Global stability of an epidemic model with stage structure and nonlinear incidence rates in a heterogeneous host population

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Abstract

In this paper, we study an epidemic model with stage structure and latency spreading in a heterogeneous host population. We show that if the disease-free equilibrium exists, then the global dynamics are determined by the basic reproduction number $R_0$. We prove that the disease-free equilibrium is globally asymptotically stable when $R_0 \leq 1$, and there exists a unique endemic equilibrium which is globally asymptotically stable when $R_0 > 1$. The global stability of the endemic equilibrium is also proved by using a graph-theoretic approach to the method of Lyapunov functions. Finally, numerical simulations are given to illustrate the main theoretical results.

Keywords: heterogeneous host; epidemic model; stage structure; nonlinear incidence rate; Lyapunov function

1 Introduction

A heterogeneous host population can be divided into several homogeneous groups according to models of transmission, contact patterns, or geographic distributions. Multi-group epidemic models have been proposed in the literature of mathematical epidemiology to describe the transmission dynamics of infectious diseases in heterogeneous host populations, such as measles, mumps, gonorrhea, HIV/AIDS, West-Nile virus and vector-borne diseases such as malaria. Various forms of multi-group epidemic models have subsequently been studied to understand the mechanism of disease transmission. One of the most important subjects in this field is to obtain a threshold that determines the persistence or extinction of a disease. Guo et al. in [1] developed a graph-theoretic approach to prove the global asymptotic stability of a unique endemic equilibrium of a multi-group epidemic model. By applying the idea, global stability of endemic equilibrium for several classes of multi-group epidemic models was investigated in [1–10].

In the real world, some epidemics, such as malaria, dengue, fever, gonorrhea and bacterial infections, may have a different ability to transmit the infections in different ages. For example, measles and varicella always occur in juveniles, while it is reasonable to consider the transmission of diseases such as typhus, diphtheria in adult population. In recent years, epidemic models with stage structure have been studied in many papers [11–17]. For some disease (for example, tuberculosis, influenza, measles), on adequate contact with an...
infective, a susceptible individual becomes exposed, that is, infected but not infective. This individual remains in the exposed class for a certain period before becoming infective (see, for example, [18–22]).

In this paper, we formulate an epidemic model with latency spreading in a heterogeneous host population. Let $S_k^{(1)}$, $S_k^{(2)}$, $E_k$, $I_k$ and $R_k$ denote the immature susceptible, mature susceptible, infected but non-infectious, infectious and recovered population in the $k$th group, respectively. The disease incidence in the $k$th group can be calculated as

$$
\sum_{i=1}^{2} s_k^{(i)} \sum_{j=1}^{n} \beta_{ij}^{(i)} G_j(I_j),
$$

where the sum takes into account cross-infections from all groups and $\beta_{ij}^{(i)}$ is the transmission coefficient between compartments $S_k^{(i)}$ and $I_j$. $G_j(I_j)$ includes some special incidence functions in the literature. For instance, $G_j(I_j) = \frac{I_j}{\delta_{ij}}$ (saturation effect). Let $d_k^{(1)}$ and $d_k^{(2)}$ represent death rates of $S_k^{(1)}$ and $S_k^{(2)}$ populations, respectively. Then we obtain the following model for a disease with latency:

$$
\begin{aligned}
\dot{S}_k^{(1)} &= \varphi_k(S_k^{(1)}) + \sum_{j=1}^{n} \beta_{ij}^{(1)} S_k^{(1)} G_j(I_j) - a_k S_k^{(1)}, \\
\dot{S}_k^{(2)} &= a_k S_k^{(1)} - \sum_{j=1}^{n} \beta_{ij}^{(2)} S_k^{(2)} G_j(I_j) - d_k^{(2)} S_k^{(2)}, \\
\dot{E}_k &= \sum_{i=1}^{2} \sum_{j=1}^{n} \beta_{ij}^{(i)} S_k^{(i)} G_j(I_j) - (d_k + \eta_k) E_k, \\
\dot{I}_k &= \eta_k E_k - (d_k + \mu_k + \gamma_k) I_k, \\
\dot{R}_k &= \gamma_k I_k - d_k R_k, \quad k = 1, 2, \ldots, n,
\end{aligned}
$$

(1)

where $\varphi_k(S_k^{(1)})$ denotes the net growth rate of the immature susceptible class in the $k$th group (a typical form of $\varphi_k(S_k^{(1)})$ is $\varphi_k(S_k^{(1)}) = b_k - d_k^{(1)} S_k^{(1)}$ with $b_k$ the recruitment constant and $d_k^{(1)}$ the natural death rate). $a_k$ is the conversion rate from an immature individual to a mature individual in group $k$. $\eta_k$ represents the rate of becoming infectious after a latent period in the $k$th group. $d_k$, $\mu_k$ and $\gamma_k$ are the natural death rate, the disease-related death rate and the recovery rate in the $k$th group, respectively. All parameter values are assumed to be nonnegative and $a_k, \eta_k, d_k^{(1)}, d_k > 0$

**Remark** The model (1) can be regarded as an SVEIR model such that $S_k^{(1)}$ is unvaccinated and $S_k^{(2)}$ is vaccinated with vaccination rate $a_k$. References studied on the SVEIR model can be seen in [23, 24] and so on.

Since the variable $R_k$ does not appear in the remaining four equations of (1), if we denote $m_k := d_k + \mu_k + \gamma_k$, then we can obtain the following reduced system:

$$
\begin{aligned}
\dot{S}_k^{(1)} &= \varphi_k(S_k^{(1)}) + \sum_{j=1}^{n} \beta_{ij}^{(1)} S_k^{(1)} G_j(I_j) - a_k S_k^{(1)}, \\
\dot{S}_k^{(2)} &= a_k S_k^{(1)} - \sum_{j=1}^{n} \beta_{ij}^{(2)} S_k^{(2)} G_j(I_j) - d_k^{(2)} S_k^{(2)}, \\
\dot{E}_k &= \sum_{i=1}^{2} \sum_{j=1}^{n} \beta_{ij}^{(i)} S_k^{(i)} G_j(I_j) - (d_k + \eta_k) E_k, \\
\dot{I}_k &= \eta_k E_k - m_k I_k, \quad k = 1, 2, \ldots, n.
\end{aligned}
$$

(2)

The initial conditions for system (2) are

$$
S_k^{(1)}(0) > 0, \quad S_k^{(2)}(0) > 0, \quad E_k(0) > 0, \quad I_k(0) > 0, \quad k = 1, 2, \ldots, n.
$$

(3)
The organization of this paper is as follows. In the next section, we prove some preliminary results for system (2). In Section 3, the main theorem of this paper is stated and proved. In the last section, a brief discussion and numerical simulations which support our theoretical analysis are given.

2 Preliminaries

We assume:

(A1) \( \varphi_k \) and \( G_k \) are Lipschitz on \([0, +\infty)\);

(A2) \( \varphi_k \) is strictly decreasing on \([0, +\infty)\), and there exists \( S^{(1)}_{k0} > 0 \) such that

\[
\varphi_k \left(S^{(1)}_{k0}\right) - a_k S^{(1)}_{k0} = 0;
\]

(A3) \( \frac{G_k(x)}{x} \) is nonincreasing on \((0, +\infty)\) and

\[
\delta_k = \lim_{x \to 0} \frac{G_k(x)}{x} > 0 \text{ exists, } k = 1, 2, \ldots, n.
\]

From our assumptions, it is clear that system (2) has a unique solution for any given initial conditions (3) and the solution remains nonnegative. If (A2) holds, then we see that system (2) has a disease-free equilibrium

\[
P_0 = \left(S^{(1)}_{10}, S^{(2)}_{10}, \ldots, S^{(1)}_{n0}, S^{(2)}_{n0}, 0, 0, \ldots, 0\right),
\]

where

\[
\varphi_k \left(S^{(1)}_{k0}\right) = d^{(2)}_k S^{(2)}_{k0}, \quad a_k S^{(1)}_{k0} = d^{(2)}_k S^{(2)}_{k0}, \quad k = 1, 2, \ldots, n. \tag{4}
\]

For two nonnegative \( n \)-square matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \), we write \( A \leq B \) if \( a_{ij} \leq b_{ij} \) for all \( i \) and \( j \), and \( A < B \) if \( A \leq B \) and \( A \neq B \). Following [25], we set matrices

\[
F := \left( \sum_{i=1}^{2} \beta^{(i)}_{kj} S^{(i)}_{k0} \delta_j \right)_{n \times n}, \quad V := \text{diag} \left( \frac{m_1(d_1 + \eta_1)}{\eta_1}, \frac{m_2(d_2 + \eta_2)}{\eta_2}, \ldots, \frac{m_n(d_n + \eta_n)}{\eta_n} \right).
\]

The next generation matrix for system (2) is

\[
Q := FV^{-1} = \left( \frac{n_k \sum_{i=1}^{2} \beta^{(i)}_{kj} S^{(i)}_{k0} \delta_j}{m_k(d_k + \eta_k)} \right)_{n \times n}
\]

\[
= \begin{bmatrix}
\frac{n_1 \sum_{i=1}^{2} \beta^{(i)}_{11} S^{(i)}_{10} \delta_1}{m_1(d_1 + \eta_1)} & \cdots & \frac{n_1 \sum_{i=1}^{2} \beta^{(i)}_{1n} S^{(i)}_{10} \delta_n}{m_1(d_1 + \eta_1)} \\
\vdots & \ddots & \vdots \\
\frac{n_n \sum_{i=1}^{2} \beta^{(i)}_{n1} S^{(i)}_{n0} \delta_1}{m_n(d_n + \eta_1)} & \cdots & \frac{n_n \sum_{i=1}^{2} \beta^{(i)}_{nn} S^{(i)}_{n0} \delta_n}{m_n(d_n + \eta_n)}
\end{bmatrix}.
\]

Thus, we obtain the basic reproduction number \( R_0 \) for system (2) as

\[
R_0 = \rho(Q),
\]

where \( \rho \) denotes the spectral radius.
Let $N_k = S^{(1)}_k + S^{(2)}_k + E_k + I_k$, $d_k = \min\{d^{(3)}_k, d^{(2)}_k, d_k, m_k\}$, $k = 1, 2, \ldots, n$. Then from (2) we have
\[
\dot{N}_k \leq \varphi_k(S^{(1)}_k) + d^{(1)}_k S^{(1)}_k - d_k N_k. \tag{5}
\]

We derive from (5) that the region
\[
\Gamma = \left\{ (S^{(1)}_1, S^{(2)}_1, \ldots, S^{(1)}_n, S^{(2)}_n, E_1, \ldots, E_n, I_1, \ldots, I_n) \in \mathbb{R}^{4n} : S^{(1)}_k \leq S^{(1)}_{k0}, S^{(2)}_k \leq S^{(2)}_{k0}, \varphi_k(S^{(1)}_k) + S^{(2)}_k + E_k + I_k \leq \frac{\varphi_k(0) + d^{(1)}_k S^{(1)}_{k0}}{d_k}, k = 1, 2, \ldots, n \right\}
\]
is positively invariant with respect to (2). Let $\Gamma^0$ denote the interior of $\Gamma$.

### 3 Main results

In the section, we study the global stability of equilibria of system (2).

**Theorem 3.1** Assume that (A1)-(A3) hold and $B = (\sum_{i=1}^{2} \beta^{(i)}_{ij})$ is irreducible.

1. If $R_0 \leq 1$, then $P_0$ is globally asymptotically stable in $\Gamma$;
2. If $R_0 > 1$, then $P_0$ is unstable and system (2) admits at least one endemic equilibrium in $\Gamma^0$.

**Proof** Let
\[
S = (S^{(1)}_1, S^{(2)}_1, \ldots, S^{(1)}_n, S^{(2)}_n), \quad S^0 = (S^{(1)}_{10}, S^{(2)}_{10}, \ldots, S^{(1)}_{n0}, S^{(2)}_{n0}),
\]
\[
I = (I_1, I_2, \ldots, I_n), \quad Q(S, I) = \left( \frac{\sum_{i=1}^{2} \sum_{j=1}^{n} \eta_k \beta^{(i)}_{ij} S^{(i)}_k G_j(I_j)}{m_k(d_k + \eta_k) I_j} \right)_{n \times n}.
\]

Notice that $B$ is irreducible, then $Q(S, I)$ and $Q$ are irreducible. By (A3), we have $0 \leq Q(S, I) \leq Q$. Hence $Q(S, I) + Q$ is also irreducible. That is, $0 \leq Q(S, I) < Q$ and $Q(S, I) + Q$ is irreducible provided that $S \neq S^0$. Thus, by [26], Corollary 1.5, p.27, $\rho(Q(S, I)) < \rho(Q)$ if $S \neq S^0$. Since $Q$ is irreducible, there exist $\omega_k > 0$, $k = 1, 2, \ldots, n$, such that
\[
(\omega_1, \omega_2, \ldots, \omega_n) \rho(Q) = (\omega_1, \omega_2, \ldots, \omega_n) Q.
\]

Consider a Lyapunov functional
\[
L = \sum_{k=1}^{n} \frac{\omega_k \eta_k}{m_k(d_k + \eta_k)} \left[ E_k + \frac{d_k + \eta_k}{\eta_k} I_k \right].
\]

Differentiating $L$ along the solution of system (2), we obtain
\[
\dot{L} = \sum_{k=1}^{n} \frac{\omega_k \eta_k}{m_k(d_k + \eta_k)} \left[ \sum_{i=1}^{2} \sum_{j=1}^{n} \eta_k \beta^{(i)}_{ij} G_j(I_j) - \frac{m_k(d_k + \eta_k)}{\eta_k} I_k \right] = \sum_{k=1}^{n} \omega_k \left[ \sum_{i=1}^{2} \sum_{j=1}^{n} \eta_k \beta^{(i)}_{ij} G_j(I_j) - \frac{m_k(d_k + \eta_k)}{\eta_k} I_k \frac{\sum_{i=1}^{2} \sum_{j=1}^{n} \eta_k \beta^{(i)}_{ij} G_j(I_j)}{m_k(d_k + \eta_k)} - I_k \right]
\]
\[ (\omega_1, \omega_2, \ldots, \omega_n)(Q(S, I)I^T - I^T) \leq (\omega_1, \omega_2, \ldots, \omega_n)(QI^T - I^T) = [\rho(Q) - 1](\omega_1, \omega_2, \ldots, \omega_n)I^T \leq 0. \]

If \( R_0 < 1 \), then \( \dot{L} = 0 \) if and only if \( I^T = 0 \). If \( R_0 = 1 \), then \( \dot{L} = 0 \) implies
\[ (\omega_1, \omega_2, \ldots, \omega_n)(Q(S, I)I^T - I^T) = 0. \]

Therefore, \( \dot{L} = 0 \) if and only if \( I = 0 \), or \( S = S^0 \). On the other hand, from the last equation in system (2), we see that \( I = 0 \) implies that \( E_k = 0 \) for \( k = 1, 2, \ldots, n \). Hence, the largest invariant subset of the set, where \( \dot{L} = 0 \), is the singleton \( \{ P_0 \} \). By LaSalle’s invariance principle, \( P_0 \) is globally asymptotically stable for \( R_0 \leq 1 \).

If \( R_0 > 1 \) and \( I \neq 0 \), then
\[ [\rho(Q) - 1](\omega_1, \omega_2, \ldots, \omega_n)I^T > 0. \]

Thus, by continuity, we have \( \dot{L} = (\omega_1, \omega_2, \ldots, \omega_n)(Q(S, I)I^T - I^T) > 0 \) in a neighborhood of \( P_0 \) in \( \Gamma^0 \). This implies that \( P_0 \) is unstable. From a uniform persistence result of [27] and a similar argument as in the proof of Proposition 3.3 of [28], we can deduce that the instability of \( P_0 \) implies the uniform persistence of system (2) in \( \Gamma^0 \). This together with the uniform boundedness of solutions of system (2) in \( \Gamma^0 \) implies that system (2) has an endemic equilibrium in \( \Gamma^0 \) (see Theorem 2.8.6 of [29] or Theorem D.3 of [30]). The proof is completed. \( \square \)

By Theorem 3.1, we have that if \( B = (\sum_{j=1}^{2} \beta_{kj}^{(i)}) \) is irreducible, (A1)-(A3) hold and \( R_0 > 1 \), then system (2) has an endemic equilibrium \( P^* \) in \( \Gamma^0 \). Let
\[ P^* = (S_1^{(1)*}, S_2^{(2)*}, \ldots, S_n^{(1)*}, S_n^{(2)*}, I_1^*, I_2^*, \ldots, I_n^*), \]

then the components of \( P^* \) satisfy
\[ \phi_k(S_k^{(1)*}) = \sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j(I_j^*) + d_k^{(2)} S_k^{(2)*}, \quad (6) \]
\[ a_k(S_k^{(1)*}) = S_k^{(2)*} \sum_{j=1}^{n} \beta_{kj}^{(2)} G_j(I_j^*) + d_k^{(2)} S_k^{(2)*}, \quad (7) \]
\[ \sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j(I_j^*) = (d_k + \eta_k)E_k^* = \frac{m_k(d_k + \eta_k)}{\eta_k} I_k^*, \quad k = 1, 2, \ldots, n. \quad (8) \]

Since \( \phi_k \) is strictly decreasing on \([0, +\infty)\), we have
\[ \left[ \phi_k(S_k^{(1)*}) - \phi_k(S_k^{(1)*}) \right] \left( 1 - \frac{S_k^{(1)*}}{S_k^{(1)*}} \right) \leq 0, \quad (9) \]

where equality holds if and only if \( S_k^{(1)*} = S_k^{(1)*}, k = 1, 2, \ldots, n \).
We further make the following assumption:

(A4) \( G_k \) is strictly increasing on \([0, +\infty)\), and

\[
\frac{G_k(x_k)I_k}{G_k(x_k)K_k} + \frac{G_k(I_k)}{G_k(x_k)} - \frac{I_k}{x_k} - 1 \leq 0, \quad k = 1, 2, \ldots, n,
\]

where \( x_k > 0 \) is chosen in an arbitrary way and equality holds if \( I_k = x_k \).

**Theorem 3.2** Assume that \( B = \{\sum_{i=1}^{2} \beta_{ij}^{(0)} \} \) is irreducible. If \( R_0 > 1 \), then \( P^* \) is globally asymptotically stable.

**Proof** Set \( \beta_{kj}^{(0)} = \sum_{i=1}^{2} \beta_{kj}^{(i)} \), \( k, j = 1, 2, \ldots, n \), and

\[
\beta = \begin{bmatrix}
\sum_{i=1}^{2} \beta_{1i}^{(0)} & -\beta_{21} & \cdots & -\beta_{m1} \\
-\beta_{12} & \sum_{i=1}^{2} \beta_{2i}^{(0)} & \cdots & -\beta_{m2} \\
\vdots & \vdots & \ddots & \vdots \\
-\beta_{1n} & -\beta_{2n} & \cdots & \sum_{i=1}^{2} \beta_{ni}^{(0)}
\end{bmatrix}.
\]

Then \( \beta \) is also irreducible. It follows from Lemma 2.1 of [1] that the solution space of the linear system

\[
\beta v = 0
\]

has dimension 1 with a basis

\[
v := (v_1, v_2, \ldots, v_n)^T = (\xi_1, \xi_2, \ldots, \xi_n)^T,
\]

where \( \xi_k \) denotes the cofactor of the \( k \)th diagonal entry of \( \beta \). Note that from (11) we have

\[
\sum_{j=1}^{n} \beta_{kj} v_k = \sum_{j=1}^{n} \beta_{kj} v_j, \quad k = 1, 2, \ldots, n.
\]

From (13), we have

\[
\sum_{k=1}^{n} v_k \sum_{j=1}^{n} \sum_{i=1}^{2} \beta_{kj}^{(i)} S_k^{(i)} G_j(I_k^{*})
\]

\[
= \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{2} \beta_{kj}^{(i)} S_k^{(i)} v_j G_k(I_k) = \sum_{k=1}^{n} \left[ \sum_{j=1}^{n} \sum_{i=1}^{2} \beta_{kj}^{(i)} S_k^{(i)} G_k(I_k^{*}) v_j \right] G_k(I_k) G_k(I_k^{*})
\]

\[
= \sum_{k=1}^{n} \left[ \sum_{j=1}^{n} (\beta_{kj} v_j) \right] G_k(I_k) G_k(I_k^{*}) = \sum_{k=1}^{n} \left[ \sum_{j=1}^{n} (\beta_{kj} v_k) \right] G_k(I_k) G_k(I_k^{*})
\]

\[
= \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \sum_{i=1}^{2} \beta_{kj}^{(i)} S_k^{(i)} G_j(I_k^{*}) G_k(I_k) G_k(I_k^{*})
\]

(14)
Consider a Lyapunov functional

\[
V = \sum_{k=1}^{n} v_k \left( \int_{I_k} \frac{E_k + \eta_k}{\eta_k} \left[ \frac{G_k(x) - G_k(I_k^*)}{G_k(x)} \right] dx \right).
\]

Differentiating \( V \) along the solution of system (2), we obtain

\[
\dot{V} = \sum_{k=1}^{n} v_k \left\{ \varphi_k(S_k^{(1)}) - d_k^{(2)} S_k^{(2)} - \frac{m_k(d_k + \eta_k)}{\eta_k} I_k \right. \\
- \frac{S_k^{(2)+}}{S_k^{(2)}} \left[ \varphi_k(S_k^{(1)}) - S_k^{(1)} \sum_{j=1}^{n} \beta_{kj} G_j(I_j) - a_k S_k^{(1)} \right] \\
- \frac{S_k^{(2)+}}{S_k^{(2)}} \left[ a_k S_k^{(1)} - S_k^{(2)} \sum_{j=1}^{2} \beta_{kj} G_j(I_j) - d_k^{(2)} S_k^{(2)} \right] \\
- \frac{E_k^*}{E_k} \left[ \sum_{j=1}^{2} \sum_{j=1}^{n} \beta_{kj} G_j(I_j) - (d_k + \eta_k) E_k \right] \\
+ \frac{G_k(I_k^*)}{G_k(I_k)} \left[ (d_k + \eta_k) E_k - \frac{m_k(d_k + \eta_k)}{\eta_k} I_k \right] \right\} \\
= \sum_{k=1}^{n} v_k \left\{ \varphi_k(S_k^{(1)}) \left( 1 - \frac{S_k^{(1)+}}{S_k^{(1)}} \right) + d_k^{(2)} S_k^{(2)} \left( 1 - \frac{S_k^{(2)+}}{S_k^{(2)+}} \right) \\
+ \frac{a_k S_k^{(1)+}}{\beta_{kj}} G_j(I_j) \sum_{j=1}^{n} \frac{S_k^{(1)+}}{S_k^{(1)+}} + \frac{S_k^{(2)+}}{S_k^{(2)+}} \right\} \\
- \frac{E_k^*}{E_k} \sum_{i=1}^{2} \sum_{j=1}^{n} \beta_{kj} G_j(I_j) \left[ (d_k + \eta_k) E_k \right] \left( 1 - \frac{E_k G_k(I_k^*)}{E_k^* G_k(I_k)} \right) \\
+ \frac{m_k(d_k + \eta_k)}{\eta_k} I_k G_k(I_k^*) - \frac{m_k(d_k + \eta_k)}{\eta_k} I_k \frac{I_k}{I_k^*}. \]

From (7) and (8), we have

\[
\dot{V} = \sum_{k=1}^{n} \left\{ \varphi_k(S_k^{(1)}) \left( 1 - \frac{S_k^{(1)+}}{S_k^{(1)}} \right) + d_k^{(2)} S_k^{(2)} \left( 1 - \frac{S_k^{(2)+}}{S_k^{(2)+}} \right) \\
- \frac{E_k^*}{E_k} \sum_{i=1}^{2} \sum_{j=1}^{n} \beta_{kj} G_j(I_j) \left[ 1 - \frac{S_k^{(1)+}}{S_k^{(1)+}} \frac{S_k^{(2)+}}{S_k^{(2)+}} \right] \\
+ \sum_{i=1}^{2} \sum_{j=1}^{n} \beta_{kj} G_j(I_j) \left[ 1 - \frac{E_k G_k(I_k^*)}{E_k^* G_k(I_k)} \right] \right\}.
\]
By (10) and (14), we obtain

\[
\dot{V} \leq \sum_{k=1}^{n} v_k \left\{ \varphi_k \left( S_k^{(1)} \right) \left( 1 - \frac{S_k^{(1)+}}{S_k^{(1)}} \right) + d_k^{(2)} \left( 1 - \frac{S_k^{(2)}}{S_k^{(2)\ast}} \right) \right. \\
- E_k^2 \sum_{i=1}^{2} S_k^{(i)} \sum_{j=1}^{n} \beta_{kj}^{(i)\ast} G_i(I_j) + \left[ \frac{2^{(2)\ast}}{S_k^{(2)\ast}} \sum_{j=1}^{n} \beta_{kj}^{(2)\ast} G_i(I_j^\ast) + d_k^{(2)} \frac{S_k^{(2)}}{S_k^{(2)\ast}} \right] \\
\times \left( 1 - \frac{S_k^{(1)+}}{S_k^{(1)}} \right) + \sum_{i=1}^{2} S_k^{(i)} \sum_{j=1}^{n} \beta_{kj}^{(i)\ast} G_i(I_j^\ast) \left[ 2 - \frac{E_k G_k(I_k^\ast)}{E_k G_k(I_k)} \right] \right\} =: B_1.
\] 

(15)

From (6), we know that

\[
\varphi_k \left( S_k^{(1)+} \right) \left( 1 - \frac{S_k^{(1)+}}{S_k^{(1)}} \right) = \left( \sum_{i=1}^{2} S_k^{(i)} \sum_{j=1}^{n} \beta_{kj}^{(i)\ast} G_i(I_j^\ast) + d_k^{(2)} \frac{S_k^{(2)}}{S_k^{(2)\ast}} \right) \left( 1 - \frac{S_k^{(1)+}}{S_k^{(1)}} \right).
\] 

(16)

By (16), we can rewrite \( B_1 \) as

\[
\dot{V} = \sum_{k=1}^{n} v_k \left\{ \left[ \varphi_k \left( S_k^{(1)} \right) - \varphi_k \left( S_k^{(1)+} \right) \right] \left( 1 - \frac{S_k^{(1)+}}{S_k^{(1)}} \right) \\
+ d_k^{(2)} \frac{S_k^{(2)}}{S_k^{(2)\ast}} \left( 3 - \frac{S_k^{(1)} + S_k^{(2)}}{S_k^{(1)+} + S_k^{(2)\ast}} \right) \right. \\
- E_k^2 \sum_{i=1}^{2} S_k^{(i)} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_i(I_j) + \left[ \frac{2^{(2)\ast}}{S_k^{(2)\ast}} \sum_{j=1}^{n} \beta_{kj}^{(2)\ast} G_i(I_j^\ast) \right] \\
\times \left( 2 - \frac{S_k^{(1)+}}{S_k^{(1)}} \right) + \sum_{i=1}^{2} S_k^{(i)} \sum_{j=1}^{n} \beta_{kj}^{(i)\ast} G_i(I_j^\ast) \left[ 2 - \frac{E_k G_k(I_k^\ast)}{E_k G_k(I_k)} \right] \left\} \right. \\
= B_2.
\] 

(17)

By (9) and the arithmetic-geometric mean, we easily see that

\[
B_1 \leq \sum_{k=1}^{n} v_k \left\{ - \frac{E_k}{E_k} \sum_{i=1}^{2} S_k^{(i)} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_i(I_j) \\
+ S_k^{(2)\ast} \sum_{j=1}^{n} \beta_{kj}^{(2)\ast} G_i(I_j^\ast) \left( 2 - \frac{S_k^{(1)+}}{S_k^{(1)}} \right) \right. \\
+ S_k^{(1)+} \sum_{j=1}^{n} \beta_{kj}^{(1)\ast} G_i(I_j^\ast) \left( 1 - \frac{S_k^{(1)+}}{S_k^{(1)}} \right) \\
\times \left( 2 - \frac{E_k G_k(I_k^\ast)}{E_k G_k(I_k)} \right) \left\} =: B_2. \right.
\]
We can rewrite $B_2$ as

$$B_2 = \sum_{k=1}^{n} v_k \left\{ 3 \sum_{j=1}^{n} \beta_{kj}^{(2)} G_j(I_j^*) \left[ 3 \frac{S_k^{(1)+} G_j(I_j^*)^2}{S_k^{(1)+} S_k^{(2)+}} - \frac{S_k^{(2)+} E_k G_j(I_j^*)}{S_k^{(2)+} E_k G_j(I_j^*)} \right] + S_k^{(1)+} \sum_{j=1}^{n} \beta_{kj}^{(1)} G_j(I_j^*) \left[ 2 - \frac{S_k^{(1)+}}{S_k^{(1)+}} \right] - \frac{S_k^{(1)+} E_k G_j(I_j^*)}{S_k^{(2)+} E_k G_j(I_j^*)} \right\} + \sum_{i=1}^{2} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j(I_j^*) \left[ 1 - \frac{E_k G_k(I_k^*)}{E_k G_k(I_k^*)} \right] \right\}. $$

By the arithmetic-geometric mean, we have that

$$B_2 \leq \sum_{k=1}^{n} v_k \left\{ 3 \sum_{j=1}^{n} \beta_{kj}^{(2)} G_j(I_j^*) \left[ 1 - \left( \frac{E_k G_j(I_j^*)}{E_k G_j(I_j^*)} \right)^{\frac{1}{2}} \right] + 2 S_k^{(1)+} \sum_{j=1}^{n} \beta_{kj}^{(1)} G_j(I_j^*) \left[ 1 - \left( \frac{E_k G_j(I_j^*)}{E_k G_j(I_j^*)} \right)^{\frac{1}{2}} \right] + \sum_{i=1}^{2} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j(I_j^*) \left[ 1 - \frac{E_k G_k(I_k^*)}{E_k G_k(I_k^*)} \right] \right\} =: B_3. $$

We can rewrite $B_3$ as

$$B_3 = \sum_{k=1}^{n} v_k \left\{ 3 \sum_{j=1}^{n} \beta_{kj}^{(2)} G_j(I_j^*) \left[ 1 - \left( \frac{E_k G_j(I_j^*)}{E_k G_j(I_j^*)} \right)^{\frac{1}{2}} + \ln \left( \frac{E_k G_j(I_j^*)}{E_k G_j(I_j^*)} \right) \right] + 2 S_k^{(1)+} \sum_{j=1}^{n} \beta_{kj}^{(1)} G_j(I_j^*) \left[ 1 - \left( \frac{E_k G_j(I_j^*)}{E_k G_j(I_j^*)} \right)^{\frac{1}{2}} + \ln \left( \frac{E_k G_j(I_j^*)}{E_k G_j(I_j^*)} \right) \right] \right\} + \sum_{i=1}^{2} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j(I_j^*) \left[ 1 - \frac{E_k G_k(I_k^*)}{E_k G_k(I_k^*)} + \ln \frac{E_k G_k(I_k^*)}{E_k G_k(I_k^*)} \right] \right\} + \sum_{i=1}^{2} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j(I_j^*) \ln \frac{E_k G_k(I_k^*)}{E_k G_k(I_k^*)} \right\}.$$

Using the fact that $1 - x + \ln x \leq 0$, where equality holds if and only if $x = 1$, we obtain

$$B_3 \leq \sum_{k=1}^{n} v_k \sum_{i=1}^{2} S_k^{(i)+} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j(I_j^*) \left[ - \ln \frac{E_k G_j(I_j^*)}{E_k G_j(I_j^*)} - \ln \frac{E_k G_k(I_k^*)}{E_k G_k(I_k^*)} \right] = \sum_{k=1}^{n} v_k \sum_{i=1}^{2} S_k^{(i)+} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j(I_j^*) \ln \frac{G_k(I_k^*) G_j(I_j^*)}{G_k(I_k^*) G_j(I_j^*)} = \sum_{k=1}^{n} \beta_{kj} \ln \frac{G_k(I_k^*) G_j(I_j^*)}{G_k(I_k^*) G_j(I_j^*)}. $$

(19)
In the following, we will show that

\[ H_n := \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} \ln \frac{G_k(I_k)G_j(I_j^*)}{G_k(I_k^*)G_j(I_j)} \equiv 0. \] (20)

We first give the proof of (20) for \( n = 2 \), which would give a reader the basic yet clear ideas without being hidden by the complexity of terms caused by larger values of \( n \). When \( n = 2 \), we have

\[ H_2 = \sum_{k=1}^{2} v_k \sum_{j=1}^{2} \beta_{kj} \ln \frac{G_k(I_k)G_j(I_j^*)}{G_k(I_k^*)G_j(I_j)}. \]

Formula (12) gives \( v_1 = \beta_{21} \) and \( v_2 = \beta_{12} \) in this case. Expanding \( H_2 \) yields

\[
H_2 = \beta_{21} \beta_{12} \ln \frac{G_1(I_1)G_2(I_2^*)}{G_2(I_2)G_1(I_1)} + \beta_{12} \beta_{21} \ln \frac{G_2(I_2)G_1(I_1^*)}{G_1(I_1)G_2(I_2)}
\]

\[ + \beta_{21} \beta_{12} \ln \frac{G_1(I_1)G_2(I_2^*)}{G_2(I_2)G_1(I_1)} + \beta_{12} \beta_{21} \ln \frac{G_2(I_2)G_1(I_1^*)}{G_1(I_1)G_2(I_2)}
\]

\[ = \beta_{12} \beta_{21} \left[ \ln \frac{G_1(I_1)G_2(I_2^*)}{G_2(I_2)G_1(I_1)} + \ln \frac{G_2(I_2)G_1(I_1^*)}{G_1(I_1)G_2(I_2)} \right] = 0.\]

For more general \( n \), by a similar argument as in the proof of \( \sum_{k,j=1}^{n} v_k \beta_{kj} \ln \frac{E_k E_j}{E_k E_j} \equiv 0 \) in [7], we obtain that

\[ \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} \ln \frac{G_k(I_k)G_j(I_j^*)}{G_k(I_k^*)G_j(I_j)} = \sum_{k,j=1}^{n} v_k \beta_{kj} \ln \frac{G_k(I_k^*)G_j(I_j)}{G_k(I_k)G_j(I_j^*)} \equiv 0. \]

From (17)-(19), we see that if \( \dot{V} = 0 \), then

\[ S_k^{(i)} = S_k^{(i)*}, \quad i = 1, 2, k = 1, 2, \ldots, n. \] (21)

If (21) holds, it follows from (2) that

\[
\begin{cases}
0 = \varphi_k(S_k^{(1)*}) - \sum_{i,j=1}^{n} \beta_{kj} S_j^{(1)*} S_k G_j(l_j) - a_k S_k^{(1)*}, \\
0 = a_k S_k^{(2)*} - \sum_{i,j=1}^{n} \beta_{kj} S_j^{(2)*} S_k G_j(l_j) - d_k S_k^{(2)*}.
\end{cases}
\]

Then we obtain that

\[ \dot{E}_k = (\varphi_k(S_k^{(1)*}) - a_k S_k^{(1)*}) + (a_k S_k^{(1)*} - d_k S_k^{(2)*}) - (d_k + \eta_k) E_k. \]

This implies that

\[
\lim_{t \to +\infty} E_k = \frac{(\varphi_k(S_k^{(1)*}) - a_k S_k^{(1)*}) + (a_k S_k^{(1)*} - d_k S_k^{(2)*})}{(d_k + \eta_k)} = E_k^*.
\] (22)

By (22) and the fourth equation of system (2), we have

\[
\lim_{t \to +\infty} I_k = \frac{\eta_k E_k^*}{m_k} = I_k^*.
\] (23)
From (21)-(23) and the characteristics of $V$, we obtain that the largest invariant subset of the set, where $\dot{V} = 0$, is the singleton \( \{ P^* \} \). By LaSalle’s invariance principle, $P^*$ is globally asymptotically stable for $R_0 > 1$. \hfill \square

4 Numerical examples

For certain sexually transmitted diseases, AIDS/HIV for example, it is natural to consider two groups of people: a group of males and a group of females. Further, it is always assumed that there are two important age stages for the susceptible, a group of immature susceptible $S^{(1)}_k$ who are less than 18 years old, and a group of mature susceptible $S^{(2)}_k$ who are more than 18 years old. Thus, we consider the following model:

\[
\begin{align*}
\dot{S}^{(1)}_k &= \varphi_k(S^{(1)}_k) - \sum_{j=1}^{2} \beta_{kj}^{(1)} S^{(1)}_k G_j(I_j) - a_k S^{(1)}_k, \\
\dot{S}^{(2)}_k &= a_k S^{(1)}_k - \sum_{j=1}^{2} \beta_{kj}^{(2)} S^{(2)}_k G_j(I_j) - d^{(2)}_k S^{(2)}_k, \\
\dot{E}_k &= \sum_{i=1}^{2} \sum_{j=1}^{2} \beta_{kj}^{(i)} S^{(i)}_k G_j(I_j) - (d_k + \eta_k) E_k, \\
\dot{I}_k &= \eta_k E_k - m_k I_k, \quad k = 1, 2,
\end{align*}
\]

where $\varphi_k(S^{(1)}_k) = b_k - d^{(1)}_k S^{(1)}_k$, $G_j(I_j) = \frac{I_j}{1 + \alpha_j I_j}$.

Clearly, (A1)-(A4) hold. We fix the parameters as follows:

\[
\begin{align*}
b_1 &= 50, \quad b_2 = 30, \quad d_1^{(1)} = 0.001, \quad d_1^{(2)} = 0.2, \quad d_2^{(1)} = 0.002, \\
d_2^{(2)} &= 0.3, \quad d_1 = 0.1, \quad d_2 = 0.2, \quad \eta_1 = 0.1, \quad \eta_2 = 0.2, \\
m_1 &= 0.5, \quad m_2 = 0.6, \quad a_1 = 0.6, \quad a_2 = 0.5, \quad \alpha_1 = \alpha_2 = 0.1.
\end{align*}
\]

Then we have $P_0 \approx (83.1947, 249.5840, 59.7610, 99.6016, 0, 0, 0, 0, 0)$.
Figure 2 Dynamic behavior of system (24) with parameter values in (25) and Case 2. $R_0 \approx 1.0941$. The initial conditions are: $S^{(1)}_1(0) = 70, S^{(2)}_1(0) = 200, S^{(1)}_2(0) = 80, S^{(2)}_2(0) = 240, E_1(0) = 1, E_2(0) = 9, I_1(0) = 3, I_2(0) = 6$.

**Case 1.** If $\beta^{(1)}_{1j} = \beta^{(2)}_{1k} = 0.002, \beta^{(1)}_{2j} = \beta^{(2)}_{2k} = 0.002, k = 1, 2, j = 1, 2$, then we obtain

$$Q \approx \begin{pmatrix} 0.6656 & 0.5546 \\ 0.3187 & 0.2656 \end{pmatrix}, \quad R_0 \approx 0.9312.$$ 

By Theorem 3.1, the disease dies out in both groups. Numerical simulation illustrates this fact (see Figure 1).

**Case 2.** If $\beta^{(1)}_{1j} = \beta^{(2)}_{1k} = 0.0025, \beta^{(1)}_{2j} = \beta^{(2)}_{2k} = 0.002, k = 1, 2, j = 1, 2$, then we have $P^* \approx (82.7845, 244.9127, 59.4787, 98.2113, 4.6734, 1.0441, 0.9347, 0.3480)$ and $R_0 \approx 1.0941$.

By Theorem 3.2, the disease persists in both groups. Numerical simulation illustrates this fact (see Figure 2).

**Competing interests**
The authors declare that they have no competing interests.

**Authors’ contributions**
All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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