The Automorphism Group of the Free Group of Rank 2 Is a CAT(0) Group

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1. Introduction

A CAT(0) metric space is a proper complete geodesic metric space in which each geodesic triangle with side lengths $a$, $b$, and $c$ is “at least as thin” as the Euclidean triangle with side lengths $a$, $b$, and $c$ (see [5] for details). We say that a finitely generated group $G$ is a CAT(0) group if $G$ may be realized as a cocompact and properly discontinuous subgroup of the isometry group of a CAT(0) metric space $X$. Equivalently, $G$ is a CAT(0) group if there exists a CAT(0) metric space $X$ and a faithful geometric action of $G$ on $X$. It is perhaps not standard to require that the group action be faithful, a point we address in Remark 1.

For each integer $n \geq 2$, we write $F_n$ for the free group of rank $n$ and $B_n$ for the braid group on $n$ strands.

In [3], Brady exhibited a subgroup $H \leq \text{Aut}(F_2)$ of index 24 that acts faithfully and geometrically on a CAT(0) 2-complex. In subsequent work [4], the same author showed that $B_4$ acts faithfully and geometrically on a CAT(0) 3-complex. It follows that $\text{Inn}(B_4)$ acts faithfully and geometrically on a CAT(0) 2-complex $X_0$ (this fact is explained explicitly by Crisp and Paoluzzi in [8]). Now, $\text{Inn}(B_n)$ has index 2 in $\text{Aut}(B_n)$ [10], and $\text{Aut}(F_2)$ is isomorphic to $\text{Aut}(B_4)$ [16, 10]; thus the result in the title of this paper is proved if we exhibit an extra isometry of $X_0$ that extends the faithful geometric action of $\text{Inn}(B_4)$ to a faithful geometric action of $\text{Aut}(B_4)$. We do this in Section 2.

In the language of [14], $X_0$ is a systolic simplicial complex. By [14, Thm. 13.1], a group that acts simplicially, properly discontinuously, and cocompactly on such a space is biautomatic. Since the action of $\text{Aut}(F_2)$ provided here is of this type, it follows that $\text{Aut}(F_2)$ is biautomatic.

Our results reinforce the striking contrast between those properties enjoyed by $\text{Aut}(F_2)$ and those enjoyed by the automorphism groups of finitely generated free groups of higher ranks. We can now say that $\text{Aut}(F_2)$ is a CAT(0) group, that it is a biautomatic group, and that it has a faithful linear representation [9; 16]; while $\text{Aut}(F_n)$ is neither a CAT(0) group [12] nor a biautomatic group [6], and it does not have a faithful linear representation [11] whenever $n \geq 3$. 

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We regard the CAT(0) 2-complex $X_0$ as a geometric companion to the Auter space (of rank 2) [13], a topological construction equipped with a group action by $\text{Aut}(F_2)$.

Let $W_3$ denote the universal Coxeter group of rank 3—that is, $W_3$ is the free product of three copies of the group of order 2. Since $\text{Aut}(F_2)$ is isomorphic to $\text{Aut}(W_3)$ (see Remark 2), we also learn that $\text{Aut}(W_3)$ is a CAT(0) group.

**Remark 1.** As pointed out in the opening paragraph, our definition of a CAT(0) group is perhaps not standard because of the requirement that the group action be faithful. We note that such a requirement is redundant when giving an analogous definition of a word hyperbolic group. This follows from the fact that word hyperbolicity is an invariant of the quasi-isometry class of a group. In contrast, the CAT(0) property is not an invariant of the quasi-isometry class of a group. Examples are known of two quasi-isometric groups, one of which is CAT(0) and the other of which is not. Examples of this type may be constructed using the fundamental groups of graph manifolds [15] and the fundamental groups of Seifert fibre spaces [1; 5, p. 258]. So the adjective “faithful” is not so easily discarded in our definition of a CAT(0) group. We do not know of two abstractly commensurable groups, one of which is CAT(0) and the other of which is not. We pose the following question.

**Question 1.** Is the property of being a CAT(0) group an invariant of the abstract commensurability class of a group?

Some relevant results in the literature show that two natural approaches to this question do not work in general. If $G$ acts geometrically on a CAT(0) space $X$ and $G'$ is a finite extension with $[G': G] = n$, then $G'$ acts properly and isometrically on the CAT(0) space $X^n$ with the product metric [7, p. 190; 18]. However, proving that this action is cocompact is either difficult or impossible in general. In [2], the authors give examples of the following type: $G$ is a group acting faithfully and geometrically on a CAT(0) space $X$, $G'$ is a finite extension of $G$, yet $G'$ does not act faithfully and geometrically on $X$. However, $G'$ may act faithfully and geometrically on some other CAT(0) space.

**Remark 2.** The fact that $\text{Aut}(F_2)$ is isomorphic to $\text{Aut}(W_3)$ appears to be well known in certain mathematical circles, but it is rarely recorded explicitly. We now outline a proof: The subgroup $E \leq W_3$ of even-length elements is isomorphic to $F_2$ and characteristic in $W_3$, and $C_{W_3}(E) = \{1\}$; it follows from [17, Lemma 1.1] that the induced homomorphism $\pi : \text{Aut}(W_3) \to \text{Aut}(E)$ is injective. One easily confirms that the image of $\pi$ contains a set of generators for $\text{Aut}(E)$, and hence $\pi$ is an isomorphism. A topological proof may also be constructed using the fact that the subgroup $E$ of even-length words in $W_3$ corresponds to the 2-fold orbifold cover of the the orbifold $S^2(2, 2, 2, \infty)$ by the once-punctured torus.

The authors would like to thank Jason Behrstock and Martin Bridson for pointing out the examples in [1; 5, p. 258; 15] and Luisa Paoluzzi for discussions regarding [8].
2. \( \text{Aut}(B_4) \) Is a CAT(0) Group

We shall describe an apt presentation of \( B_4 \), give a concise combinatorial description of \( \text{Inn}(B_4) \), describe the faithful geometric action of \( \text{Inn}(B_4) \) on \( X_0 \), and, finally, introduce an isometry of \( X_0 \) to extend the action of \( \text{Inn}(B_4) \) to a faithful geometric action of \( \text{Aut}(B_4) \).

The interested reader will find an informative, and rather more geometric, account of \( X_0 \) and the associated action of \( \text{Inn}(B_4) \) in [8].

An Apt Presentation of \( B_4 \). A standard presentation of the group \( B_4 \) is

\[
\langle a, b, c \mid aba = bab, bcb =cbc, ac = ca \rangle. \tag{1}
\]

Introducing generators \( d = (ac)^{-1}b(ac), e = a^{-1}ba, \) and \( f = c^{-1}bc \), one may verify that \( B_4 \) is also presented by

\[
\langle a, b, c, d, e, f \mid ba = ae = eb, de = ec = cd, bc = cf = fb, df = fa = ad, ca = ac, ef = fe \rangle. \tag{2}
\]

We set \( x = bac \) and write \( \langle x \rangle \subset B_4 \) for the infinite cyclic subgroup generated by \( x \). The center of \( B_4 \) is the infinite cyclic subgroup generated by \( x^4 \).

The Space \( X_0 \). Consider the 2-dimensional piecewise Euclidean CW-complex \( X_0 \) constructed as follows:

(0-S) the vertices of \( X_0 \) are in one-to-one correspondence with the left cosets of \( \langle x \rangle \) in \( B_4 \)—we write \( v_{g\langle x \rangle} \) for the vertex corresponding to the coset \( g\langle x \rangle \);

(1-S) distinct vertices \( v_{g_1\langle x \rangle} \) and \( v_{g_2\langle x \rangle} \) are connected by an edge of unit length if and only if there exists an element \( \ell \in \{a, b, c, d, e, f\} \) such that \( g_2^{-1}g_1 \ell \in \langle x \rangle \);

(2-S) three vertices \( v_{g_1\langle x \rangle}, v_{g_2\langle x \rangle}, \) and \( v_{g_3\langle x \rangle} \) are the vertices of a Euclidean (equilateral) triangle if and only if the vertices are pairwise adjacent.

The link of the vertex \( v_{g\langle x \rangle} \) in \( X_0 \), just like the link of each vertex in \( X_0 \), consists of twelve vertices (one for each of the cosets represented by elements in \( \{a, b, c, d, e, f\} \)) and sixteen edges (one for each of the distinct ways to spell \( x \) as a word of length 3 in the alphabet \( \{a, b, c, d, e, f\} \)—see [8] for more details). It can be viewed as the 1-skeleton of a Möbius strip. In Figure 1 we depict the infinite cyclic cover of the link of \( v_{g\langle x \rangle} \). Each vertex with label \( g \) in the figure lies above the vertex \( v_{g\langle x \rangle} \) in the link of \( v_{g\langle x \rangle} \). The link is formed by identifying identically labeled vertices and identifying edges with the same start and end points.

![Figure 1](image-url)
Thus the restriction of \( \rho \) above, the Brady’s Faithful Geometric Action of argument.

simply connected. We shall not digress from the task at hand to provide such an

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berly discontinuous subgroup of \( \text{Isom} \) to vertices in the link of \( X \). It is easily seen that \( X_0 \) satisfies the link condition because each injective loop in Figure 1 crosses at least six edges and each edge has length \( \pi/3 \). Thus one might show that \( X_0 \) is CAT(0) by showing that it is simply connected. We shall not digress from the task at hand to provide such an argument.

**Brady’s Faithful Geometric Action of \( \text{Inn}(B_4) \) on \( X_0 \).** We shall describe Brady’s faithful geometric action of \( \text{Inn}(B_4) \) on \( X_0 \). We shall do so by describing an isometric action \( \rho: B_4 \rightarrow \text{Isom}(X_0) \) such that the image of \( \rho \) is a properly discontinuous and cocompact subgroup of \( \text{Isom}(X_0) \) that is isomorphic to \( \text{Inn}(B_4) \).

It follows immediately from \((1-S)\) that, for each \( g \in B_4 \), the “left-multiplication by \( g \)” map on the 0-skeleton of \( X_0, g_1(x) \mapsto g g_1(x) \), extends to a simplicial isometry of the 1-skeleton of \( X_0 \). It follows immediately from \((2-S)\) that any simplicial isometry of the 1-skeleton of \( X_0 \) extends to a simplicial isometry of \( X_0 \). We write \( \phi_g \) for the isometry of \( X_0 \) determined by \( g \) in this way, and we write \( \rho: B_4 \rightarrow \text{Isom}(X_0) \) for the map \( g \mapsto \phi_g \). We compute that \( \rho(g_1 g_2)(v_{g_1(\mathbf{x})}) = v_{g_1 g_2(\mathbf{x})} = \rho(g_1)\rho(g_2)(v_{g_1(\mathbf{x})}) \) for each \( g_1, g_2, g \in B_4 \), so \( \rho \) is a homomorphism. Further, \( \phi_g(v_{g_1(\mathbf{x})}) = v_{g_2(\mathbf{x})} \) for each \( g, g \in B_4 \), so the vertices of \( X_0 \) are contained in a single \( \rho \)-orbit. It follows that \( \rho \) is a cocompact isometric action of \( B_4 \) on \( X_0 \).

To show that the image of \( \rho \) is isomorphic to \( \text{Inn}(B_4) \), it suffices to show that the kernel of \( \rho \) is exactly the center of \( B_4 \). One easily computes that \( \rho(x^4) \) is the identity isometry of \( X_0 \). Thus the kernel of \( \rho \) contains the center of \( B_4 \). It is also clear that the stabilizer of \( v_{\langle \mathbf{x} \rangle} \), which contains the kernel of \( \rho \), is the infinite subgroup \( \langle \mathbf{x} \rangle \). So to establish that the kernel of \( \rho \) is exactly the center of \( B_4 \), it suffices to show that \( \phi_{x}, \phi_{x^2}, \phi_{x^3} \) are nontrivial and distinct isometries of \( X_0 \). We achieve this by showing that these elements act nontrivially and distinctly on the link of \( v_{\langle \mathbf{x} \rangle} \) in \( X_0 \). We compute that \( x \) acts as follows on the cosets corresponding to vertices in the link of \( v_{\langle \mathbf{x} \rangle} \), where \( \delta = \pm 1 \):

\[
\begin{align*}
\phi_x^{\delta}(x) & \mapsto e^{\delta}(x) \mapsto c^{\delta}(x) \mapsto f^{\delta}(x) \mapsto a^{\delta}(x) \quad \text{and} \quad b^{\delta}(x) \leftrightarrow d^{\delta}(x).
\end{align*}
\]

Thus the restriction of \( \phi_x \) to the link of \( v_{\langle \mathbf{x} \rangle} \) may be understood, with reference to Figure 2, as translation two units to the right followed by reflection across the horizontal dotted line. It follows that \( \phi_{x}, \phi_{x^2}, \phi_{x^3} \) are nontrivial and distinct isometries of \( X_0 \), as required.

We next show that the image of \( \rho \) is a properly discontinuous subgroup of \( \text{Isom}(X_0) \). Now, the action \( \rho \) is not properly discontinuous because, as noted above, the \( \rho \)-stabilizer of \( v_{\langle \mathbf{x} \rangle} \) is the infinite subgroup \( \langle \mathbf{x} \rangle \) (so infinitely many elements of \( B_4 \) fail to move the unit ball about \( v_{\langle \mathbf{x} \rangle} \) off itself). But the image of \( \langle \mathbf{x} \rangle \) under the map \( B_4 \rightarrow \text{Inn}(B_4) \) has order 4. It follows that the image of \( \rho \) is a properly discontinuous subgroup of \( \text{Isom}(X) \).
Thus we have that the image of $\rho$ is a properly discontinuous and cocompact subgroup of $\text{Isom}(X_0)$ that is isomorphic to $\text{Inn}(B_4)$.

**EXTENDING $\rho$ BY FINDING ONE MORE ISOMETRY.** It was shown in [10] that the unique nontrivial outer automorphism of $B_n$ is represented by the automorphism that inverts each of the generators in presentation (1). Consider the automorphism $\tau \in \text{Aut}(B_4)$ determined by

$$a \mapsto a^{-1}, \quad b \mapsto b^{-1}, \quad c \mapsto c^{-1}, \quad d \mapsto b^{-1}, \quad e \mapsto f^{-1}, \quad f \mapsto e^{-1}.$$  

Note that $\tau$ is achieved by first applying the automorphism that inverts each of the generators $a, b, c$, and then applying the inner automorphism $w \mapsto (ac)^{-1}w(ac)$ for each $w \in B_4$. It follows that $\tau$ is an involution that represents the unique nontrivial outer automorphism of $B_4$. Writing $J := B_4 \rtimes \mathbb{Z}_2$, we have $\text{Aut}(B_4) \cong J/(x^4)$. We identify $B_4$ with its image in $J$. The automorphism $\tau \in \text{Aut}(B_4)$ permutes the elements of $\{a, b, c, d, e, f\}^\pm$ and maps the subgroup $\langle x \rangle$ to itself (in fact, $\tau(x) = x^{-1}$). It follows from (1-S) that the map $v_{g(\langle x \rangle)} \mapsto v_{\tau(g(\langle x \rangle))}$ on the 0-skeleton of $X_0$ extends to a simplicial isometry of the 1-skeleton of $X_0$ and hence also to a simplicial isometry $\theta$ of $X_0$. We compute that $\theta \phi_g \theta = \phi_{\tau(g)}$ for each $g \in B_4$. Thus we have an isometric action $\rho' : J \to \text{Isom}(X_0)$ given by

$$g \mapsto \phi_g \quad \text{for each} \ g \in B_4 \quad \text{and} \quad \tau \mapsto \theta.$$  

We also compute that the restriction of $\theta$ to the link of $v_{\langle x \rangle}$ may be understood as reflection across the vertical dotted line shown in Figure 2. It follows that $\theta$ is a nontrivial isometry of $X_0$ that is distinct from $\phi_1, \phi_2, \phi_3$. Thus the kernel of $\rho'$ is still the center of $B_4$, and the image of $\rho'$ is a properly discontinuous and cocompact subgroup of $\text{Isom}(X_0)$ that is isomorphic to $\text{Aut}(B_4)$. Hence we have a faithful geometric action of $\text{Aut}(B_4)$ on $X_0$, as required.

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