Information Scrambling and Loschmidt Echo

Bin Yan,1,2 Łukasz Cincio,1 and Wojciech H. Zurek1

1Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
2Department of Physics and Astronomy, Purdue University, West Lafayette, Indiana 47907
(Dated: March 20, 2019)

We establish a direct link between the out-of-time order correlator, a recently suggested measure for probing information scrambling in quantum chaotic systems, and the Loschmidt echo, a well-appreciated diagnostic that captures the dynamical aspect of quantum chaos in the time domain. More precisely, we demonstrate analytically and verify numerically that the out-of-time order correlator equals the thermal average of the Loschmidt echo signals. The relation also suggests an intriguing connection between the bound on the decay rate of the out-of-time-order correlator and the perturbation-independent (Lyapunov) decay of the Loschmidt echo.

Introduction—The study of quantum version of classically chaotic systems gave rise to the field of quantum chaos since it was realized that quantum chaotic systems share certain common characteristics [1, 2]. In particular, in spite of the absence of the tell-tale exponential sensitivity to initial conditions in unitary quantum evolution, one can use as a quantum diagnostic of chaos sensitivity of the evolution to small perturbations of the Hamiltonian [3] or its entropy production in presence of the coupling to the environment [4, 5]. These and related manifestations of quantum chaos have been by now intensively studied [6–13], using phenomena such as Loschmidt echo
tations of quantum chaos have been by now intensively studied [6–13], using phenomena such as Loschmidt echo.

The Loschmidt echo is defined as a four-point correlator with unusual time ordering: 

$$M(t) \equiv |\langle \psi | e^{iH_0 t} e^{-i(H_0+V)t} | \psi \rangle|^2.$$ (1)

This quantity incorporates the simple idea [3] that small perturbations of the Hamiltonian may also trigger dramatic changes of the dynamics in the time domain, inducing the butterfly effect.

More recently, the out-of-time-order correlator (OTOC) [16, 17], another diagnostic for quantum chaos, has been proposed and received considerable attention across many different fields in physics, including quantum information, high energy physics and condensed matter physics [18–38]. The OTOC is defined as a four-point correlator with unusual time ordering:

$$F_{\beta}(t) \equiv \langle A(t)^\dagger B(t)^\dagger A(t) B(t) \rangle_{\beta},$$ (2)

where $A$ and $B$ are typically chosen as local operators; $A(t) = e^{iHt} A e^{-iHt}$ is the Heisenberg operator evolving under total Hamiltonian $H$; and the average $\langle \rangle$ is taken over a thermal state at the inverse temperature $\beta$. In chaotic systems, the OTOC exhibits fast decay and converges to a persistent small value [17]. It was argued that under certain natural assumptions, the exponential decay rate is bounded by $\lambda \leq 2\pi/\beta$ [24, 25]. Another benefit of OTOC is that it is designed to probe the spreading of local information over the entire system. Moreover, for systems with spatial structures, information measured by the OTOC propagates ballistically with a finite velocity known as the butterfly velocity [23, 34, 35].

It is worth emphasizing that OTOC is typically understood as an intrinsic echo type quantity. For instance, when $A$ and $B$ are chosen as unitary operators, Eq. (2) can be directly measured by echo experiments [13, 39–42]. However, the precise link between OTOC and the Loschmidt echo is still missing. Establishing such a relation would be beneficial for both areas and shed new light on the whole field of quantum chaos.

In this work, we accomplish the task of connecting OTOC to the Loschmidt echo. We shall focus on 0+1 dimensional (0+1D) systems such as the Sachdev–Ye–Kitaev (SYK) model [17, 43, 44]. In such systems OTOC only exhibits a temporal decay without extra complications caused by spatial propagation. We demonstrate that the OTOC equals the thermal average of the Loschmidt echo. The coupling between the target local systems, i.e., the supports of the local operators, and the rest of the total system plays the role of a perturbation. This relation also suggests that the fundamental bound on the OTOC decay rate is intimately related to the perturbation-independent decay of the Loschmidt echo. To further support our theory, we present two numerical simulations involving a random matrix model and the fermionic SYK model [45].

Bridging out-of-time-order correlator and Loschmidt echo—For a chaotic Hamiltonian, the universal decay of the OTOC is insensitive to the form of operators $A$ and $B$ in Eq. (2), as long as they are generic, i.e., not reflecting the particular symmetries possessed by the Hamiltonian. Any generic choice of local operators, e.g., random operators, are representative for the universal decay of the OTOC. This allows us to look at the typical behavior of the OTOC by averaging all unitary operators on subsystem $S_A$ and $S_B$:

$$\overline{F_{\beta}(t)} \equiv \int dA dB \, F_{\beta}(t),$$ (3)

where the integral is performed with respect to Haar measure for unitary operators. Similar ideas have been considered in the literature [46]. However, one essential ingredient that is missing is the local structure of the operators. Here, we will assume that $A$ and $B$ are supported on distinct local subsystems. For global operators the OTOC has been shown to be closely related to the spectral form factor of the Hamiltonian [46, 47]. As will
be seen in the following, taking into account the local structure of the system is crucial to reveal the correct behavior of the OTOC. For simplicity, we focus on the OTOC at infinite temperature ($\beta = 0$). It is straightforward to generalize to finite temperature by distributing the thermal density operator over a thermal loop, e.g., using the scheme described in Ref. [24, 25, 46]. The finite temperature correction will be taken into account when discussing the temperature dependence of the decay rate.

We focus on the scenario that $A$ is an operator with support on a small local subsystem $S_A$, while operator $B$ is chosen such that its support $S_B$ is the complement of $S_A$, as illustrated in Fig. 1. It is reasonable to expect that choosing operator $B$ in such manner also captures the spreading of operator $A$ over the entire system, detected by its non-zero projection at later times on the support of operator $B$, at least in the bulk of the decay. Analysis of this particular scenario is also instructive and can be generalized in a similar way to cases where $B$ is a small local operator as well.

The Haar integral over subsystem operators can be evaluated with the aid of the formula:

$$\int dA \ A'^{iOA} = \frac{1}{d_A} \mathbb{I}_A \otimes \text{tr}_A(O),$$

where $\mathbb{I}_A$ is the identity operator, $d_A$ is the dimension of $S_A$, and $\text{tr}_A$ is the partial trace over subsystem $S_A$. The proof is given in Appendix A of Supplementary Material (SM).

The average over all random unitary operators on subsystem $S_A$ gives us:

$$\int dA \ F_{\beta=0}(t) \equiv \frac{1}{d} \int dA \ \text{tr}(A'^t(t)B^iA(t)B)$$

$$= \frac{1}{d} \int dA \ \text{tr}[\mathbb{I}_A \otimes \text{tr}_A(e^{-iHt}B^i e^{iHt}) e^{-iHt}B e^{iHt}]$$

$$= \frac{1}{d} \int dA \ \text{tr}_B[\text{tr}_A(e^{-iHt}B^i e^{iHt}) \text{tr}_A(e^{-iHt}B e^{iHt})].$$

The last line of the above equation involves the reduced dynamics of operator $B$, i.e., $B(t) = \text{tr}_A(e^{-iHt}B e^{iHt})$. In order to further perform the average over $B$, we estimate $B(t)$ in the following way. Recall that the total system Hamiltonian has the structure

$$H = \mathbb{I}_A \otimes H_B + H_A \otimes \mathbb{I}_B + H'.$$

In realistic physical systems, the coupling $H'$ is much smaller than the non-interacting part of the Hamiltonian. We can replace the effect of the coupling with an ensemble of random noises on subsystem $S_B$, namely,

$$B(t) = \text{tr}_A[e^{-iHt}Be^{iHt}]
\approx \sum_k e^{-i(H_B + V_k)t} Be^{i(H_B + V_k)t}. \quad (7)$$

Here the ensemble of random noise $V_k$ has a strength that is on the same order as the coupling $H'$.

The above claim is based on the correspondence between the symptoms of decoherence (process that involves correlation between the system and the environment) and symptoms of the suitable external noise (see, e.g. Ref. [48]). We back up this claim with an alternative and more mathematically rigorous treatment of the noise operators $V_k$. In the Appendix B of SM, it is proven that operators $V_k$ take the form

$$V_k = \langle k_A | H' | k_A \rangle. \quad (8)$$

Here, $\{ k_A \}$ is any complete basis of subsystem $S_A$.

With the aid of the alternative form for the reduced dynamics of operator $B$ in Eq. (7), averaging over operators $B$ can be further performed in the same manner as for the operators $A$. This gives the final expression for the averaged OTOC:

$$\overline{F}_{\beta=0}(t) \approx |\langle e^{iH_B t} e^{-i(H_B + V)t} \rangle_{\beta=0}|^2, \quad (9)$$

which is precisely the Loschmidt echo averaged over a thermal ensemble.

Equation (9) is the main result of this work. As has been mentioned before, the above result generalizes to finite temperature. We present the full derivation in the Appendix C of SM. It is shown there that only the strength of the noise (the average of the square of the off-diagonal matrix elements of the perturbation matrix) matters for the universal decay of the OTOC. We can then use a single averaged noise $V = \overline{V}_k + \overline{V}'$ in the derivation of the above approximation to represent the whole ensemble of $d_A$ noise operators. Note that the matrix elements of $V$ have a variance which is double of those in $V_k$.

At this point, the connection between OTOC and the thermal averaged Loschmidt echo has been established, with the coupling between subsystems playing the role as the perturbation, as given in Eq. (8). Depending on the strength of the perturbation, the Loschmidt echo can exhibit various types of behaviors. We briefly discuss those in the following four different regimes and consider their implications to the OTOC decay.
I. Perturbative regime: When the perturbation strength is smaller than the level spacing of the unperturbed Hamiltonian, the LE exhibits Gaussian decay. This regime is effectively suppressed in the thermodynamic limit since the level spacing decreases exponentially fast with the system size.

II. Fermi’s Golden rule regime: When the perturbation is increased, the LE is driven into an exponential decay regime, where the decay rate is given by the Fermi’s Golden rule, \( \Gamma_n = 2\pi \rho(n) V_{og}^2(n) \). Here, \( n \) labels the initial energy eigenstates in the LE; \( \rho(n) \) is the spectral density of the unperturbed Hamiltonian; and \( V_{og}^2(n) \) is the average of the square of the off-diagonal perturbation matrix elements at state \( |n\rangle \). This regime is valid for the decay rate \( \Gamma \) much smaller than the bandwidth of the unperturbed Hamiltonian.

III. Lyapunov regime: When the perturbation is further increased, the decay stays exponential while the rate saturates to a bounded value. This regime has been predicted by semi-classical theory and is supported by numerical simulations (See, e.g., [14] and reference therein). The quantum mechanism for the existence of this regime has not been fully understood yet. However, the OTOC-LE connection discussed in this work strongly suggests that this regime is responsible for the fundamental temperature-dependent upper bound of the OTOC decay.

IV. Gaussian regime: When the perturbation is large enough such that the exponential decay rate is comparable to the Hamiltonian band width, the global density of states (that typically has a Gaussian form [51]) starts to interfere and a Gaussian decay is finally developed [53]. Numerical simulations discussed below show that, due to the finite system size, OTOC also exhibits Gaussian decay in finite size systems, even for the SYK model, which is expected to possess a saturated exponential decay.

**Simulations and Discussions—**

*Case 1: random matrix model.* We consider first a random matrix model given by Eq. (6), where the subsystem self-Hamiltonian and the coupling are given by random matrices from the standard Gaussian Unitary Ensemble (GUE). We are mostly interested in the exponential decay of the OTOC. Thus the coupling strength of \( H' \) is further decreased by a factor \( g < 1 \) in order to tune the system into the Fermi’ golden rule decay regime.

In this case, the thermal averaged decay rate for the OTOC can be computed as a thermal average of the golden rule decay rate:

\[
\Gamma_\beta = \sum_n e^{-\beta E_n} \frac{2\pi \rho(n) V_{og}^2(n)}{\sum_n e^{-\beta E_n}}. \tag{10}
\]

**FIG. 2.** Decay rate v.s. the inverse temperature for random matrix model with structure Eq. (6). The perturbation strength is fixed at \( g = 0.1 \). The diamonds represent the exponential decay rate extracted from the numerical simulation of the OTOC for 10 qubits (The Hilbert space dimensions of subsystem \( S_A \) and \( S_B \) are 2 and \( 2^3 \), respectively) at various temperatures. The red curves are the best fit for the theoretical prediction of Eq. (12).

By adding an averaged density of states \( \rho(E) \), the discrete sum can be replaced with a smooth integral, which is evaluated further through integral by part,

\[
\Gamma_\beta = \frac{2\pi}{\beta} \int dE \frac{e^{-\beta E} \left[ \rho^2(E) V_{og}^2(E) \right]'}{\int dE e^{-\beta E} \rho(E)}. \tag{11}
\]

In the random matrix model, the perturbation is given by the random coupling matrix \( V = gH' \) according to Eq. (8), where \( H' \) is an element from the GUE. Thus \( V_{og}^2(E) = g^2 \) does not depend on the energy. Furthermore, for random matrix, the spectral density \( \rho(E) \) is given by the Wigner semi-circle law \( \rho(E) \propto \sqrt{E^2 - E_0^2} \), where \( E_0 \) is the ground state energy. In this case, the temperature dependent decay rate can be explicitly written out:

\[
\Gamma_\beta \propto g^2 \frac{\beta \cosh(E_0 \beta/2) - \sinh(E_0 \beta/2)}{E_0^{\beta/2} I_1(E_0 \beta/2)}, \tag{12}
\]

where \( I_1 \) is the modified Bessel function of the first kind.

We perform a numerical simulation of the OTOC evolution for the random matrix model with structure Eq. (6). \( A \) and \( B \) are chosen as random Hermitian operators on the corresponding subsystems. The OTOC clearly shows exponential decay for appropriate coupling strength (\( g \) from 0.05 to 0.5). Fig. 2 depicts the extracted decay rate at various temperatures, which matches our theory prediction Eq. (12) very well.

*Case 2: SYK model.* To further test our theory, we also present numerical study of the fermionic version of the SYK model proposed in [45]:

\[
H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,l=1} N J_{i,j,k,l} c_i^\dagger c_j^\dagger c_k c_l \tag{13}
\]
where $J_{i,j;k,l}$ are complex Gaussian random couplings with zero mean obeying certain symmetries. $c_i$ and $c_i^\dagger$ are fermionic annihilation and creation operators at site $i$. We compute the OTOC for operators $c_i^\dagger + c_i$ on two distinct sites. Clear Gaussian decay with very weak temperature dependence has been observed (Fig. 3), which agrees with previous numerical studies [45, 54] while contradicting the expected exponential decay with upper bound $2\pi/\beta$ [24, 25].

On the other hand, this observation fits into the theory of the present work: The OTOC decay rate is governed by the coupling strength between the target subsystem $S_A$, $S_B$ and the rest of the system. In a finite system, the coupling might be too large for the decay to be in the exponential regime. The decay rate extracted from our numerical simulation is close to the band width of the SYK Hamiltonian, which indicates that the decay is in the Gaussian regime (regime IV). Since the relative strength compared with the total Hamiltonian decreases with the system size, it is then expected that the OTOC decay would drop into the exponential regime in the large-$N$ limit. Due to the limited numerical capacity, we did not observe exponential decay up to 17 fermions, which is the largest system we are able to simulate numerically. However, we are able to observe exponential decay by manually adjusting the coupling strength. It is done by decreasing the couplings $J_{i,j;k,l}$ that involve the subsystem $S_A$ and $S_B$ by a factor $0 < g < 1$, while keeping the coupling in the rest of the system unchanged.

Figure 4 shows the exponential OTOC decay for the deformed coupling strength. The decay rate also admits quadratic dependence on the coupling factor $g$, which satisfies the Fermi’s golden rule prediction in regime II.

We would like to emphasize that the conjectured universal upper bound of the OTOC decay rate $2\pi/\beta$ is likely to be a consequence of the thermal average of the perturbation-independent decay rate of the LE (regime III).

To summarize, we have demonstrated the connection between two distinct areas of the dynamical quantum chaos, namely, the emerging field of the out-of-time-order correlator and the relatively more developed field of the Loschmidt echo. The established relation not only allows a more general understanding of the universal properties of the OTOC, e.g., the existence of Gaussian decay regime for the finite size SYK model, but also provides new insights into both subjects, e.g., the implications of the bound on the OTOC decay to the quantum mechanism behind the perturbation-independent (Lyapunov) decay of the Loschmidt echo, and vice versa. Future works could generalize the connection to higher dimensional systems with spatial structures. Higher order corrections and their implications for the early time behaviors of the OTOC are also worth further investigation.

ACKNOWLEDGMENTS

This research was supported by the DOE under the LDRD program at the Los Alamos National Laboratory. WHZ acknowledges partial support by the Foundational Questions Institute grant FQXi-1821, and Franklin Fetzer Fund, a donor advised fund of the Silicon Valley Community Foundation. LC acknowledges support by the DOE through the J. Robert Oppenheimer fellowship. BY and WHZ thank Adolfo Del Campo for helpful discussions.
[1] F. Haake, *Quantum Signatures of Chaos* (Springer-Verlag, Berlin, Heidelberg, 2006).
[2] M. Berry, “Quantum chaology, not quantum chaos,” *Phys. Scr.* 40, 335 (2006).
[3] A. Peres, ed., *Quantum Theory: Concepts and Methods*, Fundamental Theories of Physics (Springer, 2002).
[4] W. H. Zurek and J. P. Paz, “Decoherence, chaos, and the second law,” *Phys. Rev. Lett.* 72, 2508–2511 (1994).
[5] W. H. Zurek, “Decoherence, Chaos, Quantum-Classical Correspondence, and the Algorithmic Arrow of Time,” *Physica Scripta* Volume T 76, 186–198 (1998).
[6] P. R. Levstein, G. Usaj, and H. M. Pastawski, “Attenuation of polarization echoes in nuclear magnetic resonance: A study of the emergence of dynamical irreversibility in many-body quantum systems,” *J. Chem. Phys.* 108, 2718–2724 (1998).
[7] R. A. Jalabert and H. M. Pastawski, “Environment-independent decoherence rate in classically chaotic systems,” *Phys. Rev. Lett.* 86, 2490–2493 (2001).
[8] F. M. Cucchietti, D. A. R. Dalvit, J. P. Paz, and W. H. Zurek, “Decoherence and the Loschmidt echo,” *Phys. Rev. Lett.* 91, 210403 (2003).
[9] H. Kohler and C. Recher, “Fidelity and level correlations in the transition from regularity to chaos,” *EPL* 98, 10005 (2012).
[10] C. M. Sánchez, P. R. Levstein, L. Buljubasich, H. M. Pastawski, and A. K. Chattah, “Quantum dynamics of excitations and decoherence in many-spin systems detected with Loschmidt echoes: its relation to their spreading through the Hilbert space,” *Philosophical Transactions of the Royal Society of London Series A* 374, 20150155 (2016).
[11] A. Chenu, I. L. Egusquiza, J Molina-Vilaplana, and A. del Campo, “Quantum work statistics, loschmidt echo and information scrambling,” *Sci. Rep.* 8, 12634 (2018).
[12] A. Chenu, J. Molina-Vilaplana, and A. del Campo, “Work statistics, loschmidt echo and information scrambling in chaotic quantum systems,” *Quantum* 3, 127 (2019).
[13] C.M. Sánchez, A. K. Chattah, K. X. Wei, L. Buljubasich, P. Cappellaro, and H. M. Pastawski, “Emergent perturbation independent decay of the Loschmidt echo in a many-spin system studied through scaled dipolar dynamics,” (2019), arXiv:1902.06628 [quant-ph].
[14] A. Goussev, R. A. Jalabert, H. M. Pastawski, and D. Wisniacki, “Loschmidt echo,” in *Scholarpedia* (Scholarpedia, 2012) p. 7(8):11687.
[15] T. Gorin, T. Prosen, T. H. Seligman, and M. Žnidarič, “Dynamics of Loschmidt echo and fidelity decay,” *Phys. Rep.* 435, 33–156 (2006).
[16] A. Larkin and Y. N. Ovchinnikov, *Sov. Phys. JETP* 28, 1200 (1969).
[17] A. Kitaev, A simple model of quantum holography, in *Proceedings of the KITP Program: Entanglement in Strongly-Correlated Quantum Matter*, 2015 (Kavli Institute for Theoretical Physics, Santa Barbara, 2015), Vol. 7.
[18] S. H. Shenker and D. Stanford, “Black holes and the butterfly effect,” *J. High Energy Phys.* 2014, 67 (2014).
[19] S. H. Shenker and D. Stanford, “Multiple shocks,” *J. High Energy Phys.* 2014, 46 (2014).
[20] D. A. Roberts, D. Stanford, and L. Susskind, “Localized shocks,” *J. High Energy Phys.* 2015, 51 (2015).
[21] D. A. Roberts and D. Stanford, “Diagnosing chaos using Four-Point functions in Two-Dimensional conformal field theory,” *Phys. Rev. Lett.* 115, 131603 (2015).
[22] M. Blake, “Universal charge diffusion and the butterfly effect in holographic theories,” *Phys. Rev. Lett.* 117, 091601 (2016).
[23] D. A. Roberts and B. Swingle, “Lieb-Robinson bound and the butterfly effect in quantum field theories,” *Phys. Rev. Lett.* 117, 091602 (2016).
[24] J. Maldacena, S. H. Shenker, and D. Stanford, “A bound on chaos,” *J. High Energy Phys.* 2016, 106 (2016).
[25] N. Tsuji, T. Shitara, and M. Ueda, “Bound on the exponential growth rate of out-of-time-ordered correlators,” *Phys Rev E* 98, 012216 (2018).
[26] B. Swingle and D. Chowdhury, “Slow scrambling in disordered quantum systems,” *Phys. Rev. B Condens. Matter* 95, 060201 (2017).
[27] M. Campisi and J. Goold, “Thermodynamics of quantum information scrambling,” *Phys Rev E* 95, 062127 (2017).
[28] X. Chen, T. Zhou, D. A. Huse, and E. Fradkin, “Out-of-time-order correlations in many-body localized and thermal phases,” *Annalen der Physik* 529, 1600332 (2017).
[29] E. B. Rozenbaum, S. Ganeshan, and V. Galitski, “Lyapunov exponent and Out-of-Time-Order Correlator’s growth rate in a chaotic system,” *Phys. Rev. Lett.* 118, 086801 (2017).
[30] B. Dóra and R. Moessner, “Out-of-Time-Ordered Density Correlators in Luttinger Liquids,” *Phys. Rev. Lett.* 119, 026802 (2017).
[31] C.-J. Lin and O. I. Motrunich, “Out-of-time-ordered correlators in a quantum Ising chain,” *Phys. Rev. B* 97, 144304 (2018).
[32] C. J. Lin and O. I. Motrunich, “Out-of-time-ordered correlators in short-range and long-range hard-core boson models and in the Luttinger-liquid model,” *Phys. Rev. B* 98, 134305 (2018).
[33] C. W. von Keyserlingk, T. Rakovszky, F. Pollmann, and S. L. Sondhi, “Operator hydrodynamics, OTOCs, and entanglement growth in systems without conservation laws,” *Phys. Rev. X* 8, 021013 (2018).
[34] A. Nahum, S. Vijay, and J. Haah, “Operator spreading in random unitary circuits,” *Phys. Rev. X* 8, 021014 (2018).
[35] V. Khemani, A. Vishwanath, and D. A. Huse, “Operator spreading and the emergence of dissipative hydrodynamics under unitary evolution with conservation laws,” *Phys. Rev. X* 8, 031057 (2018).
[36] H. Gharibyan, M. Hanada, S. H. Shenker, and M. Tezuka, “Onset of random matrix behavior in scrambling systems,” *J. High Energy Phys.* 2018, 124 (2018).
[37] S. Xu and B. Swingle, “Locality, quantum fluctuations, and scrambling,” (2018), arXiv:1805.05376 [cond-mat.str-el].
[38] J. Tuziemski, “Out-of-time-ordered correlation functions in open systems: A Feynman-Vernon influence functional approach,” (2019), arXiv:1903.05025 [quant-ph].
[39] B. Swingle, G. Bentsen, M. Schleier-Smith, and P. Hayden, “Measuring the scrambling of quantum information,” *Phys. Rev. A* 94, 040302(R) (2016).
[40] M. Gärttner, J. G. Bohnet, A. Safavi-Naini, M. L. Wall,
J. J. Bollinger, and A. M. Rey, “Measuring out-of-time-order correlations and multiple quantum spectra in a trapped-ion quantum magnet,” Nat. Phys. 13, 781 (2017).

[41] J. Li, R. Fan, H. Wang, B. Ye, B. Zeng, H. Zhai, X. Peng, and J. Du, “Measuring Out-of-Time-Order correlators on a nuclear magnetic resonance quantum simulator,” Phys. Rev. X 7, 031011 (2017).

[42] K A Landsman, C Figgatt, T Schuster, N M Linke, B Yoshida, N Y Yao, and C Monroe, “Verified quantum information scrambling,” Nature 567, 61–65 (2019), arXiv:1806.02807 [quant-ph].

[43] S. Sachdev and J. Ye, “Gapless spin-fluid ground state in a random quantum Heisenberg magnet,” Phys. Rev. Lett. 70, 3339–3342 (1993).

[44] J. Maldacena and D. Stanford, “Remarks on the Sachdev-Ye-Kitaev model,” Phys. Rev. D 94, 106002 (2016).

[45] W. Fu and S. Sachdev, “Numerical study of fermion and boson models with infinite-range random interactions,” Phys. Rev. B 94, 035135 (2016).

[46] J. Cotler, N. Hunter-Jones, J. Liu, and B. Yoshida, “Chaos, complexity, and random matrices,” J. High Energy Phys. 2017, 118 (2017).

[47] J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, Saa, S. H. Shenker, D. Stanford, A. Streicher, and M. Tezuka, “Black holes and random matrices,” J. High Energy Phys. 2017, 118 (2017).

[48] W. H. Zurek, “Decoherence, einselection, and the quantum origins of the classical,” Rev. Mod. Phys. 75, 715–775 (2003).

[49] P. Jacquod, P. G. Silvestrov, and C. W. Beenakker, “Golden rule decay versus Lyapunov decay of the quantum Loschmidt echo,” Phys. Rev. E 64, 055203(R) (2001).

[50] F. M. Cucchietti, H. M. Pastawski, and D. A. Wisniacki, “Decoherence as decay of the Loschmidt echo in a Lorentz gas,” Phys. Rev. E 65, 045206(R) (2002).

[51] V. K. B. Kota, Embedded Random Matrix Ensembles in Quantum Physics, 1st ed., Lecture Notes in Physics (Springer International Publishing, 2014).

[52] V. V. Flambaum and F. M. Izrailev, “Excited eigenstates and strength functions for isolated systems of interacting particles,” Phys. Rev. E 61, 2539–2542 (2000).

[53] W. H. Zurek, F. H. Cucchietti, and J. P. Paz, “Gaussian decoherence and gaussian echo from spin environments,” Acta Physica Polonica B 38, 1685 (2007).

[54] P. Hosur, X.-L. Qi, D. A. Roberts, and B. Yoshida, “Chaos in quantum channels,” J. High Energy Phys. 2016, 4 (2016).
Supplementary material for “Information Scrambling and Loschmidt Echo”

Appendix A: Haar integral of subsystem operators

In this appendix we prove the formula in Eq. (4) in the main text. For a single system, \[ \int dA \ A^\dagger \otimes I_B O_{AB} A \otimes I_B \]

For an operator \( O_{AB} \) on a composed system, the integral over operators on a subsystem can be performed in a similar way, by doing the partial trace:

\[ \int dA \ A^\dagger \otimes I_B O_{AB} A \otimes I_B = \int dA \ A^\dagger \otimes I_B \sum_i O_i^A \otimes O_i^B A \otimes I_B \]

\[ = \sum_i \int dA \ O_i^A A \otimes O_i^B \]

\[ = \frac{1}{d_A} \sum_i \text{tr}(O_i^A) I_A \otimes O_i^B = \frac{1}{d_A} I_A \otimes \text{tr}_A O_{AB}. \]  

(A1)

Appendix B: Reduced dynamics for operator \( B \)

This appendix presents a more rigorous derivation of the reduced dynamical equation for the operator \( B \) in Eq. (7).

Let \( \{|k_A\} \) denote a complete eigen-basis for a subsystem \( S_A \). The reduced dynamics of \( B \) can be represented by the Kraus operators:

\[ \text{tr}_A(e^{-iHt} I_A \otimes B e^{iHt}) = \sum_l \Lambda_l B \Lambda_l^\dagger, \]

(B1)

where the Kraus operators have the representation

\[ \Lambda_l = \langle k_A | e^{-iHt} | k_A \rangle, \]

(B2)

where \( l \) runs over all pairs of \( \{k_A, k'_A\} \).

Recall the decomposition of the total Hamiltonian:

\[ H = H_A \otimes I_B + I_A \otimes H_B + H'. \]

(B3)

The system under consideration has the following structure:

\[ H_0 \equiv I_A \otimes H_B + H_A \otimes I_B \Rightarrow H', \quad H_B \gg H_A. \]

(B4)

This allows us to expand the evolution operator \( e^{-iHt} \) to the first order in \( H' \). Each first order term in such an expansion has the form \( H_0^m H'H_0^n \), with \( m, n = 0, 1, 2, \ldots \). They contribute to the Kraus operator as

\[ \langle k | H_0^m H'H_0^n | k' \rangle \approx \delta_{k, k'} H_0^m \langle k | H' | k' \rangle H_0^n. \]  

(B5)

The Kraus operators, sum up to the exponential form, have the representation

\[ \Lambda_k \approx e^{-i(H_B + V_k)t} + O(H'^2), \]

where the effective perturbation matrices are

\[ V_k = \langle k_A | H' | k_A \rangle. \]  

(B7)

This can be seen by expanding Eq. (B6) up-to the first order of \( V_k \). The resulting terms are precisely Eq. (B5). \( B(t) \) can then be expressed as

\[ B(t) = \sum_{k=1}^{d_A} e^{-i(H_B + V_k)t} B(0) e^{i(H_B + V_k)t}. \]  

(B8)

Appendix C: OTOC - Loschmidt echo connection

This appendix presents the derivation of Eq. (9) in the main text. It has been shown that after performing averaging over operator \( A \), the OTOC reads

\[ \int dA \ F_{\beta=0}(t) \equiv \int dA \ \text{tr}_B(\text{tr}_A(e^{-iHt} B^\dagger e^{iHt}) \text{tr}_A(e^{-iHt} B e^{iHt})), \]

\[ = \frac{1}{d} \int dB \ \text{tr}_B \left[ \text{tr}_A(e^{-iHt} B^\dagger e^{iHt}) \text{tr}_A(e^{-iHt} B e^{iHt}) \right]. \]

(C1)

Replacing the reduced dynamics of operator \( B \) with its alternative expression Eq. (B8) allows us to further evaluate the average over all unitary operators on subsystem \( S_B \):

\[ \bar{F}_{\beta=0}(t) = \frac{1}{d} \int dB \ \text{tr}_B \left[ \text{tr}_A(e^{-iHt} B^\dagger e^{iHt}) \text{tr}_A(e^{-iHt} B e^{iHt}) \right] \]

\[ \approx \frac{1}{d} \int dB \ \text{tr}_B \left[ \sum_{k,k'=1}^{d_A} \text{tr}_A \left[ e^{-i(H_B + V_k)t} B e^{i(H_B + V_k)t} \right] \times e^{-i(H_B + V_{k'})t} B e^{i(H_B + V_{k'})t} \right] \]

\[ = \frac{1}{d^2} \sum_{k,k'=1}^{d_A} \left| \text{tr} \left[ e^{i(H_B + V_{k'})t} e^{-i(H_B + V_k)t} \right] \right|^2 \]

\[ = \frac{1}{d^4} \sum_{k,k'=1}^{d_A} \left| \langle e^{i(H_B + V_{k'})t} e^{-i(H_B + V_k)t} \rangle \right|^2 \]

\[ \approx \left| \langle e^{iHt} e^{-i(H_B + V)t} \rangle \right|^2, \]

(C2)

As noted before, only the strength of the noise (the average of the square of the off-diagonal matrix elements of the perturbation matrix) matters for the universal decay in the Loschmidt echo. We can thus use a single averaged noise \( V = \overline{V_k} + \overline{V_{k'}} \) in the above approximation to represent the whole ensemble of \( d_A \) noise operators.

This can be generalized to the finite temperature setting, by distributing the operators at equal spacing around the thermal circle. Namely, let us define

\[ Y^4 = \frac{1}{Z} e^{-\beta H}, \]

(C3)
and evaluate the OTOC as

$$F(t) = \text{tr}[yA^\dagger(t)yB^\dagger yA(t)yB].$$ \hspace{1cm} (C4)

By absorbing the inverse temperature into the time evolution operator, the OTOC can be evaluated in the same manner as in the infinite temperature case, but in the complex time domain, $t + \beta/4$. 