DIMENSIONS OF PARAMODULAR FORMS AND COMPACT TWIST MODULAR FORMS WITH INVOLUTIONS

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Abstract. The dimension formula for algebraic modular forms of any weight associated with the binary quaternion hermitian maximal lattices in non-principal genus has been known by our previous works. Here we give a new dimension formula for those with plus or minus sign of the Atkin-Lehner involution for the prime level case. Our formula is based on a class number formula of some quinary lattices by Asai and its interpretation to the type number of quaternion hermitian forms given in our previous works. By a recent result of N. Dummigan, A. Pacetti, G. Rama and G. Tornaría, the algebraic modular forms of the above type with signs have a good correspondence with paramodular forms with signs. So together with our results above, we also give a dimension formula for paramodular forms of degree two of prime level of any weight \( \det^k \text{Sym}(j) \) with \( k \geq 3 \) with plus or minus sign of the Atkin-Lehner involution. In particular we give the complete list of primes \( p \) such that there is no paramodular cusp form of level \( p \) of weight 3 with plus sign. It is known that this has a geometric meaning on the moduli of Kummer surfaces associated to \((1, p)\) polarizations.

1. Introduction

In the introduction, we will state our results on explicit dimension formulas for algebraic modular forms and paramodular forms that have the Atkin-Lehner plus or minus sign without going into details. Precise definitions and most proofs will be given in later sections.

First we start from algebraic modular forms. Let \( B \) be the definite quaternion algebra over \( \mathbb{Q} \) with prime discriminant \( p \). For a natural number \( n \), define a positive definite quaternion hermitian form \( h(x, y) \) of \( B^n \) by

\[
h(x, y) = \sum_{i=1}^{n} \bar{x_i}y_i, \quad x = (x_i), \quad y = (y_i) \in B^n,
\]

where \( \bar{\cdot} \) is the main involution of \( B \). This is the unique positive definite quaternion hermitian form on \( B^n \) up to base change of \( B^n \) over \( B \). For

\[2020 \text{ Mathematics Subject Classification.} \ 11F46, 11F55, 11F72.
\]

This work was supported by JSPS KAKENHI Grant Number JP19K03424, JP20H00115, and AIM SQUARE “Computing paramodular forms”.
\[ g = (g_{ij}) \in M_n(B), \text{ we put } g^* = \overline{g} = (\overline{g_{ij}}). \] We denote by \( G \) the group of quaternion hermitian similitudes of degree \( n \) defined by
\[ G = \{ g \in M_n(B); gg^* = n(g)1_n, n(g) \in \mathbb{Q}_+^\times \}. \]

We denote by \( G_A \) the adelization of \( G \), and by \( G_v \) the local factor of \( G_A \) at a place \( v \). In particular, if we define the subgroup \( G_1 \) of \( G \) by taking elements with \( n(g) = 1 \), then \( G_1 \) is the compact twist \( Sp(n) \) of the split symplectic group \( Sp(n, \mathbb{R}) \subset SL_{2n}(\mathbb{R}) \) of real rank \( n \). For an irreducible representation \((\rho, V)\) of \( G_1 \) such that \( \rho(\pm 1_n) = 1 \), we define a representation of \( G_A \) by
\[ (1) \quad G_A \rightarrow G_\infty \rightarrow G_\infty/\{\text{center}\} \cong G_1/\{\pm 1_n\} \rightarrow GL(V). \]

We denote this representation also by \( \rho \) by abuse of language. For an open subgroup \( U \) of \( G_A \), we define the space \( \mathcal{M}_\rho(U) \) of modular forms on \( G_A \) with respect to \( U \) of weight \( \rho \) by
\[ \mathcal{M}_\rho(U) = \{ f: G_A \rightarrow V; f(uga) = \rho(u)f(g) \text{ for all } g \in G_A, u \in U, a \in G\}. \]

(The modular forms of this type are recently called algebraic modular forms in \cite{[10]}. Classically the Brandt matrices that appeared in the theory of Eichler (and of Jacquet-Langlands) are of this sort. For higher degrees, see also \cite{[29]}, \cite{[11]}, \cite{[12]}.) When \( n = 2 \), as was shown in \cite{[12]}, the dimension of \( \mathcal{M}_\rho(U) \) is explicitly known for any \( \rho \) and for any \( U = U_{pg}(p) \) or \( U_{npq}(p) \) corresponding to the principal genus, or the non-principal genus of maximal lattices in \( B^2 \). The irreducible representation \( \rho \) of the compact group \( G_\infty \) is defined to be the action of \( \rho \) on the space of modular forms \( \mathcal{M}_\rho(U) \).

Here in this paper we are interested in the case \( U = U_{npq}(p) \) since this case has a good correspondence with paramodular forms. Our first problem is to decompose \( \mathcal{M}_{f_1,f_2}(U_{npq}(p)) \) under the Atkin-Lehner involution for \( U_{npq}(p) \) and give a dimension formula for each eigenspace. So we explain what the involution is in this case. Let \( O \) be a maximal order of \( B \). We denote by \( \pi \) a local prime element at \( p \) of \( O_p = O \otimes \mathbb{Z}_p \).

We may assume that \( \pi^2 = -p \). Then the Atkin-Lehner involution for \( \mathcal{M}_{f_1,f_2}(U_{npq}(p)) \) is defined to be the action of the Hecke double coset \( U_{npq}\pi U_{npq} = U_{npq}\pi = \pi U_{npq} \). This action gives an involution (under a natural normalization), since \( \pi^2 = -p \). The representation matrix of the action of \( U_{npq}\pi \) on \( \mathcal{M}_{f_1,f_2}(U_{npq}) \) as a Hecke algebra is denoted by \( R_{f_1,f_2}(\pi) \). The \( +1 \) and \( -1 \) eigen subspaces of \( \mathcal{M}_{f_1,f_2}(U_{npq}) \) of the matrix \( R_{f_1,f_2}(\pi) \) will be denoted by \( \mathcal{M}_{f_1,f_2}^+(U_{npq}) \) and \( \mathcal{M}_{f_1,f_2}^-(U_{npq}) \) respectively. We would like to give dimension formulas for these spaces. By definition, we have
\[ Tr(R_{f_1,f_2}(\pi)) = \dim \mathcal{M}_{f_1,f_2}^+(U_{npq}(p)) - \dim \mathcal{M}_{f_1,f_2}^-(U_{npq}(p)). \]
and since a formula for 
\[
\dim \mathcal{M}_{f_1,f_2}(U_n p g(p)) = \dim \mathcal{M}_{f_1,f_2}^T(U_n p g(p)) + \dim \mathcal{M}_{f_1,f_2}^T(U_n p g(p))
\]
has been known in [12], all we should do is to give a formula for \( Tr(R_{f_1,f_2}(\pi)) \). This will be given below.

Before stating the result, we explain some notation that we need in the formula.

The character \( Tr(\rho_{f_1,f_2}(g)) \) of \( g \in G_\infty \) depends only on the principal polynomial \( f(x) \) of \( g/\sqrt{n}(g) \in G_\infty^1 = Sp(2) \), and by our assumption that \( \rho_{f_1,f_2}(\epsilon_2) = 1 \), the characters for \( f(x) \) and \( f(-x) \) are the same. The formula for the characters is well known and found in [49]. Here, when \( p \neq 2, 3 \), we need the following principal polynomials
\[
\begin{align*}
f_2(x) &= (x - 1)^2(x + 1)^2, \\
f_6(x) &= (x^2 + 1)^2, \\
f_9(x) &= (x^2 + x + 1)(x^2 - x + 1) = x^4 + x^2 + 1, \\
f_{11}(x) &= x^4 + 1, \\
f_{13}(x) &= x^4 + \sqrt{5}x^3 + 3x^2 + \sqrt{5}x + 1,
\end{align*}
\]
and denote by \( \chi_i \), the character of elements corresponding to \( f_i(\pm x) \).

We use this strange numbering to maintain consistency with our previous works. See also Appendix.) We denote by \( h(\sqrt{-d}) \) the class number of an imaginary quadratic field \( \mathbb{Q}(\sqrt{-d}) \) and by \( B_{2,\chi} \) the second generalized Bernoulli number for the character \( \chi \) corresponding to the real quadratic extension \( \mathbb{Q}(\sqrt{d})/\mathbb{Q} \). By definition, we have
\[
B_{2,\chi} = \frac{1}{f} \sum_{a=1}^{f} \chi(a)a^2 - \sum_{a=1}^{f} \chi(a)a,
\]
where \( f \) is the conductor of \( \chi \), i.e. \( f = p \) if \( p \equiv 1 \mod 4 \) and \( f = 4p \) if \( p \equiv 3 \mod 4 \), and the latter sum is always 0 in this case since \( \chi(-1) = 1 \) ([11]). Our first theorem is given below. Here the cases \( p = 2, 3 \) are slightly exceptional and excluded in the following Theorem [14] for simplification. But a formula for \( p = 2, 3 \) will be given as Theorem [42] in section [3] with a different proof. We put \( \delta_{a,b} = 1 \) if \( a = b \) and \( = 0 \) otherwise. We define the quadratic residue symbol \( \left( \frac{d}{p} \right) \) for a prime \( p \) to be 1, -1 and 0 if \( p \) splits unramified, remains prime, and is ramified in \( \mathbb{Q}(\sqrt{d}) \), respectively.

**Theorem 1.1.** We assume that \( n = 2 \). Then for any prime \( p > 3 \), an explicit formula for \( Tr(R_{f_1,f_2}(\pi)) \) is given for any \( f_1 \geq f_2 \geq 0 \) with \( f_1 \equiv f_2 \mod 2 \) as follows.

For \( p \equiv 1 \mod 4 \), we have
\[
TrR_{f_1,f_2}(\pi) = \frac{\chi_2}{2^3 \cdot 3} \left( 9 - 2 \left( \frac{2}{p} \right) \right) B_{2,\chi} + \frac{h(\sqrt{-p})}{2^4} \chi_6
\]
\[
+ \frac{h(\sqrt{-2p})}{2^3} \chi_{11} + \frac{h(\sqrt{-3p})}{2^2 \cdot 3} \left( 3 + \left( \frac{2}{p} \right) \right) \chi_9 + \frac{\delta_{p,5}}{5} \chi_{13}
\]
For \( p \equiv 3 \mod 4 \), we have
\[
TrR_{f_1, f_2}(\pi) = \frac{\chi^2}{2^5 \cdot 3} B_{2, \chi} + \frac{h(\sqrt{-p})}{24} \left( 1 - \left( \frac{2}{p} \right) \right) \chi_6
+ \frac{h(\sqrt{-2p})}{2^3} \chi_{11} + \frac{h(\sqrt{-3p})}{2^2 \cdot 3} \chi_9
\]
where \( \chi_i = Tr(\rho_{f_1, f_2}(q_i)) \) for any \( g_i \in G^1 \) whose principal polynomials are given by \( f_i(\pm x) \) for \( i = 2, 6, 9, 11, 13 \) defined above.

The proof of Theorem 1.1 will be given in section 3.

The explicit value of \( \chi_i \) for each \( (f_1, f_2) \) is given as follows as can be easily deduced from the classical result in [49]. We use notation \( [a_0, \ldots, a_m; m]_b \) that means the number \( a_i \) when \( b = i \mod m \).

\[
\chi_2 = \frac{(-1)^{f_1} (f_1 + 2)(f_2 + 1)}{2},
\]
\[
\chi_6 = \frac{(-1)^{(f_1 + f_2)/2}}{2} \times \begin{cases} f_1 + 2 & \text{if } f_2 \equiv 0 \mod 2, \\ -(f_2 + 1) & \text{if } f_2 \equiv 1 \mod 2. \end{cases}
\]
\[
\chi_9 = \begin{cases} [1, 0, 0, -1, 0, 0; 6]_{f_2} & \text{if } f_1 - f_2 \equiv 0 \mod 6, \\ [-1, 1, 0, 1, -1, 0; 6]_{f_2} & \text{if } f_1 - f_2 \equiv 2 \mod 6, \\ [0, -1, 0, 0, 1, 0; 6]_{f_2} & \text{if } f_1 - f_2 \equiv 4 \mod 6, \end{cases}
\]
\[
\chi_{11} = \begin{cases} (1)^{(f_1 - f_2)/4} [1, -1, 0, 0; 4]_{f_2} & \text{if } f_1 - f_2 \equiv 0 \mod 4, \\ (1)^{(f_1 - f_2 - 2)/4} [0, 1, -1, 0; 4]_{f_2} & \text{if } f_1 - f_2 \equiv 2 \mod 4, \end{cases}
\]
\[
\chi_{13} = \begin{cases} [1, 2, 1, 0, 0, -1, -2, -1, 0, 0; 10]_{f_2} & \text{if } f_1 - f_2 \equiv 0 \mod 10, \\ [2, 0, -1, 1, 0, -2, 0, 1, -1, 0, 0; 10]_{f_2} & \text{if } f_1 - f_2 \equiv 2 \mod 10, \\ [-2, -2, 2, 0, 2, 2, -2, -2, 0; 10]_{f_2} & \text{if } f_1 - f_2 \equiv 4 \mod 10, \\ [-1, 0, -2, 0, 1, -1, 0, 2; 10]_{f_2} & \text{if } f_1 - f_2 \equiv 6 \mod 10, \\ [0, -1, -2, -1, 0, 1, 2, 1, 0; 10]_{f_2} & \text{if } f_1 - f_2 \equiv 8 \mod 10. \end{cases}
\]

By definition of \( \mathfrak{M}_{f_1, f_2}^\pm (U_{npq}(p)) \), we have
\[
(2) \quad \dim \mathfrak{M}_{f_1, f_2}^+(U_{npq}(p)) = \frac{1}{2} \left( \dim \mathfrak{M}_{f_1, f_2}(U_{npq}(p)) + Tr(R_{f_1, f_2}(\pi)) \right),
\]
\[
(3) \quad \dim \mathfrak{M}_{f_1, f_2}^-(U_{npq}(p)) = \frac{1}{2} \left( \dim \mathfrak{M}_{f_1, f_2}(U_{npq}(p)) - Tr(R_{f_1, f_2}(\pi)) \right).
\]

But the dimension for \( \dim \mathfrak{M}_{f_1, f_2}(U_{npq}(p)) \) is explicitly known for any \( \rho_{f_1, f_2} \) as given in [12] [11] (reproduced in Theorem 5.1 in section 5). So as a corollary of the above theorem, we have explicit dimension formulas for \( \dim \mathfrak{M}_{f_1, f_2}^+(U_{npq}(p)) \) and \( \dim \mathfrak{M}_{f_1, f_2}^-(U_{npq}(p)) \). (Naturally by the same sort of calculation we can give an explicit formula also for \( \mathfrak{M}_{f_1, f_2}^{\pm}(U_{pq}(p)) \), but we omit it here.)

Now we proceed to our next theme. Since \( G^1_{\text{so}} = Sp(2) \) is the compact split symplectic group \( Sp(2, \mathbb{R}) \subset SL_4(\mathbb{R}) \) of real rank 2, we may expect a nice correspondence between algebraic modular forms on \( G_A \) and Siegel cusp forms of degree 2. (A general principle
by Langlands, and for this special case asked also by Y. Ihara [29].) An explicit correspondence for the non-principal genus for \( n = 2 \) was conjectured in our previous works [15], [19], [23] with precise comparison of explicit dimension formulas, and this conjecture has been proved by van Hoften in [48] and by Rösner and Weissauer in [38], independently by a completely different method. (Other parahoric cases different from the above case have been conjectured in [24], [14], [13], but this is another story.) The corresponding Siegel cusp forms here are so called paramodular forms. Now there also exists the Atkin-Lehner type involution on paramodular forms of level \( p \). Recently, Dummigan, Pacetti, Rama and Tornarà generalized the above correspondence to the case between paramodular forms and algebraic modular forms with given sign of the Atkin-Lehner involution (See [5]). (Their theorem includes some general level cases, but here we are concerned only with prime level. See also [30].) We will explain more details below. For any positive integer \( N \), we denote by \( K(N) \) the paramodular subgroup of \( \text{Sp}(2, \mathbb{Q}) \) of level \( N \) defined by

\[
K(N) = \text{Sp}(2, \mathbb{Q}) \cap \left( \begin{array}{cccc}
Z & Z & Z & Z \\
Z & Z & Z & N^{-1}Z \\
Z & NZ & Z & Z \\
NZ & NZ & NZ & Z \\
\end{array} \right).
\]

Let \( H_n \) be the Siegel upper half space of degree \( n \). For any \( g = \left( \begin{array}{cc}
A & B \\
C & D \\
\end{array} \right) \in \text{Sp}(2, \mathbb{R}) \) and a \( V_{k,j} \)-valued function \( F \) of \( Z \in H_2 \), we put

\[
F|_{k,j}[g] = \rho_{k,j}(CZ + D)^{-1}F(gZ),
\]

where \( (\rho_{k,j}, V_{k,j}) \) is the irreducible representation \( \text{det}^k \text{Sym}(j) \) of \( \text{GL}_2(\mathbb{C}) \) and \( \text{Sym}(j) \) is the symmetric tensor representation of degree \( j \). We denote by \( A_{k,j}(K(N)) \) the space of paramodular forms belonging to \( K(N) \) of weight \( \rho_{k,j} \), and by \( S_{k,j}(K(N)) \) its subspace of cusp forms. By definition, \( S_{k,j}(K(N)) \) means the vector space of \( V_{k,j} \)-valued holomorphic functions on \( H_2 \) such that \( F|_{k,j}[\gamma] = F \) for any \( \gamma \in K(N) \) and that vanish at all the cusps. When \( j = 0 \), we simply write \( A_{k,0}(K(N)) = A_k(K(N)) \) and \( S_{k,0}(K(N)) = S_k(K(N)) \). The formula for \( \dim S_{k,j}(K(N)) \) is known for square free \( N \) for any \( k \geq 3 \), \( j \geq 0 \) (See [15], [20] for \( j = 0, N = \text{prime} \), [19] for \( j > 0, N = \text{prime} \), [23] for square free \( N \) with \( j = 0, k \geq 3 \) and \( j > 0 \) and \( k > 4 \), and by Dan Petersen (colloquial communication) for \( k = 3, 4, j > 0 \)).

For a prime \( p \), we put

\[
\rho = \frac{1}{\sqrt{p}} \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & p & 0 & 0 \\
p & 0 & 0 & 0
\end{pmatrix}.
\]
Then \( \iota : F \to F|_{k,j}[p] \) induces an involution on \( S_{k,j}(K(p)) \). (This can be regarded also as the action of the Hecke operator associated with \( K(p) \).) We denote by \( S_{k,j}^{\pm}(K(p)) \subset S_{k,j}(K(p)) \) the eigenspaces of \( \iota \) belonging to eigenvalues \( +1 \) and \( -1 \), respectively. To adjust the lifting part in the correspondence, we need the space \( S_{k}(\Gamma_0(p)) \) of elliptic cusp forms of weight \( k \) belonging to the group

\[
\Gamma_0(p) = SL_2(\mathbb{Z}) \cap \left( \begin{array}{ll} \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{array} \right).
\]

We denote by \( S_{j+2}(\Gamma_0(p)) \) the eigenspaces of \( S_{j+2}(\Gamma_0(p)) \) belonging to the eigenvalues \( +1 \) and \( -1 \) of the Atkin-Lehner involution \( W_p \) defined by

\[
f(z)|_{j+2}W_p = p^{-(j+2)/2}z^{-(-j+2)}f(-1/pz)
\]
on \( S_{j+2}(\Gamma_0(p)) \). We put \( S_{j+2}^{+,new}(\Gamma_0(p)) = S_{j+2}(\Gamma_0(p)) \cap S_{j+2}^{new}(\Gamma_0(p)) \) where \( S_{j+2}^{new}(\Gamma_0(p)) \) is the space of new forms.

Now by Theorem 1.1 and the above mentioned result of Dummigan, Pacetti, Rama, Tornaria in [5], together with [48], [38], and other results, we have the following explicitly calculable formula for \( \text{dim } S_{k,j}^{\pm}(K(p)) \).

**Theorem 1.2.** Let \( p \) be any prime. For \( k \geq 3 \) and even \( j \geq 0 \), we have an explicit formula for \( \text{dim } S_{k,j}^{\pm}(K(p)) \). It is given by

\[
\begin{align*}
\text{dim } S_{k,j}^{+}(K(p)) &= \text{dim } S_{k,j}(Sp(2,\mathbb{Z})) + \text{dim } \mathfrak{M}_{j+k-3,k-3}^{-}(U_{npg}(p)) \\
&- \text{dim } S_{j+2}^{+,new}(\Gamma_0(p)) \times \text{dim } S_{2k+j-2}(SL_2(\mathbb{Z}))
\end{align*}
\]

\[
\begin{align*}
\text{dim } S_{k,j}^{-}(K(p)) &= \text{dim } S_{k,j}(Sp(2,\mathbb{Z})) - \delta_{j0} \text{dim } S_{2k-2}(SL_2(\mathbb{Z})) - \delta_{j0}\delta_{k3} \\
&+ \text{dim } \mathfrak{M}_{j+k-3,k-3}^{+}(U_{npg}(p)) - \text{dim } S_{j+2}^{-,new}(\Gamma_0(p)) \times S_{2k+j-2}(SL_2(\mathbb{Z})).
\end{align*}
\]

where the main part of RHS is explicitly given by Theorem 1.1, Theorem 4.2, and the other parts are explained below.

In particular, \( S_{3}^{+}(K(p)) \) has a geometric meaning on the moduli of the Kummer surfaces associated to \( (1,p) \) polarization (See [8]), and by the above formula we give the complete list of primes such that \( S_{3}^{+}(K(p)) = 0 \) in Proposition 6.1. This includes a partial result in [3] and [9] for primes, though they also treated composite levels.

The meaning of the dimensional relations in Theorem 1.2 and its proof will be explained later in section 2. Here we will explain why this is an explicit formula. The formula for \( \text{dim } \mathfrak{M}^{+}(U_{npg}(p)) \) is deduced by Theorem 1.1, Theorem 4.2, and (2) and (3) by virtue of [12] (reproduced as Theorem 5.1 in section 5). The formula for \( \text{dim } S_{k,j}(Sp(2,\mathbb{Z})) \) is in [27], [28], [17], [33] (not known for the case \( k = 2 \) and big \( j > 0 \) but this case is also excluded in Theorem 1.2). The dimension \( \text{dim } S_{j+2}^{+,new}(\Gamma_0(p)) \) is essentially in [51], and easily obtained by the formula (4), (5), (6), (7) explained below. The formula for \( \text{dim } S_{k}(SL_2(\mathbb{Z})) \) is well known and given by (8) below. So the above Theorem 1.2 gives a really calculable formula for any given prime \( p \), \( k \geq 3 \) and even \( j \geq 0 \).
We review the formula for $S_{j+2}^{\pm, new}(\Gamma_0(p))$ in [51] for readers’ convenience (see also [32]). For any prime $p$ and even $k \geq 2$, as well known we have

$$\dim S_k(\Gamma_0(p)) = \dim S_k^{+, new}(\Gamma_0(p)) + \dim S_k^{-, new}(\Gamma_0(p))$$

$$= \left(\frac{(p-1)(k-1)}{12}\right) + \frac{1}{4}(-1)^{k/2} + \frac{1}{3}[-1, 0, 1; 3]_k \left(1 - \left(\frac{-3}{p}\right)\right) - \delta_{k2}.$$

For $p > 3$ and even $k \geq 2$, we have

$$\dim S_k^{+, new}(\Gamma_0(p)) - \dim S_k^{-, new}(\Gamma_0(p)) = (-1)^{k/2}a_p \cdot h(\sqrt{-p}) + \delta_{k2},$$

where $h(\sqrt{-p})$ is the class number of $\mathbb{Q}(\sqrt{-p})$ and $a_p = 1$ if $p \equiv 1 \mod 4$, $a_p = 2$ if $p \equiv 7 \mod 8$, and $a_p = 4$ if $p \equiv 3 \mod 8$. We can also write $a_p h(\sqrt{-p}) = h(-p) + h(-4p)$, where $h(-d)$ denotes the class number of quadratic order of discriminant $-d$ (not necessarily maximal), regarding $h(-d) = 0$ if $-d \equiv 2, 3 \mod 4$.

The case $p = 2$ and $3$ for even $k \geq 2$ is given by

$$\dim S_k^{+, new}(\Gamma_0(2)) - \dim S_k^{-, new}(\Gamma_0(2)) = \frac{(-1)^{k/2} - (-1)^{(k-4)(k-2)/8}}{2} + \delta_{k2},$$

$$\dim S_k^{+, new}(\Gamma_0(3)) - \dim S_k^{-, new}(\Gamma_0(3)) = \delta_{k2} + \begin{cases} -1 & \text{if } k \equiv 2, 6 \mod 12, \\ 0 & \text{if } k \equiv 4, 10 \mod 12, \\ 1 & \text{if } k \equiv 0, 8 \mod 12. \end{cases}$$

It is also well known that

$$\dim S_k(SL_2(\mathbb{Z})) = \frac{k-1}{12} + \frac{1}{4}(-1)^{k/2} + \frac{1}{3}[1, 0, -1; 3]_k - \frac{1}{2} + \delta_{k2}.$$

As a special case of Theorem 1.2 for $(k, j) = (3, 0)$, we have

$$\dim S_3(\Gamma(p)) = H - T, \quad \dim S_3(\Gamma(p)) = T - 1$$

where $H$ is the class number and $T$ is the $(G$,-)type number of the non-principal genus $\mathcal{L}_{npq}$ in $B^2$ defined in [25]. The formula and numerical tables for $H$ and $T$ were given in [12] II and [26] p.218. Numerical examples of $\dim S_3^{-, new}(\Gamma(p))$ for many primes $p$ have been already given in [34] Table 4 and of course our results coincide with these values.

More numerical tables will be given in section 6.

Acknowledgement: We would like to thank Neil Dummigan for informing the interesting result in his joint work [5] and for answering author’s question in detail, as well as his big interest around the problem, and also to Cris Poor and David S. Yuen for constantly asking the
author the dimension formula with involution and sometimes assuring numerical correctness of our calculation by their examples.

2. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 since this is much shorter than the proof of Theorem 1.1. We also explain the meaning of the dimensional relations in the theorem. Of course this relation can be read as a reflection of the Hecke equivariant bijection, but since this is obvious by [48], [38] and [5], we do not explain such details.

Paramodular forms in $S_{k,j}(K(p))$ contain forms coming from $Sp(2, \mathbb{Z})$, that is, from $S_{k,j}(Sp(2, \mathbb{Z}))$ and $S_{k,j}(Sp(2, \mathbb{Z}))|_{k}[\rho] = S_{k,j}(\rho^{-1}Sp(2, \mathbb{Z})\rho) \cong S_{k,j}(Sp(2, \mathbb{Z}))$ by trace operators. These are old forms in the sense of [36] and also explained in the much earlier paper [15]. In general, old forms of $Sp(2, \mathbb{Z})$ doubly appear in paramodular forms, but the Saito-Kurokawa lift from $S_{2k-2}(SL_2(\mathbb{Z}))$ (that exists only when $j = 0$ and $k$ is even) exceptionally appears only once. Indeed, by [37] Table 1 and [41], the possibility of local representations at $p$ which have both $Sp(2, \mathbb{Z}_p)$ fixed vectors and $K(p)$ fixed vectors are (I) and (IIb) in the notation of [36]. Here (IIb) corresponds to the Saito-Kurokawa type (P), where $K(p)$ fixed vector is unique up to scalar with plus Atkin-Lehner sign, while (I) corresponds to the general type (G), where there are two $K(p)$ fixed vectors, one is of Atkin-Lehner plus and the other is of Atkin-Lehner minus. This can be seen in [36] Table A.15. (See also [7], [12], [41].) So the dimensions of new forms in $S_{k,j}^+(K(p))$ in the sense of [36] is given by

$$\begin{align*}
\dim S_{k,j}^+(K(p)) &- \dim S_{k,j}(Sp(2, \mathbb{Z})) \\
\dim S_{k,j}^-(K(p)) &- (\dim S_{k,j}(Sp(2, \mathbb{Z})) - \delta_{k,\text{even}} \dim S_{2k-2}(SL_2(\mathbb{Z}))).
\end{align*}$$

Now, [48] and [38] claim that the general part (i.e. non-lift part) of $\mathcal{M}_{k+j-3,k-3}(U_{npg}(p))$ and that of new forms of $S_{k,j}(K(p))$ have Hecke equivariant bijection. By [5] Theorem 10.1 (i), the general new part of $S_{k,j}(K(p))$ corresponds to the general part of $\mathcal{M}_{k+j-3,k-3}(U_{npg})$ (where double sign corresponds). So we have to see the Atkin-Lehner signs of the remaining lifting part. When $k$ is odd, then there is no Saito Kurokawa lift from $S_{2k-2}(SL_2(\mathbb{Z}))$ to $Sp(2, \mathbb{Z})$, but there exists an Ihara lift in [29], [16], [19] from $S_{2k-2}(SL_2(\mathbb{Z}))$ to algebraic modular forms and this lift is injective to $\mathcal{M}_{k-3,k-3}(U_{npg}(p))$ by [48] Theorem 8.2.1 (3) and [38] Proposition 12.2. By virtue of [5] Theorem 10.1 (ii), we see that this appears in $\mathcal{M}_{k-3,k-3}(U_{npg}(p))$. Also, when $k = 3$ and $j = 0$, $\mathcal{M}_{0,0}(U_{npg}(p))$ contains a constant function of $G_A$ which belongs to Atkin-Lehner plus, and this does not correspond to a cusp form. So we must subtract

$$\delta_{k,\text{odd}} \dim S_{2k-2}(SL_2(\mathbb{Z})) + \delta_j \delta_k.$$
from $\mathcal{M}^{+}_{k,j-3,k-3}(U_{np})$. Now by [38] Theorem 8.2.1 (3) and [38] Proposition 12.1, there is an injective Yoshida lifting from $S^{new}_{j+2}(\Gamma_0(p)) \times S_{2k+j-2}(SL_2(\mathbb{Z}))$ to $\mathcal{M}^{+}_{k,j-3,k-3}(U_{np})(p)$. The signature of the image of the Yoshida lift is given in [3] Theorem 10.1 (iii). This gives the following injection

\[
S^{+}_{j+2}(\Gamma_0(p)) \times S_{2k+j-2}(SL_2(\mathbb{Z})) \hookrightarrow \mathcal{M}^{+}_{k,j-3,k-3}(U_{np}), \\
S^{-}_{j+2}(\Gamma_0(p)) \times S_{2k+j-2}(SL_2(\mathbb{Z})) \hookrightarrow \mathcal{M}^{-}_{k,j-3,k-3}(U_{np})(p).
\]

Note that there is no Yoshida lift in $S_{k,j}(K(p))$ by [33] Lemma 2.2.1. The only remaining part now is a lift from $S^{new}_{2k-2}(\Gamma_0(p))$. For paramodular forms, this is the Gritsenko lift (paramodular version of Saito-Kurokawa lift) from $S^{new}_{2k-2}(\Gamma_0(p))$ to $S_{k}(K(p))$. Here we must have $\epsilon = (-1)^k$, since originally the elliptic modular forms here should correspond with Jacobi forms of index $p$ of weight $k$ and the sign of the functional equation should be $-1 = (-1)^{k-1}\epsilon$. The Atkin-Lehner sign of Gritsenko lifts is $(-1)^k$ ([7] (12), [41] Theorem 5.2 (i)). So we have the following injections.

\[
S^{+}_{2k-2}(\Gamma_0(p)) \hookrightarrow S^{+}_{k}(K(p)) \quad \text{for even } k, \\
S^{-}_{2k-2}(\Gamma_0(p)) \hookrightarrow S^{-}_{k}(K(p)) \quad \text{for odd } k.
\]

(By the way, the fact that the sign of the Saito-Kurokawa lift from $S_{2k-2}(SL_2(\mathbb{Z}))$ to $S_{k}(K(p))$ is plus can be proved also by [7] Theorem 3 Corollary.) For the compact twist, we have also a lift from $S^{new,(−1)^k}_{2k-2}(\Gamma_0(p))$ to $\mathcal{M}^{±}_{k-3,k-3}(U_{np})(p)$ by [38] Proposition 12.2. The Atkin-Lehner sign is determined by [3] Theorem 10.1 (ii), and we have

\[
S^{+}_{2k-2}(\Gamma_0(p)) \hookrightarrow \mathcal{M}^{−}_{k-3,k-3}(U_{np})(p) \quad \text{for even } k, \\
S^{−}_{2k-2}(\Gamma_0(p)) \hookrightarrow \mathcal{M}^{+}_{k-3,k-3}(U_{np})(p) \quad \text{for odd } k.
\]

Note that the sign is reversed compared with paramodular case. So in the comparison between dimensions of $S^{±}_{k,j}(K(p))$ and $\mathcal{M}^{±}_{k,j-3,k-3}(U_{np})(p)$, this part apparently need not appear. Gathering above considerations, Theorem 1.2 is proved.

3. Details of Notation and Proof of Theorem 1.1

First we explain the non-principal genus. From now on, we consider only the case $n = 2$ for simplicity. Let $B$ be the definite quaternion algebra of prime discriminant $p$ as before. We fix a maximal order $O$ of $B$. A choice of $O$ is not essential but we fix $O$ that contains a prime element $\pi$ with $\pi^2 = -p$ for the sake of simplicity (such $O$ always exists). A $\mathbb{Z}$ lattice $L \subset B^2$ (a free $\mathbb{Z}$ submodule of $B^2$ of rank 8) is said to be a left $O$ lattice if $L$ is a left $O$ module. As in Shimura [46], we define a norm $N(L)$ of $L$ as the two sided $O$ ideal spanned by $h(x,y)$ for $x, y \in L$. A left $O$ lattice $L$ is said to be maximal if any $O$-lattice $M$ with $L \subset M$ and $N(M) = N(L)$ satisfies $M = L$. For any left $O$ lattice
L and any prime \( q \), we write \( L \otimes \mathbb{Z}_q \). A genus \( \mathcal{L} \) is a set of left \( O \) lattices in \( B^2 \) such that for any \( L_1, L_2 \in \mathcal{L} \), we have \( L_{1,q} = L_{2,q} \) for some \( g_q \in G_q \) for any prime \( q \). If a lattice in \( \mathcal{L} \) is maximal, then so is any other lattice in \( \mathcal{L} \). The set of norm \( N(L) \) for \( L \in \mathcal{L} \) is determined up to a multiplication by \( \mathbb{Q}^\times \). In our case of discriminant \( p \), there are two genera of maximal lattices, one is called the principal genus \( \mathcal{L}_{pg} \) containing \( O^2 \), and the other is the one called non-principal genus \( \mathcal{L}_{npg} \) containing a maximal lattice \( L \) with \( N(L) = \pi O = O \).

Now we fix any genus \( \mathcal{L} \). For \( L \in \mathcal{L} \), a set of lattices \( \{ Lg; g \in G \} \subset \mathcal{L} \) is called a class. The number of classes in \( \mathcal{L} \) is called the class number of \( \mathcal{L} \). For a fixed left \( O \) lattice \( L \in \mathcal{L} \) and any prime \( q \), we put

\[
U(L_q) = \{ g_q \in G_q; L_q g_q = L_q \},
\]

and define an open subgroup \( U(L) \) of \( G_A \) by

\[
U = U(L) = G_{\infty} \prod_q U(L_q).
\]

For any \( g_A = (g_v) \in G_A \), we define a left \( O \) lattice \( Lg_A \subset B^2 \) by

\[
Lg_A = \bigcap_{v < \infty} (L_v g_v \cap B^2).
\]

Then the class number \( H \) is equal to the number of double cosets in \( U \backslash G_A / G \), and if we write the representatives of double cosets as

\[
G_A = \bigsqcup_{i=1}^H U g_i G \quad (\text{disjoint}),
\]

then the set \( \{ Lg_i; i = 1, \ldots, H \} \) gives a complete set of representatives of classes in \( \mathcal{L} \). If we put \( \Gamma_i = g_i^{-1} U g_i \cap G \), then these are finite groups and are (metric preserving) automorphism groups of \( Lg_i \) for \( 1 \leq i \leq H \), respectively. Let \( \rho_{f_1,f_2} \) be the irreducible representation of \( Sp(2) \) corresponding to \( (f_1, f_2) \) with \( f_1 \equiv f_2 \mod 2 \) with \( f_1 \geq f_2 \geq 0 \). We define \( \mathfrak{M}_{f_1,f_2}(U) \) as in the introduction and call an element of \( \mathfrak{M}_{f_1,f_2}(U) \) an algebraic modular form of weight \( \rho_{f_1,f_2} \) belonging to \( U \). For later use, we review another non-adelic realization of \( \mathfrak{M}_{f_1,f_2}(U) \) (see \( \text{[11, 12]} \)). Let \( V \) be a representation space of \( \rho_{f_1,f_2} \). Then we have

\[
\mathfrak{M}_{f_1,f_2}(U) \cong \bigoplus_{i=1}^H V_{\Gamma_i} \quad (\text{direct sum}),
\]

where we put

\[
V_{\Gamma_i} = \{ v \in V; \rho_{f_1,f_2}(\gamma)v = v \text{ for all } \gamma \in \Gamma_i \}.
\]

The above isomorphism is given by the mapping

\[
\mathfrak{M}_{f_1,f_2}(U) \ni f \mapsto \sum_{i=1}^H \rho(g_i)^{-1} f(g_i) \in \bigoplus_{i=1}^H V_{\Gamma_i}.
\]
Next we consider an action of the Hecke algebra. For \( g \in G_A \), we define an action of \( UgU = \bigcup_j z_j U \) (disjoint) on \( M_{f_1,f_2}(U) \) by
\[
(R_{f_1,f_2}(UgU)f)(x) = \sum_j \rho_{f_1,f_2}(z_j)f(z_j^{-1}x).
\]
(Note that the representation \( \rho_{f_1,f_2} \) is defined on \( G_A \) by \([11]\).) We interpret the action of \( UgU \) into \( H \times H \) matrix action on \( \bigoplus_{i=1}^H V_{\Gamma_i}^i \). We put \( T_{ij} = G \cap g_i^{-1}UgUg_j \). Then it is clear by definition that we have \( \Gamma_i T_{ij} \Gamma_j = T_{ij} \). So we regard \( T_{ij} \) as a formal sum
\[
T_{ij} = \sum_h \Gamma_i h \Gamma_j = \sum_m h_m \Gamma_j.
\]
Then this can be regarded as an operator of \( V_{\Gamma_j} \) to \( V_{\Gamma_i} \) by defining the action on \( v_j \in V_{\Gamma_j} \) by
\[
(10) \quad T_{ij}v_j = \sum_m \rho_{f_1,f_2}(h_m)v_j, \quad (T_{ij} = \sum_h \Gamma_i h \Gamma_j = \sum_m h_m \Gamma_j).
\]
Then by \([11]\) Lemma 1, the action of \( UgU \) is identified with the action of the matrix \( (T_{ij})_{1 \leq i,j \leq H} \) on \( \bigoplus_{i=1}^H V_{\Gamma_i} \) by \( (v_1, \ldots, v_H) \to (\sum_{j=1}^H T_{ij}v_j)_{1 \leq i \leq H} \). Here we may also write
\[
T_{ij} = \sum_{a \in G \cap g_i^{-1}UgUg_j/\Gamma_j} \rho_{f_1,f_2}(a)|V_{\Gamma_j}^i|.
\]
We describe the trace of the action of \( UgU \) in this setting. For \( x \in G \), we define \( \rho_{f_1,f_2}(x) \) by \([11]\) through the diagonal embedding \( G \to G_A \). We denote by \( Tr(\rho_{f_1,f_2}(x)) \) the trace of the representation \( \rho_{f_1,f_2}(x) \) on whole \( V \). The following Lemma 3.1 and Corollary 3.2 are almost trivial by definition and has been used many times for the calculation of dimension formulas of algebraic modular forms such as \([12],[15],[13],[19],[24]\), but to see its connection to \( SO(5) \) clearly, we explain some details for safety.

**Lemma 3.1.** For \( i = 1, \ldots, h \), define \( T_{ii} = G \cap g_i^{-1}UgUg_i \) as before. Then we have
\[
Tr(R_{f_1,f_2}(UgU)) = \sum_{i=1}^H \frac{\sum_{x \in T_{ii}} Tr(\rho_{f_1,f_2}(x))}{\#(\Gamma_i)}.
\]
where \( \#(\Gamma_i) \) is the cardinality of \( \Gamma_i \).

**Proof.** Notation being as in \([10]\), the action of \( T_{ii} \) on \( v_i \in V_{\Gamma_i} \) is given by
\[
T_{ii}v_i = \sum_m \rho_{f_1,f_2}(h_m)v_i = \sum_{m} \sum_{\gamma \in \Gamma_j} \rho_{f_1,f_2}(h_m \gamma)v_i = \sum_{x \in T_{ii}} \rho_{f_1,f_2}(x)v_i = \sum_{x \in T_{ii}} \rho_{f_1,f_2}(x)v_i = \sum_{x \in T_{ii}} \rho_{f_1,f_2}(x)v_i = \sum_{x \in T_{ii}} \rho_{f_1,f_2}(x)v_i.
\]
Now we must consider a relation between the trace of the above action of $T_{ii}$ on $V^\Gamma_i$ and $Tr(\rho_{f_1,f_2}(x))$ on $V$. We define an action of $T_{ii}$ on $v \in V$ by

$$T_{ii}v = \sum_{x \in T_{ii}} \rho_{f_1,f_2}(x)v.$$ 

Since $\Gamma,T_{ii} = T_{ii}$, we see that $T_{ii}v \in V^\Gamma_i$ for any $v \in V$. So considering the representation matrix of $T_{ii}$ with respect to a basis of $V^\Gamma_i$, it is clear that

$$\sum_{x \in T_{ii}} Tr(\rho_{f_1,f_2}(x)|V) = \sum_{x \in T_{ii}} Tr(\rho_{f_1,f_2}(x)|V^\Gamma_i)$$

By definition (11), if we put $\hat{g} = g/\sqrt{n(g)}$ for $g \in G$, then we have $Tr(\rho_{f_1,f_2}(g)) = Tr(\rho_{f_1,f_2}(\hat{g}))$. Here if $F(x)$ is the principal polynomial of $g \in G$ (defined as the characteristic polynomial of $g \in G \subset M_2(B) \subset M_4(\mathbb{C})$), the principal polynomial of $\hat{g}$ is given by $f(x) = n(g)^{-2} F(\sqrt{n(g)x})$. It is well known that the character $Tr(\rho_{f_1,f_2}(\hat{g}))$ depends only on the principal polynomial $f(x)$ of $\hat{g}$. For any such polynomial $f(x)$, we write $\chi_{f_1,f_2}(f) = Tr(\rho_{f_1,f_2}(\hat{g})) = Tr(\rho_{f_1,f_2}(g))$ where $g \in G$ is as above. Since we assumed $\rho_{f_1,f_2}(\pm 1) = 1$, we have $Tr(\rho_{f_1,f_2}(\hat{g})) = Tr(\rho_{f_1,f_2}(\bar{g}))$, so we have $\chi_{f_1,f_2}(f(x)) = \chi_{f_1,f_2}(f(-x))$. Let $G(f)$ the set of all elements $g$ of $G$ such that $f(x)$ or $f(-x)$ is the principal polynomial of $\hat{g}$.

We also put

$$Tr(R_{f_1,f_2}(UgU), f) = \sum_{i=1}^{H} \sum_{g \in T_{ii} \cap G(f)} Tr(\rho_{f_1,f_2}(g)) \frac{\#(\Gamma_i)}{\#(T_{ii})},$$

Then by Lemma 3.1 we have the following corollary.

**Corollary 3.2.** We have

$$Tr(R_{f_1,f_2}(UgU), f) = \chi_{f_1,f_2}(f) \times \sum_{i=1}^{h} \frac{\#(T_{ii} \cap G(f))}{\#(\Gamma_i)}.$$ (11)

This corollary is very useful since RHS is a product of the term depending on the representation and the term independent of the representation. We have $\chi_{0,0}(f) = 1$ for any $f$, so the general formula is reduced to the case of the trivial representation as far as the formula for $Tr(R_{0,0}(UgU))$ is given as a sum of explicit terms for each $f$. (This is usually true for any trace formula).

Now we specialize our consideration to the case when $\mathcal{L} = \mathcal{L}_{npg}$. We identify $\pi$ with an element of $G_A$ by the diagonal embedding. We consider the double coset $R(\pi) = U_{np}(p) \pi U_{np}(p) = U_{np}(p) \pi = \pi U_{np}(p)$. Our aim is to give a formula for $Tr(R_{f_1,f_2}(\pi))$ on $W_{f_1,f_2}(U_{np}(p))$. In this case, for any element $x \in T_{ii}$, we have $n(x) = p$. 

By Corollary 3.2 the main part of the calculation of $Tr(R_{f_1,f_2}(\pi))$ for $\rho_{f_1,f_2}$ is to give the value of RHS of (11) for $(f_1,f_2) = (0,0)$ for each $f$. This calculation is based on two things. One is the relation $2T = H + Tr(R_{0,0}(\pi))$ proved in [25] between the class number $H$ of $\mathcal{L}_{npg}$, the type number $T$ of $\mathcal{L}_{npg}$ (the definition will be reviewed soon), and $Tr(R_{0,0}(\pi))$. The other is an equality between the type number $T$ of $\mathcal{L}_{npg}$ and the class number of quinary lattices of some genus of det $= 2p$ proved in [26] for $p \neq 2$. (The case $p = 2$ will be treated separately in section 4.) In [26], our calculation of orders in classes of maximal orders in an algebra. But in our case, all maximal orders in $M_2(B)$ are conjugate to $M_2(O)$ since the class number of $M_2(B)$ is 1 by the strong approximation theorem of $SL_2(B)$. But here, instead of $GL_2(B)$ conjugacy, we consider the $G$ conjugacy of $R_i$. The $(G\text{-})$type number $T$ of $\mathcal{L}_{npg}$ is defined to be a number of $G$-conjugacy classes in $\{R_i\}_{1 \leq i \leq H}$. In [25], we proved that $2T = H + Tr(R_{0,0}(\pi))$. So we have $T \leq H \leq 2T$. For each principal polynomial $F$ of an element appearing in $g^{-1}(U_{npg}(p) \cup U_{npg}(p)\pi)g_i \cap G$ for some $i$, put $f(x) = n(g)^{-2}F(x/\sqrt{n(g)})$ as before. Then the contribution to $T$ of the “$f(x)$ and $f(-x)$-part” in $T = (H + Tr(R_{0,0}(\pi)))/2$ is defined to be

\begin{equation}
T(f) = \frac{1}{2} \sum_{i=1}^{H} \frac{\#(\Gamma_i \cap G(f)) + \#(g_i^{-1}U_{npg}(p)\pi g_i \cap G(f))}{\#(\Gamma_i)},
\end{equation}

where we note $\Gamma_i = g_i^{-1}U_{npg}(p)g_i \cap G$. Now we will compare this to the class number formula of quinary lattices. Let $\mathcal{M} = \mathcal{M}(1,p)$ be the genus of lattices with determinant $2p$ in the 5 dimensional positive definite quadratic space $W$ defined in [26] p. 215. We have shown in [26] that the class number of $\mathcal{M}$ is the type number $T$ of $\mathcal{L}_{npg}$ if $p \neq 2$. The class number formula for $\mathcal{M}$ can be explained as follows. We denote by $M_i$ ($i = 1, \ldots, T$) a complete set of representatives of classes in $\mathcal{M}$. Although Asai in [2] used the orthogonal group $O(W)$ to define a class, we have $O(W) = SO(W) \cup (-1_5)SO(W)$ for the special orthogonal group $SO(W)$, and the class number for both $O(W)$ and $SO(W)$ are
the same and the formulas are identical, so we explain the $SO(W)$ formulation. Let $M_1, \ldots, M_T$ be a complete set of representatives of classes in $\mathcal{M}$. We denote by $\text{Aut}(M_i)$ the group of automorphisms of $M_i$ in $SO(W)$. Then we have a trivial identity
\[
T = \sum_{i=1}^{T} \frac{\#(\text{Aut}(M_i))}{\#(\text{Aut}(M_i))}.
\]

Let \( \tilde{h}(x) \) be a principal polynomial of an element of $SO(W)$ of degree 5. It is of the shape
\[
\tilde{h}(x) = (x - 1)h(x)
\]
for some degree 4 monic reciprocal polynomial $h(x)$. We denote by $SO(W, \tilde{h})$ the set of elements of $SO(W)$ whose principal polynomial is $\tilde{h}(x)$. We put
\[
T(\tilde{h}) = \sum_{i=1}^{T} \frac{\#(\text{Aut}(M_i) \cap SO(W, \tilde{h}))}{\#(\text{Aut}(M_i))}.
\]

Naturally we have $T = \sum \tilde{h} T(\tilde{h})$. In Asai [2], he gave a formula for $T(\tilde{h})$ for each $\tilde{h}$ as a contribution of the case $C_{\pm i}$, where $C_{\pm i}$ means principal polynomials of $\pm \bar{g}$ of some element $\bar{g} \in O(W)$. Here one of $\{\bar{g}, -\bar{g}\}$ is an element of $SO(W)$ and Asai’s formula is the same as the contribution of $SO(W)$ belonging to one of $C_i$ or $C_{-i}$. Now we will interpret each $T(\tilde{h})$ into the "f-part" $T(f)$ of $T$ defined by (12). To explain this, we review some parts of [26]. The even Clifford algebra $C_2(W)$ of $W$ can be identified with $M_2(B)$, $W$ with a linear subspace of $M_2(B)$ over $\mathbb{Q}$, and the even Clifford group $\Gamma_2$ with $G$. The inner automorphism $W \ni w \rightarrow g^{-1}wg \in W$ for $g \in G$ induces an isomorphism $G/\{Q^{*12}\} \cong SO(W)$ [26 p. 210]. To compare with the formulation by $G$, we must describe $\text{Aut}(M_i)$ in terms of $G$. Let $L$ be a representative of $\mathcal{L}_{npq}$ and $R$ the right order of $L$. Then we have

Lemma 3.3 (26) Lemma 4.1 and Corollary 4.4. There exists a lattice $M \in \mathcal{M}$ such that for any $g_A = (g_v) \in G_A$, we have $g_vM_vg_v^{-1} = M_v$ if and only if $g_vR_vg_v^{-1} = R_v$ for any finite place $v$ of $\mathbb{Q}$, where we put $M_v = M \otimes \mathbb{Z} v$ and $R_v = R \otimes \mathbb{Z} v$.

For a representative $g_i = (g_{i,v})$ of the double coset in (9), we put
\[
R_i = g_i^{-1}Rg_i = \cap_{v < \infty} (g_{i,v}^{-1}R_vg_{i,v} \cap M_2(B)).
\]

This is the right order of $L_i = Lg_i$. Changing numbers $i$ if necessary, we assume that $R_1, \ldots, R_T$ are representatives of types (i.e. $G$ conjugacy classes). Then $M_i = g_iMg_i^{-1}$ $(i = 1, \ldots, T)$ are also the representatives of classes in $\mathcal{M}$. By Lemma 3.3 we see that any element of $\text{Aut}(M_i)$ comes from $g \in G$ with $gg_iRg_i^{-1}g^{-1} = g_iRg_i$. This means that $Rg_iRg_i^{-1}$ is a two sided ideal of $R$, and it is well known that any two sided ideal of
$R_v$ is spanned by $\mathbb{Q}_v^\times$ and besides $\pi$ if $v = p$ up to $R_v^\times$. By definition we have $U_{npg}(p) = G_\infty \prod_{v < \infty} (R_v^\times \cap G_v)$, and since $\mathbb{Q}_A^\times = \mathbb{Q}^\times \mathbb{R}_v^\times \prod_{v < \infty} \mathbb{Z}_v$ and $(\mathbb{R}_v^\times 1_2) \prod_{v < \infty} \mathbb{Z}_v 1_2 \subset U$, this means that $g_i(mg)g_i^{-1} \in U_{npg}(p) \cup U_{npg}(p)\pi$ for some $m \in \mathbb{Q}^\times$. Writing the projection $\iota : G \to G/\mathbb{Q}_A^\times 1_2 \cong SO(W)$ by $\iota$, we have $\iota(m1_2) = 1_5$, so we have

$$\text{Aut}(M_i) = \iota(g_i^{-1}(U_{npg}(p) \cup U_{npg}(p)\pi)g_i \cap G).$$

Here by definition we have

$$g_i^{-1}U_{npg}(p)g_i \cap G = \Gamma_i,$$

so $\iota(\Gamma_i)$ is always contained in $\text{Aut}(M_i)$. The problem is the part $g_i^{-1}U_{npg}(p)\pi g_i \cap G$. To explain this part more clearly, we review the relation of the type number and the class number of $\iota$ in (13). We fix

$$\text{the condition there exists unique} \ g \in G$$

$$\text{such that} \ g_i^{-1}U_{npg}(p)\pi \cap G \neq \emptyset.$$ 

Now we fix $i$ and see which $j \neq i$ satisfies (13). If $i \neq j$, then $g_iU_{npg}(p)g_j \cap G = \emptyset$ since $\{g_i\}_{1 \leq i \leq t}$ is a set of representatives of $U \setminus G_A/G$. The condition $g_i^{-1}U_{npg}(p)\pi g_j \cap G \neq \emptyset$ is equivalent to

$$\pi U_{npg}(p)g_j G = U_{npg}(p)\pi g_j G = U_{npg}(p)g_j G.$$ 

Since

$$U_{npg}(p)g_i G \subset G_A = \pi G_A = \prod_{i=1}^H \pi U_{npg}(p)g_i G = \prod_{i=1}^H U_{npg}(p)\pi g_i G, \quad \text{(disjoint)},$$

there exists unique $j$ that satisfies (14). This means that we have two cases. The first one is the case that $j = i$ in (14) and we have

$$U_{npg}(p)\pi g_i G = U_{npg}(p)g_i G.$$ 

This means that $g_i^{-1}U_{npg}(p)\pi g_i \cap G \neq \emptyset$. Besides, if $g, g' \in g_i^{-1}U_{npg}(p)\pi g_i \cap G$, then writing $g = g_i^{-1}u\pi g_i$ and $g' = g_i^{-1}u'\pi g_i$ for $u, u' \in U_{npg}(p)$, we have $g^{-1}g' = g_i^{-1}\pi^{-1}u^{-1}u'\pi g_i \in G$, but we have $\pi u^{-1}u' \in \pi U_{npg}(p)\pi^{-1} = U_{npg}(p)$, so $g^{-1}g' \in \Gamma_i$. This means that

$$g_i^{-1}U_{npg}(p)\pi g_i \cap G = g_0\Gamma_i$$

for some $g_0 \in G$. We may assume that the set of such $i$ is $\{1, \ldots, t\}$. Then for these $i$, and we have

$$\text{Aut}(M_i) = \iota(\Gamma_i) \cup \iota(g_0\Gamma_i).$$

This is a disjoint union in $SO(W)$. Indeed, for any element $g_0 \in G$ with $n(g_0) = p$, we have $\iota(g_0) \not\in \iota(G^1)$, because for any element $g_0' \in \mathbb{Q}^\times G^1$, we have $n(g_0') \in (\mathbb{Q}^\times)^2$. So we have $\#(\text{Aut}(M_i)) = 2\#(\iota(\Gamma_i))$. Next we
We have

\[
\text{Aut}(M_i) = \iota(G_i).
\]

We also have \(T = t + (H - t)/2\) and \(t = Tr(R_{0,0}(\pi))\). Now for a principal polynomial \(\tilde{h}\), we have

\[
2T(\tilde{h}) = 2 \sum_{i=1}^{T} \frac{\#(\iota(G_i) \cap SO(W, \tilde{h})) + \#(g_i^{-1}U_{npg}(p)\pi g_i \cap G) \cap SO(W, \tilde{h})}{\#(\iota(G_i)) + \#(g_i^{-1}U_{npg}(p)\pi g_i \cap G)}.
\]

For \(i = 1, \ldots, t\), the denominator is \(2\#(\iota(G_i))\). For \(i = t+1, \ldots, T-t\), the denominator is \(\#(\iota(G_i))\). Besides, when \(gg_iRg_i^{-1}g^{-1} = g_iRg_j^{-1}\) for some \((i, j)\) with \(1 \leq i \neq j \leq H\) and \(g \in G\), we have \(G_i \cong G_j\), so \(\#(\iota(G_i)) = \#(\iota(G_j))\). Now we compare principal polynomials \(f(x)\) of \(\tilde{g} = g/\sqrt{n(g)}\) for \(g \in G\) and \(\tilde{h}(x)\) of \(\iota(g)\). If the eigenvalues of an element of \(Sp(2)\) is \(\epsilon_1, \epsilon_2, \epsilon_1^{-1}, \epsilon_2^{-1}\), then eigenvalues of the image in \(SO(5)\) is \(1, \epsilon_1 \epsilon_2, \epsilon_1 \epsilon_2^{-1}, \epsilon_1 \epsilon_2\), so for the principal polynomial

\[
f(x) = x^4 + c_1 x^3 + c_2 x^2 + c_4 x + 1
\]

of an element \(\tilde{g} \in G_{\infty}^L\), the principal polynomial of \(\iota(g) \in SO(W)\) is given by

\[
\tilde{f}(x) = (x-1)h(x), \quad h(x) = x^4 - (c_2-2)x^3 + (c_1^2 - 2c_2 + 2)x^2 - (c_2-2)x + 1.
\]

Here for \(f(x)\) and \(f(-x)\), we have the same \(\tilde{f}(x)\). This is clear from the above calculation and also by \(Ker(\iota|Sp(2)) = \{\pm 1\}\). So noting that \(\#(G_i) = 2\#(\iota(G_i))\), \(\#(G_i \cap G(f)) = 2\#(\iota(G_i) \cap SO(W, \tilde{f}))\), and \(\#(g_i^{-1}U_{npg}(p)\pi g_i \cap G(f)) = 2\#(g_i^{-1}U_{npg}(p)\pi g_i \cap G) \cap SO(W, \tilde{f})\), we can rewrite the above formula for \(2T(\tilde{f})\) as

Lemma 3.4.

\[
2T(\tilde{f}) = \sum_{i=1}^{H} \left( \frac{\#(G_i \cap G(f))}{\#(G_i)} + \frac{\#(g_i^{-1}U_{npg}(p)\pi g_i \cap G(f))}{\#(G_i)} \right) = 2T(f)
\]

This Lemma means that \(T(f)\) can be obtained from Asai’s formula for \(T(\tilde{f})\). The principal polynomials of elements of \(G_i\) and elements of \(g_i^{-1}U_{npg}(p)\pi g_i \cap G\) are different, because the constant terms \(n(g)^2\) are different. But sometimes the principal polynomials for \(\tilde{g}\) are the same. So both contribute to the same \(T(f)\). But this does not matter, since the character of the elements depend only on \(\tilde{f}(x)\), or \(f(\pm x)\).

For more precise calculations, it would be safer to give a list of tables of principal polynomials \(F(x), f(x)\) and \(\tilde{f}(x)\) of \(g \in G, \tilde{g}\), and \(\iota(g)\). We consider possible principal polynomials of elements in \(U_{npg}(p) \cup U_{npg}(p)\tilde{x}\). The principal polynomials of elements in \(G_i\) has been listed in \([12]\) and also in section \([5]\) after Theorem 4.1. So we see the case
Lemma 3.5. For any prime \( q \neq p \), we have \( U(L_q) = G_q \cap GL_2(O_q) \). At \( p \), the prime element \( \pi \) of \( O \) we defined before is also a prime of \( O_p = O \otimes \mathbb{Z}_p \). We put

\[
G_p^* = \left\{ g \in M_2(B_p); g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g^* = n(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.
\]

Then we have \( G_p \cong G_p^* \) by \( GL_2(O_p) \) conjugation, and

\[
U_{npg}(L_p) \cong G_p^* \cap \left( O_p \pi O_p \pi^{-1} O_p \right)^x.
\]

Hence by the condition that \( g \) is conjugate to \( U_{npg} \pi \) and \( g^* = pg^{-1} \), the principal polynomial \( F(x) \) of an element \( g \) appearing in \( g^{-1} U \pi g_i \cap G \) should be of the form

\[
F(x) = x^4 + pax^3 + pbx^2 + p^2ax + p^2
\]

for some rational integers \( a, b \). The possible principal polynomials \( F \) of \( g \in U_{npg}(p) \pi \) and corresponding \( f(x) \) are as follows.

| \( p \) | \( F(x) \) | \( f(x) \) |
|-----|-----|-----|
| general \((x^2 - p)^2\) | \((x - 1)^2(x + 1)^2\) | \((x - 1)^2(x + 1)^2\) |
| general \((x^2 + p)^2\) | \((x^2 + 1)^2\) | \((x^2 + 1)^2\) |
| general \(x^4 + p^2\) | \(x^4 + 1\) | \(x^4 + 1\) |
| general \(x^4 - px^2 + p^2\) | \(x^4 - x^2 + 1\) | \(x^4 - x^2 + 1\) |
| general \(x^4 + px^2 + p^2\) | \(x^4 + x^2 + 1\) | \(x^4 + x^2 + 1\) |
| \( p = 5 \) | \(x^4 \pm 5x^3 + 15x^2 \pm 25x + 25\) | \(x^4 \pm \sqrt{5}x^3 + 5x^2 \pm \sqrt{5}x + 1\) |
| \( p = 3 \) | \(x^2 \pm 3x + 3\)^2 | \((x^2 \pm \sqrt{3}x + 1)^2\) |
| \( p = 3 \) | \((x^2 \pm 3x + 3)(x^2 + 3)\) | \((x^2 \pm \sqrt{3}x + 1)(x^2 + 1)\) |
| \( p = 2 \) | \((x^2 \pm 2x + 2)^2\) | \((x^2 \pm \sqrt{2}x + 1)^2\) |
| \( p = 2 \) | \((x^2 \pm 2x + 2)(x^2 + 2)\) | \((x^2 \pm \sqrt{2}x + 1)(x^2 + 1)\) |
| \( p = 2 \) | \(x^4 \pm 2x^3 \pm 2x^2 \pm 4x + 4\) | \(x^4 \pm \sqrt{2}x^3 + x^2 \pm \sqrt{2}x + 1\) |

This list can be obtained by using the inequalities in the following lemma.

**Lemma 3.5.** We fix \( p \). Then a polynomial

\[
F(x) = x^4 + pax^3 + pbx^2 + p^2ax + p^2
\]

is a principal polynomial of an element of \( G \) only if the following conditions are satisfied.

\[
|a| \leq 4/\sqrt{p} \\
b \leq a^2 p/4 + 2 \\
(pa^2 - 4)/2 \leq b \\
4pa^2 \leq (b + 2)^2
\]

In particular, for a fixed \( p \), there are only finitely many integers \( a, b \in \mathbb{Z} \) such that \( F(x) \) can be a principal polynomial of an element of \( G \).
Proof. Since \( x \in G \) and \( x/\sqrt{p} \in Sp(2) \), if we put
\[
h(X) = X^4 + a\sqrt{p}X^3 + bX^2 + a\sqrt{p}X + 1,
\]
then the roots of \( h(x) = 0 \) should be of absolute value 1. Writing \( Y = X + X^{-1} \), we have
\[
(15) \quad Y^2 + a\sqrt{p}Y + b - 2 = 0.
\]
Since \( |X| = 1 \), we see \( Y \) is a real number, so we have \( a^2p - 4(b - 2) \geq 0 \), which gives the second inequality. If we denote by \( \alpha, \beta \) the two roots of \( (15) \), then the equation
\[
X^2 - \alpha X + 1 = 0
\]
has imaginary roots or \( X = \pm 1 \). In both cases, we have \( \alpha^2 \leq 4 \). So we have
\[
8 \geq \alpha^2 + \beta^2 = pa^2 - 2(b - 2), \quad 0 \leq (4 - \alpha^2)(4 - \beta^2) = (b + 2)^2 - 4pa^2,
\]
which leads to the third and the fourth inequalities. \( \square \)

Now assume that \( a, b \in \mathbb{Z} \) satisfy the condition in the above lemma. For any prime, we have \( |a| \leq 4/\sqrt{p} \leq 4/\sqrt{2} \) so \( |a| = 0, 1, 2. \) If \( a = 0, \) then we have \(-2 \leq b \leq 2. \) If \( (a, b) = (0, \pm 2), \) then we have \( F(x) = x^4 \pm 2px^2 + p^2 = (x^2 \pm p)^2. \) If \( (a, b) = (0, \pm 1), \) then \( F(x) = x^4 \pm px^2 + p^2. \) If \( (a, b) = (0, 0), \) then \( F(x) = x^4 + p^2. \) If \( |a| = 1, \) then by the second and the third condition, we have \( p/2 - 2 \leq b \leq p/4 + 2, \) so we have \( p \leq 16. \) If \( p = 7, \) then \( 1.5 \leq b \leq 3.75, \) so \( b = 2 \) or \( 3. \) But the last condition \( 28 \leq (b + 2)^2 \) is not satisfied for \( b = 2 \) and \( 3. \) In the same way, for \( p = 11, \) we have \( b = 4 \) and for \( p = 13, \) we have \( b = 5, \) violating the last condition. So we have \( p = 2, 3, \) or \( 5. \) For \( p = 5, \) we have \( b = 1, 2, 3. \) By \( 20 \leq (b + 2)^2, \) we have \( b = 3. \) So we have \( F(x) = x^4 \pm 5x^3 + 15x^2 \pm 5x + 25. \) For \( p = 3, \) we have \( b = 0, 1, 2. \) Since \( 12 \leq (b + 2)^2, \) we have \( b = 2. \) This gives \( F(x) = x^4 \pm 3x^3 + 6x^2 \pm 9x + 9 = (x^2 \pm 3x + 3)(x^2 + 3). \) For \( p = 2, \) we have \( b = -1, 0, 1, 2. \) By \( 8 \leq (b + 2)^2, \) we have \( b = 1 \) or \( b = 2. \) Then \( F(x) = x^4 \pm 2^3 + 2x^2 \pm 4x + 4 \) for \( b = 1 \) and \( F(x) = (x^2 \pm 2x + x)(x^2 + 1) \) for \( b = 2. \) Finally we see the case \( |a| = 2. \) This can happen only when \( p = 2 \) or \( 3 \) since \( 2 \leq 4/\sqrt{p}. \) When \( p = 3, \) then \( b = 4 \) or \( 5. \) So \( 48 \leq (b + 2)^2, \) we should have \( b = 5. \) So \( F(x) = x^4 \pm 6x^3 + 15x^2 \pm 18x + 9 = (x^2 \pm 3x + 3)^2. \) When \( p = 2, \) we have \( b = 2, 3, 4, \) and by \( 32 \leq (b + 2)^2, \) we have \( b = 4. \) This means that \( F(x) = x^4 \pm 4x^3 + 8x^2 \pm 8x + 4 = (x^2 \pm 2x + 2)^2. \) So we have only polynomials \( F(x) \) that we have listed.

Of course this is a possible list and we are not claiming that these really occur.

Now we list all the possible principal polynomials appearing in the calculation of \( H \) and \( T \) in the following table. Here, in the first column, we write the principal polynomials \( f(x) \) of \( \tilde{g} \in G^1_{\infty} \) coming from \( g \in \mathcal{U}_{np}(p) \) and \( \mathcal{U}_{np}(p)\pi. \) The second column is the principal polynomial
Asai for $T$ do not use Asai’s result anyway. The case as can be seen, these appear only when as we noticed, he used usual dimension formula of the class number genus coming from elements of some $\Gamma$. Lemma 4.16 to indicate principal polynomials. As we noticed, he used $O(5)$ formulation instead of $SO(5)$, which causes the suffix $\pm$ of $C_{\pm i}$, and for $SO(5)$, we need one of $C_i$ or $C_{-i}$.

$$f(x), \text{Sp}(2) \quad \tilde{f}(x), \text{SO}(5) \quad \text{Asai}$$

$(x \pm 1)^4$ $(x - 1)^5$ $C_1$

$(x - 1)^2(x + 1)^2$ $(x - 1)(x + 1)^4$ $C_2$

$(x^2 + 1)$ $(x - 1)^3(x^2 + 1)^2$ $C_3$

$(x \pm 1)^2(x^2 + 1)$ $(x - 1)(x^2 + 1)^2$ $C_4$

$x^4 \pm 2\sqrt{2}x^3 + 4x^2 \pm 2\sqrt{2}x + 1$ $(x - 1)^3(x^2 + 1)^2$ $C_5$

$(x \pm 1)^2(x \mp x + 1)$ $(x - 1)(x^2 + x + 1)^2$ $C_6$

$(x^2 \pm x + 1)^2$ $(x - 1)^3(x^2 + x + 1)$ $C_7$

$(x \pm 1)^2(x^2 \pm x + 1)$ $(x - 1)(x^2 - x + 1)^2$ $C_8$

$x^4 \pm 2\sqrt{3}x^3 + 5x^2 \pm 2\sqrt{3}x + 1$ $(x - 1)^3(x^2 - x + 1)$ $C_9$

$x^4 + 1$ $(x - 1)(x^2 + 1)^2$ $C_{10}$

$x^4 - x + 1$ $(x - 1)(x^2 + 1)^2(x^2 + x + 1)$ $C_{11}$

$(x^2 + x + 1)(x^2 - x + 1)$ $(x - 1)(x^2 - x + 1)^2$ $C_{12}$

$x^4 \pm \sqrt{2}x^3 + 2x^2 \pm \sqrt{2}x + 1$ $(x - 1)(x^4 + 1)$ $C_{13}$

$x^4 \pm x^3 + x^2 \pm x + 1$ $(x - 1)(x^4 + x^3 + x^2 + x + 1)$ $C_{14}$

$x^4 \pm \sqrt{5}x^3 + 3x^2 \pm \sqrt{5}x + 1$ $(x - 1)(x^4 - x^3 + x^2 - x + 1)$ $C_{15}$

$(x^2 + 1)(x^2 \pm x + 1)$ $(x - 1)(x^4 - x^2 + 1)$ $C_{16}$

$x^4 \pm \sqrt{6}x^3 + 3x^2 \pm \sqrt{6}x + 1$ $(x - 1)(x^4 + 1)(x^2 - x + 1)$ $C_{17}$

$x^4 \pm \sqrt{2}x^3 + x^2 \pm \sqrt{2}x + 1$ $(x - 1)(x^2 + 1)(x^2 + x + 1)$ $C_{18}$

$x^4 \pm \sqrt{3}x^3 + 2x^2 \pm \sqrt{3}x + 1$ $(x - 1)(x^2 + x + 1)(x^2 - x + 1)$ $C_{19}$

The polynomials in the table below are those that do not appear in the usual dimension formula of the class number $H$ of the non-principal genus coming from elements of some $\Gamma_i$ and newly needed for the type number.

$$g \in \text{Sp}(2) \quad g \in G \quad \text{Asai}$$

$x^4 \pm 2\sqrt{2}x^3 + 4x^2 \pm 2\sqrt{2}x + 1$ $(x^2 \pm 2x + 2)^2$ $C_5$

$x^4 \pm 2\sqrt{3}x^3 + 5x^2 \pm 2\sqrt{3}x + 1$ $(x^2 \pm 3x + 3)^2$ $C_9$

$x^4 \pm \sqrt{2}x^3 + 2x^2 \pm \sqrt{2}x + 1$ $(x^2 \pm 2x + 2)(x^2 + 2)$ $C_{13}$

$x^4 \pm \sqrt{5}x^3 + 3x^2 \pm \sqrt{5}x + 1$ $x^4 \pm 5x^3 + 15x^2 \pm 25x + 25$ $C_{15}$

$x^4 \pm \sqrt{6}x^3 + 3x^2 \pm \sqrt{6}x + 1$ $x^4 + 6x^3 + 18x^2 + 36x + 36$ $C_{17}$

$x^4 \pm \sqrt{2}x^3 + x^2 \pm \sqrt{2}x + 1$ $x^4 \pm 2x^3 + 2x^2 \pm 4x + 4$ $C_{18}$

$x^4 \pm \sqrt{3}x^3 + 2x^2 \pm \sqrt{3}x + 1$ $(x^2 \pm 3x + 3)(x^2 + 3)$ $C_{19}$

As can be seen, these appear only when $p = 2, 3$ or 5. Actually, the contributions to $T$ from $C_9, C_{13}, C_{17}, C_{18}$ are known to be all zero in Asai for $d = p$ with $p \neq 2$, so we do not need these. For $p = 2$, we do not use Asai’s result anyway. The case $p = 2, 3$ will be treated
separately in the next section and the case \( p = 5 \) is included in the general theory.

**Proof of Theorem 1.1.** We are assuming here that \( p \neq 2, 3 \) and sketch a proof. We assume that \( f_1 = f_2 = 0 \) and \( \rho_{f_1, f_2} \) is the trivial representation. When \( p \neq 2 \), the formula for \( T(f) \) of \( \mathcal{L}_{npg} \) is equal \( T(\tilde{f}) \) by Lemma 3.4. For this part, we use the class number formula given in Asai [2] Theorem 4.17. The class number \( H \) has been given in Theorem in [12] II (which is reproduced in the appendix for prime discriminant case.) So all we should do is to calculate a contribution to \( Tr(R(\pi)) := Tr(R_{0,0}(\pi)) = 2T - H \) for each principal polynomial \( f(\pm x) \) or \( \tilde{f}(x) \). For example, for \( C_1 \), we have

\[
2 \frac{p^2 - 1}{5760} - \frac{p^2 - 1}{2880} = 0.
\]

For \( C_2 \), the contribution to \( T \) is

\[
\frac{1}{p} \left( 9 - 2 \left( \frac{2}{p} \right) \right) \quad \text{if } p \equiv 1 \text{ mod } 4
\]

\[
\frac{1}{2^6} B_{2x} \quad \text{if } p \equiv 3 \text{ mod } 4,
\]

and the contribution to \( H \) of finite order elements with principal polynomial \( (x - 1)^2(x + 1)^2 \) is 0 if \( p \neq 2 \), so the contribution to \( 2T - H \) is the twice of the above values. For \( C_3 \), the contribution to \( T \) is given by

\[
\frac{1}{2^4 \cdot 3} \left( p - \left( \frac{-1}{p} \right) \right) + \frac{1}{2^6 \cdot 3} \left( p \left( \frac{-1}{p} \right) - 1 \right)
\]

\[
+ h(-p) \times \begin{cases} 
\frac{1}{2^6} & \text{if } p \equiv 1 \text{ mod } 4 \\
\frac{1}{2^7} \left( 1 - \left( \frac{2}{p} \right) \right) & \text{if } p \equiv 3 \text{ mod } 4
\end{cases}
\]

The contribution to \( H \) is given by

\[
\frac{1}{2^4 \cdot 3} \left( p - \left( \frac{-1}{p} \right) \right) + \frac{1}{2^6 \cdot 3} \left( p \left( \frac{-1}{p} \right) - 1 \right).
\]

So the contribution to \( 2T - H \) is as in the term of \( h(-p) \) in the theorem. We give one more example. In the case \( C_{-15} \), the contribution to \( T \) vanishes if \( p \neq 5 \) and is \( 1/10 \) if \( p = 5 \). So the contribution to \( 2T \) is \( 1/5 \). On the other hand, the contribution to \( H \) is 0 since the polynomial \( x^4 \pm \sqrt{5}x^3 + 3x^2 \pm \sqrt{5}x + 1 \) does not appear for \( H \). We can similarly continue for all polynomials \( C_i \). All are routine calculations, so we omit further proof. \( \square \)

4. **The case \( p = 2 \) and \( 3 \)**

In this section, we give a formula for \( Tr(R_{f_1, f_2}(\pi)) \) for general \( (f_1, f_2) \) for \( p = 2 \) and \( p = 3 \) by a different method. Applying [3] and Theorem 1.2, this also gives a dimension formula for \( S_{k,j}^+(K(p)) \) for \( p = 2, 3 \) and \( k \geq 3 \) as before. For the scalar valued cases \( \dim S_{k}^+(K(2)) \) and
\text{dim } S^\pm_1(K(3)), \text{ this means that we reprove the formula that has already been known in } [18], [4], [22].

For the case \( p = 3 \), we can calculate \( Tr(R_{f_1,f_2}(\pi)) \) by the same method as in the proof of Theorem 1.1 but if \( p = 2 \), we have a problem since the correspondence between \( T \) and the class number of quinary lattices explained before is not known in [25]. So we need a different method. Here we use more direct method for both \( p = 2 \) and 3. We know that the class number \( H = 1 \) for the non-principal genus \( L_{npg} \) for both \( p = 2 \) and 3 (See [12]), so \( G_A = U_{npg}G \) and \( \Gamma_1 = G \cap U_{npg} \). We have \( \#(\Gamma_1) = 1920 \) for \( p = 2 \) and 720 for \( p = 3 \). Besides, we have \( R(\pi) = U_{npg} \pi \) for \( \pi \in O \), and since \( \pi 1_2 \in G \), we see that

\[ G \cap U_{npg} \pi = \pi(G \cap U_{npg}) = \pi \Gamma_1. \]

So in order to obtain \( Tr(R(\pi)) \), all we should do is to count the number of elements \( \gamma \in \pi \Gamma_1 \) for each fixed principal polynomial. We can concretely describe these elements by a direct calculation. First we give a table of principal polynomials and number of corresponding elements in \( \pi \Gamma_1 \), and then state our theorem

**Lemma 4.1.** For a fixed polynomial \( F(x) \), the number of elements \( \gamma \in \pi \Gamma_1 \) such that \( F(x) \) is their principal polynomial is given in the second column in the following table for \( p = 2, 3 \). We put \( f(x) = p^{-2} f(x/\sqrt{p}) \) as before. The last column is the character \( Tr(\rho_{f_1,f_2}(g)) \) of elements of the corresponding row, where the notation \( \chi_i \) is explained in section 3.

(i) The case \( p = 2 \).

| \( F(x) \)       | \( \pi \Gamma_1 \) | \( f(x) \)       | character |
|-----------------|---------------------|------------------|-----------|
| \((x^2 - 2)^2\) | 40                   | \((x - 1)^2(x + 1)^2\) | \( \chi_2 \) |
| \((x^2 + 2)^2\) | 120                  | \((x^2 + 1)^2\)   | \( \chi_6 \) |
| \(x^4 + 2x^2 + 4\) | 320              | \(x^4 + x^2 + 1\) | \( \chi_9 \) |
| \(x^4 + 4\)    | 600                  | \(x^4 + 1\)      | \( \chi_{11} \) |
| \((x^2 \pm 2x + 2)^2\) | 40               | \((x^2 \pm \sqrt{2}x + 1)^2\) | \( \chi_{14} \) |
| \(x^4 \pm 2x^3 + 2x^2 \pm 4x + 4\) | 320         | \(x^4 \pm \sqrt{2}x^3 + x^2 + \sqrt{2}x + 1\) | \( \chi_{15} \) |
| \((x^2 \pm 2x + 2)(x^2 + 2)\) | 480             | \((x^2 \pm \sqrt{2}x + 1)(x^2 + 1)\) | \( \chi_{16} \) |

(ii) The case \( p = 3 \).

| \( F(x) \)       | \( \pi \Gamma_1 \) | \( f(x) \)       | character |
|-----------------|---------------------|------------------|-----------|
| \((x^2 - 3)^2\) | 30                   | \((x - 1)^2(x + 1)^2\) | \( \chi_2 \) |
| \((x^2 + 3)^2\) | 30                   | \((x^2 + 1)^2\)   | \( \chi_6 \) |
| \((x^2 + 3x + 3)(x^2 + 3)\) | 120             | \((x^2 + \sqrt{3}x + 1)(x^2 + 1)\) | \( \chi_{17} \) |
| \((x^2 - 3x + 3)(x^2 + 3)\) | 120             | \((x^2 - \sqrt{3}x + 1)(x^2 + 1)\) | \( \chi_{17} \) |
| \(x^4 + 9\)    | 180                  | \(x^4 + 1\)      | \( \chi_{11} \) |
| \((x^4 + 3x^2 + 9)\) | 240             | \(x^4 + x^2 + 1\) | \( \chi_9 \) |

**Theorem 4.2.** For \( p = 2 \), we have

\[ Tr(R_{f_1,f_2}(\pi)) = \frac{1}{48} \chi_2 + \frac{1}{16} \chi_6 + \frac{1}{6} \chi_9 + \frac{5}{16} \chi_{11} + \frac{1}{48} \chi_{14} + \frac{1}{6} \chi_{15} + \frac{1}{4} \chi_{16}. \]
For \( p = 3 \), we have

\[
Tr(R_{f_1,f_2}(\pi)) = \frac{1}{24} \chi_2 + \frac{1}{24} \chi_6 + \frac{1}{3} \chi_9 + \frac{1}{4} \chi_{11} + \frac{1}{3} \chi_{17}.
\]

Theorem 4.2 is an easy corollary of Lemma 4.1, so we prove the lemma.

**Proof of Lemma 4.2.** When \( p = 2 \), a maximal order \( O \) is taken to be

\[
O = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z} \frac{1+i+j+k}{2}
\]

where \( i^2 = j^2 = -1 \), \( ij = -ji = k \). We have

\[
O^\times = \{\pm 1, \pm i, \pm j, \pm k, (\pm 1 \pm i \pm j \pm k)/2\}.
\]

The quaternion hermitian matrix corresponding to a lattice in the non-principal genus \( L_{npg} \) is given by

\[
gg^* = \begin{pmatrix}
2 & r \\
r & 2
\end{pmatrix}
\]

where \( r = i - k \), \( g = \begin{pmatrix} 1 & -1 \\ 0 & r \end{pmatrix} \).

The unit group \( \Gamma_1 \) is a set of matrices \( g^{-1} \epsilon g \) such that \( \epsilon \in M_2(\mathbb{Z}) \) and \( \epsilon gg^* \epsilon^* = gg^* \). This is given by the following set of elements (1) to (5) (see [14] p.592).

\[
\begin{align*}
(1) \quad & \begin{pmatrix} ar^{-1} & -aa_0r^{-1} \\ ar^{-1} & aa_0r^{-1} \end{pmatrix}, & (2) \quad & \begin{pmatrix} ar^{-1} & aa_0r^{-1} \\ -ar^{-1} & aa_0r^{-1} \end{pmatrix}, \\
(3) \quad & \begin{pmatrix} a & 0 \\ 0 & a_0 \end{pmatrix}, & (4) \quad & \begin{pmatrix} 0 & aa_0 \\ a & 0 \end{pmatrix}, \\
(5) \quad & \begin{pmatrix} (1+r^{-1}x)a & r^{-1}xa_0 \\ r^{-1}xa & (1+r^{-1}x)aa_0 \end{pmatrix},
\end{align*}
\]

where \( a \in O^\times, a_0 \in \{\pm 1, \pm i, \pm j, \pm k\} \) and \( x \in \{-i, k, (\pm 1-i-j+k)/2\} \).

For \( \pi = r = i - k \), by calculating the principal polynomials of elements in \( \pi \Gamma_1 \), we have the result in Lemma 4.1. This calculation is not so simple but anyway routine so we omit the details.

When \( p = 3 \), we can take

\[
O = \mathbb{Z} + \mathbb{Z} \frac{1+\alpha}{2} + \mathbb{Z} \beta + \mathbb{Z} \frac{(1+\alpha)\beta}{2}, \quad \alpha^2 = -3, \beta^2 = -1, \alpha \beta = -\beta \alpha.
\]

We have

\[
O^\times = \{\pm 1, \pm \beta, (\pm 1 \pm \alpha)/2, (\pm 1 \pm \alpha)\beta/2\}
\]

The quaternion hermitian matrix corresponding to a lattice in \( L_{npg} \) is given by

\[
gg^* = \begin{pmatrix} 3 & -(1+\beta)\alpha \\ (1+\beta)\alpha & 3 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 1+\beta \\ 0 & \alpha \end{pmatrix}.
\]
Then $\Gamma_1$ is given by the following elements.

\begin{align*}
(1) & \begin{pmatrix} a & 0 \\ 0 & aa_0 \end{pmatrix} \quad a \in O^\times, a_0 \in \{1, (-1 \pm \alpha)/2\}, \\
(2) & \begin{pmatrix} 0 & aa_0 \\ a & 0 \end{pmatrix} \quad a \in O^\times, a_0 \in \{1, (-1 \pm \alpha)/2\}, \\
(3) & \begin{pmatrix} \epsilon_1 & -\epsilon_1 c_2 \epsilon_2 \\ c_2 & \epsilon_2 \end{pmatrix} \quad c_2 \in O, N(c_2) = 2, \epsilon_1, \epsilon_2 \in O^\times, \\
& \quad \epsilon_1 \equiv -(1 - \beta)c_2 \mod \alpha, \epsilon_2 \equiv c_2(1 + \beta) \mod \alpha, \\
(4) & \begin{pmatrix} c_2 & \epsilon_2 \\ \epsilon_1 & -\epsilon_2 c_2 \epsilon_2 \end{pmatrix} \quad c_2 \in O, N(c_2) = 2, \epsilon_1, \epsilon_2 \in O^\times, \\
& \quad \epsilon_1 \equiv (1 + \beta)c_2 \mod \alpha, \epsilon_2 \equiv c_2(1 + \beta) \mod \alpha.
\end{align*}

The number of elements in (1), (2), (3), (4) are 36, 36, 324, 324, respectively. Then we put $\pi = \alpha$, and we calculate principal polynomials of all the elements of $\pi \Gamma_1$ and we have the result.

For $p = 2, 3$, we can compare dimensions of scalar valued paramodular cusp forms of weight $k$ and algebraic modular forms of weight $\rho_{k-3,k-3}$ ($k \geq 3$) with involutions directly by [18, 4, 22] without using [5]. We note that $S_{j+2}(\Gamma_0(p)) = 0$ for $j = 0$ and $p = 2, 3$, so the Yoshida lift part does not appear in the comparison and the relation becomes much more simple. For algebraic modular forms for $p = 2, 3$, we have the following result from the calculation of $Tr(R(\pi))$ given as above and the result in [12] (see section 5).

\begin{align*}
\sum_{f=0}^{\infty} \dim M_{f,f}(U_{npg}(2))t^f &= \frac{(1 + t^5)(1 + t^{20})}{(1 - t^4)(1 - t^6)(1 - t^8)(1 - t^{10})}, \\
\sum_{f=0}^{\infty} \dim M_{f,f}(U_{npg}(2))t^f &= \frac{1 + t^{25}}{(1 - t^4)(1 - t^6)(1 - t^8)(1 - t^{10})}, \\
\sum_{f=0}^{\infty} \dim M_{f,f}(U_{npg}(2))t^f &= \frac{t^5(1 + t^{15})}{(1 - t^4)(1 - t^6)(1 - t^8)(1 - t^{10})}, \\
\sum_{f=0}^{\infty} \dim M_{f,f}(U_{npg}(3))t^f &= \frac{(1 + t^5)(1 + t^{15})}{(1 - t^3)(1 - t^4)(1 - t^6)(1 - t^{10})}, \\
\sum_{f=0}^{\infty} \dim M_{f,f}(U_{npr}(3))t^f &= \frac{(1 + t^8)(1 + t^{15})}{(1 - t^4)(1 - t^6)^2(1 - t^{10})}, \\
\sum_{f=0}^{\infty} \dim M_{f,f}(U_{npr}(3))t^f &= \frac{(t^3 + t^5)(1 + t^{15})}{(1 - t^4)(1 - t^6)^2(1 - t^{10})}.
\end{align*}

On the other hand, for paramodular forms of level 2 and 3, the dimensions for plus and minus space (including non-cusp forms) has been given in [18, 4, 22]. The structure of cusps of paramodular varieties is well known (see for example [17]). For level $p$, there are two one-dimensional cusps isomorphic to the compactification of $H_1/SL_2(\mathbb{Z})$. 
that are crossing at one point. For a fixed $k$, the modular forms of
one variable on this boundary consist of a pair of elliptic cusp forms
of $SL_2(\mathbb{Z})$, which gives one to plus part and other to minus part
of $A_k(K(p))$, and the Eisenstein series besides, which belong to the plus
part. This fact and the surjectivity of generalized Siegel $\Phi$-operator by
Satake [39] show that $S_k^+(K(p))$ is given by

$$
\dim S_k^+(K(p)) = \dim A_k^+(K(p)) - \dim A_k(SL_2(\mathbb{Z})),
$$

$$
\dim S_k^+(K(p)) = \dim A_k^-(K(p)) - \dim S_k(SL_2(\mathbb{Z})).
$$

Here we give concrete results only for cusp forms. For the whole
space, see [22] p. 113.

$$
\sum_{k=0}^{\infty} \dim S_k^+(K(2))t^k = \frac{t^8 + t^{10} + t^{12} - t^{20} + t^{23} + t^{33}}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})},
$$

$$
\sum_{k=0}^{\infty} \dim S_k^-(K(2))t^k = \frac{t^{11} + t^{20} + t^{21} + t^{22} + t^{24} - t^{32}}{(1-t^4)(1-t^6)(1-t^8)(1-t^{12})},
$$

$$
\sum_{k=0}^{\infty} \dim S_k^+(K(3))t^k = \frac{t^6 + t^8 + t^{10} + t^{12} - t^{18} + t^{21} + t^{23} + t^{31}}{(1-t^4)(1-t^6)^2(1-t^{12})},
$$

$$
\sum_{k=0}^{\infty} \dim S_k^-(K(3))t^k = \frac{t^9 + t^{11} + t^{18} + t^{19} + t^{20} + t^{22} + t^{24} - t^{30}}{(1-t^4)(1-t^6)^2(1-t^{12})}.
$$

Observing these, we have the following dimensional relations without
assuming the results of [5]. (Actually these relations had been obtained
before independently of [5].)

**Proposition 4.3.** For $p = 2$ and $p = 3$, and any integer $k \geq 3$, we have

$$
\dim S_k^-(K(p)) = \dim S_k(Sp(2, \mathbb{Z})) + \delta_{k,\text{even}} \cdot \dim S_{2k-2}(SL_2(\mathbb{Z}))
$$

$$
- \dim M_{k-3,k-3}^+(U_{npr}(p)) - \delta_{k,3} - \delta_{k,\text{odd}} \cdot \dim S_{2k-2}(SL_2(\mathbb{Z})).
$$

$$
\dim S_k^+(K(p)) = \dim S_k(Sp(2, \mathbb{Z})) = \dim M_{k-3,k-3}^-(U_{npr}(p)),
$$

where $(\delta_{k,\text{even}}, \delta_{k,\text{odd}}) = (1, 0)$ if $k$ is even and $(0, 1)$ if $k$ is odd. Here
LHS are the dimensions of new forms in the sense of [36]. The term
$\dim S_{2k-2}(SL_2(\mathbb{Z}))$ is an adjustment for Saito-Kurokawa lift or Ihara
lift (which is the compact version of Saito-Kurokawa lift though it had
been proved much earlier).

5. **Appendix on dimensions and characters**

We review formulas for $\dim M_{f_1,f_2}(U_{npr}(p))$ from [12] II. We also give
explicit formulas of characters of irreducible representations of $Sp(2)$
for some elements of $Sp(2)$
Theorem 5.1. We assume that \( p \) is any prime including 2, 3, 5. Then we have

\[
\dim \mathfrak{M}_{k+j-3,k-3}(U_{npg}(p)) = \frac{p^2 - 1}{2880} \chi_1 + \frac{\delta_{p2}}{192} \chi_2 + \frac{\delta_{p2}}{16} \chi_3 + \frac{\delta_{p3}}{9} \chi_4 \\
+ \left( \frac{1}{2^3 \cdot 3} \left( p - \left( \frac{-1}{p} \right) \right) + \frac{1}{2^5 \cdot 3^2} \left( p - \left( \frac{-3}{p} \right) \right) \right) \chi_6 \\
+ \left( \frac{1}{2^3 \cdot 3} \left( p - \left( \frac{-3}{p} \right) \right) + \frac{1}{2^3 \cdot 3^2} \left( p - \left( \frac{-3}{p} \right) \right) \right) \chi_7 \\
+ \frac{\delta_{p2}}{6} \chi_9 + \frac{\chi_{10}}{5} \left( 1 - \left( \frac{p}{5} \right) \right) + \frac{\chi_{11}}{8} \left( 1 - \left( \frac{2}{p} \right) \right) \\
+ \frac{\chi_{12}}{24} \left( 1 - \left( \frac{3}{p} \right) + \left( \frac{-1}{p} \right) - \left( \frac{-3}{p} \right) \right)
\]

Here we understand \((n/p) = 0\) if \( p \) ramifies in \( \mathbb{Q}(\sqrt{n}) \), \(-1\) if \( p \) remains prime, and \( = 1 \) otherwise. So for example we have

\[
\left( \frac{-1}{2} \right) = \left( \frac{3}{2} \right) = \left( \frac{-3}{3} \right) = \left( \frac{5}{5} \right) = \left( \frac{2}{2} \right) = 0, \quad \left( \frac{-3}{2} \right) = -1.
\]

Here \( \chi_i \) are characters of the elements of \( Sp(2) \) whose principal polynomials are \( f_i(\pm x) \), where \( f_i(x) \) are given as follows.

\[
\begin{align*}
 f_1(x) &= (x - 1)^4 \\
 f_3(x) &= (x - 1)^2(x^2 + 1) \\
 f_5(x) &= (x - 1)^2(x^2 - x + 1) \\
 f_7(x) &= (x^2 + x + 1)^2 \\
 f_9(x) &= (x^2 + x + 1)(x^2 - x + 1) \\
 f_{11}(x) &= x^4 + 1 \\
 f_{13}(x) &= x^4 + \sqrt{5}x^3 + 3x^2 + \sqrt{5}x + 1 \\
 f_{15}(x) &= x^4 + \sqrt{2}x^3 + x^2 + \sqrt{2}x + 1 \\
 f_{17}(x) &= (x^2 + \sqrt{3}x + 1)(x^2 + 1)
\end{align*}
\]

These polynomials are possible principal polynomials of all the elements of finite order of \( G \) for \( n = 2 \) and \( g/\sqrt{p} \) with \( u(g) = p \) appearing in \( U_{npg} \). Of course the character formula is classical and written in [49], but explicit calculation is sometimes complicated and we give them below for readers’ convenience. We write the Young diagram parameter \((f_1, f_2)\) for an irreducible representation. By the general formula of [49], for a principal polynomial \( f(x) \) of any element in \( Sp(2) \), the character of \( \rho_{f_1, f_2} \) is given by

\[
p_{f_1}(p_{f_2} + p_{f_2-2}) - p_{f_2-1}(p_{f_1+1} + p_{f_1-1})
\]

where \( p_f \) is defined by

\[
\frac{1}{f(x)} = \sum_{f=0}^{\infty} p_f x^f
\]
with \( p_f = 0 \) for any \( f < 0 \).

Since it would be convenient to use the notation suitable for the calculation of paramodular forms \( S_{k,j}(K(p)) \) of weight \( \det^k \Sym(j) \), we put \((f_1, f_2) = (j + k - 3, k - 3) \ (k \geq 3, j \geq 0, j \text{ even})\) and write the characters below for these parameters \( k, j \).

Some part of the following table is up to constant the same as \( C_i(k, j) \) in [23] p. 604, and also \( \chi_i \) for \( i = 2, 6, 9, 11 \) are reproductions of those in the first section, noting that \( f_1 = k + j - 3, f_2 = k - 3 \).

\[
\begin{align*}
\chi_1 &= \frac{(j + 1)(k - 2)(j + k - 1)(j + 2k - 3)}{6}, \\
\chi_2 &= \frac{(-1)^{k-3}(k - 2)(k + j - 1)}{2}, \\
\chi_3 &= \frac{1}{2}((-1)^{j/2}(k - 2), -(j + k - 1), -(j + k - 1), -(-1)^{j/2}(k - 2), j + k - 1; 4]_k, \\
\chi_4 &= \frac{1}{3}((j + k - 1)[1, -1, 0; 3]_k + (k - 2)[1, 0, -1; 3]_j+k), \\
\chi_5 &= ((j + k - 1)[-1, -1, 0, 1, 1, 0; 6]_k + (k - 2)[1, 0, -1, 0, 1; 6]_j+k), \\
\chi_6 &= \frac{1}{2}(-1)^{(2k+j-6)/2} \times [-k + 2, j + k - 1; 2]_k, \\
\chi_7 &= \begin{cases} 
[(2k + j - 3), (2k + j - 2), (2k - 4); 3]_k/3 & \text{if } j \equiv 0 \text{ mod } 3, \\
[-(2k + j - 2), -(2k + j - 3), -(2k - 4); 3]_k/3 & \text{if } j \equiv 1 \text{ mod } 3, \\
[j + 1, -(j + 1), 0; 3]_k/3 & \text{if } j \equiv 2 \text{ mod } 3,
\end{cases} \\
\chi_8 &= \begin{cases} 
[-1, 0, 0, 1, 1, 0, 0, -1, -1, -1, 12]_k & \text{if } j \equiv 0 \text{ mod } 12, \\
[1, -1, 0, -1, 0, 0, -1, 0, 1, 1, 12]_k & \text{if } j \equiv 2 \text{ mod } 12, \\
[-1, 1, 0, 0, 1, -1, 0, 0, -1, 0, 1, 12]_k & \text{if } j \equiv 4 \text{ mod } 12, \\
[1, 0, 0, 1, -1, 0, 0, 0, -1, 1, 1, 12]_k & \text{if } j \equiv 6 \text{ mod } 12, \\
[-1, -1, 0, -1, 0, 1, 1, 0, 1, -1, 0; 12]_k & \text{if } j \equiv 8 \text{ mod } 12, \\
[1, 1, 0, 0, -1, -1, -1, 1, 0, 1, 1; 12]_k & \text{if } j \equiv 10 \text{ mod } 12,
\end{cases} \\
\chi_9 &= \begin{cases} 
[-1, 0, 0, 1, 0, 0, 6]_k & \text{if } j \equiv 0 \text{ mod } 6, \\
[1, -1, 0, -1, 1, 0; 6]_k & \text{if } j \equiv 2 \text{ mod } 6, \\
[0, 1, 0, 0, -1, 0; 6]_k & \text{if } j \equiv 4 \text{ mod } 6,
\end{cases} \\
\chi_{10} &= \begin{cases} 
[-1, 0, 0, 1, 0; 5]_k & \text{if } j \equiv 0 \text{ mod } 10, \\
[1, -1, 0, 0, 0; 5]_k & \text{if } j \equiv 2 \text{ mod } 10, \\
0 & \text{if } j \equiv 4 \text{ mod } 10, \\
[0, 0, 0, -1, 1; 5]_k & \text{if } j \equiv 6 \text{ mod } 10, \\
[0, 1, 0, 0, -1; 5]_k & \text{if } j \equiv 8 \text{ mod } 10, 
\end{cases} \\
\chi_{11} &= \begin{cases} 
[-1, 0, 0, 1; 4]_k & \text{if } j \equiv 0 \text{ mod } 8, \\
[1, -1, 0, 0; 4]_k & \text{if } j \equiv 2 \text{ mod } 8, \\
[1, 0, 0, 0; 4]_k & \text{if } j \equiv 4 \text{ mod } 8, \\
[-1, 1, 0, 0; 4]_k & \text{if } j \equiv 6 \text{ mod } 8,
\end{cases}
\]
\( \chi_{12} = (-1)^{j/2} \times \begin{cases} 
-1, 0, 0, 1, -2, 2; 6_k & \text{if } j \equiv 0 \mod 6, \\
-1, 1, 0; 3_k & \text{if } j \equiv 2 \mod 6, \\
[2, -1, 0, 0, 1, -2; 6_k] & \text{if } j \equiv 4 \mod 6,
\end{cases} \)

\( \chi_{13} = \begin{cases} 
-1, 0, 0, 1, 2, 1, 0, 0, -1, -2; 10_k & \text{if } j \equiv 0 \mod 10, \\
1, -1, 0, 2, 0, -1, 1, 0, -2, 0; 10_k & \text{if } j \equiv 2 \mod 10, \\
[0, 2, 0, -1, 1, 0, -2, 0, 1, -1; 10_k] & \text{if } j \equiv 6 \mod 10,
\end{cases} \)

\( \chi_{14} = \begin{cases} 
(-1)^{j/4}[j + k - 1, j + k - 1, k - 2, k - 2; 4_k] & \text{if } j \equiv 0 \mod 4, \\
(-1)^{(j-2)/4}[j + k - 1, k - 2, k - 2, j + k - 1; 4_k] & \text{if } j \equiv 2 \mod 4,
\end{cases} \)

\( \chi_{15} = (-1)^{[j/12]} \times \begin{cases} 
-1, 0, 0, 1, 0, -2, 1, 2, -2, -1, 2, 0; 12_k & \text{if } j \equiv 0 \mod 12, \\
1, -1, 0; 3_k & \text{if } j \equiv 2 \mod 12, \\
[0, -1, 0, 2, -1, -2, 2, 1, -2, 0, 1, 0; 12_k] & \text{if } j \equiv 4 \mod 12,
\end{cases} \)

\( \chi_{16} = \begin{cases} 
-1, 0, 0, 1, 1, 0, 0, -1; 8_k & \text{if } j \equiv 0 \mod 8, \\
1, -1, 0, 0, -1, 1, 0, 0; 8_k & \text{if } j \equiv 2 \mod 8, \\
-1, 0, 0, -1, 1, 0, 0; 8_k & \text{if } j \equiv 4 \mod 8,
\end{cases} \)

\( \chi_{17} = \begin{cases} 
-1, 0, 0, 1, 1, -1; 6_k & \text{if } j \equiv 0 \mod 12, \\
1, -1, 0; 3_k & \text{if } j \equiv 2 \mod 12, \\
-1, -1, 0, 0, 1, 1; 6_k & \text{if } j \equiv 4 \mod 12, \\
[1, 0, 0, -1, -1, 1; 6_k] & \text{if } j \equiv 6 \mod 12,
\end{cases} \)

6. Numerical Examples

We give several numerical examples of dimensions.

6.1. The case \( p = 5 \) and \( p = 7 \). First we assume \( p = 5 \) for a while. By Theorem 1.1 and 5.1 we have

\[
\sum_{f=0}^{\infty} T_{\overline{f},f}(R(\pi)) t^f = \frac{1 + t^{11}}{(1 - t^2)(1 - t^4)(1 + t^3)(1 + t^5)},
\]

\[
\sum_{f=0}^{\infty} \dim \mathcal{M}_{\overline{f},f}(U_{np(5)}) t^f = \frac{1 + t^{11}}{(1 - t^2)(1 - t^3)(1 - t^4)(1 - t^5)},
\]

so we have

\[
\sum_{f=0}^{\infty} \dim \mathcal{M}_{\overline{f},f}(U_{np(5)}) t^f = \frac{(1 + t^5)(1 + t^{11})}{(1 - t^2)(1 - t^4)(1 - t^6)(1 - t^{10})},
\]
By the way, adjusting non cusp forms by \[39\], we have

\[
\sum_{f=0}^{\infty} \dim \mathfrak{M}_{f,f}(U_{np9}(5))t^f = \frac{(t^3 + t^5)(1 + t^{11})}{(1 - t^2)(1 - t^4)(1 - t^6)(1 - t^{10})}.
\]

Next, from these result, we will give dimension formulas for paramodular forms of plus and minus signs. We have

\[
\sum_{k=0}^{\infty} \dim S_k(Sp(2, \mathbb{Z}))t^k = \frac{t^{10} + t^{12} - t^{22} + t^{35}}{(1 - t^4)(1 - t^6)(1 - t^{10})(1 - t^{12})},
\]

and

\[
\sum_{k=0}^{\infty} \dim S_{2k-2}(SL_2(\mathbb{Z}))t^k = \frac{t^7}{(1 - t^2)(1 - t^3)} = \frac{t^7 + t^{10}}{(1 - t^2)(1 - t^6)},
\]

and \(S_2(\Gamma_0(5)) = 0\), so by Theorem 1.2 we have

\[
\sum_{k=0}^{\infty} \dim S^+_k(K(5))t^k = \frac{Q^+_5(t)}{(1 - t^4)(1 - t^6)(1 - t^{10})(1 - t^{12})},
\]

\[
\sum_{k=0}^{\infty} \dim S^-_k(K(5))t^k = \frac{Q^-_5(t)}{(1 - t^4)(1 - t^6)(1 - t^{10})(1 - t^{12})},
\]

where

\[Q^+_5(t) = t^6 + 2t^8 + 3t^{10} + 3t^{12} + 2t^{14} + 2t^{16} + t^{17} + t^{18} + 2t^{19} + 2t^{21} - t^{22} + 2t^{23} + 2t^{25} + 2t^{27} + t^{29} + t^{35},\]

\[Q^-_5(t) = t^5 + t^7 + t^9 + 2t^{11} + 2t^{13} + t^{14} + 2t^{15} + t^{16} + t^{17} + t^{18} + t^{19} + 2t^{20} + t^{21} + 3t^{22} + t^{23} + 3t^{24} + t^{26} + t^{28} + t^{30} - t^{34}.\]

By the way, adjusting non cusp forms by \[39\], we have

\[
\sum_{k=0}^{\infty} \dim A^+_k(K(5))t^k = \frac{P^+_5(t)}{(1 - t^4)(1 - t^6)(1 - t^{10})(1 - t^{12})},
\]

\[
\sum_{k=0}^{\infty} \dim A^-_k(K(5))t^k = \frac{P^-_5(t)}{(1 - t^4)(1 - t^6)(1 - t^{10})(1 - t^{12})},
\]

\[
\sum_{k=0}^{\infty} \dim A_k(K(5))t^k = \frac{P_5(t)}{(1 - t^4)(1 - t^6)(1 - t^{10})(1 - t^{12})},
\]

where

\[P_+(t) = 1 + t^6 + 2t^8 + 2t^{10} + 2t^{12} + 2t^{14} + 2t^{16} + t^{17} + t^{18} + 2t^{19} + 2t^{21} + 2t^{23} + 2t^{25} + 2t^{27} + t^{29} + t^{35},\]

\[P_-(t) = t^5 + t^7 + t^9 + 2t^{11} + t^{12} + 2t^{13} + t^{14} + 2t^{15} + t^{16} + t^{17} + t^{18} + t^{19} + 2t^{20} + t^{21} + 2t^{22} + t^{23} + 2t^{24} + t^{26} + t^{28} + t^{30},\]

\[P(t) = 1 + t^6 + t^7 + 2t^8 + t^9 + 2t^{10} + t^{11} + 2t^{12} + 2t^{14} + 2t^{16} + 2t^{18} + t^{19} + 2t^{20} + t^{21} + 2t^{22} + t^{23} + 2t^{24} + t^{28} + t^{30}.\]
The last formula is easily deduced also from [15] and explicitly written in [31]. Note that the denominator for $A_k(K(5))$ is different from those for $A_k^\pm(K(5))$.

Next we consider the case $p = 7$. By using the formula in Theorem 1.1 and Theorem 5.1 for $\text{Tr}_{k-3, k-3}(R(\pi))$ and $\dim \mathcal{M}_{k-3, k-3}(U_{npq}(7))$, we have

$$\sum_{f=0}^{\infty} \dim \mathcal{M}^+(f(U_{npq}(7)))t^f = \frac{1 + t^3 + t^6 + t^{10} + t^{15} + t^{19}}{(1-t^2)(1-t^4)(1-t^6)(1-t^{10})},$$

$$\sum_{f=0}^{\infty} \dim \mathcal{M}^-(f(U_{npq}(7)))t^f = \frac{t + t^3 + t^5 + t^9 + t^{10} + t^{14} + t^{16} + t^{18}}{(1-t^2)(1-t^4)(1-t^6)(1-t^{10})}.$$ 

In the same way as before, we can calculate $\dim S^+_k(K(7))$ and $\dim A^+_k(K(7))$ for $k \geq 3$ using the above formulas. We know that $A_1(K(p)) = 0$ and $A_2(K(p)) = S_2(K(p))$ for general prime $p$. Since $\dim S_8(K(7)) = 2$ and $\dim A_6(K(7)) = 3$, we have $A_2(K(7)) = S_2(K(7)) = 0$. By $S_2(\Gamma_0(7)) = 0$, we can use Theorem 1.2 to have

$$\sum_{k=0}^{\infty} \dim S^+_k(K(7))t^k = \frac{Q^+_k(t)}{(1-t^4)^2(1-t^6)(1-t^{12})},$$

$$\sum_{k=0}^{\infty} \dim S^-_k(K(7))t^k = \frac{Q^-_k(t)}{(1-t^4)^2(1-t^6)(1-t^{12})},$$

where

$$Q^+_k(t) = t^4 + 2t^6 + 2t^8 + 2t^{10} + 2t^{12} + t^{13} + t^{14} + t^{15} + t^{17} + 2t^{19} + 2t^{21} + 2t^{23} + t^{29},$$

$$Q^-_k(t) = t^5 + 2t^7 + 2t^9 + 2t^{11} + t^{13} + t^{14} + t^{15} + 2t^{16} + t^{17} + 2t^{18} + 2t^{20} + 2t^{22} + 2t^{24} - t^{28}.$$ 

Counting the modular forms on the boundary, we have

$$\sum_{k=0}^{\infty} \dim A^+_k(K(7))t^k = \frac{P^+_k(t)}{(1-t^4)^2(1-t^6)(1-t^{12})},$$

$$\sum_{k=0}^{\infty} \dim A^-_k(K(7))t^k = \frac{P^-_k(t)}{(1-t^4)^2(1-t^6)(1-t^{12})},$$

$$\sum_{k=0}^{\infty} \dim A_k(K(7))t^k = \frac{P_k(t)}{(1-t^4)^2(1-t^6)(1-t^{12})},$$

$$P^+_k(t) = 1 + 2t^6 + 2t^8 + 2t^{10} + t^{12} + t^{13} + t^{14} + t^{15} + t^{16} + t^{17} + 2t^{19} + 2t^{21} + 2t^{23} + t^{29},$$

$$P^-_k(t) = 1 + 2t^6 + 2t^8 + 2t^{10} + t^{12} + t^{13} + t^{14} + t^{15} + t^{16} + t^{17} + 2t^{19} + 2t^{21} + 2t^{23} + t^{29},$$

$$P_k(t) = 1 + 2t^6 + 2t^8 + 2t^{10} + t^{12} + t^{13} + t^{14} + t^{15} + t^{16} + t^{17} + 2t^{19} + 2t^{21} + 2t^{23} + t^{29}.$$
\[ P_{-}^{(7)}(t) = t^5 + 2t^7 + 2t^9 + 2t^{11} + t^{12} + t^{13} + t^{14} + t^{15} + t^{16} + t^{17} + 2t^{18} + 2t^{20} + 2t^{22} + t^{24}, \]
\[ P^{(7)}(t) = 1 + t^5 + 2t^6 + 2t^7 + 2t^8 + 2t^9 + 2t^{10} + 2t^{11} + 2t^{12} + 2t^{13} + 2t^{14} + 2t^{15} + 2t^{16} + 2t^{17} + 2t^{18} + 2t^{19} + 2t^{20} + 2t^{21} + 2t^{22} + 2t^{23} + t^{24} + t^{29}. \]

The last formula is easily deduced also from [15] and explicitly written in [50].

Explicit generators of the modules \( \oplus_{k=0}^{\infty} A^\pm_k(K(p)) \) for \( p = 5 \) and 7 and their relations have been given in [50], so his results would implicitly give exact values of \( \dim A^\pm_k(K(p)) \) for \( p = 5, 7 \) for any \( k \), though the generating functions have not been explicitly given in his paper.

6.2. The case of small \( k \). For several small \( k \geq 3 \) with \( j = 0 \) and some primes, we give tables for \( \mathfrak{M}_{k-3, k-3}(U_{npq}(p)) \) and \( \dim S^\pm_k(K(p)) \).

In all the tables below, we put
\[ H = \dim \mathfrak{M}_{k-3, k-3}(U_{npq}(p)), \quad R = \text{Tr}_{k-3, k-3}(R(p)), \]
\[ M^+ = \dim \mathfrak{M}^+_{k-3, k-3}(U_{npq}(p)), \quad M^- = \dim \mathfrak{M}^-_{k-3, k-3}(U_{npq}(p)), \]
\[ S^+_k = \dim S^+_k(K(p)), \quad s^+_2 = \dim S^+_2(\Gamma_0(p)), \]
where \( k \) is fixed for each table.

Examples for the case \( k = 3 \) has been given in in [26] p. 218. We have \( \dim S^+_3(K(p)) = H - T \) and \( \dim S^-_3(K(p)) = T - 1 \). Here we determine exactly for which prime \( p \) we have \( \dim S^+_3(K(p)) = 0 \). First we explain some known facts. Let \( K(N)^* \) be the maximal extension of \( K(N) \) in \( Sp(2, \mathbb{R}) \) of order \( 2^\nu(N) \) where \( \nu(N) \) is the number of prime divisors of \( N \). In [3], it was proved that \( K(N)^* \backslash \mathcal{H} \) is the moduli space of the Kummer surfaces associated to \( (1, N) \) polarizations. In particular, elements of \( S_3(K(N)^*) \) give canonical differential forms of the Satake compactification of \( K(N)^* \backslash \mathcal{H} \) (See [4]). Now it was shown that \( \dim S_3(K(N)^*) = 0 \) for \( N \leq 40 \) in [3] and also that \( \dim S_3(K(N)^*) \geq 1 \) for \( N = 167, 173, 197, 213, 285 \) in [9]. When \( N = p \) is a prime, then we have \( S_3(K(p)^*) = S^+_3(K(p)) \). Now our new result for \( k = 3 \) is given as follows.

**Proposition 6.1.** Let \( p \) be a prime. Then we have
\[ \dim S^+_3(K(p)) = 0 \]
if and only if \( p \) is any prime such that \( p \leq 163 \) or \( p = 179, 181, 191, 193, 199, 211, 229, 241 \).

By the way, we have \( \dim S^+_3(K(p)) = 1 \) if \( p = 167, 173, 197, 223, 233, 239, 251, 271, 277, 281, 313, 331, 337 \) and \( \dim S^+_3(K(p)) = 2 \) if \( p = 227, 257, 263, 269, 283, 349, 379, 409, 421 \).
Proof of Proposition 6.1. We denote the class number and the type number for prime discriminant $p$ by $H(p)$ and $T(p)$. First by rough estimation we show that $H(p) - T(p) > 0$ for any $p \geq 3673$. Here we have $H(3673) = 5080$, $T(3673) = 3707$. Then using Theorem 1.1 and Theorem 5.1, we calculate $H(p) - T(p)$ directly for each prime $p < 3673$ and give the above result. Now we explain how to obtain the first estimation. For an imaginary quadratic field $\mathbb{Q}(\sqrt{D})$, denote by $h(\sqrt{D})$ the class number of $\mathbb{Q}(\sqrt{D})$. Let $D$ be the fundamental discriminant and by $\chi_D$ the character associated with $\mathbb{Q}(\sqrt{D})$. Then it is well known that for $p > 3$, we have 

$$L(1, \chi_D) = \frac{\pi h(\sqrt{D})}{\sqrt{|D|}}$$

On the other hand, by Siegel [45] section 15, we have 

$$|L(1, \chi_D)| < 2 \log(|D|) + \frac{1}{2}$$

so 

$$h(\sqrt{D}) < \frac{1}{\pi} \left( 2\sqrt{|D| \log |D| + \frac{1}{2} \sqrt{|D|} \right) .$$

On the other hand, denote by $D_0$ the discriminant of the real quadratic field $\mathbb{Q}(\sqrt{p})$ and by $\chi$ the character associated to $Q(\sqrt{p})$. Then we have 

$$L(2, \chi) = \frac{\pi^2}{D_0 \sqrt{D_0}} B_{2,\chi}$$

so 

$$B_{2,\chi} < \frac{p^{3/2}}{6} \quad \text{if } p \equiv 1 \mod 4$$

$$B_{2,\chi} < \frac{4p^{3/2}}{3} \quad \text{if } p \equiv 3 \mod 4$$

By these inequality we can estimate $Tr(R(\pi))$ in Theorem 1.1 from above as follows. 

$$Tr(R(\pi)) < g(p) := \frac{11p^{3/2}}{2^6 \cdot 3^2} + \frac{1}{2^{1/2}} (2\sqrt{p} \log(p) + \sqrt{p})$$

$$+ \frac{1}{2^{1/2}} (4\sqrt{2p} \log(2p) + \sqrt{3p}) + \frac{1}{3\pi} (4\sqrt{3p} \log(12p) + \sqrt{3p}) .$$

Here $Tr(R(\pi)) = 2T(p) - H(p)$. On the other hand, by Theorem 5.1 we see that 

$$H(p) > f(p) := \frac{p^2 - 1}{2880} + \frac{1}{36} (p + 1) + \frac{1}{32} (p + 1).$$

for not very small $p$. So $2(H(p) - T(p)) > f(p) - g(p) and taking the first and second derivatives of $f(p) - g(p)$, we see that $f(p) - g(p)$ is monotonously increasing for $p > 1923$ by elementary calculus. Then we see $f(3675) > g(3675)$ by numerical calculation and we are done. \qed
Next we study the cases $k = 4, 5, 6, 8$. In these cases, we have $S_k(Sp(2, \mathbb{Z})) = S_{2k-2}(SL_2(\mathbb{Z})) = 0$, so we have the following simple relations.

$$\dim S^+_k(K(p)) = \dim M^-_{k-3,k-3}(U_{npq}(p))$$
$$\dim S^-_k(K(p)) = \dim M^+_{k-3,k-3}(U_{npq}(p)).$$

The case $k = 4$.

| $p$ | 7  | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 | 43 | 47 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|
| $H$ | 1  | 1  | 2  | 2  | 3  | 3  | 4  | 6  | 8  | 7  | 9  | 8  |
| $R$ | -1 | -1 | -2 | -2 | -2 | -3 | -3 | -4 | -6 | -8 | -7 | -9 | -8 |
| $S^+_4$ | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 6 | 8 | 7 | 9 | 8 |
| $S^-_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| $p$ | 53 | 59 | 61 | 67 | 71 | 73 | 79 | 83 | 89 | 97 | 101 | 103 |
|-----|----|----|----|----|----|----|----|----|----|----|------|------|
| $H$ | 10 | 11 | 16 | 17 | 15 | 21 | 22 | 19 | 23 | 32 | 28 | 33 |
| $R$ | -10 | -11 | -16 | -17 | -15 | -21 | -22 | -17 | -23 | -32 | -26 | -31 |
| $S^+_4$ | 10 | 16 | 17 | 15 | 21 | 22 | 18 | 23 | 32 | 27 | 32 |
| $S^-_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |

A table of $\dim S^+_4(K(p))$ has been given in Table 4 in p. 28 by elaborate calculation of constructing paramodular forms, but the above table was obtained by our theoretical result independently. It is nice to see that the results coincide. (We added $p = 601$ and 607 as a small
The case $k = 5$.

| $p$  | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 | 43 | 47 | 53 | 59 | 61 | 67 | 71 |
|------|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $H$  | 1 | 2  | 3  | 4  | 5  | 5  | 9  | 10 | 14 | 15 | 16 | 16 | 22 | 24 | 31 | 33 | 33 |
| $R$  | 1 | 2  | 3  | 4  | 5  | 5  | 9  | 10 | 14 | 15 | 16 | 14 | 20 | 22 | 31 | 31 | 29 |
| $S^+_5$ | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 1  | 1  | 0  | 1  | 2  |
| $S^-_5$ | 1 | 2  | 3  | 4  | 5  | 5  | 9  | 10 | 14 | 15 | 16 | 15 | 21 | 23 | 31 | 32 | 31 |

The case $k = 6$.

| $p$  | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 | 43 | 47 |
|------|---|----|----|----|----|----|----|----|----|----|----|----|
| $H$  | 2 | 3  | 5  | 6  | 8  | 9  | 14 | 17 | 24 | 25 | 29 | 30 |
| $R$  | -2 | -3 | -5 | -6 | -8 | -9 | -14 | -17 | -24 | -23 | -27 | -24 |
| $S^+_6$ | 2 | 3  | 5  | 6  | 8  | 9  | 14 | 17 | 24 | 24 | 28 | 27 |
| $S^-_6$ | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 1  | 3  | 15 |

The case $k = 7$.

| $p$  | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 | 43 | 47 |
|------|---|----|----|----|----|----|----|----|----|----|----|----|
| $H$  | 4 | 6  | 10 | 14 | 17 | 22 | 34 | 40 | 57 | 64 | 72 | 80 |
| $R$  | -4 | -6 | -10 | -12 | -17 | -18 | -28 | -36 | -49 | -48 | -56 | -50 |
| $S^+_7$ | 4 | 6  | 10 | 13 | 17 | 20 | 31 | 38 | 53 | 56 | 64 | 65 |
| $S^-_7$ | 0 | 0  | 0  | 1  | 0  | 2  | 3  | 4  | 8  | 8  | 15 | 15 |

By the way, our formula gives $\dim S^+_8(K(277)) = 1761$ and $\dim S^-_8(K(277)) = 768$. These do not coincide with the numbers given in [34] page 31. The authors are aware of the error and will publish a correction [35].

The case $k = 8$.

| $p$  | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 | 43 | 47 |
|------|---|----|----|----|----|----|----|----|----|----|----|----|
| $H$  | 3 | 5  | 8  | 10 | 13 | 15 | 24 | 28 | 40 | 43 | 49 | 52 |
| $R$  | 3 | 5  | 8  | 10 | 13 | 13 | 22 | 26 | 36 | 37 | 41 | 36 |
| $M^+$ | 3 | 5  | 8  | 10 | 13 | 14 | 23 | 27 | 38 | 40 | 45 | 44 |
| $M^-$ | 0 | 0  | 0  | 0  | 0  | 1  | 1  | 1  | 2  | 3  | 4  | 8  |
| $s^+_2$ | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 0  | 1  | 0  |
| $s^-_2$ | 0 | 1  | 0  | 1  | 1  | 2  | 2  | 2  | 1  | 3  | 2  | 4  |
| $S^+_7$ | 0 | 0  | 0  | 0  | 0  | 1  | 1  | 1  | 1  | 3  | 3  | 8  |
| $S^-_7$ | 2 | 3  | 7  | 8  | 11 | 11 | 20 | 24 | 36 | 36 | 42 | 39 |

The case $k = 10$. We have $\dim S_{10}(Sp(2,\mathbb{Z})) = \dim S_{18}(SL_2(\mathbb{Z})) = 1$, so we have

\[
\begin{align*}
\dim S^+_7(K(p)) &= 1 + \dim M^-_{7,7}(U_{npq}(p)) - \dim S^+_{2}(\Gamma_0(p)), \\
\dim S^-_7(K(p)) &= \dim M^+_{7,7}(U_{npq}(p)) - \dim S^-_{2}(\Gamma_0(p)).
\end{align*}
\]
6.3. The case of $j > 0$. When $j = 2$, we have the following results.

$$\sum_{f=0}^{\infty} \dim M^+_{f+2,f}(U_{npq}(2))t^f = \frac{t^8(1 + t^2 + 2t^4 - t^5 - t^6 + t^9 - t^{12} + t^{13})}{(1 - t^4)(1 - t^5)(1 - t^6)(1 - t^8)},$$

$$\sum_{f=0}^{\infty} \dim M^-_{f+2,f}(U_{npq}(2))t^f = \frac{t^7}{(1 - t^2)(1 - t^4)(1 - t^5)(1 - t^8)},$$

$$\sum_{f=0}^{\infty} \dim M^+_{f+2,f}(U_{npq}(3))t^f = \frac{t^6(1 + t^2 + t^3 - t^5 + t^6 - t^7 - t^9)}{(1 - t^2)(1 - t^4)(1 - t^5)(1 - t^6)},$$

$$\sum_{f=0}^{\infty} \dim M^-_{f+2,f}(U_{npq}(3))t^f = \frac{t^5}{(1 - t^2)^2(1 - t^5)(1 - t^6)}.$$
\[
\sum_{k=0}^{\infty} \dim S_{k,2}^{+}(K(3))t^k = \frac{P_{+2}^{(3)}(t)}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})},
\]
\[
\sum_{k=0}^{\infty} \dim S_{k,2}^{-}(K(3))t^k = \frac{P_{-2}^{(3)}(t)}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})},
\]

where

\[
P_{+2}^{(2)}(t) = t^{10} + 2t^{14} + t^{15} + t^{18} + t^{19} + t^{23} - t^{24} + t^{27} - t^{29},
\]

\[
P_{-2}^{(2)}(t) = t^{11} + t^{14} + 2t^{15} + t^{16} - t^{17} + t^{18} + t^{19} + t^{22} - t^{24} + t^{26} - t^{28},
\]

\[
P_{+2}^{(3)}(t) = t^{8} + t^{10} + t^{13} + t^{14} + t^{15} + t^{16} - t^{20} + t^{21} + t^{23} - t^{25},
\]

\[
P_{-2}^{(3)}(t) = t^{9} + t^{11} + t^{12} + t^{14} + t^{15} + t^{16} + t^{22} - t^{24}.
\]

The result for \(k = 0, 1, 2\) are obtained by the dimension formula for higher \(k\). For example, we have \(\dim A_{4}(K(2)) = 1\), so if \(\dim S_{2,2}(K(2)) \neq 0\), then this contradicts to the formula \(\dim S_{6,2}(K(2)) = 0\). Arguments for the other cases are similar.

Next we consider the case \(j = 4\) and \(p = 2, 3\). There is one different point here. We have \(j + 2 = 6\) and \(2k + j - 2 = 2k + 2\). We have \(S_{0}(\Gamma_{0}(2)) = 0\) and for \(p = 2\) we have no new phenomenon and we have

\[
\dim S_{k,4}^{+}(K(2)) = \dim \mathfrak{M}_{k+1,k-3}(U_{np}(2)) + \dim S_{k,4}(Sp(2,\mathbb{Z})),
\]

\[
\dim S_{k,4}^{-}(K(2)) = \dim \mathfrak{M}_{k+1,k-3}(U_{np}(2)) + \dim S_{k,4}(Sp(2,\mathbb{Z})).
\]

But we have \(\dim S_{6}(\Gamma_{0}(3)) = \dim S_{6}^{-}(\Gamma_{0}(3)) = 1\). This means that we have the Yoshida lifting part \(S_{6}(\Gamma_{0}(3)) \times S_{2k+2}(SL_{2}(\mathbb{Z}))\) in \(M_{k+1,k-3}(U_{np}(3))\). So we have

\[
\dim S_{k,4}^{+}(K(3)) = \dim \mathfrak{M}_{k+1,k-3}(U_{np}(3)) + \dim S_{k,4}(Sp(2,\mathbb{Z})),
\]

\[
\dim S_{k,4}^{-}(K(3)) = \dim \mathfrak{M}_{k+1,k-3}(U_{np}(3)) + \dim S_{k,4}(Sp(2,\mathbb{Z})) - \dim S_{2k+2}(SL_{2}(\mathbb{Z})).
\]

By [14] and [21], we have

\[
\sum_{k=0}^{\infty} \dim S_{k,4}(Sp(2,\mathbb{Z}))t^k = \frac{t^{10} + t^{12} + t^{14} + t^{15} + t^{16} + t^{17} + t^{18} + t^{19} + t^{20} + t^{21} + t^{23} - t^{30}}{(1 - t^4)(1 - t^6)(1 - t^{10})(1 - t^{12})}.
\]

Anyway, by Theorem 4.2 and 5.1, we have

\[
\sum_{f=0}^{\infty} \dim \mathfrak{M}_{f+4,j}^{+}(U_{np}(2))t^j = \frac{t^4 + t^9}{(1 - t^2)(1 - t^4)(1 - t^6)(1 - t^8)},
\]

\[
\sum_{f=0}^{\infty} \dim \mathfrak{M}_{f+4,j}^{-}(U_{np}(2))t^j = \frac{t^5 + t^8}{(1 - t^2)(1 - t^4)(1 - t^6)(1 - t^8)},
\]
\[
\sum_{f=0}^{\infty} \dim \mathfrak{M}^+_{f+4,f}(U_{npq}(3)) t^f = \frac{t^2 + t^6}{(1 - t^2)(1 - t^3)(1 - t^4)(1 - t^6)},
\]
\[
\sum_{f=0}^{\infty} \dim \mathfrak{M}^-_{f+4,f}(U_{npq}(3)) t^f = \frac{t^3 + t^5}{(1 - t^2)(1 - t^3)(1 - t^4)(1 - t^6)}.
\]

Then we have
\[
\sum_{k=0}^{\infty} \dim S^+_{k,A}(K(2)) t^k = \frac{P^{(2)}_+(t)}{(1 - t^2)(1 - t^6)(1 - t^8)(1 - t^{12})},
\]
\[
\sum_{k=0}^{\infty} \dim S^-_{k,A}(K(2)) t^k = \frac{P^{(2)}_-(t)}{(1 - t^2)(1 - t^6)(1 - t^8)(1 - t^{12})},
\]
\[
\sum_{k=0, k \neq 2}^{\infty} \dim S^+_{k,A}(K(3)) t^k = \frac{P^{(3)}_+(t)}{(1 - t^2)(1 - t^4)(1 - t^6)(1 - t^{12})},
\]
\[
\sum_{k=0}^{\infty} \dim S^-_{k,A}(K(3)) t^k = \frac{P^{(3)}_-(t)}{(1 - t^2)(1 - t^4)(1 - t^6)(1 - t^{12})},
\]

where
\[
P^{(2)}_+(t) = t^8 + t^{10} + t^{11} + t^{12} + t^{14} + 2t^{15} + t^{16} + 2t^{19} + t^{20} + t^{22} + t^{24} - t^{26},
\]
\[
P^{(2)}_-(t) = t^7 + t^{10} + t^{11} + t^{12} + t^{14} + 2t^{15} + t^{16} + t^{19} + 2t^{20} - t^{22} + t^{24} - t^{26},
\]
\[
P^{(3)}_+(t) = t^6 + t^8 + t^9 + t^{10} + t^{11} + t^{12} + t^{14} + 2t^{15} + t^{17} + t^{20} - t^{22},
\]
\[
P^{(3)}_-(t) = 2t^9 + t^{10} + t^{11} + 2t^{12} + t^{14} + 2t^{15} + t^{17} + t^{18} + 2t^{20} - t^{21} - t^{22} - t^{24}.
\]

Here we do not know if \( S^+_{2,A}(K(3)) = 0 \) or not.

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