Dominant property for the Bel-Robinson tensor and tensor $S$

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Abstract

The Bel-Robinson tensor contains many nice mathematical properties and its dominant energy condition is desirable for describing the positive gravitational energy. The dominant property is a basic requirement for the quasi-local mass, i.e., in small sphere limit. We claim that there exists another option, a linear combination between the Bel-Robinson tensor $B$ and tensor $S$, which contributes the same dominant property. Moreover, using the 5 Petrov types as the verification, we found that this dominant property justification for the Bel-Robinson tensor can be simplified as examining $B_{0000} \geq |B_{0\alpha\beta\gamma}|$ and $B_{0000} \geq |B_{0123}|$, instead of $B_{0000} \geq |B_{\alpha\beta\lambda\sigma}|$ for all $\alpha, \beta, \lambda, \sigma = 0, 1, 2, 3$.

1 Introduction

Gravitational energy cannot be localized at a point since it is forbidden by the equivalence principle. However, the quasi-local method (i.e., small sphere) can solve out this difficulty. For describing the gravitational energy, Bel and Robinson [2, 3, 4, 5] proposed a tensor that is positive definite and satisfies the dominant property [6]. This dominant property is a relevant requirement for the quasi-local mass. The Bel-Robinson tensor also possesses other nice properties: completely symmetric, traceless and divergence free.

The quasi-local mass has been studied for a long time. There were many people attempted to give a definition for this subject: Harking [7], Penrose [8], Brown and York [9], etc. Recently, Wang and Yau [10] proposed certain requirements and one of them is the Bel-Robinson tensor in vacuum. Indeed, in order to obtain this positivity in small sphere, it is believed that it should be proportional to the Bel-Robinson tensor [11]. However, would it be the only choice? We claim that there exists another option, a linear combination between the Bel-Robinson tensor $B$ and tensor $S$, which contributes the same dominant property (i.e., see [5]).

In principle, because of the symmetry, the Bel-Robinson tensor contains 35 components. Using the 5 Petrov types [12] as the verification, we observe that the dominant property justification for $B$ can be simplified as examining $B_{0000} \geq |B_{0\alpha\beta0}|$ and $B_{0000} \geq |B_{0123}|$, instead of $B_{0000} \geq |B_{\alpha\beta\lambda\sigma}|$ for all $\alpha, \beta, \lambda, \sigma = 0, 1, 2, 3$. Because all components contain in $B$ can be written in terms of $B_{\alpha\beta0}$ and $B_{0123}$.

2 Dominant energy condition for $B + sS$

Analogy with the theory of electrodynamics, the Bel-Robinson tensor is defined as

$$B_{\alpha\beta\lambda\sigma} := C_{\alpha\xi\lambda\kappa} C_{\beta\xi\sigma}^{\kappa} + *C_{\alpha\xi\lambda\kappa} *C_{\beta\xi\sigma}^{\kappa},$$

(1)

where $C_{\alpha\beta\mu\nu}$ is the Weyl conformal tensor and its dual $*C_{\alpha\beta\mu\nu} = \frac{1}{2} \epsilon_{\alpha\beta\lambda\sigma} C_{\lambda\sigma \mu\nu}$ [13]. As the Weyl tensor and Riemann tensor are equivalent in vacuum, the energy density for this Bel-Robinson tensor becomes

$$B_{\alpha\beta\lambda\sigma} t^\alpha t^\beta t^\lambda t^\sigma = E_{ab} E^{ab} + H_{ab} H^{ab},$$

(2)

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which is non-negative and \( t^\alpha \) is the timelike unit normal. Here the Greek letters refer to spacetime and Latin stand for space. In vacuum, the Bel-Robinson tensor and tensor \( S \) can be defined as follows

\[
\begin{align*}
B_{\alpha\beta\lambda\sigma} &:= R_{\alpha\xi\lambda\sigma} R^{\xi\kappa} R_{\beta\sigma\kappa} R^{\kappa\lambda} - \frac{1}{8} g_{\alpha\beta} g_{\lambda\sigma} R^2_{\rho\tau \mu \nu}, \\
S_{\alpha\beta\lambda\sigma} &:= R_{\alpha\lambda\xi\gamma} R^{\xi\kappa} R_{\beta\sigma\kappa} R^{\kappa\lambda} + \frac{1}{4} g_{\alpha\beta} g_{\lambda\sigma} R^2_{\rho\tau \mu \nu},
\end{align*}
\]

where \( R^2_{\rho\tau \mu \nu} = R_{\rho\tau \mu \nu} R^{\rho\tau \mu \nu} \) is the Kretschmann scalar. The symmetric property for \( S \) is

\[
S_{\alpha\beta\lambda\sigma} = S_{(\alpha\beta)(\lambda\sigma)} = S_{\lambda\sigma\alpha\beta}.
\]

It is known that the Bel-Robinson tensor possesses the dominant property

\[
B_{\alpha\beta\lambda\sigma} u^\alpha v^\beta w^\lambda z^\sigma \geq 0,
\]

where \( u, v, w, z \) are any future-pointing causal vectors. This dominant energy condition is significant for defining the quasi-local mass [10]. Here we propose another option, a linear combination between \( B \) and \( S \) such that it also possesses the dominant property:

\[
(B_{\alpha\beta\lambda\sigma} + s S_{\alpha\beta\lambda\sigma}) u^\alpha v^\beta w^\lambda z^\sigma \geq 0,
\]

where \( s \) is a non-zero small constant.

For the dominant property (i.e., dominant super-energy condition), Senovilla proposed a definition (see Lemma 4.1 of [6]): “If a tensor \( T_{\mu_1...\mu_s} \) satisfies the dominant super-energy property, then \( T_{\mu_1...\mu_s} \geq |T_{\mu_1...\mu_s}|, \forall \mu_1, \ldots, \mu_s = 0, \ldots, n-1 \) in any orthonormal basis \( \{ e_\nu \} \).” For example, using the 5 Petrov types as the examination, the Bel-Robinson tensor fulfills \( B_{0000} \geq |B_{\alpha\beta\lambda\sigma}| \) requirement. Likewise, for \( B + s S \) and we found there exists a non-zero small \( s \) such that

\[
B_{0000} + s S_{0000} \geq |B_{\alpha\beta\lambda\sigma} + s S_{\alpha\beta\lambda\sigma}|.
\]

Thus, we suggest that the quasi-local mass should include this extra candidate \( B + s S \) in small sphere. Referring to Szabados’s argument [11], “Therefore, in vacuum in the leading \( r^5 \) order any coordinate and Lorentz-covariant quasi-local energy-momentum expression, which is non-spacelike and future pointing must be proportional to the Bel-Robinson ‘momentum’ \( B_{\beta\lambda\sigma\alpha} t^\beta t^\lambda t^\sigma \).” We claim that \( B + s S \) is not only satisfy the causal, but also the dominant property. However, there is a disadvantage for \( B + s S \) because we need to check \( s \) in every physical system. Nevertheless, the advantage for \( B + s S \) gives a relaxation opportunity since obtaining the pure Bel-Robinson tensor for a quasi-local expression is not easy.

Here we consider the total energy-momentum complex which accurate to zeroth order in matter and second order in empty spacetime

\[
T^\alpha_{\beta} = T^\alpha_{\beta} + t^\alpha_{\beta\lambda\sigma} x^\lambda x^\sigma,
\]

where \( T^\alpha_{\beta} \) is the stress tensor and \( t^\alpha_{\beta\lambda\sigma} \) is the gravitational pseudotensor. Note that there are 2 free indices in \( T^\alpha_{\beta} \). Confining within the small sphere region, \( t^\alpha_{\beta\lambda\sigma} x^\lambda x^\sigma \) satisfies the divergence free condition [15]: \( \partial_\alpha (t^\alpha_{\beta\lambda\sigma} x^\lambda x^\sigma) = 0 \). The gravitational energy-momentum in small sphere is

\[
\int_V t^\alpha_{\beta\lambda\sigma} x^\lambda x^\sigma d^3x = \frac{4\pi}{15} (t^\alpha_{\beta\lambda\sigma} \eta^\lambda_\sigma + t^\alpha_{\beta000}) r^5,
\]

where we used the spherical coordinates and allow the time component be constant for simplicity. Indeed \( T^\alpha_{\beta} \) is symmetric in \( \alpha, \beta \). Moreover, the dominant energy condition confined in small sphere limit is

\[
t^\alpha_{\beta000} u^\alpha v^\beta \geq 0,
\]
where \( t_{\alpha\beta\lambda\sigma} \eta^{\lambda\sigma} \) is an arbitrary constant according to the symmetry. If \( t \) is replaced by \( B \) and \( B + sS \) respectively, we have the simplified dominant property representation:

\[
B_{\alpha\beta\rho\sigma} u^\alpha u^\beta \geq 0, \quad (B_{\alpha\beta\rho\sigma} + sS_{\alpha\beta\rho\sigma}) u^\alpha u^\beta \geq 0. \tag{10}
\]

The second inequality is valid for a suitable non-zero small \( s \).

What is the criterion for selecting the small \( s \)? Here we give a concrete example by using an isotropic Schwarzschild line element in polar coordinates

\[
ds^2 = -(1 - 2Mr^{-1})dt^2 + (1 - 2Mr^{-1})^{-1}dr^2 + r^2(d \theta^2 + \sin^2 \theta d \phi^2),
\]

with the assumption that \( Mr \ll 1 \), both the gravitational constant \( G \) and speed of light \( c \) are unity. For simplicity, using the orthonormal basis, there are only three non-vanishing components \((E_{11}, E_{22}, E_{33}) = (-2, 1, 1)Mr^{-3}\). The value for the quadratic scalar is \( R^2_{\rho\tau\lambda\sigma} = 48M^2r^{-6} \). The non-vanishing components for \( B \) and \( S \) are

\[
(B_{0000}, B_{0011}, B_{0022}, B_{0033}) = (6, -2, 4, 4)M^2r^{-6},
\]

\[
(B_{1111}, B_{2222}, B_{3333}, B_{1122}, B_{1133}, B_{2233}) = (6, 6, 6, -4, -4, -2)M^2r^{-6},
\]

\[
(S_{0000}, S_{0011}, S_{0022}, S_{0033}) = (12, -28, -16, -28)M^2r^{-6},
\]

\[
(S_{0101}, S_{0202}, S_{0303}, S_{1111}, S_{2222}, S_{3333}) = (8, 2, 2, 12, 12, 24)M^2r^{-6},
\]

\[
(S_{1122}, S_{1133}, S_{2233}, S_{1212}, S_{1313}, S_{2323}) = (16, 16, 28, 28, -2, -2, -8)M^2r^{-6}. \tag{12}
\]

Obviously, the Bel-Robinson tensor fulfills the dominant energy condition. Similarly, we find that \((B + sS)\) satisfies the dominant property requires \(s \in \left[ -\frac{1}{14}, \frac{1}{3} \right]\). In particular, for the Landau-Lifschitz (LL) pseudo-tensor \([16]\), evaluated in the Riemann normal coordinates, satisfies the dominant property:

\[
\partial^2 \alpha^\beta_{\mu\nu} = \frac{1}{2} \left( 7B^\beta_{\mu\nu} + \frac{S^\beta_{\mu\nu}}{2} \right). \tag{13}
\]

### 3 Conclusion

The Bel-Robinson tensor \( B \) has the dominant energy condition and this is a requirement for describing the quasi-local mass in small sphere. We discovered that there exists an opportunity tensor \( B + sS \) such that this combination also contributes the same dominant property. As it is not easy for achieving a multiple of the pure Bel-Robinson tensor in quasi-local expression, then \( B + sS \) provides a relaxation opportunity for the dominant energy condition. Moreover, we also pointed out that the examination for the dominant property can be simplified for the Bel-Robinson tensor. Using the 5 Petrov types, instead of verifying \( B_{0000} \geq |B_{\alpha\beta\lambda\sigma}| \) for all \( \alpha, \beta, \lambda, \sigma = 0, 1, 2, 3 \), it is enough to check \( B_{0000} \geq |B_{00\alpha\beta}| \) and \( B_{0000} \geq |B_{0123}| \).

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