On the Analysis of a Generalised Rough Ait-Sahalia Interest Rate Model

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Abstract

Fractional Brownian motion with the Hurst parameter $H < \frac{1}{2}$ is used widely, for instance, to describe a 'rough' stochastic volatility process in finance. In this paper, we examine an Ait-Sahalia-type interest rate model driven by a fractional Brownian motion with $H < \frac{1}{2}$ and establish theoretical properties such as an existence-and-uniqueness theorem, regularity in the sense of Malliavin differentiability and higher moments of the strong solutions.

Keywords: Stochastic interest rate, rough volatility, fractional Brownian motion, strong solution, higher moments

1 Introduction

Over the years, SDEs driven by noise with $\alpha$-Hölder continuous random paths for $\alpha \in \left[ \frac{1}{2}, 1 \right)$ have been applied to model the dynamical behaviour of asset prices in finance. See e.g. [1, 2] and the references therein. However, in recent years, empirical evidence (see e.g. [3]) has shown that volatility paths of asset prices are more irregular in the sense of $\alpha$-Hölder continuity for $\alpha \in (0, \frac{1}{2})$ in many instances. This inadequacy actually showed the need for models based on SDEs driven by a noise of low $\alpha$-Hölder regularity with $\alpha \in (0, \frac{1}{2})$ which has been used by researchers and practitioners to describe the volatility dynamics of asset prices. These models are driven by rough signals that can capture well the 'roughness' in the volatility process of asset prices. Such rough signals arise e.g. from paths of the fractional Brownian motion (fBm). The fractional Brownian motion is a generalisation of the
ordinary Brownian motion. It is a centred self-similar Gaussian process with stationary increments which depends on the Hurst parameter $H$. The Hurst parameter lies in $(0, 1)$ and controls the regularity of the sample paths in the sense of a.e. (local) $H^{-}$-Hölder continuity. The smaller the Hurst parameter, the rougher the sample paths and vice versa. For instance, the authors in [5] employ the fractional Brownian motion with $H < \frac{1}{2}$ to model the ‘rough’ volatility process of asset prices and derive a representation of the sensitivity parameter delta for option prices. Similarly, the authors in [6] also consider an asset price model in connection with the sensitivity analysis of option prices whose correlated ‘rough’ volatility dynamics is described by means of a SDE driven by a fractional Brownian motion with $H < \frac{1}{2}$. The reader may consult [4, 15] for the coverage of properties and financial applications of the fractional Brownian motion with $H < \frac{1}{2}$. See also the Appendix.

In the context of interest rate modelling, Ait-Sahalia proposed a new class of highly nonlinear stochastic models in [8] for the evolution of interest rates through time after rejecting existing univariate linear-drift stochastic models based on empirical studies. In this model, (short term) interest rates $x_t$ have the SDE dynamics

$$dx(t) = (\alpha_1 x(t)^{\rho} - \alpha_0 + \alpha_1 x(t) - \alpha_2 x(t)^2)dt + \sigma x(t)^\theta dB_t$$

for $t \geq 0$ with initial value $x(0) = x_0$, where $\alpha_1, \alpha_0, \alpha_2 > 0$, $\sigma > 0$, $\theta > 1$ and $B_t$ is a scalar Brownian motion. This type of interest rate model has been studied by many authors (see e.g. [9, 11]).

In order to capture ”rough” (short term) interest rates, e.g. in turbulent bond markets, one may replace the driving noise $B_t$ in (1) by a fractional Brownian motion $B_t^H$ and consider an interest rate model based on the SDE

$$dx(t) = (\alpha_1 x(t)^{\rho} - \alpha_0 + \alpha_1 x(t) - \alpha_2 x(t)^2)dt + \sigma x(t)^\theta dB_t^H$$

for $t \geq 0$ with initial value $x(0) = x_0$, $t \in (0, 1]$, $H \in (0, \frac{1}{2})$ and $\rho > 1$. The stochastic integral for the fractional Brownian motion in (2) is defined via an integral concept in [4] and related to a Wick-Itô-Skorohod type of integral. See also Section 5.

On the other hand, an alternative model to (2) for the description of rough interest rate dynamics could be the following SDE, which ”preserves” the classical drift structure of the Ait-Sahalia model in (1):

$$dx(t) = (\alpha_1 x(t)^{\rho} - \alpha_0 + \alpha_1 x(t) - \alpha_2 x(t)^2)dt + \sigma x(t)^\theta dB_t^H$$

for $t \geq 0$ with initial value $x(0) = x_0$, $t \in (0, 1]$, $H \in (0, \frac{1}{2})$ and $\rho > 1$. The stochastic integral for the fractional Brownian motion in (2) is defined via an integral concept in [4] and related to a Wick-Itô-Skorohod type of integral. See also Section 5.
for $t \geq 0$ and $H \in (0, \frac{1}{2})$, where $\sigma x(t)^\theta dB^H_t$ stands for a stochastic integral in the sense of Russo and Vallois. See Section 5.

Although we also obtain an existence and uniqueness result for solutions to (3) (see Theorem 5.5), we mainly focus in this paper on the study of SDE (2).

The remainder of the paper is organised as follows: In Section 2 we introduce the fractional Ait-Sahalia interest rate model. We establish an existence and uniqueness result for solutions to SDE (2) in Section 5 by studying the properties of solutions to an associated SDE driven by an additive fractional noise (see Sections 3 and 4). In addition, we also discuss the alternative model (3) in Section 5.

2 The fractional Ait-Sahalia model

Throughout this paper unless specified otherwise, we employ the following notation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$ null sets). Denote $\mathbb{E}$ as the expectation corresponding to $\mathbb{P}$.

Suppose that $B^H_t$, $0 \leq t \leq 1$, is a scalar fractional Brownian motion (fBm) with Hurst parameter $H \in (0, \frac{1}{2})$ and $B_t$, $0 \leq t \leq 1$, is a scalar Brownian motion defined on this probability space.

In what follows, we are interested to study the SDE

$$x_t = x_0 + \int_0^t (\alpha_{-1}x_{s}^{-1} - \alpha_0 + \alpha_1x_{s} - \alpha_2s^{2H-1}x_{s}^{\rho}) ds + \int_0^t \sigma x_{s}^{\theta} dB^H_s,$$

$x_0 \in (0, \infty)$, $0 \leq t \leq 1$, where $H \in (\frac{1}{2}, 1)$, $\bar{\theta} > 0$, $\rho > 1 + \frac{1}{H\bar{\theta}}$, $\tilde{\theta} := \frac{\bar{\theta} + 1}{\bar{\theta}}$, $\sigma > 0$ and $\alpha_i > 0$, $i = -1, \ldots, 2$. Here the stochastic integral term with respect to $B^H_t$ in (1) is defined by means of an integral concept introduced by F. Russo, P. Vallois in [14], See Section 5.

As already mentioned in the introduction, solutions to the SDE (4) can be used as a model (fractional Ait-Sahalia model) for the description of the dynamics of (short term) interest rates in finance. In fact, in this paper, we aim at establishing the existence and uniqueness of strong solutions $x_t > 0$ to SDE (4). In doing so, we show that such solutions can be obtained as transformations to the SDE

$$y_t = x + \int_0^t \tilde{f}(s, y_s) ds - \tilde{\sigma} B^H_t, \quad 0 \leq t \leq 1, \quad H \in (0, \frac{1}{2}),$$

3
where
\[
\tilde{f}(s, y) = \alpha_1(-\tilde{\theta}y^{\tilde{\theta}+1}) + \alpha_0 y^{\tilde{\theta}+1} - \alpha_1 s^{2H-1} \frac{1}{\tilde{\theta}y^{\tilde{\theta}+\tilde{\theta}+1}} - \tilde{\sigma} H^{2H-1} y^{-1}(\tilde{\theta} + 1),
\]
(6)

In the sequel, we want to prove the following new properties for solutions to SDE (5):

- Existence and uniqueness of positive strong solutions (Corollary 3.1)
- Regularity of solutions in the sense of Malliavin differentiability (Theorem 4.2)
- Existence of higher moments (Theorem 4.1).

3 Existence and uniqueness of solutions to singular SDEs with additive fractional noise for \( H < \frac{1}{2} \)

In this section, we wish to analyse the following generalisation of the SDE (5) given by
\[
x_t = x_0 + \int_0^t b(s, x_s) ds + \sigma B_t^H, \quad 0 \leq t \leq 1, \quad H \in (0, \frac{1}{2}), \quad \sigma > 0.
\]
(7)

We require the following conditions

(A1) \( b \in C((0, 1) \times (0, \infty)) \) and has a continuous spatial derivative \( b' := \frac{\partial}{\partial x} b \) such that
\[
b'(t, x) \leq K_t, \quad 0 < t < 1, \quad x \in (0, \infty),
\]
where \( K_t := t^{2H-1}K \) for some \( K \geq 0 \).

(A2) There exist \( x_1 > 0, \alpha > \frac{1}{H} - 1 \) and \( h_1 > 0 \) such that \( b(t, x) \geq h_1 t^{2H-1} x^{-\alpha} \), \( t \in (0, 1], x \leq x_1 \).

(A3) There are \( x_2 > 0 \) and \( h_2 > 0 \) such that \( b(t, x) \leq h_2 t^{2H-1}(x + 1), \quad t \in (0, 1], x \geq x_2 \).
Theorem 3.1. Suppose that (A1-A3) hold. Then for all \( x_0 > 0 \) the SDE (7) has a unique strong positive solution \( x_t, 0 \leq t \leq 1 \).

Proof. Without loss of generality, let \( \sigma = 1 \). We are required to establish the following analytical properties.

(i) Uniqueness: Suppose \( x \) and \( y \) are two solutions to (7). Then

\[
x_t - y_t = \int_0^t (b(s, x_s) - b(s, y_s)) ds.
\]

So, using the product rule, the mean value theorem and (A1), we get

\[
(x_t - y_t)^2 = 2 \int_0^t (b(s, x_s) - b(s, y_s))(x_s - y_s) ds \\
\leq 2 \int_0^t K_s (x_s - y_s)^2 ds.
\]

Hence, Gronwall’s Lemma implies that

\[
x_t - y_t = 0, \quad 0 \leq t \leq 1.
\]

(ii) Existence: Let \( x_0 > 0 \). Because of the regularity assumptions imposed on \( b \), we know that the equation (7) has (path-by-path) local solutions. Define the stopping times

\[
\tau_0 := \inf\{t \in [0,1] : x_t = 0\} \quad \text{and} \quad \tau_n := \inf\{t \in [0,1] : x_t \geq n\},
\]

where \( \inf \emptyset := 1^+ \). Just as in [13], we want to prove that \( \tau_0 = 1^+ \) and \( \lim_{n \to \infty} \tau_n = 1^+ \). Here \( 1^+ \) stands for an artificially added element larger than 1. Suppose that \( \tau_0 \leq 1 \). Then there is a \( \tau_0 \in (0, \tau_0] \) such that \( x_t \leq x_1 \) for all \( (\tau_0, \tau_0] \). By (A2), we know that \( b(t, x) > 0 \) for \( x \in (0, x_1) \) and \( t > 0 \). Hence,

\[
0 = x_{\tau_0} = x_t + \int_t^{\tau_0} b(s, x_s) ds + B^{H}_{\tau_0} - B^{H}_t, \quad t \in (\tau_0, \tau_0]. \quad (8)
\]

This implies

\[
x_t \leq |B^{H}_{\tau_0} - B^{H}_t| \leq \|B^{H}\|_\beta (\tau_0 - t)^\beta, \quad t \in (\tau_0, \tau_0] \text{ for } \beta \in (0, H). \quad (9)
\]
Here $|| \cdot ||_\beta$ denotes the Hölder-seminorm given by

$$||f||_\beta = \sup_{0 \leq s < t \leq 1} \frac{|f(s) - f(t)|}{(t - s)^\beta}$$

for $\beta$-Hölder continuous functions $f$. So, we also obtain that

$$||B^H_t||_\beta(\tau_0 - t)\beta \geq |B^H_t - B^H_\tau_0| \geq \int_t^{\tau_0} b(s, x_s) ds$$

$$\geq h_1 \int_t^{\tau_0} s^{2H-1} x_s^{-\alpha} ds \geq \frac{h_1}{||B^H_t||^2_\beta} \int_t^{\tau_0} s^{2H-1} \frac{1}{(\tau_0 - s)^{\alpha \beta}} ds \geq \frac{h_1}{||B^H_t||^2_\beta} \int_t^{\tau_0} \frac{1}{(\tau_0 - s)^{\alpha \beta}} ds.$$

If $\alpha \beta \geq 1$, we get a contradiction. For $\alpha \beta < 1$, we find that

$$||B^H_t||_\beta(\tau_0 - t)\beta \geq \frac{h_1}{||B^H_t||^2_\beta} \int_t^{\tau_0} \frac{1}{(\tau_0 - s)^{\alpha \beta}} ds > 0.$$

Hence,

$$0 = \lim_{t \to \tau_0} ||B^H_t||_\beta(\tau_0 - t)\beta^{\alpha \beta} \geq \frac{h_1 \tau_0^{2H-1}}{||B^H_t||^2_\beta (1 - \alpha \beta)} > 0.$$ 

So $\tau_0 = 1^+$. Assume now that

$$\tau_\infty := \lim_{n \to \infty} \tau_n \leq 1.$$

Then we can show as in [13] by using (A3) that

$$x_t \leq x_2 + x_0 + ||B^H_t||_\beta \tau_\infty^\beta + h_2 \tau_\infty^{2H}(2H)^{-1} + h_2 \int_{\tau_1}^{t} s^{2H-1} x_s ds.$$

So, by letting

$$\alpha = x_2 + x_0 + ||B^H_t||_\beta \tau_\infty^\beta + h_2 \tau_\infty^{2H}(2H)^{-1},$$

it follows from Gronwall’s Lemma that

$$x_t \leq \alpha + \int_{\tau_1}^{t} \alpha h_2 s^{2H-1} \exp \left( \int_{s}^{t} h_2 u^{2H-1} du \right) ds$$

$$\leq \gamma + \int_{0}^{1} \gamma h_2 s^{2H-1} \exp \left( \int_{s}^{1} h_2 u^{2H-1} du \right) ds,$$

where $\gamma := x_2 + x_0 + ||B^H_t||_\beta + h_2^{2H}$. The latter estimate leads to a contradiction.
As a consequence of Theorem 3.1, we obtain the following result:

**Corollary 3.2.** Suppose that \( x \in (0, \infty) \) and \( \rho > \frac{1}{H\theta} + 1 \). Then there exists a unique strong solution \( y_t > 0 \) to SDE (3).

**Proof.** Let \( \epsilon = \frac{H}{2} \). Then

\[
\tilde{f}(s, y) = \tilde{g}_1(s, y) + \tilde{g}_2(s, y),
\]

where

\[
\tilde{g}_1(s, y) := \alpha_1(-\tilde{\theta}y^{2\tilde{\theta}+1} + \alpha_0 y^{\tilde{\theta}+1} - \alpha_1 \frac{y}{\tilde{\theta}} + \epsilon\tilde{\sigma}s^{2H-1}y^{-1}(\tilde{\theta} + 1),
\]

and

\[
\tilde{g}_2(s, y) := \alpha_2 s^{2H-1} \frac{1}{\tilde{\theta}^\rho} y^{-\tilde{\theta}p+\tilde{\theta}+1} - (H + \epsilon)\tilde{\sigma}s^{2H-1}y^{-1}(\tilde{\theta} + 1).
\]

We see that

\[
\tilde{g}_1(s, y) \geq \alpha_1(-\tilde{\theta}y^{2\tilde{\theta}+1} + \alpha_0 y^{\tilde{\theta}+1} - \alpha_1 \frac{y}{\tilde{\theta}} + \epsilon\tilde{\sigma}s^{2H-1}y^{-1}(\tilde{\theta} + 1)
\]

\[
\geq 0
\]

for all \( s \in (0, 1] \) and \( y \in (0, y_0) \) for some \( y_0 > 0 \). Since

\[-\tilde{\theta}p + \tilde{\theta} + 1 < -\frac{1}{H} + 1 < -1,\]

we also find some \( y_1 > 0 \) such that

\[
\tilde{g}_2(s, y) = s^{2H-1} \frac{1}{\tilde{\theta}^\rho} y^{-\tilde{\theta}p+\tilde{\theta}+1}(\alpha_2 \frac{1}{\tilde{\theta}^\rho} - (H + \epsilon)\tilde{\sigma}(\tilde{\theta} + 1)y^{\tilde{\theta}p-\tilde{\theta}-2})
\]

\[
\geq h_1 s^{2H-1}y^{-\alpha}
\]

for all \( s \in (0, 1] \) and \( y \in (0, y_1] \), where \( h_1 > 0 \) and \( \alpha := \tilde{\theta}p - \tilde{\theta} - 1 \). So,

\[
\tilde{f}(s, y) \geq h_1 s^{2H-1}y^{-\alpha}
\]

for all \( s \in (0, 1], y \in (0, y_1) \) for some \( y_1 > 0 \), which shows that \( \tilde{f} \) satisfies (A2). As for (A3), we see that there exists some \( y_2 \geq 1 \) such that

\[
\tilde{f}(s, y) \leq s^{2H-1}(\alpha_2 \frac{1}{\tilde{\theta}^\rho} y^{-\tilde{\theta}p+\tilde{\theta}+1} - H\tilde{\sigma}y^{-1}(\tilde{\theta} + 1)) \leq h_2 s^{2H-1}(1 + y)
\]
for all $s \in (0, 1]$, $y \in (y_2, \infty)$ and some $h_2 > 0$. We have that
$$\tilde{f}'(s, y) = f_1(s, y) + f_2(s, y),$$
where
$$f_1(s, y) := -\alpha_1 \tilde{\theta}(2\tilde{\theta} + 1)y^{2\tilde{\theta}} + \alpha_0(\tilde{\theta} + 1)y^{\tilde{\theta}} - \frac{\alpha_1}{\tilde{\theta}}$$
and
$$f_2(s, y) := s^{2H-1}\left(\alpha_2 \frac{1}{\tilde{\theta}^p}(\tilde{\theta} \rho + \tilde{\theta} + 1)y^{-\tilde{\theta}p + \tilde{\theta}} + H\tilde{\sigma}(\tilde{\theta} + 1)y^{-2}\right).$$
So, there exist $y_1, y_2 > 0$ such that
$$\tilde{f}'(s, y) \leq f_1(s, y) \leq K \leq s^{2H-1}K = K_s$$
for all $s \in (0, 1]$, $y \in (0, y_1)$ as well as
$$\tilde{f}'(s, y) \leq f_2(s, y) \leq s^{2H-1}K = K_s$$
for all $s \in (0, 1]$, $y \in (y_2, \infty)$ and some $K > 0$. On the other hand, we see that $\tilde{f}$ also satisfies (A1). Since $-B^H_t$ is a fractional Brownian motion, the proof follows.

4 Malliavin differentiability and existence of higher moments of solutions

In this section, we want to show that the solution $x$ to the SDE
$$x_t = x + \int_0^t \tilde{f}(s, x_s)ds - \tilde{\sigma}B^H_t, \quad 0 \leq t \leq 1, x > 0,$$  \hspace{1cm} (10)
is Malliavin differentiable in the direction of $B^H_t$ for $H \in (0, \frac{1}{2})$. Furthermore, we verify that solutions $x_t$ to (10) belong to $L^q$ for all $q \geq 1$. For this purpose, let $\tilde{f}_n : [0, 1] \times \mathbb{R} \to \mathbb{R}$, $n \geq 1$ be a sequence of bounded, globally Lipschitz continuous (and smooth) functions such that
(i) \( \tilde{f} |_{[\frac{1}{n}, n]} = \tilde{f} |_{[0, 1] \times [\frac{1}{n}, n]} \) for all \( n \geq 1 \),

(ii) \( \tilde{f}'_n(s, x) \leq K_s \) for all \( (s, x) \in (0, 1] \times \mathbb{R} \), \( n \geq 1 \), where \( K_s \) is defined in (A1).

So we see that

\[
\tilde{f}'_n(s, x) \longrightarrow_{n \to \infty} \tilde{f}(s, x)
\]

for all \( (s, x) \in (0, 1] \times (0, \infty) \). Denote by \( D^H \) and \( D \), the Malliavin derivative in the direction of \( B^H \) and \( W \), respectively. Here \( W \) is the Wiener process with respect to the representation

\[
B^H_t = \int_0^t K_H(t, s) dW_s, \quad t \geq 0.
\]  

(11)

See the Appendix. Since \( -B^H \) is a fractional Brownian motion, let us without loss of generality assume in (10) that \( \tilde{\sigma} = -1 \). Because of the regularity of the functions \( \tilde{f}_n \), \( n \geq 1 \), we find that the solutions \( x^n_t \) to

\[
x^n_t = x + \int_0^t \tilde{f}_n(s, x_s) ds + B^H_t, \quad x > 0, \quad 0 \leq t \leq 1
\]

are Malliavin differentiable with Malliavin derivative \( D^H u x_t \) satisfying the equation

\[
D^H u x^n_t = \int_u^t \tilde{f}'_n(s, x^n_s) D^H u x^n_s ds + \chi_{[0, t]}(u).
\]

Hence,

\[
D^H x^n_t = \chi_{[0, t]}(u) \exp \left( \int_u^t \tilde{f}'_n(s, x^n_s) ds \right) \lambda \times \text{P-a.e.}
\]

for all \( 0 \leq t \leq 1 \) ( \( \lambda \) Lebesgue measure). See [4]. Further, using the transfer principle between \( D^H \) and \( D \), we have that

\[
K_H^* D^H x_t = D x_t
\]

(12)

where \( K_H^* : \mathcal{H} \to L^2([0, T]) \) is given by

\[
(K_H^* y)(s) = K_H(T, s) y(s) + \int_s^T (y(t) - y(s)) \frac{\partial}{\partial t} K_H(t, s) dt
\]

(13)
\[ \frac{\partial}{\partial t} K_H(t, s) = c_H \left( H - \frac{1}{2} \right) \left( \frac{1}{2} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}. \]  \tag{14}

Here \( H = I_{T-}^{1/2} (L^2) \). See the Appendix. On the other hand, using (12), we also see that

\[ D_u x^n_t = \int_u^t f_n(s) D_u x^n_s ds + K_H(t, u) \]  \tag{15}

in \( L^2([0, t] \times \Omega) \) for all \( 0 \leq t \leq 1 \). Set

\[ Y^n_t(u) = D_u x^n_t - K_H(t, u). \]

Then,

\[ Y^n_t(u) = \int_u^t \left\{ f_n(s) Y^n_s(u) + f_n(s) K_H(s, u) \right\} ds. \]

Using the fundamental solution of the equation

\[ \dot{\Phi}(t) = f'(t, x^n_t) \cdot \Phi(t), \quad \Phi(u) = 1. \]

We then obtain that

\[ Y^n_t(u) = \int_u^t \exp \left( \int_u^t \tilde{f}'(r, x^n_r) dr \right) \tilde{f}_n(s) K_H(s, u) ds. \]

Hence,

\[ D_u x^n_t = \int_u^t \exp \left( \int_u^t \tilde{f}'(r, x^n_r) dr \right) \tilde{f}_n(s) K_H(s, u) ds + K_H(t, u) \]

\[ = J_1^n(t, u) + J_2^n(t, u) + K_H(t, u), \quad u < t, \quad \lambda \times \text{P-a.e.,} \]

where

\[ J_1^n(t, u) := \int_u^t \exp \left( \int_u^t \tilde{f}'(r, x^n_r) dr \right) \left( \tilde{f}_n(s) - K_s \right) K_H(s, u) ds \]

and

\[ J_2^n(t, u) := \int_u^t \exp \left( \int_u^t \tilde{f}'(r, x^n_r) dr \right) K_s \cdot K_H(s, u) ds. \]
Without loss of generality, let $T = t = 1$. Then
\[
\int_0^1 (D_u x_1^n)^2 du \leq C \left\{ \int_0^1 (J_1^n(1, u))^2 du + \int_0^1 (J_2^n(1, u))^2 du + \int_0^1 (K_H(1, u))^2 du \right\}.
\]

Using Fubini’s theorem, we get that
\[
\int_0^1 (J_1^n(1, u))^2 du = \int_0^1 \left( \int_0^1 \chi_{[u,1]}(s) \exp \left( \int_s^1 \tilde{f}(r, x_r) dr \right) \left( \tilde{f}_n(s, x^n_s) - K_s \right) K_H(s, u) ds \right)^2 du
\]
\[
= \int_0^1 \left\{ \exp \left( \int_1^s \tilde{f}(r, x_r) dr \right) \left( \tilde{f}_n(s, x^n_s) - K_s \right) \int_0^1 K_H(s, u) K_H(s, u) du \right\} ds_1 ds_2.
\]

From (11), we see for the covariance function
\[
R_H(s_1, s_2) = \mathbb{E}[B_{s_1}^H \cdot B_{s_2}^H]
\]
that
\[
R_H(s_1, s_2) = \int_0^{s_1 \wedge s_2} K_H(s_1, u) K_H(s_2, u) du.
\]

Since
\[
0 \leq R_H(s_1, s_2) = \frac{1}{2} \left( s_1^{2H} + s_2^{2H} - |s_1 - s_2|^{2H} \right) \leq 1, \quad H < \frac{1}{2}
\]
and
\[
\left( \tilde{f}_n(s, x^n_s) - K_s \right) \cdot \left( \tilde{f}_n(s, x^n_s) - K_s \right) \geq 0
\]
for $0 < s_1, s_2 \leq 1$, we find that
\[
\int_0^1 (J_1^n(1, u))^2 du \leq \left( \int_0^1 \left( \exp \left( \int_s^1 \tilde{f}(r, x_r) dr \right) \left( \tilde{f}_n(s, x^n_s) - K_s \right) ds \right) \right)^2
\]
\[
= \left\{ - \exp \left( \int_s^1 \tilde{f}(r, x_r) dr \right) \bigg|_{s=0}^1 - \int_0^1 K_s \exp \left( \int_s^1 \tilde{f}(r, x_r) dr \right) ds \right\}^2
\]
\[
\leq \left( \exp(\int_0^1 K_r dr) + \int_0^1 K_s ds \cdot \exp(\int_0^1 K_r dr) \right)^2.
\]
Similarly, we also obtain that
\[
\int_0^1 (J_n^2(1, u))^2 du \leq C(K, H)
\]
for a constant \(C(K, H) < \infty\). We also have that
\[
\int_0^1 (K_H(1, u))^2 du = \mathbb{E}[(B_1^H)^2] = 1.
\]
Altogether, we get that
\[
\mathbb{E}\left[ \int_0^1 (D_u x_n^u)^2 du \right] \leq C(K, H) \tag{17}
\]
for all \(n \geq 1\) for a constant \(C(K, H) < \infty\). Define now the stopping times \(\tau_n\) by
\[
\tau_n = \inf \left\{ 0 \leq t \leq 1; x_t \notin \left[ \frac{1}{n}, n \right] \right\} \quad (\inf \emptyset = \infty)
\]
Then we know from the proof of the existence of solutions in the previous section that \(\tau_n \nrightarrow \infty\) for \(n \to \infty\). So,
\[
x_{t \wedge \tau_n}^n - x_{t \wedge \tau_n} = \int_0^{t \wedge \tau_n} \left\{ \tilde{f}_n(s, x_s^n) - \tilde{f}(s, x_s) \right\} ds
\]
\[
= \int_0^t \chi_{[0, \tau_n)}(s) \left\{ \tilde{f}_n(s, x_{s \wedge \tau_n}^n) - \tilde{f}_n(s, x_{s \wedge \tau_n}) \right\} ds.
\]
Hence,
\[
|x_{t \wedge \tau_n}^n - x_{t \wedge \tau_n}| \leq K_n \int_0^t |x_{s \wedge \tau_n}^n - x_{s \wedge \tau_n}| ds
\]
for a Lipschitz constant \(K_n\). Then Gronwall’s Lemma implies that
\[
x_{t \wedge \tau_n}^n = x_{t \wedge \tau_n}
\]
for all \(t, n\) P-a.e. Since \(\tau_n \nrightarrow \infty\) for \(n \to \infty\) a.e. we have that
\[
x_t^n \xrightarrow{n \to \infty} x_t \tag{18}
\]
for all \( t \) \( P \)-a.e. Using the Clark-Ocone formula (see [4]), we get that

\[
x^n_t = \mathbb{E}[x^n_1] + \int_0^1 \mathbb{E}[D_s x^n_1 | \mathcal{F}_s] dW_s,
\]

where \( \{\mathcal{F}\}_{0 \leq t \leq 1} \) is the filtration generated by \( W \). It follows that

\[
\mathbb{E}[(x^n_1 - \mathbb{E}[x^n_1])^2] = \mathbb{E} \left[ \int_0^1 \left( \mathbb{E}[D_s x^n_1 | \mathcal{F}_s] \right)^2 ds \right] 
\leq \mathbb{E} \left[ \int_0^1 \mathbb{E}[(D_s x^n_1)^2 | \mathcal{F}_s] ds \right] = \int_0^1 \mathbb{E}[(D_s x^n_1)^2] ds.
\]

So, we see from (17) that

\[
\mathbb{E}[(x^n_1 - \mathbb{E}[x^n_1])^2] \leq C(K, H) < \infty
\]

for all \( n \geq 1 \). We also have that

\[
||x^n_1 - \mathbb{E}[x^n_1]| - |x_1 - \mathbb{E}[x^n_1]| | \leq |x^n_1 - x_1| \to 0
\]

because of (18). So,

\[
\lim_{n \to \infty} |x^n_1 - \mathbb{E}[x^n_1]| = \lim_{n \to \infty} |x_1 - \mathbb{E}[x^n_1]|.
\]

Suppose that \( \mathbb{E}[x^n_1], n \geq 1 \) is unbounded. Then there exists a subsequence \( n_k, k \geq 1 \) such that

\[
|\mathbb{E}[x^{n_k}_1]| \to \infty.
\]

It follows from the Lemma of Fatou and the positivity of \( x_t \) that

\[
\infty = \mathbb{E} \left[ \lim_{k \to \infty} \left( |x_1 - |\mathbb{E}[x^{n_k}_1]| | \right)^2 \right] 
\leq \mathbb{E} \left[ \lim_{k \to \infty} \left( |x_1 - \mathbb{E}[x^{n_k}_1]| \right)^2 \right] 
\leq \mathbb{E} \left[ \lim_{k \to \infty} \left( |x^{n_k}_1 - \mathbb{E}[x^{n_k}_1]| \right)^2 \right] 
\leq \lim_{k \to \infty} \mathbb{E} \left[ |x^{n_k}_1 - \mathbb{E}[x^{n_k}_1]|^2 \right] \leq C < \infty,
\]

for all \( t \)
which is a contradiction. Hence,
\[
\sup_{n \geq 1} |\mathbb{E}[x^n_1]| < \infty.
\]
Further, we also obtain from the Burkholder-Davis-Gundy inequality and (17) that
\[
\begin{align*}
\mathbb{E}[|x_1^n|^{2p}] &< C_p \left( \mathbb{E}[|x_1^n|]^{2p} + \mathbb{E}\left[ \left( \int_0^1 \mathbb{E}[D_s x_1^n | \mathcal{F}_s] dW_s \right)^{2p} \right] \right) \\
&\leq C_p \left( \mathbb{E}[|x_1^n|]^{2p} + \mathbb{E}\left[ \left( \sup_{0 \leq u \leq 1} \left| \int_0^u \mathbb{E}[D_s x_1^n | \mathcal{F}_s] dW_s \right| \right)^{2p} \right] \right) \\
&\leq C_p \left( \mathbb{E}[|x_1^n|]^{2p} + m_p \mathbb{E}\left[ \left( \int_0^1 \mathbb{E}[D_s x_1^n | \mathcal{F}_s] \right)^2 ds \right]^{p} \right) \\
&\leq C(p, K, H)
\end{align*}
\]
(19)
for \( n \geq 1 \). So it follows from (18) and the Lemma of Fatou that
\[
\mathbb{E}[|x_1|^{2p}] \leq \lim_{n \to \infty} \mathbb{E}[|x_1^n|]^{2p} \leq C(p, K, H) < \infty
\]
for all \( p \geq 1 \). So we obtain the following result:

**Theorem 4.1.** Let \( x_t, 0 \leq t \leq 1 \) be the solution to (10). Then \( x_t \in L^q(\Omega) \) for all \( q \geq 1 \) and \( 0 \leq t \leq 1 \).

In addition, we obtain from Lemma 1.2.3 in [4] in connection with the estimate (19) that \( x_1 \) is Malliavin differentiable in the direction of \( W \). The latter, in combination with (12), also entails the Malliavin differentiability of \( x_1 \) with respect to \( B^H \). Thus we have also shown the following result:

**Theorem 4.2.** The positive unique strong solution \( x_t \) to (10) is Malliavin differentiable in the direction of \( B^H \) and \( W \) for all \( 0 \leq t \leq 1 \).

## 5 Application

In this section, we aim at applying the results of the previous section to obtain a unique strong solution to \( x_t \) to the SDE
\[
\begin{align*}
x_t &= x_0 + \int_0^t \left( \alpha_{-1} x_s^{-1} - \alpha_0 + \alpha_1 x_s - \alpha_2 s^{2H-1} x_s^\theta \right) ds + \int_0^t \sigma x_s^\theta dB^H_s,
\end{align*}
\]
(20)
$0 \leq t \leq 1$, for $H \in (\frac{1}{3}, \frac{1}{2})$, $\tilde{\theta} > 0$, $\rho > 1 + \frac{1}{H\tilde{\theta}}$, $\sigma > 0$ and $\theta := \frac{\tilde{\theta} + 1}{\tilde{\theta}}$. Here, the stochastic integral with respect to $B^H_t$ is defined by

$$
\int_0^t g(x_s)dB^H_s = \int_0^t -Hs^{2H-1}g'(x_s)ds + \int_0^t g(x_s)d^0B^H_s \tag{21}
$$

for functions $g \in C^3$. See also the second Remark 5.3 below. The stochastic integral on the right hand side of (21) is the symmetric integral with respect to $B^H_\cdot$ introduced by F. Russo, P. Vallois. See e.g. [14] and the references therein. Such an integral denoted by

$$
\int_0^t Y_s d^0X_s, \quad t \in [0,1]
$$

for continuous process $X_\cdot$, $Y_\cdot$ is defined as

$$
\lim_{\epsilon \searrow 0} \frac{1}{2\epsilon} \int_0^t Y_s(X_{s+\epsilon} - X_s)ds,
$$

provided this limit exists in the ucp-topology. In order to construct a solution to (20), we need a version of the Itô formula for processes $Y_\cdot$, which have a finite cubic variation. A continuous process is said to have a finite strong cubic variation (or 3-variation), denoted by $[Y, Y, Y]$, if

$$
[Y, Y, Y] := \lim_{\epsilon \searrow 0} \frac{1}{\epsilon} \int_0^t (Y_{s+\epsilon} - Y_s)^3ds
$$

exists in ucp as well as

$$
\sup_{0<\epsilon \leq 1} \frac{1}{\epsilon} \int_0^1 (Y_{s+\epsilon} - Y_s)^3ds < \infty \quad \text{a.e.}
$$

See [14]. Using the concept of finite strong cubic variation, one can show the following Itô formula (see [14]).

**Theorem 5.1.** Assume that $Y_\cdot$ is a real valued process with finite strong cubic variation and $g \in C^3$. Then

$$
g(Y_t) = g(Y_0) + \int_0^t g'(Y_s)d^0Y_s - \frac{1}{12} \int_0^t g'''(Y_s)d[Y, Y]_s, \quad 0 \leq t \leq 1.
$$
Remark 5.2. The last term on the right hand side of the equation is a Lebesgue-Stieltjes integral with respect to the bounded variation process $[Y,Y,Y]$. 

Remark 5.3.

- We mention that for $Y = B^H$, $H \in (\frac{1}{3}, \frac{1}{2})$, $[B^H, B^H, B^H]$ is zero a.e.
- If $X = B^H$ in (21), then it follows from Theorem 6.3.1 in [15] that our stochastic integral in (21) equals the Wick-Itô-Skorohod integral. The latter also gives a justification for the definition of the stochastic integral in (21) in the general case.

Theorem 5.4. Suppose that $H \in (\frac{1}{3}, \frac{1}{2})$, $\tilde{\theta} > 0$, $\sigma > 0$ and $\rho > 1 + \frac{1}{HH}$ Let $\theta = \frac{\tilde{\theta} + 1}{\tilde{\theta}}$. Then there exists a unique strong and positive solution to the SDE (20).

Proof. Let $y_t$ be the unique strong and positive solution to

$$y_t = x + \int_0^t \tilde{f}(s, y_s)ds - \tilde{\sigma} B^H_s, \quad 0 \leq t \leq 1, \quad x > 0,$$

where $\tilde{f}$ is defined as in Section 2. Define $g \in C^3$ by $g(y) = \frac{y - \tilde{\theta}}{\theta}$. Then Theorem 5.1 entails that

$$x_t := g(y_t) = \frac{x - \tilde{\theta}}{\theta} + \int_0^t (-1)y_s^{-\tilde{\theta}+1}d^c y_s - \frac{1}{12} \int_0^t g'''(y_s)d[y,y,y]_s.$$ 

Since $[B^H, B^H, B^H]$ is zero a.e. (see Remark 5.3), we observe that $[y, y, y]$ is zero a.e. So

$$x_t = \frac{x - \tilde{\theta}}{\theta} + \int_0^t (-1)y_s^{-\tilde{\theta}+1}d^c y_s$$

$$= \frac{x - \tilde{\theta}}{\theta} + \int_0^t (-1)y_s^{-\tilde{\theta}+1} \tilde{f}(s, y_s)ds + \int_0^t \tilde{\sigma} y_s^{-\tilde{\theta}+1}d^c B^H_s$$

$$= \frac{x - \tilde{\theta}}{\theta} - \int_0^t \left\{ (-1)y_s^{-\tilde{\theta}+1} \tilde{f}(s, y_s) - H\tilde{\sigma}s^{2H-1}y_s^{-\tilde{\theta}+2}(\tilde{\theta} + 1) \right\} ds + \int_0^t \tilde{\sigma} y_s^{-\tilde{\theta}+1}dB^H_s.$$
Since we can write \((y_s)^{-(\tilde{\theta}+1)} = \tilde{\theta}^{\theta}\left(y_s^{\tilde{\theta}}\right)^{\theta}\), we now have

\[
x_t = x - \tilde{\theta} \tilde{\theta} + \int_0^t f(s, \frac{y_s}{\tilde{\theta}}) ds + \int_0^t \tilde{\sigma} y_s^{-(\tilde{\theta}+1)} dB_s^H
\]

where \(f(s, y) := \alpha_{-1} y^{-1} - \alpha_0 + \alpha_1 y - \alpha_2 s^{2H-1} y^\rho, \ s \in (0, 1], \ y \in (0, \infty).\)

So \(x\) satisfies the SDE (20) if we choose \(\tilde{\sigma} = \tilde{\theta} - \tilde{\theta} \sigma\) for \(\sigma > 0\). In order to show the uniqueness of solutions to SDE (20), one can apply the Itô formula in Theorem 5.1 to the inverse function \(g^{-1}\) given by \(g^{-1}(y) = (\tilde{\theta})^{-\frac{1}{\tilde{\theta}}} y^\frac{1}{\tilde{\theta}}\) by using the fact that \([B_s^H, B_t^H, B_t^H] = 0 \ a.e.\ for \ H \in (\frac{1}{3}, \frac{1}{2}).\)

Finally, using the same arguments as in the proof of Theorem 5.4, we also get the following result for the alternative Ait-Sahalia model (3):

**Theorem 5.5.** Retain the conditions of Theorem 5.4 with respect to \(H, \tilde{\theta}, \theta\) and \(\rho\). Then there exists a unique strong solution \(x_t > 0\) to SDE (3).

**Proof.** Just as in the proof of Theorem 5.4 we can consider the SDE (5), where the vector field \(\tilde{f}\) now is given by

\[
\tilde{f}(s, y) = \alpha_{-1} (-\tilde{\theta} y^{2\tilde{\theta}+1}) + \alpha_0 y^{\tilde{\theta}+1} - \alpha_1 \frac{y}{\tilde{\theta}} + \alpha_2 \frac{1}{\tilde{\theta}^\rho} y^{-\tilde{\theta}+\rho+1}
\]

for \(0 < y < \infty\). Then as in the proof of Corollary (3.2) one immediately verifies that \(\tilde{f}\) satisfies the assumptions of Theorem 3.1 which yields a unique strong solution \(y_t > 0\) to (3) in this case. In exactly the same way, we also obtain the results of Theorem 4.1 and Theorem 4.2 with respect to \(\tilde{f}\) in (23). Finally, we can apply the Itô formula as in the proof of Theorem 5.4 and construct a unique strong solution \(x_t > 0\) to (3) based on \(y\). \(\square\)

6 Appendix

For some of the proofs in this article we need to recall some basic concepts from fractional calculus (see [16] and [17]).
Let $a, b \in \mathbb{R}$ with $a < b$. Let $f \in L^p([a,b])$ with $p \geq 1$ and $\alpha > 0$. Then the left- and right-sided Riemann-Liouville fractional integrals are defined as

$$I_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy$$

and

$$I_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) dy$$

for almost all $x \in [a,b]$. Here $\Gamma$ is the Gamma function.

Let $p \geq 1$ and let $I_{a^+}^\alpha (L^p)$ (resp. $I_{b^-}^\alpha (L^p)$) be the image of $L^p([a,b])$ of the operator $I_{a^+}^\alpha$ (resp. $I_{b^-}^\alpha$). If $f \in I_{a^+}^\alpha (L^p)$ (resp. $f \in I_{b^-}^\alpha (L^p)$) and $0 < \alpha < 1$ then we can define the left- and right-sided Riemann-Liouville fractional derivatives by

$$D_{a^+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^{\alpha}} dy$$

and

$$D_{b^-}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(y)}{(y-x)^{\alpha}} dy.$$

The left- and right-sided derivatives of $f$ can be represented as

$$D_{a^+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x f(x) - f(y) \frac{dy}{(x-y)^{\alpha+1}} \right)$$

and

$$D_{b^-}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_x^b f(x) - f(y) \frac{dy}{(y-x)^{\alpha+1}} \right).$$

The above definitions imply that

$$I_{a^+}^\alpha (D_{a^+}^\alpha f) = f$$

for all $f \in I_{a^+}^\alpha (L^p)$ and

$$D_{a^+}^\alpha (I_{a^+}^\alpha f) = f$$

for all $f \in L^p([a,b])$ and similarly for $I_{b^-}^\alpha$ and $D_{b^-}^\alpha$.

Denote by $B^H = \{B^H_t, t \in [0,T]\}$ a $d$-dimensional fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2})$. The latter means that $B^H$ is a centered Gaussian process with a covariance function given by

$$(R_H(t,s))_{i,j} := E[B^H_t(i)B^H_s(j)] = \delta_{ij} \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right), \quad i, j = 1, \ldots, d,$$
where \( \delta_{ij} \) is one, if \( i = j \), or zero else.

In the sequel, we also shortly recall the construction of the fractional Brownian motion, which can be found in [4]. For convenience, we restrict ourselves to the case \( d = 1 \).

Denote by \( \mathcal{E} \) the class of step functions on \([0, T]\) and let \( \mathcal{H} \) be the Hilbert space which one gets through the completion of \( \mathcal{E} \) with respect to the inner product

\[
\langle 1_{[0,t]}, 1_{[0,s]} \rangle_\mathcal{H} = R_H(t, s).
\]

The latter provides an extension of the mapping \( 1_{[0,t]} \mapsto B_t \) to an isometry between \( \mathcal{H} \) and a Gaussian subspace of \( L^2(\Omega) \) with respect to \( B^H \). Let \( \varphi \mapsto B^H(\varphi) \) be this isometry.

If \( H < \frac{1}{2} \), one finds that the covariance function \( R_H(t, s) \) can be represented as

\[
R_H(t, s) = \int_0^{t \wedge s} K_H(t, u)K_H(s, u)du,
\]

where

\[
K_H(t, s) = c_H \left[ \left( \frac{t}{s} \right)^{H - \frac{1}{2}} (t - s)^{H - \frac{1}{2}} + \left( \frac{1}{2} - H \right) s^{\frac{1}{2} - H} \int_s^t u^{H - \frac{3}{2}}(u - s)^{H - \frac{1}{2}}du \right].
\]

Here \( c_H = \sqrt{\frac{2H}{(1-2H)(1-2H,H+\frac{1}{2})}} \) and \( \beta \) is the Beta function. See [4, Proposition 5.1.3].

Using the kernel \( K_H \), one can obtain via [24] an isometry \( K^*_H \) between \( \mathcal{E} \) and \( L^2([0, T]) \) such that \( (K^*_H 1_{[0,t]})(s) = K_H(t, s)1_{[0,t]}(s) \). This isometry allows for an extension to the Hilbert space \( \mathcal{H} \), which has the following representations in terms of fractional derivatives

\[
(K^*_H \varphi)(s) = c_H \Gamma \left( H + \frac{1}{2} \right) s^{\frac{1}{2} - H} \left( D^{\frac{1}{2} - H}_{T-} u^{H - \frac{1}{2}} \varphi(u) \right)(s)
\]

and

\[
(K^*_H \varphi)(s) = c_H \Gamma \left( H + \frac{1}{2} \right) \left( D^{\frac{1}{2} - H}_{T-} \varphi(s) \right)(s)
+ c_H \left( \frac{1}{2} - H \right) \int_s^T \varphi(t)(t - s)^{H - \frac{1}{2}} \left( 1 - \left( \frac{t}{s} \right)^{H - \frac{1}{2}} \right) dt.
\]
for \( \varphi \in \mathcal{H} \). One can also prove that \( \mathcal{H} = L^{1-2H}_{F-\mathcal{H}}(L^2) \). See [12] and [7, Proposition 6].

We know that \( K^*_H \) is an isometry from \( \mathcal{H} \) into \( L^2([0,T]) \). Thus, the \( d \)-dimensional process \( W = \{ W_t, t \in [0,T] \} \) defined by

\[
W_t := B^H \left( (K^*_H)^{-1}(1_{[0,t]}) \right)
\]

is a Wiener process and the process \( B^H \) has the representation

\[
B^H_t = \int_0^t \mathcal{K}_H(t,s)dW_s.
\]

References

[1] Karatzas, I., Shreve, S.E., 1998. Methods of Mathematical Finance. Springer-Verlag.

[2] Lamberton, D. and Lapeyre, B., 2011. Introduction to Stochastic Calculus Applied to Finance. CRC press.

[3] Gatheral, J., Jaisson, T. and Rosenbaum, M., 2018. Volatility is rough. Quantitative Finance, 18(6), pp.933-949.

[4] Nualart, D., 2006. The Malliavin Calculus and Related Topics (Vol. 1995, p. 317). Berlin: Springer.

[5] Amine, O., Coffie, E., Harang, F. and Proske, F., 2020. A Bismut–Elworthy–Li formula for singular SDEs driven by a fractional Brownian motion and applications to rough volatility modelling. Communications in Mathematical Sciences, 18(7), pp.1863-1890.

[6] Coffie, E., Duedahl, S. and Proske, F., 2021. Sensitivity Analysis with respect to a stochastic stock price model with rough volatility via a Bismut-Elworthy-Li Formula for Singular SDEs. arXiv preprint arXiv:2107.06022.

[7] Alòs, E., Mazet, O. and Nualart, D., 2000. Stochastic calculus with respect to fractional Brownian motion with Hurst parameter lesser than 1/2. Stochastic processes and their applications, 86(1), pp.121-139.
[8] Ait-Sahalia, Y., 1996. Testing continuous-time models of the spot interest rate. The review of financial studies, 9(2), pp.385-426.

[9] Szpruch, L., Mao, X., Higham, D.J. and Pan, J., 2011. Numerical simulation of a strongly nonlinear Ait-Sahalia-type interest rate model. BIT Numerical Mathematics, 51(2), pp.405-425.

[10] Dung, N.T., 2016. Tail probabilities of solutions to a generalized Ait-Sahalia interest rate model. Statistics and Probability Letters, 112, pp.98-104.

[11] Coffie, E. and Mao, X., 2021. Truncated EM numerical method for generalised Ait-Sahalia-type interest rate model with delay. Journal of Computational and Applied Mathematics, 383, pp.113-1372.

[12] Decreusefond, L. and Ustunel, A.S., 1998. Stochastic analysis of the fractional Brownian motion. Potential Analysis 10, pp.177-214.

[13] Zhang, S.Q. and Yuan, C., 2021. Stochastic differential equations driven by fractional Brownian motion with locally Lipschitz drift and their implicit Euler approximation. Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 151(4), pp.1278-1304.

[14] Errami, M. and Russo, F., 2003. n-covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes. Stochastic Processes and their Applications, 104(2), pp.259-299.

[15] Biagini, F., Hu, Y., Øksendal, B. and Zhang, T., 2008. Stochastic Calculus for Fractional Brownian motion and Applications. Springer Science & Business Media.

[16] Lizorkin, P.I., 2001. Fractional integration and differentiation. Encyclopedia of Mathematics, Springer.

[17] Samko, S.G., Kilbas, A.A. and Marichev, O.I., 1993. Fractional integrals and derivatives. Yverdon-les-Bains, Switzerland: Gordon and breach science publishers, Yverdon.