A cluster algorithm for Lattice Gauge Theories

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Abstract

A new algorithm for simulating compact U(1) lattice gauge theory in three dimensions is presented which is based on global changes in the configuration space. We show that this algorithm provides an effective way to extract partition functions at given external flux. As an application, we study numerically the finite temperature deconfinement phase transition.

Key words: Lattice Gauge Theory, Monte Carlo methods, Cluster algorithms

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In the presence of topological excitations local Monte Carlo simulations may become inefficient at exploring the whole configuration space. An example is provided by compact U(1) lattice gauge theory. In the Wilson formulation, the action of the model is given by

\[ S = \beta \sum_{i, \mu < \nu} \left( 1 - \cos(\theta_{\mu}(i) + \theta_{\nu}(i + \hat{\mu}) - \theta_{\mu}(i + \hat{\nu}) - \theta_{\nu}(i)) \right), \tag{1} \]

where \( \theta_{\mu}(i) \) is an angular variable defined on the link stemming from the point \( i \) in the direction \( \mu \) and \( \beta \) is related to the bare coupling \( g \) and to the lattice spacing \( a \) by \( \beta = 1/ag^2 \). We shall consider the model in three dimensions.

In this model duality transformations can be performed, which amounts to Fourier-transforming the Boltzmann weight \([1]\). The dual theory is formulated in terms of a gas of Dirac magnetic monopoles. By using the dual description, it can be proven that at zero temperature the system permanently confines static charges for any value of the coupling \([2,3]\). Using the same techniques, it can be proven that at finite temperature permanent confinement is lost at some critical value of the temperature \( T_c \), above which the system is in a deconfined phase \([4]\). The phase transition is

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expected to be of the Berezinskii-Kosterlitz-Thouless type. A numerical study of
the phase transition can be found in [5].
In the direct formulation (1) magnetic monopoles appear as topological excitations
located at the centre of cubes and are detected as sources of magnetic flux [6].
Flux conservation is provided by Dirac strings ending at monopole locations. The
theory being compact, Dirac strings are undetectable. Because of flux conservation,
Dirac strings must either be attached to a monopole or form closed loops. On a
finite periodic lattice closed Dirac strings with non-trivial topology exist, which
divide the lattice into sectors. These are labelled by integers representing the total
wrapping number of closed strings in each direction. An equivalent description of
the topological sectors can be performed in terms of fluxes [7]. In the presence of
an external flux $\phi$ across one plane the partition function is given by

$$Z[\phi] = \int_{\{\phi\}} \left( \prod_{i,\mu} D\theta_\mu(i) \right) e^{-S}, \quad (2)$$

where $\{\phi\}$ indicates the set of all configurations carrying flux $\phi$. Again because
of compactness, external fluxes differing by $2\pi n$ are indistinguishable. The integer
$n$ counts the total number of closed Dirac strings piercing the lattice in the given
direction.
In numerical simulations of the model traditional algorithms prove to be inefficient
in exploring ergodically the different topological sectors [8]. In this work we present
a cluster-like algorithm which is related to a recently introduced algorithm for sim-
ulating bosonic systems [9]. In our algorithm a proposed update consists of a flux
of randomly selected magnitude going through plaquettes pierced by a closed path
in the dual lattice (we call that path a ”worm”). The likelihood of this update stra-
tegy being more effective than local ones is based on the fact that it should be more
effective at forcing different units of flux into the system. The basic steps of our
algorithm can be summarised as follows:

1. a number $\varphi \in [\pi, \pi]$ is drawn with uniform probability;
2. a site $x_i = x_0$ of the dual lattice is randomly selected;
3. a priori probabilities to increase (respectively decrease) by $\varphi$ the flux of each
   plaquette pierced by links stemming from (resp. ending in) $x_0$ are computed;
4. a direction $\hat{i}$ is selected according to the probabilities computed in the previous
   step and its flux is updated accordingly;
5. steps 2 to 4 are iterated from the new point $x_n = x_{n-1} + \hat{i}$ until $x_n$ is equal to
   the initial point $x_i$;
6. the new configuration is accepted with a probability $P_{\text{accept}}$, related to the
   local configuration around $x_i$ in the starting and final configurations (see [9]
   for more details). In most cases the acceptance rate is around 0.95-0.98.

The proof of detailed balance is analogous to the one given in [9]. An advantage
of the “worm” algorithm over a local one is that it changes the flux through the sys-
tem by generating worms with non-trivial winding number, which should guarantee better ergodicity. Moreover, our algorithm allows to determine $Z[\phi]$ (which is proportional to the probability of having a flux $\phi$) for different values of the flux.

As an application of the “worm” algorithm, we have studied $Z[\phi]$ across the deconfinement phase transition at finite temperature. The expectation is that while in the confined phase $Z[\phi]$ is independent of $\phi$, in the deconfined phase higher fluxes through temporal planes are energetically disfavoured. In fig. 1 we plot $Z[\phi]$ (normalised to 1) in the confined and the deconfined phase. While in the former case $Z$ is independent of the flux, in the latter a clear peak at $\phi = 0$ emerges. An interesting quantity is $Z[\pi]/Z[0]$, which is one in the confined phase and goes to zero in the deconfined phase. This ratio is plotted in fig. 2 as a function of $\beta$: its sudden drop in the critical region is a clear signal of deconfinement. A finite size study of $Z[\pi]/Z[0]$ is currently in progress.

In the critical region, our algorithm proves to be more efficient than a local update at decorrelating topological sectors. For instance, on a $32^2 \times 6$ lattice it gives shorter autocorrelation times by a factor of 2-3. This factor is likely to increase on larger lattices. This result might extend also to other observables like the total number of monopoles, for which on a $64^2 \times 6$ lattice we find $\tau_{\text{worm}} \simeq 2.0 \tau_{\text{local}}$ for $\beta \simeq \beta_c$.

![Fig. 1. Probability distribution of the flux on a $32^2 \times 6$ lattice at $\beta = 1.80$ (confined phase) and $\beta = 2.50$ (deconfined phase).](image)

In conclusion, we have presented a new algorithm for simulating compact U(1) in 3d. Compared to local updates, our algorithm has the advantage of providing better ergodicity across topological sectors and a clean way for measuring ratios of partition functions. We have preliminary indication that close to the deconfinement phase transition our algorithm has shorter autocorrelation times than local updates. The autocorrelation time of the “worm” algorithm can be furtherly reduced by a more appropriate choice of the distribution of the proposed flux or by reducing the backtracking of the worm, using the concept of “directed worms” [10,11].
Fig. 2. $Z[\pi]/Z[0]$ as a function of $\beta$ on a $32^2 \times 6$. The vertical line is located at $\beta_c$ [5].

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