A New Mechanism of Spontaneous SUSY Breaking

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Abstract

We propose a new mechanism of spontaneous supersymmetry breaking. The existence of extra dimensions with nontrivial topology plays an important role. We investigate new features resulted from the mechanism in two simple supersymmetric \(Z_2\) and \(U(1)\) models. One of remarkable features is that there exists a phase in which the translational invariance for the compactified directions is broken spontaneously, accompanying the breakdown of the supersymmetry. The mass spectrum of the models appeared in reduced dimensions is a full of variety, reflecting the highly nontrivial vacuum structure of the models. The Nambu-Goldstone bosons (fermions) associated with breakdown of symmetries are found in the mass spectrum. Our mechanism also yields quite different vacuum structures if models have different global symmetries.
1 INTRODUCTION

It has been considered that the (super)string theories [1] (and/or more fundamental theory such as M-theory) are plausible candidates to govern physics at the Planck scale. In general these theories are defined in higher space-time dimensions than 4 because of the consistency of the theories.

In the region of enough low energies, however, we have already known that our space-time is 4-dimensions, so that extra dimensions must be compactified by a certain mechanism [2], and the supersymmetry (SUSY), which is usually possessed by those theories, has to be broken because it is not observed in the low energy region. At this stage, mechanisms of compactifications and SUSY breaking are not fully understood.

It may be interesting to consider quantum field theory in space-time with one of spaces being multiply connected. This is because we may shed new light on unanswered questions in the (supersymmetric) standard model and/or expect some new dynamics which is useful to seek and understand new physics beyond it. Actually, it has been reported that the flavour-blind SUSY breaking terms can be induced through compactifications by taking account of possible topological effects of multiply connected space [3, 4, 5]. Therefore, it is important to investigate the physics which must be possessed by quantum field theory considered in such the space-time.

We shall consider supersymmetric field theories in space-time with one of the space coordinates being compactified. One has to specify boundary conditions of fields for the compactified directions when the compactified space is multiply connected. Contrary to the finite temperature field theory, we do not know a priori what they should be. We shall relax the conventional periodic boundary condition to allow nontrivial boundary conditions on the fields [6, 7]. If the compactified space is topologically nontrivial, the idea of the nontrivial boundary conditions is naturally realized. Due to the topology of the space, the configuration space also has nontrivial topology. This causes an ambiguity in quantization of the theory on the space to yield an undetermined parameter in the theory [8]. The boundary conditions of the fields are twisted with the parameter.

We shall propose a new mechanism of the spontaneous SUSY breaking through compactification. Extra dimensions with nontrivial topology and nontrivial boundary conditions for the extra dimensions play a central role. Our mechanism is quite different from the known mechanisms such as the O’Raifeartaigh [9] and Fayet-Iliopoulos ones [10]. A key point of our mechanism is that solutions to $F$-term conditions exist but they are not realized as vacuum configurations because of the nontrivial boundary conditions.

As a remarkable consequence of our mechanism, there appears a nontrivial phase structure with respect to the size of the compactified space. Namely, the translational invariance for the compactified directions is broken spontaneously [11, 12], accompanying the SUSY breaking when the size of the compactified space exceeds a certain critical value. The curious vacuum structures resulted from the mechanism also have an influence on the mass spectrum of the theory. Depending on the size of the compactified space, we have a mass spectrum full of variety. The mass spectrum includes Nambu-Goldstone bosons (fermions) corresponding to the breakdown of the global symmetries of the theory.

In previous two papers, we presented our idea [13] and applied it to the simple supersymmetric $\mathbb{Z}_2$ model [14]. In those papers, however, the details have been omitted and, in particular, the mass spectrum of the model has been discussed very shortly. Since the mass spectrum has a quite nontrivial dependence on the size of the compactified space, it is worth investigating it more thoroughly. In this paper, we shall study the mass spectrum
of the $Z_2$ model in detail, including general discussions of our new mechanism. We shall also study a new model called $U(1)$ model in order to see how the mechanism works and to study new features different from those of the $Z_2$ model.

In the section 2 we will discuss a key idea of our mechanism and study a general structure of superpotentials realizing our new mechanism. In the sections 3 and 4 we will study two minimal models called $Z_2$ and $U(1)$ models in detail. The vacuum structures are determined and the mass spectrum of the models appeared in reduced dimensions are analyzed. In particular, we pay attention to the Nambu-Goldstone bosons and fermions associated with the breakdown of global symmetries, such as the translational invariance for the compactified direction and the supersymmetry. The section 5 is devoted to conclusions and discussions. We summarize mathematical tools in Appendices A and B which are used in the studies of the sections 3 and 4.

2 GENERAL DISCUSSION OF OUR MECHANISM

In this section we shall discuss a basic idea of our SUSY breaking mechanism, and clarify how to construct supersymmetric models in which our mechanism works. Two minimal models, called $Z_2$ and $U(1)$ models, will be given explicitly.

Let $W(\Phi)$ be a superpotential consisting of the chiral superfields $\Phi_j$. The scalar potential is then given by

$$V(A) = \sum_j |F_j|^2 = \sum_j \left| \frac{\partial W(A)}{\partial A_j} \right|^2,$$

where $A_j(F_j)$ denotes the lowest (highest) component of $\Phi_j$. The supersymmetry would be unbroken if there are solutions to the $F$-term conditions

$$-F^*_j = \frac{\partial W(A)}{\partial A_j} \bigg|_{A_k = \bar{A}_k} = 0 \quad \text{for all } j,$$

since the solutions lead to $V(\bar{A}) = 0$. Our idea of the supersymmetry breaking is simple: We impose nontrivial boundary conditions on the superfields for compactified directions. They must be consistent with the single-valuedness of the Lagrangian but inconsistent with the $F$-term conditions (3). Then, any solution to the $F$-term conditions will not be realized as a vacuum configuration of the model. Thus, we expect that $V(\langle A \rangle) > 0$ because $\langle A_j \rangle \neq \bar{A}_j$ and that the supersymmetry is broken spontaneously. We should notice that our mechanism is obviously different from the O’Raifeartaigh one [9] since solutions to the $F$-term conditions are allowed to exist in our mechanism.

In order to realize such the mechanism, let us consider the theory in space-time with one of the space coordinates, say, $y$ being compactified on a circle $S^1$ whose radius is $R$. Since $S^1$ is multiply connected, we have to specify boundary conditions for the $S^1$ direction. Let us impose nontrivial boundary conditions on superfields defined by

$$\Phi_j(x^\mu, y + 2\pi R) = e^{2\pi i \alpha_j} \Phi_j(x^\mu, y).$$

The phase $\alpha_j$ should be the one such that the Lagrangian density is single-valued, i.e.

$$\mathcal{L}(x^\mu, y + 2\pi R) = \mathcal{L}(x^\mu, y).$$
In other words, the phase has to be one of the degrees of freedom of symmetries of the theory. Suppose that $A_j$, which is a solution to the $F$-term conditions, is a nonzero constant for some $j$. It is easy to see that if

$$e^{2\pi i \alpha_j} \neq 1,$$

then, the vacuum expectation value $\langle A_j \rangle$ is strictly forbidden to take the nonzero constant $A_j$ because it is inconsistent with the boundary condition (3). In this way, our idea is realized by the mechanism that the nontrivial boundary conditions imposed on the fields play a role to force the vacuum expectation values of the fields not to take the solutions to the $F$-term conditions. The mechanism clearly does not depend on the space-time dimensions, so that it would work in any dimensions.

Note that the above result does not always lead us to a conclusion that $\langle A_j \rangle = 0$, which is always consistent with (3). Certainly, if the translational invariance for the $S^1$ direction is not broken, $\langle A_j \rangle$ has to vanish because of (3) with (5). If the translational invariance for the $S^1$ direction is broken, however, vacuum expectation values will no longer be constants and some of $\langle A_j \rangle$’s can depend on the coordinate of the compactified space as an energetically favourable configuration. One should include the contributions from the kinetic terms of the scalar fields to the scalar potential in order to find the true vacuum configuration.

Here, it is worth making comments on the nontrivial boundary conditions (3) imposed on the fields $\Phi_j$. The boundary conditions have to be one of the symmetry degrees of freedom of the theory in order to maintain the single-valuedness of the Lagrangian. One might think that the $U(1)_R$ symmetry, which is specific to the supersymmetric theory, is available. It is well-known that the boundary condition associated with the $U(1)_R$ symmetry explicitly breaks the supersymmetry [3, 4]. In this paper we do not consider this case, though it is known to have attractive features [5]. The boundary conditions (3) we consider in this paper are consistent with the supersymmetry. When space is multiply connected and hence has nontrivial topology, the configuration space can also have nontrivial topology. Then, it turns out that undetermined parameters like $\alpha_j$’s in (3) inevitably appear as the ambiguity in quantizing the theory on such the space and the boundary conditions are twisted according to the parameters [8]. Therefore, even though we state to impose the nontrivial boundary conditions on the fields here, such the boundary conditions are firm consequences when one studies the theory on multiply connected space.

Now, let us clarify a general structure of superpotentials in which our mechanism works. Let us restrict ourself to 4-dimensions for the illustration. According to the previous discussions, we see that the scalar potential has to be of the usual Higgs-type to exclude the origin of $A_j = 0$ as a supersymmetric vacuum. So, we require the superpotential to satisfy

$$\frac{\partial W(A)}{\partial A_j} \bigg|_{A_k=0} \neq 0 \quad \text{for some } j.$$  

(6)

It immediately follows that $W(A)$ should contain linear terms with respect to $A_j$, i.e.

$$W(A) = \sum_j \lambda_j A_j + \cdots,$$  

(7)

where some of $\lambda_j$ are nonzero constants. First, let us consider a model consisting of only one chiral superfield. From (6), the superpotential is of the form

$$W(\Phi) = \lambda \Phi + f(\Phi),$$  

(8)
where \( f(\Phi) \) stands for higher order terms. We can immediately conclude that our mechanism does not work for the superpotential (8) due to the linear term \( \Phi \). This is because the superpotential cannot possess any symmetries which are consistent with the nontrivial boundary condition of the field. The phase has to be trivial \( e^{2\pi i \alpha} = 1 \) in this case in order to keep the single-valuedness of the Lagrangian.

Let us next consider a model with two chiral superfields \( \Phi_0 \) and \( \Phi_1 \). The superpotential is written, taking account of the requirement (3), as

\[
W(\Phi_0, \Phi_1) = \lambda_0 \Phi_0 + \lambda_1 \Phi_1 + g(\Phi_0, \Phi_1),
\]

where \( g(\Phi_0, \Phi_1) \) stands for higher order interactions. If both \( \lambda_0 \) and \( \lambda_1 \) are nonzero, the superpotential does not again possess any symmetries, so that we cannot impose nontrivial boundary conditions consistent with the single-valuedness of the Lagrangian. Therefore, one of them, say \( \lambda_1 \), has to be zero, i.e.

\[
W(\Phi_0, \Phi_1) = \lambda_0 \Phi_0 + g(\Phi_0, \Phi_1).
\]

In order to maintain the single-valuedness of the Lagrangian, \( \Phi_0 \) has to obey a periodic boundary condition \( \Phi_0(x^\mu, y + 2\pi R) = \Phi_0(x^\mu, y) \). Thanks to the two chiral superfields, \( \Phi_1 \) can have a nontrivial boundary condition. We have arrived at a possible form of the superpotential, written up to cubic interactions, as

\[
W(\Phi_0, \Phi_1) = \lambda_0 \Phi_0 + m_0(\Phi_0)^2 + m_1(\Phi_1)^2 + m_0 \Phi_0 \Phi_1 + g_1(\Phi_0)^3 + g_2(\Phi_0)^2 \Phi_1 + g_3 \Phi_0(\Phi_1)^2 + g_4(\Phi_1)^3.
\]

This superpotential can have a \( Z_2 \) or \( Z_3 \) symmetry, depending on how we choose the parameters \( (\lambda_0, m_0, \cdots, g_1) \). Boundary conditions of the fields have to be consistent with the symmetry. The superpotential with the \( Z_2 \) symmetry is written as

\[
W(\Phi_0, \Phi_1)_{Z_2} = \lambda_0 \Phi_0 + m_0(\Phi_0)^2 + m_1(\Phi_1)^2 + g_1(\Phi_0)^3 + g_3 \Phi_0(\Phi_1)^2,
\]

which is invariant under \( \Phi_0 \rightarrow \Phi_0 \) and \( \Phi_1 \rightarrow -\Phi_1 \). On the other hand, the superpotential

\[
W(\Phi_0, \Phi_1)_{Z_3} = \lambda_0 \Phi_0 + m_0(\Phi_0)^2 + g_1(\Phi_0)^3 + g_4(\Phi_1)^3.
\]

has the \( Z_3 \) symmetry whose transformation is given by \( \Phi_0 \rightarrow \Phi_0 \) and \( \Phi_1 \rightarrow e^{2\pi i / 3} \Phi_1 \).

We shall first concentrate on the \( Z_2 \) symmetric case and restrict further the form of the superpotential (12) in order for our mechanism to work. Since \( W(\Phi_0, \Phi_1)_{Z_2} \) has the \( Z_2 \) symmetry we can impose the nontrivial boundary conditions on \( \Phi_1 \) defined by

\[
\Phi_0(x^\mu, y + 2\pi R) = +\Phi_0(x^\mu, y), \quad \Phi_1(x^\mu, y + 2\pi R) = -\Phi_1(x^\mu, y).
\]

The next task we have to do is to restrict the form of the superpotential (12) in such a way that there exists a nontrivial solution to the \( F \)-term conditions

\[
\frac{\partial W(A_0, A_1)}{\partial A_j} \bigg|_{A_0 = \tilde{A}_0, A_1 = \tilde{A}_1} = 0, \quad \text{with} \quad \tilde{A}_1 \neq 0.
\]

The \( F \)-term conditions for the case are

\[
0 = \frac{\partial W_{Z_2}}{\partial A_0} \bigg|_{A_0 = \tilde{A}_0, A_1 = \tilde{A}_1} = \lambda_0 + 2m_0 \tilde{A}_0 + 3g_1 \tilde{A}_0^2 + g_3 \tilde{A}_1^2,
\]

\[
0 = \frac{\partial W_{Z_2}}{\partial A_1} \bigg|_{A_0 = \tilde{A}_0, A_1 = \tilde{A}_1} = 2\tilde{A}_1 (m_1 + g_3 \tilde{A}_0).
\]
It is easy to see that the condition $\bar{A}_1 \neq 0$ requires at least $g_3 \neq 0$. Then, the second condition in (14) leads to

$$\bar{A}_1 = 0 \quad \text{or} \quad m_1 + g_3 \bar{A}_0 = 0.$$  \hspace{1cm} (17)

The first solution $\bar{A}_1 = 0$, which is actually unwanted, can be excluded by requiring $m_0 = g_1 = 0$ and $\lambda_0 \neq 0$. With these choices of the parameters, $\bar{A}_1 = 0$ is no longer a solution of (16) and the second solution in (17) is selected. As the result, the solutions of the $F$-term conditions are given by $(\bar{A}_0, \bar{A}_1) = (-m_1 g_3, \pm \sqrt{-\lambda_0 g_3})$. The solutions, however, cannot be realized as vacuum configurations because they are inconsistent with the boundary conditions (14). Thus, we have found the superpotential which satisfies our criteria (6) and (15) to realize our mechanism to be of the form

$$W(\Phi_0, \Phi_1)_{Z_2} = \lambda_0 \Phi_0 + g_3 \Phi_0 (\Phi_1)^2.$$  \hspace{1cm} (18)

Let us note that by shifting $\Phi_0$ by $-m_1/g_3$, we can also rewrite (18) as

$$W(\Phi_0, \Phi_1)_{Z_2} = \lambda_0 \Phi_0 + g_3 \Phi_0 (\Phi_1)^2$$  \hspace{1cm} (19)

up to constant. This is a minimal model to realize our mechanism. Let us note that the superpotential (19) has another global symmetry $U(1)_R$ in addition to the $Z_2$ symmetry. As we stated earlier, we do not consider any boundary conditions associated with the $U(1)_R$ symmetry degrees of freedom in this paper, concerning about the boundary conditions.

It is easy to see that the scalar potential followed from the superpotential (18) or (19) is the usual Higgs-type potential. It is essential for our mechanism that the scalar potential is the Higgs-type to have the minimum of the potential away from the origin and that nonvanishing $\bar{A}_1$, which is any solution to the $F$-term conditions, is inconsistent with the boundary condition of $A_1$.

Let us come back to the superpotential (13) with the $Z_3$ symmetry. In this case, $\Phi_0$ decouples with $\Phi_1$ completely. It is easy to see that the $F$-term condition for $A_1$ inevitably gives us $\bar{A}_1 = 0$ for any choice of the parameters in the superpotential, so that our mechanism does not work in this case.

We can also construct a supersymmetric model with a continuous symmetry. In this case the minimal model is given by introducing at least three chiral superfields. The superpotential with a global $U(1)$ symmetry is given by

$$W(\Phi_0, \Phi_1)_{U(1)} = \lambda_0 \Phi_0 + g \Phi_0 \Phi_+ \Phi_- + m \Phi_+ \Phi_-$$  \hspace{1cm} (20)

with the boundary conditions

$$\Phi_0(x^\mu, y + 2\pi R) = \Phi_0(x^\mu, y), \quad \Phi_\pm(x^\mu, y + 2\pi R) = e^{\pm 2\pi i \alpha} \Phi_\pm(x^\mu, y).$$  \hspace{1cm} (21)

The $U(1)$ charges of $\Phi_0$ and $\Phi_\pm$ are defined by 0 and $\pm 1$, respectively. By shifting $\Phi_0$ by $-m/g$ in (21), it is also written as

$$W(\Phi_0, \Phi_1)_{U(1)} = \lambda_0 \Phi_0 + g \Phi_0 \Phi_+ \Phi_-$$  \hspace{1cm} (22)

up to constant. Here, this superpotential has the $U(1)_R$ symmetry, in addition to the global $U(1)$ symmetry.

Through the discussions we have made, we finally find the conditions on the superpotential to be satisfied in order for our mechanism to work:
The origin of \( A_j = 0 \) is not the supersymmetric vacuum of the potential \( V(A) \). In other words, \( A_j = 0 \) for all \( j \) is not a solution to the \( F \)-term conditions.

Let \( \bar{A}_j \) be a configuration which satisfies the \( F \)-term conditions. Some of \( A_j \) with \( \bar{A}_j \neq 0 \) have to be non-singlets for some global symmetries of the theory.

A key point of our mechanism is that the non-singlet fields \( A_j \) with \( \bar{A}_j \neq 0 \) are required to obey nontrivial boundary conditions consistent with the single-valuedness of the Lagrangian. It should be emphasized that solutions of the \( F \)-term conditions always exist in our case but are not realized as vacuum configurations due to the nontrivial boundary conditions. Thus, our mechanism is different from other ones of the spontaneously SUSY breaking, e.g. O’Raifeartaigh models, in which there are no consistent solutions of the \( F \)-term conditions. In the following two sections, we shall study the \( Z_2 \) and \( U(1) \) models whose superpotentials are given by (18) (or (19)) and (20) (or (22)), respectively, in detail.

3 THE \( Z_2 \) MODEL

In this section we study the \( Z_2 \) model obtained in the previous section. In particular, we study the vacuum structure and the mass spectrum for bosons and fermions of the model.

3.1 Vacuum configuration

The superpotential of the \( Z_2 \) model is given by

\[
W(\Phi_0, \Phi_1) = g\Phi_0 \left[ \frac{\Lambda^2}{g^2} - \frac{1}{2}(\Phi_1)^2 \right] + \frac{\mu}{2}(\Phi_1)^2.
\]  

(23)

We have slightly changed notations of the superpotential (18) in the previous section. The parameters \( g \) and \( \Lambda \) are complex in general, but their phases can be absorbed into the redefinitions of the fields \( \Phi_0 \) and \( \Phi_1 \). Thus, we take \( g \) and \( \Lambda \) to be real without loss of generality. The model at hand is invariant under a discrete \( Z_2 \) transformation \( \Phi_0 \to \Phi_0, \Phi_1 \to -\Phi_1 \). As discussed in the previous section, the existence of the global symmetry is crucial for our mechanism because otherwise nontrivial boundary conditions cannot be imposed on the fields. The scalar potential is given by

\[
V(A_0, A_1) \equiv |F_0|^2 + |F_1|^2 = \left| \frac{\Lambda^2}{g} - \frac{g}{2}(A_1)^2 \right|^2 + |gA_0 - \mu|^2|A_1|^2.
\]

(24)

It is easy to solve the \( F \)-term conditions, and the solutions are given by

\[
\bar{A}_0 = \mu/g, \quad \bar{A}_1 = \pm \sqrt{2}\Lambda/g,
\]

(25)

at which \( V(\bar{A}_0, \bar{A}_1) \) vanishes. Therefore, one might think that the supersymmetry is unbroken while the \( Z_2 \) symmetry is broken spontaneously. This is, however, a hasty conclusion, as we will see below.

Let us consider the model on \( M^3 \otimes S^1 \). We shall denote the coordinates of \( M^3 \) and \( S^1 \) by \( x^\mu (\mu = 0, 1, 2) \) and \( y \), respectively. Since \( S^1 \) is multiply connected, we can impose the nontrivial boundary conditions on the superfields

\[
\Phi_0(x^\mu, y + 2\pi R) = +\Phi_0(x^\mu, y), \quad \Phi_1(x^\mu, y + 2\pi R) = -\Phi_1(x^\mu, y).
\]

(26)
It follows that the vacuum expectation value of $A_1$ (and also the auxiliary field $F_1$) is forced not to take a nonzero constant, i.e. $\langle A_1(x^\mu, y) \rangle \neq \text{nonzero constant}$. Therefore, the solutions (23) to the $F$-term conditions are not consistent with the boundary conditions (26) and hence are not realized as vacuum configurations. Since any supersymmetric vacuum has to satisfy the $F$-term conditions, if any, and since all the solution to the $F$-term conditions are excluded from vacuum configurations by the boundary conditions, the supersymmetry is force to be broken spontaneously, so that our mechanism does work in this model.

A configuration consistent with the boundary conditions (26) is $\langle A_0 \rangle = \mu/g$, $\langle A_1 \rangle = 0$. One may wonder whether this is a vacuum configuration or not. The configuration seems unstable because the scalar potential, which is now the Higgs-type, has a negative curvature along the $A_1$ direction at $\langle A_0 \rangle = \mu/g$. When we search vacuum configurations, we usually minimize the scalar potential. This will not, however, lead to the true vacuum configuration in the model we are considering. Since one of the space coordinates is compactified on $S^1$, the original fields may be decomposed into Kaluza-Klein modes for the $S^1$ direction. The kinetic term for the $S^1$ direction can be regarded as mass terms for the Kaluza-Klein modes from the 3-dimensional point of view. An important observation is that all the Kaluza-Klein modes for the field $A_1$ acquire nonvanishing masses since $A_1(y)$ obeys the antiperiodic boundary condition and has no zero mode. This suggests that the kinetic term of $A_1$ for the $S^1$ direction could drastically change the shape of the potential (in the 3-dimensional point of view) near the origin $A_1 = 0$ (recall that $V(A_0, A_1)$ is the Higgs-type). Therefore, we should study the scalar potential in which the contribution from the kinetic terms for the $S^1$ direction is taken into account in order to find the true vacuum configuration.

According to the above discussions, the vacuum configuration will be obtained by solving a minimization problem of the “energy” functional

$$
\mathcal{E}[A_0, A_1; R] \equiv \int_0^{2\pi R} dy \left[ \left| \frac{dA_0}{dy} \right|^2 + \left| \frac{dA_1}{dy} \right|^2 + V(A_0, A_1) \right]
$$

with the boundary conditions

$$
A_0(y + 2\pi R) = +A_0(y), \quad A_1(y + 2\pi R) = -A_1(y).
$$

We assume that the translational invariance for the 3-dimensional Minkowski space-time is not broken, so that we ignore the $x^\mu$-dependence of $A_0$ and $A_1$.

The detailed discussions on the minimization of the “energy” functional (27) are made in the Appendix A. Let us present the results here. We obtain the vacuum configuration as

$$
\begin{align*}
\text{for } R \leq R^* & \equiv \frac{1}{2\Lambda} \quad \begin{cases} 
\langle A_0(x^\mu, y) \rangle = \text{arbitrary complex constant}, \\
\langle A_1(x^\mu, y) \rangle = 0,
\end{cases} \\
\text{for } R > R^* & \equiv \frac{1}{2\Lambda} \quad \begin{cases} 
\langle A_0(x^\mu, y) \rangle = \frac{\mu}{g}, \\
\langle A_1(x^\mu, y) \rangle = \frac{2k\omega}{g} \text{sn}(\omega(y - y_0), k).
\end{cases}
\end{align*}
$$

Here, the parameter $k$ ($0 \leq k < 1$) and $y_0$ are the integration constants and $\omega \equiv \frac{\Lambda}{\sqrt{1+k^2}}$. The sn($u, k$) is the Jacobi elliptic function whose period is $4K(k)$, where $K(k)$ denotes the complete elliptic function of the first kind. Since the integration constant $y_0$, which reflects the translational invariance of the equation of motion, is irrelevant, so that hereafter we set $y_0 = 0$. 
We observe a remarkable feature that in our model there are two phases, depending on the magnitude of the radius of $S^1$. We see that the vacuum structure of the model drastically changes at the critical radius $R^*$. For $R \leq R^*$, the translational invariance for the $S^1$ direction and the $Z_2$ symmetry are unbroken, while for $R > R^*$, since $\langle A_i(x^\mu, y) \rangle$ depends on the coordinate $y$ of the extra dimension, the translational invariance for the $S^1$ direction is broken spontaneously with the breakdown of the $Z_2$ symmetry. The vacuum energy is nonzero for both $R \leq R^*$ and $R > R^*$, so that the SUSY is broken spontaneously for both regions.

As discussed in the Appendix A, the parameter $k$ ($0 \leq k < 1$) is determined through the condition

$$2\pi R = \frac{2}{\Lambda} \sqrt{1 + k^2} K(k)$$

in order for (31) to satisfy the boundary conditions (28). Knowing that $K(k)$ is a monotonically increasing function of $k$, we immediately obtain the critical radius $R^* \equiv 1/2\Lambda$, which corresponds to $k = 0$. Though we present the detailed discussion on the critical radius in the Appendix A, it may be instructive to show how the critical radius $R^*$ appears in the 3-dimensional point of view. To this end, we may expand $A_0(x^\mu, y)$ and $A_1(x^\mu, y)$ in the Fourier modes for the $y$ direction, according to the boundary conditions (28). As found in the next subsection (see also the Table III-1), the squared mass eigenvalues for the Fourier modes of $A_0$ are given by $(\pi/ R)^2$ and those for $A_1$ by $[\mu - g \langle A_0 \rangle]^2 \pm \Lambda^2 + (n \pi + \pi/ R)^2$ with $n$ being integers. It follows that that for $R \leq \frac{1}{2\Lambda}$ all the squared masses are positive semi-definite and hence the vacuum configuration (29) is stable at least locally, while for $R > \frac{1}{2\Lambda}$ negative squared masses would appear with $\langle A_0 \rangle = \frac{\mu}{g}$. This implies that the configuration $\langle A_1 \rangle = 0$ is unstable and can no longer be a vacuum for $R > \frac{1}{2\Lambda}$. The phase transition should occur at $R = R^* = \frac{1}{2\Lambda}$. As proved in the Appendix A, the vacuum configuration becomes stable by having the coordinate dependence such as (31).

### 3.2 Mass spectrum for $R \leq R^*$

Once the vacuum configurations are determined, we shall next analyze the mass spectrum of the model. We shall employ the standard prescription based on perturbation theory to obtain the mass spectrum. We expand fields around the vacuum configuration and take the quadratic terms with respect to the fluctuations. The fluctuating fields have to be expanded appropriately in modes of oscillations about the vacuum configuration. Since we are interested in the mass spectrum appeared in the 3-dimensional Minkowski space-time, we integrate over the coordinate $y$ of the extra dimension. In this way we can obtain the mass spectrum for the bosons and fermions in the model.

Let us first compute the bosonic mass spectrum. The vacuum configuration is given by (29) for $R \leq R^*$. The bosonic mass spectrum can be read from the quadratic terms of the scalar potential, including the kinetic terms of the scalar fields for the $S^1$ direction. It is given by

$$L^{(2)}_{B(R \leq R^*)} = \int_0^{2\pi R} dy \left[ - \sum_{i=0, 1} \partial_M A_i^* \partial^M A_i + \frac{\Lambda^2}{2} \left( (A_1)^2 + (A_1^*)^2 \right) - |M|^2 |A_1|^2 \right],$$

where $M \equiv \mu - g \langle A_0 \rangle$.

We find that it is convenient to introduce 4 real scalar fields defined by

$$A_i(x^\mu, y) \equiv \frac{1}{\sqrt{2}} (a_i(x^\mu, y) + ib_i(x^\mu, y)), \quad (i = 0, 1).$$

(33)
Moreover, in order to obtain the mass spectrum appeared in 3-dimensions, we expand $a_0$, $b_0$, $a_1$ and $b_1$ in Fourier series for the $S^1$ direction, according to the boundary conditions \((28)\), as

\[
a_0(x, y) = \frac{1}{\sqrt{2\pi R}} a_0^{(0)}(x) + \frac{1}{\sqrt{\pi R}} \sum_{n \in \mathbb{Z}^+} \left[ \alpha_0^{c(n)}(x) \cos\left(\frac{ny}{R}\right) + \alpha_0^{s(n)}(x) \sin\left(\frac{ny}{R}\right) \right],
\]

\[
b_0(x, y) = \frac{1}{\sqrt{2\pi R}} b_0^{(0)}(x) + \frac{1}{\sqrt{\pi R}} \sum_{n \in \mathbb{Z}^+} \left[ b_0^{c(n)}(x) \cos\left(\frac{ny}{R}\right) + b_0^{s(n)}(x) \sin\left(\frac{ny}{R}\right) \right],
\]

\[
a_1(x, y) = \frac{1}{\sqrt{\pi R}} \sum_{l \in \mathbb{Z}^+} \left[ a_1^{c(l)}(x) \cos\left(\frac{ly}{R}\right) + a_1^{s(l)}(x) \sin\left(\frac{ly}{R}\right) \right],
\]

\[
b_1(x, y) = \frac{1}{\sqrt{\pi R}} \sum_{l \in \mathbb{Z}^+} \left[ b_1^{c(l)}(x) \cos\left(\frac{ly}{R}\right) + b_1^{s(l)}(x) \sin\left(\frac{ly}{R}\right) \right].
\]

Note that the Fourier modes of the fields $a_1(x)$ and $b_1(x)$ cannot have zero modes due to the boundary condition \((28)\). Inserting these expressions into \((12)\) and integrating over the coordinate $y$, we obtain the mass spectrum of the bosons. The result is summarized in the Table III-1.

The computations of the fermion mass spectrum can be done just as in the bosonic case. The mass spectrum for the fermions can be read from the quadratic terms given by

\[
\mathcal{L}_{F(R \leq R')}^{(2)} = \int_0^{2\pi R} dy \left[ -i \sum_{i=0,1} \bar{\psi}_i \gamma^\mu \partial_\mu \psi_i - \frac{1}{2} M \psi_1 \bar{\psi}_1 - \frac{1}{2} M^* \bar{\psi}_1 \psi_1 \right].
\]

We are interested in the spectrum appeared in 3-dimensions. We find that it is convenient to introduce the Dirac spinors in 3-dimensions according to our prescription \((160)\) in the Appendix B and to decompose further the Dirac spinors into the Majorana spinors \((161)\) in 3-dimensions. In the Appendix B, we present details of the decomposition of the 4-dimensional spinors and gamma matrices into the 3-dimensional ones. According to these, we obtain

\[
\mathcal{L}_{F(R \leq R')}^{(2)} = \int_0^{2\pi R} dy \left[ -\frac{i}{2} \sum_{i=0,1} \bar{\chi}_i \gamma^\mu \partial_\mu \chi_i - \frac{i}{2} \sum_{i=0,1} \bar{\rho}_i \gamma^\mu \partial_\mu \rho_i \right.
\]

\[
- \frac{1}{2} \sum_{i=0,1} \bar{\chi}_i \partial_\mu \rho_i + \frac{1}{2} \sum_{i=0,1} \bar{\rho}_i \partial_\mu \chi_i
\]

\[
+ \frac{i}{4} (M - M^*) (\bar{\chi}_1 \chi_1 - \bar{\rho}_1 \rho_1) - \frac{1}{4} (M + M^*) (\bar{\chi}_1 \rho_1 + \bar{\rho}_1 \chi_1)),
\]

where $\chi_0$, $\rho_0$, $\chi_1$ and $\rho_1$ are the Majorana spinors in 3-dimensions. We expand $\chi_0$, $\rho_0$, $\chi_1$ and $\rho_1$ in Fourier series for the $S^1$ direction as

\[
\chi_0(x, y) = \frac{1}{\sqrt{2\pi R}} \chi_0^{(0)}(x) + \frac{1}{\sqrt{\pi R}} \sum_{n \in \mathbb{Z}^+} \left[ \chi_0^{c(n)}(x) \cos\left(\frac{ny}{R}\right) + \chi_0^{s(n)}(x) \sin\left(\frac{ny}{R}\right) \right],
\]

\[
\rho_0(x, y) = \frac{1}{\sqrt{2\pi R}} \rho_0^{(0)}(x) + \frac{1}{\sqrt{\pi R}} \sum_{n \in \mathbb{Z}^+} \left[ \rho_0^{c(n)}(x) \cos\left(\frac{ny}{R}\right) + \rho_0^{s(n)}(x) \sin\left(\frac{ny}{R}\right) \right],
\]

\[
\chi_1(x, y) = \frac{1}{\sqrt{\pi R}} \sum_{l \in \mathbb{Z}^+} \left[ \chi_1^{c(l)}(x) \cos\left(\frac{ly}{R}\right) + \chi_1^{s(l)}(x) \sin\left(\frac{ly}{R}\right) \right],
\]

\[
\rho_1(x, y) = \frac{1}{\sqrt{\pi R}} \sum_{l \in \mathbb{Z}^+} \left[ \rho_1^{c(l)}(x) \cos\left(\frac{ly}{R}\right) + \rho_1^{s(l)}(x) \sin\left(\frac{ly}{R}\right) \right].
\]
Let us note again that because of the boundary conditions (26) there are no zero modes in $\chi_1$ and $\rho_1$. Inserting these expressions into (39) and performing the $y$ integration, we obtain

$$\mathcal{L}^{(2)}_{F(R\leq R^*)} = -\frac{i}{2} \bar{\chi}_0^{(0)} \gamma^\mu \partial_\mu \chi_0^{(0)} - \frac{i}{2} \bar{\rho}_0^{(0)} \gamma^\mu \partial_\mu \rho_0^{(0)} + \frac{1}{2} \sum_{n \in \mathbb{Z} > 0} \bar{\Psi}_0^{(n)} (-i \gamma^\mu \partial_\mu 1 + \mathcal{M}_0^{(n)} ) \Psi_0^{(n)},$$

where

$$\mathcal{M}_0^{(n)} = \begin{pmatrix} 0 & 0 & 0 & -n/R \\ 0 & 0 & n/R & 0 \\ 0 & n/R & 0 & 0 \\ -n/R & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{M}_1^{(l)} = \begin{pmatrix} \frac{i}{2} (M - M^*) & 0 & -\frac{i}{2} (M + M^*) & -l/R \\ 0 & \frac{i}{2} (M - M^*) & l/R & -\frac{i}{2} (M + M^*) \\ -\frac{i}{2} (M + M^*) & l/R & -\frac{i}{2} (M - M^*) & 0 \\ -l/R & -\frac{i}{2} (M + M^*) & 0 & -\frac{i}{2} (M - M^*) \end{pmatrix},$$

and $\Psi_0^{(n)} \equiv (\chi_0^{c(n)}, \chi_0^{s(n)}, \rho_0^{c(n)}, \rho_0^{s(n)})^T, \Psi_1^{(l)} \equiv (\chi_1^{c(l)}, \chi_1^{s(l)}, \rho_1^{c(l)}, \rho_1^{s(l)})^T$. It may be sufficient to diagonalize, instead of $\mathcal{M}_0^{(n)}$ and $\mathcal{M}_1^{(l)}$, the square of the each mass matrix

$$(\mathcal{M}_0^{(n)})^2 = \left( \frac{n}{R} \right)^2 1_{4 \times 4}, \quad (\mathcal{M}_1^{(l)})^2 = \left( |M|^2 + \left( \frac{l}{R} \right)^2 \right) 1_{4 \times 4}.\quad (47)$$

We summarize the fermion mass spectrum in the Table III-2.

### 3.3 Analysis of the mass spectrum for $R \leq R^*$

Let us make several comments on the boson and fermion mass spectra obtained in the previous subsection. It is easy to see from the Table III-1 that the boson mass spectrum is positive semi-definite for $R \leq R^*$. The vacuum configuration we find is stable for $R \leq R^*$. The second comment is that the SUSY breaking scale is found, from the mass splitting, to be of order $\Lambda$. This is also understood from the fact that the vacuum energy density for the vacuum configuration (29) is of order $\Lambda$. The next comment is that we observe that the modes $A_0^{(0)} \sim a_0^{(0)} + ib_0^{(0)}$ are massless. Its physical interpretation is that the massless modes correspond to the existence of a flat direction of the scalar potential along the $A_0$ direction at the tree level. This interpretation is modified slightly if we start form the superpotential (19), instead of (18), which has an extra $U(1)_R$ symmetry. The vacuum configuration (29) breaks the $U(1)_R$ symmetry spontaneously because $A_0$ carries the $U(1)_R$ charge. We expect a Nambu-Goldstone boson associated with the breakdown of the $U(1)_R$ symmetry to appear. A part of $A_0^{(0)}$ will correspond to this massless mode and the other part of $A_0^{(0)}$ is also guaranteed to be massless by the flatness of the potential at the tree level. The last comment is that at $\langle A_0 \rangle = \mu/g$, which yields $M = 0$, we find that the squared masses of some of the bosonic modes $a_1^{c(l)}$ and $a_1^{s(l)}$ become negative for $R > R^* = \frac{1}{2\Lambda}$. This suggests that the vacuum configuration becomes unstable for $R > R^*$ and that a phase transition occurs at $R = R^*$, as discussed in the previous subsection.
Let us next make comments on the fermion mass spectrum. We can interpret \( \psi_0^{(0)} \sim \chi_0^{(0)} + i \rho_0^{(0)} \) as the Nambu-Goldstone fermions associated with the spontaneous SUSY breaking. In order to confirm this interpretation, let us consider the infinitesimal SUSY transformations

\[
\delta_S A_0(x, y) = \sqrt{2} \xi_0 \psi_0(x, y), \quad \delta_S A_1(x, y) = \sqrt{2} \xi_1 \psi_1(x, y),
\]

\[
\delta_S \psi_0(x, y) = i \sqrt{2} (\sigma^M \xi) \partial_M A_0(x, y) - \sqrt{2} \xi \left( \frac{\Lambda^2}{g} - \frac{g}{2} (A_1^*(x, y))^2 \right),
\]

\[
\delta_S \psi_1(x, y) = i \sqrt{2} (\sigma^M \xi) \partial_M A_1(x, y) - \sqrt{2} \xi (\mu - g A_0^*(x, y)) A_1^*(x, y).
\]

Nonvanishing vacuum expectation values of \( \delta_S \psi_0(x, y) \) and \( \delta_S \psi_1(x, y) \) are a signal of the SUSY breaking. They become

\[
\langle \delta_S \psi_0(x, y) \rangle = -\sqrt{2} \frac{\Lambda^2}{g} \xi, \quad \langle \delta_S \psi_1(x, y) \rangle = 0
\]

for the vacuum configuration (29). Since \( \langle \psi_0(x, y) \rangle \neq 0 \) for any nonvanishing \( \xi \), the supersymmetry is completely broken spontaneously. No supersymmetry is left in 3-dimensions. Moreover, it follows from (50) that the Nambu-Goldstone modes associated with the supersymmetry breaking should be constant modes of \( \psi_0 \) in Fourier series for the \( S^1 \) direction, i.e. they are \( \chi_0^{(0)} \) and \( \rho_0^{(0)} \). Hence, we confirm the interpretation.

Let us finally note that the mass spectra for the bosons and fermions satisfy the relations

\[
m^2_{\alpha_0^{(0)}} + m^2_{\beta_0^{(0)}} = m^2_{\chi_0^{(0)}} + m^2_{\rho_0^{(0)}},
\]

\[
m^2_{\alpha_1^{(0)}} + m^2_{\alpha_0^{(0)}} + m^2_{\beta_0^{(0)}} = m^2_{\chi_0^{(0)}} + m^2_{\chi_0^{(0)}} + m^2_{\rho_0^{(0)}} + m^2_{\rho_0^{(0)}},
\]

\[
m^2_{\alpha_1^{(l)}} + m^2_{\beta_1^{(l)}} = m^2_{\chi_1^{(l)}} + m^2_{\chi_1^{(l)}} + m^2_{\rho_1^{(l)}} + m^2_{\rho_1^{(l)}},
\]

where \( n \in \mathbb{Z} > 0 \) and \( l \in \mathbb{Z} + \frac{1}{2} > 0 \). These relations mean that the sum of the squared masses of the bosons for each mode are exactly equal to that of the fermions. This is known as the supertrace formula which is an immediate consequence of the spontaneous SUSY breaking [17], but the relations (51) are a stronger version of it. The formula given in Ref. [19] implies the equivalence between the sum of the squared masses of all the bosonic states and that of all the fermionic states (in the 3-dimensional point of view), while the relations (51) hold for each mode of the bosonic and fermionic states. Before closing this subsection, we shall summarize the mass spectra of the bosons and fermions for \( R \leq R^* \) in the Figure 1.

### 3.4 Mass spectrum for \( R > R^* \)

In this subsection, we shall compute the mass spectrum for \( R > R^* \). As we have already mentioned, the vacuum configuration is given by (30) for \( R > R^* \). The vacuum configuration is highly nontrivial in the sense that it is given by the Jacobi elliptic function which depends on the coordinate \( y \) of the extra dimension and the parameter \( k \). It means that the vacuum configuration has the \( R \)-dependence through the condition (31). This also reflects the mass spectrum, so that it is interesting to study the \( R \)-dependence of the mass spectrum for \( R > R^* \).
3.4.1 Boson sector

Let us first analyze the mass spectrum of the bosons. The quadratic terms are given, in terms of the four real scalar fields (52), by

\[
\mathcal{L}^{(2)}_{B(R>R^*)} = \int_0^{2\pi R} dy \left[ -\frac{1}{2} \partial_\mu a_0 \partial^\mu a_0 - \frac{1}{2} |g(A_1(y))|^2 (a_0)^2 \right.
\]

\[
- \frac{1}{2} \partial_\mu b_0 \partial^\mu b_0 - \frac{1}{2} |g(A_1(y))|^2 (b_0)^2 \right.
\]

\[
- \frac{1}{2} \partial_\mu a_1 \partial^\mu a_1 - \frac{1}{2} \left( \frac{3}{2} |g(A_1(y))|^2 - \Lambda^2 \right) (a_1)^2 \right.
\]

\[
- \frac{1}{2} \partial_\mu b_1 \partial^\mu b_1 - \frac{1}{2} \left( \frac{3}{2} |g(A_1(y))|^2 + \Lambda^2 \right) (b_1)^2 \right].
\]

(52)

In order to see how we can obtain the mass spectrum appeared in 3-dimensions, let us take a close look at the field \(a_0\). The quadratic terms of the Lagrangian for \(a_0\) are read from (52) as

\[
\mathcal{L}^{(2)}_{B(R>R^*)}(a_0) \equiv \int_0^{2\pi R} dy \left[ -\frac{1}{2} \partial_\mu a_0 \partial^\mu a_0 - \frac{1}{2} (gA_1(y))^2 (a_0)^2 \right]
\]

\[
= \int_0^{2\pi R} dy \left[ -\frac{1}{2} \partial_\mu a_0 \partial^\mu a_0 - \frac{1}{2} a_0 \mathcal{M}^2_{a_0}(y) a_0 \right],
\]

(53)

where

\[
\mathcal{M}^2_{a_0}(y) \equiv -\partial_y^2 + |g(a_1(y))|^2.
\]

(54)

Here, we have carried out the partial integration. Let us consider the eigenvalue equation defined by

\[
\mathcal{M}^2_{a_0}(y) \phi_{a_0}^{(i)}(y) \equiv \left[ -\frac{d^2}{dy^2} + |g(A_1(y))|^2 \right] \phi_{a_0}^{(i)}(y) = (m_{a_0}^{(i)})^2 \phi_{a_0}^{(i)}(y)
\]

(55)

with the boundary condition \(\phi_{a_0}^{(i)}(y + 2\pi R) = \phi_{a_0}^{(i)}(y)\). Since the set of \(\{ \phi_{a_0}^{(i)} \}\) is expected to form a complete set, we may expand \(a_0(x^\mu, y)\) as \(a_0(x^\mu, y) = \sum_i a_0^{(i)}(x^\mu) \phi_{a_0}^{(i)}(y)\) with \(\int_0^{2\pi R} dy \phi_{a_0}^{(i)}(y) \phi_{a_0}^{(j)}(y) = \delta_{ij}\). Then, (53) becomes

\[
\mathcal{L}^{(2)}_{B(R>R^*)}(a_0) = \sum_i \left[ -\frac{1}{2} \partial_\mu a_0^{(i)} \partial^\mu a_0^{(i)} - \frac{1}{2} (m_{a_0}^{(i)})^2 (a_0^{(i)})^2 \right].
\]

(56)

We see that the eigenvalue of the equation (55) is nothing but the squared mass for \(a_0^{(i)}\) in 3-dimensions. Therefore, finding the masses of \(a_0^{(i)}\) is equivalent to solving the eigenvalue equation (55). The field \(b_0\) satisfies the same equation with \(a_0\). In the same way, we have the eigenvalue equations for \(a_1\) and \(b_1\) as follows:

\[
\mathcal{M}^2_{a_1}(y) \phi_{a_1}^{(i)}(y) \equiv \left[ -\frac{d^2}{dy^2} + \frac{3}{2} |g(A_1(y))|^2 - \Lambda^2 \right] \phi_{a_1}^{(i)}(y) = (m_{a_1}^{(i)})^2 \phi_{a_1}^{(i)}(y),
\]

\[
\mathcal{M}^2_{b_1}(y) \phi_{b_1}^{(i)}(y) \equiv \left[ -\frac{d^2}{dy^2} + \frac{1}{2} |g(A_1(y))|^2 + \Lambda^2 \right] \phi_{b_1}^{(i)}(y) = (m_{b_1}^{(i)})^2 \phi_{b_1}^{(i)}(y).
\]

(57)

By noting \(\langle A_1(y) \rangle = \frac{2k \omega}{g} \text{sn}(\omega y, k)\) and by defining \(u \equiv \omega y\), we may recast the equation (55) into

\[
\left[ -\frac{d^2}{du^2} + 4k^2 \text{sn}^2(u, k) \right] \phi_{a_0}^{(i)}(u) = \Omega_{a_0}^{(i)} \phi_{a_0}^{(i)}(u),
\]

(58)
The boundary conditions we imposed are reduced to φ of the Lamé equation with the boundary conditions φ be classified by the four types of the eigenfunctions

\[ R > R^* \] are well understood by known properties of the Lamé equation, though the exact eigenvalues and associated eigenfunctions are known only for some special cases. We summarize important properties of the eigenvalues and the associated eigenfunctions of the Lamé equation in the Appendix B.

Before discussing the general case of 0 ≤ k < 1, it may be instructive to study the limit of k → 0 (R → R* = 1/2Λ). In this limit, the Lamé equation becomes

\[ -\frac{d^2}{du^2} \phi(u, k = 0) = \Omega(k = 0) \phi(u, k = 0). \]  

The boundary conditions we imposed are reduced to φ(u + π) = ±φ(u) because of 2K(0) = π. It is easy to solve the equation. The eigenvalues and the eigenfunctions at k = 0 are simply given as in the Table III-3. We can expand the fields a0, b0, a1 and b1 in terms of those eigenfunctions as shown in the Appendix B.

Now, let us come back to the original Lamé equation with 0 ≤ k < 1. The solutions of the Lamé equation with the boundary conditions φ(u + 2K(k)) = ±φ(u) are known to be classified by the four types of the eigenfunctions

\[ Ec^2_{N}(u, k), \ E_{s}^{2n+2}(u, k), \ E_{c}^{2n+1}(u, k), \ E_{s}^{2n+1}(u, k) \]  

with n = 0, 1, 2, \cdots. The first two eigenfunctions in are periodic under u → u + 2K(k) and the last two ones are antiperiodic. The first and the third eigenfunctions in are even under u → −u and the second and the fourth ones are odd. We denote the eigenvalues associated with Ec^2_{N} and Es^{2n+2} by α^2_{N}(k) and β^{2n+2}_{N}(k), respectively. For a positive integer N, it has been known that the lowest 2N + 1 eigenvalues and the associated eigenfunctions are exactly known as Lamé polynomials, which are written in terms of the Jacobi elliptic functions, sn(u, k), cn(u, k) and dn(u, k). For general N, solutions of the Lamé equation and even for integer N other than 2N + 1 Lamé polynomials will not be written in such the simple forms.

Let us study the mass spectrum for the bosons with the help of the known analytic properties of the Lamé equation. The Lamé equation for the field a1 corresponds to the
$N = 2$ case. We may expand $a_1(x,y)$, taking the boundary condition into account, as

$$a_1(x,y) = \sum_{n=1}^{\infty} \left[ a^{(c,2n-1)}_1(x) E^{2n-1}_c(\omega y, k) + a^{(s,2n-1)}_1(x) E^{2n-1}_s(\omega y, k) \right].$$

(64)

The squared mass eigenvalues for $a^{(c,2n-1)}_1$ and $a^{(s,2n-1)}_1$ are given by

$$(\alpha^{2n-1}_2(k) - 1 - k^2)\omega^2 \quad \text{and} \quad (\beta^{2n-1}_2(k) - 1 - k^2)\omega^2,$$

(65)

respectively. Two out of $2N + 1 = 5$ Lamé polynomials satisfy the desired antiperiodic boundary condition. The two eigenfunctions $Ec^1_1(\omega y, k)$ and $Es^1_1(\omega y, k)$ are given by the Lamé polynomials and their eigenvalues are exactly known. Those eigenfunctions and eigenvalues are explicitly given in the Table III-4. The eigenvalues $\alpha^{2n-1}_2(k)$ and $\beta^{2n-1}_2(k)$ are degenerate at $k = 0$, in fact, $\alpha^{2n-1}_2(0) = \beta^{2n-1}_2(0) = (2n - 1)^2$. They are still degenerate even for $k > 0$ except for $\alpha^1_1(k)$ and $\beta^1_1(k)$. By taking account of the mass hierarchies (164) in the Appendix B and the degeneracy among the eigenvalues, the $R$-dependence of the mass spectrum for $a^{(c,2n-1)}_1$ and $a^{(s,2n-1)}_1$ is schematically depicted in the Figure 1.

The Lamé equation for the field $b_1$ corresponds to the $N = 1$ case. We may expand the field as

$$b_1(x,y) = \sum_{n=1}^{\infty} \left[ b^{(c,2n-1)}_1(x) E^{2n-1}_c(\omega y, k) + b^{(s,2n-1)}_1(x) E^{2n-1}_s(\omega y, k) \right].$$

(66)

Then, the squared mass eigenvalues for $b^{(c,2n-1)}_1$ and $b^{(s,2n-1)}_1$ are given by

$$(\alpha^{2n-1}_1(k) + 1 + k^2)\omega^2 \quad \text{and} \quad (\beta^{2n-1}_1(k) + 1 + k^2)\omega^2,$$

(67)

respectively. Two out of $2N + 1 = 3$ Lamé polynomials satisfy the desired antiperiodic boundary condition. The two eigenfunctions $Ec^1_1(\omega y, k)$ and $Es^1_1(\omega y, k)$ are given by the Lamé polynomials and their eigenvalues are exactly known. Those eigenfunctions and eigenvalues are explicitly given in the Table III-5. The eigenvalues $\alpha^{2n-1}_1$ and $\beta^{2n-1}_1$ are degenerate at $k = 0$, and they are still degenerate even for $k > 0$ except for $\alpha^1_1(k)$ and $\beta^1_1(k)$. The $R$-dependence of the mass spectrum for $b^{(c,2n-1)}_1$ and $b^{(s,2n-1)}_1$ is schematically depicted in the Figure 1.

Although the $2N + 1$ Lamé polynomials for integer $N$ are known exactly, those exact analytical solutions to the Lamé equation are rather exceptional. The Lamé equation for the fields $a_0$ and $b_0$ corresponds to noninteger $N$, and none of the eigenfunctions and the eigenvalues are known exactly. Even though we do not know the exact solutions for this case, we can study the qualitative behaviour of the mass spectrum of $a_0$ and $b_0$. We can expand the fields $a_0$ and $b_0$ as

$$a_0(x,y) = \sum_{n=0}^{\infty} \left[ a^{(c,2n)}_0(x) E^{2n}_{N_0(\omega y, k)} + a^{(s,2n+2)}_0 E^{2n+2}_{N_0}(\omega y, k) \right],$$

$$b_0(x,y) = \sum_{n=0}^{\infty} \left[ b^{(c,2n)}_0(x) E^{2n}_{N_0(\omega y, k)} + b^{(s,2n+2)}_0 E^{2n+2}_{N_0}(\omega y, k) \right],$$

(68)

where $N_0$ satisfies $N_0(N_0 + 1) = 4$. The squared mass eigenvalue for each mode $a^{(c,2n)}_0$, $a^{(s,2n+2)}_0$, $b^{(c,2n)}_0$ and $b^{(s,2n+2)}_0$ is given by

$$a^{2n}_0(k)\omega^2, \quad b^{2n+2}_0(k)\omega^2,$$

(69)

$$a^{(c,2n)}_0, b^{(c,2n)}_0; \quad a^{2n}_0(k)\omega^2, \quad b^{2n+2}_0(k)\omega^2,$$

$$a^{(s,2n+2)}_0, b^{(s,2n+2)}_0; \quad a^{2n}_0(k)\omega^2, \quad b^{2n+2}_0(k)\omega^2.$$
Since we have already known $\alpha_{N_0}^{2n}(k = 0) = \beta_{N_0}^{2n}(k = 0) = 4n^2$, the modes $\alpha_0^{(c,0)}$ and $b_0^{(c,0)}$ are massless at $k = 0$. They are, however, expected to become massive for $k > 0$. This is concluded from the eigenvalue equation (58). The differential operator $\hat{H} \equiv -\frac{d^2}{da^2} + 4k^2sn^2(u, k)$ is positive definite for $k > 0$, so that $\langle \phi | \hat{H} | \phi \rangle > 0$ for any nontrivial eigenfunctions. Therefore, we conclude $\alpha_{N_0}^0(k), \beta_{N_0}^0(k) > 0$ for $k > 0$. The eigenvalues $\alpha_{N_0}^{2n}$ and $\beta_{N_0}^{2n}$ are degenerate at $k = 0$, but they split for $k > 0$ though we do not know their relative magnitude. The $R$-dependence of the mass eigenvalues of $a_0^{(c,2n)}, a_0^{(s,2n+2)}, b_0^{(c,2n)}$ and $b_0^{(s,2n+2)}$ is schematically depicted in the Figure 1.

Since we have studied the boson mass spectrum at the critical radius $R = R^*(k = 0)$, it may be instructive to study the behaviour of the mass spectrum for small values of $k$ in perturbation theory with respect to the parameter $k$. In the Tables III-6 and 7, the perturbative mass spectrum for the first two lowest modes of $a_0, b_0$ and for the second excited modes of $a_1, b_1$ are summarized, up to the order of $k^2$.

### 3.4.2 Fermion sector

We shall compute the mass spectrum of the fermions. The relevant part of the Lagrangian is

$$ L_{F(R>R^*)}^{(2)} = \int_0^{2\pi R} dy \left[ -i\bar{\psi}_0 \sigma^M \partial_M \psi_0 - i\bar{\psi}_1 \sigma^M \partial_M \psi_1 + (g(A_1(y))\psi_0 \psi_1 + h.c.) \right]. \quad (70) $$

As before, according to our prescription (16) given in the Appendix B, we rewrite (70) in terms of the 3-dimensional Majorana spinors (61) as follows:

$$ L_{F(R>R^*)}^{(2)} = \int_0^{2\pi R} dy \frac{1}{2} \bar{\Psi}(\gamma^\mu \partial_{\mu} 1 + M_F(y))\Psi, \quad (71) $$

where we have defined $\Psi \equiv (\chi_0, \rho_0, \chi_1, \rho_1)^T$ and

$$ M_F(y) \equiv \begin{pmatrix} 0 & -\partial_y & U(y) & 0 \\ \partial_y & 0 & 0 & -U(y) \\ U(y) & 0 & 0 & -\partial_y \\ 0 & -U(y) & \partial_y & 0 \end{pmatrix}, \quad (72) $$

where $U(y) \equiv 2k\omega \text{sn}(\omega y, k)$. The field equations for $\chi_0, \rho_0, \chi_1$ and $\rho_1$ are easily obtained. Multiplying the field equations by $(i\gamma^\nu \partial_{\nu} 1 + M_F)$, we have the second order differential equations for these fields

$$ (-\partial^\mu \partial_{\mu} 1 + M_F^2)\Psi = \begin{pmatrix} -\partial^\mu \partial_{\mu} + D_y^2 & 0 & 0 & \partial_y U(y) \\ 0 & -\partial^\mu \partial_{\mu} + D_y^2 & \partial_y U(y) & 0 \\ 0 & \partial_y U(y) & -\partial^\mu \partial_{\mu} + D_y^2 & 0 \\ \partial_y U(y) & 0 & 0 & -\partial^\mu \partial_{\mu} + D_y^2 \end{pmatrix} \Psi = 0. \quad (73) $$

Here, we have defined $D_y^2 \equiv -\partial^2_y + (U(y))^2$. It is convenient to shuffle $\chi_0, \rho_0, \chi_1$ and $\rho_1$ as follows:

$$ \eta_{\pm} \equiv \frac{1}{\sqrt{2}}(\chi_0 \pm \rho_1), \quad \zeta_{\pm} \equiv \frac{1}{\sqrt{2}}(\rho_0 \pm \chi_1). \quad (74) $$

Then, the Majorana spinors $\eta_{\pm}$ and $\zeta_{\pm}$ satisfy the equations of motion

$$ [-\partial^\mu \partial_{\mu} - \partial_y^2 + (U(y))^2 \pm \partial_y U(y)]\eta_{\pm}(x^\mu, y) = 0, \quad (75) $$

$$ [-\partial^\mu \partial_{\mu} - \partial_y^2 + (U(y))^2 \pm \partial_y U(y)]\zeta_{\pm}(x^\mu, y) = 0. $$
To find the mass spectrum, we have to be careful about the boundary conditions. The boundary conditions for \( \eta_\pm, \zeta_\pm \) are translated into

\[
\eta_\pm(x^\mu, y + 2\pi R) = \eta_\mp(x^\mu, y), \quad \zeta_\pm(x^\mu, y + 2\pi R) = \zeta_\mp(x^\mu, y), \tag{76}
\]

which are also equivalent to

\[
\eta_+(x^\mu, y + 4\pi R) = \eta_+(x^\mu, y), \quad \eta_-(x^\mu, y) = \eta_+(x^\mu, y + 2\pi R), \\
\zeta_+(x^\mu, y + 4\pi R) = \zeta_+(x^\mu, y), \quad \zeta_-(x^\mu, y) = \zeta_+(x^\mu, y + 2\pi R). \tag{77}
\]

Note that these are consistent with the fields equations because of \( U(y + 2\pi R) = -U(y) \). Once we solve the equations of motion for \( \eta_+, \zeta_+ \) with the boundary conditions (77), we automatically have solutions for \( \eta_-, \zeta_- \) through the relations (77). It follows from the above consideration that the mass spectrum of the fermions is obtained by solving the eigenvalue equation

\[
\left[ -\frac{d^2}{dy^2} + (U(y))^2 + \frac{dU(y)}{dy} \right] \phi(y) = m_F^2 \phi(y) \tag{78}
\]

with the boundary condition \( \phi(y + 4\pi R) = \phi(y) \). Changing the variables \( y \) to \( u \equiv \omega y \), we recast the above equation into

\[
\left( \frac{d}{du} + 2k \text{ sn}(u, k) \right) \left( -\frac{d}{du} + 2k \text{ sn}(u, k) \right) \phi(u) = (\frac{m_F}{\omega})^2 \phi(u) \tag{79}
\]

with \( \phi(u + 4K(k)) = \phi(u) \).

Let us first study the behavior of the fermion mass spectrum in the limit of \( k \to 0 \) (or \( R \to R^* = 1/2\Lambda \)). In this limit the above equation is reduced to

\[
-\frac{d^2}{du^2} \phi(u, k = 0) = \left( \frac{m_F(k = 0)}{\omega} \right)^2 \phi(u, k = 0) \tag{80}
\]

with \( \phi(u + 2\pi, k = 0) = \phi(u, k = 0) \). The eigenvalues and the associated eigenfunctions are easily obtained as in the Table III-8.

The properties of the eigenvalues and the eigenfunctions of the equation (78) is summarized in the Appendix B. As discussed there, in terms of the eigenfunctions denoted by \( Ec^n(u, k) \) and \( Es^{n+1}(u, k) \) \( (n = 0, 1, 2 \cdots) \), we can expand \( \eta_+(x^\mu, y) \) and \( \zeta_+(x^\mu, y) \) as

\[
\eta_+(x^\mu, y) = \sum_{n=0}^{\infty} \left[ \eta^{(c,n)}(x) Ec^n(\omega y, k) + \eta^{(s,n+1)}(x) Es^{n+1}(\omega y, k) \right], \\
\zeta_+(x^\mu, y) = \sum_{n=0}^{\infty} \left[ \zeta^{(c,n)}(x) Ec^n(\omega y, k) + \zeta^{(s,n+1)}(x) Es^{n+1}(\omega y, k) \right]. \tag{81}
\]

Then, through the relations (77), \( \eta_-(x^\mu, y) \) and \( \zeta_-(x^\mu, y) \) are given by

\[
\eta_-(x^\mu, y) = \eta_+(x, y + 2\pi R) = \sum_{n=0}^{\infty} \left[ \eta^{(c,n)}(x) Ec^n(\omega(y + 2\pi R), k) + \eta^{(s,n+1)}(x) Es^{n+1}(\omega(y + 2\pi R), k) \right], \\
\zeta_-(x^\mu, y) = \zeta_+(x, y + 2\pi R) = \sum_{n=0}^{\infty} \left[ \zeta^{(c,n)}(x) Ec^n(\omega(y + 2\pi R), k) + \zeta^{(s,n+1)}(x) Es^{n+1}(\omega(y + 2\pi R), k) \right]. \tag{82}
\]
Equipped with these, it is easy to obtain the equations of motion for \( \eta^{(c,n)}(x) \), \( \eta^{(s,n)}(x) \), \( \zeta^{(c,n)}(x) \) and \( \zeta^{(s,n)}(x) \) from the field equations (75) as follows:

\[
[-\partial_\mu \partial^\mu + (m_F^{(i,n)})^2] \eta^{(i,n)}(x) = 0, \quad [-\partial_\mu \partial^\mu + (m_F^{(i,n)})^2] \zeta^{(i,n)}(x) = 0, \quad \text{(83)}
\]

where \( i = c, s \) and we have denoted the eigenvalues of \( Ec^{n}(u, k) \) and \( Es^{n}(u, k) \) by \( \frac{(m_F^{(c,n)})^2}{\omega} \) and \( \frac{(m_F^{(s,n)})^2}{\omega} \), respectively.

Even though we do not know exact results of the eigenvalue equation (78), we can extract the behaviour of the fermion mass spectrum with respect to the radius \( R \) by applying general properties of the equation. As proven in the Appendix B, we have the hierarchies among the eigenvalues of the eigenvalue equation. They yield the mass hierarchies among the masses of the fermions

\[
\begin{align*}
    m_F^{(c,0)} &< m_F^{(c,1)} < m_F^{(c,2)} < \cdots, \\
    m_F^{(s,1)} &< m_F^{(s,2)} < m_F^{(s,3)} < \cdots, \\
    m_F^{(c,n)} &= m_F^{(s,n)}, \quad \text{if } n \neq 0. \quad \text{(84)}
\end{align*}
\]

According to these, we schematically depict the \( R \)-dependence of the fermion mass spectrum for \( \eta^{(c,n)} \), \( \eta^{(s,n)} \), \( \zeta^{(c,n)} \) and \( \zeta^{(s,n)} \) for \( R > R^* \) in the Figure 1.

We do not know the exact eigenvalues of \( (m_F^{(c,n)})^2 \) and \( (m_F^{(s,n)})^2 \) except for the lowest eigenvalue. Let us recast the eigenvalue equation (74) into the form \( \hat{a}^\dagger(u) \hat{a}(u) \phi(u) = \frac{4}{\omega} \phi(u) \), where \( \hat{a}(u) \equiv -\frac{d}{du} + 2k \text{sn}(u, k) \) and \( \hat{a}^\dagger(u) \equiv +\frac{d}{du} + 2k \text{sn}(u, k) \). It immediately follows that all eigenvalues are positive semi-definite. In fact, the lowest eigenvalue is found to be zero provided that the eigenfunction satisfies \( \hat{a}(u) Ec^0(u, k) = 0 \) with \( Ec^0(u + 4K(k), k) = Ec^0(u, k) \). The solution to the above equation is given, apart from a normalization constant, by \( Ec^0(u, k) = (k cn(u, k) - dn(u, k))^2 \). This is consistent with the previous result at \( k = 0 \). We can see from (81) that the modes \( \eta^{(c,0)}(x) \) and \( \zeta^{(c,0)}(x) \) are massless. All the other modes are massive. It is not difficult to see that these massless modes correspond to the Nambu-Goldstone fermions associated with the spontaneous SUSY breaking, as we will discuss in the next subsection.

Although we do not know the explicit forms of the eigenfunctions and their eigenvalues except for the lowest mode, we can obtain the perturbative mass spectrum for higher modes with respect to \( k \). The results for the first three excited states are given in the Table III-9.

### 3.5 Analysis of the mass spectrum for \( R > R^* \)

Let us discuss the boson and fermion mass spectra obtained in the previous subsection 3.4. We have found that the lowest mode of \( a_1 \) is massless. This mode is interpreted as the Nambu-Goldstone boson associated with the breakdown of the translational invariance for the \( S^1 \) direction. To see this, let us recall the expression (14) for \( a_1(x, y) \) in terms of the eigenfunctions of the Lamé equation with \( N = 2 \). The Nambu-Goldstone mode has to be the one proportional to \( \partial_y A_1(x, y) \) in the expansion of \( A_1(x, y) \) because of the commutation relation \([iaQ, A_1(x, y)] = a\partial_y A_1(x, y)\), where \( Q \) is a generator of the translation for the \( S^1 \) direction. Noting that \( Ec_2^0(\omega y, k) = cn(\omega y, k)dn(\omega y, k) \propto \frac{d}{dy}(A_1(x, y)) \), we find that the coefficient of \( Ec_2^0(\omega y, k) \), i.e. \( a_1^{(c,1)}(x) \) corresponds to the Nambu-Goldstone boson associated with the spontaneous breakdown of the translational invariance for the \( S^1 \) direction. Originally, the mode \( d\langle A_1 \rangle/dy \) is a local tangent specified by one parameter,
which is a consequence of the translational invariance of the equation of motion, to the space given by all the translated solutions \( \langle A_1(y - a) \rangle \), \( \alpha \in \mathbb{R} \). So, there always exists such the zero mode on the quantization around the coordinate-dependent background.

It is important to notice that the translational invariance of \( S^1 \) can be interpreted as a global \( U(1) \) invariance from the 3-dimensional point of view. In this point of view, a massless bosonic mode in the phase of the spontaneous breakdown of the translational invariance can be interpreted as an ordinary Nambu-Goldstone boson associated with the spontaneous breakdown of the global \( U(1) \) symmetry. To see this, let us expand \( A_i(i = 0, 1) \) in Fourier series, according to the boundary conditions. The commutation relations between the generator \( Q \) and the Fourier modes may be given by

\[
\begin{align*}
\left[ Q, \tilde{A}_0^{(n)}(x) \right] &= \frac{n}{R} \tilde{A}_0^{(n)}(x), \quad n \in \mathbb{Z}, \\
\left[ Q, \tilde{A}_1^{(l)}(x) \right] &= \frac{l}{R} \tilde{A}_1^{(l)}(x), \quad l \in \mathbb{Z} + \frac{1}{2}.
\end{align*}
\]

We see that the generator \( Q \), which is originally the generator of the translation of \( S^1 \), acts as if it generates global \( U(1) \) transformations under which the fields \( \tilde{A}_0^{(n)} \) and \( \tilde{A}_1^{(l)} \) carry the \( U(1) \) charges \( n/R \) and \( l/R \), respectively. For \( R \leq R^* \), the vacuum expectation values for \( A_0 \) and \( A_1 \) are an arbitrary constant and zero, respectively, so that \( \langle \tilde{A}_0^{(0)} \rangle = \) arbitrary constant, \( \langle \tilde{A}_0^{(n\neq0)} \rangle = 0 \) and \( \langle \tilde{A}_1^{(l)} \rangle = 0 \). Since \( \tilde{A}_0^{(0)} \) carries no \( U(1) \) charge, the nonvanishing vacuum expectation value of \( \tilde{A}_0^{(0)} \) does not break the \( U(1) \) symmetry. For \( R > R^* \), the vacuum expectation values are given by (85). Since \( \text{sn}(\omega y, k) \) can be expanded in Fourier series as \( \text{sn}(\omega(y - y_0), k) = \sum_{i \in \mathbb{Z} + \frac{1}{2} > 0} c_i \sin(l \frac{(y - y_0)}{R}) \) with \( c_i \neq 0 \), we obtain \( \langle \tilde{A}_0^{(0)} \rangle = \frac{k}{g} \sqrt{2\pi R}, \langle \tilde{A}_0^{(n\neq0)} \rangle = 0 \) and \( \langle \tilde{A}_1^{(l)} \rangle \neq 0 \). The \( \tilde{A}_1^{(l)} \) has a \( U(1) \) charge \( l/R \), so that the vacuum expectation value \( \langle \tilde{A}_1^{(l)} \rangle \) breaks the \( U(1) \) symmetry spontaneously. This observation is consistent with the previous discussion that the Nambu-Goldstone boson is incorporated in \( A_1(x, y) \) as \( \partial_y \langle A_1(x, y) \rangle \).

Let us discuss the Nambu-Goldstone fermions associated with the breakdown of the supersymmetry. Even though we do not obtain the fermion mass spectrum exactly, we definitely have massless modes associated with the eigenfunction \( E c^0(u, k) \). The vacuum expectation values of the SUSY transformations for the spinors in the vacuum configuration (80) are given by

\[
\begin{align*}
\langle \delta_S \psi_{0\alpha}(x, y) \rangle &= -\sqrt{2} \xi_\alpha \frac{\Lambda^2}{g} - \frac{g}{2} \langle A_1^*(y) \rangle^2, \\
\langle \delta_S \psi_{1\alpha}(x, y) \rangle &= -\sqrt{2} \xi_\alpha \frac{d \langle A_1(y) \rangle}{dy},
\end{align*}
\]

where we have used \( \bar{\xi}^\beta = (i \tau_y \xi^*)_\beta \) and \( \sigma^y = \tau_y \). Since both \( \langle \delta_S \psi_0 \rangle \) and \( \langle \delta_S \psi_1 \rangle \) do not vanish for any nontrivial \( \xi \), the supersymmetry is completely broken spontaneously and no SUSY is left in 3-dimensions. In the Appendix B we present the expressions for the expansion of \( \psi_0 \) and \( \psi_1 \) in terms of the eigenfunctions \( E c^m \) and \( E s^m \) (see (169) and (170)). Since \( E c^0 \) is found to be \( E c^0 = (k \text{cn}(u, k) - \text{dn}(u, k))^2 \), we can show that

\[
\begin{align*}
\psi_0(x, y) &= \frac{g}{\omega^2} \left( \eta^{(c,0)}(x) + i \zeta^{(c,0)}(x) \right) \frac{\Lambda^2}{g} - \frac{g}{2} \langle A_1^*(y) \rangle^2 + \cdots, \\
\psi_1(x, y) &= \frac{g}{\omega^2} \left( \eta^{(c,0)}(x) + i \zeta^{(c,0)}(x) \right) \frac{d \langle A_1(y) \rangle}{dy} + \cdots.
\end{align*}
\]
where \( \cdots \) denotes higher modes. Comparing (87) with (86) and identifying \( \xi \) in (86) with \( \frac{g}{\sqrt{2\omega}}(\eta^{(c,0)} + i\zeta^{(d,0)}) \), we can conclude that the fermion zero modes \( \eta^{(c,0)} \) and \( \zeta^{(d,0)} \) are nothing but the Nambu-Goldstone fermions associated with the spontaneous SUSY breaking.

Let us comment on the supertrace formula for the mass spectrum in the \( Z_2 \)-model for \( R > R^* \). We can immediately see from (54), (57) and (73) that the supertrace formula \( \text{Tr} M_B^2 = \text{Tr} M_F^2 \) formally holds. This relation, however, may mathematically be meaningless because the trace has to be taken in an infinite dimensional space, so that the sum of the eigenvalues of each squared mass operator would diverge. For \( R \leq R^* \), the relations (51) hold in the subspace of each Fourier mode. So, they are meaningful relations. We do not know whether there are any finite dimensional subspaces in which the relation still holds for \( R > R^* \). Finally, we summarize the phase structure of the model in the Table III-10.

4 THE U(1) MODEL

In this section we shall study the \( U(1) \) model which has a global \( U(1) \) symmetry, instead of the discrete \( Z_2 \)-symmetry. We will find interesting features different from those of the \( Z_2 \) model. We, again, study the vacuum structure and the mass spectrum for bosons and fermions.

4.1 Vacuum configuration

The superpotential of the \( U(1) \) model is given by

\[
W(\Phi_0, \Phi_\pm) = g\Phi_0 \left( \frac{\Lambda^2}{g^2} - \Phi_+ \Phi_- \right).
\]

We shall use this superpotential which has no mass term for \( \Phi_\pm \). We have slightly changed the notations of the superpotential (22). The model has two global \( U(1) \) and \( U(1)_R \) symmetries. The \( U(1) \) symmetry is defined by \( \Phi_0 \rightarrow +\Phi_0 \) and \( \Phi_{\pm} \rightarrow e^{\pm 2\pi i\alpha} \Phi_{\pm} \). This global symmetry plays an important role for our mechanism to work. The scalar potential \( V(A_0, A_{\pm}) \) is given by

\[
V(A_0, A_{\pm}) = |F_0|^2 + |F_+|^2 + |F_-|^2
= \left| \frac{\Lambda^2}{g} - gA_+ A_- \right|^2 + |g A_0|^2 \left( |A_+|^2 + |A_-|^2 \right).
\]

The solutions to the \( F \)-term conditions are easily found to be \( \bar{A}_0 = 0 \) and \( \bar{A}_+ \bar{A}_- = \frac{\Lambda^2}{g^2} \), at which the scalar potential vanishes.

Let us consider the model on \( M^3 \otimes S^1 \). The notations concerning about the space-time are the same with the previous section. Using the \( U(1) \) symmetry degrees of freedom, we can impose the nontrivial boundary conditions on the superfields \( \Phi_0(x^\mu, y) \) and \( \Phi_{\pm}(x^\mu, y) \) for the \( S^1 \) direction

\[
\Phi_0(x^\mu, y + 2\pi R) = +\Phi_0(x^\mu, y), \quad \Phi_{\pm}(x^\mu, y + 2\pi R) = e^{\pm 2\pi i\alpha} \Phi_{\pm}(x^\mu, y).
\]

It should be emphasized that we do not use the \( U(1)_R \) symmetry degrees of freedom to impose nontrivial boundary conditions, as stated earlier. In the following we restrict the phase \( \alpha \) to be the range \( 0 < \alpha \leq \frac{1}{2} \) without loss of generality.
It is easy to see that the $F$-term conditions are not consistent with the boundary conditions (90), by which the vacuum expectation values of $A_\pm$ are forced not to take any nonzero constants. Therefore, the vacuum configuration given by the solutions of the $F$-term conditions are not realized as a supersymmetric vacuum configuration. Thus, our mechanism works again in this model.

As we have done in the previous section, in order to find the true vacuum configuration, we have to solve the minimization problem of the “energy” functional defined by

$$\mathcal{E}[A_0, A_\pm; R] \equiv \int_0^{2\pi R} dy \left[ \left| \frac{dA_0}{dy} \right|^2 + \left| \frac{dA_+}{dy} \right|^2 + \left| \frac{dA_-}{dy} \right|^2 \right. \left. + \left| \frac{\Lambda^2}{g} - gA_+A_- \right|^2 + |gA_0|^2 \left( |A_+|^2 + |A_-|^2 \right) \right].$$

Then, the vacuum configuration for $A_0$ and $A_\pm$ should satisfy the field equations

$$0 = \frac{\delta \mathcal{E}[A_0, A_\pm; R]}{\delta A_0(y)} = -\frac{d^2A_0(y)}{dy^2} + g^2A_0(y)\left( |A_+(y)|^2 + |A_-(y)|^2 \right),$$

$$0 = \frac{\delta \mathcal{E}[A_0, A_\pm; R]}{\delta A_\pm(y)} = -\frac{d^2A_\pm(y)}{dy^2} - g^*A_\pm(y)\left( \frac{\Lambda^2}{g} - gA_+(y)A_-(y) \right) + g^2|A_0(y)|^2A_\pm(y).$$

It follows from the first equation that the vacuum configuration can be classified into two types of solutions as

(I); \quad \begin{cases} A_0(y) = \text{arbitrary constant} \\ A_\pm(y) = 0 \end{cases} \quad \text{(II); } \begin{cases} A_0(y) = 0 \\ A_+(y) \neq 0 \text{ and/or } A_- \neq 0. \end{cases}

For the type (I) solution, $\mathcal{E}[A_0, A_\pm; R]$ becomes

$$\mathcal{E}[A_0 = \text{const.}, A_\pm = 0; R] = \frac{2\pi R \Lambda^4}{g^2}.$$ 

On the other hand, for the type (II) solution, we can easily show that

$$\mathcal{E}[A_0 = 0, A_\pm; R] \bigg|_{\frac{\delta \mathcal{E}}{\delta A_\pm} = 0} = \frac{2\pi R \Lambda^4}{g^2} - g^2 \int_0^{2\pi R} dy |A_+(y)A_-(y)|^2;$$

$$\leq \mathcal{E}[A_0 = \text{const.}, A_\pm = 0; R].$$

An important conclusion here is that if there appears any type (II) solution with $A_\pm \neq 0$, then, the type (I) solution is no longer a vacuum configuration.

Let us next find the vacuum configuration explicitly. To do this, we redefine the fields

$$A_0(y) \equiv \tilde{A}_0(y), \quad A_\pm(y) \equiv e^{\pm i\frac{\pi y}{2}}\tilde{A}_\pm(y).$$

Note that $\tilde{A}_0(y)$ and $\tilde{A}_\pm(y)$ satisfy the periodic boundary condition. Inserting (97) into (94), we obtain

$$\mathcal{E}[A_0, A_\pm; R] = \mathcal{E}_{KE}[\tilde{A}_0; R] + \mathcal{E}_{KE}[\tilde{A}_+; R] + \mathcal{E}_{KE}[\tilde{A}_-; R] + \mathcal{E}_{PE}[\tilde{A}_0, \tilde{A}_\pm; R],$$

(98)
where

\[ E_{KE}[\tilde{A}_0; R] = \int_0^{2\pi R} dy \left| \frac{d\tilde{A}_0}{dy} \right|^2, \]

\[ E_{KE}[\tilde{A}_\pm; R] = \int_0^{2\pi R} dy \left[ \left| \frac{d\tilde{A}_\pm}{dy} \right|^2 + \frac{i\alpha}{R} \left( \frac{d\tilde{A}_\pm}{dy} - \frac{d\tilde{A}_\mp}{dy} \right) \right], \]

\[ E_{PE}[\tilde{A}_0, \tilde{A}_\pm; R] = \int_0^{2\pi R} dy \left[ \left| \frac{\Lambda^2}{g} - g\tilde{A}_\pm\tilde{A}_\mp \right|^2 + \left( g^2|\tilde{A}_0|^2 + \left( \frac{\alpha}{R} \right)^2 \right)(|\tilde{A}_\pm|^2 + |\tilde{A}_\mp|^2) \right]. \] (99)

Our strategy to find the vacuum configuration is that we first look for configurations which minimize each of \( E_{KE}[\tilde{A}_0; R], E_{KE}[\tilde{A}_\pm; R] \) and \( E_{PE}[\tilde{A}_0, \tilde{A}_\pm; R] \) and then construct configurations which minimize all of them.

Let us look for configurations which minimize each of \( E_{KE}[\tilde{A}_0; R] \) and \( E_{KE}[\tilde{A}_\pm; R] \). For that purpose, we expand the fields in Fourier series, taking account of the boundary conditions, as \( \tilde{A}_0(y) = \sum_{n \in \mathbb{Z}} a_0^{(n)} e^{iny} \) and \( \tilde{A}_\pm(y) = \sum_{n \in \mathbb{Z}} a_\pm^{(n)} e^{iny} \). Inserting them into \( E_{KE}[\tilde{A}_0; R] \) and \( E_{KE}[\tilde{A}_\pm; R] \), we obtain

\[ E_{KE}[\tilde{A}_0; R] = \frac{2\pi}{R} \sum_{n \in \mathbb{Z}} n^2 |a_0^{(n)}|^2, \quad E_{KE}[\tilde{A}_\pm; R] = \frac{2\pi}{R} \sum_{n \in \mathbb{Z}} [(n \pm \alpha)^2 - \alpha^2] |a_\pm^{(n)}|^2. \] (100)

Since \( \alpha \) is restricted to the range \( 0 < \alpha \leq \frac{1}{2} \), \( E_{KE}[\tilde{A}_0; R] \) and \( E_{KE}[\tilde{A}_\pm; R] \) are positive semi-definite. The configurations which give \( E_{KE}[\tilde{A}_0; R] = 0 \) and \( E_{KE}[\tilde{A}_\pm; R] = 0 \) are found to be

for \( 0 < \alpha < \frac{1}{2} \), \[
\begin{cases}
\tilde{A}_0(y) = a_0^{(0)}, \\
\tilde{A}_\pm(y) = a_\pm^{(0)},
\end{cases}
\] (101)

for \( \alpha = \frac{1}{2} \), \[
\begin{cases}
\tilde{A}_0(y) = a_0^{(0)}, \\
\tilde{A}_\pm(y) = a_\pm^{(0)} + a_\pm^{(\mp 1)} e^{iy/R},
\end{cases}
\] (102)

where \( a_0^{(0)}, a_\pm^{(0)} \) and \( a_\pm^{(\mp 1)} \) are arbitrary complex constants. Let us next look for configurations which minimize \( E_{PE}[\tilde{A}_0, \tilde{A}_\pm; R] \). As we saw before, we have the two types of the solutions, \( i.e. \) type (I) and (II). For the case (I), \( E_{PE}[\tilde{A}_0, \tilde{A}_\pm; R] \) becomes \( (103) \).

To find the type (II) solution we rewrite \( \tilde{A}_\pm \) into the form \( \tilde{A}_\pm(y) = \rho_+(y)e^{i\theta(y)} + \rho_-(y)e^{-i\theta(y)+i\phi(y)} \), where \( \rho_\pm(y) \) are real and positive semi-definite functions of \( y \) and \( \theta(y), \phi(y) \) are real functions of \( y \). Inserting them into \( E_{PE} \), we obtain

\[ E_{PE}[\tilde{A}_0 = 0, \tilde{A}_\pm; R] = E_{PE}[\rho_+, \theta, \phi; R] = \int_0^{2\pi R} dy \left[ \left| \frac{\Lambda^2}{g} - g\rho_+e^{i\phi} \right|^2 + \left( \frac{\alpha}{R} \right)^2 \right]. \] (103)

Since \( \rho_+ \) and \( \rho_- \) are positive semi-definite and since \( \rho_+\rho_- \neq 0 \) for the configurations which minimize \( E_{PE}[\rho_+, \theta, \phi; R] \), \( \phi \) has to vanish to minimize \( E_{PE}[\rho_+, \theta, \phi; R] \). Then, we find that the configuration minimizing \( E_{PE}[\rho_+, \theta, \phi = 0; R] \) with \( \rho_\pm \neq 0 \) is given by \( \rho_\pm(y) = \sqrt{\frac{1}{\gamma}(\Lambda^2 - \left( \frac{\alpha}{R} \right)^2)} \). Note here that this configuration is meaningful only when \( \Lambda^2 - \left( \frac{\alpha}{R} \right)^2 \geq 0 \) or \( R \geq \frac{\alpha}{\Lambda} \). The potential energy for this configuration is given by

\[ E_{PE}[\rho_+, \theta, \phi = 0; R] = \frac{2\pi RA^2}{g^2} - \frac{2\pi R}{g^2} \left( \Lambda^2 - \left( \frac{\alpha}{R} \right)^2 \right)^2 \text{ for } R \geq \frac{\alpha}{\Lambda}. \] (104)
Combining all the results, we conclude that the vacuum configuration which minimizes $E[A_0, A_\pm; R]$ is given by

\begin{align}
\text{for } R \leq R^* & \equiv \frac{\alpha}{\Lambda}, & \left\{ \langle A_0(x^\mu, y) \rangle = \text{arbitrary complex constant}, \langle A_\pm(x^\mu, y) \rangle = 0, \right. \\
\text{for } R > R^* & \equiv \frac{\alpha}{\Lambda}, & \left. \left\{ \langle A_0(x^\mu, y) \rangle = 0, \langle A_\pm(x^\mu, y) \rangle = \frac{1}{g} v e^{\pm i \frac{\alpha}{R} (y - y_0)}, \right. \right. 
\end{align}

where $v \equiv \sqrt{\Lambda^2 - (\alpha/R)^2}$. The arbitrary parameter $y_0$ reflects the $U(1)$ symmetry of the theory and also reflects the fact that the equations of motion are invariant under the translation for the $S^1$ direction. The relation between the two symmetries will be discussed later. Let us note that when $\alpha = \frac{1}{2}$ we have another choice of the vacuum configuration $\langle A_\pm(x^\mu, y) \rangle = \frac{1}{g} v e^{\pm i \frac{\alpha}{R} (y - y_0)}$ for $R > R^*$. The vacuum in this case is doubly degenerate. One of them corresponds to the right moving mode on $S^1$ and the other does to the left moving mode.

As in the $Z_2$ model, there are two phases, depending on the magnitude of the radius of $S^1$. We see that the vacuum structure drastically changes at the critical radius $R^*$. For $R < R^*$, the $U(1)_R$ symmetry is broken, while for $R > R^*$, the translational invariance for the $S^1$ direction is broken spontaneously with the breakdown of the $U(1)$ symmetry [22]. The vacuum energy is nonzero both $R \leq R^*$ and $R > R^*$, so that the supersymmetry is broken spontaneously for both regions.

### 4.2 Mass spectrum for $R \leq R^*$

Since we have obtained the vacuum configuration, let us next study the mass spectrum appeared in 3-dimensions. The mass spectrum will be obtained by the same prescription used in the previous section. Let us start to compute the boson mass spectrum, which can be read from the quadratic part of the 3-dimensional Lagrangian

\[
L^{(2)}_{B(R \leq R^*}) = \int_0^{2\pi R} dy \left[ -\sum_{i=0,\pm} \partial_M A_i^* \partial^M A_i + \Lambda^2 (A_+ A_- + A^*_+ A^*_-) - g |\langle A_0 \rangle|^2 (|A_+|^2 + |A_-|^2) \right],
\]

where the field $A_0$ has been shifted by the vacuum expectation value $\langle A_0 \rangle$. Let us expand the fields $A_0(x, y)$ and $A_\pm(x, y)$ in Fourier series for the $S^1$ direction

\[
A_0(x, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n \in \mathbb{Z}} a_0^{(n)}(x) e^{i \frac{n}{R} y}, \quad A_\pm(x, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n \in \mathbb{Z}} a_\pm^{(n)}(x) e^{i \frac{n}{R} y}.
\]

These expansions are consistent with the boundary conditions [20]. Inserting these into (107) and redefining the Fourier modes $a_\pm^{(n)}$ as $b_\pm^{(n)} \equiv \frac{1}{\sqrt{2}} (a_\pm^{(n)} \pm a_-^{(n)*})$, we have

\[
L^{(2)}_{B(R \leq R^*)} = \sum_{n \in \mathbb{Z}} \left[ -\partial_\mu a_\mu^{(n)*} \partial^\mu a_\mu^{(n)} - \sum_{i=0,\pm} \partial_\mu b_i^{(n)*} \partial^\mu b_i^{(n)} - \Phi_B^{(n)\dagger} (M_B^{(n)})^2 \Phi_B^{(n)} \right],
\]

where $\Phi_B^{(n)} \equiv (a_0^{(n)}, b_+^{(n)}, b_-^{(n)})^T$ and the squared mass matrix is given by

\[
(M_B^{(n)})^2 \equiv \begin{pmatrix}
\frac{n}{R} & 0 & 0 \\
0 & (\frac{n+\alpha}{R})^2 & |\langle A_0 \rangle|^2 - \Lambda^2 \\
0 & 0 & (\frac{n+\alpha}{R})^2 + |\langle A_0 \rangle|^2 + \Lambda^2
\end{pmatrix}.
\]
Thus, we have obtained the masses of the bosons.

Next, we shall study the fermion mass spectrum for \( R \leq R^* = \alpha/\Lambda \). It can be read from

\[
\mathcal{L}_{F(R \leq R^*)}^{(2)} = \int_0^{2\pi R} dy \left[ -i \sum_{i=0,\pm} \bar{\psi}_i \sigma^M \partial_M \psi_i + g \langle A_0 \rangle \psi_+ \psi_- + g \langle A_0 \rangle^* \bar{\psi}_+ \bar{\psi}_- \right].
\]  

(111)

We again expand the fields \( \psi_0(x, y) \) and \( \psi_\pm(x, y) \) in Fourier series for the \( S^1 \) direction

\[
\psi_0(x, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n \in \mathbb{Z}} \chi_0^{(n)}(x)e^{i \frac{2\pi n}{R} y}, \quad \psi_\pm(x, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n \in \mathbb{Z}} \chi_\pm^{(n)}(x)e^{i \frac{2\pi n}{R} y}.
\]

(112)

Inserting (112) into (111), we obtain

\[
\mathcal{L}_{F(R \leq R^*)}^{(2)} = \sum_{n \in \mathbb{Z}} \left[ \chi_0^{(n)}(x) \mathcal{M}_{\chi_0}^{(n)} \chi_0^{(n)}(x) + \frac{1}{2} \bar{\Psi}^{(n)} \mathcal{M}^{(n)}_{\chi_\pm} \Psi^{(n)} \right],
\]

(113)

where we have defined \( \Psi^{(n)} \equiv (\chi_+^{(n)}, \chi_-^{(n)}, \chi_-^{(-n)}, \chi_+^{(-n)})^T \). The matrices \( \mathcal{M}^{(n)}_{\chi_0} \) and \( \mathcal{M}^{(n)}_{\chi_\pm} \) in (113) are defined by

\[
\mathcal{M}^{(n)}_{\chi_0} \equiv -i \bar{\sigma} + \frac{n}{R} \bar{\sigma} y,
\]

(114)

\[
\mathcal{M}^{(n)}_{\chi_\pm} \equiv \begin{pmatrix} -i \sigma^\mu + \frac{n+\alpha}{R} \sigma^y & g \langle A_0 \rangle^* & 0 & 0 \\ g \langle A_0 \rangle & -i \sigma^\mu + \frac{n+\alpha}{R} \sigma^y & 0 & 0 \\ 0 & 0 & -i \sigma^\mu + \frac{n-\alpha}{R} \sigma^y & g \langle A_0 \rangle^* \\ 0 & 0 & g \langle A_0 \rangle & -i \sigma^\mu + \frac{n-\alpha}{R} \sigma^y \end{pmatrix},
\]

(115)

where \( \bar{\sigma} \equiv \sigma^\mu \partial_\mu \) and \( \sigma^\mu \partial_\mu \). The mass eigenvalues of the fermions may be extracted from the determinant of the above matrices, i.e.

\[
\det \mathcal{M}^{(n)}_{\chi_0} = \partial_\mu \partial^\mu - \left( \frac{n}{R} \right)^2,
\]

\[
\det \mathcal{M}^{(n)}_{\chi_\pm} = \left[ \partial_\mu \partial^\mu - \left( \frac{n+\alpha}{R} \right)^2 - |g \langle A_0 \rangle|^2 \right]^2 \left[ \partial_\mu \partial^\mu - \left( \frac{n-\alpha}{R} \right)^2 - |g \langle A_0 \rangle|^2 \right].
\]

(116)

Thus, we find

\[
(m_{\chi_0}^{(n)})^2 = \left( \frac{n}{R} \right)^2,
\]

\[
(m_{\chi_\pm}^{(n)})^2 = \left( \frac{n+\alpha}{R} \right)^2 + |g \langle A_0 \rangle|^2 - \left( \frac{n-\alpha}{R} \right)^2 - |g \langle A_0 \rangle|^2.
\]

(117)

We schematically depict the \( R \)-dependence of the boson and fermion mass spectra in the Figure 2.

### 4.3 Analysis of the mass spectrum for \( R \leq R^* \)

Let us study here the masses of the bosons and fermions obtained in the previous subsection. Note that all the squared masses (110) and (117) are positive semi-definite for \( R \leq \alpha/\Lambda \). The vacuum configuration (105) is stable for this region. It is easy to see from the boson mass splitting that the scale of the SUSY breaking is of order \( \Lambda \). We
observe that the mode $a_0^{(0)}$ is massless. The physical interpretation for this massless mode is that a part of it is guaranteed by the flatness of the scalar potential at the tree-level and another part of it corresponds to the Nambu-Goldstone boson associated with the breakdown of the $U(1)_R$ symmetry due to the vacuum configuration (103).

We also observe that in the mass spectrum there appears a massless fermion mode $\chi_0^{(0)}$ which is the Dirac spinor in 3-dimensions. This corresponds to the Nambu-Goldstone fermion associated with the spontaneous SUSY breaking. In order to confirm this, let us recall the infinitesimal SUSY transformations for the spinor fields

$$\delta_S \psi_0 = i\sqrt{2}(\sigma^a \xi) \partial_M A_0 - \sqrt{2}\xi(\frac{A^2}{g} - gA_+^*A_-^*),$$

$$\delta_S \psi_\pm = i\sqrt{2}(\sigma^a \xi) \partial_M A_\pm + \sqrt{2}\xi gA_0^*A_\mp^*.$$  \hspace{1cm} (118)

In the vacuum background given by (105), the vacuum expectation values of $\delta_S \psi_0$ and $\delta_S \psi_\pm$ become

$$\langle \delta_S \psi_0 \rangle = -\sqrt{2}\frac{A^2}{g}, \quad \langle \delta_S \psi_\pm \rangle = 0.$$  \hspace{1cm} (119)

Thus, for any nontrivial $\xi$, the supersymmetry is completely broken spontaneously. It follows from (119) that the mode of the Nambu-Goldstone fermion associated with the SUSY breaking is the one proportional to $\langle \delta_S \psi_0 \rangle$ in $\psi_0(x, y)$, that is, a constant mode with respect to $y$ in Fourier series. Therefore, we confirm that $\chi_0^{(0)}$ is the Nambu-Goldstone fermion associated with the breakdown of the supersymmetry.

Let us note that the mass spectrum of the bosons and fermions satisfies the following mass relations for each mode $n$:

$$(m_{e_0^{(n)}})^2 + (m_{e_0^{(-n)}})^2 = (m_{\chi_0^{(n)}})^2 + (m_{\chi_0^{(-n)}})^2,$$  \hspace{1cm} (120)

$$\sum_{i=\pm} [(m_{\psi_i^{(n)}})^2 + (m_{\psi_i^{(-n)}})^2] = \sum_{i=\pm} [(m_{\psi_i^{(n)}})^2 + (m_{\psi_i^{(-n)}})^2].$$  \hspace{1cm} (121)

These are the supertrace formulas for the squared masses in 3-dimensions, though the original proof of the supertrace formula in Ref. [13] does not necessarily imply the equivalence between the sum of the boson squared masses and that of the fermion ones for each Fourier mode.

### 4.4 Mass spectrum for $R > R^*$

In this subsection, we shall study the mass spectrum for the bosons and fermions for $R > \alpha/\Lambda$. We expand the fields around the vacuum configuration (106) and take the quadratic part of the Lagrangian to obtain the mass spectrum. The relevant terms for the boson mass spectrum are given by

$$\mathcal{L}_{B(R>R^*)}^{(2)} = \int_0^{2\pi R} dy \left[ -\sum_{i=0,\pm} \partial_M A_i^* \partial^M A_i - v^2 \left( 2A_0^*A_0 + A_+^*A_- + A_-^*A_+ \right) ight.$$

$$+ e^{-i\frac{\alpha}{R}y} A_+^*A_+ + e^{i\frac{\alpha}{R}y} A_-^*A_- + \left( \frac{\alpha}{R} \right)^2 (A_+A_- + A_+^*A_-^*) \right],$$  \hspace{1cm} (122)

where we have put $y_0 = 0$ for simplicity. Let us expand the fields $A_0(x, y)$ and $A_\pm(x, y)$ in Fourier series for the $S^1$ direction as

$$A_0(x, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n \in \mathbb{Z}} a^{(n)}_0(x)e^{i\frac{2\pi n}{R}y}, \quad A_\pm(x, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n \in \mathbb{Z}} a^{(n)}_\pm(x)e^{i\frac{2\pi n}{R}y},$$
which are consistent with the boundary conditions (120). Inserting these into (122), we obtain

\[ L^{(2)}_{B(R>R^*)} = \sum_{n \in \mathbb{Z}} \frac{1}{2} \Phi_B^{(n)} (\partial_\mu \partial^\mu - (M_B^{(n)})^2) \Phi_B^{(n)}, \]  

(123)

where \( \Phi_B^{(n)} \equiv (a_0^{(n)}, a_{-n}^{(n)}, a_+^{(n)}, a_-^{(n)}, a_0^{(-n)}, a_{-n}^{(-n)})^T \) and the squared mass matrix in (123) is given by

\[
(M_B^{(n)})^2 = \begin{pmatrix}
A & 0 & 0 & 0 & 0 & 0 \\
0 & A & 0 & 0 & 0 & 0 \\
0 & 0 & B & E & D & 0 \\
0 & 0 & E & B & 0 & D \\
0 & 0 & D & 0 & C & E \\
0 & 0 & 0 & D & E & C
\end{pmatrix}, \quad \text{where}
\]

\[
A = \left( \frac{\alpha}{R} \right)^2 + 2v^2, \quad B = \left( \frac{\alpha + \alpha}{R} \right)^2 + v^2, \quad C = \left( \frac{\alpha - \alpha}{R} \right)^2 + v^2, \quad D = v^2, \quad E = -\left( \frac{\alpha}{R} \right)^2,
\]

(124)

with \( v^2 \equiv \Lambda^2 - (\alpha/R)^2 \). The mass eigenvalues are obtained by solving the equation \( \det((M_B^{(n)})^2 - m^2 1_{6 \times 6}) = 0 \). After the straightforward calculations, the eigenvalues are found to be

\[
m^2 = \begin{cases} 
2v^2 + \left( \frac{\alpha}{R} \right)^2, \\
2v^2 + \left( \frac{\alpha}{R} \right)^2, \\
v^2 + \left( \frac{\alpha}{R} \right)^2 \pm \sqrt{v^4 + 4\left( \frac{\alpha}{R} \right)^2 v^2}, \\
v^2 + 2(\frac{\alpha}{R})^2 + (\frac{\alpha}{R})^2 \pm \sqrt{v^4 + 4(\frac{\alpha}{R})^2 v^2}.
\end{cases}
\]

(125)

Let us proceed to compute the fermion mass spectrum. It can be read from

\[
L^{(2)}_{F(R>R^*)} = \int_0^{2\pi R} dy \left[ -i \sum_{i=0,\pm} \bar{\psi}_i \gamma^\mu \partial_\mu \psi_i + v \psi_0 (e^{-i\frac{\alpha}{R} y} \psi_+ + e^{i\frac{\alpha}{R} y} \psi_-) \\
+ \bar{\psi}_0 (e^{i\frac{\alpha}{R} y} \psi_+ + e^{-i\frac{\alpha}{R} y} \psi_-) \right].
\]

(126)

We find that it is convenient to redefine the spinors so as to satisfy the periodic boundary condition, i.e. \( \psi_\pm(x, y) \equiv e^{\pm \frac{\alpha}{R} y} \psi_\pm(x, y) \), where \( \bar{\psi}_\pm(x, y + 2\pi R) = \bar{\psi}_\pm(x, y) \). Moreover, we make use of the prescription (160) given in the Appendix B. Thus, we obtain

\[
L^{(2)}_{F(R>R^*)} = \int_0^{2\pi R} dy \bar{\Psi} (-i\gamma^\mu \partial_\mu + M_F) \Psi,
\]

(127)

where \( \Psi \equiv (\psi_0^c, \bar{\psi}_+, \bar{\psi}_-)^T \) and \( \psi_0^c \) denotes the charge conjugation of \( \psi_0 \). Note that the spinors in above equation are the Dirac spinors in 3-dimensions. The mass matrix \( M_F \) in (127) is given by

\[
M_F = \begin{pmatrix}
-i\partial_y & -iv & -iv \\
v & i\partial_y - \alpha/R & 0 \\
v & 0 & i\partial_y + \alpha/R
\end{pmatrix}.
\]

(128)

We expand the spinor fields in Fourier series for the \( S^1 \) direction by taking account of the boundary condition for \( \psi_0^c \) and \( \psi_\pm \) as

\[
\psi_0^c(x, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n \in \mathbb{Z}} \chi_0^{(n)}(x) e^{in\pi y}, \quad \psi_\pm(x, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n \in \mathbb{Z}} \chi_\pm^{(n)}(x) e^{in\pi y}.
\]

(129)
Then, we find from (127)

\[ L_{F(R>R^*)}^{(2)} = \sum_{n \in \mathbb{Z}} (\bar{\chi}^{(n)}_0, \bar{\chi}^{(n)}_+, \bar{\chi}^{(n)}_-) (-i\gamma^\mu \partial_\mu + \mathcal{M}^{(n)}_F) \begin{pmatrix} \chi^{(n)}_0 \\ \chi^{(n)}_+ \\ \chi^{(n)}_- \end{pmatrix}, \]  

(130)

where

\[ \mathcal{M}^{(n)}_F = (\mathcal{M}^{(n)}_F)^\dagger = \begin{pmatrix} n/R & -iv & -iv \\ iv & -(n + \alpha)/R & 0 \\ iv & 0 & -(n - \alpha)/R \end{pmatrix}. \]  

(131)

To obtain the mass spectrum, it is sufficient to diagonalize the square of \( \mathcal{M}^{(n)}_F \). We finally have

\[ (\mathcal{M}^{(n)}_F)^2 = \begin{pmatrix} (n/R)^2 + 2v^2 & iv\frac{\alpha}{R} & -iv\frac{\alpha}{R} \\ -iv\frac{\alpha}{R} & ((n + \alpha)/R)^2 + v^2 & v^2 \\ iv\frac{\alpha}{R} & v^2 & ((n - \alpha)/R)^2 + v^2 \end{pmatrix}. \]  

(132)

We shall not here try to solve the eigenvalue equation for the fermion masses exactly, though the mass eigenvalues could be obtained by solving a cubic equation. However, some of important properties, which will be discussed in the next subsection, can easily be extracted from the expression (132). The schematic behavior of the boson and fermion masses with respect to \( R \) is found in the Figure 2.

### 4.5 Analysis of the mass spectrum for \( R > R^* \)

It is easy to see from (125) that the squared masses for the bosons are all positive semi-definite for \( R > \alpha/\Lambda \) with \( 0 < \alpha \leq \frac{1}{2} \). Since the vacuum configuration (106) breaks both the translational invariance for the \( S^1 \) direction and the global \( U(1) \) symmetry, one might expect two massless Nambu-Goldstone modes associated with the broken generators of the symmetries. If we take a look at the boson mass spectrum, however, there is only one massless mode given by

\[ m^2 = v^2 + \left( \frac{n}{R} \right)^2 - \sqrt{v^4 + 4(\frac{\alpha}{R})^2(\frac{n}{R})^2} = 0 \quad \text{for} \quad n = 0. \]  

(133)

A physical interpretation for this goes as follows: Let us consider the \( U(1)' \) transformation, which is a linear combination of the translation for the \( S^1 \) direction and the \( U(1) \) transformation, defined by

\[ U(1)': A_0(x, y) \to A_0(x, y + a), \quad A_\pm(x, y) \to e^{\mp i\frac{\alpha}{R}a} A_\pm(x, y + a). \]  

(134)

It turns out that the \( U(1)' \) transformation is an exact symmetry of the model. In fact, the vacuum configuration (106) is invariant under the modified \( U(1)' \) transformation (134). Therefore, one linear combination of the two symmetry generators survives as an unbroken generator and the massless boson corresponds to the Nambu-Goldstone mode associated with the broken generator.

It is easy to see that there is a massless mode (in the sense of the 3-dimensional Dirac spinor) in the fermion mass spectrum, which can be seen from the fact that the determinant of the matrix (132) can be zero. The determinant of the mass matrix is computed as

\[ \text{det}(\mathcal{M}_F^{(n)})^2 = \left( \frac{n}{R} \right)^2 \left( \frac{\alpha}{R} \right)^2 - \left( \frac{\alpha}{R} \right)^2 + 2v^2 \right)^2. \]  

Since \( v^2 > 0 \) and \( n^2 - \alpha^2 > 0 \) for

\( n \neq 0 \), the determinant of the matrix can be zero only when \( n = 0 \). This implies that there is a massless eigenvalue in the mass matrix. This massless mode corresponds to the Nambu-Goldstone fermion associated with the spontaneous SUSY breaking. To see this, let us consider the vacuum expectation values of the SUSY transformations for the spinors in the vacuum configuration \((106)\), i.e.

\[
\langle \delta_S \psi_0 \rangle = -\frac{\sqrt{2}}{g} \xi \left( \frac{\alpha}{R} \right)^2, \quad \langle \delta_S \psi_{\pm} \rangle = \frac{\sqrt{2}}{g} i (\sigma^y \bar{\xi}) v \left( \pm \frac{i \alpha}{R} \right) e^{\pm i \pi y}.
\]  

(135)

Since \( \langle \delta_S \psi_0 \rangle \) and \( \langle \delta_S \psi_{\pm} \rangle \) do not vanish for any nontrivial choice of \( \xi \) and \( \bar{\xi} \), the supersymmetry is spontaneously broken completely. The corresponding Nambu-Goldstone (3-dimensional Dirac) fermion is the massless mode found above.

In the \( Z_2 \) model, we have not found any simple relations between the boson and the fermion masses for \( R > R^* \). In the \( U(1) \) model, we have, however, found that the mass matrices \((124)\) and \((132)\) lead to the supertrace formula which holds for each Fourier mode between the bosons and the fermions, i.e. \( \text{Tr}(M_B^{(n)})^2 = \text{Tr}(M_F^{(n)})^2 \).

It is interesting to study the supersymmetry breaking in the \( R \to \infty \) limit. In this limit, the supersymmetry transformations for the spinor fields vanish as seen from \((135)\). Therefore, the supersymmetry is restored in this limit, which is very contrary to the \( Z_2 \) model in which the supersymmetry is broken even in this limit. The net effect of the boundary conditions of the \( U(1) \) model completely disappears and we will have the \( N = 2 \) supersymmetry in 3-dimensions in the limit of \( R \to \infty \). Finally, let us summarize the phase structure of the \( U(1) \) model in the Table IV-1.

### 5 CONCLUSIONS AND DISCUSSIONS

In this paper we have proposed a new mechanism of the spontaneous SUSY breaking. We have discussed the general criteria for the superpotential in order for our mechanism to work. We have obtained the two minimal models, \( Z_2 \) and \( U(1) \) models to realize our mechanism and studied the dynamical aspects, such as the vacuum structure and the mass spectra in the models.

The crucial point of our mechanism is that there exist the solutions to the \( F \)-term conditions but they are not realized as the vacuum configurations, as contrary to the usual cases, because of the nontrivial boundary conditions of the fields for the compactified direction. It is important to investigate the “energy” functional including the kinetic terms of the scalar fields in order to find the true vacuum configuration consistent with the nontrivial boundary conditions. A remarkable observation in our analysis is that the vacuum configurations which depends on the coordinate of the extra dimensions are energetically favorable than any coordinate-independent configurations for \( R > R^* \). In other words, there exists a phase in which the translational invariance of the compactified direction is spontaneously broken when the size of the extra dimension exceeds the critical radius, at which the phase transition occurs. On the other hand, the supersymmetry is broken spontaneously irrespective of the size of the extra dimension in both models.

The complexity of the vacuum structures reflects the mass spectra of the models, which have rich structures and depend on the size of the extra dimension in a nontrivial way. The vacuum configuration breaks some of the symmetries of the models, so that the Nambu-Goldstone bosons and/or fermions associated with the breakdown of the symmetries appear. We have actually observed those massless modes in the mass spectra.
Our mechanism can cause quite different vacuum structures if the models have different global symmetries whose degrees of freedom are available to impose nontrivial boundary conditions on the fields. This is easily observed in the two models we have studied. The vacuum structures of the two models are quite different, as seen from (29), (30), (103) and (106). Moreover, the behavior of the models in the limit of \( R \rightarrow \infty \) is also quite different. In the \( Z_2 \) model the vacuum configuration is reduced to a single kink solution, which is one of the topologically stable solutions

\[
\langle A_1(x^\mu, y) \rangle |_{R=\infty} = \frac{\sqrt{2} \Lambda}{g} \tanh \left( \frac{\Lambda}{\sqrt{2}} y \right).
\]  

(136)

The supersymmetry is still broken in this limit because the vacuum expectation values of the SUSY transformations for \( \psi_0 \) and \( \psi_1 \) do not vanish even in this background

\[
\langle \delta_S \psi_0 \rangle |_{R=\infty} = -\frac{\sqrt{2} \Lambda^2}{g} \frac{1}{\cosh^2 (\Lambda y / \sqrt{2})}, \quad \langle \delta_S \psi_1 \rangle |_{R=\infty} = i \sqrt{2} \frac{\Lambda^2}{g} \sigma^y \xi \frac{1}{\cosh^2 (\Lambda y / \sqrt{2})}.
\]  

(137)

For any nontrivial \( \xi \), there is no SUSY transformation which acts trivially on the kink solution. Therefore, the supersymmetry is still broken in the limit of \( R \rightarrow \infty \) to have no SUSY in 3-dimensions [20], while in the \( U(1) \) model, the supersymmetry is restored in the limit of \( R \rightarrow \infty \) to have the \( N = 2 \) SUSY in 3-dimensions. This is easily seen from (135). This is very contrary to the \( Z_2 \) model in which the supersymmetry is still completely broken in this limit. The effect of the boundary conditions on the fields completely disappears to be able to have a constant vacuum expectation value of \( \langle A_1 \rangle \). These are direct consequences followed from the fact that in the two models the vacuum structures have quite different dependence on the coordinate of the extra dimension. These results will not be specific to the models considered in this paper, but are expected to be general ones of the theories in which our mechanism works.

Let us comment on an effective theory appeared in 3-dimensions. If one tries to construct the effective theory in 3-dimensions by only massless modes, we have free field theories for the \( Z_2 \) and \( U(1) \) models in the region of \( R \leq R^* \). On the other hand, in the region of \( R > R^* \), we have interacting field theories with Yukawa couplings in the two models. In the limit of \( R \rightarrow \infty \), the effective \( Z_2 \) model has no supersymmetry, while the \( U(1) \) model has the 3-dimensional \( N = 2 \) supersymmetry.

There will be many ways to extend our work. It may be interesting to ask how our mechanism works on more complex manifolds, such as torus. We expect that there will appear complicated phase structures, depending on the size of their compactified spaces and on how we impose nontrivial boundary conditions on superfields. We can also study models with nonabelian global symmetries such as \( SU(N) \), instead of \( Z_2 \) and \( U(1) \) symmetries. In addition to them, it is important to study gauge theories and to see how our mechanism works and what new dynamics are hidden in them. It may also be interesting to investigate how the partial SUSY breaking occurs in the gauge theory in the connection of the well-known BPS objects. These will be reported in the near future.
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APPENDIX A

In this appendix, we show how we determine the vacuum configuration in the $Z_2$ model in detail. The vacuum configuration $\langle A_0 \rangle$ and $\langle A_1 \rangle$ have to be stable against any infinitesimal variations of $A_0$ and $A_1$. So, they have to satisfy the field equations derived from the “energy” functional $\langle 27 \rangle$

$$
0 = \frac{\delta \mathcal{E}[\langle A_0 \rangle, \langle A_1 \rangle]}{\delta A_0(y)} = -\frac{d^2}{dy^2} \langle A_0(y) \rangle + g(\langle A_0(y) \rangle - \mu) |\langle A_1(y) \rangle|^2, \quad (138)
$$

$$
0 = \frac{\delta \mathcal{E}[\langle A_0 \rangle, \langle A_1 \rangle]}{\delta A_1(y)} = -\frac{d^2}{dy^2} \langle A_1(y) \rangle - g\langle A_1(y) \rangle \left( \frac{\Lambda^2}{g} - \frac{g}{2} \langle A_1(y) \rangle^2 \right)
+ |g\langle A_0(y) \rangle - \mu|^2 \langle A_1(y) \rangle^2. \quad (139)
$$

It is convenient to separately investigate the two cases: Type (I); $\langle A_1(y) \rangle = 0$ and type (II); $\langle A_1(y) \rangle \neq 0$. Let us first consider the type (I) solution. In this case we find immediately

$$
\mathcal{E}[A_0, A_1 = 0] = \int_0^{2\pi R} dy \left[ \frac{dA_0}{dy} \right]^2 + \frac{\Lambda^4}{g^2} \geq \frac{2\pi R\Lambda^4}{g^2}. \quad (140)
$$

The equality holds only when $\frac{dA_0}{dy} = 0$ or equivalently $A_0(y) = \text{arbitrary complex constant}$. Thus, the type (I) solution is found to be $A_0^{(I)}(y) = \text{arbitrary complex constant}$ and $A_1^{(I)}(y) = 0$. It is obvious that these satisfy the equations of motion (138) and (139). For the type (I) solution the “energy” functional is given by $\mathcal{E}[A_0^{(I)}, A_1^{(I)}] = \frac{2\pi R\Lambda^4}{g^2}$.

We shall next consider the type (II) solution. Then, we may rewrite the “energy” functional $\mathcal{E}[A_0, A_1]$ as

$$
\mathcal{E}[A_0^{(II)}, A_1^{(II)}] = \mathcal{E}[A_1^{(II)}] + \Delta \mathcal{E}[A_0^{(II)}, A_1^{(II)}], \quad (141)
$$

where

$$
\mathcal{E}[A_1^{(II)}] \equiv \int_0^{2\pi R} dy \left[ \frac{dA_1^{(II)}}{dy} \right]^2 + \left[ \frac{\Lambda^2}{g} - \frac{g}{2} \langle A_1^{(II)} \rangle^2 \right] \right], \quad (142)
$$

$$
\Delta \mathcal{E}[A_0^{(II)}, A_1^{(II)}] \equiv \int_0^{2\pi R} dy \left[ \frac{dA_0^{(II)}}{dy} \right]^2 + |gA_0^{(II)} - \mu|^2 |A_1^{(II)}|^2 \right]. \quad (143)
$$

Obviously, $\Delta \mathcal{E}[A_0^{(II)}, A_1^{(II)}]$ is positive semi-definite. The equality $\Delta \mathcal{E}[A_0^{(II)}, A_1^{(II)}] = 0$ holds only when $A_0^{(II)} = \mu/g$. Since we are interested in the vacuum configuration, it turns out that it is sufficient to consider the minimization problem of $\mathcal{E}[A_1^{(II)}]$, instead of $\mathcal{E}[A_0^{(II)}, A_1^{(II)}]$, with $A_0^{(II)} = \mu/g$. 
It is useful to parametrize the field \( A^{(I)}_1(y) \) as \( A^{(I)}_1(y) \equiv \frac{1}{\sqrt{2}} (\xi(y) + i\eta(y)) \), where \( \xi(y) \) and \( \eta(y) \) are real fields satisfying \( \xi(y + 2\pi R) = -\xi(y) \) and \( \eta(y + 2\pi R) = -\eta(y) \). Inserting these representations into \( \mathcal{E}[A^{(I)}_1] \), we find

\[
\mathcal{E}[A^{(I)}_1] = \mathcal{E}[\xi] + \Delta \mathcal{E}'[\xi, \eta],
\]

where

\[
\mathcal{E}[\xi] \equiv \int_0^{2\pi R} dy \left[ \frac{1}{2} \left( \frac{d\xi}{dy} \right)^2 + \left( \frac{\Lambda^2}{g} - \frac{g}{4} \xi^2 \right)^2 \right],
\]

\[
\Delta \mathcal{E}'[\xi, \eta] \equiv \int_0^{2\pi R} dy \left[ \frac{1}{2} \left( \frac{d\xi}{dy} \right)^2 + \frac{\Lambda^2}{2} \eta^2 + \frac{g^2}{8} \xi^2 \eta^2 + \frac{g^2}{16} \eta^4 \right].
\]

Obviously, \( \Delta \mathcal{E}'[\xi, \eta] \) is positive semi-definite and the equality \( \Delta \mathcal{E}'[\xi, \eta] = 0 \) holds only when \( \eta(y) = 0 \). Therefore, we have found that the type (II) solution for the vacuum configuration should be of the form, \( A^{(II)}_0 = \mu/g \) and \( A^{(II)}_1 = \frac{1}{\sqrt{2}} \xi(y) \). Now the minimization problem of \( \mathcal{E}[A^{(II)}_0, A^{(II)}_1] \) is equivalent to that of \( \mathcal{E}[\xi] \) with the boundary condition

\[
\xi(y + 2\pi R) = -\xi(y).
\]

Here, any configurations of \( \xi(y) \) which minimize \( \mathcal{E}[\xi] \) have to satisfy the equation of motion

\[
0 = \frac{\delta \mathcal{E}[\xi]}{\delta \xi(y)} = -\frac{d^2 \xi(y)}{dy^2} - g\xi \left( \frac{\Lambda^2}{g} - \frac{g}{4} \xi^2 \right).
\]

It is important to note that if there exists any nontrivial type (II) solution, then, the type (I) solution is no longer the vacuum configuration. Actually, using (148), we can easily show that

\[
\mathcal{E}[A^{(II)}_0, A^{(II)}_1] = \frac{2\pi R \Lambda^4}{g^2} - \int_0^{2\pi R} dy \frac{g^2}{16} (\xi(y))^4 \leq \mathcal{E}[A^{(I)}_0, A^{(I)}_1].
\]

General solutions satisfying (148) are known to be given by

\[
\xi(y) = \frac{2\sqrt{2} k \omega}{g} \operatorname{sn}(\omega(y - y_0), k) \quad \text{with} \quad \omega \equiv \frac{\Lambda}{\sqrt{1 + k^2}}
\]

Here, \( \operatorname{sn}(u, k) \) is the Jacobi elliptic function whose period is given by \( 4K(k) \), where \( K(k) \) denotes the complete elliptic function of the first kind defined by

\[
K(k) \equiv \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}.
\]

The parameters \( k \) (0 \( \leq k < 1 \)) and \( y_0 \) are integration constants. Since we have found the general solutions to the equation of motion (148) (precisely speaking, we find the general solutions for periodic motions), the next task is to extract desired solutions from the general solutions, which have to satisfy the boundary condition (147). It leads to

\[
\operatorname{sn}(\omega(y + 2\pi R - y_0), k) = -\operatorname{sn}(\omega(y - y_0), k).
\]

This relation can be satisfied provided that

\[
2\pi R \omega = (2n - 1)2K(k)
\]

for some positive integer \( n \). It should be emphasized that there do not always exist solutions for any values of \( R \) and \( n \). Indeed, in order for a solution of (152) to exist, the
inequality $R \geq \frac{n-1}{\Lambda}$ has to be satisfied, where we have used $K(k) \geq K(0) = \pi/2$ and $\sqrt{1+k^2} \geq 1$.

We are, now, in a position to decide which configuration minimizes the “energy” functional (141). In the above analysis, we have found that the vacuum configuration should be given by one of the configurations

\[
\begin{align*}
type(I) & : \begin{cases} A_0^{(I)}(y) = \text{arbitrary complex constant}, \\
A_1^{(I)}(y) = 0. \end{cases} \\
type(II) & : \begin{cases} A_0^{(II)}(y) = \mu/g, \\
A_1^{(II)}(y) = \frac{2k\omega}{g} \text{sn}(\omega(y-y_0), k), \end{cases}
\end{align*}
\]

where $n \in \mathbb{Z} > 0$ and $k$ is determined by the condition (152). Let us decide which configuration is the vacuum one.

In the region of $0 < R \leq 1/2\Lambda$, the condition (152) cannot be satisfied for any $n$. So, there exists only the type (I) solution. We immediately conclude that the type (I) solution is the vacuum configuration in this region. The “energy” functional is then given by $\mathcal{E}[A_0^{(I)}, A_1^{(I)}] = \frac{2\pi R\Lambda^4}{g^2}$. In the region of $1/2\Lambda < R \leq 3/2\Lambda$, we have the type (II) solution with $n = 1$ as well as the type (I) solution. As we have already shown in (143), the vacuum configuration should be given by the type (II) solution with $n = 1$. In the same way, for the region of $(N-1/2)/\Lambda < R \leq (N+1/2)/\Lambda$, we have the type (II) solutions with $n = 1, 2, \ldots, N$ as well as the type (I) solution. The “energy” functional for each solution is given by

\[
\begin{align*}
\mathcal{E}[A_0^{(I)}, A_1^{(I)}] &= \frac{2\pi R\Lambda^4}{g^2}, \\
\mathcal{E}[A_0^{(II)}, A_1^{(II)}] &= \frac{2(2n-1)\Lambda^4}{3(1+k^2)^2} \left[ -(1-k^2)(5+3k^2)K(k) + 8(1+k^2)E(k) \right],
\end{align*}
\]

where $E(k)$ is the complete elliptic function of the second kind and $k$ is determined by the relation (152). Then, it is straightforward to show that

\[
\frac{d\mathcal{E}[A_0^{(II)}, A_1^{(II)}]}{dR} = \frac{2\pi \Lambda^4}{g^2} \left( \frac{1-k^2}{1+k^2} \right)^2 \geq 0.
\]

It follows that $\mathcal{E}[A_0^{(II)}, A_1^{(II)}]$ is a monotonically increasing function of $R$. Furthermore, we can show that $\mathcal{E}[A_0^{(II)}, A_1^{(II)}]|_{R=\frac{(n-1)\Lambda}{\sqrt{3}} \leq \mathcal{E}[A_0^{(II)}, A_1^{(II)}]|_{R} \leq \mathcal{E}[A_0^{(II)}, A_1^{(II)}]|_{R=\infty}$, which means $(2n-1)\frac{\pi \Lambda^3}{g^2} \leq \mathcal{E}[A_0^{(II)}, A_1^{(II)}]|_{R} \leq (2n-1)\frac{8\pi \Lambda^3}{3g^2}$. The $R$-dependence of $\mathcal{E}[A_0, A_1]$ is depicted in the Figure 3. Therefore, we can easily show that $\mathcal{E}[A_0^{(II)}, A_1^{(II)}] \leq \mathcal{E}[A_0^{(I)}, A_1^{(I)}]$ and $\mathcal{E}[A_0^{(I)}, A_1^{(I)}] < \mathcal{E}[A_0^{(I)}, A_1^{(I)}]$. This inequalities mean that for $R > 1/2\Lambda$ the vacuum configuration is given by $\{A_0(y) = A_0^{(I)}(y), A_1(y) = A_1^{(I)}(y)\}$. Hence, we observe the vacuum configuration of the $Z_2$-model as (29) and (30) given in the text.

The condition (152) gives

\[
2\pi R\omega = 2K(k)
\]

for the true vacuum configuration. It is also easy to see from the condition (156) that the critical radius $R^*$ is given by $R = \frac{1}{\pi \omega} K(k) \geq \frac{K(0)}{\pi \Lambda} = \frac{1}{2\Lambda} \equiv R^*$, which is nothing but the critical radius obtained in the text.
APPENDIX B

B.1 2-component spinors in 3-dimensions from 4-dimensions

Let us summarize the relations between 4-dimensional gamma matrices and 3-dimensional ones. We also present the decompositions of 4-dimensional spinors into 3-dimensional ones.

Let us define the σ-matrices in 4-dimensions by

$$
\begin{align*}
\sigma^0 &= -1_{2 \times 2} = \bar{\sigma}^0, & \sigma^1 &= \tau_z = -\bar{\sigma}^1, \\
\sigma^2 &= \tau_x = -\bar{\sigma}^2, & \sigma^3 &= \tau_y = -\bar{\sigma}^3,
\end{align*}
$$

where \(\tau_i(i = x, y, z)\) is the Pauli matrix. Then, we define the γ-matrices in 3-dimensions by \(\gamma^\mu \equiv \tau_y \bar{\sigma}^\mu (\mu = 0, 1, 2)\). The γ-matrices in 3-dimensions satisfy \(\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}\) with \(\text{diag}(\eta^{\mu\nu}) = (-, +, +, +)\), and they have properties such as

$$
\begin{align*}
(\gamma^0)^\dagger &= \gamma^0, & (\gamma^0)^T &= -\gamma^0, & (\gamma^0)^* &= -\gamma^0, \\
(\gamma^1)^\dagger &= -\gamma^1, & (\gamma^1)^T &= \gamma^1, & (\gamma^1)^* &= -\gamma^1, \\
(\gamma^2)^\dagger &= -\gamma^2, & (\gamma^2)^T &= \gamma^2, & (\gamma^2)^* &= -\gamma^2.
\end{align*}
$$

This representation is the Majorana representation. The charge conjugation in 3-dimensions is defined by \(\psi^c \equiv C\bar{\psi}^T\), where \(\bar{\psi} = \psi^\dagger \gamma_0 = -\psi^\dagger \gamma^0\). The \(C\) is the charge conjugation matrix defined by \(C \equiv \gamma^0 = -\gamma_0 = -\tau_y\). A Majorana fermion in 3-dimensions is defined by \(\psi = \psi^c = \psi^*\). According to the definitions we made above, we have the prescription used in the text. Let us denote a 2-component Dirac spinor in 3-dimensions and a 2-component Weyl spinor in 4-dimensions by \(\psi\) and \(\psi_{(4)}\), respectively. Then, we have

$$
\psi^\alpha_{(4)} = (i \tau_y \psi)^\alpha = (-i \psi^T \tau_y)^\alpha, \quad \bar{\psi}^\alpha_{(4)} = (i \tau_y \psi^*)^\alpha = (-i \psi^\dagger \tau_y)^\alpha.
$$

$$
\begin{align*}
\psi^\alpha_{(4)} \chi_{(4)} &= -i \bar{\psi}^\bar{\alpha} \chi, & \bar{\chi}_{(4)} &= i \bar{\chi} \psi^\bar{\alpha}, \\
\bar{\psi}_{(4)} (\bar{\sigma}^\mu)^{\alpha\beta} \chi_{(4)} &= \bar{\psi}_{(4)} \gamma^\mu \chi, & \chi_{(4)} (\sigma^\mu)^{\alpha\beta} \bar{\psi}_{(4)} &= \chi \gamma^\mu \psi^\bar{\alpha}, \\
\bar{\psi}_{(4)} (\bar{\sigma}^y)^{\alpha\beta} \chi_{(4)} &= -i \bar{\psi} \chi, & \chi_{(4)} (\sigma^y)^{\alpha\beta} \bar{\psi}_{(4)} &= \chi \psi^\bar{\alpha}.
\end{align*}
$$

It is useful that we express further a Dirac spinor in terms of two Majorana spinors in 3-dimensions. Let \(\psi\) and \(\chi\) be two 3-dimensional Dirac spinors. They can be expressed in terms of 3-dimensional Majorana spinors such as

$$
\psi \equiv \frac{1}{\sqrt{2}}(\rho_1 + i \rho_2), \quad \chi \equiv \frac{1}{\sqrt{2}}(\xi_1 + i \xi_2),
$$

where \(\rho_1, \rho_2, \xi_1\) and \(\xi_2\) are the Majorana spinors in 3-dimensions. The 3-dimensional Majorana spinors used in the text are defined through (161).

B.2 Eigenvalues and eigenfunctions of the Lamé equation

In this subsection, we briefly summarize general properties of the Lamé equation

$$
\left[ -\frac{d^2}{du^2} + N(N + 1)k^2 \sin^2(u, k) \right] \phi^{(i)}(u, k) = \Omega^{(i)}(k) \phi^{(i)}(u, k).
$$

The solutions to the Lamé equation with \(\phi(u + 2K(k)) = \pm \phi(u)\) are known to be classified by the four types of the eigenfunctions denoted by \(Ec^2_n(u, k), Ec^{2n+1}_n(u, k), Es^2_n(u, k)\)
and \( E_{n}^{2n+1}(u, k) \) with \( n = 0, 1, 2, \cdots \). These satisfy

\[
\begin{align*}
Ec_{n}^{2n}(-u, k) &= Ec_{n}^{2n}(u, k), & Ec_{n}^{2n}(u + 2K(k), k) &= Ec_{n}^{2n}(u, k), \\
Ec_{n}^{2n+1}(-u, k) &= Ec_{n}^{2n+1}(u, k), & Ec_{n}^{2n+1}(u + 2K(k), k) &= -Ec_{n}^{2n+1}(u, k), \\
Es_{n}^{2n+2}(-u, k) &= -Es_{n}^{2n+2}(u, k), & Es_{n}^{2n+2}(u + 2K(k), k) &= Es_{n}^{2n+2}(u, k), \\
Es_{n}^{2n+1}(-u, k) &= -Es_{n}^{2n+1}(u, k), & Es_{n}^{2n+1}(u + 2K(k), k) &= -Es_{n}^{2n+1}(u, k).
\end{align*}
\]

Let us denote the eigenvalues belonging to \( Ec_{n}^{2n}(u, k) \) and \( Es_{n}^{2n}(u, k) \) by \( \alpha_{n}^{2n}(k) \) and \( \beta_{n}^{2n}(k) \), respectively. Since the integer \( n \) corresponds to the number of nodes of the eigenfunctions, we have the following increasing sequence of the eigenvalues:

\[
\begin{align*}
\alpha_{n}^{0} &< \alpha_{n}^{1} < \alpha_{n}^{2} < \cdots, \\
\beta_{n}^{1} &< \beta_{n}^{2} < \beta_{n}^{3} < \cdots, \\
\alpha_{n}^{1} &< \beta_{n}^{1} < \alpha_{n}^{2} < \beta_{n}^{3} < \cdots, \\
\alpha_{n}^{0} &< \beta_{n}^{1} < \alpha_{n}^{2} < \beta_{n}^{3} < \cdots.
\end{align*}
\]

The following results about the degeneracy of the eigenvalue have been known:

- \( \alpha_{n}^{n} \neq \beta_{n}^{n} \) for all \( n = 0, 1, 2, \cdots \) if \( N \) is not an integer or if \( N \) is an integer and \( n = 0, 1, 2, \cdots, N \).
- \( \alpha_{n}^{n} = \beta_{n}^{n} \) if \( n \) and \( N \) are integers and \( n > N \).

For a positive integer \( N \), it has been shown that the lowest \( 2N + 1 \) eigenvalues and the associated eigenvalues are exactly known as Lamé polynomials, which are polynomials in terms of \( \text{sn}(u, k), \text{cn}(u, k) \) and \( \text{dn}(u, k) \). For general \( N \), solutions of the Lamé equation and even for integer \( N \), other than \( 2N + 1 \) Lamé polynomials will not be written in such simple forms. We explicitly present the Lamé polynomials for \( N = 1 \) and \( N = 2 \) below. The Lamé equation with \( N = 1 \) has \( 2N + 1 = 3 \) Lamé polynomials which are given, apart from normalization constants, by

| Lamé polynomial | eigenvalue |
|-----------------|------------|
| \( Ec_{1}^{0} = \text{dn}(u, k) \) | \( \alpha_{1}^{1} = k^{2} \) |
| \( Ec_{1}^{1} = \text{cn}(u, k) \) | \( \alpha_{1}^{1} = 1 \) |
| \( Es_{1}^{1} = \text{sn}(u, k) \) | \( \beta_{1}^{1} = 1 + k^{2} \) |

The Lamé equation with \( N = 2 \) has \( 2N + 1 = 5 \) Lamé polynomials which are give, apart from normalization constants, by

| Lamé polynomial | eigenvalue |
|-----------------|------------|
| \( Ec_{2}^{0} = \text{sn}^{2}(u, k) - \frac{1 + k^{2} + \sqrt{1 - k^{2} + k^{4}}}{3k^{2}} \) | \( \alpha_{2}^{0} = 2(1 + k^{2} - \sqrt{1 - k^{2} + k^{4}}) \) |
| \( Ec_{2}^{1} = \text{cn}(u, k)\text{dn}(u, k) \) | \( \alpha_{2}^{1} = 1 + k^{2} \) |
| \( Es_{2}^{1} = \text{sn}(u, k)\text{dn}(u, k) \) | \( \beta_{2}^{1} = 1 + 4k^{2} \) |
| \( Es_{2}^{2} = \text{sn}(u, k)\text{cn}(u, k) \) | \( \beta_{2}^{2} = 4 + k^{2} \) |
| \( Ec_{2}^{2} = \text{sn}^{2}(u, k) - \frac{1 + k^{2} - \sqrt{1 - k^{2} + k^{4}}}{3k^{2}} \) | \( \alpha_{2}^{2} = 2(1 + k^{2} + \sqrt{1 - k^{2} + k^{4}}) \) |
Equipped with the eigenfunctions of the Lamé equation, we may expand the fields $a_0, b_0, a_1$ and $b_1$ in (133) in terms of these eigenfunctions. By taking account of the boundary conditions the bosonic fields $a_0$ and $b_0$ can be expanded as (68) in the text. The mass eigenvalue for each mode $a_0^{(c,2n)}, b_0^{(s,2n+2)}, b_0^{(c,2n)}$ and $b_0^{(s,2n+2)}$ are given by (69) in the text. In the same way, by taking account of the boundary conditions, the field $a_1$ and $b_1$ can be expanded as (64) and (66), respectively. The eigenvalues for the each mode is given by (65) and (67).

Let us consider the limit of $k \rightarrow 0$ of the Lamé equation. This limit corresponds to the limit of $R \rightarrow R^* = 1/2\Lambda$. In this limit, the Lamé equation is reduced to $-\frac{d^2}{dx^2} \phi(u, k = 0) = \Omega(k = 0) \phi(u, k = 0)$. The boundary conditions in this limit become $\phi(u + \pi) = \pm \phi(u)$. The eigenvalues and the eigenfunctions are easily found, and they are given by

| eigenfunction | eigenvalue |
|---------------|------------|
| $Ec_n^b(u, k = 0)$ | $\alpha_n^0(k = 0) = 0$ |
| $Ec_n^a(u, k = 0)$ | $\alpha_n^a(k = 0) = n^2 \quad n = 1, 2, 3 \ldots$ |
| $Es_n^a(u, k = 0)$ | $\beta_n^a(k = 0) = n^2 \quad n = 1, 2, 3 \ldots$ |

Then, it follows that the expansions of the fields $a_0(x, y), b_0(x, y), a_1(x, y)$ and $b_1(x, y)$ at $k = 0$ become

$$
a_0(x, y) = \frac{1}{\sqrt{2\pi R}} a_0^{(0)}(x) + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} \left[ a_0^{(c,2n)}(x) \cos\left(\frac{n y}{R}\right) + a_0^{(s,2n)}(x) \sin\left(\frac{n y}{R}\right) \right],
$$

$$
b_0(x, y) = \frac{1}{\sqrt{2\pi R}} b_0^{(0)}(x) + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} \left[ b_0^{(c,2n)}(x) \cos\left(\frac{n y}{R}\right) + b_0^{(s,2n)}(x) \sin\left(\frac{n y}{R}\right) \right],
$$

$$
a_1(x, y) = \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} \left[ a_1^{(c,2n-1)}(x) \cos\left(\frac{(n - \frac{1}{2}) y}{R}\right) + a_1^{(s,2n-1)}(x) \sin\left(\frac{(n - \frac{1}{2}) y}{R}\right) \right],
$$

$$
b_1(x, y) = \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} \left[ b_1^{(c,2n-1)}(x) \cos\left(\frac{(n - \frac{1}{2}) y}{R}\right) + b_1^{(s,2n-1)}(x) \sin\left(\frac{(n - \frac{1}{2}) y}{R}\right) \right]. \quad (165)
$$

**B.3 Eigenvalue equation (78)**

Let us consider the properties of the eigenvalues and associated eigenfunctions of the equation

$$
\left[ -\frac{d^2}{dy^2} + (U(y))^2 + \frac{dU(y)}{dy} \right] \phi(y) = m_F^2 \phi(y), \quad (166)
$$

where $U(y) \equiv 2k_0\omega\sin(\omega y, k)$ and $\phi(y + 4\pi R) = \phi(y)$. Let $Ec_n^m(u, k)$ and $Es_n^m(u, k)$ be solutions of the eigenvalue equation with

$$
Ec_n^m(u, k) \xrightarrow{k \rightarrow 0} \cos(nu), \quad Es_n^m(u, k) \xrightarrow{k \rightarrow 0} \sin(nu), \quad (167)
$$

up to normalization constants. Since the eigenvalues of $Ec_n^m(u, k)$ and $Es_n^m(u, k)$ are found to be degenerate, as we will prove later, it is not sufficient to specify $Ec_n^m(u, k)$ and $Es_n^m(u, k)$ only by the conditions (166). The eigenfunctions $Ec_n^m(u, k)$ and $Es_n^m(u, k)$ may be supplemented by the condition of even/oddness under the transformation $u \rightarrow -u$:

$$
Ec_n^m(-u, k) = Ec_n^m(u, k), \quad Es_n^m(u, k) = -Es_n^m(-u, k). \quad (168)
$$

Once we have two properties (167) and (168), the eigenfunctions $Ec^n(u, k)$ and $Es^m(u, k)$ are uniquely specified up to normalization constants.

Since the set of $\{Ec^n(u, k), Es^{n+1}(u, k), n = 0, 1, 2, \cdots\}$ is expected to form a complete set, we can expand $\eta_\pm$ and $\zeta_\pm$ in terms of these eigenfunctions as done in the text (see (81) and (82)). Once we obtain the expansions for these spinor fields, the Dirac spinors $\psi_i$ ($i = 0, 1$) in 3-dimensions are expanded as

$$
\psi_0(x, y) \equiv \frac{1}{\sqrt{2}}(\chi_0(x, y) + i\rho_0(x, y))
$$

$$
= \frac{1}{2} \sum_{n=0}^{\infty} \left[ (\eta^{(c, n)}(x) + i\zeta^{(c, n)}(x))(Ec^n(\omega y, k) + Ec^n(\omega(y + 2\pi R), k)) + (\eta^{(s, n+1)}(x) + i\zeta^{(s, n+1)}(x))(Es^{n+1}(\omega y, k) + Es^{n+1}(\omega(y + 2\pi R), k)) \right],
$$

(169)

$$
\psi_1(x, y) \equiv \frac{i}{\sqrt{2}}(\chi_1(x, y) + i\rho_1(x, y))
$$

$$
= \frac{i}{2} \sum_{n=0}^{\infty} \left[ (\zeta^{(c, n)}(x) + i\eta^{(c, n)}(x))(Ec^n(\omega y, k) - Ec^n(\omega(y + 2\pi R), k)) + (\zeta^{(s, n+1)}(x) + i\eta^{(s, n+1)}(x))(Es^{n+1}(\omega y, k) - Es^{n+1}(\omega(y + 2\pi R), k)) \right].
$$

(170)

Let us discuss the relative magnitude of the eigenvalues $m_F^{(c,n)}$ and $m_F^{(s,n)}$ given by (84) in the text. The first and second relations in (84) are easy to prove. It is easy to see that at $k = 0$ the eigenfunctions $Ec^n(u, k = 0)$ and $Es^m(u, k = 0)$ have $2n$ nodes in the region $0 \leq u < 4K(0) = 2\pi$. This fact implies that the number of modes of $Ec^n(u, k)$ and $Es^m(u, k)$ even for $0 \leq k < 1$ is equal to $2n$. Since the number of nodes for $Ec^n(u, k)$ or $Es^m(u, k)$ is larger than that of $Ec^n(u, k)$ or $Es^m(u, k)$ if $n > m$, we conclude that $m_F^{(c,n)} > m_F^{(s,m)}$ ($i = c, s$). Let us next prove the third relation $m_F^{(c,n)} = m_F^{(s,n)}$ if $n \neq 0$. First, let us note that

$$
\hat{a}^\dagger(u)\hat{a}(u)Ec^n(u, k) = (m_F^{(c,n)}/\omega)^2 Ec^n(u, k),
$$

(171)

where $\hat{a}(u) \equiv -\frac{d}{du} + 2k \sin(u, k)$ and $\hat{a}^\dagger(u) \equiv \frac{d}{du} + 2k \sin(u, k)$. Since $\sin(u + 2K(k), k) = -\sin(u, k)$, $\hat{a}(u)$ and $\hat{a}^\dagger(u)$ satisfy $\hat{a}(u + 2K(k)) = -\hat{a}^\dagger(u)$ and $\hat{a}^\dagger(u + 2K(k)) = -\hat{a}(u)$. It follows that

$$
\hat{a}(u)\hat{a}^\dagger(u)Ec^n(u + 2K(k), k) = (m_F^{(c,n)}/\omega)^2 Ec^n(u + 2K(k), k).
$$

(172)

Multiplying the both sides of the above equation by $\hat{a}^\dagger(u)$, we find that $\hat{a}^\dagger(u)Ec^n(u + 2K(k), k)$ belongs to the same eigenvalue $(m_F^{(c,n)}/\omega)^2$ as $Ec^n(u, k)$. We shall now show that the eigenfunction $\hat{a}^\dagger(u)Ec^n(u + 2K(k), k)$ is indeed proportional to $Es^m(u, k)$ if $n \neq 0$. It is easy to show that

$$
\hat{a}^\dagger(u)Ec^n(u + 2K(k), k) \xrightarrow{u \to -u} \hat{a}^\dagger(-u)Ec^n(-u + 2K(k), k)
$$

$$
= -\hat{a}^\dagger(u)Ec^n(u + 2K(k), k).
$$

(173)

Thus, the eigenfunction $\hat{a}^\dagger(u)Ec^n(u + 2K(k), k)$ is odd under $u \to -u$. We can further show that

$$
\hat{a}^\dagger(u)Ec^n(u + 2K(k), k) \xrightarrow{k \to 0} \frac{d}{du}Ec^n(u + \pi, k = 0)
$$

$$
\propto \sin(nu).
$$

(174)
Therefore, the eigenfunction $\hat{a}^\dagger(u)Ec^n(u + 2K(k), k)$ has been found to satisfy the properties (167) and (168), which $Es^n(u, k)$ should satisfy, and hence $\hat{a}^\dagger(u)Ec^n(u + 2K(k), k)$ can be identified with the eigenfunction $Es^n(u, k)$, up to normalization. It follows that $Ec^n(u, k)$ and $Es^n(u, k)$ belong to the same eigenvalue, i.e. $m_{F}^{(c,n)} = m_{F}^{(s,n)}$ if $n \neq 0$.

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This observation is consistent with the Dvali-Shifman’s discussion [21] on how many SUSY is survived on the field configuration with nontrivial coordinate dependence. In our $Z_2$ model the central charge vanishes in the kink background, so that the SUSY is completely broken in $R \to \infty$ limit.

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Table Captions

| mode | (mass)$^2$ |
|------|-----------|
| $a_0^{(0)}$, $b_0^{(0)}$ | 0 |
| $a_0^{s(n)}$, $b_0^{s(n)}$, $b_0^{c(n)}$, $b_0^{(l)}$ | $(\frac{n}{R})^2$, $n \in \mathbb{Z} > 0$ |
| $a_1^{(l)}$, $a_1^{s(l)}$, $b_1^{c(l)}$, $b_1^{l(l)}$ | $|M|^2 - \Lambda^2 + (\frac{l}{R})^2$, $l \in \mathbb{Z} + \frac{1}{2} > 0$ |

Table III-1. The boson mass spectrum of the $Z_2$ model for $R \leq R^*$.

| mode | (mass)$^2$ |
|------|-----------|
| $\chi_0^{(0)}$, $\rho_0^{(0)}$ | 0 |
| $\chi_0^{s(n)}$, $\chi_0^{c(n)}$, $\rho_0^{s(n)}$, $\rho_0^{c(n)}$ | $(\frac{n}{R})^2$, $n \in \mathbb{Z} > 0$ |
| $\chi_1^{l(l)}$, $\chi_1^{s(l)}$, $\rho_1^{l(l)}$, $\rho_1^{s(l)}$ | $|M|^2 + (\frac{l}{R})^2$, $l \in \mathbb{Z} + \frac{1}{2} > 0$ |

Table III-2. The fermion mass spectrum of the $Z_2$ model for $R \leq R^*$.

| mode | (mass)$^2$ |
|------|-----------|
| $\phi_{a_0}(u, k = 0)$, $\phi_{b_0}(u, k = 0)$ | $(m_{a_0})^2, (m_{b_0})^2$ |
| $\phi_{a_0}^{(c,0)}$, $\phi_{b_0}^{(c,0)} = \frac{1}{\sqrt{2\pi R}}$ | 0 |
| $\phi_{a_0}^{(c,2n)}$, $\phi_{b_0}^{(c,2n)} = \frac{1}{\sqrt{2\pi R}} \cos(2nu)$ | $4n^2\Lambda^2$, $n \in \mathbb{Z} > 0$ |
| $\phi_{a_0}^{(s,2n)}$, $\phi_{b_0}^{(s,2n)} = \frac{1}{\sqrt{2\pi R}} \sin(2nu)$ | $4n^2\Lambda^2$, $n \in \mathbb{Z} > 0$ |
| $\phi_{a_1}(u, k = 0)$ | $(m_{a_1})^2$ |
| $\phi_{a_1}^{(c,2n-1)} = \frac{1}{\sqrt{2\pi R}} \cos(2n - 1)u$ | $((2n - 1)^2 - 1)\Lambda^2$, $n \in \mathbb{Z} > 0$ |
| $\phi_{a_1}^{(s,2n-1)} = \frac{1}{\sqrt{2\pi R}} \sin(2n - 1)u$ | $((2n - 1)^2 - 1)\Lambda^2$, $n \in \mathbb{Z} > 0$ |
| $\phi_{b_1}(u, k = 0)$ | $(m_{b_1})^2$ |
| $\phi_{b_1}^{(c,2n-1)} = \frac{1}{\sqrt{2\pi R}} \cos(2n - 1)u$ | $((2n - 1)^2 + 1)\Lambda^2$, $n \in \mathbb{Z} > 0$ |
| $\phi_{b_1}^{(s,2n-1)} = \frac{1}{\sqrt{2\pi R}} \sin(2n - 1)u$ | $((2n - 1)^2 + 1)\Lambda^2$, $n \in \mathbb{Z} > 0$ |

Table III-3. The boson mass spectrum of the $Z_2$ model in the $k \to 0$ limit.

| eigenfunction | $\Omega_{a_1}$ | $(m_{a_1})^2$ |
|---------------|----------------|----------------|
| $Ec_1(u, k)$ = $cn(u, k)dn(u, k)$ | $\alpha_1^2(k) = 1 + k^2$ | 0 |
| $Es_1(u, k)$ = $sn(u, k)dn(u, k)$ | $\beta_1^2(k) = 1 + 4k^2$ | $3k^2\omega^2$ |

Table III-4. Exact masses and eigenfunctions of some lowest modes for $a_1$.

| eigenfunction | $\Omega_{b_1}$ | $(m_{b_1})^2$ |
|---------------|----------------|----------------|
| $Ec_1^1$ = $cn(u, k)$ | $\alpha_1^1(k) = 1$ | $(2 + k^2)\omega^2$ |
| $Es_1^1$ = $sn(u, k)$ | $\beta_1^1(k) = 1 + k^2$ | $2(1 + k^2)\omega^2$ |

Table III-5. Exact masses and eigenfunctions of some lowest modes for $b_1$.

| $a_0, b_0$ | (mass)$^2$ |
|-------------|-------------|
| lowest mode | $2k^2\omega^2 + \mathcal{O}(k^4)$ |
| 1st excited mode | $4\omega^2 + \mathcal{O}(k^4)$ |

Table III-6. Perturbative mass spectrum of the first two lowest modes for $a_0$ and $b_0$. 

Table III-7. Perturbative mass spectrum of the 2nd excited modes for \( a_1 \) and \( b_1 \).

| \( a_1, b_1 \) | (mass)² |
|-----------------|---------|
| 2nd excited mode of \( a_1 \) | \((9 - \frac{2}{3}k^2)\omega^2 + \mathcal{O}(k^4)\) |
| 2nd excited mode of \( b_1 \) | \((9 - \frac{2}{3}k^2)\omega^2 + \mathcal{O}(k^4)\) |

Table III-8. Eigenvalues and eigenfunctions of \((78)\) in the limit of \( k \to 0 \).

| \( \phi(u,k=0) \) | \( \left(\frac{m_F(k=0)}{\Lambda}\right)^2 \) |
|----------------|-----------------|
| \( \cos(nu) \) | \( n^2 \) \( n \in \mathbb{Z} \geq 0 \) |
| \( \sin(nu) \) | \( n^2 \) \( n \in \mathbb{Z} > 0 \) |

Table III-9. Perturbative mass spectrum of the higher modes of the fermions

| Symmetry | \( R \leq R^* \) | \( R > R^* \) | \( R \to \infty \) |
|----------|----------------|----------------|----------------|
| \( \eta^{(c,n)} \), \( \eta^{(s,n)} \), \( \zeta^{(c,n)} \), \( \zeta^{(s,n)} \) | \( R \leq R^* \) | \( R > R^* \) | \( R \to \infty \) |
| \( n = 1 \) 1st excited state | \( (1 + \frac{14}{9}k^2)\omega^2 + \mathcal{O}(k^4) \) |
| \( n = 2 \) 2nd excited state | \( (4 + \frac{3}{30}k^2)\omega^2 + \mathcal{O}(k^4) \) |
| \( n = 3 \) 3rd excited state | \( (9 - \frac{691}{280}k^2)\omega^2 + \mathcal{O}(k^4) \) |

Table III-10. Phase structure of the \( Z_2 \) model. The \( U(1)_R \) symmetry exists only in the superpotential \((19)\).

| Symmetry | \( R \leq R^* \) | \( R > R^* \) | \( R \to \infty \) |
|----------|----------------|----------------|----------------|
| \( \eta^{(c,n)} \), \( \eta^{(s,n)} \), \( \zeta^{(c,n)} \), \( \zeta^{(s,n)} \) | \( R \leq R^* \) | \( R > R^* \) | \( R \to \infty \) |
| Supersymmetry | \( R \leq R^* \) | \( R > R^* \) | \( R \to \infty \) |
| \( \lambda \) | \( \times \) | \( \times \) | \( \times \) |
| Translational inv. | \( \circ \) | \( \times \) | \( \times \) |
| \( Z_2 \) | \( \circ \) | \( \times \) | \( \times \) |
| \( U(1)_R \) | \( \times \) | \( \circ \) | \( \circ \) |

Table IV-1. Phase structure of the \( U(1) \) model. The \( U(1)_R \) symmetry exists only in the superpotential \((22)\) and the \( U(1)' \) symmetry is a linear combination of the translational invariance and the \( U(1) \) symmetry.
Figure 1: The $R$-dependence of the mass spectrum of the $Z_2$ model with $\langle A_0 \rangle = \mu/g$ is depicted for a few lowest mass eigenstates. The solid lines correspond to bosonic states and the dashed lines to fermionic ones. The numbers in the parentheses represent the degeneracy.
Figure 2: The $R$-dependence of the mass spectrum of the $U(1)$ model with $\langle A_0 \rangle = 0$ is depicted for a few lowest mass eigenstates. The solid lines correspond to bosonic states and the dashed lines to fermionic ones. The numbers in the parentheses represent the degeneracy.
Figure 3: The $R$-dependence of $\mathcal{E}[A_0, A_1]$ in the $Z_2$ model. The lines for $n = 1, 2$ and 3 correspond to $\mathcal{E}[A_0^{\text{(II,1)}}, A_1^{\text{(II,1)}}]$, $\mathcal{E}[A_0^{\text{(II,2)}}, A_1^{\text{(II,2)}}]$ and $\mathcal{E}[A_0^{\text{(II,3)}}, A_1^{\text{(II,3)}}]$, respectively. We find the vacuum configuration for $R > R^*$ is given by $A_0 = A_0^{\text{(II,1)}}$ and $A_1 = A_1^{\text{(II,1)}}$. 