Lebesgue Spaces Norm Estimates for
Fractional Integrals and Derivatives.

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Abstract.

We study the problem estimation of classical Lebesgue-Riesz and Grand Lebesgue Norm for the fractional integrals and derivatives for the functions from the classical Lebesgue-Riesz spaces as well as from the modified Besov’s spaces.

Key words and phrases: Fractional derivatives and integrals of a Riemann-Liouville type, ordinary and generalized Riesz potential, metric measure space, disjoint function, test functions, examples and counterexamples, natural function, fundamental function for rearrangement invariant space, indicator function, upper and lower estimate, sharp estimate, Lebesgue-Riesz, Besov and Grand Lebesgue spaces (GLS), measurable set, measurable function.

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1 Notations. Statement of problem.

"Fractional derivatives have been around for centuries but recently they have found new applications in physics, hydrology and finance", see [42].

Another applications: in the theory of Differential Equations are described in [43]; in statistics see in [5], [30]; see also [14], [8]; in the theory of integral equations etc. see in the classical monograph [45].

Let $\alpha = \text{const} \in (0, 1)$; and let $g = g(x)$, $x \in R_+$ be measurable numerical function. The fractional derivative of a Riemann - Liouville type of order $\alpha$: $D^\alpha[g](x) = g^{(\alpha)}(x)$ [44], [27] is defined as follows: $\Gamma(1 - \alpha)g^{(\alpha)}(x) = 

\Gamma(1 - \alpha) \frac{d}{dx} \left( \int_0^x \frac{g(t)}{(x - t)^\alpha} dt \right). \quad (1.1)$

see, e.g. the classical monograph of S.G.Samko, A.A.Kilbas and O.I.Marichev [45], pp. 33-38; see also [43].
The case when $\alpha \in (k, k + 1)$, $k = 1, 2, \ldots$ may be considered analogously, through the suitable derivatives of integer order.

Hereafter $\Gamma(\cdot)$ denotes the ordinary Gamma function.

We agree to take $D^\alpha[g](x_0) = 0$, if at the point $x_0$ the expression $D^\alpha[g](x_0)$ does not exists.

Notice that the operator of the fractional derivative is non-local, if $\alpha$ is not integer non-negative number.

Recall also that the fractional integral $I^{(\alpha)}[\phi](x) = I^\alpha[\phi](x)$ of a Riemann-Liouville type of an order $\alpha, 0 < \alpha < 1$ is defined as follows:

$$I^{(\alpha)}[\phi](x) \overset{df}{=} \frac{1}{\Gamma(\alpha)} \int_0^x \frac{\phi(t) \, dt}{(x-t)^{1-\alpha}}, \quad x, t > 0. \quad (1.2)$$

It is known (theorem of Abel, see [45], chapter 2, section 2.1) that the operator $I^{(\alpha)}[\cdot]$ is inverse to the fractional derivative operator $D^{(\alpha)}[\cdot]$, at least in the class of absolutely continuous functions.

Another approach to the introducing of the fractional derivative, more exactly, the fractional Laplace operator leads us to the using of Fourier transform

$$F[f](t) = \int_{R^d} e^{i(t,x)} \, f(x) \, dx$$

in the space $R^d$, $d = 1, 2, \ldots$:

$$R_{\alpha,F}[f] := C_1(\alpha, d)F^{-1} \left[ |x|^{\alpha} F[f](x) \right], \quad 0 < \alpha < d,$$

which leads us in turn up to multiplicative constant to the well-known Riesz potential

$$R_{\alpha}[f] \overset{df}{=} \int_{R^d} \frac{f(y) \, dy}{|x-y|^{d-\alpha}}, \quad 0 < \alpha < d. \quad (1.3)$$

Hereafter $(t, x) = \sum_{m=1}^d t_m x_m$, $|x| = \sqrt{(x,x)}$, $t, x \in R^d$.

We consider in this short article the problem of Grand Lebesgue Norm estimation for the fractional integrals and derivatives for the functions from the classical Lebesgue-Riesz spaces as well as from the Besov spaces.

Recall that the classical Lebesgue-Riesz $L(p)$ norm $|f|_p$ of a function $f$ is defined by a formula

$$|f|_p = \left[ \int_{R^d} |f(x)|^p \, dx \right]^{1/p}, \quad 1 \leq p < \infty$$

or correspondingly

$$|f|_p = \left[ \int_{R^d} |f(x)|^p \, dx \right]^{1/p}, \quad 1 \leq p < \infty.$$
2 Fractional integral estimate in the classical Lebesgue-Riesz norm.

Let for beginning \( x \in R^d, \ d = 1, 2, \ldots, 0 < \alpha < d, \ p_+ = d/\alpha. \) Define for the value \( p = \text{const} \in (1, p_+) \) the variable \( q \) as follows

\[
\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}, \quad (2.0)
\]

then \( q \in (d/(d - \alpha), \infty) \). The relation (2.0) defines the value \( q \) as a unique defined function on \( p : \ q = q(p) \) and conversely \( p = p(q) \).

Let us investigate in this section the inequality of a form

\[
|R_\alpha[f]|_q \leq K_{R,\alpha}(p) \cdot |f|_p, \ f \in L_p(R^d) \quad (2.1)
\]
or as a particular case

\[
|\Gamma(\alpha) I^\alpha[f]|_q \leq K_{I,\alpha}(p) \cdot |f|_p, \ f \in L_p(R_+). \quad (2.1a)
\]

The equality (2.0) is necessary and sufficient for the existence and finiteness of the "constant" \( K_{R,\alpha}(p) \), see [35] - [38], [46], as well as the restriction \( 1 < p < d/\alpha \). Obviously, in the case of the fractional integration (2.1a) \( d = 1 \) and hence \( 1/q = 1/p - \alpha, \ 0 < \alpha < 1, \ 1 < p < 1/\alpha \).

This classical problem goes back to Hardy and Littlewood, see [15], [16], [17], [18]; more modern works [1], chapter 3, [2], [3], [4], [19], [35] - [38], [45], p. 64-76 etc.

We will understand in the sequel in the capacity of the value \( K_{R,\alpha}(p) \) its minimal value, namely

\[
K_{R,\alpha}(p) \overset{\text{def}}{=} \sup_{0 \neq f \in L(p)} \left[ \frac{|R_\alpha[f]|_q}{|f|_p} \right], \ 1/q = 1/p - \alpha/d, \ 1 < p < d/\alpha, \quad (2.2)
\]

and correspondingly

\[
K_{I,\alpha}(p) \overset{\text{def}}{=} \sup_{0 \neq f \in L(p)} \left[ \frac{|I^\alpha[f]|_q}{|f|_p} \right], \ 1/q = 1/p - \alpha, \ 1 < p < 1/\alpha. \quad (2.2a)
\]

In order to formulate and prove a main result of this section, we need to introduce some notations. Put

\[
\Omega(d) = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} -
\]

be a volume of Euclidean unit \( d \) - ball.

Further, the so-called maximal operator \( M[f](x), \ x \in R^d \) is defined as follows:

\[
M[f](x) \overset{\text{def}}{=} \sup_{r > 0} \left\{ \left( \Omega(d) r^d \right)^{-1} \int_{B(x, r)} |f(x)| \, dx \right\},
\]
where \( B(x, r) \) is closed Euclidean ball in the whole space \( \mathbb{R}^d \) with center at the point \( x \) and with radii \( r, \ r > 0 \).

It is known that
\[
|M[f]|_p \leq S(d) \cdot \frac{p}{p-1} \cdot |f|_p, \ p \geq 1,
\]
where the finite "constant" \( S(d) \) is named as Stein’s constant.

The first upper estimation for the value \( S(d) \) was obtained in the classical book of E.M.Stein \[46\], p. 173-188: \( S(d) \leq 2 \cdot 5^d \). It is proved in the article \[20\] that \( S(2) \leq 2 \).

In the next works of E.M.Stein \[46\] - \[49\] was obtained consequently the following estimations for \( S(d) \):
\[
S(d) \leq c_1 \sqrt{d}, \ S(d) \leq c_2
\]
with some absolute constants \( c_1 \) and \( c_2 \).

**Theorem 2.1.** We state under formulated above restrictions
\[
K_{R,\alpha}(p) \leq \frac{V_2(\alpha, d, p) \alpha^{-1}}{[(p-1)(1-\alpha p)]^{1-\alpha/d}}, \quad (2.3)
\]
where
\[
V_2(\alpha, d, p) = \Omega^{-1-\alpha/d}(d) \cdot p^{1-2\alpha/p} \cdot d^{1+(1-\alpha p)/d} \cdot S^{1-\alpha p}(d) \in (0, \infty) \quad (2.3a)
\]
is continuous function relative the variable \( p \) on the closed segment \( p \in [1, 1/\alpha] \);

\[
K_{I,\alpha}(p) \geq \frac{V_1(\alpha, p)}{[(p-1)(1-\alpha p)]^{1-\alpha}}, \quad (2.4)
\]
where the "constant" function \( p \to V_1(\alpha, p) \) is also strictly positive continuous on the closed segment \( p \in [1, 1/\alpha] \).

**Proof of the upper bound.**
It is sufficient to follow the book of D.R.Adams \[1\], chapter 3 and make accuracy computations.

Some details. We write
\[
R_\alpha[f](x) = J_1 + J_2, \quad J_1 = J_1(x) = \int_{|x-y|<\delta} \frac{f(y) \, dy}{|x-y|^{d-\alpha}},
\]
\[
J_2 = J_2(x) = \int_{|x-y|\geq\delta} \frac{f(y) \, dy}{|x-y|^{d-\alpha}}, \quad \delta > 0.
\]

We use the H"{o}lder’s inequality for the \( J_2(\cdot) \) estimation:
\[
J_2(x) \leq \Omega^{1-1/p}(d) \cdot \left( \frac{d - \alpha p}{p-1} \right)^{-1+1/p} \cdot \delta^{d-\alpha/p} \cdot |f|_p.
\]

Further, we apply lemma 3.1.4 from the book \[1\]:
\[
\int_{|x-y|<\delta} \frac{\mu(dy)}{|x-y|^{d-\alpha}} = (d - \alpha) \int_0^\delta \frac{\mu(B(x, r)) \, dr}{r^{1+d-\alpha}} \cdot \frac{\mu(B(x, \delta))}{\delta^{d-\alpha}}. \quad (2.5)
\]
Here $\mu(\cdot)$ is arbitrary Borelian measure in the space $\mathbb{R}^d$; indeed, in the considered case $\mu(dy) = |f(y)|dy$.

Then

$$\mu(B(x, \delta)) = \int_{B(x, \delta)} |f(y)| \, dy = \Omega(d) \delta^d \cdot \frac{1}{\Omega(d)} \delta^d \int_{B(x, \delta)} |f(y)| \, dy \leq \Omega(d) \delta^d M[f](x).$$

It remains to substitute into (2.5), apply the elementary equality

$$\min_{\delta > 0} \left( A \delta^\alpha + B \delta^{\alpha-d/p} \right) = \frac{d}{d-\alpha p} \cdot \left( \frac{d-\alpha p}{\alpha p} \right)^{\alpha p/d} \cdot A^{1-\alpha p/d} \cdot B^{\alpha p/d},$$

and use further the integration over $x, x \in \mathbb{R}^d$.

**Proof of the lower bound.**

**A. Case** $p \to 1 + 0$.

Let us choose the following test function

$$f_0(x) = x^{-1} I(x > 1).$$

Hereafter $I(x \in A) = 1$, if $x \in A$ and $I(x \in A) = 0$, $x \notin A$.

We derive consequently

$$|f_0|^p_p = \int_1^\infty x^{-p} \, dx = (p-1)^{-1}, \quad p > 1;$$

$$|f_0|_p = (p-1)^{-1/p}, \quad p > 1.$$

Further, when $x > 1$ and $x \to \infty$, $q \to 1/(1-\alpha)$+

$$g_0(x) := \int_0^x \frac{f_0(y)}{|x-y|^{1-\alpha}} \, dy = \int_1^x \frac{y^{-1}}{|x-y|^{1-\alpha}} \, dy = x^{\alpha-1-1} \int_{1/x}^1 \frac{z^{-1}}{|1-z|^{1-\alpha}} \, dz \sim x^{\alpha-1} \ln x;$$

$$|g_0|^q_q \sim \int_1^\infty x^{(\alpha-1)q} \ln^q(x) \, dx = \frac{\Gamma(q+1)}{[q(1-\alpha)-1]^{q+1}};$$

$$|g_0|_q \sim (q - 1/(1-\alpha))^{-1-1/q} \sim (p-1)^{-1-1/q}.$$

Therefore

$$\frac{|g_0|_q}{|f_0|^p_p} \sim \frac{(p-1)^{-1-1/q}}{(p-1)^{-1/p}} = (p-1)^{-1-1/q+1/p} = (p-1)^{-1+\alpha}. \quad (2.6)$$

**B. Case** $p \to 1/\alpha - 0$. 

Let us choose now the following test function

\[ h_\Delta = h_\Delta(x) := x^{-\alpha} |\ln x|^\Delta I(0 < x < 1/e), \quad \Delta = \text{const} > 0. \]  

Further we will take \( \Delta \to 0^+ \).

We have for the values \( x \to 0^+ \), \( p \in (1, 1/\alpha) \), \( p \to 1/\alpha - 0 \) and correspondingly \( q \to \infty \)

\[
|h_\Delta|_p^p = \int_0^{1/e} x^{-\alpha p} |\ln x|^\Delta p \, dx = (1 - \alpha p)^{-\Delta p} \int_{1-\alpha p}^{\infty} z^{\Delta p} e^{-z} \, dz \sim \]

\[
(1 - \alpha p)^{-1-\Delta p} \Gamma(\Delta p + 1); \quad |h_\Delta|_p \sim \frac{\Gamma^{1/p}(\Delta p + 1)}{(1 - \alpha p)^{\Delta+1/p}}; \]  

\[
r_\Delta(x) := \Gamma(\alpha) I^\alpha[h_\Delta](x) = \int_0^{1/e} \frac{y^{-\alpha} |\ln y|^\Delta}{|x - y|^{1-\alpha}} \, dy \sim |\ln x|^\Delta + 1; \]

\[
|r_\Delta|_q \sim e^{-1} \alpha^{-\Delta - 1} (1 - \alpha p)^{-\Delta - 1}. \]  

Therefore

\[
\frac{|r_\Delta|_q}{|h_\Delta|_p} \sim \frac{e^{-1} \alpha^{-1}}{(1 - \alpha p)^{1-\alpha}}. \]  

C. General case \( (p - 1)(1/\alpha - p) \to 0 \).

Since the functions \( f_0(\cdot) \) and \( r_\Delta(\cdot) \) are disjoint:

\[ f_0(x) \cdot r_\Delta(x) = 0, \]

we conclude

\[ |f_0(x) + r_\Delta(x)|_p^p = |f_0(x)|_p^p + |r_\Delta(x)|_p^p. \]

Choosing ultimately in the capacity of the test function \( t(x) = f_0(x) + h_\Delta(x) \), we get to the second assertion of theorem 2.1.

As a slight consequence:

**Corollary 2.1.** If \( 1 < p < 1/\alpha \), \( \alpha = \text{const} \in (0, 1) \) then

\[
\frac{V_1(\alpha, p)}{[(p - 1)(1 - \alpha p)]^{1-\alpha}} \leq K_{I, \alpha}(p) \leq \frac{V_2(\alpha, 1, p) \alpha^{-1}}{[(p - 1)(1 - \alpha p)]^{1-\alpha}}. \]  

(2.11)
3 Fractional integral estimate in the Grand Lebesgue norm.

Let \((X, Q, \nu)\) be a measurable space with non-trivial sigma finite measure \(\nu\), and let also \(\psi = \psi(q)\), \(s_1 \leq q < s_2\), \(1 \leq s_1 < s_2 \leq \infty\) be continuous on the open interval \((s_1, s_2)\) bounded from below function. By definition, a Grand Lebesgue Space (GLS) \(G\psi = G\psi(s_1, s_2)\) over our triplet \((X, Q, \nu)\) consists on all the measurable functions \(f : X \rightarrow \mathbb{R}\) with finite norm

\[
\|f\|_{G\psi} \overset{def}{=} \sup_{q \in (s_1, s_2)} \left[ \frac{|f|^q}{\psi(q)} \right].
\] (3.1)

Hereafter

\[
|f|_q = \left( \int_X |f(x)|^q \, \nu(dx) \right)^{1/q}
\]

and we will denote \((s_1, s_2) = \text{supp} \psi\).

The detail investigation of these spaces see, e.g. in [9], [10], [21], [22], [23], [31], [33], [30] etc.

The finiteness of Grand Lebesgue norm \(\|f\|_{G\psi} < \infty\) in the case when \(s_2 = \infty\) implies in particular the exponential decrease the tail \(T_f(u) = \nu\{x : |f(x)| > u\}, u \rightarrow \infty\) function of \(f\).

The fundamental function \(\phi(G\psi, \delta), \delta > 0\) for this spaces is defined as follows

\[
\phi(G\psi, \delta) = \sup_{p \in \text{supp} \psi} \left[ \frac{\delta^{1/p}}{\psi(p)} \right].
\]

This function play a very important role in the theory of operator’s interpolation, theory of Fourier series etc. [6].

In the considered here problems \(X = R^d\) or \(X = R_+\) with Lebesgue measure \(\nu; \nu(dx) = dx\).

The set of all such a functions with support \(\text{supp}(\psi) = (s_1, s_2)\) will be denoted by \(G\Psi(s_1, s_2)\).

These spaces are rearrangement invariant; and are used, for example, in the theory of Probability, theory of Partial Differential Equations, Functional Analysis, theory of Fourier series, Martingales, Mathematical Statistics, theory of Approximation etc.

Notice that the classical Lebesgue-Riesz spaces \(L_p\) are extremal case of Grand Lebesgue Spaces, see [39], [40].

Let a function \(f : X \rightarrow \mathbb{R}\) be such that

\[
\exists s_1, s_2, \ 1 \leq s_1 < s_2 \leq \infty : \forall p \in (s_1, s_2) \Rightarrow |f|_p < \infty.
\]

Then the function \(\psi = \psi(p), s_1 < p < s_2\) may be naturally defined by the following way: \(\psi(p) := |f|_p\).
Let now the (measurable) function $f : X \to \mathbb{R}$ be such that $f \in G\psi$ for some $\psi(\cdot)$ with support $\text{supp} \psi(\cdot) = (s_1, s_2)$ for which $1 \leq s_1 < s_2 \leq d/\alpha$. Of course, the function $\psi(p)$ may be picked as a natural function for the function $f(\cdot)$:

$$\psi(p) = |f|_p,$$

if it is finite for $p \in (s_1, s_2)$.

We define a new $\psi$ function, say $\psi_K = \psi_K(q)$ as follows.

$$\psi_{K,R,\alpha}(q) = K_{R,\alpha}(p(q)) \cdot \psi(p(q)), \quad p \in (s_1, s_2).$$

Recall that the variable $p$ and $q$ are closely related by the equality (2.2), which defined the variable $p$ as unique function on $q$, $p = p(q)$; the conversely is also true.

**Theorem 3.1.** We propose under formulated conditions, in particular under conditions of theorem 2.1

$$||R_\alpha[f]||_{G\psi_{K,R,\alpha}} \leq 1 \cdot ||f||_{G\psi}, \quad (3.2)$$

where the constant "1" in (3.2) is the best possible.

**Proof. Upper bound.**

Let further in this section $p \in (s_1, s_2)$, where $1 \leq s_1 < s_2 \leq d/\alpha$. We can and will suppose without loss of generality $||f||_{G\psi} = 1$. Then $|f|_p \leq \psi(p), p \in (s_1, s_2)$. We conclude after substituting into the inequality (2.3)

$$|R_\alpha[f]|_q \leq K_{R,\alpha}(p) \cdot \psi(p) = \psi_{K,R,\alpha}(q) \cdot |f|_{G\psi}. \quad (3.3)$$

The inequality (3.2) follows from (3.3) after substitution $p = p(q)$.

**Proof. Exactness.**

The exactness of the constant 1 in the proposition (3.2) follows from the theorem 2.1 in the article [40].

### 4 Fractional derivative of indicator function, with consequences.

The following example is computed and applied in [34]. Let here $X = (0, 1)$; define the function

$$g_h(x) = I(h < x), \quad x > 0, \quad 0 < h = \text{const} < 1, \quad (4.0)$$

then

$$g_h^{(\alpha)}(x) = \frac{1}{\Gamma(1 - \alpha)} \cdot I(h < x) \cdot (x - h)^{-\alpha}, \quad \alpha = \text{const} \in (0, 1). \quad (4.1)$$

It is verified also in [34] that
\[ I^\alpha \left[ g^{(\alpha)}_h \right] (x) = I(h < x) = g_h(x). \] (4.2)

It is easily to estimate \(|g_h|_p = h^{1/p}\),

\[ \Gamma(1 - \alpha) |D^\alpha g_h(\cdot)|_q \leq (1 - \alpha q)^{-1/q} \cdot h^{1/q - \alpha}, \quad 1 \leq p < \infty, 1 \leq q < 1/\alpha. \]

We investigate in this section fractional derivative for a more general indicator function

\[ g_{h_1, h_2}(x) = I(h_1 < x < h_2), \quad 0 < h_1 < h_2 < 1; \quad \Delta := h_2 - h_1. \] (4.3)

Evidently,

\[ g_{h_1, h_2}(x) = g_{h_1}(x) - g_{h_2}(x), \]

therefore

\[ \Gamma(1 - \alpha) g^{(\alpha)}_{h_1, h_2}(x) = I(x > h_1)(x - h_1)^{-\alpha} - I(x > h_2)(x - h_2)^{-\alpha}. \]

We deduce after some computations for the values \( p : 1 \leq p < 1/\alpha \)

\[ \Delta^{1/p - \alpha}(1 - \alpha p)^{-1/p} \leq |\Gamma(1 - \alpha) g^{(\alpha)}_{h_1, h_2}(\cdot)|_p \leq 3 \Delta^{1/p - \alpha}(1 - \alpha p)^{-1/p}. \] (4.4)

Let us estimate the fractional derivative \( D^\alpha [g_{h_1, h_2}] \) in the Grand Lebesgue Space norm. Denote

\[ \psi_\alpha(p) = (1 - \alpha p)^{-1/p}, \quad 1 < p < 1/\alpha. \]

Let also \( \zeta = \zeta(p), \quad 1 < p < 1/\alpha \) be any function from the set \( G\Psi(1, 1/\alpha) \), then the product function

\[ \theta(p) = \psi_\alpha(p) \cdot \zeta(p) \]

belongs also the set \( G\Psi(1, 1/\alpha) \).

**Proposition 4.1.**

\[ ||g^{(\alpha)}_{h_1, h_2}||_{G\theta} \leq 3 \Delta^{-\alpha} \phi(G\zeta, \Delta), \quad \Delta = h_2 - h_1. \] (4.5)

**Proof.** The right-hand side of the inequality (4.4) may be rewritten as follows.

\[ \Delta^\alpha |g^{(\alpha)}_{h_1, h_2}|_p \leq 3 \psi_\alpha(p) \Delta^{1/p}, \]

or equally

\[ \Delta^\alpha \frac{|g^{(\alpha)}_{h_1, h_2}|_p}{\theta(p)} \leq 3 \frac{\Delta^{1/p}}{\zeta(p)}. \]
We deduce taking supremum from both the sides of the last inequality using the direct definition of Grand Lebesgue norm and fundamental function for these spaces

$$\Delta^\alpha \|g_{h_1, h_2}^{(\alpha)}\|_{G\theta} \leq 3 \phi(G\zeta, \Delta),$$

Q.E.D.

**Definition 4.1.** A measurable function $f : (0, b) \to R$ will named *very simple* with step $h, 0 < h < 1$, write $f \in VS(h)$, if it has a form

$$f(x) = \sum_{k=1}^{n} c_k I_{A_k}(x), \quad (4.6)$$

where $A_k$ are pairwise disjoint segments of a form $A_k = (h_1(k), h_2(k))$ with $h_2(k) - h_1(k) = h = \text{const}$.

On the other words, $f(\cdot)$ is spline of zero order with constant step, stepwise function.

Obviously, if $f \in VS(h)$, then

$$|f|_p = h^{1/p} \left[ \sum_k |c_k|^p \right]^{1/p} = h^{1/p} |\vec{c}|_p, \quad p \geq 1. \quad (4.7)$$

But in the next pilcrow we impose more strictly restriction $1 \leq p < 1/\alpha$, where again $0 < \alpha < 1$. We conclude using triangle inequality and inequality (4.4) $f \in VS(h) \Rightarrow$

$$\Gamma(1 - \alpha)|f^{(\alpha)}|_p \leq 3 h^{1/p - \alpha} (1 - \alpha p)^{-1/p} \sum_k |c_k| = 3 h^{1/p - \alpha} (1 - \alpha p)^{-1/p} |\vec{c}|_1 = 3 h^{1/p - \alpha - 1} (1 - \alpha p)^{-1/p} |f|_1. \quad (4.8)$$

We obtain analogously and under conditions of proposition 4.1

**Proposition 4.2.** Let again $f \in VS(h)$, then

$$\|f^{(\alpha)}\|_{G\theta} \leq 3 h^{-\alpha - 1} \phi(G\zeta, h) |f|_1, \quad h = h_2(k) - h_1(k) = \text{const} \cdot (4.9)$$

The last estimate may be used perhaps for numerical computation of fractional derivatives via spline approximation.

The condition $h_2(k) - h_1(k) = h = \text{const}$ may be easily replaced to the following:

$$0 < C_1 < \frac{h_2(k)}{h_1(k)} < C_2, \quad C_1, C_2 = \text{const} > 0.$$
5 Fractional derivative estimate.

The problem of norm estimation for fractional derivative is more complicated. Note first of all the definition (2.2a) common with estimation (2.4) may be rewritten at least for absolutely continuous functions \( \{f\} \) such that \( f(0) = 0 \) as follows:

\[
|f|_q \leq K_{I, \alpha}(p) \cdot |D^\alpha f|_p, \quad \alpha \in (0, 1), \quad 1/q = 1/p - \alpha, \quad 1 < p < 1/\alpha
\]

Sobolev’s inequality for fractional derivatives.

Let again \( \alpha = \text{const} \in (0, 1) \); and let \( f = f(x), \ x \in (0, b), \ 0 < b = \text{const} \leq \infty \) be measurable numerical function. The fractional derivative of a Riemann-Liouville type of order \( \alpha \):

\[
D^\alpha[f](x) = f^{(\alpha)}(x)
\]

is written, e.g. in (1.1).

The next equality

\[
\Gamma(1 - \alpha) D^\alpha[f](x) = x^{-\alpha} f(x) + \alpha \int_0^x \frac{f(x) - f(t)}{(x-t)^{1+\alpha}} \, dt \overset{\text{def}}{=} x^{-\alpha} f(x) + U_\alpha[f](x), \quad (5.0)
\]

which defines the so-called Marchaud fractional derivative, is proved e.g. in [45], p. 220 - 229.

Denote as ordinary by \( \omega(f, \delta)_p \) the \( L_p \) module of continuity of the function \( f : \)

\[
\omega(f, \delta)_p \overset{\text{def}}{=} \sup_{h : |h| \leq \delta} |f(\cdot + h) - f(\cdot)|_p,
\]

where \( f(x) := 0 \), if \( x < 0 \) or if \( x > b \) in the case when \( b < \infty \).

We introduce the following modification of the classical Besov’s norm \( ||f||_{B^\alpha_p} \) and correspondent spaces \( B^\alpha_p \) as follows

\[
||f||_{B^\alpha_p} \overset{\text{def}}{=} |x^{-\alpha} f(x)|_p + \alpha \int_0^b t^{-1-\alpha} \omega(f, t)_p \, dt, \quad (5.1)
\]

\[ f(\cdot) \in B^\alpha_p \iff ||f||_{B^\alpha_p} < \infty. \]

Theorem 5.1.

\[
\forall p \in (1, 1/\alpha) \Rightarrow \sup_{0 \neq f \in B^\alpha_p} \left\{ \frac{|D^\alpha[f]|_p}{||f||_{B^\alpha_p}} \right\} = \frac{1}{\Gamma(1 - \alpha)}. \quad (5.2)
\]

Proof. The inequality

\[
| \Gamma(1 - \alpha) D^\alpha[f] |_p \leq ||f||_{B^\alpha_p}, \quad f(\cdot) \in B^\alpha_p,
\]

follows immediately from the representation (5.0), see [45], p. 220-229. In order to ground the lower bound for the fraction

\[
r_\alpha(p) := \sup_{0 \neq f \in B^\alpha_p} \left\{ \frac{|D^\alpha[f]|_p}{||f||_{B^\alpha_p}} \right\}, \quad (5.4)
\]
it is sufficient to consider the following example (counterexample) with the value \( b = 1 \);

\[
f_0(x) = g_{h_1,h_2}(x) = I(h_1 < x < h_2), \ h_1, h_2 = \text{const}, \ 0 < h_1 < h_2 < 1.
\]

Denote \( \Delta = h_2 - h_1 \), then \( \Delta \in (0,1) \) and \( \Delta \to 1 \), if \( h_1 \to 0^+, \ h_2 \to 1^- \). We estimate taking into account the restriction \( 1 \leq p < 1/\alpha \):

\[
||f_0||_p^{\beta} \geq \int_{h_1}^{h_2} x^{-\alpha p} \, dx = \frac{h_1^{1-\alpha p} - h_2^{1-\alpha p}}{1-\alpha p} \sim \Delta^{1-\alpha p},
\]

\[
||f_0||_p^{\beta} \geq \frac{\Delta^{1/p-\alpha}}{(1-\alpha p)^{1/p}}.
\]  

(5.5)

The correspondent estimate for fractional derivative \( \Gamma(1-\alpha) |f_0|_p \) is obtained in (4.4). Substituting into expression (5.4), we get to the proposition (5.2) of theorem 5.1.

**Remark 5.1.** Note that the inverse inequality for (5.3), i.e. the inequality for arbitrary function \( f : R_+ \to R \) of the form

\[
||f||_p^{\beta} \leq \tilde{K}_{I,\alpha}(p) \ \Gamma(1-\alpha) |D^{\alpha}[f]|_p, \ 0 < \alpha < 1, \ \tilde{K}_{I,\alpha}(p) < \infty
\]

is not true for any number \( p, \ p \geq 1 \). Indeed, we put \( f_\alpha(x) = x^{\alpha-1} \); then \( D^{\alpha}[f_\alpha] = 0 \), despite \( ||f_\alpha||_p^{\beta} > 0 \).

Let us estimate the fractional derivative \( D^{\alpha}[f] \) in the Grand Lebesgue Space norm. Suppose that there exists a value \( \beta, \ 1 \leq \beta \leq 1/\alpha \) such that

\[
\psi^{(\beta)}(p) := ||f||_p^{\beta} < \infty, \ 1 < p < \beta.
\]

On the other words, the function \( p \to \psi^{(\beta)}(p) \) is Besov-Grand Lebesgue Spaces natural function for the function \( f(\cdot) \).

**Proposition 5.1.**

\[
||D^{\alpha}[f]||_{G_\psi^{(\beta)}} \leq 1/\Gamma(1-\alpha).
\]  

(5.6)

**Proof** is very elementary. We use the equality (5.3) for the values \( p : 1 < p < \beta \)

\[
\Gamma(1-\alpha) |D^{\alpha}[f]|_p \leq ||f||_p^{\beta} \leq \psi^{(\beta)}(p),
\]  

(5.7)

which is equivalent to the assertion of proposition (5.1.)

6 Multidimensional case.

Let \( \alpha, \beta = \text{const} \) be two numbers such that \( 0 < \alpha, \beta < 1 \); The partial mixed fractional derivative \( D_{x,y}^{\alpha,\beta}[G](x,y) \) again of Riemann-Liouville type of order \( (\alpha,\beta) \) of a function \( G(\cdot,\cdot) \) at the positive points \( (x,y) \) is defined as follows:
or in detail for the function $G = G(x, y)$ is factorable: $H(x, y) = g_1(x)g_2(y)$ and both the functions $g_x(\cdot)$ and $g_x(\cdot)$ are "differentiable" at the points $x$ and $y$ correspondingly:

$$\exists D^\alpha[g_1](x), \quad \exists D^\beta[g_2](y),$$

then really

$$D_x^\alpha D_y^\beta[H] = D_y^\beta D_x^\alpha[H] = D_x^\alpha[g_1](x) \cdot D_y^\beta[g_2](y).$$

On the other words the (linear) operator $D_x^\alpha g_1[\cdot] = D_y^\beta g_2[\cdot]$ is the tensor product of the one-dimensional fractional derivatives

$$D_{x,y}^{\alpha,\beta} = D_x^\alpha \otimes D_y^\beta.$$

Analogously be defined the partial mixed fractional integrals $I^{\alpha,\beta}[G]$, where $0 < \alpha, \beta < 1$:

$$I_{x,y}^{\alpha,\beta} = I_x^\alpha \otimes I_y^\beta, \quad (6.2)$$

or in detail for the function $G = G(x, y), \quad x \in (0, b_1), \quad y \in (0, b_2), \quad 0 < b_{1,2} = \text{const} \leq \infty$,

$$G^{(\alpha,\beta)}(x, y) = I_{x,y}^{\alpha,\beta}[G](x, y) \overset{\text{def}}{=} \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_0^x \int_0^y \frac{G(t, s)}{(x-t)^{1-\alpha}(y-s)^{1-\beta}} \, dt \, ds. \quad (6.3)$$

Recall that the so-called mixed Lebesgue - Riesz norm $||f||_{p_1, p_2}, \quad 1 \leq p_1, p_2 < \infty$ for a function $f(x, y)$ is defined by a formula

$$||f||_{p_1, p_2} \overset{\text{def}}{=} \left\{ \int_0^{b_2} \left[ \int_0^{b_1} |f(x, y)|^{p_1} \, dx \right]^{p_2/p_1} \, dy \right\}^{1/p_2}. \quad (6.4)$$

Suppose $1 < p_1 < 1/\alpha, \quad 1 < p_2 < 1/\beta$ and define as before

$$\frac{1}{q_1} = \frac{1}{p_1} - \alpha, \quad \frac{1}{q_2} = \frac{1}{p_2} - \beta, \quad (6.5)$$

$$K_{I,\alpha,\beta}(p_1, p_2) \overset{\text{def}}{=} \sup_{0 \neq f \in L(p_1, p_2)} \frac{|I_{\alpha,\beta}[f]|_{q_1, q_2}}{||f||_{p_1, p_2}}. \quad (6.6)$$

**Proposition 6.1.** We state under necessary conditions (6.5)
Proof is very simple. The upper estimate

\[ K_{I,\alpha,\beta}(p_1, p_2) \leq K_{I,\alpha}(p_1) \cdot K_{I,\beta}(p_2) \]

may be proved analogously ones in the preprint [41], see also [35]. The lower bound for the variable \( K_{I,\alpha,\beta}(p_1, p_2) \) may be deduced by means of choice of the factorable function

\[ f_0(x, y) = g_1(x) \cdot g_2(y), \]

where the functions \( g_1(x), g_2(y) \) are extremal functions for the correspondent one-dimensional problems.

Remark 6.1. This circumstance, i.e. a phenomenon of factorability of the function \( K_{I,\alpha,\beta}(p_1, p_2) \) may seem very strange, as long as in general case the individual operators of fractional integration and differentiation are non-commutative and

\[ |f|_{p_1, p_2} \neq |f|_{p_2, p_1}. \]

Analogously may be grounded the following result.

Denote

\[ r_{\alpha,\beta}(p_1, p_2) := \sup_{0 \neq f \in B_{p_1, p_2}^{(\alpha,\beta)}} \left\{ \frac{|D^{\alpha,\beta}[f]|_{p_1, p_2}}{||f||_{B_{p_1, p_2}^{(\alpha,\beta)}}} \right\}, \]

where the double Besov’s norm \( ||f||_{B_{p_1, p_2}^{(\alpha,\beta)}} \) denotes the Bochner’s composition of two Besov’s norms for the function \( f(x, y) \) of two variables \( (x, y) \) :

\[ ||f||_{B_{p_1, p_2}^{(\alpha,\beta)}} \overset{df}{=} ||f(\cdot, y)||_{B_{x, p_1}^{(\alpha)}} ||f(\cdot, y)||_{B_{y, p_2}^{(\beta)}}. \]

Proposition 6.2. There holds under at the same necessary conditions (6.5)

\[ r_{\alpha,\beta}(p_1, p_2) = r_{\alpha}(p_1) \cdot r_{\beta}(p_2). \]

7 Concluding remarks.

A. Weight Riesz and Riemann-Liouville potential estimate.

An (linear) operator \( U_{\alpha,\beta,\gamma}[f](x) \) of a form

\[ U_{\alpha,\beta,\gamma}[f](x) = \frac{x^{-\gamma}}{\Gamma(\alpha)} \int_0^x \frac{y^{-\beta} f(y) dy}{(x-y)^{\alpha-1}}, \quad x, y \in (0, b), \ b = \text{const} \leq \infty \]
is called weight Riesz and Riemann-Liouville potential operator, or generalized Cesaro-Hardy integral operator, or fractional integral.

We refer here results about $L^p \rightarrow L^q$ estimates for norm of this operator:

$$|U_{\alpha, \beta, \gamma}[f]|_q \leq V(\alpha, \beta, \gamma; p) |f|_p,$$

when $\alpha, \beta, \gamma \in (0, 1)$, $\alpha + \beta + \gamma < 2$, $\beta^2 + \gamma^2 > 0$,

$$\frac{1}{q} = \frac{1}{p} + (\alpha + \beta + \gamma - 2), \quad 1 < p, q < \infty.$$

and as ordinary

$$V(\alpha, \beta, \gamma; p) \overset{\text{def}}{=} \sup_{0 \neq f \in L^p} \left[ \frac{|U_{\alpha, \beta, \gamma}[f]|_q}{|f|_p} \right], \quad q = q(p).$$

This case is very different from the unweighted case, see, e.g. [24], [25], [26], [13], [19], [41], [35] and so on.

It is obtained in the aforementioned articles and books the $L^p \rightarrow L^q$ sharp estimates for these operators, also in the multidimensional case, exact Grand Lebesgue norm estimates etc.

In detail, let us denote $\kappa = 2 - \alpha - \beta - \gamma$,

$$p_- = \frac{1}{1 - \beta}, \quad p_+ = \frac{1}{2 - \alpha - \beta},$$

and correspondingly

$$q_- = \frac{1}{\alpha + \gamma - 1}, \quad q_+ = \frac{1}{\gamma}.$$

Statement: if $p \in (p_-, p_+]$, or equally $q \in [q_-, q_+]$, then

$$\frac{C_1(\alpha, \beta, \gamma)}{|p - p_-|^\kappa} \leq V(\alpha, \beta, \gamma; p) \leq \frac{C_2(\alpha, \beta, \gamma)}{|p - p_-|^\kappa}, \quad 0 < C_1 \leq C_2 < \infty, \quad (7.2)$$

and $V(\alpha, \beta, \gamma; p) = \infty$ in other case.

**B. Possible generalizations on metric measure spaces.**

It is interest by our opinion to obtain our estimations, especially to derive the lower bounds, on the arbitrary metric measure spaces in the spirit of articles [28], [29] etc.; where was applied the important notion of Riesz capacity.

By definition, the Riesz potential of order $\theta$ of a measurable function $f : X \rightarrow R$, where the set $X$ is equipped by a distance function $d = d(x, y)$ and by a Borelian non-trivial measure $\tau$, is

$$R(\theta)[f](x) = \int_X \frac{f(y) \tau(dy)}{\tau(B(x, y))]^\theta}, \quad (7.3)$$

see a recent work [11] and an article [12].
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