ALMOST ALL WREATH PRODUCT CHARACTER VALUES ARE DIVISIBLE BY GIVEN PRIMES

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Abstract. For a finite group \( G \) with integer-valued character table and a prime \( p \), we show that almost every entry in the character table of \( G \wr S_N \) is divisible by \( p \) as \( N \to \infty \). This result generalizes the work of Peluse and Soundararajan on the character table of \( S_N \).

1. Introduction

Let \( S_N \) be the symmetric group on \( N \) letters. The complex irreducible characters of \( S_N \) were calculated by Frobenius in 1900; in particular, Frobenius showed that the characters are integer-valued [Fro00]. In 2019, Alex Miller investigated the distribution of the parity of entries of the character table of \( S_N \). He made the remarkable conjecture that for any prime \( p \) and exponent \( \ell \geq 1 \), the proportion of entries of the character table of \( S_N \) divisible by \( p \) (and later \( p^\ell \) for \( \ell \geq 1 \)) tends to 1 as \( N \to \infty \) [Mil19a; Mil19b]. This conjecture was recently proved by Peluse and Soundararajan in the case \( \ell = 1 \) in [PS22].

This leaves the question of investigating the distribution of residues modulo \( p \) for more general finite groups with integer-valued character tables. A natural infinite family of such is the wreath product \( G \wr S_N \) as \( N \to \infty \). When \( G \) is a fixed group with integer-valued character table, it is known that the characters of \( G \wr S_N \) are also integer-valued [Jam06, Corollary 4.4.11]. These families include the Weyl group of type \( B_N \), when \( G = \mathbb{Z}/2\mathbb{Z} \), and wreath products \( S_M \wr S_N \) of symmetric groups.

Our main result is a generalization of Peluse and Soundararajan’s theorem:

**Theorem** (see Theorem 3.8 below). Let \( G \) be a group with integer-valued character table and let \( G \wr S_N \) be the wreath product of \( G \) with \( S_N \). For all primes \( p \), the proportion of entries in the character table of \( G \wr S_N \) which are divisible by \( p \) tends to 1 as \( N \to \infty \).

The proof relies on the combinatorics of the representations of \( G \wr S_N \). If \( G \) has \( k \) conjugacy classes, then conjugacy classes and representations of \( G \wr S_N \) are both naturally labelled by \( k \)-multipartitions of \( N \). One of the key inputs is characterizing when two elements of \( G \wr S_N \) have
columns in the character table congruent modulo $p$. In Lemma 3.2, we give a combinatorial characterization directly generalizing the corresponding criterion for $S_N$.

It is known that the character tables of all Weyl groups are integer-valued. The Weyl groups of type $A$ are the symmetric groups, where our question was answered by Peluse and Soundararajan. The Weyl groups of type $B_N$ and $C_N$ are both equal to $\mathbb{Z}/2\mathbb{Z} \wr S_N$, handled by our main theorem. The only remaining infinite family of Weyl groups is that of type $D$. In Section 4, we also show that the proportion of character values of the Weyl group of type $D_N$ divisible by a prime $p$ tends to 1 as $N \to \infty$.

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2. Preliminaries

2.1. Representation Theory of the Wreath Product. Let $G$ be a finite group and let $S_N$ be the symmetric group on $N$ letters.

Definition 2.1. The wreath product of $G$ with $S_N$, denoted $G \wr S_N$, is the group of $N \times N$ permutation matrices with nonzero entries in $G$.

We begin by recalling the representation theory of $G \wr S_N$. The representation theory of wreath products was first studied in Specht’s dissertation [Spe32], anticipated by Young’s work on the case $G = \mathbb{Z}/2\mathbb{Z}$ [You30]; see also [Zel81; Jam06] for more modern treatments. If we take the representation theory of $G$ as input data and let $N$ vary, the representation theory has structural similarities to the representation theory of $S_N$, the case when $G = 1$. While representations of the symmetric group are labelled by partitions of $N$, representations of the wreath product are labelled by multipartitions:

Definition 2.2. A $k$-multipartition of an integer $N$ is $\lambda = (\lambda_1, \ldots, \lambda_k)$ where $\lambda_i$ is a partition for all $i$ such that $\sum_{i=1}^k |\lambda_i| = N$.

Suppose that $G$ has $k$ conjugacy classes. Then $k$-multipartitions of $N$ label the conjugacy classes of $G \wr S_N$. We will not need to use the specific form of this bijection in this paper; it is used in the proofs of character formulas in Propositions 2.4 and 2.10, which we omit.

Proposition 2.3 ([Jam06], Theorem 4.2.8). If $G$ has $k$ conjugacy classes, then the conjugacy classes of $G \wr S_N$ are indexed by $k$-multipartitions of $N$. Given $x \in G \wr S_N$, the multipartition $\lambda$ corresponding to $x$ is formed as follows: for each cycle in $x$ of length $\ell$, if the product of the nonzero entries in that cycle is in the $i$th conjugacy class of $G$, then add $\ell$ to $\lambda_i$. 

2
One can check the assignment of a conjugacy class to a multipartition is well-defined by checking under conjugation by $S_N$ and by diagonal matrices $G^N \subseteq G \wr S_N$. Conjugating an element of $G \wr S_N$ by $S_N$ does not change the set of cycle products at all. If $(g_1, \ldots, g_N) \in G^N$ and $(12 \cdots N)$ is an $N$-cycle, the conjugate of $(12 \cdots N)(g_1, g_2, \ldots, g_N)$ by $(g, 1, \ldots, 1)$ is $(12 \cdots N)(g_1 g^{-1}, g_2, \ldots, g g_N)$; these two elements have conjugate cycle products $g_N \cdots g_2 g_1$ and $g(g_N \cdots g_2 g_1) g^{-1}$. The general case of conjugation by $G^N$ reduces to the above case.

To find the complex irreducible representations of $G \wr S_N$, we need the complex irreducible representations of $G$ as input; call the irreducible $G$-representations $V_1, \ldots, V_k$.

**Proposition 2.4** ([Jam06], Theorem 4.4.3). If $G$ has $k$ conjugacy classes, then the irreducible representations of $G \wr S_N$ are in bijection with $k$-multipartitions of $N$. For $\lambda = (\lambda_1, \ldots, \lambda_k)$ a $k$-multipartition of $N$, let $a_i = |\lambda_i|$ and $G_a = G \wr S_a$. Then the irreducible representation of $G \wr S_N$ corresponding to $\lambda$ is

$$V^\lambda = \text{Ind}_{G_a \times \cdots \times G_{a_k}}^{G^N} \left( \bigotimes_{i=1}^k (S^{\lambda_i} \otimes V^{a_i}) \right)$$

where $S^{\lambda_i}$ is the Specht module for $S_N$ corresponding to $\lambda_i$.

Character values of wreath products can be calculated using a modified version of the Murnaghan-Nakayama rule for the symmetric group. Let $\chi^\lambda$ be the character of $V^\lambda$ and $\chi^\lambda_{\mu}$ be the value of $\chi^\lambda$ on the conjugacy class corresponding to $\mu$. Then $\chi^\lambda_{\mu}$ is calculated by decomposing the of Young diagrams of $\lambda_i$ for all $i$ using rimhooks:

**Definition 2.5.** A rimhook of a $k$-multipartition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is $k$ adjacent boxes in the Young diagram of some $\lambda_i$ such that no other boxes are remaining south or east after the rimhook has been removed and no box in the rimhook has a southeast neighbor.

![Figure 1. Examples of three invalid and one valid rimhooks in $\lambda = ((3^12^1))$.](image)

**Definition 2.6.** For $k$-multipartitions $\lambda$ and $\mu$, a rimhook decomposition of $\lambda$ by $\mu$ is obtained by repeatedly removing rimhooks in $\lambda$ with parts of $\mu$ in a fixed ordering such that after all rimhooks have been taken, there are no boxes of $\lambda$ left. All the possible ways to take rimhooks of $\lambda$ with parts of $\mu$ is the set $\text{RHD}(\lambda, \mu)$.

The Murnaghan-Nakayama rule can be modified for wreath products as follows:
Proposition 2.7 ([Jam06], Theorem 4.4.10). Let $\lambda$ and $\mu$ be $k$-multipartitions of $N$. Let $\chi^1, \chi^2, \ldots, \chi^k$ be the irreducible characters of $G$. For $\rho \in RHD(\lambda, \mu)$, let $\psi(\rho)$ be defined by

$$\psi(\rho) = \prod_{i=1}^{k} \left( \prod_{\text{rimhooks } h \text{ in } \lambda_i} \chi^i(c_h) \right)$$

where $c_h$ is the conjugacy class of $G$ associated to $h$. Then

$$\chi^\lambda_\mu = \sum_{\rho \in RHD(\lambda, \mu)} (-1)^{ht(\rho)} \psi(\rho),$$

where $ht(\rho)$ is the height of the rimhook decomposition.

The permutation module characters of wreath products form another basis for the space of class functions of $G \wr S_N$ that is easier to work with.

Definition 2.8. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a $k$-multipartition of $N$ and let $a_i = |\lambda_i|$. For each $\lambda_i$, let $S_{\lambda_i}$ be the Young subgroup of $S_{a_i}$ corresponding to $\lambda_i$ and let $G_{\lambda_i} = G \wr S_{\lambda_i}$. Then the permutation module $M^\lambda$ for $G \wr S_N$ is defined by

$$M^\lambda = \text{Ind}^{G \wr S_N}_{G_{\lambda_1} \times \cdots \times G_{\lambda_k}} \left( \bigotimes_{i=1}^{k} V_{a_i}^{\otimes a_i} \right).$$

There is a character formula for $M^\lambda$ using row decompositions instead of rimhook decompositions. It is as follows:

Definition 2.9. Let $\lambda$ and $\mu$ be $k$-multipartitions of $N$. A row decomposition of $\lambda$ by $\mu$ is a function $\rho : \{\text{rows of } \mu\} \to \{\text{rows of } \lambda\}$ such that if $r$ is a row of $\lambda$, then the rows in $\rho^{-1}(r)$ have the same total length as $r$. The set of all row decompositions of $\lambda$ by $\mu$ is denoted $RD(\lambda, \mu)$.

We will think of row decompositions of $\lambda$ by $\mu$ as a tiling of the Young diagrams of $\lambda$ by rows, where rows of $\mu$ are placed in a fixed ordering.

Figure 2. All valid row decompositions of $([\hline\hline\hline])$ by $(31\ 21)$. The numbers in the boxes indicate the order in which parts of $\mu$ are placed into rows of $\lambda$, with fixed right-to-left placement.

Proposition 2.10. Let $\lambda$ and $\mu$ be $k$-multipartitions of $N$. Let $\chi^1, \chi^2, \ldots, \chi^k$ be the irreducible characters of $G$. For $\rho \in RD(\lambda, \mu)$, let $\alpha(\rho)$ be defined by

$$\alpha(\rho) = \prod_{q=1}^{k} \left( \prod_{\text{cycles } r \text{ placed into } \lambda_q} \chi^q(c_r) \right),$$
where \( c_r \) is the conjugacy class of \( G \) associated to \( r \). Then the character for permutation module \( M^\lambda \) at \( \mu \) is

\[
M^\lambda_\mu = \sum_{\rho \in RD(\lambda, \mu)} \alpha(\rho).
\]

The proof follows from the character formula for induced representations.

We now describe the change-of-basis between irreducible and permutation characters.

**Definition 2.11.** The dominance order on \( k \)-multipartitions is defined by \( \lambda \succeq \eta \) if and only if \( \lambda_i \) dominates \( \eta_i \) for all \( i \).

**Lemma 2.12.** The matrix of multiplicities \([M^\lambda : V^\eta]\) of the irreducible representations of \( G \wr S_N \) in permutation modules is unimodular and upper-triangular with respect to dominance order.

**Proof.** Recall the Kostka numbers \( K^{\beta, \gamma} \) for \( \beta, \gamma \) partitions of \( N \) are defined by

\[
M^\beta = \text{Ind}^{S_N}_{S_\beta} 1 = \bigoplus_{\gamma} (V^\gamma)^{\otimes K^{\beta, \gamma}},
\]

where \( S_\beta \) is the Young subgroup corresponding to \( \beta \) and \( V^\gamma \) is the Specht module corresponding to \( \gamma \). Note that our notation for \( M^\beta \) and \( V^\gamma \) agrees with that of wreath products \( G \wr S_N \) when \( G = 1 \). The Kostka numbers satisfy \( K^{\beta, \beta} = 1 \) and \( K^{\beta, \gamma} > 0 \) if and only if \( \beta \succeq \gamma \) in dominance order [Mac98, p. I.6].

We claim that

\[
(1) \quad M^\lambda = \bigoplus_{\eta} (V^\eta)^{\otimes c(\lambda, \eta)}, \quad c(\lambda, \eta) = \left( \prod_{i=1}^k K^{\lambda_i, \eta_i} \right).
\]

By Definition 2.8, if \( a_i = |\lambda_i| \) for all \( i \) and \( H = G_{a_1} \times G_{a_2} \times \cdots \times G_{a_k} \), then

\[
M^\lambda = \text{Ind}_G^{G_H^\infty} \left( \bigotimes_{i=1}^k V_i^{\otimes a_i} \right) = \text{Ind}_H^{G_H^\infty} \left( \bigotimes_{i=1}^k M^\lambda_i \otimes V_i^{\otimes a_i} \right),
\]

where we make \( M^\lambda_i \otimes V_i^{\otimes a_i} \), a representation of \( G_{a_i} \), by having \( S_{a_i} \) act diagonally and \( G_{a_i} \) naturally on \( V_i^{\otimes a_i} \). Then (1) follows from multilinearity of the tensor product and linearity of induction.

Now since the matrix of Kostka numbers is unimodular and upper-triangular with respect to dominance order, the same is true of the matrix \( \{c(\lambda, \mu)\}_{\lambda, \mu} \). \( \square \)

### 2.2. Asymptotics of Partitions.

We recall a form of the Hardy-Ramanujan asymptotic for the number of partitions of \( N \), denoted \( p(N) \).

**Proposition 2.13** ([HR18], (1.36)). If \( \delta > 0 \), then

\[
\left( \frac{2 \pi}{\sqrt{6}} - \delta \right) \sqrt{N} \leq \log p(N) \leq \left( \frac{2 \pi}{\sqrt{6}} + \delta \right) \sqrt{N}
\]
for sufficiently large $N$.

Let $p_k(N)$ denote the number of $k$-multipartitions of $N$.

**Claim 2.14.** If $\delta > 0$, then

$$\left(\frac{2\pi}{\sqrt{6}} - \delta\right) \sqrt{kN} \leq \log p_k(N) \leq \left(\frac{2\pi}{\sqrt{6}} + \delta\right) \sqrt{kN}$$

for sufficiently large $N$.

This formula also appears in [Mur13]. We provide an elementary inductive proof.

**Proof.** We proceed by induction on $k$. The base case $k = 1$ is Proposition 2.13.

For $\delta > 0$, let $\delta' = \frac{\delta}{5}$. By inductive hypothesis, there exists a constant $B$ such that if $C \geq B$, then

$$\exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta'\right) \left(\sqrt{(k-1)C}\right)\right) \ll p_{k-1}(C) \ll \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right) \left(\sqrt{(k-1)C}\right)\right)$$

and

$$\exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta'\right) \left(\sqrt{C}\right)\right) \ll p(C) \ll \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right) \left(\sqrt{C}\right)\right).$$

By considering the size of the first partition in a $k$-multipartition, it follows that

$$p_k(N) = \sum_{a=0}^{N} p(a)p_{k-1}(N-a).$$

We break up the sum for $p_k(N)$ into distinct parts: let

$$D_1 = \sum_{a=0}^{B-1} p(a)p_{k-1}(N-a),$$

$$D_2 = \sum_{a=B}^{N-B} p(a)p_{k-1}(N-a),$$

$$D_3 = \sum_{a=N-B+1}^{N} p(a)p_{k-1}(N-a).$$

In $D_2$, for $B \leq a \leq N - B$, we have

$$\exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta'\right) \left(\sqrt{a + \sqrt{(k-1)(N-a)}}\right)\right) \ll p(a)p_{k-1}(N-a)$$

$$\ll \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right) \left(\sqrt{a + \sqrt{(k-1)(N-a)}}\right)\right).$$
Note that $\sqrt{a} + \sqrt{(k-1)(N-a)} \leq \sqrt{kN}$, with equality achieved at $a = \frac{N}{k}$. Summing over $a \in [B, N-B]$, we get

\[(2) \quad \exp \left( \frac{2\pi}{\sqrt{6}} \delta \sqrt{kN} \right) \ll D_2 \ll (N-2B) \exp \left( \frac{2\pi}{\sqrt{6}} + \delta' \right) \sqrt{kN}.\]

We now consider $D_1$ and $D_3$. Note that for $a \in [0, B)$, we have $p(a)p_{k-1}(N-a) \leq p(B)p_{k-1}(N)$, and for $a \in (N-B, N]$, we have $p(a)p_{k-1}(N-a) \leq p(N)p_{k-1}(B)$. Hence

\[(3) \quad 0 \leq D_1 \leq B p(B)p_{k-1}(N) \ll B \exp \left( \frac{2\pi}{\sqrt{6}} + \delta' \right) \sqrt{kN} \]

for sufficiently large $N$. Likewise,

\[(4) \quad 0 \leq D_3 \leq B p(N)p_{k-1}(B) \ll B \exp \left( \frac{2\pi}{\sqrt{6}} + \delta' \right) \sqrt{kN}.\]

Combining (2), (3), and (4), we have that

$$\exp \left( \frac{2\pi}{\sqrt{6}} - \delta \right) \sqrt{kN} \ll p_k(N) \ll N \exp \left( \frac{2\pi}{\sqrt{6}} + \delta' \right) \sqrt{kN} \ll \exp \left( \frac{2\pi}{\sqrt{6}} + \delta \right) \sqrt{kN}$$

for sufficiently large $N$. \(\square\)

Claim 2.14 implies $k$-multipartitions concentrate around having close to equal-size parts:

**Corollary 2.15.** For all $\delta > 0$, the proportion of $k$-multipartitions $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash N$ such that

$$\frac{N}{k} (1 - \delta) < |\lambda_i| < \frac{N}{k} (1 + \delta)$$

for all $1 \leq i \leq k$ goes to 1 as $N \to \infty$.

**Proof.** Pick $0 < \varepsilon < \delta$ and $1 \leq i \leq k$. The number of $k$-multipartitions of $N$ where $|\lambda_i| \notin \left( \frac{N}{k}(1-\varepsilon), \frac{N}{k}(1+\varepsilon) \right)$ is

\[(5) \quad \sum_{\lambda \text{ s.t. } |\lambda_i| \notin \left( \frac{N}{k}(1-\varepsilon), \frac{N}{k}(1+\varepsilon) \right)} p(|\lambda_i|) p_{k-1}(|\lambda_1|, \ldots, \hat{\lambda_i}|, \ldots, |\lambda_k|).\]

By Claim 2.14, the rate at which (5) approaches infinity is significantly slower than the rate at which $p_k(N)$ approaches infinity. Since $\delta > \varepsilon$, we can conclude that the number of $k$-multipartitions $\lambda$ such that $|\lambda_i| \in \left( \frac{N}{k}(1-\delta), \frac{N}{k}(1+\delta) \right)$ for all $i$ tends to 1 as $N \to \infty$. \(\square\)

3. Main Results

3.1. **Character Table Column Congruences.** Corollary 3.3 below, which we call “the mashing rule,” gives a criterion for mod $p$ congruence of two columns of the character table of $G \wr S_N$ in terms of $k$-multipartitions.
In this section, we must assume that $G$ has integer-valued character table. By [Ser77, §13.1], the group $G$ has integer-valued character table if and only if $\sigma \in G$ is conjugate to $\sigma^j$ whenever $j$ is prime to the order of $\sigma$.

**Definition 3.1.** Let $\sim_p$ be the equivalence relation on $k$-multipartitions generated by the following: $\mu \sim_p \nu$ if there is $j$ such that $\mu_i = \nu_i$ for $i \neq j$, and $\nu_j$ is formed by replacing one part of size $mp$ in $\mu_j$ with $p$ parts of size $m$ in $\nu_j$.

$$
\begin{pmatrix}
\text{\includegraphics[width=0.2\textwidth]{figure1.png}}
\end{pmatrix} \sim_3
\begin{pmatrix}
\text{\includegraphics[width=0.2\textwidth]{figure2.png}}
\end{pmatrix}
\sim_3
\begin{pmatrix}
\text{\includegraphics[width=0.2\textwidth]{figure3.png}}
\end{pmatrix}
$$

Figure 3. Example of three conjugacy classes which are congruent mod 3 in $\mathbb{Z}/2\mathbb{Z} \wr S_N$ (note that $m = 2$ in the first cycle type and $m = 1$ in the second).

**Lemma 3.2.** Let $p$ be a prime and $G$ be a group with integer-valued character table. Let $\mu = (\mu_1, \ldots, \mu_k)$ and $\nu = (\nu_1, \ldots, \nu_k)$ be $k$-multipartitions of $N$, indexing conjugacy classes of $G \wr S_N$. If $\mu \sim_p \nu$, then $M^\lambda_\mu \equiv M^\lambda_\nu \pmod{p}$ for all $k$-multipartitions $\lambda$ of $N$.

**Proof.** It suffices to show $M^\lambda_\mu \equiv M^\lambda_\nu \pmod{p}$ if there exists $j$ such that $\mu_i = \nu_i$ for all $i \neq j$, $\mu_j = (\xi, mp)$ for some $\xi$, and $\nu_j = (\xi, m^p)$. We break $RD(\lambda, \nu)$ into two cases. In case one, we consider the row decompositions of $\nu$ where $m^p$ is tiled in the same row of $\lambda$. In case two we consider the row decompositions when $m^p$ is not tiled in the same row. Recalling our formula for characters of permutation modules in Proposition 2.10, let

$$
\beta = \sum_{\rho \in RD(\lambda, \nu) \text{ s.t. } m^p \text{ is tiled in the same row}} \alpha(\rho)
$$

and

$$
\gamma = \sum_{\rho \in RD(\lambda, \nu) \text{ s.t. } m^p \text{ is not tiled in the same row}} \alpha(\rho),
$$

so that $M^\lambda_\nu = \beta + \gamma$. Case one will show $\beta \equiv M^\lambda_\mu \pmod{p}$. Case two shows $\gamma \equiv 0 \pmod{p}$. Together, these two congruences imply $M^\lambda_\mu \equiv M^\lambda_\nu \pmod{p}$.

In both cases, we break into subcases based on the ways to tile $\mu_i$ for $i \neq j$ and $\xi$. In case one, we have compatible tilings for $\mu$ and $\nu$, and in case two, we have additional tilings for $\nu$.

In case one, assume we have tiled all rows of $\mu_i$ for all $i \neq j$ and we have tiled $\xi$. We now have one row remaining. There is only one way to tile the last row for both $\mu$ and $\nu$: put
the remaining pieces into the remaining row. Let these row decompositions be denoted \( \rho_\mu \) and \( \rho_\nu \) respectively.

For \( \rho_\mu \), say that we place the final row \( r \) of size \( mp \) in the partition \( \lambda_q \). The associated cycle product is \( c_j \) because \( mp \) comes from \( \mu_j \). Then \( mp \) contributes \( \chi_q(c_j) \) to the product \( \alpha(\rho_\mu) \). Then for \( \rho_\nu \), the \( p \) rows of size \( m \) are placed into \( \lambda_q \). The conjugacy class of \( G \) associated with the \( p \) rows of size \( m \) is again \( c_j \), so \( m^p \) contributes a factor of \( \chi_q(c_j)^p \) to \( \alpha(\rho_\nu) \).

By assumption, the character values of \( G \) are integral, so by Fermat’s little theorem, \( \chi_q(c_j) \equiv \chi_q(c_j)^p \pmod{p} \). All other factors in \( \alpha(\rho_\mu) \) contributed by \( \mu_i \) for \( i \neq j \) and \( \xi \) are identical to the corresponding factors in \( \alpha(\rho_\nu) \) Hence, \( \alpha(\rho_\mu) \equiv \alpha(\rho_\nu) \pmod{p} \).

Summing over all the tilings in case one, we find \( M_\mu^\lambda \equiv \beta \pmod{p} \).

In case two, assume we have tiled all rows of \( \mu_i \) for \( i \neq j \) and \( \xi \), after which there are \( t > 1 \) remaining unfilled rows of the Young diagrams of \( \lambda \). If \( T \subseteq RD(\lambda, \nu) \) is the set of row decompositions extending our given tiling by \( \mu_i \) for \( i \neq j \) and \( \xi \), then we will show

\[
\sum_{\rho \in T} \alpha(\rho) \equiv 0 \pmod{p}.
\]

Then \( \gamma \) is the sum over all such \( T \) of \( \sum_{\rho \in T} \alpha(\rho) \), from which it will follow \( \gamma \equiv 0 \pmod{p} \).

Call the lengths of the remaining rows \( (ml_1, ml_2, \ldots, ml_t) \). Since the elements of \( T \) are in bijection with choices of placements of \( p \) cycles of length \( m \) into these rows,

\[
(8) \quad |T| = \binom{p}{\ell_1, \ell_2, \ldots, \ell_t}.
\]

Let \( \rho \in T \). Note that all pieces of \( m^p \) come from \( \mu_j \), and thus have cycle product \( c_j \), while all other cycles in \( \mu \) are in the same place in \( T \). Thus \( \alpha(\rho) = \alpha(\rho') \) for all \( \rho, \rho' \in T \). Hence \( \sum_{\rho \in T} \alpha(\rho) \) is a sum of \( |T| \) identical terms. Then \( \sum_{\rho \in T} \alpha(\rho) \equiv 0 \pmod{p} \) because \( |T| \) is divisible by \( p \).

Case one has shown that \( M_\mu^\lambda \equiv \beta \pmod{p} \), and case two has shown that \( \gamma \equiv 0 \pmod{p} \). Since \( M_\mu^\lambda = \beta + \gamma \), we conclude \( M_\mu^\lambda \equiv M_\nu^\lambda \pmod{p} \).

**Corollary 3.3 (The mashing rule).** Let \( G \) have integer-valued character table and \( k \) conjugacy classes. Let \( \mu \) and \( \nu \) be \( k \)-multipartitions of \( N \). If \( \mu \sim_p \nu \), then \( \chi_\mu^\lambda \equiv \chi_\nu^\lambda \pmod{p} \) for all irreducible characters \( \chi^\lambda \) of \( G \wr S_N \).

**Proof.** The set of irreducible characters and the set of characters of permutation modules form bases for the space of class functions on \( G \wr S_N \). Since the change of basis matrix between these two bases is unimodular and upper-triangular, as stated in Lemma 2.12, \( \chi^\lambda \) can be expressed as an integral linear combination of \( M^n \) for all \( k \)-multipartitions \( \lambda \). It follows from Lemma 3.2 that \( \mu \sim_p \nu \) implies \( \chi_\mu^\lambda \equiv \chi_\nu^\lambda \pmod{p} \). \( \square \)
3.2. **Proof of Main Theorem.** Using Corollary 3.3, the existence of one zero in the character table implies many more entries are divisible by \( p \). We proceed, following Peluse and Soundararajan in [PS22], by using Proposition 2.7 to show sufficiently many entries of the character table are zero.

**Definition 3.4.** A partition is called a *t-core* if none of the hook lengths of its Young diagram are divisible by \( t \) where \( t \in \mathbb{Z} \). For example, from Figure 4 one can see that \((4, 2, 1)\) is a 5-core.

![Figure 4. Hook-lengths for \( \lambda_i = (4, 2, 1) \)](image)

Peluse and Soundararajan proved the following estimate of the number of \( t \)-cores when \( t \) is slightly larger than the typical longest cycle in a random conjugacy class:

**Proposition 3.5 ([PS22], Proposition 1).** Let \( L \) be a positive integer, and let \( A \) be a real number with \( 1 \leq A \leq \log L / \log \log L \). Additionally suppose that \( t \) is a positive integer with

\[
 t \geq \frac{\sqrt{6}}{2\pi} \sqrt{L} (\log L) \left(1 + \frac{1}{A}\right).
\]

Then the number of partitions \( \lambda \) of \( L \) which are not \( t \)-cores is at most

\[
 O \left( p(L) \frac{\log L}{L^{\frac{1}{10}}} \right),
\]

independent of \( t \) satisfying (9).

Complementing the estimate in Proposition 3.5, Peluse and Soundararajan also estimated how many columns of the character table are congruent to a column corresponding to a partition with a large first part:

**Proposition 3.6 ([PS22], Proposition 2).** Let \( p \leq \frac{(\log L)}{(\log \log L)^{\pi}} \) be a prime. Starting with a partition \( \mu \) of \( L \), we repeatedly replace every occurrence of \( p \) parts of the same size \( m \) by one part of size \( mp \) until we arrive at a partition \( \tilde{\mu} \) where no part appears more than \( p - 1 \) times. Then the largest part of \( \tilde{\mu} \) exceeds

\[
 \frac{\sqrt{6}}{2\pi} \sqrt{L} (\log L) \left(1 + \frac{1}{5p}\right),
\]

except for at most

\[
 O \left( p(L) \exp \left(-L^{\frac{1}{15p}}\right) \right)
\]

partitions \( \mu \).
We now extend Peluse and Soundarajan’s estimate in Proposition 3.6 to $k$-multipartitions.

**Proposition 3.7.** Let $p \ll N$ be a prime. Given a $k$-multipartition $\mu = (\mu_1, \ldots, \mu_k)$ of $N$, for all $\mu_i$ with $1 \leq i \leq k$, we repeatedly replace every occurrence of $p$ parts of the same size $m$ by one part of size $mp$ until we arrive at a $k$-multipartition $\tilde{\mu}$ where no part in any $\tilde{\mu}_i$ appears more than $p - 1$ times.

Then the largest part of $\tilde{\mu}$ is of size at least

$$
\frac{\sqrt{6}}{2\pi} \sqrt{\frac{N}{k}} \left( \log \frac{N}{k} \right) \left( 1 + \frac{1}{5p} \right)
$$

except for a number of multipartitions $\mu$ which is at most

$$
O \left( \exp \left( - \left( \frac{N}{k} \right)^{\frac{1}{15p}} \right) p_k(N) \right).
$$

**Proof.** For a $k$-multipartition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ of $N$, let $\tilde{\mu}$ be as above. We will bound above the number of $k$-multipartitions $\mu$ such that $\tilde{\mu}$ has largest part less than (10).

For any $\mu$, we know that for some $1 \leq i \leq k$, $|\mu_i| \geq \frac{N}{k}$. Fix $i$ such that $\mu_i$ has size $|\mu_i| = a \geq \frac{N}{k}$. Then Proposition 3.6 tells us that the largest part of $\tilde{\mu}_i$ exceeds

$$
\frac{\sqrt{6}}{2\pi} \sqrt{a} \left( \log a \right) \left( 1 + \frac{1}{5p} \right)
$$

except for at most

$$
O \left( p(a) \exp \left( -a^{\frac{1}{15p}} \right) \right)
$$

partitions $\mu_i$ of size $a$ and therefore at most

$$
O \left( p(a) \exp \left( -a^{\frac{1}{15p}} \right) p_{k-1}(N - a) \right)
$$

total $k$-multipartitions $\mu$ with $|\mu_i| = a$. Furthermore, since $a \geq \frac{N}{k}$,

$$
\exp \left( -a^{\frac{1}{15p}} \right) \leq \exp \left( - \left( \frac{N}{k} \right)^{\frac{1}{15p}} \right),
$$

and therefore summing over all $a \geq \frac{N}{k}$, we have that the number of multipartitions $\mu$ such that $|\mu_i| \geq \frac{N}{k}$ with no part in $\tilde{\mu}_i$ exceeding (10) is at most an absolute constant times
\[
\sum_{a=\frac{N}{k}}^{N} \exp\left(-a^{\frac{1}{15p}}\right) p(a)p_{k-1}(N-a) \ll \exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right) \sum_{a=\frac{N}{k}}^{N} p(a)p_{k-1}(N-a) \\
\ll \exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right) \sum_{a=0}^{N} p(a)p_{k-1}(N-a) \\
= \exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right) p_{k}(N).
\]

Since this bound is identical for each \(i\), the number of \(k\)-multipartitions \(\mu\) such that \(\tilde{\mu}\) does not have a part of size greater than \((10)\) is at most a factor of \(k\) greater than the bound above, and therefore also at most

\[
O\left(\exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right) p_{k}(N)\right).
\]

\[\square\]

**Theorem 3.8.** Let \(G\) be a group with integer-valued character table, and let \(G \wr S_{N}\) be the wreath product of \(G\) with the symmetric group \(S_{N}\). For all primes \(p\), the proportion of entries in the character table of \(G \wr S_{N}\) divisible by \(p\) tends to 1 as \(N \to \infty\).

**Proof.** Let \(k\) be the number of conjugacy classes of \(G\). Given a \(k\)-multipartition \(\mu\), let \(\tilde{\mu}\) be the multipartition obtained by repeatedly replacing \(p\) parts of \(\mu_i\) size \(m\) with one part of size \(mp\) until no \(\mu_i\) has a part appearing more than \(p - 1\) times. For \(A = 5p\), Proposition 3.7 implies that the largest part of \(\tilde{\mu}\) has size

\[
t \geq \frac{\sqrt{6}}{2\pi} \sqrt{\frac{N}{k}} \left(\log \frac{N}{k}\right) \left(1 + \frac{1}{A}\right)
\]

for a proportion of \(\mu\) tending to 1 as \(N \to \infty\). Now pick \(A' \geq 1\) and \(\delta > 0\) such that

\[
\left(\log \frac{N}{k}\right) \left(1 + \frac{1}{A}\right) \geq \sqrt{1 + \delta} \left(\log \left(\frac{N}{k}(1 + \delta)\right)\right) \left(1 + \frac{1}{A'}\right).
\]

By Corollary 2.15, the proportion of \(k\)-multipartitions \(\lambda = (\lambda_1, \ldots, \lambda_k) \vdash N\) such that \(|\lambda_i| \in \left(\frac{N}{k}(1 - \delta), \frac{N}{k}(1 + \delta)\right)\) for all \(i\) tends to 1 as \(N \to \infty\). Thus, consider only \((\lambda, \mu)\) satisfying the above conditions.

Our choice of \(\delta\) and \(A'\) imply that

\[
\frac{\sqrt{6}}{2\pi} \sqrt{\frac{N}{k}} \left(\log \frac{N}{k}\right) \left(1 + \frac{1}{A}\right) \geq \frac{\sqrt{6}}{2\pi} \sqrt{\frac{N(1 + \delta)}{k}} \left(\log \left(\frac{N}{k}(1 + \delta)\right)\right) \left(1 + \frac{1}{A'}\right).
\]
So if \((N_1, \ldots, N_k)\) is a partition of sufficiently large \(N\), then by Proposition 3.5, the proportion of \(k\)-multipartitions \(\lambda\) with \(|\lambda_i| = N_i\) such that some \(\lambda_i\) is not a \(t\)-core is

\[
\sum_{i=1}^k O \left( \frac{\log N_i}{N_i^{\frac{1}{2}}} \right)
\]

for all \(t\) satisfying (11), independent of \(t\). Hence, over all \(k\)-multipartitions \(\lambda\) satisfying \(|\lambda_i| \in \left( \frac{N}{k} (1 - \delta), \frac{N}{k} (1 + \delta) \right)\), the proportion of \(\lambda\) such that some \(\lambda_i\) is not a \(t\)-core is

\[
O \left( \frac{\log \left( \frac{N}{k} (1 + \delta) \right)}{\left( \frac{N}{k} (1 - \delta) \right)^{\frac{1}{2}}} \right).
\]

It follows that most \((\lambda, \mu)\) satisfy that \(\lambda_i\) is a \(t\)-core for \(t\) the largest part of \(\tilde{\mu}\). Thus, for a proportion of \((\lambda, \mu)\) tending to 1 as \(N \to \infty\), we have \(\chi_{\tilde{\mu}}^{\lambda} = 0\) by Proposition 2.7 and therefore \(\chi_{\mu}^{\lambda} \equiv 0 \mod p\) by Corollary 3.3. 

\[\square\]

4. Weyl groups of type D

**Definition 4.1.** The Weyl group of type \(D_N\) is the group of \(N \times N\) signed permutation matrices with an even number of entries equal to \(-1\).

We will denote this group by \(D_N\) also (note that it is distinct from the dihedral group). \(D_N\) is a subgroup of \(\mathbb{Z}/2\mathbb{Z} \rtimes S_N\) of index two. Hence, Clifford theory determines its representations:

**Proposition 4.2.** The irreducible representations of the Weyl group of type \(D_N\) are as follows:

1. if \((\lambda, \mu)\) is a 2-multipartition of \(N\) such that \(\lambda \neq \mu\), then

\[\text{Res}_{D_N}^{B_N} V^{\lambda, \mu} = \text{Res}_{D_N}^{B_N} V^{\mu, \lambda}\]

is an irreducible representation of \(D_N\);

2. if \((\lambda, \lambda)\) is a 2-multipartition of \(N\) with equal parts, then

\[\text{Res}_{D_N}^{B_N} V^{\lambda, \lambda}\]

is the sum of two irreducible representations of \(D_N\).

3. Each irreducible representation of \(D_N\) appears exactly once in (1) or (2).

**Proof.** Let \(\psi : B_N \to \{\pm 1\}\) be the character defined by taking the product of the nonzero entries of \(B_N\). Then \(\psi \otimes V^{\lambda, \mu} = V^{\mu, \lambda}\). Now the Proposition follows from Clifford theory (see [CR81]). \[\square\]

**Corollary 4.3.** For all primes \(p\), the proportion of entries in the character table of \(D_N\) which are divisible by \(p\) tends to 1 as \(N \to \infty\).
Proof. The number of irreducible representations of $D_N$ of the form $\text{Res}_{D_N}^{B_N} V^{\lambda,\mu}$ for $\lambda \neq \mu$ equals $\frac{1}{2} (p_2(N) - p(N/2))$ when $N$ is even, and $\frac{1}{2} p_2(N)$ when $N$ is odd. The number of irreducible representations appearing as a summand of $\text{Res}_{D_N}^{B_N} V^{\lambda,\lambda}$ is $2p(N/2)$ when $N$ is even and $0$ when $N$ is odd. By Claim 2.14, we have $p_2(N) \gg p(N/2)$ for large enough $N$, so the proportion of irreducibles of the form $\text{Res}_{D_N}^{B_N} V^{\lambda,\mu}$ goes to 1 as $N \to \infty$.

Since $D_N \subseteq B_N$ is of index two, at least half of the conjugacy classes in $B_N$ intersect $D_N$. Since most entries in the character table of $B_N$ are divisible by $p$, the same is true when we restrict to the columns which intersect $D_N$, since they are at least half of the columns. Hence, the proportion of entries in the character table of $D_N$ which are divisible by $p$ goes to 1 as $N \to \infty$. \hfill \Box

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