Experimental validation of the orthogonalised reverse path method using a nonlinear beam

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Abstract. The Orthogonalised Reverse Path (ORP) method is a new algorithm of the ‘reverse path’ class but developed in the time-domain. Like the Conditioned Reverse Path (CRP) method, the ORP approach is capable of identifying the underlying linear FRF of a system or structure in the presence of nonlinearities and may well also lead to simplifications in the estimation of coefficients of nonlinear terms. The method has shown itself to be numerically robust not only for simple simulated SDOF systems but also for simulated MDOF systems. The aim of this paper is to discuss an application of the ORP method to an experimental test set-up based on a nonlinear beam rig.

1. Introduction

The Orthogonalised Reverse Path (ORP) method is a new algorithm of the Reverse Path (RP) class for nonlinear system identification [1]. The RP methods are based on the idea of removing the effects of any nonlinearity from system input and output data in such a way that one can estimate the Frequency Response Function (FRF) of the underlying linear system. Unlike the previous state-of-the-art - the Conditioned Reverse Path (CRP) method [2] - which removed nonlinear components from spectra, the ORP method removes the nonlinear components from the initial time data, thus allowing completely standard FRF estimation algorithms to be used to compute the ‘linear’ FRFs. When the ORP was introduced in [1], it was validated only on simulated data generated from a number of lumped mass Multi-Degree-of-Freedom (MDOF) systems. The objective of the current paper is to demonstrate how the ORP method fares when faced with experimental data from a continuous system. The system of interest here consists of a cantilever beam with a geometrical nonlinearity. Data from the experiments were provided by Professor Gaetan Kerschen of the University of Liege, Belgium. The Liege researchers provided an analysis of their data based on the CRP method in [3], and this will allow a comparison with the ORP results obtained in the current paper.

2. Experimental set-up and problem formulation

The description of the experimental setup is taken from [3]; it involves a clamped beam with a thin beam part at the end of the main beam as shown in Figure 1. Both beams were made of steel. The dimensions of the thick (main) beam were: length 0.7 m, width 1.4 cm, thickness 1.4cm. The
dimensions of the thin beam were: length 4 cm, width 1.4 cm, thickness 0.5 mm. Seven accelerometers were placed at regular intervals on the main beam to measure responses. In addition, there was a displacement sensor located at the end of the main beam at position 7 (Figure 1). Force was applied at the location of the third sensor. The excitation was a white-noise sequence, band limited into the 0-1000 Hz range. The shaker at position 3 excited the structure in the horizontal plane in order to minimise the effects of gravity. Different excitation levels were considered in the 1.4 to 22 N RMS range.

![Figure 1. Experimental setup taken from [3].](image)

The stiffening effect of the thin beam part at the end of main beam was modelled in [1] with a grounded symmetrical nonlinear term of the form $z_1(y) = a_1 |y|^\alpha \text{sign}(y)$, where $y$ is the displacement at the end of the beam and $a_1$ is the nonlinear coefficient term. The exponent of $\alpha$ was determined by seeking the maximum value for the spectral mean of the averaged cumulative coherence of all the seven sensors as explained in detail in [1]. Because one might argue that there are good physical grounds for believing the nonlinearity to be cubic [4], the nonlinear term used in the main analysis here took the simpler form $z_1(y) = a_1 y^3$.

3. The ORP Algorithm

The CRP algorithm has been described in numerous publications including [2] and [3] cited here, so it will not be presented here. In contrast, the ORP algorithm is only recently developed, so a basic and brief summary will be given for completeness. As previously stressed, the ORP method carries out the decorrelation procedure that removes the effects of nonlinearity directly in the time-domain. Then, in obtaining the underlying linear FRF, one uses only conventional FRF estimation approaches. For a single nonlinear term, the conditioned output of the individual responses which are rid of the nonlinearity is given as follows.

$$y_{1(-1)}(t) = y_1(t) - \int_{-\infty}^{\infty} l_{y1}(t-\tau)z_1(t)dt$$
$$y_{\sigma(t-1)}(t) = y_\sigma(t) - \int_{-\infty}^{\infty} l_{y\sigma}(t-\tau)z_1(t)dt$$

The subscripts refer to the seven measured responses from the accelerometers; where it is unlikely to cause confusion the sensor subscripts will be omitted. In general one has,
\[ y_{(-1)}(t) = y(t) - \int_{-\infty}^{\infty} l_y(t - \tau)z(t)dt \quad (2) \]

where the integrand, \( l_y(t - \tau) \) is the inverse Fourier transform of the ‘FRF’,

\[ L_y(\omega) = \frac{S_{xy}(\omega)}{S_{xx}(\omega)} \quad (3) \]

where the \( S \) denote standard cross- and autospectra. Equation (2) is then approximated by a discrete sum in order to accommodate samples \( y_i, i = -M, \cdots, M \), where \( y_i = y(i\Delta t) \) and \( \Delta t \) is the sampling interval;

\[ y_{i(-1)} = y_i - \sum_{j=-M}^{M} l_y((i-j)\Delta t)z(j\Delta t)\Delta t = y_i - \sum_{j=-M}^{M} l_{(i-j)}y_j\Delta t \]

\[ = y_i - \sum_{j=-M}^{M} c_{ij}y_j \quad (4) \]

The response data are assumed to consist of \( N \) sample points, but the number of lags (\( M \)) to be removed, or the correlating terms of \( z_1(t) \) (as only one nonlinearity exists), is assumed much smaller (\( N \gg M \)). The details can be found in [1], but essentially ORP analysis proceeds by applying a Gram-Schmidt orthogonalisation algorithm (here using the built-in Matlab function \textit{orth} [5]) to compute the orthogonal complement of the nonlinear terms within the measured response. This is an iterative process which finally yields,

\[ \{y_{(-1)}\} = \{y\} - \sum_{j=-M}^{M} a_j\{Z_j\} = \{y\} - \sum_{j=-M}^{M} b_j\{W_j\} \quad (5) \]

Where the \( \{W_j\} \) are a set of orthonormal vectors spanning the space spanned by the (nonorthogonal) \( \{Z_j\} \) generated by the nonlinear terms. To condition the input force \( x_3 \) in the time domain a similar recursion can be used,

\[ \{x_{3(-1)}\} = \{x_3\} + \sum_{j=-M}^{M} y_j\{W_j\} \quad (6) \]

The conditioned (underlying linear) FRF can now be computed using standard (unconditioned) spectral estimation methods on the conditioned input and output time-series. Note that all suspected nonlinearities must be included in the equivalent of equation (1) in order that the sequence of removals of orthogonalised nonlinear terms will work in general. The linear FRFs from the input and output time-series are estimated with the \( H_1 \) and \( H_2 \) estimators,

\[ [H_1(\omega)] = [S_{xx}(\omega)]^{-1}[S_{xy}(\omega)] \quad (7) \]
\[ [H_2(\omega)] = [S_{yx}(\omega)]^{-1}[S_{yy}(\omega)] \quad (8) \]

Once the conditioned FRF (or FRF matrix for MDOF) has been obtained, the coefficients of the nonlinear terms can be estimated as in the standard CRP method [2]. There are two ways of using the method for finding the nonlinear coefficients as used for the ORP method in this paper, the details of the algebra can be found in [6]. The first approach (denoted ORP\textsubscript{1}) yields the coefficient estimate (note that this is actually a spectrum),

\[ a_1(\omega) = \frac{S_{21x3}(\omega)H_{23}^{\ast} - S_{21y3}(\omega)}{H_{2y}^{\ast}S_{21x1}(\omega)} \quad (9) \]
where \( H_{ij} \) denotes the conditioned FRF between excitation point \( i \) and response point \( j \). The second approach (ORP\(_2\)) yields the superficially similar,

\[
\alpha_i(\omega) = \frac{H_{ij}(\omega) S_{ij}(\omega) - S_{ij}(\omega)}{H_{ij}(\omega) S_{ij}(\omega)}
\]  

(10)

4. Discussion of experimental results

There are two sets of experimental results presented here corresponding to a very low excitation, and a high level of excitation. Using different excitation levels allows one to get some idea of the influence of nonlinearity in the system. It was shown in [1] that the \( H_2 \) FRF estimator [6] proved most effective in the nonlinear system analysis, so this estimator was used throughout here, although both \( H_1 \) and \( H_2 \) estimators were initially used on the data from the lowest level of excitation. Figure 2 shows the magnitude of \( H_{ij}(\omega) \) for the lowest excitation of the beam (1.4 N RMS) using both classical estimators. The FRFs clearly identified four peaks from the system; however closer inspection showed the magnitude of the FRFs dropping when using the \( H_1 \) estimator. Figure 2 shows that influence of the nonlinearity is absent as the FRFs plot are smooth and clear. The outcomes of the results are based on the following simulation and signal processing parameters used: the sampling frequency was 2560 Hz, spectra were calculated on the basis of 20 ensemble averages of 213 (8192) FFT points, a Hanning window was used, total number of samples =163840.

![Figure 2](image)

**Figure 2.** Classical (unconditioned) \( H_1 \) and \( H_2 \) estimates for the magnitude of \( H_{ij}(\omega) \) from the lowest excitation data (1.4 N RMS).

In order to see the influence of the nonlinearity on the structure, the highest excitation level (22 N RMS) data were then used to compute the magnitude of \( H_{ij}(\omega) \) which is shown in Figure 3. It was...
found that distortions appeared in the FRF when the excitation level was increased. The third and fourth modes were not much affected by the nonlinearity, so further analysis was concentrated on the range of 0-500 Hz.

![Figure 3](image)

**Figure 3.** Classical (unconditioned) $H_2$ estimate for the magnitude of $H_{73}(\omega)$ from the highest excitation data (22 N RMS).

It now remains to illustrate the capability of the ORP method in removing the FRF distortion and also to obtain the nonlinear coefficient. The CRP results provided in [3] will be used as a benchmark. Also, the FRF for the 1.4 N RMS level estimated using the $H_2$ (known as $H_{2\text{lin}}$) will be used as a proxy for the linear FRF for comparisons with the CRP and ORP estimators later. Note that the nonlinearity considered here is the simple cubic stiffness.

Figure 4 shows the first two modes of the FRFs for the highest level of excitation of 22 N RMS. It can be observed that the first two natural frequencies from the unconditioned estimators are shifted towards higher frequencies, and this is due to the stiffening effect of the thin beam part. It is very clear that both the CRP and ORP methods purge the effects of the nonlinearity and give very close agreement with the FRF of the ‘linear’ system.

Figure 5 displays the real and imaginary parts of the estimate of the nonlinear coefficient $\alpha_4$ found from CRP and equations (9) and (10) for the ORP methods. As stated earlier, the coefficients are frequency dependent and a spectral mean has to be performed to obtain a scalar value. Table 1 presents the spectral means of the estimated nonlinear coefficients from ORP for the 22 N RMS level of excitation and also for a level intermediate between the lowest and highest. The spectral means were taken between 20 to 250 Hz to avoid the noise at low frequencies in both methods. It is worth pointing out that the imaginary part of the coefficient, which is without any physical meaning, is always at least one order of magnitude below the real part.
Figure 4. Details of first (a) and second (b) resonance peaks for FRFs computed from data at highest level of excitation (22 N RMS).

Figure 5. Real (a) and imaginary (b) parts of nonlinear coefficient spectra from data at highest level of excitation (22 N RMS).

| Excitation level (N RMS) | Nonlinear coefficient, $a_1$ (N/m$^3$) |
|-------------------------|----------------------------------------|
|                         | ORP$_1$                                 |
|                         | $7.77 \times 10^9 - i6.73 \times 10^7$ |
|                         | ORP$_2$                                 |
|                         | $7.86 \times 10^9 - i2.02 \times 10^8$ |
|                         | CRP                                    |
| 11                      | $7.72 \times 10^9 - i1.61 \times 10^8$ |
| 22                      | $7.69 \times 10^9 - i1.59 \times 10^7$ |

Table 1. Spectral mean estimates of cubic nonlinear term coefficients at two higher levels of excitation.

In order to make a comparison with reference [3], the ORP analysis was repeated with a nonlinear term of the form $z_1(y) = a_1|y|^2\text{sign}(y)$, the results of this analysis are presented in Table 2. The agreement with the results from [3] is excellent.
| Excitation level (N RMS) | Nonlinear coefficient, \(a_3\) (N/m²⁸) | ORP₁ | ORP₂ | CRP (from [3]) |
|------------------------|----------------------------------------|------|------|---------------|
|                        |                                        | 1.94x10⁹ - i1.61x10⁷ | 1.96x10⁹ - i1.93x10⁷ | 1.94x10⁹ + i1.09x10⁸ |
| 11                     |                                        | 1.98x10⁹ + i3.26x10⁶ | 2.04x10⁹ + i4.76x10⁷ | 1.96x10⁹ + i1.55x10⁸ |
| 22                     |                                        |                  |                  |               |

Table 2. Spectral mean estimates of fractional power nonlinear term coefficients at two higher levels of excitation.

Summary

Nonlinearities introduce distortions in system FRFs which can be seen with the \(H_2\) estimate at higher excitation levels. Both ORP and CRP estimates show excellent results in purging the influences of nonlinearity and yielding the FRFs of the underlying linear system. Previous work with the ORP concentrated on simulated data; the aim of this paper has been to test the ORP method by applying it in an experimental test scenario where the CRP had already been found to be successful. The results were excellent and the ORP approach has showed its ability to deal with real experimental MDOF nonlinear systems under realistic conditions. A stable value for the nonlinear coefficient was identified that agreed well with a previous CRP estimate and the imaginary part of the coefficient was several orders of magnitude below the real part.

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