The Entropy of Nonrotating Isolated Horizons in Lovelock Theory from Loop Quantum Gravity

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ABSTRACT: In this paper, we apply the method developed in loop quantum gravity to the nonrotating isolated horizons in Lovelock theory. We get the entropy that match the Wald entropy formula for this theory. We also confirm the conclusion got by Bodendorfer et al that the entropy is related to the flux operator rather than the area operator in general diffeomorphic-invariant theory.
1 Introduction

It has been realized for a long time that a black hole behaves like a thermal object. It has
temperature and entropy\[1, 2\]. The entropy is given by the famous Bekenstein-Hawking
area law
\[
S = \frac{A}{4G\hbar},
\]
(1.1)
where \(A\) is the area of the event horizon of a black hole.

Since the area law (1.1) is very simple, there are many methods to get this formula, see [3] for a review. It is a difficult task to tell which one or some actually give the right explanation for the entropy of the black hole. If considering gravity theory beyond the
Einstein theory, which often have black hole solutions, the entropy can be given by the
Wald entropy formula [4–6]. This formula has complicate form, so can’t be get easily.

Lovelock theory[7] is a natural extension of general relativity in higher dimensional
spacetime with higher derivative terms. This theory gives the second order Euler-Lagrange
equation, so can be thought as a toy model for ghost-free higher curvature gravity. It admits
a family of AdS vacua, most (but not all) of them supporting black hole solutions[8].

The entropy of black holes in Lovelock theory can be given by the Wald entropy for-
mula. Can the entropy formula has a microscopic explanation? In Ref.[9], Bodendorfer and
Neiman outline how the Wald entropy formula naturally arise in loop quantum gravity[10–
13] for this theory. The key observation is that in a loop quantization of a generalized
gravity theory, the flux operator turns out to measure the Wald entropy. But the Chern-
Simons theory they used can only be defined on odd dimensional spacetime, which is a
disadvantage.

In the previous papers, we calculated the entropy of general isolated horizons in 3
dimension AdS spacetime and nonrotating isolated horizons in 4 and higher dimensional
spacetime with the help of the BF theory. All those works are done in the category of pure
Einstein gravity theory. In this paper, those results are extended to the Lovelock theory.
In our approach it can be clearly shown that the entropy of the black hole is related to the
flux operator rather than the area operator, which is just the deep insight of the Ref.[9].
This paper is organized as follows. In section 2, we derive the symplectic form of the boundary theory, which can be seen as the same as the BF theory. In section 3 we give a microscopic calculate of the entropy of the nonrotating isolated horizons in Lovelock theory, and finally get the Wald entropy formula. Our results are discussed in section 4.

2 The symplectic form

The action of the Lovelock theory is given, in $D$ dimensional spacetime, by a sum of $K = \left\lfloor \frac{D-1}{2} \right\rfloor$ terms\cite{8},

$$I = \frac{1}{16\pi G(D-3)!} \sum_{k=0}^{K} \frac{c_k}{D-2k} I_k,$$

(2.1)

which admit a compact expression in the first order formula:

$$I_k = \int \varepsilon_{a_1 \cdots a_D} F^{a_1 a_2} \wedge \cdots \wedge F^{a_{2k-1} a_{2k}} \wedge e^{a_{2k+1}} \wedge \cdots \wedge e^{a_{D}},$$

(2.2)

where $\varepsilon_{a_1 \cdots a_D}$ is the anti-symmetric symbol, $F^{ab} := dA^{ab} + A^a_c \wedge A^{b}_c$ is the field strength 2-form of the spin connection 1-form $A^{ab}$, and $e^a$ is the covielbein 1-form. The $k = 2$ term will give the usual Gauss-Bonnet term in higher dimension.

The black hole solutions in Lovelock theory are exhaustively studied in\cite{8}. A generalization of the event horizon of the black hole is the notion of isolated horizon\cite{14–16}. The isolated horizon can also be defined in other gravity theories\cite{17}, and the properties of them remain valid in any gravity theory, since they are of geometric origin and do not involve the field equation. In this paper, we only study the nonrotating isolated horizon.

Near the isolated horizon, we adopt the Bondi-like coordinates $x^{\mu} = (u, r, \zeta^i)$ with coordinate indices $i, j = 2, \cdots, D-1$\cite{18}. The isolated horizon $\Delta$ is characterized by $r = 0$.

Following the idea of Ref.\cite{19} we choose a set of co-vielbein fields:

$$e^a_0 = \sqrt{\frac{1}{2}}(\alpha n_a + \frac{1}{\alpha}l_a), \quad e^1_a = \sqrt{\frac{1}{2}}(\alpha n_a - \frac{1}{\alpha}l_a),$$

$$e^a_A = e^a_0(dx^\mu)_a, \quad A = 2, \cdots, D-1,$$

(2.3)

where $n_a, l_a$ are null and $e^a_A$ are space-like, and $\alpha(x)$ is an arbitrary function of spacetime.

On the horizon $\Delta$, the 1-form $l_a \triangleq 0$, so the relevant co-vielbein satisfy

$$e^0_a \triangleq e^1_a, \quad e^A \triangleq \bar{e}^A,$$

(2.4)

where $\bar{e}^A$ means its value on the cross section $H$. Hereafter we denote equalities on $\Delta$ by the symbol $\triangleq$.

Define the solder field

$$\Sigma_{IJ} = \frac{1}{(D-2)!} \varepsilon_{IJK \cdots NE} e^K \wedge \cdots \wedge e^N,$$

(2.5)

then the non-zero solder fields on the horizon $\Delta$ satisfy

$$\Sigma_{01} \triangleq e^2 \wedge e^3 \wedge \cdots \wedge e^{D-1}, \quad \Sigma_{0A} \triangleq -\Sigma_{1A}.$$

(2.6)
After some straightforward calculation, the following properties for the spin connection can be get:

\[
\begin{align*}
A^{01} &\triangleq \kappa l du + d(\ln \alpha(x)), \\
A^{0A} &\triangleq A^{1A}, \\
A^{AB} &\triangleq \tilde{A}^{AB}, \\
\end{align*}
\]  

where \(\tilde{A}^{AB}\) is the connection comparable with the co-vielbein \(\tilde{e}^A\).

And the strength 2-form reads

\[
\begin{align*}
F^{01} &\triangleq 0, \\
F^{0A} &\triangleq F^{1A}, \\
F^{AB} &\triangleq \tilde{F}^{AB},
\end{align*}
\]

where \(\tilde{F}^{AB}\) is the curvature of the connection \(\tilde{A}^{AB}\).

Just like in the pure Einstein gravity theory, the symplectic current through the isolated horizon for a single \(I_k\) is

\[
\int_\Delta J(\delta_1, \delta_2) = \int_\Delta \varepsilon_{a_1 \cdots a_D} \delta_{[2} A^{a_1 a_2} \wedge \delta_{1]} (\tilde{F}^{a_3 a_4} \cdots \wedge \tilde{F}^{a_{2k-1} a_{2k}} \wedge e^{a_{2k+1}} \cdots \wedge e^{a_D}).
\]

Due to the properties of (2.4), (2.7) and (2.8) on the horizon, the only term left in the above expression is

\[
\int_\Delta J(\delta_1, \delta_2) = 2k(D-2)! \int_\Delta \delta_{[2} A^{01} \wedge \delta_{1]} (\tilde{F}^{23} \cdots \wedge \tilde{F}^{2(k-1)2k-1} \wedge e^{2k} \cdots \wedge e^{D-1}),
\]

where the coefficient \(k\) comes from the fact that there are \(k\) terms of \(\tilde{F}^{ab}\), and \((D-2)!\) comes from the property of \(\varepsilon_{01a_3 \cdots a_D}\).

Now combine all \(k\). The full symplectic current term through \(\Delta\) can be written as

\[
\int_\Delta J(\delta_1, \delta_2) = \int_\Delta \delta_{[2} A^{01} \wedge \delta_{1]} \left( \frac{1}{16\pi G(D-3)!} \sum_{k=1}^K \frac{2k(D-2)!c_k}{(D-2k)} \tilde{F}^{23} \cdots \wedge \tilde{F}^{2(k-1)2k-1} \wedge e^{2k} \cdots \wedge e^{D-1} \right) = \int_\Delta \delta_{[2} A^{01} \wedge \delta_{1]} \pi_{01},
\]

where

\[
\pi_{01} := \frac{1}{16\pi G(D-3)!} \sum_{k=1}^K \frac{2k(D-2)!c_k}{(D-2k)} \tilde{F}^{23} \cdots \wedge \tilde{F}^{2(k-1)2k-1} \wedge e^{2k} \cdots \wedge e^{D-1}
\]

is the conjugate momentum to the connection \(A_{01}\).

For an isolated horizon, we have

\[
\partial_u e^A \triangleq 0, \\
\partial_u \tilde{A}^{AB} \triangleq 0.
\]

So \(\partial_u \tilde{F}^{AB} \triangleq 0\). Then it is easy to show that

\[
d\pi_{01} \triangleq 0,
\]

thus it is a closed \((D-2)\)-form on the horizon \(\Delta\). Locally we can define a \((D-3)\)-form \(B\) which satisfy

\[
dB = \pi_{01}.
\]
From Eq. (2.7) it is easy to show
\[ dA^{01} \equiv 0. \] (2.16)

Now we get the similar formula (2.11), (2.15) and (2.16) as in 4 dimensional Einstein case. Then the boundary degrees of freedom can be described by a $SO(1,1)$ BF theory on punctured manifold, and the related fields are
\[ A^{01} \leftrightarrow A, \quad B \leftrightarrow B. \] (2.17)

The quantum theory for the BF theory on punctured manifold is given as in 4 dimension: for $n$ punctures $P = \{p_\alpha | \alpha = 1, \cdots, n\}$, the associated Hilbert space $\mathcal{H}_{BF}^n$ is spanned by basic states $\{|a_\alpha\rangle, a_\alpha \in \mathbb{R} >\$, and the full Hilbert space is $\mathcal{H}_{BF} = \oplus_n \mathcal{H}_{BF}^n$.

3 Calculation of the entropy

We want to calculate the entropy of the nonrotating isolated horizon. As in pure Einstein theory[19], the boundary condition can be chosen as
\[ dB \equiv \pi_{01} = \frac{1}{8\pi G} \sum_{k=1}^{K} \frac{k(D-2)c_k}{(D-2k)} \tilde{F}^{23} \wedge \cdots \tilde{F}^{2(k-1),2k-1} \wedge e^{2k} \wedge \cdots \wedge e^{D-1}. \] (3.1)

The relative flux constraint is
\[ \oint_H dB = \frac{1}{8\pi G} \oint_H \sum_{k=1}^{K} \frac{k(D-2)c_k}{(D-2k)} \tilde{F}^{23} \wedge \cdots \tilde{F}^{2(k-1),2k-1} \wedge e^{2k} \wedge \cdots \wedge e^{D-1} := a', \] (3.2)

Inspired by the results got in loop quantum gravity in higher dimension, the following assumption is made that the eigenvalue of the flux operator $\pi_{01}$ is quantized due to
\[ \oint_H \pi_{01}(x)|m_i, \cdots > = \beta \sum_i m_i|\lambda_i, \cdots >, \quad m_i \in \mathbb{Z}. \] (3.3)

When considering only $k = 1$ term, it return to the result in higher dimensional pure Einstein theory.

The full Hilbert space is given by $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_S$, where $\mathcal{H}_V$ is the bulk Hilbert space generated by spin network states, and $\mathcal{H}_S$ is the boundary Hilbert space given by $\mathcal{H}_{BF}$ in the last section. The quantum version of the boundary condition is
\[ (\text{Id} \otimes \oint_S dB - \oint_S \tilde{\pi}_{01} \otimes \text{Id})(\Psi_V \otimes \Psi_S) = 0. \] (3.4)

After acting on the full Hilbert space, the relation between the eigenvalues of $\hat{d}B$ and $\hat{\pi}_{01}$ is get:
\[ a_p = \beta m_p, \] (3.5)

where $a_p$ is the quantum number associated with the quantum BF theory as in the last section.
The flux constraint coming from the quantum version of the Eq. (3.2) is
\[ \sum_p |a_p| = \sum_p |m_p| = a'. \] (3.6)
or
\[ \sum_p |m_p| = \frac{a'}{\beta} = a, \quad m_p \in \mathbb{Z}, \] (3.7)
so we need the condition \( a \in \mathbb{N} \).

Then it is easy to calculate the number of the states:
\[ \mathcal{N} = \sum_{n=a}^{n=a} C_{a-1}^{n-1} 2^n = 2 \times 3^{a-1}, \] (3.8)
The entropy is given by
\[ S = \ln \mathcal{N} = a \ln 3 + \ln \frac{2}{3} = \frac{\ln 3}{8\pi G \beta} \oint_H \sum_{k=1}^{K} \frac{k(D - 2)c_k}{(D - 2k)} \tilde{F}^{23} \wedge \cdots \tilde{F}^{2(k-1), 2(k-1)} e^{2k} \wedge \cdots \wedge e^{D-1} + \ln \frac{2}{3} \] (3.9)
If \( \beta = \ln 3/(2\pi) \) is set, which is the same as in higher dimension pure Einstein gravity, we can get
\[ S = \frac{1}{4G} \oint_H \sum_{k=1}^{K} \frac{k(D - 2)c_k}{(D - 2k)} \tilde{F}^{23} \wedge \cdots \tilde{F}^{2(k-1), 2(k-1)} e^{2k} \wedge \cdots \wedge e^{D-1} + \ln \frac{2}{3} = 2\pi \oint_H \pi_{01} + \ln \frac{2}{3} \] (3.10)
This is just the Wald entropy formula for Lovelock theory in terms of the flux appeared in Ref. [9] plus a constant correction term. It has a simple physical picture: the entropy is just the total number of gravitational flux \( \pi_{01} \) through the horizon section \( H \) times \( 2\pi \).

Next let’s consider a simple example: the nonrotating isolated horizon with maximal spherical symmetry, such as Schwarzschild-type black holes. In this case, we have
\[ F^{AB} \triangleq \tilde{F}^{AB} = \frac{1}{r_0^2} e^A \wedge e^B, \] (3.11)
where \( r_0 \) is the radius of the horizon. Then \( a \) is
\[ a = \frac{2\pi}{\ln 3} \oint_H d\mathcal{B} = \frac{1}{4G \ln 3} \oint_H \sum_{k=1}^{K} \frac{k(D - 2)c_k}{(D - 2k)} \epsilon^{2k} \wedge \cdots \wedge \epsilon^{D-1} = \frac{A_H}{4G \ln 3} \sum_{k=1}^{K} \frac{k(D - 2)c_k}{(D - 2k)r_0^{2(k-1)}} \] (3.12)
where \( A_H := \oint_H \epsilon^2 \wedge \cdots \wedge \epsilon^{D-1} \) is the ‘area’ of the horizon. The entropy is given by
\[ S = a \ln 3 + \ln \frac{2}{3} = \frac{A_H}{4G} \sum_{k=1}^{K} \frac{k(D - 2)c_k}{(D - 2k)r_0^{2(k-1)}} + \ln \frac{2}{3}, \] (3.13)
which coincide with the result in [8, 20] except the constant correction term.
4 Discussion

In the previous sections, we studied the entropy of nonrotating isolated horizons in Lovelock theory. Finally we got the Wald entropy formula. Also the parameter appeared in quantized Lovelock theory have the same value as in pure Einstein theory. This fact can be consider as a consistent check of our method, since Lovelock theory can reduce to the pure Einstein theory. To get the Wald entropy formula from microscopic theory, the key assumption is the "quantized flux" assumption (3.3) in the bulk theory.

In Lovelock theory, the flux operator $\pi_{01}$, which is the conjugate momentum of the connection $A_{01}$, is not $\Sigma_{01}$ as in pure Einstein theory, but its complicate expression. This operator rather than the area operator appears in the Wald entropy formula (3.10). The $B$ field is also related to this flux operator, and through the BF theory on punctured manifold, we give the Wald entropy formula a microscopic explanation.

According to our result, the calculation of the entropy from usual Chern-Simons theory approach in 4 dimensional spacetime have to be modified. In Chern-Simons theory approach, the area constraint play an important role, and we think that this constraint should be replaced by the flux constraint, which is just the work done by Barbero et al.[21]. Our approach favors their results.

Acknowledgments

This work is supported by National Natural Science Foundation of China under the grant 11275207.

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