Dissipative coupling, dispersive coupling and its combination in simplest opto-mechanical systems

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We apply strategy of variational measurement to simplest variant of dissipative coupling (test mass displacement change transitivity of a single mirror) and compare it with simplest dispersive coupling (a single mirror as a test mass, which position changes the phase of reflected wave). We compare a ponderomotive squeezing in this two kinds of coupling. Also we analyze simplest variant of combined coupling, in which both dissipative and dispersive couplings are used, and show that it creates stable optical rigidity even in case of single pump. We demonstrate that variational measurement can be applied for combined coupling.

I. INTRODUCTION

Interaction of light in an optical cavity and a mechanical oscillator or a free mass is a subject of opto-mechanics [1]. The simple realization of so called dispersive opto-mechanical coupling is based on cavity in which a position of a mechanical body (movable mirror) changes eigen frequency of cavity and at the same time light pressure experiences a force proportional to optical power or number of optical quanta circulating in the optical cavity. Opto-mechanical systems having several degrees of freedom provide possibility of more complex interactions ranging from radiation pulling (negative radiation pressure) [2, 3], opto-mechanical interaction proportional to the quadrature of electromagnetic field [4–7] to the interaction depending on test mass speed (not the coordinate) of the mechanical system [8, 9].

Opto-mechanics are very important in precision measurements using transduction mechanism between the mechanical and optical degrees of freedom via enabling various sensors, like gravitational wave detectors [10–15], torque sensors [16], and magnetometers [17].

The accuracy of the mechanical position measurement in an opto-mechanical system usually is restricted due to quantum back action by so called standard quantum limit (SQL) [18, 19]. The SQL was studied in many systems ranging from macroscopic kilometre-sized gravitational wave detectors [7] to microcavities [20, 21]. An example of a measurement restricted by SQL is detection of a classical force acting on a mechanical degree of freedom of an opto-mechanical system. However, SQL of force measurement is not a fundamentally unavoidable limit. It can be surpassed using variational measurement [4, 7, 22], squeezed light usage [23–29], opto-mechanical velocity measurement [8, 9], and measurements in opto-mechanical systems with optical rigidity [30, 31].

There are two kinds of opto-mechanical coupling: dispersive and dissipative ones. For dispersive one displacement of mirror changes normal frequency of cavity, whereas for dissipative coupling displacement of test mass changes transparency of input mirror and, hence, relaxation rate of cavity. Dissipative coupling was proposed theoretically [32] and implemented experimentally [16, 33–35] nearly a decade ago. It was studied in a variety of opto-mechanical systems, including Fabry-Perot interferometer [16, 33–35], Michelson-Sagnac interferometer [36–38], and ring resonators [39, 40]. It was shown [41] that an opto-mechanical transducer based on dissipative coupling of optical and mechanical degrees of freedom gives possibility to realize quantum speed meter which, in turn, allows to surpass SQL.

In this paper we analyze dispersive and dissipative coupling in simplest opto-mechanical system without any cavity.

Recall dispersive coupling in cavity is characterized by dependence of normal frequency on position of test mass, for example, for Fabry-Perot cavity it is position of input or end mirror. So, we model dispersive coupling without cavity by movable mirror (it is free test mass), phase of reflected light depends on test mass position.

In turn, dissipative coupling in cavity means that its relaxation rate depends on test mass position, for Fabry Perot cavity it means that transmittance of input mirror depends on test mass position. So we model dissipative coupling without cavity as mirror which amplitude reflectivity $R$ and transmittance $T$ depend on position of test mass. In particular, it corresponds to Michelson-Sagnac interferometer (MSI) [36–38] as a generalized mirror (GM), where test mass is a movable completely reflecting mirror $M$.

We also consider opposite case of movable beam splitter (BS) in MSI and fixed position of mirror $M$ ($x_m$ is a constant) — it is a model of mirror with combined (both dispersive and dissipative) couplings.

II. MSI AS A GENERALIZED MIRROR

The detailed analysis of MSI is presented in Appendix A, MSI can be considered as GM with amplitude transmittance $T$ and reflectivity $R$ depending on displacements $x_m$, $y_{bs}$ (A2). Below we present displace-
Laser

\[ B_1 = -RB + TA \]
\[ R = \cos 2kx_m \]
\[ T = \sin 2kx_m \]

\[ A_1 = RA + TB \]
\[ B_1 = -RB + TA \]

Figure 1: Michelson-Sagnac interferometer in case of fixed position of BS \((y_0 = 0, \, x = 0)\) is a GM which transparency and reflectivity depends on position of test mass (completely reflecting mirror \(M\)) — example of dissipative coupling. In opposite case of movable BS and fixed position \(x_m = 0\) of mirror \(M\) it is a model of mirror with both dispersive and dissipative couplings.

\[ x_m = x_0 + x, \, y_{bs} = y_0 + y \]  \quad (2.1)

where \(x_0, \, y_0\) are mean constants (can be chosen) and \(x, \, y\) are small variables. Then we can expand \(R, \, T \) (A2) into series

\[ R \simeq R_0 + T_0 k [2x + \sqrt{2}y], \]  \quad (2.2a)
\[ T \simeq -T_0 + R_0 k [2x + \sqrt{2}y], \]  \quad (2.2b)
\[ R_0 = \cos k(2x_0 + \sqrt{2}y_0), \, \, T_0 = -\sin k(2x_0 + \sqrt{2}y_0), \]

where \(k = \omega_0/c, \, \omega_0\) is a carrier frequency of light waves. Below we put \(y_0 = 0\) for simplicity, then only \(x_0\) defines \(T_0, \, R_0\).

Below we present amplitudes of waves as large constant amplitude (denoted capital letter) plus small amplitudes (denoted by the same small letter) containing noise and signal. For example

\[ A = A + \hat{a}, \, \, \, A_1 = A_1 + \hat{a}_1, \, \, \, \text{and so on}. \]  \quad (2.3)

Input waves are in coherent state so operators \(\hat{a}, \, \hat{b}\) describe vacuum fluctuation wave, which commutator and correlator are the following

\[ [\hat{a}(t), \hat{a}^\dagger(t')] = [\hat{b}(t), \hat{b}^\dagger(t')] = \delta(t-t'), \]  \quad (2.4)
\[ \langle \hat{a}(t)\hat{a}^\dagger(t') \rangle = \langle \hat{b}(t)\hat{b}^\dagger(t') \rangle = \delta(t-t') \]  \quad (2.5)

Below we use Fourier transform defined as

\[ \hat{a}(t) = \int_{-\infty}^{\infty} a(\Omega) e^{-i\Omega t} \frac{d\Omega}{2\pi} \]  \quad (2.6)

and by a similar way for others values, denoting Fourier transform by the same letter but without the hat. For Fourier transform of the input fluctuation operators one can derive from (2.4) and (2.5):

\[ [a(\Omega), a^\dagger(\Omega')] = [b(\Omega), b^\dagger(\Omega')] = 2\pi \delta(\Omega - \Omega'), \]  \quad (2.7)
\[ \langle a(\Omega)a^\dagger(\Omega') \rangle = \langle b(\Omega)b^\dagger(\Omega') \rangle = 2\pi \delta(\Omega - \Omega') \]  \quad (2.8)

**III. SIMPLEST DISSIPATIVE COUPLINGS**

Let consider particular case when BS position is fixed \((y_0 = 0, \, y = 0)\), then MSI is GM as a model of dissipative coupling [36]: reflectivity and transmittance depends on position of mirror \(M\) (test mass).

Let consider the simplest particular case for mean amplitudes

\[ B = 0, \, \, A = A^*. \]  \quad (3.1)

Then using (A2), (2.2) and (2.3) we obtain for small amplitudes:

\[ \hat{a}_1 = -T_0\hat{b} + R_0\hat{a} + AT_0 2kx, \]  \quad (3.2a)
\[ \hat{b}_1 = -T_0\hat{a} - R_0\hat{b} + AR_0 2kx \]  \quad (3.2b)

Let introduce quadrature in frequency domain:

\[ a_n = \frac{a + a^\dagger}{\sqrt{2}}, \, \, \, a_p = \frac{a - a^\dagger}{i\sqrt{2}}, \, \, \, a_- = a(-\Omega) \]  \quad (3.3)

For other small amplitudes the quadratures are defined by a similar way.

We rewrite (3.2) for quadratures in frequency domain

\[ a_{1a} = -T_0b + R_0a + \sqrt{2}AT_0 2kx(\Omega), \]  \quad (3.4a)
\[ a_{1p} = -T_0b + R_0a_p, \]  \quad (3.4b)
\[ b_{1a} = -T_0a - R_0b + \sqrt{2}AR_0 2kx(\Omega), \]  \quad (3.4c)
\[ b_{1p} = -T_0a_p - R_0b_p \]  \quad (3.4d)

These equations demonstrate feature of dissipative coupling — information on displacement is in amplitude quadratures of reflected and transmitted waves. In contrast, for dispersive coupling information on displacement is in phase quadrature of only reflected wave, it is shown in Sec. IV below.

Signal \(F_s\) and fluctuation back action force (A3b) act on free test mass \(m\) (it is mass of mirror \(M\)). For particular case (3.1) we obtain in frequency domain:

\[ -m\Omega^2 x = 2\sqrt{2} \hbar A b_p + F_s \]  \quad (3.5)

In case of \(T_0 \ll R_0\) combining (3.4c, 3.4d, 3.5) we obtain

\[ b_{1a} \simeq -a - \mathcal{K} \cdot b_p - \sqrt{2}\mathcal{K} \cdot f_s, \]  \quad (3.7a)

Strictly speaking we have to take linear combination:

\[ \hat{c}_{1a} = R_0 b_{1a} + T_0 a_{1a}, \]  \quad (3.6a)
\[ \hat{c}_{1p} = -T_0 b_{1p} + R_0 a_{1p}, \]  \quad (3.6b)

In case \(T_0 \to 0\) it turns into (3.7). The accurate consideration see in Sec. V.
\[ b_1 \sim -b_p, \quad \mathcal{K} = \frac{8\hbar k^2 A^2}{m\Omega^2}, \quad f_s = \frac{F_s(\Omega)}{\sqrt{2\hbar m\Omega^2}} \] (3.7b)

Here \( \mathcal{K} \) is recalculated pump, \( f_s \) is signal force normalized to SQL. We see that information on back action and signal force is in amplitude quadrature and phase quadrature is not disturbed.

We can surpass SQL applying idea of variation measurement \([4, 7, 22]\) to compensate back action. For it we have to measure combination of amplitude and phase quadratures in transmitted wave using homodyne detection:

\[
\begin{align*}
  b_0 &= b_{1a} \cos \theta + b_{1p} \sin \theta = \\
  &= -b_a \cos \theta - b_p (\sin \theta + \mathcal{K} \cos \theta) - \cos \theta \sqrt{2\mathcal{K}} \cdot f_s,
\end{align*}
\] (3.8a)

where \( \theta \) is homodyne angle. Choosing

\[
\tan \theta = -\mathcal{K}(\Omega_0)
\] (3.9)

one can completely compensate back action, but only at previously chosen frequency \( \Omega_0 \).

Let input fields are in vacuum state — it means that correlators (2.8) are valid and single-sided power spectral densities (PSD) of quadratures are equal to PSD of noise recalculated to \( f_s \) can be easily derived from (3.8):

\[ S_{f_s}^{\text{diss}} = \frac{1}{2\mathcal{K}} + \frac{(K_0 - \mathcal{K})^2}{2\mathcal{K}}, \quad K_0 = \mathcal{K}(\Omega_0) \] (3.10)

Here \( S_{f_s}^{\text{diss}} = 1 \) corresponds to SQL sensitivity\(^2\). In case of \( K_0 \gg 1 \) minimal PSD \( S_{f_s}^{\text{min}} \approx \frac{1}{2\mathcal{K}} \) is realized in narrow bandwidth \( \Gamma \):

\[
\frac{\Gamma}{\Omega_0} \approx 2S_{f_s}^{\text{min}}
\] (3.11)

Here \( \Gamma \) is defined as \( S_{f_s}^{\text{diss}}(\Omega_0 \pm \Gamma/2) \approx 2S_{f_s}^{\text{min}} \). The relation (3.11) corresponds to known Cramer-Rao bound \([42–44]\).

\[
\begin{align*}
  \hat{b}_1 &= T_2 \hat{a} - R_2 \hat{b}.
\end{align*}
\] (4.1b)

We rewrite (4.1) for quadrature in frequency domain

\[
\begin{align*}
  a_{1a} &= T_2 b_a + R_2 a_a, \\
  a_{1p} &= T_2 b_p + R_2 a_p + \sqrt{2} AR_2 2kz(\Omega), \\
  b_{1a} &= T_2 a_a - R_2 b_a, \\
  b_{1p} &= T_2 a_p - R_2 b_p.
\end{align*}
\] (4.2a–4.4d)

We see that only phase quadrature \( a_{1p} \) of reflected wave contains information on displacement.

Signal \( F_s \) and fluctuation back action force \( (B2) \) act on free test mass \( m_2 \) (it is a mass of movable mirror \( M_2 \)). In case we (3.1) have obtain in frequency domain:

\[
-m_2\Omega^2 z = 2\sqrt{2}\hbar R_2 A \left( R_2 a_a + T_2 b_a \right) + F_s
\] (4.3)

Substituting (4.3) into (4.2) we obtain

\[
\begin{align*}
  a_{1a} &= T_2 b_a + R_2 a_a, \\
  a_{1p} &= T_2 b_p + R_2 a_p - N a_{1a} - \sqrt{2N} f_s, \\
  N &= \frac{8\hbar k^2 R_2^2 A^2}{m_2\Omega^2}
\end{align*}
\] (4.4a–4.4c)

Obviously, output quadratures \( b_{1a}, b_{1p} \) of transmitted wave do not contain any information on displacement \( z \).

We see that equations (3.7) for dissipative coupling are similar to ones (4.4) for dispersive coupling. The difference is that phase quadrature \( a_{1a} \) containing information on displacement in dissipative coupling (4.4) is replaced by amplitude quadrature \( b_{1a} \) for dissipative coupling (3.7).

The formula (3.10) is valid also for dispersive coupling but with different homodyne angle: \( \tan \theta = 1/N(\Omega_0) \).

Obviously both transforms (3.7) and (4.4) describe squeezing. The difference is illustrated on Fig. 3 if incident waves are in coherent state, which fluctuations are described by dotted circles. The reflected wave is

\[ \mathcal{I}(\theta) = \frac{\mathcal{K} \cos \theta}{\sqrt{2\mathcal{K}}} \] (4.4c)
squeezed (fluctuations denoted by ellipse): for dispersive coupling amplitude quadrature conserves and phase quadrature unsqueezes, whereas for dissipative coupling phase quadrature conserves and amplitude quadrature unsqueezes.

V. COMBINED COUPLING

Let consider combined coupling when both dissipative and dispersive coupling take place. For it we analyse the same MSI on Fig. 1 but with movable BS, it is test mass $m_{bs}$ and coordinate is $y$ (2.1) and fixed mirror $M$ ($x = 0$). Then using (A2), (2.2) and (2.3) we obtain for small amplitudes:

$$a_1 = -T_0 b_0 + R_0 a +$$
$$+ R_0 B \sqrt{2}ky + T_0 A \sqrt{2}ky - R_0 A i k \sqrt{2}y, \quad (5.1a)$$

$$b_1 = -T_0 a_0 - R_0 b +$$
$$+ R_0 A k \sqrt{2}y - T_0 B k \sqrt{2}y - R_0 B i k \sqrt{2}y. \quad (5.1c)$$

In case of one pump (3.1) we rewrite input-output relations for quadratures in frequency domain:

$$a_{1a} = -T_0 b_0 + R_0 a + T_0 A 2ky, \quad (5.2a)$$

$$a_{1p} = -T_0 b_0 + R_0 a + T_0 A 2ky, \quad (5.2b)$$

$$b_{1a} = -T_0 a_0 - R_0 b_0 + R_0 A 2ky, \quad (5.2c)$$

$$b_{1p} = -T_0 a_0 - R_0 b_0. \quad (5.2d)$$

We see that here both dissipative and dispersive coupling take place. Indeed, coordinate term in (5.2b) corresponds to dispersive coupling (compare with (4.2b)), whereas coordinate terms in (5.2a, 5.2c) — to dissipative coupling, compare with (3.4a, 3.4c).

Equations (5.2) should be supplemented by equation for mechanical degree of freedom. For coordinate $y$ we obtain using (A5) in approximation (2.1) and (3.1) in frequency domain:

$$y = \frac{2\hbar k A (-R_0^2 a + R_0 T_0 b + b_p)}{(K - m_{bs} \Omega^2)} +$$
$$+ \frac{\sqrt{2\hbar m_{bs} \Omega^2}}{(K - m_{bs} \Omega^2)} \cdot f_s, \tag{5.3}$$

$$K = 4\hbar k^2 A^2 R_0 T_0, \quad f_s = \frac{F_s}{\sqrt{2\hbar m_{bs} \Omega^2}}, \tag{5.4}$$

where $f_s$ is signal force normalized to SQL. $K$ is optical rigidity, which appears due to existence of both dissipative and dispersive coupling. In order to have positive rigidity, we should keep $T_0 R_0 > 0$, see definition (2.2). Note, rigidity $K$ is a constant, it does not depend on frequency$^3$.

The terms $\sim a$, $b$ in (5.3) correspond to back action force of dispersive coupling, whereas term $\sim b_p$ in (5.3) — to back action force of dissipative coupling.

In order to apply idea of variation measurement we have to generalize it for two output beams. One can measure in transmitted and reflected waves arbitrary quadratures by homodyne detector and then take weighted sum of results. It means that we can take arbitrary linear combination of quadratures (5.2). Coefficients of this combination can be optimized to find minimum of PSD recalculated to $f_s$ (3.7) at some predefined frequency $\Omega_0$

$$S_{f_s}^\text{comb} (\Omega) = \frac{1}{2LR_0} \times$$

$$(\frac{R_0^2 + 1}{T_0} \left[ \frac{L_0 - L}{L_0 - 1} + \frac{T_0 (L_0 - 1)^2}{(R_0^2 + 1)} \right]),$$

$$L = \frac{K}{m_{bs} \Omega^2}, \quad L_0 = \frac{K}{m_{bs} \Omega_0^2} \tag{5.6b}$$

See details in Appendix C. Recall that $S_{f_s}^\text{comb} = 1$ corresponds to SQL. At $\Omega = \Omega_0$ first term in (5.6) is equal to zero and second term defines minimum $S_{f_s}^\text{comb} (\Omega_0)$. However, at $\Omega = \Omega_0 + \Delta \Omega$ the first term increases rather rapidly and defines effective bandwidth $\Gamma$. Requiring the increase of PSD by 2 times at $\Delta \Omega = \Gamma/2$ we find:

$$\frac{\Gamma}{\Omega_0} \approx \frac{T_0 (L_0 - 1)^2}{(R_0^2 + 1) L_0}, \quad \frac{\Gamma}{\Omega_0} \approx 2R_0 S_{f_s}^\text{min} (\Omega_0) \tag{5.7}$$

Here relation between bandwidth $\Gamma$ and $S_{f_s}^\text{min}$ is similar to (3.11) and corresponds to known Cramer-Rao bound [42–44].

We see that structures of formulas for PSD $S_{f_s}^\text{comb}$ and $S_{f_s}^\text{diss}$ are similar. However, there is difference: at the same pump power one can get larger sensitivity (less

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$^3$ Strictly speaking, accurate account of Doppler effect gives tiny viscosity [45] (about $\sim F_{ps}/c$, $F_{ps}$ is constant light pressure force, $c$ is speed of light), however, here we do not take it into account.
Figure 4: Plots of amplitude spectral densities $S_{fs}^{\text{comb}}(\Omega)$ and $S_{fs}^{\text{diss}}(\Omega)$ with the same pump power ($\hbar k A^2 - \text{const}$) for parameter $L_0 = 0.75$ (top) and $L_0 = 0.9$ (bottom) for different reflectivity $R_0$.

$S_{fs}^{\text{comb}}$ for combined coupling than for dissipative one. Indeed, comparing (3.7b, 3.10) with (5.5, 5.6), we see that minimal PSD $S_{fs}^{\text{diss}}(\Omega_0)$ for dissipative coupling and $S_{fs}^{\text{comb}}(\Omega_0)$ are achieved at the same power if

$$\frac{2(L_0 - 1)^2}{R_0^2(1 + R_0^2)} = 1. \tag{5.8}$$

For $R_0 \approx 1$ it means $L_0 \approx 2$. For close to resonance case (chosen frequency $\Omega_0$ is close to resonance one, or $(L_0 - 1)^2 < 1$) combined coupling gives larger sensitivity (smaller $S_{fs}^{\text{comb}}$) as compared with dissipative (as well as dispersive) coupling. Obviously, the physical reason is optical rigidity (5.5) which takes place for combined coupling only. The plots on Fig. 4 illustrate it.

VI. CONCLUSION

We analysed the simplest (without cavity) variants of dissipative and dispersive opto-mechanical couplings and have shown that in case of dissipative coupling information on mechanical displacement as well as back action is in amplitude quadratures of reflected and transmitted waves. In contrast, for dispersive coupling information on displacement is in phase quadrature of reflected wave only, see Fig. 3.

We considered combined coupling based on Michelson-Sagnac interferometer (Fig. 1), when both dissipative and dispersive couplings takes place. For simplicity we considered the case without cavity and with one pump only. The main feature of combined coupling is optical rigidity (5.5), which appears as consequence of both kinds of couplings.

In spite of back action for pure dissipative and pure dispersive couplings acts in a different way, illustrated on Fig. 3, we have shown that variation measurement [4, 7, 22] can be applied for case of combined coupling. Moreover, at the same pump power one can surpass SQL more strongly than for pure dissipative (or dispersive) couplings. The physical reason of it is optical rigidity introduced by combined coupling.

Note, for combined coupling one has to use more complicated procedure of measurement with homodyne detection of both reflected and transmitted waves and taking optimal sum of them.

We would like to underline that we analysed simplest case of combined coupling without cavity. The case of combined coupling with cavity should be investigated separately, because of in this case we can use only one reflected wave (end mirror is assumed to be perfectly re-
fecting). For example, pure dissipative coupling, analysed in this paper, provides quantum transducer of displacement, whereas pure dissipative coupling in cavity gives quantum speed meter [41], not a displacement meter.

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Appendix A: Analysis of MSI

Here we analyse MSI shown on dashed rectangle on Fig. 1 with 50/50 movable BS (coordinate $y$), movable completely reflected mirror $M$ (coordinate $x$). We calculate input-output relations and Lebedev forces acting on mirror $M$ and BS.

Complex wave amplitudes $A$, $A_1$, $B$, $B_1$, $A_{n,e}$, $B_{n,e}$ are taken on non-shifted BS.

**Input-output relations.** We start from equations:

\[
\begin{align*}
A_e &= \frac{iB + Ae^{-i\sqrt{2}\kappa y_b}}{\sqrt{2}}, \\
A_n &= \frac{Be^{i\sqrt{2}\kappa y_b} + iA}{\sqrt{2}}, \\
A_1 &= \frac{Be^{-i\sqrt{2}\kappa y_b} + iB_n}{\sqrt{2}}, \\
B_1 &= \frac{iBe^e + Be^{i\sqrt{2}\kappa y_b}}{\sqrt{2}}.
\end{align*}
\]

Where coordinates $x_m$, $y_{bs}$ denote position of mirror $M$ and BS. For waves incident on BS from inside MSI we have

\[
\begin{align*}
B_e &= Ae^{-2ikx_m}e^{2i\phi_m}, \\
B_n &= Ae^{2ikx_m}e^{2i\phi_m},
\end{align*}
\]

where constants $\phi_{e,n}$ describe phase advance of wave travelling from BS to mirror $M$, when position of BS is $y_{bs} = 0$ and position of mirror $M$ is $x_m = 0$. Substituting (A1b) into (A1a) and putting $e^{2i\phi_m} = 1$, $e^{2i\phi_n} = -1$, we obtain

\[
\begin{align*}
A_1 &= -BT + e^{-i\sqrt{2}\kappa y_b}AR, \\
B_1 &= -AT - e^{i\sqrt{2}\kappa y_b}BR, \\
R &= \cos k(2x_m + \sqrt{2}y_{bs}), \\
T &= \sin k(2x_m + \sqrt{2}y_{bs}).
\end{align*}
\]

So we can consider MSI as a GM with amplitude reflectivity $R$ and transmittance $T$.

**Lebedev light pressure force $F_m$ acting on mirror $M$.** In general case

\[
\begin{align*}
F_m &= 2\hbar k (|A|^2 - |A_e|^2) = \\
&= 2\hbar k i \left(AB^*e^{-i\sqrt{2}\kappa y_b} - A^*Be^{i\sqrt{2}\kappa y_b}\right)
\end{align*}
\]

Here we used relations (A1).

**Lebedev light pressure force acting on BS along axes $\xi$, $\eta$ on Fig. 1 are equal to**

\[
\begin{align*}
F_{\xi BS} &= \hbar k (|B|^2 + |B_1|^2 - 2|A_e|^2), \\
F_{\eta BS} &= \hbar k (|A|^2 + |A_1|^2 - 2|A_n|^2).
\end{align*}
\]

We see that light pressure force directed along axis $y$ is equal to

\[
\begin{align*}
F_{bs} &= \frac{F_{\xi BS} - F_{\eta BS}}{\sqrt{2}} = \\
&= \sqrt{2}\hbar k \left\{ R^2 (|B|^2 - |A|^2) - \\
&- RT \left( AB^*e^{-i\sqrt{2}\kappa y_b} + A^*Be^{i\sqrt{2}\kappa y_b} \right) \right\} + \frac{\hbar}{2} i \left\{ AB^*e^{-i\sqrt{2}\kappa y_b} - A^*Be^{i\sqrt{2}\kappa y_b} \right\}
\end{align*}
\]

Here we used relations (A1, A2) and put $y_0 = 0$ (defined in (2.1)). In light pressure force (A5) terms (A5b) and (A5c) corresponds to dispersive coupling, whereas term (A5d) — to dissipative one.

Appendix B: Movable mirror

Here we analyse mirror $M_1$ with reflectivity $R_1$ and transmittance $T_1$ which can move as a free test mass along axis $z$ as shown on Fig. 2. We calculate input-output relations and Lebedev forces acting on mirror. Complex wave amplitudes $A$, $A_1$, $B$, $B_1$ are taken on non-shifted mirror.

**Input-output relations** are obvious

\[
\begin{align*}
A_1 &= BT_1 + e^{2ikz}AR_1, \\
B_1 &= AT_1 + e^{-2ikz}BR_1.
\end{align*}
\]

**Lebedev light pressure force $F_{m1}$ acting on mirror $M_1$:**

\[
\begin{align*}
F_{m1} &= \hbar k (|A|^2 - |A_1|^2 - |B|^2 - |B_1|^2) = \\
&= 2\hbar k \left( R_1^2 |A|^2 - R_1^2 |B|^2 + \\
&+ T_1 R_1 \left( AB^*e^{2ikz} + A^*Be^{-2ikz} \right) \right)
\end{align*}
\]

Here we substitute (B1) into (B2a).

Appendix C: Derivation of (5.6)

Let introduce notations for input and output amplitudes quadratures

\[
\begin{align*}
da &= R_0a - T_0b, \\
e_a &= -T_0a - R_0b, \\
g_1a &= T_0b_{1a} - R_0b_{1a}, \\
f_{1a} &= R_0a_{1a} - T_0b_{1a}
\end{align*}
\]

and rewrite (5.2) in form

\[
\begin{align*}
g_{1a} &= T_0d_a + R_0e_a + A2ky,
\end{align*}
\]
\[ j_{1a} = R_0 d_a - T_0 e_a, \]
\[ a_{1p} = -T_0 b_p + R_0 a_p - R_0 A \, 2ky, \]
\[ b_{1p} = -T_0 a_p - R_0 b_p, \]
\[ A2ky = \frac{L \{-R_0 d_a + b_p\}}{(T_0 R_0)(L - 1)} + \sqrt{\frac{2L}{T_0 R_0}} \cdot \frac{f_s}{L - 1}. \]
\[ L = \frac{T_0 R_0 4\hbar k^2 A^2}{m_B \Omega^2}, \quad f_s = \frac{F_s}{\sqrt{2m_B \Omega^2}}. \]

Obviously, new quadratures \( d_a, e_a, a_p, b_p \) are not correlated with each other, creating orthogonal basis with PSD equal to:

\[ \begin{align*}
S_{d_a} &= 1, \\
S_{e_a} &= 1, \\
S_{a_p} &= 1, \\
S_{b_p} &= 1.
\end{align*} \tag{C3} \]

Let us measure weighted sum
\[ G = C \left( g_{1a} + A_1 j_{1a} \right) + D \left( a_{1p} + A_p b_{1p} \right) = \]
\[ \left\{ C \left[ T_0 + A_1 R_0 \right] + \frac{R_0 \left( DR_0 - C \right) L}{T_0 R_0 (L - 1)} \right\} j_{1a} - \]
\[ - \left\{ D \left[ T_0 + A_p R_0 \right] + \frac{(DR_0 - C) L}{T_0 R_0 (L - 1)} \right\} b_{1p} + \]
\[ + C \left[ R_0 - A_1 T_0 \right] e_{1a} + D \left[ R_0 - A_p T_0 \right] a_{1p} + \]
\[ + \left[ C - DR_0 \right] \sqrt{\frac{2L}{T_0 R_0}} \cdot \frac{f_s}{L - 1} \quad \text{C3e} \]

where \( A_1, A_p, C, D \) are some constants to be found.

Let us require back action removal on given frequency \( \Omega_0 \). It means to equate to zero terms (C4b, C4c) in (C4) in order to find \( A_1, A_p \):

\[ A_p = -\frac{1}{DR_0} \left( \frac{DT_0 + \left( DR_0 - C \right) L_0}{T_0 R_0 (L_0 - 1)} \right). \tag{C5a} \]

Now we substitute \( A_1, A_p \) into (C4)
\[ G = \left[ \frac{C - DR_0}{(L_0 - 1)} \right] \cdot \sqrt{\frac{2L}{T_0 R_0}} \times \]
\[ \left\{ \frac{T_0 R_0}{2L_0} \left[ \frac{DR_0 L_0 - C}{R_0 (C - DR_0)} \right] e_{1a} + \right\} \]
\[ + \left\{ \frac{T_0 R_0}{2L_0} \left[ 2DR_0 L_0 - DR_0 - CL_0 \right] a_p + f_s \right\} \quad \text{C6c} \]

and calculate PSD, normalized to SQL
\[ S_{fs}(\Omega_0) = \frac{T_0}{2R_0 (C - DR_0)^2} \times \]
\[ \left\{ \left[ DR_0 \sqrt{L_0 - \frac{C}{\sqrt{L_0}}} \right]^2 + \right\} \]
\[ + \left[ \frac{(2DR_0 - C) \sqrt{L_0} - \frac{D}{\sqrt{L_0}}}{R_0} \right]^2 \quad \text{C7c} \]

Let us choose \( D, C \) to minimize \( S_{fs}(\Omega_0) \). Obviously \( S_{fs}(\Omega_0) \) depends on ratio \( D/C \) only:

\[ \frac{D}{C_{\text{opt}}} = \frac{L_0 + R_0^2}{R_0 \left( L_0 [R_0^2 + 2] - 1 \right)}. \tag{C8} \]

So with optimal choice \( D/C = (D/C_{\text{opt}}) \) PSD \( S_{fs_{\text{min}}} \) at arbitrary frequency \( \Omega \) is equal to (5.6).
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