ROT A-BAXTER MODULES TOWARD DERIVED FUNCTORS

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Abstract. In this paper we study Rota-Baxter modules with emphasis on the role played by the Rota-Baxter operators and resulting difference between Rota-Baxter modules and the usual modules over an algebra. We introduce the concepts of free, projective, injective and flat Rota-Baxter modules. We give the construction of free modules and show that there are enough projective, injective and flat Rota-Baxter modules to provide the corresponding resolutions for derived functor.

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1. Introduction

Motivated by his probability study [4], G. Baxter introduced the concept of a (Rota-)Baxter algebra in 1960. To recall its definition, let $k$ be a commutative ring with identity $1_k$ and fix a $\lambda \in k$. A Rota-Baxter algebra of weight $\lambda$ is a pair $(R, P)$ where $R$ is an algebra and $P$ is a linear operator on $R$ satisfying the Rota-Baxter axiom

\begin{equation}
P(r)P(s) = P(rP(s)) + P(P(r)s) + \lambda P(rs) \quad \text{for all } r, s \in R.
\end{equation}

In the 1960s through 1990s, this algebraic structure was studied from analytic and combinatorial viewpoints with contributions from well-known mathematicians such as Atkinson, Cartier and Rota [1, 3, 6, 16, 17]. In the Lie algebra context, it was related to the operator form of the classical Yang-Baxter equation by the Russian physicists [19]. Since the beginning of this century, this area has experienced a burst of development with broad applications ranging from number theory...
to quantum field theory [1, 2, 3, 4, 5, 6, 7]. See [8] for a survey and [1] for a more detailed treatment.

Representation theory is an important aspect in the study of any algebraic structure. A representation of a Rota-Baxter algebra is made more involved because of the Rota-Baxter operator $P$ on top of the algebra $R$. As introduced in [13] (see also [15]), a (left) Rota-Baxter module is defined to be a (left) $R$-module $M$ together with a linear operator $p$ on $M$ which satisfies the module form of Eq. (1):

$$P(r)p(m) = p(rp(m)) + p(P(r)m) + \lambda p(rm)$$ for all $r \in R, m \in M$.

Note that for any $k$-algebra $R$ and $\lambda \in k$, the scalar product operator

$$R \longrightarrow R, \quad r \mapsto -\lambda r$$ for all $r \in R,$

is a Rota-Baxter operator of weight $\lambda$. Thus any $k$-algebra can be naturally regarded as a Rota-Baxter algebra of weight $\lambda$. Likewise, any $R$-module with the same scalar product operator is a Rota-Baxter module. Thus the study of Rota-Baxter modules generalizes the study of the usual modules.

In this paper, we study Rota-Baxter modules as a first step to study their homological algebra. Thus we study the free, projective, injective and flat objects in the category of Rota-Baxter modules. We show that there are enough of these objects in this category, enabling us to define the derived functors in the category of Rota-Baxter modules.

As observed in [13], a Rota-Baxter module can be regarded as a module on the ring of Rota-Baxter operators on the Rota-Baxter algebra. Since the ring of Rota-Baxter operators in general is not yet well-understood, it is useful to study Rota-Baxter modules via a direct approach as we are taking in this paper. Further, this approach makes it easier to see the difference between Rota-Baxter modules and the usual modules. For example, a right Rota-Baxter module needs to be defined by an identity different from Eq. (2), imposing a particularly strong condition for a Rota-Baxter algebra to be a right module or a bimodule over itself. See Propositions 2.6 and 2.8. Further, contrary to the fact that an algebra is a free module over itself, a Rota-Baxter algebra is not a free Rota-Baxter module over itself, but satisfies a universal property in a restricted sense. See Theorem 2.14. Overall, even though the concepts for Rota-Baxter modules can be defined in analogue to those for modules, their constructions needs new ingredients.

Here is an outline of the paper. In Section 2, we first introduce basic notations on Rota-Baxter modules, emphasizing the difference between a right Rota-Baxter module and a left one. We then construct free operated modules and then utilize them to obtain free Rota-Baxter modules by taking quotients. Further a usual free module is characterized as a free Rota-Baxter module with an additional restriction. In Section 3 the concepts of a projective Rota-Baxter module and an injective Rota-Baxter module are defined. It is shown that there are enough projective and injective Rota-Baxter modules to obtain projective and injective resolutions of a Rota-Baxter module, allowing the definition of Rota-Baxter homology and cohomology groups. In Section 4, the concept of a tensor product over a Rota-Baxter algebra is introduced from which a flat Rota-Baxter module is defined. It is shown that free and more generally projective Rota-Baxter modules are flat Rota-Baxter modules.

Throughout the paper, all algebras, linear maps and tensor products are taken over the base ring $k$ unless otherwise stated.
2. Free Rota-Baxter modules

After introducing basic notions on Rota-Baxter modules, emphasizing their difference from modules over an algebra, we give a construction of free Rota-Baxter modules through operated modules.

2.1. Rota-Baxter modules. We first recall the notion of left Rota-Baxter modules from [13, 14] before introducing the different notion of right Rota-Baxter modules.

**Definition 2.1.** Let \((R, P)\) be a Rota-Baxter \(k\)-algebras of weight \(\lambda\).

(i) A **left (Rota-Baxter) \((R, P)\)-module** \((M, p)\) is a left \(R\)-module \(M\) together with a \(k\)-linear operator \(p : M \rightarrow M\) such that

\[
P(r)p(m) = p(P(r)m) + p(rp(m)) + \lambda p(rm) \quad \text{for all } r \in R, m \in M.
\]

(ii) For left \((R, P)\)-modules \((M, p)\) and \((M', p')\), a **left \((R, P)\)-module homomorphism** is a left \(R\)-module homomorphism \(\phi : M \rightarrow M'\) such that \(\phi \circ p = p' \circ \phi\).

(iii) A **left \((R, P)\)-module monomorphism (resp. epimorphism, isomorphism)** is defined to be an injective (resp. surjective, bijective) left \((R, P)\)-module homomorphism.

(iv) A **left \((R, P)\)-submodule** of \((M, p)\) is a submodule \(N\) of \(R\)-module \(M\) such that \(p(N) \subseteq N\), giving the pair \((N, p|_N)\).

To define a quotient module of a left Rota-Baxter module, we have

**Lemma 2.2.** Let \((R, P)\) be a Rota-Baxter \(k\)-algebra of weight \(\lambda\), \((M, p)\) a left \((R, P)\)-module and \((N, p|_N)\) a submodule of \((M, p)\). The pair \((M/N, \overline{p})\) is a left \((R, P)\)-module, where

\[
\overline{p} : M/N \rightarrow M/N, \ m + N \mapsto p(m) + N.
\]

We call \((M/N, \overline{p})\) the quotient **left \((R, P)\)-module** of \((M, p)\) by \((N, p|_N)\).

**Proof.** Since \(p(N) \subseteq N\), the prescription of \(\overline{p}\) is well-defined. Next, we verify that \(\overline{p}\) satisfies Eq. (3). For any \(r \in R, m \in M\), we have

\[
P(r)\overline{p}(m + N) = P(r)(p(m) + N)
\]
\[
= P(r)p(m) + N
\]
\[
= p(P(r)m) + p(rp(m)) + \lambda p(rm) + N
\]
\[
= (p(P(r)m) + N) + (p(rp(m)) + N) + (\lambda p(rm) + N)
\]
\[
= \overline{p}(P(r)(m + N)) + \overline{p}(r\overline{p}(m + N)) + \lambda\overline{p}(r(m + N)),
\]

as required. \(\Box\)

Denote by \(\mathcal{L}(R,P)\text{-Mod}\) the category of left \((R, P)\)-modules, with its objects the left \((R, P)\)-modules and its morphisms the \((R, P)\)-module homomorphisms.

The following are some examples of left Rota-Baxter modules.

**Example 2.3.** Let \((R, P)\) be a Rota-Baxter \(k\)-algebra of weight \(\lambda\). Then

(i) With \((R, P)\) acting on itself on the left, \((R, P)\) is a left \((R, P)\)-module.

(ii) As in the case of the usual module theory over an algebra, any left Rota-Baxter ideal \(I\) of \((R, P)\) (meaning a left ideal \(I\) of \(R\) such that \(P(I) \subseteq I\)) together with the restriction \(P : I \rightarrow I\) is a left \((R, P)\)-module. Then \((R/I, \overline{P})\) is also a left \((R, P)\)-module by Lemma 2.2.
(iii) Let \( R[x] \) be the ring of polynomials with coefficients in \( R \). Define

\[ P : R[x] \rightarrow R[x], \quad \sum_{i=0}^{n} c_{i}x^{i} \mapsto \sum_{i=0}^{n} P(c_{i})x^{i}. \]

Then \((R[x], P)\) is a Rota-Baxter \( k \)-algebra of weight \( \lambda \) \([13], [14]\), and \((R[x], P)\) is a Rota-Baxter left \((R, P)\)-module.

The difference between a module over an algebra and a Rota-Baxter module can already be observed by the concept of a Rota-Baxter right module.

**Definition 2.4.** Let \((R, P)\) be a Rota-Baxter algebra of weight \( \lambda \). A **(Rota-Baxter) right** \((R, P)\)-module \((M, p)\) is a right \( R \)-module \( M \) together with a \( k \)-linear operator \( p : M \rightarrow M \) such that

\[ p(mP(r)) = p(m)P(r) + p(p(m)r) + \lambda p(m)r \quad \text{for all } r \in R, m \in M. \]

A right \((R, P)\)-module homomorphism is defined similarly to that for left \((R, P)\)-modules.

The quite unorthodox definition of a Rota-Baxter right module originates from the Rota-Baxter operator and can be justified as follows. See Proposition 3.2 for an application of Rota-Baxter right modules.

Taking left multiplications by elements of \( R \), as well as the action of \( p \), as linear operators in \( \text{End}(M) \), then Eq. (3):

\[ P(r)p(m) = p(P(r)m) + p(rp(m)) + \lambda p(rm) \quad \text{for all } r \in R, m \in M, \]

can be rewritten as

\[ (P(r) \circ p)(m) = (p \circ P(r))(m) + (p \circ r \circ p)(m) + \lambda (p \circ r)(m), \]

regarding \( M \) as a left \( \text{End}(M) \)-module. Then the corresponding right \( \text{End}(M) \)-action on \( M \) is

\[ (m)(P(r) \circ p) = (m)(p \circ P(r)) + (m)(p \circ r \circ p) + \lambda (m)(p \circ r), \]

acting from the left to the right, which gives

\[ (mP(r))p = ((m)p)P(r) + (((m)p)r)p + \lambda ((m)p)r). \]

This is Eq. (4).

In particular, if Rota-Baxter algebra \((R, P)\) is viewed as a right Rota-Baxter module over itself, then it needs to satisfy

\[ P(rP(s)) = P(r)P(s) + P(P(r)s) + \lambda P(r)s \quad \text{for all } r, s \in R. \]

**Proposition 2.5.** Let \((R, P)\) be a Rota-Baxter \( k \)-algebra of weight \( \lambda \). Then \((R, P)\) is a right \((R, P)\)-module if and only if the Rota-Baxter operator \( P \) satisfies the relation

\[ 2P(P(r)s) + \lambda P(rs) + \lambda P(r)s = 0 \quad \text{for all } r, s \in R. \]

**Proof.** This is because, under the assumption of Eq. (1), \( P \) satisfies Eq. (3) if and only if it satisfies Eq. (4). \( \square \)

Applying this result, we next give an example of a Rota-Baxter algebra which is a right \((R, P)\)-module, as well as a left \((R, P)\)-module.
Proposition 2.6. Let \( R := k u_0 \oplus k u_1 \). Equip it with the multiplication where \( u_0 \) is the identity and \( u_1^2 = -\lambda u_1 \). Define a \( k \)-linear operator
\[
P : R \rightarrow R, \quad u_0 \mapsto u_1, \quad u_1 \mapsto -\lambda u_1.
\]
Then \((R, P)\) is a Rota-Baxter \( k \)-algebra satisfying Eq. (7) and hence is a right \((R, P)\)-module.

Proof. The cyclic \( k \)-module \( k u_1 \) with \( u_1^2 = -\lambda u_1 \) is a nonunitary \( k \)-algebra. Then \( R \) is simply the unitarization of \( k u_1 \).

We next verify that \( P \) satisfies the Rota-Baxter axiom in Eq. (7). Since \( P \) is \( k \)-linear, we only need to check it for the basis elements.

For \( r = s = u_0 \), we have
\[
P(u_0)P(u_0) = u_1 u_1 = -\lambda u_1,
\]
agreeing with
\[
P(u_0P(u_0)) + P(P(u_0)u_0) + \lambda P(u_0u_0) = 2P(u_1) + \lambda P(u_0) = -2\lambda u_1 + \lambda u_1 = -\lambda u_1.
\]
For \( r = u_0 \), \( s = u_1 \), we have
\[
P(u_0)P(u_1) = u_1(-\lambda u_1) = -\lambda u_1^2 = \lambda^2 u_1
\]
which agrees with
\[
P(u_0P(u_1)) + P(P(u_0)u_1) + \lambda P(u_0u_1) = P(-\lambda u_1) + P(u_1^2) + \lambda P(u_1) = P(u_1^2) = -\lambda P(u_1) = \lambda^2 u_1.
\]
For \( r = s = u_1 \), we have
\[
P(u_1)P(u_1) = (-\lambda u_1)(-\lambda u_1) = \lambda^2 u_1^2 = -\lambda^3 u_1,
\]
agreeing with
\[
P(u_1P(u_1)) + P(P(u_1)u_1) + \lambda P(u_1u_1) = 2P(u_1(-\lambda u_1)) + \lambda P(u_1^2) = -\lambda P(u_1^2) = \lambda^2 P(u_1) = -\lambda^3 u_1.
\]
Thus \((R, P)\) is a Rota-Baxter algebra of weight \( \lambda \).

We finally verify that \( P \) satisfies Eq. (8). Taking \( r \) and \( s \) to be the basis elements \( u_0 \) or \( u_1 \), we obtain
\[
2P(u_0)u_0 + \lambda P(u_0u_0) = 2P(u_1) + \lambda u_1 + u_1 = -2\lambda u_1 + \lambda u_1 + u_1 = 0.
\]
\[
2P(u_0)u_1 + \lambda P(u_0u_1) = 2P(u_1^2) + \lambda P(u_1) + \lambda u_1^2 = 2P(-\lambda u_1) - 2\lambda u_1^2 = 2\lambda^2 u_1 - 2\lambda u_1^2 = 0.
\]
\[
2P(u_1)u_1 + \lambda P(u_1u_1) = 2P(-\lambda u_1^2) + \lambda P(-\lambda u_1) - \lambda^2 u_1^2 = 2\lambda^2 P(u_1) + 2\lambda^3 u_1 = 0.
\]
Thus \( P \) satisfies Eq. (8). \( \square \)

We next define Rota-Baxter bimodules.

Definition 2.7. Let \((R, P)\) and \((S, \alpha)\) be Rota-Baxter algebras. An \((R, P)-(S, \alpha)\)-bimodule is a triple \((M, p^R_M, p^S_M)\) where \((M, p^R_M)\) is a left \((R, P)\)-module, \((M, p^S_M)\) is a right \((S, \alpha)\)-module and \(M\) is an \(R-S\)-bimodule over algebras, such that
\[
p^S_M(rm) = r p^S_M(m), \quad p^R_M(ms) = p^R_M(m)s, \quad p^S_M(p^R_M(m)) = p^S_M(p^S_M(m))
\]
for all \(m \in M, r \in R, s \in S\).

In general, \((R, P)\) is not its own \((R, P)-(R, P)\)-bimodule because being a Rota-Baxter bimodule implies that the operator \( P \) is \( R \)-linear, but a Rota-Baxter operator is only \( k \)-linear. Denote by \( 1_R \) the identity of \( R \). To be precise, we have
Proposition 2.8. Let \((R, P)\) be a Rota-Baxter \(k\)-algebra of weight \(\lambda\). Then \((R, P)\) is an \((R, P)\)-(R, P)-bimodule if \(P\) is \(R\)-linear on both sides, and either \(P(1_R) = 0\) or \(P(1_R) = -\lambda\). If \(R\) has no zero divisors, then the converse is also true.

Proof. Suppose that the Rota-Baxter operator \(P\) is \(R\)-linear and satisfies \(P(1_R) = 0\) or \(P(1_R) = -\lambda\). Then \(P\) is either the zero operator or the scalar operator \(P(r) = -\lambda r\) and so \(P^2(r) = \lambda^2 r^2\) for \(r \in R\). If \(P\) is the zero operator, then everything vanishes in the conditions of a Rota-Baxter bimodule. So we are done. In the latter case, the check is also simple. For example, to check Eq. (2), for every \(r, s \in R\), we have

\[
2P(P(r)s) + \lambda P(rs) + \lambda(P(r)s) = 2rsP^2(1_R) + \lambda rsP(1_R) + \lambda rsP(1_R) = 2\lambda^2 rs - \lambda^2 rs - \lambda^2 rs = 0.
\]

Conversely if \((R, P)\) is a \((R, P)-(R, P)\)-bimodule. Then \(P\) is \(R\)-linear by definition. Then by the Rota-Baxter axiom in Eq. (1) or Eq. (2), we have \(P(1_R)(P(1_R) + \lambda) = 0\). Then the assumption that \(R\) has no zero divisors implies \(P(1_R) = 0\) or \(P(1_R) = -\lambda\). \(\square\)

2.2. Free operated modules. We recall from [1] that an \textbf{operated \(k\)-algebra} is a \(k\)-algebra \(R\) equipped with a \(k\)-linear operator \(\alpha : R \to R\).

Definition 2.9. Let \((R, \alpha)\) be an operated \(k\)-algebra.

(i) A \textbf{left operated \(R\)-module} is a pair \((M, p)\) consisting of a left \(R\)-module \(M\) and a \(k\)-linear operator \(p : M \to M\).

(ii) Let \((M, p_M)\) and \((N, p_N)\) be left operated \(R\)-modules. A \textbf{left operated \(R\)-module homomorphism} \(f : (M, p_M) \to (N, p_N)\) is a left \(R\)-module homomorphism \(f : M \to N\) such that \(f \circ p_M = p_N \circ f\).

For example, left Rota-Baxter modules are left operated modules. Following the convention made in the introduction, the tensor products are all taken over \(k\), unless otherwise stated. Let \((R, \alpha)\) be an operated \(k\)-algebra and \(X\) a set. Denote \([R,X] := RX \oplus (R \otimes R)X \oplus \cdots = \oplus_{n \geq 1}(R^\otimes n)X\),

where \(R^\otimes n = R^\otimes n \otimes kX\) with \(kX\) being the free \(k\)-module on \(X\). The action of \(R\) on the left most tensor factor of \(R^\otimes n\) defines a left action of \(R\) on \([R,X]\), giving rise to a left \(R\)-module structure on \([R,X]\). Define a \(k\)-linear operator \(p_X : [R,X] \to [R,X]\) by assigning

\[(r_1 \otimes \cdots \otimes r_n)x \mapsto (1_R \otimes r_1 \otimes \cdots \otimes r_n)x \quad \text{for all } r_1, \ldots, r_n \in R, x \in X\]

for pure tensors \(r_1 \otimes \cdots \otimes r_n\) and extending by additivity.

Proposition 2.10. Let \((R, \alpha)\) be an operated \(k\)-algebra and \(X\) a set. Then, with the above notations,

(i) the pair \([R,X], p_X\) is a left operated \(R\)-module;

(ii) the pair \([R,X], p_X\), together with the natural embedding \(j_X : X \to [R,X]\), is the free left operated \(R\)-module generated by \(X\). More precisely, for any operated left \((R, \alpha)\)-module \((M, q)\) and any set map \(f : X \to M\), there exists a unique operated \(R\)-module homomorphism \(\overline{f} : [R,X] \to M\) such that \(\overline{f} \circ j_X = f\), that is the following diagram commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{j_X} & ([R,X], p_X) \\
\downarrow f & & \downarrow \overline{f} \\
(M, q) & & \\
\end{array}
\]
Proof. \(\square\) We only need to verify that \(p_X\) is \(k\)-linear which follows from the \(k\)-linearity of the tensor product:
\[
1_R \otimes (kr_1) \otimes \cdots \otimes r_n = k1_R \otimes r_1 \otimes \cdots \otimes r_n \quad \text{for all } k \in k, r_1, \ldots, r_n \in R, n \geq 1.
\]
\(\square\) We define \(\tilde{f} : M_R(X) \to M\) by defining \(\tilde{f}((r_1 \otimes \cdots \otimes r_n)x)\) for \((r_1 \otimes \cdots \otimes r_n)x \in R^\otimes X\) recursively on \(n\). For the initial step of \(n = 1\), we define
\[
(7) \quad \tilde{f}(r_1x) = r_1\tilde{f}(x) = r_1(f \circ j_X)(x) = r_1f(x).
\]
For the induction step, we define
\[
\tilde{f}((r_1 \otimes \cdots \otimes r_n)x) = r_1q(\tilde{f}((r_2 \otimes \cdots \otimes r_n)x)).
\]
By construction, \(\tilde{f}\) is a left \(R\)-module homomorphism. Note that this is also the only way to define \(\tilde{f}\) under the conditions \(\tilde{f} \circ j_X = f\) and \(q \circ \tilde{f} = \tilde{f} \circ p_X\), proving the desired uniqueness of \(\tilde{f}\) for the universal property. \(\square\)

2.3. **Free Rota-Baxter module.** We now apply free operated modules to construct free Rota-Baxter modules.

**Definition 2.11.** Let \((R, P)\) be a Rota-Baxter \(k\)-algebra of weight \(\lambda\) and \(X\) a set. A **free left \((R, P)\)-module** on \(X\) is a left \((R, P)\)-module \((F(X), p)\) together with a map \(j_X : X \to F(X)\) satisfying the following universal property: for any left \((R, P)\)-module \((M, q)\) and any set map \(f : X \to M\), there exists a unique left \((R, P)\)-module homomorphism \(\tilde{f} : F(X) \to M\) such that \(\tilde{f} \circ j_X = f\).

Now we construct free left \((R, P)\)-modules. Let \(I_X\) denote the left operated submodule of \(M_R(X)\) generated by the subset
\[
\{P(r)p_X(y) - p_X(P(r)y) - p_X(rp_X(y)) - \lambda p_X(ry) \mid r \in R, y \in M_R(X)\}.
\]
Define \(M_R(X)/I_X\) to be the quotient operated module of \(M_R(X)\) by \(I_X\) and define
\[
p : M_R(X)/I_X \to M_R(X)/I_X, \quad y + I_X \mapsto p_X(y) + I_X
\]
to be the operator on the quotient \(M_R(X)/I_X\) induced by \(p_X\). Then \((M_R(X)/I_X, p)\) is a left Rota-Baxter \((R, P)\)-module.

**Theorem 2.12.** Let \((R, P)\) be a Rota-Baxter \(k\)-algebra of weight \(\lambda\) and \(X\) a set. Then \((M_R(X)/I_X, p)\) with the natural map \(j := \pi \circ j_X : X \to M_R(X) \to M_R(X)/I_X\) is the free left \((R, P)\)-module.

**Proof.** Let \((M, q)\) be a left \((R, P)\)-module and \(f : X \to M\) a set map. From Proposition 2.11, there is a unique left operated \(R\)-module homomorphism \(\tilde{f} : (M_R(X), p_X) \to (M, q)\) such that \(\tilde{f} \circ j_X = f\), as showing in the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{j_X} & (M_R(X), p_X) \xrightarrow{\pi} (M_R(X)/I_X, p) \\
\downarrow{f} \quad & & \quad \downarrow{\tilde{f}} \\
(M, q) & &
\end{array}
\]

Now we show that \(\tilde{f}\) vanishes on the generators of \(I_X\). Indeed, let \(r \in R\) and \(y \in M_R(X)\). Then
\[
\tilde{f}(P(r)p_X(y) - p_X(P(r)y) - p_X(rp_X(y)) - \lambda p_X(ry))
\]
\[
= P(r)q(\tilde{f}(y)) - q(P(r)\tilde{f}(y)) - q(rp(\tilde{f}(y))) - \lambda q(r\tilde{f}(y))
\]
\[
= 0.
\]
So \( \tilde{f} \) induces a unique left \((R, P)\)-module homomorphism

\[
\tilde{f} : (M_R(X)/I_X, p) \rightarrow (M, q)
\]

such that \( \tilde{f} \circ \pi = \tilde{f} \). Thus

\[
\tilde{f} \circ j = \tilde{f} \circ \pi \circ j_X = \tilde{f} \circ j_X = f,
\]

as required. The uniqueness of \( \tilde{f} \) follows from the uniqueness of \( \tilde{f} \) and the uniqueness of its induced map on the quotient \( M_R(X)/I_X \). \( \Box \)

As in the case of modules, we obtain

**Corollary 2.13.**

(i) Every left Rota-Baxter module is the quotient of a free left Rota-Baxter module.

(ii) Every finitely generated left Rota-Baxter module is the quotient of a finitely generated free left Rota-Baxter module.

2.4. **Free modules as free Rota-Baxter modules.** As noted in the introduction, a Rota-Baxter algebra \((R, P)\) in general is not free as a Rota-Baxter module over itself. We now make this precise. We show that a Rota-Baxter algebra is a free Rota-Baxter module in a more restricted sense. More generally, we investigate how a free \( R \)-module behaves like a free Rota-Baxter module.

Let \((R, P)\) be a Rota-Baxter algebra of weight \( \lambda \) and \( X \) a set. For a left \((R, P)\)-module \((M, q)\), set

\[
MC(M) := \{ m \in M \mid q(rm) = P(r)m \text{ for any } r \in R \},
\]

called the set of **module constants** of \( M \) since \( m \in MC(M) \) behaves like a constant which can be taken out of the operator. Let \( F(X) \) be the free left \( R \)-module generated by \( X \):

\[
\tilde{F}(X) := \left\{ \sum_{x \in X} r_x x \mid r_x \in R \right\}.
\]

Define

\[
\tilde{p} : \tilde{F}(X) \rightarrow \tilde{F}(X), \sum_{x \in X} r_x x \mapsto \sum_{x \in X} P(r_x) x.
\]

**Theorem 2.14.** Let \((R, P)\) be a Rota-Baxter algebra of weight \( \lambda \) and \( X \) a set. Then

(i) the pair \((\tilde{F}(X), \tilde{p})\) is a left \((R, P)\)-module.

(ii) the pair \((\tilde{F}(X), \tilde{p})\), together with the natural embedding map \( \iota : X \rightarrow (\tilde{F}(X), \tilde{p}) \), is the **restricted free left \((R, P)\)-module generated by** \( X \) in the sense that, for any left \((R, P)\)-module \((M, q)\) and any set map \( f : X \rightarrow (M, q) \) with \( \text{im} f \subseteq MC(M) \), there exists a unique left Rota-Baxter homomorphism \( \tilde{f} : (\tilde{F}(X), \tilde{p}) \rightarrow (M, q) \) such that \( f = \tilde{f} \circ \iota \).

**Proof.** It is sufficient to show that \( \tilde{p} \) satisfies Eq. (3). For any \( r \in R \) and \( r_x x \in F(X) \), we have

\[
P(r)\tilde{p}(r_x x) = P(r)(P(r_x) x)
\]

\[
= P(r)P(r_x) x
\]

\[
= P(rP(r_x)) x + P(P(r) r_x) x + \lambda P(r) r_x x
\]

\[
= \tilde{p}(rP(r_x)) x + \tilde{p}(P(r) r_x) x + \lambda \tilde{p}(r) r_x x
\]

\[
= \tilde{p}(r \tilde{p}(r_x x)) + \tilde{p}(P(r)(r_x) x) + \lambda \tilde{p}(r)(r_x x),
\]

as required.
By the universal property of $F(X)$ as the free left $R$-module over $X$, there is a left $R$-module homomorphism

$$\bar{f} : (\overline{F}(X), \overline{p}) \longrightarrow (M, q), \quad \sum_{x \in X} r_x x \longmapsto \sum_{x \in X} r_x f(x).$$

Furthermore,

$$\bar{f} \circ \overline{p}(r_x, x) = \overline{f}(p_x, R) = \overline{P}(p_x) f(x) = q(r_x f(x)) = q(\bar{f}(r_x, x)) = (q \circ f)(r_x, x),$$

where the third step follows from Eq. (4) and the fourth step from $\text{im} f \subseteq MC(M)$. Thus $\bar{f} \circ \overline{p} = q \circ \bar{f}$ and so $\bar{f}$ is the desired left $(R, P)$-module homomorphism.

By the definition of $\bar{f}$, we have

$$\bar{f}(r_x, x) = r_x f(x) = r_x ((\bar{f} \circ i)(x)) = r_x f(x).$$

So $\bar{f}$ is uniquely determined by $f$. $\square$

We end this section with a condition of $MC(R) = R$ for a Rota-Baxter algebra $(R, P)$.

**Proposition 2.15.** Let $(R, P)$ be a Rota-Baxter algebra of weight $\lambda$. If $R$ has no zero divisors and $MC(R) = R$, then $P$ is right $R$-linear and either $P(1_R) = 0$ or $P(1_R) = -\lambda$

We note that this condition is different from the condition for a Rota-Baxter algebra to be a Rota-Baxter bimodule over itself.

**Proof.** From $MC(R) = R$ we have $P(r) = P(1_R) r$ for all $r \in R$. Then from Eq. (4) we obtain

$$P(1_R)^2 = 2P(1_R) + \lambda P(1_R) = 2P(1_R)^2 + \lambda P(1_R).$$

Thus $P(1_R)(P(1_R) + \lambda) = 0$ and the conclusion follows. $\square$

3. Projective and injective resolutions of Rota-Baxter modules

We now turn our attention to the Hom functor, the projectivity and the injectivity of Rota-Baxter modules.

3.1. The Hom functor. Let $\textbf{Ab}$ be the category of abelian groups. Recall that $(R, P)\text{-}\textbf{Mod}$ is the category of left $(R, P)$-modules. If $(M, p_M)$ and $(N, p_N)$ are objects of $(R, P)\text{-}\textbf{Mod}$, the set of all the homomorphisms of $(R, P)$-modules from $(M, p_M)$ to $(N, p_N)$ will be denoted by $\text{Hom}_{(R, P)}(M, N)$. Thus $\text{Hom}_{(R, P)}(M, N)$ is a subset of $\text{Hom}_R(M, N)$.

**Proposition 3.1.** Let $(R, P)$ be a Rota-Baxter $k$-algebra of weight $\lambda$ and $(M, p_M), (N, p_N) \in (R, P)\text{-}\textbf{Mod}$. Then $\text{Hom}_{(R, P)}(M, N)$ is an abelian subgroup of $\text{Hom}_R(M, N)$.

Thus $(R, P)\text{-}\textbf{Mod}$ is an abelian category.

**Proof.** First, the zero element of $\text{Hom}_R(M, N)$ is in $\text{Hom}_{(R, P)}(M, N)$. Next let $f, g \in \text{Hom}_{(R, P)}(M, N)$. The inclusion $\text{Hom}_R(M, N) \subseteq \text{Hom}_R(M, N)$ shows that $f + g$ and $-f$ are $R$-linear. Further, $f \circ p_M = p_N \circ f$ and $g \circ p_M = p_N \circ g$ give

$$(f + g) \circ p_M = p_N \circ (f + g), \quad ((-f) \circ p_M)(m) = (p_N \circ (-f))(m).$$

Thus $f + g$ and $-f$ are in $\text{Hom}_{(R, P)}(M, N)$. Therefore $\text{Hom}_{(R, P)}(M, N)$ is a sub-abelian group of $\text{Hom}_R(M, N)$. $\square$
The following are more generalizations of properties of modules to the category of Rota-Baxter modules. For simplicity, we suppress the adjective left (resp. right and bi-) from a left (resp. right or bi-) Rota-Baxter module when its meaning is clear from the context.

**Proposition 3.2.** Let \((R, P), (S, \alpha)\) and \((T, \gamma)\) be Rota-Baxter algebras.

(i) If \((M_{(R,P)}, p^R_M, p^S_M)\) and \((N_{(T,\gamma)}, p^R_N, p^S_N)\) are Rota-Baxter modules, then \((\text{Hom}_{(R,P)}(M, N), q)\) is a left \((T, \gamma)\)-module with \(q\) defined by

\[
q(f)(m) := p^T_N(f(m)), \quad f \in \text{Hom}_{(R,P)}(M, N), \quad m \in M.
\]

(ii) If \((N_{(T,\gamma)}, p^R_N, p^S_N)\) are Rota-Baxter modules, then \((\text{Hom}_{(R,P)}(M, N), q)\) is a right \((T, \gamma)\)-module with \(q\) defined by

\[
q(f)(m) := p^T_N(f(m)), \quad f \in \text{Hom}_{(R,P)}(M, N), \quad m \in M.
\]

(iii) If \((M_{(R,P)}, p^R_M, p^S_M)\) and \((N_{(T,\gamma)}, p^R_N)\) are Rota-Baxter modules, then \((\text{Hom}_{(R,P)}(M, N), q)\) is a left \((S, \alpha)\)-module with \(q\) defined by

\[
q(f)(m) := f(p^S_M(m)), \quad f \in \text{Hom}_{(R,P)}(M, N), \quad m \in M.
\]

(iv) If \((N_{(T,\gamma)}, p^R_N)\) are Rota-Baxter modules, then \((\text{Hom}_{(R,P)}(M, N), q)\) is a right \((S, \alpha)\)-module with \(q\) defined by

\[
q(f)(m) := f(p^S_M(m)), \quad f \in \text{Hom}_{(R,P)}(M, N), \quad m \in M.
\]

**Proof.** (i). The \(T\)-action on \(\text{Hom}_R(M, N)\) is defined by

\[(tf)(m) := tf(m) \quad \text{for all} \quad m \in M, f \in \text{Hom}_R(M, N), t \in T.
\]

If \(f\) is further in \(\text{Hom}_{(R,P)}(M, N)\), then \(f \circ p^R_M = p^R_N \circ f\). Thus by \(N\) being a \((T, \gamma)-(R, P)\) bimodule, we obtain

\[
(tf) \circ p^R_M(m) = tf(p^R_M(m)) = tp^R_N(f(m)) = p^R_N(tf(m)) \quad \text{for all} \quad m \in M.
\]

Thus \(\text{Hom}_{(R,P)}(M, N)\) is a left \(T\)-submodule of \(\text{Hom}_R(M, N)\).

Now we show that \(q(f)\) is in \(\text{Hom}_{(R,P)}(M, N)\). Since \(f\) and \(p^T_N\) are right \(R\)-module homomorphisms, so is their composition \(q(f)\). Likewise, since \(f \circ p^R_M = p^R_N \circ f\) from \(f \in \text{Hom}_{(R,P)}(M, N)\) and \(p^T_N \circ p^R_N = p^R_N \circ p^T_N\) from \(N\) being a \((T, \gamma)-(R, P)\)-bimodule, we have \(q(f) \circ p^R_M = p^R_N \circ q(f)\). Thus \(q(f)\) is in \(\text{Hom}_{(R,P)}(M, N)\).

We are left to prove

\[
\gamma(tq(f)) = \gamma(q(tf)) + q(tf) + \lambda q(tf) \quad \text{for all} \quad t \in T.
\]

But this follows from

\[
(\gamma(tq(f))(m) = \gamma(t)(q(f)(m))
\]

\[
= \gamma(t)p^T_N(f(m))
\]

\[
= p^T_N(\gamma(tf)(m)) + p^T_N(tp^T_N(f(m))) + \lambda p^T_N((tf)(m)) \quad \text{as} \quad (N, p^T_N) \text{is a left} \quad (T, \gamma)\text{-module}
\]

\[
= q(\gamma(tf)(m)) + q(tq(f)(m)) + \lambda q(tf)(m).
\]
The proof of Item (3) is similar.

(3). Similar to Item (1), the $S$-action on $\text{Hom}_{R,P}(M,N)$ is defined by

$$(sf)(m) = f(ms), \quad \text{for all } m \in M, s \in S, f \in \text{Hom}_{R,P}(M,N).$$

Then it follows in the same way that $q(f)$ is in $\text{Hom}_{R,P}(M,N)$. To prove

$$\alpha(s)q(f) = q(sq(f)) + q(\alpha(s)f) + \lambda q(sf) \quad \text{for all } s, f \in \text{Hom}_{R,P}(M,N),$$

we derive

$$(\alpha(s)q(f))(m) = q(f)(m\alpha(s)) \quad \text{(by the definition of } S\text{-action)}$$

$$= f(p_M^S(m\alpha(s))) \quad \text{(by the definition of } q(f))$$

$$= f(p_M^S(p_M^S(m)s) + p_M^S(m)\alpha(s) + \lambda p_M^S(m)s) \quad \text{(by the definition of Rota-Baxter right module)}$$

$$= f(p_M^S(p_M^S(m)s)) + f(p_M^S(m)\alpha(s)) + \lambda f(p_M^S(m)s)$$

$$= q(f)(p_M^S(m)s) + (\alpha(s)f)(p_M^S(m)) + \lambda(sf)(p_M^S(m))$$

$$= (sq(f))(p_M^S(m)) + (\alpha(s)f)(p_M^S(m)) + \lambda(sf)(p_M^S(m))$$

$$= q(sq(f))(m) + q(\alpha(s)f)(m) + \lambda q(sf)(m)$$

$$= (q(sq(f)) + q(\alpha(s)f) + \lambda q(sf))(m) \quad \text{(by the definition of } S\text{-action and } q(f)),$$

as required. The proof of Item (4) is similar. $\square$

3.2. Projective and injective Rota-Baxter modules. By Proposition 3.1, the category of left $(R, P)$-modules is an abelian category. By [20, § 2.5], for an abelian category with enough projective and injective resolutions, one can define derived functors of $\text{Hom}$ using projective resolutions and injective resolutions. Thus we just need to prove that there are enough projective and injective left $(R, P)$-modules.

We first give the definition of projective left Rota-Baxter modules.

**Definition 3.3.** Let $(R, P)$ be a Rota-Baxter $k$-algebra of weight $\lambda$. A left $(R, P)$-module $(V, p)$ is **projective** if, for every left $(R, P)$-module epimorphism $f : (N, p_N) \rightarrow (M, p_M)$ and every left $(R, P)$-module homomorphism $g : (V, p) \rightarrow (M, p_M)$, there exists a left $(R, P)$-module homomorphism $\overline{g} : (V, p) \rightarrow (N, p_N)$ making the following diagram commutative:

$$
\begin{array}{ccc}
(V, p) & \xrightarrow{\overline{g}} & (N, p_N) \\
\downarrow{g} & & \downarrow{f} \\
(M, p_M) & \rightarrow & 0.
\end{array}
$$

**Proposition 3.4.** A free left Rota-Baxter module is a projective left Rota-Baxter module.

**Proof.** The proof is the same as the case for left modules. We give some details for completeness. Let $(F(X), p)$ be the free left Rota-Baxter $(R, P)$-module on $X$ with the natural embedding $j_X : X \rightarrow F(X)$. Let $f : (N, p_N) \rightarrow (M, p_M)$ be a surjective $(R, P)$-module homomorphism and let $g : (F(X), p) \rightarrow (M, p_X)$
be a left Rota-Baxter module homomorphism. Since \( f \) is surjective, for each \( x \in X \), there is a \( n_x \in N \) such that \( f(n_x) = g(x) \). Define a map \( g_0 : X \to N \) by \( x \mapsto n_x \). Then by the universal property of \( F(X) \), there is a left \((R, P)\)-module homomorphism \( \overline{g} : F(X) \to N \) such that \( \overline{g} \circ j_X = g_0 \). So \( f \circ \overline{g} \circ j_X = f \circ g_0 \). Again by the universal property of \( F(X) \), we have \( f \circ \overline{g} = g \). This is what we need. \( \square \)

From Corollary 2.13 and Proposition 3.4, we obtain that there are enough projective objects in the category of Rota-Baxter modules.

We next introduce the concept of an injective Rota-Baxter module and show that there are enough injective objects in the category of left Rota-Baxter modules, namely every left Rota-Baxter module can be embedded into an injective left Rota-Baxter module. We take a similar approach as in the case of modules, but the process becomes more involved.

**Definition 3.5.** Let \((R, P)\) be a Rota-Baxter \(k\)-algebra of weight \(\lambda\). A left \((R, P)\)-module \((E, p)\) is **injective** if, whenever \(f\) is a left \((R, P)\)-module monomorphism and \(g\) is a left \((R, P)\)-module homomorphism, there exists a left \((R, P)\)-module homomorphism \(\overline{g}\) making the following diagram commutative:

\[
\begin{array}{ccc}
(E, p) & \xrightarrow{g} & (N, p_N) \\
\downarrow & & \downarrow \overline{g} \\
0 & \xrightarrow{f} & (M, p_M)
\end{array}
\]

We first recall the concept and construction of the ring of Rota-Baxter operators given in [13].

**Definition 3.6.** Let \((R, P)\) be a Rota-Baxter algebra of weight \(\lambda\) and \(k\langle R, Q \rangle\) be the free product of the \(k\)-algebras \(R\) and \(k[Q]\), where \(Q\) is a variable. The **ring of Rota-Baxter operators on** \((R, P)\), denoted by \(R_{RB}(Q)\), is defined to be the quotient

\[
R_{RB}(Q) = k\langle R, Q \rangle/I_{R,Q},
\]

where \(I_{R,Q}\) is the ideal of \(k\langle R, Q \rangle\) generated by the subset

\[
\{QrQ - P(r)Q + QP(r) + \lambda Qr \mid r \in R\}.
\]

Let \(1_{R_{RB}(Q)}\) denote the identity of \(R_{RB}(Q)\).

There is the following correspondence between Rota-Baxter modules and \(R_{RB}(Q)\)-modules [13]:

**Proposition 3.7.** If \((M, p)\) is a left \((R, P)\)-module, then the resulting left \(R\)-module \(M\) together with \(Q \cdot m := p(m), m \in M\), makes \(M\) into a left \(R_{RB}(Q)\)-module. Conversely, if \(M\) is a left \(R_{RB}(Q)\)-module, then \((M, p)\) is a left \((R, P)\)-module, where \(p : M \to M, p(m) := Qm, m \in M\). In particular, left \((R, P)\)-ideals of \(R_{RB}(Q)\) are of the form \((S, \overline{P}|_S)\) where \(S\) is a left ideal of \(R_{RB}(Q)\) and \(\overline{P} : R_{RB}(Q) \to R_{RB}(Q)\) is the left multiplication by \(Q\).

Applying this result, we next give the Rota-Baxter module version of the Baer Criterion for injectivity of modules.

**Proposition 3.8.** Let \((V, p)\) be a left \((R, P)\)-module. Then \((V, p)\) is an injective left \((R, P)\)-module if and only, for every left \((R, P)\)-ideal \((S, \overline{P}|_S)\) of \((R_{RB}(Q), \overline{P})\), every \((R, P)\)-module homomorphism \(f : (S, \overline{P}|_S) \to (V, p)\) can be extended to one from \((R_{RB}(Q), \overline{P})\).
Proof. We adapt the proof of the Baer Criterion as presented for example in [13].

Suppose that \((V, p)\) is an injective \((R, P)\)-module. Then by the definition of an injective Rota-Baxter module, every \((R, P)\)-module homomorphism \(f : (S, \overline{P}\mid_S) \rightarrow (V, p)\) can be extended to one from \((R_{RB}(Q), \overline{P})\).

Conversely, assume that, for every left \((R, P)\) ideal \((S, \overline{P}\mid_S)\) of \(R_{RB}(Q)\), every \((R, P)\)-module homomorphism \(f : (S, \overline{P}\mid_S) \rightarrow (V, p)\) can be extended to one from \((R_{RB}(Q), \overline{P})\).

Let \(f : (N, p_N) \rightarrow (M, p_M)\) be a monomorphism of left \((R, P)\)-modules and define a map
\[
\phi : (\mathcal{H}, p_{\mathcal{H}}) \rightarrow (V, p), \quad \phi = \rho \big|_{(N, p_N)}. 
\]
which is a left \((R, P)\)-module homomorphism and define a map
\[
\psi : (H + R_{RB}(Q))b \rightarrow V, \quad a + rb \mapsto h(rb) 
\]
is a well-defined \((R, P)\)-module homomorphism. By the assumption, there is an \((R, P)\)-module homomorphism \(\varphi : (R_{RB}(Q), \overline{P}) \rightarrow (V, p)\) such that \(\varphi(r) = h(rb)\) for \(r \in L\). Denote \(c := \varphi(1_{R_{RB}(Q)})\) and define a map
\[
\psi : H + R_{RB}(Q)b \rightarrow V, \quad a + rb \mapsto h(a) + rc, 
\]
which implies that \(\psi\) is well-defined. Then \(\psi\) is a left \(R_{RB}(Q)\)-module homomorphism and hence, by Proposition [57], a left \((R, P)\)-module homomorphism extending \(g\). Hence it is in \(S\) and is strictly larger than \(h\). This is a contradiction. Thus we must have \(H = M\).

Recall that an abelian group \(G\) is called a divisible abelian group, if for any \(x \in G\) and any nonzero integer \(n \in \mathbb{Z}\), there is some \(y \in G\) such that \(x = ny\).

**Proposition 3.9.** Let \((R, P)\) be a Rota-Baxter \(k\)-algebra of weight \(\lambda\) and \(D\) be a divisible abelian group. Then \((\text{Hom}_\mathbb{Z}(R_{RB}(Q), D), q)\) is an injective left \((R, P)\)-module.

**Proof.** Applying Proposition [3.8], we let \(S\) be a left ideal of \(R_{RB}(Q)\) and let \(\eta : (S, \overline{P}\mid_S) \rightarrow (R_{RB}(Q), \overline{P})\) be the embedding map. For any
\[
f : (S, \overline{P}\mid_S) \rightarrow (\text{Hom}_\mathbb{Z}(R_{RB}(Q), D), q),
\]
we extend \(f\) as in the following diagram:
\[
\begin{array}{ccc}
0 & \rightarrow & (S, \overline{P}\mid_S) \\
\eta \downarrow & & \downarrow g \\
& & (R_{RB}(Q), \overline{P}).
\end{array}
\]
Define \( \phi : S \rightarrow D \) by \( \phi(s) = f(s)(1_{R_{Q}(Q)}) \in D \). Then \( \phi \) is an \( \mathbb{Z} \)-module homomorphism. Since an abelian group is an injective \( \mathbb{Z} \)-module if and only if it is a divisible abelian group \([18]\), \( D \) is an injective \( \mathbb{Z} \)-module. Then there is a \( \mathbb{Z} \)-module homomorphism \( \psi : R_{R_{Q}(Q)} \rightarrow D \) such that \( \phi = \psi \circ \eta \).

For any \( x, y \in R_{R_{Q}(Q)} \), define \( g(x)(y) = \psi(xy) \). Then \( g \) is a map from \( R_{R_{Q}(Q)} \) to \( \text{Hom}_{\mathbb{Z}}(R_{R_{Q}(Q)}, D) \). Let \( r \in R \). Then
\[
g(rx)(y) = \psi((rx)y) = \psi((ry)x) = g(x)(yr) = (rg(x))(y)
\]
and so \( g \) is an \( R \)-module homomorphism. Since
\[
((g \circ \overline{P})(x))(y) = (g(\overline{P}(x)))(y) = \psi(y(\overline{P}(x))) = \psi(y(Qx)) = \psi((yQ)x)) = g(x)(yQ) = ((g \circ g)(x))(y),
\]
g : \( (R_{R_{Q}(Q)}, \overline{P}) \rightarrow (\text{Hom}_{\mathbb{Z}}(R_{R_{Q}(Q)}, D), q) \) is an \( (R, P) \)-module homomorphism. Let \( s \in S \). For \( x \in R \), we have
\[
((g \circ \eta)(s))(x) = g(s)(x) = \psi(xs) = \phi(xs) = f(xs)(1_{R_{Q}(Q)}) = (xf(s))(1_{R_{Q}(Q)}) = f(s)(x).
\]
For \( x = Q \), we have
\[
((g \circ \eta)(s))(Q) = g(s)(Q) = \psi(Qs) = \phi(Qs) = f(Qs)(1_{R_{Q}(Q)}) = ((f \circ \overline{P})(s))(1_{R_{Q}(Q)}) = ((q \circ f)(s))(1_{R_{Q}(Q)}) = (q(f(s)))(1_{R_{Q}(Q)}) = f(s)(Q),
\]
which implies \( g \circ \eta = f \). Hence \( (\text{Hom}_{\mathbb{Z}}(R_{R_{Q}(Q)}, D), q) \) is an injective \( (R, P) \)-module by Proposition \([18]\).

\[\square\]

**Theorem 3.10.** Let \( (R, P) \) be a Rota-Baxter \( k \)-algebra of weight \( \lambda \) and \( (V, p) \) be a left \( (R, P) \)-module. Then \( (V, p) \) can be embedded into an injective \( (R, P) \)-module.

**Proof.** Define
\[
R_{R_{Q}(Q)} \times V \rightarrow V, (r, m) \mapsto rm, (Q, m) \mapsto p(m), r \in R, m \in V.
\]
Then \( V \) is an \( R_{R_{Q}(Q)} \)-module. Now define
\[
f : (V, p) \rightarrow (\text{Hom}_{\mathbb{Z}}(R_{R_{Q}(Q)}, V), q), m \mapsto \varphi_{m},
\]
where \( \varphi_{m}(x) = xm \) for \( x \in R_{R_{Q}(Q)} \). Thus \( \varphi_{m} \) is a \( \mathbb{Z} \)-module homomorphism. For any \( r \in R \), \( x \in R_{R_{Q}(Q)} \) and \( m \in V \), we have
\[
f(rm)(x) = x(rm) = (xr)m = \varphi_{m}(xr) = r\varphi_{m}(x) = (rf(m))(x),
\]
and so \( f \) is an \( R \)-module homomorphism. Since
\[
((f \circ p)(m))(x) = f(p(m))(x) = f(Qm)(x) = x(Qm) = (xQ)m = \varphi_{m}(xQ) = f(m)(xQ) = (q(f(m)))(x) = (q \circ f(m))(x),
\]
f is an \( (R, P) \)-module homomorphism. We now show that it is a monomorphism. For any \( m, m' \in V \), if \( \varphi_{m} = \varphi_{m'} \), then \( xm = \varphi_{m}(x) = \varphi_{m'}(x) = xm' \) for all \( x \in R_{R_{Q}(Q)} \). In particular, this is true for \( x = 1_{R_{Q}(Q)} \), and so \( m = m' \).

Since every abelian group can be embedded into a divisible abelian group \([18]\), there exists an embedding map \( \eta_{1} : V \rightarrow D \). Now define \( \overline{\eta} : (\text{Hom}_{\mathbb{Z}}(R_{R_{Q}(Q)}, V), q) \rightarrow (\text{Hom}_{\mathbb{Z}}(R_{R_{Q}(Q)}, D), q') \) by \( \tau \mapsto \eta_{1} \circ \tau \). For any \( r \in R \) and \( x \in R_{R_{Q}(Q)} \), we have
\[
(\overline{\eta}(r\tau))(x) = \eta_{1}((r\tau)(x)) = (\eta_{1} \circ \tau)(xr) = r(\eta_{1} \circ \tau)(x) = (\overline{\eta}(\tau))(x),
\]
and so $\bar{\eta}$ is an $R$-module homomorphism. Moreover, since

$$((q' \circ \bar{\eta})(\tau))(x) = (q' \circ (\bar{\eta}(\tau)))(x) = \bar{\eta}(\tau)(xQ) = (\eta_1 \circ \tau)(xQ) = (\eta_1(q(\tau)))(x) = ((\bar{\eta} \circ q)(\tau))(x),$$

$\bar{\eta}$ is an $(R, P)$-module homomorphism. Therefore, $(\text{Hom}_\mathbb{Z}(R_{RB}(Q), D), q')$ is an injective $(R, P)$-module by Proposition 3.5 and

$$\bar{\eta} \circ f : (V, p) \xrightarrow{f} (\text{Hom}_\mathbb{Z}(R_{RB}(Q), V), q) \xrightarrow{\eta} (\text{Hom}_\mathbb{Z}(R_{RB}(Q), D), q'),$$

is an $(R, P)$-monomorphism, as required. \hfill \Box

4. Flat Rota-Baxter modules

We finally turn to the study of flat Rota-Baxter modules, beginning with the construction of the tensor product of two Rota-Baxter modules in the category of Rota-Baxter modules.

4.1. Tensor product of Rota-Baxter modules. We first define the tensor product of Rota-Baxter modules.

**Definition 4.1.** Let $(R, P)$ be a Rota-Baxter algebra of weight $\lambda$, $(M_{(R,P)}, \eta, p_M)$ a right $(R, P)$-module and $(\tau, P_{(R,P)}, \lambda, p_N)$ a left $(R, P)$-module.

(i) Let $G$ be an (additive) abelian group. A map $f : M \times N \longrightarrow G$ is called $(R, P)$-bilinear if for all $m, m' \in M$, $n, n' \in N$ and $r \in R$, we have

$$f(m + m', n) = f(m, n) + f(m', n),$$

$$f(m, n + n') = f(m, n) + f(m', n'),$$

$$f(mr, n) = f(m, rn),$$

$$f(p_M(m), n) = f(m, p_N(n)).$$

(ii) The tensor product $M \otimes_{(R,P)} N$ of $(M_{(R,P)}, \eta, p_M)$ and $(\tau, P_{(R,P)}, \lambda, p_N)$ over $(R, P)$ is an abelian group together with a $(R, P)$-bilinear map

$$\iota : M \times N \longrightarrow M \otimes_{(R,P)} N$$

satisfying the following universal property: for every abelian group $G$ and every $(R, P)$-bilinear map $f : M \times N \longrightarrow G$, there exists a unique abelian group homomorphism $\tilde{f} : M \otimes_{(R,P)} N \longrightarrow G$ making the following diagram commutative

$$\begin{array}{ccc}
M \times N & \xrightarrow{\iota} & M \otimes_{(R,P)} N \\
\downarrow{f} & & \downarrow{\tilde{f}} \\
G & & G
\end{array}$$

The following result gives a construction of the tensor product of Rota-Baxter modules.

**Theorem 4.2.** Let $(R, P)$ be a Rota-Baxter algebra of weight $\lambda$ and let $(M_{(R,P)}, \eta, p_M)$, $(\tau, P_{(R,P)}, \lambda, p_N)$ be Rota-Baxter modules. Let $F$ be the free abelian group on the set $M \times N$ and $I$ the subgroup of $F$ generated by all elements of $F$ of the form

$$(m + m', n) - (m, n) - (m', n), \ (m, n + n') - (m, n) - (m, n'),$$

$$(mr, n) - (m, rn), \ (p_M(m), n) - (m, p_N(n)), \ m, m' \in M, n, n' \in N, r \in R.$$

Then $F/I$ with the natural map $\iota : M \times N \rightarrow F \rightarrow F/I$ is the tensor product $M \otimes_{(R,P)} N$. 

Proposition 4.3. Let $f : M \times N \rightarrow G$ be a $(R, P)$-bilinear map. Then $f$ extends to an abelian group homomorphism $f^* : F \rightarrow G$ by additivity. Since $f^*$ vanishes on the generators of $I$, $f^*$ induces a well-defined abelian group homomorphism $\tilde{f} : F/I \rightarrow G$ such that $f^*(m, n)) = \tilde{f}(m \otimes (R, P) n)$ with $m \in M$ and $n \in N$. So we have

$$f(m, n) = f^*(m, n) = \tilde{f}(m \otimes (R, P) n) = \tilde{f} \circ i(m, n),$$

as required.

If $\tilde{f}$ satisfies the conditions, then

$$\tilde{f}(\sum_i m_i \otimes (R, P) n_i) = \sum_i \tilde{f}(m_i \otimes (R, P) n_i) = \sum_i \tilde{f}(i(m_i, n_i)) = \sum_i f(m_i, n_i).$$

So $\tilde{f}$ is uniquely determined by $f$. \hfill \Box

**Proposition 4.4.** Let $(R, P)$ be a Rota-Baxter algebra of weight $\lambda$.

(i) If $(M_{(R, P), p_M})$ is a right $(R, P)$-module, there is an additive functor $F_M : (R, P)\text{Mod} \rightarrow \text{Ab}$ defined by

$$F_M(N) = M \otimes_{(R, P)} N, \quad F_M(g) = id_M \otimes_{(R, P)} g,$$

where $(N, p_N), (L, p_L) \in (R, P)\text{Mod}$ and $g : (N, p_N) \rightarrow (L, p_L)$ is a left $(R, P)$-module homomorphism.

(ii) If $(M_{(R, P), M}, p_M)$ is a left Rota-Baxter module, there is an additive functor $G_M : \text{Mod}_{(R, P)} \rightarrow \text{Ab}$ defined by

$$G_M(N) = N \otimes_{(R, P)} M, \quad G_M(g) = g \otimes_{(R, P)} id_M,$$

where $(N, p_N), (L, p_L) \in \text{Mod}_{(R, P)}$ and $g : (N, p_N) \rightarrow (L, p_L)$ is a right $(R, P)$-module homomorphism.

**Proof.** Let $g' : (L, p_L) \rightarrow (H, p_H)$ be a left $(R, P)$-module homomorphism with $(L, p_L), (H, p_H) \in \text{Mod}_{(R, P)}$. Then

$$F_M(g \circ g') = id_M \otimes_{(R, P)} (g \circ g') = (id_M \otimes_{(R, P)} g) \circ (id_M \otimes_{(R, P)} g') = F_M(g) \circ F_M(g').$$

Since $F_M(id_N) = id_M \otimes_{(R, P)} id_N$, $F_M$ is a functor. We are left to show

$$F_M(g + h) = F_M(g) + F_M(h),$$

where $g, h : N \rightarrow L$ are left $(R, P)$-module homomorphism. Let $m \otimes_{(R, P)} n \in M \otimes_{(R, P)} N$. Then

$$F_M(g + h)(m \otimes_{(R, P)} n) = m \otimes_{(R, P)} ((g + h)(n)) = m \otimes_{(R, P)} (g(n) + h(n)) = m \otimes_{(R, P)} g(n) + m \otimes_{(R, P)} h(n) = (F_M(g) + F_M(h))(m \otimes_{(R, P)} n),$$

as required.

\hfill \Box

The proof is similar to Item (i).

**Proposition 4.4.** \textbf{(Extension of scalars)} Let $(R, P)$ and $(S, \alpha)$ be Rota-Baxter algebras of weight $\lambda$. 

(i) If \((S, \alpha) \otimes (R, P), p_M^S, p_M^R\) is a Rota-Baxter bimodule and \((R, P) N, p_N^R\) is a left \((R, P)\)-module, then \((M \otimes (R, P) N, q)\) is a left \((S, \alpha)\)-module by defining
\[
\begin{align*}
\sigma(m \otimes (R, P) n) &= (sm) \otimes (R, P) n, \\
q(m \otimes (R, P) n) &= p_M^S(m) \otimes (R, P) n, \quad \text{where } s \in S, m \in M, n \in N.
\end{align*}
\]

(ii) If \((M \otimes (R, P), p_M^R)\) is a right \((R, P)\)-module and \((R, P) N, q, p_N^S, p_N^R\) is a Rota-Baxter bimodule, then \((M \otimes (R, P) N, q)\) is a right \((S, \alpha)\)-module by defining
\[
\begin{align*}
(m \otimes (R, P) n)s &= m \otimes (R, P) (ns), \\
q(m \otimes (R, P) n) &= m \otimes (R, P) p_N^S(n), \quad \text{where } s \in S, m \in M, n \in N.
\end{align*}
\]

Proof. (i) It is straightforward to check that \(M \otimes (R, P) N\) is a left \(S\)-module. So we are left to verify Eq. (i). Let \(s \in S, m \in M, n \in N\). Then
\[
\begin{align*}
\alpha(s)q(m \otimes (R, P) n) &= \alpha(s)(p_M^S(m) \otimes (R, P) n) \\
&= (\alpha(s)p_M^S(m)) \otimes (R, P) n \\
&= p_M^S(\alpha(s)m) \otimes (R, P) n + \lambda p_M^S(sm) \otimes (R, P) n \\
&= q(\alpha(s)m) \otimes (R, P) n + q(s(p_M^S(m) \otimes (R, P) n)) + \lambda q(sm \otimes (R, P) n) \\
&= q(\alpha(s)m \otimes (R, P) n) + q(s(p_M^S(m) \otimes (R, P) n)) + \lambda q(sm \otimes (R, P) n),
\end{align*}
\]
as required.

(ii) The proof is similar to Item (i). \(\square\)

The next result shows that \(\Box \otimes (R, P) N\) and \(\text{Hom}_{(S, \alpha)}(N, \Box)\) are adjoint functors.

**Theorem 4.5.** Let \((R, P)\) and \((S, \alpha)\) be Rota-Baxter algebras of weight \(\lambda\). Let \((M \otimes (R, P), p_M^R)\) be a right \((R, P)\)-module, \((R, P) N, p_N^S, p_N^R\) a Rota-Baxter bimodule and \((L_{(S, \alpha)}, p_L^S)\) a right \((S, \alpha)\)-module. Then
\[
\text{Hom}_{(S, \alpha)}(M \otimes (R, P) N, L) \cong \text{Hom}_{(R, P)}(M, \text{Hom}_{(S, \alpha)}(N, L)).
\]

Proof. Define
\[
\tau : \text{Hom}_{(S, \alpha)}(M \otimes (R, P) N, L) \rightarrow \text{Hom}_{(R, P)}(M, \text{Hom}_{(S, \alpha)}(N, L)),
\]
\[
f \mapsto \tau(f), \text{ where } \tau(f)(m) : n \mapsto f(m \otimes (R, P) n).
\]

Then \(\tau\) is the required isomorphism. \(\square\)

4.2. **Flat Rota-Baxter modules.** As in the classical case, it is quite routine to check that the Rota-Baxter tensor product is right exact. To study the exactness of the tensor product, we introduce the flatness condition in the context of Rota-Baxter modules.

**Definition 4.6.** Let \((R, P)\) be an Rota-Baxter algebra of weight \(\lambda\). A right \((R, P)\)-module \((M, p)\) is flat if \(M \otimes (R, P) \Box\) is an exact functor, that is, whenever
\[
\begin{array}{c}
0 \rightarrow (N', p_{N'}) \xrightarrow{i} (N, p_N) \xrightarrow{j} (N'', p_{N''}) \rightarrow 0
\end{array}
\]

is an exact sequence of left \((R, P)\)-modules, then
\[
0 \longrightarrow M \otimes_{(R,P)} N' \overset{id_M \otimes i}{\longrightarrow} M \otimes_{(R,P)} N \overset{id_M \otimes j}{\longrightarrow} M \otimes_{(R,P)} N'' \longrightarrow 0
\]
is an exact sequence of abelian groups.

Since the functors \(M \otimes_{(R,P)} \square\) are right exact, we see that a right \((R, P)\)-module \((M, p_M)\) is flat if and only if, whenever \(i: (N', p_{N'}) \longrightarrow (N, p_N)\) is an injection, then \(id_M \otimes i: (M \otimes_{(R,P)} N', p') \longrightarrow (M \otimes_{(R,P)} N, p)\) is also an injection.

**Theorem 4.7.** Let \((R, P)\) be a Rota-Baxter algebra of weight \(\lambda\) and \((M, p)\) a right \((R, P)\)-module. Suppose the inclusion \(R \rightarrow R_{RB}(Q)\) gives an injective \((R, P)\)-module homomorphism \(\eta: (R, P) \longrightarrow (R_{RB}(Q), \overline{P})\). If \((M, p)\) is a flat \((R, P)\)-module, then \(M \otimes_{(R,P)} R \cong M\) as right \(R\)-modules.

**Proof.** Since \((M, p)\) is flat, the abelian group homomorphism \(id_M \otimes_{(R,P)} \eta: M \otimes_{(R,P)} R \longrightarrow M \otimes_{(R,P)} R_{RB}(Q)\) is injective. Let \(m \otimes_{R_{RB}(Q)} x\) be a pure tensor in \(M \otimes_{R_{RB}(Q)} R_{RB}(Q)\). Then
\[
(id_M \otimes \eta)(mx \otimes_{(R,P)} 1_R) = mx \otimes_{(R,P)} 1_{R_{RB}(Q)} = m \otimes_{R_{RB}(Q)} x.
\]
Thus \(id_M \otimes_{(R,P)} \eta\) is surjective and so is an abelian group isomorphism. By the extension of scalars in Proposition 4.4, \(M \otimes_{(R,P)} R\) and \(M \otimes_{(R,P)} R_{RB}(Q)\) are right \(R\)-modules. For any \(m \otimes r \in M \otimes_{(R,P)} R\) and \(r' \in R\), we have
\[
(id_M \otimes \eta)((m \otimes r')r) = (id_M \otimes \eta)(m \otimes r') = m \otimes \eta(r') = m \otimes \eta(r') = ((id_M \otimes \eta)(m \otimes r'))r
\]
and so \(id_M \otimes_{(R,P)} \eta\) is an isomorphism of right \(R\)-modules. Furthermore regard \((M, p)\) and \((R_{RB}(Q), \overline{P})\) as \(R_{RB}(Q)\)-modules by Proposition 5.7, we have \(M \otimes_{(R,P)} R_{RB}(Q) = M \otimes_{R_{RB}(Q)} R_{RB}(Q) \cong M\) as right \(R_{RB}(Q)\)-modules and also as \(R\)-modules. Hence \(M \otimes_{(R,P)} R \cong M \otimes_{(R,P)} R_{RB}(Q) \cong M\) as right \(R\)-modules. \(\square\)

Now we give an example of a Rota-Baxter algebra satisfying the conditions in Theorem 4.7.

**Example 4.8.** Let \((R, P)\) be a Rota-Baxter algebra of weight \(\lambda\) with \(P(r) = -\lambda r\) as in Proposition 2.8. Then by Definition 5.6 we have
\[
QrQ - P(r)Q + QP(r) + \lambda Qr = QrQ - \lambda Qr + \lambda Qr = QrQ + \lambda Qr = (Q + \lambda)rQ = 0
\]
and so \(Q = -\lambda\) in \(R_{RB}(Q)\). Hence for the \(\eta\) in Theorem 4.7, we get
\[
(\eta \circ P)(r) = \eta(P(r)) = \eta(-\lambda r) = -\lambda r = Qr = Q\eta(r) = (\overline{P} \circ \eta)(r)
\]
for \(r \in R\) and so \(\eta\) is an injective \((R, P)\)-module homomorphism.

Let \(\{(M_i, p_i) \mid i \in I\}\) be a family of left \((R, P)\)-modules. Then \(\bigoplus_{i \in I} M_i, \bigoplus_{i \in I} p_i\), where \(\bigoplus_{i \in I} p_i\) is defined by
\[
\left(\bigoplus_{i \in I} p_i\right)(m_i)_{i} = \left(p_i(m_i)\right)_{i},
\]
is also a left \((R, P)\)-module and is called the direct sum of \(\{(M_i, p_i) \mid i \in I\}\). It is easy to see
\[
\bigoplus_{i \in I} (M_i, p_i) = \left(\bigoplus_{i \in I} M_i, \bigoplus_{i \in I} p_i\right).
\]

For each \(i \in I\), the map \(\iota_i: (M_i, p_i) \longrightarrow \bigoplus_{i \in I} (M_i, p_i)\) is a monomorphism and satisfies \(\bigoplus_{i \in I} p_i \circ \iota_i = \iota_i \circ p_i\). The map \(\rho_i: \bigoplus_{i \in I} (M_i, p_i) \longrightarrow (M_i, p_i)\) is an epimorphism and satisfies \(p_i \circ \rho_i = \rho_i \circ (\bigoplus_{i \in I} p_i)\). Further \(\iota_i \circ \rho_i = id_{\bigoplus_{i \in I} (M_i, p_i)}\), and \(\rho_i \circ \iota_i = id_{(M_i, p_i)}\).
Lemma 4.9. Let \((R, P)\) be a Rota-Baxter \(k\)-algebra of weight \(\lambda\), and \(\{ (M_i, p_i) \mid i \in I \}, \{ (N_i, q_i) \mid i \in I \}\) be two families of \((R, P)\)-modules. Let \(\varphi_i : (M_i, p_i) \rightarrow (N_i, q_i)\) be \((R, P)\)-module homomorphisms. Then the \((R, P)\)-module homomorphism
\[
\varphi := \bigoplus_{i \in I} \varphi_i : \bigoplus_{i \in I} (M_i, p_i) \rightarrow \bigoplus_{i \in I} (N_i, q_i), \quad (m_i)_I \mapsto (\varphi_i(m_i))_I,
\]
is injective if and only if each \((R, P)\)-module homomorphism \(\varphi_i : (M_i, p_i) \rightarrow (N_i, q_i)\) is injective.

Proof. This follows from \(\ker \varphi = \oplus_{i \in I} \ker \varphi_i\).

Lemma 4.10. Let \((R, P)\) be a Rota-Baxter \(k\)-algebra of weight \(\lambda\), \(\{ (M_i, p_i) \mid i \in I \}\) be a family of left \((R, P)\)-modules, and \((L, p)\) be a right \((R, P)\)-module. Then \(L \otimes_{(R, P)} (\oplus_{i \in I} M_i) = \oplus_{i \in I} (L \otimes_{(R, P)} M_i)\).

Proof. Define group homomorphisms
\[
f : L \otimes_{(R, P)} (\oplus_{i \in I} M_i) \rightarrow \oplus_{i \in I} (L \otimes_{(R, P)} M_i), \quad \ell \otimes (m_i)_I \mapsto (\ell \otimes m_i)_I,
\]
and
\[
g : \oplus_{i \in I} (L \otimes_{(R, P)} M_i) \rightarrow L \otimes_{(R, P)} (\oplus_{i \in I} M_i), \quad (\ell_i \otimes m_i)_I \mapsto (\prod_{i \in I} \ell_i) \otimes (m_i)_I.
\]
It is easy to check that \(f \circ g = id_{\oplus_{i \in I} (L \otimes_{(R, P)} M_i)}\) and \(g \circ f = id_{L \otimes_{(R, P)} (\oplus_{i \in I} M_i)}\). Then \(L \otimes_{(R, P)} (\oplus_{i \in I} M_i) = \oplus_{i \in I} (L \otimes_{(R, P)} M_i)\).

Proposition 4.11. Let \((R, P)\) be a Rota-Baxter \(k\)-algebra of weight \(\lambda\), and \(\{ (M_i, p_i) \mid i \in I \}\) be a family of left \((R, P)\)-modules. Then the \((R, P)\)-module \(\bigoplus_{i \in I} (M_i, p_i)\) is flat if and only if each \((R, P)\)-module \((M_i, p_i)\) is flat.

Proof. Let \((L, p_L)\) and \((N, p_N)\) be two right \((R, P)\)-modules, and let \(\theta : (L, p_L) \rightarrow (N, p_N)\) be a monomorphic \((R, P)\)-module homomorphism.

Suppose that each left \((R, P)\)-module \((M_i, p_i)\) is flat. Then each group homomorphism
\[
\theta \otimes id_M : L \otimes_{(R, P)} M_i \rightarrow N \otimes_{(R, P)} M_i
\]
is injective. By Lemma 4.9, the homomorphism
\[
\bigoplus_{i \in I} (\theta \otimes id_M) : \bigoplus_{i \in I} (L \otimes_{(R, P)} M_i) \rightarrow \bigoplus_{i \in I} (N \otimes_{(R, P)} M_i)
\]
is also injective. Thus the \((R, P)\)-module \(\bigoplus_{i \in I} (M_i, p_i)\) is flat by Lemma 4.10.

Conversely, suppose that the left \((R, P)\)-module \(\bigoplus_{i \in I} (M_i, p_i)\) is flat. Then the group homomorphism
\[
\theta \otimes id_{\bigoplus_{i \in I} M_i} : L \otimes_{(R, P)} \left( \bigoplus_{i \in I} M_i \right) \rightarrow N \otimes_{(R, P)} \left( \bigoplus_{i \in I} M_i \right)
\]
is an injective map. By Lemma 4.4, we conclude that
\[
\bigoplus_{i \in I} (L \otimes_{(R, P)} M_i) \rightarrow \bigoplus_{i \in I} (N \otimes_{(R, P)} M_i)
\]
is also an injective map. By Lemma 4.9, for each \(i \in I\), the map
\[
\theta \otimes id_M : L \otimes_{(R, P)} M_i \rightarrow N \otimes_{(R, P)} M_i
\]
is injective. Thus each left \((R, P)\)-module \((M_i, p_i)\) is flat.

Theorem 4.12. Let \((R, P)\) be a Rota-Baxter \(k\)-algebra of weight \(\lambda\). Every free left \((R, P)\)-module is flat.
Proof. Let \((\mathcal{M}_R(X)/I_X, p)\) be the free left \((R, P)-\)module on \(X\) defined in Theorem \ref{thm:freeModule}. We just need to prove that if \((N', \beta, P') \rightarrow (N, \beta, P)\) is a monomorphism of right \((R, P)-\)modules, then \(N' \otimes_{(R, P)} (\mathcal{M}_R(X)/I_X) \rightarrow N \otimes_{(R, P)} (\mathcal{M}_R(X)/I_X)\) is a monomorphism of abelian groups. We prove this in several steps.

First, for each right \((R, P)-\)module \((M, p)\) and singleton \(X = \{x\}\), we have
\[
M \otimes_{(R, P)} (\mathcal{M}_R(\{x\})/I_{\{x\}}) \cong M.
\]
This can be achieved by defining
\[
f : M \otimes_{(R, P)} \left( \bigoplus_{n \geq 1} R^{\otimes n} \right) x/I_{\{x\}} \longrightarrow M, \quad v \otimes ((r_1 \otimes \cdots \otimes r_n) x + I_{\{x\}}) \mapsto vr_1 \cdots r_n,
\]
and
\[
f' : M \longrightarrow M \otimes_{(R, P)} \left( \bigoplus_{n \geq 1} R^{\otimes n} \right) x/I_{\{x\}}, \quad v \mapsto v \otimes (x + I_{\{x\}}).
\]
Then it is easy to check that \(f \circ f' = id_M\) and \(f' \circ f = id_{M \otimes_{(R, P)} (\bigoplus_{n \geq 1} R^{\otimes n} x/I_{\{x\}})}\). Thus the maps \(f, f'\) are bijective.

Second, for any set \(X\), by noting \(\mathcal{M}_R(X)/I_X \cong \bigoplus_{x \in X} \mathcal{M}_R(\{x\})/I_{\{x\}}\), we have
\[
M \otimes_{(R, P)} (\mathcal{M}_R(X)/I_X) \cong M \otimes_{(R, P)} \left( \bigoplus_{x \in X} \mathcal{M}_R(\{x\})/I_{\{x\}} \right)
\cong \bigoplus_{x \in X} (M \otimes_{(R, P)} \mathcal{M}_R(\{x\})/I_{\{x\}}) \quad \text{(by Lemma \ref{lem:freeModule})}
\cong \bigoplus_{x \in X} M.
\]

Consequently, \(N' \otimes_{(R, P)} (\mathcal{M}_R(X)/I_X) \cong \bigoplus_{x \in X} N'\), and \(N \otimes_{(R, P)} (\mathcal{M}_R(X)/I_X) \cong \bigoplus_{x \in X} N\). Then by Lemma \ref{lem:projective}, the group homomorphism \(\bigoplus_{x \in X} N' \longrightarrow \bigoplus_{x \in X} N\) is injective. So the free left \((R, P)-\)module \((\mathcal{M}_R(X)/I_X, p)\) is flat. \(\square\)

**Lemma 4.13.** Let \((R, P)\) be a Rota-Baxter \(k\)-algebra of weight \(\lambda\). Every projective left \((R, P)-\)module is a direct summand of a free \((R, P)-\)module.

**Proof.** Let \((M, p)\) be a projective \((R, P)-\)module. Let \((F(M), p')\) denote the free \((R, P)-\)module over the set \(M\). The identity map \(id_M : (M, p) \rightarrow (M, p)\), when taken as a set map, gives a \((R, P)-\)module epimorphism \(f : (F(M), p') \rightarrow (M, p)\) such that \(f|_M = id_M\). On the other hand, treating \(id_M\) as a \((R, P)-\)module homomorphism, the projectivity of \((M, p)\) gives a \((R, P)-\)module homomorphism \(\beta : (M, p) \rightarrow (F(M), p')\) such that \(f \circ \beta = id_{(M, p')}\). This gives
\[
F(M) = \text{im} \beta \oplus \ker f \cong M \oplus \ker f.
\]
Since \(\beta\) and \(f\) are \((R, P)-\)module homomorphisms, this is a direct sum of \((R, P)-\)modules. \(\square\)

By Proposition \ref{prop:projective}, Theorem \ref{thm:freeModule} and Lemma \ref{lem:projective} we obtain the following conclusion.

**Theorem 4.14.** Let \((R, P)\) be a Rota-Baxter \(k\)-algebra of weight \(\lambda\). Then every projective left \((R, P)-\)module is flat.

Theorem \ref{thm:projective} shows that there are enough flat Rota-Baxter modules, allowing us to define the Tor functors.

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