PARTICLE CREATION IF A COSMIC STRING SNAPS

A. H. Bilge*
Department of Mathematics,
TUBITAK Marmara Research Center, Gebze Kocaeli, Turkey

M. Hortaçsu
Physics Department,
TUBITAK Marmara Research Center, Gebze Kocaeli, Turkey
I.T.U., 80626, Maslak,Istanbul,Turkey

N. Özdemir
Physics Department,
I.T.U., 80626, Maslak,Istanbul,Turkey

Abstract

We calculate the Bogolubov coefficients for a metric which describes the snapping of a cosmic string. If we insist on a matching condition for all times \textit{and} a particle interpretation, we find no particle creation.

*Present Address: Department of Mathematics, Anadolu University, Eskişehir, Turkey
1. Introduction.

The possibility that cosmic strings may be the root of the mechanism explaining galaxy formation is not still ruled out [1]. We need further experimental data on the anisotropies in the cosmic microwave radiation, lensing of quasar images, or gravitational radiation stemming from the decay of strings to accept or reject this alternative to inflationary quantum fluctuations [2]. If cosmic strings exist, they may give rise to vacuum fluctuations which in turn, may result in particle production [3-8].

In the presence of a time-like Killing vector, one can define in and out states and calculate the Bogolubov coefficients to see whether particle production actually occurs. Another method would be an approximate field theory calculation [9] which has been applied to gravitational particle production during string formation in [10] with a positive result.

In this paper we investigate the particle production during the snapping of a cosmic string and contrast our result with that of [10]. The background metric is chosen as the Gleiser-Pullin solution [11], which describes the snapping of a cosmic string whose ends expand with the velocity of light.

In Section 2, we set up our notation and discuss the Klein-Gordon equation for a $C^{(0)}$ metric. In Section 3, we solve the Klein-Gordon equation (for a scalar field) and investigate whether particle production occurs via the Bogolubov coefficients.

In this analysis we have used a classical result which allows the definition of in and out states in Kasner spaces [12]. The attempts for the computation of the Bogolubov coefficients are given in Sections 3.1 and 3.2.

In Sections 3.1, we obtain a solution in terms of the Hankel functions that has a particle interpretation, and the behavior as $t \to \infty$ can be interpreted as particle production, but this solution is not bounded near the $v = 0$ hypersurface. In Section 3.2, we obtain a continuous solution in terms of hypergeometric functions but there is no particle production.

We recall that the Bogolubov coefficient method has been used in [4] to compute the particle production rates in the case of the instantanous disappearence (or appearence) of a cosmic string along the $z$-axis. In [4], the author uses a smoothed form a the $C^{(0)}$ metric and finds an exact solution for the wave equation. The in and out states correspond respectively to a spacetime with a cosmic string and to an empty spacetime. The matching condition for the solutions of the smoothed wave equation gives nonzero transmission and reflection coefficients, and this situation is interpreted as particle production. We note however that in [4] the number of particles generated and the total energy (obtained by numerical integration) are proportional to $\alpha^3(1 - \beta^2)^2$ and $\alpha^4(1 - \beta^2)^2$ where $\alpha$ is the the string formation rate. Hence they give meaningful
physical results only if the increase in the particle production rate is compensated by a decrease in the resulting conical defect. In the case of a $C^{(0)}$ metric, $\alpha \to \infty$ to start with, hence one should not expect a finite total particle number and total energy.

In the Gleiser-Pullin metric that we start with, the usual in and out states correspond to the interior of a flat expanding sphere and the exterior of this sphere with a cosmic string along the $z-$axis. Thus our in and out states (see Eq.3.1.4) should be related to Sahni’s results [4] in the limit $t \to \infty$. However we cannot obtain a continuous solution with appropriate asymptotic behavior. Even in the case we consider a discontinuous solution that matches at the top two orders as $t \to \infty$, and obtain “asymptotic Bogolubov coefficients”, these lead to a constant particle production rate, hence to an infinite number of particles. We believe that these unsatisfactory results are related to using a $C^{(0)}$ metric, since we would have similar problems in Sahni’s approach, if we use a $C^{(0)}$ metric, as discussed in the previous paragraph.

2. The scalar Klein-Gordon equation.

We start with the metric given by Gleiser and Pullin

$$ds^2 = -4dudv + (u - hv)^2 d\varphi^2 + (u + hv)^2 dz^2$$  \hspace{1cm} (2.1)

where $h = 1$ for $v > 0$ and $h = \beta^2$ for $v < 0$, $\varphi$ is a periodic coordinate, i.e $\varphi \in [0, 2\pi]$ and the points $\varphi = 0$ and $\varphi = 2\pi$ are identified, $z \in (-\infty, +\infty)$, $u > 0$, $u < v$ for $u > 0$ and $u > -\beta^2 v$ for $u < 0$. Note that $h(v)$ has a jump discontinuity, hence $h(v)v$ is continuous, thus the metric in (2.1) is $C^{(0)}$. We can equivalently work with a $C^{(\infty)}$ form of the metric by letting $h(v)$ to be a function that rises from $\beta^2$ smoothly on a small interval $v \in (-a, a)$. Then the range of $v$ is defined to be $h(v)v > u$ for $u > 0$ and $h(v)v > -u$ for $u < 0$.

The parameter $\beta$ is a number close to but less than 1, and is related to deficit angle of a cosmic string. The Ricci tensor is equal to zero, but one component of the Weyl tensor is proportional to a Dirac delta function, which indicates the existence of an impulsive wave.

The Klein-Gordon equation in these coordinates can be obtained as

$$F_{uv} - \frac{1}{(u - hv)^2} F_{\varphi\varphi} - \frac{1}{(u + hv)^2} F_{zz} + \frac{1}{2} h \left[ \frac{1}{u - hv} - \frac{1}{u + hv} \right] F_u + \frac{1}{2} \left[ \frac{1}{u - hv} + \frac{1}{u + hv} \right] F_v = 0$$ \hspace{1cm} (2.2)

We look for solutions that have $e^{i(m\varphi + kz)}$ dependence, with $m$ integer to satisfy the periodicity condition. Hence we let

$$F \to e^{im\varphi} e^{ikz} F, \quad m \text{ integer}. $$
Then (2.2) reduces to

\[
F_{uv} + \frac{m^2}{(u - hv)^2} F + \frac{k^2}{(u + hv)^2} F + \frac{1}{2} h \left[ \frac{1}{u - hv} + \frac{1}{u + hv} \right] F_u + \frac{1}{2} \left[ \frac{1}{u - hv} + \frac{1}{u + hv} \right] F_v = 0. \tag{2.3}
\]

This equation can easily be solved in the domains \( v > 0 \) and \( v < 0 \). The problem arises in matching these solutions. We would be primarily interested in continuous solutions with oscillatory behavior as \( u \to \infty \) when \( v \to 0 \). In the next section it will be seen that this is not possible: continuous solutions have wrong asymptotic behaviour and the solutions with correct asymptotic behaviour can be matched only up to two leading orders.

We can then ask whether we can have generalized solutions, i.e., the ones that can have jump discontinuities or \( \delta \) function discontinuities. Note that a solution with no \( \delta \) function discontinuity is bounded. We will show below that if \( F \) is bounded, then the \( u \) dependence of the jump discontinuity is given by Eq.(2.4).

Assume that \( F \) is bounded on \((-a, a)\) and \( F \big|_{v=-a} = \phi^- (u)\), \( F \big|_{v=a} = \phi^+ (u)\). Since \( h(v)v \) and its derivative are bounded, integrating (2.3) from \(-a\) to \( a\) (using integration by parts for the last term) and letting \( a \to 0 \), we obtain

\[
(\phi^+_u - \phi^-_u) + \frac{1}{u} (\phi^+ - \phi^-) = 0 \tag{2.4}
\]

Thus \((\phi^+ - \phi^-)\) is proportional to \(\frac{1}{u}\).

If \( F \) is allowed to have \( \delta \) function (and derivatives of \( \delta \) functions) discontinuities, then it can be seen that any jump discontinuity can be accommodated. But as the physical meaning of such solutions is not well defined, we defer the discussion of such solutions.

In Section 3.1, we obtain solutions with correct asymptotic behavior that can be matched only up to \( O(\frac{1}{t}) \). Thus the discontinuity across \( v = 0 \) hypersurface obeys (2.4) asymptotically, hence is bounded as \( t \to \infty \).

3. Computation of the Bogolubov coefficients.

3.1. Solution using Hankel functions.

We can solve the equation (2.3) for \( v \neq 0 \) in the following coordinate system.

\[
t = u + \beta^2 v \\
s = u - \beta^2 v
\]
This coordinate system results in incoming and outgoing waves for asymptotic times and it is appropriate for particle interpretation [12]. Then the equation reduces to

$$\left\{ \frac{1}{t} \frac{\partial}{\partial t} - \frac{1}{s} \frac{\partial}{\partial s} + \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2} \right\} + \frac{m^2}{\beta^2 s^2} + \frac{k^2}{\beta^2 t^2} \phi = 0$$  \hspace{1cm} (3.1.2)

with solutions

$$Z_m(as)Z_{ik}(at) \quad \text{for} \quad v > 0$$ \hspace{1cm} (3.1.3a)

$$Z_{\mp}(as)Z_{ik}(at) \quad \text{for} \quad v < 0$$ \hspace{1cm} (3.1.3b)

where $a$ is a constant parameter and the $Z$’s are generic functions obeying Bessel’s equation. We take solutions

$$F^{(in)} = \tilde{A}H_m^{(1)}(as)H_k^{(2)}(at)$$

$$F^{(out)} = \tilde{B}H_m^{(1)}(as)H_k^{(2)}(at) + \tilde{C}H_m^{(2)}(as)H_k^{(1)}(at)$$ \hspace{1cm} (3.1.4)

If we want a continuous solution we need to match the respective solutions at $t = s$. We will try to match the solutions asymptotically, we will see that we can do this only in the leading two orders.

We obtain

$$\tilde{A}e^{i\pi/2(-m+ik)} \left[ 1 - \frac{1}{2it} \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m - \frac{1}{2})} + O(t^{-2}) \right] \left[ 1 + \frac{1}{2it} \frac{\Gamma(ik + \frac{1}{2})}{\Gamma(ik - \frac{1}{2})} + O(t^{-2}) \right]$$

$$= \tilde{B}e^{i\pi/2(-m+ik)} \left[ 1 - \frac{1}{2it} \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m - \frac{1}{2})} + O(t^{-2}) \right] \left[ 1 + \frac{1}{2it} \frac{\Gamma(ik + \frac{1}{2})}{\Gamma(ik - \frac{1}{2})} + O(t^{-2}) \right]$$

$$+ \tilde{C}e^{i\pi/2(-m-ik)} \left[ 1 + \frac{1}{2it} \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m - \frac{1}{2})} + O(t^{-2}) \right] \left[ 1 - \frac{1}{2it} \frac{\Gamma(ik + \frac{1}{2})}{\Gamma(ik - \frac{1}{2})} + O(t^{-2}) \right]$$ \hspace{1cm} (3.1.5)

We define

$$A = \tilde{A}e^{i\pi/2(-m+ik)}, \quad B = \tilde{B}e^{i\pi/2(-m+ik)}, \quad C = \tilde{C}e^{-i\pi/2(-m+ik)}$$ \hspace{1cm} (3.1.6)

and

$$\omega(m, ik) = -\frac{\Gamma(m + \frac{1}{2})}{\Gamma(m - \frac{1}{2})} + \frac{\Gamma(ik + \frac{1}{2})}{\Gamma(ik - \frac{1}{2})}$$ \hspace{1cm} (3.1.7)

Then the equation reduces to

$$A \left[ 1 + \frac{1}{2it} \omega(m, ik) + O(t^{-2}) \right] = B \left[ 1 + \frac{1}{2it} \omega(m, ik) + O(t^{-2}) \right]$$

$$+ C \left[ 1 - \frac{1}{2it} \omega(m, ik) + O(t^{-2}) \right].$$ \hspace{1cm} (3.1.8)

At the first two orders we obtain

$$A \omega(m, ik) = (B - C) \omega(m, ik)$$ \hspace{1cm} (3.1.9)
Using also the condition

\[ |B|^2 - |C|^2 = 1 \]  \hspace{1cm} (3.1.10)

we solve \(|B|^2\) and \(|C|^2\) as

\[ |B|^2 = \frac{|\omega + \tilde{\omega}|^2}{|\omega + \tilde{\omega}|^2 - |\omega - \tilde{\omega}|^2} \quad |C|^2 = \frac{|\omega - \tilde{\omega}|^2}{|\omega + \tilde{\omega}|^2 - |\omega - \tilde{\omega}|^2} \]  \hspace{1cm} (3.1.11)

where \(\omega = \omega(m, ik), \tilde{\omega} = \omega(m \beta, i k \beta)\).

Using the properties of the \(\Gamma\) function it can be seen that

\[ |C|^2 = \beta^2 \left(1 - \frac{1}{\beta^2}\right)^2. \]  \hspace{1cm} (3.1.12)

Thus the transmission coefficient is constant, hence its integral cannot be finite, and we cannot obtain a finite total number of particles.

As discussed at the end of Section 1, this result is related to using a \(C^{(0)}\) metric.

### 3.2. Solution using hypergeometric functions.

In this section we obtain a solution which is continuous across \(v = 0\) hypersurface but asymptotically it goes as a power of \(t\). Here we use the following unorthodox coordinate system:

\[ u = p \cosh \theta \]  \hspace{1cm} (3.2.1)

\[ \beta^2 v = p \sinh \theta \]  \hspace{1cm} (3.2.2)

We note that, since the Fulling et.al. result [12] is shown for systems with Hankel function solutions, a positive result in this coordinate system would not mean particle production. Despite this remark, we will use our negative result as a plausibility argument, since it is in line with our previous result.

If we take solutions of the type

\[ f = e^{im\phi} e^{i\kappa z} g(p, \theta) \]  \hspace{1cm} (3.2.3)

our equation reads

\[ (-p^2 \frac{\partial^2}{\partial p^2} + 2 \coth 2\theta \frac{\partial^2}{\partial p \partial \theta} - p \frac{\partial}{\partial p} - \frac{\partial^2}{\partial \theta^2} - \frac{2k^2}{\beta^2} (\coth 2\theta - 1) + \frac{2m^2}{\beta^2} (\coth 2\theta + 1)) g = 0 \]  \hspace{1cm} (3.2.4)

where \(\kappa = ik\). An immediate solution is

\[ g = p^s (x^2 - 1)^{m/2 + \frac{k}{2}} \left[ \frac{2}{(1 + x)} \right]^{m/2 + \frac{k}{2}} e^{-\nu(m+k)} \text{$_2$F$_1$}\left[\frac{k + m}{2} - s, \frac{m - k}{2} - s; \frac{2}{1 + x}\right] \]  \hspace{1cm} (3.2.5)
where \( x = \coth 2\theta \), \( _2F_1(a, b, c, x) \) is the hypergeometric function. This solution goes as \( t \) to a power in the asymptotic region; hence lacks the particle interpretation given in [12]. When \( v = 0 \), we see that \( \theta = 0 \) and this reduces to \( p^s \).

For \( v < 0 \) we write the solution as,

\[
g = p^s(x^2 - 1)^{\frac{k + m}{2\beta}} e^{-\frac{(s-m)\theta}{2\beta}} _2F_1\left( k + m, \frac{k - m}{2\beta} - s, \frac{s - m}{2\beta} - s; \frac{2}{1 + x} \right)
\]

This solution also has the power behavior in the asymptotic region. This solution reduces to \( p^s \) when \( v = 0 \). Although we achieved the matching without problem with reasonable behaviour at the asymptotic region, we see that we can’t fix the two independent functions needed. We know that when \( v < 0 \), we get two independent solutions for the hypergeometric equation

\[
_bF_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^1 t^{a-1}(1-t)^{b-1}(1-tz)^{c-1} dt
\]

The second solution is either identically zero or singular at \( z \) equals zero depending on the sign of \( (1 - c) \). We choose the former case. Here \( c \) equals one is not allowed. In this case matching can be achieved when the second independent function is put to zero. This shows particle production can be consistently put to zero.

4. Conclusion.

We find that if we insist on a particle interpretation we can not get a consistent matching of our solutions in a specific \( C(0) \) metric. Only if we neglect terms that are \( O(t^{-2}) \) and further, we can find the matching coefficients which give infinite production. Note also that a possible matching is possible in coordinate systems where a particle interpretation can not be given.

We see that we have to be careful in applying approximate method to solutions for metrics with abrupt changes. We agree with Bernard-Duncan [13] who state that we have to be very careful in problems using \( C(0) \) metrics. We see that any of the methods used in the smoothed case [4], don’t seem to work in our case. Our result also shows that the results obtained from approximate field theory calculations [9,10] should be taken with a grain of salt.

Acknowledgement: We thank Prof. Dr. Yavuz Nutku for suggesting this problem. This work is partially supported by TÜBİTAK, The Scientific and Technical Council of Turkey under TBAG-ÇG/1.
REFERENCES

1. A. Vilenkin, Phys. Reports C121 (1985) 263.
2. C. Frenk, S. White, M. Davis and G. Estathiou, Astrophys. J. 327 (1982) 507.
3. L. Parker, Phys. Rev. Lett. 59 (1987) 1365.
4. V. Sahni, Modern Phys. Lett. A 3 (1988) 1425.
5. G. Mandell and W. Hiscock, Phys. Rev. D40 (1989) 282.
6. T. M. Helliwell and K. A. Konkowski, Phys. Rev. D34 (1986) 1908.
7. B. Linet, Phys. Rev. D33 (1986) 1833; Phys. Rev. D35 (1987) 536.
8. N. D. Birrell and P. C. W. Davies, Quantum Fields in Curved Space (Cambridge University Press).
9. J. Frieman, Phys. Rev. D39 (1989) 389;
   Ya. B. Zel’dovich and A. A. Starobinsky JETP Lett. 26 (1977) 1252.
10. V. Hussain, J. Pullin, E. Verdaguer, Phys. Lett., B232 (1989) 299.
11. R. Gleiser, J. Pullin, Class. Quantum Gravity, 6 (1989) L141.
12. S. A. Fulling, L. Parker and B. L. Hu, Phys. Rev., D12 (1974) 3905.
13. C. Bernard and A. Duncan, Annals Phys. (N.Y) 107 (1977) 201.