A NOTE ON THE $osp(1|2s)$ THERMODYNAMIC BETHE ANSATZ EQUATION

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Abstract

A Bethe ansatz equation associated with the Lie superalgebra $osp(1|2s)$ is studied. A thermodynamic Bethe ansatz (TBA) equation is derived by the string hypothesis. The high temperature limit of the entropy density is expressed in terms of the solution of the $osp(1|2s)$ version of the $Q$-system. In particular for the fundamental representation case, we also derive a TBA equation from the $osp(1|2s)$ version of the $T$-system and the quantum transfer matrix method. This TBA equation is identical to the one from the string hypothesis. The central charge is expressed by the Rogers dilogarithmic function, and identified to $s$.

1 Introduction

Recently, thermodynamics of quantum integrable spin chains related to superalgebras received attentions. In particular, there are several papers [1, 2, 3, 4, 5, 6, 7] on thermodynamic Bethe ansatz (TBA) equations [8] related to $sl(r|s)$. Namely, the TBA equation for the supersymmetric $t − J$ model, which is related to the vector representation of $sl(2|1)$, is proposed in [4]. This is also generalized [2] to the case of the fundamental representation of $sl(s|1)$. The TBA equation for a supersymmetric extended Hubbard model, which is related to the fundamental representation of $sl(2|2)$, is proposed in [3]. One can also find TBA equations beyond the fundamental representations. The TBA equation of a generalized $t − J$ model, which is based on a

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tensor representation of $sl(2|1)$, is proposed in \cite{6}. As is well known, $sl(r|s)$ admits finite dimensional representations with continuous parameters. In fact the TBA equation of the model for electrons with generalized hopping terms and Hubbard interaction, which is related to one parameter family of four dimensional representations of $sl(2|1)$, is proposed in \cite{4}. In the case of $sl(r|s)$, the TBA equation with a continuous parameter is studied in \cite{7} in relation with the continuum limit of integrable spin chains. These TBA equations are derived by the traditional string hypothesis \cite{9, 10}. One can also derive the same TBA equations by the quantum transfer matrix (QTM) method \cite{11, 12, 13, 14, 15, 5}. In fact the TBA equations for the supersymmetric $t-J$ model \cite{1} and the supersymmetric extended Hubbard model \cite{3} are also derived \cite{5} by the QTM method based on the $sl(r|s)$ version of the $T$-system \cite{16} (a system of functional relations among transfer matrices of fusion models).

On the other hand, study on TBA equations for $osp(r|2s)$ case has begun quite recently in Refs. \cite{17, 18, 19}, in which we deal with the simplest $osp(1|2)$ model from the point of view of the string hypothesis and the QTM method. The $R$-matrix of the $osp(r|2s)$ models are given as solutions of the graded Yang-Baxter equations \cite{20}, which have bosonic and fermionic degree of freedom. The $R$-matrix of the $osp(r|2s)$ models are proposed in \cite{21}. Moreover they are constructed from the point of view of automorphisms of Lie superalgebras and classical graded Yang-Baxter equations \cite{24}; quantum supergroups \cite{25}; braid-monoid algebras \cite{26}. Some special examples are also examined from various context. See, for example, Refs. \cite{25, 26, 27}. As for eigenvalue formulae of transfer matrices, $osp(1|2)$ model \cite{21} and $osp(1|2s)$ model \cite{28} are solved by the analytic Bethe ansatz; $osp(2r-1|2)$, $osp(2|2s-2)$, $osp(2r-2|2)$, $osp(1|2s)$ models \cite{29} are solved by the algebraic Bethe ansatz. These works deal with models related to fundamental representations. As for more complicated representations, we have executed an analytic Bethe ansatz \cite{30, 31} related to $C(s) = osp(2|2s-2)$ \cite{32}, $B(r|s) = osp(2r+1|2s)$ and $D(r|s) = osp(2r|2s)$ \cite{33}, in which ‘super Yangian analogue of Young supertableaux’ are proposed and $T$-systems among them are also found. The $osp(r|2s)$ integrable spin chain is related to interesting physical problems, such as the loop model which is related to statistical properties of polymers \cite{34}, and the fractional quantum Hall effect \cite{35}, etc. So it is desirable to study the $osp(r|2s)$ integrable spin chain beyond the $osp(1|2)$ case.

The purpose of this paper is to study TBA equations related to $osp(1|2s) = B(0|s)$ by the string hypotheses \cite{4, 10} and the QTM method \cite{11, 12, 13, 14, 15, 5}. In section 2, we assume a rather general Bethe ansatz equation (BAE), and derive the TBA equation by the string hypothesis. We expect this TBA
equation is the one for a higher spin analogue of the $osp(1|2s)$ integrable spin chain in Ref. [28]. In fact, the high temperature limit of the free energy is expressed by an appropriate solution of a difference equation called the $Q$-system. In particular for the fundamental representation case, we also derive the TBA equation from the QTM method in section 4. Namely, we transform the 1D quantum system to the 2D classical system by the Suzuki-Trotter mapping [11], and define a QTM, which is a transfer matrix of an inhomogeneous vertex model. We consider the QTM in the context of the fusion hierarchy of the model, and derive the functional relations among ‘fusion QTMs’ from the $osp(1|2s)$ version of the $T$-system[33]. After a dependant variable transformation, we obtain the $Y$-system (which is expected to be a system of functional relations among a solution of the TBA equation) from the $T$-system. Finally we transform the $Y$-system with certain analytical conditions (ANZC conditions: Analytic NonZero and Constant asymptotics for the large spectral parameter) into the TBA equation. Moreover we find that this TBA equation coincides with the one in section 2. This indicates the validity of the string hypothesis for the $osp(1|2s)$ model. In section 5, we evaluate the low temperature asymptotics of the specific heat. We express the central charge by the Rogers dilogarithmic function and identify it as $c = s$. This coincides with the conjecture [28] by the root density method.

2 String hypothesis

For the last several decades, many people recognized that Bethe ansatz equations (BAE) can be written in terms of the representation theoretical data of Lie algebras [36] or Lie superalgebras [21, 29]. So we assume, as our starting point, the following type of the $osp(1|2s)$ Bethe ansatz equation on complex variables \( \{v_k^{(a)}\} \):

\[
\prod_{j=1}^{N} \left( \frac{v_k^{(a)} - w_j^{(a)}}{v_k^{(a)} - w_j^{(a)} - \frac{i}{2t_a}b_j^{(a)}} \right) = -\varepsilon_a \prod_{d=1}^{s+1} \frac{Q_{\sigma(d)}(v_k^{(a)} + \frac{i}{2}B_{ad})}{Q_{\sigma(d)}(v_k^{(a)} - \frac{i}{2}B_{ad})},
\]

where $k \in \{1,2,\ldots,M_a\}$; $a \in \{1,2,\ldots,s\}$; $\sigma(d) = d$ for $1 \leq d \leq s$; $\sigma(s + 1) = s$; $t_a = 1$ for $1 \leq a \leq s - 1$; $t_s = 2$; $B_{ad} = 2\delta_{ad} - \delta_{a,d+1} - \delta_{a,d-1}$; $Q_a(v) = \prod_{k=1}^{M_a}(v - v_k^{(a)})$; $M_a \in \mathbb{Z}_{\geq 0}$; $M_0 = N$; $N$ is the number of the lattice site. $w_j^{(a)} \in \mathbb{C}$ is an inhomogeneity parameter. In this section, we consider the case $w_j^{(a)} = 0$. The parameter $\sigma$ expresses an effect of ‘a peculiar two-body self-interaction for the root $\{v_k^{(s)}\}$’ [29], which originates from the odd simple root $\alpha_s$ with $(\alpha_s|\alpha_s) \neq 0$. The parameters $(b_j^{(1)}, b_j^{(2)}, \ldots, b_j^{(s)})$ in the
Figure 1: The Dynkin diagram for the Lie superalgebra $B(0|s) = osp(1|2s)$ ($s \in \mathbb{Z}_{\geq 1}$): white circles denote even roots; a black circle denotes an odd root $\alpha_s$ with $(\alpha_s|\alpha_s) \neq 0$. The Kac-Dynkin label $b_a$ of an irreducible representation of $osp(1|2s)$ is depicted under each vertex. This irreducible representation is finite dimensional if and only if $b_a \in \mathbb{Z}_{\geq 0}$ for $a \in \{1, 2, \ldots, s - 1\}$ and $b_s \in 2\mathbb{Z}_{\geq 0}$.

BAE (2.1) represent the Kac-Dynkin label [37] (see, Figure 1). In this paper, we consider the case

$$b_j^{(a)} = b\delta_{ap}(1 - \delta_{ps}) + 2b\delta_{ap}\delta_{ps}, \quad (2.2)$$

where $b \in \mathbb{Z}_{\geq 1}, p \in \{1, 2, \ldots, s\}$. Note that $b_j^{(s)}$ is always an even number. We write the irreducible representation of $osp(1|2s)$ labeled by (2.2) as $V_{b_p}$. $\epsilon_a$ is a phase factor ($|\epsilon_a| = 1$). In particular, for $p = b = 1$ case, we have [28]

$$\epsilon_a = \begin{cases} 
(-1)^{N-M_2} & \text{if } a = 1 \\
(-1)^{M_{a-1}-M_{a+1}} & \text{if } a \in \{2, 3, \ldots, s - 1\} \\
(-1)^{M_{s-1}-M_s} & \text{if } a = s.
\end{cases} \quad (2.3)$$

Note that BAE (2.1) for $p = b = 1$ corresponds to the BAE [28, 29] of the $2s + 1$ state $osp(1|2s)$ model [22, 23, 24] (the Hamiltonian of this model is given in (4.2)).

Next we will present a function $T_{m}^{(a)}(v)$ with a spectral parameter $v \in \mathbb{C}$, which is a candidate of an eigenvalue formula (DVF) of a transfer matrix for an $osp(1|2s)$ vertex model of a fusion type for auxiliary space labeled by $(a, m)$. We consider the case where the transfer matrix is defined as the ordinary trace of a monodromy matrix. The auxiliary space $W_{m}^{(a)}$ of the corresponding transfer matrix should be an irreducible finite dimensional module of the ‘super Yangian $Y(osp(1|2s))$ ’. As an $osp(1|2s)$-module, $W_{m}^{(a)}$ is expected to contain $V_{m}^{(a)}$ as ‘top term’: any other irreducible component in $W_{m}^{(a)}$ has a highest weight lower than that of $V_{m}^{(a)}$. As for $W_{m}^{(a)}$, the multiplicity of the irreducible components may be calculated by a Kirillov-Reshetikhin [38]-like formula in Ref. [33]. The quantum space that the transfer matrix acts on is the tensor product module $(W_{b}^{(p)}) \otimes \mathcal{N}$. We can derive an explicit expression of $T_{m}^{(a)}(v)$ by modifying the vacuum part of the DVF in Ref. [33] so that the
The vacuum part is compatible with the left hand side of the BAE (2.1):

\[ T_m^{(a)}(v) = \sum_{\{d_{jk}\}} \prod_{j=1}^{m} \prod_{k=1}^{a} z(d_{jk}; v - \frac{i}{2}(m - a - 2j + 2k)), \quad (2.4) \]

where the summation is taken over \( d_{jk} \in \{1, 2, \ldots, s, 0, \bar{s}, \ldots, \bar{s}, \bar{1}\} \) \((1 < 2 < \cdots < s < 0 < \bar{s} < \cdots < \bar{s} < \bar{1})\) such that \( d_{jk} \leq d_{j+1,k} \) and \( d_{jk} < d_{j,k+1} \). The functions \( \{z(a; v)\} \) are defined as

\[
\begin{align*}
z(a; v) &= \psi_a(v) \frac{Q_{a-1}(v + \frac{i}{2}(a + 1))Q_a(v + \frac{i}{2}(a - 2))}{Q_{a-1}(v + \frac{i}{2}(a - 1))Q_a(v + \frac{i}{2}a)}, \quad \text{for} \quad a \in \{1, 2, \ldots, s\}, \\
z(0; v) &= \psi_0(v) \frac{Q_s(v + \frac{i}{2}(s - 1))Q_s(v + \frac{i}{2}(s + 2))}{Q_s(v + \frac{i}{2}(s + 1))Q_s(v + \frac{i}{2}a)}, \quad (2.5) \\
z(\bar{v}; v) &= \psi_{\bar{v}}(v) \frac{Q_{a-1}(v - \frac{i}{2}(a - 2s))Q_a(v - \frac{i}{2}(a - 2s - 3))}{Q_{a-1}(v - \frac{i}{2}(a - 2s - 2))Q_a(v - \frac{i}{2}(a - 2s - 1))}, \quad \text{for} \quad a \in \{1, 2, \ldots, s\},
\end{align*}
\]

where \( Q_0(v) := 1 \). The vacuum parts of \( \{z(a; v)\} \) are given as follows

\[
\psi_a(v) = \begin{cases} 
\zeta_a & \text{for} \quad 1 \leq a \leq p, \\
\zeta_a \phi_a(v + \frac{i}{2}(p-1)) & \text{for} \quad p + 1 \leq a \leq p + 1, \\
\zeta_a \phi_a(v + \frac{i}{2}(p-1)) \phi_a(v - \frac{i}{2}(p-2)) & \text{for} \quad 1 \leq \bar{a} \leq \bar{1},
\end{cases} \quad (2.6)
\]

where \( \phi_b(v) = \prod_{k=1}^{b} (v + \frac{i}{2}(b + 1 - 2k))^N \); \( \zeta_a \) is a phase factor \(|\zeta_a| = 1\): \( \zeta_a = \zeta_{\bar{a}} = \zeta_{\bar{1}} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_{a-1} \) for \( a \in \{1, 2, \ldots, s\} \), \( \zeta_0 = \zeta_{\bar{1}} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_s \). In particular for \( p = b = 1 \) case, the phase factor \( \zeta_a \) is given [28] as follows

\[
\zeta_a = \begin{cases} 
(-1)^{N-M_1} & \text{if} \quad a = 1, \\
(-1)^{M_a-M_1} & \text{if} \quad a \in \{2, 3, \ldots, s\}, \\
1 & \text{if} \quad a = 0, \\
(-1)^{M_{a-M_1}} & \text{if} \quad a \in \{\bar{s}, \ldots, \bar{3}, \bar{2}\}, \\
(-1)^{N-M_1} & \text{if} \quad a = \bar{1},
\end{cases} \quad (2.7)
\]

where \( \bar{a} = a \). The dress part of (2.4) is free of poles under the BAE (2.1). This is a requirement from the analytic Bethe ansatz [30, 31]. Note that \( T_1^{(1)}(v) \) for \( p = b = 1 \) corresponds to the DVF [28, 29] of the \( 2s + 1 \) state \( osp(1|2s) \) model. We may think of (2.4) as \( osp(1|2s) \) version of the Bazhanov and Reshetikhin’s eigenvalue formula [39].
We shall consider the one dimensional counterpart of the above-mentioned fusion vertex model for the case \((a, m) = (p, b)\), which is a higher spin analogue of the \(2s + 1\) state \(osp(1|2s)\) spin chain\[28\]. The energy density of the corresponding system is defined as follows:

\[
\mathcal{E} = \frac{J}{N} \int \frac{dv}{d} \log T_b^{(p)}(v) \bigg|_{v=0} = \frac{J}{N} \sum_{k=1}^{N_p} \frac{b}{(v_k^{(p)})^2 + (\frac{b}{2})^2},
\]

where \(J\) is a real coupling constant. For simplicity, the function (2.6) is normalized so that the constant term of the energy density (2.8) vanishes. The extra signs (2.7) do not appear in (2.8) explicitly. We adopt the following ordinary string solution for (2.1) in the thermodynamic limit (cf. Ref. [40]):

\[
v_{m,k}^{(a)} + \frac{i}{2} (m + 1 - 2j),
\]

where \(m \in \mathbb{Z}_{\geq 1}; a \in \{1, 2, \ldots, s\}; j \in \{1, 2, \ldots, m\}; k \in \{1, 2, \ldots, n_{m}^{(a)}\}; v_{m,k}^{(a)} \in \mathbb{R}; n_{m}^{(a)}\) is the number of color-\(a\) \(m\)-strings. Let \(\rho_{m}^{(a)}(v)\) and \(\sigma_{m}^{(a)}(v)\) be string and hole densities for each color \(a\) and length \(m\). Substituting (2.9) into (2.1), one can derive a relation between \(\rho_{m}^{(a)}(v)\) and \(\sigma_{m}^{(a)}(v)\) after some manipulation:

\[
\delta_{ap} \Psi_{m,b}(v) = \sigma_{m}^{(a)}(v) + \sum_{l=1}^{\infty} \sum_{d=1}^{s} A_{ml}^{ad} \rho_{l}^{(d)}(v),
\]

where the function \(\Psi_{m,b}(v)\) is given by

\[
\Psi_{m,b}(v) = \frac{i}{2\pi} \frac{\partial}{\partial v} \sum_{j=1}^{m} \log \left\{ \frac{v + \frac{i}{2}(m + 1 - 2j + b)}{v + \frac{i}{2}(m + 1 - 2j - b)} \right\}
\]

and the operator \(A_{ml}^{ad}\) is defined by

\[
A_{ml}^{ad} = \delta_{ad}([l - m] + 2[l - m] + 2] + \cdots + 2[l + m - 2] + [l + m]) - I_{ad}([l - m] + 1) + [l - m] + 3] + \cdots + [l + m - 1]),
I_{ad} = \delta_{a,d-1} + \delta_{a,d+1} + \delta_{ad}\delta_{as}.
\]

Here the action of \([m]\) on any function \(h(v)\) is defined as follows

\[
[m]h(v) = \begin{cases} 
(\Psi_{m,1} * h)(v) & \text{if } m \neq 0 \\
h(v) & \text{if } m = 0,
\end{cases}
\]
Thermodynamic Bethe ansatz equation

where * is a convolution

\[(\Psi_{m,1} * h)(v) = \int_{-\infty}^{\infty} dw \Psi_{m,1}(v - w) h(w). \quad (2.14)\]

In the thermodynamic limit, the energy density (2.8) reduces to

\[E = 2\pi J \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} dv \Psi_{m,b}(v) \rho_m^{(p)}(v). \quad (2.15)\]

The entropy density leads as follows

\[S = k_B \sum_{a=1}^{s} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} dv \{ (\rho_m^{(a)}(v) + \sigma_m^{(a)}(v)) \log(\rho_m^{(a)}(v) + \sigma_m^{(a)}(v)) \]

\[\quad - \rho_m^{(a)}(v) \log \rho_m^{(a)}(v) - \sigma_m^{(a)}(v) \log \sigma_m^{(a)}(v) \}, \quad (2.16)\]

where \(k_B\) denotes the Boltzmann constant. Now we shall investigate the equilibrium state. From the condition \(\frac{\delta F}{\delta \rho_m^{(a)}(v)} = 0\) for the free energy density \(F = E - TS\) (\(T: \text{temperature}\)) and the relation (2.10), we obtain a TBA equation

\[\log(1 + Y_m^{(a)}(v)) = 2\pi J \beta \Psi_{m,b}(v) \delta_{ap} + \sum_{l=1}^{\infty} \sum_{d=1}^{s} A_{ml}^{ad} \log(1 + Y_l^{(d)}(v))^{-1} \] (2.17)

and the free energy density

\[F = -\frac{1}{\beta} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} dv \Psi_{m,b}(v) \log(1 + (Y_m^{(p)}(v))^{-1}), \quad (2.18)\]

where \(Y_m^{(a)}(v) := \sigma_m^{(a)}(v)/\rho_m^{(a)}(v)\) and \(\beta := 1/(k_B T)\). We shall derive an alternative form of the TBA equation (2.17). Taking note on the relations

\[\hat{\Psi}_{m,b}(k) = \int_{-\infty}^{\infty} dv \Psi_{m,b}(v) e^{-ikv} \]

\[= \frac{\sinh(\min(b,m)k/2)}{\sinh k/2} \exp \left( -\frac{\max(b,m)|k|}{2} \right), \]

\[\sum_{m=1}^{\infty} \hat{X}_{nm} \hat{\Psi}_{m,b}(k) = \frac{\delta_{nb}}{2 \cosh k/2}, \]

\[\hat{X}_{nm} = \delta_{nm} - \frac{\delta_{n,m-1} + \delta_{n,m+1}}{2 \cosh k/2}, \quad (2.19)\]
we can rewrite (2.17) as follows:
\[
\log Y_m^{(a)}(v) = \frac{\pi J\beta \delta_{ap} \delta_{mb}}{\cosh \pi v} + K \log \left( \frac{1 + Y_{m-1}^{(a)}(1 + Y_{m+1}^{(a)})}{\prod_{d=1}^{s} (1 + (Y_{m}^{(d)})^{-1}) I_{ad}} \right)(v),
\]
where \(a \in \{1, 2, \ldots, s\}\), \(m \in \mathbb{Z}_{\geq 1}\), \(Y_0^{(a)}(v) := 0\) and the kernel is
\[
K(v) = \frac{1}{2 \cosh \pi v}.
\]
Note that (2.20) reduces to the TBA equation in Refs. [17, 18], if we set \(p = b = s = 1\).

3 High temperature limit

We shall consider the high temperature limit of the free energy density (2.18). We assume that the function \(Y_m^{(a)}(v)\) is independent of \(v\) in the limit \(T \to \infty\).

In this case, the constant solution of (2.20) obeys the constant \(Y\)-system
\[
(Y_m^{(a)})^2 = \frac{(1 + Y_{m-1}^{(a)})(1 + Y_{m+1}^{(a)})}{\prod_{d=1}^{s} (1 + (Y_{m}^{(d)})^{-1}) I_{ad}},
\]
where \(Y_0^{(a)} := 0\), \(a \in \{1, 2, \ldots, s\}\) and \(m \in \mathbb{Z}_{\geq 1}\). The solution of this constant \(Y\)-system is
\[
Y_m^{(a)} = \frac{Q_{m-1}^{(a)} Q_{m+1}^{(a)}}{\prod_{d=1}^{s} (Q_{m}^{(d)}) I_{ad}},
\]
where \(\{Q_m^{(a)}\}\) are the dependant variables of the \(Q\)-system:
\[
(Q_m^{(a)})^2 = Q_{m-1}^{(a)} Q_{m+1}^{(a)} + \prod_{d=1}^{s} (Q_{m}^{(d)}) I_{ad},
\]
where \(Q_0^{(a)} := 1\), \(a \in \{1, 2, \ldots, s\}\) and \(m \in \mathbb{Z}_{\geq 1}\). We can derive this \(Q\)-system by neglecting the vacuum part and the spectral parameter dependence of the \(T\)-system in Ref. [33] (see, [17]): a system of fusion relations among commuting transfer matrices of the \(osp(1|2s)\) vertex model. The \(Q\)-system was proposed in Ref. [38] as a system of difference equations among characters of modules of the Yangian \(Y(\mathfrak{g})\) as \(\mathfrak{g}\) modules (\(\mathfrak{g}\): simple Lie algebras). Our \(Q\)-system (3.3) is equivalent to the \(A_{2s}^{(2)}\) version of the \(Q\)-system [11, 12].
think that this coincidence originates from the correspondence \[43\] between
\( B^{(1)}(0|s) \) and \( A^{(2)}_{2\tau_c} \). There are several similar correspondences \[43\] between superalgebras and ordinary algebras. Thus, we expect that “different algebras
with the same \( Q \)-system” also occurs to several other algebras.

We can express the solution of the \( Q \)-system (3.3) as a polynomial of the
fundamental variables \( Q^{(1)}, Q^{(2)}, \ldots, Q^{(s)} \). Moreover, this polynomial has a
determinant expression:

\[
Q^{(a)}_m = \det_{1 \leq i,j \leq m} (Q_{a+i-j}),
\]

(3.4)

where \( Q_j \) satisfies

\[
Q_a = Q_{2s-a+1}
\]

(3.5)

and a condition

\[
Q_a = \begin{cases} 
0 & \text{for } a \in \mathbb{Z}_{<0} \\
1 & \text{for } a = 0 \\
Q^{(a)}_1 & \text{for } a \in \{1, 2, \ldots, s\}.
\end{cases}
\]

(3.6)

In our case, \( Q_a \) is given as follows

\[
Q_a = \left( \begin{array}{c} 2s + 1 \\ a \end{array} \right).
\]

(3.7)

In this case, \( Q^{(a)}_m \) is the number of terms in DVF (2.4), which is expected \[33\]
to be the dimension of \( W^{(a)}_m \). In particular, \( Q^{(1)}_1 \) coincides with the dimension
of \( V^{(1)}_1 \). For general case, \( Q^{(a)}_m \) is greater than or equal to the dimension
of \( V^{(a)}_m \). When solving (2.20), we assume \( Y^{(a)}(\pm\infty) \) coincide with the solution
(3.2) of the constant \( Y \)-system with (3.4)-(3.7).

Substituting (3.2) into (2.18), we find that the high temperature limit
of the entropy density is expressed in terms of a constant solution of the
\( Q \)-system (3.3):

\[
\lim_{T \to \infty} S = -\lim_{T \to \infty} \frac{F}{T} = k_B \sum_{m=1}^{\infty} \log(1 + (Y^{(p)}_m)^{-\min(b,m)}) = k_B \log Q^{(p)}_b.
\]

(3.8)

In particular for \( p = b = 1 \) case, we have

\[
\lim_{T \to \infty} S = k_B \log Q^{(1)}_1 = k_B \log(2s + 1).
\]

(3.9)

This is the logarithm of the dimension of \( W^{(1)}_1 \) or \( V^{(1)}_1 \) \[28\]. This result is
compatible with the number of the state per site of the \( osp(1|2s) \) integrable
spin chain. In closing this section, we note the fact that the solution (3.2) of the constant $Y$-system (3.1) can also be written as

$$Y_m^{(a)} = \frac{m(g + m)}{a(g - a)},$$

(3.10)

where $g = 2s + 1$. By using (3.10), one can derive an explicit expression of a constant solution of the $Q$-system (3.3)

$$Q_m^{(a)} = \left(\frac{(m + g)!m!}{(m + a)!(m + g - a)!}\right)^m \prod_{k=1}^{m} \left\{ \frac{(k + a)(k + g - a)}{k(k + g)} \right\}^k,$$

(3.11)

which will provide a dimension formula of the module $W_m^{(a)}$.

### 4 T-system and QTM method

In this section, we introduce an integrable spin chain\cite{28, 29} associated with the fundamental representation $W_1^{(1)}$, and derive the TBA equation from the point of view of the QTM method\cite{11, 12, 13, 14, 15, 5}. A more detailed explanation of the QTM method in the case of $osp(1|2)$ can be found in Ref.\cite{18}.

The $\tilde{R}$-matrix\cite{22, 23, 29} of the model is given as

$$\tilde{R}(v) = I + vP - \frac{2v}{2v - g},$$

(4.1)

where $P_{ab} = (-1)^{p(a)p(b)}\delta_{ab}\delta_{bc}$; $E_{ab} = \alpha_{ab}(\alpha^{-1})_{cd}$; $a, b, c, d \in \{1, 2, \ldots, s, 0, \overline{s}, \ldots, \overline{1}\}$; $\alpha$ is $(2s + 1) \times (2s + 1)$ anti-diagonal matrix whose non-zero elements are $\alpha_{a,\overline{a}} = 1$ for $a \in \{1, 2, \ldots, s, 0\}$ and $\alpha_{a,\overline{a}} = -1$ for $a \in \{\overline{s}, \overline{s-1}, \ldots, \overline{1}\}$; $\overline{a} = a$; $p(a) = 0$ for $a = 0$; $p(a) = 1$ for $a \in \{1, 2, \ldots, s\} \cup \{\overline{s}, \ldots, \overline{2}, \overline{1}\}$. The Hamiltonian of the present model for periodic boundary condition is given by

$$H = J \sum_{k=1}^{L} \left( P_{k,k+1} + \frac{2}{g}E_{k,k+1} \right),$$

(4.2)

where $J$ is a coupling constant: $J > 0$ and $J < 0$ correspond to the ferromagnetic and antiferromagnetic regimes respectively; $L$ is the number of the lattice sites; $P_{k,k+1}$ and $E_{k,k+1}$ act nontrivially on the $k$ th site and $k + 1$ th site. The QTM is defined as

$$T_1^{(1)}(u, v) = Tr_j \prod_{k=1}^{N} R_{a_{2k-1},a_{2k}}(u + iv)\tilde{R}_{a_{2k-1},a_{2k}}(u - iv),$$

(4.3)
where $R^{cd}_{ab}(v) = R^{cd}_{ba}(v)$; $\tilde{R}_{jk}(v) = t_k R_{kj}(v)$ ($t_k$ is the transposition in the $k$-th space); $N$ is the Trotter number and assumed to even. By using the largest eigenvalue $T^{(1)}_1(u_N, 0)$ of the QTM (4.3), the free energy density is expressed as

$$F = -\frac{1}{\beta} \lim_{N \to \infty} \log T^{(1)}_1(u_N, 0),$$  \hspace{1cm} (4.4)

where $u_N = -\frac{J\beta}{N}$. From now on, we abbreviate the parameter $u$ in $T^{(1)}_1(u, v)$. One can obtain the eigenvalue formulae of the QTM (4.3) by replacing the vacuum part (2.4) of the DVF (2.4) for $p = b = 1$ with that of the QTM. One may interpret the QTM as a transfer matrix of an inhomogeneous vertex model. In our case, the inhomogeneity parameters in the BAE (2.4) for $p = b = 1$ take the values: $w^{(a)}_j = iu \delta_{a1}$ for $j \in 2\mathbb{Z}_{\geq 1}$; $w^{(a)}_j = (-iu + \frac{ig}{2}) \delta_{a1}$ for $j \in 2\mathbb{Z}_{\geq 0} + 1$. The vacuum parts of (2.4) are given as follows

$$\psi_a(v) = \begin{cases} \zeta_1 \frac{\phi_+(v) \phi_-(v+i) \phi_+(v-\frac{2a-1}{2}i)}{\phi_+(v-\frac{2a+1}{2}i)} & \text{for } a = 1, \\ \zeta_2 \frac{\phi_+(v) \phi_-(v) \phi_+(v+\frac{2a+1}{2}i)}{\phi_+(v+\frac{2a-1}{2}i)} & \text{for } 2 \leq a \leq \overline{2}, \\ \zeta_1 \frac{\phi_-(v) \phi_+(v-i) \phi_-(v+\frac{2a+1}{2}i)}{\phi_-(v+\frac{2a-1}{2}i)} & \text{for } a = \overline{1}, \end{cases}$$  \hspace{1cm} (4.5)

where $\phi_{\pm}(v) = (v \pm iu)^{\pm N}$; $\zeta_a$ is the one in (2.7). For $a \in \{1, 2, \ldots, s\}$ and $m \in \mathbb{Z}_{\geq 1}$, we define a normalization function

$$\mathcal{N}^{(a)}_m(v) = \prod_{j=1}^{m} \prod_{k=1}^{a} \frac{\phi_-(v - \frac{2a-2j+2k}{2}i) \phi_+(v - \frac{m-a-2j+2k}{2}i)}{\phi_-(v - \frac{m-a+2j}{2}i) \phi_+(v + \frac{m-a}{2}i)}.$$  \hspace{1cm} (4.6)

We reset $T^{(a)}_m(v) / \mathcal{N}^{(a)}_m(v)$ to $T^{(a)}_m(v)$, where $T^{(a)}_m(v)$ is defined by (2.4). Since the dress part of the DVF $T^{(a)}_m(v)$ is same as the row-to-row case, $T^{(a)}_m(v)$ satisfies the following functional relation, which has essentially the same form as the $osp(1|2s)$ $T$-system in Ref. [33],

$$T^{(a)}_m(v + \frac{i}{2})T^{(a)}_m(v - \frac{i}{2}) = T^{(a)}_{m+1}(v)T^{(a)}_{m-1}(v) + T^{(a-1)}_m(v)T^{(a+1)}_m(v)$$  \hspace{1cm} (4.7)

for $a \in \{1, 2, \ldots, s-1\}$,

$$T^{(s)}_m(v + \frac{i}{2})T^{(s)}_m(v - \frac{i}{2}) = T^{(s)}_{m+1}(v)T^{(s)}_{m-1}(v) + g^{(s)}_m(v)T^{(s-1)}_m(v)T^{(s)}_m(v),$$

where

$$T^{(a)}_0(v) = \phi_-(v + \frac{a}{2}i) \phi_+(v - \frac{a}{2}i) \quad \text{for } a \in \mathbb{Z}_{\geq 1};$$

$$T^{(0)}_m(v) = \phi_-(v - \frac{m}{2}i) \phi_+(v + \frac{m}{2}i) \quad \text{for } m \in \mathbb{Z}_{\geq 1};$$

$$g^{(s)}_m(v) = \frac{\phi_-(v + \frac{m+s+1}{2}i) \phi_+(v - \frac{m+s+1}{2}i)}{\phi_-(v + \frac{m+s}{2}i) \phi_+(v - \frac{m+s}{2}i)} \quad \text{for } m \in \mathbb{Z}_{\geq 1}.$$
For $s = 1$, $g_m(v)T_m(v)$ coincides with the function $T_m(v)$ in Ref. [8]. For $m \in \mathbb{Z}_{\geq 1}$, we define the $Y$-functions:

\[
Y_m^{(a)}(v) = \frac{T_{m+1}^{(a)}(v)}{T_m^{(a-1)}(v)T_m^{(a+1)}(v)} \quad \text{for} \quad a \in \{1, 2, \ldots, s-1\},
\]

\[
Y_m^{(s)}(v) = \frac{T_{m+1}^{(s)}(v)}{g_m(v)T_m^{(s-1)}(v)T_m^{(s)}(v)}.
\]

(4.9)

By using the $T$-system (4.7), one can show that the $Y$-functions satisfy the following $Y$-system:

\[
Y_m^{(a)}(v + \frac{i}{2})Y_m^{(a)}(v - \frac{i}{2}) = \frac{(1 + Y_m^{(a+1)}(v))(1 + Y_m^{(a+1)}(v))}{\prod_{d=1}^{s}(1 + (Y_m^{(d)}(v))^{-1})},
\]

(4.10)

where $Y_0^{(a)}(v) = 0$, $a \in \{1, 2, \ldots, s\}$ and $m \in \mathbb{Z}_{\geq 1}$. A numerical analysis for finite $N, u, s$ indicates that a two-string solution (for every color) in the sector $N = M_1 = M_2 = \cdots = M_s$ of the BAE (2.1) provides the largest eigenvalue of the QTM (4.3) at $v = 0$. Moreover, we expect the following conjecture is valid for this two-string solution (see Figure 2, Figure 3, Figure 4).

**Conjecture 1** For small $u \mid u \mid \ll 1$ and $a \in \{1, 2, \ldots, s\}$, every zero of $T_m^{(a)}(v)$ is located outside of the physical strip $\text{Im}v \in [-\frac{1}{2}, \frac{1}{2}]$.

Based on this conjecture, we shall establish the ANZC property in some domain for the $Y$-functions (4.9) to transform the $Y$-system (4.10) to nonlinear integral equations. Here ANZC means Analytic NonZero and Constant asymptotics in the limit $|v| \to \infty$. One can show that the $Y$-function has the following asymptotic value

\[
\lim_{|v| \to \infty} Y_m^{(a)}(v) = \frac{m(g + m)}{a(g - a)},
\]

(4.11)

which is identified to the solution (3.10) of the constant $Y$-system (3.1). From the conjecture and (4.11), we find that the functions $1 + Y_m^{(a)}(v)$, $1 + (Y_m^{(a)}(v))^{-1}$ in the domain $\text{Im}v \in [-\delta, \delta]$ (0 $\ll \delta \ll 1$) and $Y_m^{(a)}(v)$ for $(a, m) \neq (1, 1)$ in the domain $\text{Im}v \in [-\frac{1}{2}, \frac{1}{2}]$ (physical strip) have the ANZC property. On the other hand, $Y_1^{(1)}(v)$ has zeros of order $N/2$ at $\pm i(\frac{1}{2} - u)$ if $u > 0$ ($J < 0$), poles of order $N/2$ at $\pm i(\frac{1}{2} + u)$ if $u < 0$ ($J > 0$) in the physical strip. Then we must modify $Y_1^{(1)}(v)$ as

\[
\tilde{Y}_m^{(a)}(v) = Y_m^{(a)}(v) \left\{ \tanh \frac{\pi}{2}(v + i(\frac{1}{2} \pm u)) \tanh \frac{\pi}{2}(v - i(\frac{1}{2} \pm u)) \right\}^{\pm \frac{N}{2} \frac{a+1}{(a+1)}}.
\]

(4.12)
Figure 2: Location of the roots of the BAE for \( osp(1|4) \) case \((N = 6, u = 0.05)\), which gives the largest eigenvalue of the QTM \( T^{(1)}_1(v) \). Both color 1 roots \( \{v_k^{(1)}\} \) and color 2 roots \( \{v_k^{(2)}\} \) form three two-strings.
Figure 3: Location of zeros of $T_m^{(1)}(v)$ for $osp(1|4)$ case ($m = 1, 2, 3, N = 6, u = 0.05$). The zeros recede from the physical strip $\text{Im} v \in [-\frac{1}{2}, \frac{1}{2}]$ as $m$ increases.
Figure 4: Location of zeros of $T_m^{(2)}(v)$ for $osp(1|4)$ case ($m = 1, 2, 3, N = 6, u = 0.05$). The zeros recede from the physical strip $\text{Im} v \in [-\frac{1}{2}, \frac{1}{2}]$ as $m$ increases.
where the sign $\pm$ is identical to that of $-u$. Taking note on the relation
\[
\tanh \frac{\pi}{4}(v + i)\tanh \frac{\pi}{4}(v - i) = 1, \tag{4.13}
\]
one can modify the lhs of the $Y$-system \((4.10)\) as
\[
\tilde{Y}^{(a)}_m(v - \frac{i}{2})\tilde{Y}^{(a)}_m(v + \frac{i}{2}) = \frac{(1 + Y^{(a)}_m(v))(1 + Y^{(a)}_{m-1}(v))}{\prod_{d=1}^{s}(1 + (Y^{(d)}_m)^{-1})I_{ad}}, \tag{4.14}
\]
for $m \in \mathbb{Z}_{\geq 1}$ and $a \in \{1, 2, \ldots, s\}$.

Now that the ANZC property has been established for the $Y$-system, we can transform \((4.14)\) into a system of nonlinear integral equations by a standard procedure.
\[
\log Y^{(a)}_m(v) = \mp \frac{N\delta_{a1}\delta_{m1}}{2}\log \left\{ \tanh \frac{\pi}{2}(v + i(\frac{1}{2} \pm u)) \tanh \frac{\pi}{2}(v - i(\frac{1}{2} \pm u)) \right\} \\
+ K * \log \left\{ \frac{(1 + Y^{(a)}_{m-1})(1 + Y^{(a)}_{m+1})}{\prod_{d=1}^{s}(1 + (Y^{(d)}_m)^{-1})I_{ad}} \right\}(v), \tag{4.15}
\]
where $Y^{(a)}_0(v) = 0$, $a \in \{1, 2, \ldots, s\}$ and $m \in \mathbb{Z}_{\geq 1}$. Substituting $u = -\frac{\beta J}{N}$ and taking the Trotter limit $N \to \infty$, we obtain the TBA equation \((2.20)\) for $p = b = 1$. Taking note on the relations
\[
C_{ad}(v) = \sum_{l=1}^{\min(a,d)} G_{[a-d]+2l-1}(v),
\]
\[
G_a(v) = \frac{4 \cos \left(\frac{(2s+1-2a)\pi}{4s+2}\right) \cosh \frac{2\pi v}{2s+1}}{2s \cos \left(\frac{(2s+1-2a)\pi}{4s+2}\right) + \cosh \frac{4\pi v}{2s+1}},
\]
\[
\hat{C}_{ad}(k) = \int_{-\infty}^{\infty} dv C_{ad}(v)e^{-ikv},
\]
\[
\sum_{c=1}^{s} \hat{C}_{ac}(k)\hat{D}_{cd}(k) = \delta_{ad}, \tag{4.16}
\]
\[
\hat{D}_{cd}(k) = 2\delta_{cd} \cosh \frac{k}{2} - I_{cd},
\]
one can also rewrite this TBA equation as
\[
\log Y^{(a)}_m(v) = 2\pi\beta J\delta_{m1}G_a(v) \\
+ \sum_{j=1}^{s} C_{aj} * \log \left\{ \frac{(1 + Y^{(j)}_{m-1})(1 + Y^{(j)}_{m+1})}{\prod_{d=1}^{s}(1 + Y^{(d)}_m)I_{jd}} \right\}(v), \tag{4.17}
\]
where $Y_0^{(a)}(v) = 0$, $a \in \{1, 2, \ldots, s\}$ and $m \in \mathbb{Z}_{\geq 1}$. In contrast to (2.20), (4.17) does not contain $1 + (Y_0^{(a)}(v))^{-1}$ which is not relevant to evaluate the central charge for the case $J < 0$ (see, the next section). One can derive the following relation from (4.7) for $a = 1$, (4.9) and (2.19).

\[
\log T_1^{(1)}(v) = \log \phi_-(v + i) \phi_+(v - i) \\
+ \sum_{m=1}^{\infty} \Psi_{1,m} \log (1 + (Y_1^{(1)})^{-1}(v)).
\] (4.18)

Taking the Trotter limit $N \to \infty$ with $u = -\frac{J \beta}{N}$, we find that $F = -\frac{1}{\beta} \log T_1^{(1)}(0)$ with (4.18) coincides with the free energy density (2.18) for $p = b = 1$ (if we neglect the first term in the rhs of (4.18) coming from a different normalization of the DVF). One can also derive the following relation from (4.7) for $m = 1$, (4.9) and (4.16).

\[
\log T_1^{(1)}(v) = \log \phi_-(v + i) \phi_+(v - i) + \sum_{a=1}^{s} G_a \log (1 + Y^{(a)}) \\
+ N \int_0^\infty dk \frac{2e^{-\frac{k}{2}} \sinh(2k) \cos(4k \sinh(\frac{2s-1}{4}k))}{k \cosh(\frac{2s+1}{4}k)}.
\] (4.19)

Taking the Trotter limit $N \to \infty$ with $u = -\frac{J \beta}{N}$, we obtain the free energy density without infinite sum.

\[
F = J \left\{ \frac{2}{2s + 1} \left( 2 \log 2 - \psi\left( \frac{1}{2s + 1} \right) + \psi\left( \frac{3 + 2s}{2 + 4s} \right) \right) - 1 \right\} \\
- k_B T \sum_{a=1}^{s} \int_{-\infty}^{\infty} dv G_a(v) \log (1 + Y^{(a)}(v)),
\] (4.20)

where $\psi(z)$ is the digamma function

\[
\psi(z) = \frac{d}{dz} \log \Gamma(z).
\] (4.21)

The first term in the rhs of (4.20) for $J = -1$ coincides with the grand state energy of the $osp(1|2s)$ model in [28].

5 Central charge

It was conjectured [28] that the $osp(1|2s)$ model based on the fundamental representation $W_1^{(1)}$ is governed by the $c = s$ conformal field theory. We shall
evaluate the central charge for the case $p = b = 1$ by using the result in the previous section and technique in \[14, 13\]. In this section, we assume $J = -1$ and $k_B = 1$. We define a scaling function in the low temperature limit

$$ y_{m,\pm}^{(a)}(v) = \lim_{T \to 0} Y_m^{(a)}(\pm(v + \frac{1}{v_F} \log \frac{\gamma}{T})), \quad (5.1) $$

where $\gamma$ is a constant and $v_F$ is the Fermi velocity \[28\]: $v_F = \frac{2\pi}{g}$. As $y_{m,\pm}^{(a)}(v)$ and $y_{m,\pm}^{(a)}(v)$ behave in the same way, we set $y_{m,\pm}^{(a)}(v) = y_{m,\pm}^{(a)}(v)$. From the TBA equation (4.17), $y_{m,\pm}^{(a)}(v)$ satisfies the following equation

$$ \log y_{m,\pm}^{(a)}(v) = -\frac{4v_F}{\gamma} e^{-v_F v} \sin \left( \frac{v_F a}{2} \right) + \sum_{j=1}^s C_{aj} \log \left\{ \frac{(1 + y_{m-1}^{(j)})(1 + y_{m+1}^{(j)})}{\prod_{d=1}^s (1 + y_{d}^{(d)})} \right\}, \quad (5.2) $$

where $y_0^{(a)}(v) := 0$, $a \in \{1, 2, \ldots, s\}$ and $m \in \mathbb{Z}_{\geq 1}$. The first term in rhs of (5.2) is proportional to the $a$-th component $\tau_a$ of the Perron-Frobenius eigenvector of $I_{ac}$:

$$ \sum_{c=1}^s I_{ac} \tau_c = \left( 2 \cos \frac{\pi}{g} \right) \tau_a \quad \text{for} \quad a \in \{1, 2, \ldots, s\}, $$

$$ \tau_a = \sqrt{\frac{4}{g} \sin \left( \frac{\pi a}{g} \right)}, \quad \sum_{a=1}^s \tau_a^2 = 1. \quad (5.3) $$

We assume the DVF $T_1^{(1)}(v)$ is expressed as a product of the ground state part $T_1^{(1)gs}(v)$ and the finite temperature correction part $T_1^{(1)fn}(v)$. In particular we can express the finite size correction part for small $T$ by the function (5.1).

$$ \log T_1^{(1)fn}(v) = \sum_{a=1}^s G_a \log (1 + Y_1^{(a)}(v)) \quad (5.4) $$

$$ = \frac{4v_F T}{\pi^\gamma} \cosh(v_F v) \sum_{a=1}^s \sin \left( \frac{v_F a}{2} \right) \int_{-\infty}^\infty dwe^{-v_F w} \log(1 + y_1^{(a)}(w)) + o(T). $$

Using the relation $G_a(-v) = G_a(v)$, one can derive the following relation from (5.2).

$$ \frac{4v_F^2}{\gamma} \sum_{a=1}^s \sin \left( \frac{v_F a}{2} \right) \int_{-\infty}^\infty dwe^{-v_F w} \log(1 + y_1^{(a)}(w)) $$

$$ = -\sum_{a=1}^s \sum_{m=1}^\infty \left\{ L \left( \frac{1}{1 + y_1^{(a)}(\infty)} \right) - L \left( \frac{1}{1 + y_1^{(a)}(-\infty)} \right) \right\}, \quad (5.5) $$
where \( L(x) \) is the Rogers dilogarithmic function
\[
L(x) = -\frac{1}{2} \int_0^x dy \left\{ \log \frac{y}{1-y} + \frac{\log(1-y)}{y} \right\} \quad \text{for} \quad 0 \leq x \leq 1.
\] (5.6)

For small \( T \), the leading term of the specific heat
\[
C = \frac{\partial}{\partial T} \left( T^2 \frac{\partial}{\partial T} \log T_{1{fn}}^{(1)}(0) \right)
\] (5.7)
is proportional\(^{46, 47}\) to the central charge \( c \)
\[
C = \frac{\pi c T}{3v_F} + o(T).
\] (5.8)

Then we obtain
\[
c = -\frac{6}{\pi^2} \sum_{a=1}^{s} \sum_{m=1}^{\infty} \left\{ L \left( \frac{1}{1 + y_m^{(a)}(\infty)} \right) - L \left( \frac{1}{1 + y_m^{(a)}(-\infty)} \right) \right\}.
\] (5.9)

This expression is independent of the choices of the constant \( \gamma \) and widely seen in the model related to rank \( s \) algebras\(^{48, 45}\). To proceed further, we have to evaluate the limit \( y_a^{(a)}(\pm\infty) \). At first, we expect \( \lim_{v \to \infty} y_m^{(a)}(v) \) is given by (4.11). On the other hand, the divergence from the first term in rhs of (5.2) in the limit \( v \to -\infty \) is expected to be canceled by lhs if
\[
y^{(a)}_1(v) \to +0 \quad \text{for} \quad v \to -\infty.
\] (5.10)

Then (5.2) in the limit \( v \to -\infty \) will be valid if \( y_m^{(a)}(-\infty) \) for \( m \in \mathbb{Z}_{\geq 2} \) obey the constant \( Y \)-system (3.1). Thus we expect \( y_m^{(a)}(-\infty) \) is given as the solution (3.10) of (3.1) with a shift \( m \to m - 1 \):
\[
\lim_{v \to -\infty} y_m^{(a)}(v) = \frac{(m-1)(g + m - 1)}{a(g-a)}.
\] (5.11)

Using the relations (5.9), (4.11), (5.11) and \( L(1) = \frac{\pi^2}{3} \), \( L(0) = 0 \), we find \( c = s \). This result is consistent with the conjecture\(^{28}\) by the root density method.

### 6 Discussion

In this paper, we have proposed the TBA equation for \( \text{osp}(1\mid 2s) \). It will be an interesting future problem to study the TBA equation based on a
more general orthosymplectic Lie superalgebra \(osp(r|2s)\) \((r > 1)\). As for the string hypothesis, we will need to consider more complicated strings than (2.9) as \(sl(r|s)\) case [11, 58]. On the other hand, we have only the \(T\)-system for tensor-like representations [33]. To construct complete set of the \(T\)-system which is relevant for the QTM method, we have to treat spinorial representations.

QMTs which act on \(\mathbb{Z}_2\) graded vector space are proposed in [49, 50]. In particular in [50], two types of QMTs, which give the same partition function in the Trotter limit \(N \to \infty\) are proposed. The one is defined as the supertrace of a monodromy matrix of an inhomogeneous graded vertex model. In this case, the sign of the BAE (2.1) will have the form \(\varepsilon_a = (-1)^{\deg(\alpha_a)}\) \((\deg(\alpha_a) = 0\) for the even roots; \(\deg(\alpha_a) = 1\) for the odd root) \).

The other is defined as the ordinary trace of the monodromy matrix. In this case, the sign of the BAE (2.1) will disappear \((\varepsilon_a = 1)\). As far as the ground state case is concerned, the difference among these formulation of the QTM and the ones in this paper may not be important.

A fermionic formula for \(A^{(2)}_{2s}\) coming from a different labeling of the Dynkin diagram is proposed in [51]. Taking account of the correspondence between \(A^{(2)}_{2s}\) and \(B^{(1)}(0|s)\), this formula may also be relevant to \(osp(1|2s)\) case.

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Note Added

After the submission of this paper, a preprint [52] appears, where TBA equations related to \(osp(1|2s)\) models are discussed from the point of view of integrable field theories.

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