A New Approach to Analyzing Robin Hood Hashing

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Abstract

Robin Hood hashing is a variation on open addressing hashing designed to reduce the maximum search time as well as the variance in the search time for elements in the hash table. While the case of insertions only using Robin Hood hashing is well understood, the behavior with deletions has remained open. Here we show that Robin Hood hashing can be analyzed under the framework of finite-level finite-dimensional jump Markov chains. This framework allows us to re-derive some past results for the insertion-only case with some new insight, as well as provide a new analysis for a standard deletion model, where we alternate between deleting a random old key and inserting a new one. In particular, we show that a simple but apparently unstudied approach for handling deletions with Robin Hood hashing offers very good performance even under high loads.

1 Introduction

Robin Hood hashing is a variation on open addressing hashing designed to reduce the maximum search time as well as the variance in the search time for elements in the hash table. Here we are interested in the setting where the probe sequences are random. We briefly describe the setup, starting with a setting with insertions only. We have a hash table with \( n \) cells, and \( m = \lceil \alpha n \rceil \) keys to place in the table. We refer to \( \alpha \) as the load of the table; generally, we assume \( \alpha n \) is an integer henceforth. Each key \( K_i \) has an associated infinite probe sequence \( K_{ij} \), with \( j \geq 1 \), where the \( K_{ij} \) are uniformly distributed over \([0, n-1]\). Equivalently, the \( K_{ij} \) are determined by a random hash function \( h \), where for a keyspace \( K \) the hash function has the form \( h : K \times \mathbb{N} \rightarrow [0, n-1] \). Each key will be placed according to a position in its probe sequence. If the \( i \)th element is placed in cell \( K_{ij} \), and there is no \( j' < j \) such that \( K_{ij'} = K_{ij} \), we shall say that the age of the key is \( j \). If we use the standard search process for an key, by which we mean sequentially examining cells according to the probe sequence, the age of a key in the table corresponds to the number of cells that must be searched to find it. We assume that we keep track of the age of the oldest key in the table. In the standard search process, one determines that a key not in the table is not present by sequentially examining cells according to the probe sequence until either an empty cell is found, or one has found that the key being searched for must be older than oldest key in the table. An empty cell provides a witness that the key is not in the table. We refer to a search for a key not in the system as an unsuccessful search.

For the insertion of keys in the table, we may think of the keys as being placed sequentially, using the probe sequence in the following manner. If \( K_{i1} \) is empty when the \( i \)th key is inserted into the table, the key is readily placed at cell \( K_{i1} \). Otherwise, there is a collision, and a collision resolution strategy is required. The key point of Robin Hood hashing is that it resolves collisions in favor of the key with the larger age; the key with the smaller age must continue sequentially through its probe sequence. Notice that, under Robin Hood hashing, a placed key will be displaced by the key currently being placed if the placed key’s age is smaller. In this case the placed key is moved from its current cell and becomes the item to be placed, consequentially increasing its age. Other standard conflict resolution mechanisms are first come first served and last come first served. By favoring more aged keys, Robin Hood hashing aims to reduce the maximum search time required for a key in the table.

Most of the results for Robin Hood hashing appear in the thesis of Celis [2], who provides a number of theoretical and empirical results regarding the performance of Robin Hood hashing. (See also [3].) The following results are especially worth mentioning. First, in the setting where there are only insertions, Celis analyzes the asymptotic behavior of Robin Hood hashing (in the infinite limit setting) for loads \( \alpha < 1 \). We describe this result further in

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1Alternatively, we could have each probe sequence be a random permutation of \([0, n-1]\) for each key; for our purposes, the two models are essentially equivalent, and we use the random hash function model as it is easier to work with.
Section 3.2 Second, Celis shows that the total expected insertion cost in terms of the number of probes evaluated by the standard insertion process – or equivalently the average age of keys in the table – is the same for a class of “oblivious” collision resolution strategies that do not make use of knowledge about the future values in the probe sequences that includes Robin Hood hashing (as well as first come first served and last come first served). Third, Devroye, Morin, and Viola have shown that for $\alpha < 1$ the maximum search time for Robin Hood hashing is (upper and lower) bounded by $\log_2 \log_2 n \pm O(1)$ with probability $1 - o(1)$, where the $O(1)$ terms depend on $\alpha$ [4]. This double-logarithmic behavior occurs with other seemingly quite different hashing schemes based on the power of multiple choices [11] [11]. Finally, we note that Robin Hood hashing has been also studied extensively in the setting of linear probing schemes [5] [13] [4].

In this paper, we provide a new approach for analyzing Robin Hood hashing, based on a fluid limit analysis utilizing differential equations. An interesting aspect of our analysis is that it requires using an additional level parameter, corresponding to a faster-moving Markov process (tracking the age of current key being placed) beyond the larger-scale Markov process (tracking the distribution of ages in the table). This type of analysis was previously used to study load balancing schemes with memory [7] [10]. Our analysis allows us to re-derive previous results for Robin Hood hashing, such as the asymptotic behavior for loads $\alpha < 1$, while also providing some additional novelty, such as concentration bounds for finite $n$. More importantly, our approach is amenable to studying Robin Hood hashing with deletions of random keys, an area that lacked a theoretical framework for analysis previously. We study the deletion scheme proposed by Celis in [2] under the setting of random deletions of keys and new keys being inserted (maintaining a constant load $\alpha$), and suggest and analyze an alternative deletion scheme that is simpler for practical implementations (although not necessarily more efficient).

In what follows, we provide background on the fluid limit approach we use here. We then study Robin Hood hashing in the setting of insertions only under this framework, and subsequently move on to examining how to analyze settings with random deletions.

2 Limiting Framework

For our limiting framework, we can work in the setting of finite-level finite-dimensional jump Markov chains. Here we roughly follow the exposition of [10]; further development can be found in [12]. Our discussion here is brief, and may be skipped by the uninterested reader willing to accept the more intuitive explanations that follow. However, we point out that the use of the framework of finite-level finite-dimensional jump Markov chains is surprisingly rare in studying hashing schemes; we expect the approach should be useful for studying other hashing variations.

In our setting, a chain with $D$ dimensions and $L$ levels will have the state space $\mathbb{R}^D \times \{1, 2, \ldots, L\}$. The state can be represented as a $D + L$-tuple in the natural way as follows: $(\bar{x}; m) = (x_1, \ldots, x_D; 0, 1, \ldots, 1, 0)$, where a 1 in position $D + m, 1 \leq m \leq L$, represents that the system is in level $m$. When in state $(\bar{x}; m)$ the system can make $\zeta(m)$ possible different jumps. Here we describe only unit jumps based on unit vectors, which suffices for our main application, but more general jumps are possible. The process jumps to state $(\bar{x} + \bar{e}_i(m); \bar{m}(m, i)) = (\bar{x} + e_i(m); 0, \ldots, 1 - \gamma, 0, \ldots, \gamma, \ldots, 0)$ with rate $\nu_i(\bar{x}; m)$, for $1 \leq i \leq \zeta(m)$, and $\gamma \in \{0, 1\}$. Here $e_i(m)$ is a unit vector in one of the $D$ dimensions, and, depending on value of $\gamma$, the level may change or remain as it is. The high-level idea is that here we have an underlying finite-dimensional jump Markov process, but we also have an additional associated “level” process that may drive the transition rates of the primary jump Markov process.

The generator $A$ of this Markov process, which operates on real valued functions $f : \mathbb{R}^{D+L} \rightarrow \mathbb{R}$ is defined as:

$$Af(\bar{x}; m) = \sum_{i=1}^{\zeta(m)} \nu_i(\bar{x}; m)[f(\bar{x} + \bar{e}_i(m); \bar{m}(m, i)) - f(\bar{x}; m)]$$

We now consider a scaled version of this process, with scaling parameter $n$, where the rate of each transition is scaled up by a factor of $n$ and the jump magnitude is scaled down by a factor of $n$. The state of this scaled system will be represented by $(\bar{s}_n; m) = (s_1, \ldots, s_D; 0, \ldots, 1/n, \ldots, 0)$. The associated jump vectors will be $(\bar{e}_i/n; 0, \ldots, 0, 1/n, \ldots, \gamma/m, \ldots, 0)$, with corresponding rates $n\nu_i(\bar{s}_n; m)$ for $1 \leq i \leq \zeta(m)$. The generator for the scaled Markov process is:

$$A_nf(\bar{s}_n; m) = \sum_{i=1}^{\zeta(m)} n\nu_i(\bar{s}_n; m)[f(\bar{s}_n + \bar{e}_i/m; 0, \ldots, 1/n, \ldots, \gamma/m, \ldots, 0) - f(\bar{s}_n; m)].$$

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The following theorem (Theorem 8.15 from [12]) describes the evolution of the typical path of the scaled Markov process in the limit as \( n \) grows large. The idea behind the theorem is that because the finite-level Markov chain reaches equilibrium in some finite time, for large enough \( n \) the approximation that the finite-level Markov chain is in equilibrium is sufficient to obtain Chernoff-like bounds.

**Theorem 1:** Under Conditions 1 and 2 below, for any given \( T \) and constant \( \epsilon > 0 \), there exist positive constants \( C_1, C_2(\epsilon) \) and \( n_0 \) such that for all initial positions \( \bar{s}^0 \in \mathbb{R}^D \), any initial level \( m \in \{0, 1, \ldots, L-1\} \), and any \( n \geq n_0 \),

\[
\Pr_{\bar{s}^0, m} \left( \sup_{0 \leq t \leq T} |\bar{s}_n(t) - \bar{s}_\infty(t)| > \epsilon \right) \leq C_1 \exp(-nC_2(\epsilon)).
\]

where, \( \bar{s}_\infty \) satisfies the following:

\[
\frac{d}{dt}\bar{s}_\infty(t) = \sum_{l=0}^L \Pr(m(t) = l) \sum_{i} \nu_l(\bar{s}_\infty; l)\bar{e}_i(l) \tag{4}
\]

where \( \Pr(m(t) = l) \) is the equilibrium probability of the level-process being in level \( l \) given the state \( \bar{s}_\infty(t) \).

**Condition 1:** For any fixed value of \( \bar{x} \in \mathbb{R}^D \), the Markov process evolving over the levels \( \{1, \ldots, L\} \) with transition rate \( \nu_i(\bar{x}; m) \) of going to level \( \bar{m}(m, i) \) from level \( m \), is ergodic.

**Condition 2:** The functions \( \log \nu_i(\bar{x}; y) \) are bounded and Lipschitz continuous in \( \bar{x} \) for every \( y \) (where continuity is in all the \( D \) coordinates).

We note that a limitation of this approach is that it provides bounds only on the finite-dimensional version of the process. It is thus not clear how to extend these bounds to Robin Hood of the form \( \log_2 \log_2 n \pm O(1) \) as described directly using this approach, as tracking \( \log_2 \log_2 n \) dimensions takes us outside the finite-dimensional realm. Instead, one may use these results as useful intuition for guiding non-limiting probabilistic arguments such as that derived in [4]. In return for this limitation, however, this approach provides simple and general means for generating rich, accurate numerical results that can aid in design and performance testing for real-world implementations.

## 3 Robin Hood Hashing with Insertions Only

### 3.1 Applying the Limiting Framework

We first describe the unscaled process, and we need to describe what we mean by one time step of the process. Each time step corresponds to an attempt to place a key, which can either be a new key, or a key that was not successfully placed at the last time step, or a key that was displaced by another key at the last time step.

To keep track of the state, we take advantage of the fact that keys are placed randomly into cells. Hence, for the sake of simplicity, we track the number of cells holding keys for each age; their actual position does not matter. Each time step corresponds to an attempt to place a key. Note that the number of time steps here is not equal to the number of keys placed; placing a new key can take several time steps with Robin Hood hashing, and as discussed during that process the key being placed can take the place of another key which then has to be placed. Each such placement attempt represents a time step. As is often the case with hashing schemes, we find it more useful to look at the tails of the loads rather than the loads themselves. Therefore, we let \( x_i(t) \) be the fraction of cells with a key with age at least \( i \) after \( t \) unscaled time steps. For our scaled version of the state, we let \( s_i(t) \) be the fraction of cells holding a key of age at least \( i \) after \( tn \) key placements have been tried, so that \( x_i(nt) = s_i(t) \). The level in our process will correspond to the age of key currently being placed. Fresh keys that are newly being inserted have age 1.

We note that, as described, the process is infinite-dimensional, in that we can consider the values \( s_i \) for all \( i \geq 1 \). Indeed, this is usually how we will think of the process, although as we show later we can “truncate” the system at any finite value of \( i \), which can allow us to apply Theorem 1.
As a warm-up in understanding the scaled process, note that when the load of the table is $z$, so that $zn$ cells contain a key, the number of time steps to place an element is geometrically distributed with mean $1/(1 - z)$. Hence in the limiting scaled process $\tilde{s}_\infty$, with the initial state being an empty table, the load will be $\alpha$ at time $t$

$$\int_{z=0}^{\alpha} \frac{1}{1 - z} \, dz = \ln \frac{1}{1 - \alpha}. $$

That is, we run until time $\ln \frac{1}{1 - \alpha}$, which corresponds to (in the unscaled process, asymptotically) $n \ln \frac{1}{1 - \alpha}$ time steps. Alternatively, in the limiting scaled process, at time $t$, the load is $1 - e^{-t}$.

We now turn to understanding the level process, assuming that the state of the table is fixed at certain values $s_i$. Again we find it useful to consider the tails. Thinking of the unscaled process, so $t$ again refers to discrete time steps, let $p_i(t)$ be the probability that the age of the key being placed is at least $i$. Hence $p_i(t) = 1$ for all time steps. For $i > 1$, the key being placed at time $t + 1$ will have age at least $i$ if and only if both the age of the key being placed at time $t$ has age at least $i - 1$, and cell chosen by the probe sequence at time $t$ has age at least $i - 1$. This is because, assuming an empty cell is not found, the younger of the keys will be the key being placed at the next time step. In equation form, we simply have

$$p_i(t + 1) = p_{i-1}(t) \frac{x_{i-1}(t)}{n}. $$

In the scaled time setting, this can be written as

$$p_i(t + 1/n) = p_{i-1}(t)\tilde{s}_{i-1}(t). $$

Notice that the simplicity of this equation helps justify our decision to focus on variables that represent the tails of the loads.

Since $s_1$ is bounded by the final load $\alpha$, at every placement, with probability at least $1 - \alpha$ the key is placed and the level returns to 1. Hence the Markov process over the levels is ergodic. Indeed, standard methods show that for any constant $\epsilon$ this Markov can be made $\epsilon$-close in statistical distance to its equilibrium distribution after some corresponding constant number of placement steps. We re-emphasize the intuition; in the scaled process, the $s_i$ values change significantly (that is, by $\Omega(1)$) only after $\Omega(1)$ scaled time steps (or $\Omega(n)$ unscaled time steps), while the Markov chain governing the $p_i$ converges (arbitrarily closely) to its stationary distribution in $o(n)$ unscaled time steps. Hence, it make sense in the limit to treat the $p_i$ values as in equilibrium given the current $s_i$ values.

It follows from Equation (8) that the equilibrium for the level process satisfies

$$p_i = p_{i-1}s_{i-1}$$

when we treat the $s_i$ as fixed and we use $p_i$ without the $t$ to denote the equilibrium for the $p_i$ values given the $s_i$ values at that time.

With this we turn our attention to the limiting equations for the $s_i$ in $\tilde{s}_\infty$. Note that $s_1$ increases whenever a empty cell is hit. Hence

$$\frac{ds_1}{dt} = 1 - s_1.$$

Integrating, and using $s_1(0) = 0$, this gives $s_1(t) = 1 - e^{-t}$, matching our previous warm-up analysis. For $s_i$ when $i > 1$, Equation (8) generalizes to

$$\frac{ds_i}{dt} = p_i(1 - s_i),$$

since a cell containing a key with age at least $i$ is created whenever the age of the key being placed is at least $i$ and the probe sequence finds either an empty cell or a cell containing a key with age less than $i$.

At any time $t$, let $\beta(t)$ be the corresponding load at that time. (Recall we use $\alpha$ for the “final” load.) We use $\beta$ for $\beta(t)$ where the meaning is clear. Since $\beta(t) = 1 - e^{-t}$, we can substitute using

$$\frac{d\beta}{dt} = e^{-t} = 1 - \beta.$$
At the possible risk of confusion, but to avoid conversions back and forth, we use \( s'_i(\beta) \) to represent \( s_i \) taken as a function of the load \( \beta \) instead of as a function of time. We have from the above that in the setting of the asymptotic limit \( s'_i(\beta) = s_i(-\ln(1 - \beta)) \). With the expressions for \( \frac{ds_i}{dt} \) and \( \frac{d\bar{s}_i}{dt} \) we obtain the following form for \( s'_i(\beta) \) as a function of \( \beta \) for \( i \geq 1 \).

\[
\frac{ds'_i}{d\beta} = \frac{p_i(1 - s'_i)}{1 - \beta}.
\]

Given our equations for \( p_i \), we can substitute so that all equations are in terms of the \( s_i \). Specifically, the equation for \( \frac{ds_i}{dt} \) (or \( \frac{d\bar{s}_i}{dt} \)) depends only on values \( s_j \) with \( j \leq i \). The differential equations can therefore be solved numerically for \( s_i \) values up to any desired constant \( K \). Moreover, we can truncate the infinite system of differential equations to a finite system by considering the equations for \( \frac{d\bar{s}_i}{dt} \) up to the constant \( K \). Because of this, using the large deviation theory, we may formally state the following:

**Theorem 2:** For any fixed constant \( K \) and any \( \alpha < 1 \), for \( i \leq K \) let \( s'_i(\alpha) \) be the solution for the \( s'_i \) at final load \( \alpha \) from the family of differential equations given by Equation (10) above. For \( 1 \leq i \leq K \), let \( X_{i,n} \) be the random variable denoting the fraction of cells with keys of age at least \( i \) using Robin Hood hashing at final load \( \alpha \) with \( n \) cells. Then for any \( \epsilon > 0 \), for sufficiently large \( n \)

\[
\Pr (|X_{i,n} - s'_i(\alpha)| > \epsilon) \leq C_1 \exp(-nC_2(\epsilon)),
\]

where \( C_1 \) is a constant that depends on \( K \) and \( \alpha \) and \( C_2(\epsilon) \) is a constant that depends on \( K \), \( \alpha \), and \( \epsilon \).

**Proof:** The result follows from Theorem 1. While Theorem 1 is stated in terms of time instead of load, this difference in not consequential, as we explain subsequently. Let \( Y_{i,n} \) be the random variable denoting the fraction of cells with keys of age at least \( i \) using Robin Hood hashing after \(-n\ln(1 - \alpha)\) unscaled time steps in a hash table with \( n \) cells. Then Theorem 1 gives us that

\[
\Pr (|Y_{i,n} - s_i(-\ln(1 - \alpha))| > \epsilon) \leq C_3 \exp(-nC_4(\epsilon)),
\]

where \( C_3 \) is a constant that depends on \( K \) and \( \alpha \) and \( C_4(\epsilon) \) is a constant that depends on \( K \), \( \epsilon \), and \( \alpha \). Note that this depends on our restriction of the system to be finite dimensional, and the fact that the evolution of the \( s_i \) for \( i \leq K \) only depends on the values \( s_1, s_2, \ldots, s_K \). (Again, this is another reason to write equations in terms of the tails of the loads.) The conditions of Theorem 1 are easily checked. In particular, we have noted the Markov process over levels is ergodic for any given \( s_i \) values. For the second condition, the transition rates \( \nu_i(\bar{x}; m) \) are finite size polynomials of load vector \( \bar{x} \). This implies that \( \log \nu_i(\bar{x}; m) \) is Lipschitz continuous in coordinates of \( \bar{x} \). In particular, the bound \( x_i \leq 1 \) gives an upper bound. If \( \nu_i(\bar{x}; m) = 0 \), then it means the transition is absent and we neglect it. Otherwise \( x_i \geq 1/\gamma \), giving a lower bound on the \( \log \nu_i(\bar{x}; m) \). This completes the check of condition 2.

Now note that \( s'_i(\alpha) = s_i(-\ln(1 - \alpha)) \) in the limiting system. We find \( X_{i,n} \) and \( Y_{i,n} \) differ by \( o(1) \) terms with high probability, and in fact

\[
\Pr (|Y_{i,n} - X_{i,n}| > \gamma) \leq C_5 \exp(-nC_6(\gamma)),
\]

where \( C_5 \) is a constant that depends on \( K \) and \( \alpha \) and \( C_6(\gamma) \) is a constant that depends on \( K \), \( \gamma \), and \( \alpha \). We sketch the reasoning: consider the coupling where we perform the Robin Hood process for the maximum of \(-n\ln(1 - \alpha)\) time steps and the number of time steps to reach load \( \alpha \). The load after \(-n\ln(1 - \alpha)\) time steps will be \( \alpha \pm o(1) \) with high probability, by standard martingale arguments; alternatively, the number of time steps to reach load \( \alpha \) is \(-n\ln(1 - \alpha) + o(n) \) with high probability by standard Chernoff-type bounds, since the number of time steps to place each key is an independent geometric random variable with bounded mean. The theorem statement holds by summing the probability that \( \Pr (|Y_{i,n} - X_{i,n}| > \epsilon/2) \) and \( \Pr (|Y_{i,n} - X_{i,n}| > \epsilon/2) \).

### 3.2 Implications for the Age Distribution

In [2], Celis derives the age distribution under Robin Hood hashing by providing a recurrence. We demonstrate that this result also follows from our differential equations analysis. We note that we use a different notation; the following theorem corresponds to Theorem 3.1 of [2].
Theorem 3: In the asymptotic model for an infinite Robin Hood hash table with load factor $\beta$ ($\beta < 1$), the fraction $s_i(\beta)$ of cells that contain keys of age at least $i$ is given by

$$s_i(\beta) = 1 - (1 - \beta)e^{-\sum_{j=1}^{i} s_j(\beta)}.$$  \hfill (13)

Proof: As standard techniques can be used to show that our family of differential equations has a unique solution, we show that the recurrence of Equation (13) satisfies Equation (10). We first note the following useful fact:

$$\sum_{j=1}^{i} \frac{ds_j}{d\beta} = \frac{1 - \prod_{j=1}^{i} s_j}{1 - \beta}.$$  

This follows easily from Equation (10) by induction, using that $p_i = \prod_{j=1}^{i-1} s_j = \prod_{j=i}^{i-1} s_j$, as is easily derived from Equation (7).

Now taking the derivative of Equation (13) we find

$$\frac{ds_{i+1}}{d\beta} = e^{-\sum_{j=1}^{i} s_j} - (1 - \beta) \left( \sum_{j=1}^{i} \frac{ds_j}{d\beta} \right) e^{-\sum_{j=1}^{i} s_j}$$

$$= \frac{1 - s_{i+1}}{1 - \beta} - (1 - s_{i+1}) \frac{1 - \prod_{j=1}^{i} s_j}{1 - \beta}$$

$$= \frac{\prod_{j=1}^{i} s_j}{1 - \beta}$$

$$= \frac{p_{i+1}(1 - s_{i+1})}{1 - \beta}.$$  

Hence the recurrence of Equation (13) satisfies Equation (10) as claimed. \hfill \blacksquare

3.3 Implications for Maximum Age

We now show, following an approach established in [8], that the fact that the growth of maximum age grows double logarithmically in $n$ appears as a natural consequence of the differential equations. As noted, the general large deviation results we apply only hold for finite-dimensional systems, so our result holds only for sufficiently largest constants $i$. Explicitly proving an $O(\log \log n)$ bound on the maximum age would require translating the differential equation argument to (for example) a layered induction argument, successively bounding the fraction of cells holding keys of age $i$ for each $i$. While it does not appear motivated by the differential equations approach, such a layered induction argument appears already in [4], formally providing the $O(\log \log n)$ bound. Our goal here is rather to show how the fluid limit argument provides novel insight into how these bounds arise.

Theorem 4: In the asymptotic model for an infinite Robin Hood hash table with load factor $\alpha < 1$, for sufficiently large constants $i$, the fraction $s_i(\alpha)$ of cells that contain keys of age at least $i$ satisfies

$$s_i(\alpha) \leq c_1 c_2^{2^{-c_3}}$$  \hfill (14)

for some constants $c_1, c_3 > 0$ and $c_2 < 1$ that may depend on $\alpha$.

Proof: In what follows here let $u = -\ln(1 - \alpha)$. Let $j$ be the smallest value such that $s_j(u) < u^{-1}$. Let $s_j(u) = \nu$ and $\nu \cdot u = \nu^* < 1$. We remark that while it may not be immediately clear that the $s_j$ go to 0, it is shown in [4] Lemma 4] that the $s_j$ have geometrically decreasing tails, so that $j$ is in fact a constant. (Alternatively, since the $p_j$ clearly have geometrically decreasing tails, it follows readily that the $s_j$ do as well.) Now below we use the differential equations based on time, up until time $u$, so $t \leq u$. As $p_{j+1} \leq s_j$, and $s_j$ is increasing over time

$$\frac{ds_{j+1}(t)}{dt} = p_{j+1}(1 - s_{j+1}) \leq s_j \leq \nu.$$
To reach load $\alpha$ we run for time $u = \ln \frac{1}{1-\alpha}$, and hence
\[ s_{j+1}(\alpha) = s_{j+1}(u) \leq \nu \cdot u = \nu^* . \]

Inductively, we now find by the same argument that for $k \geq 1$,
\[ p_{j+k} \leq (\nu^*)^{2^{k-1}-1} \nu ; \quad s_{j+k} \leq (\nu^*)^{2^{k-1}}, \]
and the result follows. That is,
\[ p_{j+k} = p_j \prod_{\ell=0}^{k-1} s_{j+\ell} \leq \nu \prod_{\ell=1}^{k-1} s_{j+\ell} \leq (\nu^*)^{2^{k-1}-1} \nu, \]
where the last step follows from the inductive hypothesis. Further, as
\[ \frac{ds_{j+k}(t)}{dt} \leq p_{j+k} \leq (\nu^*)^{2^{k-1}-1} \nu, \]
we have
\[ s_{j+k}(\alpha) \leq (\nu^*)^{2^{k-1}-1} \nu u = (\nu^*)^{2^{k-1}}. \]
The theorem follows.

### 3.4 Implications for Unsuccessful Search Times

For insertions-only tables, there is a simple optimization that speeds up unsuccessful searches over standard search for Robin Hood hashing. If the key being searched for is at the $i$th position in its probe sequence, and the probe yields a cell with a key with age strictly less than $i$, then the key cannot be in the table. This is because if the key were in the table it would have replaced the younger key in this cell. Making use of this fact can allow us to short-circuit an unsuccessful search early, before reaching an empty cell.

This brings up a point that we have not mentioned previously. For Robin Hood hashing, it is useful to keep track of the age of each key with each key. This may require a small number of extra bits per cell, which is not unreasonable if keys are large. As we will see, even for high loads, 3 bits may be sufficient, and 4 bits handles most insertion-only cases in practice. Of course one can always re-derive the age of a key on the fly by re-computing hash values from the key, but we expect keeping the age of the key will prove more efficient. Note that the probability that an unsuccessful search takes at least $j$ probes (up to the longest probe sequence in the system) is $\prod_{k=1}^{j-1} s_k$, since on the $k$th probe it would need to find a cell with age at least $k$ to continue. In the asymptotic limit, we have that the expected number of probes for an unsuccessful search is thus
\[ \sum_{j=1}^{\infty} j \prod_{k=1}^{j-1} s_k = \sum_{j=1}^{\infty} jp_j. \]
That is, the expected number of probes for an unsuccessful search is just the expected age according to the equilibrium distribution given by the $p_j$ at the final load $\alpha$.

### 3.5 Simulations

In this section we provide simulation results. These results serve the dual purpose of demonstrating the effectiveness of Robin Hood hashing and verifying our analysis. We note that simulation results were also presented in [2], and one might look there for further discussion on the effectiveness of Robin Hood hashing. Here, the simulation results are presented for completeness, to provide a high-level verification of the utility of theoretical framework.

Table [1] show results with a load $\alpha$ of 0.95 on the hash table. The fraction of keys in the table with a given age (up to 7) are given. The results from the differential equations were calculated using the standard Euler’s method with discrete time steps of length $10^{-6}$; that is, we calculate successive estimates of the variables from the differential equations using the derivative at the current values and advance time in steps of $10^{-6}$. We also calculate the results from the recurrence of Theorem [3]. For our simulation results, the probe sequence for all of the elements were determined.
### Handling Deletions

In this section, we demonstrate that we can analyze the deletion of keys from the system, under the assumption that deletions are of random keys. Specifically, we show that we can analyze a standard model whereby we load a number of keys into the system, and then alternate between deleting a key chosen uniformly at random from the table, and inserting a new fresh key. Our goal is to use the differential equations analysis to consider the long-term behavior and the steady state (if one exists) of such a system. We show that we can analyze the deletion method described in Celis's thesis [2], which is the only deletion scheme we know of previously proposed for Robin Hood hashing. We then introduce and analyze another deletion scheme that we believe could be more suitable in practice. Our scheme is much simpler, and interestingly, the equations we derive to model their behavior are simpler as well. While our deletion approach may not be as effective as an optimized version of the approach suggested by Celis, there are similar performance as fully random hashing in other settings may apply (e.g., [6, 9]). The challenge lies in accounting for the ages of placed keys in such an analysis.

### 4 Handling Deletions

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**Table 1: Results for Robin Hood hashing, both theoretical and from simulations.** The simulations are for \( \alpha = 0.95 \) and 1000 trials.

| Key | Differential Equations | Celis' Theorem | Sims \( n = 8192 \) | Sims \( n = 65536 \) | Sims \( n = 524288 \) |
|-----|------------------------|----------------|------------------|-----------------|-----------------|
| 1   | 0.083458328            | 0.083458403    | 0.083847726 ± 0.000491031 | 0.083431054 ± 0.001784483 | 0.083421016 ± 0.000612385 |
| 2   | 0.188976794            | 0.188976856    | 0.189753523 ± 0.000131174 | 0.189690408 ± 0.001082245 | 0.189640486 ± 0.001082245 |
| 3   | 0.323793458            | 0.323793585    | 0.323443385 ± 0.001853945 | 0.323768419 ± 0.002991134 | 0.323744102 ± 0.001026609 |
| 4   | 0.30363752             | 0.30363594     | 0.302252763 ± 0.000895214 | 0.303332755 ± 0.001566500 | 0.303420142 ± 0.001083555 |
| 5   | 0.095309269            | 0.095303242    | 0.094714212 ± 0.000959510 | 0.095374789 ± 0.002633155 | 0.095344666 ± 0.00123186 |
| 6   | 0.005092100            | 0.005092104    | 0.005060262 ± 0.000144634 | 0.005116963 ± 0.000521156 | 0.005097440 ± 0.000182278 |
| 7   | 0.000012417            | 0.0000012417   | 0.000012208 ± 0.000001596 | 0.000012271 ± 0.000001453 | 0.000012775 ± 0.000005037 |

**Table 2: Results for Robin Hood hashing, both theoretical and from simulations, but here the simulations use double hashing.**

- using the pseudo-random generator drand48; all trials were performed in a single seeded run. For the simulations, the expression \( x \pm y \) refers to the average \( x \) and standard deviation \( y \) over 1000 trials. Unsurprisingly, the differential equations and the results from Theorem 3 agree quite closely, with the discrepancy explained by our calculation method for the differential equations. The theoretical results match the simulations very closely.

These results also show the potential effectiveness of Robin Hood hashing. Even at a load of 0.95, the maximum age over these simulations was 7. The average number of probes for a successful search (of a random key in the table) is approximately 3.15; the average number of probes for an unsuccessful search is approximately 3.59. As pointed out by Celis [2], there are further possible ways to speed up searches by using procedures other than the standard search procedure. For example, since most keys have age 3 or 4 under this load, one can start the search with the third and fourth entries of the probe sequence directly.
various challenges with his approach that suggest that it is difficult to implement in practice.

4.1 Deletions with Tombstones

Celis suggests a scheme for deletions based on a standard approach of using tombstone entries. Cells with deleted keys are marked; such marked entries are called tombstones. For insertion and search purposes deleted entries are treated the same as non-deleted entries, except that when a deleted key is replaced by a younger key on an insertion, the deleted key can be discarded and does not need to be put back in the table. As an optimization, if the age of a key being inserted is equal to the age of a key in a deleted entry, it can replace the deleted entry. This maintains the property that if a key is placed in the cell given by the $i$th entry in its probe sequence, then for $1 \leq j < i$ there is a key (which may be a deleted key in a tombstone cell) with age at most $j$ in the cell corresponding to the $j$th entry of the key’s probe sequence.

This scheme has the obvious problem that the ages of keys in the table can only become larger. Hence, there is no “steady state” available for the age of keys. In the search process, Celis suggests using search approaches that are aware of the probability distribution of the ages in the current state, and search through the probe sequence using the information. To make unsuccessful searches reasonable, the table should track the largest and smallest age of keys currently in the system; keys with small ages eventually disappear, and the range of key ages to be searched remains relatively small. Celis conjectures the interval has a steady state average of $O(\ln n)$ in the deletion model we describe below. Finally, on insertions, Celis appears to describe a method where the search in the probe sequence begins at the value of the smallest key age currently in the system.

We attempt to model this deletion strategy, in the setting where we first load the system to a load $\alpha$, and then alternate between deleting a random key and inserting a new key. This is a natural model for studying deletions; for example, it was used in Celis’s thesis. We start by running the original set of differential equations to load $\alpha$, and then from the resulting state we run a new set of differential equations that take deletions into account.

We first describe changes to how we think about the state space. Our level process will require an additional state, call it state 0, which corresponds to the state when we perform a deletion. Also, while we use $s_i$ to again represent the fraction of cells in the table that have an undeleted key of age at least $i$, we also need to track the fraction of cells in the table that are tombstones containing deleted keys of age at least $i$. Let us refer to these as $u_i$.

We first consider the level process. Let us think in unscaled time steps. Let $q(t)$ be the probability of being in state 0 – that is, that we are about to perform a deletion – and as before for $i \geq 1$ let $p_i(t)$ be the probability that we are trying to place a key of age at least $i$ in the table. The state of the level process is a Markov chain assuming that the state of the table is fixed at certain values $s_i$ and $u_i$.

The equations for $p_i(t)$ and $q(t)$ are as follows:

$$p_1(t + 1) = 1 - \sum_{j=1}^{\infty} (p_j(t) - p_{j+1}(t))(1 - s_1 - u_{j+1});$$  \hspace{1cm} (15)

$$p_i(t + 1) = p_{i-1}(t)s_{i-1} + \sum_{j=i-1}^{\infty} (p_j(t) - p_{j+1}(t))u_{j+1}, \; i \geq 2;$$  \hspace{1cm} (16)

$$q(t) = \sum_{j=1}^{\infty} (p_j(t) - p_{j+1}(t))(1 - s_1 - u_{j+1}).$$  \hspace{1cm} (17)

Our equation for $p_i$ now has an additional term that takes into account that a key will be placed in a cell with a deleted key when the deleted key’s age is less than or equal to the age of key being placed. Similarly, the equation for $q_j$ is based on summing over each possible age $j$ of the key being placed the probability that it is placed successfully. This
gives the equilibrium equations:

\[ p_1 = 1 - \sum_{j=1}^{\infty} (p_j - p_{j+1})(1 - s_1 - u_{j+1}); \]  
\[ p_i = p_{i-1}s_{i-1} + \sum_{j=i-1}^{\infty} (p_j - p_{j+1})u_{j+1}, \quad i \geq 2; \]  
\[ q = \sum_{j=1}^{\infty} (p_j - p_{j+1})(1 - s_1 - u_{j+1}). \]  

We now turn to equations for \( s_i \) and \( u_i \). We find

\[ \frac{ds_i}{dt} = \sum_{j=i}^{\infty} (p_j - p_{j+1})(1 - s_i - u_{j+1}) - \frac{qs_i}{s_1}; \]  
\[ \frac{du_i}{dt} = \frac{qs_i}{s_1} - \sum_{j=i}^{\infty} p_j(u_j - u_{j+1}). \]  

Note that, in the above, the values \( s_1 \) in the denominator of the terms of the form \( \frac{qs_i}{s_1} \) can be replaced by \( \alpha \), since \( s_1 \) is \( \alpha \) once we start performing deletions under our model.

Unfortunately, this system appears inherently infinite-dimensional. A practical approach is to modify the system to consider all keys with age greater than or equal to some large constant \( L \) to be treated the same. In such cases, keys being placed with age at least \( L \) can only replace other keys with age at least \( L \) if they have been deleted; in all other cases, the process behaves the same. Intuitively, over any finite time (corresponding to \( cn \) balls), one can choose \( L \) large enough so that the gap between the systems is small enough that it can be ignored. Theorem 1 can be then be applied to the infinite-dimensional system.

Ideally, one would formalize this by bounding the infinite-dimensional system between two finite-dimensional systems, using stochastic majorization. This is the approach used in [10]. However, to this point we have not found suitable bounding systems for the Robin Hood strategy. Therefore at this point we leave it as a conjecture that the infinite system accurately models Robin Hood hashing under this deletion strategy for any finite length of time; this conjecture appears reliable based on simulations.

### 4.2 Deletions without Tombstones

We now suggest, and analyze, a simpler scheme that we believe remains effective based on our analysis and simulations. Deleted entries are simply deleted from the table; no tombstones are used. A problem with this approach is that one can no longer use an empty cell as a stopping criterion for an unsuccessful search. Similarly, we no longer have the property that on a search an occupied cell must have a key of age at most \( i \) on the \( i \)th probe, or we can declare the search unsuccessful. The only way to cope with unsuccessful searches is to keep track of largest age of any key currently in the system. This can be done by having the table keep counters of the number of keys of each age.

As before, let us first consider the level process. As a reminder we work in the model where we load the system to a load \( \alpha \), and then alternate between deleting a random key and inserting a new key. Our level process will therefore require an additional state, call it state 0, which corresponds to a deletion. Let us think in unscaled time steps. Let \( q(t) \) be the probability of being in state 0 – that is, that we are about to perform a deletion – and as before for \( i \geq 1 \) let \( p_i(t) \) be the probability that we are trying to place a key of age at least \( i \) in the table. The state is again a Markov chain, assuming that the state of the table is fixed at certain values \( s_i \).

First, let us consider \( q(t) \). When we are placing an item, we complete the placement with probability \( 1 - s_1 \), the probability of finding a cell without a key (either from deletion or from being empty). Note that \( 1 - s_1 \) is \( 1 - \alpha \) based on our model. Hence

\[ q(t + 1) = p_1(t)(1 - \alpha). \]

However, \( p_1(t) = 1 - q(t) \). Substituting gives

\[ q(t + 1) = (1 - q(t))(1 - \alpha). \]
Letting $q$ be the equilibrium probability for the chain for $q(t)$, we have

$$q = \frac{1 - \alpha}{2 - \alpha}.$$  

Note that this gives the equilibrium probability

$$p_1 = \frac{1}{2 - \alpha}.$$  

Finally, as with the original Robin Hood process, we have for $i \geq 2$ that

$$p_i = p_{i-1}s_{i-1}. \quad (23)$$

With this we turn our attention to the limiting equations for the $s_i$. Note that $s_1$ increases whenever an available cell is found for a placement, and $s_1$ decreases whenever a deletion occurs. Hence

$$\frac{ds_1}{dt} = p_1 (1 - s_1) - q. \quad (24)$$

Note that we have $\frac{ds_1}{dt} = 0$ when $s_1 = \alpha$, so this equation is consistent with our model.

For $s_i$ when $i > 1$, Equation (24) generalizes to

$$\frac{ds_i}{dt} = p_i (1 - s_1) - q(s_i/s_1) \quad (25)$$

That is, a cell containing a key with age at least $i$ is created whenever the age of the key being placed is at least $i$ and an empty cell is found by the probe sequence. A cell containing a key with age at least $i$ is removed whenever a deletion occurs, and that deletion is for a cell holding a key of age at least $i$, which occurs with probability $s_i/s_1$. Assuming we start with $s_1 = \alpha$, we can write for $i \geq 1$:

$$\frac{ds_i}{dt} = p_i (1 - \alpha) - q(s_i/\alpha) \quad (26)$$

Note that, under this model, we again have that the $s_i$ term depends only on values of $p_j$ and $s_j$ with $j \leq i$; hence we can apply Theorem 1 over finite time intervals to the truncated family of equations up to $i \leq L$ for any constant $L$ to obtain accurate values for the limiting system.

For this model, we find that there is a unique equilibrium distribution for the underlying Equations (26) and (23); this gives us an idea as to the long term performance of this approach. In equilibrium, using $s_1 = \alpha$, we find that $ds_i/dt = 0$ gives the following reasonable equation:

$$s_i = \frac{p_i}{p_i + \frac{1 - \alpha}{\alpha(2-\alpha)}}. \quad (27)$$

Equation (27) along with Equation (23) can be used to show that the $s_i$ again decrease double exponentially at some point at the equilibrium given by the family of Equations (26).

**Theorem 5:** In the asymptotic model for an infinite Robin Hood hash table with load factor $\alpha < 1$ and alternating deletions, for sufficiently large constants $i$, the value of $s_i$ at the equilibrium point where $ds_i/dt = 0$ everywhere satisfies

$$s_i \leq c_1c_2^{i-c_3} \quad (28)$$

for some constants $c_1, c_3 > 0$ and $c_2 < 1$ that may depend on $\alpha$.

**Proof:** In what follows let $s_i$ and $p_i$ refer to their values in equilibrium. Let $z = \frac{1 - \alpha}{\alpha(2-\alpha)}$. From Equation (27), $s_i < p_i/z$. If $z \geq 1$, using this and $p_i = s_{i-1}p_{i-1}$, we can induct to find

$$s_i \leq \left(\frac{\alpha}{z(2-\alpha)}\right)^{2^{i-2}}. \quad (29)$$

For case $z < 1$, we note the tails of the $p_i$ must decrease geometrically, since a key is placed in an empty cell or a cell with a deleted item with probability at least $1 - \alpha$ at each step. Let $j$ be the smallest value such that $p_j \leq z^2$. Then inductively we find

$$s_{j+k} \leq z^j. \quad \blacksquare$$
Table 3: Results for Robin Hood hashing with deletions and tombstones, both theoretical and from simulations. Here $2n$ total items are inserted. The simulations are for $\alpha = 0.90$ and 1000 trials.

| Key Age | Differential Equations | Sims $n = 8192$ | Sims $n = 65536$ | Sims $n = 524288$ |
|---------|------------------------|-----------------|-----------------|-----------------|
| 1       | 0.0000000012           | 0 ± 0           | 0 ± 0           | 0 ± 0           |
| 2       | 0.0000000088           | 0 ± 0           | 0 ± 0           | 0 ± 0           |
| 3       | 0.0000006021           | 0.0000040699 ± 0.0000004716 | 0.0000000509 ± 0.00000009272 | 0.0000000721 ± 0.00000009356 |
| 4       | 0.0000041128           | 0.0000013566 ± 0.0000004296 | 0.0000035730 ± 0.0000025999 | 0.00000447 ± 0.000009996 |
| 5       | 0.0000025165           | 0.0000016276 ± 0.00000164768 | 0.000002584 ± 0.0000062977 | 0.000002606 ± 0.0000023888 |
| 6       | 0.0000140033           | 0.0000016818 ± 0.0000047803 | 0.000013808 ± 0.0000151588 | 0.0000140804 ± 0.0000054692 |
| 7       | 0.0000115150           | 0.0000079782 ± 0.0000104372 | 0.0000080691 ± 0.0000354163 | 0.0000071820 ± 0.0000124441 |
| 8       | 0.0003302926           | 0.0003438220 ± 0.0002230370 | 0.000195890 ± 0.0000746674 | 0.0003317559 ± 0.000275909 |
| 9       | 0.0014066589           | 0.0014721280 ± 0.0005133678 | 0.0013790477 ± 0.0001734967 | 0.0014051570 ± 0.000624493 |
| 10      | 0.0054203409           | 0.0056228130 ± 0.0001261569 | 0.005517035 ± 0.0007155363 | 0.005631556 ± 0.000154173 |
| 11      | 0.0190426763           | 0.0195947376 ± 0.0003589440 | 0.018739051 ± 0.0011403881 | 0.0190677512 ± 0.0004144598 |
| 12      | 0.0364080856           | 0.0610249559 ± 0.00089433574 | 0.036704689 ± 0.0029890910 | 0.039714762 ± 0.0010159725 |
| 13      | 0.1576583321           | 0.1599762948 ± 0.00177522946 | 0.1557674884 ± 0.0061906220 | 0.1578802084 ± 0.0021976307 |
| 14      | 0.3065676742           | 0.3068089339 ± 0.0117420139 | 0.306759011 ± 0.0063584378 | 0.3067063924 ± 0.0022285521 |
| 15      | 0.3259992699           | 0.3198043883 ± 0.0176422773 | 0.325566906 ± 0.0085555114 | 0.3238149457 ± 0.0021086576 |
| 16      | 0.1187678782           | 0.1173343280 ± 0.0260781256 | 0.1218899664 ± 0.0094442642 | 0.1185390890 ± 0.0033043206 |
| 17      | 0.0079676382           | 0.008317985 ± 0.0004468123 | 0.0085186528 ± 0.0015278714 | 0.0079452633 ± 0.0005104320 |
| 18      | 0.0000292668           | 0.0000429947 ± 0.0000969077 | 0.0000348751 ± 0.0000273052 | 0.0000292015 ± 0.0000087025 |

4.3 Simulations with Deletions

We first consider the setting with tombstones. Here our goal in the simulation is simply to show that the proposed differential equations accurately model the actual system for finite periods of time correctly. In order to keep tables reasonably sized, the result of Table 3 shows results with a load $\alpha = 0.9$; here we load the table with 0.9n items, and then alternately delete and insert items until $2n$ items have been placed. For the differential equations we again use Euler’s method with discrete time steps of length $10^{-6}$. Here we show the fraction of keys in the system (not including tombstones) by age. The results are very accurate, and begins to show the effect of using tombstones; the keys of ages 1 and 2 are vanishing, and the ages of the keys in the cells are increasing over time. If we continued the simulation further, we would see ages continue to grow well beyond 18.

Table 4 shows results with the same setup for the load and deletion pattern. Here the maximum age does not increase the same way, as noted in our analysis, and the maximum age remains smaller at 12. Again, the differential equations prove highly accurate. In this case, however, we are further interested in the equilibrium distribution of the age as given by Equation (27). As a proxy we run the simulations again, but after loading the table with 0.9n items, we alternately delete and insert items until 10n items have been placed. Table 5 shows these results, compared to the calculated equilibrium distribution from Equation (27) (which was derived from the corresponding differential equations). Again, we see that the results match well, showing the utility of the differential equation. Also, as shown Theorem 5, we see the probability a cell has a certain age falls very quickly (doubly exponentially) at the tail of the distribution. The average time for a successful search naturally converges to 10; the maximum age, and hence the time for an unsuccessful search, is only around 16.

5 Conclusion

We have shown how to use the framework of Markov chains, often also called the fluid limit analysis or mean-field approach, to analyze Robin Hood hashing. In particular, we have shown that for Robin Hood hashing the analysis naturally requires the use of an additional level process. Besides providing a new way of gaining insight into previous results, we have shown that our methods lead to a simple recurrence describing the equilibrium behavior of Robin Hood hashing under a natural deletion model when not using tombstones.

Robin Hood hashing appears to perform essentially the same whether using probe sequences based on double hashing and random hashing. Proving this seems a worthwhile open question. Relatedly, the recent work of [9] applies fluid limit analysis to show that double hashing yields the same behavior as fully random hashing under the “balanced allocation” paradigm. Alternatively, one could try to extend the approach of [6] used for standard open addressing hashing.
### Table 5: Results for Robin Hood hashing with deletions and no tombstones, both theoretical and from simulations.

Here $2n$ total items are inserted. The simulations are for $\alpha = 0.90$ and 1000 trials.

| Key Age | Calculated Equilibrium | Sims $n = 8192$ | Sims $n = 65536$ | Sims $n = 524288$ |
|---------|------------------------|----------------|----------------|----------------|
| 1       | 0.0109989101          | 0.01098292070 ± 0.0000582658 | 0.0109820301 ± 0.0000383596 |
| 2       | 0.0123800001          | 0.0132215540 ± 0.0014401913 | 0.0152260213 ± 0.000535898 |
| 3       | 0.0162289897          | 0.016228928 ± 0.0016968246 | 0.0161916268 ± 0.000516735 |
| 4       | 0.0201344356          | 0.0201344356 ± 0.001696824 | 0.0201344356 ± 0.000516735 |
| 5       | 0.0258283516          | 0.0258283516 ± 0.001696824 | 0.0258283516 ± 0.000516735 |
| 6       | 0.0338343346          | 0.0338343346 ± 0.001696824 | 0.0338343346 ± 0.000516735 |
| 7       | 0.0457090363          | 0.0457090363 ± 0.001696824 | 0.0457090363 ± 0.000516735 |
| 8       | 0.0638449846          | 0.0638449846 ± 0.001696824 | 0.0638449846 ± 0.000516735 |
| 9       | 0.0921369579          | 0.0921369579 ± 0.001696824 | 0.0921369579 ± 0.000516735 |
| 10      | 0.1351844968          | 0.1351844968 ± 0.001696824 | 0.1351844968 ± 0.000516735 |
| 11      | 0.1893101510          | 0.1893101510 ± 0.001696824 | 0.1893101510 ± 0.000516735 |
| 12      | 0.269874222          | 0.269874222 ± 0.001696824 | 0.269874222 ± 0.000516735 |

Table 4: Results for Robin Hood hashing with deletions and no tombstones, both theoretical and from simulations. Here $2n$ total items are inserted. The simulations are for $\alpha = 0.90$ and 1000 trials.
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