Some properties of transversely isotropic thermoelastic layer under initial stress

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Abstract. The problem of creation and applying of materials with determined properties necessitates of complication of materials mathematical models, which are predicting their behavior under conditions of initial stress and high temperatures and are able to take into account both the complex structure of materials and the presence of coupled elastic and temperature fields. In the framework of the linearized theory of thermo-elastic waves propagation coupled boundary dynamic problem of harmonic vibrations of thermoelastic semi-infinite solid is investigated. Vibrations are induced by distributed in a region on the body surface of the oscillating heat flow. Outside this region, the upper bound is insulated. The surface is free of mechanical stress. We constructed the solution of the contact problem analytically. The Greens function of the problem poles are obtained and shown graphically. The effect of initial stress and pre-heating on the dispersion curves are investigated.

1. Introduction
The registration of mechanical stresses in solids is an important goal of modern mechanics and non-destructive control. To solving this problem, a method using the thermoelastic effect is also used. With this approach, thermoelastic deformations generated in the object by a modulated laser beam are usually considered. In this connection, in this study, a mathematical model is proposed that describes the vibrations of an transversely isotropic medium under the action of heat flow that oscillates on its surface, simulating the effect of a frequency-modulated laser beam.

Green’s function plays an important role in the studies of mechanical and physical processes in many fields of the mechanics of deformable solids, including thermoelasticity. For investigation of thermal stresses Sharma [1] obtained the fundamental solution for transversely isotropic thermoelastic media in an integral form. Yu et al. [2] used the Green’s function for the two-phase isotropic thermoelastic solids with a point heat source for the investigation of thermoelastic stresses in bimaterials. For investigation of the thermoelastic field Chen et al. [3] obtained a compact 3-D general solution for transversely isotropic thermoelastic solids. Kumara and Singh [4] studied reflection and transmission of plane waves at an imperfectly bonded interface of two orthotropic generalized thermoelastic half-spaces. Chen et. all [5] obtained exact fundamental solution for transversely isotropic elastic medium containing a penny-shaped crack and investigated the distribution of thermal stresses. In this works the initial stresses and their influence on thermal effects are not considered. Singh [6] formulated the governing equations of thermoelasticity of transversely isotropic solid half-space with initial stresses and studied reflection from a thermally insulated stress free surface. In these works of pressure and their influence on thermoeffects was not considered. Using the theory of generalized thermoelasticity Singh and Chakraborty [7] studied the problem for reflection and refraction of plane thermoelastic wave and P- and SV- and thermal wave at a solid–liquid interface in the presence of initial stress. The variations of the amplitude ratios with the initial stress have also been shown. Kalinchuk et. all [8]
considered the problem of steady-state harmonic oscillations for an inhomogeneous thermoelastic medium with initial stress. This investigation is based on the numeric construction of the Green’s function of the medium. Green’s function play an essential role at research of dynamical contact interaction and connected with it problem of the solution construction for the integral equations with oscillated kernel. Babeshko et. all [9] developed the effective method of solution for the integral equations of the contact problems of the electroelasity and thermoelasticity. This method is based on detailed analysis of an arrangement of Green’s function poles and allow with high accuracy the medium dynamic properties. It allows to investigate in details the influence of initial stresses on thermoelastic effects. Levi et. all [10] studied a dynamic problem of the harmonic vibrations of a prestressed thermoelastic layered halfspace. The effect of the initial stresses and preheating on the heat flux distribution in the reaction region is investigated.

2. Formulation of the problem

We take rectangular Lagrange coordinate system \( \{x_1, x_2, x_3\} \) associated with the natural state of the medium, \( x_3 \)-axis is pointed vertically upward outside the layer \( x_1^0, x_2^0 < x_3 \leq h \).

Transversely isotropic thermoelastic medium, which belongs to materials of hexagonal symmetry class 6mm, is considered. The layer is exposed to initial stress at uniform temperature \( t = t_0 = \text{const} \).

Layer vibration is caused by either distributed on the surface in \( \Omega \) area tension field or heat flow \( q_0 e^{-i\omega t} \) ( \( \omega \) - circular oscillation frequency, \( q_0 = [q_1, q_2, q_3, q_4] \) is an extended vector of surface load, where \( q_4 = -\lambda_{33} \frac{\partial u_4}{\partial x_3} \) is a heat flow, \( u_4 \) is a temperature change). We assume that the surface out of loading region is stress-free and thermally insulated. The bottom of layer \( x_3 = 0 \) is assumed to be rigidly coupled with non-deformable base and heat insulated. We define an extended vector of surface deformation and temperature \( u^0 = [u_1, u_2, u_3, u_4] \) for the convenience. Thus the pre-stressed thermoelastic layer motion is described by ([6]):

\[
\nabla_0 \cdot \Theta = \rho_0 \frac{\partial^2 u}{\partial t^2} + \sum_{k=1}^{3} \sum_{i=1}^{3} c_{ijkl}^* \frac{\partial^2 u_k}{\partial x_i^2} - \beta_{ij}^* u_{ij}^* \quad \forall i, j = 1, 2, 3; \tag{1}
\]

heat conduction equation

\[
\sum_{i=1}^{3} \sum_{k=1}^{3} \lambda_{ik} \frac{\partial u_k}{\partial t} = \kappa \frac{\partial^2 u_4}{\partial t^2} + \tau_4 \sum_{i=1}^{3} \sum_{k=1}^{3} \beta_{ik}^* \frac{\partial u_k}{\partial t} \quad \forall \frac{c_4}{\kappa} = \frac{c_{ijkl}^{*}}{\tau_4}. \tag{2}
\]

The subscripts after the comma denote derivatives of the corresponding coordinates. Here elastic and thermal parameters of material with existing initial stress and pre-heating are defined by next relations ([7]):

\[
c_{ijkl}^* = \frac{\delta_{ij} c_{m} (v_m^2 - 1) + 2 c_{,m} v_m^2}{2} \quad \forall \beta_{ij}^* = v_i^* \beta_{ij}^* \tag{3}, \tag{4}
\]

In these relations \( \rho_0 \) is the material density in natural state, \( t_0 \) is the uniform temperature in natural state, \( \lambda_{ij} \) are the components of the thermal conductivity tensor, \( c_e \) - specific heat capacity at constant strain, \( \beta_{ij} - \) components of the thermoelastic constants tensor, \( v_i = 1 + \delta_i \), where \( \delta_i \) (\( k = 1, 2, 3 \)) are relative fiber extensions. Following [5], further we use the normalized parameters defined by next equations:
Here $E$ - thermoelastic relation constant, $\omega^*$ - normalized medium frequency, $V_L$ - velocity of undeformed material longitudinal wave. Taking into account the quantities (5) Eqs. (1),(2) are as follows:

$$L_i u_1 + L_{i2} u_2 + L_{i3} u_3 - \beta_{11} \frac{\partial}{\partial x_1} \hat{u}_4 = 0,$$

$$L_{31} u_1 + L_{32} u_2 + L_{33} u_3 - \beta_{22} \frac{\partial}{\partial x_2} \hat{u}_4 = 0,$$

$$L_{31} u_1 + L_{32} u_2 + L_{33} u_3 - \beta_{33} \frac{\partial}{\partial x_3} \hat{u}_4 = 0,$$

$$L_{41} u_1 + L_{42} u_2 + L_{43} u_3 + L_{44} u_4 = 0,$$

where

$$L_k = c_{ikkk} \frac{\partial^2}{\partial x_1^2} + c_{ijik} \frac{\partial^2}{\partial x_2^2} + c_{ijkk} \frac{\partial^2}{\partial x_3^2} + \omega^2,$$

$$L_{mk} = c_{mnmk} \frac{\partial^2}{\partial x_m \partial x_k} + c_{mnmk} \frac{\partial^2}{\partial x_m \partial x_k}, m,k = 1,2,3,$$

$$L_4 = \frac{\lambda_1}{\partial x_1^2} + \frac{\lambda_2}{\partial x_2^2} + \frac{\lambda_3}{\partial x_3^2} + i \omega \tau_1, L_{4k} = i \omega E \beta_{kk} \frac{\partial}{\partial x_k}.$$

Since we consider the steady state vibration that propagates in harmonic order $f = f_0 e^{-i \omega t}$, the exponential factor is omitted. Asterisks have been suppressed for the convenience.

Then dimensionless boundary conditions are as follows:

at the $x_3 = h$:

$$q^* = q_0^*, (x_1, x_2) \in \Omega$$

$$q^* = 0, (x_1, x_2) \notin \Omega$$

at the $x_3 = 0$:

$$u_1 = u_2 = u_3 = 0$$

$$- \lambda_{33} u_4 = 0$$
3. Solving the problem

The solution of the boundary value problem (6) - (8) is assumed of the form [7]

\[ u(x_1, x_2, x_3, t) = \frac{1}{4\pi^2} \int \int \mathbf{U}(\alpha_1, \alpha_2, x_3, \omega) e^{-i(\xi_1 x_1 + \xi_2 x_2 - \omega t)} d\alpha_2 d\alpha_1. \]

Here \( \mathbf{U}(\alpha_1, \alpha_2, x_3, \omega) \) is a Fourier transform of \( u(x_1, x_2, x_3, t) \), \( \alpha_1, \alpha_2 \) are parameters of the Fourier transform to the coordinates \( x_1, x_2 \) respectively.

Taking a two-dimensional Fourier ([11]-[13]) transform along \( x_1 \) and \( x_2 \) axis leads Eqs. (6) to:

\[
\begin{align*}
&c_{3113}U''_1 - P_1 U_1 - \alpha_1 \alpha_2 c_1 U_2 - i\alpha_1 c_2 U_3 + i\alpha_1 \beta_1 U_4 = 0 \\
&- \alpha_1 \alpha_2 c_1 U_1' + c_{3223} U''_2 - P_2 U_2 - i\alpha_2 c_3 U_3' + i\alpha_2 \beta_2 U_4 = 0 \\
&- i\alpha_1 c_2 U_1' - i\alpha_2 c_3 U_2' + c_{3333} U''_3 - P_3 U_3 + \beta_3 U_4 = 0 \\
&\alpha_1 \beta_1 E'U_1 + \alpha_2 \beta_2 E'U_2 + i\omega E' \beta_3 U_3' + \lambda_{33} U''_4 - P_4 U_4 = 0
\end{align*}
\]

And boundary conditions are written as follows:

\[
\begin{align*}
x_3 = h: & \quad \begin{cases} c_{3113}U_1' - i\alpha_1 c_{1313} U_3 = Q_1 \\
c_{3223} U_2' - i\alpha_2 c_{2323} U_3 = Q_2 \\
c_{3333} U_3' - i\alpha_3 c_{1133} U_1 - i\alpha_2 c_{2233} U_2 - \beta_3 U_4 = Q_3 \\
U_4' = -G \\
x_3 = 0: U_1 = U_2 = U_3 = 0, U_4' = 0
\end{cases}
\end{align*}
\]

Here \( G = \lambda_{33}^{-1} Q_4 \) is a Fourier transform of heat flow \( q_\tau, E^* = E \tau \). Prime denotes partial derivative with respect to \( x_3 \). In Eqs. (10) the following quantities were defined:

\[
\begin{align*}
P_k &= c_{1kk} \alpha_1^2 + c_{2kk} \alpha_2^2 - \omega^2, \quad k = 1,2,3, \\
P_4 &= \lambda_{11} \alpha_1^2 + \lambda_{22} \alpha_2^2 - i\omega \epsilon_1, \\
c_1 &= c_{1122} + c_{1212}, \quad c_2 = c_{1333} + c_{1313}, \quad c_3 = c_{2233} + c_{2323}.
\end{align*}
\]

Solution of Eqs. (10) with boundary conditions (11)-(12) will be found in the form ([3,4]):

\[
\begin{align*}
U_p &= -i\alpha_1 \sum_{k=1}^{4} f_{pk}[d_k \sin \sigma_k x_3 + d_{k+4} \cosh \sigma_k x_3], \quad p = 1,2 \\
U_2 &= \sum_{k=1}^{4} f_{2k}[d_k \sin \sigma_k x_3 + d_{k+4} \sinh \sigma_k x_3] \\
U_4 &= \sum_{k=1}^{4} f_{4k}[d_k \sin \sigma_k x_3 + d_{k+4} \cosh \sigma_k x_3]
\end{align*}
\]

In expressions (13) \( \sigma_k \) are the roots of characteristic equation ([7])
Factors $f_{mk}$ are solutions of equations homogeneous system with matrix

$$M = \begin{pmatrix}
c_{3113}\sigma_k^2 - P_1 & -\alpha_2^2 c_1 & c_2\sigma & -\beta_{11} \\
-\alpha_1^2 c_1 & c_{3222}\sigma_k^2 - P_2 & c_3\sigma & -\beta_{22} \\
-\alpha_1^2 c_2\sigma & -\alpha_2^2 c_1\sigma & c_{3333}\sigma_k^2 - P_3 & -\beta_{33}\sigma_k \\
-i\alpha_1^2\omega E^*\beta_{11} & -i\alpha_2^2\omega E^*\beta_{22} & i\omega E^*\beta_{33}\sigma_k & \lambda_{33}\sigma_k^2 - P_4
\end{pmatrix}$$

(14)

Unknowns $d_k, k = 1,2,..,8$ are obtained by substituting of (13) in the boundary conditions (11), (12). Therefore, to find $d_k$ we solve a system of linear algebraic equations:

$$A d = Q,$$

(16)

$$A = \begin{pmatrix}
l_{c_{11}} & l_{c_{12}} & l_{c_{13}} & l_{c_{14}} & l_{s_{11}} & l_{s_{12}} & l_{s_{13}} & l_{s_{14}} \\
l_{c_{21}} & l_{c_{22}} & l_{c_{23}} & l_{c_{24}} & l_{s_{21}} & l_{s_{22}} & l_{s_{23}} & l_{s_{24}} \\
l_{s_{31}} & l_{s_{32}} & l_{s_{33}} & l_{s_{34}} & l_{c_{31}} & l_{c_{32}} & l_{c_{33}} & l_{c_{34}} \\
l_{c_{41}} & l_{c_{42}} & l_{c_{43}} & l_{c_{44}} & l_{s_{41}} & l_{s_{42}} & l_{s_{43}} & l_{s_{44}} \\
0 & 0 & 0 & 0 & f_{11} & f_{12} & f_{13} & f_{14} \\
0 & 0 & 0 & 0 & f_{21} & f_{22} & f_{23} & f_{24} \\
f_{31} & f_{32} & f_{33} & f_{34} & 0 & 0 & 0 & 0 \\
l_{41} & l_{42} & l_{43} & l_{44} & 0 & 0 & 0 & 0
\end{pmatrix},$$

(17)

where

$$l_{s_{ik}} = l_{ik} \sinh \sigma_k h, \quad l_{c_{ik}} = l_{ik} \cosh \sigma_k h$$

$$l_{pk} = c_{3pp3}\sigma_k f_{pk} + c_{p3p3}f_{3k}, \quad p = 1,2,$$

$$l_{3k} = -\alpha_2^2 c_{1133} f_{1k} - \alpha_2^2 c_{2233} f_{2k} + \sigma_k c_{3333} f_{3k} - \beta_3 f_{4k}, \quad l_{4k} = \sigma_k f_{4k}$$

In Eqs. (16) $Q = \{Q_1, Q_2, Q_3, -G, 0, 0, 0, 0\}$ - the Fourier transform of $q_0^i$. The dispersion equation of the problem is $\det A = 0$. After finding $d_k$ from (16), the solution of the boundary value problem can be written as ([4]):

$$u^r(x_1, x_2, x_3) = \frac{1}{4\pi^2} \int_\Omega \int k(x_1 - \xi, x_2 - \eta, x_3, \omega) q_0^i(\xi, \eta) d\xi d\eta,$$

(18)

where $k(s, t, x, \omega) = \int_1^2 \int_1^2 K(\alpha_1, \alpha_2, x, \omega)e^{-i(\alpha_1s + \alpha_2t)}d\alpha_1d\alpha_2$ the matrix of Green's function, it's elements are obtained from follows relations:
Here $\Delta_0$ is determinant of matrix $A$ (16), $\Delta_{jk}$ is cofactor of matrix element with subscript $jk$, $j$ is an index of loading mode $Q_j$.

Applying to (19) the inverse Fourier transform the solution of initial boundary value problem (6) - (8) can be represented as follows:

$$u^*(x_1, x_2, x_3) = \frac{1}{4\pi^2} \int_{\Omega} \int k(x_1 - \xi, x_2 - \eta, x_3, \omega) q^*(\xi, \eta) d\xi d\eta,$$

(20)

$$k(s, t, x_3, \omega) = \int_{11} \int_{12} K(\alpha_1, \alpha_2, x_3, \omega) e^{-(\alpha_1 + \alpha_2^*)} d\alpha_1 d\alpha_2.$$

(21)

4. Numerical results and discussion

The feature of this class of problems is the presence of a countable set of complex zeros and poles of function (19) elements, some of which has a small imaginary part that causes the oscillation of the kernel of an integral expression (20). In order to study the effect of initial strain and preheating on layer surface waves propagation the considered problem is assumed to be plane, i.e. all field quantities are independent of $x_2$ and $\alpha_2 = 0$.

The material chosen for numerical calculations is CdSe ([5]). The physical data for a single crystal of CdSe material is given below:

- $\tau_0 = 298.0 \, \text{K}$, $\rho_0 = 5504 \, \text{kg m}^{-3}$,
- $c_{11} = 7.41*10^{10} \, \text{N m}^{-2}$, $c_{12} = 4.52*10^{10} \, \text{N m}^{-2}$, $c_{13} = 3.93*10^{10} \, \text{N m}^{-2}$,
- $c_{33} = 8.36*10^{10} \, \text{N m}^{-2}$, $c_{44} = 1.32*10^{10} \, \text{N m}^{-2}$,
- $c_{66} = 1.445*10^{10} \, \text{N m}^{-2}$, $\beta_{11} = 0.621*10^6 \, \text{N m}^{-2} \text{K}^{-1}$,
- $\beta_{33} = 0.551*10^6 \, \text{N m}^{-2} \text{K}^{-1}$, $\lambda_{11} = 9 \, \text{W m}^{-1} \text{K}^{-1}$,
- $\lambda_{33} = 9 \, \text{W m}^{-1} \text{K}^{-1}$, $c_v = 260 \, \text{J kg}^{-1} \text{K}^{-1}$, $a = 1$, $h = 1$.

The roots $\sigma_k^j = 1, 2, 3$ of the bicubic equation have been obtained numerically by using Kardano’s method. A problem of vibrations of layer subjected to temperature on it’s surface $u_{40}(x_1) \equiv 1$ $x_1 \in [-1, 1]$ is considered.

Fig. 1-2 shows the dispersion curves of the problem calculated for a limited frequency range. In Fig. 1 are shown curves with a small imaginary part, except for the second mode, which consists of two curves: 1. Leaves from A21, goes to C1 and ends in B21. 2. It starts at A22, in C2 it approaches C1 and sharply unfolds, then ends at B22. The curves B21C1 and B22C2 lie in the real plane. The curves in Fig. 2 are part of the countable family of poles that lie in the complex plane. We can see from Fig. 1 that if we do not take into account essentially complex modes, the distribution of the poles of the Green’s function that allows the thermoelastic effects and for the purely elastic problem, differ insignificantly.
In Fig. 2 the graphs of the first real mode, illustrating the effect of initial deformations and preheating on the boundary problem Green's function poles are shown. The digits 1, 2, 3 are denote by free, initial strain \( \nu = 1.01 \), pre-heating \( \tau_1 = 301K \) states respectively. As we can see, the initial stretching shifts the first mode down, preheating shifts it up. This means that by preliminary stresses one can increase the phase velocities of the waves, by heating – reduce, which allows to compensate for their action.

5. References
[1] Kotenko N V and Leniuk M P 1974 On dynamic effects in an elastic half-space under "thermal impact" J. Appl. Math. a. Mech. 38 1055-63
[2] Boiko M S 1985 Generalized dynamic problem of thermoelasticity for a half-space heated by laser radiation J. Appl. Math. a. Mech. 49 362-66
[3] Belyankova T I, Vorovich I I, Kalinshuk V V and Puzanov Yu E 1999 The dynamic contact problem of the thermoelastic layer News of hig. Educ. inst. North Caucasus reg. Nat. sc. 4 109-10
[4] Kalinshuk V V and Belyankova T I 2000 The problem of dynamic mixed electroelasticity and thermoelasticity problems of multilayered half-space investigating News of hig. Educ. inst. North Caucasus reg. Nat. sc. 3 72-74
[5] Sharma J N, Mohinder Pal and Dayal Chand 2005 Propagation characteristics of Rayleigh waves in transversely isotropic piezothermoelastic materials *J. of S. a. Vibr.* 284 227-48

[6] Sheydakov D N, Belyankova T I, Sheydakov N E and Kalinchuk V V 2008 The pre-stressed thermoelastic medium dynamics equations *Vestnik SSC RAS* 4 9-15

[7] Belyankova T I, Kalinchuk V V and Suvorova G Yu 2012 A dynamic contact problem for a thermoelastic pre-stressed layer *J. Appl. Math. a. Mech.* 76 537-46

[8] Belyankova T I and Lyzhov V A 2012 A coupled mixed problem for a system of electrodes on the surface of a prestressed electroelasti c structurally inhomogeneous half-space *J. Appl. Math. a. Mech.* 74 637-47

[9] Babeshko V A, Belyankova T I and Kalinchuk V V 2002 The method of fictitious absorption in problems of the theory of elasticity for an inhomogeneous half-space *J. Appl. Math. a. Mech.* 66 267-74

[10] Levi G Yu and Igumnov L A 2015 Some properties of the thermoelastic prestressed medium Green function *Mat. Phys. and Mech.* 23 42-46

[11] Levi M O, Levi G Yu and Lyzhov V A 2017 Some features of ferroelectric (ferromagnetic) heterostructures *J. of Appl. Mech. and Tech. Ph.* 58 47-53

[12] Levi G Yu, Mikhailova I B and Vorovich E I 2018 The dynamic mixed problem for a pre-stressed layered thermoelastic half-space *Science in the South of Russia* 14 11-20

[13] Levi M O and Kalinchuk V V 2017 Some features of the dynamics of electro-magneto-elastic half-space with initial deformations *Dyn. of Sys., Mech. and Mach. (Dynamics)* DOI: 10.1109/Dynamics.2017.8239478

**Acknowledgments**

The work was supported by Russian Science Foundation, project 14-19-01676 and Russian Foundation for basic Research, project 16-01-00647.