Grammatic – a tool for grammar definition reuse and modularity

Andrey Breslav

St. Petersburg State University of Information Technology, Mechanics and Optics
abreslav@gmail.com

Abstract. Grammatic is a tool for grammar definition and manipulation aimed to improve modularity and reuse of grammars and related development artifacts. It is independent from parsing technology and any other details of target system implementation. Grammatic provides a way for annotating grammars with arbitrary metadata (associativity attributes, semantic actions or anything else). It might be used as a front-end for external tools like parser generators to make their input grammars modular and reusable. This paper describes main principles behind Grammatic and gives an overview of languages it provides and their ability to separate concerns and define reusable modules. Also it presents sketches of possible use cases for the tool.

1 Introduction

To follow ideas of language-oriented programming [1] or DSL-based development we need many DSLs, which may mean dozens for a medium-sized project. To be able to produce and support so many languages we need tools which strongly support reuse and modularity. We need to reuse development artifacts to produce many languages since even different ones may sometimes have much in common and it is really hard to work on a bunch of languages having common features implemented by code duplication. The primary condition for reuse is modularity therefore we will always mention them together.

Language development involves many aspects but here we will mostly concentrate on textual syntax. Structure of language syntax is usually described by a context-free grammar. Engineering aspects of grammar development did not yet receive enough attention [2]. To provide tools suitable for industry we need to consolidate and improve results achieved by different authors. We give a brief overview of existing tools in section 2.

In this paper we describe Grammatic – a tool which is aimed to provide reuse and modularity for grammar definitions which might be used for arbitrary purpose (not only parser generation but also analysis of grammars, language synthesis, layout and many other applications). Grammatic provides general languages for context-free grammar definition and annotation. We present them in section 3. These languages are designed to support modularity and reuse, we describe these features in section 4.
Grammatic is mostly dedicated to be a front-end for other tools and a framework for high-level manipulations with grammars and metadata. As a front-end Grammatic does not replace existing tools but helps to extend and join their powerful abilities. We outline some possible use cases in section 5.

2 Related work

The best developed and most widely used family of tools working with grammars are parser and compiler generators. Here we give an overview of most popular and powerful of them. Our goal is to present features supporting modularity and reuse, so we concentrate on these aspects here. Ideas implemented in some of these tools significantly influenced our approach.

2.1 Separation of concerns

The most significant modularity issues are related to separation of syntactic and semantic information in grammar definitions.

Traditional parser generators (Yacc [3], ANTLR [4], COCO/R [5] and many others) use grammar definitions with embedded semantic actions. This leads to mixing up syntactical and semantical aspects of the system into one definition file: it lacks modularity and is hard to maintain.

Tools specially dedicated to creating textual syntax for simple DSLs (xText [6], TCS [7] and others) use grammar annotations needed for their specific purposes (namely building target models) and thus also mix syntax with semantics.

SableCC [8] avoids mixing things up: it does not allow to embed semantic actions. A parser generated by SableCC creates an abstract syntax tree (AST) and the user has to write his/her own custom code to analyze it or transform into the target format. This provides a clear separation of concerns.

SDF [9] also separates semantic actions from parsing by restricting parser responsibilities to building a tree. Unlike SableCC, SDF provides much more powerful grammar definition language.

Another approach to separation of concerns is presented by LISA compiler-compiler [10]: it provides an opportunity to either embed semantic actions or attach them using AOP-like point cuts. This allows to establish different modules (aspects) even for semantic actions themselves which improves overall system modularity.

2.2 Reuse

A basic way to reuse grammar definitions in textual form is to use a C-like preprocessor (any system taking textual input allows reuse with preprocessor). This is the only way to go when using many traditional tools.

One of the most popular parser generators today is ANTLR. In version 2 it provided “grammar inheritance” with “rule overriding” and in latest version 3.1.1 it introduced grammar imports [11] which are rather restricted.
SDF has a powerful import mechanism, supports grammar composition and templates (parameterized modules).

LISA allows grammar inheritance, aspect-oriented composition and templates for semantic actions.

3 Grammatic’s languages overview

Grammatic defines a language for describing context-free grammars which does not involve anything but pure grammar productions. This language supports EBNF constructs for both lexical and syntactical rules.

Grammar is not enough to build the whole system. We need some semantics-related or other information to be added to the grammar to have a complete system description. For this purpose Grammatic defines a languages for attaching metadata to grammars and their elements.

Metadata annotations are grouped into aspects. A metadata aspect can be thought of as a description of the system made from a certain point of view: parsing, pretty printing, building target objects etc.

Below we describe these languages in more details.

3.1 Grammar definitions

A grammar definition language allows to define a grammar as a set of symbols each of which is associated to a set of productions (in concrete syntax we separate productions by “||”). Grammatic itself does not restrict grammars to any particular class (like LL(k), LALR or anything else). Here is an example from a grammar for arithmetic expressions:

```
Factor
   -> Literal
   | | ID
   | | '(' Expression ')' 
;
```

In this example characters in single quotes represent embedded lexical definitions. There is no separate notion of a lexical rule (since it is not necessarily required, see [9]) and all the regular expression operations (sequence, alternative, iteration) are available both for syntactical and lexical definitions. Here is a syntactic (nonterminal) symbol:

```
Product
   -> Factor ('*' Factor)* 
;
```

And these are “lexical” (terminal) symbols:
INT
  -> ['0'--'9']+

; REAL
  -> INT ('.' INT)? (('e' | 'E') ('+' | '-')? INT)?
  ;

Separation of lexer and parser is not needed in some cases (see [9]). By default lexical symbols do not differ from syntactical. If it is needed, we can can specify a symbol as lexical by providing corresponding metadata.

3.2 Metadata

Pure grammar definition defines a textual form of a language but there are many more things to express. Existing tools (see section 2) use some extensions to grammar definitions: embedded semantic actions, AST building instructions etc. Such things might be generally described by metadata attached to a grammar. Grammatic allows to attach arbitrary metadata to a grammar definition.

Metadata annotations might be attached to a grammar, symbol, individual production or a subexpression. Each annotation may contain several attributes (name-value pairs). Attribute values may be of different types. There are several predefined value types: identifier, string, integer, annotation and sequence of values and punctuation symbols.

```plaintext
id = someName; // the value is an identifier
str = "some string"; // string
int = 10; // integer
class = {
    name = MyClass;
    super = Object;
};
astProduction = {{ // sequence
    "(^'+' left ^('-' right 10))"
}};
```

Users may add their own types. No attribute itself has any fixed semantics. Metadata is passive, some tools (like analyzers, transformers, generators etc.) may use it according to their needs.

Even without adding custom types many things might be expressed by such annotations. The most powerful type is a sequence of values – it allows to define small embedded DSLs inside Grammatic. We use such DSLs to describe complicated custom properties when working with external tools (i.e. high-level definitions for AST structures).

3.3 Queries

How to attach metadata to a grammar? In many cases it is done by directly embedding annotations into grammar definition. Therefore different concerns
are mixed together and this results into a problem: system is not modular, is hard to understand and extend.

We employ ideas from aspect-oriented programming paradigm (AOP, see [12]) to solve this problem. We consider experience of creators of LISA [10] and extend their ideas. In Grammatic a grammar definition itself knows nothing about metadata. All the metadata is attached “from the outside”. In AOP this is done by defining join points which are described by point cuts [13]. A language of point cuts is a kind of “addressing” notation – a way to find some object. In original AOP this object may be a class or a method, in our case this will be a rule, subexpression or another object from a grammar definition. When we have found such an object we may attach metadata to it (or perform other actions, see below).

In Grammatic we have a language analogous to AOP point cuts – we call it a query language. We use structural queries with embedded variables. For example this query matches rules defining binary operations:

```
Op -> Arg (Sign Arg)* ;
```

All the names here represent variables. This query matches any of the following rules:

```
Product
  -> Factor (MultOrDiv Factor)* ;
Sum
  -> Product (PlusOrMinus Product)* ;
```

By default a variable matches a symbol but it may match a subexpression or a whole production.

```
A -> Alt=(B | C);
```

Here variables A, B and C match symbol references and Alt matches a subexpression “B | C”.

We can use wildcards in queries. The following query matches immediately left-recursive rules:

```
Rec -> Rec ..;
```

Two dots represent a wildcard which matches arbitrary subexpression. It also might be assigned to a variable:

```
Rec -> Rec Rest=..;
```

We can consider metadata in our queries. We can restrict a particular attribute to a certain type or value or require attribute’s presence or absence:
N {
    type = Nonterminal;
    operation;
    associativity : ID;
    !commutative;
}

This query matches a symbol with “type” attribute having value “Nonterminal”, “operation” attribute present, “associativity” attribute having value of type ID and “commutative” attribute not present.

### 3.4 Aspects

When a query selects some objects from a grammar definition, we can attach some metadata to them.

```
Rec -> Rec ..;
{
    Rec {
        leftRecursive;
    }
}
```

This rule adds a “leftRecursive” attribute (with no value) to all the symbols matched by Rec variable of this query. A set of such rules constitutes an aspect. Many aspects (independent or not) might be assigned to a single grammar, and even to many grammars since our queries are not tied to concrete objects but only to a grammar structure.

### 3.5 More applications of queries

Query language appears to be useful not only in attaching metadata to grammars. As in AOP (AspectJ, for example) we can use it to define some constraints for our grammars and their metadata. It’s done by defining a query and assigning an error message to it: when a query is matched an error (or warning) will be generated. The same might be defined for a case when a query does not match anything on a whole grammar.

Also we can use queries as a general mechanism for locating objects in a grammar which is useful when defining grammar transformations or some generators taking a grammar as input. Here we can consider our query language as analogous to XPath in the context of XSLT.

### 4 Reuse

One of the main purposes of Grammatic is improving reuse experience of grammar-oriented tools. An approach described above gives a good basis for it since no concerns are to be mixed together – we can describe each of them in a separate aspect.
4.1 Reusing aspects

Aspects themselves might be generally reusable – as we told above, query language does not require “hard linking” to grammar objects, these objects are located by their structural context and properties. Above we gave an example of marking all immediately left-recursive rules with a “leftRecursive” attribute. This is an example of a reusable aspect – we can use it on any grammar. Although this technique is not very powerful since we frequently define queries which simply describe whole rules with no generalization, it is still useful.

Below we describe more powerful reuse instruments defined by Grammatic.

4.2 Grammar imports and templates

First we focus on reusing grammar definitions themselves. There are many achievements in this field done by creators of LISA, SDF and other tools (see section 2).

The most popular way of reuse is importing. Some grammar definition A might be imported into some other grammar definition B. This means that all the rules of A are inserted into B. Rules of B may refer to symbols of A – this is the way two grammar definitions are connected.

Very frequently we have to customize some of the imported rules, i.e. add some more productions to the same symbols or replace existing productions. In paper [11] this is referred to as rule overriding. In Grammatic we decided to use more general form of this concept, namely templates.

A language of grammar templates allows creating grammar definitions with “placeholders” which can be replaced with actual objects upon template instantiation. Placeholders might be defined for roles of identifier, expression, production or symbol. A template instantiation might result into grammar object of the type specified by template declaration. Any objects except symbols might be used immediately in rule definitions. Symbols are treated as imported from a template instantiation. This is due to naming reasons: to avoid name duplication a template instantiation expression is a namespace and symbols from that namespace might be referred to by qualified names. Here is an example of a template and its usage.

Symbol binaryOperation<ID $name, Expression $sign, Expression $argument> {
  $name -> $argument ($sign $argument)*;
}

import binaryOperation<Product, '*' | '/', Factor>;
import binaryOperation<Sum, '+' | '-', Product>;
Factor
  -> NUMBER
  || ID
  || '(' Sum ')'
In this example we define a template named “binaryOperation” which makes up an infix binary operation out of symbol name, sign and argument expression. Then we instantiate it twice and import instantiation results into current grammar definition – so we can use new symbol “Product” to create “Sum” and “Sum” to define “Factor”. Here we did not need to use qualified names explicitly – there’s no name duplication.

Now let’s assume that we need to refer to signs of our binary operations as separate symbols. We modify the template as follows:

```plaintext
Symbol binaryOperation<ID $name, Expression $sign, Expression $argument> {
    Sign -> $sign;
    $name -> $argument (Sign $argument)*;
}
```

Now we get two symbols out of a single template instantiation: one symbol for operation and another (named “Sign”) – for its sign. To refer to these new symbols we need qualified names (and named namespaces):

```plaintext
import product = binaryOperation<Product, '*' | '/', Factor>;
import sum = binaryOperation<Sum, '+' | '-', Product>;
```

```plaintext
AnySign
    -> product.Sign | sum.Sign
    ;
```

How can we use templates for “overriding” things? We can put a customizable set of rules into a template, provide a placeholder for production or subexpression that should be replaced and then put a right thing in upon instantiation.

```plaintext
Symbol attributeValue<Production* $moreValueTypes> {
    AttributeValue
        -> STRING
        || ID
        || INT
        || Annotation
        || ValueSequence
        || $moreValueTypes
        ;
}
```

```plaintext
import attributeValue<
    ' {{{' Expression ' }}}'
>
;
```

This defines a template for “AttributeValue” symbol and instantiates it adding a new production (to use expressions as attribute values).
5 Use cases

Grammatic might be used for high level manipulation on modular grammar definitions annotated with arbitrary metadata. In practice this means that we can use it as a front-end for some external tools to provide modularity and reuse.

For example, if we need to use Yacc to generate parsers we can write a module which transforms a Grammatic grammar to a Yacc grammar to use Grammatic’s modularity and Yacc’s generator together.

But pure grammar will not allow us to generate any meaningful parser. We need to provide some additional information to generate Yacc’s semantic actions (and some other stuff which we will not consider here). The easiest way to do it is to provide string-valued attributes for Grammatic productions which simply contain semantic actions’ code. By doing so we will get modular grammar separated from semantic actions which might be transformed into a working parser using Grammatic-to-Yacc generator and then Yacc.

Another way here is to restrict functionality of the parser we are going to get and go after SDF or xText. To do so we must define attributes which describe needed actions in a declarative way: create node of this type, attach this node to this etc. And again we get a working parser by applying two transformations: Grammatic-to-Yacc and then Yacc. Although the first will be more complicated this time.

Note that Grammatic-to-Yacc transformation is to be written once and shipped as a pluggable module for Grammatic. And this might be done for many other external tools (not only parser generators).

In many cases we need to restrict an input of Grammatic-to-Something transformation: we have to ensure attribute value types, prohibit some constructs like left recursion and so on. To do so we can define queries which generate some errors or warnings (see section 3.5).

```
N {
    leftAssoc;
    rightAssoc;
};
{
    error on N : "A symbol cannot be left- and right-associative at the same time";
}
```

In this example we prohibit using attributes “leftAssoc” and “rightAssoc” on the same symbol at the same time. These checks may be put into aspects along with metadata attachment rules which makes diagnostic modular and reusable.

6 Conclusion

As we showed above Grammatic can solve many issues about grammar definition reuse and modularity. This tool is intended to be an open system so we are going
to extend it by adding support for more and more external tools. Grammatic’s
generalized grammar definition format may serve as an interchange format for
different tools which will again improve end user’s experience.

References

1. Martin P. Ward. Language-oriented programming. *Software — Concepts and Tools*, 15(4):147–161, 1994.
2. Paul Klint, Ralf Lämmel, and Chris Verhoef. Toward an engineering discipline for grammarware. *ACM Trans. Softw. Eng. Methodol.*, 14(3):331–380, 2005.
3. Steven C. Johnson. Yacc: Yet another compiler compiler. In *UNIX Programmer’s Manual*, volume 2, pages 353–387. Holt, Rinehart, and Winston, New York, NY, USA, 1979.
4. Terence Parr. *The Definitive ANTLR Reference: Building Domain-Specific Languages*. The Pragmatic Bookshelf, Raleigh, 2007.
5. Hanspeter Msemeck and Johannes Kepler. The compiler generator Coco/R – user manual, 2005.
6. Open Architecture Ware. xText. http://www.openarchitectureware.org/, 2007.
7. Frédéric Jouault, Jean Bézivin, and Ivan Kurtev. TCS: a DSL for the Specification of Textual Concrete Syntaxes in Model Engineering. In *GPCE 2006*, ACM, pages 249–254, 2006.
8. Étienne Gagnon and Laurie Hendren. An object-oriented compiler framework. In *In Proceedings of TOOLS*, pages 140–154, 1998.
9. J. Heering, P. R. H. Hendriks, P. Klint, and J. Rekers. The syntax definition formalism SDF – reference manual. *SIGPLAN Not.*, 24(11):43–75, 1989.
10. Marjan Mernik, Mitja Leni, and Enis A. Compiler/interpreter generator system LISA. In *In IEEE Proceedings of 33rd Hawaii International Conference on System Sciences*, pages 590–594, 2000.
11. Terence Parr. The reuse of grammars with embedded semantic actions. *International Conference on Program Comprehension*, 0:5–10, 2008.
12. Gregor Kiczales, John Lamping, Anurag Mendhekar, Chris Maeda, Cristina Videira Lopes, Jean marc Loingtier, John Irwin, Gregor Kiczales, John Lamping, Anurag Mendhekar, Chris Maeda, Cristina Lopes, Jean marc Loingtier, and John Irwin. Aspect-oriented programming. In *In ECOOP*. SpringerVerlag, 1997.
13. Gregor Kiczales, Erik Hilsdale, Jim Hugunin, Mik Kersten, Jeffrey Palm, and William G. Griswold. An overview of AspectJ. In *ECOOP ’01: Proceedings of the 15th European Conference on Object-Oriented Programming*, pages 327–353, London, UK, 2001. Springer-Verlag.
Preface

This textbook is intended for use by students of physics, physical chemistry, and theoretical chemistry. The reader is presumed to have a basic knowledge of atomic and quantum physics at the level provided, for example, by the first few chapters in our book *The Physics of Atoms and Quanta*. The student of physics will find here material which should be included in the basic education of every physicist. This book should furthermore allow students to acquire an appreciation of the breadth and variety within the field of molecular physics and its future as a fascinating area of research.

For the student of chemistry, the concepts introduced in this book will provide a theoretical framework for that entire field of study. With the help of these concepts, it is at least in principle possible to reduce the enormous body of empirical chemical knowledge to a few basic principles: those of quantum mechanics. In addition, modern physical methods whose fundamentals are introduced here are becoming increasingly important in chemistry and now represent indispensable tools for the chemist. As examples, we might mention the structural analysis of complex organic compounds, spectroscopic investigation of very rapid reaction processes or, as a practical application, the remote detection of pollutants in the air.

April 1995

Walter Olthoff
Program Chair
ECOOP’95
Organization

ECOOP’95 is organized by the department of Computer Science, University of Árhus and AITO (association Internationa pour les Technologie Object) in cooperation with ACM/SIGPLAN.

Executive Committee

Conference Chair: Ole Lehrmann Madsen (Århus University, DK)
Program Chair: Walter Olthoff (DFKI GmbH, Germany)
Organizing Chair: Jørgen Lindskov Knudsen (Århus University, DK)
Tutorials: Birger Møller-Pedersen (Norwegian Computing Center, Norway)
Workshops: Eric Jul (University of Kopenhagen, Denmark)
Panels: Boris Magnusson (Lund University, Sweden)
Exhibition: Elmer Sandvad (Århus University, DK)
Demonstrations: Kurt Nørdmark (Århus University, DK)

Program Committee

Conference Chair: Ole Lehrmann Madsen (Århus University, DK)
Program Chair: Walter Olthoff (DFKI GmbH, Germany)
Organizing Chair: Jørgen Lindskov Knudsen (Århus University, DK)
Tutorials: Birger Møller-Pedersen (Norwegian Computing Center, Norway)
Workshops: Eric Jul (University of Kopenhagen, Denmark)
Panels: Boris Magnusson (Lund University, Sweden)
Exhibition: Elmer Sandvad (Århus University, DK)
Demonstrations: Kurt Nørdmark (Århus University, DK)

Referees

V. Andreev Braunschweig P. Dingus
Bärwolff F.W. Büsser H. Duhm
E. Barrelet T. Carli J. Ebert
H.P. Beck A.B. Clegg S. Eichenberger
G. Bernardi G. Cozzika R.J. Ellison
E. Binder S. Dagoret Feltesse
P.C. Bosetti Del Buono W. Flanger
A. Fomenko  U. Krüger  V. Riech
G. Franke  J. Kurzhöfer  P. Robmann
J. Garvey  M.P.J. Landon  N. Sahlmann
M. Gennis  A. Lebedev  P. Schleper
L. Goerlich  Ch. Ley  Schöning
P. Goritchev  F. Linsel  B. Schwab
H. Greif  H. Lohmand  A. Semenov
E.M. Hanlon  Martin  G. Siegmon
R. Haydar  S. Masson  J.R. Smith
R.C.W. Henderso  K. Meier  M. Steenbock
P. Hill  C.A. Meyer  U. Straumann
H. Hufnagel  S. Mikocki  C. Thiebaux
A. Jacholkowska  J.V. Morris  P. Van Esch
Johannsen  B. Naroska  from Yerevan Ph
S. Kasarian  Nguyen  L.R. West
I.R. Kenyon  U. Obrock  G.-G. Winter
C. Kleinwort  G.D. Patel  T.P. You
T. Köhler  Ch. Pichler  M. Zimmer
S.D. Kolya  S. Prell
P. Kostka  F. Raupach

Sponsoring Institutions

Bernauer-Budiman Inc., Reading, Mass.
The Hofmann-International Company, San Louis Obispo, Cal.
Kramer Industries, Heidelberg, Germany
# Table of Contents

## Hamiltonian Mechanics

Hamiltonian Mechanics unter besonderer Berücksichtigung der höheren Lehranstalten ......................................................... 1  

*Ivar Ekeland, Roger Temam, Jeffrey Dean, David Grove, Craig Chambers, Kim B. Bruce, Elisa Bertino*

Hamiltonian Mechanics2 .......................................................... 7  

*Ivar Ekeland and Roger Temam*

### Author Index

| Author Index | 13 |

### Subject Index

| Subject Index | 17 |
Hamiltonian Mechanics unter besonderer Berücksichtigung der höheren Lehranstalten

Ivar Ekeland\textsuperscript{1}, Roger Temam\textsuperscript{2} Jeffrey Dean, David Grove, Craig Chambers, Kim B. Bruce, and Elsa Bertino

\textsuperscript{1} Princeton University, Princeton NJ 08544, USA, I.Ekeland@princeton.edu, WWW home page: http://users/~iekeland/web/welcome.html
\textsuperscript{2} Université de Paris-Sud, Laboratoire d’Analyse Numérique, Bâtiment 425, F-91405 Orsay Cedex, France

Abstract. The abstract should summarize the contents of the paper using at least 70 and at most 150 words. It will be set in 9-point font size and be inset 1.0 cm from the right and left margins. There will be two blank lines before and after the Abstract. . . .

1 Fixed-Period Problems: The Sublinear Case

With this chapter, the preliminaries are over, and we begin the search for periodic solutions to Hamiltonian systems. All this will be done in the convex case; that is, we shall study the boundary-value problem

\[
\dot{x} = JH'(t, x) \\
x(0) = x(T)
\]

with \( H(t, \cdot) \) a convex function of \( x \), going to \(+\infty\) when \( \|x\| \to \infty \).

1.1 Autonomous Systems

In this section, we will consider the case when the Hamiltonian \( H(x) \) is autonomous. For the sake of simplicity, we shall also assume that it is \( C^1 \).

We shall first consider the question of nontriviality, within the general framework of \((A_\infty, B_\infty)\)-subquadratic Hamiltonians. In the second subsection, we shall look into the special case when \( H \) is \((0, b_\infty)\)-subquadratic, and we shall try to derive additional information.

The General Case: Nontriviality. We assume that \( H \) is \((A_\infty, B_\infty)\)-subquadratic at infinity, for some constant symmetric matrices \( A_\infty \) and \( B_\infty \), with \( B_\infty - A_\infty \) positive definite. Set:

\[
\gamma := \text{smallest eigenvalue of } B_\infty - A_\infty \tag{1}
\]

\[
\lambda := \text{largest negative eigenvalue of } J \frac{d}{dt} + A_\infty \tag{2}
\]
Theorem 1 tells us that if \( \lambda + \gamma < 0 \), the boundary-value problem:

\[
\begin{align*}
\dot{x} &= JH'(x) \\
x(0) &= x(T)
\end{align*}
\]

has at least one solution \( \Pi \), which is found by minimizing the dual action functional:

\[
\psi(u) = \int_0^T \left[ \frac{1}{2} (A_0^{-1}u, u) + N^*(u) \right] dt
\]

on the range of \( A \), which is a subspace \( R(A)^2_L \) with finite codimension. Here

\[
N(x) := H(x) - \frac{1}{2} (A_{\infty} x, x)
\]

is a convex function, and

\[
N(x) \leq \frac{1}{2} ((B_{\infty} - A_{\infty}) x, x) + c \quad \forall x.
\]

**Proposition 1.** Assume \( H'(0) = 0 \) and \( H(0) = 0 \). Set:

\[
\delta := \liminf_{x \to 0} 2N(x) \|x\|^{-2}.
\]

If \( \gamma < -\lambda < \delta \), the solution \( \Pi \) is non-zero:

\[
\Pi(t) \neq 0 \quad \forall t.
\]

**Proof.** Condition (7) means that, for every \( \delta' > \delta \), there is some \( \varepsilon > 0 \) such that

\[
\|x\| \leq \varepsilon \Rightarrow N(x) \leq \frac{\delta'}{2} \|x\|^2.
\]

It is an exercise in convex analysis, into which we shall not go, to show that this implies that there is an \( \eta > 0 \) such that

\[
f \|x\| \leq \eta \Rightarrow N^*(y) \leq \frac{1}{2\delta'} \|y\|^2.
\]

Fig. 1. This is the caption of the figure displaying a white eagle and a white horse on a snow field.
Since \( u_1 \) is a smooth function, we will have \( \|hu_1\|_\infty \leq \eta \) for \( h \) small enough, and inequality (10) will hold, yielding thereby:

\[
\psi(hu_1) \leq \frac{h^2}{2} \frac{1}{\lambda} \|u_1\|^2 + \frac{h^2}{2} \frac{1}{\delta'} \|u_1\|^2 .
\]

(11)

If we choose \( \delta' \) close enough to \( \delta \), the quantity \( \left( \frac{1}{\lambda} + \frac{1}{\delta'} \right) \) will be negative, and we end up with

\[
\psi(hu_1) < 0 \quad \text{for } h \neq 0 \text{ small} .
\]

(12)

On the other hand, we check directly that \( \psi(0) = 0 \). This shows that 0 cannot be a minimizer of \( \psi \), not even a local one. So \( \pi \neq 0 \) and \( \pi \neq A^{-1}_0(0) = 0 \). □

**Corollary 1.** Assume \( H \) is \( C^2 \) and \((a_\infty, b_\infty)\)-subquadratic at infinity. Let \( \xi_1, \ldots, \xi_N \) be the equilibria, that is, the solutions of \( H'(\xi) = 0 \). Denote by \( \omega_k \) the smallest eigenvalue of \( H''(\xi_k) \), and set:

\[
\omega := \text{Min} \{ \omega_1, \ldots, \omega_k \} .
\]

(13)

If:

\[
\frac{T}{2\pi} b_\infty < -E \left[ -\frac{T}{2\pi} a_\infty \right] < \frac{T}{2\pi} \omega
\]

(14)

then minimization of \( \psi \) yields a non-constant \( T \)-periodic solution \( \pi \).

We recall once more that by the integer part \( E[\alpha] \) of \( \alpha \in \mathbb{R} \), we mean the \( a \in \mathbb{Z} \) such that \( a < \alpha \leq a + 1 \). For instance, if we take \( a_\infty = 0 \), Corollary 2 tells us that \( \pi \) exists and is non-constant provided that:

\[
\frac{T}{2\pi} b_\infty < 1 < \frac{T}{2\pi}
\]

(15)

or

\[
T \in \left( \frac{2\pi}{\omega}, \frac{2\pi}{b_\infty} \right) .
\]

(16)

**Proof.** The spectrum of \( \Lambda \) is \( \frac{2\pi}{T} \mathbb{Z} + a_\infty \). The largest negative eigenvalue \( \lambda \) is given by \( \frac{2\pi}{T} k_\alpha + a_\infty \), where

\[
\frac{2\pi}{T} k_\alpha + a_\infty < 0 \leq \frac{2\pi}{T}(k_\alpha + 1) + a_\infty .
\]

(17)

Hence:

\[
k_\alpha = E \left[ -\frac{T}{2\pi} a_\infty \right] .
\]

(18)

The condition \( \gamma < -\lambda < \delta \) now becomes:

\[
b_\infty - a_\infty < -\frac{2\pi}{T} k_\alpha - a_\infty < \omega - a_\infty
\]

(19)

which is precisely condition (14). □
Lemma 1. Assume that \( H \) is \( C^2 \) on \( \mathbb{R}^{2n} \setminus \{0\} \) and that \( H''(x) \) is non-degenerate for any \( x \neq 0 \). Then any local minimizer \( \tilde{x} \) of \( \psi \) has minimal period \( T \).

**Proof.** We know that \( \tilde{x}, \) or \( \tilde{x} + \xi \) for some constant \( \xi \in \mathbb{R}^{2n} \), is a \( T \)-periodic solution of the Hamiltonian system:

\[
\dot{x} = JH'(x) .
\]  
(20)

There is no loss of generality in taking \( \xi = 0 \). So \( \psi(x) \geq \psi(\tilde{x}) \) for all \( \tilde{x} \) in some neighbourhood of \( x \) in \( W^{1,2}(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n}) \).

But this index is precisely the index \( i_T(\tilde{x}) \) of the \( T \)-periodic solution \( \tilde{x} \) over the interval \((0, T)\), as defined in Sect. 2.6. So

\[
i_T(\tilde{x}) = 0 .
\]  
(21)

Now if \( \tilde{x} \) has a lower period, \( T/k \) say, we would have, by Corollary 31:

\[
i_T(\tilde{x}) = i_{kT/k}(\tilde{x}) \geq ki_{T/k}(\tilde{x}) + k - 1 \geq 1 .
\]  
(22)

This would contradict (21), and thus cannot happen. \( \square \)

**Notes and Comments.** The results in this section are a refined version of [1]; the minimality result of Proposition 14 was the first of its kind.

To understand the nontriviality conditions, such as the one in formula (16), one may think of a one-parameter family \( x_T, T \in (2\pi \omega^{-1}, 2\pi b_\infty^{-1}) \) of periodic solutions, \( x_T(0) = x_T(T) \), with \( x_T \) going away to infinity when \( T \to 2\pi \omega^{-1} \), which is the period of the linearized system at 0.

| Year       | World population |
|------------|------------------|
| 8000 B.C.  | 5,000,000        |
| 50 A.D.    | 200,000,000      |
| 1650 A.D.  | 500,000,000      |
| 1945 A.D.  | 2,300,000,000    |
| 1980 A.D.  | 4,400,000,000    |

Table 1. This is the example table taken out of *The TP\textsuperscript{X}book*, p. 246

**Theorem 1 (Ghoussoub-Preiss).** Assume \( H(t, x) \) is \((0, \varepsilon)-\text{subquadratic at infinity for all } \varepsilon > 0, \) and \( T \)-periodic in \( t \)

\[
H(t, \cdot) \text{ is convex } \forall t
\]  
(23)

\[
H(\cdot, x) \text{ is } T-\text{periodic } \forall x
\]  
(24)

\[
H(t, x) \geq n(\|x\|) \text{ with } n(s)s^{-1} \to \infty \text{ as } s \to \infty
\]  
(25)
∀ε > 0 , ∃c : \( H(t, x) \leq \frac{\varepsilon}{2} \|x\|^2 + c \). \hspace{1cm} (26)

Assume also that \( H \) is \( C^2 \), and \( H''(t, x) \) is positive definite everywhere. Then there is a sequence \( x_k, k \in \mathbb{N} \), of \( kT \)-periodic solutions of the system

\[
\dot{x} = JH'(t, x)
\]

such that, for every \( k \in \mathbb{N} \), there is some \( p_o \in \mathbb{N} \) with:

\[
p \geq p_o \Rightarrow x_{pk} \neq x_k.
\]

(28)

Example 1 (External forcing). Consider the system:

\[
\dot{x} = JH'(x) + f(t)
\]

(29)

where the Hamiltonian \( H \) is \((0, b_\infty)\)-subquadratic, and the forcing term is a distribution on the circle:

\[
f = \frac{d}{dt}F + f_o \quad \text{with} \quad F \in L^2(\mathbb{R}/TZ; \mathbb{R}^{2n}) ,
\]

(30)

where \( f_o := T^{-1} \int^T_o f(t)dt \). For instance,

\[
f(t) = \sum_{k \in \mathbb{N}} \delta_k \xi ,
\]

(31)

where \( \delta_k \) is the Dirac mass at \( t = k \) and \( \xi \in \mathbb{R}^{2n} \) is a constant, fits the prescription. This means that the system \( \dot{x} = JH'(x) \) is being excited by a series of identical shocks at interval \( T \).

Definition 1. Let \( A_\infty(t) \) and \( B_\infty(t) \) be symmetric operators in \( \mathbb{R}^{2n} \), depending continuously on \( t \in [0, T] \), such that \( A_\infty(t) \leq B_\infty(t) \) for all \( t \).

A Borelian function \( H : [0, T] \times \mathbb{R}^{2n} \to \mathbb{R} \) is called \((A_\infty, B_\infty)\)-subquadratic at infinity if there exists a function \( N(t, x) \) such that:

\[
H(t, x) = \frac{1}{2} (A_\infty(t)x, x) + N(t, x)
\]

(32)

∀t , \( N(t, x) \) is convex with respect to \( x \)

(33)

\[
N(t, x) \geq n(\|x\|) \quad \text{with} \quad n(s)s^{-1} \to +\infty \quad \text{as} \quad s \to +\infty
\]

(34)

∃c \in \mathbb{R} : \( H(t, x) \leq \frac{1}{2} (B_\infty(t)x, x) + c \quad \forall x \).

(35)

If \( A_\infty(t) = a_\infty I \) and \( B_\infty(t) = b_\infty I \), with \( a_\infty \leq b_\infty \in \mathbb{R} \), we shall say that \( H \) is \((a_\infty, b_\infty)\)-subquadratic at infinity. As an example, the function \( \|x\|^\alpha \), with \( 1 \leq \alpha < 2 \), is \((0, \varepsilon)\)-subquadratic at infinity for every \( \varepsilon > 0 \). Similarly, the Hamiltonian

\[
H(t, x) = \frac{1}{2} k\|k\|^2 + \|x\|^{\alpha}
\]

(36)

is \((k, k + \varepsilon)\)-subquadratic for every \( \varepsilon > 0 \). Note that, if \( k < 0 \), it is not convex.
Notes and Comments. The first results on subharmonics were obtained by Rabinowitz in [5], who showed the existence of infinitely many subharmonics both in the subquadratic and superquadratic case, with suitable growth conditions on $H'$. Again the duality approach enabled Clarke and Ekeland in [2] to treat the same problem in the convex-subquadratic case, with growth conditions on $H$ only.

Recently, Michalek and Tarantello (see [3] and [4]) have obtained lower bound on the number of subharmonics of period $kT$, based on symmetry considerations and on pinching estimates, as in Sect. 5.2 of this article.

References

1. Clarke, F., Ekeland, I.: Nonlinear oscillations and boundary-value problems for Hamiltonian systems. Arch. Rat. Mech. Anal. 78, 315–333 (1982)
2. Clarke, F., Ekeland, I.: Solutions périodiques, du période donnée, des équations hamiltoniennes. Note CRAS Paris 287, 1013–1015 (1978)
3. Michalek, R., Tarantello, G.: Subharmonic solutions with prescribed minimal period for nonautonomous Hamiltonian systems. J. Diff. Eq. 72, 28–55 (1988)
4. Tarantello, G.: Subharmonic solutions for Hamiltonian systems via a $\mathbb{Z}_p$ pseudoinDEX theory. Annali di Matematica Pura (to appear)
5. Rabinowitz, P.: On subharmonic solutions of a Hamiltonian system. Comm. Pure Appl. Math. 33, 609–633 (1980)
1 Fixed-Period Problems: The Sublinear Case

With this chapter, the preliminaries are over, and we begin the search for periodic solutions to Hamiltonian systems. All this will be done in the convex case; that is, we shall study the boundary-value problem

\[ \dot{x} = JH'(t, x) \]
\[ x(0) = x(T) \]

with \( H(t, \cdot) \) a convex function of \( x \), going to +\( \infty \) when \( \|x\| \to \infty \).

1.1 Autonomous Systems

In this section, we will consider the case when the Hamiltonian \( H(x) \) is autonomous. For the sake of simplicity, we shall also assume that it is \( C^1 \).

We shall first consider the question of nontriviality, within the general framework of \((A_\infty, B_\infty)-\)subquadratic Hamiltonians. In the second subsection, we shall look into the special case when \( H \) is \((0, b_\infty)-\)subquadratic, and we shall try to derive additional information.

The General Case: Nontriviality. We assume that \( H \) is \((A_\infty, B_\infty)-\)subquadratic at infinity, for some constant symmetric matrices \( A_\infty \) and \( B_\infty \), with \( B_\infty - A_\infty \) positive definite. Set:

\[ \gamma := \text{smallest eigenvalue of } B_\infty - A_\infty \]  \hfill (1)
\[ \lambda := \text{largest negative eigenvalue of } J \frac{d}{dt} + A_\infty . \]  \hfill (2)

Theorem 21 tells us that if \( \lambda + \gamma < 0 \), the boundary-value problem:

\[ \dot{x} = JH'(x) \]
\[ x(0) = x(T) \]  \hfill (3)
has at least one solution \( \varpi \), which is found by minimizing the dual action functional:

\[
\psi(u) = \int_0^T \left[ \frac{1}{2} \left( A^{-1}_o u, u \right) + N^*(-u) \right] dt
\]

on the range of \( \Lambda \), which is a subspace \( R(A)^2_L \) with finite codimension. Here

\[
N(x) := H(x) - \frac{1}{2} \left( A_\infty x, x \right)
\]

is a convex function, and

\[
N(x) \leq \frac{1}{2} \left( (B_{\infty} - A_{\infty}) x, x \right) + c \quad \forall x.
\]

**Proposition 1.** Assume \( H'(0) = 0 \) and \( H(0) = 0 \). Set:

\[
\delta := \liminf_{x \to 0} 2N(x) \|x\|^{-2}.
\]

If \( \gamma < -\lambda < \delta \), the solution \( \varpi \) is non-zero:

\[
\varpi(t) \neq 0 \quad \forall t.
\]

**Proof.** Condition (7) means that, for every \( \delta' > \delta \), there is some \( \varepsilon > 0 \) such that

\[
\|x\| \leq \varepsilon \Rightarrow N(x) \leq \frac{\delta'}{2} \|x\|^2.
\]

It is an exercise in convex analysis, into which we shall not go, to show that this implies that there is an \( \eta > 0 \) such that

\[
f \|x\| \leq \eta \Rightarrow N^*(y) \leq \frac{1}{2\delta'} \|y\|^2.
\]

**Fig. 1.** This is the caption of the figure displaying a white eagle and a white horse on a snow field.

Since \( u_1 \) is a smooth function, we will have \( \|hu_1\|_\infty \leq \eta \) for \( h \) small enough, and inequality (10) will hold, yielding thereby:

\[
\psi(hu_1) \leq \frac{h^2}{2} \frac{1}{\lambda} \|u_1\|_2^2 + \frac{h^2}{2} \frac{1}{\delta'} \|u_1\|^2.
\]
If we choose $\delta'$ close enough to $\delta$, the quantity $(\frac{1}{\lambda} + \frac{1}{\delta'})$ will be negative, and we end up with
\[
\psi(hu_1) < 0 \quad \text{for } h \neq 0 \text{ small}.
\] (12)

On the other hand, we check directly that $\psi(0) = 0$. This shows that 0 cannot be a minimizer of $\psi$, not even a local one. So $\pi \neq 0$ and $\pi \neq A_o^{-1}(0) = 0$. \(\Box\)

**Corollary 1.** Assume $H$ is $C^2$ and $(a_\infty, b_\infty)$-subquadratic at infinity. Let $\xi_1, \ldots, \xi_N$ be the equilibria, that is, the solutions of $H'(\xi) = 0$. Denote by $\omega_k$ the smallest eigenvalue of $H''(\xi_k)$, and set:
\[
\omega := \text{Min} \{\omega_1, \ldots, \omega_k\}.
\] (13)

If:
\[
\frac{T}{2\pi} b_\infty < -E\left[\frac{T}{2\pi} a_\infty \right] < \frac{T}{2\pi} \omega
\] (14)
then minimization of $\psi$ yields a non-constant $T$-periodic solution $\pi$.

We recall once more that by the integer part $E[\alpha]$ of $\alpha \in \mathbb{R}$, we mean the $a \in \mathbb{Z}$ such that $a < \alpha \leq a + 1$. For instance, if we take $a_\infty = 0$, Corollary 2 tells us that $\pi$ exists and is non-constant provided that:
\[
\frac{T}{2\pi} b_\infty < 1 < \frac{T}{2\pi}
\] (15)
or
\[
T \in \left(\frac{2\pi}{\omega}, \frac{2\pi}{b_\infty}\right).
\] (16)

**Proof.** The spectrum of $\Lambda$ is $\frac{2\pi}{T}\mathbb{Z} + a_\infty$. The largest negative eigenvalue $\lambda$ is given by $\frac{2\pi}{T} k_\omega + a_\infty$, where
\[
\frac{2\pi}{T} k_\omega + a_\infty < 0 \leq \frac{2\pi}{T}(k_\omega + 1) + a_\infty.
\] (17)

Hence:
\[
k_\omega = E\left[\frac{T}{2\pi} a_\infty \right].
\] (18)

The condition $\gamma < -\lambda < \delta$ now becomes:
\[
b_\infty - a_\infty < -\frac{2\pi}{T} k_\omega - a_\infty < \omega - a_\infty
\] (19)
which is precisely condition (14). \(\Box\)

**Lemma 1.** Assume that $H$ is $C^2$ on $\mathbb{R}^{2n} \setminus \{0\}$ and that $H''(x)$ is non-degenerate for any $x \neq 0$. Then any local minimizer $\bar{x}$ of $\psi$ has minimal period $T$. 

Proof. We know that \( \tilde{x} \), or \( \tilde{x} + \xi \) for some constant \( \xi \in \mathbb{R}^{2n} \), is a \( T \)-periodic solution of the Hamiltonian system:

\[
\dot{x} = JH'(x).
\] (20)

There is no loss of generality in taking \( \xi = 0 \). So \( \psi(x) \geq \psi(\tilde{x}) \) for all \( \tilde{x} \) in some neighbourhood of \( x \) in \( W^{1,2}(\mathbb{R}/TZ\mathbb{Z}; \mathbb{R}^{2n}) \).

But this index is precisely the index \( i_T(\tilde{x}) \) of the \( T \)-periodic solution \( \tilde{x} \) over the interval \((0,T)\), as defined in Sect. 2.6. So

\[
i_T(\tilde{x}) = 0.
\] (21)

Now if \( \tilde{x} \) has a lower period, \( T/k \) say, we would have, by Corollary 31:

\[
i_T(\tilde{x}) = i_{kT/k}(\tilde{x}) \geq k i_{T/k}(\tilde{x}) + k - 1 \geq k - 1 \geq 1.
\] (22)

This would contradict (21), and thus cannot happen. \( \square \)

Notes and Comments. The results in this section are a refined version of 1980; the minimality result of Proposition 14 was the first of its kind.

To understand the nontriviality conditions, such as the one in formula (16), one may think of a one-parameter family \( x_T, T \in (2\pi\omega^{-1}, 2\pi b^{-1}_\infty) \) of periodic solutions, \( x_T(0) = x_T(T) \), with \( x_T \) going away to infinity when \( T \to 2\pi\omega^{-1} \), which is the period of the linearized system at 0.

Table 1. This is the example table taken out of The \TeXbook, p. 246

| Year     | World population |
|----------|------------------|
| 8000 B.C.| 5,000,000        |
| 50 A.D.  | 200,000,000      |
| 1650 A.D.| 500,000,000      |
| 1945 A.D.| 2,300,000,000    |
| 1980 A.D.| 4,400,000,000    |

Theorem 1 (Ghoussoub-Preiss). Assume \( H(t,x) \) is \((0,\varepsilon)\)-subquadratic at infinity for all \( \varepsilon > 0 \), and \( T \)-periodic in \( t \)

\[
H(t,\cdot) \text{ is convex } \forall t
\] (23)

\[
H(\cdot,x) \text{ is } T-\text{periodic } \forall x
\] (24)

\[
H(t,x) \geq n(\|x\|) \text{ with } n(s)s^{-1} \to \infty \text{ as } s \to \infty
\] (25)

\[
\forall \varepsilon > 0, \ \exists c : H(t,x) \leq \frac{\varepsilon}{2} \|x\|^2 + c.
\] (26)
Assume also that $H$ is $C^2$, and $H''(t,x)$ is positive definite everywhere. Then there is a sequence $x_k$, $k \in \mathbb{N}$, of $kT$-periodic solutions of the system
\[ \dot{x} = JH'(t,x) \]  
such that, for every $k \in \mathbb{N}$, there is some $p_o \in \mathbb{N}$ with:
\[ p \geq p_o \Rightarrow x_{pk} \neq x_k . \]  
\( \Box \)

Example 1 (External forcing). Consider the system:
\[ \dot{x} = JH'(x) + f(t) \]
where the Hamiltonian $H$ is $(0,b_\infty)$-subquadratic, and the forcing term is a distribution on the circle:
\[ f = \frac{d}{dt}F + f_o \quad \text{with} \quad F \in L^2\left(\mathbb{R}/\mathbb{Z}; \mathbb{R}^{2n}\right) , \]
where $f_o := T^{-1} \int_0^T f(t)dt$. For instance,
\[ f(t) = \sum_{k \in \mathbb{N}} \delta_k \xi , \]
where $\delta_k$ is the Dirac mass at $t = k$ and $\xi \in \mathbb{R}^{2n}$ is a constant, fits the prescription. This means that the system $\dot{x} = JH'(x)$ is being excited by a series of identical shocks at interval $T$.

Definition 1. Let $A_\infty(t)$ and $B_\infty(t)$ be symmetric operators in $\mathbb{R}^{2n}$, depending continuously on $t \in [0,T]$, such that $A_\infty(t) \leq B_\infty(t)$ for all $t$.

A Borelian function $H : [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}$ is called $(A_\infty,B_\infty)$-subquadratic at infinity if there exists a function $N(t,x)$ such that:
\[ H(t,x) = \frac{1}{2} (A_\infty(t)x,x) + N(t,x) \]
\[ \forall t , \quad N(t,x) \quad \text{is convex with respect to} \quad x \]
\[ N(t,x) \geq n (\|x\|) \quad \text{with} \quad n(s)s^{-1} \to +\infty \quad \text{as} \quad s \to +\infty \]
\[ \exists c \in \mathbb{R} : \quad H(t,x) \leq \frac{1}{2} (B_\infty(t)x,x) + c \quad \forall x . \]

If $A_\infty(t) = a_\infty I$ and $B_\infty(t) = b_\infty I$, with $a_\infty \leq b_\infty \in \mathbb{R}$, we shall say that $H$ is $(a_\infty,b_\infty)$-subquadratic at infinity. As an example, the function $\|x\|^\alpha$, with $1 \leq \alpha < 2$, is $(0,\varepsilon)$-subquadratic at infinity for every $\varepsilon > 0$. Similarly, the Hamiltonian
\[ H(t,x) = \frac{1}{2} k \|k\|^2 + \|x\|^\alpha \]
is $(k,k+\varepsilon)$-subquadratic for every $\varepsilon > 0$. Note that, if $k < 0$, it is not convex.
Notes and Comments. The first results on subharmonics were obtained by Rabinowitz in 1985, who showed the existence of infinitely many subharmonics both in the subquadratic and superquadratic case, with suitable growth conditions on $H'$. Again the duality approach enabled Clarke and Ekeland in 1981 to treat the same problem in the convex-subquadratic case, with growth conditions on $H$ only.

Recently, Michalek and Tarantello (see Michalek, R., Tarantello, G. 1982 and Tarantello, G. 1983) have obtained lower bound on the number of subharmonics of period $kT$, based on symmetry considerations and on pinching estimates, as in Sect. 5.2 of this article.

References

Clarke, F., Ekeland, I.: Nonlinear oscillations and boundary-value problems for Hamiltonian systems. Arch. Rat. Mech. Anal. 78, 315–333 (1982)
Clarke, F., Ekeland, I.: Solutions périodiques, du période donnée, des équations hamiltoniennes. Note CRAS Paris 287, 1013–1015 (1978)
Michalek, R., Tarantello, G.: Subharmonic solutions with prescribed minimal period for nonautonomous Hamiltonian systems. J. Diff. Eq. 72, 28–55 (1988)
Tarantello, G.: Subharmonic solutions for Hamiltonian systems via a $\mathbb{Z}_p$ pseudoindex theory. Annali di Matematica Pura (to appear)
Rabinowitz, P.: On subharmonic solutions of a Hamiltonian system. Comm. Pure Appl. Math. 33, 609–633 (1980)
## Author Index

| Author      | Page |
|-------------|------|
| Abt I.      | 7    |
| Ahmed T.    | 3    |
| Andreev V.  | 24   |
| Andrieu B.  | 27   |
| Arpagaus M. | 34   |
| Babaev A.   | 25   |
| Bärwolff A. | 33   |
| Bán J.      | 17   |
| Baranov P.  | 24   |
| Barrelet E. | 28   |
| Bartel W.   | 11   |
| Bassler U.  | 28   |
| Beck H.P.   | 35   |
| Behrend H.-J.| 11   |
| Berger Ch.  | 1    |
| Bergstein H.| 1    |
| Bernardi G. | 28   |
| Bernet R.   | 34   |
| Besançon M.| 9    |
| Biddulph P. | 22   |
| Binder E.   | 11   |
| Bischoff A. | 33   |
| Blobel V.   | 13   |
| Borras K.   | 8    |
| Bosetti P.C.| 2    |
| Boudry V.   | 27   |
| Brasse F.   | 11   |
| Braun U.    | 2    |
| Braunschweig A.| 1    |
| Brisson V.  | 26   |
| Bünger L.   | 13   |
| Bürger J.   | 11   |
| Büscher F.W.| 13   |
| Buniatian A.| 11,37|
| Buschhorn G.| 25   |
| Campbell A.J.| 1    |
| Carli T.    | 25   |
| Charles F.  | 28   |
| Clarke D.   | 5    |
| Clegg A.B.  | 18   |
| Colombo M.  | 8    |
| Courau A.   | 26   |
| Coutures Ch.| 9    |
| Cozzika G.  | 9    |
| Criegee L.  | 11   |
| Cvach J.    | 27   |
| Dann A.W.E. | 22   |
| Dau W.D.    | 16   |
| Deffur E.   | 11   |
| Delcourt B. | 26   |
| Buono Del A.| 28   |
| Devel M.    | 26   |
| De Roeck A. | 11   |
| Dingus P.   | 27   |
| Dolifus C.  | 35   |
| Dreis H.B.  | 2    |
| Drescher A. | 8    |
| Düllmann D.| 13   |
| Dünger O.   | 13   |
| Duhm H.    | 12   |
| Ebbinghaus R.| 8    |
| Eberle M.  | 12   |
| Ebert J.   | 32   |
| Ebert T.R. | 19   |
| Efremenko V.| 23   |
| Egli S.    | 35   |
| Eichenberger S.| 35   |
| Eichner E. | 34   |
| Eisenhandler E.| 20   |
| Ellis N.N.| 3    |
| Ellison R.J.| 22   |
| Elsen E.   | 11   |
| Evrard E.  | 4    |
| Favart L.   | 4    |
| Feecken D.  | 13   |
| Felst R.   | 11   |
| Feltesse A.| 9    |
| Fensome I.F.| 3    |
| Ferrarotto F.| 31   |
| Flamm K.   | 11   |
| Flauger W. | 11   |
| Flieser M. | 25   |
| Flügge G.  | 2    |
| Name               | Number |
|--------------------|--------|
| Fomenko A.         | 24     |
| Fominykh B.        | 23     |
| Formánek J.        | 30     |
| Foster J.M.        | 22     |
| Franke G.          | 11     |
| Fretwurst E.       | 12     |
| Gabathuler E.      | 19     |
| Gamerdinger K.     | 25     |
| Garvey J.          | 3      |
| Gayler J.          | 11     |
| Gellrich A.        | 13     |
| Gennis M.          | 11     |
| Genzel H.          | 1      |
| Godfrey L.         | 7      |
| Goerlach U.        | 11     |
| Goerlich L.        | 6      |
| Gogitidze N.       | 24     |
| Goodall A.M.       | 19     |
| Gorelov I.         | 23     |
| Goritchev P.       | 23     |
| Grab C.            | 34     |
| Grässler R.        | 2      |
| Greenshaw T.       | 19     |
| Greif H.           | 25     |
| Grindhammer G.     | 25     |
| Haack J.           | 33     |
| Haidt D.           | 11     |
| Hamon O.           | 28     |
| Handschuh D.       | 11     |
| Hanlon E.M.        | 18     |
| Hapke M.           | 11     |
| Harjes J.          | 11     |
| Haydar R.          | 26     |
| Haynes W.J.        | 5      |
| Hedberg V.         | 21     |
| Heinzelmann G.     | 13     |
| Henderson R.C.W.   | 18     |
| Henschel H.        | 33     |
| Herynek I.         | 29     |
| Hildesheim W.      | 11     |
| Hill P.            | 11     |
| Hilton C.D.        | 22     |
| Hoeger K.C.        | 22     |
| Huet Ph.           | 4      |
| Hufnagel H.        | 14     |
| Huot N.            | 28     |
| Itterbeck H.       | 1      |
| Jabiol M.-A.       | 9      |
| Jacholkowska A.    | 26     |
| Jacobsson C.       | 21     |
| Jansen T.          | 11     |
| Jönsson L.         | 21     |
| Johannsen A.       | 13     |
| Johnson D.P.       | 4      |
| Jung H.            | 2      |
| Kalmus P.I.P.      | 20     |
| Kasarian S.        | 11     |
| Kaschowitz R.      | 2      |
| Kathage U.         | 16     |
| Kaufmann H.        | 33     |
| Kenyon I.R.        | 3      |
| Kermiche S.        | 26     |
| Kiesling C.        | 25     |
| Klein M.           | 33     |
| Kleinwort C.       | 13     |
| Knies G.           | 11     |
| Ko W.              | 7      |
| Köhler T.          | 1      |
| Kolanoski H.       | 8      |
| Kole F.            | 7      |
| Kolya S.D.         | 22     |
| Korbel V.          | 11     |
| Korn M.            | 8      |
| Kostka P.          | 33     |
| Kotelnikov S.K.    | 24     |
| Krehbiel H.        | 11     |
| Krücker D.         | 2      |
| Krüger U.          | 11     |
| Kubenka J.P.       | 25     |
| Kuhlen M.          | 25     |
| Kurča T.           | 17     |
| Kurzhófer J.       | 8      |
| Kuznik B.          | 32     |
| Lamarche F.        | 27     |
| Lander R.          | 7      |
| Landon M.P.J.      | 20     |
| Lange W.           | 33     |
| Lanius P.          | 25     |
| Laporte J.F.       | 9      |
| Lebedev A.         | 24     |
| Leuschner A.       | 11     |
| Levonian S.        | 11,24  |
| Lewin D.           | 11     |
| Ley Ch.            | 2      |
| Lindner A.         | 8      |
| Name              | Citation |
|-------------------|----------|
| Lindström G.      | 12       |
| Linsel F.         | 11       |
| Lipinski J.       | 13       |
| Loch P.           | 11       |
| Lohmander H.      | 21       |
| Lopez G.C.        | 20       |
| Magnussen N.      | 32       |
| Mani S.           | 7        |
| Marage P.         | 4        |
| Marshall R.       | 22       |
| Martens J.        | 32       |
| Martin A.@        | 19       |
| Martyn H.-U.      | 1        |
| Martyniak J.      | 6        |
| Masson S.         | 2        |
| Mavroidis A.      | 20       |
| McMahon S.J.      | 19       |
| Mehta A.          | 22       |
| Meier K.          | 15       |
| Mercer D.         | 22       |
| Merz T.           | 11       |
| Meyer C.A.        | 35       |
| Meyer H.          | 32       |
| Meyer J.          | 11       |
| Mikocki S.        | 6, 26    |
| Milone V.         | 31       |
| Moreau F.         | 27       |
| Moreels J.        | 4        |
| Morris J.V.       | 5        |
| Müller K.         | 35       |
| Murray S.A.       | 22       |
| Nagovizin V.      | 23       |
| Naroska B.        | 13       |
| Naumann Th.       | 33       |
| Newton D.         | 18       |
| Neyret D.         | 28       |
| Nguyen A.         | 28       |
| Niebergall F.     | 13       |
| Nisius R.         | 1        |
| Nowak G.          | 6        |
| Nyberg M.         | 21       |
| Oberlack H.       | 25       |
| Obrock U.         | 8        |
| Olsson J.E.       | 11       |
| Ould-Saada F.     | 13       |
| Pascaud C.        | 26       |
| Patel G.D.        | 19       |
| Peppel E.         | 11       |
| Phillips H.T.     | 3        |
| Phillips J.P.     | 22       |
| Pichler Ch.       | 12       |
| Pilgram W.        | 2        |
| Pitzl D.          | 34       |
| Prell S.          | 11       |
| Prosi R.          | 11       |
| Rädel G.          | 11       |
| Raupach F.        | 1        |
| Rauschnabel K.    | 8        |
| Reinshagen S.     | 11       |
| Ribarics P.       | 25       |
| Riech V.          | 12       |
| Riedlberger J.    | 34       |
| Rietz M.          | 2        |
| Robertson S.M.    | 3        |
| Robmann P.        | 35       |
| Roosen R.         | 4        |
| Royon C.          | 9        |
| Rudowicz M.       | 25       |
| Rusakov S.        | 24       |
| Rybicki K.        | 6        |
| Sahlmann N.       | 2        |
| Sanchez E.        | 25       |
| Savitsky M.       | 11       |
| Schacht P.        | 25       |
| Schiefer P.       | 14       |
| von Schlippe W.   | 20       |
| Schmidt D.        | 32       |
| Schnitz W.        | 2        |
| Schöning A.       | 11       |
| Schröder V.       | 11       |
| Schulz M.         | 11       |
| Schwab B.         | 14       |
| Schwind A.        | 33       |
| Seehausen U.      | 13       |
| Sell R.           | 11       |
| Semenov A.        | 23       |
| Shekelyan V.      | 23       |
| Shooshtari H.     | 25       |
| Shtarkov L.N.     | 24       |
| Siegmon G.        | 16       |
| Siewert U.        | 16       |
| Skillicorn I.O.   | 10       |
| Smirnov P.        | 24       |
| Smith J.R.        | 7        |
| Name          | Number |
|---------------|--------|
| Smolik L.     | 11     |
| Spitzer H.    | 13     |
| Staroba P.    | 29     |
| Steenbock M.  | 13     |
| Steffen P.    | 11     |
| Stella B.     | 31     |
| Stephens K.   | 22     |
| Stösslein U.  | 33     |
| Strachota J.  | 11     |
| Straumann U.  | 35     |
| Struczinski W.| 2      |
| Taylor R.E.   | 36,26  |
| Tchernyshov V.| 23     |
| Thiebaux C.   | 27     |
| Thompson G.   | 20     |
| Truöl P.      | 35     |
| Turnau J.     | 6      |
| Urban L.      | 25     |
| Usik A.       | 24     |
| Valkarova A.  | 30     |
| Vallée C.     | 28     |
| Van Esch P.   | 4      |
| Vartapetian A.| 11     |
| Vazdik Y.     | 24     |
| Verrecchia P. | 9      |
| Vick R.       | 13     |
| Vogel E.      | 1      |
| Wacker K.     | 8      |
| Walther A.    | 8      |
| Weber G.      | 13     |
| Wegner A.     | 11     |
| Wellisch H.P. | 25     |
| West L.R.     | 3      |
| Willard S.    | 7      |
| Winde M.      | 33     |
| Winter G.-G.  | 11     |
| Wolff Th.     | 34     |
| Wright A.E.   | 22     |
| Wulff N.      | 11     |
| Yiou T.P.     | 28     |
| Začek J.      | 30     |
| Zeitnitz C.   | 12     |
| Ziaeepour H.  | 26     |
| Zimmer M.     | 11     |
| Zimmermann W. | 11     |
Subject Index

Absorption 327
Absorption of radiation 289–292, 299, 300
Actinides 244
Aharonov-Bohm effect 142–146
Angular momentum 101–112
– algebraic treatment 391–396
Angular momentum addition 185–193
Angular momentum commutation relations 101
Angular momentum quantization 9–10, 104–106
Angular momentum states 107, 321, 391–396
Antiquark 83
α-rays 101–103
Atomic theory 8–10, 219–249, 327
Average value (see also Expectation value) 15–16, 25, 34, 357
Baker-Hausdorff formula 23
Balmer formula 8
Balmer series 125
Baryon 220, 224
Basis 98
Basis system 164, 376
Bell inequality 379–381, 382
Bessel functions 201, 313, 337
– spherical 304–306, 309, 313–314, 322
Bound state 73–74, 78–79, 116–118, 202, 267, 273, 306, 348, 351
Boundary conditions 59, 70
Bra 159
Breit-Wigner formula 80, 84, 332
Brillouin-Wigner perturbation theory 203
Cathode rays 8
Causality 357–359
Center-of-mass frame 232, 274, 338
Central potential 113–135, 303–314
Centrifugal potential 115–116, 323
Characteristic function 33
Clebsch-Gordan coefficients 191–193
Cold emission 88
Combination principle, Ritz’s 124
Commutation relations 27, 44, 353, 391
Commutator 21–22, 27, 44, 344
Compatibility of measurements 99
Complete orthonormal set 31, 40, 160, 360
Complete orthonormal system, see
Complete orthonormal set
Complete set of observables, see Complete set of operators
Eigenfunction 34, 46, 344–346
– radial 321
– calculation 322–324
EPR argument 377–378
Exchange term 228, 231, 237, 241, 268, 272
f-sum rule 302
Fermi energy 223
H$_2^+$ molecule 26
Half-life 65
Holzwarth energies 68