Temporal Means of a Metastable System of Spiking Neurons

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January 31, 2022

Abstract

We consider a stochastic system of spiking neurons which was previously proven to present a metastable behavior for a suitable choice of the parameter, in the sense that the time of extinction is asymptotically memory-less when the number of components in the system goes to $\infty$. In the present article we complete this work by showing that, previous to extinction, the system tends to stabilize in the sense that temporal means taken on an appropriate time scale converge in probability to some fixed value. This property is sometime called thermalization.

MSC Classification: 60K35; 82C32; 82C22.

Keywords: systems of spiking neurons; interacting particle systems; metastability; GL model.

1 Introduction

Informally the model is as follows. Each element in the system (a neuron) is associated to a random variable called its membrane potential. Each neuron is also associated to a point process which intensity varies across time depending on the current value of the membrane potentials, representing the spiking times, and to another point process (which is Poisson of some fixed rate $\gamma \geq 0$) representing the leakage phenomenon, that is, the drift of the membrane potential toward its resting value caused by the natural diffusion of ions through the membrane when some equilibrium has not been reached. Thus, unlike in the original model (introduced in [7]) the present one was inspired from, these leaks occurs at discrete times at which the membrane potential is reset to its resting value (conventionally set to 0). Moreover, like in the original model, the membrane potential of any given neuron is also reset to the resting value whenever this neuron spikes. Furthermore in this model all the neurons are excitatory, with the same synaptic weight (conventionally set to 1).

We give a formal definition of the model in the next section.

This model was introduced in [6] and studied for the instantiation in which there is an infinite number of neurons indexed by $\mathbb{Z}$, and each neuron is connected to its immediate neighbors, on the right and on the left. It was proven that the system then presents a phase transition with respect to the parameter $\gamma$: there exists a $0 < \gamma_c < \infty$ such that if $\gamma > \gamma_c$ then each neuron stop emitting spikes in finite time almost surely, while if $\gamma < \gamma_c$ each neuron has a positive probability of emitting spikes forever. Then it was proven in [11] that there exists a $\gamma'_c < \gamma_c$ such that if you consider a finite version of this model, in which the neurons are indexed by a finite window of $\mathbb{Z}$, and take $\gamma < \gamma'_c$, then the instant of the last spike of the system (which is almost surely finite) converges to an exponential random variable as the width of the window grows. This type of behavior is characteristic of metastable dynamics (see the seminal paper [3]). Both results will be of importance in order to derive the main theorem of the present article. The result of phase transition is restated in the next section in a slightly different form, and the result concerning the
time of extinction will be restated in Section 5, for \( \gamma < \gamma_c \), as a side consequence of the analysis that will be conducted up to that point is that \( \gamma'_c = \gamma_c \).

One of the important ideas of the proof of the main result is to consider an auxiliary process, namely the spiking rate process, which is an interacting particle system taking value in \( \{0,1\}^Z \), as well as its dual (in the sense introduced by T. Harris in [8]). Then exploiting interesting properties of this auxiliary process allows us to derive the proof of our central result.

The paper is organized as follows. In Section 2 we introduce formally our model, as well as the auxiliary process and its dual. We also summarize already obtained results concerning these processes which will be important in the sequel. In Section 3 we obtain a crucial result about the drift of the right-most component of the dual process. This result is then used in Section 4 to prove that the auxiliary process has exponentially decaying time correlations. Finally the main result is stated and proven in Section 5 using the previously established results.

2 The model, the auxiliary process and its dual

The model considered in this work is as follows. Let \( I \) be a finite or countable set representing the neurons, and to each \( i \in I \) associate a set \( V_i \subset S \) of presynaptic neurons. Each neuron \( i \in I \) has a membrane potential evolving over time, represented by a stochastic process which takes its values in the set \( \mathbb{N} \) of non-negative integers and which is denoted \( (X_i(t))_{t \geq 0} \). Moreover each neuron is also associated with a Poisson process \( (N^i(t))_{t \geq 0} \) of some parameter \( \gamma \), representing the leak times. At any of these leak times the membrane potential of the neuron concerned is reset to 0. Finally another point process \( (N^*_i(t))_{t \geq 0} \) representing the spiking times is also associated to each neuron, which infinitesimal rate at time \( t \) is given by \( \phi(X_i(t)) \), where \( \phi \) is some rate function. When a neuron spikes its membrane potential is reset to 0 and the membrane potential of all of its postsynaptic neurons is increased by one (that is the neurons of the set \( \{ j : i \in V_j \} \)). All the point processes involved are assumed to be mutually independent.

Mathematically, beside asking that \( (N^i(t))_{t \geq 0} \) be a Poisson process of some parameter \( \gamma \), this is the same as saying that \( (N^*_i(t))_{t \geq 0} \) is the point process characterized by the two following equations

\[
E(N^*_i(t) - N^*_i(s)|\mathcal{F}_s) = \int_s^t E(\phi(X_i(u))|\mathcal{F}_s)du
\]

where

\[
X_i(t) = \sum_{j \in V_i} \int_{[L_i(t), t]} dN^*_j(s),
\]

\( L_i(t) \) being the time of the last event affecting neuron \( i \) before time \( t \), that is,

\[
L_i(t) = \sup \left\{ s \leq t : N^*_i(\{ s \}) = 1 \text{ or } N^i(t)(\{ s \}) = 1 \right\}.
\]

\( (\mathcal{F}_t)_{t \geq 0} \) is the standard filtration for the point processes involved here, that is the filtration which at any time \( t \geq 0 \) is equal to the \( \sigma \)-algebra generated by the family \( \{ N^*_i(s), N^i(t)(\{ s \}), s \leq t, i \in S \} \).

In this article we continue the study initiated in [9] and [11], and we study the specific case in which the activation function is simply a hard threshold of the form \( \phi(x) = 1_{x > 0} \) and the spatial structure of the network is given by a nearest-neighbor interaction on the one-dimensional lattice, that is: \( I = \mathbb{Z} \) and \( V_i = \{ i-1, i, i+1 \} \). Then for any \( i \in I \) and \( t \geq 0 \) we write \( \xi_i(t) = 1_{X_i(t) > 0} \), and \( \xi(t) = (\xi_i(t))_{i \in \mathbb{Z}} \). The resulting auxiliary process \( (\xi(t))_{t \geq 0} \) gives the state of each neuron (active or quiescent) at any time, and it belongs to the category of interacting particle systems on \( \{0,1\}^\mathbb{Z} \) (see [10]). Its dynamic is given by the following generator

\[
\mathcal{L} f(\eta) = \gamma \sum_{i \in \mathbb{Z}} \left( f(\pi^{-1}_i(\eta)) - f(\eta) \right) + \sum_{i \in \mathbb{Z}} \eta_i \left( f(\pi_i(\eta)) - f(\eta) \right),
\]

where \( f : \{0,1\}^\mathbb{Z} \to \mathbb{R} \) is a cylinder function\(^1\) and the \( \pi^i \)'s and \( \pi_i \)'s are maps from \( \{0,1\}^\mathbb{Z} \) to \( \{0,1\}^\mathbb{Z} \) defined for any \( i \in \mathbb{Z} \) as follows:

\(^1\)Here and in the rest of the paper we call cylinder function any function \( f : \{0,1\}^\mathbb{Z} \to \mathbb{R} \) which only depends on a finite number of sites. The set \( S \subset \mathbb{Z} \) of the sites on which \( f \) depends is called the support of \( f \).
\( \pi_i(\eta) \) = \begin{cases} 0 & \text{if } j = i, \\ \eta_j & \text{otherwise,} \end{cases} \quad (2.4)

and

\( \pi_i(\eta) \) = \begin{cases} 0 & \text{if } j = i, \\ \max(\eta_k, \eta_j) & \text{if } j \in \{i-1, i+1\}, \\ \eta_j & \text{otherwise.} \end{cases} \quad (2.5)

We also consider the finite versions of the process \( (\xi(t))_{t \geq 0} \), in which the neurons are indexed on \([-N, N]\) (Here and in the sequel \([-N, N]\) is a short-hand for \(\mathbb{Z} \cap [-N, N] \)) for some integer \( N \geq 0 \) instead of the whole lattice \( \mathbb{Z} \). For any \( i \in [-N, N] \), let the set \( \mathbb{V}_i \) be equal to \( \mathbb{V} \cap [-N, N] \). Then the finite process is the one with state space \( \{0, 1\}^{2N+1} \) which is given by generator \( \mathcal{L}_i \) when you replace \( \mathbb{Z} \) by \([-N, N]\) under the summation, and where the maps \( \pi_{i,n} \) are defined with respect to \( \mathbb{V}_i \) instead of simply \( \{i-1, i+1\} \). We write \( (\xi_N(t))_{t \geq 0} \) for these finite versions.

### 2.1 Graphical construction

It is possible to use a graphical construction of the type of the construction introduced by Harris in \([8]\) to propose an alternative definition of the auxiliary process. The construction is as follows.

For any neuron \( i \in \mathbb{Z} \), let \( (N_i(t))_{t \geq 0} \) and \( (N_i^1(t))_{t \geq 0} \) be two independent homogeneous Poisson processes with respective intensity \( 1 \) and \( \gamma \). Let \( (T_{i,n})_{n \geq 0} \) and \( (T_{i,n}^1)_{n \geq 0} \) be their respective jump times. All the Poisson processes are assumed to be mutually independent and we let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be the probability space on which these Poisson processes are defined.

Now we adjoin the following structure to the time-space diagram \( \mathbb{Z} \times \mathbb{R}_+ \):

- for all \( i \in \mathbb{Z} \) and \( n \in \mathbb{N} \) put a "\( \delta \)" mark at the point \((i, T_{i,n})\),
- for all \( i \in \mathbb{Z} \) and \( n \in \mathbb{N} \) put an arrow pointing from \((i, T_{i,n})\) to \((i+1, T_{i,n})\) and another pointing from \((i, T_{i,n})\) to \((i-1, T_{i,n})\).

We obtain a random graph that we denote \( \mathcal{G} \), which consists of the time-space diagram \( \mathbb{Z} \times \mathbb{R} \) augmented by the set of "\( \delta \)" marks and horizontal arrows. Then for any \( i, j \in \mathbb{Z} \) and \( t < s \) we call a path from \((i, t)\) to \((j, s)\) on \( \mathcal{G} \) any contiguous sequence of (closed) time segments and arrows starting at \((i, t)\) and ending at \((j, s)\). Moreover we say that a path is valid if it never crosses a "\( \delta \)" mark and, when moving upward, it never crosses the rear side of an arrow. We write \((i, t) \rightarrow (j, s)\) when there is a valid path from \((i, t)\) to \((j, s)\) in \( \mathcal{G} \). More generally, for any sets \( A, B \in \mathcal{P}(\mathbb{Z}) \) we say that there is a valid path from \( A \times t \) to \( B \times s \) if there exists \( i \in A \) and \( j \in B \) such that \((i, t) \rightarrow (j, s)\).

With this construction we can easily give the following characterization of our stochastic process\(^2\). For any \( A \in \mathcal{P}(\mathbb{Z}) \), and for any \( t \geq 0 \):

\[ \xi^A(t) = \{ j \in \mathbb{Z} : (i, 0) \rightarrow (j, t) \text{ for some } i \in A \} . \]

Then \( (\xi^A(t))_{t \geq 0} \) is the process with generator given by \( \mathcal{L}_i \) and initial state \( \xi^A(0) = A \). A similar characterization can obviously be given for the finite processes \( (\xi_N(t))_{t \geq 0} \), using only the Poisson processes of the sub-diagram \([-N, N] \times \mathbb{R}_+ \), allowing a simple and useful coupling between the finite and the infinite processes.

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\(^2\)Notice that with this new definition the process is defined on \( \mathcal{P}(\mathbb{Z}) \) instead of \( \{0, 1\}^{\mathbb{Z}} \). It is of course only a matter of notation, as any element \( \eta \) of \( \{0, 1\}^{\mathbb{Z}} \) can be bijectively mapped to an element \( A \) of \( \mathcal{P}(\mathbb{Z}) \) via the relation \( A = \{ i \in \mathbb{Z} \text{ such that } \eta_i = 1 \} \). In practice we will indifferently use both ways.
Finally we define the time of extinction of the system. Indeed the state 0 on which all neurons are quiescent is an absorbing state, once it has been reached the system will remains quiescent for eternity, so that for any $A \in \mathcal{P}(\mathbb{Z})$ the time of extinction, denoted $\tau^A$, is defined as follows
\[
\tau^A = \inf\{t \geq 0 : \xi^A(t) = 0\}.
\]

Similarly we define the time of extinction of the finite process. For any $A \in \mathcal{P}([-N, N])$ let
\[
\tau^A_N = \inf\{t \geq 0 : \xi^A_N(t) = 0\}.
\]

### 2.2 Dual process

It is possible to define a dual process for $(\xi(t))_{t \geq 0}$, which is particularly useful for the study of the original process. Again, for any $i \in \mathbb{Z}$, let’s consider two independent homogeneous Poisson processes $(N_i(t))_{t \geq 0}$ and $(N_i^1(t))_{t \geq 0}$ with intensity 1 and $\gamma$ respectively, and let $(T_{i,n})_{n \geq 0}$ and $(\tilde{T}_{i,n})_{n \geq 0}$ be their respective jump times. As previously all the Poisson processes are assumed to be mutually independent.

The time-space diagram $\mathbb{Z} \times \mathbb{R}_+$ is then augmented in order to obtain the dual graph $\tilde{\mathcal{G}}$ as follows:

- for all $i \in \mathbb{Z}$ and $n \in \mathbb{N}$ put a "δ" mark at the point $(i, \tilde{T}_{i,n})$,
- for all $i \in \mathbb{Z}$ and $n \in \mathbb{N}$ put an arrow pointing from $(i, \tilde{T}_{i,n})$ to $(i, T_{i,n})$ and another pointing from $(i-1, T_{i,n})$ to $(i, \tilde{T}_{i,n})$.

Now we say that a path in $\mathcal{G}$ is a dual-valid path if it satisfies the following constraints: it never cross a "δ" mark and, when moving upward, it never cross the tip of an arrow. We write $(i, t) \xrightarrow{\text{dual}} (j, t')$ when there is a dual-valid path from $(i, t)$ to $(j, t')$ in $\tilde{\mathcal{G}}$.

Then for any $A \in \mathcal{P}(A)$ and for any $t \geq 0$ we write
\[
\eta^A(t) = \{j \in \mathbb{Z} : (i, 0) \xrightarrow{\text{dual}} (j, t) \text{ for some } i \in A\}.
\]

The process $(\eta(t))_{t \geq 0}$ thus defined is the dual of $(\xi(t))_{t \geq 0}$. We briefly explain the point of introducing a dual process, and we refer to [3] and [2] for more details about duality. The crucial point is that we can relate the initial process $(\xi(t))_{t \geq 0}$ and its dual in the following way. We fix some $s \in \mathbb{R}_+$, and for any $0 \leq t \leq s$ and $A \in \mathcal{P}((\mathbb{Z})$ we define the following random variable on $\mathcal{P}(\mathbb{Z})$ via the random graph $\mathcal{G}$ of the previous section
\[
\zeta^A(t) = \{i \in \mathbb{Z} : (s-t, i) \rightarrow (s, j) \text{ for some } j \in A\}. \tag{2.6}
\]

That way it is easy to see that $(\zeta_t)_{t \in [0, s]}$ is the dual process $(\eta_t)_{t \geq 0}$ restricted to the time interval $[0, s]$, and built on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of the orginal process $(\xi_t)_{t \geq 0}$. We call this way of defining the dual process by a coupling with the initial process—with the time reversed—the backward version of the dual process. Moreover it is straightforward to check that for any $A, B \in \mathcal{P}(\mathbb{Z})$ and any $t \geq 0$ the following holds
\[
\{\xi^A_t \cap B \neq \emptyset\} = \{\zeta^B_s \cap A \neq \emptyset\}. \tag{2.7}
\]

In particular this relation holds in probability as well, which implies the following important proposition.

**Proposition 2.1.** For any $A, B \in \mathcal{P}(\mathbb{Z})$, and $t \geq 0$ we have
\[
\mathbb{P}(\xi^A(t) \cap B \neq \emptyset) = \mathbb{P}(\eta^B(t) \cap A \neq \emptyset).
\]
For any $A \in \mathcal{P}(\mathbb{Z})$ we denote by $\sigma^A$ the time of extinction of the dual process

$$\sigma^A = \inf \{t \geq 0 : \eta^A(t) = \emptyset\}.$$  

2.3 Important properties of the auxiliary process and its dual

In this section we summarize some of the properties that have already been proven for the model and its dual. In order to avoid redundancy most of the properties are stated only for the process $(\xi(t))_{t \geq 0}$ but they hold for the dual process $(\eta(t))_{t \geq 0}$ as well. The proofs can be found in Section 4 of [1]. Below, and from now on, we will use some new notation. The time will sometimes be written as a subscript if it is more suitable, writing simply $\xi^t$ instead of $(\xi(t))_{t \geq 0}$. We sometimes write $\xi \equiv 0$ (or $\eta \equiv 0$) for the state in which all neurons are quiescent, and $\xi \equiv 1$ (or $\eta \equiv 1$) for the state in which all neurons are active. For any set $A \in \mathcal{P}(\mathbb{Z})$ and for any $i \in \mathbb{Z}$ we denote by $A + i$ the set $\{j + i \mid j \in A\}$. When the initial state is a singleton we drop the curly bracket, writing for example $\sigma^0$ instead of $\sigma^{(0)}$. When the initial state is the whole lattice $\mathbb{Z}$ we will omit the superscript, writing simply $\xi(t)$ for $\xi^t(t)$. Moreover the state space $\{0,1\}^\mathbb{Z}$ is associated with the partial order relation defined for any $\xi, \eta \in \{0,1\}^\mathbb{Z}$ by: $\xi \leq \eta$ if and only if $\xi_i \leq \eta_i$ for all $i \in \mathbb{Z}$. Whenever we say that a function on $\{0,1\}^\mathbb{Z}$ is monotonous, it is to be understood with respect to this partial order. Finally, for any probability measure $\nu$ on $\{0,1\}^\mathbb{Z}$ (associated with its standard Borel $\sigma$-algebra) and for any measurable function $f$ on $\{0,1\}^\mathbb{Z}$ we write $\nu(f) = \int f d\nu$.

(i) **Additivity:** From the graphical constructions we immediately obtain that for any $A, B \in \mathcal{P}(\mathbb{Z})$ and for any $t \geq 0$ the following holds

$$\xi^{A \cup B}(t) = \xi^A(t) \cup \xi^B(t). \quad (2.8)$$

(ii) **Monotonicity:** The previous property implies that for any $A, B \in \mathcal{P}(\mathbb{Z})$ such that $A \subset B$ and for any $t \geq 0$

$$\xi^A(t) \subset \xi^B(t). \quad (2.9)$$

(iii) **Attractiveness:** By definition an interacting particle system on $\{0,1\}^\mathbb{Z}$ with semi-group $(S(t))_{t \geq 0}$ is attractive if for any increasing function on $\{0,1\}^\mathbb{Z}$ the function $S(t)f$ is increasing for any $t \geq 0$. For any $\xi, \eta \in \{0,1\}^\mathbb{Z}$ satisfying $\xi \leq \eta$ it is immediate using monotonicity that for any increasing $f$ and for any $t \geq 0$ we have $\mathbb{E}_S(f(\xi)) \leq \mathbb{E}_S^\eta(f(\xi))$, so that our system is indeed attractive.

(iv) **Translation invariance:** It is clear from the graphical construction that the law of the process does not change if the time-space diagram is translated to the right or to the left. In particular for any $A \in \mathcal{P}(\mathbb{Z})$ and for any $i \in \mathbb{Z}$ the process $(\xi^A_{t+i})_{t \geq 0}$ has the same law as the process $(\xi^A_{t})_{t \geq 0}$.

(v) **Phase transition:** There exists a critical value $0 < \gamma_c < +\infty$ such that

$$\mathbb{P}(\sigma^0 = +\infty) > 0 \text{ if } \gamma < \gamma_c$$

and

$$\mathbb{P}(\sigma^0 = +\infty) = 0 \text{ if } \gamma > \gamma_c.$$  

(vi) **Invariant measures:** If $\gamma < \gamma_c$, then there exists a non-trivial invariant measure (in the sense that it doesn’t give mass 1 to $\xi \equiv 0$) for $(\xi^t)_{t \geq 0}$, which corresponds to the weak limit of $\xi_t$ when $t$ diverges, and which we denote $\mu$. There is an analogous invariant measure for the dual process $(\eta^t)_{t \geq 0}$ which we denote $\hat{\mu}$.

(vii) **Stochastic monotonicity:** The convergence toward $\mu$ is monotonous in the sense that for any continuous and increasing function $f : \{0,1\}^\mathbb{Z} \rightarrow \mathbb{R}$ and for $0 < s < t$ the following holds

$$\mathbb{E}(f(\xi_s)) \geq \mathbb{E}(f(\xi_t)) \geq \mu(f).$$
(viii) **Positive density**: Define the density of the system $\rho = \mu(\{\eta : \eta_0 = 1\})$. By phase transition and duality (Proposition 2.1), if $\gamma < \gamma_c$ then $\rho > 0$. The same holds for the density of the dual process $\tilde{\rho} = \tilde{\mu}(\{\eta : \eta_0 = 1\})$. While this result isn’t proven for the dual in [1], as it wasn’t explicitly needed, it is very easy to prove it using other results proven there, as shown by the following computation. Let $H \subset \{0, 1\}^Z$ and suppose $\gamma < \gamma_c$, then by Proposition 4.8 from [1]

$$\tilde{\mu}(H) \leq \tilde{\mu}(\eta \equiv 0) + \sum_{i \in \mathbb{Z}} \tilde{\mu}(\{\eta \in H : \eta_i = 1\}) \leq \sum_{i \in \mathbb{Z}} \tilde{\rho},$$

so that if $\tilde{\rho} = 0$ then $\tilde{\mu}$ shall be identically equal to 0, which is obviously a contradiction (to the fact that it shall be a probability measure for example).

(ix) **Spatial ergodicity**: The measure $\mu$ is spatially ergodic. See Theorem 4.9 in [1] for more details. While this Theorem is proven only for $\mu$ there, it is easy to check that all the arguments hold for $\tilde{\mu}$ as well.

A fact that will be important in the next two sections is that our process falls into the category of what is called growth models in [3], that is, attractive and translation invariant systems with $\emptyset$ as an absorbing state and finite range interaction.

## 3 Preliminary results

In this section we study the drift of the edge of the dual. The main result is that in the sub-critical regime the drift is linear, with a positive slope. This fact will be of importance in order to prove exponential estimates for the time of extinction and the time correlations.

For any set $A \in \mathcal{P}(\mathbb{Z})$ we write:

$$r_t^i = \max\{i \in \eta_t^A\} \quad \text{and} \quad l_t^i = \min\{i \in \eta_t^A\}.$$

Moreover we write $(\eta^-_t)_{t \geq 0}$ and $(\eta^+_t)_{t \geq 0}$ for the dual processes starting from $\eta^-_0 = \bar{\{0\}}$ and $\eta^+_0 = \bar{0, +\infty}$ respectively, and for any $t \geq 0$ we denote $r_t^- = \max\{i \in \eta_t^-\}$ and $l_t^+ = \min\{i \in \eta_t^+\}$.

Let’s start with the following lemma.

**Lemma 3.1.** For any $t \geq 0$, if $a^0 > t$ then:

(i) $r_t^0 = r_t^-$ and $l_t^0 = l_t^+$,

(ii) $\eta_t^0 = \eta_t^- \cap [l_t^0, r_t^0] = \eta_t^+ \cap [l_t^0, r_t^0],$

(iii) and $l_s^+ \leq r_s^- \quad \text{for every} \quad s \leq t.$

**Proof.** The proof of (i) follows easily from the graphical construction and is quite similar to the proofs of the three first lemmas in Section 5 of [1]. We give a quick sketch here. By set monotonicity $r_t^0 \leq r_t^-$. Moreover, as $r_t^- \in \eta_t^-$, by definition we can find a dual-valid path from some $i \in \bar{-\infty, 0}$ to $r_t^-$. If $i = 0$ then $r_t^- \in \eta_t^0$ so that $r_t^0 \geq r_t^-$ and the proof is over. Now if $i < 0$ then the dual valid-path from $i$ to $r_t^-$ has to cross the left border of $(\eta^0_t)_{t \geq 0}$ somewhere before $t$. Now the concatenation of the path following the left border from 0 to the crossing point and the path going from the crossing point to $r_t^-$ is a dual-valid path from 0 to $r_t^-$, proving again that $r_t^0 \geq r_t^-$, which is ends the proof. Of course the fact that $l_t^0 = l_t^+$ is then immediate by symmetry.

For (ii) only prove $\eta_t^0 = \eta_t^- \cap [l_t^0, r_t^0]$, the proof for the other equality being obviously identical. The fact that $\eta_t^0 \subset \eta_t^- \cap [l_t^0, r_t^0]$ is immediate by monotonicity so that it suffices to show the reverse inclusion. This is easily done by the same kind of argument as in the first item. Let

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3Notice that while $\rho$ is defined with respect to neuron 0, by translation invariance it could be define with respect to any other neuron $i \in \mathbb{Z}$. 

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$j \in \eta^{-}_t \cap [l^+_t, r^+_t]$, then there exists $i \in ]-\infty, 0[$ such that $(i,0) \stackrel{\text{dual}}{\longrightarrow} (j,t)$. If $i = 0$ then there is nothing to prove, while if $i < 0$ then the path from $i$ to $j$ has to cross $(l^+_t)_{t \geq 0}$ somewhere implying that $j \in \eta^{-}_0$.

For the last item notice that if $\eta^0_t \neq \emptyset$ then $\eta^0_t \neq \emptyset$ for every $s \leq t$ so that by (i), for any fixed $s \leq t$

$$l^+_s = l^0_s \leq r^0_s = r^+_s.$$

\[\square\]

**Remark 3.2.** It is easy to see that in the above lemma one can replace the initial state \{0\} by any set $A \subset \mathbb{Z}^-$ such that $0 \in A$ (resp. $A \subset \mathbb{Z}^+$ such that $0 \in A$) and that the results involving $(\eta^+_t)_{t \geq 0}$ (resp. $(\eta^-_t)_{t \geq 0}$) still hold.

We also have the following useful lemma.

**Lemma 3.3.** For any $A \subset \mathbb{Z}$ having finitely many positive elements and some $i \in \mathbb{Z}$ satisfying $i > \max\{j \in A\}$, the following holds for all $t \geq 0$

$$\mathbb{E}\left(r^A_t(\xi) - r^A_t\right) \geq 1.$$  

**Proof.** It is a direct consequence of the additivity of our process as well as its translation invariance. See the proof of Lemma 2.21 in [10] or the proof of Lemma 4.1 in [4].  

We have the following

**Proposition 3.4.** There exists a constant $\alpha(\gamma) \in [-\infty, \infty]$ such that the following holds

$$\frac{r^-_t}{t} \stackrel{t \to \infty}{\longrightarrow} \alpha(\gamma) \text{ almost surely.}$$

Moreover, if $\gamma < \gamma_c$ then $\alpha(\gamma) \geq 0$ and the convergence occurs in $L^1$.

**Proof.** The existence of $\alpha(\gamma)$ and the almost sure convergence follow from Theorem 2.1 in [4]. since our process falls into the category of growth models. Furthermore, if $\alpha(\gamma) < 0$ then almost surely $r^-_t \stackrel{t \to \infty}{\longrightarrow} -\infty$ and by symmetry $l^+_t \stackrel{t \to \infty}{\longrightarrow} +\infty$ so that using item (iii) in Lemma 3.1 we have

$$\mathbb{P}(\eta^0_t \neq \emptyset) \stackrel{t \to \infty}{\longrightarrow} 0.$$  

(3.10)

But by duality we know that

$$\mathbb{P}(\eta^0_t \neq \emptyset) = \mathbb{P}(\xi_t(0) = 1) \stackrel{t \to \infty}{\longrightarrow} \rho_\gamma,$$  

(3.11)

and as $\rho_\gamma > 0$ in the sub-critical regime (3.10) and (3.11) implies that $\alpha(\gamma) \geq 0$ when $\gamma < \gamma_c$. Then the $L^1$ convergence follows from the second part of Theorem 2.1 in [4].  

Using Lemma 3.3 we can prove the following proposition.

**Proposition 3.5.** For any $\gamma$ and $\lambda$ such that $0 \leq \gamma + \lambda < \gamma_c$ we have

$$\alpha(\gamma) - \alpha(\gamma + \lambda) \geq \lambda.$$

**Proof.** We adapt an argument from [4]. We build different versions of our model on the same probability space using the graphical construction. Suppose that for each integer $k \in \mathbb{Z}$ we have a Poisson process of intensity $\gamma$ and a Poisson process of intensity $\lambda$. We use both of these processes to put "delta" marks on the time-space diagram. Let $(\eta(k\gamma))_{t \geq 0}$ denote the process which use only the marks coming from the $\gamma$ Poisson processes, and let $(\eta(k(\gamma + \lambda)))_{t \geq 0}$ be the process which use
the marks of both family of Poisson processes. We let \((r^-_t(\gamma))_{t \geq 0}\) and \((r^-_t(\gamma + \lambda))_{t \geq 0}\) denote the corresponding right edge processes. Then we define the following stopping time:

\[
\sigma = \inf\{t \geq 0 : r^-_t(\gamma) < r^-_t(\gamma + \lambda)\}.
\]

We define a third process, denoted \((\tilde{\eta})_{t \geq 0}\), which use only marks coming from the Poisson processes of parameter \(\gamma\) up to time \(\sigma\), and then the marks coming from both family of Poisson processes. We require that the initial state of this process is \(\tilde{\eta}_0 = [\infty, 0]\). Moreover let \((\tilde{r})_{t \geq 0}\) denotes its right edge. For any \(t \geq 0\) we have

\[
\mathbb{E}(r^-_t(\gamma) - r^-_t(\gamma + \lambda)) \geq \mathbb{E}(\tilde{r}_t - r^-_t(\gamma + \lambda)) \geq \mathbb{E}
((\tilde{r}_t - r^-_t(\gamma + \lambda))1_{t \geq \sigma}).
\]

Furthermore \(\tilde{\eta}_\sigma\) contains finitely many positive points, \(\eta_\sigma(\gamma + \delta) \subset \tilde{\eta}_\sigma\) and \(\tilde{\eta}_\sigma\) contains at least one element further right than the right edge of \(\eta_\sigma(\gamma)\) by the definition of \(\sigma\) and the identity \(\tilde{\eta}_\sigma = \eta_\sigma(\gamma)\). Therefore, using Lemma 5.3 and the strong Markov property, we have

\[
\mathbb{E}((\tilde{r}_t - r^-_t(\gamma + \lambda))1_{t \geq \sigma}) = \mathbb{P}(t \geq \sigma)\mathbb{E}(\tilde{r}_t - r^-_t(\gamma + \lambda) \mid t \geq \sigma)
= \mathbb{P}(t \geq \sigma)\mathbb{E}(\tilde{\eta}_t - \sigma(\gamma + \lambda) - r^-_{t - \sigma}(\gamma + \lambda) \mid t \geq \sigma)
\geq \mathbb{P}(t \geq \sigma).
\]

Furthermore, if we denote by \(\hat{\sigma}\) the first time at which the rightmost element in \((\eta_t(\gamma + \lambda))_{t \geq 0}\) is affected by a mark from one of the Poisson processes of rate \(\lambda\), we have that

\[
\mathbb{P}(\sigma \leq t) \geq \mathbb{P}(\hat{\sigma} \leq t).
\]

Therefore it follows that

\[
\mathbb{E}(r^-_t(\gamma)) - \mathbb{E}(r^-_t(\gamma + \lambda)) \geq 1 - e^{-\lambda t}.
\]

Now for any integer \(n \geq 1\)

\[
\mathbb{E}(r^-_t(\gamma)) - \mathbb{E}(r^-_t(\gamma + \lambda)) \geq \sum_{k=1}^{n} \mathbb{E}\left[r^-_t\left(\gamma + \frac{k-1}{n} \lambda\right)\right] - \mathbb{E}\left[r^-_t\left(\gamma + \frac{k}{n} \lambda\right)\right]
\geq n\left(1 - e^{-\lambda t}\right),
\]

and the last term of these inequalities is equal to \(\lambda t + o(1)\) when \(n\) diverges by Taylor expansion so that by taking the limit we are left with

\[
\frac{\mathbb{E}(r^-_t(\gamma))}{t} - \frac{\mathbb{E}(r^-_t(\gamma + \lambda))}{t} \geq \lambda.
\]

We conclude by noticing that the left-hand side converges to \(\alpha(\gamma) - \alpha(\gamma + \lambda)\) using the \(L^1\) convergence part of Proposition 3.4 and the assumption that \(0 \leq \gamma + \lambda < \gamma_c\).

From there we obtain the following corollary, which was the purpose of this section and is mandatory to establish the results of the following section. It simply says that in the sub-critical regime the limit obtained in Proposition 3.4 cannot be equal to 0.

**Corollary 3.6.** If \(\gamma < \gamma_c\) then \(\alpha(\gamma) > 0\).

**Proof.** By Proposition 3.4 we have

\[
\alpha(\gamma) \geq \alpha\left(\gamma + \frac{\gamma_c - \gamma}{2}\right) + \frac{\gamma_c - \gamma}{2},
\]

and as \(\gamma + \frac{\gamma_c - \gamma}{2} < \gamma_c\) it follows from Proposition 3.4 that the first term in the right-hand side is greater or equal to 0 while the second term is strictly greater than 0, which ends the proof. \(\Box\)
4 Sub-exponential estimates

In this section we obtain sub-exponential estimates for the edge and the time of extinction of the dual, and for the time correlations of the auxiliary process. The bounds are expressed in terms of two constants \( C_1 \) and \( C_2 \) which exact value is unimportant and will change from one result to the other. In fact the value of these constants might sometimes even change from one line to the other in the course of the same proof, in order to avoid an overload in notation.

**Proposition 4.1.** If \( \gamma < \gamma_e \) then for any \( a < \alpha \) there exists positive constants \( C_1 \) and \( C_2 \) (depending on \( a \)) such that for any \( t \geq 0 \)

\[
\Pr (r_t^- < at) \leq C_1 e^{-C_2 t}.
\]

**Proof.** This result is the analogue for our system of Theorem 4 in \cite{5}. The authors prove this result for another well-known interacting particle system, namely the contact process on \( \mathbb{Z} \), using a clever construction linking the graphical characterization of the contact process (analog to the construction we gave in Section 2) to one-dependent percolation. Nonetheless, as noticed by the authors themselves at the very end of the Section 2 of their article, this construction—and therefore the proof of their Theorem 4—can be carried out without supplementary work for a larger class of systems which includes at least nearest neighbor additive growth models. This construction is thus valid for our process as well. We let it to the reader to check that all the arguments given there works as well for our system, using previously proven results. The crucial point to carry out this construction is that we have proven that in the sub-critical regime there exists some \( \alpha > 0 \) such that \( r_t^- \sim ta \) as \( t \) goes to \( \infty \) (Proposition \[5.3\] and Corollary \[5.4\]). The only other results needed to check the validity of their proof for our system are translation invariance, duality and the existence of spatially ergodic invariant measures with positive density to which \( (\eta_i)_{t \geq 0} \) and \( (\xi_i)_{t \geq 0} \) converge monotonically as \( t \) goes to \( \infty \) (see Section \[2.3\]). \( \square \)

In the course of the proof of Theorem \[5.3\] below we would like at some point to use the converse of the item (iii) in Lemma \[3.1\]. Unfortunately, it suffices to consider the event in which the only dual-valid path starting at 0 in the graphical construction immediately ends on the tips of a double arrow to see that the converse doesn’t hold. A little bit of thought though reveals that this is actually the only counter-example, so that we can obtain an assertion which is close enough to the converse we need. This is the object of the next lemma. We introduce the following notations:

\[
\tilde{T}_0 = \min_{i \in \{-1,0,1\}} \tilde{T}_{i,0}.
\]

Then we write

\[
\tilde{T}^* = \min \left( \tilde{T}_0, \tilde{T}_{0,0}^+ \right).
\]

In other words \( \tilde{T}^* \) corresponds to the time of the first event affecting neuron 0 in \( (\eta_i^0)_{t \geq 0} \).

Moreover we define the following event

\[
E = \{ \tilde{T}^* = \tilde{T}_{0,0} \}.
\]

Now we can formulate our lemma.

**Lemma 4.2.** On \( \overline{E} \), if \( \sigma^0 < \infty \) then \( l^+_{\sigma^0} > r^-_{\sigma^0} \).

**Proof.** The complementary of \( E \) can be divided into the following disjoint union

\[
\overline{E} = \left\{ \tilde{T}^* = \tilde{T}_{0,0}^+ \right\} \cup \left\{ \tilde{T}^* = \tilde{T}_{-1,0} \right\} \cup \left\{ \tilde{T}^* = \tilde{T}_{1,0} \right\}.
\] (4.12)

We take care of the different cases in (4.12) separately. First suppose that \( \{ \tilde{T}^* = \tilde{T}_{0,0}^+ \} \). Then obviously \( \sigma^0 = \tilde{T}^* \). Moreover in this case \( l^+_{\sigma^0} \geq 1 \) and \( r^-_{\sigma^0} \leq -1 \) so that \( l^+_{\sigma^0} > r^-_{\sigma^0} \).

Now let’s consider the remaining cases, that is either \( \tilde{T}^* = \tilde{T}_{-1,0} \) or \( \tilde{T}^* = \tilde{T}_{1,0} \). By symmetry it is sufficient to treat only one them. We suppose \( \tilde{T}^* = \tilde{T}_{-1,0} \). We’ve assumed that \( \sigma^0 \) is finite so that
we are allowed to define $K$ to be the random variable corresponding to the index of the last active neuron before extinction. Moreover the fact that $\hat{T}^*_i = \hat{T}_{-1,0}$ implies that $\max_{0\leq t < \sigma^0} |\eta_t^0| \geq 2$. Indeed if the first event is an activation of neuron $-1$ by neuron 0 then at the time of this activation the cardinal of the process is 2. Therefore we can define the time at which the penultimate neuron in $(\eta_t^0)_{t\geq 0}$ becomes quiescent:

$$S = \sup\{s < \sigma^0 : |\eta_t^0| = 2\}.$$ 

Again we shall distinguish between two separate cases: either neuron $K$ becomes quiescent because it encounters an event of the $(\hat{T}_{i,n})_{i\in\mathbb{Z}, n\in\mathbb{N}}$ family or because it encounters an event of the $(\hat{T}_{i,n})_{i\in\mathbb{Z}, n\in\mathbb{N}}$ family.

For the first option, we notice that by the first item of lemma 3.1 we have $r_{\sigma}^* = K = l_{\sigma}^*$ for any $s \in [S, \sigma^0]$. Therefore $r_{\sigma}^* \leq K - 1$ and $l_{\sigma}^* \geq K + 1$, so that $l_{\sigma}^* > r_{\sigma}^*$.

Now for the second option we let $L$ be the index of the penultimate neuron, which becomes quiescent at time $S$. Without loss of generality we assume $L < K$ (by symmetry the case $L > K$ can be proven using the same arguments). This last case breaks down into two more sub-cases: either $L = K - 1$ or $L < K - 1$.

If $L = K - 1$, then $L$ is not isolated at time $S$, and therefore it becomes quiescent because of a "$\eta^0$" mark. Then there is no element of the family $(\hat{T}_{L,n})_{n\in\mathbb{N}}$ on $[S, \sigma^0]$ as otherwise, either $K$ would not be the last particle alive, either $L$ would not be the penultimate. Therefore $K$ is isolated in $(\eta_t^0)_{t\in[S, \sigma^0]}$ and thus it becomes quiescent at time $\sigma^0$ in $(\eta_t^0)_{t\geq 0}$ as well. Moreover $r_{\sigma}^* = K$ for any $s \in [S, \sigma^0]$ so that $r_{\sigma}^* \leq K - 2$. Furthermore $l_{\sigma}^* \geq K$, so that we indeed have $l_{\sigma}^* > r_{\sigma}^*$ in this case.

Finally suppose that $L < K - 1$. By the second part of lemma 3.1 $\eta^0_S = \eta_S^- \cap [L, K]$. And as, by the definition of $L$ and $K$, if $i \in [L, K]$ then $i$ is quiescent in $(\eta_t^0)_{t\in[S, \sigma^0]}$; it follows that $i$ shall be quiescent in $\eta_S^-$ as well. In particular this is true for $i = K - 1$. Then the conclusion follows from the same arguments as in the previous case.

Using the two previous results we can show the following important theorem.

**Theorem 4.3.** If $\gamma < \gamma_c$, then there exists two positive constants $C_1$ and $C_2$ such that for any $t \geq 0$ and any finite set $A \in \mathcal{P}(\mathbb{Z})$

$$\mathbb{P}\{t < \sigma^A < \infty\} \leq C_1 |A|e^{-C_2 t}.$$ 

**Proof.** We first prove the result for $A = \{0\}$. Let $a$ be some real number such that $0 < a < \alpha$, and let $N \geq 0$ be some integer. Then by Proposition 3.1, the following holds for some constants $C_1$ and $C_2$

$$\mathbb{P}\{r_n^- < an \text{ for some } n \geq N\} \leq C_1 \sum_{n \geq N} (e^{-C_2 n}) = \frac{C_1}{1 - e^{-C_2}} e^{-C_2 N}.$$ 

Moreover, if $(N_t)_{t\geq 0}$ denotes an homogeneous Poisson Process of rate 1, then an obvious coupling with $(r_t^-)_{t\geq 0}$ gives

$$\mathbb{P}\{\{r_t^- < 0 \text{ for some } t \in [n, n + 1]\} \cap \{r_{n+1}^- > a(n+1)\}\} \leq \mathbb{P}(N_1 > a(n+1)),$$

and by the exponential Markov inequality

$$\mathbb{P}(N_1 > a(n+1)) \leq e^{e-1-a} e^{-an}.$$
Now
\[ P (r^{-} < 0 \text{ for some } t \geq N) \leq P (r^{-} < an \text{ for some } n \geq N) + \sum_{n \geq N} P \left( \{r^{-} < 0 \text{ for some } t \in [m, m + 1]\} \cap \{r_{m+1} > a(m + 1)\} \right). \]

Therefore, from the inequalities above it is easy to find two constants \( C'_1 \) and \( C'_2 \) such that
\[ P (r^{-} < 0 \text{ for some } t \geq N) \leq C'_1 e^{-C'_2 N}. \]

And by a slight modification of the constant \( C'_1 \) we obtain that for any \( T \in \mathbb{R}_+ \)
\[ P (r^{-} < 0 \text{ for some } t \geq T) \leq C'_1 e^{-C'_2 T}. \]

By symmetry we have the same bound for \( P (l^{+} > 0 \text{ for some } t \geq T) \), therefore
\[ P (r^{-} < l^{+} \text{ for some } t \geq T) \leq P (r^{-} < 0 \text{ for some } t \geq T) + P (l^{+} > 0 \text{ for some } t \geq T) \leq 2C'_1 e^{-C'_2 T}. \tag{4.13} \]

Finally
\[ P (t < \sigma^0 < \infty) \leq P \left( \{t < \sigma^0 < \infty\} \cap E \right) + P \left( \{\sigma^0 > t\} \cap E \right), \]
and the first element in the right hand side is less than \( 2C'_1 e^{-C'_2 t} \) by Lemma \[ \text{[4.2]} \] and \( \text{[4.13]} \), while the second element is less than \( P \left( \tilde{T}_{0,0} > t \right) = e^{-t} \), so that in the end
\[ P (t < \sigma^0 < \infty) \leq 2(C'_1 + 1)e^{-\min(1,C'_2)t}. \]

Now it only remains to generalize the result to some finite initial state \( A \) not necessarily equal to \( \{0\} \). This is easily done using additivity, monotonicity and translation invariance
\[ P (t < \sigma^A < \infty) = P \left( \bigcup_{i \in A} \{\sigma^i > t\} \cap \{\sigma^A < \infty\} \right) \leq \sum_{i \in A} P (\sigma^i > t, \sigma^A < \infty) \leq \sum_{i \in A} P (t < \sigma^i < \infty) = |A| \cdot P (t < \sigma^0 < \infty) \]

The following lemma, which state that cylinder functions on \( \mathcal{P}(\mathbb{Z}) \) are finite linear combination of indicators functions will be useful in order to prove Theorem \[ \text{[4.5]} \]

**Lemma 4.4.** Let \( f : \mathcal{P}(\mathbb{Z}) \to \mathbb{R} \) be a cylinder function. Then there exists a family \( (S_i)_{i \in [1,n]} \) of finite sets in \( \mathbb{Z} \) and a family \( (\lambda_i)_{i \in [1,n]} \) of values in \( \mathbb{R} \) such that for any \( A \in \mathcal{P}(\mathbb{Z}) \)
\[ f(A) = \sum_{i=1}^{n} \lambda_i \mathbb{I}_{A \cap S_i \neq \emptyset}. \]
Proof. Let $S \subset \mathbb{Z}$ be the support of $f$, and let $(S_i)_{i \in [1,n]}$ be the family of all the subsets of $S$. Then, for any $A \in \mathcal{P}(\mathbb{Z})$

$$f(A) = f(A \cap S) = \sum_{i=1}^{n} f(S_i) 1_{A \cap S = S_i}. \quad (4.14)$$

Moreover, for any $A,C \in \mathcal{P}(\mathbb{Z})$ the following identity holds

$$1_{A=C} = 1_{AC} - \sum_{B \subseteq C, B \neq C} 1_{A=B}, \quad (4.15)$$

and if $C$ is a singleton then

$$1_{A=C} = 1_{AC} - 1_{AC \neq \emptyset}. \quad (4.16)$$

Therefore, applying (4.15) recursively to itself—that is, to the elements in the sum of the right hand side—and using (4.16) to conclude the recursion, it follows from (4.14) that $f$ can be expressed as a finite linear combination of indicators of the form $1_{C \cap S \subseteq S_i}$. Then the conclusion follows from the fact that for any $A,C \in \mathcal{P}(\mathbb{Z})$ the following holds

$$1_{AC} = 1 - 1_{AC \neq \emptyset}. \quad \blacksquare$$

Theorem 4.5. Let $f : \mathcal{P}(\mathbb{Z}) \to \mathbb{R}$ be a cylinder function. Then there exists positive constants $C_1$ and $C_2$ (depending on $f$) such that for any $s,t \in \mathbb{R}^+$

$$|\text{Cov} (f(\xi_t), f(\xi_s))| \leq C_1 e^{-C_2 |t-s|}. \quad (4.17)$$

Proof. Let $(S_i)_{i \in [1,n]}$ and $(\lambda_i)_{i \in [1,n]}$ be as in Lemma 4.4 so that for any $t \geq 0$

$$f(\xi_t) = \sum_{i=1}^{n} \lambda_i 1_{\xi_t \cap S_i \neq \emptyset}. \quad (4.18)$$

Therefore, for any $s \geq 0$ and $t \geq 0$

$$|\text{Cov} (f(\xi_t), f(\xi_s))| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\lambda_i \lambda_j| \cdot |\text{Cov} (1_{\xi_t \cap S_i \neq \emptyset}, 1_{\xi_s \cap S_j \neq \emptyset})|. \quad (4.19)$$

Thus it is sufficient to prove that for any finite sets $A, B \in \mathcal{P}(\mathbb{Z})$ there exists some positive constants $C_1$ and $C_2$ such that for any $s \geq 0$ and $t \geq 0$

$$|\text{Cov} (1_{\xi_t \cap A \neq \emptyset}, 1_{\xi_t \cap B \neq \emptyset})| \leq C_1 e^{-C_2 |t-s|}. \quad (4.20)$$

Without loss of generality we assume $t \leq s$. Let $F$ be the event that there exists a valid path from $\mathbb{Z} \times t$ to $B \times s$. Moreover let $G = \{\xi_t \cap A \neq \emptyset\}$ and $H = \{\xi_t \cap B \neq \emptyset\}$. We have

$$|\text{Cov} (1_{\xi_t \cap A \neq \emptyset}, 1_{\xi_t \cap B \neq \emptyset})| = |\mathbb{P}(G \cap H) - \mathbb{P}(G) \mathbb{P}(H)|. \quad (4.21)$$

Furthermore it is clear from the graphical construction that $H \subset F$ so that we can replace $H$ by $H \cap F$ in the equation above. Also events depending on disjoint regions of the graph $\mathcal{G}$ are independent so that $\mathbb{P}(G \cap F) = \mathbb{P}(G) \mathbb{P}(F)$. Therefore

$$|\text{Cov} (1_{\xi_t \cap A \neq \emptyset}, 1_{\xi_t \cap B \neq \emptyset})| = |\mathbb{P}(G \cap H \cap F) - \mathbb{P}(G \cap F) + \mathbb{P}(G) \mathbb{P}(F) - \mathbb{P}(G) \mathbb{P}(H \cap F)|$$

$$= |\mathbb{P}(G \cap F) - \mathbb{P}(G \cap H \cap F) - \mathbb{P}(G) \mathbb{P}(F) - \mathbb{P}(H \cap F)|$$

$$= |\mathbb{P}(G) \mathbb{P}(F) - \mathbb{P}(G) \mathbb{P}(F) - \mathbb{P}(H \cap F)|$$

$$\leq \mathbb{P}(F \cap \overline{H}) \cdot |\mathbb{P}(G) - \mathbb{P}(G \cap F \cap \overline{H})|$$

$$\leq \mathbb{P}(F \cap \overline{H}). \quad (4.22)$$
And by the backward version of the dual process,
\[ P( F \cap H ) = P( t < \sigma^B < s ) \leq P( s - t < \sigma^B < \infty ) . \]

Then use Theorem 4.3 to conclude.

**Remark 4.6.** While the theorem above is proven only for the infinite process, by similar arguments one can easily obtain the exponentially decaying correlations for the semi-infinite processes as well.

### 5 Main Theorem

We’re aimed to prove that in the sub-critical regime, if \( n \) is big enough, then counting the number of spikes occurring in a given time interval before extinction, for a given subset of neurons of interest in the system, shall give a number which with high probability cannot be too far to some fixed value, depending only on the parameter \( \gamma \) (and of course of the number of neurons in the subset considered). This property captures the pseudo-stationarity characteristic of metastable systems.

More precisely, let \( F \subset \mathbb{Z} \) with \( |F| < \infty \). Then we define, for any \( t, R \in \mathbb{R}_+ \) and for any \( n \geq 0 \), the following quantity
\[
\hat{N}^n_{R}(t, F) = \frac{1}{R} \sum_{i \in F} N_i ([t, t + R]),
\]
where the superscript \( n \) indicates that we are considering the finite system with the set of neurons \( S = [−n, n] \). \( \hat{N}^n_{R}(t, F) \) is the average number of spikes emitted by the neurons in \( F \) on a time interval of length \( R \), starting the enumeration a time \( t \). We prove the following theorem.

**Theorem 5.1.** Suppose \( 0 < \gamma < \gamma_c \) and let \( (R_n)_{n \geq 0} \) be an increasing sequence of positive real numbers satisfying
\[
R_n \xrightarrow{n \to \infty} +\infty \quad \text{and} \quad \frac{R_n}{\mathbb{E}(\tau_n)} \xrightarrow{n \to \infty} 0.
\]

There exists some \( 0 < \rho < 1 \) (which depends only on \( \gamma \)) such that for any \( t \geq 0 \)
\[
\hat{N}^n_{R_n}(t, F) \xrightarrow{n \to \infty} |F| \cdot \rho.
\]

Before proving Theorem 5.1 per se we shall prove a similar result concerning the auxiliary process, which is the object of Section 5.2. Before doing so, we prove some lemmas and recall previous results which will be useful in the sequel.

#### 5.1 Useful results

We begin by stating a fundamental result about the time of extinction of \( (\xi_n(t))_{t \geq 0} \) which was proven in [1], and which is the the first of the two properties characterizing metastable systems we’ve listed in the introduction. It will be used repeatedly from now on up to the end of this article.

**Theorem 5.2.** Suppose \( 0 < \gamma < \gamma_c \). Then we have the following convergence
\[
\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow{n \to \infty} \delta'(1),
\]
where \( \delta' \) denotes a convergence in distribution.
Remark 5.3. In [11] this result was proven only for $\gamma < \gamma^\prime_c$, where $\gamma^\prime_c$ is the critical value for the semi-infinite process. Nonetheless, with the construction invoked in the proof of Proposition 4.4 the parameter $p$ of the percolation process can be taken as close to 1 as needed, while the coupling with $(\xi_t)_{t \geq 0}$ remains valid even if the value of $\gamma$ is kept fixed. But if $p$ is big enough then with positive probability there is a path in the percolation structure which never goes further left than 0 on the horizontal abscissa. An immediate consequence is that $\gamma_c = \gamma^\prime_c$, so that we don’t need the caveat anymore.

The second result tells us that in the sub-critical regime the expectation of the time of extinction grows at least in a linear fashion with respect to $n$. While this result will mostly be needed in the last section of this paper, for the moment it gives us a rigorous proof of a fact that might seems anyway evident: it ensures us that $E(\tau_n) \to \infty$ as $n$ grows, which in turn guarantees that there is no problem in choosing a $(R_n)_{n \geq 0}$ satisfying the conditions of Theorem 5.1.

Proposition 5.4. Suppose $0 < \gamma < \gamma_c$. Then $n/\mathbb{E}(\tau_n) \to 0$ as $n$ goes to $\infty$.

Proof. In the following we write $\frac{\gamma}{2}$ while we should sometimes write $\lfloor \frac{\gamma}{2} \rfloor$ in order to avoid a useless overload in the notation. By duality

$$
P \left( \xi_{t}^{[0, \frac{\gamma}{2}]} \neq \emptyset \right) \quad \Rightarrow \quad \tilde{\mu} \left( A : A \cap \left[ 0, \frac{n}{2} \right] \neq \emptyset \right).$$

Therefore, if we write $E_n := \{ \xi_{t}^{[0, \frac{\gamma}{2}]} \neq \emptyset \text{ for all } t \geq 0 \}$ then $E_n = \bigcap_{t \geq 0} \{ \xi_{t}^{[0, \frac{\gamma}{2}]} \neq \emptyset \}$ and thus

$$
P( E_n ) = \tilde{\mu} \left( A : A \cap \left[ 0, \frac{n}{2} \right] \neq \emptyset \right) \quad \frac{\to}{\to_{n \to \infty}} \quad 1.$$

Now we define $\kappa_n := \inf \{ t > 0 : \{-n,n\} \cap \xi_{t}^{[0, \frac{\gamma}{2}]} \neq \emptyset \}$.

From Lemma 3.1 (and translation invariance) we have $r_{t}^{[0, \frac{\gamma}{2}]} = r_{t}^{[-\infty, \frac{\gamma}{2}]}$ on $E_n$, and therefore by Proposition 3.4 and Corollary 3.6 it follows that $r_{t}^{[0, \frac{\gamma}{2}]} \to \infty$ almost surely on $E_n$ when $t$ diverges. As a consequence, on $E_n$, the stopping time $\kappa_n$ is almost surely finite and $\tau_n \geq \kappa_n$. Moreover, if $(W_k)_{k \geq 1}$ is a family of independent and identically distributed random variables such that $W_1 \sim \mathcal{E}(1)$, then an obvious coupling gives that

$$
\kappa_n \geq \frac{n}{2} \sum_{k=1}^{n/2} W_k.
$$

Therefore

$$
P( \tau_n < \frac{n}{4}, E_n ) \leq \mathbb{P} \left( \frac{1}{n/2} \sum_{k=1}^{n/2} W_k > \frac{1}{2} \right).$$

The right hand side in the inequality above goes to 0 as $n$ diverges by the law of large numbers. Thus, as $E_n$ has probability one asymptotically it implies that $P( \tau_n < n/4 )$ goes to 0 when $n$ goes to $\infty$. Then dividing by $E(\tau_n)$ in both side of the inequality and using Theorem 5.2 implies the result.

We end this subsection by stating a lemma which will be particularly useful in the next section, and which relates the finite process $(\xi_n(t))_{t \geq 0}$ to its infinite and semi-infinite counterparts. The proof, which is pretty straightforward using the natural coupling permitted by the graphical construction, can be found in [11].

Lemma 5.5. For any $t \geq 0$ the following holds on $\{ \tau_n > t \}$

(i) $\xi_n(t) = \xi(t) \cap [\min \xi_n(t), \max \xi_n(t)]$,

(ii) $\min \xi_n(t) = \min \xi_{[-n,\infty]}(t)$ and $\max \xi_n(t) = \min \xi_{[-\infty,n]}(t)$.
5.2 Times averages for the auxiliary process

For any $t, R \in \mathbb{R}_+$ and $n \geq 0$ we define the following measure on $\mathcal{P}(\mathbb{Z})$

$$A^n_R(t, \cdot) = \frac{1}{R} \int_t^{t+R} 1_{\xi_n(s) \in \cdot} ds.$$  

As usual, for any measurable function $f : \mathcal{P}(\mathbb{Z}) \to \mathbb{R}$ we write $A^n_R(t, f)$ for the integral of $f$ with respect to $A^n_R(t, \cdot)$. It is straightforward from basic measure-theoretic considerations to see that actually

$$A^n_R(t, f) = \frac{1}{R} \int_t^{t+R} f(\xi_n(s)) ds.$$  

Then the following result holds.

**Theorem 5.6.** Let $(R_n)_{n \geq 0}$ be an increasing sequence of positive real numbers satisfying the same conditions as in Theorem 5.1 Then, for any cylinder function $f : \mathcal{P}(\mathbb{Z}) \to \mathbb{R}$ we have, for any $t \geq 0$

$$A^n_{R_n}(t, f) \xrightarrow{n \to \infty} \mu(f).$$

The two main ingredients are the following lemmas, which we prove now, before entering the proof of Theorem 5.6. These are essentially a consequence of the weak convergence of $(\xi_t)_{t \geq 0}$, together with Chebyshev’s inequality and the exponential decay of the time correlations.

**Lemma 5.7.** Let $\epsilon > 0$ and $f : \mathcal{P}(\mathbb{Z}) \to \mathbb{R}$ be a cylinder function. Then there exists a constant $C > 0$ (which depends only on $f$ and $\epsilon$) such that for any fixed $t \geq 0$ and for big enough $R \in \mathbb{R}_+$ the following holds

$$\mathbb{P}\left(\left| \frac{1}{R} \int_t^{t+R} f(\xi_s) ds - \mu(f) \right| > \epsilon \right) \leq \frac{C}{R}.$$  

**Proof.** Fix some $t \geq 0$. As $\mu$ is defined as the weak limit of $(\xi_t)_{t \geq 0}$ there exists some $t_0$ such that for any $t \geq t_0$

$$|\mathbb{E}(f(\xi_t)) - \mu(f)| < \frac{\epsilon}{2}.$$  

Then, if $t \geq t_0$ we have

$$\left| \frac{1}{R} \int_t^{t+R} \mathbb{E}(f(\xi_s)) ds - \mu(f) \right| < \frac{\epsilon}{2}. \quad (5.17)$$

If $t < t_0$ then (assuming $t + R > t_0$)

$$\left| \frac{1}{R} \int_t^{t+R} \mathbb{E}(f(\xi_s)) ds - \mu(f) \right| \leq \frac{1}{R} \int_t^{t_0} \mathbb{E}(f(\xi_s)) - \mu(f) | ds + \frac{1}{R} \int_{t_0}^{t+R} \mathbb{E}(f(\xi_s)) - \mu(f) | ds,$$

and as the first element in the sum of the right hand side goes to 0 as $R$ grows while the second element is strictly less than $\epsilon/4$, the following must holds for big enough $R$

$$\mathbb{P}\left(\left| \frac{1}{R} \int_t^{t+R} f(\xi_s) ds - \mu(f) \right| > \epsilon \right) \leq \mathbb{P}\left(\left| \frac{1}{R} \int_t^{t+R} f(\xi_s) ds - \frac{1}{R} \int_t^{t+R} \mathbb{E}(f(\xi_s)) ds \right| > \frac{\epsilon}{2} \right). \quad (5.18)$$

Writing $X_{t, R} := \frac{1}{R} \int_t^{t+R} f(\xi_s) ds$ and using Fubini’s theorem, (5.18) becomes
\[
\mathbb{P}
\left(\frac{1}{R} \int_{t}^{t+R} f(\xi_s) ds - \mu(f) > \epsilon \right) \leq \mathbb{P}
\left( |X_{t,R} - \mathbb{E}(X_{t,R})| > \frac{\epsilon}{2} \right).
\] (5.19)

Again, Fubini’s theorem gives
\[
\mathbb{E}(X_{t,R}^2) = \frac{1}{R^2} \int_t^{t+R} \int_t^{t+R} \mathbb{E}(f(\xi_u) f(\xi_v)) \, du \, dv,
\]
and
\[
\mathbb{E}(X_{t,R})^2 = \frac{1}{R^2} \int_t^{t+R} \int_t^{t+R} \mathbb{E}(f(\xi_u)) \mathbb{E}(f(\xi_v)) \, du \, dv,
\]
and therefore from Theorem 4.5 we obtain the following
\[
\text{Var}(X_{t,R}) = \frac{1}{R^2} \int_t^{t+R} \int_t^{t+R} \text{Cov}(f(\xi_u), f(\xi_v)) \, du \, dv \leq \frac{4}{R^2} \int_t^{t+R} \int_t^{t+R} C_1 e^{-C_2|u-v|} \, du \, dv.
\]

Then by a simple change of variable, for any \(u \in [t, t+R]\)
\[
\int_t^{t+R} C_1 e^{-C_2|u-v|} \, dv = C_1 \left[ \int_0^{u-t} e^{-C_2v} \, dv + \int_0^{t-R-u} e^{-C_2v} \, dv \right] \leq 2C_1 \int_0^\infty e^{-C_2u} \, du.
\]

Thus
\[
\text{Var}(X_{t,R}) \leq \frac{2C_1}{C_2 R}.
\]

Then Chebyshev’s inequality and (5.18) imply that
\[
\mathbb{P}
\left(\frac{1}{R} \int_{t}^{t+R} f(\xi_s) ds - \mu(f) > \epsilon \right) \leq \frac{8C_1}{C_2 \epsilon^2 R}.
\]

\[\text{Lemma 5.8.} \text{ Let } \epsilon > 0 \text{ and } f: \mathcal{P}(\mathbb{Z}) \rightarrow \mathbb{R} \text{ be a cylinder function. Then there exists a positive constant } C \text{ (which depends only on } f \text{ and } \epsilon \text{) such that if } l \text{ is big enough, then for any fixed } t \geq 0 \text{ and for any } R \in \mathbb{R}^+ \text{ the following holds}
\]
\[
\mathbb{P}
\left( \frac{1}{R} \int_{t}^{t+R} 1_{\xi_s \cap [0,l] = \emptyset} ds > \epsilon \right) \leq \frac{C}{R}.
\]

**Proof.** This lemma is easily obtained by similar arguments as in the proof of Lemma 5.7. For any \(l \geq 0\), the function \(h_t: \xi \mapsto 1_{\xi_t \cap [0,l] = \emptyset}\) is a decreasing function, so that by stochastic monotonicity:
\[
\frac{1}{R} \int_t^{t+R} \mathbb{E}(h_t(\xi_s)) \, ds \leq \mu_{[0,\infty]}(h_t).
\]

Moreover the fact that \(\mu_{[0,\infty]}\) gives mass 0 to the empty set in the sub-critical regime implies that
\[
\mu_{[0,\infty]}(h_t) = \mu_{[0,\infty]}(\{ A: A \cap [0,l] = \emptyset \}) \rightarrow 0 \text{ as } l \rightarrow \infty.
\]
Therefore, for big enough $l$ we have
\[ P \left( \frac{1}{R} \int_t^{t+R} h_1(\xi_{[0,\infty]}(s)) \, ds > \epsilon \right) \leq P \left( \frac{1}{R} \int_t^{t+R} h_1(\xi_{[0,\infty]}(s)) - \mathbb{E} \left( h_1(\xi_{[0,\infty]}(s)) \right) \, ds > \epsilon \right). \]

Then the conclusion comes from the same arguments as in the proof of Lemma 5.7 (from Eq. (5.18) to the end), using the fact that $h_1$ is a cylinder function.

**Proof of Theorem 5.6.** Fix some $t \geq 0$. First notice that using Theorem 5.2 and the hypothesis on $(R_n)_{n \geq 0}$ (as well as the fact that $\mathbb{E}(\tau_n) \to \infty$) the following holds
\[ P \left( t + R_n < \tau_n \right) \underset{n \to \infty}{\longrightarrow} 1. \]

Hence, writing $\Omega_n = \{ t + R_n < \tau_n \}$, it will be enough to prove that, for any $\epsilon > 0$
\[ P \left( \left| A_{R_n}^n(t, f) - \mu(f) \right| > \epsilon, \; \Omega_n \right) \underset{n \to \infty}{\longrightarrow} 0. \]

We have
\[ P \left( \left| A_{R_n}^n(t, f) - \mu(f) \right| > \epsilon, \; \Omega_n \right) \leq P \left( \left| A_{R_n}^n(t, f) - \frac{1}{R_n} \int_t^{t+R_n} f(\xi_s) \, ds \right| > \epsilon, \; \Omega_n \right) + P \left( \left| \frac{1}{R_n} \int_t^{t+R_n} f(\xi_s) \, ds - \mu(f) \right| > \epsilon, \; \Omega_n \right). \]

Moreover, for any $l \in \mathbb{N}$ and for $n$ big enough, using the fact that the support of $f$ has to lie in $[-n + l, n - l]$ when $n$ is big, Lemma 5.5 yields
\[ \{ \min h_n(t) < -n + l, \max h_n(t) > n - l \} \subset \{ f(\xi_n(t)) = f(\xi_t) \}. \tag{5.20} \]

For any $l \in \mathbb{N}$ we let $h_l^n : \mathcal{P}(\mathbb{Z}) \to \mathbb{R}$ and $g_l^n : \mathcal{P}(\mathbb{Z}) \to \mathbb{R}$ be defined for any $\xi \in \mathcal{P}(\mathbb{Z})$ by
\[ h_l^n(\xi) = 1_{\xi \cap [-n, -n + l] = \emptyset}, \]
and
\[ g_l^n(\xi) = 1_{\xi \cap [n - l, n] = \emptyset}. \]

Now, using (5.20) and the fact that $|f(\xi_n(t)) - f(\xi(t))| < 2\|f\|_{\infty}$ we have
\[ \left| A_{R_n}^n(t, f) - \frac{1}{R_n} \int_t^{t+R_n} f(\xi_s) \, ds \right| \leq 2\|f\|_{\infty} \frac{1}{R_n} \int_t^{t+R_n} h_l^n(\xi_n(s)) + g_l^n(\xi_n(s)) \, ds. \tag{5.21} \]

Furthermore, for any $l < n$, and for any $0 \leq s < \tau_n$ Lemma 5.5 yields:
\[ h_l^n(\xi_n(s)) = h_l^n(\xi_{[\xi_{-n}, +\infty]}(s)) \quad \text{and} \quad g_l^n(\xi_n(s)) = g_l^n(\xi_{[-\infty, 0]}(s)). \tag{5.22} \]

Therefore on $\Omega_n$ the right hand side in (5.21) is less than
\[ 2\|f\|_{\infty} \int_t^{t+R_n} h_l^n(\xi_{[\xi_{-n}, +\infty]}(s)) + g_l^n(\xi_{[-\infty, 0]}(s)) \, ds. \]
Then, for any \( l \in \mathbb{N} \) and for \( n \) big enough the following holds (using translation invariance and symmetry)

\[
P\left( \left| A_{R_n}^a(t,f) - \mu(f) \right| > \epsilon, \Omega_n \right) \leq 2P \left[ \frac{1}{R} \int_t^{t+R} \mathbb{I}_{[0,\infty)}(s) \cap [0,\ell] = \emptyset \ ds > \frac{\epsilon}{8\|f\|_{\infty}} \right] \\
+ P \left[ \frac{1}{R} \int_t^{t+R} f(s)ds - \mu(f) \right] > \frac{\epsilon}{2}.
\]

Then the result follows from Lemma 5.7 and Lemma 5.8.

\[\square\]

5.3 Proof of the main theorem

An important element of the proof of the main theorem is the following lemma, which is the last result we prove before engaging into the proof of this theorem. It is a coarser version of Theorem 4.5, but for the finite version of our system, the exponential decay being tempered by two other terms which converge to zero when \( n \) grows. While we conjecture that the exponential decay actually holds without these two terms, we only prove this weaker version as it is sufficient for our needs.

**Lemma 5.9.** Let \( s, r \in \mathbb{R}_+ \) and let \( i \in \mathbb{Z} \). Then there exists two positive constants \( C_1 \) and \( C_2 \) such that for any \( n \geq 0 \)

\[
\left| \text{Cov} \left( \mathbb{I}_{\xi_n(s) \cap \{i\} \neq \emptyset}, \mathbb{I}_{\xi_n(t) \cap \{i\} \neq \emptyset} \right) \right| \leq C_1 e^{-C_2 |t-s|} + P \left( \tau_n < \max(s,t) \right) + \epsilon_n,
\]

where \( \epsilon_n \) is some positive quantity satisfying \( \epsilon_n \to 0 \) as \( n \to \infty \).

**Proof.** Without loss of generality we assume \( t \leq s \). Moreover we let \( G_n = \{\xi_n(t) \cap \{i\} \neq \emptyset\} \) and \( H_n = \{\xi_n(s) \cap \{i\} \neq \emptyset\} \). We also define the event \( F \) that there is a valid path on the graphical construction (of the infinite process) from \( Z \times t \) to \( \{i\} \times s \). Then by the same arguments as in the proof of Theorem 4.5 we obtain

\[
\left| \text{Cov} \left( \mathbb{I}_{\xi_n(s) \cap \{i\} \neq \emptyset}, \mathbb{I}_{\xi_n(t) \cap \{i\} \neq \emptyset} \right) \right| \leq P \left( F \cap H_n^C \right).
\]

Then

\[
P \left( F \cap H_n^C \right) \leq P \left( s - t < \sigma_i < s \right) + P \left( \tau_n \leq s \right) + P \left( \{\sigma_i \geq s\} \cap H_n^C \cap \{\tau_n > s\} \right).
\]

As in the proof of [4.5] \( P \left( s - t < \sigma_i < s \right) \leq P \left( s - t < \sigma_i < \infty \right) \leq C_1 e^{-C_2 |t-s|}, \) so it only remains to show that the last term in the sum goes to 0 as \( n \) diverges.

If \( \sigma_i \geq s \) then \( \{j\} \times 0 \to \{i\} \times s \) for some \( j \in \mathbb{Z} \). But, if \( j \notin \{-n, \ldots, n\} \), then on \( H_n^C \) this path has to cross at least one of the two frontiers \( \{-n, \ldots, n\} \times \mathbb{R} \) and \( \{-n, \ldots, n\} \times \mathbb{R}^+ \) at some point, as otherwise there would be a valid path from \( \{-n, \ldots, n\} \) to \( \{i, s\} \) that never escape \( \{-n, \ldots, n\} \times \mathbb{R}^+ \), which is not allowed on \( H_n^C \). If \( j \notin \{-n, \ldots, n\} \) of course the path crosses one of the two frontiers as well. Consequently, in both cases we can define \( s' \leq s \) to be the last time of crossing, and we have either \( (n, s') \to (i, s) \) or \( (n, s') \to (i, s) \). But in the first case \( \xi_n(s) \cap \{i, \ldots, n\} = \emptyset \), as otherwise there would be a valid path crossing the path from \( (s', n) \) to \( (s, i) \) and the concatenation would then be a valid path from \( \{-n, \ldots, n\} \times 0 \) to \( \{i, s\} \) never escaping \( \{-n, \ldots, n\} \times \mathbb{R}^+ \) (which again is not allowed on \( H_n^C \)). Similarly we have \( \xi_n(s) \cap \{-n, \ldots, i\} = \emptyset \) in the other case.

From the discussion above it follows that

\[
P \left( \{\sigma_i \geq s\} \cap H_n^C \cap \{\tau_n > s\} \right) \leq P \left( \xi_n(s) \cap \{i, \ldots, n\} = \emptyset, \tau_n > s \right) + P \left( \xi_n(s) \cap \{-n, \ldots, i\} = \emptyset, \tau_n > s \right).
\]
Then, using Lemma 5.5 this becomes
\[ P \left( \{ \sigma_i \geq s \} \cap H_n^C \cap \{ \tau_i > s \} \right) \leq P \left( \xi_{[0, +\infty)}(s) \cap [0, n - i] = \emptyset \right) + P \left( \xi_{[0, +\infty)}(s) \cap [0, n + i] = \emptyset \right) \]
and by stochastic monotonicity this is less than
\[ \epsilon_n := 2\mu_{[0, +\infty]}(A : A \cap [0, \min(n - i, n + i)] = \emptyset). \]
Finally \( \epsilon_n \xrightarrow{n \to \infty} 0 \) as \( \mu_{[0, +\infty]} \) gives mass 0 to \( \emptyset \).

With this preliminary completed, we can turn to the proof of the main theorem.

**Proof of Theorem 5.1.** We fix \( t \) and \( F \) and we let \( \epsilon > 0 \). We aim to show that
\[ P \left( \left| \hat{N}^n_{\tau_n} (t, F) - |F| \cdot \rho \right| > \epsilon \right) \xrightarrow{n \to \infty} 0. \]

For any \( \xi \in \{0, 1\}^2 \) we write \( S_F(\xi) = \sum_{i \in F} \xi_i \). Then
\[
P \left( \left| \hat{N}^n_{\tau_n} (t, F) - |F| \cdot \rho \right| > \epsilon \right) \leq P \left( \left| \hat{N}^n_{\tau_n} (t, F) - A^n_{\tau_n} (t, S_F) \right| > \frac{\epsilon}{2} \right)
+ P \left( \left| A^n_{\tau_n} (t, S_F) - |F| \cdot \rho \right| > \frac{\epsilon}{2} \right).
\]

We already know that the second element in the sum above goes to 0 by Theorem 5.6, so that it only remains to show that the first element goes to 0 as well. To do so we define, for any \( i \in F \) and \( n \geq 0 \), the following random set on \( \mathbb{R}^+ \)
\[ I_{n, i} := \{ s : t \leq s \leq t + R_n \text{ and } \xi_{n,i}(s) = 1 \}. \]

Then the following holds
\[
\left| \hat{N}^n_{\tau_n} (t, F) - A^n_{\tau_n} (t, S_F) \right| \leq \frac{1}{R_n} \sum_{i \in F} |N_i([t, t + R_n]) - \int_t^{t + R_n} \xi_{n,i}(s) ds|
= \frac{1}{R_n} \sum_{i \in F} |N_i(I_{n,i}) - \lambda(I_{n,i})|,
\]
where \( \lambda(I_{n,i}) \) denotes the Lebesgue’s measure of the set \( I_{n,i} \) (there is no problem of measurability as \( I_{n,i} \) is almost surely a finite union of intervals). Above we’ve used the fact that there can not be any spike on the set \([t, t + R_n] \cap I_{n,i}^C\). Furthermore, for any fixed \( I_{n,i}, N_i(I_{n,i}) \) corresponds to the numbers of atoms of a Poisson process of intensity 1 on a Borel set of length \( \lambda(I_{n,i}) \), so that we immediately obtain

\[ \mathbb{E} [N_i(I_{n,i})] = \lambda(I_{n,i}) \]
(5.23)

It follows that
\[
P \left( \left| \frac{1}{R_n} \sum_{i \in F} |N_i(I_{n,i}) - \lambda(I_{n,i})| \right| > \epsilon \right) \leq \sum_{i \in F} P \left( \left| N_i(I_{n,i}) - \mathbb{E} [N_i(I_{n,i})] \right| > \frac{\epsilon R_n}{2|F|} \right)
+ \sum_{i \in F} P \left( \left| \mathbb{E} [N_i(I_{n,i})] - \mathbb{E} [N_i(I_{n,i})] \lambda(I_{n,i}) \right| > \frac{\epsilon R_n}{2|F|} \right).
\]
Now, by Chebyshev’s inequality the left-hand side above is less than
\[ \frac{4|F|^3}{\epsilon^2 R_n^2} \left( \Var(N_i(I_{n,i})) + \Var(\mathbb{E}[N_i(I_{n,i})|\lambda(I_{n,i})]) \right), \]
and by the law of total variance
\[ \Var(N_i(I_{n,i})) = \mathbb{E}[\Var(N_i(I_{n,i})|\lambda(I_{n,i}))] + \Var(\mathbb{E}[N_i(I_{n,i})|\lambda(I_{n,i})]). \]
But for the same reason as for (5.23) we have \( \Var(N_i(I_{n,i})|\lambda(I_{n,i})) = \lambda(I_{n,i}) \), so that
\[ \mathbb{E}[\Var(N_i(I_{n,i})|\lambda(I_{n,i}))] \leq R_n. \]
Therefore, it only remains to bound
\[ \frac{8|F|^3}{\epsilon^2 R_n^2} \Var(\lambda_{i,n}). \]
But then, by the same computations as in the proof of Lemma 5.7 one gets
\[ \Var(\lambda_{i,n}) = \Var\left( \int_t^{t+R_n} \xi_{n,i}(s) ds \right) = \int_t^{t+R_n} \int_t^{t+R_n} \Cov(\xi_{n,i}(x), \xi_{n,i}(y)) dx dy, \]
and then, using Lemma 5.9
\[ \frac{8|F|^3}{\epsilon^2 R_n^2} \Var(\lambda_{i,n}) \leq \frac{16C_1|F|^3}{C_2\epsilon^2 R_n} + \frac{8|F|^3}{\epsilon^2} \left( \mathbb{P}(\tau_n < t + R_n) + \epsilon_n \right), \]
and this last bound goes to 0 as \( n \) diverges.
References

[1] M. Andre (2019). "A Result of Metastability for an Infinite System of Spiking Neurons". Journal of Statistical Physics, vol. 177, pp 984–1008.

[2] F. Bertein and A. Galves (1977). "Une classe de systèmes de particules stable par association". Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, Vol.41, pp 73–85.

[3] M. Cassandro, A. Galves, E. Olivieri and M.E. Vares (1984). "Metastable behavior of stochastic dynamics: A pathwise approach". Journal of Statistical Physics, Vol.35, pp 603–634.

[4] R. Durrett (1980). "On the growth of One Dimensional Contact Process". The Annals of Probability, vol. 8, pp 890–907.

[5] R. Durrett and D. Griffeath (1983). "Supercritical Contact Processes on Z". The Annals of Probability, vol. 11, pp 1–15

[6] P.A. Ferrari, A. Galves, I. Grigorescu and E. Löcherbach (2018). "Phase transition for infinite systems of spiking neurons". Journal of Statistical Physics, Vol.172, pp 1564–1575.

[7] A. Galves and E. Löcherbach (2013). "Infinite Systems of Interacting Chains with Memory of Variable Length". Journal of Statistical Physics, Vol.151, pp 896–921.

[8] T.E. Harris (1976). "On a class of set valued Markov processes". Annals of Probability, Vol.4, pp 175–194.

[9] T.E. Harris (1978). "Additive set-valued Markov processes and graphical methods". Annals of Probability, Vol.6, pp 355–378.

[10] T.M. Liggett (1985). "Interacting Particle Systems". Grundlehren der mathematischen Wissenschaften, Issue 276.

[11] R. H. Schonmann (1985). "Metastability for the Contact Process". Journal of Statistical Physics, Vol.41, pp 445–463.