We study optimal trade execution strategies in financial markets with discrete order flow. The agent has a finite liquidation horizon and must minimize price impact given a random number of incoming trade counterparties. Assuming that the order flow $N$ is given by a Poisson process, we give a full analysis of the properties and computation of the optimal dynamic execution strategy. Extensions, whereby $N$ is a Markov-modulated compound Poisson process are also considered. We derive and compare the properties of the various cases and illustrate our results with computational examples.

KEY WORDS: optimal order execution, liquidity modeling, dark pools, Markov-modulated Poisson process.

1. INTRODUCTION

One of the most important problems faced by a stock trader is how to unwind large block orders of security shares. Liquidation of a large position in a security is a challenge due to two factors: (a) possible lack of a counterparty; and (b) price impact that depresses prices by increasing supply. This occurs because the immediate market resiliency is limited and a single large order may exhaust all current buyers, bringing about dramatic price declines. Price impact implies that it is generally beneficial to split the order into several smaller blocks and sell each subblock separately. Presence of counterparties is less of a concern in traditional limit order book markets where a market maker is always quoting a price. However, trading in such markets may be disadvantageous due to information leak/privacy concerns. Indeed, by examining the order book, other participants may recognize the large trader and move against her, even if she attempts to split her trades. Thus, a recent trend involves trading in dark pool markets where there is no order book and buyers/sellers are matched up electronically without revealing any information. Such dark trades minimize information leakage and dramatically reduce risk of adverse price movement compared to conventional limit book trading. However, liquidity becomes a major concern as there is no market maker and no counterparty may be forthcoming.
We refer to trade publications such as QPL Newsletter (2008) for more information on the evolving marketplace of dark pools and their numerous specification variations.

In this paper, we propose a new framework that explicitly takes into account such liquidity features of large order trades. Thus, we replace the classical continuous trading environment with a discrete order book. In our model, incoming buy orders are represented by a Poisson process which encodes the order arrival times. To capture the empirical feature of splitting large orders into smaller pieces, we will focus on price impact and eschew consideration of actual prices. Larger trades involve a volume discount and therefore tend to carry higher spread versus the current quoted market order price. Also, smaller trades are desirable in order to maintain anonymity and mitigate information leaks. Subject to the constraint that trades are only possible at order times, the objective of the agent is to execute her large order trade within a specified time window while minimizing execution costs.

Most of the existing analysis of optimal execution has focused on limit order book markets (see for example Almgren 2003; Almgren and Lorenz 2006; Obizhaeva and Wang 2006; Alfonsi, Fruth, and Schied 2010; Schied and Schöneborn 2009, 2007). Since a market maker is always present, all cited models assume a continuous-time trading environment, with the asset price usually represented by a diffusion price process. The price impact is decomposed into temporary and permanent effects and execution strategies are specified in terms of liquidation rates per unit time. The overall problem is then translated into a continuous or singular stochastic control formulation. Conversely, in our approach all trades are discrete and therefore an execution strategy corresponds to an impulse control setting. Also, in the above literature the optimal liquidation strategies turn out to be deterministic and can be sometimes explicitly determined. In contrast, our optimal strategies are intrinsically path-dependent and will be affected by the stochastic order flow. Finally, while the above papers typically consider an infinite horizon, we assume that the agent has a hard deadline to liquidate her large block. Thus, time-to-maturity is a crucial variable in our setup; it can also be used to express time-dependencies, where, for example, the opening and closing hours have more active trading than midday. To sum up, our contribution is a new approach to modeling dark pool liquidity in terms of point processes. As we show below, our models are flexible, allow for a quick implementation, and admit fruitful probabilistic analysis.

Let us now outline the basic ingredients of our model. We assume that the order book is a Poisson process $N$ with arrival times $\sigma_i$ which denote the time-stamp of the $i$th order. In our base model we postulate that $N$ is a simple Poisson process with constant intensity $\lambda$ on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose the agent has $k$ shares (or units) to sell and an execution horizon of $T$ time epochs. We postulate that at terminal date $T$ all unsold units are immediately disposed of as one large trade, for example, through the traditional limit order book. Thus, effectively there is always one more matching order arriving at $T$. The price impact is represented in terms of a strictly increasing and strictly convex market depth function $F$, where $F(a)$ represents the cost of placing a trade of size $a$ ($F(a)$ could also represent the average cost of a random price impact, assuming this randomness is independent of everything else in the model).

Let $\mathbb{F} = (\mathcal{F}_t), \mathcal{F}_t = \sigma(N_s : 0 \leq s \leq t)$ be the filtration generated by the observation process. Then the optimization problem of the agent can be written as

\[ v(k, T) = \inf_{\xi \in \mathcal{A}_k} \mathbb{E} \left[ \sum_{i: \sigma_i \leq T} F(\xi_{\sigma_i} - \xi_{\sigma_{i-1}}) + F(\xi_T) \right] =: \inf_{\xi \in \mathcal{A}_k} v_\xi(k, T), \quad k \in \mathbb{N}_+, \; T \in \mathbb{R}_+, \]
where \(\mathcal{A}_k\) is the set of all \(\mathbb{F}\)-adapted, integer-valued, positive, and nonincreasing processes whose values change only at the time of jumps of the Poisson process \(N\) with \(\xi_0 = k\). The convexity of \(F\) is interpreted as the limited market resiliency and encourages the agent to split the large \(k\)-order into smaller pieces. However, placing a smaller trade now is risky as no more orders might come in and the trader will be left with a large leftover at \(T\) (which will carry a large associated penalty). Thus, the convexity of \(F\) also represents the impatience of the agent in terms of current versus future trading and is formally similar to the risk-aversion level in Schied and Schöneborn (2007, 2009).

In terms of the stochastic control formulation, (1.1) is related to best choice problems with Poisson processes (see for example Cowan and Zabczyk 1978; Bruss 1987). In particular, Stadje (1987, 1990) studied a similar problem for a Poisson process in the context of multi-item dynamic pricing.

The mathematical problem in (1.1) is a compromise between a tractable analytical model and real markets. In general, the execution problem with illiquid trading is not so well studied and a big challenge is to develop parsimonious models that will prescribe reasonable optimal liquidation policies. The use of a Poisson process for \(N\) allows for a comprehensive analysis of (1.1) in Section 2, however, it is clearly not very rich to capture all the intricacies of real order books. Accordingly, we consider in Section 3 several extensions to address such issues. Our base model allowed arbitrary trade sizes; in practice the agent is only able to trade up to the order size which is the second dimension of the order flow. Accordingly, in Section 3.1 we take \(N\) to be a compound Poisson process, consisting of pairs \((\sigma_i, Y_i)\) of (order times, order sizes). Correspondingly, the original problem (1.1) is modified to constrain \(\xi_{\sigma_i - \sigma_j} \leq Y_i\).

In Section 3.2 we proceed to relax the assumption that the intensity of \(N\) is constant throughout the problem horizon. Given widespread evidence that real markets experience different liquidity regimes, we extend our model to the case where \(N\) is a Markov-modulated Poisson process. Finally, in Section 3.3 we consider the limiting case where trade amounts can be continuous, which allows a dimension reduction for a certain class of depth functions \(F\). To illustrate the different models above, Section 4 presents several computational examples; finally Section 5 concludes and points to possible future extensions.

2. ANALYSIS OF THE OPTIMAL LIQUIDATION PROBLEM

In this section we analyze the properties of \(v(k, T)\) as defined in (1.1). The treatment below allows us to give a clear insight of the structure of \(v(k, T)\) and leads to a particularly simple algorithm to compute \(v\) and the associated optimal strategy, see Remark 2.7.

Our first observation is that \(v(k, T)\) satisfies the following dynamic programing equation:

\[
(2.1) \quad v(k, T) = \mathbb{E}\left[ \min_{a \in \{1, \ldots, k\}} \{v(k-a, T-\sigma_1) + F(a))\} \cdot 1_{\{\sigma_1 < T\}} + F(k) \cdot 1_{\{\sigma_1 \geq T\}} \right].
\]

A more general version of this dynamic programing principle is proved in Proposition 3.2. Let \(a(k, T)\) be the optimal order size to place when an order arrives given that one has \(k\) units remaining and \(T\) epochs until the terminal date. It follows from (2.1) that

\[
(2.2) \quad a(k, T) = \arg\min_{a \in \{1, \ldots, k\}} \{v(k-a, T) + F(a)\}.
\]
The above equation is simply the dynamic programming principle that says that the best immediate action is to sell $a$ units, such that the sum of the current cost $F(a)$ and expected future costs as represented by the value function $v(k - a, T)$ is minimized. To avoid ambiguity, we will assume that if the minimizer in (2.2) is not unique, then $a(k, T)$ is the smallest minimizer.

Before proceeding, let us make further note of the Markovian properties of our system. In parallel with the original formulation in (1.1) in terms of dynamic controls $\xi$, one may also describe Markov control strategies as $(b(k, T) : k \in \mathbb{N}, T \in \mathbb{R}_+)$, specifying the trading amount conditional on still having $k$ units left with time horizon of $T$ periods. Given such $(b(k, T))$, the corresponding dynamic unit inventory process is denoted by $\xi^{(b, k, T)}$ and satisfies

\[
d\xi^{(b, k, T)}_t = -b(\xi^{(b, k, T)}_t, T - t) dN_t, \quad \xi^{(b, k, T)}_0 = k.
\]

Economically, $\xi^{(b, k, T)}$ represents the remaining number of units at date $t$ when employing the execution strategy $b$. Using $a$ to denote the strategy characterized by (2.2), it follows that an optimal inventory process for (1.1) is of the Markovian feedback type and is given by $\xi^* \equiv \xi^{(a, k, T)}$. This result is similar to the derivation of the optimal impulse control for piecewise deterministic Markov processes (see, for example, Davis 1993).

2.1. Computing $v(k, T)$

To illustrate the problem, let us compute $v(k, T)$ for $k \leq 4$. Clearly $v(1, T) = F(1)$. For $k = 2$ and $k = 3$ due to the convexity of $F$ it is always optimal to place orders of size one. For $k = 2$, this will be possible as long as there is an arrival before date $T$, that is, $N(T) \geq 1$ (recall that the remainder, if any, is disposed of at $T$). Applying (2.1) and recalling the properties of the Poisson process yields

\[
v(2, T) = 2F(1) \cdot (1 - e^{-\lambda T}) + F(2) \cdot e^{-\lambda T}.
\]

Similar logic implies

\[
v(3, T) = F(3)\mathbb{P}(\sigma_1 > T) + \mathbb{E}[(F(1) + v(2, T - \sigma_1))1_{\{\sigma_1 < T\}}]
\]

\[
= F(3)e^{-\lambda T} + \int_0^T (F(1) + v(2, T - s))\lambda e^{-\lambda s} \, ds
\]

\[
= e^{-\lambda T}F(3) + \lambda T e^{-\lambda T} F(2) + (3 - 3e^{-\lambda T} - 2\lambda T e^{-\lambda T})F(1).
\]

The above analysis shows that $a(1, T) = a(2, T) = a(3, T) = 1$ for all $T > 0$. When $k = 4$, the order size is based on whether $v(3, T - \sigma_1) + F(1) \geq v(2, T - \sigma_1) + F(2)$ at the first arrival time $\sigma_1$. If the latter inequality is true, then one is better off selling two units, otherwise a single unit is optimal to trade. Using the above formulas for $v(2, \cdot)$ and $v(3, \cdot)$, it can be seen that there exists a critical threshold $\hat{v}_{(4,2)}$ such that $a(4, T) = 1 + 1_{\{T < \hat{v}_{(4,2)}\}}$. Hence, if little time until expiration is left, the agent becomes impatient and trades more.

We conclude this section with upper and lower bounds for $v(k, T)$. The next lemma gives an easy way to compute lower bound for the value function. Below we extend the domain of $F$ to the whole positive real line such that $F : \mathbb{R}_+ \to \mathbb{R}_+$ is still strictly convex and increasing.
Lemma 2.1. We have

\[ v(k, T) \geq \sum_{n<k-1} (n+1) F\left(\frac{k}{n+1}\right) \cdot P(N(T) = n) + k F(1) \cdot P(N(T) \geq k-1) := v(k, T). \]

Proof. Consider a genie who is affected by the randomness but for each state of the world can tell how many arrivals there will be and is further allowed to divide up her orders into nonintegral bits of size \( \geq 1 \). The right-hand side of (2.4) is the genie’s solution to (1.1).

As counterpart to Lemma 2.1, we have the following tight upper bound to the value function.

Lemma 2.2. We have

\[ v(k, T) \leq \min_{c \in \mathbb{N}_+} \mathbb{E}^c (k, T) \triangleq \min_{c \in \mathbb{N}_+} \left\{ \left(\left\lfloor \frac{k}{c} \right\rfloor \cdot F(c) + F\left( k - \left\lfloor \frac{k}{c} \right\rfloor \right) \right) \cdot P(N(T) \geq \left\lfloor \frac{k}{c} \right\rfloor ) \\
+ \sum_{n=0}^{\left\lfloor \frac{k}{c} \right\rfloor - 1} \left( n \cdot F(c) + F\left( k - n \left\lfloor \frac{k}{c} \right\rfloor \right) \right) \cdot P(N(T) = n) \right\}, \]

in which \( \left\lfloor x/c \right\rfloor \) is the largest integer smaller than \( x/c \).

Proof. The right-hand side of (2.5) is the cost of a constant \( c \)-strategy. This is the strategy where the agent insists on trading \( c \) units at each arrival time until terminal date \( T \), whence the remainder is liquidated. Although she originally optimizes over \( c \), clearly this is a suboptimal strategy. The bound in (2.5) becomes tight as \( T \to \infty \), the liquidity risk vanishes, and the optimal strategy is to always trade a single unit \( c^* = 1 \).

2.2. Properties of the Value Function

Below, we study the dependence of \( v(k, T) \) and \( a(k, T) \) on the (integer-valued) number of units to sell \( k \) and the (continuous) time to expiration \( T \). Intuitively, we expect that more units cost more for the agent to liquidate, while longer time horizon reduces costs. These observations are in fact model-free in the sense that they depend solely on the convexity of \( F \) and not on any properties of the arrival process \( N \). Below we state them as a basic lemma.

Lemma 2.3. The map \( k \to v(k, T) \) is increasing and the map \( T \to v(k, T) \) is decreasing.

Proof. Let \( \xi \) be any admissible control for \( v(k, T) \). Then \( \xi \) is also admissible for \( v(\ell, T) \) for any \( \ell \geq k \), which immediately establishes the first part of the lemma. Conversely, for any \( T' > T \), define a control \( \xi' \) via \( \xi'_t = \xi_t \) for \( t \leq T \) and \( \xi'_{\sigma_t} - \xi'_{\sigma_t-} = 1_{\{\xi_t > 0\}} \) for \( T < \sigma_t \leq T' \). Then \( \xi' \) is an admissible control for \( v(k, T') \). Moreover, due to the convexity
of $F$, the pathwise cost of $\xi'$ is at most the pathwise cost of $\xi$,
\[
\sum_{i: \sigma_i \leq T} F(\xi'_{i\sigma_i} - \xi_{\sigma_i}) + \sum_{j: T < \sigma_j \leq T} F(1) + F(\xi'_T) \leq \sum_{i: \sigma_i \leq T} F(\xi_{\sigma_i} - \xi_{\sigma_i}) + F(\xi_T) \quad \mathbb{P} \text{- a.s.,}
\]
with strict inequality if $\xi_T > 1$ and $N(T') - N(T) > 0$. It follows that $v(k, T; \xi') \leq v(k, T, \xi)$ with strict inequality as long as $\mathbb{P}(N(T) = 0, N(T') - N(T) > 0) > 0$. Note that the last statement is satisfied for $N$ a Poisson process and any $T < T'$.

A more careful analysis is furnished by the following two Propositions 2.4 and 2.5 whose proofs are deferred to the Appendix.

**Proposition 2.4.** The dependence of $v$ and $a$ on the number of units to sell $k$ is

(i) $k \to v(k, T)$ is nondecreasing, “convex” (see A.8), and $v(k + 1, T) - v(k, T) < F(k + 1) - F(k)$ for all $k, T$.

(ii) $k \to a(k, T)$ is nondecreasing and increases by jumps of size 1 only.

Proposition 2.4(i) shows that $v$ inherits the convexity of $F$ but in a milder form. Proposition 2.4(ii) shows that facing more units to sell, the agent will trade at least as much; moreover an addition of one more unit will increase the trading amount by at most 1.

Define

\[
(2.6) \quad G(k, T) \coloneqq v(k, T) - \min_{a \in \{0, 1, \ldots, k\}} [v(k - a, T) + F(a)].
\]

The quantity $G(k, T)$ in (2.6) represents the maximal gain from an immediate impending trade. As shown in our next result, $G$ is also related to the time-derivative of $v$.

**Proposition 2.5.** The dependence of $v$ and $a$ on the time to maturity $T$ is

(i) the derivative of $v$ with respect to time to maturity is

\[
(2.7) \quad \partial_T v(k, T) = -\lambda G(k, T) < 0.
\]

Moreover, $\partial_T v(k, T)$ is increasing in $k$;

(ii) there exist distinct thresholds $t^{(k,i)}$ such that

\[
(2.8) \quad a(k, T) = i, \quad \text{when} \quad t^{(k,i+1)} < T \leq t^{(k,i)}.
\]

Thus, $T \to a(k, T)$ is decreasing and right-continuous with $a(k, 0+) = \lfloor k/2 \rfloor$ and $\lim_{T \to \infty} a(k, T) = 1$.

(iii) $T \to v(k, T)$ is convex and $\partial_T^2 v(k, T)$ is continuous except at $T \in \{t^{(k,i)} : i = 1, \ldots, \lfloor k/2 \rfloor\}$.

Proposition 2.5(i) provides an implicit formula for the time-derivative of $v$; item (iii) further refines this result by studying the second-derivative. Proposition 2.5(ii) shows that as time to expiration approaches, the agent trades in larger and larger amounts; conversely when facing a very long time horizon, it is optimal to trade just one unit at a time.

In the latter sections we will relax the assumption of a constant arrival intensity $\lambda$ of $N$. To understand its effect, we state the following lemma
**Lemma 2.6.** The value function \(v(k, T)\) and optimal action \(a(k, T)\) are decreasing in \(\lambda\).

Note that we have the scaling property \(v(k, T; \lambda) = v(k, \alpha T; \lambda/\alpha)\) for any \(\alpha > 0\) since the main parameter is intensity of arrivals per effective horizon. Thus, dependence of \(v\) (and \(a\)) on \(\lambda\) is equivalent to its inverse dependence on time horizon. More generally, suppose that the arrival intensity is a deterministic function of time \(\tau\), a (strictly) increasing function \(\tau\). Then defining the strictly increasing function \(\tau(t) \triangleq \int_0^t \lambda(s)\, ds\) it follows that

\[
v(k, T; \lambda(t)) = v(k, \tau(T); 1),
\]

and \(a(k, T; \lambda(t)) = a(k, \tau(T); 1)\). Below we give a second proof of this result using the concept of coupling.

**Proof.** Consider two Poisson processes \(N_1, N_2\) with intensities \(\lambda_1(t) > \lambda_2(t)\). Then one may construct a probability space \((\Omega', \mathcal{F}', \mathbb{P}')\) and random variables \(\sigma_k^{(i)}, i = 1, 2, k = 1, 2, \ldots\) such that \(\mathbb{P}'(\sigma_k^{(i)} > i) = \exp(-\int_0^t \lambda_i(s)\, ds)\) and \(\sigma_k^{(1)} \leq \sigma_k^{(2)}\) \(\mathbb{P}'\)-almost surely. Letting \(N'_k(t) = \max(k : \sum_{j=1}^k \sigma_j^{(i)} \leq t)\) we obtain two coupled copies \(N'_1, N'_2\) of \(N_1, N_2\), such that \(\mathbb{P}'(N'_1(t) \geq N'_2(t) \forall t) = 1\). Now it is fairly obvious that \(v^{\lambda_1}(k, T) \leq v^{\lambda_2}(k, T)\) since working under \(\mathbb{P}'\), the first case has almost surely more arrivals than the second case. Formally, let us define a deterministic time-change by \(v(t) = \frac{\lambda_2(\tau(t))}{\lambda_1(\tau(t))} - 1\). Since \(\tau_1(t) > \tau_2(t), \forall t\) then \(\mathbb{P}'(\sigma_k^{(1)} \in dt) = \mathbb{P}'(\sigma_k^{(2)} \in dv(dt))\), which implies \(\mathbb{P}(N'_1(t) \leq j) = \mathbb{P}(N'_2(v(t)) \leq j)\) for all \(j\) and therefore \(v^{\lambda_1}(k, T) = v^{\lambda_2}(k, v(T))\) (map any control \(\xi\) for \(v^{\lambda_1}(k, T)\) into a control \(\xi_{v(t)}\) for \(v^{\lambda_2}(k, v(T))\)). Now, since \(v(T) > T\) it follows that \(v^{\lambda_2}(k, T) > v^{\lambda_1}(k, v(T)) = v^{\lambda_1}(k, T)\).

**Remark 2.7.** A word on the computation of the value function and the optimal action. Using the above results, one may easily compute \(v(k, T)\) for any depth function \(F(\cdot)\) by using the coupled family of first-order ordinary differential equations (2.7) over a time grid. Note that given \(v(k, T), a(k, T)\), finding the minimum in the definition of \(G(k, T + h)\) requires just one comparison since \(a(k, T + h) \in \{a(k, T), a(k, T) - 1\}\). Given \(v(k, T)\) and \(a(k, T)\) an optimal trading strategy is straightforwardly implemented using (2.3).

### 3. Extensions

Using the analysis of Section 2 as a starting point, we now consider several progressively more sophisticated versions of the original model (1.1) so as to better express the complexities of real markets.

#### 3.1. Constrained Trading

In this section we consider the modified model whereby \(N\) is a compound Poisson process with characteristics \((\lambda, \nu)\) and the agent is constrained to trade only up to the order size \(Y_i\) (which have distribution \(\nu\)). To wit, we look at the constrained value function

\[
\tilde{v}(k, T) = \inf_{\xi \in A_k} \mathbb{E}\left[ \sum_{\xi_{\sigma} \leq T} F(\xi_{\sigma} - \xi_{\sigma -}) + F(\xi_T) \right], \quad k \in \mathbb{N}_+, \ T \in \mathbb{R}_+
\]
where $\tilde{A}_k \subset A_k$ has the additional constraint that $0 \leq \xi_{a_i} - \xi_{a_j} \leq Y_j$. As a first remark, note that we trivially have the bound $\tilde{\nu}(k, T) \geq v(k, T)$.

In counterpart to the dynamic programing equation (2.1), the constrained value function $\tilde{\nu}$ is the unique fixed point of the following functional operator $\tilde{L}$:

$$
(3.1) \quad \tilde{L}\tilde{\nu}(k, T) = \mathbb{E}\left[ F(k)1_{\{\sigma_i > T\}} + \min_{a \in \{1, \ldots, \tilde{Y}_k\}} (F(a) + \tilde{\nu}(k - a, T - \sigma_i)) 1_{\{\sigma_i \leq T\}}\right].
$$

The proof of (3.1), as well as of the fact that $\tilde{L}$ has a unique fixed point is identical to that of Proposition 3.2 below and therefore deferred. Let us now define

$$
(3.2) \quad \tilde{a}(k, T) = \arg\min_{a \in \{1, \ldots, k\}} (\tilde{\nu}(k - a) + F(a)).
$$

The next proposition is analogous to Propositions 2.4 and 2.5. However, it is complicated by the fact that without proving the convexity results regarding $\tilde{\nu}(\cdot, T)$ it is not clear that

$$
\min_{a \in \{1, \ldots, \tilde{Y}_k\}} (F(a) + \tilde{\nu}(k - a, T - \sigma_i)) = F(\tilde{a}(k, T) \land \tilde{Y}_k) + \tilde{\nu}(k - (\tilde{a}(k, T) \land \tilde{Y}_k), T - \sigma_i).
$$

The above statement implies that an optimal liquidation strategy consists of placing trades of size $\tilde{a}(k, T)$ and then letting them be filled to the maximum extent by the matching incoming orders.

**Proposition 3.1.** The following hold:

(i) $k \to \tilde{\nu}(k, T)$ is nondecreasing, “convex,” and $\tilde{\nu}(k + 1, T) - \tilde{\nu}(k, T) < F(k + 1) - F(k)$;

(ii) $T \to \tilde{\nu}(k, T)$ is decreasing and convex;

(iii) denote by $\nu[\tilde{a}(k, T) \land 1] = \mathbb{P}(\tilde{Y}_k > \tilde{a}(k, T))$. Then

$$
(3.3) \quad \partial_T \tilde{\nu}(k, T) = -\lambda \left( \tilde{\nu}(k, T) - \left[\tilde{\nu}(k - \tilde{a}(k, T), T) + F(\tilde{a}(k, T))\right] \nu[\tilde{a}(k, T) \land 1] \right) \nu(k - \tilde{a}(k, T), T) + F(\tilde{a}(k, T)) \right) < 0;
$$

moreover, $\partial_T \tilde{\nu}(k, T)$ is increasing in $k$;

(iv) $k \to \tilde{a}(k, T)$ is nondecreasing and increases by jumps of size 1 only;

(v) $T \to \tilde{a}(k, T)$ is nonincreasing and right continuous with $\tilde{a}(k, 0) = \lfloor k/2 \rfloor$ and $\lim_{T \to \infty} \tilde{a}(k, T) = 1$. Moreover, its jumps are of size 1. The jumps of $T \to \tilde{a}(k, T)$ occur at the discontinuity points of $T \to \partial_T^2 \tilde{\nu}(k, T)$.

**Proof.** We will first consider an auxiliary control problem in which the agent has to submit her sell orders before seeing the size of the incoming buy orders $^1$. Let us call the corresponding value function by $\nu(k, T)$. Again, a dynamic programing principle implies that this value function is the unique fixed point of an operator $\mathcal{L}$ that is defined by

$$
\mathcal{L}\nu(k, T) = \mathbb{E}[F(k)1_{\{\sigma_i > T\}} + (F(\alpha(k, T) \land \tilde{Y}_k) + \nu(k - (\alpha(k, T) \land \tilde{Y}_k), T - \sigma_i)) 1_{\{\sigma_i \leq T\}}].
$$

$^1$ This parallels real markets where once an order is placed, it will be maximally partially filled against any incoming matching order.
in which

\[ \alpha(k, T) = \arg\min_{a \in \{1, \ldots, k\}} (V(k - a) + F(a)). \]

All the proofs in Section 2 now go through to show that the pair \((V, \alpha)\) satisfies (i)–(v) of Proposition 3.1. Now since \(V(\cdot, T)\) is convex, it follows that \(V(k - a, T) + F(a)\) is monotone on the set \(\{a \leq \alpha(k, T)\}\) and therefore the action of \(L\) and \(\tilde{L}\) from (3.1) against \(V\) is the same. Since \(V\) is a fixed point of \(L\),

\[ V = LV = \tilde{L}V. \]

But \(\tilde{L}\) has a unique fixed point, so that \(\tilde{v}(\cdot, \cdot) = V(\cdot, \cdot)\) and \(\alpha(\cdot, \cdot) = \tilde{a}(\cdot, \cdot)\), and the proof is complete. \(\square\)

### 3.2. Stochastic Liquidity

The model in Section 2 assumed a constant level of trade activity over the full time horizon. However, as practitioners know, real-life order flows experience multiple regime changes. For instance, a common intraday pattern features high level of activity in the beginning and end of the trading session and a lower trade intensity during midday. Alternatively, markets may experience liquidity crises, whereby order flow abruptly slows down. To capture such stylized features, in this section we assume that \(N\) is a regime-switching (doubly-stochastic) compound Poisson process, modulated by the market state variable \(M\). \(M\) represents the market liquidity; namely the order frequency and order sizes in the order flow book are driven by \(M\).

Formally, let \(N^{(1)}, \ldots, N^{(m)}\) be \(m\) independent compound Poisson processes with intensities and jump distributions \((\lambda_1, \nu_1), \ldots, (\lambda_m, \nu_m)\). We assume that \(M\) forms an independent finite state Markov chain with state space \(E = \{1, 2, \ldots, m\}\) and infinitesimal generator \(Q = (q_{ij})\). Then the observed order flow is given by

\[ N_t = \int_0^t \sum_{i \in E} 1_{\{M_s = i\}} dN_{s}^{(i)}, \quad t \geq 0. \]

(3.4)

By construction, the increments of \(N\) are independent conditioned on \(M\). Let \(v(k, T; i)\) represent the minimal execution costs conditional on \(M_0 = i\). Note that the lower and upper bounds derived in Lemmas 2.1 and 2.2 also bound the value function in the regime switching case.

The next proposition establishes the dynamic programing equation for \(v(k, T; i)\).

**Proposition 3.2.** The value function \(v\) satisfies the dynamic programing equation \(v = Lv\), in which \(L\) is the first jump operator given by

\[ Lv(k, T; i) = \mathbb{E}^I \left[ F(k) 1_{\{\sigma_1 > T\}} + \min_{a \in \{1, \ldots, k\}} (F(a) + v(k - a, T - \sigma_1; M_{\sigma_1})) 1_{\{\sigma_1 \leq T\}} \right]. \]

(3.5)

In fact, \(v\) is the unique fixed point of \(L\).

**Proof.** Let us introduce

\[ u_0(k, T; i) = F(k), \quad u_n(k, T; i) = Lu_{n-1}(k, T; i), \quad n \geq 1. \]

(3.6)
Following the logic of the proof of proposition 3.1 in Bayraktar and Ludkovski (2009b), we can show that

\[ u_n(k, T; i) = v_n(k, T; i) \equiv \inf_{\xi \in \mathcal{A}_n} \mathbb{E}^i \left[ \sum_{k \leq n, \sigma \leq T} F(\xi_{\sigma k} - \xi_{\sigma}) + F(\xi_T) \right], \tag{3.7} \]

which denotes the value function under the constraint that the agent only trades during the first \( n \) orders (and makes zero-trades thereafter until the close \( T \)). On the other hand \( v_n(k, T; i) = v(k, T; i) \), for \( n \geq k \) since at most \( k \) trades are needed to liquidate a position of size \( k \). Now, thanks to (3.6)

\[ v(k, T; i) = v_{k+1}(k, T; i) = Lv_k(k, T; i) = Lv(k, T; i). \]

The fact that \( v \) is the unique fixed point of \( L \), which is an increasing, continuous, and concave operator, follows from standard results in optimal control, see, for example, Zabczyk (1983) or the proof of theorem 3.1 in Bayraktar and Ludkovski (2010). □

The Hamilton-Jacobi-Bellman equation for \( v(k, T; i) \) is given by the following lemma, also compare with (2.7).

**Lemma 3.3.** Let us denote

\[ G(k, T; i) := v(k, T; i) - \min_a [v(k - a, T; i) + F(a)]. \]

Then derivative of \( v \) with respect to its second variable is

\[ \partial_T v(k, T; i) = -\lambda_i G(k, T; i) + \sum_{j \in E \setminus \{i\}} q_{ij} (v(k, T; j) - v(k, T; i)). \tag{3.8} \]

Since the proof of Proposition 2.4 is pathwise, it is unaffected by the regime-switching setting and therefore we have

**Corollary 3.4.** For a fixed \( i, k \mapsto v(k, T; i) \) is convex, and the optimal actions are nondecreasing in \( k \), \( a(k, T; i) \leq a(k + 1, T; i) \).

A further possibility is to assume that the market liquidity variable \( M \) is not observed. This is a good proxy for real markets where market participants do not know the full liquidity state. Instead, agents infer current liquidity based on observed trades. Thus, decreased frequency of trades may point to an impending liquidity crisis and force agents to preemptively place larger trades. A related model with continuous trading was considered in Almgren and Lorenz (2006).

Such a model can be tackled within the framework of stochastic control with partially observable Poisson processes, investigated by the authors in the previous papers (Bayraktar and Ludkovski 2009b, c). Namely, one postulates a Bayesian setting whereby the agent dynamically updates her beliefs about \( M \). The conditional probability process

\[ \Pi(t) \equiv (\Pi_1(t), \ldots, \Pi_m(t)) \]

with

\[ \Pi_i(t) = \mathbb{P}^{\tilde{\pi}} \{ M_t = i | \mathcal{F}_t^N \}, \quad \text{for } i \in E, \text{ and } t \geq 0 \tag{3.9} \]


becomes a new hyperstate and the partially observed execution problem can be stated as

\[
(3.10) \quad v(k, T, \tilde{\pi}) = \inf_{\xi \in \mathcal{A}_k^c} \mathbb{E} \left[ \sum_{l, \sigma_l \leq T} F(\xi(\sigma_l^-) - \xi(\sigma_l)) + F(\xi(T)) \right],
\]

where the minimization is over all \( \mathcal{F}^N \)-adapted admissible controls \( \xi \) with \( \xi_0 = k \). The model (3.10) can be approached using the methods in Bayraktar and Ludkovski (2009b, c), formulating a dynamic programing equation and characterizing the optimal strategy; we refer to Bayraktar and Ludkovski (2009a) for further details. Note that the optimal execution strategies are now more complex since they depend on the beliefs \( \hat{\Pi}(t) \).

### 3.3. Continuous Sale Amounts

A related limiting model is obtained when we allow the sale amounts to be arbitrary real numbers, rather than integers. The corresponding problem becomes

\[
(3.11) \quad \hat{u}(x, T) = \inf_{\xi \in \mathcal{A}_k^c} \mathbb{E} \left[ \sum_{l, \sigma_l \leq T} F(\xi(\sigma_l^-) - \xi(\sigma_l)) + F(\xi(T)) \right] \quad x \in \mathbb{R}_+, \ T \in \mathbb{R}_+,
\]

where \( \mathcal{A}_k^c \supseteq \mathcal{A}_k \) is now the set of all \( \mathbb{F} \)-adapted, nonincreasing processes whose values change only at the time of jumps of the Poisson process \( N \) with \( \xi_0 = x \). The value function when continuous sales are allowed is easier to work with. For instance, because the new set of admissible strategies \( \mathcal{A}_k^c \) is convex (which was not true under integer-constraints), it immediately follows that \( \hat{u} \) is convex in its first argument \( x \) (compare to Proposition 2.4(i)).

A further advantage is that the value function \( \hat{u} \) satisfies a scaling property whenever \( F \) does, which helps to reduce the dimension of the problem.

**Lemma 3.5.** Suppose the depth function \( F \) admits the following scaling property, \( F(x\beta) / F(\beta) = H(x) \), for some function \( H \) and all \( \beta > 0 \). Then \( \hat{u}(x, T) = H(x)u(T) \) in which \( u \) is the unique solution of

\[
(3.12) \quad u(T) = \mathbb{E} \left[ H(1)1_{[\sigma_1 > T]} + \min_{a \in [0, 1]} \left( H(a) + H(1 - a) \cdot u(T - \sigma_1) \right) \right] \cdot 1_{[\sigma_1 \leq T]}.
\]

In particular, suppose that \( F(x) = x^\gamma \) (i.e., \( H(x) = x^\gamma \), \( \gamma > 1 \). Then the function \( u \) in (3.12) satisfies the following nonlinear ordinary differential equation (ODE):

\[
(3.13) \quad \begin{cases}
\partial_T u(T) = \lambda u(T) \left( \frac{1}{[1 + u(T)]^{(\gamma - 1)/\gamma}} - 1 \right), & u(0) = 1; \\
\partial_T a(T) = \frac{\lambda}{\gamma - 1} a(T)(1 - a(T)) \left( (1 - a(T))^{\gamma - 1} - 1 \right) < 0, & a(0) = 1/2.
\end{cases}
\]

**Proof.** Using (3.11) and the assumption on \( F \) we can see that \( \hat{u}(x, T) = H(x)\hat{u}(1, T) \) since if \( \xi \) is a strategy for \( \hat{u}(x, T) \) then \( \xi/x \) is a strategy for \( \hat{u}(1, T) \). With the latter scaling property, the dynamic programing equation (3.12) is just the counterpart of the original (2.1).
In the case where $H(x) = x^\nu$, the dynamic programing equation (3.12) leads to the integral equation (note $H(1) = 1$)

\[
    u(T) = e^{-\lambda T} + \int_0^T \min_{a \in [0, 1]} (a^\nu + (1 - a)^\nu u(T - s)) e^{-\lambda s} ds
\]

\[
    = e^{-\lambda T} \left( 1 + \int_0^T \min_{a \in [0, 1]} (a^\nu + (1 - a)^\nu u(s)) e^{\lambda s} ds \right).
\]

The optimal action evidently satisfies

\[
    a(T) = \frac{u(T)^{1/(\nu - 1)}}{1 + u(T)^{1/(\nu - 1)}}.
\]

If we let $f(T) = e^{\lambda T}u(T)$, it can be shown that $\partial_T f(T) = \lambda f(T) \cdot [1 + (f(T)e^{-\lambda T})^{1/(\nu - 1)}]^{1-\nu}$, from which we can derive the ODE for $u$ in (3.13). Finally, the ODE for $a$ in (3.13) follows using (3.14) and the ODE for $u$. Since $a(T) \leq 1/2$, by inspection the right-hand side of the ODE for $a$ in (3.13) is negative and it can also be shown that $\partial_T^2 a(T) > 0$. □

We find that for a power depth function, Lemma 3.5 provides an excellent approximation even for moderate values $k \geq 20$. Thus, when the scaling property of $F$ is satisfied, we obtain a very fast method to compute $v(k, T) \simeq H(k)u(T)$ and $a(k, T) \simeq k \cdot a(T)$ as defined in (3.13).

4. NUMERICAL ILLUSTRATIONS

In this section we illustrate the results of our analysis with some computational examples.

We begin with the base model where we take without loss of generality $\lambda = 1$. We also take a quadratic depth function $F(a) = a^2/2$. Solving for $a(k, T)$ using Remark 2.7 we obtain Figure 4.1. As shown in Proposition 2.5, $a(k, \cdot)$ decreases by steps of size 1; at the same time as shown in Proposition 2.4, $a(\cdot, T)$ increases by steps of size 1. This surface is used in conjunction with (2.3) to react to the arrivals of orders in an optimal way.

We then proceed to study the more complex extensions of Section 3. Thus, we assume that several liquidity regimes are possible; to be concrete, we fix the liquidity regime-switching model as $M_i \in E = \{\text{High}, \text{Med}, \text{Low}\} \equiv \{1, 2, 3\}$ with infinitesimal generator

\[
    Q = \begin{pmatrix}
        -2 & 2 & 0 \\
        1 & -4 & 3 \\
        0 & 2 & -2
    \end{pmatrix}.
\]

The intensity of orders is $\lambda(M_i)$ with $\lambda = [3, 3, 1]$ and order sizes have the strictly positive Poisson distributions $\nu_i(y) = \frac{\exp(-\mu_i)}{1 - \exp(-\mu_i)} \frac{(\mu_i)^y}{y!}$, $y = 1, 2, \ldots$, with “mean” sizes $\bar{\mu} = [8, 4, 4]$.

The value function $v(k, T, i)$ is easily computed by solving the corresponding system of ODE’s in (3.8). In this context, Figure 4.2 shows the effect of constraints on optimal strategy and optimal execution cost. We observe that constraints play the largest role at medium time horizons, as on long time horizons the agent has plenty of opportunities to trade, while with very short deadlines the convexity of $F$ is the determining factor. Also, as expected the agent responds to constraints by preemptively placing marginally larger orders in the hope they will be filled.
FIGURE 4.1. Optimal sale amounts $a(k, T)$ as a function of current holding $k$ and time to maturity $T$. We take $\lambda = 1$, $F(a) = a^2/2$, and the model (1.1).

FIGURE 4.2. The effect of constraints in the regime-switching setting of Section 3.2. Left panel shows the difference between $v(k, T; i)$ and $\tilde{v}(k, T; i)$ for a fixed $i = 1$ and $k = 20$; right panel plots the difference between $a(k, T; i)$ (circular markers) and $\tilde{a}(k, T; i)$ (dashed line) for same $i = 1, k = 20$.

To compare the different models of Section 3, Table 4.1 presents a summary of the various value functions. Namely, we compare the effect of stochastic liquidity, and also of constraints. Finally, we also show the accuracy of upper and lower bounds of Lemmas 2.1 and 2.2 for this case. We see that these bounds are quite tight (relative difference of about 10–15%) and can be used to give a quick idea about $v$. The bounds are easily computed.
We consider the Regime-Switching Case with $T = 1$, $k = 20$, $F(a) = a^2/2$. The Lower Bounds $v$ are computed using Lemma 2.1 and the Upper Bound $\bar{v}$ is computed using Lemma 2.2. We also compare the Constrained $\tilde{v}$ to the Basic $v$.

| Initial regime $i$ | $v(k, T; i)$ | $v(k, T; i)$ | $\tilde{v}(k, T; i)$ | $\bar{v}(k, T; i)$ |
|------------------|--------------|--------------|-----------------|-----------------|
| 1                | 73.16        | 77.80        | 83.54           | 83.31           |
| 2                | 84.26        | 88.36        | 98.97           | 93.50           |
| 3                | 98.94        | 102.22       | 114.25          | 107.11          |

via a Monte Carlo simulation: one first simulates paths of the Markov chain $M$; the conditional distribution of $N(T)$ follows using the fact that if $M_s = j$ for $s \in [T_1, T_2]$ then $N(T_2) - N(T_1) \sim \text{Poisson}(\lambda_j(T_2 - T_1))$.

Since the formulas in Lemmas 2.2 and 2.1 are for the base case without constraints, the constrained value functions $\tilde{v}$ are typically larger than the upper bound $\bar{v}$. One could compute an adjusted $\bar{v}$ that takes into account constraints, but no simple formulas like in Lemma 2.2 appear to be forthcoming.

5. CONCLUSION

In this paper we have proposed a new model for studying the optimal trade execution problem in financial markets. Our model is directly based on a discrete order flow and therefore is specially suited to capture the features of trading in dark pools where orders are executed only when matched with a crossing counterparty.

To simplify our presentation, our analysis assumed a simple compound Poisson representation of the order flow. However, the obtained dynamic programming equations and most of the stylized properties of the value function and optimal strategy are expected to hold in much more general setups. These could include time-dependent parameters (such as price impact, order intensity, and size distribution) or further constraints on optimal strategy.

Realistic dark pool trading involves simultaneous execution on several exchanges. In particular, the trader will place trades both in the dark pool and on the regular limit order book in order to optimize the trade-off between liquidity, minimal price impact, and information content (dark pool prices are often delayed compared to the limit book). In the case where the order flows of different exchanges are independent, the problem still fits into our framework, since superposition of independent Poisson processes is another Poisson process. The only modification is that orders will now carry the tag of the associated exchange and therefore the depth function $F$ will depend on order type.

A second direction for extensions is to introduce a price dimension. Thus, each crossing trade carries a price tag and the agent maximizes total revenue (rather than minimizing slippage costs). This would extend the framework of Bertsimas and Lo (1998) to include asynchronous buy arrivals and counterparty risk. Explicit prices would allow consideration of proportional depth functions $F$. Also, if the agent herself uses limit orders, then her control is a two-dimensional (price, quantity)-pair and the probability of a
matching trade being accepted could be taken to depend on both of these variables. Such multidimensional versions of our model will be taken up in future work.

**APPENDIX**

### A.1. Proof of Proposition 2.4.

The first preparatory lemma below shows that the slope of \( v \) is smaller than that of \( F \).

**Lemma A.1.** For any \( k_1 > k_2 \) and \( t \) we have \( v(k_1, T) - v(k_2, T) < F(k_1) - F(k_2) \). Alternatively, \( F(k) - v(k, T) \) is increasing in \( k \).

**Proof.** Let \( \xi^{k_2} \) denote \( \xi^{(a, k_2, T)} \). Recall that \( v_t(k, T) \) denotes the expected performance of any control \( \xi \). Interpreting \( \xi^{k_2} \) as a suboptimal control for \( v(k_1, T) \) (which disposes of the extra \( k_1 - k_2 \) units at maturity), we have

\[
v(k_1, T) - v(k_2, T) \leq v_{\xi^{k_2}}(k_1, T) - v_{\xi^{k_2}}(k_2, T)
= \mathbb{E}\left[\sum_{i=0}^{k_2} (F(i + k_1 - k_2) - F(i))1_{\{\xi^{k_2} = i\}}\right]
< \sum_{i=0}^{k_2} \mathbb{E}\left[(F(k_1) - F(k_2))1_{\{\xi^{k_2} = i\}}\right] = F(k_1) - F(k_2),
\]

where the second inequality is due to the convexity of \( F \), whereby \( y \mapsto F(a + y) - F(y) \) is increasing. \( \square \)

The next lemma shows that if one starts with more units initially and sells them in an optimal way, then one will always have more units at any later point in time.

**Lemma A.2.** Let \( \xi_t^k \) denote \( \xi^{(a, k, T)} \), \( k \in \mathbb{N}_+ \). Then for \( \ell \geq k \) we have that \( \xi_t^\ell \geq \xi_t^k \) for all \( t \in [0, T] \).

**Proof.** First note that if at any date \( s \leq t \) we would have \( \xi_t^s = \xi_t^k \), then it follows from (2.3) and the Markov nature of \( a(k, T) \) that for all \( s' \geq s \) we will have \( \xi_t^{s'} = \xi_t^k \) as well. Thus, to have \( \xi_t^{\ell} < \xi_t^k \) on a set \( A \) of strictly positive probability there necessarily must be an arrival \( \sigma_j \) such that \( d_t := \xi_t^{\sigma_j} \geq \xi_t^k := d_k \) and \( b_t := \xi_t^{\sigma_j} < \xi_t^k := b_k \) on \( A \). By construction, \( b_t = d_t - a(d_t, T - \sigma_j) \) and \( b_k = d_k - a(d_k, T - \sigma_j) \). Moreover,

\[
\begin{align*}
a(d_t, T - \sigma_j) &= a_t = \arg\min_a\{v(d_t - a, T - \sigma_j) + F(a)\}; \\
a(d_k, T - \sigma_j) &= a_k = \arg\min_a\{v(d_k - a, T - \sigma_j) + F(a)\}.
\end{align*}
\]

Define \( c_t = d_t - d_k + a_k > a_k \), and \( c_k = d_k - d_t + a_t < a_t \). Therefore from (A.1) (and recalling that \( a_t \) is the smallest minimizer, while \( a_t > c_t \))

\[
\begin{align*}
v(d_t - a_t, T - \sigma_j) + F(a_t) &< v(d_t - c_t, T - \sigma_j) + F(c_t), \\
v(d_k - a_k, T - \sigma_j) + F(a_k) &\leq v(d_k - c_k, T - \sigma_j) + F(c_k).
\end{align*}
\]
Rearranging, we obtain
\[
\begin{align*}
&v(d_\ell - c_\ell, T - \sigma_j) - v(d_\ell - a_\ell, T - \sigma_j) > F(a_\ell) - F(c_\ell), \\
v(d_k - a_k, T - \sigma_j) - v(d_k - c_k, T - \sigma_j) \leq F(c_k) - F(a_k).
\end{align*}
\] (A.2)

However, the left-hand sides of both equations in (A.2) are the same by construction and are in fact equal to \(v(b_k, T - \sigma_j) - v(b_\ell, T - \sigma_j) > 0\). On the other hand, since \(a_\ell > c_k\) and \(c_\ell > a_k\), while
\[
a_\ell - c_\ell = c_k - a_k = a_\ell - a_k + d_k - d_\ell = b_k - b_\ell > 0,
\]
by the convexity of \(F\) we must have \(F(a_\ell) - F(c_\ell) \geq F(c_k) - F(a_k)\), contradicting (A.2).

To keep the corresponding inventory processes ordered correctly it follows that for any \(T \in \mathbb{R}_+\) and \(\ell \leq k\), we have
\[
a(k, T) - a(\ell, T) \leq k - \ell \quad \forall t \geq 0.
\] (A.3)

In particular, \(a(k + 1, T) \leq a(k, T) + 1\).

**Lemma A.3.** We have \(v(k, T)\) is “convex” in \(k\), that is, for any \(k \in \mathbb{N}_+\)
\[
v(k, T) - v(\ell, T) \geq v(k - n, T) - v(\ell - n, T), \quad \forall \ell \in \{1, \ldots, k\}, \forall n \in \{1, \ldots, \ell\}.
\] (A.4)
Also, for any \(T \in \mathbb{R}_+\) and \(\ell \leq k\),
\[
a(\ell, T) \leq a(k, T).
\] (A.5)

**Proof.** We will prove both of the above statements together by induction. Note that (A.4) holds when \(k = 1\) since \(v(0, T) = 0\). Also \(a(1, T) \geq a(0, T) = 0\). Suppose that (A.4) and (A.5) hold for some \(k \geq 1\). We will show that they are also true when \(k\) is replaced by \(k + 1\). It is enough to prove that
\[
v(k + 1, T) - v(k, T) \geq v(k, T) - v(k - 1, T),
\] (A.6)
and that \(a(k + 1, T) \geq a(k, T)\).

First, by definition \(a(k + 1, T) = \arg\min_a\{v(k + 1 - a, T) + F(a)\}\). Now suppose that \(a(k, T) > b \geq 1\). This implies that
\[
v(k - b, T) + F(b) > v(k - a(k, T), T) + F(a(k, T))
\iff v(k - b, T) - v(k - a(k, T), T) > F(a(k, T)) - F(b)
\implies v(k + 1 - b, T) - v(k + 1 - a(k, T), T) > F(a(k, T)) - F(b) > 0
\implies a(k + 1, T) \neq b,
\]
since the sale of \(b\) shares is less preferable than selling \(a(k, T)\) shares. The third line follows from the induction hypothesis since \(k + 1 - b \leq k\). Since \(a(k + 1, T) \neq b \forall b < a(k, T)\) we obtain \(a(k + 1, T) \geq a(k, T)\).
Thanks to the fact that \(a(k+1, T) \geq a(k, T)\) for all \(T \in \mathbb{R}_+\), the induction hypothesis on \(a\), and the dynamics of \(\xi^i \equiv \xi^{(a,i,T)}\) given in (2.3) we have that \(\xi^{k+1}_{\sigma_n} - \xi^{k+1}_{\sigma_n} = \xi^k_{\sigma_n} - \xi^k_{\sigma_n} + \Delta_{\sigma_n}\), where \(\Delta_{\sigma_n} \in \{0, 1\}(\Delta_{\sigma_n} \leq 1\) due to the bound in A.3). The process \(\Delta\) should be thought of as the “additional” action needed to sell one more unit starting with \(k\) units. Now, the left-hand side of (A.6) becomes

\[
(A.7) \quad v(k+1, T) - v(k, T)
\]

\[
= \mathbb{E} \left[ \sum_{n, \sigma_n \leq T} 1_{\{\Delta_{\sigma_n} > 0\}} F(\xi^{k+1}_{\sigma_n} - \xi^k_{\sigma_n}) - F(\xi^k_{\sigma_n} - \xi^k_{\sigma_n}) \right]
\]

\[
+ 1_{\sum_{n} \Delta_{\sigma_n} = 0} \left( F(\xi^k_T) - F(\xi^k_T) \right)
\]

\[
= \mathbb{E} \left[ \sum_{n, \sigma_n \leq T} 1_{\{\Delta_{\sigma_n} > 0\}} F(\xi^k_{\sigma_n} - \xi^k_{\sigma_n} + 1) - F(\xi^k_{\sigma_n} - \xi^k_{\sigma_n}) \right]
\]

\[
+ 1_{\sum_{n} \Delta_{\sigma_n} = 0} \left( F(\xi^k_T + 1) - F(\xi^k_T) \right)
\]

Let us analyze the right-hand side of (A.6). Define the control \(\xi'\) by \(\xi'_0 = k\) and \(\xi'_{\sigma_n} = \xi^k_{\sigma_n} - \xi^{k-1}_{\sigma_n} + \Delta_{\sigma_n}\). This is an admissible control for selling \(k\) units. Then,

\[
v(k, T) - v(k - 1, T) \leq \mathbb{E} \left[ \sum_{n, \sigma_n \leq T} F(\xi'_{\sigma_n} - \xi'_n) + F(\xi'_T) \right]
\]

\[
- \mathbb{E} \left[ \sum_{n, \sigma_n \leq T} F(\xi^k_{\sigma_n} - \xi'^k_{\sigma_n}) + F(\xi^k_T) \right]
\]

\[
= \mathbb{E} \left[ \sum_{n, \sigma_n \leq T} 1_{\{\Delta_{\sigma_n} > 0\}} \left( F(\xi^{k-1}_{\sigma_n} - \xi'^{k-1}_{\sigma_n} + 1) - F(\xi^{k-1}_{\sigma_n} - \xi'^{k-1}_{\sigma_n}) \right) \right]
\]

\[
+ 1_{\sum_{n} \Delta_{\sigma_n} = 0} \left( F(\xi^{k-1}_T + 1) - F(\xi^{k-1}_T) \right)
\]

\[
\leq \mathbb{E} \left[ \sum_{n, \sigma_n \leq T} 1_{\{\Delta_{\sigma_n} > 0\}} \left( F(\xi^k_{\sigma_n} - \xi^k_{\sigma_n} + 1) - F(\xi^k_{\sigma_n} - \xi^k_{\sigma_n}) \right) \right]
\]

\[
+ 1_{\sum_{n} \Delta_{\sigma_n} = 0} \left( F(\xi^k_T + 1) - F(\xi^k_T) \right)
\]

\[
= v(k + 1, T) - v(k, T).
\]

The last inequality is by the convexity of \(F\) and the induction hypothesis on \(a\) from which it follows that \(\xi^{k-1}_{\sigma_n} - \xi'^{k-1}_{\sigma_n} \leq \xi^k_{\sigma_n} - \xi^k_{\sigma_n}\). The last equality is from (A.7). This completes the proof.\[\square\]
To conclude the proof of Proposition 2.4(i), we now show by induction that

\[(A.8) \quad v(k - a - 1, T) \leq \alpha v(k - b, T) + (1 - \alpha)v(k - a, T),\]

for any \(a < k\) and any \(b \in \mathbb{N}_+\) with \(a < b \leq k\) and \(\alpha = 1/(b - a)\). Note that (A.8) or equivalently,

\[(A.9) \quad v(k - a, T) - v(k - b, T) \leq (b - a)[v(k - a, T) - v(k - a - 1, T)]\]

holds for \(b = a + 1\). Let us assume that (A.9) holds for \(b = a + n\) (in which \(n\) is such that \(a + n + 1 \leq k\)), that is, \(v(k - a, T) - v(k - a - n, T) \leq n[v(k - a, T) - v(k - a - 1, T)]\). On the other hand,

\[v(k - a - n, T) - v(k - a - n - 1, T) \leq v(k - a, T) - v(k - a - 1, T),\]

class thanks to Lemma A.3. Adding the last two inequalities, we obtain (A.9) for \(b = a + n + 1\).

The proof of Proposition 2.4(ii) follows from (A.3) and (A.5).

A.2. Proof of Proposition 2.5.

We start with a preliminary Lemma A.4 that shows that the more units the agent has, the more eager she is to sell and so the benefit of a matching order is larger.

**Lemma A.4.** The map \(k \to G(k, T)\) is nondecreasing for all \(T \in \mathbb{R}_+\).

**Proof.** Let \(k \geq \ell\). Using \(a(\ell, T) \leq a(k, T)\) and Lemma A.3 on the second line,

\[
G(k, T) \geq v(k, T) - (v(k - a(\ell, T), T) + F(a(\ell, T))) \\
\geq v(\ell, T) - (v(\ell - a(\ell, T), T) + F(a(\ell, T))) = G(\ell, T). \quad \blacksquare
\]

We now prove Proposition 2.5(ii): For \(h > 0\), let \(A = \{\sigma_1 > h\}, B = \{\sigma_1 < h, \sigma_2 > h\},\) and \(C = (A \cup B)^c\). We have that \(\mathbb{P}(A) = e^{-\lambda h}, \mathbb{P}(B) = \lambda h e^{-\lambda h},\) and \(\mathbb{P}(C) = o(h)\). Using the dynamic programming principle, we can write

\[v(k, T + h) = \mathbb{E}[v(k, T)1_A + (v(k, T) - G(k, T))1_B + X1_C],\]

in which \(X\) is a bounded random variable. Then sending \(h \to 0\) we obtain

\[(A.10) \quad \lim_{h \to 0} \frac{v(k, T + h) - v(k, T)}{h} = \lim_{h \to 0} \frac{\mathbb{E}[v(k, T)(1_A)B) - G(k, T)1_B] - v(k, T) + o(h)}{h} = \lim_{h \to 0} \frac{-\lambda h G(k, T) + o(h)}{h} = -\lambda G(k, T).\]

Next, we return to studying the properties of \(a(k, T)\).

**Lemma A.5.** Optimal trading amount decreases as the horizon becomes longer: \(a(k, S) \leq a(k, T), \forall k \in \mathbb{N}_+, \forall S > T.\)
Proof. For any $b > a(k, T)$

\[
v(k - b, T) + F(b) > v(k - a(k, T), T) + F(a(k, T))
\]

which implies that

\[
v(k - a(k, T), T) - v(k - b, T) < F(b) - F(a(k, T)).
\]

We have that $\partial_T v(k, T) \leq \partial_T v(\ell, T)$ for $\ell \leq k$, due to Lemma A.4 and (A.10). Therefore,

\[
v(k - a(k, T), S) - v(k - b, S) \leq v(k - a(k, T), T) - v(k - b, T) < F(b) - F(a(k, T))
\]

which implies that $a(k, T)$ performs strictly better than action $b$ for the minimization problem $\min_{a \in [0, t]} \{v(k - a, S) + F(a)\}$, which implies that $b \neq a(k, S)$, which is the smallest minimizer for this problem. Since this holds for any $b > a(k, T)$ we necessarily have that $a(k, T) \geq a(k, S)$. \hfill \Box

In the next lemma we shall see that $T \to a(k, T)$ decreases to 1. By construction, we also have that $a(k, 0) = \lfloor k / 2 \rfloor$.

**Lemma A.6.** \(\lim_{T \to \infty} v(k, T) = kF(1)\) and \(\lim_{T \to \infty} a(k, T) = 1\).

**Proof.** Recall from Lemma 2.2 that $v(k, T) \leq \bar{v}^1(k, T)$ where $\bar{v}^1$ denotes the performance of a constant 1-strategy that always sells a single unit. Since

\[
\bar{v}^1(k, T) = kF(1)\mathbb{P}(N(T) \geq k)
\]

\[
+ \sum_{n=0}^{k-1} (nF(k) + F(k-n))\mathbb{P}(N(T) = n) \to kF(1), \quad \text{as } T \to \infty,
\]

while $v(k, T) \geq kF(1) \forall T$, the first statement of the lemma follows.

Let us choose a positive $0 < \delta < F(2) - 2F(1)$. Fix $k > 0$; by above, for large enough $T$, we have that $v(a, T) \leq aF(1) + \delta$ for all $a \in \{1, \ldots, k\}$. Then by convexity of $F$

\[
v(k - 1, T) + F(1) \leq (k - 1)F(1) + \delta + F(1)
\]

\[
< (k - c)F(1) + F(c) \leq v(k - c, T) + F(c),
\]

for any $T \geq T$ and any $1 < c < k$. Comparing with the definition of $a(k, T)$ in (2.2), we conclude that $a(k, T) = 1$ for $T \geq T$. \hfill \Box

The existence of threshold $t^{(k,i)}$ of (2.8) follows from Lemma A.5. It remains to show that the thresholds are distinct, that is, $t^{(k,i)} < t^{(k,i-1)}$, so that as a function of $T, a(k, T)$ experiences jumps of size 1 only. Toward a contradiction, suppose that there exists $T$ and level $k$ such that $a(k, T) - a(k, T^-) > 1$. Let $a = a(k, T)$ and $b = a(k, T^-) > a + 1$. Since $1 \leq a(k, \cdot) \leq \lfloor k/2 \rfloor$ is nonincreasing and has at most $\lfloor k/2 \rfloor$ jumps, $3\delta > 0$, such that $b = a(k, T - s)$ for all $s < \delta$. By optimality of $b$ we have that

\[
v(k - b, T - s) + F(b) = v(k - a(k, T - s), T - s) + F(a(k, T - s))
\]

\[
\leq v(k - a, T - s) + F(a) \quad \forall s < \delta
\]

Therefore, by continuity of the value function in $T$, and optimality of $a$ at $T$ we must have

(A.11) \(v(k - a, T) + F(a) = v(k - b, T) + F(b)\).
Let $\alpha = 1/(b - a) \in (0, 1)$. By the strict convexity of $F$ we have that \( F(a + 1) < \alpha F(b) + (1 - \alpha) F(a) \). Similarly, by Lemma A.3, we have that \( v(k - a - 1, T) \leq \alpha v(k - b, T) + (1 - \alpha) v(k - a, T) \). Adding the two latter equations together we obtain

\[
v(k - a - 1, T) + F(a + 1) < \alpha (v(k - b, T) + F(b)) + (1 - \alpha)(v(k - a, T) + F(a)) = v(k - a, T) + F(a), \]

which contradicts the optimality of $a$.

Finally, we may obtain the properties of the second time-derivative of $v$. By (2.7), $\partial^2 TV(k, T) = -\lambda \partial_T G(k, T)$. For any $T \neq t^{(k,i)}$ we have from combining (2.6) and (2.7) that

\[(A.12) \quad \partial_T G(k, T) = -\lambda(G(k, T) - G(k - a(k, T), T)),\]

since $a(k, T)$ is constant in a neighborhood of $T$ thanks to (2.8). By Lemma A.4, $\partial_T G(k, T) > 0$ and therefore $\partial^2 TV(k, T) < 0$. When $T = t^{(k,i)}$, the right derivative of $G$ is still equal to (A.12) since $T \to a(k, T)$ is right continuous. However, $\partial_T G(k, T-) = -\lambda(G(k, T) - G(k - a(k, T) - 1, T)) < \partial_T G(k, T+)$.

### A.3. Proof of Lemma 3.3

Denote by $\tau_k$ the $k$th transition time of $M$. For $h > 0$, let $A = \{\sigma_1 > h, \tau_1 > h\}$, $B_N = \{\sigma_1 < h, \sigma_2 > h, \tau_1 > h\}$, $B_j = \{\sigma_1 > h, \tau_1 < h, \tau_2 > h, M_{ti} = j\}$, $j \in E \setminus \{i\}$, and $C = (A \cup B_N \cup \cup B_j)^c$. By conditional independence of $N$ and $M$ we have that $\mathbb{P}^i(A) = \mathbb{P}(A M_0 = i) = e^{-(\lambda - \eta_i) h}$, $\mathbb{P}^i(B_N) = \lambda h e^{-(\lambda - \eta_i) h}$, $\mathbb{P}^i(B_j) = q_{ij} h e^{-(\lambda - \eta_i) h}$, and $\mathbb{P}^i(C) = o(h)$. Using the dynamic programing principle, we can write

\[
v(k, T + h; i) = \mathbb{E}^i \left[ v(k, T; i) 1_A + (v(k, T; i) - G(k, T; i)) 1_{B_N} + \sum_{j \in E \setminus \{i\}} v(k, T; j) 1_{B_j} + X 1_C \right],
\]

in which $X$ is a bounded random variable. Taking the limit $h \to 0$ we obtain

\[
\lim_{h \to 0} \frac{v(k, T + h; i) - v(k, T; i)}{h} = \mathbb{E}^i \left[ v(k, T; i) 1_{A \cup B_N \cup \cup B_j} - G(k, T; i) 1_{B_N} + \sum_{j \in E \setminus \{i\}} (v(k, T; j) - v(k, T; i)) 1_{B_j} \right] - \frac{\lambda h G(k, T; i)}{h} + \frac{\sum_{j \in E \setminus \{i\}} q_{ij} h (v(k, T; j) - v(k, T; i))}{h} + o(h)
\]

and (3.8) follows.

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