DYNAMICS OF A NONLOCAL DIFFUSIVE LOGISTIC MODEL WITH FREE BOUNDARIES IN TIME PERIODIC ENVIRONMENT

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(Communicated by Bei Hu)

Abstract. In this paper we study a nonlocal diffusion model with double free boundaries in time periodic environment, which is the natural extension of the free boundary model in [17], where local diffusion is used to describe the population dispersal. We give the existence and uniqueness of global solution and consider the properties of principle eigenvalue of time-periodic parabolic-type eigenvalue problem. With the help of attractivity of time periodic solutions, we establish a spreading-vanishing dichotomy. The sharp criteria for spreading and vanishing are also obtained.

1. Introduction. In this paper we study the following nonlocal diffusion model with free boundaries in time periodic environment

\[
\begin{aligned}
\frac{du}{dt} &= d \int_{g(t)}^{h(t)} J(x-y) u(t,y) dy - du(t,x) + a(t,x) u - b(t,x) u^2, \quad t > 0, \quad g(t) < x < h(t), \\
h'(t) &= \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(x-y) u(t,x) dy dx, \quad t > 0, \\
g'(t) &= -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{h(t)} J(x-y) u(t,x) dy dx, \quad t > 0, \\
h(0) = -g(0) = h_0, \quad u(0,x) = u_0(x), \\
u(t,x) &= 0, \quad t \geq 0, \quad x \leq g(t) \text{ or } x \geq h(t),
\end{aligned}
\]

(1)

where \(x = g(t)\) and \(x = h(t)\) are the left and right moving boundaries to be determined together with \(u(t,x)\), which represents the population density of species. \(d > 0\) is the random diffusion rate, \(h_0\) and \(\mu\) are given positive constants. \(u_0 \in C([-h_0, h_0])\) and satisfies

\[
u_0(\pm h_0) = 0 \text{ and } u_0(x) > 0 \text{ in } (-h_0, h_0).
\]

(2)

We also assume that \((J)\): \(J \in C^1(\mathbb{R})\) is an nonnegative convolution kernel, supported on the interval \([-r_0, r_0]\), where \(0 < r_0 < +\infty\), and

\[
J(0) > 0, \quad \int_{-\infty}^{+\infty} J(x) dx = 1, \quad J \text{ is symmetric, sup } J < +\infty.
\]

2020 Mathematics Subject Classification. 35K51, 35R35, 92B05, 35B40.
Key words and phrases. Nonlocal diffusion, time periodic, free boundaries, spreading and vanishing.

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The coefficient functions $a(t, x)$ and $b(t, x)$ represent the intrinsic growth rate and self-limitation coefficient of the species, respectively, which satisfy the following conditions
\[
\begin{aligned}
(A) & \quad b \in C(\mathbb{R} \times \mathbb{R}) \text{ and is } T \text{ periodic in } t \text{ for some } T > 0, \quad a(t, x) = \alpha(t) + \beta(x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \\
& \quad \text{where } \alpha : \mathbb{R} \to \mathbb{R} \text{ is a continuous } T\text{-periodic function and } \beta : \mathbb{R}^N \to \mathbb{R} \text{ is a continuous function;} \\
(B) & \quad \text{there are positive constants } C_1, C_2, \text{ such that } C_1 \leq a(t, x), b(t, x) \leq C_2, \quad \forall \quad (t, x) \in \mathbb{R} \times \mathbb{R}; \\
(C) & \quad \text{max}_{x \in \mathbb{R}^N} a^2(x) - b^T(t) \in L^1_{{\text{loc}}} (\mathbb{R}), \text{ where } a^T(x) := \frac{1}{2} \int_0^1 a(t, x) dt.
\end{aligned}
\]

The equations governing the free boundaries are described as
\[
h'(t) = \mu \int_{y(t)}^{h(t)} J(x - y)u(t, x)dydx \quad \text{and} \quad g'(t) = -\mu \int_{y(t)}^{h(t)} J(x - y)u(t, x)dydx,
\]
whose ecological background and deduction can refer to [2, 4].

When the functions $a(t, x)$ and $b(t, x)$ are independent of time and spatial variable, problem (1) was investigated by Cao, Du, Li and Li [4]. They proved that this nonlocal problem has a unique solution defined for all time and studied its long-time dynamical behavior when the growth function is of Fisher-KPP type. Moreover, the spreading-vanishing results in the homogeneous environment were also obtained.

In the special case that $a = b = 0$, problem (1) was studied by Cortázar, Quirós and Wolanski [2]. The existence and uniqueness for this problem posed on the line, and on the half-line with constant Dirichlet data, and in the radial case in several dimensions were proved. The asymptotic behavior of the solution to (1) with $a = b = 0$ is also established in [2], which is naturally different from [4].

Furthermore, [20, 28] also considered two species nonlocal diffusion systems with free boundaries. They further developed the ideas and techniques in [4] to study this problem. Other works for nonlocal diffusion models, please refer to [1, 3, 5, 6, 7, 8, 10, 11, 22, 23, 24, 25, 27, 29, 30, 31, 32, 33] and the references therein for more details.

When the nonlocal diffusion term $\int_{y(t)}^{h(t)} J(x - y)u(t, y)dy = u(t, x)$ is replaced by the usual local diffusion term $u_{xx}$, the local diffusion periodic models with free boundary(ies) have been studied extensively. For example, [17, 37] studied the local diffusion logistic model with a free boundary in time-periodic environment. [17] proved a spreading-vanishing dichotomy, namely, either

(i): Spreading : $\lim_{t \to +\infty} h(t) = +\infty$ and
\[
\lim_{t \to +\infty} |u(t, r) - p(t, r)| = 0 \text{ locally uniformly for } r \in [0, +\infty),
\]
where $p(t, |x|)$ is the unique positive time periodic solution of problem
\[
\begin{align*}
p_t - d\Delta p &= p(a(t, |x|) - b(t, |x|)p), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^N,
\end{align*}
\]
or
(ii): Vanishing: $\lim_{t \to +\infty} h(t) \leq R$ and $\lim_{t \to +\infty} \|u(t, x)\|_{C[0, h(t)]} = 0$, where $R^* > 0$ is the unique value such that the following linear problem has a positive time-periodic solution when $R^* = R$:
\[
\begin{align*}
\phi_t - d\Delta \phi &= a(t, |x|)\phi, \quad t \in \mathbb{R}, \quad |x| < R, \\
\phi &= 0, \quad t \in \mathbb{R}, \quad |x| = R.
\end{align*}
\]
Moreover, by introducing new ideas and techniques they also determine the spreading speed. For time independent environment and the spatial environment is assumed to be asymptotically periodic at infinity in the radial direction, such free boundary problem has been investigated in [16, 18]. Ding, Du and Liang [13, 14]
studied a free boundary problem with monostable reaction term in time-space periodic media. [13] shown that the spreading-vanishing dichotomy is retained in time-space periodic environment. In particular, when spreading happens, they shown that the solution of this free boundary problem is attracted by the associated time-space periodic solution. Following the work of [13], the spreading-vanishing dichotomy is retained in time-space periodic environment. In particulary, when spreading happens, they shown that the solution of this free boundary problem is attracted by the associated time-space periodic solution. Following the work of [13], the existence of the asymptotic spreading speed when spreading happens was proved in [14]. Two-species Lotka-Volterra type competition problem with a free boundary in the heterogeneous time-periodic environment have also been studied in [12, 38]. The study of free boundary problems for other type biological model, please refer to [15, 19, 26, 35, 36, 39, 40, 41] and references cited therein.

The main intention of this paper is to analyze the long-time behavior of this nonlocal diffusion model (1). We give the existence and uniqueness results of problem (1). Furthermore, in order to discuss the long-time behavior to this problem, we study the properties of principle eigenvalue for time-periodic parabolic-type eigenvalue problem. Due to the intrinsic growth rate \(a(t, x)\) is dependent of time and spatial variable, it is difficult to get precise limit properties of the principle eigenvalue when the length of the interval tends to infinity or zero. To overcome this difficulty, we transform the parabolic-type eigenvalue problem into elliptic-type one.

To prove the long-time behavior of this nonlocal diffusion model (1), it is necessary to study attractivity of the periodic solution to associated problem. Recently, these results have been established. The attractivity of associated periodic solution in bounded domain was established in [34] (see Proposition 3.4). Moreover, the similar result of the associated Cauchy problem has been proved in [21] (see Proposition 3.5). With the help of these results, we have the following dynamic conclusions.

**Theorem 1.1.** (Spreading-vanishing dichotomy). Suppose \((J), (A), (B)\) and \((C)\) hold, and \(u_0\) satisfies (2). Let \((u, g, h)\) be the unique solution of problem (1). Then one of the following alternatives must happen for (1):

(i): Vanishing: \(\lim_{t \to +\infty} (g(t), h(t)) = (g_\infty, h_\infty)\) is a finite interval and we have
\[
\lim_{t \to +\infty} \|u(t, x)\|_{C([g(t), h(t)])} = 0.
\]

(ii): Spreading: \(\lim_{t \to +\infty} (g(t), h(t)) = \mathbb{R}\) and we have
\[
\lim_{t \to +\infty} u(t + nT, x) = w(t, x) \text{ locally uniformly in } \mathbb{R},
\]
where \(w(t, x)\) is the unique positive time-periodic solution of the equation
\[
w_t = d\int_{\mathbb{R}} J(x - y)w(t, y)dy - w(t, x) + a(t, x)w - b(t, x)w^2, \quad t \in \mathbb{R}, \; x \in \mathbb{R}.
\]

**Remark 1.1.** Theorem 1.1 extends the result of Theorem 1.2 in [17] to the nonlocal diffusion problem. In particularly, when \(a(t, x)\) and \(b(t, x)\) are time-space periodic functions, combining the Theorem 2.3 (2) and Remark 2.4 (2) of [21], we can prove that the Spreading-vanishing dichotomy is retained in time-space periodic environment using similar arguments.

It is worth mentioning that, due to \(a(t, x)\) and \(b(t, x)\) are time-periodic functions, we cannot directly use the approach in [4] to prove asymptotic behavior of solution to the problem (1) when spreading occurs. We establish the maximum principle for Cauchy problem in \(\mathbb{R}\) (see Theorem 3.3). Using this result, we prove the existence
and uniqueness of positive time-periodic solution of the equation (3). Combining this fact and Proposition 3.5, we obtain the long-time dynamical behavior in spreading case.

Moreover, we also consider the following criteria for spreading and vanishing.

**Theorem 1.2.** (Spreading-vanishing criteria). Under the conditions of Theorem 1.1,

1. If \( \min_{x \in \mathbb{R}} a_T(x) > d \), spreading always happens.
2. If \( \max_{x \in \mathbb{R}} a_T(x) < d \), there exists a unique \( l^* > 0 \) such that spreading always happens if \( 2h_0 \geq l^* \); and for \( 2h_0 < l^* \), there exists \( \mu^* > 0 \), such that vanishing happens when \( 0 < \mu \leq \mu^* \) and spreading happens when \( \mu > \mu^* \).
3. If \( \min_{x \in \mathbb{R}} a_T(x) \leq d \leq \max_{x \in \mathbb{R}} a_T(x) \), there exists \( l_* > 0 \), such that spreading always happens if \( 2h_0 \geq l_* \).

The rest of this paper is organized as follows. In Section 2, we state the existence and uniqueness results to problem (1). Moreover, some results on the principle eigenvalue of time-periodic parabolic-type eigenvalue problem are given. Section 3 is devoted to the proof of spreading-vanishing dichotomy. In section 4, some criteria for spreading and vanishing are also given there.

2. Preliminary. In this section, we prove the existence and uniqueness of the global solution to problem (1) and prepare several lemmas the results which will be used in the following sections.

For convenience, we first introduce some notations. For given \( h_0, \tau > 0 \) we define
\[
H_{h_0, \tau} := \left\{ h \in C([0, \tau]) : h(0) = h_0, \inf_{0 \leq t_1 < t_2 \leq \tau} \frac{h(t_2) - h(t_1)}{t_2 - t_1} > 0 \right\},
\]
\[
G_{h_0, \tau} := \left\{ g \in C([0, \tau]) : -g \in H_{h_0, \tau} \right\},
\]
\[
C_0([-h_0, h_0]) := \left\{ u \in C([-h_0, h_0]) : u(-h_0) = u(h_0) = 0 \right\}.
\]

For \( g \in G_{h_0, \tau}, h \in H_{h_0, \tau} \) and \( u_0 \in C_0([-h_0, h_0]) \) nonnegative, we define
\[
\Omega_{g, h} := \left\{ (t, x) \in \mathbb{R}^2 : 0 < t \leq \tau, g(t) < x < h(t) \right\},
\]
\[
X = X_{u_0, g, h} := \left\{ \phi \in C(\overline{\Omega_{g, h}}) : \phi \geq 0 \text{ in } \Omega_{g, h}, \phi(0, x) = u_0(x) \text{ for } x \in [-h_0, h_0] \text{ and } \phi(t, g(t)) = \phi(t, h(t)) = 0 \text{ for } 0 \leq t \leq \tau \right\}.
\]

We note that the growth function \( a(t, x)u(t, x) - b(t, x)u(t, x)^2 \) in (1) satisfies the conditions (f1) and (f2) of [4]. By a similar argument as Theorem 2.1 of [4], we can directly obtain the following existence and uniqueness result.

**Theorem 2.1.** Assume that (J), (A) and (B) hold. Then for any given \( h_0 > 0 \) and \( u_0(x) \) satisfying (2), problem (1) admits a unique solution \( (u(t, x), g(t), h(t)) \) defined for all \( t > 0 \). Moreover, for any \( \tau > 0 \),
\[
(u, g, h) \in X_{u_0, g, h} \times H_{h_0, \tau} \times H_{h_0, \tau},
\]
\[
0 < U(x, t) \leq C := \max \left\{ \max_{-h_0 \leq x \leq h_0} u_0(x), \frac{C_2}{C_1} \right\} \text{ for } t \in (0, \tau), \ x \in (g(t), h(t)).
\]

(4)

The following Comparison Principle will be used frequently in our analysis to follow.
Lemma 2.2. ([4] Theorem 3.1). Assume that (J), (A) and (B) hold, and \( u_0 \) satisfies (2). For \( t \in (0, +\infty) \), suppose that \( g, h \in C([0, T]) \) and \( \pi \in C(\overline{\Omega}, \overline{\Gamma}, \tau) \) satisfies

\[
\begin{align*}
\pi_t &\geq d \int_{\overline{g(t)}}^{h(t)} J(x-y) \pi(y, t) dy - d\pi(t, x) + a(t)\pi - b(t)\pi^2, \quad 0 < t \leq \tau, \ g(t) < x < h(t), \\
\pi(t, \overline{g(t)}) &\geq 0, \ \pi(t, \overline{h(t)}) \geq 0, \quad t > 0, \\
\overline{R}(t) &= \mu \int_{\overline{\pi(t)}}^{\overline{\pi(t)}} J(x-y) \overline{\pi}(t, x) dydx, \quad 0 < t \leq \tau, \\
\overline{\pi}'(t) &= -\mu \int_{\overline{\pi(t)}}^{\overline{\pi(t)}} J(x-y) \overline{\pi}(t, x) dydx, \quad 0 < t \leq \tau, \\
\overline{\pi}(0, x) &\leq u_0(x), \ \overline{\pi}(0) \geq h_0, \ \overline{\pi}(0) \leq -h_0, \quad -h_0 \leq x \leq h_0.
\end{align*}
\]

Then the unique positive solution \((u, g, h)\) of problem (1) satisfies

\[ u(t, x) \leq \overline{\pi}(t, x), \ g(t) \geq \overline{g}(t) \] and \( h(t) \leq \overline{h}(t) \) for \( 0 \leq t \leq \tau \) and \( x \in \mathbb{R} \). (6)

The triplet \((\overline{\pi}, \overline{g}, \overline{h})\) above is called an upper solution of (1). We can also define a lower solution and obtain analogous results by reversing all the inequalities (5) and (6).

Now we fix \( u_0, d \) and examine the dependence of the solution on \( \mu \), we write \((u_\mu, g_\mu, h_\mu)\) to emphasize the dependence. The following result is a direct consequence of comparison principle.

Corollary 2.3. (Corollary 3.2, [4]) Assume that (J) holds. If \( \mu_1 \leq \mu_2 \), we have \( u_\mu_1(t, x) \leq u_\mu_2(t, x) \) for \( t > 0 \), \( x \in (g_{\mu_1}(t), h_{\mu_1}(t)) \) and \( h_{\mu_1}(t) \leq h_{\mu_2}(t) \), \( g_{\mu_1}(t) \geq g_{\mu_2}(t) \) for \( t > 0 \).

In order to analyze the long-time behavior of this nonlocal diffusion model (1), we need to study the associated eigenvalue problem and analyze the properties of its principle eigenvalue. Hereafter, we assume \( \Omega \) is a finite open interval in \( \mathbb{R} \), and \(|\Omega|\) denotes its length.

The linear operator \( \mathcal{L}_\Omega + a(t, x) : C(\mathbb{R} \times \overline{\Omega}) \mapsto C(\mathbb{R} \times \overline{\Omega}) \) defined by

\[
(\mathcal{L}_\Omega + a(t, x))\phi(t, x) := -\phi_t(t, x) + d \int_\Omega J(x-y) \phi(t, y) dy - \phi(t, x) + a(t)\phi(t, x), \ (t, x) \in \mathbb{R} \times \overline{\Omega},
\]

where \( a(t, x) \) satisfies condition (A).

For convenience, we define the spaces \( X_\Omega, X^+_\Omega \) and \( X^{++}_\Omega \) as follows

\[
X_\Omega = \{ \phi \in C^{1,0}(\mathbb{R} \times \overline{\Omega}) : \phi(t + T, x) = \phi(t, x), \ (t, x) \in \mathbb{R} \times \overline{\Omega} \},
\]

\[
X^+_\Omega = \{ \phi \in X_\Omega : \phi(t, x) \geq 0, \ (t, x) \in \mathbb{R} \times \overline{\Omega} \},
\]

\[
X^{++}_\Omega = \{ \phi \in X_\Omega : \phi(t, x) > 0, \ (t, x) \in \mathbb{R} \times \overline{\Omega} \}.
\]

Define

\[
\lambda_p(-\mathcal{L}_\Omega - a(t, x)) := \sup\{ \lambda \in \mathbb{R} : \exists \phi \in X^+_\Omega \ s.t. \ (\mathcal{L}_\Omega + a(t, x))\phi + \lambda\phi \leq 0 \ in \mathbb{R} \times \overline{\Omega} \}.
\]

(7)

\[
\lambda_1(-\mathcal{L}_\Omega - a(t, x)) = \inf\{ \text{Re} \lambda : \lambda \in \sigma(-\mathcal{L}_\Omega - a(t, x)) \}
\]

where \( \sigma(-\mathcal{L}_\Omega - a(t, x)) \) is the spectrum of \(-\mathcal{L}_\Omega - a(t, x)\). Since \( a(t, x) \) satisfies (C), thus using Theorem A in [34], we obtain that \( \lambda_1(-\mathcal{L}_\Omega - a(t, x)) \) is the principle eigenvalue of \(-\mathcal{L}_\Omega - a(t, x)\) and

\[
\lambda_1(-\mathcal{L}_\Omega - a(t, x)) = \lambda_p(-\mathcal{L}_\Omega - a(t, x)).
\]
Therefore there exists a principle eigenfunction $\phi \in X^{++}_\Omega$ such that
\[
(-L_\Omega - a(t, x))[\phi] = \lambda_p(-L_\Omega - a(t, x))\phi \quad \text{for} \quad (t, x) \in \mathbb{R} \times \overline{\Omega}.
\] (8)
Since $a(t, x) = \alpha(t) + \beta(x)$. Set $\alpha_T := \frac{1}{T} \int_0^T \alpha(t) dt$, then we have $a_T(x) = \alpha_T + \beta(x)$ for $x \in \overline{\Omega}$. We define
\[
\psi(t, x) = e^{-\int_0^t (\alpha(t) - \alpha_T)\, dt} \phi(t, x) \quad \text{for} \quad (t, x) \in \mathbb{R} \times \overline{\Omega}.
\]
It is easy to see that $\psi(t, x) \in X^{++}_\Omega$. Multiplying $e^{-\int_0^t (\alpha(t) - \alpha_T)\, dt}$ on both sides of the equation (8), we obtain
\[
-\psi_t(t, x) + d \left[ \int_{\Omega} J(x - y)\psi_t(y)\, dy - \psi(t, x) \right] + a_T(x)\psi(t, x) + \lambda_p(-L_\Omega - a(t, x))\psi(t, x) = 0
\]
for $(t, x) \in \mathbb{R} \times \overline{\Omega}$.
Setting $\psi_T(x) = \frac{1}{T} \int_0^T \psi(t, x)\, dt$ for $x \in \overline{\Omega}$, and integrating the above equation over $[0, T]$ with respect to $t$ results in
\[
d \left[ \int_{\Omega} J(x - y)\psi_T(y)\, dy - \psi_T(x) \right] + a_T(x)\psi_T(x) + \lambda_p(-L_\Omega - a(t, x))\psi_T(x) = 0, \quad x \in \overline{\Omega}.
\]
That is, $-\lambda_p(-L_\Omega - a(t, x))$ is the principle eigenvalue of the following nonlocal elliptic-type operator $L_\Omega + a_T(x) : C(\overline{\Omega}) \rightarrow C(\overline{\Omega})$ defined by
\[
(L_\Omega + a_T(x))[\varphi](x) := d \left[ \int_{\Omega} J(x - y)\varphi(y)\, dy - \varphi(x) \right] + a_T(x)\varphi(x)
\] (9)
with an eigenfunction $\psi_T \in X^{++}_\Omega$. Let
\[
\lambda_p(L_\Omega + a_T(x)) := \sup \{ \lambda \in \mathbb{R} : \exists \psi_T \in X^{++}_\Omega \text{ s.t. } (L_\Omega + a_T(x))[\psi_T] \geq \lambda \psi_T \text{ in } \Omega \}
\]
be the principle eigenvalue of $L_\Omega + a_T(x)$, then we have
\[
-\lambda_p(-L_\Omega - a(t, x)) = \lambda_p(L_\Omega + a_T(x)).
\] (10)
Now, we will discuss the properties of $\lambda_p(L_\Omega + a_T(x))$. Obviously, $\lambda_p(L_\Omega + a_T(x))$ is monotone with respect to the term $a_T(x)$.

**Proposition 2.4.** Fix $\Omega$ and assume that $\overline{\sigma_T(x)} \geq \underline{a_T}(x)$, then
\[
\lambda_p(L_\Omega + \underline{a_T}(x)) \leq \lambda_p(L_\Omega + \overline{\sigma_T}(x)).
\]

**Proof.** It follows from the definition of $\lambda_p(L_\Omega + a_T(x))$. First we claim that
\[
\lambda_p(L_\Omega + \underline{a_T}(x)) \leq \lambda_p(L_\Omega + \overline{\sigma_T}(x)).
\]
It is sufficient to prove the inequality
\[
\lambda \leq \lambda_p(L_\Omega + \overline{\sigma_T}(x))
\]
for any $\lambda < \lambda_p(L_\Omega + \underline{a_T}(x))$. Let us fix $\lambda < \lambda_p(L_\Omega + \underline{a_T}(x))$. Then by definition of $\lambda_p(L_\Omega + \underline{a_T}(x))$ there exists a positive function $\phi \in X^{++}_\Omega$ such that
\[
(L_\Omega + \underline{a_T}(x))[\phi] - \lambda \phi \geq 0.
\]
Since $\overline{\sigma_T}(x) \geq \underline{a_T}(x)$, we obtain that
\[
(L_\Omega + \overline{\sigma_T}(x))[\phi] - \lambda \phi \geq (L_\Omega + \underline{a_T}(x))[\phi] - \lambda \phi \geq 0.
\]
Therefore by definition of $\lambda_p(L_\Omega + \overline{\sigma_T}(x))$ we have $\lambda \leq \lambda_p(L_\Omega + \overline{\sigma_T}(x))$. Hence $\lambda_p(L_\Omega + \underline{a_T}(x)) \leq \lambda_p(L_\Omega + \overline{\sigma_T}(x))$. \(\square\)
Combining (10) and Proposition 2.4, we have
\[ -\lambda_p(L_\Omega + \max_{x \in \Omega} a_T(x)) \leq \lambda_p(-L_\Omega - a(t, x)) \leq -\lambda_p(L_\Omega + \min_{x \in \Omega} a_T(x)). \]  
(11)

In order to get the properties of \( \lambda_p(L_\Omega + a_T(x)) \). Let us recall the following proposition.

**Proposition 2.5.** ([4] Proposition 3.4). Assume that the kernel \( J \) satisfies (J) and \( a_0 \) is a constant. Then the following hold true:

(i): \( \lambda_p(L_\Omega + a_0) \) is strictly increasing and continuous in \( |\Omega| = l_2 - l_1 \);

(ii): \( \lim_{|\Omega| \to +\infty} \lambda_p(L_\Omega + a_0) = a_0 \);

(iii): \( \lim_{|\Omega| \to 0} \lambda_p(L_\Omega + a_0) = a_0 - d \).

Combining (11) and Proposition 2.5, we obtain that
\[ -\max_{x \in \mathbb{R}} a_T(x) \leq \lim_{|\Omega| \to +\infty} \lambda_p(-L_\Omega - a(t, x)) \leq -\min_{x \in \mathbb{R}} a_T(x), \]  
(12)
and
\[ d - \max_{x \in \mathbb{R}} a_T(x) \leq \lim_{|\Omega| \to 0} \lambda_p(-L_\Omega - a(t, x)) \leq d - \min_{x \in \mathbb{R}} a_T(x). \]  
(13)

3. Long time behavior of free boundary problem. This section is devoted to prove the asymptotic behavior. It follows from Theorem 2.1 that \( x = g(t) \) is monotonic decreasing and \( x = h(t) \) is monotonic increasing. Therefore, there exist \( g_\infty \in [-\infty, 0) \) and \( h_\infty \in (0, +\infty] \) such that \( \lim_{t \to +\infty} g(t) = g_\infty \) and \( \lim_{t \to +\infty} h(t) = h_\infty \).

We first present two Maximum Principle, which is an analogue of and Lemma 3.3 and Lemma 2.2 in [4]. We omitted the proof here.

**Lemma 3.1.** ([4], Lemma 3.3). Suppose that (J) holds, and \( h_0, \tau > 0 \). Assume that \( u(t, x) \) as well as \( u(t, x) \) are continuous in \( \Omega_0 := [0, \tau] \times [-R, R] \), and for some \( c \in L^\infty(\Omega_0) \),
\[
\begin{align*}
& \begin{cases} 
 & u_t \geq d \int_{-R}^{R} J(x-y)u(t, y)dy - du(t, x) + c(t, x)u, \\
 & u(0, x) \geq 0,
\end{cases} \quad t \in (0, \tau], \ x \in [-R, R], \ (14)
\end{align*}
\]

Then \( u(t, x) \geq 0 \) for all \( 0 \leq t \leq \tau \) and \( x \in [-R, R] \). Moreover, if \( u(0, x) \neq 0 \) in \([-R, R]\), then \( u(t, x) > 0 \) in \((0, \tau] \times [-R, R]\).

**Lemma 3.2.** ([4], Lemma 2.2). Suppose that (J) holds, and \((g, h) \in G_{h_0, \tau} \times H_{h_0, \tau}\) for some \( h_0, \tau > 0 \). Assume that \( u(t, x) \) as well as \( u(t, x) \) are continuous in \( \overline{\Omega_{g, h}} \) and satisfies for some \( c \in L^\infty(\Omega_{g, h}) \),
\[
\begin{align*}
& \begin{cases} 
 & u_t \geq d \int_{g(t)}^{h(t)} J(x-y)u(t, y)dy - du(t, x) + c(t, x)u, \\
 & u(t, g(t)) \geq 0, \ u(t, h(t)) \geq 0, \\
 & u(0, x) \geq 0, \\
\end{cases} \quad 0 < t \leq \tau, \ g(t) < x < h(t), \ (15)
\end{align*}
\]

Then \( u(t, x) \geq 0 \) for all \( 0 \leq t \leq \tau \) and \( x \in [g(t), h(t)] \). Moreover, if \( u(0, x) \neq 0 \) in \([-h_0, h_0] \), then \( u(t, x) > 0 \) in \( \Omega_{g, h} \).

Next we prove the following strong maximum principle for Cauchy problem in \( \mathbb{R} \) which will be used in our main theorem.
Theorem 3.3. Assume that $u(t, x)$ as well as $u_t(t, x)$ are continuous in $[0, +\infty) \times \mathbb{R}$ and satisfy, for some $c \in L^\infty([0, +\infty) \times \mathbb{R})$,
\begin{align}
\begin{cases}
u_t \geq d \int J(x-y)u(t,y)dy - du(t,x) + c(t,x)u, & t > 0, \ x \in \mathbb{R}, \\
u(0, x) \geq 0, & x \in \mathbb{R}.
\end{cases}
\tag{16}
\end{align}

Then $u(t, x) \geq 0$ for all $t > 0$ and $x \in \mathbb{R}$. Moreover, if $u(0, x) \neq 0$ in $\mathbb{R}$, then $u(t, x) > 0$ for all $t > 0$ and $x \in \mathbb{R}$.

Proof. Since $c(t, x) \in L^\infty([0, +\infty) \times \mathbb{R})$, there exists constant $\gamma > 0$ such that $\gamma + c(t, x) - d > 0$ in $[0, +\infty) \times \mathbb{R}$. Let $w := e^{\gamma t}u$, then $w$ satisfies
\begin{equation}
w_t \geq d \int J(x-y)w(t,y)dy - dw(t,x) + [c(t,x) + \gamma]w \quad \text{for } t > 0, \ x \in \mathbb{R}. \tag{17}
\end{equation}

Let us decompose $w = w_+ - w_-$, where $w_+$ is $w$ if $w \geq 0$ otherwise zero and $w_-$ is $-w$ if $w \leq 0$ otherwise zero. Multiplying the negative part $w_-$ on both sides of equation (17) and integrating over $\mathbb{R}$, we have
\begin{equation}
- \int_{\mathbb{R}} w_tw_-dx + d \int_{\mathbb{R}} \int J(x-y)[w(t,y) - w(t,x)]w_-(t,x)dydx + \int_{\mathbb{R}} [c(t,x) + \gamma]w_- dx \leq 0.
\end{equation}

Since $w_+w_- = 0$, we obtain that
\begin{equation}
\int_{\mathbb{R}} (w_-)_t w_-dx + d \int_{\mathbb{R}} \int J(x-y)[w(t,y) + w(t,x)]w_-(t,x)dydx - \int_{\mathbb{R}} [c(t,x) + \gamma]w_-^2 dx \leq 0. \tag{18}
\end{equation}

In view of $w = w_+ - w_-$ and $J$ satisfies (J), by calculation, we have
\begin{align*}
&\int_{\mathbb{R}} \int_{\mathbb{R}} J(x-y)[w(t,y) + w_-(t,x)]w_-(t,x)dydx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} J(x-y)[w_+(t,y)w_-(t,x) - w_-(t,y)w_-(t,x) + w_2(t,x)]dydx \\
&\geq \int_{\mathbb{R}} \int_{\mathbb{R}} J(x-y)[w_+(t,y)w_-(t,x) + w_2(t,x)]dydx - \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} J(x-y)[w_2(t,y) + w_2(t,x)]dydx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} J(x-y)[w_+(t,y)w_-(t,x) + w_2(t,x)]dydx - \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} J(x-y)[w_2(t,y) + w_2(t,x)]dydx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} J(x-y)[w_+(t,y)w_-(t,x)]dydx \geq 0.
\end{align*}

Substituting above inequality in (18) yields
\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}} (w_-)^2 dx \leq 2[\gamma + \|c\|_\infty] \int_{\mathbb{R}} (w_-)^2 dx. \tag{19}
\end{equation}

By Gronwall’s inequality, we immediately get $w_- \equiv 0$. Hence, $w$ is nonnegative. Therefore $u(t, x) \geq 0$ for all $t > 0$ and $x \in \mathbb{R}$.

Now assume that $u(0, x) \neq 0$ in $\mathbb{R}$. To complete the proof, it suffices to show that $u > 0$ for $[0, +\infty) \times \mathbb{R}$. Suppose that there is a point $(t^*, x^*) \in [0, +\infty) \times \mathbb{R}$ such that $u(t^*, x^*) = 0$.

First, we claim that $w(t^*, x) = 0$ for $x \in \mathbb{R}$. Otherwise, there exists $\tilde{x} \in \{x \in \mathbb{R}: w(t^*, x) = 0\}$.

Then at $(t^*, \tilde{x})$, $u(t^*, \tilde{x}) = 0$ and by (17), we get
\begin{equation*}
0 \geq w_t(t^*, \tilde{x}) \geq d \int J(\tilde{x} - y)w(t^*, y)dy > 0
\end{equation*}
according to (J). This is impossible. Thus \( w(t^*, x) = 0 \) for all \( x \in \mathbb{R} \). Hence, by (17), for \( x \in \mathbb{R} \), we have

\[
-w(0, x) = w(t^*, x) - w(0, x) \\
\geq d \int_0^{t^*} \int_\Omega J(x-y)w(t,y)dy + \int_0^{t^*} [c(t,x) + \gamma - d]w(t,x)dt \geq 0.
\]

This implies that \( w(0, x) \equiv 0 \) in \( \mathbb{R} \), which is a contradiction.

Next we state some important results that will be needed to complete the proof of the main theorem.

**Proposition 3.4.** ([34] Theorem B) Assume that (J), (A), (B) and (C) hold. Let \( u(t,x) \) be a solution of

\[
\begin{cases}
  u_t = d[\int_\Omega (J(x-y)u(t,y)dy - u(t,x))] + a(t,x)u - b(t,x)u^2, & t > 0, x \in \Omega, \\
  u(0, x) = u_0(x), & x \in \Omega,
\end{cases}
\]

where \( u_0 \in C(\Omega) \) is non-negative and not identically zero. The following statements hold: If \( \lambda_p(-L-a(t,x)) < 0 \), then the equation

\[
u_t = d[\int_\Omega (J(x-y)u(t,y)dy - u(t,x))] + a(t,x)u - b(t,x)u^2, \quad t \in \mathbb{R}, \quad x \in \Omega,
\]

admits a unique solution \( w^*(t,x) \in X_1^+ \) and there holds

\[
\|u(t, \cdot ; u_0) - w^*(t, \cdot)\|_{C(\Omega)} \to 0 \text{ as } t \to \infty.
\]

**Proposition 3.5.** ([21] Theorem 2.3). Assume that (J), (A), (B) and (C) hold. Then the equation (3) has exactly one T-periodic solution

\[
w^*(t, \cdot) \in X_1^+ := \{ u(x) \in C(\mathbb{R}) \mid u > 0 \text{ is uniformly continuous and bounded } \}
\]

Moreover, the positive T-periodic solution \( w^*(t,x) \) is globally asymptotically stable in the sense that

\[
\|w(t, \cdot ; w_0) - w^*(t, \cdot)\|_{C(\mathbb{R})} \to 0 \text{ as } t \to \infty
\]

for any initial value \( w_0 \in X_1^+ \) with \( \inf_{x \in \mathbb{R}} w_0(x) > 0 \), where \( w(t,x; w_0) \) is a solution of (3) with initial value \( w_0 \).

### 3.1. Vanishing case

In this subsection, we study the asymptotic behavior of solution to problem (1) when vanishing occurs \( (h_\infty - g_\infty < +\infty) \).

**Theorem 3.6.** Let \( (u, g, h) \) be any solution of (1). If \( h_\infty - g_\infty < +\infty \), we have

\[
\lim_{t \to +\infty} \|u(t,x)\|_{C([g(h(t)), h(t)])} = 0,
\]

and \( \lambda_p(-L_{(g_\infty, h_\infty)} - a(t,x)) \geq 0. \)

**Proof.** We first prove that \( \lambda_p(-L_{(g_\infty, h_\infty)} - a(t,x)) \geq 0. \) Suppose that \( \lambda_p(-L_{(g_\infty, h_\infty)} - a(t,x)) < 0. \) Then \( \lambda_p(-L_{(g_\infty + \varepsilon, h_\infty - \varepsilon)} - a(t,x)) < 0 \) for small \( \varepsilon > 0 \), say \( \varepsilon \in (0, \varepsilon_1). \)

Moreover, for such \( \varepsilon \), there exists \( T_\varepsilon > 0 \) such that

\[
h(t) > h_\infty - \varepsilon, \quad g(t) < g_\infty + \varepsilon \text{ for } t > T_\varepsilon.
\]
Let $w_{\varepsilon}(t, x)$ be the unique solution of the following problem

$$
\begin{cases}
    w_t = d \left[ \int_{g_{\infty} + \varepsilon}^{h_{\infty} - \varepsilon} J(x-y)w(t,y)dy - w(t,x) \right] \\
    + a(t,x)w - b(t,x)w^2, & t > T_\varepsilon, \ x \in [g_\infty + \varepsilon, h_\infty - \varepsilon], \\
    w(T_\varepsilon,x) = u(T_\varepsilon,x), & x \in [g_\infty + \varepsilon, h_\infty - \varepsilon].
\end{cases}
$$

(23)

Since $\lambda_p(-L_{(g_\infty + \varepsilon, h_\infty - \varepsilon)} - a(t,x)) < 0$, Proposition 3.4 indicates that $w_{\varepsilon}(t, x)$ satisfies

$$
\|w_{\varepsilon}(t, \cdot; u_0) - w_{\varepsilon}^*(t, \cdot)\|_{C([g_\infty + \varepsilon, h_\infty - \varepsilon])} \to 0, \ \text{as} \ t \to \infty,
$$

where $w_{\varepsilon}^*(t,x) \in X^{+ +}_{[g_\infty + \varepsilon, h_\infty - \varepsilon]}$ is the unique solution of

$$
w_t = d \left[ \int_{g_{\infty} + \varepsilon}^{h_{\infty} - \varepsilon} J(x-y)w(t,y)dy - w(t,x) \right] + a(t)w - b(t)w^2, \ t \in \mathbb{R}, \ x \in [g_\infty + \varepsilon, h_\infty - \varepsilon].
$$

By a simple comparison argument we have

$$
u(t,x) \geq w_{\varepsilon}(t,x) \text{ in } [T_\varepsilon, +\infty) \times [g_\infty + \varepsilon, h_\infty - \varepsilon].
$$

(24)

Hence, there exists $T_\varepsilon > T_\varepsilon$ such that

$$u(t,x) \geq \frac{1}{2} w_{\varepsilon}^*(t,x) > 0 \ \text{for all} \ t > T_2, x \in [g_\infty + \varepsilon, h_\infty - \varepsilon].$$

Note that since $J(0) > 0$, there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that $J(x) > \delta_0$ if $|x| < \varepsilon_0$. Thus for $0 < \varepsilon < \varepsilon_0/2$ and $t > T_2$, we have

$$h'(t) = \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(x-y)u(t,y)dydx \geq \mu \int_{h_{\infty} - \varepsilon}^{h_{\infty} - \varepsilon} \int_{h_{\infty}}^{+\infty} J(x-y)u(t,x)dydx
$$

$$\geq \mu \int_{h_{\infty} - \varepsilon_0/2}^{h_{\infty} - \varepsilon_0/2} \int_{h_{\infty}}^{+\infty} \delta_0 \frac{1}{2} w_{\varepsilon}^*(t,x)dydx > 0.
$$

This implies $h_\infty = +\infty$ a contradiction to the assumption that $h_\infty - g_\infty < +\infty$. Therefore we must have

$$\lambda_p(-L_{(g_\infty, h_\infty)} - a(t,x)) \geq 0.$$
Combining with (B), (25) and the continuity of \( \alpha(t) \), thus there exist some positive constants \( M_1, M_2 \) such that
\[
M_1 \leq b(t, x)e^{\int_0^t (\alpha(\tau) - \alpha_T) d\tau} \leq M_2.
\]

By Lemma 3.1 (comparison principle) we have \( U(t, x) \leq \overline{U}(t, x) \), where \( \overline{U}(t, x) \) is the unique solution of
\[
\begin{aligned}
&\frac{\partial U}{\partial t} = d \int_{g(t,h)}^{h(t)} J(x-y)U(t, y)dy - \overline{U}(t, x) + a_T(x)\overline{U} - M_1U^2, & t > 0, \ x \in [g_\infty, h_\infty], \\
&\overline{U}(0, x) = \overline{u}_0(x), & x \in [g_\infty, h_\infty].
\end{aligned}
\]

In view of (10) we have
\[
\lambda_p(-\mathcal{L}_{(g_\infty, h_\infty)} - a(t, x)) = -\lambda_p(\mathcal{L}_{(g_\infty, h_\infty)} + a_T(x)) \geq 0,
\]
where \( \mathcal{L}_{(g_\infty, h_\infty)} + a_T(x) \) is defined in (9). By Theorem 1.7 in [9], we yield that
\[
\lim_{t \to +\infty} \overline{U}(t, x) = 0 \text{ uniformly in } [g_\infty, h_\infty].
\]

It implies that
\[
\lim_{t \to +\infty} \overline{\pi}(t, x) = 0 \text{ uniformly in } [g_\infty, h_\infty].
\]

By Lemma 3.2, we obtain that \( u(t, x) \leq \pi(t, x) \) for \( t > 0 \) and \( x \in [g(t), h(t)] \). Hence
\[
\lim_{t \to +\infty} \|u(t, x)\|_{C([g(t), h(t)])} = 0.
\]

The proof is complete. \( \square \)

3.2. Spreading case \( (h_\infty - g_\infty = +\infty) \). In this subsection, we discuss the spreading case \( h_\infty - g_\infty = +\infty \). We first provide a lemma which asserts the equivalence of \( h_\infty - g_\infty = +\infty \) and \( g_\infty = -\infty, h_\infty = +\infty \).

**Lemma 3.7.** \( h_\infty < +\infty \) if and only if \( -g_\infty < +\infty \).

**Proof.** Arguing indirectly, we assume, without loss of generality, that \( h_\infty = +\infty \) and \( -g_\infty < +\infty \). By (12), there exists large \( h_1 > 0 \) such that
\[
\lambda_p(-\mathcal{L}_{(0, h_1)} - a(t, x)) < 0.
\]

Moreover, for any small \( \varepsilon > 0 \), there exists \( T_1 > 0 \) such that \( h(t) > h_1, g(t) < g_\infty + \varepsilon < 0 \) for \( t > T_1 \). We now consider
\[
\begin{aligned}
&\underline{u}_t = d \int_{g_\infty + \varepsilon}^{h_1} J(x-y)\underline{u}(t, y)dy - \underline{u}(t, x) + a(t, x)\underline{u} - b(t, x)\underline{u}^2, & t > T_1, \ g_\infty + \varepsilon \leq x \leq h_1, \\
&\underline{u}(T_1, g_\infty + \varepsilon) = \underline{u}(T_1, h_1) = 0, & t > T_1, \\
&\underline{u}(T_1, x) = \underline{u}(T_1, x), & g_\infty + \varepsilon \leq x \leq h_1.
\end{aligned}
\]

Since \( \lambda_p(-\mathcal{L}_{(g_\infty + \varepsilon, h_1)} - a(t, x)) < \lambda_p(-\mathcal{L}_{(0, h_1)} - a(t, x)) < 0 \), Proposition 3.4 indicates that the solution \( \underline{u}(t, x) \) of (26) satisfies
\[
\|\underline{u}(t, \cdot; u_0) - \underline{u}_s^*(t, \cdot)\|_{C([g_\infty + \varepsilon, h_1])} \to 0, \text{ as } t \to \infty,
\]
where \( \underline{u}_s^*(t, x) \in X_{[g_\infty + \varepsilon, h_1]}^{++} \) is the unique solution of
\[
\begin{aligned}
&\underline{u}_t = d \int_{g_\infty + \varepsilon}^{h_1} J(x-y)\underline{u}(t, y)dy - \underline{u}(t, x) + a(t, x)\underline{u} - b(t, x)\underline{u}^2, & t \in \mathbb{R}, \ g_\infty + \varepsilon \leq x \leq h_1.
\end{aligned}
\]

Moreover, by a simple comparison argument we have
\[
u(t, x) \geq \underline{u}(t, x) \text{ for } t > T_1 \text{ and } g_\infty + \varepsilon \leq x \leq h_1.
\]
Thus, there exits $T_2 > T_1$ such that
\[
u(t, x) \geq \frac{1}{2} \omega^*(t, x) > 0 \text{ for } t > T_2 \text{ and } g_\infty + \varepsilon \leq x \leq h_1.
\]

Note that since $J(0) > 0$, there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that $J(x) > \delta_0$ if $|x| < \varepsilon_0$. Thus for $0 < \varepsilon < \varepsilon_0/2$ and $t > T_2$, we have
\[
g'(t) = -\mu \int_{-\infty}^{\delta_0 \omega^*(t, x) dy dx} J(x-y)u(t, x) dy dx \leq -\mu \int_{g_\infty + \varepsilon/2}^{g_\infty} J(x-y)u(t, x) dy dx
\]
\[
\leq -\mu \int_{g_\infty - \varepsilon/2}^{g_\infty} \delta_0 \frac{1}{2} \omega^*(t, x) dy dx < 0.
\]

This is a contradiction to $-g_\infty < +\infty$. \hfill \Box

In the following we investigate the spreading phenomenon. To this aim, we first study the existence and uniqueness of positive $T$-periodic solutions to problem (3).

**Theorem 3.8.** Assume that (A), (B) and (C) hold. Then problem (3) has a unique positive time periodic solution $\varphi(t, x)$.

**Proof.** First, consider the following problem
\[
\begin{cases}
  w_t = d \int_{-R}^{R} J(x-y)w(t, y)dy - w(t, x) + a(t, x)w - b(t, x)w^2, & t \in \mathbb{R}, \ x \in [-R, R], \\
  w(t, x) = 0, & x \notin [-R, R].
\end{cases}
\]

(27)

By (12) there exists $R^* > 0$ such that $\lambda_p(-\mathcal{L}(-R, R) - a(t, x)) < 0$ if $R > R^*$. For such $R > R^*$, it follows from Lemma 3.4 that the equation (27) admits a unique positive $T$-periodic solution $w_R(t, x)$. Now we show that, for any $R_2 > R_1 > R^*$, there holds
\[
w_{R_2}(t, x) \geq w_{R_1}(t, x) \text{ for all } t \in \mathbb{R}, \ x \in [-R_1, R_1].
\]

(28)

To do so, we choose a non-null nonnegative function $u_{R_1, 0} \in C([-R_1, R_1])$ with $u_{R_1, 0}(-R_1) = u_{R_1, 0}(R_1) = 0$ such that $u_{R_1, 0}(x) \leq w_{R_2}(0, x)$ for all $x \in [-R_1, R_1]$. Thus, $w_{R_2}(t, x)$ is a supersolution to the problem
\[
\begin{cases}
  u_t = d \int_{-R_1}^{R_1} J(x-y)u(t, y)dy - u(t, x) + a(t, x)u - b(t, x)u^2, & t > 0, \ x \in [-R_1, R_1], \\
  u(t, x) = 0, & t > 0, \ x \notin [-R_1, R_1], \\
  u(0, x) = u_{R_1, 0}(x), & x \in [-R_1, R_1].
\end{cases}
\]

(29)

Since $\lambda_p(-\mathcal{L}(-R_1, R_1) - a(t, x)) < 0$, by Proposition 3.4, we obtain that the solution $u_{R_1}(t, x)$ of (29) satisfies
\[
limit_{n \to \infty} u_{R_1}(t + nT, x) = w_{R_1}(t, x) \text{ for } t > 0, \ x \in [-R_1, R_1].
\]

It follows from comparison principle that
\[
u_{R_1}(t + nT, x) \leq w_{R_2}(t + nT, x) = w_{R_2}(t, x) \text{ for all } t > 0, \ x \in [-R_1, R_1], \ n \in \mathbb{N}.
\]

Hence (28) holds.

Choose a sequence $\{R_i\}_{i \in \mathbb{N}} \subset (R^*, +\infty)$ with $R_i \nearrow +\infty$ as $i \to \infty$, and let $w_{R_i}(t, x)$ be the positive $T$-periodic solution to (27) with $R = R_i$. Since $a(t, x)$ and $b(t, x)$ satisfy (A) and (B), there exists $M > 1$, such that $a(t, x)M - b(t, x)M^2 \leq 0$ for $(t, x) \in \mathbb{R} \times \mathbb{R}$, hence the positive constant $M$ is a supersolution to problem
(27) with \( R = R_i \). Since \( w_i(t, x) = 0 \) for \( x \notin [-R_i, R_i] \), by the above arguments we obtain
\[
0 < w_R(t, x) \leq w_{R+1}(t, x) \leq M \text{ for } t \in \mathbb{R}, \ x \in \mathbb{R}, \ i \in \mathbb{N}.
\]
By monotone convergence theorem, we deduce that
\[
\lim_{i \to \infty} w_{R_i}(t, x) = w(t, x) \text{ for } t \in \mathbb{R}, \ x \in \mathbb{R}.
\]
It is clear that \( w(t, x) \) is a positive solution to problem (3). Hence, we show the existence of positive bounded \( T \)-periodic solution \( w(t, x) \) for problem (3).

The uniqueness follows directly from Theorem 3.3. Assume that \( w_1(t, x) \) and \( w_2(t, x) \) are two positive bounded \( T \)-periodic solution of (3). By Theorem 3.3, we have \( w_1 \leq w_2 \). As \( w_1 \) and \( w_2 \) play a symmetric role, one has \( w_1 \equiv w_2 \). This proof is complete. \( \square \)

**Theorem 3.9.** Let \((u, g, h)\) be the solution of (1). If \( h_\infty = -g_\infty = \infty \), then
\[
\lim_{n \to \infty} u(t + nT, x) = w(t, x)
\]
locally uniformly in \( \mathbb{R} \), where \( w(t, x) \) is the unique positive \( T \)-periodic solution of equation (3).

**Proof.** To prove (30), we first consider the following problem
\[
\begin{aligned}
\begin{cases}
v_i = d \left[ \int_{\mathbb{R}} J(x-y)v(t, y)dy - v(t, x) \right] + a(t, x)v - b(t, x)v^2, & t \in \mathbb{R}, \ x \in [-R, R], \\
v(t, x) = 0, & t \in \mathbb{R}, \ x \notin [-R, R].
\end{cases}
\end{aligned}
\]
(31)
By (12), there exists \( R^* > 0 \) such that \( \lambda_p(-\mathcal{L}_{(-R, R)} - a(t, x)) < 0 \) if \( R > R^* \). For such \( R > R^* \), it follows from Proposition 3.4 that the equation (31) admits a unique positive \( T \)-periodic solution \( v_R(t, x) \). Since (3) has a unique positive \( T \)-periodic solution \( w(t, x) \), similarly to the proof of Theorem 3.8, then we have
\[
v_R(t, x) \to w(t, x) \text{ locally uniformly for } (t, x) \in \mathbb{R} \times \mathbb{R} \text{ as } R \to \infty.
\]
(32)
Choose a sequence \( \{R_i\}_{i \in \mathbb{N}} \subset (R^*, +\infty) \) with \( R_i \to \infty \) as \( i \to \infty \), we have \( \lambda_p(-\mathcal{L}_{(-R_i, R_i)} - a(t, x)) < 0 \) for all \( i \in \mathbb{N} \). Moreover, we can find \( T_i > 0 \) such that \( h(t) \geq R_i \) and \( g(t) \leq -R_i \) for \( t \geq T_i \). Let \( V_i(t, x) \) be a solution of following problem
\[
\begin{aligned}
\begin{cases}
V_i = d \left[ \int_{-R_i}^{R_i} J(x-y)V(t, y)dy - V(t, x) \right] + a(t, x)V - b(t, x)V^2, & t \geq T_i, \ x \in [-R_i, R_i], \\
V(T_i, x) = u(T_i, x), & x \in [-R_i, R_i].
\end{cases}
\end{aligned}
\]
(33)
Since \( \lambda_p(-\mathcal{L}_{(-R_i, R_i)} - a(t, x)) < 0 \), by Proposition 3.4, we obtain that
\[
V_i(t + nT, x) \to v_{R_i}(t, x), \text{ as } n \to +\infty
\]
uniformly for \((t, x) \in \mathbb{R} \times [-R_i, R_i]\), where \( v_{R_i}(t, x) \) is the positive \( T \)-periodic solution to (31) with \( R = R_i \). By Lemma 3.1, we have
\[
V_i(t, x) \leq u(t, x) \text{ for } t \geq T_i, \ x \in [-R_i, R_i].
\]
Therefore,
\[
\liminf_{n \to \infty} u(t + nT, x) \geq v_{R_i}(t, x) \text{ uniformly for } (t, x) \in \mathbb{R} \times [-R_i, R_i].
\]
Setting \( i \to \infty \), in view of (32) we obtain
\[
\liminf_{n \to \infty} u(t + nT, x) \geq w(t, x) \text{ locally uniformly for } (t, x) \in [0, T] \times \mathbb{R}.
\]
(34)
On the other hand, we will prove
\[ \limsup_{n \to \infty} u(t + nT, x) \leq w(t, x) \text{ locally uniformly for } (t, x) \in [0, T] \times \mathbb{R}. \]
Let \( \tilde{w}(t, x) \) be a solution of
\[
\begin{cases}
  w_t = d \int_{\mathbb{R}} J(x - y)w(t, y)dy - w(t, x) + a(t, x)w - b(t, x)w^2, & t > 0, \ x \in \mathbb{R}, \\
  w(0, x) = \|u_0\|_{\infty}, & x \in \mathbb{R}.
\end{cases}
\]
By Lemma 3.2, we have \( \tilde{w}(t, x) \geq u(t, x) \) for \( t > 0, \ x \in (g(t), h(t)) \). Moreover, according to Proposition 3.5, we obtain that
\[ \tilde{w}(t + nT, x) \to w(t, x), \text{ as } n \to +\infty \]
uniformly for \( (t, x) \in \mathbb{R} \times \mathbb{R} \). Therefore,
\[ \limsup_{n \to \infty} u(t + nT, x) \leq w(t, x) \text{ locally uniformly for } (t, x) \in [0, T] \times \mathbb{R}. \] (35)
Clearly, (30) is a consequence of (34) and (35).

Combining Theorem 3.6 and 3.9, we immediately obtain Theorem 1.1.

4. The criteria governing spreading and vanishing. In this section, we look for criteria guaranteeing spreading or vanishing for (1). In view of (13) we see that if
\[ \min_{x \in \mathbb{R}} a_T(x) > d, \] (36)
then \( \lambda_p(-L_{(-R,R)} - a(t, x)) < 0 \) for any finite interval \( (-R, R) \). Combining this with Theorem 3.6 we immediately obtain the following conclusion.

Theorem 4.1. When (36) holds, spreading always happens for (1).

Next we consider the case
\[ \max_{x \in \mathbb{R}} a_T(x) < d. \] (37)
In this case, by (12) and (13), there exists \( l^* > 0 \) such that
\[ \lambda_p(-L_{\Omega} - a(t, x)) = 0 \text{ if } |\Omega| = l^*, \lambda_p(-L_{\Omega} - a(t, x)) < 0 \text{ if } |\Omega| > l^*, \lambda_p(-L_{\Omega} - a(t, x)) > 0 \text{ if } |\Omega| < l^*, \]
where \( \Omega \) stands for a finite open interval in \( \mathbb{R} \), and \( |\Omega| \) denotes its length.

Theorem 4.2. Suppose that (37) holds. If \( 2h_0 \geq l^* \) then \( h_\infty - g_\infty = +\infty \). If \( 2h_0 < l^* \), then there exists \( \mu \geq 0 \) such that \( h_\infty - g_\infty = +\infty \) when \( \mu > \mu \).

Proof. If \( 2h_0 \geq l^* \) and \( h_\infty - g_\infty < +\infty \), then \([g_\infty, h_\infty]\) is a finite interval with length strictly bigger than \( 2h_0 \geq l^* \). Therefore \( \lambda_p(-L_{(g_\infty, h_\infty)} - a(t, x)) < 0 \), contradicting conclusion in Theorem 3.6. Thus when \( 2h_0 \geq l^* \), \( h_\infty - g_\infty = +\infty \). By Lemma 3.7, \( 2h_0 \geq l^* \) implies \( g_\infty = -\infty \) and \( h_\infty = +\infty \).

We now consider the case \( 2h_0 < l^* \), suppose that for any \( \mu \geq 0 \), \( h_\infty - g_\infty < +\infty \), we will derive a contradiction.

In view of Theorem 3.6, we have \( \lambda_p(-L_{(g_\infty, h_\infty)} - a(t, x)) \geq 0 \). This implies that \( h_\infty - g_\infty \leq l^* \).

To stress the dependence on \( \mu \), let \((u_\mu, g_\mu, h_\mu)\) denote the solution of (1). By Corollary 2.3, \( u_\mu, -g_\mu, h_\mu \) are increasing in \( \mu > 0 \). Also denote
\[ h_{\mu, \infty} := \lim_{t \to +\infty} h_\mu(t), g_{\mu, \infty} := \lim_{t \to +\infty} g_\mu(t). \]
Obviously, both $h_{\mu,\infty}$ and $-g_{\mu,\infty}$ are increasing in $\mu$. Denote

$$H_\infty := \lim_{\mu \to +\infty} h_{\mu,\infty}, \quad G_\infty := \lim_{\mu \to +\infty} g_{\mu,\infty}.$$ 

Recall that since $J(0) > 0$, there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that $J(x) > \delta_0$ if $|x| < \varepsilon_0$. Then exist $\mu_1, t_1$ such that for $\mu \geq \mu_1$, $t \geq t_1$, we have $g_{\mu,\infty} < G_\infty + \varepsilon_0/4$ and $h_{\mu,\infty} > H_\infty - \varepsilon_0/4$. Thus it follows that

$$\mu = \left( \int_{t_1}^{t_1+1} \int_{h_{\mu_1}(\tau)}^{h_{\mu_1}(\tau)+\varepsilon_0/4} J(x-y)u_{\mu_1}(\tau,x)dyd\tau \right) \geq \left( \int_{t_1}^{t_1+1} \int_{h_{\mu_1}(\tau)}^{h_{\mu_1}(\tau)+\varepsilon_0/4} J(x-y)u_{\mu_1}(\tau,x)dyd\tau \right) - \left( \int_{t_1}^{t_1+1} \int_{h_{\mu_1}(\tau)+\varepsilon_0/4}^{h_{\mu_1}(\tau)+\varepsilon_0/4} J(x-y)u_{\mu_1}(\tau,x)dyd\tau \right)$$

which is clearly a contradiction. The above argument also shows that if

$$\mu \geq \mu := \max \left\{ 1, \left( \int_{t_1}^{t_1+1} \int_{h_{\mu_1}(\tau)}^{h_{\mu_1}(\tau)+\varepsilon_0/2} u_{\mu_1}(\tau,x)dyd\tau \right) \right\},$$

then $h_\infty - g_\infty = +\infty$. Therefore $h_\infty - g_\infty = +\infty$ for $\mu > \mu_\infty$. 

**Theorem 4.3.** Suppose that (37) holds. If $2h_0 < l^*$, then there exists $\overline{\mu} > 0$ such that $h_\infty - g_\infty < +\infty$ if $\mu \leq \overline{\mu}$

**Proof.** Since $2h_0 < l^*$, we fix $h_1 \in (h_0, l^*/2)$, and consider the following problem

$$\begin{cases}
w_t = d \int_{-h_1}^{h_1} J(x-y)w(t,y)dy - w(t,x) = a(t,x)w - b(t,x)w^2, & t > 0, x \in [-h_1, h_1], \\
w(0,x) = u_0(x), & x \in [-h_1, h_1], \\
w(0,0) = 0, & x \in [-h_1, h_0] \cup (h_0, h_1],
\end{cases}$$

and denote its unique solution by $\hat{w}(t,x)$. The choice of $h_1$ guarantees that

$$\hat{\lambda}_p := \lambda_p(-\mathcal{L}(-h_1, h_1) - a(t,x)) > 0.$$ 

Let $\phi_1 > 0$ be the corresponding normalized eigenfunction of $\hat{\lambda}_p$, namely $\|\phi_1\|_\infty = 1$

and

$$(\mathcal{L}(-h_1, h_1) + a(t,x))[\phi_1](t,x) + \hat{\lambda}_p \phi_1(t,x) = 0 \text{ for } t > 0, x \in [-h_1, h_1].$$

By direct calculations, we obtain

$$\hat{w}_t(t,x) = d \int_{-h_1}^{h_1} J(x-y)\hat{w}(t,y)dy - d\hat{w}(t,x) + a(t,x)w - b(t,x)w^2$$

$$\leq d \int_{-h_1}^{h_1} J(x-y)\hat{w}(t,y)dy - d\hat{w}(t,x) + a(t,x)w.$$
On the other hand, for $C_1 > 0$ to be determined later and $W_1 = C_1 e^{-\tilde{\lambda}_p t/4} \phi_1$, it is easy to check that
\[
\int_{-h_1}^{h_1} J(x-y)W_1(t,y)dy - dW_1(t,x) + a(t,x)W_1 - (W_1)_t
= C_1 e^{-\tilde{\lambda}_p t/4} \left[ \int_{-h_1}^{h_1} J(x-y)\phi_1(y)dy - d\phi_1 + a(t,x)\phi_1 + \frac{\lambda_1}{4} \phi_1 \right]
= -\frac{3\tilde{\lambda}_p}{4} C_1 e^{-\tilde{\lambda}_p t/4} \phi_1 < 0.
\]
Choose $C_1 > 0$ large such that $C_1 \phi_1(x) > u_0(x)$ in $[-h_1, h_1]$. Then we can apply Lemma 3.1 to $W_1 - \hat{w}$ to deduce
\[
\hat{w}_t(t,x) \leq W_1(t,x) = C_1 e^{-\tilde{\lambda}_p t/4} \phi_1 \leq C_1 e^{-\tilde{\lambda}_p t/4} \text{ for } t > 0 \text{ and } x \in [-h_1, h_1]. \tag{39}
\]
Now define
\[
\hat{h}(t) = h_0 + 2\mu h_1 C_1 \int_0^t e^{-\tilde{\lambda}_p s/4} ds \text{ and } \hat{g}(t) = -\hat{h}(t) \text{ for } t \geq 0.
\]
We claim that $(\hat{w}, \hat{g}, \hat{h})$ is an upper solution of \((1)\). Firstly, we compute that for any $t > 0$,
\[
\hat{h}(t) = h_0 + 2\mu h_1 C_1 \frac{4}{\bar{\lambda}_p} (1 - e^{-\tilde{\lambda}_p t/4}) < h_0 + 2\mu h_1 C_1 \frac{4}{\bar{\lambda}_p} \leq h_1
\]
provided that
\[
0 < \mu \leq \overline{\mu} := \frac{\tilde{\lambda}_p(h_1 - h_0)}{8h_1 C_1}.
\]
Similarly, $\hat{g}(t) > -h_1$ for all $t > 0$. Thus (38) gives that
\[
\hat{w}_t(t,x) \geq \int_{\hat{g}(t)}^{\hat{h}(t)} J(x-y)\hat{w}(t,y)dy - d\hat{w}(t,x) + a(t,x)\hat{w} - b(t,x)\hat{w}^2 \text{ for } t > 0, x \in [\hat{g}(t), \hat{h}(t)].
\]
Secondly, due to (39), it is easy to check that
\[
\int_{\hat{g}(t)}^{\hat{h}(t)} \int_{\hat{h}(t)}^{+\infty} J(x-y)\hat{w}(t,x)dydx < 2h_1 C_1 e^{-\tilde{\lambda}_p t/4}.
\]
Thus
\[
\hat{h}'(t) = 2\mu h_1 C_1 e^{-\tilde{\lambda}_p t/4} > \int_{\hat{g}(t)}^{\hat{h}(t)} \int_{\hat{h}(t)}^{+\infty} J(x-y)\hat{w}(t,x)dydx.
\]
Similarly, one has
\[
\hat{g}'(t) < -\mu \int_{\hat{g}(t)}^{\hat{h}(t)} \int_{\hat{h}(t)}^{+\infty} J(x-y)\hat{w}(t,x)dydx.
\]
By the above argument, it is clear that $(\hat{w}, \hat{g}, \hat{h})$ is an upper solution of \((1)\). Hence, by Lemma 2.2, we have
\[
u(t,x) \leq \hat{w}(t,x), \ g(t) \geq \hat{g}(t) \text{ and } \hat{h}(t) \leq \hat{h}(t) \text{ for } t > 0, x \in [\hat{g}(t), \hat{h}(t)].
\]
Therefore
\[
h_\infty - g_\infty \leq \lim_{t \to +\infty} \left( \hat{h}(t) - \hat{g}(t) \right) \leq 2h_1 < +\infty.
\]
Hence $h_\infty - g_\infty < +\infty$ if $\mu \leq \overline{\mu}$. \hfill \ensuremath{\Box}

**Theorem 4.4.** Suppose that \((37)\) holds and $2h_0 < l^*$. Then there exists $\mu^* > 0$ such that $h_\infty - g_\infty = +\infty$ if $\mu > \mu^*$ and $h_\infty - g_\infty < +\infty$ if $0 < \mu \leq \mu^*$. 

Proof. The proof is similar to Theorem 3.14 of [4]. We omit the details here.

Next we consider the case

$$\min_{x \in \mathbb{R}} a_T(x) \leq d \leq \max_{x \in \mathbb{R}} a_T(x).$$  \hspace{1cm} (40)

In this case, by (12) and (13), there exists $l_*>0$ such that

$$\lambda_p(-\mathcal{L}_I + a(t,x)) < 0 \text{ if } |I| > l_*.$$  

By similar arguments as Lemma 4.2, we have the following lemma.

**Theorem 4.5.** If $2h_0 \geq l_*$ then $h_\infty - g_\infty = +\infty.$

When $2h_0 < l_*$, whether the principle eigenvalue is positive or negative is not clear, so we can’t get the previous result.

Combining Theorem 4.1, 4.2, 4.3 and 4.5, we immediately obtain Theorem 1.2.

**Acknowledgments:** The authors would like to express their sincere thanks to the anonymous reviewers for their helpful comments and suggestions. The work is partially supported by National Natural Science Foundation of China (11771380) and Natural Science Foundation of Jiangsu Province (BK20191436).

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