Self–dual tensors and gravitational anomalies in $4n + 2$ dimensions

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Abstract

Starting from a manifestly Lorentz– and diffeomorphism–invariant classical action we perform a perturbative derivation of the gravitational anomalies for chiral bosons in $4n + 2$ dimensions. The manifest classical invariance is achieved using a newly developed method based on a scalar auxiliary field and two new bosonic local symmetries. The resulting anomalies coincide with the ones predicted by the index theorem. In the two–dimensional case, moreover, we perform an exact covariant computation of the effective action for a chiral boson (a scalar) which is seen to coincide with the effective action for a two–dimensional complex Weyl–fermion. All these results support the quantum reliability of the new, at the classical level manifestly invariant, method.

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1 Introduction

For long time the main problem related with (anti)self–dual antisymmetric tensors of rank $2n + 1$ in $4n + 2$ dimensions (chiral bosons), was the absence of a manifestly Lorentz–invariant action principle. Some time ago for two–dimensional chiral bosons Siegel [1] proposed a manifestly invariant lagrangian which is, however, plagued by a quantum anomaly whose proper handling constitutes still a problematic feature of his approach. Apart from this feature, the most inconvenient aspect of this method, also in higher dimensions [2], is that it produces the square of the self–duality equation of motion rather than the self–duality condition itself. This leads at the quantum level to problematic aspects due to the presence of second class constraints which have to be appropriately handled [3].

Manifestly covariant actions for chiral bosons have also been constructed in [4] using an infinite tower of Lagrange multipliers, in which case the problem is shifted to a consistent truncation of this tower.

Another class of actions, which avoid all these problems, has been presented in [3, 5, 4, 4, 8]. The principal drawback of these actions is that they are not manifestly Lorentz–invariant, although being invariant under a set of modified transformations which satisfy the Lorentz algebra. When one couples these actions to gravity, due to this feature, they lack also manifest invariance under diffeomorphisms and a detailed analysis is needed to establish it [3]. Clearly this non manifest invariance becomes even more problematic at the quantum level. Nevertheless, in [7, 8], using these actions the gravitational anomalies for chiral bosons have been derived and shown to coincide with the expected results [10, 11].

In this paper we rederive the gravitational anomalies for chiral bosons in $4n + 2$ dimensions using a newly developed lagrangian approach [12] which, at the classical level, is manifestly invariant under Lorentz–transformations. This feature makes a manifestly diffeomorphism invariant coupling to gravity trivial: the minimal prescription just works. The method itself is based on a single scalar auxiliary field $a(x)$ and on two new bosonic symmetries whose physical interpretation is very simple: the first symmetry allows to eliminate the auxiliary field and the second reduces the second order equation of motion for the antisymmetric tensor to the first order self–duality condition.

At the classical level this new method appears to be very general. It proved compatible with all known symmetries, e.g. with global [13] and local supersymmetry [14], with $\kappa$–symmetry [15] and with manifest duality symmetry between $p$–form potentials in a generic space–time dimension $D$ and their Hodge duals, $(D – p – 2)$–forms [14, 16, 17]; it is also consistent with (self)interacting chiral bosons [18, 19] and, as we will see in detail in section two, the two bosonic symmetries allow a simple control of the physical propagating degrees of freedom of the chiral bosons. Moreover, if one chooses a gauge fixing for the scalar field such that $a(x) = n_m x^m$, where $n_m$ is a constant vector, one recovers the non–manifestly covariant formulations mentioned above: $n_m = (1, 0, \ldots, 0)$
leads to $[3, 5, 6, 7]$ and $n_m = (0, 0, \ldots, 1)$ leads to $[8]$.

Due to these successes of the method at the classical level the most compelling question which remains is if it has some reliability also at the quantum level. For dimensions greater or equal than six, of course, this question is somehow academic due to non-renormalizability of the actions; in two dimensions, on the other side, the question is perfectly well suited and the expected results are known: e.g. the effective action for a chiral boson coupled to gravity should equal the effective action of a complex Weyl fermion. Moreover, the derivation of gravitational anomalies is a well suited issue also in higher dimensions, since anomalies are always finite, and the expected results for chiral bosons are known [10, 11].

The purpose of this paper is to perform a quantum check of the new method in these directions. In the next section we present the classical covariant action for chiral bosons in $D = 4n + 2$ dimensions, interacting with an external gravitational field, and show that it gives rise, as equation of motion, to the self-duality condition for the field-strength. In this section we give also a self-contained account of the new covariant method itself. To avoid writing indices we will use mainly the language of forms. Section three is dedicated to a detailed analysis for chiral bosons in $D = 2$. We perform an exact covariant computation for the effective action and show that it equals, modulo local terms, the effective action of a complex Weyl fermion, computed in a diffeomorphism preserving framework (i.e. where the anomaly is shifted from diffeomorphisms to local Lorentz- and Weyl-transformations). A perturbative one-loop computation of the $D = 2$ anomaly is also performed to prepare the anomaly derivation in higher dimensions. In section four we derive the gravitational anomalies for chiral bosons in higher dimensions showing that the effective Feynman rules associated to our classical action coincide with the Feynman rules conjectured in [10] which, in turn, led to the results predicted by the index theorem [11]. This procedure requires, in particular, an appropriate gauge fixing of the two new bosonic symmetries mentioned above. Section five is devoted to some concluding remarks.

2 The classical action for chiral bosons

The language we will use mainly in this paper is the language of differential forms. This will allow to keep the formulae compact, i.e. without writing explicitly indices, and to perform the relevant computations using only the algebra satisfied by the exterior differential $d$, the hodge-dual $\ast$ and the interior product $i_v$ of a vector $v$ with a $p$-form (see below).

In particular we will write our classical action for chiral bosons in $D = 2k + 2$ dimensions, with $k$ even, as an integral over a $D$-form. Before doing that we state our conventions on forms and present the relevant identities.
We define the components of a $p$–form $\phi_p$ as
\[ \phi_p = \frac{1}{p!} dx^{n_1} \cdots dx^{n_p} \phi_{n_p \cdots n_1}, \]
and correspondingly the exterior differential $d = dx^n \partial_n$ begins to act on the right. The product between forms will always be the wedge product and the symbol $\wedge$ will be omitted.

The interior product of a vector field $v = v^m \partial_m$ with a $p$–form is defined by
\[ i_v \phi_p = \frac{1}{(p-1)!} dx^{n_1} \cdots dx^{n_{p-1}} v^{n_p} \phi_{n_p \cdots n_1}, \]
and satisfies the same distribution law,
\[ i_v (\phi_p \phi_q) = \phi_p i_v \phi_q + (-)^q v^i (\phi_p) \phi_q, \]
as $d$.

Introducing a metric $g^{mn}$ on the space our convention for the Hodge–dual is
\[ \ast(\phi_p) \equiv \frac{1}{(D-p)!} dx^{n_1} \cdots dx^{n_{D-p}} (\ast \phi)_{n_{D-p} \cdots n_1}, \]
where
\[ (\ast \phi)_{n_1 \cdots n_{D-p}} = \frac{1}{p!} g_{n_1 m_1} \cdots g_{n_{D-p} m_{D-p}} \varepsilon^{m_1 \cdots m_{D-p} n_1 \cdots n_p} \phi_{m_1 \cdots m_p} \sqrt{g} \]
and $g = -detg_{mn}$. Our flat metric is $\eta_{mn} = (1, -1, \cdots, -1)$ and $\varepsilon^{01 \cdots D-1} = +1$.

The delta–operator, which sends a $p$–form in a $(p-1)$–form, is defined as usual by
\[ \delta = \ast d \ast. \]

On a $p$–form the square of the Hodge–dual, in an even dimensional space–time as the ones considered here, satisfies
\[ \ast^2 = (-)^{p+1}. \tag{2.1} \]

Using these properties and definitions one can prove the following operatorial identities, which hold on any $p$–form and for any vector field $v$, and which will be frequently used in what follows:

\[ \ast i_v = v \ast \]
\[ i_v \ast = - \ast v \]
\[ i_v v - \ast v = (-)^p v^2 \]
\[ \Box_g = D_m g^{mn} D_n = \delta d + d \delta. \tag{2.5} \]

With the one–form $v$ we mean here
\[ v = dx^n g_{mn} v^m, \]
and $v^2 = g_{mn} v^m v^n$. Particularly useful will be the following decompositions of the identity $I$ and of the $\ast$–operator which follow from these identities:

\[ I = \frac{1}{v^2} \left( (-)^{p+1} v i_v + \ast v i_v \ast \right) \tag{2.6} \]
\[ \ast = (-)^{p+1} \frac{1}{v^2} (\ast i_v + v i_v \ast) \tag{2.7} \]
\[ (1 - \ast) v i_v (1 - \ast) = v^2 (1 + (-)^p) + v i_v (1 + (-1)^p). \tag{2.8} \]
(2.6) allows, in particular, to decompose every $p$–form uniquely into a component along $v$ and a component orthogonal to $v$.

We turn now to the construction of the action [12]. Chiral bosons are described by a $k$–form $B$ whose curvature, a $(k+1)$–form,

$$H = dB$$

satisfies as equation of motion the self–duality condition

$$H = *H. \quad (2.9)$$

For definiteness we treat here the case of a self–dual field strength, for an antiself–dual field strength the procedure is completely analogous.

As anticipated in the introduction, in order to write an action in addition to $B$ we introduce also an auxiliary scalar field $a(x)$ and define the related one-form

$$v = \frac{da}{\sqrt{g^{mn} \partial_m a \partial_n a}} \equiv dx^a v^a,$$

which satisfies

$$v^2 = g_{mn} v^m v^n = 1;$$

this leads, in particular, in (2.6) – (2.8) to the disappearance of the factor $v^2$.

The action for a chiral boson in an external gravitational field depends now also on $a$ and is given by

$$S[B, a, g] = \frac{1}{2} \int v h H = \frac{1}{4} \int (H * H - h * h), \quad (2.10)$$

where we defined the $k$–form

$$h = i_v (H - *H).$$

The equality of the two expressions in this formula can be inferred from the definition of $h$ and from the decomposition (2.6), which leads to

$$H - *H = vh - *vh. \quad (2.11)$$

From the second way of writing $S$ one sees that this action equals the action for non–chiral bosons, modulo a term which is proportional to the square of the self–duality condition (2.9). This particular form of the action is dictated by the symmetries it has. Under generic variations of $B$ and $a$ we get, in fact

$$\delta S = \int \left( v h d \delta B - \frac{1}{2} \frac{1}{\sqrt{g^{mn} \partial_m a \partial_n a}} v h d \delta a \right). \quad (2.12)$$

From this formula it is easy to see that the action is invariant under the following local transformations ($\delta g^{mn} = 0$):

1) \hspace{1cm} $\delta B = \frac{1}{\sqrt{g^{mn} \partial_m a \partial_n a}} h \varphi$, \hspace{0.5cm} $\delta a = \varphi$, \hspace{0.5cm} (2.13)

2) \hspace{1cm} $\delta B = \Lambda_{k-1} da$, \hspace{0.5cm} $\delta a = 0$, \hspace{0.5cm} (2.14)

3) \hspace{1cm} $\delta B = d \Sigma_{k-1}$, \hspace{0.5cm} $\delta a = 0$. \hspace{0.5cm} (2.15)
Here $\Sigma_{k-1}$ and $\Lambda_{k-1}$ are $(k-1)$–form transformation parameters and $\varphi$ is a scalar transformation parameter; under 2) and 3) the action is, actually, also invariant under finite transformations.

The transformation 3) is nothing else than the usual gauge invariance for $k$–forms, $B$ appears in (2.10), in fact, only through its field strength $dB$. The transformation 1) says that the auxiliary field $a$ is a non propagating ”pure gauge” field in that it can be transformed to any arbitrary value; due to its non–polynomial appearance in the action, however, mainly the appearance of the factor $g^{mn}\partial_m a \partial_n a$ at the denominator, it can not be set to zero. The transformation 2), instead, allows to reduce the equation of motion for $B$ to (2.9). To see this we read the equations of motion for $a$ and $B$ respectively from (2.12)

$$d \left( \frac{1}{\sqrt{g^{mn}\partial_m a \partial_n a}} v h h \right) = 0, \quad (2.16)$$
$$d(v h) = 0. \quad (2.17)$$

It can be directly checked that (2.16) is a consequence of (2.17), as follows also from the fact that $a$ is pure gauge.

The general solution of (2.17), on the other hand, is given by

$$v h = d\tilde{\Lambda}_{k-1}da$$

for some $(k-1)$–form $\tilde{\Lambda}_{k-1}$. Since under a finite transformation 2) we find

$$v h \to v h + d\Lambda_{k-1}da,$$

choosing $\Lambda_{k-1} = \tilde{\Lambda}_{k-1}$ this transformation can be used to reduce (2.18) to

$$h = 0.$$

Due to (2.11) this is then equivalent to the self–duality equation of motion.

This concludes the proof that our action describes correctly the dynamics of classical chiral bosons interacting with an external gravitational field. The (gauge–fixed) equation of motion we got, $H = *H$, is manifestly invariant and $a$–independent and has been obtained using the equations of motion and the symmetries of the action. In particular, for the symmetry 2) we used a rather unconventional ”gauge-fixing”. On the other hand, at the quantum level, in a functional integral approach, one can not make direct use of the equations of motion and needs conventional gauge–fixings, i.e. gauge fixings of the type $f(B, a) = 0$. In preparation of the quantum developments of sections three and four we present here such a set of gauge fixings and show that it leads to the correct number of physical degrees of freedom carried by chiral bosons in $2k + 2$ dimensions.

We choose a flat metric $g^{mn} = \eta^{mn}$ and fix the symmetries 1)–3) according to

$$1') \quad a(x) = n_i x^i$$
$$2') \quad i_n B = 0 \quad \leftrightarrow \quad n^{i_1} B_{i_1...i_k} = 0$$
$$3') \quad \delta B = 0 \quad \leftrightarrow \quad \partial^{i_1} B_{i_1...i_k} = 0,$$
where \( n_i \) is a constant vector, normalized such that \( n^2 = n_i n_j \eta^{ij} = 1 \). The gauge fixing for the symmetry 1) implies, in particular, that \( v = \frac{\alpha}{\sqrt{n^2}} = dx^i n_i \equiv n \). The choice 1′) appears the most simplest and treatable one and breaks manifest Lorentz invariance; manifestly invariant gauge fixings for this symmetry do not seem to exist.

For what concerns 2′) we observe that (2.6) allows to decompose \( B \) as

\[
B = -n_i n B + n_i * B
\]

and that the symmetry 2) shifts the component along \( n \), the first one, by an arbitrary amount leaving the second one invariant. The choice 2′) amounts then just to setting the component along \( n \) to zero. The gauge fixing for 3) is just the usual covariant Lorentz gauge.

The gauge fixings 2′) and 3′) leave a “residual” invariance for which

\[
\delta_{\text{res}} B = n \Lambda_{k-1} + d \Sigma_{k-1}
\]

with the constraint

\[
i_n (n \Lambda_{k-1} + d \Sigma_{k-1}) = 0 = d * (n \Lambda_{k-1} + d \Sigma_{k-1}).
\]

Using only 1′) the equation of motion for \( B \), (2.17), becomes now

\[
(T \partial_n + T^2) B = 0,
\]

where \( \partial_n = n^i \partial_i \) operates only on the components of a form and the operator \( T \), which sends a \( k \)-form in a \( k \)-form, is defined by

\[
T = * n d = * d n = -i_n * d.
\]

This operator, which on the components of a \( k \)-form acts as a \( k \times k \) antisymmetric tensor, will play a central role in section four, so we present here its main properties. Viewed as a tensor, \( T \) is symmetric in the interchange of the two branches of \( k \) antisymmetric indices. Its square, using the algebra given above, can be computed to be (on an even form)

\[
T^2 = (\partial_n^2 - \Box) + (n \partial_n + d) \delta - (n \delta + \partial_n) d i_n.
\]

Applying \( T \) again to this formula, due to \((i_n)^2 = d^2 = \delta^2 = 0\), only the first bracket contributes and one gets

\[
T^3 = (\partial_n^2 - \Box) T,
\]

which is the main formula. Also, on forms which satisfy 2′) and 3′) we have

\[
T^2 B = (\partial_n^2 - \Box) B,
\]

and (2.22) reduces to

\[
(\partial_n^2 - \Box) B = -T \partial_n B.
\]
Squaring the operators appearing in this relation on the left and on the right hand side and using again (2.26) we obtain

$$\Box(\partial_n^2 - \Box)B = 0.$$  \hspace{1cm} (2.28)

The solution \((\partial_n^2 - \Box)B = 0\) would imply \(T\partial_n B = 0\) and the solutions of this equation are "pure gauge", in the sense that they can be eliminated using the residual invariances given above. We remain therefore with the equation

$$\Box B = 0,$$

which describes massless excitations as expected. In this case (2.27) reduces to a constraint on the polarizations

$$TB = -\partial_n B.$$  \hspace{1cm} (2.29)

Going to momentum space, \(B_{i_1\cdots i_k}(x) \rightarrow b_{i_1\cdots i_k}(p)\), and choosing for example

\[n^i = (1, 0, \cdots, 0),\]

we split our indices into \(i = (0, \alpha)\). Then 2') implies that only space–like indices survive in the polarizations and our solution is characterized by

\[p_ip^i = 0\]  \hspace{1cm} (2.30)

\[p^{\alpha_1}b_{\alpha_1\cdots \alpha_k} = 0\]  \hspace{1cm} (2.31)

\[b_{\alpha_1\cdots \alpha_k} = \frac{1}{k!}\varepsilon_{\alpha_1\cdots \alpha_k\beta_1\cdots \beta_{k+1}}\frac{p^\beta_1}{|p|}b^{\beta_2\cdots \beta_{k+1}}.\]  \hspace{1cm} (2.32)

The third condition is just (2.29) in momentum space. One can easily count the number of independent polarizations which remain undetermined by these equations and the result is \(\frac{1}{2}(2k)!/(k!)^2\), which is the correct number.

This was just a check of the appropriateness of the the gauge fixings 1’–3’); they will prove to be very convenient also at the quantum level as we will see in section four.

3 Chiral bosons in \(D = 2\)

In the two–dimensional case it is convenient to work with light–cone indices and to introduce light–cone zweibeins \(e^i_{\pm}\) to describe the metric

\[g^{ij} = \frac{1}{2}(e^i_-e^j_+ + e^i_+e^j_-)\]  \hspace{1cm} (3.1)

\[\varepsilon^{ij}/\sqrt{g} = \frac{1}{2}(e^i_-e^j_+ - e^i_+e^j_-).\]  \hspace{1cm} (3.2)

All vector indices can then be transformed to local Lorentz indices through \(V_\pm = e^i_\pm V_i\). For example \(\partial_\pm = e^i_\pm \partial_i, \partial_\pm a = e^i_\pm \partial_i a\) and so on.
The action \((2.10)\) can then be rewritten, in two dimensions, also as

\[
S[B, a, e] = \frac{1}{2} \int d^2 x \sqrt{g} \left( \partial_+ B \partial_- B - \frac{\partial_+ a}{\partial_- a} \partial_- B \right),
\]

(3.3)

where \(B(x)\) is now a scalar field. This action is also invariant under local Weyl rescalings of the metric, as it should.

The self–duality equation becomes in this case simply

\[
\partial_- B = 0.
\]

(3.4)

In the two–dimensional case the symmetry 3) becomes a global one, just the shift by a constant, and the symmetry 2) assumes a slightly different form. The action \((3.3)\) is, in fact, invariant under the following two transformations:

1) \(\delta B = \frac{\partial_- B}{\partial_- a} \varphi, \quad \delta a = \varphi,\)

(3.5)

2) \(\delta B = \Lambda(a), \quad \delta a = 0,\)

(3.6)

(3.3) is just the transformation 1) of the previous section, written in light–cone indices. In the transformation \((3.6)\) \(\Lambda(a)\) is an arbitrary function of \(a\), so this is not a local symmetry but rather an infinite set of global symmetries and at the quantum level it does not need a gauge–fixing. Nevertheless, it is needed at the classical level to obtain the self–duality relation \((3.4)\). To see this we observe that \((2.17)\) becomes in two dimensions (the equation for \(a\) is again a consequence of this one)

\[
\varepsilon^{ij} \partial_i (v_j v_+ \partial_- B) = 0,
\]

whose general solution is \(\partial_- B = \partial_- \tilde{\Lambda}(a)\). Performing a transformation 2), with \(\Lambda(a) = \tilde{\Lambda}(a)\), we get \(\partial_- B = 0\).

We want now compute the effective action associated to \((3.3)\), in an external gravitational field. Since only the symmetry 1) needs a gauge fixing, formally this is given by

\[
e^{-\Gamma[e]} = \int \mathcal{D} B \mathcal{D} a \, e^{-S[B,a,e]} \delta(a - a_0),
\]

(3.7)

where we introduced an arbitrary gauge–fixing function \(a_0(x)\), and no Faddeev–Popov determinant arises.

We evaluate \(\Gamma[e]\) in two steps. First we perform the functional integral over \(B\)

\[
e^{-\Gamma[e,a]} = \int \mathcal{D} B \, e^{-S[B,a,e]}.
\]

(3.8)

Since, as we will see, the symmetry 1) is anomaly free\footnote{The BRST cohomology, associated to the symmetry 1), in the sector with ghost number one - which is the one related to possible anomalies - is presumably trivial.} one could expect that \(\delta_1 \Gamma[e, a] = \int d^2 x \frac{\delta^\Gamma}{\delta a} \varphi = 0\), which would imply that \(\Gamma[e, a]\) is, actually, independent of \(a\). But in a
generic symmetry breaking regularization scheme\(^3\) one can have trivial anomalies, i.e. anomalies which have to be eliminated by subtracting finite local counterterms. This will indeed be necessary in our case. After this subtraction we will perform the final \(a\)-integration, which will then become trivial as we will see.

The main point is to show that \(\Gamma[e]\) is, modulo local terms, independent of \(a_0\) and coincides with the effective action of a complex Weyl fermion. To this end we recall now some known results regarding the determinants and effective actions for two-dimensional non-chiral bosons and complex Weyl fermions.

The principal relation, for a generic metric \(e^i\_\pm\), is the following

\[
- \frac{1}{2} \ln \text{det} (\sqrt{g} \partial_\pm) = \ln \text{det} (\sqrt{g} \partial_-) + \ln \text{det} (\sqrt{g} \partial_+) + \text{loc.}
\]  

(3.9)

The l.h.s. is the effective action for a non-chiral boson and at the r.h.s. we have the sum of the effective actions for a left-handed and a right-handed complex Weyl fermion. The local terms depend on the regularizations. In a diffeomorphism preserving framework we have also the explicit expressions

\[
\Gamma\_\pm[e] \equiv \ln \text{det} (\sqrt{g} \partial_\pm) = \frac{1}{96\pi} \int d^2 x \sqrt{g} \left( D\_\pm \Omega\_\pm \frac{1}{\sqrt{g}} \partial_\pm \right),
\]  

(3.10)

where

\[
D\_\pm = \partial\_\pm \pm \Omega\_\pm
\]  

(3.11)

\[
\Omega\_\pm = \pm \frac{1}{\sqrt{g}} \partial_i \left( \sqrt{g} e^i\_\pm \right).
\]  

(3.12)

In this framework the local terms in (3.9) are proportional to \(\int d^2 x \sqrt{g} \Omega_+ \Omega_-\).

One more information we need is the anomaly carried by the chiral determinants (3.10) under finite Weyl rescalings

\[
\Gamma\_\pm[\lambda e] = \Gamma\_\pm[e] + \frac{1}{96\pi} \int d^2 x \sqrt{g} \ln \lambda (\sqrt{g} \ln \lambda \mp 2 \ln \lambda D\_\pm \Omega\_\pm). \]  

(3.13)

The unique feature of this relation we will need is that under a finite rescaling of the metric \(\Gamma\_\pm[e]\) changes by terms which are local in the scaling parameter and in the metric itself.

The expected result for the effective action for chiral bosons is

\[
\Gamma[e] = \Gamma\_-[e],
\]  

(3.14)

modulo local terms.

To prove this result we proceed as follows. We begin by rewriting the action (B.3), in terms of a fictitious metric, as

\[
S[B, a, e] = \frac{1}{2} \int d^2 x \sqrt{G} G^{ij} \partial_i B \partial_j B,
\]  

(3.15)

\(^3\)We will use a diffeomorphism preserving regularization which breaks local Lorentz transformations, Weyl transformations and the symmetry 1).
where the metric $G^{ij} = \frac{1}{2} \left( E_i^i E_j^j + E_i^j E_j^i \right)$ is defined by

\begin{align}
E_i^i &= e_i^i \\
E_i^j &= e_i^j - \partial_+ a \varepsilon^j_\alpha e^-_\alpha = -\frac{2}{\partial_- a} \varepsilon^{ij} \frac{\sqrt{g}}{j} a \\
\sqrt{G} &= \sqrt{g}.
\end{align}

The fact that the determinants of the two metrics coincide is a consequence of the general relation (3.2). The intermediate effective action $\Gamma[e, a]$ can therefore be written in terms of the determinant of the laplacian associated to the fictitious metric $G^{ij}$

\begin{align}
\Gamma[e, a] &= -\frac{1}{2} ln \det \left( \sqrt{G} e_i \right) \\
\Gamma[e, a] &= ln \det \left( \sqrt{G} E^- \partial_i \right) + ln \det \left( \sqrt{G} E^+ \partial_i \right) \\
\Gamma[e, a] &= \Gamma_- [e] + \Gamma_+ [E].
\end{align}

Here we used the general decomposition (3.9), applied to the metric $G^{ij}$, and the relations (3.16),(3.18). It remains to show that $\Gamma_+ [E]$ is local in the fields $a, e$. This can be shown using the fact that for a finite rescaling $\Gamma_+ [E]$ changes by local terms

\begin{align}
\Gamma_+ [E] = \Gamma_+ [E^* = \lambda E] + \text{loc}.
\end{align}

Choosing $\lambda = \frac{1}{\partial_- a}$ we get

\begin{align}
E^+_i &= \frac{1}{\partial_- a} E_i^i = -\frac{2}{(\partial_- a)^2} \varepsilon^{ij} \frac{\sqrt{g}}{j} a \\
\sqrt{G^*} &= (\partial_- a)^2 \sqrt{G} = (\partial_- a)^2 \sqrt{g} \\
\Omega^*_+ &= \frac{1}{\sqrt{G^*}} \partial_i \left( \sqrt{G^*} E^+_i \right) = -\frac{2}{\sqrt{G^*}} \partial_i \left( \varepsilon^{ij} \partial_+ a \right) = 0.
\end{align}

Since $\Gamma_+ [E^*]$ is quadratic in $\Omega^*_+$, see (3.10), it vanishes, actually. We have therefore

\begin{align}
\Gamma[a, e] = \Gamma_- [e] + \Gamma_{\text{loc}} [a, e].
\end{align}

This proves, in particular, that the symmetry 1) is anomaly free. The symmetry breaking term $\Gamma_{\text{loc}} [a, e]$ is, in fact, local and has to be subtracted, as mentioned above. After this subtraction the integration over $a$ in (3.7) is now trivial since the integrand is $a$–independent, apart from $\delta(a - a_0)$, and the result is $\Gamma[e] = \Gamma_- [e]$.

In practise, the functional integral in (3.7) can be evaluated as

\begin{align}
e^{-\Gamma[e]} = \int \mathcal{D}B e^{-S[B,a_0,e]},
\end{align}

where finally one has to subtract all (local) terms which depend on $a_0$.

This concludes the proof of (3.14) at a non–perturbative level and ensures, therefore, also that the gravitational anomaly carried by the chiral boson equals that of a complex
Weyl fermion, which is associated to the invariant polynomial $\frac{1}{96\pi} tr(RR)$, in agreement with the index theorem for $D = 2$ chiral bosons. [11].

Since in higher dimensions exact results for the effective actions are not available it is also instructive to perform a perturbative derivation of $\Gamma[e]$. This will allow us to gain some insight in the ingredients which are essential for a perturbative evaluation of the contribution to the effective action which is responsible for the anomaly also in higher dimensions. For the perturbative analysis a convenient starting point is (3.25); so we will also gain a better understanding of how to deal with the non-manifestly covariant gauge fixings related with the particular choice of $a_0(x)$.

For the perturbative expansion, like we did in the preceding section, we choose a class of gauge fixings parametrized by a constant vector $n_i$ satisfying

$$n_i n_j \delta^{ij} = 1,$$

and set

$$a_0(x) = n_i x^i.$$

In this case, since the effective action depends on $a_0$ only through $\partial_i a_0 = n_i$, the term $\Gamma_{\text{loc}}[a_0, e]$ becomes a local functional of only the metric, and $\Gamma[e]$ in (3.25) can depend on $n_i$ only through local terms.

We expand the metric around the flat one, $\eta^{ij} = \delta^{(i}\delta^{j)}$, and expand the action in powers of $h^{\pm\pm}$ (from now on all light-cone indices are flat, e.g. $n_{\pm} = \delta^{i}_{\pm} n_i$, $\partial_{\pm} = \delta^{i}_{\pm} \partial_i$)

$$e^{i}_{\pm} = \delta^{i}_{\pm} + h^{i}_{\pm} \hspace{1cm} (3.26)$$

$$h^{i}_{+} = \frac{1}{2} \left( \delta_{-}^{i} h^{++} + \delta_{+}^{i} h^{+-} \right) \hspace{1cm} (3.27)$$

$$h^{i}_{-} = \frac{1}{2} \left( \delta_{+}^{i} h^{-+} + \delta_{-}^{i} h^{--} \right) \hspace{1cm} (3.28)$$

$$\sqrt{g} = 1 - \frac{1}{2} (h^{++} + h^{+-}) + o(h^2), \hspace{1cm} (3.29)$$

and expand the action in powers of $h^{\pm\pm}$

$$S[B, a_0, e] = -\frac{1}{2} \int d^2 x B \partial_- \left( \partial_+ - n^2_+ \partial_- \right) B$$

$$+ \frac{1}{4} \int d^2 x \left( \partial_+ B - n^2_+ \partial_- B \right)^2 h_{--}$$

$$+ o(h^2). \hspace{1cm} (3.30)$$

In the first line we have the kinetic term and in the second the interaction term with the metric. Since we are only interested in the $h-h$ two-point function the higher order terms are not needed. We notice that from the interaction terms $h^{++}$ and $h^{+-}$ dropped out – this is a consequence of Weyl invariance – and that also $h^{+-}$ decoupled. This is due to the fact that left-handed chiral bosons, are not coupled to this field in that its equation of motion is $e^{j}_{-} \partial_j B = \partial_- B + h^{j}_{-} \partial_j B = 0$. The action $S[B, a_0, e]$ itself depends, actually, also on $h^{++}$, through the $o(h^2)$-terms, but the effective action depends only locally on it.
From (3.31) we can read vertices and propagators

\[ B(k) - B(k) \text{ propagator} \quad \frac{i}{k_- (k_+ - n_+^2 k_-)} \quad (3.31) \]

\[ B(k) - h_{--}(p) - B(l) \text{ vertex} \quad -\frac{i}{2} (k_+ - n_+^2 k_-)(l_+ - n_+^2 l_-) \equiv -\frac{i}{2} W(k)W(l). \quad (3.32) \]

In the computation of a one–loop Feynman diagram with \( N \) external \( h_{--} \) fields these vertices and propagators appear always in the sequence \( \cdots V \cdot P \cdot V \cdot P \cdots \), and from (3.31) and (3.32) one sees that the factor \((k_+ - n_+^2 k_-)\) in the denominator of a propagator cancels always against a corresponding factor in an adjacent vertex. This means that the effective propagator reduces simply to \( \frac{1}{k_-} \) which is the appropriate propagator for a chiral field in two dimensions. Moreover, since the vertex is factorized, the above sequence can also be seen as a sequence of building blocks of the form

\[ B(k) = -\frac{i}{2} W(k) - \frac{i}{2} W(k) = \frac{1}{2} \frac{k_+ (k_+ - n_+^2 k_-)}{k_- (k_+ - n_+^2 k_-)} = \frac{1}{2} \left( \frac{k_+^2}{k_-} - n_+^2 \right). \quad (3.33) \]

We see in particular that in the \( n \)-dependent term of this basic block the pole has cancelled. In the leading anomaly diagram, in the present case the two–point function, this implies, for dimensional reasons, that the \( n \)-dependence will occur only in local terms. For diagrams with more than two external \( h_{--} \)–legs the \( n \)-dependence will occur also in non–local terms; but, since these terms have one or more poles less, they will be cancelled by diagrams which originate from the \( o(h^2) \) terms in (3.31). For example, in the three–point function \( n \)-dependent terms in which one pole cancelled lead to a contribution to the effective action of the form \( n_+^2 \int d^2x d^2y h_{--}(x) G(x - y) h_{--}(y) \), and these cancel against contributions from the diagram with one vertex of the type (3.32) and one vertex of the type \( B(x) h_{--}(x) B(x) \).

The two–point function can now be easily evaluated. It is given by a Feynman diagram with just two of the above building blocks (3.33),

\[ \Gamma_2(p) = \frac{1}{8} \int \frac{d^2k}{(2\pi)^2} \cdot \frac{k_+ (k_+ - n_+^2 k_-)l_+ (l_+ - n_+^2 l_-)}{k_- l_-}, \quad (3.34) \]

where \( l = p - k \), and can be evaluated with standard methods. A quick way to do it is to change coordinates from \( k_\pm \) to \( K_- = k_- \), \( K_+ = k_+ - n_+^2 k_- \), and to define \( P_- = p_- \), \( P_+ = p_+ - n_+^2 p_- \). This leads to

\[ \Gamma_2(p) = \frac{1}{8} \int \frac{d^2 K}{(2\pi)^2} \cdot \frac{K_+^2 (P - K)^2}{K_- (K - K_+)^2}, \quad (3.35) \]

which gives, in dimensional regularization,

\[ \Gamma_2(p) = -\frac{1}{192\pi} \frac{P_+^3}{P_-} = -\frac{1}{192\pi} \left( \frac{p_+^3}{p_-} - 3p_+^2 n_+^2 + 3n_+^4 p_-^2 - n_+^6 p_-^2 \right). \]
As anticipated above, the $n$–dependence is only in the local terms, which have to be subtracted, and the non–local term amounts to a contribution to the effective action given by

$$\Gamma_2 = \frac{1}{4} \cdot \frac{1}{96\pi} \int d^2x \partial_+ \partial_- h_-- \frac{1}{\Box} \partial_+ \partial_- h_--.$$

This coincides with the expansion of $\Gamma_- [e]$, see \(3.10\), up to local terms, since

$$D_+ \Omega_- = -\frac{1}{2} (\partial_+ \partial_- h_-- - \Box h_-) + o(h^2).$$

This concludes the perturbative and non perturbative analysis of the effective action for chiral bosons based on the classical action \(3.3\). Many of the features appearing in the two–dimensional case will arise also in higher dimensions, as we will see in the next section. One of the main points will be the determination of a convenient basic block, analogous to \(3.33\), and the determination of effective Feynman rules. We will also encounter a cancellation of factors between propagators and vertices, as happened with the ones in \(3.31\) and \(3.32\).

The appearance of the common factor \((k_+ - n^2 k_-)\) is, actually, a consequence of the symmetry 2). Once one has chosen \(a_0(x) = n_i x^i\), the symmetry 2) reduces to a symmetry of the action \(S[B, a_0, e]\). This action is now still invariant under $\delta B = \Lambda$, but only for $\Lambda$'s which depend on $x$ only through $n_i x^i$. The unique first order derivative operator which is invariant under such transformations, due to $n_+ n_- = 1$, is indeed $(\partial_+ - n^2 \partial_-) B$, and this is the reason why it appears in \(3.30\) in the kinetic and interaction terms.

A similar role will be played in higher dimensions by the transformation 2) of the preceding section, which becomes then a true local symmetry.

## 4 Chiral bosons in $4n+2$ dimensions

In this section we want to show that the leading anomaly diagram in $D = 2k + 2$ \((k\) even) dimensions, computed from our classical action \(2.10\), coincides with the diagram, based on conjectured Feynman rules, which has been used by Alvarez–Gaumé and Witten in \[10\] to determine the anomaly. This is a one–loop diagram with $k + 2$ external gravitons.

We outline first the procedure which has been adopted in \[10\] to conjecture these Feynman rules. One starts from an action for non–chiral bosons in $D$ dimensions interacting with a gravitational field. This is simply given by (as an integral over a $D$–form)

$$S[B, g] = -\frac{1}{2} \int H \ast H, \quad (4.1)$$

where $H = dB$. First one derives the Feynman rules for non–chiral bosons writing the metric as

$$g^{ij} = \eta^{ij} + h^{ij}, \quad (4.2)$$

$$h \equiv \eta_{ij} h^{ij}, \quad (4.3)$$
and expanding the action in powers of $h^{ij}$. For notational reasons it is convenient to parametrize the symmetric matrix $h^{ij}$ in terms of $D$ vectors $M_\alpha$, $\alpha = (1, \cdots, D)$ such that

$$h^{ij} = \sum_\alpha M_\alpha^i M_\alpha^j.$$ 

This allows to introduce $D$ one–forms $M_\alpha = dx^i M_\alpha^i$, and we use the same notation $M_\alpha$ for the associated vectors $M_\alpha^i \partial_i$, since no confusion could arise. The indices $i, j$ are now raised and lowered with the flat metric and in what follows the sum over $\alpha$ will always be understood. This will allow us to write compact expressions for vertices and propagators.

With these notations the action (4.1) can be expanded as follows

$$S[B, g] = \frac{1}{2} \int (B \ast \Box B + \delta B \ast \delta B)$$

(4.4)

$$+ \frac{1}{2} \int dB \left( M_i M - \frac{1}{2} h \right) \ast dB$$

(4.5)

$$+ o(h^2).$$

(4.6)

The action (4.1) is invariant under the usual gauge transformations for $k$–form potentials; these can be fixed by adding to the action the "free" i.e. metric–independent term $- \frac{1}{2} \int \delta B \ast \delta B$, and the propagator becomes then simply

$$\text{propagator} = - \frac{1}{\Box}.$$ 

(4.7)

For non–chiral bosons the $B h B$–vertex could be read from (4.5). For chiral bosons the authors of [10] conjectured Feynman rules for which the propagator is still given by (4.7) while, for what concerns the vertex, they inserted in (4.3) on $dB$ the projector $\frac{1}{2}(1 + \ast)$, i.e. they took as interaction term, instead of (4.3) the expression

$$\frac{1}{2} \int \frac{(1 + \ast)}{2} dB \left( M_i M - \frac{1}{2} h \right) \frac{(1 + \ast)}{2} dB.$$ 

(4.8)

From this expression one can read the $B(k)-h(p)-B(l)$ vertex which, schematically, is given by an expression of the form (it is convenient to keep the external leg $h^{ij}$ inserted)

$$k^i \left( M_i M - \frac{1}{2} h \right) l^j,$$

which is a $k \times k$ antisymmetric tensor. In a one–loop diagram the sequence $\cdots V \cdot P \cdot V \cdot P \cdots$ can then be written as

$$\cdots k^i \left( M_i M - \frac{1}{2} h \right) l^j \cdot \frac{1}{l^2} \cdot l^r \left( M_i M - \frac{1}{2} h \right) q^s \cdots.$$

From this one can extract a building block which depends only on a single momentum, say $l$

$$\left( M_i M - \frac{1}{2} h \right) l^i \cdot \frac{1}{l^2} \cdot l^r.$$
This is now a $(k+1) \times (k+1)$ antisymmetric tensor, so, turning to configuration space, it can be represented as a linear operator which sends a $(k+1)$–form in a $(k+1)$–form. Reinserting the appropriate contraction of indices this operator is given by

$$B_{AGW} = - \left( M_{iM} - \frac{1}{2} h \right) \frac{1}{2} (1 + \ast) \frac{d \ast d}{\Box} \frac{1}{2} (1 - \ast).$$

(4.9)

Since we have the operatorial identity

$$\frac{1}{2} (1 + \ast) \left( M_{iM} - \frac{1}{2} h \right) \frac{1}{2} (1 + \ast) = 0,$$

(4.10)

and the one–loop diagram is now a chain of blocks (4.9), the last projector in the block can be omitted.

In the remaining part of this section we want now show that the action (2.10) leads to the same building block $B_{AGW}$.

Starting from this action, the effective action $\Gamma[g]$ is obtained via a functional integral over $B$ and $a$ upon gauge fixing the local symmetries 1)–3). For the symmetry 1) we proceed as in the perturbative treatment of the two–dimensional case, inserting the $\delta$–function $\delta(a - n_i x^i)$. Since also in higher dimensions the symmetry 1) is expected to be anomaly free, the effective action will depend on $n$ only through local terms.

For what concerns the symmetries 2) and 3), we use the gauge–fixings $2')$ and $3'$ of section two, but now with an appropriate weighting function $f(b_2, b_3)$:

$$e^{-\Gamma[g]} = \int DB \, Da \, e^{-S[B,a,g]} \int Db_2 \, Db_3 \, \delta(a - n_i x^i) \delta(i_n B - b_2) \delta(\delta B - b_3) \, e^{-\frac{1}{2} \int f(b_2, b_3)}$$

$$= \int DB \, e^{-S_n[B,g]},$$

(4.11)

where

$$S_n[B, g] = S[B, n_i x^i, g] + \frac{1}{2} \int f(i_n B, \delta B).$$

(4.12)

Here $b_2$ and $b_3$ are $(k-1)$–forms, $f(b_2, b_3)$ is a quadratic metric–independent function of these fields which parametrizes the gauge fixing terms (contractions are made with the flat metric), and in the $\delta$–functions appearing in (4.11) the terms $i_n B$ and $\delta B$ are also constructed with the flat metric. This implies that the gauge fixing term in (4.12), $\frac{1}{2} \int f(i_n B, \delta B)$, is metric independent and that no Faddeev–Popov determinants arise. For the moment $f$ is left undetermined, we will make a convenient choice below.

The gauge–fixed action $S_n[B, g]$ can now be expanded in powers of $h^{ij}$ and, using the same notation as above, one gets

$$S_n[B, g] = - \frac{1}{2} \int \left( B \ast \left[ T \partial_n + T^2 \right] B - f(i_n B, \delta B) \right)$$

$$+ \frac{1}{2} \int TB \ast \left[ M_{iM} + \frac{1}{2} h - (n \cdot M)^2 + (n \cdot M) \ast Mn \right] TB$$

$$+ o(h^2).$$

(4.13, 4.14, 4.15)
Here with \((n \cdot M)\) we mean \(n_i M^i\) and the operator \(T\) has been defined in section two, (2.23). It plays the same role as the differential operator \((\partial_+ - n_+^2 \partial_-)\) in the two-dimensional case; for \(D = 2\) it reduces, actually, apart from a constant, to this operator. Once one has chosen \(a_0 = n_i x^i\), the symmetry 2) reduces, indeed, to \(\delta B = n \Lambda_{k-1}\) and it is precisely the combination \(TB = *n dB\) which is invariant under this reduced local symmetry and under the usual gauge transformation 3).

We choose now the function \(f\) such that the kinetic operator in (4.13) becomes as simple as possible. The explicit expression of the operator \(T^2\) is given in (2.24) and it can be reduced simply to \(\partial n - \Box\) upon choosing

\[
f(b_2, b_3) = db_2 * db_2 + b_3 * b_3 + 2b_2 * \partial_n b_3.
\]

With this choice one obtains for the kinetic term

\[
-\frac{1}{2} \int (B * [T \partial_n + T^2] B - f(i_n B, \delta B)) = -\frac{1}{2} \int B * \Omega B,
\]

where the gauge–fixed kinetic operator is

\[
\Omega = \partial^2_n - \Box + \partial_n T.
\]

It sends a \(k\)-form in a \(k\)-form and becomes, in momentum space, a \(k \times k\) antisymmetric tensor. The \(B-B\) propagator \(P\) is just the inverse and can be easily computed using algebraic methods. The essential ingredient is the identity (2.25) which implies that every power series in \(T\), like \(\Omega^{-1}\), can be reduced (in momentum space) to a polynomial in \(\Pi, T\) and \(T^2\). In configuration space the result is

\[
P = \Omega^{-1} = \frac{1}{\partial^2_n - \Box} \left( \Pi + \frac{T \partial_n}{\Box} - \frac{T^2 \partial^2_n}{\Box \left( \partial^2_n - \Box \right)} \right).
\]

Since in the interaction term (4.14) \(B\) appears always as \(TB\), what is actually needed in a Feynman diagram is the combination

\[
TPT = (\partial_n - T) T^1_\Box,
\]

which leads finally to the cancellation of the unphysical pole \(\frac{1}{\partial^2_n - \Box}\) and to the appearance of the massless physical pole \(\frac{1}{\Box}\).

Now we step to the problem of individuating a convenient building block for a one-loop Feynman diagram with a certain number of external gravitons, the one responsible for the gravitational anomaly carrying \(k + 2\) of them.

If we indicate the operator between square brackets in (4.14) with \(W \equiv W(h)\), it sends a \(k\)-form in a \(k\)-form, the interaction term can be written as

\[
\frac{1}{2} \int TB * WTB,
\]

(4.20)
and the vertex–propagator sequence becomes, due to (4.19),
\[ \cdots [TWT] P [TWT] \cdots = \cdots TW (\partial_n - T) T \frac{1}{\Box} WT \cdots \] (4.21)

We can extract as building block
\[-W (T - \partial_n) T \frac{1}{\Box} = -W (T - \partial_n) * d \frac{1}{\Box} n,\]
or, equivalently, due to cyclicity
\[ B = -n W (T - \partial_n) * d \frac{1}{\Box} \]
(4.22)
\[ = -n W (T - i_n d - di_n) * d \frac{1}{\Box} \]
(4.23)
\[ = -n W (T - i_n d) * d \frac{1}{\Box} - n \left[ W * d \frac{1}{\Box} \right] n. \]
(4.24)

We used the identity \( \partial_n = i_n d + di_n \) in the first line and (2.3) to get the last line.

In the sequel we will make repeated use of the identities (2.2)-(2.5) with \( v = n, v^2 = 1 \).

This block is written as an operator which sends a \((k+1)\)-form in a \((k+1)\)-form, as is realized by direct inspection, so in momentum space it becomes a \((k+1) \times (k+1)\) antisymmetric tensor, as does \( B_{AGW} \). In a one-loop diagram these blocks are multiplied by themselves; the term between square brackets in the last line carries a factor of \( n \) on each side, so one of these factors encounters necessarily another factor of \( n \) and, due to antisymmetry (or, due to the fact that the square of a one–form is zero), these terms drop all out. We remain therefore only with the first term in (4.24). Inserting the definition of \( T \), this can be written as
\[ B = n W i_n (1 + *) d \frac{1}{\underline{\Box}}. \]

Due to the appearance of the projector \((1 + *)\), one can now eliminate the unique \(*\)–operator contained in \( W \). After some algebra one finds
\[ B = -n i_n \left( Mi_M - \frac{1}{2} h \right) (1 + *) d \frac{1}{\underline{\Box}}. \]
(4.25)

The appearance of the combination \( (Mi_M - \frac{1}{2} h) \) in this formula, as well as in (4.3), is due to Weyl–invariance at the linearized level, i.e. invariance under \( \delta h^{ij}(x) = \lambda(x) \eta^{ij} \).

In (4.25), due to the identity (4.10), one can insert the projector \( \frac{1}{2}(1 - *) \) after the operator \( n i_n \) and, due to cyclicity, one can replace (4.25) with
\[ B = - \left( Mi_M - \frac{1}{2} h \right) \frac{1}{2}(1 + *) d \frac{1}{\underline{\Box}} \frac{n i_n}{2}(1 - *) \]
(4.26)
\[ = - \left( Mi_M - \frac{1}{2} h \right) \frac{1}{2}(1 + *) \left( d \frac{1}{\underline{\Box}} \frac{1}{2}(1 - *) + \frac{1}{2} \right) n i_n \frac{1}{2}(1 - *) \]
(4.27)
\[ = - \left( Mi_M - \frac{1}{2} h \right) \frac{1}{2}(1 + *) d \frac{1}{\underline{\Box}} \frac{1}{2}(1 - *) - \left( Mi_M - \frac{1}{2} h \right) \frac{1}{2}(1 + *) n i_n \frac{1}{2}(1 - *). \]
(4.28)
In the second line we used (2.5), for a flat metric, and in the third the identity (2.8) with \( p = k + 1 \), which is odd.

This is the formula for the building block which generalizes (3.33) to a generic dimension, for \( D = 2 \) it reduces actually to that formula.

Again we see that in the \( n \)-dependent term the pole (propagator) \( \frac{1}{\xi^2} \) cancelled out, so, as in the two–dimensional case, the \( n \)-dependence in these terms has to cancel against diagrams which contain vertices of the type \( B(h^{ij})^p B \). For what concerns the leading part of the anomaly, these diagrams have one (or more) propagators less and they can not contribute to the anomaly \([7],[10]\).

The first term in (4.28) coincides with (4.9), and hence the gravitational anomalies derived from the classical action (2.10) coincide with the ones computed in \([10]\).

5 Final remarks

In this paper we proved that the gravitational anomalies derived from the classical manifestly invariant action for chiral bosons in \( 4n + 2 \) dimensions, proposed in \([12]\), coincide with the expected ones. This supports the quantum reliability of the new method itself at the perturbative level. On the other hand, as all lagrangian formulations of theories with chiral bosons, the method is expected to be insufficient for what concerns the quantization of these actions on manifolds with non trivial topology \([20]\); see, however, also \([21]\).

In this paper we were concerned with diffeomorphism anomalies of the ABBJ–type, which are non trivial cocycles of the corresponding BRST operator, and exist only in \( 4n + 2 \) dimensions (clearly they can be shifted to Lorentz–anomalies). In a generic even dimension, however, there exists also another class of diffeomorphism cocycles\(^4\) of the “Weyl–type”, which can be eliminated at the expense of Weyl–anomalies \([22]\), if the corresponding theory is classically Weyl invariant (otherwise they become simply trivial diffeomorphism cocycles). The resulting inequivalent Weyl–cocycles, in four and six dimensions, have been determined through a cohomological analysis in \([23]\), see also \([24, 25]\).

In four dimensions there are three of them and in six dimensions there are four.

Our classical action for chiral bosons (2.10) is indeed invariant under local Weyl rescalings, \( g^{ij} \rightarrow e^{\lambda}g^{ij} \), \( B \rightarrow B \), \( a \rightarrow a \); therefore one expects that, as in the two–dimensional case, the effective action \( \Gamma[g] \) is plagued also by diffeomorphism anomalies of the Weyl–type, or, equivalently, by the Weyl–anomalies discussed in \([23]\). Having at our disposal a classically manifestly invariant action principle for chiral bosons in an external gravitational field could be essential in the determination of these anomalies, which, to our knowledge, for \( D > 2 \) are still unknown. In six dimensions, for example, this would amount to the determination of the coefficients of the four non trivial Weyl–cocycles mentioned above. In higher dimensions, even the form (and the number) of the non trivial cocycles

\(^4\)None of these cocycles contains the \( \varepsilon \)–tensor.
is unknown.

The fact that the action (2.10) gave rise to the correct ABBJ gravitational anomalies carried by chiral bosons, makes us hope that it can also prove useful to make some progress for what concerns the determination of their Weyl anomalies.

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