 7.23 of [12] we can estimate using Hölder’s inequality

$$
eq c\delta^{-1} \int_{T_r(x_0)} (\varepsilon(u)|(x + h e_\alpha) + |\varepsilon(u)|(x))^p \varepsilon(\psi_h^\alpha)|^2 \, dx
\leq c\delta^{-1} \left[ \int_{T_r(x_0)} |\varepsilon(\psi_h^\alpha)|^p \, dx \right]^2 \left[ \int_{T_r(x_0)} |\nabla u|^p \, dx \right]^{1-\frac{2}{p}}
\leq c\delta^{-1} r^{-2} \int_{T_r(x_0)} |\nabla u|^p \, dx.
$$

We apply a similar reasoning to the first term on the r.h.s. of (4.4) and get from (4.4)–(4.7)

$$
\int_{B_r(x_0)} \eta^2 (|\varepsilon(u)|^p - (x + h e_\alpha) + |\varepsilon(u)|^p - (x))^p |\Delta_h^\alpha \varepsilon(u)| \, dx
\leq \delta \int_{T_r(x_0)} (|\varepsilon(u)|^p - (x + h e_\alpha) + |\varepsilon(u)|^p - (x))^p |\Delta_h^\alpha \varepsilon(u)| \, dx
\leq \delta \int_{T_r(x_0)} (|\varepsilon(u)|^p - (x + h e_\alpha) + |\varepsilon(u)|^p - (x))^p |\Delta_h^\alpha \varepsilon(u)| \, dx
+ c\delta^{-1} r^{-2} \int_{T_r(x_0)} |\nabla u|^p \, dx + c\left[ |T_2| + |S_2| \right]
$$
with $T_2$ defined in (4.3) for the choice $\varphi = \eta^2 \Delta_h^\alpha u$ and $S_2$ from (4.5). Let us look at $T_2$: we have

$$T_2 = \int_{B_{2r}(x_0)} \Delta_h^\alpha (u^k \partial_k u^i) \eta^2 \Delta_h^\alpha u^i \, dx$$

$$= \int_{B_{2r}(x_0)} \Delta_h^\alpha u^k \partial_k u^i \Delta_h^\alpha u^i \eta^2 \, dx + \int_{B_{2r}(x_0)} u^k \partial_k (\Delta_h^\alpha u^i) \Delta_h^\alpha u^i \eta^2 \, dx$$

$$= \int_{B_{2r}(x_0)} \Delta_h^\alpha u^k \partial_k u^i \Delta_h^\alpha u^i \eta^2 \, dx - \frac{1}{2} \int_{B_{2r}(x_0)} u^k (\Delta_h^\alpha u \cdot \Delta_h^\alpha u) \partial_k \eta^2 \, dx,$$

hence

$$|T_2| \leq c \left[ \int_{B_{2r}(x_0)} (\Delta_h^\alpha u \cdot \Delta_h^\alpha u) |\nabla u| \, dx + \frac{1}{r} \int_{R_r(x_0)} (\Delta_h^\alpha u \cdot \Delta_h^\alpha u) |u| \, dx \right].$$

(4.9)

For estimating $S_2$ we again use the properties of $\psi_h^{\alpha}$ as already done after (4.6):

$$S_2 = - \int_{R_r(x_0)} u^k \partial_k u \cdot \Delta_h^\alpha \psi_h^{\alpha} \, dx$$

$$\leq \left[ \int_{R_r(x_0)} |\nabla \psi_h^{\alpha}|^2 \, dx \right] \left[ \int_{R_r(x_0)} |u|^2 |\nabla u|^2 \, dx \right]^{\frac{1}{2}}$$

$$\leq c r^{-1} \left[ \int_{R_r(x_0)} |\nabla u|^2 \, dx \right] \left[ \int_{R_r(x_0)} |u|^2 |\nabla u|^2 \, dx \right]^{\frac{1}{2}},$$

thus

$$|S_2| \leq c \left[ r^{-1} \int_{R_r(x_0)} |\nabla u|^2 \, dx + r^{-1} \int_{R_r(x_0)} |u|^2 |\nabla u|^2 \, dx \right].$$

(4.10)

Inserting (4.9) and (4.10) into (4.8) and using the $\delta$-Lemma A.4 with suitable functions $f$, $f_j$ and $g$ (replacing the domain of integration $R_r(x_0)$ through $B_{2r}(x_0)$ on the r.h.s. of the inequalities under consideration), we deduce

$$\int_{B_r(x_0)} (|\varepsilon(u)|^{p-2}(x + he_\alpha) + |\varepsilon(u)|^{p-2}(x)) |\Delta_h^\alpha \varepsilon(u)|^2 \, dx \leq c(r, u) < \infty$$

(4.11)

for a constant $c(r, u)$ being independent of $h$. Now it is easy to see (cf. Lemma A.5, i)) that

$$\Delta_h^\alpha W(\varepsilon(u)) : \Delta_h^\alpha W(\varepsilon(u))$$

can be bounded from above by the quantity

$$(|\varepsilon(u)|^{p-2}(x + he_\alpha) + |\varepsilon(u)|^{p-2}(x)) |\Delta_h^\alpha \varepsilon(u)|^2,$$

so that (4.11) implies

$$W(\varepsilon(u)) \in W^{1, \text{loc}}(\mathbb{R}^2, \mathbb{R}^{2 \times 2}).$$

(4.12)
At the same time we can deduce from (4.8) and the subsequent estimates by taking from now on the sum w.r.t. $\alpha$ (letting $W = W(\varepsilon(u))$ and using the formulas for $T_2, S_2$)
\[
\int_{B_r(x_0)} \Delta_h^\alpha W : \Delta_h^\alpha W \, dx \\
\leq \delta \int_{B_2r(x_0)} \Delta_h^\alpha W : \Delta_h^\alpha W \, dx + c \left[ \delta^{-1} r^{-2} \int_{T_r(x_0)} |\nabla u|^p \, dx \right]^{\frac{q}{q-2}} + r^{-1} \int_{T_r(x_0)} |u| (\Delta_h^\alpha u \cdot \Delta_h^\alpha u) \, dx \\
+ r^{-1} \int_{T_r(x_0)} |u|^2 \, dx + r \int_{T_r(x_0)} |u| |\Delta_h^\alpha \psi_h|^2 \, dx.
\]
Here the third and the fourth integral on the r.h.s. correspond to $T_2$, whereas the last two ones are produced by breaking up $S_2$ with the help of Young’s inequality. Using the properties of $\psi_h^\alpha$ we can estimate the last integral on the r.h.s. of (4.13) by Hölder’s inequality in order to get for any $q > 2$
\[
\int_{T_r(x_0)} |u| |\Delta_h^\alpha \psi_h|^2 \, dx \leq \left[ \int_{T_r(x_0)} |u|^{\frac{q}{q-2}} \right]^{\frac{q-2}{q}} \left[ \int_{T_r(x_0)} |\Delta_h^\alpha \psi_h|^q \, dx \right]^{\frac{2}{q}} \\
\leq cr^{-2} \left[ \int_{T_r(x_0)} |u|^{\frac{q}{q-2}} \right]^{\frac{q-2}{q}} \left[ \int_{T_r(x_0)} |\nabla u|^q \, dx \right]^{\frac{2}{q}},
\]
If we insert this estimate into (4.13), we obtain after passing to the limit $h \to 0$ (using $\partial_\alpha u^k \partial_k u^i \partial_\alpha u^i \equiv 0$)
\[
\int_{B_r(x_0)} |\nabla W(\varepsilon(u))|^2 \, dx \leq \delta \int_{B_2r(x_0)} |\nabla W(\varepsilon(u))|^2 \, dx \\
+ c \left[ \delta^{-1} r^{-2} \int_{T_r(x_0)} |\nabla u|^p \, dx + r^{-1} \int_{T_r(x_0)} |u| |\nabla u|^2 \, dx \right]^{\frac{q}{q-2}} \left[ \int_{T_r(x_0)} |\nabla u|^q \, dx \right]^{\frac{2}{q}},
\]
and (4.14) holds for all $\delta > 0$, all disks $B_r(x_0)$ and for any $q > 2$. Hence, with (4.14) our claim (4.1) is established. \hfill \Box

We also need a substitute for Lemma 3.3.

**Lemma 4.2.** Suppose that $v \in C^1(\mathbb{R}^2)$ satisfies $\int_{\mathbb{R}^2} |\nabla v|^p \, dx < \infty$ for some $p \in (2, \infty)$. Then we have
\[
\limsup_{R \to \infty} \frac{1}{R^{3-\frac{3}{p}}} \int_{B_R(0)} |v| \, dx < \infty.
\]

**Proof of Lemma 4.2.** From the proof of Lemma 3.3 we recall the inequality
\[
\varphi(R) - \varphi(1) \leq \left[ \int_1^R \int_0^{2\pi} |f_r(r, \theta)|^p \, d\theta \, dr \right]^{\frac{1}{p}} \left[ \int_1^R r^{-\frac{1}{p} + \frac{3}{p}} \, dr \right]^{1 - \frac{1}{p}}.
\]
being valid also for $p \geq 2$. In place of (3.16) we obtain (recalling $|f_r(r, \theta)| \leq |\nabla v(r e^{i\theta})|$

$$
\varphi(R) \leq \varphi(1) + c(p)R^{\frac{p-2}{p}} \left[ \int_{B_R(0) - B_1(0)} |\nabla u|^p \, dx \right]^{\frac{1}{p}},
$$

provided we choose $R \geq 1$. Using the finiteness of the energy we get after passing to the limit $\gamma \to 0$

$$
\sup_{R \geq 1} R^{2-p} \int_{0}^{2\pi} |v(R \cos(\theta), R \sin(\theta))|^p \, d\theta < \infty.
$$

This estimate implies for $R \geq 1$

$$
\int_{B_R(0)} |v|^p \, dx = \int_{0}^{R} \int_{0}^{2\pi} |v(r \cos(\theta), r \sin(\theta))|^p r \, d\theta \, dr
$$

$$
\leq c + \int_{1}^{R} \int_{0}^{2\pi} |v(r \cos(\theta), r \sin(\theta))|^p r \, d\theta \, dr
$$

$$
\leq c(1 + R^p) \leq cR^p.
$$

Finally we make use of Hölder’s inequality

$$
\int_{B_R(0)} |v| \, dx \leq c \left[ \int_{B_R(0)} |v|^p \, dx \right]^{\frac{1}{p}} R^{2(1 - \frac{1}{p})},
$$

hence our claim follows by inserting the previous estimate. \qed

Next we give the

**Proof of Theorem 1.4.** W.l.o.g. let $u_\infty = 0$. Let us further assume that

$$
\sup_{|x| \geq R} |u(x)||x|^{-\gamma} \to 0 \quad \text{as} \quad R \to \infty
$$

(4.15)

for some $\gamma \in [-1/3, 0)$, hence we have for all $R \geq 1$:

$$
|u(x)| \leq \Theta(R) R^\gamma \quad \text{for all} \quad R \leq |x| \leq 2R
$$

(4.16)

with some function $\Theta$ such that $\Theta(R) \to 0$ as $R \to \infty$. From (4.1) we deduce choosing $q = p$ and applying Young’s inequality ($W := W(\varepsilon(u))$)

$$
\int_{B_{r}(x_0)} |\nabla W|^2 \, dx \leq \delta \int_{B_{2r}(x_0)} |\nabla W|^2 \, dx + c \left[ \delta^{-1} r^{-2} \int_{B_{2r}(x_0)} |\nabla u|^p \, dx \right. + \left. r^{-1} \left[ \int_{B_{2r}(x_0)} |u|^\frac{p}{p-2} \, dx \right]^{\frac{1}{p}} \left[ \int_{B_{2r}(x_0)} |\nabla u|^p \, dx \right]^{\frac{2}{p}} \right]
$$

$$
\leq \delta \int_{B_{2r}(x_0)} |\nabla W|^2 \, dx + c \left[ \delta^{-1} r^{-2} \int_{B_{2r}(x_0)} |\nabla u|^p \, dx \right. + \left. r^{-1} \left[ \int_{B_{2r}(x_0)} |\nabla u|^p \, dx + \tau^{-\frac{2}{p-2}} \int_{B_{2r}(x_0)} |u|^\frac{p}{p-2} \, dx \right] \right]$$

$$
+ \tau^{-\frac{1}{p-2}} \left[ \int_{B_{2r}(x_0)} |\nabla u|^p \, dx \right]^{\frac{p}{2}} + \tau^{-\frac{1}{p-2}} \left[ \int_{B_{2r}(x_0)} |u|^\frac{p}{p-2} \, dx \right]^{\frac{p}{2}}$$
for any disk $B_r(x_0)$. Let $\tau := r^\kappa$ for some $\kappa \in (0, 1)$. The $\delta$-Lemma A.4 yields for any disk $B_r(x_0)$

$$
\int_{B_r(x_0)} |\nabla W|^2 \, dx \leq c \left[ r^{-2} \int_{B_{2r}(x_0)} |\nabla u|^p \, dx + r^{-1+\kappa} \int_{B_{2r}(x_0)} |\nabla u|^p \, dx \right.
\left. + r^{-\frac{2\kappa}{p-2} - 1} \int_{B_{2r}(x_0)} |u|^\frac{p}{p-2} \, dx \right].
$$

(4.17)

We choose $x_0 = 0$, $r = R > 1$ and insert (2.5) in (4.17), where the last integral on the r.h.s. of (4.17) is handled with the condition $|u| \leq c$. We arrive at

$$
\int_{B_R(0)} |\nabla W|^2 \, dx \leq c \left[ R^{-2+1+3\gamma} + R^{-1+\kappa+1+3\gamma} + R^{-\frac{2\kappa}{p-2} - 1} R^2 \right]
\leq c \left[ R^{\kappa+3\gamma} + R^{1-\frac{2\kappa}{p-2}} \right],
$$

i.e. we have with some $\nu < 1$ (w.l.o.g. $\nu > 0$)

$$
\int_{B_R(0)} |\nabla W|^2 \, dx \leq c R^\nu \quad \text{for all } R \geq 1.
$$

(4.18)

Next we choose $\mu \in (\nu, 1)$ and apply (4.1) with $q = p$ and $\delta = R^{-\mu}$ to obtain

$$
\int_{B_R(0)} |\nabla W|^2 \, dx \leq c \left[ R^{-\mu+\nu} + R^{\mu-2+1+3\gamma} + R^{-1} R^2 - \frac{4}{p} \sup_{R \leq |x| \leq 2R} |u| R^{(1+3\gamma)\frac{2}{p}} \right].
$$

(4.19)

By the choice of the above parameters, the first two terms on the r.h.s. of (4.19) converge to zero as $R \to \infty$ and it remains to discuss the quantity (recall (4.16))

$$
\zeta_R := R^{1-\frac{4}{p}} \Theta(R) R^{7\gamma} R^{(1+3\gamma)\frac{2}{p}} = \Theta(R) R^{1-\frac{4}{p} + (1+3\gamma)\frac{2}{p}},
$$

where we have to distinguish the three different cases of Theorem 1.4.

Case 1. For $2 < p < 6$ we may choose $\gamma = (2 - p)/(p + 6)$ in (4.15), where we note that

$$
\gamma > -\frac{1}{3} \iff p < 6.
$$

This particular choice of $\gamma$ gives

$$
1 - \frac{2}{p} + \gamma \left( 1 + \frac{6}{p} \right) = 0
$$

which implies $\zeta_R \to 0$ as $R \to \infty$, hence the first part of the theorem is established.

Case 2. For $p = 6$ we have by assumption

$$
|u(x)| \leq c R^{-\frac{1}{3}} \quad \text{for all } |x| \geq R
$$

and for all $R \geq 1$. Since the condition $\Theta(R) \to 0$ as $R \to \infty$ is not needed for deriving (4.18), we obtain (4.18) as before. Moreover, (2.5) gives

$$
\int_{\mathbb{R}^2} |\nabla u|^p \, dx < \infty.
$$

(4.20)

As above we let $q = p$ and $\delta = R^{-\mu}$ in (4.1) to obtain (recall (4.18))

$$
\int_{B_R(0)} |\nabla W|^2 \, dx \leq c \left[ R^\mu - \mu - 2 + R^{-1} \left( \int_{T_R(0)} |u|^{\frac{2}{3}} \, dx \right)^{\frac{3}{2}} \left( \int_{T_R(0)} |\nabla u|^p \, dx \right)^{\frac{1}{2}} \right].
$$

(4.21)
Here we observe
\[
R^{-1} \left[ \int_{T_R(0)} |u|^\frac{3}{2} \, dx \right]^{\frac{2}{3}} \leq c R^{-\frac{1}{3}} R^{2\frac{3}{2}} \leq c
\]
and by (4.20) the last integral of (4.21) converges to 0 as \( R \to \infty \) which completes the proof in the second case of Theorem 1.4.

Case 3. In the case \( p > 6 \) we again have by assumption the global energy estimate (4.20). We recall (2.15) of Sect. 2, choose \( \delta = 1/2 \) in this inequality and observe that by the boundedness of \( u \)
\[
R^{-p} \int_{T_R(0)} |u|^p \, dx \to 0 \quad \text{as} \quad R \to \infty.
\]
Moreover we have
\[
|T_3| + |T_4| \leq c R \left[ \sup_{R \leq |x| \leq 2R} |u| \right]^3 \to 0 \quad \text{as} \quad R \to \infty.
\]
As a consequence we see
\[
\int_{\mathbb{R}^2} |\varepsilon(u)|^p \, dx \leq \frac{1}{2} \int_{\mathbb{R}^2} |\varepsilon(u)|^p \, dx
\]
which means \( \varepsilon(u) \equiv 0 \), hence \( u \) is a rigid motion and \( u = \text{const} \) by the decay assumption. This completes the proof of Theorem 1.4. \( \square \)

We finish this section with the

**Proof of Theorem 1.3.** Let \( 2 < p \leq 3 \). As above we have (4.17), where we know in the situation at hand that
\[
\int_{\mathbb{R}^2} |\nabla u|^p \, dx < \infty,
\]
hence for any \( R \geq 1 \) (\( W := W(\varepsilon(u)) \))
\[
\int_{B_R(0)} |\nabla W|^2 \, dx \leq c \left[ R^{-1+\kappa} + R^{-\frac{2\kappa}{p-2} - 1} \int_{B_{2R}(0)} |u|^\frac{p}{p-2} \, dx \right]. \tag{4.22}
\]
We insert (4.22) in the r.h.s. of (4.1) choosing \( q = p \) there and get for any \( \delta > 0 \)
\[
\int_{B_R(0)} |\nabla W|^2 \, dx \leq \delta \left[ R^{-1+\kappa} + R^{-\frac{2\kappa}{p-2} - 1} \int_{B_{2R}(0)} |u|^\frac{p}{p-2} \, dx \right] + c \left[ \delta^{-1} R^{-2} \int_{T_R(0)} |\nabla u|^p \, dx \right]
+ R^{-1} \left[ \int_{T_R(0)} |u|^\frac{p}{p-2} \, dx \right]^\frac{p-2}{p} \left[ \int_{T_R(0)} |\nabla u|^p \, dx \right]^\frac{2}{p}. \tag{4.23}
\]
Let
\[
A := \int_{B_{2R}(0)} u \, dx
\]
and observe
\[
\int_{B_{2R}(0)} |u|^{\frac{p}{p-2}} \, dx \leq c \left[ \int_{B_{2R}(0)} |u - A|^{\frac{p}{p-2}} \, dx + R^2 |A|^{\frac{p}{p-2}} \right]
\]
\[
\leq c \left[ \int_{B_{2R}(0)} |u - A|^{\frac{p}{p-2}} \, dx + \left( R^{-2+\frac{2}{p}} \right) \int_{B_{2R}(0)} u \, dx \right]^{\frac{p}{p-2}}. \tag{4.24}
\]

To the first integral on the r.h.s. of (4.24) we apply the Sobolev–Poincaré inequality, which is possible on account of \( p/(p-2) > 2 \): letting \( 1 < q := \frac{2p}{3p-4} \) and observing \( q < p \) on account of \( p > 2 \), we find
\[
\left[ \int_{B_{2R}(0)} |u - A|^{\frac{p}{p-2}} \, dx \right]^{\frac{p-2}{p}} \leq c \left[ \int_{B_{2R}(0)} |\nabla u|^q \, dx \right]^{\frac{1}{q}}
\]
\[
\leq c \left[ \int_{B_{2R}(0)} |\nabla u|^p \, dx \right]^{\frac{2}{p}} \left( R^{2(1-\frac{2}{p})} \right)^{\frac{1}{q}}
\]
\[
= c R^{\frac{2}{p} - \frac{2}{p}} \left[ \int_{B_{2R}(0)} |\nabla u|^p \, dx \right]^{\frac{1}{p}}, \tag{4.25}
\]
where we also made use of Hölder’s inequality. With (4.24) and (4.25) we find
\[
\xi_1 := R^{-1} \left[ \int_{T_R(0)} |u|^{\frac{p}{p-2}} \, dx \right]^{\frac{p-2}{p}} \left[ \int_{T_R(0)} |\nabla u|^p \, dx \right]^{\frac{2}{p}}
\]
\[
\leq R^{-1} \left[ \int_{B_{2R}(0)} |u|^{\frac{p}{p-2}} \, dx \right]^{\frac{p-2}{p}} \left[ \int_{T_R(0)} |\nabla u|^p \, dx \right]^{\frac{2}{p}}
\]
\[
\leq c \left[ R^{-1} R^{\frac{2}{p} - \frac{2}{p}} \left[ \int_{T_R(0)} |\nabla u|^p \, dx \right]^{\frac{2}{p}} \left[ \int_{B_{2R}(0)} |\nabla u|^p \, dx \right]^{\frac{1}{p}} \right]
\]
\[
+ R^{-1} R^{-2+\frac{2}{p}} \left[ \int_{B_{2R}(0)} u \, dx \right]^{\frac{2}{p}} \left[ \int_{T_R(0)} |\nabla u|^p \, dx \right]^{\frac{2}{p}}
\]
\[
= c \left[ R^{2-\frac{6}{p}} \left[ \int_{T_R(0)} |\nabla u|^p \, dx \right]^{\frac{2}{p}} \left[ \int_{B_{2R}(0)} |\nabla u|^p \, dx \right]^{\frac{1}{p}} \right]
\]
\[
+ R^{1-\frac{4}{p}} \left[ \int_{B_{2R}(0)} u \, dx \right]^{\frac{2}{p}} \left[ \int_{T_R(0)} |\nabla u|^p \, dx \right]^{\frac{2}{p}}, \tag{4.26}
\]
and since
\[ \lim_{R \to \infty} \int_{T_R(0)} |\nabla u|^p \, dx = 0 \]
it follows
\[ \lim_{R \to \infty} \xi_1 = 0 \tag{4.27} \]
on account of \( p \leq 3 \) and by quoting Lemma 4.2. Using (4.24) and (4.25) one more time we obtain
\[ \xi_2 := \delta R^{-\frac{2\kappa}{p-2}-1} \int_{B_{2R}(0)} |u|^{\frac{p}{p-2}} \, dx \]
\[ \leq c \delta R^{-\frac{2\kappa}{p-2}-1} \left[ R^{\left(\frac{2}{\kappa} - \frac{2}{p} \right) \frac{p}{p-2}} \int_{B_{2R}(0)} |\nabla u|^p \, dx \right]^{\frac{p}{p-2}} + \left[ R^{-2+2\frac{p-2}{p}} \int_{B_{2R}(0)} u \, dx \right]^{\frac{p}{p-2}} \]
\[ = c \delta R^{-\frac{2\kappa}{p-2}-1} \left[ R^3 \left[ \int_{B_{2R}(0)} |\nabla u|^p \, dx \right]^{\frac{1}{p-2}} + \left[ R^{-\frac{4}{p}} \int_{B_{2R}(0)} u \, dx \right]^{\frac{p}{p-2}} \right]. \tag{4.28} \]
Since \( p \leq 3 \), it holds
\[ -\frac{2\kappa}{p-2} - 1 + 3 = 2 - \frac{2\kappa}{p-2} \leq 2 - 2\kappa. \]
Recalling that \( \kappa \in (0,1) \) is arbitrary, we may fix, e.g., \( \kappa = 3/4 \), hence \( 2 - 2\kappa = 1/2 \). Finally we choose \( \delta = 1/R \) in (4.23). This implies
\[ \delta R^{-\frac{2\kappa}{p-2}-1} R^3 \left[ \int_{B_{2R}(0)} |\nabla u|^p \, dx \right]^{\frac{1}{p-2}} \to 0 \]
as \( R \to \infty \) and at the same time by Lemma 4.2
\[ \delta R^{-\frac{2\kappa}{p-2}-1} R^{-\frac{4}{p}} \int_{B_{2R}(0)} u \, dx \left[ \int_{B_{2R}(0)} u \, dx \right]^{\frac{p}{p-2}} \to 0 \]
as \( R \to \infty \), hence
\[ \lim_{R \to \infty} \xi_2 = 0. \tag{4.29} \]
Inserting (4.26)–(4.29) into (4.23) and passing to the limit \( R \to \infty \), we have shown that \( \nabla W = 0 \) on \( \mathbb{R}^2 \), hence \( u \) is affine and the finiteness of the \( p \)-energy implies the constancy of \( u \). \( \square \)

5. Proof of Theorem 1.5

Let \( u \) denote an entire solution of (1.1) satisfying (1.14). Introducing the vorticity
\[ \omega := \partial_2 u^1 - \partial_1 u^2 \]
we have for \( q, l \in \mathbb{N} \) sufficiently large with \( \eta \in C_0^\infty(\mathbb{R}^2) \)

\[
\int_{\mathbb{R}^2} \omega^{2q} \eta^{2l} \, dx = \int_{\mathbb{R}^2} (\partial_2 u^1 - \partial_1 u^2) \omega^{2q-1} \eta^{2l} \, dx = \int_{\mathbb{R}^2} \text{div}(-u^2, u^1) \omega^{2q-1} \eta^{2l} \, dx
\]

\[
= - \int_{\mathbb{R}^2} (-u^2, u^1) \cdot \nabla \left[ \omega^{2q-1} \eta^{2l} \right] \, dx = (2q - 1) \int_{\mathbb{R}^2} \nabla \cdot (u^2, -u^1) \omega^{2q-2} \eta^{2l} \, dx
\]

\[
+ 2l \int_{\mathbb{R}^2} (u^2, -u^1) \cdot \nabla \eta \omega^{2q-1} \eta^{2l-1} \, dx ,
\tag{5.1}
\]

and from \( \text{div} u = 0 \) we infer

\[
\int_{\mathbb{R}^2} u \cdot \nabla \omega^{2q-3} \eta^{2l} \, dx = \frac{1}{2q - 2} \int_{\mathbb{R}^2} u \cdot \nabla \omega^{2q-2} \eta^{2l} \, dx
\]

\[
= - \frac{1}{2q - 2} \int_{\mathbb{R}^2} u \cdot \nabla \eta \omega^{2q-2} \, dx .
\tag{5.2}
\]

Recall that

\[
\Delta \omega - u \cdot \nabla \omega = 0 \quad \text{on } \mathbb{R}^2 ,
\]

hence

\[
\int_{\mathbb{R}^2} \nabla \omega \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^2} u \cdot \nabla \omega \varphi \, dx = 0
\]

for \( \varphi \in C_0^1(\mathbb{R}^2) \). We specify \( \varphi = \eta^{2l} \omega^{2q-3} \) and get

\[
\int_{\mathbb{R}^2} \eta^{2l} (2q - 3) |\nabla \omega|^2 \omega^{2q-4} \, dx = - \int_{\mathbb{R}^2} \nabla \omega \cdot \nabla \eta^{2l} \omega^{2q-3} \, dx - \int_{\mathbb{R}^2} u \cdot \nabla \omega \omega^{2q-3} \eta^{2l} \, dx .
\tag{5.3}
\]

By Young’s inequality, the first term on the r.h.s. of (5.3) is estimated through

\[
\delta \int_{\mathbb{R}^2} |\nabla \omega|^2 \omega^{2q-4} \eta^{2l} \, dx + c(\delta, l) \int_{\mathbb{R}^2} |\nabla \eta|^2 \eta^{2l-2} \omega^{2q-2} \, dx ,
\]

to the second term on the r.h.s. of (5.3) we apply (5.2). This yields after appropriate choice of \( \delta \)

\[
\int_{\mathbb{R}^2} |\nabla \omega|^2 \omega^{2q-4} \eta^{2l} \, dx \leq c(l, q) \left[ \int_{\mathbb{R}^2} \omega^{2q-2} \eta^{2l-2} |\nabla \eta|^2 \, dx + \int_{\mathbb{R}^2} |u||\nabla \eta|^2 |\omega^{2q-2} \, dx \right] .
\tag{5.4}
\]

Now we return to (5.1) and estimate

\[
\int_{\mathbb{R}^2} \omega^{2q} \eta^{2l} \, dx \leq (2q - 1) \int_{\mathbb{R}^2} |\nabla \omega||u|\omega^{2q-2} \eta^{2l} \, dx + 2l \int_{\mathbb{R}^2} |u||\nabla \eta|\omega^{2q-1} \eta^{2l-1} \, dx
\]

\[
\leq \delta \int_{\mathbb{R}^2} \omega^{2q} \eta^{2l} \, dx + c(\delta, q) \int_{\mathbb{R}^2} |\nabla \omega|^2 |u|^2 \omega^{2q-4} \eta^{2l} \, dx + 2l \int_{\mathbb{R}^2} |u||\nabla \eta|\omega^{2q-1} \eta^{2l-1} \, dx ,
\]

hence for \( \delta \) sufficiently small

\[
\int_{\mathbb{R}^2} \eta^{2l} \omega^{2q} \, dx \leq c(l, q) \left[ \int_{\mathbb{R}^2} |\nabla \omega|^2 |u|^2 \omega^{2q-4} \eta^{2l} \, dx + \int_{\mathbb{R}^2} |u||\nabla \eta|\omega^{2q-1} \eta^{2l-1} \, dx \right] .
\tag{5.5}
\]
Next we specify \( \eta \): let \( R \geq 1 \) and choose \( \eta = 1 \) on \( B_R(0) \), \( 0 \leq \eta \leq 1 \), \( \text{spt} \eta \subset B_{2R}(0) \), \( |\nabla \eta| \leq c/R \). From (1.14) we get (w.l.o.g. we assume \( \alpha > 0 \))

\[
|u(x)| \leq cR^\alpha \quad \text{for all } x \in B_R(0).
\]

We use (5.6) on the r.h.s. of (5.5) and get

\[
\int_{B_{2R}(0)} \eta^2 l^2 \omega^{2l} \, dx \leq c(l, q) \left[ R^{2\alpha} \int_{B_{2R}(0)} |\nabla \omega|^{2q} \eta^{2l - 2} |\nabla \eta| \, dx + R^{3\alpha} \int_{B_{2R}(0)} |\nabla \eta|^{2l} \omega^{2q - 2} \, dx \right],
\]

and if we apply (5.4) on the r.h.s. quoting (5.6) one more time it follows

\[
\int_{B_{2R}(0)} \eta^{2l} \omega^{2q} \, dx \leq c(l, q) \left[ R^{2\alpha} \int_{B_{2R}(0)} \omega^{2q - 2} \eta^{2l - 2} |\nabla \eta|^{2l - 1} \, dx + \int_{B_{2R}(0)} \omega^{2q - 1} |\nabla \eta|^{2l - 1} \, dx \right]
\]

\[
= c(l, q)[T_1 + T_2 + T_3].
\]

Young’s inequality yields

\[
T_1 \leq \int_{B_{2R}(0)} \omega^{2q - 2} \eta^{2l - 2} R^{2\alpha - 2} \, dx
\]

\[
\leq \delta \int_{B_{2R}(0)} \omega^{2q} \eta^{(2l - 2)/2} R^{2q - 2} \, dx + c(\delta) R^{2 + q(2\alpha - 2)}
\]

and

\[
T_2 \leq \int_{B_{2R}(0)} \omega^{2q - 2} \eta^{2l - 1} R^{3\alpha - 1} \, dx
\]

\[
\leq \delta \int_{B_{2R}(0)} \omega^{2q} \eta^{(2l - 1)/2} R^{2q - 2} \, dx + c(\delta) R^{2 + q(3\alpha - 1)}
\]

as well as

\[
T_3 \leq c \int_{B_{2R}(0)} \omega^{2q - 1} \eta^{2l - 1} R^{\alpha - 1} \, dx
\]

\[
\leq \delta \int_{B_{2R}(0)} \omega^{2q} \eta^{(2l - 1)/2} R^{2q - 1} \, dx + c(\delta) R^{2 + 2q(\alpha - 1)}.
\]

Moreover, for \( l \gg 1 \) we have

\[
2l \leq \frac{(2l - 2)2q}{2q - 2} \quad \text{and} \quad 2l \leq \frac{(2l - 1)2q}{2q - 1},
\]

hence, for \( \delta \) small enough, we obtain from (5.7)

\[
\int_{B_{2R}(0)} \eta^{2l} \omega^{2q} \, dx \leq c(l, q) \left[ R^{2 + q(2\alpha - 2)} + R^{2 + q(3\alpha - 1)} + R^{2 + 2q(\alpha - 1)} \right].
\]
Recall that \( \alpha < 1/3 \). Therefore we can fix a sufficiently large exponent \( q \) with the property that
\[
2 + q(3\alpha - 1) < 0,
\]
and (5.8) shows
\[
\int_{B_R(0)} \omega^{2q} \, dx \leq c(l, q) R^{2+q(3\alpha-1)} \to 0 \quad \text{as} \quad R \to 0,
\]
hence \( \omega = 0 \) on \( \mathbb{R}^2 \). This together with \( \text{div} \, u = 0 \) shows that \( u \) is harmonic and the constancy of \( u \) then follows from (1.14) and results concerning entire harmonic functions. \( \square \)

Appendix. Helpful Tools

The following lemma is a well known result. A proof together with further comments can be found in [8], Chapter III, Section 3. Our formulation is taken from [1], Lemma 2.5.

**Lemma A.1.** Suppose that we are given numbers \( 1 < p_1 \leq p \leq p_2 < \infty \).

Then there exists a constant \( c = c(p_1, p_2) \) as follows: if \( f \in L^p(B_r(x_0)) \) satisfies \( \int_{B_r(x_0)} f \, dx = 0 \),
then there exists a field \( v \) in the space \( \dot{W}^1_p(B_r(x_0), \mathbb{R}^2) \) satisfying \( \text{div} \, v = f \) on the disk \( B_r(x_0) \) together with the estimate
\[
\int_{B_r(x_0)} |\nabla v|^s \, dx \leq c \int_{B_r(x_0)} |f|^s \, dx
\]
for any exponent \( s \in [p_1, p] \). The same is true if the disk is replaced by the annulus \( T_r(x_0) = B_{2r}(x_0) - B_r(x_0) \).

Our next tool is a collection of Korn-type inequalities. We refer the reader to Lemma 3.0.1 in [4], where a list of references is given. We note that the last statement follows from the first one by applying i) to \( \eta v \), where \( \eta \) is a suitable cut-off function.

**Lemma A.2.** Let \( 1 < p < \infty \). Then there exists a constant \( c(p) \) such that the following inequalities hold.

i) For all \( v \in \dot{W}^1_p(B_r(x_0), \mathbb{R}^2) \) we have
\[
||\nabla v||_{L^p(B_r(x_0))} \leq c(p) ||\varepsilon(v)||_{L^p(B_r(x_0))}.
\]

ii) For all \( v \in W^1_p(B_r(x_0), \mathbb{R}^2) \) we have
\[
||\nabla v||_{L^p(B_r(x_0))} \leq c(p) \left[ ||\varepsilon(v)||_{L^p(B_r(x_0))} + r^{-1} ||v||_{L^p(B_r(x_0))} \right].
\]

iii) For all \( v \in W^1_p(B_{2r}(x_0), \mathbb{R}^2) \) we have letting \( T_r(x_0) = B_{2r}(x_0) - B_r(x_0) \)
\[
||\nabla v||_{L^p(T_r(x_0))} \leq c(p) \left[ ||\varepsilon(v)||_{L^p(B_{2r}(x_0))} + r^{-1} ||v||_{L^p(T_r(x_0))} \right].
\]

The following lemma originates from the work of Ladyzhenskaya (see [15], Lemma 1, p. 8). Actually it is a local variant of Ladyzhenskaya’s lemma established as Lemma 2.6 in part B of [19].

**Lemma A.3.** Suppose that \( u \in W^1_2(B_r(x_0)), B_r(x_0) \subset \mathbb{R}^2 \). Then there is a constant \( c \) independent of \( u, x_0 \) and \( r \) such that
\[
\int_{B_r(x_0)} |u|^4 \, dx \leq c \left[ \int_{B_r(x_0)} |u|^2 \, dx \int_{B_r(x_0)} |\nabla u|^2 \, dx + r^{-2} \left( \int_{B_r(x_0)} |u|^2 \, dx \right)^2 \right].
\]

The next lemma goes back to Giaquinta and Modica (see [10], Lemma 0.5). We state a small extension presented in [5] as Lemma 3.1.
Lemma A.4. Let \( f, f_1, \ldots, f_l \) denote non-negative functions from the space \( L^1_{\text{loc}}(\mathbb{R}^2) \). Suppose further that we are given exponents \( \alpha_1, \ldots, \alpha_l > 0 \).

Then we can find a number \( \delta_0 > 0 \) (depending on \( \alpha_1, \ldots, \alpha_l \)) as follows: if for \( \delta \in (0, \delta_0) \) it is possible to calculate a constant \( c(\delta) > 0 \) such that the inequality

\[
\int_{B_r(x_0)} f \, dx \leq \delta \int_{B_{2r}(x_0)} f \, dx + c(\delta) \sum_{j=1}^l r^{-\alpha_j} \int_{B_{2r}(x_0)} f_j \, dx
\]  

(A.1)

holds for any choice of \( B_r(x_0) \subset \mathbb{R}^2 \), then there is a constant \( c \) with the property

\[
\int_{B_r(x_0)} f \, dx \leq c \sum_{j=1}^l r^{-\alpha_j} \int_{B_{2r}(x_0)} f_j \, dx
\]  

(A.2)

for all disks \( B_r(x_0) \subset \mathbb{R}^2 \).

Finally we recall some well known inequalities.

Lemma A.5. Let \( p > 2 \).

i) With suitable positive constants \( c_1 < c_2 \) it holds

\[
c_1 \left[ |\xi|^{p-2} + |\eta|^{p-2} \right] |\xi - \eta|^2 \leq \left| |\xi|^{\frac{p-2}{2}} \xi - |\eta|^{\frac{p-2}{2}} \eta \right|^2 \leq c_2 \left[ |\xi|^{p-2} + |\eta|^{p-2} \right] |\xi - \eta|^2
\]

for any \( \xi, \eta \in \mathbb{R}^M, M \geq 1 \).

ii) There exists a constant \( c > 0 \) such that

\[
\left( |\xi|^{p-2} \xi - |\eta|^{p-2} \eta \right) : (\xi - \eta) \geq c \left[ |\xi|^{p-2} + |\eta|^{p-2} \right] |\xi - \eta|^2
\]

for all \( \xi, \eta \in \mathbb{R}^M, M \geq 1 \).

Proof. i) follows from inequality (2.4) in [11] by letting \( \mu = 0, \delta = p - 2 \) in this reference.

For proving ii) we let \( F(\xi) = |\xi|^{p-2} \xi \) and observe that

\[
\left( F(\xi) - F(\eta) \right) : (\xi - \eta) = \int_0^1 \frac{d}{dt} F(\eta + t(\xi - \eta)) \, dt : (\xi - \eta)
\]

\[
= \int_0^1 |\eta + t(\xi - \eta)|^{p-2} \, dt |\xi - \eta|^2 + A,
\]

where \( A \) is easily seen to be non-negative. From Lemma 2.2 in [7] we therefore deduce

\[
\left( F(\xi) - F(\eta) \right) : (\xi - \eta) \geq c |\xi - \eta|^2 \left[ |\xi - \eta|^{p-2} + |\eta|^p \right],
\]

and our claim immediately follows from this estimate by considering the cases \( |\xi| \geq 2|\eta| \) and \( |\xi| < 2|\eta| \), respectively.

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