SINGULAR SOLUTIONS TO $k$-HESSIAN EQUATIONS WITH FAST-GROWING NONLINEARITIES

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Abstract. We study a class of elliptic problems, involving a $k$-Hessian and a very fast-growing nonlinearity, on a unit ball. We prove the existence of a radial singular solution and obtain its exact asymptotic behavior in a neighborhood of the origin. Furthermore, we study the multiplicity of regular solutions and bifurcation diagrams. An essential ingredient of this study is analyzing the number of intersection points between the singular and regular solutions for rescaled problems. In the particular case of the exponential nonlinearity, we obtain the convergence of regular solutions to the singular and analyze the intersection number depending on the parameter $k$ and the dimension $d$.

Keywords: $k$-Hessian equation; singular solution; multiplicity of regular solutions; number of intersection points; bifurcation diagrams; Liouville-Bratu-Gelfand problem

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1. Introduction

In this paper, we are concerned with the following Dirichlet problem:

\[
\begin{cases}
S_k(D^2w) = \lambda F(-w) & \text{in } \Omega, \\
w < 0, & \text{in } \Omega, \\
w = 0, & \text{on } \partial\Omega,
\end{cases}
\]

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, $F$ is a positive function increasing to infinity, whose exact properties will be specified later, and $\lambda > 0$ is a parameter. Above, $1 \leq k \leq d$ is an integer, where $d > 2$ is the dimension, and $S_k(D^2w)$ is the $k$-Hessian, i.e., the symmetric elementary function of the $k$-th order of the eigenvalues of $D^2w$.

We prove the existence of a specific radial singular solution to (1) in the case when $\Omega$ is a unit ball and obtain its exact asymptotic behavior in a neighborhood of the origin. We furthermore demonstrate that this specific singular solution allows to study properties of radial regular solutions to (1) such as bifurcation diagrams and multiplicity, where an important tool is the number of intersection points between the singular and regular solutions for rescaled problems.

1.1 Elliptic problems with nonlinearities increasing to infinity

In recent years, semilinear elliptic problems of the form

\[
\begin{cases}
-\Delta u = \lambda G(u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial\Omega,
\end{cases}
\]

where $G$ is smooth and increasing to infinity, have been of significant interest; see, e.g., [1, 2, 3, 7, 9, 10, 11, 12, 18, 19, 24, 28, 30, 32]. This list of references is by no means complete, so we refer the reader to [1, 3, 7, 19, 32], and monograph [10] for a survey of results on this subject. The following important points are usually addressed when dealing with this type problems: 1) the existence of a critical parameter $\lambda^#$ such that for each $\lambda \in (0, \lambda^#)$ there exists a regular solution $(u_\lambda, \lambda)$ to problem (2), and there are no
solutions to (2) for $\lambda > \lambda^#$; 2) the existence of a parameter $\lambda^* \in (0, \lambda^#]$ such that problem (2) possesses a singular solution for $\lambda = \lambda^*$. Since a tremendous amount of papers (see, e.g., the above-cited ones and references therein) has been dedicated to problems of type (2) and, in particular, to answering the above questions, this topic can be regarded as fundamental.

In this work, we are concerned with a related problem; namely, with problem (1) involving a $k$-Hessian and a function $F$ under quite flexible imposed assumptions. $k$-Hessian equations constitute an important class of fully nonlinear PDEs; so they have been studied by many authors [6, 8, 33, 34, 35, 38, 39, 40, 41]. When restricted to so-called $k$-admissible solutions (for the definition see e.g. [38]), these equations are elliptic and their solutions enjoy properties similar to those of solutions to semilinear equations involving the Laplacian. Furthermore, $k$-Hessian equations are of interest in geometric PDEs [15] and differential geometry [36].

Problem (1) is in turn a larger generalization, compared to those that exist in the literature [12, 18, 24, 28], of the classical Gelfand problem. The latter deals with positive solutions to the system

$$\begin{cases}
\Delta u + \lambda e^u = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$

Problem (3) is a fundamental problem that arises in many theories, such as non-linear diffusions generated by non-linear sources [20, 21, 23], thermal ignition of a mixture of chemically active gases [11], unstable membrane [4], and gravitational equilibrium of polytropic stars [5, 17], among others. For problem (3), one is mainly interested in the existence and multiplicity of regular solutions, as well as bifurcation diagrams. Most of authors were concentrated on the case of a unit ball, e.g., [11, 19], since in this case, it is known that all solutions are radial [14] and problem (3) has the following simple representation:

$$\begin{cases}
u'' + \frac{d-1}{r}v' + \lambda e^v = 0, \\
u'(0) = u(1) = 0.
\end{cases}$$

In [11], Gelfand proved the existence of the value $\lambda$ for which (4) has an infinite number of solutions. In [19], the authors completely described the relation between the multiplicity of solutions to (4) and the dimension $d$ of the ball.

Problem (4) was further generalized by various authors in the following two directions: replacing $e^u$ with a more general function while keeping the radial Laplacian [12, 24, 28, 30]; and replacing the Laplacian with the radial operator

$$L(u) = r^{-\gamma}(|u'| \beta u')'$$

while preserving the term $e^u$ [18]. More specifically, in [12], the term $e^u$ was replaced with the $n$-times iterated exponential (henceforth denoted by $\exp^n(u)$), in [24], it was replaced with $e^{nw}$, and in [28], this term was replaced with $e^{u} + "lower-order term"$.

1.2 Our setting: operator $L$ and function $F$

In this work, we are concerned with negative radial solutions to problem (1) on a unit ball. At the same time, our methods allow to study the convergence of positive regular solutions to the singular for the equation $L(u) + \lambda e^u = 0$. To hit both of these goals, in (1), we make the substitution $w = -u$, and by this, we reduce problem (1) to its radial form (see Appendix for details) as follows:

$$\begin{cases}
L(u) + \lambda e^f(u) = 0, & 0 \leq r < 1, \\
u > 0, & 0 \leq r < 1, \\
u(1) = 0.
\end{cases}$$
where $L$ is defined by (5) and $f = \ln F$ satisfies the assumptions below. For the sake of clarity, we formulate these assumptions (in terms of $f$ and the inverse function $g = f^{-1}$) in a simplified manner. For the actual and more general set of assumptions, we redirect the reader to Subsection 2.1.

1. $f$ is smooth and increasing to infinity;
2. $|g''(t)|$ monotonically decreases to zero;
3. $\lim_{t \to +\infty} \frac{\log(1+\varepsilon(t))}{g(t+\varepsilon(t))} = 1$, where $\varepsilon(t) = O(g''(s))$;
4. As $t \to +\infty$, $\frac{g'(t)}{g'(t+\alpha(t))} \to 1$ and $\frac{g''(t)}{g''(t+\alpha(t))} = O(1)$;
5. $\lim_{u \to +\infty} f'(u) = +\infty$, $\lim_{u \to +\infty} f''(u) = 0$;
6. $\alpha > \beta + 1$ (i.e., $d > 2k$ for the $k$-Hessian); $\beta \geq 0$; $\gamma + 1 > \alpha$.

We emphasize that problem ($P_{\lambda}$) is an extension of problem (4) in the two aforementioned directions simultaneously, i.e., replacing the Laplacian with the operator (5), which can be roughly regarded as the radial case of a $k$-Hessian, and replacing $\exp(u)$ with a rather general term $\exp(f(u))$, which encompasses exponential-type nonlinearities previously considered in the literature [12, 24, 28], such as $\exp(u)$, $\exp(w)$, the iterated exponential $\exp^n(u)$, $\exp(u) + \text{lower-order term}$ (see Subsection 2.2). In addition, due to flexibility of our assumptions, $\exp(f(u))$ provides a plenty of new examples of nonlinearities which have not been considered in the literature, but can be treated within our framework.

We allow the parameters $\alpha$, $\beta$, and $\gamma$ to take a larger range of values than it is required for a $k$-Hessian since it does not give us extra work in the proofs and allows to include the important case of a $p$-Laplacian. The exact values of $\alpha$, $\beta$, and $\gamma$ for a $k$-Hessian and a $p$-Laplacian are summarized in the table below. The table includes the parameter $\theta = \gamma + 2 + \beta - \alpha$ whose role will become clear later.

|          | $\alpha$ | $\beta$ | $\gamma$ | $\theta$ |
|----------|----------|----------|----------|----------|
| $k$-Hessian | $d-k$    | $k-1$    | $d-1$    | $2k$     |
| $p$-Laplacian | $d-1$    | $p-2$    | $d-1$    | $p$      |

Remark 1.1. Remark that since the solutions of interest to problem (1) are negative, according to [38], the operator $S_k(D^2w)$ (regarded as a $d \times d$ matrix) is elliptic on those solutions. On the other hand, if $u$ is a solution to problem (P), the operator $S_k(D^2u(\cdot))$ may not be elliptic since $S_k(D^2u) = (-1)^k S_k(D^2w)$. Its ellipticity depends on the parity of $k$.

1.3 Contributions of this work and comparison with other results

One of the main results of this work is a construction of a negative radial singular solution to problem (1) on a unit ball, which is reduced to a construction of a positive radial singular solution to problem ($P_{\lambda}$) (see Appendix), with an exact asymptotic behavior in a neighborhood of the origin. By a positive singular solution to problem (1) in a ball, we understand a $C^2$-function which solves problem (1) at all points of the ball except for the origin, and tends to the negative infinity as the argument goes to 0. By a singular solution to problem (P), we understand a $C^2$-function which solves problem (P) at all points of (0,1] and tends to the positive infinity as the argument goes to 0+.

Radial singular solutions on a unit ball have attracted significant interest from many researchers [2, 12, 13, 24, 25, 27, 28, 29, 30, 31]. Most of these works deal however with semilinear equations involving the Laplacian. Papers [13] and [31] study radial singular solutions to equations with the operator (5), but they do not discuss asymptotics for these solutions, and furthermore, they study different types of nonlinearities. Among applications of singular solutions are bifurcation diagrams for elliptic problems [12, 24, 28, 29, 30] and blow-up behavior of solutions to parabolic problems [26, 37]. Asymptotic representations
for radial singular solutions on a unit ball were previously obtained in [12, 24, 27, 28]. However, these papers deal with the radial Laplacian and very particular types of the function \( f(u) \). Moreover, the asymptotics of singular solutions obtained in [12, 24, 28] are included in ours as particular cases. Our assumptions on \( f \) are quite flexible, so examples of \( f \) include not only \( \exp^{\alpha}(u) \), \( u^p \), and \( u + \text{"lower-order term"} \) (cf. [12, 24, 28]), but also many others. As such, we show that \( f(u) = \exp^{\alpha}(u^p) \), satisfies our assumptions, encompassing both \( \exp^{\alpha}(u) \) from [12] and \( u^p \) from [24].

We also remark that, compared to [25, 29, 30], our results are valid for the operator \( L \), given by (5) and therefore generalizing the Laplacian, and, on the whole, our work contains a sharp asymptotic representation of a singular solution which is not present in the aforementioned works.

We state our main theorem on the asymptotic representation of a radial singular solution as follows:

**Theorem 1.1.** Assume (1)–(6). Then, there exists \( \lambda^* > 0 \) such that problem \((P_\lambda)\) with \( \lambda = \lambda^* \) possesses a radial singular solution of the form

\[
\begin{align*}
  u^*_\rho(r) &= g(Z(r)) + O(g''(\ln r^{-\theta})) , \quad \text{where} \\
  Z(r) &= \ln \left\{ \theta^{\beta+1}(\alpha - \beta - 1) \right\} - \ln \lambda^* - \theta \ln r \\
  &\quad + (\beta + 1) \ln g'(\tau) + (\beta + 1) g''(\tau) \ln g'(\tau) \quad \text{with} \quad \tau = \ln \left\{ \frac{\beta + 1}{\ln r^{-\theta}} \right\}.
\end{align*}
\]

The important particular case is the exact form for the singular solution to problem \((P_\lambda)\) in the case \( f(u) = u \). Namely, the following corollary holds.

**Corollary 1.1.** Assume (6) and consider problem \((P_\lambda)\) with \( f(u) = u \). Then, \((u^*_\rho, \lambda^*)\), given by

\[
  u^*_\rho(x) = -\theta \ln |x|, \quad \lambda^* = \theta^{\beta+1}(\alpha - \beta - 1),
\]

is a radial singular solution to problem \((P_\lambda)\).

We furthermore use the singular solution \((u^*_\rho, \lambda^*)\), constructed in Theorem 1.1, to study properties of regular solutions such as bifurcation diagrams and multiplicity. Specifically, we obtain the following result.

**Theorem 1.2.** Assume (1)–(6) and let \( \alpha - \beta - 1 < \frac{4d}{\beta+1} \) (\( d < 2k + 8 \)). Further, we let \((u_{\lambda(\rho)}, \lambda(\rho))\) be the family of regular solutions to problem \((P_\lambda)\) parametrized by \( \rho = \sup_{r \leq 1} |u_{\lambda}(r)| \), and let \((u^*_\rho, \lambda^*)\) be the singular solution constructed in Theorem 1.1.

Then, as \( \rho \to +\infty \), \( \lambda(\rho) \) oscillates around \( \lambda^* \). In particular, there are infinitely many regular solutions to problem \((P_\lambda)\) for \( \lambda(\rho) = \lambda^* \).

An important tool for obtaining the above result is the number of intersection points between the regular and singular solutions to the equation

\[
L(u) + e^{f(u)} = 0.
\]

Note that \( u^*(r) = u^*_\rho(r(\lambda^*)^{-\frac{1}{\theta}}) \) is a singular solution to (7). In general, the change of variable

\[
u(r) = u_{\lambda}(r(\lambda^*)^{-\frac{1}{\theta}})
\]

brings the equation in \((P_\lambda)\), containing \( \lambda \), to equation (7). As such, the radial regular solution \( u(\cdot, \rho) \) to (7) is obtained from the radial regular solution \((u_{\lambda(\rho)}, \lambda(\rho))\) to \((P_\lambda)\) by the change of variable (8). We have the following result.

**Theorem 1.3.** Assume (1)–(6) and let \( \alpha - \beta - 1 < \frac{4d}{\beta+1} \). Then, for any \( \delta > 0 \), it holds that

\[
\lim_{\rho \to +\infty} \mathcal{L}_{\rho}(u(\cdot, \rho) - u^*(\cdot)) = +\infty,
\]

where \( u \) and \( u^* \) are regular and singular solutions to (7) and \( \mathcal{L}_{\rho}(u - u^*) \) is the number of zeros of \( u - u^* \) on the interval \((0, \delta)\).
Furthermore, we turn our attention to a study of radial regular solutions. The result that we obtain in this direction is the following.

**Theorem 1.4.** Assume (1), (5), and (6). Then, there exists a parameter $\lambda^\# > 0$, depending on $\alpha$, $\beta$, and $f$, such that any regular solution $(u, \lambda)$ to problem $(P_\lambda)$ satisfies $\lambda \leq \lambda^\#$. Moreover, for any $\lambda < \lambda^\#$, there exists a regular solution to $(P_\lambda)$, and for $\lambda = \lambda^\#$, the solution can be either regular or singular.

The actual set of assumptions for Theorem 1.4 is, again, somewhat weaker, and the reader is redirected to Subsection 3.3.

Finally, we study the equation

$$L(u) + e^u = 0$$

In [18], the authors considered problem $(P_\lambda)$ with the nonlinearity $e^u$ and obtained the result of Theorem 1.2 along with the convergence $\lambda(\rho) \to \lambda^*$ as $\rho \to +\infty$. In the current work, we complement the aforementioned results by studying the number of intersection points between the regular and singular solutions to (9), depending on $\alpha$, $\beta$, and $\gamma$, and by proving the convergence of regular solutions to the singular as $\rho \to +\infty$. Namely, we have the following results.

**Theorem 1.5.** Assume (6). Suppose $0 < \alpha - \beta - 1 < \frac{4\theta}{\beta+1} (2k < d < 2k + 8)$. Then, $\mathcal{L}_{(0, +\infty)}(u - u^*) = +\infty$.

**Theorem 1.6.** Assume (6). Suppose $\alpha - \beta - 1 \geq 4\theta(\beta + 1) (d \geq 2k + 8k^2)$. Then, $\mathcal{L}_{(0, +\infty)}(u - u^*) = 0$.

**Theorem 1.7.** Assume (6). Then,

$$\lim_{\rho \to +\infty} u(\cdot, \rho) = u^* \quad \text{and} \quad \lim_{\rho \to +\infty} (u_{\lambda(\rho)}, \lambda(\rho)) = (u^*, \lambda^*) \quad \text{in} \quad C_{loc}(0, +\infty).$$

**Remark 1.2.** Remark that the results on the intersection number, similar to those of Theorems 1.5 and 1.6, are known for the equation $L(u) + |u|^{p-1}u = 0$ (see e.g. [29]). Here they are obtained for the exponential counterpart of this equation.

### 1.4 Methods

In this work, we use combinations of newly developed methods along with extensions, from the Laplacian to the $k$-Hessian, of existing techniques. New methods are used for construction of a singular solution and obtaining its sharp asymptotic (Theorem 1.1), for proving the convergence of regular solutions to the singular (Theorem 1.7), and for showing that the regular and singular solutions to equation (9) do not intersect each other if $\alpha - \beta - 1 \geq 4\theta(\beta + 1)$ (Theorem 1.5). To be specific, in the proof of Theorem 1.1, we arrive at the equation

$$v'' - (\hat{\alpha} - 1)v' = -\frac{e^{-\theta t + f(v)}}{|v|^\beta}$$

which is equivalent to the equation in $(P_\lambda)$, but the variable $t$ varies in a neighbourhood of infinity. The difficulty arises due to the fact that the denominator on the right-hand side contains $|v|^\beta$, the term which is not present in the Laplacian case ($\beta = 0$). This does not allow to apply methods known for semilinear equations with the Laplacian (see, e.g., [12, 24, 28]). In particular, the singular solution $v^*$ to (11) takes the form

$$v^*(t) = g(\theta t + \varphi_1(t) + \varphi_2(t)) + \eta(t),$$

where

$$\varphi_1 = \ln\left\{\theta^\beta (g'(\theta t) + (\beta + 1) g''(\theta t) \ln g'(\theta t))^{\beta}\right\},$$

$$\varphi_2 = \ln \left\{\theta \left(\frac{\alpha}{\beta + 1} - 1\right)\right\} + \ln\left\{g'(\theta t) + (\beta + 1) g''(\theta t) \ln g'(\theta t)\right\}$$

and

$$\eta(t) = \rho^\alpha \ln \rho + \left(\frac{\alpha}{\beta + 1} - 1\right) \frac{\rho^\alpha}{\beta + 1} + \left(\theta - 1\right) \frac{\rho^\alpha}{\beta + 1} + C_1.$$
and \(\eta(t) = O(g''(\theta t))\). Remark that the term \(\varphi_1\) does not appear in the Laplacian case; it was for the first time introduced in this work. Its role is to make certain cancelations with the denominator \(|v'v''|^{\beta}\) to ensure the possibility to reduce (11) to a pair of semilinear equations with respect to \(\eta\) and \(\eta'\). The order of the reminder \(\eta\), in the general setting, was also first established in this work. Additional technical difficulties arise due to the fact that the function \(f\) (unlike [12, 24, 28]) is arbitrary, so the asymptotic representation (6) is expressed via the derivatives of \(f^{-1}\).

Furthermore, in the proof of Theorem 1.7, we use the result of Section 3 on the relative compactness of regular solutions on each compact interval of \((0, +\infty)\) (along with the result of [18]). To the best of our knowledge, the aforementioned technique is new. Also, it is worth to mention Theorem 1.6 whose proof does not appear to have analogs in the literature; while the result itself, together with Theorem 1.5, can be regarded as fundamental. Indeed, Theorems 1.5 and 1.6 study the number of intersection points between the regular and singular solutions for the canonical equation (9), the result which did not appear in the literature prior to this work.

Furthermore, many of our methods were previously unknown in the context of \(k\)-Hessian equations. They represent non-trivial extensions, from the Laplacian to the \(k\)-Hessian, of existing techniques. For example, in the proof of Theorem 1.3, dealing with the intersection number between the regular and singular solutions, one obtains a certain equation via the following transformation of equation (7):

\[
\tilde{u}(s, \rho) = F_1^{-1}(1) \frac{\rho}{\rho^+} F(u(\varepsilon, \rho)), \quad \varepsilon = \left(1 + \frac{F(\rho)}{F(1)}\right)^{(\beta + 1)/\beta},
\]

where \(F(u) = \int_0^{+\infty} \exp \left\{ -\int_{\rho}^{u} \right\} ds \) and \(F_1(u) = (\beta + 1) e^{-\frac{\rho}{\rho^+}}\). We believe that in the \(k\)-Hessian case, this transformation, in its exact form (14), was first used in this work. A version of this transformation, slightly different from (14), was introduced in [13]. The “Laplacian” analog of (14) can be found in, e.g., [30]. The important difference with the Laplacian case is the following: when we apply this transformation, the first two terms \(L(\tilde{u}) + e^\alpha\) of the transformed equation are exactly as in the canonical equation (9); however, the additional term \(s^\gamma (\tilde{u}')^{\beta+2} \left\{ \frac{I(u(\varepsilon, \rho))}{\beta+1} - 1 \right\}\), where \(I(u) = F(u)f'(u) \exp \left\{ \frac{f(u)}{\beta+1} \right\}\), has a more complicated structure and its convergence to zero is rather difficult to show. This convergence however is crucial to show the convergence of \(\tilde{u}\) to the regular solution of the canonical (limit) equation (9). Remark that even in the Laplacian case, considered in [30], the aforementioned convergence was announced without a proof. It is important to mention that in order to show the convergence of singular solutions, which is also crucial for proving Theorem 1.3, we employ our novel asymptotic (6) obtained in Theorem 1.1. Thus, we reinforce that the proof of Theorem 1.3 on the intersection number differs significantly from the Laplacian case and requires new tools developed in this work.

Furthermore, the proof of Theorem 1.2 on bifurcation diagrams relies heavily on Theorem 1.3 discussed above. In addition, the aforementioned proof contains a much more detailed analysis on the oscillation of \(\lambda(\rho)\) arround \(\lambda^*\) compared to [30] and preceding works.

Finally, we mention Proposition 3.1 that deals with upper and lower bounds for \(u(\cdot, \rho)^{-1}(B), B > 0\), whenever \(\rho\) is sufficiently big. In the Laplacian case, this result was obtained in [25] (Theorem 2.4) and then was widely used in the same paper. However, an extension of Theorem 2.4 to the operator \(L\) is as important for \(k\)-Hessian equations as the aforementioned theorem is important for semilinear equations with the Laplacian. Besides of being a non-trivial extension from the Laplacian setting, Proposition 3.1 is an important tool in the absence of which many of the results of this work could not be obtained. As such, Proposition 3.1 is used in the proof of the relative compactness of regular solutions to equation (7) (Proposition 3.2) which is used to obtain the convergence (10) in Theorem 1.7. Next, Proposition 3.1 is used to obtain Theorem 1.4. Finally, Proposition 3.1 is used
in the following chain of results leading to one of the main results of this work. Namely, it is directly applied in Lemma 4.2 implying Proposition 4.1 on the convergence of the singular and regular solutions of the equation obtained via the transformation (14) to the respective solutions of the canonical equation (9). This, in turn, implies Theorem 1.3 on the number of intersection points while the latter is used in the proof of Theorem 1.2 on bifurcation diagrams.

1.5 Structure of this work

Our paper is structured as follows. In Section 2, we prove the existence of a singular solution to \( (P_\lambda) \) and obtain its exact asymptotic behavior in a neighborhood of the origin. The main result of this section is Theorem 1.1. In Section 3, we prove Theorem 1.4. In the same section, we obtain the existence and relative compactness of regular solutions to problem \( (P_\lambda) \). As a byproduct, we obtain a version of Pohozaev’s identity for the operator (5). In Section 4, we study bifurcation diagrams and the number of intersection points between the regular and singular solutions for the case \( 0 < \alpha - \beta - 1 \leq \frac{4\theta}{\beta + 1} \). The main results of this section are Theorems 1.2 and 1.3. It is important to mention that we employ here the transformation (14) that reduces equation (7) to an equivalent equation whose limit results of this section are Theorems 1.2 and 1.3. This, in turn, implies Theorem 1.3 on singular and regular solutions of the equation obtained via the transformation (14) to the respective solutions of the canonical equation (9). The aforementioned equation is then studied in Section 5, where we obtain Theorems 1.5, 1.6, 1.7.

2. Singular solution to Problem \( (P_\lambda) \)

In this section, we will construct a positive radial singular solution to \( (P_\lambda) \).

2.1 Standing assumptions and useful lemmas

We start by introducing the general set of assumptions implying assumptions (1)–(6) from Subsection 1.2. Recall that \( g \) denotes the inverse function for \( f \).

(A1) \( f \in C^4([0, +\infty)), f' > 0, \) and \( \lim_{u \to +\infty} f(u) = +\infty \).

(A2) \( \lim_{t \to +\infty} g''(t) = 0 \) and \( \lim_{t \to +\infty} \frac{f'(g(t) + \varepsilon(t))}{t} = 1 \), where \( \varepsilon(t) = O(g''(t)) \).

(A3) \( \lim_{u \to +\infty} \frac{g'(t)}{g'(t + o(t))} = 1 \).

(A4) \( \lim_{u \to +\infty} \frac{g'(t)}{g(t)} \ln g'(t) = 0 \).

(A5) As \( t \to +\infty \),

(a) \( g'''(t) \ln g'(t)^n = O(g''(t)) \), \( n = 0, 1, 2 \), \( g^{(iv)}(t) \ln g'(t) = O(g''(t)) \),

(b) \( \sup_{t \geq 1} |g''(s)| = O(g''(t)) \), \( g''(t + o(t)) = O(g''(t)) \).

(A6) \( \alpha > \beta + 1 \) (i.e., \( d > 2k \) for the \( k \)-Hessian); \( \beta \geq 0; \) \( \theta > 0 \).

Remark 2.1. It will be rather convenient to have some alternative equivalent conditions at hand for (A2) and (A4). Namely, (A2) and (A4) are equivalent to

(A2′) \( \lim_{u \to +\infty} \frac{f''(u)}{f'(u)^3} = 0 \) and \( \lim_{u \to +\infty} \frac{f(u + \tilde{\varepsilon}(u))}{f(u)} = 1 \), where \( \tilde{\varepsilon}(u) = O(f''(u)/f'(u)^3) \).

(A4′) \( \lim_{u \to +\infty} \frac{f''(u)}{f'(u)^2} \ln f'(u) = 0 \).

We start showing (A4′). A straightforward computation gives

\[
\frac{f''(u)}{f'(u)^2} \ln f'(u) = \frac{g''(t)}{g'(t)} \ln g'(t), \quad \text{where} \quad u = g(t).
\]

To show the equivalence of (A2) and (A2′), we note that \( g''(f(u)) = f''(u)/f'(u)^3 \), and hence, the first limits in (A2) and (A2′) are equivalent. Substituting \( t = f(u) \), we represent \( \eta(t) \) as \( \phi(u)g''(f(u)) \), where \( \phi(u) \) is a bounded function. Defining \( \tilde{\eta}(u) = \phi(u) f''(u)/f'(u)^3 \), we obtain the desired equivalence.
Remark 2.2. For the $p$-Laplacian, the condition $\alpha > \beta + 1$ is interpreted as $p < d$.

Remark 2.3. Note that under (A4) and (A2), $\lim_{t \to +\infty} \frac{g''(t)}{g(t)} = 0$. Indeed, assume the opposite. Then, there exists a sequence $t_n \to +\infty$ such that $\left| \frac{g''(t_n)}{g(t_n)} \right| > M$ for some constant $M > 0$. By (A4), it should hold that $\lim_{n \to +\infty} g'(t_n) = 1$, which, in turn, implies that $\lim_{n \to +\infty} \frac{g''(t_n)}{g(t_n)} = 0$ by (A2).

Remark 2.4. Some of examples of the function $\phi(t) = g''(t)$ satisfying the first expression in (A5)-(b) are monotone functions and functions which can be represented as $\phi(t) = \phi_1(t)\phi_2(t)$, where $\phi_1$ is decreasing to zero as $t \to +\infty$, and $\phi_2$ is bounded from above and below, with an infinite number of maximum and minimum points, such as $2 + \sin t$ or $e^{\sin t}$.

Remark 2.5. In what follows, we will need the following facts: as $t \to +\infty$, 

$$e^{\frac{\pi}{4}}g'(t) = O(t e^{\frac{\pi}{4t}}), \quad e^{-\frac{\pi}{4}}g'(t) = O(te^{-\frac{\pi}{4t}}).$$

The above expressions obviously hold by (A2). Indeed, $g''(t)$ is bounded, and therefore, by Taylor’s formula, $g'(t)$ has at most linear growth.

It turns out that some of the conditions (A5)-(a) are fulfilled if $\lim_{u \to +\infty} f'(u) = +\infty$, or, which is the same, $\lim_{t \to +\infty} g'(t) = 0$.

Lemma 2.1. Suppose $\lim_{u \to +\infty} f'(u) = +\infty$. Then, conditions (A5)-(a), except for $n = 2$, are fulfilled. If, in addition, $\lim_{u \to +\infty} \frac{f''(u)}{f'(u)}(\ln f'(u))^2 = 0$, then all conditions in (A5)-(a) are fulfilled.

Proof. Note that $g''(t)$ cannot be identically equal to zero in a neighborhood of $+\infty$. Indeed, in this case $g'(t) = 0$ in this neighborhood which is a contradiction.

Note that, by (A4), $\lim_{t \to +\infty} g''(t) \ln g'(t) = 0$. By L'Hopital’s rule,

$$\lim_{t \to +\infty} \frac{g''(t)}{g''(t)} \ln g'(t) = \lim_{t \to +\infty} g''(t) \ln g'(t) - \lim_{t \to +\infty} \frac{g''(t)}{g'(t)} = 0.$$ 

The above inequality implies, in particular, that $\lim_{t \to +\infty} g'''(t) \ln g'(t) = 0$. Again, by L'Hopital’s rule,

$$\lim_{t \to +\infty} \frac{g^{(4)}(t)}{g''(t)} \ln g'(t) = \lim_{t \to +\infty} \frac{g^{(4)}(t)}{g''(t)} \ln g'(t) - \lim_{t \to +\infty} \frac{g''(t)}{g'(t)} = 0.$$ 

Note that $\lim_{t \to +\infty} \frac{g^{(4)}(t)}{g''(t)} = 0$. Therefore,

$$\lim_{t \to +\infty} \frac{g^{(4)}(t)}{g''(t)} \ln g'(t) = \lim_{t \to +\infty} \frac{g^{(4)}(t)}{g''(t)} \ln g'(t) \lim_{t \to +\infty} \frac{g''(t)}{g'(t)} = 0.$$ 

Finally, a straightforward computation shows that

$$\frac{f''(u)}{f'(u)}(\ln f'(u))^2 = \frac{g''(t)}{g'(t)}(\ln g'(t))^2, \quad \text{where} \quad u = g(t).$$ 

The first identity in (A5)-(a) for $n = 2$ follows now by L'Hopital’s rule. \qed

Remark 2.6. Remark that Lemma 2.1 justifies condition (5) in Subsection 1.2.

Lemma 2.2. If $\zeta(t)$ is a differentiable function $\mathbb{R}_+ \to \mathbb{R}$ such that $\lim_{t \to +\infty} \zeta'(t) = 0$, then $\lim_{t \to +\infty} \frac{\zeta(t)}{t} = 0$.

Proof. We have

$$\zeta(0) - \int_0^t |\zeta'(s)| ds \leq \zeta(t) \leq \zeta(0) + \int_0^t |\zeta'(s)| ds.$$ 

The integral $\int_0^t |\zeta'(s)| ds$ is either bounded or tends to $+\infty$ as $t \to +\infty$. In the first case the conclusion of the lemma is obvious. In the second case it follows from L'Hopital’s rule. \qed
2.2 Examples of nonlinearities

Here, we give examples of functions \( f(u) \) in \( (P_k) \) satisfying (A1)–(A5).

Example 1. \( f(u) = \exp^\circ n(u^p) \) \( (p > 0, n \geq 1) \). First, introduce the notation for the iterated exponential and the iterated logarithm:

\[
\exp^\circ n(t) = \underbrace{\exp \circ \exp \circ \cdots \circ \exp}_n(t); \quad \ln^\circ n(t) = \underbrace{\ln \circ \ln \circ \cdots \circ \ln}_n(t).
\]

The inverse function to \( f \) is \( g(t) = [\ln^\circ n(t)]^\frac{1}{p} \). We prove that (A1)–(A5) are fulfilled for the pair \( f, g \).

Remark 2.7. Remark that in [12], \( f(u) = \exp^\circ n(u) \), and in [24], \( f(u) = u^p \), both are particular cases of \( f(u) = \exp^\circ n(u^p) \) discussed in this example. In addition, we remark that [12] and [24] deal with the classical radial Laplacian, while our partial differential operator takes the form (5).

First, we compute

\[
g'(t) = \frac{1}{p} \frac{1}{\ln^\circ n(t)^{1 - \frac{1}{p}}} \frac{1}{\ln^\circ (n-1)(t) \ln^\circ (n-2)(t) \cdots t},
\]

\[
g''(t) = \frac{1}{p} \left( \frac{1}{p} - 1 \right) \frac{1}{\ln^\circ n(t)^{2 - \frac{1}{p}}} \frac{1}{\ln^\circ (n-1)(t)^2 \cdots t^2} - \ln^\circ n(t)^{1 - \frac{1}{p}} \ln^\circ (n-1)(t)^2 \cdots t^2 - \cdots - \ln^\circ n(t)^{1 - \frac{1}{p}} \ln^\circ (n-1)(t)^2 \cdots t^2.
\]

Note that \( \lim_{t \to +\infty} g''(t) = 0 \). Moreover, \( \lim_{t \to +\infty} \frac{g''(t)}{g'(t)} (\ln g'(t))^2 = 0 \). To see the latter, we note that \( \frac{g''(t)}{g'(t)} = O(\frac{1}{t}) \) and \( \ln g'(t) = O(\ln t) \). Therefore, by Lemma 2.1, (A5)-(a) is fulfilled. Also, \( |g''(t)| \) is decreasing, so the first identity in (A5)-(b) is fulfilled. The above argument also implies (A4).

A verification of (A1) is straightforward. To verify (A3), it is sufficient to show that for the iterated logarithm, it holds that \( \lim_{t \to +\infty} \ln^\circ n(t) \ln^\circ (n+o(t)) = 1 \). We prove the following statement by induction on \( n \): \( \ln^\circ n(t + o(t)) = \tau + o(\tau) \), where \( \tau = \ln^\circ n(t) \). For \( n = 1 \), it is obvious since \( \ln(t + o(t)) = \ln t + o(1) \). Suppose we proved the statement for \( n - 1 \), i.e., \( \ln^\circ (n-1)(t + o(t)) = \tau + o(\tau) \) with \( \tau = \ln^\circ (n-1)(t) \). Then, \( \ln^\circ n(t) = \ln(\tau + o(\tau)) = \ln t + o(\ln(\tau)) = \ln^\circ n(t) + o(\ln^\circ n(t)) \), which implies (A3). The same argument implies the second expression in (A5)-(b).

It remains to verify (A2). Here, without loss of generality we can set \( p = 1 \). Indeed, define the function \( \tilde{g}(t) = \ln^\circ n(t) \) so that \( g(t)^p = \tilde{g}(t) \). We have

\[
(g(t) + O(g''(t)))^p = \tilde{g}(t) \left( 1 + O\left( \frac{g''(t)}{g(t)} \right) \right)^p = \tilde{g}(t) + O(g''(t))g(t)^{p-1} = \tilde{g}(t) + O(\tilde{g}'(t))
\]

since \( \tilde{g}'(t) = \frac{p g''(t) g(t)^{p-1}}{g(t)^p} + (p-1) \cdot \frac{g''(t)^2}{g(t) g'''(t)} = g''(t) g(t)^{p-1} O(1) \). Indeed, \( \frac{g''(t)}{g'(t)} = \frac{1}{t} + o(\frac{1}{t}) \) and \( \frac{g'(t)}{g(t)} = O(\frac{1}{t}) \). We further argue by induction on \( n \). For \( n = 1 \), (A2) is clearly satisfied. Suppose (A2) is true for \( n - 1 \). We have

\[
\exp(\tilde{g}(t) + O(\tilde{g}''(t))) = \ln^\circ (n-1)(t) \exp\{\ln^\circ (n-1)(t) (1 + O(\tilde{g}''(t)))\} = \tilde{g}(t) + O(\tilde{g}''(t)), \quad \text{where} \quad \tilde{g}(t) = \ln^\circ (n-1)(t).
\]

Indeed, one immediately verifies that \( \tilde{g}''(t) \ln^\circ (n-1)(t) = \tilde{g}''(t) + o(\tilde{g}''(t)) \). (A2) now follows from the induction hypothesis.
Example 2. \( f(u) = u^p, \ p > \frac{1}{2} \). We restrict here the range of \( p \) to \( (\frac{1}{2}, +\infty) \) in order to guarantee that \( \lim_{t \to +\infty} g''(t) = 0 \). Indeed, \( g''(t) = \frac{1}{p} \left( \frac{1}{p} - 1 \right) t^{\frac{1}{p} - 2} \). The verification of (A1)–(A5) is straightforward.

Example 3. Perturbed Gelfand’s problem. Take \( f(u) = u + \vartheta(u) \), where \( \vartheta(u) \) is a \( C^4 \)-smooth function with the following properties:

\begin{itemize}
\item[(i)] \( \lim_{u \to +\infty} \vartheta^{(n)}(u) = 0 \) for \( n = 0, 1, 2 \);
\item[(ii)] \( u + \vartheta(u) \) is increasing on \([0, +\infty)\);
\item[(iii)] \( \vartheta'''(u) = O(\vartheta''(u)), \vartheta^{(iv)}(u) = O(\vartheta''(u)), \vartheta''(u) = O(\vartheta''(u)), \vartheta'(u) = O(\vartheta''(u)) \), and \( \vartheta''(u) + o(u) = O(\vartheta''(u)) \).
\end{itemize}

Let us check (A1)–(A5). (A1) is clear. (A2') and (A4'), which are equivalent to (A2) and, respectively, (A4) by Remark 2.1, follow immediately from (i).

To check (A3), note that \( g(t) = t + \vartheta(t) \), where \( \vartheta(t) = \vartheta(u) \) with \( u = g(t) \). We further note that \( g'(t) = 1 + \vartheta'(t) = \frac{1}{f(u)} = \frac{1}{1+\vartheta(u)} \), which shows that \( \lim_{t \to +\infty} \vartheta'(t) = 0 \) and immediately implies (A3). It remains to verify (A5). It is straightforward to compute

\[ \vartheta''(t) = -\frac{\vartheta''(u)}{(1 + \vartheta(u))^2} \text{ and } \vartheta'''(t) = \frac{4\vartheta'''(u)}{(1 + \vartheta(u))^5} - \frac{\vartheta''''(u)}{(1 + \vartheta(u))^5}, \]

which, by (iii), implies the first identity in (A5)-(a). Computing the fourth derivative of \( \vartheta(t) \), it is straightforward to obtain the second identity in (A5)-(a). The expression for \( \vartheta''(t) \) implies (A5)-(b) if we note that \( t + o(t) = f(u + o(u)) \).

2.3 Frequently used notation

For the reader’s convenience, we introduce a list of symbols that will be frequently used throughout the paper:

\[ \theta = \gamma + 2 + \beta - \alpha \; ; \; \hat{\alpha} = \frac{\alpha}{\beta + 1}; \; \hat{\theta} = \frac{\theta}{\beta + 1}. \]

\((u_\lambda, \lambda)\) and \((u_\gamma^*, \gamma^*)\) denote regular and singular solutions to problem \((P_\lambda)\).

\( u \) and \( u^* \) are rescaled, by (8), regular and singular solutions that solve (7).

\( g' \), \( g'' \), \( g''' \), \( g^{(iv)} \) denote derivatives of the respective orders w.r.t. their arguments.

2.4 Singular solution in a neighborhood of the origin

One of the main results of this work is the representation of a singular solution announced in Theorem 1.1. The statement of this theorem holds, however, under the more general assumptions (A1)–(A6).

Theorem 2.1. Assume (A1)–(A6). Then, the statement of Theorem 1.1 holds true.

Let \( \Gamma(t) = F^{-1}(t) \); namely, \( \Gamma(t) = g(\ln t) \). The result of Theorem 2.1 can be rewritten as follows.

Corollary 2.1. Assume (A1)–(A6). Then, there exists \( \lambda^* > 0 \) such that problem \((P_{\lambda^*})\) possesses a singular radial solution of the form

\[ u^*_\lambda(r) = \Gamma(\bar{Z}(\tau)) + O(\Gamma''(\tau)\tau + \Gamma''(\tau)\tau^2), \quad \text{where} \quad \tau = \frac{\beta + 1}{\lambda^* + \tau}, \]

and \( \bar{Z}(\tau) = \theta^{\beta+1}(\alpha - 1)\tau^{\beta+2}[\Gamma'(\tau) + (\beta + 1)\ln\{\Gamma'(\tau)\tau\} + \Gamma''(\tau)\tau^2]^{\beta+1} \).

Proof. Note that \( g'(\ln t) = \Gamma'(\tau)\tau \) and \( g''(\ln t) = \Gamma'(\tau)\tau + \Gamma''(\tau)\tau^2 \). Therefore,

\[ g\left( \ln \left\{ \theta^{\beta+1}(\alpha - 1) + \ln \tau + (\beta + 1)\ln\{g'(\ln t) + (\beta + 1)g''(\ln t)\ln g'(\ln t)\} \right\} \right) = g\left( \ln\left\{ \theta^{\beta+1}(\alpha - 1)\tau g'(\ln t) + (\beta + 1)g''(\ln t)\ln g'(\ln t)\}^{\beta+1} \right\} \]

\[ = \Gamma(\theta^{\beta+1}(\alpha - 1)\tau^{\beta+2}[\Gamma'(\tau) + (\beta + 1)\ln\{\Gamma'(\tau)\tau\} + \Gamma''(\tau)\tau^2]^{\beta+1}). \]

This completes the proof of the corollary. \( \square \)
In Corollary 2.2, we obtain asymptotic representations for singular solutions when \( e^{f(u)} \) is one of the nonlinearities considered in \([12, 24, 28]\). The respective representations from \([12, 24, 28]\) agree with ours when \( L \) is the radial Laplacian.

**Corollary 2.2.** Assume (A6) and consider the following three particular cases of \( f \):

(a) \( f(u) = e^u \); (b) \( f(u) = u^p \), \( p > \frac{1}{2} \); (c) \( f(u) = u + \vartheta(u) \), where \( \vartheta(u) \) satisfies assumptions (i), (ii), (iii) from Example 3 in Subsection 2.2 and such that \( |\vartheta(u)| + |\vartheta'(u)| \leq C_0 e^{-\beta u} \).

Then,

\[
(a) \quad u^*_r \left( \frac{r}{\lambda^{*1/p}} \right) = \ln \left\{ \theta \ln \frac{1}{r} + \frac{\alpha - 1}{\ln \theta} + (\beta + 1) \ln \left( 1 + \frac{\ln \theta}{\ln r} \right) \right\} + O \left( \ln \left( \frac{1}{r} \right)^{\frac{\beta - 2}{2}} \right);
\]

\[
(b) \quad u^*_r \left( \frac{r}{\lambda^{*1/p}} \right) = \left( \theta \ln \frac{1}{r} + (\beta + 1) \ln k + \frac{\ln (\alpha - \beta - 1) \theta^{\beta + 1}}{\theta^{\beta + 1}} \right) + \frac{1}{\ln \theta} + (\beta + 1) \ln \left( 1 + \frac{\ln \theta}{\ln r} \right) \right\} \right\} + O \left( \ln \left( \frac{1}{r} \right)^{\frac{\beta - 2}{2}} \right);
\]

\( (c) \quad u^*_r (r) = \ln \left\{ \theta^{\beta + 1} (\alpha - \beta - 1) \right\} - \ln \lambda^* - \theta \ln r + O((\theta^\alpha) \right). \)

Above, \( k = (1 + \beta)^{\frac{1}{2}}, \ \tau = \theta \ln \frac{1}{\theta}. \)

**Proof.** (a) and (b) follow immediately from representation (6). To show (c), we note that in a neighborhood of infinity, by (iii), \( \vartheta''(u) \) either equals zero or \( \vartheta''(u) = O(1) \). By L’Hopital’s rule and (i), \( \limsup_{u \to +\infty} \left| \vartheta''(u) / \vartheta(u) \right| \) is finite, which means that \( |\vartheta''(u)| \leq C_1 e^{-\delta u} \) for some constant \( C_1 > 0 \). Furthermore, the representations for \( \hat{\vartheta}, \hat{\vartheta}', \) and \( \hat{\vartheta}'' \) imply that in a neighborhood of infinity, \( |\hat{\vartheta}(t)| + |\hat{\vartheta}'(t)| + |\hat{\vartheta}''(t)| \leq C_2 e^{-\delta t} \) for some constant \( C_2 > 0 \). Representation (c) follows now from (6).

**Remark 2.8.** In example (b), we deal with the same nonlinear term as in [24]. We would like to emphasize the following fact. In [24], the asymptotic representation for the singular solution (Theorem 1.1), in the case \( 0 < p < 1 \), is given up to a rest term (not computed explicitly) which does not tend to zero. Unlike [24], our asymptotic representation includes additional explicit terms, tending to infinity, so our rest term is of a higher order than in [24] and goes to zero.

The proof of Theorem 2.1 is divided into the steps outlined below.

### 2.4.1 Equivalence problem in a neighborhood of infinity

It is convenient to rewrite problem \((P_\lambda)\) by using the change of variable \( t = \ln (\frac{\zeta}{r}) \), where \( \kappa > 0 \) is a constant to be specified later. First, we note that the equation in \((P_\lambda)\) can be rewritten as follows:

\[
u''(r)|u'(r)|^\beta + \hat{\alpha} \frac{u''(r)|u'(r)|^\beta}{r} + \lambda r^{-\alpha} e^f(u) = 0.
\]

Let us show that for a singular solution to \((P_\lambda)\), it holds that \((u^*_r)'(r) < 0\) for all \( r > 0 \). Note that if \( \beta > 0 \), from the above equation, it follows that a singular solution cannot have local maximum or minimum points, since in those points \( u'(r) = 0 \). Suppose \( \beta = 0 \). Then, every local extremum point, is necessarily a local maximum point, which contradicts to the fact that \( \lim_{r \to 0^+} u^*_r (r) = +\infty \).

By doing the change of variable \( t = \ln (\frac{\zeta}{r}) \) and defining \( v(t) = u(r) \), we transform \((P_\lambda)\) to the following problem:

\[
\begin{align*}
\nu'' - (\hat{\alpha} - 1) v' + \lambda \kappa^\beta e^{-\theta} e^f(v) &= 0, \quad t \in (\ln \kappa, +\infty), \\
v(\ln \kappa) &= 0.
\end{align*}
\]
By choosing $\kappa$ in such a way that $\frac{1}{\beta+1} = \kappa$, we arrive at equation (11) with the boundary condition $v(\ln \kappa) = 0$, where $\kappa = (\frac{\beta+1}{\alpha})^\frac{1}{\beta}$. This problem is equivalent to $(P_\lambda)$, but given in a neighborhood of infinity.

**Proposition 2.1.** Assume (A1)–(A6). Then, there exists $T > 0$ such that on $[T, +\infty)$, equation (11) possesses a singular solution of the form (12), where $\eta(t) = O(g''(\theta t))$ is a $C^2$-function.

**Remark 2.9.** Remark that by Lemma 2.2 and Remark 2.3, $\lim_{t \to +\infty} \ln g'(\theta t) = 0$. Therefore, $(\varphi_1 + \varphi_2)(t) = o(t)$.

**Proof.** Step 1. Transformation to a fixed-point problem. The idea is to obtain a semilinear equation with respect to $\eta$ by substituting $v^* = g + \eta$ into (11) and showing that the difference of singular terms that occur on the both sides of the resulting equation make a regular function. More specifically, we are going to show that the original problem is equivalent to a couple semilinear equations with respect to $\eta$ and $\zeta = \eta'$

\[
\begin{cases}
\eta''(t) - (\alpha - \beta - 1) \eta'(t) + \theta (\hat{\alpha} - 1) \eta(t) = \Phi(t, \eta, \zeta), \\
\zeta'(t) - (\alpha - \beta - 1) \zeta(t) = \Phi(t, \eta, \zeta) - \theta (\hat{\alpha} - 1) \eta(t),
\end{cases}
\]

(15)

where the function $\Phi$ is expected to be regular. For simplicity of notation, define

$$\varphi = \varphi_1 + \varphi_2,$$

where $\varphi_1$ and $\varphi_2$ are defined by (13). We start by transforming $\exp \{-\theta t + f(v^*)\} = \exp \{-\theta t + f(g(\theta t + \varphi) + \eta)\}$, taking into account that $g = f^{-1}$. It holds that

$$f(g(\theta t + \varphi) + \eta) = \theta t + \varphi + f'g(\theta t + \varphi) \vartheta(t, \eta)\eta,$$

where $\vartheta(t, \eta)$ takes values between 0 and 1. Let us transform the term $f'g(\theta t + \varphi) + \vartheta(t, \eta)$.

Define

$$\eta(t) = \theta t + \varphi + \frac{\theta (\hat{\alpha} - 1) \eta(t)}{\eta g'(\theta t + \varphi + \xi)} = \theta t + \varphi + \frac{\eta}{g'(\theta t + o(t))}.$$

Therefore,

\[
\exp \{-\theta t + f(v^*)\} = \frac{e^{\varphi_1}}{|v^*(t)|^\beta} e^{\varphi_2 + \eta g'(\theta t + o(t))},
\]

(17)

Let us transform the first factor on the right-hand side. Defining $\vartheta_1(t) = \theta^{-1} \varphi'(t)$, we obtain

\[
\frac{e^{\varphi_1}}{|v^*(t)|^\beta} = \frac{\theta^\beta (g'(\theta t) + (\beta + 1)g''(\theta t) \ln g'(\theta t))^{\frac{\beta}{2}}}{\theta^\beta |g'(\theta t + \varphi) + \vartheta_1(t)|^{\frac{\beta}{2}} + \theta^{-1} \eta^{\frac{\beta}{2}}}
\]

\[
= \frac{1}{1 + \frac{\eta}{g'(\theta t + \varphi) + \vartheta_1(t)}} \frac{1}{(1 + \vartheta_1)^\frac{\beta}{2}} \frac{1}{(1 + \vartheta_2)^\beta},
\]

(18)

where $\vartheta_2$ is defined through the formula

$$g'(\theta t + \varphi) = (g'(\theta t) + (\beta + 1)g''(\theta t) \ln g'(\theta t))(1 + \vartheta_2(t)).$$

Let us obtain expressions for $\vartheta_1$ and $\vartheta_2$ convenient for future computations. By (A4) and (A5),
(19) \( \vartheta_1(t) = \theta^{-1} \varphi'(t) = (\beta + 1) \frac{g''(\theta t) + (\beta + 1)g'''(\theta t) \ln g'(\theta t) + (\beta + 1) g''(\theta t)^2}{g'(\theta t) + (\beta + 1) g''(\theta t) \ln g'(\theta t)} = O\left(\frac{g''(\theta t)}{g'(\theta t)}\right). \)

Since

(20) \( \varphi(t) = (\beta + 1) \ln g'(\theta t) + O(1), \)

by using the expansion of \( g'(\theta t + \varphi) \) by Taylor’s formula around \( \theta t \), we obtain

(21) \( \vartheta_2(t) = \frac{g'(\theta t + \varphi) - g'(\theta t) - (\beta + 1) g''(\theta t) \ln g'(\theta t)}{g'(\theta t) + (\beta + 1) g''(\theta t) \ln g'(\theta t)} \)

\[ = \frac{g''(\theta t) O(1) + \int_0^1 (1 - r) g''(\theta t + r \varphi) d\varphi}{g'(\theta t) + (\beta + 1) g''(\theta t) \ln g'(\theta t)} = O\left(\frac{g''(\theta t)}{g'(\theta t)}\right). \]

The last identity holds by (A4), (A5), and the following argument. First of all, we note that by Taylor’s formula around \( \theta t \) and (20),

\[ \ln g'(\theta t + \varphi) = \ln g'(\theta t) + \ln \left(1 + O\left(\frac{g''(\theta t)}{g'(\theta t)}\right) + O\left(\frac{g''(\theta t)}{g'(\theta t)} \ln g'(\theta t)\right)\right) = \ln g'(\theta t) + o(1), \]

which, by Remark 2.9, implies that

\[ g'''(\theta t + r \varphi) \varphi^2 = O(g''(\theta t + \varphi)) = O(g''(\theta t)). \]

Note that since we are interested in solving equation (11) in a neighborhood of infinity, the expressions \( \frac{1}{1 + \frac{1}{\theta t}} \) and \( \frac{1}{1 + \frac{2}{\theta t}} \) are well-defined.

Next, by the results of Subsection 2.4.1, for any singular solution to \( (P_\lambda) \), it holds that \( (u^*_\lambda)'(r) < 0 \) for all \( r > 0 \). This implies that \( v^*(t) > 0 \) for all \( t \). Then, by Taylor’s formula, (18) can be transformed as follows:

\[ \frac{e^{\varphi_1}}{v^*(t)^\beta} = \left(1 - \frac{\beta \eta'}{\theta g'(\theta t)} + o(1) \frac{\eta'}{g'(\theta t)} + \frac{\eta'^2}{g'(\theta t)^2} \sigma(t, \eta')\right) \left(1 + O\left(\frac{g''(\theta t)}{g'(\theta t)}\right)\right), \]

where

(22) \( \sigma(t, \eta') = O(1) \int_0^1 \frac{(1 - r)}{(1 + rx)^{\beta + 2}} dr \quad \text{with} \quad x = \frac{\eta'}{\theta g'(\theta t + \varphi) (1 + \varphi_1)}. \)

Recalling that \( \frac{g''(\theta t)}{g'(\theta t)} \to 0 \) as \( t \to +\infty \) (see Remark 2.3), the last expression can be rewritten as

(23) \[ \frac{e^{\varphi_1}}{v^*(t)^\beta} = 1 - \frac{\beta \eta'}{\theta g'(\theta t)} + o(1) \frac{\eta'}{g'(\theta t)} + O(1) \frac{\eta'^2}{g'(\theta t)^2} \sigma(t, \eta') + O\left(\frac{g''(\theta t)}{g'(\theta t)}\right). \]

Taking into account that \( e^{\varphi_2} = \theta(\alpha - 1) g'(\theta t)(1 + o(1)) \), by (17), (19), and (23), expression (17) can be transformed as follows:

\[ \exp\left\{\frac{-\theta t + f(v^*)}{v^*(t)^\beta}\right\} = e^{\frac{\eta}{v^*(t)^\beta} + \frac{\eta}{v^*(t)^\beta} - \frac{\eta}{v^*(t)^\beta}} = e^{\frac{\eta}{v^*(t)^\beta}} \frac{e^{\varphi_2}}{v^*(t)^\beta} = e^{\frac{\eta}{v^*(t)^\beta}} \left[ e^{\varphi_2 - \beta(\alpha - 1) \eta' + o(1) \eta'} + O\left(\frac{\eta'^2}{g'(\theta t)^2} \sigma(t, \eta') + O\left(\frac{g''(\theta t)}{g'(\theta t)}\right)\right) \right]. \]

Taking into account that \( e^{\varphi_2} = \theta(\alpha - 1) g'(\theta t)(1 + o(1)) \), by (17), (19), and (23), expression (17) can be transformed as follows:

\[ \exp\left\{\frac{-\theta t + f(v^*)}{v^*(t)^\beta}\right\} = e^{\frac{\eta}{v^*(t)^\beta} + \frac{\eta}{v^*(t)^\beta} - \frac{\eta}{v^*(t)^\beta}} = e^{\frac{\eta}{v^*(t)^\beta}} \left[ e^{\varphi_2 - \beta(\alpha - 1) \eta' + o(1) \eta'} + O\left(\frac{\eta'^2}{g'(\theta t)^2} \sigma(t, \eta') + O\left(g''(\theta t)\right)\right) \right] = e^{\varphi_2} \left( e^{\frac{\eta}{v^*(t)^\beta} - \frac{\eta}{g'(\theta t + o(t)) - 1} + \frac{\eta}{g'(\theta t + o(t))} + e^{\varphi_2 - \beta(\alpha - 1) \eta'} + F_1(t, \eta, \eta') + F_2(t, \eta) \right). \]
where
\[
F_1(t, \eta, \eta') = \left[ o(1)\eta' + O(1) \frac{\eta'^2}{g'(\theta t)} \sigma(t, \eta') \right] \left( \frac{\eta}{g'(\theta t + o(t))} \right) - \beta (\hat{\alpha} - 1) \eta' \left( \frac{g(\theta t)(1 + o(1))}{g'\theta t + o(t)} - 1 \right) + O(g''(\theta t)) + O\left( e^{\frac{\sigma(t, \eta')}{n}} - 1 \right) + O\left( \frac{\eta}{g'(\theta t + o(t))} - 1 \right);
\]
\[
F_2(t, \eta) = O\left( g''(\theta t) \right).
\]

Now we are ready to evaluate differences of suitable singular terms that would result in regular functions. More specifically, we aim to obtain (15) from (11) by substituting \( v^*(t) = g(\theta t + \varphi) + \eta(t) \) into (11). We have
\[
e^{\varphi \eta} - (\hat{\alpha} - 1) \frac{d}{dt} g(\theta t + \varphi) = \theta(\hat{\alpha} - 1) \left( g'(\theta t) + (\beta + 1)g''(\theta t) \ln g'(\theta t) - g'(\theta t + \varphi)(1 + \theta_1) \right) = O(g''(\theta t)).
\]
In (25), we used (A5) and representation (19) for \( \theta_1 \). Next,
\[
\frac{d^2}{dt^2} g(\theta t + \varphi) = g''(\theta t + \varphi)(\theta + \varphi')^2 + g'(\theta t + \varphi)\varphi'' = O(g''(\theta t)).
\]
This holds by (A3), (A5), formula (19) for \( \varphi' \), and the following expression for \( \varphi'' \):
\[
\varphi''(t) = \theta(\beta + 1) \frac{g''(\theta(t) + g'(\theta t)\ln g'(\theta t) + \frac{3g''(\theta t)g'(\theta t)}{g'(\theta t)} - g''(\theta t)^3}{g'(\theta t) + g''(\theta t)\ln g'(\theta t)} - (\varphi')^2.
\]
Thus, we obtain (15) with
\[
\Phi(t, \eta, \zeta) = -\left( F_1(t, \eta, \zeta) + F_2(t, \eta) + F_3(t, \eta) + F_4(t) + F_5(t, \eta) \right),
\]
where \( \eta' \) is substituted by \( \zeta \) on the right-hand side of the first equation of (15) and in the second equation. Furthermore, \( F_3 \) is defined by (24), \( F_4(t) = O(g''(\theta t)) \) comes due to (25) and (26), and \( F_5 \) is defined as follows:
\[
F_5(t, \eta) = e^{\varphi \eta} \left( \frac{\eta}{g'(\theta t + o(t))} - \frac{\eta}{g'(\theta t + o(t))} \right) - \theta(\hat{\alpha} - 1) \int_0^1 (1 - r) e^{\frac{\sigma(t, \eta')}{n}} dr \frac{g'(\theta t)O(1)}{g'(\theta t + o(t))^2} \eta^2.
\]
We will be searching for pairs \((\eta, \zeta)\) that solve (15) in a certain metric space which we denote by \( X_T \). If \( g''(t) \) is not an identical zero in a neighborhood of infinity, we define
\[
X_T = \{ x \in C[T, \infty] : x(t) = O(g''(\theta t)) \text{ as } t \to +\infty \}.
\]
Note that under (A5)-(b), either \( g''(\theta t) \) does not take zero values in a neighborhood of infinity (case (i)), or identically equals zero in a neighborhood of infinity (case (ii)). In the case (i), for sufficiently large \( T \), the following norm in \( X_T \) is well-defined:
\[
\| x \|_{X_T} = \sup_{t \geq T} [ |g''(\theta t)|^{-1} |x(t)| ].
\]
In the case (ii), we define \( X_T = C_b([T, +\infty)) \) with the supremum norm. Here, \( C_b \) stands for the space of bounded continuous functions.

In \( X_T \), (15) uniquely transforms to the following couple of fixed-point equations:
\[
\eta = \Psi(\eta, \zeta), \quad \zeta = \hat{\Psi}(\eta, \zeta).
\]
Since $\delta > 1$, the explicit formula for $\hat{\Psi}$ is the following:

\begin{equation}
\hat{\Psi}(\eta, \zeta)(t) = \int_{t}^{\infty} e^{2\delta(t-s)} (\Phi(s, \eta, \zeta) - \theta(\hat{\alpha} - 1)(\eta(s))) \, ds,
\end{equation}

where $\delta = \frac{\alpha - \beta - 1}{2}$. The explicit formula for $\Psi$ depends on the sign of the discriminant $D = 4(\delta^2 - \theta(\hat{\alpha} - 1))$ of the corresponding characteristic equation. In the case $D < 0$, we have

\begin{equation}
\Psi(\eta, \zeta)(t) = \frac{1}{\mu} \int_{t}^{\infty} e^{\delta(t-s)} \sin(\mu(s-t)) \Phi(s, \eta, \zeta) \, ds,
\end{equation}

where $\mu = \sqrt{|D|} / 2$.

In the case $D > 0$, we have

\begin{equation}
\Psi(\eta, \zeta)(t) = \frac{1}{2\mu} \int_{t}^{\infty} e^{(\delta - \mu)(t-s)} (1 - e^{2\mu(t-s)}) \Phi(s, \eta, \zeta) \, ds.
\end{equation}

Finally, if $D = 0$, we have

\begin{equation}
\Psi(\eta, \zeta)(t) = \int_{t}^{\infty} e^{\delta(t-s)} (s-t) \Phi(s, \eta, \zeta) \, ds.
\end{equation}

**Step 2. Existence of a fixed point.** Let us show that we can find constants $M, \tilde{M} > 0$ and a number $T > 0$ such that the map $(\Psi, \hat{\Psi})$ acts from $\Sigma$ to $\Sigma$, where the latter is defined as a complete metric space of the following form:

$$
\Sigma = \Sigma(M, \tilde{M}) = \{(\eta, \zeta) \in X_T \times X_T : \|\eta\|_{X_T} \leq M \text{ and } \|\zeta\|_{X_T} \leq \tilde{M}\}.
$$

We define the metric in $\Sigma$ as follows:

$$
\rho(\{(\eta_1, \zeta_1), (\eta_2, \zeta_2)\} = \|\eta_1 - \eta_2\|_{X_T} + \|\zeta_1 - \zeta_2\|_{X_T}.
$$

Moreover, we show that $T$ can be chosen in such a way that $(\Psi, \hat{\Psi})$ is a contraction map $\Sigma \to \Sigma$.

The integral operator applied to $\Phi$ and defined by (31), (32), or (33) splits into five terms (which we denote by $I_1, I_2, I_3, I_4,$ and $I_5$), corresponding to the decomposition of $\Phi$ by formula (27). We evaluate each of these terms. First, we compute the integrals

\begin{align*}
&\frac{1}{\mu} \int_{t}^{\infty} e^{\delta(t-s)} \, ds = \frac{1}{\mu \delta}; \\
&\frac{1}{2\mu} \int_{t}^{\infty} e^{(\delta - \mu)(t-s)} \, ds = \frac{1}{2\mu(\delta - \mu)}; \\
&\int_{t}^{\infty} e^{\delta(t-s)} (s-t) \, ds = \frac{1}{\delta^2};
\end{align*}

and define

$$
\varkappa = \max \left\{ \frac{1}{\mu \delta}, \frac{1}{2\mu(\delta - \mu)}, \frac{1}{2\mu}, \frac{1}{\delta^2} \right\}.
$$

The expression for $\varkappa$ involves the values of the above integrals and the term $\frac{1}{\delta^2}$ which appears from the integration of $e^{2\delta(t-s)}$ in (30). We make analysis separately in the case (i), when $g''(\theta t) \neq 0$ in a neighborhood of infinity, and in the case (ii), when $g''(\theta t)$ identically equals zero in a neighborhood of infinity. We start from the case (i). Introduce constants $M_1, M_2,$ and a number $T_1 > 0$ such that

\begin{equation}
\sup_{s \geq t} |g''(\theta s)| \leq M_1 |g''(\theta)|, \quad t \geq T_1;
\end{equation}

\begin{equation}
|F_i(t)| \leq M_2 |g''(\theta)|, \quad i = 2, 4, \quad t \geq T_1.
\end{equation}

Let us fix the metric space for the fixed point argument. Take $M > 5\varkappa M_1 M_2$ and $\tilde{M} = \frac{8\varkappa M_1 M_2}{1 + \frac{1}{M}} + M$. Choose $T > T_1$ in such a way that if the a priori estimates $|\eta(t)| \leq M |g''(\theta t)|$ and $|\zeta(t)| < \tilde{M} |g''(\theta t)|$ hold, then,

\begin{equation}
|F_i(t)| \leq M_2 |g''(\theta)|, \quad i = 1, 3, 5, \quad t \geq T.
\end{equation}

We can always do this since for $\eta$ and $\zeta$ with the above property, $F_1, F_3,$ and $F_5$ are $o(g''(t))$. Now take $(\eta, \zeta) \in \Sigma(M, \tilde{M}, T)$ and note that each of the terms $I_i, i = 1, 2, 3, 4, 5$, ...
becomes smaller than $\varepsilon M_1 M_2$, which is smaller than $\frac{M}{5}$. Thus, we have proved that $|\Psi(\eta, \zeta)(t)| \leq M|g''(t)|$. Next, we represent $\hat{\Psi}$ as $\Psi_1 + \Psi_2$, where

$$
\Psi_1(\eta, \zeta)(t) = \int_t^{+\infty} e^{2\delta(t-s)}\Phi(s, \eta(s), \zeta(s))ds,
$$

$$
\Psi_2(\eta)(t) = -\theta(\alpha - 1) \int_t^{+\infty} e^{2\delta(t-s)}\eta(s)ds.
$$

From the above argument, we obtain that $|\Psi_1(\eta, \zeta)(t)| \leq M|g''(t)|$. Furthermore, $|\Psi_2(\eta)(t)| \leq \frac{2 M_1 M_2}{1 + \beta} |g''(t)|$. Therefore, by the choice of $M$, we obtain that $|\Psi(\eta, \eta')(t)| \leq M|g''(t)|$. Thus, we proved that $\Psi(\eta, \zeta)$ is a map $\Sigma \to \Sigma$.

Let us show that if the number $T$ is sufficiently large, then $\Psi : \Sigma \to \Sigma$ is a contraction map. Let $\chi(s-t)$ denote one of the factors $\sin\{\mu(s-t)\}$, $1 - e^{2\mu(t-s)}$, or $s-t$ which occur in the integrals (31), (32), and (33). By (A5)-(b) and the integrability of the exponential factor, multiplied by $\chi$, it suffices to observe that for two elements $(\eta_1, \zeta_1)$ and $(\eta_2, \zeta_2)$ from $\Sigma$, and for each $t \in (T, +\infty)$,

$$
|F_i(t, \eta_1, \zeta_1) - F_i(t, \eta_2, \zeta_2)| \leq e(t)(|\eta_1 - \eta_2| + |\zeta_1 - \zeta_2|),
$$

$$
|F_i(t, \eta_1) - F_i(t, \eta_2)| \leq e(t)|\eta_1 - \eta_2|, \quad i = 3, 5,
$$

where $e$ is a positive function tending to 0 as $t \to +\infty$. Inequalities (36) follow immediately from the expressions for $F_i$, $i = 1, 3, 5$, if we take into account the estimates $|\eta_i(t)| \leq M|g''(t)|$, $|\zeta_i(t)| \leq M|g''(t)|$, $i = 1, 2$, Remark 2.3, and assumption (A3). We also observe that, by (22), $\sigma_1(t, \zeta)$ is Lipschitz in its second argument for sufficiently large $t$. By (34) and (36),

$$
\sup_{t \geq T} g''(\theta t)^{-1}|\Psi(\eta_1, \zeta_1)(t) - \Psi(\eta_2, \zeta_2)(t)|
$$

$$
\leq 5 \sup_{t \geq T} \left\{ \left( g''(\theta t)^{-1} \right) \int_t^{+\infty} e^{\delta(t-s)}\chi(t-s)\epsilon(s)(|\eta_1(s) - \eta_2(s)| + |\zeta_1(s) - \zeta_2(s)|)ds \right\}
$$

$$
\leq 5 M_1 \sup_{t \geq T} e(t) \int_t^{+\infty} e^{\delta(t-s)}\chi(t-s)|g''(\theta s)|^{-1}(|\eta_1(s) - \eta_2(s)| + |\zeta_1(s) - \zeta_2(s)|)ds.
$$

Thus, by choosing $T$ sufficiently large, we can make the factor in front of the integral small, so that there exists $r > 0$ sufficiently small (whose exact bound will be specified later) such that

$$
\|\Psi(\eta_1, \zeta_1) - \Psi(\eta_2, \zeta_2)\|_{X_T} \leq r(\|\eta_1 - \eta_2\|_{X_T} + \|\zeta_1 - \zeta_2\|_{X_T}).
$$

By (30),

$$
\|\hat{\Psi}(\eta_1, \zeta_1) - \hat{\Psi}(\eta_2, \zeta_2)\|_{X_T} \leq \|\Psi_1(\eta_1, \zeta_1) - \Psi_1(\eta_2, \zeta_2)\|_{X_T} + \|\Psi_2(\eta_1) - \Psi_2(\eta_2)\|_{X_T},
$$

where $\Psi_1$ and $\Psi_2$ are given by (35). The first term in the above identity is evaluated exactly as in (37). For the second term, we have

$$
\|\Psi_2(\eta_1) - \Psi_2(\eta_2)\|_{X_T} \leq R\|\eta_1 - \eta_2\|_{X_T}
$$

with $R = \frac{\theta M_1}{1 + \beta}$ (recall that $2\delta = \alpha - \beta - 1$). Therefore,

$$
\|\hat{\Psi}(\eta_1, \zeta_1) - \hat{\Psi}(\eta_2, \zeta_2)\|_{X_T} \leq (R + r)\|\eta_1 - \eta_2\|_{X_T} + r\|\zeta_1 - \zeta_2\|_{X_T}.
$$

Now let $(\hat{\Psi}(2), \hat{\Psi}(2)) = (\Psi, \hat{\Psi}) \circ (\Psi, \hat{\Psi})$. We show that one can find a number $r \in (0, 1)$ such that $(\hat{\Psi}(2), \hat{\Psi}(2))$ is a contraction map. We have

$$
\|\hat{\Psi}(2)(\eta_1, \zeta_1) - \hat{\Psi}(2)(\eta_2, \zeta_2)\|_{X_T} \leq r(2r + R)\|\eta_1 - \eta_2\|_{X_T} + 2r^2\|\zeta_1 - \zeta_2\|_{X_T},
$$

$$
\|\hat{\Psi}(2)(\eta_1, \zeta_1) - \hat{\Psi}(2)(\eta_2, \zeta_2)\|_{X_T} \leq 2r(r + R)\|\eta_1 - \eta_2\|_{X_T} + r(2r + R)\|\zeta_1 - \zeta_2\|_{X_T}.
Therefore,
\[\|\Psi^{(2)}(\eta_1, \zeta_1) - \Psi^{(2)}(\eta_2, \zeta_2)\|_{X_T} + \|\Phi^{(2)}(\eta_1, \zeta_1) - \Phi^{(2)}(\eta_2, \zeta_2)\|_{X_T} \leq (4r^2 + 3Rr)(\|\eta_1 - \eta_2\|_{X_T} + \|\zeta_1 - \zeta_2\|_{X_T}).\]

Picking \(r > 0\) in such a way that \(4r^2 + 3Rr < 1\), we obtain that \((\Psi^{(2)}, \Phi^{(2)})\) is a contraction map \(\Sigma \to \Sigma\). Let \((\eta, \zeta)\) be its unique fixed point. Then, \((\eta, \zeta)\) is also a fixed point of \((\Psi, \Phi)\), while \(\eta\) determines solution (12) with the prescribed singular part given by the first term. To see this, it suffices to note that the solution \((\eta, \zeta)\), that we just found, satisfies the property \(\eta' = \zeta\). Indeed, \(\zeta = \eta'\) also satisfies the second equation in (15) which implies that
\[\eta'(t) = \hat{\Psi}(\eta, \zeta)(t) + Ce^{(\alpha-\beta-\gamma-1)t},\]
where \(C\) is a constant. On the other hand, (31), (32), or (33) imply that
\[\eta'(t) = o(1).\]
Therefore, \(C = 0\) and \(\zeta = \eta'\).

Thus, we obtained a singular solution \(\nu(t) = g(\theta t + \varphi(t)) + \eta(t)\) to (11).

Consider now the case (ii), i.e., \(g''(\theta t) = 0\) on \([\tilde{T}, +\infty)\) for some \(\tilde{T} > 0\). In this case, the terms \(F_2\) and \(F_3\) are equal to zero. The non-zero terms are \(F_1\), \(F_3\), and \(F_5\). Since \(F_1(t, 0, 0) = F_3(t, 0, 0) = F_5(t, 0, 0) = 0\), \(\eta = 0\) is a solution to
\[\eta'(t) = (\alpha - \beta - 1)\eta'(t) + \theta(\alpha - 1)\eta(t) = \Phi(t, \eta, \eta').\]

Hence, in the case (ii), \(\nu(t) = g(\theta t + \varphi(t))\) is an exact singular solution to (11). \(\Box\)

Now we extend the singular solution to (11), that we constructed in Proposition 2.1, to a solution to the same equation on the entire real line.

**Proposition 2.2.** In the assumptions of Proposition 2.1, the solution \(\nu^*\) can be extended to a solution of (11) on \((-\infty, +\infty)\). Moreover, the extended solution \(\nu^*\) is strictly increasing and there exists a number \(T^* \in (-\infty, T)\) such that \(\nu^*(T^*) = 0\).

**Proof.** By Proposition 2.1, there exists a singular solution \(\nu^*\) to (11) on some interval \([T, +\infty)\). Equation (11) can be rewritten as follows:
\[(e^{(\beta+1-\alpha)t} |\nu'(t)|^\beta \nu'(t))' = -(\beta + 1)e^{-(\gamma+1)t} e^{\nu}(v).\]

Substituting \(\nu^*\) into (38) and integrating it from \(t\) to \(+\infty\), we obtain
\[e^{(\beta+1-\alpha)t} |\nu^*(t)|^\beta \nu^*(t) = (\beta + 1) \int_t^{+\infty} e^{-(\gamma+1)s} e^{\nu}(v) ds.\]

Indeed, since \(\lim_{t \to +\infty} g''(\theta t) = 0\), we conclude that \(g''(t)\) has at most linear growth. Furthermore, since \(\nu^*(t) = g'(\theta t + \varphi(t)) + \eta'(t)\), where \(\varphi = o(t), \varphi' = O(\frac{g''(\theta t)}{g'(\theta t)})\) (see formula (19)) and \(\eta' = O(g''(\theta t))\), we obtain that \(\nu^*(t)\) also has at most linear growth on \([T, +\infty)\). Since \(\beta + 1 - \alpha < 0\), the expression on left-hand side of (39) goes to zero as \(t \to +\infty\), and hence, (39) holds. Recall that \(\nu^*(t) > 0\) (as we showed in the proof of Proposition 2.1) for any extension of \(\nu^*\). Let \(w(t) = e^{(\beta+1-\alpha)t} \nu^*(t)^{\beta+1}\). We then obtain the following system with respect to \(w\) and \(\nu^*\):
\[
w'(t) = -(\beta + 1)e^{-(\gamma+1)t} e^{\nu}(v),
\nu^*(t) = e^{(\alpha-1)t} w^{\frac{1}{\alpha-1}}.
\]

Pick \(S > T\) and define \(c_0 = \nu^*(S), c_1 = w(S)\). Furthermore, we note that on \((-\infty, S]\), \(\eta(t) \leq \nu(t) \leq c_1 + c_2 e^{-(\gamma+1)t} = c_1\) for some constant \(c_2 > 0\). Define \(\nu = 1_{(-\infty, c_0+\varepsilon]} * \rho_\varepsilon\) and \(\zeta = 1_{[c_1-\varepsilon, c_1+\varepsilon]} * \rho_\varepsilon\), where \(\rho_\varepsilon, \varepsilon < c_1\), is a standard mollifier supported on the ball of
radius \( \varepsilon \). Note that \( \nu(\cdot) = 1 \) on \( (-\infty, c_0) \) and \( \zeta_1(\cdot) = 1 \) on \([c_1, \varepsilon_i] \). Instead of (40), consider the system of first-order ODEs on \( (-\infty, S) \)

\[
\begin{cases}
w'(t) = -(\beta + 1)e^{-(\gamma+1)t}e^{\nu(t)f(v)}, \\
v^*(t) = e^{(\alpha-1)t}(\zeta(\nu(t)))^{\frac{1}{\gamma+1}}.
\end{cases}
\]

On \([T, S] \), \((v^*(t), w(t))\) is also a solution to (41), since over this interval \( \nu(v^*(t)) = 1 \) and \( \zeta(\nu(t)) = 1 \). Next, since \( e^{\nu(t)f(v)} \leq e^{(\nu_0+2\varepsilon)} \) and \( 0 < (\zeta(\nu(t)))^{\frac{1}{\gamma+1}} \leq (\varepsilon_i + 2\varepsilon)^{\frac{1}{\gamma+1}} \), one can extend \((v^*, w)\) to \(( -\infty, S) \) (see, e.g., [16], Chapter 2, §6), obtaining by this a solution \((\tilde{v}, \tilde{w})\) to (41) which coincides with \((v^*, w)\) on \([T, S]\). Since on \(( -\infty, S]\), \( \nu(\tilde{v}) = 1 \) and \( \zeta(\tilde{w}) = 1 \), \((\tilde{v}, \tilde{w})\) is also a solution to (38), and therefore, to (11). We then use the same notation for this extended solution, i.e., we write \((v^*, w)\) instead of \((\tilde{v}, \tilde{w})\). There are two possible situations: either \( v^* \) is strictly positive over \(( -\infty, S]\), or there exists \( T^* \in \mathbb{R} \) such that \( v^* > 0 \) on \([T^*, S]\) and \( v^*(T^*) = 0 \). Let us show that the first situation cannot be realized. If \( v^* \) is strictly positive over \(( -\infty, S]\), then there exists a finite limit \( L = \lim_{t \to -\infty} v^*(t) \). Integrating (38) from \(-R\) to 0 (where \( R > 0 \) is sufficiently large) and taking into account that \( e^{\nu(t)} \geq e^{\nu(L)} \), we obtain that

\[
v^*(R)^{\beta+1} \geq e^{-(\alpha+1)R}v^*(0)^{\beta+1} + \frac{\beta+1}{\gamma+1} e^{\nu(L)} \left( e^{\beta R} - e^{-(\alpha+1)R} \right)
\]

which shows that \( v^*(R) \to +\infty \) as \( R \to +\infty \). The latter implies that \( v^*(-R) = v^*(0) - \int_{-R}^{0} v^*(t) dt \to -\infty \) as \( R \to +\infty \). This contradicts to the fact that \( \lim_{t \to -\infty} v^*(t) = L < +\infty \).

Thus, we conclude that solution (12) can be extended to \(( -\infty, +\infty) \) in such a way that there exists a finite number \( T^* \) such that \( v^*(T^*) = 0 \). \( \square \)

### 2.4.2 Proof of Theorem 2.1

The proof of Theorem 2.1 now follows from the inverse change of variable.

**Proof of Theorem 2.1.** Let \( T^* \) be as in Proposition 2.2. Define \( \lambda^* = (\beta + 1)e^{-\eta T^*} \) and

\[
\kappa = \left( \frac{\beta+1}{\lambda^*} \right)^{\frac{1}{\beta+1}}.
\]

Further define \( u^*_\frac{\lambda}{\lambda}(r) = v^*(\ln(\kappa/r)) \). By the argument in paragraph 2.4.1, \((u^*_\frac{\lambda}{\lambda}, \lambda^*)\) is a singular solution to problem \((P_\lambda)\). \( \square \)

### 2.5 Exact singular solution to the Gelfand problem for a \( k \)-Hessian

Consider the problem

\[
\begin{cases}
-L(u) = \lambda e^u \text{ in } B, \\
u = 0 \text{ on } \partial B,
\end{cases}
\]

where \( L \) can be a \( k \)-Hessian with \( k < \frac{d}{2} \), a \( p \)-Laplacian with \( p < d \), or, in general, the operator (5) with \( \alpha, \beta, \gamma \) satisfying (A6). Here, we prove Corollary 1.1, stating the exact form of the singular solution constructed in Theorem 2.1. Furthermore, we “decode” the formula for a singular solution announced in Corollary 1.1 for the cases of a \( k \)-Hessian and a \( p \)-Laplacian. Namely, we have the following corollary.

**Corollary 2.3.** Assume (A6) and consider problem (42) for the case when \( L \) is a \( k \)-Hessian with \( k < \frac{d}{2} \). Then, the singular solution constructed in Theorem 2.1 takes the form

\[
u_\frac{\lambda}{\lambda}(x) = -2k \ln|x|, \quad \lambda^* = (2k)^k(d - 2k).
\]

If \( L \) is a \( p \)-Laplacian with \( p < d \), then, the aforementioned singular solution is

\[
u_\frac{\lambda}{\lambda}(x) = -p \ln|x|, \quad \lambda^* = p^{d-1}(d - p).
\]
Proof of Corollaries 1.1 and 2.3. Note that for problem (42), \( f(t) = g(t) = t \). Therefore, 
\[ g'(t) = 1, \quad g''(t) = 0. \] 
By Proposition 2.1, in a neighborhood of \(+\infty\), the solution (12) takes the form
\[ u^*(t) = \theta t + \ln\{\theta^{\beta+1} (\hat{\alpha} - 1)\}. \]
It is straightforward to check that (45) verifies (11) with \( f(v) = v \) on the entire real line. Furthermore, \( v(t) = 0 \) if \( t = T^* = -\frac{\ln(\theta^{\beta+1} (\hat{\alpha} - 1))}{g}\). Therefore,
\[ \lambda^* = (\beta + 1) e^{-\theta T^*} = (\beta + 1) \theta^{\beta+1} (\hat{\alpha} - 1) = \theta^{\beta+1} (\alpha - \beta - 1). \]
Recall that \( u^*_r(x) = v^*(t) \) with \( t = \ln \left\{ \left( \frac{\beta+1}{\lambda^*} \right)^{\frac{1}{\beta}} \right\} \) \(-\frac{1}{\beta} \ln\{\theta^{\beta+1} (\hat{\alpha} - 1)\} - \ln|x|\).
Comparing it with (45), we obtain that \( u^*_r(x) = -\theta \ln|x|\).
In the case of a \( k \)-Hessian with \( k < \frac{d}{2} \), we have \( \theta = 2k \), \( \beta + 1 = k \), \( \alpha - \beta - 1 = d-2k \), which implies (43).
In the case of a \( p \)-Laplacian with \( p < d \), we have \( \theta = p \), \( \beta + 1 = p - 1 \), \( \alpha - \beta - 1 = d-p \), which implies (44). □

3. Regular solutions to problem \( (P_\lambda) \)

Here, we will be interested in regular solutions to problem \( (P_\lambda) \). For regular solutions, problem \( (P_\lambda) \) should be stated as follows:
\[ (P_{\lambda}) \]
\[ \begin{cases} 
-L(u_\lambda(r)) = \lambda e^{f(u_\lambda(r))}, \\
u_\lambda'(0) = u_\lambda(1) = 0.
\end{cases} \]
Indeed, from Lemma 3.1 below, it follows that any solution to \( (P_\lambda) \) is decreasing in \( r \in [0, +\infty) \) and achieves the maximum at \( r = 0 \). Moreover, since the actual solution \( u_\lambda \) should be regular in a ball, we add the condition \( u_\lambda'(0) = 0 \).

**Lemma 3.1.** Let \((u_\lambda, \lambda)\) be a regular radial solution to \( (P_{\lambda}) \). Then, \( u_\lambda'(r) < 0 \) on \((0, +\infty)\).

**Proof.** The equation in \( (P_{\lambda}) \) implies
\[ r^\alpha |u_\lambda'(r)|^\beta u_\lambda'(r) = - \int_0^r s^\gamma e^{f(u_\lambda(s))} ds + C. \]
Since \( u_\lambda'(0) = 0 \), we obtain that \( C = 0 \). This implies that \( u'(r) < 0 \) for all \( r > 0 \). □

**Remark 3.1.** If \( \gamma + 1 > \alpha \) (as in the case of the \( k \)-Hessian), then in \( (P_{\lambda}) \), we do not need to state that \( u'(0) = 0 \) if we are interested in a regular solution. This assumption is fulfilled automatically. Indeed, by L’Hospital’s rule,
\[ (-u_\lambda'(0))^{\beta+1} = \lim_{r \to 0} \frac{\alpha^{-1} r^{\gamma+1 - \alpha} e^{f(u_\lambda(r))}}{u_\lambda'(0)} = 0. \]

3.1 Standing assumptions: regular solutions

(B1) \( f \in C^1(\mathbb{R}, \mathbb{R}), \ f' > 0, \) and \( \lim_{u \to +\infty} f(u) = +\infty \).
(B2) \( \lim_{u \to +\infty} u^{-1} \exp \left\{ \frac{f(u)}{\beta+1} \right\} = \lim_{u \to +\infty} u f''(u) = +\infty \).
(B3) \( \alpha > \beta + 1 > 0; \ \theta > 0 \).

**Remark 3.2.** Remark that the second limit in (B2) is implied by (A4) and the first condition of (A2) \( \lim_{t \to +\infty} g'(t) = 0 \). We have
\[ \lim_{u \to +\infty} u f''(u) = \lim_{t \to +\infty} \frac{g(t)}{g'(t)} \geq \lim_{t \to +\infty} \int_0^t |g''(s)| ds + g'(0) = +\infty. \]
Indeed, we know that \( \lim_{t \to +\infty} g(t) = +\infty \). If \( \lim_{t \to +\infty} \int_0^t |g''(s)| ds < +\infty \), then (46) is straightforward. Otherwise, if \( \lim_{t \to +\infty} \int_0^t |g''(s)| ds = +\infty \), then (46) follows from
L'Hopital's rule and Remark 2.3. The first expression in (B2) tends to $+\infty$ by L'Hopital's rule and Remark 2.5.

3.2 Existence and uniqueness of a regular solution to \((P'_\lambda)\)

First of all, we note that by the change of variable (8), problem \((P'_\lambda)\) can be transformed to

\[
\begin{aligned}
-\mathcal{L}(u) &= e^{f(u)}, \\
u(0) &= \rho, \\
u'(0) &= 0.
\end{aligned}
\]

Above, $\rho$ and $\lambda$ are connected through the identity

\[
\rho = \lambda \pi^+ \int_0^1 t^{-\pi^+} \left( \int_0^t s^\gamma e^{f(u_{\lambda}(s))} ds \right)^{\pi^+} dt,
\]

where $\{u_{\lambda}, \lambda\}$ is the solution to \((P'_\lambda)\). Note that problem \((P_\rho)\) is equivalent to the integral equation

\[
u(r) = \rho - \int_0^r t^{-\pi^+} \left( \int_0^t s^\gamma e^{f(u(s))} ds \right)^{\pi^+} dt.
\]

We have the following result on the existence and uniqueness of solution to (47).

**Lemma 3.2.** Assume \((B1)\) and \((B3)\). Then, equation (47) has a unique solution.

**Proof.** Let the map $\Gamma(u)$ be given by the right-hand side of (47). Fix an arbitrary interval $[0, T]$. Note that for any solution $u$ of (47), on $[0, T]$, it holds that

\[ u \leq \rho \quad \text{and} \quad u \geq \rho - e^{f(\rho)} (\gamma + 1)^{-\pi^+} \hat{\theta}^{-1} T \hat{\theta}.
\]

Let $A_{\rho, T}$ denote the constant on the right-hand side of the second estimate. Introduce the complete metric space

\[
\Sigma = \{ u \in C[0, T] : A_{\rho, T} \leq u \leq \rho \}
\]

with the supremum norm as the metric, and take two functions $u_1$ and $u_2$ from $\Sigma$. It is easy to see that $\Gamma(u_1)$ and $\Gamma(u_2)$ are functions $\Sigma \to \Sigma$. Furthermore,

\[
\Gamma(u_1)(r) - \Gamma(u_2)(r) = \frac{1}{\beta + 1} \int_0^r t^{-\frac{\alpha}{\beta + 1}} \left( \int_0^1 \Lambda(t, \rho, \lambda)^{-\frac{\beta}{\beta + 1}} d\lambda \right) \int_0^t s^\gamma (e^{f(u_2(s))} - e^{f(u_1(s))}) ds dt,
\]

where $\Lambda(t, \rho, \lambda) = \int_0^t s^\gamma (\lambda e^{f(u_1(s))} + (1 - \lambda) e^{f(u_2(s))}) ds$, and we have that

\[ e^{f(A_{\rho, T})} (\gamma + 1)^{-1} T \leq \Lambda(t, \rho, \lambda) \leq e^{f(\rho)} (\gamma + 1)^{-1} T \hat{\theta}.
\]

Taking into account that $|u_1|$ and $|u_2|$ are bounded, we obtain that there exists a constant $K = K(\rho, T, \gamma, \beta, \hat{\theta})$ such that

\[
\sup_{[0, r]} |\Gamma(u_1) - \Gamma(u_2)| \leq K \int_0^r t^{-\frac{\alpha + \gamma + 1}{\beta + 1}} \sup_{[0, t]} |u_1 - u_2| ds.
\]

Since $\frac{\alpha + \gamma + 1}{\beta + 1} = \hat{\theta} - 1$, we obtain that for the map $\Gamma^{(n)} = \Gamma \circ \Gamma \circ \cdots \circ \Gamma$, it holds that

\[
\sup_{[0, r]} |\Gamma^{(n)}(u_1) - \Gamma^{(n)}(u_2)| \leq \frac{(Kr^{\hat{\theta} - 1})^n}{n!} \sup_{[0, r]} |u_1 - u_2| ds.
\]

This implies that $\Gamma^{(n)}$ is a contraction for some $n$. Clearly, the unique fixed point of $\Gamma^{(n)}$ is also the unique fixed point of $\Gamma$, the solution to (47).
3.3 Properties of regular solutions

We start by deriving a version of Pohozaev’s identity suitable for our applications.

**Lemma 3.3.** Let \( f \in C(\mathbb{R}, \mathbb{R}) \), \( a \in \mathbb{R} \) be an arbitrary constant, and \( u \in C^2[0, +\infty) \) be a solution to problem \((P_\rho)\). Then, for all \( r > 0 \),

\[
(48) \quad r^\alpha |u'|^{\beta+2} \left( a - \frac{\alpha - \beta - 1}{\beta + 2} \right) + r^\gamma \left\{ (\gamma + 1) \int_0^u e^{f(t)} dt - au e^{f(u)} \right\} \\
= \frac{d}{dr} \left\{ r^\alpha \left( \frac{\beta + 2}{\beta + 1} |u'|^{\beta+2} + r^{\gamma-\alpha} \int_0^u e^{f(t)} dt + ar^{-1} |u'|^\beta u' \right) \right\}.
\]

**Proof.** In [22], a version of Pohozaev’s identity was obtained for solutions to equations whose particular type is

\[
(49) \quad r^{-(d-1)} (r^{d-1} |u'(r)|^\beta u'(r))' + h(r, u) = 0,
\]
where \( d \) is the space dimension and \( h \in C(\mathbb{R}_+ \times \mathbb{R}) \). As it follows from the proof of Theorem 2.1, exposed in paragraph 2 of [22], instead of \( d - 1 \), one can use any number \( \alpha > 0 \). More specifically, the fact that \( d - 1 \) is an integer which differs from the space dimension by one, does not play any role in the computation on p. 545 of [22]. Multiplying the both parts of the equation in \((P_\rho)\) by \( r^{\alpha-\gamma} \), we bring this equation to a form similar to (49):

\[
r^{-(\alpha-\gamma)} (r^\alpha |u'(r)|^\beta u'(r))' + r^{\gamma-\alpha} e^{f(u)} = 0.
\]

We then apply Theorem 2.1 from [22] to the above equation, substituting \( d \) by \( \alpha + 1 \) and taking into account that \( u'(0) = 0 \). Referring to the notation of the aforementioned theorem, we have \( A(\rho) = \rho^\beta \) and \( E(\rho) = (\beta + 1)\rho^\beta \). It is then straightforward to verify the assumptions of Theorem 2.1 from [22] as well as to make necessary computations leading to equation (48). \( \square \)

Let \( u(\cdot, \rho) \) be the solution to problem \((P_\rho)\). Our next result, given by Proposition 3.1, is about upper and lower bounds for the number

\[
(50) \quad R(B, \rho) = u(\cdot, \rho)^{-1}(B), \quad 0 \leq B \leq \rho.
\]

**Proposition 3.1.** Assume (B1)–(B3). Then, for each \( B \geq 0 \), the number \( R(B, \rho) \) is well-defined. Moreover, there exist two positive constants \( \check{R}(B) \) and \( \hat{R}(B) \), and a number \( \rho_0(B) > B \), all depending only on \( B \), such that for all \( \rho \geq \rho_0(B) \),

\[
\underline{R}(B) \leq R(B, \rho) \leq \overline{R}(B).
\]

Moreover, the second inequality holds for all \( \rho \geq B \).

**Proof.** Fix a number \( B \geq 0 \). We start by showing that \( R(B, \rho) \) is well-defined for \( \rho > B \). This is equivalent to the fact that a zero of \( u(r, \rho) - B \) exists. Suppose this is not the case. Since \( u(r, \rho) \) is decreasing, then \( u(r, \rho) > B \) for all \( r \geq 0 \). From \((P_\rho)\), we obtain that \( u(r, \rho) \leq \rho - e^{\frac{f(u)}{1+\tau}} (\gamma + 1)^{-\frac{1}{\gamma+1}} \beta^{-1} e^{-\gamma} \to -\infty \), as \( r \to +\infty \). This is a contradiction. Hence, there exists a unique \( r \) such that \( u(r, \rho) = B \).

Let us prove that the function \( [B, +\infty) \to \mathbb{R}, \rho \mapsto R(B, \rho) \) is bounded from above. Rescale problem \((P_\rho)\) by setting \( \lambda^{\frac{\beta}{\beta+1}} = R(B, \rho) \) as follows:

\[
\begin{cases}
-L(u(\lambda^{\frac{\beta}{\beta+1}} r)) = \lambda e^{f(u(\lambda^{\frac{\beta}{\beta+1}} r))}, \\
u(0) = \rho, \quad u'(0) = 0
\end{cases}
\]

which is equivalent to

\[
u(\lambda^{\frac{\beta}{\beta+1}} r) = B + \int_0^1 \lambda^{\frac{\beta}{\beta+1}} t^{-\frac{\beta}{\beta+1}} \left( \int_0^t s^\gamma e^{f(u(\lambda^{\frac{\beta}{\beta+1}} s)))} ds \right)^{-\frac{1}{\gamma+1}} dt.
\]
For $r \in (0, 1)$, it holds that
\[
u(\lambda^\frac{1}{r} r) - B \geq \lambda^\frac{1}{1-r} \int_r^1 t^{-\frac{1}{1-r}} dt \left( \int_0^r s^\gamma e^{f(\nu(\lambda^\frac{1}{r} r))} ds \right)^{\frac{1}{1-r}}.
\]
This implies that for $r \in (0, 1)$, we have the estimate
\[
\sup_{u \geq B} \{(u - B) e^{-\frac{f(u)}{u}}\} \geq (u(\lambda^\frac{1}{r} r) - B) e^{-\frac{f(u(\lambda^\frac{1}{r} r))}{u(\lambda^\frac{1}{r} r)}} \geq K_1 \lambda^\frac{1}{1-r} (r^{\frac{1}{1-r}} - r^{\frac{1}{1-r} + \epsilon}),
\]
where $K_1 > 0$ is a constant. By (B2), $0 < \sup_{u \geq B} (u - B) e^{-\frac{f(u)}{u}} < +\infty$. Hence, the right-hand side of (51) is bounded uniformly in $r \in (0, 1)$. Evaluating it (say) at $r = \frac{1}{2}$, we obtain that for all $B \geq 0$ and $\rho \geq B$,
\[
R(B, \rho) = \lambda^\frac{1}{2} \leq K_2 \sup_{u \geq B} \{(u - B) e^{-\frac{f(u)}{u}}\}^{\frac{1}{2}} = \tilde{R}(B),
\]
where $K_2$ is a constant. Now we prove the existence of a lower bound $\tilde{R}(B)$ by using identity (48). By L'Hopital's rule and (B2),
\[
\lim_{u \to +\infty} \frac{u e^{f(u)}}{\int_u^\infty e^{f(t)} dt} = \lim_{u \to +\infty} \frac{e^{f(u)} + u e^{f(u)} f'(u)}{e^{f(u)}} = 1 + \lim_{u \to +\infty} u f'(u) = +\infty.
\]
As we show below, it suffices to prove the existence of $\tilde{R}(B)$ for those $B > 0$ which make the following inequality hold:
\[
(\gamma + 1) \int_0^u e^{f(t)} dt < au e^{f(u)} \quad \text{for all } u \geq B,
\]
where $a \in \mathbb{R}$ is to be fixed later. Indeed, suppose the existence of $\tilde{R}(B)$ is obtained for those $B$ which make inequality (53) fulfilled for $u \geq B$. If $B \geq 0$ is arbitrary, we find a number $\tilde{B} > B$ such that (53) holds for $u \geq \tilde{B}$. Then, $\tilde{R}(B) \leq R(\tilde{B}, \rho) \leq R(B, \rho)$ for $\rho \geq \rho_0(\tilde{B})$, and we can define $\tilde{R}(B) = R(\tilde{B})$ and $\rho_0(B) = \rho_0(\tilde{B})$. Thus, without loss of generality, we assume that $B$ is sufficiently large so that (53) holds.

The number $a$ is chosen as follows. Note that for any two positive numbers $x$ and $y$, it holds that $(x + y)^{-\frac{1}{\beta+1}} < x^{-\frac{1}{\beta+1}} + y^{-\frac{1}{\beta+1}}$ if $\beta \geq 0$ and $(x + y)^{-\frac{1}{\beta+1}} \leq 2^{-\frac{\beta}{\beta+1}} (x^{\frac{1}{\beta+1}} + y^{\frac{1}{\beta+1}})$ if $-1 \leq \beta < 0$. We set $A_\beta = \max\{1, 2^{-\frac{\beta}{\beta+1}}\}$ and fix $\hat{a} \in (0, A_\beta^{-1} - \frac{a(\beta+1)^2}{(\beta+2)(\beta-\beta-1)}).$ The number $a > 0$ is then has to be chosen in such a way that $\frac{a(\beta+1)^2}{(\beta+2)(\beta-\beta-1)} < A_\beta^{-1}.$ In addition, we choose $\hat{a}$ to satisfy $a < \hat{a}^{-\frac{1}{\beta+1}}$ so that the first term on the left-hand side of (48) is negative. By (48) and (53), on $(0, \tilde{R}(B))$, it holds that
\[
\frac{\beta + 2}{\beta + 1} |u'|^{\beta+2} + ar^{-1} |u'|^{\beta} u' < 0.
\]

Since $u' < 0$, for $r \in (0, \tilde{R}(B)]$,
\[
r |u'(r)| < a \frac{\beta + 1}{\beta + 2} u(r).
\]
Next, we define
\[
D = B \left(1 - \hat{a} A_\beta - \frac{A_\beta a(\beta+1)^2}{(\beta+2)(\beta-\beta-1)}\right)^{-1} \text{ and } \tilde{R}(B) = (\hat{a} D \hat{\theta}(\gamma + 1)^{\frac{1}{\gamma+1}} e^{-\frac{f(0)}{\gamma+1}})^{\frac{1}{2}}.
\]
We aim to prove that $R(B, \rho) \geq \tilde{R}(B)$ for all $\rho \geq D$. If, for some $\rho \geq D$, $R(D, \rho) \geq \tilde{R}(B)$, then (since $D > B$) $R(B, \rho) \geq \tilde{R}(B)$. Otherwise, if $R(D, \rho) < \tilde{R}(B)$, consider the following problem on $[R(D, \rho), \tilde{R}(B)]$:
\[
L(v) = -e^{f(D)}, \quad v(R(D, \rho)) = D, \quad u'(R(D, \rho)) = u'(R(D, \rho), \rho).
\]
To simplify notation, define $\hat{D} = e^{f(D)}$ and $\hat{R} = R(D, \rho)$. Since
\begin{equation}
(56) \quad r^\alpha |v'|^{\beta + 1} = -\hat{D} \int_{\hat{R}} s^\gamma ds - \hat{R}^\alpha |u'(\hat{R})|^{\beta + 1},
\end{equation}
we obtain that on $[\hat{R}, \hat{R}(B)]$, $v'(r) < 0$. Taking into account this observation, by (54) and (56), for $r \in [\hat{R}, \hat{R}(B)]$, we obtain
\begin{align*}
v(r) &= D - \int_{\hat{R}}^{r} \left( \frac{\hat{D}}{\gamma + 1} (s^{\gamma + 1} - \hat{R}^{\gamma + 1}) + \hat{R}^\alpha |u'(\hat{R})|^{\beta + 1} \right)^{\frac{1}{\beta + 1}} s^{\alpha} ds \\
&\geq D - A_\beta \left[ \left( \frac{\hat{D}}{\gamma + 1} \right)^{\frac{1}{\beta + 1}} \int_{\hat{R}}^{r} s^{\frac{\gamma}{\gamma + 1}} s^{-\frac{\alpha}{\gamma + 1}} ds + \frac{\hat{R}^\alpha |u'(\hat{R})|^{\beta + 1}}{\alpha - 1} - \frac{R(u'(\hat{R}))}{\alpha - 1} \right] \\
&> D - A_\beta \left( \frac{\hat{D}}{\gamma + 1} \right)^{\frac{1}{\beta + 1}} \frac{R(B)^\beta}{\bar{\theta}} - A_\beta a(\beta + 1) D = B,
\end{align*}
where the last identity follows from the definition of $R(B)$ and $D$.

It remains to show that on $[\hat{R}, \hat{R}(B)]$, $u(r, \rho) \geq v(r)$. Indeed, by (55), for $r \in [\hat{R}, \hat{R}(B)]$, we have
\begin{align*}
(r^\alpha |u'|^{\beta + 1})' &= r^\gamma e^{f(D)} \geq r^\gamma e^{f(u)} = (r^\alpha |u'|^{\beta + 1})' \quad \text{and} \\
nu r^\alpha |u'|^{\beta + 1} &\geq r^\alpha |u'|^{\beta + 1}.
\end{align*}

Since the derivatives $u'$ and $v'$ are negative, again by (55), we obtain that on $[\hat{R}, \hat{R}(B)]$, $u(r, \rho) \geq v(r) \geq B$. Therefore, $R(B, \rho) \geq R(B)$, and we can take $\rho_0(\partial) = D$. The proof is now complete. \hfill \Box

**Proposition 3.2.** Assume (B1)–(B3). Then, for any interval $[r_1, r_2] \subset (0, \infty)$, the family $\{u(\cdot, \rho)\}_{\rho \geq 0}$ is relatively compact in $C([r_1, r_2])$. Furthermore, each limit point of this family is a solution to (7), regular or singular.

**Proof.** Fix an interval $[r_1, r_2] \subset (0, +\infty)$. Let $B > 0$ and let $R(B), \hat{R}(B)$, and $\rho_0(B)$ be as they were defined in Proposition 3.1, so we have that $R(B, \rho) \in [R(B), \hat{R}(B)]$ for all $\rho > \rho_0(B)$, where $R(B, \rho)$ is defined by (50). By (B2), $\lim_{B \to +\infty} \hat{R}(B) = 0$. Choose a number $B > 0$ such that $r_1 > \hat{R}(B)$, and note that on $[r_1, r_2]$, $u(r, \rho) < B$ for all $\rho \geq B$. By (47), it holds that for $r \geq \hat{R}(B)$,
\begin{equation}
(57) \quad u(r, \rho) = B - \int_{R(B, \rho)}^{r} t^{-\frac{1}{\beta + 1}} \left( \int_{t}^{\hat{R}} s^\gamma e^{f(u(s, \rho))} ds \right)^{\frac{1}{\beta + 1}} dt \\
\geq B - e^{-\frac{\beta}{\beta + 1}} \int_{R(B, \rho)}^{r} t^{-\frac{1}{\beta + 1}} \left( \int_{0}^{t} s^\gamma ds \right)^{\frac{1}{\beta + 1}} dt \geq B - e^{-\frac{\beta}{\beta + 1}} (\gamma + 1)^{-\frac{1}{\beta + 1}} \bar{\theta}^{-1} r^{\beta}.
\end{equation}

Therefore, the family $\{u(r, \rho)\}_{\rho \geq B}$ is uniformly bounded on $[r_1, r_2]$.

Let us show that the family $\{u(r, \rho)\}_{\rho \leq B}$ is uniformly bounded on $[0, r_2]$. Note that this family is bounded from above by $B$. It is also bounded from below by the same argument as (57). Namely,
\begin{align*}
u r_2, \rho = \rho - \int_{0}^{r} t^{-\frac{1}{\beta + 1}} \left( \int_{0}^{t} s^\gamma e^{f(u(s, \rho))} ds \right)^{\frac{1}{\beta + 1}} dt \geq -e^{-\frac{\beta}{\beta + 1}} (\gamma + 1)^{-\frac{1}{\beta + 1}} \bar{\theta}^{-1} r^{\beta}.
\end{align*}

Furthermore, (47) implies the uniform continuity of the family $\{u(r, \rho)\}_{\rho \geq 0}$ on $[r_1, +\infty)$. Indeed, for all $r', r''$, such that $r_1 < r' < r''$, and $\rho \geq 0$,
\begin{equation}
(58) \quad 0 < u(r', \rho) - u(r'', \rho) = \int_{r'}^{r''} t^{-\frac{1}{\beta + 1}} \left( \int_{0}^{t} s^\gamma e^{f(u(s, \rho))} ds \right)^{\frac{1}{\beta + 1}} dt \\
\leq e^{-\frac{\beta}{\beta + 1}} (\gamma + 1)^{-\frac{1}{\beta + 1}} \bar{\theta}^{-1} (r''^\beta - r'^\beta).
\end{equation}
By the Arzelà-Ascoli theorem, the family \( \{u(r, \rho)\} \) is relatively compact on \([r_1, r_2]\).

Let us prove now that each limit point of the family \( \{u(r, \rho)\}_{\rho \geq 0} \) is either regular or singular solution. Let \( \{u(r, \rho_n)\} \) converge to a function \( u^{**}(r) \) in \( C([r_1, r_2]) \). We claim that the sequence \( \partial_r u(r_1, \rho_n) \) is bounded. Suppose it is not the case, that is, we can find a subsequence \( \partial_r u(r_1, \rho_{n_k}) \to -\infty \) as \( k \to \infty \). Without loss of generality, we assume that \( \lim_{n \to \infty} \partial_r u(r_1, \rho_n) = -\infty \). For \( r \in [r_1, r_2] \), it holds that,

\[
u(r, \rho_n) = u(r_1, \rho_n) - \int_{r_1}^{r} t^{-\beta+1} \left( r_1^\beta (\partial_r u(r_1, \rho_n))^{\beta+1} + \int_{r_1}^{t} \gamma e^{f(u(s, \rho_n))} ds \right) \frac{1}{\gamma+1} dt.
\]

The last term in this identity converges to \( \infty \) which contradicts to the convergence of \( \{u(r, \rho_n)\} \) in \( C([r_1, r_2]) \). Therefore, \( \partial_r u(r_1, \rho_n) \) is bounded. Let us show that, in the first case, \( \lim_{n \to \infty} \partial_r u(r_1, \rho_n) = -\infty \). For \( r \in [r_1, r_2] \), we obtain

\[
u^{**}(r) = u^{**}(r_1) - \int_{r_1}^{r} t^{-\beta+1} \left( k(r_1) + \int_{r_1}^{t} \gamma e^{f(u^{**}(s))} ds \right) \frac{1}{\gamma+1} dt.
\]

From the above identity, we obtain that \( u^{**}(r) \) is differentiable in \( r \), \( u^{**}'(r) < 0 \), \( k(r_1) = r_1^\beta (u^{**}'(r_1))^{\beta+1}, \) and \( u^{**}(r) \) solves (7) on \([r_1, r_2]\). By extracting further subsequences from \( \{u(r, \rho_n)\} \) and repeating the argument on compact subintervals of \((0, +\infty)\), we can extend \( u^{**}(r) \) to a solution of (7) on \((0, +\infty)\). Note that \( \{u(r, \rho_n)\} \) contains a subsequence that converges pointwise to \( u^{**}(r) \) on \((0, +\infty)\).

Let us show now the following fact. Suppose \( u(r, \rho_n) \) converges to \( u^{**}(r) \) pointwise on \((0, +\infty)\). If \( \rho_n \) is unbounded, then \( u^{**}(r) \) is a singular solution, if \( \rho_n \) is bounded, then it is a regular solution. Let us show that, in the first case, \( \lim_{r \to +\infty} u^{**}(r) = +\infty \). Suppose this is not the case, we can find a sequence \( r_n \to 0 \) as \( n \to +\infty \) such that \( \lim_{n \to \infty} u^{**}(r_n) = B < +\infty \). Since \( u^{**}'(r) < 0 \), \( \sup_{r \geq r_n} u^{**}(r) \leq u^{**}(r_n) \), which implies that \( \sup_{r \geq 0} u^{**}(r) \leq B \). Let \( \rho_{n_k} \to +\infty \) be a subsequence of \( \{\rho_n\} \). By Proposition 3.1, we can find \( R(2B) \) and \( \rho_0(2B) \) such that whenever \( \rho_{n_k} > \rho_0(2B) \), \( u(r, \rho_{n_k}) > 2B \) on \((0, R(2B))\). Therefore, \( u^{**}(r) \geq 2B \) on \((0, R(2B))\). This contradicts to the previously obtained inequality \( \sup_{r \geq 0} u^{**}(r) \leq B \). Hence, \( \lim_{r \to +\infty} u^{**}(r) = +\infty \).

Suppose now \( \rho_n \), introduced above, is bounded. Then, we can extract a subsequence \( \lim_{k \to +\infty} \rho_{n_k} = \bar{\rho} < +\infty \). Let us prove that \( u^{**}(r) \) is a regular solution with \( u^{**}(0) = \bar{\rho} \), i.e., \( u^{**}(r) = u(r, \bar{\rho}) \). It follows from (47), that \( u(r, \rho) \) is continuous in \( \rho \) uniformly in \( r \) varying over compact intervals. Indeed,

\[
sup_{[0, r]} |u(\cdot, \rho_1) - u(\cdot, \rho_2)| \leq |\rho_1 - \rho_2| + \frac{1}{\beta+1} \int_0^r s^{-\beta} \left( \int_0^s (\Lambda (s, \lambda)^{-\beta \lambda}) d\lambda \right) \int_0^r \gamma e^{f(u(t, \rho_2))} - e^{f(u(t, \rho_1))} dt ds
\]

\[
\leq |\rho_1 - \rho_2| + K_{r, \rho_1, \rho_2} \int_0^r s^{\delta-1} \sup_{[0, s]} |u(\cdot, \rho_1) - u(\cdot, \rho_2)| ds,
\]

where \( K_{r, \rho_1, \rho_2} > 0 \) is a constant that only depends on \( r, \rho_1, \rho_2 \). Furthermore, above,

\[
\Lambda (s, \lambda) = \int_0^s t^\gamma (\lambda e^{f(u(t, \rho_1))} + (1 - \lambda) e^{f(u(t, \rho_2)))} dt,
\]

\[
K^{(1)}_{r, \rho_1, \rho_2} s^{\gamma+1} \leq \Lambda (s, \lambda) \leq K^{(2)}_{r, \rho_1, \rho_2} s^{\gamma+1}.
\]

where \( K^{(1)}_{r, \rho_1, \rho_2} \) and \( K^{(2)}_{r, \rho_1, \rho_2} \) are positive constants. Now by Gronwall’s inequality, one can find another constant \( K^{(3)}_{r, \rho_1, \rho_2} \), such that

\[
sup_{[0, r]} |u(\cdot, \rho_1) - u(\cdot, \rho_2)| \leq K^{(3)}_{r, \rho_1, \rho_2} |\rho_1 - \rho_2|.
\]
This implies that \( \lim_{k \to +\infty} u(r, \rho_n) = u(r, \bar{\rho}) \) on compact intervals of \([0, +\infty)\), and hence, \( u^{**}(r) = u(r, \bar{\rho}) \).

From the proof of Proposition 3.2, we obtain the following corollary.

**Corollary 3.1.** Under assumptions of Proposition 3.2, any limit point of the family \( \{ u(\cdot, \rho) \}_{\rho \geq 0} \) that can be obtained as a limit of a subsequence with \( \rho_n \to +\infty \), is a singular solution. Otherwise, it is a regular solution.

The theorem below is the main result of this subsection.

**Theorem 3.1.** Assume (B1)–(B3). Then, the result of Theorem 1.4 holds true.

**Proof.** Since we know that there exists a unique solution \( u(r, \rho) \) to problem \((P_\rho)\), there is also a unique solution \((u_\lambda, \lambda)\) with \( u_\lambda(r) = u(\lambda^{\frac{1}{2}} r, \rho) \) to problem \((P_\lambda)\). For \( u \) and \( u_\lambda \), related through this identity, it holds that the zero \( R(0, \rho) \) of \( u \) is expressed via \( \lambda \) as follows:

\[
R(0, \rho) = \lambda^{\frac{1}{2}}.
\]

By Proposition 3.1, \( R(0, \rho) \) has an upper bound. Therefore, the set of \( \lambda \) with the property that \((u_\lambda, \lambda)\) is a solution to \((P_\lambda)\) is bounded. Let \( \lambda^# \) be the exact upper bound for this set. Let us show that for all \( \lambda < \lambda^# \), there exists a regular solution to problem \((P_\lambda)\). Indeed, \( R(0, \rho) \) is the unique solution to the equation \( u(r, \rho) = 0 \) with respect to \( r \) for each fixed \( \rho > 0 \). By the implicit function theorem, the function \( R(0, \cdot) : (0, +\infty) \to \mathbb{R} \) is continuous. Moreover, \( \lim_{\rho \to 0} R(0, \rho) = 0 \) which is implied by the inequality

\[
\rho = u(0, \rho) - u(R(0, \rho), \rho) = \int_0^{R(0, \rho)} t^{-\frac{\gamma + 1}{\gamma}} \left( \int_0^t s^\gamma e^{f(u(s, \rho))} \, ds \right)^{\frac{1}{\gamma + 1}} \, dt
\]

As a continuous function of \( \rho \), \( \lambda = R(0, \rho)^{\frac{1}{2}} \) takes all the values on the interval \([0, \lambda^#]\).

Consider the case \( \lambda = \lambda^# \). We prove that there exists a solution to \((7)\), singular or regular, such that \((\lambda^#)^{\frac{1}{2}} \) is its zero. Find a sequence \( \{ \rho_n \} \) such that \( \lambda_n = R(0, \rho_n)^{\frac{1}{2}} \) converges to \( \lambda^# \). Let \( u^{**} \) be a limit point of \( u(\cdot, \rho_n) \) on an interval \([a, b] \subset (0, +\infty)\) such that \((\lambda^#)^{\frac{1}{2}} \in (a, b)\), i.e., there exists a subsequence \((u(\cdot, \rho_{n_k})) \to u^{**}\), as \( k \to \infty \), uniformly on \([a, b]\). By Corollary 3.1, if \( \rho_{n_k} \) is unbounded, then \( u^{**} \) is a singular solution. Otherwise, it is a regular solution. Without loss of generality, we write \( \rho_n \) for \( \rho_{n_k} \). Let \( R^*(0) = (\lambda^#)^{\frac{1}{2}} \).

We have

\[
|u^{**}(R^*(0))| = |u(R(0, \rho_n), \rho_n) - u^{**}(R^*(0))| \\
\leq |u(R(0, \rho_n), \rho_n) - u(R^*(0), \rho_n)| + |u(R^*(0), \rho_n) - u^{**}(R^*(0))|.
\]

By the uniform continuity of the family \( u(\cdot, \rho)_{\rho \geq 0} \), the first term on the right-hand side of (60) goes to zero. The other term goes to zero by the uniform convergence \( u(\cdot, \rho_n) \to u^{**} \) on \([a, b]\). This implies that \( u^{**}((\lambda^#)^{\frac{1}{2}}) = 0 \), which completes the proof.

**4. Study of regular solutions by means of the singular solution**

Here we demonstrate how the asymptotic representation (6) can be used to study regular solutions to problem \((P_\lambda)\).

The change of variable (8) transforms problem \((P_\lambda)\) to the following:

\[
\begin{cases}
-L(u) = e^{f(u)}, & 0 \leq r \leq \lambda^{\frac{1}{2}}, \\
u(\lambda^{\frac{1}{2}}) = 0.
\end{cases}
\]

In the above problem, one can actually exclude \( \lambda \) out of consideration keeping in mind that \( \lambda^{\frac{1}{2}} \) is the zero of the solution (by Proposition 3.1, we know that a zero of the solution
exists). In Subsections 4.1 and 4.2, we deal with the singular solution \( u^*(r) = u^*_r((\lambda^*)^{-\frac{s}{r}} r) \) and regular solutions \( u(r, \rho) = u_s(\lambda^{-\frac{s}{r}} r, \rho) \) to equation (7). In the case of regular solutions, equation (7) is complemented with the initial conditions \( u(0, \rho) = \rho, \partial_r u(0, \rho) = 0 \).

### 4.1 Transformation of problem \((P_\rho)\) and its limit equation

For simplicity of notation, in this section, we define the function

\[
\psi = e^f.
\]

Note that if \( \lim_{t \to +\infty} g''(t) = 0 \), then, by Remark 2.5, \( \mathcal{F}(u) \), involved in (14), is finite for all \( u \). Indeed, by the change of variable \( s = g(t) \), we obtain

\[
\int_u^{+\infty} e^{-\frac{t}{\sigma^2}} ds = \int_{\mathcal{F}(u)}^{+\infty} e^{-\frac{t}{\sigma^2}} g'(t) dt < +\infty.
\]

**Remark 4.1.** Sometimes we write \( \rho \) as the second parameter of \( \tilde{u} \). However, we do not mean that \( \rho \) is its maximal value, unlike the notation \( u(r, \rho) \). Note that

\[
\tilde{u}(s, \rho) = \mathcal{F}_1^{-1}\left\{ \mathcal{F}_1(1) \frac{\mathcal{F}(u(r, \rho))}{\mathcal{F}(\rho)} \right\} \leq \mathcal{F}_1^{-1}(\mathcal{F}_1(1)) = 1, \quad s \geq 0,
\]

and

\[
\tilde{u}(0, \rho) = 1.
\]

We just want to emphasize that \( \tilde{u} \) actually depends on \( \rho \). Moreover, we will study the limit of \( \tilde{u} \) as \( \rho \) goes to infinity.

**Remark 4.2.** Sometimes we skip the dependence on \( \rho \) in both \( u \) and \( \tilde{u} \) to simplify notation.

**Lemma 4.1.** Assume (61) is fulfilled. Let \( u(r) \) be a \( C^2 \)-solution to problem \((P_\rho)\). Then, the function \( \tilde{u}(s) \), defined by (14), is a solution to the problem

\[
L(\tilde{u}) + e^\theta + s^{-\gamma}|\tilde{u}|^{\beta+2}\left( \frac{I(u(\epsilon_s s))}{\beta + 1} - 1 \right) = 0,
\]

\[
\tilde{u}(0) = 1, \quad \tilde{u}'(0) = 0,
\]

where \( \tilde{u} \) is everywhere evaluated at \( s \) and \( I(u) = \mathcal{F}(u)f'(u)\psi(u)\frac{1}{s^{\gamma+1}} \).

**Proof.** First, we note that the \( \tilde{u}(s) \), defined by (14), satisfies the initial conditions in (62). Define \( G(u) = \mathcal{F}_1^{-1}\epsilon_{\rho}^{-\theta} \mathcal{F}(u) \), where \( \theta = \frac{1}{2s} \). Then, \( u(r) = G^{-1}(\tilde{u}(s)) \), where \( r = \epsilon_s s \).

Computing \( L(u) \) by using the previous expression, we obtain

\[
L(u(r)) = \epsilon_{\rho}^{-\theta} s^{-\gamma}(s^a \tilde{u}'(s))^\beta s\tilde{u}'(s)G'(u(r))^{-\beta-1}
\]

\[
= \epsilon_{\rho}^{-\theta} L(\tilde{u}(s))G'(u(r))^{-\beta-1} - \epsilon_{\rho}^{-\theta} (\beta + 1)s^{a-\gamma}|\tilde{u}'(s)|^{\beta+2}G'(u(r))^{-\beta-3}G''(u(r)).
\]

Problem \((P_\rho)\) is then transformed to the following:

\[
L(\tilde{u}(s)) + \epsilon_{\rho}^{-\theta} \psi(u(r))G'(u(r))^{\beta+1} - (\beta + 1)s^{a-\gamma}|\tilde{u}'(s)|^{\beta+2}G''(u(r)) \frac{G''(u(r))}{G'(u(r))^2} = 0.
\]

By the definition of \( \mathcal{F}(u) \) and \( \mathcal{F}_1(u) \) in (14),

\[
G'(u(r)) = \epsilon_{\rho}^{-\theta} e^{\tilde{u}(s)} \frac{\mathcal{F}(u)}{s^{\gamma+1}},
\]

which transforms the second term into \( e^{\tilde{u}(s)} \). Furthermore,

\[
\frac{G''(u)}{G'(u)^2} = \frac{(\ln G'(u))'}{G'(u)} = \frac{1}{\beta + 1} \frac{G'(u) - f'(u)}{G'(u)}.
\]
In (64), \( u \) is everywhere evaluated at \( r \). Recall that \( \tilde{u}(s) = G(u(r)) \). For simplicity of notation, we skip the dependence on \( s \) in \( \tilde{u} \) and, with a slight abuse of notation, simply write \( \tilde{u} = G(u) \). By (63),

\[
\frac{f'(u)}{G'(u)} = \frac{\psi'(u)\psi(u) - \frac{\beta}{u} \tau^\tau F(u)}{\varepsilon^\theta F(u) e^{\frac{\beta}{u} \tau^\tau}} = \frac{I(u)}{\beta + 1}
\]

since \( \varepsilon^\theta F(u) = F_1(\tilde{u}) \) and \( F_1(\tilde{u}) = (\beta + 1)e^{-\frac{\beta}{u} \tau^\tau} \). The above computations imply (62).

**Lemma 4.2.** Assume (B1)–(B3). Then, \( \lim_{\rho \to +\infty} u(\varepsilon \rho, s) = +\infty \) in \( C_{\text{loc}}[0, +\infty) \).

**Proof.** Fix an arbitrary interval \([0, T]\). From Proposition 3.1 it follows that for any \( B > 0 \), there exists a number \( R(B) > 0 \) and a constant \( \rho_0 > 0 \) such that for all \( \rho > \rho_0 \) and \( r \in (0, R(B)) \), \( u(r, \rho) > B \). Note that \( \lim_{\rho \to +\infty} \varepsilon \rho = 0 \). This proves that \( \lim_{\rho \to +\infty} \sup_{s \in [0, T]} u(\varepsilon \rho, s) = +\infty \).

**Lemma 4.3.** Assume (61), (B1)–(B3), and that \( \lim_{t \to +\infty} \frac{g'(t)}{g''(t)} = 0 \). Then, \( \lim_{t \to +\infty} I(u(\varepsilon \rho, s)) = \beta + 1 \) in \( C_{\text{loc}}[0, +\infty) \).

**Proof.** We have

\[
I(u) = \int_u^{+\infty} e^{-f'(s)} ds f'(u) = \int_{f(u)}^{+\infty} e^{-\frac{\beta}{u} \tau^\tau} g'(t) dt.
\]

Let \( v = f(u) \). By L'Hopital's rule,

\[
\lim_{v \to +\infty} \int_v^{+\infty} e^{-\frac{\beta}{u} \tau^\tau} g'(t) dt = \lim_{v \to +\infty} \frac{e^{-\frac{\beta}{u} \tau^\tau} g'(v)}{(\beta + 1)^{-1} e^{-\frac{\beta}{u} \tau^\tau} g'(v) - e^{-\frac{\beta}{u} \tau^\tau} g''(v)} = \lim_{v \to +\infty} \frac{(\beta + 1)}{1 - (\beta + 1) \frac{g''(v)}{g'(v)}} = \beta + 1.
\]

The statement now follows from Lemma 4.2.

**Lemma 4.4.** Assume (A1)–(A6). Then, the following representation holds for the singular solution (12) of equation (11):

\[
v^*(t) = F^{-1}(A^{-1} e^{-\hat{\theta}t}(1 + \varepsilon(t))),
\]

where \( A = \hat{\theta}(\hat{\alpha} - 1) \tau^\tau \) and \( \lim_{t \to +\infty} \varepsilon(t) = 0 \).

**Proof.** Identity (65) is equivalent to \( \varepsilon(t) = A e^{\hat{\theta}t} F(v^*(t)) - 1 \). Let us show that \( \lim_{t \to +\infty} e^{\hat{\theta}t} F(v^*(t)) = A^{-1} \). Recall that \( v^*(t) = g(\hat{\theta}t + \varphi(t)) + \eta(t) \), where \( \varphi = \varphi_1 + \varphi_2 \) is defined by (13), \( \eta = O(g''(\hat{\theta}t)) \), and \( \eta' = O(g''(\hat{\theta}t)) \). By L'Hopital's rule,

\[
\lim_{t \to +\infty} \frac{F(v^*(t))}{e^{-\hat{\theta}t}} = \lim_{t \to +\infty} \frac{e^{-\frac{f(v^*(t))}{\hat{\theta}e^{-\theta}}} (g'(\hat{\theta}t + \varphi)(\theta + \varphi') + \eta')}{\hat{\theta} e^{-\hat{\theta}t}}.
\]

Transforming \( f(v^*(t)) \) by formula (16), we obtain that the limit on the right-hand side equals to

\[
\lim_{t \to +\infty} \hat{\theta}^{-1} e^{-\frac{\beta}{u} \tau^\tau} g'(\hat{\theta}t + \varphi)(\theta + \varphi') = \lim_{t \to +\infty} \frac{g'(\hat{\theta}t + \varphi)(\theta + \varphi')}{\hat{\theta} (\hat{\alpha} - 1) \tau^\tau g'(\hat{\theta}t)} = A^{-1}.
\]

We used representation (13) for \( \varphi = \varphi_1 + \varphi_2 \) and the fact that \( \lim_{t \to +\infty} \frac{\eta}{g''(\hat{\theta}t + \varphi(t))} = 0 \). Finally, \( \varphi' \) goes to zero by (19). This concludes the proof.
Lemma 4.5. Assume (A1)–(A6). Then, the singular solution \( u^\ast \) to (7) has the representation
\[
u^\ast(r) = F^{-1}(K r^\theta (1 + \delta(r))),
\]
where \( K = \frac{\beta + 1}{\theta} (\alpha - \beta - 1)^{- \frac{1}{\theta + 1}}, \delta(r) = \varepsilon (\ln \kappa - \ln r), \kappa = (\beta + 1)^{\frac{1}{\theta}}. \)

Proof. The proof follows immediately from Lemma 4.4 by the substitution \( t = \ln \kappa - \ln r. \)

Lemma 4.5 allows to obtain a representation for the singular solution \( \tilde{u}^\ast \) to (62).

Lemma 4.6. Assume (A1)–(A6). Then, the singular to equation (62) \( \tilde{u}^\ast(s) = F_1^{-1} A^{-1}(\varepsilon^\rho(s)) \) can be represented as follows:
\[
u^\ast(s) = \ln\{\theta^{\beta + 1}(\alpha - \beta - 1)\} - \theta \ln s - (\beta + 1) \ln \{1 + \delta(\varepsilon^\rho(s))\}.
\]

Proof. Formula (66) is the result of application of transformation (14) to \( u^\ast(\varepsilon^\rho(s)). \) Indeed, \( \tilde{u}^\ast(s) = F_1^{-1}(K s^\theta (1 + \delta(\varepsilon^\rho(s)))) \), where \( F_1^{-1}(t) = -(\beta + 1) \ln \frac{t}{\pi_{\theta + 1}}. \) This immediately implies (66).

Proposition 4.1. Assume (A1)–(A6). Then, for the singular and regular solutions to (62), it holds that
\[
\lim_{\rho \to +\infty} \tilde{u}(s) = w(s) \quad \text{and} \quad \lim_{\rho \to +\infty} \tilde{u}^\ast(s) = w^\ast(s)
\]
in \( C_{loc}(0, +\infty) \) and \( C_{loc}(0, +\infty), \) respectively. Here, \( w(s) \) is the regular solution to the problem
\[
L(w) + e^w = 0, \quad w(0) = 1, \quad w'(0) = 0,
\]
and \( w^\ast(s) \) is the singular solution to the equation in (68) given explicitly as follows:
\[
w^\ast(s) = \ln\{\theta^{\beta + 1}(\alpha - \beta - 1)\} - \theta \ln s = \ln\{\theta^{\beta + 1}(\alpha - \beta - 1)\} + \theta \ln \frac{\kappa}{s}.
\]

Remark 4.3. The last expression in (69) agrees with (45) with \( t = \ln \frac{\kappa}{s}. \)

For the proof of Proposition 4.1, we need the following lemma.

Lemma 4.7. Under assumptions of Lemma 4.3, problem (68) is the limit problem for (62) as \( \rho \to +\infty. \) More specifically, the last term on the right-hand side of (62) goes to zero in \( C_{loc}(0, +\infty) \) as \( \rho \to +\infty. \)

Proof. Let \( \hat{w}(s) = |\tilde{u}'(s)|^{\beta + 1}. \) Recall that by Lemma 3.1, \( u'(r) < 0. \) Therefore, by (14), \( |\tilde{u}'(s)| = -\tilde{u}'(s). \) Fix an arbitrary interval \([0, T]\) and introduce the notation,
\[
\mathcal{I}_\rho(s) = \frac{I(u(\varepsilon^\rho(s)))}{\beta + 1} - 1.
\]
Then, \( \hat{w}'(s) \) satisfies the following first-order differential inequality:
\[
\hat{w}'(s) = s^{\gamma - \alpha} e^{\tilde{u}'} - \alpha s^{-1} \hat{w} + \tilde{w} \mathcal{I}_\rho(s) + C + \varepsilon \hat{w} + \varepsilon \hat{w}^2.
\]
Here, \( C = T^{\gamma - \alpha} e^h. \) Recall that \( \mathcal{I}_\rho(s) < \varepsilon \) for sufficiently large \( \rho. \) Finally, it holds that \( \hat{w} \mathcal{I}_\rho(s) \leq \hat{w} + \hat{w}^2. \)

We aim to estimate \( \hat{w}'(s) \) via the solution \( \omega(s) \) to the problem
\[
\omega' = C + \varepsilon \omega + \varepsilon \omega^2, \quad \omega(0) = 0.
\]
It is known that \( \omega = \frac{-1}{\varepsilon y} u' \) is a solution to the ODE in (71) if \( y \) is the solution to
\[
y'' - \varepsilon y' + C \varepsilon y = 0.
\]
For sufficiently small \( \varepsilon \), the both roots of the characteristic equation are complex, and therefore, the solution takes the form
\[
y(s) = Ke^{2s}(\cos(Ds) + K \sin(Ds)), \quad \text{where} \quad D = \frac{\sqrt{4C\varepsilon - \varepsilon^2} - \varepsilon}{2}.
\]

It is straightforward to obtain now the solution \( \omega \) to problem (71), computing \( K \) from the condition \( \omega(0) = 0 \):
\[
\omega(s) = \frac{\varepsilon^2 + 4D^2}{4D\varepsilon} \frac{1}{\operatorname{ctg}(Ds) - \frac{s^2}{4D}}.
\]

Note that the solution \( \omega(s) \) is well-defined on the interval \( [0, \frac{2\arctg(\frac{s}{\sqrt{4C\varepsilon - \varepsilon^2}})}{\varepsilon}] \). When \( \varepsilon \) is sufficiently small, the numerator of the right endpoint is close to \( \pi \); however, the denominator is small. Thus, we can choose \( \varepsilon \) sufficiently small (which makes \( \rho \) sufficiently large) such that \( \omega(s) \) is well-defined on \( [0, T] \). Therefore, on \( [0, T] \), it holds that
\[
|\tilde{w}'(s, \rho)|^{\beta+1} \leq \omega(s)
\]
for sufficiently large \( \rho \). This, along with Lemma 4.3, proves the lemma. \( \square \)

**Proof of Proposition 4.1.** Everywhere in the proof, \( K_i, i = 1, 2, 3 \), are positive constants. Fix an arbitrary interval \([0, T]\). We have
\[
w(r) - \tilde{w}(r) = \frac{1}{\beta + 1} \int_0^r t^{\beta} \frac{\alpha^2}{\pi \tau} \left( \int_0^1 \Lambda(t, \rho, \lambda)^{-\frac{\beta}{2}} d\lambda \right) \int_0^t (s^\gamma (e^{\tilde{w}} - e^w) + s^\alpha |\tilde{w}'(s)|^{\beta+2} I_\rho(s)) ds dt,
\]
where \( \Lambda(t, \rho, \lambda) \) is defined by the first identity below and can be estimated by the expression on the right-hand side:
\[
\Lambda(t, \rho, \lambda) = \lambda \int_0^1 \left( s^\gamma e^{\tilde{w}} + s^\alpha |\tilde{w}'(s)|^{\beta+2} I_\rho(s) \right) ds + (1-\lambda) \int_0^t s^\gamma e^w ds \geq (1-\lambda) e^{w(T)} \int_0^t s^\gamma ds.
\]
Indeed, it holds that
\[
\int_0^t (s^\gamma e^{\tilde{w}} + s^\alpha |\tilde{w}'(s)|^{\beta+2} I_\rho(s)) ds = t^\alpha |\tilde{w}'(t)|^{\beta+1} \geq 0.
\]
By Gronwall’s inequality,
\[
t^\alpha |\tilde{w}'(t)|^{\beta+1} \leq e^{\int_0^t |\tilde{w}'(s) I_\rho(s)| ds} \int_0^t s^\gamma e^\tilde{w} ds \leq K_1 t^{\gamma+1},
\]
where the last estimate holds by (72) for sufficiently large \( \rho \). Furthermore, estimate (73) implies
\[
\sup_{[0, r]} |\tilde{u} - w| \leq K_2 \int_0^r t^{\frac{\beta+1}{\gamma+1}} \left\{ \sup_{[0, t]} |\tilde{u} - w| + \sup_{[0, r]} |I_\rho| \right\} dt,
\]
where, in particular, it is taken into account that \( \int_0^1 (1-\lambda)^{-\frac{\beta}{2}} d\lambda = \frac{\beta+1}{2} \). By Gronwall’s inequality, \( \sup_{[0, r]} |\tilde{u} - w| \leq K_3 \sup_{[0, r]} |I_\rho| r^\beta e^{K_3 t^{\gamma+1}} \). This implies that \( \sup_{[0, T]} |\tilde{u} - w| \to 0 \) as \( \rho \to +\infty \).

We prove now the second convergence in (67). It follows from the proof of Corollary 1.1 that the singular solution to (68) takes the form (69). If, in (66), we pass to the limit as \( \rho \) goes to \( +\infty \), we obtain exactly (69). The convergence holds in \( C_{\text{loc}}(0, +\infty) \) since the convergence \( \lim_{\rho \to +\infty} \delta(\varepsilon \rho s) = 0 \) holds in this space. \( \square \)
4.2 Number of intersection points. Case \( \alpha - \beta - 1 < \frac{4\theta}{d+1} \) \((d < 2k+8)\).

In this subsection, we first study the number of intersection points between the regular and singular solutions \( w \) and \( w^* \) to the equation in (68), and then use this result to study the limit, as \( \rho \to +\infty \), of the number of intersection points between \( u(\cdot, \rho) \) and \( u^* \). Recall that the number of zeros of \( w - w^* \) on an interval \((R,S)\) is denoted by \( \mathcal{L}_{(R,S)}(w - w^*) \).

Although Theorem 4.1 has its own interest, we prove it in this subsection as an auxiliary result.

**Theorem 4.1.** Assume \((A6)\). Suppose \( \alpha - \beta - 1 < 4\theta \). Then, for any \( R > 0 \), \( w \) and \( w^* \) have an infinite number of intersection points on \((R, +\infty)\), i.e., \( \mathcal{L}_{(R, +\infty)}(w - w^*) = +\infty \). Furthermore, \( \mathcal{L}_{(0,R]}(w - w^*) \) is finite.

**Proof.** Define \( v(t) = w(s) \) and \( \omega(t) = v'(t)^{\beta+1} \), where \( t = \ln\{\frac{\rho}{s}\} \) with \( \kappa = (\beta + 1)^{\beta+1} \). By Lemma 3.1, \( v'(t) > 0 \), and hence, by (38),

\[
(\omega^{\beta+1 - \alpha})' = -((\beta + 1)e^{-((\gamma+1)t)\kappa})e^\nu.
\]

Now let \( w^*(s) \) be given by (69) and \( v^*(t) = w^*(s) \), where \( t = \ln\{\frac{\rho}{s}\} \). Then,

\[
v^*(t) = \theta t + \ln\{\theta^{\beta+1}(\alpha - 1)\}.
\]

Define \( x(t) = (\frac{\rho}{s})\kappa e^\nu \) and \( y(t) = \omega(t) - \omega^*(t) \), where \( \omega^*(t) = v^*(t)^{\beta+1} - \theta^{\beta+1} \).

Equation (74) implies that \( x \) and \( y \) satisfy the following system of ODEs

\[
\begin{cases}
y' = (\alpha - \beta - 1)(y - \theta^{\beta+1}(e^\nu - 1)), \\
x' = (y + \theta^{\beta+1})\frac{\rho}{s} - \theta,
\end{cases}
\]

where the last equation holds due to the fact that \( v' = \omega \frac{s}{\rho} \). Note that system (75) has a unique steady state \((0,0)\). We are going to determine its type. The linearization of (75) has the form

\[
\begin{cases}
y' = (\alpha - \beta - 1)(y - \theta^{\beta+1}x), \\
x' = \frac{w}{(\beta+1)\theta^\nu}.
\end{cases}
\]

Since \( 0 < \alpha - \beta - 1 < 4\theta \), the roots of the characteristic equation for the linearized system are complex numbers with the positive real part \( \frac{\alpha - \beta - 1}{2} \), and hence, \((0,0)\) is an unstable focus. This means that as \( t \) goes to \(-\infty \), \( x(t) \) crosses zero infinitely many times at any neighborhood of \(-\infty \). This implies that for any \( R > 0 \), \( \mathcal{L}_{(-\infty, -R]}(v - v^*) \) is \( \mathcal{L}_{(\kappa e^\nu, +\infty)}(w - w^*) = +\infty \).

Suppose \( \mathcal{L}_{(0,R]}(w - w^*) = +\infty \) for some \( R > 0 \). Then, there exists an accumulation point, say \( s_0 \), of zeros of \( w - w^* \) belonging to \([0,R] \). Note that \( (w - w^*)(s_0) = 0 \) which implies that \( s_0 \in (0,R] \). Furthermore, \( (w - w^*)'(s_0) = 0 \). This implies that \( v(t_0) = v^*(t_0) \) and \( \omega(t_0) = \omega^*(t_0) \) for \( t_0 = \ln\kappa - \ln s_0 \), which is a contradiction.

**Corollary 4.1.** Under assumptions of Theorem 4.1, there exists \( R > 0 \) such that \( w - w^* \) changes the sign at each zero point on \((R, +\infty)\).

**Proof.** The proof follows immediately from the fact that \((0,0)\) is a focus for (75). \(\square\)

**Theorem 4.2.** Assume \((A1)-(A6)\) and \( \alpha - \beta - 1 < 4\theta \). Let \( u(\cdot, \rho) \) be the regular solution to problem \((P)\) and \( u^* \) be the singular solution to (7) defined by (8) via \( (u_\delta^*, \lambda^*) \).

Then, for any \( \delta > 0 \), it holds that

\[
\lim_{\rho \to +\infty} \mathcal{L}_{(0,\delta)}(u(\cdot, \rho) - u^*(\cdot)) = +\infty.
\]

**Proof.** Recall that \( \tilde{u} - \tilde{u}^* \to w - w^* \) as \( \rho \to +\infty \) uniformly on any interval \([R,S] \subset (0, +\infty) \). Note that for a sufficiently large \( \rho \), \( \mathcal{L}_{[R,S]}(\tilde{u} - \tilde{u}^*) \) is not smaller than \( \mathcal{L}_{[R,S]}(w - w^*) - 2 \), where the latter number is finite by Theorem 4.1. Recall that \( u(\cdot, \rho, s) = \mathcal{F}_1^{-1}e_\rho^\beta \mathcal{F}(\tilde{u}(s, \rho)) \) with \( \lim_{\rho \to +\infty} e_\rho = 0 \). Fix \( N \in \mathbb{N} \). There exist \( R, S > 0 \) and a number \( \rho_{R,S} > 0 \) such that \( \mathcal{L}_{[R,S]}(\tilde{u} - \tilde{u}^*) \geq N \) for \( \rho > \rho_{R,S} \). Note that for each zero point \( s \in [R,S] \) of \( \tilde{u} - \tilde{u}^* \), there
exists exactly one zero point \( r \in [\varepsilon, R, \varepsilon] \) of \( u(\cdot, \rho) - u^*(\cdot) \). Choose \( \rho_5 > \rho_{R,S} \) such that for all \( \rho > \rho_5 \), \([\varepsilon, R, \varepsilon] \subset (0, \delta) \). Therefore, for any \( N \in \mathbb{N} \), one can find \( \rho_5 \) such that for all \( \rho \geq \rho_5 \), \( \mathcal{L}_{(0,\delta)}(u(\cdot, \rho) - u^*(\cdot)) \geq N \). The theorem is proved.

### 4.3 Bifurcation diagram. Case \( \alpha - \beta - 1 < \frac{4\theta}{\theta + 1} \) \( (d < 2k + 8) \).

**Theorem 4.3.** Assume (A1)–(A6) and \( \alpha - \beta - 1 < 4\theta \). Let \((u_{\lambda(\rho)}, \lambda(\rho))\) be the family of regular solutions to problem (P_\lambda) and let \((u^*, \lambda^*)\) be the singular solution constructed in Theorem 2.1. Then, as \( \rho \to \infty \), \( \lambda(\rho) \) oscillates around \( \lambda^* \). In particular, there are infinitely many regular solutions to problem (P_\lambda) for \( \lambda(\rho) = \lambda^* \).

**Proof.** Let \( u(\cdot, \rho) \) and \( u^* \) be the regular and singular solutions to (7), as in Theorem 4.2. Define \( \hat{v}(r, \rho) = u(r, \rho) - u^* \) and show that

\[
\hat{v}(\tau, \rho) = 0 \quad \text{whenever} \quad \hat{v}(\rho, \rho) = 0.
\]

Suppose \( \hat{v}(r, \rho) = 0 \) for \( r \in (0, r_0) \), we have

\[
u(\cdot, \rho) - u^*(\cdot) = \int_{r_0}^{\infty} s^{-\frac{\beta + 1}{\beta + 1}} \left[ \left( r_0^{\alpha} |u'(r_0)|^{\beta + 1} \right) - \int_{s}^{\infty} \xi^{\gamma} e^{f(u(\xi))} d\xi \right]^{\frac{1}{\beta + 1}} ds

= \frac{1}{\beta + 1} \int_{r_0}^{\infty} s^{-\frac{\beta}{\beta + 1}} \left( \int_{s}^{\infty} \xi^{\gamma} e^{f(u(\xi))} d\xi \right)^{\frac{1}{\beta + 1}} ds,
\]

where

\[
\Lambda(s, \lambda) = r_0^{\alpha} |u'(r_0)|^{\beta + 1} - \int_{s}^{\infty} \xi^{\gamma} (\lambda e^{f(u(\xi))} + (1 - \lambda) e^{f(u(\xi))}) d\xi = s^{\alpha} |u'(s)|^{\beta + 1} + (1 - \lambda) |u^*(s)|^{\beta + 1}.
\]

Take \( \varepsilon \in (0, r_0) \). For \( \tau \in [\varepsilon, r_0] \), we have

\[
\inf_{s \in [\varepsilon, r_0]} (s^{\alpha} \min\{|u'(s)|^{\beta + 1}, |u^*(s)|^{\beta + 1}\}) \leq \Lambda(s, \lambda) \leq r_0^{\alpha} |u'(r_0)|^{\beta + 1}.
\]

Therefore, there exists a constant \( K > 0 \) such that for \( r \in [\varepsilon, r_0] \),

\[
\sup_{r \in [\varepsilon, r_0]} |u - u^*| \leq K \sup_{r \in [\varepsilon, r_0]} |u(r) - u^*(r)| ds.
\]

By Gronwall’s inequality, \( \sup_{r \in [\varepsilon, r_0]} |u - u^*| = 0 \). Since \( \varepsilon \in (0, r_0) \) is arbitrary, we obtain that \( u = u^* \) on \((0, r_0)\) which is a contradiction. Thus, (77) is true.

Recall that \( \lambda(\rho) \hat{\tau} \) and \( \lambda^* \hat{\tau} \) are zeros of \( u(\cdot, \rho) \) and \( u^* \), respectively. Since \( u \) and \( u^* \) are decreasing in \( r \), they do not have other zeros. Define \( r(\rho) = \min\{\lambda(\rho) \hat{\tau}, \lambda^* \hat{\tau}\} \) and

\[
z(\rho) = L_{(0,r(\rho))}(\hat{v}(\cdot, \rho)).
\]

Note that if \( \rho \) is sufficiently large, then, by Proposition 3.1, we can find two positive constants \( \tau_1 \) and \( \tau_2 \) such that \( \tau_1 < r(\rho) < \tau_2 \). Furthermore, we note that \( z(\rho) \) is positive and finite. Indeed, \( z(\rho) > 0 \) since, by Theorem 4.2, \( \lim_{\rho \to +\infty} L_{(0,\tau_1)}(\hat{v}(\cdot, \rho)) = +\infty \). Assuming that \( z(\rho) \) is infinite, we can find a sequence \( \{r_n\} \) of zeros of \( \hat{v} \) converging to a point \( \hat{r} \in [0, r(\rho)] \). One has \( \hat{v}(\hat{r}, \rho) = \hat{v}(\hat{r}, \rho) = 0 \) which contradicts to (77). Thus, we conclude that for each fixed \( \rho \), the equation \( \hat{v}(r, \rho) = 0 \) has a finite number of solutions on \((0, r(\rho))\). By the same argument, for each fixed \( \rho > 0 \), the equation \( \hat{v}(r, \rho) = 0 \) has a finite number of solutions on any finite interval \([0, R]\), and therefore, there is a countable number of zeros of \( \hat{v}(r, \rho) \) on \((0, +\infty)\).

Fix \( \rho_0 > 0 \) and take an arbitrary zero \( r_0 \) of \( \hat{v}(\cdot, \rho_0) \). By (77), \( \partial_r \hat{v}(r_0, \rho_0) \neq 0 \). By the implicit function theorem, there exists an interval \((\rho_1, \rho_2)\), containing \( \rho_0 \), and a unique continuous function \( \tau(\rho) \) solving the equation \( \hat{v}(r, \rho) = 0 \) on \((\rho_1, \rho_2)\) and such that \( \tau(\rho_0) = r_0 \). Without loss of generality, \((\rho_1, \rho_2)\) can be regarded as the maximal
interval to which \( r(\rho) \) can be extended. This extension will be unique by (77) and the implicit function theorem. The curve \( r(\rho) \) may cross \( r(\rho) \) at some point from \((\rho_1, \rho_2)\) or may not. When it happens a zero enters or exits the interval \((0, r(\rho))\). Remark that \( \rho_1 \) can be either zero or positive; \( \rho_2 \) can be either finite or \(+\infty\). Let us show that if \( \rho_2 \) is finite, then \( \lim_{\rho \to \rho_2} r(\rho) = +\infty \). Suppose \( \rho_2 \) is finite, but \( \lim_{\rho \to \rho_2} r(\rho) \neq +\infty \). Then, we can find a sequence \( \rho^n \to \rho_2 \) as \( n \to +\infty \) such that \( r(\rho^n) \) converges to a number \( r_2 \in (0, +\infty) \). By continuity, \( \hat{v}(r_2, \rho_2) = 0 \). If \( r_2 \) is the limit of \( r(\rho) \) as \( \rho \to \rho_2 \), then, by the implicit function theorem, we can find an extension of \( r(\rho) \) beyond \( \rho_2 \) which is a contradiction. Therefore, the limit at \( \rho_2 \) does not exist. Then, we can find a neighborhood \( U = U(r_2, \rho_2) = (\rho_2 - \delta, \rho_2 + \delta) \times (\rho_2 - \delta, \rho_2 + \delta) \), where 1) the equation \( \hat{v}(r, \rho) = 0 \) is uniquely solvable for each fixed \( \rho \in (\rho_2 - \delta, \rho_2 + \delta) \) and, furthermore, determines \( r \) as a continuous function of \( \rho \) living inside \( U \); and 2) \( r(\rho) \) exits and enters \( U \) infinitely many times. This contradicts to the implicit function theorem. Hence, if \( \rho_2 \) is finite, we have that \( \lim_{\rho \to \rho_2} r(\rho) = +\infty \). By the similar argument, one can show that if \( \rho_1 > 0 \), then \( \lim_{\rho \to \rho_1} r(\rho) = +\infty \). Also remark that by the implicit function theorem and (77), two functions \( t_1(\rho) \) and \( t_2(\rho) \) representing two different solutions of \( \hat{v}(r, \rho) = 0 \) cannot exit or enter the interval \((0, r(\rho))\) through its right end at the same point \( \rho \). One can extend this observation stating that two functions \( t_1(\rho) \) and \( t_2(\rho) \) representing two different solutions of \( \hat{v}(r, \rho) = 0 \) never intersect each other, which means that two zeros cannot merge. The above arguments allow us to conclude that \( z(\rho) \) is positive, finite, and changes only by 1 when a zero of \( \hat{v}(\cdot, \rho) \) enters or exists the interval \((0, r(\rho))\) through its right end.

Let us prove now that \( \lambda(\rho) \) oscillates infinitely around \( \lambda^* \) as \( \rho \to +\infty \). First, we note that \( \hat{v}(0, \rho) = -\infty \) for all \( \rho > 0 \), i.e., does not change the sign as \( \rho \) varies. Then, \( z(\rho) \) changes only if \( \hat{v}(r(\rho), \rho) \) changes the sign. Suppose there exists \( \sigma_1 > 0 \) such that for all \( \rho > \sigma_1 \), \( \lambda(\rho) \geq \lambda^* \). Then, \( r(\rho) = \lambda^{\frac{1}{\gamma}} \) and we have that \( \hat{v}(r(\rho)) = u(\lambda^{\frac{1}{\gamma}}, \rho) \geq 0 \) for all \( \rho > \sigma_1 \). Likewise, if we assume that there exists \( \sigma_2 > 0 \) such that for all \( \rho > \sigma_2 \), \( \lambda(\rho) \leq \lambda^* \), then \( r(\rho) = \lambda^{\frac{1}{\gamma}} \) and \( \hat{v}(r(\rho)) = -u^{\ast}(\lambda^{\frac{1}{\gamma}}, \rho) \leq 0 \) for all \( \rho > \sigma_2 \). In both situations, i.e., \( \lambda(\rho) \geq \lambda^* \) for all \( \rho > \sigma_1 \) or \( \lambda(\rho) \leq \lambda^* \) for all \( \rho > \sigma_2 \), a zero of \( \hat{v} \) cannot enter the open interval \((0, r(\rho))\) through its right end (however it can reach \( r(\rho) \) and then turn back) when \( \rho > \max\{\sigma_1, \sigma_2\} \), which contradicts to Theorem 4.2 stating that \( \lim_{\rho \to +\infty} L(0, r_{1\ast})(\hat{v}(\cdot, \rho)) = +\infty \).

5. **Liouville-Bratu-Gelfand problem**

In this section, we consider the equation

\[
L(u) + e^u = 0.
\]

As before, when we deal with regular solutions, we complete equation (78) with the initial conditions \( u(0) = \rho \) and \( u'(0) = 0 \); the regular solution will be also denoted by \( u(r, \rho) \). Furthermore, we let \( u^{\ast} \) be the singular solution to (78) given by (69).

In this section, we aim to complement the results of [18], where the authors studied bifurcation diagrams for problem (42), with a study of the number of intersection points between the regular and singular solutions to equation (78) and the convergence of regular solutions to the singular. Recall that equations (42) and (78) are equivalent by the change of variable (8).

5.1 **Number of intersection points between the regular and singular solutions**

We already studied the number of intersection points between the regular and singular solutions \( u(\rho, \cdot) \) and \( u^{\ast} \) to (78) in Subsection 4.2. Namely, we obtained Theorem 4.1 whose statement we repeat here for completeness of the presentation.

**THEOREM 4.1.** Assume (A6). Suppose \( \alpha - \beta - 1 < 4\hat{\theta} \) \((2k < d < 2k + 8)\). Then, \( L(0, \infty)(u - u^{\ast}) = +\infty \).
Theorem 5.1. Assume (A6). Suppose \( \alpha - \beta - 1 \geq 4\theta(\beta + 1) \) \((d \geq 2k + 8k^2)\). Then, \( \mathcal{L}_{(0, +\infty)}(u - u^*) = 0 \).

Proof. First of all, we note that \( \alpha - \beta - 1 \geq 4\theta(\beta + 1) \geq 4\hat{\theta} \). We keep the same notation as in the proof of Theorem 4.1 in Subsection 4.2, except for the notation for the regular and singular solutions. Namely, we define the same objects with \( w = u \) and \( w^* = u^* \).

The unique steady state \((0, 0)\) of (76), and therefore of (75), is then an unstable node. We further note that the eigenvectors for the linearized system (76) point to the first quadrant of the phase plane. Therefore, in a small neighborhood of the origin, the integral curves are located at the first and the third quadrants. As \( t \) increases, the motion along the curves occurs clockwise. Note that the solution to (75) satisfies the following conditions:

\[
\lim_{t \to +\infty} x(t) = -\infty \quad \text{and} \quad \lim_{t \to +\infty} y(t) = -\theta^{\beta + 1}.
\]

From (75), we will evaluate the derivative \( \frac{dy}{dx} \) at the points of the curve \( y = \mathcal{C}(x) = (\ell x + \delta)^{\beta + 1} - \theta^{\beta + 1} \), which is the same as \((y + \theta^{\beta + 1})\frac{1}{1 + \theta} - \theta = \ell x\), where the number \( \ell \) is to be fixed later. Introducing the notation \( \delta = \alpha - \beta - 1 \), we obtain that in the area \( \{x < 0, -\theta^{\beta + 1} < y < 0\} \), the following estimate holds:

\[
\frac{dy}{dx} = \delta \left( \frac{y}{(y + \theta^{\beta + 1}) \frac{1}{1 + \theta} - \theta} - \frac{\theta^{\beta + 1} e^{\ell x} - 1}{\ell} \right) \geq \delta \frac{\delta(1 - \frac{\theta}{\ell})}{1 + \theta} \geq (\beta + 1)\ell,
\]

where the number \( \ell \) is to be chosen in such a way that the last inequality is satisfied. To see why the second inequality is true, we note that the function \( y \mapsto \frac{y}{(y + \theta^{\beta + 1}) \frac{1}{1 + \theta} - \theta} \) is non-decreasing on \((-\theta^{\beta + 1}, 0)\) (a straightforward computation shows that it has a positive derivative if \( \beta \geq 0 \)), and therefore, on \((-\theta^{\beta + 1}, 0)\), it is greater than \( \theta \). Now solving the inequality

\[
\delta \left(1 - \frac{\theta}{\ell}\right) \geq (\beta + 1)\ell
\]

with respect to \( \ell \), we obtain that under the assumption \( \delta \geq 4\theta(\beta + 1) \), it has at least one solution. Next, we remark that \( \mathcal{C}(x) = (\ell x + \delta)^{\beta + 1} - \theta^{\beta + 1} \) is a convex increasing function on \( [-\frac{\alpha}{\theta}, 0] \) whose graph connects the points \((-\frac{\alpha}{\theta}, -\theta^{\beta + 1})\) and \((0, 0)\). We further note that the integral curves of (75) cannot intersect the curve \( \mathcal{C} \) when approaching it from the right. Indeed, \( \frac{dy}{dx} = (\beta + 1)(\ell x + \delta)^{\beta + 1} < (\beta + 1)\ell \) when \( x < 0 \). Assuming that such an intersection occurs, by (80), we obtain that \( \frac{dy}{dx} \geq \frac{dy}{dx} \ell \). Therefore, the only possibility for (79) to hold is when the integral curve, solving (75), never leaves the third quadrant. This means that this curve never intersects the \( y \)-axis, and therefore \( v(t) \neq v^*(t) \) for all \( t \). This implies that \( u(r, \rho) \neq u^*(r) \) for all \( r > 0 \).

\( \square \)

5.2 Convergence of regular solutions

Here, we prove Theorem 1.7.

Proof of Theorem 1.7. Fix an arbitrary interval \([r_1, r_2] \subset (0, +\infty)\) and find a sequence \( u(\cdot, \rho_n) \), \( \lim_n \rho_n = +\infty \), converging uniformly on \([r_1, r_2] \) to a singular solution \( u^* \) to (78). Let us prove that \( u^* = u^* \), where \( u^* \) is the singular solution to (78) given by (69).

For any \( B \geq 0 \), we define \( R^*(B) = (u^*)^{-1}(B) \). Recall that \( R^*(0) = (\lambda^*)^\frac{1}{d} \). From [18] (Theorem 3.1), we know that \( R(0, \rho_n) \to R^*(0) \).

Note that \( u_B(\cdot, \rho_n) = u(\cdot, \rho_n) - B \) solves the equation \( Lu(u) + e^B e^u = 0 \). This implies that \( u_B(\cdot, \rho_n) = u_B(e^{-\frac{B}{d}} r, \rho_n) \) satisfies equation (78) and the initial conditions \( u_B(0, \rho_n) = \rho_n - B, \partial_t u_B(0, \rho_n) = 0 \). Therefore, \( u_B(e^{-\frac{B}{d}} r, \rho_n) = u(r, \rho_n - B) \). Hence, \( u_B(e^{-\frac{B}{d}} R(0, \rho_n) - B, \rho_n) = 0 \) which is the same as \( u(e^{-\frac{B}{d}} R(0, \rho_n - B), \rho_n) = B \). Thus, as
\[ R(B, \rho_n) = e^{-\frac{\theta}{n}} R(0, \rho_n - B) \to e^{-\frac{\theta}{n}} R^*(0) = R^*(B), \]

where the convergence holds by Theorem 3.1 in [18]. To see why \( R^*(B) = e^{-\frac{\theta}{n}} R^*(0) \), we observe that \( u_\beta^*(e^{-\frac{\theta}{n}} r) = \lambda^* - \theta \ln r = u^*(r) \) (a verification is straightforward). This implies that \( u_\beta^*(e^{-\frac{\theta}{n}} R^*(0)) = 0 \), which is the same as \( u^*(e^{-\frac{\theta}{n}} R^*(0)) = B \). Hence,

\[ u^{-1}(\cdot, \rho_n) \to (u^*)^{-1} \quad \text{in } C_{\text{loc}}(0, +\infty). \]

For simplicity, we write \( u_n \) for \( u(\cdot, \rho_n) \). Since \( u_n \) and \( u^* \) are decreasing in \( r \) and \( u_n \rightharpoonup u^* \) on \([r_1, r_2] \), for a small \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \([u_n(r_2), u_n(r_1)] \subset [u^*(r_2) - \varepsilon, u^*(r_1) + \varepsilon] \) for \( n \geq N \). Since \( u_n^{-1} \rightharpoonup (u^*)^{-1} \) on \([u^*(r_2) - \varepsilon, u^*(r_1) + \varepsilon] \), we obtain that uniformly on \([r_1, r_2] \),

\[ |r - (u^*)^{-1}(u_n(r))| = |u_n^{-1}(u_n(r)) - (u^*)^{-1}(u_n(r))| \to 0 \quad \text{as } n \to \infty. \]

By continuity of \( u^* \),

\[ |u^*(r) - u_n(r)| \to 0 \quad \text{as } n \to \infty \]

uniformly on \([r_1, r_2] \). Therefore, \( u^* = u^* \).

Let us obtain the first limit in (10). Suppose it is not true. Then, one can find an interval \([r_1, r_2] \subset (0, +\infty) \), a sequence \( \rho_n \to +\infty \), and a number \( \varepsilon > 0 \) such that \(|u(r, \rho_n) - u^*(r)| > \varepsilon \) on \([r_1, r_2] \) for all \( n \). On the other hand, we can find a subsequence \( u(\cdot, \rho_{n_k}) \) which converges to a solution of (78) uniformly on \([r_1, r_2] \). By Corollary 3.1, this solution is singular. By what was proved, this singular solution is \( u^* \). This is a contradiction. Thus, the first limit in (10) is obtained.

To show the second equality in (10), we note that by Theorem 3.1 from [18], \( \lambda(\rho) \to \lambda^* \) as \( \rho \to +\infty \). Since \( u_{\lambda(\rho)}(r) = u(\lambda(\rho)^{\frac{1}{d}} r, \rho) \) and \( u_{\lambda(\rho)}^*(r) = u^*(\lambda^{\frac{1}{d}} r) \), the result follows. \( \square \)

**Corollary 5.1.** Let assumptions of Theorem 1.7 be fulfilled and let \( \lambda^d \) be as in Theorem 1.4. Then, if \( 0 < \alpha - \beta - 1 < 4\hat{\theta} \) (\( 2k < d < 2k + 8 \)), \( (u_{\lambda^d}, \lambda^d) \) is a regular solution; if \( \alpha - \beta - 1 \geq 4\hat{\theta} \) (\( d \geq 2k + 8 \)), then \( \lambda^d = \lambda^* \).

**Proof.** If \( 0 < \alpha - \beta - 1 < 4\hat{\theta} \), the statement follows from Theorems 4.3 and 1.7. If \( \alpha - \beta - 1 \geq 4\hat{\theta} \), the statement follows from Theorem 3.1 (case III) in [18]. \( \square \)

**Open problems**

In this work, some of the results were obtained for the case \( f(u) = u \). It would be interesting to obtain those results, along with the convergence \( \lambda(\rho) \to \lambda^* \), for problem (P\( \lambda \)). More specifically, the results of interest are

(i) For problem (P\( \lambda \)), to show that \( \lambda(\rho) \to \lambda^* \) as \( \rho \to +\infty \).

(ii) For the same problem, to obtain the result of Theorem 1.7 about the convergence of regular solutions to the singular.

(iii) For equation (7), to find a condition on \( \alpha, \beta, \gamma \) so that the singular and regular solutions do not intersect each other. Such a result would be a generalization of Theorem 1.6.

These problems are the subject of our future research.

**Appendix**

Remark that the radial form of problem (1) is the following:

\[
\begin{cases}
\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + k w(r)^3 = \lambda e^{(-w)}, & 0 \leq r < 1, \\
w < 0, & 0 \leq r < 1, \\
w(1) = 0.
\end{cases}
\]

(81)
We show that for regular radial solutions, (81) can be reduced to \((P_\lambda)\), and vice versa. Since the solutions of interests are radial in the ball, we have that \(w'(0) = 0\). Therefore, the integral form of the above equation is

\[
r^{d-k} w'(r) = \frac{\lambda d}{C_d} \int_0^r s^{d-1} e^{f(-w(s))} ds.
\]

This shows that \(w'(r)\) cannot take zero values inside the ball except for the origin. Since \(w(1) = 0\) and \(w'(0) = 0\), \(w'(r)\) is strictly increasing, and therefore, \(w'(r) > 0\) for \(0 < r \leq 1\).

By doing the substitution \(u = -w\), we rewrite problem (81) as follows:

\[
\begin{align*}
-\frac{C_d^2}{d} r^{1-d} (r^{d-k} |u'|(r)^{k-1} u'(r))' &= \lambda e^{f(u)}, & 0 \leq r < 1, \\
u > 0, & 0 \leq r < 1, \\
u(1) &= 0.
\end{align*}
\]

By this, in the context of regular solutions, problem (1) is equivalent to problem \((P_\lambda)\) with the operator \(L\) defined by (5) and \(\alpha = d - k\), \(\beta = k - 1\), \(\gamma = d - 1\).

On the other hand, as we showed in Subsection 2.4.1, for any singular solution to \((P_\lambda)\), it holds that \((u_\lambda^*)' < 0\). This implies that \(w_\lambda^* = -u_\lambda^*\) is a singular solution to (81).

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