Abelian Chern-Simons theory on the torus and physical views on the Hecke operators

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Abstract

In the previous paper arXiv:1711.07122, we show that a holomorphic zero-mode wave function in abelian Chern-Simons theory on the torus can be considered as a quantum version of a modular form of weight 2. Motivated by this result, in this paper we consider an action of a Hecke operator on such a wave function from a gauge theoretic perspective. This leads us to obtain some physical views on the Hecke operators in number theory.
1 Introduction

It has been known for a long time that holomorphic part of zero-mode wave functions in abelian Chern-Simons (CS) theory on the torus can be expressed in terms of a Jacobi theta function in the context of geometric quantization [1, 2, 3]. (For foundations of the geometric quantization, see, e.g., [4, 5, 6].) Since any complex functions on the torus can, by definition, be expressed in terms of elliptic functions, this result sounds natural. Indeed it is well-known that the so-called Jacobi elliptic functions can be defined in terms of the Jacobi theta functions. Strictly speaking, however, the resultant form of the holomorphic wave function is not invariant under doubly periodic translations. This implies that we can not make a smooth transition from classical functions on the torus to quantum wave functions on the same manifold. One may interpret this matter as a reflection of ambiguities buried in a quantization process. We would, however, expect to make a quantization such that the doubly periodicity also holds in the quantum wave function. In fact, such a quantization has been reported previously by the author [7, 8]. One of the main purposes of this paper is to deliver a detailed review of this quantization procedure.

In order to clarify the issue, we now state some technical aspects of the situation as follows. In the geometric quantization the transition from a classical theory to a quantum counterpart is realized by imposing a polarization condition on a prequantum wave function parametrized by canonical coordinates on a symplectic manifold. The polarization condition can straightforwardly be implemented by use of holomorphicity if the symplectic manifold also holds complex structure, that is, the manifold is Kähler. This is relevant to the present case since the tours is a Kähler manifold. The holomorphic coordinates \((z, \bar{z})\) of the tours are therefore a suitable choice to parametrize wave function. This choice is, however, not compatible with a classical picture of the doubly periodic functions which are basically parametrized by \(\text{Re}z\) and \(\text{Im}z\). This is a main reason for the above-mentioned discrepancy between classical and quantum wave functions on the torus. The discrepancy may be solved if we describe the prequantum wave function in terms of \(\text{Re}z\) and \(\text{Im}z\). Upon the quantization, however, we need to introduce the holomorphic coordinates \((z, \bar{z})\) otherwise we can not suitably impose the polarization condition. We thus need to define the wave function in accord with these requirements. This is exactly what we shall carry out in section 3 of the
This paper is also motivated by an interest in applications of quantum field theory to number theory. According to the modularity theorem, or the formerly-called Taniyama-Shimura-Weil conjecture, an elliptic curve over rational number \( \mathbb{Q} \) and a modular form of weight 2 are in one-to-one correspondence at the level of \( L \)-functions. On the other hand, the elliptic curves, extended to the field of complex number \( \mathbb{C} \), can be described by the elliptic functions, as typically represented by the Weierstrass \( \wp \) function. As mentioned above the very elliptic functions can be considered as complex functions on the torus. These interrelations at least suggest a possibility to connect the holomorphic zero-mode wave functions in abelian CS theory on the torus with the modular forms weight 2. For introduction to the modular forms and related subjects such as elliptic curves and \( L \)-functions, see, \( e.g.\), [9, 10, 11, 12, 13]. The online database [14] on the \( L \)-functions and the modular forms is also useful.

Partly motivated by these thoughts, in the previous paper [15] we argue that the holomorphic zero-mode wave function can quantum theoretically considered as the modular form of weight 2. Another purpose of the present paper arises from a natural extension of this result, that is, we like to make use of this result so as to find a physical perspective on a problem in number theory. Particularly, we are interested in a physical interpretations of a Hecke operator acting on the modular forms. In the literature, physical views on the Hecke operator have been considered previously. In [16] it is discussed that finding simultaneous eigenvectors of the Hecke operator is analogous to determining simultaneous eigenfunctions of a hermitian operator in quantum mechanical systems; in particular, connection between eigenvalues of the Hecke operators and spectra of hermitian random matrices at a certain limit has been suggested. The Hecke operators also appear in the study of elliptic genera in superconformal field theories [17, 18] as well as in the context of the geometric Langlands program [19]. These topics are beyond the scope of this paper; we here simply focus on an interpretation of the Hecke operator as a hermitian operator acting on the holomorphic zero-mode wave function in abelian CS theory on the torus.

The organization of this paper is as follows. In the next section we review the geometric quantization of abelian CS theory on the torus, following Nair’s formulation [2, 3]. In section 3, as mentioned above, we show that the holomorphic zero-mode wave function can obey the doubly periodic condition when the level number of the abelian CS theory is even. We derive this relation by imposing gauge invariance on the zero-mode wave function where the gauge transformations are induced by the doubly periodic translations of the zero-mode variable. In section 4 we review main results in the previous paper [15]. In section 5 we briefly introduce some basic facts on the Hecke operators and the \( L \)-functions for the modular forms, including related topics such as level \( N \) congruence subgroups of the modular group and corresponding \( L \)-functions. We then consider how the Hecke operators act on the holomorphic wave function constructed in section 3. We argue that the action of the Hecke operator can be interpreted as a sum over the above-mentioned gauge transformations of the holomorphic wave function. This automatically explains that the holomorphic wave function is an eigenform of the Hecke operator. We argue that the notion of the level which is inherent in the modular forms also appears in the holomorphic wave function. Other speculative physical views on the Hecke
operators are also discussed in this section. Lastly, in section 6 we present brief conclusions.

2 Geometric quantization of abelian CS theory on the torus

In this section we briefly review geometric quantization of abelian Chern-Simons (CS) theory on the torus, following [1, 2, 3]. The geometric quantization is carried out on the zero-mode part of the abelian CS gauge field. A guiding concept of the quantization is a Kähler form of the torus on which the zero-mode variable is defined. In the following we see that all the key ingredients of the geometric quantization, such as a Kähler potential, a symplectic potential and a zero-mode wave function, can be derived from the Kähler form.

The torus can be described in terms of two real coordinates $\xi_1, \xi_2$, satisfying the periodicity condition $\xi_r \rightarrow \xi_r + \text{(integer)}$ where $r = 1, 2$. In other words, $\xi_r$ take real values in $0 \leq \xi_r \leq 1$, with the boundary values 0, 1 being identical. Complex coordinates of the torus can be parametrized as $z = \xi_1 + \tau \xi_2$ where $\tau \in \mathbb{C}$ is the modular parameter of the torus. By definition, we can impose the doubly periodic condition on $z$. Namely, functions of $z$ are invariant under the doubly periodic translations

$$z \rightarrow z + m + n\tau$$

where $m$ and $n$ are integers. Notice that we can absorb the real part of $\tau$ into $\xi_1$ without losing generality. In the following, we then assume $\text{Re}\tau = 0$, i.e.,

$$\tau = \text{Re}\tau + i\text{Im}\tau = i\text{Im}\tau := i\tau_2$$

with $\tau_2 > 0$.

The torus has a holomorphic one-form $\omega = \omega(z)dz$, satisfying

$$\int_\alpha \omega = 1, \quad \int_\beta \omega = \tau = i\tau_2$$

(2.3)

where the integrals are made along two non-contractible cycles on the torus, which are conventionally labeled as $\alpha$ and $\beta$ cycles. The one-form $\omega$ is a zero mode of the anti-holomorphic derivative $\partial_\bar{z} = \frac{\partial}{\partial \bar{z}}$. We can assume $\omega(z) = 1$. In terms of $\omega$ the gauge potential of CS theory on the torus can be parametrized as

$$A_z = \partial_z \theta + \frac{\pi \bar{\omega}}{\tau_2} a$$

(2.4)

where $\theta$ is a complex function $\theta(z, \bar{z})$ and $a$ is a complex number corresponding to the value of $A_z$ along the zero mode of $\partial_\bar{z}$. The abelian gauge transformations can be represented by

$$\theta \rightarrow \theta + \chi$$

(2.5)
where $\chi$ is a complex constant or a phase factor of the $U(1)$ theory. With a suitable choice of $\chi$ we can parametrize the gauge potential solely by the zero-mode contributions, $a$ and its complex conjugate $\bar{a}$:

$$A_z = \frac{\pi \omega}{\tau_2} a, \quad A_{\bar{z}} = \frac{\pi \bar{\omega}}{\tau_2} a.$$  \hspace{1cm} (2.6)

Since the complex variable $a$ is defined on the torus it is natural to require that physical observables of the zero modes are invariant under the doubly periodic translations

$$a \rightarrow a + m + in\tau_2$$  \hspace{1cm} (2.7)

where $m$ and $n$ correspond to the winding numbers along the $\alpha$ and $\beta$ cycles, respectively. It is known that the complex torus can be embedded into a complex projective space, that is, the zero-mode variable $a$ may satisfy the scale invariance

$$a \sim \lambda a$$  \hspace{1cm} (2.8)

where $\lambda$ is a complex constant.

Geometric quantization as a Kähler-form program

All the important ingredients in geometric quantization of abelian Chern-Simons theory on the torus are derived from a Kähler form of the torus parametrized by the zero-mode variable $a$. From (2.6) the zero-mode Kähler form is defined as

$$\Omega^{(\tau_2)} = \frac{l}{2\pi} da \wedge d\bar{a} \int_{z,\bar{z}} \frac{\pi \bar{\omega}}{\tau_2} \wedge \frac{\pi \omega}{\tau_2} = \frac{i\pi l}{\tau_2} da \wedge d\bar{a}$$  \hspace{1cm} (2.9)

where the integral is taken over $dzd\bar{z}$ and $l$ is the level number associated to the abelian Chern-Simons theory. We here use the normalization of $\omega$ and $\bar{\omega}$ given by

$$\int_{z,\bar{z}} \bar{\omega} \wedge \omega = i2\tau_2.$$  \hspace{1cm} (2.10)

A Kähler potential $K(a, \bar{a})$ associated with the zero-mode Kähler form $\Omega^{(\tau_2)}$ is defined as

$$\Omega^{(\tau_2)} = i\partial\bar{\partial}K(a, \bar{a})$$  \hspace{1cm} (2.11)

where $\partial, \bar{\partial}$ denote the Dolbeault operators. The is definition leads to

$$K(a, \bar{a}) = \frac{\pi l}{\tau_2} a\bar{a} + u(a) + v(\bar{a})$$  \hspace{1cm} (2.12)

where $u(a)$ and $v(\bar{a})$ are purely holomorphic and anti-holomorphic functions, respectively. These functions represent ambiguities in the choice of $K(a, \bar{a})$.

A symplectic potential (or a canonical one-form) $A^{(\tau_2)}$ corresponding to the Kähler form $\Omega^{(\tau_2)}$ is defined as

$$\Omega^{(\tau_2)} = dA^{(\tau_2)}.$$  \hspace{1cm} (2.13)
In the program of geometric quantization a quantum wave function $\Psi[A \bar{z}]$ generally satisfies the so-called polarization condition
\[
\left( \partial_a + \frac{1}{2} \partial_{\bar{a}} K \right) \Psi[A \bar{z}] = 0 \tag{2.14}
\]
where $K = K(a, \bar{a})$ is the zero-mode Kähler potential in (2.12). The polarization condition leads to the specific form
\[
\Psi[A \bar{z}] = e^{-\frac{1}{2} \psi} 
\tag{2.15}
\]
where $\psi[A \bar{z}]$ is a holomorphic function of $A \bar{z}$. In the present case the physical variables are given by $a$ and $\bar{a}$ so that the wave function can be expressed as
\[
\Psi[A \bar{z}] := \Psi[a, \bar{a}] = e^{-\frac{1}{2} K(a, \bar{a})} f(a) \tag{2.16}
\]
where $f(a)$ is a function of $a$. We call $f(a)$ a holomorphic zero-mode wave function. Notice that we here define $\Psi[a, \bar{a}]$ with $K(a, \bar{a})$ including the above-mentioned ambiguities in its choice.

An inner product of the zero-mode wave functions $\Psi[a, \bar{a}]$ in (2.16) can be expressed as
\[
\langle \Psi | \Psi' \rangle = \int d\mu(a, \bar{a}) \Psi[A \bar{z}] \Psi'[A \bar{z}]
= \int d\mu(a, \bar{a}) e^{-\frac{1}{2} K(a, \bar{a})} \overline{f(a)} f'(a) \tag{2.17}
\]
where $\overline{f(a)}$ is the complex conjugate of $f(a)$ and $d\mu(a, \bar{a})$ denotes the integral measure of the zero-mode variable on the torus. The meaning of the integral is therefore the same as those in (2.9) and (2.10).

### 3 Double periodicity in holomorphic wave functions

Since the zero-mode variable $a$ is defined on the torus, the holomorphic zero-mode wave function $f(a)$ is expected to be invariant under the doubly periodic translations $a \rightarrow a + m + in\tau_2$ ($m, n \in \mathbb{Z}$). Indeed, as shown in [7, 8], we can argue that such an expectation is true for $l \in 2\mathbb{Z}$. To be more specific, a gauge invariance condition on the zero-mode wave function $\Psi[a, \bar{a}]$ with certain choices of $A[^{\tau_2}]$ and $K$ (where the gauge transformations are induced by the doubly periodic translations) leads to the relation $f(a + m + in\tau_2) = e^{i\pi lm} f(a)$. In this section we shall carry a careful review of this relation.

**Change of variables and the symplectic structure of the torus**

From our setting $\tau = i\tau_2$ in (2.2), the complex coordinate $z$ on the torus is written as $z = \xi_1 + i\tau_2 \xi_2$. It is then useful to choose coordinates on the torus as $z_1 := \bar{z} - z = -2i\tau_2 \xi_2$ and $z_2 := \tau \bar{z} - \bar{z} = 2i\tau_2 \xi_1$ so that we can parametrize the torus in terms of the real variables along the non-contractible cycles. This parametrization does not mean that the torus loses
the complex structure, of course. It is, however, useful to connect the coordinates directly to the doubly periodic transformations in (2.1). In fact, this parametrization corresponds to a conventional definition of a double periodic complex function; \( f(x, y) = f(x + m, y + n) \) where \( z = x + iy \) and \( m, n \in \mathbb{Z} \).

We then reparametrize the complex zero-mode variables \((a, \bar{a})\) by

\[
\begin{align*}
    a_1 &= \bar{a} - a, \\
    a_2 &= \tau \bar{a} - \bar{\tau} a = i\tau_2 (\bar{a} + a).
\end{align*}
\]

Notice that \( a_1 = -i2(\text{Im} a), \) \( a_2 = i2\tau_2(\text{Re} a) \). These variables essentially represent the real and imaginary parts of the complex variable \( a \). In terms of these the integral measure \( d\mu(a, \bar{a}) \) may be expressed as \( d(\text{Re} a)d(\text{Im} a) \). On the other hand, the complex structure of the torus and, hence, a holomorphic quantization program should be described in terms of \((a, \bar{a})\). Thus, for the sake of the geometric quantization, \( \text{per se} \), it is not appropriate to use the variables \((a_1, a_2)\). In other words, for the construction of zero-mode wave functions we still need to keep using the polarization condition (2.14). As seen in a moment, we can, however, express a symplectic two-form of the torus in terms of \((a_1, a_2)\). A corresponding symplectic potential and an analog of the Kähler potential can then be expressed in terms of \((a_1, a_2)\). This suggests that at least at a level of choosing the Kähler potential (2.12) we can use the canonical coordinates \((a_1, a_2)\). In what follows we shall clarify these points by reviewing some results in [7, 8].

In terms of \((a_1, a_2)\) the zero-mode part of the abelian CS gauge potentials are written as

\[
\begin{align*}
    A_{\xi_1} &= \frac{\pi \omega_2 a_1}{\tau_2}, \\
    A_{\xi_2} &= \frac{\pi \omega_1 a_2}{\tau_2}.
\end{align*}
\]

where the associated one-forms \( \omega_1, \omega_2 \) are defined as

\[
\begin{align*}
    \omega_1 &= \frac{dz_1}{2i} = -\tau_2 d\xi_2, \\
    \omega_2 &= \frac{dz_2}{2i} = \tau_2 d\xi_1.
\end{align*}
\]

The normalization for \( \omega_1 \) and \( \omega_2 \) can be given by

\[
\int_{z, \bar{z}} \frac{\omega_1}{\tau_2} \wedge \frac{\omega_2}{\tau_2} = 1.
\]

In terms of these the holonomies of the torus (2.3) are simplified as

\[
\oint_{\alpha_r} \omega_s = \epsilon_{rs} \tau_2
\]

where \( \epsilon_{rs} (r, s = 1, 2) \) denotes the rank-2 Levi-Civita symbol and \( \alpha_1, \alpha_2 \) correspond to the \( \alpha \) and \( \beta \) cycles, respectively.

A zero-mode symplectic two-form is then expressed as

\[
\Omega^{(\tau_2)} = \frac{l}{2\pi \tau_2} da_1 \wedge da_2 \int_{z, \bar{z}} \frac{\pi \omega_2}{\tau_2} \wedge \frac{\pi \omega_1}{\tau_2} = -\frac{\pi l}{2\tau_2} da_1 \wedge da_2.
\]
The corresponding symplectic potential (or the canonical one-form) can be written as

$$A^{(\tau_2)} = \frac{\pi l}{4\tau_2^2} \int_{z, \bar{z}} \left( \frac{\omega_2 a_1}{\tau_2} \wedge \frac{\omega_1}{\tau_2} da_2 - \frac{\omega_1 a_2}{\tau_2} \wedge \frac{\omega_2}{\tau_2} da_1 \right) = -\frac{\pi l}{4\tau_2^2} \left( a_1 da_2 + a_1 da_2 \right).$$  \hspace{1cm} (3.8)

The explicit form of the symplectic two-form $\Omega^{(\tau_2)}$ in (3.7) means that $a_1$ and $a_2$ can be served as canonical coordinates of the torus. Although these are not complex conjugate to each other, it is suggestive that we can define an analog of a Kähler potential corresponding to $\Omega^{(\tau_2)}$ in (3.7):

$$W(a_1, a_2) = i \frac{\pi l}{2\tau_2} a_1 a_2 = -\frac{\pi l}{2\tau_2} (\bar{a}^2 - a^2).$$  \hspace{1cm} (3.9)

This is in a form of separation of holomorphic and antiholomorphic parts. We are thus allowed to rewrite the Kähler potential $K(a, \bar{a})$ in (2.12) as

$$K(a, \bar{a}) = K_0 + W + u + \bar{v}$$  \hspace{1cm} (3.10)

where

$$K_0 = \frac{\pi l}{\tau_2} a\bar{a}, \quad W = W(a_1, a_2), \quad u = u(a), \quad \bar{v} = v(\bar{a}).$$  \hspace{1cm} (3.11)

As before, $u(a)$ and $v(\bar{a})$ denote holomorphic and antiholomorphic functions, respectively.

The zero-mode wave function (2.16) is written as

$$\Psi[a, \bar{a}] = e^{-\frac{K_0 + u + \bar{v}}{2}} F[a, \bar{a}]$$  \hspace{1cm} (3.12)

where

$$F[a, \bar{a}] = e^{-\frac{W}{2}} f(a).$$  \hspace{1cm} (3.13)

The inner product (2.17) is then expressed as

$$\langle \Psi | \Psi' \rangle = \int d\mu(a, \bar{a}) e^{-(K_0 + u + \bar{v})} F(a) f'(a)$$  \hspace{1cm} (3.14)

where we use $F[a, \bar{a}] = e^{-\frac{W}{2}} f(a)$, $F'[a, \bar{a}] = e^{-\frac{W}{2}} f'(a)$. The wave function $F[a, \bar{a}]$ is relevant to the choice of the symplectic form $\Omega^{(\tau_2)}$ in (3.7) or the choice of the canonical coordinates $(a_1, a_2)$. The factor of $e^{-\frac{W}{2}}$ in (3.13) is then appropriate one in the definition of $F[a, \bar{a}]$. We now argue on this point briefly.

If we consider the Kähler form in (2.9), the potential $W$ can be absorbed into the ambiguities in the choice of the Kähler potential since, as explicitly shown in (3.9), $W$ is given in a form of separation of holomorphic and antiholomorphic parts. On the other hand, in terms of $(a_1, a_2)$ the potential $K_0$ can be expressed as

$$K_0 = \frac{\pi l}{\tau_2} a\bar{a} = -\frac{\pi l}{4\tau_2} \left( a_1^2 + \left( \frac{a_2}{\tau_2} \right)^2 \right).$$  \hspace{1cm} (3.15)

Thus it is given in a form of $u(a_1) + v(a_2)$ as well. This implies that $W(a_1, a_2) = i \frac{\pi l}{2\tau_2} a_1 a_2$ in (3.9) is considered to be a counter part of $K_0 = \frac{\pi l}{\tau_2} a\bar{a}$ in the $(a_1, a_2)$ representation. As
mentioned elsewhere, \((a_1, a_2)\) are canonical coordinates of the phase space of interest whose symplectic form is given by (3.7). Thus, classically, we can describe physical observables in terms of \((a_1, a_2)\). As far as the Kähler potential and the symplectic potential are concerned, we can then define these in terms of \((a_1, a_2)\). Quantum theoretically, however, we need to impose the polarization condition (2.7) on the zero-mode wave function. We therefore require that \(f(a)\) as defined in (3.12, 3.14) should be a holomorphic function of \(a\).

**Gauge transformations induced by doubly periodic translations**

As mentioned earlier, holomorphic functions on the torus in general obey the double periodicity condition. Thus we can naturally assume \(f(a) = f(a + m + in\tau_2)\). In what follows, we show how this assumption can be understood by imposing an invariance on the wave function \(F[a, \bar{a}] = e^{-\frac{W}{2}}f(a)\) under a gauge transformation of \(A^{(\tau_2)}\) in (3.8) where the gauge transformation is induced by the doubly periodic translations \(a \rightarrow a + m + in\tau_2\).

Under \(a \rightarrow a + m + in\tau_2\), \(a_1\) and \(a_2\) vary as
\[
a_1 \rightarrow a_1 - 2in\tau_2, \quad a_2 \rightarrow a_2 + 2im\tau_2. \tag{3.16}
\]
The symplectic potential (3.8) transforms as
\[
A^{(\tau_2)} \rightarrow A^{(\tau_2)} + d\Lambda_{m,n} \tag{3.17}
\]
where
\[
\Lambda_{m,n} = -i\frac{\pi l}{2\tau_2}(ma_1 - na_2). \tag{3.18}
\]
The gauge invariance of the wave function \(F[a, \bar{a}] = e^{-\frac{W}{2}}f(a)\) is then realized by the relation
\[
e^{i\Lambda_{m,n}}F[a, \bar{a}] = F[a + m + in\tau_2, \bar{a} + m - in\tau_2]. \tag{3.19}
\]

Explicit forms of the left and right-hand sides are given by
\[
(l.h.s) = \exp \left[\frac{\pi l}{2\tau_2}(ma_1 - na_2) - i\frac{\pi l}{4\tau_2}a_1a_2\right]f(a), \tag{3.20}
\]
\[
(r.h.s) = \exp \left[-i\frac{\pi l}{4\tau_2}(a_1 - 2in\tau_2)(a_2 + 2im\tau_2)\right]f(a + m + in\tau_2). \tag{3.21}
\]
The gauge invariance condition (3.19) thus leads to the relation
\[
e^{i\pi lm n}f(a) = f(a + m + in\tau_2). \tag{3.22}
\]
The consequence of the gauge invariance (3.19) is the following; the holomorphic zero-mode wave function \(f(a)\) is invariant under \(a \rightarrow a + m + in\tau_2\), given that the level number \(l\) is quantized by even integers, i.e.,
\[
l \in 2\mathbb{Z}. \tag{3.23}
\]
This level-number quantization condition has been known for the abelian CS theory on the torus \([1, 2]\). The relation (3.22) was first reported in [7] and further developed in [8].
but in these papers the ambiguities in the choice of the Kähler potential has not been
carefully treated. We here argue that the use of the particular wave function \( F[a, \bar{a}] \) in
(3.13) appropriately leads to the relation (3.22).

Lastly, we would like to comment on the irreducibility of the symplectic potential \( \mathcal{A}(r_2) = -\frac{\pi l}{4\tau_2^2} (a_1 da_2 + a_1 da_2) \). As mentioned before, \( a_1 \) and \( a_2 \) can serve as the canonical coordinates of a physical system. So it may be possible to impose a “polarization” condition on this \( \mathcal{A}(r_2) \). The doubly periodic translations, however, involve the both winding numbers \((m, n)\), corresponding to the variations of \( a_2 \) and \( a_1 \), respectively, as shown in (3.16). Thus, in order to express the gauge transformation (3.17) such that these winding numbers are
impartially treated, we need to define \( \mathcal{A}(r_2) \) in terms of both \( a_1 \) and \( a_2 \) explicitly. In this sense (3.8) provides an irreducible representation for the symplectic potential whose gauge transformations are induced by the doubly periodic translations.

4 Holomorphic wave functions as modular forms of weight 2

In this section we briefly present the main results in the previous paper [15]. The upshot of the previous paper is that under the modular \( S\)- and \( T\)-transformations a holomorphic zero-mode wave function \( f(a) \) in abelian Chern-Simons theory on the torus varies as

\[
S : f\left(-\frac{1}{a}\right) = a^2 f(a), \quad (4.1)
\]

\[
T : f(a+1) = f(a), \quad (4.2)
\]

given that \( f(a) \) is quantum theoretically characterized by the operative relation

\[
\frac{\partial}{\partial a} f(a) = \frac{\pi l}{\tau_2} \bar{a} f(a) \quad (4.3)
\]

and the inner product of the zero-mode wave functions

\[
\langle \Psi | \Psi' \rangle = \int d\mu(a, \bar{a}) e^{-K(a, \bar{a})} \overline{f(a)} f'(a) \quad (4.4)
\]

where the zero-mode wave function is defined as \( \Psi[a, \bar{a}] = e^{-K(a, \bar{a})} f(a) \). The operative relation (4.2) is guaranteed as long as the Kähler potential is given in the form of \( K(a, \bar{a}) = K_0 + u(a) + v(\bar{a}) \) where \( K_0 = \frac{\pi l}{\tau_2} a\bar{a} \). Notice that our choice (3.10) in the previous section falls within this form.

The modular transformations of our interest are generated by

\[
S : (a, \tau_2) \rightarrow \left(-\frac{1}{a}, \frac{\tau_2}{|a|^2}\right), \quad (4.5)
\]

\[
T : (a, \tau_2) \rightarrow (a+1, \tau_2). \quad (4.6)
\]
These imply that the quantity $\frac{|da|^2}{\tau^2}$, or the number density of the zero modes per unit area, is preserved under the modular transformations of $a$. Note that under the $S$-transformation the area element $d\alpha d\bar{\alpha}$ changes as $|da|^2 \to \frac{|da|^2}{|a|^4}$. Thus from (4.5) we find that the number density $\frac{|da|^2}{\tau^2}$ is preserved under the $S$-transformation. The modular $T$-invariance of the number density is obvious.

**Basics of modular forms**

We review some basic facts on the modular forms. In general, the modular form $f(z)$ of weight $k$ is defined by

$$f \left( \frac{\alpha z + \beta}{\gamma z + \delta} \right) = (\gamma z + \delta)^k f(z)$$

where $\alpha, \beta, \gamma, \delta$ are matrix elements of the modular group

$$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right| \alpha, \beta, \gamma, \delta \in \mathbb{Z}, \alpha \delta - \beta \gamma = 1 \} := \Gamma.$$  

(4.8)

The modular forms are defined on the upper-half plane $\mathbb{H} = \{ z \in \mathbb{C} | \text{Im} z > 0 \}$. Accordingly, to be rigorous, the modular group is defined as $PSL(2, \mathbb{Z}) := SL(2, \mathbb{Z})/\{\pm I\}$, with $I$ the identity matrix. The fundamental domain $\mathcal{F}$ for the action of $SL(2, \mathbb{Z})$ generators on $\mathbb{H}$ is given by

$$\mathcal{F} = \left\{ z \in \mathbb{C} \middle| \text{Im} z > 0, |z| \geq 1, |\text{Re} z| \leq \frac{1}{2} \right\}.$$  

(4.9)

The generators of the modular group is given by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The definition of the modular form in (4.7) is then obtained from the conditions

$$f(z + 1) = f(z),$$

$$f \left( -\frac{1}{z} \right) = z^k f(z).$$

(4.10) (4.11)

The first condition simply means that $f(z)$ can be expressed in a form of the Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n q^n$$

(4.12)

where $q = e^{i2\pi z}$ and $a_n$ is the Fourier coefficient. If $a_0 = 0$, the modular form $f(z)$ is called the cusp form. The vector space formed by the cusp forms of weight $k$ is denoted by $S_k(\Gamma)$, i.e.,

$$S_k(\Gamma) := \left\{ f : \mathbb{H} \to \mathbb{C} \middle| f \left( -\frac{1}{z} \right) = z^k f(z), f(z) = \sum_{n=1}^{\infty} a_n q^n \right\}.$$  

(4.13)

Let $f(z), g(z) \in S_k(\Gamma)$, then the Petersson inner product is defined as

$$\langle f, g \rangle = \frac{1}{\text{vol} \mathcal{F}} \int_{\mathcal{F}} f(z) \overline{g(z)} y^k \frac{dxdy}{y^2}$$

(4.14)
where $z = x + iy$. Notice that this inner product represents a manifestly modular invariant integral.

From (4.1, 4.2) and (4.10, 4.11) we can quantum theoretically identify $f(a)$ as a modular form of weight 2. Consequently, we may define the zero-mode variable on the fundamental domain $\mathcal{F}$ in (4.7). The inner product of the zero-mode wave functions (2.17) can then be rewritten as

$$\langle \Psi | \Psi' \rangle = \int d\mu(a, \bar{a}) e^{-K(a, \bar{a})} \overline{f(a)} f'(a)$$

$$\rightarrow \frac{1}{\text{vol} \mathcal{F}} \int_{\mathcal{F}} d(\text{Re} a) d(\text{Im} a) e^{-K(a, \bar{a})} \overline{f(a)} f'(a)$$

(4.15)

where we express the integral measure $d\mu(a, \bar{a})$ as $d(\text{Re} a) d(\text{Im} a)$ in the second line. This inner product can be considered as a quantum version of the Petersson inner product (4.14) for the modular forms of weight 2.

5 Physical views on the Hecke operators

Basics on the Hecke operators

We now introduce basic ideas of Hecke operators acting on the modular forms of weight $k$, following mathematical textbooks, e.g., [9, 10, 11, 12, 13]. In order to define the Hecke operators it is useful to view the modular forms as functions on complex lattices. Let $f(L)$ be a function on a lattice $L$ in $\mathbb{C}$. Then a Hecke operator $T_m$ acting on $f(L)$ is defined as

$$T_m f(L) = \sum_{[L:L'] = m} f(L')$$

(5.1)

where the sum is taken over all sublattices $L' \subset L$ of index $m$. Note that a sublattice $L' \subset L$ has index $m$ if the quotient $L/L'$ has order dividing $m$ so that $mL \subset L' \subset L$ and

$$L'/mL \subset L/mL = (\mathbb{Z}/m\mathbb{Z})^2.$$ 

(5.2)

This means that the sublattices of index $m$ correspond to the subgroups of order $m$ in $(\mathbb{Z}/m\mathbb{Z})^2$. If $m$ is a prime number $p$, there are $p + 1$ such subgroups, since the number of the subgroups corresponds to the number of nonzero vectors in $\mathbb{F}_p$ modulo scalar equivalence, and there are $(p^2 - 1)/(p - 1) = p + 1$ such vectors [12].

It is known that there is a one-to-one correspondence between sublattices $L' \subset L$ of index $m$ and matrices $\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$ with $\alpha, \beta, \delta \in \mathbb{Z}$, $\alpha \delta = m$ and $0 \leq \beta \leq \delta - 1$; for details and a proof, see [12]. The choice of the index-$m$ sublattices is then reduced to that of the matrix
elements in the above matrix with determinant $m$. By use of this fact, one finds that the Hecke operator acting on $f(z) \in S_k(\Gamma)$ can be defined as

\[
T_m f(z) = m^{k-1} \sum_{\delta} \sum_{\alpha=0}^{\delta-1} \sum_{\beta=0}^{\delta-k} \left( \frac{\alpha z + \beta}{\delta} \right) f \left( \frac{\alpha z}{\delta} \right)
\]

\[
= \sum_{n=1}^{\infty} \left( \sum_{\alpha \mid (m,n)} \alpha^{k-1} a_{\frac{mn}{\alpha^2}} \right) q^n
\]

(5.3)

where the Fourier expansion of $f(z) = \sum_{n \geq 1} a_n q^n$ is given by (4.13) with $q = e^{i2\pi z}$, and $(m,n)$ denotes $\text{gcd}(m,n)$. This means that the Hecke operator $T_m$ preserves the space of modular forms of a given weight, $T_m : S_k(\Gamma) \to S_k(\Gamma)$. When the index $m$ is prime $m = p$, the above expression simplifies as

\[
T_p f(z) = \sum_{n=1}^{\infty} \left( a_{pn} + p^{k-1} a_{\frac{n}{p^2}} \right) q^n.
\]

(5.4)

The Hecke operator forms an abelian algebra

\[
T_m T_n = T_n T_m = \sum_{\alpha \mid (m,n)} \alpha^{k-1} T_{\frac{mn}{\alpha^2}}
\]

(5.5)

From this relation we find $T_m T_n = T_n T_m = T_{mn}$ if $(m,n) = 1$. Suppose $f(z) \in S_k(\Gamma)$ is a simultaneous eigenfunction of the Hecke operators $T_m$ for all $m = 1, 2, \cdots$, i.e., if there exists a set of eigenvalues $\lambda_m$ such that

\[
T_m f(z) = \lambda_m f(z),
\]

(5.6)

then the following form of a Dirichlet series can be expressed as an Euler product:

\[
L(s, f) = \sum_{n \geq 1} \lambda_n n^{-s} = \prod_{p: \text{prime}} \frac{1}{1 - \lambda_p p^{-s} + p^{k-1-2s}}
\]

(5.7)

where $s \in \mathbb{C}$. This function of $s$ is called the $L$-function of the modular form $f(z)$.

As described in (4.14), the vector space of the cusp forms $f(z), g(z) \in S_k(\Gamma)$ has the Petersson inner product $\langle f, g \rangle$. It is well known that in terms of this inner product the Hecke operator $T_m$ is a hermitian operator; see [9] for details. In other words, we have $\langle T_m f, g \rangle = \langle f, T_m g \rangle$ and, accordingly, the eigenvalue is a real constant $\lambda_m \in \mathbb{R}$. In fact, from (5.3) and (5.6) we can show that

\[
\lambda_m = a_m
\]

(5.8)

with the normalization of $f(z)$ by $a_1 = 1$. This can easily be seen by expanding $T_m f(z) = \lambda_m f(z)$ as $T_m f(z) = \sum_{n \geq 1} b_n q^n$ and reading off the coefficient $b_1$ from (5.3), which leads to $b_1 = a_m = \lambda_m a_1$. 

13
Level $N$ congruence subgroup $\Gamma_0(N)$ of $\Gamma$

A level $N$ congruence subgroup $\Gamma_0$ of the modular group $\Gamma$ is defined as

$$\Gamma_0(N) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z}) \mid \gamma \equiv 0 \pmod{N} \right\} \tag{5.9}$$

There exist modular forms corresponding to $\Gamma_0(N)$, i.e., those that satisfy the definition (4.7) by the matrix elements in (5.9). In terms of such modular forms a congruence subgroup of the vector space $S_k(\Gamma)$ in (4.13) is similarly defined and is conventionally denoted by $S_k(\Gamma_0(N))$. Obviously, $\Gamma_0(N)$ and $S_k(\Gamma_0(N))$ reduce to $\Gamma$ and $S_k(\Gamma)$, respectively, at $N = 1$.

Introducing a Dirichlet character $\chi$ modulo $N$, we can define a relevant cusp form $f(z) \in S_k(\Gamma_0(N), \chi)$ by

$$f\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) = \chi(\delta) (\gamma z + \delta)^k f(z) \tag{5.10}$$

where $\alpha, \beta, \gamma, \delta$ are matrix elements of $\Gamma_0(N)$. In analogy to (5.3), the action of the Hecke operator $T_m$ on $f(z) = \sum_{n \geq 1} a_n q^n \in S_k(\Gamma_0(N), \chi)$ can be defined by [10]

$$T_m f(z) = \sum_{n=1}^{\infty} \left( \sum_{\alpha | (m,n)} \chi(\alpha) \alpha^{k-1} a_{\frac{mn}{m}} \right) q^n \tag{5.11}$$

Suppose that $f(z) \in S_k(\Gamma_0(N), \chi)$ is a simultaneous eigenfunction of the above Hecke operators $T_m$ for all $m = 1, 2, \cdots$, that is, we have $T_m f(z) = \lambda_m f(z)$. Since $T_m f(z) \in S_k(\Gamma_0(N), \chi)$ as well, we can expand it as $T_m f(z) = \sum_{n \geq 1} b_n q^n$. From (5.11) and the identity of the Dirichlet character $\chi(1) = 1$, we then find

$$b_1 = a_m = \lambda_m a_1 = \lambda_m \tag{5.12}$$

with the normalization $a_1 = 1$. In this case the corresponding $L$-function can be expressed as

$$L(s, f) = \sum_{m \geq 1} \lambda_m m^{-s}$$

$$= \prod_{p \nmid N} \frac{1}{1 - a_p p^{-s}} \prod_{p | N} \frac{1}{1 - a_p p^{-s} + \chi(p)p^{k-1-2s}} \tag{5.13}$$

where, as in (5.7), $p$ denotes the prime numbers. Notice that the Dirichlet character $\chi(p)$ of modulo $N$ vanishes whenever $p \mid N$; thus we can define the above $L$-function simply as $L(s, f) = \prod_p (1 - a_p p^{-s} + \chi(p)p^{k-1-2s})^{-1}$ without splitting into factors of $p \mid N$ and $p \nmid N$.

Hecke operators acting on the holomorphic wave function $f(a)$

In the previous sections we have argued that the holomorphic zero-mode wave function $f(a)$ in abelian Chern-Simons theory on the torus can be considered as a quantum version of a modular form of weight 2. In the following we think of how the Hecke operator arises as
an action to \( f(a) \). To begin with, we recall that the gauge invariance condition (3.19) for the zero-mode wave function leads to the relation \( e^{i\pi mn} f(a) = f(a + m + in\tau_2) \) in (3.22). For \( l \in 2\mathbb{Z} \), \( f(a) \) satisfies the doubly periodic condition and we can identify \( f(a) \) as a holomorphic function defined on a complex lattice. For \( l \) being an odd integer, say \( l = 1 \), we have

\[
(5.14)
\]

Regarding the factor \((-1)^{mn}\) as a “phase” factor, we can also consider \( f(a) \) as a function on the complex lattice. Thus the above-mentioned definition of the Hecke operator (5.1)-(5.5) applies to \( f(a) \) as well. One of the peculiarities in \( f(a) \), distinguished from classical functions on the torus, is given by the relation (5.14). In order to investigate quantum properties of \( f(a) \) we fix \( l \) at \( l = 1 \), while keeping \( \tau_2 \) finite, in the following.

From (5.1) we see that the Hecke operator \( T_M \) acting on a function \( f(L) \) on a complex lattice \( L \) is defined as the sum over sublattices \( L' \subset L \) of index \( M \). In terms of the holomorphic wave function \( f(a) \) this can be expressed as

\[
T_M f(L) = \sum_{[L,L'] = M} f(L')
\]

\[
\rightarrow T_M f(a) = \sum_{m \in \mathbb{F}_M} \sum_{n \in \mathbb{F}_M} f(a + m + in\tau_2)
\]

\[
= \sum_{m,n \in \mathbb{F}_M} (-1)^{mn} f(a)
\]

where we use (5.2), (5.14) and \( \mathbb{F}_M = \mathbb{Z}/M\mathbb{Z} \). Naively, this means that \( f(a) \) is an eigenfunction of the Hecke operator, \( T_M f(a) = \lambda_M f(a) \) where the eigenvalue is given by

\[
(5.16)
\]

As reviewed earlier, the eigenvalue corresponds to the Fourier coefficient of \( f(a) \) and defines the \( L \)-function of interest. Thus it is intriguing if we can compute this value. Although the expression (5.16) suggests that \( \lambda_M \) are integers, this expression is rather intuitive and not well-defined compared to that of (5.3). For example, the sum over \( m, n \in \mathbb{F}_M \) means a change of fields for \( m, n \) since these are initially defined as integers, corresponding to the winding numbers along \( \alpha \) and \( \beta \) cycles on the torus, respectively. Within the interpretation of the doubly periodic translations \( a \rightarrow a + m + in\tau_2 \) as a combination of modular transformations, this implies that we change the matrix elements of the modular group from integer to finite field, i.e., \( SL(2,\mathbb{Z}) \rightarrow SL(2,\mathbb{Z}/M\mathbb{Z}) \). Thus, the notion of the level for \( f(a) \) naturally arises from an interpretation of (5.16). In other words, in order to compute the value of (5.16) it would be suitable to consider \( f(a) \) as level \( M \) cusp forms of weight 2, \( f(a) \in S_2(\Gamma_0(M)) \).

A speculative connection to the Legendre symbol

As a digression, we now briefly discuss a speculative idea on the interpretation of the factor \((-1)^{mn}\). The scale invariance of the zero-mode coordinate in (2.8) suggests an implicit
condition \( \gcd(m, n) = 1 \). One of the simplest choices would be \((m, n) = (p, q)\) where \(p, q\) are (odd) prime numbers. Such a choice reminds us of mathematical analogies between primes and knots [20]. Previously in [8], we argue that the factor \((-1)^{mn}\) acting on \(f(a)\) can be interpreted as \((-1)^{lk(\alpha, \beta)}\) where \(lk(\alpha, \beta)\) denotes a linking number of the \(\alpha\) and \(\beta\) cycles along the torus. With the choice of \((m, n) = (p, q)\), the linking number becomes \((-1)^{\lambda(p, q)}\). Then, by use of mathematical analogies between linking numbers and Legendre symbols [20], we have

\[
(-1)^{\lambda(p, q)} \leftrightarrow \lambda_p(q) \tag{5.17}
\]

where \(\lambda_p(q)\) denotes the Legendre symbol, with \(p\) and \(q\) being odd primes. In terms of the conventional notation this can also be expressed as

\[
\lambda_p(q) = \left( \frac{q}{p} \right) = \left( \frac{p^*}{q} \right) = \lambda_q(p^*) \tag{5.18}
\]

where \(p^* = (-1)^{\frac{p-1}{2}}p\) and we have used the reciprocity law of the Legendre symbol \(\left( \frac{a}{p} \right) = \left( \frac{p^*}{q} \right)\).

Since the Legendre symbol gives a map \(\lambda_p(q) : \mathbb{F}_p \to \mathbb{C}\) it is natural to consider an action of it to the holomorphic wave function \(f(a)\) in terms of its Fourier transform:

\[
\hat{\lambda}_p := \sum_{q \in \mathbb{F}_p} \lambda_p(q) e^{\frac{2\pi i q}{p}} = \sqrt{p^*} \tag{5.19}
\]

which is known as the Gauss sum. Once we choose and fix the pair \((p, q)\) and bear in mind the above analogies, we may speculate that the action of \(\sum_{m,n}(-1)^{mn}\) on \(f(a)\) defined in the \(\mathbb{C}\)-space would be described by the Gauss sum or a normalized value of it. To make this statement a bit clearer, let us compute the eigenvalues \(\lambda_N\) for the level \(N\) cusp forms of weight 2, \(f(a) \in S_2(\Gamma_0(N))\) with \(N\) being odd primes. According to [14], such cusp forms become dimension 1 only for \(N = 11, 17, 19\) and in each case the coefficient \(\lambda_N\) for the corresponding \(L\)-function is given by \(\lambda_N = 1/\sqrt{N}\). Note that these values can be read off from a list of analytically normalized \(L\)-functions [14]. Notice also that the corresponding coefficient \(a_N\) of the \(q\)-expansion of the cusp form \(f(a) \in S_2(\Gamma_0(N))\) is given by \(a_N = 1\) for \(N = 11, 17, 19\). Thus the factor of \(1/\sqrt{N}\) may be interpreted as an overall normalization factor but other coefficients \(\lambda_m (m \neq N)\) of the \(L\)-function as a Dirichlet series are expressed as \(\lambda_m = \mathbb{Z}/\sqrt{m}\) [14]. Thus we may not consider \(1/\sqrt{N}\) as an overall normalization factor for the \(L\)-functions of \(f(a) \in S_2(\Gamma_0(N))\). Following the above discussion, we find that these values \((\lambda_N = 1/\sqrt{N})\) may be interpreted as a normalized Gauss sum \(\sqrt{N}/N = 1/\sqrt{N}\) with the choice of \(p = N^*\). We have tried to develop these ideas to understand other coefficients \(\lambda_m (m \neq N = 11, 17, 19)\) of the \(L\)-functions for \(f(a) \in S_2(\Gamma_0(N))\) but, at the present, we do not have any satisfactory explanations for these values.

**Physical interpretation of the Hecke operator**

In this section, we have considered how the Hecke operators act on the holomorphic zero-mode wave function \(f(a)\) in abelian Chern-Simons theory on the torus by use of the
relation (5.14). We first introduce the formal definition (5.1) of the Hecke operators acting on a function on a complex lattice. Applying this definition to \( f(a) \), we find the expression (5.15). This naturally gives rise to the notion of the level for \( f(a) \) as the modular form. As mentioned in (3.19), the relation (5.14) arises from invariance under gauge transformations induced by the doubly periodic translations \( a \rightarrow a + m + i \tau_2 (m, n \in \mathbb{Z}) \) of the zero-mode variable \( a \in \mathbb{C} \). Therefore, from a gauge theoretic perspective, it is straightforward that \( f(a) \) is an eigenform of the Hecke operator. In this context we can interpret the action of the Hecke operator on \( f(a) \) as a sum of the possible gauge transformations of \( f(a) \) induced by the doubly periodic translations. If \( f(a) \) is an eigenform of the Hecke operator, then there automatically exists a corresponding \( L \)-function for \( f(a) \), with the Dirichlet characters \( (\lambda_m \text{ in (5.13)}) \) given by the eigenvalues of \( f(a) \). Thus it is intriguing if we can understand the eigenvalues (5.16) from a physical perspective. We briefly sketch that part of such values may be computed by use of mathematical analogies between linking numbers and Legendre symbols [20].

6 Conclusion

In the previous paper [15] we show that the holomorphic zero-mode wave function \( f(a) \) in abelian Chern-Simons theory on the torus can be considered as a quantum version of a modular form of weight 2. Motivated by this result, in this paper we consider how a Hecke operator acts on \( f(a) \), in hope of obtaining physical interpretations of the Hecke operators and corresponding \( L \)-functions in number theory. The Hecke operators are formally defined as a sum of sublattices on which modular forms in general are defined. The modular forms can be considered as holomorphic functions on a complex lattice or a (complex) torus. Such functions generally satisfy the doubly periodic conditions.

In the first half of this paper we review that the holomorphic wave function \( f(a) \) satisfies the doubly periodic condition \( f(a) = f(a + m + i \tau_2) \) (with \( m, n \in \mathbb{Z} \) and \( \tau_2 > 0 \)) when the level number \( l \) of the Chern-Simons theory is even. To be more precise, we show that the gauge invariance condition (3.19) for the zero-mode wave function leads to the relation \( e^{i \pi l mn} f(a) = f(a + m + i \tau_2) \) in (3.22). We can then interpret \( f(a) \) as a holomorphic function defined on the complex lattice as well, with the factor of \( e^{i \pi l mn} \) representing quantum effects. Nontrivial quantum effects are given by \( l \) being an odd integer.

In the latter half of the paper, we consider an action of the Hecke operator acting on \( f(a) \) with \( l = 1 \). From the formal definition of the Hecke operator (5.1) we argue that the action of it on \( f(a) \) can be described as a sum of the gauge transformations of \( f(a) \) induced by \( a \rightarrow a + m + i \tau_2 \). In order to make sense of the resultant expression (5.15) we also argue that the notion of the level naturally arises for \( f(a) \) as a modular form. Our interpretation of the Hecke operator, i.e., as a sum over possible gauge transformations of \( f(a) \), automatically indicates that \( f(a) \) is an eigenform of the Hecke operator. This, on the other hand, guarantees the existence of the corresponding \( L \)-function for \( f(a) \) where \( f(a) \) can be seen as a level \( N \) cusp form of weight 2, \( f(a) \in S_2(\Gamma_0(N)) \).

We also present a speculative idea that eigenvalues \( \lambda_N \) (with \( N \) being an odd prime)
for such $f(a) \in S_2(\Gamma_0(N))$ may be computed by use of mathematical analogies between linking numbers and Legendre symbols [20]. According to [14], we have $\dim[S_2(\Gamma_0(N))] = 1$ for $N = 11, 17, 19$ and in these particular cases $\lambda_N$ as the Dirichlet characters of the corresponding $L$-functions are given by $\lambda_N = 1/\sqrt{N}$. We observe that these values may be interpreted as normalized versions of the Gauss sum $\hat{\lambda}_N^* = \sqrt{N}$ which can be seen as a Fourier transform of the corresponding Legendre symbol. Unfortunately, these ideas are still at a speculative stage but, hopefully, would shed some new light on physical approaches to problems in number theory.

References

[1] M. Bos and V. P. Nair, “U(1) Chern-Simons Theory and c=1 Conformal Blocks,” Phys. Lett. B 223, 61 (1989). doi:10.1016/0370-2693(89)90920-9
[2] V. P. Nair, Quantum Field Theory: A Modern Perspective, Springer (2004), see pp.515-522.
[3] V. P. Nair, “Elements of Geometric Quantization and Applications to Fields and Fluids,” arXiv:1606.06407 [hep-th].
[4] N. M. J. Woodhouse, “Geometric quantization,” Second edition, Clarendon Press, Oxford (1992) [Oxford mathematical monographs].
[5] Matthias Blau, Symplectic Geometry and Geometric Quantization, [http://www.blau.itp.unibe.ch/lecturesGQ.ps.gz] (ps.gz file).
[6] S. T. Ali and M. Englis, “Quantization methods: A Guide for physicists and analysts,” Rev. Math. Phys. 17, 391 (2005) doi:10.1142/S0129055X05002376 [math-ph/0405065].
[7] Y. Abe, “On the deconfining limit in (2+1)-dimensional Yang-Mills theory,” Nucl. Phys. B 828, 215 (2010) [arXiv:0804.3125 [hep-th]].
[8] Y. Abe, “Application of abelian holonomy formalism to the elementary theory of numbers,” J. Math. Phys. 53, 052303 (2012) [arXiv:1005.4299 [hep-th]].
[9] N. Koblitz, Introduction to Elliptic Curves and Modular Forms, Springer (1993).
[10] Ken Ono, The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series, CBMS Regional Conference Series in Mathematics Volume 102, American Mathematical Society (2004).
[11] Nobushige Kurokawa, Masato Kurihara and Takeshi Saito, Number Theory II — Iwasawa Theory and Modular Forms (in Japanese), Iwanami-shoten (2005) Tokyo.
[12] William Stein, Modular Forms, a Computational Approach, Graduate Studies in Mathematics Volume 79, American Mathematical Society (2007), [http://wstein.org/books/modform/modform/].
[13] Á. Lozano-Robledo, Elliptic Curves, Modular Forms, and Their L-functions, Student Mathematical Library, Volume 58, IAS/Park City Mathematical Subseries, American Mathematical Society (2011).
[14] The LMFDB Collaboration, The L-functions and Modular Forms Database, [http://www.lmfdb.org/].
[15] Y. Abe, “Wave functions in abelian Chern-Simons theory on the torus as modular forms of weight two,” arXiv:1711.07122 [hep-th].

[16] S. G. Rajeev, “New classical limits of quantum theories,” hep-th/0210179.

[17] R. Dijkgraaf, G. W. Moore, E. P. Verlinde and H. L. Verlinde, “Elliptic genera of symmetric products and second quantized strings,” Commun. Math. Phys. 185, 197 (1997) doi:10.1007/s002200050087 [hep-th/9608096].

[18] S. Gukov, E. Martinec, G. W. Moore and A. Strominger, “An Index for 2-D field theories with large N = 4 superconformal symmetry,” hep-th/0404023.

[19] A. Kapustin and E. Witten, “Electric-Magnetic Duality And The Geometric Langlands Program,” Commun. Num. Theor. Phys. 1, 1 (2007) doi:10.4310/CNTP.2007.v1.n1.a1 [hep-th/0604151].

[20] M. Morishita, “Analogies between Knots and Primes, 3-Manifolds and Number Rings,” arXiv:0904.3399 [math.GT]; Knots and Primes (in Japanese), Springer-Japan (2009).