Optimization over Young diagrams

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Abstract
We consider the problem of finding a Young diagram minimizing the sum of evaluations of a given pair of functions on the parts of the associated pair of conjugate partitions. While there are exponentially many diagrams, we show it is polynomial time solvable.

Keywords Young diagram · Number partition · Discrete optimization

1 Introduction

For a Young diagram \( \Lambda \), let \( n = |\Lambda| \) be the number of cells, let \( \lambda \vdash n \) be the partition of \( n \) whose \( i \)-th part \( \lambda_i \) is the number of cells in the \( i \)-th row, and let \( \lambda^* \vdash n \) be the conjugate partition of \( n \) whose \( j \)-th part \( \lambda_j^* \) is the number of cells in the \( j \)-th column, so that \( \lambda_j^* = |\{ i : \lambda_i \geq j \}| \). For function \( f : [n] \rightarrow \mathbb{Z} \) let \( f(\lambda) = \sum_{i} f(\lambda_i) \) be the sum of evaluations of \( f \) on the parts of \( \lambda \). See [1] for more information on Young diagrams and partitions and their many applications.

We consider here the following algorithmic problem.

**Optimization over Young diagrams** Given \( n \) and functions \( f, f^* : [n] \rightarrow \mathbb{Z} \), find a Young diagram \( \Lambda \) which minimizes \( f(\lambda) + f^*(\lambda^*) \). Equivalently, solve \( \min \{ f(\lambda) + f^*(\lambda^*) : \lambda \vdash n \} \).

**Example 1.1** Let \( n = 6 \) and \( f(k) = f^*(k) = k^2 \). Then, there are 11 Young diagrams \( \Lambda \) with

\[
\lambda = (6), (5, 1), (4, 2), (4, 1^3), (3^2), (3, 2, 1), (3, 1^3), (2^3), (2^2, 1^2), (2, 1^4), (1^6), \\
\lambda^* = (1^6), (2, 1^4), (2^2, 1^2), (3, 1^3), (2^3), (3, 2, 1), (4, 1^2), (3^2), (4, 2), (5, 1), (6).
\]
Computing the objective function $f(\lambda) + f^*(\lambda^*)$ exhaustively for all, we find that the unique optimal one is the self-conjugate $\lambda = \lambda^* = (3, 2, 1)$ with value $(3^2 + 2^2 + 1^2) + (3^2 + 2^2 + 1^2) = 28$.

The number of Young diagrams is exponential in $n$ and so solution by exhaustive search in general is prohibitive. Nonetheless, we show that the problem is polynomial time solvable.

**Theorem 1.2** Optimization over Young diagrams can be done in time polynomial in $n$.

**2 Proof**

We begin with a construction of Young diagrams which will be necessary for our purposes. The *type* of a Young diagram $\Lambda$ and of the associated partition $\lambda$ is the number of distinct parts of $\lambda$. It is easy to see that $\lambda$ and $\lambda^*$ have the same type which, assuming that the diagram is drawn according to the English convention, is equal to the number of “southeast corners” of $\Lambda$. If $|\Lambda| = n$ and $\Lambda$ has type $k$ so $\lambda = (r_1^{c_1}, \ldots, r_k^{c_k})$ for some $r_1 > \cdots > r_k \geq 1$ and some $c_1, \ldots, c_k \geq 1$, then $n = \sum_{i=1}^k c_i r_i \geq \sum_{i=1}^k i$ so $k < \sqrt{2n}$. For instance, $\lambda = (19, \ldots, 2, 1^{11})$ is a partition of $n = 200$ of maximum type $k = 19 < 20 = \sqrt{2n}$.

Let $1 \leq k < \sqrt{2n}$ and let $n \geq r_1 > \cdots > r_k > r_{k+1} = 0$ and $0 = c_0 < c_1 < \cdots < c_k \leq n$. These numbers define the Young diagram $\Lambda$ of type $k$ which, for $i = 1, \ldots, k$, has $c_i - c_{i-1}$ rows with $r_i$ cells and $r_i - r_{i+1}$ columns with $c_i$ cells, with partition and conjugate partition

$$\lambda = (r_1^{c_1-c_0}, \ldots, r_k^{c_k-c_{k-1}}), \quad \lambda^* = (c_k^{r_k-r_{k+1}}, \ldots, c_1^{r_1-r_2}).$$

Note that $|\Lambda| = \sum_{i=1}^k (c_i - c_{i-1})r_i = \sum_{i=1}^k (r_i - r_{i+1})c_i$ is not necessarily equal to $n$, but any diagram with $|\Lambda| = n$ does arise that way for a unique choice of type $k$ and such $r_i$ and $c_j$.

Let now $n$, $f$, $f^*$ be given. Fix any $1 \leq k < \sqrt{2n}$. We reduce the problem of finding a diagram $\Lambda$ with $|\Lambda| = n$ of type $k$ with minimum $f(\lambda) + f^*(\lambda^*)$ to finding a shortest directed path in a directed graph $D$ where each edge has a length. We construct $D$ as follows.

There are two vertices $s, t$, and vertices labeled by quadruples of integers $(i, c_i, r_{i+1}, n_i)$ for $0 \leq i \leq k$, with $1 \leq c_i, r_i, n_i \leq n$ for $1 \leq i \leq k$, $c_0 = n_0 = r_{k+1} = 0$, and $n_k = n$.

There are edges $[s, (0, 0, r_1, 0)]$ for $1 \leq r_1 \leq n$ and edges $[(k, c_k, 0, n), t]$ for $1 \leq c_k \leq n$, all of length 0, and there are edges $[(i-1, c_{i-1}, r_i, n_{i-1}), (i, c_i, r_{i+1}, n_i)]$ for $1 \leq i \leq k$, $c_i > c_{i-1}$, $r_{i+1} < r_i$, $n_i = n_{i-1} + (c_i - c_{i-1})r_i$, of length $(c_i - c_{i-1})f(r_i) + (r_i - r_{i+1})f^*(c_i)$.
Consider any directed path from $s$ to $t$ in $D$, which by the definition of $D$ looks like
\[
\begin{align*}
s &\longrightarrow (0, c_0 = 0, r_1, n_0 = 0) \longrightarrow (1, c_1, r_2, n_1) \longrightarrow \cdots \cdots \\
&\longrightarrow (k, c_k, r_{k+1} = 0, n_k = n) \longrightarrow t.
\end{align*}
\]
By definition of $D$, we have $0 = c_0 < c_1 < \cdots < c_k \leq n$ and $n \geq r_1 > \cdots > r_k > r_{k+1} = 0$, giving a Young diagram $\Lambda$ of type $k$ as explained above, with partition and conjugate partition
\[
\lambda = \left( \frac{c_1-c_0}{r_1}, \ldots, \frac{c_k-c_{k-1}}{r_k} \right), \quad \lambda^* = \left( \frac{r_k-r_{k+1}}{r_1}, \ldots, \frac{r_1-r_2}{c_1} \right).
\]
Moreover, we have $|\Lambda| = \sum_{i=1}^{k} (c_i - c_{i-1}) r_i = n_k = n$, and the length of the path is
\[
\sum_{i=1}^{k} (c_i - c_{i-1}) f(r_i) + (r_i - r_{i+1}) f^*(c_i) = f(\lambda) + f^*(\lambda^*).
\]
Conversely, it is clear that any diagram $\Lambda$ with $|\Lambda| = n$ of type $k$ gives such a directed path, of length $f(\lambda) + f^*(\lambda^*)$. So a shortest $s-t$ path gives an optimal Young diagram of type $k$.

Now, the number of vertices of $D$ is bounded by $2 + 2n + (k - 1)n^2 = O(n^{3.5})$ and hence by a polynomial in $n$. So a shortest directed path from $s$ to $t$ in $D$ can be obtained in polynomial time, see, e.g., [2], by the following simple algorithm. For $i = 1, \ldots, k$, compute for every vertex $v = (i, c_i, r_{i+1}, n_i)$ the length of a shortest $s-v$ path and an edge entering it on such a shortest path, using the values already computed for $i-1$, and then do the same for $t$.

Now, repeat the above procedure for $k = 1, \ldots, \lfloor \sqrt{2n} \rfloor$ and output the best diagram. □

As a simple example, for $n = 2$ and $k = 1$, the directed graph has exactly two $s-t$ paths,
\[
\begin{align*}
s &\overset{0}{\longrightarrow} (0, 0, 1, 0) \overset{2 f(1)+f^*(2)}{\longrightarrow} (1, 2, 0, 2) \overset{0}{\longrightarrow} t, \quad \text{of length } 2 f(1) + f^*(2), \\
s &\overset{0}{\longrightarrow} (0, 0, 2, 0) \overset{f(2)+2 f^*(1)}{\longrightarrow} (1, 1, 0, 2) \overset{0}{\longrightarrow} t, \quad \text{of length } f(2) + 2 f^*(1),
\end{align*}
\]
which correspond to the two diagrams $\Lambda$ and their partitions $\lambda = (1^2)$ and $\lambda = (2)$, respectively.

We note that the above proof in fact shows that we can solve in polynomial time the more refined problem where we search for a best diagram among those of a prescribed type $k$.

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