Mathematical and numerical models of buckling of three-dimensional structures taking into account the thermal effect

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Abstracts. A mathematical model of buckling of three-dimensional structures is considered, considering the thermal effect on them. The proposed methodology for studying stability consists in solving a system of three problems: thermal conductivity, quasi-static linear thermoelasticity, and, in fact, the problem of buckling theory. The first two tasks correspond to the basic state of the structure, and the third to the varied one, which occurs when the structure loses its stability. The approach described in the work allows considering the influence on the buckling of the structure not only of three-dimensional phenomena (holes, joined local element etc.) but also the influence of thermal stresses. In the study, an algorithm for solving the buckling problem is considered and a variational statement is formulated. The presented model was tested on the rod buckling problem, which showed good agreement with the results obtained in CAE ANSYS software. The modelling presented was carried out in the SMCM software, developed by SIMPLEX Scientific and Educational Centre of Bauman Moscow State Technical University.

1. Introduction
Stability calculation problems play an important role in the design of thin-walled composite structures. Usually, the calculation of the stability of thin-walled structures is carried out on the basis of one-dimensional or two-dimensional theories of bars, plates or shells [1-9] However, for many important problems, it is necessary to take into account three-dimensional phenomena that can affect critical buckling loads. An example of such problems is the calculation of the stability of non-stationary uneven heating of structures. In these problems, transverse effects are essential - interlayer shear stresses and transverse normal stresses [10]. An accurate calculation of these stresses requires the use of three-dimensional numerical methods, and, accordingly, three-dimensional problem statements [11]. Calculation of the stability of thin-walled structures with uneven heating also requires the development of a three-dimensional theory of stability. It should be noted that in commercial software packages, for example ANSYS, there is usually no information about the formulation of problems used in the three-dimensional calculation of structural stability problems. In works [12-14], a new original formulation of three-dimensional problems of the theory of stability of structures was developed. In [15-18], a numerical finite element method for solving these problems was developed. The purpose of this work is to further develop the proposed method for the case of taking into account thermal expansion and non-uniform non-stationary heating of thin-walled structures.

2. Three-dimensional model of the theory of stability of structures, taking into account the thermal effect
Using the concept of constructing problems of the three-dimensional theory of stability under small deformations, proposed in [12-14], we introduce two configurations of the structure: basic (deformed) and varied. In the basic configuration, 2 problems are considered - the problem of...
unsteady heat conduction and the three-dimensional problem of linear thermoelasticity. Let’s consider the formulation of these tasks.

The first of them is the problem of thermal conductivity for a linear thermoelastic medium:

$$\rho^0 c_v \frac{\partial \theta^0}{\partial t} = -\nabla \cdot q^0,$$

$$q^0 = -\Lambda \cdot \nabla \theta^0,$$

$$n \cdot q^0 \big|_{\Sigma_v} = q_v^0, \quad \theta^0 \big|_{\Sigma_v} = \theta_v^0, \quad t = 0: \quad \theta^0 = \theta_v^0,$$

where $\theta^0$ – temperature; $\rho^0$ - density; $c_v$ – heat capacity; $q^0$ – vector of the heat flux; $\Lambda$ – thermal conductivity tensor. Hereinafter, it is assumed that the values with the superscript 0 refer to the actual configuration.

The second problem is a quasi-static linear thermoelastic problem:

$$\nabla \cdot \sigma^0 = 0,$$

$$\sigma^0 = 4C \cdot (\varepsilon^0 - \lambda_5 \varepsilon^f), \quad \varepsilon^0 = \alpha (\theta - \theta_v), \quad \varepsilon^f = \frac{1}{2} (\nabla \otimes u^0 + \nabla \otimes u^0^T),$$

$$n \cdot \sigma^0 \big|_{\Sigma_v} = \lambda_1 S_v, \quad u^0 \big|_{\Sigma_v} = \lambda_2 u_v,$$

where $\sigma^0$ – stress tensor; $\varepsilon^0$ – small strain tensor; $\varepsilon^f$ – thermal strain tensor; $u^0$ – vector of displacement; $4C$ – elastic modulus tensor; $\alpha$ – thermal expansion tensor; $S_v$ - vectors of external surface forces, $u_v$ – vectors of external displacements; $\lambda_1$ and $\lambda_2$ – task parameters.

The procedure for calculating tensors $4C$, $\alpha$ and $\Lambda$ for composite materials with different structures of fiber reinforcement is given in [10].

The stress tensor in (2) can be considered as a function of parameters $\lambda_1$ and $\lambda_2$:

$$\sigma^0 = \sigma^0 (\lambda_1, \lambda_2).$$

Then we introduce stress tensors:

$$\sigma^0_1 = \sigma^0 (1,0), \quad \sigma^0_2 = \sigma^0 (0,1),$$

where $\sigma^0 (1,0)$ is the solution to the problem (2) for $\lambda_1 = 1$ and $\lambda_2 = 0$, and $\sigma^0 (0,1)$ is the solution to the problem (2) for $\lambda_1 = 0$ and $\lambda_2 = 1$. $\sigma^0 (0,1)$ represents the true thermal stress, and the general solution (2), due to linearity, can be represented as:

$$\sigma^0 = \lambda_1 \sigma^0_1 + \lambda_2 \sigma^0_2.$$

Because of the foregoing, the third problem – the actual buckling problem theory in the presence of thermal effects – can be written as:

$$\nabla \cdot (\lambda_1 \sigma^0_1 + \sigma^0_2) \cdot (B \cdot \varepsilon) = 0,$$

$$\sigma = 4C \cdot \varepsilon, \quad B = \nabla \otimes \omega, \quad \omega = \frac{1}{2} (\varepsilon \cdot \Omega),$$

$$\varepsilon = \frac{1}{2} (\nabla \otimes \varepsilon + \nabla \otimes \varepsilon^T), \quad \Omega = \frac{1}{2} (\nabla \otimes \omega - \nabla \otimes \omega^T)$$

$$n \cdot (\sigma - (\lambda_1 \sigma_1^0 + \sigma_2^0) \cdot \varepsilon \cdot \Omega) \big|_{\Sigma_v} = 0, \quad w \big|_{\Sigma_v} = 0,$$

where $\sigma$ – stress tensor; $\omega$ – displacement vector of varied configuration; $\varepsilon$ – small strain tensor; $\varepsilon^f$ – Levi-Civita tensor, and it is assumed that $\lambda = \lambda_1$, $\lambda_2 = 1$.

Problems (2) – (5) are solved by means of following algorithm:

1. Solve the heat conductivity problem (1).
2. Calculate the tensor field \( \sigma_0^0 \) by solving the problem (2) under the assumption \( \lambda_1 = 1 \) and \( \lambda_2 = 0 \), i.e. without taking into account thermal stresses.

3. Calculate the tensor field \( \sigma_2^0 \) by solving the problem (2) under the assumption \( \lambda_1 = 0 \) and \( \lambda_2 = 1 \), i.e. excluding external loads and displacements.

4. Calculate the field of the stress tensor \( \sigma^0 \). It is assumed that any other value of the parameter \( \lambda = \lambda_1 \) corresponds to the field of the stress tensor \( \sigma^0_1(\lambda) = \lambda \sigma^0_1(1) \) at given thermal stresses.

5. Substituting the field \( \sigma^0 \) in (5), obtain the buckling theory problem (the problem for eigenvalues).

6. Find a minimal eigenvalue \( \lambda \) and correspondent eigenfunction \( w \).

3. A computational method for solving the buckling theory problem taking into account a thermal effect

To solve problems (1) and (2), describing the actual state of the structure, the finite elements method is used. For this, the variational formulations must be formulated initially, which can be found in [18]. Let us consider in more detail the solution to the problem (5), which describes the varied state.

As shown in [15, 16], in the case of buckling problem (5), the variational equation has the form:

\[
\int_V \left( [C \cdot \varepsilon(w)] + \sigma^0 \cdot \Omega(w) \right) \cdot \partial V \otimes w^T dV = 0.
\]

Taking into account the presence of thermal stresses, this formulation can be rewritten as:

\[
\int_V \left( [C \cdot \varepsilon(w)] + \left( \lambda \sigma_1^0 + \sigma_2^0 \right) \cdot \Omega(w) \right) \cdot \partial V \otimes w^T dV = 0.
\]

Equation (7) is an eigenvalue problem in which it is required to find the eigenvalues \( \lambda \) and the corresponding eigenfunctions \( w \). Of greatest practical interest is the smallest eigenvalue \( \lambda_{\text{min}} \), since it corresponds to the critical load \( S_{\text{cr}} \) leading to the first form of buckling. All other values \( \lambda \) will correspond to other forms of the structure buckling.

To obtain a numerical setting, we also use the finite element method [17, 18]. We will assume that a tetrahedral element with three degrees of freedom at each node is used to triangulate the computational domain. Let us introduce the following notations:

\[
\{ \sigma \}^T = \left( \sigma_{11} \, \sigma_{22} \, \sigma_{33} \, \sigma_{12} \, \sigma_{13} \, \sigma_{23} \right)
\]

– vector of stress tensor components in a varied configuration;

\[
\{ \varepsilon \}^T = \left( \varepsilon_{11} \, \varepsilon_{22} \, \varepsilon_{33} \, 2\varepsilon_{12} \, 2\varepsilon_{13} \, 2\varepsilon_{23} \right)
\]

– vector of components of the tensor of small deformations in a varied configuration;

\[
\{ W \}^T = \left( W_1 \, W_2 \, W_3 \right)
\]

– vector of displacement in a varied configuration;

\[
\{ w \}^T = \left( w_{11} \, w_{12} \, w_{13} \, w_{21} \, w_{22} \, w_{23} \, w_{31} \, w_{32} \, w_{33} \, w_{41} \, w_{42} \, w_{43} \right)
\]

– the displacement vector of the finite element nods for in varied configuration;
The matrix of stress tensor components at $\alpha = 1$ and the matrix of thermal stresses at $\alpha = 2$ in the actual state:

$$\{ R \}_{9} = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{21} & R_{22} & R_{23} & R_{31} & R_{32} & R_{33} \end{pmatrix}$$ (13)

– a vector of $R_{ij} = \frac{\partial W_{j}}{\partial x_{i}}$ ;

$$[T]_{9 \times 9} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ (14)

– transposition matrix.

Second equations in (5) (generalized Hooke’s) law can be written in the following matrix form

$$\{ \sigma \} = \begin{pmatrix} C \end{pmatrix} \{ \varepsilon \},$$ (15)

and the third equations in (5) (the Cauchy relations) take the matrix form

$$\{ \varepsilon \} = \begin{pmatrix} L \end{pmatrix}_{6 \times 3} \{ W \},$$ (16)

here the differential operator $\begin{pmatrix} L \end{pmatrix}_{6 \times 3}$ is introduced:

$$\begin{pmatrix} \frac{\partial}{\partial x_{1}} & 0 & 0 & \frac{\partial}{\partial x_{2}} & 0 & \frac{\partial}{\partial x_{3}} \end{pmatrix}^{T}$$ (17)

The displacement vector in the finite element is related to the displacement vector of its nodes as follows

$$\{ W \} = \begin{pmatrix} N \end{pmatrix}_{3 \times 12} \{ w \},$$ (18)

Here $\begin{pmatrix} N \end{pmatrix}_{3 \times 12}$ is the matrix of form functions.
We calculate the variations of vectors $\{\varepsilon\}$ and $\{R\}$
\[
\begin{align*}
\{\delta\varepsilon\} &= \left[ L_3 \right] \{\delta W\}, \\
\{\delta R\} &= \left[ L_9 \right] \{\delta W\}, \\
\{\delta W\} &= \left[ N \right] \{\delta w\},
\end{align*}
\] (19)
where $\left[ L_3 \right]$ is the matrix of partial derivatives is introduced [16].

We transform the integrand in (6) taking into account the generalized Hooke's law and the expression for the tensor $\Omega(\mathbf{w})$:
\[
\left( ^t \mathbf{C} \cdot \varepsilon(\mathbf{w}) + \sigma^0 \cdot \Omega(\mathbf{w}) \right) \cdot \partial \mathbf{N} \otimes \mathbf{w}^T =
\]
\[
= \sigma \cdot \partial \mathbf{N} \otimes \mathbf{w}^T + \frac{1}{2} \sigma^0 \cdot \nabla \otimes \mathbf{w} \cdot \partial \mathbf{N} \otimes \mathbf{w}^T - \frac{1}{2} \sigma^0 \cdot \nabla \otimes \mathbf{w}^T \cdot \partial \mathbf{N} \otimes \mathbf{w}^T =
\]
\[
= G^{(1)} + G^{(2)} + G^{(3)},
\] (20)
where we have introduced the notations for the terms:
\[G^{(1)} = \sigma \cdot \partial \mathbf{N} \otimes \mathbf{w}^T, \quad G^{(2)} = \frac{1}{2} \sigma^0 \cdot \nabla \otimes \mathbf{w} \cdot \partial \mathbf{N} \otimes \mathbf{w}^T, \quad G^{(3)} = -\frac{1}{2} \sigma^0 \cdot \nabla \otimes \mathbf{w}^T \cdot \partial \mathbf{N} \otimes \mathbf{w}^T.
\]

With the notations introduced above we get:
\[
G^{(1)} \Box \{\sigma\}^T \left[ B_6 \right] \{\delta w\}, \quad G^{(2)} \Box \frac{1}{2} \{\delta w\}^T \left[ B_9 \right] \{\sigma^0\} \left[ B_3 \right] \{\mathbf{w}\}, \quad -G^{(3)} \Box \frac{1}{2} \{\delta w\}^T \left[ B_9 \right] \{\sigma^0\} \left[ T \right] \left[ B_3 \right] \{\mathbf{w}\}.
\] (21)

where $\left[ B_6 \right] = \left[ L_3 \right]{\left[ N \right]}$ and $\left[ B_9 \right] = \left[ L_9 \right]{\left[ N \right]}$ are matrices of derivatives of form functions and a matrix
\[
\left[ \sigma^0 \right] = \lambda \left[ \sigma^0 \right]^+ + \left[ \sigma^0 \right]^\circ
\]
is introduced. The product of symmetric matrices $\left[ \sigma^0 \right]$ and $\left[ T \right]$ gives in the general case an asymmetric matrix. However, one can perform symmetrization using the properties of the quadratic form and assuming, due to the arbitrariness of the variation of $\{\delta w\}^T$, that $\{\delta w\}^T = \{\mathbf{w}\}^T$.

Then we have:
\[
G^{(2)} + G^{(3)} = -\frac{1}{2} \{\delta w\}^T \left[ B_9 \right] \left\{ \sigma^0 \right\} \left[ B_3 \right] \{\mathbf{w}\},
\] (22)

where $\left[ \Sigma^0 \right] = \left[ \Sigma^0 \right]^+ + \lambda \left[ \Sigma^0 \right]^\circ$ – symmetric matrix, and
\[
\left[ \Sigma^0 \right]^\circ = \frac{1}{2} \left( \left[ \sigma^0 \right]^T \left[ \sigma^0 \right]+ \left[ \sigma^0 \right]^T \right), \quad \alpha = 1, 2, \quad \left[ T \right] = \left[ T \right] - \left[ E \right],
\]

$\left[ E \right]$ – identity matrix.

Suppressing relations (15) - (21) into the variational equation (7), we obtain the final homogeneous system of linear algebraic equations for finding the eigenvalues
\[
\left[ K \right] \{\mathbf{w}\} = \lambda \left[ S \right] \{\mathbf{w}\},
\] (23)

where:
\[
\left[ K \right] = \left[ K_1 \right]^+ \left[ K_2 \right],
\]
\[
\left[ K_1 \right] = \frac{1}{2} \int \left[ B_1 \right]^T \left[ C \right] \left[ B_1 \right] dV, \quad \left[ K_2 \right] = \frac{1}{4} \int \left[ B_2 \right]^T \left[ B_2 \right] dV,
\]
\[
\left[ S \right] = \frac{1}{4} \int \left[ B_3 \right]^T \left[ \Sigma^0 \right] \left[ B_3 \right] dV.
\]
Here matrix \( \mathbf{K}_{12} \) determines the contribution of thermal stresses to the buckling of the structure.

After performing the ensemble procedure, the resulting generalized eigenvalue problem can be solved by using Subspace iterative methods or the Lanczos method.

4. Results of modeling

Let us consider the application of the presented model by the example of calculating the buckling of a molybdenum rod with dimensions \( 1 \text{ m} \times 0.05 \text{ m} \times 0.05 \text{ m} \). In the ground state, one of the ends of the rod was rigidly fixed, and a longitudinal compressive load of 0.1 GPa was applied to the opposite end. A temperature of 800 K acted on two adjacent sides of the rod, and a temperature of 293 K acted on two other adjacent sides. The results were obtained based on using the SMCM CAE Software, developed at the SIMPLEX Scientific and Educational Centre of Bauman Moscow State Technical University and are shown in the figure. 1. Figure 2 shows the results of solving a similar problem using the CAE ANSYS tools. Comparison of the obtained eigenvalues is shown in table 1.

![Figure 1. The results of solving the buckling problem obtained in SMCM CAE Software. The following components are shown: a) \( w_1 \); b) \( w_2 \); c) \( w_3 \); d) \( \sigma_{11} \); e) \( \sigma_{22} \); f) \( \sigma_{33} \); g) \( \sigma_{12} \); h) \( \sigma_{13} \); i) \( \sigma_{23} \) - rerun]
Figure 2. The results of solving the buckling problem obtained in CAE ANSYS. The following components are shown:

a) \( w_1 \); b) \( w_2 \); c) \( w_3 \); d) \( \sigma_{11} \); e) \( \sigma_{22} \); f) \( \sigma_{33} \); g) \( \sigma_{12} \); h) \( \sigma_{13} \); i) \( \sigma_{23} \)

Table 1. Comparison of the loading parameters (eigenvalues) values obtained in the CAE Manipula and in CAE ANSYS

|                  | CAE Manipula | CAE ANSYS | Relative error, % |
|------------------|--------------|-----------|-------------------|
|                  | 1,0912       | 1,0911    | 0,009             |

As can be seen from the above results, the solution obtained in the SMCM CAE Software is in good agreement with the solution from CAE ANSYS.

5. Conclusions
The paper proposes a mathematical model for 3D calculation of the stability of linear elastic structures taking into account non-uniform non-stationary heating. A numerical algorithm is proposed for taking into account temperature deformations when calculating eigenvalues and critical loads. The developed model was tested on a 3D problem of calculating the stability of a composite anisotropic rod, which showed good agreement with the results obtained using CAE ANSYS.

References
[1] Timoshenko S P and Gere J M 1961 Theory of elastic stability. 2nd ed. (New York/Toronto/London: McGraw-Hill) p 356
[2] Donnell L G 1982 Beams, plates and covers (Moscow: Nauka Publ.) p 568
[3] Simitses G J 1976 An introduction to the elastic stability of structures (Prentice Hall, NJ) p 256
[4] Iyengar N.G.R. 1986 Structural stability of columns and plates (New Delhi: Affiliated East-West Press) p 284
[5] Bazant Z P and Cedolin L 1990 Stability of structures (Oxford: Oxford University Press) p 316
[6] Zhukov A E and Volkov V M 2007 Experimental and theoretical research of stability of plates with cracks Problems of durability and plasticity: Interuniversity collection. 69, pp.
13-17

[7] Pikul V V 2008 Current state of the theory of stability of covers *Bulletin of the Far Eastern Branch of the Russian Academy of Sciences* 3 pp. 3-9

[8] Van’ko V I 2014 *Sketches according to the theory of stability of elements of designs* (Moscow: Izd-vo MGTU im. N.E. Baumana Publ.) p 220

[9] Solomonov Yu S Georgievskij V P Nedbaj A Ya and Andryushin V A 2014 *Applied problems of mechanics of composite cylindrical covers* (Moscow: Fizmatlit Publ.) p 408

[10] Dimitrienko Yu I 1999 *Thermomechanics of Composites under High Temperatures* (Springer) p 370

[11] Kohanenko Yu V 2009 Three-dimensional stability of the cylinder at a non-uniform initial state *Reports of the National Academy of Sciences of Ukraine* 1, pp. 60-62

[12] Dimitrienko Yu I 2013 Generalized three-dimensional theory of elastic bodies stability. Part 1: finite deformations *Herald of the Bauman Moscow State Tech. Univ., Nat. Sci.* 4 (51) pp. 79-95

[13] Dimitrienko Yu I 2013 *Continuum mechanics* vol. 4. *Fundamentals of solid mechanics* (Moscow: BMSTU Publ.) p 624

[14] Dimitrienko Yu I 2014 Generalized three-dimensional theory of elastic bodies stability. Part 2: small deformations *Herald of the Bauman Moscow State Tech. Univ., Nat. Sci.* 1 pp. 17-26

[15] Dimitrienko Yu I and Bogdanov I O 2016 Finite-element method for three-dimensional problems of elastic structures buckling theory *Herald of the Bauman Moscow State Technical University, Series Natural Sciences* 6 pp 73-92

[16] Dimitrienko Yu I and Bogdanov I O 2020 Numerical simulation of the stability of three-dimensional elastic composite structures based on the finite element method *IOP Conf. Series: Materials Science and Engineering* 934 (1) 012011. DOI: 10.1088/1757-899X/934/1/012011

[17] Dimitrienko Yu I Gabareva E A and Yurin Yu V 2015 Finite element modeling of processes of thermocreep on the basis of Runge-Kutta's methods *Science and education. Electronic magazine* 3, pp 296-312. DOI: 10.7463/0315.0759406 http://technomag.bmstu.ru/doc/759406.html

[18] Dimitrienko Yu I and Yurin Yu V 2015 Finite element modeling of the intense deformed condition of rocks taking into account creep *Mathematical modeling and numerical methods* 3 pp 101-118