Stochastic pumping of particles with zero-range interactions

Shahaf Asban and Saar Rahav

1 Faculty of Physics, Technion—Israel Institute of Technology, Haifa 32000, Israel
2 Schulich Faculty of Chemistry, Technion—Israel Institute of Technology, Haifa 32000, Israel
E-mail: rahavs@tx.technion.ac.il

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Abstract

We consider a model of a stochastic pump in which many particles jump between sites along a ring. The particles interact with each other via zero-range interactions. We argue that for slow driving the dynamics can be approximated by a nonlinear diffusion equation. This nonlinear equation is then used to derive a current decomposition formula, expressing the current as a sum of two contributions. The first is from the momentary steady state while the second arises due to the variation of the density with time, and is identified as the pumped current. The dynamics is found to satisfy the no-pumping theorem for cyclic processes, in agreement with recent results in discrete systems.

1. Introduction

In our macroscopic world we are used to encounter machines and motors which are designed for specific tasks. Such machines work in a predictable and ordered manner. It is fascinating to realize that microscopic systems may also operate as machines, even though their dynamics exhibit a large measure of randomness, due to bombardment by their environment. Many biological cellular processes are assisted by proteins which act as motors [1]. This has been used as a motivation to design and synthesize artificial molecular machines for specific objectives [2–6]. Such artificial machines can function at conditions in which their biological counterparts fail, for instance to operate at a much wider temperature range. Moreover, they can be driven by new driving mechanisms. One such driving mechanism exhibits similarity to that of macroscopic pumps. In this mechanism the system is driven away from equilibrium by a periodic variation of externally controllable parameters. For each value of the parameters the system may satisfy detailed balance. However, when external parameters are varied in time the system’s probability distribution lags behind the momentary equilibrium distribution; this may result in the generation of some directed motion after a full cycle is completed. Systems driven by this mechanism are often referred to as stochastic pumps.

Stochastic pumps have generated some interest, partially due to their resemblance to quantum pumps [7, 8]. When driven adiabatically their dynamics can also be described geometrically in parameter space [9, 10]. Other topics that were studied included the appearance of quantization of currents at low temperature [11], perturbative approaches for the calculation of their state [12–14], detailed studies of simple models [15], and the efficiency of stochastic pumps [16]. More information can be found in the reviews of Sinitsyn [10] and Dean Astumian [17].

An intriguing aspect of stochastic pumps is that the relation between the driving cycle and the resulting pumped current is not trivial. One may even ask under what conditions driving the system does not result in directed motion. It turns out that a seemingly natural way of driving such systems will not lead to directed motion. Consider a thermally activated system with transition rates of the form \( R_{ij} \propto \exp\left[ \frac{(E_i - E_j)}{k_B T} \right] \), where \( E_i \) are local site energies, and \( B_{ij} = B_{ji} \) are energy barriers. Driving the system by a periodic variation of \( E_i(t) \), while keeping the barriers constant, does not lead to directed motion even when the variation of site energies during the cycle has a preferred direction. The net flux accumulated during a full cycle of driving vanishes. One has to vary both \( E_i \) and \( B_{ij} \) in time to generate directed motion [18]. This result was termed no-pumping theorem (NPT). The NPT was originally motivated by a beautiful experiment in which catenane based artificial molecular machine was realized [19]. The adiabatic limit of this system was studied by Dean Astumian [13].
However, the NPT holds also for non-adiabatic driving. The NPT was generalized to account for the topology of the network by Chernyak and Sinitsyn [20]. Due to its appealing simplicity, the NPT has generated some interest [21–25].

In the experiment [19], the Leigh group understood that a single particle stochastic pump does not result in directed motion. Ingeniously, they have induced directionality by harnessing particle–particle interactions into the system. They synthesized two molecules interlocked onto a larger ringlike molecule. The presence of one molecule in a binding site was assumed to prevent the other from entering, thereby exhibiting exclusion interactions. This interaction resulted in the generation of directed motion when the local site energies, $E_i$, were varied periodically. The exclusion interaction as depicted in figure 1(a), forbids more than one particle per site. Models satisfying exclusion interaction have been extensively studied as microscopic models for many particles systems away from equilibrium [26–28]. A natural question that may arise is whether the addition of any interaction is enough to violate the NPT? In a recent paper we considered stochastic pump of particles interacting via zero-range interaction [29]. Surprisingly, we have found that the NPT holds for zero-range interactions. The zero range process (ZRP) describes hopping of particles between neighboring sites with transition rates that depend in the most general form on the local potential and the number of particles at the departure site, see figure 1(b). These results suggest an interesting and fundamental difference between local interactions (zero-range) and nearest-neighbor type interactions such as exclusion. Mandal has shown that this NPT of interacting particles applies also to open systems [30].

Exclusion [28, 31] and zero-range [32, 33] interactions are the paradigmatic interactions used to investigate non-trivial steady states of many particle systems away from equilibrium. One extensively studied aspect of the dynamics of many particles and sites, be it a zero-range or exclusion process, is their behavior on long wavelength and time-scales. On such time-scales the evolution is expected to follow a diffusion-like equation, often referred to as the hydrodynamic equation [26, 31, 34]. In absence of interactions the hydrodynamic equation is simply the diffusion equation. Horowitz and Jarzynski have studied the diffusion of driven non-interacting particles on a ring, and derived the NPT for this case [21]. The existence of NPT with zero-range interactions on a network [29], and for non-interacting diffusing particles [21] suggests that the hydrodynamic limit of the ZRP should also exhibit an NPT.

Proceeding with this line, several question naturally emerge; what is the form of the hydrodynamic equation describing a slowly driven ZRP? What is the structure of the corresponding NPT in this hydrodynamic description? In this paper we argue that the hydrodynamic limit of the ZRP, when it exists, is described by a nonlinear diffusion equation with a particular structure. Furthermore, we show that due to this structure, the NPT derivation by Horowitz and Jarzynski [21] can be modified to apply for the case of driven particles with zero-range interactions. Finally, we give a heuristic argument explaining why a derivation along similar lines would fail for exclusion processes.

The organization of the paper is as follows. In section 2 we offer a heuristic derivation for the hydrodynamic equation of a slowly driven ZRP. The resulting evolution due to the nonlinear diffusion equation is compared to a corresponding kinetic Monte-Carlo (KMC) simulation. In section 3.1 a formal expression for the current, expressed in terms of the momentary steady state current, and the time derivative of the density is derived. The resulting expression is used to investigate the conditions for no-pumping in section 3.2. In section 4 we discuss the implications of these results emphasizing the differences between exclusion and zero-range interactions.
2. The hydrodynamic equation of a ZRP

When studying a system with many particles, a substantial simplification can be achieved by considering the locally averaged density instead of a description based on the precise position of all particles. The equation that describes such coarse-grained evolution is referred to as the hydrodynamical equation. This type of hydrodynamic description requires that the dynamics of a system display local equilibrium, namely, when local thermalization has a significantly shorter time-scale than the equilibration time of the whole system. In this section we present a derivation for the hydrodynamic equation of one dimensional ZRP with potential that varies slowly in space and time. We then compare the evolution of this equation with a numerical simulation of a jump process.

2.1. Analytical derivation

Consider a system of  \( \mathcal{M} \gg 1 \) sites on a 1D lattice of length of \( L \) endowed in a ring topology (periodic). The system contains \( N \gg 1 \) particles. Let this one dimensional system obey the dynamics of the ZRP with the transition rates of the form

\[
R_{i \rightarrow i+1} = \varphi_i \chi_{i,i+1} \omega_{n_i},
\]

where \( \varphi_i \) is the local potential (corresponding to local minima in figure 2). We interpret \( \varphi_i \) as depending on local energy that may generally be varied in time. For the discrete ZRP with thermal activation rates, considered in [29], \( \varphi_i = \exp(E_i/T) (k_B = 1) \). Similarly, \( \chi_{i,j} = \chi_{j,i} \) is the inter-site barrier, analogous to the potential barriers in [29], where \( \chi_{i,j} = \exp(-B_{i,j}/T) \) (inter-well separation in figure 2). Finally \( \omega_{n_i} = n_i \eta(n_i) \) is the part of the transition rate that depends on the number of particles in site \( i \), where \( \eta(n_i) \) includes the contribution of the local interactions. In the absence of interactions \( \eta(n) = 1 \); hence the factor \( n_i \) expresses the fact that \( n_i \) particles at site \( i \) are equally likely to jump. In [29] \( \eta(n) = \exp[(\bar{U}(n)/T)], \) where \( \bar{U}(n) = U(n) - U(n-1) \) is the difference between the local interaction energy containing \( n_i \) and \( n_i - 1 \) particles. This ensures that in the absence of driving the system relaxes to the Boltzmann distribution with the correct many body energy, i.e. \( E = \sum n_i E_i + \bar{U}(n_i) \). We demand that \( \varphi_i(t) \) and \( \chi_{i,j}(t) \) vary slowly spatially, such that they change considerably only when \( |i - j| \approx \mathcal{M} \), namely on length-scales comparable to the system size. Such spatial slow variation of parameters make it possible to approximate differences by derivatives in the hydrodynamic limit.

The ZRP is characterized by a simple steady state probability distribution, namely the product measure [26, 33]. In this measure the occupation probability at site \( i \) is given by the single site marginals

\[
P_{\varphi_i}(n_i) = \frac{\phi_i^{n_i}}{Z_i} \prod_{k=1}^{n_i} [k \eta(k)]^{-1},
\]

where the empty product \( (n_i = 0) \) is one by definition. Strictly speaking, by employing equation (2) we describe the system using a grand-canonical ensemble, allowing the number of particles to fluctuate. Depending on the precise form of the transition rates, the ZRP may have a condensate phase in which particles accumulate at one site while the average occupancy at the rest of the sites is fixed at a critical density \( \rho_c \) [35]. Away from the condensate regime of the ZRP, namely for all repulsive and some attractive interactions, the grand canonical measure (2) is expected to be a good approximation for systems with \( N \gg 1 \) particles. We assume that this is the case in what follows.
Time dependent ZRP’s are not as studied as their steady-state counterparts. Nevertheless, when the time-scale of the drive is longer than the local equilibration time-scale, we expect equation (2) to be a good approximation of the particle occupation distribution. $\phi_i$ is a bare fugacity which determines the average occupation through

$$\langle n_i \rangle (\phi_i) = \sum_{n_i} n_i P_{\phi_i}(n_i). \quad (4)$$

The term bare fugacity is used since we omitted $\{\phi_i, \chi_{i+1}\}$ from equation (2). Direct substitution of the product measure in the master equation leads to an equilibrium between sites when $\phi_i = \phi_{i+1}$.

The product measure (2) can be used to derive the hydrodynamic equation describing the slow dynamics of the ZRP. Its structure allows one to easily calculate local (ensemble) averages defined via

$$\mathcal{O}_i = \langle n_i \rangle = \sum_{n_i} n_i P_{\phi_i}(n_i). \quad (5)$$

The local average of the current between two successive sites is given by

$$J_{i\rightarrow i+1} = \sum_{n_i} P_{\phi_i}(n_i) R_{i\rightarrow i+1} - \sum_{n_{i+1}} P_{\phi_{i+1}}(n_{i+1}) R_{i+1\rightarrow i}$$

$$= \chi_{i+1} \left( \phi_i \phi_i - \phi_{i+1} \phi_{i+1} \right). \quad (6)$$

Particles conservation results in the continuity equation

$$\frac{d}{dt} \langle n_i \rangle = J_{i\rightarrow i-1} - J_{i\rightarrow i+1}. \quad (7)$$

To complete the derivation, we aim to replace the discrete equations (6) and (7) by continuous differential equations. Specifically we replace the site index $i$ by a continuous variable $x$ and the local occupation average $\langle n_i \rangle$ by the continuous density $\rho(x)$. This can be done by considering a limit where one increases the number of sites (particles) $\mathcal{M}(N)$, and reduces the distance between them $\Delta x$. This continuum limit is constructed so that $\Delta x \to 0$ and $\mathcal{M}(N) \to N$ while $N/\mathcal{M}(N) \to N'$ and $\mathcal{M} \Delta x \to L$. A slight complication may arise from the fact that the right-hand side of equation (6) depends on the bare fugacity, $\phi_i$ and not on the occupations $\langle n_i \rangle$. To formally derive the hydrodynamic equation, we exploit the relation between the local fugacity and average occupation number $\langle n_i \rangle_{\phi} = \phi_i \partial_{\phi_i}$ in $Z_\phi \langle \phi_i \rangle$, and invert it to obtain $\phi_i (\langle n_i \rangle)$. This procedure can always be done, since $\langle n_i \rangle_{\phi}$ is monotonically increasing. In the continuum limit, $\phi_i (\langle n_i \rangle)$ is replaced by a continuous function $h[\rho(x)]$, and equation (6) transforms to

$$J(x) = -\chi(x) \frac{d}{dx} \{\phi(x, t) h[\rho(x, t)]\}, \quad (8)$$

while the continuity equation is recast as

$$\frac{\partial}{\partial t} \rho(x, t) + \frac{d}{dx} J(x, t) = 0. \quad (9)$$

We finally combine equations (8) and (9) to obtain the nonlinear diffusion equation

$$\frac{\partial}{\partial t} \rho(x, t) = \frac{d}{dx} \left\{ \chi(x, t) \frac{d}{dx} \{\phi(x, t) h[\rho(x, t)]\} \right\}. \quad (10)$$

Equation (10) is the hydrodynamical equation which describes the long length and time scales of the ZRP. For stochastic pumps with many sites and particles equation (10) represents an effective dynamics in which pumping is achieved by cyclic variation of $\chi$, $\phi$ in time. In the case of thermal activation considered in [29], $\phi(x, t) = \exp [E(x, t)/T]$ and $\chi(x, t) = \exp [-B(x, t)/T]$. Furthermore, in the absence of interactions, $h[\rho(x, t)] = \rho(x, t)$ and equilibrium is achieved for $\rho(x) \propto \exp [-E(x)/T]$ as expected.

Equation (10) is consistent with prior studies of the hydrodynamic limit of time-independent ZRP [36]; there it was demonstrated that the interaction results in a hydrodynamic equation in which the fugacity replaces the density. The heuristic derivation above suggests that the same equation can be applied to time dependent systems as long as local equilibration is a good approximation. We are not aware of work discussing the conditions for deviations from local equilibration. Nevertheless, it is reasonable to expect that local equilibration is a good approximation when the time scale for equilibration is much shorter than the period of the driving.

2.2. Comparison with numerical simulations

To test the nonlinear diffusion equation (10), we compared its predictions to a simulation of a system with many particles and sites. The evolution was computed via straightforward numerical integration of (10), with periodic...
boundary conditions and a uniform initial distribution at \( t = 0 \). The results were compared to a direct numerical simulation of the jump process, comprising of 1000 particles and 100 sites. This many-particle system was also initially uniformly distributed, and the simulation followed its relaxation towards equilibrium. The local potential was set to 
\[ E(x) = -\pi \cos(2\pi x/L) + Bx, \]
leading to nonuniform equilibrium profiles. The interaction term was chosen to satisfy 
\[ U(n) = \gamma \ln n. \]
The simulation was realized with a KMC using the BKL algorithm \([37, 38]\). The resulting profiles were averaged over a few hundreds of realizations to obtain a smoother density profile. The resulting profile is compared to the one predicted by equation (10) in figure 3. It is clear that there is a good agreement between the evolution predicted by equation (10) and the simulation.

Based on the analytical arguments above, and on the numerical correspondence, we view equation (10) as representing the hydrodynamics of a slowly driven ZRP. This equation will therefore be the basis of our investigation of the no-pumping conditions in the hydrodynamic limit of ZRP.

3. NPT for the hydrodynamic limit of the ZRP

Our goal is to investigate the conditions under which a periodic variation of \( \chi(x, t) \) and \( \phi(x, t) \) will not lead to a flux of particles around the ring. This no-pumping condition can be derived from a formal expression that decomposes the current in the system into a sum of two contributions \([18, 21]\). The first is due to the steady state current at the momentary value of the external parameters (here given by \( \chi, \phi \)). The second is an excess current which results from the time variation of the density. In section 3.1 we derive this current decomposition formula for systems evolving according to equation (10). This derivation follows similar approach to that of \([21]\), with modifications due to the inter-particle interactions. The current decomposition formula is then used to study the conditions for no-pumping in section 3.2.

3.1. Current decomposition formula

Equation (10) can also be written as
\[ \frac{\partial \rho}{\partial t} = \mathcal{L}_{FP} h[\rho(x, t)], \tag{11} \]
where \( \mathcal{L}_{FP} = -\partial_x F(x, t) + \partial_x^2 D(x, t) \) is the Fokker–Planck operator. Here \( F(x, t) = (\partial_x \chi) \phi \) and \( D(x, t) = \phi \chi \). \( F(x, t) \) therefore corresponds to the force field felt by the particles, while \( D(x, t) \) is the diffusion coefficient.

The current decomposition formula is obtained by formally considering equation (11) as an equation in which the Fokker–Planck operator maps a function \( h[\rho(x, t)] \) to a function \( \partial_t \rho(x, t) \), ignoring the fact that this is also an evolution equation. One may then try to invert the equation and express \( h[\rho(x, t)] \) as \( \mathcal{L}_{FP}^{-1} \partial_t \rho(x, t) \).
This would have allowed one to calculate the current using \( J(x, t) = F(x, t)h[\rho(x, t)] - \partial_x D(x, t)h[\rho(x, t)] \) —which is easily derived using the continuity equation—resulting in an expression for the excess current which depends on \( \partial_\rho \). This formal prescription is complicated by the fact that \( \mathcal{L}_\mathrm{FP} \) has a zero eigenvalue and therefore is not invertible. The right eigenfunction corresponding to this vanishing eigenvalue is the momentary steady-state \( \rho^\alpha \), satisfying the relations \( \mathcal{L}_\mathrm{FP} h[\rho^\alpha] = 0 \). We assume that this steady state is unique. The corresponding left eigenfunction is the constant function on the circle.

Since \( \mathcal{L}_\mathrm{FP} \) is not invertible, the current decomposition formula is obtained using the pseudoinverse of \( \mathcal{L}_\mathrm{FP} \). This pseudoinverse projects out the subspace of the steady state and at the same time acts as an inverse in the orthogonal subspace. In [21], Horowitz and Jarzynski have found the pseudoinverse of \( \mathcal{L}_\mathrm{FP} \) and derived the current decomposition formula for non-interacting particles satisfying (11), but with \( h[\rho] = \rho \). A similar pseudoinverse can be used here, with some care due to modifications originating from the nonlinear nature of \( h[\rho] \). The details of this calculation can be found in the appendix. The resulting current decomposition formula is

\[
J(x, t) = \beta(t)J^\infty + \int_0^L \mathrm{d}x' \nu(x, x'; t)\rho(x', t),
\]

where

\[
\beta(t) = \frac{\int_0^L \mathrm{d}x h[\rho(x, t)]}{\int_0^L \mathrm{d}x h[\rho^\alpha(x, t)]}
\]

is a renormalization resulting from the nonlinear nature of \( h[\rho] \). The appearance of this factor is the main difference between the nonlinear diffusion considered here and the linear problem studied in [21]. It emerges since although \( \int_0^L \mathrm{d}x \rho(x, t) \) is a constant of motion, \( \int_L \mathrm{d}x h[\rho(x, t)] \) is not. For a non-interacting system \( \beta(t) \) is identically equal to 1.

The kernel \( \nu(x, x'; t) \) in (12) takes the form

\[
\nu(x, x'; t) = [1 + \beta(t)J^\infty \tau_L] \pi(x', t) + \theta(x' - x) + \beta(t)J^\infty \tau(x', t)
\]

where \( \theta(x' - x) \) is the Heaviside function, while \( \pi(x, t) \) and \( \tau(x, t) \) are given by

\[
\pi(x, t) = \frac{\int_x^L \mathrm{d}y e^{-\int_x^y \frac{\nu_j(\tau)}{\nu_j(x)}}}{\int_0^L \mathrm{d}y e^{-\int_0^y \frac{\nu_j(\tau)}{\nu_j(x)}}},
\]

and

\[
\tau(x, t) = \int_0^x \mathrm{d}y \int_0^L \mathrm{d}z D^{-1}(z).
\]

\( \pi(x, t) \) can be interpreted as the splitting probability, expressing the likelihood that a particle in the system who was initially at \( x \) leaves the region \( (0, L) \) through the left \( (x = 0) \). \( \tau(x, t) \) is the corresponding mean exit time out of this domain, while \( \tau_L = \tau(L, t) \). Note that the expression for the kernel is not unique and exhibits some gauge freedom such that \( \nu(x, x'; t) \to \nu(x, x'; t) + \lambda(x) \) results in the same current for any \( \lambda(x) \).

In order to discuss the condition for no-pumping we now focus on systems that satisfy detailed balance at each instance of the driving cycle. \( \tau_L = 0 \) for such systems, and the kernel simplifies to

\[
\nu(x, x'; t) = \pi(x', t) + \theta(x' - x).
\]

The only contribution for the current is due to the time variation of the probability density. We find

\[
J(x, t) = \int_0^L \mathrm{d}x \left[ \pi(x', t) + \theta(x' - x) \right] \frac{\partial \rho}{\partial t}.
\]

We have verified the validity of equation (17) by numerically solving the dynamics of (11) and then calculating separately both sides of (17) assuming the system has settled into its asymptotic periodic state (see section 3.2). The results are depicted in figure 4. The parameters used are \( \varphi(x, t) = \exp \{ -4 + \cos[2\pi(x/L - t/T)] \} \), \( g(x, t) = \exp \{ 1 + \cos[2\pi(x/L)] \} \), \( \eta(n) = \exp \{ U(n) \} \) and \( U(n) = \gamma \log n \) with \( \gamma = T = 1 \). It is clear that there is excellent agreement with the decomposition formula (17).

### 3.2. No pumping condition

When the system is driven periodically with period \( \tau \) it eventually settles into an asymptotic state in which \( \rho(x, t) = \rho(x, t + \tau) \). During a cycle the particles may slosh one way or another at different times. An indicator...
of the generation of directed motion is whether there is a net flux of particles after a full cycle of the dynamics. One is therefore interested in the integrated flux

$$\Phi = \frac{\partial}{\partial \tau} \int_{0}^{L} \nu(x', t; t) \frac{\partial \rho(x', t)}{\partial t} \, dx \, dt,$$

where we assumed detailed balance. Here $\oint_{\tau} \rho \partial \tau dt$ denotes a time integral over a full period of the driving cycle. For a system on a ring with periodic boundary conditions $\Phi(x)$ will not depend on $x$ in the asymptotic state. When $\Phi \neq 0$ some directed motion is generated while for $\Phi = 0$ clearly there is no directed motion.

A non-trivial, but not necessarily the most general, condition for no-pumping is that the kernel $\nu(x', t)$ is time independent. In this case the periodicity of $\rho$ implies that $\oint \partial \rho dt \partial t = \partial \rho(x, t + \tau) - \partial \rho(x, t) = 0$, thus $\Phi = 0$. From equations (14) and (16) it is clear that the kernel $\nu(x, x'; t)$ is time independent when the ratio $F(x, t)/D(x, t)$ is time independent for any $x$. Substituting the expressions for $F$ and $D$ we find the no-pumping condition

$$\frac{\partial}{\partial \tau} \left[ \frac{\partial}{\partial x} \ln \chi(x, t) \right] = 0.$$

This condition states that there is no pumping of excess current for time independent barriers, in agreement with previous results. No directed motion will be generated also if the time dependence of $B$ is through a spatial independent term. We note that there is another no-pumping condition, which is not obtained from this derivation. Specifically, there is no net flux when the barriers are time dependent but the local energies are not. In this case the system will relax to an equilibrium distribution where $J(x, t) \equiv 0$, and will not exhibit directed motion.

Figure 5 depicts the integrated currents as a function of time in the asymptotic state of two different driving protocols, with time dependent and time independent $\chi$. The currents were calculated by numerical integration of (11) with $\nu(x) = \exp[-4 + \cos(2\pi(x/L - t/\tau))]$, $\chi(x) = \exp[1 + \Delta \cdot \cos(2\pi(x/L - t/\tau))]$ and $U(n) = \gamma \log n$. The interaction strength was chosen to be $\gamma = 3/2$. For time independent $\chi(\Delta = 0)$, the integrated flux after a full period ($t = \tau$) vanishes as predicted by the no-pumping condition (19). In contrast, for time dependent $\chi(\Delta = 1)$, one observes a finite flux after a full period of driving as expected.
4. Discussion

Small systems can be driven out of equilibrium in a variety of ways. Most biological motors and engines are driven by autonomous thermodynamic affinities. However, systems can also be driven out of equilibrium via a cyclic variation of parameters, just like a pump. Under such driving the probability distribution lags behind the momentary steady-state distribution, and this lag may result in an excess or pumped current.

For stochastic pumps composed of non-interacting particles it was found that the current can be decomposed into a sum of a contribution from the momentary steady-state current, \( J_{ss} \), and an excess current, originating from the time dependence of the probability distribution [18, 21]. For systems which satisfy detailed balance this current decomposition formula was used to identify pumping cycles which do not result in directed motion. Recently it was found, using a different approach, that zero-range interactions do not violate the NPT. It is of interest then to find out how zero-range interaction modify the current decomposition formula.

In this paper we considered a slowly driven system with many particles and sites. In many cases it is useful to approximately describe the dynamics in terms of an equation for a coarse-grained density field. For systems of non-interacting particles this hydrodynamic equation is simply the diffusion equation. The current decomposition formula and the NPT for this case were studied by Horowitz and Jarzynski [21]. Here, we have found that the inclusion of zero-range interactions modifies the hydrodynamic equation. The linear diffusion equation is replaced by a nonlinear one, see equation (10). The effects of the interactions are expressed through the appearance of the nonlinear function \( \rho_h \) in (10).

How does the interaction affect the validity of the current decomposition formula? Crucially, the same nonlinear function of \( \rho \) appears in the drift and diffusion terms of equation (11). As a result the pseudoinverse method used in [21] can be also applied for the interaction considered here. We obtained a modified current decomposition, given by equations (12) and (13). While the current is decomposed into a momentary steady-state and excess contributions, the former is renormalized. This is a direct result of the nonlinear nature of \( \rho_h \), and is the most prominent difference from the non-interacting case. For systems satisfying detailed balance \( (J^* = 0) \) the current decomposition formula can be used to identify the conditions for no-pumping. We find that the no-pumping theorem holds for zero-range interactions, in agreement with previous results obtained for models with a discrete set of sites [29, 30] and with the results of [21] for non-interacting particles.

Is the zero-range interaction special? Will non-local interactions break the NPT? A commonly studied interaction is exclusion, in which a particle is prevented from entering an already occupied site. This is clearly a non-local effect. The hydrodynamic equation for the weakly asymmetric exclusion process has the form [31]

\[
\frac{\partial \rho_x}{\partial t} = -\frac{\partial}{\partial x}\left[ \frac{\partial H}{\partial \rho_x} (1 - \rho_x) \right] + \frac{1}{2} \frac{\partial^2 \rho_x}{\partial x^2}.
\]

One immediately notes that this equation exhibits different dependence on \( \rho \) in the drift and diffusion terms. The pseudoinverse approach employed here can not be used in such cases. Moreover, previous studies of discrete
pumps with exclusion are known to violate the NPT [13, 19, 39]. These violations of the NPT highlights yet another difference between inclusion and exclusion interactions, two of the paradigmatic interactions used in stochastic models of nonequilibrium systems. Understanding precisely which property of the interaction cause the violation of the NPT will require further research.

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Appendix. Derivation of the current decomposition formula

In this appendix we present the derivation of the current decomposition formula (12) and the associated kernel (13). For non-interacting systems the current decomposition formula was derived by Horowitz and Jarzynski [21]. The derivation below follows similar lines, but is modified due to the nonlinear form of \( h \rho \). For completeness we present most of the details of the derivation. We look for a pseudoinverse operator, defined by \( Rf = \int dx' g(x', x; t)f(x') \) where \( g(x', x; t) \) is the instantaneous modified Green’s function [40]. This pseudoinverse is required to satisfy the relation

\[
R\mathcal{L}_{FP} h[\rho] = h[\rho] - \beta(t) h[\rho^s].
\]  

(A.1)

As a pseudoinverse \( R \) acts as an inverse for all functions which are orthogonal to the subspace corresponding to the vanishing eigenvalue of the operator \( \mathcal{L}_{FP} \). The eigenfunctions corresponding to this vanishing eigenvalue are mapped to zero. The rhs of (A.1) ensures that the pseudoinverse projects out the steady-state, \( \rho^s \). Indeed, when \( \rho = \rho^s \), \( \beta(t) = 1 \) and \( \mathcal{L}_{FP} h[\rho^s] = 0 \). In addition, \( \int dx [h[\rho] - \beta(t) h[\rho^s]] = 0 \), ensuring that for any \( h \) and \( \rho \) the rhs has a vanishing bi-orthogonal product with the left eigenfunction corresponding to the eigenvalue 0. This latter condition is the reason for the inclusion of the renormalization factor \( \beta(t) \) and is one of the major differences between the pseudoinverse of the nonlinear problem considered here, and the one of non-interacting where \( \int dx \rho = \int dx \rho^s = N \) and \( \beta(t) = 1 \).

Equation (11) is defined with a periodic boundary conditions for \( \rho, \varphi, \chi \). One can then find a pseudoinverse with periodic modified Green’s function \( g(x, x'; t) \) by substituting the integral expression for \( R \) in (A.1). It is easy to verify that (A.1) is satisfied if the kernel is the solution of the boundary value problem

\[
\mathcal{L}_{FP}^+ g(x, x') = \delta(x' - x) - \int_y^y dy h[\rho^s(y, t)],
\]  

(A.2)

here \( \mathcal{L}_{FP}^+ g(x, x') = F(x, t) \partial_x + D(x, t) \partial_x^2 \) is the adjoint of \( \mathcal{L}_{FP}^+ \). The boundary conditions are \( g(x, 0) = g(x, L) \) and \( \partial_{x'} g(x, x') \bigg|_{x'=0} = \partial_{x'} g(x, x') \bigg|_{x'=L} \). As pointed in [21] for the purpose of finding the current decomposition formula one can solve for an equation which is simpler than (A.2) and nevertheless allows to obtain the current. One defines a current operator via

\[
\mathcal{J}(x, t) \alpha(x) = F(x, t) \alpha(x) - \partial_x (D(x, t) \alpha(x, t)),
\]  

(A.3)

such that \( \mathcal{J}(x, t) \mathcal{J}[\rho] = \mathcal{J}[\rho] - \beta(t) \mathcal{J}[\rho^s] \). One then operate on equation (A.2) with the current operator, yielding

\[
\int dx' \nu(x', x; t) \mathcal{L}_{FP} h[\rho] = \mathcal{J}(x, t) - \beta(t) \mathcal{J}[\rho^s],
\]  

(A.4)

where \( \nu(x, x'; t) \equiv \mathcal{J}(x, t) g(x, x'; t) \). The kernel \( \nu \) is the solution of

\[
\mathcal{L}_{FP}^+ \nu(x, x') = \mathcal{J}(x) \delta(x' - x) - \int_0^L \frac{dy}{\rho^s(y, t)} \mathcal{J}[\rho^s(y, t)],
\]  

(A.5)

with the boundary conditions \( \nu \bigg|_{x'=0} = \nu \bigg|_{x'=L} \). Equation (A.5) is slightly simpler than equation (A.2) since \( \mathcal{J}[\rho^s] \) does not depend on \( x \). To solve equation (A.5), one builds the solution from known solutions of simple homogeneous and inhomogeneous equations. The homogeneous boundary value problem is given by

\[
\mathcal{L}_{FP}^+ \pi(x) = 0; \pi(0) = 1, \pi(L) = 0,
\]  

(A.6)
while the inhomogeneous equation is defined by
\[ \mathcal{L}_{\beta} \tau(x) = -1; \quad \tau(0) = 0, \quad \frac{\partial \tau}{\partial x} \bigg|_{x=L} = 0. \] (A.7)

The solution for the source term is obtained by
\[ \mathcal{L}_{\beta} (x') \theta(x' - x) = \left[ F(x', t) \partial_{x'} + D(x', t) \partial_{x'}^2 \right] \theta(x' - x) = \mathcal{J}(x) \delta(x' - x). \] (A.8)

The solutions of (A.6) and (A.7) are given by equations (14) and (15). It is easy to verify that equation (13) is a solution for equation (A.5), and that the coefficients \( \beta(t) \), \( f^{\alpha} \) and \( \tau(L) \) are such that the boundary conditions on \( \nu \) are satisfied. The boundary condition on the derivatives of \( \nu \) is then automatically satisfied with this choice of \( \beta(t) \). The proof is based on the fact that \( f^{\alpha} = \mathcal{J} \rho \) is spatially independent. We note that \( \nu \) is not unique and in general will provide the same current under the transformation \( \nu(x, x'; t) \rightarrow \nu(x, x'; t) + \lambda(x) \).

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