PREQUANTIZATION OF THE MODULI SPACE OF FLAT PU(p) BUNDLES
WITH PRESCRIBED BOUNDARY HOLOMOMIES

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Abstract. Using the framework of quasi-Hamiltonian actions, we compute the obstruction to prequantization for the moduli space of flat PU(p)-bundles over a compact orientable surface with prescribed holonomies around boundary components, where p > 2 is prime.

1. Introduction

Let \( G \) be a compact connected simple Lie group and \( \Sigma \) a compact oriented surface with \( s \) boundary components. Given conjugacy classes \( C_1, \ldots, C_s \), let \( \mathcal{M} = M_G(\Sigma; C_1, \ldots, C_s) \) denote the moduli space of flat \( G \)-bundles on \( \Sigma \) with prescribed boundary holonomies in the conjugacy classes \( C_j \). Recall that \( \mathcal{M} \) is a (possibly singular) symplectic space, where the symplectic form is defined by a choice of invariant inner product on the Lie algebra \( \mathfrak{g} \) of \( G \) [4]. This paper considers the obstruction to the existence of a prequantization of \( \mathcal{M} \) (i.e. prequantum line bundle \( L \to \mathcal{M} \)), by expressing the corresponding integrality condition on the symplectic form in terms of the choice of inner product on the simple Lie algebra \( \mathfrak{g} \), which is hence a certain multiple \( k \) of the basic inner product.

If the underlying structure group \( G \) is simply connected, the moduli space \( \mathcal{M} \) is connected and the obstruction to prequantization is well known—a prequantization exists if and only if \( k \in \mathbb{N} \) and each conjugacy class \( C_j \) corresponds to a level \( k \) weight (e.g. see [2, 5, 13]). If \( G \) is not simply connected, \( \mathcal{M} \) may have multiple components. Moreover integrality of \( k \) is not sufficient to guarantee a prequantization even in the absence of markings/prescribed boundary holonomies: if \( \Sigma \) is closed and has genus at least 1, then \( k \) must be a multiple of an integer \( l_0(G) \) (computed in [10] for each \( G \)). If \( \Sigma \) has boundary with prescribed holonomies, only the case \( G = \text{SO}(3) \cong \text{PU}(2) \) has been fully resolved [12].

In this paper, we describe the connected components of \( \mathcal{M} \) for non-simply connected structure groups \( G/Z \) in Corollary 4.2 and Proposition 4.3 (where \( G \) is simply connected and \( Z \) is a subgroup of the centre of \( G \)). The decomposition into components makes use of an action of the centre \( Z(G) \) on a fundamental Weyl alcove \( \Delta \) in \( \mathfrak{t} \), the Lie algebra of a maximal torus. The action is described concretely in [17] for classical groups and Appendix A records the action for the two remaining exceptional cases.

Finally, we compute the obstruction to prequantization in Theorem 5.6 in the case \( G = \text{PU}(p) \) (\( p > 2 \), prime) for any number of boundary components \( s \). We work within the theory of quasi-Hamiltonian group actions with group-valued moment map [1], where the moduli space \( \mathcal{M} \) is a central example. In quasi-Hamiltonian geometry, quantization is defined as a certain element of the twisted \( K \)-theory of \( G \) [14], analogous to \( \text{Spin}^c \) quantization for Hamiltonian group actions on symplectic manifolds. In this context, the obstruction to the existence of a prequantization is a cohomological obstruction (see Definition 5.1). The obstruction for other cases of non-simply connected structure group does not follow from the approach here (see Remark 5.5) and will be considered elsewhere.

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2. Preliminaries

Notation. Unless otherwise indicated, $G$ denotes a compact, simply connected, simple Lie group with Lie algebra $\mathfrak{g}$. We fix a maximal torus $T \subset G$ and use the following notation:

- $\mathfrak{t}$ - Lie algebra of $T$;
- $\mathfrak{t}^*$ - dual of the Lie algebra of $T$;
- $W = N(T)/T$ - Weyl group;
- $I = \ker \exp_T$ - integer lattice;
- $P = I^* \subset \mathfrak{t}^*$ - (real) weight lattice;
- $Q \subset \mathfrak{t}^*$ - root lattice;
- $Q^\vee \subset \mathfrak{t}$ - coroot lattice;
- $P^\vee \subset \mathfrak{t}$ - coweight lattice.

Recall that since $G$ is simply connected, $I = Q^\vee$. Moreover, the coroot lattice and weight lattice are dual to each other, as are the root lattice and coweight lattice. A choice of simple roots $\alpha_1, \ldots, \alpha_l$ (with $l = \text{rank}(G)$) spanning $Q$, determines the fundamental coweights $\lambda_1^\vee, \ldots, \lambda_l^\vee$ spanning $P^\vee$, defined by $\langle \alpha_i, \lambda_j^\vee \rangle = \delta_{i,j}$.

We let $\langle -,- \rangle$ denote the basic inner product, the invariant inner product on $\mathfrak{g}$ normalized to make short coroots have length $\sqrt{2}$. With this inner product, we will often identify $\mathfrak{t} \cong \mathfrak{t}^*$.

Given a subgroup $Z$ of the centre $Z(G)$ of $G$, we shall abuse notation and denote by $q : G \to G/Z$ the resulting covering(s).

Finally, let $\{e_1, \ldots, e_n\}$ denote the standard basis for $\mathbb{R}^n$, equipped with the standard inner product that will also be denoted with angled brackets $\langle -,- \rangle$.

Quasi-Hamiltonian group actions. We recall some basic definitions and facts from [1]. (For the remainder of this section, we may take $G$ to be any compact Lie group with invariant inner product $\langle -,- \rangle$ on $\mathfrak{g}$.) Let $\theta^L, \theta^R$ denote the left-invariant, right-invariant Maurer-Cartan forms on $G$, and let $\eta = \frac{1}{12} \langle \theta^L, [\theta^L, \theta^L] \rangle$ denote the Cartan 3-form on $G$. For a $G$-manifold $M$, and $\xi \in \mathfrak{g}$, let $\xi^t$ denote the generating vector field of the action. The Lie group $G$ is itself viewed as a $G$-manifold for the conjugation action.

Definition 2.1. [1] A quasi-Hamiltonian $G$-space is a triple $(M, \omega, \Phi)$ consisting of a $G$-manifold $M$, a $G$-invariant 2-form $\omega$ on $M$, and an equivariant map $\Phi : M \to G$, called the moment map, satisfying:

\begin{enumerate}
\item $d\omega + \Phi^* \eta = 0$,
\item $\iota_{\xi^t} \omega + \frac{1}{2} \Phi^* ([\theta^L + \theta^R] \cdot \xi) = 0$ for all $\xi \in \mathfrak{g}$,
\item at every point $x \in M$, $\ker \omega_x \cap \ker d\Phi_x = \{0\}$.
\end{enumerate}

We will often denote a quasi-Hamiltonian $G$-space $(M, \omega, \Phi)$ simply by the underlying space $M$ when $\omega$ and $\Phi$ are understood from the context.

The fusion product of two quasi-Hamiltonian $G$-spaces $M_j$ with moment maps $\Phi_j : M_j \to G$ ($j = 1, 2$) is the product $M_1 \times M_2$, with the diagonal $G$-action and moment map $\Phi : M_1 \times M_2 \to G$ given by composing $\Phi_1 \times \Phi_2$ with multiplication in $G$.

The symplectic quotient of a quasi-Hamiltonian $G$-space is the symplectic space $M//G = \Phi^{-1}(1)/G$, which is a symplectic orbifold whenever the group unit 1 is a regular value. If 1 is a singular value, then the symplectic quotient is a singular symplectic space as defined in [15].

The conjugacy classes $C \subset G$, with moment map the inclusion into $G$, are basic examples of quasi-Hamiltonian $G$-spaces. Another important example is the double $D(G) = G \times G$, equipped with diagonal $G$-action and moment map $\Phi(g,h) = ghg^{-1}h^{-1}$, the group commutator. These two
families of examples form the building blocks of the moduli space of flat $G$-bundles over a surface $\Sigma$ with prescribed boundary holonomies. (See Section 1 for a sketch of this construction.)

3. Conjugacy classes invariant under translation by central elements

This section describes the set of conjugacy classes in $G$ that are invariant under translation by a subgroup of the centre $Z(G)$ of $G$. We begin with a Lemma that relates the central subgroups leaving conjugacy classes (of $G$) invariant with conjugacy classes in $G/Z$ with $Z \subset Z(G)$.

**Lemma 3.1.** Let $Z$ be a subgroup of the centre $Z(G)$ of $G$ and let $\mathcal{C} \subset G/Z$ be a conjugacy class. For any conjugacy class $\mathcal{D} \subset G$ covering $\mathcal{C}$, the restriction $q|_{\mathcal{D}} : \mathcal{D} \to \mathcal{C}$ is the universal covering projection and hence the fundamental group $\pi_1(\mathcal{C}) \cong \mathcal{Z}_D = \{z \in Z : z\mathcal{D} = \mathcal{D}\}$.

**Proof.** The inverse image $q^{-1}(\mathcal{C})$ is a disjoint union of conjugacy classes in $G$ that cover $\mathcal{C}$. Since conjugacy classes in a compact simply connected Lie group are simply connected and $Z_D$ acts freely on $\mathcal{D}$, the lemma follows. □

Recall that every element in $G$ is conjugate to a unique element $\exp \xi$ in $T$ where $\xi$ lies in a fixed (closed) alcove $\Delta \subset t$ of a Weyl chamber. Therefore, the set of conjugacy classes in $G$ is parametrized by $\Delta$. Since the $Z(G)$-action commutes with the conjugation action, we obtain an action $Z(G) \times \Delta \to \Delta$. Next we identify this description of the action of $Z(G)$ on an alcove $\Delta$ with a more concrete description of a $Z(G)$-action on $\Delta$ given in [17, Section 4.1]. (See also [6, Section 3.1] for a similar treatment.)

Let $\{\alpha_1, \ldots, \alpha_i\}$ be a basis of simple roots for $t^*$, with highest root $\tilde{\alpha} = -\alpha_0$. Let $\Delta \subset t$ be the alcove

$$\Delta = \{\xi \in t : \langle \xi, \alpha_j \rangle \geq 0, \langle \xi, \tilde{\alpha} \rangle \leq 1\}.$$

Recall that the centre $Z(G) \cong P^\vee/Q^\vee$ (induced by the exponential map), and that the non-zero elements of the centre have representatives $\lambda^\vee_i \in P^\vee$ given by minimal dominant coweights. By [17, Lemma 2.3] the non-zero minimal dominant coweights $\lambda^\vee_i$ are dual to the special roots $\alpha_i$, which are those roots with coefficient 1 in the expression $\tilde{\alpha} = \sum m_i \alpha_i$. In Proposition 4.1.4 of [17], Toledano-Laredo provides a $Z(G)$-action on $\Delta$ defined by

$$z \cdot \xi = w_i \xi + \lambda^\vee_i$$

where $z = \exp \lambda^\vee_i$, and $w_i \in W$ is a certain element of the Weyl group. The element $w_i \in W$ is the unique element that leaves $\Delta \cup \{\alpha_0\}$ invariant (i.e. induces an automorphism of the extended Dynkin diagram) and satisfies $w_i(\alpha_0) = \alpha_i$ (see [17, Proposition 4.1.2]). The following Proposition shows these actions coincide.

**Proposition 3.2.** The translation action of $Z(G)$ on $G$ induces an action $Z(G) \times \Delta \to \Delta$ and is given by the formula $z \cdot \xi = w_i \xi + \lambda^\vee_i$, where $z = \exp \lambda^\vee_i$ and $w_i$ is the unique element in $W$ that leaves $\Delta \cup \{\alpha_0\}$ invariant and satisfies $w_i(\alpha_0) = \alpha_i$.

**Proof.** Observe that for any element $w$ in $W$, $w\lambda^\vee_i - \lambda^\vee_i \in I = Q^\vee$ since $w \exp \lambda^\vee_i = \exp \lambda^\vee_i$. Therefore, $w_i \xi + \lambda^\vee_i = w_i(\xi + \lambda^\vee_i + (w_i^{-1}(\lambda^\vee_i - \lambda^\vee_j)))$. In other words, $w_i \xi + \lambda^\vee_i = \hat{w}(\xi + \lambda^\vee_i)$ for some $\hat{w}$ in the affine Weyl group. Letting $z = \exp \lambda^\vee_i$, this shows that $z \cdot \xi = \exp(\xi + \lambda^\vee_i)$ is conjugate to $\exp(w_i \xi + \lambda^\vee_i)$, which proves the Proposition. □

In fact, as the next Proposition shows, the automorphism of the Dynkin diagram induced by $w_i$ encodes the resulting permutation of the vertices of the alcove $\Delta$.

**Proposition 3.3.** Let $v_0, \ldots, v_l$ denote the vertices of $\Delta$ with $v_j$ opposite the facet parallel to $\ker \alpha_j$. Then $\exp \lambda^\vee_i \cdot v_j = v_k$ whenever $w_i \alpha_j = \alpha_k$, where $w_i$ is as in Proposition 3.2.
Proof. Let $v_0, \ldots, v_l$ denote the vertices of $\Delta$ where the vertex $v_j$ is opposite the facet (codimension 1 face) parallel to $\ker \alpha_j$. That is, $v_0 = 0$ and for $j \neq 0$, $v_j$ satisfies:

$$\langle \alpha_0, v_j \rangle = -1, \quad \text{and} \quad \langle \alpha_r, v_j \rangle = 0 \quad \text{if and only if} \quad 0 \neq r \neq j.$$  

(Hence, for $j \neq 0$ we have $\langle \alpha_j, v_j \rangle = \frac{1}{m_j}$ where $m_j$ is the coefficient of $\alpha_j$ in the expression $\hat{\alpha} = \sum m_i \alpha_i$.)

Suppose that $w_i \alpha_0 = \alpha_i$ and let $w_i \alpha_j = \alpha_k$ (where $k$ depends on $j$).

Consider $\exp \lambda_i^\vee \cdot v_0$. Since $\langle \alpha_0, w_i v_0 + \lambda_i^\vee \rangle = \langle \alpha_0, \lambda_i^\vee \rangle = -1$, and (for $r \neq 0$) $\langle \alpha_r, w_i v_0 + \lambda_i^\vee \rangle = \langle \alpha_r, \lambda_i^\vee \rangle = \delta_{r,k}$ we have $\exp \lambda_i^\vee \cdot v_0 = v_i$.

Next, consider $\exp \lambda_i^\vee \cdot v_j = w_i v_j + \lambda_i^\vee$ where $j \neq 0$. If $k = 0$ so that $w_i \alpha_j = \alpha_0$ then $\alpha_j = w_i^{-1} \alpha_0$ is a special root (i.e. $m_j = 1$) since $w_i^{-1} = w_j$. Therefore, $\langle \alpha_0, w_i v_j + \lambda_i^\vee \rangle = \langle w_i^{-1} \alpha_0, v_j \rangle - 1 = \langle \alpha_j, v_j \rangle - 1 = 0$. And if $r \neq 0$,

$$\langle \alpha_r, w_i v_j + \lambda_i^\vee \rangle = \langle w_i^{-1} \alpha_r, v_j \rangle + \langle \alpha_r, \lambda_i^\vee \rangle.$$  

If $r \neq i$, $w_i^{-1} \alpha_r$ is a simple root other than $\alpha_j$; therefore, each term above is 0. Moreover, if $r = i$, then the above expression becomes $\langle \alpha_0, v_j \rangle + \langle \alpha_i, \lambda_i^\vee \rangle = -1 + 1 = 0$. Hence we have $\exp \lambda_i^\vee \cdot v_j = v_0$ whenever $w_i \alpha_j = \alpha_0$.

On the other hand, if $k \neq 0$ so that $w_i \alpha_j = \alpha_k$ is a simple root, then $\langle \alpha_0, w_i v_j + \lambda_i^\vee \rangle = \langle w_i^{-1} \alpha_0, v_j \rangle + \langle \alpha_0, \lambda_i^\vee \rangle = 0 - 1 = -1$ since the simple root $w_i^{-1} \alpha_0 \neq \alpha_j$. And if $r \neq 0$, we consider again the expression (1) and find (for the same reason as above) that (1) is trivial whenever $r \neq k$. If $r = k$, (1) becomes $\langle \alpha_k, w_i v_j + \lambda_i^\vee \rangle = \langle w_i^{-1} \alpha_k, v_j \rangle + \langle \alpha_k, \lambda_i^\vee \rangle = \langle \alpha_j, v_j \rangle \neq 0$. Hence we have that $\exp \lambda_i^\vee \cdot v_j = v_k$, as required.  

The $Z(G)$-action on $\Delta$ is explicitly described in [17] for all classical groups. (In Appendix A we record the action of the centre on the alcove for the exceptional groups $E_6$ and $E_7$, the remaining compact simple Lie groups with non-trivial centre.)

Conjugacy classes in $SU(n)$. We now specialize to the case $G = SU(n)$ and consider the action of the centre on the alcove. Identify $t \cong t^* \subset \mathbb{R}^n$ as the subspace $\{x = \sum x_j e_j; \sum x_j = 0\}$ and recall that the basic inner product coincides with (the restriction of) the standard inner product on $\mathbb{R}^n$. The roots are the vectors $e_i - e_j$ with $i \neq j$. Taking the simple roots to be $\alpha_i = e_i - e_{i+1}$ ($i = 1, \ldots, n-1$) and the resulting highest root $\hat{\alpha} = e_1 - e_n$ gives the alcove

$$\Delta = \{x \in t; x_1 \geq x_2 \geq \cdots \geq x_n, x_1 - x_n \leq 1\}.$$  

Its vertices are

$$v_0 = 0 \quad \text{and} \quad v_j = \sum_{i=1}^{j} e_i - \frac{j}{n} \sum_{i=1}^{n} e_i \quad (j = 1, \ldots, n).$$

The centre $Z(SU(n)) \cong \mathbb{Z}/n\mathbb{Z}$ is generated by (exp of) the minimal dominant coweight $\lambda_i^\vee = e_1 - \frac{1}{n} \sum_{i=1}^{n} e_i$ corresponding to the special root $\alpha_1 = e_1 - e_2$. Since the element $w_1$ inducing an automorphism of the extended Dynkin diagram for $SU(n)$ satisfies $w_1 \alpha_0 = \alpha_1$, by Proposition 3.3 the permutation of the vertices of $\Delta$ induced by the action of exp $\lambda_1^\vee$ is the $n$-cycle $(v_0, v_1 \cdots v_{n-1})$ (since $v_j$ is the vertex opposite the facet parallel to $\ker \alpha_j$).

It follows that the only point in $\Delta$ fixed by the action of $Z(G)$ is the barycenter

$$\zeta_s = \frac{1}{n} \sum_{j=0}^{n-1} v_j = \frac{n - 1}{2n} e_1 + \frac{n - 3}{2n} + \cdots + \frac{1 - n}{2n} e_n.$$  

Hence there is a unique conjugacy class in $SU(n)$ that is invariant under translation by the centre—namely, matrices in $SU(n)$ with eigenvalues $z_1, \ldots, z_n$, the distinct $n$-th roots of $(-1)^{n+1}$. As the next Proposition shows, however, restricting the action to a proper subgroup $Z \cong \mathbb{Z}/\nu\mathbb{Z}$ ($\nu|n$) of the centre results in larger $Z$-fixed point sets in $\Delta$. 

Proposition 3.4. Let $n = \nu m$ and consider the subgroup $\mathbb{Z}/\nu\mathbb{Z} \subset \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}(\text{SU}(n))$. The $\mathbb{Z}/\nu\mathbb{Z}$-fixed points in the alcove $\Delta$ for $\text{SU}(n)$ consist of the convex hull of the barycenters of the faces spanned by the orbits of the vertices $v_0, \ldots, v_{m-1}$ of $\Delta$.

Proof. Write $x = \sum t_i v_i$ in $\Delta$ in barycentric coordinates (with $t_i \geq 0$ and $\sum t_i = 1$). Then a generator of $\mathbb{Z}/\nu\mathbb{Z}$ sends $x$ to $\sum t'_i v_i$, with $t'_i = t_i - m \mod n$. Therefore $x$ is fixed if and only if $t_i = t'_i - m \mod n$, and in this case we may write,

$$x = t_0 \nu^1 \sum_{j=0}^{\nu-1} v_{jm} + t_1 \nu^1 \sum_{j=0}^{\nu-1} v_{1+jm} + \cdots + t_m \nu^1 \sum_{j=0}^{\nu-1} v_{m-1+jm}$$

which exhibits a fixed point in the desired form. □

To illustrate, consider the subgroup $Z \cong \mathbb{Z}/2\mathbb{Z}$ of the centre $\mathbb{Z}(\text{SU}(4)) \cong \mathbb{Z}/4\mathbb{Z}$, which acts by transposing the vertices $v_0 \leftrightarrow v_2$ and $v_1 \leftrightarrow v_3$. The barycenters $\zeta_0, \zeta_1$ of the edges $v_0 v_2$ and $v_1 v_3$, respectively, are fixed and thus the $Z$-fixed points are those on the line segment joining $\zeta_0$ and $\zeta_1$. (See Figure 1).

![Figure 1. Alcove for SU(4). The indicated line segment through the barycenter parametrizes the set of conjugacy classes invariant under translation by $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Z}(\text{SU}(4))$.](image)

4. Components of the moduli space with markings

In this section we recall the quasi-Hamiltonian description of the moduli space of flat bundles over a compact orientable surface with prescribed boundary holonomies. We refer to the original article [1] for the details regarding the construction sketched below.

Let $\Sigma$ be a compact, oriented surface of genus $h$ with $s$ boundary components. For conjugacy classes $C_1, \ldots, C_s$ in $G/Z$, let $M_{G/Z}(\Sigma; C_1, \ldots, C_s)$ be the moduli space of flat $G/Z$-bundles over $\Sigma$ with prescribed boundary holonomies lying in the conjugacy classes $C_j$ ($j = 1, \ldots, s$). Points in $M_{G/Z}(\Sigma; C_1, \ldots, C_s)$ are (gauge equivalence classes of) principal $G/Z$-bundles over $\Sigma$ equipped with a flat connection whose holonomy around the $j$-th boundary component lies in the conjugacy class $C_j$. This moduli space is an important example in the theory of quasi-Hamiltonian group actions, where it is cast a symplectic quotient of a fusion product,

$$M_{G/Z}(\Sigma; C_1, \ldots, C_s) = (D(G/Z)^h \times C_1 \times \cdots \times C_s) /\! / (G/Z),$$

which may have several connected components if $Z$ is non-trivial. Extending the discussion in [12, Section 2.3], we describe the connected components of (2) as symplectic quotients of an auxiliary quasi-Hamiltonian $G$-space.
As in [12] Section 2.2, given a quasi-Hamiltonian $G/Z$-space $N$ with group-valued moment map $\Phi : N \to G/Z$, let $\tilde{N}$ be the fibre product defined by the Cartesian square,

\[
\begin{array}{ccc}
\tilde{N} & \xrightarrow{\Phi} & G \\
\downarrow & & \downarrow \\
N & \xrightarrow{\Phi} & G/Z
\end{array}
\]

Then $\tilde{N}$ is naturally a quasi-Hamiltonian $G$-space with moment map $\Phi$. The following Proposition from [12] and its Corollary summarize some properties of this construction.

**Proposition 4.1.** [12 Prop. 2.2] Let $\tilde{N}$ be the fibre product defined by (3) where $\Phi : N \to G/Z$ is a group-valued moment map.

(i) We have a canonical identification of symplectic quotients $\tilde{N}/G \cong N/(G/Z)$.

(ii) For a fusion product $N = N_1 \times \cdots \times N_r$ of quasi-Hamiltonian $G/Z$-spaces, the space $\tilde{N}$ is a quotient of $N_1 \times \cdots \times N_r$ by the group $\{(c_1, \ldots, c_r) \in \mathbb{Z}^r \mid \prod_{j=1}^r c_j = e\}$.

(iii) If $\Phi : N \to G/Z$ lifts to a moment map $\Phi' : N \to G$, thus turning $N$ into a quasi-Hamiltonian $G$-space then $\tilde{N} = N \times Z$.

**Corollary 4.2.** Let $\tilde{N}$ be the fibre product defined by (3) where $\Phi : N \to G/Z$ is a group-valued moment map, and write $\tilde{N} = \bigsqcup X_j$ as a union of its connected components. Then the components of $N/(G/Z)$ can be identified with the symplectic quotients $X_j/G$.

**Proof.** The restrictions $\tilde{\Phi}_j = \Phi|_{X_j}$ are $G$-valued moment maps whose fibres are connected by [11 Theorem 7.2]. Since $\tilde{\Phi}_j^{-1}(e) = \bigcup \tilde{\Phi}_j^{-1}(e)$, it follows that $\tilde{N}/G = \Phi^{-1}(e)/G = \bigcup \tilde{\Phi}_j^{-1}(e)/G = \bigsqcup X_j/G$. The result follows from Proposition 4.1 (i). □

Hence to identify the components of (2), it suffices to identify the components of $\tilde{N}/G$, where $N = D(G/Z)^h \times C_1 \times \cdots \times C_s$—namely, $X_j/G$ where $X_j$ ranges over the components of $\tilde{N}$. With this in mind, choose conjugacy classes $D_j \subset G$ covering $C_j$ ($j = 1, \ldots, s$) and let

$$\tilde{N} = D(G)^h \times D_1 \times \cdots \times D_s.$$ 

Let

$$\Gamma = \{(\gamma_1, \ldots, \gamma_s) \in Z_{D_1} \times \cdots \times Z_{D_s} : \prod \gamma_j = 1\} \subset \mathbb{Z}^s$$

(cf. Lemma 3.1). We show next that the components of $\tilde{N}$ are all homeomorphic to $\tilde{N}/(Z^{2h} \times \Gamma)$.

**Proposition 4.3.** Let $N = D(G/Z)^h \times C_1 \times \cdots \times C_s$ for conjugacy classes $C_j \subset G/Z$ ($j = 1, \ldots, s$) and let $\tilde{N}$ be the fibre product defined by (3). Then $\tilde{N}$ may be written as a union of its connected components,

$$\tilde{N} \cong \bigsqcup_{Z/(Z_{D_1} \cdots Z_{D_s})} D(G/Z)^h \times (D_1 \times \cdots \times D_s)/\Gamma$$

where $D_j \subset G$ are conjugacy classes covering $C_j$ ($j = 1, \ldots, s$) and $\Gamma$ is as in (4).

**Proof.** This is a straightforward application of the properties (3) and (3) listed in Proposition 4.1. By property (3), $D(G/Z)^h = D(G/Z)^h \times Z$, and by Lemma 3.1, $\hat{C}_j = D_j \times Z/Z_{D_j}$. Therefore, by property (3),

$$\tilde{N} \cong D(G/Z)^h \times (Z \times D_1 \times Z/Z_{D_1} \times \cdots \times D_s \times Z/Z_{D_s})/\Lambda$$

where $\Lambda = \{(c_0, \ldots, c_s) \in \mathbb{Z}^{s+1} : c_0 \cdots c_s = 1\}$. Since

$$(Z \times D_1 \times Z/Z_{D_1} \times \cdots \times D_s \times Z/Z_{D_s})/\Lambda \cong (Z \times D_1 \times \cdots \times D_s)/\Gamma'$$
where $\Gamma' = \{(\gamma_0, \ldots, \gamma_s) \in Z \times Z_{D_1} \times \cdots \times Z_{D_s} : \prod \gamma_j = 1\}$, we see that the components of $\bar{N}$ are in bijection with $Z/(Z_{D_1} \cdots Z_{D_s})$.

Consider the component corresponding to $\bar{z} \in Z/(Z_{D_1} \cdots Z_{D_s})$ in which each point is of the form $(\bar{g}, [(z, x_1, \ldots, x_s)]_{r'})$, where $\lfloor \ ] \rfloor$ denotes a $\Gamma'$-orbit. (Note that there is always a representative of this form with $z$ in the first coordinate.) This component is homeomorphic to $D(G/Z)^h \times (D_1 \times \cdots D_s)/\Gamma$ by the map $(\bar{g}, [(z, x_1, \ldots, x_s)]_{r'}) \mapsto (\bar{g}, [(z, x_1, \ldots, x_s)]_{r'})$.

For the case $G/Z = SU(p)/(\mathbb{Z}/p\mathbb{Z}) = PU(p)$, where $p$ is prime, the decomposition above simplifies. In particular, there is only one conjugacy class $\mathcal{D}_s = SU(p) \cdot \exp \zeta_s$, corresponding to the barycenter $\zeta_s \in \Delta$, invariant under the action of the centre. Let $\mathcal{C}_s = q(\mathcal{D}_s)$ be the corresponding conjugacy class in $PU(p)$. Therefore, we obtain the following Corollary (cf. [12, Lemma 2.3]).

**Corollary 4.4.** Let $p$ be prime and let $N = D(PU(p))^h \times C_1 \times \cdots \times C_s$ for conjugacy classes $\mathcal{C}_j \subset PU(p)$ ($j = 1, \ldots, s$) and let $\bar{N}$ be the fibre product defined by (3). Then,

$$\bar{N} \cong \begin{cases} D(PU(p))^h \times (D_1 \times \cdots \times D_s)/\Gamma & \text{if } \exists j: \mathcal{C}_j = \mathcal{C}_s; \\ D(PU(p))^h \times D_1 \times \cdots \times D_s \times Z & \text{otherwise.} \end{cases}$$

where $\mathcal{D}_j \subset SU(p)$ are conjugacy classes covering $\mathcal{C}_j$ ($j = 1, \ldots, s$) and $\Gamma$ is as in (4).

In particular, if (after re-labelling) $\mathcal{C}_j = \mathcal{C}_s$ for all $j \leq r$ ($r > 0$), then we obtain

$$\bar{N} \cong D(PU(p))^h \times (D_s)^r / \Gamma \times D_{r+1} \times \cdots \times D_s,$$

where, in this case, $\Gamma = \{(\gamma_1, \ldots, \gamma_r) \in Z^r : \prod \gamma_j = 1\}$.

5. **Obstruction to prequantization**

5.1. **Prequantization for quasi-Hamiltonian group actions.** We recall some definitions and properties regarding prequantization of quasi-Hamiltonian group actions. Recall that the Cartan 3-form $\eta \in \Omega^3(G)$ is integral—in fact, $[\eta] \in H^3(G; \mathbb{R})$ is the image of a generator $x \in H^3(G; \mathbb{Z}) \cong \mathbb{Z}$ under the coefficient homomorphism induced by $\mathbb{Z} \to \mathbb{R}$. Condition (1) in Definition 2.1 says that the pair $(\omega, \eta)$ defines a relative cocycle in $\Omega^3(\Phi)$, the algebraic mapping cone of the pull-back map $\Phi^*: \Omega^*(G) \to \Omega^*(M)$, and hence a cohomology class $[(\omega, \eta)] \in H^3(\Phi; \mathbb{R})$. (See [7] Ch. I, Sec. 6) for the definition of relative cohomology.)

**Definition 5.1.** [10,14] Let $k \in \mathbb{N}$. A level $k$ prequantization of a quasi-Hamiltonian $G$-space $(M, \omega, \Phi)$ is an integral lift $\alpha \in H^3(\Phi; \mathbb{Z})$ of the class $k[(\omega, \eta)] \in H^3(\Phi; \mathbb{R})$.

The definition of prequantization in 5.1 uses the assumption in this paper that $G$ is simply connected. The general definition of prequantization [14, Definition 3.2] (with $G$ semi-simple and compact) requires an integral lift in $H^2(\Phi; \mathbb{Z})$ of an equivariant extension of the class $k[(\omega, \eta)]$. When $G$ is simply connected, [10, Proposition 3.5] shows that the definition above is equivalent.

We list some basic properties level $k$ prequantizations that we shall encounter.

(a) If $M_1$ and $M_2$ are pre-quantized quasi-Hamiltonian $G$-spaces at level $k$, then their fusion product $M_1 \times M_2$ inherits a prequantization at level $k$. Conversely, a prequantization of the product induces prequantizations of the factors. See [10, Proposition 3.8].

(b) A level $k$ prequantization of $M$ induces a prequantization of the symplectic quotient $M//G$, equipped with the $k$-th multiple of the symplectic form.

(c) The long exact sequence in relative cohomology gives a necessary condition $k\Phi^*(x) = 0$ for the existence of a level $k$-prequantization. If $H^2(M; \mathbb{R}) = 0$, $k\Phi^*(x) = 0$ is also sufficient [10, Proposition 4.2] to conclude a level $k$-prequantization exists.

The following examples relate to the moduli space of flat bundles with prescribed boundary holonomies.
Example 5.2. The double \( D(G) = G \times G \) with moment map \( \Phi : D(G) \to G \) equal to the group commutator admits a prequantization at all levels \( k \in \mathbb{N} \). For non-simply connected groups, the double \( D(G/Z) \) with moment map \( \Phi : D(G/Z) \to G \) the canonical lift of the group commutator admits a level \( k \)-prequantization if and only if \( k \) is a multiple of \( l_0 \in \mathbb{N} \), where \( l_0 \) is a positive integer depending on \( G/Z \) computed for all compact simple Lie groups in \([10]\). For \( G/Z = \text{PU}(n) \), \( l_0 = n \).

Example 5.3. Conjugacy classes \( D \subset G \) admitting a level \( k \)-prequantization are those \( D = G \cdot \exp \xi (\xi \in \Delta) \) with \( (k\xi)^t \in P \) \([13]\), where \( (k\xi)^t = (k\xi, -) \) (i.e. a level \( k \) weight). For simply laced groups (such as \( G = \text{SU}(n) \)), under the identification \( t \cong t^* \), \( P^\vee \cong P \). Therefore, in this case, \( D \) admits a level \( k \)-prequantization if and only if \( k\xi \in P^\vee \). Since \( \exp^{-1} Z(G) = P^\vee \), we see that \( D \) admits a level \( k \)-prequantization if and only if \( g^k \in Z(G) \) for all \( g \in D \). (So in particular if \( k \) is a multiple of the order of \( D \) \([5]\) Definition 5.76), then \( D \) admits a level \( k \) prequantization.)

5.2. The obstruction to prequantization for the moduli space of \( \text{PU}(p) \) bundles, \( p \) prime.

Let \( p \) be an odd prime. In this section we obtain the obstruction to prequantization for the quasi-Hamiltonian \( \text{SU}(p) \)-space \( \hat{N} \), where \( N = D(\text{PU}(p))^h \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_s \) for conjugacy classes \( \mathcal{C}_j \subset \text{PU}(p) \) \((j = 1, \ldots, s) \). Let \( M \subset \hat{N} \) be a connected component (by Corollary \([4,4]\)),

\[
M = D(\text{PU}(p))^h \times (\mathcal{D}_1 \times \cdots \times \mathcal{D}_s) / \Gamma
\]

where \( \Gamma \) is as in \([4]\). As we shall see in the proof of Theorem 5.6 we will find Property (a) in Section 5.1 very useful in order to proceed ‘factor by factor,’ using the decomposition \([5]\).

To begin, we establish the following Proposition which allows us to use Property (c) in Section 5.1 to compute the obstruction to prequantization for the factor \((\mathcal{D}_s)^r/\Gamma \) in \([4]\).

Proposition 5.4. Let \( \mathcal{D}_s \subset \text{SU}(p) \) denote the conjugacy class of the barycenter \( \zeta_* \) of the alcove \( \Delta \) and let \( \Gamma = \{ (\gamma_1, \ldots, \gamma_r) \in Z^r : \prod \gamma_j = 1 \} \) with \( r > 1 \). Then \( H^2((\mathcal{D}_s)^r/\Gamma; \mathbb{R}) = 0 \).

Proof. Since \((\mathcal{D}_s)^r \to (\mathcal{D}_s)/\Gamma \) is a covering projection, \( H^2((\mathcal{D}_s)^r/\Gamma; \mathbb{R}) \cong H^2((\mathcal{D}_s)^r; \mathbb{R})^\Gamma \). By the Küneth Theorem, \( H^2((\mathcal{D}_s)^r; \mathbb{R}) \cong \bigoplus H^2(\mathcal{D}_s; \mathbb{R}). \) Since the \( \Gamma \)-action factors through \( Z^m \), \( H^2((\mathcal{D}_s)^r; \mathbb{R})^\Gamma = \bigoplus H^2(\mathcal{D}_s; \mathbb{R})^Z \).

Recall that since \( \zeta_* \) lies in the interior of the alcove, the centralizer \( \text{SU}(p)_{\exp \zeta_*} = T \) and hence \( \mathcal{D}_s \cong \text{SU}(p)/T \). Moreover, we have \( H^*(\mathcal{D}_s; \mathbb{R}) \cong \mathbb{R}[t_1, \ldots, t_p]/(\sigma_1, \ldots, \sigma_p) \), where \( \sigma_i \)'s are the elementary symmetric polynomials. In particular, we may write

\[
H^2(\mathcal{D}_s; \mathbb{R}) \cong (\mathbb{R}t_1 \oplus \cdots \oplus \mathbb{R}t_p)/(t_1 + \cdots + t_p = 0).
\]

The \( Z \)-action on \( \mathcal{D}_s \) corresponds to an action on \( \text{SU}(p)/T \) by a cyclic subgroup of the Weyl group (e.g. see the proof of Proposition 5.2). Since the Weyl group (i.e. symmetric group \( \Sigma_p \)) acts by permuting the \( t_i \), \( Z \) acts by a \( p \)-cycle on the \( t_i \). Therefore, \( H^2(\mathcal{D}_s; \mathbb{R})^Z = 0 \), which establishes the result.

Remark 5.5. The analogue of Proposition 5.4 for the factors \((\mathcal{D}_1 \times \cdots \times \mathcal{D}_s)/\Gamma \) that appear in the decomposition in Proposition 4.3 need not hold when considering other non-simply connected structure groups \( G/Z \).

Theorem 5.6. The quasi-Hamiltonian \( \text{SU}(p) \)-space \( M = D(\text{PU}(p))^h \times (\mathcal{D}_1 \times \cdots \times \mathcal{D}_s) / \Gamma \) admits a level \( k \)-prequantization if and only if the following conditions are satisfied:

(i) If \( h \geq 1 \), then \( k \in p\mathbb{N} \);

(ii) \( g^k \in Z(\text{SU}(p)) \) for every \( g \in \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_s \).

Proof. By Property (a) in Section 5.1 \( M \) admits a level \( k \)-prequantization if and only if each factor does. Since \( D(\text{PU}(p)) \) admits a level \( k \)-prequantization if and only if Condition (i) is satisfied (see Example 5.2), we may assume from now on \( h = 0 \).
We first verify the necessity of Condition (ii). A prequantization of $M = (D_1 \times \cdots \times D_s)/\Gamma$ induces a prequantization of its universal cover $\tilde{M} = D_1 \times \cdots \times D_s$, and hence each $D_j$ must admit a prequantization, which is equivalent to condition (ii).

Next we verify that Condition (ii) is sufficient for a level $k$-prequantization of $M$ (with $h = 0$). As in the decomposition (5), write (possibly after re-labelling)

$$M = (D_s \times \cdots \times D_s)/\Gamma \times D_{r+1} \times \cdots \times D_s$$

Using Property (a) in Section 5.1 again, it suffices to consider the case $1 < r = s$. (Note that if $s = r = 1$, $\Gamma$ is trivial.) In this case, Condition (ii) is simply that $D_s$ admit a level $k$-prequantization. Since $D_s$ consists of matrices in $SU(p)$ conjugate to

$$\exp \zeta_s = \text{diag}(\exp(\frac{p-1}{p} \pi \sqrt{-1}), \exp(\frac{p-3}{p} \pi \sqrt{-1}), \cdots, \exp(\frac{1-p}{p} \pi \sqrt{-1}))$$

$D_s$ admits a level $k$-prequantization if and only if $(\exp \zeta_s)^k$ is a scalar matrix; and only if $k$ is a multiple of $p$. By Property (c) in Section 5.1 and Proposition 5.4, it suffices to show that $p \cdot \Phi^*_w = 0$, where $\Phi : M \to SU(p)$ is the group-valued moment map.

By Corollary 7.6 in [3], $h^\vee \Phi^*_w = W_3(M)$, the third integral Stiefel-Whitney class, where $h^\vee$ denotes the dual Coxeter number. Recall that $W_3(M) = \beta w_2(M)$, where $\beta : H^2(M; \mathbb{Z}/2\mathbb{Z}) \to H^3(M; \mathbb{Z})$ is the (integral) Bockstein homomorphism and $w_2(M)$ is the second Stiefel-Whitney class. Since $\Gamma$ has odd order, $H^2(M; \mathbb{Z}/2\mathbb{Z}) \cong H^2((D_8)^c; \mathbb{Z}/2\mathbb{Z})^T$, which is trivial (by an argument similar to the proof of Proposition 5.4). Since $h^\vee = p$, this completes the proof.

\[\square\]

**Appendix A. The action of the centre on the alcove of exceptional Lie groups**

Below we record the action of the centre $Z(G)$ on an alcove for the exceptional Lie groups $G = E_6$ and $G = E_7$. (The action for classical groups appears in [17].)

The vertices of the alcove were obtained using polymake [9], which outputs the vertices of a polytope presented as an intersection of half-spaces. The relevant Weyl group element from Proposition 3.2—one which gives an automorphism of the extended Dynkin diagram—was found with the help of John Stembridge’s coxeter-weyl package for Maple [16]; a direct calculation then shows that this element has the desired properties in Proposition 3.2.

Let $\{e_1, \ldots, e_8\}$ denote the standard basis in $\mathbb{R}^8$, equipped with the usual inner product. Given a vector $\alpha$ in $\mathbb{R}^8$, $s_\alpha : \mathbb{R}^8 \to \mathbb{R}^8$ denotes reflection in the subspace orthogonal to $\alpha$, $s_\alpha(v) = v - \frac{2\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$. The notation used below is consistent with that found in [3, Planches V-VI].

$G = E_6$. Let $t^* \cong t^* \cong \{(x_1, \ldots, x_8) \in \mathbb{R}^8 : x_6 = x_7 = -x_8\}$. The simple roots $\alpha_1, \ldots, \alpha_6$ and highest root $\alpha$ determine the half-spaces whose intersection is the alcove $\Delta \subset t^*$. The vertices of $\Delta$ (opposite the facets parallel to the corresponding root hyperplanes) are given in Table 11.

The non-zero elements of the centre $Z(E_6) = \mathbb{Z}/3\mathbb{Z}$ are given by $(\exp \text{of})$ the minimal dominant coweights $\lambda_7^\vee = \frac{3}{2}(e_8 - e_7 - e_6)$ and $\lambda_6^\vee = e_5 + \frac{3}{2}(e_8 - e_7 - e_6)$. The corresponding elements $w_1$ and $w_6$ of the Weyl group (as in Proposition 3.2), inducing automorphisms of the extended Dynkin diagram are:

$$w_1 = s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_6} s_{\alpha_5} s_{\alpha_6} s_{\alpha_5} s_{\alpha_6};$$
$$w_6 = s_{\alpha_5} s_{\alpha_5} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_5} s_{\alpha_5} s_{\alpha_2} s_{\alpha_5} s_{\alpha_5} s_{\alpha_2} s_{\alpha_5} s_{\alpha_5} s_{\alpha_5} s_{\alpha_5} s_{\alpha_2} s_{\alpha_5} s_{\alpha_2} s_{\alpha_5} s_{\alpha_2} s_{\alpha_5} s_{\alpha_2} s_{\alpha_5} s_{\alpha_2} s_{\alpha_5}.$$

The permutation of the vertices induced by the action of $\exp(\lambda_7^\vee)$ (encoded by the automorphism $w_1$ of the underlying extended Dynkin diagram) is shown schematically in Figure 2.
facets parallel to the corresponding root hyperplanes) are given in Table 1.

\[
\begin{array}{c|c}
\text{Simple or dominant root} & \text{Opposite vertex} \\
\hline
\alpha_1 = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) & v_1 = (0, 0, 0, 0, 0, -\frac{2}{3}, -\frac{2}{3}) \\
\alpha_2 = (1, 1, 0, 0, 0, 0, 0) & v_2 = \left(\frac{1}{4}, 1, 1, 1, 1, 1, -\frac{1}{4}, -\frac{1}{4}\right) \\
\alpha_3 = (-1, 1, 0, 0, 0, 0, 0) & v_3 = (-\frac{1}{4}, 1, 1, 1, 1, 1, -\frac{5}{12}, -\frac{5}{12}) \\
\alpha_4 = (0, -1, 1, 0, 0, 0, 0) & v_4 = (0, 0, 1, 1, 1, -\frac{1}{3}, -\frac{1}{3}) \\
\alpha_5 = (0, 0, -1, 1, 0, 0, 0) & v_5 = (0, 0, 0, 1, -\frac{1}{3}, -\frac{1}{3}) \\
\alpha_6 = (0, 0, 0, -1, 1, 0, 0) & v_6 = (0, 0, 0, 0, 1, -\frac{1}{3}, -\frac{1}{3}) \\
\tilde{\alpha} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) & v_0 = 0 \\
\end{array}
\]

\textbf{Table 1. Alcove data for } E_6. 

\begin{figure}
\centering
\includegraphics[width=0.7\textwidth]{alcove}\caption{Permutation induced by action of } \exp \lambda^V_1 \text{ on the vertices of the alcove for } E_6.\end{figure}

\( G = E_7. \) Let \( t \cong t^* \cong \{ (x_1, \ldots, x_8) \in \mathbb{R}^8 : x_7 = -x_8 \}. \) The simple roots \( \alpha_1, \ldots, \alpha_7 \) and highest root \( \tilde{\alpha} \) determine the half-spaces whose intersection is the alcove \( \Delta \subset t. \) The vertices of \( \Delta \) (opposite the facets parallel to the corresponding root hyperplanes) are given in Table \[1\]

\[
\begin{array}{c|c}
\text{Simple or dominant root} & \text{Opposite vertex} \\
\hline
\alpha_1 = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) & v_1 = (0, 0, 0, 0, 0, 0, -\frac{1}{2}, \frac{1}{2}) \\
\alpha_2 = (1, 1, 0, 0, 0, 0, 0) & v_2 = \left(\frac{1}{4}, 1, 1, 1, 1, 1, \frac{1}{4}, -\frac{1}{4}\right) \\
\alpha_3 = (-1, 1, 0, 0, 0, 0, 0) & v_3 = (-\frac{1}{4}, 1, 1, 1, 1, 1, -\frac{1}{4}, -\frac{1}{4}) \\
\alpha_4 = (0, -1, 1, 0, 0, 0, 0) & v_4 = (0, 0, 1, 1, 1, -\frac{1}{2}, -\frac{1}{2}) \\
\alpha_5 = (0, 0, -1, 1, 0, 0, 0) & v_5 = (0, 0, 0, 1, -\frac{1}{2}, -\frac{1}{2}) \\
\alpha_6 = (0, 0, 0, -1, 1, 0, 0) & v_6 = (0, 0, 0, 0, 1, -\frac{1}{2}, -\frac{1}{2}) \\
\alpha_7 = (0, 0, 0, 0, -1, 1, 0) & v_7 = (0, 0, 0, 0, 0, 1, -\frac{1}{2}, -\frac{1}{2}) \\
\tilde{\alpha} = (0, 0, 0, 0, 0, 0, -1, 1) & v_0 = 0 \\
\end{array}
\]

\textbf{Table 2. Alcove data for } E_7.
The non-zero element of the centre $Z(E_7) \cong \mathbb{Z}/2\mathbb{Z}$ is given by \((\exp \text{ of})\) the minimal dominant coweight $\lambda_7^\vee = e_6 + \frac{1}{2}(e_8 - e_7)$. The corresponding element $w_7$ of the Weyl group (as in Proposition 3.2), inducing an automorphism of the extended Dynkin diagram is:

$$w_7 = s_{\alpha_7}s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_3}s_{\alpha_1}s_{\alpha_7}s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_2}s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_6}s_{\alpha_7}.$$  

The permutation of the vertices induced by the action of $\exp(\lambda_7^\vee)$ (encoded by the automorphism $w_7$ of the underlying extended Dynkin diagram) is shown schematically in Figure 3.

**Figure 3.** Permutation induced by action of $\exp \lambda_7^\vee$ on the vertices of the alcove for $E_7$.

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