ON SHIMURA SUBVARIETIES
GENERATED BY FAMILIES OF ABELIAN
COVERS OF \( \mathbb{P}^1 \)

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Abstract

We study the locus of abelian Galois covers of \( \mathbb{P}^1 \) in \( A_g \) and the problem of occurrence of Shimura (special) subvarieties generated by these covers in the Torelli locus \( T_g \) inside \( A_g \). We first investigate the existence of Shimura subvarieties in the mentioned locus by some computational methods based on Moonen-Oort works and then exclude many cases using both characteristic \( p \) methods and monodromy computations.

0. Introduction

Let \( M_g \) be the (coarse) moduli space of non-singular complex curves of genus \( g \) and \( A_g \) the moduli space of principally polarized complex abelian varieties of dimension \( g \). Let \( j : M_g \to A_g \) be the Torelli map. i.e. the map which associates to a curve it’s Jacobian. The Torelli theorem asserts that this map is injective. We call the image of this map, the open Torelli locus and denote it by \( T_g^0 \). By it’s definition, it consists of the Jacobians of smooth curves of genus \( g \). The closure of the open Torelli locus inside \( A_g \) is called the closed Torelli locus, which we denote by \( T_g \). The closed Torelli locus is the locus of Jacobians of stable curves of compact type. The quest of Shimura subvarieties (or special subvarieties) contained generically in the
Torelli locus $T_g$ and not fully contained in the boundary, is a longstanding problem which can be traced back at least to Shimura (see [S]). Although not realized then, more than two decades later, Coleman formulated his famous conjecture suggesting that for $g \geq 4$, there are only finitely many curves $C$ of genus $g$ over the complex numbers whose Jacobian $J(C)$ is an abelian variety of CM type. See [C]. Shortly after that, de Jong and Noot disproved that conjecture by finding examples of families of curves which give rise to Shimura subvarieties in $A_g$ lying generically in the (open) Torelli locus and intersecting the (closed) Torelli locus non-trivially. The families that they found -which were also found in the aforementioned article of Shimura- were all families of cyclic coverings of $\mathbb{P}^1$. Their examples were of fiber genus 4 and 6 and they did explain the relation between their examples and the Coleman conjecture. Later, examples of higher genera ($g = 5$ and $g = 7$), again for families of cyclic coverings of $\mathbb{P}^1$, were found by Rohde [R]. Finally Moonen completed the list of Shimura subvarieties arising from families of cyclic coverings of $\mathbb{P}^1$ in [M1] by showing that there are no more Shimura subvarieties in this locus. The fiber genus of all of these families were bounded by 8. Based on the above examples, one can reformulate the Coleman conjecture as follows:

**The (corrected) Coleman conjecture.** For $g \geq 8$, there are only finitely many smooth projective curves $C$ over $\mathbb{C}$ of genus $g$ such that $J(C)$ is an abelian variety of CM type.

Bearing in mind that every Shimura variety contains a dense subset of CM points, we see that the Coleman conjecture is related to the following conjecture and in fact disproving this conjecture will disprove the Coleman conjecture:

**Conjecture ([O], §5).** For large $g$ (in any case $g \geq 8$), there is no positive-dimensional Shimura subvariety contained in $T_g$ such that $Z \cap T_g$ is non-empty.

We just remark that if one assumes the André-Oort conjecture to be true, then the above conjecture is even equivalent to the Coleman conjecture. Also note that the condition that our subvariety meets $T_g$ is a key condition. Otherwise one can easily construct a lot of Shimura subvarieties contained fully in the boundary of $T_g$ for every $g \geq 2$. There are also several other results
on the occurrence of Shimura subvarieties in the Torelli locus. Viehweg and Zuo, for example studied in [VZ1] the occurrence of Shimura curves in the moduli stack of principally polarized abelian varieties relating it to Arekov equalities. See also [MVZ1], [MVZ2]. The recent work of Lu and Zuo [LZ] has made some progress in the proof of Coleman’s conjecture. They show that there do not exist Shimura curves in Torelli locus of hyperelliptic curves of genus greater than 7. They also show that Shimura curves with maximal Higgs field (they are either self product of universal family of elliptic curves or Shimura curves of Mumford type) can not be contained in Torelli locus of curves of genus greater than 5.

In [MO] Oort and Moonen asked whether one can obtain further Shimura subvarieties in the Torelli locus by taking families of abelian coverings of $\mathbb{P}^1$ with a non-cyclic Galois group. They also gave some examples of families of abelian covers of $\mathbb{P}^1$ which give rise to Shimura subvarieties in $T_g$. In this article we try to generalize their methods and classify Shimura subvarieties arising from families of abelian covers of the projective line. We fix integers $N \geq 2$ and $s \geq 4$ and an $s$-tuple $(z_1, ..., z_s)$ and consider a family of abelian covers $Y_t \to \mathbb{P}^1$ with an abelian Galois group $G$ which is isomorphic to the column span of the matrix $A$ and hence is a subgroup of the group $(\mathbb{Z}/N\mathbb{Z})^m$. By varying the branch points we obtain a subvariety whose closure $Z$ (inside $A_g$) lies in $T_g$. Of course $Z$ will be of dimension $s - 3$ where $s$ is the number of branch points of the covering and it lies in the Torelli locus non-trivially. We then try to classify the cases where $Z$ is a Shimura subvariety. Our method here is a generalization of that of Moonen-Oort [MO] and Moonen [M1]. The Jacobians in our families admit an action of the group ring $\mathbb{Z}[\mu_G]$. This action defines a Shimura subvariety $S(\mu_G)$ in $A_g$. This subvariety is in fact the Hodge locus given by the Hodge classes that are exactly elements of $\mathbb{Z}[\mu_G]$ viewed as endomorphisms of the Jacobians in our family. It follows that $Z \subseteq S(\mu_G)$ and therefore $s - 3 \leq \text{dim}S(\mu_G)$. When $\text{dim}S(\mu_G) = s - 3$, or equivalently when $Z = S(\mu_G)$, one concludes that the subvariety $Z$ is a Shimura subvariety. The dimension of $S(\mu_G)$ can be calculated in terms of dimensions of certain eigenspaces with respect to the Galois group action and the equality $\text{dim}S(\mu_G) = s - 3$ is then easy to check by a computer and by using a simple computer program we list the examples in a table (see table 1). Note that this table appears also in [MO], where they present them as a set of examples which are found of families of abelian covers. Their method - based on the only example that they give- is to decompose the Jacobian of the fibers by representing them as cyclic covers and then compute the multiplicity of the corresponding group action on the tangent space of the Jacobian. Also, they do not claim that
this table contains all examples for which \( \text{dim} S(\mu_G) = s - 3 \) and do not claim that there are no more examples outside of the table. Here we use a systematic approach based on an explicit formula in terms of the dimension of the eigenspaces and then we compute also these dimensions explicitly. We also clarify that there can be no more examples at least for \( s = 4 \) for which \( \text{dim} S(\mu_G) = s - 3 \) holds and investigate all cases for \( s, N \) and \( m \). Of course when \( s - 3 < \text{dim} S(\mu_G) \), one can not conclude that \( Z \) is not a Shimura subvariety. It can very well happen that \( Z \) is still a smaller Shimura subvariety contained in \( S(\mu_G) \). In other words, there could be Hodge classes in our family of Jacobians that are not contained in \( \mathbb{Z}[\mu_G] \) and hence we do not see them in the construction of \( S(\mu_G) \). In [M1], Moonen shows that this does not happen for families of cyclic covers of \( \mathbb{P}^1 \). This is what we also try to show for families of abelian coverings. That is, we are able to exclude many non-Shimura examples in what follows. The methods that Moonen uses for families of cyclic covers, can be transformed and translated for families of abelian covers, but not completely. We will see that some of his arguments are not anymore true in the case of families of abelian coverings of the projective line. In any case we will enhance some of the methods and arguments in order to exclude further examples of families of abelian covers. We achieve this by using an obstruction of Dwork and Ogus and another method based on computing the monodromy of a family of curves. Our main results are Theorem 2.1, Proposition 5.1.1 and Theorem 5.2.7.

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1. Construction of abelian covers of \( \mathbb{P}^1 \) and their families

An abelian Galois cover is determined by a collection of equations in the following way: Consider an \( m \times s \) matrix \( A = (r_{ij}) \) whose entries \( r_{ij} \) are in \( \mathbb{Z}/N\mathbb{Z} \) for some \( N \geq 2 \). Set

\[
  w_i^N = \prod_{j=1}^s (z - z_j)^{\tilde{r}_{ij}} \text{ for } i = 1, \ldots, m
\]
Where \( \tilde{r}_{ij} \) is a lift of \( r_{ij} \) to \( \mathbb{Z} \cap [0, N) \). Denote by \( \tilde{A} = (\tilde{r}_{ij}) \) and call it the \textit{lifted matrix} of \( A \). We impose the condition that the sum of the columns are zero. This implies that the cover is not ramified over infinity. The matrix \( A \) will be called the matrix of the covering. Note that our notations here are mostly that of [W]. Also we consider the row and column spans of the matrix \( A \) as modules over the ring \( \mathbb{Z}/N\mathbb{Z} \) and so all of the operations with rows and columns will be carried out in the ring \( \mathbb{Z}/N\mathbb{Z} \), i.e. it will be considered modulo \( N \). The abelian Galois group \( G \) of the covering is isomorphic to the column span of the matrix \( A \) and hence can be considered as a subgroup of \( (\mathbb{Z}/N\mathbb{Z})^m \) (denoted also by \( \mathbb{Z}_{N}^m \)), see [W], 2.2. Sometimes for preventing confusions we use the symbol \([\cdot]_N\) to show in which ring we are working and for example write \([r_{ij}]_N\) instead of \( r_{ij} \). For each \( 1 \leq i \leq m \), the function \( w_i \) is an element of \( \mathbb{C}(z) \). The abelian cover, is then the Riemann surface with function field \( \mathbb{C}(z)[w_1, ..., w_m] \). It can easily be seen that any Galois cover of \( \mathbb{P}^1 \) with abelian Galois (or deck) group is obtained in this way from a certain matrix \( A \). The local monodromy at the branch point \( z_j \) is given by the column vector \((r_{1j}, ..., r_{mj})^t\) and thus the order of ramification at \( z_j \) is \( \frac{N}{\gcd(N, \tilde{r}_{1j}, ..., \tilde{r}_{mj})} \). Using this and the Riemann-Hurwitz formula, the genus \( g \) of the cover is then given by:

\[
g = 1 + d \left( \frac{s^2 - 2}{2} - \frac{1}{2N} \sum_{j=1}^{s} \gcd(N, \tilde{r}_{1j}, ..., \tilde{r}_{mj}) \right)
\]

Where \( d \) is the degree of the covering. Note that the degree \( d \) of the covering (or equivalently the order of the Galois group) can be realized as the size of the row span (equivalently column span) of the matrix \( A \), see [W].

Next we turn to define families of curves which are abelian coverings of the projective line.

Let \( U \subset (\mathbb{A}^1)^s \) be the complement of the big diagonals. i.e. \( U = \mathcal{P}_s = \{(z_1, ..., z_s) \in (\mathbb{A}^1)^s \mid z_i \neq z_j \forall i \neq j \} \). Over this open affine set, we define a family of abelian covers of \( \mathbb{P}^1 \) to have the equation:

\[
w_i^N = \prod_{j=1}^{s} (z - z_j)^{\tilde{r}_{ij}} \text{ for } i = 1, \ldots, m.
\]

Where the tuple \((z_1, ..., z_s)\) varies in \( U \) defined above and \( \tilde{r}_{ij} \) is a lift of
If \( f : C \rightarrow T \) is a family of abelian covers constructed as above, we write \( J \rightarrow T \) for the relative Jacobian of \( C \) over \( T \). This family gives a natural map \( \phi : T \rightarrow A_g \). Let \( Z = Z(N, s, r) = \phi(T) \) be the closure of \( \phi(T) \) in \( A_g \). Such a family therefore gives rise to a closed subvariety \( Z = Z(N, s, r) \) in the moduli space \( A_g \) and we have \( \dim Z = s - 3 \). This is because any three points on \( \mathbb{P}^1 \) can be moved to the points 0, 1, \( \infty \). We call the subvariety \( Z \), the moduli variety associated to the family \( f : C \rightarrow T \).

**Remark 1.1.** Note that a Galois cover \( f : C \rightarrow \mathbb{P}^1 \) branched above the points \( S = \{ z_1, ..., z_s \} \) corresponds to a surjection \( \phi : \pi_1(\mathbb{P}^1 \setminus S) \rightarrow G \) (See [V], Theorem 5.14). The fundamental group \( \pi_1(\mathbb{P}^1 \setminus S) \) is generated by the loops \( \gamma_j \) around \( z_j \) with only the condition that \( \gamma_1...\gamma_s = 1 \) and the local monodromy around \( z_j \) is given by \( \phi(\gamma_j) \).

### 1.1. The local system associated to an abelian cover

In this section we are going to represent an alternative construction of abelian coverings using line bundles and local systems. This construction resembles that of [EV] in the case of cyclic coverings of algebraic varieties. Let \( G \) be a finite abelian group. We denote by \( \mu_G \) the group of the characters of \( G \), i.e. \( \mu_G = \text{Hom}(G, \mathbb{C}^*) \). Consider a Galois covering \( \pi : X \rightarrow \mathbb{P}^1 \) with Galois group \( G \). The group \( G \) acts naturally on the sheaves \( \pi_*\mathcal{O} \) and \( \pi_*(\mathcal{C}) \) via its characters i.e. \( f(gx) = \chi(g)f(x) \) for \( \chi \in \mu_G \). Under this action the sheaf decomposes as direct sum of the eigensheaves corresponding to the characters of \( G \). Let \( L_\chi^{-1} = \pi_*\mathcal{O}_X |_\chi \) and \( \mathbb{C}_\chi = \pi_*\mathcal{C}_\chi \) denote the eigensheaves corresponding to the character \( \chi \). \( L_\chi \) is a line bundle and outside of the branch locus of \( \pi \), \( \mathbb{C}_\chi \) is a local system of rank 1. We will look more closely on these sheaves and describe them in detail. Consider an abelian cover given by the equations above which is branched above \( s \) points. For \( j = 1, 2, ..., s \), let \( G_j \) be the corresponding inertia subgroup of \( z_j \). It is the
subgroup of $G$ consisting of elements that pointwise fix the elements of the inverse image $\pi^{-1}(z_j)$. It is a cyclic subgroup of $G$ and it’s order is equal to the ramification order of $z_j$ which we have seen is equal to $N/\gcd(N, \tilde{r}_{ij}, \ldots, \tilde{r}_{mj})$. Let $g_j$ be the generator of $G_j$. Since $G$ is a finite abelian group, we can consider each element of $G$ as a root of unity and therefore we get that $g_j$ can be identified with $\alpha_j = e^{2\pi i \mu_j}$ where $\mu_j = \frac{\gcd(\tilde{r}_{1j}, \ldots, \tilde{r}_{mj})}{N}$. We call the $\alpha_j$ the local monodromy data around $z_j$. Now we can describe the eigensheaves $C^\chi$ and $L^\chi$.

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Theorem 1.1.1. The monodromy representation of the local system $\pi_*C|_{\mathbb{P}^1 \setminus S}$ is given by:

$$\rho : \pi_1(\mathbb{P}^1 \setminus S) \to GL_N(\mathbb{C})$$

$$\gamma_k \mapsto \text{diag}(e^{2\pi i \mu_j} | j = 0, 1, \ldots, N - 1).$$

Following [P1], we call the bundles $L^\chi$ and $z_j, j = 1, \ldots, s$ considered as divisors in $\mathbb{P}^1$ the building data of the cover. The reason is that these data determine the cover completely. Let us now give a more explicit description of the building data for an abelian cover given by the equations above.

Lemma 1.1.2. Let $\tilde{A} = (\tilde{r}_{ij})$ be the lifted matrix of $A$. Let $a = (a_1, \ldots, a_m) \in G \subseteq \mathbb{Z}_N^m$ and consider $a.\tilde{A} = (a_1, \ldots, a_s)$. In this product, the $a_i$ are regarded as integer between $0$ and $N$ and hence the $\alpha_i$ are regarded as integers. The sheaf $\pi_*(\omega)$ also decomposes with respect to the Galois group action. For the line bundles $L^\chi$ corresponding to the character $\chi$ associated to the element $a \in G$ and $\pi_*(\omega)_\chi$ we have:

$$L^\chi = \mathcal{O}_{\mathbb{P}^1}(\sum_1^s < \frac{a_i}{N} >),$$

where $< x >$ denotes the fractional part of the real number $x$.

and $\pi_*(\omega)_\chi = \omega_{\mathbb{P}^1} \otimes L^\chi^{-1} = \mathcal{O}_{\mathbb{P}^1}(-2 + \sum_1^s < -\frac{a_i}{N} >)$
Proof. Note that since the sum of the columns of the matrix $A$ is zero, the above sum is an integer. One can easily see that each section of the line bundle $\mathcal{O}_{\mathbb{P}^1}(\sum^i_{l,} < \frac{P_l}{P} >)$ is a function on which the Galois group acts as $\chi$ and conversely any such section must be a function of the above form. The rest of the lemma is [P2], Proposition 1.2.

Remark 1.1.3. Note that in the case of a cyclic cover of $\mathbb{P}^1$, the bundles $L_\chi$ and local systems $\mathbb{C}_\chi$ coincide with the bundles $L^{(j)}$ and the local systems $\mathbb{L}_j$ in [R], as one expects naturally.

1.2. Two Shimura varieties containing $Z$

In this subsection we describe two naturally constructed Shimura subvarieties of $A_g$ associated to a family $f : C \to T$ of abelian covers that contain the moduli variety $Z$. The construction that we describe below and Lemma 1.2.1 are based on the works in [M1] and [MO] which we just reformulate for abelian covers. For the sake of completeness, we give the proof of Lemma 1.2.1 in detail.

Let $\mu_G$ be the group of characters of the abelian group $G$. The Jacobians in the family $J \to T$ admit naturally an action of the group ring $\mathbb{Z}[\mu_G]$. This action defines a Shimura subvariety of PEL type $S(\mu_G)$ in $A_g$ that contains $Z$. More precisely, by fixing a base point $t \in T$, there is a Hodge structure on $V = H_1(C_t, \mathbb{Z})$ which corresponds to a point $y \in \mathcal{H}_g$ in the Siegel space. On the vector space $V_Q$ there is a natural action of $F = \mathbb{Q}[\mu_G]$ and so $V_Q$ has also the structure of an $F$-module. $F$ is equipped with a natural involution $\ast$. The polarization $\phi$ on $V_Q$ satisfies.

$$\phi(bu, v) = \phi(u, b^*v)$$

for all $b \in F$ and $u, v \in V$.

Define the subgroup $M$ as in [MO]:

$$M = Gsp(V_Q, \phi) \cap GL_F(V_Q)$$

If $h_0 : S \to Gsp_{2g,\mathbb{R}}$ is the Hodge structure on $V = H_1(C_t, \mathbb{Z})$ corresponding to the point $y \in \mathcal{H}_g$, then by the above $F$-action, this homomorphism factors through the subvariety $M_{\mathbb{R}}$. Define
\[ Y_M = \{ x : S \to Gsp_{\mathbb{R}} | x \text{ factors through } M_{\mathbb{R}} \} \]

Where each homomorphism \( x : S \to Gsp_{\mathbb{R}} \) is in the conjugacy class of the homomorphism \( h_0 \). By the above, the point \( y \) lies in \( Y_M \) and there is a connected component \( Y^+ \subseteq Y_M \) containing \( y \) and \( S(\mu_G) \) is equal to the image of the quotient map \( \mathcal{H}_g \to Gsp(V, \phi) \backslash \mathcal{H}_g \cong A_g \).

Since \( Z \subseteq S(\mu_G) \), we have that \( s - 3 \leq \text{dim} S(\mu_G) \). Therefore if \( \text{dim} S(\mu_G) = s - 3 \), it follows that \( Z = S(\mu_G) \) and hence \( Z \) will be a Shimura subvariety of \( A_g \). We can find the dimension of the variety \( S(\mu_G) \) by finding the tangent space to it at an arbitrary point. To do this we have to consider the eigenspaces of the action of the group \( G \) on cohomology. The group \( G \) acts naturally on the cohomology \( H^1(C_t, \mathbb{C}) \). There is also a natural action on \( H^1(0, C_t, V, \phi) \). By the action of \( G \), for every \( n \in G \), there is an eigenspace \( H^0(0, C_t, V, \phi)(n) \). Put \( d_n = \text{dim}_{\mathbb{C}} H^0(0, C_t, V, \phi)(n) \). We have:

**Lemma 1.2.1** \( \text{dim} S(\mu_G) = \sum_{2n \neq 0} d_n d_{-n} + \frac{1}{2} \sum_{2n=0} d_n (d_n + 1) \).

Note that 2.0 = 0 in \( G \) and \( d_0 = 0 \), so in fact the second sum in the right hand side of the above equality is always meaningful and if \( |G| \) is an odd number it will be zero.

**Proof.** We calculate \( \text{dim} T_y(Y_M) \) at the point \( y \in \mathcal{H}_g \). The dimension of tangent space of \( S(\mu_G) \) at the point \( y \) will be equal to this number. to compute \( \text{dim} T_y(\mathcal{H}_g) \), we note that:

\[
T_y(\mathcal{H}_g) = \text{Hom}^{sym}(F^{1,0}, V_{\mathbb{C}}/F^{1,0}) := \{ \beta : F^{1,0} \to V_{\mathbb{C}}/F^{1,0} \mid \phi(v, \beta(v')) = \phi(v', \beta(v)) \forall v, v' \in F^{1,0} \}
\]

i.e. the elements of \( T_y(\mathcal{H}_g) \) are given by the symmetric homomorphisms with respect to \( \beta \) from \( F^{1,0} \) to \( V_{\mathbb{C}}/F^{1,0} \), which means that each \( \beta \) is its own dual via the isomorphisms induced by \( \phi \). The subspace \( T_y(Y_M) \subseteq T_y(\mathcal{H}_g) \) is therefore given by the elements \( \beta \in \text{Hom}^{sym}(F^{1,0}, V_{\mathbb{C}}/F^{1,0}) \), that respect the \( F \)-action on \( V \), that is, are \( F_{\mathbb{C}} \)-linear. Any such \( \beta \) can be written as the sum \( \sum \beta_n \), where \( \beta_n : F^{1,0}_{C, n} \to F^{0,1}_{C, n} \) is the induced action on the eigenspaces. These \( \beta_n \) should satisfy the relation

\[
\phi_n(v, \beta_{-n}(v')) = \phi_{-n}(v', \beta_n(v))
\]
Note that the map $\tilde{\phi}_n$ induced by the polarization $\phi$, gives a duality between $F^+_{C,n}$ and $F^-_{C,(-n)}$. So we have a duality between $\beta_n$ and $\beta_{-n}$ if $n \neq -n$ in $G$. If $n = -n$ in $G$, i.e. if $2n = 0$ in $G$, this gives a self duality for $\beta_k$. Therefore $dim T_y(Y_M)$ is equal to:

$$\sum_{2n \neq 0} d_n d_{-n} + \frac{1}{2} \sum_{2n=0} d_n (d_n + 1)$$

\[\square\]

**Construction 1.2.2.** The construction of the second Shimura subvariety that contains $Z$, is in fact Mumford’s construction of ”variety of Hodge type”. Namely, let $M$ be the generic Mumford-Tate group of the family $f : C \to T$. For the definition and construction of generic Mumford-Tate group, look at [M2] or [R]. Note that $M$ is a reductive $\mathbb{Q}$-algebraic group and let $S_f$ be the natural Shimura variety associated to $M$. $S_f$ is in fact the smallest Shimura subvariety that contains $Z$ and it’s dimension depends on the real adjoint group $M^{ad}_R$. Namely, if $M^{ad}_R = Q_1 \times ... \times Q_r$ is the decomposition of $M^{ad}_R$ to $\mathbb{R}$-simple groups, then $dim S_f = \sum \delta(Q_i)$. If $Q_i(\mathbb{R})$ is not compact, then $\delta(Q_i)$ is the dimension of the real group $Q_i$ which can be read from table V of [H]. If $Q_i(\mathbb{R})$ is compact i.e. if $Q_i$ is anisotropic, we set $\delta(Q_i) = 0$. We remark that for $Q = PSU(p,q)$, $\delta(Q) = pq$ and for $Q = Sp_{2p}$, $\delta(Q) = \frac{p(p+1)}{2}$. In particular, $Z$ is a Shimura subvariety if and only if $\sum \delta(Q_i) = s - 3$, i.e. if and only if $dim Z = dim S_f = s - 3$.

**Computation of $d_n$.** We have seen that the dimension of the Shimura variety $S(\mu_G)$ can be expressed in terms of the dimension of the eigensapces of Galois action on the cohomology of the fibers. We will now try to compute these dimensions. Let $n = (a_1, ..., a_m) \in G$ be an element. Since $G$ is a finite abelian group, the groups $G$ and $\mu_G$ are isomorphic and hence their elements correspond. Let $\chi \in \mu_G$ be the character corresponding to $n$ and $\tilde{A}$ be the lifted matrix of $A$, the matrix of the abelian cover, and the $\alpha_i$ as in Lemma 1.1.2. We have the following result:

**Proposition 1.2.3.** For $C$ an abelian cover of $\mathbb{P}^1$, we have:

$$d_n = h^1_{\chi} \cdot (C) = -1 + \sum_{i} \langle -\frac{\alpha_i}{N} \rangle$$
Proof. By Lemma 1.1.2, we have that:

\[ \pi_*(\omega) = \omega_{\mathbb{P}^1} \otimes L_{\chi^{-1}} = \mathcal{O}_{\mathbb{P}^1}(-2 + \sum s_i < -\frac{\alpha_i}{N}) \]

It follows that:

\[ h^{1,0}_\chi(C) = h^0(\pi_*(\omega)) = -1 + \sum s_i < -\frac{\alpha_i}{N}. \]

Note that it naturally follows that

\[ d_{-n} = h^{1,0}_{\chi^{-1}}(C) = h^0_{\chi}(C) = -1 + \sum s_i < \frac{\alpha_i}{N}. \]

Remark 1.2.4. There are other methods to compute the dimension of the eigenspaces \( d_n \). Note that the abelian Galois group \( G \) of the covering is a (possibly proper) subgroup of \( \mathbb{Z}^m_N \) and therefore we can show an element of \( G \) as an \( m \)-tuple \( n = (a_1, \ldots, a_m) \). The space of differential forms with respect to the character \( n \), is generated over \( \mathbb{C}(z) \) by the form \( \prod(z - z_j)^{-t_j(-n)}dz \), where \( t_j(n) =< \sum a_i \tilde{r}_{ij} > \) and \( < . > \) denotes the fractional part of a real number. It is then straightforward to check that a meromorphic form \( p(z) \prod(z - z_j)^{-t_j(-n)}dz \) is holomorphic if and only if \( p(z) \) is a polynomial of degree at most \( t(n) = \sum t_j(-n) - 1 \). So that the dimension of \( H^{1,0}_n \) is equal to \( t(n) \) (see [W], Lemma 2.6). Alternatively, one can use the Chevalley-Weil formula to compute the the dimension of the eigenspaces. See [CW].

2. Examples of Shimura varieties arising from abelian covers

In [M1], Moonen completed the list of Shimura subvarieties generated by families of cyclic covers of \( \mathbb{P}^1 \) and proved that in the locus of cyclic covers of \( \mathbb{P}^1 \), there is no more Shimura varieties. The fiber genus of these families is bounded by 8, confirming the bound given by the corrected version of Coleman conjecture, see page 2. In [MO], Oort and Moonen give a table of 7 examples of abelian non-cyclic Galois covers of \( \mathbb{P}^1 \), that generate Shimura subvarieties in \( A_g \). All of these examples satisfy the equality \( \dim S(\mu_G) = s - 3 \). Their method to obtain these examples is based on analyzing the decomposition of Jacobians up to the isogeny under the action of group ring \( \mathbb{Q}[\mu_G] \). Our argument here is based on checking the equality of Lemma 1.2.1.
Table 1: Monodromy data of families of abelian coverings that generate Shimura subvarieties.

| genus | Galois group | N  | monodromy data                          |
|-------|--------------|----|-----------------------------------------|
| 1     | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | 2  | $\{(1,0)(1,0)(0,1)(0,1)\}$             |
| 2     | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | 2  | $\{(1,0)(1,0)(1,1)(0,1)\}$             |
| 3     | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ | 4  | $\{(2,0)(2,1)(0,1)(0,2)\}$             |
| 3     | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ | 4  | $\{(2,0)(2,2)(0,1)(0,1)\}$             |
| 3     | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | 2  | $\{(1,0)(1,1)(1,1)(0,1)(0,1)\}$        |
| 4     | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ | 6  | $\{(3,0)(3,1)(0,2)(0,3)\}$             |
| 4     | $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ | 3  | $\{(1,0)(1,0)(1,2)(0,1)\}$             |

Checking whether this equality holds is something that can be checked by a computer and by using a computer program we have checked the examples which satisfy this equality. However, our computer search did not provide a further example satisfying $\dim S(\mu_G) = s - 3$. We are able, however to prove that for $s = 4$, the table contains all examples with $\dim S(\mu_G) = s - 3 = 1$ (see Theorem 2.1 below). We moreover study the families that do not appear in the table above i.e. those that do not satisfy the equality $\dim S(\mu_G) = s - 3$. In this case, $Z \neq S(\mu_G)$ but it does not imply that $Z$ is not a Shimura subvariety: it could still be a smaller Shimura subvariety (inside $S(\mu_G)$) or in other words, there might be Hodge classes, that are not given by the action of $\mathbb{Z}[\mu_G]$. We are able to show that some large classes of families, including all families with $s = 4$, do not give rise to Shimura subvarieties in $A_g$ provided that the family satisfies an irreducibility condition (see page 19).

**Theorem 2.1.** The families in table 1, give rise to Shimura subvarieties in $A_g$. Moreover for $s=4$ this table contains all examples for which $\dim S(\mu_G) = s - 3 (= 1)$ and contains all such examples for $5 \leq s \leq 7$, $2 \leq m \leq 5$ and $N \leq 20$.

**Proof.** One can compute the dimensions $d_n$ of eigenspaces with the aid of the formula in Proposition 1.2.3. It is straightforward to check that in
all of these cases $\dim S(\mu_G) = \dim Z = s - 3$ and therefore $Z = S(\mu_G)$ is a Shimura subvariety of $A_g$. For $s = 4$, note that if $\dim S(\mu_G) = 1$, then by using Lemma 1.2.1, there is a unique $a \in G$, such that $d_a = d_{-a} = 1$ and for all other $n \in G$, $d_n d_{-n} = 0$. We may therefore assume that the first row of the matrix $A$ satisfies this equality. By results of [R] we know that there are only finitely many of these with $N \leq 12$ and by the aforementioned computer program we may check that the above examples are the only ones which satisfy $\dim S(\mu_G) = 1$ and $N \leq 12$. This means that table 1 contains all examples with $\dim S(\mu_G) = 1$. By using the same computer program we see that this table contains all examples with $\dim S(\mu_G) = s - 3$ and with conditions as above. \qed

3. The Dwork-Ogus obstruction

As we remarked earlier, we are going to exclude further examples of Shimura subvarieties arising from families of abelian covers. To do this we will need an obstruction introduced by Dwrok and Ogus in [DO]. Although we will encounter cases that we can not use this obstruction, it remains a crucial tool in proving that a certain variety is not a Shimura subvariety. The construction of the obstruction is as follows:

Let the $f : C \to T$ be a family of smooth projective curves with an irreducible base scheme $T$. We denote the sheaf of relative differentials with $\omega_{C/T}$ and the Hodge bundle $E = E(C/T) = f_* \omega_{C/T}$. Consider the Kodaira-Spencer map $\kappa : \text{Sym}^2(E) \to \Omega^1_T$ (usually the dual of this map is defined to be the Kodaira-spencer map, but as we will mainly work with this map, rather than the original Kodaira-Spencer, we name it as such). The multiplication map $\text{mult} : \text{Sym}^2(E) \to f_*(\omega_{C/T}^\otimes 2)$ induces the following sheaves:

$$\mathcal{K} = \text{Ker}(\text{mult}) = \text{Ker}(\text{Sym}^2(E) \to f_*(\omega_{C/T}^\otimes 2))$$

$$\mathcal{L} = \text{Coker}(\text{mult}^\vee) = \text{Coker}((f_* \omega_{C/T}^\otimes 2)^\vee \to \text{Sym}^2(E)^\vee),$$

If the fibers are not hyperelliptic, by a famous result of Max Noether (see [OS]), $\text{mult}$ is surjective and $\mathcal{K}$ is dual to $\mathcal{L}$. 13
Now, if $C$ is a smooth projective curve over a field $k$ of positive characteristic with an ordinary Jacobian (we call such a curve an ordinary curve) and a principal polarization $\lambda$, Serre-Tate theory (see [K1] or [DO]) guarantees that there exists a canonical lifting $J^{can}$ of $J$ to the Witt ring $W(k)$. The question of whether the canonical lifting of a Jacobian $J$ is again a Jacobian has been of main interest and Dwok and Ogus have shown in [DO] that even over the Witt ring of length 2, this is a very restrictive condition and in general is not true. Their method consists of constructing an obstruction $\beta$, such that $\beta = 0$ if and only if the canonical lifting $J^{can}$ is a Jacobian. They then show that this obstruction is generically non-zero. We recall the construction of $\beta$ in short. The curve $C$ is called $W_2$-canonical if the canonical lifting $(J^{can}, \lambda^{can})$ over $W_2(k)$ is isomorphic to the Jacobian of a smooth projective curve $Y$ as a principally polarized abelian variety. According to Dwork-Ogus theory, the obstruction $\beta_C$ to the existence of such $Y$, is the restriction of an element $\beta_C \in Sym^2 (F^1) \text{sym}^2 (E)$ to the kernel ker$(\mu_C)$ of the multiplication map. This obstruction can be generalized to an obstruction for families $f: C \rightarrow T$ of ordinary curves to give an obstruction $\tilde{\beta}_{C/T}$ which is a global section of $F_{T}^*L(C/T)$ where $F_T : T \rightarrow T$ denotes the absolute Frobenius map and the value of $\tilde{\beta}_{C/T}$ at $t \in T$ is equal to $F^*_k(\beta_{C/t})$. Note that since the family is assumed to have ordinary fibers, the inverse Cartier operator $\gamma : F_T^*E \rightarrow E$ is an $O_T$-linear map and in fact it is the inverse transpose of the Frobenius action on $R^1f_*O_C$. By a result of Katz in [K2], the pull-back $F_{T}^*L(C/T)$ comes equipped with a natural flat connection:

$$\nabla : F_{T}^*L \rightarrow F_{T}^*L \otimes \Omega^1_{T/k}.$$ 

For the Dwork-Ogus obstruction $\tilde{\beta}_{C/T}$, it holds that $-\nabla \tilde{\beta}_{C/T} : F_T^*\mathcal{K} \rightarrow \Omega^1_{T/k}$ is equal to the composition

$$F_T^*\mathcal{K} \rightarrow F_T^*Sym^2(E) \xrightarrow{S^2(\gamma)} Sym^2(E) \xrightarrow{\kappa} \Omega^1_{T/k} \ (\ast)$$

In the above sequence $\gamma : F_T^*E \rightarrow E$ is the inverse Cartier operator i.e. the inverse transpose of the Frobenius action on $R^1f_*O_C$. The matrix of this map will be called the Hasse-Witt matrix of the family. The map $\kappa : Sym^2(E) \rightarrow \Omega^1_{T/k}$ is the Kodaira-Spencer map associated to the family $f : C \rightarrow T$.

Using the above description of $\nabla \tilde{\beta}_{C/T}$, this gives us something computable which we will use later to show that the obstruction is not zero for our families.
Let \( f : C \to T \) be as usual a family of abelian covers as in section 1. We can choose a prime number \( p \equiv 1 \pmod{N} \) and an open subset \( U \) of \( T \otimes \mathbb{F}_p \) such that for all \( t \in U \), the fibers are ordinary curves in characteristic \( p \). This is possible for example by results of [B]. For such \( p \) and \( U \), consider the restricted family \( C_U \to U \). The abelian group \( G \) also acts on the sheaves \( \mathcal{L}(C_U/U) \) and gives the eigensheaf decomposition \( \mathcal{L}(C_U/U) = \bigoplus_{n \in G} \mathcal{L}(n) \). The same is true for \( \mathbb{E}_U = \mathbb{E}(C_U/U) \) and \( \mathcal{K}_U = \mathcal{K}(C_U/U) \). This in turn, gives us the decomposition \( \tilde{\beta}_{C_U/U} = \sum_n \tilde{\beta}_n \). Here \( \tilde{\beta}_n \) is considered as a section of \( F^*_U \mathcal{L}_n \).

The main observation here is that if the family gives rise to a Shimura subvariety in \( A_g \), then the Dwork-Ogus obstruction vanishes:

**Lemma 3.1.** For prime number \( p \) and open subset \( U \) as above, if the family gives rise to a Shimura subvariety \( Z \subseteq A_g \), then for any \( t \in U \) we have that the Jacobian \( J_t \) is pre-\( W_2 \)-canonical and in particular \( \tilde{\beta}_{C_U/U} = 0 \).

**Proof.** This follows from [M3] or [N]. In fact if the moduli variety \( Z \) is a Shimura subvariety, and \( t \in T \) is an ordinary point (i.e. it’s pre-image is an ordinary curve), then the canonical lifting \( J_t^{\text{can}} \) of \( J_t \) is a \( W(k) \)-valued point of \( Z \). This means in particular that it is a Jacobian and hence \( J_t \) is pre-\( W_2 \)-canonical. By Dwork-Ogus theory, this forces \( \tilde{\beta}_{C_U/U} \) to be zero.

Now assuming that the fibers of the family are ordinary over \( U \) and the family gives rise to a Shimura subvariety in \( A_g \), it follows from the Lemma 3.1 that \( \tilde{\beta}_{C_U/U} = 0 \) and hence \( \nabla \tilde{\beta}_{C_U/U} = 0 \). This shows that the composition map \((*)\) should vanish identically.

From now on we just work with the restricted family \( C_U/U \) whose fibers are all ordinary instead of \( C/T \) and denote it simply as \( C/U \). Next we remark that the sequence \((*)\) factors through the map

\[
\text{Sym}^2(\mathbb{E}) \to \text{Sym}^2(\mathbb{E})(0) \xrightarrow{\text{mult}(0)} \mathcal{F}_* (\omega_{C/U}^2)(0)
\]

Where by the index \((0)\) we mean the subspace of invariant elements under the action of \( G \), i.e. the subspace on which \( G \) acts with the trivial character.
Note that $f_*(\omega_{C/U}^{\otimes 2})(0)$ is a locally free sheaf of rank $N-3$, see [FK], V.2.2. The above factorization follows from the general fact that the fiber of $\text{Sym}^2(E)(0)$ at $t$ can be identified with (dual of) the space of $G$-equivariant deformations of $C_t$, i.e. the deformations for which the $G$-action also deforms along. Since there is a $G$-action on our whole family, the Kodaira-Spencer map should factor through the above map. The last map in the above sequence is just multiplication of forms.

**Proposition 3.2.** With notations as above, the map

$$F^*_U K(0) \rightarrow F^*_U \text{Sym}^2(E_U)(0) \xrightarrow{S^2(\gamma)} \text{Sym}^2(E_U)(0) \xrightarrow{\text{mult}(0)} f_*(\omega_{C/U}^{\otimes 2})(0)$$

vanishes identically, provided that the family gives rise to a Shimura subvariety in $A_g$.

**Proof.** Let us first note that the induced Kodaira-Spencer map $\kappa(0) : f_*(\omega_{C/T}^{\otimes 2})(0) \rightarrow \Omega^1_T$ is injective. We remark that this map is in fact the dual of the usual Kodaira-Spencer map $\kappa : \Theta_T \rightarrow H^1(\Theta_{C/T})$. Therefore its injectivity means the surjectivity of the Kodaira-Spencer map i.e. the versality (or completeness) of our family. Now if $D/k[\epsilon]$ is a $G$-equivariant first order deformation of the fiber $C_t$, the versality means that $D/k[\epsilon]$ can be obtained by pull-back from our family. But this is true, because in this case $D/G$ is isomorphic to $\mathbb{P}^1_{k[\epsilon]}$ and so as an abelian cover of $\mathbb{P}^1$, it can be obtained by pull-back from our family. If the fibers are non-hyperelliptic, the vanishing of the above map follows directly from the theory of Dwork-Ogus, see [DO], together with the injectivity of $\kappa(0)$ discussed above. In fact, according to [DO], the exact sequence (*) being equal to $-\nabla \beta_{C/U}$ vanishes identically (Lemma 3.1 above). Injectivity of $\kappa(0)$ then gives the vanishing of the claimed map. So we may assume that the fibers are hyperelliptic curves. From this point on, everything goes like [M1], Proposition 5.8. Namely, with $\iota \in \text{Aut}(C/U)$ being the hyperelliptic isomorphism we conclude from results of [OS] that although the multiplication map $\text{Sym}^2(E_U) \rightarrow f_*(\omega_{C/U}^{\otimes 2})$ is no longer surjective, the induced map $\text{mult}_\iota : \text{Sym}^2(E_U)_\iota \rightarrow f_*(\omega_{C/U}^{\otimes 2})_\iota$ on the the sheaves of invariants of $\iota$ is again surjective. Since our family is contained in the hyperelliptic locus, this implies that the map $\text{mult}(0)$ is also surjective and this forces $\beta(0)$ to be an $\mathcal{O}_U$-linear map. If $\beta(0)$ is zero, of course $\nabla \beta(0)$ will be also zero. To complete the proof one can check that 0-component analogue of the exact
sequence (*) also holds true for $\nabla \tilde{\beta}_{(0)}$.

\[ \nabla \tilde{\beta}_{(0)} \]

\section{The generalization of a lemma}

For our classification purposes, we will need the generalization to the abelian case of a lemma in [B] that concerns only with cyclic coverings, see [B], Lemma 5.1.i. This lemma allows us to compute explicitly the Hasse-Witt matrix of an abelian covering which considering the above constructions will be needed to compute the obstruction $\tilde{\beta}_{C/U}$. Let $a = (a_1, ..., a_m) \in G \subseteq \mathbb{Z}_N^m$ be an element in the Galois group of the abelian covering. Let $\widetilde{A} = (\tilde{r}_{ij})$ be the matrix whose entries $\tilde{r}_{ij}$ are lifts of the $r_{ij}$ to $\mathbb{Z} \cap [0, N)$.

Next take a prime number $p$ such that $p \equiv 1 \pmod{N}$ and let $q = \frac{p - 1}{N}$.

**Lemma 4.1.** With the notations as above, the $(h_{\nu\iota})$ entry of the Hasse-Witt matrix of the abelian covering $Y$ is given by the formula:

\[
\sum \sigma \left( \binom{\nu - \alpha_1}{l_1} \cdots \binom{\nu - \alpha_s}{l_s} \right) z_1^{l_1} \cdots z_s^{l_s}
\]

Where $\Sigma = (d_n - \iota)(p - 1) + (\nu - \iota)$ and $\left( \binom{a}{b} \right) = \frac{a!}{b!(a-b)!}$ and $[\alpha]_N$ denotes the representative of the integer $\alpha$ modulo $N$ in $\{0, 1, .., N - 1\}$.

**Proof.** Like in [B] take $U_1 = \mathbb{P}^1 - \{\infty\}$ and $U_2 = \mathbb{P}^1 - \{0\}$ and $V_i = \pi^{-1}U_i$ for $i = 1, 2$. Let

\[
v_a = w_1^{a_1} \cdots w_m^{a_m} (z - z_1)^{-[\alpha_1]} \cdots (z - z_s)^{-[\alpha_s]}
\]

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Then we have:

\[
\Gamma(V_1, \mathcal{O}_{V_1}) = \bigoplus_{a \in G} k[z]v_a
\]

\[
\Gamma(V_2, \mathcal{O}_{V_2}) = \bigoplus_{a \in G} k[z^{-1}]z^{-|a|-1}v_a
\]

\[
\Gamma(V_1 \cap V_2) = \bigoplus_{a \in G} k[z, z^{-1}]v_a
\]

Defining \(\xi_j = z^{-j}v_a\) for \(j = 1, \ldots, |a|\), with \(|a| = \text{dim}H^1(Y, \mathcal{O}_Y)_a\), we see that the \(\xi_j\) form a basis for \(H^1(Y, \mathcal{O}_Y)_a = \frac{\Gamma(V_1 \cap V_2)_a}{\Gamma(V_1)_a + \Gamma(V_2)_a}\).

If \(B_a\) is the matrix of the Hasse-Witt map \(F: H^1_a \to H^1_a\), then the \((i, j)\) entry of \(B_a\) is given by the coefficient of \(\xi_i\) in \(\xi_j^p\). This follows from the fact that the \(\xi_j \otimes 1\) determine a local basis for the bundle \(F^*_T(R^1f_\ast \mathcal{O}_C)_a\) and the Hasse-Witt operator \(\gamma: F^*_T(R^1f_\ast \mathcal{O}_C)_a \to (R^1f_\ast \mathcal{O}_C)_a\) with respect to these bases is given by the \(p\)-th power endomorphism of \(\mathcal{O}_C\). This is because of the fact that the \(p\)-linear composite map

\[
R^1f_\ast \mathcal{O}_C \to F^*_T(R^1f_\ast \mathcal{O}_C) \to (R^1f_\ast \mathcal{O}_C)
\]

is induced by the \(p\)-th power endomorphism of \(\mathcal{O}_C\) (cf.[K], 2.3.4.1.4). Now one sees that the coefficient of \(\xi_i\) in this polynomial is as claimed above:

\[
\sum \sum_{l_i=\Sigma} (\text{q.}_{\lfloor -\alpha_1 \rfloor_N})_{l_1} \cdots (\text{q.}_{\lfloor -\alpha_s \rfloor_N})_{l_s}z_{l_1} \cdots z_{l_s}.
\]

Where \(\binom{n}{b} = \frac{\alpha!}{b!(\alpha-b)!}\) and where \([\alpha]_N\) denotes the residue of an integer \(\alpha\) modulo \(N\). Occasionally, we just drop \([\alpha]_N\) and write only \(\alpha\). \(\square\)

5. Excluding non-Shimura examples

At this point we are going to exclude the families of abelian covers of the projective line that do not give rise to Shimura subvarieties in \(A_g\). For some
technical reasons, working with families with 4 branch points is different from families with more branch points and it should be noted that this is in some sense the most important case, as most of the examples of Shimura families that have been found in [M1] or [MO] are obtained from families with 4 branch points. We therefore distinguish between this case and other cases. Before we state our results, we give an ”irreducibility condition” that we assume all of the families till end of these notes satisfy:

**Condition (§).** We say that the family satisfies the condition (§) if the rows of the associated matrix are linearly independent over $\mathbb{Z}/N\mathbb{Z}$. For families of cyclic covers, this implies that the family is irreducible. For families of abelian covers it implies that all of the intermediate cyclic covers are irreducible. As we said earlier, we assume that all of the families till end of this note satisfy this condition.

### 5.1. The case of four branch points

**Theorem 5.1.1.** Let $Y \to T$ be a family of abelian covers with $s = 4$, i.e. with 4 branch points. Then the associated subvariety $Z \subseteq A_g$ is a Shimura subvariety if and only if $Z = S(\mu_G)$. i.e. if and only if it appears in table 1.

**Proof.** Clearly the statement holds if $Z = S(\mu_G)$ as $S(\mu_G)$ is a Shimura variety of PEL type. Now assume on the contrary that $Z \neq S(\mu_G)$ but $Z$ is a Shimura subvariety and we will derive a contradiction. By Theorem 2.1, the assumption $Z \neq S(\mu_G)$ implies that $\dim S(\mu_G) > 1$. Note that in this theorem we have classified all cases where $\dim Z = \dim S(\mu_G) = 1$, i.e. the cases for which $s = 4$ and $Z = S(\mu_G)$. The fact that $\dim S(\mu_G) > 1$ shows that there are pairs $a, a' \in G$ with $a' \neq \pm a$ such that $d_a = d_{-a} = 1$ and $d_{a'} = d_{-a'} = 1$. Therefore for every $l \in \{\pm a, \pm a'\}$, the Hasse-Witt matrix $A_l$ is a polynomial in $\mathbb{F}_p[z_1, \ldots, z_4]$. Note that according to the discussion just before Lemma 3.1, there is an open subset $U$ of $T$ and a suitable prime number $p$, such that all fibers above $U$ are ordinary after reduction mod $p$. Now since the fibers are all ordinary curves in $U$, we conclude that the Hasse-Witt operator is an isomorphism and so $A_l$ is invertible as a section of $O_U$. Note that $\omega_a \cdot \omega_{-a} = \omega_{a'} \cdot \omega_{-a'}$ is a non-zero section of the bundle $f_*(\omega^{\otimes 2})$ and so we must have:
\[ A_a \cdot A_{-a} = A_{a'} \cdot A_{-a'} \]

as polynomials.

We will show that this identity can not happen with the above conditions. The polynomials \( A_l \) are given by the above lemma and we could set \( B_l = A_l \mid z_1 = 0 \). It means that we have:

\[
B_l = \sum_{j_1 + j_2 + j_3 = N - 1} \left( \frac{q^{|-\alpha_2|N}}{j_2} \right) \cdots \left( \frac{q^{|-\alpha_4|N}}{j_4} \right) z_2^{j_2} \cdots z_4^{j_4}.
\]

Let \( r_a(l) \) be the largest integer \( r \) such that \( B_l \) is divisible by \( t_r l \). We have that

\[
r_a(l) = \max\{0, q.\alpha_1 + q.\alpha_2 - (N - 1)\}.
\]

Similarly let \( r_{\pm a}(l) \) be the largest integer \( r \) such that \( B_a \cdot B_{-a} \) is divisible by \( t_r l \). We have:

\[
r_{\pm a}(l) = q.\max\{\alpha_1 + \alpha_l, \alpha_k + \alpha_\lambda\} - (N - 1).
\]

Now the equality \( A_a \cdot A_{-a} = A_{a'} \cdot A_{-a'} \) implies that \( r_{\pm a}(l) = r_{\pm a'} \) and so we get the following equality:

\[
\{\alpha_1 + \alpha_l, \alpha_k + \alpha_\lambda\} = \{\alpha'_1 + \alpha'_l, \alpha'_k + \alpha'_\lambda\}.
\]

By an easy lemma in [M1] (Lemma 6.3) we conclude that there exists an even permutation \( \sigma \in A_4 \) of order 2, such that \( \alpha_i = \alpha'_{\sigma(i)} \). We first claim that \( \sigma \neq 1 \). This in fact follows from the above technical condition (*) which ensures that \( \sigma \) is not trivial i.e. that \( \alpha_i \) and \( \alpha'_i \) are not all the same. Furthermore, without loss of generality we can assume that \( \alpha_i = r_{1i} \) for all \( i = 1, \ldots, 4 \). That is, we may consider \( (\alpha_1, \ldots, \alpha_4) \) as the first row of the matrix \( A \) of the abelian covering. We set \( a_i = \alpha_i \) instead of \( r_{1i} \) for simplicity. Now since \( \alpha_i \) and \( \alpha'_i \) are different by the above argument, we may again without loss of generality suppose that:

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\[ \alpha_1' = a_2, \alpha_2' = a_1 \]

\[ \alpha_3' = a_4, \alpha_4' = a_3 \]

by our assumptions on \( a_i \) and \( a_i' \), we have that

\[ \sum[a_i]_N = \sum[a_i']_N = 2N \]

Suppose that \([a_1]_N + [a_2]_N = [a_3]_N + [a_4]_N = N\), or in other words, \([a_2]_N = -[a_1]_N\) and \([a_4]_N = -[a_3]_N\) in \(\mathbb{Z}/N\mathbb{Z}\). This means that the two rows \(n = (a_1, \ldots, a_4)\) and \(n' = (a_1', \ldots, a_4')\) are linearly dependent and this contradicts condition (*)). So the above equality does not hold and we may assume that \(a_1 + a_2 < N\) and \(a_3 + a_4 > N\). Now consider the row vector

\[ n + n' = (a_1, \ldots, a_4) + (a_1', \ldots, a_4') = (a_1 + a_2, a_1 + a_2, a_3 + a_4, a_3 + a_4) \]

Note that condition (*) assures that \(n + n' \neq \pm n\) and one can easily verify that this row vector also satisfies the conditions for \(a_i\) and \(a_i'\) (in fact
\[ 2([a_1 + a_2] + [a_3 + a_4]) = 2([(N - 1)(a_1 + a_2)] + [(N - 1)(a_3 + a_4)]) = 2N \]
and so we may replace the second row \((a_1', \ldots, a_4') = (a_2, a_1, a_4, a_3)\) by this row vector and the equality \(A_n . A_{-n} = A_{n'} . A_{-n'}\) should hold for this row vector as \(n'\) and \((a_1, \ldots, a_4)\) as \(n\). We show that this is impossible. In fact, if this equality holds, it is easy to see that the left hand side must contain a monomial of the form \(z_2^\alpha z_3^\beta\) and also a monomial of the form \(z_1^\gamma z_4^\delta\). This means that \(a_2 + a_3 = a_1 + a_4 = a_1 + a_3 = a_2 + a_4 = N\) which is exactly to say that \(n = n' = (a_1, a_1, -a_1, -a_1)\). This is against our assumptions and this contradiction completes the proof. \[\square\]

5.2. The cases where \(s \geq 5\)

In this section we restrict our attention to families with such Galois groups and from now on we assume that the family has Galois group of the form \(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}\) i.e. that the matrix \(A\) is a \(2 \times s\) matrix.
In [M1] Moonen uses the Dwork-Ogus obstruction also in order to prove that further examples of Shimura families of cyclic covers of \( \mathbb{P}^1 \) do not exist for \( s > 4 \). The core observation in his proof is then that for a minimal Shimura family not existing in his table, there exists two integers \( n \) and \( n' \), such that \( d_n = d_{-n'} = 1 \) and \( d_{-n} = d_{-n'} = s - 3 \). To deduce the existence of these two elements, he argues that for \( n \in (\mathbb{Z}/m\mathbb{Z})^* \), \( d_n + d_{-n} = s - 2 \). This point, as we will see explicitly later, does not remain necessarily true for families of abelian coverings. It could very well be that for all \( n \in G \), \( d_n + d_{-n} < s - 2 \). Example I \( \star \) below (see Proposition 5.2.6) is the simplest example of such a family. In fact by what we have seen so far, if this equality holds and the family is Shimura, then \( N = 2, 3, 4, 5, 6 \) and \( s \leq 6 \). So in order to exclude further examples first we will use another method based on the monodromy of a family of curves. We first need a definition:

**Definition 5.2.1.** Let \( f : Y \rightarrow T \) be a family of abelian Galois covers of \( \mathbb{P}^1 \) as constructed in section 1. Then \( \mathcal{L} = R^1f_*\mathbb{C} \) is a polarized variation of Hodge structures (PVHS) of weight 1. This PVHS decomposes according to the action of the abelian Galois group \( G \) and the eigenspaces \( \mathcal{L}_i \) (or \( \mathcal{L}_\chi \) where \( i \in G \) corresponds to character \( \chi \in \mu_G \)) are again variations of Hodge structures and we are mainly interested in these. Take a \( t \in T \) and assume that \( h^{1,0}((\mathcal{L}_i)_t) = a \) and \( h^{0,1}((\mathcal{L}_i)_t) = b \). Then the polarization equips \( (\mathcal{L}_i)_t \) with a Hermitian form of signature \( (a, b) \) (see [DM], 2.21 and 2.23). This implies that \( \text{Mon}^0(\mathcal{L}_i) \subseteq U(a, b) \). In this case, we say that \( \mathcal{L}_i \) is of type \( (a, b) \). The above observations are key to our further analysis. Let us first prove a lemma:

**Lemma 5.2.2.** Let \( \mathcal{L}_i \) be an eigenspace as discussed above of type \( (a, b) \) with \( ab \neq 0 \). Then \( \text{Mon}^0(\mathcal{L}_i) = SU(a, b) \), unless when \( |G| = 2l \) is even and \( i \) is of order 2 in \( G \), in which case there is a surjection from \( \text{Mon}^0(\mathcal{L}_i) \) to \( SU(n, n) = Sp_{2n} \). Where \( n = d_i \).

**Proof.** Let \( t \in T \) and let \( \chi \in \text{Hom}(G, \mathbb{C}^*) \) be the character corresponding to \( i \). Consider the cover \( f_{\chi,t} : Y_{\chi,t} \rightarrow \mathbb{P}^1 \) with group \( \chi(G) \) branched above the points \( z_j \) with local monodromy \( \chi(\phi(\gamma_j)) \) about \( z_j \). Where \( \phi \) is the surjection in Remark 1.1. Note that \( \chi(G) \) is a cyclic group and so \( f_{\chi,t} \) is in fact a cyclic cover with group \( \chi(G) \). Varying \( t \in T \), we get a family of cyclic covers of \( \mathbb{P}^1 \). The eigenspace \( \mathcal{L}_i \) is exactly the eigenspace corresponding to this family (or in other words, it is the \( \mathcal{L}_1 \) of this family of cyclic covers). Unless when
|G| = 2l is even and i is of order 2 in G, Theorem 5.1.1 of [R] applies and we get that $Mon^0(L_i) = SU(a, b)$. If |G| = 2l is even and i is of order 2 in G by taking quotient of the family $f_{x,t}$, we obtain a family of hyperelliptic curves of the form $w^2 = (z - z_1)...(z - z_{2n+2})$. Note that in this case it follows from the formulas of Proposition 1.2.3 that there are $2n + 2$ odd powers in the equation of $f_{x,t}$ for $n = d_i$. Now it is well-known that $Mon^0$ of a family of hyperelliptic curves is the full symplectic group and so the proof is complete.

**Remark 5.2.3.** Assume that $Y \to T$ is a family of curves and let $M$ be the generic Mumford-Tate group of this family. Recall from construction 1.2.2, that there is a natural Shimura variety $S_f = Sh(M,Y)$ associated to $M$ (which is a reductive group) and the dimension of $S_f$ only depends on $M^\text{ad}$. The Shimura datum comes from the Hodge structures of the fibers in the family. This Shimura variety is the smallest Shimura subvariety in $A_g$ which contains $Z$. Our purpose is to show that for families of abelian covers with a big $s$, we have that $\sum \delta(Q_i) > s - 3$ therefore the family is not a Shimura family. Here $\delta(Q_i)$ is as in construction 1.2.2. The following remark is well-known but very important for our goals:

**Remark 5.2.4.** If the family $f : Y \to T$ gives rise to a Shimura subvariety in $A_g$, then the connected monodromy group $Mon^0$ is a normal subgroup of the generic Mumford-Tate group $M$ (in fact in this case, $Mon^0 = M^\text{der}$, see for example [M2] or [R]). Consequently, if $M_{\text{g}}^\text{ad} = \prod_i Q_i$, as a product of simple Lie groups, then there exists a subset $K \subseteq \{1,...,l\}$, such that $Mon_{0,\text{ad}}^0 = \prod_{i \in K} Q_i$.

Our strategy is to show that for large $s$, there are eigenspaces $L_i$ of types $(a_i,b_i)$ with $\{a_i,b_i\} \neq \{a_j,b_j\}$ for $i \neq j$ and such that $\sum \delta(L_i) > s - 3$. Then by the above Remark 5.2.4, we conclude that $dimS_f > s - 3$. Note that this property is far from being true for the families of cyclic covers of $\mathbb{P}^1$. For those families it can happen that all of the eigenspaces are either unitary (i.e. $a_i = 0$ or $b_i = 0$) or of the same type. Take for example the family $(11,(1,1,1,1,7))$. In this case all of eigenspaces are either of type $(3,0)$ (or $(0,3)$) and hence unitary, or of type $(1,2)$.

**Remark 5.2.5.** An important observation is that if in the family, one row, say the first row, does not have any 0 entry, then the cyclic cov
ing arising from this row is either a Shimura family (of cyclic covers) or the whole family will not be a Shimura family. This is true because if this family is not a Shimura family then by the above notations and observations, there are $\mathbb{R}$-simple factors $Q_i$, in the decomposition of $M_{1,\mathbb{R}}^{ad}$ such that $\sum \delta(Q_i) > s - 3$. Where $M_1$ is the Mumford-Tate group associated to this family of cyclic covers. As this is a sub-Hodge structure, we know from [VZ2], that $M_1$ is a quotient of $M$ and therefore the factors $Q_i$ also occur in decomposition of $M^{ad}$ (note that $M^{ad}$ is semi-simple group with trivial center) and so $\text{dim} S_f > s - 3$ i.e. the family is not a Shimura family. On the other hand, if this cyclic family is a Shimura family it must be one of the families in [R] (or [M1]) and therefore, $N$ is one the 10 numbers in table 1 of [M1] or [R]. Of course this leaves only finitely many possibilities to investigate. So, if in one of the rows all of the entries are non-zero, according to the table in [M1], $N = 3, 4, 5, 6$ and we will exclude these in what follows, but if there are 0 entries in the rows, there are only three possibilities: $I \begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 \\ 0 & 0 & 0 & b_2 & b_3 \end{pmatrix}$, $II \begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 \\ 0 & 0 & b_1 & b_2 & b_3 \end{pmatrix}$, $III \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & b_1 & b_2 & b_3 \end{pmatrix}$ or $IV \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & 0 & b_1 & b_2 \end{pmatrix}$.

Also consider the families $I^* \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$, $III^* \begin{pmatrix} 2 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$ with $N = 3$ which is special cases of families $I$ and $III$. Then the following theorem holds:

The following proposition serves as an example for the methods and arguments based on the above remark which we will use later in the proof of Theorem 5.2.7:

**Proposition 5.2.6.** A family $Y \to T$ of abelian covers of $\mathbb{P}^1$ with $s = 5$ branch points does not give rise to a Shimura variety in $A_g$ except possibly the families $I^*$ and $III^*$.

**Proof.** Recall from the beginning of section 5, that all of the families we consider in this section are assumed to satisfy condition (*). Following the discussions in Remark 5.2.5, we see that the families $\begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 \\ 0 & 0 & 0 & b_2 & b_3 \end{pmatrix}$ are either not Shimura or can only have $N = 3, 4, 5, 6$ because for these
families the eigenspace \( \mathcal{L}_{(1,1)} \) is that of a Shimura family of cyclic covers with 5 branch points which according to [M1] (or [R]), table 1 leaves only these 4 possibilities for \( N \). An argument similar to the argument below shows that except possibly for the family \( I^* \) none of these families can give rise to a Shimura subvariety. Therefore we need only to consider families of the form

\[
\begin{pmatrix}
a_1 & a_2 & a_3 & 0 & 0 \\
0 & 0 & b_1 & b_2 & b_3
\end{pmatrix}
\] or

\[
\begin{pmatrix}
a_1 & a_2 & a_3 & a_4 & 0 \\
0 & 0 & b_1 & b_2 & b_3
\end{pmatrix}
\] or

\[
\begin{pmatrix}
a_1 & a_2 & a_3 & a_4 & 0 \\
0 & 0 & 0 & b_1 & b_2
\end{pmatrix}.
\]

We exclude the first family which is easier and our method can more easily be elucidated with this example. In this case, the eigenspace associated to the element \( (1, 1) \) is given by \((a_1, a_2, a_3 + b_1, b_2, b_3)\) so we must have \( a_3 + b_1 = 0 \) (in \( \mathbb{Z}/N\mathbb{Z} \)), otherwise by Remark 5.2.5, \( N = 3, 4, 5, 6 \), which can be excluded in each case using the same argument as the last part of the proof below (i.e. by finding eigenspaces \( \mathcal{L}_i \) with \( \sum \delta(\mathcal{L}_i) \geq 3 \)). Likewise \( a_3 - b_1 = 0 \) and so we have that \( a_3 = b_1 = \frac{N}{2} \). Consider the eigenspace associated to the element \( (2, 1) \) given by the cyclic cover \((2a_1, 2a_2, \frac{N}{2}, b_2, b_3)\) since none of \( a_1 \) and \( a_2 \) is zero, we have also that \( 2a_1 \neq 0 \) and \( 2a_2 \neq 0 \) (note that we assume that \( \sum a_i = N \), otherwise we can replace \( a_i \) with \(-a_i\) and we get an isomorphic cover for which \( \sum a_i = N \)). By what we said earlier, this implies that \( N = 4, 6 \) which we have to exclude now. For \( N = 4 \), the only possible families are

\[
\begin{pmatrix}
1 & 1 & 2 & 0 & 0 \\
0 & 0 & 2 & 1 & 1
\end{pmatrix}
\] and

\[
\begin{pmatrix}
2 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 2
\end{pmatrix}
\] and

\[
\begin{pmatrix}
1 & 1 & 2 & 0 & 0 \\
0 & 0 & 1 & 1 & 2
\end{pmatrix}.
\]

All of the 3 families are not Shimura as we will show here. Take for example the first family. By Proposition 1.2.3, the eigenspace \( \mathcal{L}_{(1,2)} \) has type \((1, 2)\) and the eigenspace \( \mathcal{L}_{(1,3)} \) has type \((1, 1)\). This shows by the above remarks that \( \text{dim} S_f = \sum \delta(Q_i) \geq 3 \). Therefore \( Z \neq S_f \) and the family is not Shimura. In the same way one sees easily that the other two families are not Shimura too. Also by the same method one can conclude that for \( N = 6 \), there does not exist any Shimura family. For \( N = 3 \) there is only one family, namely the family

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1
\end{pmatrix}
\] which is not Shimura again because there is an eigenspace of type \((1, 2)\) and another one of type \((1, 1)\) which forces \( \text{dim} S_f \geq 3 \). Now the only exceptions that do not follow the regulation above are families \( I^* \) and \( III^* \). For these two families, all of the eigenspaces are either of type \((a, 0)\) or of the same type (for \( I^* \) of type \((1, 2)\) and for \( III^* \) of type \((1, 1)\)). Note that in general if two eigenspaces are of the same type \((a, b)\), then one can not claim that they correspond to different \( Q_i \) in the decomposition of \( M^d \), because both of them may correspond to the same \( Q_i \). Hence for our dimension-argument we have to find eigenspaces of different types. \( \square \)
As we explained earlier, if one row of the family has no non-zero entries and the family is a Shimura family, then $N$ can be one of the numbers 3, 4, 5, 6. Therefore if we have a Shimura family with $N$ other than the numbers above, the rows must always contain zero. Let the number of these zeros in the first row be $l$, i.e. the family has the form

\[
\begin{pmatrix}
  a_{11} & \cdots & a_{1l} & a_{1l+1} & \cdots & a_{1r} & 0 & \cdots & 0 \\
  0 & \cdots & 0 & b_{1l+1} & \cdots & b_{1r} & b_{1r+1} & \cdots & b_s
\end{pmatrix}.
\]

Furthermore suppose that $\sum a_i > 2N$ and $\sum (-a_i) = (\sum [-a_i]_N) > 2N$ or equivalently that $d_{(1,0)} > 1$ and $d_{(-1,0)} > 1$. For $s > 19$, we prove that:

**Theorem 5.2.7.** With the above condition and if $s > 19$, then the family

\[
\begin{pmatrix}
  a_{11} & \cdots & a_{1l} & a_{1l+1} & \cdots & a_{1r} & 0 & \cdots & 0 \\
  0 & \cdots & 0 & b_{1l+1} & \cdots & b_{1r} & b_{1r+1} & \cdots & b_s
\end{pmatrix}
\]

does not give rise to a Shimura subvariety in $A_g$.

**Proof.** Assume that the associated eigenspaces are of types $(k_1, r-k_1-2)$ and $(k_2, s-l-k_2-2)$. If these two types are different, we will have:

\[
dim S_f \geq k_1(r-k_1-2) + k_2(s-l-k_2-2) \geq 2(r-4) + 2(s-l-4)
\]

Now for $s \geq 19$, one sees that $2(r-4) + 2(s-l-4) > s-3$ and hence $\dim S_f > s-3$. It remains to treat the case where the two types are the same which implies that $s = r + l$. We have:

\[
dim S_f \geq k_2(s-l-k_2-2) \geq 2(s-l-4)
\]

The right hand side is strictly greater than $s-3$ if and only if $r-l > 5$. We may therefore assume that $r-l \leq 5$. As explained in the proof of Theorem 5.2.6 we need to find eigenspaces of different types for which $\sum \delta(L_i) > s-3$. Note that in any case the above inequality implies that $\dim S_f \geq s - l - 3$. We have:
\[ t(-1, 1) = \sum_{l}^{l'} (-a_i) + \sum_{j=1}^{l'} (b_j - a_j) + \sum_{k}^{s} b_k \geq (l - 1)N. \]

That is, for the eigenspace associated to the element \((-1, 1)\), we have that \( t(-1, 1) \geq (l - 1) \) and consequently \( d_{(-1, 1)} \geq l - 2 \). Similarly one sees that \( d_{(1, -1)} \geq l - 2 \). Note that this eigenspace (i.e. the eigenspace associated with the element \((-1, 1)\)) can not be of the type \((k_2, s - l - k_2 - 2)\) (equivalently of the type \((k_1, r - k_1 - 2)\)). For otherwise we will have \( k_2 \geq l - 2 \) and \( s - l - k_2 - 2 \geq l - 2 \). The first inequality says that \( k_2 > 5 \) (because \( s > 19 \) and hence \( l > 7 \)) and the second one implies that \( k_2 \leq s - 2l \leq 5 \). This contradiction shows that we have an eigenspace of a new type. Now since \( d_{(-1, 1)}d_{(1, -1)} \geq (l - 2)^2 > l \), we conclude that:

\[ \dim S_f > s - 3 \]

and the claim follows. \( \square \)

Note that in the proof of this theorem we do not have to assume that the family satisfies condition (*)..

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