Localized Matter and Geometry of the Dirac Field.

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Within the framework of classical field theory, the connection between the Dirac field as the field of matter and the spacetime metric is discussed. Polarization structure of the Dirac field is shown to be rich enough to determine the spacetime metric locally and to explain the emergence of observed matter as localized waveforms. The localization of the waveforms is explained as the result of the local time slowdown and the Lorentz contraction as a dynamic re-shaping of the waveforms in the course of their acceleration. A definition of mass as a limiting curvature of the spinor-induced metric is proposed. A view of the vacuum as a uniformly distributed unit invariant density of the Dirac field with an explicitly preserved invariance of the light cone is brought forward. Qualitative explanation of the observed charge asymmetry as the consequence of the dynamics of localization is given. The classical pion field is obtained as a manifestation of stresses, mass and charge flux in localized waveforms of the Dirac field. Some implications of the finite size of colliding objects for high-energy processes are discussed. A hypothesis that known internal degrees of freedom are the local spacetime (angular) coordinates that have no precise counterparts in Riemannian geometry is proposed.

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I. INTRODUCTION

The purpose of this paper is to show that the classical theory of the Dirac field, considered as a primary form of matter can explain those important properties of the observed matter which so far remain a mystery when viewed from the perspective of quantum field theory. In the first place, these properties are localization of elementary objects and the origin of their mass and finite size. Another, not less intriguing question is the origin of the observed charge asymmetry of normal matter – we find only small, heavy, positively charged protons (nuclei) and light, negatively charged, poorly localized electrons as the only stable particles around us. It seems that a possibility to answer these big questions has been overlooked at the early stage of field theory.

An idea that the fields \( \psi(x) \) of matter themselves can immediately define the metric tensor \( g_{\mu\nu}(x) \) was brought forward by Wigner \[1\] and Sakharov \[2\]. From the physics perspective, this idea is extremely sound; coordinates can be measured only through positions and shapes of material bodies. [In quoted works, tensor indices of \( g_{\mu\nu} \) were due to the derivatives \( \partial_\mu \psi(x) \).] It appears that the Dirac field builds up the metric of spacetime without resorting to \textit{ad hoc} derivatives and it does this in such a way that the time slows down in the domains of magnified invariant density. This observation alone leads to a natural and startlingly elegant answer to these big questions. Merely in the spirit of Huygens principle, this fact leads to self-localization of the Dirac wave forms into small, heavy, positively charged objects of finite size while leaving a negatively charged fraction of matter in the form of an agile substance surrounding these small and heavy objects. It also changes the image of the Dirac sea as the vacuum – a uniformly distributed unit invariant density \( \mathcal{R} = 1 \) is identified with \( g_{00} = 1/\mathcal{R}^2 = 1 \) and replaces a continuum of oscillators with an unbound energy spectrum.

From the perspective of the present work, the Dirac field is important, not as a special representation of the Lorentz group, but as a field that accurately describes the hydrogen atom. Lagrangian formalism is not used and no symmetry is assumed \textit{a priori}. The main focus is on the possibility of deriving the most important properties of observed stable matter and its motion starting from the basic properties of the Dirac field and its equation of motion. No significant attempt to develop a formal perturbation theory that could have dealt with the finite size of particles as the \textit{in-} and \textit{out-} states of the quantum scattering process has been made so far. Once localized wave forms are found they immediately can be used as a basis for second quantization and their fields can serve as Heisenberg operators. The best prospect of this study is connected with the possibility to bridge the gap between point-like particles of classical electrodynamics and the plane waves of the quantum theory of scattering.

The logic of the present work can be outlined as follows:

The stage is set in Sec.IIA beginning from a review of well-known properties of the bilinear forms of the Dirac field with emphasis on their purely algebraic origin. These forms are empirically verified to be affine Lorentz tensors at a generic point and they are further used to build a quadruple of orthogonal Lorentz unit vectors (tetrad). The possibility of treating these unit vectors as the tangent vectors of the coordinate lines of a usual holonomic coordinate system and thus to define the Riemannian metric as a descendant of the Dirac field is

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considered in Sec. III. It appears that certain conditions of integrability should be met and that these conditions are controlled by the Dirac equation. A key observation that the matter-induced metric must be equivalent to a long range interaction is made.

The rules of differential calculus for the Dirac field in curved spacetime are reviewed in Appendix A mostly following V. Fock [3]. A greater generality than in [3] is intentionally admitted – and there was no possibility to truncate it later. Sec. III thoroughly investigates if various differential identities, derived from the Dirac equation, can be put in the form of tensor equations, thus being independent of a particular choice of coordinate system. For identities that involve the energy-momentum $T^{a}_{\mu}$, the conclusion is negative with the following facts firmly established: (i) The normal covariant form of the energy-momentum conservation cannot be assembled when the coordinate system is normal. (ii) The tetrad components $T^{a}_{\mu}$ of energy-momentum are not the invariants of tensors. (iii) Being formally translated into coordinate form, the identity of energy-momentum balance keeps an explicit dependence on tetrad vectors. It does not reproduce the equation for a geodesic line in a given metric background. (iv) An explicit expression for the force of gravity (inertia) is derived from the constraint, which accounts for the interplay between scalar and pseudoscalar quantities in the course of a physical acceleration of a Dirac object. Only in a crude approximation of a point-like object is the standard metric expression recovered. To account for the flux of momenta in spacelike directions (pressure) in a localized waveform, a new stress tensor, $P^{a}_{\mu}$, is introduced and studied in the same detail in Appendix B. A connection with the theory of Nambu and Jona-Lasinio is traced. The wave equation for the pseudoscalar density (pion field) with the source that has the structure of the axial anomaly is derived in Section III.

An unremovable dependence on tetrad vectors prompted a detailed investigation (in Sec. IV) of the geometric properties of vector and axial currents and constraints that affect the integrability of differential equations for their lines. It is shown that for the on-mass-shell Dirac field, the timelike congruence of lines of the vector current always is normal so that there always exists a system of hypersurfaces of a constant time. The key result reads as $dt = R ds_{0}$, where $R = \sqrt{\mathbf{J}^{2}}$ is the invariant density of the Dirac field; $dt$ and $ds_{0}$ are the intervals of the world and proper time, respectively. It immediately predicts a general trend of self-localization for the Dirac field and a Lorentz contraction of accelerated elementary objects as a physical process. Investigation of constraints connected with identities for the axial current (which, having a source, determines a radial direction) brought about another result – the maximal curvature of the 2-d surface of constant radius cannot exceed $m$, the mass parameter in the Dirac equation. A set of relations that connect bending of coordinate lines with the distribution of the axial current is obtained. These relations prompt a strong parallel between the local dynamics of the Dirac field and systems of inertial navigation – linear acceleration inevitably causes a precession and vice versa. Sec. V deals with the intuitively appealing (and possibly not realistic) case of normal radial coordinate, in which behavior of the angular coordinates can be studied analytically.

With the metric explicitly depending on the field of matter the Dirac equation becomes nonlinear in a unique way, which leads to self-localization as an intrinsic property of the Dirac field. Different forms of this equation along with an analysis of individual terms are the subject of Sec. VI and Appendix C.

In Sec. VI a major conjecture regarding the nature of electric charge is made. Maxwell equations are introduced and it is shown that a stable Dirac waveform cannot interact with its own electric field. Furthermore, two such forms cannot intersect each other in spacetime. The origin of electromagnetic radiation is explicitly traced back to the loss of simultaneity between the Dirac waveform and its Coulomb field.

We conclude in Sec. VII with a short list of the existing data and experiments that are in line with or can serve as the tests for our predictions.

The results of this work, if looked at as a launch-pad for further investigations, are striking in their anticipated mathematical complexity and physical transparency. It seems, however, that the former is the inevitable toll for the latter. The nonlinearity of the Dirac equation makes finding its explicit solutions a formidable task. But this nonlinearity is not artificial – no ad hoc nonlinear terms were added to the basic Dirac Lagrangian in order to simulate any experimentally found patterns of matter behavior, symmetry, etc. On the contrary, the discovered generic structure corresponds to the perfectly understood phenomenon of localization, which is due to the local time slowdown, and then the loss of certain elements of spatial symmetry due to localization. Despite being genuinely nonlinear, these phenomena are so natural for any kind of wave propagation that only a minimal amount of information about the physical nature of the waves is needed to not only understand the whole picture qualitatively, but even to make semi-quantitative estimates. In the text we also outline how the existence of the pion field or how the known properties of the neutrino can be inferred from the concept of a localized Dirac waveform.

II. DIRAC FIELD AND RIEMANNIAN GEOMETRY.

The first attempts to bring the Dirac equation into the framework of General Relativity (GR) was made by V. Fock [3] and H. Weyl [4] in a series of papers in 1929. This study (and many other studies of that year) was in line with the basic concept of Einstein’s GR that, in the local limit (inertial reference frame), one has to reproduce the results of special relativity; it was established earlier that
spinors do indeed provide a linear representation of the Lorentz group. Somewhat later, E. Cartan pointed to an insurmountable difficulty — there are no representations of the general linear group of transformations \( GL(4) \) that are similar to spinor representations of the Lorentz group of rotations. Cartan stated the following theorem, which vetoed spinors in Riemannian geometry:

"With the geometric sense given to the word “spinor” it is impossible to introduce spinors into classical Riemannian technique; i.e., having chosen an arbitrary system of co-ordinates \( x^\alpha \) for space, it is impossible to represent spinor by any finite number of components \( \psi \), such that \( \psi \) have covariant derivatives of the form \( \partial_\mu \psi_i + \Gamma^j_{i\mu} \psi_j \), where \( \Gamma^j_{i\mu} \) are determinate functions of \( x^\mu \).

Of these two underscored reservations of Cartan, the first one was investigated by Ne'eman et al [6], who proposed to overcome the veto by resorting to the infinite-dimensional representations of the Lorentz group. The present study explores the window, which is left open by the second reservation. As long as natural coordinates for the Dirac field are anholonomic (also in the sense of the theory of dynamical systems) the "connections" \( \Gamma^j_{i\mu} \) and, eventually, the metric tensor \( g_{\mu\nu}(x) \) appear to be functions of the Dirac field (which are determined by Eqs. (4.7), (4.11), and (4.21), (4.22) below) and not determinate functions of \( x^\mu \).

In this section we set the stage by demonstrating that the two key issues of geometry, direction and distance, can be separated in a "physical way" by associating the field of directions with the Dirac field of matter. The Riemannian metric will then be associated with the propagation of signals.

A. Algebraic properties of the Dirac Field.

All observables associated with the Dirac field are bilinear forms built with the aid of Dirac matrices \( \alpha^a \) and \( \beta \), which satisfy the commutation relations ( \( \alpha^a = (1, \alpha^i) \); \( \alpha = 0, 1, 2, 3; i = 1, 2, 3 \))

\[
\alpha^a \beta^b + \alpha^b \beta^a = 2 \beta \eta^{ab},
\]

(2.1)

where \( \eta^{ab} = \text{diag}(1, -1, -1, -1) \). We begin with a review of the properties of the Dirac field \( \psi(x) \) which hold at a point, without a precise definition of the coordinates \( x \). For now, \( \psi \) will stand for a column of four complex numbers \( \psi_\alpha \).

There are sixteen linearly independent \( 4 \times 4 \) Hermitian matrices all of which can be constructed from the four matrices \( \alpha^i \) and \( \beta \). The Dirac matrices, \( \rho_i \) ( \( \rho_1 = \beta \), \( \rho_3 = -i \alpha^3 \alpha^3 \), \( \rho_2 = -i \beta \rho_3 \)), and \( \sigma_i = \rho_i \rho_4 \) satisfy the same commutation relations as the Pauli matrices, and all \( \sigma \) matrices commute with the \( \rho \) matrices: \( \sigma_i \sigma_k = \delta_{ik} + i \epsilon_{ikl} \sigma_l, \rho_a \rho_b = \delta_{ab} + i \epsilon_{abc} \rho_c, \sigma_i \rho_j = -\rho_j \sigma_i = 0 \). The matrices \( -\rho_3 \) and \( \rho_1 \) are commonly known as \( \gamma^0 \) and \( \gamma^1 \), respectively. Below, these matrices are used in the spinor representation,

\[
\alpha_i = \begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

\[
\rho_2 = \begin{pmatrix} 0 & -i \cdot 1 \\ i \cdot 1 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

where \( \tau_i \) are the \( 2 \times 2 \) Pauli matrices. Employing the Dirac matrices, we can define the four components of the "vector current" \( j^a = \psi^+ \alpha^a \psi \equiv \psi \gamma^a \psi \), the four components of the "axial current" \( J^a = \bar{\psi} \gamma^a \psi \equiv \psi \gamma^a \gamma^5 \psi \), the "scalar" \( S = \psi^+ \rho_1 \psi \equiv \psi \psi \) and "pseudoscalar" \( \mathcal{P} = \psi^+ \rho_2 \psi \equiv -i \psi \gamma^5 \psi \), and the six components of the skew-symmetric "tensor" \( M^{ab} = (i/2) \psi^+ [\alpha^a \rho_1 \alpha^b - \alpha^b \rho_1 \alpha^a] \psi \).

The similarity of these quantities to the Lorentz tensors can be verified in a purely algebraic way. Indeed, if the Dirac field \( \psi \) is transformed by means of a substitution \( \psi \to S \psi \) (or the matrices are transformed as \( \alpha^a \to S^+ \alpha^a S \), etc.), with the matrix \( S \) depending on four complex parameters,

\[
S = \begin{pmatrix} \lambda & 0 \\ 0 & (\lambda^+)^{-1} \end{pmatrix}, \quad \lambda = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},
\]

\[
det |S| = \alpha \delta - \beta \gamma = 1,
\]

(2.2)

then the components of \( j^a, J^a, S, \mathcal{P} \) and \( M^{ab} \) experience a four-dimensional Lorentz rotation at angles which are uniquely determined by these parameters. For example, if we take \( \alpha = e^{-i \phi/2}, \beta = \gamma = 0, \) and \( \delta = e^{i \phi/2} \), then the transformation \( S = e^{-i \phi \sigma_3 / 2} \) is unitary and the components of \( j^a \) are \( \psi^+ S^+ \alpha^a S \psi \) are

\[
\begin{align*}
 j^0 &= j^0, \\
 j^1 &= j^1 \cos \phi - j^2 \sin \phi, \\
 j^2 &= j^1 \sin \phi + j^2 \cos \phi, \\
 j^3 &= j^3,
\end{align*}
\]

(2.3)

which corresponds to the rotation of the vector \( j^a \) at an angle \( \phi \) around the axis \( \gamma^3 \). In exactly the same way, if we take \( \alpha = e^{-i \eta / 2}, \beta = \gamma = 0, \) and \( \delta = e^{i \eta / 2} \) then \( S = e^{-i \eta \alpha_3 / 2} \); the components of \( j^a \) will be

\[
\begin{align*}
 j^0 &= j^0 \cosh \eta - j^3 \sinh \eta, \\
 j^1 &= j^1, \\
 j^2 &= j^2, \\
 j^3 &= -j^0 \sinh \eta + j^3 \cosh \eta.
\end{align*}
\]

(2.4)

This transformation of \( \psi \) corresponds to a Lorentz boost in the "third" direction and is not unitary. Similar correct relations are immediately verified for the scalars \( S \) and \( \mathcal{P} \), the vector \( J^a = \psi^+ S^+ \rho_3 \alpha^a S \psi \), etc. Therefore, for example, the quantities

\[
\begin{align*}
 j^2 &= \eta_{ab} j^a j^b = j_0^2 - j_1^2, \\
 J^2 &= \eta_{ab} J^a J^b, \\
 j \cdot J &= \eta_{ab} j^a j^b = j_0 J_0 - j_1 \vec{J}.
\end{align*}
\]

(2.5)

1 We consciously refrain from using the anti-hermitian matrices \( \gamma^i = \rho_1 \alpha^i \) and the Pauli-conjugated spinors \( \bar{\psi} = \psi^+ \rho_1 \). In their terms, the formulae of parallel transport (see Appendix A) would be much less transparent and unnecessarily complicated.
where we assumed that $\det|e_\beta^\alpha| \neq 0$. Then, a simple algebra verifies that the objects
\begin{equation}
 g_{\alpha\beta} = \eta_{\alpha\beta}e_{(a)}^\alpha e_{(b)}^\beta, \quad g^{\alpha\beta} = \eta^{\alpha\beta}e^{(a)}_\alpha e^{(b)}_\beta \tag{2.12}
\end{equation}
can be used to move the Greek indices up and down, for example,
\begin{equation}
 g_{\alpha\beta}e^{\beta}_{(b)} = e^{(a)}_\alpha e^{(a)}_\beta e^{(b)}_{\alpha} = \delta^\alpha_\beta e^{(a)}_\alpha = e_{(a)}^\beta.
\end{equation}

It is also evident that the repeated upper and lower Greek indices are contracted.

B. Dirac currents and Riemannian geometry.

From now on, we look at the $\psi_\rho(x)$ as the physical Dirac field, the continuous functions of the arbitrarily parameterized points $x^\mu = (x^0, x^1, x^2, x^3)$ of the spacetime. So far, we have verified that the algebraic structure of bilinear forms of the Dirac field naturally contains an orthogonal quadruple of unit Lorentz vectors at a generic point, thus defining spacetime directions at that point.

In a sense, linear transformations (2.2) of the Dirac field generate a group of homogeneous linear transformations for vectors, thus associating with every point a local centered affine space. Vectors of this quadruple are thought of as smooth functions of $\psi(x)$ and it is tempting to make it treat as a quadruple of the vector fields. But these transformations of the Dirac field have nothing to do with the general transformation of coordinates, which are arguments of $\psi(x)$. For a given fixed $\lambda$, we can consider $x^\lambda = \text{const}$ as the equation of a coordinate hypersurface and the lines along which all coordinates but $x^\lambda$ are constant as coordinate lines. Tangent vectors of these lines (which are gradients of the linear function $\varphi(x) = x^\lambda$) are $h^\mu_\lambda = \partial x^\mu / \partial x^\lambda = \delta^\mu_\lambda$; therefore, this coordinate system is an holonomic one, but it has no metric and there is no way to determine if its coordinate lines are orthogonal. One may replace $x^\mu$ by smooth functions of other coordinates $y^\mu$, $x^\mu = f^\mu(y)$, thus redefining coordinate lines and surfaces, but such a change does not alter $\psi(x(y))$ and has nothing to do with affine Lorentz transformations (2.2).

To bridge the gap between the abstract field of directions determined by the Dirac field and the given above definition of the holonomic coordinates, it is necessary to know in advance that four systems of differential equations (for the unknown $x^\mu$),
\begin{equation}
 \frac{dx^0}{e^0_{(a)}(x)} = \frac{dx^1}{e^1_{(a)}(x)} = \frac{dx^2}{e^2_{(a)}(x)} = \frac{dx^3}{e^3_{(a)}(x)} = ds^a, \tag{2.13}
\end{equation}
for congruences of lines (labeled by the ordinal numbers $(a)$) are solvable and thus determine a coordinate net. In other words, if these equations are integrable, then the system
\begin{equation}
dx^\alpha = e^\alpha_{(a)}ds_a, \quad \mu = 0, 1, 2, 3, \tag{2.14}
\end{equation}
are invariants of the $\psi \rightarrow S\psi$ transformations. Notably, the Minkowski signature matrix of Eq. (2.4), $\eta^{ab} \equiv \eta_{\alpha\beta}\delta^\alpha_\beta$, and its inverse $\eta^{ab}$, $\eta^{ab} = \delta^\alpha_\beta$, came up here in a purely algebraic way, without even mentioning the Lorentz symmetry and we will use it right away in order to preserve the usual convention about contraction of repeated upper and lower indices. It is also just an algebraic exercise to check that $R^2 = j^a_ja = -J^\alpha J_\alpha = S^2 + P^2 > 0$ and that $j \cdot J = 0$. The latter relation means that if the vector current of the transformed field is of the form $j^a = (R, 0)$ then the axial current can only be of the form $J^\alpha = (0, \vec{J})$ and that it can be further “rotated” to $J^a = (0, 0, 0, R)$. Therefore, at a generic point $x$, the vectors $e_{(0)}^a(x) = J^a / R$ and $e_{(3)}^a(x) = J^a / R$ are the orthogonal timelike and spacelike unit vectors, respectively, and they can be reduced to $e_{(0)}^a \equiv \delta^a_0$ and $e_{(3)}^a \equiv \delta^a_3$. (In what follows, the symbol $(\pm)$ is used in equations that imply such a particular reduction.)

The components of the tensor $M^{ab}$ and its dual $\tilde{M}^{ab}$ are
\begin{equation}
 M^{(0)} = K_\lambda = e^+ \rho_\lambda \psi, \quad \tilde{M}^{ij} = e^{0ijm}K_m,
\end{equation}
\begin{equation}
 \tilde{M}^{0i} = L_\lambda = e^+ \rho_\lambda \psi, \quad \tilde{M}^{ij} = e^{0ijm}L_m. \tag{2.6}
\end{equation}

Because $M^{ab} \tilde{M}^{bc} = (\vec{L} \cdot \vec{K}) \delta^a_3$, these two tensors can be used to build two couples of vectors which are spacelike, orthogonal to $e_{(0)}^a$ and $e_{(3)}^a$, to each other,
\begin{equation}
 E_c = (J^a / R)M^{ab}[\delta^b_c + J^b \vec{J} / R^2] \equiv (0, K_1, K_2, 0), \quad \vec{E}_c = (J^a / R)M^{ab}[\delta^b_c - J^b \vec{J} / R^2] \equiv (0, K_2, -K_1, 0),
\end{equation}
\begin{equation}
 H_c = (J^a / R)M^{ab}[\delta^b_c - J^b \vec{J} / R^2] \equiv (0, -L_2, L_1, 0), \quad \vec{H}_c = (J^a / R)\tilde{M}^{ab}[\delta^b_c + J^b \vec{J} / R^2] \equiv (0, L_1, L_2, 0). \tag{2.7}
\end{equation}

A full set of easily verifiable identities between invariants of the transformations (2.2) is given by
\begin{equation}
 R^2 = j^a_ja = -J^\alpha J_\alpha = S^2 + P^2, \quad \vec{J} \cdot j^a = 0, \quad S^2 - P^2 = \vec{L}^2 - \vec{K}^2, \quad SP = \vec{L} \cdot \vec{K}. \tag{2.8}
\end{equation}

The scalars allow for the following parameterizations,
\begin{equation}
 S = R \cos \Upsilon, \quad P = R \sin \Upsilon, \tag{2.9}
\end{equation}
where both $R$ and $\Upsilon$ are functions of the Dirac field. It is important that the absolute values of $S$ and $P$ do not exceed $R$. A similar assertion is true for the second line of Eq. (2.8),
\begin{equation}
 S^2 - P^2 = \vec{L}^2 - \vec{K}^2 = R^2 \cos 2\Upsilon, \quad 2SP = 2\vec{L} \cdot \vec{K} = R^2 \sin 2\Upsilon. \tag{2.10}
\end{equation}

Concluding the discussion of the algebraic properties of bilinear forms of the Dirac field at a generic point $x$, let us introduce, along with the orthogonal system $e_{(a)}^\beta \{\psi\}$, a reciprocal (in algebraic sense) system $e^\alpha_{(a)} \{\psi\}$,
\begin{equation}
 \sum_{\alpha} e^\alpha_{(a)} e^{(b)}_{\alpha} = \delta^b_a, \quad \sum_{\alpha} e^\alpha_{(a)} e^{(a)}_{\alpha} = \delta^b_3, \tag{2.11}
\end{equation}
will represent lines, which at every point \( x \) have a determinate direction \( c^{\alpha}_{(a)}(\psi) \), and only one line of the congruence \((a)\) passes through each point in spacetime. The tetrad \( c^{\alpha}_{(a)} \) will be a Lorentz vector and a coordinate vector. A change of the vector variables \( x^\alpha = f^\alpha(y) \) in Eq. (2.14) will result in the transformation of the differential
\[
\frac{dx^\alpha}{dy^\gamma} = \frac{\partial x^\alpha}{\partial y^\gamma}, \quad \frac{dx^\alpha}{dy^\gamma} = \frac{\partial x^\alpha}{\partial y^\gamma} = \delta^\alpha_\beta, \quad \text{(2.15)}
\]
and the equation for the same congruence in new coordinates will read as
\[
dy^\gamma = e^{\sigma}_{(a)}(y) ds_a, \quad e^{\sigma}_{(a)}(y) \equiv \frac{\partial y^\gamma}{\partial x^\sigma} \cdot c^{\alpha}_{(a)}(\psi). \quad \text{(2.15)}
\]
A new element here is that tangent vectors depend on the coordinates via the coordinate dependence of the Dirac field which, in its turn, is constrained by the equations of motion. Therefore, the problem of integrability of Eqs. (2.13) - (2.15) cannot be addressed solely within Riemannian geometry; at least some properties of congruences must be controlled by the Dirac equation. For “temporal” and “radial” congruences, the Dirac equation indeed yields a set of constraints with a clear physical meaning. The properties of congruences of angular arcs (including their symmetry), in general, not only explicitly depend on particular solutions of the Dirac equation but there may even be no meaningful holonomic coordinates associated with these arcs. Nevertheless, even keeping such a difficult perspective in mind, let us consider all four tetrads as contravariant vectors of Riemannian geometry.

Traditional approaches assume solving the Dirac equation in a determinate metric field \( g_{\mu\nu}(x) \) of spacetime. The polarization properties of the Dirac field prompt the opposite direction of thinking. Namely, the field \( \psi(x) \) must be the solution of the Dirac equation, which explicitly depends on a resulting metric \( g_{\mu\nu}(\psi(x)) \) given by Eqs. (2.12). In such a context, the Minkowski form of the metric in the local limit is associated not with an imaginable local inertial frame but rather with the algebraic properties of the Dirac field and (complementary to the latter) the hyperbolic character of the Dirac equation. Thus, it is possible to overcome Cartan’s veto in two major points. First, there is no arbitrary coordinates for spacetime (modulo a trivial change of variables). Second, the connections, \( \Gamma^\nu \), are no longer determinate functions of \( x \); they become functions of the Dirac field. In this framework, as is shown below, the hypersurfaces of a constant temporal coordinate naturally emerge; their existence is a prerequisite for the quantization of the nonmass-shell Dirac field in curved spacetime. The proper time slows down in domains of a higher matter density, which points to self-localization as an intrinsic property of the Dirac field. This effect also clarifies the nature of electric charge and of charge asymmetry of the empirically known stable matter. Along with localized matter, there always exists a preferred system of orthogonal congruences determined by the internal polarization structure of physical objects. In general, it can be considered as the net of the anholonomic coordinate system [7].

Let us follow the key idea of intrinsic geometry to associate tensor fields with mutual invariants of tensors and parameters (tangent vectors \( e^\mu_{(a)}(x) \)) of a system of congruences. Furthermore, let us read Eqs. (2.11) and (2.12) as
\[
g_{\nu\mu}(x)e^\mu_{(a)}(x)e^\nu_{(b)}(x) = \eta_{ab} \\
\eta_{ab}e^\mu_{(a)}(x)e^\nu_{(b)}(x) = g_{\nu\mu}(x), \quad \text{(2.16)}
\]
and consider this \( g_{\nu\mu}(x) \), as a primary choice of the spacetime metric. Only by virtue of Eqs. (2.13) can we translate the first of equations Eq. (2.16) into
\[
ds^2 = \eta_{ab} ds^a ds^b = g_{\nu\mu}(x)e^\mu_{(a)}(x)e^\nu_{(b)}(x) ds^a ds^b = g_{\nu\mu}(x) dx^\mu dx^\nu, \quad \text{(2.17)}
\]
and reconcile Eqs. (2.12) (inspired by the algebra of the Dirac matrices) with the measure of length postulated in Riemannian geometry.

When \( e^\mu_{(a)} \) is a vector with the law of transformation (2.13) and \( g_{\nu\mu}(x) \) is a tensor (not necessarily determining a metric) then the covariant derivative \( \nabla^\nu e^\mu_{(a)} \) with respect to \( g_{\nu\mu} \) is also a tensor [8]. Therefore, one can introduce a system of invariants (the Ricci coefficients of rotation of a system of congruences)
\[
\omega_{bca} = e^\mu_{(a)}(\nabla^\nu e^\nu_{(b)} e^\rho_{(c)}) = -\omega_{cba} \quad \text{(2.18)}
\]
For a given \( c \), six parameters \( \omega_{abc} ds \) determine an infinitesimal rotation of the pyramid of tetrads in the “plane” \( (ab) \) when the vertex of the pyramid is displaced by \( ds \) along a line of congruence \( c \). Equation
\[
\nabla^\mu e^\rho_{(b)} = \omega_{bca} e^\rho_{(c)} e^\mu_{(a)} \quad \text{(2.19)}
\]
is the inverse of (2.13). Using Eqs. (2.18) and (2.19), it is straightforward to check that if \( g_{\nu\mu}(x) \) has the form (2.17) then \( \nabla^\mu g_{\nu\mu} = 0 \). Consequently, the vector connections \( \Gamma^\rho_{\nu\mu} \) coincide with the Christoffel symbols of the metric \( g_{\nu\mu} \).

In an ideal geometric world (i.e. when all four holonomic coordinates exist) the necessary conditions for integrability of Eqs. (2.19) are given by the Ricci identities [3]:
\[
(\nabla^\rho \nabla^\rho - \nabla^\lambda \nabla^\mu) e^\rho_{(a)\nu} = e^\nu_{(a)} R^\lambda_{\sigma\nu\mu\lambda} \quad \text{(2.20)}
\]
where \( R^\rho_{\sigma\nu\mu\lambda} \) is the Riemann curvature tensor. These equations can be cast in the form,
\[
e^\sigma_{(a)} e^\nu_{(b)} e^\mu_{(c)} e^\lambda_{(d)} R^\rho_{\sigma\nu\mu\lambda} = R_{abcd}, \quad \text{(2.21)}
\]
where
\[
R_{abcd} \equiv \partial_\rho \omega_{abc} - \partial_\gamma \omega_{abd} \quad \text{(2.22)}
\]
\[
+ \sum_f \eta_{f} [\omega_{fad} \omega_{fbc} - \omega_{fca} \omega_{fbd} + \omega_{abf} (\omega_{fcd} - \omega_{fde})],
\]
\[
\]
is a system of invariants, which is then known as the tetrad representation of the Riemann tensor. Since at least some of the Ricci coefficients of rotation will appear to be functions of the Dirac field, this dependence will be carried through onto the Riemann and Ricci tensors. The Einstein equations for the metric field \( g_{\mu\nu}(x) \) that describes motion of macroscopic objects may appear to be descendants of the constraints stemming from the Dirac equation.

To summarize, if in spacetime, with arbitrarily chosen holonomic coordinates, \( x^\mu \), the Dirac field \( \psi(x) \) is defined and at each point the 16 quantities, \( \epsilon^{(a)}_\nu(\psi(x)) \), are computed (along with the algebraically reciprocal system \( \epsilon^{(a)}_\nu(\psi(x)) \)) then the metric \( g_{\mu\nu}(x) \) of spacetime is given by Eq.(2.10) and the interval by Eq.(2.17). This metric depends on the Dirac field and is not defined \( a \) priori. From the physical perspective, its existence seems to be a privilege of exceptional solutions rather than a rule.

It is important to realize that the material Dirac field defines a system of the unit vector fields \( \epsilon^{(a)}_\mu(x) \) — therefore, the effect of such a matter-induced metric should be equivalent to a long-range interaction between localized objects.

### III. DIFFERENTIAL IDENTITIES FOR TENSORS.

In order to find limitations on the metric of spacetime, which can host the localized configurations of the Dirac field, we begin with the examination of various identities that are consequences of the Dirac equation. The question is, whether differentials of various bilinear forms of the Dirac field, which are considered as the physical observables, can be translated into covariant derivatives of tensors. We use this question as a test of the roots of the discrepancies that could have led to Cartan’s veto. It appears that these discrepancies correspond to the clearly understood physical processes.

Following Fock and Weyl, we postulate that the equation of motion of the Dirac field and its conjugate are

\[
\begin{align*}
\alpha^a D_a \psi &= -i m \rho_1 \psi, \\
\psi^+ D_a^+ \alpha^a &= i m \psi^+ \rho_1,
\end{align*}
\]  

(3.1)

where the covariant derivative \( D_a \psi = (\partial_a - \Gamma_a)\psi \) of the Dirac field is defined in Appendix A. The object \( D_\mu \psi = e^a_\mu D_a \psi = (\partial_\mu - \Gamma_\mu)\psi \) will be used only as a symbol, since it has no clear geometrical meaning. The mass parameter in these equation is \( a \) priori arbitrary. Because the Dirac field has the property of self-localization, every stable localized waveform will determine the corresponding value of \( m \).

A. Identities for vector and axial currents.

From the equations of motion (3.1) and (3.2) one immediately derives two well-known identities. One of them,

\[
D_\alpha j^\alpha = \nabla_\mu j^\mu = \frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} \psi^+ \alpha^\mu \psi] = 0,
\]  

(3.3)

clearly indicates conservation of the vector (probability) current of the on-mass-shell Dirac field, while the second one indicates that the axial current is not conserved,

\[
D_\alpha J^\alpha = \nabla_\mu J^\mu = 2m \mathcal{P},
\]  

(3.4)

and has the pseudoscalar density as a source. The significance of Eq.(3.4) is due to the pseudoscalar density \( \mathcal{P} \) on the r.h.s. Since \( \mathcal{P} \) is localized not less than \( \mathcal{R} \) and the vector \( J^\mu \) is spacelike, the unit axial vector \( e^{(3)}_\mu \) defines the outward radial direction. The existence of such a direction is a distinct characteristic of a localized object.

B. Flux of momenta: not tensors.

Consider now a more complicated object \( T^a_b = i \psi^+ \alpha^a D_b \psi \), the Hermitian part of which is normally regarded as the energy-momentum tensor of the Dirac field\(^2\). Its components are interpreted as the flux of components \( i D_B \) of the momentum in the direction of congruence of lines of the vector current \( j^\mu \). Because the vector current is \( timelike \), this tensor is well-suited to describe the flux of momenta carried by massive particles. The Lagrangian \( L_D \) of the Dirac field vanishes for the on-mass-shell configurations and \( T^a_b \) does not have a diagonal term, \( -L_D \delta^a_b \), which could have been responsible for the flux of momentum in the spacelike direction (e.g., the pressure). Since the spacelike radial direction is controlled by the axial current, we are led to consider another object, the stress tensor \( P^a_b = i \psi^+ \rho_3 \alpha^a D_b \psi \), which accounts for the flux of momenta in the radial direction. For stable localized wave forms, there must be no flux of any observables in the spacelike outward direction. However, if we decide to investigate a particle’s Lorentz contraction as a dynamic process or the decay of a long-lived waveform (considered as a particle), then we are led to consider the spacelike flux of momenta due to the “phase shifts” inside the wave form. Regardless of how adequate this intuitive physical interpretation of \( T^a_b \) or \( P^a_b \) is, or even without any physical interpretation, they both can

---

\(^2\) The reader should not be confused by how the standard name “energy-momentum tensor” is used. In the context of the present work, the invariants \( T^a_b \) and \( P^a_b \) are the auxiliary objects. We are interested only in identities that can be derived from the Dirac equation in tetrad form and then translated, if possible, into the tensor form. Only Hermitian part of these objects enters the equations that allow for a physical interpretation.
be used to derive various useful identities, which allow one to compute the rotation coefficients \( \omega_{ab} \) as functions of the Dirac field and thus constrain the possible metric.

In this section, we study \( T^a_b \) in detail. The stress tensor \( P^a_b \) is studied in Appendix B along the same guidelines.

One would expect that the absolute differential of \( T_{ab} \), being computed according to the Leibnitz rule, will be as follows,

\[
D_c T_{ab} = \partial_c T_{ab} - \omega_{adc} T_{db} - \omega_{bdc} T_{ad} = \nabla_c T_{ab} \quad .
\] (3.5)

If this expectation turns out to be justified then the usual covariant derivative will be immediately reproduced as

\[
\partial_T T_{\sigma \mu} - \Gamma^\nu_{\sigma \lambda} T_{\nu \mu} - \Gamma^\nu_{\mu \lambda} T_{\sigma \nu} = c^\nu_{a} c^\lambda_{b} \psi^e \psi^f \psi^g \nabla_c T_{ab} = \nabla_T T_{\sigma \mu} \quad .
\] (3.6)

Contrary to the expectation of (3.5), the answer reads

\[
D_a [\psi^+ a^a \bar{D}_b \psi] = \partial_a [\psi^+ a^a \bar{D}_b \psi] - \psi^+ [\Gamma^+_e a^e + a^e \bar{D}_e] \bar{D}_b \psi = \partial_a [\psi^+ a^a \bar{D}_b \psi] - \omega_{adc} \psi^+ a^d \bar{D}_b \psi, \tag{3.7}
\]

with the last term of Eq. (3.5) missing, and no hope to recover the full expression (3.6) of the covariant derivative of the tensor! Contracting indices \( a \) and \( c \) we arrive at the expression,

\[
D_a [\psi^+ a^a \bar{D}_b \psi] = \partial_a [\psi^+ a^a \bar{D}_b \psi] + \omega_{abc} \psi^+ a^c \bar{D}_b \psi = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial \nu^a} \left( \sqrt{-g} e^\nu_{[a]} (\psi^+ a^a \bar{D}_b \psi) \right) , \tag{3.8}
\]

which is exactly what one may wish to have as the l.h.s. of a conservation law for the energy-momentum of the Dirac field. The missing term is exactly the one that does not let one interpret equations like \( \nabla_a T^a \) as conservation of anything. However, at the moment, a covariance can not yet be explicitly visible; it may occur that the r.h.s. of an expected conservation law recovers the covariance of the resulting identity as a whole. The r.h.s. must be determined using the equations of motion. Let us first rewrite the l.h.s. of (3.8) as

\[
D_a [\psi^+ a^a \bar{D}_b \psi] = \psi^+ a^a [\bar{D}_a \bar{D}_b - \bar{D}_b \bar{D}_a] \psi + \psi^+ \bar{D}_a a^a \bar{D}_b \psi + D_\psi (\psi^+ a^a \bar{D}_a \psi) - \psi^+ \bar{D}_b a^a \bar{D}_a \psi.
\] (3.9)

By virtue of the equations of motion (and due to the Leibnitz rule) the last three terms exactly cancel out and the final result is

\[
D_a [\psi^+ a^a \bar{D}_b \psi] = \psi^+ a^a [\bar{D}_a \bar{D}_b - \bar{D}_b \bar{D}_a] \psi + \psi^+ \bar{D}_a a^a \bar{D}_b \psi = \partial_a [\psi^+ a^a \bar{D}_b \psi] = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial \nu^a} \left( \sqrt{-g} e^\nu_{[a]} (\psi^+ a^a \bar{D}_b \psi) \right) . \tag{3.10}
\]

Using Eqs. (3.10) and (3.11) to separate the terms with and without derivatives in the commutator and comparing with (3.8) we find that the Dirac equation yields the identity:

\[
\partial_a T_{ab} - \omega_{abc} T_{ac} = \omega_{bac} T_{ac} - \omega_{cab} T_{ac} = \psi^+ \partial a \omega_{ab} \psi , \tag{3.10}
\]

where the commutator \( D_{ab} \) does not contain derivatives of \( \psi \). After moving the term \( \omega_{bac} T_{ac} \) from the right to the left, the l.h.s. becomes, according to (3.5) and (3.6), the system of mutual invariants of a usual covariant divergence of the tensor \( T^a_b \) and congruences \( e^a_{ac} \). It can be transformed either into coordinate form (3.6), which has no explicit dependence on tetrad vectors or into a non-coordinate form. Unfortunately, this coordinate independence of a fragment of identity (3.11) is useless, because there remains an abnormal term \( \omega_{abc} T_{ac} \) on the right. Being translated into a coordinate form, this term becomes \( \nabla_a e^a_{ac} T^c_{ac} \). It explicitly depends on tetrad vectors (on how the coordinate lines are bending).

The abnormal term \( \omega_{abc} T_{ac} \) in Eq. (3.10) enters another identity that follows from the Dirac equation. It arises after contracting indices \( a \) and \( b \) in Eq. (3.7),

\[
D_a [\psi^+ a^a \bar{D}_a \psi] = \partial_a [\psi^+ a^a \bar{D}_a \psi] - \omega_{abc} \psi^+ a^b \bar{D}_a \psi. \tag{3.11}
\]

Eq. (3.11) reveals one more inconsistency, which is similar to the one observed in Eq. (3.8). The quantity \( D_a T^a \) is derivative of the trace of a tensor, i.e. of a scalar. However the result has an additional term with a connection, which is one more piece of evidence that the quantities, \( T^a_a \), are not the invariants of a tensor. By the same token, the l.h.s. of Eq. (3.11) is not a covariant derivative of a scalar.

By virtue of the Dirac equation, the first term on the r.h.s. of (3.11) becomes \( \partial_a [-i m \psi^+ \rho_1 \psi] \). Alternatively, one can immediately use the equations of motion on the l.h.s. and only then differentiate,

\[
D_a [\psi^+ a^a \bar{D}_a \psi] = -i m D_a [\psi^+ \rho_1 \psi] = -i m \partial_a [\psi^+ \rho_1 \psi] + i m \psi^+ \partial_1 \rho_1 + i m \partial_1 \Gamma_1 \psi. \tag{3.12}
\]
Comparing the last two equations and using (A.4) we finally get

$$ \omega_{acb} \cdot T_{ca} = 2mgPRb. \quad (3.13) $$

Using Eq. (3.13), one can then rewrite Eq. (3.10) in a formally covariant form,

$$ \nabla_\sigma T^\sigma_\nu = i\psi^+ \alpha^\mu [D_\mu, D_\nu] \psi + 2mgPR_{\nu}, \quad (3.14) $$

where \( \omega_\nu = e_{(a} \omega_{\nu b)} \) and the commutator \( D_\mu D_\nu - D_\nu D_\mu = [D_\mu, D_\nu] = -e_\rho^\nu e_\mu^\rho D_{ab} \) on the r.h.s. has no derivatives. For the sake of completeness we mention that the imaginary part of \( T^\sigma_\nu \) is a tensor; it is the covariant derivative \((i/2)\nabla_\nu j^\sigma\). Because the vector current is conserved, the imaginary part of Eq. (3.14) is just an identity [3], \( i\nabla_\sigma (\nabla_\nu j^\sigma) = iR_{\sigma\nu j}^\sigma \), where \( R_{\sigma\nu} \) is the Ricci curvature (contracted Riemann tensor of curvature).

An attempt to make \( \omega_\nu = 0 \) leads to the main result of Fock’s paper [3], which was derived entirely in the coordinate representation (using \( D_\mu \) as a well-defined operator) and interpreted, with the reference to the correspondence principle, as the equation of a geodesic line. Since Eqs. (3.9) and (3.13) are nothing but two identities corresponding principle, as the equation of a geodesic line (as it was conjectured in \[ 3 \]) then the term \( \Gamma^\nu_{\mu\sigma} \) is symmetric, which was not an \textit{a priori} requirement. Since, in general, the Ricci coefficients are not zero and the “tensor” \( T_{\sigma\mu} \) is not symmetric (except for a plane-wave solution), we cannot argue that the r.h.s. of (3.13) must be zero for whatever reason. If, in addition to (A.4), we unconditionally required that \( \delta S = \delta P = 0 \), then arriving at (3.13) we would generate controversy.

An \textit{ad hoc} choice of an orthogonal coordinate system (where \( \omega_{abc} = \omega_{acb} \)) can serve only as a crude approximation. To be consistent, we have to replace \( \omega_{acb} T_{ca} \) by \( \omega_{abc} T_{ca} \) in Eqs. (3.11) and Eq. (3.13) simultaneously. Then, the latter can be rewritten as \( \omega_{bca} T_{cb} = -2mgPRb \), so that the symmetric (and only the symmetric) part of \( T_{ba} \) matters. This transmutation indicates that we implicitly employed the approximation of a material point when internal deformations, that are bringing an object into a new state of motion, are discarded. In this case, the unit vector \( e_{(0)} \) plays the role of the 4-velocity \( u^\mu \) of a small object as a whole. Since the first term in brackets in Eq. (3.16) can be dropped, we may write

$$ \frac{\partial}{\partial x^\mu} \left[ \sqrt{-g} \Re(T^\mu_\nu) \right] = e \sqrt{-g} j^\mu F_{\mu\nu}, \quad (3.17) $$

$$ \Gamma^\nu_{\mu\sigma} \Re(T^\mu_\nu) = -2mgPR_{\mu\nu}, \quad (3.18) $$

which is a perfect expression for the energy-momentum conservation complemented by the constraint (3.16). If the equation \( \nabla_\sigma T^\sigma_\nu = 0 \) is considered as a prototype for the equation of a geodesic line (as it was conjectured in [3]) then the term \( \Gamma^\nu_{\mu\sigma} T^\sigma_\nu = \Gamma^\nu_{\mu\sigma} T^\sigma_\nu \) in it is connected with \( N_\mu \) through Eq. (3.18). Depending on the nature of the physical process, this term is responsible either for the gravitational force or for the force of inertia. These forces are real and one cannot set \( N_\mu \) or \( P \) to zero without losing them. An estimate of the coordinate dependence of \( N_\mu \) yields Newton’s approximation for the metric tensor. At large distances, we have \( N \propto 1/r^2 \), as it follows from Eq. (4.13). A startling connection of the field \( P \) with the localization of the Dirac field and origin of its mass is discussed in Sec. LV.

The physical origin of these forces can be understood from another perspective. The r.h.s. of Eq. (3.16) can be rewritten as \( m(\partial_\mu S - D_\mu S) \). The first term accounts only for propagation of the wave form considered as an object. The second term also accounts for the internal polarization of the wave field. The difference between them is a force, which is due to internal polarization. This fact motivates the view on coordinates, as descendants of the polarization structure of the Dirac field, which was proposed in Sec. IIA. Its dynamics are described by Eq. (5.10). An immediate consequence of the existence of an internal dynamic in a localized Dirac waveform is a view of pions as one of polarizations of the Dirac field, which can be resolved, e.g., as a 2γ-resonance (see Appendix B).
IV. DIRAC FIELD AND CONGRUENCES OF CURVES.

In this section, we closely follow the ideas of the intrinsic geometry of Ricci and Levi-Civita as they are presented in the monograph [6]; the metric properties of the spacetime are expressed in terms of rotations of the local coordinate pyramid. The main subject of the analysis are Eqs. (3.13) and (B.4), which are the differential identities that follow from the Dirac equations. Eq. (3.13) is trivially satisfied only for plane waves, i.e., when $P = 0$ and the tensor $T_{ab}$ is symmetric. These solutions are employed in scattering theory and they do not represent particles. In such a context, equations like (3.13) and (B.4) cannot even be derived. Unlike the commonly known identities (6.9) and (6.4), Eqs. (3.14) and (B.1) are not covariant in the sense that they explicitly depend on congruences, which are the physical characteristics of the Dirac field. It appears that these identities impose important limitations on the properties of the metric, which is compatible with the localized solutions of the Dirac equations. These limitations are studied in the following section.

A. Vector current and timelike congruence.

The Ricci coefficients are real-valued and skew-symmetric in the first two indices. The tensor $T_{ab}$ is neither real nor symmetric. The r.h.s. of Eq. (3.13) is real. Therefore, the imaginary part of Eq. (3.13) is just $\text{Im}(T_{ac} - T_{ca}) = D_c(\psi^a \alpha_1 \psi) - D_a(\psi^a \alpha_1 \psi) = \nabla c_j a_n - \nabla a_j c_n = 0$, and it should be considered together with the equation $\nabla a_j a_n = 0$ of the vector current conservation. Since $\nabla a_j b_n$ are the invariants of a true tensor, $\nabla_{\mu} j_{\nu}$, we have two tensor equations,

$$\nabla_{\mu} j_{\nu} - \nabla_{\nu} j_{\mu} = 0 \quad (4.1)$$

and

$$\nabla_{\mu} j_{\nu} = 0 \quad (4.2)$$

The vector field $j^{\mu}(x)$ is globally timelike; its tangent unit vector is $e^{\mu}_{(0)}(x)$, $j^{\mu} = R e^{\mu}_{(0)}$, where $R = \sqrt{j^2} > 0$ is the invariant density of the Dirac (spinor) matter. Therefore, Eq. (4.1) becomes

$$\nabla_{\mu} e^{(0)}_{\nu} - \nabla_{\nu} e^{(0)}_{\mu} + e^{(0)}_{\mu} \partial_{\nu} \ln R - e^{(0)}_{\nu} \partial_{\mu} \ln R = 0. \quad (4.3)$$

Contracting this equation with $e^{(0)}_{(a)} e^{\mu}_{(b)}$, $a, b = 1, 2, 3$ and recalling Eqs. (2.18) we find that

$$\omega_{30 a} - \omega_{30 a} = 0 , \quad a, b = 1, 2, 3 \quad (4.4)$$

which is a necessary and sufficient condition for the congruence $e^{(0)}_{(0)}$ to be normal [8, 9]. Namely, there exists such a function, $T(x)$, that the vector field $e^{(0)}_{(0)}(x)$ is orthogonal to the family of surfaces $T(x) = \text{const}$,

$$\partial_{\mu} T(x) = f(x) e^{(0)}_{(0)}(x), \quad (4.5)$$

where $f(x)$ is a coordinate scalar. Contracting Eq. (4.3) with $e^{(0)}_{(0)}$ we get

$$\partial_{\mu} \ln R = e^{(0)}_{(0)} \partial_{(0)} \ln R - \omega_{000} e^{(0)}_{(0)}, \quad (4.6)$$

where $\partial_{(0)} \ln R = e^{(0)}_{(0)} \partial_{(0)} \ln R$, is the derivative in the direction of the arc $s_0$. Contraction of Eq. (4.3) with $e^{(0)}_{(0)} e^{\mu}_{(a)}$ yields

$$\partial \ln R / \partial s_0 = -\omega_{000} , \quad a = 1, 2, 3, \quad (4.7)$$

which indicates that congruences of lines, defined by the system of equations (2.13), $dx^\mu / ds_0 = e^{(0)}_{(0)}$, must experience permanent bending (acceleration) whenever the invariant density $R(x)$ of the Dirac field is not uniformly distributed. The spatial gradient of $R(x)$ cannot vanish for any localized state. Even more, the congruence of lines of the Dirac current is not a geodesic congruence, since, for geodesic lines, the vector of geodesic curvature would have vanished, i.e., $\omega_{000} = 0$.

Additional information can be extracted from Eq. (4.2). From definition (2.13) it follows that

$$\nabla_{\mu} e^{\nu}_{(0)} = - \partial \ln R / \partial s_0 = \sum_a \eta_{(a)} \omega_{00 a} . \quad (4.8)$$

Hence, we can rewrite (4.6) as

$$\partial_{(0)} \ln R = - e^{(0)} \sum_a \eta_{(a)} \omega_{00 a} - \omega_{000} e^{(0)}, \quad (4.9)$$

which shows that the r.h.s. of Eq. (4.2), which contains only geometric objects, is a component of a gradient. Together with condition (4.4) this constitutes a necessary and sufficient condition that the function $T(x)$, defined by Eq. (4.5), is an harmonic function [8],

$$\nabla = g^{\mu \nu} \nabla_{\mu} \nabla_{\nu} T = 0. \quad (4.10)$$

We may take the parameter $t$ of $T(x) = t = \text{const}$ as a definition of the world time. For the harmonic function, $T(x)$, the conditions of integrability for the system (4.5) of partial differential equations reads as [8]

$$\partial_{(a)} f = - e^{(0)} \sum_a \eta_{(a)} \omega_{00 a} - \omega_{000} e^{(0)} . \quad (4.9)$$

Comparing it with (4.9) we find that $f(x) = R$, so that the world time $t$ and the “proper time” $s_0$ are related by

$$dt = R ds_0 = ds_0 / \sqrt{g_{00}}. \quad (4.11)$$

Hence, we can draw the major conclusion that: The proper time, $s_0$, flows more slowly than the world time, $t$, whenever localized Dirac matter is present. If Dirac matter is in a stable configuration (on mass shell), then there is a well defined time and one can consistently speak of a (quantum) state of the Dirac field.

Since the congruence $e^{(0)}_{(0)}$ appeared to be normal, the hypersurfaces $T(x) = t = \text{const}$ represent space at different times $t$. The three other vectors $e^{(i)}_{(0)}(x)$, $i = 1, 2, 3$
of local tetrad are spacelike, orthogonal to \( e_{(i)}^\mu (x) \) and thus belong to such hypersurfaces (by the definition, \( e_{(i)}^\mu \partial_\mu \mathcal{T} = 0 \)). The interval becomes as

\[
 ds^2 = g_{00}dt^2 + g_{ik}dx^i dx^k. \tag{4.12}
\]

Accordingly, \( e_{(0)}^\mu = (1/\sqrt{g_{00}}, 0) \), \( e_{(0)}^0 = (\sqrt{g_{00}}, 0) \).

Equation (3.3) of the vector current conservation now reads as

\[
 \partial_\mu (\sqrt{-g} e_{(0)}^\mu) = \partial_\mu (\sqrt{-g} g^{00}) = 0. \tag{4.13}
\]

This can be recognized as the condition for the coordinate \( x^\mu \) to be harmonic,

\[
 \Box \varphi = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left( \sqrt{-g} g^{\mu \nu} \partial_\nu \varphi \right) = 0,
\]

which is specified for the normal coordinate \( \varphi = \mathcal{T} = x^0 \) (see, e.g. [12], §41). From (4.13), there follows one more (very intuitive) form of the current conservation,

\[
 \partial_\mu (R \sqrt{-g}) = 0, \tag{4.14}
\]

where \( g = \text{det}[g_{ik}] \) is the determinant of the spatial metric. This form means that when the density \( R \) grows and local time slows down, then the measure of space volume shrinks. Since the variation of \( R \) and time slowdown both are intimately connected with acceleration, the last equation unites them and the Lorentz contraction of a localized object in one physical process. One should not even refer to a spacelike interval between two events on the opposite sides of an elementary object.

B. Axial current and spatial part of metric.

The axial current \( J^\mu \) is spacelike and orthogonal to the vector \( j^\mu \). According to Eq. (3.4), the axial current has a source \( 2m\mathcal{P} \), which is localized together with the invariant density \( R \). Since there is no flux of vector current in this direction (the amount of matter inside a closed surface remains the same), we associate the radial direction \( e_{(3)}^\mu (x) \) with the axial current, \( J^\mu = Re_{(3)}^\mu \). Then Eq. (4.11) takes form

\[
 \nabla_\mu e_{(3)}^\mu + e_{(3)} e_\mu \partial_\mu \ln R = 2m\mathcal{P} / R = 2m \sin \mathcal{T}. \tag{4.15}
\]

On the one hand, by definition,

\[
 \nabla_\mu e_{(3)}^\mu = \sum_a \eta(a) \omega_{3aa} = \omega_{300} - \omega_{311} - \omega_{322}.
\]

On the other hand, according to Eq. (4.7), we have

\[
 e_{(3)}^\mu \partial_\mu \ln R = \partial \ln R / \partial s_3 = -\omega_{300}.
\]

Substituting these expressions into Eq. (4.15) we obtain an extremely important relation,

\[
 \omega_{131} + \omega_{232} = 2m \sin \mathcal{T}. \tag{4.16}
\]

This can be read in different ways. First and foremost, it expresses the curvature of the two-dimensional surface (\( s_1, s_2 \)) of angular coordinates via the local parameter \( \sin \mathcal{T} = \mathcal{P} / \mathcal{R} \) of the Dirac field. Vice versa, once geodesic curvatures \( \omega_{131} \) and \( \omega_{232} \) are known in advance (e.g., from an alleged symmetry, experiment, etc.) then \( \mathcal{T}(x) \) is known as an explicit function of spacetime coordinates and there remains no freedom of “chiral” transformations, like in Eq. (A.9).

Second, the l.h.s. of (4.16) must be a well-defined geometric object (at least when the congruence \( e_{(3)}^\mu \) is normal and the radial coordinate is well-defined). In this case, we must have \( D_a \mathcal{T} = 0 \) because the covariant differential operator \( D_a \) is defined only by its action on the Dirac field. Consequently, by virtue of Eq. (A.8), \( D_a \mathcal{T}[\psi] = \partial_a \mathcal{T} + 2g\mathcal{R}_a \), we have

\[
 2g\mathcal{R}_a = -\partial_a \mathcal{T}, \tag{4.17}
\]

i.e. the field \( \mathcal{R}_a \) must be a gradient. If the congruence \( e_{(3)}^\mu \) is not normal, then any symmetry of any explicit solutions of the Dirac equation with respect to areas \( s_1 \) and \( s_2 \) should be considered a dynamical internal symmetry.

Third, for a concave surface the curvature is positive so that \( 0 < \mathcal{T} < \pi / 2 \). For the normal orthogonal spherical coordinates we have \( \omega_{131} = \omega_{232} = 1/r \) and if such a coordinate system were possible we would immediately know that

\[
 \mathcal{T}[\psi] = \arcsin(1/mr), \quad mr > 1 \quad 2g\mathcal{R}_3[\psi] = -\partial_3 \mathcal{T} = \frac{1}{r\sqrt{mr^2 - 1}}. \tag{4.18}
\]

Obviously, this simple formula cannot be exact; rather it predicts the correct asymptotic behavior at large distances.

Fourth, the condition \( \sin \mathcal{T}(x) < 1 \) defines the mass parameter \( m \) as the upper limit of a possible curvature, which is, in fact, a definition of mass from the perspective of the internal structure of a Dirac particle. (In spherical case we would have \( mr > 1 \); the radius must exceed the Compton length!) This result is also in agreement with the kinematic Lorentz contraction of special relativity. An accelerated particle is Lorentz contracted and the maximal curvature become \( \propto 1/\sqrt{1 - v^2} \).

In order to facilitate further analysis of the real part of Eq. (4.15), let us rewrite its l.h.s in terms of the axial current. Let us use the dual representation of the axial current as \( e^{stua} J_a = i\psi^+ \alpha^a \rho_1 \alpha^b \rho_1 \alpha^u \psi \). \( (s, t, u, \neq) \), and differentiate it. The result reads as

\[
 D_a e^{stua} J_a = i \sum_{u \neq s,t} D_a (\psi^+ \alpha^a \rho_1 \alpha^b \rho_1 \alpha^u \psi). \tag{4.19}
\]

If we extend here the sum over all values of \( u \) (this sum vanishes by virtue of the equations of motion) and subtract the terms with \( u = s \) and \( u = t \), the result will be

\[
 D_a e^{stua} J_a = -i\psi^+ \alpha^a \bar{T}^a \psi + i\psi^+ \alpha^a \bar{T}^a \psi -i\psi^+ \bar{T}_s^a \alpha^a \psi + i\psi^+ \bar{T}_t^a \alpha^a \psi,
\]
where the r.h.s is four times the anti-symmetric Hermitian part of the energy momentum tensor. Therefore, the real part of Eq. (4.13) reads as

\[(1/4)\omega_{ab}e^{acst} \cdot \nabla_s J_t = 2mgPN_b, \quad (4.20)\]

and can be immediately recognized as the dual to Eq. (1.5), derived for the stress tensor,

\[(1/2)\omega_{ab}e^{bcst}\nabla_c J_a = -2gmSN_b. \quad (B.5)\]

These two equations clearly indicate that any motion of the Dirac field follows the path of a helix. The underlying reason for such a motion is the non-vanishing curl of the spacelike axial current. The boost \(\omega_{0ab}\) in the direction \(\dot{s}_i\) is inevitably accompanied by the spatial rotation \(e^{jk}\omega_{jka}\) in the plane perpendicular to \(s_i\). In plain words, the Dirac field cannot be accelerated without causing a rotation thus behaving as a (relativistic) system of inertial navigation.

In Eqs. (4.20) and (B.5), \(\nabla_s J_t = e^{\mu}_s e^{\nu}_{(t)} \nabla_{\nu} J_\mu\). Since \(J_\mu = R e_\mu^{(3)}\), we further have

\[D_s J_t = Re^{\mu}_s e^{\nu}_{(t)} \nabla_{\nu} e_\mu^{(3)} + \delta_3^\mu \delta^a_\mu \partial_a R = R[\omega_{3ts} + \delta_3^\mu (\partial \ln R/\partial s_a)].\]

Finally, using Eqs. (4.17) and (4.18), which define \(\omega_{0ab}\) and \(\omega_{0aa}\) as the functionals of the Dirac field, we arrive at

\[(1/4)\omega_{ab}e^{acst}[\omega_{3ts} - \delta_3^t\delta_0^s - \delta_0^s \delta_3^t]\omega_{0aa} = m \sin \Upsilon \cdot 2gN_b, \quad (4.21)\]

\[(1/2)\omega_{sab}[\omega_{3ts} - \delta_3^t\delta_0^s - \delta_0^s \delta_3^t] = -m \cos \Upsilon \cdot 2gN_b. \quad (4.22)\]

At this point, we can conclude that the parameters \(N_b\) in the connection \(\Gamma_\mu [4,3]\) are totally defined by the bending of the system of congruences. Despite that fact that the Lagrangian for the Dirac equations (3.1) and (3.2) includes the term \(J^\mu N_\mu\), which can be interpreted as an interaction between the axial current and the field \(N_a\), it cannot be viewed as an independent field that is governed by an additional equation of motion. (Otherwise, such equations must be invented, which we, so far, tried to avoid.)

**C. The case of the normal radial coordinate.**

**Qualitative consequences of the localization.**

Even for the localized waveforms, the existence of the surface of a constant distance from a center is not given gratis. Such a surface must be orthogonal, at every point, to the tangent vectors \(e_\mu^{(3)}\) of the congruence of lines of the axial current. Unlike the previously studied case of the timelike congruences \(e_\mu^{(1)}\), the corresponding conditions for integrability do not universally follow from the equations of motion. Most likely, the radial coordinate cannot be normal. However, sometimes (mostly for long-lived particles) empirical data may hint that such a normal hypersurface of a constant distance \(s_3\) from a center, which is spanned by the “angular” arcs \((s_1, s_2)\) with tangent unit vectors \((2.7)\) may be a good approximation. This is what we intuitively expect in a one-body problem and we have to verify that this assumption is consistent with the equations of motion and the established earlier constraints. In what follows, we consider Eqs. (4.17) as the criterion of the spherical symmetry and try to profit from the fact that Eqs. (4.21) and (4.22) significantly simplify under the assumption that the congruence \(e_\mu^{(3)}\) is a normal congruence (which, possibly, can be a first approximation in a sequence of iterations). This will also allow us to qualitatively understand the trends in critical behavior of the tetrad vectors near the limit surface, \(\sin \Upsilon \to 1\), and rediscover some well known properties of matter (which cannot be done in a picture of matter as plane waves).

Since the congruence \(e_\mu^{(3)}\) is set normal, there should exist a function \(\mathcal{L}(x)\) such that

\[\partial_\mu \mathcal{L}(x) = \zeta(x)e_\mu^{(3)}(x), \quad (4.23)\]

where \(\zeta(x)\) is a coordinate scalar. The hypersurfaces \(\mathcal{L}(x) = \tau = \text{const}\) are the surfaces of radius \(r\), i.e.,

\[dr = \zeta ds_3 = ds_3/\sqrt{3s_3}. \quad (4.24)\]

From the integrability condition for Eq. (4.23) it is straightforward to derive the equations (which are similar to equations for the the function \(f(x)\) (cf. 4.3))

\[\partial_\mu \ln \zeta = e_\mu^{(3)}\partial_\mu \ln \zeta - \omega_{a33}e_\mu^{(3)}(a), \quad (4.25)\]

\[(\partial \ln \zeta/\partial s_a) = \omega_{a33}, \quad (a = 0, 1, 2); \quad (4.26)\]

but, we have no constraint that would express \(\zeta\) as a function of the Dirac field. For the normal congruence \(e_\mu^{(3)}\) we have \(\omega_{0ab} = \omega_{3ba}\) for \(a, b \neq 3\). \(a \neq b\), as a necessary and sufficient condition [3, 9] and, consequently, the first term in brackets in Eqs. (4.21) and (4.22) simplifies, \(\omega_{sab}e^{acst} = -\omega_{a33}\omega_{3st} + \omega_{a33}\omega_{33} + \omega_{230}\omega_{233}\) and \(\omega_{acb}e^{acst} = \omega_{126}\omega_{33} + \omega_{236}\omega_{33} + \omega_{016}\omega_{233}\). In the second term, the sum includes only \(a, c = 1, 2\). Because
we assume a stable object, the third term in brackets cancels, \( \partial \ln(R\zeta) / \partial s \approx 0 \). Hence, the system of Eqs. (4.21) and (4.22) can be cast as

\[
\begin{align*}
(\omega_1\omega_{02} - \omega_2\omega_{01}) &= 2mgR_0 \sin \Upsilon, \\
(\omega_1\omega_{31b} + \omega_2\omega_{32b}) &= 2mgR_0 \cos \Upsilon,
\end{align*}
\]

(4.27)

where

\[
\omega_j = \frac{\omega_{j0} + \omega_{j33}}{2} = \frac{\partial \ln(\zeta/R)}{\partial s_j}, \quad j = 1, 2.
\]

Then, Eqs. (4.27) with \( b = 0, 1, 2 \) (and \( \mathbb{R} = \mathbb{R}_1 = \mathbb{R}_2 = 0 \)) yield a set of six equations,

\[
\omega_1 = -\frac{\omega_{30}}{\omega_{310}} = \frac{\omega_{011}}{\omega_{021}} = \frac{\omega_{012}}{\omega_{022}} = \frac{\omega_{321}}{\omega_{131}} = \frac{\omega_{232}}{\omega_{312}} = \pm 1,
\]

(4.28)

where the rightmost equation immediately follows from \( \omega_{312}^2 = \omega_{311}^2 \omega_{232}^2 = 1/r^2 \) and \( \omega_{311} = \omega_{232} = 1/r \). In spherical case, \( \mathbb{R}_3 \) is given by Eqs. (4.18) and, consequently,

\[
2\omega_1\omega_{023} = \frac{1}{r^2 \sqrt{m^2 r^2 - 1}}, \quad \omega_1(\omega_{313} \pm \omega_{323}) = \frac{1}{r^2},
\]

\[
\omega_2 = \pm \omega_1, \quad \omega_{013} = \mp \omega_{023}, \quad \omega_{312} = \omega_{321} = \pm 1/r.
\]

(4.29)

As one could expect, in the case of normal angular conjugu-

tes, there is a full symmetry between conjugu-

ges of arcs \( ds_1 \) and \( ds_2 \). At large distances we generally have \( mr \gg 1 \) so that spatial rotations dominate. **Vice versa**, near the inner boundary \( mr = 1 \) the boosts \( \omega_{013} ds_3 \) and \( \omega_{023} ds_3 \) in tangent directions (as well as boosts \( \omega_{031} ds_1 \) and \( \omega_{032} ds_2 \) in radial directions) become infinite. When, starting from a generic point \( x_0 \), we move along lines of conjugu-
ges \( e_{(1)}^\mu (x) \) approaching \( r \sim m^{-1} \), then local tetrad rotate (with respect to the \( e_{(1)}^\mu (x_0) \)) in such a way that all directions become nearly lightlike, so that the tangent velocities \( v_i = s_i \rightarrow c \). These observations explain the formally derived inner boundary \( r = 1/m \) (generally, \( \sin \Upsilon = 1 \)) of the Dirac particle as the caustic of the lines of the Dirac currents.

In fact, we have two interconnected mechanisms of the time slowdown (due to the amplified \( R \) and because the vector currents tend to approach the lightlike directions), which cannot be separated. From the perspective of an "external observer", the time flow literally stops at the critical surface of a stable Dirac waveform. Therefore, the sharp interaction of the deeply inelastic scattering always resolves an apparently static object in a random config-

igration determined by the foregoing causal evolution (cf. discussion of evolution equations of QCD in Ref. [12]). Certain patterns of symmetry observed in such processes most likely correspond to the symmetry of (the critical points of) projection of the actual currents onto a surface determined by the collision axis, the collision plane, etc. These patterns well may have very little to do with the internal dynamics of the stable waveform.

When \( mr \gg 1 \), we generally have sin \( \Upsilon \rightarrow 0 \) and, according to Eq. (2.10), the pseudoscalar density nearly van-

ishes and \( S \approx R \). Magnetic polarization of the Dirac field is greater than the electric one, \( |\hat{L}_1| \geq |\hat{K}| \). **Vice versa**, at the shortest possible distances, when \( \Upsilon \rightarrow \pi/2 \), the Lorentz boosts play a major role. Accordingly, the pseudo-

doscalar density \( \mathbb{P} \approx R \) is large and electric polarization is dominant, \( \hat{K}^2 - \hat{L}^2 \approx R^2 \gg 1 \). These two sectors are separated by the two-dimensional surface \( \Upsilon[\psi] = \pi/4 \). When \( \Upsilon[\psi] < \pi/4 \) (large distances) an appropriate choice for \( e_1^\mu(1) \) and \( e_2^\mu(2) \) will be vectors \( H_i \) and \( \hat{E}_i \), \n
defined by a system of boosts, which are large along with the pseudoscalar density.

In fact, this domain is also responsible for the magnetic moment of the Dirac particle. Indeed, using the classical formula for the magnetic moment, \( \mu = (e/2c)[r \times v] \), with the limit values, \( r = \lambda = \hbar/mc \) and \( v = c \), which follow from the Eqs. (4.20), we obtain

\[
\mu = \frac{e}{2c} \cdot \frac{\hbar}{mc} \cdot c = \frac{eh}{2mc}.
\]

(4.30)

This is a well-known result corresponding to the gyro-

magnetic ratio \( g = 2 \).

The impossibility to introduce a normal orthogonal co-

ordinate system in the presence of the axial potential \( \mathbf{K} \) in the equations of motion is explicitly illustrated in Appendix C. An attempt to separate the angular variables in the Dirac equation in the presence of only the radial component \( \mathbb{R}_r(r) \) is made. In this case, the radial coordinate, \( r \), is a well defined normal coordinate. It appears that even in such a simplest case there are no operators with eigenvalues of angular momentum that commute with the Hamiltonian. At the same time, angular variables can be explicitly separated in the equations of motion. The only possible explanation of this fact is that the formally introduced angles do not represent arcs of the usual spatial angular coordinates. There is no reason to require that solutions of the Dirac equations must be single-valued along these arcs. However, equations (C.8) for angular functions are clearly the equations for spherical harmonics. These harmonics can be interpreted only as elements of an internal symmetry in the space of polar-

izations of the Dirac field. In order to find out what can be the non-geometric integrals of motion the angular and radial functions must be studied together.

V. NONLINEAR DIRAC EQUATION.

In this section, following the programme outlined in Sec. II B, we will incorporate the nonlinear effects, which so far were found as constraints, into the Dirac equation. Following Fock [3], let us rewrite the operator \( \alpha^a \Omega_a \) as

\[
\alpha^b \Omega_b = (1/2)\omega_{aca} \alpha^c - (i/4)\epsilon_{abcd} \omega_{acbd} \alpha^d.
\]

(5.1)
Then, the Dirac equation reads as
\[ \alpha^b \left[ \frac{\partial}{\partial s_b} + ieA_b + ig\rho_3 \nu_b - \frac{1}{2} w_{ab} - \frac{i}{2} w_b \rho_3 \right] \psi = -im\rho_1 \psi, \]  
(5.2)
where \( \nu_b \) and \( w_b = -(1/2)e_{abcd}w_{ad} \) are the sets of invariants; the latter differ from zero whenever spacetime does not admit a coordinate net of all normal congruences. In general, the invariants \( \nu_b \) do not vanish and they are complementary to the invariants \( \nu_b \) given by Eqs. (4.21) and (4.22). [Since congruence \( \epsilon_{(0)}^a \) is normal, we have \( w_1 = -\omega_{230}, w_2 = -\omega_{310}, w_3 = -\omega_{120} \).]

According to Eqs. (4.17) and (4.18), each sum \( \sum_a \eta(a) \omega_{ab} \) includes either the terms \( \omega_{00} = \partial \ln \mathcal{R}/\partial s_i \) (\( i = 1, 2, 3 \)) or \( \sum_i \omega_{0i} = \partial \ln \mathcal{R}/\partial s_0 \), all of which are bilinear forms of \( \psi^+ \) and \( \psi \) and, geometrically, are the geodesic curvatures. At first glance, the presence of these curvatures makes the Dirac equation extremely non-linear. This genuine nonlinearity, however, can be effectively alleviated after the following “normalization” of the Dirac wave function. If we observe that
\[ \frac{\partial \psi}{\partial s_a} - \frac{1}{2} \frac{\partial \ln \mathcal{R}}{\partial s_a} \psi = \sqrt{\mathcal{R}} \frac{\partial}{\partial s_a} \left( \frac{\psi}{\sqrt{\mathcal{R}}} \right), \]  
(5.3)
and assume Eq. (4.17) be true, then we arrive at a much simpler equation for the normalized function \( \xi = \psi/\sqrt{\mathcal{R}} = (g_{00} |\psi|)^{1/4} \psi : 

\[ \left[ i \frac{\partial}{\partial s_0} - eA_0 - \frac{1}{2} \rho_3 [w_0 + \partial_0 Y] - \sum_{i=1}^3 \alpha^i \left( i \frac{\partial}{\partial s_i} + i \frac{k_i}{2} - eA_i + \frac{1}{2} \rho_3 (w_i + \partial_i Y) \right) - m\rho_1 \right] \xi = 0, \]  
(5.4)
where \( k_i = \sum_{j \neq i} \omega_{ij} \). The nonlinear Dirac equation \( (5.2) \) now looks like a linear equation for the normalized function \( \xi \). Once \( m^{-1} \) is accepted as a measure of length, this equation is dimensionless and does not change under a similarity transformation. At the first glance, the term \( \rho_3 \partial_0 Y \) in it is always nonlinear; however, the constraint \( k_3 = 2m \sin \sigma = 2m(\xi^+ \rho_2 \xi) \) (cf. Eq. (4.10)) eliminates the non-linearity whenever the curvature \( k_3 \) can be determined as a function of coordinates from geometric considerations. In the one body problem this curvature always has the meaning of the inverse radius of the enveloping convex surface. At least at large distances (at the scale of \( 1/m \)) we have \( Y \propto 1/mr^2 \) and \( \partial_0 Y \propto -1/mr^3 \), which brings in a singular potential \( \propto 1/r^2 \) into the Dirac equation and a Newton’s force into Eqs. (3.13) - (3.16). Such a construct is equivalent to the Newton’s approximation in general relativity and it may serve as the first step of an iterative procedure for the Dirac equation.

In general, there is no direct connection between the radius of curvature and the distance to any distinct point inside a localized object; the gradients \( \partial \mathcal{P} \) can be arbitrary large. Most likely, the solutions of Eq. (5.2) have multiple caustics where the invariant density \( \mathcal{R} \) is large and \( \mathcal{P} \) dominates to the extent that \( \mathcal{R} \approx \mathcal{P} \). So far, we did not find in mathematical literature regular methods to study equations like (5.4). The methods of contact geometry [14] seem to be most relevant.

The most important source of the nonlinearity of the Dirac equation resides in that fact that evolution, in terms of proper time \( s_0 \), has a different rate at different points of the localized object; the rate of evolution, \( \partial/\partial s_0 \), along this time cannot yield anything like the energy of this object as a whole. Fortunately, we have proved that a stable object does have a well defined hypersurface of a constant world time \( t \). Therefore, a meaningful evolution scale for a stable object as associated with the physical time \( t \). According to Eq. (4.11), we have \( dt = \mathcal{R} ds_0 \). Hence,
\[ \left[ i \mathcal{R} \frac{\partial}{\partial t} - eA_0 - \frac{1}{2} \rho_3 [w_0 + \partial_0 Y] - \sum_{i=1}^3 \alpha^i \left( i \frac{\partial}{\partial s_i} + i \frac{k_i}{2} - eA_i + \frac{1}{2} \rho_3 (w_i + \partial_i Y) \right) - m\rho_1 \right] \psi/\sqrt{\mathcal{R}} = 0, \]  
(5.5)
and now this equation has a scale fixing factor \( \mathcal{R} \) in front of the \( \partial/\partial t \). This effect is a major one because it corresponds to a clearly understood physical effect, refraction of the Dirac waves. It is of the same physical origin as self-focusing in nonlinear optics and acoustics or deflection of light in the gravitational field of a star. In fact, \( \mathcal{R} \) plays a role of “refractive index” depending on the amplitude. Since the phase velocity decreases with increasing amplitude, the field tends to auto-localize. This mechanism of concentration is the most distinctive property of gravity which may signal its role in matter formation from fields at all scales. The spatially uniform solutions of the Dirac equation just cannot be stable.

VI. LOCALIZED DIRAC MATTER, ELECTRIC CHARGE AND RADIATION.

Our major perception regarding the vacuum is the absence of matter. Since matter inevitably is localized, this means that in the vacuum, \( \mathcal{R} \) is constant and the spacetime metric of the Lorentz vacuum has \( g_{00} = c^2 = 1 \). At
the same time, we have Eq. (4.11),
\[ dt = R ds_0 = ds_0/\sqrt{g_{00}} . \] (4.11)
Therefore, the empirical $g_{00} = 1$ of an empty space corresponds to $R = 1$. This state cannot be stable. Due to a very special nonlinearity of Eq. (5.2) Dirac waves tend to refract towards domains where $R - 1 > 0$ amplifying $R$ there until some saturation level (or caustic) is reached and an external boundary is formed. The opposite trend must be observed in domains where $R - 1 < 0$; the Dirac waves tend to escape them. This conjecture can be phrased more precisely as:

"Identification of the sign of $(R - 1)$ with the sign of electric charge leads to a dynamic picture of an empirically known charge-asymmetric world in which stable positively charged on-mass-shell Dirac objects are highly localized (and presumably heavy) while negatively charged objects tend to be poorly localized (and presumably light)."

The best prospect of this idea is that these objects are the protons (or nuclei) and the electrons of the real world. When electric forces come into play, the electrons become somewhat localized around heavy objects, thus forming electrically neutral matter. For atomic electrons, which are smeared over distances much exceeding the Compton length and held weakly localized near nuclei by electric forces, the effect of $R - 1 \ll 1$ on the metric must be negligible. However, the view of a vacuum as a classical Dirac field with the standard level $R = 1$ and propagating in it waveforms, instead of an ensemble of quantum oscillators with an unbound spectrum that interact with plane waves, may alleviate the problem of ultraviolet divergences of the standard perturbation series and its renormalization.

Our major conclusion about the nature of localization is drawn from the analysis of the on-mass-shell waveforms. However, it relies on basic properties of the wave propagation so that it seems reasonable to apply them to at least the long-lived waveforms. From this perspective, any positively charged particle should have a slightly longer lifetime and be more localized than its negative partner. While the proton is small and stable, the antiproton should not have an as well defined outward boundary as the proton has. The lifetime of the anti-hydrogen may not be long even when it is completely isolated from normal matter. We refrain here from speculating about the possible mode of its decay.

The next important question is the interaction of the localized Dirac waveforms with the electromagnetic field. Empirically, the field $A_\mu$ in the connection (A.3) is the electromagnetic field, which is responsible for the Lorentz force. The system of invariants $R_\mu$ in $\Gamma_\alpha$ is determined either by geometric properties of congruences like (4.16) or by nonlinear constraints (4.21) and (4.22). As long as the field $R_\mu$ is a gradient, the r.h.s. of Eq. (5.3) is a system of invariants, which include, except for the geometric terms, the term $\epsilon_\mu^a F_{ab}$. In the coordinate representation (3.12), this part is translated into $\epsilon_\mu^a F_{\mu\nu}$, and this is the only term on the r.h.s. of the real part of Eq. (3.9). This term is known as the Lorentz force of the field $F_{\mu\nu} = \nabla_\nu A_\mu - \nabla_\mu A_\nu$ acting on a charged particle. The vector field $A_\mu = \epsilon_\mu^a A_a$ originates from the connection $\Gamma_\alpha$ and thus is an external field.

For the real part of the energy-momentum tensor, $T_{ab} = (i/2)[\psi^+ a^a \bar{D}_b \psi - \psi^+ \bar{D}_b a^a \psi]$, and in the artificial normal coordinates, discussed in Sec III, we had equation (3.17),
\[ \frac{\partial}{\partial x^\mu} [\sqrt{-g} T^\mu_\nu] = e \sqrt{-g} \epsilon_\mu^a F_{\mu\nu} . \] (6.1)
As it should be, the kinematic acceleration $w^\mu = \epsilon_\mu^a \nabla_\nu \epsilon_\nu^a$ of a charged particle does not include a gravitational part (see [12], §63).

The field $F_{\mu\nu}$ is a tensor and it satisfies the identity (the first couple of Maxwell equations),
\[ F_{\mu\sigma\nu} \equiv \nabla_\mu F_{\sigma\nu} + \nabla_\sigma F_{\mu\nu} + \nabla_\nu F_{\mu\sigma} = 0 . \] (6.2)
The divergence of this tensor, $\nabla_\mu F_{\mu\nu}$, can be cast in the following form,
\[ - \Box F_{\mu\nu} + R^\kappa_\mu F_{\kappa\nu} - R^\kappa_\nu F_{\kappa\mu} - R_{\mu\nu\kappa\sigma} F_{\kappa\sigma} = (\nabla_\mu [\nabla_\nu F_{\sigma\nu}] - \nabla_\nu [\nabla_\sigma F_{\mu\sigma}]) , \] (6.3)
where $\nabla_\sigma = g^{\alpha\lambda} \nabla_\lambda$. The l.h.s. of this equation is the wave operator for the field $F_{\mu\nu}$. The two terms in the r.h.s. can be transformed further after we postulate the second couple of Maxwell equations,
\[ \nabla_\mu F_{\mu\nu} = e J^\nu , \quad \nabla_\mu J^\mu = 0 , \] (6.4)
with the Dirac’s vector current $e J^\mu$ in the r.h.s. This amounts to the second (in fact, independent) definition of electric charge as the divergence of the electric field, and a few reservations must be made. First, without a good reason, the same coupling constant, as in the connection $\Gamma_\alpha$, is postulated. Only gauge invariance as an independent principle can provide for an unquestionable equality of these constants. Second, the Dirac field in Eq. (6.5) is assumed to be a stable configuration and the field $\nabla_\mu$ is considered a gradient. Otherwise, the conservation of the vector current and the non-conservation of the axial current will conflict with Maxwell equations. Third, the interaction between the electromagnetic field and the spacelike axial current or pseudoscalar density, which are present in Eq. (6.10), and affect the balance of momenta inside the Dirac object, are disregarded. Only those interactions that are responsible for the change of timelike components of the momentum of a stable particle are allowed to be sources of electromagnetic field. Then Eq. (6.2) becomes,
\[ - \Box F_{\mu\nu} + R^\kappa_\mu F_{\kappa\nu} - R^\kappa_\nu F_{\kappa\mu} - R_{\mu\nu\kappa\sigma} F_{\kappa\sigma} = e Q_{\mu\nu} , \quad Q_{\mu\nu} = (\nabla_\mu J_\nu - \nabla_\nu J_\mu) . \] (6.5)
where $Q_{\mu\nu}$ is a convenient intermediate notation.

The crucial question is, if $J_\mu$ in this equation is (or can be) the current $j^\mu$ of Eq. (6.1), which was derived
as a consequence of the Dirac equation with the potential \( A_c \) in the connection \( \Gamma_c \). Evidently, the answer is no because then, according to Eq. (6.1), we must have \( Q_{\mu \nu} = 0 \). Therefore, an on-mass-shell object cannot be a source of an electromagnetic field that may result in the Lorentz force of self-interaction. The definition (6.4) must be complemented by Eq. (6.1), which then determines the measured acceleration of another charge that senses the field of the first one. The potential \( A_c \) in the Dirac equation that describes a localized object must be “external”; its source can be only the current of another object. The problems of mass and charge (including the problem of electromagnetic mass) are not a single-body problem. One further implication of this observation is that if one can simultaneously identify two localized objects, then the Dirac fields of these objects cannot overlap in space-time. Since, for a stable object, one cannot set \( e_j \nu = \nabla_\mu F^\mu_\nu \) in the expression for the Lorentz force, it is also impossible to express this force as the divergence of the energy-momentum tensor, i.e., as \( e_j \nu F^\mu_\nu = \nabla_\sigma F^{\sigma \mu} F^\mu_\nu = -\nabla_\mu \Theta^\nu_\mu (F) \) and claim that \( \Theta^\nu_\mu \) is the energy density of the electromagnetic field. This is not surprising since one cannot convert this energy into any other form without a second object.

For an isolated stable charged object the condition that it cannot interact with its own electric field means that the only “potential” in the wave equation [6.2] is \( \nabla^2 \propto 1/r^2 \). The wave equation with such a steep potential may have a strongly localized solution regardless of the sign of this potential. The difference from the commonly studied cases is that now the signs of this potential for the left- and right-handed components are opposite and that, for a stable wave form, the region \( mr < 1 \) is cut off by Eq. (6.10) as unphysical. Therefore, a precursor of a localized state is present in the Dirac equation even before the universal nonlinear mechanism of the time slowdown takes over.

The field which is measured via the Lorentz force (6.1) always is a “field in vacuum”. The wave equation (6.5) for the \( F_{\mu \nu} \) from Eq. (6.1) is a homogeneous equation, which depends on the Dirac field of (6.1) only parametrically, via derivatives of \( g_{\mu \nu} (\psi) \) in the Riemann tensor. The electromagnetic sector of the theory turns out to be entirely in the form required by Riemannian geometry. This sector is responsible for the propagation of signals that are used to synchronize macroscopic clocks (which is unambiguous only in special relativity). A stable wave-form of the Dirac field with \( Q_{\mu \nu} = 0 \) neither interacts with its own electromagnetic field nor can it emit an electromagnetic field, as a signal, by itself. This is yet further evidence that the object is in a stationary state. This property is in line with the well known fact that equation of the Coulomb’s law is a constraint and not an equation of motion. The longitudinal part of the electromagnetic field does not propagate; the Coulomb field is simultaneous with its source. The field of radiation emerges only when this simultaneity is lost. This is yet another view of the realm of the well-known phenomena of transient processes where the proper field of a particle is truncated. What if it occurred possible to trace an observed radiation field back to the current in the interior of the localized object (so that \( J_\mu = j_\mu \)) and, e.g. by a precise analysis of radiation, to learn that \( Q_{\mu \nu} \neq 0 \) there? Then Eq. (6.1) must be replaced by the second equation of (6.5). Proceeding as previously, we get

\[
\nabla_\mu e^\nu_{\nu} - \nabla_\nu e^\nu_{\mu} + e^\nu_{\nu} \partial_\mu \ln \mathcal{R} - e^\nu_{\nu} \partial_\mu \ln \mathcal{R} = -(1/\mathcal{R}) Q_{\mu \nu} .
\]

(6.6)

Contracting this equation with spacelike \( e^\nu_{(i)} e^\mu_{(j)} \) \( (i, j = 1, 2, 3) \) and recalling Eqs. (2.18) we find that

\[
\omega_{0ij} - \omega_{0ji} = -(1/\mathcal{R}) Q_{ij} .
\]

(6.7)

If \( Q_{ij} \neq 0 \) starting from some time moment \( t_0 \), then at \( t > t_0 \) the congruence \( e^\nu_{(0)} \) of lines of the vector current cannot be a normal congruence. The family of spacelike surfaces \( t = \text{const} \), orthogonal to the vector current, vanishes. This means that Eq. (4.13) cannot be obtained and Dirac field cannot form a stable object. The electromagnetic fields produced by such an object are not just longitudinal (Coulomb) fields and the object must start to radiate solely because the electromagnetic field around it is not simultaneous with its source. Since the Dirac equation is of the hyperbolic type, the changes of the Dirac field must propagate also, having a light cone as a leading wave front.

Contracting Eq. (6.6) with \( e^\nu_{(0)} e^\mu_{(1)} \) we obtain another equation,

\[
\omega_{100} = -(\partial \ln \mathcal{R}/\partial s_i) - (1/\mathcal{R}) Q_{i0} .
\]

(6.8)

---

3 From perspective of the second quantization, when eigenfunctions of the Dirac equation are associated with different states, this means that the Fock operators of these two states must anticommute!

4 It is important to emphasize that the l.h.s. of Eq. (6.5) is given in terms of measurable electric and magnetic fields; therefore we indeed are dealing with a signal that may have leading and rear fronts. Since the world time \( t \) (1.10) is a harmonic function it can be discontinuous along characteristics. The time of emission of a photon, in principle, is not defined. A photon does not constitute a signal.

5 This is yet another way to view two seemingly different phenomena, the Meissner effect and precession of the spin in magnetic field. In order to be in a stable quantum state, the superconductor expels magnetic field from its interior or confines it into vortices, thus defining a common time across its whole volume. In the same way, electron with the magnetic moment (4.30), being placed in magnetic field, moves in precession with the Larmor frequency. Therefore, in rotating frame the magnetic field vanishes and the electron still can have the same world time across its volume (staying in a certain quantum state).
that accounts for the effect of the electric field, which adds a boost in the direction of the congruence $e^{\mu}_{(\nu)}$. Interaction with the electric field alone (which can be the case only when this field is longitudinal) does not destroy the hypersurfaces of constant time of a localized object, which allows it to stay intact. The most important effect of acceleration in an electric field is altering the shape of a charged object which leads to an increase of its internal energy and of the local charge density.

Referring to the above qualitative analysis we may go further and discuss a qualitative picture of some transient processes. If the Dirac waveform is not stable (as is in the case of $\mu^+$) then the development of instability (and the lifetime) must still be stretched, due to the nonlinear effect of the time slowdown. When the limit of stability (bifurcation) at $r \sim \lambda_\mu = 1.86 \cdot 10^{-13} cm$ is reached, then the previous dynamic regime suddenly breaks up and the field begins to evolve towards a new configuration of a smaller mass $m_e$ and a larger $\lambda_e = 3.86 \cdot 10^{-11} cm$. The Dirac field of the $\mu^+$, which originally occupied the interval $\lambda_\mu < r < \lambda_e$, must be radiated. Since the Dirac equation is hyperbolic, the sharp front of the Dirac field, as any precursor, must propagate along a characteristic (at the speed of light), in the outward direction. This can only be the right component of the Dirac spinor (which inherits its lightlike current from the caustic), the $\bar{\nu}_\mu$. The final state of a $e^+$ emerges not earlier than a new caustic at $r = \lambda_e$ is formed. This requires yet another transient process of the violent collapse of the Dirac matter onto a new caustic and radiation of a similar precursor, but with the opposite phase velocity, $\nu_e$. The leptonic number is preserved dynamically in both processes. Remarkably, it is exactly the existence of a well-defined (by the spacelike axial vector) outward direction that eliminates the illusion of the reflection symmetry of a plane wave and thus predetermines a unique polarization of the spinor precursors. As it was noted by Wigner [10], it is only a theoretical idea of mirror symmetry (expressed in terms of polar vectors) that hints of the possible existence of the second polarization for lightlike spinor waveforms.

VII. CONCLUSION.

The nonlinear Dirac equation, with its capricious interplay of the many polarization degrees of freedom, poses a tough mathematical challenge for theory. Its explicit solutions may well yield various "magic numbers" that are currently known only from experiment. Even before regular mathematical methods are developed, one may rely on various qualitative consequences of the finite size of the Dirac waveforms to re-analyze existing data.

1. The conjectured connection between the mechanism of self-localization and the sign of the electric charge of the Dirac waveform also assumes that positively charged particles, which are not perfectly stable, must have a somewhat longer lifetime than their negatively charged anti-particles. The ratio $\chi = (\tau_+ - \tau_-)/\tau_{av}$ was measured for the most long-lived species as a test of CPT-invariance. According to the Particle Data Group, the difference in lifetime is indeed always positive, $\chi(K^{\pm}) = (0.11 \pm 0.09) \cdot 10^{-2}$, $\chi(\pi^{\pm}) = (5.5 \pm 7.1) \cdot 10^{-4}$, $\chi(\mu^{\pm}) = (2 \pm 11) \cdot 10^{-5}$, being the largest for the heaviest specie.

2. One of the predicted manifestations of charge asymmetry is the existence of the particle’s external size. The internal radius is universally limited from the below by the Compton length $\lambda = h/mc$. Due to the time slowdown in domains of large Dirac density, the positively charged species must have smaller external size than their negatively charged partners. Possibly, $e^+$ has reasonably well defined external boundary, which then may explain its relatively long lifetime in the environment of normal matter. There may well exist observed differences in the dynamics of electrons and positrons (or $p$ and $\bar{p}$, e.g., in accelerators) that are currently attributed to technical issues.

3. There are certain coincidences of numbers that may prompt another look at well known phenomena. We know that, $\lambda_e = 386 fm$, $\lambda_\mu = 1.86 fm$. The radius of the proton, as estimated via its electromagnetic form-factors, is $r_p = 1 fm$ and it grows for nuclei as $A^{1/3}$. This may well tell us something about the high rate of $\mu^-$ capture by light nuclei versus the low rate of the inverse $\beta$-decay by even heavy nuclei. The correlation between the capture rate and size of a nucleus may be a useful test.

4. The failure to keep anti-hydrogen molecule in the cold atom trap for an indefinitely long time may establish the limits of stability of the anti-proton in anti-matter surroundings. One may think of the capture of $e^+$ by a poorly localized $\bar{p}$ as a possible mechanisms of instability.

5. The Lorentz contraction of the accelerated waveforms is a dynamic effect, which leads to the accumulation of energy in an extremely small volume and its release in the course of a collision. The 5-8 Gev electrons and positrons are compressed to a size about $10^{-1} fm$ having a density higher than a proton. Upon colliding, they stop and create a sharp peak of invariant density, having a density higher than a proton. Upon colliding, they stop and create a sharp peak of invariant density, which is very far from a stable configuration and rapidly decays. The $B^\pm$ lifetime is reasonably long, $1.7 \cdot 10^{-12} s$, and its size is about $10^{-1} fm$. The time slowdown at $R \gg 1$ may result in two effects: (i) an abnormally long lifetime of the resonance and (ii) an exotic trend to further decay into compact heavy objects rather than to decay into lighter objects according to the usual spectator model. The mode $B^+ \rightarrow K^+ X(3872) \rightarrow J/\psi \pi^+ \pi^-$ seems to be a candidate for this kind of the process because the width of $X(3872)$ is very small.

APPENDIX A: PARALLEL TRANSPORT OF DIRAC FIELD.

In order to derive a measure for comparison of the fields $\psi(x_1)$ and $\psi(x_2)$ at two close points let us require,
following Fock [3], that the components \( j^a = \psi^+ \alpha^a \psi \) are the invariants of the vector \( j^\mu(x) \) and the congruences \( e^\mu_{(a)}(x) \) at the point where vector is defined, \( j^a = e^a_{(a)} j^\mu \). For now, we assume that a set of four orthogonal congruences is fixed in advance and that Dirac matrices \( \alpha^a \) are either invariants or covariantly constant objects. When the invariant \( j_a \) is parallel-transported by \( ds_b \) along an arc of congruence \( (b) \), then, solely because the local pyramid is being rotated, it must change by \( \delta j_a = \omega_{acb} j^b = \omega_{acb} \psi^+ \alpha^e ds^b \). The invariants \( \omega_{abc} \) are defined by Eq.(2.11). Let matrix \( \Gamma_a \) (the connection) define the change of the Dirac field components in the course of the same infinitesimal displacement, \( \delta \psi = \Gamma_a \psi ds^a \). Let differential of the product \( \psi \alpha^a \psi \) obey Leibnitz rule. This gives yet another expression for \( \delta j_a \),

\[
\delta j_a = \psi^+ (\Gamma^b_a \alpha_a + \alpha_a \Gamma_b) \psi ds^b . \tag{A.1}
\]

The two forms of \( \delta j_a \) must be identical. Hence, the equation that defines \( \Gamma_a \) is

\[
\Gamma^b_a \alpha_a + \alpha_a \Gamma_b = \omega_{abc} \alpha^c , \tag{A.2}
\]

and it has the most general solution,

\[
\Gamma_b(x) = i e A_b(x) + i g \rho_3 N_b(x) - \frac{1}{2} \omega_{0kb}(x) \rho_3 \sigma_k - i \frac{1}{2} \epsilon_{0kim} \omega_{imb}(x) \sigma_k , \tag{A.3}
\]

where the last two terms can be compacted as, \( \Omega_b = (1/4) \omega_{cd0} \rho_3 \alpha^c \rho_1 \alpha^d = (1/4) \omega_{cd0} \gamma^{-1} \tau^d \). These two terms correspond to an infinitesimal boost of \( \sigma^a \) along the spatial \( k \)-axis with parameter \( \omega_{0kb} \) and an infinitesimal rotation of \( \sigma^a \) in the \( (im) \)-plane with parameter \( \omega_{imb} \), respectively. This analogy, however, is limited. While Eqs.(A.1)-A.3 do imply some measure for the length of an arc (and of an angle as the ratio of the two lengths), Eq.(2.2) does not. The first two terms are due to an intrinsic indeterminacy that arises when one has to compare Dirac fields at two different points relying only on the properties of the vector forms \( e^\mu_{(a)} \psi \). The first term is readily associated with the electromagnetic potential. The second one would not appear at all if, following Fock [3], we required that \( \delta(\psi^+ \rho_1 \psi) = 0 \) and \( \delta(\psi^+ \rho_2 \psi) = 0 \). This decision was motivated by that kind of invariance of the Dirac equation in Minkowski space (local Lorentz invariance), which is not inherited by the Dirac field in Riemannian geometry. The position of \( N_b \) in connection (A.3) may lead to the impression that it can well be a “next field”, which interacts with the axial current \( J_b \) of the Dirac field and is governed by an independent equation of motion. At least for the on-mass-shell Dirac field, this is not true.

The connection (A.3) commutes with the matrix \( \rho_3 \), so that Eq.(A.2) remains the same when \( \alpha_a \rightarrow \rho_3 \alpha_a \). It neither commutes nor anti-commutes with \( \rho_1 \) and \( \rho_2 \).

\[
\begin{align*}
\Gamma^b_a \rho_1 + \rho_1 \Gamma_b &= 2g \rho_3 N_b , \\
\Gamma^b_a \rho_2 + \rho_2 \Gamma_b &= -2g \rho_1 N_b .
\end{align*}
\tag{A.4}
\]

From now on, we postulate that invariant derivative of the Dirac field is \( D_a \psi = (\partial_a - \Gamma_a) \psi \) where \( \partial_a = e^a_{(a)} \partial_a \) is the derivative in the direction of a curve of congruence \( (a) \). Assuming the Leibnitz rule for \( D_a \) and considering all Dirac matrices as constants we readily reproduce the reference point of Eqs.(A.1) and (A.2) as

\[
D_b j_a = \partial_b j_a - \omega_{acb} j_c \equiv \nabla_b j_a . \tag{A.5}
\]

The result (A.5) for \( D_b j_a \) is a warrant that after projecting the r.h.s. into coordinate space we must recover the covariant derivative \( \nabla_a j_b \). Indeed, \( e^a_{(a)} e^b_{(b)} \nabla_b j_a = \partial_a j_b - \Gamma_a j_b = \nabla_a j_b \), and we shall consider this as a proof that \( j_a \) is an invariant of the vector \( j_a \) and congruence \( e^a_{(a)} \). (There is no one-to-one correspondence between the two terms of \( \nabla_b j_a \) and \( \nabla_a j_b \)).

In exactly the same way we may verify that the invariants \( D_b j_a \) of the axial current are of the form \( D_b j_a = \nabla_b j_a \) and conclude that \( e^a_{(a)} e^b_{(b)} \nabla_b j_a = \nabla_a j_b \). Vice versa, the quantities \( D_b j_a = e^a_{(a)} e^b_{(b)} \nabla_b j_a \) are invariants of a tensor and a system of congruences. It is straightforward to verify (computing all derivatives as functions of \( \psi \)) that equations like (A.5) hold not only for \( j^a \) and \( J^a \) but for \( e^a_{(a)} \psi = j^a / R \) and \( e^a_{(a)} \psi = J^a / R \). Recalling discussion of Sec.II, we may view this as complementary to (2.13), i.e., proof that the unit vectors \( e^a_{(a)} \) and \( e^a_{(a)} \) are vectors of Riemannian geometry. For the same reason, the projectors \( [\partial^a - J^b j_a / R^2] \) and \( [\partial^a + J^a j_b / R^2] \) are the tensors.

Using the same technique of differentiating and by virtue of Eqs.(A.4) we obtain

\[
D_a S = \partial_a S - 2g P N_a , \quad D_a P = \partial_a P + 2g S N_a . \tag{A.6}
\]

As one can see, there is no immediate correspondence between the algebraic and differential properties of the scalars. However, the quantities from the first line of (2.8) (like \( R \)) are differentiated as true scalars. The same behavior is observed for the components of the skew-symmetric tensor \( M_{ab} \). Instead of the anticipated \( e^a_{(a)} e^b_{(b)} D_a M_{ab} = \nabla_a M_{ab} \) we encounter one more disagreement with the differential criterion (A.5),

\[
D_a M_{ab} = \nabla_a M_{ab} - 2g N_a \nabla M_{ab} , \tag{A.7}
\]

\[
D_a \nabla M_{ab} = \nabla_a \nabla M_{ab} + 2g N_a \nabla M_{ab} .
\]

We leave open the question of if and when \( e^a_{(1)} \) and \( e^a_{(2)} \) of Eqs.(2.7) are the vectors with the same degree of confidence as \( e^a_{(0)} \) and \( e^a_{(3)} \). In most cases, \( e^a_{(1)} \) and \( e^a_{(2)} \) correspond to angular coordinates, which are physically uncertain without external fields (other objects nearby). The additional terms in Eqs.(A.7) indicate that these congruences can be normal only in very rare cases of solutions

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6 Schouten (7), Ch.II, §9; Ch.III, §9) considers equations like (A.3) as a condition that fixes the components of a vector with respect to anholonomic coordinate system.
with a dynamically generated symmetry. The expected one-to-one match with geometry is spoiled by the extra (with respect to generator of Lorentz transformations in the connection $\Gamma_a$) matrices $\rho_1$ and $\rho_2$ responsible for the “mixing” between right and left components. From this perspective, the Sakharov’s idea [2], regarding the topological nature of elementary charges, seems be closer to reality that it initially appeared. A further analysis of this issue seems to be impossible without an explicit solution of the equations of motion.

Recalling Eqs. (2.9) and using (A.6) we can compare the covariant derivative $D_a\mathcal{P}$ computed in two ways, as $D_a\mathcal{P} = \partial_a(\mathcal{R} \cdot \sin \Upsilon) + 2g\mathcal{R} \cos \Upsilon \mathcal{N}_a$ or, alternatively, as $D_a\mathcal{P} = \partial_a\mathcal{R} \cdot \sin \Upsilon + \mathcal{R} \cos \Upsilon D_a\Upsilon$. The result reads as

$$D_a\Upsilon[\psi] = \partial_a\Upsilon + 2g\mathcal{N}_a, \quad (A.8)$$

which is invariant under the simultaneous transformations, $\Upsilon \to \Upsilon + \Upsilon(x)$ and $2g\mathcal{N}_a \to 2g\mathcal{N}_a - \partial_a\Upsilon(x)$. A supposed freedom of such (chiral) transformations is not permissible, since these transformations change the observables in the r.h.s. of Eqs. (3.13) and (B.4) without altering the l.h.s.

The commutator $[D_a, D_b]$ still contains derivatives. Indeed,

$$[D_a, D_b]\psi = (\partial_a\partial_b - \partial_b\partial_a)\psi - [\partial_a\Gamma_b - \partial_b\Gamma_a - \Gamma_a\Gamma_b + \Gamma_b\Gamma_a]\psi. \quad (A.9)$$

However, for practical purposes it is important that $[D_a, D_b]\psi$ can be split into two parts, with and without derivatives. Since $\psi$ is a coordinate scalar and, in general, derivatives along arcs do not commute, we have

$$(\partial_a\partial_b - \partial_b\partial_a)\psi = (\omega_{ab} - \omega_{ba})\partial_c\psi. \quad \text{Now we can reassemble \([D_a, D_b]\psi\) as follows,}$$

$$[\vec{D}_a, \vec{D}_b]\psi = (\omega_{cab} - \omega_{cba})\vec{D}_c\psi - \vec{D}_{ab}, \quad (A.10)$$

where the term $C_{cab}\Gamma_c = (\omega_{cab} - \omega_{cba})\Gamma_c$ is added and subtracted to replace $\partial_c\psi$ by the covariant derivative $\vec{D}_c\psi$. The matrix operator $\vec{D}_{ab} = -\epsilon^c\epsilon^d[D_{\mu}\, \vec{D}_c]$ in the r.h.s. does not contain derivatives and can be explicitly found,

$$\vec{D}_{ab} = -\frac{1}{4}R_{abcd}\rho_1\partial_c\alpha^d + ieF_{ab} + ig\rho_3\omega_{ab}, \quad (A.11)$$

$$F_{ab} = \partial_a\mathbf{A}_b - \partial_b\mathbf{A}_a - (\omega_{ab} - \omega_{ba})A_c, \quad U_{ab} = \partial_a\mathbf{N}_b - \partial_b\mathbf{N}_a - (\omega_{ab} - \omega_{ba})\mathbf{N}_c,$$

where invariants of the Riemann curvature tensor are defined by Eq. (2.22) and $F_{ab}$ and $U_{ab}$ are invariants of the electromagnetic tensor $F_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu$ and the field tensor $U_{\mu\nu} = \partial_\mu\mathbf{N}_\nu - \partial_\nu\mathbf{N}_\mu$, respectively.

Using the cyclic symmetry of the Riemann curvature tensor it is straightforward to show [3] that

$$\alpha^a\vec{D}_{ab} = \frac{1}{2}\alpha^aR_{abcd}\rho_1\partial_c\alpha^d + ie\alpha^aF_{ab} + ig\rho_3\alpha^a\omega_{ab}. \quad (A.12)$$

When field $\mathcal{N}$ in the connection $\Gamma_a$ is a gradient, the tensor $U_{\mu\nu}$ vanishes identically, which is assumed throughout this paper except for Eq. (B.10), where a possible connection with electroweak theory is mentioned.

**APPENDIX B: STRESS TENSOR AND PION FIELD.**

1. Internal flux of mass and stress in the Dirac field.

In this section we study the stress tensor $P_\mu^a = i\psi^+\rho_3\alpha^aD_\mu\psi$, mostly following the same logic as for the energy momentum tensor $T_\mu^a = i\psi^+\alpha^aD_\mu\psi$ in Sec.IIIB, starting from its covariant derivative. We find that

$$D_c[\psi^+\rho_3\alpha^a\vec{D}_b\psi] = \partial_c[\psi^+\alpha^a\vec{D}_b\psi] - \omega_{abc}\psi^+\rho_3\alpha^d\vec{D}_b\psi. \quad (B.1)$$

Once again, the last term of Eq. (B.13) is missing, and thus we have no confidence that the covariant derivative is a tensor. This time, let us begin by contracting indices $a$ and $b$ in Eq. (B.1),

$$D_c[\psi^+\rho_3\alpha^a\vec{D}_a\psi] = \partial_c[\psi^+\rho_3\alpha^a\vec{D}_a\psi] - \omega_{abc}\psi^+\rho_3\alpha^b\vec{D}_a\psi. \quad (B.2)$$

By virtue of the Dirac equations, the first term in the r.h.s. of (B.2) becomes $\partial_c[m\psi^+\rho_2\psi]$. Alternatively, we can immediately use the equations of motion in the l.h.s. and only then differentiate (matrices $\rho_3$ and $\alpha^a$ commute),

$$D_c[\psi^+\rho_3\alpha^a\vec{D}_a\psi] = mD_c[\psi^+\rho_2\psi] = m\partial_c[\psi^+\rho_2\psi] + m \cdot 2g\mathcal{N}_c. \quad (B.3)$$

Comparing the last two equations we finally get the equation,

$$\omega_{acb} \cdot P_{ca} = -2igm\mathcal{N}_b, \quad (B.4)$$

which is complementary to Eq. (4.10). The imaginary part in the l.h.s. is due to $(1/2)[P_{ca} - P_{ca}^+] = (i/2)D_c\mathcal{J}_a$. Since the axial current is a vector, we can rewrite the last equation as

$$(1/2)\omega_{acb} \nabla_c\mathcal{J}_a = -2igm\mathcal{N}_b, \quad (B.5)$$
which is complementary (dual) to Eq.(120). The skew-symmetric Hermitian part, \((P_{ca} + P_{ac}^+) - (P_{ac} + P_{ca}^+))\), must vanish since the r.h.s. of Eq.(124) is an imaginary quantity. This yields the equation, which duplicates Eq. (4.1),

\[
i[\psi^+ \rho_3 \alpha_a \bar{D}_a \psi - \psi^+ \bar{D}_c^+ \alpha_a \rho_3 \psi - \psi^+ \rho_3 \alpha_c \bar{D}_a \psi + \psi^+ \bar{D}_a^+ \alpha_c \rho_3 \psi] = \epsilon_{acdl} D_a j_t = 0. \tag{B.6}
\]

and thus indicates that we still are dealing with a stable waveform.

Contracting (in Eq.(B.2)) indices \(a\) and \(c\) we arrive at the expression, which is similar to Eq.(3.3),

\[
D_a [\psi^+ \rho_3 \alpha^a \bar{D}_b \psi] = \partial_a [\psi^+ \rho_3 \alpha^a \bar{D}_b \psi] + \omega_{ac} \psi^+ \rho_3 \alpha^a \bar{D}_b \psi = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left[ \sqrt{-g} e^{e^a_{(a)}} (\psi^+ \rho_3 \alpha^a \bar{D}_b \psi) \right]. \tag{B.7}
\]

Let us first rewrite the l.h.s. of Eq. (B.7) as

\[
D_a [\psi^+ \rho_3 \alpha^a \bar{D}_b \psi] = \psi^+ \rho_3 \alpha^a [\bar{D}_a \bar{D}_b - \bar{D}_b \bar{D}_a] \psi + \psi^+ \bar{D}_a^+ \psi + D_b (\psi^+ \rho_3 \alpha^a \bar{D}_a \psi) - \psi^+ \bar{D}_b^+ \rho_3 \alpha^a \bar{D}_a \psi. \tag{B.8}
\]

Because of an obvious change of signs (caused by an extra \(\rho_3\)), the last three terms of (B.8) do not cancel. Instead of (3.9) we have

\[
D_a P^a_\mu = i \psi^+ \rho_3 \alpha^a [\bar{D}_a \bar{D}_b - \bar{D}_b \bar{D}_a] \psi + i m D_b P + im [\psi^+ \rho_2 \bar{D}_b \psi - \psi^+ \bar{D}_b^+ \rho_2 \psi]. \tag{B.9}
\]

By splitting the commutator according to (A.10), we can assemble the covariant derivative in the l.h.s. as

\[
\nabla_\mu P^\mu_\nu = \nabla_\mu \text{Re}(P^\mu_\nu) + \frac{i}{2} \nabla_\mu \nabla_\nu \mathcal{J}^\mu = \nabla_\mu \text{Re}(P^\mu_\nu) + \frac{i}{2} \mathcal{J}^\mu R_{\mu\nu} + im \partial_\nu \mathcal{P},
\]

leaving on the r.h.s a remainder, \(e^{\partial}_b [\omega_{ab} P^a_\mu + e \mathcal{J}^\mu F_{ab} + g j^\mu U_{ab} + (i/2) \mathcal{J}^\mu R_{ab}]\). By virtue of Eqs.(B.4) and (A.6) the imaginary terms on both sides exactly cancel each other and the remaining real part reads as

\[
\nabla_\mu \text{Re}(P^\mu_\nu) = e \mathcal{J}^\mu F_{\mu\nu} + g j^\mu U_{\mu\nu} + im [\psi^+ \rho_2 \bar{D}_b \psi - \psi^+ \bar{D}_b^+ \rho_2 \psi]. \tag{B.10}
\]

The flux of momentum in the spacelike direction is determined by the Lorentz force that acts on the electric axial current \(e \mathcal{J}^\mu\), but also by the Lorentz force of the field \(\mathcal{J}_\mu\) that acts on the vector current \(g j^\mu\). This part is logical to write down in terms of the mixed fields that act on lightlike left and right Dirac currents. It describes the interplay between the right and left components of a Dirac particle and belongs to the area of electro-weak interactions. Being interested only in stable states, we do not consider it in any details here. The last term is due to convection transport of the pseudoscalar mass density \(m \mathcal{P}\) in an electromagnetic field. An importance of axial electric forces for the processes of pion electro-production was noticed by Nambu and Shrauner [15]. The convection term can be cast as

\[
m[\psi^+ \rho_2 (i \partial_\nu \psi^- - (i \partial_\nu \psi^+)) \rho_2 \psi + 2e A_\nu \mathcal{P} + (1/2) \omega_{cde} \mathcal{M}^{cde}]\]

which repeats the familiar pattern of Gordon’s decomposition of the Dirac (vector) current with the replacement \(\rho_1 \rightarrow \rho_2\), \(e \rightarrow m\), and where the long derivative includes only the electromagnetic potential. The pseudoscalar density \(\mathcal{P}\) is one of many polarization degrees of freedom of the Dirac field and is not an independent field. In some approximation, the charged pseudoscalar flux inside Dirac waveforms (e.g., nuclei) can be viewed as the interaction of highly localized nucleons via soft

2. Pion field and \(\pi^0 \leftrightarrow 2\gamma\) processes.

The flux of charge, mass and momentum carried by the pseudoscalar density in the interaction between Dirac nucleons is commonly attributed to the pion field. Pions can also be detected as sufficiently long lived particles. Therefore, \(\mathcal{P}\) should satisfy the Klein-Gordon equation, which we will derive below.

The first step is to put the axial current in a form with separated convection and polarization currents, as
is done in Gordon’s decomposition of the vector current, \( J^a = \frac{1}{2m} \eta_{i\alpha} D_a \mathcal{P} + \frac{1}{2m} f^a; \) \( I^a = -\frac{1}{2} \psi^\dagger \psi \alpha^a \rho_2 \alpha^b \overline{D}_b \psi - \psi^\dagger \overline{D}_b \psi \alpha^a \rho_2 \alpha^b \psi \). where \( \alpha^a \ldots \alpha^b \) stands for \( \alpha^a \ldots \alpha^b \ldots \alpha^a \). (Now, the entire convection term is reduced to the derivative of \( \mathcal{P} \)) Computing the covariant derivative of both sides of the last equation and using (A.4) we obtain

\[
D_a \mathcal{P} = 2 \psi^\dagger \overline{D}_a \rho_2 \overline{D}_a \psi - 2m^2 \mathcal{P} - \text{Re}[\psi^\dagger \psi \alpha^a \rho_2 \alpha^b (D_a D_b - D_b D_a) \psi].
\]

(B.12)

For the on-mass-shell Dirac field of the nucleon with a large mass \( M \), we may take the Dirac field in non-relativistic approximation, \( \psi \propto e^{i S/\hbar} \) so that the first two terms in the r.h.s. constitute the classical Hamilton-Jacobi equation for the eikonal \( S \). If this equation is satisfied with a sufficient accuracy (the waveform behaves as a classical particle and the resonance is sufficiently narrow), then in the r.h.s. remains only the last term. By virtue of Eqs. (A.10) and (A.11) this term becomes nothing but \( e F_{ab} \chi^{ab} + (R_0/2) \mathcal{P} - (1/2) C_{cab} \psi^\dagger \alpha^c \rho_2 \alpha^b \overline{D}_c \psi \), where \( R_0 = R_{[\psi^\dagger, \psi] \text{ is the scalar Riemannian curvature (with dimension } m^2) \), which is a functional of the Dirac waveform. As a result, we arrive at the independent wave equation for the pseudoscalar density \( \mathcal{P} \) (the pion field),

\[
D_a^2 \mathcal{P} - \frac{R_x}{2} \mathcal{P} \approx -C_{cab} \text{Re}[\psi^\dagger \psi \alpha^a \rho_2 \alpha^b \overline{D}_c \psi] + e F_{ab} \chi^{ab},
\]

(B.13)
a Heisenberg equation of motion with a variable mass defined by the negative scalar Riemannian curvature \( R_x \) outside the stable nucleons\(^7\). Quite surprisingly, exactly this \( \mathcal{P} \) enters the r.h.s. of Eq. (B.13) that defines the force of gravity/inertia in the same approximation of a material point.

The source in the r.h.s. is Hermitian. Its first term is “geometric” and accounts for the flux of momentum and twist of the tetrad basis which are needed in order that the pion could be born. The second term is more related to the pion decay and it can be rewritten in two ways. On the one hand it can be presented as \( 2 e \psi^\dagger [\rho_1 \overline{E} + \rho_2 \overline{B}] \partial_\tau \psi \). When \( F_{ab} \) is the field of a standing transverse electromagnetic wave it has a simple representation in terms of the spin interaction with two waves of circular polarizations,

\[
e F_{ab} \chi^{ab} = 4 e \sum_k \frac{-i \sqrt{e_{\eta k}}}{\sqrt{2(2 \pi)}} \left( C_k e^{-ikx} - C_k e^{ikx} \right)
\]

\[
\times \left[ e_L \cdot \psi_R \partial_\tau \psi_L + e_R \cdot \psi_L \partial_\psi_R \right],
\]

(B.14)

where \( e_L \parallel \psi_R \) are the vectors of the two circular polarizations, \( \psi_L, \psi_R \) are the left and right components of the Dirac spinor field, \( \omega_k = E_k \), is the “photon’s energy” and \( C_k \) is the Fourier component of the initial or final (possibly, coherent) state of the Heisenberg field \( F_{ab} \). This form of the source of the pion field allows one to qualify \( \pi^0 \) as a resonance in the system of the Dirac field and a standing electromagnetic wave formed by the two circular polarizations, which causes the simultaneous flip of helicity of both components of the Dirac field. On the other hand the source in the wave equation (B.13) has the structure of the axial anomaly. This could be an exact correspondence if there was a simple proportionality between \( \mathcal{M}^{ab} \) and \( F^{ab} \). Then, \( e F_{ab} \chi^{ab} = C e F_{ab} F^{ab} \), where the explicit value of \( C \) must comply with the observed rate of the \( \pi^0 \rightarrow 2 \gamma \) decay. It is instructive that the wave equation for the pseudoscalar meson field \( \mathcal{P} \) (that yields the pole in the pion propagator) was derived exactly from the original equation Eq. (B.14). The term which was \textit{ad hoc} added to this equation by S. Adler\(^1\) (in order to save the Ward identity for the axial vertex in triangle graph) has naturally appeared as the source in the wave equation. A uniform method to derive wave equations for meson fields is beyond the scope of this paper.\(^2\)

In the first approximation, the value of mass of the Dirac field is not important. For this particular resonance, the mass term in the l.h.s. of Eq. (B.13) can be confidently identified with and measured as \( (2 E_{\pi})^2 \). In fact, \( m_{\pi}^2 \sim -R_x/2 \) is an independent of the Dirac mass \( m \) measure of the “metric elasticity” in the ground state of the Dirac field, when balance between left and right is probed by electromagnetic field, thus being a fundamental constant. The geometry of currents inside \( \pi^0 \) as a finite-sized object is not clear so far. The dynamic quantity \( m_{\pi} \) is meaningful only for \( \approx 10^{-16} \text{s} \) of the resonance spike of the pseudoscalar density. The form of the source (sink) in the Eq. (B.13) indicates that this spike is a parametric resonance which can be created in a reaction like \( \gamma p \rightarrow \pi^0 p \rightarrow \gamma \gamma \pi^0 \) or, more precisely, \( \gamma^\ast \gamma^\ast \rightarrow \pi^0 \rightarrow \gamma \gamma \). It decays due to axial polarization currents (eventually producing two photons of the opposite circular polarization) of the nucleon. The nucleon is needed solely to trigger the process – to locally determine the difference between left and right or inward and outward. Another way to excite the \( \pi^0 \) resonance is \( e^+ e^- \)-annihilation at high energy, etc.

For the sake of completeness mention that the source in Eq. (B.13) is complementary to the interaction of the axial current with the electromagnetic field given by the

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\)

\(^7\) The negative curvature is typical for the geometry of the expanding universe. The pions (and mostly pions) are abundantly created in the high-energy collisions where strong contraction of the colliding particles is translated into the rapidity plateau in the distribution of pions. At any given moment of time \( t \), the proper time at a distance \( x \) from a generic point \( x = 0 \) corresponds to the earlier proper time \( \tau \) and has a larger density \( R(x, t) \). Therefore, as can be perceived from any point, the proper time flows more slowly with larger distance \( x \) from this point. As a result, Dirac waves tend to refract into distant spatial regions. The issue of stability of the expanding Dirac field and of its localization as pions and muons will be discussed elsewhere.
term $eJ^\mu F_{\mu\nu}$ in Eq. [B.10], which describes the balance of momentum when the waveform changes its shape.

The processes with the charged pions can be considered in a similar manner [20]. The dynamic footing for the phenomenological formalism of isospin symmetry, as it is presented in various texts (e.g., [21]), can be naturally built from the basic properties of the Dirac field. Pions are identified as one specific (pseudoscalar) polarization degree of freedom of the Dirac waveforms. The totally dynamic origin of the pion mass term, which is determined by the curvature $R_{\mu}[\psi^+, \psi]$ of the matter-induced metric, explains the diversity of faces that pions may reveal in different situations. This variety ranges from soft pion glue within nuclei (when DIS cannot resolve pion's structure functions) and up to free propagation of the massive pions (tracks) at distances that allow for the pion interferometry (which is sensitive to the microscopic dynamics of the pions emission [22]).

**APPENDIX C: ANGULAR VARIABLES IN DIRAC EQUATION.**

Let us examine the properties of the solutions of the Dirac equation [53], neglecting nonlinear terms, in the presence of the radial field $2q\kappa_r = -\partial_t Y$ and in a perfectly spherical symmetry. Since we will focus on the nature of angular variables, an explicit dependence $\kappa_\ell(r)$ is not essential. [One may think of Eq.[4.18] as an example.] The only non-vanishing components of the Ricci rotation coefficients are $\omega_{212} = (1/r)\cot \theta$ and $\omega_{231} = \omega_{232} = 1/r$. Solely as a reference, assume that there is an external electromagnetic Coulomb field $A_\ell(r)$. Then the Dirac equation is

$$\left[ i\partial_0 - eA_0 + g\sigma_3 \kappa_r - i\rho_3\sigma_3 (\partial_\rho + \frac{1}{r} \partial_\theta - i\frac{\rho_2}{r} \cot \theta) - m\rho_1 \right] \psi = 0 \; . \quad (C.1)$$

In terms of a new unknown function, $\tilde{\psi}(r,\theta,\varphi) = r\sqrt{\sin \theta} \psi$, this equation becomes

$$\left[ i\partial_0 - eA_0 + g\sigma_3 \kappa_r + \rho_3\sigma_3 (-i\partial_\theta) + \frac{\rho_3}{r}(-i\sigma_1 \partial_\theta - i\frac{\sigma_2}{\sin \theta} \partial_\varphi) - m\rho_1 \right] \tilde{\psi} = 0 \; . \quad (C.2)$$

Equation (C.1) is the Dirac equation in the tetrad basis. In order to find its solution one has to separate the angular and radial variables. This is known to be a somewhat tricky problem, even in the standard problem with a radial Coulomb field (when $\kappa_r = 0$). The Hermitian operators in Eq. (C.2) are the tetrad components of the momenta $p_3 = -i\partial_\varphi$, $p_1 = -i r^{-1} \partial_\theta p_2 = -i (r \sin \theta)^{-1} \partial_\varphi$. The operators $p_1$ and $p_2$ are clearly associated with the angular motion. If the coefficients in this equation where not matrices, it would have already been an equation with separated variables, which would match the perfect spherical symmetry of the external fields $A_\ell$ and $\kappa_r$. The problem is that the operators of the radial and angular momenta do not commute (they anti-commute, $[\alpha_3 p_3, (\alpha_1 p_1 + \alpha_2 p_2)] + = 0$). A regular way to avoid this obstacle is as follows [17, 18]: One attempts to construct a minimal set of operators that commute with the Hamiltonian. For example, one can check that the commutator $[\alpha_3 p_3, (\alpha_1 p_1 + \alpha_2 p_2)] = 0$ and take the operator $\rho_1 (\alpha_1 p_1 + \alpha_2 p_2)$ as a generator of the conserved quantum number. This trick works when $\kappa = 0$ and it is very instructive to see the details of its failure when $\kappa \neq 0$.

The conventional operator of angular momentum is $\mathbf{L} = [\mathbf{r} \times \mathbf{p}] + \frac{\sigma}{2}$. An additional operator $\mathbf{L} = \mathbf{\sigma} + \mathbf{L} - 1/2$ commutes with the orbital momentum, $[\mathbf{L}, (\mathbf{r} \times \mathbf{\sigma})] = 0$, and has the properties, $L(\mathbf{L} - 1) = [\mathbf{r} \times \mathbf{p}]^2$ and $L^2 = \mathbf{L}^2 + 1/4$. Therefore, if $\kappa$ is an eigenvalue of operator $\mathbf{L}$ we obviously have $\kappa(\kappa - 1) = l(l + 1)$ and $\kappa^2 > 0$. On the other hand, if $\mathbf{L}_A = \rho_1 \mathbf{L}$, then $(\mathbf{L}_A)^2 = \mathbf{L}^2$ and these operators have the same sets of eigenvalues. In the tetrad basis, these generators of the angular quantum numbers are

$$\mathbf{L}_A = \rho_1 \left( -i\sigma_2 \partial_\theta + \frac{i\sigma_1}{\sin \theta} \partial_\varphi \right), \quad \mathbf{L}_3 = -i\partial_\varphi + \frac{1}{2}\sigma_3 \; . \quad (C.3)$$

In terms of the auxiliary operator $\mathbf{L}_A$ (which has the same set of quantum numbers as the operator of the angular momentum but is a different operator) and the projection $\mathbf{L}_3$ of angular momentum, the Dirac equation becomes

$$\left[ i\partial_0 - eA_0 + g\sigma_3 \kappa_r + \rho_3\sigma_3 (-i\partial_\theta) - \frac{\rho_3}{r}(-i\sigma_1 \partial_\theta - i\frac{\sigma_2}{\sin \theta} \partial_\varphi) - m\rho_1 \right] \tilde{\psi} = 0 \; . \quad (C.4)$$

When the operator $\mathbf{L}_A$ commutes with all terms of Hamiltonian (which is the case when $\kappa = 0$) we can require the wave function be an eigenfunction of the Hamiltonian and these two operators

$$\mathbf{L}_A \tilde{\psi} = \kappa \tilde{\psi}, \quad \text{and} \quad \mathbf{L}_3 \tilde{\psi} = (m \frac{1}{2} + \frac{1}{2}) \tilde{\psi} \; . \quad (C.5)$$

Since the $\mathbf{L}_A$ anti-commutes with $g\sigma_3 \kappa_r$ we have no obvious solution for the separation of variables.

Because the presence of the component $\kappa_\ell(r)$ at least apparently preserves the spherical symmetry, we can try to look for a general solution of the following form ($\xi$ and $\eta$ are the left and right components of the Dirac field in spinor representation, respectively),

$$\tilde{\xi} = \left( u_L(r,t) \mathbf{Y}(\theta, \varphi) \right), \quad \tilde{\eta} = \left( d_L(r,t) \mathbf{Z}(\theta, \varphi) \right) \; . \quad (C.6)$$
As a first step, we may try to substitute the Dirac spinor into Eqs. (C.5). One can immediately see that the angular variables in (C.5) can be separated only when \( u_L = d_R \) and \( u_R = d_L \). At the same time, by inspection of the complete system of four Dirac equations,\[ (i\partial_0 - eA_0 - g\mathbb{N}_r - i\partial_r)u_L = m d_R Y + d_L \frac{i\Lambda_+}{r} Z, \]
\[ (i\partial_0 - eA_0 + g\mathbb{N}_r + i\partial_r)u_L = m d_R Z + u_L \frac{i\Lambda_-}{r} Y, \]
\[ (i\partial_0 - eA_0 - g\mathbb{N}_r + i\partial_r)u_R = m u_L Y - d_R \frac{i\Lambda_-}{r} Z, \]
\[ (i\partial_0 - eA_0 + g\mathbb{N}_r - i\partial_r)u_R = m d_R Y - u_R \frac{i\Lambda_+}{r} Y, \] (C.7)
where \( \Lambda_{\pm} = (\partial_0 \pm \frac{i}{\sin \theta} \partial_\phi) \), (C.8)
one can see that the condition \( u_L = d_R \) and \( u_R = d_L \) is inconsistent with the presence of the field \( \mathbb{N}_r \). The Dirac equation breaks up into two systems of equations for only two radial functions which are incompatible unless \( \mathbb{N}_r = 0 \).

Nevertheless, just by inspection, one can see that the angular functions \( Y_{k,m}(\theta,\phi) \) and \( Z_{k,m}(\theta,\phi) \) that satisfy the equations,
\[ \Lambda_- Z_{k,m}(\theta,\phi) = -k Y_{k,m}(\theta,\phi), \]
\[ \Lambda_+ Y_{k,m}(\theta,\phi) = k Z_{k,m}(\theta,\phi), \] (C.9)
do separate the angular variables in the Dirac equation (and do not separate them in Eq. (C.5)). With this separation of angular variables, Eqs. (C.7) yield a system of four differential equations for the four radial functions. Eqs. (C.3) are the equations for spherical harmonics but \( \theta \) and \( \phi \) are not the angles of the spatial angular coordinates.

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