LOCALLY PI BUT NOT PI DIVISION RINGS OF ARBITRARY
GK-DIMENSION

J. C. MCCONNELL AND A. R. WADSWORTH

Abstract. We give examples of locally PI but not PI division rings of GK-dimension $n$ for every positive integer $n$.

INTRODUCTION

In this note we give a construction of division algebras which are locally PI but not PI of GK-dimension $n$ for any positive integer $n$, with center a field $F$ which may be countable or uncountable, of any characteristic. The example is the quotient division ring $D = q(R)$ of a twisted group ring $R = K \# G$, where $K$ is a $\mathbb{Z}_p$-extension field of $F$, and $G$ is a suitable free abelian subgroup of rank $n$ of the Galois group $\mathcal{G}(K/F)$. (Terminology will be defined below.) $R$ can also be described as an $n$-fold iterated twisted Laurent polynomial ring over $K$, with center $Z(R) = F$.

In [M, Prop. 4.1], McConnell gave an example of a simple Noetherian domain $R$ that is locally PI but not PI and of GK-dimension 1 over its center. That $R$ is a twisted group ring $K \# G$ where $K$ is an infinite-degree cyclotomic extension of the rational numbers $\mathbb{Q}$ and $G$ is an infinite cyclic group of automorphisms of $K$. It was later observed that the quotient division ring $D = q(R)$ is also locally PI but not PI and still has GK-dimension 1 over the center. Some years later, in 1996, Zhang gave in [Z, Ex. 5.7] an example of a locally PI but not PI division algebra of GK-dimension 2 over its center. Meanwhile, McConnell showed how to obtain GK-dimension $n$ examples as the quotient rings of tensor products of variants of his GK-1 ring $R$. He sketched out his approach in a note written to Lance Small in 1997. Later, in 2012, Small asked his colleague Wadsworth to look over McConnell’s note to clarify the infinite Galois theory being used. He did so, and in May, 2012 wrote a note “McConnell’s Example” on the example with complete proofs. The present article is based on that 2012 note.

In 2018 the authors learned of the preprint [DBH] with gives a different construction of locally PI but not PI division algebras of GK-dimension any positive integer $n$. We felt it would be worthwhile to make McConnell’s construction available to the mathematical community, and that has led to the present article.

1. $\mathbb{Z}_p$ FIELD EXTENSIONS

Let $p$ be any prime number, and let $\mathbb{Z}_p$ denote the additive group of $p$-adic integers. Thus, $\mathbb{Z}_p = \lim_{\leftarrow n \in \mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$. The usual topology on $\mathbb{Z}_p$ as a complete metric space coincides with its topology as a profinite group. In this section we recall for the reader’s convenience some well-known facts about infinite degree Galois field extensions with Galois group $\mathbb{Z}_p$. For general background on the Galois theory of infinite-degree algebraic field extensions and the topological structure of the associated Galois groups, see, e.g., [G] §1.6.
Let $F \subseteq K$ be fields. We say that $K$ is a $\mathbb{Z}_p$-extension of $F$ if there is a chain of intermediate fields $F \subseteq L_1 \subseteq L_2 \subseteq \ldots \subseteq K$ such that $K = \bigcup_{m=1}^{\infty} L_m$ and each $L_m$ is Galois over $F$ with $[L_m:F] = p^m$ and cyclic Galois group $G(L_m/F) \cong \mathbb{Z}/p^m\mathbb{Z}$. Note the following properties of a $\mathbb{Z}_p$-extension:

(i) $K$ is algebraic over $F$ with $[K:F] = \infty$.
(ii) If $L$ is a field with $F \subsetneq L \subsetneq K$, then $L = L_m$ for some $m$.
(iii) $K$ is Galois over $F$ (since it is a direct limit of finite-degree Galois extensions of $F$). Moreover, $G(K/F) \cong \mathbb{Z}_p$, a topological group isomorphism. Indeed, since $K = \bigcup_{m=1}^{\infty} L_m$, we have

$$G(K/F) = \lim_{\longrightarrow} G(L_m/F) \cong \lim_{\longrightarrow} \mathbb{Z}/p^m\mathbb{Z} = \mathbb{Z}_p.$$  

(iv) The fixed field $K^{G(K/F)} = F$, since $K$ is Galois over $F$.

(v) If $\tau \in G(K/F)$, then its restriction $\tau|_{L_m}$ lies in $G(L_m/F)$ for each $m$, since $L_m$ is Galois over $F$.

(vi) The group $G(K/F)$ is abelian, uncountable, and torsion-free. This is immediate from (iii) above. (Here is a direct proof that $G(K/F)$ is torsion-free: If $\tau \in G(K/F)$ with $\tau \neq \text{id}_K$, then there is an $m$ with $\tau|_{L_m} \neq \text{id}_{L_m}$. Say $\tau|_{L_m}$ has order $p^j$ in $G(L_m/F)$, with $1 \leq j \leq m$. Then, for the fixed field $L_m^\tau$ we have $[L_m:L_m^\tau] = p^j$, so $L_m^\tau = L_{m-j}$ (using (ii) above). For any $k \geq m$, we have $\tau|_{L_k}$ fixes $L_{m-j}$ but not $L_{m-j+1}$. Hence, $L_k^\tau = L_{m-j}$, so $\tau|_{L_k}$ has order $[L_k:L_k^\tau] = p^{k-(m-j)}$, which tends to infinity as $k \to \infty$. Thus, $\tau$ has infinite order.)

(vii) $G(K/F)$ is topologically cyclic. That is, there is $\sigma \in G(K/F)$ with fixed field $K^\sigma = F$. For example, take any nonidentity $\rho \in G(L_1/F)$, and let $\sigma$ be any extension of $\rho$ to $K$ (which exists as $K$ is normal over $F$). Then $K^\sigma$ doesn’t contain $L_1$, so we must have $K^\sigma = F$ by (ii) above. While $\langle \sigma \rangle \neq G(K/F)$ (as $G(K/F)$ is uncountable), its closure $\overline{\langle \sigma \rangle}$ is all of $G(K/F)$.

Examples.

(i) Let $F$ be a field with $F \neq F^p$ such that $F$ contains $p^m$ different $p^m$-th roots of unity for every positive integer $m$. Take any $a \in F \setminus F^p$, let $L_m = F(\sqrt[p^m]{a})$ in some algebraic closure of $F$, and let $K = \bigcup_{m=1}^{\infty} L_m$. We have $F \subseteq L_1 \subseteq L_2 \subseteq \ldots \subseteq K$, and by Kummer theory each $L_m$ is Galois over $F$ with $G(L_m/F) \cong \mathbb{Z}/p^m\mathbb{Z}$ (since $aF^{p^m}$ has order $p^m$ in $F^*/F^{p^m}$). More specifically, let $k$ be any field of characteristic not $p$ such that $k$ contains all $p^m$-th roots of unity for all $m$. ($k$ could be countable or uncountable.) Let $F = k(x)$, where $x$ is transcendental over $k$, then $x \notin F^p$, so we could take $K = \bigcup_{m=1}^{\infty} F(\sqrt[p^m]{x})$.

(ii) Let $F$ be any finite field. In an algebraic closure $\overline{F}$ of $F$ there is a unique extension field $L_m$ of $F$ with $[L_m:F] = p^m$, and $L_m$ is cyclic Galois over $F$. Moreover, $L_1 \subseteq L_2 \subseteq \ldots$. Let $K = \bigcup_{m=1}^{\infty} L_m$. Then, $K$ is a $\mathbb{Z}_p$-extension of $F$.

(iii) Suppose $K$ is a $\mathbb{Z}_p$-extension of $F$. If $E$ is a purely transcendental field extension of $F$, then the field $K \otimes_F E$ is a $\mathbb{Z}_p$-extension of $E$.

(iv) Let $F = \mathbb{Q}$, the rational numbers, and let $p$ be any odd prime number. For any $m \in \mathbb{N}$, let $\mathbb{Q}_{p^m}$ be the $p^m$-th cyclotomic extension of $\mathbb{Q}$, i.e., $\mathbb{Q}_{p^m} = \mathbb{Q}(\omega_{p^m})$, where $\omega_{p^m}$ is a primitive $p^m$-th root of unity in $\mathbb{C}$. Then, $[\mathbb{Q}_{p^m} : \mathbb{Q}] = \varphi(p^m) = p^m - 1$ and $\mathbb{Q}_{p^m}$ is Galois over $\mathbb{Q}$ with $G(\mathbb{Q}_{p^m}/\mathbb{Q}) \cong (\mathbb{Z}/p^m\mathbb{Z})^\ast$, the group of units of the ring $\mathbb{Z}/p^m\mathbb{Z}$; this is a cyclic group, as $p$ is odd. Since $G(\mathbb{Q}_{p^m}/\mathbb{Q}) \cong \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p^m-1\mathbb{Z}$, the field $\mathbb{Q}_{p^m}$ has unique subfields $E_m$ and $N_m$ with $[E_m : \mathbb{Q}] = p - 1$ and $[N_m : \mathbb{Q}] = p^{m-1}$. (In fact, $E_m = \mathbb{Q}_p$, and
Then we can form the twisted group ring $K$. Indeed, $K$ has a unique subfield of degree $p^m$ over $Q$. Thus, $K = \bigcup_{m=1}^{\infty} L$ is a $Z_p$-extension of $Q$. (For $p = 2$, and $m \geq 2$, one can show that each $Q_{2^m} \cap R$ is a cyclic Galois extension of $Q$ of degree $2^{m-2}$; hence $(\bigcup_{m=1}^{\infty} Q_{2^m}) \cap R$ is a $Z_2$-extension of $Q$.)

2. Twisted group rings

Let $K$ be a field, let $G$ be an abelian group, and let $\psi : G \to \text{Aut}(K)$ be a group homomorphism. Then we can form the twisted group ring $K\#G$,

$$K\#G = \{ \sum_{g \in G} c_g g \mid \text{each } c_g \in K \text{ and almost all } c_g = 0 \},$$

with addition given by $\sum c_g g + \sum d_g g = \sum (c_g + d_g) g$ and multiplication determined by

$$(c g)(d h) = (c \psi(g)(d)) gh.$$

We are interested here only in the case where $G$ is a free abelian group of finite rank $n \geq 1$. Then, $K\#G$ can be viewed as an $n$-fold iterated twisted Laurent polynomial ring,

$$K\#G = K[x_1, x_1^{-1}, \sigma_1; \ldots; x_n, x_n^{-1}, \sigma_n],$$

where $\{x_1, \ldots, x_n\}$ is a base of $G$ as a free $Z$-module and each $\sigma_i$ is given by $\psi(x_i)$ on $K$, then extended to $K[x_1, x_1^{-1}, \sigma_1; \ldots; x_n, x_n^{-1}, \sigma_n]$ by setting $\sigma_i(x_j) = x_j$ for all $j < i$. Thus, $K\#G$ is a left and right Noetherian Ore domain (see, e.g., [MR, Th. 4.5, p. 21]). Note also that $K\#G$ has a natural grading indexed by $G$:

$$K\#G = \bigoplus_{g \in G} (K\#G)_g \quad \text{where} \quad (K\#G)_g = Kg.$$

Indeed, $K\#G$ is a “graded division ring,” i.e., every nonzero homogeneous element is a unit. Furthermore, every unit of $K\#G$ is homogeneous. (To see this, choose some total ordering on $G$ to make it an ordered abelian group. Then observe that if $r, s$ are any inhomogeneous elements of $K\#G$, then $rs$ is also inhomogeneous, since its lowest-degree term is the product of the lowest-degree terms of $r$ and $s$, and likewise for the highest-degree term.) Thus, for the group of units,

$$(K\#G)^* = \bigcup_{g \in G} K^*g. \quad (*)$$

If $\psi(G) = \{\text{id}_K\}$, then $K\#G$ is the (untwisted) group ring $K[G]$, which is commutative. Otherwise, $K\#G$ is noncommutative.

**Lemma.** Assume that $G$ is a free abelian group of finite rank.

(i) $Z(K\#G) = E[H]$, where $E = K^\psi(G)$, the fixed field of $K$ under the action of $\psi(G)$, and $H = \ker(\psi)$.

(ii) If $\psi$ is injective, then $Z(K\#G) = K^\psi(G)$.

(iii) If $|\psi(G)| = k < \infty$, then $K\#G$ is a free $Z(K\#G)$-module of rank $k^2$.

(iv) If $\psi$ is injective, then $K\#G$ is a simple ring.

**Proof.** (i) Since $G$ is abelian, every homogeneous component of a central element of $K\#G$ is also central. Hence, $Z(K\#G)$ is a graded subring of $K\#G$. If a homogeneous element $cg$ of $K\#G$ is central ($c \in K, g \in G$), then $c$ commutes with all $h \in G$, so $c \in K^\psi(G) = E$; also, $g$ commutes with all $d \in K$, so $g \in \ker(\psi)$. Thus, $Z(K\#G) \subseteq E\#H = E[H]$, and the reverse inclusion is clear.
(ii) is immediate from (i).

(iii) Let \( E = K^\psi(G) \). If \( |\psi(G)| = k < \infty \), then \( |G : H| = |G : \ker(\psi)| = k \) and \( |K : E| = |\psi(G)| = k \). Let \( \{c_1, \ldots, c_k\} \) be a base of \( K \) as an \( E \)-vector space, and let \( g_1, \ldots, g_k \in G \) be a set of representatives for the cosets of \( H \) in \( G \). Then, \( \{c_1 g_j\}_{i=1}^k \) is a base of \( K\#G \) as a free \( E[H] \)-module.

(iv) Let \( \{x_1, \ldots, x_n\} \) be a base of the free abelian group \( G \). Take subgroups \( G_0 = \{\text{id}_K\}, G_1 = \{x_1\}, \ldots, G_i = \{x_1, \ldots, x_i\}, \ldots, G_n = G \), and let \( R_i = K \# G_i \). So \( R_0 = K \) and \( R_i = R_{i-1}[x_i, x_i^{-1}; \sigma_i] \) for \( i = 1, 2, \ldots, n \) where the automorphism \( \sigma_i \) of \( R_{i-1} \) is given by \( \psi(x_i) \) on \( K \) and \( \sigma_i(x_j) = x_j \) for \( j < i \). So, \( R_n = K \# G \). Of course, \( R_0 = K \) is a simple ring. For \( i \geq 1 \), if \( R_{i-1} \) is simple, then its twisted Laurent polynomial ring \( R_i \) is simple by \([\text{MR}] \text{ Th. 1.8.5, p. 35}\), since no power of \( \sigma_i \) is an inner automorphism of \( R_{i-1} \). (Since \( R_{i-1} = \bigcup_{g \in G_{i-1}} K^\psi g \) by (\ast) above, every inner automorphism of \( R_{i-1} \) acts on \( K \) by an element of \( \psi(G_{i-1}) \), while \( \sigma_i \) acts on \( K \) by \( \psi(x_i) \); no power of \( \psi(x_i) \) lies in \( \psi(G_{i-1}) \), as \( \psi \) is injective.) Hence, by induction, \( K \# G = R_n \) is simple.  

\[
\square
\]

3. The example

Let \( p \) be any prime number, and let \( F \subseteq K \) be any \( \mathbb{Z}_p \)-extension of fields, as described in §1. Take any \( \sigma_1 \in G(K/F) \) such that \( F = K^{\sigma_1} \) (see (vii) in §1). Now fix any positive integer \( n \), and take any \( \sigma_2, \ldots, \sigma_n \in G(K/F) \) such that \( \sigma_1, \ldots, \sigma_n \) are \( \mathbb{Z} \)-independent in \( G(K/F) \). This is possible since \( G(K/F) \) is an uncountable torsion-free abelian group. Let \( G = \langle \sigma_1, \ldots, \sigma_n \rangle \subseteq G(K/F) \), so \( G \cong \mathbb{Z}^n \), and let \( \psi: G \hookrightarrow G(K/F) \) be the inclusion map. Let \( R = K \# G \), and let \( D \) be its quotient division ring, \( D = q(R) \).

**Proposition.** \( D \) has center \( Z(D) = Z(R) = F \). Moreover, \( D \) is locally PI but not PI, and \( D \) has GK-dimension \( n \) as an \( F \)-algebra.

**Proof.** For any positive integer \( k \), let \( L_k \) be the field with \( F \subseteq L_k \subseteq K \) and \( [L_k:F] = p^k \). Since \( L_k \) is Galois over \( F \), the \( \sigma_i \) restrict to automorphisms of \( L_k \), so we can view the twisted group ring \( S_k = L_k \# G \) as a subring of \( R \). Then \( S_1 \subseteq S_2 \subseteq \ldots \) and \( R = \bigcup_{k=1}^\infty S_k \). Hence, \( D = q(R) = \bigcup_{k=1}^\infty q(S_k) \).

Consider \( S_k \). Since \( K^{\sigma_1} = F \), we also have \( L_k^{\sigma_1} = F \). Hence, \( \psi_k: G \to G(L_k/F) \) (given by \( \tau \mapsto \tau|_{L_k} \)) is surjective. Let \( H_k = \ker(\psi_k) \), which is a subgroup of \( G \) with \( |G : H_k| = |G(L_k/F)| = p^k \). So, like \( G \), the subgroup \( H_k \) is free abelian of rank \( n \). By the Lemma, \( Z(S_k) = F[H_k] \), which is a Laurent polynomial ring in \( n \) variables over \( F \). Therefore, \( q(Z(S_k)) \) is a rational function of transcendence degree, so GK-dimension, \( n \) over \( F \). By the Lemma, \( S_k \) is a free \( Z(S_k) \)-module of rank \( p^{2k} \). Hence, the division ring \( q(S_k) = S_k \otimes_{Z(S_k)} q(Z(S_k)) \) has dimension \( p^{2k} \) over its center \( q(Z(S_k)) \). Thus, \( q(S_k) \) has PI-degree \( p^k \) (see \([\text{MR}] \text{ Th. 13.3.6, p. 455}\)). Also, \( q(S_k) \) has GK-dimension \( n \) over \( F \), since it is finite-dimensional over \( q(Z(S_k)) \), which has GK-dimension \( n \) over \( F \).

Since \( D = \bigcup_{k=1}^\infty q(S_k) \), a nested union, and each \( q(S_k) \) has GK-dimension \( n \) over \( F \), \( D \) also has GK-dimension \( n \) over \( F \).

Every finitely-generated \( F \)-subalgebra or finitely-generated division subalgebra \( T \) of \( D \) lies in some \( q(S_k) \). So, \( T \) is PI. Hence, \( D \) is locally PI. But since the PI degree of \( q(S_k) \) tends to infinity with \( k \), \( D \) cannot be PI (see, e.g., \([\text{MR}] \text{ §13.3, pp. 454–456}\)).

By the Lemma, \( Z(R) = K^{\psi(G)} = F \). The Lemma also shows that \( R \) is a simple ring. Therefore, \( Z(D) = Z(R) \), as one can see by considering the denominator ideal of any element of \( Z(D) \) (as in \([\text{MR}] \text{ Prop. 2.1.16(viii), p. 48}\)).  

\[
\square
\]
REFERENCES

[DBH] Trinh Thanh Deo, Mai Hoang Bien, and Bui Xuan Hai, On weakly locally finite division rings, preprint, 2018.

[G] K. Gruenberg, Profinite Groups, Ch. V, pp. 116–127, in J. W. S. Cassels and A. Fröhlich, Algebraic Number Theory, Academic Press, London, 1967.

[M] J. C. McConnell, Representations of solvable Lie algebras. V. On the Gelfand-Kirillov dimension of simple modules, J. Algebra, 76 (1982), 489–493.

[MR] J. C. McConnell and J. C. Robson, Noncommutative Noetherian Rings, Wiley, Chichester, 1987.

[Z] James J. Zhang, On Gelfand-Kirillov transcendence degree, Trans. Amer. Math. Soc., 348 (1996), 2867–2899.

E-mail address: arwadsworth@ucsd.edu