Spontaneous Collapse of Unstable Quantum Superposition State: A Single-Particle Model of Modified Quantum Dynamics

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We propose a modified dynamics of quantum mechanics, in which classical mechanics of a point mass derives intrinsically in a massive limit of a single-particle model. On the premise that a position basis plays a special role in wavefunction collapse, we deduce to formalize spontaneous localization of wavefunction on the analogy drawn from thermodynamics, in which a characteristic energy scale and a time scale are introduced to separate quantum and classical regimes.

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There is a significant recent increase of interest in the foundations of quantum mechanics (QM), owing to the technological progress particularly in the field of quantum optics to investigate individual stochastic behaviors of quantum systems. Nevertheless, it would be fair to state that conceptual and philosophical difficulties of QM are still left behind persistently. Indeed, we are so accustomed to the classical notion of definite position in real life that we can scarcely imagine what a superposition of macroscopic states localized at different positions would look like. On the one hand, there are many who advocate that such a problematic state must be excluded in terms of a proper interpretation adapted to within the present formalism of QM. On the other hand, there are researchers questing for the clearcut solution of the so-called “measurement” problem on the basis of a realistic viewpoint, by modifying the dynamical formalism of QM

\[ \Psi(r) \equiv \langle r \rangle \]

From the latter viewpoint, the aim of this paper is to make such an attempt to formalize wavefunction collapse without being involved in many-particle features of a macroscopic body, so that the conceptual gap between classical and quantum mechanics may be bridged mathematically within a single-particle formalism.

Ghirardi, Rimini and Weber (GRW) made a breakthrough with a many-particle model postulating discontinuous collapse, which was soon developed into a continuous model by Pearle. Our proposal presented below has similar physical implications as the original discontinuous GRW model, where also two parameters are introduced. In the GRW model, a many-particle wavefunction is subjected to spontaneous localizations in a multidimensional configuration space of the micro-constituents, of which the frequency and the localization function, or the localization length, are postulated to be universal, \( \lambda \approx 10^{-15} \text{sec}^{-1} \) and \( a \approx 10^{-5} \text{cm} \), respectively. The GRW localization mechanism is such that the collapse rate effectively increases as the number \( N \) of the constituents increases, so that a macroscopic object comprising an Avogadro number of constituents collapses extremly rapidly, at a rate of \( N \lambda \approx 10^7 \text{sec}^{-1} \). In contrast, we shall postulate the universal collapse rate \( \gamma_0 = \tau_0^{-1} \) independently of system size. On the other side, instead of positing the universal length, we scheme that the localization length scale for a given particle state is variationally determined case by case by the mass \( m \) of the particle, an external potential, and another constant \( T_0 \) of the dimensions of energy. In this respect, our proposal may seem more complicated than the GRW model. However, we argue that the complication is compensated by our simple premise that, unlike the previous works, we intend to cope with the measurement problem without resort to many-particle treatment. In effect, we regard the problem essentially as a single-particle problem of the modified dynamics. This must be the simplest and non-trivial option to attack the problem, which however seems not to be undertaken thoroughly so far. Specifically, in essence, we shall attach no fundamental relevance to the number \( N \) of particles comprising a system. Therefore, we focus directly and solely on the effective single-particle model proper, of which we still stress the wide potential applicability to offset the primitive status of the present proposal compared with the highly developed many-particle treatments.

To begin with, the special status we assign below (in \( \Psi(r) \)) to the position variable \( r \) of wavefunction \( \Psi(r) \) is apparent from its role in the classical limit. According to Ehrenfest’s theorem, the Newton equation

\[
\frac{d^2 \bar{r}}{dt^2} = -\nabla V(\bar{r})
\]

for the point mass \( \bar{r} \equiv \langle r \rangle \) derives from the Schrödinger equation

\[
i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi = \left(-\frac{\hbar^2}{2m} \nabla^2 + V(r)\right) \Psi,
\]

in the limit \( (\Delta r)^2 = \langle r^2 \rangle - \bar{r}^2 \to 0 \). However, even if the limit should hold true at a moment, it cannot remain...
so for ever according to (2), or we cannot justify the general validity of (1) on the basis of (2) alone. Hence, we stipulate a modified intrinsic dynamics of $\Psi(r)$ to keep $|\delta r|$ finite so that classical mechanics (1) of the mass point $r(t)$ derives approximately and holds for good. Here, as a premise, we do not resort to any other system, degrees of freedom, and whatever cause but collapse to establish. It is stressed that a matter of our interest is to follow time development of a particular system individually (instead of statistically).

In order to describe a mode of wavefunction collapse, we introduce a set of $N$ real functions $P_n(r)$ ($n = 1, 2, \cdots, N$) corresponding to the $N$ potential results of collapse. Given the normalized wavefunction $\Psi_t(r)$ of a particle at time $t$,

$$\int dr \rho(r) = 1, \quad \rho(r) = |\Psi_t(r)|^2,$$

our idea is to fix the localization functions $P_n(r)$ so as to minimize the spatial extension $(\delta r)^2$ of the ensemble $\Phi_n(r) \propto P_n \Psi$ realized by the spontaneous collapse. There are various ways to express this mathematically. Postulating that the collapse outcomes are given by

$$\Phi_n = \frac{1}{\sqrt{w_n}} P_n \Psi_t,$$

we propose that $P_n(r)$ are determined so as to maximize

$$S' = \sum_n w_n \int |\Phi_n|^2 (\log |\Phi_n|^2 - 1) dr,$$

under the constraint

$$T_0 \Delta S = \Delta E,$$

where $\Delta S$ and $\Delta E$ represent the physical changes of entropy and energy accompanied by the collapse $\Psi \rightarrow \Phi_n$, which, from our standpoint, is regarded as a real process of nature. In (7), we introduced the model parameter $T_0$ of the dimensions of energy. The functions $P_n$ should be fixed variationally under the constraint

$$\sum_n P_n(r)^2 = 1, \quad 0 \leq P_n(r) \leq 1.$$

Thus we divide the whole space $r$ into numbers of patches indexed by $n$, in one of which the particle is found as a result of spontaneous collapse. The entropy $S = -\text{Tr}(\rho \log \rho)$ and the energy $E = \text{Tr}(\rho \hat{H})$ to be used for (7) are customarily defined in terms of the density matrix $\rho$ of the collapse outcomes,

$$\langle r|\rho|r'\rangle = \sum_n w_n \Phi_n(r)^* \Phi_n(r').$$

In particular, for $\hat{H}$ defined in (2), we obtain

$$\Delta E = \frac{\hbar^2}{2m} \int dr \rho(r) \sum_n (\nabla P_n(r))^2 > 0. \quad (10)$$

In terms of $\{P_n\}$, and a constant $\gamma_0$ of the dimensions of frequency, we postulate the following modified dynamics to relate $\Psi_{t+\Delta t}(r)$ with $\Psi_t(r)$, where $\Delta t$ is an infinitesimal time interval. The continuous deterministic evolution (i) applies only with probability $1 - \gamma_0 \Delta t$, while the discontinuous stochastic evolution (ii) or (ii') applies with probability $\gamma_0 \Delta t$: (i) The Schrödinger time evolution:

$$\Psi_{t+\Delta t} = \left(1 - \frac{1}{\hbar} \hat{H} \Delta t \right) \Psi_t. \quad (11)$$

(ii) Stochastic collapse: If $\{P_n\}$ is uniquely determined, then the $n$-th result from among the $N$ possibilities (1) is realized with the probability $w_n$ (the absolute probability $w_n \gamma_0 \Delta t$),

$$\Psi_{t+\Delta t} = \Phi_n. \quad (12)$$

(ii') If the set $\{P_n\}$ is not uniquely determined from the given $\rho(r)$, one set is chosen at random with equal a priori probability, then apply (ii). We have the two parameters $T_0$ and $\tau_0 \equiv \gamma_0^{-1}$ to characterize the present collapse model. One can take the limit $\Delta t \rightarrow 0$ without changing the physical consequences, viz., the temporal sequence of $\Psi_t(r)$. Note that the probability to keep following (i) during $N$ steps of $\Delta t = \tau/N$ is given by $(1 - \frac{\gamma_0 \tau}{N})^N \rightarrow \exp (-\gamma_0 \tau)$ as $N \rightarrow \infty$. Accordingly, the quantum state $\Psi(r)$ is always subjected to spontaneous collapse with the constant decay rate $\gamma_0$. In particular, we recover (2) in the limit $\gamma_0 \rightarrow 0$, irrespective of $T_0$. It may be of interest to note the methodological similarity of the present dual dynamics (i) and (ii) with the so-called Monte Carlo wavefunction approach to solve the optical Bloch equations (3-10), in which the decay rate of an unstable state plays the counterpart of our $\gamma_0$. The latter approach follows the conventional procedure to invoke “measurement” to realize collapse.

The probability $w_n$ defined in (6) adds up to unity, $\sum_n w_n = 1$, owing to (8) and (9), while the collapse outcomes $\Phi_n(r)$ are normalized by $w_n$ in (4), $\int dr |\Phi_n(r)|^2 = 1$. In general, however, the outcomes $\Phi_n$ are not orthogonal, or $P_n(r) P_m(r) \neq 0$ for $n \neq m$. In the special case of the strict localization by the step function $P_n(r)^2 = P_n(r)$, for which $P_n(r) = 0$ or 1, the postulate (6) reproduces the conventional Born formula, $w_n = |\langle \Phi_n |\Psi \rangle|^2 = \int P_n(r) = 1 \int dr |\Psi(r)|^2$. Consequently, Born's probability rule is reproduced approximately when we find $P_n(r) P_m(r) \approx 0$ for $n \neq m$ (cf. below (17)).

It is always possible to find a non-trivial set $\{P_n\}$ for arbitrary $\rho(r)$. To give a simple example, one may assume the trial functions

$$P_\pm(r)^2 = \frac{1}{2} \left(1 \pm \tanh \left|\langle r|\cdot|X\rangle\rangleight|\right) \quad (13)$$
to satisfy \(\mathbf{S}\), fix the direction of the unit vector \(\mathbf{i}\), and vary the parameters \(X\) and \(t\) to maximize \(\mathbf{S}' \simeq \log |\delta \mathbf{r}|^{-3}\) under the condition \(\mathbf{R}\). Thus, in principle, there is no problem to implement our modified dynamics for numerical use in place of the collapse-free Schrödinger equation \(\mathbf{E}\). With respect to the case \((\mathbf{ii}')\), we have in mind the special situations where symmetry possessed by \(\mathbf{H}\) and \(\rho(\mathbf{r})\) is spontaneously broken by collapse, or by a specific choice of \(\{\rho_n(\mathbf{r})\}\). This holds, for example, in the reduction of a spherically symmetric outgoing wave \(\mathbf{W}(\mathbf{r})\) under the spherically symmetric potential \(V(|\mathbf{r}|)\).

The above model is not so artificial as it may first look. In fact, the condition \(\mathbf{R}\) as well as \((\mathbf{ii}')\) suggest themselves naturally on the analogy of the second law of thermodynamics. It is the established fact that a thermodynamic state immersed in the heat bath at temperature \(T_0\) suffers the spontaneous irreversible change under the condition \(T_0 \Delta S \geq \Delta E\). Although there is no logical necessity that the law governing the microscopic irreversible process of our concern, wavefunction collapse, must bear resemblance to that of thermodynamics, we argue that the proposed postulates would be justified if only by relying on the analogy, and the consequences must be worth close inspection. In short, the postulates are physically interpreted as follows: Wavefunction collapse tends to reduce the spatial extent \(|\delta \mathbf{r}|^2\) of the wavepacket at intervals (spontaneous localization\(\mathbf{F}\)), while respecting the “thermodynamic” constraint \(\mathbf{R}\). To help understand how \((\mathbf{ii}')\) works, some examples of \(\rho(x)\) and \(P_n(x)^2\) are schematically displayed in Fig. \(\mathbf{F}\).

Without detailed calculation, we can make an order estimate of how small \(|\delta \mathbf{r}|^2\) is reduced. According to \(\mathbf{R}\), even in the absence of the external potential \(V(\mathbf{r}) = 0\), the energy scale \(T_0\) introduces a length scale

\[
\lambda_0 \equiv \hbar \sqrt{\frac{2\pi}{mT_0}},
\]

which is nothing but the thermal de Broglie wavelength at “temperature” \(T_0\). Accordingly, we obtain the finite wavepacket width of order \(|\delta \mathbf{r}| \simeq \lambda_0 \propto m^{-1/2}\), which brings about our purpose to reproduce classical mechanics of the point mass \(\mathbf{r}(t)\) in the limit \(\lambda_0 \to 0\), or \(m \to \infty\). The finite wavepacket width \(\mathbf{F}\) from \(\mathbf{E}\) is just as expected by the thermodynamic analogy mentioned above. In effect, the degree of localization \(|\delta \mathbf{r}|^2\) depends on the energy \(\Delta E > 0\) required for the localization. If it were not for the constraint \(\mathbf{F}\), or if we let \(T_0 \to \infty\), the hypothetical spontaneous collapse would reduce \(|\delta \mathbf{r}|^2\) without limit, and the established results of quantum mechanics at a short length scale must be spoiled altogether. Thus, in the present model, the crossover length scale presumed between classical and quantum regimes is regulated by \(T_0\) in a controlled and universal manner.

To put it concretely, in order to guarantee \(\lambda_0 > 10^9\) for electron while \(\lambda_0 < 1\mu\text{m}\) for a tiny particle of a nanogram (either too large or small for the crossover regime to be detected unexpectedly), we estimate

\[
10^{-49} \text{ J} < T_0 < 10^{-39} \text{ J}.
\]

Without being affected by the collapse, the wavepacket evolves according to \(\mathbf{E}\) for the duration of order \(\tau_0\). In the meantime, the width \(|\delta \mathbf{r}|^2 \simeq \lambda_0^2\) grows up to \(\lambda_0^2 (1 + (T_0/4\pi)^2)\), so that the second length scale must be introduced if \(T_0\tau_0 \gg 4\pi\hbar\). However, with the above estimate taken for granted, this limit must be rejected physically, because we hardly accept \(\tau_0\) as long as \(\hbar/T_0 > 10^9\) second on physical grounds. Hence we assume the limit

\[
T_0\tau_0 \ll 4\pi\hbar
\]

without invalidating the model, and consequently, the single length scale \(\mathbf{F}\) in free space. Specifically, the collapse time \(\tau_0\) must be longer than characteristic time scales of microscopic phenomena because the postulated collapse must more or less affect conventional quantum results following from the Schrödinger equation. At the same time, \(\tau_0\) must be short enough to ensure that we have never observe such a problematical superposition of the “classical” states with “distinct” spatial configurations. Therefore, physically, it is the vital point of the model that experiments do not exclude the non-trivial scale \(\tau_0\). By way of illustration, the density profile \(\rho(x, t)\) of a free wavepacket expected in the modified dynamics is schematically displayed in Fig. \(\mathbf{F}\).

Let us discuss a particle under the external potential, where another length scale is brought about by the potential \(V(\mathbf{r})\). In the limit \(m \to 0\) where \(\lambda_0\) exceeds the length scale, the non-trivial effects depending on \(T_0\) become negligible. This must be the case for well-established quantum mechanical phenomena of microscopic systems where wavefunction collapse is not apparently resorted to for their explanation. For example, for the atom with \(V(\mathbf{r}) = -e^2/\mathbf{r}\), we have the ground state with \(|\delta \mathbf{r}| \simeq a_B = \hbar^2/me^2\), the Bohr radius. It is straightforward to check that the collapse \((\mathbf{ii})\) will not affect the bound state \(\mathbf{W}(\mathbf{r})\) in the limit \(|\delta \mathbf{r}| \simeq a_B \ll \lambda_0\).
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