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American Journal of Mathematics, Volume 132, Number 2, April 2010, pp. 257-295 (Article)

Published by Johns Hopkins University Press

DOI: https://doi.org/10.1353/ajm.0.0104

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MINIMAL SURFACES IN THE THREE SPHERE BY DOUBLING
THE CLIFFORD TORUS

By NIKOLAOS KAPOULEAS and SEONG-DEOG YANG

Abstract. We construct embedded closed minimal surfaces in the round three-sphere \( \mathbb{S}^3(1) \), resembling two parallel copies of the Clifford torus, joined by \( m^2 \) small catenoidal bridges symmetrically arranged along a square lattice of points on the torus.

1. Introduction.

Historical background and the general idea. An interesting possible general construction for minimal surfaces is motivated by examples of minimal surfaces which resemble two copies of a minimal surface joined together with many catenoidal bridges. Karcher, Pinkall, and Sterling have constructed [17] minimal surfaces resembling roughly an equatorial sphere in \( \mathbb{S}^3(1) \) which has been “doubled”, and the two sheets have been connected by necks arranged at the vertices of a Platonic solid, with the corresponding symmetry imposed. The examples constructed this way are finitely many, because the Platonic solids are finitely many and the size of the bridges is determined by the neck configuration (the number and positions of the catenoidal bridges). Pitts and Rubinstein have discussed [20] constructions for discrete families of minimal surfaces, where the size of the catenoidal bridges used can be arbitrarily small, and then the number of the bridges (and hence the genus) tends to infinity, while the surfaces tend to a limit varifold. These constructions are highly symmetrical. Some of the constructions have a limit varifold which is a minimal surface counted with multiplicity two. Wohlgemuth [23] constructed minimal surfaces which resemble two catenoids connected by a “ring” of catenoidal bridges. These examples are also highly symmetrical, and in the limit as the size of the bridges tends to 0, their number tends to infinity, the minimal surfaces tend to a doubly covered catenoid, and the bridges concentrate around a circle on the limit catenoid.

We call such constructions “doubling constructions” as suggested in [14]. A motivation for general doubling constructions is problem 88 in the list of open problems proposed by S.-T. Yau in 1982 [26]. In this problem it is required to establish that there are infinitely many minimal surfaces in any three-dimensional
Riemannian manifold. A general doubling construction would reduce this question to the existence of a single minimal surface satisfying the appropriate necessary conditions. The doubling construction then would allow doubling this minimal surface to produce infinitely many minimal surfaces which would tend as varifolds to a double covering of the given minimal surface, as the number of the catenoidal bridges introduced tends to infinity, and the size of the bridges tends to 0. In more detail the ingredients for such a doubling would be the minimal surface $\Sigma$ in the Riemannian three-manifold, two nearby copies of $\Sigma$, $\Sigma_1$ and $\Sigma_2$, and a set of points $L \subset \Sigma$. $\Sigma_1$ and $\Sigma_2$ can be thought of as the graphs of two functions $\phi_1$ and $\phi_2$ on $\Sigma$. $\phi_1$ and $\phi_2$ are assumed to be small and with small derivatives. The minimal surface constructed would consist of a region $M_\Sigma$ which approximates $\Sigma_1$ and $\Sigma_2$ minus small discs, and a collection of regions which approximate small truncated catenoids. The discs removed are centered at the points on $\Sigma_1$ and $\Sigma_2$ corresponding to the points of $L$. The catenoidal regions serve as bridges connecting to $M_\Sigma$ at the boundaries of the removed discs. We call directions perpendicular to $\Sigma$ “vertical” and directions along $\Sigma$ “horizontal”. The axes of the catenoidal regions would be approximately vertical.

Since a Riemannian manifold at small scale is approximately Euclidean, we can use horizontal and vertical (approximate) translations to find balancing obstructions to the existence of such surfaces. More precisely we can consider the force $F$ exerted by the region close to $\Sigma_1$ to a catenoidal bridge, and the force $F_c$ exerted through the waist of the bridge to the part of the catenoidal bridge closer to $\Sigma_1$, by the other part. The vertical component of $F_c$ is approximately equal to the length of its waist. (Balancing for minimal surfaces is based simply on the first variation formula [18], [22]. For a general discussion in the current context see [14].) If $F$ is intercepted at a suitable curve which can be approximated by a curve on $\Sigma_1$ enclosing a domain $\Omega \subset \Sigma_1$, then the vertical component of $F$ can be approximated by the integral of the mean curvature of $\Sigma_1$ on $\Omega$. Because of the smallness assumptions for $\phi_1$, we ignore the nonlinear terms and the derivatives, and then the mean curvature is approximated by $(|A|^2 + \text{Ric} (\nu, \nu)) \phi_1$, and the vertical component of $F$ by the area of $\Omega$ times the value of $(|A|^2 + \text{Ric} (\nu, \nu)) \phi_1$ at the corresponding point of $L$. This quantity has to balance the vertical component of $F_c$, which is approximately $2\pi \tau$, where $\tau$ is the size (radius of the waist) of the catenoidal bridge. We have then

\begin{equation}
\text{Area}(\Omega) \left( |A|^2 + \text{Ric} (\nu, \nu) \right) \phi_1 = 2\pi \tau.
\end{equation}

The above heuristic argument suggests that a necessary condition for a doubling construction is that the mean curvature of the parallel surfaces to $\Sigma$ points away from $\Sigma$, which in general amounts to

\begin{equation}
|A|^2 + \text{Ric} (\nu, \nu) > 0 \quad \text{on} \quad \Sigma.
\end{equation}
This condition then ensures that the vertical components of $F$ and $F_c$ point in opposite directions. Equivalently it ensures that the mean curvature of $\Sigma_1$ (or $\Sigma_2$), has the opposite sign from the sign of the mean curvature introduced by the extra bending of the catenoidal bridge needed to attach it smoothly to $\Sigma_1$ (or $\Sigma_2$).

Vertical component balancing considerations, as in 1.1, relate the size of $\phi_1$ and $\phi_2$ with the size of the catenoidal bridge and the area of $\Omega$. Since the matching of the catenoidal bridge to $\Sigma_1$ and $\Sigma_2$ gives further relations between $\phi_1$, $\phi_2$, and the area of $\Omega$ can be guessed from $L$, the construction should be determined completely by $L$. Horizontal force considerations should further restrict the possible configurations $L$ and the sizes of the catenoidal bridges.

Developing in detail such a general construction is beyond the scope of this paper. Instead we present a particular doubling construction where $\Sigma = \mathbb{T}$, the Clifford torus in the unit three-sphere $S^3(1)$, and the neck configuration $L$ is a square lattice of points on $\mathbb{T}$. Because of the high symmetry involved the construction simplifies significantly, in particular we do not need to consider horizontal forces. This construction has been outlined in [14]. The method used is a gluing Partial Differential Equations method. The particular kind of methods used relates most closely to the methods developed in [6], [21], especially as they evolved and were systematized in [9], [10], [11]. We refer the reader to [14] for a general discussion of this methodology. We state now a rough version of the Main Theorem 5.4 of this paper.

**Theorem 1.3.** If $m \in \mathbb{N}$ is large enough, then there exists a closed minimal surface $M_m$ of genus $m^2 + 1$, embedded in the unit sphere $S^3(1) \subset \mathbb{C}^2$, which can be obtained by a “doubling” construction as outlined above, from the Clifford torus $\mathbb{T} = S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2}) \subset S^3(1)$, by using $m^2$ catenoidal bridges centered at the points of an $m \times m$ lattice $L \subset \mathbb{T}$, to connect two nearby copies of $\mathbb{T}$. $M_m$ is symmetric under the group $G$ of isometries of $S^3(1)$ which fix $L$. As $m \to \infty$, $M_m$ tends as a varifold to the Clifford torus $\mathbb{T}$ with multiplicity two.

We hope that this paper is a significant first step towards understanding the possibility of general doubling constructions. A way to pursue this goal would be by relaxing gradually the symmetry assumptions. The simplest first generalization would be to replace the square lattice $L$ with rectangular lattices $m \times km$ or $k \times m$ for $m$ large enough in terms of $k$. Such a configuration does not allow symmetries exchanging the two sides of $\mathbb{T}$, or equivalently the two copies of $\mathbb{T}$ being used. This implies that the dimension of the kernel involved is higher, and therefore the initial surfaces will depend on more parameters, making the construction more complicated (see also the discussion before 4.14). A further step where horizontal forces start playing a role would be to consider constructions with the symmetries of a lattice $L$ as before, but where more catenoidal bridges located on the segments joining nearby lattice points are being used.

Another motivation for the construction in this paper is that it is nontrivial to obtain new examples of closed minimal surfaces in $S^3(1)$. In the case of genus
one examples, there has been significant progress by using integrable system methods [1], [4]. In the case of genus higher than one, integrable system methods have not been successful so far and the list of known examples, especially embedded ones, is limited [15], [16], [17], [19]. Besides the embedded minimal surfaces $M_m$ constructed in this paper, a more complicated class of minimal embedded closed surfaces can be constructed by applying a general desingularization theorem which has been announced in [14, Theorem F], to appropriate finite collections $\{M_{m_i}\}$ of the minimal surfaces constructed here [14, Theorem G]. The $m_i$'s are such that the corresponding $m_i \times m_i$ lattices are nested. The symmetry group of the desingularization construction is the group of symmetries $G$ corresponding to the smallest $m_i$. The collection of the surfaces $\{M_{m_i}\}$ close to a point common to all lattices resembles intersecting coaxial catenoids (but without the rotational invariance) as in [12]. In order to establish this result in particular, the geometric principle is used to ensure that there is no exceptional kernel on $M_{m_i}$ invariant under $G$. This work will appear elsewhere, where the related question of the equivariant index of $M_m$, under its symmetry group or the symmetry group of sublattices, will also be discussed.

Outline of the construction. Our construction is facilitated by the existence of a simple coordinate system which is well adjusted to our purposes. We study this coordinate system in Section 2. We call the corresponding coordinates $(x, y, z)$. The surface $\{z = 0\}$ in $S^3(1)$ is the Clifford torus $T$ on which the doubling construction is based. The surfaces parallel to $T$ are the surfaces of constant $z$.

In Section 3 we construct the initial surfaces $M$. The construction is based on a square lattice $L \subset T$ (see 3.2) which consists of $m^2$ points. The construction of the minimal surfaces in the main theorem works when $m$ is large. The surfaces constructed have genus $m^2 + 1$ because they amount to two tori connected by $m^2$ handles. The size of the catenoidal bridges $\tau$ can be predicted by using 1.1 (see [14]). This calculation depends on the height of the surface at $\partial \Omega$ where the vertical component of the force $F$ is calculated. The height however cannot be predicted precisely because the surface is modified later when it is corrected to minimality by solving a Partial Differential Equation. We can therefore only predict $\tau$ to be $\tau := m^{-1}e^{-m^2/4\pi}$ up to a factor which we call $e^\xi$ (see 3.4). The factor $e^\xi$ can be large or small but uniformly controlled independently of $m$ (see 3.3). The formula for $\tau$ is justified in this paper because it implies 3.10 which allows us to prove—see 5.3 and the proof of the main theorem 5.4—that there is a $\tau$ in the range determined by 3.3 for which the corresponding initial surface can be corrected to minimality.

The construction of $M$ is carried out in parallel with a similar construction of a surface $\hat{M}$, which would give a doubling of the plane in three-dimensional Euclidean space. By the maximum principle, $\hat{M}$ cannot be perturbed to minimality, in contrast with $M$ which by the main theorem of the paper 5.4 can (for a certain $\tau$). This is consistent with 1.2, since $|A|^2 + Ric(\nu, \nu) = 4 > 0$ on $T$, while
$|A|^2 + Ric(\nu, \nu) = 0$ on the plane and the mean curvature vanishes on its parallel surfaces which are planes themselves. Actually the conormal on a perturbed $\hat{M}$ on the vertical planes of reflectional symmetry (that is on $\partial \hat{D}$, see 3.1) is horizontal, so the force $F$ in the discussion above vanishes providing an alternative proof that $\hat{M}$ cannot be corrected.

As is often the case in such constructions [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [24], [25], it is convenient to define two more metrics on the initial surfaces $M$, $h$ and $\chi$, besides the induced metric $g$. $h$ and $\chi$ are conformal to $g$. $h$ allows us to write the linearized equation with uniformly bounded coefficients. Moreover, it allows us to understand the spectrum and the approximate kernel. In the usual terminology $M$ modulo the symmetries has two standard regions, which when viewed with respect to $h$ tend to a planar square and a unit sphere. The square corresponds to a fundamental domain of $T$ and the unit sphere to the catenoidal bridge. There is only one (modulo the symmetries) transition region $\Lambda$ connecting the standard regions. $(\Lambda, \chi)$ is approximately isometric to a standard cylinder of length $m^2/4\pi$ up to lower order terms. The geometric quantities of $M$ are discussed in 3.18. These estimates are important because they allow us to ensure later that we can perturb to minimality with an appropriately small perturbation. Finally in 3.29 we quantify the limiting behavior of the standard regions in the $h$ metric as $m \to \infty$.

In Section 4 we develop the linear theory needed. The main results from this section we use later are 4.30 and 4.28. In 4.28 we simply extract from the information we have on the mean curvature from 3.18, the relevant estimate we can use according to 4.30. In 4.30 we provide a solution modulo the substitute kernel for the linear problem with appropriate decay estimates. The construction leading to 4.30 follows the general methodology of [9], [10], [11]. It is simpler than usual however, because of the small number of standard and transition regions and the one-dimensionality of the substitute kernel, which can serve also as extended substitute kernel (see [14] for a general discussion). The one-dimensionality of the approximate and (hence) the substitute kernel follows from the fact that the symmetries kill the first harmonics of the Laplacian on the spherical standard region corresponding to the catenoidal bridge, and therefore the only eigenfunctions allowed in the kernel in the limiting configuration as $m \to \infty$, are the constants on the square (see 4.14). It turns out that the substitute kernel is enough for arranging the decay we need (see 4.23 and 4.24), and hence there is no need for extra “extended substitute kernel”.

Finally in Section 5 we prove the Main Theorem. To do so, we first provide in 5.1 an estimate of the nonlinear terms consistent with the decay estimates we have. This estimate is based on a general estimate which can be derived from general principles (see A.3) and which we present in appendix A. Next we calculate in detail the forces in the spirit of the discussion earlier (see 5.3), and use that information to ensure that there is some initial surface $M$ which can be perturbed to minimality. This is consistent with the Geometric Principle (see [9],
[10], [11], [14]), because effectively creation of substitute kernel is achieved by repositioning the copies of $T$ used in the construction at varying distances $a\tau$ from $T$. Finally we state and prove the Main Theorem 5.4 by using as usual the Schauder fixed point theorem [3, Theorem 11.1] to minimize the required estimates. We remark that the minimal surfaces we find are consistent with the description of the surfaces in Example 12 in [20, page 306].

**Notation and conventions.** In this paper we use weighted H"older norms. The definition we use is given by

\[(1.4) \quad \|\phi\colon C^{k,\beta}(\Omega, g, f)\| := \sup_{x \in \Omega} \frac{\|\phi\colon C^{k,\beta}(\Omega \cap B_x, g)\|}{f(x)},\]

where $\Omega$ is a domain inside a Riemannian manifold $(M, g), f$ is a weight function on $\Omega$, $B_x$ is a geodesic ball centered at $x$ and of radius the minimum of 1 and half the injectivity radius at $x$.

We will be using extensively cut-off functions, and for this reason we adopt the following notation: We fix a smooth function $\Psi\colon \mathbb{R} \to [0, 1]$ with the following properties:

(i) $\Psi$ is nondecreasing.
(ii) $\Psi \equiv 1$ on $[1, \infty]$ and $\Psi \equiv 0$ on $(-\infty, -1]$.
(iii) $\Psi - \frac{1}{2}$ is an odd function.

Given then $a, b \in \mathbb{R}$ with $a \neq b$, we define a smooth function $\psi[a, b]$: $\mathbb{R} \to [0, 1]$ by

\[(1.5) \quad \psi[a, b] = \Psi \circ L_{a,b},\]

where $L_{a,b}$: $\mathbb{R} \to \mathbb{R}$ is the linear function defined by the requirements $L(a) = -3$ and $L(b) = 3$.

Clearly then $\psi[a, b]$ has the following properties:

(i) $\psi[a, b]$ is weakly monotone.
(ii) $\psi[a, b] = 1$ on a neighborhood of $b$ and $\psi[a, b] = 0$ on a neighborhood of $a$.
(iii) $\psi[a, b] + \psi[b, a] = 1$ on $\mathbb{R}$.

We will denote the span of vectors $e_1, \ldots, e_k$ with coefficients in a field $\mathbb{F}$ by $\langle e_1, \ldots, e_k \rangle_{\mathbb{F}}$.

**Acknowledgments.** The authors would like to thank Rick Schoen for his constant interest and support and insightful discussions and suggestions. N. K. would like to thank the Mathematics Department and the MRC at Stanford University for providing a stimulating mathematical environment and generous financial support during the fall of 2006.

We would also like to thank the referee for his careful refereeing and his many suggestions which significantly improved the presentation.
2. A coordinate system on $S^3(1)$.

The parametrization $\Phi$. Our construction is facilitated by the existence of a simple coordinate system which is ideally suited to describing the Clifford torus and its parallel surfaces. We proceed to describe this coordinate system and the local parametrization which is its inverse. To simplify the notation we identify $\mathbb{R}^4 \cong \mathbb{C}^2 \supset S^3(1)$. We define the parametrization $\Phi$, which covers the unit sphere with two orthogonal circles removed, that is $S^3(1) \setminus \{(z_1, z_2) \in \mathbb{C}^2: z_1 = 0 \text{ or } z_2 = 0\}$, by the following:

\begin{equation}
(2.1) \quad \Phi: \text{Dom} \Phi \rightarrow S^3(1) \subset \mathbb{R}^4 \cong \mathbb{C}^2, \quad \text{where} \quad \text{Dom} \Phi := \mathbb{R} \times \mathbb{R} \times \left( -\frac{\pi}{4}, \frac{\pi}{4} \right),
\end{equation}

\[
\Phi(x, y, z) = \sin (z + \frac{\pi}{4}) e^{\sqrt{2}x i} \hat{e}_1 + \cos (z + \frac{\pi}{4}) e^{\sqrt{2}y i} \hat{e}_2,
\]

where $\hat{e}_1 = (1, 0)$ and $\hat{e}_2 = (0, 1)$ form the standard basis of $\mathbb{C}^2$.

Symmetries of $\Phi$. To study the symmetries of the parametrization $\Phi$, we first define for $c \in \mathbb{R}$ translations $\hat{X}_c$, $\hat{Y}_c$, and reflections $\hat{X}_c$, $\hat{Y}_c$, $\hat{X}_0 := \hat{X}_0$, $\hat{Y}_0 := \hat{Y}_0$, and $\hat{Z}_c$ of its domain $\text{Dom} \Phi$, by

\begin{equation}
(2.2) \quad \hat{X}_c(x, y, z) = (x + c, y, z), \quad \hat{Y}_c(x, y, z) = (x, y + c, z),
\end{equation}

\[
\hat{X}_c(x, y, z) = (2c - x, y, z), \quad \hat{Y}_c(x, y, z) = (x, 2c - y, z),
\]

\[
\hat{Z}(x, y, z) = (y, x, -z).
\]

We also define corresponding rotations $X_c$, $Y_c$, and reflections $X_c$, $Y_c$, $X_0 := X_0$, $Y_0 := Y_0$, and $Z$ of $\mathbb{C}^2 \supset S^3(1)$ by

\begin{equation}
(2.3) \quad X_c(z_1, z_2) = (e^{\sqrt{2}c i}z_1, z_2), \quad Y_c(z_1, z_2) = (z_1, e^{\sqrt{2}c i}z_2),
\end{equation}

\[
X(z_1, z_2) = (\bar{z}_1, z_2), \quad Y(z_1, z_2) = (z_1, \bar{z}_2),
\]

\[
X_c := X_2c \circ X, \quad Y_c := Y_2c \circ Y,
\]

\[
Z(z_1, z_2) = (z_2, z_1).
\]

Note that $X_c$, $Y_c$, and $Z$ are reflections with respect to the 3-planes $(e^{\sqrt{2}c i} \hat{e}_1, \hat{e}_2, i\hat{e}_2) \mathbb{R}$, $(\hat{e}_1, i\hat{e}_1, e^{\sqrt{2}c i} \hat{e}_2) \mathbb{R}$, and the 2-plane $\{z_1 = z_2\}$ respectively. $Z$ exchanges the two sides of the Clifford torus and also interchanges its parallels with its meridians. $X_{\sqrt{2} \pi}$ and $Y_{\sqrt{2} \pi}$ are the identity map. We record the symmetries of $\Phi$ in the following lemma:

**Lemma 2.4.** $\Phi$ is a covering map onto $S^3(1) \setminus \{(z_1, z_2) \in \mathbb{C}^2: z_1 = 0 \text{ or } z_2 = 0\}$. Moreover the following hold:

(i) The group of covering transformations is generated by $\hat{X}_{\sqrt{2} \pi}$ and $\hat{Y}_{\sqrt{2} \pi}$, in particular $\Phi = \Phi \circ \hat{X}_{\sqrt{2} \pi} = \Phi \circ \hat{Y}_{\sqrt{2} \pi}$. 

\[ X_c \circ \Phi = \Phi \circ \tilde{X}_c, \quad Y_c \circ \Phi = \Phi \circ \tilde{Y}_c, \quad \text{and} \quad Z \circ \Phi = \Phi \circ \tilde{Z}_c. \]

\[(ii) \quad X_c \circ \Phi = \Phi \circ \tilde{X}_c, \quad Y_c \circ \Phi = \Phi \circ \tilde{Y}_c, \quad \text{and} \quad Z \circ \Phi = \Phi \circ \tilde{Z}_c. \]

**Proof.** (ii) and (iii) follow from the definitions. (i) follows from (iii) and the observation that \(X_{\sqrt{2}} \pi\) and \(Y_{\sqrt{2}} \pi\) are the identity map. □

The coordinates \(xyz\). The local inverses of \(\Phi\) provide us with local coordinate systems. We denote the corresponding coordinates by \(x, y, z\). A straightforward calculation shows that

\[
\begin{align*}
\partial_x &= \sqrt{2} \sin(z + \frac{\pi}{4}) i e^{\sqrt{2}x_i} \tilde{e}_1, \\
\partial_y &= \sqrt{2} \cos(z + \frac{\pi}{4}) i e^{\sqrt{2}y_i} \tilde{e}_2, \\
\partial_z &= \cos(z + \frac{\pi}{4}) e^{\sqrt{2}x_i} \tilde{e}_1 - \sin(z + \frac{\pi}{4}) e^{\sqrt{2}y_i} \tilde{e}_2.
\end{align*}
\]

By calculating further we obtain

\[\Phi^*g = (1 + \sin 2z) dx^2 + (1 - \sin 2z) dy^2 + dz^2,\]

where \(g\) is the induced metric on the unit sphere \(S^3(1)\). Moreover the only non-vanishing Christoffel symbols for the \((x, y, z)\)-coordinate system are given by

\[
\begin{align*}
\Gamma_{13}^{1} &= \Gamma_{31}^{1} = \frac{\cos 2z}{1 + \sin 2z}, & \Gamma_{32}^{2} &= - \frac{\cos 2z}{1 - \sin 2z}, \\
\Gamma_{11}^{3} &= - \cos 2z, & \Gamma_{22}^{3} &= \cos 2z.
\end{align*}
\]

The level surface with \(z = 0\) is the Clifford torus \(T := \Phi(\{z = 0\}) = \{(z_1, z_2) \in S^3(1) \subset \mathbb{C}^2 : |z_1| = |z_2| = 1/\sqrt{2}\}\).

The level surfaces \(\Phi(\{z = c\}) (c \in (-\frac{\pi}{4}, \frac{\pi}{4}))\) are tori of constant mean curvature \(H = 2 \tan 2c\), parallel at signed distance \(c\) to the Clifford torus \(T\), with \(\partial_z\) as their unit normal vector field. Note also that for \(c \in \mathbb{R}\), we have the level surfaces

\[
\begin{align*}
\Phi(\{x = c\}) &= S^3(1) \cap \{t_1 e^{\sqrt{2}c_i} \tilde{e}_1 + t_2 \tilde{e}_2 + t_3 i \tilde{e}_2 : t_1 \in \mathbb{R}^+, t_2, t_3 \in \mathbb{R}\}, \\
\Phi(\{y = c\}) &= S^3(1) \cap \{t_1 \tilde{e}_1 + t_2 i \tilde{e}_1 + t_3 e^{\sqrt{2}c_i} \tilde{e}_2 : t_1, t_2 \in \mathbb{R}, t_3 \in \mathbb{R}^+\},
\end{align*}
\]

which are equatorial half-two-spheres orthogonal to the parallel tori. These three families of level surfaces are orthogonal. The intersections of the last two are great semicircles orthogonal to the tori. Finally a calculation shows that \(\det[\Phi, \Phi_x, \Phi_y, \Phi_z] = \cos 2z > 0\).
**Killing fields.** Clearly $\partial_x$ and $\partial_y$ are Killing fields generating the rotations in the $\langle \vec{e}_1, i \vec{e}_1 \rangle_\mathbb{R}$ and $\langle \vec{e}_2, i \vec{e}_2 \rangle_\mathbb{R}$ planes respectively. However $\partial_z$ is not a Killing field. For this reason we consider the Killing field $\tilde{K}$ which agrees with $\partial_z$ on $\{x = y = 0\}$ and is defined on $\mathbb{C}^2 \supset S^3(1)$ by

$$\tilde{K}\big|_{(z_1, z_2)} := \text{Re} \, z_2 \, \vec{e}_1 - \text{Re} \, z_1 \, \vec{e}_2. \tag{2.9}$$

$\tilde{K}$ generates the rotations in the $\langle \vec{e}_1, \vec{e}_2 \rangle_\mathbb{R}$ plane. A straightforward calculation shows that

$$\tilde{K} = -\frac{1}{\sqrt{2}} \cot (z + \pi/4) \sin \sqrt{2} x \cos \sqrt{2} y \partial_x + \frac{1}{\sqrt{2}} \tan (z + \pi/4) \cos \sqrt{2} x \sin \sqrt{2} y \partial_y + \cos \sqrt{2} x \cos \sqrt{2} y \partial_z. \tag{2.10}$$

3. **The initial surfaces.** In this section we define and discuss the initial surfaces. The genus and the geometry of the initial surfaces depend on $m \in \mathbb{N}$ which we fix now and is assumed to be as large as needed. The number of catenoidal bridges used to connect the two parallel copies of the Clifford torus is $m^2$ and the genus of the resulting surface $m^2 + 1$. These bridges are arranged with maximal symmetry at the points of a square lattice. To describe the symmetry involved we have the following (recall Section 2):

**Definition 3.1.** We denote by $\hat{G}$ the group of diffeomorphisms of $\text{Dom}_\Phi$ generated by the reflections $\hat{X}, \hat{X}_\pi/\sqrt{2} m, \hat{Y}, \hat{Y}_\pi/\sqrt{2} m$, and $\hat{Z}$. We denote by $G$ the group of isometries of $S^3(1)$ generated by the reflections $X, X_\pi/\sqrt{2} m, Y, Y_\pi/\sqrt{2} m$, and $Z$. We also define $\tilde{D} \subset S^3(1)$ and $\hat{D} \subset \text{Dom}_\Phi$ by $\tilde{D} := \Phi(\hat{D})$ and

$$\hat{D} := \left\{ (x, y, z) \in \text{Dom}_\Phi : |x| \leq \frac{\pi}{\sqrt{2} m}, |y| \leq \frac{\pi}{\sqrt{2} m} \right\}.$$

The reflections $\hat{X}, \hat{X}_\pi/\sqrt{2} m, \hat{Y}, \hat{Y}_\pi/\sqrt{2} m$, and $\hat{Z}$ generating $\hat{G}$ are with respect to the planes $\{x = 0\}, \{x = \frac{\pi}{\sqrt{2} m}\}, \{y = 0\}, \{y = \frac{\pi}{\sqrt{2} m}\}$, and the line $\{x = y, z = 0\}$ respectively. Clearly $\hat{X}_\pi/\sqrt{2} m, \hat{Y}_\pi/\sqrt{2} m \in \hat{G}$ and $X_\pi/\sqrt{2} m, Y_\pi/\sqrt{2} m \in G$. $\hat{D}$ is a fundamental domain for the action of the translations in $\hat{G}$ and is invariant under the action of $\hat{X}, \hat{Y}$ and $\hat{Z}$. Similarly (recall 2.4) $\tilde{D}$ is a fundamental domain for the action of the rotations in $G$ and is invariant under the action of $X, Y$ and $Z$.

The role of $Z$ is especially important because it exchanges the two sides of the Clifford torus $T$, or equivalently the two copies of $T$ being used in the construction. This simplifies the construction considerably. Since this requires (recall the definition of $\hat{Z}$ in 2.2) an exchange of the $x$ and $y$ coordinates, it forces the lattices we use to be square: We define square lattices $\hat{L}$ on the plane $\{z = 0\}$.
and \( L \) on \( \mathbb{T} \) (recall 2.4 and 2.8) by

\[
\hat{L} := \mathcal{G}(0, 0, 0), \quad L := \Phi(\hat{L}) = \mathcal{G}(0, 0, 0).
\]

\( L \) consists of \( m^2 \) points which will be the centers of the catenoidal bridges we use.

The initial surfaces form a smooth one-parameter family. We call the parameter of the family \( \zeta \). \( m \) (which we have already fixed) and \( \zeta \) control the size of the catenoidal bridges, so that we can later ensure that for the right choice of \( \zeta \) balancing implies the minimality of the appropriately corrected initial surface. The range of \( \zeta \) is determined by

\[
|\zeta| \leq \epsilon,
\]

where \( \epsilon \) is a constant which is large but independent of \( m \) and which will be chosen later in the proof of the Main Theorem 5.4. It has to be chosen large enough so that certain error terms depending on other uniform constants are overwhelmed by \( \epsilon \) in the proof of 5.4.

The size (radius of the waist) of the catenoidal bridges, which we call \( \tau \), cannot be guessed precisely; but by using the balancing considerations outlined in the introduction (or see [14]), we can determine that it should be up to a (uniformly controlled independently of \( m \)) factor equal to the constant \( \overline{\tau} \) given below in 3.4. We write the factor by which \( \tau \) differs from \( \overline{\tau} \) as \( e^{\zeta} \). By 3.3 the factor does not have to be close to one but it is uniformly controlled independently of \( m \). \( \tau \) may be larger or smaller than \( \overline{\tau} \) depending on the sign of \( \zeta \). We define

\[
\overline{\tau} := m^{-1}e^{-m^2/4\pi}, \quad \tau := e^{\zeta} \overline{\tau}.
\]

Note that \( \overline{\tau} \) depends only on \( m \), but \( \tau \) on both \( m \) and the parameter \( \zeta \). The following simplifies the presentation although occasionally we emphasize that \( m \) is chosen large enough in terms of various constants.

**Convention 3.5.** We assume through the paper that \( m \) is as large as needed in terms of constants which do not depend on \( m \). By 3.4 and 3.3 this is equivalent to \( \overline{\tau} \) or \( \tau \) being small enough in terms of such constants.

Recall that a catenoid of size \( \tau \) in Euclidean three-space can be parametrized conformally on a cylinder by

\[
\tilde{X}(t, \theta) := (\tau \cosh t \cos \theta, \tau \cosh t \sin \theta, \tau t) = (r(t) \cos \theta, r(t) \sin \theta, z(t)),
\]

where \( r(t) := \tau \cosh t \), \( z(t) := \tau t \).
Alternatively the part above the waist can be given as a radial graph of
\[ \varphi_{cat}(r) := \tau \arccosh \frac{r}{\tau} = \tau \left( \log r - \log \tau + \log \left( 1 + \sqrt{1 - \tau^2 r^{-2}} \right) \right), \]
where \( r \) is the polar coordinate defined by
\[ r := \sqrt{x^2 + y^2}. \]

To define the catenoidal bridge \( \hat{M}_{cat} \) (see Figure 1), we truncate the catenoid at radius \( r = 1/m \) or equivalently the parametrizing cylinder at \( t = \pm a. \ a > 0 \) is defined then by
\[ r(a) = \frac{1}{m} \iff \cosh a = \frac{1}{m \tau} \iff a = \frac{1}{\tau} \varphi_{cat} \left( \frac{1}{m} \right). \]

Using the last expression and 3.7, together with 3.4 and 3.3, it follows that
\[ \left| a + \zeta - \frac{m^2}{4\pi} - \log 2 \right| < \tau. \]

For future reference we define \( \alpha \) to be the value of \( a \) when \( \zeta = 0. \) We have then (compare with 3.9)
\[ \cosh \alpha = \frac{1}{m \tau}. \]

We define now the catenoidal bridge by
\[ \hat{M}_{cat} := \hat{X}([-a, a] \times S^1) \subset \hat{D}. \]
We also define a region of a horizontal plane (corresponding under $\Phi$ to a parallel surface to $T$) together with a gluing region by

$$M_{\text{tor}} := \{(x, y, z) \in \hat{D} : z = \varphi(r), \quad m^{-1} \leq r\},$$

(3.13)

where $\varphi(r) := \varphi_{\text{cat}}(r) + \psi_{[-m+1,2m-1]}(r) (\varphi_{\text{cat}}(m^{-1}) - \varphi_{\text{cat}}(r))$,

where $\psi_{[m^{-1},2m^{-1}]}$ is a cut-off function defined as in 1.5. Notice that $\varphi$ transits then smoothly from being $\varphi_{\text{cat}}$ in a neighborhood of $r = 1/m$, to being the constant

$$\varphi_{\text{cat}}(1/m) = \tau a$$
for $r \geq 2/m$ (note that $2 < \pi/\sqrt{2}$). Correspondingly $M_{\text{tor}}$ extends smoothly $M_{\text{cat}}$ close to its inner boundary circle and transits to the plane $z = \varphi_{\text{cat}}(1/m)$ close to its outer boundary.

The stabilizer of $M_{\text{cat}}$ under the action of $\hat{G}$ is generated by $\hat{X}$, $\hat{Y}$ and $\hat{Z}$. The stabilizer of $M_{\text{tor}}$ is generated by $\hat{X}$ and $\hat{Y}$. Note that $M_{\text{cat}}$ was defined to include both its top and bottom halves. $M_{\text{tor}}$ was defined for convenience not to include its bottom half because the top and the bottom half are different connected components. This way $\hat{Z}$ defines a symmetry of $M_{\text{cat}}$ but not of $M_{\text{tor}}$ which it maps to its bottom counterpart. We define now (see Figure 1) smooth embedded surfaces $\hat{M} \subset \text{Dom}_\Phi$ and $M_{\text{cat}}, M_{\text{tor}}, M \subset S^3(1)$ by

$$\hat{M} := \hat{\mathcal{G}}(M_{\text{cat}} \cup M_{\text{tor}}), \quad M_{\text{cat}} := \Phi(M_{\text{cat}}), \quad M_{\text{tor}} := \Phi(M_{\text{tor}}), \quad M := \Phi(M) = \mathcal{G}(M_{\text{cat}} \cup M_{\text{tor}}).$$

(3.15)

$\Phi|_{\hat{M}} : \hat{M} \to M$ is clearly a covering map, and $M$ is a closed connected embedded smooth surface of genus $m^2 + 1$. We take $M$ to be our initial surface, and we will prove in the Main Theorem that for some value of $\zeta$, it can be perturbed to a nearby minimal surface.

Geometric quantities on the initial surfaces. We start by discussing some of the metrics we use. We denote by $\hat{g}$ the standard Euclidean metric on $\text{Dom}_\Phi$ and by $g$ the standard metric on the round sphere $S^3(1)$. Since $\Phi$ is a covering map, these metrics induce metrics on the range and the domain of $\Phi$ respectively, which we denote by slight abuse of notation by the same symbols. We also use the same symbols to denote the metrics induced on $M, \hat{M}$ and (by using $\hat{X}$) on the cylinder $S^1 \times [-a,a]$. We define a smooth function $\rho$ on $M$ (or $\hat{M}$), by requiring it is invariant under the action of $\mathcal{G}$ (or $\hat{\mathcal{G}}$) and on $\hat{D} \cap M$ (or $\hat{D} \cap \hat{M}$), it satisfies

$$\rho = \frac{1}{r} + \psi_{[m^{-1},2m^{-1}]}(r) \left(\frac{m}{2} - \frac{1}{r}\right).$$

(3.16)
We define then smooth metrics $\chi$ and $\hat{\chi}$ on our surfaces by

\begin{equation}
\chi := \rho^2 g, \quad \hat{\chi} := \rho^2 \hat{g}.
\end{equation}

We denote by $\nu$ the unit normal on $M$ which satisfies $\langle \nu, \partial z \rangle > 0$ on $M_{\text{tor}}$, $A$ the second fundamental form induced by $g$, $|A|^2$ its square length, and $H$ the mean curvature. We use a hat to denote the corresponding geometric quantities induced by $\hat{g}$. We will need precise quantitative control of the various geometric quantities throughout the paper. This is provided by the following lemma. The reader may wish to skip its proof at first reading but the estimates themselves will be vital for the rest of the paper. Note that the constants do not depend on $c$ as required to ensure that certain error terms can be overwhelmed by choosing $c$ large enough in the proof of the Main Theorem 5.4. The reason we can achieve this is that the estimates involve $\tau$ rather than $\overline{\tau}$. Note also that in the estimates $\hat{\chi}$ and $\chi$ are interchangeable because of (iii).

**Lemma 3.18.** Assuming that $m$ is large enough in terms of $k \in \mathbb{N}$ the following hold:

(i) $\|\rho^{\pm 1} : C^k(M, \hat{\chi}, \rho^{\pm 1})\| \leq C(k)$.

(ii) $\|z : C^k(M, \hat{\chi}, |z| + \tau)\| \leq C$ and $|z| + \tau < m^2 \tau$ on $M$.

(iii) $\|\chi - \hat{\chi} : C^k(M, \hat{\chi}, |z| + \tau)\| \leq C(k)$. On $\hat{M}_{\text{cat}}$ we have $\hat{\chi} = dt^2 + d\theta^2$.

(iv) $\|ho^{-2}H : C^k(M, \chi, (\tau + \rho^{-2})(|z| + \tau))\| \leq C(k)$.

(v) $\||A|^2 - 2\tau^2 \rho^4 : C^k(M, \chi, 1 + \tau \rho^2)\| \leq C(k)$. Moreover on $\hat{M}_{\text{cat}}$ we have $|A|^2 = 2\tau^2 \rho^4$.

**Proof:** We first check these estimates on $M_{\text{cat}}$. For convenience we adopt the notation $O(f)$ to denote the product $O(f) = fT$ of a function $f$ and a function or more generally a tensor field $T$ which satisfies the inequality

\[ \|T : C^k(M_{\text{cat}}, \hat{\chi})\| \leq C(k). \]

Using 3.6 we have by straightforward calculation that

\[ \hat{X}_t = \tau(\sinh t \cos \theta, \sinh t \sin \theta, 1), \]
\[ \hat{X}_\theta = \tau \cosh t(-\sin \theta, \cos \theta, 0), \]

which together with 2.6 imply that

(a) $\hat{g} = r^2(dt^2 + d\theta^2)$ and $g = r^2(dt^2 + d\theta^2 + O(z))$, or, more precisely,
\[ g = \hat{g} + \tau^2 \sin 2z(\sin^2 t \cos 2\theta d^2 - \sin 2t \sin 2\theta dt d\theta - \cosh^2 t \cos 2\theta d^2 - \cosh^2 t \cos 2\theta d^2). \]

Using (a), the expression for $z(t)$ in 3.6, 3.10, and the definitions, we conclude that (i), (ii), and (iii) hold on $M_{\text{cat}}$. 


Using 2.6 it is straightforward to check that the unit normal is given by

\[ \nu = \frac{1 + O(\tau^2 r^{-2} z)}{\cosh t} \left( -\frac{\cos \theta}{1 + \sin 2z} \partial_x - \frac{\sin \theta}{1 - \sin 2z} \partial_y + \sinh t \partial_z \right). \]

To calculate the second fundamental form, we rename our coordinates as \((x^1, x^2) := (t, \theta)\) and use the formula

\[ A_{\alpha\beta} = (\hat{X}^k_{\alpha;\beta} + \Gamma^k_{lm} \hat{X}_l^m \nu^n g_{kn}) \partial_x^\alpha \partial_y^\beta, \]

where Greek indices take the values 1 and 2, and Latin indices take the values 1, 2, 3, corresponding to the coordinates \(x, y, z\). Let \( C_{\alpha\beta} := (\Gamma^k_{lm} \hat{X}_l^m \nu^n g_{kn}) \partial_x^\alpha \partial_y^\beta \) \( k = 1 \). We have then by using 2.7 that

\[ (b) \quad C_{\alpha\beta} = (O(1)(\hat{X}^1_{\alpha;\beta} + \hat{X}^1_{\beta;\alpha}), O(1)(\hat{X}^2_{\alpha;\beta} + \hat{X}^2_{\beta;\alpha}), (1 + O(z^2))(-\hat{X}^1_{\alpha;\beta} + \hat{X}^2_{\beta;\alpha}), \]

Using the above we calculate that

\[ \hat{X}_{11} = \tau \cosh t \left( \cos \theta, \sin \theta, 0 \right), \]
\[ \hat{X}_{12} = \tau \sinh t \left( -\sin \theta, \cos \theta, 0 \right), \]
\[ \hat{X}_{22} = \tau \cosh t \left( -\cos \theta, -\sin \theta, 0 \right), \]

\[ C_{11} = \tau^2 (O(1) \sinh t, O(1) \sinh t, -(1 + O(z^2)) \sinh^2 t \cos 2\theta), \]
\[ C_{12} = \tau^2 (O(1) \cosh t, O(1) \cosh t, (1 + O(z^2)) \sinh t \cosh t \sin 2\theta), \]
\[ C_{22} = \tau^2 (0, 0, (1 + O(z^2)) \cosh^2 t \cos 2\theta), \]

\[ (g_{kn} \nu^n)^3_{k=1} = \frac{1 + O(\tau^2 r^{-2} z)}{\cosh t} (-\cos \theta, -\sin \theta, \sinh t). \]

It is easy to modify the above to calculate the second fundamental form \( \hat{A} \) induced by the Euclidean metric \( \hat{g} \). The Christoffel symbols of \( \hat{g} \) in our coordinate system vanish, and it is straightforward to find that

(c) \( \hat{A} = \tau (-dt^2 + d\theta^2) \).

We also define

(d) \( \hat{A} := \hat{I}^2 (\cos 2\theta (-dt^2 + d\theta^2) + 2 \sin 2\theta \, dt \, d\theta) \).

It is straightforward to calculate then that on \( M_{cat} \cap \{ t \geq 0 \} \)

\[ \hat{X}^k_{\alpha;\beta} g_{kn} \nu^n dx^\alpha dx^\beta = (1 + O(\tau^2 r^{-2} z)) \hat{A}, \]
\[ C_{\alpha;\beta} g_{kn} \nu^n dx^\alpha dx^\beta = \hat{A} + O(\tau^2) + O(\tau^2 z^2), \]

where we used also that \( \sinh t = (1 + O(\tau^2 r^{-2})) \cosh t \). We conclude then

(e) \( A = A + \hat{A} + O(\tau z) + O(\tau^2 z) \).

Since the traces of \( \hat{A} \) and \( \hat{A} \) with respect to \( dt^2 + d\theta^2 \) vanish, (iv) follows from (a), (e), and the symmetry (for \( t \leq 0 \)). (v) follows also because clearly \( \tau z < Cr^2 \).
It remains to check that the estimates hold on $M_{tor}$. We define scaled up coordinates by $x^1 := mx$, $x^2 := my$, and $\tilde{r} := mr$. By 3.16 and the properties of $\psi[a,b]$ as defined in 1.5, we have that

$$\rho = m \left[ \frac{1}{\tilde{r}} + \psi[1,2](\tilde{r}) \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right) \right].$$

This implies

(f) $\| \rho^{\pm 1} \| C^k(M_{tor}, (dx^1)^2 + (dx^2)^2) \| \leq C(k) m^{\pm 1}$.

By 3.14 and 3.10 we conclude

(g) $\frac{1}{64} m^2 \tau \leq \varphi_{cat}(1/m) = \tau a < \frac{1}{2} m^2 \tau$.

By 3.7 and the definition of $\varphi$ in 3.13, we have

$$\varphi(\tilde{r}) - \varphi_{cat}(1/m) = \psi[2,1](\tilde{r}) \tau \left( \log \tilde{r} + \log \left( 1 + \sqrt{1 - m^2 \tau^2 \tilde{r}^{-2}} \right) - \log \left( 1 + \sqrt{1 - m^2 \tau^2} \right) \right),$$

which implies that

(h) $\| \varphi - \varphi_{cat}(1/m) \| C^k(M_{tor}, (dx^1)^2 + (dx^2)^2) \| \leq C(k) \tau$.

Since $M_{tor}$ is the graph of $\varphi$ (recall 3.13), we have

$$\tilde{X}_{1} = ( m^{-1}, 0, \varphi_{1} ),$$
$$\tilde{X}_{2} = ( 0, m^{-1}, \varphi_{2} ),$$

from which we conclude (j), and by using also 2.6, (k):

(j) $\hat{g} = m^{-2}((dx^1)^2 + (dx^2)^2) + \varphi_{,\alpha} \varphi_{,\beta} dx^\alpha dx^\beta = dx^2 + dy^2 + \varphi_{,\alpha} \varphi_{,\beta} dx^\alpha \ dx^\beta$,

(k) $g - \hat{g} = m^{-2} \sin 2 \varphi ((dx^1)^2 - (dx^2)^2)$.

Using (h) we conclude that

(l) $\| \hat{g} - (dx^2 + dy^2) \| C^k(M_{tor}, (dx^1)^2 + (dx^2)^2) \| \leq C(k) \tau^2$.

Using (l) we can replace $(dx^1)^2 + (dx^2)^2$ in (f) and (h) with $m^2 \hat{g}$. By using the modified (f) then we can further replace $m^2 \hat{g}$ with $\tilde{\chi}$ to conclude (m) and (n):

(m) $\| \rho^{\pm 1} \| C^k(M_{tor}, \tilde{\chi}) \| \leq C(k) m^{\pm 1}$.

(n) $\| \varphi - \varphi_{cat}(1/m) \| C^k(M_{tor}, \tilde{\chi}) \| \leq C(k) \tau$.

(m) implies (i), and (n) with (g) imply (ii). (k) implies

$$\chi - \tilde{\chi} = \rho^2 m^{-2} \sin 2 \varphi ((dx^1)^2 - (dx^2)^2),$$

which by (g), (m), and (n), implies

$$\| \chi - \tilde{\chi} \| C^k(M_{tor}, (dx^1)^2 + (dx^2)^2) \| \leq C(k) m^2 \tau.$$

As before we can replace $(dx^1)^2 + (dx^2)^2$ with $\tilde{\chi}$, and then (iii) follows by (g) and (n). Similarly using (j), (k), (g), and (n) we conclude that
(o) \( g = m^{-2}((dx^1)^2 + (dx^2)^2 + O(m^2 \tau)) \), or equivalently
\[
\|g - (dx^2 + dy^2) : C^k(M_{tor}, \hat{\chi})\| \leq C(k) \tau.
\]

By extending to \( M_{tor} \) the earlier notation for \( M_{cat} \) and recalling (b), we have
\[
\hat{X}_{\alpha\beta} = (0, 0, \varphi_{\alpha\beta}),
\]
\[
C_{11} = m^{-2} (mO(\varphi_1), 0, -(1 + O(z^2))),
\]
\[
C_{12} = m^{-2} (mO(\varphi_2), mO(\varphi_1), 0),
\]
\[
C_{22} = m^{-2} (0, mO(\varphi_2), 1 + O(z^2)).
\]

By using 2.7, it is easy to check that the unit normal \( \nu \) satisfies
\[
(3.19) \quad (g_{nmn})_{n=1}^3 = (1 + m^2 (O(\varphi_1^2) + O(\varphi_2^2)))
\times(-m\varphi_{,1}(1 + O(z)), -m\varphi_{,2}(1 + O(z)), 1).
\]

Combining the above we conclude that

(p) \( A = m^{-2}(- (dx^1)^2 + (dx^2)^2 + O(m^2 \tau)) \) or equivalently
\[
\|A + dx^2 - dy^2 : C^k(M_{tor}, \hat{\chi})\| \leq C(k) \tau.
\]

(o) and (p) imply then (iv) and (v) on \( M_{tor} \).

Remark 3.20. Notice that in the last proof \(-dx^2 + dy^2\) on \( M_{tor} \) corresponds to \( \tilde{A} \) on \( M_{cat} \). \( \tilde{A} \) (defined as the second fundamental form induced by the Euclidean metric \( \hat{g} \)) is not trace-free subject to \( \hat{g} \) on the gluing region, as it was on \( M_{cat} \), and therefore we do not record it separately on the expansion of \( A \) in (p). In the gluing region, \( \tilde{A} \) contributes to the mean curvature at the same order as \( g - \hat{g} \), in agreement with the heuristics used to choose \( \tau \).

Remark 3.21. Although we do not carry a careful study of where the umbilics of the minimal surfaces constructed in this paper are located, it is worth noting that if we ignore the corrections and concentrate on \( \tilde{A} + \hat{A} \), we get exactly two “umbilics” on \( M_{cat} \cap \{z \geq 0\} \), at \((r, \theta) = (\sqrt{\tau}, \pm \pi/2)\), where \( \tilde{A} + \hat{A} = 0 \). This is consistent with the symmetries of the surface and the Lawson result which implies that there are four umbilics in each fundamental region of the final minimal surface (\(4m^2\) on the whole surface).

Standard and transition regions. We proceed to define the various regions on the initial surface \( M \) (see Figure 2) in the usual fashion of [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]. Modulo the symmetries imposed, there are only
two standard regions which we denote by $S[0]$ (corresponding to the catenoidal bridge) and $S[1]$ (corresponding to the torus), and only one transition region we denote by $Λ$. The extended standard regions $\tilde{S}[0]$ and $\tilde{S}[1]$ are the standard regions augmented by the transition region.
In order to ensure uniformity with respect to different values of the parameter \( \zeta \), we define and use a variant \( t \) of the parameter \( t \) by

\[
\tau = \frac{a}{\bar{a}} t, \tag{3.22}
\]

where \( a \) is defined in 3.9 for the current value of \( \zeta \), and \( \bar{a} \) is defined in the same way when \( \zeta = 0 \), and hence \( \tau = \tau \) (recall 3.4). This way the range of values of \( t \) on \( M_{\text{cat}} \) is \([ -a, a ]\) and depends only on the choice of some \( m \gg 1 \) via 3.4. Note also that by 3.10, we have

\[
|t - \bar{t}| \leq C \zeta \quad \text{on } M_{\text{cat}}, \quad \left| \frac{a}{\bar{a}} - 1 \right| \leq C \zeta m^{-2}. \tag{3.23}
\]

We extend \( t \) to the whole of \( M \cap D \) by taking \( t = a \) on \( M_{\text{tor}} \) and \( t = -a \) on \( Z(M_{\text{tor}}) \). Notice that \( t \) is odd with respect to \( Z \). In the following definitions we use a constant \( b \) to prescribe the exact extent of the standard and transition regions. The larger \( b \) is, the larger \( S[0] \) and \( S[1] \) are, and the smaller \( \Lambda \) is. We need to choose \( b \) independently of \( m \), but large, so that we ensure the smallness of various quantities on \( \Lambda \) (see for example 4.7(ii)). This way we ensure that the linearized equation and its spectrum can be properly understood. On the other hand, this increases the distortions on \( S[0] \) and \( S[1] \) compared to their limiting geometry as \( \tau \to 0 \). This however can be remedied by choosing \( m \) large enough in terms of \( b \) (recall 3.5).

We use subscripts \( x \) and \( y \) to modify the extent and boundary of the standard and transition regions. In particular each \( S_x[n] \) is a neighborhood of \( S[n] \), while \( \tilde{S}_x[n] \) or \( \Lambda_x \) is \( S[n] \) or \( \Lambda \) with an appropriate neighborhood of its boundary excised.

**Definition 3.24.** We define the following (see Figure 2):

\[
\begin{align*}
S_x[0] &:= M \cap D \cap \{ t \in [-b - x, b + x] \}, \\
S_x[1] &:= M \cap D \cap \{ t \geq a - b - x \}, \\
\tilde{S}_x[0] &:= M \cap D \cap \{ t \in [-a + b + x, a - b - x] \}, \\
\tilde{S}_x[1] &:= M \cap D \cap \{ t \geq b + x \}, \\
\Lambda_{x,y} &:= M \cap D \cap \{ t \in [b + x, a - b - y] \}, \\
C_x[0] &:= M \cap D \cap \{ t = b + x \}, \\
C_x[1] &:= M \cap D \cap \{ t = a - b - x \}, \\
C_{\partial} &:= \partial D \cap \partial M_{\text{tor}}
\end{align*}
\]

where \( b \) is a large constant independent of \( m \) chosen in 4.29 and \( x, y \in [0, 4] \). When \( x = y = 0 \), we drop the subscripts. We also write \( \Lambda_x \) for \( \Lambda_{x,x} \).
Although sometimes we will repeat ourselves, we have the following to streamline the presentation (see also 4.29 and recall 3.5):

**Convention 3.26.** From now on we assume that $b$ is as large as needed in absolute terms.

The limiting behavior of the standard regions and the linearized operator on them, as $m \to \infty$, is best understood in the $h$ metric which is defined on our surfaces by

$$h := \frac{|A|^2 + m^2}{2}g.$$  \hfill (3.26)

We define the map $\varpi: \mathbb{D} \to \mathbb{R}^2$ by

$$\varpi(x, y, z) := \frac{m}{\sqrt{2}}(x, y).$$  \hfill (3.27)

The following lemma describes the limiting behavior as $m \to \infty$:

**Lemma 3.29.** If $m$ is large enough in terms of $b$, then the following hold, where $C(b)$ denotes a constant which depends only on $b$:

(i) $\|h - \hat{\nu}^* g_{S^2} : C^5(S_x[0], \hat{\nu}^* g_{S^2})\| \leq C(b) \tau$, where $\hat{\nu}^* g_{S^2}$ is the pullback of the standard metric $g_{S^2}$ of the unit sphere $S^2(1)$ by $\hat{\nu}$ and satisfies

$$\hat{\nu}^* g_{S^2} = \frac{1}{2} |A|^2 g = \tau^2 r^{-4} g = \tau^2 r^{-2} \tilde{\chi}.$$  

Moreover

$$\hat{\nu}(S_x[0]) = \{(x, y, z) \in S^2(1): x^2 + y^2 \geq \tilde{R}_x^2\},$$

where $\tilde{R}_x = 1/\cosh [(b + x)a/g]$.

(ii) $\|h - \varpi^* g_0 : C^5(S_x[1], \varpi^* g_0)\| \leq C(b)/m^2$, where $\varpi^* g_0$ is the pullback of the standard Euclidean metric $g_0$ on $\mathbb{R}^2$ by $\varpi$ (restricted to $S_x[1]$). Moreover there is $\tilde{R}_x$ such that

$$\varpi(S_x[1]) = \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2: |\tilde{x}| \leq \frac{\pi}{2}, |\tilde{y}| \leq \frac{\pi}{2}, \tilde{x}^2 + \tilde{y}^2 \geq \tilde{R}_x^2\}$$

and $|\tilde{R}_x - 2^{-1/2} e^{-(b+x)a/g}| \leq \tau$.

**Proof.** Since the catenoid is a minimal surface, it follows from standard theory that $\hat{\nu}^* g_{S^2} = \frac{1}{2} |A|^2 \hat{g}$, and the expressions in terms of $r$ follow from 3.18(v) and the definitions. This implies that the length of $\hat{\nu}(C_x[0])$ is $2\pi \tau / \tau(t) = 2\pi / \cosh [(b + x)a/g]$, which implies that $\hat{\nu}(S_x[0])$ is as stated. Since

$$h - \hat{\nu}^* g_{S^2} = \frac{1}{2}(|A|^2 + m^2 - 2\tau^2 \rho^4) \rho^{-2} \chi + \tau^2 \rho^2 (\chi - \tilde{\chi}),$$

...
we conclude by using 3.18 that
\[ \| h - \hat{\nu}^* g_0 : C^5(S_4[0, \tilde{\chi}])\| \leq C(b) (m^2 \tau^2 + \tau) \leq C(b) \tau. \]

This implies the desired estimate (recall \( x \in [0, 4] \)) and completes the proof of (i).

The second part of (ii) follows easily from the expression for \( r(t) \) in 3.6, 3.9, and the observation that \( \tilde{R}_x = (m/\sqrt{2}) r(a - (b + x)a/g) \).

By 3.27 we have \( \tilde{\omega}^* g_0 = \frac{1}{2} m^2 (d x^2 + d y^2) \). Using also 3.26, we calculate that
\[ h - \tilde{\omega}^* g_0 = \frac{|A|^2 + m^2}{2} (g - (d x^2 + d y^2)) + \frac{|A|^2}{m^2} \tilde{\omega}^* g_0. \]

Arguing as for (f) in the proof of 3.18, we have (recall \( x \in [0, 4] \))
\[ \| \rho^{\pm 1} : C^k(S_4[1], \tilde{\omega}^* g_0) \| \leq C(k, b) m^{\pm 1}. \]

This implies the equivalence of \( \chi \) and \( \tilde{\omega}^* g_0 \) on \( S_4[1] \). Arguing as for (o) in the proof of 3.18 we have
\[ \| g - (d x^2 + d y^2) : C^k(S_4[1], \tilde{\omega}^* g_0)\| \leq C(k, b) \tau. \]

By 3.18(v) we have
\[ \| |A|^2 : C^k(S_4[1], \tilde{\omega}^* g_0)\| \leq C(k). \]

Combining the above we conclude the proof. \( \square \)


c 4. The Linearized Equation.

**Introduction and conventions.** In this section we study the linearized equation on \( M \) which can be stated in any of the following equivalent formulations,

\[ (4.1) \quad \mathcal{L}_\chi u = E, \quad \text{or} \quad \mathcal{L} u = \rho^2 E, \quad \text{or} \quad \mathcal{L}_h u = \frac{2\rho^2}{|A|^2 + m^2} E, \]

where the corresponding linear operators are given by

\[ (4.2) \quad \mathcal{L}_\chi := \Delta_\chi + \rho^{-2}(|A|^2 + 2), \quad \mathcal{L}_h := \Delta_h + 2\frac{|A|^2 + 2}{|A|^2 + m^2}, \]

\[ \mathcal{L} := \Delta_g + |A|^2 + 2 = \rho^2 L_\chi = \frac{|A|^2 + m^2}{2} L_h. \]

**Convention 4.3.** From now on, we fix constants \( \beta, \gamma \in (0, 1) \). \( \beta \) will be the fractional coefficient of the Holder norms \( C^{k, \beta} \) we will be using, \( \gamma \) will control the exponential decay of the solutions. Since \( \beta \) and \( \gamma \) are fixed, we will ignore
any dependence of the various constants on them. There is one occasion (in the proof of 4.30) where we have to consider a faster decay with $\gamma$ replaced by $\gamma' = \frac{\gamma + 1}{2}$. We can still assume in this instance that the constants $C$ we are using are valid as needed for $\gamma'$ decay as well.

The linearized equation on the transition region. In this subsection, we consider the linearized equation on the transition region $\Lambda_{x,y}$ defined as in (3.25e), where we assume that $x,y \in [0,4]$. For simplicity in this subsection, we will denote the neck under consideration by $\Lambda$, and its boundary circles $C_x[0]$ and $C_y[1]$ by $C$ and $\bar{C}$ respectively. We next define $x, x, x: \Lambda \to \mathbb{R}$ to measure the $t$-coordinate distance from $C, C, C \partial \Lambda = C \cup \bar{C}$ respectively:

$$b + x + \bar{x} = \ell, \quad a - b - y - \bar{x} = \ell, \quad \bar{x} := \min (x, \bar{x}). \quad (4.4)$$

Note that we can use $\Phi \circ \tilde{X}$ to identify $\Lambda$ with the cylinder $[(b + x)a/a, a - (b + y)a/a] \times S^1$. We define $\bar{\ell}$ to be the $t$-coordinate length of the cylinder and $\ell$ to be the $t$-coordinate length of the cylinder, so that

$$\bar{\ell} = a - 2b - x - y, \quad \ell = a - (2b + x + y)a/a. \quad (4.5)$$

Using 3.10, 3.23, and our assumption that $x, y \in [0,4]$, we estimate

$$\left| \ell + 2b + \frac{m^2}{4\pi} \right| < 10. \quad (4.6)$$

Our understanding of the linear equations on the transition region is based on the comparison with $\Delta \chi$, which is based on the following lemma where we interpret earlier estimates in the form needed here. Note also for later use that the area of $\Lambda$ with respect to the $h$ metric can be assumed as small as needed by assuming $b$ large enough, as easily follows from 4.7(ii).

**Lemma 4.7.** The following hold on $\Lambda$:

(i) $\|\chi - \hat{\chi}: C^5(\Lambda, \hat{\chi})\| \leq Cm^2\tau$.

(ii) $\|\rho^{-2}(|A|^2, m^2): C^5(\Lambda, \chi, e^{-3\tau/2})\| \leq Ce^{-3b/2}$.

**Proof.** (i) is a straightforward consequence of 3.18(ii) and (iii). To prove (ii), we use the multiplicative properties of the Holder norms and 3.18(i) and (v) to conclude that

$$\|\rho^{-2}|A|^2: C^k(\Lambda, \chi, \tau^2r^{-2} + r^2)\| \leq C(k),$$

where we also used the inequality $\tau \leq \tau^2r^{-2} + r^2$ and the definition of $\rho$. Similarly using 3.18(i), we have

$$\|m^2\rho^{-2}: C^k(\Lambda, \chi, m^2r^2)\| \leq C(k).$$
It is enough then to establish that $\tau r^{-1} \leq e^{-3(b+2)/4}$ and $m r \leq e^{-3(b+3)/4}$ on $\Lambda$ (recall 4.4). This follows from the exponential decays implied by the expression for $r(t)$ in 3.6 and 3.23, and the proof is complete.

**Proposition 4.8.** If $b$ and $m$ are large enough, then the lowest eigenvalue of the Dirichlet problem for $L \chi$ on $\Lambda$ is $> C \ell^{-2}$.

**Proof.** The proof is similar to the arguments leading to Proposition 2.28 in [10]. It is easy to prove that for $\phi \in L^2(\Lambda)$ with $L^2$ derivatives and $\phi = 0$ on $\partial \Lambda$, we have

$$\int_{\Lambda} e^{-3s/2} \phi^2 d\tilde{\chi} \leq C \int_{\Lambda} |\nabla \phi|_\chi^2 d\tilde{\chi},$$

which together with 4.7, implies

$$\int_{\Lambda} |\nabla \phi|_\chi^2 d\chi - \int_{\Lambda} \rho_{-2} (|A|^2 + 2) \phi^2 d\chi \geq \left( \frac{2}{3} - Ce^{-3b/2} \right) \int_{\Lambda} |\nabla \phi|_\chi^2 d\tilde{\chi}. $$

Using the variational characterization of eigenvalues and assuming $b$ large enough, the result follows since the smallest eigenvalue for $\Delta \tilde{\chi}$ is $> C \ell^{-2}$.

**Corollary 4.9.** (i) The Dirichlet problem for $L \chi$ on $\Lambda$ for given $C^{2,\beta}$ Dirichlet data has a unique solution.

(ii) For $E \in C^{0,\beta}(\Lambda)$, there is a unique $\varphi \in C^{2,\beta}(\Lambda)$ such that $L \chi \varphi = E$ on $\Lambda$ and $\varphi = 0$ on $\partial \Lambda$. Moreover $\|\varphi : C^{2,\beta}(\Lambda, \chi)\| \leq C \ell^{\delta} \|E : C^{0,\beta}(\Lambda, \chi)\|$.

**Proof.** (i) follows trivially and (ii) by using standard linear theory.

All our constructions have to respect the symmetries imposed, in particular we only consider functions on $M$ which are invariant under the action of $G$. $\Lambda$ is not invariant under $G$ but it is invariant under $X$ and $Y$. Under the identification of $\Lambda$ with a cylinder as discussed above, $X$ corresponds to $\theta \to \pi - \theta$, and $Y$ corresponds to $\theta \to -\theta$. We use the subscript “$S$” to specify subspaces of functions on $\Lambda$ which are invariant under these symmetries. In the next proposition and its corollary, we study the Dirichlet problem when we are allowed to modify the lowest harmonic on the boundary data in order to have decay estimates appropriate for our purposes:

**Proposition 4.10.** There is a linear map $R_{\Lambda} : C^{0,\beta}_S(\Lambda) \to C^{2,\beta}_S(\Lambda)$ such that the following hold for $E \in C^{0,\beta}_S(\Lambda)$ and $V := R_{\Lambda} E$:

(i) $L \chi V = E$ on $\Lambda$.

(ii) $V$ is constant on $\mathcal{C}$ and vanishes on $\mathcal{C}$.

(iii) $\|V : C^{2,\beta}(\Lambda, \chi, e^{-\gamma T})\| \leq C \|E : C^{0,\beta}_S(\Lambda, \chi, e^{-\gamma T})\|$.

(iv) $R_{\Lambda}$ depends continuously on $\tau$.

The proposition still holds if the roles of $\mathcal{C}$ and $\overline{\mathcal{C}}$ are exchanged in (ii) and $\overline{\chi}$ is replaced by $\chi$ in (iii). Another possibility is to allow $V$ to be constant on each of $\mathcal{C}$ and $\overline{\mathcal{C}}$ in (ii), while $\overline{\chi}$ is replaced by $\chi$ in (iii).
Proof. The proposition follows by standard theory if $L_\chi$ is replaced by $\Delta \hat{\chi}$.

We denote the corresponding linear map and solution in the $\Delta \hat{\chi}$ case by $\tilde{R}_\Lambda$ and $\tilde{V}$ respectively. Using then 4.7, we have

$$\|L_\chi \tilde{V} : C^0, \beta(\Lambda, \hat{\chi}, e^{-\gamma x})\| \leq (m^2 \tau + e^{-3b/2}) C \|E: C^0, \beta(\Lambda, \hat{\chi}, e^{-\gamma x})\|.$$ 

By assuming $b$ and $m$ large enough, we can ensure that the coefficient in the right-hand side is small. The proposition then follows by an iteration where we treat $L_\chi$ and $R_\Lambda$ as small perturbations of $\Delta \hat{\chi}$ and $\tilde{R}_\Lambda$.

We will only need the next statement with $\varepsilon_1 = 1$:

**Corollary 4.11.** Assuming $b$ large enough in terms of given $\varepsilon_1 > 0$, there is a linear map

$$R_{\partial} : \{ u \in C^2, \beta(\mathcal{C}): \int_{\mathcal{C}} u d\theta = 0 \} \rightarrow C^2, \beta(\Lambda)$$

such that the following hold for $u$ in the domain of $R_{\partial}$ and $V := R_{\partial}u$:

(i) $L_\chi V = 0$ on $\Lambda$.

(ii) $V$ vanishes on $\mathcal{C}$ and $V - u$ is constant on $\bar{\mathcal{C}}$ where it satisfies (iii) below.

(iii) $|V|_{\bar{\mathcal{C}}} \leq \varepsilon_1 \|u : C^2, \beta(\bar{\mathcal{C}}, d\theta^2)\|$.  

(iv) $\|V : C^2, \beta(\Lambda, \chi, e^{-\gamma x})\| \leq C \|u : C^2, \beta(\bar{\mathcal{C}}, d\theta^2)\|$.

(v) $R_{\partial}$ depends continuously on $\tau$.

The proposition still holds if the roles of $\bar{\mathcal{C}}$ and $\mathcal{C}$ are exchanged and $\bar{x}$ is replaced by $\hat{x}$.

Proof. By standard theory there is a linear map

$$\tilde{R}_{\partial} : \{ u \in C^2, \beta(\bar{\mathcal{C}}): \int_{\bar{\mathcal{C}}} u d\theta = 0 \} \rightarrow C^2, \beta(\Lambda)$$

such that for $u$ in the domain and $\tilde{V} = \tilde{R}_{\partial}u$ the following hold:

(a) $\Delta \hat{\chi} \tilde{V} = 0$ on $\Lambda$.

(b) $\tilde{V} = u$ on $\bar{\mathcal{C}}$ and $\tilde{V}$ vanishes on $\mathcal{C}$.

(c) $\|\tilde{V} : C^2, \beta(\Lambda, \chi, e^{-\gamma x})\| \leq C(\beta, \gamma) \|u : C^2, \beta(\bar{\mathcal{C}}, d\theta^2)\|$.

The corollary then follows by defining

$$R_{\partial}u := \tilde{R}_{\partial}u - R_\Lambda L_\chi \tilde{R}_{\partial}u,$$

applying the proposition, and using 4.7. $\square$
COROLLARY 4.12. If \( u \in C^{2,\beta}_S(\Lambda) \) satisfies \( \mathcal{L}_\chi u = 0 \) on \( \Lambda \), then
\[
\| u : C^{2,\beta}_S(\Lambda, \chi) \| \leq C \| u : C^{2,\beta}_S(\partial \Lambda, \chi) \|.
\]

Proof. Because of 4.11 and 4.9 it is enough to prove the Corollary when \( u \) is constant on each boundary circle. Let \( \tilde{V} \) be the solution of
\[
\Delta \chi \tilde{V} = 0 \text{ on } \Lambda, \quad \tilde{V} = 1 \text{ on } \mathcal{C}, \quad \tilde{V} = 0 \text{ on } \mathcal{C}.
\]
By 4.7 we can write \( \mathcal{L}_\chi \tilde{V} = E_1 + E_2 \) where \( \| E_1 : C^{2,\beta}_S(\Lambda, \chi, e^{-\gamma \chi}) \| \leq C e^{-3b/2} \) and \( \| E_2 : C^{2,\beta}_S(\Lambda, \chi, e^{-\gamma \chi}) \| \leq C e^{-3b/2} / \ell \). By applying twice 4.10 and assuming \( b \) large enough we obtain \( \tilde{V} \in C^{2,\beta}_S(\Lambda, \chi) \) such that \( \mathcal{L}_\chi \tilde{V} = 0 \) on \( \Lambda \), \( \tilde{V} \) is constant on each boundary circle of \( \Lambda \), \( \| \tilde{V} : C^{2,\beta}_S(\Lambda, \chi) \| \leq C \), \( |\tilde{V} - 1| \leq 1/9 \) on \( \mathcal{C} \), and \( |\tilde{V}| \leq 1/9 \) on \( \mathcal{C} \). By exchanging \( \mathcal{C} \) with \( \mathcal{C} \) we obtain \( V \) instead of \( \tilde{V} \). By considering linear combinations of \( \tilde{V} \) and \( V \) we complete the proof.

The approximate kernel. We proceed now to discuss the approximate kernel of \( L_h \) on the extended standard regions, cf. [10, Prop. 2.22]. By approximate kernel we mean the span of eigenfunctions whose eigenvalues are close to 0. Since we have to take into account the symmetries imposed, note that the stabilizer of \( \tilde{S}[0] \) with respect to the action of \( \mathcal{G} \) is generated by the reflections \( \mathcal{X}, \mathcal{Y}, \) and \( \mathcal{Z} \), and the stabilizer of \( \tilde{S}[1] \) by \( \mathcal{X} \) and \( \mathcal{Y} \). Therefore we have to restrict our attention to functions on the extended standard regions which are invariant under the action of these subgroups. Moreover the functions on \( \tilde{S}[1] \) should extend smoothly in the appropriate class to \( \mathcal{G}\tilde{S}[1] \). This implies by the symmetries under \( \mathcal{G} \) the second statement in the following definition.

Definition 4.13. We call functions which satisfy the above conditions appropriately symmetric and we use the subscript “\( \text{sym} \)” to denote subspaces of appropriately symmetric functions. Note that appropriately symmetric functions on \( \tilde{S}[1] \) satisfy Neumann boundary conditions on \( C_\partial \subset \partial \tilde{S}[1] \).

We understand the approximate kernel in the next proposition by comparing it to the kernel of the operator \( \Delta + 2 \) on the round sphere \( \mathbb{S}^2(1) \), and \( \Delta \) on the square \( [-\pi/2, \pi/2] \times [-\pi/2, \pi/2] \) with Neumann boundary conditions on the boundary. Because of the symmetries the former is trivial and the latter one-dimensional. Note that without the \( \mathcal{Z} \) symmetry we would have a one-dimensional kernel on the round sphere corresponding to vertical translations. We would have two squares which would not be equivalent by a symmetry, and each would carry a one-dimensional kernel. We would have then three-dimensional approximate and substitute kernels. This is why a construction based on a rectangular lattice where \( \mathcal{Z} \) cannot be a symmetry, would be more complicated, and we would need a three-parameter family of initial surfaces.
Proposition 4.14. Assuming $b$ large enough in absolute terms, and $\tau$ small enough (equivalently $m$ large enough) in terms of a given $\varepsilon > 0$, the following hold:

(i) $L_h$ acting on appropriately symmetric functions on $\widetilde{S}[0]$ with vanishing Dirichlet conditions has no eigenvalues in $[-1, 1]$, and the corresponding approximate kernel is trivial.

(ii) $L_h$ acting on appropriately symmetric functions on $\widetilde{S}[1]$, with Dirichlet boundary conditions on $C[0] \subset \partial \widetilde{S}[1]$ and Neumann on $C[0] \subset \partial \widetilde{S}[1]$, satisfies the following. It has exactly one eigenvalue $\lambda_0$ in $[-\varepsilon, \varepsilon]$, and no other eigenvalues in $[-1/2, 1/2]$, and therefore the corresponding approximate kernel is one-dimensional. Moreover the approximate kernel is spanned by a function $f_0 \in C^\infty_{\text{sym}}(\widetilde{S}[1])$, which depends continuously on $\zeta$, and satisfies

$$\|f_0 - 1: C^{2, \beta}(S[1])\| < \varepsilon, \quad \|f_0: C^{2, \beta}(\widetilde{S}[1], \chi)\| < C.$$ 

Proof. The proof is based on the results of [6, Appendix B] which are based on basic facts about eigenvalues and eigenfunctions [2]. Before using those results, we remark the following: First, the first inequality in [6, B.1.6] should read

$$\|Ff\|_\infty \leq 2\|f\|_\infty$$

instead. Second, the spaces of functions can be constrained to satisfy appropriate symmetries, as indeed was the case in some of the constructions in [6], and will be the case here. Third, obvious modifications should be made to allow for the fact that we are dealing here with operators of the form $\Delta + q$, where $q$ is a uniformly bounded smooth function. Fourth, the only use of the Sobolev inequality [6, B.1.5] is to establish supremum bounds for the eigenfunctions. These in our case can instead be established by using the uniformity of geometry of $S_5[n]$ to obtain interior estimates on $S_1[n]$, and then using a variant of 4.12 to obtain estimates on the transition regions. More precisely the eigenvalue equation under consideration is $L_h u + \lambda u = 0$, which is equivalent to

$$L_{\chi, \lambda} u = 0 \quad \text{where} \quad L_{\chi, \lambda} = L_{\chi} + \frac{|A|^2 + m^2}{2\rho^2} \lambda.$$ 

We are only interested in eigenvalues $\lambda$ which satisfy $|\lambda| < 3$ because the smallest eigenvalue on the sphere or the square is $-2$ and anything above $3$ is beyond the range of the approximate kernel. Since the modified part of the operator, $\frac{|A|^2 + m^2}{2\rho^2} \lambda$, satisfies the same estimates by 4.7 as $\rho^{-2}(|A|^2 + 2)$, we can repeat the arguments leading to 4.12 to establish the same estimate under the modified assumption that $L_{\chi, \lambda} u = 0$ on $\Lambda$. 
For (i) we compare with the following:

\[ N[0] = S^2(1) \cup \left\{ (x, y, z) \in \mathbb{R}^2 : x^2 + y^2 \leq \tilde{R}_\varepsilon^2 \right\}, \]

where \( \tilde{R}_\varepsilon \) was defined in 3.29. The action of \( X, Y, \) and \( Z \) on \( N[0] \) should be consistent with their action on \( M \) (recall 2.2) and the maps \( \tilde{v} \) and \( \varpi \). We define for \( (x, y, z) \in S^2(1) \) and \( (i, \bar{x}, \bar{y}) \in \{1, -1\} \times \tilde{D}(\tilde{R}_0) \)

\begin{align*}
X(x, y, z) &= (-x, y, z), & X(i, \bar{x}, \bar{y}) &= (i, -\bar{x}, \bar{y}), \\
Y(x, y, z) &= (x, -y, z), & Y(i, \bar{x}, \bar{y}) &= (i, \bar{x}, -\bar{y}), \\
Z(x, y, z) &= (y, x, z), & Z(i, \bar{x}, \bar{y}) &= (-i, \bar{y}, \bar{x}).
\end{align*}

We consider the Dirichlet problem on \( N[0] \) where the operator is \( \Delta + 2 \) on \( S^2(1) \) and the standard Laplacian \( \Delta \) on \( \{1, -1\} \times \tilde{D}(\tilde{R}_0) \). By standard theory then, there are no eigenvalues in \( [-1, 1] \) because the symmetries do not allow the first harmonics on \( S^2(1) \), and \( \tilde{R}_0 \) is small enough so that the smallest eigenvalue on the discs is \( \geq 2 \).

For (ii) we compare with the following:

\[ N[1] = \tilde{D} \cup \left\{ (-\pi/2, \pi/2) \times [-\pi/2, \pi/2] \right\}, \]

where \( \tilde{D} = \left\{ (x, y, z) \in S^2(1) : x^2 + y^2 \leq \tilde{R}_0^2, \quad z \geq 0 \right\}, \)

where \( \tilde{R}_0 = 1 / \cosh (ab/g) \) (recall 3.29). The action of \( X, Y \) on \( N[1] \) should be consistent again with their action on \( M \) (recall 2.2) and the maps \( \tilde{v} \) and \( \varpi \): We define for \( (x, y, z) \in \tilde{D} \) and \( (\bar{x}, \bar{y}) \in [-\pi/2, \pi/2] \times [-\pi/2, \pi/2] \)

\begin{align*}
X(x, y, z) &= (-x, y, z), & X(\bar{x}, \bar{y}) &= (-\bar{x}, \bar{y}), \\
Y(x, y, z) &= (x, -y, z), & Y(\bar{x}, \bar{y}) &= (\bar{x}, -\bar{y}),
\end{align*}

As before the operator on \( \tilde{D} \subset S^2(1) \) is \( \Delta + 2 \) and on \( [-\pi/2, \pi/2] \times [-\pi/2, \pi/2] \subset \mathbb{R}^2 \) is the standard Laplacian \( \Delta \). The boundary conditions are the Dirichlet condition on \( \partial \tilde{D} \) and the Neumann condition—more precisely extendibility to \( \mathbb{R}^2 \) by reflections across the lines \( \{ x = n\pi/2 \} \) and \( \{ y = n\pi/2 \} \) \( (n \in \mathbb{Z}) \)—for the boundary of the square \( [-\pi/2, \pi/2] \times [-\pi/2, \pi/2] \subset \mathbb{R}^2 \). The smallness of \( \tilde{R}_0 \) and our knowledge of the eigenvalues on the square imply the only eigenvalue in \( [-2/3, 2/3] \) is 0, with corresponding eigenfunctions the functions which are constant on the square and vanish on \( \tilde{D} \).

To complete the proof, we use \( \tilde{v}, \varpi \), and the logarithmic cut-off function \( \psi[2d, d] \circ \varpi \) on \( \Lambda \) to define the maps \( F_1 \) and \( F_2 \) required by [6, B.1.4] as usual. \( d \) is taken to be large enough in terms of \( \varepsilon \). It is straightforward then to check the required assumptions by using 4.7, and then the results of [6, Appendix B] apply.
We upgrade the $L^2$ estimates for $f_0 - 1$ to $C^{2,\beta}$ estimates on $S_5[1]$ by using the uniformity of the geometry of $S_4[1]$ (see 3.29) and standard linear theory interior estimates. Applying then the variant of 4.12 we discussed earlier, we estimate $f_0$ on $\Lambda$ and complete the proof.

**The (extended) substitute kernel.** As we have already mentioned in the introduction, the extended substitute kernel in this case is particularly simple since it is one-dimensional. This reflects the fact that the approximate kernel, and hence the substitute kernel also, are one-dimensional. Moreover decay can be ensured by using the substitute kernel and so no further extended substitute kernel is required.

Following the general methodology (see [14] for a general discussion), we need to define an appropriately symmetric function (recall 4.13) on $M$ which is supported on $\tilde{S}[1]$ and spans the substitute kernel. The main required property of such a function is that as $\tau \to 0$ this function should tend to a function on the “limit” of $(\tilde{S}[1], h)$ which is not orthogonal to the kernel. Recall that by 3.29 the limit of $(\tilde{S}[1], h)$ is a square in the Euclidean plane. The linearized operator tends to the standard Laplacian on the square acting with Neumann data on the boundary. The kernel consists of the constant functions on the square and 4.14 makes the comparison between kernel in the limit and approximate kernel on the initial surface precise. When solving the linear equation with an inhomogeneous term 4.1, we will be modifying appropriately the inhomogeneous term, by adding functions of the substitute kernel.

We define now the function under discussion spanning the substitute kernel to be

$$2\rho^2 |A|^2 + m^2 w,$$

where $w \in C^\infty_\text{sym}(M)$ is defined by requiring that on $M \cap D$ it satisfies

$$w := \psi[m^{-1}, 2m^{-1}](r). \quad (4.18)$$

Note that the factor in front of $w$ is included to simplify the linear equation as in 4.23(i), 4.24(i), and 4.30(i) later. The main use of the substitute kernel is to allow us to make the inhomogeneous term orthogonal to the approximate kernel according to the following lemma.

**Lemma 4.19.** Given $E \in C^0_\text{sym}(\tilde{S}[1])$, there is a unique $\mu \in \mathbb{R}$ such that \[ \frac{2\rho^2}{|A|^2 + m^2} (E + \mu w) \] is $L^2(\tilde{S}[1], h)$-orthogonal to $f_0$, where $f_0$ is the eigenfunction in 4.14. Moreover

$$|\mu| \leq C \left\| \frac{2\rho^2}{|A|^2 + m^2} E \right\|_{L^2_\text{sym}(\tilde{S}[1], h)}.$$
Proof. Using 3.16 and 3.18(v) we conclude that \( \frac{1}{C} \leq \frac{2\rho^2}{|A|^2 + m^2} \leq C \) on the support of \( w \), which together with 3.29, implies the result. \( \square \)

Following the general methodology again, we need to define a function \( v \in C_\infty(\tilde{S}[1]) \), whose main property is that it approximates a constant on \( C_1[1] \), and is used to ensure appropriate decay for the solutions of the linear equation 4.1. This is done by modifying the solutions by multiples of \( v \) and correspondingly modifying the inhomogeneous term by elements of the extended substitute kernel. In our case, the substitute kernel is sufficient if we define \( v \) carefully enough along the lines of [10, Lemma 6.6]. Following this \( v \) is defined as a modification of the eigenfunction \( f_0 \) spanning the approximate kernel (recall 4.14): We define \( v \in C_\infty(\tilde{S}[1]) \) by

\[
v := f_0 + u,
\]

where \( u \in C_\infty(\tilde{S}[1]) \) is the solution on \( \tilde{S}[1] \) to

\[
L \chi u = -L \chi f_0 + \mu_v w, \quad \text{or equivalently,} \quad L_h u = E_v,
\]

with vanishing Dirichlet data on \( C[0] \subset \partial \tilde{S}[1] \), where

\[
E_v := \frac{2\rho^2}{|A|^2 + m^2} (-L \chi f_0 + \mu_v w) = \lambda_0 f_0 + \mu_v \frac{2\rho^2}{|A|^2 + m^2} w,
\]

and \( \mu_v \in \mathbb{R} \) is determined by the requirement (recall 4.19) that \( E_v \) is \( L^2(\tilde{S}[1], h) \)-orthogonal to \( f_0 \). Note that the orthogonality condition together with 4.19 implies the existence of a unique \( u \) which we estimate in the proof of 4.23 below. We record now the properties of \( v \):

**Lemma 4.23.** \( v \in C_\infty(\tilde{S}[1]) \) satisfies the following, where \( \varepsilon \) is as in 4.14:

(i) \( L \chi v = \mu_v w \) on \( \tilde{S}[1] \) for some \( \mu_v \in \mathbb{R} \), and therefore \( L \chi v = 0 \) on \( \Lambda \).

(ii) \( v = 0 \) on \( C[0] \subset \partial \tilde{S}[1] \) and \( v \) satisfies Neumann conditions on \( C_\theta \subset \partial \tilde{S}[1] \).

(iii) \( \|v : C^{0,1}([\tilde{S}[1], \chi]) \| \leq C \).

(iv) \( |\mu_v| \leq C\varepsilon \).

(v) \( \|v - 1 : C^{2,0}(C_1[1], d\theta^2) \| \leq C(b) \varepsilon \).

**Proof.** (i) and (ii) follow from the definitions. We apply now 4.19 with \( E = -L \chi f_0 \). We have then \( \frac{2\rho^2}{|A|^2 + m^2} E = -L_h f_0 = \lambda_0 f_0 \) which implies by 4.19 that

\[
|\mu_v| \leq C |\lambda_0| \|f_0 : L^2(\tilde{S}[1], h)\|.
\]

This together with the estimates in 4.14 implies (iv) and that

\[
\|E_v : L^2(\tilde{S}[1], h)\| \leq C\varepsilon.
\]
Standard elliptic estimates for 4.21 imply then
\[ \|u : C^0(S_4[1])\| \leq C(b) \varepsilon, \]
which further implies
\[ \|u : C^{2,\beta}(S_3[1], \chi)\| \leq C(b) \varepsilon. \]
This together with the estimates for \( f_0 \) from 4.14 implies
\[ \|v - 1 : C^{2,\beta}(S_3[1], \chi)\| \leq C(b) \varepsilon, \]
which implies (v), and together with (i) and 4.12 applied to \( v \), (iii).

**Solving the linearized equation semi-locally.** In this subsection, we solve and estimate the linear equation on the extended standard regions. We can assume the inhomogeneous term \( E \) to vanish on \( \Lambda_1 \), because in the proof of 4.30, we use first 4.10 to solve for the part of the inhomogeneous term which is supported there. In the case of \( \tilde{S}[1] \) we have nontrivial approximate kernel, and therefore we have to adjust the inhomogeneous term appropriately by using \( w \). \( w \) can also be used so that appropriate exponential decay can be arranged for the solution:

**Lemma 4.24.** There is a linear map
\[ \mathcal{R}_{\tilde{S}[1]} : \{ E \in C^{0,\beta}_{sym}(\tilde{S}[1]) : E \text{ is supported on } S_1[1] \} \rightarrow C^{2,\beta}_{sym}(\tilde{S}[1]) \times \mathbb{R}, \]
such that the following hold for \( E \) in the domain of \( \mathcal{R}_{\tilde{S}[1]} \) above and \((\varphi, \mu) = \mathcal{R}_{\tilde{S}[1]}(E)\):
(i) \( \mathcal{L}_\chi \varphi = E + \mu w \) on \( \tilde{S}[1] \).
(ii) \( \varphi \) vanishes on \( C[0] \subset \partial \tilde{S}[1] \) and satisfies appropriate Neumann boundary conditions on \( C[0] \subset \partial \tilde{S}[1] \) (recall (3.25h)).
(iii) \( |\mu| + ||\varphi : C^{2,\beta}_{sym}(\tilde{S}[1], \chi)|| \leq C(b) \|E : C^{0,\beta}_{sym}(S_1[1], \chi)||. \)
(iv) \( ||\varphi : C^{2,\beta}_{sym}(\Lambda, \chi, e^{-\gamma^\beta})|| \leq C(b) \|E : C^{0,\beta}_{sym}(S_1[1], \chi)||. \)
(v) \( \mathcal{R}_{\tilde{S}[1]} \) depends continuously on \( \zeta \).

**Proof:** We fix \( b \) to be large enough so that 4.14 and 4.11 with \( \varepsilon_1 = 1 \) apply. Recall that by 3.17 and 3.26 we have \( \chi = \frac{2\rho^2}{|A|^{1+m^2}} h \) and moreover by 3.18 we have that the conformal factor and its inverse have their \( C^k \) norms on \( S_3[1] \) with respect to \( h \) or \( \chi \) bounded by a uniform constant which depends only on \( k \) and \( b \). By applying 4.19 and using that \( E \) is supported on \( S_1[1] \), where the factor \( \frac{2\rho^2}{|A|^{1+m^2}} \) satisfies a bound depending only on \( b \), we obtain \( \mu_1 \) such that
\[ |\mu_1| \leq C(b) \|E : C^{0,\beta}_{sym}(S_1[1], \chi)||. \]
and \(\frac{2\rho^2}{|A|^{1/2}}(E + \mu_1w)\) is \(L^2(\overline{S}[1], h)\)-orthogonal to \(f_0\). There is a unique solution \(\varphi_1 \in C^{2,\beta}_{\text{sym}}(\overline{S}[1])\) which is \(L^2(\overline{S}[1], h)\)-orthogonal to \(f_0\), vanishes on \(C[0] \subset \partial\overline{S}[1]\) and satisfies Neumann conditions on \(C_{\partial}\), and satisfies on \(\overline{S}[1]\)
\[
\mathcal{L}_\chi \varphi_1 = E + \mu_1w, \quad \text{or equivalently} \quad \mathcal{L}_h \varphi_1 = \frac{2\rho^2}{|A|^2 + m^2}(E + \mu_1w).
\]

Using interior estimates for the second equation we have
\[
\|\varphi_1\|_{C^0(S[1])} \leq C(b) \|E\|_{C^0}.
\]

Interior estimates for the first equation then give
\[
\|\varphi_1\|_{C^{2,\beta}_{\text{sym}}(S[1], \chi)} \leq C(b) \|E\|_{C^{0,\beta}_{\text{sym}}(S[1], \chi)}.
\]

We apply now 4.11 on \(\Lambda_{0,1}\) (recall (3.25e)) with \(u = v - \text{avg } v\) on \(C_1[1] \subset \partial\Lambda_{0,1}\) to obtain
\[
V_v := \mathcal{R}_\partial(v|_{C_1[1]} - \text{avg } v|_{C_1[1]}),
\]
and once more with \(u = \varphi_1 - \text{avg } \varphi_1\) on \(C_1[1] \subset \partial\Lambda_{0,1}\) to obtain
\[
V_{\varphi_1} := \mathcal{R}_\partial(\varphi_1|_{C_1[1]} - \text{avg } \varphi_1|_{C_1[1]}).
\]

Combining 4.23(v) with 4.11(iii) we conclude that the constant \((V_v - v)|_{C_1[1]}\) is close to \(-1\). Hence there is a unique constant \(\mu_2\) such that \(\varphi := \varphi_1 + \mu_2v\) agrees with \(V_{\varphi_1} + \mu_2V_v\) on \(\partial\Lambda\), and therefore by 4.9 also on \(\Lambda\). By taking \(\mu := \mu_1 + \mu_2\mu v\) and using the available estimates from 4.11 and 4.23 we complete the proof.

The corresponding statement for \(\overline{S}[0]\) is simpler, reflecting the triviality of the approximate kernel there and that we do not need exponential decay either:

**Lemma 4.25.** There is a linear map
\[
\mathcal{R}_{\overline{S}[0]} : \{ E \in C^{0,\beta}_{\text{sym}}(\overline{S}[0]): E \text{ is supported on } S[0] \} \rightarrow C^{2,\beta}_{\text{sym}}(\overline{S}[0]),
\]

such that the following hold for \(E\) in the domain of \(\mathcal{R}_{\overline{S}[0]}\) above and \(\varphi = \mathcal{R}_{\overline{S}[0]}(E)\):

(i) \(\mathcal{L}_\chi \varphi = E\) on \(\overline{S}[0]\).
(ii) \(\varphi\) vanishes on \(\partial\overline{S}[0]\).
(iii) \(\|\varphi\|_{C^{2,\beta}_{\text{sym}}(\overline{S}[0], \chi)} \leq C(b) \|E\|_{C^{0,\beta}_{\text{sym}}(S[0], \chi)}\).
(iv) \(\mathcal{R}_{\overline{S}[0]}\) depends continuously on \(\zeta\).
Proof. By 4.14 there are no small eigenvalues, and so we can solve and obtain $L^2(h)$ estimates, which together with interior estimates on $S^2[0]$ and 4.12, imply the result.

Solving the linearized equation globally. In order to solve the linearized equation 4.1 globally on $M$ and provide estimates for the solutions, we paste together the semi-local solutions on $\Lambda$, $\tilde{S}[1]$, and $\tilde{S}[0]$, provided by 4.10, 4.24, and 4.25, to obtain a global solution in the proof of 4.30. Before we state the proposition we define appropriate norms:

**Definition 4.26.** For $k \in \mathbb{N}$ and $\beta, \gamma \in (0, 1)$ as before, we define norms by

$$\|\phi\|_{k,0,\gamma} := \|\phi: C^k_{\text{sym}}(M, \chi, \tilde{f})\|, \quad \|\phi\|_{k,\beta,\gamma} := \|\phi: C^k_{\text{sym}}(M, \chi, \tilde{f})\|,$$

where the weight function $\tilde{f}$ is defined by requesting that it is invariant under the action of $G$, $\tilde{f} = 1$ on $S[1]$, $\tilde{f} = e^{-\gamma x}$ on $\Lambda$, and $\tilde{f} = e^{-(a-2b)\gamma} = e^{-\gamma x}|_{C[0]}$ on $S[0]$ (recall 4.5).

Note that $\tilde{f}$ is continuous on $M$ and decreasing as a function of $\overline{x}$ on $\Lambda$. Similarly $\rho\tilde{f}$ is continuous on $M$, and by a straightforward calculation using 4.4, 3.6, 3.16, and 3.22, increasing as a function of $\overline{x}$ on $\Lambda$. This implies that the values of $\tilde{f}$ and $\rho\tilde{f}$ attained on $S[0]$ are their global minimum and maximum on $M$ respectively. By using 3.10 and absorbing constants by small powers of $\overline{z}$, we obtain the bounds

$$\overline{z}^{\gamma + \frac{1}{2}} \leq \tilde{f} \quad \text{and} \quad \rho\tilde{f} \leq \overline{z}^{\gamma - 1} \quad \text{on} \quad M,$$

which we use to control $\tilde{f}$ in the estimates later.

Recall that the linearized equation for the mean curvature when written in the form $L^\chi \phi = E$ has the inhomogeneous term $E = -\rho^{-2}H$. In order to obtain estimates with decay later, we will need to control $\|E\|_{0,\beta,\gamma}$. Given the estimates in 3.18 this is easy to achieve as follows. Note that the estimate obtained depends on $\tau$ and not $\overline{z}$. Because of this the constant $C$ does not depend on $\overline{c}$. This allows us later in 5.3 and in the proof of 5.4 to control the force term uniformly independently of $\overline{c}$ and close the argument by choosing $c$ large enough.

**Lemma 4.28.** If $m$ is large enough, we have on $M$ the estimate

$$\|\rho^{-2}H\|_{0,\beta,\gamma} \leq C\tau.$$

**Proof.** By 3.18(iv) it is enough to prove that on $M$ we have

$$(\tau + \rho^{-2})(|z| + \tau) \leq C\tilde{f}\tau.$$
By 3.18(ii) it is enough to prove the inequalities
\[ m^2 \tau \leq \tilde{f} \quad \text{and} \quad m^2 \rho^{-2} \leq C\tilde{f}. \]

The first one follows from 4.27 and the observation that 1 is a higher power than \( \frac{8}{9} \gamma + \frac{1}{2} \). The second inequality clearly holds on \( S[1] \) by 3.16. Both sides of the inequality decay exponentially on \( \Lambda \) as functions of \( \tau \), but the left hand side at a faster rate. This completes the proof.

From now on there are no instances where \( b \) may be needed to be chosen any larger than before. For this reason we adopt the following:

**Convention 4.29.** From now on we fix a \( b \) large enough in absolute terms so that the earlier results apply (4.11 with \( \varepsilon_1 = 1 \)). We will not mention anymore the dependence of constants \( C \) on \( b \).

**Proposition 4.30.** There is a linear map \( R_M : C^{0,\beta}(M) \to C^{0,\beta}(M) \times \mathbb{R} \) such that for \( E \in C^{0,\beta}(M) \) and \( (\varphi, \mu) = R_M E \) the following hold:

(i) \( \mathcal{L}_\chi \varphi = E + \mu \omega \) on \( M \).
(ii) \( |\mu| + \| \varphi \|_{2,\beta,\gamma} \leq C \| E \|_{0,\beta,\gamma} \).
(iii) \( R_M \) depends continuously on \( \zeta \).

**Proof.** We decompose \( E = E_{S[0]} + E_{S[1]} + E_\Lambda \) by requesting that \( E_{S[0]}, E_{S[1]}, \) and \( E_\Lambda \), are invariant under \( \mathcal{G} \) and satisfy

\[
E_{S[0]} := E \psi[1,0] \circ \chi, \\
E_{S[1]} := E \psi[1,0] \circ \tau, \\
E_\Lambda := E \psi[0,1] \circ \chi,
\]
on \( \Lambda \), \( E_{S[0]} := E, E_{S[1]} := 0, E_\Lambda := 0 \) on \( S[0] \), and \( E_{S[0]} := 0, E_{S[1]} := E, E_\Lambda := 0 \) on \( S[1] \). Using 4.10 we define \( V_\Lambda \in C^{2,\beta}(M) \) by \( V_\Lambda = 0 \) on \( S[0] \cup S[1] \) and \( V_\Lambda = \psi[0,1] \circ \chi \mathcal{L}_\chi V_\Lambda - E_\Lambda \) on \( \Lambda \). \( \mathcal{L}_\chi V_\Lambda - E_\Lambda \) is supported on \( \Lambda \setminus \Lambda_1 \), and can be decomposed as \( \mathcal{L}_\chi V_\Lambda - E_\Lambda = E + \overline{E} \) where \( \overline{E} \) is supported on \( \{ \chi \leq 1 \} \) and \( \overline{E} \) is supported on \( \{ \chi \leq 1 \} \).

Using 4.25 we define \( V_{S[0]} \in C^{2,\beta}(M) \) by requesting \( V_{S[0]} = 0 \) on \( S[1] \) and \( V_{S[0]} = \psi[0,1] \circ \tau \mathcal{L}_\tau V_{S[0]}(E_{S[0]} - E) \) on \( S[0] \). We define also \( E_1 \) by \( \mathcal{L}_\chi V_{S[0]} + E_1 = E_{S[0]} - E \). Using the estimates provided by 4.10 and 4.25, and using also 4.26 to estimate \( E_{S[0]} \) and \( E \) in terms of \( \| E \|_{0,\beta,\gamma} \), we obtain in particular that

\[ \| E_1 : C^{2,\beta}(M, \chi) \| \leq C e^{-Q^{-2}b} \| E \|_{0,\beta,\gamma}. \]
Notice that $E_1$ is supported on $\mathcal{G}(S_1[1] \cap \Lambda)$, and there we have $e^{\gamma} \leq C$. Also, we have $e^{-2b}\gamma \leq \frac{\gamma}{2}$. We conclude that

$$\|E_1\|_{0,\beta,\gamma} \leq C\frac{\gamma}{2}\|E\|_{0,\beta,\gamma}.$$ 

We apply now 4.24 with $\gamma' = \frac{\gamma + 1}{2}$ in place of $\gamma$ to define $V_{S[1]} \in C^2_{\text{sym}}(M)$ by requesting $V_{S[1]} = 0$ on $S[0]$ and $V_{S[1]} = \psi[0,1] \circ x V'_{S[1]}$ on $S[1]$, where

$$(V'_{S[1]},\mu_1) = \mathcal{R}_{S[1]}^-(E_{S[1]} - E).$$

We define also $E_1$ by $L\chi V_{S[1]} + E_1 = E_{S[1]} - E + \mu_1 w$. Using the estimates provided by 4.10 and 4.24, and using also 4.26 to estimate $E_{S[1]}$ and $E$ in terms of $\|E\|_{0,\beta,\gamma}$, we obtain in particular that

$$\|E_{1}: C^2_{\text{sym}}(\Lambda,\chi, e^{-\gamma'})\| \leq C\|E\|_{0,\beta,\gamma}.$$ 

Notice that $E_1$ is supported on $S_1[0] \cap \Lambda$, and there we have $e^{\gamma - \gamma'} \leq \frac{\gamma'}{2}$. Using 4.26 we conclude that

$$\|E_1\|_{0,\beta,\gamma} \leq C\frac{\gamma}{2}\|E\|_{0,\beta,\gamma}.$$ 

Note that the factor $\frac{\gamma'}{2}$ is necessary to ensure that the norm of $E_1$ is better than the norm of $E$ so we can iterate later. This is the reason we applied 4.24 with $\gamma$ replaced by $\gamma'$.

We define now $\varphi_1 := V_{S[0]} + V_{S[1]}$ and $E_1 := E_{S[1]} + E_{1}$. We have then $L\chi \varphi_1 + E_1 = E + \mu_1 w$ on $M$. Using the estimates above and the ones provided by 4.10, 4.24, and 4.25, and assuming that $\tau$ is small enough in terms of $b$, we conclude that

$$|\mu_1| + \|\varphi_1\|_{2,\beta,\gamma} \leq C\|E\|_{0,\beta,\gamma}, \quad \|E_1\|_{0,\beta,\gamma} \leq \frac{1}{2}\|E\|_{0,\beta,\gamma}.$$ 

We iterate then with $E_1$ instead of $E$ and so on. We complete the proof then by defining $\varphi := \sum_{n=1}^{\infty} \varphi_n$ and $\mu := \sum_{n=1}^{\infty} \mu_n$. \qed

5. The main results.

The nonlinear terms. If $\phi \in C^1_{\text{sym}}(M)$ is appropriately small, we denote by $M_\phi$ the perturbation of $M$ by $\phi$, defined as $I_\phi(M)$ in the notation of Appendix A, where $I: M \to \mathbb{S}^3(1)$ is the inclusion map of $M$. Clearly then $M_\phi$ is invariant under the action of $\mathcal{G}$ on the sphere $\mathbb{S}^3(1)$. Using then rescaling and Proposition A.3, we prove a global estimate of the nonlinear terms for the mean curvature of $M_\phi$ as follows:
Lemma 5.1. If \( \phi \in C^{2,\beta}_{\text{sym}}(M) \) satisfies \( \|\phi\|_{2,\beta,\gamma} \leq \tau^{-\frac{3}{4}} \), then \( M_\phi \) is well defined as above and satisfies

\[
\|\rho^{-2}H_\phi - H - \mathcal{L}_\chi \phi\|_{0,\beta,\gamma} \leq \tau^{\frac{3}{4}} - \frac{3}{4} \gamma \|\phi\|_{2,\beta,\gamma}^2,
\]

where \( H_\phi \) is the mean curvature of \( M_\phi \) (pulled back to \( M \) by \( I_\phi \)), and \( H \) is the mean curvature of \( M \).

Proof. Let \( D \) be a disc of radius 1 and center at some point \( p \in M \) with respect to the \( \chi \) metric. If we magnify the metric of the sphere \( S^3(1) \) by a factor \( \rho(p) \), it is easy to arrange for the hypothesis A.1 to be satisfied so that we can apply A.3 with some universal \( c_1 \) to conclude

\[
\| (\rho(p))^{-1}(H_\phi - H - \mathcal{L}_\phi) : C^{0,\beta}(D, \chi) \| \leq C \frac{1}{\epsilon(c_1)} \|\phi\|: C^{2,\beta}(D, \chi) \|^2,
\]

where the factors \( \rho(p) \) correspond to the scaling of the quantities involved. By using 3.18(i) and the multiplicative properties of the Holder norms, we conclude

\[
\|\rho^{-2}(H_\phi - H - \mathcal{L}_\phi) : C^{0,\beta}(D, \chi) \| \leq C \frac{\rho(p)}{\epsilon(c_1)} \|\phi\|: C^{2,\beta}(D, \chi) \|^2.
\]

By 4.26 we conclude

\[
\frac{1}{f(p)} \|\rho^{-2}(H_\phi - H - \mathcal{L}_\phi) : C^{0,\beta}(D, \chi) \| \leq C \frac{\rho(p)f(p)}{\epsilon(c_1)} \|\phi\|_{2,\beta,\gamma}^2.
\]

This implies the result by using 4.27.

The vertical force and balancing. If \( \phi \in C^{1}_{\text{sym}}(M) \), \( M_\phi \), and \( H_\phi \) are as in the previous subsection, we define \( F \) by

\[
F := \int_{M_\phi \cap \tilde{D}_+} H_\phi \left\langle \nu, \tilde{K} \right\rangle \ dg = \int_{M_\phi \cap \partial \tilde{D}_+} \left\langle \tilde{\eta}, \tilde{K} \right\rangle \ dg,
\]

where \( \tilde{D}_+ := D \cap \{ z \geq 0 \} \), \( \nu \) the unit normal chosen so that \( \langle \nu, \partial_z \rangle > 0 \) on \( \tilde{M}_{\text{tor}} \), \( \tilde{K} \) is the Killing field defined in 2.9, and \( \tilde{\eta} \) the outward conormal to \( \partial(M_\phi \cap \tilde{D}_+) = M_\phi \cap \partial \tilde{D}_+ \) tangent to \( M_\phi \). Note that the second equality in 5.2 follows from the first variation formula \[22\], \[18\]. We have then the following:

Lemma 5.3. If \( \|\phi\|_{1,0,\gamma} \leq \tau^{1-\frac{1}{2}} \), then there is a universal constant \( C \) such that

\[
\left| \frac{m^2}{8\pi^2} F + \zeta \right| \leq C \left( 1 + \frac{1}{\tau} \|\phi\|_{1,0,\gamma} \right).
\]
Proof. Let $d := \pi / \sqrt{2} m$ and decompose

$$\partial(M_\phi \cap \mathbb{D}^+) = M_\phi \cap \partial \mathbb{D}^+ = \partial_{+1} \cup \partial_{-1} \cup \partial_{+2} \cup \partial_{-2} \cup \partial_0,$$

where $\partial_{+1} \subset \{ x = d \}$, $\partial_{-1} \subset \{ x = -d \}$, $\partial_{+2} \subset \{ y = d \}$, $\partial_{-2} \subset \{ y = -d \}$, and $\partial_0 \subset \{ z = 0 \}$. Recall that in a neighborhood of $M \cap \partial \mathbb{D}^+ \cap \{ z > 0 \}$, $M$ lies on the surface $\{ z = a \tau \}$ whose unit normal is $\nu = \partial_z$, and therefore $M_\phi$ can be described as the graph of $z = a \tau + \phi$ over the Clifford torus. This implies that on $\partial_{\pm1}$ the normal to $M_\phi$ is close to $\partial_z$ and the tangent to $\partial_{\pm1}$ is close to $\partial_y$. Moreover the normal is tangent to $\{ x = \pm d \}$ because $M_\phi$ is symmetric with respect to $X_{\pm \pi / \sqrt{2} m}$, and so is the tangent to $\partial_{\pm1}$ because the whole curve lies on $\{ x = \pm d \}$. We conclude that the conormal $\vec{\eta}$, which is by definition normal to both, is normal to $\{ x = \pm d \}$, and therefore by 2.6

$$\vec{\eta} = \pm (1 + \sin 2z)^{-1/2} \partial_x.$$

Using then 2.6 and 2.10 we calculate that on $\partial_{\pm1}$ we have

$$\langle \vec{\eta}, \vec{K} \rangle = -\frac{1}{\sqrt{2}} \sqrt{1 + \sin 2z} \cot \left( z + \frac{\pi}{4} \right) \sin \sqrt{2} d \cos \sqrt{2} y,$$

$$dg = \sqrt{1 - \sin 2z + \phi^2_y} dy.$$

Combining the above and using the trigonometric identity $\cot (z + \frac{\pi}{4}) = (1 - \sin 2z) / \cos 2z$, we obtain

$$\int_{\partial_{\pm1}} \langle \vec{\eta}, \vec{K} \rangle \, dg = -\frac{1}{\sqrt{2}} \int_{-d}^{d} (1 - \sin 2z + O(|z|^2 + |\phi_y|^2)) \sin \sqrt{2} d \cos \sqrt{2} y \, dy,$$

where in this proof we use the big-$O$ notation to denote terms $O(A)$ which satisfy $|O(A)| \leq CA$ for some universal constant $C$. Similarly

$$\int_{\partial_{\pm2}} \langle \vec{\eta}, \vec{K} \rangle \, dg = \frac{1}{\sqrt{2}} \int_{-d}^{d} (1 + \sin 2z + O(|z|^2 + |\phi_x|^2)) \cos \sqrt{2} x \sin \sqrt{2} d \, dx.$$

Since as argued earlier on $\partial_{+1} \cup \partial_{-1} \cup \partial_{+2} \cup \partial_{-2}$ we have $z = \tau a + \phi$, by combining the above, we conclude that

$$\int_{\partial_{+1} \cup \partial_{-1} \cup \partial_{+2} \cup \partial_{-2}} \langle \vec{\eta}, \vec{K} \rangle \, dg = (8a \tau + O(\|\phi\|^2 + a^2 \tau^2)) \sin^2 \sqrt{2} d,$$

where $\|\phi\| := \|\phi\|_{1,0,\gamma}$. Similarly

$$\int_{\partial_0} \langle \vec{\eta}, \vec{K} \rangle \, dg = -2\pi \tau (1 + O(\tau^2 + \tau^{-1} \|\phi\|^2)).$$
Combining and substituting $a = \frac{m^2}{4\pi} - \zeta + O(1)$ (see 3.10), we conclude

$$F = -\frac{8\pi^2 \tau}{m^2} \zeta + \frac{1}{m^2} O(\tau + \|\phi\|),$$

which implies the result.

**The main theorem.** We have now all the information we need to state and prove the main theorem of the paper:

**Theorem 5.4.** There are absolute constants $c_0, C > 0$ such that if $m$ is large enough, then there is $\zeta_1 \in [-c_0, c_0]$ such that on the corresponding initial surface $\mathcal{M}$, there is $\phi \in C^{\infty}_{\text{sym}}(\mathcal{M})$ with $\|\phi\|_{2,\beta,\gamma} \leq C \tau$ (with $\tau$ defined as in 3.4), such that $\mathcal{M}_\phi$ is a genus $m^2 + 1$ embedded minimal surface in $\mathbb{S}^3(1)$ invariant under the action of $\mathcal{G}$.

**Proof.** We will use a subscript $\zeta$ to specify the initial surface $\mathcal{M}_\zeta$ which is constructed as in the discussion preceding 3.15. We need to be able to compare functions on different $\mathcal{M}_\zeta$’s in order to set up the standard fixed point theorem approach. For this we need to define diffeomorphisms $F_\zeta: \mathcal{M}_\zeta \to \mathcal{M}_0$ which respect the symmetries, depend continuously on $\zeta$, and do not distort much the $\chi$ metrics. These requirements are satisfied if we define $F_\zeta$ so that the $(t, \theta)$ coordinates (recall 3.6 and 3.22) of corresponding points on $\mathcal{M}_\text{cat} \subset \mathcal{M}_\zeta$ and $\mathcal{M}_\text{cat} \subset \mathcal{M}_0$ are the same, while corresponding points on $\mathcal{M}_\text{tor} \subset \mathcal{M}_\zeta$ and on $\mathcal{M}_\text{tor} \subset \mathcal{M}_0$ map to the same point by $\varpi$ (recall 3.27). The two conditions proposed do not match exactly however, so we have to use a cut-off function to transit from one condition to the other as follows.

To facilitate the discussion, we denote by $X_\zeta: [-a, a] \times \mathbb{S}^1 \to \mathcal{M}_\zeta$ the parametrization of $\mathcal{M}_\text{cat} \subset \mathcal{M}_\zeta$ corresponding to the $(t, \theta)$ coordinates. We have then by 3.6, 3.22, and 3.27, that

$$\varpi \circ X_\zeta(t, \theta) = \frac{m\tau}{\sqrt{2}} \cosh (at/a) \left( \cos \theta, \sin \theta \right).$$

This is independent of $\zeta$ and $\tau$ at $t = a$ by 3.9, and it is easy to check then using 3.23 that a unique $\tilde{t}: [a-2, a-1] \to [a-3, a]$ can be defined by the requirement that for $(t, \theta) \in [a-2, a-1] \times \mathbb{S}^1$, we have $\varpi \circ X_\zeta(t, \theta) = \varpi \circ X_0(\tilde{t}(t), \theta)$. Moreover $\tilde{t}$ is close to the inclusion map but not equal to it (for $\zeta \neq 0$). We can think of it as expressing the mismatch between the two conditions above. We define now the diffeomorphism $F_\zeta: \mathcal{M}_\zeta \to \mathcal{M}_0$ by requiring that it is equivariant under the action of $\mathcal{G}$, it satisfies $\varpi \circ F_\zeta = \varpi$ on $\mathcal{M}_\text{tor} \cup (\mathcal{M}_\text{cat} \cap \{ t > a-1 \})$, and that for $(t, \theta) \in [-1, a-1] \times \mathbb{S}^1$, we have

$$F_\zeta \circ X_\zeta(t, \theta) = X_0 \left( \tilde{t}(t) \psi[a-2, a-1](t) + \tilde{t}(t) - t, \theta \right).$$
We define now a map $J : B \rightarrow B$ where

$$B := \{ u \in C^{2,\beta}_{\text{sym}}(M_0) : \|u\|_{2,\beta,\gamma} \leq \tau^{\frac{2}{\gamma} + 1} \} \times [-\varepsilon, \varepsilon]$$

as follows: We assume $(u, \zeta) \in B$ given. Let $\phi \in C^{2,\beta}(M_\zeta)$ be defined by $\phi := u \circ F_\zeta + \varphi$ where $(\varphi, \mu) = \mathcal{R}_{M_\zeta}(-\rho^2 H)$ as in 4.30. We have then

(a) $L_\chi \varphi + \rho^{-2} H = \mu w$, or equivalently $L \varphi + H = \mu \rho^2 w$.

(b) By 4.30 and 4.28 we have $|\mu| + \|\phi\|_{2,\beta,\gamma} \leq C \tau^{\frac{3}{\gamma} - \frac{1}{2}}$.

Note that the constant $C$ does not depend on $\varepsilon$ which allows us to close the argument later.

Applying 4.30 again and using 5.1, we obtain $(v, \mu') := R_{M_\zeta}(-\rho^{-2} H \varphi - L_\chi \phi)$ which satisfies the following:

(c) $L_\chi v + \rho^{-2} H \phi - \rho^{-2} H - L_\chi \phi = \mu' w$.

(d) $|\mu| + \|v\|_{2,\beta,\gamma} \leq C \tau^{\frac{5}{\gamma} - 1} \|\phi\|_{2,\beta,\gamma}^2$.

Combining (a) and (c) with the definition of $\phi$, we obtain

(e) $L_\chi (v - u \circ F_\zeta) + \rho^{-2} H \phi = (\mu + \mu') w$.

This motivates us to define

$$J(u, \zeta) = \left( v \circ (F_\zeta)^{-1}, \frac{m^2}{8\tau^2 \pi^2} F + \zeta \right),$$

where $F$ is defined as in 5.2. By using (b), (d), and 5.3, and by choosing $\varepsilon$ large enough in terms of an absolute constant, it is straightforward to check that $J(B) \subset B$. $B$ is clearly a compact convex subset of $C^{2,\beta}_{\text{sym}}(M_0) \times \mathbb{R}$ for $\beta' \in (0, \beta)$, and it is easy to check that $J$ is a continuous map in the induced topology. By Schauder’s fixed point theorem [3, Theorem 11.1] then, there is a fixed point of $J$. Using (e) then, we conclude that for the corresponding $\zeta$ and $\phi$, we have

$$H_\phi = (\mu + \mu') \rho^2 w, \quad F = 0.$$

It is easy to check as follows, that on the support of $w$ in $(M_\zeta)_{\phi} \cap \mathbb{D}_+$, we have $\langle \nu, \bar{K} \rangle > 0$. The smallness of $\|\phi\|_{2,\beta,\gamma}$ implies that the unit normal $\nu$ to $(M_\zeta)_{\phi}$ is close to the unit normal of $M_\zeta$. The support of $w$ under consideration is contained then in $M_{tor}$, where by 3.19, $g_{un}^0 k^3 x_1$ is close to $(0, 0, 1)$. By 2.10 and because the coordinates $(x, y, z)$ are close to $(0, 0, 0)$, we have that $\bar{K}$ is close to $\partial_z$, which allows us to conclude that $\langle \nu, \bar{K} \rangle = g_{un}^0 \nu^I \bar{K}^k$ is close to 1. $F = 0$ implies then by 5.2 that $\mu + \mu' = 0$, and hence $(M_\zeta)_{\phi}$ is a minimal surface. The smoothness of $\phi$ follows then by standard regularity theory. The embeddedness of $(M_\zeta)_{\phi}$ follows from the smallness of $\|\varphi\|_{2,\beta,\gamma}$ and the size (by 3.10) of $a \tau$. \qed
Appendix A. The mean curvature of a perturbed surface. We assume given an immersion $X : D \to U$, where $D$ is a disc of radius 1 in the Euclidean plane $\mathbb{R}^2$, and $U$ is an open cube in $\mathbb{R}^3$ equipped with a metric $g$ whose components are functions $g_{ij} : U \to \mathbb{R}$. We assume that $dist_g(X(D), \partial U) > 1$ and the following holds for some $c_1 > 0$:

\[(A.1) \quad \|\partial X : C^{2,\beta}(D, g_0)\| \leq c_1, \quad \|g_{ij}, g^{ij} : C^{4,\beta}(U, g_0)\| \leq c_1, \quad g_0 \leq c_1 X^* g,\]

where $\partial X$ are the partial derivatives of the coordinates of $X$, $g^{ij}$ denotes the components of the inverse of $g_{ij}$, and $g_0$ denotes the standard Euclidean metric on $U$ or $D$ respectively. Note that A.1 can be arranged by first appropriately magnifying the target (see for example 5.1). We also choose a unit normal $\nu : D \to \mathbb{R}^3$ for the immersion $X$ with respect to the $g$ metric. Given a function $\phi : D \to \mathbb{R}$ which is small enough, we define $X_\phi : D \to U$ by

\[(A.2) \quad X_\phi(p) := \exp_{X(p)}(\phi(p) \nu(p)),\]

where $\exp$ is the exponential map with respect to the $g$ metric. We have then the following:

**Proposition A.3.** There exists a (small) constant $\epsilon(c_1) > 0$ such that if $X$ is an immersion satisfying A.1 and the function $\phi : D \to \mathbb{R}$ satisfies

\[\|\phi : C^{2,\beta}(D, g_0)\| < \epsilon(c_1),\]

then $X_\phi : D \to U$ is a well-defined immersion by A.2 and satisfies

\[\|H_\phi - H - (\Delta_g + |A|^2 + \text{Ric}(\nu, \nu))\phi : C^{0,\beta}(D, g_0)\| \leq \frac{1}{\epsilon(c_1)} \|\phi : C^{2,\beta}(D, g_0)\|^2,\]

where $H = \text{tr}_g A$ is the mean curvature of $X$, defined as the trace with respect to $X^* g$ of the second fundamental form $A$, $H_\phi$ is the mean curvature of $X_\phi$, $\Delta_g$ is the Laplacian with respect to $X^* g$, and $\text{Ric}$ is the Ricci curvature of $(U, g)$.

**Proof.** That the linear terms are as stated is well known and follows by a straightforward calculation we omit. The nonlinear terms are given by expressions of monomials consisting of contractions of derivatives of $X, g_{ij}, g^{ij}$, the exponential map, and $\phi$. This implies both the existence results and the estimate on the nonlinearity. \qed

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