The Hartogs Extension Phenomenon in Toric Varieties

Sergey Feklistov · Alexey Shchuplev

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Abstract
We study the Hartogs extension phenomenon in non-compact toric varieties and its relation to the first cohomology group with compact support. We show that a toric variety admits this phenomenon if at least one connected component of the fan complement is concave, proving by this an earlier conjecture M. Marciniak.

Keywords Hartogs phenomenon · Toric variety · Holomorphic extension

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1 Introduction

The classical Hartogs extension theorem states that for every domain $D \subset \mathbb{C}^n \ (n > 1)$ and a compact set $K \subset D$ such that $D \setminus K$ is connected, any holomorphic function $f$ on $D \setminus K$ extends holomorphically to $D$. A natural question arises if this is true for complex analytic spaces.

Definition 1 We say that a connected complex space $X$ admits the Hartogs phenomenon if for any domain $D \subset X$ and a compact set $K \subset D$ such that $D \setminus K$ is connected, the restriction homomorphism

$$H^0(D, \mathcal{O}) \to H^0(D \setminus K, \mathcal{O})$$

is an isomorphism.
In this or a similar formulation this phenomenon has been extensively studied in many situations, including Stein manifolds and spaces, \((n-1)\)-complete normal complex spaces and so on \([1–5,9,12,14,15,19,21,23]\).

Our goal is to study the Hartogs phenomenon in toric varieties. Toric varieties is a class of algebraic varieties that are rather easy to construct. Locally, they are algebraic sets defined by systems of binomial equations which are glued together via monomial mappings. This allows to describe their structure and properties in purely combinatorial terms; each \(p\)-dimensional toric variety \(X\) is encoded by a fan \(\Sigma\)—a collection of cones in \(\mathbb{R}^p\) with the common apex that may intersect only along their common face. The toric geometry arises naturally in constructions involving monomial functions or curves, notably in resolution of singularities or asymptotics, e.g. in the theory of residue currents (see, e.g., \([25–27]\)).

In the context of toric varieties the Hartogs phenomenon was first studied by Marciniak in her thesis \([16]\) and a paper in this journal \([17]\). She was able to prove the global version of the Hartogs phenomenon (with \(D = X\)):

**Theorem** Let \(X_\Sigma\) be a smooth toric surface or the total space of a smooth toric line bundle over a compact base. If the support \(|\Sigma|\) of \(\Sigma\) is a strictly convex cone then for any compact set \(K \subset X_\Sigma\) such that \(X_\Sigma \setminus K\) is connected, the restriction homomorphism

\[
H^0(X_\Sigma, \mathcal{O}) \to H^0(X_\Sigma \setminus K, \mathcal{O})
\]

is an isomorphism.

To prove the theorem one needs to know the transition functions between coordinate charts to construct holomorphic extensions of a function given in one chart to all others. Since the transition functions are given by the cones of maximal dimensions in the fan, this yields the restrictions on \(\Sigma\). However, this approach can hardly be generalised to higher dimensions because of increasing combinatorial difficulties. Nevertheless, Marciniak has formulated the following

**Conjecture** (\([16]\)) Let \(X_\Sigma\) be a smooth toric variety. If the complement of \(|\Sigma|\) has at least one concave connected component then \(X_\Sigma\) admits the global Hartogs phenomenon.

We shall follow a more general approach that goes back to Serre \([24]\). First we prove the result about vanishing cohomology

**Theorem A** Let \(X_{\Sigma'}\) be a \(p\)-dimensional toric variety with the fan \(\Sigma'\). Assume that the complement of the fan’s support \(C := \mathbb{R}^p \setminus |\Sigma'|\) is connected, then \(H^1_c(X_{\Sigma'}, \mathcal{O}) = 0\) if and only if \(\text{conv}(C) = \mathbb{R}^p\).

This allows us to specify what concavity in the conjecture formulated above means. Let \(\Sigma\) be a fan in \(\mathbb{R}^p\), and \(\mathbb{R}^p \setminus |\Sigma|\) be its complement. The complement may have several connected components \(\mathbb{R}^p \setminus |\Sigma| = \bigsqcup_j C_j\).
Definition 2 A complement component $C_j$ is called concave if

$$\text{conv}(C_j) = \mathbb{R}^p.$$  

Then we use the toric version of Serre’s theorem

Theorem B Let $X_\Sigma$ be a non-compact normal toric variety with the complement $\mathbb{R}^p \setminus |\Sigma|$ being connected. The cohomology group $H^1_c(X_\Sigma, \mathcal{O})$ is trivial if and only if $X_\Sigma$ admits the Hartogs phenomenon.

to prove the main result from which Marciniak’s conjecture follows

Theorem C Let $X_\Sigma$ be a normal non-compact toric variety with the fan $\Sigma$ whose complement is $\mathbb{R}^p \setminus |\Sigma| = \bigcup_{j=1}^n C_j$. Then

- if at least one of $C_j$’s is concave then $X_\Sigma$ admits the Hartogs phenomenon.
- if $n = 1$ then the converse is also true, i.e. if $X_\Sigma$ admits the Hartogs phenomenon then $\mathbb{R}^p \setminus |\Sigma|$ is concave.

2 Toric Varieties

In this section we briefly review the necessary elements of the theory of toric varieties. Here we follow [10,20].

Lattices and cones.
Consider a lattice $N$ of rank $r$ and the dual lattice $M := \text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ with the canonical $\mathbb{Z}$-bilinear product

$$\langle -, - \rangle : M \times N \rightarrow \mathbb{Z}, \langle l, x \rangle := l(x).$$

We define scalar extensions

$$N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}, M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$$

and obtain the canonical $\mathbb{R}$-bilinear product

$$\langle -, - \rangle : M_\mathbb{R} \times N_\mathbb{R} \rightarrow \mathbb{R}, \langle l, x \rangle := l(x).$$

Definition 3 A subset $\sigma \subset N_\mathbb{R}$ is a rational polyhedral cone with the apex at the origin $O \in N_\mathbb{R}$ if there is a finite number of $n_1, n_2, \ldots, n_s \in N$ such that

$$\sigma = \{a_1n_1 + \cdots + a_sn_s \mid a_i \in \mathbb{R}, a_i \geq 0, \forall i = 1, \ldots, s\}.$$ 

A rational polyhedral cone $\sigma$ is strictly convex if $\sigma \cap (-\sigma) = \{O\}.$
Throughout the paper, unless stated explicitly, we shall refer to rational polyhedral cones simply as cones. A cone generated by the list \( \{n_1, n_2, \ldots, n_s\} \) we denote as \( \text{Cone}(n_1, n_2, \ldots, n_s) \).

**Definition 4**  
– The dimension \( \dim \sigma \) of the cone \( \sigma \) is the dimension of the smallest subspace \( \mathbb{R}(\sigma) \subset \mathbb{N}_\mathbb{R} \) containing \( \sigma \).

– The dual cone \( \sigma^\vee \subset M_\mathbb{R} \) for \( \sigma \) is defined as

\[
\sigma^\vee := \{ x \in M_\mathbb{R} | \langle x, y \rangle \geq 0, \forall y \in \sigma \}.
\]

– A face \( \tau \) of \( \sigma \) (denoted as \( \tau < \sigma \)) is the set

\[
\tau := \sigma \cap H_{m_0}
\]

for some \( m_0 \in \sigma^\vee \) where \( H_{m_0} := \{ y \in \mathbb{N}_\mathbb{R} | \langle m_0, y \rangle = 0 \} \).

A strictly convex rational polyhedral cone \( \sigma \subset \mathbb{N}_\mathbb{R} \) defines a semigroup \( S_{\sigma} := \sigma^\vee \cap M \). Its properties are summarized in the following

**Proposition 1** ([20, Proposition 1.1]) Let \( \sigma \subset \mathbb{N}_\mathbb{R} \) be a strictly convex rational polyhedral cone. Then the following hold

1. \( S_{\sigma} \subset M \) is an additive subsemigroup containing the origin, i.e. \( O \in S_{\sigma} \) and for all \( m, m' \in S_{\sigma} \) we have \( m + m' \in S_{\sigma} \).  
2. \( S_{\sigma} \) is finitely generated as an additive semigroup, i.e. there exist \( m_1, \ldots, m_p \in S_{\sigma} \) such that \( S_{\sigma} = \mathbb{Z}_{\geq 0}\langle m_1, \ldots, m_p \rangle \).  
3. \( S_{\sigma} + (-S_{\sigma}) = M \).  
4. \( S_{\sigma} \) is saturated, i.e. if \( cm \in S_{\sigma} \) for \( m \in M \) and \( c \in \mathbb{Z}_{>0} \) then \( m \in S_{\sigma} \).

Conversely, for any additive semigroup \( S \subset M \) satisfying 1–4 there exists a unique strictly convex rational polyhedral cone \( \sigma \in \mathbb{N}_\mathbb{R} \) such that \( S = S_{\sigma} \).

**Fans and toric varieties.**  
Consider an algebraic torus \( T_N \) associated with the lattice \( N \) (and the dual lattice \( M \))

\[
T_N := \text{Hom}_\mathbb{Z}(M, \mathbb{C}^*) = N \otimes_\mathbb{Z} \mathbb{C}^*.
\]

Each element \( m \in M \) gives rise to a character (a group homomorphism)

\[
\chi^m : T_N \to \mathbb{C}^*, \chi^m(t) := t(m).
\]

The characters form a multiplicative group \( T_N \) which may be identified with the lattice \( M \). Each element \( n \in N \) defines a one-parametric subgroup

\[
\lambda^n : \mathbb{C}^* \to T_N, \lambda^n(t)(m) := t^{(m,n)}.
\]
The multiplicative group of all one-parametric subgroups of the torus $T_N$ may be identified with the lattice $N$.

Let $\{n_1, \ldots, n_r\}$ be a $\mathbb{Z}$-basis of the lattice $N$ and $\{m_1, \ldots, m_r\}$ be the dual $\mathbb{Z}$-basis of $M$. Denoting $u_j = \chi^{m_j}$, we get an isomorphism

$$T_N \cong (\mathbb{C}^*)^r,$$

$t \mapsto (u_1(t), \ldots, u_r(t))$.

Now, if $m = \sum_{j=1}^r a_j m_j$ then $\chi^m(t) = u_1^{a_1} \cdots u_r^{a_r}$; if $n = \sum_{j=1}^r b_j n_j$ then $\lambda^n(t) = (t^{b_1}, \ldots, t^{b_r})$.

For any cone $\sigma$ we construct a normal complex space $U_\sigma$ as follows

**Proposition 2** ([20, Proposition 1.2]) Let

$$S_\sigma = \mathbb{Z}_{\geq 0} \langle m_1, \ldots, m_p \rangle \subset M$$

be a finitely generated subsemigroup determined by a strictly convex rational polyhedral cone $\sigma \subset N_\mathbb{R}$. Let

$$U_\sigma := \{u : S_\sigma \to \mathbb{C} \mid u(O) = 1, u(m + m') = u(m)u(m'), \forall m, m' \in S_\sigma\}$$

and $\chi^m(u) := u(m)$ for $m \in S_\sigma$, $u \in U_\sigma$. Then

- the mapping $(\chi^{m_1}, \ldots, \chi^{m_p}) : U_\sigma \to \mathbb{C}^p$ is injective.
- identifying $U_\sigma$ with its image we get an algebraic subset in $\mathbb{C}^p$ given by a system of equations of the form

$$z_1^{a_1} \cdots z_p^{a_p} - z_1^{b_1} \cdots z_p^{b_p} = 0$$

for $(a_1, \ldots, a_p), (b_1, \ldots, b_p) \in \mathbb{Z}_p^p$ such that $\sum_{i=1}^p a_i m_i = \sum_{i=1}^p b_i m_i$.
- the structure of an $r$-dimensional irreducible normal complex space on $U_\sigma$ is induced by the complex analytic structure of $\mathbb{C}^p$ and does not depend on the choice of the system of generators $\{m_1, \ldots, m_p\}$ for the subsemigroup $S_\sigma$.
- each $m \in S_\sigma$ defines a polynomial function $\chi^m$ on $U_\sigma$ which is holomorphic with respect to the induced complex structure.

Any face $\tau$ of a cone $\sigma$ is again a cone, hence, it defines a complex space $U_\tau$. The relationship between $U_\sigma$ and $U_\tau$ is given by

**Proposition 3** ([20, Proposition 1.3])

1. For a strictly convex rational polyhedral cone $\sigma \subset N_\mathbb{R}$ its dual cone $\sigma^\vee \subset M_\mathbb{R}$ is a rational polyhedral cone.
2. If $\tau < \sigma$ then there exists $m_0 \in M \cap \sigma^\vee$ such that

   - $\tau = \sigma \cap H_{m_0}$.
Fig. 1 A fan with concave complement components

- \( S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-m_0) \).
- \( U_\tau = \{ u \in U_\sigma \mid u(m_0) \neq 0 \} \) is an open subset in \( U_\sigma \).

If \( \sigma_1, \sigma_2 \) are two cones with a common face \( \tau = \sigma_1 \cap \sigma_2 \), then \( U_\tau \) is an open subset of both \( U_{\sigma_1} \) and \( U_{\sigma_2} \) and we can glue them together along \( U_\tau \) to obtain a complex space. A fan is a collection of cones specifying how to glue all \( U_\sigma \)'s together.

**Definition 5** A fan is a pair \((\Sigma, N)\) where \( \Sigma \) is a finite set of strictly convex rational polyhedral cones \( \sigma \subset N_{\mathbb{R}} \) with the properties:

- each face of any cone \( \sigma \in \Sigma \) is in \( \Sigma \).
- for any cones \( \sigma, \sigma' \in \Sigma \) their intersection \( \sigma \cap \sigma' \) is a face of both.

**Example** Consider a collection of the following 2-dimensional cones (Fig. 1)

- Cone\(((1, 1, 1), (1, -1, -1))\)
- Cone\(((1, -1, -1), (-1, -1, 1))\)
- Cone\(((1, 1, 1), (1, 1, 1))\)
- Cone\(((1, -1, 1), (-1, 1, -1))\)
- Cone\(((1, 1, 1), (1, 1, 1))\)

This collection is a fan, both its complement components are concave.

**Proposition 4** ([20, Theorem 1.4]) For a fan \((\Sigma, N)\) the result of gluing \(\{U_\sigma \mid \sigma \in \Sigma\}\) together as described above is an irreducible normal complex space \(X_{\Sigma, N}\) of dimension \( r = \text{rk} N \).

**Definition 6** The complex space \(X_{\Sigma, N}\) is called a toric variety associated to the fan \((\Sigma, N)\).

When the lattice \(N\) is fixed we omit mentioning it in the notation, writing simply \(\Sigma\) and \(X_{\Sigma}\). We also denote by \(\Sigma(1)\) the set of all 1-dimensional cones in \(\Sigma\).

Smoothness and compactness of a normal toric variety can also be described in terms of its fan.

**Definition 7** We call a cone \(\sigma\) smooth if there exists a \(\mathbb{Z}\)-basis \(\{n_1, \ldots, n_r\}\) of the lattice \(N\) and \(s \leq r\) such that \(\sigma = \text{Cone}(n_1, \ldots, n_s)\).
Theorem [20, Theorem 1.10] A toric variety $X_\Sigma$ is smooth (i.e. is a smooth complex manifold) if and only if every cone $\sigma \in \Sigma$ is smooth.

Definition 8 A fan $\Sigma$ is called complete if its support $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma$ coincides with $N_{\mathbb{R}}$.

Theorem [20, Theorem 1.11] A toric variety $X_{\Sigma}$ is compact if and only if its fan $\Sigma$ is complete.

Torus action Let $(\Sigma, N)$ be a fan; it always contains a zero-dimensional cone $\{O\}$—the origin. Then $S_{\{O\}} = M$, and $U_{\{O\}} = T_N$. This cone is a face of every other cone in $(\Sigma, N)$, therefore $T_N$ is an open subset in each $U_\sigma$. Hence, a toric variety $X_\Sigma$ contains an algebraic torus $T_N$ as an open subset.

1. Let $t \in T_N$, it gives a group homomorphism $t : M \to \mathbb{C}^*$;
2. If $u \in U_\sigma$, then we have a mapping $u : S_\sigma \to \mathbb{C}$ with the property $u(O) = 1$, $u(m + m') = u(m)u(m')$, $\forall m, m' \in S_\sigma$;
3. Define $tu : S_\sigma \to \mathbb{C}$ as $(tu)(m) := t(m)u(m)$ for $m \in S_\sigma$;
4. $tu \in U_\sigma$, since $(tu)(O) = 1$ and $(tu)(m + m') = t(m + m')u(m + m') = t(m)t(m')u(m)u(m') = (tu)(m)(tu)(m')$;
5. Since $(t_1t_2)(u) = t_1(t_2(u))$, we get an action of $T_N$ on each $U_\sigma$ and $X_\Sigma$.

Thus, a toric variety $X_\Sigma$ contains an algebraic torus $T_N$ with an algebraic action on itself which extends to an action on the whole $X_\Sigma$. The converse is also true.

Theorem [20, Theorem 1.5] Suppose the algebraic torus $T_N$ acts algebraically on an irreducible normal algebraic variety $X$ locally of finite type over $\mathbb{C}$. If $X$ contains an open orbit isomorphic to $T_N$, then there exists a unique fan $(\Sigma, N)$ such that $X$ is equivariantly isomorphic to $X_{\Sigma}$.

For each cone $\sigma \in \Sigma$ we can consider an algebraic torus $O(\sigma) := \{\text{group homomorphisms } u : M \cap \sigma^\perp \to \mathbb{C}^*\}$ as a $T_N$-orbit in $X_\Sigma$. In terms of the fan $\Sigma$ we can describe the $T_N$-orbits as follows:

Proposition 5 ([20, Proposition 1.6])

1. Each $T_N$-orbit is of the form $O(\sigma)$.
2. The fan $\Sigma$ is in a 1-1 correspondence with the set of $T_N$-orbits in $X_\Sigma$.
3. $O(\{O\}) = U_{\{O\}} = T_N$.
4. For $\sigma \in \Sigma$, the complex dimension $\dim(O(\sigma)) = r - \dim(\sigma)$.
5. For $\tau, \sigma \in \Sigma$, $\tau \prec \sigma$ if and only if $O(\sigma) \subseteq O(\tau)$.
6. For $\sigma \in \Sigma$, the orbit $O(\sigma)$ is the only closed $T_N$-orbit in $U_\sigma$, and $U_\sigma = \bigcup_{\tau < \sigma} O(\tau)$. 

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7. Let \( n \in \mathbb{N} \) and \( \sigma \in \Sigma \). Then \( n \in \sigma \) if and only if for the one-parameter subgroup \( \lambda^n \) corresponding to \( n \) there exists the limit \( \lim_{t \to 0} \lambda^n(t) \) in \( U_\sigma \). In this case, the limit \( \lim_{t \to 0} \lambda^n(t) \) coincides with the identity element of the algebraic torus \( O(\tau) \), where \( \tau \) is a face of \( \sigma \) which contains \( n \) in its relative interior.

Equivariant compactification and mappings. In what follows, we consider toric varieties with incomplete fans (i.e. non-compact toric varieties). It turns out that such varieties can be compactified such that the resulting variety is again a toric variety. This is given by Sumihiro’s theorem

**Theorem** ([20, p. 17]) Suppose a connected linear algebraic group \( G \) acts algebraically on an irreducible normal algebraic variety \( X \) of finite type over \( \mathbb{C} \). Then \( X \) can be embedded as a \( G \)-invariant open subset of a complete irreducible normal algebraic variety \( \widehat{X} \) on which \( G \) acts algebraically.

We have

**Corollary 1** For any fan \( (\Sigma, N) \) there exists a complete fan \( (\widehat{\Sigma}, N) \) such that \( X_{\widehat{\Sigma}} \) is a \( T_N \)-equivariant compactification of \( X_{\Sigma} \).

Holomorphic mappings compatible with torus actions can also be described in terms of fans. Let \( (\Sigma, N) \) and \( (\Sigma', N') \) be two fans.

**Definition 9** A fan morphism \( \phi : (\Sigma', N') \rightarrow (\Sigma, N) \) is a \( \mathbb{Z} \)-linear homomorphism of lattices \( \phi : N' \rightarrow N \) such that the corresponding scalar extension \( \phi_{\mathbb{R}} : N'_{\mathbb{R}} \rightarrow N_{\mathbb{R}} \) has the property: for each \( \sigma' \in \Sigma' \) there exists \( \sigma \in \Sigma \) such that \( \phi_{\mathbb{R}}(\sigma') \subset \sigma \).

The following theorem describes morphisms of toric varieties

**Theorem** ([20, Theorem 1.13])

- A fan morphism \( \phi : (\Sigma', N') \rightarrow (\Sigma, N) \) gives rise to a holomorphic mapping

\[
\phi_* : X_{\Sigma', N'} \rightarrow X_{\Sigma, N}
\]

whose restriction to the open subset \( T_{N'} \) coincides with the homomorphism of algebraic tori

\[
\phi \otimes 1 : T_{N'} \rightarrow T_N.
\]

Besides, \( \phi_* \) is equivariant with respect to the actions of \( T_{N'} \) and \( T_N \) on the toric varieties.

- Conversely, let \( f' : T_{N'} \rightarrow T_N \) be a homomorphism of algebraic tori, and \( f : X_{\Sigma', N'} \rightarrow X_{\Sigma, N} \) be a holomorphic mapping equivariant with respect to \( f' \). Then there exists a unique \( \mathbb{Z} \)-linear homomorphism

\[
\phi : N' \rightarrow N,
\]

which gives rise to a fan morphism

\[
\phi : (\Sigma', N') \rightarrow (\Sigma, N)
\]
such that $f = \phi_\star$.

In other words, an equivariant holomorphic mapping of normal toric varieties in local coordinates is given by monomial functions.

**Resolution of singularities and subdivisions of fans.**

**Definition 10** – Let $(\Sigma, N)$ and $(\Sigma', N)$ be two fans. A fan $(\Sigma', N)$ is called a subdivision of a fan $(\Sigma, N)$ if each cone of $\Sigma'$ is contained in a cone of $\Sigma$, and $|\Sigma'| = |\Sigma|$.

– We call a subdivision $(\Sigma', N)$ smooth if each cone $\sigma' \in \Sigma'$ is smooth.

Consider the identity mapping of the lattice $\text{id}: N \to N$. This mapping is a fan morphism $\text{id}: (\Sigma', N) \to (\Sigma, N)$, since $\Sigma'$ is a subdivision of $\Sigma$, and we get an equivariant holomorphic mapping

$$id_\star: X_{\Sigma'} \to X_{\Sigma}.$$
Proof One inclusion is obvious, the converse one follows from the Riemann extension theorem.

A valuation $v_{\rho}$ for a $\rho \in \Sigma(1)$ may be identified with a point $u_{\rho}$ in $N$.

Lemma 2 ([10, prop. 4.1.1]) $v_{\rho}(t^I) = \langle u_{\rho}, I \rangle$, where $u_{\rho}$ is the minimal integer generator of $\rho \cap N$.

For any Laurent polynomial $f = \sum_{I \in A} a_I t^I$ we have $v_{\rho}(f) = \min_{I \in A} \langle u_{\rho}, I \rangle$, which is attained for $I = I_0 \in A$. Indeed,

$$f = t^{I_0} \left( a_{I_0} + \sum_{I \in A, I \neq I_0} a_I t^{I-I_0} \right)$$

and

$$v_{\rho}(f) = \langle u_{\rho}, I_0 \rangle + v_{\rho} \left( a_{I_0} + \sum_{I \in A, I \neq I_0} a_I t^{I-I_0} \right).$$

Since $v_{\rho}(t^{I-I_0}) > 0$ and $a_{I_0} \neq 0$, $a_{I_0} + \sum_{I \in A, I \neq I_0} a_I t^{I-I_0}$ is a holomorphic function at every point of $V(\rho)$ and non vanishing on $V(\rho)$. Hence, $v_{\rho}(f) = \langle u_{\rho}, I_0 \rangle$.

For series we have

Lemma 3 If $f \in \mathcal{O}(D \cap T_N)$ is given by a Laurent series $\sum_{I \in A} a_I t^I$ and $v_{\rho}(f) \geq 0$ for all $\rho \in \Sigma(1)$, then $A \subset |\Sigma|^\vee \cap M$, where $|\Sigma|^\vee$ is the intersection of supports of all cones dual to cones in $\Sigma$.

Proof The space $M_{\mathbb{R}}$ can be exhausted by compact sets. This induces an exhaustion of $A$ by finite sets, i.e. $A = \bigcup A_n$, $A_n \subset A_{n+1}$. The series $f = \sum_{I \in A} a_I t^I$ is the limit of partial sums $f_n = \sum_{I \in A_n} a_I t^I$ in the topology of uniform convergence on compact sets.

Assume that there exists $I \in A$ such that $\langle u_{\rho}, I \rangle < 0$. The monomial $t^I$ is a term in a partial sum $f_n$ for some $n_0 \in \mathbb{N}$, then $v_{\rho}(f_{n_0}) = \min_{I \in A_{n_0}} \langle u_{\rho}, I \rangle < 0$. Moreover, for all $N > n_0$ we have $v_{\tau}(f_N) < 0$. This means that for all $N > n_0$ partial sums $f_N$ have a pole on $V(\tau)$. This contradiction proves the statement.

Let $D \subset X_{\Sigma}$ be a domain such that $D \cap V(\rho) \neq \emptyset$ for all $\rho \in \Sigma(1)$.

Corollary 2 If $f \in \mathcal{O}(D \cap T_N)$ is given by a Laurent series $\sum_{I \in A} a_I t^I$ and $v_{\rho}(f) \geq 0$ for all $\rho \in \Sigma(1)$, then $A \subset |\Sigma|^\vee \cap M$, where $|\Sigma|^\vee$ is the intersection of supports of all cones dual to cones in $\Sigma$.

Proof The proof follows from the fact that $|\Sigma|^\vee = \{ I \in M_{\mathbb{R}} \mid \langle u_{\rho}, I \rangle \geq 0 \forall \rho \in \Sigma(1) \}$.  

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Exhaustion of toric varieties by compact sets.

**Definition 12** Let $X$ be a topological space. A collection of compact subsets $\{V_n\}_{n=1}^{\infty}$ in $X$ is called an exhaustion of $X$ if

1. for any $n$ we have $V_n \subset \text{int}(V_{n+1})$;
2. $X = \bigcup_{n=1}^{\infty} V_n$.

We shall prove the existence of a certain exhaustion for toric varieties satisfying an additional property.

**Lemma 4** Let $X_{\Sigma}$ be a non-compact normal toric variety, and the complement $\mathbb{R}^p \setminus |\Sigma|$ of its fan be connected. Then there exists an exhaustion of $X_{\Sigma}$ by compact sets $\{V_n\}$ such that $X_{\Sigma} \setminus V_n$ are connected.

**Proof** Let $X_{\Sigma'}$ be a compactification of $X_{\Sigma}$, then $Z := X_{\Sigma'} \setminus X_{\Sigma}$ is a compact algebraic set. According to Proposition 5 we have

$$Z = \bigcup_{\tau \in \Sigma', \text{relint}(\tau) \subset \mathbb{R}^p \setminus |\Sigma|} O(\tau),$$

where relint$(\tau)$ is a relative interior of the cone $\tau$. Due to connectedness of the complement of the fan, the set $Z$ is connected.

Let $U \subset X_{\Sigma'}$ be a connected open neighborhood of $Z$. Since $U$ is also a normal space and $Z$ is a thin set in $U$, by the criterion of connectedness [13, p. 81] we obtain $U \setminus Z \subset X_{\Sigma}$ is a connected set. Then the set $V = X_{\Sigma'} \setminus U$ is a compact subset in $X_{\Sigma}$, and $X_{\Sigma} \setminus V$ is connected.

A sequence of nested connected neighborhoods $\{U_n\}_{n=1}^{\infty}$ of $Z$ with the property $U_{n+1} \subset U_n$ induces a sequence of compact sets $\{V_n\}_{n=1}^{\infty}$ in $X_{\Sigma}$ giving an exhaustion of $X_{\Sigma} \setminus V_n$. Existence of such a sequence $\{U_n\}_{n=1}^{\infty}$ follows from paracompactness of $X_{\Sigma'}$. Indeed, consider a covering of $X_{\Sigma'}$ by two open sets $X_{\Sigma'} = X_{\Sigma} \cup U_1$ where $U_1$ is some connected neighborhood of $Z$. Then there exists a covering of $X_{\Sigma'} = V_1 \cup V_2$ such that $V_1 \subset X_{\Sigma}$, $V_2 \subset U_1$ ([8, p. 37]). It is obvious that $Z$ lies in $V_2$, then we set $U_2 := V_2$. Now, consider a covering of $X_{\Sigma'}$ by two open sets $X_{\Sigma'} = X_{\Sigma} \cup U_2$, then there exists $U_3 \subset U_2$ and so on. 

\[\square\]

3 Cohomology with Compact Supports

First we recall some facts in sheaf theory [7,11], and then describe the group $H^1_c(X_{\Sigma'}, \mathcal{O})$.

**Sheaf cohomology.** Let $\mathcal{A}$ be a sheaf of Abelian groups on a paracompact topological space $X$. Denote by $\mathcal{A}|_F$ the restriction of $\mathcal{A}$ to a subset $F$ of $X$, i.e. the preimage of $F$ under the local homeomorphism of the étalé space to the base.
Theorem 1 ([7, Theorem 9.5]) Let $F$ be a closed subset of $X$. Then

$$H^0(F, \mathcal{A}|_F) = \lim_{U \supseteq F} H^0(U, \mathcal{A}),$$

where $U$ ranges over neighborhoods of $F$.

In other words, a section $s \in H^0(F, \mathcal{A}|_F)$ can be thought of as a section $s' \in H^0(U, \mathcal{A})$, where $U$ is some neighborhood of $F$.

Definition 13 A family $\Phi$ of closed subsets of $X$ is called a family of supports if any closed subset of an element of $\Phi$ belongs to $\Phi$ and if a union of two elements of $\Phi$ belongs to $\Phi$.

For a family of supports $\Phi$ one defines cohomology with supports in $\Phi$ (see, e.g., [7, Sect. 2, §2]). If $\Phi$ is the family of compact subset of $X$ such cohomology is called cohomology with compact supports and denoted $H^*_c(X, \mathcal{A})$. For a subset $Y$ of $X$ one can consider the groups $H^*_c(Y, \mathcal{A}|_Y)$, in this case we shall write simply $H^*_c(Y, \mathcal{A})$.

If $\Phi$ is the family of closed subsets of a compact subset $K$ of $X$, we get cohomology with supports in $K$ and denote it by $H^*_c(K, \mathcal{A})$.

If $\{V_n\}_{n=1}^\infty$ is a compact exhaustion of $X$ then there is the following relation:

Proposition 7 ([5, p. 11]) There is canonical isomorphism

$$\lim_{V_n} H^p_{V_n}(X, \mathcal{F}) \cong H^p_c(X, \mathcal{F}).$$

For later use we need two long exact sequences.

Theorem 2 ([7, Sect. 2, §10]) Let $X$ be locally compact, $F \subset X$ be closed, and $U = X \setminus F$. Then the following sequence is exact

$$\cdots \rightarrow H^p_c(U, \mathcal{A}) \rightarrow H^p_c(X, \mathcal{A}) \rightarrow H^p_c(F, \mathcal{A}) \rightarrow H^{p+1}_c(U, \mathcal{A}) \rightarrow \cdots$$

Theorem 3 ([5, pp. 52–53]) Let $K$ be a closed subset of $X$, then the following sequence is exact

$$0 \rightarrow H^0_K(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{F}) \rightarrow H^0(X \setminus K, \mathcal{F}) \rightarrow H^1_K(X, \mathcal{F}) \rightarrow \cdots$$

The group $H^1_c(X_{\Sigma'}, \mathcal{O})$. Before giving a description of this cohomology group, we prove two lemmas.

Lemma 5 Let $Z \subset \mathbb{C}^n$ be a union of coordinate subspaces, and $U$ be its neighborhood. Then there exists a complete Reinhardt domain $W$ centered at the origin such that $Z \subset W \subset U$. 

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Proof For a non-empty proper subset $J \subset [n] = \{1, \ldots, n\}$ we denote by $C^{|J|}_{z_J}$ the $|J|$-dimensional coordinate subspace with coordinates $z_J = (z_i \mid i \in J)$. Then $Z = \bigcup_{J \in F} C^{|J|}_{z_J}$ where $F$ is some set of subsets of $[n]$.

Consider a point $z = (z_1, \ldots, z_n) \in Z$, then $z \in C^{|J|}_{z_J} \subset U$ for some $J \in F$. There exists an open polycylinder

$$W_z := \{|w_i| < e_i \mid i \in [n] - J\} \times \{|w_i| < |z_i| \mid i \in J\}$$

such that for sufficiently small $e_i$ this polycylinder is contained in $U$.

Indeed, let $A := \{|w_i| < |z_i| \mid i \in J\} \subset C^{|J|}_{z_J} \subset \mathbb{C}^n$ and consider

$$\rho(A, \partial U) = \inf_{a \in A, b \in \partial U} \|a - b\|$$

(here $\|a - b\| := \sqrt[2]{\sum_i |a_i - b_i|^2}$ is the Euclidean metric in $\mathbb{C}^n$). Then we let $e_i = \frac{\rho(A, \partial U)}{2\sqrt{n-|J|}}$ $\forall i \in [n] - J$.

Assume that $w \in W_z$ but $w \notin U$, then $\rho(A, \partial U) \leq \rho(A, w)$ (indeed, $\rho(A, \partial U) \leq \rho(A, b)$ for all points $b \in \partial U$, in particular we have $\rho(A, \partial U) \leq \rho(A, \partial U \cap L) \leq \rho(A, w)$ where $L$ is the perpendicular to the $A$ from the point $w$). We have $w = (w_{[n]-J}, w_J) \in C^{|J|}_{z_{[n]-J}} \times C^{|J|}_{z_J}$ then $w = w_1 + w_2$ where $w_1 = (w_{[n]-J}, 0)$, $w_2 = (0, w_J)$. Since $\|w - a\| \leq \|w_1\| + \|w_2 - a\|$, we obtain

$$\inf_{a \in A} \|w - a\| \leq \inf_{a \in A} (\|w_1\| + \|w_2 - a\|) = \|w_1\|$$

$$= \sqrt{\sum_{i \in [n] - J} |w_i|^2} \leq \sqrt{\sum_{i \in [n] - J} e_i^2} \leq \frac{\rho(A, \partial U)}{2}.$$ 

Therefore $\rho(A, \partial U) \leq \frac{\rho(A, \partial U)}{2}$, so we have arrived at a contradiction, and $W_z \subset U$.

Having done the same for all $z \in Z$, we get a family of polycylinders $\{W_z\}_{z \in Z}$ centered at the origin. Their union $W := \bigcup_{z \in Z} W_z$ is an open complete Reinhardt domain centered at the origin that contains the set $Z$. \qed

This implies that if $f \in \mathcal{O}(U)$ then in a smaller neighbourhood $U' \subset W \subset U$ the function is represented by an absolutely convergent Taylor series.

Now we turn to the description of $H^1_c(X_{\Sigma'}, \mathcal{O})$. Here we use a long exact sequence of a pair, as Marciniak did in [18] for smooth toric surfaces.

Let $N = \mathbb{Z}^p$, $\Sigma'$ be a fan whose complement is connected, and $X_{\Sigma'}$ be the corresponding toric variety. According to Corollary 1, the fan $\Sigma'$ has a completion $\Sigma''$. Denote by $\Sigma$ the fan that consists of those cones of $\Sigma''$ whose supports lie in the closed set $\Sigma'' \setminus \text{int}(\Sigma')$.

The variety $X_{\Sigma'}$ is an open subspace the compact complex space $X_{\Sigma''}$. The complement $Z := X_{\Sigma''} \setminus X_{\Sigma'}$ is a connected compact $T_N$-invariant analytic set in $X_{\Sigma''}$.

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By Theorem 2 we have an exact sequence

\[ 0 \rightarrow H^0_c(X_{\Sigma'}, \mathcal{O}) \rightarrow H^0_c(X_{\Sigma''}, \mathcal{O}) \rightarrow H^0_c(Z, \mathcal{O}) \rightarrow \cdots \]

Hence, \( H^0_c(X_{\Sigma'}, \mathcal{O}) = 0 \) since \( X_{\Sigma'} \) is not compact. Moreover, on a compact space \( X_{\Sigma''} \) we have \( H^*_{c}(X_{\Sigma''}, \mathcal{O}) = \mathbb{C} \) in dimension 0 and vanishes for higher dimensions \([20, \text{Corollary 2.9}].\) Therefore, \( H^0_c(X_{\Sigma''}, \mathcal{O}) = \mathbb{C} \) and \( H^1_c(X_{\Sigma''}, \mathcal{O}) = 0.\)

Thus, \( H^1_c(X_{\Sigma'}, \mathcal{O}) \cong H^0(Z, \mathcal{O})/\mathbb{C}.\) On the other hand, by Theorem 1

\[ H^0(Z, \mathcal{O}) = \lim_{U \ni Z} H^0(U, \mathcal{O}). \]

Since the set \( Z \) together with its neighborhood lies in \( X_{\Sigma} \), we can work with \( X_{\Sigma} \) instead of \( X_{\Sigma''}.\)

By Proposition 5

\[ Z = \bigcup_{\tau \in \Sigma, \text{relint}(\tau) \in \text{int}(|\Sigma|)} O(\tau) \subset X_{\Sigma}, \]

where \( O(\tau) \) is the orbit of the action of \((\mathbb{C}^*)^n\) on \( X_{\Sigma} \) that corresponds to a cone \( \tau.\)

Assume that \( X_{\Sigma} \) is smooth and consider the equivalence class \([f, V]\) \( \in H^0(Z, \mathcal{O}),\) that is a function \( f \) holomorphic in a neighborhood \( V \) of \( Z.\)

Recall that

\[ Z \cap U_\sigma = \bigcup_{\tau < \sigma, \text{relint}(\tau) \in \text{int}(|\Sigma|)} O(\tau) \subset X_{\Sigma}, \]

where \( U_\sigma \) is an affine chart corresponding to \( \sigma \in \Sigma,\) i.e. \( Z \cap U_\sigma \) is a union of coordinate subspaces.

By Lemma 5, for \( V \cap U_\sigma \) there exists a complete Reinhardt domain \( W_\sigma \) such that \( Z \cap U_\sigma \subset W_\sigma \subset V \cap U_\sigma.\) Therefore in \( W_\sigma \) the function \( f \) is given by a convergent power series. Choose a neighborhood \( D \) of \( Z \) such that \( D \subset \bigcup_{\sigma} W_\sigma \) and \( D \cap U_\sigma \subset W_\sigma.\)

In \( D \cap U_\sigma \) the function \( f \) is given by the same series as in \( W_\sigma.\) All these series written in the coordinates \( t \) of the torus \( T_N \) are one and the same power series; thus, \( f = \sum_{I \in A} a_I t^I.\)

By Corollary 2 we get \( A \subset |\Sigma|^\vee \cap M.\) Thus, an equivalence class \([f, V]\) can be written

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as \( \left[ \sum_{I \in A} a_I t^I, D \right] \), provided the series converges in \( D \). Denote now \( C := \mathbb{R} \setminus |\Sigma'| \), then \( C = |\Sigma| \). In this notation we have

\[
H^0(Z, \mathcal{O}) = \left\{ \left[ \sum_{I \in A} a_I t^I, D \right] \mid \text{the series converges in } D, \ A \subset C^\vee \cap M \right\}.
\]

Assume now that \( X_\Sigma \) is an arbitrary (normal) toric variety, not necessarily smooth. We reduce this case to the smooth one. Let \( \Sigma_0 \) be a smooth subdivision of \( \Sigma \) and \( \pi : X_{\Sigma_0} \to X_\Sigma \) be an equivariant resolution of singularities. Note that \( \pi^{-1}(Z) = \bigcup_{\tau \in \Sigma_0, \text{relint}(\tau) \in \text{int}(|\Sigma_0|)} O(\tau) \subset X_{\Sigma_0} \).

Denote this set by \( Z_0 \).

For structure sheaves \( O_\Sigma \) and \( O_{\Sigma_0} \) of \( X_\Sigma \) and \( X_{\Sigma_0} \), respectively, we have a natural mapping

\[
H^0(Z, O_\Sigma) \to H^0(Z_0, O_{\Sigma_0}), \ [f, V] \to [f \circ \pi, \pi^{-1}(V)].
\]

This is an isomorphism. Injectivity is obvious, let us show its surjectivity. Since the exceptional set of resolution \( \pi \) is contained in \( Z_0 \), for a class \( [g, U] \in H^0(Z_0, O_{\Sigma_0}) \) there exists a neighborhood \( V \subset X_\Sigma \) of \( Z \) such that \( \pi^{-1}(V) \subset U \), therefore \( [g, U] = [g|_{\pi^{-1}(V)}, \pi^{-1}(V)] \). Using the fact that \( \pi_* O_{\Sigma_0} \cong O_\Sigma \) (this follows from normality of \( X_\Sigma \)), we see that \( g|_{\pi^{-1}(V)} = f \circ \pi \) for some \( f \in O_\Sigma(V) \).

This proves the following

**Lemma 6** Let \( X_{\Sigma'} \) be a (normal) toric variety with the set \( C := \mathbb{R}^p \setminus |\Sigma'| \) being connected. Then

\[
H^1_c(X_{\Sigma'}, \mathcal{O}) \cong \left\{ \left[ \sum_{I \in A} a_I t^I, D \right] \mid A \subset C^\vee \cap M \right\} / \mathcal{C}
\]

where the series converges in some neighborhood \( D \supset Z \) of the smooth variety \( X_\Sigma \).

Recall that a set \( A \subset \mathbb{R}^n \) with the property that for any point \( x \in A \) and \( \lambda \geq 0 \) one has \( \lambda x \in A \) is called non-negatively homogeneous or simply a cone (not necessarily convex), see [6, p. 25]).

**Lemma 7** ([6, p. 53]) Let \( A \) be a non-negatively homogeneous set, then

\[
\text{conv}(A) = A^{\vee \vee}
\]

In particular, if \( A \) is closed then \( \text{conv}(A) = A^{\vee \vee} \).

This gives us the following result
Theorem A Let $X_{\Sigma'}$ be a $p$-dimensional toric variety with the fan $\Sigma'$. Assume that the complement of the fan’s support $C := \mathbb{R}^p \setminus |\Sigma'|$ is connected, then $H^1_c(X_{\Sigma'}, \mathcal{O}) = 0$ if and only if $\text{conv}(C) = \mathbb{R}^p$.

Proof Since $C$ is a non-negatively homogeneous set, by Lemma 7 $\text{conv}(C) = C^{\vee \vee}$. Therefore $\text{conv}(C) = \mathbb{R}^p$ if and only if $C^{\vee} = O$. By Lemma 6, $C^{\vee} = O$ if and only if $H^1_c(X_{\Sigma'}, \mathcal{O}) = 0$. ⊓⊔

4 The Hartogs Phenomenon

Toric Serre’s theorem

Serre has formulated the cohomological condition for the Hartogs phenomenon in complex manifolds [24]. We give a proof of this statement for toric varieties. First, recall the excision lemma [5, pp. 52-53].

Lemma 8 Let $X$ be a topological space. If $Y$ is an open subset in $X$ and $A$ is a closed subset of $Y$, then we have canonical isomorphisms $H^n_A(X, F) \cong H^n_{A\cap Y}(Y, F)$ for any $n$.

Now, we can prove toric Serre’s theorem.

Theorem B Let $X_{\Sigma}$ be a non-compact normal toric variety with the complement $\mathbb{R}^p \setminus |\Sigma|$ being connected. The cohomology group $H^1_c(X_{\Sigma}, \mathcal{O})$ is trivial if and only if $X_{\Sigma}$ admits the Hartogs phenomenon.

Proof By Lemma 4 there exists an exhaustion of $X_{\Sigma}$ by compact sets $\{V_n\}$ such that $X_{\Sigma} \setminus V_n$ is connected. Then, by Proposition 7 it follows that $H^1_c(X_{\Sigma}, \mathcal{O}) \cong \lim_{\rightarrow V_n} H^1(V_n, X_{\Sigma}, \mathcal{O})$.

Suppose that $H^1_c(X_{\Sigma}, \mathcal{O}) = 0$ and let $X_{\Sigma'}$ be a toric compactification of $X_{\Sigma}$. By Theorem 3 we have

$$0 \longrightarrow H^0(V_n, X_{\Sigma'}, \mathcal{O}) \longrightarrow H^0(X_{\Sigma'}, \mathcal{O}) \longrightarrow H^0(X_{\Sigma'} \setminus V_n, \mathcal{O}) \longrightarrow H^1(V_n, X_{\Sigma'}, \mathcal{O}) \longrightarrow H^1(X_{\Sigma'}, \mathcal{O}) \longrightarrow 0$$

The uniqueness theorem implies $H^0(V_n, X_{\Sigma'}, \mathcal{O}) = 0$, and by the vanishing theorem for compact toric varieties [20, Corollary 2.9] we have

$$H^0(X_{\Sigma'}, \mathcal{O}) = \mathbb{C}, \quad H^1(X_{\Sigma'}, \mathcal{O}) = 0.$$

All this together with Lemma 8 yields a short exact sequence for any $n$

$$0 \longrightarrow \mathbb{C} \longrightarrow H^0(X_{\Sigma'} \setminus V_n, \mathcal{O}) \longrightarrow H^1(V_n, X_{\Sigma}, \mathcal{O}) \longrightarrow 0.$$ (1)
Using homological algebra machinery (see, e.g. [11, Sect. 4, §1]), we get the following commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & C & \rightarrow & H^0(X_{\Sigma'} \setminus V_{n-1}, \mathcal{O}) & \rightarrow & H^1_{V_{n-1}}(X_{\Sigma}, \mathcal{O}) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & C & \rightarrow & H^0(X_{\Sigma'} \setminus V_{n}, \mathcal{O}) & \rightarrow & H^1_{V_{n}}(X_{\Sigma}, \mathcal{O}) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & C & \rightarrow & H^0(X_{\Sigma'} \setminus V_{n+1}, \mathcal{O}) & \rightarrow & H^1_{V_{n+1}}(X_{\Sigma}, \mathcal{O}) & \rightarrow & 0 \\
\end{array}
\]

Here \( \text{id}_n \) is the identity homomorphism, \( p_n \) is the restriction homomorphism and \( q_n \) is the homomorphism induced by the embedding \( V_n \subset V_{n+1} \).

Since \( \lim_{\rightarrow} H^1_{V_n}(X_{\Sigma}, \mathcal{O}) = 0 \), taking direct limits of the diagram, we obtain

\[
\lim_{\rightarrow} H^0(X_{\Sigma'} \setminus V_n, \mathcal{O}) \cong C.
\]

Also we have \( H^0(X_{\Sigma'} \setminus V_n, \mathcal{O}) = H^0(U_n, \mathcal{O}) \), where \( Z := X_{\Sigma'} \setminus X_{\Sigma} \) and \( U_n := X_{\Sigma'} \setminus V_n \) is a connected neighborhood of \( Z \).

Now, let \( f \in H^0(U_n, \mathcal{O}) \). Since \( \lim_{\rightarrow} H^0(U_n, \mathcal{O}) \cong C \), we have \( f \equiv \text{const} \) in \( U_n \) by the uniqueness theorem. It follows that \( H^0(U_n, \mathcal{O}) = C \) and for any compact set \( V_n \) we obtain \( H^1_{V_n}(X_{\Sigma}, \mathcal{O}) = 0 \). Thus, from (1) we get \( H^0(X_{\Sigma'} \setminus V_n, \mathcal{O}) = C \).

Let \( K \) be a compact set in the domain \( D \subset X_{\Sigma} \) such that \( D \setminus K \) is connected. There exists a compact set \( V_n \) such that \( K \subset V_n \). Therefore

\[
H^0(X_{\Sigma'} \setminus K, \mathcal{O}) \subset H^0(X_{\Sigma'} \setminus V_n, \mathcal{O}),
\]

and we see that

\[
H^0(X_{\Sigma'} \setminus K, \mathcal{O}) = C.
\]

Now, for a compact set \( K \) in \( X_{\Sigma} \) we have an exact short sequence similar to (1), from which we conclude that \( H^1_{V_n}(X_{\Sigma}, \mathcal{O}) = 0 \), and by Lemma 8 we obtain \( H^1_{V_n}(D, \mathcal{O}) = 0 \). By Theorem 3 the restriction map \( H^0(D, \mathcal{O}) \rightarrow H^0(D \setminus K, \mathcal{O}) \) is an isomorphism, i.e. the Hartogs phenomenon holds in \( X_{\Sigma} \).

Now, suppose that the Hartogs phenomenon holds in \( X_{\Sigma} \), i.e. for any domain \( D \subset X_{\Sigma} \) and for any compact set \( K \subset D \) such that \( D \setminus K \) is connected, the restriction
The Hartogs Extension Phenomenon in Toric Varieties

map

\[ H^0(D, \mathcal{O}) \to H^0(D \setminus K, \mathcal{O}) \]

is an isomorphism.

We have a commutative diagram where all morphisms are restriction maps:

\[
\begin{array}{ccc}
H^0(X_{\Sigma'}, \mathcal{O}) & \to & H^0(X_{\Sigma'} \setminus K, \mathcal{O}) \\
\downarrow & & \downarrow \\
H^0(D, \mathcal{O}) & \to & H^0(D \setminus K, \mathcal{O})
\end{array}
\]

By assumption, the lower arrow of the diagram is an isomorphism. Let \( f \in H^0(X_{\Sigma'} \setminus K, \mathcal{O}) \) be a holomorphic function, and let \( h := f|_{D \setminus K} \). There exists \( g \in H^0(D, \mathcal{O}) \) such that \( g \equiv h \) on \( D \setminus K \). Since \( X_{\Sigma'} = (X_{\Sigma'} \setminus K) \cup D \) and \( D \setminus K = (X_{\Sigma'} \setminus K) \cap D \), there exists \( F \in H^0(X_{\Sigma'}, \mathcal{O}) \) such that \( F \equiv f \) on \( X_{\Sigma'} \setminus K \) and \( F \equiv g \) on \( D \). But \( H^0(X_{\Sigma'}, \mathcal{O}) = \mathbb{C} \), therefore \( H^0(X_{\Sigma'} \setminus K, \mathcal{O}) = \mathbb{C} \) and \( H^1_X(X_{\Sigma'}, \mathcal{O}) = 0 \).

Now we take an exhaustion of \( X_{\Sigma} \) by compact sets \( \{V_n\} \) such that each \( X_{\Sigma} \setminus V_n \) is connected. Replacing \( K \) by \( V_n \) in above, we obtain \( H^1_{V_n}(X_{\Sigma}, \mathcal{O}) = 0 \). By Proposition 7 it follows that \( H^1_{V_n}(X_{\Sigma}, \mathcal{O}) \cong \lim_{\to \mathcal{V}_n} H^1_{V_n}(X_{\Sigma}, \mathcal{O}) = 0 \).

\[ \square \]

The Hartogs phenomenon for normal toric varieties.

We can now prove our main result from which Marciniak’s conjecture follows.

**Theorem C** Let \( X_{\Sigma} \) be a normal non-compact toric variety with the fan \( \Sigma \) whose complement is \( \mathbb{R}^p \setminus |\Sigma| = \bigcup_{j=1}^n C_j \). Then

- if at least one of \( C_j \)'s is concave then \( X_{\Sigma} \) admits the Hartogs phenomenon.
- if \( n = 1 \) then the converse is also true, i.e. if \( X_{\Sigma} \) admits the Hartogs phenomenon then \( \mathbb{R}^p \setminus |\Sigma| \) is concave.

**Proof** Let \( C_1 \) be a concave component. The toric variety \( X_{\Sigma} \) can be embedded into a toric variety \( X_{\Sigma' \setminus |\Sigma'|} \) such that \( \mathbb{R}^p \setminus |\Sigma'| = C_1 \). By Theorem A we have \( H^1_{c}(X_{\Sigma'}, \mathcal{O}) = 0 \). Therefore, by toric Serre’s theorem (Theorem B), the Hartogs phenomenon holds in \( X_{\Sigma'} \). Since \( X_{\Sigma} \subset X_{\Sigma'} \) is an open subvariety, this is true in \( X_{\Sigma} \) also.

If \( n = 1 \) then by toric Serre’s theorem, \( H^1_{c}(X_{\Sigma}, \mathcal{O}) = 0 \). It remains to apply Theorem A to finish the proof.

\[ \square \]

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