ON SINGULAR VALUES OF HANKEL OPERATORS ON BERGMAN SPACES

M. BOURASS, O. EL-FALLAH, I. MARRHICH, AND H. NAQOS

Abstract. In this paper, we study the behavior of the singular values of Hankel operators on weighted Bergman spaces $A^2_{\omega\varphi}$, where $\omega\varphi = e^{-\varphi}$ and $\varphi$ is a subharmonic function. We consider compact Hankel operators $H_{\varphi}$ with anti-analytic symbols $\varphi$, and give estimates of the trace of $h(|H_{\varphi}|)$ for any convex function $h$. This allows us to give asymptotic estimates of the singular values $(s_n(H_{\varphi}))_n$ in terms of decreasing rearrangement of $|\varphi'|/\sqrt{\Delta\varphi}$. For the radial weights, we first prove that the critical decay of $(s_n(H_{\varphi}))_n$ is achieved by $(s_n(H_{z}))_n$. Namely, we establish that if $s_n(H_{\varphi}) = o(s_n(H_{z}))$, then $H_{\varphi} = 0$. Then, we show that if $\Delta\varphi(z) \asymp \frac{1}{(1-|z|^2)^{1+\beta}}$ with $\beta \geq 0$, then $s_n(H_{\varphi}) = O(s_n(H_{z}))$ if and only if $\varphi'$ belongs to the Hardy space $H^p$, where $p = \frac{2(1+\beta)}{2+\beta}$. Finally, we compute the asymptotics of $s_n(H_{\varphi})$ whenever $\varphi' \in H^p$.

1. Introduction

Hankel operators are one of the most important classes of bounded linear operators acting on spaces of analytic functions. They have many connections with function theory, harmonic analysis, approximation theory, moment problems, spectral theory, orthogonal polynomials, stationary Gaussian processes, $\partial$-operator ... etc. The book of V. Peller [24] is an acknowledged reference in the classical theory of Hankel operators and their various applications. We are interested in the behavior of the singular values of Hankel operators with anti-analytic symbols acting on Bergman spaces. The first work in this subject is due to Axler [3] who described boundedness and compactness of such operators in the classical Bergman space. Right after, Arazy, Fischer and Peetre [2] studied the membership for such operators in Schatten classes. They highlighted the existence of a cut-off by proving that these operators can not be of finite trace. In [13], Engliš and Rochberg described, using the Boutet de Monvel-Guillemin theory, all Hankel operators with anti-analytic symbols that belongs to the Dixmier class and gave an explicit formula for Dixmier Trace in this case. Several authors considered the same problems in other spaces of analytic functions [20, 17, 6, 26, 23, 29, 22]. In this paper, we give the asymptotic behavior of the singular

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values of compact Hankel operators, we describe the class of symbols for which the associated Hankel operators have the critical decay and compute the asymptotics of singular values of Hankel operators associated with this class of symbols.

The space of all holomorphic functions on the unit disc $D$ in the complex plane $\mathbb{C}$ will be denoted by $\text{Hol}(D)$. The Lebesgue measure on $\mathbb{C}$ is denoted by $dA$. The standard Bergman space $A^2_\alpha$ with $\alpha > -1$, consists of holomorphic functions $f$ on $D$ such that

$$\|f\|^2_\alpha := \int_D |f(z)|^2 dA_\alpha(z) < \infty,$$

where $dA_\alpha(z) := \frac{(\alpha+1)}{\pi}(1 - |z|^2)^\alpha dA(z)$.

Recall that $A^2_\alpha$ is a reproducing kernel Hilbert space with the kernel

$$K(z, w) = \frac{1}{(1 - \overline{w}z)^{\alpha+2}}, \quad z, w \in D.$$

The orthogonal projection of $L^2_\alpha := L^2(dA_\alpha)$ onto $A^2_\alpha$ will be denoted by $P_\alpha$. Given $g \in L^1(dA_\alpha)$, the linear transformation

$$H_g f = gf - P_\alpha(gf)$$

is a densely defined operator of $A^2_\alpha$ into $L^2_\alpha \ominus A^2_\alpha$. The operator $H_g$ is called the (big) Hankel operator with symbol $g$. For general facts concerning Hankel operators on Bergman spaces we refer to [2, 29]. In this paper we are interested in Hankel operator $H_\phi$ with anti-analytic symbol $\phi$. In [2], J. Arazy, S. Fisher and J. Peetre proved that $H_\phi$ is bounded (resp. compact) on $A^2_\alpha$ if and only if $\phi$ belongs to the Bloch space $\mathcal{B}$ (resp. $\phi$ belongs to the little Bloch space $\mathcal{B}_0$). This result was first proved by S. Axler [3] in the case $\alpha = 0$. The membership in Schatten classes of Hankel operators $H_\phi$ was also studied by J. Arazy, S. Fisher and J. Peetre in [2]. They proved that $H_\phi \in S_p(A^2_\alpha)$, for $p > 1$, if and only if $\phi$ belongs to the Besov space $\mathcal{B}_p$ defined by

$$\mathcal{B}_p = \{ \phi \in \text{Hol}(D), \quad \int_D |\phi'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty \}.$$

For $p \leq 1$, they proved that $H_\phi \in S_p(A^2_\alpha)$ if and only if $H_\phi = 0$ (which means that $\phi$ is a constant). These results were extended by Galanopoulos and Pau in [15] to large Bergman spaces associated with radial weights.

Let $\omega := e^{-\varphi}$ be a weight on $D$ such that $\varphi$ is a regular subharmonic function and let $A^2_\omega$ be the weighted Bergman space given by

$$A^2_\omega = \{ f \in \text{Hol}(D) : \|f\|_\omega := \left( \int_D |f(z)|^2 dA_\omega(z) \right)^{1/2} < \infty \}, \quad dA_\omega := \omega dA.$$

As before, the Hankel operator with anti-analytic symbol $\overline{\phi}$ is the operator $H_{\overline{\phi}} : L^2(dA_\omega) \to L^2(dA_\omega) \ominus A^2_\omega$ given by

$$H_{\overline{\phi}} f = \overline{\phi} f - P_\omega(\overline{\phi} f),$$
where $P_\omega$ is the orthogonal projection from $L^2(dA_\omega)$ onto $A^2_\omega$.

For a class of radial rapidly decreasing weights $\omega(z) = e^{-\varphi(|z|)}$, Galanopoulos and Pau proved in [15] that $H_\varphi$ is bounded (resp. compact) if and only if $\frac{|\phi'|^2}{\Delta \varphi}$ is bounded (resp. $\lim_{|z| \to 1^+} \frac{|\phi'(z)|^2}{\Delta \varphi(z)} = 0$). They also prove, for $p > 1$, that $H_\phi \in S_p$ if and only if $\frac{|\phi'|^2}{\Delta \varphi} \in L^{p/2}(\Delta \varphi dA)$.

Our goal in this paper is to study the asymptotic behavior of the singular values of $H_\varphi$. We will consider the class of weights $W$ which covers all previous examples (see Section 2.1). In order to state our main results, we introduce some notations. The reproducing kernel of $A^2_\omega$ is denoted by $K$,

$$
\tau_\omega(z) = \frac{1}{\omega^{1/2}(z)\|K_z\|} \quad \text{and} \quad d\lambda_\omega = \frac{dA}{\tau_\omega^2}.
$$

It should be noted that in several, but not all, situations $\tau_\omega^2$ is comparable to $1/\Delta \varphi$. For more information, see the examples given in Section 2.1.

By analogy with the standard case, we write $\mathcal{B}_\omega$ (resp. $\mathcal{B}_0^\omega$) for the space of analytic functions $\phi$ on $\mathbb{D}$ such that $\sup_{z \in \mathbb{D}} \tau_\omega(z)|\phi'(z)| < \infty$ (resp. $\lim_{|z| \to 1^-} \tau_\omega(z)|\phi'(z)| = 0$).

The following theorem will play an important role in the sequel.

**Theorem 1.1.** Let $\omega \in W^*$ and let $\phi \in \mathcal{B}_0^\omega$. Let $h : [0, +\infty] \to [0, +\infty]$ be an increasing convex function such that $h(0) = 0$. Then there exists $B > 0$, which depends only on $\omega$, such that

$$
\int_{\mathbb{D}} h\left(\frac{1}{B} \tau_\omega(z)|\phi'(z)|\right) d\lambda_\omega(z) \leq \text{Tr} \left(h(|H_\varphi|)\right) \leq \int_{\mathbb{D}} h\left(B \tau_\omega(z)|\phi'(z)|\right) d\lambda_\omega(z).
$$

Let us denote by $\mathcal{R}_{\phi,\omega}^+$ the decreasing rearrangement of the function $\tau_\omega|\phi'|$ with respect to $d\lambda_\omega$. Namely,

$$
\mathcal{R}_{\phi,\omega}^+(x) := \sup\{t \in (0, \|\tau \phi'\|_\infty) : \mathcal{R}_{\phi,\omega}(t) \geq x\},
$$

where $\mathcal{R}_{\phi,\omega}$ is the distribution function given by

$$
\mathcal{R}_{\phi,\omega}(t) := \lambda_\omega(\{z \in \mathbb{D} : \tau_\omega(z)|\phi'(z)| > t\}).
$$

As a first consequence of Theorem 1.1 we prove that if $\rho$ is an increasing function such that $\rho(x)/x^\gamma$ is decreasing for some $\gamma < 1$, then

$$
s_n(H_\varphi) \asymp 1/\rho(n) \iff \mathcal{R}_{\phi,\omega}^+(n) \asymp 1/\rho(n).
$$

The next result is motivated by a problem raised in [2, 1]. We prove in the following theorem that the critical decay of $(s_n(H_\varphi))^n$ is achieved by the symbol $\phi = z$.

**Theorem 1.2.** Let $\omega \in W^*$ be such that $\tau_\omega$ is equivalent to a radial function and let $\phi \in \mathcal{B}_0^\omega$. Then
Note that, in general, it is not difficult to estimate $\tau_{\omega}^{2}(z) \asymp (1 - |z|^{2})^{2+\beta}$ with $\beta \geq 0$, then $R_{z,\omega}^{+}(n) \asymp n^{-2+\beta}$. Thus, Theorem 1.2 implies that

$$s_{n}(H_{\phi}^{2}) = o\left(n^{-2+\beta}\right) \implies H_{\phi}^{2} = 0.$$ 

Now our goal is to describe the class of functions $\phi \in B_{0}^{\omega}$ such that $(s_{n}(H_{\phi}^{2}))_{n}$ has the critical decay. For simplicity, we state our result only in the case $\tau_{\omega}^{2}(z) \asymp (1 - |z|^{2})^{2+\beta}$. As usual, the Hardy space $H^{p}$, $p \geq 1$, consists of analytic functions $f$ on $\mathbb{D}$ such that

$$\|f\|_{H^{p}}^{p} := \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^{p}d\theta < \infty.$$ 

**Theorem 1.3.** Let $\omega \in W^{*}$ be such that $\tau_{\omega}^{2}(z) \asymp (1 - |z|^{2})^{2+\beta}$ with $\beta \geq 0$. Set $p = \frac{2(1+\beta)}{2+\beta}$ and let $\phi \in B_{0}^{\omega}$. We have

$$s_{n}(H_{\phi}^{2}) = O(1/n^{1/p}) \iff \phi' \in H^{p}.$$ 

Furthermore, if $\omega$ is radial and $s_{n}(H_{\phi}) \sim \gamma/n^{\frac{3}{2}}$ for some $\gamma \in (0, \infty)$, then

$$s_{n}(H_{\phi}^{2}) \sim \|\phi'\|_{H^{p}}\frac{\gamma}{n^{p}}, \quad \phi' \in H^{p}.$$ 

It should be noted that if $\tau_{\omega}^{2}(z) \asymp (1 - |z|^{2})^{2+\beta} \log^{-\alpha}(2/1 - |z|^{2})$ with $\alpha \geq 0$, then $R_{z,\omega}^{+}(n) \asymp \frac{\log^{\alpha/1+\beta}(n)}{n^{1/p}}$. We prove in Theorem 6.2 that if $\beta > 0$, then

$$s_{n}(H_{\phi}^{2}) = O\left(\frac{\log^{\alpha/1+\beta}(n)}{n^{1/p}}\right) \iff \phi' \in H^{p}.$$ 

However, if $\beta = 0$ and $\alpha > 0$, curiously, the previous result is not valid. This is the subject of Proposition 6.3.

For the standard Bergman spaces $A_{\alpha}^{2}$ we have $\tau_{\omega_{\alpha}}^{2}(z) \asymp (1 - |z|^{2})^{2}$. It is known, and can be easily seen, that $s_{n}(H_{\phi}) \sim \frac{\sqrt{1+\alpha}}{n+1}$. Hence, Theorem 1.3 says that if $\phi \in B_{0}$ then

$$s_{n}(H_{\phi}^{2}) = O(1/n) \iff \phi' \in H^{1}.$$ 

And in this case $s_{n}(H_{\phi}^{2}) \sim \frac{\sqrt{1+\alpha}}{n+1} \|\phi'\|_{H^{1}}$. This last result improves the results obtained in [8, 13]. Another example is given in Section 8.

The paper is organized as follows: In Section 2 we introduce all definitions and notations that are used in the rest of the paper. In section 3, we give a description of boundedness and compactness of Hankel operators $H_{\phi}^{2}$ on $A_{\omega}^{2}$. We establish, in Section 4, an upper and a lower estimates of the Trace of $h([H_{\phi}^{2}])$. The upper estimate is obtained from H"{o}rmander type $L_{\alpha}^{2}$ estimates for $\bar{\partial}$- equation and from recent estimates obtained by El-Fallah and El
Ibbaoui for Toeplitz operators \[10, 9\]. The lower estimate is obtained through a sort of local Berezin transform of \( H_{\phi} \). Two direct consequences of trace estimates are obtained by a suitable choice of the convex function \( h \). The first one gives a sharp asymptotic estimates of the singular values of compact operators of \( H_{\phi} \). The second, presented in Section 5, shows that the critical decay of the sequence \( (s_n(H_{\phi}))_n \) is achieved by the symbol \( \phi = z \). In Section 6, we prove the first assertion of Theorem 1.3. The proof is based on Theorem 1.1 and on estimates of some maximal non-tangential functions. The second part of Theorem 1.3 is given in Section 7. The proof of this result is based on the first part of Theorem 1.3 and on a result on asymptotic orthogonality due to A. Pushnitski (see Appendix). Section 8 is devoted to an explicit example.

Throughout the paper, the notation \( A \lesssim B \) means that there is a constant \( c \) independent of the relevant variables such that \( A \leq cB \). We write \( A \asymp B \) if both \( A \lesssim B \) and \( B \lesssim A \).

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2. Preliminaries

2.1. The class of weights \( \mathcal{W}^* \). Throughout this paper, \( \omega \) will denote a function from \( \mathbb{D} \) into \( ]0, \infty[ \) which is integrable with respect to the Lebesgue measure. The associated Bergman space will be denoted by \( A^2_\omega \). We will also assume that \( \omega \) is bounded below by a positive constant on each compact set of \( \mathbb{D} \). This implies that \( A^2_\omega \) is a reproducing kernel Hilbert space. The kernel of \( A^2_\omega \) will be denoted by \( K \).

The orthogonal projection from \( L^2_\omega := L^2(\mathbb{D}, dA_\omega) \) onto \( A^2_\omega \) will be designated by \( P_\omega \). It can be represented as follows

\[
P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)K(z, \zeta)dA_\omega(\zeta), \quad f \in L^2_\omega.
\]

So, the domain of \( P_\omega \) can be extended to all functions \( f \) such that \( fK_z \in L^1_\omega := L^1(\mathbb{D}, dA_\omega) \), for all \( z \in \mathbb{D} \). Let \( g \in L^2_\omega \) such that \( gK_z \in L^2_\omega \), for all \( z \in \mathbb{D} \). The (big) Hankel operator \( H_g \) with symbol \( g \) is the densely defined operator on \( A^2_\omega \) defined by

\[
H_gf = gf - P_\omega(gf),
\]

where \( f = \sum_{1 \leq i \leq n} c_i K_{z_i} \), with \( c_i \in \mathbb{C} \) and \( z_i \in \mathbb{D} \).

The explicit formula for \( H_g \) is

\[
H_g f(z) = \int_{\mathbb{D}} (g(z) - g(w))f(w)K(z, w)dA_\omega(w), \quad z \in \mathbb{D}.
\]
We are interested in this paper in anti-analytic symbols on $\mathbb{D}$, $g = \overline{\phi}$. In this case a direct computation gives the following useful formula
\begin{equation}
(H_{\overline{\phi}}K_a)(z) = (\overline{\phi}(z) - \overline{\phi}(a))K_a(z), \quad z, a \in \mathbb{D}.
\end{equation}

First, we recall the definition of the class of weights $W$ introduced in [11]. Let
\begin{equation}
\tau_\omega(z) = \frac{1}{\omega^{1/2}(z) \|K_z\|_\omega}, \quad z \in \mathbb{D}.
\end{equation}
Suppose that the reproducing kernel $K$ satisfies the following conditions
\begin{equation}
\lim_{|z| \to 1^-} \|K_z\| = \infty \text{ and for every } \zeta \in \mathbb{D}, \quad |K(\zeta, z)| = o(\|K_z\|), \quad |z| \to 1^-.
\end{equation}
We will suppose that $\tau_\omega$ is such that
\begin{equation}
\tau_\omega(z) = O\left(1 - |z|\right), \quad z \in \mathbb{D},
\end{equation}
and that there exists constant $\eta > 0$ such that for $z, \zeta \in \mathbb{D}$ satisfying $|z - \zeta| \leq \eta \tau_\omega(z)$, we have
\begin{equation}
\tau_\omega(z) \asymp \tau_\omega(\zeta) \text{ and } \|K_z\|_\omega \|K_\zeta\|_\omega \lesssim \|K(\zeta, z)\|.
\end{equation}
If the weight $\omega$ satisfies all the previous conditions, we shall say that the weight $\omega$ belongs to the class $W$. Note that $W$ contains all standard weights. For more information, see the examples listed in [11].

The Laplace operator $\Delta$ is given by $\Delta = \partial \overline{\partial}$, with
\begin{equation}
\partial := \frac{1}{2}(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}), \quad \overline{\partial} := \frac{1}{2}(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}).
\end{equation}
Let $\omega = e^{-\varphi} \in W$ such that $\varphi \in C^2(\mathbb{D})$. We shall suppose that
\begin{equation}
\tau_\omega^2(z) \Delta \varphi(z) \gtrsim 1, \quad z \in \mathbb{D},
\end{equation}
or, there exist a subharmonic function $\psi : \mathbb{D} \to \mathbb{R}^+$ and constants $\delta > 0$ and $t \in (-1, 0)$ such that for all $z \in \mathbb{D}$ we have
\begin{equation}
\tau_\omega^2(z) \Delta \psi(z) \geq \delta, \quad \Delta \varphi(z) \geq t \Delta \psi(z) \text{ and } |\partial \psi(z)|^2 \leq \Delta \psi(z).
\end{equation}

**Definition 2.1.** We say that $\omega \in W^\ast$ if $\omega \in W$ and satisfies (3) or (6).

2.1.1. **Examples.** In this subsection, we give three examples which will be our references throughout this paper.

- **Standard Bergman spaces $A_\alpha^2$:** These spaces are associated with $\omega_\alpha(z) = \frac{\alpha + 1}{\alpha} (1 - |z|^2)^\alpha$, with $\alpha > -1$. The reproducing kernel of $A_\alpha^2$ is $K(z, w) = \frac{1}{(1 - wz)^{\alpha + 1}}$. Then
\begin{equation}
\tau_{\omega_\alpha}(z) = \sqrt{\frac{\pi}{\alpha + 1}} (1 - |z|^2).
\end{equation}
Clearly, $\omega_\alpha \in W$. Note that if $\alpha > 0$, then $\tau_{\omega_\alpha}(z) \asymp 1/\sqrt{\Delta \varphi}$, where $\varphi = \log 1/\omega_\alpha$. Then $\omega_\alpha$ satisfies (3) and $\omega_\alpha \in W^\ast$. It is also clear that if $\alpha \in (-1, 0]$, then $\omega_\alpha$ satisfies (6) with $t = \alpha$ and $\psi(z) = \log(1/1 - |z|^2)$. Then $\omega_\alpha \in W^\ast$. 


• Harmonically weighted Bergman spaces: In this case we suppose that \( \omega \) is a positive harmonic weight. It is proved in [12], that \( \tau_\omega(z) \approx 1 - |z|^2 \) and \( \omega \in W \). One can easily see that \( \varphi = \log(1/\omega) \) is subharmonic and that \( \omega \) satisfies (6), with \( t \in (-1, 0) \) and \( \psi(z) = \log(1 - |z|^2) \). It should be noted that in general \( \tau_\omega^2 \) and \( 1/\Delta \varphi \) are not comparable.

• Large Bergman spaces: The following class of weights was introduced by Hu, Lv and Schuster in [16]. It includes the classes considered in [7, 18, 21]. Let \( \mathcal{L}_0 \) be the class of functions \( \tau \in \text{lip}(\mathbb{D}, \mathbb{R}) \) such that \( \lim_{|z| \to 1^-} \tau(z) = 0 \). We denote \( \mathcal{W}_0 \) the set of weights \( \omega = e^{-\varphi} \), where \( \varphi \in C^2 \) is strictly subharmonic and for which there exists \( \tau \in \mathcal{L}_0 \) such that \( \tau \approx 1/\sqrt{\Delta \varphi} \). One can directly see, from [16], that if \( \omega = e^{-\varphi} \in \mathcal{W}_0 \), then \( \omega = e^{-\varphi} \in \mathcal{W}^* \) and \( \tau_\omega \approx 1/\sqrt{\Delta \varphi} \).

2.2. Some inequalities involving convex functions. The following elementary lemma is proved in [9].

**Lemma 2.2.** Let \((a_n)_{n \geq 1}, (b_n)_{n \geq 1}\) be two decreasing sequences such that \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 0 \) and suppose that there exists \( \gamma \in (0, 1) \) such that \( (n^\gamma b_n) \) is increasing. Suppose that there exists \( B > 0 \) such that

\[
\sum_{n \geq 1} h\left(\frac{b_n}{B}\right) \leq \sum_{n \geq 1} h(a_n) \leq \sum_{n \geq 1} h(Bb_n),
\]

for all increasing convex function \( h \). Then \( a_n \approx b_n \).

We also need the following elementary result. For completeness, we include the proof.

**Lemma 2.3.** Let \( A, p \) be two real numbers such that \( 0 < A < p \). Let \( \rho \) be an increasing function such that \( \rho(x)/x^A \) is decreasing and let \((a_n)_{n \geq 1}\) be a decreasing sequence such that

\[
\sum_{n \geq 1} h(a_n) \leq \sum_{n \geq 1} h\left(\frac{1}{\rho(n)}\right),
\]

for all increasing function \( h \) such that \( h(t^p) \) is convex. Then

\[
a_n \leq C(p, A)/\rho(n).
\]

**Proof.** Let \( \delta > 0 \) and consider \( h(t) = (t^{1/p} - \delta^{1/p})^+ \). By hypothesis,

\[
\sum_{n \geq 1} (a_n^{1/p} - \delta^{1/p})^+ \leq \sum_{n \geq 1} (1/\rho^{1/p}(n) - \delta^{1/p})^+.
\]

Then we have

\[
c(p) \sum_{a_n \geq 2\delta} a_n^{1/p} \leq \sum_{\rho(n) \leq 1/\delta} 1/\rho^{1/p}(n).
\]

Using the fact that \( n^{A/p}/\rho^{1/p}(n) \) is increasing and the fact that \( A/p < 1 \), we obtain

\[
\delta^{1/p}\text{Card}\{n : a_n \geq 2\delta\} \leq C(p, A)\delta^{1/p}\text{Card}\{n : \rho(n) \leq 1/\delta\}.
\]

This implies the result. \( \square \)
The following elementary lemma will be useful in the proof of Theorem 1.1.

**Lemma 2.4.** Let $H$, $K$ be two Hilbert spaces and let $T : H \to K$ be a compact operator. Suppose there exist $(u_n)_{n \geq 1} \subset H$, $(v_n)_{n \geq 1} \subset K$ such that $\|u_n\| = \|v_n\| = 1$ and
\[
\sum_n |\langle u_n, u \rangle|^2 \leq C \|u\|^2 \quad \text{and} \quad \sum_n |\langle v_n, v \rangle|^2 \leq C \|v\|^2 \quad (u \in H, v \in K),
\]
for some $C > 0$. Then for any increasing convex function $h$ such that $h(0) = 0$, we have
\[
\sum_n h(|\langle Tu_n, v_n \rangle|) \leq C \sum_n h(s_n(T)).
\]

**Proof.** It suffices to use the spectral decomposition of $T$. \hfill \Box

3. **Boundedness and compactness of Hankel operators on $A^2_\omega$**

We need the following $L^2$-estimates of solutions of the $\bar{\partial}$-equation due to B. Berndtsson (\cite{Berndtsson}, Theorem 3.1).

**Theorem 3.1 (B. Berndtsson).** Let $\Omega$ be a domain in $\mathbb{C}$ and let $\varphi$ and $\psi$ be subharmonic functions in $\Omega$. Assume that $\psi$ satisfies
\[
|\partial \psi(z)|^2 \leq \Delta \psi(z), \quad z \in \Omega. \tag{7}
\]
Let $s \in (0, 1)$. Then, for any function $g$ on $\Omega$, there exists a solution $u$ to the equation $\bar{\partial}u = g$ such that
\[
\int_\Omega |u|^2 e^{-\varphi + s\psi} dA \lesssim \int_\Omega |g|^2 e^{-\varphi + s\psi} dA.
\]

**Lemma 3.2.** Let $\omega \in \mathcal{W}^\ast$. There exists $C > 0$ such that for any function $g$ on $D$, there exists a solution $u$ to the equation $\bar{\partial}u = g$ such that
\[
\int_D |u(z)|^2 \omega(z) dA(z) \leq C \int_D \tau^2(z) |g(z)|^2 \omega(z) dA(z).
\]

**Proof.** Note that if $\omega$ satisfies \cite{Berndtsson}, then the result comes directly from $\bar{\partial}$–Hörmander’s theorem. On the other hand, suppose that $\omega$ satisfies condition \cite{Berndtsson}, that is there exist a subharmonic function $\psi : D \to \mathbb{R}^+$ and constant $t \in (-1, 0)$ such that for all $z \in D$ we have
\[
\frac{1}{\Delta \psi(z)} \lesssim \tau_\omega^2(z), \quad t \Delta \psi(z) \leq \Delta \varphi(z) \quad \text{and} \quad |\partial \psi(z)|^2 \leq \Delta \psi(z).
\]
Then by Theorem 3.1 applied to the couple of subharmonic functions $(\varphi - t\psi, \psi)$ with $s = -t$, there exists a solution $u$ to the equation $\bar{\partial}u = g$ such that
\[
\int_D |u|^2 e^{-\varphi} dA \lesssim \int_D |g|^2 e^{-\varphi} dA.
\]
Then, we get
\[
\int_D |u(z)|^2 \omega(z) dA(z) \lesssim \int_D |g(z)|^2 \tau_\omega^2(z) \omega(z) dA(z).
\]
In the sequel $k_z = \frac{K_z}{\|K_z\|}$ will denote the normalized reproducing kernel of $A^2_\omega$. The following result describes the boundedness of $H_\phi$ on $A^2_\omega$ when $\omega \in W^*$.

**Theorem 3.3.** Let $\omega \in W^*$. Then, the Hankel operator $H_\phi$ is bounded on $A^2_\omega$ if and only if $\phi \in B^2$. In this case $\|H_\phi\| \asymp \sup_{z \in \mathbb{D}} \tau_\omega(z)|\phi'(z)|$, where the implied constants depend only on $\omega$.

**Proof.** Suppose that $H_\phi$ is bounded on $A^2_\omega$. Fix $a$ in $\mathbb{D}$ and let $\delta \leq \eta$ ($\eta$ is the constant which appears in (4)). By the formula (1), we have

$$
\int_{\mathbb{D}} |H_\phi k_a(z)|^2 \omega(z) dA(z) = \int_{\mathbb{D}} |\phi(z) - \phi(a)|^2 |k_a(z)|^2 \omega(z) dA(z)
\geq \int_{D(a,\delta \tau_\omega(a))} |\phi(z) - \phi(a)|^2 |k_a(z)|^2 \omega(z) dA(z).
$$

Using the fact that $|k_a(z)|^2 \asymp K(z, z)$ and the fact that $\tau_\omega(z) \asymp \tau_\omega(a)$ when $z \in D(a, \delta \tau_\omega(a))$, we have

$$
\int_{D(a,\delta \tau_\omega(a))} |\phi(z) - \phi(a)|^2 \|k_z\|^2 \omega(z) dA(z) \asymp \frac{1}{\tau_\omega^2(a)} \int_{D(a,\delta \tau_\omega(a))} |\phi(z) - \phi(a)|^2 dA(z).
$$

Then we obtain

$$
\int_{\mathbb{D}} |H_\phi k_a(z)|^2 \omega(z) dA(z) \gtrsim \frac{1}{\tau_\omega^2(a)} \int_{D(a,\delta \tau_\omega(a))} |\phi(z) - \phi(a)|^2 dA(z).
$$

By Cauchy’s representation formula, we get

$$
\tau_\omega(a)|\phi'(a)| \lesssim \frac{1}{\tau_\omega^2(a)} \int_{D(a,\delta \tau_\omega(a))} |\phi(z) - \phi(a)| dA(z).
$$

It follows that

$$
\tau_\omega^2(a)|\phi'(a)|^2 \lesssim \frac{1}{\tau_\omega^2(a)} \int_{D(a,\delta \tau_\omega(a))} |\phi(z) - \phi(a)|^2 dA(z)
\lesssim \int_{\mathbb{D}} |H_\phi k_a(z)|^2 \omega(z) dA(z)
= \|H_\phi k_a\|^2
\leq \|H_\phi\|^2.
$$

Hence, $\phi \in B^2$.

Suppose now that $\sup_{z \in \mathbb{D}} \tau_\omega(z)|\phi'(z)| < \infty$ and let $f \in A^2_\omega$. By Lemma 3.2 there exists a solution $u$ to the equation

$$
\overline{\partial} u = \overline{\phi} f
$$

satisfying

$$
\int_{\mathbb{D}} |u(z)|^2 \omega(z) dA(z) \lesssim \int_{\mathbb{D}} \tau_\omega^2(z)|\phi'(z)|^2 |f(z)|^2 \omega(z) dA(z).
$$
Therefore, since $H_\phi f$ is the $L^2_\omega$—minimal solution to the equation (8), we have
\[
\int_B |H_\phi f(z)|^2 \omega(z) dA(z) \lesssim \int_B \tau_\omega(z) |\phi'(z)|^2 |f(z)|^2 \omega(z) dA(z). \tag{9}
\]
Hence
\[
\|H_\phi\| \lesssim \sup_{z \in \Omega} \tau_\omega(z) |\phi'(z)| < \infty.
\]

Note that from equation (9), if $(f_n)_n$ is an orthonormal basis of $A^2_\omega$ then
\[
\sum_n \|H_\phi f_n\|^2 \lesssim \sum_n \int_B |f_n(z)|^2 \tau_\omega^2(z) |\phi'(z)|^2 \omega(z) dA(z)
\leq \int_B \|K_z\|^2 \tau_\omega^2(z) |\phi'(z)|^2 \omega(z) dA(z)
= \int_B |\phi'(z)|^2 dA(z).
\]
Consequently, if $\phi \in B_2$ then $H_\phi \in S_2$. We will see in the next paragraph that the converse is also true.

We have the following description of the compactness of Hankel operators.

**Theorem 3.4.** Let $\omega \in W^*$. Then, the Hankel operator $H_\phi$ is compact on $A^2_\omega$ if and only if $\phi \in B^*_0$.

**Proof.** Suppose that $H_\phi$ is compact. Since $\omega \in W^*$, $k_z = \frac{K_z}{\|K_z\|}$ converges weakly to 0 when $|z| \to 1^-$. Then $\lim_{|z| \to 1^-} \|H_\phi k_z\| = 0$. By the proof of Theorem 3.3 we have
\[
\tau(z) |\phi'(z)| \lesssim \|H_\phi k_z\|.
\]
This implies that $\phi \in B^*_0$.

Conversely, let $\phi \in B^*_0$ and let $\phi_r(z) = \phi(rz)$. Clearly, $\phi_r$ converges to $\phi$ in $B^\omega$ as $r \to 1^-$. Then by Theorem 3.3 $H_{\phi_r}$ converges to $H_\phi$. Now since $\phi_r \in B_2$, $H_{\phi_r}$ is compact. Hence, $H_\phi$ is compact. \qed

4. **Trace estimates for Hankel operators.**

This section is devoted to the proof of Theorem 3.1. Before starting the proof, we recall the following covering lemma.

**Lemma 4.1.** (III) Let $X$ be a subset of $\mathbb{C}$ and let $\tau : X \to (0, \infty)$ be a bounded function. Suppose that there are two constants $\gamma, C > 0$ such that, for every $z, w \in X$ with $|z - w| < \gamma \tau(z)$, we have $\frac{\tau(z)}{C} \leq \tau(w) \leq C \tau(z)$. Set $B = C + 1$ and let $\delta \leq \gamma / B$. There exists a sequence $(z_n)_{n \geq 1} \subset X$ such that
\begin{enumerate}
  \item $X \subset \bigcup_{n > 1} D(z_n, \delta \tau(z_n))$.
  \item $D(z_n, \delta \tau(z_n)/2C) \cap D(z_m, \delta \tau(z_m)/2C) = \emptyset$ for $n \neq m$.
  \item For $z \in D(z_n, \delta \tau(z_n))$ we have $D(z, \delta \tau(z)) \subset D(z_n, B \delta \tau(z_n))$.
\end{enumerate}
The following lemma will be used in the proof of the lower estimate of $\text{Tr}$.

Such sequences will be called a $(\tau, \delta)$–lattice of $X$.

We say that $(R_n)_n \in \mathcal{L}_\omega$ if $(R_n)_n = (D(z_n, \delta \tau(z_n)))_n$ and satisfies the conditions $(1)-(4)$ of the above lemma. In the sequel we fix $(R_n)_n = (D(z_n, \delta \tau(z_n)))_n \in \mathcal{L}_\omega$ and $b > 1$ such that $(bR_n)_n =: (D(z_n, b\delta \tau(z_n)))_n$ is a covering of $\mathbb{D}$ of finite multiplicity.

For a positive Borel measure $\mu$ on $\mathbb{D}$, the Toeplitz operator associated with $\mu$ and defined on $A^2_\omega$ is given by

$$T_\mu f(z) = \int_{\mathbb{D}} f(w) K(z, w) \omega(w) d\mu(w), \quad z \in \mathbb{D}.$$ 

It is easy to verify that $T_\mu$ satisfies the following remarkable formula

$$\langle T_\mu f, f \rangle = \int_{\mathbb{D}} |f(z)|^2 \omega(z) d\mu(z).$$

For more properties of Toeplitz operators we refer to [19, 9, 10, 11].

The following lemma will be used in the proof of the lower estimate of $\text{Tr} h(|H_\omega^\tau|)$.

**Lemma 4.2.** Let $\omega \in \mathcal{W}^*$ and let $\phi \in B^{\omega}$. For any increasing convex function $h$ such that $h(0) = 0$, we have

$$\sum_n h(\|\chi_{bR_n} H_{\omega(\phi) k_{z_n}}\|) \lesssim \sum_n h(s_n(H_{\omega(\phi)}),$$

where the implied constant depends only on $(bR_n)_n$ and $\omega$.

**Proof.** Let $u_n = k_{z_n}$, $v_n = \frac{\chi_{bR_n} H_{\omega(\phi) k_{z_n}}}{\|\chi_{bR_n} H_{\omega(\phi) k_{z_n}}\|}$ and remark that $\|\chi_{bR_n} H_{\omega(\phi) k_{z_n}}\| = \langle H_{\omega(\phi)} u_n, v_n \rangle$. So, it suffices to prove that the conditions of Lemma 2.4 are satisfied. Indeed, let $f \in A^2_\omega$ and let $d\mu = \sum_n \frac{1}{\|K_{z_n}\|^2 \omega(z_n)} d\delta_{z_n}$. Since $\omega \in \mathcal{W}$, $T_\mu$ is bounded [11]. Then we have

$$\sum_n \langle u_n, f \rangle^2 = \sum_n \frac{|f(z_n)|^2}{\|K_{z_n}\|^2} = \langle T_\mu f, f \rangle \leq \|T_\mu\| \|f\|^2.$$

Now let $g \in L^2(\omega dA)$. By Holder inequality, we have

$$\sum_n |\langle v_n, g \rangle|^2 = \sum_n |\langle \frac{\chi_{bR_n} H_{\omega(\phi) k_{z_n}}}{\|\chi_{bR_n} H_{\omega(\phi) k_{z_n}}\|}, \chi_{bR_n} g \rangle|^2 \leq \sum_n \|\chi_{bR_n} g\|^2 \lesssim \|g\|^2.$$

This completes the proof.\hfill \Box

**Proof of Theorem 1.7** First, we prove that $\sum_n h(s_n(H_{\omega(\phi)})) \leq \int_{\mathbb{D}} h(C|\phi'(z)| \tau_\omega(z)) \ d\lambda_\omega(z)$. The equation (9) implies that

$$H_{\omega(\phi)} H_{\omega(\phi)} \lesssim T_{\mu_\phi}.$$ (10)

Then, by the monotonicity Weyl’s lemma, we have

$$s_n^2(H_{\omega(\phi)}) = \lambda_n(H_{\omega(\phi)} H_{\omega(\phi)}) \lesssim \lambda_n(T_{\mu_\phi}).$$ (11)
Let $\tilde{h}(t) = h(\sqrt{t})$. By Theorem 4.5 of [9], we have
\[
\sum h(s_n(H_{\sigma})) = \sum \tilde{h}(s_n^2(H_{\sigma})) \\
\leq \sum \tilde{h}(C\lambda_n(T_{\mu_\phi})) \\
\leq \sum \tilde{h}(C\mu_\phi(R_n)/A(R_n)) \\
= \sum \tilde{h} \left( C \frac{1}{A(R_n)} \int_{R_n} \tau^2_\sigma(z)|\phi'(z)|^2dA(z) \right) \\
\leq \sum \tilde{h} \left( C \int_{R_n} |\phi'(z)|^2dA(z) \right).
\]

By subharmonicity, we have for all $z \in \mathbb{R}^n$
\[
|\phi'(z)| \lesssim \frac{1}{A(R_n)} \int_{bR_n} |\phi'(\zeta)|dA(\zeta).
\]

Then,
\[
\int_{R_n} |\phi'(z)|^2dA(z) \lesssim \frac{1}{A(R_n)} \left( \int_{bR_n} |\phi'(\zeta)|dA(\zeta) \right)^2 \sim \left( \int_{bR_n} |\phi'(\zeta)|\tau_\omega(\zeta)d\lambda_\omega(\zeta) \right)^2.
\]

Since $h$ is convex we obtain
\[
\tilde{h} \left( C \int_{R_n} |\phi'(z)|^2dA(z) \right) \leq \tilde{h} \left( C \left( \int_{bR_n} |\phi'(\zeta)|\tau_\omega(\zeta)d\lambda_\omega(\zeta) \right)^2 \right) \\
\leq h \left( C \int_{bR_n} |\phi'(\zeta)|\tau_\omega(\zeta)d\lambda_\omega(\zeta) \right) \\
\leq \int_{bR_n} h(C|\phi'(\zeta)|\tau_\omega(\zeta))d\lambda_\omega(\zeta).
\]

Combining these inequalities and the fact that $(bR_n)_n$ is of finite multiplicity, we get
\[
\sum h(s_n(H_{\sigma})) \leq \int_{\mathbb{R}^n} h(C|\phi'(z)|\tau_\omega(z))d\lambda_\omega(z).
\]

Now we prove the lower inequality by using Lemma 4.2. We have
\[
\|\chi_{bR_n}H_{\sigma}\kappa_{z_n}\| \asymp \left( \int_{bR_n} |\phi(z) - \phi(z_n)|^2||K_z||^2\omega(z)dA(z) \right)^{1/2} \\
\asymp \left( \int_{bR_n} |\phi(z) - \phi(z_n)|^2d\lambda_\omega(z) \right)^{1/2}.
\]

By Cauchy’s formula we have
\[
\tau_\omega(z)|\phi'(z)| \lesssim \left( \int_{bR_n} |\phi(\zeta) - \phi(z_n)|^2d\lambda_\omega(\zeta) \right)^{1/2} \asymp \|\chi_{bR_n}H_{\sigma}\kappa_{z_n}\|, \ z \in R_n.
\]
Then
\[
\int_{\mathbb{D}} h(\tau_\omega(z)|\phi'(z)|) d\lambda_\omega(z) \gtrsim \sum_n \int_{R_n} h(\tau_\omega(z)|\phi'(z)|) d\lambda_\omega(z) \leq \sum_n h(C\|\chi_{bR_n} H_{\gamma}^{k_{zn}}\|).
\]
By Lemma 4.2, we obtain the desired result.

Let us denote by
\[
B^{\omega,p} = \{ \phi \in B^\omega : \int_{\mathbb{D}} (|\phi'(z)|\tau_\omega(z))^p d\lambda_\omega(z) < \infty \} \quad (p > 0).
\]
Note that 
\[
B^{\omega,2} = B_2
\]
is the classical Dirichlet space and it doesn’t depend on the weight \(\omega\). Note that since \(\tau_\omega(z) = O((1 - |z|^2))\), \(B^{\omega,p} = \{0\}\) whenever \(p \leq 1\). Using standard arguments, one can easily prove that an analytic function \(f\) on \(\mathbb{D}\) belongs to \(B^{\omega,p}\) if and only if
\[
\sum_n \left( \frac{\mu(R_n)}{A(R_n)} \right)^p < \infty, \text{ where } d\mu(z) = \tau_\omega(z)|f'(z)| dA(z) \text{ and } (R_n)_n \in L_\omega.
\]
In particular, this implies that the family \((B^{\omega,p})_p\) is increasing.

The following result, which extends the main results in [2, 15], is a direct consequence of Theorem 1.1.

**Corollary 4.3.** Let \(p \geq 1\) and let \(\omega \in W^*\). Let \(\phi \in B^{\omega}_0\). Then
\[
H_{\tilde{\phi}} \in S_p(A^2_\omega) \iff \phi \in B^{\omega,p}.
\]

As a consequence of Theorem 4.1, we give some estimates of the singular values of compact Hankel operators. To this end, let us recall that \(R_{\phi,\omega}(t) := \lambda_\omega(\{ z \in \mathbb{D} : \tau_\omega(z)|\phi'(z)| > t \})\). Let \(R_{\phi,\omega}^+\) be the increasing rearrangement of the function \(\tau_\omega|\phi'|\). For any increasing function \(h\), by a standard computation, we have
\[
\int_{\mathbb{D}} h(\tau_\omega(z)|\phi'(z)|) d\lambda_\omega(z) = \int_0^\infty R_{\phi,\omega}(t) dh(t).
\]
Then, there exists \(B > 0\) which depends only on \(\omega\) such that
\[
\sum_{n \geq 0} h\left( \frac{1}{B} R_{\phi,\omega}^+(n) \right) \leq \int_{\mathbb{D}} h(\tau_\omega(z)|\phi'(z)|) d\lambda_\omega(z) \leq \sum_{n \geq 0} h\left( B R_{\phi,\omega}^+(n) \right). \quad (12)
\]
As a consequence of Theorem 4.1 we obtain the following result.

**Theorem 4.4.** Let \(\omega \in W^*\) and let \(\phi \in B^\omega_0\). Let \(\rho\) be an increasing function such \(\rho(x)/x^\gamma\) is decreasing for some \(\gamma \in (0, 1)\). Then
\[
s_n(H_{\tilde{\phi}}) \asymp 1/\rho(n) \iff R_{\phi,\omega}^+(n) \asymp 1/\rho(n).
\]

**Proof.** By Theorem 4.1 and inequalities (12), there exists \(B > 0\) such that for every increasing convex function \(h\) we have
\[
\sum_{n \geq 0} h\left( \frac{1}{B} R_{\phi,\omega}^+(n) \right) \leq \sum_{n \geq 0} h(s_n(H_{\tilde{\phi}})) \leq \sum_{n \geq 0} h\left( B R_{\phi,\omega}^+(n) \right).
\]
Let $b > \sum B > (11)$ it suffices to prove that $\lambda \rho$. We obtain from the convexity of $h$

First, recall that $\lambda \rho$. Proof. (12).

By Lemma 2.2, we obtain the desired result.

Then

$$s_n(H_{\phi}) = O(1/\rho(n)), \quad n \to \infty.$$ 

We cite as examples, radial weights $\omega \phi \in W_0$ such that $\Delta \varphi$ is equivalent to a radial function.

5. The cut-off

In this section we consider weights $\omega \in W^*$ such that $\tau_\omega$ is equivalent to a radial function. We cite as examples, radial weights $\omega \in W^*$, positive harmonic weights and weights $\omega = e^{-\varphi} \in W_0$ such that $\Delta \varphi$ is equivalent to a radial function.

Proof of Theorem 4.5. Suppose that $s_n(H_{\phi}) = o(s_n(H_{\phi})).$ Let $\delta \in (0, 1)$ and let $h_\delta(t) = (t - \delta)^+$. By Theorem 3.2 we have

$$\int_{\mathbb{D}} \frac{1}{B} \tau_\omega(z)|\phi'(z)| \, d\lambda_\omega(z) \leq \sum_{n \geq 1} h_\delta(s_n(H_{\phi})).$$
Let $\rho \in (1/2, 1)$ and put $K = \int_0^{2\pi} |\phi'(pe^{it})| \frac{dt}{2\pi}$. By Jensen’s inequality we have
\[
\int h_\delta \left( \frac{1}{B} \tau_\omega(z)|\phi'(z)| \right) d\lambda_\omega(z) \geq \frac{K}{2B} \int \sum_{n \geq 1} h_\delta(s_n(H_{\overline{\phi}})) + \sum_{n \geq 1} h_\delta(\varepsilon s_n(H_{\overline{\phi}})) d\lambda_\omega(z)
\]
\[
\leq \sum_{n < N} h_\delta(s_n(H_{\overline{\phi}})) + \sum_{n \geq 1} h_\delta(\varepsilon s_n(H_{\overline{\phi}}))
\]
\[
\leq N\|H_{\overline{\phi}}\| + \int h_\delta(\varepsilon B \tau_\omega(z)) d\lambda_\omega(z)
\]
\[
\leq N\|H_{\overline{\phi}}\| + 2\varepsilon B \int \sum_{n < N} h_\delta(s_n(H_{\overline{\phi}})) + \sum_{n \geq 1} h_\delta(\varepsilon s_n(H_{\overline{\phi}})) d\lambda_\omega(z)
\]
\[
= \int h_\delta \left( \frac{1}{B} \tau_\omega(z)|\phi'(z)| \right) d\lambda_\omega(z) - \delta + \frac{dr}{\tau_\omega^2(r)}.
\]

Suppose that $\phi' \neq 0$, then $K > 0$. For $\tau_\omega(r) \geq \frac{2B}{K}$, we have $(\frac{K}{B} \tau_\omega(r) - \delta)^+ \geq \frac{K}{2B} \tau_\omega(r)$. Then we obtain
\[
\int h_\delta \left( \frac{1}{B} \tau_\omega(z)|\phi'(z)| \right) d\lambda_\omega(z) \geq \frac{K}{2B} \int \frac{dr}{\tau_\omega^2(r)}.
\]

Let $\varepsilon \in (0, K/2B^2)$ and let $N$ be such that for $n \geq N$ we have $s_n(H_{\overline{\phi}}) \leq \varepsilon s_n(H_{\overline{\phi}})$. Using Theorem 1.1, we have
\[
\sum_{n \geq 1} h_\delta(s_n(H_{\overline{\phi}})) \leq \sum_{n < N} h_\delta(s_n(H_{\overline{\phi}})) + \sum_{n \geq 1} h_\delta(\varepsilon s_n(H_{\overline{\phi}}))
\]
\[
\leq N\|H_{\overline{\phi}}\| + \int h_\delta(\varepsilon B \tau_\omega(z)) d\lambda_\omega(z)
\]
\[
\leq N\|H_{\overline{\phi}}\| + 2\varepsilon B \int \sum_{n < N} h_\delta(s_n(H_{\overline{\phi}})) + \sum_{n \geq 1} h_\delta(\varepsilon s_n(H_{\overline{\phi}})) d\lambda_\omega(z)
\]
\[
= \int h_\delta \left( \frac{1}{B} \tau_\omega(z)|\phi'(z)| \right) d\lambda_\omega(z) - \delta + \frac{dr}{\tau_\omega^2(r)}.
\]

Since $1/\varepsilon B > 2B/K$, we obtain
\[
\sum_{n \geq 1} h_\delta(s_n(H_{\overline{\phi}})) \leq N\|H_{\overline{\phi}}\| + 2\varepsilon B \int \frac{dr}{\tau_\omega^2(r)}.
\]

Combining inequalities (13) and (14), we obtain
\[
\frac{K}{2B} \int \frac{dr}{\tau_\omega(r)} \leq N\|H_{\overline{\phi}}\| + 2\varepsilon B \int \frac{dr}{\tau_\omega(r)}.
\]

Since $\int \frac{dr}{\tau_\omega(r)} = \infty$, when $\delta$ goes to 0, we obtain $\frac{K}{2B} \leq 2\varepsilon B$. This gives a contradiction. The second assertion is obtained by using the same argument.

\begin{corollary}
Let $\omega \in \mathcal{W}^*$ be such that $\tau_\omega^2(z) \asymp (1 - |z|^2)^{2+\beta} \nu^2(\log(\frac{1}{1-|z|^2}))$, where $\beta \geq 0$ and $\nu$ is a monotone function such that $\nu(2t) \asymp \nu(t)$. Let $\phi \in \mathcal{B}_0$ and let $p = \frac{2(1+\beta)}{2+\beta}$. If
\[
s_n(H_{\overline{\phi}}) = o \left( \frac{1}{n^{p} \nu^{1+\beta}(\log n)} \right)
\]
then $\phi' = 0$.
\end{corollary}
Proof. It is not difficult to verify that 
\[ \int_{\delta}^{1} \frac{dt}{t^{2+\beta}\nu^{2}(\log 1/t)} \preceq \frac{1}{\delta^{1+\beta}\nu^{2}(\log 1/\delta)}. \]
Then, since \( \tau_{\omega}^{2}(z) \asymp (1 - |z|^{2})^{2+\beta}\nu^{2}(\log(1 - |z|^{2})) \), we have
\[ R_{z,\omega}(t) = \int_{\{z \in \mathbb{D}: \; \tau_{\omega}(z) \geq t\}} \frac{dA(z)}{(1 - |z|^{2})^{2+\beta}\nu^{2}(\log 1/(1 - |z|^{2}))} \preceq \frac{1}{t^{2(1+\beta)\nu^{2}(\log 1/t)}}. \]
This implies that \( R_{z,\omega}^{+}(n) \asymp \frac{1}{n^{p}\nu^{1+\beta}(\log n)} \). By the second assertion of Theorem 1.2, we get \( \phi' = 0 \). □

Note that if \( \omega \) is radial then
\[ H_{\omega}^{*}H_{\omega}\left(\frac{z^{n}}{\|z^{n}\|}\right) = \frac{(\|z^{n+1}\|^{2} - \|z^{n}\|^{2})\|z^{n}\|^{2}}{(\|z^{n-1}\|^{2})\|z^{n}\|} =: m_{\omega}^{2}(n)\frac{z^{n}}{\|z^{n}\|}, \quad n \geq 1. \]
So, the sequence of the singular values of \( H_{\omega} \) is the decreasing rearrangement of the sequence \( (m_{\omega}(n))_{n \geq 1} \).

For the standard Bergman spaces \( A_{\alpha}^{2} \), it is easy to see that
\[ \|z^{n}\|^{2} = \frac{\Gamma(n+1)\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)} \quad \text{and} \quad m_{\omega_{n}}^{2}(n) = \frac{\alpha + 1}{(n + \alpha + 1)(n + \alpha + 2)}. \]
where \( \Gamma(x) = \int_{0}^{\infty} t^{x-1}e^{-t}dt \) denotes the Gamma function.

Then
\[ s_{n}(H_{\omega}) \sim \frac{\sqrt{\alpha + 1}}{n + 1}. \]
For larger Bergman spaces we have the following result.

**Proposition 5.2.** Let \( \omega \in \mathcal{W}^{*} \) and let \( \beta > 0 \). Suppose that \( \tau_{\omega}^{2}(z) \asymp (1 - |z|^{2})^{2+\beta}\nu^{2}(\log(1 - |z|^{2})) \), where \( \nu \) is a monotone function which satisfies \( \nu(2t) \asymp \nu(t) \). Then
\[ s_{n}(H_{\omega}) \asymp \frac{1}{n^{p}\nu^{1+\beta}(\log n)}, \quad \text{where} \quad p = \frac{2(1 + \beta)}{2 + \beta}. \]

**Proof.** From the hypothesis, we have \( R_{z,\omega}^{+}(n) \asymp \frac{1}{n^{p}\nu^{1+\beta}(\log n)} \). By Theorem 4.4 we obtain the result. □
6. Critical decay

In this section we describe the class of functions $\phi \in \mathcal{B}_0^\omega$ such that $s_n(H_\phi) = O(s_n(H_z))$. This kind of problem was first examined by Arazy, Fisher and Peetre for standard Bergman spaces [2]. They proved that if $\phi$ belongs to the Besov space $B_1 : = \{ f \in \text{Hol}(\mathbb{D}), \int_\mathbb{D} |f''(z)|dA(z) < \infty \}$, then

$$s_n(H_\phi) = O(\log(n+1)), \text{ where } s_n(H_\phi) := \sum_{j=1}^n s_j(H_\phi).$$

They also proved, in the same paper, that the converse is false. In [13], M. Engliš and R. Rochberg gave a complete answer to this problem for the classical Bergman space. They proved, by using Boutet de Monvel-Guillemin theory, that $s_n(H_\phi) = O(1/n)$ if and only if $\phi' \in H^1$. This result was extended by R. Tytgat [27, 28] to standard Bergman spaces $A_2^\alpha$.

In what follows we study this problem in more general situations. Our approach is based on Theorem 1.1.

**Proposition 6.1.** Let $\omega \in \mathcal{W}^*$ be a weight such that $\tau_\omega^2(z) \asymp (1-|z|^2)^{2+\beta}\nu^2(\log(1-|z|^2))$ where $\beta \geq 0$ and $\nu$ is a monotone function such that $\nu(2t) \asymp \nu(t)$. Let $\phi \in \mathcal{B}_0^\omega$ and let $p = \frac{2(1+\beta)}{2+\beta}$. Then

$$s_n(H_\phi) = O\left(\frac{1}{n^p\nu^{1+\beta}(\log(n))}\right) \implies \phi' \in H^p.$$  

**Proof.** Suppose that $s_n(H_\phi) \leq \frac{C}{n^p\nu^{1+\beta}(\log(n))}$. Since $\tau_\omega^2(z) \asymp (1-|z|^2)^{2+\beta}\nu^2(\log(1-|z|^2))$, we have

$$\mathcal{R}_{z,\omega}(t) \asymp \frac{1}{t^p\nu^{2+\beta}(\log(1/t))} \quad \text{and} \quad \mathcal{R}_{z,\omega}^+(n) \asymp \frac{1}{n^p\nu^{1+\beta}(\log(n))}.$$  

Then for every increasing function $h$, we have

$$\sum_{n=1}^\infty h\left(\frac{1}{B_1n^p\nu^{1+\beta}(\log(n))}\right) \leq \int_\mathbb{D} h(\tau_\omega(z))d\lambda_\omega(z) \leq \sum_{n=1}^\infty h\left(\frac{B_1}{n^p\nu^{1+\beta}(\log(n))}\right).$$  

where $B_1$ doesn’t depend on $h$. If in addition $h$ is convex and $h(0) = 0$, then by Theorem 1.1 we obtain

$$\int_\mathbb{D} h(\tau_\omega(z)|\phi'(z)|)d\lambda_\omega(z) \leq \sum_{n=1}^\infty h\left(Bs_n(H_\phi)\right) \leq \sum_{n=1}^\infty h\left(BC\mathcal{R}_{z,\omega}^+(n)\right) \leq \int_\mathbb{D} h(B_1BC\tau_\omega(z))d\lambda_\omega(z).$$
Let \( \varepsilon \in (0,1) \) and put \( p_\varepsilon = (1-\varepsilon)p + \varepsilon \). Note that if \( \beta = 0 \) (i.e. \( p = 1 \)) then \( p_\varepsilon = 1 \) and if \( \beta > 0 \) then \( 1 < p_\varepsilon < p \). Note that in the two cases we have

\[
\int_1^\infty \frac{dr}{r^{2-p_\varepsilon}(r)} = \infty.
\]

The last inequality, with \( h(t) = h_\delta(t^{p_\varepsilon}) \), becomes

\[
\int_D h_\delta \left( \tau^{p_\varepsilon}_\omega(z) \phi'(z)^{p_\varepsilon} \right) d\lambda_\omega(z) \lesssim \int_D h_\delta \left( K^{p_\varepsilon} \tau^{p_\varepsilon}_\omega(z) \right) d\lambda_\omega(z), \quad \text{where } K = CBB_1.
\]

Using the convexity of \( h_\delta \), we get

\[
\int_0^1 h_\delta \left( \tau^{p_\varepsilon}_\omega(r) \right) \left( \int_0^{2\pi} \phi'(re^{it}) \frac{dt}{2\pi} \right) \frac{dr}{\tau^{2-p_\varepsilon}_\omega(r)} \leq \int_D h_\delta \left( \tau^{p_\varepsilon}_\omega(z) \phi'(z)^{p_\varepsilon} \right) d\lambda_\omega(z) \lesssim \int_D h_\delta \left( K^{p_\varepsilon} \tau^{p_\varepsilon}_\omega(z) \right) d\lambda_\omega(z).
\]

Suppose that there exists \( \rho \in (0,1) \) such that \( \|\phi'_\rho\|_{p_\varepsilon} \geq 2K \). We have

\[
\int_\rho^1 h_\delta \left( \tau^{p_\varepsilon}_\omega(r) \|\phi'_\rho\|_{p_\varepsilon} \right) \frac{dr}{\tau^{2-p_\varepsilon}_\omega(r)} \lesssim \int_D h_\delta \left( K^{p_\varepsilon} \tau^{p_\varepsilon}_\omega(z) \right) d\lambda_\omega(z).
\]

Now, using the fact that \( h_\delta(t) \geq t/2 \) if \( t \geq 2\delta \), we get

\[
\|\phi'_\rho\|_{p_\varepsilon} \int_{\{r \in (\rho,1): \tau^{p_\varepsilon}_\omega(r) \|\phi'_\rho\|_{p_\varepsilon} \geq 2\delta\}} \frac{dr}{\tau^{2-p_\varepsilon}_\omega(r)} \lesssim K^{p_\varepsilon} \int_{\{r \in (0,1): \tau^{p_\varepsilon}_\omega(r) K^{p_\varepsilon} \geq \delta\}} \frac{dr}{\tau^{2-p_\varepsilon}_\omega(r)}.
\]

Since \( \|\phi'_\rho\|_{p_\varepsilon} \geq 2K \), \( \{r \in (\rho,1): \tau^{p_\varepsilon}_\omega(r) K^{p_\varepsilon} \geq \delta\} \subset \{r \in (0,1): \tau^{p_\varepsilon}_\omega(r) \|\phi'_\rho\|_{p_\varepsilon} \geq 2\delta\} \). Then

\[
\|\phi'_\rho\|_{p_\varepsilon} \int_{\{r \in (\rho,1): \tau^{p_\varepsilon}_\omega(r) K^{p_\varepsilon} \geq \delta\}} \frac{dr}{\tau^{2-p_\varepsilon}_\omega(r)} \lesssim K^{p_\varepsilon} \int_{\{r \in (0,1): \tau^{p_\varepsilon}_\omega(r) K^{p_\varepsilon} \geq \delta\}} \frac{dr}{\tau^{2-p_\varepsilon}_\omega(r)}.
\]

Recall that \( \int_1^\infty \frac{dr}{r^{2-p_\varepsilon}(r)} = \infty \). We obtain, when \( \delta \) goes to \( 0^+ \), that \( \|\phi'_\rho\|_{p_\varepsilon} \lesssim K \). When \( \varepsilon \) goes to 0, we get \( \|\phi'_\rho\|_{p} \lesssim K \). This proves that \( \phi' \in H^p \).

**Proof of the first assertion of Theorem 1.3** Let \( p = p_\beta := \frac{2(1+\beta)}{2+\beta} \). Suppose that \( s_n(H^{-p}_\phi) = O(1/n^{1/p}) \). By Proposition 6.1, with \( \nu = 1 \), we have \( \phi' \in H^p \).

For the converse, by Theorem 4.5 it suffices to prove that \( \mathcal{R}^+_\phi(z) = O(1/z^{1/p}) \), whenever \( \phi' \in H^p \). To this end, let \( U \) be the non-tangential maximal function of \( \phi' \). Since \( \phi' \in H^p \)
we have $U \in L^p$ and $\|U\|_p \lesssim \|\phi'\|_p$ (even if $p = 1$). We have

$$R_{\phi, \omega}(t) = \lambda_{\omega}(\{z \in \mathbb{D} : \tau_{\omega}(z)|\phi'(z)| \geq t\})$$

$$\leq \lambda_{\omega}(\{re^{i\theta} \in \mathbb{D} : \tau_{\omega}(r)U(e^{i\theta}) \geq t\})$$

$$\lesssim \int_{\{re^{i\theta} \in \mathbb{D} : \tau_{\omega}(r)U(e^{i\theta}) \geq t\}} \frac{drd\theta}{\pi(1 - r^2)^{2+\beta}}$$

$$\lesssim \frac{1}{t^p} \int_0^{2\pi} U^p(e^{i\theta}) \frac{d\theta}{2\pi}$$

$$\lesssim \frac{1}{t^p} \|\phi'\|_{p'}^p.$$ 

This is equivalent to $R_{\phi, \omega}^+(x) \lesssim \|\phi'\|_{x^{1/p'}}$. The proof is complete.

Now, we study the converse of Proposition 6.1, when $\tau_{\omega}(z) \asymp (1 - |z|^2)^{2+\beta}\nu^2(\log(\frac{1}{1-|z|^2}))$. In the following result we consider the case $\beta > 0$. The case $\beta = 0$, will be discussed right after.

**Theorem 6.2.** Let $\omega \in W^*$ be a weight such that $\tau_{\omega}^2(z) \asymp (1 - |z|^2)^{2+\beta}\nu^2(\log(\frac{1}{1-|z|^2}))$, where $\beta > 0$ and $\nu$ is a monotone function such that $\nu(2t) \asymp \nu(t)$. Let $\phi \in B_{\omega}^0$ and let $p = \frac{2(1+\beta)}{2+\beta}$. Then

$$s_n(H_{\phi}) = O\left(\frac{1}{n^{\frac{1}{p}\nu^{-1/\beta}(\log n)}}\right) \iff \phi' \in H^p.$$ 

In this case we have

$$s_n(H_{\phi}) \lesssim \frac{1}{n^{\frac{1}{p}\nu^{-1/\beta}(\log n)}} \|\phi'\|_p,$$

where the involved constant doesn’t depend on $\phi$.

**Proof.** By Proposition 6.1, it remains to prove that if $\phi' \in H^p$ then $s_n(H_{\phi}) = O\left(\frac{1}{n^{\frac{1}{p}\nu^{-1/\beta}(\log n)}}\right)$. We will proceed as in the proof of Theorem 1.3. Without loss of generality, suppose that $\|\phi'\|_p = 1$. Our goal is to show that

$$R_{\phi, \omega}(t) = O\left(\frac{1}{t^{p}\nu^{2+\beta}(\log(1/t))}\right),$$

where the implied constant doesn’t depend on $\phi$.

We have

$$|\phi'(r\zeta)| \leq \frac{1}{(1 - r)^{1/p}} = \frac{1}{(1 - r)^{2+\beta(1+\beta)}}, \quad r \in (0, 1) \text{ and } \zeta \in \mathbb{T}. \quad (15)$$
This implies that
\[ |\phi'(r\zeta)|\tau_\omega(r) \lesssim (1 - r)^{(2 + \beta)\beta}(\log(1/1 - r)).\]
Then there exists \( r_0 \in (0, 1) \), which depends only on \( \omega \), such that
\[ |\phi'(r\zeta)|\tau_\omega(r) \leq (1 - r)^{(2 + \beta)\beta}, \quad r \in (r_0, 1).\]
So, if \( |\phi'(r\zeta)|\tau_\omega(r) \geq t \) then \( r \leq r_t \) where \( r_t \) is given by \((1 - r_t)^{(2 + \beta)\beta} = t\).

Let \( U_t \) be the non-tangential maximal function associated with \( z \in \mathbb{D} \to \phi'(r_tz) \). By inequality (15), we have
\[ |\phi'(r\zeta)| \leq U_t(\zeta) \leq \frac{1}{(1 - r_t)^{1/\beta}} = \frac{1}{t^{2/\beta}}, \quad r \leq r_t. \tag{16} \]
For \( t \) small enough, we have
\[ \mathcal{R}_{\phi,\omega}(t) = \lambda_\omega(\{re^{i\theta} \in \mathbb{D} : \tau_\omega(r)|\phi'(re^{i\theta})| \geq t \}) \]
\[ \leq \int_{\{re^{i\theta} : r \leq r_t, \tau_\omega(r)|U_t(e^{i\theta})| \leq t\}} \frac{dr}{\tau_\omega^2(r)} d\theta. \]
\[ \leq \int_0^{2\pi} \mathcal{R}_{z,\omega} \left( \frac{t}{U_t(e^{i\theta})} \right) d\theta. \]
Since \( \tau_\omega^2 \asymp (1 - r^2)2 + \beta \nu^2 (\log(1/1 - r^2)) \), we have \( \mathcal{R}_{z,\omega}(t) \asymp \frac{1}{t^{p\nu + 2/\beta}(\log 1/t)} \). First, note that if \( U_t(e^{i\theta}) \leq t^{1/2} \) then \( \mathcal{R}_{z,\omega} \left( \frac{t}{U_t(e^{i\theta})} \right) \leq \mathcal{R}_{z,\omega}(t^{1/2}) \leq \mathcal{R}_{z,\omega}(t) \). Otherwise, from equation (16), we get
\[ \mathcal{R}_{z,\omega} \left( \frac{t}{U_t(e^{i\theta})} \right) \asymp \frac{U_t^p(e^{i\theta})}{t^{p\nu + 2/\beta}(\log U_t^p(e^{i\theta})/t)} \lesssim \frac{U_t^p(e^{i\theta})}{t^{p\nu + 2/\beta}(\log 1/t)} . \]
Combining these inequalities we obtain
\[ \mathcal{R}_{\phi,\omega}(t) \lesssim \frac{||U_t||^p_{B^\nu}}{t^{p\nu + 2/\beta}(\log 1/t)} \lesssim \frac{1}{t^{p\nu + 2/\beta}(\log 1/t)} , \]
where the involved constant doesn’t depend on \( \phi' \). By Theorem 4.5, we obtain the desired result. \( \square \)

Now, we will prove that the previous result is not true when \( \beta = 0 \) which is somewhat unexpected.

**Proposition 6.3.** Let \( \omega \in \mathcal{W}^* \) such that \( \tau_\omega(z) \asymp \frac{(1 - |z|)}{\log^\alpha(1/1 - |z|)} \) with \( \alpha > 0 \). Then for \( \nu \in ]\frac{\alpha + 1}{2\alpha + 1}, 1[ \), there exists \( \phi \in \mathcal{B}_0^\omega \) such that \( \phi' \in H^1 \) and
\[ s_n(\overline{H}_\sigma) \asymp \frac{1}{n^\nu} . \]
Proof. Let $\gamma \in (1, \alpha + 1)$ be such that $\nu = \frac{\alpha + \gamma}{2\alpha + 1}$. Let $\phi$ be such that

$$\phi'(z) = \frac{1}{(1 - z) \log^\gamma \left( \frac{e}{1 - z} \right)}, \quad z \in \mathbb{D}.$$ 

Since $\gamma > 1$, $\phi' \in H^1$. Write $R_{\phi, \omega} = R_1 + R_2$, where

$$R_1(t) = \lambda_{\omega}(\{r \iota^\theta \in \mathbb{D} : 1 - r > |\theta| \text{ and } \tau_{\omega}(r)|\phi'(re^{i\theta})| \geq t\})$$

and

$$R_2(t) = \lambda_{\omega}(\{re^{i\theta} \in \mathbb{D} : 1 - r \leq |\theta| \text{ and } \tau_{\omega}(r)|\phi'(re^{i\theta})| \geq t\}).$$

Clearly, we have

$$R_1(t) \asymp \lambda_{\omega}(\{r \iota^\theta \in \mathbb{D} : 1 - r > |\theta| \text{ and } \frac{1}{\log^{\alpha + \gamma}(\nu/(1 - r))} \geq t\}) \asymp \frac{1}{t^{\frac{2\alpha + 1}{\alpha + \gamma}}}.$$ 

Similarly,

$$R_2(t) \asymp \lambda_{\omega}(\{re^{i\theta} \in \mathbb{D} : 1 - r \leq |\theta| \text{ and } \frac{1 - r}{\log^{\alpha + \gamma}(r|\theta|)} \geq t\}) \asymp \frac{1}{t^{\frac{2\alpha + 1}{\alpha + \gamma}}}.$$ 

Then, $R_{\phi, \omega}(t) \asymp \frac{1}{t^{\frac{2\alpha + 1}{\alpha + \gamma}}}$. By Theorem 4.4 we obtain that $s_n(H_{\phi}) \asymp \frac{1}{n^{\nu}}$. The proof is complete. \qed

7. Asymptotics

Now, we will precise the results obtained in Section 5 when $s_n(H_{\phi})$ is regular. The main result of this section is the following theorem.

**Theorem 7.1.** Let $\omega \in \mathcal{W}^*$ be a radial weight such that $\tau_{\omega}^2(z) \asymp (1 - |z|^2)^{2 + \beta} \nu (\log(\frac{1}{1 - |z|^2})), \quad \omega \in \mathcal{W}^*$,

where $\beta > 0$ and $\nu$ is a monotone function such that $\nu(2t) \asymp \nu(t)$. Suppose that

$$s_n(H_{\phi}) \sim \frac{\gamma}{n^p \nu^{\frac{1}{p}}(\log n)},$$

where $\gamma > 0$ and $p = \frac{2(1 + \beta)}{2 + \beta}$. Then, for $\phi' \in H^p$, we have

$$s_n(H_{\phi}) \sim \frac{\gamma}{n^p \nu^{\frac{1}{p}}(\log n)} \|\phi'\|_p.$$ 

To prove this result, we will introduce the following functionals (see [5, 25]). Let $T$ be a compact operator between two Hilbert spaces. Let $n(s, T)$ be the singular values counting function given by

$$n(s, T) = \#\{n : s_n(T) \geq s\}, \quad s > 0.$$ 

The class of strictly increasing continuous functions $\psi : (0, +\infty) \to (0, +\infty)$ such that $\psi(0) = 0$ and such that

$$\psi(\alpha t) \sim \alpha^p \psi(t), \quad (t \to 0^+),$$

for some $p > 0$, will be denoted by $\mathcal{C}_p$. Let

$$D_\psi(T) := \limsup_{s \to 0^+} \psi(s)n(s, T) \quad \text{and} \quad d_\psi(T) := \liminf_{s \to 0^+} \psi(s)n(s, T).$$
From these definitions it is easy to see that if \( \psi \in C_p \), then
\[
d_\psi(T) = D_\psi(T) \in (0, \infty) \implies s_n(T) \sim D_\psi(T)^{1/p} \psi^{-1}(1/n).
\]
For more definitions and properties of these functionals see Section 8.

First, we need some results on the operator \( HzP_\omega M_g \), where \( M_g \) denotes the multiplication operator defined on \( L^2(dA_\omega) \).

For an arc \( \delta = \{e^{i\theta} : \theta_1 \leq \theta < \theta_2 \} \subset \mathbb{T} \), \( R_N^\delta \) will denote
\[
R_N^\delta = \{z = re^{i\theta} : 0 < 1 - r \leq 2\pi/N, \theta_1 \leq \theta < \theta_2 \}, \quad N = 1, 2, ...
\]
The proof of the following results is similar to that one given by A. Pushnitski in [25].

**Lemma 7.2.** Let \( \omega \in W^* \) be a radial weight. Let \( \psi \in C_p \) such that \( D_\psi(Hz) \) is finite. The following are true

1. Let \( g \) be a bounded function on \( D \). We have
   \[
   D_\psi(HzP_\omega M_g) \leq \|g\|_\infty^p D_\psi(Hz).
   \]

2. Let \( \delta_1, \delta_2 \) be two arcs of \( \mathbb{T} \) such that \( \overline{\delta_1} \cap \overline{\delta_2} = \emptyset \). Then
   \[
   \left(HzP_\omega M_{\chi_{R_N^\delta_1}}\right)^* \left(HzP_\omega M_{\chi_{R_N^\delta_2}}\right) \in \cap_{p>0} S_p.
   \]

3. Let \( \delta \subset \partial D \) be an arc such that \( |\delta| < 2\pi \). Then
   \[
   D_\psi(HzP_\omega \chi_{R_N^\delta}) \leq |\delta| D_\psi(Hz).
   \]

4. Let \( \delta_1, \delta_2 \) be two arcs of \( \mathbb{T} \) such that \( \overline{\delta_1} \cap \overline{\delta_2} \) is reduced to one point. Then
   \[
   \left(HzP_\omega M_{\chi_{R_N^\delta_1}}\right)^* \left(HzP_\omega M_{\chi_{R_N^\delta_2}}\right) \in \Sigma_{\psi, \chi}^0.
   \]

**Proof.** Since \( g \) is bounded, \( M_g \) is bounded on \( L^2_\omega \) and \( \|M_g\| = \|g\|_\infty \). Then \( n(s, HzP_\omega M_g) \leq n(s, \|M_g\|Hz) = n(s, \|g\|_\infty Hz) \), for all \( s > 0 \). Hence, by Proposition 8.1 we get
\[
D_\psi(HzP_\omega M_g) \leq D_\psi(\|g\|_\infty Hz) = \|g\|_\infty^p D_\psi(Hz).
\]
This proves the first assertion.

The same proof as that proposed by Pushnitski in [25], gives the second assertion.

To prove the third assertion, let \( N \in \mathbb{N}^* \) such that \( \frac{2\pi}{N+1} \leq |\delta| < \frac{2\pi}{N} \) and let
\[
\delta_n := e^{2\pi in/N} \delta, \quad n = 1, 2, ..., N.
\]
Let $R^0_N := R^{\delta_0}_N$. Since $g := \sum_{n=1}^N \chi_{R^0_N} \leq 1$, we have
\[
D_{\psi}(H_\pi) \geq D_{\psi}(H_\pi P_\omega M_g) = D_{\psi} \left( \sum_{n=1}^N H_\pi P_\omega M_{\chi_{R^0_N}} \right).
\]
For $i \neq j$ we have $\left( H_\pi P_\omega M_{\chi_{R^0_N}} \right) \left( H_\pi P_\omega M_{\chi_{R^0_N}} \right)^* = 0$. Since the closures of $R^i_N$ and $R^j_N$ are disjoint, we have
\[
\left( H_\pi P_\omega M_{\chi_{R^0_N}} \right)^* \left( H_\pi P_\omega M_{\chi_{R^0_N}} \right) \in \Sigma^0_{\psi e \sqrt{\tau}}.
\]
Further, since $\omega$ is radial, the operators $H_\pi P_\omega M_{\chi_{R^0_N}}$ are unitarily equivalent. It follows from Theorem 8.3 that
\[
D_{\psi}(H_\pi) \geq D_{\psi} \left( \sum_{n=1}^N H_\pi P_\omega M_{\chi_{R^0_N}} \right) = ND_{\psi}(H_\pi P_\omega M_{\chi_{R^0_N}}).
\]
Therefore, $D_{\psi}(H_\pi P_\omega M_{\chi_{R^0_N}}) \leq 1/ND_{\psi}(H_\pi) \leq |\delta|D_{\psi}(H_\pi)$.

The last assertion is a consequence of the second and the third assertion. Indeed, let $\varepsilon > 0$ small enough. Suppose that $\delta_1 = \{ e^{i\theta} : \theta_1 \leq \theta < \theta_2 \}$ and $\delta_2 = \{ e^{i\theta} : \theta_2 \leq \theta < \theta_3 \}$ and let $\delta_1(\varepsilon) = \{ e^{i\theta} : \theta_1 \leq \theta < \theta_2 - \varepsilon \}$. By (2), $\left( H_\pi P_\omega M_{\chi_{R^0_N}} \right)^* \left( H_\pi P_\omega M_{\chi_{R^0_N}} \right) \in \cap_{p>0} S_p$. By Corollary 8.2, we get
\[
D_{\psi e \sqrt{\tau}} \left( \left( H_\pi P_\omega M_{\chi_{R^0_N}} \right)^* \left( H_\pi P_\omega M_{\chi_{R^0_N}} \right) \right) = D_{\psi e \sqrt{\tau}} \left( \left( H_\pi P_\omega M_{\chi_{R^0_N}} \right)^* \left( H_\pi P_\omega M_{\chi_{R^0_N}} \right) \right).
\]
Applying Proposition 8.1, we have
\[
D_{\psi e \sqrt{\tau}} \left( \left( H_\pi P_\omega M_{\chi_{R^0_N \setminus R^0_N}} \right)^* \left( H_\pi P_\omega M_{\chi_{R^0_N \setminus R^0_N}} \right) \right) \leq 2D_{\psi}(H_\pi P_\omega M_{\chi_{R^0_N \setminus R^0_N}})D_{\psi}(H_\pi P_\omega M_{\chi_{R^0_N \setminus R^0_N}}) \lesssim \varepsilon D_{\psi}(H_\pi).
\]
Letting $\varepsilon$ to 0, we obtain $D_{\psi e \sqrt{\tau}} \left( \left( H_\pi P_\omega M_{\chi_{R^0_N \setminus R^0_N}} \right)^* \left( H_\pi P_\omega M_{\chi_{R^0_N \setminus R^0_N}} \right) \right) = 0$, as required.

**Proposition 7.3.** Let $\omega \in \mathcal{W}^*$ be a radial weight. Let $\psi \in C_p$ such that $0 < d_\psi(H_\pi) \leq D_\psi(H_\pi) < \infty$.

- Let $\delta \subset \partial \mathbb{D}$ be an arc such that $|\delta| = \frac{2\pi}{N}$, where $N \in \mathbb{N}^*$. Then
  \[
  D_\psi(H_\pi P_\omega M_{\chi_{R^0_N}}) = \frac{1}{N} D_\psi(H_\pi) \quad \text{and} \quad d_\psi(H_\pi P_\omega M_{\chi_{R^0_N}}) = \frac{1}{N} d_\psi(H_\pi).
  \]
- Let $g$ be a continuous function on $\overline{\mathbb{D}}$. We have
  \[
  \|g\|^p_p d_\psi(H_\pi) \leq d_\psi(H_\pi P_\omega M_g) \leq D_\psi(H_\pi P_\omega M_g) \leq \|g\|^p_p D_\psi(H_\pi).
  \]
Proof. Let
\[ R_N = \{ z = re^{i\theta} \in \mathbb{D} : \ r > 1 - 1/N \} , \]
\[ R^k_N = \{ z = re^{i\theta} \in R_N : \ 2\pi k/N \leq \theta < \frac{2\pi (k+1)}{N} \} , \ k = 0, 1, \ldots, N - 1. \]
And let
\[ h := \sum_{k=0}^{N-1} \chi_{R^k_N} = 1 - \chi_{\mathbb{D} \setminus R_N}. \]
Hence, by Corollary 8.2, Lemma 7.2 and Theorem 8.3, we have
\[ D_{\psi}(H_{\tau}) = D_{\psi}(H_{\tau}P_\omega M_h) \]
\[ = D_{\psi} \left( \sum_{k=0}^{N-1} H_{\tau}P_\omega M_{\chi_{R^k_N}} \right) \]
\[ = \sum_{k=0}^{N-1} D_{\psi}(H_{\tau}P_\omega M_{\chi_{R^k_N}}) \]
\[ = ND_{\psi}(H_{\tau}P_\omega M_{\chi_{R^k_N}}). \]
Then \( D_{\psi}(H_{\tau}P_\omega M_{\chi_{R^k_N}}) = \frac{1}{N} D_{\psi}(H_{\tau}). \) Similarly, we obtain \( d_{\psi}(H_{\tau}P_\omega M_{\chi_{R^k_N}}) = \frac{1}{N} d_{\psi}(H_{\tau}). \)
Let \( \xi_k \) be the center of the arc \( R^k_N \cap \partial \mathbb{D} \) and let
\[ g_N = \sum_{k=0}^{N-1} g(\xi_k) \chi_{R^k_N}. \]
By Proposition 8.1, we have
\[ D^{1/p+1}_{\psi}(H_{\tau}P_\omega M_g) \leq D^{1/p+1}_{\psi}(H_{\tau}P_\omega M_{g_N}) + D^{1/p+1}_{\psi}(H_{\tau}P_\omega M_{g-g_N}) \]
On one hand, by Lemma 7.2, we have
\[ D_{\psi}(H_{\tau}P_\omega M_{g-g_N}) \leq \| g - g_N \|^p_{\infty} D_{\psi}(H_{\tau}). \]
On the other hand, by Lemma 7.2, Theorem 8.3 and the first assertion of this proposition, we obtain
\[ D_{\psi}(H_{\tau}P_\omega M_{g_N}) \leq \sum_{k=0}^{N-1} D_{\psi}(g(\xi_k) H_{\tau}P_\omega M_{\chi_{R^k_N}}) \]
\[ = \sum_{k=0}^{N-1} |g(\xi_k)|^p D_{\psi}(H_{\tau}P_\omega M_{\chi_{R^k_N}}) \]
\[ = \frac{1}{N} \sum_{k=0}^{N-1} |g(\xi_k)|^p D_{\psi}(H_{\tau}). \]
Combining these inequalities and letting \( N \) going to \( \infty \), we obtain
\[ D_{\psi}(H_{\tau}P_\omega M_g) \leq \| g \|^p_{\infty} D_{\psi}(H_{\tau}). \]
The lower estimates can be obtained by the same arguments. \( \square \)
Let \( \Phi : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{C} \) be an analytic function, where \( R > 1 \). Let \( A_\Phi : A^2_\omega \to L^2_\omega \) be the operator defined by

\[
A_\Phi f(z) = \int_\mathbb{D} (z - \bar{\xi})^2 K(z, \xi) \Phi(z, \xi) \omega(\xi) dA(\xi), \quad z \in \mathbb{D}.
\]

And let \( A \) be the operator given by

\[
Af(z) = \int_\mathbb{D} (z - \bar{\xi})^2 K(z, \xi) f(\xi) \omega(\xi) dA(\xi), \quad z \in \mathbb{D}.
\]

Now, we introduce notations which will be used in the proof of the next lemma. Let \( r \in (0, 1) \) and let \( J_r \) be the embedding operator from \( A^2_\omega \to L^2_\omega (\omega\chi_r \mathbb{D})dA \). Note that \( T_r := J_r^* J_r \) is the Toeplitz operator with symbol \( \chi_r \mathbb{D} \). Namely,

\[
T_r f = \int_{r \mathbb{D}} f(\zeta) K(., \zeta) \omega(\zeta) dA(\zeta).
\]

By [9], there exists \( C = C(\omega) \) such that

\[
\sum_{n \geq 1} \lambda_n^p(T_r) \leq C \frac{\int_0^r d\rho}{\tau_\omega^2(\rho)}, \quad p \in (0, 1).
\]

This implies that

\[
\lambda_n(T_r) \leq \inf_{p \in (0, 1)} \left( \frac{C}{pn} \int_0^r \frac{d\rho}{\tau_\omega^2(\rho)} \right)^{\frac{1}{p}} = \exp \left( -\frac{e^{-1}}{C} \int_0^r \frac{d\rho}{\tau_\omega^2(\rho)} \right)^n.
\]

If we suppose that \( \tau_\omega^2(z) \asymp (1 - |z|^2)^{2+\beta} \nu^2(\log(1/|z|^2)), \) then

\[
\lambda_n(T_r) \leq \exp \left( -C(\omega)(1-r)^{1+\beta} \nu^2(\log(1/1-r))n \right).
\]

In particular we have

\[
\lambda_n(T_r) = O \left( \exp \left( -C(\omega, \varepsilon)(1-r)^{1+\beta-\varepsilon} n \right) \right), \quad \forall \varepsilon > 0.
\]

**Lemma 7.4.** Let \( \omega \in W^* \) be a radial weight such that \( \tau_\omega^2(z) \asymp (1 - |z|^2)^{2+\beta} \nu^2(\log(1/|z|^2)), \) where \( \beta \geq 0 \) and \( \nu \) is a monotone function such that \( \nu(2t) \asymp \nu(t) \). Then, for all \( \varepsilon > 0 \), we have

\[
s_n(A_\Phi) = O \left( \frac{1}{n^{2/p-\varepsilon}} \right), \quad p = \frac{2(1+\beta)}{2+\beta}.
\]

**Proof.** First, we prove the result for \( A \) which corresponds to \( \Phi = 1 \). Remark that

\[
Af = H_{\bar{z}} f - 2H_z \frac{\omega f}{\bar{\omega}} \in A^2_\omega.
\]
Then $Af$ is the $L^2_\omega$-minimal solution of $\bar{\partial}u = 2H_z f$. Applying Lemma 3.2 twice, we get

$$\|Af\|^2 \lesssim \int_{\mathbb{D}} |2H_z f(z)|^2 \tau_\omega^2(z) \omega(z) dA(z)$$

$$\lesssim \int_{\mathbb{D}} |H_z f(z)|^2 \omega(z) dA(z) + \tau_\omega^2(r) \int_{\mathbb{D}\setminus\mathbb{D}} |H_z f(z)|^2 \omega(z) dA(z)$$

$$\lesssim \int_{\mathbb{D}} |f(z)|^2 \omega(z) dA(z) + \int_{\mathbb{D}} |P_\omega \bar{z} f(z)|^2 \omega(z) dA(z) + \tau_\omega^2(r) \int_{\mathbb{D}} |f(z)|^2 \tau_\omega^2(z) \omega(z) dA(z)$$

$$\lesssim \|J_r f\|^2 + \|J_r P_\omega \bar{z} f\|^2 + \tau_\omega^2(r) \langle T_{T_{r_\omega^2}} f, f \rangle.$$ 

Then we obtain

$$A^* A \lesssim T_r + (P_\omega \bar{z})^* T_r P_\omega \bar{z} + \tau_\omega^2(r) T_{T_{r_\omega^2}}$$

This implies that

$$s_{3n}^2(A) \lesssim \lambda_n(T_r) + \tau_\omega^2(r) \lambda_n(T_{T_{r_\omega^2}}). \tag{20}$$

Since $\int r^1 \frac{dr}{\tau_\omega^2 - r} < \infty$, by [3], $T_{T_{r_\omega^2}} \in S_{\frac{1}{2} + \varepsilon}$ for every $\varepsilon > 0$. Then,

$$\lambda_n(T_{T_{r_\omega^2}}) = O \left( \frac{1}{n^{\frac{1}{2} - \varepsilon}} \right), \quad \text{for all } \varepsilon > 0. \tag{21}$$

Combining inequalities (19), (20) and (21), we obtain

$$s_{3n}^2(A) \lesssim \inf_{r(0,1)} \left( \exp \left( -C(\omega, \varepsilon)(1 - r)^{1+\beta - \varepsilon} n \right) + \frac{\tau_\omega^2(r)}{n^{\frac{1}{2} - \varepsilon}} \right), \quad \text{for all } \varepsilon > 0.$$ 

For a suitable choice of $r$, we obtain the result for $A$.

To complete the proof of the Lemma we use the argument given by M. Dostanic in [8]. Suppose that $\Phi$ is analytic and bounded by $M$ on $R\mathbb{D} \times R\mathbb{D}$. Clearly, we have

$$\Phi(z, \zeta) = \sum_{k \geq 0} \frac{\Phi^{(k)}(z, 0)}{k!} \zeta^k \quad \text{and} \quad \frac{|\Phi^{(k)}(z, 0)|}{k!} \leq \frac{M}{R^k}.$$ 

And since $A_{\Phi}f = \sum_{k \geq 0} \frac{\Phi^{(k)}(z, 0)}{k!} A P_\omega \bar{z}^k f$, we obtain

$$s_{(N+2)m}(A_{\Phi}) \leq \sum_{k \geq 0} \left| \frac{\Phi^{(k)}(z, 0)}{k!} \right| s_m(A) + \sum_{k > N} \left| \frac{\Phi^{(k)}(z, 0)}{k!} \right| \|A\| \leq \frac{M}{R - 1} \left( s_m(A) + \|A\| \right).$$

Now for $N m = n$ and $m \sim n^{1-\varepsilon}$, we get the result.

Proof of Theorem 7.1 Let $\psi(t) = \frac{1}{\gamma_p} t^{p/2^\gamma} (\log t)^{1/\beta}$. Since $s_n(H_\phi) \sim \frac{\psi^{(p/2^\gamma)/\gamma}}{n^{p/2^\gamma}} (\log n)$, $D_\psi(H_\phi) = d_\psi(H_\phi) = 1$. Note that

$$s_n(H_\phi) \sim s_n(H_\phi) \|\phi\|_p \iff D_\psi(H_\phi) = d_\psi(H_\phi) = \|\phi\|_p^p.$$
First, suppose that $\phi$ is analytic in a neighborhood of $\overline{\mathbb{D}}$. Then there exist $R > 1$ and an analytic bounded function $\Phi$ on $R\mathbb{D} \times R\mathbb{D}$ such that
\[
\phi(z) - \phi(w) = (z - w)\phi'(w) + (z - w)^2\Phi(z, w), \quad z, w \in R\mathbb{D}.
\]
We have
\[
H_{\phi'}P_\omega = H_\pi P_\omega \overline{\phi'} + A_\Phi P_\omega.
\]
By Lemma 7.4
\[
s_n(A_\Phi P_\omega) = o(s_n(H_\pi P_\omega)), \quad n \to \infty.
\]
So,
\[
D_\psi(A_\Phi P_\omega) = 0.
\]
By Corollary 8.2 we deduce that
\[
D_\psi(H_{\phi'}) = D_\psi(H_\pi P_\omega \overline{\phi'}) \quad \text{and} \quad d_\psi(H_{\phi'}) = d_\psi(H_\pi P_\omega \overline{\phi'}).
\]
We obtain, by Proposition 7.3 that
\[
d_\psi(H_{\phi'})\|\phi'\|_p^p \leq d_\psi(H_{\phi'}) \leq D_\psi(H_{\phi'}) \|\phi'\|_p^p.
\]
Since $D_\psi(H_{\phi'}) = d_\psi(H_{\phi'}) = 1$, we obtain
\[
d_\psi(H_{\phi'}) = D_\psi(H_{\phi'}) = \|\phi'\|_p^p.
\]
Now, suppose that $\phi' \in H^p$. By Theorem 6.2 we have
\[
s_n(H_{\phi'-\phi_r}) \leq C\|\phi' - \phi_r\|_p s_n(H_{\phi}), \quad n \geq 1.
\]
This implies that $D_\psi(H_{\phi'-\phi_r}) \leq C^p\|\phi' - \phi_r\|_p^p D_\psi(H_{\phi})$. Then we have
\[
|D_\psi(H_{\phi'})^\frac{1}{p+1} - (D_\psi(H_{\phi'}))^\frac{1}{p+1}|^\frac{1}{p+1} \leq D_\psi(H_{\phi'-\phi_r})^\frac{1}{p+1} \leq (C^p\|\phi' - \phi_r\|_p^p D_\psi(H_{\phi}))^\frac{1}{p+1}.
\]
When $r \to 1^-$, we obtain $D_\psi(H_{\phi'}) = D_\psi(H_{\phi'})\|\phi'\|_p^p = \|\phi'\|_p^p$. With the same arguments we have $d_\psi(H_{\phi'}) = d_\psi(H_{\phi'})\|\phi'\|_p^p$. The proof is complete. \qed

Similarly, one can prove the following result which corresponds to the situation $\beta = 0$ and $\nu \asymp 1$ in Theorem 7.1.

**Theorem 7.5.** Let $\omega \in \mathcal{W}^*$ be a radial weight such that $\tau_\omega^2(z) \asymp (1 - |z|^2)^2$. Suppose that
\[
s_n(H_{\phi'}) \sim \frac{\gamma}{n},
\]
where $\gamma > 0$. Then, for $\phi' \in H^1$, we have
\[
s_n(H_{\phi'}) \sim \frac{\gamma}{n} \|\phi'\|_1.
\]
**Proof of the second assertion of Theorem 1.3.** It suffices to combine Theorem 7.5 and Theorem 7.1 with $\nu = 1$. \qed
8. An example

For a radial weight $\omega$, the sequence of the singular values of the Hankel operator $H_\omega$ on $A^2_\omega$ is the decreasing rearrangement of the sequence

$$
\left( \left( \frac{||z^{n+1}||^2}{||z^n||^2} - \frac{||z^n||^2}{||z^{n-1}||^2} \right)^{1/2} \right)_n.
$$

Recall that for the standard case $\omega_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha$, we have $s_n(H_\omega) \sim \sqrt{\frac{\alpha+1}{n}}$. In this section, we consider the weight

$$
\omega(z) = \exp\left( -\frac{\alpha}{(\log \frac{1}{|z|^2})^\beta} \right), \quad \alpha, \beta > 0.
$$

We have

$$
||z^n||^2 = \int_0^1 |z|^{2n} \exp\left( -\frac{\alpha}{(\log \frac{1}{|z|^2})^\beta} \right) dA(z)
$$

$$
= \int_0^1 r^{2n} \exp\left( -\frac{\alpha}{(\log \frac{1}{r^2})^\beta} \right) 2rdr
$$

$$
= \int_0^{+\infty} \exp\left( -(n+1)x - \frac{\alpha}{x^\beta} \right) dx.
$$

Let $x_n := \left( \frac{\alpha\beta}{n+1} \right)^{1/1+\beta}$ be the minimum of the function $(n+1)x + \frac{\alpha}{x^\beta}$. After the change of variable $u = \frac{x}{x_n}$, we get

$$
||z^n||^2 = x_n \exp\left( -(n+1)x_n - \frac{\alpha}{x_n^\beta} \right) \int_{-1}^{+\infty} \exp\left( -\frac{\alpha}{x_n^\beta} h(u) \right) du,
$$

where $h(u) = \beta u + \frac{\frac{1}{(1+u)^\beta}}{1+u^\beta} - 1$.

In the sequel, we will use Laplace method \[14\] to get an expansion of $||z^n||^2$. By Laplace Theorem, we have

$$
H(t) = \sqrt{\frac{th''(0)}{2\pi}} \int_{-1}^{+\infty} \exp(-th(u)) du \sim 1, \quad t \to +\infty.
$$

Let $\eta > 0$. We have

$$
\sqrt{\frac{th''(0)}{2\pi}} \int_{[-1,-\eta] \cup [\eta,+\infty]} \exp(-th(u)) du = O(e^{-c(\eta)t}), \quad t \to +\infty.
$$
Let
\[
H^+_{\eta}(t) = \sqrt{\frac{th''(0)}{2\pi}} \int_{0 < u < \eta} \exp(-th(u)) \, du,
\]
and
\[
H^-_{\eta}(t) = \sqrt{\frac{th''(0)}{2\pi}} \int_{-\eta < u < 0} \exp(-th(u)) \, du.
\]
Using the change of variable \( v = h(u) \), when \( \eta \) is small enough, one can write
\[
H^+_{\eta}(t) = 2^N \sum_{j=0}^{2N} c_j t^{j/2} + o\left(\frac{1}{t^N}\right),
\]
where \( c_0, \ldots, c_{2N} \in \mathbb{R} \). The same is also true for \( H^-_{\eta} \). Then
\[
H(t) = 2^N \sum_{j=0}^{2N} d_j t^{j/2} + o\left(\frac{1}{t^N}\right),
\]
where \( d_0 = 1 \) and \( d_1, \ldots, d_{2N} \in \mathbb{R} \). Let
\[
A_n = \sqrt{\frac{2\pi}{h''(0)}} x_n^1 x_n^2 \exp\left(- (n+1)x_n - \frac{\alpha}{x_n}\right).
\]
We have
\[
||z^n||^2 = A_n H\left(\frac{\alpha}{x_n}\right).
\]
By a direct calculation, we get
\[
\frac{||z^n||^2}{||z^{n-1}||^2} = 1 - \frac{a}{n^{1/\beta}} + \frac{b}{n} + \frac{c}{n^{1+1/\beta}} + o(1/n^{1+1/\beta}),
\]
where \( a = (\alpha \beta)^{1/1+\beta} \). Then, we have
\[
\frac{||z^{n+1}||^2}{||z^n||^2} - \frac{||z^n||^2}{||z^{n-1}||^2} \sim \frac{\gamma^2}{n^{3/2+1}},
\]
where \( \gamma = \sqrt{\frac{(\alpha \beta)^{1/1+\beta}}{1+\beta}} \). Finally, we obtain
\[
s_n(H_{\pi}) \sim \frac{\gamma}{n^{2/\beta+1}}.
\]

**APPENDIX: ASYMPTOTIC ORTHOGONALITY**

Let \( T \) be a compact operator between two complex Hilbert spaces. The decreasing sequence of singular values of \( T \) will be denoted by \((s_n(T))_n\). The counting function of the singular values of \( T \) is denoted by
\[
n(s, T) = \#\{n : s_n(T) \geq s\}, \quad s > 0.
\]
Recall that \( C_\text{p} \) denotes the class of increasing continuous function \( \psi : (0, +\infty) \rightarrow (0, +\infty) \) satisfying
\[
\psi(\alpha t) \sim \alpha^p \psi(t), \quad (t \rightarrow 0^+),
\]
As before, $D\psi(T), d\psi(T)$ are given by

$$D\psi(T) := \lim_{s \to 0^+} \sup \psi(s)n(s, T) \quad \text{and} \quad d\psi(T) := \lim_{s \to 0^+} \inf \psi(s)n(s, T).$$

The goal of this Annex is to extend the results obtained for $\psi(t) = t^p, \ [5, 25]$, to the class $C_p$. We give here the proof for completeness.

**Proposition 8.1.** Let $T$ and $V$ be compact operators and $\lambda \in \mathbb{C}$. We have

1. $D\psi(\lambda T) = |\lambda|^p D\psi(T)$ and $d\psi(\lambda T) = |\lambda|^p d\psi(T)$.
2. $D\psi(T + V)^\frac{1}{p + 1} \leq D\psi(T)^\frac{1}{p + 1} + D\psi(V)^\frac{1}{p + 1}$.
3. $d\psi(T + V)^\frac{1}{p + 1} \leq d\psi(T)^\frac{1}{p + 1} + D\psi(V)^\frac{1}{p + 1}$.
4. $D\psi(TV) \leq 2D\psi^2(T)D\psi^2(V)$.

**Proof.** 1. For $\lambda \neq 0$, we have

$$D\psi(\lambda T) = \lim_{s \to 0^+} \sup \psi(s)n(s, \lambda T) = \lim_{s \to 0^+} \sup \psi(s)n(s/|\lambda|, T)$$

$$= \lim_{s \to 0^+} \psi(|\lambda|s)n(s, T)$$

$$= |\lambda|^p \lim_{s \to 0^+} \sup \psi(s)n(s, T)$$

$$= |\lambda|^p D\psi(T).$$

Similarly, we have $d\psi(\lambda T) = |\lambda|^p d\psi(T), \forall \lambda \in \mathbb{C}$.

2. Let $x \in (0, 1)$. For all $s > 0$, we have

$$n(s, T + V) = n(xs + (1 - x)s, T + V) \leq n(xs, T) + n((1 - x)s, V)$$

$$= n\left(s, \frac{1}{x}T\right) + n\left(s, \frac{1}{1 - x}V\right).$$

Hence, $D\psi(T + V) \leq D\psi\left(\frac{1}{x}T\right) + D\psi\left(\frac{1}{1 - x}V\right)$. By the first assertion we obtain

$$D\psi(T + V) \leq x^{-p} D\psi(T) + (1 - x)^{-p} D\psi(V).$$

It follows that

$$D\psi(T + V) \leq \min_{x \in (0, 1)} \left\{x^{-p} D\psi(T) + (1 - x)^{-p} D\psi(V)\right\},$$

and 2. is obtained.

By the same way we obtain also the third assertion.

3. For all $\alpha, s > 0$, we have

$$n(s, TV) = n(\alpha \sqrt{s}, \alpha^{-1} \sqrt{s}, TV) \leq n(\sqrt{s}, T) + n(\alpha^{-1} \sqrt{s}, V)$$

$$= n\left(\sqrt{s}, \frac{1}{\alpha}T\right) + n\left(\sqrt{s}, \alpha V\right).$$
Hence
\[ \psi(s)n(s, TV) \leq \psi(s)n(\sqrt{s}, \frac{1}{\alpha} T) + \psi(s)n(\sqrt{s}, \alpha V) \]
\[ = \tilde{\psi}(\sqrt{s})n(\sqrt{s}, \frac{1}{\alpha} T) + \tilde{\psi}(\sqrt{s})n(\sqrt{s}, \alpha V), \]
with \( \tilde{\psi} = \psi(t^2) \). Then
\[ D\psi(TV) \leq D\tilde{\psi}\left(\frac{1}{\alpha} T\right) + D\tilde{\psi}(\alpha V) \]
\[ = \alpha^{-2p}D\tilde{\psi}(T) + \alpha^{2p}D\tilde{\psi}(V). \]

It follows that
\[ D\psi(TV) \leq \min_{\alpha > 0} \left\{ \alpha^{-2p}D\tilde{\psi}(T) + \alpha^{2p}D\tilde{\psi}(V) \right\} \]
\[ \leq 2D\tilde{\psi}(T)D\tilde{\psi}(V). \]

Let us denote
\[ \Sigma_\psi := \{ T \in S_\infty : \psi(s)n(s, T) = O(1) \} \]
and
\[ \Sigma^0_\psi : = \{ T \in S_\infty : \psi(s)n(s, T) = o(1) \}. \]
The second and the third assertions of the Proposition implies the following corollary.

**Corollary 8.2.** Let \( T \) and \( V \) be compact operators. Suppose that \( V \in \Sigma^0_\psi \), then
\[ D\psi(T + V) = D\psi(T) \quad \text{and} \quad d\psi(T + V) = d\psi(T). \]

Once we have that, we can state the Theorem of A. Pushnitski [25, Theorem 2.2] in the following form

**Theorem 8.3.** Let \( T_1, T_2, \ldots, T_n \) be a compact operators such that
\[ T_i^*T_j \in \Sigma^0_{\psi \circ \sqrt{\cdot}} \quad \text{and} \quad T_iT_j^* \in \Sigma^0_{\psi \circ \sqrt{\cdot}}, \quad \forall i \neq j. \]
Then
\[ D\psi\left(\sum_{k=1}^{n} T_k\right) = \limsup_{s \to 0^+} \psi(s)\sum_{k=1}^{n} n(s, T_k) \quad \text{and} \quad d\psi\left(\sum_{k=1}^{n} T_k\right) = \liminf_{s \to 0^+} \psi(s)\sum_{k=1}^{n} n(s, T_k). \]
In particular, if for all \( s > 0 \), we have \( n(s, T_k) = n(s, T_1), k = 1, 2, \ldots, n \), then
\[ D\psi\left(\sum_{k=1}^{n} T_k\right) = nD\psi(T_1) \quad \text{and} \quad d\psi\left(\sum_{k=1}^{n} T_k\right) = nd\psi(T_1). \]
Proof. Let \( T = \sum_{k=1}^{n} T_k \). Since

\[
TT^* = \sum_{i=1}^{n} T_i T_i^* + \sum_{i,j=1,i\neq j}^{n} T_i T_j^*,
\]

and \( T_i T_j^* \in \Sigma^0_{\psi_0 \varphi^*} \); for \( i \neq j \), we have

\[
D_{\psi}(T) = D_{\psi_0 \varphi^*}(TT^*) = D_{\psi_0 \varphi^*}\left( \sum_{i=1}^{n} T_i T_i^* \right)
= D_{\psi_0 \varphi^*}((JA)(JA)^*)
= D_{\psi_0 \varphi^*}((JA)^*(JA)).
\]

Where

\[
J : H^n \longrightarrow H \quad \text{and} \quad A : H^n \longrightarrow H^n
\]

\[
(f_1, \ldots, f_n) \longrightarrow \sum_{i=1}^{n} f_i \quad \text{and} \quad (f_1, \ldots, f_n) \longrightarrow (T_1 f_1, T_2 f_2, \ldots, T_n f_n)
\]

The matrix of \((JA)^*(JA)\) is

\[
\begin{pmatrix}
T_1^* T_1 & T_1^* T_2 & \cdots & T_1^* T_n \\
T_2^* T_1 & T_2^* T_2 & \cdots & T_2^* T_n \\
\vdots & \vdots & \ddots & \vdots \\
T_n^* T_1 & T_n^* T_2 & \cdots & T_n^* T_n
\end{pmatrix}
= \begin{pmatrix}
T_1 T_1^* & 0 & \cdots & 0 \\
T_2 T_2^* & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
T_n T_n^* & 0 & \cdots & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & T_1 T_2^* & \cdots & T_1 T_n^* \\
T_2 T_1^* & 0 & \cdots & T_2 T_n^* \\
\vdots & \vdots & \ddots & \vdots \\
T_n T_1^* & T_n T_2^* & \cdots & 0
\end{pmatrix}
=: T' + T''.
\]

Since \( T_i^* T_j^* \in \Sigma^0_{\psi_0 \varphi^*} \); for \( i \neq j \), we deduce that

\[
D_{\psi}(T) = D_{\psi_0 \varphi^*}(T' + T'') = D_{\psi_0 \varphi^*}(T')
= D_{\psi_0 \varphi^*}(T_1^* T_1, T_2^* T_2, \ldots, T_n^* T_n).
\]

Let now \( T_0 \) the operator

\[
T_0 := \begin{pmatrix}
T_1 & 0 \\
T_2 & \ddots \\
0 & \ddots \\
0 & \ddots & T_n
\end{pmatrix}
\]

We have

\[
D_{\psi}(T) = D_{\psi_0 \varphi^*}(T_1^* T_1, T_2^* T_2, \ldots, T_n^* T_n)
= D_{\psi_0 \varphi^*}(T_0^* T_0)
= D_{\psi}(T_0)
= \limsup_{s \to 0^+} \psi(s) \sum_{k=1}^{n} n(s, T_k).
\]
In the same way we prove that
\[ d_\psi \left( \sum_{k=1}^{n} T_k \right) = \liminf_{s \to 0^+} \psi(s) \left( \sum_{k=1}^{n} n(s, T_k) \right). \]

\[ \square \]

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