ON TANGENT CONES AND PARALLEL TRANSPORT IN WASSERSTEIN SPACE

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ABSTRACT. If $M$ is a smooth compact Riemannian manifold, let $P(M)$ denote the Wasserstein space of probability measures on $M$. If $S$ is an embedded submanifold of $M$, and $\mu$ is an absolutely continuous measure on $S$, we compute the tangent cone of $P(M)$ at $\mu$. We describe a geometric construction of parallel transport of tangent cones along certain geodesics in $P(M)$. We show that when everything is smooth, the geometric parallel transport agrees with earlier formal calculations.

1. Introduction

In optimal transport theory, a displacement interpolation is a one-parameter family of measures that represents the most efficient way of displacing mass between two given probability measures. Finding a displacement interpolation between two probability measures is the same as finding a minimizing geodesic in the space of probability measures, equipped with the Wasserstein metric $W_2$ [11, Proposition 2.10]. For background on optimal transport and Wasserstein space, we refer to Villani’s book [17].

If $M$ is a compact connected Riemannian manifold with nonnegative sectional curvature then $P(M)$ is a compact length space with nonnegative curvature in the sense of Alexandrov [11, Theorem A.8], [16, Proposition 2.10]. Hence one can define the tangent cone $T_\mu P(M)$ of $P(M)$ at a measure $\mu \in P(M)$. If $\mu$ is absolutely continuous with respect to the volume form $d\text{vol}_M$ then $T_\mu P(M)$ is a Hilbert space [11, Proposition A.33]. More generally, one can define tangent cones of $P(M)$ without any curvature assumption on $M$, using Ohta’s 2-uniform structure on $P(M)$ [13]. Gigli showed that $T_\mu P(M)$ is a Hilbert space if and only if $\mu$ is a “regular” measure, meaning that it gives zero measure to any hypersurface which, locally, is the graph of the difference of two convex functions [6, Corollary 6.6]. It is natural to ask what the tangent cones are at other measures.

A wide class of tractable measures comes from submanifolds. Suppose that $S$ is a smooth embedded submanifold of a compact connected Riemannian manifold $M$. Suppose that $\mu$ is an absolutely continuous probability measure on $S$. We can also view $\mu$ as an element of $P(M)$. For simplicity, we assume that $\text{supp}(\mu) = S$.

Theorem 1.1. We have

\begin{equation}
T_\mu P(M) = H \oplus \int_{s \in S} P_2(N_s M) \, d\mu(s),
\end{equation}

where

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• $H$ is the Hilbert space of gradient vector fields $\text{Im}(\nabla) \subset L^2(TS, d\mu)$,
• $N_sM$ is the normal space to $S \subset M$ at $s \in S$ and
• $P_2(N_sM)$ is the metric cone of probability measures on $N_sM$ with finite second moment, equipped with the $2$-Wasserstein metric.

The homotheties in the metric cone structure on $P_2(N_sM)$ arise from radial rescalings of $N_sM$. The direct sum and integral in (1.2) refer to computing square distances.

The proof of Theorem 1.1 amounts to understanding optimal transport starting from a measure supported on a submanifold. This seems to be a natural question in its own right which has not been considered much. Gangbo and McCann proved results about optimal transport between measures supported on hypersurfaces in Euclidean space [5]. McCann-Sosio and Kitagawa-Warren gave more refined results about optimal transport between two measures supported on a sphere [9, 12]. Castillon considered optimal transport between a measure supported on a submanifold of Euclidean space and a measure supported on a linear subspace [4].

In the setting of Theorem 1.1, a Wasserstein geodesic $\{\mu_t\}_{t \in [0, \epsilon]}$ starting from $\mu$ consists of a family of geodesics shooting off from $S$ in various directions. The geometric meaning of Theorem 1.1 is that the tangential component of these directions is the gradient of a function on $S$. To motivate this statement, in Section 2 we give a Benamou-Brenier-type variational approach to the problem of optimally transporting a measure supported on one hypersurface to a measure supported on a disjoint hypersurface, through a family of measures supported on hypersurfaces. One finds that the only constraint is the aforementioned tangentiality constraint. The rigorous proof of Theorem 1.1 is in Section 3.

If $\gamma : [0, 1] \to M$ is a smooth curve in a Riemannian manifold then one can define the (reverse) parallel transport along $\gamma$ as a linear isometry from $T_{\gamma(1)}M$ to $T_{\gamma(0)}M$. If $X$ is a finite-dimensional Alexandrov space then the replacement of a tangent space is a tangent cone. If one wants to define a parallel transport along a curve $c : [0, 1] \to X$, as a map from $T_{c(1)}X$ to $T_{c(0)}X$, then there is the problem that the tangent cones along $c$ may not look much alike. For example, the curve $c$ may pass through various strata of $X$. One can deal with this problem by assuming that $c$ is in the interior of a minimizing geodesic. In this case, Petrunin proved the tangent cones along $c$ are mutually isometric, by constructing a parallel transport map [15]. His construction of the parallel transport map was based on passing to a subsequential limit in an iterative construction along $c$. It is not known whether the ensuing parallel transport is uniquely defined, although this is irrelevant for Petrunin’s result.

In the case of a smooth curve $c : [0, 1] \to P^\infty(M)$ in the space of smooth probability measures, one can do formal Riemannian geometry calculations on $P^\infty(M)$ to write down an equation for parallel transport along $c$ [10, Proposition 3]. It is a partial differential equation in terms of a family of functions $\{\eta_t\}_{t \in [0, 1]}$. Ambrosio and Gigli noted that there is a weak version of this partial differential equation [11 (5.9)]. By a slight extension, we will define weak solutions to the formal parallel transport equation; see Definition 4.13.

Petrunin’s construction of parallel transport cannot work in full generality on $P(M)$, since Juillet showed that there is a minimizing Wasserstein geodesic $c$ with the property that the tangent cones at measures on the interior of $c$ are not all mutually isometric [8].
However one can consider applying the construction on certain convex subsets of $P(M)$. We illustrate this in two cases. The first and easier case is when $c$ is a Wasserstein geodesic of $\delta$-measures (Proposition 5.1). The corresponding tangent cone at a point of $c$ comes from Theorem 1.1 when $S$ is a point. The second case is when $c$ is a Wasserstein geodesic of absolutely continuous measures, lying in the interior of a minimizing Wasserstein geodesic, and satisfying a regularity condition. The corresponding tangent cone at a point of $c$ comes from Theorem 1.1 when $S = M$. Suppose that $\nabla \eta_1 \in T_{c(1)}P(M)$ is an element of the tangent cone at the endpoint. Here $\nabla \eta_1 \in L^2(TM, dc(1))$ is a square-integrable gradient vector field on $M$ and $\eta_1$ is in the Sobolev space $H^1(M, dc(1))$. For each sufficiently large integer $Q$, we construct a triple

\[
(\nabla \eta_Q, \nabla \eta_Q(0), \nabla \eta_Q(1)) \in L^2([0, 1]; L^2(TM, dc(t))) \oplus L^2(TM, dc(0)) \oplus L^2(TM, dc(1))
\]

with $\nabla \eta_Q(1) = \nabla \eta_1$, which represents an approximate parallel transport along $c$.

Theorem 1.4. Suppose that $M$ has nonnegative sectional curvature. A subsequence of $\{(\nabla \eta_Q, \nabla \eta_Q(0), \nabla \eta_Q(1))\}_{Q=1}^{\infty}$ converges weakly to a weak solution $(\nabla \eta_\infty, \nabla \eta_{\infty,0}, \nabla \eta_{\infty,1})$ of the parallel transport equation with $\nabla \eta_{\infty,1} = \nabla \eta_1$. If $c$ is a smooth geodesic in $P_\infty(M)$, $\eta_1$ is smooth, and there is a smooth solution $\eta$ to the parallel transport equation (4.6) with $\eta(1) = \eta_1$, then $\lim_{Q \to \infty}(\nabla \eta_Q, \nabla \eta_Q(0), \nabla \eta_Q(1)) = (\nabla \eta, \nabla \eta(0), \nabla \eta(1))$ in norm.

Remark 1.5. In the setting of Theorem 1.4 we can say that $\nabla \eta_{\infty,0}$ is the parallel transport of $\nabla \eta_1$ along $c$ to $T_{c(0)}P(M)$.

Remark 1.6. We are assuming that $M$ has nonnegative sectional curvature in order to apply some geometric results from [15]. It is likely that this assumption could be removed.

Remark 1.7. Based on Theorems 1.1 and 1.4 it seems likely that Petrunin’s construction could be extended to define parallel transport along Wasserstein geodesics of absolutely continuous measures on submanifolds of $M$. We have done this in the extreme cases when the submanifolds have dimension zero or codimension zero.

Remark 1.8. A result related to Theorem 1.4 was proven by Ambrosio and Gigli when $M = \mathbb{R}^n$ [1, Theorem 5.14], and extended to general $M$ by Gigli [7, Theorem 4.9]. As explained in [1, 7], the construction of parallel transport there can be considered to be extrinsic, in that it is based on embedding the (linear) tangent cones into a Hilbert space and applying projection operators to form the approximate parallel transports. Although we instead use Petrunin’s intrinsic construction, there are some similarities between the two constructions; see Remark 5.32. We use some techniques from [1], especially the idea of a weak solution to the parallel transport equation.

The structure of this paper is as follows. In Section 2 we give a formal derivation of the equation for optimal transport between two measures supported on disjoint hypersurfaces of a Riemannian manifold. The derivation is based on a variational method. In Section 3 we prove Theorem 1.1. In Section 4 we discuss weak solutions to the parallel transport equation. In Section 5 we prove Theorem 1.4.

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2. Variational approach

Let $M$ be a smooth closed Riemannian manifold. Let $S$ be a smooth closed manifold and let $S_0, S_1$ be disjoint codimension-one submanifolds of $M$ diffeomorphic to $S$. Let $\rho_0 \, d\text{vol}_{S_0}$ and $\rho_1 \, d\text{vol}_{S_1}$ be smooth probability measures on $S_0$ and $S_1$, respectively. We consider the problem of optimally transporting $\rho_0 \, d\text{vol}_{S_0}$ to $\rho_1 \, d\text{vol}_{S_1}$ through a family of measures supported on codimension-one submanifolds $\{S_t\}_{t \in [0,1]}$. We will specify the intermediate submanifolds to be level sets of a function $T$, which in turn will become one of the variables in the optimization problem.

We assume that there is a codimension-zero submanifold-with-boundary $U$ of $M$, with $\partial U = S_0 \cup S_1$. We also assume that there is a smooth submersion $T : U \to [0,1]$ so that $T^{-1}(0) = S_0$ and $T^{-1}(1) = S_1$. For $t \in [0,1]$, put $S_t = T^{-1}(t)$. These are the intermediate hypersurfaces.

We now want to describe a family of measures $\{\mu_t\}_{t \in [0,1]}$ that live on the hypersurfaces $\{S_t\}_{t \in [0,1]}$. It is convenient to think of these measures as fitting together to form a measure on $U$. Let $\mu$ be a smooth measure on $U$. In terms of the fibering $T : U \to [0,1]$, decompose $\mu$ as $\mu = \mu_t \, dt$ with $\mu_t$ a measure on $S_t$. We assume that $\mu_0 = \rho_0 \, d\text{vol}_{S_0}$ and $\mu_1 = \rho_1 \, d\text{vol}_{S_1}$.

Let $V$ be a vector field on $U$. We want the flow $\{\phi_s\}$ of $V$ to send level sets of $T$ to level sets. Imagining that there is an external clock, it’s convenient to think of $S_t$ as the evolving hypersurface at time $t$. Correlating the flow of $V$ with the clock gives the constraint

$$ V T = 1. $$

Then $\phi_s$ maps $S_t$ to $S_{t+s}$.

We also want the flow to be compatible with the measures $\{\mu_t\}_{t \in [0,1]}$ in the sense that $\phi_s^* \mu_{t+s} = \mu_t$. Now $\phi_s^* dT = d\phi_s^* T = d(T + s) = ds$, so it is equivalent to require that $\phi_s^*$ preserves the measure $\mu = \mu_t \, dt$. This gives the constraint

$$ \mathcal{L}_V \mu = 0. $$

In particular, each $\mu_t$ is a probability measure.

To define a functional along the lines of Benamou and Brenier [2], put

$$ E = \frac{1}{2} \int_U |V|^2 \, d\mu = \frac{1}{2} \int_0^1 \int_{S_t} |V|^2 \, d\mu_t \, dt. $$

We want to minimize $E$ under the constraints $\mathcal{L}_V \mu = 0$, $VT = 1$, $\mu_0 = \rho_0 \, d\text{vol}_{S_0}$ and $\mu_1 = \rho_1 \, d\text{vol}_{S_1}$. Let $\phi$ and $\eta$ be new functions on $U$, which will be Lagrange multipliers for the constraints. Then we want to extremize

$$ \mathcal{E} = \int_U \left[ \frac{1}{2} |V|^2 \, d\mu + \phi \mathcal{L}_V d\mu + \eta (VT - 1) d\mu \right] $$

with respect to $V$, $\mu$, $\phi$ and $\eta$. 
We will use the equations

\[(2.5) \quad \int_U \phi \mathcal{L}_V d\mu = \int_U [\mathcal{L}_V (\phi d\mu) - (\mathcal{L}_V \phi) d\mu] = -\int_U (V \phi) d\mu + \int_{S_1} \phi(1) d\mu_1 - \int_{S_0} \phi(0) d\mu_0 \]

and

\[(2.6) \quad \int_U \eta V T d\mu = \int_U \left[ \mathcal{L}_V (T \eta d\mu) - T \mathcal{L}_V (\eta d\mu) \right] = -\int_U T \mathcal{L}_V (\eta d\mu) + \int_{S_1} \eta(1) d\mu_1. \]

The Euler-Lagrange equation for \(V\) is

\[(2.7) \quad V - \nabla \phi + \eta \nabla T = 0. \]

The Euler-Lagrange equation for \(\mu\) is

\[(2.8) \quad \frac{1}{2} |V|^2 - V \phi = 0. \]

Varying \(T\) gives

\[(2.9) \quad 0 = \mathcal{L}_V (\eta d\mu) = (V \eta) d\mu, \]

so the Euler-Lagrange equation for \(T\) is

\[(2.10) \quad V \eta = 0. \]

One finds that \(\phi\) and \(T\) must satisfy

\[(2.11) \quad \langle \nabla \phi, \nabla T \rangle = 1 - |\nabla \phi| \cdot |\nabla T|. \]

Then \(V\) is given in terms of \(\phi\) and \(T\) by

\[(2.12) \quad V = \nabla \phi + \frac{|\nabla \phi|}{|\nabla T|} \nabla T \]

and must satisfy

\[(2.13) \quad V \frac{|\nabla \phi|}{|\nabla T|} = 0, \]

which is equivalent to

\[(2.14) \quad \frac{1}{2} |V|^2 = 0. \]

Equation \((2.14)\) says that \(V\) has constant length along its flowlines.

The function \(\eta\) is expressed in terms of \(\phi\) and \(T\) by

\[(2.15) \quad \eta = -\frac{|\nabla \phi|}{|\nabla T|}. \]

The measure \(\mu\) must still satisfy the conservation law \((2.2)\).
From (2.8), the evolution of $\phi$ between level sets is given by

$$V\phi = \frac{1}{2}|V|^2 = \frac{1}{2}|
abla\phi|.$$  

The normal line to a level set $S_t$ is spanned by $\nabla T$. It follows from (2.7) that the tangential part of $V$ is the gradient of a function on $S_t$:

$$V_{\text{tan}} = \nabla_{S_t} \left( \phi |_{S_t} \right).$$

The normal part of $V$ is

$$V_{\text{norm}} = \frac{\langle V, \nabla T \rangle}{|\nabla T|^2} \nabla T = \frac{1}{|\nabla T|^2} \nabla T,$$

as must be the case from (2.1).

The conclusion is that the tangential part of $V$ on $S_t$ is a gradient vector field on $S_t$, while the normal part of $V$ on $S_t$ is unconstrained.

3. Tangent cones

3.1. Optimal transport from submanifolds. Let $M$ be a smooth closed Riemannian manifold. Let $i : S \to M$ be an embedding.

Let $\pi : TM \to M$ be the projection map. Given $\epsilon > 0$, define $E_{\epsilon} : TM \to TM$ by $E_{\epsilon}(m, v) = (\exp_m(\epsilon v), d(\exp_m)_{e^\epsilon v} e^\epsilon v)$. We define $\pi^S$ and $E^S_{\epsilon}$ similarly, replacing $M$ by $S$.

Put $T_S M = i^*TM$, a vector bundle on $S$ with projection map $\pi_{T_S M} : T_S M \to S$. There is an orthogonal splitting $T_S M = TS \oplus N_S M$ into the tangential part and the normal part. Let $\pi_{N_S M} : N_S M \to S$ be the projection to the base of $N_S M$. Given $v \in TS$, let $v^T \in TS$ denote its tangential part and let $v^\perp \in NS$ denote its normal part. Let $p^T : T_S M \to TS$ be orthogonal projection.

A function $F : S \to \mathbb{R} \cup \{\infty\}$ is semiconvex if there is some $\lambda \in \mathbb{R}$ so that for all minimizing constant-speed geodesics $\gamma : [0, 1] \to S$, we have

$$F(\gamma(t)) \leq tF(\gamma(1)) + (1 - t)F(\gamma(0)) - \frac{1}{2}\lambda t(1 - t)d(\gamma(0), \gamma(1))^2$$

for all $t \in [0, 1]$.

Suppose that $F$ is a semiconvex function on $S$. Then $(s, w) \in TS$ lies in the subdifferential set $\nabla^- F$ if for all $w' \in T_S S$,

$$F(s) + \langle w, w' \rangle \leq F(\exp_s w') + o(|w'|).$$

Define the cost function $c : S \times M \to \mathbb{R}$ by $c(s, x) = \frac{1}{2}d(s, x)^2$. Given $\eta : M \to \mathbb{R} \cup \{-\infty\}$, its $c$-transform is the function $\eta^c : S \to \mathbb{R} \cup \{\infty\}$ given by

$$\eta^c(s) = \sup_{x \in M} \left( \eta(x) - \frac{1}{2}d^2(s, x) \right).$$
Given $\psi : S \to \mathbb{R} \cup \{\infty\}$, its $c$-transform is the function $\psi^c : M \to \mathbb{R} \cup \{-\infty\}$ given by
\begin{equation}
\psi^c(x) = \inf_{s \in S} \left( \psi(s) + \frac{1}{2} d^2(s, x) \right). \tag{3.4}
\end{equation}
A function $\psi : S \to \mathbb{R} \cup \{\infty\}$ is $c$-convex if $\psi = \psi^c$ for some $\eta : M \to \mathbb{R} \cup \{-\infty\}$. A function $\eta : M \to \mathbb{R} \cup \{-\infty\}$ is $c$-concave if $\eta = \psi^c$ for some $\psi : S \to \mathbb{R} \cup \{\infty\}$.

From [17, Proposition 5.8], a function $F : S \to \mathbb{R} \cup \{-\infty\}$ is $c$-convex if and only if $F = (F^c)^c$, i.e. for all $s \in S$,
\begin{equation}
F(s) = \sup_{s' \in M} \inf_{s'' \in S} \left( F(s') + \frac{1}{2} d^2(s', x) - \frac{1}{2} d^2(s, x) \right). \tag{3.5}
\end{equation}

The next lemma appears in [6, Lemma 2.9] when $S = M$.

**Lemma 3.6.** If $F : S \to \mathbb{R} \cup \{\infty\}$ is a semiconvex function then there is some $\epsilon > 0$ so that $\epsilon F$ is $c$-convex.

**Proof.** Clearly
\begin{equation}
\epsilon F(s) \geq \sup_{s' \in M} \inf_{s'' \in S} \left( \epsilon F(s') + \frac{1}{2} d^2(s', x) - \frac{1}{2} d^2(s, x) \right), \tag{3.7}
\end{equation}
as is seen by taking $s' = s$ on the right-hand side of (3.7). Hence we must show that for suitable $\epsilon > 0$, for all $s \in S$ we have
\begin{equation}
\epsilon F(s) \leq \sup_{s' \in M} \inf_{s'' \in S} \left( \epsilon F(s') + \frac{1}{2} d^2(s', x) - \frac{1}{2} d^2(s, x) \right). \tag{3.8}
\end{equation}
For this, it suffices to show that for each $s \in S$, there is some $x \in M$ so that
\begin{equation}
\epsilon F(s) \leq \inf_{s' \in S} \left( \epsilon F(s') + \frac{1}{2} d^2(s', x) - \frac{1}{2} d^2(s, x) \right). \tag{3.9}
\end{equation}
That is, it suffices to show that for each $s \in S$, there is some $x \in M$ so that for all $s' \in S$, we have
\begin{equation}
\epsilon F(s) \leq \epsilon F(s') + \frac{1}{2} d^2(s', x) - \frac{1}{2} d^2(s, x), \tag{3.10}
\end{equation}
i.e.
\begin{equation}
\epsilon F(s) + \frac{1}{2} d^2(s, x) \leq \epsilon F(s') + \frac{1}{2} d^2(s', x). \tag{3.11}
\end{equation}

We know that $F$ is $K$-Lipschitz for some $K < \infty$ [17, Theorem 10.8 and Proposition 10.12]. Hence if $v \in \nabla_x F$ then $|v| \leq K$. Given $s$, choose $v \in \nabla_x F$ and put $x = \exp_s(\epsilon v) \in M$. Then $d(s, x) \leq \epsilon K$.

Put $G(s') = \epsilon F(s') + \frac{1}{2} d^2(s', x)$. We want to show that $G(s) \leq G(s')$ for all $s' \in S$. Suppose not. Let $s'$ be a minimum point for $G$; then $G(s') < G(s)$.

We claim first that $s' \in B_{4\epsilon K}(s)$. To see this, if $d(s, s') \geq 4\epsilon K$ then since
\begin{equation}
d(s', x) \geq d(s, s') - d(s, x) \geq d(s, s') - \epsilon K, \tag{3.12}
\end{equation}
we have
\begin{equation}
\frac{1}{2} d^2(s', x) - \frac{1}{2} d^2(s, x) \geq \frac{1}{2} (d(s, s') - \epsilon K)^2 - \frac{1}{2} (\epsilon K)^2
\end{equation}
\begin{align*}
&= \frac{1}{2} (d(s, s') - 2\epsilon K) \cdot d(s, s') \\
&\geq \epsilon K d(s, s') \geq \epsilon (F(s) - F(s')),
\end{align*}
which contradicts that $G(s') < G(s)$. This proves the claim.

If $10\epsilon K$ is less than the injectivity radius of $M$ then there is a unique minimizing geodesic from $s$ to $x$, and its tangent vector at $s$ is $\nu v$. It follows that $0 \in \nabla^- s G$. Finally, since $d(s, x) \leq \epsilon K$, we can choose an $\epsilon$ (depending on $K, S$ and $M$) to ensure that $G$ is strictly convex on $B_{\epsilon K}(s)$, with the latter being a totally convex set. Considering the function $G$ along a minimizing geodesic from $s$ to $s'$, we obtain a contradiction to the assumed strict convexity of $G$, along with the facts that $0 \in \nabla^- s G$ and $0 \in \nabla^- s' G$.

Thus $G$ is minimized at $s$, which implies (3.11). \hfill \square

Let $\nu$ be a compactly-supported probability measure on $T_S M \subset TM$. Let $L < \infty$ be such that the support of $\nu$ is contained in \{ $v \in T_S M : |v| \leq L$ \}. Put $\mu_c = \pi_s(E_c)_\ast \nu$.

**Proposition 3.14.** a. Let $f$ be a semiconvex function on $S$. Suppose that $\nu$ is supported on \{ $v \in T_S M : v^T \in \nabla^- f$ \}. Then there is some $\epsilon > 0$ so that the 1-parameter family of measures \{ $\mu_t \}_{t \in [0, \epsilon]}$ is a Wasserstein geodesic.

b. Given $\nu$, suppose that for some $\epsilon > 0$, the 1-parameter family of measures \{ $\mu_t \}_{t \in [0, \epsilon]}$ is a Wasserstein geodesic. Then there is a semiconvex function $f$ on $S$ so that $\nu$ is supported on \{ $v \in T_S M : v^T \in \nabla^- f$ \}.

**Proof.** a. For $t > 0$, define $\eta_t : M \to \mathbb{R}$ by $\eta_t = (tf)^c$. From Lemma 3.6, if $t$ is small enough then $tf$ is $c$-convex. It follows from [17] Proposition 5.8 that $(\eta_t)^c = tf$.

From [17] Theorem 5.10], if a set $\Gamma_t \subset S \times M$ is such that $\eta_t(x) = tf(s) + \frac{1}{2} d^2(s, x)$ for all $(x, s) \in M \times S$ then any probability measure $\Pi_t$ with support in $\Gamma_t$ is an optimal transport plan. We take
\begin{equation}
\Gamma_t = \{(s, x) \in S \times M : \eta_t(x) = tf(s) + \frac{1}{2} d^2(s, x)\}.
\end{equation}

Now $\eta_t(x) = tf(s) + \frac{1}{2} d^2(s, x)$ if for all $s' \in S$, we have
\begin{equation}
\tag{3.16}
tf(s) + \frac{1}{2} d^2(s, x) \leq tf(s') + \frac{1}{2} d^2(s', x).
\end{equation}

To prove part a. of the proposition, it suffices to show that for all sufficiently small $t$, equation (3.16) is satisfied for $s, s' \in S$ and $x = \text{exp}_s(tv)$, where $v \in T_s M$ lies in the support of $\nu$ and satisfies $v^T \in \nabla^- f$.

Given $s$ and $v$, we know that $d(s, x) \leq t L$. Put $G(s') = tf(s') + \frac{1}{2} d^2(s', x)$. Let $s'$ be a minimum point of $G$ and suppose, to get a contradiction, that $G(s') < G(s)$. 


Let $K < \infty$ be the Lipschitz constant of $f$. We claim first that $s' \in B_{t(2K+2L)}(s)$. To see this, if $d(s, s') \geq t(2K + 2L)$ then
\begin{equation}
(3.17) \quad d(s', x) \geq d(s, s') - d(s, x) \geq d(s, s') - tL
\end{equation}
and
\begin{align*}
(3.18) \quad \frac{1}{2} d^2(s', x) - \frac{1}{2} d^2(s, x) & \geq \frac{1}{2} (d(s, s') - tL)^2 - (tL)^2 \\
& = \frac{1}{2} (d(s, s') - 2tL) \cdot d(s, s') \\
& \geq tKd(s, s') \geq t(f(s) - f(s')),
\end{align*}
which is a contradiction and proves the claim.

There is some $\epsilon > 0$ (depending on $L$, $S$ and $M$) so that if $t \in [0, \epsilon]$ then we are ensured that there is a unique minimizing geodesic from $s$ to $x$, and its tangent vector at $s$ is $tv$. It follows that $0 \in \nabla^- G$. Finally, since $d(s, x) \leq \epsilon L$, we can choose $\epsilon$ (depending on $K$, $L$, $S$ and $M$) to ensure that $G$ is strictly convex on $B_{\epsilon L}(s)$, the latter being totally convex. Considering the function $G$ along a minimizing geodesic from $s$ to $s'$, we obtain a contradiction to the assumed strict convexity of $G$, along with the facts that $0 \in \nabla^- G$ and $0 \in \nabla^- G$. This proves part (a) of the proposition.

Now suppose that $\{\mu_t\}_{t \in [0, \epsilon]}$ is a Wasserstein geodesic. From [17, Theorem 5.10], there is a $c$-convex function $\epsilon f$ on $S$ so that if we define its conjugate $(\epsilon f)^c$ using (3.4) then $\{(s, \exp_s(\epsilon v))\}_{(s, \nu) \in \text{supp}(\nu)}$ is contained in
\begin{equation}
(3.19) \quad \Gamma_{\epsilon} = \left\{(s, x) \in S \times M : (\epsilon f)^c(x) = \epsilon f(s) + \frac{1}{2} d^2(s, x)\right\}.
\end{equation}
That is, for all $s' \in S$,
\begin{equation}
(3.20) \quad \epsilon f(s) + \frac{1}{2} d^2(s, \exp_s(\epsilon v)) \leq \epsilon f(s') + \frac{1}{2} d^2(s', \exp_s(\epsilon v)).
\end{equation}

Without loss of generality, we can shrink $\epsilon$ as desired. Define a curve in $S$ by $s'(u) = \exp_s(-uu')$ where $u' \in T_s S$, $u$ varies over a small interval $(-\delta, \delta)$ and $\exp_s$ denotes here the exponential map for the submanifold $S$. Let $\{\gamma_u : [0, \epsilon] \to M\}_{u \in (-\delta, \delta)}$ be a smooth 1-parameter family with $\gamma_0(t) = \exp_s(tv)$, $\gamma_u(0) = s'(u)$ and $\gamma_u(\epsilon) = \exp_s(\epsilon v)$. Let $L(u)$ be the length of $\gamma_u$. Then
\begin{equation}
(3.21) \quad \epsilon f(s'(u)) + \frac{1}{2} d^2(s'(u), \exp_s(\epsilon v)) \leq \epsilon f(s'(u)) + \frac{1}{2} L^2(u).
\end{equation}
By the first variation formula,
\begin{equation}
(3.22) \quad \frac{d}{du} \bigg|_{u=0} \frac{1}{2} L^2(u) = \epsilon \langle v^T, w' \rangle.
\end{equation}
It follows that $\epsilon v^T \in \nabla^- \epsilon f$, so $v^T \in \nabla^- f$. \qed

\textbf{Remark 3.23.} The phenomenon of possible nonuniqueness, in the normal component of the optimal transport between two measures supported on convex hypersurfaces in Euclidean space, was recognized in [3, Proposition 4.3].
Example 3.24. Put $M = S^1 \times \mathbb{R}$. (It is noncompact, but this will be irrelevant for the example.) Let $F \in C^\infty(S^1)$ be a positive function. Put $S = \{(x, F(x)) : x \in S^1\}$. Define $p : S \to S^1 \times \{0\}$ by $p(x, F(x)) = (x, 0)$. Let $\mu_0$ be a smooth measure on $S$. Put $\mu_1 = p_*\mu_0$. The Wasserstein geodesic from $\mu_0$ to $\mu_1$ moves the measure down along vertical lines. Defining $f$ on $S$ by $f(x, F(x)) = -\frac{1}{2} (F(x))^2$, one finds that $\nu^T = \nabla f$. Compare with [4, Corollary 2.6].

3.2. Tangent cones. If $X$ is a complete length space with Alexandrov curvature bounded below then one can define the tangent cone $T_x X$ at $x \in X$ as follows. Let $\Sigma'_x$ be the space of equivalence classes of minimal geodesic segments emanating from $x$, with the equivalence relation identifying two segments if they form a zero angle at $x$ (which means that one segment is contained in the other). The metric on $\Sigma'_x$ is the angle. By definition, the space of directions $\Sigma_x$ is the metric completion of $\Sigma'_x$. The tangent cone $T_x X$ is the union of $\mathbb{R}^+ \times \Sigma_x$ and a “vertex” point, with the metric described in [3 §10.9].

If $X$ is finite-dimensional then one can also describe $T_x X$ as the pointed Gromov-Hausdorff limit $\lim_{\lambda \to \infty} (\lambda X, x)$. This latter description doesn’t make sense if $X$ is infinite-dimensional, whereas the preceding definition does.

If $M$ is a smooth compact connected Riemannian manifold, and it has nonnegative sectional curvature, then $P(M)$ has nonnegative Alexandrov curvature and one can talk about a tangent cone $T_\mu P(M)$ [11 Appendix A]. If $M$ does not have nonnegative sectional curvature then $P(M)$ will not have Alexandrov curvature bounded below. Nevertheless, one can still define $T_\mu P(M)$ in the same way [13 Section 3].

As a point of terminology, what is called a tangent cone here, and in [11], is called the “abstract tangent space” in [6]. The linear part of the tangent cone is called the “tangent space” in [1] and the “space of gradients” or “tangent vector fields” in [6].

A minimal geodesic segment emanating from $\mu \in P(M)$ is determined by a probability measure $\Pi$ on the space of constant-speed minimizing geodesics

$$\Gamma = \{\gamma : [0, 1] \to M : L(\gamma) = d_M(\gamma(0), \gamma(1))\},$$

which has the property that under the time-zero evaluation $e_0 : \Gamma \to M$, we have $(e_0)_*\Gamma = \mu$ [11 Section 2]. The corresponding geodesic segment is given by $\mu_t = (e_t)_*\Pi$, where $e_t : \Gamma \to M$ is time-$t$ evaluation.

Using this characterization of minimizing geodesic segments, one can describe $T_\mu P(M)$ as follows. With $\pi : TM \to M$ being projection to the base, put

$$P_2(TM)_\mu = \{\nu \in P_2(TM) : \pi_*\nu = \mu\},$$

where $P_2$ refers to measures with finite second moment. Given $\nu^1, \nu^2 \in P_2(TM)_\mu$, decompose them as

$$\nu^i = \int_M \nu^i_m \, d\mu(m),$$

with $\nu^i_m \in P_2(T_m M)$. Define $W_\mu(\nu^1, \nu^2)$ by

$$W_\mu^2(\nu^1, \nu^2) = \int_M W_2^2(\nu^1_m, \nu^2_m) \, d\mu(m).$$
Let \( \text{Dir}_\mu \) be the set of elements \( \nu \in P_2(TM)_\mu \) with the property that \( \{ \pi_*(E_t)_*\nu \}_{t \in [0, \varepsilon]} \) describes a minimizing Wasserstein geodesic for some \( \varepsilon \). Then \( T_\mu P(M) \) is isometric to the metric completion of \( \text{Dir}_\mu \) with respect to \( W_\mu \) [9, Theorem 5.5].

We note that since \( M \) is compact, any element of \( \text{Dir}_\mu \) has compact support. This is because for \( \nu \)-almost all \( v \in TM \), the geodesic \( \{ \exp_{\pi(v)} tv \}_{t \in [0, \varepsilon]} \) must be minimizing [11, Proposition 2.10], so \( |v| \leq \varepsilon^{-1} \text{diam}(M) \).

**Proof of Theorem 1.1:** From Proposition 3.14, \( \text{Dir}_\mu \) is the set of compactly-supported measures \( \nu \in P(TS M) \subset P(TM) \) so that \( \pi_*\nu = \mu \) and there is a semiconvex function \( f \) on \( S \) such that \( \nu \) has support on \( \{ v \in TS M : v^T \in \nabla^{-} f \} \). Because \( \nu \) has full support on \( S \) by assumption, \( \nabla^{-} f \) is single-valued at \( \mu \)-almost all \( s \in S \). Equivalently, there is a compactly-supported \( \nu^N \in P(N_S M) \), which decomposes under \( \pi_{N_S M} : N_S M \rightarrow S \) as \( \nu^N = \int_S \nu^N_s \, d\mu(s) \) with \( \nu^N_s \in P_2(N_s M) \), so that for all \( F \in C(TS M) = C(TS \oplus N_S M) \), we have

\[
(3.29) \quad \int_{TS M} F \, d\nu = \int_S \int_{N_s M} F(\nabla^{-} f(s), w) \, d\nu^N_s(w) \, d\mu(s).
\]

Given two such measures \( \nu^1, \nu^2 \), it follows that

\[
(3.30) \quad W_\mu^2(\nu^1, \nu^2) = \int_S \langle \nabla^{-} f^1, \nabla^{-} f^2 \rangle \, d\mu + \int_S W_2^2(\nu^1_s, \nu^2_s) \, d\mu(s).
\]

Upon taking the metric completion of \( \text{Dir}_\mu \), the tangential term in (3.30) gives the closure of the space of gradient vector fields in the Hilbert space \( L^2(TS, d\mu) \) of square-integrable sections of \( TS \) [11 Proposition A.33]. The normal term gives \( \int_{S \times S} P_2(N_s N_{s'}) \, d\mu(s, s') \), where the metric comes from the last term in (3.30). This proves the theorem. \( \Box \)

**Remark 3.31.** In Section 2 we considered transports in which the intermediate measures were supported on hypersurfaces. This corresponds to Wasserstein geodesics starting from \( \mu \) for which the initial velocity, as an element of \( T_\mu P(M) \), comes from a section of \( TS M \). In terms of Theorem 1.1, this means that the data for the initial velocity consisted of a gradient vector field \( \nabla \phi \) on \( S \) and a section \( \mathcal{N} \) of \( N_S M \), with the element of \( P_2(N_s M) \) being the delta measure at \( \mathcal{N}(s) \).

### 4. Weak solutions to the parallel transport equation

Let \( M \) be a compact connected Riemannian manifold. Put

\[
(4.1) \quad P^\infty(M) = \{ \rho \, d\text{vol}_M : \rho \in C^\infty(M), \rho > 0, \int_M \rho \, d\text{vol}_M \, = \, 1 \}.
\]

Given \( \phi \in C^\infty(M) \), define a vector field \( V_\phi \) on \( P^\infty(M) \) by saying that for \( F \in C^\infty(P^\infty(M)) \),

\[
(4.2) \quad (V_\phi F)(\rho \, d\text{vol}_M) = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} F \left( \rho \, d\text{vol}_M - \epsilon \nabla^i (\rho \nabla^i \phi) \, d\text{vol}_M \right).
\]
The map $\phi \mapsto V_\phi$ passes to an isomorphism $C^\infty(M)/\mathbb{R} \to T_{\rho \text{dvol} M} P^\infty(M)$. Otto’s Riemannian metric on $P^\infty(M)$ is given \cite{14} by

\begin{equation}
\langle V_\phi_1, V_\phi_2 \rangle (\rho \text{dvol} M) = \int_M \langle \nabla \phi_1, \nabla \phi_2 \rangle \rho \text{ dvol}_M
\end{equation}

In view of (4.2), we write $\delta_{V_\phi} \rho = -\nabla^i (\rho \nabla_i \phi)$. Then

\begin{equation}
\langle V_\phi_1, V_\phi_2 \rangle (\rho \text{dvol} M) = \int_M \phi_1 \delta_{V_\phi_2} \rho \text{ dvol}_M = \int_M \phi_2 \delta_{V_\phi_1} \rho \text{ dvol}_M.
\end{equation}

To write the equation for parallel transport, let $c : [0, 1] \to P^\infty(M)$ be a smooth curve. We write $c(t) = \mu_t = \rho(t) \text{dvol}_M$ and define $\phi(t) \in C^\infty(M)$, up to a constant, by $\frac{dc}{dt} = V_{\phi(t)}$. This is the same as saying

\begin{equation}
\frac{\partial \rho}{\partial t} + \nabla^j (\rho \nabla_j \phi) = 0.
\end{equation}

Let $V_{\eta(t)}$ be a vector field along $c$, with $\eta(t) \in C^\infty(M)$. The equation for $V_{\eta}$ to be parallel along $c$ \cite{10} Proposition 3 is

\begin{equation}
\nabla_i \left( \rho \left( \nabla^i \frac{\partial \eta}{\partial t} + \nabla_j \phi \nabla^i \nabla^j \eta \right) \right) = 0.
\end{equation}

**Lemma 4.7.** \cite{10} Lemma 5 If $\eta, \overline{\eta}$ are solutions of (4.6) then $\int_M \langle \nabla \eta, \nabla \overline{\eta} \rangle \text{d}\mu_t$ is constant in $t$.

**Lemma 4.8.** Given $\eta \in C^\infty(M)$, there is at most one solution of (4.6) with $\eta(1) = \eta_1$, up to time-dependent additive constants.

**Proof.** By linearity, it suffices to consider the case when $\eta_1 = 0$. From Lemma 4.7, $\nabla \eta(t) = 0$ and so $\eta(t)$ is spatially constant. \hfill $\square$

For consistency with later notation, we will write $C^\infty([0, 1]; C^\infty(M))$ for $C^\infty([0, 1] \times M)$.

**Lemma 4.9.** (c.f. \cite{11} (5.8)) Given $f \in C^\infty([0, 1]; C^\infty(M))$, if $\eta$ satisfies (4.6) then

\begin{equation}
\frac{d}{dt} \int_M \langle \nabla f, \nabla \eta \rangle \text{d}\mu_t = \int_M \langle \nabla \frac{\partial f}{\partial t}, \nabla \eta \rangle \text{d}\mu_t + \int_M \text{Hess}_f(\nabla \eta, \nabla \phi) \text{d}\mu_t.
\end{equation}

**Proof.** We have

\begin{equation}
\frac{d}{dt} \int_M \langle \nabla f, \nabla \eta \rangle \text{d}\mu_t = \frac{d}{dt} \int_M \langle \nabla f, \nabla \eta \rangle \rho \text{ dvol}_M
\end{equation}

\begin{align*}
= \int_M \langle \nabla \frac{\partial f}{\partial t}, \nabla \eta \rangle \rho \text{ dvol}_M + \int_M \langle \nabla f, \nabla \frac{\partial \eta}{\partial t} \rangle \rho \text{ dvol}_M + \\
\int_M \langle \nabla f, \nabla \eta \rangle \frac{\partial \rho}{\partial t} \text{ dvol}_M.
\end{align*}
Then

\begin{align}
\frac{d}{dt} \int_M \langle \nabla f, \nabla \eta \rangle \, d\mu_t - \int_M \langle \nabla \frac{\partial f}{\partial t}, \nabla \eta \rangle \, d\mu_t \\
= \int_M (\nabla_i f) \left( \nabla^i \frac{\partial \rho}{\partial t} \right) \rho \, d\text{vol}_M - \int_M (\nabla_i f) (\nabla^i \eta) \nabla^j (\rho \nabla_j \phi) \, d\text{vol}_M \\
= - \int_M f \nabla_i \left( \rho \nabla^i \frac{\partial \rho}{\partial t} \right) \rho \, d\text{vol}_M - \int_M (\nabla_i f) (\nabla^i \eta) \nabla^j (\rho \nabla_j \phi) \, d\text{vol}_M \\
= \int_M f \nabla_i \left( \rho (\nabla_j \phi) (\nabla^i \eta) \right) \, d\text{vol}_M + \int_M \nabla^j ((\nabla_i f) (\nabla^i \eta)) (\nabla_j \phi) \rho \, d\text{vol}_M \\
= - \int_M (\nabla_i f) (\nabla_j \phi) (\nabla^i \eta) \rho \, d\text{vol}_M \\
+ \int_M \nabla^j ((\nabla_i f) (\nabla^i \eta)) (\nabla_j \phi) \rho \, d\text{vol}_M \\
= \int_M (\nabla_j \nabla_i f) (\nabla^i \eta) (\nabla_j \phi) \rho \, d\text{vol}_M \\
= \int_M \text{Hess}_f(\nabla \eta, \nabla \phi) \, d\mu_t .
\end{align}

This proves the lemma. \qed

We now weaken the regularity assumptions. Let \( P^{ac}(M) \) denote the absolutely continuous probability measures on \( M \) with full support. Suppose that \( c : [0, 1] \to P^{ac}(M) \) is a Lipschitz curve whose derivative \( c'(t) \in T_{c(t)} P(M) \) exists for almost all \( t \). We can write \( c'(t) = V_{\phi(t)} \) with \( \nabla \phi(t) \in L^2(TM, dc(t)) \). By the Lipschitz assumption, the essential supremum over \( t \in [0, 1] \) of \( \|\nabla \phi(t)\|_{L^2(TM, dc(t))} \) is finite. As before, we write \( c(t) = \mu_t \).

**Definition 4.13.** Let \( c : [0, 1] \to P^{ac}(M) \) be a Lipschitz curve whose derivative \( c'(t) \in T_{c(t)} P(M) \) exists for almost all \( t \). Given \( \nabla \eta_0 \in L^2(TM, d\mu_0) \), \( \nabla \eta_1 \in L^2(TM, d\mu_1) \) and \( \nabla \eta \in L^2([0, 1]; L^2(TM, d\mu_t)) \), we say that \( (\nabla \eta, \nabla \eta_0, \nabla \eta_1) \) is a weak solution of the parallel transport equation if

\begin{align}
\int_M \langle \nabla f(1), \nabla \eta_1 \rangle \, d\mu_1 - \int_M \langle \nabla f(0), \nabla \eta_0 \rangle \, d\mu_0 = \\
\int_0^1 \int_M \left( \langle \nabla \frac{\partial f}{\partial t}, \nabla \eta \rangle + \text{Hess}_f(\nabla \eta, \nabla \phi) \right) \, d\mu_t \, dt 
\end{align}

for all \( f \in C^\infty([0, 1]; C^\infty(M)) \).

**Remark 4.15.** In what follows, there would be analogous results if we replaced \( C^\infty([0, 1]; C^\infty(M)) \) everywhere by \( C^0([0, 1]; C^2(M)) \cap C^1([0, 1]; C^1(M)) \). We will stick with \( C^\infty([0, 1]; C^\infty(M)) \) for concreteness.

From Lemma \[\text{Lemma 4.13}\] if \( c \) is a smooth curve in \( P^\infty(M) \) and \( \eta \in C^\infty([0, 1]; C^\infty(M)) \) is a solution of \( (4.6) \) then \( (\nabla \eta, \nabla \eta(0), \nabla \eta(1)) \) is a weak solution of the parallel transport equation. We now prove the converse.
Lemma 4.16. Suppose that \( c \) is a smooth curve in \( P^\infty(M) \). Given \( \eta_0, \eta_1 \in C^\infty(M) \) and \( \eta \in C^\infty([0, 1]; C^\infty(M)) \), if \((\nabla \eta, \nabla \eta_0, \nabla \eta_1)\) is a weak solution of the parallel transport equation then \( \eta \) satisfies \((4.6)\), \( \eta(0) = \eta_0 \) and \( \eta(1) = \eta_1 \) (modulo constants).

Proof. In this case, equation \((4.14)\) is equivalent to

\[
\int_M \langle \nabla f(1), \nabla \eta_1 \rangle \, d\mu_1 - \int_M \langle \nabla f(0), \nabla \eta_0 \rangle \, d\mu_0 = \\
\int_M \langle \nabla f(1), \nabla \eta(1) \rangle \, d\mu_1 - \int_M \langle \nabla f(0), \nabla \eta(0) \rangle \, d\mu_0 + \\
\int_0^1 \int_M f \nabla_i \left( \nabla^i \frac{\partial \eta}{\partial t} + \nabla_j \phi \nabla^i \eta \right) \, d\mu_i \, dt.
\]

Taking \( f \in C^\infty([0, 1]; C^\infty(M)) \) with \( f(0) = f(1) = 0 \), it follows that \((4.6)\) must hold. Then taking all \( f \in C^\infty([0, 1]; C^\infty(M)) \), it follows that \( \nabla \eta_0 = \nabla \eta(0) \) and \( \nabla \eta_1 = \nabla \eta(1) \). Hence \( \eta(0) = \eta_0 \) and \( \eta(1) = \eta_1 \) (modulo constants). \(\square\)

Lemma 4.18. Suppose that \( c \) is a smooth curve in \( P^\infty(M) \). Given \( \nabla \eta_0 \in L^2(TM, d\mu_0) \), \( \nabla \eta_1 \in L^2(TM, d\mu_1) \), \( \nabla \eta \in L^2([0, 1]; L^2(TM, d\mu_t)) \) and \( f \in C^\infty([0, 1]; C^\infty(M)) \), suppose that

1. \((\nabla \eta, \nabla \eta_0, \nabla \eta_1)\) is a weak solution to the parallel transport equation,
2. \( f \) satisfies \((4.6)\),
3. \( \nabla f(1) = \nabla \eta_1 \),
4. \((4.19)\)

\[
\int_M |\nabla \eta_0|^2 \, d\mu_0 \leq \int_M |\nabla \eta_1|^2 \, d\mu_1
\]

and

5. \((4.20)\)

\[
\int_0^1 \int_M |\nabla \eta|^2 \, d\mu_i \, dt \leq \int_M |\nabla \eta_1|^2 \, d\mu_1
\]

Then \( \nabla f(0) = \nabla \eta_0 \), and \( \nabla f(t) = \nabla \eta(t) \) for almost all \( t \).

Proof. From \((4.6)\) (applied to \( f \)) and \((4.14)\), we have

\[
\int_M \langle \nabla f(0), \nabla \eta_0 \rangle \, d\mu_0 = \int_M \langle \nabla f(1), \nabla \eta_1 \rangle \, d\mu_1 = \int_M \langle \nabla \eta_1, \nabla \eta_1 \rangle \, d\mu_1.
\]

From Lemma 4.7

\[
\int_M \langle \nabla f(0), \nabla f(0) \rangle \, d\mu_0 = \int_M \langle \nabla f(1), \nabla f(1) \rangle \, d\mu_1 = \int_M \langle \nabla \eta_1, \nabla \eta_1 \rangle \, d\mu_1.
\]

Then

\[
\int_M |\nabla(\eta_0 - f(0))|^2 \, d\mu_0 = \int_M |\nabla \eta_0|^2 \, d\mu_0 - \int_M |\nabla \eta_1|^2 \, d\mu_1 \leq 0.
\]

Thus \( \nabla f(0) = \nabla \eta_0 \) in \( L^2(TM, d\mu_0) \).
Next, replacing $f$ by $tf$ in (4.14) gives
\begin{equation}
\int_0^1 \int_M \langle \nabla f, \nabla \eta \rangle \, d\mu_t \, dt = \int_M \langle \nabla f(1), \nabla \eta_1 \rangle \, d\mu_1 = \int_M \langle \nabla \eta_1, \nabla \eta_1 \rangle \, d\mu_1.
\end{equation}

Then
\begin{equation}
\int_0^1 \int_M |\nabla f - \nabla \eta|^2 \, d\mu_t \, dt = \\
\int_0^1 \int_M |\nabla f|^2 \, d\mu_t \, dt - 2 \int_0^1 \int_M \langle \nabla f, \nabla \eta \rangle \, d\mu_t \, dt + \int_0^1 \int_M |\nabla \eta|^2 \, d\mu_t \, dt = \\
\int_M |\nabla f(1)|^2 \, d\mu_1 - 2 \int_M |\nabla \eta_1|^2 \, d\mu_1 + \int_0^1 \int_M |\nabla \eta|^2 \, d\mu_t \, dt = \\
\int_0^1 \int_M |\nabla \eta|^2 \, d\mu_t \, dt - \int_M |\nabla \eta_1|^2 \, d\mu_1 \leq 0.
\end{equation}

Thus $\nabla f(t) = \nabla \eta(t)$ in $L^2(TM, d\mu_t)$, for almost all $t$. □

5. Parallel transport along Wasserstein geodesics

5.1. Parallel transport in a finite-dimensional Alexandrov space. We recall the construction of parallel transport in a finite-dimensional Alexandrov space $X$.

Let $c : [0, 1] \to X$ be a geodesic segment that lies in the interior of a minimizing geodesic. Then $T_{c(t)}X$ is an isometric product of $\mathbb{R}$ with the normal cone $N_{c(t)}X$. We want to construct a parallel transport map from $N_{c(1)}X$ to $N_{c(0)}X$.

Given $Q \in \mathbb{Z}^+$ and $0 \leq i \leq Q - 1$, define $c_i : [0, 1] \to X$ by $c_i(u) = c \left( \frac{i+u}{Q} \right)$. We define an approximate parallel transport $P_i : N_{c_i(1)}X \to N_{c_i(0)}X$ as follows. Given $v \in N_{c_i(1)}X$, let $\gamma : [0, \epsilon] \to X$ be a minimizing geodesic with $\gamma(0) = c_i(1)$ and $\gamma'(0) = v$. For each $s \in (0, \epsilon)$, let $\mu_s : [0, 1] \to X$ be a minimizing geodesic with $\mu_s(0) = c_i(0)$ and $\mu_s(1) = \gamma(s)$. Let $w_s \in N_{c_i(0)}X$ be the normal projection of $\frac{1}{s} \mu_s'(0) \in T_{c_i(0)}X$. After passing to a sequence $s_i \to 0$, we can assume that $\lim_{i \to \infty} w_{s_i} = w \in N_{c_i(0)}X$. Then $P_i(v) = w$. If $X$ has nonnegative Alexandrov curvature then $|w| \geq |v|$.

In [15], the approximate parallel transport from an appropriate dense subset $L_Q \subset N_{c(1)}X$ to $N_{c(0)}X$ was defined to be $P_0 \circ P_1 \circ \ldots \circ P_{Q-1}$. It was shown that by taking $Q \to \infty$ and applying a diagonal argument, in the limit one obtains an isometry from a dense subset of $N_{c(1)}X$ to $N_{c(0)}X$. This extends by continuity to an isometry from $N_{c(1)}X$ to $N_{c(0)}X$.

If $X$ is a smooth Riemannian manifold then $P_i$ is independent of the choices and can be described as follows. Given $v \in N_{c_i(1)}X$, let $j_v(u)$ be the Jacobi field along $c$ with $j_v(0) = 0$ and $j_v(1) = v$. (It is unique since $c$ is in the interior of a minimizing geodesic.) Then $P_i(v) = j_v^t(0)$.

5.2. Construction of parallel transport along a Wasserstein geodesic of delta measures. Let $M$ be a compact connected Riemannian manifold. Let $\gamma : [0, 1] \to M$ be a geodesic segment that lies in the interior of a minimizing geodesic. Let $\Pi : T_{\gamma(1)}M \to T_{\gamma(0)}M$ be (reverse) parallel transport along $\gamma$. Put $c(t) = \delta_{\gamma(t)} \in P(M)$. Then $\{c(t)\}_{t \in [0,1]}$ is a
Wasserstein geodesic that lies in the interior of a minimizing geodesic. We apply Petrunin’s construction to define parallel transport directly from the tangent cone $T_{c(1)}P(M)$ to the tangent cone $T_{c(0)}P(M)$ (instead of the normal cones).

**Proposition 5.1.** The parallel transport map from $T_{c(1)}P(M) \cong P_2(T_{c(1)}M)$ to $T_{c(0)}P(M) \cong P_2(T_{c(0)}M)$ is the map $\mu \to \Pi_\ast \mu$.

**Proof.** Given $Q \in \mathbb{Z}^+$ and $0 \leq i \leq Q - 1$, define $\gamma_i : [0, 1] \to M$ by $\gamma_i(u) = \gamma \left( \frac{i + u}{Q} \right)$ and $c_i : [0, 1] \to P(M)$ by $c_i(u) = \delta_{\gamma_i(u)}$. We define an approximate parallel transport $P_i : T_{c_i(1)}P(M) \to T_{c_i(0)}P(M)$ as follows.

Given $s \in \mathbb{R}^+$ and a real vector space $V$, let $R_s : V \to V$ be multiplication by $s$. Let $\nu$ be a compactly-supported element of $P(T_{\gamma_1(1)}M)$. For small $\epsilon > 0$, there is a Wasserstein geodesic $\sigma : [0, \epsilon] \to P(M)$, with $\sigma(0) = c_i(1)$ and $\sigma'(0)$ corresponding to $\nu \in T_{c_i(1)}PM$, given by $\sigma(s) = (\exp_{\gamma_1(1)} \circ R_s)^\ast \nu$. Given $s \in (0, \epsilon]$, let $\mu_s : [0, 1] \to P(M)$ be a minimizing geodesic with $\mu_s(0) = c_i(0) = \delta_{\gamma_1(0)}$ and $\mu_s(1) = \sigma(s)$. There is a compactly-supported measure $\tau_s \in P(T_{\gamma_0(0)}M) = T_{c_i(0)}P(M)$ so that for $v \in [0, 1]$, we have $\mu_s(v) = (\exp_{\gamma_1(0)} \circ R_s)^\ast \tau_s$. If $Q$ is large and $\epsilon$ is small then all of the constructions take place well inside a totally convex ball, so $\tau_s$ is unique and can be written as $\tau_s = (\exp_{\gamma_1(0)} \circ \exp_{\gamma_1(1)} \circ R_s)^\ast \nu$. Then $\lim_{s \to 0} \frac{1}{s} \tau_s$ exists and equals $(d \exp_{\gamma_1(0)})^{-1} \ast \nu$. Thus $P_i = (d \exp_{\gamma_1(0)})^{-1}$.

Now

$$P_0 \circ P_1 \circ \ldots \circ P_{Q-1} = \left( (d \exp_{\gamma_0(0)})^{-1} \circ (d \exp_{\gamma_1(0)})^{-1} \circ \ldots \circ (d \exp_{\gamma_{Q-1}(0)})^{-1} \right)^\ast.$$  
Taking $Q \to \infty$, this approaches $\Pi_\ast$. \hfill $\Box$

### 5.3. Construction of parallel transport along a Wasserstein geodesic of absolutely continuous measures

Let $M$ be a compact connected Riemannian manifold with nonnegative sectional curvature. Then $(P(M), W_2)$ has nonnegative Alexandrov curvature.

Let $c : [0, 1] \to P^{ac}(M)$ be a geodesic segment that lies in the interior of a minimizing geodesic. Write $c'(t) = V_{\phi(t)}$. Since $\phi(t)$ is defined up to a constant, it will be convenient to normalize it by $\int_M \phi(t) \, d\mu_t = 0$. We assume that

$$\sup_{t \in [0, 1]} \| \phi(t) \|_{C^2(M)} < \infty.$$  
In particular, this is satisfied if $c$ lies in $P^\infty(M)$.

Let $N_{c(t)}P(M)$ denote the normal cone to $c$ at $c(t)$. We want to construct a parallel transport map from $N_{c(1)}P(M)$ to $N_{c(0)}P(M)$.

Given $Q \in \mathbb{Z}^+$ and $0 \leq i \leq Q - 1$, define $c_i : [0, 1] \to P(M)$ by $c_i(u) = c \left( \frac{i + u}{Q} \right)$. Correspondingly, write $\mu_{i,u} = \mu_{\frac{i+u}{Q}}$. We define an approximate parallel transport $P_i : N_{c_i(1)}P(M) \to N_{c_i(0)}P(M)$, using Jacobi fields, as follows.
Let us write $c'_i(u) = V_{\phi_i(u)}$, i.e. $\phi_i(u) = \frac{1}{q} \phi \left( \frac{i+u}{q} \right)$. The curve $c_i$ is given by $c_i(u) = (F_{i,u})_* c_i(0)$, where $F_{i,u}(x) = \exp_x (u \nabla_x \phi_i(0))$. That is, for any $f \in C^\infty(M)$,

$$\int_M f \, dc_i(u) = \int_M f(F_{i,u}(x)) \, d\mu_{i,0}(x).$$

(5.4)

If $\sigma_i$ is a variation of $\phi_i(0)$, i.e. $\delta \phi_i(0) = \sigma_i$, then taking the variation of (5.4) gives

$$\int_M f \, d\delta c_i(u) = \int_M \langle \nabla f, d \exp_{u \nabla_x \phi_i(0)}(u \nabla_x \sigma_i) \rangle_{F_{i,u}(x)} \, d\mu_{i,0}(x)$$

$$= u \int_M \langle \nabla f, W_{\sigma_i}(u) \rangle \, d\mu_{i,u}.$$ 

(5.5)

Here

$$W_{\sigma_i}(u)y = d \exp_{u \nabla_x \phi_i(0)}(\nabla_x \sigma_i),$$

with $y = F_{i,u}(x)$. The corresponding tangent vector at $c_i(u)$ is represented by $L_{\sigma_i}(u) = \Pi_{c_i(u)} W_{\sigma_i}(u)$, where $\Pi_{c_i(u)}$ is orthogonal projection on $\Im \nabla \subset L^2(TM, d\mu_{i,u})$. We can think of $J_{\sigma_i}(u) = u L_{\sigma_i}(u)$ as a Jacobi field along $c_i$. If $v = J_{\sigma_i}(1) = L_{\sigma_i}(1) = \Pi_{c_i(1)} W_{\sigma_i}(1)$ then its approximate parallel transport along $c_i$ is represented by $w = J_{\sigma_i}'(0) = L_{\sigma_i}(0) = \nabla \sigma_i \in \overline{\Im \nabla} \subset L^2(TM, d\mu_{i,0})$.

Next, using (5.6), for $f \in C^\infty(M)$ we have

$$\frac{d}{du} \int_M \langle V_f, L_{\sigma_i} \rangle_{d\mu_{i,u}} = \frac{d}{du} \int_M \langle V_f, W_{\sigma_i} \rangle \, d\mu_{i,u} = \frac{d}{du} \int_M \langle \nabla f, d \exp_{u \nabla_x \phi_i(0)}(\nabla_x \sigma_i) \rangle_{F_{i,u}(x)} \, d\mu_{i,0}(x)$$

$$= \int_M \Hess_{F_{i,u}(x)}(f) \left( d \exp_{u \nabla_x \phi_i(0)}(\nabla_x \phi_i(0)), d \exp_{u \nabla_x \phi_i(0)}(\nabla_x \sigma_i) \right) \, d\mu_{i,0}(x) +$$

$$\int_M \langle \nabla f, D_{\partial_u} d \exp_{u \nabla_x \phi_i(0)}(\nabla_x \sigma_i) \rangle_{F_{i,u}(x)} \, d\mu_{i,u}(x)$$

$$= \int_M \Hess(f) \left( \nabla \phi_i(u), W_{\sigma_i}(u) \right) \, d\mu_{i,u} +$$

$$\int_M \langle \nabla f, D_{\partial_u} W_{\sigma_i}(u) \rangle \, d\mu_{i,u}.$$ 

(5.7)

Here $\partial_u$ is the vector at $F_{i,u}(x)$ given by

$$\partial_u = \frac{d}{du} F_{i,u}(x) = d \exp_{u \nabla_x \phi_i(0)}(\nabla_x \phi_i(0)).$$

If instead $f \in C^\infty([0, 1]; C^\infty(M))$ then

$$\frac{d}{du} \int_M \langle V_f, L_{\sigma_i} \rangle_{d\mu_{i,u}} = \int_M \langle \nabla \sigma_i, \partial_u L_{\sigma_i} \rangle \, d\mu_{i,u} +$$

$$\int_M \Hess(f) \left( \nabla \phi_i(u), W_{\sigma_i}(u) \right) \, d\mu_{i,u} +$$

$$\int_M \langle \nabla f, D_{\partial_u} W_{\sigma_i}(u) \rangle \, d\mu_{i,u}.$$ 

(5.8)
We will need to estimate \( \int_M |W_{\sigma_i}(u) - L_{\sigma_i}(u)|^2 \, d\mu_{i,u} \).

**Lemma 5.10.** For large \( Q \), there is an estimate

\[
\int_M |W_{\sigma_i}(u) - L_{\sigma_i}(u)|^2 \, d\mu_{i,u} \leq \text{const.} \| \text{Hess}(\phi_i(\cdot)) \|_{L^\infty([0,1] \times M)}^2 \| L_{\sigma_i}(0) \|^2_{L^2(TM, d\mu_{i,0})}.
\]

Here, and hereafter, const. denotes a constant that can depend on the fixed Riemannian manifold \((M, g)\).

**Proof.** Since \( \Pi_{i,u}(u) \) is projection onto \( \text{Im}(\nabla) \subset L^2(TM, d\mu_{i,u}) \), and \( \nabla(\sigma_i \circ F_{i,u}^{-1}) \in \text{Im}(\nabla) \), we have

\[
\int_M |W_{\sigma_i}(u) - L_{\sigma_i}(u)|^2 \, d\mu_{i,u} \leq \int_M |W_{\sigma_i}(u) - \nabla(\sigma_i \circ F_{i,u}^{-1})|^2_g \, d\mu_{i,u}
\]

\[
= \int_M |(dF_{i,u})^{-1}_* W_{\sigma_i}(u) - \nabla \sigma_{i,u}^2|_{F_{i,u}^* g} \, d\mu_{i,0}.
\]

(Compare with [1] Proposition 4.3.) Defining \( T_{i,t,x} : T_x M \to T_x M \) by

\[
T_{i,t,x}(z) = (dF_{i,u})^{-1}_* (d\exp_u \nabla \phi_i(0)(z)),
\]

we obtain

\[
\int_M |W_{\sigma_i}(u) - L_{\sigma_i}(u)|^2 \, d\mu_{i,u} \leq \left( \sup_{x \in M} \| (dF_{i,u})^{-1}_* dF_{i,u}(x) \| : \| T_{i,t,x} - I \|^2 \right) \| L_{\sigma_i}(0) \|^2_{L^2(TM, d\mu_{i,0})}.
\]

Since \( \sup_{t \in [0,1]} \| \nabla \phi(t) \|_{C^0(M)} < \infty \), if \( Q \) is large then \( \| \nabla \phi_i(0) \|_{C^0(M)} \) is much smaller than the injectivity radius of \( M \). In particular, the curve \( \{ F_{i,u}(x) \}_{u \in [0,1]} \) lies well within a normal ball around \( x \). Now \( T_{i,t,x} \) can be estimated in terms of Hess(\( \phi_i \)). In general, if a function \( h \) on a complete Riemannian manifold satisfies Hess(\( h \)) = 0 then the manifold isometrically splits off an \( \mathbb{R} \)-factor and the optimal transport path generated by \( \nabla h \) is translation along the \( \mathbb{R} \)-factor. In such a case, the analog of \( T_{i,t,x} \) is the identity map. If Hess(\( h \)) \( \neq 0 \) then the divergence of a short optimal transport path from being a translation can be estimated in terms of Hess(\( h \)). Putting in the estimates gives (5.11). \( \square \)

Using Lemma 5.11 we have

\[
\left| \int_M \text{Hess}(f)(\nabla \phi_i(u), W_{\sigma_i}(u)) \, d\mu_{i,u} - \int_M \text{Hess}(f)(\nabla \phi_i(u), L_{\sigma_i}(u)) \, d\mu_{i,u} \right| \leq \text{const.} \| \text{Hess}(f) \|_{C^0(M)} \| \text{Hess}(\phi_i(\cdot)) \|_{L^\infty([0,1] \times M)} \| \nabla \phi_i(0) \|_{L^2(TM, d\mu_{i,0})} \| L_{\sigma_i}(0) \|^2_{L^2(TM, d\mu_{i,0})}.
\]

Next, given \( x \in M \), consider the geodesic

\[
\gamma_{i,x}(u) = F_{i,u}(x).
\]

Put

\[
j_{\gamma_{i,x}}(u) = u(W_{\sigma_i}(u))_{\gamma_{i,x}(u)} \in T_{\gamma_{i,x}(u)}M.
\]
Then \( j_{\sigma_i,x} \) is a Jacobi field along \( \gamma_{i,x} \), with \( j_{\sigma_i,x}(0) = 0 \) and \( j'_{\sigma_i,x}(0) = \nabla_x \sigma_i \). Jacobi field estimates give

\[
\| D_{\partial u} W_{\sigma_i}(u) \|_{L^2(TM,d\mu_{i,u})} \leq \text{const.} \| \nabla \sigma_i \|_{L^2(TM,d\mu_{i,u})} \| \nabla \phi_i(\cdot) \|_{L^\infty([0,1] \times M)},
\]

again for \( Q \) large.

**Lemma 5.19.** Define \( A_i : \left( \text{im}(\nabla) \subset L^2(TM,d\mu_{i,0}) \right) \rightarrow \left( \text{im}(\nabla) \subset L^2(TM,d\mu_{i,1}) \right) \) by

\[
(5.20) \quad A_i(\nabla \sigma_i) = L_{\sigma_i}(1).
\]

Then for large \( Q \), the map \( A_i \) is invertible for all \( i \in \{0, \ldots, Q-1\} \).

**Proof.** Define \( B_i : \left( \text{im}(\nabla) \subset L^2(TM,d\mu_{i,1}) \right) \rightarrow \left( \text{im}(\nabla) \subset L^2(TM,d\mu_{i,0}) \right) \) by

\[
(5.21) \quad B_i(\nabla f) = \nabla (f \circ F_{i,1}).
\]

Then whenever \( \nabla f \in L^2(TM,d\mu_{i,1}) \), we have

\[
(5.22) \quad (A_i B_i)(\nabla f) = A_i(\nabla (f \circ F_{i,1})) = L_{f \circ F_{i,1}}(1),
\]

so whenever \( \nabla f' \in L^2(TM,d\mu_{i,1}) \), for large \( Q \) we have

\[
(5.23) \quad \langle \nabla f', (A_i B_i - I) (\nabla f) \rangle_{L^2(TM,d\mu_{i,1})} = \langle \nabla f', W_{f \circ F_{i,1}}(1) - \nabla f \rangle_{L^2(TM,d\mu_{i,1})} \leq \text{const.} \| \text{Hess}(\phi_i(\cdot)) \|_{L^\infty([0,1] \times M)} \| \nabla f' \|_{L^2(TM,d\mu_{i,1})} \| \nabla f \|_{L^2(TM,d\mu_{i,1})}.
\]

Hence \( \| A_i B_i - I \| = o(Q) \), so for large \( Q \) the map \( A_i B_i \) is invertible and a right inverse for \( A_i \) is given by \( B_i(A_i B_i)^{-1} \). This implies that \( A_i \) is surjective.

Now suppose that \( \nabla \sigma \in \text{Ker}(A_i) \) is nonzero, with \( \sigma \in H^1(M,d\mu_{i,0}) \). After normalizing, we may assume that \( \nabla \sigma \) has unit length. Then

\[
(5.24) \quad 0 = \langle \nabla(\sigma \circ F_{i,1}), A_i(\nabla \sigma) \rangle_{L^2(TM,d\mu_{i,1})} = \langle \nabla(\sigma \circ F_{i,1}), L_{\sigma_i}(1) \rangle_{L^2(TM,d\mu_{i,1})} = \langle \nabla(\sigma \circ F_{i,1}), W_\sigma(1) \rangle_{L^2(TM,d\mu_{i,1})} = \langle \nabla \sigma, (dF_{i,1})^{-1} W_\sigma(1) \rangle_{L^2(TM,d\mu_{i,0})} = 0 - \langle \nabla \sigma, \nabla (dF_{i,1})^{-1} W_\sigma(1) \rangle_{L^2(TM,d\mu_{i,0})} \geq 1 - \text{const.} \| \text{Hess}(\phi_i(\cdot)) \|_{L^\infty([0,1] \times M)},
\]

for large \( Q \). If \( Q \) is sufficiently large then this is a contradiction, so \( A_i \) is injective. \( \square \)

Fix \( \mathcal{V}_1 \in N_{c(1)} P(M) \). If \( \mathcal{V}_1 \neq 0 \) then after normalizing, we may assume that it has unit length. For \( Q \in \mathbb{Z}^+ \) large and \( t \in [0,1] \), define \( \mathcal{V}_Q(t) \in N_{c(t)} P(M) \) as follows. First, using Lemma 5.19 find \( \sigma_{Q-1} \) so that \( \mathcal{V}_1 = L_{\sigma_{Q-1}}(1) \). For \( t \in \left[ \frac{Q-1}{Q}, 1 \right] \), put

\[
(5.25) \quad \mathcal{V}_Q(t) = L_{\sigma_{Q-1}}(Qt - (Q - 1)).
\]

Doing backward recursion, starting with \( i = Q - 2 \), using Lemma 5.19 we find \( \sigma_i \) so that \( L_{\sigma_i}(1) = L_{\sigma_{i+1}}(0) = \nabla \sigma_{i+1} \). For \( t \in \left[ \frac{i}{Q}, \frac{i+1}{Q} \right] \), put

\[
(5.26) \quad \mathcal{V}_Q(t) = L_{\sigma_i}(Qt - i).
\]

Decrease \( i \) by one and repeat. The last step is when \( i = 0 \).
From the argument in [15] Lemma 1.8,
\[
\lim_{Q \to \infty} \sup_{t \in [0,1]} \|V_Q(t)\| - 1 = 0.
\]

We note that the proof of [15, Lemma 1.8] only uses results about geodesics in Alexandrov spaces, it so applies to our infinite-dimensional setting. It also uses the assumption that \(c\) lies in the interior of a minimizing geodesic. After passing to a subsequence, we can assume that
\[
\lim_{Q \to \infty} (V_Q(0), V_Q(1)) = (V_\infty, V_{\infty,0}, V_{\infty,1})
\]
in the weak topology on \(L^2([0,1]; L^2(TM, d\mu_t)) \oplus L^2(TM, d\mu_0) \oplus L^2(TM, d\mu_1)\). Note that \(V_{\infty,1} = V_1\).

From (5.9), (5.15) and (5.18), for a fixed \(f \in C^\infty([0,1]; C^\infty(M))\), on each interval \(\left[\frac{i}{Q}, \frac{i+1}{Q}\right]\)
we have
\[
\frac{d}{dt} \int_M \langle V_f, V_Q \rangle \ d\mu_t = \int_M \left\langle \nabla \frac{\partial f}{\partial t}, V_Q(t) \right\rangle \ d\mu_t + \int_M \text{Hess}(f)(\nabla \phi(t), V_Q(t)) \ d\mu_t + o(Q).
\]

It follows that \((V_\infty, V_{\infty,0}, V_{\infty,1})\) is a weak solution of the parallel transport equation. As the limiting vector fields are gradient vector fields, we can write \((V_\infty, V_{\infty,0}, V_{\infty,1}) = (\nabla \eta_\infty, \nabla \eta_{\infty,0}, \nabla \eta_{\infty,1})\) for some \((\eta_\infty, \eta_{\infty,0}, \eta_{\infty,1}) \in L^2([0,1]; H^1(M, d\mu_t)) \oplus H^1(M, d\mu_0) \oplus H^1(M, d\mu_1))\).

Suppose that \(c\) is a smooth geodesic in \(P^\infty(M)\), that \(V_1\) (and hence \(\eta_{\infty,1}\)) is smooth and that there is a smooth solution \(\eta\) to the parallel transport equation (4.6) with \(\nabla \eta(1) = \nabla \eta_{\infty,1}\). By Lemma 4.17 \(\|\nabla \eta(t)\|\) is independent of \(t\). By Lemma 4.18 \((\nabla \eta_\infty, \nabla \eta_{\infty,0}, \nabla \eta_{\infty,1}) = (\nabla \eta, \nabla \eta(0), \nabla \eta(1))\). We claim that
\[
\lim_{Q \to \infty} (\nabla \eta_Q, \nabla \eta_Q(0), \nabla \eta_Q(1)) = (\nabla \eta, \nabla \eta(0), \nabla \eta_{\infty,1})
\]
in the norm topology on \(L^2([0,1]; L^2(TM, d\mu_t)) \oplus L^2(TM, d\mu_0) \oplus L^2(TM, d\mu_1)\). This is because of the general fact that if \(\{x_i\}_{i=1}^\infty\) is a sequence in a Hilbert space \(H\) with \(\lim_{i \to \infty} |x_i| = 1\), and there is some unit vector \(x_\infty \in H\) so that every weakly convergent subsequence of \(\{x_i\}_{i=1}^\infty\) has weak limit \(x_\infty\), then \(\lim_{i \to \infty} x_i = x_\infty\) in the norm topology.

In particular,
\[
\lim_{Q \to \infty} \nabla \eta_Q(0) = \nabla \eta(0)
\]
in the norm topology on \(L^2(TM, d\mu_0)\).

This proves Theorem 1.4.

**Remark 5.32.** The construction of parallel transport in [1, Section 5] and [7, Section 4] is also by taking the limit of an iterative procedure. The underlying logic in [1, 7] is different than what we use, which results in a different algorithm. The iterative construction in [1, 7] amounts to going forward along the curve \(c\) applying certain maps \(P_i\), instead of going backward along \(c\) using the inverses of the \(A_i\)'s as we do. In the case of \(\mathbb{R}^n\), the map \(P_i\) is the same as \(A_i\), but this is not the case in general. The map \(P_i\) is nonexpanding, which
helps the construction in [17]. In contrast, $A_i^{-1}$ is not nonexpanding. In order to control its products, we use the result (5.27) from [15].

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