Some properties of the Yamabe soliton and the related nonlinear elliptic equation

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Abstract

We will prove the non-existence of positive radially symmetric solution of the nonlinear elliptic equation \( \frac{n-1}{m} \Delta v^m + av + \beta x \cdot \nabla u = 0 \) in \( \mathbb{R}^n \) when \( n \geq 3, 0 < m \leq \frac{n-2}{n} \), \( \alpha < 0 \) and \( \beta \leq 0 \). Let \( n \geq 3 \) and \( g = v^{\frac{4}{n+2}} dx^2 \) be a metric on \( \mathbb{R}^n \) where \( v \) is a radially symmetric solution of the above elliptic equation in \( \mathbb{R}^n \) with \( m = \frac{n-2}{n+2} \), \( \alpha = \frac{2\beta + \rho}{1-m} \) and \( \rho \in \mathbb{R} \). For \( n \geq 3, m = \frac{n-2}{n+2} \), we will prove that \( \lim_{r \to \infty} r^2 v^{1-m}(r) = \frac{(n-1)(n-2)}{\rho} \) if \( \beta > \frac{\rho}{n-2} > 0 \), the scalar curvature \( R(r) \to \rho \) as \( r \to \infty \) if either \( \beta > \frac{\rho}{n-2} > 0 \) or \( \rho = 0 \) and \( \alpha > 0 \) holds, and \( \lim_{r \to \infty} R(r) = 0 \) if \( \rho < 0 \) and \( \alpha > 0 \). We give a simple different proof of a result of P. Daskalopoulos and N. Sesum [DS2] on the positivity of the sectional curvature of rotational symmetric Yamabe solitons \( g = v^{\frac{4}{n+2}} dx^2 \) with \( v \) satisfying the above equation with \( m = \frac{n-2}{n+2} \). We will also find the exact value of the sectional curvature of such Yamabe solitons at the origin and at infinity.

Key words: non-existence, nonlinear elliptic equation, asymptotic behaviour, scalar curvature, sectional curvature, Yamabe soliton

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1 Introduction

In this paper we will study various properties of the solutions of the following nonlinear degenerate elliptic equation,

\[ \frac{n-1}{m} \Delta v^m + \alpha v + \beta x \cdot \nabla v = 0, \quad v > 0, \quad \text{in } \mathbb{R}^n \tag{1.1} \]

where

\[ 0 < m \leq \frac{n-2}{n} \quad \text{and} \quad n \geq 3. \tag{1.2} \]

It is proved in [DS2] by P. Daskalopoulos and N. Sesum that if a metric \( g \) is a complete locally conformally flat Yamabe gradient soliton with positive sectional curvature, then \( g = v^{4\frac{n-2}{n+2}} dx^2 \) where \( v \) is a radially symmetric solution of (1.1) in \( \mathbb{R}^n \) with \( m = \frac{n-2}{n+2} \) for some \( \beta \geq 0 \) and \( \alpha = \frac{2\beta + \rho}{1-m} \) where \( \rho > 0, \rho = 0 \) or \( \rho < 0 \) depending on whether \( g \) is a Yamabe shrinking, steady or expanding soliton. A similar result on the rotational symmetry of complete, noncompact, gradient Yamabe solitons with positive Ricci curvature without assuming the local conformally flatness of the solitons is also proved recently by G. Catino, C. Mantegazza, and L. Mazzieri [CMM]. It is also proved in [DS2] that if \( v \) is a radially symmetric solution of (1.1) in \( \mathbb{R}^n \) with \( m = \frac{n-2}{n+2} \) for some \( \beta \geq 0 \) and \( \alpha = \frac{2\beta + \rho}{1-m} > 0 \), then the metric \( g = v^{4\frac{n-2}{n+2}} dx^2 \) is a Yamabe gradient shrinking, steady or expanding soliton on \( \mathbb{R}^n \) depending on whether \( \rho > 0, \rho = 0 \) or \( \rho < 0 \).

On the other hand suppose \( v \) is a solution of (1.1) with \( 0 < m < 1 \) and \( n \geq 3 \). Then as observed by B.H. Gilding and L.A. Peletier [GP], P. Daskalopoulos and N. Sesum [DS1], [DS2], M. del Pino and M. Sáez [PS], J.L. Vázquez [V1], [V2], and others, the function

\[ u_1(x,t) = t^{-\alpha} v(xt^{-\beta}) \]

is a solution of

\[ u_t = \frac{n-1}{m} \Delta u^m \quad \text{in } \mathbb{R}^n \times (0,T) \tag{1.3} \]

if

\[ \alpha = \frac{2\beta - 1}{1-m} \]

and for any \( T > 0 \) the function

\[ u_2(x,t) = (T-t)^\alpha v(x(T-t)^\beta) \]

is a solution of (1.3) in \( \mathbb{R}^n \times (0,T) \) if

\[ \alpha = \frac{2\beta + 1}{1-m} > 0 \]

and the function

\[ u_3(x,t) = e^{-\alpha t} v(xe^{-\beta t}) \]
is an eternal solution of (1.3) in $\mathbb{R}^n \times (-\infty, \infty)$ if

$$\alpha = \frac{2\beta}{1-m}.$$  

The equation (1.1) also appears in the study of the extinction behaviour of the solution of (1.3) near the extinction time [DS1]. Hence in order to understand the solutions of (1.3) and the local conformally flat Yamabe solitons, it is important to study the various properties of the solutions of (1.1). Note that $v$ is a radially symmetric solution of (1.1) if and only if

$$\frac{n-1}{m} \left( (v^m)'' + \frac{n-1}{r}(v^m)' \right) + \alpha v + \beta rv' = 0, \quad v > 0,$$ (1.4)
in $(0, \infty)$ with

$$v'(0) = 0, \quad v(0) = \eta$$ (1.5)
for some constant $\eta > 0$. In [H] I have proved that for any $n, m$, satisfying (1.2) and

$$\alpha \leq \frac{\beta(n-2)}{m} \quad \text{and} \quad \beta > 0,$$

there exists a unique global solution of (1.4), (1.5). In this paper we will prove the following non-existence results for radially symmetric solutions of (1.1).

**Theorem 1.1.** Let $m, n$, satisfy (1.2) and let $\eta > 0, \alpha < 0$ and $\beta \leq 0$. Then (1.4), (1.5), has no solution in $(0, \infty)$.

We will also prove the following property of Yamabe solitons.

**Theorem 1.2.** Let $n \geq 3$, $m = \frac{n-2}{n+2}$, $\alpha = \frac{2\beta+\rho}{1-m}$, and let $v$ be a radially solution of (1.1). Let $R(x)$ be the scalar curvature of the metric $g = v^{\frac{4}{n-2}}dx^2$ on $\mathbb{R}^n$. Then the following holds.

(i) If either

$$\begin{cases} 
\rho = 0 \\
\beta > 0 
\end{cases} \quad \text{or} \quad \beta > \rho/(n-2) > 0$$ (1.6)

holds, then

$$\lim_{|x| \to \infty} R(x) = \rho.$$ (1.7)

(ii) If $\rho < 0$ and $\alpha > 0$, then

$$\lim_{|x| \to \infty} R(x) = 0.$$ (1.8)

(iii) If $\beta > \frac{\rho}{n-2} > 0$, then

$$\lim_{|x| \to \infty} |x|^2v^{1-m}(x) = \frac{(n-1)(n-2)}{\rho}$$ (1.9)

and

$$0 < |x|^2v^{1-m}(x) < \frac{(n-1)(n-2)}{\rho} \quad \forall x \neq 0.$$ (1.10)
We will give a simple different proof of the result of P. Daskalopoulos and N. Sesum [DS2] on the positivity of the sectional curvature of Yamabe solitons. We also find the exact value of the sectional curvature of such Yamabe solitons at the origin and at infinity. More precisely we will prove the following theorem.

**Theorem 1.3.** Let $g = v^{\frac{4}{n+2}}dx^2$ be a rotationally symmetric metric such that $v$ satisfies (1.1) with $m = \frac{n-2}{n+2}$, $\alpha = \frac{2\beta + \rho}{1-m}$, which is either a steady Yamabe soliton ($\rho = 0$) with $\alpha > 0$, or a Yamabe expanding soliton ($\rho < 0$) with $\alpha > 0$, or a Yamabe shrinking soliton ($\rho > 0$) with $\beta > \rho/(n-2)$. Then $g$ has strictly positive sectional curvature. If $K_0$ and $K_1$ are the sectional curvatures of the 2-planes perpendicular to and tangent to the spheres $\{x\} \times S^{n-1}$ respectively, then

$$K_0(0) = K_1(0) = \frac{2\beta + \rho}{n(n-1)},$$

$$\lim_{r \to \infty} K_0(r) = 0,$$

and

$$\lim_{r \to \infty} K_1(r) = \begin{cases} \frac{\rho}{(n-1)(n-2)} & \text{if } g \text{ is a shrinking Yamabe soliton} \\ 0 & \text{if } g \text{ is a steady or expanding Yamabe soliton.} \end{cases}$$

Note that the results (1.11), (1.12) and (1.13), are entirely new. The plan of the paper is as follows. We will prove the non-existence of solutions of (1.4), (1.5), and prove various properties of solutions of (1.4), (1.5), in section two. In section three we will prove the asymptotic behaviour of the metric and the scalar curvature of the Yamabe solitons. In section four we will give another proof of the positivity of the sectional curvature of Yamabe solitons. We will also prove the asymptotic behaviour of the sectional curvature of the Yamabe solitons as $|x| \to \infty$.

2 Non-existence and properties of solutions of the non-linear elliptic equation

For any $\beta \in \mathbb{R}$, $\alpha \neq 0$, let $k = \beta/\alpha$. We first recall a result of [H].

**Lemma 2.1.** (Lemma 2.1 of [H]) Let $m$, $\alpha \neq 0$, $\beta \neq 0$, satisfy (1.2) and

$$\frac{m\alpha}{\beta} \leq n - 2.$$  \hspace{1cm} (2.1)

For any $R_0 > 0$ and $\eta > 0$, let $v$ be the solution of (1.4), (1.5), in $(0, R_0)$. Then

$$v + krv'(r) > 0 \quad \text{in } [0, R_0)$$  \hspace{1cm} (2.2)

and

$$\begin{cases} v'(r) < 0 & \text{in } (0, R_0) \quad \text{if } \alpha > 0 \\ v'(r) > 0 & \text{in } (0, R_0) \quad \text{if } \alpha < 0. \end{cases}$$  \hspace{1cm} (2.3)
Proof of Theorem 1.1: Suppose there exists a solution $v$ for (1.4), (1.5), in $(0, \infty)$. Multiplying (1.4) by $r^{n-1}$ and integrating we have

$$\frac{n-1}{m} r^{n-1} (v^{m})'(r) = \alpha \int_{0}^{r} z^{n-1} v(z) \, dz - \beta \int_{0}^{r} z^{n} v'(z) \, dz$$

$$= -\beta r^{n} v(r) + (n\beta - \alpha) \int_{0}^{r} z^{n-1} v(z) \, dz \quad \forall r > 0. \quad (2.4)$$

We now divide the proof into three cases.

**Case 1:** $0 > n\beta > \alpha$.

By (2.4),

$$\frac{n-1}{m} r^{n-1} (v^{m})'(r) \geq |\beta| r^{n} v(r) \quad \Rightarrow \quad (n-1) v^{m-2} v'(r) \geq |\beta| r \quad \forall r > 0.$$

**Case 2:** $0 > \alpha \geq n\beta$.

By (1.2), (2.1) holds. Since $\alpha < 0$, by Lemma (2.1) $v'(r) > 0$ for any $r > 0$. Then by (2.4),

$$\frac{n-1}{m} r^{n-1} (v^{m})'(r) \geq -\beta r^{n} v(r) - \frac{\alpha - n\beta}{n} r^{n} v(r) = \frac{|\alpha|}{n} r^{n} v(r)$$

for any $r > 0$. Hence

$$(n-1) v^{m-2} v'(r) \geq \frac{|\alpha|}{n} r \quad \forall r > 0.$$

By case 1 and case 2,

$$v^{m-2} v'(r) \geq C_{1} r \quad \forall r > 0 \quad (2.5)$$

where

$$C_{1} = \frac{1}{n-1} \min \left( \frac{|\alpha|}{n}, |\beta| \right).$$

Integrating (2.5) over $(0, r)$,

$$\frac{1}{1-m} (\eta^{m-1} - v(r)^{m-1}) \geq \frac{C_{1}}{2} r^{2} \quad \forall r > 0$$

$$\Rightarrow \quad v^{1-m}(r) \geq \left( \eta^{m-1} - \frac{(1-m)C_{1}}{2} r^{2} \right)^{-1} \rightarrow \infty \quad \text{as} \quad r \nearrow \sqrt{2/(C_{1}(1-m))} \eta^{m-1}. $$

Contradiction arises. Hence (1.4), (1.5), has no global solution when either case 1 or case 2 holds.

**Case 3:** $\alpha < 0$ and $\beta = 0$.

By (2.4),

$$\frac{n-1}{m} r^{n-1} (v^{m})'(r) = |\alpha| \int_{0}^{r} s^{n-1} v(s) \, ds > 0 \quad \forall r > 0$$

$$\Rightarrow \quad 0 < \frac{n-1}{m} r^{n-1} (v^{m})'(r) \leq \frac{|\alpha|}{n} r^{n} v(r) \quad \forall r > 0$$

$$\Rightarrow \quad 0 < \frac{n-1}{r} (v^{m})'(r) \leq \frac{m|\alpha|}{n} v(r) \quad \forall r > 0. \quad (2.6)$$
Hence by (1.4) and (2.6),
\[
|\alpha|v = \frac{n-1}{m} \left( (v^m)'(r) + \frac{n-1}{r} (v^m)(r) \right) \\
\leq \frac{n-1}{m} \left( (v^m)'(r) + \frac{m|\alpha|}{n} v(r) \right) \quad \forall r > 0
\]
\[\Rightarrow \quad (v^m)'(r) \geq \frac{m|\alpha|}{n(n-1)} v(r) \quad \forall r > 0 \quad (2.7)
\]
\[\Rightarrow \quad (v^m)^2(r) \geq \frac{2m^2|\alpha|}{n(n-1)(1+m)} (v^{1+m}(r) - \eta^{1+m}) \quad \forall r > 0. \quad (2.8)
\]

Since \(v(r) > v(0)\) for all \(r > 0\), by (2.7) \(v(r) \to \infty\) as \(r \to \infty\). Hence there exists a constant \(R_1 > 0\) such that
\[v^{1+m}(r) > 2\eta^{1+m} \quad \forall r \geq R_1. \quad (2.9)
\]

By (2.8) and (2.9), there exists a constant \(C_2 > 0\) such that
\[
(v^m)^2(r) \geq C_2^2 m^2 v^{1+m}(r) \quad \forall r \geq R_1
\]
\[\Rightarrow \quad v^{-\frac{1}{1+m}} v'(r) \geq C_2 \quad \forall r \geq R_1
\]
\[\Rightarrow \quad v^{\frac{1}{1+m}}(r) \geq \left( v(R_1)^{\frac{1}{1+m}} - \frac{(1-m)C_2}{2} (r - R_1) \right)^{-1} \to \infty \quad \text{as} \quad r \to R_1 + \frac{2v(R_1)^{\frac{1}{1+m}}}{(1-m)C_2}.
\]

Contradiction arises. Hence (1.4), (1.5), has no global solution in case 3 and the theorem follows.

\[\square\]

Note that if \(\alpha = \beta = 0\), then the constant function \(v = \eta\) is a solution of (1.4), (1.5). Hence Theorem 1.1 is sharp. As a result of Theorem 1.1 we get the following result of [DS2].

**Corollary 2.2.** (cf. Claim 3.1 of [DS2]) Let \(n \geq 3\) and \(\eta > 0\). Suppose \(g = v^{\frac{4}{n+2}}dx^2\) is a complete locally flat gradient Yamabe soliton with radially symmetric \(v\) satisfying (1.4), (1.5), with \(m = \frac{n-2}{n+2}\), \(\alpha = \frac{2\rho+\beta}{1-m}\), where \(\rho = 0\) for Yamabe steady soliton and \(\rho < 0\) for Yamabe expanding soliton respectively. Then \(\beta > 0\) if \(\rho < 0\) and \(\beta \geq 0\) if \(\rho = 0\).

**Lemma 2.3.** Let \(n \geq 3\), \(0 < m < 1\), \(\alpha \geq n\beta\) and \(\alpha > 0\), and \(\eta > 0\). Suppose \(v\) is a solution of (1.4), (1.5). Then
\[
0 < r^2 v^{1-m}(r) \leq \frac{2n(n-1)}{\alpha(1-m)} \quad \forall r > 0 \quad (2.10)
\]
and
\[
\frac{\alpha}{n(n-1)} r^2 v^{1-m}(r) + \frac{\alpha' r}{v(r)} \leq 0 \quad \forall r > 0. \quad (2.11)
\]
Hence \(v'(r) < 0\) for all \(r > 0\).
Proof: By (2.4),
\[
\frac{n-1}{m} \lim_{r \to 0} \frac{(v^m)'(r)}{r} = -\beta \lim_{r \to 0} v(r) - (\alpha - n\beta) \lim_{r \to 0} \frac{1}{r^m} \int_0^r z^{n-1}v(r) \, dz
\]
\[
= -\beta v(0) - \frac{\alpha - n\beta}{n} v(0)
\]
\[
= -\frac{\alpha}{n} v(0) < 0.
\]
Hence there exists \( R_0 > 0 \) such that \( (v^m)'(r) < 0 \) or \( v'(r) < 0 \) for all \( 0 < r \leq R_0 \). Let \((0, R_1)\) be the maximal interval such that \( v'(r) < 0 \) for all \( 0 < r \leq R_1 \). Then by (2.4),
\[
\frac{n-1}{m} r^{n-1}(v^m)'(r) \leq -\beta r^n v(r) - (\alpha - n\beta) \int_0^r z^{n-1}v(r) \, dz = -\frac{\alpha}{n} r^2 v(r) \quad \forall 0 < r < R_1
\]
\[
\Rightarrow v^{m-2}(r)v'(r) \leq -\frac{\alpha}{n(n-1)} r \\
\forall 0 < r < R_1. \tag{2.12}
\]
If \( R_1 < \infty \), then by (2.12) there exists a constant \( R_2 > R_1 \) such that \( v'(r) < 0 \) for all \( 0 < r \leq R_2 \).
This contradicts the maximality of \( R_1 \). Hence \( R_1 = \infty \). Thus \( v'(r) < 0 \) for all \( r > 0 \). Then (2.12) holds in \((0, \infty)\) and (2.11) follows. Integrating (2.12) over \((0, r)\) and simplifying,
\[
v(r) \leq \left( \eta^{m-1} + \frac{\alpha(1-m)}{2n(n-1)} r^2 \right)^{-\frac{1}{m-1}} \leq \left( \frac{2n(n-1)}{\alpha(1-m)} r^2 \right)^{-\frac{1}{m}} \quad \forall r > 0
\]
and (2.10) follows. \( \square \)

**Remark 2.4.** If \( n \geq 3, 0 < m \leq \frac{n-2}{n} \) and \( m\alpha \geq \beta(n-2) \), then \( \alpha \geq n\beta \).

3 Asymptotic behaviour of Yamabe solitons

We will assume from now on that \( n \geq 3, m = \frac{n-2}{n+2} \), and \( k = \beta/\alpha \) for any \( \alpha \neq 0 \) and \( \beta \in \mathbb{R} \) for the rest of the paper. In this section we will study the asymptotic behaviour of locally conformally flat Yamabe solitons with positive sectional curvature. By the results of [DS2] we can write the metric of such soliton as \( g = v^{\frac{1}{2}} dx^2 \) where \( v \) is a radially symmetric solution of (1.1) in \( \mathbb{R}^n \) with \( m = \frac{n-2}{n+2} \) for some \( \beta \geq 0 \) and \( \alpha = \frac{2\beta + \rho}{1-m} \) where \( \rho > 0, \rho = 0 \) or \( \rho < 0 \), depending on whether \( g \) is a Yamabe shrinking, steady or expanding soliton. We will use \( R \) to denote the scalar curvature of the metric \( g \) and let
\[
w(r) = r^2 \sqrt{1-m}(r), \quad s = \log r, \quad \bar{w}(s) = w(r).
\]

**Lemma 3.1.** Let \( g = v^{\frac{1}{2}} dx^2 \) be a rotational symmetric Yamabe soliton with \( v \) satisfying (1.1) with \( m = \frac{n-2}{n+2} \) and \( \alpha = \frac{2\beta + \rho}{1-m} \) for some constant \( \rho \in \mathbb{R} \). Then the following holds.

(i) If \( g \) is a Yamabe steady soliton or a Yamabe expanding soliton, then \( R > 0 \) if \( \alpha > 0 \).
(ii) If \( g \) is a Yamabe expanding soliton, then \( R < 0 \) if \( \alpha < 0 < \beta \).

(iii) If \( g \) is a Yamabe shrinking soliton with \( \beta > \rho/(n-2) \), then \( R > \rho \).

(iv) If \( g \) is a Yamabe shrinking, steady, or expanding soliton with \( \alpha > 0 \), then
\[
0 \leq R \leq \alpha(1-m). \tag{3.1}
\]

(v) Suppose \( g \) is either a Yamabe shrinking soliton with \( \beta > \rho/(n-2) \) or a Yamabe steady or expanding soliton with \( \alpha > 0 \). Then
\[
1 + \frac{1-m}{2} \frac{rv'(r)}{v(r)} > 0 \quad \forall r \geq 0 \tag{3.2}
\]
and \( w'(r) > 0 \) for any \( r > 0 \).

Proof: (i), (ii) and (iii) is proved in Proposition 4.1 of [DS2]. For the sake of completeness we will give a simple different proof of (i) and (ii) here. As observed in [DS2] since the scalar curvature satisfies (P.184 of [SY]),
\[
R = -\frac{4(n-1)}{n-2} \cdot \frac{\Delta v^m}{v}, \tag{3.3}
\]
by (1.1),
\[
R(r) = (1-m) \left( \alpha + \beta \frac{rv'(r)}{v(r)} \right) = \alpha(1-m) \left( 1 + k \frac{rv'(r)}{v(r)} \right). \tag{3.4}
\]
Note that for Yamabe steady soliton and Yamabe expanding soliton we have \( \alpha = \frac{2\beta + \rho}{1-m} \) with \( \rho = 0 \) and \( \rho < 0 \) respectively. Hence (2.1) holds for Yamabe steady soliton with \( \alpha > 0 \) and for Yamabe expanding soliton with either \( \alpha > 0 \) or \( \alpha < 0 < \beta \). Thus by Lemma 2.1 (2.2) holds for all \( r > 0 \) in both cases (i) and (ii). By (2.2) and (3.4), (i) and (ii) of the lemma follows.

To prove (iv) we first observe that by the result of [DS2] if \( g \) is a Yamabe shrinking soliton, then \( R \geq 0 \). This together with (i) imply the first inequality \( R \geq 0 \) of (3.1). We next observe that under the hypothesis of (iv) by Lemma 2.1 Lemma 2.3 and Remark 2.4 we have \( v'(r) < 0 \) for all \( r > 0 \). Hence by (3.4) we get (3.1).

Suppose \( g \) is either a Yamabe shrinking soliton with \( \beta > \rho/(n-2) \) or a Yamabe steady or expanding soliton with \( \alpha > 0 \). Then by (i), (iii), and (3.4),
\[
\rho + 2\beta \left( 1 + \frac{1-m}{2} \frac{rv'(r)}{v(r)} \right) > \rho \quad \forall r \geq 0
\]
and (3.2) follows. Hence
\[
w'(r) = 2r v^{1-m} \left( 1 + \frac{1-m}{2} \frac{rv'(r)}{v(r)} \right) > 0 \quad \forall r > 0 \tag{3.5}
\]
and (v) follows. \( \square \)
Lemma 3.2. Let \( n \geq 3, m = \frac{n-2}{n-1}, \eta > 0, \beta > \frac{\rho}{n-2} > 0, \alpha = \frac{2\beta+\rho}{1-\beta}, \) be such that \( n\beta > \alpha. \) Suppose \( \nu \) is a solution of (1.4), (1.5). Then

\[
0 < r^2\nu^{1-m}(r) \leq \frac{(n-1)(n-2)}{\rho} \quad \forall r > 0
\]  

and for any \( 0 < \delta < \frac{\rho}{n(1-m)-2} \) there exists a constant \( R_1 > 1 \) such that

\[
\frac{1}{n-1}\left[\frac{\rho}{n(1-m)-2} - \delta\right] r^2\nu^{1-m}(r) + r\nu'(r) \leq 0 \quad \forall r \geq R_1.
\]  

Proof: We first claim that

\[
\limsup_{r \to \infty} \frac{\int_0^r z^{n-1}\nu(z) \, dz}{r^n\nu(r)} \leq \frac{1-m}{n(1-m)-2}.
\]  

By (v) of Lemma 3.1 there exists a constant \( C > 0 \) such that \( \nu(r) \geq C \) for any \( r > 1. \) Hence

\[
r^n\nu(r) = r^{n-\frac{2}{n-1}}w^{\frac{1}{n-1}}(r) \geq Cr^{n-\frac{2}{n-1}} \quad \forall r > 1
\]  

\[
\Rightarrow \quad r^n\nu(r) \to \infty \quad \text{as} \quad r \to \infty.
\]  

We now divide the claim into two cases.

Case 1: \( \int_0^\infty z^{n-1}\nu(z) \, dz < \infty. \)

By (3.10) we get (3.8).

Case 2: \( \int_0^\infty z^{n-1}\nu(z) \, dz = \infty. \)

Since by (3.9) and (v) of Lemma 3.1

\[
\frac{d}{dr}(r^n\nu(r)) = \left(n - \frac{2}{1-m}\right)r^{n-1}\nu(r) + \frac{1}{1-m}r^{n-\frac{2}{n-1}}w^{\frac{1}{n-1}}(r)w'(r) \geq \left(n - \frac{2}{1-m}\right)r^{n-1}\nu(r) \quad \forall r > 0,
\]  

by the l’Hospital rule,

\[
\limsup_{r \to \infty} \frac{\int_0^r z^{n-1}\nu(z) \, dz}{r^n\nu(r)} = \limsup_{r \to \infty} \frac{r^{n-1}\nu(r)}{\left(n - \frac{2}{1-m}\right)r^{n-1}\nu(r) + \frac{1}{1-m}r^{n-\frac{2}{n-1}}w^{\frac{1}{n-1}}(r)w'(r)} \leq \left(n - \frac{2}{1-m}\right)^{-1}
\]  

and (3.8) follows. Let \( 0 < \delta < \frac{\rho}{n(1-m)-2} \) and let \( \epsilon > 0 \) be given by

\[
\frac{2(n-1)}{1-m} \cdot \left[\frac{\rho}{n(1-m)-2} - \delta\right]^{-1} = \frac{2(n-1)(n(1-m)-2)}{(1-m)r} + \epsilon = \frac{(n-1)(n-2)}{\rho} + \epsilon.
\]  

Then \( \epsilon \to 0 \) as \( \delta \to 0. \) By the above claim there exists a constant \( R_1 > 1 \) such that

\[
\frac{\int_0^r z^{n-1}\nu(z) \, dz}{r^n\nu(r)} < \frac{(1-m)}{n(1-m)-2} + \frac{\delta}{n\beta - \alpha} \quad \forall r \geq R_1
\]  

\[
\Rightarrow \quad \int_0^r z^{n-1}\nu(z) \, dz \leq \left(\frac{(1-m)}{n(1-m)-2} + \frac{\delta}{n\beta - \alpha}\right)r^n\nu(r) \quad \forall r \geq R_1.
\]  

(3.12)
By (2.4) and (3.12),
\[
\frac{n-1}{m} r^{n-1} (v^m)'(r) \leq -\beta r^m v(r) + \left( \frac{(m^\beta - \alpha)(1-m)}{n(1-m) - 2} + \delta \right) r^m v(r)
\]
\[
\leq -\left( \frac{\rho}{n(1-m) - 2} - \delta \right) r^m v(r) \quad \forall r \geq R_1
\]
\[
\Rightarrow (n-1) v^{m-2} v'(r) \leq -\left( \frac{\rho}{n(1-m) - 2} - \delta \right) r \quad \forall r \geq R_1
\]
(3.13)

and (3.7) follows. Integrating (3.13) over \((R_1, r)\) and simplifying, by (3.11),
\[
v^{1-m}(r) \leq \left( v(R_1)^{n-1} + \frac{1-m}{2(n-1)} \cdot \left( \frac{\rho}{n(1-m) - 2} - \delta \right) r^2 \right)^{-1} \quad \forall r \geq R_1
\]
\[
\leq \left( \frac{(n-1)(n-2)}{\rho} + \varepsilon \right) r^{-2} \quad \forall r \geq R_1
\]
\[
\Rightarrow r^2 v^{1-m}(r) \leq \frac{(n-1)(n-2)}{\rho} + \varepsilon \quad \forall r \geq R_1.
\]

Hence by (v) of Lemma 3.1,
\[
w(r) = r^2 v^{1-m}(r) \leq \frac{(n-1)(n-2)}{\rho} + \varepsilon \quad \forall r > 0.
\]
(3.14)

Letting \(\varepsilon \to 0\) in (3.14) we get (3.6) and the lemma follows. \(\square\)

**Lemma 3.3.** Let \(n \geq 3, m = \frac{n-2}{n+2}, \alpha = \frac{2^{\beta+\rho}}{1-m} \) and \(\beta > \frac{\rho}{n-2} > 0\). Suppose \(v\) is a radially symmetric solution of (1.1). Then
\[
a_0 = \lim_{r \to \infty} r^2 v^{1-m}(r)
\]
exists and \(0 < a_0 < \infty\).

**Proof:** By Lemma 2.3 (2.10) holds if \(\alpha \geq n\beta\) and by Lemma 3.2 (3.6) holds if \(n\beta > \alpha\). Hence by (v) of Lemma 3.1 \(r^2 v^{1-m}(r)\) converges to some positive number as \(r \to \infty\) and the lemma follows. \(\square\)

**Lemma 3.4.** Let \(g = v^{\frac{2}{n-2}} dx^2\) be a rotationally symmetric metric which satisfies (1.1) with \(m = \frac{n-2}{n+2}, \alpha = \frac{2^{\beta+\rho}}{1-m}\), which is either a steady Yamabe soliton \((\rho = 0)\) with \(\alpha > 0\), or a Yamabe expanding soliton \((\rho < 0)\) with \(\alpha > 0\), or a Yamabe shrinking soliton \((\rho > 0)\) with \(\beta > \frac{\rho}{n-2}\). Then
\[
\lim_{r \to \infty} \frac{rv'(r)}{v(r)} = \begin{cases} 
-\frac{2}{1-m} & \text{if } g \text{ is a shrinking or steady Yamabe soliton} \\
-\frac{1}{k} & \text{if } g \text{ is an expanding Yamabe soliton.}
\end{cases}
\]
(3.15)
Proof: By Lemma 2.1, Lemma 2.3, Remark 2.4 and Lemma 3.1

\[- \frac{2}{1 - m} \leq \frac{rv'(r)}{v(r)} \leq 0 \quad \forall r > 0. \quad (3.16)\]

Let \( \{r_i\}_{i=1}^\infty \) be a sequence such that \( r_i \to \infty \) as \( i \to \infty \). Then by (3.16) \( \{r_i\}_{i=1}^\infty \) has a subsequence which we may assume without loss of generality to be the sequence \( \{r_i\}_{i=1}^\infty \) itself such that \( \frac{rv'(r_i)}{v(r_i)} \) converges to some number \( a \) as \( i \to \infty \). Let \( a_0 \) be given by Lemma 3.3. We now divide the proof into three cases.

Case 1: \( g \) is a Yamabe shrinking soliton with \( \beta > \frac{\rho}{n-2} \).

Then by the l'Hospital rule,

\[
a_0 = \lim_{i \to \infty} r_i^2 v^{1-m}(r_i) = \lim_{i \to \infty} \frac{v^{1-m}(r_i)}{r_i^{-2}} = \lim_{i \to \infty} \frac{(1 - m)v^{-m}(r_i)v'(r_i)}{-2r_i^{-3}}
\]

\[
= - \frac{1 - m}{2} \lim_{i \to \infty} r_i^2 v^{1-m}(r_i) \cdot \lim_{i \to \infty} \frac{r_i v'(r_i)}{v(r_i)} = - \frac{1 - m}{2} \cdot a_0 a
\]

\[
\Rightarrow a = - \frac{2}{1 - m}.
\]

Since the sequence \( \{r_i\}_{i=1}^\infty \) is arbitrary, we get

\[
\lim_{r \to \infty} \frac{rv'(r)}{v(r)} = - \frac{2}{1 - m}. \quad (3.17)
\]

Case 2: \( g \) is a Yamabe steady soliton with \( \alpha > 0 \).

By Theorem 1.3 of [H],

\[
\lim_{r \to \infty} \frac{r^2 v^{1-m}(r)}{\log r} = a_1
\]

where \( a_1 = 2(n-1)(n-2-mn)/[\beta(1-m)] \). Then by the l'Hospital rule,

\[
a_1 = \lim_{i \to \infty} \frac{v^{1-m}(r_i)}{-r_i^{-2} \log r_i} = \lim_{i \to \infty} \frac{(1 - m)v^{-m}(r_i)v'(r_i)}{-2r_i^{-3} \log r_i + r_i^{-3}} = - \frac{1 - m}{2} \lim_{i \to \infty} \frac{r_i v^{1-m}(r_i)}{\log r_i} \cdot \lim_{i \to \infty} \frac{r_i v'(r_i)}{v(r_i)}
\]

\[
= - \frac{1 - m}{2} \cdot a_1 a.
\]

Hence

\[
a = - \frac{2}{1 - m}.
\]

Since the sequence \( \{r_i\}_{i=1}^\infty \) is arbitrary, we get (3.17).

Case 3: \( g \) is a Yamabe expanding soliton with \( \alpha > 0 \).

By Theorem 1.6 of [H] (cf. Theorem 3.2 of [V1]),

\[
\lim_{r \to \infty} r^2 v^{2k}(r) = a_2
\]
for some constant $0 < a_2 < \infty$. Then by the l’Hospital rule,

$$a_2 = \lim_{i \to \infty} \frac{v^{2k}(r_i)}{r_i^{-2}} = \lim_{i \to \infty} \frac{2k \cdot v^{2k-1}(r_i)v'(r_i)}{-2r_i^3} = -k \lim_{i \to \infty} r_i^2 v^{2k}(r_i) \cdot \lim_{i \to \infty} \frac{r_i v'(r_i)}{v(r_i)} = -ka_2a.$$

Hence

$$a = -\frac{1}{k}.$$

Since the sequence $\{r_i\}_{i=1}^{\infty}$ is arbitrary, we get

$$\lim_{r \to \infty} rv'(r) = \frac{1}{k}.$$

By case 1, case 2 and case 3, the lemma follows. \qed

We are now ready for the proof of Theorem 1.2.

**Proof of Theorem 1.2**: Suppose first (1.6) holds. Then by (3.4) and Lemma 3.4,

$$R = \rho + 2\beta \left( 1 + \frac{1 - m rv'(r)}{2v(r)} \right) \to \rho \quad \text{as} \quad r \to \infty \quad (3.18)$$

and (i) of Theorem 1.2 follows. If $\rho < 0$ and $\alpha > 0$, then by (3.4) and Lemma 3.4 we get (ii) of Theorem 1.2.

We next assume that $\beta > \frac{\rho}{n-2} > 0$. Let $a_0$ be as in Lemma 3.3. Then by Lemma 2.3, Lemma 3.2, (3.5) and (3.18),

$$\tilde{w}_s(s) = rv'(r) = \frac{\tilde{w}(s)}{\beta}(R - \rho) \to 0 \quad \text{as} \quad s = \log r \to \infty. \quad (3.19)$$

Let $\{r_i\}_{i=1}^{\infty}$ be such that $r_i \to \infty$ as $i \to \infty$ and $s_i = e^{r_i}$. As proved in [DS2] and [H] $\tilde{w}$ satisfies

$$\tilde{w}_{ss} = \frac{1 - 2m}{1 - m} \cdot \frac{\tilde{w}_s^2}{\tilde{w}} - \frac{\beta}{n-1} \tilde{w}_s \tilde{w}_s - \frac{\rho}{n-1} \tilde{w}^2 + \frac{2(n - 2 - nm)}{1 - m} \tilde{w} \quad (3.20)$$

in $(-\infty, \infty)$. Hence by Lemma 3.3 (3.19) and (3.20),

$$a_3 = \lim_{s \to \infty} \tilde{w}_s(s) = -\frac{a_0^2 \rho}{n-1} + \frac{2(n - 2 - nm)}{1 - m} a_0$$

exists. Suppose $a_3 \neq 0$. Without loss of generality we may assume that $a_3 > 0$. Then

$$\lim_{s \to \infty} \tilde{w}_s(s) = \infty$$

which contradicts (3.19). Hence $a_3 = 0$. Thus

$$\frac{a_0^2 \rho}{n-1} = \frac{2(n - 2 - nm)}{1 - m} a_0 \quad \Rightarrow \quad a_0 = \frac{2(n - 1)(n - 2 - nm)}{(1 - m) \rho} = \frac{(n - 1)(n - 2)}{\rho}$$

and (1.9) follows. By (1.9) and (v) of Theorem 3.1 we get (1.10) and the theorem follows. \qed
4 Positivity and asymptotic behaviour of the sectional curvature of Yamabe solitons

In this section we will give a simple proof on the positivity of the sectional curvature of rotational symmetric Yamabe solitons of the form \( g = v^{\frac{4}{n-2}} dx^2 \) in \( \mathbb{R}^n \), \( n \geq 3 \), where \( v \) satisfies (1.1) with \( m = \frac{n-2}{n+2} \). We also find the exact value of the sectional curvature of such Yamabe solitons at the origin and at infinity. We first prove the following improvement of Corollary 4.2 of [DS2].

**Lemma 4.1.** (cf. Corollary 4.2 of [DS2]) Let \( g = v^{\frac{4}{n-2}} dx^2 \) be a rotationally symmetric Yamabe soliton in \( \mathbb{R}^n \), \( n \geq 3 \), such that \( v \) satisfies (1.1) with \( m = \frac{n-2}{n+2} \) and \( \alpha = \frac{2\beta + \rho}{1-m} \). Suppose \( \beta > \frac{\rho}{n-2} > 0 \) if \( g \) is a Yamabe shrinking soliton, and \( \alpha > 0 \) if \( g \) is a Yamabe steady or expanding soliton. Then the scalar curvature \( R(r) \) is a strictly decreasing function of \( r > 0 \) and \( R'(r) < 0 \) for all \( r > 0 \).

**Proof:** As proved in [C] and [DS2], \( R \) satisfies

\[
(n-1)\Delta R + \beta (x \cdot \nabla) v^{1-m} + R(R - \rho) v^{1-m} = 0 \quad \text{in} \quad \mathbb{R}^n
\]

where \( \Delta, \nabla \), are the laplacian and gradient with respect to the Euclidean metric in \( \mathbb{R}^n \). Hence

\[
R''(r) + \frac{n-1}{r} R'(r) + \frac{\beta}{n-1} v(r)^{1-m} R'(r) = -\frac{1}{n-1} v(r)^{1-m} R(r)(R(r) - \rho) \quad \forall r > 0
\]

\[
\Rightarrow \frac{d}{dr} \left( (n-1)^{-1} e^{\frac{\beta}{n-1}} \int_0^{\tau(v(r))^{1-m}} dt R'(r) \right) = -\frac{n-1}{n-1} v(r)^{1-m} R(r)(R(r) - \rho) e^{\frac{\beta}{n-1}} \int_0^{\tau(v(r))^{1-m}} dt \quad \forall r > 0
\]

\[
\Rightarrow R'(r) = -\int_0^r e^{\frac{\beta}{n-1}} \int_0^{\tau(v(z))^{1-m}} dz \quad \forall r > 0.
\]

By (i) and (iii) of Lemma 3.1, the term \( R(r)(R(r) - \rho) \) is always positive. Hence \( R'(r) < 0 \) for all \( r > 0 \) and the lemma follows. \( \square \)

We are now ready for the proof of Theorem 1.3.

**Proof of Theorem 1.3** Positivity of the sectional curvature of Yamabe soliton is proved in [DS2]. Since the proof in [DS2] is hard, we will give a simple proof of this result in this paper. Similar to [DS2] we let

\[
\bar{s} = \int_{-\infty}^s \bar{w}(\tau)^{\frac{1}{n}} d\tau \quad \text{and} \quad \psi(s) = \bar{w}(s)^{\frac{2}{n}} \quad \forall s > -\infty.
\]

Then

\[
(log \bar{w}(s))_s = 2\psi_s, \quad \psi_{ss} = \frac{(log \bar{w}(s))_{ss}}{2\bar{w}(s)^{\frac{2}{n}}}
\]

and \( K_0, K_1 \), are given by (P.25 of [DS2]),

\[
K_0 = -\frac{\psi_{ss}}{\psi}, \quad K_1 = \frac{1 - \psi_s^2}{\psi^2}.
\]
By (3.19) and Lemma 4.1,
\[ \beta(\log \tilde{w}(s))ss = R_s = rR_r < 0 \quad \forall s > -\infty. \] (4.4)
By (4.2), (4.3) and (4.4), we get
\[ K_0 > 0 \quad \forall r > 0 \]
and
\[ K_0 = -\frac{R_s}{2\beta \tilde{w}} = -\frac{R_r}{2\beta r^{1-m}(r)}. \] (4.5)
Let \( Q(r) = v(r)^{1-m}R(r)(R(r) - \rho)/(n - 1) \). Then by (3.4), (4.1) and (4.5),
\[ K_0(0) = \lim_{r \to 0} \int_0^r z^{n-1}Q(z)e^{\frac{\beta}{n-1} \int_0^z \tau v(\tau)^{1-m}d\tau} dz \]
\[ = \frac{v^{m-1}(0)}{2\beta} \lim_{r \to 0} \int_0^r z^{n-1}Q(z)e^{\frac{\beta}{n-1} \int_0^z \tau v(\tau)^{1-m}d\tau} dz \]
\[ = \frac{v^{m-1}(0)}{2\beta} \lim_{r \to 0} \frac{r^{n-1}Q(r)e^{\frac{\beta}{n-1} \int_0^z \tau v(\tau)^{1-m}d\tau} dz}{n!} \]
\[ = \frac{R(0)(R(0) - \rho)}{2\beta n(n - 1)} \]
\[ = \frac{2\beta + \rho}{n(n - 1)}. \] (4.6)
Hence \( K_0(r) > 0 \) for all \( r \geq 0 \). We will now show that \( K_1 \) is strictly positive on \( \mathbb{R}^n \). We first claim that \( \psi_s(0) = 1 \). This result is stated without proof in [AK] and [DS2]. For the sake of completeness we will give a short simple proof here. By direct computation,
\[ \psi_s = 1 + \frac{1 - m}{2} \cdot \frac{rv_r(r)}{v(r)} \]
\[ \Rightarrow \psi_s(0) = 1 \]
and the claim follows. We next observe that by (4.2) and (4.4), \( \psi_{\tilde{s}} < 0 \). Hence
\[ \psi_{\tilde{s}}(\tilde{s}) < \psi_{\tilde{s}}(0) = 1 \quad \forall \tilde{s} > 0. \] (4.8)
By (4.7) and (v) of Lemma 3.1,
\[ \psi_{\tilde{s}}(\tilde{s}) > 0 \quad \forall \tilde{s} \geq 0. \] (4.9)
By (4.3), (4.8) and (4.9),
\[ K_1(r) > 0 \quad \forall r > 0. \]
By (4.3) and the l'Hospital rule,
\[ K_1(0) = \lim_{\tilde{s} \to 0} \frac{1 - \psi_{\tilde{s}}^2}{\psi^2} = -\lim_{\tilde{s} \to 0} \frac{\psi_{\tilde{s}}}{\psi} = K_0(0). \] (4.10)
By (4.6) and (4.10), we get (1.11) and $K_1 > 0$ for all $r \geq 0$. We will now prove (1.12). By (iii) of Theorem 1.2 and the result of [H] there exists a constant $C > 0$ such that

$$r^2 v^{1-m} \geq \begin{cases} 
C & \text{for } g \text{ is a Yamabe shrinking soliton with } \beta > \frac{\rho}{n-2} \\
C \log r & \text{for } g \text{ is a Yamabe steady soliton with } \alpha > 0 \\
C \beta^{p/\beta} & \text{for } g \text{ is a Yamabe expanding soliton with } \alpha > 0.
\end{cases} \quad (4.11)$$

Hence we always have

$$r^n v^{1-m}(r) \to \infty \quad \text{as } r \to \infty. \quad (4.12)$$

We now divide the proof of (1.12) into two cases.

**Case 1:** $\int_0^\infty z^{n-1} Q(z)e^{\frac{\beta}{r}} \int_0^{\tau(r)} v^{1-m} dt \, dz < \infty$.

Then by (4.12),

$$\lim_{r \to \infty} K_0(r) = \lim_{r \to \infty} \int_0^r z^{n-1} Q(z)e^{\frac{\beta}{r}} \int_0^{\tau(r)} v^{1-m} dt \, dz = 0. \quad (4.13)$$

**Case 2:** $\int_0^\infty z^{n-1} Q(z)e^{\frac{\beta}{r}} \int_0^{\tau(r)} v^{1-m} dt \, dz = \infty$.

Then by (4.12) and the l'Hospital rule,

$$\lim_{r \to \infty} K_0(r) = \lim_{r \to \infty} \int_0^r z^{n-1} Q(z)e^{\frac{\beta}{r}} \int_0^{\tau(r)} v^{1-m} dt \, dz = \frac{1}{2 \beta} \lim_{r \to \infty} \frac{r^{n-1} Q(r)}{E}. \quad (4.14)$$

where

$$E = (r^2 v(r)^{1-m})_r, r^{n-2} + r^2 v(r)^{1-m} \left[ (n-2)r^{n-3} + r^{n-2} \cdot \frac{\beta}{n-1} r v(r)^{1-m} \right]. \quad (4.15)$$

Since by (v) of Lemma 3.1 and Lemma 3.1 of [H], $(r^2 v^{1-m}(r))_r > 0$ for all $r > 0$, by (4.15),

$$E \geq r^{n-1} v(r)^{1-m} \left( (n-2) + \frac{\beta}{n-1} r^2 v(r)^{1-m} \right). \quad (4.16)$$

By (4.14) and (4.16),

$$0 \leq \lim_{r \to \infty} K_0(r) \leq \frac{1}{2 \beta} \lim_{r \to \infty} \frac{Q(r)}{v^{1-m}(r)(n-2 + (\beta/(n-1))r^2 v(r)^{1-m})} \frac{R(r)(R(r) - \rho)}{R(r)(R(r) - \rho)} \leq \frac{1}{2 \beta^2} \lim_{r \to \infty} \frac{R(r)(R(r) - \rho)}{r^2 v(r)^{1-m}}. \quad (4.17)$$

By Theorem 1.2 (iv) of Lemma 3.1 and (4.11), the right hand side of (4.17) tends to zero as $r \to \infty$ and (1.12) follows. We will now prove (1.13). We first observe that by (4.3),

$$0 \leq K_1(r) \leq \frac{1}{r^2 v(r)^{1-m}} \quad \forall r > 0. \quad (4.18)$$
If $g$ is a Yamabe steady or expanding soliton with $\alpha > 0$, then by (4.11) the right hand side of (4.18) tends to zero as $r \to \infty$ and hence

$$\lim_{r \to \infty} K_1(r) = 0.$$ 

Suppose now $g$ is a Yamabe shrinking soliton with $\beta > \frac{\rho}{n-2}$. Then by (3.19), (4.2) and Theorem 1.2,

$$\psi_s(s) = \frac{R - \rho}{2\beta} \to 0 \quad \text{as} \quad s = \log r \to \infty. \quad (4.19)$$

By (4.3), (4.19) and Theorem 1.2,

$$\lim_{r \to \infty} K_1(r) = \frac{\rho}{(n-1)(n-2)}$$

and the theorem follows.

□

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