Quantum Locality, Rings a Bell?:
Bell’s inequality meets local reality and true determinism.

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Abstract

By assuming a deterministic evolution of quantum systems and taking realism into account, we carefully build a hidden variable theory for Quantum Mechanics based on the notion of ontological states proposed by ’t Hooft[1]. We view these ontological states as the ones embedded with realism and compare them to the (usual) quantum states that represent superpositions, viewing the latter as mere information of the system they describe.

Such a deterministic model puts forward conditions for the applicability of Bell’s inequality: the usual inequality cannot be applied to the usual experiments. We build a Bell-like inequality that can be applied to the EPR scenario and show that this inequality is always satisfied by Quantum Mechanics.

In this way we show that Quantum Mechanics can indeed have a local interpretation, and thus meet with the causal structure imposed by the Theory of Special Relativity in a satisfying way.

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I. INTRODUCTION

Since Bell showed his impossibility proof of local hidden variables\(^2\), the quest for realism has taken a wide variety of paths\(^1\)\(^3\)\(^6\). No agreement has been reached, however. Is a quantum state real, or is it a carrier of information? Is the wave function only a mathematical construct, even when we can see wave-like interference patterns in Young’s double slit experiment? Quantum states in superposition cannot be observed (the dead-and-alive cat, for instance) suggesting that they merely embody statistical restrictions on measurement results. Yet we think of them as describing physical systems that evolve in time in accordance to well given mathematical equations.

This evolution, we picture, takes place in physical spacetime, and this spacetime is endowed with a locally causal structure. But there is a violation of causality embedded in Quantum Mechanics; so much so that many interpretations have been given as to what this violation might physically mean\(^7\)\(^9\). Local causality is imposed on spacetime by Special Relativity: a sequence of cause and effect that constitutes, we believe, a fundamental principle on which we think about and do our scientific work.

This means that we need a better understanding of the most basic phenomena of Quantum Mechanics. Several no-go theorems have shut the door for realism and locality\(^3\)\(^10\)\(^12\); but in which way?, with what assumptions?, is the door really locked? In this work we will start to examine these questions by proposing a realist hidden variable interpretation of Quantum Mechanics: factuality. Within this perspective we will analyse the first and most important of the no-go theorems: Bell’s inequality\(^1\). This is only a first step towards developing a local deterministic formulation of Quantum Mechanics.

II. CONSTRUCTION

We will begin by revisiting the tools of Quantum Mechanics that are necessary for the construction of our proposal. To do so, we will make a general statement that will be applied to the particular case of a spin degree of freedom for fermions (which might be extended to

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\(^1\) Although the inequality that is experimentally tested\(^13\)\(^14\) is the variation of Bell’s inequality formulated by Clauser \textit{et. al.}\(^12\), we will revisit Bell’s original construction, given that the analysis we make rests on the common ground of both, and it is easier to look at the original one.
polarization for photons).

A. Tools and ontology

Quantum Mechanics (QM) is a wave theory in that it associates wave properties to particles. But it actually reduces all mechanics to the mechanics of particles themselves. The wave nature (as in the double-slit experiment) arises when one observes the collective behaviour of a large ensemble of particles, just as ripples in water arise from a statistical behaviour of many water particles, or electromagnetic waves, in quantum theory, are the result of a large collection of photons. We see the phenomenon of superposition in waves, but not in the individual particles which are the building blocks (physical entities) in QM. With this in mind, we can then say that:

Physical entities do not appear in superposed states, that is, nature in its fundamental parts does not emerge as a superposition of states. The superposition principle is a mathematical construct which can then be applied to the individual parts of an ensemble as a statistical description of the ensemble, not to each individual entity as a realistic description of the latter.

With this in hand we define:

Quantum states as the states generated from linear combinations of eigenstates of some observable, and denote them with $|\psi\rangle$.

Ontological states as the eigenstates of any observable, and denote them with $|\Omega\rangle$.

It is important to notice that, if we have a quantum state description of a one-body system, we can always perform a basis transformation so that this description becomes an ontological state description. For example: $rac{1}{\sqrt{2}} [ |\uparrow\rangle^z + |\downarrow\rangle^z ]$ is a quantum state description, but acquires an ontological meaning when we switch to the $\hat{\sigma}_x$-diagonal basis, resulting in $|\uparrow\rangle^x = \frac{1}{\sqrt{2}} [ |\uparrow\rangle^z + |\downarrow\rangle^z ]$. So every one-body system can always be described as an ontological state, and this description is the real state of the system (at a given time).

On the other hand, ensembles of individual particles might be described either as pure states or as mixed states. Of course, each description depicts different ensembles. In a pure state description we regard the ensemble as if every one of its components were in precisely that pure state, while in a mixed state description we regard the ensemble as one where
different components of the ensemble are in different pure states, with a certain probability
distribution. We will denote these two descriptions as $\rho$ and $\tilde{\rho}$ respectively, i.e.,

$$\rho = |\Omega\rangle\langle\Omega|$$

$$\tilde{\rho} = \sum \limits_i c_i |\Omega_i\rangle\langle\Omega_i|$$

When we get into two-body systems and ensembles of two-body systems, things start to
get a bit more complicated, then we have to carefully define what different kinds of states
might physically mean.

We again have quantum states and ontological states, following the definition from above.
But a big difference emerges now. Not all quantum states can take on an ontological state
description. These specific states are **entangled states**.

So, for example, the quantum state

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}} \left[ |\uparrow\uparrow\rangle^{\hat{z}} + |\uparrow\downarrow\rangle^{\hat{z}} \right]$$

can be described as an ontological state,

$$|\Omega\rangle_{AB} = |\uparrow\uparrow\rangle^{\hat{x}}$$

whilst the entangled state

$$|\phi\rangle_{AB} = \frac{1}{\sqrt{2}} \left[ |\uparrow\downarrow\rangle^{\hat{z}} + |\downarrow\uparrow\rangle^{\hat{z}} \right]$$

can not.

With this in mind, we will focus on two kinds of states: ontological pairs and entangled
states.

**Ontological pairs** are those which emerge due to the interaction between $A$ and $B$, two
physical entities (be them $A$, an electron and $B$, a measurement device; or $A$ and $B$ two
electrons in a spin state $S = 0$; or any two particles $A$ and $B$ that come together at time
$t = t_0$).

An ontological pair is the complete and known description of a system at a given time,
$t_0$. For example, in a measurement of any given property, what we describe (and know) is
the ontological pair of the system [particle]-[measurement device].

Ontological pairs give rise to pure states that are separable, while entangled states give
rise to non-separable pure states. These will be denoted by:

$$A_\rho B = \left[|\Omega\rangle\langle\Omega|\right]_{AB}$$
$$\tilde{\rho}_{\tilde{B}} = \left[|\phi\rangle\langle\phi|\right]_{\tilde{A}B}$$

respectively.

Along the same line in which quantum states emerge only as a mathematical description of a system, mixed states only represent a statistical description of an ensemble that is comprised of many entities, each one in a pure state.

In two-body systems, using the language above, the mathematical description of an entangled state is to be regarded as a statistical description of an ensemble whose constituents are ontological pairs that evolve according to a function of time and a hidden variable. Entanglement, or the correlations that arise from an entangled state, are due to the hidden variables shared by each ontological pair.

That is to say, no individual two-body system is in an entangled state per se, it is actually an ontological pair determined from time to time by a function of hidden variables, whose state only becomes evident when a measurement takes place. Nevertheless, an ensemble of such pairs can be successfully described by an entangled (quantum) state vector.

B. Evolution

In our construction, beyond the realism embedded in the ontological state description that we put forward above, we must have a deterministic evolution of the ontological pairs, and thus we should give a function that governs such evolution.

We are analysing Bohm’s thought experiment\[15\], so we work with an ensemble of two-body systems described by an entangled state

$$|\phi\rangle_{AB} = \frac{1}{\sqrt{2}} \left[ |\uparrow\downarrow\rangle^{\tilde{z}} + |\downarrow\uparrow\rangle^{\tilde{z}} \right]$$

Each system divides into its two components and each of these reaches a detector, where its spin projection is measured.

In our view each individual system is an ontological pair, which will evolve according to a function of a hidden variable $\lambda$ and time $t$. Since we are only focusing on the projection of the spin degree of freedom of each component of the pair, such function, when evaluated at any given value of the hidden variable $\lambda$ and any given time $t$, will result in the direction of the spin projection of the two components of the ontological pair. That is:

$$\theta : \Lambda \times \mathbb{R} \rightarrow \mathbb{R}^3 \times \mathbb{R}^3$$
where Λ is the set of values that the hidden variable can take, i.e., it is the domain of Λ. So, given λ ∈ Λ and t ∈ R,
\[ θ(λ, t) = (\hat{o}_A, \hat{o}_B) \]
where \( \hat{o}_A \) and \( \hat{o}_B \) are the spin projection orientations of each component of the pair, and they themselves are functions of λ and t, \( \hat{o}_A(λ, t) \) and \( \hat{o}_B(λ, t) \). Note that these functions are absolutely deterministic, and a direct consequence of this is the fact that the orientation of the detectors is also encoded in λ. There is no *what would have happened if the detector had not been in such and such orientation?* The detector will have only one true orientation, determined by all the previous conditions accessible to it. This is what a truly deterministic scenario entails. A detector in a different orientation will have different values of λ at all earlier times.

Going back to our description, the spin degree of freedom of a two-body system is always an ontological pair state given by:
\[ |↑↑⟩^{θ(λ,t)} \]
So for example, an initial state with \( S = 0 \) would be
\[ |↑↑⟩^{(\hat{z}, -\hat{z})} \]
and the evolution of this pair, when separated into its two components, would look like that of Figure 1 (right hand side). What we see in the left hand side of Figure 1 is just the pictorial evolution we assign to an ontological pair when the two components of the pair are put together.

These are all the tools we need for an ontological and deterministic description of reality. In the next section we will talk about locality conditions and the mechanism for entanglement.

III. LOCALITY

We have constructed a description of entanglement that is implicitly local, given the introduction of hidden variables. We make, though, one statement about deterministic evolution that was not made when hidden variable interpretations were first introduced[2] and then we put forward the mechanism for entanglement.
We affirm that the evolution function $\theta(\lambda, t)$ must satisfy a condition we call **factuality**. Mathematically, this condition is no news: for any given function, different outcomes of the function must come from different inputs. So, once the values of hidden variable and time are given, our function $\theta(\lambda, t)$ can only acquire a certain value ($\hat{o}_A^\lambda$, $\hat{o}_B^\lambda$). Physically, this is the factuality condition: if a system evolved in time ($t_0 \rightarrow t_1$) to a particular state, it is because only this state was accessible to it given the initial condition ($\lambda, t_0$) and, therefore, different states at time $t_1$ must come from different values of hidden variables $\lambda_i$. This is only a consequence of determinism.

We have already proposed that non-local correlations emerge from the deterministic evolution of a shared hidden variable between two components of an ontological pair. Entanglement arises every time two (or more) physical entities share hidden variables. This suffices for the time being, and for the example we work below. In what follows, we will analyse the emergence of Bell’s inequality within our proposed description of reality.

**IV. BELL’S INEQUALITY**

Suppose a pair of entangled electrons in a singlet state is split into two electrons at time $t = t_0$. If the spin of electron $A$ is measured at a later time in the $\hat{z}$ direction and we get, for
example, $|\uparrow\rangle_A$, then we can be sure that the spin of electron $B$ is $|\downarrow\rangle_B$. Locality associates, with the spin of each electron, a hidden variable quality, that is: $A(\hat{a}, \lambda) = \pm 1$ where $A$, the value of the spin of particle $A$, is a function of the direction of the detector $\hat{a}$ and of a hidden variable $\lambda$. Same for $B$ in any direction $\hat{b}$. The expectation value of the correlation between $A$ (measured in the direction $\hat{a}$) and $B$ (measured in the direction $\hat{b}$) naturally arises,

$$E(\hat{a}, \hat{b}) = \int_{\Lambda} A(\hat{a}, \lambda)B(\hat{b}, \lambda)\rho(\lambda)d\lambda$$

Bell shows[2] that if such functions $A$ and $B$ exist, the expectation value of the correlation between them must satisfy:

$$|E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c})| \leq 1 + E(\hat{b}, \hat{c})$$

where $\hat{a}$, $\hat{b}$ and $\hat{c}$ are three alternative directions of the detectors used to measure the spin projection of the electrons.

A. Under factuality

First we need to find the relation between the functions $A(\hat{j}, \lambda)$ and $B(\hat{k}, \lambda)$ and our deterministic evolution function $\theta(\lambda, t)$.

Functions $A$ and $B$ are both results of a measurement, so they must be related to the function $\theta$ when the latter is evaluated at the time of measurement, say $t = t_1$.

Now, $\theta(\lambda, t_1)$ gives a pair of orientations, $(\hat{\theta}_A, \hat{\theta}_B)$. These two orientations are those of the spin projection for particles $A$ and $B$ respectively at the time of measurement, and it is important to recall that the orientation of the two detectors is also encoded in the value of the hidden variable $\lambda$.

Function $A(\hat{j}, \lambda)$ asks the question, “given a detector device with orientation $\hat{j}$ and a hidden variable $\lambda$, is the electron’s spin orientation $\hat{j}$ or $-\hat{j}$?”. So for this question to be posed, the electron’s spin orientation must be $\hat{j}$ or $-\hat{j}$. Analogously for function $B(\hat{k}, \lambda)$. Then, these two questions can be posed iff $\theta(\lambda, t_1) = (\pm \hat{j}, \pm \hat{k})$.

**Fact 1** functions $A(\hat{j}, \lambda)$ and $B(\hat{k}, \lambda)$ are simultaneously well defined iff $\theta(\lambda, t_1) = (\pm \hat{j}, \pm \hat{k})$.

Now we will impose the factuality condition on three deterministic scenarios (Figure[2]). In the left hand side scenario of that figure, the measurement outcome can be any of four
different possibilities, \((\hat{a}, \hat{b}), (\hat{a}, -\hat{b}), (-\hat{a}, \hat{b})\) and \((-\hat{a}, -\hat{b})\), that is \(\theta(\lambda, t_1) = (\pm\hat{a}, \pm\hat{b})\). In the second scenario, \(\theta(\lambda, t_1) = (\pm\hat{a}, \pm\hat{c})\) and in the right hand scenario \(\theta(\lambda, t_1) = (\pm\hat{b}, \pm\hat{c})\).

![Diagram](image)

**FIG. 2:** Three ontological pairs whose spin degree of freedom is described by the function \(\theta(\lambda, t)\), each pair subject to a different set of measurements at time \(t = t_1\).

Under the factuality condition, each of these sets of outcomes must come from a different set of hidden variables, that is:

\[
\theta(\lambda, t_1) = (\pm\hat{a}, \pm\hat{b}) \leftrightarrow \lambda \in \Lambda_1 \\
\theta(\lambda, t_1) = (\pm\hat{a}, \pm\hat{c}) \leftrightarrow \lambda \in \Lambda_2 \\
\theta(\lambda, t_1) = (\pm\hat{b}, \pm\hat{c}) \leftrightarrow \lambda \in \Lambda_3
\]

Furthermore, \(\Lambda_1 \cap \Lambda_2 = \Lambda_1 \cap \Lambda_3 = \Lambda_2 \cap \Lambda_3 = \emptyset\), which can be seen by the simple reasoning:

If \(\lambda \in \Lambda_1\), then \(\theta(\lambda, t_1) = (\pm\hat{a}, \pm\hat{b}) \neq (\pm\hat{a}, \pm\hat{c})\), then \(\lambda \notin \Lambda_2\); etc.

Then, from **Fact 1** and equations (1), (2) and (3):

**Fact 2.1** functions \(A(\hat{a}, \lambda)\) and \(B(\hat{b}, \lambda)\) are simultaneously well defined iff \(\lambda \in \Lambda_1\).

**Fact 2.2** functions \(A(\hat{a}, \lambda)\) and \(B(\hat{c}, \lambda)\) are simultaneously well defined iff \(\lambda \in \Lambda_2\).

**Fact 2.3** functions \(A(\hat{b}, \lambda)\) and \(B(\hat{c}, \lambda)\) are simultaneously well defined iff \(\lambda \in \Lambda_3\).
So, if we were to follow Bell’s steps to derive his inequality, we would start by comparing the expectation values,

\[ E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c}) = \int_{\Lambda_1} A(\hat{a}, \lambda) B(\hat{b}, \lambda) \rho(\lambda) d\lambda - \int_{\Lambda_2} A(\hat{a}, \lambda) B(\hat{c}, \lambda) \rho(\lambda) d\lambda \]

where we have explicitly written the integration domains imposed by Fact 2.1 and Fact 2.2. Since \( \Lambda_1 \cap \Lambda_2 = \emptyset \) we cannot carry on to Bell’s next step in order to derive his inequality, so:

**Fact 3** In a local deterministic scenario, governed by factuality, Bell’s inequality cannot be derived, therefore the violation of his inequality by experiments does not show that the assumption of locality in this scenario is incorrect.

The statement above begs the question, in which scenario can Bell’s inequality be derived? And, do experiments violate this inequality in such scenario? We will take a look at these questions in the next subsection.

**B. Building Bell’s inequality**

In the previous sections we have worked with a deterministic view of reality, in which the spin degree of freedom of an ontological pair is governed by a function \( \theta(\lambda, t) \).

This view can take on two possible paths when describing three different scenarios (cf. Figure 2), they are:

- Each different scenario can be governed by a different function, \( \theta_i(\lambda, t), i = 1, 2, 3 \).
- On each different scenario the measurement can take place at a different time, so the final state could be described by \( \theta(\lambda, t_i), i = 1, 2, 3 \).

When taking any of these two paths, Bell’s steps can be followed further than we could on the last subsection. As before, we will start by identifying the functions \( A \) and \( B \) used by Bell with our function \( \theta \).

We can directly see that functions \( A(\hat{a}, \lambda) \) and \( B(\hat{b}, \lambda) \) can only be simultaneously identified with \( \theta_1(\lambda, t_1) \) (in the first path) or \( \theta(\lambda, t_1) \) (in the second path). We will take on the first path (the second path is shown in \( \boxed{A} \)).
We know that
\[ \theta_1(\lambda, t_1) = (\hat{o}_{A_1}(\lambda, t_1), \hat{o}_{B_1}(\lambda, t_1)) = (\pm \hat{a}, \pm \hat{b}) \]
then we can define
\[ A_1(\hat{a}, \lambda) \equiv \text{sign}(\hat{o}_{A_1}(\lambda, t_1)) \]
and
\[ B_1(\hat{b}, \lambda) \equiv \text{sign}(\hat{o}_{B_1}(\lambda, t_1)) \]
Note that we carried the subscript 1 to distinguish these functions from the ones defined by \( \theta_2(\lambda, t_1) \). In this second case we have:
\[ \theta_2(\lambda, t_1) = (\hat{o}_{A_2}(\lambda, t_1), \hat{o}_{B_2}(\lambda, t_1)) = (\pm \hat{a}, \pm \hat{c}) \]
and we can simultaneously define
\[ A_2(\hat{a}, \lambda) \equiv \text{sign}(\hat{o}_{A_2}(\lambda, t_1)) \]
and
\[ B_2(\hat{c}, \lambda) \equiv \text{sign}(\hat{o}_{B_2}(\lambda, t_1)) \]
And in the third case:
\[ \theta_3(\lambda, t_1) = (\hat{o}_{A_3}(\lambda, t_1), \hat{o}_{B_3}(\lambda, t_1)) = (\pm \hat{b}, \pm \hat{c}) \]
so
\[ A_3(\hat{b}, \lambda) \equiv \text{sign}(\hat{o}_{A_3}(\lambda, t_1)) \]
and
\[ B_3(\hat{c}, \lambda) \equiv \text{sign}(\hat{o}_{B_3}(\lambda, t_1)) \]

We have three possibilities for the domains of the functions \( \theta_1, \theta_2 \) and \( \theta_3 \): they can be identical, they can intersect, or their intersection can be the empty set. If we took the third option, we would not get further than in the last subsection, so we will discard that as an option, and start with option 1.
1. Three functions with identical domains

Let us go back to Bell’s first step,

\[ E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c}) = \int_\Lambda A_1(\hat{a}, \lambda) B_1(\hat{b}, \lambda) \rho(\lambda) d\lambda - \int_\Lambda A_2(\hat{a}, \lambda) B_2(\hat{c}, \lambda) \rho(\lambda) d\lambda \]

where we have implicitly written the subscripts that define each function in terms of the deterministic evolution of each different experiment.

Now, his second step is

\[ \left| E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c}) \right| = \left| \int_\Lambda [A_1(\hat{a}, \lambda) B_1(\hat{b}, \lambda) - A_2(\hat{a}, \lambda) B_2(\hat{c}, \lambda)] \rho(\lambda) d\lambda \right| \]  \hspace{1cm} (4)

where we have highlighted \( A_1(\hat{a}, \lambda) \) and \( A_2(\hat{a}, \lambda) \) to stress the fact that for his next step, Bell takes these two functions to be identical. This is his first assumption (out of three).

We will analyse what can be said about the quantity \( \left| E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c}) \right| \) in two cases: while taking Bell’s three assumptions, and while taking none of them.

Within Bell’s assumptions

Bell’s three assumptions are (shown in B):

\[ A_1(\hat{a}, \lambda) = A_2(\hat{a}, \lambda) \]
\[ B_1(\hat{b}, \lambda) = -A_3(\hat{b}, \lambda) \]
\[ B_2(\hat{c}, \lambda) = B_3(\hat{c}, \lambda) \]

These are constraints on the functions \( \theta_i(\lambda, t) \) that have to be met in order for Bell’s inequality to be derived. So, the applicable domain of his inequality is the one that behaves according to these constraints, that is, the deterministic functions \( \theta_i(\lambda, t) \) that govern the three experiments built to test Bell’s inequality have to be so that these constraints are satisfied.

This has an implication on the expectation values of the correlation between measurements. If these three constraints are satisfied, the predicted expectation values result in:

\[ E(\hat{a}, \hat{b}) = -\cos \theta_{ab} \]
\[ E(\hat{a}, \hat{c}) = -\cos \theta_{ac} \]
\[ E(\hat{b}, \hat{c}) = -\cos \theta_{ab} \cos \theta_{ac} \]

which is caused by the fact that the given constraints tamper with the probabilities of getting \((\pm \hat{a}), (\pm \hat{b})\) or \((\pm \hat{c})\) in the measurements performed. The derivation of these results is given in \[C\].

Now, this result leads to two conclusions:

The first one is: if the experiments were to satisfy the constraints necessary to build Bell’s inequality, then the expectation values would be such that when plugged into the inequality one would get:

\[ | - \cos \theta_{ab} + \cos \theta_{ac} | \leq 1 - \cos \theta_{ab} \cos \theta_{ac} \]  \hspace{1cm} (5)

and, as shown in \[D\] this inequality is always satisfied.

The second conclusion is: the experiments used to test Bell’s inequality do not result in an expectation value given by a product of cosines \((- \cos \theta_{ab} \cos \theta_{ac})\), so they do not behave according to the constraints necessary to build Bell’s inequality, so they do not have to satisfy such an inequality and the violation of the inequality by the experiments does not show that reality cannot behave in a local deterministic way.

**Without Bell’s assumptions**

We will now go back to his second step and build a Bell-like inequality without his assumptions.

\[
\left| E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c}) \right| = \left| \int_{\Lambda} [A_1(\hat{a}, \lambda)B_1(\hat{b}, \lambda) - A_2(\hat{a}, \lambda)B_2(\hat{c}, \lambda)]\rho(\lambda)d\lambda \right|
\]

\[= \sum_{i=1}^{8} \left| \int_{\tilde{\Lambda}_i} [A_1(\hat{a}, \lambda)B_1(\hat{b}, \lambda) - A_2(\hat{a}, \lambda)B_2(\hat{c}, \lambda)]\rho(\lambda)d\lambda \right|\]

where we build the sets \(\tilde{\Lambda}_i, i = 1, \ldots, 8\), in terms of the different relations that the functions \(A_1, A_2, A_3, B_1, B_2\) and \(B_3\) hold between them. These 8 sets \(\tilde{\Lambda}_i\) are defined as:

\[\tilde{\Lambda}_1 = \{ \lambda \mid A_1 = A_2, B_1 = -A_3, B_2 = B_3 \}\]

\[\tilde{\Lambda}_2 = \{ \lambda \mid A_1 = A_2, B_1 = -A_3, B_2 = -B_3 \}\]

\[\tilde{\Lambda}_3 = \{ \lambda \mid A_1 = A_2, B_1 = A_3, B_2 = B_3 \}\]

\[\tilde{\Lambda}_4 = \{ \lambda \mid A_1 = A_2, B_1 = A_3, B_2 = -B_3 \}\]

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\[ \tilde{\Lambda}_5 = \{ \lambda \mid A_1 = -A_2, B_1 = -A_3, B_2 = B_3 \} \]
\[ \tilde{\Lambda}_6 = \{ \lambda \mid A_1 = -A_2, B_1 = -A_3, B_2 = -B_3 \} \]
\[ \tilde{\Lambda}_7 = \{ \lambda \mid A_1 = -A_2, B_1 = A_3, B_2 = B_3 \} \]
\[ \tilde{\Lambda}_8 = \{ \lambda \mid A_1 = -A_2, B_1 = A_3, B_2 = -B_3 \} \]

So
\[
\left| \int_{\tilde{\Lambda}_1} [A_1(\hat{a}, \lambda)B_1(\hat{b}, \lambda) - A_2(\hat{a}, \lambda)B_2(\hat{c}, \lambda)]\rho(\lambda)d\lambda \right|
\leq \int_{\tilde{\Lambda}_1} [1 + A_3(\hat{b}, \lambda)B_3(\hat{c}, \lambda)]\rho(\lambda)d\lambda
\]
\[ = Z(\tilde{\Lambda}_1) - Z(\tilde{\Lambda}_1) \cos \theta_{ab} \cos \theta_{ac} \]

where \( Z(\tilde{\Lambda}_1) \) is the measure of the set \( \tilde{\Lambda}_1 \), and the integral of the product \( A_3B_3\rho(\lambda) \) results in \(-Z(\tilde{\Lambda}_1) \cos \theta_{ab} \cos \theta_{ac}\) given that, when \( \lambda \) belongs to \( \tilde{\Lambda}_1 \), the functions \( A_3 \) and \( B_3 \) are correlated precisely by the constraints used to build the tables II - V.

Since we are not as familiar with the constraints in \( \tilde{\Lambda}_2 \) as those in \( \tilde{\Lambda}_1 \) we will perform the next integral in more detail.

\[
\left| \int_{\tilde{\Lambda}_2} [A_1(\hat{a}, \lambda)B_1(\hat{b}, \lambda) - A_2(\hat{a}, \lambda)B_2(\hat{c}, \lambda)]\rho(\lambda)d\lambda \right|
= \left| \int_{\tilde{\Lambda}_2} A_1(\hat{a}, \lambda)B_1(\hat{b}, \lambda)[1 + A_3(\hat{b}, \lambda)B_2(\hat{c}, \lambda)]\rho(\lambda)d\lambda \right|
\]
given the first two constraints, which turns to
\[
\leq \int_{\tilde{\Lambda}_2} [1 - A_3(\hat{b}, \lambda)B_3(\hat{c}, \lambda)]\rho(\lambda)d\lambda
\]
by use of the last constraint. We can determine this integral by using the correlations given in Table I.
TABLE I: Joint probabilities of $A_3(\hat{b}, \lambda)$ and $B_3(\hat{c}, \lambda)$, when $\lambda$ belongs to $\tilde{\Lambda}_2$.

| $A_2(\hat{a}, \lambda)$ | 1 | -1 |
|-------------------------|---|----|
| 1                       | $\frac{1}{2}$ sin² $\frac{\theta_{ab}}{2}$ | $\frac{1}{2}$ cos² $\frac{\theta_{ab}}{2}$ |
| -1                      | $\frac{1}{2}$ cos² $\frac{\theta_{ab}}{2}$ | $\frac{1}{2}$ sin² $\frac{\theta_{ab}}{2}$ |

| $-B_3(\hat{c}, \lambda)$ | 1 | -1 |
|---------------------------|---|----|
| 1                         | $\frac{1}{2}$ sin² $\frac{\theta_{ac}}{2}$ | $\frac{1}{2}$ cos² $\frac{\theta_{ac}}{2}$ |
| -1                        | $\frac{1}{2}$ cos² $\frac{\theta_{ac}}{2}$ | $\frac{1}{2}$ sin² $\frac{\theta_{ac}}{2}$ |

obtaining, once again,

$$\int_{\tilde{\Lambda}_2} [1 - A_3(\hat{b}, \lambda) B_3(\hat{c}, \lambda)] \rho(\lambda) d\lambda = Z(\tilde{\Lambda}_2) - Z(\tilde{\Lambda}_2) \cos \theta_{ab} \cos \theta_{ac}$$

Following the same procedure, one can verify that

$$\left| \int_{\tilde{\Lambda}_i} [A_1(\hat{a}, \lambda) B_1(\hat{b}, \lambda) - A_2(\hat{a}, \lambda) B_2(\hat{c}, \lambda)] \rho(\lambda) d\lambda \right| \leq Z(\tilde{\Lambda}_i) - Z(\tilde{\Lambda}_i) \cos \theta_{ab} \cos \theta_{ac}$$

$\forall i$. Adding all these integrals over $i$, and normalising to the volume of $\Lambda$, i.e.

$$\sum_{i=1}^{8} Z(\tilde{\Lambda}_i) = 1$$

yields the value $1 - \cos \theta_{ab} \cos \theta_{ac}$.

This shows that the inequality the two expectation values must satisfy, when assuming no specific relation between functions $A$ and $B$, is:

$$\left| E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c}) \right| \leq 1 - \cos \theta_{ab} \cos \theta_{ac} \quad (6)$$

where $\cos \theta_{ab} \cos \theta_{ac}$ is just a quantity, not an expectation value of a specific scenario.

We have already shown that Quantum Mechanics’ predictions and experimental results always satisfy inequality (6).
V. CONCLUSIONS

Local realistic interpretations can have a place in the Quantum Mechanical model. We conclude this by showing that:

The usual application of Bell’s inequality to experiments is not a proof of the non-local nature of reality, in that Bell’s inequality cannot be derived for the conditions of the built experiments.

There is an inequality that can be experimentally tested, that is the Bell-like inequality we constructed in subsection [IV B 1] (eq. (6)). This inequality is always satisfied by Quantum Mechanics’ predictions, and thus by the known experimental results.

It seems, then, that a local interpretation of QM may be built. More work is in order, particularly on entanglement within this scenario. This is under current consideration.

Appendix A

We start from:

\[
\theta(\lambda, t_1) = (\hat{o}_A(\lambda, t_1), \hat{o}_B(\lambda, t_1)) = (\pm \hat{a}, \pm \hat{b})
\]

and we define:

\[
A_1(\hat{a}, \lambda) \equiv \text{sign}(\hat{o}_A(\lambda, t_1))
\]

and

\[
B_1(\hat{b}, \lambda) \equiv \text{sign}(\hat{o}_B(\lambda, t_1))
\]

Where we carried the subscript 1 to distinguish these functions from the ones defined by \(\theta(\lambda, t_2)\). Now:

\[
\theta(\lambda, t_2) = (\hat{o}_A(\lambda, t_2), \hat{o}_B(\lambda, t_2)) = (\pm \hat{a}, \pm \hat{c})
\]

so we can simultaneously define:

\[
A_2(\hat{a}, \lambda) \equiv \text{sign}(\hat{o}_A(\lambda, t_2))
\]

and

\[
B_2(\hat{c}, \lambda) \equiv \text{sign}(\hat{o}_B(\lambda, t_2))
\]

And finally:

\[
A_3(\hat{b}, \lambda) \equiv \text{sign}(\hat{o}_A(\lambda, t_3))
\]
and

\[ B_3(\hat{c}, \lambda) \equiv \text{sign}(\hat{o}_B(\lambda, t_3)) \]

Now, of course functions \( A_i \) and \( B_i \) defined this way are not necessarily identical to those defined by the first path, just because \( \hat{o}_A(\lambda, t_3) \) is not necessarily the same as \( \hat{o}_{A_3}(\lambda, t_1) \), etc. The thing is that, once one defines a set of functions \( \{A_1, B_1, A_2, B_2, A_3, B_3\} \), function \( A_1(\hat{a}, \lambda) \) can be different from \( A_2(\hat{a}, \lambda) \) (and so forth) and this is the argument we use in the rest of our development.

**Appendix B**

Bell parts from equation (4):

\[ \left| E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c}) \right| = \left| \int_{\Lambda} [A_1(\hat{a}, \lambda)B_1(\hat{b}, \lambda) - A_2(\hat{a}, \lambda)B_2(\hat{c}, \lambda)]\rho(\lambda)d\lambda \right| \]

and makes his first assumption:

\[ A_1(\hat{a}, \lambda) = A_2(\hat{a}, \lambda) \]

then equation* (4) turns to:

\[ \left| E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c}) \right| = \left| \int_{\Lambda} A_1(\hat{a}, \lambda)B_1(\hat{b}, \lambda) \left[ 1 - B_1(\hat{b}, \lambda)B_2(\hat{c}, \lambda) \right] \rho(\lambda)d\lambda \right| \]

where he uses the fact that \( B_1B_1 = 1 \). Now, taking the absolute value function into the integral and using the fact that \( |A_1B_1| = 1 \) his last equation turns to:

\[ \left| E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c}) \right| \leq \int_{\Lambda} \left| 1 - B_1(\hat{b}, \lambda)B_2(\hat{c}, \lambda) \right| \rho(\lambda)d\lambda \quad (B1) \]

but what is inside the absolute value function is always positive, so he just discards the bars. Next comes his second assumption:

\[ B_1(\hat{b}, \lambda) = -A_3(\hat{b}, \lambda) \]

so equation (B1) becomes:

\[ \left| E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c}) \right| \leq \int_{\Lambda} \left[ 1 + A_3(\hat{b}, \lambda)B_2(\hat{c}, \lambda) \right] \rho(\lambda)d\lambda \quad (B2) \]

And finally, he takes a third assumption:

\[ B_2(\hat{c}, \lambda) = B_3(\hat{c}, \lambda) \]
then equation (B2) turns to:

\[ |E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c})| \leq \int_{\Lambda} \left[ 1 + A_3(\hat{b}, \lambda) B_3(\hat{c}, \lambda) \right] \rho(\lambda) d\lambda \]  

(B3)

which takes him to his final step,

\[ \int_{\Lambda} \left[ 1 + A_3(\hat{b}, \lambda) B_3(\hat{c}, \lambda) \right] \rho(\lambda) d\lambda = 1 + E(\hat{b}, \hat{c}) \]

concluding,

\[ |E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{c})| \leq 1 + E(\hat{b}, \hat{c}) \]

Appendix C

We begin by building a table of probabilities for the first scenario (detector settings \( \hat{a} \) and \( \hat{b} \)), under the following knowledge: the probability of getting either +1 or −1 when measuring the spin projection of particle A is \( \frac{1}{2} \), but once one of those is guaranteed, say +1, the probability of getting +1 when measuring the spin projection of particle B is \( \sin^2 \frac{\theta_{ab}}{2} \) and the probability of getting −1 is \( \cos^2 \frac{\theta_{ab}}{2} \). So we have the joint probabilities shown in Table II.

| \( B_1(\hat{b}, \lambda) \) | 1 | -1 |
|----------------|---|----|
| \( A_1(\hat{a}, \lambda) \) | 1/2 | \( \sin^2 \frac{\theta_{ab}}{2} \) | 1/2 | \( \cos^2 \frac{\theta_{ab}}{2} \) |
| -1 | 1/2 | \( \cos^2 \frac{\theta_{ab}}{2} \) | 1/2 | \( \sin^2 \frac{\theta_{ab}}{2} \) |

Now, the assumption \( A_1(\hat{a}, \lambda) = A_2(\hat{a}, \lambda) \) invites us to substitute \( A_1 \) for \( A_2 \) and the assumption \( B_1(\hat{b}, \lambda) = -A_3(\hat{b}, \lambda) \) allows us to substitute \( B_1 \) for \( -A_3 \), turning Table II into Table III.
TABLE III: Joint probabilities under the assumptions $A_1(\hat{a}, \lambda) = A_2(\hat{a}, \lambda)$ and $B_1(\hat{b}, \lambda) = -A_3(\hat{b}, \lambda)$.

| $-A_3(\hat{b}, \lambda)$ | 1 | -1 |
|--------------------------|---|----|
| $A_2(\hat{a}, \lambda)$ | \(\frac{1}{2}\) & \(\sin^2 \frac{\theta_{ab}}{2}\) & \(\frac{1}{2}\) & \(\cos^2 \frac{\theta_{ab}}{2}\) |
| -1 | \(\frac{1}{2}\) & \(\cos^2 \frac{\theta_{ab}}{2}\) & \(\frac{1}{2}\) & \(\sin^2 \frac{\theta_{ab}}{2}\) |

The joint probabilities for experiment 2 are built accordingly and result in the left hand side of Table IV. Taking into account the assumption $B_2(\hat{c}, \lambda) = B_3(\hat{c}, \lambda)$ one gets the right hand side of Table IV.

TABLE IV: LHS: joint probabilities for experiment 2. RHS: same, under the assumption $B_2(\hat{c}, \lambda) = B_3(\hat{c}, \lambda)$.

| $B_2(\hat{c}, \lambda)$ | 1 | -1 |
|-------------------------|---|----|
| $A_2(\hat{a}, \lambda)$ | \(\frac{1}{2}\) & \(\sin^2 \frac{\theta_{ac}}{2}\) & \(\frac{1}{2}\) & \(\cos^2 \frac{\theta_{ac}}{2}\) |
| -1 | \(\frac{1}{2}\) & \(\cos^2 \frac{\theta_{ac}}{2}\) & \(\frac{1}{2}\) & \(\sin^2 \frac{\theta_{ac}}{2}\) |

| $B_3(\hat{c}, \lambda)$ | 1 | -1 |
|-------------------------|---|----|
| $A_2(\hat{a}, \lambda)$ | \(\frac{1}{2}\) & \(\sin^2 \frac{\theta_{ac}}{2}\) & \(\frac{1}{2}\) & \(\cos^2 \frac{\theta_{ac}}{2}\) |
| -1 | \(\frac{1}{2}\) & \(\cos^2 \frac{\theta_{ac}}{2}\) & \(\frac{1}{2}\) & \(\sin^2 \frac{\theta_{ac}}{2}\) |

Table V just brings together Table III and the right hand side of Table IV. We will use this to compute the joint probabilities of $A_3(\hat{b}, \lambda)$ and $B_3(\hat{c}, \lambda)$. 

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TABLE V: Joint probabilities of $A_3(\hat{b}, \lambda)$ and $B_3(\hat{c}, \lambda)$.

| $A_3(\hat{b}, \lambda)$ | 1   | -1   | $B_3(\hat{c}, \lambda)$ | 1   | -1   |
|--------------------------|-----|------|--------------------------|-----|------|
| $A_2(\hat{a}, \lambda)$ |     |      | $A_2(\hat{a}, \lambda)$ |     |      |
| 1                        | $\frac{1}{2}$ | $\sin^2 \frac{\theta_{ab}}{2}$ | 1                        | $\frac{1}{2}$ | $\sin^2 \frac{\theta_{ac}}{2}$ |
| -1                       | $\frac{1}{2}$ | $\cos^2 \frac{\theta_{ab}}{2}$ | -1                       | $\frac{1}{2}$ | $\cos^2 \frac{\theta_{ac}}{2}$ |

The procedure is as follows:

$A_2(\hat{a}, \lambda) = +1$ for $\lambda$ in a certain set, say $\Lambda_+$, and from Table V if $\lambda \in \Lambda_+$, then the probability that $A_3(\hat{b}, \lambda) = 1$ is $\cos^2 \frac{\theta_{ab}}{2}$ and the probability that $A_3(\hat{b}, \lambda) = -1$ is $\sin^2 \frac{\theta_{ab}}{2}$, while the probability that $B_3(\hat{c}, \lambda) = 1$ is $\sin^2 \frac{\theta_{ac}}{2}$ and the probability that $B_3(\hat{c}, \lambda) = -1$ is $\cos^2 \frac{\theta_{ac}}{2}$. So, for $\lambda \in \Lambda_+$ the probability of getting the same sign in both functions $A_3$ and $B_3$ is:

$$P(A_3 \cdot B_3 = 1) = \cos^2 \frac{\theta_{ab}}{2} \sin^2 \frac{\theta_{ac}}{2} + \sin^2 \frac{\theta_{ab}}{2} \cos^2 \frac{\theta_{ac}}{2}$$

and the probability of getting opposite signs is:

$$P(A_3 \cdot B_3 = -1) = \sin^2 \frac{\theta_{ab}}{2} \sin^2 \frac{\theta_{ac}}{2} + \cos^2 \frac{\theta_{ab}}{2} \cos^2 \frac{\theta_{ac}}{2}$$

Similarly, if $\lambda \in \Lambda_-$,

$$P(A_3 \cdot B_3 = 1) = \cos^2 \frac{\theta_{ab}}{2} \sin^2 \frac{\theta_{ac}}{2} + \sin^2 \frac{\theta_{ab}}{2} \cos^2 \frac{\theta_{ac}}{2}$$

and

$$P(A_3 \cdot B_3 = -1) = \sin^2 \frac{\theta_{ab}}{2} \sin^2 \frac{\theta_{ac}}{2} + \cos^2 \frac{\theta_{ab}}{2} \cos^2 \frac{\theta_{ac}}{2}$$

But at the same time, the probability that $\lambda \in \Lambda_+$ is $\frac{1}{2}$ as is the probability that $\lambda \in \Lambda_-$. Altogether,

$$P(A_3 \cdot B_3 = 1) = \cos^2 \frac{\theta_{ab}}{2} \sin^2 \frac{\theta_{ac}}{2} + \sin^2 \frac{\theta_{ab}}{2} \cos^2 \frac{\theta_{ac}}{2}$$  \hspace{1cm} (C1)

and

$$P(A_3 \cdot B_3 = -1) = \sin^2 \frac{\theta_{ab}}{2} \sin^2 \frac{\theta_{ac}}{2} + \cos^2 \frac{\theta_{ab}}{2} \cos^2 \frac{\theta_{ac}}{2}$$  \hspace{1cm} (C2)
Functions with the probability distributions given in equations (C1) and (C2) describe an experiment in which the expectation value of the correlation between these two functions would be:

\[
E(\hat{b}, \hat{c}) = \mathcal{P}(A_3 \cdot B_3 = 1) - \mathcal{P}(A_3 \cdot B_3 = -1) \\
= \cos^2 \frac{\theta_{ab}}{2} \sin^2 \frac{\theta_{ac}}{2} + \sin^2 \frac{\theta_{ab}}{2} \cos^2 \frac{\theta_{ac}}{2} \\
- \sin^2 \frac{\theta_{ab}}{2} \sin^2 \frac{\theta_{ac}}{2} - \cos^2 \frac{\theta_{ab}}{2} \cos^2 \frac{\theta_{ac}}{2} \\
= \left(\cos^2 \frac{\theta_{ab}}{2} - \sin^2 \frac{\theta_{ab}}{2}\right) \left(\sin^2 \frac{\theta_{ac}}{2} - \cos^2 \frac{\theta_{ac}}{2}\right) \\
= - \cos \theta_{ab} \cos \theta_{ac}
\]

Appendix D

The inequality to be analysed is:

\[| - \cos \theta_{ab} + \cos \theta_{ac}| \leq 1 - \cos \theta_{ab} \cos \theta_{ac}\]

which turns to:

\[\cos \theta_{ab} \cos \theta_{ac} - 1 \leq - \cos \theta_{ab} + \cos \theta_{ac} \leq 1 - \cos \theta_{ab} \cos \theta_{ac}\]

These two inequalities are satisfied iff

\[\cos \theta_{ab} \cos \theta_{ac} + \cos \theta_{ab} \leq \cos \theta_{ac} + 1\]

or, equivalently, \(\cos \theta_{ab}(\cos \theta_{ac} + 1) \leq \cos \theta_{ac} + 1\)

and

\[\cos \theta_{ac} + \cos \theta_{ab} \cos \theta_{ac} \leq 1 + \cos \theta_{ab}\]

or, equivalently, \(\cos \theta_{ac}(1 + \cos \theta_{ab}) \leq 1 + \cos \theta_{ab}\)

And these are true iff

\[\cos \theta_{ab} \leq 1\]

\[\cos \theta_{ac} \leq 1\]

which always holds. So inequality (5) is always satisfied.

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