ON GRADED DIVISION RINGS

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Abstract. We develop the theory of group graded division ring parallel to the one by P. Cohn for (ungraded) division rings.

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Introduction

Let $R$ be a commutative ring. It is well known the prime ideals or $R$ classify the homomorphisms from $R$ to division rings. Indeed, for any prime ideal $P$ of $R$, we obtain a homomorphism from $R$ to a division ring via the natural homomorphism $R 	o Q(R/P)$, where $Q(R/P)$ denotes the field of fractions of $R/P$. Conversely, if $\varphi : R \to D$ is a homomorphism from $R$ to a division ring $D$, then $P = \ker \varphi$ is a prime ideal of $R$, $\varphi$ factors through $R \to Q(R/P)$ and therefore the division subring of $D$ generated by the image of $R$ is $R$-isomorphic to $Q(R/P)$. Moreover, let $P \subseteq P'$ be prime ideals of $R$. The localization of $R/P$ at the prime ideal $P'/P$ yields a local subring of $Q(R/P)$ with residue field isomorphic to $Q(R/P')$. This implies that any fraction $ab^{-1} \in Q(R)$ which is defined in $Q(R/P')$ it is also defined in $Q(R/P)$. Also, looking at the determinants of matrices, one sees that any matrix

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with entries in $R$ that becomes invertible in $Q(R/P')$ then it also becomes invertible in $Q(R/P)$.

If the ring $R$ is not commutative, prime ideals no longer classify the homomorphisms to division rings. It may even possible that $R$ has infinitely many different “fields of fractions”, see for example [18] Section 9.

Let $R$ be any ring. An epic $R$-division ring is a ring homomorphism $R \to K$ where $K$ is a division ring generated by the image of $R$. In [6], P. M. Cohn showed that the epic $R$-division rings are characterized up to $R$-isomorphism by the collection of square matrices over $R$ which are carried to matrices singular over $K$. This set of matrices is called the singular kernel of $R \to K$. He also gave the precise conditions for a set of square matrices over $R$ to be a singular kernel, calling such a collection a prime matrix ideal of $R$. The name comes from the fact that, if we endow the set of square matrices over $R$ with certain two operations of sum and product (one of them partial), those sets have a similar behaviour to prime ideals. These operations are defined so that, when defined on square matrices over a commutative ring, the determinant of the sum of matrices equals the product of the determinants. Also in [6], Cohn showed that if $P, P'$ are prime matrix ideals of $R$ and $P \subseteq P'$ if and only if there exists a local subring of $K_P$ containing the image of $R$ with residue class division ring $R$-isomorphic to $K_{P'}$. We say that there exists a specialization from $K_P$ to $K_{P'}$. Furthermore, if a rational expression built up from elements of $R$ makes sense in $K_{P'}$, then it can also be evaluated in $K_P$. P. M. Cohn also provided conditions on square matrices over $R$ which functions from the class of finitely presented right $R$-modules with values in $R$ are rank functions induced from epic $R$-division rings. Another alternative way of determining epic $R$-division rings described by P. Malcolmson is induced from the notion of dimension. More precisely, if $R \to K$ is an epic $R$-division ring, we can associate with each finitely presented right $R$-module $M$ the number $\dim_K(M \otimes_R K) \in \mathbb{N}$. He described which functions from the class of finitely presented right $R$-modules with values in $\mathbb{N}$ are induced from epic $R$-division rings as dimensions.

Another important feature of rank functions is that, theoretically speaking, it is easy to know when there exists a specialization from an epic $R$-division ring to another in terms of rank functions as defined by P. Malcolmson. In [28], A. Schofield gave another equivalent notion to that of epic $R$-division rings in terms of a rank function that satisfies certain natural conditions. This time it is a function from the class of homomorphisms between finitely generated projective right $R$-modules with values in $\mathbb{N}$. We would like to remark that Sylvester rank functions with values in $\mathbb{R}_+$ have proved useful in many different situations [1] [2] [9] [10] [11] [15] [16] [28].

The theory of group graded rings has played an important role in Ring Theory (see for example [12], [29]) and many of the results in classical ring theory have a mirrored version for group graded rings. Furthermore, if $R$ is a filtered ring, it has proved fruitful to study the associated graded ring, which usually is a simpler object, in order to obtain information about the original ring.
The main aim of this article is to develop Cohn’s theory on division rings in the context of group graded rings. More precisely, let \( \Gamma \) be a group and \( R = \bigoplus_{\gamma \in \Gamma} R_{\gamma} \) be a \( \Gamma \)-graded ring. A \( \Gamma \)-graded epic \( R \)-division ring is a homomorphism of \( \Gamma \)-graded rings \( R \to K \) where \( K \) is a \( \Gamma \)-graded division ring generated by the image of \( R \). Matrices over \( R \) represent homomorphisms between finitely generated free \( R \)-modules. Homomorphisms of \( \Gamma \)-graded modules between \( \Gamma \)-graded free \( R \)-modules are given by (what we call) **homogeneous matrices**. These are \( m \times n \) matrices \( A \) for which there exist \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in \Gamma \) such that each \((i, j)\) entry of \( A \) belongs to \( R_{\alpha_i \beta_j} \). We show that \( \Gamma \)-graded epic \( R \)-division rings \( R \to K \) are characterized, up to \( R \)-isomorphism of \( \Gamma \)-graded rings, by the collection of homogeneous matrices which are carried to singular matrices over \( K \). These sets are called the gr-singular kernel of \( R \to K \). We give the precise conditions under which a collection of homogeneous matrices over \( R \) is a gr-singular kernel and thus defining the concept of gr-prime matrix ideal. If \( P, P' \) are gr-prime matrix ideals of \( R \) and \( R \to K_P, R \to K_{P'} \) are the corresponding \( \Gamma \)-graded epic \( R \)-division rings, then \( P \subseteq P' \) if and only if there exists a \( \Gamma \)-graded local subring of \( K_P \) that contains the image of \( R \) with residue class \( \Gamma \)-graded division ring \( R \)-isomorphic to \( K_{P'} \) as \( \Gamma \)-graded rings. Furthermore, if a homogeneous rational expression obtained from elements of \( R \) make sense in \( K_{P'} \) then it can also be evaluated in \( K_{P} \). We then provide conditions on the set of square homogeneous matrices over \( R \) that characterize when there exists an (injective) homomorphism of \( \Gamma \)-graded rings from \( R \) to a \( \Gamma \)-graded division ring and when there exists a best \( \Gamma \)-graded epic \( R \)-division ring. We also provide the graded concepts corresponding to the different rank functions defined by Malcolmson and Schofield. We show they give alternative ways of determining \( \Gamma \)-graded epic \( R \)-division rings in terms of rank functions from the set of homogeneous matrices, from the class of \( \Gamma \)-graded finitely presented modules and from the class of \( \Gamma \)-graded homomorphisms between \( \Gamma \)-graded projective \( R \)-modules, respectively, all of them with values in \( \mathbb{N} \).

In the study of division rings, one of the pioneering works carrying the information from the associated graded ring to the original filtered ring was \[15\]. P. M. Cohn showed that if a ring \( R \) endowed with a valuation with values in \( \mathbb{Z} \) is such that its associated graded ring is a (graded) Ore domain, then \( R \) can be embedded in a division ring. Other proofs of this result can be found in \[20\] and in \[3\] together with \[19\]. More recently, a generalization of the result by Cohn has been given by A. I. Valitskas \[29\]. We believe that our work could be helpful in order to generalize the result by Cohn to a greater extent than has been done by Valitskas.

An elementary application of our theory is as follows. Suppose that \( R \) is a ring graded by a group \( \Gamma \). As an immediate consequence of \[26\] Proposition 1.2.2, one obtains that if there exists an (injective) homomorphism from \( R \) to a division ring, then there exists an (injective) homomorphism of \( \Gamma \)-graded rings from \( R \) to a \( \Gamma \)-graded division ring. Thus if one shows that there do not exist (injective) homomorphisms of \( \Gamma \)-graded rings from \( R \) to \( \Gamma \)-graded division rings, then there do not exist (injective) homomorphisms from \( R \) to division rings. See section \[4\] for other similar results.

We end this introduction by showing that the existence of an (injective) homomorphism from a \( \Gamma \)-graded ring \( \bar{R} \) to a division ring is not equivalent to the existence of a homomorphism of \( \Gamma \)-graded rings from \( R \) to a \( \Gamma \)-graded division ring. For that we produce an easy example of a graded ring for which there does not exist a homomorphism to a division ring but it is embeddable in a graded division ring. Let \( T \) be the ring obtained as localization of \( \mathbb{Z} \) at the prime ideal \( 3\mathbb{Z} \). Let \( R \) be the ring \( T[x] \subseteq \mathbb{C} \). Let \( C_2 = \langle x \rangle \) be the cyclic group of order two, and let
σ: C₂ → Aut(R) be the homomorphism of groups which sends x to the automorphism induced by the complex conjugation. Set now S = R[C₂; σ]. That is, S is the skew group ring of G over R induced by σ. Hence S is a C₂-graded ring, S = S₀ + S₁, where S₀ = R and S₁ = Rx and the product is determined by xr = Tr for all r ∈ R. Clearly S is embeddable in the C₂ graded division ring \( \mathbb{Q}[C₂; σ] \).

Suppose that there exists a homomorphism of rings from S to a division ring K. Let \( ϕ: S → K \) be such homomorphism. Since \((1 - x)(1 + x) = 0\), then either \( ϕ(1 + x) = 0 \) or \( ϕ(1 - x) = 0 \). If \( ϕ(1 + x) = 0 \), then \( 0 = ϕ(1 + x) = 1 + ϕ(x) \).

Thus \( ϕ(x) = -1 \). But then \((-1)ϕ(i) = ϕ(xi) = ϕ(-ix) = -ϕ(i)(-1) = ϕ(i) \). Since \( ϕ(i) ≠ 0 \), then K has characteristic two. This is a contradiction because ϕ induces a homomorphism from \( R = S \) to \( K \) and 2 is invertible in \( R \). In the same way, it can be shown that if \( ϕ(1 - x) = 0 \), then \( ϕ(x) = 1 \) and, again, it implies that the characteristic of K is 2, a contradiction.

In Section 1 we introduce some of the notation that will be used throughout the paper and provide a short survey about the results on graded rings that will be used.

Let \( Γ \) be a group. A Γ-almost graded division ring is a (not necessarily graded) homomorphic image of a Γ-graded division ring. For example, let \( K \) be a field and consider the group ring \( K[Γ] \). It is a Γ-graded division ring, and the augmentation map \( K[Γ] → K \), which is not a homomorphism of Γ-graded rings, endows \( K \) with as structure of Γ-almost graded division ring. In the nongraded context, this concept is not necessary because a nontrivial image of a division ring is again a division ring. In Section 2 we show that if \( R \) is a Γ-graded ring, \( ϕ: R → D \) is a homomorphism of Γ-graded rings with \( D \) a Γ-graded division ring and \( ψ: D → E \) is a ring homomorphisms where \( E \) is a nonzero ring, then the homogeneous matrices over \( R \) that become invertible via \( ϕ \) and via \( ψϕ \) are the same. Thus (a posteriori) \( ψ(D) \) determines a Γ-graded epic \( R \)-division ring.

The main results in Section 3 are as follows. Let \( ϕ: R → D \) be a homomorphism of Γ-graded rings and let \( Σ \) be a set of square homogeneous matrices with entries in \( R \). Suppose that the matrices of \( Σ \) become invertible in \( D \) via \( ϕ \). Then, under certain natural conditions on \( Σ \), the entries of the inverses of the matrices in \( Σ \) are the homogeneous elements of a Γ-graded subring of \( R \). Moreover, if \( D \) is a Γ-graded division ring and \( Σ \) the set of homogeneous matrices that become invertible under \( ϕ \), then any homogeneous element of \( D \) is an entry of the inverse of some matrix in \( Σ \).

Section 4 begins showing that the universal localization \( R_Σ \) of the Γ-graded ring \( R \) at a set of homogeneous matrices is again a Γ-graded ring. Then it is shown that a homomorphism of Γ-graded rings \( ϕ: R → D \), where \( D \) is Γ-graded division ring, is an epimorphism in the category of Γ-graded rings if and only if \( D \) is generated by the image of \( ϕ \). If this is the case, we say that \((K, ϕ)\) is a Γ-graded epic \( R \)-division ring and we prove that if \( Σ \) is the set of square homogeneous matrices that become invertible in \( D \) via \( ϕ \), then \( R_Σ \) is a Γ-graded local ring with Γ-graded residue division ring \( R \)-isomorphic to \( D \). Then the concept of gr-specialization between Γ-graded epic \( R \)-division rings is defined. The section ends showing that the existence of a gr-specialization from \((K, ϕ)\) to another Γ-graded epic \( R \)-division ring \((K', ϕ')\) is equivalent to say that all the homogeneous rational expressions (from elements of \( R \)) that make sense in \((K', ϕ')\) make sense in \((K, ϕ)\) too, and that it is also equivalent to fact that any homogeneous matrix over \( R \) that becomes invertible in \((K', ϕ')\) becomes invertible in \((K, ϕ)\) too.

Section 5 is devoted to the proof of the graded version of the so called Malcolmson’s criterion [25] and an important consequence. This criterion determines the kernel of the natural homomorphism from \( R \) to the universal localization \( R_Σ \) of
At certain sets of homogeneous matrices. As a corollary one obtains a sufficient condition for the ring \( R \); not to be the zero ring. Both results play a key role in the following section, but the proof of Malcolmson’s criterion is very long and technical.

The concept of gr-prime matrix ideal is given in Section 6 and it is shown that the different \( \Gamma \)-graded epic \( R \)-division rings are determined by the gr-prime matrix ideals up to \( R \)-isomorphism of \( \Gamma \)-graded rings.

In Section 7 the concepts of a gr-matrix ideal and of the radical of a gr-matrix ideal are defined and it is characterized how is the gr-matrix ideal generated by a set of homogeneous square matrices. Then it is proved that gr-prime matrix ideals behave like prime ideals in a commutative ring. All these concepts are used to provide necessary and sufficient condition for the existence of homomorphisms (embeddings) of \( \Gamma \)-graded rings to \( \Gamma \)-graded division rings.

The basic theory of Sylvester rank functions in the graded context with values in \( \mathbb{N} \) is developed in Section 8. The main difference with the ungraded case stems from the fact that, in the graded case, the same homogeneous matrix can define more than one homomorphism between \( \Gamma \)-graded free modules. As far as we know, this is the first paper where Sylvester rank functions are considered for graded objects.

In Section 9 we deal with a new situation that appears in the graded context. If \( \Gamma \) is a group and \( R \) is a \( \Gamma \)-graded ring, then the ring \( R \) can be considered as a \( \Gamma/\Omega \)-graded ring for any normal subgroup \( \Omega \) of \( \Gamma \). Thus there are \( \Gamma \)-graded and \( \Gamma/\Omega \)-graded versions of the concepts studied before. In this section, we try to relate them. Note that when \( \Omega = \Gamma \), a \( \Gamma/\Omega \)-graded epic \( R \)-division ring is simply an \( R \)-division ring and thus one can relate the theory of \( \Gamma \)-graded division rings and the theory of division rings as developed by Cohn.

The last section is devoted to identify inverse limits in the category of \( \Gamma \)-graded epic \( R \)-division rings with specializations as morphisms with certain ultraprod ucts of \( \Gamma \)-graded epic \( R \)-division rings. In the context of division rings, a similar result was given in [21, Section 7], but our proof is more direct and general even when specialized to the ungraded case.

A second paper is in the works where we deal, among other topics, with the graded versions of weak algorithm, (semi)firs and (pseudo-)Sylveste r domains.

We would like to finish this introduction by pointing out that most of the techniques used in this paper are adaptations of the ones from the works by P. M. Cohn and P. Malcolmson. We just take credit for realizing that they can be applied in the more general setting of group graded rings.

### 1. Basic definitions and notation

Rings are supposed to be associative with 1. We recall that a domain is a nonzero ring such that for elements \( x, y \) of the ring, the equality \( xy = 0 \) implies that either \( x = 0 \) or \( y = 0 \). A division ring is a nonzero ring such that every nonzero element is invertible. For a ring \( R \), we define \( \mathbb{M}(R) \) the set of all square matrices of any size. Also, for each \( i \) with \( 1 \leq i \leq n \), let \( e_i \) denote the column

\[
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\]

in which the \( i \)-th entry is 1 and the other entries are zero.

Let \( A \in M_n(R) \). We say that \( A \) is full if whenever \( A = PQ \), with \( P \in M_{n \times r}(R) \) and \( Q \in M_{r \times n}(R) \), then \( r \geq n \). If we think of \( A \) as an endomorphism of the free (right) \( R \)-module \( R^n \), it means that \( A \) does not factor through \( R^r \) with \( r < n \). We
say that $A$ is hollow if it has an $r \times s$ block of zeros where $r + s > n$. It is well known that a hollow matrix is not full.

Let $S$ be a ring and $f : R \to S$ be a ring homomorphism. For each matrix $M$ with entries in $R$, we denote by $M^f$ the matrix whose entries are the images of the entries of $M$ by $f$, that is, if $a_{ij} \in R$ is the $(i, j)$-entry of $M$, then the $(i, j)$-entry of $M^f$ is $f(a_{ij})$. Given a set of matrices $\Sigma$, we denote $\Sigma^f = \{M^f : M \in \Sigma\}$. We say that the ring homomorphism $f : R \to S$ is $\Sigma$-inverting if the matrix $M^f$ is invertible in $S$ for each $M \in \Sigma$.

We proceed to give some basics on group graded rings that can be found in \cite{26} and \cite{12}, for example.

If $\Gamma$ is a group, the identity element of $\Gamma$ will be denoted by $e$.

Let $\Gamma$ be a group. A ring $R$ is called a $\Gamma$-graded ring if $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ where each $R_\gamma$ is an additive subgroup of $R$ and $R_\gamma R_\delta \subseteq R_{\gamma \delta}$ for all $\gamma, \delta \in \Gamma$. The support of $R$ is defined as the set $\text{supp } R = \{\gamma \in \Gamma : R_\gamma \neq \{0\}\}$. The set $h(R) = \bigcup_{\gamma \in \Gamma} R_\gamma$ is called the set of homogeneous elements of $R$. It is well known that the identity element $1 \in R$ belongs to $R_0$, that $R_0$ is a subring of $R$ and that if $x \in R_\gamma$ is invertible in $R$, then $x^{-1} \in R_{\gamma^{-1}}$. A (two-sided) ideal $I$ of $R$ is called a graded ideal if $I = \bigoplus (I \cap R_\gamma)$. Thus $I$ is a graded ideal if and only if for any $x \in I$, $x = \sum x_i$, where $x_i \in h(R)$, implies that $x_i \in I$. Observe that if $X \subseteq h(R)$, then the ideal of $R$ generated by $X$ is a graded ideal. If $I$ is a graded ideal, then the quotient ring $R/I$ is a $\Gamma$-graded ring with $R/I = \bigoplus_{\gamma \in \Gamma} (R/I)_\gamma$, where $(R/I)_\gamma = (R_\gamma + I)/I$.

A $\Gamma$-graded domain is a nonzero $\Gamma$-graded ring such that if $x, y \in h(R)$, the equality $xy = 0$ implies that either $x = 0$ or $y = 0$. A $\Gamma$-graded division ring is a nonzero $\Gamma$-graded ring such that every nonzero homogeneous element is invertible. A commutative $\Gamma$-graded division ring is a $\Gamma$-graded field. Clearly, any $\Gamma$-graded division ring is a $\Gamma$-graded domain.

A $\Gamma$-graded ring $R$ is called a $\Gamma$-graded local ring if the two-sided ideal $m$ generated by the noninvertible homogeneous elements is a proper ideal. In this case, the $\Gamma$-graded ring $R/m$ is a $\Gamma$-graded division ring and it will be called the residue class $\Gamma$-graded division ring of $R$.

For $\Gamma$-graded rings $R$ and $S$, a homomorphism of $\Gamma$-graded rings $f : R \to S$ is a ring homomorphism such that $f(R_\gamma) \subseteq S_\gamma$ for all $\gamma \in \Gamma$. An isomorphism of $\Gamma$-graded rings is a homomorphism of $\Gamma$-graded rings which is bijective. Notice that the inverse is also an isomorphism of $\Gamma$-graded rings.

Let $\Omega$ be a normal subgroup of $\Gamma$. Consider the $\Gamma$-graded ring $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$. It can be regarded as a $\Gamma/\Omega$-graded ring as follows

$$R = \bigoplus_{\alpha \in \Gamma/\Omega} R_\alpha, \quad \text{where } R_\alpha = \bigoplus_{\gamma \in \alpha} R_\gamma.$$

Let $R$ be a $\Gamma$-graded ring. A $\Gamma$-graded (right) $R$-module $M$ is defined to be a right $R$-module with a direct sum decomposition $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$, where each $M_\gamma$ is an additive subgroup of $M$ such that $M_\lambda M_\gamma \subseteq M_{\lambda \gamma}$ for all $\lambda, \gamma \in \Gamma$.

A submodule $N$ of $M$ is called a graded submodule if $N = \bigoplus_{\gamma \in \Gamma} (N \cap M_\gamma)$. In this case, the factor module $M/N$ forms a $\Gamma$-graded $R$-module with $M/N = \bigoplus_{\gamma \in \Gamma} (M/N)_\gamma$, where $(M/N)_\gamma = (M_\gamma + N)/N$.

For $\Gamma$-graded $R$-modules $M$ and $N$, a homomorphism of $\Gamma$-graded $R$-modules $f : M \to N$ is a homomorphism of $R$-modules such that $f(M_\gamma) \subseteq N_\gamma$ for all $\gamma \in \Gamma$. In this case, $\ker f$ is a graded submodule of $M$ and $\text{Im } f$ is a graded submodule of $N$. 


If $\Omega$ is a normal subgroup of $\Gamma$, then a $\Gamma$-graded $R$-module $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ can be regarded as a $\Gamma/\Omega$-graded over the $\Gamma/\Omega$-graded ring $R$ as follows

$$M = \bigoplus_{\alpha \in \Gamma/\Omega} M_\alpha, \quad \text{where } M_\alpha = \bigoplus_{\gamma \in \alpha} M_\gamma.$$  

Moreover, a homomorphism of $\Gamma$-graded $R$-modules is also a homomorphism of $\Gamma/\Omega$-graded $R$-modules.

Let $\{M_i: i \in I\}$ be a set of $\Gamma$-graded $R$-modules. Then $\bigoplus_{i \in I} M_i$ has a natural structure of $\Gamma$-graded $R$-module given by $\bigoplus_{i \in I} M_i\gamma$.

Let $M$ be a $\Gamma$-graded right $R$-module and $N$ be a $\Gamma$-graded left $R$-module. Then the tensor product $M \otimes_R N$ has a natural structure of $\Gamma$-graded $R$-module where $(M \otimes_R N)\gamma = \{\sum m_i \otimes n_i; m_i \in M_\gamma', n_i \in N_\gamma''; \gamma'\gamma'' = \gamma\}$.

Let $M$ be a $\Gamma$-graded $R$-module. For $\delta \in \Gamma$, we define the $\delta$-shifted $\Gamma$-graded $R$-module $M(\delta)$ as

$$M(\delta) = \bigoplus_{\gamma \in \Gamma} M(\delta)\gamma, \quad \text{where } M(\delta)\gamma = M_{\delta\gamma}.$$  

A $\Gamma$-graded $R$-module $F$ is called a $\Gamma$-graded free $R$-module if $F$ is a free $R$-module with a homogeneous basis. It is well known that the $\Gamma$-graded free $R$-modules are of the form

$$\bigoplus_{i \in I} R(\delta_i), \quad \text{where } I \text{ is an indexing set and } \delta_i \in \Gamma.$$  

If $I = \{1, \ldots, n\}$, then $\bigoplus_{i \in I} R(\delta_i) = R(\delta_1) \oplus \cdots \oplus R(\delta_n)$, will also be denoted by $R^n(\overrightarrow{\delta})$ where $\overrightarrow{\delta} = (\delta_1, \ldots, \delta_n) \in \Gamma^n$.

A $\Gamma$-graded $R$-module $P$ is called a $\Gamma$-graded projective module if for any diagram of $\Gamma$-graded $R$-modules and homomorphisms of $\Gamma$-graded modules

$$\begin{array}{ccc}
P & \xrightarrow{b} & M \\
\downarrow{h} & & \downarrow{g} \\
0 & & N
\end{array}$$

there is a graded $R$-module homomorphism $h: P \to M$ with $gh = u$. As in the ungraded case, the following statements are equivalent ways of saying that $P$ is a $\Gamma$-graded projective module

1. $P$ is $\Gamma$-graded and projective as an $R$-module.
2. Every short exact sequence of homomorphisms of $\Gamma$-graded $R$-modules $0 \to L \to M \to P \to 0$ splits via a homomorphism of $\Gamma$-graded $R$-modules.
3. $P$ is isomorphic, as $\Gamma$-graded $R$-module, to a direct summand of a $\Gamma$-graded free $R$-module.

Let $P$ be a $\Gamma$-graded projective $R$-module and let $\Omega$ be a normal subgroup of $\Gamma$. If we regard $P$ as a $\Gamma/\Omega$-graded $R$-module, then $P$ is also projective as a $\Gamma/\Omega$-graded $R$-module.

Let $\Gamma$ be a group and $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ be a $\Gamma$-graded ring. Following [12], for $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \Gamma^m$ and $\overrightarrow{\beta} = (\beta_1, \ldots, \beta_n)$, set

$$M_{m \times n}(R)[\overrightarrow{\alpha}][\overrightarrow{\beta}] = \begin{pmatrix}
R_{\alpha_1\beta_1^{-1}} & R_{\alpha_1\beta_2^{-1}} & \cdots & R_{\alpha_1\beta_n^{-1}} \\
R_{\alpha_2\beta_1^{-1}} & R_{\alpha_2\beta_2^{-1}} & \cdots & R_{\alpha_2\beta_n^{-1}} \\
\vdots & \vdots & \ddots & \vdots \\
R_{\alpha_m\beta_1^{-1}} & R_{\alpha_m\beta_2^{-1}} & \cdots & R_{\alpha_m\beta_n^{-1}}
\end{pmatrix}.$$
That is $M_{m \times n}(R)[\alpha][\beta]$ consists of the matrices whose $(i, j)$-entry belongs to $R_{\alpha_i \beta_j}^{-1}$.

Such a matrix $A \in M_{m \times n}(R)[\alpha][\beta]$ gives a homomorphism of $\Gamma$-graded rings

$$R^n(\beta) \to R^m(\alpha), \quad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

and in this way $M_{m \times n}(R)[\alpha][\beta]$ can be identified with the set of all homomorphisms of $\Gamma$-graded $R$-modules $R^n(\beta) \to R^m(\alpha)$.

By $A \in \mathfrak{M}_{m \times n}(R)$, we mean that $A \in M_{m \times n}(R)[\alpha][\beta]$ of some $\alpha, \beta \in \Gamma^n$. It is important to note that, for a matrix $A \in \mathfrak{M}_{m \times n}(R)$, it is possible that $A \in M_{m \times n}(R)[\alpha][\beta] \cap M_{m \times n}(R)[\alpha'][\beta']$ even if $\alpha \neq \alpha'$ or $\beta \neq \beta'$. The matrix $A$ belongs to that intersection if whenever the $(i, j)$-entry of $A$ is not zero, then $\alpha_i \beta_j^{-1} = \alpha'_i \beta'_j^{-1}$.

We set

$$\mathfrak{M}_n(R) = \bigcup_{m,n} \mathfrak{M}_{m \times n}(R).$$

We remark that if $A \in M_{m \times n}(R)[\alpha][\beta]$ and $B \in M_{m \times n}(R)[\beta'][\varepsilon]$ then $AB \in M_{m \times n}(R)[\alpha][\varepsilon]$. We will say that $A, B$ are compatible.

When $m = n$, we will write $M_n(R)[\alpha][\beta]$ and $\mathfrak{M}_n(R)$. The set of all such matrices will be denoted by $\mathfrak{M}(R)$, that is,

$$\mathfrak{M}(R) = \bigcup_n \mathfrak{M}_n(R).$$

If $A \in M_n(R)[\alpha][\beta]$ is an invertible matrix, then $A^{-1} \in M_n(R)[\beta][\alpha]$.

If $\Sigma \subseteq \mathfrak{M}(R)$, we will write $\Sigma_n[\alpha][\beta]$ to denote the set $\Sigma \cap M_n(R)[\alpha][\beta]$.

A matrix $A \in \mathfrak{M}_n(R)$ is gr-full if every time that $A = PQ$ for some matrices $P \in M_{n \times r}(\alpha)[\gamma], Q \in M_{r \times n}(\beta)[\delta], r \geq n$. If we think of $A$ as a homomorphism of $\Gamma$-graded modules between two $\Gamma$-graded free $R$-modules, it means that for all $\alpha, \beta \in \Gamma^n$, such that $A$ defines a graded homomorphism $R^n(\beta) \to R^m(\alpha)$, then it never factors by any graded homomorphism $R^n(\beta) \to R^r(\gamma)$ with $r < n$.

Suppose that $A \in M_n(R)[\alpha][\beta]$, $E \in M_n(R)$ is a permutation matrix obtained permuting the rows of $I_n$ according to the permutation $\sigma \in S_n$. Then $E \in M_n(R)[\sigma][\sigma]$, where $\sigma = (\alpha(1), \ldots, \alpha(n))$, and $EA \in M_n(R)[\sigma'][\beta]$. Similarly the matrix $E \in M_n(R)[\beta'][\delta']$, where $\delta' = (\beta'(1), \ldots, \beta'(n))$, and $AE \in M_n(R)[\alpha][\delta']$. Hence, for permutation matrices $E, F$ of appropriate size, a matrix $A \in \mathfrak{M}(R)$ is gr-full if, and only if, $EAF$ is gr-full.

A hollow matrix $A \in \mathfrak{M}(R)$ is not gr-full. Indeed, suppose that $A$ has an $r \times s$ block of zeros. There exist permutation matrices $E, F$ such that $EAF = \begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix}$, that is, the block of $r \times s$ zeros is in the north-east corner. Then

$$\begin{pmatrix} T & 0 \\ U & V \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ U & V \end{pmatrix},$$

where $T \in M_{r \times (n-s)}(R)[\alpha][\gamma], U \in M_{(n-r) \times (n-s)}(R)[\beta][\delta], V \in M_{(n-r) \times n}(R)[\delta][\varepsilon]$ for some sequences $\alpha, \beta, \gamma$ of elements of $\Gamma$. The result now follows because $\begin{pmatrix} T & 0 \\ 0 & V \end{pmatrix} \in M_{r \times (n-s)}(R)[\alpha][\gamma], \begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix} \in M_{(n-r) \times (n-s)}(R)[\beta][\delta], \begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix} \in M_{(n-r) \times n}(R)[\delta][\varepsilon]$.

Let $D$ be a $\Gamma$-graded division ring and $M$ be a $\Gamma$-graded $D$-module. As in the ungraded case, the following assertions hold true

1. Any $\Gamma$-graded $D$-module is graded free.
2. Any $D$-linearly independent subset of $M$ consisting of homogeneous elements can be extended to a homogeneous basis of $M$.
3. Any two homogeneous bases of $M$ over $D$ have the same cardinality.
4. If $N$ is a $\Gamma$-graded submodule of $M$, then $\dim_D(N) + \dim_D(M/N) = \dim_D(M)$. 
We remark that, over a $\Gamma$-graded division, the concepts of gr-full matrix and of invertible matrix coincide.

Let $D$ be a $\Gamma$-graded division ring. Let $A \in \mathfrak{M}_{m \times n}(D)$. The elementary homogeneous row (column) operations on $A$ are

1. Interchange two rows (columns) of $A$.
2. Multiply a row on the left (a column on the right) by a nonzero homogeneous element.
3. Suppose that $A \in M_{m \times n}(D)[\mathcal{O}][\mathcal{F}]$. Multiply row $i$ on the left by an element of $R_{\alpha_i\alpha_i^{-1}}$ and add the result to row $j$ (multiply column $i$ on the right by an element of $R_{\beta_i\beta_i^{-1}}$ and add the result to column $j$).

Notice that these three operations on the rows (columns) can be obtained multiplying $A$ on the left (right) by an invertible matrix in $M_m(D)[\mathcal{O}][\mathcal{F}]$ (in $M_n(D)[\mathcal{O}][\mathcal{F}]$).

The rank of $A$ is the dimension of the right $D$-module spanned by its columns. The matrix $A$ can be regarded as a $D$-linear map of right $D$-modules $R(\beta_1) \oplus \cdots \oplus R(\beta_n) \rightarrow R(\alpha_1) \oplus \cdots \oplus R(\alpha_n)$. The rank of $A$ coincides with the dimension of the image of $A$. The rank of $A$ can also be computed reducing the matrix $A$ to column echelon form by homogeneous column operations. It is the number of nonzero columns of the column echelon form.

The rank of $A$ equals also the dimension of the left free $D$-module spanned by its rows. The matrix $A$ can be regarded as a $D$-linear map of left $D$-modules $R(\alpha_1^{-1}) \oplus \cdots \oplus R(\alpha_m^{-1}) \rightarrow R(\beta_1^{-1}) \oplus \cdots \oplus R(\beta_n^{-1})$. The rank of $A$ coincides with the dimension of the image of $A$. The rank of $A$ can also be computed reducing the matrix $A$ to row echelon form by homogeneous row operations. It is the number of nonzero rows of the row echelon form.

Furthermore, the rank of $A$ coincides with the size of a largest invertible square submatrix (obtained by eliminating rows and/or columns). We will denote the rank of $A$ by $\text{rank}(A)$.

2. Almost graded division rings

Throughout this section, let $\Gamma$ be a group.

We say that a ring $R$ is a $\Gamma$-almost graded ring if there is a family $\{R_\gamma : \gamma \in \Gamma\}$ of additive subgroups $R_\gamma$ of $R$ such that $1 \in R_e$, $R = \sum_{\gamma \in \Gamma} R_\gamma$, and $R_\gamma R_{\gamma'} \subseteq R_{\gamma \gamma'}$, for all $\gamma, \gamma' \in \Gamma$. The name of almost graded rings was chosen to be compatible with the definition of almost strongly graded rings given in [26] p.14. We define $\text{supp} R = \{\gamma \in \Gamma : R_\gamma \neq \{0\}\}$. Given two $\Gamma$-almost graded rings $R$ and $S$, a ring homomorphism $f : R \rightarrow S$ is a homomorphism of $\Gamma$-almost graded rings if $f(R_\gamma) \subseteq S_\gamma$, for all $\gamma \in \Gamma$. Clearly, any $\Gamma$-graded ring $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ is a $\Gamma$-almost graded ring in the natural way. Given two $\Gamma$-graded rings $R, S$, a homomorphism of $\Gamma$-almost graded rings is in fact a homomorphism of $\Gamma$-graded rings.

Let $R$ be a $\Gamma$-graded ring, $S$ be ring and $f : R \rightarrow S$ be a ring homomorphism. Then $\text{Im} f$ is a $\Gamma$-almost graded ring with $(\text{Im} f)_\gamma = f(R_\gamma)$ and the restriction $f : R \rightarrow \text{Im} f$ is a homomorphism of $\Gamma$-almost graded rings. Furthermore, any $\Gamma$-almost graded ring can be regarded in this way. More precisely, suppose that $R = \sum_{\gamma \in \Gamma} R_\gamma$ is a $\Gamma$-almost graded ring. Set $\tilde{R}_\gamma$ to be a disjoint copy of $R_\gamma$. If $a \in R_\gamma$, denote by $\tilde{a} \in \tilde{R}_\gamma$ the disjoint copy of $a \in R_\gamma$. Consider the $\Gamma$-graded additive group $\tilde{R} = \bigoplus_{\gamma \in \Gamma} \tilde{R}_\gamma$. Define $\tilde{R}_\gamma \times \tilde{R}_{\gamma'} \rightarrow \tilde{R}_{\gamma \gamma'}$ by $\tilde{(a,b)} \mapsto \tilde{a} \tilde{b}$, and extend it by distributivity to $\tilde{R} \times \tilde{R} \rightarrow \tilde{R}$. This endows $\tilde{R}$ with a structure of $\Gamma$-graded
ring such that $\text{supp} \, \tilde{R} = \text{supp} \, R$ and $\varphi: \tilde{R} = \bigoplus_{\gamma \in \Gamma} \tilde{R}_{\gamma} \to R = \bigoplus_{\gamma \in \Gamma} R_{\gamma}$ determined by $\tilde{a} \mapsto a$ for all $a \in R_{\gamma}$, $\gamma \in \Gamma$, is a homomorphism of $\Gamma$-almost graded rings such that $\varphi(\tilde{R}_{\gamma}) = R_{\gamma}$. $\text{Im} \varphi = R$.

Another important example is as follows. If $S = \bigoplus_{a \in \Gamma} S_{a}$ is a $\Gamma/\Omega$-graded ring, then $S$ can be endowed with a structure of $\Gamma$-almost graded ring defining $S_{\gamma} = S_{a}$ for all $\gamma \in a$. Suppose that $R = \bigoplus_{\gamma \in \Gamma} R_{\gamma}$ is a $\Gamma$-graded ring and that $\Omega$ is a normal subgroup of $\Gamma$. The ring $\tilde{R}$ is a $\Gamma/\Omega$-graded ring defining $R_{a} = \bigoplus_{\gamma \in a} R_{\gamma}$ for each $a \in \Gamma/\Omega$. If $f: \tilde{R} \to S$ is a homomorphism of $\Gamma/\Omega$-graded rings, then it is a homomorphism of $\Gamma$-almost graded rings. Let $\Gamma'$ be the subgroup of $\Gamma$ generated by $\text{supp} \, R$. Observe that if $\Omega$ is a normal subgroup of $\Gamma'$ (instead of $\Gamma$), $S$ is a $\Gamma'/\Omega$-graded ring and $f: \tilde{R} \to S$ a homomorphism of $\Gamma'/\Omega$-graded rings, then $f$ is a homomorphism of $\Gamma'$-almost graded rings.

We say that a nonzero ring $E$ is a $\Gamma$-almost graded division ring if $E$ is a $\Gamma$-almost graded ring such that every nonzero element $x \in E_{\gamma}$, $\gamma \in \Gamma$, is invertible with inverse $x^{-1} \in E_{1-\gamma}$. Note that if $E$ is a $\Gamma$-almost graded division ring, then $\tilde{E}$ is a $\Gamma$-graded division ring.

The following easy result tells us that $\Gamma$-almost graded division rings are graded division rings although not necessarily of type $\Gamma$.

**Lemma 2.1.** Let $E$ be a $\Gamma$-almost graded division ring. The following assertions hold true.

1. If $0 \neq b \in E_{\gamma}$, then $bE_{\gamma'} = E_{\gamma + \gamma'}$ and $E_{\gamma}b = E_{\gamma + \gamma'}$ for $\gamma \in \Gamma$.
2. $E_{\gamma} \cdot E_{\gamma'} = E_{\gamma + \gamma'}$ for all $\gamma, \gamma' \in \Gamma$.
3. $\text{supp} \, E$ is a subgroup of $\Gamma$.
4. There exists a normal subgroup $N$ of $\text{supp} \, E$ such that $E$ is a $\text{supp} \, E/N$-graded division ring.

**Proof.** If $u \in E_{\gamma + \gamma'}$, then $b^{-1}u = u$ where $b^{-1}u \in E_{\gamma}$. The other part is analogous. Thus (1) is proved.

(2) is a consequence of (1).

Since $1 \in E_{e}$, then (3) follows from (2).

(4) First note that, for each $\gamma \in \Gamma$, the condition $E_{\gamma} \cap E_{e} \neq \{0\}$ implies that $E_{\gamma} = E_{e}$. Indeed, if $0 \neq b \in E_{\gamma} \cap E_{e}$, (1) implies that $E_{\gamma} = bE_{e} = E_{e}$.

Define $N = \{\gamma \in \Gamma: E_{\gamma} = E_{e}\}$. We show that $N$ is a normal subgroup of $\text{supp} \, \Gamma$.

Clearly $e \in N$. If $\gamma, \gamma' \in N$, then $E_{\gamma + \gamma'} = E_{\gamma}E_{\gamma'} = E_{e}E_{e} = E_{e}$. Now, if $\gamma \in N$, then $E_{\gamma} = E_{e}E_{\gamma^{-1}} = E_{e}E_{\gamma^{-1}}$. Thus $\gamma^{-1} \in N$. Suppose $\gamma \in N$ and $\sigma \in \text{supp} \, \Gamma$. Then $E_{\gamma \sigma^{-1}} = E_{\gamma}E_{\sigma^{-1}} = E_{e}E_{\sigma^{-1}} = E_{e}$. Thus $\sigma \gamma \sigma^{-1} \in N$.

Now let $\gamma, \gamma' \in \text{supp} \, \Gamma$. Then

$$\gamma^{-1} \gamma' \in N \Leftrightarrow E_{\gamma^{-1} \gamma'} = E_{e} \Leftrightarrow E_{\gamma} = E_{\gamma'}.$$

And the result is proved. □

Let $R$ be a $\Gamma$-graded ring, $S$ be a ring and $f: R \to S$ be a ring homomorphism. For each $\gamma \in \Gamma$, define

$$(S_{0})_{\gamma} = f(R_{\gamma}).$$

If $n \geq 0$, and $(S_{n})_{\gamma}$ has been defined for each $\gamma \in \Gamma$, define

$$(T_{n+1})_{\gamma} = \{y^{-1}: y \in (S_{n})_{\gamma^{-1}} \text{ and } y \text{ is invertible in } S\},$$

$$(S_{n+1})_{\gamma} = \text{Additive subgroup of } S \text{ generated by } \gamma,$$

$$\{x_{1}x_{2} \cdots x_{r}: r \in \mathbb{N}, x_{i} \in (S_{n})_{\gamma_{i}} \cup (T_{n+1})_{\gamma_{i}}, \gamma_{1}\gamma_{2} \cdots \gamma_{n} = \gamma\}.$$
Now set \((\text{DC}(f))_\gamma = \text{Subgroup generated by } \bigcup_{n \geq 0} (S_n)_\gamma\). Then the subring of \(S\) defined by

\[
\text{DC}(f) = \text{Additive subgroup generated by } \bigcup_{\gamma \in \Gamma} (\text{DC}(f))_\gamma
\]

is the \textit{almost graded division closure of } \(f: R \to S\). Note that \(\text{DC}(f)\) is a \(\Gamma\)-almost graded ring such that if \(x \in (\text{DC}(f))_\gamma\) and \(x\) is invertible in \(S\), then \(x^{-1} \in (\text{DC}(f))_{\gamma^{-1}}\). It is the least subring of \(S\) that contains \(\text{Im} f\) and is closed under inversion of almost homogeneous elements.

If \(\text{DC}(f) = S\) and \(\text{DC}(f)\) is a \(\Gamma\)-almost graded division ring, we say that \(S\) is the \(\Gamma\)-almost graded division ring generated by \(\text{Im} f\).

Note also that if \(S\) is a division ring, then \(\text{DC}(f)\) is a \(\Gamma\)-almost graded division ring.

Note that if \(S\) is a \(\Gamma\)-graded ring, and \(f: R \to S\) is a homomorphism of \(\Gamma\)-graded rings, then \((S_n)_\gamma \subseteq S_\gamma\) for each \(n \geq 0\). Therefore \((\text{DC}(f))_\gamma \subseteq S_\gamma\) and \(\text{DC}(f)\) is a \(\Gamma\)-graded subring of \(S\). It is the least subring of \(S\) that contains \(\text{Im} f\) and is closed under inversion of homogeneous elements. Moreover if \(S\) is a \(\Gamma\)-graded division ring, then \(\text{DC}(f)\) is a \(\Gamma\)-graded division subring of \(S\). In this case, if \(S = \text{DC}(f)\) we say that \(S\) is the \(\Gamma\)-graded division ring generated by \(\text{Im} f\).

**Proposition 2.2.** Let \(\Gamma\) be a group, \(D = \bigoplus_{\gamma \in \Gamma} D_\gamma\) be a \(\Gamma\)-graded division ring, and let \(f: D \to S\) be a ring homomorphism with \(S\) a nonzero ring. The following assertions hold true.

1. \(\text{DC}(f)\) is a \(\Gamma\)-almost graded division ring with

   \[
   (\text{DC}(f))_\gamma = (\text{Im} f)_\gamma = \{f(x) : x \in D_\gamma\}
   \]

   and \(D = \widetilde{\text{DC}(f)}\).

2. The sets

   \[
   \Upsilon = \{A \in M(D) : A \text{ is invertible over } D\},
   \Sigma = \{A \in M(D) : A^f \text{ is invertible over } S\}
   \]

   coincide.

3. If \(R\) is a \(\Gamma\)-graded ring and \(\varphi: R \to D\) is a homomorphism of \(\Gamma\)-graded rings then the sets

   \[
   \Upsilon_\varphi = \{A \in M(R) : A^\varphi \text{ is invertible over } D\},
   \Sigma_\varphi = \{A \in M(R) : A^{f\varphi} \text{ is invertible over } S\}
   \]

   coincide.

**Proof.** (1) has already been proved.

(2) Clearly, if \(A \in \Upsilon\), then \(A \in \Sigma\). Suppose now that \(A \in M_n(R)[\bar{\gamma}][\delta]\) such that \(A \notin \Upsilon\). Then there exists a nonzero homogeneous column \(
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{pmatrix}
\)

such that \(A^f \begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{pmatrix} = 0\). Note that \(\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{pmatrix} \neq 0\) because \(D\) is a graded division ring and \(S\) is not the zero ring. Thus \(A^f \begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{pmatrix} = 0\), which implies that \(A \notin \Sigma\).

(3) follows from (2) because \(\Sigma_\varphi = \{A \in M(R) : A^\varphi \in \Sigma\}\) and \(\Upsilon_\varphi = \{A \in M(R) : A^{f\varphi} \in \Upsilon\}\). \(\square\)
3. Graded rational closure

Throughout this section, let $\Gamma$ be a group.

We begin this section introducing some important notation that will be used throughout.

Let $\mathbf{\pi} = (\alpha_1, \ldots, \alpha_n) \in \Gamma^n$, $\mathbf{\pi'} = (\alpha'_1, \ldots, \alpha'_m) \in \Gamma^m$ and $\delta \in \Gamma$, then we define

$$\mathbf{\pi} \ast \mathbf{\pi'} := (\alpha_1, \ldots, \alpha_n, \alpha'_1, \ldots, \alpha'_m) \in \Gamma^{n+m}.$$  

Let $R$ be a $\Gamma$-graded ring and $S$ be a ring.

For each sequence $\mathbf{\pi} = (\alpha_1, \ldots, \alpha_n)$ $\in \Gamma^n$, the last element $\alpha_n$ will be denoted $\alpha_\infty$, and $(\alpha_1, \ldots, \alpha_{n-1})$ will be denoted by $\alpha_*$. Thus $\alpha = \alpha_* \ast \alpha_\infty$

For $u \in M_{n \times 1}(S)$, the last entry of $u$ will be denoted by $u_\infty$ and the $(n-1) \times 1$ column consisting of the remaining entries will be denoted by $u_*$. We will write $A$ as desired.

We remark that if $n = 1$, then $A_*, \alpha_*, u_*$ are empty and thus $A = A_\infty$, $\alpha = \alpha_\infty$, and $u = u_\infty$.

If $A \in \mathfrak{M}_{n \times (n+1)}(R)$, we will denote by $A_0$ its first column, by $A_\infty$ its last column and by $A_*$ the matrix consisting of the other $n-1$ columns, that is, we will write $A = (A_0 \ A_* \ A_\infty)$. We will call the matrix $(A_0 \ A_*)$ the numerator of $A$ and the matrix $(A_* \ A_\infty)$ the denominator of $A$. If $A \in M_{m \times (n+1)}(R)[\mathbf{\pi}][\mathbf{\beta}]$, we suppose $\beta$ is divided as $\beta_0 \ast \beta_* \ast \beta_\infty$. If $u \in M_{(n+1) \times 1}(S)$, we will write $u = \left(\begin{array}{c} u_0 \\ u_* \\ u_\infty \end{array}\right)$.

Again, we remark that if $n = 1$, then $A_*, \beta_*, u_*$ are empty and thus $A = (A_0 \ A_\infty)$, $\beta = (\beta_0 \ \beta_\infty)$ and $u = \left(\begin{array}{c} u_0 \\ u_* \\ u_\infty \end{array}\right)$.

Let $R = \bigoplus_{\gamma \in \Gamma} R_{\gamma}$ be a $\Gamma$-graded ring and $\Sigma \subseteq \mathfrak{M}(R)$.

We say that the subset $\Sigma$ of $\mathfrak{M}(R)$ is $gr$-lower semimultiplicative if it satisfies the following two conditions:

(i) $1 \in \Sigma$, i.e. the identity matrix of size $1 \times 1$ belongs to $\Sigma$.
(ii) If $A \in \Sigma_n[\mathbf{\pi}][\mathbf{\beta}]$ and $B \in \Sigma_m[\mathbf{\pi}'][\mathbf{\beta}']$, then the matrix $\left(\begin{array}{cc} A & 0 \\ C & B \end{array}\right) \in \Sigma$ for any $C \in M_{m \times n}(R)[\mathbf{\pi}][\mathbf{\pi'}][\mathbf{\beta}][\mathbf{\beta'}]$.

An gr-upper semimultiplicative subset of $\mathfrak{M}(R)$ is defined analogously.

A subset $\Sigma$ of $\mathfrak{M}(R)$ is $gr$-multiplicative if it satisfies the following two conditions

(i) $\Sigma$ is lower gr-semimultiplicative.
(ii) If $A \in \Sigma$, then $EAF \in \Sigma$ for any permutation matrices $E, F$ of appropriate size.

Remark 3.1. We remark that if $\Sigma$ is $gr$-multiplicative then it is also an upper gr-semimultiplicative subset of $\mathfrak{M}(R)$. Indeed, suppose that $A \in \Sigma_n[\mathbf{\pi}][\mathbf{\beta}]$, $B \in \Sigma_m[\mathbf{\pi}'][\mathbf{\beta}']$ and $C \in M_{m \times n}(R)[\mathbf{\pi}][\mathbf{\pi'}][\mathbf{\beta}][\mathbf{\beta'}]$. Then, since $\Sigma$ is lower gr-semimultiplicative, $\left(\begin{array}{c} B \\ A \end{array}\right) \in \Sigma$. But now $\left(\begin{array}{cc} A & 0 \\ C & B \end{array}\right) = E^{-1} \left(\begin{array}{cc} B & 0 \\ C & A \end{array}\right) E \in \Sigma$ for some permutation matrix $E$, as desired.

Proposition 3.2. Let $R$ be a $\Gamma$-graded ring, $S$ be a ring and $f : R \rightarrow S$ be a ring homomorphism. Then the set

$$\Sigma = \{M \in \mathfrak{M}(R) : M' \text{ is invertible over } S\}$$

is $gr$-multiplicative.
Proof. Clearly the $1 \times 1$ matrix $(1) \in \Sigma$.

Let $A \in \Sigma_m[\alpha][\beta]$, $B \in \Sigma_m[\gamma][\beta]$ and $C \in M_{m,n}[\alpha]$. Then the matrix
\[
\begin{pmatrix}
A & 0 \\
C & B
\end{pmatrix}
\]
belongs to $\Sigma$ because it is invertible with inverse
\[
\begin{pmatrix}
(A^f)^{-1} & 0 \\
-(B^f)^{-1}(C^f)(A^f)^{-1} & (B^f)^{-1}
\end{pmatrix}.
\]
Notice that if $E, F$ are permutation matrices, then $E^f, F^f$ are also permutation matrices. Hence if $A \in \Sigma$, then $(EAF)^f$ is invertible with inverse $(F^f)^{-1}(A^f)^{-1}(E^f)^{-1}$.

Note that if $S$ is a $\Gamma$-graded ring, $f: R \to S$ is a graded homomorphism and $A \in M_n(R)[\alpha][\beta]$, then $A^f \in M_n(S)[\alpha][\beta]$. Moreover, if $A^f$ is invertible, then $(A^f)^{-1} \in M_n(S)[\alpha][\beta]$, and the $(j, i)$-entry of $(A^f)^{-1}$ belongs to $R_{\beta_j\alpha_i}$. With this in mind, we make the following definition.

Let $R = \bigoplus_{\gamma \in \Gamma} R_{\gamma}$ be a $\Gamma$-graded ring and $\Sigma \subseteq \mathfrak{M}(R)$. Let $S$ be a ring (not necessarily graded) and $f: R \to S$ be a $\Sigma$-inverting ring homomorphism. Fix $\gamma \in \Gamma$. We define the homogeneous rational closure of degree $\gamma$ as the set $(Q_f(\Sigma))_\gamma$, consisting of all $x \in S$ such that there exist $\alpha, \beta \in \Gamma^n$ and $A \in \Sigma_n[\alpha][\beta]$ such that $\gamma = (\alpha, \beta)_\gamma^i = \beta_j\alpha_i$ and $x$ is the $(j, i)$-entry of $(A^f)^{-1}$ (for some positive integer $n$ and $i, j \in \{1, \ldots, n\}$). The homogeneous rational closure is the set
\[
Q_f(\Sigma) = \bigcup_{\gamma \in \Gamma} (Q_f(\Sigma))_\gamma.
\]
The graded rational closure, denoted by $R_f(\Sigma)$, is the additive subgroup of $S$ generated by $Q_f(\Sigma)$.

When the set $\Sigma$ is gr-lower semimultiplicative, the graded rational closure $R_f(\Sigma)$ is a subring of $S$ as the following results show.

**Lemma 3.3.** Let $R$ be a $\Gamma$-graded ring and $\Sigma$ be a $\Gamma$-lower semimultiplicative subset of $\mathfrak{M}(R)$. Let $S$ be a ring and $f: R \to S$ be a $\Sigma$-inverting ring homomorphism. Fix $\gamma \in \Gamma$. For $x \in S$, the following conditions are equivalent.

1. $x \in (Q_f(\Sigma))_\gamma$.
2. There exist $\alpha, \beta \in \Gamma^n$ and $A \in \Sigma_n[\alpha][\beta]$ such that $\alpha_i = e$, $\beta_j = \gamma$ and $x$ is the $(j, i)$-entry of $(A^f)^{-1}$.
3. There exist $\alpha, \beta \in \Gamma^n$, $A \in \Sigma_n[\alpha][\beta]$ and $u \in M_{n \times 1}(S)$ such that $\alpha_i = e$, $\beta_j = \gamma$, $u_j = x$ and $A^f u = e_i$.
4. There exist $\alpha, \beta \in \Gamma^n$, $A \in \Sigma_n[\alpha][\beta]$, $\alpha \in M_{n \times 1}(R)[\gamma][\alpha]$ and $u \in M_{n \times 1}(S)$ such that $\beta_j = \gamma$, $u_j = x$ and $A^f u = e_i$.
5. There exist $\alpha, \beta \in \Gamma^n$, $A \in \Sigma_n[\alpha][\beta]$, $\alpha \in M_{n \times 1}(R)[\gamma][\alpha]$ and $u \in M_{n \times 1}(S)$ such that $\beta_\infty = \gamma$, $u_\infty = x$ and $A^f u = e_i$.
6. There exist $\alpha, \beta \in \Gamma^n$, $A \in \Sigma_n[\alpha][\beta]$, $b \in M_{1 \times n}(R)[\gamma][\beta]$ and $c \in M_{n \times 1}(R)[\gamma][\alpha]$ such that $x = b \cdot (A^f)^{-1} e_i$.
7. There exist $\alpha, \beta \in \Gamma^n$, $A \in \Sigma_n[\alpha][\beta]$ such that $x$ is the $(j, i)$-entry of $(A^f)^{-1}$ and there exist $i, j$ such that $\gamma = (\alpha_i, \beta_j)^{-1} = \beta_j\alpha_i$. Then $A$ can be regarded as a matrix in $A \in \Sigma_n[\alpha \cdot \alpha^{-1}][\beta \cdot \alpha^{-1}]$ and thus (2) follows.

Proof. (1)$\Rightarrow$(2) Let $A \in \Sigma_n[\alpha][\beta]$ such that $x$ is the $(j, i)$-entry of $(A^f)^{-1}$ and there exist $i, j$ such that $\gamma = (\alpha_i, \beta_j)^{-1} = \beta_j\alpha_i$. Then $A$ can be regarded as a matrix in $A \in \Sigma_n[\alpha \cdot \alpha^{-1}][\beta \cdot \alpha^{-1}]$ and thus (2) follows.

(2)$\Rightarrow$(3) Suppose that (2) holds. Let $u$ be the $i$th column of $(A^f)^{-1}$. Then $A^f u = e_i$, as desired.

(3)$\Rightarrow$(4) It is clear because $e_i \in M_{n \times 1}(R)[\gamma][\alpha]$ and $e_i^f = e_i$. 

(4)⇒(5) Let \( A \in \Sigma, \ i, j, a, b, c \) and \( u \) be as in (4). Suppose that \( A^f u = a^f \) with \( u_j = x \). The matrix \( \begin{pmatrix} A^f & 0 \\ -c^f_j & 1 \end{pmatrix} \in \Sigma_n[\overline{\alpha} * \beta_j][\overline{\beta} * \beta_j] \). Notice that it belongs to \( \Sigma \) because \( \Sigma \) is gr-lower semimultiplicative. The matrix \( \begin{pmatrix} u \\ 0 \end{pmatrix} \in M_{(n+1) \times 1}(R)[\overline{\alpha} * \beta_j][\overline{\beta}] \). Now (5) follows from the following equality
\[
\begin{pmatrix} A^f & 0 \\ -c^f_j & 1 \end{pmatrix} \begin{pmatrix} u \\ 0 \end{pmatrix} = (\alpha^f \cdot \beta_j)^{-1} = (a^f \cdot \beta_j)^{-1}.
\]

(5)⇒(6) From (5) we obtain that \( u = (A^f)^{-1}a^f \). Hence
\[
x = (e^f_n)^f u = (e^f_n)^f (A^f)^{-1}a^f.
\]
Now (6) follows because \( e_n^f \in M_{1 \times n}(R)[\gamma][\overline{\beta}] \).

(6)⇒(1) Let \( A, b, c \) and \( e \) as in (6). Then
\[
\begin{pmatrix} 1 & 0 & 0 \\ c & A & 0 \\ 0 & b & 1 \end{pmatrix} \in \Sigma_{(n+2) \times (n+2)}[\gamma * \Gamma][\overline{\Gamma} * \gamma].
\]
Moreover
\[
\begin{pmatrix} 1 & 0 & 0 \\ c & A & 0 \\ 0 & b & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -b^f(A^f)^{-1}c \beta \beta \end{pmatrix}.
\]
Thus \( x = b^f(A^f)^{-1}c \beta \beta \) belongs to \( (Q_f(\Sigma))_{\gamma} \).

(5)⇒(7) Suppose \( A \in M_n(R)[\overline{\alpha}] [\overline{\beta}] \) with \( \beta_\infty = \gamma \), \( a \in M_{n \times 1}(R)[\overline{\alpha}][\overline{\beta}] \) and \( u \in M_{1 \times 1}(S) \) with \( u_\infty = x \). Then the equality \( A^f u = a^f \) is equivalent to the equality
\[
\begin{pmatrix} -a^f & A^f \\ 1 & u \end{pmatrix} = 0.
\]
Notice that \( -a^f \in M_{n \times (n+1)}(R)[\overline{\alpha}][\overline{\beta}] \).

**Theorem 3.4.** Let \( R \) be a \( \Gamma \)-graded ring and \( \Sigma \) be a gr-lower semimultiplicative subset of \( \mathfrak{M}(R) \). Let \( S \) be a ring and \( f : R \to S \) a \( \Sigma \)-inverting ring homomorphism. Then

1. For each \( \gamma \in \Gamma \), \( f(R_\gamma) \subseteq (Q_f(\Sigma))_{\gamma} \).
2. If \( \gamma, \delta \in \Gamma \) and \( x, y \in (Q_f(\Sigma))_{\gamma} \), then \( x + y \in (Q_f(\Sigma))_{\gamma} \).
3. If \( x, y \in (Q_f(\Sigma))_{\gamma} \), then \( xy \in (Q_f(\Sigma))_{\gamma} \).

Hence \( R_f(\Sigma) \) is a \( \Gamma \)-almost graded ring (which is a subring of \( S \)) that contains \( \text{im}(f) \). Furthermore

4. The restriction \( f : R \to R_f(\Sigma) \) is a ring epimorphism.
5. If \( S \) is a \( \Gamma \)-graded ring and \( f : R \to S \) is a homomorphism of \( \Gamma \)-graded rings, then \( (Q_f(\Sigma))_{\gamma} \subseteq S_{\gamma} \) for each \( \gamma \in \Gamma \) and \( R_f(\Sigma) = \bigoplus_{\gamma \in \Gamma} (Q_f(\Sigma))_{\gamma} \) is a \( \Gamma \)-graded subring of \( S \) such that \( h(R_f(\Sigma)) = Q_f(\Sigma) \).

**Proof.**

1. (1) Let \( r \in R_\gamma \). Then \( f(1)f(r) = f(r) \) where \( 1 \in M_1(R)[\gamma][\gamma] \) and \( r \in M_1(R)[\gamma][\gamma] \). Then Lemma 3.3(5) implies that \( f(r) \in (Q_f(\Sigma))_{\gamma} \).

2. Let \( x, y \in (Q_f(\Sigma))_{\gamma} \). By Lemma 3.3(5), there exist \( \overline{\alpha}, \overline{\beta} \in \Gamma^n, A \in \Sigma_n[\overline{\alpha}][\overline{\beta}], a \in M_{n \times 1}(R)[\overline{\alpha}][\overline{\beta}] \) and \( u \in M_{1 \times 1}(S) \) such that \( \beta_\infty = \gamma, u_\infty = x \) and
\[
A^f u = (A^f \cdot A^f_{\infty})(\begin{pmatrix} 1 \\ x \end{pmatrix}) = a^f.
\]
There also exist $B \in \Sigma_{n'}[\alpha',\beta']$, $b \in M_{n' \times 1}(R)[\alpha'][e]$ and $v \in M_{n' \times 1}(S)$ such that $\beta'_{\infty} = \gamma$, $v_{\infty} = y$ and

$$B^f v = (B^f_u B^f_v) \left( \begin{array}{c} u \\ v \end{array} \right) = b^f.$$

Then the matrix

$$\left( \begin{array}{cc} A^f_u & A^f_v \\ 0 & -B^f_\infty \end{array} \right) \in \Sigma_{n' + n}[\alpha' \ast \alpha'][\beta' \ast \beta']$$

is invertible. Hence $x + y \in (Q_f(\Sigma))_{\gamma \beta}$.

(3) Let $x \in (Q_f(\Sigma))_{\gamma}$ and $y \in (Q_f(\Sigma))_{\delta}$. There exist $A \in \Sigma_n[\alpha',\beta']$, $a \in M_{n \times 1}(R)[\alpha'][e]$ and $u \in M_{n \times 1}(S)$ such that $\beta_{\infty} = \gamma$, $u_{\infty} = x$ and $A^f u = a^f$. There also exist $B \in \Sigma_{n'}[\alpha'][\beta']$, $b \in M_{n' \times 1}(R)[\alpha'][e]$ and $v \in M_{n' \times 1}(S)$ such that $\beta_{\infty} = \delta$, $v_{\infty} = y$ and $B^f v = b^f$. Now

$$\left( \begin{array}{cc} B^f_u & B^f_v \\ 0 & -a^f_\infty \end{array} \right) \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} a^f \\ b^f \end{array} \right).$$

Hence $\gamma \beta \delta$.

From (1)–(3), it is easy to show that $R_f(\Sigma)$ is a $\Gamma$-almost graded ring and a subring of $S$.

(4) Let $g,h : R_f(\Sigma) \to T$ be ring homomorphisms. If $x \in (Q_f(\Sigma))_{\gamma}$, then $x$ is an entry of a square matrix $B$ which is the inverse of $A^f$ for some $A \in \Sigma$. From $A^f B = B A^f = I$, it follows that $A^h B^g = B^g A^h = I$ and $A^{h f} B^{g f} = B^g A^{h f} = I$. Thus $B^g = B^h$, and $g(x) = h(x)$. Since $R_f(\Sigma)$ is generated by $(Q_f(\Sigma))_{\gamma}$, $\gamma \in \Gamma$, then $f : R \to R_f(\Sigma)$ is a ring epimorphism.

(5) Now suppose that $S$ is a $\Gamma$-graded ring and $f : R \to S$ is a homomorphism of $\Gamma$-graded rings. Let $x \in (Q_f(\Sigma))_{\gamma}$. There exist $A \in \Sigma_n[\alpha',\beta']$, $a \in M_{n \times 1}(R)[\alpha'][e]$ and $u \in M_{n \times 1}(S)$ such that $\beta_{\infty} = \gamma$, $u_{\infty} = x$ and $A^f u = a^f$. Notice that $A^f \in M_n(\Sigma)[\alpha][\beta]$ is an invertible matrix and that $a^f \in M_{n \times 1}(\Sigma)[\alpha'][e]$. The matrix $(A^f)^{-1} \in M_n(\Sigma)[\beta][\alpha']$. Now $(A^f)^{-1}$ and $a^f$ are compatible and $u = (A^f)^{-1} a^f$. Then $x = u_{\infty} \in S_{\beta_{\infty}}$, that is $x \in S_{\gamma}$.

By (1)–(3), it is easy to prove that $R_f(\Sigma)$ is a graded subring of $S$ whose set of homogeneous elements equals $Q_f(\Sigma)$. \hfill \Box

Lemma 3.5 (Cramer’s rule). Let $R$ be a $\Gamma$-graded ring and $\Sigma$ be a subset of $\mathfrak{M}(R)$. Let $S$ be a ring and $f : R \to S$ be a $\Sigma$-inverting ring homomorphism.

Let $\gamma \in \Gamma$ and $x \in (Q_f(\Sigma))_{\gamma}$. Suppose that $\alpha \in \Gamma^n$, $\beta \in \Gamma^{n+1}$, $A \in M_{n \times (n+1)}(R)[\alpha][\beta]$ and $u \in M_{n \times 1}(S)$ such that $\beta_0 = e$, $\beta_{\infty} = \gamma$, $u_0 = 1$, $u_{\infty} = x$, $(\mathfrak{A}_u, \mathfrak{A}_v) \in \Sigma$ and $A^f u = 0$. Then the following assertions hold true.

(1) $x$ is invertible in $S$ if, and only if, the matrix $(A^f_0 A^f_1)$ is invertible in $M_n(S)$.
(2) $x$ is a regular element of $S$ if, and only if, the matrix $(A^f_0 A^f_1)$ is a regular element of $M_n(S)$.\hfill \Box
(3) If \( x = 0 \), then the matrix \((A_0, A_\bullet)^f\) is not full over \( S \). Furthermore, if \( S \) is a \( \Gamma \)-graded ring and \( f: R \to S \) is a homomorphism of graded rings, then matrix \((A_0, A_\bullet)^f \in M_n(S)[\overline{\beta}][\overline{\beta}]\) is not gr-full over \( S \).

Proof. First note the equality
\[
(A^f_\bullet - A^f_0) = (A^f_\bullet \ A^f_\infty) \begin{pmatrix} I & u_\bullet \\ 0 & x \end{pmatrix}.
\]
Also notice that the homogeneous matrix \((A^f_\bullet, A^f_\infty)\) is invertible in \( M_n(S) \) because \( f \) is \( \Sigma \)-inverting.

(1) Suppose that \( x \) is invertible in \( S \). Then
\[
\begin{pmatrix} I & u_\bullet \\ 0 & x \end{pmatrix}
\]
is invertible in \( M_n(S) \). Hence \((A^f_\bullet - A^f_0)\) is invertible, and therefore \((A^f_0, A^f_\bullet)\) is invertible in \( M_n(S) \).

Conversely, suppose that \((A^f_0, A^f_\bullet)\) is invertible in \( M_n(S) \). Hence the fact that \((A^f_\bullet - A^f_0)\) is invertible and (3) imply that
\[
\begin{pmatrix} I & u_\bullet \\ 0 & x \end{pmatrix}
\]
is invertible in \( M_n(S) \). Thus there exists \((v \ y \ w) \in M_n(S)\) such that
\[
\begin{pmatrix} I & u_\bullet \\ 0 & x \end{pmatrix} \begin{pmatrix} v & w \\ y & z \end{pmatrix} = I,
\]
Thus \( xz = 1, yI = 0 \) and \( yu_\bullet + zx = 1 \). Therefore \( x \) is invertible in \( S \).

(2) Follows easily from (3).

(3) Suppose that \( x = 0 \). Then (3) can be expressed as
\[
(A^f_\bullet - A^f_0) = (A^f_\bullet \ A^f_\infty) \begin{pmatrix} I & u_\bullet \\ 0 & x \end{pmatrix}
\]
which implies that \((A^f_\bullet - A^f_0)\) is not full and therefore \((A^f_0, A^f_\bullet)\) is not full.

If moreover \( f: R \to S \) is a homomorphism of \( \Gamma \)-graded rings, then \((A_\bullet - A_0)^f \in M_n(S)[\overline{\beta}][\overline{\beta}]\) and \((I \ u_\bullet) \in M_{n(\bullet - n) \times [\overline{\beta}]}[\overline{\beta}][\overline{\beta}]\). It implies that the matrix \((A_\bullet - A_0)^f\) is not gr-full which in turn implies that \((A_0 \ A_\bullet)^f\) is not gr-full, as desired.

\(\square\)

Given \( A \) and \( x \) as in Lemma 3.3, we say that \((A_0, A_\bullet)\) is the numerator of \( x \) and \((A_\bullet, A_\infty)\) is the denominator of \( x \). Thus, \( x \) is invertible in \( S \) if and only if its numerator is invertible in \( M_n(S) \).

**Theorem 3.6.** Let \( R \) be a \( \Gamma \)-graded ring. Let \( S \) be a ring and \( f: R \to S \) be a ring homomorphism. Set
\[
\Sigma = \{ A \in \mathfrak{M}(R): A^f \text{ is invertible over } S \}.
\]
If \( x \in (Q_f(\Sigma))_\gamma \) is invertible in \( S \), then \( x^{-1} \in (Q_f(\Sigma))_{\gamma - 1} \).

Moreover, if \( S \) is a \( \Gamma \)-almost graded division ring and \( f: R \to S \) is a homomorphism of \( \Gamma \)-almost graded rings, then \( R_f(\Sigma) \) is a \( \Gamma \)-almost graded division subring of \( S \).
Proof. Let \( x \in (Q_f(\Sigma))_\gamma \). By Lemma 3.3(7), there exist \( A \in M_{n \times (n+1)}(R) \) such that \( \beta_0 = e, \beta_\infty = \gamma, u_0 = 1, u_\infty = x, (A_*, A_\infty) \in \Sigma \) and \( AFu = (A_0^\prime A_*^\prime A_\infty^\prime) \left( \begin{array}{c} u_x \\ \ast \end{array} \right) = 0 \). Equivalently \( A_0^\prime + A_* u_* + A_\infty u_\infty = 0 \). Hence \( A_0^\prime x^{-1} + A_* u_* x^{-1} + A_\infty = 0 \), or equivalently

\[
\begin{pmatrix} A_0^\prime & A_* & A_\infty \\
0 & 0 & -B_\infty \\
B_* & B_\infty \end{pmatrix} \begin{pmatrix} 1 \\ u_x \\ x \end{pmatrix} = 0.
\]

Since \( x \) is invertible, Cramer’s rule implies that the matrix \( (A_0^\prime A_* A_\infty) \) is invertible over \( S \), and therefore \( (A_*, A_\infty) \in \Sigma \). Moreover, notice that \( (A_\infty A_* A_0) \in M_{n \times (n+1)}(R) \big[ \beta_\infty, \beta_0 \big] \). This can also be expressed as \( A \in M_{n \times (n+1)}(R) \big[ \beta_\infty, \beta_0 \big] \big[ \beta_\infty, \beta_0 \big] \big[ \beta_\infty, \beta_0 \big] \big[ \beta_\infty, \beta_0 \big] \). By Lemma 3.3(7), and observing the equality \( \beta_\infty \beta_\infty + \beta_0 \beta_\infty + \beta_0 \beta_\infty = e \), we get that \( x^{-1} \) in \( (Q_f(\Sigma))_\gamma \).

The second part follows because, by Theorem 3.4, \( R_f(\Sigma) \) is a \( \Gamma \)-almost graded subring of \( S \) which, by the foregoing, is closed under inverses of homogeneous elements.

\[ \square \]

**Corollary 3.7.** Let \( R \) be a \( \Gamma \)-graded ring, \( K \) be a \( \Gamma \)-graded division ring and \( f : R \to K \) be a homomorphism of \( \Gamma \)-graded rings. If

\[ \Sigma = \{ A \in \mathfrak{N}(R) : A_f \text{ invertible over } K \} \]

and \( K \) is generated as a \( \Gamma \)-graded division ring by the image of \( f \), then \( K = R_f(\Sigma) \).

\[ \square \]

We end this section with an interesting result, but that will not be used in later sections. We show that two elements (and by induction any finite number of elements) can be brought to a common denominator.

**Lemma 3.8.** Let \( R \) be a \( \Gamma \)-graded ring and \( \Sigma \) be a \( \gamma \)-lower semimultiplicative subset of \( \mathfrak{N}(R) \). Let \( S \) be a ring and \( f : R \to S \) a \( \Sigma \)-inverting ring homomorphism.

If \( x \in (Q_f(\Sigma))_\gamma \) and \( y \in (Q_f(\Sigma))_\delta \) for some \( \gamma, \delta \in \Gamma \), then they can be brought to a common denominator.

**Proof.** Let \( x \in (Q_f(\Sigma))_\gamma \) and \( y \in (Q_f(\Sigma))_\delta \). There exist \( A \in M_{n \times (n+1)}(R) \) and \( u \in M_{(n+1) \times 1}(S) \) such that \( \beta_0 = e, \beta_\infty = \gamma, u_0 = 1, u_\infty = x, (A_*, A_\infty) \in \Sigma \) and \( AFu = (A_0^\prime A_*^\prime A_\infty^\prime) \left( \begin{array}{c} 1 \\ u_x \\ x \end{array} \right) = 0 \). There also exist \( B \in M_{n \times (n+1)}(R) \) and \( v \in M_{(n+1) \times 1}(S) \) such that \( \beta_0 = e, \beta_\infty = \delta, u_0 = 1, u_\infty = y, (B_*, B_\infty) \in \Sigma \) and \( BFv = (B_0^\prime B_*^\prime B_\infty^\prime) \left( \begin{array}{c} 1 \\ u_y \\ v_x \end{array} \right) = 0 \). Then

\[
\begin{pmatrix} A_0 & A_* & A_\infty \\
0 & 0 & -B_\infty \\
B_* & B_\infty \end{pmatrix} \begin{pmatrix} 1 \\ u_x \\ x \end{pmatrix} = 0,
\]

\[
\begin{pmatrix} 0 & A_* & A_\infty \\
0 & 0 & -B_\infty \\
B_* & B_\infty \end{pmatrix} \begin{pmatrix} 1 \\ u_y \\ v_x \end{pmatrix} = 0.
\]

\[ \square \]
Now \( \begin{pmatrix} A_0 & A_\infty \\ 0 & -B_\infty \end{pmatrix} \) \( \in M_{(n+n') \times (n+n'+1)}(R)[\overline{\alpha}\overline{\beta}{_{\infty}}^{-1}\beta_{\infty}][\overline{\beta}*\overline{\tau}] \) where \( \overline{\tau} = (\beta'_{\infty} \overline{\beta}{_{\infty}}^{-1} \beta_{\infty}, \beta_{\infty}) \). The matrix \( \begin{pmatrix} 0 & A_\infty \\ B_0 & -B_\infty \end{pmatrix} \) belongs to \( M_{(n+n') \times (n+n'+1)}(R)[\overline{\beta}{_{\infty}}^{-1}\beta_{\infty}][\overline{\alpha}\overline{\beta}{_{\infty}}^{-1}\beta_{\infty}][\overline{\beta}*\overline{\tau}] \) where \( \overline{\tau} = (\beta'_{\infty} \overline{\beta}{_{\infty}}^{-1} \beta_{\infty}, \beta_{\infty}, \beta_{\infty}'). \) □

4. The category of graded \( R \)-division rings and GR-specializations

This section is an adaptation of \cite[Section 7.2]{1} to the graded situation.

Throughout this section, let \( \Gamma \) be a group. Let \( R = \bigoplus_{\gamma \in \Gamma} R_{\gamma} \) be a \( \Gamma \)-graded ring.

A \( \Gamma \)-graded \( R \)-ring is a pair \((K, \varphi)\) where \( K \) is a \( \Gamma \)-graded ring and \( \varphi: R \rightarrow K \) is a homomorphism of graded rings. A \textit{graded \( R \)-subring} of \((K, \varphi)\) is a graded subring \( L \) of \( K \) such that \( \varphi(R) \subseteq L \).

A \( \Gamma \)-graded \( R \)-division ring is a \( \Gamma \)-graded \( R \)-ring \((K, \varphi)\) such that \( K \) is a \( \Gamma \)-graded division ring. If \( K = DC(\varphi) \), that is, \( K \) is the \( \Gamma \)-graded division ring generated by \( \varphi \). Now we say that \((K, \varphi)\) is a \( \Gamma \)-graded epic ring.

A \textit{homomorphism of graded \( R \)-rings} between \( \Gamma \)-graded \( R \)-rings \((K, \varphi)\) and \((K', \varphi')\) is a homomorphism of graded rings \( f: K \rightarrow K' \) such that \( \varphi' = f \circ \varphi \). If, moreover, \( f: K \rightarrow K' \) is an isomorphism of \( \Gamma \)-graded rings, we say that \( f \) is an \textit{isomorphism of graded \( R \)-rings}.

Let now \( \Sigma \subseteq \mathfrak{M}(R) \). The \textit{universal localization of \( R \) at} \( \Sigma \) is a pair \((R_{\Sigma}, \lambda)\) where \( R_{\Sigma} \) is a ring and \( \lambda: R \rightarrow R_{\Sigma} \) is a \( \Sigma \)-inverting homomorphism such that for any other \( \Sigma \)-inverting ring homomorphism \( f: R \rightarrow S \) there exists a unique ring homomorphism \( F: R_{\Sigma} \rightarrow S \) with \( f = F \lambda \).

Now we give some important properties of \( R_{\Sigma} \).

**Proposition 4.1.** Let \( R \) be a \( \Gamma \)-graded ring and let \( \Sigma \subseteq \mathfrak{M}(R) \). Then the following statements hold true:

1. There exists the universal localization \((R_{\Sigma}, \lambda)\) of \( R \) at \( \Sigma \).
2. \( \lambda: R \rightarrow R_{\Sigma} \) is a ring epimorphism.
3. The ring \( R_{\Sigma} \) is a \( \Gamma \)-graded ring, \( \lambda: R \rightarrow R_{\Sigma} \) is a homomorphism of \( \Gamma \)-graded rings, and \((R_{\Sigma}, \lambda)\) is unique up to isomorphism of \( \Gamma \)-graded \( R \)-rings.
4. Suppose that \( S \) is a \( \Gamma \)-graded ring, \( f: R \rightarrow S \) is a \( \Sigma \)-inverting homomorphism of \( \Gamma \)-graded rings and \( F: R_{\Sigma} \rightarrow S \) is the unique homomorphism of rings such that \( f = F \lambda \). Then \( F: R_{\Sigma} \rightarrow S \) is a homomorphism of \( \Gamma \)-graded \( R \)-rings. Moreover, if \( \Sigma \) is gr-lower semimultiplicative, then \( \text{Im} F = R_{f}(\Sigma) \).

**Proof.** First we construct a free ring \( \mathbb{Z}(X) \) where \( X \) is constructed as follows. For each \( \gamma \in \Gamma \) and \( r \in R_{\gamma} \), consider a symbol \( x^\gamma_{r} \). For each matrix \( A = (a_{ij}) \in \Sigma \), fix \((\overline{\alpha}, \overline{\beta})\) such that \( A \in M_{n}(R)[\overline{\alpha}, \overline{\beta}] \) and consider a matrix \( A^* \) whose entries are symbols \( A^* = (a^*_{ij}) \). Then let \( X \) be the disjoint union

\[ X = \{ x^\gamma_{r} : r \in R_{\gamma}, \gamma \in \Gamma \} \cup \{ a^*_{ij} : a_{ij} \text{ is the } (i, j)-\text{entry of } A \in \Sigma \}. \]

Now we turn \( \mathbb{Z}(X) \) into a \( \Gamma \)-graded ring by giving degrees to the elements of \( X \). If \( r \in R_{\gamma} \), we set \( x^\gamma_{r} \) to be of degree \( \gamma \). If \( A = (a_{ij}) \in \Sigma \) with fixed \((\overline{\alpha}, \overline{\beta})\), then \( a_{ij} \in R_{n_{\gamma, n_{\delta}}^{-1}} \), thus we let \( a^*_{ij} \) of degree \( \beta_{\gamma} \alpha_{\delta}^{-1} \). Notice that \( A^* \in M_{n}(\mathbb{Z}(X))[\overline{\beta}, \overline{\alpha}] \).

Let \( I \) be the ideal of \( \mathbb{Z}(X) \) generated by the homogeneous elements of any of the following forms:

- \( x^\gamma_{r+s} - x^\gamma_{r} - x^\gamma_{s} \) for \( r, s \in R_{\gamma} \).
- \( x^\gamma_{r} x^\delta_{s} - x^\delta_{s} x^\gamma_{r} \) for \( r \in R_{\gamma} \) and \( s \in R_{\delta} \).
- \( x^\gamma_{1} - 1 \).
\[
\begin{align*}
\sum_k x_{ak,k} a_{k}^{\gamma} - \delta_{ij} & \text{ for } A \in \Sigma. \\
\sum_k a_{k}^{\gamma} x_{ak,j} - \delta_{ij} & \text{ for } A \in \Sigma.
\end{align*}
\]

Set \( R_{\Sigma} = \mathbb{Z}(X) / I \) and \( \lambda : R \to R_{\Sigma} \) be the homomorphism of \( \Gamma \)-graded rings determined by \( \lambda(r) = x^{\gamma} \) for each \( r \in R_{\gamma}, \gamma \in \Gamma \). Since \( I \) is a graded ideal of \( \mathbb{Z}(X) \), then \( R_{\Sigma} \) is a \( \Gamma \)-graded ring and \( \lambda \) is a homomorphism of graded rings.

Suppose that \( f : R \to S \) is a \( \Sigma \)-inverting ring homomorphism. For each \( A = (a_{ij}) \in \Sigma \) with fixed \( (\bar{a}, \bar{b}) \), suppose that \( (A^f)^{-1} = (b_{ij}) \). Notice that \( A^f \in M_{n}(\mathbb{Z}[\bar{a}][\bar{b}]) \) and \( (A^f)^{-1} \in M_{n}(\mathbb{Z}[\bar{a}][\bar{b}]) \). Then there exists a unique homomorphism of \( \Gamma \)-graded rings \( F_{\gamma} : \mathbb{Z}(X) \to S \) such that \( F_{\gamma}(x^{\gamma}) = f(r) \) for each \( r \in R_{\gamma}, \gamma \in \Gamma \), and \( F_{\gamma}(a_{ij}) = b_{ij} \). Note that \( I \subseteq \ker F \), and let \( F : R_{\Sigma} \to S \) be the induced homomorphism. Hence \( F\lambda = f \), as desired. To prove the uniqueness and the fact that \( \lambda : R \to R_{\Sigma} \) is a ring epimorphism, notice that from \( F\lambda = f \), we obtain that \( F(x^{\gamma}) = f(r) \), and now the same argument of Theorem 3.3(4) shows that \( F(a_{ij}) = b_{ij} \).

Now our aim is to show that if \( (K, \varphi) \) is a \( \Gamma \)-graded epic \( R \)-field, then \( \varphi : R \to K \) is in fact an epimorphism of \( \Gamma \)-graded rings. For the sake of completeness, we preferred to give the proof of the following lemma, but this could be shown as a direct consequence of [7, Proposition 7.2.1] and the fact that if \( f : R \to S \) is a homomorphism of \( \Gamma \)-graded rings that is an epimorphism in the category of \( \Gamma \)-graded rings, then it is an epimorphism in the category of rings. The proof of this fact is as follows, if \( g_{1}, g_{2} : S \to T \) are homomorphisms of rings such that \( g_{1}f = g_{2}f \), there exist homomorphisms of \( \Gamma \)-graded rings \( \bar{g}_{1} : S \to \text{Im} g_{1}, \bar{g}_{2} : S \to \text{Im} g_{2} \) and homomorphism of rings \( \pi : \text{Im} g_{1}f \to T \) such that \( \bar{g}_{1}f = \bar{g}_{2}f \) and \( g_{1} = \pi \bar{g}_{1}, g_{2} = \pi \bar{g}_{2} \). Since \( f \) is an epimorphism of \( \Gamma \)-graded rings, then \( \bar{g}_{1} = \bar{g}_{2} \). Thus \( g_{1} = g_{2} \).

**Lemma 4.2.** Let \( R, S \) be \( \Gamma \)-graded rings and \( f : R \to S \) be a homomorphism of \( \Gamma \)-graded rings. The following statements are equivalent.

(1) \( f \) is an epimorphism of \( \Gamma \)-graded rings.

(2) In the \( \Gamma \)-graded \( S \)-bimodule \( S \circ_{R} S, x \circ 1 = 1 \circ x \) for all \( x \in S \).

(3) The natural map \( f : S \circ_{R} S \to S \) determined by \( f(x \circ y) = xy \) is an isomorphism of graded \( S \)-bimodules.

**Proof.** (1) \( \Rightarrow \) (2) Consider the \( \Gamma \)-graded additive group \( M = S \circ (S \circ_{R} S) \). It can be endowed with a structure of \( \Gamma \)-graded ring via the multiplication \( (x, u)(y, v) = (xy, xv + uy) \). Notice that if \( (x, u) \in M_{\gamma} \) and \( (y, v) \in M_{\delta} \), then \( x, u \) have degree \( \gamma \) and \( y, v \) have degree \( \delta \). Hence \( xy \) and \( xv + uy \) have degree \( \gamma \delta \).

Consider the homomorphisms of \( \Gamma \)-graded rings \( g, h : S \to M \) defined by \( g(x) = (x, 0) \) and \( h(x) = (x, x \circ 1 - 1 \circ x) \). Since \( gf = hf \) and \( f \) is an epimorphism of graded rings, then \( x \circ 1 = 1 \circ x \).

(2) \( \Rightarrow \) (1) Let \( g, h : S \to T \) be homomorphisms of \( \Gamma \)-graded rings such that \( gf = hf \). Then there exists a well defined map \( F : S \circ_{R} S \to T, x \circ y \to g(x)h(y) \). For each \( x \in S \), since \( x \circ 1 = 1 \circ x \), we obtain that \( g(x) = F(x \circ 1) = F(1 \circ x) = h(x) \). Thus \( g = h \), as desired.

(2) \( \Rightarrow \) (3) First note that \( f \) is surjective. Now, since \( f \left( \sum x_{i} \otimes y_{i} \right) = \sum x_{i}y_{i} \), injectivity follows from the fact that \( \sum x_{i} \otimes y_{i} = \sum x_{i}(1 \otimes y_{i}) = \sum x_{i}(y_{i} \circ 1) = \sum x_{i}y_{i} \circ 1 = (\sum x_{i}y_{i}) \otimes 1 \).
Let \( f \) be an epimorphism of \( \Gamma \)-graded rings, if, and only if, \( K = \text{DC}(f) \).

\[ K \cong K \otimes_{\text{DC}(f)} K \cong \left( \bigoplus_{b \in B} b \text{DC}(f) \right) \otimes_{\text{DC}(f)} K \cong \bigoplus_{b \in B} (b \text{DC}(f) \otimes_{\text{DC}(f)} K) \]

for some \( \gamma_b \in \Gamma \). Hence \( B \) must consist of just one element.

Conversely, suppose that \( \text{DC}(f) = K \). Let

\[ \Sigma = \{ A \in \mathfrak{M}(R) : A^I \text{ is invertible over } K \} \]

Then \( K = \text{DC}(f) = R_f(\Sigma) \). By Proposition 4.1(4), \( f : R \to K \) is a ring epimorphism, and therefore an epimorphism of \( \Gamma \)-graded rings.

\[ \Sigma = \{ A \in \mathfrak{M}(R) : A^I \text{ is invertible over } K \} \]

The following assertions hold true.

(a) \( R_{\Sigma} \) is a \( \Gamma \)-graded local ring.

(b) If \( m \) is the maximal graded ideal of \( R_{\Sigma} \), then \( R_{\Sigma}/m \) is a \( \Gamma \)-graded epic \( R \)-division ring.

\[ \Sigma = \{ A \in \mathfrak{M}(R) : A^I \text{ is invertible over } K \} \]

(1) If \( \Sigma \subseteq \mathfrak{M}(R) \) is such that the universal localization \( R_{\Sigma} \) is a \( \Gamma \)-graded local ring with maximal graded ideal \( m \), then \( R_{\Sigma}/m \) is a \( \Gamma \)-graded epic \( R \)-division ring.

(2) Let \( K \) be a \( \Gamma \)-almost graded division ring and \( f : R \to K \) be a homomorphism of \( \Gamma \)-almost graded rings such that \( \text{DC}(f) = K \). Let

\[ \Sigma = \{ A \in \mathfrak{M}(R) : A^I \text{ is invertible over } K \} \]

The following assertions hold true.

(a) \( R_{\Sigma} \) is a \( \Gamma \)-graded local ring.

(b) If \( m \) is the maximal graded ideal of \( R_{\Sigma} \), then \( R_{\Sigma}/m \) is a \( \Gamma \)-graded epic \( R \)-division ring satisfying the following statements.

(i) There exists a surjective homomorphism of \( \Gamma \)-almost graded rings

\[ \tilde{F} : R_{\Sigma}/m \to K \]

such that the following diagram is commutative

\[ R \xrightarrow{\lambda} R_{\Sigma} \xrightarrow{\pi} R_{\Sigma}/m \]

\[ f \]

\[ K \]

\[ \tilde{F} \]

(ii) If \( K \) is a \( \Gamma \)-graded division ring, then \( \tilde{F} : R_{\Sigma}/m \to K \) is an isomorphism of \( \Gamma \)-graded epic \( R \)-division rings.

Proof. (1) The homomorphism \( \lambda : R \to R_{\Sigma} \) is a ring epimorphism by Proposition 4.1(2). The natural homomorphism \( \pi : R_{\Sigma} \to R_{\Sigma}/m \) is surjective. Therefore \( \pi \lambda : R \to R_{\Sigma}/m \) is a ring epimorphism, thus a \( \Gamma \)-graded epic \( R \)-division ring.

(2) Let \( \lambda : R \to R_{\Sigma} \) be the canonical homomorphism. Hence there exists a unique homomorphism of \( \Gamma \)-almost graded \( R \)-rings \( F : R_{\Sigma} \to K \) such that \( F \lambda = f \). Set \( m = (\ker F)_2 \), in other words, \( m = \bigoplus_{\gamma \in \Gamma} (\ker F \cap (R_{\Sigma})_{\gamma}) \subseteq \ker F \). Let \( x \in (R_{\Sigma})_\gamma \setminus m \). Then \( F(x) \neq 0 \) and then \( F(x) \in K_\gamma \) is invertible in \( K \). By Proposition 4.1(4),
\( R_{\Sigma} = R_{\Lambda}(\Sigma) \). Thus, there exist \( \overline{\alpha} \in \Gamma^n, \overline{\beta} \in \Gamma^{n+1}, \ A \in M_{n \times (n+1)}(R)[\overline{\alpha}][\overline{\beta}] \) and \( u \in M_{(n+1) \times 1}(S) \) such that \( \beta_0 = e, \beta_\infty = \gamma, \ u_0 = 1, \ u_\infty = x, \ (A_\bullet, A_\infty) \in \Sigma \) and

\[
(A_0^\Lambda A_\infty^\Lambda) \begin{pmatrix} u & v \\ x & u \end{pmatrix} = 0.
\]

Applying \( F \) to the entries of the matrices involved we obtain

\[
(A_0^f A_\infty^f) \begin{pmatrix} u & v \\ F(x) & u \end{pmatrix} = 0.
\]

Since \( F(x) \) is invertible, by Cramer’s rule, \( (A_0^f A_\infty^f) \) is invertible in \( K \). Therefore \( (A_0 A_\bullet) \in \Sigma \) and \( (A_0^\Lambda A_\infty^\Lambda) \) is invertible in \( R_{\Sigma} \). Again by Cramer’s rule, \( x \) is invertible in \( R_{\Sigma} \). Hence \( R_{\Sigma} \) is a \( \Gamma \)-graded local ring where \( m \) is the ideal generated by the noninvertible homogeneous elements of \( R_{\Sigma} \), and (a) is proved. The ring \( R_{\Sigma}/m \) is a \( \Gamma \)-graded division ring and, by (1), (b) follows.

(i) and (ii) follow, respectively, because \( m \subseteq \ker F \) and \( m = \ker F \) if \( K \) is a \( \Gamma \)-graded division ring. \( \square \)

Now we proceed to define the category of graded epic \( R \)-division rings and gr-specializations.

Let \( R \) be a \( \Gamma \)-graded ring.

Suppose that \( (K, \varphi), (L, \psi) \) are \( \Gamma \)-graded epic \( R \)-division rings and set

\[ \Sigma = \{ A \in \mathfrak{M}(R) : A^\varphi \text{ is invertible over } L \}. \]

If there exists a homomorphism of \( \Gamma \)-graded \( R \)-rings \( \Phi: R_{\Sigma} \rightarrow K \), we define the core of \( L \) in \( K \) as \( \mathcal{C}_L(K) = \Phi(R_{\Sigma}) \). We remark that, if it exists, it is unique and observe that, by Proposition 4.4(1), \( \mathcal{C}_L(K) = \Phi(R_{\Sigma}) \). By Theorem 4.4(2)(a), \( R_{\Sigma} \) is a \( \Gamma \)-graded local ring. Therefore \( \mathcal{C}_L(K) \) is a \( \Gamma \)-graded local local subring of \( K \) that contains \( R \). Moreover, the natural homomorphism of \( \Gamma \)-graded \( R \)-rings \( \Psi: R_{\Sigma} \rightarrow L \) factors through \( \mathcal{C}_L(K) \) in a unique way, because \( L \cong R_{\Sigma}/m \) where \( m \) is the maximal graded ideal of \( R_{\Sigma} \).

A \textit{gr-subhomomorphism} is a homomorphism of \( \Gamma \)-graded \( R \)-rings \( f: K_f \rightarrow L \) where \( K_f \) is a \( \Gamma \)-graded local subring of \( K \) such that \( x^{-1} \in K_f \) for each \( x \in \ker f \). Note that \( K_f \) is a \( \Gamma \)-graded local subring of \( K \) because any homogeneous element not in the graded ideal \( \ker f \) is invertible. Hence \( K_f/\ker f \) is a \( \Gamma \)-graded \( R \)-division ring contained in \( L \). This implies that \( f \) is a surjective homomorphism of \( \Gamma \)-graded \( R \)-rings and that \( K_f/\ker f \cong L \) is a \( \Gamma \)-graded epic \( R \)-division ring. For each \( A \in \Sigma, \) consider \( A^\varphi \) which belong to \( \mathfrak{M}(K) \). Since \( K_f \) is a \( \Gamma \)-graded local \( R \)-ring whose residue graded division ring is \( L \), we get that \( A^\varphi \) is invertible in \( K_f \).

Thus there exists a unique homomorphism of graded \( R \)-rings \( \Phi: R_{\Sigma} \rightarrow K_f \subseteq K \) and a commutative diagram of homomorphisms of graded \( R \)-rings

\[
\begin{array}{c}
R_{\Sigma} \\
\phi \searrow \downarrow f \nearrow \psi \\
K_f & \leftarrow & L
\end{array}
\]

Thus \( \mathcal{C}_L(K) \) is contained in the domain of any subhomomorphism from \( K \) to \( L \), it is a \( \Gamma \)-graded local \( R \)-subring of \( K \), the restriction of any subhomomorphism to \( \mathcal{C}_L(K) \) is a subhomomorphism and all such restrictions coincide in \( \mathcal{C}_L(K) \), because of the commutativity of (4.1).

Now we give another description of \( \mathcal{C}_L(K) \). Let \( f: K_f \rightarrow L \) be a gr-subhomomorphism between the \( \Gamma \)-graded epic \( R \)-fields \( (K, \varphi), (L, \psi) \). For each \( \gamma \in \Gamma \) define \( (c(f)_{0})_\gamma = \)

\( \varphi(R_\gamma) \), and if \( n \geq 0 \), set

Additive subgroup of \( K \) generated by

\[
(c(f)_{n+1})_\gamma = \{ x_1 \cdots x_r : r \geq 1, x_i \in (c(f)_n)_{\gamma_i}, \text{ or } x_i = y_i^{-1} \text{ where } y_i \in (c(f)_n)_{\gamma_i}^{-1} \setminus \ker f, \gamma_1 \cdots \gamma_r = \gamma \}
\]

Then define \( c(f)_\gamma = \bigcup_{n \geq 0} (c(f)_n)_\gamma \), and \( C_L(K) = \bigoplus_{\gamma \in \Gamma} c(f)_\gamma \). Note that \( C_L(K) \) is a \( \Gamma \)-graded local \( R \)-subring of \( K_f \) with maximal graded ideal \( C_L(K) \cap \ker f \) and such that the restriction \( f : C_L(K) \to L \) is a gr-subhomomorphism. If we take \( K_f = \mathfrak{E}_L(K) \), then we obtain that \( C_L(K) \subseteq \mathfrak{E}_L(K) \), but since \( \mathfrak{E}_L(K) \) is contained in the domain of any gr-subhomomorphism, we get that \( C_L(K) = \mathfrak{E}_L(K) \). Roughly speaking, this equality means that any rational homogeneous expression obtained from the elements of \( (\text{the image of}) \ R \) in \( L \) makes sense in \( K \) and the elements obtained with those rational expressions from the elements of \( (\text{the image of}) \ R \) in \( K \) form \( \mathfrak{E}_L(K) \).

Because if there exist gr-subhomomorphisms between the \( \Gamma \)-graded epic \( R \)-division rings \( (K, \varphi) \) and \( (L, \psi) \), then they all coincide in the core, we make the following definition. A gr-specialization is the unique homomorphism of \( \Gamma \)-graded \( R \)-rings \( f : \mathfrak{E}_L(K) \to L \).

Suppose that \( (K, \varphi), (L, \psi) \) and \( (M, \phi) \) are \( \Gamma \)-graded epic \( R \)-division rings. If \( f : K_f \to L \) and \( g : L_g \to M \) are gr-subhomomorphisms, then the restriction \( g\chi : P = f^{-1}(L_g) \to M \) is a gr-subhomomorphism which will be called the composition gr-subhomomorphism of \( f \) and \( g \). Indeed, suppose that \( z \in h(P) \setminus \ker(g\chi) \). Since \( g(f(z)) \neq 0 \), then \( f(z)^{-1} \in L_g \). As \( f(z) \neq 0 \), and thus \( z^{-1} \in K_f \), then \( z^{-1} \in P \). We define the composition of the corresponding gr-specializations, as the gr-specialization corresponding to the composition gr-subhomomorphism of \( f \) and \( g \). In other words, the unique homomorphism of \( \Gamma \)-graded \( R \)-rings \( \mathfrak{E}_{Mf}(K) \to M \). It follows that the composition of gr-specializations is associative.

Note that the only subhomomorphism from the \( \Gamma \)-graded epic \( R \)-division ring \( (K, \varphi) \) to \( (K, \varphi) \) is the identity map on \( K \). Therefore \( \mathfrak{E}_K(K) = K \) and the corresponding specialization is the identity map.

We define the category \( \mathfrak{E}_R \) as the category whose objects are the \( \Gamma \)-graded epic \( R \)-division rings and whose morphisms are the gr-specializations. We remark that there is at most one morphism between two objects in this category and that isomorphisms correspond to isomorphisms of \( \Gamma \)-graded \( R \)-rings. Indeed, if the composition of two gr-specializations \( f \) and \( g \) is the identity gr-specialization, then they have to be isomorphisms of \( \Gamma \)-graded \( R \)-rings.

An initial object \( (K, \varphi) \) in the category \( \mathfrak{E}_R \) is a universal \( \Gamma \)-graded epic \( R \)-division ring. In other words, there exists a gr-specialization from \( (K, \varphi) \) to any other \( \Gamma \)-graded epic \( R \)-division ring \( (L, \psi) \). If moreover, \( \varphi : R \to K \) is injective, we say that this initial object is a universal \( \Gamma \)-graded epic \( R \)-division ring of fractions of \( R \).

Now we give the following important result.

**Theorem 4.5.** Let \( R \) be a \( \Gamma \)-graded ring and let \( (K_1, \varphi_1), (K_2, \varphi_2) \) be \( \Gamma \)-graded epic \( R \)-division rings. Set

\[
\Sigma_i = \{ A \in \mathfrak{M}(R) : A^{\varphi_i} \text{ is invertible over } K_i \}, \; i = 1, 2.
\]

The following statements are equivalent.

1. There exists a gr-specialization from \( (K_1, \varphi_1) \) to \( (K_2, \varphi_2) \).
2. \( \Sigma_2 \subseteq \Sigma_1 \).
3. There exists a homomorphism \( R_{\Sigma_2} \to R_{\Sigma_1} \) of \( \Gamma \)-graded \( R \)-rings.
Furthermore, if there exists a gr-specialization from \((K_1, \varphi_1)\) to \((K_2, \varphi_2)\) and another gr-specialization from \((K_2, \varphi_2)\) to \((K_1, \varphi_1)\), then \(K_1\) and \(K_2\) are isomorphic \(\Gamma\)-graded \(R\)-rings.

Proof. (1) \(\Rightarrow\) (2) By definition, there exists a homomorphism of \(\Gamma\)-graded \(R\) rings \(\mathcal{E}_{K_2}(K_1) \to K_2\). By definition of \(\mathcal{E}_{K_2}(K_1)\), any matrix in \(\Sigma_2\) is invertible over \(\mathcal{E}_{K_1}(K_1) \subseteq K_1\). Thus \(\Sigma_2 \subseteq \Sigma_1\).

(2) \(\Rightarrow\) (3) If \(\Sigma_2 \subseteq \Sigma_1\), the universal property of \(R_{\Sigma_2}\) implies the existence of a homomorphism of \(\Gamma\)-graded \(R\)-rings \(R_{\Sigma_2} \to R_{\Sigma_1}\).

(3) \(\Rightarrow\) (1) Consider the unique homomorphisms of \(\Gamma\)-graded \(R\)-rings \(\mathcal{E}_{K_1}(K_1) \to K_1\). Then the composition \(gf\) gives a gr-specialization from \(K_1\) in itself. Thus it has to be the identity. Similarly the composition \(fg\) gives a gr-specialization from \(K_2\) in itself. Hence, \(f\) is an isomorphism in the category \(\mathcal{E}_R\) of \(\Gamma\)-graded epic \(R\)-division rings. Therefore, \(f\) is an isomorphism of graded \(R\)-rings.

\(\square\)

**Corollary 4.6.** Let \(R\) be a \(\Gamma\)-graded ring. Suppose that there exists \(\Omega \subseteq \mathfrak{M}(R)\) such that \((R_{\Omega}, \lambda)\), where \(\lambda: R \to R_{\Omega}\) is the canonical homomorphism, is a \(\Gamma\)-graded (epic) \(R\)-division ring. Then the only gr-specializations to \((R_{\Omega}, \lambda)\) are isomorphisms of \(\Gamma\)-graded \(R\)-rings.

Proof. Suppose there exists a gr-specialization from the \(\Gamma\)-graded epic \(R\)-division ring \((K, \varphi)\) to \((R_{\Omega}, \lambda)\). By Theorem 4.5.3, then there exists a (unique) homomorphism of \(\Gamma\)-graded \(R\)-rings \(R_{\Omega} \to R_{\Sigma} \to K\), where

\[\Sigma = \{A \in \mathfrak{M}(R): A^\varphi \text{ is invertible over } K\}\]

Now, since \(R_{\Omega}\) and \(K\) are \(\Gamma\)-graded epic \(R\)-division rings, the image of \(R_{\Omega}\) must be \(K\) and therefore they are isomorphic as \(\Gamma\)-graded \(R\)-rings.

\(\square\)

**Corollary 4.7.** Let \(R\) be a \(\Gamma\)-graded ring with a universal \(\Gamma\)-graded epic \(R\)-division ring \((U, \rho)\). Suppose that \(\Sigma \subseteq \mathfrak{M}(R)\) is such that there exists a homomorphism of \(\Gamma\)-graded rings \(R_{\Sigma} \to L\) for some \(\Gamma\)-graded division ring \(L\). Then \((U, \rho)\) is a universal \(\Gamma\)-graded epic \(R_{\Sigma}\)-division ring.

Proof. Consider the canonical homomorphism \(\lambda: R \to R_{\Sigma}\). Let \(f: R_{\Sigma} \to L\) be a homomorphism of \(\Gamma\)-graded rings with \(L\) a \(\Gamma\)-graded division ring. Then \((\text{DC}(f), f\lambda)\) is a \(\Gamma\)-graded epic \(R\)-division ring such that the matrices in \(\Sigma\) become invertible. Hence, by Theorem 4.5.3, \(\Sigma^\rho\) consists of invertible matrices in \(U\). Thus there exists a unique homomorphism of \(\Gamma\)-graded rings \(\psi: R_{\Sigma} \to U\) and \((U, \psi)\) is a \(\Gamma\)-graded epic \(R_{\Sigma}\)-division ring.

Consider a \(\Gamma\)-graded epic \(R_{\Sigma}\)-division ring \((K, \varphi)\). The composition \(\varphi\lambda: R \to K\) is an epimorphism of \(\Gamma\)-graded rings, because \(\lambda\) and \(\varphi\) are. Hence \((K, \varphi\lambda)\) is a \(\Gamma\)-graded epic \(R\)-division ring and therefore there exists a specialization from \((U, \rho)\) to \((K, \varphi\lambda)\).

\(\square\)

Adapting [7, p.426] to the graded context, we give some examples to illustrate the concepts of universal graded division ring and graded division rings that are universal localizations.

Let \(R = \bigoplus_{\gamma \in \Gamma} R_{\gamma}\) be a commutative \(\Gamma\)-graded domain. Then the localization of \(R\) at the set \(h(R) \setminus \{0\}\) of nonzero homogeneous elements yields a \(\Gamma\)-graded epic
$R$-field $(F, \varphi)$. We point out that $F = \bigoplus_{\gamma \in \Gamma} F_\gamma$ is a $\Gamma$-graded field with 

$$F_\gamma = \{ab^{-1} | a \in R_S, b \in R, \delta e^{-1} = \gamma\}$$

for each $\gamma \in \Gamma$. Furthermore, if $(K, \psi)$ is a $\Gamma$-graded epic $R$-division ring, then $\ker \psi$ is a graded prime ideal of $R$. That is, $\ker \varphi \neq R$ and if $x, y \in \ker(\psi) \cap \ker(\psi)$ with $xy \in \ker \psi$, then $x \in \ker \psi$ of $y \in \ker \psi$. Hence $h(R) \setminus \ker \psi$ is a multiplicative subset of $R$. Then the localization of $R$ at $h(R) \setminus \ker \psi$ is a $\Gamma$-graded local subring of $F$ with $\Gamma$-graded residue division ring $R$-isomorphic to $K$. Therefore $(F, \varphi)$ is a $\Gamma$-graded universal $R$-division ring of fractions that is a universal localization.

Let $S = E \times F$ be the direct product of two $\Gamma$-graded fields $E = \bigoplus_{\gamma \in \Gamma} E_\gamma$ and $F = \bigoplus_{\gamma \in \Gamma} F_\gamma$. Then $S = \bigoplus_{\gamma \in \Gamma} S_\gamma$ is a $\Gamma$-graded ring with $S_\gamma = E_\gamma \times F_\gamma$. Suppose $(D, \rho)$ is a $\Gamma$-graded epic $S$-division ring. Since $(1, 1) = (1, 0) + (0, 1)$ and $(1, 0)(0, 1) = (0, 0)$, then either $\rho(1, 0) = 0$ or $\rho(0, 1) = 0$. If $\rho(1, 0) = 0$, then $\rho(E \times \{0\}) = 0$ and if $\rho(0, 1) = 0$, then $\rho(\{0\} \times F) = 0$. Hence $S$ has only two epic $S$-division rings which are $E$ and $F$. Note that none of them is a universal $\Gamma$-graded epic $R$-division ring. On the other hand, both are universal localizations. For example $E$ is the universal localization of $S$ at $(\{(a, 0) | a \in h(E) \setminus \{0\}\})$.

Let now $E$ be a $\Gamma$-graded field. Then the polynomial ring $E[x] = \bigoplus_{\gamma \in \Gamma} E[x]_\gamma$ is a $\Gamma$-graded ring with 

$$E[x]_\gamma = E_\gamma[x] = \{a_0 + a_1x + \cdots + a_nx^n | a_i \in E_\gamma, n \in \mathbb{N}\}.$$ 

The ideal $(x^2)$ is a graded ideal of $E[x]$. Hence $T = E[x]/(x^2)$ is a $\Gamma$-graded local ring with maximal graded ideal $(x)/(x^2)$. Then $E$ is the unique $\Gamma$-graded epic $T$-division ring, and thus $E$ is a universal $\Gamma$-graded epic $T$-division ring. Notice that $E$ is not a universal localization at matrices in $\mathfrak{M}(R)$ because the matrices which become invertible in $E$ are already invertible in $T$ since $E$ is the $\Gamma$-graded residue division ring of $T$.

The ring $U = T \times F$ with $T$ as before and $F$ a $\Gamma$-graded field has $E$ and $F$ as $\Gamma$-graded epic $U$-division rings, but only $F$ is a universal localization.

5. Malcolmson’s Criterion

Throughout this section, let $\Gamma$ be a group.

In this section, we show that the natural extension of the results and arguments of the paper by P. Malcolmson [25] work for $\Gamma$-graded rings. The main results of this section, and the only ones that will be used later, are Theorem 5.1 and Corollary 5.2. The proof of Theorem 5.1 is very technical and most of this long section is devoted to prove it.

In this section, for the ease of exposition, we use the following notation. By the expression $A$ is a homogenous matrix, we mean $A \in \mathfrak{M}_n(R)$. We will also use the terms homogeneous row, homogeneous column to emphasize that the matrix in question is a row or a column, respectively. If $A \in \mathfrak{M}_{m \times n}(R)[\overline{\beta}]$, but we do not want to make reference to the size of $A$, we will say $A$ is a homogenous matrix of distribution $(\alpha, \beta)$. Also, the sequence $\overline{\alpha}_\gamma$ will be denoted by $\alpha_\gamma$ for each $\overline{\alpha} \in \Gamma^n$ and $\gamma \in \Gamma$.

Theorem 5.1. Let $R$ be a $\Gamma$-graded ring and $\Sigma$ be a gr-lower semimultiplicative subset of $\mathfrak{M}(R)$. Consider the canonical homomorphism of $\Gamma$-graded rings $\lambda : R \rightarrow R_\Sigma$. For $\gamma \in \Gamma$, a homogeneous element $\gamma \in R_\Sigma$ belongs to $\ker \lambda$ if and only if there exist $L, M, P, Q \in \Sigma$, homogeneous rows $J, U$ and homogeneous columns $W, V$ such

$$T \times F$$
that
\[
\begin{pmatrix}
L & 0 & W \\
0 & M & 0 \\
0 & J & r
\end{pmatrix} = \begin{pmatrix}
P \\
Q \\
V
\end{pmatrix}
\]
where \(P, U, Q, V\) have distributions \((\pi, \omega), (\gamma, \omega), (\omega, \theta), (\omega, e)\) respectively.

**Corollary 5.2.** Let \(\Gamma\) be a group, \(R\) be a \(\Gamma\)-graded ring and \(\Sigma\) be a gr-multiplicative subset of \(\mathcal{M}(R)\) consisting of gr-full matrices. Then \(R_\Sigma\) is a nonzero \(\Gamma\)-graded ring.

**Proof.** It is enough to prove that \(1 \in R_e\) is not in the kernel of the canonical homomorphism of graded rings \(\lambda: R \rightarrow R_\Sigma\). Suppose that \(1 \in \ker \lambda\). Then, by Theorem 5.1, there exist \(L, M, P, Q \in \Sigma\), homogeneous rows \(J, U\) and homogeneous columns \(W, V\) such that
\[
\begin{pmatrix}
L & 0 & W \\
0 & M & 0 \\
0 & J & r
\end{pmatrix} = \begin{pmatrix}
P \circ \Sigma \\
Q \circ \Sigma \\
V \circ \Sigma
\end{pmatrix}
\]
where \(P, U, Q, V\) have distributions \((\pi, \omega), (\gamma, \omega), (\omega, \theta), (\omega, e)\) respectively. Making elementary column operations, we obtain
\[
\begin{pmatrix}
L & -W J & W \\
0 & M & 0 \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
P \circ \Sigma \\
Q \circ \Sigma \\
V \circ \Sigma
\end{pmatrix}
\]
where \(P, U, Q, V\) have distributions \((\pi, \omega), (\gamma, \omega), (\omega, \theta), (\omega, e)\) respectively. Since \(\Sigma\) is gr-multiplicative, it is also upper gr-semimultiplicative by Remark 3.1. Thus, the matrix \(\begin{pmatrix}
L & -W J & W \\
0 & M & 0 \\
0 & 0 & 1
\end{pmatrix}\) is not gr-full, a contradiction. Therefore, \(1 \notin \ker \lambda\).

\(\square\)

5.1. **Equivalence relation.** Let \(\Gamma\) be a group and \(R\) be a \(\Gamma\)-graded ring. Let \(\Sigma\) be a gr-lower semimultiplicative subset of \(\mathcal{M}(R)\).

For \(\gamma \in \Gamma\), let \((T_\Sigma)\gamma\) be the set of 5-tuples \((F, A, X, \alpha, \beta)\) where \(A \in \Sigma\) of distribution \((\alpha, \beta)\), \(F\) is a homogeneous row of distribution \((\gamma, \beta)\), and \(X\) is a homogeneous column of distribution \((\alpha, e)\).

Let \((F, A, X, \alpha, \beta), (G, B, Y, \delta, \epsilon) \in (T_\Sigma)\gamma\). We say that
\[
(F, A, X, \alpha, \beta) \sim (G, B, Y, \delta, \epsilon),
\]
if and only if there exist \(L, M, P, Q \in \Sigma\), homogeneous rows \(J, U\) and homogeneous columns \(W, V\) such that
\[
\begin{pmatrix}
A & 0 & 0 & 0 & X \\
0 & B & 0 & 0 & Y \\
0 & 0 & L & 0 & W \\
0 & 0 & 0 & M & 0 \\
F & -G & 0 & 0 & J
\end{pmatrix} = \begin{pmatrix}
P \circ \Sigma \\
Q \circ \Sigma \\
V \circ \Sigma
\end{pmatrix}
\]
where \(P, U, Q, V\) have distributions \((\pi, \omega), (\gamma, \omega), (\omega, \theta), (\omega, e)\) respectively, and, if we think of
\[
\pi = \pi_1 \ast \pi_2 \ast \pi_3 \ast \pi_4 \quad \text{and} \quad \theta = \theta_1 \ast \theta_2 \ast \theta_3 \ast \theta_4
\]
then \(\pi_1 = \alpha, \pi_2 = \delta, \theta_1 = \beta, \theta_2 = \epsilon\).

The right hand side of (5.1) will also be denoted by
\[
\begin{pmatrix}
P_{11} & P_{12} & P_{13} & P_{14} \\
P_{21} & P_{22} & P_{23} & P_{24} \\
P_{31} & P_{32} & P_{33} & P_{34} \\
P_{41} & P_{42} & P_{43} & P_{44}
\end{pmatrix}
\begin{pmatrix}
Q_{11} & Q_{12} & Q_{13} & Q_{14} & V_1 \\
Q_{21} & Q_{22} & Q_{23} & Q_{24} & V_2 \\
Q_{31} & Q_{32} & Q_{33} & Q_{34} & V_3 \\
Q_{41} & Q_{42} & Q_{43} & Q_{44} & V_4
\end{pmatrix}
\]
Lemma 5.3. Let \((F, A, X, \alpha, \beta), (G, B, Y, \delta, \varepsilon) \in (T_\Sigma)_n\) such that there is a factorization as a product of homogeneous matrices of any of these forms with \(L, M, P, Q \in \Sigma\) and with the corresponding distributions

\[
\begin{pmatrix}
A & 0 & | & X \\
0 & B & | & Y \\
F & -G & | & 0
\end{pmatrix} = \begin{pmatrix}
P & | & V \\
Q & | & 0
\end{pmatrix}
\]

(1)

\[
\begin{pmatrix}
A & 0 & | & X \\
0 & B & | & Y \\
0 & 0 & | & M \\
F & -G & | & 0
\end{pmatrix} = \begin{pmatrix}
P & | & V \\
Q & | & 0
\end{pmatrix}
\]

(2)

\[
\begin{pmatrix}
A & 0 & | & X \\
0 & B & | & Y \\
0 & 0 & | & L \\
F & -G & | & J
\end{pmatrix} = \begin{pmatrix}
P & | & V \\
Q & | & 0
\end{pmatrix}
\]

(3)

then \((F, A, X, \alpha, \beta) \sim (G, B, Y, \delta, \varepsilon)\).

Proof. (1) Suppose \(P, U, Q, V\) have distributions \((\pi, \omega), (\gamma, \omega), (\omega, \theta), (\omega, \varepsilon)\) where \(\pi_1 = \alpha, \pi_2 = \delta, \theta_1 = \beta\) and \(\theta_2 = \varepsilon\), and that we have the factorization

\[
\begin{pmatrix}
A & 0 & | & X \\
0 & B & | & Y \\
F & -G & | & 0
\end{pmatrix} = \begin{pmatrix}
P_{11} & P_{12} & | & V_1 \\
P_{21} & P_{22} & | & V_2 \\
U_1 & U_2 & | & 0
\end{pmatrix}
\]

Thus, we have the factorization

\[
\begin{pmatrix}
A & 0 & 0 & 0 & | & X \\
0 & B & 0 & 0 & | & Y \\
0 & 0 & 1 & 0 & | & 1 \\
0 & 0 & 0 & 1 & | & 0 \\
F & -G & 0 & 1 & 0 & | & 0
\end{pmatrix} = \begin{pmatrix}
P_{11} & P_{12} & 0 & 0 & | & V_1 \\
P_{21} & P_{22} & 0 & 0 & | & V_2 \\
0 & 0 & 1 & 0 & | & 1 \\
0 & 0 & 0 & 1 & | & 0
\end{pmatrix}
\]

where the factors of the right hand side have distributions

\((\alpha \ast \delta \ast \varepsilon \ast \gamma \ast \gamma, \omega_1 \ast \omega_2 \ast \varepsilon \ast \gamma \ast \gamma)\), \((\omega_1 \ast \omega_2 \ast \varepsilon \ast \gamma \ast \gamma)\), \((\beta \ast \varepsilon \ast \gamma \ast \gamma \ast \varepsilon)\).

(2) Suppose \(P, U, Q, V\) have distributions \((\pi, \omega), (\gamma, \omega), (\omega, \theta), (\omega, \varepsilon)\) where \(\pi_1 = \alpha, \pi_2 = \delta, \theta_1 = \beta\) and \(\theta_2 = \varepsilon\) and we have the factorization:

\[
\begin{pmatrix}
A & 0 & 0 & | & X \\
0 & B & 0 & | & Y \\
0 & 0 & L & | & W \\
F & -G & 0 & 1 & | & 0
\end{pmatrix} = \begin{pmatrix}
P_{11} & P_{12} & P_{13} & | & V_1 \\
P_{21} & P_{22} & P_{23} & | & V_2 \\
P_{31} & P_{32} & P_{33} & | & V_3 \\
U_1 & U_2 & U_3 & | & 0
\end{pmatrix}
\]

Thus, we have the following equality

\[
\begin{pmatrix}
A & 0 & 0 & 0 & | & X \\
0 & B & 0 & 0 & | & Y \\
0 & 0 & L & 0 & | & W \\
0 & 0 & 0 & 1 & | & 1 \\
F & -G & 0 & 1 & 0 & | & 0
\end{pmatrix} = \begin{pmatrix}
P_{11} & P_{12} & P_{13} & 0 & | & V_1 \\
P_{21} & P_{22} & P_{23} & 0 & | & V_2 \\
P_{31} & P_{32} & P_{33} & 0 & | & V_3 \\
0 & 0 & 0 & 0 & 1 & | & 0 \\
U_1 & U_2 & U_3 & 1 & | & 0
\end{pmatrix}
\]

where the factors of the right hand side have distributions

\((\alpha \ast \delta \ast \pi_3 \ast \gamma \ast \gamma, \omega_1 \ast \omega_2 \ast \omega_3 \ast \gamma \ast \gamma)\), \((\omega_1 \ast \omega_2 \ast \omega_3 \ast \gamma \ast \gamma)\), \((\beta \ast \varepsilon \ast \theta_3 \ast \gamma \ast \gamma)\).
(3) Suppose \( P, U, Q, V \) have distributions \((\pi, \omega), (\gamma, \omega), (\omega, \theta), (\omega, \varepsilon)\) where \( \pi_1 = \alpha, \pi_2 = \delta, \theta_1 = \beta \) and \( \theta_2 = \varepsilon \) and that we have the factorization

\[
\begin{pmatrix}
A & 0 & 0 & X \\
0 & B & 0 & Y \\
0 & 0 & M & 0 \\
F & -G & J & 0
\end{pmatrix} =
\begin{pmatrix}
P_{11} & P_{12} & P_{13} & 0 \\
P_{21} & P_{22} & P_{23} & 0 \\
P_{31} & P_{32} & P_{33} & 0 \\
U_1 & U_2 & U_3 & J
\end{pmatrix}
\begin{pmatrix}
Q_{11} & Q_{12} & Q_{13} & V_1 \\
Q_{21} & Q_{22} & Q_{23} & V_2 \\
Q_{31} & Q_{32} & Q_{33} & V_3 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

Thus, we have the factorization

\[
\begin{pmatrix}
A & 0 & 0 & 0 & X \\
0 & B & 0 & 0 & Y \\
0 & 0 & M & 0 & 0 \\
F & -G & 0 & J & 0
\end{pmatrix} =
\begin{pmatrix}
P_{11} & P_{12} & P_{13} & 0 \\
P_{21} & P_{22} & P_{23} & 0 \\
P_{31} & P_{32} & P_{33} & 0 \\
U_1 & U_2 & U_3 & M
\end{pmatrix}
\begin{pmatrix}
Q_{11} & Q_{12} & Q_{13} & V_1 \\
Q_{21} & Q_{22} & Q_{23} & V_2 \\
Q_{31} & Q_{32} & Q_{33} & V_3 \\
0 & 0 & -1 & 1
\end{pmatrix}
\]

where the factors of the right hand side have distribution

\[(\alpha \ast \omega \ast \beta \ast \varepsilon \ast \gamma, \; \omega_1 \ast \omega_2 \ast \omega_3 \ast \theta_3, \; \beta \ast \varepsilon \ast \theta_3 \ast \pi_3 \ast \pi_3 \ast \varepsilon)\), respectively.

\[\square\]

**Lemma 5.4.** For each \( \gamma \in \Gamma \), the relation \( \sim \) defined in \((T_\Sigma)_\gamma\) is an equivalence relation.

**Proof.** Let \((F, A, X, \alpha, \beta), (G, B, Y, \delta, \varepsilon), (H, C, Z, \zeta, \eta) \in (T_\Sigma)_\gamma\).

The relation \( \sim \) is symmetric. Indeed, we have the factorization

\[
\begin{pmatrix}
A & 0 & X \\
0 & A & X \\
F & -G & J
\end{pmatrix} =
\begin{pmatrix}
I & 0 & 0 \\
0 & I & -A \\
0 & 0 & F
\end{pmatrix}
\begin{pmatrix}
A & 0 & X \\
F & -F & 0
\end{pmatrix}
\]

where the factors are homogeneous matrices that have distributions \((\alpha \ast \alpha \ast \gamma, \; \alpha \ast \beta)\) and \((\alpha \ast \beta, \; \beta \ast \beta \ast \varepsilon)\), respectively. This shows that \((F, A, X, \alpha, \beta) \sim (F, A, X, \alpha, \beta)\).

Suppose now that \((F, A, X, \alpha, \beta) \sim (G, B, Y, \delta, \varepsilon)\). Then \(L, M, P, Q \in \Sigma\), homogeneous rows \(J, U\) and homogeneous columns \(W, V\) such that

\[
\begin{pmatrix}
A & 0 & 0 & 0 & X \\
0 & B & 0 & 0 & Y \\
0 & 0 & L & 0 & W \\
0 & 0 & 0 & M & 0 \\
F & -G & 0 & J & 0
\end{pmatrix} =
\begin{pmatrix}
P_{11} & P_{12} & P_{13} & P_{14} & 0 \\
P_{21} & P_{22} & P_{23} & P_{24} & 0 \\
P_{31} & P_{32} & P_{33} & P_{34} & 0 \\
P_{41} & P_{42} & P_{43} & P_{44} & 0 \\
U_1 & U_2 & U_3 & U_4 & J
\end{pmatrix}
\begin{pmatrix}
Q_{11} & Q_{12} & Q_{13} & Q_{14} & V_1 \\
Q_{21} & Q_{22} & Q_{23} & Q_{24} & V_2 \\
Q_{31} & Q_{32} & Q_{33} & Q_{34} & V_3 \\
Q_{41} & Q_{42} & Q_{43} & Q_{44} & V_4
\end{pmatrix}
\]

where \(P, U, Q, V\) have distributions \((\pi, \omega), (\gamma, \omega), (\omega, \theta), (\omega, \varepsilon)\), respectively, and \(\pi_1 = \alpha, \pi_2 = \delta, \theta_1 = \beta, \theta_2 = \varepsilon\). Then we have the factorization

\[
\begin{pmatrix}
B & 0 & 0 & 0 & 0 & X \\
0 & A & 0 & 0 & 0 & 0 \\
0 & 0 & B & 0 & 0 & 0 \\
0 & 0 & 0 & L & 0 & W \\
0 & 0 & 0 & 0 & M & 0 \\
G & -F & 0 & -J & 0 & G
\end{pmatrix} =
\begin{pmatrix}
I & 0 & 0 & 0 & 0 & 0 \\
0 & P_{11} & P_{12} & P_{13} & P_{14} & 0 \\
0 & P_{21} & P_{22} & P_{23} & P_{24} & 0 \\
0 & P_{31} & P_{32} & P_{33} & P_{34} & 0 \\
0 & P_{41} & P_{42} & P_{43} & P_{44} & 0 \\
-U_1 & -U_2 & -U_3 & -U_4 & G
\end{pmatrix}
\begin{pmatrix}
B & 0 & 0 & 0 & 0 & 0 & X \\
Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{16} & 0 \\
Q_{22} & Q_{23} & Q_{24} & Q_{25} & Q_{26} & 0 \\
Q_{32} & Q_{33} & Q_{34} & Q_{35} & Q_{36} & 0 \\
Q_{42} & Q_{43} & Q_{44} & Q_{45} & Q_{46} & 0 \\
0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix}
\]

where the factors have distributions \((\delta \ast \alpha \ast \delta \ast \pi_3 \ast \pi_3 \ast \delta \ast \gamma, \; \delta \ast \omega_1 \ast \omega_2 \ast \omega_3 \ast \omega_4 \ast \varepsilon)\) and \((\delta \ast \omega_1 \ast \omega_2 \ast \omega_3 \ast \omega_4 \ast \varepsilon, \; \varepsilon \ast \beta \ast \varepsilon \ast \theta_3 \ast \theta_4 \ast \varepsilon \ast \varepsilon)\) respectively. Hence, \((G, B, Y, \delta, \varepsilon) \sim (F, A, X, \alpha, \beta)\), and the symmetric property of the relation \( \sim \) is proved.

Now we proceed to prove that \( \sim \) satisfies the transitive property. Suppose that \((F, A, X, \alpha, \beta) \sim (G, B, Y, \delta, \varepsilon)\) and \((G, B, Y, \delta, \varepsilon) \sim (H, C, Z, \zeta, \eta)\). Hence, there exist \(L, M, P, Q \in \Sigma\), homogeneous rows \(J, U\) and homogeneous columns \(W, V\) as in...
and there exist \( L', M', P', Q' \in \Sigma \), homogeneous rows \( J', U' \) and homogeneous columns \( W', V' \) such that

\[
\begin{pmatrix}
B & 0 & 0 & 0 & Y \\
0 & C & 0 & 0 & Z \\
0 & 0 & L' & 0 & W' \\
0 & 0 & 0 & M' & 0 \\
G - H & 0 & J' & 0
\end{pmatrix} = \begin{pmatrix}
P'_{11} & P'_{12} & P'_{13} & P'_{14} \\
P'_{21} & P'_{22} & P'_{23} & P'_{24} \\
P'_{31} & P'_{32} & P'_{33} & P'_{34} \\
P'_{41} & P'_{42} & P'_{43} & P'_{44} \\
U'_1 & U'_2 & U'_3 & U'_4
\end{pmatrix} \begin{pmatrix}
Q'_{11} & Q'_{12} & Q'_{13} & Q'_{14} \\
Q'_{21} & Q'_{22} & Q'_{23} & Q'_{24} \\
Q'_{31} & Q'_{32} & Q'_{33} & Q'_{34} \\
Q'_{41} & Q'_{42} & Q'_{43} & Q'_{44}
\end{pmatrix} \begin{pmatrix}
V' \\
W \\
M' \\
0
\end{pmatrix},
\]

where \( P', U', Q', V' \) have distributions \((\pi', \omega'), (\gamma, \omega'), (\alpha', \theta'), (\omega', \epsilon)\), respectively, and \( \pi' = \delta, \pi' = \zeta, \theta' = \epsilon, \theta' = \eta \). Then we have the factorization of the matrix

\[
\begin{pmatrix}
C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & L & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & L' & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & C & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & M & 0 \\
H - F & 0 & 0 & 0 & 0 & -G & H & 0 & -J' & J
\end{pmatrix}
\]

as a product of the matrices

\[
\begin{pmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & P'_{11} & P'_{12} & P'_{13} & P'_{14} & 0 & 0 & 0 & 0 \\
0 & P'_{21} & P'_{22} & P'_{23} & P'_{24} & 0 & 0 & 0 & 0 \\
0 & P'_{31} & P'_{32} & P'_{33} & P'_{34} & 0 & 0 & 0 & 0 \\
0 & P'_{41} & P'_{42} & P'_{43} & P'_{44} & 0 & 0 & 0 & 0 \\
0 -P'_{31}-P'_{22}-P'_{33}-P'_{24} & P'_{31} & P'_{32} & P'_{33} & P'_{34} & 0 & 0 & 0 & 0 \\
0 -P'_{41}-P'_{32}-P'_{43}-P'_{34} & P'_{41} & P'_{42} & P'_{43} & P'_{44} & 0 & 0 & 0 & 0 \\
0 -P'_{11}-P'_{22}-P'_{13}-P'_{24} & P'_{11} & P'_{12} & P'_{13} & P'_{14} & 0 & 0 & 0 & 0 \\
0 & -P'_{11} & -P'_{22} & -P'_{13} & -P'_{24} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I
\end{pmatrix}
\]

where the factors have distributions

\[
(\zeta \ast \alpha \ast \delta \ast \pi_3 \ast \pi_4 \ast \pi_5 \ast \theta_4), \qquad (\zeta \ast \omega_1 \ast \omega_2 \ast \omega_3 \ast \omega_4 \ast \omega_5 \ast \theta_4).
\]

This factorization implies that \((F, A, X, \alpha, \beta) \sim (H, C, Z, \pi, \eta)\), as desired. 

\[\square\]

### 5.2. Operations

Let \( \gamma \in \Gamma \).

If \((F', A', X', \alpha', \beta') \in (T_{\Sigma})_{\gamma}\), then we define

\[
(F', A', X', \alpha', \beta') + (F, A, X, \alpha, \beta) = \left( F', \begin{pmatrix} A' & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} X' \end{pmatrix}, \alpha' \ast \alpha, \beta' \ast \beta \right).
\]

Note that it belongs to \((T_{\Sigma})_{\gamma}\).

If \((F', A', X', \alpha', \beta') \in (T_{\Sigma})_{\gamma'}\) and \((F, A, X, \alpha, \beta) \in (T_{\Sigma})_{\gamma}\), then we define

\[
(F', A', X', \alpha', \beta') \cdot (F, A, X, \alpha, \beta) = \left( \begin{pmatrix} 0 & F' \\ 0 & X' \end{pmatrix}, \begin{pmatrix} A & 0 \\ -X'F & A' \end{pmatrix}, \begin{pmatrix} X \end{pmatrix}, \alpha \ast \alpha', \beta \ast \beta' \right).
\]

Note that this element belongs to \((T_{\Sigma})_{\gamma \ast \gamma'}\) because the homogeneous matrix \((0 F')\) has distribution \((\gamma' \gamma, \beta \ast \beta' \gamma)\) and the homogeneous matrix \((X \ 0)\) has distribution \((\alpha \ast \alpha', \gamma, \epsilon)\).
If \((F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma\), we define
\[-(F, A, X, \alpha, \beta) = (-F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma.\]

Finally, if \(r \in R_\gamma\), we define
\[\mu(r) = (r, 1, 1, e, e) \in (T_\Sigma)_\gamma.\]

Now we prove a series of lemmas that show the compatibility of the operations just defined and the equivalence relation \(\sim\).

**Lemma 5.5.** The following assertions hold true.

1. If \((F', A, X, \alpha, \beta), (F, A, X, \alpha, \beta) \in (T_\Sigma)_\beta\), then
\[(F', A, X, \alpha, \beta) + (F, A, X, \alpha, \beta) \sim (F' + F, A, X, \alpha, \beta).\]
2. If \((F, A, X', \alpha, \beta), (F, A, X, \alpha, \beta) \in (T_\Sigma)_\beta\), then
\[(F, A, X', \alpha, \beta) + (F, A, X, \alpha, \beta) \sim (F, A, X' + X, \alpha, \beta).\]
3. If \(r \in R_\gamma\) and \((F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma\), then
\[\mu(r) \cdot (F, A, X, \alpha, \beta) \sim (rF, A, X, \alpha, \beta) \in (T_\Sigma)_{\gamma_r}.\]
4. If \((F', A', X', \alpha', \beta') \in (T_\Sigma)_{\gamma'}\) and \(r \in R_\gamma\), then
\[(F', A', X', \alpha', \beta') \cdot (r) \sim (F', A', X', r, \alpha', \beta') \in (T_\Sigma)_{\gamma \gamma'}.\]

**Proof.** (1) It follows from the following factorization
\[
\begin{pmatrix}
A & 0 & 0 & X
0 & A & 0 & X
0 & 0 & A & X
F' & F & -F' & -F
\end{pmatrix}

= \begin{pmatrix}
I & 0 & 0 & X
I & A & 0 & X
I & 0 & A & X
0 & F & -F' & -F
\end{pmatrix}

\begin{pmatrix}
A & 0 & 0 & X
-1 & 1 & 0 & 0
-1 & 0 & 1 & 0
\end{pmatrix}
\]

where the factors of the right hand side have distributions \((\alpha \ast \alpha' \ast \alpha \ast \gamma, \alpha \ast \beta \ast \beta)\) and \((\alpha \ast \beta \ast \beta, \beta \ast \beta \ast \beta \ast e)\), respectively.

(2) It follows from the equality
\[
\begin{pmatrix}
A & 0 & 0 & X'
0 & A & 0 & X'
0 & 0 & A & X' + X
F' & F & -F & -F
\end{pmatrix}

= \begin{pmatrix}
I & 0 & 0 & X'
0 & I & 0 & X'
0 & 0 & I & X'
0 & 0 & -F & -F
\end{pmatrix}

\begin{pmatrix}
A & 0 & 0 & X'
0 & A & 0 & X'
0 & 0 & -F & -F
-1 & -1 & 1 & 0
\end{pmatrix}
\]

where the factors of the right hand side have distributions \((\alpha \ast \alpha \ast \alpha \ast \gamma, \alpha \ast \alpha \ast \beta)\) and \((\alpha \ast \alpha \ast \beta, \beta \ast \beta \ast \beta \ast e)\), respectively.

(3) It follows from the factorization
\[
\begin{pmatrix}
A & 0 & 0 & X
-F & 1 & 0 & 0
0 & 0 & A & X
0 & 0 & r & -rF
\end{pmatrix}

= \begin{pmatrix}
I & 0 & 0 & X
0 & 0 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & r & -rF
\end{pmatrix}

\begin{pmatrix}
A & 0 & 0 & X
-F & 1 & 0 & 0
-1 & 0 & 0 & 0
\end{pmatrix}
\]

where the factors of the right hand side have distributions \((\alpha \ast \gamma \ast \alpha \ast \gamma \ast \gamma, \alpha \ast \gamma \ast \beta)\) and \((\alpha \ast \gamma \ast \beta, \beta \ast \gamma \ast \beta \ast e)\), respectively.

(4) It follows from the factorization
\[
\begin{pmatrix}
1 & 0 & 0 & 1
-A' & 0 & 0 & 0
0 & A' & X' & 0
0 & -A' & X' & -F
\end{pmatrix}

= \begin{pmatrix}
1 & 0 & 0 & 1
0 & I & 0 & 0
X' & I & A' & 0
-1 & -I & 0 & 0
\end{pmatrix}

\begin{pmatrix}
1 & 0 & 0 & 1
-A' & 0 & 0 & 0
0 & -I & 0 & 0
\end{pmatrix}
\]

where the factors of the right hand side have distributions \((e \ast \alpha' \gamma \ast \alpha' \gamma \ast \gamma \ast \gamma, e \ast \alpha' \gamma \ast \beta' \gamma)\) and \((e \ast \alpha' \gamma \ast \beta' \gamma, e \ast \beta' \gamma \ast \beta' \gamma \ast e)\), respectively. \qed
Lemma 5.6. The relation \(\sim\) is compatible with the operations defined on the 
\((T_\Sigma)\gamma\). More precisely, the following assertions hold true.

1. For \(x', x \in (T_\Sigma)\gamma\), then \(x + x' \sim x' + x\).
2. For \(x', x, y \in (T_\Sigma)\gamma\), such that \(x \sim y\), then \(x' + x \sim x' + y\) and \(x + x' \sim y + x'\).
3. For \(x, y \in (T_\Sigma)\gamma\) and \(x' \in (T_\Sigma)\gamma\), such that \(x \sim y\), then \(x' x \sim x' y\) and \(xy \sim x'y'\).
4. For \(x, y \in (T_\Sigma)\gamma\), such that \(x \sim y\), then \(-x \sim -y\).

Proof. (1) Let \((F', A', X', \alpha', \beta')\), \((F, A, X, \alpha, \beta) \in (T_\Sigma)\gamma\). The equality

\[
\begin{pmatrix}
A' & 0 & 0 & 0 & X' \\
0 & A & 0 & 0 & X \\
0 & 0 & A & 0 & X \\
0 & 0 & 0 & A' & x \\
F' & F & -F & -G & 0
\end{pmatrix}
= 
\begin{pmatrix}
I & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 \\
0 & I & A & 0 & 0 \\
0 & I & 0 & A' & 0 \\
-1 & I & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
A' & 0 & 0 & 0 & X' \\
0 & A & 0 & 0 & X \\
0 & 0 & A' & 0 & x \\
0 & 0 & 0 & A & 0 \\
-1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

where the factors of the right hand side have distributions \((\alpha' \ast \alpha \ast \alpha' \ast \gamma, \alpha' \ast \alpha' \ast \gamma, \alpha' \ast \alpha' \ast \gamma')\) and \((\alpha' \ast \alpha \ast \beta' \ast \beta' \ast \gamma, \beta' \ast \beta \ast \beta' \ast \gamma' \ast \varepsilon)\), respectively, shows (1).

(2) First note that, by (1), it is enough to prove that \(x' + x \sim x' + y\). Now let \(\gamma, \gamma' \in \Gamma\), let \((F', A', X', \alpha', \beta') \in (T_\Sigma)\gamma\) and let \((F, A, X, \alpha, \beta), (G, B, Y, \delta, \varepsilon) \in (T_\Sigma)\gamma\), be such that \((F, A, X, \alpha, \beta) \sim (G, B, Y, \delta, \varepsilon)\). Thus, there exist \(L, M, P, Q \in \Sigma\), homogenous rows \(J, U\), and homogeneous columns \(W, V\) as in \([5.1]\). The result follows because the matrix

\[
\begin{pmatrix}
A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A' & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & L & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M & 0 & 0 \\
F' & F & -F & -G & 0 & F & -G & 0 & J & 0 & 0 & 0
\end{pmatrix}
\]

can be expressed as the product of the homogenous matrices

\[
\begin{pmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A' & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & L & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F' & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & U_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & U_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & U_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & U_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

that have distributions \((\alpha' \ast \alpha \ast \alpha' \ast \delta \ast \pi_3 \ast \alpha \ast \delta \ast \pi_3 \ast \pi_4 \ast \gamma, \alpha' \ast \alpha \ast \delta \ast \pi_3 \ast \pi_4 \ast \gamma)\) and \((\alpha' \ast \alpha \ast \beta' \ast \delta \ast \pi_3 \ast \pi_4 \ast \gamma, \beta' \ast \beta \ast \gamma \ast \varepsilon \ast \theta_3 \ast \varepsilon \ast \theta_3 \ast \varepsilon \ast \theta_4 \ast \varepsilon)\), respectively.

(3) Let \(\gamma, \gamma' \in \Gamma\), let \((F', A', X', \alpha', \beta') \in (T_\Sigma)\gamma\) and let \((F, A, X, \alpha, \beta), (G, B, Y, \delta, \varepsilon) \in (T_\Sigma)\gamma\), be such that \((F, A, X, \alpha, \beta) \sim (G, B, Y, \delta, \varepsilon)\). Thus, there exist \(L, M, P, Q \in \Sigma\), homogenous rows \(J, U\), and homogeneous columns \(W, V\) as in \([5.1]\).

We prove first that

\[
(F', A', X', \alpha', \beta') \cdot (F, A, X, \alpha, \beta) \sim (F', A', X', \alpha', \beta') \cdot (G, B, Y, \delta, \varepsilon).
\]
It follows because the following homogeneous matrix
\[
\begin{pmatrix}
A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \\
-X'F & A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y \\
0 & 0 & -X'G & A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W \\
0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -X'F & X'G & 0 & -XJ & A' & 0 & 0 & 0 \\
0 & F' & 0 & -F' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
has the factorization, as a product of homogeneous matrices,
\[
\begin{pmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -I & 0 & 0 & 0 & 0 & P_{11} & P_{21} & P_{31} & P_{41} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & P_{21} & P_{22} & P_{32} & P_{42} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -I & 0 & 0 & P_{31} & P_{32} & P_{33} & P_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -I & 0 & P_{41} & P_{42} & P_{43} & P_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & I & 0 & -X'U_1 -X'U_2 -X'U_3 -X'U_4 A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0X \\
-X'F & A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -X'G & A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -X'F & X'G & 0 & -XJ & A' & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & F' & 0 & -F' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
where the factors have distributions
\[
(\alpha * \alpha' \gamma * \delta * \alpha' \gamma * \pi_3 * \alpha * \delta * \pi_3 * \pi_4 * \alpha' \gamma * \gamma' * \gamma \cdot \alpha * \alpha' \gamma * \delta * \alpha' \gamma * \pi_3 * \omega_1 * \omega_2 * \omega_3 * \omega_4 * \beta' * \gamma),
\]
\[
(\alpha * \alpha' \gamma * \delta * \alpha' \gamma * \pi_3 * \omega_1 * \omega_2 * \omega_3 * \omega_4 * \beta' * \gamma \cdot \beta' * \gamma * \epsilon * \beta' * \gamma * \theta_3 * \beta * \epsilon * \theta_3 * \theta_4 * \beta' * \gamma * \epsilon),
\]
respectively.

Now let \( \gamma, \gamma' \in \Gamma \), let \( (F, A, X, \alpha, \beta) \in (T_\Sigma)_{\gamma} \) and let \( (F', A', X', \alpha', \beta') \), \( (G', B', Y', \delta', \epsilon') \in (T_\Sigma)_{\gamma'} \) be such that \( (F', A', X', \alpha', \beta') \sim (G', B', Y', \delta', \epsilon') \). Thus, there exist \( L', M', P, Q \in \Sigma \), homogenous rows \( J', U \), and homogenous columns \( W', V \) such that
\[
\begin{pmatrix}
A' & 0 & 0 & 0 & X' \\
B' & 0 & 0 & 0 & Y' \\
0 & 0 & L' & 0 & W' \\
0 & 0 & 0 & M' & 0 \\
F' - G' & 0 & J' & 0 & 0 \\
\end{pmatrix}
= \begin{pmatrix}
P_{11} & P_{12} & P_{13} & P_{14} \\
P_{21} & P_{22} & P_{23} & P_{24} \\
P_{31} & P_{32} & P_{33} & P_{34} \\
P_{41} & P_{42} & P_{43} & P_{44} \\
U_1 & U_2 & U_3 & U_4 \\
\end{pmatrix}
\begin{pmatrix}
Q_{11} & Q_{12} & Q_{13} & Q_{14} & V_1 \\
Q_{21} & Q_{22} & Q_{23} & Q_{24} & V_2 \\
Q_{31} & Q_{32} & Q_{33} & Q_{34} & V_3 \\
Q_{41} & Q_{42} & Q_{43} & Q_{44} & V_4 \\
\end{pmatrix},
\]
where \( P, U, Q, V \) have distributions \( (\pi_1', \omega'), (\gamma', \omega'), (\omega', \theta'), (\omega', \epsilon) \), respectively, and \( \pi_1' = \alpha', \pi_2' = \delta', \theta_1' = \beta', \theta_2' = \epsilon' \). We show that
\[
(F', A', X', \alpha', \beta') \cdot (F, A, X, \alpha, \beta) \sim (G', B', Y', \delta', \epsilon') \cdot (F, A, X, \alpha, \delta).
\]
It follows because the following homogeneous matrix
\[
\begin{pmatrix}
A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X
\end{pmatrix}
\]
\[
\begin{pmatrix}
-A'F & A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X
\end{pmatrix}
\]
\[
\begin{pmatrix}
0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X
\end{pmatrix}
\]
\[
\begin{pmatrix}
0 & 0 & -Y'F & B' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X
\end{pmatrix}
\]
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X
\end{pmatrix}
\]
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & -W'F & L' & 0 & 0 & 0 & 0 & 0 & 0 & X
\end{pmatrix}
\]
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M'
\end{pmatrix}
\]
can be expressed as the following product of homogeneous matrices
\[
\begin{pmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

where the factors have distributions \((\alpha \ast \alpha' \ast \beta \ast \beta' \ast \gamma \ast \omega)\) and \((\alpha' \ast \beta' \ast \beta \ast \beta \ast \gamma \ast \omega)\) respectively.

(4) Let \((F, A, X, \alpha, \beta), (G, B, Y, \delta, \epsilon) \in (T_2)_{\gamma}\) such be that \((F, A, X, \alpha, \beta) \sim (G, B, Y, \delta, \epsilon)\) and, thus, there exist \(L, M, P, Q \in \Sigma\), homogeneous rows \(J, U\), and homogeneous columns \(W, V\) as in \((5, 1)\). The result follows because we have the factorization
\[
\begin{pmatrix}
A & 0 & 0 & 0 & X
\end{pmatrix}
\begin{pmatrix}
P_{11} & P_{12} & P_{13} & P_{14}
\end{pmatrix}
\begin{pmatrix}
Q_{11} & Q_{12} & Q_{13} & Q_{14}
\end{pmatrix}
\begin{pmatrix}
V_1
\end{pmatrix}
\begin{pmatrix}
B & 0 & 0 & Y
\end{pmatrix}
\begin{pmatrix}
P_{21} & P_{22} & P_{23} & P_{24}
\end{pmatrix}
\begin{pmatrix}
Q_{21} & Q_{22} & Q_{23} & Q_{24}
\end{pmatrix}
\begin{pmatrix}
V_2
\end{pmatrix}
\begin{pmatrix}
0 & L & 0 & W
\end{pmatrix}
\begin{pmatrix}
P_{31} & P_{32} & P_{33} & P_{34}
\end{pmatrix}
\begin{pmatrix}
Q_{31} & Q_{32} & Q_{33} & Q_{34}
\end{pmatrix}
\begin{pmatrix}
V_3
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & M
\end{pmatrix}
\begin{pmatrix}
P_{41} & P_{42} & P_{43} & P_{44}
\end{pmatrix}
\begin{pmatrix}
Q_{41} & Q_{42} & Q_{43} & Q_{44}
\end{pmatrix}
\begin{pmatrix}
V_4
\end{pmatrix}
\begin{pmatrix}
-F & G & 0 & -J
\end{pmatrix}
\begin{pmatrix}
-U_1 & -U_2 & -U_3 & -U_4
\end{pmatrix}
\]

where the factors have distributions \((\alpha' \ast \beta \ast \beta' \ast \gamma \ast \omega)\) and \((\alpha' \ast \beta \ast \beta' \ast \gamma \ast \omega)\) respectively.

\[\square\]

5.3. Graded ring structure. We define \((\mathcal{R}_2)_{\gamma}\) as the set of equivalence classes in \((T_2)_{\gamma}\) under the equivalence relation \(\sim\). The equivalent class of \((F, A, X, \alpha, \beta) \in (T_2)_{\gamma}\) will be denoted by \([F, A, X, \alpha, \beta]\).

In Section \((5, 2)\) we proved that the operation \(+\) is well defined in \((\mathcal{R}_2)_{\gamma}\) for each \(\gamma \in \Gamma\).

Lemma 5.7. Let \(\gamma \in \Gamma\). Then \((\mathcal{R}_2)_{\gamma}\) is an abelian group with sum defined by
\[
[F', A', X', \alpha', \beta'] + [F, A, X, \alpha, \beta] = \left((F' \quad F), \begin{pmatrix} A' & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} X' \\ X \end{pmatrix}, \alpha' \ast \alpha, \beta' \ast \beta\right)
\]
Proof. The operation is well defined and commutative by Lemma 5.6(2) and (1).
Now we show that the operation is associative. Let \([F'', A'', X'', \alpha'', \beta'']\), \([F', A', X', \alpha', \beta']\), \([F, A, X, \alpha, \beta]\) \((T_\Sigma)_\gamma\). Then
\[
[F'', A'', X'', \alpha'', \beta''] + ([F', A', X', \alpha', \beta'] + [F, A, X, \alpha, \beta])
\]
\[
= [F'', A'', X'', \alpha'', \beta''] + \left(\begin{array}{ccc}
A'' & 0 & 0 \\
0 & A' & 0 \\
0 & 0 & A
\end{array}\right) \cdot \left(\begin{array}{ccc}
X'' & 0 \\
0 & X' \\
0 & 0 & X
\end{array}\right) \cdot \left(\begin{array}{ccc}
\alpha' & \alpha'' & \beta' & \beta'' \\
0 & 0 & 1 & 0
\end{array}\right)
\]
\[
= \left(\begin{array}{ccc}
A'' & 0 & 0 \\
0 & A' & 0 \\
0 & 0 & A
\end{array}\right) \cdot \left(\begin{array}{ccc}
X'' & 0 \\
0 & X' \\
0 & 0 & X
\end{array}\right) \cdot \left(\begin{array}{ccc}
\alpha'' & \alpha' & \beta'' & \beta'
\end{array}\right)
\]
\[
= \left(\begin{array}{ccc}
0 & 0 \\
0 & 0 \\
0 & 0
\end{array}\right)
\]

as desired.

The element \(\mu(0) = [0, 1, 1, e, e]\) is the zero element. Indeed, let \((F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma\). Then we have the following factorization
\[
\left(\begin{array}{ccc}
A & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & A
\end{array}\right) \cdot \left(\begin{array}{ccc}
X & 0 \\
0 & X \\
0 & 0
\end{array}\right) = \left(\begin{array}{ccc}
I & 0 & 0 \\
I & 0 & 0 \\
0 & 0 & -F
\end{array}\right) \cdot \left(\begin{array}{ccc}
A & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)
\]
by Lemma 5.5 where the factors have distributions \((\alpha * e * \alpha * \gamma, \alpha * e * \beta)\) and \((\alpha * e * \beta, \beta * e * e)\), respectively. Thus, \([F, F, A, X, \alpha, \beta] + [0, 1, 1, e, e] = [F, A, X, \alpha, \beta]\).

Given \((F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma\), the element \([-F, A, X, \alpha, \beta]\) is well defined by Lemma 5.6(4). We claim that it is the additive inverse of \([F, A, X, \alpha, \beta]\) in \(R_\gamma\).

Thus, consider the following factorization
\[
\left(\begin{array}{ccc}
A & 0 & 0 \\
0 & A & 0 \\
0 & 0 & 1
\end{array}\right) \cdot \left(\begin{array}{ccc}
X & 0 \\
0 & X \\
0 & 0
\end{array}\right) = \left(\begin{array}{ccc}
I & 0 & 0 \\
I & 0 & 0 \\
0 & 0 & -F
\end{array}\right) \cdot \left(\begin{array}{ccc}
A & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)
\]
where the factors have distributions \((\alpha * e * \alpha * \gamma, \alpha * \beta * e)\) and \((\alpha * \beta * e, \beta * \beta * e * e)\), respectively. It shows that \([F, A, X, \alpha, \beta] + [-F, A, X, \alpha, \beta] = [0, 1, 1, e, e]\), as claimed. \(\square\)

In Section 5.2 we showed that the product functions \((R_\Sigma)_\gamma \times (R_\Sigma)_\gamma \rightarrow (R_\Sigma)_\gamma \gamma\) are well defined. Now we define \(R_\Sigma = \bigoplus_{\gamma \in \Gamma}(R_\Sigma)_\gamma\). By the foregoing lemma, it is an additive group. We now prove that it is a \(\Gamma\)-graded ring with the induced product.

Lemma 5.8. \(R_\Sigma\) is a \(\Gamma\)-graded ring with the product determined by the rule
\[
[F', A', X', \alpha', \beta'] \cdot [F, A, X, \alpha, \beta] = \left(\begin{array}{ccc}
0 & 0 \\
-A' F & A'
\end{array}\right) \cdot \left(\begin{array}{ccc}
X & 0 \\
0 & 0
\end{array}\right) \cdot \left(\begin{array}{ccc}
\alpha & \alpha' & \beta' \gamma
\end{array}\right)
\]
for any \((F', A', X', \alpha', \beta') \in (T_\Sigma)_\gamma\) and \((F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma\).

Proof. By Lemma 5.6(3), the product is well defined.

By Lemma 5.5(3) and (4), the identity element is \([1, 1, 1, e, e]\).

Now we proceed to show that the product is associative. Let \((F'', A'', X'', \alpha'', \beta'') \in (T_\Sigma)_\gamma\), \((F', A', X', \alpha', \beta') \in (T_\Sigma)_\gamma\) and \((F, A, X, \alpha, \beta) \in (T_\Sigma)_\gamma\).
which shows that the product is associative.

It remains to show that the distributive laws are satisfied. Let \((F', A', X', \alpha', \beta')\), \((G', B', Y', \delta', \varepsilon') \in (T_\Sigma)_{\gamma}\), and \((F, A, X, \alpha, \beta) \in (T_\Sigma)_{\gamma}\). First note that

\[
([F', A', X', \alpha', \beta'] + [G', B', Y', \delta', \varepsilon']) \cdot [F, A, X, \alpha, \beta] =
\begin{bmatrix}
A \\
-X'F \\
-Y'F
\end{bmatrix}
\begin{bmatrix}
0 \\
A' \\
B'
\end{bmatrix}
\begin{bmatrix}
X \\
0 \\
0
\end{bmatrix}
\alpha \ast \alpha' \gamma \ast \delta' \gamma \ast \beta \ast \beta' \gamma \ast \delta' \varepsilon' \gamma.
\] (5.4)

Second observe that

\[
[F', A', X', \alpha', \beta'] \cdot [F, A, X, \alpha, \beta] + [G', B', Y', \delta', \varepsilon'] \cdot [F', A', X', \alpha', \beta'] =
\begin{bmatrix}
A \\
-X'F \\
-Y'F
\end{bmatrix}
\begin{bmatrix}
A \\
A' \\
0
\end{bmatrix}
\begin{bmatrix}
X \\
0 \\
0
\end{bmatrix}
\alpha \ast \alpha' \gamma \ast \alpha' \delta' \gamma \ast \beta' \gamma \ast \beta \ast \varepsilon' \gamma.
\] (5.5)

The fact that (5.3) equals (5.5) follows because the homogeneous matrix

\[
\begin{pmatrix}
A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \\
-X'F & A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-Y'F & 0 & B' & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B' \\
0 & 0 & -F' & G' & 0 & -F' & 0 & -G' & 0
\end{pmatrix}
\]

factorizes as the product of homogeneous matrices

\[
\begin{pmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -X'F & A' & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 \\
0 & 0 & 0 & -Y'F & B' & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \\
-X'F & A' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-Y'F & 0 & B' & 0 & 0 & 0 & 0 & 0 & 0 \\
-I & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & -I & 0 & 0 & I & 0 & 0 & 0 & 0 \\
-I & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & -I & 0 & 0 & 0 & I & 0 & 0
\end{pmatrix}
\]

where the factors have distributions

\[
(\alpha \ast \alpha' \gamma \ast \delta' \gamma \ast \alpha \ast \alpha' \gamma \ast \alpha \ast \delta' \gamma \ast \delta' \gamma \ast \beta \ast \beta' \gamma \ast \beta \ast \beta' \gamma \ast \delta' \varepsilon' \gamma),
\]

\[
(\alpha \ast \alpha' \gamma \ast \delta' \gamma \ast \beta \ast \beta' \gamma \ast \beta \ast \delta' \gamma \ast \delta' \varepsilon' \gamma \ast \beta \ast \beta' \gamma \ast \beta \ast \delta' \gamma \ast \delta' \varepsilon' \gamma),
\]

respectively.
Let now $\langle F', A', X', \alpha', \beta' \rangle \in (T_{\Sigma})_\gamma$, and $\langle F, A, X, \alpha, \beta \rangle, \langle G, B, Y, \delta, \varepsilon \rangle \in (T_{\Sigma})_\gamma$. First note that

$$\langle F', A', X', \alpha', \beta' \rangle \cdot \langle F, A, X, \alpha, \beta \rangle + \langle G, B, Y, \delta, \varepsilon \rangle =$$

$$\begin{pmatrix} 0 & 0 & F' \end{pmatrix}, \begin{pmatrix} A & 0 & 0 & 0 \\ -X'F & A' \end{pmatrix}, \begin{pmatrix} X \\ Y \\ 0 \end{pmatrix}, \alpha \ast \delta \ast \alpha', \beta \ast \varepsilon \ast \beta' \gamma .$$

Second observe that

$$\langle F', A', X', \alpha', \beta' \rangle \cdot \langle F', A', X', \alpha', \beta' \rangle \cdot \langle G, B, Y, \delta, \varepsilon \rangle =$$

$$\begin{pmatrix} 0 & 0 & F' \\ F' & F' \end{pmatrix}, \begin{pmatrix} A & 0 & 0 & 0 \\ -X'F A' \end{pmatrix}, \begin{pmatrix} X \\ 0 \\ 0 \end{pmatrix}, \alpha \ast \alpha' \gamma \ast \delta \ast \alpha' \gamma \ast \beta \ast \beta' \gamma \ast \varepsilon \ast \beta' \gamma .$$

The fact that (5.6) equals (5.7) follows because the homogeneous matrix

$$\begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 & 0 & X \\ -X'F A' & 0 & 0 & 0 & 0 & 0 & 0 & Y \\ 0 & 0 & B & 0 & 0 & 0 & 0 & X \\ 0 & 0 & 0 & A & 0 & 0 & 0 & Y \\ 0 & 0 & 0 & 0 & B & 0 & 0 & X \\ 0 & 0 & 0 & 0 & 0 & B' & 0 & 0 \\ -I' & 0 & 0 & 0 & 0 & 0 & -I' & 0 \\ 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -F' & 0 \end{pmatrix}$$

factorizes as the product of homogeneous matrices

$$\begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \end{pmatrix}, \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 & 0 & X \\ -X'F A' & 0 & 0 & 0 & 0 & 0 & 0 & Y \\ 0 & 0 & B & 0 & 0 & 0 & 0 & X \\ 0 & 0 & 0 & A & 0 & 0 & 0 & Y \\ 0 & 0 & 0 & 0 & B & 0 & 0 & X \\ 0 & 0 & 0 & 0 & 0 & B' & 0 & 0 \\ -I' & 0 & 0 & 0 & 0 & 0 & -I' & 0 \\ 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -F' & 0 \end{pmatrix}$$

where the factors have distributions

$$(\alpha \ast \alpha' \gamma \ast \delta \ast \alpha' \gamma \ast \alpha \ast \delta \ast \alpha' \gamma \ast \gamma' \gamma), \quad (\alpha \ast \alpha' \gamma \ast \delta \ast \alpha' \gamma \ast \beta \ast \varepsilon \ast \beta' \gamma), \quad (\alpha \ast \alpha' \gamma \ast \delta \ast \alpha' \gamma \ast \beta \ast \varepsilon \ast \beta' \gamma \ast \varepsilon \ast \beta' \gamma \ast \varepsilon \ast \beta' \gamma),$$

respectively. \hfill \Box

5.4. Universal localization property.

**Proposition 5.9.** Consider the map $\mu : R \to R_{\Sigma}$ determined by $\mu(r) = [r, 1, 1, e, e]$ for all $r \in R_{\gamma}$, $\gamma \in \Gamma$. Then the pair $(R_{\Sigma}, \mu)$ is the universal localization of $R$ at $\Sigma$.

**Proof.** By Lemma (5.5(1)) and (3), $\mu$ is a homomorphism of $\Gamma$-graded rings.

By $E_i$ we will denote the column matrix consisting of 1 as its $i$th-entry and all the other entries are zero, and by $E_i^T$ its transpose, the row matrix consisting of 1 as its $i$th-entry and all other entries are zero. T

Let $A = (a_{ij}) \in \Sigma$ an $n \times n$ homogeneous matrix of distribution $(\alpha, \beta)$.

We claim that the $n \times n$ matrix $B = (E_i^T, A, E_j, \alpha, \beta)_{ij}$ is the inverse of $A^n$.\hfill \Box
As before, we show that \([E_i^T, A, E_j, \alpha, \beta] \in R_{\beta \alpha_j^{-1}}\) because \(E_i^T\) has distribution \((\beta \alpha_j^{-1}, \beta \alpha_j^{-1})\), \(A\) has distribution \((\alpha \alpha_j^{-1}, \alpha \alpha_j^{-1})\) and \(E_j\) has distribution \((\alpha \alpha_j^{-1}, e)\). Thus, \([(E_i^T, A, E_j, \alpha, \beta)]_{ij}\) is homogeneous of distribution \((\beta, \alpha)\).

Second, using Lemma 5.5(3) and (1), we obtain that the product of the \(i\)-th line of \(A^\mu\) with the \(j\)-th column of \(B\) equals

\[
\sum_k \mu(a_{i,k})[E_k^T, A, E_j, \alpha, \beta] = \sum_k [a_{i,k}E_k^T, A, E_j, \alpha, \beta] = [\sum_k a_{i,k}E_k^T, A, E_j, \alpha, \beta] = [E_i^T A, A, E_j, \alpha, \beta] \in R_{\alpha \alpha_j^{-1}}.
\]

Third, we show that

\[
[E_i^T A, A, E_j, \alpha, \beta] = \mu(\delta_{ij}) = [\delta_{ij}, 1, 1, e, e] = \begin{cases} [1, 1, 1, e, e] & \text{if } i = j, \\ [0, 1, 1, e, e] & \text{if } i \neq j. \end{cases}
\]

It follows from the following factorization

\[
\begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ E_i^T A & -\delta_{ij} & 0 \end{pmatrix} E_j = \begin{pmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ E_i^T & -\delta_{ij} & 1 \end{pmatrix} \begin{pmatrix} A & 0 & E_j \\ 0 & 1 & 0 \end{pmatrix}
\]

where the factors have distributions \((\alpha \alpha_j^{-1} \ast e \ast \alpha \alpha_j^{-1}, \alpha \alpha_j^{-1} \ast e)\) and \((\alpha \alpha_j^{-1} \ast e, \beta \alpha_j^{-1} \ast e \ast e)\), respectively. Therefore \(B\) is the right inverse of \(A^\mu\).

Now we proceed to prove that \(B\) is the left inverse of \(A^\mu\). Using Lemma 5.5(4) and (2), we obtain that the product of the \(i\)-th line of \(B\) with the \(j\)-th column of \(A^\mu\) equals

\[
\sum_k [E_i^T, A, E_k, \alpha, \beta] \mu(a_{k,j}) = \sum_k [E_i^T, A, E_k a_{k,j}, \alpha, \beta] = [E_i^T, A, \sum_k E_k a_{k,j}, \alpha, \beta] = [E_i^T, A, A E_j, \alpha, \beta] \in R_{\beta \alpha_j^{-1}}.
\]

As before, we show that \([E_i^T, A, A E_j, \alpha, \beta] = \mu(\delta_{ij})\). It follows from

\[
\begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ E_i^T & -\delta_{ij} & 0 \end{pmatrix} A E_j = \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ E_i^T & -\delta_{ij} & 1 \end{pmatrix} \begin{pmatrix} I & 0 & E_j \\ 0 & 1 & 0 \end{pmatrix}
\]

where the factors have distributions \((\alpha \beta_j^{-1} \ast e \ast \beta \beta_j^{-1}, \beta \beta_j^{-1} \ast e)\) and \((\beta \beta_j^{-1} \ast e, \beta \beta_j^{-1} \ast e)\), respectively.

Therefore, the claim is proved.

It remains to prove that \(\mu: R \to \mathcal{R}_{\Sigma}\) is universal.

Note that if \((F, A, X, \alpha, \beta) \in (T_{\Sigma})_\gamma\), and we suppose that \(F = (f_1, \ldots, f_n)\) and

\[
X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},
\]

then

\[
F^\mu (A^\mu)^{-1} X^\mu = \sum_{i,j} \mu(f_{ij})[E_i^T, A, E_j, \alpha, \beta] \mu(x_{ij}) = \sum_{i,j} [f_{ij}E_i^T, A, E_j x_{ij}, \alpha, \beta] = [\sum_i f_i E_i^T, A, \sum_j E_j x_{ij}, \alpha, \beta] = [F, A, X, \alpha, \beta]. \tag{5.8}
\]

Let now \(S\) be a \(\Gamma\)-graded ring and \(\varphi: R \to S\) be a \(\Sigma\)-inverting homomorphism of graded rings. We define \(\Phi: \mathcal{R}_{\Sigma} \to S\) as follows. Let \((F, A, X, \alpha, \beta) \in (T_{\Sigma})_\gamma\), then

\[
\Phi([F, A, X, \alpha, \beta]) = F^\varphi (A^\varphi)^{-1} X^\varphi \in S_{\gamma}.
\]
Now we show that $\Phi$ is well defined. Let $(F, A, X, \alpha, \beta), (G, B, Y, \delta, \epsilon) \in (T_2)_{\gamma}$ be such that $(F, A, X, \alpha, \beta), (G, B, Y, \delta, \epsilon)$. Then there exist $L, M, P, Q \in \Sigma$, homogeneous rows $J, U$ and homogeneous columns $W, V$ such that

$$
\begin{pmatrix}
A & 0 & 0 & 0 & X \\
0 & B & 0 & 0 & Y \\
0 & 0 & L & 0 & W \\
F & -G & 0 & J & 0
\end{pmatrix} = \begin{pmatrix} P \\ U \end{pmatrix} \begin{pmatrix} Q & V \end{pmatrix}
$$

where $P, U, Q, V$ have distributions $(\pi, \omega), (\gamma, \omega), (\omega, \theta), (\omega, \epsilon)$, respectively, and $\pi_1 = \alpha, \pi_2 = \delta, \theta_1 = \beta, \theta_2 = \epsilon$. Then

$$
0 = U^\varphi V^\varphi = U^\varphi Q^\varphi (Q^\varphi)^{-1} (P^\varphi)^{-1} P^\varphi V^\varphi = (UQ)^\varphi ((PQ)^\varphi)^{-1} (PV)^\varphi
$$

which shows that $\Phi$ is well defined.

Let now $(F', A', X', \alpha', \beta') \in (T_2)_{\gamma}$. Then

$$
\Phi([F', A', X', \alpha', \beta'] + [F, A, X, \alpha, \beta]) = (F' F)^\varphi \left( \begin{pmatrix} A' \\ 0 \\ A \end{pmatrix} \right)^{-1} \left( \begin{pmatrix} X' \\ X \end{pmatrix} \right)^\varphi = (F' F)^\varphi \left( \begin{pmatrix} A' \varphi^{-1} \\ 0 \\ (A')^{-1} \end{pmatrix} \right) \left( \begin{pmatrix} X' \varphi \\ X \varphi \end{pmatrix} \right) = F' (A')^{-1} X' \varphi - G^\varphi (B')^{-1} Y' \varphi + 0 (L')^{-1} W' \varphi + J' (M')^{-1} 0 = F' (A')^{-1} X' \varphi - G^\varphi (B')^{-1} Y' \varphi
$$

Thus, $\Phi$ is an additive map.

Let $(F', A', X', \alpha', \beta') \in (T_2)_{\gamma}$, and $(F, A, X, \alpha, \beta) \in (T_2)_{\gamma}$. Then

$$
\Phi([F', A', X', \alpha', \beta'] \cdot [F, A, X, \alpha, \beta]) =
$$

$$
= (0 \ F')^\varphi \left( \begin{pmatrix} A' \\ -X'F \ A' \end{pmatrix} \right)^{-1} \left( \begin{pmatrix} X' \\ X \end{pmatrix} \right)^\varphi = (0 \ F')^\varphi \left( \begin{pmatrix} A' \varphi^{-1} \\ -X'F \ A' \varphi^{-1} \end{pmatrix} \right) \left( \begin{pmatrix} X' \varphi \\ 0 \end{pmatrix} \right) = F' (A')^{-1} X' \varphi - F^\varphi (A')^{-1} X' \varphi
$$

Hence, $\Phi$ is a homomorphism of graded rings. Clearly $\Phi \mu = \varphi$. The uniqueness of $\Phi$ now follows from (5.8). $\square$
5.5. **Proof of Theorem 5.1** By Proposition 5.9, \( \mu : R \rightarrow \mathcal{R}_\Sigma \) is the universal localization of \( R \) at \( \Sigma \). Thus \( \lambda(r) = 0 \) if and only if \( \mu(r) = 0 \).

Hence, suppose that \( r \in \mathcal{R}_\Sigma \) is such that \( \mu(r) = 0 \). It means that \( [r, 1, 1, e, e] \sim [0, 1, 1, e, e] \). Thus there exist \( L, M, P, Q \in \Sigma \), homogeneous lines \( J, U \) and homogeneous columns \( W, V \) such that

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & L & 0 & W \\
0 & 0 & 0 & M & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & J & 0
\end{pmatrix}
= \begin{pmatrix}
P_{11} & P_{12} & P_{13} & P_{14} \\
P_{21} & P_{22} & P_{23} & P_{24} \\
P_{31} & P_{32} & P_{33} & P_{34} \\
P_{41} & P_{42} & P_{43} & P_{44} \\
U_1 & U_2 & U_3 & U_4
\end{pmatrix}
\begin{pmatrix}
Q_{11} & Q_{12} & Q_{13} & Q_{14} & V_1 \\
Q_{21} & Q_{22} & Q_{23} & Q_{24} & V_2 \\
Q_{31} & Q_{32} & Q_{33} & Q_{34} & V_3 \\
Q_{41} & Q_{42} & Q_{43} & Q_{44} & V_4
\end{pmatrix}
\]

where \( P \) has distribution \((\pi, \omega)\), \( U \) has distribution \((\gamma, \omega)\), \( Q \) has distribution \((\omega, \theta)\) and \( V \) has distribution \((\omega, e)\). Now the following equality

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & L & 0 & W \\
0 & 0 & 0 & M & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & J & 0
\end{pmatrix}
= \begin{pmatrix}
P_{11} & P_{12} & P_{13} & P_{14} & 0 \\
P_{21} & P_{22} & P_{23} & P_{24} & 0 \\
P_{31} & P_{32} & P_{33} & P_{34} & 0 \\
P_{41} & P_{42} & P_{43} & P_{44} & 0 \\
1 & 0 & 0 & 0 & 1 \\
-1 & -1 & -1 & -1 & 1
\end{pmatrix}
\]

where the homogeneous matrices of the right hand side have distributions \((e * e * \pi_1 * \pi_4 * e \gamma, \omega_1 * \omega_2 * \omega_3 * \omega_4 * e)\) and \((\omega_1 * \omega_2 * \omega_3 * \omega_4 * e, e * e * \theta_3 * \theta_4 * e * e)\), respectively, shows the result.

Conversely, suppose there exist \( L, M, P, Q \in \Sigma \), homogeneous rows \( J, U \) and homogeneous columns \( W, V \) such that

\[
\begin{pmatrix}
L & 0 & W \\
0 & M & 0 \\
0 & J & 0
\end{pmatrix}
= \begin{pmatrix}
P & U \\
Q & V
\end{pmatrix}
\]

where \( P, U, Q, V \) have distributions \((\pi, \omega), (\gamma, \omega), (\omega, \theta), (\omega, e)\), respectively. It follows that \([0, 1, 1, e, e] \sim [r, 1, 1, e, e]\) because

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & L & 0 & W \\
0 & 0 & 0 & M & 0 \\
0 & -r & 0 & 0 & J
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & P \\
0 & 0 & Q \\
0 & 0 & 1
\end{pmatrix}
\]

where the factors of the right hand side have distributions \((e * e * \pi * \gamma, e * e * \omega)\) and \((e * e * \omega, e * e * \theta * e)\), respectively.

6. **A gr-prime matrix ideal yields a graded division ring, and vice versa**

This Section is the adaptation to the graded context of the first part of [7, Section 7.3] and the second part of [7, Section 7.4]. For the proof of the main result Theorem 6.3, instead of using an analog of the first part of [7, Section 7.4], we use Corollary 6.2. Theorem 6.5 could also have been proved via a graded version of [23] that can be found in [17].

Throughout this section, let \( \Gamma \) be a group.

Let \( R \) be a \( \Gamma \)-graded ring. If \((K, \varphi)\) is a graded epic \( R \)-field, the set

\[
\{ A \in \mathcal{M}(R) : A^\varphi \text{ is not invertible over } K \}
\]

will be called the singular kernel of \((K, \varphi)\). Now we show that gr-singular kernels are gr-prime matrix ideals. The aim of this section is to show that gr-singular
kernels determine graded epic $R$-division rings in a similar way as commutative $R$-fields are determined by prime ideals of $R$.

Given an $n \times n$ matrix $A$ with entries in $R$, if we write $A = \begin{pmatrix} A_1 & A_2 & \ldots & A_n \end{pmatrix}$ we understand that $A_1, \ldots, A_n$ are the columns of $A$. And if we write $A = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}$ we understand that $A_1, \ldots, A_n$ are the rows of $A$.

Given two matrices $A, B \in \mathfrak{M}(R)$, we define the **diagonal sum** of $A$ and $B$ as

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$ 

Notice that if $A \in M_{m}(R)[\alpha][\beta]$ and $B \in M_{n}(R)[\alpha'][\beta']$, then $A \oplus B \in M_{m+n}(R)[\alpha*\alpha'][\beta*\beta']$.

Let $A, B \in M_{n}(R)[\alpha][\beta]$. If they differ at most in the $i$-th column, then we define the **determinantal sum** of $A$ and $B$ with respect to the $i$-th column as

$$A \nabla B = (A_1 \ldots A_i + B_i \ldots A_n).$$ 

Similarly, if they differ at most in the $i$-th row we define the determinantal sum of $A$ and $B$ with respect to the $i$-th row as

$$A \nabla B = \begin{pmatrix} A_1 \\ \vdots \\ A_i + B_i \\ \vdots \\ A_n \end{pmatrix}.$$ 

The matrix $A \nabla B$, when defined, has the same distribution as $A$ and $B$.

Note that the operation $\oplus$ is associative. On the other hand, the operation $\nabla$ is not always defined, and as a consequence it is not associative.

Note that distributive laws are satisfied. More precisely, if $C$ is another homogeneous matrix, then $C \oplus (A \nabla B) = (C \oplus A) \nabla (C \oplus B)$ and $(A \nabla B) \oplus C = (A \oplus C) \nabla (B \oplus C)$ whenever $A \nabla B$ is defined.

On the other hand, if $B, C \in M_{n}(R)[\gamma][\delta]$ which differ on at most one column, $A \in M_{n}(R)[\alpha][\gamma]$, $D \in M_{n}(R)[\beta][\delta]$, then may happen that

$$A(B \nabla C) \neq AB \nabla AC, \quad (B \nabla C)D \neq BD \nabla CD,$$

because, for example, $AB$ and $AC$ ($BD, BC$) may differ on more than 1 row/column. But in some cases we can apply the distributive law. Let $X \in \mathfrak{M}(R)$ and suppose that either $X$ is a diagonal matrix, or $X$ is a permutation matrix, then

$$X(B \nabla C) = XB \nabla XC, \quad (B \nabla C)X = BX \nabla CX.$$ 

Moreover, we can regard $X \in M_{n}(R)[\alpha'][\gamma] \cap M_{n}(R)[\beta'][\delta]$ for some $\alpha', \beta' \in \Gamma^n$. Thus $X(B \nabla C) \in M_{n}(R)[\alpha'][\beta']$ and $(B \nabla C)X \in M_{n}(R)[\alpha'][\delta]$.

Let $\Gamma$ be a group and let $R$ be a $\Gamma$-graded ring. A subset $\mathcal{P}$ of $\mathfrak{M}(R)$ is a **gr-prime** matrix ideal if the following conditions are satisfied.

(PM1) $\mathcal{P}$ contains all the homogeneous matrices that are not gr-full;
(PM2) If $A, B \in \mathcal{P}$ and their determinantal sum (with respect to a row or column) exists, then $A \nabla B \in \mathcal{P}$;
(PM3) If $A \in \mathcal{P}$, then $A \oplus B \in \mathcal{P}$ for all $B \in \mathfrak{M}(R)$;
(PM4) For $A, B \in \mathfrak{M}(R)$, $A \oplus B \in \mathcal{P}$ implies that $A \in \mathcal{P}$ or $B \in \mathcal{P}$; 
(PM5) $1 \notin \mathcal{P}$;
(PM6) If $A \in \mathcal{P}$ and $E, F$ are permutation matrices of appropriate size, then $EAF \in \mathcal{P}$. 
We remark that when $\Gamma = \{1\}$, that is, the ungraded case, $(PM6)$ is a consequence of $(PM1)$–$(PM5)$ as shown in [7] (g) in p.431. We have not been able to obtain $(PM6)$ from the others in the general graded case.

**Proposition 6.1.** Let $R$ be a $\Gamma$-graded ring. Let $K$ be a $\Gamma$-almost graded division ring and $\varphi: R \to K$ be a homomorphism of $\Gamma$-almost graded rings. Then

$$\mathcal{P} = \{A \in \mathfrak{m}(R): A^\varphi \text{ is not invertible}\}$$

is a gr-prime matrix ideal. Therefore, the following assertions hold true.

1. If $(K, \varphi)$ is a $\Gamma$-graded epic $R$-division ring, then the gr-singular kernel of $(K, \varphi)$ is a $\Gamma$-gr-prime matrix ideal.

2. Let $N$ be a normal subgroup of $\Gamma$ and consider $R$ as a $\Gamma/N$-graded ring. Let $(K, \varphi)$ be a $\Gamma/N$-graded epic $R$-division ring. Then

$$\mathcal{P} = \{A \in \mathfrak{m}_R(R): A^\varphi \text{ is not invertible}\}$$

is a $\Gamma$-gr-prime matrix ideal.

**Proof.** Let $K$ be a $\Gamma$-almost graded division ring and $\varphi: R \to K$ be a homomorphism of $\Gamma$-almost graded rings.

First suppose that $K = DC(\varphi)$ and let

$$\Sigma = \mathfrak{m}(R) \setminus \mathcal{P} = \{A \in \mathfrak{m}(R): A^\varphi \text{ is invertible over } K\}.$$  

By Theorem 4.4(2), $R_\Sigma$ is a local ring. If $m$ is the maximal graded ideal of $R_\Sigma$, there exists a surjective homomorphism of $\Gamma$-almost graded rings $\Phi: R_\Sigma/m \to K$ such that the following diagram is commutative

$$\begin{CD}
R @>\lambda>> R_\Sigma @>\pi>> R_\Sigma/m \\
| @VV\varphi V @V\Phi VV |
K @>\bar{\Phi}>> K
\end{CD}$$

By Proposition 2.2(3), the sets $\{A \in \mathfrak{m}(R): A^{(\pi, \lambda)}$ is invertible over $R_\Sigma/m\}$ and $\{A \in \mathfrak{m}(R): A^{(\bar{\Phi}, \lambda)}$ is invertible over $K\}$ are equal. Because this last set equals $\Sigma$, we get that $\Sigma = \{A \in \mathfrak{m}(R): A^{(\pi, \lambda)}$ is invertible over $R_\Sigma/m\}$. Now, since $(R_\Sigma/m, \pi, \lambda)$ is a $\Gamma$-graded epic $R$-division ring, it is enough to prove (1). Thus, suppose that $K$ is a $\Gamma$-graded division ring and $(K, \varphi)$ is a $\Gamma$-graded epic $R$-division ring. Let $\mathcal{P} = \{A \in \mathfrak{m}(R): A^\varphi \text{ is not invertible over } K\}$.

If $A \in \mathfrak{m}(R)$ is not gr-full, then $A^\varphi$ is not gr-full. Since $K$ is a $\Gamma$-graded division ring, $A^\varphi$ is not invertible over $K$. Thus, $(PM1)$ is satisfied.

Let now $A, B \in \mathfrak{P}_{\mathfrak{m}(R)}$ such that $A\nabla B$ is defined. We may suppose that $A, B$ differ on the first column. Hence $A = (A_1, C_2, \ldots, C_n)$ and $B = (B_1, C_2, \ldots, C_n)$. Since $A^\varphi$ and $B^\varphi$ are not invertible over $K$, the columns of $A^\varphi$ and $B^\varphi$ are right linearly dependent over $K$. If the columns $C_2^\varphi, \ldots, C_n^\varphi$ are right linearly dependent over $K$, then the columns of $(A\nabla B)^\varphi$ are right linearly dependent over $K$ and thus $A\nabla B \in \mathcal{P}$. Hence we can suppose that there exist homogeneous elements $a_1, \ldots, a_n, b_1, \ldots, b_n \in K$, with $a_1, b_1 \neq 0$, such that

$$A_1a_1 + C_2a_2 + \cdots + C_na_n = 0, \quad B_1b_1 + C_2b_2 + \cdots + C_nb_n = 0.$$ 

But then

$$A_1 + B_1 + C_2(a_2a_1^{-1} + b_2b_1^{-1}) + \cdots + C_n(a_na_1^{-1} + b_nb_1^{-1}) = 0,$$

which shows that $A\nabla B \in \mathcal{P}$. Thus $(PM2)$ is proved.

Let $A \in \mathcal{P}$ and $B \in \mathfrak{m}(R)$, then $A^\varphi$ is not invertible over $K$, but then $A^\varphi \oplus B^\varphi = (A \oplus B)^\varphi$ is not invertible over $K$. It implies $(PM3)$. 

Now suppose that \( A, B \in \mathfrak{M}(R) \) are such that \( A \oplus B \in \mathcal{P} \). It means that the homogeneous matrix \( A^\varepsilon \oplus B^\varepsilon \) is not invertible over \( K \). It implies that either \( A^\varepsilon \) or \( B^\varepsilon \) is not invertible. That is, \( A \in \mathcal{P} \) or \( B \in \mathcal{P} \) and (PM4) follows.

Clearly, (PM5) is satisfied.

Let \( A \in \mathcal{P} \) and \( E, F \) be permutation matrices with entries in \( R \). Notice that \( E^\varepsilon, F^\varepsilon \) are permutation matrices with entries in \( K \). Thus, if \( (EAF)^\varepsilon = E^\varepsilon A^\varepsilon F^\varepsilon \) were invertible over \( K \), then \( A^\varepsilon = (E^\varepsilon)^{-1}(EAF)^\varepsilon(F^\varepsilon)^{-1} \) would be invertible over \( K \), a contradiction. Thus \( EAF \in \mathcal{P} \) and (PM6) is shown. \( \square \)

**Lemma 6.2.** Let \( R \) be a \( \Gamma \)-graded ring and \( \mathcal{P} \) be a gr-prime matrix ideal. Let \( A, B \in \mathfrak{M}(R) \). The following assertions hold true.

1. If \( A \) and \( B \) are such that \( C = A \nabla B \) exists and \( B \) is not gr-full. Then \( A \in \mathcal{P} \) if and only if \( C \in \mathcal{P} \).
2. Let \( A \in \mathcal{P} \). The result of adding a suitable right multiple of one column of \( A \) to another column again lies in \( \mathcal{P} \). More precisely, if \( A \in M_n(R)[\bar{\alpha}][\bar{\beta}] \) and \( a \in R \), then \( (A_1 \cdots A_{j-1} A_j + A_{j+1} A_{j+2} \cdots A_n) \) belongs to \( \mathcal{P} \).
3. If \( A \oplus B \in \mathcal{P} \), then \( B \oplus A \in \mathcal{P} \).
4. Suppose that \( A \in M_m(R)[\bar{\beta}][\bar{\gamma}] \) and \( B \in M_n(R)[\bar{\delta}][\bar{\varepsilon}] \). For \( C \in M_{m\times n}(R)[\bar{\alpha}][\bar{\beta}] \),

\[
\begin{pmatrix}
A & 0 \\
C & B
\end{pmatrix} \in \mathcal{P} \text{ if and only if } \begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix} \in \mathcal{P}
\]

Similarly, for \( C \in M_{m\times n}(R)[\bar{\delta}][\bar{\nu}] \),

\[
\begin{pmatrix}
A & C \\
0 & B
\end{pmatrix} \in \mathcal{P} \text{ if and only if } \begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix} \in \mathcal{P}
\]

5. The set \( \mathfrak{M}(R) \setminus \mathcal{P} \) is gr-multiplicative.
6. No identity matrix belongs to \( \mathcal{P} \).
7. Suppose that \( A \in M_n(R)[\bar{\alpha}][\bar{\beta}] \) and \( B \in M_n(R)[\bar{\beta}][\bar{\gamma}] \). Then \( AB \in \mathcal{P} \) if, and only if, \( A \oplus B \in \mathcal{P} \).
8. No invertible matrix in \( \mathfrak{M}(R) \) belongs to \( \mathcal{P} \).
9. Suppose that \( A \) and \( B \) are such that \( C = A \nabla B \) exists and \( B \in \mathcal{P} \). Then \( A \in \mathcal{P} \) if, and only if, \( C \in \mathcal{P} \).

**Proof.**
(1) By (PM1) and (PM2), if \( A \in \mathcal{P} \), then \( C \in \mathcal{P} \). Conversely, suppose that \( C \in \mathcal{P} \). Clearly \( C = \mathcal{C} \nabla \mathcal{B} \) where \( \mathcal{B} \) is obtained from \( \mathcal{B} \) changing the sign of a row or column. Now \( A \in \mathcal{P} \) because \( \mathcal{B} \) is not gr-full.

(2) Suppose that \( \bar{\beta} = \beta_1 \times \bar{\beta}_1 \) and \( c \in R_{\beta_2 \beta_2} \). If \( A = (A_1 A_2 \cdots A_n) \), then

\[
(A_1 + A_2 c A_2 \cdots A_n) = (A_1 A_2 \cdots A_n) \nabla (A_2 A_2 A_2 \cdots A_n)
\]

\[
= A \nabla (A_2 A_2 A_3 \cdots A_n) \begin{pmatrix}
c & 1 & 0 \\
0 & \ddots & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Thus, the right hand side is a determinantal sum of \( A \) and \( (A_2 A_2 A_2 \cdots A_n) \), which is not gr-full. Indeed, it is the product of \( (A_2 A_3 \cdots A_n) \in M_{(n-1)\times n}(R)[\bar{\beta}][\bar{\nu}] \) and

\[
\begin{pmatrix}
c & 1 & 0 \\
0 & \ddots & 0 \\
0 & 0 & 1
\end{pmatrix} \in M_{(n-1)\times n}(R)[\bar{\beta}][\bar{\nu}],
\]

respectively.

(3) It follows from (PM6).

(4) We show the first statement, the other can be proved analogously. If we write \( A = (A_1 A') \) and \( C = (C_1 C') \), then

\[
\begin{pmatrix}
A & 0 \\
C & B
\end{pmatrix} \begin{pmatrix}
0 & A' \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
A & 0 \\
C' & B
\end{pmatrix} \begin{pmatrix}
C_1 & A' \\
0 & 0
\end{pmatrix}
\]
The second matrix of the right hand side is a matrix with a submatrix that is a block of zeros of size \( m \times (n + 1) \). Since \( m + n + 1 > m + n \), that matrix is hollow and therefore not gr-full. By (1),

\[
\begin{pmatrix}
A & 0 \\
C & B
\end{pmatrix} \in \mathcal{P}
\text{ if and only if }
\begin{pmatrix}
A_1 & A' & 0 \\
0 & C' & B
\end{pmatrix} \in \mathcal{P}
\]

Similarly, one can repeat the argument applied to columns of \( A' \) and \( C' \) and so on, to obtain the desired result.

(5) Let \( \Sigma = 2\mathfrak{M}(R) \setminus \mathcal{P} \). By (PM5), \( 1 \in \Sigma \). By (PM4), \( A \oplus B \in \Sigma \) if \( A, B \in \Sigma \). Now (4) implies that \( \Sigma \) is lower gr-semimultiplicative. Finally, (PM6) shows that \( \Sigma \) is gr-multiplicative.

(6) It follows from (PM4) and (PM5).

(7) First notice that, by (6) and (PM4), a matrix \( C \in 2\mathfrak{M}(R) \) belongs to \( \mathcal{P} \) if and only if \( C \oplus I \in \mathcal{P} \) for the identity matrix \( I \) of the same size as \( C \).

We claim that \( C \in \mathcal{P} \) if and only if \( -C \in \mathcal{P} \). Indeed,

\[
\begin{pmatrix}
C & 0 \\
0 & I
\end{pmatrix} \in \mathcal{P} \iff \begin{pmatrix}
-C & 0 \\
0 & I
\end{pmatrix} \in \mathcal{P} \iff \begin{pmatrix}
0 & -C \\
-I & I
\end{pmatrix} \in \mathcal{P},
\]

and the claim is proved. Then

\[
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix} \in \mathcal{P} \iff \begin{pmatrix}
A & 0 \\
I & B
\end{pmatrix} \in \mathcal{P} \iff \begin{pmatrix}
A & -AB \\
I & 0
\end{pmatrix} \in \mathcal{P}
\]

and, by the claim, the result follows.

(8) If \( A \in M_n(R)[\alpha][\beta] \) is invertible, then \( A^{-1} \in M_n(R)[\beta][\alpha] \). Since \( AA^{-1} = I \notin \mathcal{P} \), (7) implies that \( A \oplus A^{-1} \notin \mathcal{P} \). Now (PM3) shows that \( A \notin \mathcal{P} \).

(9) By (PM2), if \( A \in \mathcal{P} \), then \( C \in \mathcal{P} \). Conversely, suppose that \( C \in \mathcal{P} \). Clearly \( A = C \nabla B' \) where \( B' \) is obtained from \( B \) changing the sign of a row or column. More precisely, \( B' \) is the product of \( B \) by a diagonal matrix \( D \) whose diagonal elements are 1 or \(-1\). Now \( B' \oplus D \in \mathcal{P} \) because \( B \in \mathcal{P} \). Thus \( B' \in \mathcal{P} \) by (7). Therefore \( A \in \mathcal{P} \) by (PM2). \( \square \)

The proof of Lemma 6.2 is very similar to the one for the ungraded case, see for example [7, p. 430–431]. The main difference is that we were not able to show [7, (d) p. 430] because not every multiple of a column can be added to another column so that the matrix remains homogeneous. As a consequence the proof of Lemma 6.2(7) is also different.

The following result is well known and can be found, for example, in [12, Proposition 1.1.31].

**Lemma 6.3.** Let \( R \) be a \( \Gamma \)-graded ring. Then \( R \) is a \( \Gamma \)-graded local ring if and only if \( R_\alpha \) is a local ring. \( \square \)

The proof of the following lemma follows the one given in the ungraded result in [7, Proposition 7.2.6].

**Lemma 6.4.** Let \( R \) be a \( \Gamma \)-graded ring, \( \Sigma \) be a gr-multiplicative subset of \( 2\mathfrak{M}(R) \) and \( \lambda : R \to R_\Sigma \) be the natural homomorphism of \( \Gamma \)-graded rings. Then \( R_\Sigma \) is a \( \Gamma \)-graded local ring if and only if it satisfies the following two conditions:

1. \( R_\Sigma \neq \{0\} \);
(2) For a matrix $A \in \Sigma_n[\alpha^* e][\beta^* e]$, if $B$, the $(n,n)$-minor of $A$, is such that $B^\lambda$ is not invertible over $R_\Sigma$, then $(A - e_{nn})^\lambda$ is invertible over $R_\Sigma$, where $e_{nn}$ denotes the matrix with 1 in the $(n,n)$ entry and zeros everywhere else.

Proof. Consider the canonical homomorphism of $\Gamma$-graded local rings $\lambda: R \rightarrow R_\Sigma$.

Suppose that $R_\Sigma$ is a $\Gamma$-graded local ring with maximal graded ideal $m$ and canonical homomorphism $\pi: R_\Sigma \rightarrow R_\Sigma/m$. Since $R_\Sigma$ is graded local, by definition, $R_\Sigma \neq \{0\}$. Recall that any matrix $C \in \mathfrak{M}(R_\Sigma)$ is invertible if and only if $C^\pi$ is invertible over $R/m$. Let $A \in \Sigma_n[\alpha^* e][\beta^* e]$ such that its $(n,n)$-minor $B$ is not invertible over $R_\Sigma$. It is enough to show that $(A - e_{nn})^\pi$ is invertible. Some non-trivial left linear combination (over the graded division ring $R/m$) with homogeneous coefficients of the rows of $B^\pi$ is zero. If we take the corresponding left linear combination of the first $n - 1$ rows of $A^\pi$, we obtain $(0,0,\ldots,0,c)$ where $c$ is homogeneous and $c \neq 0$, because $A^\pi$ is invertible. We now subtract from the last row of $A$, $c^{-1}$ times this combination of the other rows and obtain the matrix $A - e_{nn}$, which is therefore invertible in $R_\Sigma$ because it is the product of the matrix corresponding to those elementary operations on $A^\pi$ times $A^\pi$.

Conversely, suppose now that conditions (1) and (2) are satisfied. By Lemma 6.2, it is enough to prove that $(R_\Sigma)_c$ is a local ring. Let $x \in (R_\Sigma)_c$. By Lemma 5.3, there exist $\mathfrak{p}, \mathfrak{q} \in \Gamma^\ast$, $A \in \Sigma_n[\mathfrak{p}][\mathfrak{q}]$ and $x \in M_{n \times 1}(R_\Sigma)$ such that $a_1 = c$, $\beta_i = c$, $u_i = x$ and $A^\lambda u = e_i$. Since $\Sigma$ is $\Gamma$-multiplicative, we may suppose that $A \in \Sigma_n[\alpha^* e][\beta^* e]$, $u_i = x$ and $A^\lambda u = e_i$. Suppose $x$ is not invertible in $R_\Sigma$. Equivalently, by Lemma 5.3, the matrix $(A^\lambda e_i^\mathfrak{p})$ is not invertible in $R_\Sigma$. This implies that the $(n,n)$-minor of $(A^\lambda e_i^\mathfrak{p})$, which is the $(n,n)$-minor of $A$ is not invertible in $R_\Sigma$. Hence, $(A - e_{nn})^\lambda$ is invertible over $R_\Sigma$. Then the matrix $(A^\lambda)^{-1}(A - e_{nn})^\lambda = I - (A^\lambda)^{-1}e_{nn}$ is invertible in $R_\Sigma$. Since this matrix is of the form

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 & * \\
0 & 1 & \cdots & 0 & * \\
\vdots & \vdots & \ddots & 1 & * \\
0 & 0 & \cdots & 1 & 1 - x
\end{pmatrix},
\]

we obtain that $1 - x$ is invertible in $R_\Sigma$, as desired. \hfill \Box

Let $R$ be a graded ring and let $\mathcal{P}$ be a gr-prime matrix ideal. The universal localization of $R$ at the set $\Sigma = \mathfrak{M}(R) \setminus \mathcal{P}$ will be denoted by $R_\mathcal{P}$ (instead of $R_\Sigma$).

Theorem 6.5. Let $\Gamma$ be a group and $R$ be a $\Gamma$-graded ring. The following assertions hold true:

1. If $\mathcal{P}$ is any gr-prime matrix ideal of $R$, then the localization $R_\mathcal{P}$ is a graded local ring. Moreover, its residue class $\Gamma$-graded division ring is a $\Gamma$-graded epic $R$-division ring such that its gr-singular kernel equals $\mathcal{P}$.
2. If $(K, \varphi)$ is a $\Gamma$-graded epic $R$-division ring, with gr-singular kernel $\mathcal{P}$, then $\mathcal{P}$ is a gr-prime matrix ideal and the $\Gamma$-graded local ring $R_\mathcal{P}$ has residue class graded division ring $R$-isomorphic to $K$.

Proof. (1) First note that $\Sigma = \mathfrak{M}(R) \setminus \mathcal{P}$ is a gr-multiplicative subset of $\mathfrak{M}(R)$ by Lemma 6.2(5). By Corollary 5.2, since $\Sigma$ consists of gr-full matrices, we get that $R_\mathcal{P}$ is a nonzero $\Gamma$-graded ring.

Consider now an $n \times n$ matrix $A \in \Sigma_n[\alpha^* e][\beta^* e]$ and let $B$ be its $(n,n)$-minor. Suppose that $B^\lambda$ is not invertible over $R_\Sigma$. Hence, $B$ belongs to $\mathcal{P}$. Write $A = (A' A_n)$ where $A_n$ is the last column of $A$. Then

\[A - e_{nn} = (A' A_n) \nabla (A' - e_n).\]

Now note that $A = (A' A_n) \in \Sigma$, and that $(A' - e_n) \in \mathcal{P}$ because $B \in \mathcal{P}$ and $\mathcal{P}$ is gr-lower semimultiplicative. By Lemma 6.2(9), $A - e_{nn} \not\in \mathcal{P}$, or equivalently
$A - e_{nn} \in \Sigma$. Therefore $(A - e_{nn})^\lambda$ is invertible over $R_{\Sigma}$. Now Lemma 6.4 implies that $R_{\Sigma}$ is a $\Gamma$-graded local ring.

By Theorem 4.4(2)(a), the residue class graded division ring is a graded epic $R$-division ring. By construction, the singular kernel equals $\mathcal{P}$.

(2) By Proposition 5.1 $\mathcal{P}$ is a gr-prime matrix ideal.

By (1), $R_{\mathcal{P}}$ is a $\Gamma$-graded local ring and its residue class graded division ring is a graded epic $R$-division ring with singular kernel $\mathcal{P}$. Then, by Theorem 4.4(b)(ii), $K$ and the residue class graded division ring of $R_{\Sigma}$ are isomorphic $\Gamma$-graded $R$-rings.

The following is Theorem 4.5 but expressed in terms of gr-prime matrix ideals.

**Corollary 6.6.** Let $R$ be a $\Gamma$-graded ring, $(K_i, \varphi_i)$, $i = 1, 2$, be $\Gamma$-graded epic $R$-division rings with singular kernels $\mathcal{P}_i$, respectively. The following statements are equivalent.

1. There exists a gr-specialization from $K_1$ to $K_2$.
2. $\mathcal{P}_1 \subseteq \mathcal{P}_2$.
3. There exists a homomorphism $R_{\Sigma_2} \to R_{\Sigma_1}$ of $\Gamma$-graded $R$-rings.

Furthermore, if there exists a gr-specialization from $K_1$ to $K_2$ and another gr-specialization from $K_2$ to $K_1$, the $K_1$ and $K_2$ are isomorphic graded $R$-rings.

**Corollary 6.7.** Let $R$ be a $\Gamma$-graded ring and $(K, \varphi)$ be a graded epic $R$-division ring with singular kernel $\mathcal{P}$. Suppose that $\gamma \in \Gamma$. Consider the universal localization $\lambda: R \to R_{\mathcal{P}}$ and let $\Phi: R_{\mathcal{P}} \to K$ be the homomorphism of $\Gamma$-graded rings such that $\varphi = \Phi \lambda$.

1. Let $x \in K_{\gamma}$. Then $x = 0$ if and only if its numerator belongs to $\mathcal{P}$.
2. Let $x \in (R_{\mathcal{P}})_{\gamma}$. Then $x \in \ker \Phi$ if and only if its numerator belongs to $\mathcal{P}$.

**Proof.** Suppose that $(A_0 A_*)$ is the numerator of $x$.

(1) By Lemma 3.3(1), $x$ is invertible if and only if $(A_0 A_*)^\gamma$ is invertible over $K$. That is, if and only if $(A_0 A_*)$ belongs to $\mathcal{P}$.

(1) By Lemma 3.3(1), $x$ is invertible if and only if $(A_0 A_*)^\lambda$ is invertible over $R_{\mathcal{P}}$. Since $R_{\mathcal{P}}$ is a local ring with residue class graded division ring $R$-isomorphic to $K$, $x$ is invertible if and only if $(A_0 A_*)^\Phi \lambda$ is invertible over $K$. That is, $x \in \ker \Phi$ if and only if $(A_0 A_*)$ belongs to $\mathcal{P}$.

**Corollary 6.8.** Let $R$ and $R'$ be $\Gamma$-graded rings with gr-prime matrix ideals $\mathcal{P}$ and $\mathcal{P}'$, respectively, with corresponding graded epic $R$-division rings $(K, \varphi)$ and $(K', \varphi')$ respectively. Let $f: R \to R'$ be a homomorphism of $\Gamma$-graded rings. The following assertions hold true.

1. $f$ extends to a gr-specialization if, and only if, $\mathcal{P}^f \subseteq \mathcal{P}'$.
2. $f$ extends to a homomorphism $K \to K'$ if, and only if, $\mathcal{P}^f \subseteq \mathcal{P}'$ and $\Sigma^f \subseteq \Sigma'$, where $\Sigma = \mathfrak{M}(R) \setminus \mathcal{P}$ and $\Sigma' = \mathfrak{M}(R') \setminus \mathcal{P}'$.

**Proof.** (1) First note that the set $\mathcal{P}'' = \{ A \in \mathfrak{M}(R): A^f \in \mathcal{P}' \}$ is a gr-prime matrix ideal whose corresponding graded epic $R$-division ring is $\varphi'f: R \to DC(\varphi'f)$.

By Corollary 6.6 there exists a specialization from $(K, \varphi)$ to $(DC(\varphi'f))$ if, and only if, $\mathcal{P} \subseteq \mathcal{P}''$.

(2) If $\mathcal{P}^f \subseteq \mathcal{P}'$ and $\Sigma^f \subseteq \Sigma'$, then $\mathcal{P} = \mathcal{P}''$, and therefore the gr-specialization of (1) is in fact an isomorphism by Corollary 6.6.

7. Gr-matrix ideals

In this section, the concepts, arguments and proofs are an adaptation of the ones in [7, Section 7.3] to the graded context.
Throughout this section, let $\Gamma$ be a group.

Let $R$ be a $\Gamma$-graded ring. A subset $\mathcal{I}$ of $\mathfrak{M}(R)$ is a gr-matrix pre-ideal if the following conditions are satisfied.

(I1) $\mathcal{I}$ contains all the homogeneous matrices that are not gr-full;

(I2) If $A,B \in \mathcal{I}$ and their determinantal sum (with respect to a row or column) exists, then $A\nabla B \in \mathcal{I}$;

(I3) If $A \in \mathcal{I}$, then $A \oplus B \in \mathcal{I}$ for all $B \in \mathfrak{M}(R)$;

(I4) If $A \in \mathcal{I}$ and $E,F$ are permutation matrices of appropriate size, then $EAF \in \mathcal{I}$.

If, moreover, we have

(I5) For $A \in \mathfrak{M}(R)$, if $A \oplus 1 \in \mathcal{I}$, then $A \in \mathcal{I}$,

we call $\mathcal{I}$ a gr-matrix ideal.

Clearly, $\mathfrak{M}(R)$ is a gr-matrix ideal. A proper gr-matrix ideal is a gr-matrix ideal different from $\mathfrak{M}(R)$.

**Lemma 7.1.** Let $R$ be a $\Gamma$-graded ring and $\mathcal{I}$ be a gr-matrix pre-ideal. Let $A,B \in \mathfrak{M}(R)$. The following assertions hold true.

1. If $A$ and $B$ are such that $C = AB$ exists and $B$ is not gr-full. Then $A \notin \mathcal{I}$ if and only if $C \notin \mathcal{I}$.
2. Let $A \in \mathcal{I}$. The result of adding a suitable right multiple of one column of $A$ to another column again lies in $\mathcal{I}$. More precisely, if $A \in M_n(R)[\overline{\mathfrak{p}}][\overline{\mathfrak{q}}]$ and $a \in R_{\beta,\beta^{-1}}$, then $(A_1 \ldots A_{j-1} A_j + A_{\alpha} A_{j+1} \ldots A_n)$ belongs to $\mathcal{I}$.
3. If $A \oplus B \in \mathcal{I}$, then $B \oplus A \in \mathcal{I}$.
4. Suppose that $A \in M_m(R)[\overline{\mathfrak{p}}][\overline{\mathfrak{q}}]$ and $B \in M_n(R)[\overline{\mathfrak{p}}][\overline{\mathfrak{q}}]$. For $C \in M_{m \times n}(R)[\overline{\mathfrak{p}}][\overline{\mathfrak{q}}]$, 
\[
\begin{pmatrix}
A & 0 \\
C & B
\end{pmatrix} \in \mathcal{I} \text{ if and only if } \begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix} \in \mathcal{I}
\]

Similarly, for $C \in M_{m \times n}(R)[\overline{\mathfrak{p}}][\overline{\mathfrak{q}}]$, 
\[
\begin{pmatrix}
A & C \\
0 & B
\end{pmatrix} \in \mathcal{I} \text{ if and only if } \begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix} \in \mathcal{I}
\]

If, moreover, $\mathcal{I}$ is a gr-matrix ideal, then the following assertions hold true.

5. Suppose that $A \in M_n(R)[\overline{\mathfrak{p}}][\overline{\mathfrak{q}}]$, $B \in M_n(R)[\overline{\mathfrak{p}}][\overline{\mathfrak{q}}]$. Then $AB \in \mathcal{I}$ if and only if $A \oplus B \in \mathcal{I}$.
6. If $A$ and $B$ are such that $C = AB$ exists and $B \in \mathcal{I}$. Then $A \in \mathcal{I}$ if and only if $C \in \mathcal{P}$.
7. If an identity matrix $I_n$, $n \geq 1$, belongs to $\mathcal{I}$, then $\mathcal{I} = \mathfrak{M}(R)$

**Proof.** Note that (I1),(I2),(I3), (I4) are the same as (PM1),(PM2),(PM3),(PM6).

Hence (1)–(6) follow in exactly the same way as in Lemma 6.2.

To prove (7), note that if $I_n \in \mathcal{I}$, for some $n \geq 1$, an application of (I5), shows that $1 \times 1$ matrix $1 \in \mathcal{I}$. By (I3), any identity matrix $I_m$, $m \geq 1$, belongs to $\mathcal{I}$. Again, using (I3), $I_m \oplus A \in \mathcal{I}$ for any positive integer $m$ and matrix $A \in \mathfrak{M}(R)$. By (5), any $A \in \mathfrak{M}(R)$ belongs to $\mathcal{I}$, as desired.

One could think of defining a gr-prime matrix ideal as a gr-matrix ideal $\mathcal{I}$ such that the following two conditions are satisfied.

(I6) $\mathcal{I}$ is a proper gr-matrix ideal.

(I7) $\mathcal{I}$ satisfies (PM4).
We proceed to show that both definitions are equivalent. Let $P$ be a gr-prime matrix ideal, i.e. (PM1)–(PM6) in page 40 are satisfied. Clearly, $P$ satisfies (I1)–(I4) and (I7). By (PM5), $1 \not\in P$. Therefore, by (PM4), if $A \oplus 1 \in P$, then $A \in P$ for any $A \in \mathfrak{M}(R)$. Hence (I5) is satisfied. Again by (PM5), $P$ is a proper gr-matrix ideal. Conversely, suppose that $I$ satisfies (I1)–(I7). Clearly (PM1)–(PM4), (PM6) are satisfied. By Lemma (7.1) and (I6), (PM5) is satisfied, as desired.

It is easy to prove that any intersection of gr-matrix (pre-)ideals is again a gr-matrix (pre-)ideal. Thus, given a subset $S \subseteq \mathfrak{M}(R)$, we define the gr-matrix (pre-)ideal generated by $S$ as the intersection of gr-matrix (pre-)ideals $I$ that contain $S$. That is, $\bigcap_{S \subseteq I} I$. Note that this gr-matrix (pre-)ideal is contained in any gr-matrix (pre-)ideal that contains $S$.

Now we fix some notation that will be used in what follows.

Let $W \subseteq \mathfrak{M}(R)$. We say that a matrix $C \in \mathfrak{M}(R)$ is a determinantal sum of elements of $W$ if there exist $A_1, \ldots, A_m \in W$, $m \geq 1$, such that $A_1 \nabla A_2 \nabla \ldots \nabla A_m$ exists for some choice of parenthesis and equals $C$.

We will write $\mathcal{N}$ to denote the subset of $\mathfrak{M}(R)$ consisting of the matrices which are not gr-full.

We will denote the set of all identity matrices by $\mathcal{I}$.

If $X \subseteq \mathfrak{M}(R)$, we denote by $\mathcal{D}(X)$ the set of all matrices in $\mathfrak{M}(R)$ which are of the form $E(X \oplus A)F$ where $X \in X$, $A \in \mathfrak{M}(R)$ and $E, F$ are permutation matrices of appropriate sizes. We remark that we allow $A$ to be the empty matrix $\emptyset$.

Lemma 7.2. Let $R$ be a $\Gamma$-graded ring and $A$ be a gr-matrix pre-ideal. Suppose that $\Sigma \subset \mathfrak{M}(R)$ satisfies the following two conditions

(i) $1 \in \Sigma$;
(ii) if $P, Q \in \Sigma$, then $P \oplus Q \in \Sigma$.

Then the following assertions hold true

(1) The set $A/\Sigma := \{ A \in \mathfrak{M}(R) : A \oplus P \in A \text{ for some } P \in \Sigma \}$ is a gr-matrix ideal containing $A$.

(2) The gr-matrix ideal $A/\Sigma$ is proper if and only if $A \cap \Sigma = \emptyset$.

(3) The gr-matrix ideal $A/\mathcal{I}$ is the gr-matrix ideal generated by $A$.

Proof. (1) Let $A \in A$. By (I3), $A \oplus 1 \in A$. Since $1 \in \Sigma$, $A \in A/\Sigma$. Hence $A \subseteq A/\Sigma$ and, by (I1), all non gr-full matrices belong to $A$. Therefore $A/\Sigma$ satisfies (I1).

Let $A, B \in A/\Sigma$ be such that $A \nabla B$ is well defined. There exist $P, Q \in \Sigma$ such that $A \oplus P, B \oplus Q \in A$. By (I3), $A \oplus P \oplus Q$ and $B \oplus Q \oplus P$ belong to $A$. By (I4), $B \oplus P \oplus Q \in A$. Now $(A \nabla B) \oplus P \oplus Q = (A \oplus P \oplus Q) \nabla (B \oplus P \oplus Q) \in A$ by (I2). Hence $A \nabla B \in A/\Sigma$ and $A/\Sigma$ satisfies (I2).

Let $A \in A/\Sigma$ and $B \in \mathfrak{M}(R)$. There exists $P \in \Sigma$ such that $A \oplus P \in A$. By (I3), $A \oplus P \oplus B \in A$. Now (I1) implies that $A \oplus B \oplus P \in A$. Hence $A \oplus B \in A/\Sigma$, and $A/\Sigma$ satisfies (I3).

Let $A \in A/\Sigma$ and $E, F$ be permutation matrices of the same size as $A$. There exists $P \in \Sigma$ such that $A \oplus P \in A$. Since $E \oplus I$ and $F \oplus I$ are also permutation matrices, (I4) implies that $(E \oplus I)(A \oplus P)(F \oplus I) = EAF \oplus P \in A$. Hence $EAF \in A/\Sigma$ and (I4) is satisfied.

Let now $A \in \mathfrak{M}(R)$ such that $A \oplus 1 \in A/\Sigma$. Thus there exists $P \in \Sigma$ such that $A \oplus 1 \oplus P \in A$. Since $1 \oplus P \in \Sigma$, then $A \in A/\Sigma$ and $A/\Sigma$ satisfies (I5).

(2) Suppose that $A \cap \Sigma \neq \emptyset$. Let $P \in A \cap \Sigma$ and $M \in \mathfrak{M}(R)$. Then $P \oplus M \in A$ by (I3). By (I4), $M \oplus P \in A$. Hence $M \in A/\Sigma$. Therefore, $A/\Sigma = \mathfrak{M}(R)$.

Conversely, suppose that $A/\Sigma = \mathfrak{M}(R)$. Thus, $1 \in A/\Sigma$ and there exists $P \in \Sigma$ such that $1 \oplus P \in A$. Notice that $1 \oplus P \in \Sigma$, by (i) and (ii). Therefore $A \cap \Sigma \neq \emptyset$. 
Proof. (1) Clearly $\mathcal{I}$ satisfies conditions (i) and (ii). Thus $A/\mathcal{I}$ is a gr-matrix ideal that contains $A$ by (1). Let now $B$ be a gr-matrix ideal such that $A \subseteq B$. If $A \in A/\mathcal{I}$, then there exists $n \geq 1$ such that $A \oplus I_n \in A \subseteq B$. By applying (15) repeatedly, we obtain that $A \in B$, as desired. $\square$

Lemma 7.3. Let $R$ be a $\Gamma$-graded ring and let $\mathcal{X} \subseteq \mathfrak{M}(R)$. Let $\mathcal{A}(\mathcal{X})$ be the subset of $\mathfrak{M}(R)$ consisting of all the matrices that can be expressed as determinantal sum of elements of $\mathcal{N} \cup \mathcal{D}(\mathcal{X})$. The following assertions hold true.

1. $\mathcal{A}(\mathcal{X})$ is the gr-matrix pre-ideal generated by $\mathcal{X}$.
2. $\mathcal{A}(\mathcal{X})/\mathcal{I}$ is the gr-matrix ideal generated by $\mathcal{X}$.
3. The gr-matrix ideal generated by $\mathcal{X}$ is proper if and only if $\mathcal{A}(\mathcal{X}) \cap \mathcal{I} = \emptyset$.

Proof. (1) $\mathcal{X} \subseteq \mathcal{A}(\mathcal{X})$ because $X = I(X \oplus \mathcal{O})I$ for all $X \in \mathcal{X}$. By definition of $\mathcal{A}(\mathcal{X})$, every homogeneous matrix that is not gr-full belongs to $\mathcal{A}(\mathcal{X})$. By the same reason, if $A, B \in \mathcal{A}(\mathcal{X})$ and $A \cap B$ is defined, then $A \cap B \in \mathcal{A}(\mathcal{X})$.

Let $A \in \mathcal{A}(\mathcal{X})$ and $B \in \mathfrak{M}(R)$. That $A \oplus B \in \mathcal{A}(\mathcal{X})$ follows from the following three facts. First, for any $U, V \in \mathfrak{M}(R)$, when defined $(U \nabla V) \oplus M = (U \oplus M) \nabla (V \oplus M)$. Second, for $X \in \mathcal{X}$ and $U, M \in \mathfrak{M}(R)$ and permutation matrices $E, F$ of suitable size $E(X \oplus U)F \oplus M = (E \oplus I)(X \oplus U \oplus M)(F \oplus I)$. Third, if $U$ is not gr-full, then, for all $M \in \mathfrak{M}(R)$, $U \oplus M$ is not full for all $M \in \mathfrak{M}(R)$. Indeed, if $U = U_1 U_2$, then $U \oplus M = (U_1 \oplus M)(U_2 \oplus I)$.

If $A \in \mathcal{A}(\mathcal{X})$ and $E, F$ are permutation matrices of appropriate size, then $EAF \in \mathcal{A}(\mathcal{X})$. This follows from the following facts. First, if $U, V \in \mathfrak{M}(R)$ and $E, F$ are permutation matrices such that $E(A \nabla B)F$ is defined, then $E(A \nabla B)F = EAF \nabla EBF$.

Second, for $X \in \mathcal{X}$, $U \in \mathfrak{M}(R)$ and permutation matrices $E, F, P, Q$ of appropriate sizes then $P(E(X \oplus U)F)Q = (PE)(X \oplus U)(FQ)$. Third, if $U \in \mathfrak{M}(R)$ is not gr-full, and $E, F$ are permutation matrices of appropriate size, then $EUF$ is not gr-full. Indeed, if $U = U_1 U_2$, then $EUF = (EU_1)(U_2 F)$.

Therefore, $\mathcal{A}(\mathcal{X})$ is a gr-matrix pre-ideal that contains $\mathcal{X}$.

Let now $\mathcal{B}$ be a gr-matrix pre-ideal such that $\mathcal{X} \subseteq \mathcal{B}$. By (11), $\mathcal{X} \subseteq \mathcal{B}$. By (13) and (14), $E(X \oplus A)F \in \mathcal{B}$ for all $X \in \mathcal{X}$, $A \in \mathfrak{M}(R)$ and permutation matrices $E, F$ of appropriate size. By (12), $\mathcal{A}(\mathcal{X}) \subseteq \mathcal{B}$.

(2) Any gr-matrix ideal containing $\mathcal{X}$, must contain $\mathcal{A}(\mathcal{X})$. By Lemma 7.2(3), the result follows.

(3) By (2), the gr-matrix ideal generated by $\mathcal{X}$ equals $\mathcal{A}(\mathcal{X})/\mathcal{I}$. By Lemma 7.2(2), $\mathcal{A}(\mathcal{X})/\mathcal{I}$ is proper if and only if $\mathcal{A}(\mathcal{X}) \cap \mathcal{I} = \emptyset$. $\square$

Corollary 7.4. Let $R$ be a $\Gamma$-graded ring. The set $\mathcal{A}(\mathcal{N})/\mathcal{I}$ is the least gr-matrix ideal. Hence $R$ has proper gr-matrix ideals if and only if no matrix of $\mathcal{I}$ can be expressed as a determinantal sum of matrices of $\mathcal{N}$.

Proof. The set $\mathcal{N}$ is contained in each gr-matrix ideal. By Lemma 7.2(2), $\mathcal{A}(\mathcal{N})/\mathcal{I}$ is the gr-matrix ideal generated by $\mathcal{N}$. Thus all gr-matrix ideals contain the gr-matrix ideal $\mathcal{A}(\mathcal{N})/\mathcal{I}$.

Since any matrix in $\mathfrak{M}(R)$ of the form $E(X \oplus A)F$ where $X \in \mathcal{N}$, $A \in \mathfrak{M}(R)$ and $E, F$ are permutation matrices of appropriate sizes, again belongs to $\mathcal{N}$, then $\mathcal{D}(\mathcal{N}) = \mathcal{N}$. Thus, $\mathcal{A}(\mathcal{N})$ consists of the matrices in $\mathfrak{M}(R)$ that can be expressed as a determinantal sum of matrices from $\mathcal{N}$.

Now $R$ has proper gr-matrix ideals if and only if $\mathcal{A}(\mathcal{N})/\mathcal{I}$ is proper. By Lemma 7.2(3), this is equivalent to $\mathcal{A}(\mathcal{N}) \cap \mathcal{I} = \emptyset$. In other words, no matrix of $\mathcal{I}$ can be expressed as a determinantal sum of matrices of $\mathcal{N}$. $\square$

Lemma 7.5. Let $R$ be a $\Gamma$-graded ring, $\mathcal{I}$ be a gr-matrix ideal and $Z \subseteq \mathfrak{M}(R)$. Then the set $\mathcal{I}Z = \{A \in \mathfrak{M}(R): A \oplus Z \in \mathcal{I} \text{ for all } Z \in \mathcal{Z}\}$ is a gr-matrix ideal.
Proof. Let \( A \in \mathfrak{M}(R) \) and suppose it is not gr-full. If \( A = BC \), then \( A \oplus Z = (B \oplus Z)(C \oplus I) \) for all \( Z \in Z \). Thus \( A \in \mathcal{I}_Z \) and (11) is satisfied.

Let \( A, B \in \mathcal{I}_Z \) and suppose that \( A \vee B \) exists. Then \( (A \vee B) \oplus Z = (A \oplus Z)(B \oplus Z) \) for all \( Z \in Z \). Since \( A \oplus Z, B \oplus Z \in \mathcal{I}_Z \), then \( (A \vee B) \oplus Z \in \mathcal{I}_Z \) for all \( Z \in Z \). Hence \( A \vee B \in \mathcal{I}_Z \), and (12) is satisfied.

Let \( A \in \mathcal{I}_Z \) and \( B \in \mathfrak{M}(R) \). Since \( A \oplus Z \subseteq I \) for all \( Z \in Z \) and \( I \) is a gr-matrix ideal, then \( A \oplus Z \oplus B \in \mathcal{I}_Z \) for all \( Z \in Z \). By (14), \( A \oplus B \oplus Z \in \mathcal{I}_Z \) for all \( Z \in Z \). Therefore \( A \oplus B \in \mathcal{I}_Z \) and (13) is satisfied.

If \( A \in \mathcal{I}_Z \), \( Z \in Z \) and \( E, F \) are permutation matrices of appropriate size, then
\[
EAF \oplus Z = (E \oplus I)(A \oplus Z)(F \oplus I).
\]
It shows that \( EAF \in \mathcal{I}_Z \) and (14) is satisfied.

Suppose now that \( A \in \mathfrak{M}(R) \) and that \( A \oplus 1 \in \mathcal{I}_Z \). Hence \( A \oplus 1 \oplus Z \in \mathcal{I}_Z \) for all \( Z \in Z \). By (14), \( A \oplus Z \oplus 1 \in \mathcal{I}_Z \) for all \( Z \in Z \). Now, by (15), \( A \oplus Z \in \mathcal{I}_Z \) for all \( Z \in Z \), which shows that \( A \in \mathcal{I}_Z \). Therefore (15) is satisfied. \( \square \)

Let \( A_1, A_2 \) be two gr-matrix ideals of a \( \Gamma \)-graded ring \( R \). The product of \( A_1 \) and \( A_2 \), denoted by \( A_1 A_2 \), is the gr-matrix ideal generated by the set
\[
\{ A_1 \oplus A_2 : A_1 \in A_1, A_2 \in A_2 \}.
\]

A helpful description of \( A_1 A_2 \) is given in the following lemma.

Lemma 7.6. Let \( R \) be a \( \Gamma \)-graded ring and \( X_1, X_2 \subseteq \mathfrak{M}(R) \). Set
\[
X = \{ X_1 \oplus X_2 : X_1 \in X_1, X_2 \in X_2 \}.
\]

Let \( A_1 \) be the gr-matrix ideal generated by \( X_1 \), \( A_2 \) be the gr-matrix ideal generated by \( X_2 \) and \( A \) be the gr-matrix ideal generated by \( X \). Then \( A = A_1 A_2 \).

As a consequence, for any \( A, B \in \mathfrak{M}(R) \), \( \langle A \rangle \langle B \rangle = \langle A \oplus B \rangle \), where \( \langle A \rangle \) denotes the gr-matrix ideal generated by \( \{ A \} \).

Proof. First, \( A \subseteq A_1 A_2 \) because \( X_1 \oplus X_2 \subseteq A_1 A_2 \) for all \( X_1 \in X_1, X_2 \in X_2 \).

Now observe that \( X_1 \oplus X_2 \subseteq A \) for all \( X_1 \in X_1, X_2 \in X_2 \). By (14), \( X_2 \oplus X_1 \in X \subseteq A \) for all \( X_1 \in X_1, X_2 \in X_2 \). Hence \( X_2 \) is contained in the gr-matrix ideal \( A X_1 \). Thus, \( A_2 \subseteq A X_1 \). It implies that \( A_2 \oplus X_1 \subseteq A \) for all \( A_2 \in A_2 \) and \( X_1 \in X_1 \). Again by (14), \( X_1 \oplus A_2 \subseteq A \) for all \( A_2 \in A_2 \) and \( X_1 \in X_1 \). Therefore \( X_1 \) is contained in the gr-matrix ideal \( A_2 \). Thus \( A_1 \subseteq A_2 \). This means that \( A_1 \oplus A_2 \subseteq A \) for all \( A_1 \in A_1 \) and \( A_2 \in A_2 \). Therefore \( A_1 A_2 \subseteq A \). \( \square \)

Now we show that gr-prime matrix ideals behave like graded prime ideals of graded rings.

Proposition 7.7. Let \( R \) be a \( \Gamma \)-graded ring. For a proper gr-matrix ideal \( \mathcal{P} \), the following are equivalent
\begin{enumerate}
\item \( \mathcal{P} \) is a gr-prime matrix ideal,
\item For gr-matrix ideals \( A_1, A_2 \), if \( A_1 A_2 \subseteq \mathcal{P} \), then \( A_1 \subseteq \mathcal{P} \) or \( A_2 \subseteq \mathcal{P} \).
\item For gr-matrix ideals \( A_1, A_2 \) that contain \( \mathcal{P} \), if \( A_1 A_2 \subseteq \mathcal{P} \), then \( A_1 = \mathcal{P} \) or \( A_2 = \mathcal{P} \).
\end{enumerate}

Proof. Suppose (1) holds true. Let \( A_1, A_2 \) be gr-matrix ideals such that \( A_1 \not\subseteq \mathcal{P} \) and \( A_2 \not\subseteq \mathcal{P} \). Hence there exist \( A_1 \in \mathcal{A}_1 \setminus \mathcal{P} \) and \( A_2 \in \mathcal{A}_2 \setminus \mathcal{P} \). Hence \( A_1 \oplus A_2 \not\subseteq \mathcal{P} \). It implies that \( A_1, A_2 \not\subseteq \mathcal{P} \). Therefore (2) holds true.

Clearly (2) implies (3).

Suppose (3) holds true and let \( A_1, A_2 \in \mathfrak{M}(R) \) be such that \( A_1 \oplus A_2 \subseteq \mathcal{P} \). Let \( A_1, A_2 \) be the gr-matrix ideals generated by \( \mathcal{P} \cup \{ A_1 \} \) and \( \mathcal{P} \cup \{ A_2 \} \), respectively. Notice that \( X_1 \oplus X_2 \in \mathcal{P} \) for \( X_1 \in A_1, X_2 \in A_2 \). Hence \( A_1 A_2 \subseteq \mathcal{P} \). By (3), either \( A_1 = \mathcal{P} \) or \( A_2 = \mathcal{P} \). Hence \( A_1 \in \mathcal{P} \) or \( A_2 \in \mathcal{P} \), and (1) is satisfied. \( \square \)
Let $A$ be a gr-matrix ideal. The radical of $A$ is defined as the set
\[ \sqrt{A} = \{ A \in \mathfrak{M}(R) : \oplus^r A \in A \text{ for some positive integer } r \}. \]

We say that a proper gr-matrix ideal $A$ is gr-semiprime if $\sqrt{A} = A$.

**Lemma 7.8.** Let $R$ be a $\Gamma$-graded ring and let $A$ be a gr-matrix ideal. The following assertions hold true.

1. $\sqrt{A}$ is a gr-matrix ideal that contains $A$.
2. $\sqrt{\sqrt{A}} = \sqrt{A}$.
3. If $A$ is a gr-prime matrix ideal, then $\sqrt{A} = A$.

**Proof.** (1) If $A \subseteq A$, then, for $r = 1$, we obtain that $A = \oplus^1 A \subseteq A$. Hence $A \subseteq \sqrt{A}$. In particular, all homogeneous matrices which are not gr-full belong to $\sqrt{A}$. Thus $\sqrt{A}$ satisfies (I1).

Let $A, B \in \sqrt{A}$ such that $A \nabla B$ exists. There exist $r, s \geq 1$ such that $\oplus^r A$, $\oplus^s B \in A$. Set $n = r + s + 1$. To prove that $\sqrt{A}$ satisfies (I2), it is enough to show that $\oplus^n (A \nabla B) \in A$. For that aim, using $(A \nabla B) \oplus P = (A \oplus P) \nabla (B \oplus P)$, one can prove by induction on $n$ that $\oplus^n (A \nabla B)$ is a determinantal sum of elements of the form
\[ C_1 \oplus C_2 \oplus \cdots \oplus C_n \] (7.1)

where each $C_i$ equals $A$ or $B$. By the choice of $n$, there are at least $r$ $C_i$’s equal to $A$ or at least $s$ $C_i$’s equal to $B$. Either case, there exist permutation matrices $E$, $F$ of appropriate size such that
\[ C_1 \oplus C_2 \oplus \cdots \oplus C_n = \begin{cases} E((\oplus^r A) \oplus C_{r+1} \oplus \cdots \oplus C_n)F & \text{if } n \text{ is even} \\ E((\oplus^s B) \oplus C_{s+1} \oplus \cdots \oplus C_n)F & \text{if } n \text{ is odd} \end{cases} \]

It implies that the elements in (7.1) belong to $A$ by (I3). Now (I2) implies that $\oplus^n (A \nabla B) \in A$, as desired.

Let now $A \in \sqrt{A}$ and $B \in \mathfrak{M}(R)$. There exists $r \geq 1$ such that $\oplus^r A \in A$. The equality $\oplus^r (A \oplus B) = E((\oplus^r A) \oplus (\oplus^r B))F$ holds for some permutation matrices $E$, $F$. Hence $\oplus^r (A \oplus B) \in A$. Thus $A \oplus B \in \sqrt{A}$ and $\sqrt{A}$ satisfies (I3).

Let $A \in \sqrt{A}$ such that $\oplus^r A \in A$. For permutation matrices $E$, $F$ of appropriate size
\[ \oplus^r (EAF) = (\oplus^r E)(\oplus^r A)(\oplus^r F) \in A. \]

Therefore $EAF \in \sqrt{A}$ and $\sqrt{A}$ satisfies (I4).

If $X \in \mathfrak{M}(R)$ is such that $X \oplus 1 \in \sqrt{A}$, then there exists $t \geq 1$ such that $\oplus^t (X \oplus 1) \in A$. But now $(\oplus^t X) \oplus 1 = E((\oplus^t X) \oplus 1)F \in A$. Applying (I5), we get that $\oplus^t X \in A$, and therefore $X \in \sqrt{A}$. Hence $\sqrt{A}$ satisfies (I5).

(2) By (1), $\sqrt{A} \subseteq \sqrt{\sqrt{A}}$. Let now $A \in \sqrt{\sqrt{A}}$. It means that $\oplus^r A \in \sqrt{A}$ for some positive integer $r$. Hence there exists a positive integer $s$ such that $\oplus^s (\oplus^r A) \in A$. Thus, $\oplus^r A = \oplus^s (\oplus^r A) \in A$. Therefore $A \in \sqrt{A}$, as desired.

(3) Suppose $A$ is a gr-prime matrix ideal and let $A \in \sqrt{A}$. Hence $\oplus^r A \in A$. By (PM4), $A \in A$, as desired. \qed

**Proposition 7.9.** Let $R$ be a $\Gamma$-graded ring. Suppose that the nonempty subset $\Sigma$ of $\mathfrak{M}(R)$ and the gr-matrix ideal $A$ satisfy the following two conditions.

(i) $A \oplus B \subseteq A$ for all $A, B \in \Sigma$;
(ii) $A \cap \Sigma = \emptyset$.

Then the set $W$ of gr-matrix ideals $B$ such that $A \subseteq B$ and $B \cap \Sigma = \emptyset$ has maximal elements and each such maximal element is a gr-prime matrix ideal.
Proof. Let \((C_i)_{i \in I}\) be a nonempty chain in \(W\). Set \(C = \bigcup_{i \in I} C_i\). It is not difficult to show that \(C\) is a gr-matrix ideal. Then clearly \(A \subseteq C_i \subseteq C\) and \(C \cap \Sigma = (\bigcup_{i \in I} C_i) \cap \Sigma = \bigcup_{i \in I} (C_i \cap \Sigma) = \emptyset\). By Zorn’s lemma, \(W\) has maximal elements. Suppose that \(P\) is a maximal element of \(W\). Since \(P \cap \Sigma = \emptyset\), \(P\) is a proper gr-matrix ideal. Let \(A_1, A_2\) be gr-matrix ideals such that \(P \subseteq A_1, P \subseteq A_2\). Since \(P\) is maximal in \(W\), there exist \(A_1 \in A_1 \cap \Sigma, A_2 \in A_2 \cap \Sigma\). Then \(A_1 + A_2 \in \Sigma\) and \(A_1 + A_2 \notin P\). Therefore \(A_1, A_2 \notin P\).

Corollary 7.10. Let \(R\) be a \(\Gamma\)-graded ring. Let \(A\) be a proper gr-matrix ideal. Then there exist maximal gr-matrix ideals \(P\) with \(A \subseteq P\), and such maximal gr-matrix ideals are gr-prime matrix ideals. In particular, if there are proper gr-matrix ideals, then gr-prime matrix ideals exist.

Proof. By Lemma 7.11, no identity matrix belongs to \(A\). Apply now Proposition 7.9 to \(A\) and \(\Sigma = \emptyset\).

Proposition 7.11. Let \(R\) be a \(\Gamma\)-graded ring. For each proper gr-matrix ideal \(A\), the radical \(\sqrt{A}\) is the intersection of all gr-prime matrix ideals that contain \(A\).

Proof. Let \(P\) be a prime matrix ideal such that \(A \subseteq P\). If \(A \in \sqrt{A}\), then \(\oplus^r A \in A \subseteq P\) for some positive integer \(r\). By (PM4), \(A \in P\). Thus \(\sqrt{A} \subseteq P\).

Let now \(A \in \sqrt{R} \setminus \sqrt{A}\). Notice that such \(A\) exists because \(\sqrt{A} \subseteq P\). If we apply Proposition 7.9 to \(A\) and \(\Sigma = \{\oplus^r A : r \text{ positive integer}\}\), we obtain a gr-prime matrix ideal \(P\) such that \(A \subseteq P\), \(P \cap \Sigma = \emptyset\). Therefore \(A\) does not belong to the intersection of the gr-prime matrix ideals that contain \(A\).

Corollary 7.12. Let \(R\) be a \(\Gamma\)-graded ring. A proper gr-matrix ideal is gr-semiprime if and only if it is the intersection of gr-prime matrix ideals.

Let \(R\) be a \(\Gamma\)-graded ring. By Corollary 7.4, \(A(N)/\mathfrak{N}\) is the least gr-matrix ideal. We define the gr-matrix nilradical of \(R\) as the gr-matrix ideal \(\mathfrak{N} = \sqrt{A(N)}/\mathfrak{N}\).

Theorem 7.13. Let \(R\) be a \(\Gamma\)-graded ring. The following assertions are equivalent.

(1) There exists a \(\Gamma\)-graded epic \(R\)-division ring \((K, \varphi)\).

(2) There exists a homomorphism of \(\Gamma\)-almost graded rings from \(R\) to a \(\Gamma\)-almost graded division ring.

(3) The gr-matrix nilradical is a proper gr-matrix ideal.

(4) No identity matrix can be expressed as a determinantal sum of elements of \(N\).

Proof. (1) is equivalent to (2) by Theorem 4.4(2)(b). One could also argue as follows. By Proposition 6.1, (2) implies the existence of gr-prime matrix ideals, and therefore of \(\Gamma\)-graded epic \(R\)-division rings by Theorem 6.5.

If (1) holds, the gr-singular kernel of \(\varphi\) is a gr-prime matrix ideal by Theorem 6.5. Thus (3) holds.

If (3) holds, then \(A(N)/\mathfrak{N}\) is a proper gr-matrix ideal. By Corollary 7.10, (4) holds.

Suppose that (4) holds true. Again by Corollary 7.4, there exist proper gr-matrix ideals. By Corollary 7.10 gr-prime matrix ideal exists. Now Theorem 6.5 implies (1).

Theorem 7.14. Let \(R\) be a \(\Gamma\)-graded ring. There exists a universal \(\Gamma\)-graded epic \(R\)-division ring if and only if the gr-matrix nilradical is a gr-prime matrix ideal.

Proof. By Corollary 6.3, the existence of a universal \(\Gamma\)-graded epic \(R\)-division ring is equivalent to the existence of a least gr-prime matrix ideal \(P\). Hence the least gr-matrix ideal \(A(N)/\mathfrak{N} \subseteq P\) is proper. By Proposition 7.11 \(\mathfrak{N}\) is the intersection of all gr-prime matrix ideals. Hence \(\mathfrak{N} = P\).
Conversely, if \( \mathcal{R} \) is a gr-prime matrix ideal, then \( \mathcal{A}(\mathcal{N})/\mathcal{I} \) is proper and, by Proposition 7.11, \( \mathcal{R} \) is the intersection of all gr-prime matrix ideals. Therefore \( \mathcal{R} \) is the least gr-prime matrix ideal. \( \square \)

**Proposition 7.15.** Let \( R \) be a \( \Gamma \)-graded ring and let \( P, Q \in \mathcal{M}(R) \). There exists a homomorphism of \( \Gamma \)-graded rings \( \varphi: R \to K \) to a \( \Gamma \)-graded division ring \( K \) such that \( P^\varphi \) is invertible over \( K \) and \( Q^\varphi \) is not invertible over \( K \) if and only if no matrix of the form \( I \oplus (\oplus P) \) can be expressed as a determinantal sum of matrices of \( \mathcal{N} \cup \mathcal{D}(\{Q\}) \).

**Proof.** The existence of such \((K, \varphi)\) is equivalent to the existence of gr-prime matrix ideals \( P \) such that \( Q \notin P \) and \( P \notin P \). The existence of such gr-prime matrix ideals is equivalent to the condition \( P \notin \sqrt{\{Q\}} \), where \( \{Q\} \) denotes the gr-matrix ideal generated by \( Q \). Hence it is equivalent to the condition that no matrix of the form \( \oplus P \in \{Q\} \). By Lemma 7.12, \( \{Q\} \) is of the form \( \mathcal{A}(\{Q\})/\mathcal{I} \). Therefore, by Lemmas 7.2 and 7.3, everything is equivalent to the condition that no matrix of the form \( I \oplus (\oplus P) \) can be expressed as a determinantal sum of matrices of \( \mathcal{N} \cup \mathcal{D}(\{Q\}) \), as desired. \( \square \)

**Corollary 7.16.** Let \( R \) be a \( \Gamma \)-graded ring and let \( P, Q \in \mathcal{M}(R) \). The following assertions hold true.

1. There exists a \( \Gamma \)-graded epic \( R \)-division ring \((K, \varphi)\) such that \( P^\varphi \) is invertible over \( K \) if and only if no matrix of the form \( I \oplus (\oplus P) \) can be expressed as a determinantal sum of matrices of \( \mathcal{N} \).
2. There exists a \( \Gamma \)-graded epic \( R \)-division ring \((K, \varphi)\) such that \( Q^\varphi \) is not invertible over \( K \) if and only if no identity matrix can be expressed as a determinantal sum of matrices of \( \mathcal{N} \cup \mathcal{D}(\{Q\}) \).

**Proof.** (1) In Proposition 7.15, let \( Q = 0 \).

(2) In Proposition 7.15, let \( P = 1 \). \( \square \)

**Theorem 7.17.** Let \( R \) be a \( \Gamma \)-graded ring. The following assertions are equivalent.

1. There exists a \( \Gamma \)-graded epic \( R \)-division ring of fractions \((K, \varphi)\).
2. There exists a homomorphism of \( \Gamma \)-almost graded rings \( \varphi: R \to K \) with \( K \) a \( \Gamma \)-almost graded division ring such that \( \varphi(x) \neq 0 \) for each \( x \in h(R) \setminus \{0\} \).
3. \( R \) is a \( \Gamma \)-graded domain and no matrix of the form \( aI \) with \( a \in h(R) \setminus \{0\} \) can be expressed as a determinantal sum of matrices of \( \mathcal{N} \).
4. No diagonal matrix with nonzero homogeneous elements on the main diagonal can be expressed as a determinantal sum of matrices of \( \mathcal{N} \).

**Proof.** (1) and (2) are equivalent by Theorem 4.11b).

Suppose that (1) holds true. Then, for each diagonal matrix \( A \) as in (4), \( A^\varphi \) is invertible. Thus, \( A \notin \mathcal{P} \), the gr-prime matrix ideal given as the gr-singular kernel of \( \varphi \). In particular, \( A \) cannot be expressed as the determinantal sum of matrices in \( \mathcal{N} \). Thus (4) holds.

Suppose (4) holds. Clearly no matrix of the form \( aI \) with \( a \in h(R) \setminus \{0\} \) can be expressed as a determinantal sum of matrices of \( \mathcal{N} \). Thus to prove (3), it remains to show that \( R \) is a \( \Gamma \)-graded domain. Thus, let \( a, b \in h(R) \) of degrees \( \gamma, \delta \in \Gamma \), respectively. If \( ab = 0 \), then \((a_{0 \, 0}) \in M_2(R)((\gamma, \delta))[[e, \delta^{-1}]]\). Then we can express \((a_{0 \, 0}) = (a_{1 \, 0}) \times (a_{0 \, 1}^{-1}) \) as a determinantal sum of matrices in \( M_2(R)((\gamma, \delta))[[e, \delta^{-1}]] \). Note that \((a_{1 \, 0})\) is hollow, and hence it is not gr-full. Furthermore, \((a_{1 \, 0}) = (a_{1 \, 0}) (a_{1 \, 1}) \), where the factors belong to \( M_{2 \times 2}(R)((\gamma, \delta))[[e, \delta^{-1}]] \) and \( M_{1 \times 2}(R)((e, \delta^{-1}))[[e, \delta^{-1}]] \), respectively. Hence \((a_{1 \, 0})\) cannot be expressed as a determinantal sum of matrices from \( \mathcal{N} \). By (4), either \( a = 0 \) or \( b = 0 \). Hence \( R \) is a \( \Gamma \)-graded domain and (3) holds.
Suppose now that (3) holds. If there does not exist a $\Gamma$-graded epic $R$-division ring of fractions, then, by Corollary [10.2], there exists nonzero $a \in \mathcal{R}(R)$ such that $a^n$ is not invertible for every homomorphism of $\Gamma$-graded rings $\phi : R \to K$ with $K$ a $\Gamma$-graded division ring. Hence the $1 \times 1$ homogeneous matrix $(a)$ belongs to the intersection of all gr-$\mathcal{P}$-prime matrix ideals, i.e. $(a) \in \mathcal{R}$. Hence $\oplus^\circ (a) \in \mathcal{A}(\mathcal{N})/\mathcal{J}$. Thus $I_a \oplus (\oplus^\circ (a)) = I_a \oplus aI_r$ can be written as a determinantal sum of matrices of $\mathcal{N}$. Then, since $aI_a \oplus aI_r \in \mathcal{M}(R)$ and it is diagonal, $aI_{r+s} = (aI_a \oplus aI_r)(I_a \oplus aI_r)$ is a determinantal sum of matrices of $\mathcal{N}$; a contradiction. Therefore (1) holds. \qed

8. GR-Sylvester rank functions

Throughout this section, let $\Gamma$ be a group.

The aim of this section is to show that the different definitions of gr-Sylvester rank functions (with values in $\mathbb{N}$) given below are equivalent between them and with the definition of a gr-prime matrix ideal, and thus they uniquely determine homomorphisms to graded division rings. We will adapt the definitions, results and proofs of [24] Sections 1 and 3 and [28] p.94–98 to the graded situation. In defining gr-Sylvester rank functions, the main difference with the ungraded case stems from the the fact that, in the graded case, the same matrix $A \in \mathcal{M}_n(R)$ can define more than one homomorphism between $\Gamma$-graded free modules. This is reflected in properties (MatRF4), (ModRF4) and (MapRF5) below.

We begin this section providing the different definitions of gr-Sylvester rank functions for a $\Gamma$-graded ring (with values in $\mathbb{N}$), together with some of its basic properties.

Let $R$ be a $\Gamma$-graded ring. A $\textbf{gr-Sylvester matrix rank function}$ for $R$ is a map $r : \mathcal{M}_\bullet(R) \to \mathbb{N}$ that satisfies the following conditions

(MatRF1) $r((1)) = 1$, where $(1)$ is the identity matrix of size $1 \times 1$.
(MatRF2) $r(AB) \leq \min\{r(A), r(B)\}$ for all compatible matrices $A, B \in \mathcal{M}_\bullet(R)$.
(MatRF3) $r(\begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}) = r(A) + r(B)$ for all $A, B \in \mathcal{M}_\bullet(R)$.
(MatRF4) $r(\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}) \geq r(A) + r(B)$ for all $A, B, C \in \mathcal{M}_\bullet(R)$ such that $A$ has distribution $(\overline{\alpha}, \overline{\beta})$, $B$ has distribution $(\overline{\delta}, \overline{\tau})$ and $C$ has distribution $(\overline{\pi}, \overline{\varphi})$ for some finite sequences $\overline{\alpha}, \overline{\beta}, \overline{\delta}, \overline{\tau}, \overline{\pi}, \overline{\varphi}$ of elements of $\Gamma$.

Let $r_1, r_2$ be two gr-Sylvester matrix rank functions for $R$. We say that $r_1 \leq r_2$ if $r_1(A) \leq r_2(A)$ for all $A \in \mathcal{M}_\bullet(R)$. In this way, there is defined a partial order in the set of gr-Sylvester matrix rank functions for $R$.

The following lemma describes some useful properties of gr-Sylvester matrix rank functions.

**Lemma 8.1.** Let $R$ be a $\Gamma$-graded ring and $r : \mathcal{M}_\bullet(R) \to \mathbb{N}$ be a gr-Sylvester matrix rank function. Let $A \in M_{m \times n}(R)[\overline{\alpha}][\overline{\beta}]$. The following assertions hold true.

1. $r(I_n) = n$ for all positive integers $n$.
2. $r(Z) = 0$ for all zero matrices of any size.
3. The condition $r(A) \geq 0$ follows from (MatRF1)–(MatRF4).
4. $r(PA) = r(PQ)$ for all invertible matrices $P \in M_n(R)[\overline{\alpha}][\overline{\alpha}]$ and $Q \in M_n(R)[\overline{\beta}][\overline{\beta}]$.
5. If $P \in \mathcal{M}(R)$ is invertible of size $n \times n$, then $r(P) = n$.
6. $r(A) \leq \min(m, n)$.
7. $r(A) \leq r(\begin{pmatrix} A & B \\ 0 & C \end{pmatrix})$ for all $B \in M_{m \times n}(R)[\overline{\alpha}][\overline{\beta}]$ and $C \in M_{m \times n}(R)[\overline{\pi}][\overline{\varphi}]$.

**Proof.** (1) follows from (MatRF1) and (MatRF3).
(2) We denote the zero matrix of size $m \times n$ by $0_{m \times n}$. From
\[
(1 0_{1 \times n}) \begin{pmatrix} 1 & 0 \\ 0 & 0_{m \times n} \end{pmatrix} \begin{pmatrix} 0_{m \times 1} & 1 \\ 0 \end{pmatrix} = (1), \quad (1) (1 0_{1 \times n}) = \begin{pmatrix} 1 & 0 \\ 0 & 0_{m \times n} \end{pmatrix},
\]
applying (MatRF1)–(MatRF3), we obtain that $1 + r(0_{m \times n}) \geq 1$ and $1 \geq 1 + r(0_{m \times n})$, respectively.

(3) Let $Z$ be a zero matrix of appropriate size. By (2) and (MatRF2), $0 = r(Z) = r(AZ) \leq \min\{r(A), r(Z)\} \leq r(A).

(4) By (MatRF2),
\[
r(PA) \leq \min\{r(P), r(A)\} \leq r(A),
\]
and hence $r(A) = r(PA)$. The other case is shown in the same way.

(5) By (4), $r(I) = r(PI) = r(P)$.

(6) Since $A$ is $m \times n$, using (1), we obtain $r(A) = r(I_m A) \leq \min\{r(I_m), r(A)\} \leq m$
and $r(A) = r(AI_n) = \min\{r(A), r(I_n)\} \leq n$.

(7) follows from
\[
r(A) = r \left( \begin{pmatrix} I_m & 0_{m \times m'} \\ 0_{n \times m'} & A \end{pmatrix} \right) \leq r \left( \begin{pmatrix} A & B \end{pmatrix} \right),
\]
\[
r(A) = r \left( \begin{pmatrix} A & C \end{pmatrix} \right) \leq r \left( \begin{pmatrix} A' & B' \end{pmatrix} \right).
\]

□

Let $R$ be a $\Gamma$-graded ring. We will denote the forgetful functor from the category of finitely presented $\Gamma$-graded $R$-modules to the category of finitely presented $R$-modules by $\mathcal{F}$.

A $gr$-$Sylvester$ $module$ $rank$ $function$ for $R$ is a function $d$ on the class of finitely presented $\Gamma$-graded (right) $R$-modules with values on $\mathbb{N}$ such that

(ModRF1) $d(R) = 1$, where $R$ is the $\Gamma$-graded ring $R$ considered as a (right) $R$-module in the natural way.

(ModRF2) $d(M_1 \oplus M_2) = d(M_1) + d(M_2)$.

(ModRF3) For any exact sequence $M_1 \to M_2 \to M_3 \to 0$ of graded homomorphisms between finitely presented $\Gamma$-graded $R$-modules,
\[
d(M_3) \leq d(M_2) \leq d(M_1) + d(M_3).
\]

(ModRF4) Let $f : R^n(\overline{\alpha}) \to R^m(\overline{\gamma})$ and $f' : R^n(\overline{\alpha'}) \to R^m(\overline{\gamma'})$ be homomorphisms of $\Gamma$-graded $R$-modules. If $\mathcal{F}(f) = \mathcal{F}(f')$, then $d(\text{coker } f) = d(\text{coker } f')$.

Let $d_1, d_2$ be two $gr$-$Sylvester$ module rank functions for $R$ with values in $\mathbb{N}$. We say that $d_1 \leq d_2$ if $d_1(M) \leq d_2(M)$ for all finitely presented $\Gamma$-graded $R$-modules $M$. In this way, there is defined a partial order in the set of $gr$-$Sylvester$ module rank functions for $R$.

The following easy but important remarks are in order.

**Lemma 8.2.** Let $R$ be a $\Gamma$-graded ring and $d$ be a $gr$-$Sylvester$ module rank function.

(1) $d(0) = 0$, where 0 denotes the zero module.

(2) $d(M) \geq 0$ for all finitely presented $\Gamma$-graded $R$-modules follows from (ModRF1)–(ModRF4).

(3) If $M$ and $N$ are isomorphic as $\Gamma$-graded $R$-modules, then $d(M) = d(N)$.

(4) $d(\varnothing) = 1$ for all $\varnothing \in \Gamma$.

(5) In (ModRF4), the condition $\mathcal{F}(f) = \mathcal{F}(f')$ implies that the lengths of the sequences $\overline{\alpha}$ and $\overline{\alpha'}$ (respectively $\overline{\beta}$ and $\overline{\beta'}$) coincide.

(6) $d(R^n(\overline{\delta})) = d(R^n(\overline{\delta'}))$ for any finite sequences $\overline{\delta}$, $\overline{\delta'}$ of the same length.
Proof. (1) follows from (ModRF3) applied to $0 \to 0 \oplus 0 \oplus 0 \to 0 \to 0$.
(2) follows applying (ModRF3) to $M \to 0 \to 0 \to 0$.
(3) Make $M_1 = 0$, $M_2 = M$ and $M_3 = N$ in (ModRF3).
(4) By (ModRF4), the natural inclusions of $R$ in the first component $R \to R \oplus R(\theta)$, $R \to R \oplus R$, implies that $d(R(\theta)) = d(R)$.
(5) Trivial.
(6) holds true by (ModRF2) and the foregoing.

Let $R$ be a $\Gamma$-graded ring. A gr-Sylvester map rank function for $R$ is a function $\rho$ on the class of all homomorphisms of $\Gamma$-graded (right) $R$-modules between finitely generated $\Gamma$-graded projective $R$-modules with values on $\mathbb{N}$ such that

(1) $(\text{MapRF1})$ $\rho(1_R) = 1$, where $1_R$ denotes the identity map on $R$.
(2) $(\text{MapRF2})$ $\rho(gf) \leq \min\{\rho(f), \rho(g)\}$.
(3) $(\text{MapRF3})$ $\rho\left(\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right) = \rho(f) + \rho(g)$.
(4) $(\text{MapRF4})$ $\rho\left(\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right) \geq \rho(f) + \rho(g)$.
(5) $(\text{MapRF5})$ Let $f : R^n(\overline{\theta}) \to R^n(\overline{\theta})$ and $f' : R^n(\overline{\theta}) \to R^n(\overline{\theta})$ be homomorphisms of $\Gamma$-graded $R$-modules. If $\mathcal{F}(f) = \mathcal{F}(f')$, then $\rho(f) = \rho(f')$.

Let $\rho_1, \rho_2$ be two gr-Sylvester map rank functions for $R$. We say that $\rho_1 \leq \rho_2$ if $\rho_1(f) \leq \rho_2(f)$ for all homomorphism $f$ of $\Gamma$-graded modules between finitely generated $\Gamma$-graded projective modules. In this way, there is defined a partial order in the set of gr-Sylvester map rank functions for $R$.

The proof of the following remarks can be proved very much as in Lemma 8.1.

**Lemma 8.3.** Let $R$ be a $\Gamma$-graded ring and $\rho$ be a gr-Sylvester map rank function. Let $f : P \to Q$ be a homomorphism of $\Gamma$-graded rings between the finitely generated $\Gamma$-graded projective modules $P$ and $Q$. The following assertions hold true.

(1) $r(1_R(\overline{\theta})) = n$ where $1_R(\overline{\theta})$ denotes the identity homomorphism of the $\Gamma$-graded free $R$-module $R^n(\overline{\theta})$.
(2) $r(0) = 0$ where $0$ denotes the zero homomorphism between any finitely generated $\Gamma$-graded projective $R$-modules.
(3) The condition $r(f) \geq 0$ follows from (MapRF1)–(MapRF4).
(4) $r(f) = r(gf) = r(fh)$ for all isomorphisms of $\Gamma$-graded $R$-modules between finitely generated $\Gamma$-graded projective $R$-modules $g : Q \to Q'$, $h : P' \to P$.
(5) If $f$ is invertible, then $\rho(f) = \rho(1_P) = \rho(1_Q)$.
(6) $\rho(f) \leq \min\{\rho(1_P), \rho(1_Q)\}$.
(7) $\rho(f) \leq \rho\left(\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right)$ and $\rho(f) \leq \rho\left(\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right)$ for all homomorphisms of $\Gamma$-graded $R$-modules $g : P \to Q'$ and $h : P' \to Q$ with $P', Q'$ being finitely generated $\Gamma$-graded projective $R$-modules.

Proof. (1) Notice that the identity maps $1_{R(\theta)} : R(\theta) \to R(\theta)$ and $1_R : R \to R$ are such that $\mathcal{F}(1_R) = \mathcal{F}(1_{R(\theta)})$. Hence $\rho(1_{R(\theta)}) = 1$ for all $\theta \in \Gamma$. Now apply (MapRF3) to $1_{R^n(\overline{\theta})}$ for any $\overline{\theta} \in \Gamma^n$.
(2) We denote by $0_{MN}$ the zero homomorphism $M \to N$ between the $\Gamma$-graded $R$-modules $M$ and $N$. From the equalities

$$
\begin{pmatrix}
1_R & 0_{PR} \\
0_{RQ} & 0_{PQ}
\end{pmatrix}
\begin{pmatrix}
1_R \\
0_{RQ}
\end{pmatrix} = \begin{pmatrix}
1_R \\
0_{RQ}
\end{pmatrix} = (1_R),
$$

$$
\begin{pmatrix}
1_R \\
0_{RQ}
\end{pmatrix} (1_R) = \begin{pmatrix}
1_R \\
0_{RQ}
\end{pmatrix} = \begin{pmatrix}
1_R \\
0_{RQ}
\end{pmatrix},
$$

we obtain that $1 + \rho(0_{PQ}) = \rho\left(\begin{smallmatrix} 1_R & 0_{PR} \\ 0_{RQ} & 0_{PQ} \end{smallmatrix}\right) \geq \rho(1_R) = 1$ and $1 = \rho(1_R) \geq \rho\left(\begin{smallmatrix} 1_R & 0_{PR} \\ 0_{RQ} & 0_{PQ} \end{smallmatrix}\right) = 1 + \rho(0_{PQ})$. Hence $\rho(0_{PQ}) = 0$. 

\[\blacksquare\]
(3) By (2) and (MapRF2), \(0 = \rho(0_{PQ}) = \rho(0_{QQ} f) \leq \min\{\rho(0_{QQ}), \rho(f)\} \leq \rho(f)\).
(4) By (MapRF2),
\[
\rho(g f) \leq \min\{\rho(g), \rho(f)\} \leq \rho(f),
\]
\[
\rho(f) = \rho(g^{-1} g f) \leq \min\{\rho(g^{-1}), \rho(gf)\} \leq \rho(gf).
\]
Hence \(\rho(f) = \rho(gf)\). The other case is shown in the same way.
(5) By (4), \(\rho(1_P) = \rho(f 1_P) = \rho(f)\) and similarly \(\rho(f) = \rho(1_Q)\).
(6) Using (MapRF2), we obtain \(\rho(f) = \rho(1_Q f) \leq \min\{\rho(1_Q), \rho(f)\} \leq \rho(1_Q)\) and 
\(\rho(f) = \rho(f 1_P) = \min\{\rho(f), \rho(1_P)\} \leq \rho(1_P)\).
(7) follows from
\[
\rho(f) = \rho\left(\begin{pmatrix} 1_Q & 0_{QQ} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}\right) \leq \rho\left(\begin{pmatrix} f \\ g \end{pmatrix}\right),
\]
\[
\rho(f) = \rho\left(\begin{pmatrix} f & h \end{pmatrix} \begin{pmatrix} 1_P \\ 0_{PP'} \end{pmatrix}\right) \leq \rho\left(\begin{pmatrix} f & h \end{pmatrix}\right).
\]
\[\square\]

### 8.1. Equivalence between gr-prime matrix ideals and gr-Sylvester matrix rank functions

The proofs and arguments contained in this subsection are the easy and natural adaptation of the ones in [24, Section 3].

Let \(D\) be a \(\Gamma\)-graded division ring. It is not difficult to show that \(\mathfrak{M}_\ast(D) \rightarrow \mathbb{N},\) \(A \mapsto \text{rank}(A),\) satisfies (MatRF1)–(MatRF4).

Let \(R\) be a \(\Gamma\)-graded ring and let \((K, \varphi)\) be a \(\Gamma\)-graded epic \(R\)-division ring.
One can induce a gr-rank function \(r\varphi\) for \(R\) defining \(r\varphi(A) = \text{rank}(A^\varphi)\) for all \(A \in \mathfrak{M}_\ast(R)\). Note that non-isomorphic \(\Gamma\)-graded epic \(R\)-division rings induce different gr-rank functions because the gr-singular kernels do not coincide, by Theorem 6.5.

The aim of this section is to show that there are no other gr-rank functions for \(R\).

**Lemma 8.4.** Let \(R\) be a \(\Gamma\)-graded ring. Suppose that \(r: \mathfrak{M}_\ast(R) \rightarrow \mathbb{N}\) is a gr-rank function and that \(r(A) = n\) for some \(A \in \mathfrak{M}_\ast(R)\). The following assertions hold true.

1. Suppose that \(A'\) is obtained as the result of eliminating a column (a row) of \(A\), then 
\(r(A) - 1 \leq r(A') \leq r(A)\).

2. If the result \(A'\) of eliminating any of the columns (rows) of \(A\) is such that 
\(r(A') < r(A)\), then \(A\) has exactly \(n\) columns (rows).

3. \(n\) is the largest natural number such that there is a square submatrix \(B\) of \(A\) with \(r(B) = \text{size of } B\).

**Proof.**

1. Suppose that \(A = (A' a)\), where \(a\) is the last column of \(A\). By Lemma 8.17, 
\(r(A') \leq r(A)\). Moreover
\[
r(A) = r(A' a) \leq r\left(\begin{pmatrix} A' & a \\ 0 & 1 \end{pmatrix}\right) = r\left(\begin{pmatrix} A' & 0 \\ 0 & 1 \end{pmatrix}\right) = r(A') + 1,
\]
where we have used Lemma 8.14 on the second equality. When the column we eliminate is not the last one, the result follows by Lemma 8.14.

When \(A'\) is obtained by eliminating some row, the result can be proved analogously.

2. We claim that \(A'\) obtained as the result of eliminating \(m\) of the columns of \(A\) satisfies \(r(A') = n - m\). We prove the claim by induction on \(m\).

If \(m = 1\), the result follows from (1). Suppose that the claim holds true for \(m \leq k - 1\). Let \(A'\) be the result of eliminating any \(k\) columns and \(a_1, a_2\) be two
The main result of this subsection is a theorem which relates the size of a matrix to the size of its submatrices. This is a useful result in the study of matrix ranks.

**Theorem 8.5.** Let $R$ be a $\Gamma$-graded ring. There is an anti-isomorphism of partially ordered sets

$$\{\text{gr-prime matrix ideals of } R\} \leftrightarrow \{\text{gr-Sylvester matrix rank functions for } R\}$$

defined as follows.

1. If $\mathcal{P}$ is a gr-prime matrix ideal of $R$ and $A \in \mathfrak{M}_*(R)$, then $r_\mathcal{P}(A)$ is the size of the largest square submatrix of $A$ which is not in $\mathcal{P}$. Equivalently, if $(K, \varphi_\mathcal{P})$ is a $\Gamma$-graded epic $R$-division ring with singular kernel $\mathcal{P}$, then $r_\mathcal{P}(A) = \text{rank}(A^\varphi_\mathcal{P})$.

2. Conversely, given a gr-Sylvester matrix rank function $r: \mathfrak{M}_*(R) \rightarrow \mathbb{N}$, then the set

$$\mathcal{P}_r = \{A \in \mathfrak{M}(R): r(A) < \text{size of } A\}$$

is a gr-prime matrix ideal.

**Proof.** Theorem 8.5 proves that the two ways of describing the correspondence $\mathcal{P} \mapsto r_\mathcal{P}$ are equivalent. By the comment at the beginning of Section 8.1, $r_{\mathcal{P}_r}$ is a gr-rank function for $R$. Moreover, if $\mathcal{P} \subseteq \mathcal{Q}$ are gr-prime matrix ideals, then $r_\mathcal{P} \geq r_\mathcal{Q}$ because there are more possible square submatrices which are not in $\mathcal{P}$.

(a) Let now $r$ be a gr-Sylvester matrix rank function for $R$. Let $\mathcal{P}_r$ be defined as (b) of the statement of the theorem. We have to show that $\mathcal{P}_r$ is a gr-prime matrix ideal.

Let $A \in \mathfrak{M}(R)$ be an $n \times n$ non gr-full matrix. There exist $\pi, \delta \in \Gamma^n$ such that $A \in M_n(R)[\pi][\delta]$, $\delta \in \Gamma^{-1}$ and matrices $B \in M_{n \times (n-1)}(R)[\pi][0]$, $C \in M_{(n-1) \times n}(R)[0][\delta]$ such that $A = BC$. By Lemma 8.1(6), $r(B) \leq n - 1$ and $r(C) \leq n - 1$. By (MatRF2), $r(A) = r(BC) \leq \min\{r(B), r(C)\} < n$. Thus $A \in \mathcal{P}_r$. and $\mathcal{P}_r$ satisfies (PM1).

Let $A \in \mathcal{P}_r$, $B \in \mathfrak{M}(R)$. Since $r(A) < \text{size of } A$, then $r(A \oplus B) = r(A) + r(B) < \text{size of } A \oplus B$. Thus (PM3) follows.

Let $A \in \mathfrak{M}(R)$ and suppose that $A \oplus 1 \in \mathcal{P}_r$. It means that $r(A) + 1 = r(A \oplus 1) < 1 + \text{size of } A$. Hence $r(A) < \text{size of } A$. Therefore $A \in \mathcal{P}_r$ and (PM5) is satisfied.
By definition of gr-Sylvester matrix rank function, \((1) \not\in \mathcal{P}_r\) and thus (PM5) holds.

By Lemma 5.14, (PM6) follows. It remains to show (PM2).

Let \(A, A' \in \mathcal{P}_r\) such that \(A \nabla A'\) exists with respect to the last column. Suppose that \(A = (B\, c)\), \(A' = (B\, c')\). Then \(A \nabla A' = (B\, c + c')\). We claim that \(r(A \nabla A') \leq \max\{r(A), r(A')\}\). This claim implies that \(A \nabla A' \in \mathcal{P}_r\). Then the case of the determinantal sum with respect to any other column follows from Lemma 5.14 and the claim.

Now we prove the claim.

\[
\begin{align*}
\text{r}(A) + \text{r}(A') &= \text{r}(B\, c) + \text{r}(B\, c') \\
&= r\begin{pmatrix} B & c & 0 & 0 \\ 0 & 0 & B & c' \end{pmatrix} = r\begin{pmatrix} B & c & 0 & 0 \\ -B & 0 & B & c' \end{pmatrix} \\
&= r\begin{pmatrix} B & c & 0 & 0 \\ 0 & c & B & c' \end{pmatrix} = r\begin{pmatrix} B & c & 0 & c \\ 0 & c & B & c + c' \end{pmatrix} \\
&\geq r(B) + r(c\, B\, c + c').
\end{align*}
\]

(8.1)

If \(r(A) \leq r(B)\), then \(r(A) = r(B)\) by Lemma 5.17. By (8.1),

\[
\text{r}(A') \geq r(c\, B\, c + c') \geq r(B\, c + c') = r(A \nabla A'),
\]
as desired.

If \(r(A) > r(B)\), then

\[
\text{r}(A) \geq r(B) + 1 = r\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} = r\begin{pmatrix} B & c + c' \\ 0 & 1 \end{pmatrix} \geq r(B\, c + c') = r(A \nabla A').
\]

Interchanging the roles of \(A\) and \(A'\), we get that if \(r(A') \leq r(B)\), then \(r(A) \geq r(A \nabla A')\) and that if \(r(A') > r(B)\), then \(r(A') \geq r(A \nabla A')\). Thus, the claim is proved.

It remains to show that the maps \(\mathcal{P} \mapsto r_{\mathcal{P}}\) and \(r \mapsto \mathcal{P}_r\) are inverse one of the other.

If \(\mathcal{P}\) is a gr-prime matrix ideal, the gr-prime matrix ideal that corresponds to \(r_{\mathcal{P}}\) is the set of matrices \(A \in \mathfrak{M}(R)\) such that \(r_{\mathcal{P}}(A) < \text{size of } A\). That is, the set of matrices \(A \in \mathfrak{M}(R)\) whose largest square submatrix that is not in \(\mathcal{P}\) is less than the size of \(A\). In other words, the matrices \(A \in \mathcal{P}\). Therefore \(r_{\mathcal{P}} \mapsto \mathcal{P}\). On the other hand, let now \(r\) be a gr-rank function for \(R\). Let \(r_{\mathcal{P}_r}\) be the associated gr-rank function associated to \(\mathcal{P}_r\). If \(A \in \mathfrak{M}^{\ast}(R)\), then \(r_{\mathcal{P}_r}(A)\) equals the size of a largest square submatrix of \(A\) which is not in \(\mathcal{P}_r\). That is, the size of a largest square submatrix of \(B\) of \(A\) such that \(r(B) = \text{size of } B\). Hence we have to show that \(r(A) = n\) if and only if \(n\) is the size of a largest square submatrix of \(A\) such that \(r(B) = n\). But this now follows from Lemma 5.13. \(\square\)

8.2. Equivalence between gr-Sylvester rank functions. The following result can be proved in exactly the same way as in [24] Lemma 2] where the ungraded case is shown.

Lemma 8.6. Let \(R\) be a \(\Gamma\)-graded ring. If \(0 \rightarrow K \rightarrow Q \rightarrow M \rightarrow 0\) and \(0 \rightarrow K' \rightarrow Q' \rightarrow M \rightarrow 0\) are exact sequences of \(\Gamma\)-graded \(R\)-modules with \(Q\) and \(Q'\) \(\Gamma\)-graded projective \(R\)-modules and with \(K \subseteq Q\) and \(K' \subseteq Q'\), then there is an automorphism of \(\Gamma\)-graded \(R\)-modules of \(Q \oplus Q'\) which maps \(K \oplus K'\) onto \(Q \oplus K'\). \(\square\)

The next result was first stated in [28] p. 97] for the ungraded case. Our proof follows the one of [24] Theorem 4]. We would like to remark that the fact that \(\mathbb{N}\) is the set of values of Sylvester rank functions is not used in the proof.
Theorem 8.7. Let $R$ be a $\Gamma$-graded ring. There is an anti-isomorphism of partially ordered sets

$$
\begin{align*}
\left\{ \text{gr-Sylvester module rank functions for } R \right\} & \quad \to \quad \left\{ \text{gr-Sylvester map rank functions for } R \right\} \\
d & \quad \mapsto \quad \rho_d \\
d \rho & \quad \mapsto \quad \rho
\end{align*}
$$

defined as follows.

(a) If $d$ is a gr-Sylvester module rank function for $R$ and $f: P \to Q$ is a homomorphism of $\Gamma$-graded $R$-modules with $P, Q$ finitely generated $\Gamma$-graded projective $R$-modules, then $\rho_d(f) = d(Q) - d(\text{coker } f)$.

(b) Conversely, let $\rho$ be a gr-Sylvester map rank function and suppose that $f: P \to Q$ is a homomorphism of $\Gamma$-graded $R$-modules with $P, Q$ finitely generated $\Gamma$-graded projective $R$-modules such that $\text{coker } f = M$. Then $d\rho(M) = \rho(1_Q) - \rho(f)$.

Proof. First we show that the correspondence is an anti-isomorphism of partially ordered sets. Let $\rho_1 \leq \rho_2$ be two gr-Sylvester map rank functions. Let $M$ be a finitely presented $\Gamma$-graded $R$-module and suppose that $R^n(\overline{\mathcal{S}}) \xrightarrow{\alpha} R^m(\overline{\mathcal{T}}) \to M \to 0$ is a presentation of $M$ as $\Gamma$-graded $R$-module. Then $\rho_1(1_{R^n(\overline{\mathcal{S}})}) = \rho_2(1_{R^n(\overline{\mathcal{S}})}) = m$ and $\rho_1(f) \leq \rho_2(f)$. Hence $d\rho_1(M) = \rho_1(1_{R^n(\overline{\mathcal{S}})}) - \rho_1(f) \geq \rho_2(1_{R^n(\overline{\mathcal{S}})}) - \rho_2(f) = d\rho_2(M)$.

Secondly, we show that the correspondences are one inverse of the other. Let $d$ be a gr-Sylvester module rank function. Let $M$ be a finitely presented $\Gamma$-graded $R$-module. Then, given a graded presentation of $M$, $P \xrightarrow{f} Q \to M \to 0$, $d\rho_1(M) = \rho_1(1_Q) - \rho_1(f) = d(Q) - d(M) - d(\text{coker } f)$. Conversely, let $\rho$ be a gr-Sylvester map rank function. Let $f: P \to Q$ be a homomorphism of $\Gamma$-graded $R$-modules with $P, Q$ finitely generated $\Gamma$-graded projective $R$-modules. Then $\rho_1(f) = d\rho(Q) - d\rho(\text{coker } f) = \rho(1_Q) - d(\rho(1_Q) - \rho(f)) = \rho(f)$, as desired.

(a) Suppose that $d$ is a gr-Sylvester module rank function. For a homomorphism of $\Gamma$-graded $R$-modules $f: P \to Q$ between finitely generated $\Gamma$-graded projective $R$-modules define $\rho_d(f) = d(Q) - d(\text{coker } f)$. Notice it is clearly well defined. We must show that $\rho_d$ satisfies (MapRF1)–(MapRF5).

Clearly $\rho_d(1_R) = d(R) - d(0) = d(R) = 1$. Thus (MatRF1) is satisfied.

Let $g: Q \to T$ be a graded homomorphism between finitely generated $\Gamma$-graded $R$-modules. By definition, $\rho_d(gf) = d(T) - d(\text{coker } (gf))$. From the natural homomorphism of graded modules $T/\text{Im}(gf) \to T/\text{Im } g \to 0 \to 0$, by (ModRF3), we get that $d(\text{coker } g) \leq d(\text{coker } (gf))$. Thus $\rho_d(gf) \leq d(T) - d(\text{coker } g) = \rho_d(g)$. Let $\pi_1: Q \to Q/\text{Im } f$ and $\pi_2: T \to T/\text{Im } (gf)$ be the natural homomorphisms (of $\Gamma$-graded $R$-modules), and consider $\overline{g}: Q/\text{Im } f \to T/\text{Im } (gf)$ induced from $g: Q \to T$. It is not difficult to show that the following sequence of homomorphisms of $\Gamma$-graded $R$-modules is exact

$$
Q \xrightarrow{\overline{g}} T \oplus Q/\text{Im } f \xrightarrow{(-\pi_2, g)} T/\text{Im } (gf) \to 0.
$$

Hence, by (ModRF3), $d(\text{coker } (gf)) \leq d(T) + d(\text{coker } f) \leq d(\text{coker } (gf)) + d(Q)$. Now, subtracting $d(\text{coker } (gf)) + d(\text{coker } f)$ on both sides of the second inequality we obtain $\rho_d(gf) = d(T) - d(\text{coker } f) \leq d(Q) - d(\text{coker } f) = \rho_d(f)$. Thus (MapRF2) is satisfied.
Let $f: P \to Q$, $f': P' \to Q'$ be homomorphisms of $\Gamma$-graded $R$-modules where $P, Q, P', Q'$ are $\Gamma$-graded projective $R$-modules. Now

$$\rho_d \left( \begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix} \right) = d(Q \oplus Q') - d \left( \text{coker} \left( \begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix} \right) \right) = d(Q) + d(Q') - d(\text{coker} f \oplus \text{coker} f') = d(Q) - d(\text{coker} f) + d(Q') - d(\text{coker} f') = \rho_d(f) + \rho_d(f').$$

Thus (MapRF3) is satisfied.

Let $h: P' \to Q$ be a homomorphism of $\Gamma$-graded $R$-modules and consider the homomorphism of $\Gamma$-graded $R$-modules $\left( \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \right): P \oplus P' \to Q \oplus Q'$. Let $\pi: Q / \text{Im} f \to Q' / \text{Im} f'$ be induced from the natural projection $Q \to Q + Q'$. Then $\pi: \frac{Q \oplus Q'}{\text{Im} \left( \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \right)} \to Q' / \text{Im} f'$ is induced from the natural inclusion $Q \to Q \oplus Q'$. Let $(\text{ModRF1}) - (\text{ModRF4})$ be satisfied $\pi$. Hence $\rho_d(f) = \rho_d(f')$ and (MapRF5) is satisfied. Therefore $\rho_d$ is a gr-Sylvester map rank function.

(b) Suppose that $\rho$ is a gr-Sylvester map rank function. If $M$ is a finitely presented $\Gamma$-graded $R$-module and $f: P \to Q$ is a homomorphism of $\Gamma$-graded $R$-modules with $P, Q$ finitely generated $\Gamma$-graded projective $R$-modules such that $\text{coker} f = M$, then we define $d_{\rho}(M) = \rho(1_{Q}) - \rho(f)$. We must show that $d$ is well defined and satisfies (ModRF1) - (ModRF4).

We begin showing that $d_{\rho}$ is well defined. Suppose that $P \xrightarrow{\lambda} Q \to M \to 0$ and $P' \xrightarrow{\lambda'} Q' \to M \to 0$ are two graded presentations with $P, P', Q, Q'$ finitely generated $\Gamma$-graded $R$-modules. By Lemma 5,8 there exists an automorphism $h$ of $\Gamma$-graded $R$-modules of $Q \oplus Q'$ which maps $\text{Im} f \oplus \text{Im} f'$ onto $Q \oplus \text{Im} f'$. Since $Q \oplus Q'$ is a $\Gamma$-projective $R$-module, we obtain the following diagram of homomorphisms of $\Gamma$-graded $R$-modules

$$\begin{array}{c}
\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix} \\
\text{coker} \left( \begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix} \right) \\
\end{array} \xrightarrow{d} \\
\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix} \\
\text{coker} \left( \begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix} \right)
\end{array}$$

0
Hence we obtain \( h\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1_{Q'} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \) and \( h^{-1}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} f & 0 \\ 0 & 1_{Q'} \end{smallmatrix}\right) \) \( u' \) and, by (MapRF2),

\[
\rho(f) + \rho(1_Q) = \rho\left(h\left(\begin{smallmatrix} f & 0 \\ 0 & 1_{Q'} \end{smallmatrix}\right)\right) \leq \rho\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) = \rho(f') + \rho(1_Q).
\]

\[
\rho(f') + \rho(1_Q) = \rho\left(h^{-1}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) \leq \rho\left(\left(\begin{smallmatrix} f & 0 \\ 0 & 1_{Q'} \end{smallmatrix}\right)\right) = \rho(f) + \rho(1_Q).
\]

It implies \( \rho(1_{Q'}) - \rho(f') \leq \rho(1_Q) - \rho(f) \) and \( \rho(1_Q) - \rho(f) \leq \rho(1_{Q'}) - \rho(f') \). Therefore \( d_{\rho}(M) = \rho(1_Q) - \rho(f) \) is well defined.

Consider the exact sequence \( 0 \to R \to Q \to 0 \). By definition, \( d_{\rho}(R) = \rho(1_R) - \rho(0) = 1 - 0 = 1 \). Thus (ModRF1) is satisfied.

Let \( M_1, M_2 \) be finitely presented \( \Gamma \)-graded \( R \)-modules. Let \( P_1 \xrightarrow{f_1} Q_1 \to M_1 \to 0 \) and \( P_2 \xrightarrow{f_2} Q_2 \to M_2 \to 0 \) be exact sequences of homomorphisms of \( \Gamma \)-graded \( R \)-modules with \( P_1, P_2, Q_1, Q_2 \) finitely generated \( \Gamma \)-graded projective \( R \)-modules.

Then \( P_1 \oplus P_2 \xrightarrow{\left(\begin{smallmatrix} f_1 & 0 \\ 0 & f_2 \end{smallmatrix}\right)} Q_1 \oplus Q_2 \to M_1 \oplus M_2 \to 0 \) is a \( \Gamma \)-graded presentation of \( M_1 \oplus M_2 \). By definition and (MapRF3), \( d_{\rho}(M_1 \oplus M_2) = \rho(1_{Q_1} \oplus Q_2) - \rho\left(\left(\begin{smallmatrix} f_1 & 0 \\ 0 & f_2 \end{smallmatrix}\right)\right) = \rho(1_{Q_1}) - \rho(f_1) + \rho(1_{Q_2}) - \rho(f_2) = d_{\rho}(M_1) + d_{\rho}(M_2) \). Hence (ModRF2) is satisfied.

Let \( M_1 \xrightarrow{u} M_2 \xrightarrow{v} M_3 \to 0 \) be an exact sequence of homomorphisms of \( \Gamma \)-graded \( R \)-modules with \( M_1, M_2, M_3 \) finitely presented \( \Gamma \)-graded \( R \)-modules. Let \( P_1 \xrightarrow{f_1} Q_1 \xrightarrow{g_1} M_1 \to 0 \) and \( P_2 \xrightarrow{f_2} Q_3 \xrightarrow{g_3} M_3 \to 0 \) be exact sequences of homomorphisms of \( \Gamma \)-graded \( R \)-modules with \( P_1, Q_1, P_3, Q_3 \) finitely generated \( \Gamma \)-graded projective \( R \)-modules. By diagram chasing, it is easy to obtain the following commutative diagram, with exact rows and columns, of homomorphisms of \( \Gamma \)-graded \( R \)-modules where \( \iota \) and \( \pi \) are the natural inclusion and projection, respectively.

\[
\begin{array}{ccccccccc}
0 & & & & & & & & 0 \\
& \downarrow & & & & & & \downarrow & \\
\ker g & & \pi & \ker g & & \ker g_3 & & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & & Q_1 & & \pi & \downarrow & & Q_3 & & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
M_1 & & u & & M_2 & & v & & M_3 & & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

Since \( \ker g \) is a finitely generated \( \Gamma \)-graded \( R \)-module, there exist a finitely generated \( \Gamma \)-graded projective \( R \)-module and a surjective homomorphism of \( \Gamma \)-graded \( R \)-modules \( P \xrightarrow{f} \ker g \). In this way, we obtain a commutative diagram, with exact
rows and columns, of homomorphisms of $\Gamma$-graded $R$-modules

Note that $\iota f_1(P_1) \subseteq \ker g$ and that $f: P \to Q_1 \oplus Q_3$ is of the form $\left( \begin{array}{c} \lambda' \\ 0 \\ 0 \end{array} \right)$ for some homomorphism of $\Gamma$-graded $R$-modules $\lambda: P \to Q_1$. Thus we can modify the foregoing diagram to obtain

$$
\begin{array}{cccccccc}
P_1 \oplus P & \overset{\pi}{\longrightarrow} & P \\
\downarrow & & \downarrow \\
(Q_1 \oplus Q_3) & \overset{\pi}{\longrightarrow} & Q_3 \\
\downarrow & & \downarrow \\
M_1 & \overset{u}{\longrightarrow} & M_2 & \overset{v}{\longrightarrow} & M_3 & \to 0
\end{array}
$$

Then, by (MapRF4),

$$
d_\rho(M_2) = \rho(1_{Q_1 \oplus Q_3}) - \rho\left( \begin{array}{c} f_1 \\ 0 \\ 0 \end{array} \right) \\
\leq \rho(1_{Q_1}) + \rho(1_{Q_3}) - \rho(f_1) - \rho(f'_3) \\
= d_\rho(M_1) + d_\rho(M_3).
$$

Moreover,

$$
d_\rho(M_2) = \rho(1_{Q_1 \oplus Q_3}) - \rho\left( \begin{array}{c} f_1 \\ 0 \\ 0 \end{array} \right) \\
\geq \rho(1_{Q_1}) + \rho(1_{Q_3}) - \rho(f'_3) - \rho((f_1, \lambda)) \\
= d_\rho(M_1) + \rho(1_{Q_1}) - \rho((f_1, \lambda)) \\
\geq d_\rho(M_3),
$$

where we have used the fact that $\left( \begin{array}{c} f_1 \\ 0 \\ 0 \end{array} \right) = \left( \begin{array}{c} f_1 \\ 0 \\ 0 \end{array} \right) \left( \begin{array}{c} 1_{f_3} \\ 0 \\ 0 \\ 1_{f_3} \end{array} \right)$, and properties (MapRF2), (MapRF3) on the first inequality, and Lemma 8.3(6) on the second inequality. Therefore (ModRF3) is satisfied.

(ModRF4) follows easily. Indeed, let $f: R^n(\overline{\beta}) \to R^m(\overline{\alpha})$, $f': R^n(\overline{\beta'}) \to R^m(\overline{\alpha'})$ be homomorphisms of $\Gamma$-graded $R$-modules such that $F(f) = F(f')$. By (MapRF5), $\rho(f) = \rho(f')$ and therefore $d_\rho(\text{coker } f) = \rho(1_{R^m(\overline{\alpha})}) - \rho(f) = \rho(1_{R^m(\overline{\alpha'})}) - \rho(f') = d_\rho(\text{coker } f')$. \hfill \Box

The next result was given in [24, Theorem 4] in the ungraded context.
Theorem 8.8. Let $R$ be a $\Gamma$-graded ring. There is an anti-isomorphism of partially ordered sets
\[
\begin{align*}
\{ \text{gr-Sylvester matrix rank functions for } R \} & \quad \rightarrow \quad \{ \text{gr-Sylvester module rank functions for } R \} \\
\ r & \quad \mapsto \quad \ d_r \\
\ r_d & \quad \mapsto \quad d
\end{align*}
\]
defined as follows.

(a) If $r$ is a gr-Sylvester matrix rank function for $R$ and $M$ is a finitely presented $\Gamma$-graded $R$-module with presentation $R^n(\overline{\beta}) \xrightarrow{\Delta} R^m(\overline{\alpha}) \to M \to 0$, where $A \in M_{m \times n}(R)[\overline{\alpha}][\overline{\beta}]$, we define $d_r(M) = m - r(A)$.

(b) Conversely, let $d$ is a gr-Sylvester module rank function for $R$. If $A \in M_{m \times n}(R)[\overline{\alpha}][\overline{\beta}]$, we consider $A$ as a homomorphism of $\Gamma$-graded $R$-modules $R^n(\overline{\beta}) \to R^m(\overline{\alpha})$ and define $r(A) = m - d(R^m(\overline{\alpha})/A(R^n(\overline{\beta})))$.

Proof. First we show that the correspondence is an anti-isomorphism of partially ordered sets. Let $r_1 \leq r_2$ be two gr-Sylvester matrix rank functions. Let $M$ be a finitely presented $\Gamma$-graded $R$-module and suppose that $R^n(\overline{\beta}) \xrightarrow{\Delta} R^m(\overline{\alpha}) \to M \to 0$ is a presentation of $M$ as $\Gamma$-graded $R$-module. Then $r_1(1_{R^m(\overline{\alpha})}) = r_2(1_{R^m(\overline{\alpha})}) = m$ and $r_1(A) \leq r_2(A)$. Hence $d_{r_1}(M) = r_1(1_{R^m(\overline{\alpha})}) - r_1(A) = r_2(1_{R^m(\overline{\alpha})}) - r_2(A) = d_{r_2}(M)$.

Secondly, we show that the correspondences are one inverse of the other. Let $d$ be a gr-Sylvester module rank function. Let $M$ be a finitely presented $\Gamma$-graded $R$-module. Then, given a graded presentation of $M$, $R^n(\overline{\alpha}) \xrightarrow{\Delta} R^m(\overline{\beta}) \to M \to 0$, $d_{r_1}(M) = r_1(1_{R^m(\overline{\alpha})}) - r_1(A) = d(R^m(\overline{\alpha})) - 0 - (d(R^m(\overline{\beta})) - d(M)) = d(M)$.

Conversely, let $r$ be a gr-Sylvester matrix rank function. Let $A \in M_{m \times n}(R)[\overline{\alpha}][\overline{\beta}]$. Then $r_{d}(A) = d(r(M)) - d_r(coker A) = r(1_{R^m(\overline{\alpha})}) - 0 - (r(1_{R^m(\overline{\alpha})}) - r(A)) = r(A)$, as desired.

(a) Suppose that $d$ is a Sylvester module rank function for $R$. Let $A \in M_{m \times n}(R)[\overline{\alpha}][\overline{\beta}]$ and consider $A$ as a homomorphism of $\Gamma$-graded $R$-modules $R^n(\overline{\beta}) \to R^m(\overline{\alpha})$. Define $r_{d}(A) = d(R^m(\overline{\alpha})) - d(coker A) = m - d(R^m(\overline{\alpha})) - d(R^m(\overline{\beta}))/A(R^n(\overline{\beta}))$. The fact that $r_{d}$ is well defined follows from (ModRF4). That is, the matrix $A$ may define different homomorphisms of $\Gamma$-graded modules, but the value of $r_{d}(A)$ is the same. The proof that $r_{d}$ satisfies (MatRF1)–(MatRF4) follows in the same way as the proof that $r_{d}$ satisfies (MapRF1)–(MapRF5) in Theorem 5.7.

(b) Suppose now that $r$ is a gr-Sylvester matrix rank function for $R$. Let $M$ be a finitely presented $\Gamma$-graded $R$-module with presentation $R^n(\overline{\beta}) \xrightarrow{\Delta} R^m(\overline{\alpha}) \to M \to 0$, where $A \in M_{m \times n}(R)[\overline{\alpha}][\overline{\beta}]$, we define $d_{r}(M) = m - r(A)$. One can show that $d_{r}$ is well defined and satisfies (ModRF1)–(ModRF4) in the same way that one proves that $d_{r}$ is well defined and satisfies (ModRF1)–(ModRF4). There is another way to prove that $d_{r}$ is well defined and satisfies (ModRF1)–(ModRF4). There is no way to prove that $d_{r}$ satisfies (ModRF1)–(ModRF4). By Theorem 5.7 let $(K, \varphi)$ be the corresponding $\Gamma$-graded epic $R$-division ring with $r$. Then, for a finitely presented $\Gamma$-graded $R$-module $M$, the $K$-module $M \otimes_R K$ is a $\Gamma$-graded free $K$-module. One can define $d(M) = \dim_K(M \otimes_R K)$. It is not difficult to show that $d$ satisfies (ModRF1)–(ModRF4). Now let $M$ be a finitely presented $\Gamma$-graded $R$-module with presentation $R^n(\overline{\beta}) \xrightarrow{\Delta} R^m(\overline{\alpha}) \to M \to 0$, where $A \in M_{m \times n}(R)[\overline{\alpha}][\overline{\beta}]$. Then, by Theorem 5.7 $\dim(M \otimes_R K)$ equals $m$ minus the number of columns of $A^\Gamma$ which are right linearly independent over $K$ and that is exactly $m - r(A) = d_{r}(M)$.

It is worth noting the following corollary. It is just a re-writing of parts of Theorems 4.5, 8.7, 8.8 and Corollary 6.6.
Corollary 8.9. Let $R$ be a $\Gamma$-graded ring. Suppose that $(K_1, \varphi_1)$ and $(K_2, \varphi_2)$ are $\Gamma$-graded epic $R$-division rings with corresponding gr-prime matrix ideals $P_1, P_2,$ respectively, and $\Gamma$-Sylvester rank functions $r_1, r_2, d_1, d_2, \rho_1, \rho_2$, respectively. The following assertions are equivalent

1. There exists a gr-specialization from $(K_1, \varphi_1)$ to $(K_2, \varphi_2)$.
2. $P_1 \subseteq P_2$.
3. $r_1 \geq r_2$.
4. $d_1 \leq d_2$.
5. $\rho_1 \geq \rho_2$.

9. GR-PRIME SPECTRUM

Throughout this section, let $\Gamma$ be a group and $\Omega \subseteq \Omega'$ be normal subgroups of $\Gamma$.

Let $R$ be a $\Gamma$-graded ring. It can be considered as a $\Gamma/\Omega$-graded ring too. Now we introduce some notation in order to clarify which structure of graded object is being considered. We will denote by $\mathcal{M}^{\Omega}(R)$, $\mathcal{M}_1^{\Omega}(R)$ and by $\mathcal{M}^{\Omega/\Omega}(R)/\mathcal{M}_1^{\Omega}(R)$, $\mathcal{M}_1^{\Omega/\Omega}(R)$ the corresponding sets of matrices. Notice that $\mathcal{M}^{\Omega}(R) \subseteq \mathcal{M}^{\Omega/\Omega}(R)$ and $\mathcal{M}_1^{\Omega}(R) \subseteq \mathcal{M}^{\Omega/\Omega}(R)$. We denote by $\text{Spec}_{\Gamma}(R)$ the set of all $\Gamma$-gr-prime matrix ideals and by $\text{Spec}_{\Gamma/\Omega}(R)$ the set of all $\Gamma/\Omega$-gr-prime matrix ideals. If $\Omega = \Gamma$, we will write $\text{Spec}(R)$ instead of $\text{Spec}_{\Gamma/\Omega}(R)$. Note that $\text{Spec}(R)$ is the usual set of prime matrix ideals.

It follows directly from the definition that if $\mathcal{P}$ is a $\Gamma/\Omega$-gr-prime matrix ideal, then $\mathcal{P} \cap \mathcal{M}^{\Omega}(R)$ is a $\Gamma$-gr-prime matrix ideal. Hence, there exists a map

$$\text{Spec}_{\Gamma/\Omega}(R) \to \text{Spec}_{\Gamma}(R), \quad \mathcal{P} \mapsto \mathcal{P} \cap \mathcal{M}^{\Omega}(R).$$

Suppose now that $r^{\Gamma/\Omega}: \mathcal{M}_1^{\Gamma/\Omega}(R) \to \mathbb{N}$ is a Sylvester $\Gamma/\Omega$-gr-matrix rank function. It follows from the definition that the restriction of $r^{\Gamma/\Omega}$ to $\mathcal{M}_1^{\Omega}(R)$ induces a $\Gamma$-gr-matrix rank function $r^{\Gamma}: \mathcal{M}_1^{\Omega}(R) \to \mathbb{N}$. In this way, there is a function

$$\{ \Gamma/\Omega\text{-gr-Sylvester matrix rank functions for } R \} \quad \to \quad \{ \Gamma\text{-gr-Sylvester matrix rank functions for } R \}, \quad t^{\Gamma/\Omega} \mapsto t^{\Gamma}.$$

Similarly, since any $\Gamma$-graded (finitely generated, finitely presented, projective) $R$-module is also a $\Gamma/\Omega$-graded (finitely generated, finitely presented, projective) $R$-module and any homomorphism between $\Gamma$-graded $R$-modules is also a homomorphism of $\Gamma/\Omega$-graded $R$-modules, by restriction, we obtain functions

$$\{ \Gamma/\Omega\text{-gr-Sylvester module rank functions for } R \} \quad \to \quad \{ \Gamma\text{-gr-Sylvester module rank functions for } R \}, \quad d^{\Gamma/\Omega} \mapsto d^{\Gamma}.$$

$$\{ \Gamma/\Omega\text{-gr-Sylvester map rank functions for } R \} \quad \to \quad \{ \Gamma\text{-gr-Sylvester map rank functions for } R \}, \quad \rho^{\Gamma/\Omega} \mapsto \rho^{\Gamma}.$$

Considering $R$ as a $\Gamma/\Omega$-graded ring and $\Omega'/\Omega$ as a normal subgroup of $\Gamma/\Omega$, we obtain maps $\text{Spec}_{\Gamma/\Omega'}(R) \to \text{Spec}_{\Gamma/\Omega}(R)$ for each pair of such normal subgroups of $\Gamma$. Hence if $\text{Spec}_{\Gamma/\Omega}(R)$ is empty, then $\text{Spec}_{\Gamma/\Omega'}(R)$ is also empty. In other words, if there does not exist a $\Gamma/\Omega$-graded epic $R$-division ring, then there does not exist a $\Gamma/\Omega'$-graded epic $R$-division ring. In particular, for $\Omega' = \Gamma$ we obtain maps $\text{Spec}(R) \to \text{Spec}_{\Gamma/\Omega'}(R), \mathcal{Q} \mapsto \mathcal{Q} \cap \mathcal{M}^{\Omega/\Omega}(R)$, for each normal subgroup $\Omega$ of $\Gamma$. Therefore, if there exist a normal subgroup $\Omega$ of $\Gamma$ such that there does not exist a $\Gamma/\Omega$-graded epic $R$-division ring, then there does not exist an epic $R$-division ring.

Let $\mathcal{Q}' \in \text{Spec}_{\Gamma/\Omega'}(R)$ and let $\mathcal{Q} = \mathcal{Q}' \cap \mathcal{M}^{\Omega/\Omega}(R)$ be the corresponding element in $\text{Spec}_{\Gamma/\Omega}(R)$. Let $(K_{\mathcal{Q}'}, \varphi_{\mathcal{Q}'})$ be the $\Gamma/\Omega'$-graded epic $R$-division ring determined by $\mathcal{Q}'$, and let $(K_{\mathcal{Q}}, \varphi_{\mathcal{Q}})$ be the $\Gamma/\Omega$-graded epic $R$-division ring determined by $\mathcal{Q}$. Let $x$ be a homogeneous element of $R$ considered as a $\Gamma/\Omega$-graded ring. Notice it is
also a homogeneous element of $R$ considered as a $\Gamma/\Omega'$-graded ring. If $x \notin \ker \varphi_Q'$, then $x \in \mathfrak{M}^{F/(\Omega')}(R) \setminus Q'$, and $x \notin \ker \varphi_Q$. Hence if $\varphi_Q'$ is injective, then $\varphi_Q$ is also injective. In other words, if $(K_Q', \varphi_Q')$ is a $\Gamma/\Omega'$-graded epic $R$-division ring of fractions, then $(K_Q, \varphi_Q)$ is also a $\Gamma/\Omega$-graded epic $R$-division ring of fractions. Therefore, if there exists a normal subgroup $\Omega$ of $R$ such that there does not exist a $\Gamma/\Omega$-graded epic $R$-division ring of fractions, then there does not exist an epic $R$-division ring of fractions.

Let $\mathcal{P}' \in \text{Spec}_{\Gamma/\Omega'}(R)$ and set $\mathcal{P} = \mathcal{P}' \cap \mathfrak{M}^{F/\Omega}(R) \in \text{Spec}_{\Gamma/\Omega}(R)$. If $\mathcal{P}' \subseteq Q'$, then $\mathcal{P} \subseteq Q$. Hence a specialization from $(K_{\mathcal{P}'}, \varphi_{\mathcal{P}'})$ to $(K_Q, \varphi_Q)$, implies the existence of a specialization from $(K_{\mathcal{P}}, \varphi_{\mathcal{P}})$ to $(K_Q, \varphi_Q)$ by Corollary 6.6. Notice that it could happen that $Q = \mathcal{P}$.

Also, if the map $\text{Spec}_{\Gamma/\Omega'}(R) \to \text{Spec}_{\Gamma/\Omega}(R)$ is surjective and $R$ has a universal $\Gamma/\Omega'$-graded epic $R$-division ring (of fractions), then $R$ has a universal $\Gamma/\Omega$-graded epic $R$-division ring (of fractions).

Suppose that for each $\Gamma$-graded epic $R$-division ring $D$ there exist ring homomorphisms to division rings. Then $\text{Spec}_{\Gamma/\Omega}(R) \to \text{Spec}_{\Gamma}(R)$ is surjective for each $\Omega < \Gamma$. Let $(D, \varphi)$ be a $\Gamma$-graded epic $R$-division ring with $\Gamma$-singular kernel $\mathcal{P}$. Let $\phi: D \to E$ be a ring homomorphism with $E$ a division ring. Consider the composition $\phi \circ \varphi: R \to E$. It is a homomorphism of $\Gamma/\Omega$-almost graded rings with $E$ a $\Gamma/\Omega$-almost graded division ring. By Theorem 4.3.2(b), there exists $\psi: R \to D'$ a $\Gamma/\Omega$-graded epic $R$-division ring, and a homomorphism $\rho: D' \to E$ such that $\phi \psi = \rho \psi$. By Proposition 2.2,

\[
\{ A \in \mathfrak{M}^{F}(R): A^{(\phi \psi)} \text{ invertible over } E \} = \{ A \in \mathfrak{M}^{F}(R): A^{\rho} \text{ invertible over } D' \},
\]
\[
\{ A \in \mathfrak{M}^{F/\Omega}(R): A \text{ invertible over } D' \} = \{ A \in \mathfrak{M}^{F/\Omega}(R): A^{(\rho \psi)} \text{ invertible over } E \}.
\]

Now, since $\mathfrak{M}^{F}(R) \subseteq \mathfrak{M}^{F/\Omega}(R)$, we get that
\[
\{ A \in \mathfrak{M}^{F}(R): A^{\rho} \text{ inverts over } D \} = \{ A \in \mathfrak{M}^{F/\Omega}(R): A \text{ inverts over } D' \} \cap \mathfrak{M}^{F}(R).
\]

Hence, if $\mathcal{P}'$ is the $\Gamma/\Omega$-singular kernel of $(D', \psi)$, then $\mathcal{P} = \mathcal{P}' \cap \mathfrak{M}^{F}(R)$.

We gather together what we have just proved in the following result.

**Theorem 9.1.** Let $R$ be a $\Gamma$-graded ring. The following assertions hold true.

1. If there does not exist a $\Gamma/\Omega$-graded epic $R$-division ring (of fractions), then there does not exist a $\Gamma/\Omega'$-graded epic $R$-division ring (of fractions). Therefore, if there exists a normal subgroup $\Omega$ of $\Gamma$ such that there does not exist a $\Gamma/\Omega'$-graded epic $R$-division ring (of fractions), then there does not exist an epic $R$-division ring (of fractions).

2. Let $(K_{\mathcal{P}'}, \varphi_{\mathcal{P}'})$, $(K_Q', \varphi_Q')$ be $\Gamma/\Omega'$-epic $R$-division rings, such that there exists a specialization from $(K_{\mathcal{P}'}, \varphi_{\mathcal{P}'})$ to $(K_Q', \varphi_Q')$, then there exists a specialization between the corresponding $\Gamma/\Omega$-graded epic $R$-division rings.

3. If the map $\text{Spec}_{\Gamma/\Omega'}(R) \to \text{Spec}_{\Gamma}(R)$, $Q' \mapsto Q' \cap \mathfrak{M}^{F/\Omega}(R)$, is surjective, then the existence of a universal $\Gamma/\Omega'$-graded epic $R$-division ring implies the existence of a universal $\Gamma/\Omega$-graded epic $R$-division ring. Therefore, if $\text{Spec}(R) \to \text{Spec}_{\Gamma/\Omega}(R)$, $Q' \mapsto Q' \cap \mathfrak{M}^{F/\Omega}(R)$, is surjective, the existence of a universal $\Gamma/\Omega$-division ring implies the existence of a universal $\Gamma/\Omega$-graded epic $R$-division ring.

4. If for each $\Gamma$-graded epic $R$-division ring there exist ring homomorphisms to division rings, then $\text{Spec}_{\Gamma/\Omega}(R) \to \text{Spec}_{\Gamma}(R)$, $Q \mapsto Q \cap \mathfrak{M}^{F}(R)$, is surjective. □

Note that there is a corresponding statement of Theorem 9.1 in terms of the different gr-Sylvester rank functions instead of gr-prime matrix ideals.
Let $R = \bigoplus_{\gamma \in \Gamma} R_{\gamma}$ be a $\Gamma$-graded ring. In the foregoing, we gave a correspondence from the set of $\Gamma/\Omega$-graded epic $R$-division rings to the set of $\Gamma$-graded epic $R$-division rings. We proceed to give a more down to earth description of such correspondence. Recall that $R$ can be regarded as a $\Gamma/\Omega$-graded ring making $R = \bigoplus_{\gamma \in \Gamma/\Omega} R_{\gamma}$ where $R_{\gamma} = \bigoplus_{\alpha \in \gamma} R_{\alpha}$ for each $\alpha \in \Gamma/\Omega$.

Let $E = \bigoplus_{\alpha \in \Gamma/\Omega} E_{\alpha}$ be a $\Gamma/\Omega$-graded division ring. Consider the group ring $E[\Gamma] = \bigoplus_{\gamma \in \Gamma} E_{\gamma}$. We construct a $\Gamma$-graded division ring $D = \bigoplus_{\gamma \in \Gamma} D_{\gamma}$ which is a $\Gamma$-graded subring of $E[\Gamma]$ in the same way as in [26, Proposition 1.2.2]. For each $\gamma \in \Gamma$, there exists a unique $\alpha \in \Gamma/\Omega$ such that $\gamma \in \alpha$. Set $D_{\gamma} = E_{\alpha \gamma} \subseteq E_{\gamma}$. Note that

$$D_{\gamma}D_{\gamma'} = E_{\alpha \gamma}E_{\alpha' \gamma'} = E_{\alpha E_{\gamma} \alpha' \gamma'} \subseteq E_{\alpha \alpha' \gamma \gamma'} = D_{\gamma \gamma'}.$$ 

Hence $D$ is a $\Gamma$-graded ring. Since $E$ is a $\Gamma/\Omega$-graded division ring, any nonzero homogeneous element of $D$ is invertible. Thus $D$ is a $\Gamma$-graded division ring.

Suppose that $(E, \varphi)$ is a $\Gamma/\Omega$-graded epic $R$-division ring. Let $\gamma \in \Gamma$ and $\alpha \in \Gamma/\Omega$ such that $\gamma \in \alpha$. For each $a_{\gamma} \in R_{\gamma}$, $\varphi(a_{\gamma}) \in E_{\alpha}$. Then define $\varphi(a_{\gamma}) = \varphi(a_{\gamma}) \in D_{\gamma}$. In this way, we obtain a homomorphism of $\Gamma$-graded rings $\varphi: R \rightarrow D$.

Let $A = (a_{ij}) \in M_{\gamma}(R[\Gamma])$. We claim that $A^\varphi$ is invertible in $E$ if and only if $A^\varphi$ is invertible in $D$. Indeed, let $\alpha_i, \beta_j \in \Omega$ be such that $\delta_i \in \alpha_i$, $\varepsilon_j \in \beta_j$. Then $A^\varphi = (b_{ij})$ with $b_{ij} \in E_{\alpha_i \beta_j}$ and $(A^\varphi)^{-1} = (c_{ij})$ with $c_{ij} \in E_{\alpha_i \beta_j}$. Then $A^\varphi = (b_{ij} \alpha_i \beta_j^{-1})$ is invertible in $D$ with inverse $(A^\varphi)^{-1} = (c_{ij} \beta_j \alpha_i^{-1})$. Conversely, if $A^\varphi$ is invertible with inverse $(A^\varphi)^{-1} = (d_{ij} \beta_i \alpha_j^{-1})$ where $d_{ij} \in E_{\beta_i \alpha_j}$, then $(A^\varphi)^{-1} = (d_{ij})$. Hence, let $P \in \text{Spec}_{\Gamma/\Omega}(R)$. If $(E, \varphi)$ is the $\Gamma/\Omega$-graded epic $R$-division ring associated to $P$, then the $\Gamma$-graded epic $R$-division ring associated to $P \cap 2\Omega(R)$ is determined by the $\Gamma$-graded division ring $\varphi: R \rightarrow D$. That is, the $\Gamma$-graded epic $R$-division ring $\varphi: R \rightarrow D'$ where $D'$ the graded division ring generated by $\text{Im} \varphi$.

Now we proceed to give an important family of examples of Theorem 9.1 (4). Let $(\Gamma, <)$ be an ordered group. Let $D = \bigoplus_{\gamma \in \Gamma} D_{\gamma}$ be a $\Gamma$-graded division ring. Given a map $f: \Gamma \rightarrow D$, let supp $f = \{ \gamma \in \Gamma: f(\gamma) \neq 0 \}$. We will write $f$ as a series. Thus, $f = \sum_{\gamma \in \Gamma} a_{\gamma}$ means that $f(\gamma) = a_{\gamma} \in D$ for each $\gamma \in \Gamma$. Consider the set

$$D((\Gamma; <)) = \left\{ f = \sum_{\gamma \in \Gamma} a_{\gamma} : a_{\gamma} \in D_{\gamma} \text{ for all } \gamma \in \Gamma, \text{ supp } f \text{ is well ordered} \right\}.$$ 

$D((\Gamma; <))$ is an abelian group under the natural sum. That is, for $f = \sum_{\gamma \in \Gamma} a_{\gamma}$, $f' = \sum_{\gamma \in \Gamma} a'_{\gamma}$, then

$$f + f' = \sum_{\gamma \in \Gamma} (a_{\gamma} + a'_{\gamma}).$$

One can then define the product in $D((\Gamma; <))$ as

$$ff' = \sum_{\gamma \in \Gamma} \left( \sum_{\delta \epsilon \gamma} a_{\delta}a'_{\epsilon} \right).$$

These operations endow $D((\Gamma; <))$ with a ring structure. We regard $D$ as a subring of $D((\Gamma; <))$ identifying $D$ with the series of $D((\Gamma; <))$ of finite support. Malcev and Neumann independently showed that $D((\Gamma; <))$ is in fact a division ring [22, 27]. Hence, we have just shown that for every $\Gamma$-graded division ring there exists a homomorphism of rings to a division ring.
Now we proceed to show that every $D((\Gamma; <))$ contains a $\Gamma/\Omega$-graded division ring and that it corresponds to $D$ via $\text{Spec}_{\Gamma/\Omega}(R) \to \text{Spec}_{\Gamma}(R)$. Let $\Omega$ be a normal subgroup of $\Gamma$. Consider $D$ as a $\Gamma/\Omega$-graded ring. For each $\alpha \in \Gamma/\Omega$, define the subset of $D((\Gamma; <))$

$$E_{\alpha} = \left\{ f = \sum_{\gamma \in \Gamma} a_{\gamma} \in D((\Gamma; <)) : \text{supp} f \subseteq \alpha \right\}.$$ 

Note that $E_{\alpha}$ is an additive subgroup of $D((\Gamma; <))$. Let $\alpha, \beta \in \Gamma/\Omega$. Suppose that $f = \sum_{\gamma \in \Gamma} a_{\gamma} \in E_{\alpha}$ and $f' = \sum_{\gamma \in \Gamma} a'_{\gamma} \in E_{\beta}$. Then

$$ff' = \sum_{\gamma \in \Gamma} \left( \sum_{\beta \in \gamma} a_{\beta}a'_{\gamma} \right) \in E_{\alpha\beta}.$$ 

Hence $E_{\alpha}E_{\beta} \subseteq E_{\alpha\beta}$. Moreover, if $\alpha \in \Gamma/\Omega$,

$$E_{\alpha} \cap \left( \sum_{\beta \in \Gamma/\Omega, \beta \neq \alpha} E_{\beta} \right) = \{0\},$$

because $\Gamma$ is the disjoint union $\Gamma = \bigcup_{\beta \in \Gamma/\Omega} \beta$. Hence $E(\Omega) = \bigoplus_{\alpha \in \Gamma/\Omega} E_{\alpha}$ is a $\Gamma/\Omega$-graded ring. Furthermore, let $f = \sum_{\gamma \in \Gamma} a_{\gamma} \in E_{\alpha}$, $f \neq 0$. Then $f$ is invertible in $D((\Gamma; <))$ with inverse

$$f^{-1} = \left( \sum_{n \geq 0} (-1)^{n} g^{n} \right) a_{\gamma_{0}}^{-1}$$

where $\gamma_{0} = \min \text{supp} f$ and $g = \sum_{\gamma \in \Gamma} a_{\gamma}^{-1}a_{\gamma}$. Since $\text{supp} \gamma \subseteq \alpha$, $\gamma_{0} \in \alpha$ and $\gamma_{0}^{-1} \in \alpha^{-1}$, then $\text{supp} g \subseteq E_{e}$ where $e$ denotes the identity element in $\Gamma/\Omega$. Hence $\text{supp} g^{n} \subseteq E_{e}$ for each integer $n \geq 0$ and $\text{supp}(\sum_{n \geq 0} (-1)^{n} g^{n}) \subseteq E_{e}$. Thus, $\text{supp} f^{-1} \subseteq \alpha^{-1}$. Therefore $E(\Omega)$ is a $\Gamma/\Omega$-graded division ring and the embedding $\phi_{\Omega}: D \to E(\Omega)$ is a homomorphism of $\Gamma/\Omega$-graded rings. Let $D(\Omega)$ be the $\Gamma/\Omega$-graded division subring of $E(\Omega)$ generated by $D$. Then $(D(\Omega), \phi_{\Omega}: D \to D(\Omega))$ is a $\Gamma/\Omega$-graded epic $D$-division ring.

Let $R = \bigoplus_{\gamma \in \Gamma} R_{\gamma}$ be a $\Gamma$-graded ring, where $(\Gamma, <)$ is an ordered group. Let $P \in \text{Spec}_{\Gamma}(R)$ with corresponding epic $R$-division ring $(K, \phi)$. Consider $K((\Gamma, <))$. Then for each $\Omega < \Gamma$, we get that $\text{Spec}_{\Gamma/\Omega}(R) \to \text{Spec}_{\Gamma}(R)$ is surjective. Indeed, if $Q \in \text{Spec}_{\Gamma/\Omega}(R)$ be the corresponding $\Gamma/\Omega$-graded prime matrix ideal to the $\Gamma/\Omega$-graded epic $R$-division ring $(K(\Omega), \phi_{\Omega} \phi)$, then $Q \mapsto P$ by Proposition \[2.3\].

We would like to remark that $D(\Gamma)$, the division subring of $D((\Gamma; <))$ generated by $D$, does not depend on the order $<$ of $\Gamma$ by \[13\] or \[8\]. Hence, since $D(\Omega)$ is just $DC(\phi_{\Omega})$, then $D(\Omega)$ does not depend on the order $<$ of $\Gamma$.

We end this section with a concrete application of the results in this section. Let $K$ be a field, $X$ be a nonempty set and $K(X)$ be the free $K$-algebra on $X$. It is well known that $K(X)$ has a universal division ring of fractions \[17\] Section 7.5. Let now $\Gamma$ be a group and $X \to \Gamma$, $x \mapsto x_{x}$, be a map. Then $K(X) = \bigoplus_{\gamma \in \Gamma} K(X)_{\gamma}$ is a $\Gamma$-graded ring where $K(X)_{\gamma}$ is the $K$-vector space spanned by the monomials $x_{1}x_{2} \ldots x_{r}$ such that $x_{1}x_{2} \ldots x_{r} = \gamma$. If $(\Gamma, <)$ is an ordered group, then $K(X)$ has a $\Gamma$-graded universal division ring of fractions by the foregoing example and Theorem \[0.13\],(4).
10. Inverse limits and ultraproducts in the category of graded epic
R-division rings

For details on filters and ultrafilters we refer the reader to [4].

Let $I$ be a nonempty set. A filter on $I$ is a set $\mathfrak{F}$ of subsets of $I$ which has the following properties

(F1) Every subset of $I$ that contains a set of $\mathfrak{F}$ belongs to $\mathfrak{F}$.
(F2) Every finite intersection of sets of $\mathfrak{F}$ belongs to $\mathfrak{F}$.
(F3) The empty set is not in $\mathfrak{F}$.

The set of filters on $I$ is partially ordered by inclusion. An ultrafilter on $I$ is a maximal filter. By [4] Theorem 1, p.60], each filter is contained in an ultrafilter. An ultrafilter $\mathfrak{U}$ on $I$ has the following property: if $J, K$ are subsets of $I$ such that $J \cup K = I$, then either $J \in \mathfrak{U}$ or $K \in \mathfrak{U}$.

The concrete ultrafilters we will be dealing with are constructed as follows. Let $(I, \leq)$ be a directed preorder set. For each $i \in I$, the set $S(i) = \{ j \in I : i \leq j \}$ is called a section of $I$ relative to $i$. The set $S$ consisting of all sections relative to elements of $I$ is a filter base and there exists a filter containing $S$ [4] Proposition 2, p.59). Therefore there exists an ultrafilter of $I$ containing $S$.

Let $\Gamma$ be a group.

Let $I$ be a set and $\mathfrak{U}$ be an ultrafilter on $I$. For each $i \in I$, let $R_i = \bigoplus_{\gamma \in \Gamma} R_{i\gamma}$ be a $\Gamma$-graded ring. We proceed to define the graded ultraproduct of the family $\{ R_i \}_{i \in I}$ following [14]. Consider the ring $P = \prod_{i \in I} R_i$ and consider the following subset $S$ of $P$,

$$S = \bigoplus_{\gamma \in \Gamma} \left( \prod_{i \in I} R_{i\gamma} \right).$$

Note that $S$ is a subring of $P$ which is $\Gamma$-graded with $S_\gamma = \prod_{i \in I} R_{i\gamma}$. For each $\gamma \in \Gamma$, if $x = (x_i)_{i \in I} \in S_\gamma$, let $z(x) = \{ i \in I : x_i = 0 \}$. The set $Z_i = \{ x \in S_\gamma : z(x) \in \mathfrak{U} \}$ is an additive subgroup of $S_\gamma$. Moreover, if $y \in S_\delta$ and $x \in Z_\gamma$, then $xy \in Z_{\gamma \delta}$ and $xy \in Z_{\delta \gamma}$. Therefore $Z = \bigoplus_{\gamma \in \Gamma} Z_\gamma$ is a graded ideal of $S$. Then the $\Gamma$-graded ring $U = S/Z$ is called the graded ultraproduct of the family of $\Gamma$-graded rings $\{ R_i \}_{i \in I}$.

A homogeneous element element $x \in U_\gamma$ is the class of an element $(x_i)_{i \in I} \in S_\gamma$, where each $x_i \in R_{i\gamma}$. We will write $x = [(x_i)_{i \in I}]_\mathfrak{U}$. Observe that if $x = [(x_i)_{i \in I}]_\mathfrak{U}$ and $y = [(y_i)_{i \in I}]_\mathfrak{U}$, then $x = y$ if and only if the set $\{ i \in I : x_i = y_i \} \in \mathfrak{U}$.

Suppose that $(R_i, \varphi_i)$ is a $\Gamma$-graded $R$-ring for each $i \in I$. Hence $\varphi_i : R \to R_i$ is a homomorphism of $\Gamma$-graded rings. Then there exists a unique homomorphism of rings $\varphi : R \to \prod_{i \in I} R_i$ such that $\pi_i \varphi = \varphi_i$ for each $i \in I$. Observe that $\text{Im} \varphi \subseteq S$. Composing with the natural homomorphism $S \to S/Z = U$, we obtain a homomorphism of $\Gamma$-graded rings $\varphi : R \to U$. Hence $U$ is a $\Gamma$-graded $R$-ring in a natural way. This fact and the following lemma will be very useful in this section.

**Lemma 10.1.** Let $\Gamma$ be a group. Let $I$ be a nonempty set and $\mathfrak{U}$ be an ultrafilter on $I$.

1. If $R_i$ is a $\Gamma$-graded division ring for each $i \in I$, then the ultraproduct $U$ of the family $\{ R_i \}_{i \in I}$ is a $\Gamma$-graded division ring.
2. If $R_i$ is a $\Gamma$-graded local ring with graded maximal ideal $m_i$ for each $i \in I$, then the ultraproduct $U$ of the family $\{ R_i \}_{i \in I}$ is a $\Gamma$-graded local ring with residue $\Gamma$-graded division ring $V$, the ultraproduct of the family of $\Gamma$-graded division rings $\{ R_i/m_i \}_{i \in I}$.
Proof. (1) Let \( x \in U_\gamma \). Then \( x = [(x_j)_{j} \in \mathcal{I}] \) for some \( x_j \in R_\gamma \). If \( x \) is nonzero, then \( J = \{ i : x_i \neq 0 \} \in \mathcal{I} \). For each \( i \in I \), define

\[
x_i' = \begin{cases} x_i^{-1} & \text{if } i \in J \\ 0 & \text{if } i \notin J \end{cases}.
\]

Notice that \( x_i' \in R_{\gamma^{-1}} \) for each \( i \in I \). Then \( x' = [(x_i')_{i} \in \mathcal{I}] \in U_{\gamma^{-1}} \) and \( xx' = x'x = 1 \), as desired.

(2) A homogeneous element \( x = [(x_i)_{i} \in \mathcal{I}] \) is invertible in \( U \) if and only if the set \( \{ i \in I : x_i \text{ is invertible in } U \} \in \mathcal{I} \) if and only if \( \{ i \in I : x_i \notin \mathfrak{m}_i \} \in \mathcal{I} \). Therefore, the ideal \( \mathfrak{m}_{i,j} \) generated by the homogeneous noninvertible elements, that is, the set \( \{(x_i)_{i} \in \mathcal{I} \in h(U) : \{ i \in I : x_i \in \mathfrak{m}_i \} \in \mathcal{I} \} \), is a proper ideal of \( U \). Hence \( U \) is a graded local ring.

It is not difficult to prove that the projections \( R_i \to R_i/\mathfrak{m}_i, a \mapsto a \), induce a surjective homomorphism of \( \Gamma \)-graded rings \( U \to V, [(x_i)_{i} \in \mathcal{I}] \mapsto [(x_i')_{i} \in \mathcal{I}] \). Note that a homogeneous element \([(x_i)_{i} \in \mathcal{I}] \) is in the kernel if and only if it belongs to \( \mathfrak{m} \).

The following corollary was used in Section 7.

Corollary 10.2. Let \( R \) be a \( \Gamma \)-graded domain. Suppose that, for each \( a \in h(R) \setminus \{0\} \), there exists a homomorphism of \( \Gamma \)-graded rings \( \varphi_a : R \to K_a \), where \( K_a \) is a \( \Gamma \)-graded division ring such that \( \varphi_a(a) \neq 0 \). Then there exists a \( \Gamma \)-graded epic \( R \)-division ring of fractions.

Proof. Let \( I = h(R) \setminus \{0\} \). For each \( a \in I \), let \( I_a = \{ \lambda \in I : \varphi_\lambda(a) \neq 0 \} \). Let \( E = \{ a_1, \ldots, a_n \} \) be a finite subset of \( I \). Then \( \bigcap_{i=1}^n I_{a_i} \neq \emptyset \), because \( \varphi_{a_1} \cdots \varphi_{a_n}(a_i) \neq 0 \) for each \( i = 1, \ldots, n \). Hence the set \( \mathfrak{B} = \{ I_a : a \in I \} \) is a set of subsets of \( I \) such that no finite subset of \( \mathfrak{B} \) has empty intersection. By [4, Proposition 1,p.58], there exists a filter on \( I \) containing \( \mathfrak{B} \). By [4, Theorem 1, p.60], there exists an ultrafilter \( \mathfrak{U} \) on \( I \) containing \( \mathfrak{B} \). By Lemma 10.1, the ultraproduct \( U \) of the family \( \{ K_a \}_{a \in I} \) is a \( \Gamma \)-graded division ring and there exists a homomorphism of \( \Gamma \)-graded rings \( \varphi : R \to U \), defined by \( \varphi(x) = [(\varphi_a(x))_{a \in \mathcal{I}}] \). Since the set \( I_\mathfrak{U} \in \mathcal{I} \), then \( \varphi(x) \neq 0 \) for each \( x \in h(R) \setminus \{0\} \). Therefore \( \varphi \) is injective.

Let \( R \) be a \( \Gamma \)-graded ring. Consider the category \( \mathcal{E}_R \) of \( \Gamma \)-graded epic \( R \)-division rings with specializations as morphisms defined in Section 4.

First we look at how inverse systems are in this category. An inverse system in \( \mathcal{E}_R \) is a pair \((K_i, \varphi_i)_{i \in I}, (\psi_{i,j})_{i,j \geq i}\) where \((I, \leq)\) is a directed preordered set, \((K_i, \varphi_i)\) is a \( \Gamma \)-graded epic \( R \)-division ring for each \( i \in I \), and \( \psi_{i,j} \) is a specialization from \((K_i, \varphi_i)\) to \((K_j, \varphi_j)\), \( i \geq j \), such that

\[
\psi_{j,k} \circ \psi_{i,j} = \psi_{i,k} \quad \text{for all } i, j, k \in I, i \geq j \geq k.
\]

Observe that (10.1) is superfluous because since the specializations already exist, and there is at most one specialization between graded epic \( R \)-division rings, the equality in (10.1) holds trivially.

Now we look at inverse limits. An inverse limit of the inverse system \((\{ (K_i, \varphi_i)_{i \in I}, (\psi_{i,j})_{i,j \geq i} \})\) in \( \mathcal{E}_R \) is a pair \((K, \varphi), (\psi_i)_{i \in I}\) where \((K, \varphi)\) is a \( \Gamma \)-graded epic \( R \)-division ring and \( \psi_i \) is a specialization from \((K, \varphi)\) to \((K_i, \varphi_i)\) for each \( i \in I \) such that the following properties are satisfied:

(i) \( \psi_{i,j} \circ \psi_i = \psi_j \) for all \( i, j \in I \), \( i \geq j \).

(ii) For each pair \((\{K', \varphi'\}, (\psi'_i)_{i \in I})\) that satisfies (i), i.e. \( \psi_{i,j} \circ \psi'_i = \psi'_j \) for \( i \geq j \),

then there is a specialization \( \phi : L \to K \) such that \( \psi_i \circ \phi = \psi'_i \) for all \( i \in I \).

Again note that (i) and the equality of specializations in (ii) are superfluous.
Theorem 10.3. Let $R$ be a $\Gamma$-graded ring. Let $((K_i, \varphi_i)_{i \in I}, (\psi_{i,j})_{i,j \geq j})$ be an inverse system in $E_R$ indexed on the directed nonempty preordered set $(I, \leq)$. Consider an ultrafilter $\mathcal{U}$ on $I$ that contains all the sections $\mathcal{S}(i)$, $i \in I$, of $I$. Set

$$
\Sigma_i = \{ A \in \mathfrak{M}(R): A^{\varphi i} \text{ is invertible over } K_i \}, \quad i \in I.
$$

The following assertions hold true.

1. There exists a $\Gamma$-graded epic $R$-division ring $(K, \varphi)$ which is the inverse limit of $((K_i, \varphi_i)_{i \in I}, (\psi_{i,j})_{i,j \geq j})$.

2. If $\Sigma = \{ A \in \mathfrak{M}(R): A^{\varphi} \text{ is invertible over } K \}$, then $\Sigma = \bigcup_{i \in I} \Sigma_i$.

3. The $\Gamma$-graded epic $R$-division ring determined by the ultraproduct $U$ of the family $\{(K_i, \varphi_i)_{i \in I}\}$ equals $(K, \varphi)$.

4. The ultraproduct $V$ of the family $\{R_{\Sigma_i}\}_{i \in I}$ is a $\Gamma$-graded local ring with residue $\Gamma$-graded division ring equal to $U$.

5. $R_{\Sigma}$ embeds in $V$.

Proof. For each $i \in I$, let $P_i$ be the singular kernel of $\varphi_i: R \rightarrow K_i$.

1. Since there exists a specialization $\psi_{i,j}: K_i \rightarrow K_j$ for $i, j \in I$, $i \geq j$, then $P_i \subseteq P_j$ by Corollary 6.6. Thus, the family of prime matrix ideals $\{P_i\}_{i \in I}$ is directed from below. Set $P = \bigcap_{i \in I} P_i$. It is not difficult to prove that $P$ satisfies (PM1), (PM3), (PM5) and (PM6) in the definition of gr-prime matrix ideal. To show that $P$ satisfies (PM4), let $A, B \in \mathfrak{M}(R) \setminus P$. There exist $i, j \in I$ such that $A \notin P_i$ and $B \notin P_j$. Since $I$ is directed, there exists $k \in I$ such that $i \leq k$, $j \leq k$. Thus $P_k \subseteq P_i$ and $P_k \subseteq P_j$, and both $A$ and $B$ do not belong to the gr-prime matrix ideal $P_k$. Hence $A + B \notin P_k$, and therefore $A + B / P \notin P$.

Let $(K, \varphi)$ be the $\Gamma$-graded epic $R$-division ring corresponding to $P$. Since $P \subseteq P_i$ for all $i \in I$, there exists a unique specialization $\psi_i: K \rightarrow K_i$. Consider now a pair $((L, \varphi'), (\psi_i')_{i \in I})$ where $L$ is a $\Gamma$-graded epic $R$-division ring and $\psi_i'$ is a gr-specialization from $(L, \varphi')$ to $(K_i, \varphi_i)$ for each $i \in I$. Let $Q$ be the singular kernel of $\varphi': R \rightarrow L$. Then $Q \subseteq P_i$ for each $i \in I$. Hence $Q \subseteq P$. Therefore, there exists a specialization $\psi: L \rightarrow K$ by Corollary 6.6.

2. By (1), the singular kernel of $(K, \varphi)$ equals $P = \bigcap_{i \in I} P_i$. Hence, $\Sigma = \mathfrak{M}(R) \setminus P = \bigcup_{i \in I} \Sigma_i$.

3. Let $U$ be the graded ultraproduct of the family $\{(K_i, \varphi_i)_{i \in I}\}$ of $\Gamma$-graded epic $R$-division rings, and let $\varphi: R \rightarrow U$ be the canonical homomorphism of $\Gamma$-graded rings. Let $L$ be the $\Gamma$-graded division subring of $U$ generated by the image of $\varphi$. Consider the $\Gamma$-graded epic $R$-division ring $(L, \varphi)$.

Let $A \in \bigcup_{i \in I} \Sigma_i$ be such that $A \in M_n(R)[\overline{t}][\overline{t}]$ for some $\overline{t}, \overline{t} \in \Gamma^n$. By Theorem 4.5, note that $\Sigma_j \subseteq \Sigma_i$ for all $i, j \geq j$. Let $t \in I$ be such that $A \in \Sigma_i$. Then $A \in \Sigma_i$ for all $i \in \mathcal{S}(t)$. Define $M_i = \begin{cases} (A^{\varphi i})^{-1} & \text{if } i \in \mathcal{S}(t) \\ 0 & \text{if } i \notin \mathcal{S}(t) \end{cases}$ If $M_i$ has $(u, v)$-entry $m_{uv}$ in $(K_i)_{\beta_{uv}}^{-1}$, let $M = (m_{uv}) \in M_n(L)$ where $m_{uv} = [(m_{uv}'))_{i \in I}]_{U}$. Note now that $A^{\varphi} M = MA^{\varphi} = I$. Indeed, if $AM = ([c_{uv}'])_{U}$, then $[(c_{uv}')_{U}]_{U} \in U_{e}, \mathcal{S}(t) \subseteq \{ i \in I: c_{uv}' = 1 \}$ and $\mathcal{S}(t) \subseteq \{ i \in I: c_{uv}' = 0, u \neq v \}$. Similarly one can show that $MA = I$.

Conversely, suppose that $A \in \Sigma_i[\overline{t}][\overline{t}]$. Let $(A^{\varphi})^{-1} = (m_{uv})$, $u, v = 1, \ldots, n$, where $m_{uv} = [(m_{uv}')]_{U}$. For each $i \in I$, let $M_i = M_n(K_i)[\overline{t}][\overline{t}]$ be the matrix with $(u, v)$-entry $m_{uv}$. Since $A^{\varphi}(A^{\varphi})^{-1}$ and $(A^{\varphi})^{-1}A^{\varphi}$ equal the identity matrix, the set

$$
\mathcal{U} = \{ i \in I: M_i \text{ is the inverse of } A^{\varphi i} \text{ over } K_i \} \subseteq I.
$$

In particular $\mathcal{U} \neq \emptyset$. If $i \in \mathcal{U}$, then $A \in \Sigma_i$, and therefore $A \in \bigcup_{i \in I} \Sigma_i$. Hence, the singular kernel of $(L, \varphi)$ is $P = \mathfrak{M}(R) \setminus (\bigcup_{i \in I} \Sigma_i)$.

4. is Lemma 10.12.
(5) First observe that $R_{\Sigma} = \lim_{\Sigma} R_{\Sigma,i}$. Let $\tau_i: R_{\Sigma,i} \rightarrow R_{\Sigma}$ and $\lambda_i: R \rightarrow R_{\Sigma,i}$ be the natural homomorphism of $\Sigma$-graded rings. By (3), there exists a natural homomorphism of $\Sigma$-graded rings $\rho: R_{\Sigma,i} \rightarrow U$. Since $V$ is $\Sigma$-graded local with residue $\Sigma$-graded division ring equal to $U$, the universal property of $R_{\Sigma,i}$ induces a homomorphism of $\Sigma$-graded rings $\rho: R_{\Sigma} \rightarrow V$. We must prove that $\rho$ is injective. Suppose that $x \in \ker \rho$ is homogeneous. Then $x$ is the $(u,v)$-entry of the inverse of a matrix $A^v$ with $A \in \bigcup_{i \in I} \Sigma_i$. Let $i \in I$ be such that $A \in \Sigma_i$. Note that $A \in \Sigma_i$ for all $I \in I$, $i \leq l$. Let $x_{uv}^l$ be the $(u,v)$-entry of $A^{v_i}$, $i \leq l$. Then $\tau_i(x_{uv}^l) = x$.

Then $\rho(x) = \{z_{uv}^l \in I \mid I \} \in U$ where $z_{uv}^l = \left\{ \begin{array}{ll}
 x_{uv}^l & \text{if } i \leq l \\
 0 & \text{otherwise} \end{array} \right.$ Since $\rho(x) = 0$, then $\{z_{uv}^l \in I \mid I \} = 0$ and the set $\{ j \in I : z_{uv}^j = 0 \} \in U$. It implies that $\tau_i(x_{uv}^j) = 0$. Therefore $x = 0$.

Note that the proof of (3) shows also (1) in a more elementary way.

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