LIOUVILLE TYPE THEOREMS FOR THE STATIONARY COMPRESSIBLE 3D MHD EQUATIONS

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Abstract. In this paper, we investigate the three dimensional stationary compressible 3D MHD equations, and obtain Liouville type theorems if a smooth solution \((\rho, u)\) satisfies some suitable conditions. In particular, our results improve and generalize the corresponding result.

Keywords: Liouville type theorem; compressible 3D MHD equations; Lorentz space; (local) Morrey space

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1. Introduction

The system of barotropic compressible magnetohydrodynamic (MHD) equations in \(\mathbb{R}^3\) can be read as follows:

\[
\begin{align*}
\text{div}(\rho u) &= 0, \\
-\nu \Delta u - (\lambda + \nu)\nabla \text{div} u + \text{div}(\rho u \otimes u) - \text{div}(b \otimes b) + \nabla (P + \frac{|b|^2}{2}) &= 0, \\
-\Delta b + \text{div}(\rho u \otimes b) - \text{div}(b \otimes u) &= 0, \\
\text{div} b &= 0,
\end{align*}
\]

where the vector \(u\) denotes the flow velocity field and the scalar function \(\rho\) represents the density of the fluid. Without loss of generality, we consider the flows with \(\gamma\)-law pressure:

\[
P(\rho) := a\rho^\gamma, \quad a > 0, \quad \gamma > 1.
\]

The shear viscosity \(\nu\) and bulk viscosity \(\lambda\) are both constants and satisfy

\[
\nu > 0, \quad \lambda + \frac{2}{3}\nu > 0.
\]

We consider the initial value problem of (1.1), which requires initial conditions

\[
\begin{align*}
(1.2) \quad u(x, 0) &= u_0(x) \quad \text{and} \quad b(x, 0) = b_0(x), \quad x \in \mathbb{R}^3.
\end{align*}
\]

Among them, we briefly recall the previous results concerned with the compressible 3D MHD equations. The local strong solutions to the compressible MHD with large initial data were obtained, by Vol’pert and Khudiaev [27] as the initial density is strictly positive and also by Fan and Yu [5] as the initial density may contain vacuum, respectively. Hu and Wang [6, 7, 8, 9] proved the global existence and uniqueness of classical solutions to the Cauchy problem with smooth initial data which are of small energy but possibly large oscillations. They obtain the global existence of weak solutions for initial data which may be discontinuous and contain vacuum states, and where the initial temperature is allowed to be zero. Later, Li, Xu and Zhang [15] shown if the initial energy is sufficiently small, the global existence of a regular solution is shown. This solution is classical with arbitrarily large oscillations, and contain vacuum states.

On the contrary, to the best of one’s knowledge, There is not much result related to the equations (1.1). Yang et. al. [29] prove existence and uniqueness of strong solutions to
barotropic compressible magnetohydrodynamic (see also [28] and [22]). Comparing to the steady incompressible MHD equations, we refer to [32] and [34]. The long-time behavior of a small global solution can be found in [9], [31].

Regarding Liouville theorems for compressible 3D Navier-Stokes equations, Chae [2] showed if the smooth solution \((\rho, u, b)\) satisfies
\[
\|\rho\|_{L^\infty(\mathbb{R}^3)} + \|\nabla u\|_{L^2(\mathbb{R}^3)} + \|u\|_{L^\frac{2}{3}(\mathbb{R}^3)} < \infty,
\]
then \(u \equiv 0\) and \(\rho = \text{constant}\). After that, through the suitable decomposition for pressure (Lemma 2.3 below), Li and Yu in [20] shown
\[
\|\rho\|_{L^\infty(\mathbb{R}^3)} + \|\nabla u\|_{L^2(\mathbb{R}^3)} + \|u\|_{L^\frac{2}{3}(\mathbb{R}^3)} < \infty.
\]
Later, in the Lorentz framework, Li and Niu [17] proved if \((\rho, u)\) is a smooth solution with \(\rho \in L^\infty(\mathbb{R}^3), \nabla u \in L^2(\mathbb{R}^3)\) and \(u \in L^{p,q}(\mathbb{R}^3)\) for \(3 < p < \frac{9}{2}, 3 \leq q \leq \infty\) or \(p = q = 3\). Then \(u \equiv 0\) and \(\rho = \text{constant}\) on \(\mathbb{R}^3\). It is a relaxation result to the corresponding result [20]. In mathematics analysis, Liouville theorems is classical problem. However, it has not been resolved so far in a fluid equations.

In present paper, we study Liouville Theorem for the compressible 3D MHD equations in a whose space in viewpoint of Lorentz space or Morrey space based on the Cacciopoli-type estimates. In our analysis, in particular, we use pressure decomposition [21] based on Lemma 2.3 below, which is a essential tool to deal with the pressure.

Before we look at the main results, we give some definitions for functional spaces. Given \(1 \leq p < \infty, 1 \leq q \leq \infty\), we say that a measurable function \(f \in L^{p,q}(\mathbb{R}^3)\) if \(\|f\|_{L^{p,q}(\mathbb{R}^3)} < \infty\), where
\[
\|f\|_{L^{p,q}(\mathbb{R}^3)} := \begin{cases} \left( \int_0^\infty t^{q-1} \{ \{ x \in \mathbb{R}^3 : |f(x)| > t \}^{\frac{q}{p}} \} \frac{dt}{t} \right)^{\frac{1}{q}}, & \text{if } q < +\infty, \\ \sup_{t > 0} t^{\frac{q}{p}} \{ \{ x \in \mathbb{R}^3 : |f(x)| > t \}^{\frac{1}{p}} \}, & \text{if } q = +\infty. \end{cases}
\]
The space satisfies the continuous embedding (see e.g. [21])
\[
L^p(\mathbb{R}^3) = L^{p,p}(\mathbb{R}^3) \hookrightarrow L^{p,q}(\mathbb{R}^3) \hookrightarrow L^{p,\infty}(\mathbb{R}^3), \quad p \leq q < \infty.
\]
In this direction, parallel to the result of Li and Niu, first result is stated as

**Theorem 1.1.** Suppose that \((\rho, u, b)\) is a smooth solution to (1.1) with \(\rho \in L^\infty(\mathbb{R}^3), \nabla u, \nabla b \in L^2(\mathbb{R}^3)\) and \(u, b \in L^{p,q}(\mathbb{R}^3)\) for \(3 < p < \frac{9}{2}, 3 \leq q \leq \infty\) or \(p = q = 3\). Then \(u \equiv 0\) and \(\rho = \text{constant}\) on \(\mathbb{R}^3\).

In light of [16] Theorem 1.2, for \(p \geq \frac{9}{2}\), we are easily checked to obtain a following results and omit a proof.

**Theorem 1.2.** Let \(C(R/2, R) = \{ x \in \mathbb{R}^3 : R/2 < |x| < R \} \) and
\[
M_{p,q}(R) := R^{\frac{3}{p} - \frac{2}{q}} \left( \|u\|_{L^{p,q}(C(R/2, R))} + \|b\|_{L^{p,q}(C(R/2, R))} \right).
\]
Suppose that \((\rho, u, b)\) is a smooth solution to (1.1) with \(\rho \in L^\infty(\mathbb{R}^3)\) and \(\nabla u, \nabla b \in L^2(\mathbb{R}^3)\). For \(p \geq \frac{9}{2}, 3 \leq q \leq \infty\), assume that
\[
\liminf_{R \to \infty} M_{p,q}(R) < \infty,
\]
then
\[
D(u, b) := \int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla b|^2 \, dx \leq C \liminf_{R \to \infty} M_{3,p,q}(R).
\]
If moreover assume
\[
\liminf_{R \to \infty} M_{3,p,q}(R) \leq \delta D(u)
\]
for some \(0 < \delta < \frac{1}{C}\), then \(u \equiv 0\) and \(\rho = \text{constant}\) on \(\mathbb{R}^3\).
Next, we also introduce the definition of the (homogeneous) Morrey and local Morrey spaces (see e.g. [14] for more details). Let $1 < p < r < +\infty$, the homogeneous Morrey space $M_{p,r}^{\beta}(\mathbb{R}^3)$ is the set of functions $f \in L_{\text{loc}}^p(\mathbb{R}^3)$ such that

$$
\|f\|_{M_{p,r}^{\beta}} = \sup_{R > 0, x_0 \in \mathbb{R}^3} R^\beta \left( \frac{1}{R^3} \int_{B(x_0,R)} |f(x)|^p dx \right)^{\frac{1}{p}} < +\infty,
$$

where $B(x_0,R)$ denotes the ball centered at $x_0$ and with radio $R$. It is well known $L^r(\mathbb{R}^3) \subset L^{r,q}(\mathbb{R}^3) \subset M_{p,r}^{\beta}(\mathbb{R}^3)$, where, for $r \leq q \leq +\infty$.

For $\gamma \geq 0$ and $1 < p < r < +\infty$, we define the local Morrey space $M_{\gamma}^{p}(\mathbb{R}^3)$ as the Banach space of functions $f \in L_{\text{loc}}^p(\mathbb{R}^3)$ such that

$$
\|f\|_{M_{\gamma}^{p}} = \sup_{R \geq 1} \left( \frac{1}{R^\gamma} \int_{B(0,R)} |f(x)|^p dx \right)^{1/p} < +\infty.
$$

Moreover,

$$
\gamma_1 \leq \gamma_2 \Rightarrow M_{\gamma_1}^{p}(\mathbb{R}^3) \subset M_{\gamma_2}^{p}(\mathbb{R}^3),
$$

and also

$$
1 < p < r < +\infty \Rightarrow M_{p,r}^{\beta}(\mathbb{R}^3) = M_{p,3(1-p/r)}^{\beta}(\mathbb{R}^3) \subset M_{\gamma}^{p}(\mathbb{R}^3),
$$

where $\gamma > 0$ is constant with $3(1 - p/r) < \gamma$.

Finally, we define the space $M_{\gamma,0}^{p}(\mathbb{R}^3)$ as the set of functions $f \in M_{\gamma}^{p}(\mathbb{R}^3)$ such that

$$
\lim_{R \to +\infty} \left( \frac{1}{R^\gamma} \int_{C(R/2,R)} |f(x)|^p dx \right)^{1/p} = 0.
$$

Relationship between these spaces [13] is simply known that for $3 < r < 9/2$ and for $3(1 - 3/r) < \delta < 1$,

$$
L^r(\mathbb{R}^3) \subset L^{r,\infty}(\mathbb{R}^3) \subset M^{3,r}(\mathbb{R}^3) \subset M_{3}^{p}(\mathbb{R}^3) \subset M_{1,0}^{p}(\mathbb{R}^3),
$$

and for $r = 9/2$ and $9/2 < q < +\infty$, $L^{9/2}(\mathbb{R}^3) \subset L^{9/2,q}(\mathbb{R}^3) \subset M_{1,0}^{p}(\mathbb{R}^3)$.

In the framework of Morrey space, third result is followed as

**Theorem 1.3.** Let $u, b \in L_{\text{loc}}^2(\mathbb{R}^3)$ be a weak solution of (1.1). If $u, b \in M_{p,r}^{\beta}(\mathbb{R}^3)$ with $3 \leq p < r < \frac{9}{2}$, then $u \equiv 0 \equiv b$ in $\mathbb{R}^3$.

In the framework of local Morrey space, third result is followed as

**Theorem 1.4.** Let $(p, u, b)$ be a smooth solution of (1.1).

1) Let $\gamma = 1$. If $u \in M_{1,0}^{3}(\mathbb{R}^3)$ and $\nabla b \in M_{1}^{3}(\mathbb{R}^3)$ then we have $u = 0$ and $\nabla b = 0$.

2) Let $1 < \gamma < 3/2$. If $u \in M_{\gamma,0}^{3}(\mathbb{R}^3)$ and $\nabla b \in M_{\gamma}^{3}(\mathbb{R}^3)$, and moreover if the velocity $u$ verifies:

$$
\lim_{R \to +\infty} R^{\gamma - 1} \left( \frac{1}{R^3} \int_{C(R/2,R)} |u(x)|^3 dx \right)^{1/3} = 0,
$$

then we have $u = 0$ and $b = 0$.

When $\rho = \text{constant}$, the system of (1.1) reduce as follows:

$$
\begin{cases}
-\nu \Delta u + \text{div}(u \otimes u) + \nabla \pi = 0, \\
-\Delta b + \text{div}(u \otimes b) - \text{div}(b \otimes u) = 0,
\end{cases}
$$

(1.6)

$$
\text{div} u = 0 = \text{div} b,
$$

where $\pi$ is a total pressure.
For the incompressible 3D MHD equations (1.6), Schulz [26] proved a Liouville theorem without the finite Dirichlet integral, which is if the smooth solution

$$(u, b) \in (L^p \cap BMO^{-1})(\mathbb{R}^3), \quad p \in (2, 6],$$

then $u = b = 0$. Recently, Yuan and Xiao [30] shown if

$$(u, b) \in L^p(\mathbb{R}^3), \quad 2 \leq p \leq 9/2,$$

then $u = b = 0$. For a interesting results, we also refer to [19] in spite of results in [11] and [16] inspired by the work in [4] in the Lorentz framework.

In this directions, two results are stated in the framework of Lebesgue and Lorentz spaces.

**Theorem 1.5.** Let $u, b \in L^2_{\text{loc}}(\mathbb{R}^3)$ be a weak solution of (1.6).

1) If $u, b \in L^{9/2,q}(\mathbb{R}^3)$, with $9/2 \leq q < +\infty$, then we have $u = 0 = b$.
2) If $u \in L^{r,q}$ with $9/2 < r \leq q < +\infty$ and $b \in L^{9/2,q}(\mathbb{R}^3)$, with $9/2 \leq q < +\infty$, and in addition, if

$$\sup_{R > 1} R^{2-\frac{2}{r}} \|u\|_{L^{r,q}(C(\mathbb{R}/4,2R))} < +\infty,$$

then we have $u = 0 = b$.

Investigating solutions of the equations (1.1) that may decay to zero in different rates as $|x| \to \infty$ with respect to different directions, we introduce a new functional space [24]. For two given numbers $q, r \in (1, \infty)$ and $f : \mathbb{R}^3 \to \mathbb{R}$ is a measurable function, the mixed-norm Lebesgue space $L_{q,r}(\mathbb{R}^3)$ is the space that is equipped with the following norm

$$\|f\|_{L_{q,r}(\mathbb{R}^3)} = \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} |f(x_1, x_2, x_3)|^q dx_1 dx_2 \right)^{r/q} dx_3 \right]^{1/r}.$$

**Theorem 1.6.** Let $q, r \in [3, \infty)$ be two numbers satisfying

$$\frac{2}{q} + \frac{1}{r} \geq \frac{2}{3},$$

and $(u, b) \in H^1_{\text{loc}}(\mathbb{R}^3)$ be a weak solution of (1.6) and assume that $(u, b) \in L_{q,r}(\mathbb{R}^3)$. Then $u \equiv 0 \equiv b$ in $\mathbb{R}^3$.

**Remark 1.1.** The result given in [7,9] is of particular interest since this result can be regarded as an improvement of the results given in [10] and [25].

**Remark 1.2.** In case of the compressible fluids, we don’t know if Theorem 1.5 or 1.6 are true yet. For a proof of Theorem 1.3, because it is necessary to get the estimate of $\|\nabla u\|_{L^p}$. More speaking, to control a estimate of $\|\nabla u\|_{L^p}$, it needs a higher regularity of $\rho$, in succession, which yields that it needs a higher regularity of $u$. On the other hands, for a proof of Theorem, because it is necessary to use Lemma 1.4 based on the weight function.

Lastly, we make a few comments.

**Remark 1.3.** The present paper focuses on Liouville theorem for the compressible MHD equation, not Hall-MHD equation. We think it is secondary to consider the Hall term in the system (1.1). Considering the Hall term, curl $\times (\text{curl} b \times b)$ in (1.1), namely, (compressible) Hall-MHD equations, we can see that additional conditions are required for the magnetic filed $b$ in light of the references [33], [30] and [18]. Proofs of this part is not difficult, and thus leave it to the readers. For this equation, we refer to [3] and [34] for interested readers.

**Remark 1.4.** For a various coupled equation with (compressible) Naiver-Stokes equations, the above Theorems are applicable through Cacciopol type estimate.
2. Proof of Theorem 1.1

The following inequalities in Lorentz spaces are useful.

**Lemma 2.1** ([23]). Let \( f \in L^{p_1,q_1}(\mathbb{R}^3) \) and \( g \in L^{p_2,q_2}(\mathbb{R}^3) \) with \( 1 \leq p_1, p_2 \leq \infty, 1 \leq q_1, q_2 \leq \infty \). Then \( fg \in L^{p,q}(\mathbb{R}^3) \) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \) and

\[
\|fg\|_{L^{p,q}(\mathbb{R}^3)} \leq C \|f\|_{L^{p_1,q_1}(\mathbb{R}^3)} \|g\|_{L^{p_2,q_2}(\mathbb{R}^3)}
\]

for a constant \( C > 0 \).

**Lemma 2.2** ([?]). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n, 1 < p < \infty, 1 < q \leq \infty \) and \( f \in L^{p,q}(\Omega) \). Then

\[
\|\nabla^2(-\Delta)^{-1}f\|_{L^{p,q}(\Omega)} \leq C \|f\|_{L^{p,q}(\Omega)},
\]

where the constant \( C > 0 \) is independent of \( \Omega \).

To obtain the estimate for the pressure term, we recall the following lemma from [20].

**Lemma 2.3.** Let \( P \in L^{\infty}(\mathbb{R}^3), p_1 \in L^{r_1}(\mathbb{R}^3), p_2 \in L^{r_2}(\mathbb{R}^3) \) with \( 1 \leq r_1, r_2 < \infty \). Suppose that \( P - p_1 - p_2 \) is weakly harmonic, that is

\[
\Delta(P - p_1 - p_2) = 0
\]

in the sense of distribution, then there exists a constant \( c \) such that

\[
P - p_1 - p_2 = c \quad \text{a.e.} \quad x \in \mathbb{R}^3
\]

If furthermore \( P(x) \geq 0 \) a.e., then we also have \( c \geq 0 \).

Taking the \( \text{div} \) operation on both sides of (1.1), we have \( \Delta(P - p_1 - p_2) = 0 \), where

\[
p_1 := (-\Delta)^{-1}\partial_i\partial_j(\rho u_i u_j) \quad \text{and} \quad p_2 := (\lambda + 2\nu) \text{div} \, u.
\]

Using the assumption \( \nabla u \in L^2(\mathbb{R}^3) \) and the Sobolev embedding \( H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \), it follows

\[
p_1 \in L^3(\mathbb{R}^3), \quad p_2 \in L^2(\mathbb{R}^3).
\]

Due to Lemma 2.3, there exists a constant \( c \geq 0 \) such that

\[
a \rho^{\gamma} = P = c + p_1 + p_2.
\]

Considering the function \( P_1 := \rho^{\gamma - 1} \left( \frac{\alpha}{\rho} \right)^{\gamma - 1} = (\frac{\alpha}{\rho} + \frac{\alpha + \beta}{\rho})^{\gamma - 1} - (\frac{\alpha}{\rho})^{\gamma - 1} \), we have

\[
\nabla P = \nabla(a \rho^{\gamma}) = \frac{a \gamma}{\gamma - 1} \rho \nabla(\rho^{\gamma - 1}) = \frac{a \gamma}{\gamma - 1} \rho \nabla P_1
\]

and

\[
|P_1 \rho| \leq (|p_1| + |p_2|).
\]

**Proof of Theorem 1.1** Following the same arguments in [18] or [20], we are aimed to show a suitable Caccioppoli-type inequality. For the convenience of our readers, we give a sketch of the proof. Let \( \varphi \in C_0^\infty(\mathbb{R}^3) \) be a radial cut-off function satisfying

\[
\varphi(|x|) = \begin{cases} 
1, & \text{if } |x| < \frac{1}{2}, \\
0, & \text{if } |x| > 1,
\end{cases}
\]

and \( 0 \leq \varphi(|x|) \leq 1 \) for \( \frac{1}{2} \leq |x| \leq 1 \). For each given \( R > 0 \), we define \( \varphi_R(x) := \varphi\left(\frac{|x|}{R}\right) \) satisfying

\[
\|\nabla^k \varphi_R\|_{L^\infty} \leq CR^{-k}, \quad k = 0, 1, 2
\]

for some constant \( C > 0 \) independent of \( x \in \mathbb{R}^3 \).
Taking the inner product of $[1, 1]_2$ with $u \varphi_R$ and $[1, 1]_3$ with $b \varphi_R$ and after integrating by parts, it follows that
\begin{align}
(2.4) \quad \int_{B_{2R}} \left( \nu |\nabla u|^2 + |\nabla b|^2 \right) \varphi_R \, dx + (\lambda + \nu) \int_{B_{2R}} |\operatorname{div} u|^2 \, dx \\
= -\nu \int_{C(R/2, R)} (|u|^2 + |b|^2) \Delta \varphi_R \, dx - 2(\lambda + \nu) \int_{C(R/2, R)} \operatorname{div} u \cdot \nabla \varphi_R \, dx \\
- \int_{C(R/2, R)} |u|^2 \nabla \varphi_R \cdot \rho u \, dx + \int_{C(R/2, R)} |b|^2 \nabla \varphi_R \cdot u \, dx - \int_{C(R/2, R)} \nabla P \cdot u \varphi_R \, dx := \sum_{i=1}^6 I_i,
\end{align}
where we use $\operatorname{div}(\rho u) = 0$ and by the integration by parts and chain rule,
\[-2\nu \int_{\mathbb{R}^3} \nabla u : u \otimes \nabla \varphi_R \, dx = \nu \int_{\mathbb{R}^3} |u|^2 \Delta (\varphi_R) \, dx.\]

Now, we estimate $I_i$ term by term. We assume $p > 3$, $3 \leq q \leq \infty$ or $p = q = 3$. For $I_1$, since
\[
\int_{C(R/2, R)} |u|^2 |\Delta \varphi_R| \, dx \leq R^{1-\frac{6}{p}} \left( \|u\|_{L^{p,q}(C(R/2, R))}^2 + \|b\|_{L^{p,q}(C(R/2, R))}^2 \right),
\]
we have
\[
|I_1| \leq R^{1-\frac{6}{p}} \left( \|u\|_{L^{p,q}(C(R/2, R))}^2 + \|b\|_{L^{p,q}(C(R/2, R))}^2 \right).
\]
For $I_2$, by Hölder and Young inequalities, it yields
\[
|I_2| \leq C(\lambda + \nu) \int_{C(R/2, R)} |\operatorname{div} u||u||\nabla \varphi_R| \, dx \\
\leq \frac{(\lambda + \nu)}{2} \|\operatorname{div} u\|_{L^2(\mathbb{R}^3)}^2 + CR^{1-\frac{6}{p}} \|u\|_{L^{p,q}(C(R/2, R))}^2.
\]
For $I_3$, we have
\[
|I_3| \lesssim R^{-1} \|\rho\|_{L^\infty} \|u\|_{L^{p,q}(C(R/2, R))}^3 \|u\|_{L^{p,q}(C(R/2, R))} \leq CR^{2-\frac{6}{p}} \|u\|_{L^{p,q}(C(R/2, R))}^3.
\]
In a same manner as $I_3$, by the integration by parts, $I_4$ and $I_5$ are rewritten by
\[
|I_4| + |I_5| \lesssim R^{2-\frac{6}{p}} \left( \|u\|_{L^{p,q}(C(R/2, R))}^3 + \|b\|_{L^{p,q}(C(R/2, R))}^3 \right).
\]
Integrating by parts with (2.1) and (2.2), $I_6$ becomes
\[
I_6 = -\int_{\mathbb{R}^3} \frac{a}{\gamma - 1} \rho \nabla P_1 \cdot u \varphi_R \, dx \\
= \frac{a}{\gamma - 1} \int_{\mathbb{R}^3} P_1 \rho \nabla (\varphi_R) \, dx + \frac{a}{\gamma - 1} \int_{\mathbb{R}^3} P_1 \rho u \cdot \nabla (\varphi_R) \, dx \\
= \frac{2a}{\gamma - 1} \int_{\mathbb{R}^3} P_1 \rho u \cdot (\varphi_R \nabla \varphi_R) \, dx,
\]
and thus we have
\[
|I_6| \leq R^{-1} \int_{C(R/2, R)} (|p_1| + |p_2|) |u| \, dx.
\]
By Lemmas 2.1 and 2.2, we have
\[
R^{-1} \int_{C(R/2, R)} |p_1| |u| \, dx \leq CR^{-1} \|p_1\|_{L^{p,q}(C(R/2, R))} \|u\|_{L^{p,q}(C(R/2, R))} \|1\|_{L^{p,q}(C(R/2, R))} \\
\leq CR^{2-\frac{6}{p}} \|u\|_{L^{p,q}(C(R/2, R))}^3.
\]
Therefore,

\[ |\mathcal{I}_6| \lesssim R^{2-\frac{2}{p}} \|u\|^3_{L^{p,q}(C(R/2, R))} + CR^{1-\frac{6}{p}} \|u\|^2_{L^{p,q}(C(R/2, R))}. \]

Substituting all estimates of \( \mathcal{I}_1-\mathcal{I}_6 \) into (2.4) leads to

\[ \nu \int_{|x| \leq R} \left| \nabla u \right|^2 + \frac{(\lambda + \nu)}{2} \int_{|x| \leq R} |\text{div } u|^2 \]  

\[ \lesssim \left( R^{1-\frac{6}{p}} \|u\|^2_{L^{p,q}(C(R/2, R))} + \|b\|^2_{L^{p,q}(C(R/2, R))} \right) + R^{2-\frac{2}{p}} \left( \|u\|^3_{L^{p,q}(C(R/2, R))} + \|b\|^3_{L^{p,q}(C(R/2, R))} \right). \]

for all \( R > 0 \).

Passing \( R \to +\infty \) in (2.5), due to the assumptions, we have

\[ \lim_{R \to +\infty} \left( \int_{B^R_{2}} \left| \nabla u \right|^2 + \int_{B^R_{2}} |\text{div } u|^2 \right) = 0, \]

which implies that \( u \equiv 0 \equiv b \). On the other hand, we know that \( \nabla(ap^\gamma) = 0 \), which implies that \( \rho = \text{constant on } \mathbb{R}^3 \) by means of (1.1). The proof of Theorem 1.1 is ended. \( \square \)

3. Morrey Space - Proof of Theorem 1.3

From (2.5) with the relationship between spaces (1.3), for all \( R > 0 \), we know

\[ \nu \int_{|x| \leq R} \left| \nabla u \right|^2 + \frac{(\lambda + \nu)}{2} \int_{|x| \leq R} |\text{div } u|^2 \]  

\[ \lesssim \left( R^{1-\frac{6}{p}} \|u\|^2_{L^{p,q}(C(R/2, R))} + \|b\|^2_{L^{p,q}(C(R/2, R))} \right) + R^{2-\frac{2}{p}} \left( \|u\|^3_{L^{p,q}(C(R/2, R))} + \|b\|^3_{L^{p,q}(C(R/2, R))} \right). \]

Using the definition of Morrey space, the inequality (3.1) becomes

\[ \int_{B^R_{2}} \left( \left| \nabla u \right|^2 + |\nabla b|^2 \right) \varphi_R \, dx + \int_{B^R_{2}} |\text{div } u|^2 \, dx \]

\[ \lesssim R^{2-\frac{2}{p}} \left( \|u\|^2_{M^{p,r}} + \|b\|^2_{M^{p,r}} \right) \|u\|_{M^{p,r}} + R^{1-\frac{6}{p}} \left( \|u\|^2_{M^{p,r}} + \|b\|^2_{M^{p,r}} \right), \]

which implies \( u \equiv 0 \equiv b \) in \( \mathbb{R}^3 \) as \( R \to \infty \). The proof of Theorem 1.3 is complete.

4. Proof of Theorem 1.4

From (2.5), for all \( R > 0 \), we know

\[ \nu \int_{|x| \leq R} \left| \nabla u \right|^2 + \frac{(\lambda + \nu)}{2} \int_{|x| \leq R} |\text{div } u|^2 \]  

\[ \lesssim \left( R^{1-\frac{6}{p}} \|u\|^2_{L^{p,q}(C(R/2, R))} + \|b\|^2_{L^{p,q}(C(R/2, R))} \right) + R^{2-\frac{2}{p}} \left( \|u\|^3_{L^{p,q}(C(R/2, R))} + \|b\|^3_{L^{p,q}(C(R/2, R))} \right). \]
Note that by definition of the local Morrey spaces $M^2_{\gamma,0}(\mathbb{R}^3)$, we have $u, b \in L^3_{\text{loc}}(\mathbb{R}^3)$ and thus, letting $p = 3$, the estimate (4.2) becomes

$$
\int_{B_{R/2}} (|\nabla u|^2 + |\nabla b|^2) \varphi_R \, dx + \int_{R/2} |\text{div} u|^2 \, dx
\leq R^{-1} (\|u\|^3_{L^2(C(R/2,R))} + \|b\|^3_{L^2(C(R/2,R))}) + CR^{-1} (\|u\|^2_{L^3(C(R/2,R))} + \|b\|^2_{L^3(C(R/2,R))}).
$$

And thus, we again write for $1 \leq \gamma < 3/2$

$$
\int_{B_{R/2}} (|\nabla u|^2 + |\nabla b|^2) \varphi_R \, dx + \int_{R/2} |\text{div} u|^2 \, dx
\leq R^{-1} (\|u\|^2_{L^2(C(R/2,R))} + \|b\|^2_{L^2(C(R/2,R))}) + \left(\frac{1}{R^{\gamma}} \int_{C(R/2,R)} |u|^3 \, dx\right)^{2/3} R^{\gamma-1} \left(\frac{1}{R^{\gamma}} \int_{C(R/2,R)} |b|^3 \, dx\right)^{1/3}
= R^{-1} (\|u\|^2_{L^2(C(R/2,R))} + \|b\|^2_{L^2(C(R/2,R))})
+ R^{\gamma-1} \left[\|u\|^3_{M^2_{\gamma}} \left(\frac{1}{R^{\gamma}} \int_{C(R/2,R)} |u|^3 \, dx\right)^{1/3} + \|b\|^3_{M^2_{\gamma}} \left(\frac{1}{R^{\gamma}} \int_{C(R/2,R)} |b|^3 \, dx\right)^{1/3}\right].
$$

We shall study each term in the right-hand side. For the first and second term in (4.3), we have

$$
\lim_{R \to +\infty} \frac{c}{R^2} \int_{C(R/2,R)} |u|^2 \, dx = 0, \quad \text{and} \quad \lim_{R \to +\infty} \frac{c}{R} \left(\int_{C(R/2,R)} |u|^3 \, dx\right)^{2/3} = 0.
$$

Indeed,

$$
\frac{c}{R^2} \int_{C(R/2,R)} |u|^2 \, dx \leq c R^{-1} \left(\int_{C(R/2,R)} |u|^3 \, dx\right)^{2/3} \leq c R^{2\gamma-1} \|u\|^2_{M^2_{\gamma}}.
$$

and also through a same computation, we have

$$
R^{-1} \|u\|^2_{L^2(C(R/2,R))} \leq c R^{2\gamma-1} \|u\|^2_{M^2_{\gamma}}.
$$

Due to $\frac{2}{\gamma} - 1 < 0$, hence (4.4) follows. Also as $R \to \infty$, the remains term in the estimate above vanish depending on the parameter $\gamma$ (see [13] for a detailed proof). Hence, we have the identities $u = 0$ and $b = 0$ and therefore, Theorem 1.4 is proven.

5. PROOF OF THEOREM 1.5

Before proving Theorem 1.5, we see the following Caccioppoli type estimate:

**Lemma 5.1.** Let $C(R/2,R) = \{x \in \mathbb{R}^3 : R/2 < |x| < R\}$. If the solution $(u, b)$ verifies $u \in L^p_{\text{loc}}(\mathbb{R}^3)$ and $\nabla u \in L^p_{\text{loc}}(\mathbb{R}^3)$ with $3 \leq p < +\infty$, then for all $R > 1$ we have

$$
\int_{B_{R/2}} |\nabla u|^2 \, dx + \int_{B_{R/2}} |\nabla b|^2 \, dx \leq \left(\int_{C(R/2,R)} |\nabla u|^2 \, dx\right)^{\frac{2}{p}} R^{2 - \frac{2}{p}} \left(\int_{C(R/2,R)} |u|^p \, dx\right)^{\frac{1}{p}}
+ \left(\int_{C(R/2,R)} |\nabla b|^2 \, dx\right)^{\frac{2}{p}} R^{2 - \frac{2}{p}} \left(\int_{C(R/2,R)} |b|^p \, dx\right)^{\frac{1}{p}}
+ \left[\left(\int_{C(R/2,R)} |u \otimes u|^2 \, dx\right)^{\frac{2}{p}} + \left(\int_{C(R/2,R)} |b \otimes b|^2 \, dx\right)^{\frac{2}{p}}\right] R^{2 - \frac{2}{p}} \left(\int_{C(R/2,R)} |u|^p \, dx\right)^{\frac{1}{p}}.
$$
Proof. Note that

$$- \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \partial_{ij} v_i \cdot v_i \varphi_R \, dx = \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \partial_{ij} v_i \cdot \partial_{ij} \varphi_R \, dx + \int_{\mathbb{R}^3} \partial_{ij} v_i \cdot v_i \partial_{ij} \varphi_R \, dx$$

(5.5)

$$= \int_{\mathbb{R}^3} |\nabla v|^2 \varphi_R \, dx + \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \partial_{ij} v_i \cdot v_i \partial_{ij} \varphi_R \, dx.$$  

Using (5.5),

$$\int_{B_{R/2}} (|\nabla u|^2 + |\nabla b|^2) \, dx = - \sum_{i,j=1}^{3} \int_{B_{R}} \left[ (\partial_{ij} u_i)(\partial_{ij} \varphi_R) u_i + (\partial_{ij} b_i)(\partial_{ij} \varphi_R) b_i \right] \, dx$$

$$- 2(\lambda + \nu) \int_{C(R/2,R)} \text{div} \, u u \cdot \nabla \varphi_R \, dx + \int_{B_{R}} (u \cdot \nabla) u \cdot \varphi_R \, dx$$

$$- \int_{C(R/2,R)} |u|^2 \nabla \varphi_R \cdot \rho u \, dx + \int_{C(R/2,R)} |b|^2 \nabla \varphi_R \cdot u \, dx,$$

(5.6)

$$- \int_{C(R/2,R)} \nabla P \cdot u \varphi_R \, dx := \sum_{i=1}^{6} M_i.$$  

We study now these four terms above. In term $I_1$ remark that we have the function $\partial_i \varphi_R$, but since the test function $\varphi_R$ verifies $\varphi_R(1)$ if $|x| < \frac{R}{2}$ and $\varphi_R(x) = 0$ if $|x| > R$ then we have $\text{supp}(\nabla \varphi_R) \subset C(R/2,R)$, and thus we can write

$$M_1 = - \sum_{i,j=1}^{3} \int_{C(R/2,R)} \partial_{ij} u_i(\partial_{ij} \varphi_R) u_i \, dx - \sum_{i,j=1}^{3} \int_{C(R/2,R)} \partial_{ij} b_i(\partial_{ij} \varphi_R) b_i \, dx.$$  

Then, applying the Hölder inequality, we have

$$\int_{C(R/2,R)} \partial_{ij} u_i(\partial_{ij} \varphi_R) u_i \, dx \lesssim \left( \int_{C(R/2,R)} |\nabla u|^\frac{p}{2} \, dx \right)^{\frac{p}{2}} \frac{1}{R} \left( \int_{C(R/2,R)} |u|^q \, dx \right)^{\frac{1}{q}}$$

$$\lesssim \left( \int_{C(R/2,R)} |\nabla u|^\frac{p}{2} \, dx \right)^{\frac{p}{2}} R^{2 - \frac{p}{q}} \left( \int_{C(R/2,R)} |u|^p \, dx \right)^{\frac{1}{p}}.$$  

With this estimate at hand $M_1$ is bounded by

$$\left( \int_{C(R/2,R)} |\nabla u|^\frac{p}{2} \, dx \right)^{\frac{p}{2}} R^{2 - \frac{p}{q}} \left( \int_{C(R/2,R)} |u|^p \, dx \right)^{\frac{1}{p}} + \left( \int_{C(R/2,R)} |\nabla b|^\frac{p}{2} \, dx \right)^{\frac{p}{2}} R^{2 - \frac{p}{q}} \left( \int_{C(R/2,R)} |b|^p \, dx \right)^{\frac{1}{p}}.$$  

In a same manner as $M_1$, $M_2$ is estimated by

$$\left( \int_{C(R/2,R)} |\nabla u|^\frac{p}{2} \, dx \right)^{\frac{p}{2}} R^{2 - \frac{p}{q}} \left( \int_{C(R/2,R)} |u|^p \, dx \right)^{\frac{1}{p}}.$$  

We study the term $M_4$ and $M_5$ in (5.6). we have

$$M_3 + M_4 + M_5 \lesssim \left[ \left( \int_{C(R/2,R)} |u \otimes u|^\frac{p}{2} \, dx \right)^{\frac{p}{2}} + \left( \int_{C(R/2,R)} |b \otimes b|^\frac{p}{2} \, dx \right)^{\frac{p}{2}} \right] R^{2 - \frac{p}{q}} \left( \int_{C(R/2,R)} |u|^p \, dx \right)^{\frac{1}{p}}.$$  

Here, for the estimate $M_3$, in addition, we use $\|\rho\|_{L^\infty} < \infty$. Finally, From (2.1) and (2.2), we recall that

$$M_6 = \frac{2\alpha\gamma}{\gamma - 1} \int_{\mathbb{R}^3} P_1 \rho u \cdot (\varphi_R \nabla \varphi_R) \, dx,$$
Similarly, we can know
\[ |P_1 \rho| \leq C(a, \|\rho\|_{L^\infty})(|p_1| + |p_2|). \]
in a similar way,
\[
\mathcal{M}_6 \lesssim \left( \left( \int_{C(R/2,R)} |\nabla u|^2 \! \, dx \right)^{\frac{2}{p}} + \left( \int_{C(R/2,R)} \|u^2\|^2 \! \, dx \right)^{\frac{2}{p}} \right)^{\frac{1}{2}} \left( \int_{C(R/2,R)} |u|^p \! \, dx \right)^{\frac{1}{p}}.
\]
Collecting all \( \mathcal{M}_i \), we proved Lemma (5.11).

**Proof of Theorem 7.3**
Following the arguments in [12], for \( 1 < p < r \leq q < +\infty \) and for \( R > 1 \) we have
\[
(5.7) \quad \int_{B_R} |u|^p \! \, dx \leq c \, R^{3(1-\frac{2}{p})} \|u\|_{L^{r,\infty}}^p \leq c \, R^{3(1-\frac{2}{p})} \|u\|^{p}_{L^{r,\infty}}.
\]
On the other hand, since
\[
\|u\|_{L^{r,\infty}} = \frac{1}{\Delta} \left( \mathbb{P} \left( (u \cdot \nabla)u \right) - \mathbb{P} \left( (b \cdot \nabla)b \right) \right),
\]
we have for \( i = 1, 2, 3 \)
\[
\partial_i u = - \sum_{j=1}^{3} \frac{1}{\Delta} \left( \mathbb{P} (\partial_i \partial_j (u_j u)) - \mathbb{P} (\partial_i \partial_j (B_j b)) \right) = \sum_{j=1}^{3} \mathbb{P} \left( R_i \mathcal{R}_j(u_j u) - R_i \mathcal{R}_j(B_j b) \right),
\]
where \( \mathcal{R}_i = \frac{\partial}{\partial x_i} \) denotes the i-th Riesz transform. Thus,
\[
(5.8) \quad \int_{B_R} |\nabla u|^\frac{p}{2} \! \, dx \leq c \, R^{3(1-\frac{2}{p})} \left( \|u \otimes u\|_{L^{\frac{p}{2}, \frac{p}{2}}}^\frac{p}{2} + \|b \otimes b\|_{L^{\frac{p}{2}, \frac{p}{2}}}^\frac{p}{2} \right), \quad 3 \leq p < +\infty.
\]
Similarly, we can know
\[
(5.9) \quad \int_{B_R} |\nabla b|^\frac{p}{2} \! \, dx \leq c \, R^{3(1-\frac{2}{p})} \left( \|u \otimes b\|_{L^{\frac{p}{2}, \frac{p}{2}}}^\frac{p}{2} + \|b \otimes u\|_{L^{\frac{p}{2}, \frac{p}{2}}}^\frac{p}{2} \right), \quad 3 \leq p < +\infty.
\]
Through Lemma (5.1) we write for all \( R > 1 \)
\[
(5.10) \quad \int_{B_{R/2}} |\nabla u|^2 \! \, dx + \int_{B_{R/2}} |\nabla b|^2 \! \, dx \lesssim G_u P_u + G_b P_b + (S_u + S_b) P_u,
\]
where
\[
G_u := c \left( R^2 \left( \frac{1}{R^3} \int_{C(R/2,R)} |\nabla \otimes u|^\frac{p}{2} \! \, dx \right)^\frac{2}{p} \right), \quad S_u := R^2 \left( \frac{1}{R^3} \int_{C(R/2,R)} |u \otimes u|^\frac{p}{2} \! \, dx \right)^\frac{2}{p},
\]
\[
G_b := c R^2 \left( \frac{1}{R^3} \int_{C(R/2,R)} |\nabla \otimes b|^\frac{p}{2} \! \, dx \right)^\frac{2}{p}, \quad S_b := R^2 \left( \frac{1}{R^3} \int_{C(R/2,R)} |b \otimes b|^\frac{p}{2} \! \, dx \right)^\frac{2}{p},
\]
\[
P_u := R^{2-\frac{q}{p}} \left( R^2 \left( \frac{1}{R^3} \int_{C(R/2,R)} |u|^p \! \, dx \right)^\frac{1}{p} \right), \quad P_b := R^{2-\frac{q}{p}} \left( R^2 \left( \frac{1}{R^3} \int_{C(R/2,R)} |b|^p \! \, dx \right)^\frac{1}{p} \right).
\]
For this we introduce the cut-off function \( \theta_R \in C^\infty_0(\mathbb{R}^3) \) such that \( \theta_R = 1 \) on \( C(R/2,R) \), \( \text{supp} \ (\theta_R) \subset C(R/4,2R) \) and \( \|\nabla \theta_R\|_{L^\infty} \leq \frac{1}{R} \). In the same approach in [12] with (5.7)–(5.9), (5.10) yields
\[
\int_{B_{R/2}} \left( |\nabla u|^2 \! \, dx + |\nabla b|^2 \! \, dx \right) \leq c \left( \|\theta_R (\nabla u)\|_{L^{\frac{p}{2}, \frac{p}{2}}}^2 + \|\theta_R u\|_{L^{r,\infty}}^2 \right) R^{2-\frac{q}{p}} \|u\|_{L^{r,\infty}(C(R/4,R))},
\]
\[
+ c \|\theta_R (\nabla b)\|_{L^{\frac{p}{2}, \frac{p}{2}}}^2 R^{2-\frac{q}{p}} \|b\|_{L^{r,\infty}(C(R/4,R))} + c \|\theta_R b\|_{L^{r,\infty}(C(R/4,R))}^2 R^{2-\frac{q}{p}} \|u\|_{L^{r,\infty}(C(R/4,R))}.
\]
This implies \( u \equiv 0 \equiv b \) in \( \mathbb{R}^3 \) under the assumptions in Theorem (1.5).
\[ \square \]
For each $q \in [1, \infty)$, a non-negative measurable function $\omega : \mathbb{R}^n \to \mathbb{R}$ is said to be in the Muckenhoupt $A_q(\mathbb{R}^n)$-class if

$$[\omega]_{A_q} := \sup_{R > 0, x_0 \in \mathbb{R}^n} \left( \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} \omega(x) dx \right) \left( \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} \omega(x)^{\frac{1}{q-1}} dx \right)^{q-1} < \infty$$

for $q \in (1, \infty)$, and

$$[\omega]_{A_1} := \sup_{R > 0, x_0 \in \mathbb{R}^n} \left( \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} \omega(x) dx \right) \| \frac{1}{\omega} \|_{L_\infty(B_R(x_0))} < \infty,$$

where $B_R(x_0)$ denotes the ball in $\mathbb{R}^3$ with the radius $R$ and centered at $x_0 \in \mathbb{R}^3$.

We introduce the following lemma on weighted mixed norm estimates for the pressure of the equations (1.1). This lemma is an key ingredient in the paper.

**Lemma 6.1.** Let $q, r \in (2, \infty)$ and $M_0 \geq 1$. Assume that $u, b \in L_{q,r}(\mathbb{R}^3, \omega)$ with $\omega(x) = \omega_1(x')\omega_2(x_3)$ for all a.e. $x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$, and for $\omega_1 \in A_2^{q}(\mathbb{R}^2)$, $\omega_2 \in A_2^{q}(\mathbb{R})$ with

$$[\omega_1]_{A_2^{q}(\mathbb{R}^2)} \leq M_0, \quad [\omega_2]_{A_2^{q}(\mathbb{R})} \leq M_0.$$

Then, there exists $N = N(q, r, M_0) > 0$ such that

$$\left\| P \right\|_{L_{2,q,r}^{\omega}(\mathbb{R}^3, \omega)} \leq N \left( \left\| u \right\|_{L_{q,r}(\mathbb{R}^3, \omega)}^2 + \left\| b \right\|_{L_{q,r}(\mathbb{R}^3, \omega)}^2 \right),$$

where $P$ is defined as

$$(6.11) \quad P = \sum_{i,j=1}^{3} \left( \mathcal{R}_i \mathcal{R}_j(u_iu_j) + \mathcal{R}_i \mathcal{R}_j(b_ib_j) \right),$$

in which $\mathcal{R}_i$ denotes the $i$-th Riesz transform.

**Proof.** This proof is almost similar to that in [24, Theorem 1.4] and so we omit the proof. □

**Proof of Theorem 1.6.** Let $\phi \in C_0^\infty(\mathbb{R})$ be a standard cut-off function with $0 \leq \phi \leq 1$ and

$$\phi = 1 \quad \text{on} \quad [-1/2, 1/2] \quad \text{and} \quad \phi = 0 \quad \text{on} \quad \mathbb{R} \setminus [-1, 1].$$

For each $R > 0$, let $Q_R = [-R, R]^3$ and

$$\phi_R(x) = \phi \left( \frac{x_1}{R} \right) \phi \left( \frac{x_2}{R} \right) \phi \left( \frac{x_3}{R} \right), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Then, we see that $\phi_R(x) = 1$ on $Q_{R/2}$, $\phi = 0$ on $\mathbb{R}^3 \setminus Q_R$. Then, there is a constant $C > 0$ independent on $R$ such that $|\nabla^k \phi_R| \leq \frac{C}{R^k}$, $k = 0, 1, 2$. Testing $u\phi_R$ and $b\phi_R$ as a test function for the system (1.1), respectively, we have

$$\int_{Q_R/2} (|\nabla u|^2 + |\nabla b|^2) dx$$

$$\leq \frac{1}{2} \int_{Q_R \setminus Q_R/2} (|u|^2 + |b|^2) \Delta \phi_R dx + \frac{1}{2} \int_{Q_R/2} (|u|^3 + |b|^3) |\nabla \phi_R| dx$$

$$+ \int_{Q_R \setminus Q_R/2} |p||u||\nabla \phi_R| dx,$$

where we use

$$\int_{Q_R \setminus Q_R/2} |u||b|^2 |\nabla \phi_R| dx \leq \frac{1}{2} \int_{Q_R \setminus Q_R/2} (|u|^3 + |b|^3) |\nabla \phi_R| dx.$$
From the arguments in \[24\], we can show that the three terms in (6.12) becomes to 0 as \( R \to \infty \) under the assumptions and Lemma 6.11. Indeed, the proof of this part is almost same that in \[24\] and thus we skip a proof. Hence, we obtain
\[
\int_{\mathbb{R}^3} \left( |\nabla u|^2 + |\nabla b|^2 \right) dx = \lim_{R \to \infty} \int_{Q_{R/2}} \left( |\nabla u|^2 + |\nabla b|^2 \right) dx = 0.
\]
Therefore, \((u, b)\) is a constant function in \( \mathbb{R}^3 \). From this and the fact that \((u, b) \in L_{q,r}(\mathbb{R}^3)\), we conclude that \( u \equiv 0 \equiv b \) in \( \mathbb{R}^3 \). The proof is then completed. \(\square\)

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