\(N=2\) Supersymmetric Black Attractors in Six and Seven Dimensions

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Abstract

Using a quaternionic formulation of the moduli space \(M (IIA/K3)\) of 10D type IIA superstring on a generic K3 complex surface with volume \(V_0\), we study extremal \(N = 2\) black attractors in 6D space-time and their uplifting to 7D. For the 6D theory, we exhibit the role played by 6D \(N = 1\) hypermultiplets and the \(Z^m\) central charges isotriplet of the 6D \(N = 2\) superalgebra. We construct explicitly the special hyperKahler geometry of \(M (IIA/K3)\) and show that the \(SO(4) \times SO(20)\) invariant hyperKahler potential is given by \(H = H_0 + \text{Tr} \ln (1 - V_0^{-1} S)\) with Kahler leading term \(H_0 = \text{Tr} \ln V_0\) plus an extra term which can be expanded as a power series in \(V_0^{-1}\) and the traceless and symmetric \(3 \times 3\) matrix \(S\). We also derive the holomorphic matrix prepotential \(G\) and the flux potential \(G_{BH}\) of the 6D black objects induced by the topology of the RR field strengths \(F_2 = dA_1\) and \(F_4 = dA_3\) on the K3 surface and show that \(G_{BH}\) reads as \(Q_0 + \sum_{m=1}^3 q^m Z^m\). Moreover, we reveal that \(Z^m = \sum_{i=1}^{20} Q_i \left( \int_{C^i} J^m \right)\) where the isotriplet \(J^m\) is the hyperKahler 2-form on the K3 surface. It is found as well that the uplifting to seven dimensions is quite similar to 4D/5D correspondence for back hole potential.

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considered in arXiv 0707.0964 [hep-th]. Then we study the \( \mathcal{N} = 2 \) black object attractors in 6D and 7D obtained respectively from type IIA string and M-theory on K3.

**Key words:** Type IIA superstring on the K3 surface, Special hyperKahler geometry, 6D/7D \( \mathcal{N} = 2 \) black objects, Attractor mechanism.

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1 Introduction

Large distance of compactified ten dimensional (10D) type II superstrings and 11D M-theory down to lower space time dimensions have recently known a great revival of interest in connection with topological string [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11], black objects and attractor mechanism [12, 13, 14, 4, 15, 16, 17, 18, 19, 20]. In the context of 4D black holes, Calabi-Yau compactification of 10D type IIA superstring (11D M-theory) down to 4D (5D) can be approximated at large distances by 4D (5D) $\mathcal{N} = 2$ supergravity whose low energy theory include, in addition to the 4D (5D) $\mathcal{N} = 2$ supergravity multiplet, matter organized into a number $n_V$ of 4D (5D) $\mathcal{N} = 2$ vector multiplets and $n_H$ hypermultiplets. In the 4D $\mathcal{N} = 2$ black hole attractor mechanism, the hypermultiplets decouple and one is left with gravity coupled to vector multiplets whose vacuum configuration is nicely described by special Kahler (real) geometry [21, 22, 23, 24, 25, 26, 27]. There, special complex (real) geometry plays a crucial role in the study of 4D (5D) $\mathcal{N} = 2$ extremal black holes and in the understanding of their attractor mechanisms.
Motivated by the exploration of the attractor mechanism for black objects in 6D and 7D space-time as well as the basic role that have to play 6D $\mathcal{N} = 1$ hypermultiplets in the attractor issue, we first study extremal $\mathcal{N} = 2$ black objects in six dimensional space time and analyse the special $d$-geometry of underlying 6D $\mathcal{N} = 2$ supergravity describing the low energy limit of type superstring IIA on the K3 surface. Then, we consider the uplifting of the 6D model to 7D which can be understood as the compactification of 11D M-theory on the K3 surface. For the 6D case, we take advantage of the hyperkahler structure of the moduli space $M$ (IIA/K3) of 10D type IIA superstring on a generic K3 surface, with fixed string coupling constant, to introduce a convenient basis of local quaternionic coordinates \{${w^I}$, $0 \leq I \leq 20$\} built as,

\[ w^I = \int_{C_I^2} \left( B^{NS} + i \sum_{s=0, \pm 1} \sigma^s \Omega^{(1+s,1-s)} \right), \quad I = 1, \ldots, h_{K3}^{(1,1)}, \]

where the basis set \{${C_I^2}$\} refer to the twenty real 2- cycles of K3 dual to the Hodge $(1,1)$-forms. In this relation, the $2 \times 2$ matrices $\sigma^{0,\pm}$ are the usual Pauli matrices in the Cartan basis, the $(p,q)$- form $\Omega^{(p,q)}$, with $p + q = 2$, are the Hodge 2-forms on the K3 surface and

\[ J = B^{NS} I_{2 \times 2} + i \sum_{s=0, \pm 1} \sigma^s \Omega^{(1+s,1-s)}, \]

is the "quaternionified" Kahler 2-form on the K3 surface. This 2- form is valued in the $2 \times 2$ matrix algebra generated, in addition to the identity $I_{2 \times 2}$, by the three Pauli matrices $\sigma^m$ and should be compared with the complexified Kahler form

\[ B^{NS} + i \Omega^{(1,1)}, \]

we encounter in the study of 10D type IIA superstring on Calabi-Yau threefolds. The local coordinate basis \{${w^I}$\} allows to approach the underlying special hyperkahler $d$-geometry of $M$ (IIA/K3) in quite similar manner as do the complexified Kahler moduli

\[ z^I = \int_{C_I^2} \left( B^{NS} + i \Omega^{(1,1)} \right), \]

in the study of 10D type IIA superstring on Calabi-Yau threefolds. It is a basis set of non abelian $2 \times 2$ matrices which permit to use the power of the algebra of matrices to study the quaternionic geometry of $M$ (IIA/K3). This matrix formulation has several special properties mainly governed by the quaternionic structure of $M$ (IIA/K3) and captured in practice by the spin $\frac{1}{2}$ representation of the $SU(2)$ R- symmetry of the 6D $\mathcal{N} = 2$ superalgebra and by the standard Clifford algebra identities of the $2 \times 2$ Pauli matrices. Matrix formulation of the special quaternionic $d$-geometry of $M$ (IIA/K3) allows in particular to:

(1) exhibit manifestly the $SO(4) \times SO(20)$ gauge symmetry of the moduli space $M$ (IIA/K3)
with fixed string coupling constant.

(2) permit to treat explicitly the computation of the hyperKahler potential $\mathcal{H}$, the holomorphic matrix prepotential $G$ and the flux potential $G_{BH}$ of the $6D \mathcal{N} = 2$ black object attractors.

(3) get the explicit moduli realization of the central charges of BPS states of the $6D \mathcal{N} = 2$ superalgebra.

(4) provide a natural way to deal with $7D$ supersymmetric black objects by uplifting in the same spirit as recently done in [21] for studying extremal black hole attractor mechanism in $4D/5D$ extended supergravities.

The organization of this paper is as follows. In section 2, we review the compactification of $10D$ type IIA superstring on the K3 surface. Then we give comments on the $6D \mathcal{N} = 2$ supersymmetric low energy limit; in particular the aspect regarding the structure of central charges and $6D$ supersymmetric BPS states. In section 3, we develop a matrix formulation to analyse the corresponding moduli space $M_{\text{IIA/K3}}$. In section 4, we introduce the quaternionified 2-form $J$ and develop the special quaternionic d-geometry in the matrix formalism. In section 5, we compute the hyperKahler potential $\mathcal{H}$ as a power series in the inverse of the volume of the K3 surface and derive the matrix holomorphic prepotential $G$ using the 2-cycles intersection matrix $d_{IJ}$ of generic K3. In section 6, we study the $6D \mathcal{N} = 2$ black object attractors and their $7D$ uplifting by using type IIA D-branes wrapping cycles of the K3 surface. Extending the idea on the flux compactification of $10D$ type IIA superstring on Calabi-Yau 4-folds given in [28] to the case of the K3 surface, we derive, amongst others, the flux potential for $6D \mathcal{N} = 2$ supergravity theory. In section 7, we study the effective scalar potential and the attractor mechanism for the $6D$ and $7D$ black objects. In section 8, we give our conclusion and make a discussion. In section 9, we give an appendix on $\mathcal{N} = 2$ supersymmetry in six dimensions.

2 Type II Superstrings on K3

Low energy dynamics of $10D$ type II superstrings on the K3 complex surface is described by $6D \mathcal{N} = 2$ supergravity coupled to superYang-Mills $[29, 30, 31, 32, 33]$. One distinguishes two six dimensional models A and B depending on whether one started from $10D$ type IIA or $10D$ type IIB superstrings. These models are respectively given by the usual $6D$ non chiral $\mathcal{N} = (1, 1)$ and $6D$ chiral $\mathcal{N} = (2, 0)$ models and have different moduli spaces $M_{\text{IIA/K3}}$ and $M_{\text{IIB/K3}}$. In present paper, we mainly consider the large distance limit described by non chiral $6D \mathcal{N} = (1, 1)$ ($\mathcal{N} = 2$ for short) supergravity and
focus on the hyperKahler structure of $M_{\text{IIA}/K3}$. The moduli space $M_{\text{IIA}/K3}$ namely
\[ M_{\text{IIA}/K3} = \frac{S0(5,21)}{S0(5) \times S0(21)}, \] (2.1)
hasn’t however such hyperKahler structure. The local coordinates of $M_{\text{IIA}/K3}$ are in the bi- fundamental of $S0(5) \times S0(21)$ and so $M_{\text{IIA}/K3}$ has a real dimension multiple of 5 rather than a multiple of 4 which is a necessary condition for having quaternionic geometry.

### 2.1 Type IIA superstring on K3

#### 2.1.1 Moduli space $M_{\text{IIA}/K3}$

In 10D type superstring IIA on a generic K3 surface, which is known to be dual to the 10D heterotic superstring on $T^4$, the moduli space $M_{\text{IIA}/K3}$ of metric deformations and stringy vacuum field configurations is given by the non compact real space
\[ M_{\text{IIA}/K3} = M \times SO(1,1), \] (2.2)
with $M$ given by the homogeneous space
\[ M = \frac{SO(4,20)}{SO(4) \times SO(20)}, \] (2.3)
and where the extra $SO(1,1)$ stands for the dilaton (i.e the string coupling $g_s$). For fixed $g_s$, $M_{\text{IIA}/K3}$ reduces to $M$ and so the restricted moduli space $M$ has a real eighty dimension capturing a quaternionic structure. In addition to the NS-NS B-field 2-form $B^{\text{NS}}$ whose role will be discussed later on, the hyper- structure of $M$ should be described by a real $SU(2)$ isotriplet of 2-forms
\[ J^i = (J^1, J^2, J^3), \quad J^i = (J^i)^{\dagger}, \quad dJ^i = 0, \] (2.4)
rotated under the adjoint representation of $SU(2)$ isometry group of the K3 surface as shown below
\[ [D^i, J^j] = i\epsilon^{ijk}J^k, \quad i, j, k = 1, 2, 3, \]
\[ [D^i, D^j] = i\epsilon^{ijk}D^k. \] (2.5)

In the above relations, the 2-form $J^3$, to be denoted often as $\Omega^{(1,1)}$ or sometimes as $J^0$, is the usual hermitian Kahler 2-form one encounters in generic complex $n$- dimensional Kahler manifold. The two others are given\(^1\) by
\[ J^1 = \text{Re} \, \Omega^{2,0}, \quad J^2 = \text{Im} \, \Omega^{2,0}, \] (2.6)

\(^1\)Given a generic Hodge $(p,q)$-form $\Omega^{(p,q)}$ on a Kahler manifold, one can associate to it two integers: the highest weight $h = \frac{p+q}{2}$, of an underlying $SU(2)$ group representation that classify the forms, and the isospin $s = \frac{p-q}{2}$. A Hodge multiplet consists of those $(p,q)$-form $\Omega^{(p,q)}$ with isospins as $-h \leq s \leq h$. For $h = 1$, the corresponding Hodge multiplet is an isotriplet with isospins $s = 0, \pm 1$. 

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where $\Omega^{(2,0)}$ is the complex holomorphic 2-form on the K3 surface with complex conjugate $\overline{\Omega^{(2,0)}} = \Omega^{(0,2)}$. The $D^i$'s denote the generators of $SU(2)$ symmetry group rotating the $J_i$'s and $\epsilon^{ijk}$ is the associated completely antisymmetric structure constants.

For later use, we recall that in the K3 compactification of 10D type II superstrings, the local Lorentz group $SO(1,9)$ of the 10D space time breaks down to $SO(1,5) \times SU(2)$. The latter, which is required by supersymmetry and K3 holonomy, is contained in $SO(1,5) \times SO(4)$. We also recall that the homomorphism $SO(4) \sim SU(2) \times SU'(2)$ allowing to put the 4-vector representation of $SO(4)$ as a $2 \times 2$ hermitian matrix.

### 2.1.2 Central charges in 6D supersymmetric theories

We first introduce the central charges as they appear in the standard Haag-Lopuszanski-Sohnius (HLS) superalgebra. Then, we consider its extension by implementing p-branes, $p > 1$ [34].

**A. Geometric central charges:**

There are different ways to introduce central charges $Z_m$ of extended supersymmetric algebras. A tricky way is to use their geometric realization as translations along the transverse directions of 10D superstrings compactified down to lower dimensions. Below, we describe this realization for the case of supersymmetric field theory in six dimension space time.

$Z_m$ as translation operators:

First, recall that a generic ten dimensional real vector $V_M$ splits generally into a real 6D space time vector $V_\mu$ and four real space time scalars $f_m$. In particular, this is valid for the 10D space time coordinates $x^M \rightarrow (x^\mu, y^m)$; but also for generic 10D Maxwell gauge fields $A_M$ which then split as

$$A_M \rightarrow (A_\mu, \phi_m).$$

The same reduction can be done for the 10D energy momentum vector $P_M$ which decomposes into 6D energy momentum $P_\mu$ vector and four central charges $P_m$ as shown below:

$$P_M \rightarrow (P_\mu, P_m), \quad m = 6, 7, 8, 9.$$  

By central charge, we mean that, viewed as operators, the $P_m$'s commute with all generators of the 6D supersymmetric Poincaré algebra $SP_{6D}^{N=2}$ given in appendix and which will be considered later on. The $P_m$'s commute in particular with the bosonic $P_\mu$ and $M_{[\mu \nu]} = x_{[\mu} P_{\nu]}$ generators of the 6D Poincaré algebra $P_{6D}$:

$$[P_\mu, P_m] = 0, \quad [M_{[\mu \nu]}, P_m] = 0, \quad [P_m, P_n] = 0, \quad (2.9)$$
In quantum physics where 10D energy momentum operator $P_M$ is realized as $\hbar i \partial_M$, the central charge operators $P_m$ are given by $\hbar i \partial_m$. From this representation, we learn that central charges $P_m$ may geometrically be interpreted as translations along the $y^m$ coordinates of the 4D transverse space.

$Z_m$ and the sgauginos $\phi_m$:
The above description can be pushed further by establishing a close link between the central charges $Z_m$ and the scalar gauge fields $\phi_m$ eq.(2.7). The idea is to use the same trick for the abelian gauge field strength $F_{MN} = \partial_M A_N - \partial_N A_M$. We have the decomposition

$$F_{MN} \rightarrow (F_{\mu\nu}, F_{\mu m} = \partial_\mu \phi_m),$$

where we have used the anzats $\partial_m A_\mu = 0 = \partial_m \phi_n$; that is the fields $A_\mu$ and $\phi_n$ have no dependence on the internal coordinates $y^m$:

$$A_\mu = A_\mu(x), \quad \phi_n = \phi_n(x).$$

Now, using the fact that $F_{MN}$ is also the curvature of the gauge covariant derivative $D_M = \partial_M + A_M$, i.e

$$F_{MN} = [D_M, D_N]$$

we learn the two following:

(i) the real four sgauginos $\phi_m$ are just "gauge fields" along the transverse directions which appear in the covariantization of the central charges

$$\partial_m \rightarrow D_m = \partial_m + \phi_m. \quad (2.13)$$

(ii) By introduction of the gauge fields, the commutator between the covariantized space time and transverse translations get a non zero curvature given by the gradient $\partial_\mu \phi_m$ of the sgauginos, $F_{\mu m} = [D_\mu, D_m]$. The square scalar $\sum_{\mu,m} (F_{\mu m} F^{\mu m})$ gives just the kinetic energy of the four real sgauginos.

**Supersymmetric algebra:**
Under the compactification on the K3 surface, the 32 supersymmetries of 10D $N = 2$ superalgebra reduce to 16 supersymmetries forming a 6D $N = 2$ superalgebra with central charges. To get the defining relations of this algebra, it is interesting to express space time and transverse space translations $P_\mu$ and $P_m$ in terms of spinor representations of the $SO(1,5) \times SO(4)$ group. Using $SO(1,5)$ spinor representations, the 6D vector $P_\mu$ can be written as $P_{[\alpha\beta]}$ (roughly speaking as a $SU(4)$ antisymmetric representation). The central charges are scalars under $SO(1,5)$. Similarly, the four real central charges $P_m$ can

2In 6D $N = 2$ supersymmetric gauge theory, vector multiplets have, in addition to gauginos, the usual 6D vector field $A_\mu$; but also four scalars $\phi_m$. 
be also put as $Z^{ia}$; that is as a spin $(\frac{1}{2}, \frac{1}{2})$ representation of $SO(4) \sim SU(2) \times SU'(2)$. As spin $\frac{1}{2}$ representations are complex, we need to impose the reality condition
\[ \overline{Z^{ia}} = \epsilon_{ab} \epsilon_{ij} Z^{jb}, \quad i, j = 1, 2, \quad a, b = 1, 2, \tag{2.14} \]
where $\epsilon_{ij}$ ($\epsilon_{ab}$) is the metric tensor of $SU(2)$ ($SU'(2)$). By identifying the two $SU(2)$ factors of $SO(4)$, the $2 \times 2$ component matrix $Z^{ia}$ becomes $Z^{ij}$ and can be decomposed as a real isosinglet $Z_0$ and a real isotriplet $Z^{ij}$ as given below
\[ Z^{ij} = Z_0 \epsilon^{ij} + Z^{(ij)}, \quad i, j = 1, 2. \tag{2.15} \]
We will show later, when we go into the details of 10D type IIA superstring compactification on the K3 surface, that $Z_0$ can be related to the NS-NS B-field and $Z^{ij}$ to the hyperKahler structure on K3.

In the appendix eq(9.13), see also eqs(2.23), we show that four real central charges are allowed by 6D $\mathcal{N} = (1, 1)$ supersymmetric algebra. This non chiral superalgebra is generated by two kinds of fermionic generators $Q^i_\alpha$ and $S^\alpha_i$ whose basic graded commutation relations are given by,
\[ \{ Q^i_\alpha, S^\beta_j \} = \epsilon^{ij} P_{[\alpha \beta]}, \quad \alpha, \beta = 1, \ldots, 4 \]
\[ \{ S^\alpha_i, S^\beta_j \} = \epsilon_{ij} P^{[\alpha \beta]}, \quad i, j = 1, 2 \]
\[ \{ Q^i_\alpha, S^\beta_j \} = \delta^\alpha_\beta Z^i_j \tag{2.16} \]
\[ [P_{[\alpha \beta]}, Q^i_j] = [P_{[\alpha \beta]}, S^\alpha_j] = 0 \]
\[ [P_{[\alpha \beta]}, Z^i_j] = [Z^i_\alpha, Q^i_j] = [Z^i_\alpha, S^\beta_j] = 0. \]
The four central charge components $Z^{ij}$ appearing in above eqs can be also given an interpretation from both the view of $SO(1, 5) \times SU(2)$ group theoretical representation and the view of extended $\mathcal{N} = 4$ and $\mathcal{N} = 2$ supersymmetry in 4D.

From the $SO(1, 5) \times SU(2)$ view, the general form of the anticommutation relation of the two $SO(1, 5)$ space time spinors $Q^i_\alpha$ and $S^\alpha_j$ should be as,
\[ \{ Q^i_\alpha, S^\beta_j \} = Z^{i\beta}_{j\alpha}, \tag{2.17} \]
where $Z^{i\beta}_{j\alpha}$ is constrained to commute with all other generators of the 6D $\mathcal{N} = 2$ superalgebra. However space time interpretation which demand that the central charges to have no space time index; that is $SO(1, 5)$ invariant eqs(2.8,2.9), requires then
\[ Z^{i\alpha}_{j\beta} = \delta^{\alpha}_\beta Z^i_j, \tag{2.18} \]
This restriction should be understood as associated with the space time singlet (1, 4) in the following tensor product
\[ (4, 2) \times (4, 2) = (1, 4) \oplus (15, 4), \tag{2.19} \]
where \((4, 2)\) stands for \(Q^i_{\dot{a}}\) and \((\bar{I}, 2)\) for \(S^\alpha_j\). The \((15, 4)\) extra term in the above decomposition transforms non trivially under space time symmetry. By roughly thinking about \(SO(1,5)\) as the Euclidean \(SO(6) \simeq SU(4)\), we see that the term \((15, 4)\) transforms in the adjoint representation of \(SU(4)\) space time symmetry. It also corresponds to the antisymmetric component in the reduction of the tensor product of two \(SO(1,5)\) vectors namely: \(6 \otimes 6 = 1 + 15 + 20\).

From the view of 4D \(\mathcal{N} = 4\) superalgebra, it is interesting to recall first that there, one has six central charges,

\[ Z^{[ab]}, \quad a, \ b = 1, \ldots, 4, \quad (2.20) \]

transforming in the 6- dimensional representation of \(SU(4) \sim SO(6)\) R- symmetry. Under the breaking of the \(SO(6)\) R-symmetry as \(SO(2) \times SO(4)\), the 6 central charges \(Z^{[ab]}\) of the 4D \(\mathcal{N} = 4\) superalgebra split as \(6 = 4 + 2\) where the four central charges are precisely given by \(Z^{ij}\), the same as in eqs\((2.16)\), and the extra two, which can be denoted as \(Z_0 + iZ_e\), are generated by the compactification from 6D down to 4D. To fix the ideas on the parallel between 6D \(\mathcal{N} = 2\) and 4D \(\mathcal{N} = 2\) superalgebras, we recall herebelow the 4D \(\mathcal{N} = 2\) supersymmetric algebra and the way in which the two central charges \(Z^{\mathcal{N}=2}_{4D}\) enter in the game:

\[
\begin{align*}
Q^i_{\dot{a}}Q^j_a + Q^\dot{a}_iQ^a_j &= \delta^i_j P^a_{\dot{a}}, \quad a, \ \dot{a} = 1, 2, \\
Q^i_{\dot{a}}Q^j_a + Q^i_{\dot{a}}Q^j_a &= \epsilon^{ij} \epsilon_{ab} Z^{\mathcal{N}=2}_{4D}, \quad i, \ j = 1, 2, \\
Q^i_{\dot{a}}Q^j_a + Q^i_{\dot{a}}Q^j_a &= \epsilon^{i\dot{a}} \epsilon_{j\dot{a}} Z^{\mathcal{N}=2}_{4D}, \\
[Z^{\mathcal{N}=2}_{4D}, Q^i_{\dot{a}}] &= [Z^{\mathcal{N}=2}_{4D}, Q^j_a] = [Z^{\mathcal{N}=2}_{4D}, P^a_{\dot{a}}] = 0. 
\end{align*}
\]

We also recall that the positivity of the norm of the 10D energy momentum vector \((\sum_0^9 P_M P^M = (E^2 - P^2) - \sum_m (P_m)^2 \geq 0)\) puts a strong constraint on the \(Z^{ij}\) central charges of 6D \(\mathcal{N} = 2\) supersymmetric representations. We have

\[ Z_0^2 + Z^{(ij)} Z_{(ij)} \leq M_6^2 \quad (2.22) \]

where \(M_6\) is the mass of the 6D \(\mathcal{N} = 2\) supermultiplet; that is energy \(E = P_0\) in the rest frame of particles of the supermultiplet. The 6D \(\mathcal{N} = 2\) BPS corresponds to supersymmetric states with sutured bound; i.e \(Z_0^2 + Z^{(ij)} Z_{(ij)} = M^2\). We will turn later to this relation when we consider black attractors in six dimensions.

For completeness of the study of the 6D \(\mathcal{N} = 2\) superalgebra, it is useful to recall as well that a generic 10D Majorana-Weyl spinor with 16 components \(Q_A\) generally

\[ \text{Notice that we have been using two kinds of } SO(6) \text{ symmetry groups which should not be confused. We have: (1) the usual } SO(6) \text{ R- symmetry group of the compactification of 10D space time down to 4D. (2) the Euclidean version of the 6D space time group } SO(1,5). \]
decomposes, in six dimensions, into four $SO(1,5)$ Weyl spinors $Q^i_\alpha$ and $Q^\dagger_\alpha$, constrained by a reality condition, according to the reduction rule $16 = (4,2) + (4',2')$. In the particular case of the K3 compactification where only half of supersymmetries survive, a 10D Majorana-Weyl spinor $Q_A$ reduces down to $8 = (4,2)$ real object $Q^i_\alpha$, $\alpha = 1, ..., 4$, $i = 1, 2$, and satisfies the following reality condition \[ (Q^\dagger_\alpha) = \epsilon_{ij}B^j_\alpha Q^i_\beta, \quad \text{with} \quad B^+B = -1. \] (2.23)

The two real $Q^i_\alpha$'s can be complexified as $Q^i_\alpha = Q^i_\alpha + iQ^2_\alpha$ to give one complex 4- Weyl spinor. We will turn to eqs (2.15) and (2.23) later on when we consider the supersymmetric algebra in six dimensions (see section 4 and appendix). For completeness, it is also convenient to introduce the following basis for the Kahler 2-forms

\[ J^+ = J^1 + iJ^2, \quad J^- = J^1 - iJ^2, \quad J^0 = J^3 \]

\[ (J^\pm)^\dagger = J^\mp, \quad dJ^\pm = 0, \] (2.24)

where the charges 0 and ± stand for the usual $U_C(1)$ Cartan charge of the $SU(2)$. Obviously $J^+ = \Omega^{(2,0)}$ and $J^- = \Omega^{(0,2)}$; see also footnote 1.

**B. Implementing p- branes**

In the study of black objects of supergravity theories, in particular in 6D $\mathcal{N} = 2$ non chiral supergravity, one has to distinguish between two kinds of central charges:

1. the four usual central charges $Z^a = \sigma^a_{ij}Z^i_j$ considered above and transforming as scalars under space time rotations. These central charges are strongly related with the gauge fields $\{A^i_{\mu j}, i, j = 1, 2\}$ belonging to the $\mathcal{N} = 2$ non chiral supergravity multiplet

\[ (g_{\mu\nu}, B^\pm_{\mu\nu}, A^{ij}_\mu, \sigma; \psi^i_{\pm\mu\alpha}, \chi^i_{\pm\alpha}) \] (2.25)

These charges (dressed by scalars fields) appear in the supersymmetric transformations of the gravitinos $\psi^i_{\pm\mu\alpha}$ and gravi-gauginos $\chi^i_{\pm\alpha}$ (gravi-photinos/gravi-dilatinos)).

2. central charges $Z_{\Lambda}$ that are associated with (dressed) electric and/or magnetic charges of the $(p+2)$- form gauge field strengths (and their duals) corresponding to the p- branes and $(D - p - 4)$- branes) within the supergravity theory. These central charges, which have space time indices, do not appear in the standard HLS superalgebra; but rather in its extended version \[ 34, 35, 36, 37, 38 \].

In 6D $\mathcal{N} = 2$ non chiral supergravity we are interested in here, we have in addition to the four 1- form gauge fields $A^{ij}_\mu$ of the supergravity multiplet, other gauge fields that contribute as well to the full spectrum of the central charges of the theory and so to its effective scalar potential that will be considered in section 7. These charges are given
by:

(i) Two central charges $Z_+$ and $Z_-$ associated with the 3-form field strengths $\mathcal{H}_3^\pm \sim dB_2^\pm$

$$Z_\pm \sim \int_{S^3} \mathcal{H}_3^\pm,$$  \hspace{1cm} (2.26)

where $B_2^+$ and $B_2^-$ are respectively the self dual and anti-self dual NS-NS B-field of the $\mathcal{N} = 2$ non chiral supergravity in six dimensional space time.

(ii) Twenty central charges $Z'_I$, associated with the gauge fields $A_{\mu}^I$ of the twenty 6D $\mathcal{N} = 2$ Maxwell multiplets, that follow from the compactification of the 10D type IIA superstring on K3.

$$Z'_I \sim \int_{S^2} F_2^I, \quad F_2^I = dA^I.$$ \hspace{1cm} (2.27)

Notice that the point-like states associated with fields are obtained by wrapping D2-branes on the $h^{1,1}$ 2-cycles of K3, $h^{1,1}(K3) = 20$. More details on the relations (2.26-2.27) as well as others will be given in section 7.

### 2.2 \textit{SO} (4) $\times$ \textit{SO} (20) invariance

A natural way to study the geometry of the moduli space $M = \text{SO} (4, 20)/\text{SO} (4) \times \text{SO} (20)$ eq(2.3) is to use the real local coordinate system

$$\{x^{aI}\}, \quad a = 1, \ldots, 4, \quad I = 1, \ldots, 20.$$ \hspace{1cm} (2.28)

These coordinates transform in the bi-fundamental

$$x^{aI} \sim (4, 20)$$ \hspace{1cm} (2.29)

of the $\text{SO} (4) \times \text{SO} (20)$ gauge symmetry as follows

$$x^{aI} \rightarrow \tilde{x}^{aI} = \left( \sum_{b=1}^4 \Lambda_b^a \right) \left( \sum_{j=1}^{20} \Gamma_j^I \right) x^{bJ}.$$ \hspace{1cm} (2.30)

where $\Lambda_b^a$ and $\Gamma_j^I$ are rotation matrices. The $\text{SO} (20)$ symmetry corresponds to the arbitrariness in the choice of the basis of Kahler deformations while $\text{SO} (4) \sim \text{SU}^2 (2)$ corresponds to the rotation symmetry of the hyperKahler 2-form isotriplet.

The $\text{SO} (4)$ symmetry of the moduli space (2.3) plays an important role in the study of 6D supersymmetric field theory limit of 10D type IIA superstring on the K3 surface. An immediate goal is to implemented this symmetry in the formalism as a manifest covariance. A priori, one can imagine different, but equivalent, ways to do it. Two methods seem particularly interesting especially for the study of black objects in 6D dimensions and the corresponding attractor mechanism. These methods are given by:
(1) Matrix formulation which has been motivated by the use of quaternions to deal with eq(2.3). In this method, the $SU(2)$ symmetry is captured by the Pauli matrices

$$\sigma^1, \sigma^2, \sigma^3,$$

which, as it is well known, obey as well as a 2D Clifford algebra that is used to realize the three complex structures $i, j$ and $k = i \wedge j$ of the quaternions. This algebraic method will be developed in the present paper.

(2) Geometric method based on the geometrization of $SU(2)$ symmetry. Instead of the Pauli matrices, the generators of the $SU(2)$ algebra (2.5) are realized in this method as follows

$$D^{++} = 2 \sum_{i=1}^{2} u^{+i} \frac{\partial}{\partial u^{-i}},$$
$$D^{--} = 2 \sum_{i=1}^{2} u^{-i} \frac{\partial}{\partial u^{+i}},$$
$$D^{0} = 2 \sum_{i=1}^{2} \left( u^{+i} \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}} \right).$$

This approach has been motivated by the harmonic superspace method used for the study of 4D $\mathcal{N} = 2$ supersymmetric field theories [40, 41, 42, 43]. We suspect that this method to be the natural framework to deal with the study of special hyperKahler $d$-geometry and the corresponding hyperKahler metrics building [46, 47]. This method is powerful; but it requires introducing more technicalities. This will be considered elsewhere [57].

3 Matrix Formulation

The key idea of the algebraic method we will develop in what follows to deal with special hyperkahler geometry may be summarized as follows:

First identify the $SU(2)$ and $SU(2)'$ subgroup factors of $SO(4)$ so that the real 4-vector of $SO(4)$ splits under the $1/2 \times 1/2$ spin representation of $SU(2)$ as a isosinglet and isotriplet; i.e

$$4 = 1 \oplus 3.$$  \hspace{1cm} (3.1)

Then think about $SU(2)$ as an algebraic structure that allows to put the isotriplet $\mathbf{J} = (J^1, J^2, J^3)$ into a traceless hermitian $2 \times 2$ matrix

$$\mathbf{J} = \mathbf{J} - \left( \frac{1}{2} \text{Tr} \mathbf{J} \right) \mathbf{I}_2,$$
$$\text{Tr} \mathbf{J} = 0.$$

This method has several advantages mainly given by:

(i) the similarity with the usual analysis of $\mathcal{N} = 2$ black holes in 4D [15, 16].
(ii) the power of the spin $\frac{1}{2}$ representation of the $su(2)$ algebra to deal with $su(2)$ tensor analysis.

This method has also dis-advantages mainly associated with the fact that the basic objects, in particular the moduli space variables, are non commuting matrices. This property is not a technical difficulty; it captures in fact the novelties brought by the hyperKahler geometry with respect to the standard Kahler one.

### 3.1 Quaternionized HyperKahler 2-Form

The main lines of the matrix formulation towards the study of the special hyperKahler $d$- geometry may be summarized in the three following points:

1. **Represent the three closed Kahler 2-forms $J^1$, $J^2$ and $J^3$ by a traceless hermitian $2 \times 2$ matrix as shown below**

   $J = \begin{pmatrix} J^3 & J^1 + iJ^2 \\ J^1 - iJ^2 & -J^3 \end{pmatrix}$,  
   \hspace{1cm} (3.3)

   or equivalently

   $J = \sum_{i=1}^{3} J_i \sigma^i$  \hspace{1cm} (3.4)

   with $J^+ = J$, $dJ = 0$, and where we have set

   $\Omega^{(2,0)} = J^1 + iJ^2$, \hspace{0.5cm} $\Omega^{(0,2)} = J^1 - iJ^2$, \hspace{0.5cm} $\Omega^{(1,1)} = J^3$.  \hspace{1cm} (3.5)

   The $\sigma^i$'s in eq(3.4) are the standard $2 \times 2$ Pauli matrices given by:

   $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, \hspace{0.5cm} $\sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, \hspace{0.5cm} $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.  \hspace{1cm} (3.6)

2. **Quaternionify the isotriplet 1- form $J$ by the implementation of the NS-NS B- field on the 2-cycles. Setting**

   $B^{NS}_{2\times2} = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$,  \hspace{1cm} (3.7)

   where $I_{2\times2}$ is the $2 \times 2$ identity matrix, which will be dropped now on for simplicity of notations, the * quaternionified* closed hyperkahler 2-form reads as follows

   $J^+ = B^{NS} + i \left( \sum_{i=1}^{3} J^i \sigma^i \right)$,  \hspace{1cm} (3.8)

   or in a condensed form ($B^{NS} = B$) as

   $J^+ = B + i\sigma J$,  \hspace{1cm} (3.9)
with \( d\mathcal{J}_+ = 0 \). We also have
\[
\mathcal{J}_- = B - i\sigma \mathbf{J}, \quad (\mathcal{J}_+)^+ = \mathcal{J}_-
\] (3.10)
as well as
\[
B = \frac{1}{2} \text{Tr} (\mathcal{J}_\pm), \quad J^i = \mp \frac{i}{2} \text{Tr} (\mathcal{J}_\pm \sigma^i).
\] (3.11)
Note in passing that along with eqs (3.9), one may also quaternionify the hyperKahler 2-form \( \mathbf{J} \) by using the two following real quantities,
\[
\mathcal{K}_0 = B + \sigma \mathbf{J}, \quad \mathcal{L}_0 = B - \sigma \mathbf{J}, \quad \sigma \mathbf{J} = \sum_{i=1}^3 J^i \sigma^i,
\] (3.12)
that is without the complex number \( i \) in front of \( \sigma \mathbf{J} \). These two extra objects are self adjoints and lead basically to a vector like theory with strong constraints on the Kahler potential. Though interesting for a complete study, we will restrict our discussion to considering only \( \mathcal{J}_+ \) and \( \mathcal{J}_- \) for the following reasons:

(i) \( \mathcal{J}_\pm \) are enough to define the components of the quaternion (3.11).

(ii) \( \mathcal{J}_\pm \) are enough to have a reality condition for building the (hyper) Kahler potential.

(iii) \( \mathcal{J}_\pm \) exhibit a striking parallel with the complex Kahler geometry of 10D type IIA superstring on the Calabi-Yau threefolds. There, the complexified Kahler 2-form is given by
\[
\mathcal{K}_\pm = B^{NS} \pm i\Omega^{(1,1)}.
\] (3.13)

(iv) \( \mathcal{J}_\pm \) allow to define a "holomorphic" prepotential \( \mathcal{G} \) in same manner as in 10D type IIA superstring on Calabi-Yau threefolds.

However a complete analysis would also take into account the relations (3.12) as well.
We will give a comment on the effect of implementing the quantities (3.12) in the game later on.

(3) Then require that observable quantities such as hyperkahler potential \( \mathcal{H} \), and the "holomorphic" prepotential \( \mathcal{G} \), to be invariant under the following SU(2) gauge transformations,
\[
\mathcal{J}_\pm' = \mathcal{U}^+ \mathcal{J}_\pm \mathcal{U},
\] (3.14)
where \( \mathcal{U} \) is an arbitrary unitary 2 × 2 matrix of SU(2). In other words, thinking about the hyperkahler potential \( \mathcal{H} \) as a hermitian function \( \mathcal{H}(w^+, w^-) \) on the quaternionic moduli
\[
w^\pm = \int_{C_2} \mathcal{J}_\pm,
\] (3.15)
to be introduced with details later on, we should have
\[
\mathcal{H}(w'^+, w'^-) = \mathcal{H}(w^+, w^-),
\] (3.16)
where \( w^{±′} \) are the transform of \( w^± \) under \( SU(2) \) group. A similar statement can be said about the holomorphic prepotential \( G \).

Notice that, though very interesting, this method has also some weak points. One of the difficulties of this method is that the derivatives \( \partial_{-I} = \frac{\partial}{\partial w^+} \) and \( \partial_{+I} = \frac{\partial}{\partial w^-} \) with respect to \( w^+ \) and \( w^- \) have non trivial torsions. They make the building of the hyperkahler metric following from the potential \( \mathcal{H} \) a non easy task. The point is that by defining the derivatives with respect to \( w^+ \) and \( w^- \) as,

\[
\partial_{-I} = \frac{\partial}{\partial w^+} = \frac{1}{2} \left( \frac{\partial}{\partial y^I} - \frac{i}{3} \sigma^I \cdot \nabla I \right), \quad \partial_{+I} = \frac{\partial}{\partial w^-} = \frac{1}{2} \left( \frac{\partial}{\partial y^I} + \frac{i}{3} \sigma^I \cdot \nabla I \right),
\]

satisfying

\[
\frac{\partial w^{+J}}{\partial w^+} = \delta^j_i, \quad \frac{\partial w^{-J}}{\partial w^-} = 0
\]

and so on, we can check that their commutators \([\partial_{-I}, \partial_{+J}]\) and \([\partial_{±I}, \partial_{±J}]\) are non zero. This property puts strong restrictions on the differential analysis using \( w^+ \) and \( w^- \) matrices as basic variables. This difficulty, which is due to the use of Pauli matrices that do not commute,

\[
\sigma^i \sigma^j - \sigma^j \sigma^i = i \sum_{k=1}^{3} \epsilon^{ijk} \sigma^k,
\]

can be overcome by using a geometric representation of \( SU(2) \) relying on the duality \( SU(2) \sim S^3 \). However, this method is beyond the scope of present study. We refer to [57] for more details.

Before going ahead, we would like to comment on the parallel between the Kahler and hyperKahler geometries. This study will be helpful when we consider the building of the potential \( \mathcal{H} \) and the "holomorphic" prepotential \( G(w) \).

### 3.2 Kahler/HyperKahler correspondence

We start by recalling that hyperKahler manifolds form a particular subset of complex 2\( n \) dimensional Ricci flat Kahler ones. To study special hyperKahler \( d\)-geometry of the moduli space of 10D type IIA superstring on the K3 surface, we shall then use the matrix formulation presented above and mimic the method made for the case of special Kahler \( d\)-geometry of type IIA superstring on Calabi-Yau threefolds. Here we first give a general correspondence between Kahler and hyperKahler geometries. Then we make comments on their geometric interpretations, in particular the issue regarding the realization of the central charges of the 4D/6D \( \mathcal{N} = 2 \) superalgebras, the D2-brane wrapping 2-cycles and corresponding potentials.
3.2.1 Type IIA superstring on CY3

In studying the moduli space of 10D type IIA superstring on Calabi-Yau threefolds, we have few geometric objects that play a central role. Some of these basic quantities and their corresponding physical interpretations are collected in the following table:

\[
\begin{align*}
\text{Kahler 2-form: } & \Omega^{(1,1)} = \overline{\Omega^{(1,1)}} \\
\text{Complexified: } & \mathbf{K}_\pm = B^{NS} \pm i \Omega^{(1,1)} \\
\text{Volume form } & \Omega^{(3,0)} \wedge \overline{\Omega^{(0,3)}}
\end{align*}
\]

Here \( \mathbf{K}_+ = B^{NS} + i \Omega^{(1,1)} \) is the complexification of the Kahler 2-form \( \Omega^{(1,1)} \) and \( \mathbf{K}_- \) is its complex conjugate \( (\mathbf{K}_+)^* \). We also have the usual globally defined complex holomorphic 3-form of the CY3 \( \Omega^{(3,0)} \) together with its antiholomorphic partner \( \Omega^{(0,3)} \). These quantities play an important role in type IIB superstring on the mirror of the Calabi-Yau threefold \[44\].

In the type IIA picture we are considering here, the Kahler deformations of the CY3 metric are parameterized by complex numbers

\[
z^I = y^I + i x^I
\]

which on the physical side describe the vevs of the scalar fields in the 4D \( \mathcal{N} = 2 \) Maxwell gauge supermultiplets. The real numbers \( x^I \) are the area of the 2-cycle \( C^I_2 \) inside the CY3 and \( y^I \) are the real fluxes of the NS-NS B-field through the \( C^I_2 \). Notice that on the moduli space of 10D type IIA superstring on CY3, the Kahler form \( \mathbf{J} = \Omega^{(1,1)} \), the NS-NS B-field \( B^{NS} \) and the complexified Kahler form \( \mathbf{K}_+ \) can be respectively defined as follows:

\[
\begin{align*}
\mathbf{J} & = \sum_{I=1}^{h^{(1,1)}_{\text{CY3}}} x^I J_I, \\
B^{NS} & = \sum_{I=1}^{h^{(1,1)}_{\text{CY3}}} y^I J_I, \\
\mathbf{K}_+ & = \sum_{I=1}^{h^{(1,1)}_{\text{CY3}}} z^I J_I,
\end{align*}
\]

where \( \{J_I\} \) is a canonical basis of real 2-forms normalized as

\[
\int_{C^I_2} J_K = \delta^I_K, \quad I, K = 1, \ldots, h^{(1,1)}_{\text{CY3}}.
\]

Along with these objects, we also define three more interesting quantities:

(i) the particular real 2-cycle \( C_2 \) given by the following integral linear combination

\[
C_2 = \sum_I q_I C^I_2,
\]

with \( q_I \) integers. A D2-brane wrapping \( C_2 \) splits in general as the sum of \( q_I \) components wrapping the basis \( C^I_2 \). The charge \( q_I \) is then interpreted as the number of D2-branes...
wrapping $C'_2$.

(ii) the real geometric area of $C_2$ given by

$$Z_e = \int_{C_2} \Omega^{(1,1)}, \quad (3.25)$$

which, by using above relations, can be put in the explicit form

$$Z_e = \sum_{I=1}^{b_{CY3}^{1,1}} q_I x^I. \quad (3.26)$$

Up to an overall factor of the tension, this $Z_e$ is just the mass of a D2-brane wrapping $C_2$ and is interpreted as the electric central charge of the 4D $\mathcal{N} = 2$ superalgebra.

(iii) in type IIB superstring set up, the Kahler potential $K(x,\overline{x})$ reads in terms of the holomorphic 3- forms as follows

$$K(x,\overline{x}) = i \int_{CY3} \Omega^{(3,0)} \land \Omega^{(0,3)}. \quad (3.27)$$

By using the usual symplectic basis $(A^\mu, B^\mu)$ of real 3-cycles within the Calabi-Yau threefold and the complex moduli

$$x^\mu = \int_{A^\mu} \Omega^{(3,0)}, \quad F_\mu = \int_{B_\mu} \Omega^{(3,0)}, \quad (3.28)$$

together with their complex conjugates $(\overline{x}^\mu)$ and $(\overline{F}_\mu)$, we can put $K(x,\overline{x})$ in the following explicit form

$$K(x,\overline{x}) = i \sum_{\mu=1}^{b_{CY3}^{1,1}} \left( x^\mu \overline{(F_\mu)} - \overline{(x^\mu)} F_\mu \right). \quad (3.29)$$

In type IIA superstring on CY3, the Kahler potential $K(z,\overline{z})$ reads in terms of the Kahler form

$$K(z,\overline{z}) = \int_{CY3} \Omega^{(1,1)} \land \Omega^{(1,1)} \land \Omega^{(1,1)}, \quad (3.30)$$

where now the $z$’s stand for the 2-cycles area. We also have $2i\Omega^{(1,1)} = (K_+ - K_-)$.

### 3.2.2 Type IIA superstring on K3

The relations given above have quite similar partners for the case the moduli space of 10D type IIA superstring on the K3 surface. For the analog of the table eq(3.20), we have

\[
\begin{align*}
\text{HyperKahler- form:} & \quad \overline{J^{(ij)}} = \epsilon_{ik} \epsilon_{jl} J^{(kl)} \\
\text{Quaternionified:} & \quad J = B^{NS} + i \sigma J \\
\text{Volume form:} & \quad \sigma J \land \sigma J \\
\text{real moduli:} & \quad x \sigma = \int_{C_2} \sigma J \\
\text{Quaternion:} & \quad w = \int_{C_2} J, \\
\text{Hyperpotential} & \quad \mathcal{H}(w, \overline{w})
\end{align*}
\]  \quad (3.31)
where we have used the $2 \times 2$ matrix representation

$$\sigma \cdot J = \sigma^1 J^1 + \sigma^2 J^2 + \sigma^3 J^3,$$  \hfill (3.32)

to represent the real isotriplet 2-form $J^{(ij)}$ and where $B^{NS} + i\sigma \cdot J$ is the quaternionified hyperKahler 2-form. Notice that $B^{NS} + i\sigma \cdot J$ can in general be written as a spin $(\frac{1}{2}, \frac{1}{2})$ representation $J^{ia}$ of the group $SU(2) \times SU'(2)$. Notice also that on the moduli space of 10D type IIA superstring on K3, the hyperKahler form $\sigma \cdot J$, the NS-NS B-field $B^{NS}$ and the quaternionified hyperKahler form $J$ can be defined as follows:

$$\sigma \cdot J = \sum_{I=1}^{h_{K3}^{(1,1)}} J_I x^I \sigma,$$

$$B^{NS} = \sum_{I=1}^{h_{CY}^{(1,1)}} y^I J_I,$$

$$J_+ = \sum_{I=1}^{h_{CY}^{(1,1)}} (y^I + i\sigma \cdot x^I) J_I,$$

where $\{J_I\}$ is a canonical basis of real 2-forms normalized as:

$$\int_{C^2_I} J_K = \delta^I_K, \quad I, K = 1, \ldots, h_{K3}^{(1,1)}. \hfill (3.34)$$

Like for eqs (3.24-3.26), we define the integral 2-cycle $C_2$ of the complex surface K3,

$$C_2 = \sum_I q_I C^I_2,$$  \hfill (3.35)

where $q_I$ is the number of a D2-brane wrapping $C^I_2$. This special cycle allows to compute the central charge isotriplet $Z_{e}^{(ij)}$. Indeed, notice first that like the $J^{(ij)}$ hyperKahler 2-form, we can use the Pauli matrices to define $Z_{e}^{(ij)}$ isotriplet as a hermitian traceless $2 \times 2$ matrix $\sigma \cdot Z_{e}$ eq (3.32). So we have

$$Z_{e}^{(ij)} = \int_{C_2} J^{(ij)} \quad \iff \quad \sigma \cdot Z_{e} = \int_{C_2} \sigma \cdot J,$$

which up on integration leads to

$$\sum_{m=1}^{3} Z_{e}^m \sigma^m = \sum_{m=1}^{3} \left( \sum_{I=1}^{h_{K3}^{(1,1)}} q_I x^m I \right) \sigma^m,$$  \hfill (3.37)

from which one reads the explicit expression of $Z_{e}^m$ namely

$$Z_{e}^m = \sum_{I=1}^{h_{K3}^{(1,1)}} q_I x^m I, \quad \iff \quad Z_{e} = \sum_{I=1}^{h_{K3}^{(1,1)}} q_I x^I.$$  \hfill (3.38)
Notice that in presence of the NS-NS B-field, one has extra contribution associated with the isosinglet $Z^0$ as shown below:

$$Z^0 = \sum_{I=1}^{h_{K3}^{(1,1)}} q_I y^I. \quad (3.39)$$

The isosinglet $Z_0$ and the isotriplet $Z^{(ij)}$ can be combined into the real four dimensional quantity

$$Z = \int_{c_2} J. \quad (3.40)$$

Using the moduli

$$w_{ia} = \int_{c_2} J^a, \quad G^I_{ia} = \int_{K3} J^a \wedge J^I, \quad i, a = 1, 2, \quad (3.41)$$

where the $C^I_{c_2}$'s is a real basis of real 2-cycles of the K3 surface, one can write down the basic relations for the special hyperKahler geometry.

Because of the $SU(2) \times SU'(2)$ tensor structure, the generalized hyperKahler 2-form is captured by the real 2-form $J^a$ and so the general object we can write down is given by

$$\mathcal{H}^{ijab}(w, G) = \int_{K3} (J^a \wedge J^b) \quad (3.42)$$

or equivalently

$$\mathcal{H}^{ijab}(w, G) = \sum_{I=1}^{h_{K3}^{(1,1)}} \left( w_{ia} G^j_{I} + w_{jb} G^i_{I} \right). \quad (3.43)$$

Notice that $\mathcal{H}^{ijab}$ is a reducible tensor and can be decomposed in four components as follows:

$$\mathcal{H}^{ijab} = \epsilon^{ij} \epsilon^{ab} \mathcal{H}_0 + \epsilon^{ij} \mathcal{H}^{(ab)}_1 + \epsilon^{ab} \mathcal{H}^{(ij)}_2 + \mathcal{H}^{(ij)(ab)}_3 \quad (3.44)$$

with $\mathcal{H}_0 = 2 \sum_{I=1}^{h_{K3}^{(1,1)}} w_{ia} G^j_{IaI}$ and,

$$\mathcal{H}^{(ab)}_1 = \sum_{I=1}^{h_{K3}^{(1,1)}} \left( w_{ia} G^b_{IaI} + w_{ib} G^a_{IaI} \right),$$

$$\mathcal{H}^{(ij)}_2 = \sum_{I=1}^{h_{K3}^{(1,1)}} \left( w_{ia} G^j_{IaI} + w_{ja} G^i_{IaI} \right), \quad (3.45)$$

$$\mathcal{H}^{(ij)(ab)}_3 = \sum_{I=1}^{h_{K3}^{(1,1)}} \left( w_{ia} G^j_{IaI} + w_{ib} G^a_{IaI} + w_{ja} G^i_{IaI} + w_{jb} G^a_{IaI} \right),$$

By identifying the two $SU(2)$ group factors, $\mathcal{H}^{ijab}$ becomes $\mathcal{H}^{ijkl}$ and can be reduced as follows

$$\mathcal{H}^{ijkl} = \epsilon^{ij} \epsilon^{kl} \mathcal{H}_0 + \left( \epsilon^{ij} \mathcal{H}^{(kl)}_1 + \epsilon^{(ij} \mathcal{H}^{(kl)}_2 \right) + \mathcal{H}^{(ijkl)}_3 \quad (3.46)$$
We will develop further these issues later by using the $2 \times 2$ matrix formulation; see section 5, all we need is to keep in mind the two following:

(i) the correspondence between Kahler and hyperKahler manifolds, eqs[3.20][3.31],
(ii) the role played by the leading $SU(2)$ representations in specifying the hyperKahler geometry, eq[3.46].

We end this subsection by noting that in the case of moduli space of $10D$ type IIA superstring compactification on Calabi-Yau manifolds, the connection between supersymmetry and complex geometry reads for the Higgs and Coulomb branches of $4D$ and $6D$ supersymmetric field theory limits as follows:

| Moduli space $\mathcal{M}$ | $4D$ $\mathcal{N}=1$ | $4D$ $\mathcal{N}=2$ | $6D$ $\mathcal{N}=1$ | $6D$ $\mathcal{N}=(1,1)$ |
|----------------------------|----------------------|----------------------|----------------------|----------------------|
| Higgs Branch               | Kahler               | hyperkahler          | hyperkahler          |                      |
| Coulomb Branch             | -                    | Kahler               | -                    | hyperkahler          |

Notice in passing that in present study we are interested in the hyperKahler structure of $6D$ $\mathcal{N}=2$ theory. This is also given by the Higgs branch of $6D$ $\mathcal{N}=1$ supersymmetric theory in the same spirit as does the Kahler geometry in $4D$ $\mathcal{N}=2$ and $4D$ $\mathcal{N}=1$ theories.

4 Quaternionic Geometry in Matrix Formulation

We start by setting up the problem of special hyperkahler $d$-geometry for the moduli space of type IIA superstring on the K3 surface. Then we consider the way to solve it by using the $2 \times 2$ matrix formulation.

4.1 Moduli Space $M_{\text{IIA/K3}}$

We first give some general on the moduli space $M_{\text{IIA/K3}} = \mathcal{M}$ of the type IIA superstring on the K3 surface with fixed string coupling constant $[29, 30, 31, 32, 33]$. Then we describe $\mathcal{M}$ by using the quaternionic moduli.

4.1.1 Spectrum 10D Type IIA on the K3 surface

To begin recall that the spectrum of 10D Type IIA superstring has two sectors: perturbative (fields) and non perturbative (D2$n$-branes, $n = 0,1,2,3$ sources of RR fields).
10D Supermultiplet

In the perturbative massless sector, the bosonic fields of 10D type IIA superstring spectrum are given by

\[ \phi_{\text{dil}}, \quad G_{(MN)}, \quad B_{[MN]}, \quad A_M, \quad C_{MNK}, \]

(4.1)

where \( M, N, K = 0, \cdots, 9 \) are vector indices of \( SO(1,9) \). Altogether have a total number of 128 on shell degrees of freedom. Their fermionic partners involve two 10D-gravitinos and two 10D-gauginos. These are 10D Majorana-Weyl spinors. For simplicity, we shall deal with the bosonic sector only.

The compactification of type IIA superstring on the K3 surface is obtained by breaking space-time symmetry \( SO(1,9) \) down to the subgroup \( SO(1,5) \times SU(2) \) which is contained in \( SO(1,5) \times SO(4) \). Degrees of freedom of type IIA string on the K3 surface are determined by retaining only half of the original ones since K3 preserves only half of the 32 original supersymmetries.

6D Supermultiplets

In six dimensions, the spectrum of bosonic fields with non zero spin is formally given by,

\[ G_{\mu\nu}, \quad B_{\mu\nu}, \quad A_{\mu}, \quad C_{\mu\nu\rho}, \quad C_{\mu\nu}, \quad \]

where \( G_{\mu\nu} \) is the 6D metric, \( B_{\mu\nu} \) and \( C_{\mu\nu\rho} \) are the 6D antisymmetric gauge fields. We also have the 6D gravi-photon \( A_{\mu} \) and the Maxwell gauge fields \( C_{\mu\nu} \) following from the compactification of the RR 3-form on the real 2-cycles of K3 surface. These real 6D 1-forms have two indices on the K3 surface and should be thought of as

\[ C_{\mu\nu} \equiv C_{I\mu}, \quad I = 1, \cdots, 22. \]

(4.3)

For the spectrum of the 6D scalars, we have in addition to the 6D dilaton \( \phi_{\text{dil}} \), the following

\[ \phi_{\text{dil}}, \quad B_{mn}, \quad G_{mn}^{\text{kahler}}, \quad G_{mn}^{\text{complex}}, \quad G_{mn}^{\text{complex}}. \]

(4.4)

They describe the scalars \( B_{mn} \) resulting from the compactification of the NS-NS 2-form on the real 2-cycles of the K3 surface. These \( B_{mn} \)'s and the field moduli \( G_{mn} = \{ G_{mn}^{\text{kahler}}, \ G_{mn}^{\text{complex}} \} \) following from the compactification of metric field should be understood as follows:

\[ G_{mn}^{\text{kahler}} \equiv G_{I\text{kahler}}, \quad I = 1, \cdots, 20, \]

\[ G_{mn}^{\text{complex}} \equiv G_{J\text{complex}}, \quad J = 1, \cdots, 19, \]

\[ B_{mn} \equiv B^K, \quad K = 1, \cdots, 22. \]

(4.5)
Notice that the field moduli $G^I_{\text{kahler}}$ and the complex $G^I_{\text{complex}}$ stand for Kahler and complex deformations of K3 metric with SU(2) holonomy group. On the other hand, viewed from $6D\, \mathcal{N} = 2$ supergravity low energy limit, these fields combine as follows:

(i) **Gravity:**

(32 + 32) on shell degrees of freedom for the $6D\, \mathcal{N} = 2$ gravity supermultiplet whose bosonic sector is as follows:

$$G_{\mu\nu}, B_{\mu\nu}, \mathcal{A}^{(ij)}_{\mu}, C_{\mu\nu\rho}, \phi_{\text{dil}}, \quad (4.6)$$

involving, besides the 3-form $H = dB$, the gauge field strengths:

$$\mathcal{F}^{(ij)}_2 = d\mathcal{A}^{(ij)}_1, \quad \mathcal{F}^{(ij)}_4 = *\mathcal{F}^{(ij)}_2$$

$$\mathcal{F}^0_4 = dC_3, \quad \mathcal{F}^0_2 = *\mathcal{F}^0_4. \quad (4.7)$$

Notice that $\mathcal{A}^{(ij)}_\mu$, $\mathcal{F}^{(ij)}_2$ and $*\mathcal{F}^{(ij)}_2$ are isotriplets and all remaining others singlets.

(ii) **Coulomb:**

$20 \times (8 + 8)$ on shell degrees of freedom for the six dimensional $\mathcal{N} = 2$ supersymmetric Maxwell multiplets with bosonic sector

$$\mathcal{C}^I_\mu, \ w^{\pm I}, \quad I = 1, \cdots, 20. \quad (4.8)$$

Setting $\mathcal{C}^I_1 = dx^\mu \mathcal{C}^I_\mu$, the corresponding gauge field strengths and their 6D duals are as follows:

$$\mathcal{F}^I_2 = d\mathcal{C}^I_1, \quad \mathcal{F}^I_4 = *\mathcal{F}^I_2, \quad I = 1, \cdots, 20. \quad (4.9)$$

These $6d\, \mathcal{N} = 2$ supermultiplets are $SU(2)$ singlets but transforms as a vector under $SO(20)$. They can be nicely described in the $6d\, \mathcal{N} = 1$ superspace formalism by still maintaining $SU(2)$ isometry. Below, we give some details.

### 4.1.2 Supersymmetry in 6D

In six dimensional space-time, one distinguishes two supersymmetric algebras: (i) the $6D\, \mathcal{N} = (2,0)$ chiral superalgebra and (ii) the $6D\, \mathcal{N} = (1,1)$ ($6D\, \mathcal{N} = 2$) non chiral one given by eq(2.10). More details on the graded commutation relations of these superalgebras including central extensions are exhibited in the appendix. What we need here is the physical representations of the $6D\, \mathcal{N} = 2$ superalgebra and the field theoretical way to deal with them.

In the language of $6D\, \mathcal{N} = 1$ supersymmetric representations, $6D\, \mathcal{N} = 2$ supermultiplets ($R^{\mathcal{N}=2}_{6D}$) split into pairs of $6D\, \mathcal{N} = 1$ representations $[\{45\}]$ as given below,

$$R^{\mathcal{N}=2}_{6D} = R^{\mathcal{N}=1}_{6D} \oplus R^{\mathcal{N}=1}_{6D}. \quad (4.10)$$
To fix the idea, we consider here after the $6D\,\mathcal{N}=2$ gauge multiplet $V_{6D}^{N=2}$ which, as given above, consists of the following on shell degrees of freedom,

$$V_{6D}^{N=2} = \left(1, \frac{1}{2}, 0^4\right),$$

(4.11)

where $1, \frac{1}{2}$ and $0$ stand for the spin of the component field content and the powers for their numbers. The existence of four scalar gauge fields (sgauginos) within the $6D\,\mathcal{N}=2$ reflects in some sense the four central charges $Z^{ij}$ of the underlying superalgebra eq(2.16). In the $6D\,\mathcal{N}=1$ formalism, the $V_{6D}^{N=2}$ vector multiplet splits into a vector $V_{6D}^{N=1}$ and a hypermultiplet $H_{6D}^{N=1}$ as follows

$$V_{6D}^{N=2} = V_{6D}^{N=1} \oplus H_{6D}^{N=1},$$

(4.12)

where

$$V_{6D}^{N=1} = \left(1, \frac{1}{2}\right), \quad H_{6D}^{N=1} = \left(\frac{1}{2}, 0^4\right).$$

(4.13)

Notice that $V_{6D}^{N=1}$ has no scalar in six dimensions while $H_{6D}^{N=1}$ has four scalars capturing the hyperkahler structure of the Coulomb branch of $6D\,\mathcal{N}=2$ supersymmetry. Notice also that in the $4D\,\mathcal{N}=2$ language, these multiplets reduce generally to

$$V_{4D}^{N=2} = \left(1, \frac{1}{2}, 0^2\right), \quad H_{4D}^{N=2} = \left(\frac{1}{2}, 0^4\right).$$

(4.14)

Notice moreover that in the case of toroidal compactification from $6D$ down to $4D$, $6D\,\mathcal{N}=2$ supersymmetry leads to $4D\,\mathcal{N}=4$ supersymmetry. The gauge multiplet $V_{6D}^{N=2}$ gives $V_{4D}^{N=4}$ with on shell degrees of freedom,

$$V_{4D}^{N=4} = \left(1, \frac{1}{2}, 0^6\right),$$

(4.15)

corresponding precisely to the combination of $V_{4D}^{N=2}$ and $H_{4D}^{N=2}$ multiplets. The six scalars involved in $V_{4D}^{N=4}$ are exactly the six dimensional vector moduli that we encounter in the study of the moduli space

$$M_{het/T^6} = \frac{SO(6, 22)}{SO(6) \times SO(22)},$$

(4.16)

defined by the toroidal compactification of $10D$ heterotic superstring on $T^6$.

The splitting of $6D\,\mathcal{N}=2$ representations into pairs of $6D\,\mathcal{N}=1$ ones has a remarkable parallel with the reduction $4D\,\mathcal{N}=2$ into $4D\,\mathcal{N}=1$ representations. This is a crucial technical point that can be used for the study of the Coulomb branch $4D\,\mathcal{N}=2$ theories and black holes in type II superstrings on CY3. This parallel can be learnt on the following correspondence,
vector multiplet in 4D $\mathcal{N}=2$ $\rightarrow$ chiral multiplet in 4D $\mathcal{N}=1$

vector multiplet in 6D $\mathcal{N}=(1,1)$ $\rightarrow$ hypermultiplet in 6D $\mathcal{N}=1$

It is interesting to note that one can still exhibit manifestly the $SU(2)$ symmetry within $6D \mathcal{N}=1$ supersymmetry. This property follows from the use of by the supercharges $Q^i_\alpha$, eq(2.23) as the generators of the $6D \mathcal{N}=1$ algebra whose anticommutation relations can be directly read from eq(2.16) as shown below,

$$\{Q^i_\alpha, Q^j_\beta\} = \epsilon^{ij} P_{[\alpha\beta]}, \quad \alpha, \beta = 1, \ldots, 4$$
$$[P_{[\alpha\beta]}, Q^i_\gamma] = 0, \quad i, j = 1, 2. \quad (4.17)$$

Here $P_{[\alpha\beta]} = \sum_{\mu=0}^5 (\Gamma^\mu)_{\alpha\beta} P_\mu$ is the 6D- energy momentum vector and the $\Gamma^\nu$’s are the 6D gamma matrices.

One can also develop a $6D \mathcal{N}=1$ superspace formalism to describe supersymmetric multiplets in terms of superfields $\Phi = \Phi(x^\mu, \theta^{\alpha i})$. For instance the $6D \mathcal{N}=1$ Maxwell superfield in the Wess- Zumino gauge is defined, using 6D Grassmann variables $\theta^{\alpha i}$, by a real isotriplet superfield $V^{(ij)}$ with $\theta$- expansion given, in the Wess-Zumino gauge, by

$$V^{(ij)} = \theta^{(i}\theta^{j)} A_{[\alpha\beta]} + \theta^{(i}|(\theta^4) D^{(ij)}(i), \quad (4.18)$$

where $(\theta^4) D^{(ij)}(i)$ stands for $\epsilon_{\alpha\beta\gamma\delta} \theta^{(i}\theta^{j}\theta^{k}\theta^{l}\delta} D_{(kl)}$. Here $A_{[\alpha\beta]} \sim A_\mu$ is the 6D gauge field, $\lambda^i$ its gaugino partners and $D^{(kl)}$ an isotriplet of auxiliary fields. Quite similar quantities can be written down for the hypermultiplets which are also described by superfields in $SU(2)$ representations; for details see [39, 40, 45]. Note finally that a more convenient way to deal with $6D \mathcal{N}=1$ vector and hypermultiplet representations ($6D \mathcal{N}=2$ vector multiplet) is to use $4D \mathcal{N}=2$ superspace obtained by decomposing 6D vectors in 4D vectors and 2 scalars and 6D spinors $\theta^{\alpha i}$ into a 4D Weyl spinor $\theta^{ai}$ and its complex conjugate $\overline{\theta}^{ai}$,

$$(\theta^{\alpha i})_{1 \leq \alpha \leq 4} \rightarrow (\theta^{ai}, \overline{\theta}^{ai})_{a=1,2}. \quad (4.19)$$

It is an interesting task to study this reduction in the framework of the harmonic superspace formalism [40] where hyperkahler geometry is nicely described in terms of $4D \mathcal{N}=2$ hypermultiplets couplings.

### 4.2 Quaternionic Moduli in 6D $\mathcal{N}=2$ Gauge Theory

From the set up of 10D type IIA superstring on a generic K3 surface, the real 80 degrees of freedom of $M$ (23) are arranged as follows:

(i) Twenty (20) Kahler deformations to be denoted as $x^{0I}$. They can be expressed in
terms of the Kahler structure of the K3 surface captured by the Kahler 2-form $\Omega^{(1,1)} = J^0$, as follows,

$$x^{0I} = \int_{C^I_2} J^0, \quad I = 1, ..., 20. \quad (4.20)$$

Here $C^I_2 \in H_2 (K3, R)$ is a real basis of 2-cycles of the K3 surface. So $x^{0I}$ can be thought of as the real area of the 2-cycle $C^I_2$. Notice that using the basis $\{J^0_I\}$ of real 2-forms normalized as

$$\delta J^I = \int_{C^I_2} J^0_J, \quad (4.21)$$

one can invert the previous relation as follows

$$J^0 = \sum x^{0I} J^0_I. \quad (4.22)$$

Note in passing that the 20 Kahler moduli should be split as $20 = 1 + 19$. From SU(2) representation theory, only one of these degrees of freedom is an isosinglet. The remaining 19 ones are components belonging to 19 isotriplets. The missing $2 \times 19$ moduli come from the complex deformations to be considered in moment. Notice also that one can express the real volume of the K3 surface as follows

$$\mathcal{V}_{K3} = \int_{K3} J^0 \wedge J^0 = d_{IJ} x^{0I} x^{0J}, \quad d_{IJ} = \int_{K3} J^0_I \wedge J^0_J \quad (4.23)$$

(ii) Nineteen (19) complex deformations to be denoted as $x^+I$ and $x^-I$, that is thirty eight (38) real moduli,

$$x^+I = x^{1I} + ix^{2I}, \quad x^-I = x^{1I} - ix^{2I}, \quad I = 1, ..., 19. \quad (4.24)$$

They can be expressed in terms of deformations of the holomorphic and antiholomorphic 2-forms $J^+ = \Omega^{(2,0)}$ and $J^- = \Omega^{(0,2)}$ as follows:

$$x^+I = \int_{C^I_2} J^+, \quad x^-I = \int_{C^I_2} J^-, \quad I = 1, ..., 19. \quad (4.25)$$

Using the 2-form basis $\{J_I\}$, these relations can be also inverted as

$$J^+ = \sum x^+I J_I, \quad J^- = \sum x^-I J_I. \quad (4.26)$$

The $x^{\pm I}$ moduli may be interpreted as holomorphic volumes of the 2-cycles $C^I_2$. Indeed computing the volume of the K3 surface but now using the relation

$$\mathcal{V}_{K3} = \int_{K3} J^+ \wedge J^- = 2 \sum_{I,J=1}^2 d_{IJ} x^{+I} x^{-J}. \quad (4.27)$$

Then equating it with (4.23), we learn that the volume of $\mathcal{V}_{K3}$ can be rewritten as,

$$\mathcal{V}_{K3} = d_{IJ} \left( \frac{1}{2} x^{+I} x^{-J} + \frac{1}{2} x^{-I} x^{+J} + x^{0I} x^{0J} \right). \quad (4.28)$$
showing that $V_{K3}$ is an $su(2)$ invariant and can be put in the form $V_{K3} = d_{IJ} (x^I \cdot x^J)$.

(iii) $22 = (20 + 2)$ real moduli coming from the values of the $B_{NS}$ field on 2-cycles of the K3 surface. It is given by the Betti number $b_2 (K3) = h^{(2,0)} + h^{(0,2)} + h^{(1,1)}$ which is equal to $1 + 1 + (4 + 16)$. The first $(4 + 16) = 20$ moduli have same nature as Kahler deformations and are written as

$$y^{0I} = \int_{C^2} B_{NS}^0, \quad I = 1, \ldots, 20.$$  \hspace{1cm} (4.29)

The two extra ones should viewed as the 20-th complex moduli, that is

$$y^0_{21} + iy^0_{22} = x^{+20}, \quad y^0_{21} - iy^0_{22} = x^{-20}. \hspace{1cm} (4.30)$$

Notice that nineteen of the $y^{0I}$, say $I = 1, \ldots, 19$, are isosinglets, and the remaining three namely $(y^0_{20}, y^0_{21}, y^0_{22})$ can be combined as in eq (4.30) to form altogether an isotriplet; see also previous discussion concerning the Kahler form. This is a crucial point which we will encounter when we consider the uplifting to 7D.

To exhibit the hyperkahler structure of the moduli space of 10D type IIA superstring on the K3 surface and keeping in mind the above discussion, we can use the $SU(2)$ spin $\frac{1}{2} \times \frac{1}{2}$ representations to split the real 80 moduli as $20$ isosinglets \{ $y^{0I}$ \} plus $20$ isotriplets \{ $x^{-I}, x^{0I}, x^{+I}$ \}:

$$80 = 20 \times 1 + 20 \times 3. \hspace{1cm} (4.31)$$

As such the real 80 moduli can be grouped into 20 quaternions $w^{+I}$ as shown below

$$w^+_I = y^0_I + i x_I \cdot \sigma, \quad I = 1, \ldots, 20, \hspace{1cm} (4.32)$$

together with there adjoint conjugates $w^{-I} = (w^{+I})^\dagger$. Here $y^0_I$ should be thought of as $y^0_I$ times the identity matrix $I_{2 \times 2}$ and $x_I \cdot \sigma$ as

$$x_I \cdot \sigma = x^+_I \sigma^+ + x^0_I \sigma^0 + x^-_I \sigma^-,$$  \hspace{1cm} (4.33)

with $\sigma^{0,\pm}$ being the usual $2 \times 2$ Pauli matrices satisfying both $SU(2)$ commutation relations and the 2D Clifford algebra anticommutations. In particular we have:

$$(i \sigma^1)^2 = (i \sigma^2)^2 = (i \sigma^3)^2 = -1,$$

$$\sigma^\pm = \sigma^1 \pm i \sigma^2, \quad \sigma^0 = \sigma^3. \hspace{1cm} (4.34)$$

For later use, we define the adjoint conjugate\footnote{Notice that quaternions have three complex structures $i, j$ and $k$ related as $i \wedge j = k$. As such, one should have three kinds of complex conjugations, say one for $i \ (i = -i)$, an other for $j \ (j = -j)$ and the combined one for $k \ (k = -k, k^* = -k, k = k)$. An aspect of this feature will be considered when we introduce harmonic space.} of the $w_I$ quaternions as $w_I^+ = y_I^0 - i x_I \cdot \sigma$.

So the "real" and "imaginary" parts (see also footnote) of the $w_I^\pm$ quaternions as given
by,

\[ y_I^0 = \frac{w_I + w_I^+}{2}, \]
\[ x_I \sigma = \frac{w_I - w_I^+}{2i}. \]  

(4.35)

With these tools, it is not difficult to see that the moduli \((4.20-4.25)\) can be rewritten as

\[ x_I \sigma = \int_{C^I_{2}} \sigma J, \quad \sigma J = \sum_{m=1}^{3} \sigma^{-m} J m. \]  

(4.36)

Introducing the following "quaternionified" 2-form, in analogy with the usual complexified 2-form in the Kahler geometry

\[ J_+ = B_{NS} + i \sigma J, \quad J_- = B_{NS} - i \sigma J, \quad J_+ = (J_+)^\dagger \]  

(4.37)

we see that the \(\{y_I^0, x_I^{-1}, x_I^{0I}, x_I^{+I}\}\) moduli can be grouped altogether in quaternions as shown below

\[ w^I = \int_{C^I_{2}} J_+, \quad \overline{w}^I = \int_{C^I_{2}} J_-, \]  

(4.38)

where \(w^I\) stand for \(w^{+I}\) and \(\overline{w}^I = w^{-I}\) for its adjoint conjugate. For later use, it is interesting to introduce the basis of quaternionic 2-forms \(\{J_I\}\) on the \(C^I_{2}\)-cycles. With these objects, the above relations may be rewritten as

\[ J_+ = \sum w^{+I} J_I, \quad J_- = \sum w^{-I} J_I, \]  

(4.39)

where we have used \(\int_{C^I_{2}} J_J = \delta^I_J\).

### 4.3 7D Uplifting

Large distance 7D \(\mathcal{N} = 2\) supergravity theory limit of 10D superstring and 11D M-theory compactifications can be obtained in different, but equivalent, ways. In particular, they are obtained by the three following routes:

(1) compactification of eleven dimensional M-theory on the K3 surface.

(2) compactification of 10D heterotic superstring on a real 3-torus.

(3) uplifting 6D \(\mathcal{N} = 2\) theory to seven dimensions.

The last method is useful for studying \(\mathcal{N} = 2\) extremal black objects in seven dimensions from six dimensional view. This way is in the same spirit used in the uplifting of 4D \(\mathcal{N} = 2\) theory to five dimensions. The 7D uplifting moduli space is obtained by putting appropriate constraint eqs on the 6D one. These constraint eqs, that remain to be worked out, have to reduce the real \textit{eighty one} dimension moduli space of type IIA superstring on the K3 surface, namely

\[ \frac{SO(4,20)}{SO(4) \times SO(20)} \times SO(1,1), \]  

(4.40)
down to
\[ \frac{SO(3,19)}{SO(3) \times SO(19)} \times SO(1,1). \] (4.41)

This reduction corresponds to fix \((81 - 58) = 23\) real moduli.

In our matrix formulation, the 7D uplifting corresponds to put adequate constraint eqs on the quaternionic moduli \(w^{\pm I}\). Using the analysis of subsection 4.2, it is not difficult to see that the real 23 constraint equations are obtained by:

(i) killing the real 22 moduli coming from the NS-NS B-field on the 2-cycles of K3.

(ii) fixing the real volume of the K3 surface.

(iii) breaking the \(SO(4) \sim SU(2) \times SU'(2)\) down to \(SO(3) \sim SU(2)\).

This is achieved as follows: Since nineteen of the real 22 moduli the NS-NS B-field come as isosinglets, see discussion after eq(4.30), they are killed by the conditions

\[ \text{Tr} \left( w^{\pm I} \right) = 0, \quad I = 1, \ldots, 19. \] (4.42)

The remaining three moduli appear as a isotriplet and are then killed by the vectorial condition

\[ \text{Tr}_{SU(2)} \left( \int_{K3} \left( \sigma w^+ w^- \right) \right) = 0, \] (4.43)

which, up on integration, gives:

\[ \sum_{I,J=1}^{20} d_{IJ} \text{Tr} \left( \sigma^i w^+ w^- \right) = 0, \quad i = 1, 2, 3. \] (4.44)

The 23-rd modulus is fixed by imposing a condition on the volume of the K3 surface as shown below,

\[ \sum_{I,J=1}^{20} \text{Tr} \left( w^+ w^- d_{IJ} \right) = \mathcal{V}_{K3} = \text{constant}, \] (4.45)

where the constant can be taken as \(\mathcal{V}_{K3} = 1\). Notice that the constraint relations (4.43, 4.45) can be equivalently stated as follows:

\[ \sum_{I,J} w^+ w^- d_{IJ} \right) = \frac{\mathcal{V}_{K3}}{2} I_2, \] (4.46)

that is as a hermitian \(2 \times 2\) matrix condition fixing four real degrees of freedom.

5 Computing the Potentials

The special hyperkahler potential \(\mathcal{H}\), whose \(SU(2)\) tensor structure has been obtained earlier eqs(3.44-3.46), and the prepotential \(\mathcal{G}(w)\) will be obtained as follows:

(1) Using the matrix formulation developed above; in particular the quaternionified 2-forms \(\mathcal{J}_\pm\) and the corresponding quaternionic moduli \(w^{\pm I}\).

(2) Mimicking the analysis for computing the Kahler potential of the moduli space of the 10D type IIA superstring on Calabi-Yau threefolds.
5.1 Special hyperkahler potential $\mathcal{H}$

Extending the analysis made for the special Kahler geometry, one discovers that the special hyperKahler potential $\mathcal{H}$ of the 10D type IIA superstring on the K3 surface can be given in term of the integral on the volume 4-form

$$ \mathbf{J} \wedge \mathbf{J}, $$

(5.1)

on the K3 surface. A priori, $\mathcal{H}$ is not a simple real number since, from SU(2) group theoretical view, the tensor product $\mathbf{J} \otimes \mathbf{J}$ has real nine dimensions which decomposes in terms of SU(2) irreducible representations as

$$ J^n \otimes J^m = \delta^{nm} (J^n J^m) \oplus J^{[n} J^{m]} \oplus J^{(n} J^{m)}, \quad n, m = 1, 2, 3. $$

(5.2)

These factors are precisely the components given by eqs(3.4 4-3.46). Moreover, notice that since the wedge product $\mathbf{J} \wedge \mathbf{J}$ is symmetric in the interchange of the 2-form isotriplets $\mathbf{J}$, the antisymmetric part in the above decomposition should not contribute. As such $\mathbf{J} \wedge \mathbf{J}$ contains an isosinglet which we set as $V_0$ and a quintet which we represent by a traceless symmetric $3 \times 3$ matrix $S$. With these objects, we can show that the hyperKahler potential reads as

$$ \mathcal{H} = \text{Tr} [\ln (V_0 - S)]. $$

(5.3)

Let us give details on the way of building $\mathcal{H}$.

Using Pauli matrices, the volume 4-form $\mathbf{J} \wedge \mathbf{J}$ can be also written like $\sigma \mathbf{J} \wedge \sigma \mathbf{J}$. Introducing the hermitian "volume" matrix,

$$ V = \int_{K3} \sigma \mathbf{J} \wedge \sigma \mathbf{J}, $$

(5.4)

we can state the problem of the determination of the potential $\mathcal{H}$ as follows,

$$ \mathcal{H} = f (V), $$

(5.5)

where $f$ is some function that remains to be specified. It should be invariant under $SO (4) \times SO (20)$. Before proceeding further, let us make three comments:

(1) The hyperKahler 2-form $\mathbf{J}$ on the K3 surface is an isotriplet. The hermitian quantity $V$ is valued in the tensor algebra of two SU(2) isotriplets which generally decompose as follows

$$ 3 \times 3 = 1 + 3 + 5. $$

(5.6)

These irreducible components correspond, in the language of the tensor algebra of Pauli matrices, to

$$ \sigma \otimes \sigma = (1) \oplus (\sigma) \oplus (\sigma \otimes \sigma)_{sym} $$

(5.7)
where \((\sigma \otimes \sigma)_{\text{sym}}\) stands for traceless symmetric product. They are respectively associated with isosinglet, isotriplet and isoquintet.

(2) Using eq. (4.37),

\[
\sigma \cdot J = \frac{1}{2i} (J_+ - J_-),
\]

we can express \(V\) as

\[
V = \frac{1}{4} (H_0 - H_{++} - H_{--})
\]

with

\[
H_{--} = \int_{K^3} (J_- \wedge J_-),
\]
\[
H_0 = \int_{K^3} (J_+ \wedge J_- + J_- \wedge J_+),
\]
\[
H_{++} = \int_{K^3} (J_+ \wedge J_+).
\]

Then using the expansions \(J_\pm = \sum w^\pm I J_I\) and integrating over the 2-cycles of \(K^3\), we get on one hand

\[
H_{++} = \sum_{I,J} w^+ I d_{IJ} w^+ J,
\]
\[
H_0 = \sum_{I,J} (w^+ I d_{IJ} w^- J + w^- J d_{IJ} w^+ J),
\]
\[
H_{--} = \sum_{I,J} w^- I d_{IJ} w^- J,
\]

where \(d_{IJ} = \int_{K^3} (J_I \wedge J_J)\) stands for the intersection matrix numbers of the 2-cycles of the \(K^3\) surface encountered earlier. It reads for a generic \(K^3\) surface as follows:

\[
d_{IJ} = \begin{pmatrix}K(E_8) & 0_{8 \times 8} & 0_{8 \times 2} & 0_{2 \times 2} \\ 0_{8 \times 8} & K(E_8) & 0_{8 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 8} & 0_{2 \times 8} & \Delta_2 & 0_{2 \times 2} \\ 0_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} & \Delta_2 \end{pmatrix}_{20 \times 20}
\]

where \(K(E_8)\) is the Cartan matrix of ordinary \(E_8\) Lie algebra and where

\[
\Delta_2 = \begin{pmatrix}0 & 1 \\ 1 & 0 \end{pmatrix}
\]

generates the intersections of the \(T^2\)-cycles within \(T^4\) in the orbifold limit construction of \(K^3\) of the \(K^3\) surface.

On the other hand, using the moduli \(w^\pm I = \left( \int_{C^2} J_\pm \right)\) we have:

\[
V = \frac{1}{2} \sum_{I,J} (w^+ I - w^- I) d_{IJ} (w^+ J - w^- J).
\]
This relation can be also rewritten as follows

\[ V = \frac{1}{2} \sum_{I,J} \left( \sum_{n,m=1}^{3} \sigma_n \sigma_m x^n_I d_{IJ} x^m_J \right) = \sum_{n,m=1}^{3} \sigma_n \sigma_m V^{nm}. \]  

(5.15)

Notice that since \( d_{IJ} \) is symmetric, the \( SU(2) \) tensor

\[ V^{nm} = \frac{1}{2} \sum d_{IJ} x^n_I x^m_J, \]  

(5.16)

which reads as well as

\[ \frac{1}{4} \sum d_{IJ} \left( x^n_I x^m_J + x^n_J x^m_I \right), \]  

(5.17)

is also symmetric; \( V^{nm} = V^{mn} \). So there is no contribution coming from the isotriplet component of the formal expansion \( (5.6) \) since

\[ V^{[nm]} = \frac{1}{4} \sum d_{IJ} \left( x^n_I x^m_J - x^m_I x^n_J \right), \]  

(5.18)

vanishes identically. As such, it is interesting to split the \( 3 \times 3 \) matrix \( V^{nm} \) as follows

\[ V^{nm} = V_0 \delta^{nm} - S^{nm}, \]  

(5.19)

where

\[ V_0 = \frac{1}{3} \text{Tr} (V^{nm}), \]  

(5.20)

which reads in terms of

\[ V_{K3} = \sum_{I,J} x^I d_{IJ} x^J, \]  

(5.21)

the volume of K3, as

\[ V_0 = \frac{1}{3} \sum_{n=1}^{3} \left( \frac{1}{2} \sum_{I,J} x^n_I d_{IJ} x^n_J \right) = \frac{1}{6} \sum_{I,J} x^I d_{IJ} x^J, \]  

(5.22)

and where the traceless matrix

\[ S^{nm} = V^{nm} + V_0 \delta^{nm}. \]  

(5.23)

By substituting eq(5.19) back into

\[ V = \sum_{n,m=1}^{3} \sigma_n \sigma_m V^{nm}, \]  

(5.24)

we can put the volume matrix as follows

\[ V = V_0 I_3 - S. \]  

(5.25)

\( I_3 \) stands for the \( 3 \times 3 \) identity matrix which will be dropped now on and is given by,

\[ S = \sum_{n,m=1}^{3} \sigma_n \sigma_m S^{nm}. \]  

(5.26)
Notice that while $V_0$ is invariant under $SO(4) \times SO(20)$, the matrix $S$ is invariant under $SO(4)$ but still transforms as quintet under $SU(2) \subset SO(4)$.

Pushing further the similarity with the Kahler geometry, one can then define the hyperKahler potential $\mathcal{H}$ in term of the logarithm of the volume form $V$ as follows

$$\mathcal{H} = Tr(\ln V) = Tr[\ln (V_0 - S)],$$

which can be put in the form

$$\mathcal{H} = Tr[\ln V_0] + Tr[\ln (1 - V_0^{-1}S)]$$

and then expanded into in power series in the inverse of the volume $V_0$ of the K3 surface. Notice that the leading term $Tr[\ln V_0]$ appears as the Kahler component which is independent of $S$. The next leading term given by the expansion of $\ln (1 - V_0^{-1}S)$ namely

$$Tr[V_0^{-1}S] = V_0^{-1}Tr[S]$$

vanishes identically due to the property $Tr[S] = 0$. This feature may be associated with the Ricci flat property of hyperKahler manifolds.

### 5.2 Matrix prepotential $\mathcal{G}(w)$

In this subsection, we want to derive the "holomorphic" matrix prepotential $\mathcal{G}(w)$ of type IIA superstring on K3 surface. Holomorphicity should be understood in terms of the matrix formulation. In other words $\mathcal{G}(w)$ is a holomorphic $2 \times 2$ matrix which do not depend on $\pi$,

$$\partial \mathcal{G}(w) / \partial \pi = 0.$$

As we will show, $\mathcal{G}(w)$ and its adjoint conjugate $\mathcal{G}(\bar{\pi})$ are prepotentials involved in the building of the volume matrix $V$ eq(5.4). Notice that we have used the terminology holomorphic because of the analog role of $\mathcal{G}(w)$ with the usual classical holomorphic prepotential $\mathcal{F}(z)$ of 10D type IIA superstring on CY3.

#### 5.2.1 Prepotential $\mathcal{F}(z)$ in type IIA on CY3

Recall that in type IIA superstring on CY3, the classical expression of the prepotential $\mathcal{F}(z)$ is given in term of the local complex coordinates

$$z^I = \int_{C^I_2} K_+, \quad I = 1, \ldots, h^{(1,1)}_{CY3} (CY3),$$

as follows:

$$\mathcal{F}(z) = \frac{1}{3!} \sum_{I,J,K=1}^{h^{(1,1)}_{CY3}} d_{IJK} z^I z^J z^K.$$
In these relations \( K_\pm = B^{NS} \pm i\Omega^{(1,1)} \) is the complexified Kahler 2-form and

\[
d_{IJK} = \int_{CY^3} (\mathcal{J}_I \wedge \mathcal{J}_J \wedge \mathcal{J}_K),
\]

(5.33)
is the triple intersection of 2-cycles within the Calabi-Yau treefold. The prepotential \( \mathcal{F}(z) \) is a cubic holomorphic function playing a crucial role in the characterization of the Kahler potential of the Kahler special \( d \)-geometry for type IIA superstring on CY3. We also have for the Kahler potential

\[
\mathcal{K}(z, \bar{z}) = \ln \left( \int_{CY^3} (K_+ - K_-) \wedge (K_+ - K_-) \wedge (K_+ - K_-) \right).
\]

(5.34)

By integration, we get precisely

\[
\mathcal{K}(z, \bar{z}) = \ln \left( \sum_{I,J,K=1}^{h^{(1,1)}_{CY^3}} d_{IJK} (z^I - \bar{z}^I) (z^J - \bar{z}^J) (z^K - \bar{z}^K) \right),
\]

(5.35)

which can be also rewritten as

\[
\mathcal{K}(z, \bar{z}) = \ln \left( \sum_{I,J,K=1}^{h^{(1,1)}_{CY^3}} d_{IJK} (z^I - \bar{z}^I) (z_I - \bar{z}_I) \right),
\]

(5.36)

where we have set

\[
z_I = \int_{CY^3} K_+ \wedge K_+ \wedge J_I,
\]

(5.37)

and similarly for its complex conjugates \( \bar{z}_I \).

### 5.2.2 Prepotential \( G(w) \) in type IIA on the K3 surface

In the case of 10D type IIA superstring on the K3 surface, the situation is a little bit subtle but we do still have quite similar relations. More precisely, using the local quaternionic coordinates

\[
w^{+I} = \int_{C^I_2} \mathcal{J}_+, \quad I = 1, \ldots, 20,
\]

\[
w^+_I = \int_{C^I_2} \mathcal{J}_+ \wedge \mathcal{J}_I,
\]

(5.38)

with \( \mathcal{J}_+ = (B^{NS} + i\sigma.J) \) which can be expanded as

\[
\mathcal{J}_+ = \sum_{I=1}^{20} w^{+I} \mathcal{J}_I,
\]

(5.39)

and

\[
\int_{C^I_2} \mathcal{J}_K = \delta^I_K,
\]

(5.40)
as well as

\[ w_I^+ = d_{IJ}w^{+J}, \quad (5.41) \]

the matrix prepotential \( \mathcal{G}(w^+) \) is given by

\[ G^{++} = \frac{1}{2} \sum_{I,J} w^{+I} d_{IJ} w^{+J}. \quad (5.42) \]

Notice that \( G^{++} \), which is invariant under \( SO(20) \) transformations, is holomorphic in \( w^+ \) in the sense that it does not depend on \( w^- \). We also have

\[ G^{--} = \frac{1}{2} w^{-I} d_{IJ} w^{-J}, \quad G^{++} = \frac{1}{2} w^{+I} d_{IJ} w^{+J}, \quad (5.43) \]

and

\[ G_I^+ = \frac{\partial G^{++}}{\partial w^{+I}} = d_{IJ} w^{+J}, \]
\[ G_I^- = \frac{\partial G^{--}}{\partial w^{-I}} = d_{IJ} w^{-J}. \quad (5.44) \]

Thinking about \( d_{IJ} \) as a tensor metric, we can also set \( G_I^+ = w_I^+ \) and \( G_I^- = w_I^- \). Using these relations, we can rewrite eq\((5.14)\) as follows:

\[ V = \frac{1}{2} \sum_I \left( w^{+I} - w^{-I} \right) \left( w_I^+ - w_I^- \right), \quad (5.45) \]

which should be compared with eq\((5.35)\).

6 Black objects in 6D/7D

We first study black objects in 6D space time in the context of 10D type IIA superstring compactified on the K3 surface. Then we consider the uplifting of these black objects to 7D.

6.1 Black objects in 6D

We start by noting that in six dimensional space-time, the electric/magnetic duality requires that if we have an electrically charged \( D_1 \)-brane, the magnetically charged dual object is a \( D_2 \)-brane such that

\[ D_1 + D_2 = 2. \quad (6.1) \]

It follows that there are essentially three kinds of black objects in 6D with the following horizon geometries:

1. \( D_1 = 2 \) and \( D_2 = 0 \) with an \( AdS_2 \times S^4 \) horizon geometry describing a magnetically charged 6D black hole. It is associated with the 6D \( \mathcal{N} = 2 \) supergravity limit of 10D
type IIA superstring on the K3 surface or equivalently to 10D hererotic superstring on 4-torus $T^4$. Notice also that such extremal 6D black holes can be also recovered from the flux compactification of 11D M-theory on $K3 \times S^1$. The explicit D-brane configurations of these black holes will be described later.

(2) $AdS_4 \times S^2$ horizon geometry describing an electrically charged 6D black 2-brane. This black object, which solves the above eq as $D_1 = 0$ and $D_2 = 2$, is dual to the previous 6D $\mathcal{N} = 2$ black hole with $D_1 = 2$ and $D_2 = 0$.

(3) $AdS_3 \times S^3$ horizon geometry describing a dyonic black F-string. Black D-string should, a priori, be described in the context of 10D type IIB superstring moving on the K3 surface.

Below, we consider the first two configurations; their attractor mechanism will be considered in section 7. Dyonic black F-string ant its attractor mechanism will be studied in section 7 and black D-string will be considered elsewhere \cite{58}.

### 6.1.1 GVW potential for the 6D black hole

By 6D black hole, we mean the background of the 6 $\mathcal{D} N = 2$ supergravity describing the large distance limit of type IIA superstring on the K3 surface with the near horizon geometry $AdS_2 \times S^4$. This black hole can produced by a system of D0-D2-D4-branes wrapping the appropriate cycles of the K3 surface with a Gukov-Vafa-Witten (GVW) type prepotential $G_{BH}$ induced by RR fluxes on it. Below, we determine the prepotential $G_{BH}$ which is given by flux contributions in the K3 compactification. To that purpose, we shall first consider the non zero flux compactification of type IIA superstring on the K3 surface. Then, we consider switching on the fluxes.

**Compactification with non zero fluxes**

To start recall that to fully specify the vacuum background of 10D type IIA superstring on complex $n$-dimensional Calabi-Yau manifold $X_n$ ($n = 2, 4$; but in present discussion $n = 2$), one must specify not just the geometry of $X_n$, but also the topological classes $\zeta$ and $\xi$ of the 1- form and 3- form gauge fields $A_1$ and $A_3$, i.e

$$\zeta = \left[ \frac{\mathcal{F}_2}{2\pi} \right], \quad \xi = \left[ \frac{\mathcal{F}_4}{2\pi} \right].$$

(6.2)

Here $\mathcal{F}_2 = dA_1$ and $\mathcal{F}_4 = dA_3$ are the field strengths of the gauge fields $A_1$ and $A_3$ \cite{28}. Topologically, these $A$- fields are classified by the characteristic classes $\zeta \in H^2(X_n; \mathbb{Z})$

---

\footnote{The Gukov-Vafa-Witten (GVW) type superpotential $G_{BH}$ should not be confused with the Weinhold scalar potential $V_{eff}$ that describe the attractor mechanism of the 6D $\mathcal{N} = 2$ black objects; see section 7 for details. The GVW superpotential deals with type II A compactification on K3 in presence of fluxes.}
and \( \xi \in H^4(X, \mathbb{Z}); \). In the case of type IIA superstring on the K3 surface, the characteristic classes \( \zeta \) and \( \xi \) are non trivial. They are respectively given by the twenty two 2- cocycles \( \{ \zeta^a, a = 1, ..., 22 \} \) and the volume form \( \Omega^{(2,2)} \) of K3. According to topology, one should then distinguishes the two following situations:

1. Compactification with \( \mathcal{F}_2 = \mathcal{F}_4 = 0 \).
2. Compactification with \( \mathcal{F}_2 \neq 0, \mathcal{F}_4 \neq 0 \).

In the first case where the fluxes are turned off, \( \mathcal{F}_2 = \mathcal{F}_4 = 0 \), the variations of the K3 metric is generated by the two following:

(a) **Complex deformations**: the variations of the complex structure of complex dimension Calabi-Yau manifold are described by the complex moduli

\[
z^I = \int_{C^n} \Omega^{(n,0)}, \quad I = 1, ..., h^{(n-1,1)}, \quad n = 2.
\]

(6.3)

From the 4D \( \mathcal{N} = 1 \) superfield theory language, the \( z^I \)'s are just the leading scalar component fields of 4D \( \mathcal{N} = 1 \) chiral superfields \( Z^I(y, \theta) \) with \( \theta \)-expansion is given by

\[
Z^I(y, \theta) = z^I(y) + \theta \psi^I(y) + \theta^2 F^I(y), \quad I = 1, ..., h^{(n-1,1)}.
\]

(6.4)

Here \( y = x + i \theta \sigma \theta \) with \( x \) being the usual 4D space time coordinates. We also have the antichiral superfields

\[
\overline{Z}^I(\overline{y}, \overline{\theta}) = \overline{\psi}^I(\overline{y}) + \overline{\theta} \overline{\psi}^I(\overline{y}) + \overline{\theta}^2 \overline{F}^I(\overline{y}), \quad I = 1, ..., h^{(n-1,1)}.
\]

(6.5)

Notice that in the particular case of K3 surface \( (n = 2) \), these superfields which for later use we rewrite as a \( 2 \times 2 \) matrix as follows,

\[
\begin{pmatrix}
0 & Z^I(y, \theta) \\
\overline{Z}^I(\overline{y}, \overline{\theta}) & 0
\end{pmatrix},
\]

(6.6)

are not yet all what we need. These superfields \( Z^I \) capture in fact just half of the degrees of freedom of the 6D \( \mathcal{N} = 1 \) hypermultiplet \( H^I \) which is same as 4D \( \mathcal{N} = 2 \) hypermultiplet; that is two different types of 4D \( \mathcal{N} = 1 \) chiral superfields together with their complex conjugates. Notice also that in eq(6.3), we have added the complex dimension \( n \) just to make contact with higher dimensional Calabi-Yau manifolds and also to exhibit the specific feature for \( n = 2 \). In this particular case, complex \( \Omega^{(2,0)} \), its complex conjugate \( \Omega^{(0,2)} \) and Kahler \( \Omega^{(1,1)} \) structures combine to give a hyperKahler structure on the moduli space of 10D type IIA superstring on the K3 surface. This is exactly what we need to reproduce the hypermultiplets as described below.

(b) **Kahler deformations**: the variation of the (complexified) Kahler structure is described by the complex moduli,

\[
t^I = \int_{C^2} \left( B^{NS} + i \Omega^{(1,1)} \right), \quad I = 1, ..., h^{(1,1)}.
\]

(6.7)
From the language of $4D \mathcal{N} = 1$ superfield theory, the parameters $t^I$ get promoted to $4D \mathcal{N} = 1$ chiral superfields as shown below

$$T^I \left( \bar{y}, \bar{\theta} \right) = t^I + \bar{\theta} \psi^I + \bar{\theta}^2 \tilde{F}^I. \quad (6.8)$$

A similar expression is valid for the antichiral superfield $\bar{T}^I = T^I \left( \bar{y}, \bar{\theta} \right)$. The $6D \mathcal{N} = 1$ supersymmetric hypermultiplets, which as noted before is equivalent to two $4D \mathcal{N} = 1$ chiral superfields, have then the superspace structure

$$H^I = H^I \left( y, \theta; \bar{y}, \bar{\theta}; \bar{\bar{y}}, \bar{\bar{\theta}} \right). \quad (6.9)$$

They are made of the superfields $Z^I$ and $T^I$. Using the representation (6.6), the $H^I$ superfields can be represented by the following

$$H^I = \begin{pmatrix} T^I & Z^I \\ Z^I & \bar{T}^I \end{pmatrix}, \quad (6.10)$$

where the lowest component fields are precisely the quaternionic moduli $w^{+I}$.

**c) Deformations in the presence of RR- fluxes:**

First note that in the case where $\mathcal{F}_2 = \mathcal{F}_4 = 0$, the expectation values of the $T^I$ and $Z^I$ are arbitrary in the supergravity approximation to $10D$ type superstring IIA on the K3 surface. However, in the case where the fluxes are turned on, i.e

$$\mathcal{F}_2 \neq 0, \quad \mathcal{F}_4 \neq 0, \quad (6.11)$$

the situation become more subtle since we must adjust the complex and Kahler structures of the K3 surface so that to stabilize the fluxes. This adjustment is achieved by adding a potential $\mathcal{G}_{BH}$ whose variation with respect to complex and Kahler moduli fixes the fluxes to zero. This will be discussed in the next paragraph.

**GVW superpotential $\mathcal{G}_{BH}$**

We first consider the adjustment of the complex structure and next the adjustment of the Kahler one. Obviously in the present case dealing with $10D$ type IIA superstring on the K3 surface, these two adjustments can be done altogether in $SU(2)$ covariant way. We will turn to this property later on.

To adjust the complex structure of $10D$ type IIA superstring on the K3 surface, we should impose the following constraint relations:

$$\frac{\delta \mathcal{G}_{BH}}{\delta \Omega^{(2,0)}} = \mathcal{F}^{(0,2)} = 0, \quad \frac{\delta \mathcal{G}_{BH}}{\delta \Omega^{(0,2)}} = \mathcal{F}^{(2,0)} = 0. \quad (6.12)$$

The field strengths $\mathcal{F}^{(0,2)}$ and $\mathcal{F}^{(2,0)}$ are the holomorphic and antiholomorphic components appearing the decomposition of the 2-form $\mathcal{F}_2$. We recall that the real 2-form $\mathcal{F}_2$
can be usually decomposed as an isotriplet as shown below, see also footnote 1,
\[ F_2 = F^{(1,1)} \oplus F^{(0,2)} \oplus F^{(2,0)}. \quad (6.13) \]

To adjust the Kahler structure of 10D type IIA superstring on the K3 surface, one has to impose the following constraint relation
\[ \frac{\delta G_{BH}}{\delta \Omega^{(1,1)}} = F^{(1,1)} = 0. \quad (6.14) \]

Notice that the 2-form \( F^{(1,1)} \) appearing above comes precisely from the field strength \( F_2 \) decomposition \((6.13)\). From the 6D \( \mathcal{N} = 1 \) supersymmetric field theory, the relations \((6.12-6.14)\) can be viewed as superfield equations of motion following from the variation of the potential,
\[ G_{BH} = \frac{1}{2} \left( \int_{K3} F^{(2,0)} \wedge \Omega^{(0,2)} + F^{(0,2)} \wedge \Omega^{(2,0)} \right) + \int_{K3} F_4, \quad (6.15) \]

where we have added the integral constant \( q = \int_{K3} F_4 \). Notice that the variation of \( G_{BH} \) with respect to \( \Omega^{(0,2)} \), \( \Omega^{(2,0)} \) and \( \Omega^{(1,1)} \), one recovers precisely eqs \((6.12-6.14)\). Notice also that up on rewriting
\[ F^{(0,2)} \wedge \Omega^{(2,0)} = (i) F^{(0,2)} \wedge (-i) \Omega^{(2,0)}, \]
\[ \Omega^{(1,1)} = \frac{1}{2} K_- + \frac{1}{2} K_+, \quad (6.16) \]

where we have used
\[ K_\pm = B^{NS} \pm i\Omega^{(1,1)}, \quad (6.17) \]

we can put \( G_{BH} \) as,
\[ G_{BH} = \frac{i}{2} \left( \int_{K3} F^{(1,1)} \wedge K_- \right) - \frac{i}{2} \left( \int_{K3} F^{(1,1)} \wedge K_+ \right) + \frac{1}{2} \left( \int_{K3} (i) F^{(2,0)} \wedge (-i) \Omega^{(0,2)} \right) + \frac{1}{2} \left( \int_{K3} (i) F^{(0,2)} \wedge (-i) \Omega^{(2,0)} \right) + \int_{K3} F_4. \quad (6.18) \]

Then setting
\[ F_2 = \begin{pmatrix} F^{(1,1)} & F^{(2,0)} \\ F^{(0,2)} & -F^{(1,1)} \end{pmatrix}, \quad (6.19) \]

which reads also in terms of Pauli matrices as
\[ F_2 = \sigma^0 F^{(1,1)} + \sigma^+ F^{(2,0)} + \sigma^- F^{(0,2)}, \quad (6.20) \]
we can bring $G_{BH}$ to the following SU(2) covariant formula,

$$G_{BH} = \int_{K3} F_4 + \frac{i}{2} \text{Tr} \left( \int_{K3} F_2 \wedge J_- \right)$$  \hspace{1cm} (6.21)

where we have used

$$J_- = \begin{pmatrix} K_- & -i\Omega^{(2,0)} \\ -i\Omega^{(0,2)} & K_+ \end{pmatrix} = B - i\sigma \cdot J.$$  \hspace{1cm} (6.22)

Now, substituting $J_-$ by its expansion $\sum w^{-I}J_I$, doing the same for $F_2$

$$F_2 = \sum J \sigma \cdot q^J J,$$  \hspace{1cm} (6.23)

with $q^J = (q^{1J}, q^{2J}, q^{3J})$ and then factorizing the $q^J$ integer 3-vectors as follows

$$q^{1J} = p^1 p^J, \quad q^{2J} = p^2 p^J, \quad q^{3J} = p^3 p^J$$  \hspace{1cm} (6.24)

or equivalently in a condensed form as

$$q^J = pp^J,$$  \hspace{1cm} (6.25)

we get

$$G_{BH} = q + \text{Tr} \left[ \sigma \cdot p \left( i \sum_{I,J=1}^{20} w^{-I} d_{IJ} p^J \right) \right].$$  \hspace{1cm} (6.26)

This relation reads also as

$$G_{BH} = q + \text{Tr} \left[ \sigma \cdot p \left( i \sum_{I,J=1}^{20} w^{-I} p_I \right) \right],$$  \hspace{1cm} (6.27)

where we have set

$$p_I = \sum d_{IJ} p^J.$$  \hspace{1cm} (6.28)

Expanding $w^{-I} = y^{0I} - i x^I \sigma$ and computing trace, we obtain the general form of the GVW potential for type IIA compactification on K3,

$$G_{BH} = q + p \cdot \left( \sum x^I p_I \right).$$  \hspace{1cm} (6.29)

Using eq(3.37), one can re-express the above relation, in terms of the isotriplet central charge $Z = \sum x^I p_I$, as follows:

$$G_{BH} = q + p \cdot Z,$$  \hspace{1cm} (6.30)

together with

$$p \cdot Z = \sum_{i=1}^{3} p^m Z^m.$$  \hspace{1cm} (6.31)

Notice finally that in the above relations, $q$ is the number of D4-branes wrapping the K3 surface, and the $p_I$’s give the number of D2-branes wrapping 2-cycles of the K3 surface.
6.1.2 6D Black 2-Brane

Similarly as for 6D \( \mathcal{N} = 2 \) black hole, the 6D \( \mathcal{N} = 2 \) black 2-brane is given by the 6D \( \mathcal{N} = 2 \) supergravity describing the large distance limit of 10D type IIA superstring on the K3 surface with near horizon geometry \( AdS_4 \times S^2 \). We can produce these 6D \( \mathcal{N} = 2 \) black 2-brane by considering a D-brane system consisting of:

1. a D4-brane wrapping the \( C^I_2 \) 2-cycles in K3 and the extra directions filling two dimensions \( (t,x) \) in space time, say the first \( x^0 = t \) and second \( x^1 = x \) dimensions.
2. a D6-brane wrapping K3 and the remaining two others in the first and second of space time.

The potential induced by RR fluxes on K3 for 6D black 2-brane potential has a similar structure as for 6D black hole except that now that the black object is electrically charged. Similar computations lead to

\[
G_{B2-brane} = p + q_i Z_i, \tag{6.32}
\]

with

\[
Z = \sum_i q_i x^i, \tag{6.33}
\]

and

\[
q_i Z = \sum_{i=1}^{3} q_i Z^m. \tag{6.34}
\]

In eq(6.32), the integer \( p \) gives the number of wrapped of D6-branes on K3 and the \( q_i \)'s give the number of D4 wrapping the \( I \)-th 2-cycles of the K3 surface.

6.2 Black Objects in 7D

The simplest way to describe 7D \( \mathcal{N} = 2 \) supergravity is to think about it as the large distance limit of 11D M-theory compactified on the K3 surface. The 7D \( \mathcal{N} = 2 \) component field action \( S_{7D} \) can be obtained by starting from the 11D \( \mathcal{N} = 1 \) supergravity action and performing the compactification on the K3 surface. The bosonic part action of the 11D supergravity is given by

\[
S_{11D} = -\frac{1}{2\kappa_{11}^2} \int \left( \mathcal{R} \ (\ast_{11}) + \frac{1}{2} \mathcal{F}_4 \wedge (\ast \mathcal{F})_7 + \frac{1}{3!} \mathcal{F}_4 \wedge \mathcal{F}_4 \wedge \mathcal{C}_3 \right), \tag{6.35}
\]

where \( \mathcal{R} \) is the usual scalar curvature in 11D, the 4-form field strength is \( \mathcal{F}_4 = d\mathcal{C}_3 \) and the 7-form \( (\ast \mathcal{F})_7 \) is the Hodge dual of \( \mathcal{F}_4 \). The coupling constant \( \kappa_{11}^2 \) is related to Newton’s constant as \( \kappa_{11}^2 = 8\pi G_D \). By compactifying on the K3 surface preserving half of the 32 supersymmetries, we can get the explicit expression of \( S_{7D} \). As there is no 1-cycle nor 3-cycle within the K3 surface, the moduli space of the 7D \( \mathcal{N} = 2 \) theory is
just the geometric one given by,

\[ M_{7D} \times SO(1,1), \quad M_{7D} = \frac{SO(3,19)}{SO(3) \times SO(19)}, \]  

(6.36)

where \( SO(1,1) \) stands for the dilaton. The other 57 moduli parameterizing the homogeneous space \( M_{7D} \) are associated with scalars within the 7D \( \mathcal{N} = 2 \) multiplets. Let us comment below these 7D \( \mathcal{N} = 2 \) multiplets; we have:

(1) One 7D \( \mathcal{N} = 2 \) gravity multiplet \( \mathcal{G}^{N=2}_{7D} \) consisting of 40 + 40 on shell degrees of freedom. The component fields of its bosonic sector are given by

\[ \mathcal{G}^{N=2}_{7D} : (\phi, g_{\mu\nu}, C_{\mu\nu\rho}, A_0^\mu, A_\pm^\mu) \].

(6.37)

They describe, in addition to the dilaton, a spin 2 field, one 3-form gauge field and three 1-form gauge fields. It involves then one scalar only.

(2) Fifty seven (57) 7D \( \mathcal{N} = 2 \) vector multiplet

\[ (\mathcal{V}^{N=2}_{7D})_I, \quad I = 1, ..., 19. \]  

(6.38)

Each vector multiplet \( \mathcal{V}^{N=2}_{7D} \) has 8 + 8 on shell degrees of freedom. The bosonic sector of \( \mathcal{V}^{N=2}_{7D} \) contains a seven dimensional gauge field \( C_\mu \) and three scalars

\[ \mathbf{x} = (x^1, x^2, x^3). \]

(6.39)

The bosonic sector of the 57 supermultiplets of the seven dimensional \( \mathcal{N} = 2 \) vector superfields \( (\mathcal{V}^{N=2}_{7D})_I \) read then as

\[ (\mathcal{V}^{N=2}_{7D})_I : (C^I_\mu, \mathbf{x}^I), \quad I = 1, ..., 19 \]  

(6.40)

where the \( \mathbf{x}^I \)'s \( ((x^{1I}, x^{2I}, x^{3I})) \) refer to the 57 isotriplets of 7D scalars. They describe the metric deformations of the K3 surface. Notice that the gauge fields \( C^I_\mu \) belong to the representation \( (1,19) \) of the group \( SO(3) \times SO(19) \) and the scalars \( \mathbf{x}^I \) are in the \( (3,19) \) one.

\[ C^I_\mu \sim (1,19) \in SO(3) \times SO(19) \]

\[ \mathbf{x}^I \sim (3,19) \in SO(3) \times SO(19) \]  

(6.41)

Now, we move to discuss \( \mathcal{N} = 2 \) black objects in seven dimensions from the view of 10D type IIA superstring on the K3 surface and their uplifting to 7D. This is achieved by starting from the 6D results, in particular the expression of the central charges in 6D, eqs (6.31) and (6.34),

\[ Z^m = \sum_{I=1}^{20} Q_I x^{mI}, \]

(6.42)
and minimizing $Z^m = Z^m(x)$ by taking into account the constraint eqs. (4.43, 4.45) fixing the volume of the K3 surface to a constant (V_{K3} = \text{constant}), i.e.

$$
\sum_{n=1}^{3} \left( \sum_{i,j=1}^{20} x^{nI} d_{IJ} x^{nI} \right) = \text{constant}. \quad (6.43)
$$

The electric/magnetic duality equation requires that the space dimensions $D_1$ and $D_2$ of two dual objects in 7D should be as:

$$
D_1 + D_2 = 3. \quad (6.44)
$$

According to 10D type II superstrings view, there are four kinds of 7D black objects: Two of them can be described in type 10D IIA superstring and the two others in 10D type IIB superstring. The near horizon of these geometries are classified as follows:

(i) $AdS_2 \times S^5$ describing the near horizon geometry of 7D black holes.

(ii) $AdS_4 \times S^3$ describing 7D black 2-branes.

These black objects are quite similar to the 6D case discussed perviously. They are described in the context of uplifting of 10D type IIA superstring on the K3 surface. The two others should be described in the context of uplifting of 10D type IIB superstring on the K3 surface. Their near horizon geometries are as follows:

(iii) $AdS_3 \times S^4$ associated with 7D black strings.

(iv) $AdS_5 \times S^2$ associated with 7D black 3-branes.

The electric/magnetic duality in seven dimensions can be understood as describing the following interchange

$$
AdS_2 \times S^5 \leftrightarrow AdS_5 \times S^2
\quad AdS_3 \times S^4 \leftrightarrow AdS_4 \times S^3. \quad (6.45)
$$

It interchanges the A and B-models and maps 7D black holes to 7D black 3-branes and 7D black strings to 7D black 2-branes. This interchange may have a T-duality interpretation connecting type IIA and type IIB in odd dimensional space-time; in particular in 7D. As previously mentioned $AdS_3 \times S^3$ is a dyonic black string in 6D, and its uplift to 7D describes a 7D black string with the horizon geometry $AdS_3 \times S^4$ and a 7D black 2-brane with the horizon geometry $AdS_4 \times S^3$.

On the other hand, $AdS_2 \times S^4$ and $AdS_4 \times S^2$ is uplifted respectively to $AdS_3 \times S^4$ and $AdS_4 \times S^3$ describing type IIA black objects in 7D. From M-theory compactification view, these 7D black brane configurations can be reproduced by wrapping M2 and M5-branes on appropriate cycles in the K3 surface. Note that in M-theory background there is no RR 2-form flux since we have no 1-form. We have rather a 4-form flux where quite similar computations can be done.
7 Effective potential and Attractor mechanism in 6D and 7D

In this section, we study the effective potential and attractor mechanism of the 6D (7D) supersymmetric black objects. This study requires considering all the scalar field moduli of the 6D (7D) non chiral supergravity theory including the dilaton field that we have freezed before. In 6D (7D) space time, there are 81 (58) scalars distributed as follows:

(i) the dilaton $\sigma$ belonging to the 6D (7D) $\mathcal{N} = 2$ gravity multiplet. It has been neglected before; but below it will be taken as a dynamical variable. For 7D, the field $\sigma$ has a geometric interpretation in term of the volume of K3.

(ii) the eighty (fifty seven) other moduli $\omega_{aI}$ ($\rho_{aI}$) belonging to the 6D (7D) $\mathcal{N} = 2$ Maxwell multiplets.

To fix the ideas, we consider in the next two subsections 7.1 and 7.2 the 6D $\mathcal{N} = 2$ supergravity and study the effective scalar potential $V_{\text{eff}} = V_{\text{eff}} (\sigma, \omega_{aI})$ of the 6D space time black objects. In subsection 7.3, we give the results for 7D.

The effective scalar potential $V_{\text{eff}}$ of the 6D black objects is given by the Weinhold potential expressed in terms of the dressed charges,

$$ (Z_+, Z_-, Z_a, Z_I), \quad a = 1, \ldots, 4, \quad I = 1, \ldots, 20. \quad (7.1) $$

These central charges appear in the supersymmetric transformations of the fields of the 6D supergravity theory; in particular in [34]:

(i) the supersymmetric transformations of the two 6D gravitinos and the four graviphotinos/dilatinos of the supergravity multiplet.

(ii) the supersymmetric transformations of the twenty photinos of the $U^{20} (1)$ gauge supermultilet that follow from the compactification of 10D type IIA superstring on K3.

At the event horizon of the 6D black objects, the potential $(V_{\text{eff}})_{\text{black}}$ attains the minimum. The real $(\sigma, \omega_{aI})$ moduli parameterizing $\tilde{G} = SO(1, 1) \times G$ with $G = SO(4, 20) / SO(4) \times SO(20)$ are generally fixed by the charges

$$ g^+, \ g^-, \ g^a, \ h^I, \ q_a, \ p_I, \quad (7.2) $$

of the $\mathcal{N} = 2$ 6D supergravity gauge field strengths

$$ H^+_3, \ H^-_3, \ F^a_2, \ F^I_2, \ F^a_4, \ F^I_4. \quad (7.3) $$

The attractor equations of the 6D $\mathcal{N} = 2$ black objects will be obtained from the minimization of the $(V_{\text{eff}})_{\text{black}}$. Once we write down these attractor eqs, we pass to examine
their solutions at the horizon of the black attractors. These solutions fix the moduli \((\sigma, \omega_a I)\) in terms of the charges \((7.2)\),

\[
(\sigma)_{\text{horizon}} = \sigma (g, e), \quad (\omega)_{\text{horizon}} = \omega (g_a ; p_I), \tag{7.4}
\]

and give the expression of the black objects entropies in terms of these charges.

Notice by the way that there are different ways to derive the gauge invariant effective scalar potential \(V_{\text{eff}}\) of the 6D \(N = 2\) supersymmetric BPS black objects \([34, 48, 49, 50]\). Here, we shall follow the approach used in the works \([51, 52, 53, 54]\). The effective scalar potential \(V_{\text{eff}}\) of the 6D black object is expressed as a quadratic form of the central charges \((7.1)\).

Notice moreover that from the field spectrum of the 6D \(N = 2\) non chiral supergravity, one learns that two basic situations should be distinguished:

(1) 6D black F-string (BFS) with near horizon geometry \(AdS_3 \times S^3\). This is a 6D dyonic black F-string solution. The electric/magnetic charges involved here are those of the gauge invariant 3-form field strengths

\[
H_3^+ = \frac{1}{2} (H_3 + \star H_3), \quad H_3^- = \frac{1}{2} (H_3 - \star H_3), \tag{7.5}
\]

associated with the usual 6D 2-form antisymmetric \(B^{\pm}_{\mu \nu}\) fields. The \(\star\) conjugation stands for the usual Poincaré duality interchanging \(n\)-forms with \((6 - n)\) ones.

(2) 6D black hole (BH) and its black 2-brane (B2B) dual. Their near horizon geometries were discussed in previous sections. The field strengths involved in these objects are related by the Poincaré duality in 6D space time which interchanges the 2- and 4-form field strengths.

Below, we study separately these two configurations.

### 7.1 Black F-string in 6D

The black BPS object of the 6D \(N = 2\) non chiral theory is a dyonic string charged under both the self dual \(H_3^+\) and antiself dual \(H_3^-\) field strengths of the NS-NS \(B^{\pm}\)-fields. Using the following bare magnetic/electric charges,

\[
g^\pm = \int_{S^3} H_3^\pm, \quad g = \frac{1}{2} (g \pm e), \tag{7.6}
\]

where \(g = \int_{S^3} H_3\) and \(e = \int_{S^3} \star H_3\), one can write down the physical charges in terms of the dressed charges.

#### 7.1.1 Dressed charges

The dressed charges play an important role in the study of supergravity theories. They appear in the supersymmetric transformations of the Fermi fields (here gravitinos), and
generally read like

\[ Z^+ = X^+_+g^+ + X^+_g^- \]
\[ Z^- = X^-_+g^+ + X^-_g^- , \]  

(7.7)

where the real \( 2 \times 2 \) matrix

\[ X = \begin{pmatrix} X^+_+ & X^+_g^- \\ X^-_+ & X^-_g^- \end{pmatrix}, \]  

(7.8)

parameterizes the \( SO(1, 1) \) factor of the moduli space \( \hat{G} \).

Taking the \( \eta_{ns} \) flat metric as \( \eta = diag (1, -1) \), we can express all the four real parameters \( X^\pm \) and \( X^\mp \) in terms of the dilaton \( \sigma = \sigma (x) \) by solving the constraint eqs \( X^t \eta X = \eta \) which split into four constraint relations like

\[ X^+_+X^+_+ - X^-_+X^-_+ = 1 \]
\[ X^+_+X^+_+ - X^-_+X^-_+ = 0 \]
\[ X^+_+X^+_+ - X^-_+X^-_+ = 0 \]
\[ X^-_+X^-_+ - X^+_+X^+_+ = 1 . \]

These eqs can be solved by,

\[ X^+_+ = X^-_+ = \cosh (2\sigma), \quad X^-_+ = X^+_+ = \sinh (2\sigma) . \]  

(7.10)

Putting these solutions back into the expressions of the central charges \( Z^+ \) and \( Z^- \), we get the following dilaton dependent quantities

\[ Z^+ = \frac{1}{2} [g \exp (-2\sigma) + e \exp (+2\sigma)] \]
\[ Z^- = \frac{1}{2} [g \exp (-2\sigma) - e \exp (2\sigma)] . \]  

(7.11)

Notice that these dressed charges have no dependence on the \( \omega_{ai} \) field moduli of the coset \( SO(4, 20) / SO(4) \times SO(20) \). This is because the NS-NS B- fields is not charged under the isotropy group of the above coset manifold.

### 7.1.2 BFS potential

With the dressed charges \( Z^+ \) and \( Z^- \), we can write down the gauge invariant effective scalar potential \( \mathcal{V}_{BFS} \). It is given by the so called Weinhold potential,

\[ \mathcal{V}_{BFS} = (Z^+)^2 + (Z^-)^2 . \]  

(7.12)

Notice that, as far symmetries are concerned, one also have the other "orthogonal" combination namely \((Z^+)^2 - (Z^-)^2\). This quantity cannot, however, be interpreted as
a supersymmetric BPS potential. First it is not positive definite and second it has
an interpretation in terms of eqs[7.9]. We will show later that this gauge invariant
combination corresponds just to the electric/magnetic charge quantization condition.
By substituting eq(7.7) into the relation (7.12), we get the following form of the potential,

\[ V_{BFS} = (g^+, g^-) \mathcal{M} \left( \begin{array}{l} g^+ \\ g^- \end{array} \right), \]  

(7.13)

with

\[ \mathcal{M} = \left( \begin{array}{cc} \left( X_+^2 \right)^2 + \left( X_-^2 \right)^2 & 2X_+X_-^+ \\ 2X_+X_-^- & \left( X_-^2 \right)^2 + \left( X_+^2 \right)^2 \end{array} \right). \]  

(7.14)

From this matrix and using the transformations given in [52], we can read the gauge field
coupling metric \( N^+ \) and \( N^- \) that appear in the 6D \( \mathcal{N} = 2 \) supergravity component field
Lagrangian density

\[ \mathcal{L}_{6D}^{N=2 \text{ sugra}} = \mathcal{R}_6 + \left( \frac{1}{2} N_+ H^+ \wedge H^- + \frac{1}{2} N_- H^- \wedge H^+ \right) + \cdots \]  

(7.15)

In this eq, \( \mathcal{R}_6 \) is the usual 6D scalar curvature and \( g = \det (g_{\mu\nu}) \). By further using (7.11),
we can put the potential \( V_{BFS} \) into the following form

\[ V_{BFS} (\sigma) = \frac{g^2}{2} \exp (-4\sigma) + \frac{e^2}{2} \exp (4\sigma). \]  

(7.16)

Notice that the self and anti- self duality properties of the field strengths \( H_3^+ \) and \( H_3^- \)
imply that the corresponding magnetic/electric charges are related as follows

\[ g^+ = e^+, \quad g^- = -e^-. \]  

(7.17)

Using the quantization condition for the dyonic 6D black F- string namely,

\[ (e^+ g^+ + g^- e^-) = 2\pi k, \quad k \text{ integer}, \]  

(7.18)

one gets,

\[ (g^+ g^+ - g^- g^-) = eg = 2\pi k. \]  

(7.19)

Then the the quantity \( (Z^+)^2 - (Z^-)^2 \) becomes

\[ (Z^+)^2 - (Z^-)^2 = 2eg, \]  

(7.20)

being just the quantization condition of the electric/magnetic charges of the F-string in
6D space time.
7.1.3 BFS attractor equation

The attractor condition on the 6D field $\sigma$ modulus at the horizon geometry of the 6D black F-string is obtained by minimizing the potential $V_{\text{BFS}}$. The corresponding attractor eq reads then as follows:

$$
\left( \frac{dV_{\text{BFS}}}{d\sigma} \right) = 0,
\left( \frac{d^2V_{\text{BFS}}}{d\sigma^2} \right) > 0.
$$

(7.21)

By help of eq(7.16), we then have the following condition on the field modulus $\sigma$ at the horizon geometry of the BFS:

$$
\frac{g^2}{2} \left[ \exp (-4\sigma) \right]_{\text{horizon}} = \frac{e^2}{2} \left[ \exp (4\sigma) \right]_{\text{horizon}}.
$$

(7.22)

Its solution is given by

$$
\left[ \exp (4\sigma) \right]_{\text{horizon}} = \frac{g}{e} \geq 0.
$$

(7.23)

Notice that in 6D non chiral $\mathcal{N} = 2$ supergravity the value of the dilaton is no longer infinite as it is the case in $\mathcal{N} = (1,0)$ and $\mathcal{N} = (2,0)$ chiral supergravities with dilaton field belonging to the tensor multiplet. Indeed switching off the magnetic charge $g$ (or equivalently the electric charge $e$); i.e $g = 0$ ($e = 0$), we recover the standard picture

$$
\left[ e^{4\sigma} \right]_{\text{horizon}} = 0, \quad g = 0,
$$

(7.24)

which leads to the quite well known result

$$
\left[ \sigma \right]_{\text{horizon}} = -\infty.
$$

(7.25)

Now putting the solution eq(7.23) back into the central charge relations (7.11), we obtain:

$$
(Z^+)_\text{horizon} = \frac{1}{2} \left[ g \sqrt{\frac{e}{g}} + e \sqrt{\frac{g}{e}} \right] = \sqrt{eg},
$$

$$
(Z^-)_\text{horizon} = \frac{1}{2} \left[ g \sqrt{\frac{e}{g}} - e \sqrt{\frac{g}{e}} \right] = 0.
$$

(7.26)

Therefore the value of the potential at the horizon is,

$$
(V_{\text{BFS}})_{\text{horizon}} = \frac{g^2 e}{2 g} + \frac{e^2 g}{2 e} = eg
$$

(7.27)

which, up on using the quantization condition, can be expressed in terms of the positive definite integer $k$ of eq(7.19). The analogous of the Bekenstein-Hawking entropy $S_{\text{entropy}}$ of the 6D BFS is then proportional to $eg$,

$$
S_{\text{entropy}} \sim G_N^{-\frac{3}{4}} \times \mathcal{A}_{\text{area}} \sim eg,
$$

(7.28)

where $\mathcal{A}_{\text{area}}$ is the 3d- horizon area and $G_N$ is the Newton constant in 6D. $S_{\text{entropy}}$ vanishes for $e = 0$ or $g = 0$ as predicted by chiral supergravity theories in 6D.

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*6We thank S. Ferrara and A. Marrani for drawing our attention to this point.*
7.2 6D Black Hole

Contrary to the dyonic BFS, the 6D black hole is magnetically charged under the $U^4(1) \times U^{20}(1)$ gauge group symmetry generated by the gauge transformations of the $(4 + 20)$ gauge fields of the 6D $N = 2$ gravity fields spectrum.

Recall that in 6D, the electric charges are given, in terms of the field strength $F_{4a}$ and $F_{4I}$, by,

$$q_a = \int_{S^4} F_{4a}, \quad a = 1, ..., 4,$$
$$p_I = \int_{S^4} F_{4I}, \quad a = 1, ..., 20.$$  \hspace{1cm} (7.29)

The corresponding magnetic duals, which concern the black 2-brane, involve the 2-form field strengths $F_{2a}$ integrated over 2-sphere,

$$g^a = \int_{S^2} F_{2a}, \quad a = 1, ..., 4,$$
$$h^I = \int_{S^2} F_{2I}, \quad a = 1, ..., 20.$$  \hspace{1cm} (7.30)

Like for BFS, the charges $Q_{\Lambda} = (q_a, p_I)$ are not the physical ones. The physical charges; to be denoted like

$$Z_a, \quad Z_I,$$  \hspace{1cm} (7.31)

appear dressed by the 6D scalar fields $\omega^{aI}$ parameterizing the moduli space of the 10D type IIA superstring on K3. Recall that the charges $Z_a$ and $Z_I$ appear respectively in the supersymmetric transformations of the four gravi-photinos/dilatinos and the twenty photinos of the $U^{20}(1)$ Maxwell multiplet of the gauge-matter sector.

7.2.1 Dressed charges

The dressing of the twenty four electric charges $(q^a, p^I)$ of the gauge fields $(A^a_{\mu}, A^I_{\mu})$ read as follows:

$$Z_a = e^{-\sigma} \left( Y_{ab} q^b + \omega_{aJ} p^J \right),$$
$$Z_I = e^{-\sigma} \left( V_{ib} q^b + Y_{IJ} p^J \right).$$  \hspace{1cm} (7.32)

Using the real $24 \times 24$ matrix $M_{\Lambda\Sigma}$,

$$M_{\Lambda\Sigma} = e^{-\sigma} \times L_{\Lambda\Sigma}, \quad L_{\Lambda\Sigma} = \begin{pmatrix} Y_{ab} & \omega_{aJ} \\ V_{ia} & Y_{IJ} \end{pmatrix},$$  \hspace{1cm} (7.33)

that defines the moduli space $\hat{G}$, the dressed charges $Z_{\Lambda} = (Z_a, Z_I)$ can be put in the condensed form

$$Z_a = M_{a\Sigma} Q^\Sigma = e^{-\sigma} L_{a\Sigma} Q^\Sigma,$$
$$Z_I = M_{I\Sigma} Q^\Sigma = e^{-\sigma} L_{I\Sigma} Q^\Sigma.$$  \hspace{1cm} (7.34)
Obviously not all the parameters carried by $L_{\Lambda \Sigma}$ are independent. The extra dependent degrees of freedom will be fixed by imposing the $SO(4, 20)$ orthogonality constraint eqs and requiring gauge invariance under $SO(4) \times SO(20)$. The factor $e^{-\sigma}$ of eq(7.32) is then associated with the non compact abelian factor $SO(1, 1)$ considered previously. Taking the $\eta_{\Lambda \Sigma}$ flat metric of the non compact group $SO(4, 20)$ as $\eta_{\Lambda \Sigma} = diag (4 (+), 20 (-))$,

$$\eta_{\Lambda \Sigma} = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & -\delta_{IJ} \end{pmatrix},$$

(7.35)

we can express all the $24 \times 24 = 576$ real parameters $L_{\Lambda \Sigma}$ in terms of eighty of them only; say $\omega_{aI}$; i.e

$$Y_{cd} = f(\omega_{aI}), \quad a, b = 1, ..., 4,$$

$$Y_{JK} = g(\omega_{aI}), \quad I, J = 1, ..., 20,$$

$$V_{Jb} = h(\omega_{aI}),$$

(7.36)

where $f(\omega_{aI})$, $g(\omega_{aI})$ and $h(\omega_{aI})$ are some (non linear) functions that can be worked out explicitly by solving the constraint eqs on $L_{\Lambda \Sigma}$ orthogonal matrix. Indeed by solving the constraint eqs

$$L^t \eta L = \eta,$$

(7.37)

for the $SO(4, 20)$ group elements, we obtain the following identities,

$$Y_{ca} Y^{cb} - V_{Ka} V^{Kb} = \delta_{ab}^c,$$

$$Y_{KI} Y^{KJ} - \omega_{cI} \omega^{cJ} = \delta_{IJ}^I,$$

(7.38)

and

$$Y_{ca} \omega^{cI} = V_{Ja} Y^{JI}.$$  

(7.39)

Notice that the last eq gives the relation between $\omega_{cI}$ and $Y_{Ja}$. By introducing the inverse matrices $E^{ab}$ and $E_{IK}$

$$Y_{ca} E^{ab} = \delta_{c}^b,$$

$$Y^{JI} E_{IK} = \delta_{JK}^I,$$

(7.40)

we have either

$$V_{Ja} = \omega_{cI} (Y_{ca} E^{JI}),$$

(7.41)

or

$$\omega_{cI} = V_{Ja} (Y^{JI} E_{ca}).$$

(7.42)

The other constraint relations (7.38) can be used to fix 300 parameters of the matrix $L_{\Lambda \Sigma}$ leaving then 276 parameters. Moreover using the isotropy symmetry $SO(4) \times SO(20)$ of the moduli space one can reduce further this number to

$$276 - 6 - 190 = 80$$

(7.43)
This gauge fixing is done by taking $Y_{ab}$ and $Y_{IJ}$ as given by $4 \times 4$ and $20 \times 20$ symmetric matrices respectively:

$$Y_{ab} = Y_{ba}, \quad Y_{IJ} = Y_{JI}. \quad (7.44)$$

In this gauge, one can solve eqs (7.38) as follows,

$$Y_{a}^{b} = \sqrt{\delta_{a}^{b} + \sum_{K=1}^{20} V_{K a} V^{K b}}, \quad Y_{I}^{J} = \sqrt{\delta_{I}^{J} + \sum_{c=1}^{4} \omega_{c I} \omega^{c J}}. \quad (7.45)$$

Notice moreover that setting,

$$Z_{a} = e^{-\sigma} R_{a}, \quad R_{a} = (L_{a \Sigma} Q^{\Sigma})$$
$$Z_{I} = e^{-\sigma} R_{I}, \quad R_{I} = (L_{I \Sigma} Q^{\Sigma}) \quad (7.46)$$

as well as

$$L_{\Sigma}^{T} \cdot E_{F}^{\Sigma} = L_{a}^{T} \cdot E_{a}^{a} - L_{I}^{T} \cdot E_{I}^{I} = \delta_{I}^{T}, \quad (7.47)$$

one can compute a set of useful relations. In particular we have

$$dL_{I} = L_{I \Lambda} \cdot (dL_{\Sigma}^{T}) \cdot P_{I}^{\Sigma},$$
$$\nabla Z_{a} = (D_{H_{1}} Z_{a} + Z_{a} d\sigma),$$
$$\nabla Z_{I} = (D_{H_{2}} Z_{I} + Z_{I} d\sigma), \quad (7.48)$$

where

$$D_{H_{1}} Z_{a} = (dZ_{a} - \Omega_{a}^{b} Z_{b}), \quad H_{1} = O(4),$$
$$D_{H_{2}} Z_{I} = (dZ_{I} - \Omega_{I}^{J} Z_{J}), \quad H_{2} = O(4) \quad (7.49)$$

and where $\Omega_{a}^{b}$ and $P_{a}^{I}$ are given by

$$\Omega_{a}^{b} = E_{a}^{\Sigma} \cdot (dL_{\Sigma}^{b}), \quad P_{a}^{I} = E_{a}^{\Sigma} \cdot (dL_{\Sigma}^{I}), \quad (7.50)$$

together with similar relation for $\Omega_{I}^{J}$ and $P_{I}^{a}$.

Using (7.48), we can write down the Maurer-Cartan eqs for the dressed charge. They read as follows,

$$\nabla Z_{a} = P_{a}^{I} Z_{I}, \quad \nabla Z_{I} = P_{I}^{a} Z_{a}. \quad (7.51)$$

Notice in passing that $Z_{I} = 0$ is a solution of $\nabla Z_{a} = 0$. The same property is valid for $Z_{a} = 0$ which solves $\nabla Z_{I} = 0$. These properties will be used when we study the minimization of the black hole potential.
7.2.2 Effective black hole potential

Using the dressed charges (7.32-7.34), we can write down the gauge invariant effective scalar potential $V_{BH}$. Following [53], this effective potential is given by the Weinhold potential,

$$V_{BH} (\sigma, L) = (Z_a Z^a) + (Z_I Z^I),$$

which can be also put in the form

$$V_{BH} (\sigma, L) = e^{-2\sigma} [(R_a R^a) + (R_I R^I)].$$

Clearly $V_{BH}$, which is positive, is manifestly gauge invariant under both:

(a) the $U^4(1) \times U^{20}(1)$ gauge transformations since the vectors $Z_a$ and $Z_I$ depend on the electric charges of the field strengths only which, as we know, are gauge invariant.

(b) the gauge transformations of the $SO(4) \times SO(20)$ isotropy group of the moduli space. $V_{BH}$ is given by scalar products of the vectors $Z_a$ and $Z^a$ (resp $Z_I$ and $Z^I$).

Using eqs (7.32), we can express the black hole potential as follows:

$$V_{BH} = e^{-2\sigma} (q^a N_{ab} q^b + q^a N_{aJ} p^J + p^I N_{ib} q^b + p^I N_{IJ} p^J),$$

or in a condensed manner like,

$$V_{BH} = e^{-2\sigma} Q^A N_{\Lambda \Sigma} Q^\Sigma,$$

with

$$N_{\Lambda \Sigma} = \begin{pmatrix} N_{ab} & N_{aJ} \\ N_{aJ} & N_{IJ} \end{pmatrix}$$

and

$$N_{ab} = Y_{ca} Y_{b}\,V_b^c + V_{Ka} V_b^K = N_{ba}$$

$$N_{aJ} = Y_{ca} \omega^c J + V_{aJ} Y_{Ji} = N_{Ja}$$

$$N_{ib} = \omega^c I Y_{cb} + Y_{iJ} V_b^J = N_{bi}$$

$$N_{IJ} = Y_{Kl} Y^{KJ} + \omega_{cI} \omega^{cJ} = N_{JI},$$

together with the constraint relations (7.38-7.39). Notice that, like for BFS, $N_{\Lambda \Sigma}$ has a 6D field theoretical interpretation in terms of the gauge coupling of the gauge field strengths $F^A_{\mu \nu}$; i.e a term like $\frac{1}{4} \sqrt{-g} N_{\Lambda \Sigma} F^A_{\mu \nu} F^{\mu \nu \Sigma}$ appears in the component fields of the 6D $\mathcal{N} = 2$ supergravity Lagrangian density.

7.2.3 6D black hole attractors

The attractor condition on the 6D field moduli at the horizon geometry of the 6D black hole is obtained by minimizing the potential $V_{BH} (\sigma, L)$ with respect to the field
moduli \((\sigma, L)\). The variation of the potential \(V_{BH}\) can be put in the nice form,

\[
\delta V_{BH} = \left( \frac{\partial V_{BH}}{\partial Z_a} \right) \nabla Z_a + \left( \frac{\partial V_{BH}}{\partial Z_I} \right) \nabla Z_I,
\]

(7.58)

where we have used eqs\([7.48,7.49]\). Moreover, using eq(7.52), we also have

\[
\delta V_{BH} = 2Z^a \nabla Z_a + 2Z^I \nabla Z_I,
\]

(7.59)

together with the constraint relations

\[
\nabla Z_a = P_I^a Z_I, \quad \nabla Z_I = P_I^a Z_a,
\]

(7.60)

following from Maurer-Cartan relations \([7.51]\), and

\[
Z_a Z^a - Z_I Z^I = e^{-2\sigma} Q^2.
\]

(7.61)

In the above relation, we have set

\[
Q^2 = Q^\Lambda \eta_{\Lambda \Sigma} Q^\Sigma = q^2 - p^2,
\]

\[
q^2 = \sum_{i=1}^{4} q_a^2, \quad p^2 = \sum_{I=1}^{20} p_I p^I.
\]

(7.62)

Eq(7.61) follows from the orthogonality condition \((L^i \eta L)_{\Lambda \Sigma} = \eta_{\Lambda \Sigma}\). Up on multiplying both sides of this condition by \(Q^\Lambda Q^\Sigma\), that is,

\[
Q^\Lambda (L^i \eta L)_{\Lambda \Sigma} Q^\Sigma = Q^\Lambda \eta_{\Lambda \Sigma} Q^\Sigma,
\]

(7.63)

we end with eq(7.61). Notice that eq(7.61) has an indefinite sign since it can be either positive, null or negative in agreement with the sign of the number \((q^2 - p^2)\).

The attractor condition for black hole read therefore

\[
[Z^a (\nabla Z_a)]_{\text{horizon}} = 0,
\]

\[
[Z^I (\nabla Z_I)]_{\text{horizon}} = 0,
\]

(7.64)

together with the constraint eqs,

\[
[\nabla Z_a]_{\text{horizon}} = [P^I_a Z_I]_{\text{horizon}},
\]

\[
[\nabla Z_I]_{\text{horizon}} = [P^a_I Z_a]_{\text{horizon}},
\]

(7.65)

as well as

\[
[Z_a Z^a - Z_I Z^I]_{\text{horizon}} = (e^{-2\sigma})_{\text{horizon}} Q^2.
\]

(7.66)

By substituting \(Z_a\) and \(Z_I\) by eqs\([7.46,7.48,7.49]\), we can also express these conditions in terms of the fields of the moduli space.
Solutions

Clearly one distinguishes three main classes of solutions minimizing the black hole potential. These are given by

\begin{align*}
(1) & : \quad Z_a = 0, \quad Z_I = 0, \\
(2) & : \quad Z_a = 0, \quad \nabla Z_I = 0, \\
(3) & : \quad Z_I = 0, \quad \nabla Z_a = 0. 
\end{align*}

The case \( \nabla Z_a = 0 \) and \( \nabla Z_I = 0 \) is the same as the case (1) because of the constraint relations \[ (7.65) \].

(1) **Case** \( Z_a = Z_I = 0 \)

This solution corresponds to the two following possibilities:

(a) the field \( \sigma \to \infty \) whatever the other moduli fields \( L_{\Lambda \Sigma} \) are.

(b) the fields \( L_{\Lambda \Sigma} = Q_{\Lambda} Q_{\Sigma} \) with \( Q^2 = 0 \).

In both cases (a) and (b), there is no attractor solution. The value of the potential at the horizon is

\[ (V_{BH})_{\text{horizon}} = 0, \quad (7.68) \]

and so there is no black hole entropy.

(2) **Case** \( Z_a = 0, \nabla Z_I = 0 \)

In this case, a non zero entropy solution can be given by taking the value dilaton \( \sigma \) arbitrary but finite; and the \( L_{\Lambda \Sigma} \) moduli like,

\[ (L_{\Lambda \Sigma})_{\text{horizon}} Q^2 = 0. \quad (7.69) \]

A remarkable candidate for the value of the dilaton \( (\sigma)_{BH}^{\text{horizon}} \) consists to take it the same as the value of the black hole horizon of the BFS

\[ (\sigma)_{BH}^{\text{horizon}} = [\exp (-2\sigma)]_{\text{horizon}}^{\text{Black-F-string}} = \sqrt{\frac{e}{g}}. \quad (7.70) \]

The remaining \( L_{\Lambda \Sigma} \) moduli, which solve \( L_{\Lambda \Sigma} Q^2 = 0 \), are as follows,

\[ [L_{\Lambda \Sigma}]_{\text{horizon}} = \left( \frac{Q_{\Lambda} Q_{\Sigma}}{Q^2} - \eta_{\Sigma \Lambda} \right), \quad Q^2 \neq 0. \quad (7.71) \]

Moreover, since \( Z_a = 0 \), the constraint relation

\[ [Z_a Z_a - Z_I Z_I]_{\text{horizon}} = Q^2 (e^{-2\sigma})_{\text{horizon}}, \quad (7.72) \]

reduces to,

\[ [Z_I Z_I]_{\text{horizon}} = -Q^2 (e^{-2\sigma})_{\text{horizon}}, \]

\[ (Z_I)_{\text{horizon}} = \frac{p_I (e^{-\sigma})_{\text{horizon}}}{\sqrt{\sum_{J=1}^{20} p_J p_J}} \sqrt{|Q^2|}. \quad (7.73) \]
So the $q_a$ and $p_I$ charges should be constrained like
\[
Q^2 = (q^2 - p^2) < 0. \tag{7.74}
\]
Therefore the value of the potential at the BH horizon is
\[
(V_{BH})_{\text{horizon}} = |Q^2| (e^{-2\sigma})_{\text{horizon}} = (p^2 - q^2) (e^{-2\sigma})_{\text{horizon}} > 0. \tag{7.75}
\]
It is proportional to the norm $|Q^2|$ of the central charge vector $Q = (q, p)$. Using eq(7.70), we get
\[
(V_{BH})_{\text{horizon}} = \left| \sum_{l=1}^{20} p_I^2 - \sum_{a=1}^{4} q_a^2 \right| \sqrt{|e/g|}. \tag{7.76}
\]
(3) Case $Z_I = 0$, $\nabla Z_a = 0$
This situation is quite analogous to the previous one. Non zero entropy solution corresponds to finite values of the dilaton $(\sigma)_{\text{horizon}}$ which can be taken as in the BFS eq(7.70); $(\sigma)_{\text{horizon}} = \sqrt{|e/g|}$. Moreover, because of the constraint relation,
\[
[Z_a Z^a - Z_I Z^I]_{\text{horizon}} = Q^2 (e^{-2\sigma})_{\text{horizon}}, \tag{7.77}
\]
which reduces to
\[
[Z_a Z^a]_{\text{horizon}} = Q^2 (e^{-2\sigma})_{\text{horizon}}, \tag{7.78}
\]
the $q_a$ and $p_I$ charges should be like
\[
Q^2 = |Q^2| = (q^2 - p^2) > 0. \tag{7.79}
\]
The solution for $Z_a$ in terms of the central charges is given by
\[
(Z_a)_{\text{horizon}} = \frac{q_a (e^{-\sigma})_{\text{horizon}}}{\sqrt{\sum_{b=1}^{4} q_a q_b \sqrt{|Q^2|}}} \tag{7.80}
\]
As expected the value of the potential on the black hole horizon is $(V_{BH})_{\text{horizon}} = |Q^2| (e^{-2\sigma})_{\text{horizon}}$. 

Beyond Weinhold potential
We end this study by making two comments:
(1) the Weinhold potential we have been using above has a natural extension given by,
\[
V_{BH} (\sigma, L) = (Z_a Z^a) + (Z^I d_{IJ} Z^J) \tag{7.81}
\]
where the $20 \times 20$ matrix $d_{IJ}$ stand for the intersection matrix of the $h^{1,1}$ cycles of K3.
Since $d_{IJ}$ can be also defined as the scalar of some real 20- dimensional vector basis $\{\alpha_I\}$ like $d_{IJ} = \alpha_I \cdot \alpha_J$, the black hole potential $V_{BH} (\sigma, L)$ is positive.
The constraint relations that go with eq(7.81) can be obtained by using the relations
given for the Weinhold potential and substitute the $\delta_{IJ}$ metric by $d_{IJ}$.

(2) The results for the 6D black hole derived above are valid as well for the potential eq(7.81); all one has to do is to replace the metric $\eta_{\Lambda\Sigma}$ (7.35) by the new one,

$$G_{\Lambda\Sigma} = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & -d_{IJ} \end{pmatrix},$$  \hspace{1cm} (7.82)

and think about $Q^2$ as given by $Q^2 = Q^{\Lambda} G_{\Lambda\Sigma} Q^\Sigma = q^a \delta_{ab} q^b - p^I d_{IJ} p^J$. The generic formula for 6D black hole entropy reads as

$$S_{\text{entropy}}^{\text{BH}} \sim \left| \left( \sum_{I=1}^{20} p^I d_{IJ} p^J - \sum_{a=1}^{4} q_a^2 \right) \right| \sqrt{\frac{e}{g}},$$  \hspace{1cm} (7.83)

where $e$ and $g$ are as before.

### 7.3 7D black attractors

Here we discuss briefly the effective scalar potential and attractor mechanism of the black objects in 7D. This study is quite similar to the previous 6D analysis.

Recall that the moduli space of this theory is given by

$$SO(3,19) \times SO(3) \times SO(19) \times SO(1,1).$$  \hspace{1cm} (7.84)

Here an interesting property emerges, the dilaton $\sigma$ parameterizing $SO(1,1)$ has an interpretation in term of the volume of the K3 surface. Recall also that in 7D space time, the bosonic fields content of the $\mathcal{N} = 2$ supergravity multiplet is given by

$$\left( g_{\mu\nu}, \ B_{[\mu\nu]}, \ A^a_{\mu}, \ \sigma \right), \hspace{0.5cm} a = 1, 2, 3, \hspace{0.5cm} \mu, \nu, \rho = 0, ..., 6,$$  \hspace{1cm} (7.85)

where $B_{[\mu\nu]}$ is dual to a 3- form gauge field $C_{[\mu\nu\rho]}$. There is also nineteen U(1) Maxwell with the following 6D bosons:

$$\left( A^I_{\mu}, \ \rho^a I \right), \hspace{0.5cm} a = 1, 2, 3, \hspace{0.5cm} I = 1, ..., 19,$$  \hspace{1cm} (7.86)

where $\rho^a I$ capture $3 \times 19$ degrees of freedom. The gauge invariant $(p + 2)$- forms of the 7D $\mathcal{N} = 2$ supergravity are given by

$$H_3 \sim dB_2, \hspace{0.5cm} \mathcal{F}_2^a \sim dA^a, \hspace{0.5cm} \mathcal{F}_2^I \sim dA^I.$$  \hspace{1cm} (7.87)

Extending the above 6D study to the 7D case, one distinguishes:

(i) 7D black 2- brane (black membrane BM)

The effective scalar potential of the BM is

$$V_{BM}^{7D} (\sigma) \sim Z^2 = e^{-4 \sigma} g^2,$$  \hspace{1cm} (7.88)
with
\[ g = \int_{S^3} H_3. \quad (7.89) \]
The extremum of this potential is given by \( \sigma = \infty \). The value of the potential at the minimum is
\[ [\mathcal{V}_{BM}^D(\infty)]_{\text{min}} = 0, \quad (7.90) \]
and so the entropy vanishes identically.

(ii) 7D black hole:
The Weinhold potential of this black hole is given by
\[ \mathcal{V}_{BH}^D(\sigma, L) = \sum_{a=1}^{3} Z_a Z^a + \sum_{I=1}^{19} Z_I Z^I, \quad (7.91) \]
where
\[ Z_a = e^{-\sigma} L_{a\Lambda} y^\Lambda, \]
\[ Z_I = e^{-\sigma} L_{a\Lambda} y^\Lambda, \quad (7.92) \]
satisfying the constraint relation,
\[ \sum_{a=1}^{3} Z_a Z^a - \sum_{I=1}^{19} Z_I Z^I = Q^2, \quad \left( \sum_{a=1}^{3} q_a q^a - \sum_{I=1}^{19} p_I p^I \right) = Q^2 \quad (7.93) \]
and \( Q^\Lambda = (q^a, p^I) \) with
\[ q^a = \int_{S^2} f^a_2, \quad a = 1, 2, 3, \]
\[ p^I = \int_{S^2} f^I_2, \quad I = 1, \ldots, 19. \quad (7.94) \]
The real \( 22 \times 22 \) matrix
\[ L_{a\Lambda} = \begin{pmatrix} L_{ab} & \rho_{aI} \\ V_{ta} & L_{IJ} \end{pmatrix}, \quad (7.95) \]
is associated with the group manifold \( SO(3, 19) / SO(3) \times SO(19) \). It is an orthogonal matrix satisfying
\[ L^\prime \eta L = \eta, \quad \eta = \text{diag} [3 (+), 19 (-)]. \quad (7.96) \]
The \( SO(3) \times SO(19) \) symmetry can be used to choose \( L_{ab} \) and \( L_{IJ} \) matrices as follows:
\[ L_{ab} - L_{ba} = 0, \quad L_{IJ} - L_{JI} = 0. \quad (7.97) \]
Putting the relations (7.92) back into (7.91), we get
\[ \mathcal{V}_{BH}^D(\sigma, L) = e^{-2\sigma} Q^\Lambda N_{\Lambda\Sigma} Q^\Sigma, \quad (7.98) \]
The attractor equations following from the extremum of the $V_{BH}^7 (\sigma, L)$,

$$\delta V_{BH}^7 = 2 \sum_{a=1}^{3} Z_a \nabla Z^a + 2 \sum_{I=1}^{19} Z_I \nabla Z^I = 0,$$

have the following solutions:

(i) Case 1:
In this case we have $\sigma = \infty$ whatever $L_{\Sigma \Lambda}$ is or $\sigma = \sigma_0 = \text{finite}$ and $L_{\Sigma \Lambda} = Q_{\Lambda} Q_{\Sigma}$ with $Q^2 = 0$. In both cases, we have

$$(V_{BH}^7 (\sigma = \infty, L))_{\min} = 0,$$

and

$$(V_{BH}^7 (\sigma_0, L_{\Sigma \Lambda} = Q_{\Lambda} Q_{\Sigma}))_{\min} = 0$$

Since $\sigma$ is just the volume of K3, the solution (7.101) corresponds to a large volume limit of K3 and the physically un-interesting.

(ii) Case 2: \(\sigma = \sigma_0 = \text{finite}\) and

$$Z^a = 0, \quad \nabla Z^I = 0, \quad Z^I \neq 0 : \quad L_{\Sigma \Lambda}^a Q^{\Lambda} = 0.$$

(iii) Case 3: \(\sigma = \sigma_0 = \text{finite}\) and

$$Z^a \neq 0, \quad \nabla Z^a = 0, \quad Z^I = 0 : \quad L_{\Sigma \Lambda}^a Q^{\Lambda} = 0$$

Like in the case of 6D, we have

Case 2: $$(V_{BH}^7)_{\min} = -Z_I Z^I = e^{-2\sigma_0} |Q|^2,$$
Case 3: $$(V_{BH}^7)_{\min} = Z_a Z^a = e^{-2\sigma_0} |Q|^2,$$

which depend on $|Q|^2 = |q_a q^a - p_I p^I|$; but also on the inverse of the volume of K3.

## 8 Conclusion and Discussion

In this paper we have studied six dimensional $\mathcal{N} = 2$ supersymmetric black attractors (black hole, black F-string, black 2-brane) and their uplifting to seven dimensions. These backgrounds arise as large distance limits of 10D type IIA superstring (11D M-theory) on K3 and may encode informations on $\mathcal{N} = 4$ black hole attractors in four dimensions considered recently in [59, 60]. After revisiting some general results on 10D type II superstrings compactified on the K3 surface and $\mathcal{N} = 2$ supersymmetry in
six dimensional space-time, we have developed a matrix method to exhibit manifestly the special quaternionic structure of 10D type IIA superstring on the K3 surface. This matrix formulation relies on the homomorphism between the 4-vector representation of $SO(4)$ and the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation of $SU(2) \times SU'(2)$. By identification of the two $SU(2)$ factors, the $SO(4)$ becomes $SU^2 (2)$, the 4-vector representation, which reads now as $2 \otimes 2$, split as $1 \oplus 3$ and the $SO(4)$ invariance of the moduli space of type IIA superstring on the K3 surface is then completely captured by representations of the basic $SU(2)$ symmetry.

Our $2 \times 2$ matrix formulation has been shown to be an adequate method to deal with the underlying special quaternionic geometry of the moduli space of the 10D type IIA superstring on the K3 surface. The non abelian $2 \times 2$ matrix formalism with typical moduli

$$w = y + i \sum_{m=0,\pm 1} \sigma^m x^m,$$  \hspace{1cm} (8.1)

where $\sigma^m$ are the usual Pauli $2 \times 2$ matrices and

$$y = \int_{C_2} B^{NS}, \qquad x^m = \int_{C_2} \Omega^{(1+m,1-m)},$$  \hspace{1cm} (8.2)

have a formal similarity with the usual complex formalism

$$w \leftrightarrow z,$$  \hspace{1cm} (8.3)

with

$$z = y + i x^0, \quad y = \int_{C_2} B^{NS}, \quad x^0 = \int_{C_2} \Omega^{(1,1)},$$  \hspace{1cm} (8.4)

being the usual complex Kahler moduli of the special Kahler geometry of 10D type IIA superstring on Calabi-Yau threefolds. The matrix method developed in present study has allowed us to compute explicitly the hyperKahler potential

$$\mathcal{H} = \text{Tr} [\ln (V_0 - S)]$$  \hspace{1cm} (8.5)

in terms of the "volume" $V_0$ of the K3 surface and an isoquintet described by a $3 \times 3$ traceless symmetric matrix $S$, see also the discussion around eq(5.28). The hermitian hyperKahler potential $\mathcal{H}$ has remarkable properties which deserves more analysis. Let us comment briefly some of these specific features herebelow:

(1) $\mathcal{H}$ is given by the trace of $3 \times 3$ matrix $(V_{03 \times 3} - S)$. The real number $V_0$, which is proportional to the volume of K3 ($V_{K3} = 6V_0$), has the following realization in term of the geometric moduli $x^{mI}$,

$$V_0 = \frac{1}{6} \sum_{m=1}^{3} \left( \sum_{I,J=0}^{n_{K3}^{1,1}} x^{mI} d_{IJ} x^{mJ} \right),$$  \hspace{1cm} (8.6)
where \( d_{IJ} \) is the intersection matrix of 2-cycles in the K3 surface. The \( 3 \times 3 \) matrix \( S \) is an isoquintet captured by the symmetrization of the tensor product \( \sigma^m \sigma^n \) of the Pauli matrices and has the following realization in terms of the geometric moduli:

\[
S = \frac{1}{6} \left( \sum_{I,J=0}^{h^1}_{K3} \sum_{k=1}^{3} x^{kI} d_{IJ} x^{kJ} \right) \delta^{mn} - \frac{1}{2} \sum_{m,n=1}^{3} \sigma^m \sigma^n \left( \sum_{I,J=0}^{h^1}_{K3} x^{mI} d_{IJ} x^{nJ} \right).
\]  

(8.7)

Notice also that the trace in eq(8.5) is required by gauge invariance under the SO(4) = SU(2) isometry of the moduli space \( SO(4,20)/SO(4) \times SO(20) \).

(2) The quantities \( V_0 \) and \( S \) are also invariant under the \( SO(20) \) gauge symmetry which rotates the twenty Kahler moduli of K3.

(3) The form (8.5) of the hyperKahler potential \( \mathcal{H} \) of the special hyperKahler geometry of type IIA superstring on the K3 surface can be expanded into a power series as follows

\[
\mathcal{H} = \text{Tr}[\ln V_0] - \sum_{n \geq 1} \frac{g^n}{n} \text{Tr}(S^n),
\]

(8.8)

where the coupling parameter \( g \) is given by the inverse of the volume of the K3 surface:

\[
g = \frac{1}{V_0},
\]

(8.9)

Notice that the leading term \( \text{Tr}[\ln V_0] \) appears as just the Kahler component and next leading vanishes identically since \( \text{Tr}(S) = 0 \). This property could be interpreted as capturing the Ricci flat condition. Higher terms might be interpreted as the geometric correction one has to have in order to bring a Kahler manifold to a hyperKahler one.

(4) The expansion can be also viewed as a perturbative definition of the potential of HyperKahler geometry in terms the potential Kahler geometry \( \text{Tr}[\ln V_0] \) plus extra contribution captured by the trace of powers of isoquintet representation \( S \).

We have also computed the corresponding "holomorphic" matrix prepotential \( \mathcal{G}(w) \) which is found to be quadratic in the quaternionic matrix moduli

\[
\mathcal{G}(w) = \sum_{I=1}^{20} w^I d_{IJ} w^J,
\]

(8.10)

in agreement with the structure of the intersection matrix of the 2-cycles within the K3 surface. The potential \( \mathcal{G}(w) \) is invariant under \( SO(20) \) gauge invariance; but transforms in the adjoint of \( SU(2) \). We have studied as well the vacuum background of 10D type IIA superstring on K3 by considering also extra contributions coming from the non zero topological classes \( [\frac{F_2}{2\pi}] \) and \( [\frac{F_4}{2\pi}] \) associated with the fields strengths \( \mathcal{F}_2 = dA_1 \) and \( \mathcal{F}_4 = dA_3 \) of the 1- form and 3- form gauge fields \( A_1 \) and \( A_3 \). These non zero fluxes on the K3 surface have allowed us to determine the potential of the 6D supersymmetric
black attractors and to get the explicit moduli expression of the isotriplet $Z^m$ of central charges of the 6D $\mathcal{N} = 2$ supersymmetric algebra given by eqs. (6.30-6.33).

We end this study by making four more comments:

The first comment concerns the attractor mechanism of the 6D black objects. The entropy relations following from solving the attractor eqs of the black f-string and black hole are respectively given by

$$S_{\text{entropy}}^{\text{black f-string}} = \frac{1}{4} |eg|,$$

$$S_{\text{entropy}}^{\text{black hole}} = \frac{1}{4} |Q^2| e^{-\sigma_0},$$

where $e$ and $g$ are the electric charge and magnetic charge of the 6D dyonic F-string and $Q^2 = (\sum_{a=1}^{4} q_a^2 - \sum_{I=1}^{20} p_I^2)$. The charge vector $(q_a, p_I)$ defines the magnetic charge of the 6D black hole. Notice that $e^{-\sigma_0}$ may be a finite number and can be taken as $e^{-\sigma_0} = \sqrt{|e/g|}$. Notice also that value $S_{\text{entropy}}^{\text{black hole}}$ can be zero for $Q^2 = 0$ and or $\sigma_0 = \infty$.

The second comment concerns the link with the analysis on c-map developed in [55, 56]. From the relation between Higgs branch of 10D type II superstrings on $\text{CY}3 \times S^1$ and 6D $\mathcal{N} = (1,0)$ hypermultiplets, we suspect that the matrix formulation developed in the present paper could be also used to approach the quaternion-Kahler geometry underlying the hypermultiplet moduli space of type II superstrings on $\text{CY}3 \times S^1$ and the c-map considered in the above mentioned references.

The third comment deals with the harmonic space formulation of the special hyperKahler structure of the moduli space of 10D type IIA superstring on the K3 surface. There, one uses the identification $SU(2) \sim S^3$ to "geometrize" the $SU(2)$ R-symmetry. The 6D Minkowski space-time $M_{6D}$ with local coordinates $(x^\mu)$ and $SO(1,5)$ symmetry gets promoted to

$$M_{6D} \times S^3, \quad S^3 \sim S^2 \times S^1,$$

with local coordinates $(x^\mu, u_i^\pm)$. Here the complex isodoublets $u_i^\pm$ are harmonic variables parameterising $S^2 \times S^1$ whose defining equation as a real 3-dimensional hypersurface in the complex plane $C^2$ is

$$\sum_{i,j=1}^{2} \frac{1}{2} \epsilon_{ij} (u^{+i} u^{-j} - u^{+j} u^{-i}) = 1.$$

In terms of these variables, the quaternionified 2-form $\mathcal{J}$ is given by a special a function on $S^2$ as shown below:

$$\mathcal{J} (u) = u_i^+ u_j^- \mathcal{J}^{ij} = u_i^+ u_j^- \mathcal{J}^{[ij]} + u_i^+ u_j^- \mathcal{J}^{(ij)},$$

where one recognizes the antisymmetric part associated with NS-NS B-field contribution and the geometric one associated with the symmetric part. With this harmonic 2-form
together with other tools that can be found in [57], one can go ahead and study the special hyperKahler geometry of the moduli space of 10D type IIA superstring on the K3 surface.

The fourth comment concerns the study of 6D supersymmetric black string. As noted before, this should be described in the framework of 10D type IIB superstring on the K3 surface with the field theoretical limit given by 6D $\mathcal{N} = (2, 0)$ chiral supersymmetric gauge theory and the moduli space given by eq(2.1). However a quick inspection of the graded commutation relations of the 6D $\mathcal{N} = (2, 0)$ chiral superalgebra reveals that the 6D $\mathcal{N} = (2, 0)$ chiral superalgebra is very special. It allows no central charges which are isosinglets under space-time symmetry; see eqs(9.10). As such, there is apparently no flux potential of the kind we have obtained in the context of 6D black holes and 6D black 2-branes of 10D type IIA on K3. The fact that the 6D $\mathcal{N} = (2, 0)$ chiral superalgebra is special can be also viewed on its vector representation. Indeed, recall that the on shell degrees of freedom of the gauge multiplet in the chiral 6D $\mathcal{N} = (2, 0)$ supersymmetry are given, in terms of the $(j_1, j_2)$ representations of the $SO(4)$ group transverse to $SO(1, 1)$ with $SO(1, 5)$, as follows

$$
(1, 0), \quad \left(\frac{1}{2}, 0\right)^4, \quad (0, 0)^5,
$$

where there is no standard 1-form gauge field. The representation $(1, 0)$, which captures three on shell degree of freedom, is described by a self dual antisymmetric tensor field $B^{\mu\nu}$. The above multiplet should be compared with the gauge multiplet of the non chiral 6D $\mathcal{N} = (1, 1)$ superalgebra

$$
\left(\frac{1}{2}, \frac{1}{2}\right), \quad \left(\frac{1}{2}, 0\right)^2, \quad \left(0, \frac{1}{2}\right)^2, \quad (0, 0)^4,
$$

where $\left(\frac{1}{2}, \frac{1}{2}\right)$ stands for the gauge field and where the four scalar have been interpreted in section 2, eq(2.13) as sgauginos along the central charge directions. It is then an interesting task to study 6D black D-string attractor mechanism in the context of type IIB superstring. Progress in this direction will be reported elsewhere.

9 Appendix: $\mathcal{N} = 2$ supersymmetry in 6D

Because of the self conjugation property of Weyl spinors in 6D, one distinguishes two kinds of $\mathcal{N} = 2$ supersymmetric algebras in six dimensional space time:

(1) the chiral $\mathcal{N} = (2, 0)$ supersymmetric algebra.

(2) the non chiral $\mathcal{N} = (1, 1)$ supersymmetric algebra.

These superalgebras appear as world volume supersymmetries in NS-NS 5-branes of type...
IIA and IIB superstring theories. They are also graded symmetries of 6D supersymmetric
gauge theories describing the low energy limit of the compactification of 10D type IIA
and IIB superstrings on K3.

6D $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ can be obtained by special reductions of the 10D IIA
and IIB superalgebras down to 6D by keeping only half of the 32 original supercharges.
They can be also obtained by taking the tensor product of two 6D $\mathcal{N} = (1,0)$ ($\mathcal{N} =
(0,1)$) chiral superalgebras. Recall that 6D $\mathcal{N} = (1,0)$ supersymmetry is the basic
superalgebra in six space time dimensions, it is the symmetry of the field theoretical
limit of 10D heterotic and type I superstrings on K3. Below, we give the main relations
of these superalgebras.

9.1 6D $\mathcal{N} = (p,q)$ superalgebra

To begin, we consider $\mathcal{N} = 1$ supersymmetry in ten dimensional space time and
work out the reduction down to six dimensions. Then, we specify to the 10D $\mathcal{N} = 2$ case
on K3 where $p$ and $q$ positive integers to $p + q = 2$.

9.1.1 Dimensional reduction

In the compactification of 10D space time $\mathcal{M}_{10}$ down to 6D space time $\mathcal{M}_6$, ($\mathcal{M}_{10} \sim
\mathcal{M}_{10} \times K_4$, where $K_4$ is a real four compact manifold), the $SO(1,9)$ Lorentz group gets
reduced down to

$$SO(1,9) \rightarrow SO(1,5) \times SU(2) \times SU'(2),$$

(9.1)

where we have substituted $SO(4)$ by $SU(2) \times SU'(2)$. The 10D vector and the two 10D
Majorana-Weyl spinors 16 and 16' decompose as follows:

$$
\begin{align*}
10 &= (6,1,1') \oplus (1,2,2'), \\
16 &= (4,2,1') \oplus (4',1,2'), \\
16' &= (4,1,2') \oplus (4',2,1'),
\end{align*}
$$

(9.2)

where (*, 2, 1') and (*, 1, 2') denote the two fundamental isospinors of $SO(4) \sim SU(2) \times
SU'(2)$. Notice that each of the term $(4,2,1')$ and $(4',1,2')$ has eight real components
coming into four real isodoublets. Up on breaking the $SU(2)$ groups down to $U(1) \times
U'(1)$, the real doublet $(4',2,1')$ gives rise to a $SU(4)$ complex 4- vector which we denote
as $4_+$ together with its complex conjugate $\overline{4}_- = (4_+^*)$. Similarly $(4',2,1')$ gives $4_-$
and $\overline{4}_+ = (4_-^*)$. Notice also that the $4_+$ nor $4_-$ can obey a Majorana condition and so are
the smallest spinor objects one can have in 6D.
9.1.2 Compactification from 10D down to 4D

In the toroidal compactification of 10D space time down to 6D, \((K_4 = T^4)\), all the original supersymmetries in 10D are preserved in 6D. Then, 10D \(\mathcal{N} = 1\) supersymmetry, generated by a single 16- component Majorana-Weyl spinor, gives therefore \(\mathcal{N} = (1, 1)\) supersymmetry in 6D; and 10D \(\mathcal{N} = 2\) supersymmetry generated by two superscharges, gives 6D \(\mathcal{N} = (2, 2)\) supersymmetry.

The situation is different for the compactification on the K3 surface where half of the original 10D supersymmetric charges are broken. There, unbroken supersymmetries depend on where K3 holonomies lie. If the holonomy lies in the \(SU'(2)\) factor, then a constant spinor 2 is also covariantly constant and gives rise to unbroken supersymmetry in 6D. As such, we have the following reductions

\[
16 \rightarrow 8 = (4, 2, 1'), \\
16' \rightarrow 8' = (4', 2, 1').
\]  

(9.3)

In type 10D IIA superstring compactification on the K3 surface where one starts with \(16 \oplus 16'\), we end with \(8 \oplus 8'\) and so a non chiral 6D \(\mathcal{N} = (1, 1)\) supersymmetric algebra. For 10D IIB superstring on the K3 surface, we start with \(16 \oplus 16\) and end with a 6D \(\mathcal{N} = (2, 0)\) superalgebra. For 10D type I and heterotic superstrings one starts with a 10D \(\mathcal{N} = 1\) supersymmetry and ends with a 6D \(\mathcal{N} = (1, 0)\) algebra. Let us give below some useful details on the explicit commutation relations of these superalgebras.

9.2 \(6D \mathcal{N} = (2, 0)\) superalgebra

The chiral \(\mathcal{N} = (2, 0)\) supersymmetric algebra is generated by two identical copies of \(\mathcal{N} = (1, 0)\) supersymmetric algebras with supergenerators denoted as \(Q^i_{1\alpha}\) and \(Q^j_{2\alpha}\). Each one of these supercharges has eight real components \((\alpha = 1, \ldots, 4; i = 1, 2)\) belonging to the \((4, 2)\) representation of the \(SO(1, 5) \times SU(2)\) group symmetry. Since

\[
Q^i_{1\alpha} \sim (4, 2) \in SO(1, 5) \times SU(2)
\]

(9.4)

are complex, we have to impose the reality condition

\[
\overline{(Q^i_{1\alpha})} = \epsilon_{ij}B^{\beta}_{\alpha}Q^j_{1\beta}.
\]

(9.5)

In this relation, \(B\) is a \(4 \times 4\) matrix constrained as \(B^+B = -1\). This condition can be checked by computing the complex conjugate of eq(9.5), i.e:

\[
\overline{(\epsilon_{ij}B^{\beta}_{\alpha}Q^j_{1\beta})}.
\]

(9.6)
and using the identity $\epsilon_{ij} = \epsilon_{ji}$. One can also use complex 4-component notations; But we will not use it here since it breaks the $SU(2)$ symmetry of (9.4) we have been interested in throughout this study.

To get the graded commutation relations of the chiral $\mathcal{N} = (2,0)$ superalgebra, it is interesting to compute the reduction of the tensor product of $Q_{1\alpha}^i$ and $Q_{2\alpha}^i$. From representation group theoretical language, we have

$$ (4, 2) \times (4, 2) = (16, 4), \quad 16 \times 4 = 64, \quad (9.7) $$

which decompose into the following irreducible components,

$$ (16, 4) = (6, 1) \oplus (6, 3) \oplus (10, 1) \oplus (10, 3). \quad (9.8) $$

This decomposition deserves two comments:

First it involves two $SU(2)$ isosinglet components: (i) The $(6, 1)$ representation that captures the energy momentum tensor $P_{[\alpha\beta]}$ and should be associated with

$$ \{ Q_{1\alpha}^i, Q_{1\beta}^j \} = \epsilon^{ij} P_{[\alpha\beta]} \quad (9.9) $$

(ii) The $(10, 1)$ representation associated with the commutator $[Q_{1\alpha}^i, Q_{1\beta}^j]$ which as the tensor structure $\epsilon^{ij} D_{(\alpha\beta)}$. This representation does not concern directly the defining graded commutation relations.

Second, the decomposition (9.8) involves no space time singlet and priori the chiral $\mathcal{N} = (2,0)$ superalgebra allows no central charge. The graded commutation relations of the $\mathcal{N} = (2,0)$ superalgebra read then as follows:

$$\begin{align*}
\{ Q_{1\alpha}^i, Q_{1\beta}^j \} &= \epsilon^{ij} P_{[\alpha\beta]}; \\
\{ Q_{2\alpha}^i, Q_{2\beta}^j \} &= \epsilon^{ij} P_{[\alpha\beta]}; \\
\{ Q_{1\alpha}^i, Q_{2\beta}^j \} &= 0, \quad (9.10) \\
[P_{[\alpha\beta]}, Q_{1\gamma}^i] &= 0, \\
[P_{[\alpha\beta]}, Q_{2\gamma}^i] &= 0.
\end{align*}$$

Notice that one can also define this superalgebra by using complex Weyl spinor $Q_{\pm\alpha}^i = Q_{1\alpha}^i \pm iQ_{2\alpha}^i$. The smallest representation of this superalgebra is given by eq (8.16). The chiral $6D \mathcal{N} = (2,0)$ supergravity multiplet is given by

$$ \text{Bosons} : (1, 1) \oplus (0, 1)^5, \quad \text{Fermions} : \left(\frac{1}{2}, 1\right)^4, \quad (9.11) $$

with $3 \times (3 + 5) = 24$ bosonic on shell degrees of freedom and $4 \times (2 \times 3) = 24$ fermionic ones.
### 9.3 6D \( \mathcal{N} = (1, 1) \) superalgebra

The non chiral \( \mathcal{N} = (1, 1) \) supersymmetric algebra consists of two copies of 6D \( \mathcal{N} = 1 \) superalgebra with opposite chiralities; that is the tensor product of the \( \mathcal{N} = (1, 0) \) and \( \mathcal{N} = (0, 1) \) supersymmetric algebras. Denoting by \( Q^i_\alpha \) and \( S^\alpha_\bar{a} \) the fermionic generators of these superalgebras and using eq\((9.8)\), its complex conjugates as well as the following decomposition

\[
(4, 2) \times (\bar{1}, 2') = (1, 2 \times 2') \oplus (15, 2 \times 2'), \tag{9.12}
\]

one can write down the graded commutation relations. From the above representation group reduction, one sees that we do have space-time singlet given by the term \((1, 2 \times 2')\). So non trivial central charges \( Z^i_a \) are allowed by non chiral 6D \( \mathcal{N} = (1, 1) \) supersymmetric algebra. Therefore, the graded commutation relations following from eqs\((9.8-9.12)\) read as

\[
\begin{align*}
\{Q^i_\alpha, Q^j_\beta\} &= \epsilon^{ij} P_{[\alpha\beta]}, \\
\{S^\alpha_a, S^\beta_b\} &= \epsilon_{ab} P^{[\alpha\beta]}, \\
\{Q^i_\alpha, S^\alpha_a\} &= \delta^\alpha_\beta Z^i_a \\
[P_{[\alpha\beta]}, Q^i_\gamma] &= [P_{[\alpha\beta]}, S^\alpha_a] = 0 \\
[P_{[\alpha\beta]}, Z^i_a] &= [Z^i_a, Q^j_\gamma] = [Z^i_a, S^\alpha_b] = 0 \tag{9.13}
\end{align*}
\]

where \( P^{[\alpha\beta]} = \epsilon_{\alpha\beta\gamma\delta} P_{[\gamma\delta]} \) and where \( \epsilon_{\alpha\beta\gamma\delta} \) is the completely antisymmetric tensor in real four dimensions. Notice that under the identification of the \( SU(2) \) automorphism groups of the two \( \mathcal{N} = (1, 0) \) and \( \mathcal{N} = (0, 1) \) sectors, the four central charges split as

\[
Z^{ij} = Z_0 \epsilon^{ij} + Z^{(ij)} \tag{9.14}
\]

which should be compared with eq\((2.15)\). The smallest representation of this algebras is given by eq\((8.17)\). The 6D \( \mathcal{N} = (1, 1) \) supergravity multiplet is given by

\[
\begin{align*}
\text{Bosons} : \quad & (1, 1) \oplus (1, 0) \oplus (0, 1) \oplus \left( \frac{1}{2}, \frac{1}{2} \right)^4 \oplus (0, 0), \\
\text{Fermions} : \quad & \left( \frac{1}{2}, \frac{1}{2} \right)^2 \oplus \left( \frac{1}{2}, 1 \right)^2 \oplus \left( \frac{1}{2}, 0 \right)^2 \oplus \left( 0, \frac{1}{2} \right)^2. \tag{9.15}
\end{align*}
\]

The carry \( 9 + 3 + 3 + 16 + 1 = 32 \) bosonic on shell degrees of freedom and \( 6 \times 4 + 4 \times 2 = 32 \) fermionic ones.

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