Nonlinearly charged dyonic black holes

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In this paper, we investigate the thermodynamics of dyonic black holes in the presence of Born-Infeld electromagnetic field. We show that electric-magnetic duality reported for dyonic solutions with Maxwell field is omitted in case of Born-Infeld generalization. We also confirm that generalization to nonlinear field provides the possibility of canceling the effects of cosmological constant. This is done for nonlinearity parameter with $10^{-33} \text{ eV}^2$ order of magnitude which is high nonlinearity regime. In addition, we show that for small electric/magnetic charge and high nonlinearity regime, black holes would develop critical behavior and several phases. In contrast, for highly charged case and Maxwell limits (small nonlinearity), black holes have one thermal stable phase. We also find that the pressure of the cold black holes is bounded by some constraints on its volume while hot black holes’ pressure has physical behavior for any volume. In addition, we report on possibility of existences of triple point and reentrant of phase transition in thermodynamics of these black holes. Finally, We show that if electric and magnetic charges are identical, the behavior of our solutions would be Maxwell like (independent of nonlinear parameter and field). In other words, nonlinearity of electromagnetic field becomes evident only when these black holes are charged magnetically and electrically different.

I. INTRODUCTION

In the past two decades, the progresses in thermodynamics of black holes provided us with new deep insights into physics of black holes as well as its applications in other fields. Among these insights/applications, one can point out to development of the AdS/CFT correspondence [1], principles of holography [2, 3], connections to hydrodynamics [4], introduction of van der Waals like behavior [5], reentrant of the phase transition [6], existence of the triple point [7, 8], analogous heat engines [9, 10] and, connection between quasinormal modes and thermodynamical phase transition [11, 12].

Among different classes of black holes and their thermodynamics, dyonic black holes are of special interest/importance. This is because this family of the magnetically charged static black hole solutions were proven to have wide applications in different fields of the physics. To name a few, one can mention: the Hall conductivity and zero momentum hydrodynamic response functions in the context of AdS/CFT [13], correspondence between large dyonic black holes and stationary solutions of the equations of relativistic magnetohydrodynamics [14], inducing external magnetic field effects on superconductors [15], investigation of Hall conductance, DC longitudinal conductivity [16] and paramagnetism/ferromagnetism phase transitions [17, 18]. So far, dyonic black holes were obtained in effective [19] and hetroctic [20] string theories, supergravity [21, 22], massive gravity [18], gravity’s rainbow [23] and dilatonic gravity [24, 25]. Their thermodynamics were investigated in Refs. [26, 27]. In this paper, we investigate the thermodynamics of the dyonic black holes in the presence of Born-Infeld (BI) nonlinear electromagnetic field.

BI theory is one of the well established theories which generalizes Maxwell field in a nonlinear framework. The primary motivation of this theory was to remove self energy of a point like charge in Maxwell theory [28]. Later on, it was shown that this electromagnetic field have features such as: the absence of shock waves, birefringence phenomena [29] and electric-magnetic duality [30]. So far, different classes of black holes in the presence of BI were constructed and investigated including: BTZ dilatonic [31], Lovelock–Born–Infeld-scalar gravity [32], massive gravity [33], gravity’s rainbow [34] and iBorn-Infeld [35]. Previously, dyonic black holes in the presence of the BI field were obtained in Ref. [36] and their holographic complexity were investigated [37]. In this paper, we focus more on the thermodynamics of dyonic black holes in the presence of BI nonlinear electromagnetic field. We study the thermal phases of these black holes, their behaviors in different regimes, limitation on having physical solutions, existence of phenomena such as reentrant of phase transitions and triple point. In addition, we confirm that solutions could exhibit linearly charge dyonic black holes’ behavior if certain limits, which are not expected, are meet.

The structure of the paper is as follows: first, we present the action governing these black holes, obtain the metric function and investigate the geometrical properties. Then, thermodynamical quantities are calculated. In addition, thermodynamical phases available for these black holes, the possibility of phase transition and restrictions for having

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critical behavior are studied. Next, the conditions for having linearly charge dyonic black holes’ behavior are extracted and discussed. The paper is concluded by some closing remarks.

II. BLACK HOLE SOLUTIONS

In this paper, we intend to construct 4-dimensional topological black holes in the presence of BI nonlinear electromagnetic field with dyonic charge. To do so, we consider the following action

$$\mathcal{I} = -\frac{1}{16\pi} \int d^4x \sqrt{-\mathcal{g}} \left[ \mathcal{R} - 2\Lambda + L(\mathcal{F}) \right],$$

(1)

where $g$ is the trace of metric tensor, $\mathcal{R}$ is the scalar curvature, $\Lambda$ is the cosmological constant and $L(\mathcal{F})$ is the Lagrangian of BI theory given by

$$L(\mathcal{F}) = 4\beta^2 \left( 1 - \sqrt{1 + \frac{\mathcal{F}^2}{\beta^2}} \right),$$

(2)

in which $\beta$ is the nonlinearity parameter. The Maxwell invariant is $\mathcal{F} = F_{\mu\nu} F^{\mu\nu}$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor and $A_\mu$ is the gauge potential.

Variation of the action (1) with respect to the metric tensor $g_{\mu\nu}$ and the Faraday tensor $F_{\mu\nu}$, leads to

$$G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} L(\mathcal{F}) - \frac{2F_{\mu\lambda}F^\lambda_{\nu}}{\sqrt{1 + \frac{\mathcal{F}^2}{\beta^2}}} = 0,$$

(3)

$$\partial_\mu \left( \frac{\sqrt{-g} F^{\mu\nu}}{\sqrt{1 + \frac{\mathcal{F}^2}{\beta^2}}} \right) = 0,$$

(4)

where $G_{\mu\nu}$ is the Einstein tensor.

Since we are interested in topological black holes, we consider the metric to be

$$ds^2 = -\psi(r)dt^2 + \frac{dr^2}{\psi(r)} + r^2 \left\{ \begin{array}{ll} d\theta^2 + \sin^2 \theta d\varphi^2, & k = 1 \\ d\theta^2 + d\varphi^2, & k = 0 \\ d\theta^2 + \sinh^2 \theta d\varphi^2, & k = -1 \end{array} \right.,$$

(5)

The constant $k$ indicates that the boundary of $t = \text{constant}$ and $r = \text{constant}$ can be a negative (hyperbolic), zero (flat) or positive (elliptic) constant curvature hypersurface.

To obtain the electric-magnetic matter field, we employ the following gauge potential

$$A = h(r) dt + H(\theta) d\varphi.$$

(6)

Such gauge potential should create a radial dependent electric field and spatial dependent magnetic field. One can show that by using Maxwell equation (4), metric (5) and gauge potential (6), two set of solutions are found for electric-magnetic matter field

$$h(r) = \int \frac{qE_0}{\sqrt{4k^2 + r^2 + \beta^2}} dr, \quad H(\theta) = qM,$$

$$h(r) = \int \frac{qE_0}{\sqrt{4k^2 + \beta^2 r^2 + \beta^2}} dr, \quad H(\theta) = qM \left\{ \begin{array}{ll} \sin \theta, & k = 1 \\ \theta, & k = 0 \\ \sinh \theta, & k = -1 \end{array} \right.,$$

(7)

where $qE$ and $qM$ are integration constants related to total electric and magnetic charges. The first set of solutions indicates that the magnetic part of gauge potential is constant. This results into vanishing magnetic component of
In linearly charged dyonic black holes, there is a symmetry for swapping near-dyonic regimes, far away from Maxwell regime. Another interesting issue is the absence of electric-magnetic duality.

In order to investigate the second condition, we use Kretschmann scalar given by

\[ \lim_{\beta \to \infty} h'(r) = \frac{q_E}{r^2} - \frac{q_E^2}{2r^2\beta^2} + O\left(\frac{1}{\beta^4}\right). \]  

where \( \psi'(r) = \frac{d\psi(r)}{dr} \) and \( \psi''(r) = \frac{d^2\psi(r)}{dr^2} \). The metric function is obtained as

\[
\psi(r) = k - \frac{m}{r} + \frac{2 \left( q_E^2 + \beta^2 r^4 \right)}{r^2} \sqrt{\frac{q_M^2 + \beta^2 r^4}{q_E^2 + \beta^2 r^4}} + \frac{(2\beta^2 - \Lambda)r^2}{3} \frac{4\beta^2 r^2 \left( 6\beta^2 r^4 F_1 \left( \frac{7}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2}; \frac{11}{4}; -\frac{r^4 \beta^2}{q_E^2}, -\frac{r^4 \beta^2}{q_M^2} \right) + 7 \left( q_E^2 + q_M^2 \right) F_1 \left( \frac{3}{4}, \frac{1}{2}, \frac{1}{2}; \frac{7}{4}; -\frac{r^4 \beta^2}{q_E^2}, -\frac{r^4 \beta^2}{q_M^2} \right) \right)}{21 q_E q_M},
\]

in which \( m \) is integration constant related to total mass and known as geometrical mass, and \( F_1 \) is Appell Hypergeometric function [38].

The term \( 2\beta^2 - \Lambda \) enables us to remove the cosmological constant’s effects. This indicates that effective behavior of the cosmological constant (a gravitational correction) could be canceled out by the presence of nonlinear electromagnetic field (a matter field correction). This results into absence of observational evidences of cosmological constant’s contributions in the behavior of such black holes. The cosmological constant is associated to vacuum energy and according to the latest observations, its value is about \( 10^{-66} \, \text{eV}^2 \) in natural unit. To cancel out the effects of the cosmological constant, the nonlinearity parameter should be in order of \( 10^{-33} \, \text{eV}^2 \). This corresponds to high nonlinearity regimes, far away from Maxwell regime. Another interesting issue is the absence of electric-magnetic duality. In linearly charged dyonic black holes, there is a symmetry for swapping \( E \) and \( M \) integration constants [14] [15]. This is called electric-magnetic duality. Here, we see that generalization to nonlinear electromagnetic results into vanishing of such symmetry, hence absence of electric-magnetic duality. Finally, it should be noted that in the limit of \( q_E \to 0 \), \( q_M \to 0 \) and \( \beta \to 0 \), the metric function [11] reduces to Schwarzschild solutions in the presence of cosmological constant

\[
\psi(r) = k - \frac{m}{r} - \frac{\Lambda r^2}{3}.
\]

The solutions could be interpreted as black hole ones if two conditions are satisfied: I) Existence of event horizon for solutions. This is done by finding positive and real valued roots for the metric function. This is a crucial requirement that must be met under any circumstances. II) Existence of irremovable divergency for the solutions. This is done by investigating the curvature scalars. To address the first condition, we use numerical method and plot diagrams in Fig. [1]. Evidently, the solutions could admit up to two roots if suitable values are considered for different parameters. It should be noted that there is a possibility of absence of root, hence event horizon. In this case, if an irremovable divergency exists, it is called naked singularity. Here, we are not interested in such solutions.

In order to investigate the second condition, we use Kretschmann scalar given by
\[ K = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \left( \frac{d^2 \psi(r)}{dr^2} \right) + \frac{4}{r^2} \left( \frac{d\psi(r)}{dr} \right)^2 + \frac{8}{r^4} (\psi(r) - k)^2, \]  

(13)

where \( R_{\alpha\beta\gamma\delta} \) is Riemann tensor. It is a matter of calculation to show that limiting behavior of this scalar for small and large radius are given by

\[ \lim_{r \to 0} K \to \infty, \]  

(14)

\[ \lim_{r \to \infty} K \propto -\frac{2\Lambda}{3} + \frac{8\Lambda^2}{3}. \]  

(15)

where confirm two facts: a) Our solutions contains an irremovable divergence at the origin covered by event horizon(s). b) Solutions are asymptotically (A)dS depending on signature of the cosmological constant. It is worthwhile to mention that Kretschmann scalar is regular everywhere outside of the event horizon (see Fig. 2).

III. THERMODYNAMICS

In this section, we calculate thermodynamical quantities and investigate the thermodynamical behavior of the solutions.
A. First law of black hole thermodynamics

Entropy is the first quantity that we will calculate. In general, the calculation of the entropy depends on the gravity under consideration. In this paper, we obtained black holes in the presence of Einstein gravity. Therefore, we can use Hawking and Bekenstein area law \[39, 40\] to obtain entropy as

\[
S = \frac{1}{4} \int d^2x \sqrt{\gamma} = \frac{r_+}{4},
\]

where \( \gamma \) is the induced metric on the boundary with constant temporal and radial coordinates, and \( r_+ \) is the event horizon of the black hole. Evidently, there is no direct contributions from matter field on entropy and the area law in its original format is resulted (without any modification).

In next step, we use surface gravity in the following form \[41\] to calculate the temperature

\[
T = \frac{\kappa}{2\pi} = \frac{1}{2\pi} \sqrt{-\frac{1}{2} (\nabla_{\mu} \chi_{\nu})(\nabla^{\mu} \chi^{\nu})},
\]

where \( \chi^{\nu} \) is the time-like Killing vector. The spacetime in this paper have a time-like Killing vector of \( \chi = \partial_t \). Consequently, the surface gravity is related to first order derivation of metric function with respect to radial coordinate, \( k = \left. \frac{d(\psi(r))}{dr} \right|_{r=r_+} \). To obtain the final form of the temperature, one should also calculate the geometrical mass, \( m \). We can do this by evaluating the metric function on horizon (\( \psi(r = r_+) = 0 \)) which results into

\[
m = kr_+ + 2 \left( \frac{q_E^2 + \beta^2 r^4}{r_+} \right) \sqrt{\frac{q_M^2 + \beta^2 r^4}{q_E^2 + \beta^2 r^4} + \frac{(2\beta^2 - \Lambda)r^3}{3}} - \frac{4\beta^2 r^3}{3} \left(6\beta^2 r^4 F_1 \left( \frac{7}{4}; \frac{3}{2}, \frac{7}{4}; -\frac{r^4\beta^2}{q_M^2}, -\frac{r^4\beta^2}{q_E^2} \right) + 7 \left( \frac{q_E^2 + q_M^2}{2} \right) F_1 \left( \frac{3}{2}; \frac{3}{2}, \frac{7}{4}; -\frac{r^4\beta^2}{q_M^2}, -\frac{r^4\beta^2}{q_E^2} \right) \right) \frac{21q_E q_M}{16\pi^2 r_+^4}.
\]

Finally, by replacing the geometric mass with Eq. (18) in \( T = \frac{1}{4\pi} \left( \frac{d(\psi(r))}{dr} \right)_{r=r_+} \), we find the temperature as

\[
T = \frac{kr_+ - r_+^4 (\Lambda - 2\beta^2) - 2 \left( \frac{q_M q_E}{2r_+^3} \right) \sqrt{\frac{q_M^2 + \beta^2 r^4}{q_E^2 + \beta^2 r^4}}}{4\pi r_+^3}.
\]

The positivity of the temperature is one of the physical restrictions on our solutions. To meet this end, we investigate the limiting behaviors of the temperature as

\[
\lim_{r_+ \to 0} T = -\frac{q_M q_E}{2\pi r_+^4} + \frac{k}{4\pi r_+} + O(r_+), \tag{20}
\]

\[
\lim_{r_+ \to \infty} T = -\frac{\Lambda r_+}{4\pi} + \frac{k}{4\pi r_+} + O(\frac{1}{r_+^4}), \tag{21}
\]

where the \( r_+ \to 0 \) is related to high energy limit and \( r_+ \to \infty \) investigates the asymptotic behavior of the temperature. For AdS black holes: I) At least one root exists for temperature. II) For small (large) black holes, temperature is always negative (positive) and it is governed by matter (gravitational) field contributions. Medium black holes’s behaviors are determined by topological factor.

For dS black holes: a) Temperature could be completely negative (for hyperbolic horizon and \( \Lambda \leq 2\beta^2 \)), with one root, or with two roots (for spherical black holes). b) Small and large black holes have negative temperature, while the medium black hole’s temperature is determined by horizon of the black holes.

Next thermodynamical quantity of interest is the total mass of black holes. To calculate this property, one can use Arnowitt-Deser-Misner approach \[42\] which yields

\[
M = \frac{m}{8\pi}. \tag{22}
\]
Using obtained geometrical mass \([18]\), we find the mass as

\[
M = \frac{kr_+}{8\pi} + \frac{q_E^2}{4\pi r_+} \left( \frac{q_E^3 + \beta r_+^4}{q_E^3 + \beta r_+^4} \right) - \frac{\beta^4 r_+^7 F_1 \left( \frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}; \frac{r_+^4 \beta^2}{q_E^2}, \frac{r_+^4 \beta^2}{q_M^2} \right)}{7\pi q_E q_M} + \frac{\beta^2 r_+^3 \left( 3q_E q_M \sqrt{q_E^2 + \beta r_+^4} - 2 \left( q_E^2 + q_M^2 \right) F_1 \left( \frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}; \frac{r_+^4 \beta^2}{q_E^2}, \frac{r_+^4 \beta^2}{q_M^2} \right) \right)}{12\pi q_E q_M}. \tag{23}
\]

In order to calculate the total electric and magnetic charges of black holes, one can use Gauss law \([25]\). For the total electric charge, this law is given by

\[
Q_E = \frac{1}{4\pi} \int_{r \to \infty} \sqrt{-g} F_{tr} d^2 x, \tag{24}
\]

which gives the total electric charge per unit volume as

\[
Q_E = \frac{q_E}{4\pi}. \tag{25}
\]

The same method could be applied to find total magnetic charge per unit volume as

\[
Q_M = \frac{q_M}{4\pi}. \tag{26}
\]

In order to calculate the electric and magnetic potentials, one can use the free energy approach \([17, 25]\). The free energy is given by

\[
W = \frac{I_{on \ shell}}{\zeta}, \tag{27}
\]

where \(I_{on \ shell}\) is on shell action and \(\zeta\) is the inverse of temperature. It is a matter of calculation to find electric and magnetic potentials as \([17, 25]\)

\[
U_E = -\frac{dW}{dQ_E} = -\frac{\Omega}{21q_E q_M r_+}, \tag{28}
\]

\[
U_M = \frac{dW}{dQ_M} = -\frac{\Omega}{21q_M q_E r_+}, \tag{29}
\]

where

\[
\Omega = 7\beta^2 r_+^4 \left( 2q_E^2 + q_M^2 \right) F_1 \left( \frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}; \frac{r_+^4 \beta^2}{q_E^2}, \frac{r_+^4 \beta^2}{q_M^2} \right) + 9\beta^4 r_+^8 F_1 \left( \frac{7}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}; \frac{r_+^4 \beta^2}{q_E^2}, \frac{r_+^4 \beta^2}{q_M^2} \right) - \frac{21q_E q_M \sqrt{(q_E^2 + \beta r_+^4) (q_M^2 + \beta^2 r_+^4)}}{21q_E q_M (q_E^2 + \beta r_+^4) (q_M^2 + \beta^2 r_+^4)}. \tag{30}
\]

Our final investigation in this section is checking the validation of the first law of black hole thermodynamics. In our black holes as thermodynamical systems, mass plays the role of internal energy, and entropy, total electric and magnetic charges are extensive quantities, whereas temperature, total electric and magnetic potentials are intensive quantities. Therefore, the first law of black holes thermodynamics is given by

\[
dM = TDdS + U_E dQ_E + U_M dQ_M, \tag{31}
\]
where
\[ T = \left( \frac{\partial M}{\partial S} \right)_{Q_M, q_M} \quad \& \quad U_E = \left( \frac{\partial M}{\partial Q_E} \right)_{S, q_M} \quad \& \quad U_M = \left( \frac{\partial M}{\partial Q_M} \right)_{S, q_E}. \] (32)

It is a matter of calculation to show that obtained temperature [19], electric [28] and magnetic [29] potentials coincide with calculated ones using the first law of black holes thermodynamics. This confirms the validation of first law of black hole thermodynamics.

B. Thermal stability

In the next step, we study the contributions of nonlinear electromagnetic field on thermal stability of the solutions. The aim is to investigate thermodynamical phases available for the solutions, their stability/instability and type of transition between them. To do so, we use heat capacity obtained as

\[ C = T \left( \frac{dS}{d r_+} \right)_{q_M, q_E} = \frac{k r_+^2 - r_+^4 (\Lambda - 2 \beta^2) - 2 (\beta^2 r_+^4 + q_E^2) \sqrt{q_E^2 + \beta^2 r_+^4}}{2 r_+^2 (2 \beta^2 - \Lambda) - 2 k + \frac{4 (\beta^2 r_+^2 (q_E^2 + \beta^2 r_+^4) + 3 q_E^2 q_M^2 \beta^2 - \beta^4 r_+^4)}{r_+^4 (q_M^2 + \beta^4 r_+^4)} \sqrt{q_E^2 + \beta^4 r_+^4}}. \] (33)

The negativity/positivity of heat capacity indicates black holes having unstable/stable thermal phase. Usually, different phases are separated by transition points which could be roots and/or divergencies of the heat capacity. In the case of divergence, the transition between phases is thermodynamical phase transition.

The obtained heat capacity is more complex to calculate its roots and divergencies analytically. Therefore, we use numerical method to investigate the effects of different parameters on heat capacity in Fig. [3]. In order to have a more precise picture, we have distinguished regions with negative temperature as “non-physical” in diagrams.

![Diagram](image_url)

(a) \( q_M = 0.1, \beta = 0.1 \) and \( \Lambda = -1 \) (b) \( q_E = 0.1, \beta = 0.1 \) and \( \Lambda = -1 \) (c) \( q_E = 0.1, q_M = 0.3 \) and \( \Lambda = -1 \) (d) \( q_E = 0.1, q_M = 0.3 \) and \( \beta = 0.1 \).

FIG. 3: Heat capacity versus horizon radius for \( k = 1 \).

Evidently, for highly charged electric (Fig. [3a]) and magnetic (Fig. [3b]) fields and also large nonlinearity parameter (Fig. [3c]), there are two phases available for these black holes: small black holes which are non-physical and large thermally stable ones. These two phases are separated by the root of temperature/heat capacity. The existence of critical behavior is limited by an upper bound coming from nonlinearity parameter, electric and magnetic charges. There are specific critical values for these parameters where black holes would have three phases: small non-physical black holes, medium and large stable ones. The medium and large phases are separated by divergence point which is known as critical point. Therefore, here we have a thermal phase transition between two stable phases. For lightly electrically and magnetically charged cases and small nonlinearity parameter, there will be four phases: very small non-physical black holes, small stable ones, medium unstable ones and large stable ones. The transition between three phases of stable and unstable takes place through divergencies, hence thermal phase transitions. These effective
behaviors are observed for $AdS$ black holes. For $dS$ case (Fig. 3d), there could be four phases available for black holes as well. But here, very small and large black holes are non-physical, medium black holes are unstable and only small black holes are stable. The transition between these two physical phases takes place through a thermal phase transition.

In conclusion, we see that if our black hole solutions move toward situation with small electric and/or magnetic charge, it would critically become active and several phases with specific thermal phase transitions would be observed for them. In context of nonlinearity parameter, small values of it indicates high strength for nonlinear nature of the electromagnetic field. Therefore, we see for high degrees of nonlinearity, system would acquire critical behavior. In contrast, if electromagnetic field behaves like Reissner-Nordström black holes with dyonic charge, the critical points would vanish and thermally stable black holes would be resulted.

C. Extended phase space thermodynamics

Next, we consider the negative branch of cosmological constant to be a thermodynamical quantity, pressure. Such a proposal could be used to show the presence of van der Waals like behavior for black holes [5]. The cosmological constant and pressure are related to each other with following relation

$$\Lambda = -8\pi P.$$  \hspace{1cm} (34)

If we use this relation, we can recognize the obtained temperature (19) as equation of state and find the pressure as

$$P = \frac{4\pi r^4_+ T - k r^2_+ - 2\beta^2 r^4_+ + (2\beta^2 r^4_+ + 2q^2_E) \sqrt{\frac{q^2_M + \beta^2 r^2_+}{q^2_E + \beta^2 r^2_+}}}{8\pi r^4_+}.$$  \hspace{1cm} (35)

To understand the pressure in more details, we find its limiting behaviors as follows

$$\lim_{r_+ \rightarrow 0} P = -\frac{q_M q_E}{2\pi r^2_+} + \frac{k}{8\pi r^2_+} + \frac{T}{2r^4_+} + \frac{(q_E - q_M)^2 \beta^2}{8\pi q_M q_E} + O(r^3_+),$$ \hspace{1cm} (36)

$$\lim_{r_+ \rightarrow \infty} P = \frac{T}{2r^4_+} - \frac{k}{4\pi r^4_+} + \frac{q^2_E + q^2_M \beta^2}{8\pi r^4_+} - \frac{(q_E - q_M)^2}{32\pi \beta^2 r^8_+} + O\left(\frac{1}{r^9_+}\right).$$  \hspace{1cm} (37)

Evidently, for large and small black holes, the pressure would be positive. In contrast, for medium black holes, the pressure could vanish and/or be negative. According to classical thermodynamics, pressure should be positive valued and a decreasing function of the volume. The region where pressure is an increasing function of volume is not accessible for the thermodynamical system and thermal phase transition takes place over it.

The volume could be obtained by extended version of first law black holes thermodynamics given by

$$dM = T dS + V dP + U_E dQ_E + U_M dQ_M,$$  \hspace{1cm} (38)

which gives us the volume as

$$V = \left(\frac{dM}{dP}\right)_{q_M, q_E} = \frac{r^3_+}{3}.$$ \hspace{1cm} (39)

Since the volume and horizon radius are cubically related to each other, one can use the horizon radius instead of volume for investigating thermodynamical properties of black holes. If one wants pressure to be a decreasing function of the volume (horizon radius), the derivation of pressure with respect to horizon radius should be negative ($P' = \frac{dP}{d r_+} < 0$). To have a full picture regarding the effects of different parameters on behavior of the pressure and distinguish physical regions, we have plotted the following diagrams in Fig. 3.

Thermodynamically acceptable behaviors are where $P > 0$ and $P' < 0$. Accordingly, the roots of pressure and $P'$ could be used as starting points to describe (non)physical regions and the effects of different parameters on them.

Evidently, for high temperature, nonlinearity parameter, electric and magnetic charged, black holes’s pressure is positive while $P'$ is negative. Therefore, for any volume (horizon radius), black hole’s pressure has a sound behavior,
hence solutions are physical. In contrast, for small values of these parameters, pressure and $P'$ could acquire first one and then two roots. This indicates that for small temperature, nonlinearity parameter, electric and magnetic charged, black holes develop regions in which pressure could be negative and/or $P'$ is positive. Since these are physically forbidden, one can draw the conclusion that these regions do not admit black hole solutions. In case where both pressure and $P'$ have two roots, the nonphysical region is determined by smaller root of the pressure and larger root of $P'$. Interestingly, when pressure has one root and $P'$ have two roots, the root of the pressure coincide with smaller root of $P'$.

The observed behaviors confirm that: For Maxwell like electromagnetic field (large nonlinearity parameter), there is no bound on the pressure as a function of the volume. In contrast, for highly nonlinear regime, pressures behavior would be bounded by conditions coming from its sign and being a decreasing function of volume. From electric and magnetic charge perspectives, restrictive behavior for pressure is observed for lightly charged cases. The super magnetic and electric charged black holes would have no restriction on their pressures. Surprisingly, the bounded behavior for pressure could be seen for cold black holes while the hot ones prove to have no limitation on their pressure.

It should be noted, one can write the equation of state (35) in form of thermodynamical quantities as

$$ P = \frac{32\pi S^{3/2}T - 4kS - 32\beta^2 S^2}{128\pi S^2} + \frac{(32\pi^2 Q_E^2 + 32\beta^2 S^2) \sqrt{16\pi^2 Q_M^2 + 16q^2 S^2}}{128\pi S^2}, \quad (40) $$

or replace the entropy with volume and find

$$ P = \frac{12\pi TV - 3^{2/3}k V^{2/3} - 6\sqrt{3}\beta^2 V^{4/3}}{24\sqrt{3}\pi V^{4/3}} + \frac{(32\pi^2 Q_E^2 + 6\sqrt{3}\beta^2 V^{4/3}) \sqrt{16\pi^2 Q_M^2 + 3\sqrt{3}\beta^2 V^{4/3}}}{24\sqrt{3}\pi V^{4/3}}, \quad (41) $$

in any case, the discussion regarding the effects of different parameters on thermodynamical behavior of the system would yield results whether one uses Eq. (35) or Eqs. (40) and/or (41).

In next step, we investigate the type of thermal phase transition that these black holes could have. To do so, we use the equation of state (35) and the concept of the inflection point given by

$$ \left( \frac{\partial P}{\partial r_+} \right)_{T,q_M,q_E} = \left( \frac{\partial^2 P}{\partial r_+^2} \right)_{T,q_M,q_E} = 0, \quad (42) $$

where it would yield the following equation for calculating critical horizon radius (volume)

$$ kr_+^2 - 2\left( \frac{q_M^2 + \beta^2 r_+^4}{4\pi r_+^3} \right)^{3/2} \left( 6q_M^2 + 3\beta^2 r_+^4 (q_M^2 + q_M^3) + 9\beta^2 q_M^2 q_M^4 + 6q_M^2 q_M^4 + 3\beta^2 r_+^4 (q_M^2 + 16q_M^2 q_M^3 + q_M^4) \right) \left( q_M^2 + \beta^2 r_+^4 \right)^3 = 0. \quad (43) $$
Due to complexity of this relation, it is not possible to obtain critical horizon radius analytically, therefore, we use numerical method. As an example, we have plotted diagrams in Fig. 5. The presences of subcritical isothermal bars and isobars show that the type of phase transition is van der Waals like. For pressures larger than critical pressure, black holes have uniform phases without any phase transitions. In contrast, when the pressure of the black holes becomes smaller than critical pressure, black holes develop a region where physical behavior is not observed. Therefore, a phase transition would take place between two different volumes (horizon radius) with same pressure (see Fig. 5a). The same could be stated for temperature as well (see Fig. 5b). These are the characteristics of van der Waals like phase transition of liquid-gas.

To understand the critical behavior of the system in more details, we use numerical method to obtain critical horizon radius as a function of different parameters. The results are given in Fig. 6.

The following points could be understood from obtained diagrams:

I) Irrespective of choices for different parameters, for large magnetic charge and nonlinearity parameter, black holes have only one critical horizon radius. This shows that for highly magnetized solutions and Maxwell-like case, there is only one critical point where the usual van der Waals like phase transition would take place.

II) In the absence of nonlinearity parameter (Fig. 6a), the critical horizon radius has an exponential growth as a function of magnetic charge. In contrast, for large nonlinearity parameter, it would be a linear growth. For the medium nonlinearity parameter, there are two specific magnetic charges \((q_{M1} \text{ and } q_{M2})\) where for \(q_{M1} \leq q_M \leq q_{M2}\), there would be two or three critical horizon radii for the same magnetic charge. This indicates that for specific values of different parameters, our critical equations yield two or three distinguishable critical horizon radii, temperatures and pressures. This is a phenomena in thermally critical systems known as the reentrant of phase transition.
III) In the absence of electric charge (Fig. 6b), there is a minimum for magnetic charge ($q_{M_{\text{min}}}$) where for $q_M < q_{M_{\text{min}}}$, there is no critical horizon radius, hence no critical behavior. In contrast, for $q_{M_{\text{min}}} \leq q_M \leq q_M$, there would be two critical horizon radii for each magnetic charge and finally in case of $q_M < q_M$, the critical horizon radius has a linear growth as a function of magnetic charge. For medium electric charge, the critical horizon radius has a cubic growth as a function of magnetic charge. Interestingly, for small and large electric charges, similar to the medium nonlinearity parameter case, there are two specific magnetic charges ($q_{M_1}$ and $q_{M_2}$) where for $q_{M_1} \leq q_M \leq q_{M_2}$, there would be two or three critical horizon radii for the same magnetic charge.

IV) In the absence of magnetic charge (Fig. 6c), there is a minimum for nonlinearity parameter ($\beta_{\text{min}}$) where for $\beta < \beta_{\text{min}}$, there is no critical horizon radius, hence no critical behavior. In contrast, for $\beta_{\text{min}} \leq \beta \leq \beta_f$, there would be two critical horizon radii for each nonlinearity parameter and finally in case of $\beta_f < \beta$, the critical horizon radius has a linear growth as a function of nonlinearity parameter. For small magnetic charge, for each nonlinearity parameter, there is one critical horizon radius. In case of medium magnetic charge, there are two critical horizon radii, ($r_{C_1}$ and $r_{C_2}$) where for $r_{C_1} \leq r_C \leq r_{C_2}$, there are two values of nonlinearity parameter which yield same critical horizon radius. This is the reminiscence of phenomena known as triple point. It should be noted that such values of nonlinear parameter are small, hence they are in high nonlinearity regime. Finally, for large values of the magnetic charge (Fig. 6d), there are two specific nonlinear parameters ($\beta_1$ and $\beta_2$) where for $\beta_1 \leq \beta \leq \beta_2$, there would be two or three critical horizon radii for nonlinear parameter. Here as well, $\beta_1$ and $\beta_2$ are small, therefore, such case takes place in high nonlinearity regime.

In conclusion, we observe that the black holes under consideration here could have a thermodynamical phenomena known as the reentrant of phase transition. As we noticed, such phenomena is taking place in $a)$ high nonlinearity regime, $b)$ large magnetic/electric charge, $c)$ absence of magnetic/electric charge and $d)$ small electric charge. In addition, we observed that black holes could have another thermodynamical phenomena known as triple point. It should be noted that such point was observed in high nonlinearity regime and medium magnetic charge.

IV. REISSNER-NORDSTRÖM LIKE BEHAVIOR

Our final discussion in this paper is reporting on an interesting phenomena observed for these black holes. Through our investigation, we made distinction between electric and magnetic charges, and we did not assume that they would have identical values. If one consider electric and magnetic charge to be identical ($q_M = q_E$), the metric function (11) would reduce to

$$
\psi(r) = k - \frac{m}{r} + \frac{2q^2}{r^2} - \frac{\Lambda r^2}{3}.
$$

Accordingly, if one applies the $q_M = q_E$ to temperature (19) and mass (23), the following temperature and mass would be obtained, respectively

$$
T = \frac{\sqrt{k r^2 + 2 q^2 + \Lambda r^4}}{4 \pi r^3},
$$

$$
M = \frac{3 k r^2 + 6 q^2 - \Lambda r^4}{24 \pi r^3}.
$$

Based on the calculated temperature (45) and entropy (16), the heat capacity would be

$$
C = \frac{-k r^4 + 2 q^2 r^2 + \Lambda r^4}{2 \left( k r^2 + 6 q^2 + \Lambda r^4 \right)},
$$

and pressure is given by

$$
P = \frac{-k r^2 + 2 q^2 + 4 \pi r^3 T}{8 \pi r^4}.
$$

The first noticeable issue is the absences of nonlinear parameter and traces of nonlinearity behavior in metric function and thermodynamical quantities. In fact, the obtained metric function and thermodynamical quantities are almost
identical to those calculated for Reissner-Nordström black holes with a different factor for magnetic/electric charge. Careful examination of the factor of magnetic/electric charge shows that we can break it down into $g_M + g_E = 2g_E$. Therefore, we see that this is indeed Reissner-Nordström with a magnetic charge (linearly charged dyonic solutions). Usually, in order to remove the effects of nonlinearity, one should consider $r \to \infty$ or $\beta \to \infty$. But here, we see that due to presence of the magnetic charge and resultant nonlinear form, we have yet another limit for obtaining the Maxwell like behavior. The main issue is that such limit is independent from nonlinearity parameter and $r$, and it only depends on the amount of magnetic and electric charges. This shows that if magnetic and electric charges become identical in these black holes, although in essence electromagnetic field has nonlinear nature, its nonlinearity would be totally screened by geometrical/thermodynamical structure/properties of the black holes and they would behave like linearly charged dyonic black holes. The effective nonlinear nature of the electromagnetic field becomes evident only when these black holes are charged magnetically and electrically different.

V. CONCLUSION

In this paper, we investigated nonlinearly charged dyonic black holes in the presence of Born-Infeld electromagnetic field. The metric function was obtained and it was shown that these black holes have an irremovable singularity located at the origin and asymptotically $(A)dS$ behavior. The thermodynamical quantities and behavior were obtained and studied. In dyonic black holes with Maxwell field, solutions has electric-magnetic duality. Here, we showed that generalization to Born-Infeld omits such duality. In addition, it was shown that it is possible to cancel the effects of cosmological constant with nonlinearity parameter. In addition, we showed that in highly electric/magnetic charged regimes and Maxwell limits (small nonlinearity), black holes would develop only one thermal stable phase. In contrast, for small electric/magnetic charge and high nonlinearity regime, black holes become critically active and several phases were observed. In studying the pressure, we showed that in case of cold black holes, the pressure is bounded by some constraints on its volume whereas, hot black holes’ pressure has physical behavior for any given volume.

The critical behavior of these black holes showed that generalization to Born-Infeld would enrich phase space of these black hole and introduce phenomena absent for Maxwell case. Among these phenomena, we reported on possibility of existences of triple point, reentrant of phase transition and van der Waals like behavior.

Finally, we showed that if electric and magnetic charges are identical, the effects of the nonlinearity would be completely removed and our solutions would behave like linearly charged dyonic black holes (Maxwell like ones). This indicated that nonlinearity of electromagnetic field has effective behavior which becomes evident only when these black holes are charged magnetically and electrically different.

In the next step, one can employ the obtained results here to investigate the AdS/CFT duality. Specially, one can focus on diamagnetic/paramagnetic behavior on the boundary of the solutions and modification of the Curie law due to nonlinear electromagnetic field. In addition, it would be interesting to see whether it is possible to see the effects of nonlinear electromagnetic field, if electric and magnetic charges are identical, through other properties of the black holes not investigated in this paper.

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