Yoonweon Lee

March 5, 1995

Abstract. The purpose of this note is to provide a short cut presentation of a Mayer-Vietoris formula due to Burghelea-Friedlander-Kappeler for the regularized determinant in the case of elliptic operators of Laplace Beltrami type in the form typically needed in applications to torsion.

1. Statement of Mayer-Vietoris Formula for Determinants

Let \((M,g)\) be a closed oriented Riemannian manifold of dimension \(d\) and \(\Gamma\) be an oriented submanifold of codimension 1. We denote by \(\nu\) the unit normal vector field along \(\Gamma\). Let \(M_\Gamma\) be the compact manifold with boundary \(\Gamma^+ \sqcup \Gamma^-\) obtained by cutting \(M\) along \(\Gamma\), where \(\Gamma^+\) and \(\Gamma^-\) are copies of \(\Gamma\) and denote by \(p: M_\Gamma \to M\) the identification map. The vector field \(\nu\) has the lift on \(M_\Gamma\) which we denote by \(\tilde{\nu}\) again. Denote by \(\Gamma^+\) the component of the boundary where the lift of \(\nu\) points outward. Given a smooth vector bundle \(E \to M\), denote by \(E_\Gamma\) the pull back of \(E \to M\) to \(M_\Gamma\) by \(p\). Let \(A: C^\infty(E) \to C^\infty(E)\) be an elliptic, essentially self-adjoint, positive definite differential operator of Laplace-Beltrami type, where we say that \(A\) is of Laplace-Beltrami type if \(A\) is an operator of order 2 whose principal symbol is \(\sigma_L(x,\xi) = \|\xi\|^2 Id_x\), \(Id_x \in \text{End}_{x}(E_x,E_x)\). We denote by \(A_{\Gamma,B}: C^\infty(E_\Gamma) \to C^\infty(E_\Gamma) \oplus C^\infty(E_\Gamma |_{\Gamma^+ \sqcup \Gamma^-})\) the extension of \(A\) to smooth sections of \(E_\Gamma\).

Consider Dirichlet and Neumann boundary conditions \(B, C\) on \(\Gamma^+ \sqcup \Gamma^-\) defined as follows:

\[
B: C^\infty(E_\Gamma) \to C^\infty(E_\Gamma |_{\Gamma^+ \sqcup \Gamma^-}), B(f) = f |_{\Gamma^+ \sqcup \Gamma^-}
\]

\[
C: C^\infty(E_\Gamma) \to C^\infty(E_\Gamma |_{\Gamma^+ \sqcup \Gamma^-}), C(f) = \nu(f) |_{\Gamma^+ \sqcup \Gamma^-}.
\]

Consider \(A_{\Gamma,B}^{-1} = (A_{\Gamma,B})^{-1} : C^\infty(E_\Gamma) \oplus C^\infty(E_\Gamma |_{\Gamma^+ \sqcup \Gamma^-}) \to 0\). From the properties of \(A\) it follows that \(A_{\Gamma,B}^{-1}\) is invertible. Therefore we can define the corresponding Poisson operator \(P_B\) as the restriction of \(A_{\Gamma,B}^{-1}\) to \(0 \oplus C^\infty(E_\Gamma |_{\Gamma^+ \sqcup \Gamma^-})\). Denote by \(A_B\) the restriction of \(A_{\Gamma,B}^{-1}\) on \(\{u \in C^\infty(E_\Gamma) \mid B(u) = 0\}\). Then \(A_B\) is also essentially self-adjoint and positive definite (cf Lemma 3.1). This (using standard analytic continuation technique due to Seeley (cf.[Se])) allows us to define

\[
\log\text{Det}(A) = -\frac{d}{ds} \bigg|_{s=0} tr \frac{1}{2\pi i} \int_{\gamma} \lambda^{-s}(\lambda - A)^{-1} d\lambda
\]
\[ \log \text{Det}(A_\Gamma, B) = -\frac{d}{ds} \bigg|_{s=0} \frac{1}{2\pi i} \int_\gamma \lambda^{-s}(\lambda - A_B)^{-1} d\lambda, \]

where \( \gamma \) is a path around the negative real axis,

\[ \{\rho \mathrm{e}^{i\pi} \mid \infty > \rho \geq \epsilon\} \cup \{\epsilon \mathrm{e}^{i\theta} \mid \pi \geq \theta \geq -\pi\} \cup \{\rho \mathrm{e}^{-i\pi} \mid \epsilon \leq \rho < \infty\} \]

with \( \epsilon > 0 \) chosen sufficiently small to ensure that \( \Gamma \) does not separate the spectrum.

We define the Dirichlet to Neumann operator, associated to \( A, B \) and \( C \),

\[ R : C^\infty(E \mid \Gamma) \to C^\infty(E \mid \Gamma) \]

by the composition of the following maps

\[ C^\infty(E \mid \Gamma) \xrightarrow{\Delta_{\alpha}} C^\infty(E \mid \Gamma^+) \oplus C^\infty(E \mid \Gamma^-) \xrightarrow{P_B} C^\infty(E_{\Gamma}) \xrightarrow{C} C^\infty(E \mid \Gamma^+) \oplus C^\infty(E \mid \Gamma^-) \xrightarrow{\Delta_{if}} C^\infty(E \mid \Gamma), \]

where \( \Delta_{\alpha}(f) = (f, f) \) is the diagonal inclusion and \( \Delta_{if}(f, g) = f - g \) is the difference operator. Then \( R \) is an essentially self-adjoint, positive definite, elliptic operator of order 1 (cf Lemma 3.5).

**Theorem 1.1 (Mayer-Vietoris Type Formula for Determinants [BFK]).**

Let \( (M, g) \) be a closed oriented Riemannian manifold of dimension \( d \) and \( A \) be an elliptic, essentially self-adjoint, positive definite differential operator of Laplace-Beltrami type acting on smooth sections of a vector bundle \( E \to M \). Then \( A_B \) and \( R \) are essentially self-adjoint, positive definite elliptic operators and

\[ \text{Det}(A) = c \text{Det}(A_\Gamma, B) \text{Det}(R), \]

where \( c \) is a local quantity which can be computed in terms of the symbols of \( A, B \) and \( C \) along \( \Gamma \).

Remark: The above result can be extended to manifolds with boundary. E.g. consider an oriented, compact, smooth manifold \( M \) whose boundary \( \partial M \) is a disjoint union of two components \( \partial_+ M \) and \( \partial_- M \) with \( \Gamma \cap \partial M = \emptyset \), an operator \( A \) of Laplace-Beltrami type and differential elliptic boundary conditions \( B_+ \) respectively \( B_- \) for \( A \) on \( \partial_+ M \) respectively \( \partial_- M \). Denote by \( A^{(0)} \) the operator \( A \) with domain \( \{u \in C^\infty(E) \mid B_+ u = 0, B_- u = 0\} \). Then Theorem 1.1 remains true with \( A \) replaced by \( A^{(0)} \).

2. The Asymptotics of Determinants of Elliptic Pseudodifferential Operators with Parameter

Let \( V \) be an open angle in the complex \( \lambda \)-plane and \( \mathcal{P}(\lambda), \lambda \in V \), a family of \( \Psi \text{DO} \)'s of order \( m \), \( m \) a positive integer, acting on smooth sections of a vector bundle \( E \to M \) of rank \( \nu \), where \( M \) denotes a closed smooth Riemannian manifold of dimension \( d \).

**Definition 2.1.** (cf.[Sh]) The family \( \mathcal{P}(\lambda), \lambda \in V \), is said to be a \( \Psi \text{DO} \) with parameter of weight \( \chi > 0 \) if in any coordinate neighborhood \( U \) of \( M \), \( M \) necessarily connected, and for an arbitrarily fixed \( \lambda \in V \), the complete symbol \( p(\lambda; x, \xi) \) of \( P \) is in \( C^\infty(U \times \mathbb{R}^d, \text{End}(\mathbb{C}^\nu)) \) and, moreover, for any multiindices \( \alpha \) and \( \beta \), there exists a constant \( C_{\alpha, \beta} \) such that

\[ |\partial^\alpha_{\xi} \partial^\beta_{\xi} p(\lambda; x, \xi) | \leq C_{\alpha, \beta} (1 + |\xi| + |\lambda|^{\frac{1}{d}})^{m-|\alpha|}. \]
Definition 2.2. \(P(\lambda)\) is called classical if in any chart the complete symbol 
\[ p(\lambda; x, \xi) \] 
admits an expansion of the form 
\[ p(\lambda; x, \xi) \sim p_m(\lambda; x, \xi) + p_{m-1}(\lambda; x, \xi) + \cdots, \]
where \(p_j(\tau^x\lambda; x, \tau\xi) = \tau^j p_j(\lambda; x, \xi)(\tau > 0, j \leq m)\). The family \(P(\lambda)\) is said to be 
elastic with parameter if \(p_m(\lambda; x, \xi)\) is invertible for all \(x \in M, \xi \in T^*_x(M)\) and 
\(\lambda \in V\) satisfying \(|\xi| + |\lambda|^{-\frac{1}{2}} \neq 0\).

Definition 2.3. Let \(Q\) be an elliptic \(\Psi DO\). The angle \(\pi\) is called an Agmon angle 
for \(Q\) if for some \(\epsilon > 0\), \(\text{spec}(Q) \cap \Lambda_\epsilon = \emptyset\), where \(\text{spec}(Q)\) denotes the spectrum of 
\(Q\) and \(\Lambda_\epsilon = \{z \in \mathbb{C} | \pi - \epsilon < \arg(z) < \pi + \epsilon \text{ or } |z| < \epsilon\}\).

Theorem 2.4. Let \(P(\lambda)\) be an essentially self-adjoint, positive definite, classical 
\(\Psi DO\) of order \(m \in \mathbb{N}\) with parameter \(\lambda \in V\) of weight \(\chi > 0\) such that
(i) \(P(\lambda)\) is elliptic with parameter and
(ii) for each \(\lambda \in V\), \(P(\lambda)\) has \(\pi\) as an Agmon angle.

Then \(\log \text{Det}P(\lambda)\) admits an asymptotic expansion for \(\lambda \in V\), \(|\lambda| \to \infty\), of the form
\[ \log \text{Det}P(\lambda) \sim \sum_{j=-d}^{\infty} \pi_j |\lambda|^{-\frac{j}{\chi}} + \sum_{j=0}^{d} q_j |\lambda|^{\frac{j}{\chi}} \log |\lambda|. \]

The coefficients \(\pi_j\) and \(q_j\) can be evaluated in terms of the symbol of \(P\) and \(\frac{\lambda}{|\lambda|}\). In 
particular, \(\pi_0\) is independent of perturbations by lower order operators, whose 
orders differ at least by \(d + 1\) from the order of \(P(\lambda)\).

For the convenience of the reader we include the proof of this theorem which can be 
found in the appendix of [BFK].

Proof of Theorem 2.4 We divide the proof into several steps.

Step 1 By a standard procedure we construct a parametrix for 
\(R(\mu, \lambda) = (\mu - P(\lambda))^{-1}(\mu \leq 0)\).

Step 2 Define \(R_N(\mu, \lambda)\) to be a conveniently chosen approximation of \(R(\mu, \lambda)\) and 
write \(P(\lambda)^{-s} = P_N(\lambda; s) + \tilde{P}_N(\lambda; s)\), where \(P_N(\lambda; s) = \frac{1}{2\pi i} \int_\gamma \mu^{-s}R_N(\mu, \lambda)d\mu\) 
and where \(\gamma\) denotes a contour around the negative axis, enclosing the origin in 
clockwise orientation. Then for \(s \in \mathbb{C}\) with \(\text{Re} s\) sufficiently large, 
\(\zeta(s) = \zeta_N(s) + \tilde{\zeta}_N(s)\), where \(\zeta(s) = \text{tr}P(\lambda)^{-s}, \zeta_N(s) = \text{tr}P_N(\lambda, \lambda), \text{and } \tilde{\zeta}_N(s) = \text{tr}\tilde{P}_N(\lambda, \lambda)\).

Step 3 Describe an asymptotic expansion of \(\frac{\partial}{\partial s} |_{s=0} \zeta_N(\lambda, s)\) as \(\lambda \to \infty\).

Step 4 Provide an estimate for the remainder term \(\frac{\partial}{\partial s} |_{s=0} \tilde{\zeta}_N(\lambda, s)\) as \(\lambda \to \infty\).

Step 5 Provide a formula for \(\pi_0\).

Step 1 We want to construct a parametrix for \(R(\mu, \lambda) = (\mu - P(\lambda))^{-1}(\mu \leq 0)\). Consider the equation 
\((\mu - P(\lambda; x, \xi)) \circ r(\mu, \lambda; x, \xi) = \text{Id}\), where \(\circ\) denotes 
multiplication in the algebra of symbols.

Introduce, for \(\alpha = (\alpha_1, \cdots, \alpha_d)\), the standard notation \(\alpha! = \alpha_1! \cdots \alpha_d!\), 
\(\partial^\alpha = (\frac{\partial}{\partial \xi})^{\alpha_1} \cdots (\frac{\partial}{\partial \xi})^{\alpha_d}\) and 
\(D^\alpha = (\frac{1}{i})^\alpha \partial^\alpha \xi\).

Write \(r(\mu, \lambda; x, \xi) \sim r_{-m}(\mu, \lambda; x, \xi) + r_{-m-1}(\mu, \lambda; x, \xi) + \cdots\), where \(r_j(\mu, \lambda; x, \xi)\) 
is positive homogeneous of degree \(-j\) in \((\xi, \mu^\frac{1}{N}, \lambda^\frac{1}{N})\). Then we obtain the following 
formula:
\[ r_{-j}(\mu, \lambda; x, \xi) = (\mu - p_j(\lambda; x, \xi))^{-1}. \]
and for \( j \geq 1, \)
\[
r_{-m-j}(\mu, \lambda; x, \xi)
= -(\mu - p_m(\lambda; x, \xi))^{-1} \sum_{k=0}^{j-1} \sum_{|\alpha|+l+k=j} \frac{1}{\alpha!} \partial_{\xi}^\alpha p_{m-l}(\lambda; x, \xi) D_x^\alpha r_{-m-k}(\mu, \lambda; x, \xi).
\]

The functions \( r_j(\mu, \lambda; x, \xi) \) satisfy the following homogeneity condition
\[
r_j(\tau^m \mu, \tau^\lambda \lambda; x, \tau \xi) = \tau^j r_j(\mu, \lambda; x, \xi)(\tau > 0).
\]

By a standard procedure, \( \sum_{j \geq 0} r_{-m-j}(\mu, \lambda; x, \xi) \) gives rise to a \( \Psi \)DO with parameter, called a parametrix for \( R(\mu, \lambda). \)

**Step 2** Introduce a finite cover \( (U_j) \) of \( M \) by open charts and take a partition of unity \( \varphi_j \), subordinate to \( U_j \). Choose \( \psi_j \in C_0^\infty(U_j) \) such that \( \psi_j \equiv 1 \) in some neighborhood of \( \text{supp} \varphi_j \). Let us fix local coordinates in every \( U_j \) and define the operators
\[
(R^{(j)}_{N, \mu, \lambda} f)(x) = \psi_j(x) \cdot \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dy \left( r^{(N)}(\mu, \lambda; x, \xi) e^{i(x-y) \cdot \xi} \varphi_j(y) f(y) \right),
\]
where \( r^{(N)}(\mu, \lambda; x, \xi) = \sum_{j=0}^{N-1} r_{-m-j}(\mu, \lambda; x, \xi) \) in the local coordinates of \( U_j \). The approximation \( R_N(\mu, \lambda) \) of the resolvent \( R(\mu, \lambda) \) is defined by \( R_N(\mu, \lambda) = \sum_j R^{(j)}_{N, \mu, \lambda} \). We need an estimate of \( R(\mu, \lambda) - R_N(\mu, \lambda) \) in trace norm. The latter is denoted by \( \| \cdot \| \).

**Lemma 2.5.** Choose \( N > \frac{3d}{2} + m \). Then for \( \lambda \in V_1 \) and \( \mu \in \mathbb{R}^- \) with \( |\mu| \) sufficiently large
\[
\| R(\mu, \lambda) - R_N(\mu, \lambda) \| < C_N(1+|\lambda|)^{(N-\frac{3d}{2}-m)}(1+|\lambda|)^{-2},
\]
where \( V_1 \) is an angle whose closure is contained in \( V, V_1 \ll V \).

**Proof.** Define \( T_N(\mu, \lambda) \) by \( (\mu - P(\lambda)) R_N(\mu, \lambda) = Id - T_N(\mu, \lambda) \). From \( (\mu - P(\lambda)) - R(\mu, \lambda) = Id - T_N(\mu, \lambda) \), we then conclude that \( R(\mu, \lambda) - R_N(\mu, \lambda) = R(\mu, \lambda) T_N(\mu, \lambda) \).

The claimed estimate of the lemma follows, once we have proved that for some \( \tau > d \)
\[
\| R(\mu, \lambda) \|_{L^2 \to L^2} \leq C(1+|\mu|)^{-1} \tag{2.1}
\]
\((\lambda \in V_1 \ll V, \mu \in \mathbb{R}^-, |\mu| \) sufficiently large\) and
\[
\| T_N(\mu, \lambda) \|_{L^2 \to H^\tau} \leq C(1+|\lambda|^{\frac{1}{\lambda}} + |\mu|^{\frac{1}{\mu}})^{-N+\tau} \tag{2.2}
\]
because, from (2.2) we can conclude that \( T_N(\mu, \lambda) \) is a \( \Psi \)DO of order \(-\tau < -d\) and hence of trace class, when considered as an operator on \( L^2\)-sections of \( E \to M \). The estimate (2.1) is standard (cf.e.g.[Sh]) and (2.2) follows from the fact that the symbol \( t_N(\mu, \lambda; x, \xi) \) of \( T_N(\mu, \lambda) \) satisfies
\[
| D_x^\alpha D_\xi^{\beta} t_N(\mu, \lambda; x, \xi) | \leq C_{\alpha\beta}(1+|\xi| + |\lambda|^{\frac{1}{\lambda}} + |\mu|^{\frac{1}{\mu}})^{-N+|\beta|}.
\]
Thus the norm of \( T_N(\mu, \lambda) \) as an operator from \( H^s \) to \( H^{s+\tau} \) is
\[
O(1+|\lambda|^{\frac{1}{\lambda}} + |\mu|^{\frac{1}{\mu}})^{-N+\tau},
\]
Therefore

\[ \| T_N \| = O(1 + |\lambda|^{d/2} + |\mu|^{d/2})^{-N + \frac{2d}{2}.} \]

\[ \square \]

**Step 3** Next we study the asymptotic expansion of \( \frac{\partial}{\partial s} |_{s=0} \zeta_N(\lambda, s) \) as \( \lambda \to +\infty \). Recall that \( P_N(\lambda, s) = \frac{1}{2\pi i} \int_{\gamma} d\mu \mu^{-s} R_N(\mu, \lambda) \). Its Schwarz kernel is given by

\[ P_N(\lambda, s; x, y) = \sum_j \psi_j(x) \varphi_j(y) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi e^{i(x-y) \xi} \frac{1}{2\pi i} \int_{\gamma} d\mu \mu^{-s} \sum_{k=0}^{N-1} r_{m-k}. \]

As a consequence

\[ P_N(\lambda, s; x, x) = \sum_j \varphi_j(x) \frac{1}{(2\pi)^d} \sum_{k=0}^{N-1} I_k(s, \lambda, x) = \frac{1}{(2\pi)^d} \sum_{k=0}^{N-1} I_k(s, \lambda, x), \]

where \( I_j(s, \lambda, x) = \frac{1}{2\pi i} \int_{\mathbb{R}^d} d\xi \int_{\gamma} d\mu \mu^{-s} r_{m-j}(\mu, x, \xi) \). By the change of variables \( \xi = |\lambda|^{\frac{1}{m'}} \xi', \mu = |\lambda|^{\frac{1}{m}} \mu' \) and by using the homogeneity of \( r_{m-j} \), we obtain

\[ I_j(s, \lambda, x) = \frac{1}{2\pi i} |\lambda| \int_{\mathbb{R}^d} d\xi \int_{\gamma} d\mu \mu^{-s} r_{m-j}(\mu, \lambda^{1/m} x; \xi). \]

We need to investigate \( J_k(s, \omega; x) := \frac{1}{2\pi i} \int_{\mathbb{R}^d} d\xi \int_{\gamma} d\mu \mu^{-s} r_{m-k}(\mu, \lambda^{1/m} x; \xi) \).

**Lemma 2.6.** Let \( \omega \in V \) with \( |\omega| = 1 \). Then \( J_k(s, \omega; x) \) is holomorphic in \( s \) in the half plane \( \text{Res} > \frac{d-k}{m} \) and it admits a meromorphic continuation in the complex \( s \)-plane. The point \( s=0 \) is always regular and \( J_k(0, \omega; x) = 0 \) if \( k > d \).

**Proof.** (i) Integrating by parts with respect to \( \mu \), one obtains

\[ J_k(s, \omega; x) = \int_{\mathbb{R}^d} d\xi \frac{1}{(1-s)\cdots(l-s)} \frac{1}{2\pi i} \int_{\gamma} (-1)^l d\mu \mu^{-s+l} \frac{\partial^l}{\partial \mu^l} r_{m-k}(\mu, \omega; x, \xi). \]

If \( l > \text{Res} - 1 \), the contour integral reduces to

\[ -\frac{\sin \pi s}{\pi} \int_0^{\infty} d\mu \mu^{-s+l} \frac{\partial^l}{\partial \mu^l} r_{m-k}(-\mu, \omega; x, \xi). \]

Further, the matrix \( \frac{\partial^l}{\partial \mu^l} r_{m-k} \) can be estimated

\[ |\frac{\partial^l}{\partial \mu^l} r_{m-k}| \leq C(1 + |\mu|^{\frac{1}{m}} + |\xi|^{\frac{m}{m-k}})^{m-k-m}. \]

Thus, for \( \text{Res} > \frac{d-k}{m} \), the integral

\[ \int d\xi \int_0^{\infty} d\mu \mu^{-s+l} \left( \frac{\partial^l}{\partial \mu^l} r_{m-k} \right)(-\mu, \omega; x, \xi). \]
converges absolutely and therefore is a holomorphic function in $s$. Moreover, 
\[-\frac{\sin\pi s}{\pi(1-s)\cdots(l-s)}\] is entire. In all, we have proved that $J_k(s,\lambda;x)$ is holomorphic in $Res > \frac{d-m}{k}$.

(ii) Next let us prove that $J_k(s,\lambda;x)$ can be meromorphically continued to the entire complex $s$-plane. To keep the exposition simple let us assume that $P(\lambda)$ is a scalar $\Psi$DO. The expressions $r_{-m-k}(\mu,\omega;x,\xi)$ have been defined in a recursive fashion and are sums of terms of the form $(\mu - p_m(\omega;x,\xi))^{-l}q_{l,k}(\omega;x,\xi)$ with $l \geq 1$, where $\text{ord}(q_{l,k}) = -m-k+ml$ and $q_{l,k}$ is an expression, independent of $\mu$, involving only the symbols $p_{m-j}(\omega;x,\xi)$ and their derivatives with $0 \leq j \leq k$.

It follows from the recursive definition of the $r_{-m-k}$ that $l$ has to satisfy $l \geq k+1$ and thus, in the case $k \geq 1$, $J_k$ consists of a sum of terms of the form

$$
\int_{\mathbb{R}^d} d\xi q_{l,k}(\omega;x,\xi) \frac{1}{2\pi i} \int_{\gamma} d\mu \mu^{-s}(\mu - p_m(\omega;x,\xi))^{-l}
$$

$$
= \left( \int_{\mathbb{R}^d} d\xi q_{l,k}(\omega;x,\xi)(p_m(\omega;x,\xi))^{-s-l+1} \right) \left( \frac{(-1)^{l-1}}{(l-1)!} s(s+1)\cdots(s+l-2) \right),
$$

where after integration by parts, we used Cauchy’s formula. As $|\omega|=1$, it follows from Definition 2.1 that the integrand $q_{l,k}(\omega;x,\xi)p_m(\omega;x,\xi)^{-s-l+1}$ is absolutely integrable in $|\xi| \leq 1$. Thus one only needs to consider the integral over $|\xi| > 1$. For $\omega$ fixed, the symbols $q_{l,k}$ and $p_m$ are classical and admit an asymptotic expansion in $\xi$-homogeneous functions. Consider two cases:

Case 1: $k = 0$.

Using that $r_{-m}(\mu,\omega;x,\xi) = (\mu - p_m(\omega;x,\xi))^{-1}$ we conclude that

$$
J_0(s,\omega;x) = \int_{\mathbb{R}^d} d\xi \frac{1}{2\pi i} \int_{\gamma} d\mu \mu^{-s}(\mu - p_m(\omega;x,\xi))^{-1} = \int_{\mathbb{R}^d} d\xi(p_m(\omega;x,\xi))^{-s}.
$$

Recall that $\omega$ with $|\omega|=1$ is fixed and thus $p_m(\omega;x,\xi)$ defines an elliptic $\Psi$DO $P_m(\omega;x,D)$ and we can apply the standard theory of complex powers of elliptic operators (cf.e.g.[Se]) to conclude that $\int_{\mathbb{R}^d} d\xi p_m(\omega;x,\xi)^{-s}$ has a meromorphic continuation in the whole complex $s$-plane, with at most simple poles and that $s=0$ is a regular point. The poles are located at $s_j = \frac{d-j}{m}$ with $j \in \{0,1,2,\ldots\} \setminus \{d\}$.

Case 2: $k \geq 1$.

As it was observed by Guillemin [Gu] and Wodzicki [Wo] in the context of non-commutative residues, $\int_{\mathbb{R}^d} q_{l,k}(\omega;x,\xi)(p_m(\omega;x,\xi))^{-s-l+1}d\xi$ admits a meromorphic continuation to the whole complex $s$-plane with at most simple poles. Thus $s \cdot \int_{\mathbb{R}^d} d\xi q_{l,k}(\omega;x,\xi)(p_m(\omega;x,\xi))^{-s-l+1}$ must be regular at $s=0$. This shows that $J_k(s,\omega;x)(k \geq 1)$ is meromorphic and that $s=0$ is a regular point.

(iii) Let $k > d+1-m$. Observe that

$$
| r_{-m-k}(\mu,\omega;x,\xi) | \leq C_k(1+|\mu|^{\frac{1}{d}} + |\xi|)^{-m-k}.
$$

As $m \geq 1$, the integral $J_k(s,\omega;x) = \frac{1}{2\pi i} \int_{\mathbb{R}^d} d\xi \int_{\gamma} d\mu \mu^{-s} r_{-m-k}(\mu,\omega;x,\xi)$ converges absolutely at $s=0$. Evaluating at $s=0$, one obtains $\int_{\gamma} d\mu r_{-m-k}(\mu,\omega;x,\xi) = 0$ and thus $J_k(0,\omega;x) = 0$ for $k > d$. □

By the above lemma, we see that

$$
P_N(\lambda,s;x,x) = \frac{1}{(2\pi)^d} \sum_{k=1}^{N-1} \lambda^{\left(\frac{d-m-k}{1+k}\right)} J_k(s,\frac{\lambda}{1+k};x).
$$
Hence, with $N^* = \min(N - 1, d)$,

$$\frac{\partial}{\partial s} \mid_{s=0} P_N(\lambda, s; x, x) = \frac{1}{(2\pi)^d} \sum_{k=0}^{N-1} \lambda^{d-k} \frac{\partial}{\partial s} J_k(s, \frac{\lambda}{|\lambda|}; x) \mid_{s=0} -$$

$$\frac{m}{\chi} \frac{1}{(2\pi)^d} \sum_{k=0}^{N^*} \lambda^{d-k} \log |\lambda| \cdot J_k(0, \frac{\lambda}{|\lambda|}; x).$$

**Step 4** We have to estimate $\text{tr} \tilde{P}_N(\lambda, s)$, where

$$\tilde{P}_N(\lambda, s) = P(\lambda)^{-s} - P_N(\lambda, s) = \frac{1}{2\pi i} \int_\gamma d\mu \mu^{-s}(R(\mu, \lambda) - R_N(\mu, \lambda)).$$

The estimate of Lemma 2.5 implies that

$$|\text{tr} \frac{\partial}{\partial s} \tilde{P}_N(\lambda, s) \mid_{s=0} \leq C(1 + |\lambda|)^{-(N - \frac{d-m}{2})} \int_0^\infty d\mu \cdot \frac{|\log \mu|}{1 + |\mu|^2}$$

and thus the asymptotic expansion given in Theorem 2.4 is proved.

**Step 5** In the notation introduced above, we obtain the following formula for $\pi_0$:

$$\pi_0 = \sum_j \frac{\partial}{\partial s} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dvol(x) J_d(s, \lambda; x, x) \mid_{s=0},$$

where $J_d(s, \lambda; x) = \frac{1}{2\pi i} \int_{\mathbb{R}^d} d\xi \int_\gamma d\mu \mu^{-s} r_{m-d}^{-s}(\mu, \lambda; x, \xi)$.

As $r_{m-d}(\mu, \frac{\lambda}{|\lambda|}; x, \xi)$ is defined recursively by ($j \geq 1$)

$$r_{m-j}(\mu, \lambda; x, \xi) =$$

$$-(\mu - p_m(\lambda; x, \xi))^{-1} \sum_{k=0}^{j-1} \sum_{|\alpha| + l + k = j} \frac{1}{\alpha!} \partial^\alpha_x p_{m-l}(\lambda; x, \xi) D_x^\alpha r_{m-k}(\mu, \lambda; x, \xi),$$

we conclude that $\pi_0$ only depends on $p_m(\lambda; x, \xi)$ for $0 \leq j \leq d$ and its derivatives up to order $d$. □

The following result is due to Voros [Vo] and Friedlander [Fr]. For the convenience of the reader we include Voros’ proof.

**Proposition 2.7.** Let $\{\lambda_k\}_{k \geq 1}$ be a sequence in $V_0, \pi = \{z \in \mathbb{C} \mid -\frac{\pi}{2} + \epsilon < \arg(z) < \frac{\pi}{2} - \epsilon\}$, possibly with multiplicities, arranged in such a way that $0 < \text{Re}\lambda_1 \leq \text{Re}\lambda_2 \leq \ldots$. Assume that the heat trace, $\theta(t) := \sum_{k=0}^\infty e^{-t\lambda_k} (t > 0)$ admits an asymptotic expansion for $t \to 0$ of the form

$$\theta(t) \sim \sum c_n t^n,
where \( i_0 < 0 \) and \( i_0 < i_1 < i_2 < \cdots \to +\infty \). For \( \lambda \) with \( \text{Re} \lambda > 0 \), let \( \zeta(s, \lambda) = \sum_{k=0}^{\infty} (\lambda_k + \lambda)^{-s} \). Then \( \pi_0 = 0 \), where \( \pi_0 \) is the constant term in the asymptotic expansion of \( -\frac{d}{ds} \zeta(s, \lambda) \big|_{s=0} \) for \( |\lambda| \to +\infty \).

**Proof.** It is well known that for \( \text{Re} \lambda > 0 \), \( \zeta(s, \lambda) \) is well defined and \( \zeta(s, 0) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \theta(t) \cdot t^{s-1} dt \). Let

\[
\eta(s, \lambda) = \int_{0}^{\infty} \sum_{k=0}^{\infty} e^{-\lambda k} e^{-\lambda t} t^{s-1} dt.
\]

Then \( \zeta(s, \lambda) = \frac{1}{\Gamma(s)} \eta(s, \lambda) \). For \( \text{Re} \lambda > -i_0 \), \( \eta(s, \lambda) \) can be expanded in \( \lambda \) for \( |\lambda| \to \infty \)

\[
\eta(s, \lambda) \sim \sum_{n \geq 0} c_n \lambda^{-s-i_n} \int_{0}^{\infty} t^{n+s-1} e^{-t} dt = \sum_{n \geq 0} c_n \Gamma(s+i_n) \lambda^{-s-i_n}.
\]

Thus

\[
\zeta(s, \lambda) = \frac{1}{\Gamma(s)} \eta(s, \lambda) \sim \frac{1}{\Gamma(s)} \lambda^{-s} \sum_{n \geq 0} c_n \Gamma(s+i_n) \lambda^{-i_n}.
\]

All functions involved are meromorphic functions of \( s \). Moreover \( s = 0 \) is a regular point of \( \zeta(s, \lambda) \) and thus \( \frac{d}{ds} \zeta(s, \lambda) \big|_{s=0} \) admits an asymptotic expansion in \( \lambda \) of the form

\[
\frac{d}{ds} \zeta(s, \lambda) \big|_{s=0} \sim \sum_{i_n \notin Z^- \cup \{0\}} c_n \Gamma(i_n) \lambda^{-i_n} + \sum_{i_n \in Z^-} c_n \frac{1}{\Gamma(s+i_n) \cdots (s-1)} \big|_{s=0} \lambda^{-i_n} - \sum_{i_n \in Z^- \cup \{0\}} \frac{c_n}{(i_n) \cdots (-1)} \lambda^{-i_n} \log \lambda,
\]

where \( Z^- \) is the set of all negative integers. This expansion shows that \( \pi_0 = 0 \). \( \square \)

3. **Auxiliary Results for the Proof of Theorem 1.1**

We begin by collecting a number of results about operators related to \( A \) and \( \Gamma \). Denote by \( H^\epsilon(E_\Gamma) \) the Sobolev spaces of \( E_\Gamma \)-valued sections. Throughout section 3 and section 4 we assume that \( A \) satisfies the hypothesis of Theorem 1.1 and fix \( \epsilon > 0 \), so that the spectrum of \( A \) is bounded from below by \( \epsilon \).

**Lemma 3.1.** (i) The operator \( A_B : \{ u \in C^\infty (E_\Gamma) \mid B(u) = 0 \} \to C^\infty (E_\Gamma) \) has a self-adjoint extension \( \bar{A}_B \) with domain \( D(\bar{A}_B) := \{ u \in H^2(E_\Gamma) \mid B(u) = 0 \} \).

(ii) The operator \( \bar{A}_B \) is positive definite and its spectrum is bounded below by \( \epsilon \).

(iii) The operator

\[
(\bar{A}_\Gamma, B) : C^\infty (E_\Gamma) \to C^\infty (E_\Gamma) \oplus C^\infty (E_\mid_{\Gamma^+ \cap \Gamma^-})
\]

defined by \( (\bar{A}_\Gamma, B)(u) = (A_\Gamma(u), B(u)) \) can be extended to an invertible operator \( (\bar{A}_\Gamma, B)^\dagger \)

\[
(A_\Gamma, B)_{\Gamma^+} : H^2(E_\Gamma) \to L^2(E_\Gamma) \oplus H^{2-\frac{1}{2}}(E_\mid_{\Gamma^+}),
\]

\[
(A_\Gamma, B)_{\Gamma^-} : H^2(E_\Gamma) \to L^2(E_\Gamma) \oplus H^{2-\frac{1}{2}}(E_\mid_{\Gamma^-}),
\]

\[
(A_\Gamma, B)_{\Gamma^0} : H^2(E_\Gamma) \to L^2(E_\Gamma) \oplus H^{2-\frac{1}{2}}(E_\mid_{\Gamma^0}),
\]

\[
(A_\Gamma, B)_{\Gamma} : H^2(E_\Gamma) \to L^2(E_\Gamma) \oplus H^{2-\frac{1}{2}}(E_\mid_{\Gamma}).
\]
Proof. (i) Using a partition of unity and integration by parts one shows that $A_B$ is symmetric. Clearly, $\bar{A}_B$ is well defined and by a standard argument self-adjoint. To prove (ii) one first notices that for any $u \in C^\infty(E_\Gamma)$ with $u \mid_{\Gamma+\Gamma^-} = 0$, one can find a sequence $\{\phi_n\}$ such that $\text{supp}(\phi_n) \subset M - \Gamma$ and $\phi_n$ converges to $u$ in $H^1(E_\Gamma)$. Observe that $\langle A_B \phi_n, \phi_n \rangle = \langle A \phi_n, \phi_n \rangle \geq \epsilon \|\phi_n\|^2$ and, integrating by parts, one concludes that

$$\langle A_B u, u \rangle = \lim_{n \to \infty} \langle A \phi_n, \phi_n \rangle \geq \epsilon \|u\|^2.$$ 

Thus (ii) follows.

(iii) As $A_B$ is injective, so is the extension $(A_\Gamma, B)$. To prove that this extension is onto, consider $f \in L^2(E_\Gamma)$ and $\varphi \in H^{2-\frac{2}{d}}(E_\Gamma \mid_{\Gamma+\Gamma^-})$. Choose any section $v \in H^2(E_\Gamma)$ so that $Bv = \varphi$. As $\bar{A}_B$ is invertible, there exists $w \in H^2(E_\Gamma)$ satisfying $\bar{A}_Bw = f - \bar{A}_Bv$ and the boundary conditions $Bw = 0$. Therefore $u = w + v$ is an element in $H^2(E_\Gamma)$ with $(A_\Gamma, B)u = (f, \varphi)$. Altogether one concludes that $(A_\Gamma, B)$ is an isomorphism. $\square$

Set $\alpha_k = e^{-\frac{s+2k\pi}{d}}$ for $0 \leq k \leq d - 1$, where $d = \text{dim}(M)$.

**Lemma 3.2.** The following operators are invertible for $0 \leq k \leq d - 1$ and $t \geq 0$

$$(A_\Gamma - \alpha_k t, B) : C^\infty(E_\Gamma) \to C^\infty(E_\Gamma) \oplus C^\infty(E_\Gamma \mid_{\Gamma+\Gamma^-}).$$

Proof. As $\alpha_k \in \mathbb{C} \setminus \mathbb{R}^+$ and thus, for $t \geq 0$, $\alpha_k t \notin \text{Spec}(A_B)$, the operator $(A_\Gamma - \alpha_k t, B)$ is injective. To prove that this operator is onto one argues as in the proof of Lemma 3.1 (iii). $\square$

Since $(A_\Gamma - \alpha_k t, B)$ is invertible, we can define the Poisson operator $P(\alpha_k t)$ associated to $(A_\Gamma - \alpha_k t, B)$, $P(\alpha_k t) : C^\infty(E_\Gamma \mid_{\Gamma+\Gamma^-}) \to C^\infty(E_\Gamma)$, i.e. for $\varphi \in C^\infty(E_\Gamma \mid_{\Gamma+\Gamma^-})$, $u = P(\alpha_k t) \varphi$ is the solution in $C^\infty(E_\Gamma)$ of $(A_\Gamma - \alpha_k t)u = 0$ with boundary conditions $u \mid_{\Gamma+\Gamma^-} = \varphi$.

Let $R(\alpha_k t) : C^\infty(E \mid_\Gamma) \to C^\infty(E \mid_\Gamma)$ be the Dirichlet to Neumann operator corresponding to $A_\Gamma - \alpha_k t, B$ and $C$. Then the following result holds:

**Lemma 3.3.** For $0 \leq k \leq d - 1$, and $t \geq 0$, $R(\alpha_k t)$ is an invertible classical $\Psi DO$ of order 1, which is elliptic with parameter $t$ of weight 1.

Proof. In a sufficiently small collar neighborhood $U$ of $\Gamma$, choose coordinates $x = (x', s)$ such that $(x', 0) \in \Gamma$ and $\frac{\partial}{\partial s} \mid_{(x', 0)} = \nu(x', 0)$. Let $\xi = (\xi', \eta)$ be coordinates in the cotangent space corresponding to the coordinates $(x', s)$. Let $D_s = \frac{1}{2} \frac{\partial^2}{\partial s^2}$ and write $(A - \alpha_k t) = A_2 D_s^2 + A_1 D_s + A_0$, where the $A_j$’s are differential operators of order at most $2 - j$. The $A_j$’s induce, when restricted to $\Gamma$, differential operators, again denoted by $A_j$, $A_j : C^\infty(E \mid_\Gamma) \to C^\infty(E \mid_\Gamma)$. Since $\sigma_L(x, (\xi', \eta)) = \| (\xi', \eta) \|^2$ and since $\nu(x', 0)$ is the unit normal to $\Gamma$ at $(x', 0)$, one has $A_2(x) = Id_{x} \in \text{End}_{x}(E_x, E_x)$ on $\Gamma$.

For any $\varphi \in C^\infty(E \mid_\Gamma)$ and $t \geq 0$, we can choose $u \in C^\infty(E_\Gamma) \cap C(E)$ such that $(A - \alpha_k t)u = 0$ on $M - \Gamma$ and $u \mid_{\Gamma} = \varphi = u \mid_{\Gamma^-}$. Then $\frac{\partial u}{\partial s}(x', s)$ has a jump across $\Gamma$, which is $-R(\alpha_k t)(\varphi)(x')$. Hence

$$\frac{\partial u}{\partial s}(x', s) = -R(\alpha_k t)(\varphi)(x') H(s) + \nu(x', s).$$
where $v(x', s) \in C^\infty(E_{\Gamma} \mid U) \cap C(E \mid U)$ and $H(s)$ is the Heavyside function. Therefore, on $U$,

$$(A - \alpha_k t)u = A_2 R(\alpha_k t)(\varphi) \otimes \delta_{\Gamma} - A_2 \frac{\partial v}{\partial s} + \frac{1}{i} A_1 \frac{\partial u}{\partial s} + A_0 u.$$ 

Since $(A - \alpha_k t)u = 0$ on $M - \Gamma$, we conclude that, on $U \cap (M \setminus \Gamma)$,

$$-A_2 \frac{\partial v}{\partial s} + \frac{1}{i} A_1 \frac{\partial u}{\partial s} + A_0 u = 0.$$ 

As $-A_2 \frac{\partial v}{\partial s} + \frac{1}{i} A_1 \frac{\partial u}{\partial s} + A_0 u \in L^2(E \mid U)$, it follows that

$$(A - \alpha_k t)u = A_2 \cdot (\cdot \otimes \delta_{\Gamma}) \cdot R(\alpha_k t)\varphi.$$ 

Using that $A_2 = Id$ on $\Gamma$, one therefore obtains $Id = J \cdot (A - \alpha_k t)^{-1} \cdot (\cdot \otimes \delta_{\Gamma}) \cdot R(\alpha_k t)$ where $J$ is the restriction operator to $\Gamma$. From this identity it follows that $R(\alpha_k t)$ is invertible. Moreover, setting $\phi = R(\alpha_k t)\varphi$,

$$R(\alpha_k t)^{-1} \phi = J \cdot (A - \alpha_k t)^{-1} \cdot (\phi \otimes \delta_{\Gamma})$$

$$= J \cdot \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} e^{ix' \cdot s}(\xi', \eta)[(A - \alpha_k t)^{-1}(\phi \otimes \delta_{\Gamma})](\xi', \eta) d\eta d\xi'$$

$$= \int_{\mathbb{R}^{d-1}} e^{ix' \cdot s} \int_{\mathbb{R}} \sigma((A - \alpha_k t)^{-1}(x', 0, \xi', \eta) \hat{\phi}(\xi') \cdot \frac{1}{\sqrt{2\pi}} d\eta d\xi'.$$

Hence $R(\alpha_k t)^{-1}$ is a classical $\Psi$DO of order -1 with symbol

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sigma((A - \alpha_k t)^{-1}(x', 0, \xi', \eta) d\eta,$$

and therefore $R(\alpha_k t)$ is a classical $\Psi$DO of order 1 with parameter $t$ of weight 1. The ellipticity with parameter of $R(\alpha_k t)$ follows from the explicit formula of the symbol. $\square$

**Lemma 3.4.** For $\epsilon' < \frac{\pi}{d}$ sufficiently small and $0 \leq k \leq d - 1$, the operator $R(\alpha_k t)$ does not have any eigenvalues in $\Lambda_{\epsilon'}$, where $\Lambda_{\epsilon'} = \{ z \in \mathbb{C} \mid \pi - \epsilon' < \arg(z) < \pi + \epsilon' \text{ or } |z| < \epsilon' \}$. Hence $R(\alpha_k t)$ has $\pi$ as an Agmon angle.

**Proof.** By assumption, $A : C^\infty(E) \to C^\infty(E)$ is essentially self-adjoint and positive definite. Let $\{\psi_j\}_{j \geq 1}$ be a complete orthonormal system of eigensections of $A$ with corresponding eigenvalues $\{\lambda_j\}_{j \geq 1}$. Then $(A - \alpha_k t)^{-1}\psi_j = (\lambda_j - \alpha_k t)^{-1}\psi_j$.

Moreover, for any $\varphi \in C^\infty(E \mid \Gamma), \varphi \otimes \delta_{\Gamma}$ is an element in $H^{-1}(E)$ and $(A - \alpha_k t)^{-1}(\varphi \otimes \delta_{\Gamma}) \in L^2(E)$. Therefore

$$\langle (A - \alpha_k t)^{-1}\varphi \otimes \delta_{\Gamma}, \psi_j \rangle =$$

$$\langle \varphi \otimes \delta_{\Gamma}, (A - \alpha_k t)^{-1} \psi_j \rangle = (\lambda_j - \alpha_k t)^{-1} \int_{\Gamma} (\varphi, \psi_j) d\mu_{\Gamma},$$

where $d\mu_{\Gamma}$ is the volume form on $\Gamma$ induced from the metric on $M$ and $(\cdot, \cdot)$ is the Hermitian inner product on $E$.

Since $R(\alpha_k t)^{-1} = J \cdot (A - \alpha_k t)^{-1} \cdot (\cdot \otimes \delta_{\Gamma})$, one obtains for $\varphi_1, \varphi_2 \in C^\infty(E \mid \Gamma)$

$$\langle R(\alpha_k t)^{-1}\varphi_1, \varphi_2 \rangle = \sum_{j = 1}^{\infty} (\lambda_j - \alpha_k t)^{-1} \int_{\Gamma} (\varphi_1, \psi_j) d\mu_{\Gamma} \int_{\Gamma} (\psi_j, \varphi_2) d\mu_{\Gamma}.$$ 

Together with Lemma 3.3 this implies that $\Lambda_{\epsilon'}$ has an empty intersection with $\text{Spec} R(\alpha_k t)$. $\square$

Using the above formula, one obtains as an immediate consequence the following.
Corollary 3.5. The operator $R = R(0)$ is essentially self-adjoint and positive definite.

Next we are collecting a number of results about operators involving the $d$–th power of $A$ and the submanifold $\Gamma$.

Consider the families of operators $A^d + t^d$ and $A^d_t + t^d$ for nonnegative real numbers $t$. Then $A^d + t^d$ and $A^d_t + t^d$ are elliptic differential operators with parameter, where the weight of $t$ is 2. Note that

$$A^d_t + t^d = (A_\Gamma - te^{i\pi})(A_\Gamma - te^{3i\pi}) \cdots (A_\Gamma - te^{i\pi(d-1)}).$$

Let us introduce the boundary conditions $B_d(t), C_d(t)$ by setting

$$B_d(t) = (B, B(A_\Gamma - \alpha_0 t), B(A_\Gamma - \alpha_1 t)(A_\Gamma - \alpha_0 t), \cdots, B(A_\Gamma - \alpha_{d-2} t)(A_\Gamma - \alpha_0 t),$$

and

$$C_d(t) = (C, C(A_\Gamma - \alpha_0 t), C(A_\Gamma - \alpha_1 t)(A_\Gamma - \alpha_0 t), \cdots, C(A_\Gamma - \alpha_{d-2} t)(A_\Gamma - \alpha_0 t)).$$

It follows from Lemma 3.2 that the following operator is invertible

$$(A^d_t + t^d, B_d(t)) : C^\infty(E_\Gamma) \to C^\infty(E_\Gamma) \oplus (\oplus_d C^\infty(E_\Gamma |_{\Gamma^+ \cup \Gamma^-})).$$

Therefore the corresponding Poisson operator $\tilde{P}_d(t) : \oplus_d C^\infty(E |_{\Gamma^+ \cup \Gamma^-}) \to C^\infty(E_\Gamma)$ is well defined.

Lemma 3.6. The Poisson operator $\tilde{P}_d(t)$ associated to $(A^d_t + t^d, B_d(t))$ is given by

$$\tilde{P}_d(t)(\varphi_0, \cdots, \varphi_{d-1}) = P(\alpha_0 t)\varphi_0 + (A_\Gamma - \alpha_0 t)^{-1}P(\alpha_1 t)\varphi_1 + \cdots +$$

$$(A_\Gamma - \alpha_d t)^{-1}(A_\Gamma - \alpha_1 t)^{-1}\cdots (A_\Gamma - \alpha_{d-2} t)^{-1}P(\alpha_{d-1} t)\varphi_{d-1},$$

where $(A_\Gamma - \alpha t)_B$ is the restriction of $A_\Gamma - \alpha t$ to $\{u \in C^\infty(E_\Gamma) | Bu = 0\}$.

Proof. Denoting the right hand side of the claimed identity by $Q_d(t)(\varphi_0, \cdots, \varphi_{d-1})$ one obtains

$$(A^d_t + t^d) \cdot Q_d(t)(\varphi_0, \cdots, \varphi_{d-1}) = 0.$$

Moreover, for $0 \leq k \leq d - 1, Q_d(t)(\varphi_0, \cdots, \varphi_{d-1})$ satisfies the boundary conditions

$$B(A_\Gamma - \alpha_{k-1} t)(A_\Gamma - \alpha_{k-2} t) \cdots (A_\Gamma - \alpha_{0} t)Q_d(t)(\varphi_0, \cdots, \varphi_{d-1}) =$$

$$B(A_\Gamma - \alpha_{k-1} t) \cdots (A_\Gamma - \alpha_{0} t)P(\alpha_{0} t)\varphi_0 + \cdots +$$

$$B(A_\Gamma - \alpha_{k-1} t) \cdots (A_\Gamma - \alpha_{0} t)(A_\Gamma - \alpha_{0} t)^{-1} \cdots (A_\Gamma - \alpha_{k-1} t)^{-1}P(\alpha_{k} t)\varphi_k + \cdots +$$

$$B(A_\Gamma - \alpha_{k-1} t) \cdots (A_\Gamma - \alpha_{0} t)(A_\Gamma - \alpha_{0} t)^{-1} \cdots (A_\Gamma - \alpha_{d-2} t)^{-1}P(\alpha_{d-1} t)\varphi_{d-1} = \varphi_k,$$

since $(A_\Gamma - \alpha_j t)P(\alpha_j t) = 0$ and $B(A_\Gamma - \alpha_j t)^{-1}B = 0$. These two properties of $Q_d(t)$ establish the claimed identity. □
Further let us consider the boundary conditions \( B_d(t) \) and \( C_d(t) \) for \( t = 0 \). Note that
\[
B_d(0) = (B, BA_\Gamma, \cdots, BA^{d-1}_\Gamma); C_d(0) = (C, CA_\Gamma, \cdots, CA^{d-1}_\Gamma).
\]
Let \( \Omega(t) \) be the following lower triangular \( d \times d \) matrix
\[
\Omega(t) = \begin{pmatrix}
\alpha_0 t & 0 & \cdots & 0 \\
\alpha_0 t^2 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_0^{d-1} t^{d-1} & t^{d-2} & \cdots & 1
\end{pmatrix}.
\]
Then \( B_d(0) = \Omega(t) B_d(t) \) as well as \( C_d(0) = \Omega(t) C_d(t) \). Let \( P_d(t) := \tilde{P}_d(t) \Omega(t)^{-1} \) and notice that \( P_d(t) \) is the Poisson operator corresponding to \( (A^d_\Gamma + t^d, B_d(0)) \).

Consider the Dirichlet to Neumann operator \( \tilde{R}_d(t) = \triangle_i f \cdot C_d(t) \cdot \tilde{P}_d(t) \cdot \triangle_i a \) corresponding to \( A^d_\Gamma + t^d, B_d(t) \) and \( C_d(t) \). Then
\[
\tilde{R}_d(t)(\varphi_0, \cdots, \varphi_{d-1}) = \triangle_i f \cdot (C, C(A_\Gamma - \alpha_0 t), \cdots, C(A_\Gamma - \alpha_{d-2} t) \cdots (A_\Gamma - \alpha_0 t)).
\]

Thus \( \tilde{R}_d(t) : \oplus_d C^\infty(E |_\Gamma) \to \oplus_d C^\infty(E |_\Gamma) \) can be represented by a \( d \times d \) matrix of upper triangular form,
\[
\begin{pmatrix}
R(\alpha_0 t) & \triangle_i f C(A_\Gamma - \alpha_0 t)^{-1} P(\alpha_1 t) \triangle_i a & \cdots & \triangle_i f C(A_\Gamma - \alpha_0 t)^{-1} P(\alpha_{d-1} t) \triangle_i a \\
0 & R(\alpha_1 t) & \cdots & \triangle_i f C(A_\Gamma - \alpha_1 t)^{-1} P(\alpha_{d-1} t) \triangle_i a \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & R(\alpha_{d-1} t)
\end{pmatrix},
\]
where \( R(\alpha_k t) \) is the Dirichlet to Neumann operator corresponding to \( A_\Gamma - \alpha_k t, B \) and \( C \) defined earlier. In particular, we conclude that \( \tilde{R}_d(t) \) is invertible and has \( \pi \) as an Agmon angle.

Finally introduce the Dirichlet to Neumann operator \( R_d(t) \) associated to \( A^d_\Gamma + t^d, B_d(0) \) and \( C_d(0) \). Then
\[
\tilde{R}_d(t) = \triangle_i f \cdot C_d(t) \cdot \tilde{P}_d(t) \cdot \triangle_i a = \triangle_i f \cdot \Omega(t)^{-1} \cdot C_d(0) \cdot P_d(t) \cdot \Omega(t) \cdot \triangle_i a = \Omega(t)^{-1} \cdot \triangle_i f \cdot C_d(0) \cdot P_d(t) \cdot \triangle_i a \cdot \Omega(t) = \Omega(t)^{-1} \cdot R_d(t) \cdot \Omega(t).
\]
As a consequence, \( R_d(t) \) has the same spectrum as \( \tilde{R}_d(t) \) and therefore, \( R_d(t) \) is invertible, has \( \pi \) as an Agmon angle and satisfies \( \log \text{Det}(R_d(t)) = \log \text{Det}(\tilde{R}_d(t)) \).

In view of the fact that \( \tilde{R}_d(t) \) is of upper triangular form one has
\[
\log \text{Det}(\tilde{R}_d(t)) = \sum_{k=0}^{d-1} \log \text{Det}(R(\alpha_k t)).
\]
As $A$ is positive and essentially selfadjoint, the operator $A^d + t^d : C^\infty(E) \to C^\infty(E)$ is invertible for $t \geq 0$. Using the kernel $k_t(x,y)$ of $(A^d + t^d)^{-1}$ this operator can be extended to $C^\infty(E_\Gamma)$ by setting $(u \in C^\infty(E_\Gamma))$

$$((A^d + t^d)^{-1})_\Gamma u(x) = \int_{M_\Gamma} k_t(x,y)u(y)dy.$$ 

It follows from Lemma 3.2 that

$$\left(A^d_\Gamma + t^d, B_d(t)\right) : C^\infty(E_\Gamma) \to C^\infty(E_\Gamma) \oplus (\oplus_d C^\infty(E_\Gamma |_{\Gamma^+ \cup \Gamma^-})) \,.$$

is invertible. Thus, since $B_d(0) = \Omega(t)B_d(t)$, we conclude that $(A^d_\Gamma + t^d, B_d(0))$ is invertible as well. Denote by $(A^d_\Gamma + t^d)_{B_d(0)}$ the restriction of $A^d_\Gamma + t^d$ to $\{u \in C^\infty(E_\Gamma) \mid B_d(0)u = 0\}$ and let $(A^d_\Gamma + t^d)^{-1}_{B_d(0)}$ be its inverse.

**Lemma 3.7.** $(A^d_\Gamma + t^d)^{-1}_{B_d(0)} = ((A^d + t^d)^{-1})_\Gamma - P_d(t) \cdot B_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma$

**Proof.** Denote by $Q(t)$ the right hand side of the claimed identity. One verifies that for $u \in C^\infty(E_\Gamma)$

$$(A^d_\Gamma + t^d)Q(t)u = u$$

and

$$B_d(0)Q(t)u = B_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma u - B_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma u = 0.$$ 

These two identities imply that $Q(t) = (A^d_\Gamma + t^d)^{-1}_{B_d(0)}$.

**Lemma 3.8.** (i) $\frac{d}{dt}P_d(t) = -dt^{-1}(A^d_\Gamma + t^d)^{-1}_{B_d(0)} \cdot P_d(t)$

(ii) $R_d(t)^{-1} \cdot \frac{d}{dt}R_d(t) = -dt^{-1}R_d(t)^{-1} \cdot \triangle_if \cdot C_d(0) \cdot (A^d_\Gamma + t^d)^{-1}_{B_d(0)} \cdot P_d(t) \cdot \triangle_{ia}$.

In particular, $d$ being the dimension of $M$, $R_d(t)^{-1} \cdot \frac{d}{dt}R_d(t)$ is of trace class.

**Proof.** (i) Derive $(A^d_\Gamma + t^d) \cdot P_d(t) = 0$ with respect to $t$ to obtain

$$(A^d_\Gamma + t^d) \cdot \frac{d}{dt}P_d(t) = - \frac{d}{dt}(A^d_\Gamma + t^d)^{-1}_{B_d(0)} \cdot P_d(t) = -dt^{-1}P_d(t).$$

Similarly, deriving $B_d(0) \cdot P_d(t) = Id$ with respect to $t$ yields $B_d(0) \frac{d}{dt}P_d(t) = 0$. Hence

$$(A^d_\Gamma + t^d)_{B_d(0)} \cdot \frac{d}{dt}P_d(t) = -dt^{-1}P_d(t)$$

and therefore

$$\frac{d}{dt}P_d(t) = -dt^{-1}(A^d_\Gamma + t^d)^{-1}_{B_d(0)} \cdot P_d(t).$$

(ii) follows from the definition of $R_d(t)$ and (i).
Corollary 3.9. 

\[ R_d(t)^{-1} \cdot \frac{d}{dt} R_d(t) = dt^{d-1} Pr_\Gamma \cdot B_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma \cdot P_d(t) \cdot \Delta_{ia} \]

Proof. By Lemma 3.7 and 3.8

\[ R_d(t)^{-1} \cdot \frac{d}{dt} R_d(t) = -dt^{d-1}(\Delta_{if} \cdot C_d(0) \cdot P_d(t) \cdot \Delta_{ia})^{-1}. \]

\[ \Delta_{if} \cdot C_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma \cdot P_d(t) \cdot B_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma \cdot P_d(t) \cdot \Delta_{ia}. \]

Clearly \( \Delta_{if} \cdot C_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma (u) = 0 \) for \( u \in C^\infty(E) \)
and thus \( \Delta_{if} \cdot C_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma \cdot P_d(t) \cdot \Delta_{ia} = 0 \). Therefore
\( R_d(t)^{-1} \cdot \frac{d}{dt} R_d(t) = dt^{d-1}(\Delta_{if} \cdot C_d(0) \cdot P_d(t) \cdot \Delta_{ia})^{-1} \cdot \Delta_{if} \cdot C_d(0) \cdot P_d(t) \cdot B_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma \cdot P_d(t) \cdot \Delta_{ia}. \)

Note that for any \( u \in C^\infty(E_\Gamma) \), the boundary values of \( ((A^d + t^d)^{-1})_\Gamma u \) on \( \Gamma^+ \) and \( \Gamma^- \) are the same, i.e.
\[ B_d(0)((A^d + t^d)^{-1})_\Gamma u \big|_{\Gamma^+} = B_d(0)((A^d + t^d)^{-1})_\Gamma u \big|_{\Gamma^-}. \]

Hence
\[ B_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma \cdot P_d(t) \cdot \Delta_{ia} = \Delta_{ia} \cdot Pr_\Gamma \cdot B_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma \cdot P_d(t) \cdot \Delta_{ia}. \]

As \( (A^d + t^d)^{-1} (A^d + t^d)^{-1} \big|_{B_d(0)} \) and \( R_d(t)^{-1} \frac{d}{dt} R_d(t) \) are of trace class we can apply
the well known variational formula for regularized determinants:

Lemma 3.10. Let \( Q(t) \) denote any of the operators \( A^d + t^d \), \( (A^d + t^d)_{B_d(0)} \) or \( R_d(t) \). Then, for any \( t \geq 0 \),
\[ \frac{d}{dt} \log \text{Det} Q(t) = \text{tr} (Q(t)^{-1} \frac{d}{dt} Q(t)). \]

4. Proof of the Theorem 1.1

In the case where the operator \( A^{-1} \) is of trace class, the proof of Theorem 1.1 is considerably simpler. Unfortunately, this is only the case if the dimension \( d \) of \( M \) is equal to 1. Our strategy is to first prove a version of Theorem 1.1 for \( A^d \) (Lemma 4.1), using the fact that \( (A^d)^{-1} \) is of trace class. Together with the auxiliary results of section 3 and the asymptotic expansion derived in section 2, the proof of Theorem 1.1 is then completed.

Lemma 4.1. Let \( A^d + t^d \) and \( (A^d + t^d, B_d(0)) \) be as above. Then, for \( t \geq 0 \),
\[ \frac{d}{dt} \left( \log \text{Det}(A^d + t^d) - \log \text{Det}(A^d + t^d, B_d(0)) \right) = \frac{d}{dt} \log \text{Det} R_d(t). \]

Proof. Define \( u(t) := \frac{d}{dt} (\log \text{Det}(A^d + t^d) - \log \text{Det}(A^d + t^d, B_d(0))). \) By Lemma 3.10 and Lemma 3.7
if

Statement (i) follows from the fact that

\[ \text{Setting } \]

\[ \text{Proof of Theorem 1.1.} \]

By Theorem 2.4.

Using the result of Theorem 2.4, Theorem 1.1 follows.

\[ \Box \]

On the other hand, by Lemma 3.10, Corollary 3.9 and the commutativity of the trace,

\[ \frac{d}{dt} \log \det R_d(t) = tr\left( \frac{d}{dt} R_d(t) \right) \cdot R_d(t)^{-1} \]

\[ = dt^{d-1} \text{tr}\left( P \Gamma \cdot B_d(0) \cdot ((A^d + t^d)^{-1}) \cdot \Gamma \right) \]

Combining the above two identities shows that

\[ \begin{align*}
\log \det (A^d + t^d) - \log \det ((A_d^d + t^d), B_d(0)) &= \tilde{c} + \sum_{k=0}^{d-1} \log \det R(\alpha_k t), \tag{4.1}
\end{align*} \]

where \( \tilde{c} \) is independent of \( t \).

Note that \( \log \det (A^d + t^d), \log \det (A^d + t^d), B_d(0) \) and \( \log \det R(\alpha_k t)(0 \leq k \leq d-1) \) have asymptotic expansions as \( t \to +\infty \). Since the eigenvalues of \( A^d + t^d \) and \( (A_d^d + t^d)B_d(0) \) satisfy the condition in Proposition 2.7, the constant terms in the asymptotic expansions of \( \log \det (A^d + t^d) \) and \( \log \det ((A_d^d + t^d), B_d(0)) \) are zero. Let \( \pi_0(R(\alpha_k t)) \) be the constant term in the asymptotic expansion of \( \log \det (R(\alpha_k t)) \).

Then \( \tilde{c} = -\sum_{k=0}^{d-1} \pi_0(R(\alpha_k t)), \) which is computable in terms of the symbol of \( R(\alpha_k t) \) by Theorem 2.4.

**Lemma 4.2.** (i) \( \det(A^d, B_d(0)) = (\det(A^d, B))^d \); (ii) \( \det(A^d) = (\det A)^d \).

**Proof.** Statement (i) follows from the fact that \( \lambda \) is an eigenvalue of \( A_B \) if and only if \( \lambda^d \) is an eigenvalue of \( (A_d^d)^{B_d(0)} \) and (ii) is proved in the same way. \( \Box \)

**Proof of Theorem 1.1.** Setting \( t = 0 \) in (4.1), one obtains

\[ \log \det A^d - \log \det (A_d^d, B_d(0)) = \tilde{c} + \log \det R_d(0). \]

By Lemma 4.2, \( \log(\det A)^d - \log(\det(A_d^d, B))^d = \tilde{c} + \log(\det R)^d \). Hence

\[ \log \det A = \log(c) + \log(\det A^d, B) + \log \det R, \text{ where } \log(c) = -\frac{1}{d} \sum_{k=0}^{d-1} \pi_0(R(\alpha_k t)). \]

Using the result of Theorem 2.4, Theorem 1.1 follows. \( \Box \)
References

[BFK] D.Burghelia, L.Friedlander, T.Kappeler, *Mayer-Vietoris Type Formula for Determinants of Elliptic Differential Operators*, J. of Funct. Anal. **107** (1992), 34-65.

[Fr] L.Friedlander, *The asymptotic of the determinant function for a class of operators*, Proc.Amer.Math.Soc. **107** (1989), 169-178.

[Gu] V.Guillemin, *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues*, Adv.Math. **55** (1985), 131-160.

[Se] R.Seely, *Complex powers of elliptic operators*, Proceedings of Symposia on Singular Integrals, Amer.Math.Soc., Providence, RI **10** (1967), 288–307.

[Sh] M.A.Shibin, *Pseudodifferential Operators and Spectral Theory* (1985), Springer-Verlag, Berlin/New York.

[Vo] A.Voros, *Spectral function, special functions and Selberg zeta function*, Comm.Math.Phys. **110** (1987), 439-465.

[Wo] M.Wodzicki, *Noncommutative residue in K-theory, Arithmetic and Geometry*, Lecture Notes in Mathematics (Y.Manin, eds.), vol. 1289, Springer-Verlag,Berlin/New York, 1987.