On $q$-analogues of quadratic Euler sums

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Abstract
In this paper we study $q$-analogues of Euler sums and present a new family of identities by using the method of Jackson $q$-integral representations of series. We then apply it to obtain a family of identities relating quadratic Euler sums to linear sums and $q$-polylogarithms. Furthermore, we use certain stuffle products to evaluate several $q$-series with $q$-harmonic numbers. Some interesting new results and illustrative examples are considered. Finally, if $q$ tends to 1, we obtain some explicit relations for the classical Euler sums.

Keywords $q$-harmonic number · $q$-Euler sum · $q$-polylogarithm function

Mathematics Subject Classification 05A30 · 65B10 · 33D05 · 11M99 · 11M06 · 11M32

1 Introduction

For positive integers $m$ and $k$, let $H_m^{(k)}$ and $\overline{H}_m^{(k)}$ denote the $m$-th generalized harmonic number and the $m$-th generalized alternating harmonic number defined by

$$H_m^{(k)} := \sum_{j=1}^{m} \frac{1}{j^k}, \quad \overline{H}_m^{(k)} := \sum_{j=1}^{m} \frac{(-1)^{j-1}}{j^k},$$

respectively (see, e.g., [10]). If $k > 1$, the generalized harmonic number $H_m^{(k)}$ converges to the (Riemann) zeta value $\zeta(k)$:

$$\lim_{m \to \infty} H_m^{(k)} = \zeta(k).$$

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When \( k = 1 \), \( H^{(1)}_m = H_m \) (resp. \( \overline{H}^{(1)}_m = \overline{H}_m \)) is the classical harmonic number (resp. the classical alternating harmonic number).

Let \( n \) be a positive integer, \( k_1, \ldots, k_n \) be nonzero integers, and let \( k \) be a positive integer with \( k \geq 2 \). The classical Euler sums are defined by the convergent series

\[
S(k_1, \ldots, k_n; k) := \sum_{m=1}^{\infty} \frac{X_m(k_1) \cdots X_m(k_n)}{m^k},
\]

\[
\overline{S}(k_1, \ldots, k_n; k) := \sum_{m=1}^{\infty} \frac{X_m(k_1) \cdots X_m(k_n) (-1)^{m-1}}{m^k},
\]

where

\[
X_m(k) := \begin{cases} 
H^{(k)}_m & \text{if } k \geq 1, \\
\overline{H}^{(-k)}_m & \text{if } k \leq -1.
\end{cases}
\]

Here we call \( |k_1| + \cdots + |k_n| + k \) the weight, and \( n \) the depth. Throughout the paper, for a positive integer \( k \), we use \( k \) to denote the negative entry \(-k\). For example, we have

\[
S(1, 2, 3; 4) = S(1, -2, 3; 4), \quad \overline{S}(1, 2, 3; 4) = \overline{S}(-1, -2, 3; 4).
\]

It is clear that every Euler sum of weight \( w \) and depth \( n \) is a \( \mathbb{Q} \)-linear combination of multiple zeta values or multiple zeta star values (that is, values of multiple zeta functions or multiple zeta star functions at integer arguments) of weight \( w \) and depth less than or equal to \( n + 1 \). In other words, multiple zeta (star) values are "atomic" quantities into which Euler sums decompose. The multiple zeta and zeta star values are defined by

\[
\zeta(k) \equiv \zeta(k_1, \ldots, k_n) := \sum_{m_1 \geq \cdots \geq m_n \geq 1} \frac{\text{sgn}(k_1)^{m_1} \text{sgn}(k_2)^{m_2} \cdots \text{sgn}(k_n)^{m_n}}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}},
\]

\[
\zeta^*(k) \equiv \zeta^*(k_1, \ldots, k_n) := \sum_{m_1 \geq \cdots \geq m_n \geq 1} \frac{\text{sgn}(k_1)^{m_1} \text{sgn}(k_2)^{m_2} \cdots \text{sgn}(k_n)^{m_n}}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}},
\]

where for convergence \( |k_1| + \cdots + |k_j| > j \) for \( j = 1, 2, \ldots, n \), and

\[
\text{sgn}(k_j) := \begin{cases} 
1 & \text{if } k_j > 0, \\
-1 & \text{if } k_j < 0.
\end{cases}
\]

see [12,13,32,33]. Here, we call \( l(k) := n \) and \( |k| := \sum_{j=1}^{n} |k_j| \) the depth and the weight of multiple zeta values, respectively.

Euler sums and multiple zeta values have connections with many branches of mathematics; see especially Zagier [31]. The evaluation of Euler sums also has been useful in various areas of theoretical physics, and also in support of Feynman diagram calculations and in resolving open questions on Feynman diagram contributions and relations among special functions [6,7], including the dilogarithm, Clausen function, and generalized hypergeometric function. An array of harmonic number sums and multiple zeta values is required in calculations of high energy physics. These quantities appear for instance in developing the scattering theory of massless quantum electrodynamics [2]. Broadhurst (see Borwein and Girgensohn [4]) encountered them in relation with Feynman diagrams and associated knots in perturbative quantum field theory.
Several papers on Euler sums have focused on the problem of determining when complicated sums can be expressed in terms of simpler sums. Thus, researchers are interested in determining which sums can be expressed in terms of other sums of lesser depth. The origin of the study of Euler sums goes back to the correspondence of Euler with Goldbach in 1742–1743 (see [14]) and Euler’s paper [9] that appeared in 1776. Euler studied linear (or double) Euler sums and established some important formulas for them. For example, he proved that (see [1,10])

\[ S(1; k) = \frac{1}{2} \left\{ (k + 2)\zeta(k + 1) - \sum_{i=1}^{k-2} \zeta(k - i)\zeta(i + 1) \right\}. \]

Moreover, Euler proved that the linear sums \( S(l; k) \) \((l \geq 1, k \geq 2)\) are reducible to zeta values whenever \( k + l \) is less than 7 or when \( k + l \) is odd and less than 13. Furthermore, he conjectured that the linear sums \( S(l; k) \) would be reducible to zeta values whenever \( k + l \) is odd, and even proposed the general formula. In [3], D. Borwein, J. M. Borwein and R. Girgensohn proved the conjecture, and in [1], Bailey, Borwein and Girgensohn demonstrated that it is “very likely” that the linear sums \( S(l; k) \) with \( k + l > 7 \) and \( k + l \) even, are not reducible. After that many different methods, including partial fraction expansions, Eulerian Beta integrals, summation formulas for generalized hypergeometric functions and contour integrals, have been used to evaluate these sums (see [1,3,10]). For example, Flajolet and Salvy [10] evaluated Euler sums in an entirely different way, namely using contour integration and the residue theorem. In this way they managed to prove, for example, that the sums \( S(1, 1; k) \) with \( k = 2, 3, 4, 6 \) can be evaluated in terms of zeta values. The quadratic Euler sum \( S(1, 1; k) \) can be evaluated in terms of linear sums and zeta values,

\[
S(1, 1; k) - S(2; k) = kS(1; k + 1) - \frac{(k - 2)(k + 3)}{6}\zeta(k + 2) + \zeta(2)\zeta(k) - \sum_{j_1 + j_2 = k, j_1, j_2 \geq 1} \zeta(j_1 + 1)\zeta(j_2 + 1) + \frac{1}{3} \sum_{j_1 + j_2 + j_3 = k - 1, j_1, j_2, j_3 \geq 1} \zeta(j_1 + 1)\zeta(j_2 + 1)\zeta(j_3 + 1).
\]

There are also a lot of recent contributions on nonlinear Euler sums (depth \( \geq 2 \)), see [24,25,28,29]. For example, in [28], we proved that all Euler sums of the form \( S(k_1, k_2; k) \) with weight 4, 5, 6, 7, 9 are expressible polynomially in terms of zeta values. For weight 8, all such sums are the sum of a polynomial in zeta values and a rational multiple of \( S(2; 6) \).

And all weight 10 quadratic sums \( S(1, l; k) \) are reducible to \( S(2; 6) \) and \( S(2; 8) \). Wang et al. [24] showed that all Euler sums of weight \( \leq 9 \) are reducible to zeta values and linear sums.

So far, surprisingly little work has been done on \( q \)-analogues of Euler sums and multiple zeta values. Actually, there are many possible ways to \( q \)-extend the Euler sums and multiple zeta values. Here we recall one \( q \)-analogue. Let \( q \) be a fixed real number with \( 0 < q < 1 \). Let \( n \) be a positive integer. For a sequence \( k = (k_1, \ldots, k_n) \) of positive integers, a sequence \( x = (x_1, \ldots, x_n) \) of variables with \( -1 \leq x_i \leq 1 \), a positive integer \( k \) and a variable \( x \) with \( -1 < x < 1 \), we set

\[
S \left[ \begin{array}{c} k \\ x \end{array} \right] = S \left[ \begin{array}{c} k_1, \ldots, k_n \\ x_1, \ldots, x_n \end{array} \right] = \sum_{m=1}^{\infty} \frac{\zeta_m[k_1, x_1] \cdots \zeta_m[k_n, x_n]}{[m]^k} x^m,
\]

(1.1)
where \([b]\) denotes the \(q\)-analogue of a real number \(b\), defined by

\[
[b] \equiv [b]_q := \frac{1 - q^b}{1 - q},
\]

and \(\zeta_m[k, x]\) is the partial sum of the \(q\)-polylogarithm function \(L_i[x]\), which is called the \(q\)-harmonic number, defined as

\[
\zeta_m[k, x] := \sum_{j=1}^{m} \frac{x^j}{[j]^k}, \tag{1.2}
\]

Here the \(q\)-polylogarithm function \(L_i[x]\) is defined by

\[
L_i[x] := \sum_{m=1}^{\infty} \frac{x^m}{[m]^k}, \quad (-1 < x < 1).
\]

Note that

\[
\ln[1 - x] := - L_1[x]
\]

is the \(q\)-analogue of the natural logarithm function. If \(n = 0\) in (1.1), we set

\[
S \left[ \emptyset \kern 0em \emptyset \kern 0em k \kern 0em x \right] := L_i[x].
\]

When taking the limit \(q \to 1\) and \(x \to 1\) with \(x_j = 1\) in (1.1), we get

\[
\lim_{q \to 1} S \left[ \begin{array}{c} k_1, \ldots, k_n \\ 1, \ldots, 1 \end{array} \mid 1 \right] = S(k_1, \ldots, k_n; k).
\]

There are not many results for sums of the type (1.1). Some related results for \(q\)-Euler type sums may be seen in [8, 11, 18, 20, 22, 23, 30, 33] and references therein. The second author jointly with M. Zhang and W. Zhu [30] proved that for a positive integer \(k \geq 2\), the \(q\)-linear sum

\[
S \left[ \begin{array}{c} 1 \\ 1 \kern 0em q \end{array} \right]
\]

can be expressed as a rational linear combination of products of \(q\)-polylogarithms, the quadratic sum

\[
S \left[ \begin{array}{c} 1, 1 \\ 1 \kern 0em 1 \kern 0em q \end{array} \right]
\]

and the cubic combination sum

\[
S \left[ \begin{array}{c} 1, 1, 1 \\ 1, 1, 1 \kern 0em q \end{array} \right] - 3S \left[ \begin{array}{c} 1, 2 \\ 1, 1 \kern 0em q \end{array} \right]
\]

are reducible to \(q\)-linear sums and to polynomials in \(q\)-polylogarithms. Some simple examples are

\[
S \left[ \begin{array}{c} 1 \\ 1 \kern 0em q \end{array} \right] = L_3[1] + L_3[2],
\]

\[
S \left[ \begin{array}{c} 1 \\ 1 \kern 0em 3 \kern 0em q \end{array} \right] = \frac{3}{2} L_4[2] + L_4[1] - \frac{1}{2} L_3^2[1].
\]
\[
S \left[ \frac{1}{q}, \frac{1}{q}, \frac{2}{q} \right] = \frac{7}{2} \text{Li}_4 \left[ q^2 \right] + 2 \text{Li}_4 [q] - \frac{1}{2} \text{Li}_2^2 [q] - (1 - q) \left( \text{Li}_3 \left[ q^2 \right] + \text{Li}_3 [q] \right).
\]

Similarly, the multiple zeta (star) values also have many \( q \)-extensions. For example, a definition of \( q \)-multiple zeta (star) values is the following:

\[
\zeta [k_1, k_2, \ldots, k_n] := \sum_{m_1 > m_2 > \cdots > m_n \geq 1} \frac{q^{m_1 + m_2 + \cdots + m_n}}{[m_1]^{k_1} [m_2]^{k_2} \cdots [m_n]^{k_n}}, \quad (n, k_i \in \mathbb{N}),
\]

\[
\zeta^* [k_1, k_2, \ldots, k_n] := \sum_{m_1 \geq m_2 \geq \cdots \geq m_n \geq 1} \frac{q^{m_1 + m_2 + \cdots + m_n}}{[m_1]^{k_1} [m_2]^{k_2} \cdots [m_n]^{k_n}}, \quad (n, k_i \in \mathbb{N}).
\]

Another particularly well-behaved \( q \)-analogue of the multiple zeta functions is defined in [33] by Zhao, generalizing the Riemann \( q \)-zeta function studied by Kaneko et al. [16]. It is very important to understand the relations between their special values; see [5] for some relevant results. Recently, Kh. and T. Hessami Pilehrood proved a \( q \)-analogue of the two-one formula in [11]. For instance, from [11], we have for positive integer \( m \),

\[
\zeta^* \left[ \{2\}_m \right] = \sum_{r=1}^{\infty} (-1)^{r-1} q^{r(r+2m-1)/2} \frac{(1 + q^r)}{[r]_q^{2m}}.
\]

Here \( \{l\}_m \) denotes the sequence \( l, \ldots, l \). By taking \( q \to 1 \) in the identity above we obtain the well-known result

\[
\zeta^* \left( \{2\}_m \right) = 2 \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^{2m}} = 2 \left( 1 - 2^{1-2m} \right) \zeta (2m).
\]

However, we do not consider \( q \)-extensions of multiple zeta values. We continue the study of \( q \)-Euler sums in this paper. Our work is motivated by the recently discovered results in [27,28]. In [27], the second author used the integrals

\[
\int_0^x \frac{H_m (t, a) H_p (t, b)}{t} \, dt, \quad x \in (-1, 1)
\]

and

\[
\int_0^x \frac{H_m (t, a) H_p (t, b)}{t (1 - t)} \, dt, \quad x \in (-1, 1)
\]

to establish many relations involving the digamma function, the Hurwitz zeta function, parametric linear and quadratic Euler sums. Here the function \( H_k (x, a) \) is defined by

\[
H_k (x, a) = \lim_{q \to 1} H_k [x, a],
\]

and \( H_k [x, a] \) is defined for a real number \( a \neq -1, -2, \ldots \) by

\[
H_k [x, a] := \sum_{m=1}^{\infty} \frac{x^{m+a}}{(m+a)^k}, \quad k \in \mathbb{N}, \ x \in (-1, 1).
\]

In [28], we used the Tornheim type series

\[
\sum_{n,k=1}^{\infty} \frac{H_n^{(m)}}{kn(n+k)}, \quad k, p, m \in \mathbb{N}
\]
to prove the result that the combined quadratic sums

\[ (-1)^{p-1} S (1, m; p + 1) + (-1)^{m-1} S (1, p + 1; m) \]

are reducible to linear sums and zeta values. We find that the methods and results in [27,28] are easily extended to our q-Euler sums. In this paper we will give some extended results on q-analogues of Euler sums; see Theorems 1.1–1.3. Moreover, we use certain stuffle products to obtain a general formula of product of any q-polylogarithms.

The purpose of the paper is to prove the following theorems.

**Theorem 1.1** Let k, l be positive integers, a, b, x be real numbers with a, b, a + b ≠ -1, -2, . . . and let |x| < 1. Then the following identity holds:

\[
(-1)^{k-1} \sum_{m=1}^{\infty} \frac{q^{(m+b)k}}{[m+b]^{k+l}} \sum_{j=1}^m \frac{x^{j+a+b}}{[j+a+b]} - (-1)^{l-1} \sum_{m=1}^{\infty} \frac{q^{(m+a)l}}{[m+a]^{k+l}} \sum_{j=1}^m \frac{x^{j+a+b}}{[j+a+b]} = \sum_{j=1}^{l-1} \sum_{m=1}^{j-1} H_{k+j}[q^{j-1}x, a]H_{l+j}[b, H_{k+j-1}] + (-1)^{l-1} (H_k[x, b]H_{k+l}[q^{k-1}x, a] - H_1[x, a + b]H_{k+l}[q^k, a]) - (-1)^{k-1} (H_k[x, b]H_{k+l}[q^{k-1}x, b] - H_1[x, a + b]H_{k+l}[q^k, b]).
\]

**Theorem 1.2** Let k, l be positive integers, s, h, x be real numbers with l > s ≥ 0, k > h ≥ 0 and let |x| < 1. Then we have

\[
(-1)^{k-1} S \left[ \frac{l^q}{q}, \frac{h^q}{q} \right] - (-1)^{l-1} S \left[ \frac{k^q}{q}, \frac{l^q}{q} \right] = \sum_{j=1}^{l-1} (-1)^{j-1} Li_{j+1-k}[q^{j-1}x]S \left[ \frac{k^q}{q}, \frac{j^q}{q} \right] - \sum_{j=1}^{k-1} (-1)^{j-1} Li_{k+1-j}[q^h]S \left[ \frac{l^q}{q}, \frac{j^q}{q} \right] + (-1)^{l-1} \ln[1 - q^s x] \left( S \left[ \frac{k^q}{q}, \frac{l^q}{q} \right] - S \left[ \frac{k^q}{q}, \frac{l^q}{q} \right] \right) - (-1)^{k-1} \ln[1 - q^h x] \left( S \left[ \frac{l^q}{q}, \frac{k^q}{q} \right] - S \left[ \frac{l^q}{q}, \frac{k^q}{q} \right] \right).
\]

**Theorem 1.3** For positive integers k and l, we have

\[
(-1)^{k-1} S \left[ \frac{l + 1}{q^l}, \frac{1}{q} \right] + (-1)^{l-1} S \left[ \frac{k}{q^k-1}, \frac{1}{q} \right] = Li_{l+1}[q^l]Li_{k+1}[q^k] + \sum_{j=1}^{k-1} (-1)^{j-1} Li_{k+1-j}[q^{k-j}]S \left[ \frac{j^q}{q}, \frac{l + 1}{q} \right] + (-1)^{k-1} Li_{l+1}[q^l]S \left[ \frac{1}{q}, \frac{k}{q} \right] - \sum_{j=1}^{k-1} (-1)^{j-1} Li_{k+1-j}[q^{k-j}]Li_{l+j+1}[q^{l+j+1}] - \sum_{j=1}^{l-1} (-1)^{j-1} Li_{l+1-j}[q^{l-j}]S \left[ \frac{k}{q^k-1}, \frac{j + 1}{q} \right].
\]
Theorem 1.4 Let $n, k_1, \ldots, k_n$ be positive integers and $x_1, \ldots, x_n$ be real numbers with $|x_j| < 1$. Then we have

$$
\prod_{j=1}^{n} \text{Li}_{k_j}[x_j] = \sum_{j=0}^{n-1} \sum_{1 \leq i_1 < \cdots < i_j \leq n} (-1)^{n-1-j} S \left[ \begin{array}{c} k_{i_1}, \ldots, k_{i_j} \\ x_{i_1}, \ldots, x_{i_j} \end{array} \right] \frac{(k_1 + \cdots + k_n) - (k_{i_1} + \cdots + k_{i_j})}{(x_1 \cdots x_n)/(x_{i_1} \cdots x_{i_j})}.
$$

We prove Theorems 1.1–1.3 in Sect. 2 by calculating the Jackson $q$-integral of $q$-polylogarithm functions, and prove Theorem 1.4 in Sect. 3 algebraically. In Sect. 4, we give some interesting identities (known or new) involving harmonic numbers.

## 2 Proofs of Theorems 1.1–1.3

We prove Theorems 1.1–1.3 in this section by calculating the Jackson $q$-integral of $q$-polylogarithm functions.

### 2.1 Jackson $q$-integral

The Jackson $q$-integral and $q$-derivative are defined by

$$
\int_a^x f(t) dq t := (1-q) \sum_{i=0}^{\infty} q^i \left[ t f(q^i t) - a f(q^i a) \right], \quad D_q f(x) := \frac{f(qx) - f(x)}{qx - x},
$$

respectively; see, e.g. [15]. For example, we have

$$
D_q (H_k[x,a]) = \frac{H_{k-1}[x,a]}{x}, \quad D_q (x^m) = [m]x^{m-1},
$$

and it is easy to verify that

$$
D_q (f(x)g(x)) = g(qx)D_q (f(x)) + f(x)D_q (g(x)) = f(qx)D_q (g(x)) + g(x)D_q (f(x)),
$$

$$
D_q \left( \int_a^x f(t) dq t \right) = f(x), \quad \int_a^x D_q (f(t)) dq t = f(x) - f(a),
$$

$$
\int_a^x f(t)D_q (g(t)) dq t = \left[ f(t)g(t) \right]_a^x - \int_a^x g(qt)D_q (f(t)) dq t.
$$

In particular,

$$
\int_0^x t^{k-1} dq t = \frac{x^k}{[k]}, \quad k \in \mathbb{N}.
$$

### 2.2 Proof of Theorem 1.1

To prove Theorem 1.1, we need the following lemma.

Lemma 2.1 Let $m, k$ be positive integers and $a, b, c, x$ be real numbers with $a, b, a + b \neq -1, -2, \ldots$ and $|cx| < 1$. Then the following identity holds:
\[
\int_0^x H_k[ct, a] t^{m+b-1} dt = \sum_{j=1}^{k-1} (-1)^{j-1} \frac{q^{(m+b)(j-1)} x^{m+b}}{[m+b]^j} H_{k+1-j}[cx, a] \\
+ (-1)^{k-1} \frac{q^{(m+b)(k-1)}}{[m+b]^k} \left( x^{m+b} H_1[cx, a] - (q/c)^{m+b} H_1[cx, a + b] \right) \\
+ (-1)^{k-1} \frac{(q^k/c)^{m+b}}{[m+b]^k} \sum_{j=1}^{m} \frac{(cx)^{j+a+b}}{[j+a+b]}. \tag{2.5}
\]

**Proof** Denote the left hand-side of (2.5) by \( I_k \). Then by (2.3) and (2.4) we have

\[
I_k = \frac{1}{[m+b]} \int_0^x H_k[ct, a] D_q(t^{m+b}) dt = \frac{x^{m+b}}{[m+b]} H_k[cx, a] - \frac{q^{m+b}}{[m+b]} I_{k-1},
\]

and

\[
I_1 = \frac{x^{m+b}}{[m+b]} H_1[x, a] - \frac{q^{m+b}}{[m+b]} \int_0^x \frac{x^{m+a+b}}{1-t} dt = \frac{x^{m+b}}{[m+b]} H_1[cx, a] + \frac{(q/c)^{m+b}}{[m+b]} \sum_{j=1}^{m} \frac{(cx)^{j+a+b}}{[j+a+b]} - \frac{(q/c)^{m+b}}{[m+b]} H_1[cx, a + b].
\]

Hence we get (2.5) by induction on \( k \).

In particular, if \( a = b = 0 \), then (since \( H_k[x, 0] = Li_k[x] \))

\[
\int_0^x t^{m-1} Li_k(ct) dt = \sum_{j=1}^{k-1} (-1)^{j-1} \frac{q^{m(j-1)} x^m}{[m]^j} Li_{k+1-j}[cx] + (-1)^{k-1} \frac{q^{m(k-1)} x^m}{[m]^k} (x^m - (q/c)^{m}) Li_1[cx] \\
+ (-1)^{k-1} \frac{(q^k/c)^m}{[m]^k} \zeta_m[1, cx]. \tag{2.6}
\]

**Proof of Theorem 1.1** By considering the Jackson \( q \)-integral

\[
\int_0^x H_k[t, a] H_l[t, b] \frac{dt}{t} = \sum_{m=1}^{\infty} \frac{1}{[m+a]^k} \int_0^x H_l[t, b] t^{m+a-1} dt d_q t = \sum_{m=1}^{\infty} \frac{1}{[m+b]^l} \int_0^x H_k[t, a] t^{m+b-1} dt d_q t,
\]

and using (2.5), we get (1.3). \( \square \)

Setting \( x = q \) and \( k = l = 1 \) in Theorem 1.1, we get

\[
\sum_{m=1}^{\infty} \frac{q^{m+b}}{[m+b]^2} \sum_{j=1}^{m} \frac{q^{j+a+b}}{[j+a+b]} - \sum_{m=1}^{\infty} \frac{q^{m+a}}{[m+a]^2} \sum_{j=1}^{m} \frac{q^{j+a+b}}{[j+a+b]}
\]

\[
= [a] H_2[q, a] \sum_{m=1}^{\infty} \frac{q^{m+b}}{[m+b][m+a+b]} - [b] H_2[q, b] \sum_{m=1}^{\infty} \frac{q^{m+a}}{[m+a][m+a+b]}.
\]

\( \square \) Springer
2.3 Proof of Theorem 1.2

This proof is similar to the one of Theorem 1.1.

**Proof of Theorem 1.2** Using (2.5) with \(a = b = 0, c = q^h, q^s\) or using (2.6) with \(c = q^h, q^s\), we compute the Jackson \(q\)-integral

\[
\int_0^x \frac{\text{Li}_1[q^s t] \text{Li}_k[q^h t]}{t(1-t)} \, dq t = \sum_{m=1}^{\infty} \frac{\zeta_m[l, q^s]}{m} \int_0^t t^{m-1} \text{Li}_k[q^h t] \, dq t
\]

\[
= \sum_{j=1}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^h x] \sum_{m=1}^{\infty} \frac{\zeta_m[l, q^s]}{m^j} (q^{j-1} x)^m
\]

\[
+ (-1)^{k-1} \text{Li}_1[1 - q^h x] \sum_{m=1}^{\infty} \frac{\zeta_m[l, q^s]}{m^k} (q^{k-m} - (q^{k-1} x)^m)
\]

\[
+ (-1)^{k-1} \sum_{m=1}^{\infty} \frac{\zeta_m[l, q^s] \zeta_m[1, q^h x]}{m^k} q^{(k-m)}
\]

which gives (1.4).

Setting \(x \to 1\) in Theorem 1.2, we obtain

\[
(-1)^{k-1} S \left[ k, \frac{1}{q^s}, \frac{1}{q^{k-h}} \right] - (-1)^{l-1} S \left[ k, \frac{1}{q^h}, \frac{1}{q^{l-s}} \right]
\]

\[
= \sum_{j=2}^{l-1} (-1)^{j-1} \text{Li}_{l+1-j}[q^s] S \left[ k, \frac{1}{q^h}, \frac{1}{q^{j-1}} \right] - \sum_{j=2}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^h] S \left[ l, \frac{1}{q^s}, \frac{1}{q^{j-1}} \right]
\]

\[
+ (-1)^{l-1} \text{Li}_1[1 - q^s] \left( S \left[ k, \frac{1}{q^h}, \frac{1}{q^{l-s}} \right] - S \left[ k, \frac{1}{q^h}, \frac{1}{q^{l-1}} \right] \right)
\]

\[
- (-1)^{k-1} \text{Li}_1[1 - q^h] \left( S \left[ l, \frac{1}{q^s}, \frac{1}{q^{k-h}} \right] - S \left[ l, \frac{1}{q^s}, \frac{1}{q^{k-1}} \right] \right)
\]

\[
+ \sum_{m=1}^{\infty} \text{Li}_1[q^s] \zeta_m[k, q^h] - \text{Li}_k[q^h] \zeta_m[l, q^s] \tag{2.7}
\]

To evaluate the last sum on the right-hand side of (2.7), we use...
Theorem 2.2 Let $k, l$ be positive integers and $x, y, z$ be real numbers with $|x|, |y|, |z| < 1$. Then we have
\[
\sum_{m=1}^{\infty} \frac{\text{Li}_k[x] \zeta_m[l, y] - \text{Li}_l[y] \zeta_m[k, x]}{[m]} z^m
= \text{Li}_l[y] S \left[ \frac{1}{z}, \frac{1}{x} \right] - \text{Li}_k[x] S \left[ \frac{1}{z}, \frac{l}{y} \right] + \text{Li}_k[x] \text{Li}_{l+1}[z y] - \text{Li}_l[y] \text{Li}_{k+1}[z x].
\]
(2.8)

Taking $x = q^h$, $y = q^s$ and $z \rightarrow 1$ in (2.8), and using (2.7), we have

Corollary 2.3 Let $s, h$ be positive reals and $k, l$ be positive integers with $l > \max\{s, 1\}$ and $k > \max\{h, 1\}$. Then we have
\[
(−1)^{k−1} S \left[ \frac{l}{q^s}, \frac{1}{q^h} \right] - (−1)^{l−1} S \left[ \frac{k}{q^h}, \frac{1}{q^s} \right] q^{l−s}
\]
\[
= \sum_{j=2}^{l+1} (−1)^{j−1} \text{Li}_{l+1−j}[q^s] S \left[ \frac{k}{q^h}, \frac{j}{q^{l−j}} \right] - \sum_{j=2}^{k−1} (−1)^{j−1} \text{Li}_{k+1−j}[q^h] S \left[ \frac{l}{q^s}, \frac{j}{q^{l−j}} \right]
\]
\[
+ (−1)^{l−1} \ln[1 − q^s] \left( S \left[ \frac{k}{q^h}, \frac{l}{q^{l−s}} \right] − S \left[ \frac{k}{q^h}, \frac{l}{q^{l−1}} \right] \right)
\]
\[
− (−1)^{k−1} \ln[1 − q^h] \left( S \left[ \frac{l}{q^s}, \frac{k}{q^{k−h}} \right] − S \left[ \frac{l}{q^s}, \frac{k}{q^{k−1}} \right] \right)
\]
\[
+ \text{Li}_k[q^h] S \left[ \frac{l}{1}, \frac{1}{q^s} \right] - \text{Li}_l[q^s] S \left[ \frac{k}{1}, \frac{1}{q^h} \right] + \text{Li}_l[q^s] \text{Li}_{k+1}[q^h] - \text{Li}_k[q^h] \text{Li}_{l+1}[q^s].
\]

Finally, we prove Theorem 2.2.

Proof of Theorem 2.2 We compute the $N$-th partial sum of the series of the left-hand side of (2.8) as follows:
\[
\sum_{m=1}^{N} \frac{\text{Li}_k[x] \zeta_m[l, y] - \text{Li}_l[y] \zeta_m[k, x]}{[m]} z^m
\]
\[
= \text{Li}_k[x] \sum_{m=1}^{N} \frac{\zeta_m[l, y]}{[m]} z^m - \text{Li}_l[y] \sum_{m=1}^{N} \frac{\zeta_m[k, x]}{[m]} z^m
\]
\[
= \text{Li}_k[x] \sum_{m=1}^{N} \sum_{j=1}^{m} \frac{y^j z^m}{[m][j]^k} - \text{Li}_l[y] \sum_{m=1}^{N} \sum_{j=1}^{m} \frac{x^j z^m}{[m][j]^k}
\]
\[
= \text{Li}_k[x] \sum_{j=1}^{N} \sum_{m=j}^{N} \frac{y^j z^m}{[m][j]^k} - \text{Li}_l[y] \sum_{j=1}^{N} \sum_{m=j}^{N} \frac{x^j z^m}{[m][j]^k}
\]
\[
= \text{Li}_k[x] \sum_{j=1}^{N} \frac{\zeta_N[1, z] - \zeta_{j−1}[1, z]}{[j]^k} y^j - \text{Li}_l[y] \sum_{j=1}^{N} \frac{\zeta_N[1, z] - \zeta_{j−1}[1, z]}{[j]^k} x^j
\]
\[
= \zeta_N[1, z] \text{Li}_k[x] \zeta_N[l, y] - \text{Li}_l[y] \zeta_N[k, x] + \text{Li}_l[y] \sum_{j=1}^{N} \frac{\zeta_{j}[1, z]}{[j]^k} x^j - \text{Li}_k[x] \sum_{j=1}^{N} \frac{\zeta_{j}[1, z]}{[j]^k} y^j
\]
\[
− \zeta_N[k + 1, z x] + \zeta_N[l + 1, z y].
\]

Letting $N$ tend to infinity, we get (2.8).
2.4 Proof of Theorem 1.3

To prove Theorem 1.3 we need the following lemmas.

Lemma 2.4 For positive integers $k$ and $i$,\[
\sum_{m=1}^{\infty} \frac{\zeta_m[k, q^{k-1}]}{[m][m+i]} q^m = \frac{1}{[i]} \left\{ \text{Li}_{i+1}[q^k] + (-1)^{k-1} \sum_{j=1}^{i-1} \frac{\zeta_j[1, q]}{[j]^k} q^j + \sum_{j=1}^{k-1} (-1)^{j-1} \text{Li}_{i+j-1} [q^{k-j}] \zeta_{i-1}[j, q] \right\} \quad (2.9)
\]
holds.

Proof By using the Cauchy product of power series and the definition of $q$-harmonic numbers (1.2), we have\[
\sum_{m=1}^{\infty} \zeta_m[k, q^l] x^m = \frac{\text{Li}_k[q^l x]}{1 - x},
\]
where $l$ is any positive integer and $x$ is any real number with $|x| < 1$. Multiplied by $x^{1-2i}$ and $q$-integrated over $(0, q)$, the above equation yields\[
[i] \sum_{m=1}^{\infty} \frac{\zeta_m[k, q^l]}{[m][m+i]} q^m = \text{Li}_{k+1}[q^{l+1}] + \sum_{j=1}^{i-1} \sum_{m=1}^{\infty} \frac{q^{l+j} m+j}{[m][m+j]}.
\]
Taking $l = k - 1$ in the above equation, we obtain\[
[i] \sum_{m=1}^{\infty} \frac{\zeta_m[k, q^{k-1}]}{[m][m+i]} q^m = \text{Li}_{k+1}[q^k] + \sum_{j=1}^{k-1} \sum_{m=1}^{\infty} \frac{q^{k-1}}{[m][m+j]} q^m
\]
which together with the formula\[
\sum_{m=1}^{\infty} \frac{q^{km}}{[m][m+j]} = \sum_{p=1}^{k-1} \frac{(-1)^{p-1}}{[j]^p} \text{Li}_{k-p+1}[q^{k-p}] + (-1)^{k-1} \frac{\zeta_j[1, q]}{[j]^k} \quad (2.10)
\]
(see [21]) yields the desired result. □

Lemma 2.5 For any positive integers $k_1, k_2$ and any real numbers $x, y$ with $|x|, |y| < 1$, we have\[
S \left[ \begin{array}{cc} k_1 & k_2 \\ x & y \end{array} \right] + S \left[ \begin{array}{cc} k_2 & k_1 \\ y & x \end{array} \right] = \text{Li}_{k_1}[x] \text{Li}_{k_2}[y] + \text{Li}_{k_1+k_2}[xy]. \quad (2.11)
\]

Proof We consider the generating function\[
F_2[x, y, z] := \sum_{m=1}^{\infty} (\zeta_m[k_1, x] \zeta_m[k_2, y] - \zeta_m[k_1 + k_2, xy]) z^{m-1},
\]
where $|z| < 1$. By the definition of $\zeta_m[k, x]$, we have\[
F_2[x, y, z] = \sum_{m=0}^{\infty} \left\{ \left( \zeta_m[k_1, x] + \frac{x^{m+1}}{[m+1]^2} \right) \left( \zeta_m[k_2, y] + \frac{y^{m+1}}{[m+1]^2} \right) \right\} z^m
\]
\[
= z F_2[x, y, z] + \sum_{m=0}^{\infty} \left( \frac{\zeta_m[k_1, x]}{[m+1]^2} y^{m+1} + \frac{\zeta_m[k_2, y]}{[m+1]^2} x^{m+1} \right) z^m
\]

where

Replacing clear that (2.11) and (2.12) are immediate corollaries of Theorem 1.4.

and applying the same arguments as in the proof of (2.11), we may deduce the formula

\[ F_2[x, y, z] = \sum_{m=1}^{\infty} \left( \frac{\xi_m[k_1, x]}{[m]^{k_2}} y^m + \frac{\xi_m[k_2, y]}{[m]^{k_1}} x^m - 2 \frac{x^m y^m}{[m]^{k_1+k_2}} \right) z^{m-1}. \]

Hence, we obtain

\[ F_2[x, y, z] = \sum_{m=1}^{\infty} \left( \frac{\xi_m[k_1, x]}{[j]^{k_2}} y^j + \frac{\xi_m[k_2, y]}{[j]^{k_1}} x^j - 2 \frac{x^j y^j}{[j]^{k_1+k_2}} \right) z^{m-1}. \]

Then equating coefficients of \( z^{m-1} \), we establish the relation

\[ \sum_{j=1}^{m} \left( \frac{\xi_j[k_1, x]}{[j]^{k_2}} y^j + \frac{\xi_j[k_2, y]}{[j]^{k_1}} x^j - 2 \frac{x^j y^j}{[j]^{k_1+k_2}} \right) = \xi_m[k_1, x] \xi_m[k_2, y] + \xi_m[k_1 + k_2, x y]. \]

Letting \( m \) tend to infinity in above equation, we deduce (2.11).

\[ \square \]

**Remark 2.6** Similarly, considering the function

\[ F_3[x, y, z, t] := \sum_{m=1}^{\infty} (\xi_m[k_1, x] \xi_m[k_2, y] \xi_m[k_3, z] - \xi_m[k_1 + k_2 + k_3, xyz]) t^{m-1}, \]

and applying the same arguments as in the proof of (2.11), we may deduce the formula

\[ S \left[ k_1, k_2 \mid k_3 \right] + S \left[ k_1, k_3 \mid k_2 \right] + S \left[ k_2, k_3 \mid k_1 \right] \]

\[ = S \left[ k_1 \mid k_2 + k_3 \right] + S \left[ k_2 \mid k_1 + k_3 \right] + S \left[ k_3 \mid k_1 + k_2 \right] + \mathrm{Li}_{k_1}[x] \mathrm{Li}_{k_2}[y] \mathrm{Li}_{k_3}[z] - \mathrm{Li}_{k_1+k_2+k_3}[xyz], \]

(2.12)

where \( k_1, k_2, k_3 \) are positive integers and \( x, y, z \) are real number with \(|x|, |y|, |z| < 1\). It is clear that (2.11) and (2.12) are immediate corollaries of Theorem 1.4.

**Lemma 2.7** For any positive integers \( k_1, k_2 \) and any real numbers \( x, y, z \) with \(|x|, |y|, |z| < 1\), we have

\[ \sum_{m=1}^{\infty} \frac{x^m}{[m]^{k_1}} \sum_{j=1}^{m} \frac{z^j}{[j]^{k_2}} \xi_j[1, y] + S \left[ k_1, 1 \mid k_2 \mid z \right] = \mathrm{Li}_{k_1}[x] S \left[ 1 \mid y \mid k_2 \right] + S \left[ 1 \mid y \mid k_1 + k_2 \right]. \]

(2.13)

**Proof** Replacing \( y \) by \( zt \) in (2.11), then dividing it by \( 1 - t \) and \( q \)-integrating over the interval \((0, y)\), we can deduce that the left hand side is equal to

\[ \int_0^y S \left[ k_1 \mid k_2 \mid zt \right] + S \left[ k_2 \mid zt \mid k_1 \right] \frac{dt}{1 - t} \]

\[ = \mathrm{Li}_{1}[y] \left( S \left[ k_1 \mid k_2 \mid z \right] + S \left[ k_2 \mid z \mid k_1 \right] \right) - S \left[ k_1, 1 \mid k_2 \mid z \right] - \sum_{m=1}^{\infty} \frac{x^m}{[m]^{k_1}} \sum_{j=1}^{m} \frac{z^j}{[j]^{k_2}} \xi_j[1, y], \]

\[ \square \]
while the right-hand side is equal to
\[ \int_0^y \frac{\text{Li}_{k_1}[x] \text{Li}_{k_2}[y] + \text{Li}_{k_1+k_2}[xy]}{1-t} \, dq \, dt = \text{Li}_{k_1}[x] \left( S \left[ \frac{k_2}{z} \left| \begin{array}{c} 1 \end{array} \right. \right] - \text{Li}_{k_2+1}[yz] \right) + S \left[ \frac{k_1 + k_2}{xz} \left| \begin{array}{c} 1 \end{array} \right. \right] - \text{Li}_{k_1+k_2+1}[xyz]. \]
Then with the help of (2.11) we easily deduce the desired result. \(\square\)

Now we are in position to prove Theorem 1.3.

**Proof of Theorem 1.3** Set
\[ \sum = \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} \frac{\zeta_m[k, q^{k-1}]}{[m][m+i][i]} q^{m+i}. \]
On the one hand, using (2.9), we have
\[
\sum = \sum_{i=1}^{\infty} \frac{q^{li}}{[i]^{l+1}} \left\{ \text{Li}_{k+1}[q^k] + (-1)^{k-1} \sum_{j=1}^{i-1} \frac{\zeta_j[1,q]}{[j]^k} q^j \right. \\
+ \left. \sum_{j=1}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^{k-j}] \zeta_{i-1}[j, q] \right\}
\]
\[
= (-1)^{k-1} \sum_{m=1}^{\infty} \frac{q^{lm}}{[m]^{l+1}} \sum_{j=1}^{m} \frac{\zeta_j[1,q]}{[j]^k} q^j - (-1)^{k-1} S \left[ \frac{1}{q} \left| \begin{array}{c} k+l+1 \\
q^{l+1} \end{array} \right. \right]
\]
\[
+ \sum_{j=1}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^{k-j}] S \left[ \frac{j}{q} \left| \begin{array}{c} l+1 \\
q^{l+1} \end{array} \right. \right] + \text{Li}_{i+1}[q^l] \text{Li}_{k+1}[q^k]
\]
\[
- \sum_{j=1}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^{k-j}] \text{Li}_{i+j+1}[q^{l+1}].
\]
Setting \(k_1 = l + 1, k_2 = k, x = q^l, y = z = q \) in (2.13), we get
\[
\sum_{m=1}^{\infty} \frac{q^{lm}}{[m]^{l+1}} \sum_{j=1}^{m} \frac{q^{lj}}{[j]^k} \zeta_j[1,q] - S \left[ \frac{1}{q} \left| \begin{array}{c} k+l+1 \\
q^{l+1} \end{array} \right. \right] = \text{Li}_{i+1}[q^l] S \left[ \frac{1}{q} \left| \begin{array}{c} k \\
q^{l+1} \end{array} \right. \right] - S \left[ \frac{l+1, 1}{q^l, q} \right].
\]
which implies that
\[
\sum = (-1)^{k-1} \text{Li}_{i+1}[q^l] S \left[ \frac{1}{q} \left| \begin{array}{c} k \\
q \end{array} \right. \right] - (-1)^{k-1} S \left[ \frac{l+1, 1}{q^l, q} \right]
\]
\[
+ \sum_{j=1}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^{k-j}] S \left[ \frac{j}{q} \left| \begin{array}{c} l+1 \\
q^{l+1} \end{array} \right. \right] + \text{Li}_{i+1}[q^l] \text{Li}_{k+1}[q^k]
\]
\[
- \sum_{j=1}^{k-1} (-1)^{j-1} \text{Li}_{k+1-j}[q^{k-j}] \text{Li}_{i+j+1}[q^{l+1}]. \quad (2.14)
\]
On the other hand, using (2.10), we get
\[
\sum = \sum_{m=1}^{\infty} \frac{\zeta_m[k, q^{k-1}]}{[m]} q^m \sum_{i=1}^{\infty} \frac{q^{li}}{[i]^l[m+i]}
\]
Comparing (2.14) and (2.15), we get the desired result (1.5).

\[\text{3 Proof of Theorem 1.4}\]

In this section, we use the stuffle product to give a proof of Theorem 1.4.

For a sequence \(k = (k_1, \ldots, k_n)\) of positive integers, the weight and the depth of \(k\) are defined by

\[\text{wt}(k) = k_1 + \cdots + k_n, \quad \text{dep}(k) = n,\]

respectively. For an empty sequence, we set \(\text{wt}(\emptyset) = \text{dep}(\emptyset) = 0\). We call \(l\) a subsequence of \(k\) if there exist integers \(m, i_1, \ldots, i_m\) with \(0 \leq m \leq n\) and \(1 \leq i_1 < \cdots < i_m \leq n\), such that \(l = (k_{i_1}, \ldots, k_{i_m})\). Let \(\text{Sub}(k)\) be the set of all subsequences of \(k\). If \(x = (x_1, \ldots, x_n)\) is a sequence of variables, we set \(|x| = x_1 \cdots x_n\). And for any \(l = (k_{i_1}, \ldots, k_{i_m}) \in \text{Sub}(k)\), we set

\[x_l = (x_{i_1}, \ldots, x_{i_m}).\]

Note that \(|\emptyset| = 1\), \(x_{\emptyset} = \emptyset\) and \(x_k = x\). Therefore Theorem 1.4 may be rewritten as

**Theorem 3.1** Let \(n\) be a positive integer, \(k = (k_1, \ldots, k_n)\) be a sequence of positive integers and \(x = (x_1, \ldots, x_n)\) be a sequence of real numbers with \(|x_j| < 1\). We have

\[
\prod_{j=1}^{n} \text{Li}_{k_j}[x_j] = \sum_{l \in \text{Sub}(k), l \neq k} (-1)^{n - \text{dep}(l) - 1} S \left[ \frac{l}{x_l}, \frac{\text{wt}(k) - \text{wt}(l)}{|x|/|x_l|} \right].
\]

(3.1)

To prove Theorem 3.1 we use the stuffle product. Similar as in [13], let

\[\mathcal{M} := \left\{ \left[ \begin{array}{c} k \\ x \end{array} \right] \mid (k, x) \in \mathbb{N} \times (-1, 1) \right\},\]

which we consider as an alphabet. Let \(\mathcal{M}^*\) be the set of all words generated by \(\mathcal{M}\), which contains the empty word \(1_{\mathcal{M}}\). We denote a nonempty word \(\left[ \begin{array}{c} k_1 \\ x_1 \end{array} \cdots \begin{array}{c} k_n \\ x_n \end{array} \right]\) simplify by \(k_1, \ldots, k_n, x_1, \ldots, x_n\). Let \(\mathfrak{h}^1 = \mathbb{Q}(\mathcal{M})\) be the noncommutative polynomial algebra over \(\mathbb{Q}\) generated by \(\mathcal{M}\). As a rational vector space, \(\mathfrak{h}^1\) has basis \(\mathcal{M}^*\).

We now define the stuffle product \(*\) on the algebra \(\mathfrak{h}^1\), which is \(\mathbb{Q}\)-bilinear, and satisfies the following axioms:

1. \(1_{\mathcal{M}} \ast w = w \ast 1_{\mathcal{M}} = w\) for any \(w \in \mathcal{M}^*\);
2. \(a u \ast b v = a(u \ast b v) + b(a \ast v) - (a \circ b)(u \ast v)\) for any \(a, b \in \mathcal{M}\) and any \(u, v \in \mathcal{M}^*\).
Here we set
\[
\begin{bmatrix} k \end{bmatrix} \circ \begin{bmatrix} l \end{bmatrix} := \begin{bmatrix} k + l \\ x y \end{bmatrix}.
\]
Then by [13,19], the product \( \ast \) is commutative and associative.

For any \( w \in \mathcal{H}^1 \), we define a function \( \text{Li}^*[w] \) by \( \mathbb{Q} \)-linearity, \( \text{Li}^*[1_{\mathcal{M}}] = 1 \) and
\[
\text{Li}^* \left[ \begin{array}{c} k_1, \ldots, k_n \\ x_1, \ldots, x_n \end{array} \right] := \sum_{m_1 \geq \cdots \geq m_n \geq 1} \frac{x_1^{m_1} \cdots x_n^{m_n}}{[m_1]^{k_1} \cdots [m_n]^{k_n}}.
\]
Then we have
\[
\text{Li}^* \left[ \begin{array}{c} k \\ x \end{array} \right] = \text{Li}_k[x].
\]

Immediately from the definitions, we obtain the following lemma.

**Lemma 3.2**
1. For any \( w_1, w_2 \in \mathcal{H}^1 \), we have
\[
\text{Li}^*[w_1 \ast w_2] = \text{Li}^*[w_1] \text{Li}^*[w_2].
\]
2. Let \( n \) be a positive integer and \( w_1 = \left[ \begin{array}{c} k_1 \\ x_1 \end{array} \right], \ldots, w_n = \left[ \begin{array}{c} k_n \\ x_n \end{array} \right] \in \mathcal{M} \). Then we have
\[
S \left[ \begin{array}{c} k_1, \ldots, k_n \\ x_1, \ldots, x_n \end{array} \right] \text{Li}^*[w] = \text{Li}^*[w(w_1 \ast \cdots \ast w_n)].
\]

**Proof** One can prove (1) similarly as in [13,19], and prove (2) similarly as in [17]. \( \square \)

We prove the corresponding equation of (3.1) in the algebra \( \mathcal{H}^1 \).

**Theorem 3.3** Let \( n \) be a positive integer and \( w_1 = \left[ \begin{array}{c} k_1 \\ x_1 \end{array} \right], \ldots, w_n = \left[ \begin{array}{c} k_n \\ x_n \end{array} \right] \in \mathcal{M} \). If we set \( k = (k_1, \ldots, k_n) \) and \( x = (x_1, \ldots, x_n) \), then we have
\[
w_1 \ast \cdots \ast w_n = \sum_{l=(k_{i_1}, \ldots, k_{i_m}) \in \text{Sub}(k) \atop l \neq k} (-1)^{n-m-1} \left[ \begin{array}{c} \text{wt}(k) - \text{wt}(l) \\ x/|x| \end{array} \right] (w_{i_1} \ast \cdots \ast w_{i_m}) \tag{3.2}.
\]

**Proof** We proceed by induction on \( n \). The case \( n = 1 \) is trivial. Now assume that (3.2) is proved for \( k \) and \( x \). For any \( w_{n+1} = \left[ \begin{array}{c} k_{n+1} \\ x_{n+1} \end{array} \right] \in \mathcal{M} \), set \( k' = (k_1, \ldots, k_n, k_{n+1}) \) and \( x' = (x_1, \ldots, x_n, x_{n+1}) \). Using the induction hypothesis and the definition of the stuffle product, we have
\[
w_1 \ast \cdots \ast w_n \ast w_{n+1} = \sum_{l=(k_{i_1}, \ldots, k_{i_m}) \in \text{Sub}(k) \atop l \neq k} (-1)^{n-m-1} w_{n+1} \ast \left[ \begin{array}{c} \text{wt}(k) - \text{wt}(l) \\ x/|x| \end{array} \right] (w_{i_1} \ast \cdots \ast w_{i_m})
\]
\[
= \sum_{l=(k_{i_1}, \ldots, k_{i_m}) \in \text{Sub}(k) \atop l \neq k} (-1)^{n-m-1} w_{n+1} \ast \left[ \begin{array}{c} \text{wt}(k) - \text{wt}(l) \\ x/|x| \end{array} \right] (w_{i_1} \ast \cdots \ast w_{i_m})
\]
Proof of Theorem 3.1
we get the result.

Since any

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\[ (i) \]

\[ x \rightarrow \pm 1, q \rightarrow 1, \]

this proves (3.2) for \( k' \) and \( x' \).

\[ \square \]

**Proof of Theorem 3.1** Applying \( Li^* \) on both sides of (3.2), and with the help of Lemma 3.2, we get the result.

\[ \square \]

4 Some identities on Euler sums

From Theorems 1.2–1.4, taking \( x \rightarrow \pm 1, q \rightarrow 1, \) we get the following corollaries.

**Corollary 4.1** [27] For positive integers \( k > 1 \) and \( l > 1, \) it holds

\[ (-1)^{k-1} S(1, l; k) - (-1)^{l-1} S(1, k; l) \]

\[ = \sum_{j=2}^{l-1} (-1)^{j-1} \xi(l+1-j)S(k; j) - \sum_{j=2}^{k-1} (-1)^{j-1} \xi(k+1-j)S(l; j) \]

\[ + \xi(k)S(1; l) - \xi(l)S(1; k) + \xi(l)\xi(k+1) - \xi(k)\xi(l+1). \]

**Proof** Letting \( x \rightarrow 1, q \rightarrow 1 \) in Theorem 1.2 yields the desired result.

\[ \square \]

**Corollary 4.2** [28] For positive integers \( k > 1 \) and \( l, \) it holds

\[ (-1)^{l-1} S(1, k; l+1) + (-1)^{k-1} S(1, l+1; k) \]

\[ = \xi(k+1)\xi(l+1) + \sum_{j=1}^{k-1} (-1)^{j-1} \xi(k+1-j)S(j; l+1) + (-1)^{k-1} \xi(l+1)S(1; k) \]

\[ - \sum_{j=1}^{k-1} (-1)^{j-1} \xi(k+1-j)\xi(l+j+1) - \sum_{j=1}^{l-1} (-1)^{j-1} \xi(l+1-j)S(k; j+1). \]
Proof Letting \( x \to 1, \ q \to 1 \) in Theorem 1.3 yields the desired result. \( \square \)

Corollary 4.3 Let \( l \geq 2 \) and \( k \geq 0 \) be integers. Then we have
\[
(-1)^j \left[ S(\bar{1}, l + 2k + 1; l) + S(\bar{1}, l; l + 2k + 1) \right]
= \sum_{j=1}^{l+2k-1} (-1)^{j-1} \zeta(l + 2k + 2 - j) S(l; \bar{j})
\]
\[
- \sum_{j=1}^{l-1} (-1)^{j-1} \zeta(l + 1 - j) S(l + 2k + 1; \bar{j})
+ (-1)^j \ln 2 \left[ S(l + 2k + 1; l) + S(l; l + 2k + 1) \right]
+ (-1)^j \ln 2 \left[ S(l + 2k + 1; \bar{l}) + S(l; \bar{l} + 2k + 1) \right].
\]

Proof The result follows from Theorem 1.2 with \( x \to -1, \ q \to 1 \) and \( k \mapsto l + 2k + 1 \). \( \square \)

Corollary 4.4 For integers \( l \in \mathbb{N} \setminus \{1\} \) and \( k \in \mathbb{N} \cup \{0\} \), we have
\[
(-1)^j \left[ S(\bar{1}, l + 2k; l) - S(\bar{1}, l; l + 2k) \right]
= \sum_{j=1}^{l+2k-1} (-1)^{j-1} \zeta(l + 2k + 1 - j) S(l; \bar{j})
\]
\[
- \sum_{j=1}^{l-1} (-1)^{j-1} \zeta(l + 1 - j) S(l + 2k; \bar{j})
+ (-1)^j \ln 2 \left[ S(l + 2k; l) - S(l; l + 2k) \right]
+ (-1)^j \ln 2 \left[ S(l + 2k; \bar{l}) - S(l; \bar{l} + 2k) \right].
\]

Proof The result follows from Theorem 1.2 with \( x \to -1, \ q \to 1 \) and \( k \mapsto l + 2k \). \( \square \)

From Theorem 1.4, we find that for a positive integer \( l > 1 \), it holds
\[
\zeta^4(l) = 4S([l]_3; l) - 6S([l]_2; 2l) + 4S(l; 3l) - \zeta(4l),
\]
\[
\zeta(2l)\zeta^2(l) = 2S(l, 2l; l) + S([l]_2; 2l) - S(2l; 2l) - 2S(l; 3l) + \zeta(4l),
\]
\[
\zeta(3l)\zeta^2(l) = 2S(l, 3l; l) + S([l]_2; 3l) - S(3l; 2l) - 2S(l; 4l) + \zeta(5l),
\]
\[
\zeta^5(l) = 5S([l]_4; l) - 10S([l]_3; 2l) + 10S([l]_2; 3l) - 5S(l; 4l) + \zeta(5l),
\]
\[
\zeta(2l)\zeta^3(l) = S([l]_3; 2l) + 3S([l]_2; 2l; l) - 3S([l]_2; 3l) - 3S(l; 2l; 2l)
+ 3S(l; 4l) + S(2l; 3l) - \zeta(5l).
\]

Corollary 4.5 For integers \( l \in \mathbb{N} \setminus \{1\} \) and \( k \in \mathbb{N} \cup \{0\} \), the following identity holds:
\[
S(\bar{1}, l + 2k + 1; l) + S(\bar{1}, l; l + 2k + 1) + S(l, l + 2k + 1; \bar{l})
= S(l; \bar{l} + 2k + 2) + S(\bar{1}; 2l + 2k + 1) + S(l + 2k + 1; \bar{l} + \bar{1})
+ \zeta(l + 2k + 1) \ln 2 - \bar{\zeta}(2l + 2k + 2).
\]

Proof Taking \( q \to 1, \ (x_1, x_2, x_3) \to (-1, 1, 1) \) and \( (k_1, k_2, k_3) \to (1, l + 2k + 1, l) \) in Theorem 1.4 gives the result. \( \square \)
Hence, from Corollaries 4.3 and 4.5, we obtain the following description of quadratic Euler sums.

**Corollary 4.6** For \( l \in \mathbb{N} \setminus \{1\} \) and \( k \in \mathbb{N} \cup \{0\} \), the alternating quadratic sums

\[
S(l, l + 2k + 1; \bar{1}) = \sum_{n=1}^{\infty} \frac{H_n^{(l)} H_n^{(l+2k+1)}}{n} (-1)^{n-1}
\]

are reducible to linear sums.

A simple example is as follows:

\[
S(2, 3; \bar{1}) = -\frac{161}{64} \zeta(6) + \frac{31}{16} \zeta(5) \ln 2 + \frac{9}{32} \xi^2(3) + \frac{3}{8} \zeta(2) \xi(3) \ln 2 + 2 \zeta(2) \text{Li}_4 \left( \frac{1}{2} \right) \\
- \frac{5}{4} \zeta(4) \ln^2 2 + \frac{1}{12} \zeta(2) \ln^4 2 + S(2; \bar{4}) - S(3; 3).
\]

In fact, proceeding similarly as in the evaluation of Theorem 1.2 and Corollary 4.6, it is possible to evaluate other Euler sums involving harmonic numbers and alternating harmonic numbers. For example, in the same way as in the proof of Corollary 4.6, we also prove that the alternating quadratic sums (see [26])

\[
S(l, l + 2k + 1; \bar{1}) = \sum_{n=1}^{\infty} \frac{H_n^{(l)} H_n^{(l+2k+1)}}{n} (-1)^{n-1}
\]

are reducible to linear sums, for \( l \in \mathbb{N} \setminus \{1\} \) and \( k \in \mathbb{N} \cup \{0\} \). As a special case we have

\[
S(\bar{2}, \bar{3}; \bar{1}) = -\frac{163}{128} \zeta(6) - \frac{31}{16} \zeta(5) \ln 2 + \frac{3}{16} \xi^2(3) - \frac{3}{4} \zeta(2) \xi(3) \ln 2 - \zeta(2) \text{Li}_4 \left( \frac{1}{2} \right) \\
+ \frac{5}{8} \zeta(4) \ln^2 2 - \frac{1}{24} \zeta(2) \ln^4 2 + S(\bar{2}; 4) + S(3; 3).
\]

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