SOME ROBUST IMPROVED GEOMETRIC AGGREGATION OPERATORS UNDER INTERVAL-VALUED INTUITIONISTIC FUZZY ENVIRONMENT FOR MULTI-CRITERIA DECISION-MAKING PROCESS

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ABSTRACT. The objective of this manuscript is to present some new interactive geometric aggregation operators for the interval-valued intuitionistic fuzzy numbers (IVIFNs). In order to achieve it, firstly the shortcomings of the existing operators have been highlighted and then resolved it by defining new operational laws based on the pairs of hesitation degree between the membership functions. By using these improved laws, some geometric aggregation operators, namely interval-valued intuitionistic fuzzy Hamacher interactive weighted and hybrid geometric labeled as IIFHIIWG and IIFHIHGR operators, respectively have been proposed. Furthermore, desirable properties corresponding to these operators have been stated. Finally, a decision-making method based on the proposed operator has been illustrated to demonstrate the approach. A computed result is compared with the existing results.

1. Introduction. Intuitionistic fuzzy set (IFS) [2] and interval-valued intuitionistic fuzzy set (IVIFS) [1] theories are one of the most successful theory to handle and describe the uncertainties in the data in terms of defining their membership and non-membership grades corresponding to each element in the universe of discourse. Since then, these theories have been widely used by the researchers in the different disciplines to handle the uncertainties [6, 8, 11, 17, 23, 28, 29]. With the growing complexities of the systems day-by-day, it is difficult, if not impossible, to the decision maker to make a correct decision within a reasonable time. However, the key issue in the decision making process is to find the suitable attribute weights and the process to aggregate the decision makers’ (DM) preferences so as to make the right decision within a reasonable time. In order to achieve it, an information aggregation process is one of the important and interesting topic in the phase of the decision making and are receiving more and more attention. For instance, the weighted geometric aggregation operators under the IVIFS environment has been studied by Wei and Wang [26] by using algebraic norm operations while Wang and Liu [24] studied it by using Einstein norm operations. Further, Garg [5] presented a generalized intuitionistic fuzzy interactive geometric aggregation operators using...
Einstein t-norm and t-conorm operations. Garg [13] proposed some series of interactive aggregations operators for IFSs. Garg [8] developed a new generalized improved score function for ranking the IVIFSs. Xu and Chen [27]; Xu [30] developed some averaging and geometric aggregation operators for aggregating the interval-valued intuitionistic fuzzy information while Liu [20] presented an aggregation operator based on Hamacher operations on IVIFSs and their corresponding applications to the decision making problems. Later on, Garg [7, 9] extended the aggregation operators under IFS environment to the Pythagorean fuzzy set (PFSs) environment and presented a generalized geometric as well as averaging aggregation operators for solving the decision-making problems. Garg [10, 12], further, presented an novel accuracy function as well as the correlation coefficient measures for the PFSs, respectively. Apart from that, various researchers have presented the different types of an aggregation operators under IFSs and/or IVIFSs environment [3, 4, 14, 15, 18, 19, 21, 22, 25, 31, 32, 33, 34].

Since, the above work on the aggregation operators are widely used by the researcher but in some situations, they are not capable not capable to be used for most of the purposes. For instance, consider the two IVIFSs such that membership degrees of any one of the set is zero then by the multiplication operations of IVIFSs we get the aggregated membership degree is zero. From it, we concluded that other degrees of the membership functions does not play any significant role during the aggregation process. Similarly, if we consider any one of the degree of non-membership function is zero then by addition operations between them, we get the overall degree of non-membership value is zero which means that other degrees have no effects on the aggregation process. Therefore, it has been concluded that there exists an undesirable feature of the existing operators. Furthermore, the operation laws of IVIFSs does not contain any dependency relation between the degree of the membership and non-membership functions. Thus, the existing operators are sometimes in capable to rank the alternative and hence there is a need to enhance these operators. Therefore, in this article, we focus on developing some new operations laws and their corresponding aggregation operators based on IVIFSs for overcoming the shortcoming of the existing laws.

Therefore, the objective of this work is to present a series of improved geometric aggregation operators under the IVIFSs environment. In order to design it, firstly the shortcoming of the existing operators has been tackled by defining new operational laws on IVIFSs by considering the pairs of hesitation degree between the membership functions. Based on these new laws, the main research contents are summarized as following three parts: (i) some geometric interactive weighted operators based on Hamacher norm operations; (ii) Hamacher geometric interactive hybrid weighted operators; (iii) group decision making method based on the above operators. In these aggregation operators, the impact of the membership function is more than the non-membership function which shows the nature of decision making to be optimistic. Finally, a multi-criteria decision-making (MCDM) method has been presented based on these operators for finding the best alternative(s).

The remainder of the paper is organized as follows. Section 2 describes the basic definition related to IVIFS and the existing aggregation operators along with their shortcomings. In section 3, new operational laws and their corresponding aggregation operators have been proposed. Some remarkable properties of these operators are also being investigated in it. In section 4, a method for solving MCDM problem has been presented where attribute values are represented in the
forms of IVIFSs. The proposed approach is illustrated with a numerical example in section 5. Finally, in section 6 the paper is ended up with some remarks.

2. Basic concepts on IVIFSs. In this section, some basic concepts on IVIFSs and their relevant terms have been defined.

2.1. Intuitionistic and interval-valued intuitionistic fuzzy set. An IFS \( A \) is defined as a set of ordered pairs over a universal set \( X \) given by [2]

\[
A = \{ (x, \mu_A(x), \nu_A(x)) \mid x \in X \}
\]

where \( \mu_A(x) \) and \( \nu_A(x) \) respectively the grades of membership and non-membership of an element \( x \) with the conditions that \( 0 \leq \mu_A(x), \nu_A(x) \leq 1 \), and \( \mu_A(x) + \nu_A(x) \leq 1 \), while an IVIFS is defined as [1]

\[
A = \{ (x, [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]) \mid x \in X \}
\]

where \( 0 \leq \mu_A^L(x) \leq \mu_A^U(x) \leq 1, 0 \leq \nu_A^L(x) \leq \nu_A^U(x) \leq 1 \) and \( \mu_A^L(x) + \nu_A^L(x) \leq 1 \). Clearly, for every \( x \in X \), if

\[
\mu_A(x) = \mu_A^L(x) = \mu_A^U(x), \quad \nu_A(x) = \nu_A^L(x) = \nu_A^U(x)
\]

then, IVIFS reduces to an IFS. For an IVIFS, this pair is often denoted by \( \alpha = \langle [a, b], [c, d] \rangle \), where \( [a, b] \subseteq [0, 1] \), \( [c, d] \subseteq [0, 1] \), and \( b + d \leq 1 \) is called interval-valued intuitionistic fuzzy number (IVFN). In order to rank the different IVFNs, score \( (S) \) and accuracy \( (H) \) functions [30] are defined as \( S(\alpha) = \frac{a+b-c-d}{2} \) and \( H(\alpha) = \frac{a+b+c+d}{2} \). Based on these functions, an order relation between two IVFNs, \( \alpha_1 \) and \( \alpha_2 \), are defined as follows [30, 31].

(i) If \( S(\alpha_1) > S(\alpha_2) \) then \( \alpha_1 \triangleright \alpha_2 \).

(ii) If \( S(\alpha_1) = S(\alpha_2) \) then

- If \( H(\alpha_1) > H(\alpha_2) \) then \( \alpha_1 \triangleright \alpha_2 \); 
- If \( H(\alpha_1) = H(\alpha_2) \) then \( \alpha_1 = \alpha_2 \).

2.2. Hamacher t-norm and t-conorm. Hamacher [16] proposed a generalized t-norm \( t(x, y) = \frac{xy}{x+(1-\gamma)(x+y-xy)} \) and t-conorm \( T(x, y) = \frac{x+y-xy-(1-\gamma)xy}{1-(1-\gamma)xy} \) and based on it, the arithmetic operations for two IVFNs \( \alpha_1 = \langle [a_1, b_1], [c_1, d_1] \rangle \) and \( \alpha_2 = \langle [a_2, b_2], [c_2, d_2] \rangle \) are defined as follows:

\[
\alpha_1 \oplus \alpha_2 = \left[ \begin{array}{c}
\frac{a_1 + a_2 - a_1 a_2 - (1-\gamma)a_1 b_2 + b_1 b_2 - b_1 b_2 - (1-\gamma)b_1 b_2}{1 - (1-\gamma)b_1 b_2}, \\
\gamma + (1-\gamma)(c_1 + c_2 - c_1 c_2), \\
\gamma + (1-\gamma)(d_1 + d_2 - d_1 d_2)
\end{array} \right]
\]

\[
\alpha_1 \otimes \alpha_2 = \left[ \begin{array}{c}
\frac{a_1 a_2}{b_1 b_2}, \\
\gamma + (1-\gamma)(a_1 + a_2 - a_1 a_2), \\
\gamma + (1-\gamma)(b_1 + b_2 - b_1 b_2)
\end{array} \right]
\]

Based on these operations, Liu [20] proposed different aggregation operators for aggregating the IVFNs \( \alpha_i \)'s by using weight vector \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) such that \( \omega_i > 0 \) and \( \sum_{i=1}^{n} \omega_i = 1 \) as
(a) The interval-valued intuitionistic fuzzy (IIF) Hamacher weighted geometric (IIFHWG) operator

\[
\text{IIFHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha_1^\omega_1 \otimes \alpha_2^\omega_2 \otimes \ldots \otimes \alpha_n^\omega_n
\]

\[
= \left\langle \frac{\gamma \prod_{i=1}^{n} a_i^\omega_i}{\prod_{i=1}^{n} (1 + (\gamma - 1)(1 - a_i))^\omega_i + (\gamma - 1) \prod_{i=1}^{n} a_i^\omega_i}, \frac{\gamma \prod_{i=1}^{n} b_i^\omega_i}{\prod_{i=1}^{n} (1 + (\gamma - 1)(1 - b_i))^\omega_i + (\gamma - 1) \prod_{i=1}^{n} b_i^\omega_i} \right\rangle,
\]

\[
= \left\langle \frac{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^\omega_i - \prod_{i=1}^{n} (1 - c_i)^\omega_i}{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^\omega_i + (\gamma - 1) \prod_{i=1}^{n} (1 - c_i)^\omega_i}, \frac{\prod_{i=1}^{n} (1 + (\gamma - 1)d_i)^\omega_i - \prod_{i=1}^{n} (1 - d_i)^\omega_i}{\prod_{i=1}^{n} (1 + (\gamma - 1)d_i)^\omega_i + (\gamma - 1) \prod_{i=1}^{n} (1 - d_i)^\omega_i} \right\rangle
\]

(b) IIF Hamacher hybrid weighted geometric (IIFHHWG) operator

\[
\text{IIFHHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \hat{\alpha}_{\sigma(1)}^\omega \otimes \hat{\alpha}_{\sigma(2)}^\omega \otimes \ldots \otimes \hat{\alpha}_{\sigma(n)}^\omega
\]

\[
= \left\langle \frac{\gamma \prod_{i=1}^{n} \hat{a}_{\sigma(i)}^\omega}{\prod_{i=1}^{n} (1 + (\gamma - 1)(1 - \hat{a}_{\sigma(i)})^\omega_i + (\gamma - 1) \prod_{i=1}^{n} \hat{a}_{\sigma(i)}^\omega_i}, \frac{\gamma \prod_{i=1}^{n} \hat{b}_{\sigma(i)}^\omega}{\prod_{i=1}^{n} (1 + (\gamma - 1)(1 - \hat{b}_{\sigma(i)})^\omega_i + (\gamma - 1) \prod_{i=1}^{n} \hat{b}_{\sigma(i)}^\omega_i} \right\rangle,
\]

\[
= \left\langle \frac{\prod_{i=1}^{n} (1 + (\gamma - 1)\hat{c}_{\sigma(i)})^\omega_i - \prod_{i=1}^{n} (1 - \hat{c}_{\sigma(i)})^\omega_i}{\prod_{i=1}^{n} (1 + (\gamma - 1)\hat{c}_{\sigma(i)})^\omega_i + (\gamma - 1) \prod_{i=1}^{n} (1 - \hat{c}_{\sigma(i)})^\omega_i}, \frac{\prod_{i=1}^{n} (1 + (\gamma - 1)\hat{d}_{\sigma(i)})^\omega_i - \prod_{i=1}^{n} (1 - \hat{d}_{\sigma(i)})^\omega_i}{\prod_{i=1}^{n} (1 + (\gamma - 1)\hat{d}_{\sigma(i)})^\omega_i + (\gamma - 1) \prod_{i=1}^{n} (1 - \hat{d}_{\sigma(i)})^\omega_i} \right\rangle
\]

where \( \hat{\alpha}_{\delta(i)} \) is the \( \delta \)th largest of the weighted intuitionistic fuzzy values \( \hat{\alpha}_i \) where \( \hat{\alpha}_i = \alpha_i^{\omega_i}, \ i = 1, 2, \ldots, n \).

2.3. Shortcoming of the existing operators. Based on the above operators, we have analyzed that they have some sort of deficiencies during aggregating the different preferences, which have been highlighted as follow:
Example 2.1. Consider a set of four IVIFNs $\alpha_1 = \langle [0, 0], [0.35, 0.40] \rangle$, $\alpha_2 = \langle [0.32, 0.37], [0.55, 0.61] \rangle$, $\alpha_3 = \langle [0.25, 0.28], [0.43, 0.47] \rangle$, and $\alpha_4 = \langle [0.65, 0.69], [0.21, 0.25] \rangle$ and $\omega = (0.2, 0.3, 0.4, 0.1)^T$ is the standardized weight vector of these numbers. Now, if we utilize IIFHWG operator to aggregate these IVIFNs corresponding the membership degrees of $\alpha_0$. Then, based on IIFHWG operator we get the aggregated IVIFN as $\langle \gamma_0, \beta_0 \rangle$. Hence, it has been seen that the degree of membership of aggregated number is zero due to zero membership degree of $\alpha_1$ and hence the other degree of membership plays an insignificant role during the aggregation process. In other words, there is no role of the membership degrees of $\alpha_2, \alpha_3, \alpha_4$ on the aggregation of these IVIFNs. Therefore, it has been concluded that the existing operator, as proposed by Liu [20] are invalid to rank the alternatives.

Example 2.2. Let $\alpha_1 = \langle [0.23, 0.27], [0.45, 0.48] \rangle$, $\alpha_2 = \langle [0.65, 0.69], [0.23, 0.29] \rangle$, $\alpha_3 = \langle [0.71, 0.75], [0.17, 0.22] \rangle$ and $\alpha_4 = \langle [0.19, 0.24], [0.62, 0.65] \rangle$ be four IVIFNs and $\omega = (0.2, 0.3, 0.4, 0.1)^T$ is the standardized weight vector of these numbers. Then, based on IIFHWG operator we get the aggregated IVIFN as $\langle [0.5026, 0.5508], [0.2989, 0.3462] \rangle$ corresponding to $\gamma = 2$. On the other hand, if we replace non-membership degree of second and third IVIFNs as $\beta_2 = \langle [0.65, 0.69], [0.12, 0.15] \rangle$ and $\beta_3 = \langle [0.71, 0.75], [0.22, 0.24] \rangle$ then the aggregate IVIFN by using IIFHWG operator becomes $\langle [0.5026, 0.5508], [0.2868, 0.3144] \rangle$.

Therefore, the existing aggregation operators are incapable to select the alternatives and hence some new aggregation operators need to be imposed on IVIFS environment. To manage these, a new operational law and their corresponding aggregation operators has been introduced here by taking the pairs of dependency between the membership functions.

3. Improved operational laws on interval-valued intuitionistic fuzzy numbers. The above shortcoming has been overcome by defined some new operational laws on IFNs, which has been defined as below.

Definition 3.1. Let $\alpha = \langle [a, b], [c, d] \rangle$ and $\alpha_i = \langle [a_i, b_i], [c_i, d_i] \rangle$, $(i = 1, 2, \ldots, n)$ be an IVIFN and $\lambda > 0$ be a real number then the new operational rules based on Hamacher norms are defined as below.

(i) $\alpha_1 \odot \alpha_2 \odot \ldots \odot \alpha_n$

$$
\begin{array}{c}
\left( \prod_{i=1}^{n} \left( 1 + (\gamma_1 - 1) a_i \right) - \gamma \prod_{i=1}^{n} (1 - a_i) \right) \prod_{i=1}^{n} \left( 1 + (\gamma - 1) b_i \right) - \gamma \prod_{i=1}^{n} (1 - b_i) \\
\gamma \prod_{i=1}^{n} (1 - a_i) - \gamma \prod_{i=1}^{n} (1 - a_i - c_i) \\
\prod_{i=1}^{n} \left( 1 + (\gamma_1 - 1) a_i \right) + (\gamma - 1) \prod_{i=1}^{n} (1 - a_i) \\
\prod_{i=1}^{n} \left( 1 + (\gamma - 1) b_i \right) + (\gamma - 1) \prod_{i=1}^{n} (1 - b_i) \\
\prod_{i=1}^{n} (1 + (\gamma_1 - 1) a_i) + (\gamma - 1) \prod_{i=1}^{n} (1 - a_i) \\
\prod_{i=1}^{n} \left( 1 + (\gamma - 1) b_i \right) + (\gamma - 1) \prod_{i=1}^{n} (1 - b_i) \\
\prod_{i=1}^{n} (1 + (\gamma_1 - 1) a_i) + (\gamma - 1) \prod_{i=1}^{n} (1 - a_i)
\end{array}
$$

(ii) $\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_n$

$$
\begin{array}{c}
\left( \prod_{i=1}^{n} \left( 1 - c_i \right) - \gamma \prod_{i=1}^{n} (1 - a_i - c_i) \right) \prod_{i=1}^{n} \left( 1 - d_i \right) - \gamma \prod_{i=1}^{n} (1 - b_i - d_i) \\
\gamma \prod_{i=1}^{n} (1 - c_i) - \gamma \prod_{i=1}^{n} (1 - a_i - c_i) \\
\prod_{i=1}^{n} \left( 1 + (\gamma_1 - 1) a_i \right) + (\gamma - 1) \prod_{i=1}^{n} (1 - a_i) \\
\prod_{i=1}^{n} \left( 1 + (\gamma - 1) b_i \right) + (\gamma - 1) \prod_{i=1}^{n} (1 - b_i) \\
\prod_{i=1}^{n} (1 + (\gamma_1 - 1) a_i) + (\gamma - 1) \prod_{i=1}^{n} (1 - a_i) \\
\prod_{i=1}^{n} \left( 1 + (\gamma - 1) b_i \right) + (\gamma - 1) \prod_{i=1}^{n} (1 - b_i) \\
\prod_{i=1}^{n} (1 + (\gamma_1 - 1) a_i) + (\gamma - 1) \prod_{i=1}^{n} (1 - a_i)
\end{array}
$$
if $a$ been proposed as follows for a collection of IVIFNs $a$

Based on these operational laws, weighted geometric aggregated operators have been proposed as follows for a collection of IVIFNs $a_1, a_2, \ldots, a_n$.

### 3.1. IIF Hamacher interactive weighted geometric (IIFHIWG) operator.

**Definition 3.2.** Let $a_i, (i = 1, 2, \ldots, n)$ be an IVIFN, and IIFHIWG : $\Omega^n \rightarrow \Omega$, if

$$\text{IIFHIWG}(a_1, a_2, \ldots, a_n) = a_1^{\omega_1} \otimes a_2^{\omega_2} \otimes \cdots \otimes a_n^{\omega_n}$$

where $\Omega$ is the set of IVIFNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $a_i$ with $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$ then IIFHIWG is called an IIF Hamacher interactive weighting geometric operator.

**Theorem 3.1.** Let $a_i = ([a_i, b_i], [c_i, d_i]), (i = 1, 2, \ldots, n)$ be an IVIFN, then

$$\text{IFHIWG}(a_1, a_2, \ldots, a_n) =$$

\[
\left[ \begin{array}{c}
\frac{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i) - \prod_{i=1}^{n} (1 - c_i)}{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i) + (\gamma - 1) \prod_{i=1}^{n} (1 - c_i)} \\
\frac{\prod_{i=1}^{n} (1 + (\gamma - 1)d_i) - \prod_{i=1}^{n} (1 - d_i)}{\prod_{i=1}^{n} (1 + (\gamma - 1)d_i) + (\gamma - 1) \prod_{i=1}^{n} (1 - d_i)}
\end{array} \right]
\]

As it has been clearly observed from the above definition that the sum of IVIFNs become more optimistic than the existing sum because the non-membership degree of $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ contains the pairs of membership and non-membership i.e., $a_i \cdot c_i$ and $b_i \cdot d_i$ while membership function does not. Hence the attitude towards the decision is more optimistic. Similarly, in $a_1 \otimes a_2 \otimes \cdots \otimes a_n$, the decision is more pessimistic.
Proof. When \( n = 1, \omega_1 = 1 \), we have

\[
\text{IIFHIWG}(\alpha_1) = \alpha_1^\omega_1 = \langle [a_1, b_1], [c_1, d_1] \rangle = \\
\left\langle \frac{\gamma\left\{ (1 - c_1)^1 - (1 - a_1 - c_1)^1 \right\}}{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} (1 - c_i)^{\omega_i}} \right\rangle,
\]

Thus, result is true for \( n = 1 \). Assume that result given in Eq. (1) holds for \( n = k \), i.e.,

\[
\text{IIFHIWG}(\alpha_1, \alpha_2, \ldots, \alpha_k) = \\
\left\langle \frac{\gamma\left\{ \prod_{i=1}^{k} (1 - c_i)^{\omega_i} - \prod_{i=1}^{k} (1 - a_i - c_i)^{\omega_i} \right\}}{\prod_{i=1}^{k} (1 + (\gamma - 1)c_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{k} (1 - c_i)^{\omega_i}} \right\rangle,
\]

Then, for \( n = k + 1 \) by using Definition 3.1, we have

\[
\text{IIFHIWG}(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}) = \bigotimes_{i=1}^{k+1} \alpha_i^{\omega_i}
\]

\[
= \text{IIFHIWG}(\alpha_1, \alpha_2, \ldots, \alpha_k) \otimes \alpha_{k+1}^{\omega_{k+1}}
\]

\[
= \left\langle \frac{\gamma\left\{ \prod_{i=1}^{k} (1 - c_i)^{\omega_i} - \prod_{i=1}^{k} (1 - a_i - c_i)^{\omega_i} \right\}}{\prod_{i=1}^{k} (1 + (\gamma - 1)c_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{k} (1 - c_i)^{\omega_i}} \right\rangle.
\]
\[
\frac{\gamma\left\{ \frac{\prod_{i=1}^{k} (1 - d_i)^{\omega_i} - \prod_{i=1}^{k} (1 - b_i - d_i)^{\omega_i}}{\prod_{i=1}^{k} (1 + (\gamma - 1)d_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{k} (1 - d_i)^{\omega_i}} \right\}}{\prod_{i=1}^{k} (1 + (\gamma - 1)d_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{k} (1 - d_i)^{\omega_i},}
\]

\[
\frac{\prod_{i=1}^{k} (1 + (\gamma - 1)c_i)^{\omega_i} - \prod_{i=1}^{k} (1 - c_i)^{\omega_i}}{\prod_{i=1}^{k} (1 + (\gamma - 1)c_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{k} (1 - c_i)^{\omega_i},}
\]

\[
\frac{\prod_{i=1}^{k} (1 + (\gamma - 1)d_i)^{\omega_i} - \prod_{i=1}^{k} (1 - d_i)^{\omega_i}}{\prod_{i=1}^{k} (1 + (\gamma - 1)d_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{k} (1 - d_i)^{\omega_i}}
\]

\[
\gamma \left\{ \frac{\prod_{i=1}^{k+1} (1 - c_i)^{\omega_i} - \prod_{i=1}^{k+1} (1 - a_i - c_i)^{\omega_i}}{\prod_{i=1}^{k+1} (1 + (\gamma - 1)c_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{k+1} (1 - c_i)^{\omega_i}} \right\}
\]

\[
\frac{\prod_{i=1}^{k+1} (1 + (\gamma - 1)c_i)^{\omega_i} - \prod_{i=1}^{k+1} (1 - c_i)^{\omega_i}}{\prod_{i=1}^{k+1} (1 + (\gamma - 1)c_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{k+1} (1 - c_i)^{\omega_i}}
\]

\[
\frac{\prod_{i=1}^{k+1} (1 + (\gamma - 1)d_i)^{\omega_i} - \prod_{i=1}^{k+1} (1 - d_i)^{\omega_i}}{\prod_{i=1}^{k+1} (1 + (\gamma - 1)d_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{k+1} (1 - d_i)^{\omega_i}}
\]

which is true for \( n = k + 1 \). Thus by principle of mathematical induction, we get the required result for all \( n \in \mathbb{Z}^+ \).
Corollary 1. If $a_i + c_i = 1$ and $b_i + d_i = 1$ for any $i$ then the proposed IIFHIWG operator becomes

$$IIFHWG(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left[ \gamma \prod_{i=1}^{n} a_i^{\omega_i} \right. $$

$$\left. \frac{\gamma \prod_{i=1}^{n} a_i^{\omega_i}}{\prod_{i=1}^{n} (1 + (\gamma - 1)(1 - a_i))^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} a_i^{\omega_i}} \right]$$

$$\left. \frac{\gamma \prod_{i=1}^{n} b_i^{\omega_i}}{\prod_{i=1}^{n} (1 + (\gamma - 1)(1 - b_i))^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} b_i^{\omega_i}} \right]$$

$$\frac{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^{\omega_i} - \prod_{i=1}^{n} (1 - c_i)^{\omega_i}}{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} (1 - c_i)^{\omega_i}}$$

$$\frac{\prod_{i=1}^{n} (1 + (\gamma - 1)d_i)^{\omega_i} - \prod_{i=1}^{n} (1 - d_i)^{\omega_i}}{\prod_{i=1}^{n} (1 + (\gamma - 1)d_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} (1 - d_i)^{\omega_i}}$$

Thus, the IIFHWG operator [20] is a special case of our proposed operator. In other words, the proposed operator is a more generalized version of the existing IIFHWG operator.

Now, based on this proposed operator, it has been analyzed that it successfully overcome the shortcoming of the existing operators as described in section 2.3.

Example 3.1. If we utilize IIFHIWG operator to aggregate all the IVIFNs as given in Example 2.1 then the result corresponding to it has been summarized as.

$$\prod_{i=1}^{4} (1 - c_i)^{\omega_i} = (1 - 0.35)^{0.2}(1 - 0.55)^{0.3}(1 - 0.43)^{0.4}(1 - 0.21)^{0.1} = 0.5632$$

$$\prod_{i=1}^{4} (1 - d_i)^{\omega_i} = (1 - 0.40)^{0.2}(1 - 0.61)^{0.3}(1 - 0.47)^{0.4}(1 - 0.25)^{0.1} = 0.5131$$

$$\prod_{i=1}^{4} (1 - a_i - c_i)^{\omega_i} = (1 - 0 - 0.35)^{0.2}(1 - 0.32 - 0.55)^{0.3}(1 - 0.25 - 0.43)^{0.4}(1 - 0.65 - 0.21)^{0.1} = 0.2591$$

$$\prod_{i=1}^{4} (1 - b_i - d_i)^{\omega_i} = (1 - 0 - 0.40)^{0.2}(1 - 0.37 - 0.61)^{0.3}(1 - 0.28 - 0.47)^{0.4}(1 - 0.69 - 0.25)^{0.1} = 0.1210$$

$$\prod_{i=1}^{4} (1 + c_i)^{\omega_i} = (1 + 0.35)^{0.2}(1 + 0.55)^{0.3}(1 + 0.43)^{0.4}(1 + 0.21)^{0.1} = 1.4242$$
for $\gamma = 1$, we have
\[
\text{IIFHIWG}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(0.5632 - 0.2591, 0.5131 - 0.1210\right), \left[1 - 0.5632, 1 - 0.5131\right]
\]
and for $\gamma = 2$, we have
\[
\text{IIFHIWG}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(0.3041, 0.3921\right), \left[0.4368, 0.4869\right)
\]
Therefore, it has been concluded that the membership grades of the aggregated value, are non-zero and hence there is a significant role of the other grades of membership function during the aggregation process. Thus, the proposed aggregation operator can be suitably utilized to aggregate the different IVIFNs even when at least one of the membership grade is zero.

**Example 3.2.** If we utilize the IIFHIWG operator to Example 2.2 then their corresponding aggregated values are IIFHIWG($\alpha_1, \alpha_2, \alpha_3, \alpha_4$) = $\langle 0.5460, 0.6069, 0.2989, 0.3462 \rangle$ and IIFHIWG($\beta_1, \beta_2, \beta_3, \beta_4$) = $\langle 0.5610, 0.6291, 0.2868, 0.3144 \rangle$ for $\gamma = 2$. From this, it has been shown that by changing the grades of membership and non-membership, the overall aggregated membership grades is different and hence the other membership values of IVIFNs $\alpha_2$ and $\alpha_3$ will play a significant role during the aggregation process. In other words, we can say that there is a proper dependency relationship between the grades of membership functions during the aggregation process.

**Lemma 3.2.** [28] Let $\alpha_i = \langle [a_i, b_i], [c_i, d_i] \rangle$ be an IVIFN and $\omega_i > 0$ for $i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} \omega_i = 1$, then
\[
\prod_{i=1}^{n} \alpha_i^{\omega_i} \leq \sum_{i=1}^{n} \omega_i \alpha_i
\]
with equality holds if and only if $\alpha_1 = \alpha_2 = \ldots = \alpha_n$.

**Corollary 2.** For a collection of an IVIFN $\alpha_i (i = 1, 2, \ldots, n)$, the proposed IIFHIWG operators as well as the existing IIFHWG operators satisfies the following inequality.
\[
\text{IIFHIWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) \geq \text{IIFHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n)
\]

**Proof.** For a collection of IVIFNs, let IIFHIWG($\alpha_1, \alpha_2, \ldots, \alpha_n$) = $\langle [a^p_1, b^p_1], [c^p_1, d^p_1] \rangle = \alpha^p$ (say) and IIFHWG ($\alpha_1, \alpha_2, \ldots, \alpha_n$) = $\langle [a_1, b_1], [c_1, d_1] \rangle = \alpha$ (say), then
\[
\prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} (1 - c_i)^{\omega_i} \leq \sum_{i=1}^{n} \omega_i (1 + (\gamma - 1)c_i) + (\gamma - 1) \sum_{i=1}^{n} \omega_i (1 - c_i) = \gamma
\]

\[
\Rightarrow \frac{\gamma \{ \prod_{i=1}^{n} (1 - c_i)^{\omega_i} - \prod_{i=1}^{n} (1 - a_i - c_i)^{\omega_i} \}}{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} (1 - c_i)^{\omega_i}} \geq \frac{\prod_{i=1}^{n} (1 - c_i)^{\omega_i} - \prod_{i=1}^{n} (1 - a_i - c_i)^{\omega_i}}{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} (1 - c_i)^{\omega_i}}
\]

Therefore, \(a^p_\alpha \geq a_\alpha\) and \(c^p_\alpha = c_\alpha\). Similarly, \(b^p_\alpha \geq b_\alpha\) and \(d^p_\alpha = d_\alpha\), where equality holds if and only if \(a_1 = a_2 = \ldots = a_n\) and \(c_1 = c_2 = \ldots = c_n\). Thus,

\[
S(\alpha^p) = \frac{a^p_\alpha + b^p_\alpha - c^p_\alpha - d^p_\alpha}{2} \geq \frac{a_\alpha + b_\alpha - c_\alpha - d_\alpha}{2} = S(\alpha)
\]

If \(S(\alpha^p) > S(\alpha)\) then for every \(\omega\), we have

\[\text{IFFHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) > \text{IFFHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n)\]

If \(S(\alpha^p) = S(\alpha)\) i.e., \(a^p_\alpha + b^p_\alpha - c^p_\alpha - d^p_\alpha = a_\alpha + b_\alpha - c_\alpha - d_\alpha\), then by the condition \(a^p_\alpha \geq a_\alpha\) and \(b^p_\alpha \geq b_\alpha\), we have \(a^p_\alpha = a_\alpha\), \(b^p_\alpha = b_\alpha\), \(c^p_\alpha = c_\alpha\) and \(d^p_\alpha = d_\alpha\). Then by definition of accuracy function, we have \(H(\alpha^p) = \frac{a^p_\alpha + b^p_\alpha + c^p_\alpha + d^p_\alpha}{2} = \frac{a_\alpha + b_\alpha + c_\alpha + d_\alpha}{2} = H(\alpha)\). Thus in this case,

\[\text{IFFHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \text{IFFHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n)\]

Hence,

\[\text{IFFHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) \geq \text{IFFHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n)\]

where that equality holds if and only if \(\alpha_1 = \alpha_2 = \ldots = \alpha_n\). \(\Box\)

Thus, it has been concluded from the Corollary 2 that IFFHWG operator shows pessimistic attitude than IIFHWG operator in the aggregation process.

**Theorem 3.3.** If \(\alpha_i = [\alpha_i, b_i, [c_i, d_i]]\) be an IVIFN, \(i = 1, 2, \ldots, n\), then the aggregate value by IFFHWG operator is also an IVIFN i.e., \(\text{IFFHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \text{IVIFN}\)

**Proof.** Since \(\alpha_i = [\alpha_i, b_i, [c_i, d_i]]\), \(i = 1, 2, \ldots, n\) be an IVIFN, then we have

\[0 \leq a_i, b_i, c_i, d_i \leq 1\] and \(b_i + d_i \leq 1\)

Take, IFFHWG(\(\alpha_1, \ldots, \alpha_n\)) = \([a_{\text{IFFHWG}}, b_{\text{IFFHWG}}, [c_{\text{IFFHWG}}, d_{\text{IFFHWG}}]]\)

As,

\[
\frac{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^{\omega_i} - \prod_{i=1}^{n} (1 - c_i)^{\omega_i}}{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} (1 - c_i)^{\omega_i}} = 1 - \frac{\gamma \prod_{i=1}^{n} (1 - c_i)^{\omega_i}}{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} (1 - c_i)^{\omega_i}} \leq 1 - \prod_{i=1}^{n} (1 - c_i)^{\omega_i} \leq 1
\]
Also

\[
1 + (\gamma - 1)c_i \geq (1 - c_i) \iff \prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^{\omega_i} - \prod_{i=1}^{n} (1 - c_i)^{\omega_i} \geq 0
\]

\[
\iff \frac{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^{\omega_i} - \prod_{i=1}^{n} (1 - c_i)^{\omega_i}}{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} (1 - c_i)^{\omega_i}} \geq 0.
\]

Thus,

\[
0 \leq c_{\text{IFHIWG}} \leq 1
\]

On the other hand,

\[
\gamma \left\{ \frac{\prod_{i=1}^{n} (1 - c_i)^{\omega_i} - \prod_{i=1}^{n} (1 - a_i - c_i)^{\omega_i}}{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} (1 - c_i)^{\omega_i}} \right\} \leq \frac{\gamma \prod_{i=1}^{n} (1 - c_i)^{\omega_i}}{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} (1 - c_i)^{\omega_i}} \leq \prod_{i=1}^{n} (1 - c_i)^{\omega_i} \leq 1
\]

Also

\[
\prod_{i=1}^{n} (1 - c_i)^{\omega_i} - \prod_{i=1}^{n} (1 - a_i - c_i)^{\omega_i} \geq 0
\]

\[
\iff \frac{\gamma \left\{ \prod_{i=1}^{n} (1 - c_i)^{\omega_i} - \prod_{i=1}^{n} (1 - a_i - c_i)^{\omega_i} \right\}}{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} (1 - c_i)^{\omega_i}} \geq 0
\]

Thus,

\[
0 \leq a_{\text{IFHIWG}} \leq 1
\]

Moreover,

\[
a_{\text{IFHIWG}} + c_{\text{IFHIWG}} = 1 - \frac{\gamma \prod_{i=1}^{n} (1 - a_i - c_i)^{\omega_i}}{\prod_{i=1}^{n} (1 + (\gamma - 1)c_i)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} (1 - c_i)^{\omega_i}} \leq 1 - \prod_{i=1}^{n} (1 - a_i - c_i)^{\omega_i} \leq 1
\]

Similarly, \(0 \leq b_{\text{IFHIWG}} \leq 1, \ 0 \leq d_{\text{IFHIWG}} \leq 1\) and \(b_{\text{IFHIWG}} + d_{\text{IFHIWG}} \leq 1\). Hence, IIFHIWG \(\in [0, 1]\). Therefore, the aggregated number obtained by IIFHIWG operator is again an IVIFN.

Now, we have presented some properties of the IIFHIWG operator for a collection of IVIFNs \(\alpha_i = [a_i, b_i, c_i, d_i] (i = 1, 2, \ldots, n)\).

**Property 3.1.** (Idempotency:) If \(\alpha_i = \alpha_0 = [a_0, b_0, c_0, d_0]\) for all \(i\), then

\[
\text{IIFHIWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha_0
\]
Proof. Since \( \alpha_i = \alpha_0 = \langle [a_0, b_0], [c_0, d_0] \rangle (i = 1, 2, \ldots, n) \), then by Theorem 3.1, we have

\[
\text{IFHIWG}(\alpha_1, \alpha_2, \ldots, \alpha_n)
= \left\langle \left[ \frac{\gamma \left\{ \prod_{i=1}^{n} (1 - c_0)^{\omega_i} - \prod_{i=1}^{n} (1 - a_0 - c_0)^{\omega_i} \right\}}{\prod_{i=1}^{n} \left( 1 + (\gamma - 1)c_0 \right)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} (1 - c_0)^{\omega_i}} \right], \left[ \frac{\gamma \left\{ \prod_{i=1}^{n} (1 - d_0)^{\omega_i} - \prod_{i=1}^{n} (1 - b_0 - d_0)^{\omega_i} \right\}}{\prod_{i=1}^{n} \left( 1 + (\gamma - 1)d_0 \right)^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} (1 - d_0)^{\omega_i}} \right] \right\rangle

\]

\[
= \left\langle \left[ \frac{\gamma \left\{ \sum_{i=1}^{n} \omega_i (1 - c_0) - \sum_{i=1}^{n} \omega_i (1 - a_0 - c_0) \right\}}{\left( 1 + (\gamma - 1)c_0 \right) \sum_{i=1}^{n} \omega_i + (\gamma - 1) \sum_{i=1}^{n} \omega_i} \right], \left[ \frac{\gamma \left\{ \sum_{i=1}^{n} \omega_i (1 - d_0) - \sum_{i=1}^{n} \omega_i (1 - b_0 - d_0) \right\}}{\left( 1 + (\gamma - 1)d_0 \right) \sum_{i=1}^{n} \omega_i + (\gamma - 1) \sum_{i=1}^{n} \omega_i} \right] \right\rangle

\]

\[
= \left\langle \left[ \frac{\gamma \left\{ \sum_{i=1}^{n} \omega_i (1 - c_0) - \sum_{i=1}^{n} \omega_i (1 - a_0 - c_0) \right\}}{\left( 1 + (\gamma - 1)c_0 \right) \sum_{i=1}^{n} \omega_i + (\gamma - 1) \sum_{i=1}^{n} \omega_i} \right], \left[ \frac{\gamma \left\{ \sum_{i=1}^{n} \omega_i (1 - d_0) - \sum_{i=1}^{n} \omega_i (1 - b_0 - d_0) \right\}}{\left( 1 + (\gamma - 1)d_0 \right) \sum_{i=1}^{n} \omega_i + (\gamma - 1) \sum_{i=1}^{n} \omega_i} \right] \right\rangle

\]

\[
= \langle [a_0, b_0], [c_0, d_0] \rangle
= \alpha_0
\]
Property 3.2. (Boundedness:) Let $\alpha^- = \min(\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha^+ = \max(\alpha_1, \alpha_2, \ldots, \alpha_n)$ then

$$\alpha^- \leq \text{IIIFIWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+$$

Proof. Let $f(x) = \frac{1-x}{1+(\gamma-1)x}, x \in [0, 1]$ then $f'(x) = \frac{-\gamma}{(1+(\gamma-1)x)^2} < 0$; thus, $f(x)$ is decreasing function. Since $c_{i, \min} \leq c_i \leq c_{i, \max}$, for all $i = 1, 2, \ldots, n$ then $f(c_i) \leq f(c_{i, \min})$ for all $i$, i.e., $\frac{1-c_{i, \max}}{1+(\gamma-1)c_{i, \max}} \leq \frac{1-c_i}{1+(\gamma-1)c_i} \leq \frac{1-c_{i, \min}}{1+(\gamma-1)c_{i, \min}}$ for all $i$. Hence, we have

$$\left(1 - c_{i, \max}\right)^{\gamma_i} \leq \left(1 - c_i\right)^{\gamma_i} \leq \left(1 - c_{i, \min}\right)^{\gamma_i}.$$

Thus, $(\gamma-1)\left(1 + (\gamma-1)c_{i, \min}\right) \leq \gamma - (\gamma-1)\left(1 + (\gamma-1)c_{i, \min}\right) \leq \gamma \leq (\gamma-1)\left(1 + (\gamma-1)c_{i, \min}\right)$

$$\leq (\gamma-1)\left(1 - c_{i, \min}\right)^{\gamma_i} \leq 1 + (\gamma-1)c_{i, \min} \leq 1 + (\gamma-1)c_{i, \max}.$$

Then

$$\Rightarrow (\gamma-1)c_{i, \min} \leq \frac{\prod_{i=1}^{n} (1 + (\gamma-1)c_i)^{\omega_i} - \prod_{i=1}^{n} (1 - c_i)^{\omega_i}}{\prod_{i=1}^{n} (1 + (\gamma-1)c_i)^{\omega_i} + (\gamma-1)\prod_{i=1}^{n} (1 - c_i)^{\omega_i}} \leq c_{i, \max} \quad (2)$$

Similarly,

$$d_{i, \min} \leq \frac{\prod_{i=1}^{n} (1 + (\gamma-1)d_i)^{\omega_i} - \prod_{i=1}^{n} (1 - d_i)^{\omega_i}}{\prod_{i=1}^{n} (1 + (\gamma-1)d_i)^{\omega_i} + (\gamma-1)\prod_{i=1}^{n} (1 - d_i)^{\omega_i}} \leq d_{i, \max} \quad (3)$$

On the other hand, let $g(y) = \frac{\gamma-(\gamma-1)y}{\gamma+y}, y \in [0, 1]$ then $g'(y) = -\gamma/y^2 < 0$ so $g(y)$ is decreasing function. Since $1 - c_{i, \min} \leq 1 - c_i \leq 1 - c_{i, \min}$ for all $i$ then $g(1 - c_{i, \min}) \leq g(1 - c_i) \leq g(1 - c_{i, \min})$ i.e., $\gamma \frac{(\gamma-1)(1-c_{i, \min})}{1-c_{i, \min}} \leq \gamma \frac{(\gamma-1)(1-c_i)}{1-c_i} \leq \gamma \frac{(\gamma-1)(1-c_{i, \min})}{1-c_{i, \min}}$

$$\Rightarrow \frac{1+(\gamma-1)c_{i, \min}}{1-c_{i, \min}} \leq \prod_{i=1}^{n} \frac{1+(\gamma-1)c_i}{1-c_i} \leq (\gamma-1) \prod_{i=1}^{n} \frac{1+c_{i, \max}}{1-c_{i, \max}} \leq 1 - c_{i, \min}.$$

Also, $1 - a_{i, \max} - c_{i, \min} \leq 1 - a_i - c_i \leq 1 - a_{i, \min} - c_{i, \min} \leq 1 - a_{i, \max} - c_{i, \max} \leq 1 - a_{i, \min} - c_{i, \min}$

$$\Rightarrow \frac{-c_{i, \max} + a_{i, \max} - c_{i, \min}}{1-c_{i, \max}} \leq \frac{1 - a_i - c_i}{1-c_i} \leq \frac{-c_{i, \min} + a_{i, \max} + c_{i, \max}}{1-c_{i, \min}} \leq -c_{i, \min} + a_{i, \max} + c_{i, \max} \leq -c_{i, \min} + a_{i, \max} + c_{i, \max} \leq 0.$$
Proof. As that \( \alpha \) be collections of IVIFNs, then follows from above property.

\[ a_{i,\min} \leq \frac{\gamma \left\{ \prod_{i=1}^{n} (1 - c_i) + \prod_{i=1}^{n} (1 - a_i) \right\}}{\prod_{i=1}^{n} (1 - (\gamma - 1)c_i) + (\gamma - 1) \prod_{i=1}^{n} (1 - c_i)} \leq a_{i,\max} \tag{4} \]

Similarly,

\[ b_{i,\min} \leq \frac{\gamma \left\{ \prod_{i=1}^{n} (1 - d_i - b_i) + \prod_{i=1}^{n} (1 - d_i) \right\}}{\prod_{i=1}^{n} (1 - (\gamma - 1)d_i) + (\gamma - 1) \prod_{i=1}^{n} (1 - d_i)} \leq b_{i,\max} \tag{5} \]

Take \( a_{\min} = \min_i \{a_i\} \), \( a_{\max} = \max_i \{a_i\} \), \( b_{\min} = \min_i \{b_i\} \), \( b_{\max} = \max_i \{b_i\} \), \( c_{\min} = \min_i \{c_i\} \), \( c_{\max} = \max_i \{c_i\} \), \( d_{\min} = \min_i \{d_i\} \) and \( d_{\max} = \max_i \{d_i\} \).

Let \( IIFHIWG(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha = \{a_\alpha, b_\alpha, [c_\alpha, d_\alpha]\} \) then Eqs. (2), (3), (4) and (5) are transformed into the following forms, respectively

\[ a_{\min} \leq a_\alpha \leq a_{\max} \quad \text{and} \quad c_{\min} \leq c_\alpha \leq c_{\max} \]
\[ b_{\min} \leq b_\alpha \leq b_{\max} \quad \text{and} \quad d_{\min} \leq d_\alpha \leq d_{\max} \]

Thus,

\[ S(\alpha) = \frac{a_\alpha + b_\alpha - c_\alpha - d_\alpha}{2} \leq \frac{a_{\max} + b_{\max} - c_{\min} - d_{\min}}{2} = S(\alpha^+) \]
and \[ S(\alpha) = \frac{a_\alpha + b_\alpha - c_\alpha - d_\alpha}{2} \geq \frac{a_{\min} + b_{\min} - c_{\max} - d_{\max}}{2} = S(\alpha^-) \]

If \( S(\alpha) < S(\alpha^+) \) and \( S(\alpha) > S(\alpha^-) \) then by order relation between two IVIFNs, we have \( \alpha^- < IIFHIWG(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+ \).

\[ \square \]

Property 3.3. (Monotonicity:) Let \( \alpha_i \) and \( \beta_i \) be two collections of IVIFNs such that \( \alpha_i \leq \beta_i \) for all \( i \), then

\[ IIFHIWG(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq IIFHIWG(\beta_1, \beta_2, \ldots, \beta_n) \]

Proof. Follows from above property.

\[ \square \]

Property 3.4. Let \( \alpha_i = \{a_i, b_i, [c_i, d_i]\} (i = 1, 2, \ldots, n) \) and \( \beta = \{a_\beta, b_\beta, [c_\beta, d_\beta]\} \) be collections of IVIFNs, then

\[ IIFHIWG(\alpha_1 + \beta, \alpha_2 + \beta, \ldots, \alpha_n + \beta) = IIFHIWG(\alpha_1, \alpha_2, \ldots, \alpha_n) + \beta \]

Proof. As \( \alpha_i, \beta \in \text{IVIFNs} \), so

\[ \alpha_i + \beta = \left\{ \gamma \left( (1 - c_i)(1 - c_\beta) - (1 - a_i - c_i)(1 - a_\beta - c_\beta) \right) \over \left( 1 + (\gamma - 1)c_i \right) \left( 1 + (\gamma - 1)c_\beta \right) + (\gamma - 1)(1 - c_i)(1 - c_\beta) \right\} \]
\[ \times \left\{ \gamma \left( (1 - d_i)(1 - d_\beta) - (1 - b_i - d_i)(1 - b_\beta - d_\beta) \right) \over \left( 1 + (\gamma - 1)d_i \right) \left( 1 + (\gamma - 1)d_\beta \right) + (\gamma - 1)(1 - d_i)(1 - d_\beta) \right\}, \]
\[ \left\{ \left( 1 + (\gamma - 1)c_i \right) \left( 1 + (\gamma - 1)c_\beta \right) - (1 - c_i)(1 - c_\beta) \right\} \]
\[ \left\{ \left( 1 + (\gamma - 1)d_i \right) \left( 1 + (\gamma - 1)d_\beta \right) - (1 - d_i)(1 - d_\beta) \right\}. \]
Therefore,
\[ \text{HIFHIWG}(\alpha_1 \oplus \beta, \alpha_2 \oplus \beta, \ldots, \alpha_n \oplus \beta) = \left[ \gamma \left\{ \prod_{i=1}^{n} \left( 1 - c_i \right) \prod_{i=1}^{n} \left( 1 - a_i - c_i \right) \right\}^{\omega_i} - \prod_{i=1}^{n} \left( 1 - a_i - c_i \right) \right] \]

\[ \prod_{i=1}^{n} \left( 1 - b_i - d_i \right) \left[ \gamma \left\{ \prod_{i=1}^{n} \left( 1 - d_i \right) \left( 1 - d_{i'} \right) \right\}^{\omega_i} - \prod_{i=1}^{n} \left( 1 - d_i \right) \left( 1 - d_{i'} \right) \right] \]

\[ \prod_{i=1}^{n} \left( 1 - a_i - c_i \right) \left[ \gamma \left\{ \prod_{i=1}^{n} \left( 1 - c_i \right) \left( 1 - c_{i'} \right) \right\}^{\omega_i} - \prod_{i=1}^{n} \left( 1 - c_i \right) \left( 1 - c_{i'} \right) \right] \]

\[ \prod_{i=1}^{n} \left( 1 - c_i \left( 1 - c_{i'} \right) \right) \left[ \gamma \left\{ \prod_{i=1}^{n} \left( 1 - d_i \right) \left( 1 - d_{i'} \right) \right\}^{\omega_i} - \prod_{i=1}^{n} \left( 1 - d_i \right) \left( 1 - d_{i'} \right) \right] \]

\[ \prod_{i=1}^{n} \left( 1 - a_i - c_i \right) \left( 1 - a_{i'} - c_{i'} \right) \left[ \gamma \left\{ \prod_{i=1}^{n} \left( 1 - c_i \right) \left( 1 - c_{i'} \right) \right\}^{\omega_i} - \prod_{i=1}^{n} \left( 1 - c_i \right) \left( 1 - c_{i'} \right) \right] \]

\[ \prod_{i=1}^{n} \left( 1 - a_i - c_i \right) \left( 1 - b_i - d_i \right) \left( 1 - b_{i'} - d_{i'} \right) \left[ \gamma \left\{ \prod_{i=1}^{n} \left( 1 - d_i \right) \left( 1 - d_{i'} \right) \right\}^{\omega_i} - \prod_{i=1}^{n} \left( 1 - d_i \right) \left( 1 - d_{i'} \right) \right] \]

\[ \prod_{i=1}^{n} \left( 1 - a_i - c_i \right) \left( 1 - a_{i'} - c_{i'} \right) \left( 1 - b_i - d_i \right) \left( 1 - b_{i'} - d_{i'} \right) \left[ \gamma \left\{ \prod_{i=1}^{n} \left( 1 - d_i \right) \left( 1 - d_{i'} \right) \right\}^{\omega_i} - \prod_{i=1}^{n} \left( 1 - d_i \right) \left( 1 - d_{i'} \right) \right] \]

\[ \prod_{i=1}^{n} \left( 1 - a_i - c_i \right) \left( 1 - a_{i'} - c_{i'} \right) \left( 1 - b_i - d_i \right) \left( 1 - b_{i'} - d_{i'} \right) \left( 1 - c_i \left( 1 - c_{i'} \right) \right) \left[ \gamma \left\{ \prod_{i=1}^{n} \left( 1 - d_i \right) \left( 1 - d_{i'} \right) \right\}^{\omega_i} - \prod_{i=1}^{n} \left( 1 - d_i \right) \left( 1 - d_{i'} \right) \right] \]

\[ \prod_{i=1}^{n} \left( 1 - a_i - c_i \right) \left( 1 - a_{i'} - c_{i'} \right) \left( 1 - b_i - d_i \right) \left( 1 - b_{i'} - d_{i'} \right) \left( 1 - c_i \left( 1 - c_{i'} \right) \right) \left( 1 - d_i \left( 1 - d_{i'} \right) \right) \left[ \gamma \left\{ \prod_{i=1}^{n} \left( 1 - d_i \right) \left( 1 - d_{i'} \right) \right\}^{\omega_i} - \prod_{i=1}^{n} \left( 1 - d_i \right) \left( 1 - d_{i'} \right) \right] \]

\[ \prod_{i=1}^{n} \left( 1 - a_i - c_i \right) \left( 1 - a_{i'} - c_{i'} \right) \left( 1 - b_i - d_i \right) \left( 1 - b_{i'} - d_{i'} \right) \left( 1 - c_i \left( 1 - c_{i'} \right) \right) \left( 1 - d_i \left( 1 - d_{i'} \right) \right) \left( 1 - e_i \left( 1 - e_{i'} \right) \right) \left[ \gamma \left\{ \prod_{i=1}^{n} \left( 1 - d_i \right) \left( 1 - d_{i'} \right) \right\}^{\omega_i} - \prod_{i=1}^{n} \left( 1 - d_i \right) \left( 1 - d_{i'} \right) \right] \]

\[ \prod_{i=1}^{n} \left( 1 - a_i - c_i \right) \left( 1 - a_{i'} - c_{i'} \right) \left( 1 - b_i - d_i \right) \left( 1 - b_{i'} - d_{i'} \right) \left( 1 - c_i \left( 1 - c_{i'} \right) \right) \left( 1 - d_i \left( 1 - d_{i'} \right) \right) \left( 1 - e_i \left( 1 - e_{i'} \right) \right) \left( 1 - f_i \left( 1 - f_{i'} \right) \right) \left[ \gamma \left\{ \prod_{i=1}^{n} \left( 1 - d_i \right) \left( 1 - d_{i'} \right) \right\}^{\omega_i} - \prod_{i=1}^{n} \left( 1 - d_i \right) \left( 1 - d_{i'} \right) \right] \]
3.2. IIF Hamacher interactive hybrid weighted geometric (IIFHIHWG) operator.

**Definition 3.3.** Consider a family of an IVIFN, \( \alpha_i = ([a_i, b_i], [c_i, d_i]), (i = 1, 2, \ldots, n) \) and IIFHIHWG : \( \Omega^n \rightarrow \Omega \), if

\[
\text{IIFHIHWG}(\alpha_1, \ldots, \alpha_n) = \tilde{\alpha}_{\sigma(1)}^{\omega_1} \otimes \tilde{\alpha}_{\sigma(2)}^{\omega_2} \otimes \cdots \otimes \tilde{\alpha}_{\sigma(n)}^{\omega_n}
\]

then IIFHIHWG is called IIF Hamacher interactive hybrid weighted geometric operator where \( \tilde{\alpha}_i = \alpha_i^{n\omega_i}, i = 1, 2, \ldots, n \); \( w = (w_1, w_2, \ldots, w_n)^T \) is the weight vector of \( \alpha_i \) with \( w_i > 0 \), \( \sum_{i=1}^{n} w_i = 1 \) and \( (\sigma(1), \sigma(2), \ldots, \sigma(n)) \) is a permutation of \( (1, 2, \ldots, n) \), such that \( \tilde{\alpha}_{\sigma(i)}^{\omega_i} \geq \tilde{\alpha}_{\sigma(i)}^{\omega_i} \) for \( i = 2, 3, \ldots, n \).

**Theorem 3.4.** Suppose \( \alpha_i = ([a_i, b_i], [c_i, d_i]), (i = 1, 2, \ldots, n) \) be a family of IVIFNs then aggregate value by IIFHIHWG operator can be expressed as

\[
\text{IIFHIHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left[ \prod_{i=1}^{n} \left( 1 + (\gamma - 1) \tilde{c}_{\sigma(i)}^{\omega_i} + (\gamma - 1) \prod_{i=1}^{n} \left( 1 - \tilde{c}_{\sigma(i)}^{\omega_i} \right) \right) \right]^\gamma \prod_{i=1}^{n} \left( 1 - \tilde{b}_{\sigma(i)}^{\omega_i} \right) - \prod_{i=1}^{n} \left( 1 - \tilde{b}_{\sigma(i)}^{\omega_i} \right) \right]^\gamma \prod_{i=1}^{n} \left( 1 - \tilde{d}_{\sigma(i)}^{\omega_i} \right) - \prod_{i=1}^{n} \left( 1 - \tilde{d}_{\sigma(i)}^{\omega_i} \right) \right]
\]

**Proof.** The proof is similar to Theorem 3.1. \( \square \)

**Corollary 3.** The IIFHIHWG and IIFHHWG operators have the following inequality for a collection of an IVIFN \( \alpha_i, (i = 1, 2, \ldots, n) \)

\[
\text{IIFHIHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) \geq \text{IIFHHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n)
\]

**Property 3.5.** Let \( \alpha_i, \beta_i, (i = 1, 2, \ldots, n) \) and \( \beta \) be collection of IVIFNs then we have the following properties.

(i) Idempotency: If all \( \alpha_i, (i = 1, 2, \ldots, n) \) are equal i.e., \( \alpha_i = \alpha_0 \) for all \( i \), then

\[
\text{IIFHIHWG}(\alpha_1, \ldots, \alpha_n) = \alpha_0
\]

(ii) Boundedness:

\[
\alpha_{\min} \leq \text{IIFHIHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha_{\max}
\]

where \( \alpha_{\min} = \min(\alpha_1, \alpha_2, \ldots, \alpha_n) \) and \( \alpha_{\max} = \max(\alpha_1, \alpha_2, \ldots, \alpha_n) \)
(iii) Monotonicity: If \( \alpha_i \leq \beta_i \) then for every weight vector \( \omega \), we have
\[
\text{IFHIFHWG}(\alpha_1, \ldots, \alpha_n) \leq \text{IFHIFHWG}(\beta_1, \ldots, \beta_n)
\]

(iv) Shift-invariance:
\[
\text{IFHIFHWG}(\alpha_1 \oplus \beta, \alpha_2 \oplus \beta, \ldots, \alpha_n \oplus \beta) = \text{IFHIFHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) \oplus \beta
\]

The proof of this properties is similar to that of IIFHIHWG operator properties and hence it is omitted here.

**Example 3.3.** Let \( \alpha_1 = \langle [0.22, 0.31], [0.23, 0.54] \rangle \), \( \alpha_2 = \langle [0.04, 0.21], [0.35, 0.46] \rangle \) and \( \alpha_3 = \langle [0.25, 0.27], [0.23, 0.40] \rangle \) be three IVIFNs, \( w = (0.314, 0.355, 0.331)^T \) is the weight vector of \( \alpha_i(i = 1, 2, 3) \) and \( \omega = (0.25, 0.50, 0.25)^T \) is the position weighted vector, i.e., \( a_1 = 0.22, a_2 = 0.04, a_3 = 0.25, b_1 = 0.31, b_2 = 0.21, b_3 = 0.27, c_1 = 0.23, c_2 = 0.35, c_3 = 0.23 \) and \( d_1 = 0.54, d_2 = 0.46, d_3 = 0.40 \). Assume \( \gamma = 2 \), calculating \( \dot{\alpha}_i = c_i^2 \frac{\partial}{\partial \omega} \langle \alpha_i \rangle \) for \( i = 1, 2, 3 \) we get

\[
\dot{\alpha}_1 = \left\langle \frac{2\left\{ (1 - 0.23)^3 \times 0.314 - (1 - 0.22 - 0.23)^3 \times 0.314 \right\}}{(1 + (2 - 1) \times 0.23)^3 \times 0.314 + (2 - 1) \times (1 - 0.23)^3 \times 0.314}, \ldots \right\rangle
\]

Similarly, we get \( \dot{\alpha}_2 = \langle [0.0412, 0.2102], [0.3707, 0.4851] \rangle \) and \( \dot{\alpha}_3 = \langle [0.2491, 0.2697], [0.2284, 0.3975] \rangle \). Thus, their corresponding score values are \( S(\dot{\alpha}_1) = -0.1014, S(\dot{\alpha}_2) = -0.3022 \) and \( S(\dot{\alpha}_3) = -0.0536 \) and hence ranking of IVIFNs are \( \dot{\alpha}_3 > \dot{\alpha}_1 > \dot{\alpha}_2 \). Therefore, \( \dot{\alpha}_{\sigma(1)} = \langle [0.2491, 0.2697], [0.2284, 0.3975] \rangle \), \( \dot{\alpha}_{\sigma(2)} = \langle [0.2127, 0.3164], [0.2171, 0.5147] \rangle \) and \( \dot{\alpha}_{\sigma(3)} = \langle [0.0412, 0.2102], [0.3707, 0.4851] \rangle \) and hence based on its, we have

\[
\prod_{i=1}^{3}(1 + (\gamma - 1)\dot{a}_{\sigma(i)})^{\omega_i} = 1.1760, \quad \prod_{i=1}^{3}(1 - \dot{a}_{\sigma(i)})^{\omega_i} = 0.8174
\]
\[
\prod_{i=1}^{3}(1 + (\gamma - 1)\dot{b}_{\sigma(i)})^{\omega_i} = 1.2774, \quad \prod_{i=1}^{3}(1 - \dot{b}_{\sigma(i)})^{\omega_i} = 0.7205
\]
\[
\prod_{i=1}^{3}(1 + (\gamma - 1)\dot{c}_{\sigma(i)})^{\omega_i} = 1.2567, \quad \prod_{i=1}^{3}(1 - \dot{c}_{\sigma(i)})^{\omega_i} = 1.4772
\]
\[
\prod_{i=1}^{3}(1 - \dot{\alpha}_{\sigma(i)})^{\omega_i} = 0.7386, \quad \prod_{i=1}^{3}(1 - \dot{\dot{\alpha}}_{\sigma(i)})^{\omega_i} = 0.5199
\]
\[
\prod_{i=1}^{3}(1 - \dot{\alpha}_{\sigma(i)})^{\omega_i} = 1.7404, \quad \prod_{i=1}^{3}(1 - \dot{\dot{\alpha}}_{\sigma(i)})^{\omega_i} = 0.5622
\]
\[
\prod_{i=1}^{3}(1 - \hat{b}_{\sigma(i)} - \hat{d}_{\sigma(i)})^{\omega_i} = 0.2319 \quad \prod_{i=1}^{3}(1 + (\gamma - 1)(1 - \hat{d}_{\sigma(i)}))^{\omega_i} = 1.5213
\]

So,

\[\text{IIFHIHWG}(\alpha_1, \alpha_2, \alpha_3) = \langle [0.1768, 0.2884], [0.2597, 0.4793] \rangle\]

and \[\text{IIFHHWG}(\alpha_1, \alpha_2, \alpha_3) = \langle [0.1512, 0.2764], [0.2597, 0.4793] \rangle\]

Thus, it is clear from these results that \[\text{IIFHIHWG}(\alpha_1, \alpha_2, \alpha_3) > \text{IIFHHWG}(\alpha_1, \alpha_2, \alpha_3)\]

4. **MCDM method based on the proposed operators.** An approach has been presented to investigate the MCDM problem based on the proposed operators in which preferences related to the different alternatives are taken in the form of IVIFNs and their corresponding attribute weights are of real numbers. For this, assume that \(X = \{X_1, X_2, \ldots, X_m\}\) be a discrete set of alternatives which are evaluated by the decision-makers under the set of different criteria \(C = \{C_1, C_2, \ldots, C_n\}\) whose weight vectors suggested by decision maker is \(\omega_j, \omega_j > 0\) and \(\sum_{j=1}^{n} \omega_j = 1\). Let \(\alpha_{ij} = \langle [a_{ij}, b_{ij}], [c_{ij}, d_{ij}] \rangle (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)\), \(0 \leq a_{ij}, b_{ij}, c_{ij}, d_{ij} \leq 1\) and \(b_{ij} + d_{ij} \leq 1\) be the values provided by the decision maker for the alternative \(X_i \in X\) with respect to the criteria \(C_j \in C\) in the form of IVIFNs such that \([a_{ij}, b_{ij}]\) represents the degree of acceptance of the \(i^{th}\) alternative under the \(j^{th}\) criteria given by the decision-makers while \([c_{ij}, d_{ij}]\) represents the degree of rejection of the alternative. Then the procedure for computing the best alternative based on the proposed operators are summarized as follows.

**(Step 1.) Construction of IVIF decision-making matrix:** The preference given by the decision-makers towards the different alternatives are summarized in the form of IVIF decision-matrix \(D = \langle \alpha_{ij} \rangle_{m \times n}\) which are expressed as

\[
D = \\
\begin{bmatrix}
\langle [a_{11}, b_{11}], [c_{11}, d_{11}] \rangle & \langle [a_{12}, b_{12}], [c_{12}, d_{12}] \rangle & \ldots & \langle [a_{1n}, b_{1n}], [c_{1n}, d_{1n}] \rangle \\
\langle [a_{21}, b_{21}], [c_{21}, d_{21}] \rangle & \langle [a_{22}, b_{22}], [c_{22}, d_{22}] \rangle & \ldots & \langle [a_{2n}, b_{2n}], [c_{2n}, d_{2n}] \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle [a_{m1}, b_{m1}], [c_{m1}, d_{m1}] \rangle & \langle [a_{m2}, b_{m2}], [c_{m2}, d_{m2}] \rangle & \ldots & \langle [a_{mn}, b_{mn}], [c_{mn}, d_{mn}] \rangle
\end{bmatrix}
\]

**(Step 2.) Compute normalized decision matrix:** Since the given parameters is divided into two categories namely cost \((C)\) and benefit \((B)\) and hence corresponding to it, rating values are normalized by using the following formula:

\[
r_{ij} = \begin{cases} 
\alpha_{ij}^c ; & j \in C \\
\alpha_{ij}^b ; & j \in B 
\end{cases}
\]

(6)

where \(\alpha_{ij}^c = \langle [c_{ij}, d_{ij}], [a_{ij}, b_{ij}] \rangle\) is the complement of \(\alpha_{ij} = \langle [a_{ij}, b_{ij}], [c_{ij}, d_{ij}] \rangle\).
(Step 3.) **Compute the aggregated value of alternatives.** The overall rating values \( r_i \) corresponding to each alternative \( X_i, (i = 1, 2, \ldots, m) \) is computed either by using IIFHIWG operator

\[
 r_i = \text{IIFHIWG}(r_{i1}, r_{i2}, \ldots, r_{in}), (i = 1, 2, \ldots, m)
\]

or by using IIFHIHWG operator

\[
 r_i = \text{IIFHIHWG}(\tilde{r}_{i(1)}, \tilde{r}_{i(2)}, \ldots, \tilde{r}_{i(n)}), (i = 1, 2, \ldots, m)
\]

(Step 4.) **Compute the assessment:** Rank all of the alternatives based on the value of the score values \( S(r_i) \).

(Step 5.) **Ranking the alternative:** Choose the best alternative with respect to the maximum value of \( S(r_i) \).

5. **Numerical examples.** The above proposed operators and their corresponding approach has been tested on a multi-criteria decision-making problem by considering the five different alternatives \( X_i, (i = 1, 2, \ldots, 5) \). These alternatives have to be evaluated by the decision maker(s) according to the six different criteria \( C_j, (j = 1, 2, \ldots, 6) \) and hence gave their preferences to each alternative as in the form of IVIFNs which are summarized in the standardized form in Table 1. Let \( \gamma = 2, \omega = (0.20, 0.10, 0.25, 0.10, 0.15, 0.20)^T \) is the weight vector corresponding to operators and \( w = (0.20, 0.10, 0.15, 0.25, 0.10, 0.20)^T \) is the weight vector of \( \alpha_i \).

| \( X_i \) | \( C_1 \) | \( C_2 \) | \( C_3 \) | \( C_4 \) | \( C_5 \) | \( C_6 \) |
|---------|---------|---------|---------|---------|---------|---------|
| \( x_1 \) | \([0.2, 0.3], [0.4, 0.5] \| [0.6, 0.7], [0.2, 0.3] \| [0.4, 0.5], [0.2, 0.4] \| [0.7, 0.8], [0.1, 0.2] \| [0.1, 0.3], [0.5, 0.6] \| [0.5, 0.7], [0.2, 0.3] \) | \( x_2 \) | \([0.6, 0.7], [0.2, 0.3] \| [0.5, 0.6], [0.1, 0.3] \| [0.6, 0.7], [0.2, 0.3] \| [0.6, 0.7], [0.1, 0.2] \| [0.3, 0.4], [0.5, 0.6] \| [0.4, 0.7], [0.1, 0.2] \) | \( x_3 \) | \([0.4, 0.5], [0.3, 0.4] \| [0.7, 0.8], [0.1, 0.2] \| [0.5, 0.6], [0.1, 0.4] \| [0.6, 0.7], [0.1, 0.4] \| [0.4, 0.5], [0.3, 0.4] \| [0.3, 0.5], [0.1, 0.3] \) | \( x_4 \) | \([0.6, 0.7], [0.2, 0.3] \| [0.5, 0.6], [0.1, 0.3] \| [0.7, 0.8], [0.1, 0.2] \| [0.3, 0.4], [0.1, 0.2] \| [0.5, 0.6], [0.1, 0.3] \| [0.7, 0.8], [0.1, 0.2] \) | \( x_5 \) | \([0.5, 0.6], [0.3, 0.4] \| [0.1, 0.4], [0.3, 0.5] \| [0.6, 0.7], [0.1, 0.3] \| [0.6, 0.8], [0.1, 0.2] \| [0.6, 0.7], [0.2, 0.3] \| [0.5, 0.6], [0.2, 0.4] \) |

5.1. **By IIFHIWG operator.** If we utilize the IIFHIWG operator to aggregate all these performance values \( r_{ij}, (j = 1, 2, \ldots, 6) \) corresponding to each alternative \( X_i, (i = 1, 2, \ldots, 5) \) and hence get the overall performance value \( r_i \) of the \( i^{th} \) alternative

\[
 r_1 = \text{IIFHIWG}(r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16})
\]

\[
 = \left\{ \begin{array}{c}
 2 \left( (0.6)^{0.20} (0.8)^{0.10} (0.8)^{0.25} (0.9)^{0.10} (0.5)^{0.15} (0.8)^{0.20} \\
  - (0.4)^{0.20} (0.2)^{0.10} (0.4)^{0.25} (0.2)^{0.10} (0.4)^{0.15} (0.3)^{0.20} \\
  (1.4)^{0.20} (1.2)^{0.10} (1.2)^{0.25} (1.1)^{0.10} (1.5)^{0.15} (1.2)^{0.20} \\
  + (0.6)^{0.20} (0.8)^{0.10} (0.8)^{0.25} (0.9)^{0.10} (0.5)^{0.15} (0.8)^{0.20} \end{array} \right),
\]

\[
 2 \left( (0.5)^{0.20} (0.7)^{0.10} (0.6)^{0.25} (0.8)^{0.10} (0.4)^{0.15} (0.7)^{0.20} \\
  + (0.5)^{0.20} (0.7)^{0.10} (0.6)^{0.25} (0.8)^{0.10} (0.4)^{0.15} (0.7)^{0.20} \right),
\]
Similarly, we have

\[ r_2 = \langle [0.5186, 0.6770], [0.2106, 0.3230] \rangle \quad ; \quad r_3 = \langle [0.4774, 0.6484], [0.2221, 0.3516] \rangle \]

\[ r_4 = \langle [0.6141, 0.7543], [0.1202, 0.2457] \rangle \quad ; \quad r_5 = \langle [0.5348, 0.6473], [0.1963, 0.3527] \rangle \]

Thus, by using the score function, we get the score values of each alternative as \( S(r_1) = 0.1454, S(r_2) = 0.3310, S(r_3) = 0.2760, S(r_4) = 0.5013 \) and \( S(r_5) = 0.3166 \) and hence \( S(r_4) > S(r_2) > S(r_3) > S(r_5) > S(r_1) \). Therefore, the ranking order of the five alternatives is \( X_4 > X_2 > X_5 > X_3 > X_1 \) and hence \( X_4 \) is the most desirable one while \( X_1 \) is the least one.

In order to analyze the effect of \( \gamma \) on its score values and their corresponding ranking of the alternative, we performed an experiment by taking the different values of \( \gamma \), say \( \gamma = 1, 2, 3 \) and the results corresponding to these values are summarized in Table 2 and compared the results with the existing operators results [20, 24, 26]. From this table, it has been concluded that the score values of each alternative is better than the existing approach values and hence their corresponding obtained aggregated IVIFNs are more pessimistic behavior than the existing ones. Furthermore, the overall ranking values coincide with the existing methodology results for taking a decision. Thus, the proposed IIFHIWG operator is a more promisingly operator for aggregating the different preferences of the decision makers.

**Table 2. Effect of the parameter \( \gamma \) on the ranking of the alternatives by IIFHIWG and the existing operators**

| \( \gamma = 1 \) | \( \gamma = 2 \) | \( \gamma = 3 \) |
|------------------|------------------|------------------|
| \begin{array}{l}
\text{Wen and Wang [24] Proposed} \\
\text{Wang and Liu [24] Proposed} \\
\text{Liu [24] Proposed}
\end{array} | \begin{array}{l}
\text{Score value} \\
\text{Score value} \\
\text{Score value}
\end{array} | \begin{array}{l}
\text{Score value} \\
\text{Score value} \\
\text{Score value}
\end{array} |
| \begin{array}{l}
X_1 \\
X_2 \\
X_3 \\
X_4 \\
X_5
\end{array} | \begin{array}{l}
0.1049 \\
0.2874 \\
0.0129 \\
0.1493 \\
0.0856
\end{array} | \begin{array}{l}
0.1344 \\
0.1714 \\
0.2713 \\
0.3197 \\
0.2139
\end{array} |
| \begin{array}{l}
\text{Score value} \\
\text{Score value} \\
\text{Score value}
\end{array} | \begin{array}{l}
0.1634 \\
0.3119 \\
0.2986 \\
0.3769 \\
0.3478
\end{array} | \begin{array}{l}
0.1177 \\
0.1399 \\
0.1274 \\
0.1413 \\
0.1073
\end{array} |

**5.2. By IIFHIHWG operator.** Alternatively, if we utilize IIFHIHWG operator to aggregate the IVIFNs then we firstly find \( \tilde{\alpha}_{ij} = (\alpha_{ij})^{w_j} \), where \( w = (0.20, 0.10, 0.15, 0.25, 0.10, 0.20)^T \) as

\[
\tilde{\alpha}_{11} = \left[ \frac{2 \{(0.6)^{0.20} - (0.4)^{0.20}\}}{(1.4)^{0.20} + (0.6)^{0.20}}, \frac{2 \{(0.5)^{0.20} - (0.2)^{0.20}\}}{(1.5)^{0.20} + (0.5)^{0.20}} \right].
\]
Thus, based on its score function, permutation of these IVIFNs are
\[
\hat{r}_\sigma(11) = \langle 0.7615, 0.7049, 0.1494, 0.2951 \rangle, \quad \hat{r}_\sigma(12) = \langle 0.4964, 0.8164, 0.1210, 0.1836 \rangle, \\
\hat{r}_\sigma(13) = \langle 0.5267, 0.6448, 0.2386, 0.3552 \rangle, \quad \hat{r}_\sigma(14) = \langle 0.3804, 0.5093, 0.1805, 0.3638 \rangle, \\
\hat{r}_\sigma(15) = \langle 0.0854, 0.3425, 0.3181, 0.3935 \rangle, \quad \hat{r}_\sigma(16) = \langle 0.2047, 0.2816, 0.4687, 0.5778 \rangle
\]

Hence, we can get the overall decision matrix \( \hat{R}_\sigma(ij) \) as

\[
\begin{pmatrix}
\langle 0.7615, 0.7049, 0.1494, 0.2951 \rangle \\
\langle 0.4964, 0.8164, 0.1210, 0.1836 \rangle \\
\langle 0.5267, 0.6448, 0.2386, 0.3552 \rangle \\
\langle 0.3804, 0.5093, 0.1805, 0.3638 \rangle \\
\langle 0.0854, 0.3425, 0.3181, 0.3935 \rangle \\
\langle 0.2047, 0.2816, 0.4687, 0.5778 \rangle
\end{pmatrix}
\]

Now, based on these matrix information and their corresponding weight vector
\[
\omega = (0.20, 0.10, 0.25, 0.10, 0.15, 0.20)^T,
\]
the overall preference value \( r_i \) corresponding
Therefore, based on it, the ranking order of these five alternatives is $X_4 \succ X_2 \succ X_5 \succ X_3 \succ X_1$ i.e., $X_4$ is the most desirable one while $X_1$ is the least one.

In order to compare the ranking of these alternatives with other aggregation operators [20, 24, 26] by proper assigning the value of $\gamma$ to a desired number. Their corresponding score values as well as an overall aggregated IVIFN by the existing methodology aggregated values for taking a decision. From this table, it has been concluded that the proposed operators results coincides with the existing methodology results and obtained aggregated IVIFN is more pessimistic than existing methodology aggregated values for taking a decision.

**Table 3. Effect of the parameter $\gamma$ on the ranking of the alternatives by using IIFHIWG and the existing operators**

| $\gamma = 1$ | $\gamma = 2$ | $\gamma = 3$ |
|--------------|--------------|--------------|
| Wei and Wang [24] | Proposes | Wei and Liu [24] | Proposes | Liu [24] | Proposes |
| Score value | Score value | Score value | Score value | Score value | Score value |
| $X_1$ | 0.1241 | 0.4068 | 0.2141 | 0.2904 | 0.1159 | 0.4051 |
| $X_2$ | 0.2384 | 0.3711 | 0.3443 | 0.2714 | 0.0522 | 0.4705 |
| $X_3$ | 0.2535 | 0.3819 | 0.3040 | 0.3848 | 0.0802 | 0.3113 |
| $X_4$ | 0.2573 | 0.4572 | 0.2652 | 0.4815 | 0.3000 | 0.4426 |
| $X_5$ | 0.1341 | 0.3414 | 0.1514 | 0.3519 | 0.3203 | 0.5000 |

**5.3. Effect of $\gamma$ on ranking.** To analyze the effect of $\gamma$ on the most desirable alternatives on the given attributes, an experiment has been conducted for different values of $\gamma$'s, ($\gamma = 0.1, 0.5, 1, 2, 5, 10, 25$). The score values for each attribute $X_i (i = 1, 2, 3, 4, 5)$ by taking different operators IIFHIWG and IIFHIHWG along with their ranking order are computed and are summarized in Table 4. From this analysis, it has been concluded that the ranking of the given alternative is symmetric with respect to the operators and found that the most desirable attribute is $X_4$ and $X_1$ is the least one for different values of $\gamma$'s corresponding to different operators. Thus, the proposed results corresponding to different values of $\gamma$ will offer the various choices for the decision maker for assessing the decisions.

**6. Conclusion.** The objective of this manuscript is to present some geometric aggregation operators in the environment where preferences related to different alternatives are given by the decision makers in terms of IVIFNs. As from the former works, it has been concluded that their aggregation operators are unable to aggregate the information if at least one of the grades of membership is zero. Also, their operators do not consider the effect of the membership function on to the non-membership function. These shortcomings have been improved in the presented manuscript by defining the new operational laws on IVIFSs based on Hamacher t-norm and t-conorm. Based on these new operations, some geometric aggregation operators have been proposed for different IVIFNs and showed that they are more pessimistic nature than the existing operators. Some desirable properties corresponding to these operators have also been investigated. Furthermore, it has been observed from the proposed operator that when we fix the parameter $\gamma = 1$
Table 4. Ordering of the attributes for different γ

| γ  | By IFHFWG | By IFHHFWG |
|----|-----------|------------|
|    | Aggregated IVIFN Score values | Aggregated IVIFN Score values |
|    |                          |                          |
| 0.1 | {0.3771, 0.5753}, [0.2996, 0.4247] | {0.4562, 0.6029}, {0.2818, 0.3971} |
|     | 0.1140 | 0.1901 |
| 0.5 | {0.5042, 0.6545}, [0.2324, 0.3455] | {0.5016, 0.6889}, [0.1971, 0.3111] |
|     | 0.2904 | 0.3411 |
| 1   | {0.4719, 0.6441}, [0.2310, 0.3559] | {0.4808, 0.6605}, [0.2187, 0.3395] |
|     | 0.2646 | 0.2916 |
| 2   | {0.6130, 0.7519}, [0.1218, 0.2481] | {0.5404, 0.7670}, [0.1097, 0.2330] |
|     | 0.4975 | 0.4824 |
| 3   | {0.5306, 0.6405}, [0.2027, 0.3505] | {0.5338, 0.6603}, [0.1876, 0.3937] |
|     | 0.3645 | 0.3334 |

Then our operator reduces to the IIFWG operator [26] and when γ = 2 then it reduces to IIFEWG operators [24]. Therefore, these operators are the special cases of the proposed operators. Finally, the proposed operators have been applied to...
solve the MCDM problem and compare their results with the existing operators results. From the studies it has been concluded that the proposed results coincide with the ones, shown in existing approaches. In addition, the proposed results corresponding to different values of $\gamma$ will offer the various choices for the decision maker for assessing the decisions.

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