Nearly Tight Lower Bounds for Succinct Range Minimum Query

Mingmou Liu
School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore

Abstract

Given an array of distinct integers \(A[1 \ldots n]\), the Range Minimum Query (RMQ) problem requires us to construct a data structure from \(A\), supporting the RMQ query: given an interval \([a, b] \subseteq [1, n]\), return the index of the minimum element in subarray \(A[a \ldots b]\), i.e. return \(\arg\min\limits_{i \in [a, b]} A[i]\). The fundamental problem has a long history. The textbook solution which uses \(O(n)\) words of space and \(O(1)\) time by Gabow, Bentley, Tarjan (STOC 1984) and Harel, Tarjan (SICOMP 1984) dates back to 1980s. The state-of-the-art solution is presented by Fischer, Heun (SICOMP 2011) and Navarro, Sadakane (TALG 2014). The solution uses \(2n - \frac{5}{2}\log n + \frac{n}{(\log n)^{\Omega(1)}}\) bits of space and \(O(t)\) query time, where the additive \(\tilde{O}(n^{3/4})\) is a pre-computed lookup table used in the RAM model, assuming the word-size is \(\Theta(\log n)\) bits. On the other hand, the only known lower bound is proved by Liu and Yu (STOC 2020). They show that any data structure which solves \(\text{RMQ}\) in \(t\) query time must use \(2n - \frac{5}{2}\log n + \frac{n}{(\log n)^{\Omega(1)}}\) bits of space, assuming the word-size is \(\Theta(\log n)\) bits. In this paper, we prove nearly tight lower bound for this problem. We show that, for any data structure which solves \(\text{RMQ}\) in \(t\) query time, \(2n - \frac{5}{2}\log n + \frac{n}{(\log n)^{\Omega(1)}}\) bits of space is necessary in the cell-probe model with word-size \(\Theta(\log n)\) bits. We emphasize that, in terms of time complexity, our lower bound is tight up to a polylogarithmic factor.

From the perspective of data structure lower bound, our proof is a new approach to proving higher worst-case lower bound for succinct data structure problem whose datapoints are “non-evenly” spread over the universe like the \(\text{RMQ}\) problem, beating the state-of-the-art proofs by compression-based information-theoretical contradiction introduced by Pătraşcu, Viola (SODA 2010) and refined by Liu, Yu (STOC 2020).

2012 ACM Subject Classification Theory of computation → Cell probe models and lower bounds

Keywords and phrases range minimum query, data structure lower bound, succinct data structure, round elimination, predecessor search

Funding Mingmou Liu was supported by Singapore Ministry of Education (AcRF) Tier 2 grant MOE2018-T2-1-013.

Acknowledgements Mingmou Liu thanks Yi Li and Huacheng Yu for helpful discussions.

1 Introduction

Give an array of distinct integers \(A[1 \ldots n]\), the Range Minimum Query (RMQ) problem requires us to construct a data structure from \(A\), supporting the RMQ query: given an interval \([a, b] \subseteq [1, n]\), return the index of the minimum element in subarray \(A[a \ldots b]\), i.e. return \(\text{RMQ}(a, b) \triangleq \arg\min\limits_{i \in [a, b]} A[i]\).

The data structures to \(\text{RMQ}\) problem play significant roles in numerous areas of computer science, such as graph problems [10], [3], [15], [2], [27], text processing [11], [32], [12], [11], [32], [48], [6], [13], [20], [9] and other areas of computer science [44], [46], [7].

A popular textbook solution to \(\text{RMQ}\) problem is the sparse table algorithm. The algorithm prepares answers to all the \(\text{RMQ}\) queries whose lengths are power of two (for simplicity, we ignore the rounding issue), thus the algorithm uses \(O(n \log n)\) memory words of \(\Omega(\log n)\)
Nearly Tight Lower Bounds for RMQ

bits. Observe that any RMQ query can be solved by comparing the elements in array \( A \) corresponding to the answers to at most two prepared overlapping RMQ queries. Therefore the algorithm can answer RMQ query in \( O(1) \) time. Gabow, Bentley, Tarjan [13] and Harel, Tarjan [19] further show that we can compress the space usage to \( O(n) \) memory words while enjoy the \( O(1) \) query time by reducing to a problem named ±1RMQ.

A trend in the theory of data structures is the succinct data structure [21]. From the perspective of the succinct data structure, the classical solution is far from being optimal. It turns out that the Cartesian tree \([50]\) of the input array \( A[1... n] \) precisely captures all the information about the answers to all RMQ queries. The Cartesian tree is a rooted binary tree of \( n \) nodes, hence the number of different inputs is the \( n \)-th Catalan number \( C_n \triangleq \binom{2n}{n}/(n+1) \), and the information-theoretical optimal number of bits needed to store the input is \( 2n - 1.5 \log n + O(1) \). However the above classical solution consumes \( O(n) \) words \( \approx O(n \log n) \) bits.

The first “truly” linear space (which is called a compact data structure in the language of succinct data structure) solution is presented by Sadakane [41]. Sadakane proposed a RMQ data structure using \( \Omega(4n) \) bits of space and \( O(1) \) query time, assuming the word size is \( \Theta(\log n) \) bits. Later, the space cost is improved to \( 2n - 1.5 \log n + O(n \log \log n / \log n) \) bits by Fischer and Heun [10, 11]. The state-of-the-art solution [43, 33] is based on the succinct augmented B-tree [13] (which is called segment tree by competitive programming participants [22]). Assuming the word size is \( \Theta(\log n) \) bits, the solution solves RMQ with \( 2n - 1.5 \log n + n/(\log 2)^t \) bits of space for database, extra \( O(n^{3/4}) \) bits of space for a pre-computed lookup table to help the computation in the RAM model, and \( O(t) \) query time, where \( t > 1 \) is an arbitrary parameter. On the other hand, the only known lower bound for RMQ is presented by Liu and Yu [29]. They proved that any data structure which solves RMQ in \( t \) time must use \( 2n - 1.5 \log n + n/(\log n)^{O(t \log^2 t)} \) bits of space, assuming the word size is \( \Theta(\log n) \) bits. It is easy to see that, in terms of time complexity, the gap between the state-of-the-art upper bound and the lower bound is quadratic: if the space cost is limited to \( 2n - 1.5 \log n + r \) bits, then the query time upper bound is \( t = O\left(\frac{\log(n/r)}{\log \log n/\log n}\right) \), while the query time lower bound is \( t = \tilde{\Omega}\left(\sqrt{\frac{\log n}{\log \log n}}\right) \). In particular, when \( r = O(1) \), the upper bound is \( t = O(\log n) \) meanwhile the lower bound is \( t = \tilde{\Omega}(\sqrt{\log n}) \). Thus a question is raised from [33, 29]: what is the optimal time-space trade-off for RMQ data structures?

1.1 Our contribution

In this paper, we present the following improved lower bound for the Range Minimum Query problem.

**Theorem 1.** Given an array \( A[1... n] \), for any data structure supporting RMQ queries using \( 2n - 1.5 \log n + r \) bits of space and query time \( t \), we must have

\[
r = \frac{n}{w}^{O(t \log^2 t)},
\]

in the cell-probe model with word-size \( w = \Omega(\log n) \).

We emphasize that, for any \( r \), we present a lower bound of \( t = \Omega(t_{opt} / \log^3 t_{opt}) \), where \( t_{opt} \) is the optimal time cost when the data structure is allowed to consume \( 2n - 1.5 \log n + r \) bits of space. Hence our lower bound is nearly tight. For example, assuming the data
structure consumes $2n - 1.5 \log n + O(1)$ bits, then we have lower bound $t = \Omega \left( \frac{\log n}{\log \log n} \right)$, meanwhile the known upper bound is $t = O(\log n)$.

In the area of data structure lower bound, the state-of-the-art technique is a variant of round-elimination from [39]. However, the original technique [39] works for proving average-case lower bound. Specifically, the original technique works in the case that a random query consumes high time cost with high probability. In contrast, for the RMQ problem, a few answers are sufficient to answer a substantial fraction of RMQ queries among the $\binom{n}{2}$ queries, so a random query can be solved by accessing a few memory cells which contain the aforementioned answers with high probability. [29] fixes this issue by executing round-elimination only on a few hard queries, which highly rely on the database. Nevertheless the technique in [29] is awkward in dealing with problems whose datapoints are “non-evenly” spread over the universe like RMQ. As a result, one round of the round-elimination in [29] can eliminate at most $O(1/(t \log^2 t))$ times of memory accessing in expectation, which causes the quadratic gap. To circumvent this problem, we present a better round-elimination, so that one round of round-elimination eliminates at least $\Omega(1/\log^2 t)$ times of memory accessing in expectation. See Section 2 and Section 4 for more details.

1.2 Related works

The richness technique and the round-elimination are two major techniques used to prove data structure lower bounds. Both of the two techniques are derived from [30, 31].

The richness technique was improved by Pătraşcu and Thorup [36] (direct-sum for richness) and by Panigrahy, Talwar and Wieder [34] (cell-sampling). The cell-sampling technique was applied to static data structures [47, 34, 24, 18, 52], dynamic data structures [23, 8, 26], streaming [25], and succinct data structures [15]. The cell-sampling technique requires a property that a few obliviously random queries reveal a lot of information about the database, whereas it does not hold for RMQ problem as we discussed earlier.

The round-elimination technique was improved and refined for several times. It has application to data structures with polynomial space usage [4, 45, 5, 28], data structures with near-linear space usage [37, 38], as well as succinct data structures [39, 29]. We compare our technique with the ones in [39, 29] in next subsection.

Golynski [17] developed a different technique to prove lower bound for succinct data structures, and proved lower bounds for three problems. The technique requires a pair of opposite queries (the “forward query” and the “inverse query”) which can “verify” each other, for example $\pi(i) = j$ and $\pi^{-1}(j) = i$ for some permutation $\pi$.

Recently, Viola [49] proposed a new technique to prove lower bounds for both samplers and succinct data structures. Viola’s technique does not require encoding arguments. Furthermore, Viola presented a new proof of $r = n/w^{O(1)}$ lower bounds for the rank problem [39] as well as the colored-multi-predecessor problem [49]. We emphasize that Viola’s technique only works for data structure problems whose datapoints are “evenly” spread, such as the rank problem and the colored-multi-predecessor problem. However the input for the RMQ problem in the worst case is far from being “even”, which is precisely the nature of the difficulty prevents us from proving higher lower bound for the RMQ problem. Hence Viola’s contribution is totally orthogonal to ours. We elaborate this at the end of Section 2.2.

---

1 Notably, before the cell-sampling technique was refined and named in [34], it was applied in proving lower bounds for some special data structure problems [47, 15].
2 Technique Overview

All of [39, 29] and our technique work in the cell-probe model with published bits.

The classical cell-probe model [51] is almost identical with the RAM model, except that all operations are free except memory access. In particular, assuming the database \( A \) is from some universe \( \mathcal{B} \), a solution in cell-probe model consists of a code \( T : \mathcal{B} \rightarrow (\{0,1\}^w)^s \) and a query algorithm \( A \), where \( w \) is the word size and \( s \) is the number of memory cells consumed by the data structure. Given database \( A \), we construct a table which consists of \( s \) cells of \( w \) bits with the code \( T \). Given a query \( i \), we execute the algorithm \( A \) to adaptively probe (i.e. access) \( t \) cells: at first we choose the first cell and probe it according to the query \( i \), then we choose the second cell and probe it according to the query \( i \) and content of the first cell, and we go on and on. After collected the contents of \( t \) cells, we output the answer with query \( i \) and the information revealed by the \( t \) cells.

The cell-probe model with published bits further provides additional \( p \) published bits to the algorithm \( A \). informally, the published bits can be considered as the CPU cache, so that the CPU can access them for free at any point of time. In particular, the code becomes \( T : \mathcal{B} \rightarrow (\{0,1\}^w)^s \times \{0,1\}^p \), and the query algorithm can access the \( p \) published bits for free, and can take the \( p \) bits as advice during both the procedures of cell-probing and outputting. Given a succinct data structure which uses \( \log |\mathcal{B}| + r \) bits of space and \( t \) query time in the worst case, we can easily construct a data structure which uses \( \log |\mathcal{B}| \) bits of memory space, \( r \) published bits and \( t \) query time by simply moving the last \( r \) bits of the data structure to the published bits, then simulate the original query algorithm. Therefore it suffices to prove lower bound in the cell-probe model with published bits.

2.1 Pătraşcu and Viola’s technique

Pătraşcu and Viola [39] developed a new round-elimination technique to prove lower bound for rank data structure. In the rank problem, we are required to construct a data structure from a set of elements \( A \subseteq [n]^2 \) supporting rank query: \( \text{rank}(i) \triangleq \# \{x \in A : x \leq i \} \).

Pătraşcu and Viola proved that any rank data structure must use at least \( n + n/(t(w + \log n))^p \) bits of space, where the \( w \) is the word size and the \( t \) is the query time. The result is tight due to the upper bound [53]. Their idea is to show there is a set of query \( Q_{pub} \) such that a substantial number of queries have to probe the cells probed by \( Q_{pub} \) — we denote by \( \text{Probe}(Q_{pub}) \) the set of cells probed by queries \( Q_{pub} \). (See Section 3.3 for formal definition.) We publish (i.e. concatenate) the addresses together with the contents of \( \text{Probe}(Q_{pub}) \) to the published bits, so that a lot of queries can skip one time of cell-probing with the published bits. We then apply this argument recursively. After all the cell-probing have been eliminated, we can recover the database from the published bits, hence there have to be at least \( n \) published bits at the end.

The key argument lies in choosing proper \( Q_{pub} \) and showing the overlap between \( \text{Probe}(Q_{pub}) \) and \( \text{Probe}(\{q\}) \) for an average query \( q \). Pătraşcu and Viola evenly choose \( Q_{pub} \) of size \( \Theta(p) \), slightly larger than the number of published bits, from \( [n] \), and prove the overlap by a compression-based information-theoretical contradiction. They show that at least \( \Omega(p) \) bits of information is shared between the answers to random query set \( Q \) of size \( \Theta(p) \) and the answers to \( Q_{pub} \). If most of queries in \( Q \) do not probe any cell in \( \text{Probe}(Q_{pub}) \), the two sets of cells are almost disjoint. Thus, we can compress the contents of \( \text{Probe}(Q) \) or \( \text{Probe}(Q_{pub}) \)

\footnote{\([n] \triangleq \{1, \ldots, n\} \).}
to save $\Omega(p)$ bits. However the original data structure consumes $n + p$ bits, the new data structure will consume $n - \Omega(p)$ bits, which is impossible. Consequently, after one round of bit-publishing, the number of published bits will be multiplied by a factor of $O(t(w + \log n))$ and the query time will be subtracted by $\Omega(1)$. Putting everything together, we conclude $r(t(w + \log n))^{O(t)} \geq n$, assuming the data structure consumes $n + r$ bits of space.

However, the original strategy fails to prove lower bound for RMQ. Because the information contained in answers to obliviously random set of queries is small. As we discussed earlier, a few answers are sufficient to answer a substantial fraction of RMQ queries.

### 2.2 Liu and Yu’s technique

Liu and Yu [29] proved a space lower bound of $2n - 1.5 \log n + n/(t(w + \log n))^{O(t^2 \log^2 t)}$ bits for RMQ problem, where the $w$ is the word size and the $t$ is the query time. In order to simplify the proof, Liu and Yu reduce the RMQ problem from pred-z, a variant of predecessor search problem (see Section 3 for formal definition). Although the tight lower bound for predecessor search in the regime of linear space is known due to [37], the hard instance in [37] is different from the input distribution of pred-z induced by the RMQ problem. It turns out that the pred-z problem still has an efficient average-case solution under the induced input distribution.

To prove a worst-case lower bound for pred-z, Liu and Yu consider a set of hard queries $Q$, which highly relies on the database, and try to eliminate the cell-probing of $Q$. To this end, they find a set of queries $Q_{pub}$ of size $p \log^4 n$ such that the answers to $Q_{pub}$ reveal $\Theta(p \log^2 n)$ bits of information about the database. The last piece of the puzzle is to show the overlap between $\text{Probe}(Q_{pub})$ and $\text{Probe}\{q\}$ for an average query $q$ in the set of hard queries $Q$. Whereas the idea in [37] cannot be applied in this case: we do not even know what $Q$ is [3], thus we can neither compress the contents of $\text{Probe}(Q_{pub})$ with the contents of $\text{Probe}(Q)$, nor compress the contents of $\text{Probe}(Q)$ with the contents of $\text{Probe}(Q_{pub})$. To circumvent this, they compress the contents of $\text{Probe}(Q_{pub})$ with the information contained in the contents of all the cells excluding $\text{Probe}(Q_{pub})$. It may work because the latter set of cells is a superset of $\text{Probe}(Q)$ if $\text{Probe}(Q_{pub})$ and $\text{Probe}(Q)$ are disjoint.

The approach has another issue. The information shared between the two disjoint sets of memory cells may be contained in the addresses of the two sets. In other words, the addresses of the cells probed by $Q_{pub}$ may reveal too much information about the answers to $Q_{pub}$. Let $\text{Probe}_l(Q_{pub})$ denote the set of cells which are probed in $l$-th round of cell-probing, and $\text{Probe}_{<l}(Q_{pub}) \triangleq \cup_{i<l} \text{Probe}_i(Q_{pub})$. Their solution is derived from a simple observation: given set of queries $Q_{pub}$, $\text{Probe}_l(Q_{pub})$ is determined by the contents of $\text{Probe}_{<l}(Q_{pub})$. In other words, the addresses of the $l$-th probed cells do not contain any information so long as the contents of $\text{Probe}_{<l}(Q_{pub})$ are known. Hence they pick a good $l$, then show that, if the overlap between $\text{Probe}(Q_{pub})$ and $\text{Probe}\{q\}$ for an average query $q$ in $Q$ is too small, we can compress the contents of $\text{Probe}_l(Q_{pub})$ with the contents of the remaining cells to save too much space.

Now we briefly discuss how does [29] obtain the lower bound and show the bottleneck in the proof. The $Q_{pub}$ break the universe of datapoints into $\approx p \log^4 n$ intervals of equal length, and the most useful information revealed by the answers to $Q_{pub}$ is the emptiness of the

---

3 In fact, the information contained in the answers to $Q$ precisely is “what is $Q$”. So, any attempt to encode $Q$ does no work.
intervals, i.e. the set of non-empty intervals. Note that it is easy to check whether an interval is empty with the operation of predecessor search. [24] chooses exact the input datapoints (i.e. all the possible answers to predecessor search, given the database) as the set of hard queries Q. Thus, if an interval is non-empty but we cannot learn this with the information contained in the contents of all the cells excluding \textit{Probe}_i(Q_{pub}), then all the hard queries contained in the interval must probe some cell in \textit{Probe}_i(Q_{pub}). Otherwise, we can enumerate all the queries and try to execute the query algorithm on them with the contents of all the cells, excluding \textit{Probe}_i(Q_{pub}), to learn this. For an average l, the contents of \textit{Probe}_i(Q_{pub}) reveal \(\Theta((p \log^2 n)/t)\) bits of information about the set of non-empty intervals. Using this information bound, [24] then shows that:

1. There are \(\Omega(1/(t \log t))\)-fraction of non-empty intervals in expectation, such that every hard query contained in the intervals has to probe some cell in \textit{Probe}_i(Q_{pub});
2. The non-empty intervals contain \(\Omega(1/(t \log t)^2)\)-fraction of hard queries in expectation, so \(\Omega(1/(t \log t)^2)\)-fraction of hard queries probe some cell in \textit{Probe}_i(Q_{pub}) in expectation;
3. There are \(t \textit{Probe}_i(Q_{pub})\)'s, so an average hard query probes \(\Omega(1/(t \log^2 t))\) cells in \textit{Probe}(Q_{pub}) in expectation.

Therefore, one round of bit-publishing eliminates \(\Omega\left(\frac{1}{(t \log^2 t)}\right)\) times of cell-probing in expectation. Putting everything together, we conclude \(r(t(w + \log n))^{\Theta(r \log^2 t)} \geq 2n - 1.5 \log n\), assuming the data structure consumes \(2n - 1.5 \log n + r\) bits of space.

Liu and Yu’s technique is awkward in dealing with input distribution which spreads the datapoints “non-evenly” over the universe. As we discussed earlier, the input distribution is “non-even”: \(O(1/(t \log t))\)-fraction of non-empty intervals may contain at most \(O(1/(t^2 \log^2 t))\)-fraction of the hard queries in expectation. Even worse, we can neither assume that there is some l such that the contents of \textit{Probe}_i(Q_{pub}) reveal \(\omega((p \log^2 n)/t)\) bits of information about the set of non-empty intervals to improve step (1), nor assume that the union of the sets of hard queries which probe some cell in \textit{Probe}_i(Q_{pub}) for different l is large to improve step (2) together with step (3). As a result, none of the three steps can be improved easily.

In contrast, the worst case input distributions to the \textit{rank} problem [39] and the \textit{colored-multi-predecessor} problem [49] guarantee that the number of datapoints contained in any interval of length \(\Omega(\log n)\) is in proportion to the length of the interval with high probability. It is why [39] [49] can prove a lower bound of \(r(t(w + \log n))^{\Theta(t)} \geq n\), assuming the space cost is \(n + r\) bits.

### 2.3 Our technique

Our major contribution is an improvement to a very core lemma of the proof in [24]. We show that, in fact, the efficiency of the round-elimination procedure in [24] is much better than the one they proved. Recall the procedure to prove the efficiency of the round-elimination:

1. There are \(\Omega(1/(t \log t))\)-fraction of non-empty intervals in expectation, such that every hard query contained in the intervals has to probe some cell in \textit{Probe}_i(Q_{pub});
2. The non-empty intervals contain \(\Omega(1/(t \log t)^2)\)-fraction of hard queries in expectation, so \(\Omega(1/(t \log t)^2)\)-fraction of hard queries probe some cell in \textit{Probe}_i(Q_{pub}) in expectation;
3. There are \(t \textit{Probe}_i(Q_{pub})\)'s, so an average hard query probes \(\Omega(1/(t \log^2 t))\) cells in \textit{Probe}(Q_{pub}) in expectation.

In this paper, instead of examining a single \textit{Probe}_i(Q_{pub}), we examine the whole \textit{Probe}(Q_{pub}) to improve the whole procedure. This is our major contribution. The new procedure becomes simpler now:
(1) There are $\Omega(1/\log t)$-fraction of non-empty intervals in expectation, such that every hard query contained in the intervals has to probe some cell in $\text{Probe}(Q_{\text{pub}})$;

(2) The non-empty intervals contain $\Omega(1/\log^2 t)$-fraction of hard queries in expectation, so $\Omega(1/\log^2 t)$-fraction of hard queries probe some cell in $\text{Probe}(Q_{\text{pub}})$ in expectation.

In the light of this improvement, we have a lower bound of $r ((t(w + \log n))^\omega(t \log^2 t) \geq 2n$ by providing our result as a black box to the proof in [29].

We introduce some new notions to prove step (1). Let $E$ be the set of non-empty intervals, $K$ the set of non-empty intervals $f$ such that there is a query $q$ which does not probe any cell in $\text{Probe}(Q_{\text{pub}})$ but the answer to $q$ is contained in $I$. In other words, $K$ is the set of non-empty intervals we can recover with the answers to all the queries $q$ such that $q$ does not probe any cell in $\text{Probe}(Q_{\text{pub}})$. (See Algorithm 3 for formal definition of $K$.) Observe that every hard query contained in $E \setminus K$ probes some cell in $\text{Probe}(Q_{\text{pub}})$. Thus every hard query contained in $E \setminus K$ saves at least one probe if we allow the query algorithm to access $\text{Probe}(Q_{\text{pub}})$ for free. Consequently it suffices to bound $E[|E \setminus K|]$ from below.

We adopt an information-theoretical approach to bound $E[|E \setminus K|]$. Instead of bounding $E[|E \setminus K|]$ directly, we bound the amount of the information contained in $E$, given $K$ (we denote this by $H(E \mid K)$, i.e. the entropy of $E$ conditioned on $K$). $E[|E \setminus K|]$ is large as long as $H(E \mid K)$ is large, since there must be a lot of elements in $E \setminus K$ are spread out randomly. Let $H(E)$ denote the amount of information contained in $E$ (i.e. the entropy of $E$). We try to bound $H(E \mid K)$ from below with $H(E)$. Let $\text{Foot}_{<i}(Q_{\text{pub}}), \text{Foot}_{<i}(Q_{\text{pub}})$ denote the contents of $\text{Probe}_{<i}(Q_{\text{pub}}), \text{Probe}_{<i}(Q_{\text{pub}})$ respectively. To obtain a good lower bound for $H(E \mid K)$, we would like to examine the amount of information about $E$ contained in each of $\text{Foot}_{<i}(Q_{\text{pub}})$, given $\text{Foot}_{<i}(Q_{\text{pub}})$, learned by the cell-probing algorithm. A straightforward idea is to examine the amount of information shared by $E$ and $\text{Foot}_{<i}(Q_{\text{pub}})$, given $\text{Foot}_{<i}(Q_{\text{pub}})$ (i.e. the mutual information $I(E : \text{Foot}_{<i}(Q_{\text{pub}}) | \text{Foot}_{<i}(Q_{\text{pub})})$). The approach does not work, since the subadditivity for conditional entropy does not hold in this case (i.e. inequality $I(A : B | C) \geq I(A : B | C, D)$ is not always true), and the inequality is necessary for our approach. A workaround is to define $\{E_i\}$ so we can approximate the aforementioned quantity $H(E_i | E_1, \ldots, E_{i-1})$ (i.e. $H(E_i | E_1, \ldots, E_{i-1}) \approx I(E : \text{Foot}_{<i}(Q_{\text{pub}}) | \text{Foot}_{<i}(Q_{\text{pub}}))$).

But the $\{E_i\}$ is not always well-defined. Our solution is to bound $H(\text{Foot} \mid | \text{Probe}_{<i}(Q_{\text{pub}})$ from below with $H(\text{Foot}(Q_{\text{pub}}))$, then we bound $H(\text{Foot}(Q_{\text{pub}}) \mid K)$ and $H(\text{Foot}(Q_{\text{pub}}))$ with $H(E \mid K)$ and $H(E)$ respectively.

To obtain a good lower bound for $H(\text{Foot}(Q_{\text{pub}}) \mid K)$, we take a closer look at the query algorithm. We would like to rewrite $H(\text{Foot}(Q_{\text{pub}}) \mid K)$ as summation of $H(\text{Foot}(Q_{\text{pub}}) \mid K, \text{Foot}_{<i}(Q_{\text{pub}}))$ by the chain rule, then bound each term from below with $H(\text{Foot}_{<i}(Q_{\text{pub}}) \mid \text{Foot}_{<i}(Q_{\text{pub}}))$. To this end, we introduce some useful notions $K_1, \ldots, K_t$ and $K_1', \ldots, K_t'$. Being similar with $K, K_i$ is the set of non-empty intervals we can recover with the answers to all the queries $q$ such that $q$ does not probe any cell in $\text{Probe}_{<i+1}(Q_{\text{pub}})$. Obviously $K_i \subseteq K_{i+1}$, so we let $K_i' = K_i \setminus K_{i-1}$ to measure the information the query algorithm learns from $\text{Foot}_{<i}(Q_{\text{pub}})$. It turns out that we can bound $H(\text{Foot}_{<i+1}(Q_{\text{pub}}) \mid K_i)$ from below with $H(\text{Foot}_{<i}(Q_{\text{pub}}) \mid K_i, K_i' \setminus K_{i-1})$ and $H(K_i, K_i' \setminus K_{i-1})$. Note that $H(\text{Foot}(Q_{\text{pub}}) \mid K) = H(\text{Foot}_{<i+1}(Q_{\text{pub}}) \mid K_i)$.

Then we recursively apply the inequality to bound $H(\text{Foot}(Q_{\text{pub}}) \mid K)$ with $H(\text{Foot}(Q_{\text{pub}}))$ and the summation of $H(K_i, K_i' \setminus K_{i-1})$'s.

Bounding $H(\text{Foot}(Q_{\text{pub}}) \mid K)$ and $H(\text{Foot}(Q_{\text{pub}}))$ with $H(E \mid K)$ and $H(E)$ is easy. Thus the last piece of the puzzle is to bound $H(E)$, $H(E \mid K)$, and the summation of $H(K_i, K_i' \setminus K_{i-1})$'s respectively. $H(K_i, K_i' \setminus K_{i-1})$ can be bounded easily since $(K_i, K_i' \setminus K_{i-1})$ is a partition of set $K_{i-1}$. For $H(E \mid K)$ and $H(E)$, [29] provides a lot of useful lemmas, so we can bound the two entropies with them easily. Putting everything together, one round of
the bit-publishing eliminates at least $\Omega\left(\frac{1}{\log^2 t}\right)$ times of cell-probing in expectation. As a result, we have lower bound of $r(t(w + \log n))^{O(r\log^2 t)} \geq 2n$, assuming the data structure consumes $2n - 1.5\log n + r$ bits of space.

See Section 4 for formal proof.

3 Lower Bound for Succinct Range Minimum Query

We reduce RMQ from a variant of predecessor search problem, which is named pred-z by [29].

Given parameters $d, B, u, Z$, the problem asks us to preprocess input $S_1, \ldots, S_d \subseteq [B]$ of size $u$ together with input $z$ into a data structure, such that the following queries are supported:

- $\text{pred}(i, x)$: return predecessor of $x$ in sub-database $S_i$, and return 0 if all the elements are larger than $x$;
- $\text{query-z}()$: return $z$.

We are interested in the space cost of the data structure and the time cost on answering query pred in the worst case. Note that we do not care about the time cost on answering query-z, and it could take arbitrarily long time to answer query-z. By [29], it suffices to prove lower bound for pred-z under distribution $D$ induced by the maximum entropy input distribution for RMQ. The distribution $D$ is defined as follows: $S_1, \ldots, S_d$ are mutually independent; sample $S_i$ with probability

$$\Pr[S_i = \{s_1, \ldots, s_u\}] = \frac{\prod_{j=0}^{u} C_{s_{j+1} - s_j - 1}}{\sum_{s'_1, \ldots, s'_u} \prod_{j=0}^{u} C_{s'_{j+1} - s'_j - 1}},$$

where $0 = s_0 < s_1 < \cdots < s_u < s_{u+1} = B + 1$ and $C_n$ is the $n$-th Catalan number; finally sample $z$ uniformly from $\left[\frac{Z \cdot \prod_{i=1}^{d} \prod_{j=0}^{u} C_{s^{(i)}_{j+1} - s^{(i)}_j - 1}}{}\right]$.

Our major contribution is the following lemma.

\begin{lemma}
Suppose there is a data structure using $p$ published bits for pred-z under distribution $D$, and its worst-case pred query time is $t = o(\log B)$. Then there exists a set of queries $Q_{pub}$ such that $\mathbb{E}[|E \setminus K|] = \Omega(p \log^2 B/\log t)$.
\end{lemma}

We present the formal definition of $E$ and $K$ in Section 3.1. Basically, given Lemma 2 we can prove the desired lower bounds by following the proofs in [29]. For completeness, in Section 3.2 we show how to obtain the lower bounds for RMQ with lower bounds for pred-z; in Section 3.3 we show how to obtain lower bound for pred-z with Lemma 2. We prove Lemma 2 in Section 4.

3.1 Selecting $Q_{pub}$ and defining $E, K$

At first we explicitly select $Q_{pub}$. To do so, at first we break the sub-databases $S_i$ evenly into consecutive disjoint blocks such that every block contains $p/d$ elements from each set $S_i$, then we select poly log $n$ queries evenly over each block such that the distance between adjacent selected queries are equal. More specifically, let $m \triangleq ud/p$, and recall that $S_i = \{s^{(i)}_1, \ldots, s^{(i)}_u\}$, we let

$$S_{pt}^{(i)} = \left\{s^{(i)}_{m+i}, s^{(i)}_{2m+i}, s^{(i)}_{3m+i}, \ldots, s^{(i)}_u\right\}$$

be the set of elements used to partition sub-databases into consecutive disjoint blocks so that each block contains precise $m$ elements from $S_i$. In other words, the set of blocks
a sub-database \( S_i \) is \([1, s_m^{(i)}], [s_m^{(i)} + 1, s_{2m}^{(i)}], \ldots \). Note that the blocks may have different lengths since the elements are randomly spread over the whole \([B]\). We are going to focus on the good blocks which are of length approximately \( m^2 \):

\[
S_{good} \triangleq \left\{ (i, [x, y]) : i \in [d], \frac{1}{2} m^2 \leq y - x \leq 2m^2, \exists l \in [p/d] (x = s_m^{(i)} + 1, y = s_{(l+1)m}^{(i)}) \right\}.
\]

Finally we evenly select approximately \( \log^4 B \) queries over each of good blocks. Recall that a block contains at least \( m \triangleq \frac{ud}{p} \) elements and we assume \( p < \frac{du}{\log^4 B} \), thus the selecting is always possible. Formally speaking, let \( L \triangleq \frac{m^2}{\log^4 B} \), and \( \Delta \) a random variable uniformly sampled from \([L]\),

\[
Q_{pub} \triangleq \bigcup_{(i, [x, y]) \in S_{good}} \left\{ (i, x + j \cdot L + \Delta) : j \geq 1, x + j \cdot L + \Delta < y \right\}.
\]

Note that there are at most \( p \) good blocks and any good block is of length at most \( 2L \log^4 B \), hence \( |Q_{pub}| = O(p \log^4 B) \).

\( Q_{pub} \) partitions the set of queries into consecutive disjoint intervals. The answers to \( Q_{pub} \) reveal a lot of information about the database. To characterize this, we define

\[
E_j^{(i, [x, y])} \triangleq \mathbf{1}_{\text{pred}(i, x + j \cdot L + \Delta) \neq \text{pred}(i, x + (j+1) \cdot L + \Delta)},
\]

indicating if the intervals induced by \( Q_{pub} \) are empty, i.e. if there is a datapoint between the adjacent queries in \( Q_{pub} \).

For any fixed database and a query \( q \), we let \( \text{Probe}(q) \) denote the set of memory cells probed by the query algorithm in order to answer query \( q \). Note that \( \text{Probe}(q) \) is a random variable since it depends on the database. Furthermore, for any \( l \in [l] \), we let \( \text{Probe}_l(q) \) denote the set consists of the \( l \)-th memory cell probed by the query algorithm in order to answer query \( q \), and let \( \text{Probe}_{<l}(q) \triangleq \bigcup_{j \leq l} \text{Probe}_j(q) \). \( \text{Probe}_{<l}(q) \triangleq \bigcup_{j \leq l} \text{Probe}_j(q) \). \( \text{Probe}_0(q) \triangleq \emptyset \). For any set of queries \( Q \), we define \( \text{Probe}(Q) \triangleq \bigcup_{q \in Q} \text{Probe}(q) \), \( \text{Probe}_l(Q) \triangleq \bigcup_{q \in Q} \text{Probe}_l(q) \), \( \text{Probe}_{<l}(Q) \triangleq \bigcup_{q \in Q} \text{Probe}_{<l}(q) \), \( \text{Probe}_{<l}(Q) \triangleq \bigcup_{q \in Q} \text{Probe}_{<l}(q) \), \( \text{Probe}_0(Q) \triangleq \emptyset \).

Let \( \mathcal{E} \) denote the set of non-empty intervals contained in the good blocks, \( \mathcal{K} \subseteq \mathcal{E} \) denote the set of non-empty intervals which contain at least one hard query \( q \) such that \( \text{Probe}([q]) \cap \text{Probe}(Q_{pub}) = \emptyset \) and \( \text{pred}(q') = q \) for some query \( q' \). Formally, \( \mathcal{K} \) is computed by the Algorithm 1:

**Algorithm 1** Algorithm to compute \( \mathcal{K} \).

\[
\begin{align*}
\mathcal{K} & \leftarrow \emptyset; \\
\text{foreach} \ (j, [x, y]) & \text{ is good block do} \\
& \quad \text{for } k \in [x, y] \text{ do} \\
& \quad \quad \text{if } \text{Probe}(j, k) \cap \text{Probe}(Q_{pub}) = \emptyset \text{ and } \text{pred}(j, k) \geq x \text{ then} \\
& \quad \quad \quad \mathcal{K} \leftarrow \mathcal{K} \cup \{ \text{the interval contains the answer to } \text{pred}(j, k) \}; \\
& \quad \text{end} \\
& \quad \text{end} \\
\text{return } \mathcal{K};
\end{align*}
\]
3.2 Reduction from pred-z

In the remainder of Section 3, we prove our main theorem, a lower bound for succinct RMQ.

**Theorem 1.** Given an array \(A[1 \ldots n]\), for any data structure supporting RMQ queries using 
\(2n - 1.5 \log n + r\) bits of space and query time \(t\), we must have
\[
r = n/w^{O(t \log^2 t)},
\]
in the cell-probe model with word-size \(w = \Omega(\log n)\).

With our improved Lemma 3, we follow the identical proof of Theorem 1 in \[29\] to prove the lower bound. It guarantees that
\[
1 \leq z \leq Z \cdot \prod_{j=0}^{d} \prod_{i=1}^{u} C_{s_j^{(i)} - s_j^{(i)} - 1},
\]
where \(S_i = \{s_1^{(i)}, \ldots, s_u^{(i)}\}\) such that \(s_j^{(i)} < s_{j+1}^{(i)}\) for \(j \in [u - 1]\), and \(C_n = \binom{2n}{n}/(n + 1)\) is the \(n\)-th Catalan number.

By the definition of distribution \(D\), the total number of possible inputs to \(\text{pred-z}\) is
\[
Z \cdot \left( \sum_{0 < s_1 < \cdots < s_u < B+1} \prod_{j=0}^{u} C_{s_{j+1} - s_j - 1} \right)^d.
\]
We denote the minimum number of bits needed to store the input by
\[
H_{d,u,B,Z} \triangleq d \cdot \log \left( \sum_{0 < s_1 < \cdots < s_u < B+1} \prod_{j=0}^{u} C_{s_{j+1} - s_j - 1} \right) + \log Z,
\]
which is \(d \cdot (2B - u - \Theta(\log B)) + \log Z\) by \[29\] Lemma 2. We are going to apply the following setting throughout this paper to construct the reduction:
\[
d \triangleq 2r, \quad B \triangleq \left\lceil \frac{n}{d} \right\rceil - 1, \quad u \triangleq \lfloor \sqrt{B} \rfloor, \quad Z \triangleq \binom{2u}{u}^r.
\]
Moreover, \(H_{d,u,B,Z} = 2n - O(d \log B)\) under this setting. In particular, by the proof of \[29\] Theorem 1, there is a data structure for \(\text{pred-z}\) using \(H_{d,u,B,Z} + O(d \log B)\) bits of space and \(\text{pred}\) query time \(t\) if there is a data structure for RMQ using \(2n - 1.5 \log n + r\) bits of space and query time \(t\).

**Lemma 3 (Improved \[29\] Lemma 3).** For any parameters \(d, u, B, Z\) satisfying \(u = \Theta(\sqrt{B})\), any data structure solves the \(\text{pred-z}\) problem under distribution \(D\) that uses at most \(H_{d,u,B,Z} + O(d \log B)\) bits of space and answers \(\text{pred}\) queries in time \(t\) must have
\[
(wt \log B)^{O(t \log^2 t)} \geq B,
\]
in the cell-probe model with word-size \(w\).

Theorem 1 can be easily proved by following the identical proof of the Theorem 1 in \[29\] with Lemma 3, which is an improved version of the Lemma 3 in \[29\], so we omit the proof in this paper. We give a proof sketch here for completeness.

**Proof sketch of Theorem 1.** Observe that the range minimum query becomes predecessor search if one end of the range is fixed. The basic idea is to break the universe \([n]\) for RMQ into \(d = 2r\) blocks of length \(B = \lfloor n/d \rfloor - 1\), and to embed the each subdatabase of \(\text{pred-z}\) into each block, so that we can invoke the range minimum query to solve the predecessor search. Recall that the database of RMQ is a binary tree of size \(n\). Then we let the large integer \(z\) to encode the remaining structure of the binary tree. ▶
3.3 Lower bound for \texttt{pred-z}

In this subsection, we prove Lemma 3 with Lemma 2. The proof is identical with the proof of Lemma 3 in [29], which is based on a round-elimination argument suggested by [39]. Let \( Q \triangleq \{ \texttt{pred}(i, s(i)) : i \in [d], j \in [u] \} \) be the set of queries on all input datapoints. We are going to prove an expected average-case lower bound for the time cost of all the queries in \( Q \) under input distribution \( D \). We emphasize that \( Q \) is random since it precisely is the set of all input datapoints.

We adopt a round-elimination strategy to prove Lemma 3. To this end, we are going to work with data structures in the cell-probe model with published bits, where the model was introduced at the beginning of Section 2. At the beginning of each round, we have a data structure which answers queries faster (i.e. probes fewer memory cells) but consumes the same memory space and more published bits.

Proof of Lemma 4. Suppose there is a data structure using \( p \) published bits for \texttt{pred-z} under distribution \( D \), and its worst-case \texttt{pred} query time is \( t = o(\log B) \). Then there exists a set \( Q_{pub} \) of \( p \log^4 B \) \texttt{pred} queries, possibly random and depending on the input, such that

\[
\mathbb{E} \left[ \frac{1}{|Q|} \sum_{q \in Q} \text{Probe}(q) \cap \text{Probe}(Q_{pub}) \right] = \Omega \left( \frac{1}{\log^2 t} \right),
\]

where the expectation is taken over the random input data \((\{S_1, \ldots, S_d\}, z) \sim D\) and the choice of \( Q_{pub} \).

Lemma 3 can be proved by following the identical proof of Lemma 3 in [29] with our Lemma 2 and by noting that the lower bound of Eq(1) is always weaker than \( t = \Omega(\log B) \) (which is equivalent to \( r = n/\exp(O(t)) \)). So we give a proof sketch here for completeness as well.

Proof sketch of Lemma 3. We just recursively apply Lemma 2 in each round, we begin with a data structure with \( p \) published bits; then we apply Lemma 4 to find \( Q_{pub} \) and append the addresses and contents of all cells in \( \text{Probe}(Q_{pub}) \) to the published bits; by Lemma 2, publishing the cells takes \( O(p \cdot w t \log^4 B) \) bits, and the new published bits save the expected average query time of \( Q \) by \( \Omega(1/\log^2 t) \). At the end of the recursion, we have a data structure with \( O(d \log B \cdot (w t \log^4 B)^{O(1/\log^2 t)}) \) published bits and the data structure can answer all the queries in \( Q \) only with the published bits, so it must hold that \( O(d \log B \cdot (w t \log^4 B)^{O(1/\log^2 t)}) \geq du/\log^4 B \).

We prove Lemma 4 with Lemma 2 and Lemma 5.

\textbf{Lemma 2.} Suppose there is a data structure using \( p \) published bits for \texttt{pred-z} under distribution \( D \), and its worst-case \texttt{pred} query time is \( t = o(\log B) \). Then there exists a set of queries \( Q_{pub} \) such that \( \mathbb{E}[|\mathcal{E} \setminus \mathcal{K}|] = \Omega(p \log^2 B/\log t) \).

\textbf{Lemma 5 ([29 Lemma 15])}. For \( l \leq m/\log^2 B = O(\sqrt{L}) \), in the good blocks, the expected number of intervals that have between \( l/2 \) and \( l \) elements is at most \( O(p \log^2 B/m + p) \).

Proof of Lemma 4. Recall that we suppose that \( t = o(\log B) \), then \( \mathbb{E}[|\mathcal{E} \setminus \mathcal{K}|] = \Omega(p \log^2 B/\log t) \) by Lemma 2. We further consider a set \( \mathcal{K} \subset \mathcal{E} \) such that \( \mathcal{K} \) is the set of non-empty intervals.
which contain at least one hard query \( q \) such that \( \text{Probe}(\{q\}) \cap \text{Probe}(Q_{\text{pub}}) = \emptyset \). Note that \( \mathcal{E} \setminus \mathcal{K}' \) is the set of non-empty intervals \( I \) such that it holds \( \text{Probe}(\{q\}) \cap \text{Probe}(Q_{\text{pub}}) \neq \emptyset \) for every hard query \( q \in I \). In other words, every hard query contained in \( \mathcal{E} \setminus \mathcal{K}' \) saves at least one probe if we allow the query algorithm to probe \( \text{Probe}(Q_{\text{pub}}) \) for free. Indeed \( \mathcal{K}' \subseteq \mathcal{K} \subseteq \mathcal{E} \), thus it holds that \( \mathbb{E}||\mathcal{E} \setminus \mathcal{K}'|| \geq \mathbb{E}||\mathcal{E} \setminus \mathcal{K}|| = \Omega(p \log^2 B / \log t) \).

Suppose the hidden constants in Lemma 2 and Lemma 5 are \( c_i, c_l \) respectively. By Lemma 3 in the good blocks, the expected number of intervals which contains at most \( l \) elements is at most

\[
\sum_{i=0}^{\log t} c_i p 2^i \log^4 B/m + p \log l \leq 2plc_l \log^4 B/m + p \log l.
\]

Hence we set \( t \) to a proper value \( \Theta(m/(\log^2 B \log t)) \) according to \( c_i, c_l \) to balance the two terms, such that \( 2lc_i p \log^4 B/m + p \log l \leq (pc_i \log^2 B)/(2 \log t) \) and \( pl \log^4 B/m = \Omega(p \log^2 B / \log t) \). Therefore, in the good blocks, the expected number of intervals which contains \( \Omega(m/(\log^2 B \log t)) \) elements is at least \( (c_p \log^2 B)/(2 \log t) \). Thus the expected number of hard queries \( q \) such that \( \text{Probe}(q) \cap \text{Probe}(Q_{\text{pub}}) \neq \emptyset \) is at least \( \Omega(pm/\log^2 t) \). The lemma then follows from the fact that there are exact \( ud = pm \) hard queries.

\[ \blacksquare \]

4 Analyzing the Input Distribution

In this section, we prove Lemma 2 to complete the proof. For simplicity, we assume the \( p \) published bits, \( S_p, \Delta, Q_{\text{pub}} \) are known in advance from now on. In other words, we will ignore the four random variables in the conditions of conditional entropies and conditional mutual informations in this section and Appendix A, Appendix B.

Recall that we let \( \mathcal{E} \) denote the set of non-empty intervals contained in the good blocks, let \( \mathcal{K} \subseteq \mathcal{E} \) denote the set of non-empty intervals in the good blocks that we can recover with the answers to all the queries \( q \) such that \( \text{Probe}(\{q\}) \cap \text{Probe}(Q_{\text{pub}}) = \emptyset \). Formally, \( \mathcal{K} \) can be computed by Algorithm 1.

\[ \blacktriangleleft \] Lemma 2. Suppose there is a data structure using \( p \) published bits for \( \text{pred-z} \) under distribution \( D \), and its worst-case \( \text{pred} \) query time is \( t = o(\log B) \). Then there exists a set of queries \( Q_{\text{pub}} \) such that \( \mathbb{E}[|\mathcal{E} \setminus \mathcal{K}|] = \Omega(p \log^2 B / \log t) \).

Before we present the formal proof, we introduce some useful notions.

We adopt the notion of footprint from [39]. For a query \( q \), we let \( \text{Foot}_i(q) \in \{0, 1\}^w \) denote the content of the \( i \)-th memory cell probed by the query algorithm when answering query \( q \). Furthermore, we let \( \text{Foot}_{\leq i}(q) \triangleq \text{Foot}_1(q) \cdot \text{Foot}_2(q) \cdot \ldots \cdot \text{Foot}_i(q) \in \{0, 1\}^{w_i} \) denote the concatenation of the first \( i \) memory cells probed by the query algorithm when answering \( q \), and let \( \text{Foot}_{< i}(q) \triangleq \text{Foot}_{\leq i-1}(q), \text{Foot}(q) \triangleq \text{Foot}_{< i}(q), \text{Foot}_0(q) \triangleq \emptyset \). For a set of queries \( Q \), we let \( \text{Foot}_{\leq i}(Q) \in \{0, 1\}^{w_i} |Q| \) denote the concatenation of the contents of the \( i \)-th memory cells probed by the query algorithm when answering \( Q \), in the lexicographical order of the queries in \( Q \). Similarly, we let \( \text{Foot}_{\leq i}(Q) \triangleq \text{Foot}_1(Q) \cdot \text{Foot}_2(Q) \cdot \ldots \cdot \text{Foot}_{i}(Q) \in \{0, 1\}^{w_i} \), let \( \text{Foot}_{< i}(Q) \triangleq \text{Foot}_{\leq i-1}(Q), \text{Foot}(Q) \triangleq \text{Foot}_{< i}(Q), \text{Foot}_0(Q) \triangleq \emptyset \). Note that \( \text{Probe}_{\leq i}(Q_{\text{pub}}) \) can be known from \( \text{Foot}_{< i}(Q_{\text{pub}}) \) together with \( Q_{\text{pub}} \). Let \( \text{Probe}_{< i}(Q_{\text{pub}}) \) denote the set of all the cells excluding \( \text{Probe}_{\leq i}(Q_{\text{pub}}) \). Note that the \( \text{Probe}_{< i}(Q_{\text{pub}}) \) is known so long as \( \text{Foot}_{< i}(Q_{\text{pub}}) \) together with \( Q_{\text{pub}} \) are known, since \( \text{Probe}_{\leq i}(Q_{\text{pub}}) \) is known now. Let \( \text{rest}_i(Q_{\text{pub}}) \in \{0, 1\}^{w_{\text{Probe}_{\leq i}(Q_{\text{pub}})}} \) denote the binary string obtained by concatenating all the \( w \)-bit) contents of cells in \( \text{ Probe}_{-i}(Q_{\text{pub}}) \) in the order of their addresses.
Let $\mathcal{K}_i$ denote the set of non-empty intervals which contain at least one hard query $q$ such that $\text{Probe}_{\{q'\}}(\mathcal{P}_{\text{pub}}) \cap \text{Probe}_{\leq i}(\mathcal{P}_{\text{pub}}) = \emptyset$ and $\text{pred}(q') = q$ for some query $q'$. (See Algorithm 2 for formal definition.) Note that $\mathcal{K}_i$ is known as $\mathcal{P}_{\text{pub}}, \text{Foot}_{\leq i}(\mathcal{P}_{\text{pub}})$, and $\text{rest}_{i}(\mathcal{P}_{\text{pub}})$ are known. Also note that $\mathcal{K}_i = \mathcal{E}, \mathcal{K}_{i+1}$ denote the additional known non-empty intervals by further allowing the query algorithm to probe the cells in $\text{Probe}_{\leq i}(\mathcal{P}_{\text{pub}}) \setminus \text{Probe}_{\leq i+1}(\mathcal{P}_{\text{pub}}) = \text{Probe}_{i+1}(\mathcal{P}_{\text{pub}}).

\textbf{Algorithm 2} Algorithm to compute $\mathcal{K}_i$.

\begin{algorithm}
\begin{footnotesize}
\begin{verbatim}
$\mathcal{K}_i \leftarrow \emptyset$;
\textbf{foreach} $(j, [x, y])$ is good block \textbf{do}
\hspace{1em} \textbf{for} $k \in [x, y]$ \textbf{do}
\hspace{2em} \textbf{if} $\text{Probe}(j, k) \cap \text{Probe}_{\leq i}(\mathcal{P}_{\text{pub}}) = \emptyset$ and $\text{pred}(j, k) \geq x$ \textbf{then}
\hspace{3em} $\mathcal{K}_i \leftarrow \mathcal{K}_i \cup \{\text{the interval contains the answer to pred}(j, k)\}$;
\hspace{2em} end
\hspace{1em} end
\end{verbatim}
\end{footnotesize}
\end{algorithm}

\textbf{Proof of Lemma 2.} By the chain rule of entropy, for any $0 < i \leq t$,

$$
H(\text{Foot}_{\leq i}(\mathcal{P}_{\text{pub}})|\mathcal{K}_i) = H(\text{Foot}_{\leq i}(\mathcal{P}_{\text{pub}})|\text{Foot}_{\leq i}(\mathcal{P}_{\text{pub}}), \mathcal{K}_i) + H(\text{Foot}_{\leq i}(\mathcal{P}_{\text{pub}})|\mathcal{K}_i).
$$

\textbf{Claim 6.} For any $0 < i \leq t$,

$$
H(\text{Foot}_{\leq i}(\mathcal{P}_{\text{pub}})|\text{Foot}_{\leq i}(\mathcal{P}_{\text{pub}}), \mathcal{K}_i) = H(\text{Foot}_{\leq i}(\mathcal{P}_{\text{pub}})|\text{Foot}_{\leq i}(\mathcal{P}_{\text{pub}})) - O(p \log B).
$$

\textbf{Claim 7.} For any $0 < i \leq t$,

$$
H(\text{Foot}_{\leq i}(\mathcal{P}_{\text{pub}})|\mathcal{K}_i) \geq H(\text{Foot}_{\leq i}(\mathcal{P}_{\text{pub}})|\mathcal{K}_{i-1}) - H(\mathcal{K}_i, \mathcal{K}_{i-1}|\mathcal{K}_{i-1}).
$$

The proofs of the two claims are easy. If Claim 6 does not hold, then there is a too-good-to-be-true compression scheme. Claim 7 can be proved by a standard information argument. We defer the formal proofs of the two claims to Appendix A and Appendix B. Recall that $\text{Foot}(\mathcal{P}_{\text{pub}}) = \text{Foot}_{\leq t}(\mathcal{P}_{\text{pub}})$ and $\mathcal{K} = \mathcal{K}_t$. By recursively applying the above two claims and the chain rule of entropy, we have

$$
H(\text{Foot}(\mathcal{P}_{\text{pub}})|\mathcal{K}) = \sum_{i=1}^{t} H(\text{Foot}_{\leq i}(\mathcal{P}_{\text{pub}})|\text{Foot}_{\leq i}(\mathcal{P}_{\text{pub}})) - O(tp \log B) - \sum_{i=1}^{t} H(\mathcal{K}_i, \mathcal{K}_{i-1}|\mathcal{K}_{i-1})
$$

$$
= H(\text{Foot}(\mathcal{P}_{\text{pub}})) - O(tp \log B) - \sum_{i=1}^{t} H(\mathcal{K}_i, \mathcal{K}_{i-1}|\mathcal{K}_{i-1}).
$$

Hence, we have

$$
\sum_{i=1}^{t} H(\mathcal{K}_i, \mathcal{K}_{i-1}|\mathcal{K}_{i-1}) = I(\text{Foot}(\mathcal{P}_{\text{pub}}) : \mathcal{K}) - O(tp \log B) \geq I(\mathcal{E} : \mathcal{K}) - O(tp \log B)
$$

$$
\implies \sum_{i=1}^{t} H(\mathcal{K}_i, \mathcal{K}_{i-1}|\mathcal{K}_{i-1}) + H(\mathcal{E}|\mathcal{K}) + O(tp \log B) \geq H(\mathcal{E}). \tag{2}
$$

We apply the following claims to complete the proof.
Nearly Tight Lower Bounds for RMQ

▷ Claim 8. \[ \sum_{i=1}^{t} H(K_i, K'_{i-1}|K_{i-1}) = O\left(E[|E \setminus K|] \log(t \cdot E[|E|]/E[|E \setminus K|])\right). \]

▷ Claim 9. \[ H(E|K) = O\left(E[|E \setminus K|] \log(E[|E|]/E[|E \setminus K|]) + p(\log \log B)^2\right). \]

▷ Claim 10. \[ H(E) = \Omega(p \log^2 B) \text{ and } E[|E|] = \Theta(p \log^2 B). \]

Therefore, we have
\[ E[|E \setminus K|] \leq E[|E \setminus K|] \log(t \cdot E[|E|]/E[|E \setminus K|]) + tp \log B = \Omega(p \log^2 B). \]

Hence, one of the following inequalities must be true: (i) \( E[|E \setminus K|] = \Omega(p \log^2 B / \log t) \), (ii) \( E[|E \setminus K|] = \Omega(p \log^2 B) \), (iii) \( t = \Omega(\log B) \).

Now we prove Claim 8, Claim 9 and Claim 10 respectively.

▷ Claim 11. For any two joint distributed random real number \( X, Y > 0 \), \( E[X \log(Y/X)] \leq E[X] \log(E[Y]/E[X]). \)

\[
\sum_{i \in [t]} E[|K'_{i-1}| \log(|K_{i-1}|/|K'_{i-1}|)] \\
\leq \sum_{i \in [t]} E[|K'_{i-1}| \log(E[|K_{i-1}|]/E[|K'_{i-1}|])] \\
= tE_{i \sim [t]}[E[|K'_{i-1}| \log(E[|K_{i-1}|]/E[|K'_{i-1}|])] \\
\leq E[|E \setminus K|] \log(t \cdot E[|E|]/E[|E \setminus K|]),
\]

where we let \( i \sim [t] \) denote that \( i \) is sampled from \([t]\) uniformly at random.

▷ Claim 9. \[ H(E|K) = O\left(E[|E \setminus K|] \log(E[|E|]/E[|E \setminus K|]) + p(\log \log B)^2\right). \]

Claim 9 is an easy corollary of the following lemma: \( K \) together with \( E(K) \) determine \( E \), so \( H(E|K) \leq E[|E(K)|] \). The proof of the following lemma is almost identical with the Lemma 7 in [29], therefore we present the proof in Appendix D.

▷ Lemma 12 ([29] Lemma 7). There is a prefix-free binary string \( E(K) \) such that \( E(K) \) and \( K, Q_{pub} \) together determine \( E \). Moreover, we have the following bound on the length of \( E(K) \):

\[
E[|E(K)|] = O\left(E[|E \setminus K|] \log(E[|E|]/E[|E \setminus K|]) + p \log^2 \log B\right).
\]

▷ Claim 10. \[ H(E) = \Omega(p \log^2 B) \text{ and } E[|E|] = \Theta(p \log^2 B). \]
Claim 10 can be obtained by combining Lemma 13, Lemma 14, Lemma 15, and the fact there are at most $p$ good blocks. In particular, Lemma 13 ensures $\Theta(p)$ good blocks in expectation; Lemma 14 guarantees that the entropy of the set of non-empty intervals inside a good block is $\Omega(\log^2 B)$; Lemma 14 together with Lemma 15 ensure that the expected number of non-empty intervals inside a good block is $\Theta(\log^2 B)$.

▶ Lemma 13 ([29 Lemma 6]). For every integers $i \in [d], l \in [p/3d, 2p/3d]$, we have

$$\Pr[\text{block}(i, [s^i_{lm} + 1, s^{i+1}_l]) \text{ is good}] = \Omega(1).$$

▶ Lemma 14 ([29 Lemma 5]). In a good block $(i, [x, y])$, let $E_{i,[x,y]}$ denote the set of non-empty intervals inside the good block. Then

$$H(E_{i,[x,y]}) = \Omega(\log^2 B).$$

Moreover, for any good block $(i, [x, y])$, we have $\mathbb{E}[|E_{i,[x,y]}|] = \Omega(\log^2 B)$.

▶ Lemma 15 ([29 Lemma 17]). In a good block $(i, [x, y])$, $\mathbb{E}[|E_{i,[x,y]}|] = O(\log^2 B)$.

References

1. Amihood Amir, Gad M. Landau, and Uzi Vishkin. Efficient pattern matching with scaling. J. Algorithms, 13(1):2–32, 1992.
2. Michael A. Bender, Martin Farach-Colton, Giri G. Premasani, Steven Skiena, and Pavel Sumazin. Lowest common ancestors in trees and directed acyclic graphs. J. Algorithms, 57(2):75–94, 2005.
3. Omer Berkman and Uzi Vishkin. Recursive star-tree parallel data structure. SIAM J. Comput., 22(2):221–242, 1993.
4. Amit Chakrabarti, Bernard Chazelle, Benjamin Gum, and Alexey Lvov. A lower bound on the complexity of approximate nearest-neighbor searching on the hamming cube. In Proceedings of the Thirty-First Annual ACM Symposium on Theory of Computing, STOC ’99, page 305–311, New York, NY, USA, 1999. Association for Computing Machinery. doi: 10.1145/301250.301325.
5. Amit Chakrabarti and Oded Regev. An optimal randomized cell probe lower bound for approximate nearest neighbor searching. SIAM Journal on Computing, 39(5):1919–1940, 2010.
6. Gang Chen, Simon J. Puglisi, and W. F. Smyth. Lempel–Ziv factorization using less time & space. Mathematics in Computer Science, 1(4):605–623, Jun 2008.
7. Kuan-Yu Chen and Kun-Mao Chao. On the range maximum-sum segment query problem. Discrete Applied Mathematics, 155(16):2043–2052, 2007.
8. Raphaël Clifford, Allan Grønlund, and Kasper Green Larsen. New unconditional hardness results for dynamic and online problems. In IEEE 56th Annual Symposium on Foundations of Computer Science, FOCS 2015, Berkeley, CA, USA, 17-20 October, 2015, pages 1089–1107, 2015.
9. Maxime Crochemore, Costas S. Iliopoulos, Marcin Kubica, M. Sohel Rahman, German Tischler, and Tomasz Walen. Improved algorithms for the range next value problem and applications. Theor. Comput. Sci., 434:23–34, 2012.
10. Johannes Fischer and Volker Heun. A new succinct representation of rmq-information and improvements in the enhanced suffix array. In Combinatorics, Algorithms, Probabilistic and Experimental Methodologies, First International Symposium, ESCAPE 2007, Hangzhou, China, April 7-9, 2007, Revised Selected Papers, pages 459–470, 2007.
11. Johannes Fischer and Volker Heun. Space-efficient preprocessing schemes for range minimum queries on static arrays. SIAM J. Comput., 40(2):465–492, 2011.
12 Johannes Fischer, Volker Heun, and Stefan Kramer. Optimal string mining under frequency constraints. In Knowledge Discovery in Databases: PKDD 2006, 10th European Conference on Principles and Practice of Knowledge Discovery in Databases, Berlin, Germany, September 18-22, 2006, Proceedings, pages 139–150, 2006.

13 Johannes Fischer, Veli Mäkinen, and Gonzalo Navarro. Faster entropy-bounded compressed suffix trees. Theor. Comput. Sci., 410(51):5354–5364, 2009.

14 Harold N. Gabow, Jon Louis Bentley, and Robert Endre Tarjan. Scaling and related techniques for geometry problems. In Proceedings of the 16th Annual ACM Symposium on Theory of Computing, April 30 - May 2, 1984, Washington, DC, USA, pages 135–143, 1984.

15 Anna Gál and Peter Bro Miltersen. The cell probe complexity of succinct data structures. Theor. Comput. Sci., 379(3):405–417, 2007.

16 Loukas Georgiadis and Robert Endre Tarjan. Finding dominators revisited: extended abstract. In Proceedings of the Fifteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2004, New Orleans, Louisiana, USA, January 11-14, 2004, pages 869–878, 2004.

17 Alexander Golynski. Cell probe lower bounds for succinct data structures. In Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2009, New York, NY, USA, January 4-6, 2009, pages 625–634, 2009.

18 Allan Grønlund and Kasper Green Larsen. Towards tight lower bounds for range reporting on the RAM. In 43rd International Colloquium on Automata, Languages, and Programming, ICALP 2016, July 11-15, 2016, Rome, Italy, pages 92:1–92:12, 2016.

19 Dov Harel and Robert Endre Tarjan. Fast algorithms for finding nearest common ancestors. SIAM J. Comput., 13(2):338–355, 1984.

20 Wing-Kai Hon, Rahul Shah, and Jeffrey Scott Vitter. Space-efficient framework for top-k string retrieval problems. In 50th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2009, October 25-27, 2009, Atlanta, Georgia, USA, pages 713–722, 2009.

21 Guy Joseph Jacobson. Succinct Static Data Structures. PhD thesis, Carnegie Mellon University, Pittsburgh, PA, USA, 1988.

22 Antti Laaksonen. Range Queries, pages 119–129. Springer International Publishing, Cham, 2017. doi:10.1007/978-3-319-72547-5_9

23 Kasper Green Larsen. The cell probe complexity of dynamic range counting. In Proceedings of the 44th Symposium on Theory of Computing Conference, STOC 2012, New York, NY, USA, May 19 - 22, 2012, pages 85–94, 2012.

24 Kasper Green Larsen. Higher cell probe lower bounds for evaluating polynomials. In 53rd Annual IEEE Symposium on Foundations of Computer Science, FOCS 2012, New Brunswick, NJ, USA, October 20-23, 2012, pages 293–301, 2012.

25 Kasper Green Larsen, Jelani Nelson, and Huy L. Nguyën. Time lower bounds for nonadaptive turnstile streaming algorithms. In Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing, STOC 2015, Portland, OR, USA, June 14-17, 2015, pages 803–812, 2015.

26 Kasper Green Larsen, Omri Weinstein, and Huacheng Yu. Crossing the logarithmic barrier for dynamic boolean data structure lower bounds. In Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2018, Los Angeles, CA, USA, June 25-29, 2018, pages 978–989, 2018.

27 Hsiao-Fei Liu and Kun-Mao Chao. Algorithms for finding the weight-constrained k longest paths in a tree and the length-constrained k maximum-sum segments of a sequence. Theor. Comput. Sci., 407(1-3):349–358, 2008.

28 Mingmou Liu, Xiaoyin Pan, and Yitong Yin. Randomized approximate nearest neighbor search with limited adaptivity. ACM Trans. Parallel Comput., 5(1), June 2018. doi:10.1145/3209884

29 Mingmou Liu and Huacheng Yu. Lower bound for succinct range minimum query. In Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing, STOC 2020, page 1402–1415, New York, NY, USA, 2020. Association for Computing Machinery. doi:10.1145/3357713.3384260
Peter Bro Miltersen, Noam Nisan, Shmuel Safra, and Avi Wigderson. On data structures and asymmetric communication complexity. In Proceedings of the Twenty-Seventh Annual ACM Symposium on Theory of Computing, STOC ’95, page 103–111, New York, NY, USA, 1995. Association for Computing Machinery. \(\text{doi:10.1145/225058.225093}\)

Peter Bro Miltersen, Noam Nisan, Shmuel Safra, and Avi Wigderson. On data structures and asymmetric communication complexity. Journal of Computer and System Sciences, 57(1):37–49, 1998. URL: https://www.sciencedirect.com/science/article/pii/S002200009891577X, doi:https://doi.org/10.1006/jcss.1998.1577

S. Muthukrishnan. Efficient algorithms for document retrieval problems. In Proceedings of the Thirteenth Annual ACM-SIAM Symposium on Discrete Algorithms, January 6-8, 2002, San Francisco, CA, USA., pages 657–666, 2002.

Gonzalo Navarro and Kunihiko Sadakane. Fully functional static and dynamic succinct trees. ACM Trans. Algorithms, 10(3):16:1–16:39, 2014.

Rina Panigrahy, Kunal Talwar, and Udi Wieder. Lower bounds on near neighbor search via metric expansion. In 51th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2010, October 23-26, 2010, Las Vegas, Nevada, USA, pages 805–814, 2010.

Mihai Pătraşcu. Succincter. In 49th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2008, October 25-28, 2008, Philadelphia, PA, USA, pages 305–313, 2008.

Mihai Pătraşcu and Mikkel Thorup. Higher lower bounds for near-neighbor and further rich problems. In 2006 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS’06), pages 646–654. IEEE, 2006.

Mihai Pătraşcu and Mikkel Thorup. Time-space trade-offs for predecessor search. In Proceedings of the thirty-eighth annual ACM symposium on Theory of computing, pages 232–240. ACM, 2006.

Mihai Pătraşcu and Mikkel Thorup. Randomization does not help searching predecessors. In Proc. 18th ACM/SIAM Symposium on Discrete Algorithms (SODA), pages 555–564, 2007.

Mihai Pătraşcu and Emanuele Viola. Cell-probe lower bounds for succinct partial sums. In Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2010, Austin, Texas, USA, January 17-19, 2010, pages 117–122, 2010.

Vijaya Ramachandran and Uzi Vishkin. Efficient parallel triconnectivity in logarithmic time. In VLSI Algorithms and Architectures, 3rd Aegean Workshop on Computing, AWOC 88, Corfu, Greece, June 28 - July 1, 1988, Proceedings, pages 33–42, 1988.

Kunihiko Sadakane. Compressed suffix trees with full functionality. Theory Comput. Syst., 41(4):589–607, 2007.

Kunihiko Sadakane. Succinct data structures for flexible text retrieval systems. J. Discrete Algorithms, 5(1):12–22, 2007.

Kunihiko Sadakane and Gonzalo Navarro. Fully-functional succinct trees. In Proceedings of the 2010 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 134–149, 2010. URL: https://epubs.siam.org/doi/abs/10.1137/1.9781611973075.13, \text{arXiv:https://epubs.siam.org/doi/pdf/10.1137/1.9781611973075.13 doi:10.1137/1.9781611973075.13}

Sanjeev Saxena. Dominance made simple. Inf. Process. Lett., 109(9):419–421, 2009.

Pranab Sen and S. Venkatesh. Lower bounds for predecessor searching in the cell probe model. Journal of Computer and System Sciences, 74(3):364–385, 2008. Computational Complexity 2003. URL: https://www.sciencedirect.com/science/article/pii/S0022000007000839, doi:https://doi.org/10.1016/j.jcss.2007.06.016

Tetsuo Shibuya and Igor Kurochkin. Match chaining algorithms for cdna mapping. In Algorithms in Bioinformatics, Third International Workshop, WABI 2003, Budapest, Hungary, September 15-20, 2003, Proceedings, pages 462–475, 2003.

Alan Siegel. On universal classes of fast high performance hash functions, their time-space tradeoff, and their applications. In Proceedings of the 30th Annual Symposium on Foundations
We can decode the input database from the data structure, thus the codeword must be of whole data structure, together with all the published bits, from the codeword. Furthermore, of the cells in the partition is known as long as the addresses together with the contents of the cells in disjoint sets:

Recall that $H(Foot_i(Q_{pub})|Foot_{<i}(Q_{pub}), K_i) \geq H(Foot_i(Q_{pub})|Foot_{<i}(Q_{pub}), rest_i(Q_{pub}))$ since $K_i$ is known from $Foot_{<i}(Q_{pub})$ together with $rest_i(Q_{pub})$, $Q_{pub}$. Thus it suffices to prove

$$H(Foot_i(Q_{pub})|Foot_{<i}(Q_{pub}), rest_i(Q_{pub})) = H(Foot_i(Q_{pub})|Foot_{<i}(Q_{pub})) - O(p \log B).$$

For any $i \in [l]$, the following encoding scheme always exists:

1. write down the $p$ published bits;
2. write down $S_{pt}$;
3. sample $\Delta \in [L]$ uniformly at random, write down $\Delta$;
4. write down the contents of cells in $\text{Probe}_{<i}(Q_{pub})$ using $w \cdot |\text{Probe}_{<i}(Q_{pub})|$ bits;
5. write down the contents of cells in $\text{Probe}_{=i}(Q_{pub})$, i.e. the contents of all the cells excluding the cells in $\text{Probe}_{\leq i}(Q_{pub})$, in the increasing order of their addresses, using $|rest_i(Q_{pub})|$ bits;
6. write down the contents of the cells in $\text{Probe}_{i}(Q_{pub}) \setminus \text{Probe}_{<i}(Q_{pub})$ in the increasing order of their addresses, using the optimal expected $H(Foot_i(Q_{pub}) | rest_i(Q_{pub}), Foot_{<i}(Q_{pub}))$ bits.

Recall that $Q_{pub}$ is known from $S_{pt}$ together with $\Delta$. It is easy to see that we can decode the whole data structure, together with all the published bits, from the codeword. Furthermore, we can decode the input database from the data structure, thus the codeword must be of length at least $H_{u,v,B,Z}$ bits in expectation.

According to [29] Proof of Lemma 4, the first three steps consume $O(p \log B)$ bits.

The encoding scheme partitions the set of all the cells of the data structure into three disjoint sets: $\text{Probe}_{<i}(Q_{pub})$, $\text{Probe}_{i}(Q_{pub}) \setminus \text{Probe}_{<i}(Q_{pub})$, and $\text{Probe}_{=i}(Q_{pub})$. Note that the partition is known as long as the addresses together with the contents of the cells in $\text{Probe}_{=i}(Q_{pub})$, i.e. $Foot_{<i}(Q_{pub})$, are known. The data structure represents the contents of the cells in $\text{Probe}_{i}(Q_{pub}) \setminus \text{Probe}_{<i}(Q_{pub})$ using $w \cdot E[|\text{Probe}_{i}(Q_{pub}) \setminus \text{Probe}_{<i}(Q_{pub})|] \geq H(Foot_i(Q_{pub}) | Foot_{<i}(Q_{pub}))$ bits in expectation. On the other hand, the above encoding
scheme represents the contents of the cells in $\text{Probe}_{\lambda}(Q_{\text{pub}}) \setminus \text{Probe}_{<\lambda}(Q_{\text{pub}})$ using expected $H(\text{Foot}_{\lambda}(Q_{\text{pub}}) \setminus \text{rest}_{\lambda}(Q_{\text{pub}}), \text{Foot}_{<\lambda}(Q_{\text{pub}}))$ bits at cost of writing down extra $O(p \log B)$ bits. If $H(\text{Foot}_{\lambda}(Q_{\text{pub}}) \setminus \text{Foot}_{<\lambda}(Q_{\text{pub}})) - H(\text{Foot}_{\lambda}(Q_{\text{pub}}) \setminus \text{rest}_{\lambda}(Q_{\text{pub}}), \text{Foot}_{<\lambda}(Q_{\text{pub}})) = \omega(p \log B)$, the last step will save $\omega(p \log B)$ bits in writing down the contents of the cells in $\text{Probe}_{\lambda}(Q_{\text{pub}}) \setminus \text{Probe}_{<\lambda}(Q_{\text{pub}})$. Which is impossible, since we do not waste any bit in writing down the contents of the cells in $\text{Probe}_{<\lambda}(Q_{\text{pub}})$ and $\text{Probe}_{\lambda}(Q_{\text{pub}})$. In other words, the codeword is of length $H_{d,u,b,z} - \omega(p \log B)$ bits in expectation, if Eq. (1) does not hold.

Therefore, $H(\text{Foot}_{\lambda}(Q_{\text{pub}}) | \text{Foot}_{<\lambda}(Q_{\text{pub}}), \text{rest}_{\lambda}(Q_{\text{pub}})) = H(\text{Foot}_{\lambda}(Q_{\text{pub}}) | \text{Foot}_{<\lambda}(Q_{\text{pub}})) - O(p \log B)$.

B Proof of Claim 7

Note that $H(\text{Foot}_{<\lambda}(Q_{\text{pub}}) | K_{\lambda}) \geq H(\text{Foot}_{<\lambda}(Q_{\text{pub}}) | K_{\lambda}, K_{\lambda - 1}, K_{\lambda - 1}')$ and $H(\text{Foot}_{<\lambda}(Q_{\text{pub}}) | K_{\lambda - 1}) - H(\text{Foot}_{<\lambda}(Q_{\text{pub}}) | K_{\lambda - 1}, K_{\lambda - 1}, K_{\lambda - 1}') = I(\text{Foot}_{<\lambda}(Q_{\text{pub}}) : K_{\lambda}, K_{\lambda - 1} | K_{\lambda}) \leq H(K_{\lambda - 1}^i | K_{\lambda - 1})$.

C Proof of Fact 11

\[
E[X \log(Y/X)] = E[(X/Y) \log(Y/X)] = \sum_y \Pr[Y = y] \cdot y \cdot E[X/Y] | Y = y
\]

By the convexity of $x \log(1/x)$,

\[
\leq \sum_y \Pr[Y = y] \cdot y \cdot E[X/Y | Y = y] \log(1/E[X/Y | Y = y])
\]

\[
= E[Y] \sum_y (\Pr[Y = y] \cdot y/E[Y]) \log(1/E[X/Y | Y = y])
\]

Observe that $(\Pr[Y = y] \cdot y/E[Y])$ is a probability function with respect to $y$. Let $\mu$ denote the distribution,

\[
= E[Y] E_{y \sim \mu}[E[X/Y | Y = y] \log(1/E[X/Y | Y = y])]
\]

Note that $E_{y \sim \mu}[E[X/Y | Y = y]] = E[X]/E[Y]$, by the convexity of function $x \log(1/x)$,

\[
\leq E[X] \log(E[Y]/E[X]).
\]

D Proof of Lemma 12

First note that $K \subseteq \mathcal{E}$. Hence it suffices to encode in $\mathcal{E}(K)$, the set $\mathcal{E} \setminus K$. At first we present a encoding scheme which encodes set $\mathcal{E}_{j,(x,y)} \setminus K_{j,(x,y)}$ in prefix-free binary string \( \mathcal{E}_{j,(x,y)} \setminus K_{j,(x,y)} \) for some good block $(j, [x, y])$, where $\mathcal{E}_{j,(x,y)}$ is the non-empty intervals in good block $(j, [x, y])$ and $K_{j,(x,y)} \triangleq K \cap \mathcal{E}_{j,(x,y)}$. Then we show that we can efficiently encode $\mathcal{E} \setminus K$ block by block.
A encoding scheme works within good block

D.1.1 Encode $\mathcal{E}_{(j,[x,y])}$ given $\mathcal{K}_{(j,[x,y])}$, $Q_{pub}$

Let $K_{ne} \triangleq |\mathcal{E}_{(j,[x,y])}|$, $K_{un} \triangleq |\mathcal{E}_{(j,[x,y])} \setminus \mathcal{K}_{(j,[x,y])}|$. Let $I_1, I_2, \ldots, I_{K_{ne}} \in \mathcal{E}_{(j,[x,y])}$ be all the elements in the increasing order. Let $I_{i_1}, I_{i_2}, \ldots, I_{i_{K_{un}}} \in \mathcal{E}_{(j,[x,y])} \setminus \mathcal{K}_{(j,[x,y])}$ be all the elements to be encoded, where $i_1 < i_2 < \cdots < i_{K_{un}}$. We first write down $K_{ne}$, then for $a = 1, \ldots, K_{un}$, we do the following:

1. write down $i_a - i_{a-1}$ ($i_0$ is assumed to be 0);
2. write down $I_a - I_{a-1}$.

All integers are encoded using the folklore prefix-free encoding which takes $O(\log N)$ bits to encode a non-negative integer $N$. This completes $\mathcal{E}_{(j,[x,y])}(\mathcal{K}_{(j,[x,y])})$.

D.1.2 Decode $\mathcal{E}_{(j,[x,y])}$ given $\mathcal{E}_{(j,[x,y])}(\mathcal{K}_{(j,[x,y])})$ and $\mathcal{K}_{(j,[x,y])}$, $Q_{pub}$

We read $K_{ne}$ (which, together with $|\mathcal{K}_{(j,[x,y])}|$, $Q_{pub}$, determines $K_{un}$), and for $a = 1, \ldots, K_{un}$, do the following:

1. read the next integer and recover $i_a$;
2. for $i = i_a - 1 + 1, \ldots, i_a - 1$, let $I_i$ be the next element in $\mathcal{E}_{(j,[x,y])}$ ($i_0$ is assumed to be 0);
3. read the next integer and recover $I_{i_a}$.

Finally, for $i = i_{K_{un}} + 1, \ldots, K_{ne}$, let $I_i$ be the next element in $\mathcal{E}_{(j,[x,y])}$. This recovers all $I_1, \ldots, I_{K_{ne}}$, hence, decodes $\mathcal{E}_{(j,[x,y])}$.

D.1.3 The length of $\mathcal{E}_{(j,[x,y])}(\mathcal{K}_{(j,[x,y])})$

Next, we analyze the expected length of $\mathcal{E}_{(j,[x,y])}(\mathcal{K}_{(j,[x,y])})$. $K_{ne}$ takes $O(\log \log B)$ bits to encode. Then for $a = 1, \ldots, K_{un}$, $(i_a - i_{a-1})$ takes $O(\log (i_a - i_{a-1}))$ bits to encode. Since all these integers sum up to (at most) $K_{ne}$, by concavity of log, the total number of bits used to encode $\{i_a - i_{a-1}\}$ is at most

$$O(K_{un} \cdot \log (K_{ne} / K_{un})).$$

Then by Fact 11 the expected encoding length of all $i_a - i_{a-1}$ is at most

$$O\left(\mathbb{E}[|\mathcal{E}_{(j,[x,y])} \setminus \mathcal{K}_{(j,[x,y])}|] \log \frac{\mathbb{E}[|\mathcal{E}_{(j,[x,y])}|]}{\mathbb{E}[|\mathcal{E}_{(j,[x,y])} \setminus \mathcal{K}_{(j,[x,y])}|]}\right).$$

Next, the value $I_{i_a} - I_{i_{a-1}}$ takes $O(\log (I_{i_a} - I_{i_{a-1}}))$ bits to encode. For all $a$ such that $I_{i_a} - I_{i_{a-1}} \leq \left(\frac{\mathbb{E}[|\mathcal{E}_{(j,[x,y])}|]}{\mathbb{E}[|\mathcal{E}_{(j,[x,y])} \setminus \mathcal{K}_{(j,[x,y])}|]}\right)^2$, their total encoding length is at most

$$O\left(K_{un} \cdot \log \frac{\mathbb{E}[|\mathcal{E}_{(j,[x,y])}|]}{\mathbb{E}[|\mathcal{E}_{(j,[x,y])} \setminus \mathcal{K}_{(j,[x,y])}|]}\right),$$

and its expectation is at most

$$O\left(\mathbb{E}[|\mathcal{E}_{(j,[x,y])} \setminus \mathcal{K}_{(j,[x,y])}|] \log \frac{\mathbb{E}[|\mathcal{E}_{(j,[x,y])}|]}{\mathbb{E}[|\mathcal{E}_{(j,[x,y])} \setminus \mathcal{K}_{(j,[x,y])}|]}\right).$$

(4)
Lemma 16 (Lemma 17). In a good block, for any \( t \geq 1 \), the expected number of \( \mathcal{E} \setminus \mathcal{K} \) that have between \( t - 1 \) and \( 2t \) empty intervals and no non-empty interval in between is at most \( O \left( \frac{\sqrt{(\log^4 B)} + 1}{t} \right) \).

For all \( a \) such that \( I_{a_n} - I_{a_{n-1}} > \left( \frac{E[|E(j,x,y)|]}{E[|E(j,x,y)| \setminus K_{j,x,y}]} \right)^2 \), by Lemma 16, the summation of their expected encoding lengths is at most

\[
\sum_{t=2^k \cdot \left( \frac{E[|E(j,x,y)|]}{E[|E(j,x,y)| \setminus K_{j,x,y}]} \right)^2, b \geq 0, t \leq \log^4 B} O \left( \left( \frac{\sqrt{(\log^4 B)} + 1}{t} \right) \log t \right) \leq O \left( \left( \frac{\log^2 B}{E[|E(j,x,y)|]} \right) \setminus K_{j,x,y} \right) \cdot \log \left( \frac{E[|E(j,x,y)|]}{E[|E(j,x,y)| \setminus K_{j,x,y}]} \right) + \log^2 \log B \right). \]

Since \( E[|E(j,x,y)|] = \Omega(\log^2 B) \) by Lemma 14, it is at most

\[
O \left( \left( \frac{\log^2 B}{E[|E(j,x,y)|]} \right) \setminus K_{j,x,y} \right) \cdot \log \left( \frac{E[|E(j,x,y)|]}{E[|E(j,x,y)| \setminus K_{j,x,y}]} \right) + \log^2 \log B \right). \tag{6}
\]

Finally, summing up (4), (5) and (6), the expected length of \( E(j,x,y) \) is at most

\[
O \left( \left( \frac{\log^2 B}{E[|E(j,x,y)|]} \right) \setminus K_{j,x,y} \right) \cdot \log \left( \frac{E[|E(j,x,y)|]}{E[|E(j,x,y)| \setminus K_{j,x,y}]} \right) + \log^2 \log B \right).
\]

D.2 Encoding \( E \setminus K \)

To encode \( E \) given \( K \), we just enumerate the good blocks in lexicographical order and write down the \( E(j,x,y) \setminus K_{j,x,y} \) one by one. Let \( \mathcal{G} \) denote the set of good blocks. We calculate the expected cost to complete the proof:

\[
\mathbb{E} \left[ \sum_{(j,x,y) \in \mathcal{G}} \left( \mathbb{E}[|E(j,x,y)| \setminus K_{j,x,y}] \right) \cdot \log \left( \frac{\mathbb{E}[|E(j,x,y)|]}{\mathbb{E}[|E(j,x,y)| \setminus K_{j,x,y}]} \right) + \log^2 \log B \right].
\]

By applying the law of total expectation, we can restrict the distribution to the event that \( \mathcal{G} \neq \emptyset \). We let \( (j,x,y) \sim \mathcal{G} \) denote we sample \( (j,x,y) \) from \( \mathcal{G} \) uniformly at random.

\[
\mathbb{E} \left[ \mathbb{E}[|E(j,x,y)| \setminus K_{j,x,y}] \cdot \log \left( \frac{\mathbb{E}[|E(j,x,y)|]}{\mathbb{E}[|E(j,x,y)| \setminus K_{j,x,y}]} \right) + \log^2 \log B \right].
\]

Observe that \( \mathbb{E}[|E(j,x,y)| \setminus K_{j,x,y}] = E[|E \setminus K|] \) and \( E[|E(j,x,y)|] = \mathbb{E}[|E|] - |\mathcal{G}| \) by Fact 11

\[
\leq \mathbb{E}[|E \setminus K|] \cdot \log(E[|E|]/\mathbb{E}[|E \setminus K|]) + \mathbb{E}[|\mathcal{G}|] \cdot \log^2 \log B.
\]

The lemma then follows from the fact that \( |\mathcal{G}| \leq p \).