The best-known question in the area of movable/rigid configurations deals with the FCC and HCP configurations of 12 unit balls touching a central unit ball; see Figure 1.

The question is whether one can roll the 12 balls over the central ball in such a way that they cease to touch (or kiss) each other. (The number of kissing points of the outer spheres is 24 for both configurations.) When the answer is positive, we say that the configuration can be unlocked.

The fact that the configuration FCC can be unlocked is well known. One unlocking move is described in [3]. The situation with the HCP cluster is less well known. In [4] it is stated that the configuration HCP is rigid. But such is not in fact the case, and the unlocking moves for both FCC and HCP are presented in [8]. In this paper, we present some quite simple moves that unlock FCC and HCP.

The FCC configuration. The balls of the FCC configuration are centered at the vertices of the cuboctahedron. During the move, the top three balls and the bottom three balls remain fixed. Consider the three “triangles of balls” sharing one ball with the top triangle. (One of them, the triangle BCD, is visible in Figure 2(a).) Now rotate each of these three triangles with the same velocities around their top vertices; e.g., the triple C, B, D is rolled over the central ball, as a solid, with the ball C fixed (Figure 2(a)).

To see that the balls do not hit each other during the move, consider the ball B, which kisses the four balls A, C, D, and E. The triple BCD moves as a solid, so B keeps kissing C and D, and there is no conflict here. The ball B, being rotated around C, goes below the equator, while A, being rotated around F, goes above the equator, with the same equatorial projection of speeds, so the distance between them increases. The balls C, B, and E lie on the same great circle, and neither C nor E moves, so the distance BE also increases.

These considerations suffice due to the rotational symmetry (through the angle $2\pi/3$ around the vertical axis) of the cuboctahedron.

The HCP configuration. The balls of the HCP configuration are centered at the vertices of the Johnson solid called the triangular orthobicupola. They can be seen as three “rhombic” configurations, one of which is $ABCD$ in Figure 2(b). Here $A$, $B$, $C$, $D$ are the centers of the corresponding four balls. The move consists in rolling each of three rhombi around their centers, with equal velocities. That is, one applies to the four balls of each rhombus the rotation around the axis connecting the center of the rhombus (the midpoint of the segment $BD$ for the rhombus $ABCD$) and the origin.
To see the absence of conflicts here, we observe that the balls $E$ and $C$ move downward, with the same horizontal projections of velocities, so their mutual distance increases (since their common distance to the north pole increases). The ball $B$ goes up, while the ball $F$ goes down, so they avoid each other. The rest again follows from the same rotational $\mathbb{Z}_3$ symmetry of the orthobicupola.

In both cases FCC and HCP, the twelve balls can be moved apart after unlocking and positioned at the vertices of the regular icosahedron. Then some free space appears between the unit balls; their radii can be blown up to the value

$$r = \left(\frac{\sqrt{5} + \sqrt{15}}{2} - 1\right)^{-1} \approx 1.10851,$$

at which point they finally begin to kiss, making 30 kissing points. Yet this extra space is not large enough to incorporate the 13th unit ball, as was shown in [14], settling the famous discussion begun by Newton and Gregory.

**Cylinders**

Our interest in the case of cylinders began with a question posed by Włodzimierz Kuperberg [7]. One can easily put six unit cylinders around a central unit ball $B$; see Figure 3(a). The question is whether one can arrange seven such cylinders around $B$ in a nonintersecting way. The question seems an insult to intuition, but to this day we do not have a rigorous answer. One cannot, however, arrange eight unit cylinders around $B$; see [1]. The configuration shown in Figure 3(b) looks quite solid, and though it is not rigid, it looks pretty tight.
So it came as a big surprise that there are configurations of six unit cylinders around the unit ball in which the cylinders do not touch each other. Such configurations were found by Moritz Firsching [5]. In his example, he was able to position six cylinders of radius \( r = 1.049659 \) around the unit ball. This example was obtained by a numerical exploration of the corresponding 18-dimensional configuration manifold.

In our paper [9], we found a way to unlock the configuration \( C_m \) in a symmetric (with respect to the dihedral group \( D_3 \)) manner, which enabled us to improve Firsching’s 1.049659 to the value

\[
r_m = \frac{1}{3} \left( 3 + \sqrt{33} \right) \approx 1.093070331.
\]

(1)

The corresponding \( D_3 \)-symmetric configuration \( C_m \) is shown in Figure 4.

Here is what the unlocking process looks like. Assume that the cylinders on the left in Figure 4 (the configuration \( C_0 \)) point toward the north. We first describe a three-dimensional family of moves. The triple of pink cylinders first go upward by \( \varphi \) and then “horizontally” by \( -\alpha \), and finally, the three vectors \( \uparrow \) are rotated (around the axes joining the origin and the tangent points, counterclockwise if viewed from the tips) by \( \delta \). The three remaining cylinders go downward by \( \varphi \), then “horizontally” by \( \alpha \), and the three vectors \( \uparrow \) are rotated, in the same way as the upper ones, by \( \delta \). Now optimizing \( \alpha \) and \( \delta \) for each \( \varphi \), we obtain a curve \( \gamma \) in the moduli space of six cylinders. For each point on the trajectory \( \gamma \), the configuration is \( D_3 \)-symmetric. Its symmetries are a 120° rotation around the north–south axis and 180° around a perpendicular axis. The common radius of the cylinders grows as the configuration moves along \( \gamma \) up to a certain point that corresponds to the configuration \( C_m \) and then decreases. It turns out that the trajectory \( \gamma \) is an algebraic curve (given by the relations (25)–(27) in [9]), and this is the reason for the exact equality (1). Yoav Kallus made a movie featuring our path [6].

It is interesting to note that all the angles describing the configuration \( C_m \) are pure geodetic, in the sense of [2]: an angle \( \alpha \) is pure geodetic if the square of its sine is rational.

We believe that our configuration \( C_m \) is the record configuration, and in any configuration, the radii of six equal nonintersecting cylinders tangent to the unit ball are less than or equal to \( r_m \). But we have no proof of this maximality. We can prove, however, a local version of this statement: if \( C \) is a small proper perturbation of the configuration \( C_m \), then the radii \( r \) of its cylinders satisfy \( r < r_m \). Here “proper” means that \( C \) is not a rotation of \( C_m \), and “small” means that the six tangent points and six directions of the (equal) cylinders that constitute \( C \) are close to those of \( C_m \).

The proof of this statement can be found in our paper [10]. The proof is quite involved, since the function \( r \) on our 18-dimensional configuration manifold is not smooth. It turns out that the differentials of the distances between the tangent (to the central unit ball) generatrices of the cylinders obey a convex linear dependence \( \Lambda \) at the point \( C_m \) on the curve \( \gamma \).

Let \( E \) be the linear subspace of the tangent space on which all differentials vanish. Then we have proved that a sufficient condition for a local maximum is as follows: the same linear combination \( \Lambda \) of the second differentials is negatively defined on \( E \). We were able to check that this sufficient condition holds at the point \( C_m \) [9].

**Remark 1.** Let \( F \) denote the function \( \max(F_1, \ldots, F_k) \), where \( F_1, \ldots, F_k \) are analytic functions on a vector space, and \( F_i(x_0) = \cdots = F_k(x_0) \) at some point \( x_0 \). The condition that the restriction of \( F \) to any straight line passing through the point \( x_0 \) have a local maximum at \( x_0 \) is insufficient to guarantee that the function \( F \) has a local maximum at the point \( x_0 \). A simple example of just one \( C^\infty \) function \( \Phi \) whose restriction to any analytic path passing through \((0,0) \in \mathbb{R}^2 \) has a local minimum at the origin, while the origin is not a local minimum of \( \Phi \). Consider the function

\[
\psi(x) = \begin{cases}
\exp\left(-\frac{1}{x}\right) & \text{if } x > 0, \\
0 & \text{if } x \leq 0,
\end{cases}
\]

and let \( \eta(x) \) be any \( C^\infty \) function on \( \mathbb{R}^1 \) with support on the segment \([-\frac{1}{2}, \frac{1}{2}]\) that is even, decaying on \([0, \infty)\), and satisfies \( \eta(0) = 1 \). We define the \( C^\infty \) function \( \Phi \) on \( \mathbb{R}^2 \) by

\[
\Phi(x,y) = \begin{cases}
\exp\left(-\frac{1}{x^2}\right)\eta\left(\frac{y-x^2-1}{x}\right) & \text{for } x > 0, \\
0 & \text{for } x \leq 0.
\end{cases}
\]

The function \( \Phi \) is indeed of class \( C^\infty \). The only problematic point is the origin \((0,0)\). Every partial derivative (as \( x \to 0^+ \), \( y \to 0 \)) of \( \Phi \) belongs to the vector space (invariant with respect to partial derivatives) of finite linear combinations of functions \( e^{-x^2} \) with \( x \to 0 \), \( x \to 1 \) for some \( a_1, a_2, a_3, a_4 \in \mathbb{Z}_{\geq 0} \) and hence vanishes.

Let us show that for every analytic path \( \gamma : [0,1] \to \mathbb{R}^2 \), \( \gamma(0) = (0,0) \), the function \( \Phi(\gamma(t)) \) vanishes on some segment \( 0 \leq t < u_f \), so \( 0 \) is a local minimum of the function \( \Phi(\gamma(t)) \) on \([0,1]\), while the point \((0,0) \in \mathbb{R}^2 \) is evidently not a local minimum of the function \( \Phi \).

To see this, note that the support of the function \( \Phi \) lies inside the “beck” \( \beta = \{(x,y) : x \geq 0, \frac{1}{2}\psi(x) \leq y \leq 2\psi(x)\} \subset \mathbb{R}^2 \).

Let \( \gamma(t) = (x(t), y(t)) \) be an analytic path, defined by two analytic functions \( x(t), y(t) \). Then \( x(t) = a_1 t^{k_1} + O\left(t^{k_1+1}\right) \), \( y(t) = a_2 t^{k_2} + O\left(t^{k_2+1}\right) \) for some real \( a_1, a_2 \) and integer \( k_1, k_2 \), and therefore the path \( \gamma \) in the vicinity of the origin is a graph of a function \( y_{\gamma}(x) = bx^{k_1} + o\left(x^{k_1} \right) \) for some real \( b \). (Without loss of generality, we can assume that...
The function $\Phi$ vanishes on that piece of $\Omega$, because the function $\exp\left(-\frac{1}{x}\right)$ is smaller than any power of $x$ in the appropriate vicinity of 0. Therefore, the function $\Phi$ vanishes on that piece of $\gamma$.

Apparently, to probe some $C^\infty$ function, one needs all $C^\infty$ paths, and not just analytic paths.

The local maximality of the configuration $C_m$ implies that it is rigid, i.e., cannot be unlocked. It seems to play the role of the icosahedral configuration of 12 kissing balls above, while the six unit cylinders can be rolled away from each other and create some free space between them on the sphere. Yet whether this space is sufficient for the seventh unit cylinder to be squeezed in (equivalent to the Kuperberg question) is unknown, in contrast to the 13 unit balls problem. The thesis of Osman Yardimci [15] contains the theorem, proven together with Andras Bezdek, that one cannot put seven cylinders of Firsching radius 1.049659 in contact with the unit central ball in a nonintersecting way.

In addition to the configurations in Figure 3, Kuperberg pointed out yet another configuration of six unit cylinders around the unit ball, shown in Figure 5. It looks rigid, and in our paper [11] we have proved this rigidity: for every small proper perturbation of the configuration $O_6$, the radii $r$ of its cylinders satisfy $r < 1$.

In particular, the configuration $O_6$ also cannot be unlocked. The proof turned out to be even more involved. The differentials of the distances between the tangent (to the central unit ball) generatrices of the cylinders satisfy three convex linear dependencies $\Lambda_1, \Lambda_2,$ and $\Lambda_3$. Let $E$ again denote the linear subspace of the tangent space on which all differentials vanish and denote by $q_1, q_2, q_3$ the same linear combinations $\Lambda_1, \Lambda_2,$ and $\Lambda_3$ of the second derivatives. We have checked that the system $q_1(x) > 0,$ $q_2(x) > 0, q_3(x) > 0$ of inequalities has no solution on $E$ and proved that this is a sufficient condition for a local maximum [11, 13]. Interestingly, no convex linear combination of the forms $q_1, q_2, q_3$ is negatively defined on $E$, so we could not just apply techniques such as the Sylvester criterion. We think that it would be a beautiful program to develop the theory (examples, criteria, classification) of tuples $q_1, \ldots, q_6$ of quadratic forms on a vector space for which the system $q_1(x) > 0, \ldots, q_6(x) > 0$ of inequalities has no solution.

Sometimes, this configuration $O_6$ is called octahedral, likely because the points of tangency lie at the vertices of the regular octahedron. We present an interpretation of the configuration $O_6$ that relates it to the configuration of rotated edges of the regular tetrahedron. This interpretation, which shows that it rather deserves to be called the tetrahedral configuration, is as follows. Consider the configuration of lines tangent to the unit sphere that are continuations of the edges of the regular tetrahedron. The points of the sphere through which tangent lines pass are the edge centers of the tetrahedron; see Figure 6.

Then each edge is rotated about the diameter of the unit sphere, passing through the midpoint of the edge, through an angle $\delta$. In Figure 6, the point $A$ (in green) is the midpoint of the edge $UV$. The line passing through the point $A$ and rotated through the angle $\delta$ is shown in red. The lines passing through the other five midpoints of the edges are rotated through the same angle $\delta$, in accordance with the group $H_3$ of the proper symmetries of the tetrahedron. We call this motion the $\delta$-process.

Let us replace each rotated line by a cylinder of radius $r$ tangent to the sphere in such a way that the line is its generatrix and (some of) the cylinders are kissing, which uniquely defines $r$ as a function $r(\delta)$ of the angle $\delta$. Then for $\delta = 0, \pi/2$, the radius $r$ is equal to zero, while it is maximal at $\delta = \pi/4$, with $r(\pi/4) = 1$. This is precisely the configuration $O_6$.

Following this interpretation, we have introduced in [12] configurations of tangent cylinders for the two remaining pairs of dual Platonic solids, that is, for the octahedron/cube ($O/C$) and icosahedron/dodecahedron ($I/D$). For the
The number of cylinders in our configurations is equal to the number of edges of either of the Platonic solids in the pair, that is, twelve for the pair octahedron/cube and thirty for the pair icosahedron/dodecahedron. The corresponding radii \( r_{O/C}, r_{I/D} \) of the cylinders are obtained by maximizing the functions \( r_{O/C}(\delta), r_{I/D}(\delta) \) over the rotation angle \( \delta \).

For the pair \( O/C \), the optimal configuration is as shown in Figure 7. Each cylinder touches four other cylinders. The corresponding value \( \delta_{O/C} \) satisfies \( \tan(\delta_{O/C}) = \frac{3^{1/4}}{\sqrt{2}} \) (approximately, \( \delta_{O/C} \approx 0.23856 \approx 0.74946 \)). The corresponding radius of touching cylinders is

\[
r_{O/C} = \frac{\sqrt{3} - 1}{1 + 2\sqrt{2} - \sqrt{3}} \approx 0.3492.
\]

For the pair \( I/D \), the optimal configuration is as shown in Figure 8. Here each cylinder touches eight other cylinders. The optimal angle \( \delta_{I/D} \) is given by

\[
\delta_{I/D} = \arctan \left( \sqrt{t_0} \right),
\]

where \( t_0 \approx 0.694356 \) is a root of the polynomial

\[
5t^6 - 80t^5 + 190t^4 - 4t^3 - 84t + 9.
\]

Approximately,

\[
\delta_{I/D} \approx 0.694707.
\]

The corresponding radius of cylinders is approximately

\[
r_{I/D} \approx 0.115558.
\]

Another image of this configuration is shown in Figure 9. We conjecture that both record configurations \( O/C \) and \( I/D \) are rigid. This probably can be checked using our machinery developed in [10] and [11], but the computations are quite lengthy, and we have not attempted to perform them.

Figure 8. Maximal \( I/D \) configuration, view from the tip of a 3-fold axis.

Figure 9. Maximal configuration of \( I/D \), view from the tip of a fivefold axis.

Figure 10. Minina of \( I/D \).
Another interesting phenomenon occurs during the $\delta$-process at the angle values at which the function $r_{I/D}(\delta)$ vanishes. This means that some of the 30 lines are intersecting. There are three values of $\delta$ at which this happens. The patterns emerging are ten linked triangles, five linked tetrahedra, and six linked pentagonal stars (Figure 10).

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