Topological inheritance in half-SSH Hubbard models

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The interplay between interparticle interactions and topological features may result in unusual phenomena. Interestingly, interactions may induce topological features in an originally trivial system, as we illustrate for the case of a one-dimensional two-component Hubbard model in which one component is subjected to Su-Schrieffer-Heeger (SSH) dimerization, whereas the other one is not. We show that due to inter-component interactions the topological properties of one component are induced in the originally trivial one. Although for large interactions topological inheritance may be readily explained by on-site pairing, we show that the threshold for full inheritance occurs at weak interactions, for which the components are not yet paired. We illustrate this inheritance by discussing both bulk and edge properties, as well as dynamical observables as mean chiral displacement and charge pumping.

Symmetry protected topological (SPT) phase transitions are a class of phase transitions which do not come under the well-known Landau-Ginzburg paradigm associated with symmetry breaking. After the seminal observation of the fractional quantum Hall effect in condensed-matter systems [1,6], research on SPT phases in disparate systems ranging from exotic materials, ultracold neutral atoms, trapped ions, or photonic systems has earned enormous attention in recent years [7,11].

One of the simplest models exhibiting an SPT phase transition is the Su-Schrieffer-Heeger (SSH) model, which was first discussed in the context of solitons in polyacetylene [15]. The SSH model is a two-band one-dimensional tight-binding model with dimerized hoppings, which exhibits a topologically trivial to non-trivial transition through a gapless point. The non-trivial phase possesses zero-energy edge modes and a non-zero quantized Zak phase [16]. The non-interacting SSH model has been extensively analysed as a paradigmatic model to understand topological phenomena in various systems [17-23]. It has been recently implemented in quantum gas and photonic experiments [14,24,27] and, in particular, Thouless topological charge pumping [28] has been observed [29,35]. Interactions lead to exciting new physics due to the competing interplay of correlation effects, particle statistics, and lattice topology, as recently discussed theoretically for both Bose- [23,30,37] and Fermi-Hubbard models [38,39].

In this paper, we show how, due to interactions, a topological system may induce topological features in a non-topological one. In particular, we consider a two-component system, in which one of the components experiences dimerized hopping, and hence an SSH model, whereas the other presents non-dimerized hopping, and would be hence topologically trivial if considered alone. We show that a finite interaction (attractive or repulsive) coupling the two species maps topological properties into the a-priori non-topological component, resulting in the formation of strongly-correlated edge pairs.

**Model.** We consider two Bose components \{↑, ↓\}, such that ↑ experiences SSH dimerization, whereas ↓ is not dimerized. The interacting many-body system is described by the model:

\[
\mathcal{H} = - t_1 \sum_{\sigma} \sum_{i \in \text{odd}} (c_{i\sigma}^\dagger c_{i+1\sigma} + \text{H.c.}) - t_2 \sum_{i \in \text{even}} (c_{i\sigma}^\dagger c_{i+1\downarrow} + \text{H.c.}) - t \sum_{i} (c_{i\uparrow}^\dagger c_{i+1\downarrow} + \text{H.c.}) + U \sum_{i} n_{i\uparrow} n_{i\downarrow}
\]

where \(c_{i\sigma}^\dagger\) and \(c_{i\sigma}\) are the creation and annihilation operators for \(\sigma = \uparrow, \downarrow\) at site \(i\), \(n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma}\) are the number operators, \(t_1\) and \(t_2\) are the tunneling rates of ↑ from odd and even sites, respectively, \(t\) is the hopping rate of ↓, and \(U\) characterizes the inter-component interaction.

We consider the hard-core constraint, \(n_{i\uparrow} n_{i\downarrow} \leq 1\). Note that model (1) can be mapped to a Fermi mixture, which presents the same spectrum and diagonal correlations.

In absence of interactions (\(U = 0\)) model (1) reduces to two uncoupled models, an SSH model for ↑, and a trivial Hubbard model for ↓. The quantized Zak phase of ↑ is zero for \(t_1 > t_2\) (trivial phase) and \(\pi\) for \(t_1 < t_2\) (topological phase). While the bulk remains gapped for both phases, only the latter possesses zero energy edge modes. In our density matrix renormalization group (DMRG) calculations below, we consider half-filling, in which the number of particles in both components \(N_{\uparrow, \downarrow} = L/2\), for
a lattice with $L$ sites. We set $t_2 = t = 1$ as energy unit, and fix $t_1 = 0.2$, within the non-trivial regime for $\uparrow$.

**Pairing.** For a sufficiently strong $U < 0$ ($> 0$) a particle of one component pairs on-site with a particle (hole) of the other. Pairing is best monitored by $\eta = \frac{i}{L} \sum_i \left( \langle n_{i\uparrow} n_{i\downarrow} \rangle - \frac{1}{2} \right)$, see Fig. 2(a). For large-enough $|U| > 8$, $|\eta| \approx 1$ indicating strongly localized on-site pairing. For strong particle-particle pairing (particle-hole pairing is treated analogously), the system is described by an effective SSH model for on-site hard-core pairs:

$$\mathcal{H} = \frac{2t_1 t}{U} \sum_{i \text{ odd}} P_i^\dagger P_{i+1} + \frac{2t_2 t}{U} \sum_{i \text{ even}} P_i^\dagger P_{i+1} + \text{H.c.},$$

with $P_i = c_{i\uparrow}^\dagger c_{i\downarrow}$. Model 2 is topological if the SSH model for $\uparrow$ is topological. Hence, the pairs (and with them the second component) inherit the topology of the first component. Interestingly, as discussed below, much weaker pairing $\eta \ll 1$, and hence very moderate $|U|$, already suffices to induce full topological inheritance.

**Induced bulk properties.** In absence of interactions, the $\downarrow$ component is a gapless superfluid, whereas due to hopping dimerization the bulk of the $\uparrow$ component is in a gapped dimer phase with a finite dimer structure factor, $S_{\sigma\sigma}(k) = \frac{1}{L} \sum_{i,j} e^{ikr_i} \langle D_{i\sigma} D_{j\sigma} \rangle$, with $D_{i\sigma} = c_{i\sigma}^\dagger c_{i+1\sigma} + \text{H.c.}$. The dotted blue (solid red) curve in Fig. 2(b) depicts the charge gap $\Delta_{\sigma} = E(L, N_\sigma - 1) - 2E(L, N_\sigma)$ and dashed blue (dot-dashed red) curve depicts the dimer structure factor for both components. Strong pairing asymptotically results for large $|U|$ in an exact replication of the bulk properties of the $\uparrow$ component on the $\downarrow$ component. Note, however, that any $U \neq 0$ (either repulsive or attractive) results in a finite bulk gap and dimer order in the spin-$\downarrow$ component. Any finite interaction drives the bulk of the $\downarrow$ component into a gapped dimer phase.

**Induced edge states** For $t_1 < t_2$ the $\uparrow$ component is non-trivial, possessing doubly-degenerate edge modes. Sufficiently attractive (repulsive) interactions induced correlated (anti-correlated) edge states in $\downarrow$, see Fig. 2(c). The inheritance of the edge states by the $\downarrow$ component is best monitored by the polarization $P_{\sigma} = \frac{1}{L} \sum_{i=0}^{L-1} \langle \sigma_i \rangle$ for the ground state $|\psi\rangle$, which we depict in Fig. 2(d) for $L = 120$ for different values of $U$. Note that $P_{\uparrow} = -1/2$ due to the topological character of the $\uparrow$ component. The $\downarrow$ component shows maximally polarized edges, $|P_{\downarrow}| \approx 1/2$, already for $|U| \approx 1$. Hence, the $\downarrow$ component fully inherits the topological edge modes for values of $|U|$ well below those needed for strong pairing.

**Mean chiral displacement.** Topological inheritance may be easily probed experimentally by monitoring the dynamical evolution after a quench. The mean-chiral displacement (MCD), recently utilized in the context of quantum random walk on a graph, can be utilized to measure the topological winding number in photonic and ultracold atomic systems [44–47]. The MCD is defined as $C_\sigma(\tau) = \langle \Psi(0) | \Gamma_\sigma m_\sigma | \Psi(\tau) \rangle$, where $\Gamma_\sigma$ and $m_\sigma$ are the chiral and unit cell operators, respectively, and $|\Psi(\tau)\rangle$ is the time-evolved state. The MCD displays an oscillatory behavior, but after a sufficiently large time its time averaging converges to the winding number $\omega$ [48].

We analyze the MCD by considering an initial state $|\Psi(0)\rangle = O|\Psi_{GS}\rangle$, with $|\Psi_{GS}\rangle$ the ground-state, and $O = c_{0\uparrow} c_{0\downarrow} (O = c_{0\uparrow}^\dagger c_{0\downarrow})$, i.e. pair annihilation (spin flip) at the central site, for $U < 0$ ($> 0$). Quantum walks from these initial states provide insights about the

![Figure 2](image)

**Figure 2.** (Color online) (a) Pairing $\eta$ as a function of $U$; (b) Bulk gaps $\Delta_{\sigma}$ (blue dotted and red continuous), and dimer structure factors $S_{\sigma\sigma}(\pi)$ (blue dashed and red dot-dashed) for $\uparrow$ and $\downarrow$ components, respectively, as a function of $U$. The observables are extrapolated to $L = \infty$ using our DMRG results for system sizes of length $L = 40, 60, 80, 100$ and $120$; (c) Edge states for both component for $U = -5$ for a system with $L = 120$; (d) Polarization $P_{\sigma}$ for $\uparrow$ (blue squares) and $\downarrow$ (red circles) as a function of $U$, obtained for $L = 120$.

![Figure 3](image)

**Figure 3.** (Color online) (a) MCD for a system with $L = 6$ for $U = 0$ for $\uparrow$ (filled symbols) and $\downarrow$ (hollow symbols); (b) Same for $U = -15$ (red circles) and $U = -5$ (black triangles). Here solid and empty symbols are corresponding to $\uparrow$ and $\downarrow$ respectively. For $|U| = 5(15)$ the MCD evolution curves are marked by symbol triangle(circle). In all cases we employ as initial state $|\Psi(0)\rangle = c_{0\uparrow} c_{0\downarrow} |\Psi_{GS}\rangle$. 


charge and spin winding numbers, respectively. Figures show our results of $C_\sigma(\tilde{t})$, evaluated with the pair annihilation, for a system of size $L = 6$ with open boundary conditions. For $U_{\uparrow,\downarrow} = 0$ (Fig. 3(a)), $C_\uparrow(\tilde{t})$ (filled squares) oscillates around the winding number $-1$, as expected from the topological character of the $\uparrow$ component. In contrast, $C_\downarrow(\tilde{t})$ oscillates around zero (open squares), showing no topological behavior. When increasing the inter-component interaction, the MCD of both components becomes identical oscillating around the winding number (Fig. 3(b)) revealing the inheritance by the $\downarrow$ component of the topological properties of the $\uparrow$ component. Note that the deviations from the true winding numbers can be attributed to the finite size and interaction effects.

**Thouless charge pumping.** Alternatively, topological inheritance may be dynamically probed by investigating Thouless charge pumping, i.e. the transport of quantized charge as a result of an adiabatic periodic modulation of the system parameters. We consider that only the $\uparrow$ component is driven following a Rice-Mele (RM) model. The pumping is best evaluated by correlation $\eta$ which can be obtained by computing $p_\sigma = \int_0^1 d\tau \partial_\tau P_\sigma(\tau)$. As expected, $P_\uparrow(\tau)$ (Fig. 4(a)) shows the robust pumping of one particle, indicating the existence of edge states. While no pumping occurs in $\downarrow$ for $|U| = 0$, increasing $|U|$ leads to a finite pumping (Fig. 4(b)), as the fact that only $\uparrow$ is externally modulated. For $|U| \gtrsim 1$ the pumping of a full $\downarrow$ particle marks the complete topological inheritance.

**Inheritance threshold.** The previous results show that the $\downarrow$ component fully inherits the topological properties of the $\uparrow$ component (edge states, winding number, MCD, full-particle Thouless pumping) for inter-particle interactions beyond a given threshold. Such an inheritance threshold is not only revealed by the edge polarization (Fig. 4(d)) and the change of character of the charge pumping (Figs. 4(b) and (c)), but also by the analysis of the fidelity susceptibility $\chi(\downarrow) = \lim_{U \rightarrow -U'} \frac{-2 \ln |\langle \psi(\uparrow) | \psi(\uparrow') \rangle|}{(U - U')^2}$. As shown in Fig. 4(d), $\chi/L$ shows a clear maximum, that marks the inheritance threshold. Such a threshold approaches asymptotically $|U| \approx 1.06$ for growing $L$. At the threshold, the pairing correlation $\eta \approx 0.2$ (Fig. 4(a)), and hence, remarkably, the on-set of full topological inheritance occurs when the components are not yet paired.

**Conclusions.** Due to interactions, a topological system may induce topological features in a non-topological one, as we have illustrated for a Hubbard model in which a component experiences SSH dimerization and the other not. Although, for strong interactions topological inheritance may be readily understood from the formation of on-site localized inter-component pairs which experience an effective SSH model, we have shown that, interestingly, the threshold for full topological inheritance occurs for much weaker interactions, for which the two components are not yet paired.

A possible experimental realization of Model (1) may be achieved by employing two neighboring 1D lattices of hardcore dipolar bosons, which act like the two components $\{\uparrow, \downarrow\}$. Hopping dimerization in one of the chains can be introduced by using a secondary lattice on an already isolated ladder obtained using the primary lasers, as sketched in Fig. 3(a). The Hubbard interaction can be simulated by aligning the dipoles in individual chains at the so-called magic angle, such that in-leg interactions vanish, and only inter-leg interactions are relevant. Alternatively, state dependent optical lattices may allow for a direct experimental realization of Model (1). The latter may be extended to more than two components, for which topological inheritance may occur as well. As an example, Fig. 4(b) shows our results for a three-component system (assuming SU(3)-symmetric all-to-all interactions, $U_{\mathbb{SU}(3)}$), in which only one component is topological. As for the two-component case, the trivial components fully inherit the topological properties for relatively weak interactions.
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The modified limit of integration and the two different twisting angles take care of the two different components \[37\]. Our exact numerical calculation for \( L = 8 \) shows that the winding number in the topological phase is appropriately captured as \( \omega = 1 \) for \(|U| = 5\) with the choice of twist angle \( \theta_\uparrow = \pm \theta_\downarrow \).

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