Algebraic Quarks from the Tangent Bundle: Methodology

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To Bernd Schmeikal, for his pioneering work
on $SU(3)$ as spacetime symmetry

Abstract. In a previous paper, we developed a table of components of algebraic solutions of a system of equations generated by an inhomogeneous proper-value equation involving Kähler’s total angular momentum. This table looks as if it were a representation of real-life quarks. We did not consider all options for solutions of the system of equations that gave rise to it. We shall not, therefore, claim that the present distribution of those components as a well-ordered table has strict physical relevance. It, however, is of great interest for the purpose of developing methodology, which may then be used for other solutions.

We insert into our present table concepts that parallel those of the phenomenology of high energy physics (HEP): generations, color, flavor, isospin, etc. Breaking then loose from that distribution, we consider simpler alternatives for algebraic “quarks” of primary color (The mathematics speaks of each generation having its own primary color). We use them to show how stoichiometric argument allows one to reach what appear to be esthetically appealing idempotent representation of particles for other than electrons and positrons (Kähler already provided these half a century ago with idempotents similar to our hypothetical quarks). We then use neutron decay to obtain formulas for also hypothetical algebraic neutrinos and the $Z_0$, and use pair annihilation to obtain formulas for gamma particles.

We finally go back to the aforementioned system of equations and start to develop an alternative option. We solve the system of equations for this new option but stop short of studying it along the lines of the present paper. This would thus be an easy entry point to this theory by HEP physicists. Their knowledge of the phenomenology will allow them to go faster and further.
1 Introduction

The present paper becomes the fourth in a series \cite{1}, \cite{2}, \cite{3} on idempotents that enter solutions with symmetry of exterior systems. This was first considered by É. Kähler \cite{4}, but only for only for idempotents involving only scalar-valued differential forms. We have studied Clifford-valued ones \cite{2}, \cite{3}. This permits a formidable enrichment in the theory of such solutions.

In \cite{1}, we reproduced a little geometric calculation by É. Cartan \cite{5}. It shows that modern differential geometry amounts to \textit{only} a theory of moving frames. We solved this problem through a canonical Kaluza-Klein space (KK) where the fifth dimension is propertime. No compactification is involved. Readers who see it difficult to conceive of propertime as a fifth dimension need only accept that a fifth coordinate becomes propertime on curves, where all non-null differential forms are multiples of just one differential 1-form.

Time and propertime remain connected here. The interaction of these two concepts takes place (a) in the product of two time-space-propertime Clifford algebra structures (resulting in Clifford-valued Kähler differential forms in five dimensions), followed by (b) algebra of privileged elements in that product structure and (c) finally, idempotents in space-propertime subspace.

The orthogonality of time-space frames that characterizes Special Relativity (SR) —and more generally the Lorentz transformations (LTs)— is not present in its associated space-propertime, which is to be viewed as the high energy physics (HEP) subspace. As argued in \cite{1}, there is a specific alternative to SR that preserves orthogonality in space-propertime at a most fundamental level, and still involves the LTs at a practical level. The role of these transformations in what might in principle be a non-relativistic context but with length contraction and time dilation has been known to philosophers of science and interested physicists for many decades, in the context of the issue of conventionality of synchronizations. Theirs has been such an extreme view that they have actually claimed that there is no new theory in that alternative, a statement which goes too far. The LTs are inescapable in this “para-Lorentzian” (PL) alternative, but the SR and PL bundles of frames are different, which is of the essence in our argument. Unexpectedly, the space-propertime of that alternative is like the time-space of SR, thus orthogonal. This is not the case when the bundle of spacetime frames is the set of orthonormal frames associated with the LTs.

Because of that structural equivalence of relativistic time-space and PL space-propertime and in order to postpone anything controversial, we devel-
oped mathematical theory in [2] and [3] as if we were dealing with the time-space of SR [We, however, dealt extensively with the physical differences in our paper “U(1) × SU(2) from the tangent bundle” [1]. We connect here with those papers by first giving names (quarks, flavor, color, generations) to concepts that had previously emerged [3]. This name-giving demands that one assigns specific directions in space to the generations, since there has to be a reason why the masses of “alike particles” differ as a function of generation. But consideration of this issue far exceeds the reach of this paper. Suffice to say for illustration that we have in mind the likes of the direction of the velocity with respect to the preferred frame of the center of mass of a system of colliding particles.

The contents of the paper is organized as follows. In section 2, we proceed to discuss the limitations of group theory in physics, with the retrospective view emerging in our study. We similarly ignore that the connections that enter Yang-Mills theory live in auxiliary bundles not directly related to the time-space manifold. Consider $U(1)$ on its own. It does not represent all 1-dimensional groups, but $u(1)$ does. Hence there is no problem in thinking of it as pertaining to time translations. Even less of a problem lies in thinking of $SU(2)$ and, therefore, also of $su(2)$, as pertaining to rotations. But $SU(3)$, or $su(3)$ for that matter, is not directly connected with geometric motions like, say, displacements. If we, however, look at algebraic structure in the manifold of solutions, we find idempotents that look like an algebraic representation of quarks related to symmetry under translations and rotations [2]. Hence, though groups have certainly been very helpful in theoretical physics, they may not be the best road to structure in high energy physics (HEP). The best venue for HEP may lie in algebra of solutions of equations rather than in group theory approach to the symmetry of those equations (See section 2). If we discount antiparticles, which Kähler showed to be a concomitant of time translation symmetry [6], the 3-D Euclidean group of symmetries suffices to get us to an algebraic palette of quarks.

Apart from the work of Kähler on idempotent solutions of his “Kähler-Dirac equation” [4] —to which he referred simply as Dirac equation in spite of its paradigm changing character—a major inspiration for our work is a paper on algebraic quarks by Schmeikal [7]. The part of his work that occupies us in subsection 3.1 is of algebraic nature. But our route to a (different) algebraic representation of quarks is very different from his, as he uses only tangent Clifford algebra. In subsection 3.2, we summarize in simple terms the nature of the structure in which we expand Kähler’s use of idempotents in solutions.
with symmetry of equations.

In section 4, the algebraic results with which we concluded paper [3] are connected with phenomenology. Color and flavor become algebraic concepts directly tied to geometry. In section 5, we show how to combine stoichiometry with particle reaction phenomenology. In doing so, we implicitly make the case that quark stoichiometry may have a great potential for improving present fundamental particle theory, in addition to connecting it to tangent bundle geometry. In section 6, we study solutions with proper value different from zero of the algebraic system that gave rise to the palette of quarks that has occupied us in this paper.

To conclude this introduction, let us bring to the attention of conservative readers that a new perspective on the LTs and concomitant new perspective on flat time-space structure may be a very low price to pay in order to make algebraic progress in high energy-physics (HEP). Hypothetical deviations from the LTs at ultra-relativistic speeds, which is sometimes intimated in the HEP literature, may not be necessary at all for theoretical progress in certain areas of HEP. One simply may need to accept that the LTs are the facade of something deeper. That is no problem; there is no great building without a facade.

2 Precursor Issues

2.1 Clifford-algebra related issues

2.1.1 Background field: a main difference between Dirac and Kähler

Where Dirac uses a tangent structure to deal with quantum physics with electromagnetic coupling, Kähler uses an algebra of scalar-valued differential forms. We shall refer to the latter as Kähler algebra. This leaves the tangent Clifford algebra for use as valuedness algebra. Unlike the Dirac equation, which involves spinors, Kähler’s equation is in first instance about a “background field”. Spinors are solutions with symmetry emerging in that background field, which then is to be considered as primordial. Probability densities must be considered as a derived concept. This is compatible in principle with Einstein’s view of particles as regions of high concentration of the field. But it was naive of Einstein to think that the type of equations he considered would yield particles.
Related to quantum equations being about a non-spinorial field, operators are not in general as fundamental in Kähler’s scheme. Except those for energy and angular momentum, whose relevance is already known from group theoretic arguments, operators also should be considered as emerging concepts. As an example, consider Kähler’s equation with electromagnetic coupling [4], [6]. Momenta and generalized momenta operators emerge while obtaining a classical Hamiltonian with electromagnetic coupling by the expedient process of considering the mass of the electron as the dominant energy, which is introduced in the exponent of a phase factor [8].

Concepts like strangeness, charm, etc. are results of the inventiveness of physicists. They have been created to represent reality, but there is always room for better representations. In what concerns our algebraic representation, our idempotents speak of where the field wants to go but does not quite get because the particles that the ternary idempotents would embody (read quarks) start their collapse in the same process as they building their defining properties.

In Kähler’s calculus (KC), idempotents become the principal factor in the products that constitute solutions with symmetry of equations (See Eq. 11). Those solutions are composed of pieces, the set of which constitutes a highly structured palette, unlike anything that the solutions of the Dirac theory have to offer.

In HEP based on Dirac’s theory, the structure of the set of quarks is reached through phenomenology. In contrast, the “primordial field” of Kähler’s quantum mechanics can materialize into a rich variety of linear combinations of spacetime related idempotents, making superfluous the resorting to internal symmetries and auxiliary bundles. Given the Kähler ascendancy of those idempotents, they should be taken seriously.

2.1.2 Idempotents in the commutative algebra arising from geometrization of units imaginary

Kähler considered the Clifford algebra of scalar-valued differential forms based on the equations
\[ dx^\mu dx^\nu + dx^\nu dx^\mu = 2\eta^{\nu\mu} \]  

(1a)

for the Lorentzian signature. Again, we refer to it as Kähler algebra, regardless of signature and dimension. He also considered generalizations where
the differential forms are tensor-valued. He produced several physical applications for scalar-valued differential forms, but none for more general valuedness.

Clifford-valued clifforms are the elements of the product of two Clifford algebras. One of them is Kähler’s algebra. The other one is its associated tangent Clifford algebra, defined by

\[ a_\mu a_\nu + a_\nu a_\mu = 2 \eta_{\mu\nu}, \]  

specifically

\[ a_0 \cdot a_0 = -1, \quad dt \cdot dt = -1 \]  

and

\[ a_i \cdot a_i = 1, \quad dx^i \cdot dx^i = 1. \]  

We reached expressions of the type \( \varepsilon^\pm I^\pm_{ij} P^\pm_i \), where

\[ I^\pm_{ij} = \frac{1}{2} (1 \pm w_{ij}), \]  

with

\[ w_{ij} = w_k a_i a_j, \quad w_k \equiv dx^{ij} \equiv dx^i dx^j, \]  

and

\[ P^\pm_i = \frac{1}{2} (1 \pm dx^i a_i) \quad \text{(no sum, } i = 1, 2, 3) \]  

and

\[ \varepsilon^\pm = \frac{1}{2} (1 \mp dt a_0). \]  

Notice the absence of the unit imaginary, its role being played on a case by case basis by different elements of square is minus one in the tangent Clifford algebra defined by the first equations (2) and (3).

In the product structure, there is a subset of what we called mirror elements [2]. They play a privileged role. They constitute a commutative algebra under correlated products in the individual factor algebras that make that product structure [1], [2], [3].

For a short history of products of structures similar to those, we go as far back as É. Cartan when he spoke of the curvatures of Euclidean connections as being bivector-valued differential 2–forms [9]. In 1986, Oziewicz proposed a structure which looks like our product of two Clifford algebras [10]. Of special importance is Helmstetter’s work on Clifford algebra, because he not
only has a product of such algebras but also a subset of privileged elements [1]. See [2] for our superficial report of some of his related work. But the total picture still is more sophisticated than what we have just presented, owing to an argument to be summarized in the next section. Briefly speaking, we shall extend these ideas to five dimensions but in such a way that the usual relativistic metric is maintained, reinterpreted as a null metric in five dimensions, and where equations like (1a) and (1b) lose their independence from each other.

2.1.3 Of the metric in Kaluza-Klein space

The concept of metric has evolved with the growth of differential geometry. Metric and distance were in essence the same differential invariant in Riemannian geometry since each of them implies the other. This is not the case in Finsler geometry. It suffices to refibrate a Euclidean bundle as a Finslerian bundle, where the distance simply is $\omega^0$, mod $\omega^m$, with $\omega^\mu = (\omega^0, \omega^m)$. There are infinite metrics ($\sum (\omega^\mu)^2 = g_{\mu\nu}(x, u)dx^\mu dx^\nu$ on the same distance).

Let us now look at the metric from the perspective of the 5-D canonical Kaluza-Klein (KK) space of a spacetime. In [1], we considered the KK space endowed with translation element $d\wp = \omega^\mu e_\mu + d\tau u$, signature $(-1, 1, 1, 1, -1)$ and $0 = d\wp(\cdot, \cdot)d\wp$. We had

$$d\wp(\cdot, \cdot)d\wp = 0 = \eta_{\mu\nu}\omega^\mu \cdot \omega^\nu - d\tau \cdot d\tau + 2(\omega^\mu \cdot d\tau)u \cdot e_\mu.$$ (8)

If we set $\omega^\mu \cdot d\tau = 0$,

$$\omega^0 \cdot \omega^0 - \sum_i \omega^i \cdot \omega^i = d\tau \cdot d\tau.$$ (9)

On curves, all differentials are multiples of just one. We define $\gamma$ by $d\tau = \gamma^{-1}dt$ and obtain

$$\gamma^{-2}dt \cdot dt = dt \cdot dt - \sum_i (u^i)^2 dt \cdot dt.$$ (10)

Hence $\gamma^{-2} = 1 - \sum_i (u^i)^2$. Condition $\omega^\mu \cdot d\tau = 0$, however, is too strong since it implies $0 = \gamma^{-1}dx^i \cdot dt$ and $0 = \gamma^{-1}dt \cdot dt$, which are obviously wrong. The appropriate treatment of the metric in KK contest was given in Section 5 of [1]. In that argument $u^2 = -1$, which was denoted as $w^2 = -1$. In this paper, we reserve the symbol $w$ for other purposes.

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2.2 Group related issues

2.2.1 On role and relevance of group theory in high energy physics

Gauge symmetry has to do with the form of the Dirac and Kähler’s equations. The wave function is a spinor in the first equation, but not in general in the second one. As Kähler’s treatment of symmetry shows [4, 6], the $U(1)$ group and its Lie algebra may not be the best representation of the dynamics of the electromagnetic interaction. In fact, gauge symmetry is mentioned in just one line in Kähler’s most comprehensive papers in spite of the fact that he gets algebraic representations of electrons and positrons.

It is well known that not all 1–dimensional groups are isomorphic, but all 1-dimensional Lie algebras are, since the Lie algebra is about transported tangent spaces. $u(1)$ may, therefore, represent translations. The idempotent (read ideal, read spinor) for time translations is associated with $u(1)$, though not with $U(1)$.

As for $SU(2)$, it is intimately related to rotations, and so is therefore $su(2)$. Like translations, rotations are displacements.

On the other hand, a group like $SU(3)$ does not reveal a geometrical interpretation of any type, nor does $U(1) \times SU(2) \times SU(3)$. But our algebraic representation of quarks based on tangent bundle related geometry does. In a hypothetical future paradigm of the physics, the Lie algebra of the group $U(1) \times SU(2) \times SU(3)$ will still be present in one way another, but it might not be at the fore of its description. A direct product of groups should be a consequence of something more fundamental, not an opportunistic representation of reality.

The point that group theory is a useful technique but no substitute for physics was already made by Giorgi [12]

“I think that group theory, perhaps because it seems to give information for free, has been more misused in contemporary particle physics than any other branch of mathematics, except geometry. Students should learn the difference between physics and mathematics from the start”.

In physics, a group theory argument usually is an insert into some other theory, rather than an organic part of it. If, as we claim, KC is “the calculus for physics”, much of quantum mechanics is simply standard mathematical argument within that calculus, following a few basic physical assumptions.
The claim is substantiated by the depth of results that Kähler himself obtained in using it in relativistic quantum mechanics [4], [6], by our derivation of the electromagnetic Hamiltonian without resort to Foldy-Wouthuysen transformations [8] and by what we are presently doing in connection with quark phenomenology. Of special interest in this regard is his treatment of total angular momentum [6], which we have adapted to our commutative algebra [2], [3]. Readers are asked to compare it with, for instance, the group theoretic treatment by Gilmore [13]. To conclude, group theory is very relevant for physics, when there is nothing better.

2.2.2 Groups, SU(3) symmetry and our road to quarks

In Kähler’s sophisticated mathematical treatment, solutions with symmetry of differential systems are more structured than in the physical paradigm. Kähler gave the form of solutions with time translation and rotational symmetry as

\[ u = p(\rho, z, d\rho, dz) e^{im\phi - iEt} \epsilon^\pm I_{xy}^*. \] (11)

The idempotents \( \epsilon^\pm \) and \( I_{xy}^* \) are precursors of the idempotents with similar notation given in subsubsection 2.1.2.

The relevance of phase factors is well known from the paradigm. With equal or greater reason, idempotent factors share that relevance. But whereas \( m \) and \( E \) in \( e^{im\phi - iEt} \) depend on what equation one is solving and what energy and angular momentum is available, \( \epsilon^\pm I_{xy}^* \) is fixed for corresponding symmetries; one does not even need to know which are the equations that have solutions of that form.

When the symmetries are time translation (or propertime translation, see below) jointly with rotational symmetry, the factor \( p \) in (11) depends only on \( \rho, z, d\rho \) and \( dz \); the dependence on \( dt \) (later \( d\tau \), where \( \tau \) is proper time) and \( d\phi \) (in disguised form, since \( p\rho d\phi = dxdy \)) is taken care of by the idempotent factors, and all the dependence on \( t \) (or \( \tau \)) and \( \phi \) is in the phase factors. Hence a limited form of \( u(1) + su(2) \) is more present in those solutions of Kähler’s equation with electromagnetic coupling than \( u(1) \) is. This presence is embodied in both \( e^{im\phi - iEt} \) and \( \epsilon^\pm I_{xy}^* \).

The statements just made raise the well known issue of whether a tangent bundle symmetry — if this were the case with \( SU(2) \) — can be put together with an auxiliary bundle symmetry, as might be thought to be the case with \( U(1) \times SU(2) \times SU(3) \). Unlike \( SU(2) \), there is not any geometry in \( SU(3) \), whether in the tangent bundle or in an auxiliary bundle. This author submits
that the solution to such quandaries lies in algebra pertaining to solutions, not in group elements pertaining to equations. Those are all the group considerations that we need to make.

Our road to quarks takes place through a commutative algebra associated with space-propertime \((x^i, \tau)\), not \((t, x^i)\) \([1]\). Once propertime translation and rotational symmetry have been jointly implemented into a solution, there is not the option of implementing \(z\)-translation symmetry, due to the fact that \(d\tau\) and \(dz\) do not commute.

In that commutative algebra, \(z\)-translation symmetry has priority over “\(\rho\)-translation” symmetry since, whereas the first one is a Euclidean symmetry, the second one is not. Because of commutativity, there will be room for the \(z\)-translation idempotent, but the corresponding “phase shift” will not be adequate because the square of the geometric factor in the exponent will not be minus one. As a consequence, quarks in this theory are unstable solutions to the point that they die as they try to become; the exponential factor makes quarks in the making overextend themselves through the non-decaying (actually growing with distance, hence the problem) exponential factor. No other mechanism is necessary. In modern physics, one should be careful with whether we are trying to solve a physical problem or one created by the limitations of the theory. Phenomenology tells us nevertheless that they live enough to have a mass ascribed to them.

### 2.2.3 \(U(1) \times SU(2)\) and \(SU(3)\) versus geometry

The physics of the paradigm did not know how many generations to expect. It knows now, but not yet why there are three and only three generations and colors, and why \(e, \mu\) and \(\tau\) are associated with the generations. The algebra that we are advocating has answers for these questions.

In our replacement of scalar-valued with Clifford-valued Kähler algebra, the idempotent for conservation of third component of angular momentum is just one of the three idempotents with which \(SU(2)\) contributes. And the symmetries present in solutions of the form (11) are a truncated form of \(U(1) \times SU(2)\). Since we might have chosen any of the three rotational idempotents, the three generations are already there. Those idempotents commute like everything else in our superseding, commutative algebra.

Concepts like roots and weights are based on Cartan’s subalgebra spanned by the subset of commutative group generators. But the commutative algebra to which we have referred changes all that. Hence we must be aware of
the fact that some limitations of theory may simply be artifacts of a less than adequate mathematical description. Consider, for instance, spin. In section 27 of [6], Kähler obtains a second spin term which may cancel or equate the standard one, thus changing the perspective one may have of the gyromagnetic ratio. He actually emphasizes this difference at the end of his comprehensive 1962 paper [6]. This ratio does not appear in the derivation of the Pauli term from the Kähler equation with electromagnetic coupling [8], and yet one obtains the right equation, and the spectrum of the hydrogen atom is not affected at all [6]. The hyperfine corrections to the gyromagnetic ratio should then be viewed as limitations of theory, rather than involving the unnecessary concept of gyromagnetic ratio.

Group theory does not provide a geometric interpretation for \( SU(3) \), not even in the auxiliary bundles which are the domain of present gauge theory. Thus the concept of group as an organizing concept in high energy physics is inferior ab initio to theory where there is geometric interpretation through algebra for quarks, and where other concepts are superfluous. One should not stop getting help from group theory whenever available, but it should not be considered more than a succedaneum for other, more attractive options. The true understanding of the unity of the non-gravitational interactions may lie less in groups than in algebras and geometry.

In this paper, we might still make reference to \( U(1) \times SU(2) \times SU(3) \), but just as a succedaneum for eliciting the right response from readers familiar with the paradigm’s terminology. But, in reality, \( U(1) \times SU(2) \times SU(3) \) is made esoteric by the naturalness of the alternative: displacements and time translations as symmetries of solutions whose algebraic factor are the aforementioned idempotents.

3 Precursor Works

3.1 Schmeikal’s quarks

Early attempts at relating \( SU(3) \) to Clifford algebra are due to Chisholm and Farwell [15] and [16]. But it was Schmeikal’s relating of idempotents to quarks that inspired the present author’s connecting of Kähler’s solutions of exterior systems with quarks [7]. Schmeikal exploits his virtuosity in algebra. The present author, on the other hand, develops further Kähler’s treatment of symmetry. One more difference is that he is restricted by commutativity
of idempotents. In his algebra, one has to look for it, which is not needed in our commutative structure. We proceed to reproduce some of his findings.

The issue that would interest particle physicists is: show me the representation of quarks, of operators and of their action on quarks. Schmeikal answers as follows. Six color spinor spaces are defined by the pairs \{e_1, e_{24}\}, \{e_1, e_{34}\}, \{e_2, e_{34}\}, \{e_2, e_{14}\}, \{e_3, e_{24}\}, \{e_3, e_{14}\}; the \epsilon_{\mu\nu} being the bivectors \epsilon_{\mu} \wedge \epsilon_{\nu} formed from the orthonormal \{\epsilon_{\mu}\} basis of tangent vectors. The primitive idempotents associated with what he calls the first color space are

\[ f_{11,12} = \frac{1}{2}(1 + e_1)\frac{1}{2}(1 \pm e_{24}), \quad f_{13,14} = \frac{1}{2}(1 - e_1)\frac{1}{2}(1 \pm e_{24}). \quad (12) \]

Notice similarities and differences with Kähler’s primitive idempotents [4]. Primitive idempotents for the other color spinor spaces are similarly constructed.

Schmeikal gives his version of the standard strong interaction operators:

\[ t_z = \frac{1}{4}(e_{24} - e_{124}) \quad (13a) \]
\[ y = \frac{1}{6}(-2e_1 + e_{24} + e_{124}) \quad (13b) \]
\[ q = \frac{1}{6}(-e_1 + 2e_{24} - e_{124}) \quad (13c) \]
\[ s = -f_{12} \quad (13d) \]
\[ b = \rho - s = \frac{1}{12}(3 - e_1 - e_{24} - e_{124}), \quad (13e) \]

where \epsilon_{klm} is a trivector and where \( s \) and \( b \) stand for strangeness and baryon number. He goes on to provide the following scheme where \( |v> \) stands for neutrinos and where \( |s>, |u> \) and \( |d> \) stand for quarks.

To conclude, Chrisholm, Farwell, Schmeikal and this author believe in our own ways that the auxiliary bundles and internal spaces are unnecessary concepts.

### 3.2 Clifford-valued Solutions Endowed with Symmetry

Our approach, like Kähler’s, is based on solutions with symmetry of exterior systems [4]. But we differ from him in that our idempotents are Clifford-valued [11] and that instead of considering just binary idempotents —i.e.
products of just two monary ones as is [4], [6]— we consider ternary idempotents [1], [2], [3].

In solutions with symmetry like (11), the idempotents are rigid, unlike the exponentials, which are flexible, meaning that there is room in principle for different values of the coefficients in the exponents. Although it is not a most pertinent example here for not being quantum mechanical, the standard computation of the Compton effect illustrates that the energies and momenta of the particles emerging from the collision take whatever values are determined by the conservation laws and the input values. Flexibility does not mean irrelevance. Exponential factors may still forbid a reaction. Such is the case with antiproton decay, allowed by crossing of neutron decay, but energetically forbidden. But the nature of the particles in the output of a HEP collision is primarily determined by the idempotents, whose rigidity determines what final products can emerge from the collisions. So, the algebra of idempotents should constitute the first approximation in the study of particle phenomenology.

The third factor in (11), can be any function of the type $p(\rho, d\rho, z, dz)$ needed in each case to make a solution of whatever equation it pertains.

In view of those considerations as to what is fixed in solutions with symmetry, we proceed to discuss idempotents, for they constitute their least flexible factor. We shall have all three options for $I^\pm_{ij}$ and all three options for $P^\pm_m$, in any combination. But we shall have no more than one $I^\pm$ factor and not more than one $P^\pm$ factor in each product, since anything else would not make sense. All these idempotents will have to be accompanied by corresponding geometric phase factors, i.e.

$$e^{m\phi a_i a_j}, e^{E n a_i a_j}, e^{\lambda x^i a_i P^\pm},$$

with no sums over repeated indices. The symbol $u$ is more representative.
than $a_4$ \[1\], but we can ignore that for the moment.

It should be noticed that the exponentials for space-translation symmetries, unlike the one for time-translation symmetry, are not phase factors by virtue of the fact that the square of $a_i$ is not minus one. Time translation should thus have pre-eminence over the other translation symmetries. Hence, the difference in character between the exponential factors for space and propertime translations may be the reason why quarks do not behave like normal particles.

In HEP phenomenology, operators do not follow the same pattern in the electron generation as in the other generations. See, for instance, the $I_3$ of the Gellmann-Nishijima equation, where, in addition, there are too many operators. To make matters still worse, components are not invariants. What should matter is that hadrons should be viewed as composites with proper value of total angular momentum (or something along those lines) and not that individual quarks have proper valued different from zero for one of many operators, and zero for all other operators.

4 A Provisional Palette of Algebraic Quarks

4.1 Source palette of ternary idempotents

In paper \[3\], we derived the following table:

| $u/d$ | Subscript 1 | Subscript 2 | Subscript 3 |
|-------|-------------|-------------|-------------|
| $u^3$ | $\epsilon^+ I_{12}^+ P_1^+$ | $\epsilon^+ I_{12}^+ P_1^-$ | $-\epsilon^+ I_{12}^+$ |
| $d^3$ | $\epsilon^+ I_{12}^- P_2^+$ | $\epsilon^+ I_{12}^- P_2^-$ | $-\epsilon^+ I_{12}^-$ |
| $\bar{d}^3$ | $\epsilon^+ I_{12}^+ P_2^+$ | $\epsilon^+ I_{12}^+ P_2^-$ | $-\epsilon^+ I_{12}^+$ |
| $\pi^3$ | $\epsilon^- I_{12}^- P_1^+$ | $\epsilon^- I_{12}^- P_1^-$ | $-\epsilon^- I_{12}^-$ |

The superscript 3 in the first column correlates with the absence of the subscript 3 in the other three columns. This table is for the generation of the electron, proton and neutron, precisely because of that superscript. In historical retrospect, one should have used first component rather than third component of angular momentum in dealing with the electromagnetic interaction. In that way that generation would have been associated with $I_{23}$ and the superscript 1 of $u$, rather than with $I_{12}^+$ and the superscript 3.
We also indicated the obvious way to make parallel tables for superscripts 1 and 2 of $u$ and $d$. We now proceed to put all those tables together and make the idempotents correspond to actual quarks, but only as an illustrative example of the methodology. The palette of quarks of Table 3 follows.

Table 3. Provisional Palette of Quarks

|        | Color 1       | Color 2       | Color 3       |
|--------|---------------|---------------|---------------|
| $t = u^1$ | $-e^+ I_{23}^+$ & $e^+ I_{23}^+ P^+_2$ & $e^+ I_{23}^+ P^-_2$ |
| $b = d^1$ | $-e^+ I_{23}^-$ & $e^+ I_{23}^- P^-_3$ & $e^+ I_{23}^- P^+_3$ |
| $\bar{b} = d^1$ | $-e^- I_{23}^+$ & $e^- I_{23}^+ P^+_2$ & $e^- I_{23}^+ P^-_2$ |
| $t = \bar{u}^1$ | $-e^- I_{23}^-$ & $e^- I_{23}^- P^-_3$ & $e^- I_{23}^- P^+_3$ |
| $c = u^2$ | $e^+ I_{31}^+ P^-_3$ & $-e^+ I_{31}^+ P^+_3$ & $e^+ I_{31}^+ P^-_3$ |
| $s = d^2$ | $e^+ I_{31}^- P^+_1$ & $-e^+ I_{31}^- P^-_1$ & $e^+ I_{31}^- P^+_1$ |
| $\bar{c} = u^2$ | $e^- I_{31}^+ P^+_1$ & $-e^- I_{31}^+ P^-_1$ & $e^- I_{31}^+ P^+_1$ |
| $u = u^3$ | $e^+ I_{12}^+ P^+_1$ & $e^+ I_{12}^+ P^-_1$ & $-e^+ I_{12}^+ P^+_1$ |
| $d = d^3$ | $e^+ I_{12}^- P^+_2$ & $e^+ I_{12}^- P^-_2$ & $-e^+ I_{12}^- P^+_2$ |
| $\bar{d} = d^3$ | $e^- I_{12}^+ P^+_2$ & $e^- I_{12}^+ P^-_2$ & $-e^- I_{12}^+ P^+_2$ |
| $\bar{u} = \bar{u}^3$ | $-e^- I_{12}^+ P^-_1$ & $-e^- I_{12}^+ P^+_1$ & $-e^- I_{12}^+ P^-_1$ |

The algebraic interpretation of quarks as idempotents speaks clearly of why there are three and only three generations, and two families per generation. This might be thought to be a peculiarity of the solution that we are using, but we shall see in section (6) that the alternative solution fits a similar pattern. The relation of the number of colors to the dimensionality of the 3-D configuration space is less obvious. We shall make clear in the next subsection that we can make $P_1$, $P_2$ and $P_3$ appear explicitly in columns for colors 1, 2 and 3 respectively.

### 4.2 Algebraic color

In table 2, the subscript of $P$ does not always correspond to the heading of the column where the idempotents are placed. Yet color still is closely related to that subscript, as we now explain.
In each generation use the term primary color to refer to the one (among
colors \( k = 1, 2, 3 \)) which is different from the subscripts \( i \) and \( j \) in \( I_{ij} \). It is
thus 1, 2 and 3 for the respective generations of \((t, b)\), \((c, s)\) and \((u, d)\). The
other two colors will be called secondary. The following equalities apply to
the secondary colors of the different generations:

\[
e^\pm I_{ij}^k P_i^\pm = e^\pm I_{ij}^j P_j^\pm, \quad e^\pm I_{ij}^- P_i^\mp = e^\pm I_{ij}^- P_j^\mp.
\] (15)

Notice the inversion of superscripts of \( P \) that accompanies the change in its
subscript in the second of those equations.

The expression in this table of hypothetical algebraic quarks of primary
color resulted from the sum 
\[
e^\pm I_{ij}^k P_i^\pm + e^\pm I_{ij}^j P_j^\pm.
\]
We, however, wrote \( e^\pm I_{ij}^l \) and \( e^\pm I_{ij}^- \) instead of \( e^\pm I_{ij}^k \) and \( e^\pm I_{ij}^- \) in order to signify that, in principle,
\( e^\pm I_{ij}^k \) and \( e^\pm I_{ij}^- \) will multiply different factors and, therefore, the sum
is not justified except as an approximation which might work in certain cir-
cumstances.

In view of these considerations, the row for, say, quark \( c \) can now be
written

\[
e^\pm I_{31}^k P_1^\pm, \quad -(e^\pm I_{31}^k P_2^\pm \oplus e^+ I_{31}^k P_3^-), \quad e^\pm I_{31}^k P_3^+,
\] (16)

the symbol \( \oplus \) being chosen simply to remind ourselves of the remark just
made about the lack of justification of adding idempotents indiscriminately.
Of course, more serious is the remark already made that this is not at all the
solution that we should be considering, were it not for the specific purpose of
demonstrating the methodology regardless of specific claim as to the solution
to which quarks would correspond.

One would assume that the notation for secondary colors illustrated in
(16) is better than the one in table 3. But this may not be so, as we now explain.

We compare

\[
\begin{array}{ccc}
  u_{1,2} & 1 & 2 \\
  u & e^+ I_{12}^1 P_1^+ & e^+ I_{12}^2 P_1^- \\
  \bar{u} & e^- I_{12}^1 P_1^- & e^- I_{12}^2 P_1^+
\end{array}
\]

with

\[
\begin{array}{ccc}
  u_{1,2} & 1 & 2 \\
  u & e^+ I_{12}^1 P_2^+ & e^+ I_{12}^2 P_2^- \\
  \bar{u} & e^- I_{12}^1 P_2^- & e^- I_{12}^2 P_2^+
\end{array}
\]

Because of (15), the two \( u \) rows coincide, but the \( \bar{u} \) rows do not. The question
then arises of which of these two equivalent representations of \( u_{1,2} \) should
we go by for the purpose of obtaining antiparticles, sure enough through reversion of the superscripts. In the second representation $\bar{u}_1$ and $\bar{u}_2$ come out equal to each other by virtue of (15)). We thus disregard this option.

The sum $u_1 + u_2 + u_3$ yields zero, ignoring again exponential factors. But $u_1 + u_2 + d_3$ yields $e^+ dx^{12}$ under the same circumstances. This would lead us to consider proper values of total operators for $u_1 + u_2 + d_3$ angular momenta, if we had to consider this palette seriously.

### 4.3 Algebraic flavor

The charge of quarks and their composites is governed by the Gell-Mann-Nishijima formula. But this formula should be considered as having limitations typical of phenomenological formulas. One such limitation is the profusion of concepts, like the potentially unnecessary large number of operators. In order to establish notation, we write the formula as

$$Q = \frac{B}{2} + Y = \frac{B}{2} + I_3 + \frac{C + S + T + B'}{2},$$

(17)

the meaning of each symbol then being easily inferred.

The asymmetry between generations is obvious in this formula. In order to close the gap between phenomenology and algebra, we replace the operator $I_3$ of the paradigm with $I_3$ defined as

$$I_3 = \frac{I_{xy}^+ - I_{xy}^-}{2},$$

(18)

If we define operators $U$ and $D$ by

$$U \equiv I_{xy}^+, \quad D \equiv -I_{xy}^-,$$

(19)

we then have

$$\frac{U + D}{2} = \frac{dx dy}{2} = I_3.$$

(20)

and momentarily view the Gell-Mann-Nishijima equation as

$$Q = \frac{B}{2} + \frac{U + D + C + S + T + B'}{2}.$$

(21)

The proper values of $u$ under the action of $U$ and $D$ are 1 and 0, and those for $d$ are 0 and $-1$. 

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Now that all flavors are in the same footing, we proceed to view $C$, $S$, $T$ and $B'$ algebraically:

$$C \equiv I_{yz}^+, \quad S \equiv -I_{yz}^-, \quad T \equiv I_{yz}^+, \quad B \equiv -I_{yz}^-.$$ (22)

and, therefore,

$$\frac{C + S}{2} \equiv \frac{I_{yz}^+ - I_{yz}^-}{2} = \frac{dydz}{2} = I_1, \quad \frac{T + B'}{2} \equiv \frac{I_{xx}^+ - I_{xx}^-}{2} = \frac{dxdy}{2} = I_2.$$ (23)

When dealing with ternary idempotents, we shall refer to the $I_{ij}^\pm$ as isospin idempotents, and the $dx^{ij}$ as isospin operators. It is certainly the case that $u_3$ and $d_3$ are proper functions of $dx^{12}$ with proper values $\pm 1$. But the quarks of secondary colors in the nucleons generation are not proper functions of this operator.

In HEP phenomenology, operators are created and proper values are assigned ad hoc. The specific proper values for so many operators — $B$, $U$, $D$, $C$, $S$, $T$ and $B'$ — makes it unlikely that an algebraic representation for them exists. Replacement of equations (19) and (22) in (21) is not meaningful. Hence, in this paper, we shall content ourselves with a hybrid treatment lying between phenomenology and what should be an algebraic representation. We thus build the algebraic table for the generation of electrons and nucleons, the operators acting only on the primary color. With $B$ as $-d\tau/3$, we have the following scheme of proper values:

| Quarks | $-d\tau/3$ | $dx^{ij}y$ | $B/2$ | $I_3$ | $B/2 + I_3$ |
|--------|------------|-------------|-------|-------|-------------|
| $u_3$  | $\epsilon^+ I_{12}^\tau$ | 1/3 | 1 | 1/6 | 1/2 | 2/3 |
| $d_3$  | $\epsilon^+ I_{12}^-\epsilon^-$ | 1/3 | -1 | 1/6 | -1/2 | -1/3 |
| $d_3$  | $\epsilon^- I_{12}^\tau$ | -1/3 | 1 | -1/6 | 1/2 | 1/3 |
| $u_3$  | $\epsilon^- I_{12}^-\epsilon^-$ | -1/3 | -1 | -1/6 | -1/2 | -2/3 |

Recall that the subscripts of $u$ and $d$ are for color. Their superscripts have been omitted. For other generations we would replace $dx^{ij}y$ by cyclic permutation. We have connected with the phenomenology of the paradigm, but much remains to be done.

In the second column, we left out the factors $P$ because they do not influence the action of those operators. So, the different colors are not represented. There is nothing wrong with that. But, since the $\epsilon^\pm I_{12}^\tau$ by themselves
represent leptons and the charges than one gets are not those of leptons, one has to conclude that the GellMann-Nishijima equation does not represent any deep truth of nature but a construction that puts together ad hoc operators.

5 Stoichiometry

Stoichiometry may be used to explain reactions or, with the help of phenomenology, find algebraic representations of different types of particles. There is a very large number of simple reactions that may help determine the formula for all particles. We start with a few ones. Readers need to be aware that there are steps where a little bit of guess work is involved, specially but not only because dealing only with idempotents is only a first step towards a more thorough treatment. We are thus illustrating possibilities.

We start with neutron decay. It is interpreted as a $d$ going into a $u$. One wonders whether the other $d$ and the $u$ of the neutron just stand by while the decay happens. When we let color enter the representation of the decay, a more sophisticated process takes place, as we now show.

5.1 Decay of $d$ quarks of secondary color

Since the odd man out in the proton and neutron respectively are the $d$ and the $u$ quarks, we reserve for these quarks the subscript 3, i.e. the label of their generation. Thus, a neutron will be given by $(d_1, d_2, u_3)$ and the proton by $(u_1, u_2, d_3)$. The decay of just a $d$ into a $u$ does not explain neutron decay, for we would then have, say, $(u_1, d_2, u_3)$ instead of $(u_1, u_2, d_3)$. That is simply ad hoc interpretation when one does not take color into account.

We proceed by reverse engineering to address the issue of a $d$ quark going to a $u$ quark. We start with $u_1$ from which we want to reach some $d$. A first step is motivated by the fact that neutron decay generates an electron, whose representation was amply discussed in [1]. We get:

\[ u_1 = \varepsilon^+ I_{12}^+ P_1^+ = I_{12}^+ P_1^+ - \varepsilon^- I_{12}^+ P_1^- = I_{12}^+ P_1^+ - \varepsilon^- I_{12}^+ + \varepsilon^- I_{12}^+ P_1^- . \] (24)

$\varepsilon^- I_{12}^+$ is an electron with spin/chirality “plus”, which we thus move to the left hand side to accompany $u_1$:

\[ u_1 + \varepsilon^- I_{12}^+ = I_{12}^+ P_1^+ + \varepsilon^- I_{12}^+ P_1^- = \]

\[ = I_{12}^+ P_1^+ + I_{12}^+ P_1^- - \varepsilon^+ I_{12}^- P_1^+ = I_{12}^+ - \varepsilon^+ I_{12}^- P_1^- . \] (25)
The last term will now be decomposed so that the minus sign at the front of the ternary idempotent will go into a plus sign, while remaining on the same side of the equation,

$$-\varepsilon^+ I_{12}^+ P_1^- = -\varepsilon^+ P_1^- + \varepsilon^+ I_{12}^- P_1^-. \quad (26)$$

Since $\varepsilon^+ I_{12}^- P_1^-$ equals $\varepsilon^+ I_{12}^- P_2^+$, we read from our palette of quarks that this is $d_1$. Combining equations (21)-(23), one gets

$$d_1 = u_1 + \varepsilon^- I_{12}^+ + \varepsilon^+ P_1^- - I_{12}^+. \quad (27)$$

We similarly obtain

$$d_2 = u_2 + \varepsilon^- I_{12}^+ + \varepsilon^+ P_1^- - I_{12}^+. \quad (28)$$

Thus the $u_3$ of the neutron and the $d_3$ of the proton should exchange places in neutron decay. So, in this mathematically induced HEP-like theory, the decay of the neutron is a collective phenomenon, as it involves all three quarks of each nucleon.

### 5.2 Decay of the neutron

Addition of (24) and (25) yields

$$d_1 + d_2 = u_1 + u_2 + ^+ e + \varepsilon^- I_{12}^+ + \varepsilon^+ I_{12}^- - I_{12}^+. \quad (29)$$

where we have chosen to write one of the two $\varepsilon^- I_{12}^+$ terms as $^+ e$ to signify an electron with a defined chirality (left superscript +). We now add $u_3$ to both sides of (29). We also add zero in the form $d_3 - d_3$ on the right hand side. We thus get

$$d_1 + d_2 + u_3 = (u_1 + u_2 + d_3) + ^+ e + [u_3 - d_3 + \varepsilon^- I_{12}^+ - I_{12}^- - I_{12}^+]. \quad (30)$$

Let us name the six terms in the square bracket as (1) to (6) in order of appearance. We have

$$\begin{align*}
(4) + (5) &= -\varepsilon^+ I_{12}^+; \\
(3) + (4) + (5) &= \varepsilon^+ I_{12}^-; \\
(3) + ... + (6) &= \varepsilon^+ I_{12}^- - I_{12}^+.
\end{align*} \quad (31, 32, 33)$$
With N = neutron, P = proton (We reserve p for positrons), Eq. (30) can be written as

$$N = P + e^- + e^+ I_{12}^+ + I_{12}^- - I_{12}^+, \quad (34)$$

after replacing $u_3$ and $d_3$ with their algebraic expressions. Hence, if we ignore the exponential factors hidden in $\pm I_{12}^+$, this yields

$$\bar{\nu}_e = -\epsilon^+ I_{12}^+ + \epsilon^+ I_{12}^- - I_{12}^+ - (\epsilon^+ - 1) I_{12}^+. \quad (35)$$

This is a highly dubious result. We shall not discuss why, given the caveats already mentioned about its lack of relevance except for illustration of the methodology.

5.3 “Neutrinos” and generation-independent formulas

We now assume –just for illustrative purposes with the same generation— that we had obtained for the quarks $u$ and $d$ of primary color the expressions $\epsilon^+ I_{12}^+ P_3^+$ and $\epsilon^+ I_{12}^- P_3^-$ respectively. For further illustration, we shall study all four combinations of signs and compare results. Instead of the last four terms on the right hand side of (29), we would have the following four combinations:

$$\epsilon^+ I_{12}^+ P_3^+ - \epsilon^+ I_{12}^- P_3^+ + \epsilon^+ I_{12}^- - I_{12}^+. \quad (36)$$

The asterisk as a superscript signifies that the sign in $P_3^*$ is independent of what sign we have in the superscript of $P_3$ in the first term. Let us compute for the four combinations of signs,

$$u_3 - d_3 = \epsilon^+ I_{12}^+ P_3^+ - \epsilon^+ I_{12}^- P_3^+. \quad (37)$$

We so have the following four options

$$\epsilon^+ I_{12}^+ P_3^+ - \epsilon^+ I_{12}^- P_3^+ = \epsilon^+ P_3^+ dx dy. \quad (38)$$

$$\epsilon^+ I_{12}^+ P_3^- - \epsilon^+ I_{12}^- P_3^- = \epsilon^+ P_3^- dx dy. \quad (39)$$

$$\epsilon^+ I_{12}^+ P_3^+ - \epsilon^+ I_{12}^- P_3^- = \epsilon^+ P_3^+ (I_{12}^+ + I_{12}^-) - \epsilon^+ I_{12}^+ = \epsilon^+ P_3^+ - \epsilon^+ I_{12}^+. \quad (40)$$

$$\epsilon^+ I_{12}^+ P_3^- - \epsilon^+ I_{12}^- P_3^+ = \epsilon^+ I_{12}^- P_3^- + \epsilon^+ I_{12}^- (P_3^- - 1) = \epsilon^+ P_3^- - \epsilon^+ I_{12}^-. \quad (41)$$

Replacement of the pair (38)-(39) in (36) yields

$$\epsilon^+ P_3^+ dx dy + \epsilon^+ I_{12}^- - I_{12}^+ = \epsilon^+ \frac{1}{2}(1 \pm dx dy dz) - I_{12}^+, \quad (42)$$

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whose right hand side should thus be identified with $\bar{\nu}_e$ on account of (29) and well known phenomenology. If we define

$$w^\pm \equiv \frac{1}{2}(1 \pm dx dy dz),$$

we have

$$\bar{\nu}_e = e^+ w^+ - I^+_{12}, \quad \nu_e = e^- w^- - I^-_{12}. \quad (44)$$

The generation independent equation

$$Z^0 = \nu_e + \bar{\nu}_e = w^+ - 1 = -(\frac{1}{2} \pm w), \quad (45)$$

follows. These two formulas for $Z_0$ correspond to

$$(u_3, d_3) = (e^+ I^+_{12} P^+_3, e^+ I^-_{12} P^+_3) \quad \text{and} \quad (u_3, d_3) = (e^+ I^+_{12} P^-_3, e^+ I^-_{12} P^-_3) \quad (46)$$

Using now the options (37)-(38), we get, by similar process,

$$\bar{\nu}_e = e^+ P^+_3 - I^+_{12}, \quad \nu_e = e^- P^-_3 - I^-_{12}, \quad (47)$$

$$Z^0 = \nu_e + \bar{\nu}_e = e^+ P^+_3 + e^- P^-_3 - 1 = -(\frac{1}{2} \pm dz d\tau), \quad (48)$$

and

$$(u_3, d_3) = (e^+ I^+_{12} P^+_3, e^+ I^-_{12} P^-_3) \quad \text{and} \quad (u_3, d_3) = (e^+ I^+_{12} P^-_3, e^+ I^-_{12} P^+_3). \quad (49)$$

By virtue of the fact that $Z^0$ can decay by similar channels of different generations, this option would not look as attractive as the previous one. But we still would not discard it (if this were the real thing) since it might represent other viable channels. The phenomenology is very rich and could help sort out these matters until the theory reaches the point that it can stand on its own.

### 5.4 Introduction to Photons

This subsection is independent of any specific representation of quarks. The previous manipulations may help readers feel more comfortable than if introduced at an earlier stage.
Consider the decay of electron-positron pairs into two photons. This corresponds to the combinations $\epsilon^{-} I_{12}^{+} + \epsilon^{+} I_{12}^{-}$ and $\epsilon^{-} I_{12}^{-} + \epsilon^{+} I_{12}^{+}$. One readily gets

$$\epsilon^{-} I_{12}^{+} + \epsilon^{+} I_{12}^{-} = \epsilon^{-}(I_{12}^{+} - I_{12}^{-}) + I_{12}^{-} = dxdy(\epsilon^{-} - \frac{1}{2}) + \frac{1}{2} = \frac{1}{2}(1 + d\tau dxdy), \quad (50)$$

$$\epsilon^{-} I_{12}^{-} + \epsilon^{+} I_{12}^{+} = \epsilon^{-}(I_{12}^{-} - I_{12}^{+}) + I_{12}^{+} = -dxdy(\epsilon^{-} - \frac{1}{2}) + \frac{1}{2} = \frac{1}{2}(1 - d\tau dxdy), \quad (51)$$

Solutions for the photons corresponding to (50) appear to be

$$\gamma_1 = \frac{1}{2}(1 + dx)\frac{1}{2}(1 + d\tau ddy), \quad \gamma_2 = \frac{1}{2}(1 - dx)\frac{1}{2}(1 - d\tau ddy), \quad (52)$$

and, for (51):

$$\gamma_1 = \frac{1}{2}(1 + dx)\frac{1}{2}(1 - d\tau ddy), \quad \gamma_2 = \frac{1}{2}(1 - dx)\frac{1}{2}(1 + d\tau ddy). \quad (53)$$

All these formulas correspond to defined helicities. It is then clear that linear polarizations must correspond to combinations of the two helicities. We do not enter into the subtle issue of opposite directions of the photons. Let us just intimate the following. These are space-propertime configurations of the field pertaining to photons. They would be used in reactions representative of the exchange of a photon. This is a tricky issue. We have avoided issues like this because it involves taking risks, whether one is right or wrong. This author believes that it is more productive and less controversial to obtain results that are close to the mathematics, minimizing to any possible extent controversial issues.

Let us make, however, a remark of a general nature about trajectories. A trajectory is a curve, thus a 1-dimensional manifold, in time-space. All differential 1-forms must be multiples of just one, say $dt$. The $dx^i$ in $dx$, $dy$ and $dz$ become multiples of $dt$ through the natural lifting conditions $dx^i - u^i dt = 0$. But the treatment of the relation between time-space and space-propertime equations is not obvious when anything going at the speed of light is involved. We leave this problem for a future paper. These two options differ by a reversal of the sign of the superscript of the P’s. We do not know at this point the implications of this difference.

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6 Path to the Palette of Algebraic Quarks

The present paper and the previous one [3] has its origin in the study of proper values of what we have called total operators. We use this term to refer to products of operators where one of them is total angular momentum. Total is used here in the sense not only that spin is included, but specially in the sense that all three components enter the operator at the same time and on an equal footing. Total operators do not have as proper functions the ternary idempotents that are proper functions of components of angular momentum [3]. In that study [3], one had to deal with combinations of ternary idempotents, and where the equations not only had a proper value term on the right hand side, but also a “covalue” term. Those linear combinations are the sums of the lines in table 3. The terms in each line add up to zero, but they will certainly not when multiplied by factors, different from one term to another. Having said that, let us stay at the level of idempotents, disregarding those factors. There is much to learn from them. The terms that constituted those “zero sums splits” led us to the considerations of the previous two sections. They are full of HEP flavor, in more than one sense of the term.

6.1 Zero Proper Values for Total Operators

Recall the equation what we had in [3]:

\[ [(K + 1)dr]X_A = \mu'X_A + \pi_A, \]  

(54)

where \((K + 1)\) is Kähler’s total angular momentum operator. The \(dr\) is the usual differential element of translation. We labeled the ternary idempotents as \(X_A\). We tried to make sums of \(X_A\)’s belonging to the same generation and proper value \(\mu'\), allowing for the extra terms \(\pi_A\). This rather than being an equation is an implicit definition of terms. \(\pi_A\) will not be a number in general. We tried to make linear combinations of \(X_A\)’s such that the sum of the \(\pi_A\)’s for suitable \(\mu'\) would are numbers.

We defined

\[ \mu := -\frac{\mu'}{4} \]  

(55)

so that Eq. (54) could be given the form

\[ [(K + 1)dr + 4\mu]X_A = \pi_A. \]  

(56)
The factor $-1/4$ in (56) was chosen to minimize clutter.

We formed linear combinations $\Sigma_A \lambda_A X_A$. We then proceeded to compute

$$\lambda_A[(K + 1)dr + 4\mu]X_A$$

(57)

for the 8 ternary idempotents of the generation (excluding antiquarks) and summed all of that up:

$$\sum_A \lambda_A[(K + 1)dr + 4\mu]X_A = \sum_A \pi_A,$$

(58)

Hence $\sum \pi_A$ became the covalue of the equation

$$[(K + 1)dr][\sum_1^8 \lambda_A X_A] = \mu' \sum \lambda_A \chi_A + \pi,$$

(59)

where $\pi$ is $\sum \pi_A$.

We got the following equations

$$(\lambda_1 + \lambda_2) + (\lambda_5 + \lambda_6) + \mu[(\lambda_1 - \lambda_2) + (\lambda_3 - \lambda_4)] = 0,$$

(60)

$$(\lambda_1 + \lambda_2) + (\lambda_5 + \lambda_6) + \mu[(\lambda_1 - \lambda_2) - (\lambda_3 - \lambda_4)] = 0,$$

(61)

or the simpler equivalent pair

$$\lambda_4 = \lambda_3$$

(62)

$$(\lambda_1 + \lambda_2) + (\lambda_5 + \lambda_6) + \mu(\lambda_1 - \lambda_2) = 0.$$  

(63)

We also got the equations

$$[(\lambda_1 - \lambda_2) + (\lambda_3 - \lambda_4)] + (\lambda_5 - \lambda_6) = 0,$$

(64)

and

$$\frac{1}{2}[(\lambda_1 - \lambda_2) - (\lambda_3 - \lambda_4)] + (\lambda_5 - \lambda_6) = 0.$$  

(65)

In view of (62), they both became

$$(\lambda_1 - \lambda_2) + 2(\lambda_5 - \lambda_6) = 0.$$  

(66)

We then obtained

$$(\lambda_1 + \lambda_2) - (\lambda_3 + \lambda_4) + (\lambda_5 + \lambda_6) - (\lambda_7 + \lambda_8) + 2\mu[(\lambda_5 - \lambda_6) - (\lambda_7 - \lambda_8)] = 0,$$

(67)
\[(\lambda_1 + \lambda_2) + (\lambda_3 + \lambda_4) + (\lambda_5 + \lambda_6) + (\lambda_7 + \lambda_8) + 2\mu[(\lambda_5 - \lambda_6) + (\lambda_7 - \lambda_8)] = 0. \] (68)

Adding and subtracting (67) and (68), we got
\[(\lambda_1 + \lambda_2) + (\lambda_5 + \lambda_6) + 2\mu(\lambda_5 - \lambda_6) = 0 \quad (69)\]
\[(\lambda_3 + \lambda_4) + (\lambda_7 + \lambda_8) + 2\mu(\lambda_7 - \lambda_8) = 0. \quad (70)\]

Finally, we also got
\[\begin{align*}
(\lambda_1 - \lambda_2) + \frac{1}{2}(\lambda_5 - \lambda_6) - \frac{1}{2}(\lambda_7 - \lambda_8) + \\
\mu[(\lambda_1 + \lambda_2) - (\lambda_3 + \lambda_4) + (\lambda_5 + \lambda_6) - (\lambda_7 + \lambda_8)] &= 0. \quad (71)
\end{align*}\]

We proceeded to solve this system of equations. It follows from (63) and (69) that
\[\mu(\lambda_1 - \lambda_2) - 2\mu(\lambda_5 - \lambda_6) = 0. \quad (72)\]

Hence, either \(\mu = 0\) or
\[\lambda_1 - \lambda_2 = 2(\lambda_5 - \lambda_6). \quad (73)\]

In [3], we solved the option \(\mu = 0\), though not exhaustively. We shall come back to this later. We shall now deal with the option \(\mu \neq 0\), (73).

### 6.2 Non-zero proper values

From (64) and (73), using (62), we have
\[\lambda_6 = \lambda_5, \quad \lambda_2 = \lambda_1. \quad (74)\]

Then, from (63),
\[\lambda_5 = -\lambda_1. \quad (75)\]

From (70),
\[2\lambda_3 + (\lambda_7 + \lambda_8) + 2\mu(\lambda_7 - \lambda_8) = 0. \quad (76)\]

And from (71),
\[\frac{1}{2}(\lambda_7 - \lambda_8) + 2\mu\lambda_3 + \mu(\lambda_7 + \lambda_8)] = 0. \quad (77)\]

The last two together yield \(\mu = \pm 1/2\). We thus get
\[\begin{align*}
\mu &= \frac{1}{2}, \\
\lambda_7 &= -\lambda_3, \\
\lambda_6 &= \lambda_1, \lambda_3, -\lambda_1, -\lambda_3, \lambda_8. \quad (78)
\end{align*}\]
Similarly

$$\mu = -\frac{1}{2}, \quad \lambda_8 = -\lambda_3,$$
$$\lambda_A = \lambda_1, \lambda_1, \lambda_3, \lambda_3, -\lambda_1, -\lambda_1, \lambda_7, -\lambda_3.$$  \quad (79)

Consider (78). Two obvious choices are $\lambda_1 = 0$ and $\lambda_3 = 0$ respectively. The pattern for $\mu = 0$ repeats itself if we choose $\lambda_3 = 0$, meaning that the two idempotents $X_5$ and $X_6$ combine. On the other hand, if we chose $\lambda_1 = 0$ and also $\lambda_8 = 0$, we are going to have a third color that will be more like in the subsection of neutrinos, which gave rise to appealing results. We thus suggest focusing on

$$\mu = \frac{1}{2}, \quad \lambda_A = 0, 0, \lambda_3, \lambda_3, 0, 0, -\lambda_3, 0.$$  \quad (80)

Similarly for (79),

$$\mu = -\frac{1}{2}, \quad \lambda_A = 0, 0, \lambda_3, \lambda_3, 0, 0, 0, -\lambda_3.$$  \quad (81)

The idempotents $X_3$, $X_4$, $X_7$ and $X_8$ were defined as

$$X_3 = I_{12}P_1^+, \quad X_4 = I_{12}P_1^-, \quad X_7 = I_{12}P_3^+, \quad X_8 = I_{12}P_3^-,$$  \quad (82)

(ignoring at this point the factor $\varepsilon$, since $(K + 1)dr$ does not act on it). We would proceed to build the palette of quarks from these solutions as we did for $\mu = 0$. We would then compute $\pi$ from the last column of a table like the present Table 3 [3].

\section{Conclusion}

We have found that there are three proper values, 1/2, 0 and -1/2. These options have suboptions. Some look more interesting than other, but all of them might be relevant. The stage is now set for being more subtle with the study of solutions. In particular, we chose $\lambda_2 = 0$ in [3]. We would not do so now.

As we said in the abstract, we are here at a relatively easy point for entry to this theory by HEP physicists, given their knowledge of the phenomenology. Two issues, however, will require deep knowledge of the Kähler calculus.
rather than deep knowledge of the phenomenology. One of them is the long standing issue of the spin of nucleons, starting from the spin of the quarks. The other issue is the GellMann-Nishijima formula. We have said enough about it in the subsection on algebraic flavor to understand, at least in the present context, that there should be an equation dealing with the same subject but with fewer operators. We also suspect that it is a disguised form of a value and covalue equation.

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