ON P-POWER FREENESS IN POSITIVE CHARACTERISTIC

HIROMU TANAKA

Abstract. In this note, we study base point freeness up to taking $p$-power, which we will call $p$-power freeness. We first establish some criteria for $p$-power freeness as analogues of criteria for semi-ampleness. We then apply these results to three-dimensional birational geometry.

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1. Introduction

It is a fundamental problem to study torsion line bundles in algebraic geometry. If $L$ is an $m$-torsion line bundle on an algebraic variety $X$ of characteristic zero, then $L$ induces an étale cover of degree $m$. The same statement holds in characteristic $p > 0$ when $m$ is not divisible by $p$. However, if $m$ is divisible by $p$, then the resulting cyclic cover $Y \rightarrow X$ is no longer étale. Therefore $p^e$-torsion line bundles have different feature from $\ell$-torsion line bundles for $\ell \in \mathbb{Z}_{\geq 0} \setminus p\mathbb{Z}$.

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To observe another phenomenon on $p^e$-torsion line bundles, let us recall the Lefschetz theorem for the local Picard groups (cf. [BdJ14, Theorem 0.1]). Let $(A, \mathfrak{m})$ be an excellent normal local ring containing a field. For $f \in \mathfrak{m} \setminus \{0\}$, $V := \text{Spec} A \setminus \{\mathfrak{m}\}$, and $V_0 := \text{Spec} (A/f) \setminus \{\mathfrak{m}\}$, the restriction map $\rho : \text{Pic}(V) \to \text{Pic}(V_0)$ satisfies the following properties.

(0) Assume that $A$ is of characteristic zero. If depth$_{\mathfrak{m}}(A/f) \geq 2$, then $\rho$ is injective.

(p) Assume that $A$ is of positive characteristic. Then $\rho$ is injective up to taking $p$-power, i.e. if $\rho(L) \simeq \mathcal{O}_{V_0}$, then $L \otimes p^e \simeq \mathcal{O}_V$ for some $e \in \mathbb{Z}_{>0}$.

In this theorem, we need to care $p^e$-torsion line bundles for the case of positive characteristic, whilst there does not appear such non-trivial torsion line bundles in the corresponding statement of characteristic zero. One of the purposes of this note is to observe a similar phenomenon that appears in birational geometry.

To this end, we first study the behaviour of $p^e$-torsion line bundles in general settings. For flexibility, we study a wider notion: base point freeness up to $p$-power, which we will call $p$-power freeness.

**Definition 1.1.** Let $f : X \to Y$ be a proper morphism of noetherian $\mathbb{F}_p$-schemes. Let $L$ be an invertible sheaf on $X$.

1. We say that $L$ is $f$-free if the induced homomorphism

$$f^* f_* L \to L$$

is surjective. If $Y = \text{Spec} k$ for a field $k$, then we simply say that $L$ is free.

2. We say that $L$ is $p$-power $f$-free or $p$-power free over $Y$ if there exists a positive integer $e \in \mathbb{Z}_{>0}$ such that $L \otimes p^e$ is $f$-free. If $Y = \text{Spec} k$ for a field $k$, then we simply say that $L$ is $p$-power free.

There are some criteria for semi-ampleness which hold only in positive characteristic. A typical result is Keel’s theorem [Kee99, Theorem 0.2]. Another example is the equivalence between the relative semi-ampleness and the fibrewise semi-ampleness [CT, Theorem 1.1]. It is remarkable that these two semi-ampleness criteria have analogous statements for $p$-power freeness as follows.

**Theorem 1.2 (Theorem 3.4).** Let $f : X \to Y$ be a projective morphism of noetherian $\mathbb{F}_p$-schemes. Let $L$ be an invertible sheaf on $X$. If $L|_{X_y}$ is $p$-power free for any point $y \in Y$, then $L$ is $p$-power $f$-free.
Theorem 1.3 (Theorem 3.6). Let \( f : X \to Y \) be a projective morphism of noetherian \( \mathbb{F}_p \)-schemes. Let \( L \) be an \( f \)-nef invertible sheaf on \( X \) and let \( g : E_f(L) \to X \to Y \) be the induced morphism. Then \( L \) is \( p \)-power \( f \)-free if and only if \( L|_{E_f(L)} \) is \( p \)-power \( g \)-free.

We then apply these criteria to birational geometry in positive characteristic. Let us first recall the Kawamata–Shokurov base point free theorem in characteristic zero [KMM87, Theorem 3-1-1].

Theorem 1.4 (Kawamata–Shokurov). Let \( k \) be a field of characteristic zero. Let \((X, \Delta)\) be a klt pair over \( k \) and let \( f : X \to Z \) be a projective \( k \)-morphism to a quasi-projective \( k \)-scheme \( Z \). Let \( L \) be an \( f \)-nef Cartier divisor such that \( L - (K_X + \Delta) \) is \( f \)-nef and \( f \)-big. Then there exists a positive integer \( m_0 \) such that \( mL \) is \( f \)-free for any integer \( m \geq m_0 \).

It is known that the same statement is no longer true in positive characteristic [Tan, Theorem 1.2]. More specifically, over an algebraically closed field \( k \) of characteristic \( p \in \{2, 3\} \), there exist a three-dimensional klt pair \((X, \Delta)\), a projective morphism \( f : X \to Z \) to a smooth curve \( Z \), and an \( f \)-numerically trivial Cartier divisor \( L \) on \( X \) such that \( L - (K_X + \Delta) \) is \( f \)-ample and \( L \not\sim_f 0 \). On the other hand, it holds that \( pL \sim_f 0 \) for this example. Then it is tempting to hope that \( L \) is \( p \)-power free in the case of positive characteristic.

Conjecture 1.5. Let \( k \) be a field of characteristic \( p > 0 \). Let \((X, \Delta)\) be a klt pair over \( k \) and let \( f : X \to Z \) be a projective \( k \)-morphism to a quasi-projective \( k \)-scheme \( Z \). Let \( L \) be an \( f \)-nef Cartier divisor on \( X \) such that \( L - (K_X + \Delta) \) is \( f \)-nef and \( f \)-big. Then \( L \) is \( p \)-power \( f \)-free, i.e. there exists a positive integer \( e \) such that \( p^eL \) is \( f \)-free.

Remark 1.6. Assume that \( \text{dim} X = 2 \). Then the same statement of Theorem 1.4 holds for the case when \( k \) is a perfect field of characteristic \( p > 0 \) (Lemma 4.1). However, if \( k \) is allowed to be an imperfect field, then the same statement of Theorem 1.4 no longer holds [Tan, Theorem 1.4]. On the other hand, Conjecture 1.5 is known to hold when \( X \) is a surface over an imperfect field (cf. Remark 4.2).

In this note, we establish the following partial solution (Theorem 1.7) as an application of our criteria for \( p \)-power freeness. Note that our proof depends on the recent development of minimal model program for threefolds of characteristic \( p > 5 \) ([HX15], [CTX15], [Bir16], [BW17], [Wal18], [HNT]).

Theorem 1.7 (Theorem 4.4). Let \( k \) be a perfect field of characteristic \( p > 5 \). Let \((X, \Delta)\) be a three-dimensional klt pair and let \( f : X \to Z \)
be a projective surjective $k$-morphism to a quasi-projective $k$-scheme $Z$. Let $L$ be an $f$-nef Cartier divisor on $X$ such that $L - (K_X + \Delta)$ is $f$-nef and $f$-big. Assume that either

1. $\dim Z \geq 1$, or
2. $\dim Z = 0$ and $L \not\equiv 0$.

Then $L$ is $p$-power $f$-free.

**Remark 1.8.** Under the same assumptions as the ones of Theorem 1.7, Bernasconi proves that if $p \gg 0$, $\dim Z \geq 1$, and $L$ is a Cartier divisor on $X$ such that $L \equiv_f 0$, then $L \sim_f 0$ [Ber].

1.1. **Overviews of proofs.** We now overview how to prove the main theorems. Let us first discuss the criteria for $p$-power freeness (Theorem 1.2, Theorem 1.3). Theorem 1.3 follows from the same argument as the proof of Keel’s theorem given by [CMM14]. Theorem 1.2 is the $p$-power free version of [CT], Theorem 1.1. Although the same argument as in [CT, Theorem 1.1] does not work for our case, Theorem 1.2 follows from a combination of [CT, Theorem 1.1] and a recent result by Bhatt–Scholze [BS17].

We now give an overview of how to show Theorem 1.7. The first step is to establish a birational version as follows.

**Theorem 1.9 (Theorem 4.3).** Let $k$ be a perfect field of characteristic $p > 5$. Let $f : X \to Y$ be a projective birational $k$-morphism of quasi-projective normal threefolds over $k$. Assume that there exists an effective $\mathbb{R}$-divisor $\Delta_Y$ on $Y$ such that $(Y, \Delta_Y)$ is klt. Let $L$ be a Cartier divisor on $X$ such that $L \equiv_f 0$. Then there exist a positive integer $e$ and a Cartier divisor $L_Y$ on $Y$ such that $L \sim p^e L_Y$.

Roughly speaking, the proof of Theorem 1.9 is done by resolution of singularities and minimal model program. Thus, a crucial part is the case when $f$ is an extremal contraction of pl-type (cf. Step 1 of Theorem 4.3). In this case, the problem is reduced to the case of dimension two by using Theorem 1.3.

In order to prove Theorem 1.7 we utilise a variant of Theorem 1.2 (Theorem 3.5), so that it is enough to show that $L|_{X_z}$ is $p$-power free for any closed point $z \in Z$. Theorem 1.9 enables us to take birational model changes. Then we may assume that $T := (X_z)_{\text{red}}$ is a surface of Fano type, i.e. there exists an effective $\mathbb{Q}$-divisor $\Delta_T$ such that $(T, \Delta_T)$ is klt and $-(K_T + \Delta_T)$ is ample (cf. Step 1 of Theorem 4.3). Then $L|_{(X_z)_{\text{red}}}$ is trivial, hence also $p^e L|_{X_z}$ is trivial for some $e \in \mathbb{Z}_{>0}$. In particular, $L|_{X_z}$ is $p$-power free. For more details, see Section 4.
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2. Preliminaries

2.1. Notation.

(1) We will freely use the notation and terminology in [Har77] and [Kol13].

(2) For a scheme $X$, its reduced structure $X_{\text{red}}$ is the reduced closed subscheme of $X$ such that the induced morphism $X_{\text{red}} \to X$ is surjective.

(3) For a field $k$, we say that $X$ is a variety over $k$ or a $k$-variety if $X$ is an integral scheme that is separated and of finite type over $k$. We say that $X$ is a curve over $k$ or a $k$-curve (resp. a surface over $k$ or a $k$-surface, resp. a threefold over $k$) if $X$ is a $k$-variety of dimension one (resp. two, resp. three).

(4) Let $f : X \to Y$ be a morphism of noetherian schemes. We say that $f$ is projective if there exists a closed immersion $X \hookrightarrow \mathbb{P}^n_Y$ over $Y$ for some $n \in \mathbb{Z}_{>0}$. This definition coincides with the one in [Har77, page 103], but differs from the one given by Grothendieck [Gro61, Définition 5.5.2]. On the other hand, their definitions coincide in many cases (cf. [FGI+05, Section 5.5.1]).

(5) A morphism $f : X \to Y$ of schemes has connected fibres if $X \times_Y \text{Spec} L$ is either empty or connected for any field $L$ and any morphism $\text{Spec} L \to Y$.

2.2. Properties of divisors. Let $f : X \to Y$ be a proper morphism of noetherian schemes and let $L$ be an invertible sheaf on $X$.

(1) $L$ is $f$-nef if for any field $K$, morphism $\text{Spec} K \to Y$, and a curve $C$ on $X \times_Y \text{Spec} K$, the inequality $\alpha^* L \cdot C \geq 0$ holds for the induced morphism $\alpha : X \times_Y \text{Spec} K \to X$.

(2) $L$ is $f$-numerically trivial if both $L$ and $L^{-1}$ are $f$-nef.

(3) $L$ is $f$-free if the natural homomorphism $f^* f_* L \to L$ is surjective. In particular, if $L$ is $f$-free then it induces a morphism $X \to \mathbb{P}(f_* L)$ over $Y$.

(4) $L$ is $f$-very ample if it is $f$-free and the induced morphism $X \to \mathbb{P}(f_* L)$ is a closed immersion.

(5) $L$ is $f$-semi-ample (resp. $f$-ample) if $L^\otimes m$ is $f$-free (resp. $f$-very ample) for some positive integer $m$. 
(6) $L$ is $f$-weakly big if there exist an $f$-ample invertible sheaf $A$ on $X$ and a positive integer $m$ such that if $g : X_{\text{red}} \to Y$ denotes the induced morphism, then
$$ g_*((L^\otimes m \otimes \mathcal{O}_X A^{-1})|_{X_{\text{red}}}) \neq 0. $$
Assume that $X$ is normal. $L$ is $f$-big if, for any connected component $X'$ of $X$, the restriction $L|_{X'}$ is $f'$-weakly big, where $f' : X' \to Y$ denotes the induced morphism.

(7) If $L$ is $f$-nef, the $f$-exceptional locus of $L$, denoted by $\mathcal{E}_f(L)$, is defined as the union of all the reduced closed subschemes $V \subset X$ such that $L|_V$ is not $f|_V$-weakly big. It is known that $\mathcal{E}_f(L)$ is a closed subset of $X$ [CT, Lemma 2.18]. We consider $\mathcal{E}_f(L)$ as a reduced closed subscheme of $X$.

3. $p$-POWER FREENESS

In this section, we first introduce $p$-power freeness for invertible sheaves (Subsection 3.1). In Subsection 3.2, we prove that the relative $p$-power freeness is equivalent to the fibrewise $p$-power freeness (Theorem 3.4). In Subsection 3.3, we establish the $p$-power free version of Keel’s theorem (Theorem 3.6).

3.1. Definition.

**Definition 3.1.** Let $f : X \to Y$ be a proper morphism of noetherian $\mathbb{F}_p$-schemes. Let $L$ be an invertible sheaf.

1. We say that $L$ is $p$-power $f$-free or $p$-power free over $Y$ if there exists a positive integer $e \in \mathbb{Z}_{>0}$ such that $L^\otimes p^e$ is $f$-free (for definition of being $f$-free, see Subsection 2.2(3)). If $Y = \text{Spec} \ k$ for a field $k$, then we simply say that $L$ is $p$-power free.

2. The $p$-power $f$-base locus of $L$ is defined as the following closed subset of $X$:
$$ \mathcal{B}_f^p(L) = \bigcap_{e=0}^{\infty} \text{Supp} \ Coker(f^* f_* L^\otimes p^e \to L^\otimes p^e). $$
Note that $L$ is $p$-power $f$-free if and only if $\mathcal{B}_f^p(L) = \emptyset$.

**Lemma 3.2.** Let
$$ X' \xrightarrow{\alpha} X \xrightarrow{f} Y \quad Y' \xrightarrow{\beta} Y $$
be a cartesian diagram consisting of morphisms of noetherian $\mathbb{F}_p$-schemes, where $f$ is proper and $\beta$ is faithfully flat. Let $L$ be an invertible sheaf on $X$. Then $L$ is $p$-power $f$-free if and only if $\alpha^*L$ is $p$-power $f'$-free.

**Proof.** The assertion follows from the corresponding statement for usual freeness. □

**Lemma 3.3.** Let

$$f' : X' \xrightarrow{\pi} X \xrightarrow{f} Y$$

be proper morphisms of noetherian $\mathbb{F}_p$-schemes. Assume that $\pi$ is a surjective morphism which have connected fibres. Let $L$ be an invertible sheaf on $X$. Then $L$ is $p$-power $f$-free if and only if $\pi^*L$ is $p$-power $f'$-free.

**Proof.** Taking the Stein factorisation of $\pi$, we may assume that $\pi_*\mathcal{O}_{X'} = \mathcal{O}_X$ or $\pi$ is finite. If $\pi_*\mathcal{O}_{X'} = \mathcal{O}_X$, then the assertion follows from the corresponding statement for usual freeness. Thus we may assume that $\pi$ is finite. In this case, $\pi$ is a finite universal homeomorphism. Assuming that $\pi^*L$ is $p$-power $f'$-free, it suffices to show that $L$ is $p$-power $f$-free. It follows from [Kol97, Proposition 6.6] that there exists a positive integer $e$ such that the $e$-th iterated absolute Frobenius morphism $F^e : X \rightarrow X$ factors through $\pi$:

$$F^e : X \xrightarrow{\rho} X' \xrightarrow{\pi} X.$$

Since $\pi^*L$ is $p$-power free over $Y$, so is its pullback

$$L \otimes^{p^e} = (F^e)^*L = \rho^*\pi^*L.$$

Hence, $L$ is $p$-power free over $Y$. □

### 3.2. Fibrewise $p$-power freeness

In this subsection, we prove that the relative $p$-power freeness is equivalent to the fibrewise $p$-power freeness (Theorem 3.4). If the base field is uncountable, then we obtain a stronger criterion (Theorem 3.5).

**Theorem 3.4.** Let $f : X \rightarrow Y$ be a projective morphism of noetherian $\mathbb{F}_p$-schemes. Let $L$ be an invertible sheaf on $X$. If $L|_{X_y}$ is $p$-power free for any point $y \in Y$, then $L$ is $p$-power $f$-free.

**Proof.** It follows from [CT, Theorem 1.1] that $L$ is $f$-semi-ample. Thus, there exist a positive integer $m$, projective morphisms

$$f : X \xrightarrow{g} Z \xrightarrow{h} Y,$$

and an ample invertible sheaf $A_Z$ on $Z$ such that $L^\otimes m \simeq g^*A_Z$ and $g_*\mathcal{O}_X = \mathcal{O}_Z$. Since $L|_{X_y}$ is $p$-power free for any point $y \in Y$, $L_{X_z}$ is $p$-power free for any point $z \in Z$. By noetherian induction, we
can find $e \in \mathbb{Z}_{>0}$ such that $L \otimes p^e|_{X_z}$ is free for any $z \in Z$. Since $L \otimes p^e|_{X_z}$ is numerically trivial, we obtain $L \otimes p^e|_{X_z} \simeq \mathcal{O}_{X_z}$. Therefore, by [BS17, Theorem 1.3], there exists $d \in \mathbb{Z}_{>0}$ and an invertible sheaf $L_Z$ on $Z$ such that $L \otimes p^d \simeq g^*L_Z$. Since $L_Z$ is ample over $Y$, $L_Z$ is $p$-power free over $Y$. Hence, also its pullback $L \otimes p^d \simeq g^*L_Z$ is $p$-power free over $Y$. Therefore, $L$ is $p$-power free over $Y$. □

Although the proof of the following theorem is very similar to the one of [CT, Lemma 6.1], we give a proof for the sake of completeness.

**Theorem 3.5.** Let $k$ be an uncountable field of characteristic $p > 0$. Let $f : X \to Y$ be a projective $k$-morphism of schemes which are of finite type over $k$. Let $L$ be an invertible sheaf on $X$. If $L|_{X_y}$ is $p$-power free for any closed point $y \in Y$, then $L$ is $p$-power free.

**Proof.** We prove the assertion by induction on $\dim Y$. If $\dim Y = 0$, then there is nothing to show. Thus, we may assume that $\dim Y > 0$ and that the assertion holds if the dimension of the base is smaller than $\dim Y$. By Theorem 3.4 it is enough to show that $L|_{X_\xi}$ is $p$-power free for the generic point $\xi \in Y$ of an irreducible component of $Y$. Replacing $Y$ by an open neighbourhood of $\xi \in Y$, the problem is reduced to the case when $Y$ is an affine irreducible scheme such that $f$ is flat.

By the semi-continuity theorem [Har77, Ch. III, Theorem 12.8], if $e$ is a positive integer, then there exist a positive integer $d_e$ and a non-empty affine open subset $U_e \subset Y$ such that the equation

$$d_e = \dim_{k(y)} H^0(X_y, L \otimes p^e|_{X_y})$$

holds for any point $y \in U_e$. Since $k$ is uncountable, there exists a closed point

$$z \in \bigcap_{m \in \mathbb{Z}_{>0}} U_m.$$

As $L|_{X_z}$ is $p$-power free, there exists a positive integer $e_0$ such that $L \otimes p^{e_0}|_{X_z}$ is free. By [Har77, Ch. III, Corollary 12.9], the restriction map

$$H^0(f^{-1}(U_{e_0}), L \otimes p^{e_0}|_{f^{-1}(U_{e_0})}) \to H^0(X_z, L \otimes p^{e_0}|_{X_z})$$

is surjective. Since the base locus of $L \otimes p^{e_0}$ is a closed subset of $X$, it is disjoint from $X_z$. In particular, $L \otimes p^{e_0}|_{X_\xi}$ is free, as desired. □

3.3. **Keel’s theorem.** We have the $p$-power free version of Keel’s theorem on semi-ampleness (Theorem 3.6). For a later use, we also establish a variant (Proposition 3.7).
Theorem 3.6. Let $f : X \to Y$ be a projective morphism of noetherian $\mathbb{F}_p$-schemes. Let $L$ be an $f$-nef invertible sheaf on $X$ and let $g : \mathbb{E}_f(L) \hookrightarrow X \xrightarrow{f} Y$ be the induced morphism. Then the equation
\[ B^p_f(L) = B^p_g(L|_{\mathbb{E}_f(L)}) \]
holds. In particular, $L$ is $p$-power $f$-free if and only if $L|_{\mathbb{E}_f(L)}$ is $p$-power $g$-free.

Proof. We may apply the same argument as in [CT, Proposition 2.20] after replacing the stable base loci $B(\cdot)$ by the $p$-power base loci $B^p(\cdot)$ (for the definition of $B^p(\cdot)$, see Definition 3.1).

Proposition 3.7. Let $k$ be a field of characteristic $p > 0$. Let $f : X \to Y$ be a projective $k$-morphism from a normal $k$-variety $X$ to a scheme $Y$ which is of finite type over $k$. Let $L$ be an $f$-nef Cartier divisor. Assume that there exist an $f$-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $A$ and an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $E$ such that $L \equiv_f A + E$. If $L|_{\text{Supp} E}$ is $p$-power free over $Y$, then $L$ is $p$-power free over $Y$.

Proof. After replacing $A$ by $A' := L - E$, we may assume that the equation $L = A + E$ of $\mathbb{Q}$-divisors holds. It is enough to show the inclusion $\mathbb{E}_f(L) \subset \text{Supp} E$, which follows from [CT, Lemma 2.18(1)].

4. Application to threefolds

As applications of results in Section 3, we prove Theorem 4.3 and Theorem 4.4. We start with a result on surfaces.

Lemma 4.1. Let $k$ be a perfect field of characteristic $p > 0$. Let $(X, \Delta)$ be a two-dimensional klt pair and let $f : X \to Y$ be a projective $k$-morphism to a quasi-projective $k$-scheme $Y$. Let $L$ be an $f$-nef Cartier divisor on $X$ such that $L - (K_X + \Delta)$ is $f$-nef and $f$-big. Then there exists a positive integer $m_0$ such that $mL$ is $f$-free for any integer $m \geq m_0$. In particular, $L$ is $p$-power $f$-free.

Proof. We may assume that $k$ is algebraically closed and $f_* \mathcal{O}_X = \mathcal{O}_Y$. If either $\dim Y \geq 1$ or $\dim Y = 0$ and $L \not\equiv 0$, then the assertion follows from [Tan18, Theorem 4.2]. Thus, we may assume that $\dim Y = 0$ and $L \equiv 0$. In this case, the assertion follows from [Tan15, Corollary 3.6].

Remark 4.2. If we allow $k$ to be an imperfect field, then the same statement as in Lemma 4.1 no longer holds [Tan, Theorem 1.4]. On the other hand, even if $k$ is an imperfect field, the $p$-power freeness of $L$ holds. Indeed, if either $\dim Y \geq 1$ or $\dim Y = 0$ and $L \not\equiv 0$, then the
assertion follows from [Tan18, Theorem 4.2]. If \( \dim Y = 0 \) and \( L \equiv 0 \),
then we obtain \( p^e L \sim 0 \) by [BT, Theorem 1.3]. We do not use this fact
in this paper.

**Theorem 4.3.** Let \( k \) be a perfect field of characteristic \( p > 5 \). Let \( f : X \to Y \)
be a projective birational \( k \)-morphism of quasi-projective normal threefolds over \( k \).
Assume that there exists an effective \( \mathbb{R} \)-divisor \( \Delta_Y \) on \( Y \) such that \( (Y, \Delta_Y) \)
is klt. Let \( L \) be a Cartier divisor on \( X \) such that \( L \equiv_f 0 \). Then there exist a positive integer \( e \)
and a Cartier divisor \( L_Y \) on \( Y \) such that \( L \sim p^e L_Y \).

**Proof.** The proof consists of three steps.

**Step 1.** The assertion of Theorem 4.3 holds if \( Y \) is \( \mathbb{Q} \)-factorial.

**Proof of Step 1.** Taking a log resolution of \((Y, \Delta_Y)\) which dominates \( X \), we may assume that \( f : X \to Y \) is a log resolution of \((Y, \Delta_Y)\). Let \( E \) be the reduced \( f \)-exceptional divisor such that \( \text{Supp } E = \text{Ex}(f) \). Set \( \Delta := f^{-1}_* \Delta_Y + E \).
Then we have
\[
K_X + \Delta = K_X + f^{-1}_* \Delta_Y + E \sim_{\mathbb{R}} f^*(K_Y + \Delta_Y) + E'
\]
where \( E' \) is an \( f \)-exceptional effective \( \mathbb{R} \)-divisor on \( X \) such that \( \text{Supp } E' = \text{Ex}(f) \). By [HNT, Theorem 1.1], there exists a \((K_X + \Delta)\)-MMP over \( Y \) that terminates:
\[
X =: X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_N.
\]
The negativity lemma implies that \( f_N : X_N \to Y \) is small. Since \( Y \) is \( \mathbb{Q} \)-factorial, \( f_N \) is an isomorphism.

Therefore, it is enough to treat the case when \((X, \Delta)\) is a \( \mathbb{Q} \)-factorial dlt pair, \(-(K_X + \Delta)\) is \( f \)-ample, \( \rho(X/Y) = 1 \), and \(-S\) is \( f \)-ample for some prime divisor \( S \) contained in \( \text{Supp } \Delta \). It holds that \( A := L - S \) is an \( f \)-ample \( \mathbb{Q} \)-Cartier \( \mathbb{Z} \)-divisor. Since \( L|_S \) is \( p \)-power free over \( Y \) (Lemma 3.3), it follows from Proposition 3.7 that \( L \) is \( p \)-power free over \( Y \). This completes the proof of Step 1.

**Step 2.** The assertion of Theorem 4.3 holds if there exists an effective \( \mathbb{Q} \)-divisor \( \Delta \) on \( X \) such that \((X, \Delta)\) is klt and \(-(K_X + \Delta)\) is \( f \)-nef and \( f \)-big.

**Proof of Step 2.** Set \( \mathcal{L} := \mathcal{O}_X(L) \). Taking the base change to an uncountable algebraically closed field, we may assume that \( k \) is an uncountable algebraically closed field (Lemma 3.2). For an arbitrary closed point \( y \in Y \), it is enough to show that \( \mathcal{L}|_{X_y} \) is \( p \)-power free
(Theorem 3.5). By [GNT19, Proposition 2.15], there exists a commu-
tative diagram consisting of projective birational morphisms of quasi-
projective normal threefolds

\[
\begin{array}{ccc}
W & \xrightarrow{\psi} & X' \\
\downarrow{\varphi} & & \downarrow{f'} \\
X & \xrightarrow{f} & Y
\end{array}
\]

and an effective \( \mathbb{Q} \)-divisor \( \Delta' \) on \( X' \) which satisfy the following properties.

1. \( (X', \Delta') \) is a \( \mathbb{Q} \)-factorial plt threefold and \( -(K_{X'} + \Delta') \) is \( f' \)-ample.
2. \( -\Delta' \) is \( f' \)-nef and \( f'^{-1}(z)_{\text{red}} = \Delta' \).
3. For \( T := \Delta' \), there exists an effective \( \mathbb{Q} \)-divisor \( \Delta_T \) on \( T \) such that \( (T, \Delta_T) \) is klt and \( -(K_T + \Delta_T) \) is ample.

Set \( L_W := \varphi^*L \). After replacing \( L \) by \( L^{\otimes e} \) for some \( e \in \mathbb{Z}_{>0} \), Step 1 enables us to find an invertible sheaf \( L' \) on \( X' \) such that \( L_W \cong \psi^*L' \). Then we have that \( L'|_{(X')_{\text{red}}} = L'|_T \) is p-power free (Lemma 4.1). It follows from Lemma 3.3 that also \( L'|_{X'_{\text{red}}} \) is p-power free. Since \( L_W|_{W_y} \) is the pullback of \( L'|_{X'_{\text{red}}} \), it holds that \( L_W|_{W_y} \) is p-power free. Finally, since \( W_y \to X_y \) has connected fibres, it follows again from Lemma 3.3 that \( L|_{X_y} \) is p-power free. This completes the proof of Step 2. □

Step 3. The assertion of Theorem 4.3 holds without any additional assumptions.

Proof of Step 3. By the same argument as in Step 1, the problem is reduced to the case when \( X \) is \( \mathbb{Q} \)-factorial and \( f \) is a small birational morphism. In particular, for \( \Delta := f^{-1}_*\Delta_Y \), it holds that \( K_X + \Delta = f^*(K_Y + \Delta_Y) \). In particular, \( -(K_X + \Delta) \) is \( f \)-nef and \( f \)-big. Then we may apply Step 2. This completes the proof of Step 3. □

Step 3 completes the proof of Theorem 4.3. □

Theorem 4.4. Let \( k \) be a perfect field of characteristic \( p > 5 \). Let \( (X, \Delta) \) be a three-dimensional klt pair and let \( f : X \to Z \) be a projective surjective k-morphism to a quasi-projective k-scheme \( Z \). Let \( L \) be an \( f \)-nef Cartier divisor on \( X \) such that \( L - (K_X + \Delta) \) is \( f \)-nef and \( f \)-big. Assume that either

1. \( \dim Z \geq 1 \), or
2. \( \dim Z = 0 \) and \( L \neq 0 \).

Then \( L \) is p-power \( f \)-free.

Proof. The proof consists of two steps.
**Step 1.** The assertion of Theorem 4.4 holds if (1) holds.

**Proof of Step 1.** Set $L := O_X(L)$. Then, by Lemma 3.2, we may assume that $k$ is an uncountable algebraically closed field.

Fix a closed point $z \in Z$. By [GNT19, Proposition 2.15], there exists a commutative diagram consisting of projective morphisms of quasi-projective normal varieties

$$
\begin{array}{ccc}
W & \xrightarrow{\psi} & X' \\
\downarrow \varphi & & \downarrow f' \\
X & \xrightarrow{f} & Z
\end{array}
$$

and an effective $\mathbb{Q}$-divisor $\Delta'$ on $X'$ which satisfy the following properties.

1. $(X', \Delta')$ is a $\mathbb{Q}$-factorial plt threefold and $-(K_{X'} + \Delta')$ is $f'$-ample.
2. $-\Delta'_{\varphi}$ is $f'$-nef and $f'^{-1}(z)_{\text{red}} = \Delta'_{\varphi}$. 
3. $W$ is a smooth threefold, and both of $\varphi$ and $\psi$ are projective birational morphisms.
4. For $T := \Delta'_{\varphi}$, there exists an effective $\mathbb{Q}$-divisor $\Delta_T$ on $T$ such that $(T, \Delta_T)$ is klt and $-(K_T + \Delta_T)$ is ample.

Then, by the same argument as in Step 2 of Theorem 4.3, it holds that $L|_{X_z}$ is $p$-power free for any closed point $z \in Z$. Then it follows from Theorem 3.5 that $L$ is $p$-power $f$-free. This completes the proof of Step 1. $\square$

**Step 2.** The assertion of Theorem 4.4 holds if (2) holds.

**Proof of Step 2.** It follows from [GNT19, Theorem 2.9] that $L$ is $f$-semi-ample. Hence, there exist a positive integer $m$, projective morphisms

$$
f : X \xrightarrow{g} Y \xrightarrow{f_Y} Z,
$$

and an ample Cartier divisor $A$ on $Y$ such that $g_*O_X = O_Y$ and $mL \sim g^*A$. In particular, we have $L \equiv_g 0$. By (2), it holds that $\dim Y > 0$. By Step 1 there exist $e \in \mathbb{Z}_{>0}$ and an $f_Y$-ample Cartier divisor $A'$ on $Y$ such that $p^eL \sim g^*A'$. This completes the proof of Step 2. $\square$

Step 1 and Step 2 complete the proof of Theorem 4.4. $\square$

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