SIGNED CLASP NUMBERS AND FOUR-GENUS BOUNDS

CHARLES LIVINGSTON

Abstract. There exist knots having positive and negative four-dimensional clasp numbers zero but having four-genus, and hence clasp number, arbitrarily large. Such examples were first constructed by Allison Miller, answering a question of Juhász–Zemke. Further examples are constructed here, complementing those of Miller in that they are of infinite order in the concordance group, rather than being two-torsion.

1. Introduction

Let $q = p^2$, where $p$ is an odd prime integer. We consider the two-bridge knot $B(q, 2)$, abbreviated $B_q$, which can also be described as the $k$-twisted positive Whitehead double of the unknot, $D_+(U, k)$, where $k = (q - 1)/4$. Figure 1 is an illustration of $B_q$ in which the $k$ in the box denotes $k$ full right-handed twists. If $k = 1$ in the diagram, the resulting knot is the figure eight. We prove the following theorem, which holds in the smooth and topological locally-flat categories.

**Theorem 1.** If $p \geq 5$ is prime and $q = p^2$, then there exists a real number $c_p > 0$ such that the four-genus satisfies $g_4(nB_q) \geq c_p n$ for all $n > 0$.

![Figure 1. The knot $B_q$, where $k = (q - 1)/4$ denotes full right-handed twists.](image)

Finding this result was motivated by a question of Juhász–Zemke [7] concerning signed four-dimensional clasp numbers. Let $c(K)$ denote the *four-dimensional clasp number*: this is the minimum value of $m$ for which $K$ bounds a smooth immersed disk in $B^4$ with $m$ double points. Let $c^+(K)$ denote the minimum value of $m$ for which $K$ bounds a smooth disk in $B^4$ with $m$ positive double points, and define $c^-(K)$ similarly, minimizing negative double points. In [7] it was asked whether $c(K) - (c^+(K) + c^-(K))$ can be arbitrarily large.

Miller [10] provided the first examples answering the Juhász–Zemke question positively. It follows from Theorem 1 that the knots $B_q$ are also examples. From the diagram it is clear that $B_q$, and hence $nB_q$, can be unknotted using only positive, or only negative, crossing changes. Hence, $c^+(nB_q) = 0$. If $nB_q$ bounds a disk in $B^4$ with a double points, then those double points could be resolved to form an embedded surface of genus $a$; it follows that $g_4(nB_q) \leq c(nB_q)$ and thus Theorem 1 implies $c(nB_q) \geq c_p n$.

The examples of this paper are complementary to Miller’s. The examples of [10] are all amphichiral knots and thus satisfy $c^+(2K) = c^-(2K) = c(2K) = 0$; stated differently, the knots are of order two in the knot concordance group. In contrast, the fact that $c(nB_q) > 0$ for all $n > 0$ implies that $B_q$ is of infinite order in the concordance group.

**Remarks.**

This work was supported by a grant from the National Science Foundation, NSF-DMS-1505586.
Acknowledgements. This is essentially the key numeric computation of [1]. The invariant restrict attention to even values of \( H \) generator of \( \text{im} \) the immediate consequence of [1, Theorem 3]. Theorem 3. We isolate the result we need. In this statement, \( \sigma_2 \) permits Jiang’s result to be improved to give a linearly increasing genus bound. The proof of Theorem 1 provides a specific value of \( c_p \) that is close to, but always less than, 1/2. For instance, we find \( c_5 = 1/4 \) and \( c_7 = 5/14 \). Finding any genus one knot \( K \) for which \( c^+(K) = c^-(K) = 0 \) and \( g_4(nK) \geq n/2 \) for all \( n \geq 1 \) appears to be especially challenging.

The standard Seifert surface \( F_q \) for \( B_q \) contains simple closed curves of framing \( \frac{q-1}{2} \) and \( -1 \) that are unknotted in \( S^3 \). Using these curves we can construct an unknotted essential curve \( \alpha \) on the Seifert surface \( G \) for \( B_q \# \frac{q-1}{2} B_q = \frac{q^2+3}{2} B_q \) of Seifert framing 0. Surgery can be performed on \( G \) along \( \alpha \) to produce a surface \( B^4 \) bounded by \( \frac{q^2+3}{2} B_q \) of genus one less than the genus of \( G \); that is, it is of genus \( \frac{q-1}{2} \). Thus, \( g_4(\frac{q^2+3}{2} B_q) \leq \frac{q-1}{2} \). It follows that \( g_4(nB_q) \) is asymptotically bounded above by \( (\frac{q-1}{2})n \).

The invariants studied here are all knot concordance invariants. From a modern perspective it would be interesting to prove the analog of Theorem 1 for a family of topologically slice knots.

Acknowledgements. I appreciate helpful feedback from Pat Gilmer and Allison Miller. Comments of a referee of an early version of this paper led to significant improvements.

2. Casson-Gordon invariants and four-genus bounds

For a knot \( K \), let \( M_2(K) \) denote its 2-fold branched cover and let \( \chi: H_1(M_2(K)) \to \mathbb{C}^* \) be a character taking values in the group of units generated by \( e^{2\pi i/q} \), where \( q \) is a prime power. (Such characters are naturally identified with homomorphism \( \chi: H_1(M_2(K)) \to \mathbb{Z}_q \).) In [1], Casson and Gordon defined two rational-valued invariants, \( \sigma(K, \chi) \) and \( \sigma_1 \tau(K, \chi) \). The first is more readily computable in the case that \( M_2(K) \) is a lens space; the second provides an obstruction to the knot being slice. They are related by the following result, an immediate consequence of [1] Theorem 3.

**Theorem 2.** If \( M_2(K) \) is a lens space and \( \chi: H_1(M_2(K)) \to \mathbb{Z}_q \) is a nontrivial character, then

\[
|\sigma(K, \chi) - \sigma_1 \tau(K, \chi)| \leq 1.
\]

**Proof.** This is essentially the key numeric computation of [1]. The invariant \( \sigma(K, \chi) \) is defined in terms of signatures of Hermitian forms and is thus symmetric: \( \sigma(K, \chi^*) = \sigma(K, \chi^{−1}) = \sigma(K, \chi^{p−1}). \) This permits us to restrict attention to even values of \( r \): \( \{\sigma(B_q, \chi^{2r})\}_{0 < r < p/2} \). In [1] it is shown that \( \sigma(B_q, \chi^{2r}) = 4r^2 - 2pr + 1 \) for \( 0 < r < p/2 \). (The result appears on page 196, with the values “m” and “n” there having the value \( p \) in our application.)

2.1. Computing \( \sigma(B_q, \chi^r) \) for \( r \neq 0 \mod p \). We have the following result.

**Theorem 3.** Let \( q = p^2 \), where \( p \) is an odd prime. Let \( \chi \) denote a character that takes value \( e^{2\pi i/p} \) on some generator of \( H_1(M_2(B_q)) \cong \mathbb{Z}_q \). Then

\[
\{\sigma(B_q, \chi^r)\}_{0 < r < p} = \{4r^2 - 2pr + 1\}_{0 < r < p/2}.
\]

**Proof.** This is essentially the key numeric computation of [1]. The invariant \( \sigma(K, \chi) \) is defined in terms of signatures of Hermitian forms and is thus symmetric: \( \sigma(K, \chi^*) = \sigma(K, \chi^{−1}) = \sigma(K, \chi^{p−1}). \) This permits us to restrict attention to even values of \( r \): \( \{\sigma(B_q, \chi^{2r})\}_{0 < r < p/2} \). In [1] it is shown that \( \sigma(B_q, \chi^{2r}) = 4r^2 - 2pr + 1 \) for \( 0 < r < p/2 \). (The result appears on page 196, with the values “m” and “n” there having the value \( p \) in our application.)

2.2. Computing \( \sigma_1 \tau(B_q, \chi^0) \). In general, there are few methods available for computing \( \sigma_1 \tau(K, \chi) \). However, in the case that \( K \) is of three-genus one and is algebraically slice, the invariant is determined by the Levine-Tristram signature functions of certain knots formed as simple closed curves on a genus one Seifert surface. This is a consequence of results related to companionship proved independently by Cooper [2], Gilmer [3], and Litherland [8]. The paper [5] presents a more recent exposition. We isolate the result we need. In this statement, \( \sigma_K(\omega) \) denotes the Tristram-Levine signature function defined on the unit circle in \( \mathbb{C}^* \).

**Theorem 4.** Suppose that \( K \) bounds a genus one Seifert surface \( F \) and \( H_1(M_2(K)) \cong \mathbb{Z}_q \) with \( q = p^2 \) for some prime \( p \). Suppose that \( \alpha \) is an essential simple closed curve on \( F \) for which the value \( V(\alpha, [\alpha]) = 0 \), where \( V \) is the Seifert form of \( F \). Then for \( \chi: H_1(M_2(K)) \to \mathbb{Z}_p \subset \mathbb{C}^* \),

\[
\sigma_1 \tau(K, \chi) = 2\sigma_\alpha(\zeta^r),
\]

for some \( r \), where \( \chi(x) = \zeta \in \mathbb{C}^* \) for a generator \( x \) in \( H_1(M_2(K)) \).
The Levine-Tristram signature function satisfies \( \sigma_K(1) = 0 \) for all \( K \). Thus we have the following corollary when applied to \( \chi^0 \), which is trivial.

**Corollary 5.** Suppose that \( K \) bounds a genus one Seifert surface \( F \), \( H_1(M_2(K)) \cong \mathbb{Z}_q \), and \( V(\alpha, \alpha) = 0 \) for a simple closed curve \( \alpha \) representing a nontrivial homology class, \([\alpha] \in H_1(F)\). Then \( \sigma_1 \tau(K, \chi^0) = 0 \) for all \( \chi \).

### 2.3. Bounds on \( \sigma_1 \tau(B_q, \chi^r) \).

**Theorem 6.** Assume \( q = p^2 \) where \( p \geq 5 \) is an odd prime.

- There exists a generator \( \chi \) of the group of order \( p \) characters on \( H_1(M_2(B_q)) \) such that \( \sigma_1 \tau(B_q, \chi) \leq \frac{9p^2}{4} \).
- \( \sigma_1 \tau(B_q, \chi^r) \leq 0 \) for all \( r \).

**Proof.** We consider the function \( f(r) = 4r^2 - 2pr + 1 \) that appears in Theorem \( 5 \) as a real quadratic in the variable \( r \). Its minimum occurs at \( p/4 \). The closest integer point to \( p/4 \) is either \( (p - 1)/4 \) or \( (p + 1)/4 \) depending on whether \( p \equiv 1 \) mod 4 or \( p \equiv 3 \) mod 4. In both cases the value at this point is \((5 - p^2)/4 < -1\). Since \( \sigma(B_q, \chi^r) \) and \( \sigma_1 \tau(B_q, \chi^r) \) differ by at most one, we have the first statement.

For integers \( r \) with \( 1 \leq r \leq p/2 \), the maximum value of the quadratic \( f(r) \) must be at an endpoint, either \( r = 1 \) or \( r = (p - 1)/2 \). We compute \( f(1) = (5 - 2p) \) and \( f((p - 1)/2) = 2 - p \). The larger of the two is \( 2 - p < 1 \). Even upon adding 1, this is negative. Thus, if \( \sigma_1 \tau(B_q, \chi^r) \) were to be positive for some \( r \), it would have to be at \( r = 0 \), where the value was shown to be 0 in Corollary \( 5 \).

\[ \square \]

### 3. The genus bound

The proof of Theorem \( 1 \) depends on the following special case of a theorem of Gilmer \( 3 \). Theorem 1] that relates values of \( \sigma_1 \tau(K, \chi) \) to \( g_4(K) \).

**Theorem 7.** Let \( K \) be a knot for which \( H_1(M_2(K)) \cong (\mathbb{Z}_q)^n \), where \( q \) is a prime power. If \( g_4(K) \leq n/2 \) and the classical signature of \( K \) satisfies \( \sigma(K) = 0 \), then there is a subgroup \( M \subset (\mathbb{Z}_q)^n \subset H_1(M_2(K)) \) of order at least \( q^{(n-2g_4(K))/2} \) such that for all \( \chi \in M \),

\[ |\sigma_1 \tau(K, \chi)| \leq 4g_4(K) \]

We will refer to the subgroup \( M \) as a *metabolizer*. In the statement of Gilmer’s theorem in \( 3 \) there is an additional term \( \mu(K, \chi) \), but prior to the statement of that theorem he points out that \( \mu(K, \chi) = 0 \) in the case of characters \( \chi \) of prime power order.

### 3.1. Proof of Theorem 1.

The continuing assumption is that \( q = p^2 \) where \( p \geq 5 \) is a prime. Here is a restatement of the theorem with the value of \( c_p \) specified.

**Theorem 1** For every odd prime \( p \geq 5 \), let \( c_p = (\frac{1}{2} - \frac{8}{p^2 + 7})n \). Then for \( q = p^2 \), \( g_4(nB_q) \geq c_p n \).

**Proof.** We will first assume that \( n \) is such that \( g_4(nB_q) < n/2 \) and find a value of \( c_p < 1/2 \) for which \( g_4(nB_q) \geq c_p n \) for all such \( n \). Then, in any cases that \( g_4(nB_q) \geq n/2 \) we will certainly also have that \( g_4(nB_q) \geq c_p n \).

We abbreviate \( g_4(nB_q) = g \). We have the \( H_1(nB_q) \cong (\mathbb{Z}_q)^n \). The metabolizer \( M \) given by Theorem \( 7 \) has order at least \( p^{(n-2g)/2} \). Since each element in \( M \) has order at most \( p^2 \), an independent set of generators of \( M \) must have at least \( p^{(n-2g)/2} \) elements. Since the value is an integer, we can take the ceiling and let \( d = \lceil \frac{p^{(n-2g)/2}}{2} \rceil \).

Represent a set of generators of \( M \) as vectors in \((\mathbb{Z}_q)^n \). Together these can be used to form the rows of a matrix with at least \( d \) rows. Row operations and column interchanges can convert this into a matrix for which the top left \( d \times d \) block is an upper triangular matrix with nonzero diagonal entries and with the further property that rows corresponding to diagonal entries divisible by \( p \) have all their entries divisible by \( p \). We can multiply each of the elements of \( M \) that correspond to these rows by some element in \( \mathbb{Z}_q \) so that the first non-zero entry is \( p \). Further row operations can transform this so that the top left \( d \times d \) block is diagonal with all entries \( p \).

The sum of the vectors formed from the first \( d \) rows of that matrix is an element of \( M \) for which the first \( d \) values are all \( p \). This element can be multiplied by \( k \) so that the \( d \) diagonal entries are \( kp \). We can choose
k so that the character $\chi$ on $H_1(B_q)$ that corresponds to $kp$ is the same as $\chi$ given in Theorem 6 for which $
abla_1(B_q, \chi) \leq (9 - p^2)/4$. The vector corresponds to a character $\chi$: $H_1(M_3(nB_q)) \to \mathbb{C}^*$ taking values among $p$-roots of unity.

Applying the fact that $\nabla_1(B_q, \chi) \leq 0$ for all $\chi$, along with the additivity of $\nabla_1$ (see 4), after taking absolute values we have

$$d(\frac{p^2 - 9}{4}) = \frac{(n - 2g)(p^2 - 9)}{8} \leq |\nabla_1(nB_q, \chi)| \leq 4g,$$

where the second inequality comes from Theorem 7.

Solving for $g$ we find

$$g \geq \left( \frac{p^2 - 9}{2p^2 + 14} \right) n = \left( \frac{1}{2} - \frac{8}{p^2 + 7} \right) n.$$

4. Observations and questions

(1) The stable clasp number. A function $f: \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called subadditive if $f(a + b) \leq f(a) + f(b)$ for all $a$ and $b$. For any such function, $\lim_{n \to \infty} f(n)/n$ exists. In [9] this is used to define the stable four-genus of a knot $K$: $g_s(K) = \lim_{n \to \infty} g_4(nK)/n$. In the exact same way, one can define the stable clasp number of a knot $K$ to be $c_s(K) = \lim_{n \to \infty} c(nK)/n$. For Miller’s examples [10], $c_s(K) = 0$. We have

$$\frac{q - 9}{2q - 14} \leq c_s(B_q) \leq 1.$$

Problems. Determine $c_s(B_q)$ exactly. Find any knot $K$ for which $c_s(K)$ for which $c_s(K) \notin \mathbb{Z}$. I

(2) Find topologically slice knots $K_n$ for which $c(K_n) - (c^+(K_n) + c^-(K_n))$ goes to infinity as $n$ increases. Can such example be found for which $c^+(K_n) = 0 = c^-(K_n)$ for all $n$?

(3) The examples in this paper and those in [10] depended on estimates of the four-genus. Are there examples of knots $K$ for which $c^+(K) = 0 = c^-(K)$ and $c(K) > g_4(K)$?

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Charles Livingston: Department of Mathematics, Indiana University, Bloomington, IN 47405

Email address: livingst@indiana.edu