Elimination of imaginaries in multivalued fields

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Abstract

In this paper we study elimination of imaginaries in some classes of henselian valued fields of equicharacteristic zero and residue field algebraically closed. The results are sensitive to the complexity of the value group. We focus first in the case where the ordered abelian group has finite spines, and then prove a better result for the dp-minimal case. In [25] it was shown that an ordered abelian with finite spines weakly eliminates imaginaries once we add sorts for the quotient groups $\Gamma/\Delta$ for each definable convex subgroup $\Delta$, and sorts for the quotient groups $\Gamma/\Delta + l\Gamma$ where $\Delta$ is a definable convex subgroup and $l \in \mathbb{N}_{\geq 2}$. We refer to these sorts as the quotient sorts. In [27] F. Janke, P. Simon and E. Walsberg characterized dp-minimal ordered abelian groups as those that for every prime $p$ satisfy that $[\Gamma : p\Gamma] < \infty$.

We prove the following two theorems:

**Theorem.** Let $K$ be a valued field of equicharacteristic zero, residue field algebraically closed and value group with finite spines. Then $K$ admits weak elimination of imaginaries once we add codes for all the definable $\mathcal{O}$-submodules of $K^n$ for each $n \in \mathbb{N}$, and the quotient sorts for the value group.

**Theorem.** Let $K$ be a henselian valued field of equicharacteristic zero, residue field algebraically closed and dp-minimal value group. Then $K$ eliminates imaginaries once we add codes for all the definable $\mathcal{O}$-submodules of $K^n$ for each $n \in \mathbb{N}$, the quotient sorts for the value group and constants to distinguish representatives of the cosets of $\Delta + l\Gamma$ in $\Gamma$, where $\Delta$ is a convex definable subgroup and $l \in \mathbb{N}_{\geq 2}$.

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1 Introduction

The model theory of henselian valued fields has been a major topic of study during the last century, it was initiated by Robinson’s model completeness results for algebraically closed valued fields in [17]. Remarkable work has been achieved by Haskell, Hrushovski and Macpherson to understand the model theory of algebraically closed valued fields. In a sequence of papers [2] and [3] they developed the notion of stable domination, that rather than being a new form of stability should be understood as a way to apply techniques of stability in the setting of valued fields. Further work of Ealy, Haskell and Maficová in [1] for the setting of real closed convexly valued fields, suggested that the notion of having a stable part of the structure was not fundamental to achieve domination results and indicated that the right notion should be residue field domination or domination by the sorts internal to the residue field. Our main motivation for the present document arises from the natural question of how much further a notion of residue field domination could be extended to broader classes of valued fields to gain a deeper model theoretic insight of henselian valued fields, and the first step is finding a reasonable language where the valued field will eliminate imaginaries.

The starting point in this project relies on the Ax-Kochen theorem, which states that the first order theory of a henselian valued field of equicharacteristic zero or unramified mixed characteristic with perfect residue field is completely determined by the first order theory of its valued group and its residue field. A natural principle follows from this theorem: model theoretic questions about the valued field itself can be understood by reducing them to its residue field, its valued group and their interaction in the field.

A fruitful application of this principle has been achieved to describe the class of definable sets. For example, in [4] Pas proved field quantifier elimination relative to the residue field and the valued group once angular component maps are added in the equicharacteristic case. Further studies of Basarab and F.V. Kuhlmann show a quantifier elimination relative to the $RV$ sorts [see [5], [7] respectively].

The question of whether a henselian valued field eliminates imaginaries in a given language is of course subject to the complexity of its value group and its residue field as both are interpretable structures in the valued field itself. The case for algebraically closed valued fields was finalized by Haskell, Hrushovski and Macpherson in their important work [3], where elimination of imaginaries for $ACVF$ is achieved once the geometric sorts $S_n$ (codes for the $O$-lattices of rank $n$) and $T_n$ (codes for the residue classes of the elements in $S_n$) are added. This proof was later significantly simplified by Will Johnson in his PhD Thesis [12] by using a criterion isolated by Hrushovski [see [16]].

Recent work has been done to achieve elimination of imaginaries in some other examples of henselian valued fields, as the case of separably closed valued fields in [18], the $p$-adic case in [19] or enrichments of $ACVF$ in [20].

However, the above results are all obtained for particular instances of henselian valued fields while the more general approach of obtaining a relative statement for broader classes of henselian valued fields is still a very interesting open question. Following the Ax-Kochen style principle, it seems natural to first attempt to solve this question by looking at the problem in two orthogonal directions: one by making the residue field as docile as possible and studying which troubles would the value group bring into the picture, or by making the value group tame and understanding the difficulties that the residue field would contribute to the problem. Hils and Rideau [21] had proved that under the assumption of having a definably complete value group and requiring that the residue field eliminates the $\exists^\infty$ quantifier, then any definable set admits a code once the geometric sorts and the linear sorts are added to the language. Any definably complete ordered abelian group is either divisible or a $\mathbb{Z}$-group (i.e. a model of Presburger Arithmetic).

This paper is addressing the first approach in the setting of henselian valued fields of equicharacteristic zero. We suppose the residue field to be algebraically closed and we obtain results which are sensitive to the complexity of the value group. We first analyze the case where the value group has finite spines. An ordered abelian with finite spines weakly eliminates imaginaries once we add sorts for the quotient groups $\Gamma/\Delta$ for each definable convex subgroup $\Delta$, and sorts for the quotient groups $\Gamma/\Delta + l\Gamma$ where $\Delta$ is a definable convex subgroup and $l \in \mathbb{N}_{>0}$. We refer to these sorts as the quotient sorts. The first result that we obtain is:
Theorem 1.1. Let $K$ be a valued field of equicharacteristic zero, residue field algebraically closed and value group with finite spines. Then $K$ admits weak elimination of imaginaries once we add codes for all the definable $\mathcal{O}$-submodules of $K^n$ for each $n \in \mathbb{N}$, and the quotient sorts for the value group.

Later, we prove a better result for the dp-minimal case, this is:

Theorem 1.2. Let $K$ be a henselian valued field of equicharacteristic zero, residue field algebraically closed and dp-minimal value group. Then $K$ eliminates imaginaries once we add codes for all the definable $\mathcal{O}$-submodules of $K^n$ for each $n \in \mathbb{N}$, the quotient sorts for the value group and constants to distinguish representatives of the cosets of $\Delta + l\Gamma$ in $\Gamma$, where $\Delta$ is a convex definable subgroup and $l \in \mathbb{N}_{\geq 2}$.

This document is organized as follows:

- **Section 2**: We introduce the required background, including quantifier elimination statements, the state of art of the model theory of ordered abelian groups and some results about valued vector spaces.

- **Section 3**: We study definable $\mathcal{O}$-modules of $K^n$. We focus on the 1-dimensional case and the existence of unary codes.

- **Section 4**: We start by presenting Hrushovski’s criterion to eliminate imaginaries. We introduce the stabilizer sorts, where the $\mathcal{O}$-submodules of $K^n$ can be coded. We finish this section by giving a complete description of each of the stabilizers.

- **Section 5**: We show that the conditions of Hrushovski’s criterion hold. This is the density of definable types in definable sets in 1-variable $X \subseteq K$. We continue proving that any definable type can be encoded in the stabilizer sorts and $\Gamma^{eq}$. We conclude this section proving the weak elimination of imaginaries of any henselian valued field of equicharacteristic zero, residue field algebraically closed and value group with finite spines down to the stabilizer sorts.

- **Section 6**: We show a complete elimination of imaginaries statement when the value group is dp-minimal. We prove that any finite set of tuples in the stabilizer sorts can be coded.

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2 Preliminaries

Quantifier Elimination for valued fields of equicharacteristic zero and residue field algebraically closed

In this section we recall several results relevant for our statement, in particular we state a quantifier elimination relative to the value group in the canonical three sorted language $\mathcal{L}_{\text{val}}$ for the class of valued fields of equicharacteristic zero and residue field algebraically closed.

The three-sorted language $\mathcal{L}_{\text{val}}$

We consider valued fields as three sorted structures $(K,k_\kappa,\Gamma)$. The first two sorts are equipped with the language of fields $\mathcal{L}_{\text{fields}} = \{0,1,+,\cdot,\leq,\geq,\prec,\succ,\ll,\gg\}$, we refer to the first one as the main field sort while we call the second one as the residue field sort. The third sort is supplied with the language of ordered abelian groups $\mathcal{L}_{\text{OAG}} = \{0,\prec,\succ,\ll,\gg\}$, and we refer to it as the value group sort. We also add constants $\infty$ to the second sort and the third sort. We introduce a function symbol $\text{v} : K \rightarrow \Gamma \cup \{\infty\}$, interpreted as the valuation and, we add a map $\text{res} : K \rightarrow k_\kappa \cup \{\infty\}$, where $\text{res} : \mathcal{O}_\kappa \rightarrow k_\kappa$ is interpreted as a surjective homomorphism of rings, and for any element $x \in K \setminus \mathcal{O}_\kappa$ we have $\text{res}(x) = \infty$. We denote this language as $\mathcal{L}_{\text{val}}$.

The extension Theorem

Let $\mathcal{K} = (K,k,\Gamma)$ be a valued field. A triple $\mathcal{E} = (E,k_E,\Gamma_E)$ is a substructure of $\mathcal{K}$ if $E$ is a subfield of $K$, $k_E$ is a subfield of $k$, $\Gamma_E$ is a subgroup of $\Gamma$, $\text{v}(E^*) \subseteq \Gamma_E$ and $\text{res}(O_E) \subseteq k_E$, where $O_E = O \cap E$.

**Definition 2.1.** Let $\mathcal{K}_1 = (K_1,k_1,\Gamma_1)$ and $\mathcal{K}_2 = (K_2,k_2,\Gamma_2)$ be valued fields of equicharacteristic zero with residue field algebraically closed.

Let $\mathcal{E} = (E,k_E,\Gamma_E)$ be a substructure of $\mathcal{K}_1$ a triple $(f,f_r,f_v) = \mathcal{E} \rightarrow \mathcal{K}_2$ is said to be an admissible embedding if:

- $f : E \rightarrow K_2$ is a field embedding,
- $f_r : k_E \rightarrow k_2$ is a field embedding,
- $f_v : \Gamma_E \rightarrow \Gamma_2$ is a partial elementary map between $\Gamma_1$ and $\Gamma_2$, i.e. for every $\mathcal{L}_{\text{OAG}}$ formula $\phi(x_1,\ldots,x_n)$ and tuple $e_1,\ldots,e_n \in \Gamma_E$ we have that $\vDash \phi(e_1,\ldots,e_n)$ if and only if $\vDash \phi(f_v(e_1),\ldots,f_v(e_n))$.
- for any $a \in O_E$, $f_r(\text{res}(a)) = \text{res}(f(a))$ and
- for any $a \in E$, $f_v(\text{v}(a)) = \text{v}(f(a))$.

Let $\kappa = \max\{|k_E|,|\Gamma_E|\}$, if $\mathcal{K}_2$ is $\kappa^+$-saturated we say that $(f,f_r,f_v)$ is an admissible map with small domain.

**Theorem 2.2.** [The Extension Theorem] The theory of henselian valued fields of equicharacteristic zero with residue field algebraically closed admits quantifier elimination relative to the value group in the language $\mathcal{L}_{\text{val}}$. That is given $\mathcal{K}_1$ and $\mathcal{K}_2$ henselian valued fields of equicharacteristic zero with residue field algebraically closed, given $\mathcal{E}$ a substructure of $\mathcal{K}_1$ and $(f,f_r,f_v) : \mathcal{E} \rightarrow \mathcal{K}_2$ an admissible map with small domain, for any $b \in \mathcal{K}_1$ there is an admissible maps $\tilde{f}$ extending $f$ whose domain contains $b$.

**Proof.** This is straightforward using the standard techniques to obtain elimination of field quantifiers already present in the area. We refer the reader for example to [8, Theorem 5.21]. The unique step that requires the presence of an angular component map is when for a subfield $E \subseteq K_1$ we want to add an element $\gamma$ to $\text{v}(E^*)$ and there is some prime number $p$ such that $p\gamma \in \text{v}(E^*)$. For this, take $a \in E$ be such that $\text{v}(a) = p\gamma$, and $c \in K_1$ such that $\text{v}(c) = \gamma$. We first aim to find $b_1 \in K_1$ that is a root of the polynomial $Q(x) = x^p - \frac{c}{\gamma} \in O_E(x)$. Let $d = \text{res}(\frac{c}{\gamma})$, because $K_1$ is algebraically closed there is some $z \in k_1$ such that $z^p - d = 0$. Let $\alpha \in K_1$ be such that $\text{res}(\alpha) = z$, then $\text{v}(Q(\alpha)) > 0$ while $\text{v}(Q'(\alpha)) = 0$, because $p \neq \text{char}(k_1)$. In fact, $Q'(x) = px^{p-1}$ and $z \neq 0$ as $d \neq 0$.

By henselianity we can find $b_1 \in K_1$ that is a root of $Q(x)$. Then $x_1 = (b_1 \cdot c)$ is a $p$-root of $a$. 

□
The following is an immediate consequence of the quantifier elimination.

**Corollary 2.3.** The residue field and the value group are both stably embedded and orthogonal to each other.

### Some results on the model theory of ordered abelian groups

In this Subsection we summarize many interesting results about the model theory of ordered abelian groups. We start by recalling the following folklore fact.

**Fact 2.4.** Let \( (\Gamma, \preceq, +, 0) \) be a non-trivial ordered abelian group. Then the topology induced by the order in \( G \) is discrete if and only if \( G \) has a minimal positive element. In this case we say that \( G \) is discrete, otherwise we say that it is dense.

The first results about the model completions of ordered abelian groups appear in [6] (1960), where the notion of \( n \)-regularity was isolated.

**Definition 2.5.**  
1. Let \( \Gamma \) be an ordered abelian group and \( \gamma \in \Gamma \), we say that \( \gamma \) is \( n \)-divisible if there is some \( \beta \in \Gamma \) such that \( \gamma = n\beta \).

2. Let \( n \geq 2 \), an ordered abelian group \( \Gamma \) is said to be \( n \)-regular if any interval with at least \( n \)-points contains an \( n \)-divisible element.

3. Let \( \Gamma \) be an ordered abelian group, we say that it is regular if it is \( n \)-regular for all \( n \in \mathbb{N} \).

In [6] Robinson and Zakon characterized completely the possible completions of regular groups and proved that each regular group is elementarily equivalent to some archimedian group. Later, Belegradek studied a larger class of poly-regular ordered abelian groups in [11], and characterized them as those that are elementarily equivalent to some subgroup of the lexicographical ordered group \( \mathbb{R}^\gamma \). Belegradek proved that an ordered abelian group is poly-regular if and only if it has at most \( n \)-proper definable convex subgroups, all definable over the empty-set. A quantifier elimination statement was achieved by Weispfenning for poly-regular groups in [10]. In [23] Cluckers and Halupczok introduce a language \( L_{qe} \) to obtain a quantifier elimination for ordered abelian groups relative to the auxiliary sorts. This language is similar in the spirit of the one introduced by Gurevich and Schmitt in [24], but has been lately preferred by the community as it is more in line with the many-sorted language of Shelah’s expansion. The description of the auxiliary sorts \( S_n, T_n \) and \( T_n^+ \) can be found in [23, Definition 1.5]. Schmitt does not distinguish between the sorts \( S_n, T_n \) and \( T_n^+ \), instead for each \( n \in \mathbb{N} \) he works with a single sort \( S_{p,n}(\Gamma) \) called the \( n \)-spine of \( \Gamma \), whose description can be found in Section 2 in [24]. In [23, Section 1.5] it is explained how the auxiliary sorts of Cluckers and Halupczok are related to the \( n \)-spines \( S_{p,n}(G) \) of Gurevich-Schmitt.

### Ordered abelian groups with bounded regular rank

Between the class of poly-regular ordered abelian groups and the more general case of order abelian groups, there is a tamer class which are those ordered abelian groups with finite \( n \)-regular rank for all \( n \in \mathbb{N} \).

**Definition 2.6.** An ordered abelian group \( \Gamma \) has \( n \)-regular rank equal to \( m \) iff there are \( \Delta_0, \ldots, \Delta_m \) convex subgroups of \( \Gamma \), such that:

1. \( \{0\} = \Delta_0 \leq \Delta_1 \leq \cdots \leq \Delta_m = \Gamma \),

2. for each \( 0 \leq i < m \), the quotient group \( \Delta_{i+1}/\Delta_i \) is \( n \)-regular,

3. the quotient group \( \Delta_{i+1}/\Delta_i \) is not \( n \)-divisible for \( 0 < i < m \).

In this case we denote a the set of \( n \)-regular jumps \( RJ_n(\Gamma) = \{\Delta_0, \ldots, \Delta_{m-1}\} \). If for all \( n \in \mathbb{N} \), \( \Gamma \) has finite \( n \)-regular rank, we denote as \( RJ(\Gamma) = \bigcup\limits_{n \geq 2} RJ_n(\Gamma) \).

The following is [22, Proposition 2.3].

**Proposition 2.7.** Let \( \Gamma \) be an ordered abelian group. The following are all equivalent:

1. \( \Gamma \) has finite \( p \)-regular rank for each prime \( p \),
Moreover, in this case $RJ(\Gamma)$ is the collection of all proper definable convex subgroups of $\Gamma$ and all are definable without parameters.

In view of (3) of the previous Proposition we will say that an ordered abelian group has **bounded regular rank** if it satisfies any (equivalently all) the conditions of Proposition 2.7. In Section 2 [22], it is shown that an ordered abelian group $\Gamma$ has bounded regular rank if and only if all the $n$-spines are finite, and $Sp_n(\Gamma) = RJ_n(\Gamma)$. In this case, we define the regular rank of $\Gamma$ as $|RJ(\Gamma)|$, which is either finite or $\aleph_0$. Now we would like to introduce a language $L_b$, in which any ordered abelian group of bounded regular rank has quantifier elimination.

We define the **Presburger Language** $L_{Pres} = \{0, 1, +, -, <, (\equiv_m)_{m \in \mathbb{N}_{\geq 2}}\}$. Given an ordered abelian group $\Gamma$, we naturally see it as a $L_{Pres}$-structure. The symbols $\{0, +, -, <\}$ take their obvious interpretation. If $\Gamma$ is discrete, the constant symbol 1 is interpreted as the least positive element of $\Gamma$, and by 0 otherwise. For each $m \in \mathbb{N}_{\geq 2}$, the binary relation symbol $\equiv_m$ is interpreted as the equivalence modulo $m$, i.e. for any $g, h \in \Gamma \quad g \equiv_m h$ if and only if $g - h \in m\Gamma$.

**Definition 2.8.** *(The language $L_b$)* Let $\Gamma$ be an ordered abelian group with bounded regular rank, we view $\Gamma$ as a multi-sorted structure where:

1. We add a sort for the ordered abelian group $\Gamma$, and we equip it with a copy of the language $L_{Pres}$ extended with predicates to distinguish each of the convex subgroups $\Delta \in RG(\Gamma)$. We refer to this sort as the main sort.

2. We add a sort for each of the ordered abelian groups $\Gamma/\Delta$, equipped with a copy of the language $L_{Pres}^{\Delta} = \{0^\Delta, 1^\Delta, +^\Delta, -^\Delta, <^\Delta, (\equiv_m^\Delta)\}$. We add as well a map $\rho_\Delta := \Gamma \to \Gamma/\Delta$, interpreted as the natural projection map.

The following statement is a direct consequence of [28, Proposition 3.4].

**Theorem 2.9.** Let $\Gamma$ be an ordered abelian group with bounded regular rank. Then $\Gamma$ admits quantifier elimination in the language $L_b$.

We will consider an extension of this language that we will denote as $L_{bd}$, where for each natural number $n \geq 2$ and $\Delta \in RJ(\Gamma)$ we add a sort for the quotient group $\Gamma/(\Delta + n\Gamma)$ and a map $\pi_\Delta := \Gamma \to \Gamma/(\Delta + n\Gamma)$. We will refer to the sorts in the language $L_{bd}$ as quotient sorts. The following is [25, Theorem 5.1].

**Theorem 2.10.** Let $\Gamma$ be an ordered abelian group with bounded regular rank. Then $\Gamma$ admits weak elimination of imaginaries in the language $L_{bd}$, i.e. once all the quotient sorts are added.

**Definable end-segments in ordered abelian groups with bounded regular rank**

**Definition 2.11.** 1. A set $S \subset \Gamma$ is said to be an end-segment (respectively an initial segment) if for any $x \in S$ and $y \in \Gamma$, $x < y$ we have that $y \in S$. (Respectively if $x \in S$, $y \in \Gamma$ and $y < x$ then $y \in S$.)

2. Let $n \in \mathbb{N}$, $\Delta \in RJ(\Gamma)$, $\beta \in \Gamma \cup \{-\infty\}$ and $\square \in \{\geq, >\}$. The set:

$$S_{\Delta}(\beta) := \{\eta \in \Gamma \mid n\eta + \Delta \nabla \beta + \Delta\}$$

defines an end-segment of $\Gamma$. We call any of the end-segments of this form as divisibility end-segments. Likewise we can relativize these notions to initial segments if $\square \in \{\leq, <\}$.

3. Let $S \subseteq \Gamma$ be an end-segment and $\Delta \in RJ(\Gamma)$. We consider the projection map $\rho_\Delta := \Gamma \to \Gamma/\Delta$, and we denote as $S_\Delta = \rho_\Delta(S)$. This is a definable end-segment of $\Gamma/\Delta$. 

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4. Let $\Delta \in RJ(\Gamma)$ and $S \subseteq \Gamma$ an end-segment. We say that $S$ is $\Delta$-decomposable if it is a union of $\Delta$-cosets.

5. Let $X$ and $Y$ be definable sets. We say that $Y$ is coinitial in $X$ if for any $y \in X$ there is some element $z \in X \cap Y$ such that $z \leq y$.

**Definition 2.12.** Let $\Gamma$ be an ordered abelian group. Let $S, S' \subseteq \Gamma$ be definable end-segments. We say that $S$ is a translate of $S'$ if there some $\beta \in \Gamma$ such that $S = \beta + S'$. Given a family $S$ of definable end-segments we say that $S$ is complete if every definable end-segment is a translate of some $S' \in S$.

**Fact 2.13.** Let $\Gamma$ be an ordered abelian group with bounded regular rank. Let $\beta, \gamma \in \Gamma$, $\Delta \in RJ(\Gamma)$ and $n \in \mathbb{N}_{>2}$. If $\beta - \gamma \in \Delta + n\Gamma$ then $S_n^\Delta(\gamma)$ is a translate of $S_n^\Delta(\beta)$.

The following is [25, Proposition 3.3].

**Proposition 2.14.** Let $\Gamma$ be an ordered abelian group of bounded regular rank. Any definable end-segment is a divisibility end-segment.

An immediate consequence of this Proposition is the following corollary.

**Corollary 2.15.** Let $\Gamma$ be an ordered abelian group with bounded regular rank. For each $n \in \mathbb{N}$ and $\Delta \in RG(\Gamma)$, let $C_n^\Delta$ be a complete set of representatives of the cosets $\Delta + n\Gamma$ in $\Gamma$. Define $S^\Delta_n := \{S_n(\beta) \mid \beta \in C_n^\Delta\}$. Then $S = \bigcup_{\Delta \in RJ(\Gamma), n \in \mathbb{N}} S_n^\Delta$ is a complete family.

**Definition 2.16.** Let $S \subseteq \Gamma$ be a definable end-segment. Let

$$\Sigma^\text{gen}_S(x) := \{ x \in S \} \cup \{ x \in B \mid B \not\subseteq S \text{ and } B \text{ is a definable end-segment} \}.$$  

We refer to this partial type as the generic type in $S$.

**Definition 2.17.**

1. A basic positive congruence is a formula of the form $z \rho_\Delta(x) \equiv_\Gamma^\Delta \beta + k^\Delta$ where $\beta \in \Gamma/\Delta$, $z, k \in \mathbb{Z}$, $l \in \mathbb{N}_{>2}$, where $k^\Delta = k \cdot 1_\Delta$ where $1_\Delta$ is the minimal element of $\Gamma/\Delta$ if it exists. Likewise we say that a basic negative formula if it is a formula of the form $z \rho_\Delta(x) \not\equiv_\Gamma^\Delta \beta + k^\Delta$. A basic congruence formula is either a basic positive congruence formula or a basic negative formula.

2. A finite congruence restriction is a finite conjunction of basic congruence formulas.

The following statement is a particular instance of [25, Fact 3.7].

**Fact 2.18.** Let $\Gamma$ be an ordered abelian group with bounded regular rank and $S \subseteq \Gamma$ be a definable end-segment. Let $X$ be a definable set such that $S \cap X \neq \emptyset$. Then there is an end-segment $S'$ such that:

- for any $x \in \Gamma$, $x \in S \cap X$ if and only if $x \in S' \cap X$,
- $X$ is coinitial and cofinal in $S'$. 

**Lemma 2.19.** Let $\Gamma$ be an ordered abelian group of bounded regular rank and $S \subseteq \Gamma$ be a definable end-segment. Let $X$ be a congruence restriction which is co-initial in $S$. Then there is a $\text{acl}^q(S \cap X)$-definable global type $p(x) \models x \in S \cap X$ extending $\Sigma_S^\text{gen}(x)$.

**Proof.** Let $(\Sigma_n(x) \mid n \in \mathbb{N})$ be the sequence of partial types given by [25, Lemma 4.5]. Let $p(x)$ be any completion of $\Sigma_n(x) = \bigcup \Sigma_n(x)$. By the quantifier elimination (Theorem 2.9) the type $p(x)$ is completely determined by the data in $\Sigma_\infty(x)$, which is definable over $\text{acl}^q(S \cap X)$.

**The dp-minimal case**

In 1984 the classification of the model theoretic complexity of ordered abelian groups was initiated by Gurevich and Schmitt, who proved that any ordered abelian group does not have the independence property. During the last years finer classifications have been achieved, in particular dp-minimal ordered abelian groups have been characterized in [27].

**Definition 2.20.** Let $\Gamma$ be an ordered abelian group and let $p$ be a prime number. We say that $p$ is a singular prime if $[\Gamma : p\Gamma] = \infty$. If $\Gamma$ does not have singular primes we call it non-singular.
The following result corresponds to [27, Proposition 5.1].

**Proposition 2.21.** Let $\Gamma$ be an ordered abelian group, the following conditions are equivalent:

1. $\Gamma$ does not have singular primes,
2. $\Gamma$ is dp-minimal.

**Definition 2.22.** [The language $\mathcal{L}_{dp}$] Let $\Gamma$ be a dp-minimal ordered abelian group. We consider the language extension $\mathcal{L}_{dp}$ of $\mathcal{L}_b$ [see Definition 2.8] where for each $n \in \mathbb{N}_{\geq 2}$ and $\Delta \in RJ(\Gamma)$ we add a set of constant $\Omega^\Delta_n$ in the main sort $\Gamma$ such that $\{\rho_\Delta(d) \mid d \in \Omega^\Delta_n\}$ is a complete set of representatives of each of the cosets of $n(\Gamma/\Delta)$ in $\Gamma/\Delta$.

The following is [25, Corollary 5.2].

**Corollary 2.23.** Let $\Gamma$ be a dp-minimal ordered abelian group. Then $\Gamma$ admits elimination of imaginaries in the language $\mathcal{L}_{dp}$.

The following will be a very useful fact.

**Fact 2.24.** Let $\Gamma$ be a dp-minimal ordered abelian group and let $S \subseteq \Gamma$ be a definable end-segment. Then there is a $'S'$ definable type $q(x) \vdash x \in S$ extending the generic type $\Sigma^\text{gen}_S(x)$.

**Proof.** Let $\Sigma^\text{gen}_S(x)$ be the generic type of $S$ and $q(x)$ be any completion then $q(x)$ is definable over $'S'$. Indeed, for each $\Delta \in RJ(\Gamma)$ and $n \in \mathbb{N}_{\geq 2}$ let $c \in \Omega^\Delta_n$ be such that $\rho_\Delta(x) - \rho_\Delta(c) \in n(\Gamma/\Delta)$. Then $\{\beta \in \Gamma \mid \rho_\Delta(x) - \rho_\Delta(\beta) + k^\Delta \in n(\Gamma/\Delta) \in q(x)\}$ is defined by the formula $\rho_\Delta(\beta) + k^\Delta - \rho_\Delta(c) \in n(\Gamma/\Delta)$. □

**Corollary 2.25.** Let $\Gamma$ be a dp-minimal ordered abelian group. For each $\Delta \in RJ(\Gamma)$ and $n \in \mathbb{N}_{\geq 2}$ let $\mathcal{S}_n^\Delta := \{\mathcal{S}_n^\Delta(d) \mid d \in \Omega^\Delta_n\}$. The set $\mathcal{S}_{dp} = \bigcup_{\Delta \in RJ(\Gamma), n \in \Gamma} \mathcal{S}_n^\Delta$ is a complete family whose elements are all definable over $\emptyset$.

**Henselian valued fields of equicharacteristic zero with residue field algebraically closed and value group with bounded regular rank**

The main goal of this section is to describe the 1-definable subsets $X \subseteq K$, where $K$ is a valued field with residue field algebraically closed and with value group of bounded regular rank.

The language $\mathcal{L}$

Let $K$ be a valued field of equicharacteristic zero, residue field algebraically closed and value group with bounded regular rank. We will view this valued field as an $\mathcal{L}$-structure, where $\mathcal{L}$ is the language extending $\mathcal{L}_{val}$ in which the value group sort is equipped with the language $\mathcal{L}_b$ described in Definition 2.8. Let $T$ be the $\mathcal{L}$-theory of henselian valued fields with residue field algebraically closed of characteristic zero and value group with bounded regular ran in the language $\mathcal{L}_b$ described in Definition 2.8.

**Corollary 2.26.** The first order theory $T$ admits quantifier elimination in the language $\mathcal{L}$.

**Proof.** This is a direct consequence of Theorem 2.2 and Theorem 2.9. □

**Description of definable sets in 1-variable**

In this Subsection we give a description of the definable subsets in 1-variable $X \subseteq K$, where $K \models T$. We denote as $\mathcal{O}$ its valuation ring.

**Definition 2.27.** Let $(K, \mathcal{O})$ be a henselian valued field of equicharacteristic zero and let $\Gamma$ be its value group. Let $\Delta$ be a convex subgroup of $\Gamma$ then the map:

$$v_\Delta := \begin{cases} K & \to \Gamma/\Delta \\ x & \mapsto v(x) + \Delta, \end{cases}$$

is a henselian valuation on $K$ and it is commonly called as the coarsened valuation induced by $\Delta$. Note that $v_\Delta = \rho_\Delta \circ v$. 


The following is a folklore fact.

**Fact 2.28.** There is a one-to-one correspondence between the $\mathcal{O}$-submodules of $K$ and the definable end-segments of $\Gamma$. In fact, given $M \subseteq K$ an $\mathcal{O}$-submodule, we have that $S_M := \{v(x) \mid x \in M\}$ is a definable end-segment of $\Gamma$. We refer to $S_M$ as the end-segment induced by $M$. And given an end-segment $S \subseteq \Gamma$, the set $M_S := \{x \in K \mid v(x) \in S\}$ is an $\mathcal{O}$-submodule of $M$.

**Definition 2.29.**
1. Let $M$ and $N$ be $\mathcal{O}$-submodules of $K$, we say that $M$ is a scaling of $N$ if there is some $b \in K$ such that $M = bN$.
2. A family $\mathcal{F}$ of definable $\mathcal{O}$-submodules of $K$ is said to be complete if any definable submodule $M \subseteq K$ is a translate of some $\mathcal{O}$-submodule $N \in \mathcal{F}$.

**Fact 2.30.** Let $\mathcal{F} = \{M_S \mid S \in S\}$, where $S$ is the complete family of end-segments described in Corollary 2.15. Then $\mathcal{F}$ is a complete family of $\mathcal{O}$-submodules of $K$.

**Definition 2.31.**
1. A 1-torsor of $K$ is a set of the form $a + bI$ where $a, b \in K$ and $I \in \mathcal{F}$.
2. Let $w : K \to \Gamma_w$ be a valuation, $\gamma \in \Gamma_w$ and $a \in K$. The closed ball of radius $\gamma$ centered at $a$ according to the valuation $w$ is the set of the form $B_\gamma(a) = \{x \in K \mid \gamma \leq w(x - a)\}$, and the open ball of radius $\gamma$ centered at $a$ according to the valuation $w$ is the set of the form $B_\gamma(a) = \{x \in K \mid \gamma < w(x - a)\}$.
3. A swiss cheese according to the valuation $w$ is a set of the form $A \setminus (B_1 \cup \cdots \cup B_n)$ where for each $i \leq n$, $B_i \subseteq A$ and the $B_i$ and $A$ are balls according to the original valuation $w : K \to \Gamma_w$.
4. A generalized swiss cheese is a set of the form $A \setminus (B_1 \cup \cdots \cup B_n)$ where for each $i \leq n$, $B_i \subseteq A$ and the $B_i$ and $A$ are 1-torsors.
5. A basic positive congruence formula in the valued field is a formula of the form $zv_\Delta(x - \alpha) - \rho_\Delta(\beta) + k\Delta \in n(\Gamma/\Delta)$, where $k, z \in \mathbb{Z}$, $\alpha, \beta \in K$, $\gamma \in \Gamma$, $n \in \mathbb{N}_{\geq 2}$ and $k\Delta = k \cdot 1^\Delta$, where $1^\Delta$ is the minimal positive element of $\Gamma/\Delta$ if it exists.
6. A basic negative congruence formula in the valued field is a formula of the form $zv_\Delta(x - \alpha) + \rho_\Delta(\beta) + k\Delta \in (\Gamma/\Delta)$, where $k, z \in \mathbb{Z}$, $\alpha, \beta \in K$, $\gamma \in \Gamma$, $n \in \mathbb{N}_{\geq 2}$ and $k\Delta = k \cdot 1^\Delta$, where $1^\Delta$ is the minimal positive element of $\Gamma/\Delta$ if it exists.
7. A basic congruence formula in the valued field is either a basic positive congruence formula in the valued field or a basic negative congruence formula in the valued field.
8. A finite congruence restriction in the valued field is a finite conjunction of basic congruence formulas in the valued field.
9. A nice set is a set defined by $S \wedge C$ where $S$ defines is a generalized swiss cheese and $C$ is a finite congruence restriction in the valued field.

To describe completely the definable subsets of $K$ we will need the following lemmas, which permit us to reduce the valuation of a polynomial into the valuation of linear factors for the form $v(x - a)$. We recall a definition and some results present in Flenner’s PhD Thesis [13] that will be useful for this purpose.

**Definition 2.32.** Let $(K, w)$ be a valued field, $\alpha \in K$ and $S$ a swiss cheese. Let $p(x) \in K[x]$, we define:

$$m(p, \alpha, S) := \max\{i \leq d \mid \exists x \in S \forall j \leq d \ (w(a_i(x - \alpha)^j) \leq w(a_j(x - \alpha)^j))\},$$

where the $a_i$ are the coefficients of the expansion of $p$ around $\alpha$ and $p(x) = \sum_{i \leq d} a_i(x - \alpha)^i$.

Thus $m(p, \alpha, S)$ is the highest order term in $p$ centered at $\alpha$ which can have minimal valuation (among the other terms of $p$) in $S$.

The following is [13, Proposition 2.2.2].

**Proposition 2.33.** Let $K$ be a valued field of characteristic zero. Let $p(x) \in K[x]$ and $S$ be a swiss cheese in $K$. Then there are (disjoint) sub-swiss cheeses $T_1, \ldots, T_n \subseteq S$ and $\alpha_1, \ldots, \alpha_n \in K$ such that $S = \bigcup_{1 \leq i \leq n} T_i$, where for all $x \in T_i$, $w(p(x)) = w(a_{im}(x - \alpha_i)^m_i)$, where $p(x) = \sum_{n=0}^{d} a_{in}(x - \alpha_i)^n$ and $m_i = m(p, \alpha_i, T_i)$. Furthermore, $\alpha_1, \ldots, \alpha_k$ can be taken algebraic over the subfield of $K$ generated by the coefficients of $p(x)$.
Though the preceding proposition is stated for a single polynomial, the same result will hold for any finite number of polynomials $\Sigma$. To obtain the desired decomposition, simply apply the proposition to each $p(x) \in \Sigma$, then intersect the resulting partitions to get one that works for all $p(x) \in \Sigma$, using the fact that intersection of two swiss cheeses is again a swiss cheese.

**Fact 2.34.** Let $(K,w)$ be a henselian valued field of equicharacteristic zero, and $Q_1(x), Q_2(x) \in K[x]$ be two polynomials in a single variable. There is a finite union of swiss cheeses $K = \bigcup_{i \in k} T_i$, coefficients $\epsilon_i \in K$, elements $\gamma_i \in \Gamma$ and integers $z_i \in \mathbb{Z}$ such that for any $x \in T_i$, $w(Q_1(x)) - w(Q_2(x)) = \gamma_i + z_i w(x - \epsilon_i)$.

**Proof.** The statement is a straightforward computation after applying Proposition 2.33, and it is left to the reader.

**Proposition 2.35.** Let $K \equiv T$, for each $\Delta \in RJ(\Gamma)$ let $v_\Delta := K \to \Gamma/\Delta$ be the coarsened valuation induced by $\Delta$. Let $X \subseteq K$ be a set defined by a formula of the form:

$$\gamma \leq^\Delta v_\Delta(Q_1(x)) - v_\Delta(Q_2(x)) \text{ or } v_\Delta\left(\frac{Q_1(x)}{Q_2(x)}\right) - \gamma \in n (\Gamma/\Delta);$$

where $Q_1(x), Q_2(x) \in K[x]$, $\gamma \in \Gamma/\Delta$ and $n \in \mathbb{N}$. Then $X$ is a finite union of nice sets.

**Proof.** This is a straightforward computation after applying Fact 2.34, and it is left to the reader.

We conclude this section by characterizing the definable sets in 1-variable.

**Theorem 2.36.** Let $K \equiv T$ and $X \subseteq K$ be a definable set. Then $X$ is a finite union of nice sets.

**Proof.** By Corollary 2.26, $X$ is a boolean combination of sets defined by formulas of the form $\gamma \leq^\Delta v_\Delta(Q_1(x)) - v_\Delta(Q_2(x))$ or $v_\Delta\left(\frac{Q_1(x)}{Q_2(x)}\right) - \gamma \in n (\Gamma/\Delta)$, where $\Delta \in RJ(\Gamma)$, $\gamma \in \Gamma/\Delta$ and $n \in \mathbb{N}_2$. By Proposition 2.35 each of these formulas defines a finite union of nice sets. As intersection of two generalized swiss cheeses is again a generalized swiss cheese, the statement follows.

**O-modules and homomorphisms in maximal valued fields**

In this section we recall some results about modules over maximally complete valued fields. We follow ideas of Kaplansky in [14] to characterize the $O$-submodules of finite dimensional $K$-vector spaces.

**Definition 2.37.**

1. Let $K$ be a valued field and $O$ its valuation ring. We say that $K$ is maximal, if whenever $\alpha_r \in K$ and (integral or fractional) ideals $I_r$ are such that the congruences $x - \alpha_r \in I_r$ are pairwise consistent, then there exists in $K$ a simultaneous solution of all the congruences.

2. Let $K$ be a valued field and $M \subseteq K^n$ be an $O$-module. We say that $M$ is maximal if whenever ideals $I_r \subseteq O$ and elements $s_r \in M$ are such that $x - s_r \in I_r M$ is pairwise consistent in $M$, then there exists in $M$ a simultaneous solution of all the congruences.

3. Let $N \subseteq K^n$ be an $O$-submodule. Let $x \in N$ we say that $x$ is $\alpha$-divisible in $N$ if there is some $n \in N$ such that $x = \alpha n$.

We start by recalling a very useful fact,

**Fact 2.38.** maxext Let $K$ be a henselian valued field of equicharacteristic zero, then there is an elementary extension $K \prec K'$ that is maximal.

**Proof.** Let $K$ be a henselian valued field of equicharacteristic zero and $T$ its $\mathcal{L}_{val}$-first order theory and let $\mathcal{C}$ its monster model. By [8, Lemma 4.30] there is some maximal immediate extension of $K \subseteq F \subseteq \mathcal{C}$. Then, $K \prec F$ by [8, Theorem 7.12].

The following is [14, Lemma 5].

**Lemma 2.39.** Let $K$ be a maximal valued field, then any (integral or fractional) ideal $I$ of $O$ is maximal. Moreover, any finite direct sum of maximal $O$-modules is also maximal.
**Fact 2.40.** Let $N \subseteq K$ be a non-trivial $\mathcal{O}$-submodule. Let $n \in N$ then $N = nI$ where $I$ is a copy of $K$, $\mathcal{O}$ or an (integral or fractional) ideal of $\mathcal{O}$.

**Definition 2.41.** Let $K$ be a field and $n \in \mathbb{N}$, we say that a set $\{a_1, \ldots, a_n\}$ is an upper triangular basis of the vector space $K^n$ if it is a $K$-linearly independent set and the matrix $[a_1, \ldots, a_n]$ is upper triangular.

**Theorem 2.42.** Let $K$ be a maximal valued field and $n \in \mathbb{N}_{\geq 2}$. Let $N \subseteq K^n$ be an $\mathcal{O}$-submodule then $N$ is maximal, and $N$ is definably isomorphic to a direct sum of copies of $K$, $\mathcal{O}$ and (integral or fractional) ideals of $\mathcal{O}$. Moreover, if $N \cong \bigoplus_{i \subseteq n} I_i$ where each $I_i$ is either a copy of $K$, $\mathcal{O}$ and (integral or fractional) ideals of $\mathcal{O}$ one can find an upper triangular basis $\{a_1, \ldots, a_n\}$ of $K^n$ such that $N = \{a_1x_1 + \ldots + a_nx_n \mid x_i \in I_i\}$. In this case we say that $[a_1, \ldots, a_n]$ is a representation matrix for the module $N$.

**Proof.** We proceed by induction on $n$, the base case is given by Fact 6.36 and Lemma 2.39. For the inductive step, let $\pi : K^{n+1} \to K$ be the projection into the last coordinate and let $M = \pi(N)$. We consider the inductive sequence of $\mathcal{O}$-modules $0 \to N \cap (K^n \times \{0\}) \to N \to M \to 0$. By induction, $N \cap (K^n \times \{0\})$ is maximal and of the required form. And there is an upper triangular basis $\{a_1, \ldots, a_n\}$ of $K^n \times \{0\}$ such that $[a_1, \ldots, a_n]$ is a representation matrix for $N \cap (K^n \times \{0\})$. If $M = \{0\}$ we are all set, so we may take $m \in M$ such that $m \neq 0$.

**Claim:** There is some element $x \in N$ such that $\pi(x) = m$ and $x$ satisfies the same divisibilities than $m$, i.e. for any $\alpha \in \mathcal{O}$, $x$ is $\alpha$-divisible in $N$ if and only if $m$ is $\alpha$-divisible in $M$.

**Proof.** Let $J = \{\alpha \in \mathcal{O} \mid m$ is $\alpha$-divisible in $M\}$. For each $\alpha \in J$, let $m_\alpha \in M$ be such that $m = \alpha m_\alpha$ and take $n_\alpha \in \pi^{-1}(m_\alpha) \cap N$. Fix an element $y \in N$ satisfying $\pi(y) = m$ and let $s_\alpha = y - \alpha m_\alpha \in N \cap (K^n \times \{0\})$. Consider $\mathcal{S} = \{x - s_\alpha \in \alpha N \cap (K^n \times \{0\}) \mid \alpha \in J\}$ this is system of congruences in $N \cap (K^n \times \{0\})$. We will argue that it is pairwise consistent. Let $\alpha, \beta \in \mathcal{O}$, then either $\frac{\alpha}{\beta} \in \mathcal{O}$ or $\frac{\beta}{\alpha} \in \mathcal{O}$ (or both). Without loss of generality assume that $\frac{\alpha}{\beta} \in \mathcal{O}$, then:

$$s_\alpha - s_\beta = (y - \alpha m_\alpha) - (y - \beta m_\beta) = \beta m_\beta - \alpha m_\alpha = \beta \left( m_\beta - \frac{\alpha}{\beta} n_\alpha \right).$$

Thus $s_\alpha$ is a solution to the system $\{x - s_\alpha \in \alpha N \cap (K^n \times \{0\})\} \cup \{x - s_\beta \in \beta N \cap (K^n \times \{0\})\}$. By maximality of $N \cap (K^n \times \{0\})$ we can find an element $z \in N \cap (K^n \times \{0\})$ such that $z$ is a simultaneous solution to the whole system of congruences in $\mathcal{S}$. Let $x = y - z \in N$, then $x$ satisfies the requirements. In fact, for each $\alpha \in J$, we had chosen $z - s_\alpha \in \alpha N \cap (K^n \times \{0\})$, so $z = s_\alpha + \omega x$ for some $w \in N \cap (K^n \times \{0\})$. Thus, $x = y - z = y - s_\alpha - \omega x = y - (y - \alpha m_\alpha) - \omega x = \alpha(n_\alpha - w) \in \alpha N$, as desired.

Let $s : M \to N$ be the map sending an element $\alpha m$ to $\alpha x$, where $\alpha \in K$. As $N$ is a torsion free module, $s$ is well defined. One easily verifies that $s$ is an homomorphism such that $\pi \circ s = id_M$. Thus, $N$ is the direct sum of $N \cap (K^n \times \{0\})$ and $s(M)$, so it is maximal by Lemma 2.39. Moreover, $[a_1, \ldots, a_n, x]$ is a representation matrix for $N$, as required.

**Proposition 2.43.** Let $K$ be a maximal valued field. Let $M \subseteq K$ be a $\mathcal{O}$-module. For any homomorphism $h : M \to K/N$ there is some $a \in K$ such that for any $x \in M$, $h(x) = ax + N$.

**Proof.** By Fact 6.36, $M$ is either $K$ or $M = Bi$, where $I \subseteq \mathcal{F}$. It is sufficient to prove the statement for $b = 1$. Let $S_I = \{v(y) \mid y \in I\}$ the end-segment induced by $I$. Let $\{\gamma_\alpha \mid \alpha \in \kappa\}$ be a co-initial decreasing sequence in $S_I$. Choose an element $x_\alpha \in K$ such that $\gamma(x_\alpha) = \gamma_\alpha$, then for each $\alpha < \beta < \kappa$, $x_\beta \mathcal{O} \subseteq x_\alpha \mathcal{O}$ and $I = \bigcup_{\alpha < \kappa} x_\alpha \mathcal{O}$.

**Claim 1:** for each $\alpha \in \kappa$, there is an element $a_\alpha \in K$ such that for all $x \in x_\alpha \mathcal{O}$ we have $h(x) = a_\alpha x + N$.

For each $\alpha$, choose an element $y_\alpha$ such that $h(x_\alpha) = y_\alpha + N$ and let $a_\alpha = x_\alpha^{-1} y_\alpha$. Fix an element $x \in x_\alpha \mathcal{O}$, then:

$$h(x) = h(x_\alpha (x_\alpha^{-1} x)) = (x_\alpha^{-1} x) \cdot h(x_\alpha) = (x_\alpha^{-1} x) \cdot (a_\alpha x_\alpha + N) = a_\alpha x + N.$$ 

**Claim 2:** Given $\beta < \alpha < \kappa$, then $a_\beta - a_\alpha \in x_\beta^{-1} N$.

Note that $x_\beta \in x_\beta \mathcal{O} \subseteq x_\alpha \mathcal{O}$, by Claim 1 we have $h(x_\beta) = a_\alpha x_\beta + N = a_\beta x_\beta + N$, then $(a_\alpha - a_\beta)x_\beta \in N$. Hence, $(a_\alpha - a_\beta) \in x_\beta^{-1} N$.
Claim 3: Without loss of generality we may assume that for any $\alpha < \kappa$ there is some $\alpha < \alpha' < \kappa$ such that for any $\alpha'' < \alpha' < \kappa$, $a_\alpha - a_{\alpha''} \notin x_{\alpha''}^{-1} N$.

Suppose the statement is false. Then there is some $\alpha$ such that for any $\alpha < \alpha'$ we can find $\alpha' < \alpha''$ such that $a_\alpha - a_{\alpha''} \notin x_{\alpha''}^{-1} N$. Define:

$$h^* := \begin{cases} I & \to K/N \\ x & \to a_\alpha x + N. \end{cases}$$

We will show that for any $x \in I$, $h(x) = h^*(x)$. Fix an element $x \in I$, since $< \gamma_\alpha | \alpha \in \kappa >$ is coinitial and decreasing in $S_I$ we can find an element $\alpha' > \alpha$ such that $v(x) > \gamma_{\alpha'}$, so $x \in x_{\alpha'} \mathcal{O} \subseteq x_{\alpha''} \mathcal{O}$. Then

$$(a_\alpha - a_{\alpha''})x = (a_\alpha - a_{\alpha''} + a_{\alpha''} - a_{\alpha'})x = \varepsilon_{x_{\alpha''}^{-1}xN \subseteq N} \varepsilon_{x_{\alpha''}^{-1}xN \subseteq N}$$

we conclude that $(a_\alpha - a_{\alpha''})x \in N$. By Claim 1 we have $h(x) = a_\alpha x + N$, thus $h^*(x) = h(x)$ and $h^*$ witnesses the conclusion of the statement.

Claim 4: There is a subsequence $< b_\alpha | \alpha \in \text{cof}(\kappa) >$ of $< a_\alpha | \alpha \in \kappa >$ that is pseudoconvergent.

We construct such sequence by transfinite recursion, building a strictly increasing function $f : \text{cof}(\kappa) \to \kappa$ satisfying the following conditions:

1. For all $\alpha < \kappa$, $b_\alpha = a_f(\alpha)$.
2. For any $\alpha < \kappa$, the sequence $(b_\eta | \eta < \alpha)$ is pseudoconvergent. This is, given $\eta_1 < \eta_2 < \eta_3 < \alpha$, $v(b_{\eta_3} - b_{\eta_2}) > v(b_{\eta_2} - b_{\eta_1})$.
3. For any $\eta < \alpha < \kappa$, and $f(\alpha) < \eta' < \kappa$ we have $v(a_{\eta'} - b_{\alpha}) > v(b_{\eta_2} - b_{\alpha})$ and $a_{\eta'} - b_{\alpha} \notin x_{\eta'}^{-1} N$.

For the base case, set $b_0 = a_0$ and $f(0) = 0$. Assume now that for $\mu < \kappa$, $f | \mu$ has been defined and $< b_\eta | \eta < \mu >$ has been constructed. Let $\mu^* = \text{sup}\{f(\eta) | \eta < \mu\}$, if $\mu^* = \kappa$ we are done. If $\mu^* < \kappa$ by Claim 3 there is some $\mu' < v < \kappa$ such that for any $\nu' > v$ and $\eta \leq \mu'$, $b_\eta - a_{\nu'} \notin x_{\nu'}^{-1} N$. Set $f(\mu) = v$ and $b_\mu = a_v$. We check that $(b_\eta | \eta \leq \mu)$ is a pseudoconvergent sequence. Fix $\eta_1 < \eta_2 < \mu$ we must show that $v(b_{\mu} - b_{\eta_2}) > v(b_{\eta_2} - b_{\eta_1})$.

By construction $b_\mu = a_{f(\mu)} = a_v$ and $f(\mu) = v > \mu^* \geq f(\eta_2)$. Since the third condition holds for $\eta_2$ we must have $v(a_v - b_{\eta_2}) > v(b_{\eta_2} - b_{\eta_1})$, as required.

Finally we just need to guarantee that given $\eta < \mu$ and $v = f(\mu) < \eta'$, $v(a_{\eta'} - b_{\mu}) > v(b_{\mu} - b_{\eta})$. Suppose by contradiction that this inequality does not hold, then $\frac{b_{\eta'} - b_{\eta}}{a_{\eta'} - b_{\mu}} \notin \mathcal{O}$. Since $\mu > \eta$, then $v = f(\mu) > f(\eta)$ by the third condition $b_\mu - b_{\eta} = a_v - a_{f(\eta)} \notin x_{\eta'}^{-1} N$. By the second Claim $a_{\eta'} - b_\mu = a_{\eta'} - a_v \in x_{\eta'}^{-1} N$, then

$$b_\mu - b_{\eta} = \frac{b_{\eta} - b_{\mu}}{a_{\eta'} - b_{\mu}} (a_{\eta'} - b_\mu) \in x_{\eta'}^{-1} N,$$

which leads us to a contradiction.

Since $K$ is maximal there is some $a \in K$ that is a pseudolimit of $< b_\alpha | \alpha \in \text{cof}(\kappa) >$. We aim to prove that $h(x) = ax + N$ for $x \in I$. By construction of the function $f$, we can find some $\alpha \in \text{cof}(\kappa)$ such that $x \in x_{f(\alpha)} \mathcal{O} \subseteq I$. By the first claim $h(x) = a_{f(\alpha)} x + N$, hence it is sufficient to prove that $(a - a_{f(\alpha)}) x \in N$. As $x \in x_{f(\alpha)} \mathcal{O}$ it is enough to show that $(a - a_{f(\alpha)}) = (a - b_\alpha) \notin x_{f(\alpha)}^{-1} N$. Let $\alpha < \beta < \kappa$, by the second Claim $(b_{\beta} - b_{\alpha}) = (a_{f(\beta)} - a_{f(\alpha)}) \notin x_{f(\alpha)}^{-1} N$. Also, $v(a - a_{f(\alpha)}) = v(a_{f(\beta)} - a_{f(\alpha)})$ thus $(a - a_{f(\alpha)}) = u(a_{f(\beta)} - a_{f(\alpha)})$ for some $u \in \mathcal{O}^x$, thus $(a - a_{f(\alpha)}) \in x_{f(\alpha)}^{-1} N$, as desired.

**Valued vector spaces**

We introduce valued vector spaces and some facts that will be required through this paper. An interested reader can consult [15] for a more exhaustive presentation.

**Definition 2.44.** Let $(K, \Gamma, v)$ be a valued field and $V$ a $K$-vector space. A tuple $(V, \Gamma(V), \text{val}, +)$ is a valued vector space structure if:

1. $\Gamma(V)$ is a linear order,
2. there is an action $+: \Gamma \times \Gamma(V) \to \Gamma(V)$ which is order preserving in each coordinate,
Proof. This is [12, Remark 6.1.2].

**Fact 2.46 (Separated basis).** Let $(V, \Gamma(V), val, +)$ be a valued vector space. Assume that $V$ is a $K$-vector space of dimension $n$. A basis $\{v_1, \ldots, v_n\} \subseteq V$ is a separated basis if for any $\alpha_1, \ldots, \alpha_n \in K$ we have that $val(\sum_{i \leq n} \alpha_i v_i) = \min\{val(\alpha_i v_i) | i \leq n\}$.

**Definition 2.47.** Let $(V, \Gamma(V), v, +)$ be a valued vector space and let $W$ be a subspace of $V$. We say that $W \leq V$ has the optimal approximation property if for any $v \in V \setminus W$ the set $\{val(v - w) | w \in W\}$ attains a maximum.

The following is a folklore fact. See for example [15, Chapter 1, Lemma 1.1.5].

**Fact 2.48.** Let $(V, \Gamma(V), val, +)$ be a valued vector space, and $W$ a subspace of $V$ the following statements are equivalent:

1. $W$ is maximal in $V$,
2. $W$ has the optimal approximation property.

### 3 Modules, torsors and unary codes

Let $(K, v)$ be a henselian valued field of equicharacteristic zero with residue field algebraically closed and value group with bounded regular rank. Let $O$ its valuation ring and $T$ be its $\mathcal{L}$-first order theory. In this section we study the definable $O$-modules and torsors. We recall that we write as $\mathcal{F}$ to refer to the complete family of $O$-submodules of $K$ described in Fact 2.30.

**Definition 3.1.** Let $K \models T$, a definable torsor is a coset in $K^n$ of a definable $O$-submodule of $K^n$.

**Corollary 3.2.** Let $K \models T$ and $N$ be a definable $O$-submodule of $K^n$. Then $N$ is definably isomorphic to a direct sum of copies of $K$, $O$, or (integral or fractional) ideals of $O$. Moreover, if $N \cong \oplus_{i \leq n} I_i$, there is some upper triangular basis $\{a_1, \ldots, a_n\}$ of $K^n$ such that $[a_1, \ldots, a_n]$ is a representation matrix of $N$.

**Proof.** Let $K'$ be an elementary extension of $K$ that is maximal and apply Theorem 2.42. As the statement that we are trying to show are first order expressible, it must hold as well in $K$. 

**Corollary 3.3.** Let $K \models T$ and $N \subseteq K$ be a definable $O$-submodule. Let $I \in \mathcal{F}$, $a \in K$ and $h : aI \to K/N$. Then there is some $b \in K'$ satisfying that for any $y \in aI$, $h(y) = by + N$.

**Proof.** The statement that we are trying to prove is first order expressible. By Fact ?? we can find an elementary extension of $K'$ that is maximal and apply Proposition ??.

**Definition 3.4.** Let $(I_1, \ldots, I_n) \in \mathcal{F}^n$ a fixed tuple.

1. An $O$-module $M \subseteq K^n$ is of type $(I_1, \ldots, I_n)$ if $M \cong \bigoplus_{i \leq n} I_i$.
2. An $O$-module $M \subseteq K^n$ of type $(O, \ldots, O)$ is said to be an $O$-lattice of rank $n$.
3. A torsor $Z$ is of type $(I_1, \ldots, I_n)$, if $Z = \mathcal{O} + M$ where $M \subseteq K^n$ is an $O$-submodule of $K^n$ of type $(I_1, \ldots, I_n)$.

**Proposition 3.5.** Let $Z$ be a torsor of type $(I_1, \ldots, I_n)$. Then there is some $O$-module $L \subseteq K^{n+1}$ of type $(O, I_1, \ldots, I_n)$ such that $'Z'$ and $'L'$ are interdefinable.
Lemma 3.9. Let \( N \subseteq K^n \) be the \( \mathcal{O} \)-submodule and some \( \bar{d} \in K^n \) be such that \( Z = \bar{d} + N \). Let \( N_2 = \{0\} \times N \) which is an \( \mathcal{O} \) submodule of \( K^{n+1} \) and \( \bar{b} = \left[ \frac{1}{d} \right] \). Define the \( \mathcal{O} \)-module of \( K^{n+1} \) \( L_{\bar{d}} = \bar{b} \mathcal{O} + N_2 \). By a standard computation, one can verify that the definition of \( L_{\bar{d}} \) is independent of the choice of \( d \), i.e. if \( \bar{d} - \bar{d}' \in N \) then \( L_{\bar{d}} = L_{\bar{d}'} \). So we can denote \( L = L_{\bar{d}} \), and we aim to show that \( L \) and \( Z \) are interdefinable. It is clear that \( 'L' \in \text{dcl}(Z) \), while \( 'Z' \in \text{dcl}(L) \) because \( Z = \pi_{25n}(L) \) where \( \pi_{25n+1} : K^{n+1} \to K^n \) is the projection into the last \( n \)-coordinates.

1-definable modules, 1-torsors and unary codes

We now focus on the 1-dimensional case.

Notation 3.6. Let \( M \subseteq K \) be a definable \( \mathcal{O} \)-module. We denote by \( S_M := \{v(x) \mid x \in M\} \) the end-segment induced by \( M \). We recall as well that we denote by \( \mathcal{F} \) the complete family of \( \mathcal{O} \)-submodules of \( K \) from Fact 2.30.

Definition 3.7. A definable 1-module is an \( \mathcal{O} \)-module which is definably isomorphic to a quotient of a 1-definable \( \mathcal{O} \)-submodule of \( K \) by another, i.e. something of the form \( a_1/IbJ \) where \( a, b \in K \) and \( I, J \in \mathcal{F} \).

The following operation between \( \mathcal{O} \)-modules will be particularly useful in our setting.

Definition 3.8. Let \( N, M \) be \( \mathcal{O} \)-submodules of \( K \), we define the colon module \( \text{Col}(N : M) = \{x \in K \mid xM \subseteq N\} \). It is a well known fact from Commutative Algebra that \( \text{Col}(N : M) \) is also an \( \mathcal{O} \)-module.

Lemma 3.9. Let \( K = T \). Let \( A \) be a 1-definable \( \mathcal{O} \)-module. We suppose that \( A = A_1/A_2 \), where \( A_2 \subseteq A_1 \) are \( \mathcal{O} \)-submodules of \( K \). Let \( I \in \mathcal{F} \) and \( a \in K \) be such that \( A_1 = aI \). Then the \( \mathcal{O} \)-module \( H_{\mathcal{O}}(A, A) \) is definably isomorphic to the 1-definable \( \mathcal{O} \)-module \( \text{Col}(I : I)/\text{Col}(A_2 : A_1) \).

Proof. By Fact ?? without loss of generality we may assume \( K \) to be maximal, because the statement is first order expressible. Let

\[
B = \{f : A_1 \to A_1/A_2 \mid f \text{ is a homomorphism and } \ker(f) = A_2\}.
\]

\( B \) is canonically in one-to-one correspondence with \( \text{Hom}_{\mathcal{O}}(A, A) \). By Corollary 3.3, for every homomorphism \( f \in B \) there is some \( b_f \in K \) satisfying that for any \( x \in A_1 \), \( f(x) = b_fx + A_2 \). We consider now the map \( \phi : B \to \text{Col}(I : I)/\text{Col}(A_2 : A_1) \) that sends each homomorphism \( f \) to the coset \( b_f + \text{Col}(A_2 : A_1) \). By a straight forward computation one can verify that \( \phi \) is a well-defined \( \mathcal{O} \)-homomorphism, so the lemma follows.

Lemma 3.10. Let \( n \in \mathbb{N}_{\geq 2} \) and \( M \subseteq K^n \) be an \( \mathcal{O} \)-module.

1. Let \( \pi^{n-1} : K^n \to K^{n-1} \) be the projection into the first \((n-1)\)-coordinates and \( B_{n-1} = \pi^{n-1}(M) \). Take \( A_1 \subseteq K \) be the \( \mathcal{O} \)-module such that \( \ker(\pi^{n-1}) = N \cap ((0)^{n-1} \times K) = ((0)^{n-1} \times A_1) \).

2. Let \( \pi_n := K^n \to K \) be the projection into the last coordinate and \( B_1 = \pi_n(M) \). Let \( A_{n-1} \subseteq K^{n-1} \) be the \( \mathcal{O} \)-module such that \( \ker(\pi_n) = N \cap (K^{n-1} \times \{0\}) = (A_{n-1} \times \{0\}) \).

Then \( A_{n-1} \leq B_{n-1} \) and both lie in \( K^{n-1} \), and \( A_1 \leq B_1 \) and both lie in \( K \). Then the map \( \phi : B_{n-1} \to K/A_1 \) given by \( b \mapsto a + A_1 \) where \( (b, a) \in M \), is a well defined homomorphism of \( \mathcal{O} \)-modules whose kernel is \( A_{n-1} \). In particular, \( B_{n-1}/A_{n-1} \cong B_1/A_1 \).

Furthermore if \( M \) is definable, \( \phi \) is also definable.

Proof. Let \( \bar{m} \in A_{n-1} \), then \((\bar{m}, 0) \in M \) thus \( \pi^{n-1}(\bar{m}, 0) = \bar{m} \in B_{n-1} \). We conclude that \( A_{n-1} \) is a submodule of \( B_{n-1} \), likewise \( A_1 \subseteq B_1 \). For the second part of the statement, it is a straightforward computation to verify that the map \( \phi : B_{n-1} \to K/A_1 \) (defined as in the statement) , is a well defined surjective homomorphism of \( \mathcal{O} \)-modules whose kernel is \( A_{n-1} \). Lastly, the definability of \( \phi \) follows immediately by the definability of \( M \).

Definition 3.11. Let \( e \in K^{eq} \), a sequence \((a_1, \ldots, a_m)\) of elements in \( K^{eq} \) is a unary code of \( e \) over \( C \) if:

1. \( \text{dcl}^{eq}(Ce) = \text{dcl}^{eq}(a_1, \ldots, a_m, C) \),

2. for any \( i = 1, \ldots, m \) the element \( a_i \) is an element in a 1-definable module defined over \( \text{dcl}^{eq}(Ca_1, \ldots, a_{i-1}) \).
Proposition 3.12. Let $K$ be a model of $T$ and let $N \subseteq K^n$ be a definable $\mathcal{O}$-submodule. Then $\langle N \rangle$ has a unary code.

Proof. We will show by induction on $n$ the existence of a unary code. The base case is clear. We want to show that the statement holds for $n$. Let $\pi^{n-1} = K^n \rightarrow K^{n-1}$ be the projection into the first $(n-1)$-coordinates and $B_{n-1} = \pi^{n-1}(N)$. Take $A_1 \subseteq K$ be the $\mathcal{O}$-module such that $\ker(\pi^{n-1}) = N \cap (\{0\}^{n-1} \times K) = (\{0\}^{n-1} \times A_1)$. Let $\pi_n = K^n \rightarrow K$ be the projection into the last coordinate and $B_1 = \pi_n(N)$. Let $A_{n-1} \subseteq K^{n-1}$ be the $\mathcal{O}$-module such that $\ker(\pi_n) = N \cap (K^{n-1} \times \{0\}) = (A_{n-1} \times \{0\})$. By the induction hypothesis, we can find unary codes $c_1,b_1,c_{n-1},b_{n-1}$ for $A_1,B_1,A_{n-1}$ and $B_{n-1}$ respectively.

Claim: The sequence $(c_1,b_1,c_{n-1},b_{n-1},e)$ is a unary code for $e$.

It is clear that $\text{dcl}^{eq}(c_1,b_1,c_{n-1},b_{n-1},e,C) = \text{dcl}^{eq}(eC)$, as all the $\mathcal{O}$-modules $A_1,B_1,A_{n-1},B_{n-1}$ are definable over $N$. Let $Y = \mathcal{Y}(c_1,b_1,c_{n-1},b_{n-1})$ be the set of codes of the $\mathcal{O}$-submodules $M \subseteq K^n$ satisfying that:

- $\pi^{n-1}(M) = B_{n-1}$ and $\ker(\pi^{n-1}) = M \cap (\{0\}^{n-1} \times K) = (\{0\}^{n-1} \times A_1)$,
- $\pi_n(M) = B_1$ and $\ker(\pi_n) = M \cap (K^{n-1} \times \{0\}) = (A_{n-1} \times \{0\})$.

Let $D = \text{Hom}_\mathcal{O}(B_{n-1}/A_{n-1},B_1/A_1)$, and $D' \leq D$ be the $\mathcal{O}$-submodule of isomorphisms between $B_{n-1}/A_{n-1}$ and $B_1/A_1$. First we argue that there is a one to one correspondence between $Y$ and $D'$. Consider the map:

$$\phi : D' \rightarrow Y$$

$$f \rightarrow \langle M_f \rangle,$$

where $M_f = \{(x,y) \in B_{n-1} \times B_1 \mid f(x + A_{n-1}) = y + A_1\}$.

It is a standard computation to verify that $\phi$ is a well defined injective function. While, surjectivity follows by Lemma 3.10. We now show that $D'$ is isomorphic to a 1-definable module. By Lemma 3.10, $D = \text{Hom}_\mathcal{O}(B_{n-1}/A_{n-1},B_1/A_1)$ is definably isomorphic to $\text{Hom}_\mathcal{O}(A,A)$ for $A = B_1/A_1$. By Lemma 3.9 $\text{Hom}_\mathcal{O}(A,A)$ is definably isomorphic to a 1-definable $\mathcal{O}$-module $\tilde{Z} = Z_1/Z_2$. Let $\psi : D \rightarrow \tilde{Z}$ be such isomorphism and let $X = \psi(D')$. Then $X$ is an $\mathcal{O}$-submodule of $\tilde{Z}$, which is a 1-definable submodule. Then the function $\psi \circ \phi^{-1} : Y \rightarrow X$ is a definable isomorphism between $Y$ and a 1-definable $\mathcal{O}$-module and $e \in Y$, this concludes the proof of the statement. \hfill \square

4 The Stabilizer sorts

An abstract criterion to eliminate imaginaries

We start by recalling Hrushovski's criterion.

Theorem 4.1. Let $T$ be a first order theory with home sort $K$ (meaning that $\mathfrak{M}^eq = \text{dcl}^eq(K)$). Let $\mathcal{G}$ be some collection of sorts. If the following conditions all hold, then $T$ has weak elimination of imaginaries in the sorts $\mathcal{G}$.

1. Density of definable types: for every non-empty definable set $X \subseteq K$ there is an $\text{acl}^eq(\langle X \rangle)$-definable type in $X$.

2. Coding definable types: every definable type in $K^n$ has a code in $\mathcal{G}$ (possibly infinite). This is, if $p$ is any (global) definable type in $K^n$, then the set $\langle p \rangle$ of codes of the definitions of $p$ is interdefinable with some (possibly infinite) tuple from $\mathcal{G}$.

Proof. See [12, Theorem 6.3]. Note that the first part of the proof shows weak elimination of imaginaries as it is shown that for any imaginary element $e$ we can find a tuple $a \in \mathcal{G}$ such that $e \in \text{dcl}^eq(a)$ and $a \in \text{acl}^eq(e)$. \hfill \square

We start by describing the sorts that are required to be added to apply this criterion and show that any valued field of equicharacteristic zero, with residue field algebraically closed and value group with bounded regular rank admits weak elimination of imaginaries.

Definition 4.2. For each $n \in \mathbb{N}$, let $\{e_1,\ldots,e_n\}$ be the canonical basis of $K^n$ and $(I_1,\ldots,I_n) \in \mathcal{F}^n$.

1. Let $C_{(I_1,\ldots,I_n)} = \{ \sum_{1 \leq i \leq n} x_i e_i \mid x_i \in I_i \}$, we refer to this module as the canonical $\mathcal{O}$-submodule of $K^n$ of type $(I_1,\ldots,I_n)$. 15
2. We denote as $B_n(K)$ the multiplicative group of $n \times n$-upper triangular and invertible matrices.
3. We define the subgroup $\text{Stab}(I_1, \ldots, I_n) = \{ A \in B_n(K) \mid AC(I_1, \ldots, I_n) = C(I_1, \ldots, I_n) \}$.
4. Let $'\Lambda(I_1, \ldots, I_n)' := \{ 'M' \mid M \subseteq K^n \text{ is an } \mathcal{O}\text{-module of type } (I_1, \ldots, I_n) \}$.
5. Let $\mathcal{U}_n \subseteq (K^n)^n$ be the set of $n$-tuples $(b_1, \ldots, b_n)$, such that $B = [b_1, \ldots, b_n]$ is an invertible upper triangular matrix. We define the equivalence relation $E(I_1, \ldots, I_n)$ on $\mathcal{U}_n$ as:
   $$E(I_1, \ldots, I_n)(a_1, \ldots, a_n; b_1, \ldots, b_n) \text{ holds if and only if } (\bar{a}_1, \ldots, \bar{a}_n) \text{ and } (\bar{b}_1, \ldots, \bar{b}_n)$$
   generate the same $\mathcal{O}$-module of type $(I_1, \ldots, I_n)$ that is:  
   $$\{ \sum_{1 \leq i \leq n} x_i \bar{a}_i \mid x_i \in I_i \} = \{ \sum_{1 \leq i \leq n} x_i \bar{b}_i \mid x_i \in I_i \}.$$
6. We denote as $\tilde{\rho}(I_1, \ldots, I_n)$ the canonical projection map:
   $$\tilde{\rho}(I_1, \ldots, I_n) := \begin{cases} \mathcal{U}_n & \rightarrow \mathcal{U}_n/E(I_1, \ldots, I_n) \\ (\bar{a}_1, \ldots, \bar{a}_n) & \rightarrow [(\bar{a}_1, \ldots, \bar{a}_n)]_{E(I_1, \ldots, I_n)} \end{cases}.$$

**Remark 4.3.** 1. The set $'\Lambda(I_1, \ldots, I_n)'$ can be canonically identified with $B_n(K)/\text{Stab}(I_1, \ldots, I_n)$. Indeed, by Corollary 3.2 given any $\mathcal{O}$-module $M$ of type $(I_1, \ldots, I_n)$ we can find an upper triangular basis $(a_1, \ldots, a_n)$ of $K^n$ such that $[a_1, \ldots, a_n]$ is a matrix representation of $M$. The code $'M'$ is interdefinable with the coset $[a_1, \ldots, a_n]_{\text{Stab}(I_1, \ldots, I_n)}$.
2. Fix some $n \in \mathbb{N}$ and let $(I_1, \ldots, I_n)$ be a fixed tuple. The sort $B_n(K)/\text{Stab}(I_1, \ldots, I_n)$ is in definable bijection with the equivalence classes of $\mathcal{U}_n/E(I_1, \ldots, I_n)$. In fact we can consider the $\mathcal{O}$-definable map:
   $$f := \begin{cases} \mathcal{U}_n/E(I_1, \ldots, I_n) & \rightarrow B_n(K)/\text{Stab}(I_1, \ldots, I_n) \\ [(\bar{a}_1, \ldots, \bar{a}_n)]_{E(I_1, \ldots, I_n)} & \rightarrow [\bar{a}_1, \ldots, \bar{a}_n]_{\text{Stab}(I_1, \ldots, I_n)} \end{cases}.$$

We denote as $\rho(I_1, \ldots, I_n) = \mathcal{U}_n \rightarrow B_n(K)/\text{Stab}(I_1, \ldots, I_n)$ the composition maps $\rho(I_1, \ldots, I_n) = f \circ \tilde{\rho}(I_1, \ldots, I_n)$.

**Definition 4.4.** [The stabilizer sorts] We consider the language $\mathcal{L}_G$ extending the three sorted language $\mathcal{L}$ (see Subsection 2), where:
1. We equipped the value group with the entire $\mathcal{L}_{\text{bg}}$ structure, where $\mathcal{L}_{\text{bg}}$ is the language described in subsection 2.
2. For each $n \in \mathbb{N}$ we consider the parametrized family of sorts $B_n(K)/\text{Stab}(I_1, \ldots, I_n)$ and maps $\rho(I_1, \ldots, I_n)$:
   $$\mathcal{U}_n \rightarrow B_n(K)/\text{Stab}(I_1, \ldots, I_n) \text{ where } (I_1, \ldots, I_n) \in \mathcal{F}^n.$$ We refer to the sorts in the language $\mathcal{L}_G$ as the stabilizer sorts. We denote as $\mathcal{G}$ their union, i.e.
   $$K \cup k \cup \Gamma \cup \{ \Gamma/\Delta \mid \Delta \in RJ(\Gamma) \} \cup \{ \Gamma/\Delta + n\Gamma \mid \Delta \in RJ(\Gamma), n \in \mathbb{N}_{\geq 2} \} \cup \{ B_n(K)/\text{Stab}(I_1, \ldots, I_n) \mid n \in \mathbb{N}, (I_1, \ldots, I_n) \in \mathcal{F}^n \}.$$

**Remark 4.5.** The geometric sorts for the case of ACVF are a particular instance of the stabilizer sorts. Let $S_n$ denotes the set of $\mathcal{O}$-lattices of $K^n$ of rank $n$, these are simply the $\mathcal{O}$-modules of type $(O, \ldots, O)$. For each $\Lambda \in S_n$, let $\text{res}(\Lambda) = \Lambda \otimes_O k = \Lambda/M\Lambda$, which is a $k$-vector space of dimension $n$.
Let $T_n = \bigcup_{\Lambda \in S_n} \text{res}(\Lambda) = \{ (\Lambda, x) \mid \Lambda \in S_n, x \in \text{res}(\Lambda) \}$. Each of these torsors is considered in the stabilizer sorts as the code of an $\mathcal{O}$ module of type $(O, M_1, \ldots, M)$, because any torsor of the form $a + M\Lambda$ for some $\Lambda \in S_n$ can be identified with an $\mathcal{O}$-module of type $(O, M_1, \ldots, M)$ (see Proposition 3.5).
An explicit description of the stabilizer sorts

The following statements are directed to obtain a more explicit description of each of the sorts $Stab(t_1,\ldots,t_n)$.

**Definition 4.6.** Let $S \subseteq \Gamma$ be a definable end-segment.

1. Let $\Delta \in RJ(\Gamma)$ and $S \subseteq \Gamma$ an end-segment. We say that $S$ is $\Delta$-decomposable if it is a union of $\Delta$-cosets.

2. We denote as $\Delta_S$ the stabilizer of $S$, i.e. $\Delta_S := \{ \eta \in \Gamma \mid \eta + S = S \}$.

The following is [25, Fact 4.1].

**Fact 4.7.** Let $S \subseteq \Gamma$ be a definable end-segment. Then $\Delta_S$ is a definable convex subgroup of $\Gamma$, therefore $\Delta_S \in RJ(\Gamma)$. Furthermore, $\Delta_S = \bigcup_{\Delta \in C} \Delta$, where $C = \{ \Delta \in RJ(\Gamma) \mid S$ is $\Delta$-decomposable $\}$.

**Notation 4.8.** For each $\Delta \in RJ(\Gamma)$ we denote as $O_\Delta$ the valuation ring of $K$ of the coarsened valuation $v_\Delta : K^\times \to \Gamma/\Delta$ induced by $\Delta$.

**Fact 4.9.** Let $I \in \mathcal{F}$ and let $S_I = \{v(x) \mid x \in I\}$. Then $Stab(I) = O_{\Delta_{S_I}} = \{x \in K \mid v(x) \in \Delta_{S_I}\}$.

**Proof.** This is an immediate consequence of Fact 4.7.

**Proposition 4.10.** Let $n \in \mathbb{N}$, and $(I_1,\ldots,I_n) \in \mathcal{F}^n$. Then

$$Stab(t_1,\ldots,t_n) = \{(a_{ij})_{1 \leq i,j \leq n} \in B_n(K) \mid a_{ii} \in O_{\Delta_{S_{I_i}}} \wedge a_{ij} \in Col(I_i, I_j) \text{ for each } 1 \leq i < j \leq n \}$$

**Proof.** This is a straightforward computation and it is left to the reader.

5 Weak Elimination of imaginaries for henselian valued field with value group with bounded regular rank

This section is devoted to showing the two conditions required by Hrushovski’s criteria to obtain weak elimination of imaginaries down to the stabilizer sorts. Let $T$ be the $L_G$-first order theory of a henselian valued field of equicharacteristic zero, residue field algebraically closed and value group with bounded regular rank. We start by proving the density of the definable types. Later, we show that each definable type can be coded in the stabilizer sorts.

5.1 Density of definable types

**Definition 5.1.** Let $K \models T$ and let $C$ be a finite congruence restriction in the valued field.

1. Let $A$ be a 1-torsor, we say that $C$ is co-initial in $A$ if for any element $y \in A$ there is some $x \in A \cap C$ such that $v(x) \leq v(y)$.

2. Let $S$ be a generalized swiss cheese, say of the form $S = A \setminus (B_1 \cup \cdots \cup B_n)$ where $A,B_i$ are 1-torsors and each $B_i \subseteq A$. We say that $C$ is co-initial in $S$ if $C$ is co-initial in $A$.

**Fact 5.2.** Let $K \models T$ and $X$ be a nice set of the form $S \cap C$ where $S$ is a generalized swiss cheese and $C$ is the set defined by some finite congruence restriction. Then there is some generalized swiss cheese $S'$ such that the set $X' := S' \cap C$ satisfies the following conditions:

1. $x \in X$ if and only if $x \in X'$,

2. $C$ is co-initial in $X'$.

**Proof.** Suppose $S = A \setminus B_1 \cup \cdots B_n$ where $B_i \subseteq A$ are 1-torsors. Without loss of generality we may assume $A$ to be an $O$-submodule of $K$, because given $c \in X$ we have that the definable affine map $f_c : K \to K$ that sends each element $x$ to $x - c$ and $f_c(A)$ is an $O$-submodule of $K$. The statement now follows by Fact 2.18 and the fact that each coarsened valuation $v_\Delta$ is locally constant.

**Theorem 5.3.** For every non-empty definable set $X$, there is a $acl^d(X')$-definable global type $p(x) \vdash x \in X$. 

Proof. By Theorem 2.36 there is a finite partition of \( X = \bigcup_{i \leq l} T_i \) where each \( T_i \) is a nice set, and \( ^\prime T_i \) \( \epsilon \) acl\(^{eq}(X)\). For each nice set \( T_i \) we construct a acl\(^{eq}(T_i)\) definable type \( p_{T_i}(x) \vdash x \epsilon T_i \). Fix some \( i \leq l \) and suppose \( T_i = S_i \cap C_i \) where \( S_i \) is a generalized swiss cheese and \( C_i \) is a finite congruence restriction in the field. We denote as \( C_i^+ (x) \) the conjunction of the basic positive congruence formulas in the field mentioned in \( C_i(x) \) while we denote as \( C_i^-(x) \) the conjunction of the basic negative congruence formulas in the field mentioned in \( C_i(x) \). In view of Fact 5.2, we can assume that the set of realizations \( C_i \) of the congruence restriction \( C_i \) is co-initial in \( T_i \). Suppose that \( S_i = A_i \setminus B_{i_1} \cup \cdots \cup B_{i_n} \) where \( B_{i_j} \subseteq A_i \) are 1-torsors and let \( S_{A_i} := \{ v(x) \mid x \epsilon A_i \} \) be the definable end-segment induced by \( A_i \).

Without loss of generality we may assume \( A_i \) to be an \( O \)-submodule of \( K \). For each basic congruence restriction in the field of the form \( \phi(x) := \gamma v_{\Delta}(x-a) - \beta \epsilon n(\Delta/\Gamma) \) we let \( \hat{\phi}(y) := \gamma y - \beta + k_{\Delta} \epsilon n(\Delta/\Gamma) \), and we define:

\[
Y_i := \{ \hat{\phi}(y) \mid \phi(x) \epsilon C_i^+(x) \} \cup \{ \neg \phi(x) \epsilon C_i^-(x) \}.
\]

Then \( Y_i \) is co-initial in \( S_{A_i} \). By Proposition 2.19 there is a complete global type \( q(y) \vdash y \epsilon S_{A_i} \land Y_i \epsilon acl\(^{eq}(T_i)\) \) definable over acl\(^{eq}(S_{A_i} \land Y_i) \subseteq acl\(^{eq}(T_i)\) \). Let \( \gamma \epsilon \) a realization of \( q(y) \) and \( a \epsilon \) be an element of the field such that \( v(a) = \gamma \). Let \( p_{T_i}(x) := tp(a/\mathfrak{M}) \), this is a global type concentrated in \( T_i \) and definable over acl\(^{eq}(T_i)\), as required.

### 5.2 Coding definable types

In this subsection we prove that any definable type can be coded in the stabilizer sorts \( \mathcal{G} \). Let \( x = (x_1, \ldots, x_k) \) be a tuple of variables in the main field sort. By the quantifier elimination any definable type \( p(x) \) over a model \( K \) is completely determined by the boolean combinations formulas of the form:

- \( Q_1(x) = 0 \),
- \( v_{\Delta}(Q_1(x)) \leq v_{\Delta}(Q_2(x)) \),
- \( v_{\Delta} \left( \frac{Q_1(x)}{Q_2(x)} \right) - k_{\Delta} \epsilon n(\Delta/\Gamma) \),
- \( v_{\Delta} \left( \frac{Q_1(x)}{Q_2(x)} \right) = k_{\Delta} \).

where \( Q_1(x), Q_2(x) \epsilon K[X_1, \ldots, X_k], n \epsilon \mathbb{N}_{\geq 2}, \Delta \epsilon RJ(\Gamma), k \epsilon \mathbb{Z} \) and \( k_{\Delta} = k \cdot 1_{\Delta} \) where \( 1_{\Delta} \) is the minimal element of \( \Gamma/\Delta \) if it exists. We will approximate such a type by considering for each \( l \epsilon \mathbb{N} \) the definable vector space \( D_l/I_l \), where \( D_l \) is the set of polynomials of degree at most \( l \) and \( I_l \) is the subspace of \( D_l \) of polynomials \( Q(x) \) such that \( Q(x) = 0 \) is a formula in \( p(x) \). The formulas of the first kind, essentially just turns \( D_l/I_l \) into a valued vector space with all the coarsest valuations, while the formulas of the second kind simply impose some equivalence relations in the linear order \( \Gamma(D_l/I_l) \). This philosophy reduces the problem of coding definable types simply into finding a way to code the possible valuations that could be induced over some power of \( K \) while taking care as well for the congruences.

The following is [12, Lemma 6.3.3].

**Fact 5.4.** Let \( K \) be any field. Let \( V \) be a subspace of \( K^n \) then \( V \) can be coded by a tuple of \( K \), and \( V \) and \( K^n/V \) have a \(^V\)-definable basis.

We start by coding the \( O \)-submodules of \( K^n \).

**Lemma 5.5.** Let \( K = T \) and \( M \subseteq K^n \) be a definable \( O \)-submodule. Then the code \(^M \) can be coded in the stabilizer sorts.

**Proof.** Let \( V^+ \) the span of \( M \) and \( V^- \) the maximal \( K \)-subspace of \( K^n \) contained in \( M \). By Fact 5.4 the subspaces \( V^+ \) and \( V^- \) can be coded by a tuple \( c \) in \( K \), and the quotient vector space \( V^+/V^- \) admits a c-definable basis. Hence \( V^+/V^- \) can be identified over \( c \) with some power \( K^m \). And \( M \) in interdefinable over \( c \) with the image of \( N/V^- \) in \( K^m \). But this image is an \( O \)-submodule of \( K^m \) of type \( (I_1, \ldots, I_m) \epsilon \mathcal{F}^m \) so it admits a code in \( B_m/\text{Stab}(I_1, \ldots, I_m) \). So \( M \) admits a code in the stabilizer sorts, as required.
Definition 5.6. [Valued relation] Let $K = T$, and $\Gamma$ be its value group. Let $V$ be some finite dimensional $K$-vector space and $R \subseteq V \times V$ be a definable subset, we say that $R$ is a valued relation if there is an interpretable valued vector structure $(V, \Gamma(V), \text{val}, +)$ in $K$ such that $(v, w) \in R$ if and only if $\text{val}(v) \leq \text{val}(w)$.

In fact, given a relation $R \subseteq V \times V$ satisfying that :

- for all $v, w \in V$, $(\min\{v, w\}, v + w) \in R$,
- for all $v \in V$, $(v, v) \in R$,
- for all $v, w \in V$ and $\alpha \in K$, if $(v, w) \in R$ then $(\alpha v, \alpha w) \in R$,

we can define an equivalence relation $E_R$ over $V$ as:

$$E_R(v, w) \iff (v, w) \in R \land (w, v) \in R.$$ 

The set $\Gamma(V) = V/E_R$ is therefore interpretable in $K$ and we call it as the linear order induced by $R$. Let $\text{val} : V \to \Gamma(V)$ be the canonical projection map that sends each vector to its class. We can naturally define an action of $\Gamma$ over $\Gamma(V)$ as:

$$+: \Gamma \times \Gamma(V) \to \Gamma(V) \quad \text{where } a \in K \text{ such that } v(a) = \alpha.$$ 

This is a well defined map by the third condition imposed over $R$. The structure $(V, \Gamma(V), \text{val}, +)$ is an interpretable valued vector structure over $V$ and we refer to it as the valued vector structure induced by $R$.

Lemma 5.7. Let $K$ be a model of $T$ and let $R \subset K^n \times K^n$ be a binary relation inducing a valued vector structure $(K^n, \Gamma(K^n), \text{val}, +)$ over $K^n$. Then we can find a basis $\{v_1, \ldots, v_n\}$ of $K^n$ such that:

1. It is a separated basis for $\text{val}$, this is given any set of coefficients $\lambda_1, \ldots, \lambda_n \in K$,

   $$\text{val}\left( \sum_{i \leq n} \lambda_i v_i \right) = \min\{v(\lambda_i) + \text{val}(v_i) \mid i \leq n\}.$$ 

2. for each $i \leq n$, $\gamma_i = \text{val}(v_i) \in \text{dcl}(\text{val}(R))$.

Proof. Because the statement we are proving is first order expressible, by Fact ?? we may assume that $K$ is maximal. We proceed by induction on $n$. For the base case, note that $K = \text{span}_K(1)$ then $\gamma = \text{val}(1) \in \text{dcl}(\text{val}(R))$. We assume the statement for $n$ and we want to prove it for $n + 1$. Let $W = K^n \times \{0\}$, $\text{val}_W = v \restriction_W$, $\Gamma(W) = \{\text{val}(w) \mid w \in W\}$, and $R_W = R \cap (W \times W)$. Then $(W, \Gamma(W), \text{val}_W, +)$ is a valued vector structure over $W$ and $\text{dcl}(\text{val}(R_W)) \subseteq \text{dcl}(\text{val}(R))$. The subspace $W$ admits an $\mathcal{S}$-definable basis, so it can be canonically identified with $K^n$. By the induction hypothesis we can find $\{w_1, \ldots, w_n\}$ a separated basis of $W$ such that $\text{val}_W(w_i) \in \text{cl}(\text{val}(R_W)) \subseteq \text{cl}(\text{val}(R))$. As $W$ is finite dimensional it is maximal by Lemma 2.39. By Fact 2.48 $W$ has the optimal approximation property. We can therefore define the valuation over the quotient space $K^{n+1}/W$ as follows:

$$\text{val}_{K^{n+1}/W} := \left\{ \left( \frac{K^{n+1}}{W} \right) \to K^n \right\} v + W \mapsto \text{max}\{\text{val}(v + w) \mid w \in W\}.$$ 

Define $R_{K^{n+1}/W} = \{(w_1 + W, w_2 + W) \mid \text{val}_{K^{n+1}/W}(w_1 + W) \leq \text{val}_{K^{n+1}/W}(w_2 + W)\}$, which is a valued relation over the quotient space $K^{n+1}/W$. As $K^{n+1}/W = K^{n+1}/(K^n \times \{0\})$ is definably isomorphic over the $\mathcal{S}$-sets to $K$, we can find a non zero coset $v + W$ such that $\text{val}_{K^{n+1}/W}(v + W) \in \text{cl}(\text{val}(R_{K^{n+1}/W})) \subseteq \text{cl}(\text{val}(R))$. Let $w^* \in W$ be the vector where the maximum of $\{\text{val}_{K^{n+1}/W}(v + w) \mid w \in W\}$ is attained, i.e. $\text{val}_{K^{n+1}/W}(v + W) = \text{val}(v + w^*)$. It is sufficient to show that $\{w_1, \ldots, w_n, v + w^*\}$ is a separated basis for $K^{n+1}$. Let $\alpha \in K$, we show that for any $w \in W$ $\text{val}(v + w^* + \alpha w) = \text{min}\{\text{val}(v + w^*), \text{val}(\alpha w)\}$. If $\text{val}(v + w^*) \neq \text{val}(\alpha w)$ then $\text{val}(v + w^* + \alpha w) = \text{min}\{\text{val}(v + w^*), \text{val}(\alpha w)\}$. So let’s assume that $\gamma = \text{val}(v + w^*) = \text{val}(\alpha w)$, by the ultrametric inequality $\text{val}(v + w^* + \alpha w) \geq \gamma$. By the maximal choice of $w^*$, we have that $\text{val}(v + w^* + \alpha w) = \text{val}(v + w^*) = \gamma$. So $\text{val}(v + w^* + \alpha w) = \text{min}\{\text{val}(v + w^*), \text{val}(\alpha w)\}$ as required. 

\[\square\]
\textbf{Theorem 5.8.} Let $K$ be a model of $T$ and $\Gamma$ be its value group. Let $R$ be a definable valued relation over $K^n$ and $(K^n, \Gamma(K^n), \text{val}, +)$ be the valued vector structure induced by $R$. Then $'R'$ is interdefinable with a tuple of elements in the stabilizer sorts and there is an $'R'$- definable bijection $\Gamma(K^n)$ and some power of $\Gamma$.

\textit{Proof.} As the statement that we are trying to prove is first order expressible, without loss of generality we may assume that $K$ is maximal. Let $R$ be a valued relation over $K^n$ and let $(K^n, \Gamma(K^n), \text{val}, +)$ be the valued vector structure induced by $R$. By Lemma 5.7, we can find a separated basis $\{v_1, \ldots, v_n\}$ of $K^n$, such that for each $i \leq n$, $\text{val}(v_i) \in \text{dcl}^q('R')$. Let $\{\gamma_1, \ldots, \gamma_s\} \subseteq \{\text{val}(v_i) \mid i \leq n\}$ be a complete set of representatives of the orbits of $\Gamma$ over the linear order $\Gamma(K^n)$, this is:

$$\Gamma(K^n) = \bigcup_{i \leq s} \Gamma + \gamma_i.$$ 

For each $i \leq s$, we define $B_i := \{x \in K^n \mid \text{val}(x) \geq \gamma\}$. Each $B_i$ is an $\mathcal{O}$-submodule of $K^n$, so by Lemma 5.5 it admits a code $'B_i'$ in the stabilizer sorts. The valued vector structure over $K^n$ is completely determined by the closed balls containing 0, and each of these ones is of the form $\alpha B_i$ for some $\alpha \in K$ and $i \leq s$. Thus the code $'R'$ is interdefinable with the tuple $('B_1', \ldots, 'B_s')$. We conclude that $'R'$ can be coded in the stabilizer sorts.

For the second part of the statement, consider the map:

$$\begin{align*}
\left\{ f : \bigcup_{i \leq s} \Gamma + \gamma_i &\rightarrow \Gamma^s \\
\alpha + \gamma_i &\mapsto (0, \ldots, 0, \underbrace{\alpha}_{i-th\ coordinate}, 0, \ldots, 0).
\right.
\end{align*}$$

As $\{\gamma_1, \ldots, \gamma_s\} \subseteq \text{dcl}^q('R')$ this is a $'R'$-definable bijection between $\Gamma(K^n)$ and the power $\Gamma^s$.

\textbf{Theorem 5.9.} Let $p(x)$ be a definable global type in $\mathcal{M}^n$. Then $p(x)$ can be coded in $\mathcal{G} \cup \Gamma^q$.

\textit{Proof.} Let $p(x)$ be a definable global type, and let $K$ be a small model where $p(x)$ is defined. Let $q(x) = p(x) \upharpoonright_K$ it is sufficient to code $q(x)$.

For each $\ell \in \mathbb{N}$ let $D_\ell$ be the space of polynomials in $K[X_1, \ldots, X_n]$ of degree less or equal than $\ell$. This is a finite dimensional $K$-vector space with an $\varnothing$-definable basis. Let $I_\ell := \{Q(\bar{x}) \in D_\ell \mid Q(\bar{x}) = 0 \in q(\bar{x})\}$, this is a subspace of $D_\ell$. Let $R_\ell := \{(Q_1(\bar{x}), Q_2(\bar{x})) \in D_\ell \times D_\ell \mid v(Q_1(\bar{x})) \leq v(Q_2(\bar{x})) \in q(\bar{x})\}$, this relation induces a valued vector structure on the quotient space $V_\ell = D_\ell / I_\ell$. Let $(V_\ell, \Gamma(V_\ell), \text{val}_\ell, +_\ell)$ be the valued vector structure induced by $R_\ell$ over $V_\ell$.

For each $\Delta \in RJ(\Gamma)$ and $k \in \mathbb{Z}$, a formula of the form $v_\Delta(Q_1(\bar{x})) = v_\Delta(Q_2(\bar{x})) + k_\Delta$ determines a definable relation $\phi_\Delta^k \subseteq \Gamma(V_\ell)^2$, defined as:

$$(\text{val}_\ell(Q_1(\bar{x})), \text{val}_\ell(Q_2(\bar{x}))) \in \phi_\Delta^k \text{ if and only if } v_\Delta(Q_1(\bar{x})) = v_\Delta(Q_2(\bar{x})) + k_\Delta \in q(\bar{x}).$$

Similarly, for each $\Delta \in RJ(\Gamma)$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}_{\geq 2}$ we consider the definable relation $\psi_\Delta^{k,n} \subseteq \Gamma(V_\ell)^2$ determined as:

$$(\text{val}_\ell(Q_1(\bar{x})), \text{val}_\ell(Q_2(\bar{x}))) \in \psi_\Delta^{k,n} \text{ if and only if } v_\Delta(Q_1(\bar{x})) - v_\Delta(Q_2(\bar{x})) + k_\Delta \in n(\Gamma/\Delta) \in q(\bar{x}).$$

Likewise, for each $\Delta \in RJ(\Gamma)$ and $k \in \mathbb{Z}$ we consider the definable relations $\theta_\Delta^k \subseteq \Gamma(V_\ell)^2$ defined as:

$$(\text{val}_\ell(Q_1(\bar{x})), \text{val}_\ell(Q_2(\bar{x}))) \in \theta_\Delta^k \text{ if and only if } v_\Delta(Q_1(\bar{x})) < v_\Delta(Q_2(\bar{x})) + k_\Delta \in q(\bar{x}).$$

Let

$$S_\ell = \{\phi_\Delta^k \mid \Delta \in RJ(\Gamma), k \in \mathbb{Z}\} \cup \{\psi_\Delta^{k,n} \mid \Delta \in RJ(\Gamma), k \in \mathbb{Z}, n \in \mathbb{N}_{\geq 2}\}$$

$$\cup \{\theta_\Delta^k \mid \Delta \in RJ(\Gamma), k \in \mathbb{Z}\}$$

We denote as $V_\ell = (V_\ell, \Gamma(V_\ell), \text{val}_\ell, +_\ell, S_\ell)$ the valued vector space over $V_\ell$ with the enriched structure over the linear order $\Gamma(V_\ell)$. By the quantifier elimination, the type $q(x)$ is completely determined by boolean combinations of formulas of the form:

- $Q_1(x) = 0,$

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\[ v_\Delta(\frac{Q_1(x)}{Q_2(x)}) < v_\Delta(\frac{Q_2(x)}{Q_2(x)}) \]
\[ v_\Delta\left(\frac{Q_1(x)}{Q_2(x)}\right) - k_\Delta \in n(\Gamma/\Delta) \]
\[ v_\Delta\left(\frac{Q_1(x)}{Q_2(x)}\right) = k_\Delta. \]

where \( Q_1(x), Q_2(x) \in K[X_1, \ldots, X_k], n \in \mathbb{N}_{>2}, \Delta \in RJ(\Gamma), k \in \mathbb{Z} \) and \( k_\Delta = k \cdot 1_\Delta \) where \( 1_\Delta \) is the minimal element of \( \Gamma/\Delta \) if it exists. Hence the type \( p(x) \) is entirely determined (and determines completely) by the sequence of valued vector spaces with enriched structure \((V_\ell | \ell \in \mathbb{N})\).

By Fact 5.4 for each \( \ell \in \mathbb{N} \) we can find codes \( 'I_\ell' \) in the home sort for the \( I'_k \)'s. After naming these codes, each quotient space \( V_\ell = D_\ell f_\ell \) has a definable basis, so it can be definably identified with some power of \( K \).

Therefore, without loss of generality we may assume that the underlying set of the valued vector space with enriched structure \( V_\ell \) is some power of \( K \). By Theorem 5.8, the relation \( R_\ell \) admits a code \( 'R_\ell' \) in the stabilizer sorts. Moreover, there is a \( 'R_\ell' \) definable bijection \( f := \Gamma(V_\ell) \to \Gamma^q \), where \( s \in \mathbb{N}_{>2} \) is the number of \( \Gamma \)-orbits over \( \Gamma(V_\ell) \).

In particular, for each \( \Delta \in RJ(\Gamma), n \in \mathbb{N} \) and \( k \in \mathbb{Z} \) the definable relations \( \phi^k_\Delta, \psi^k_\Delta \) and \( \theta^k_\Delta \) are interdefinable over \( 'R' \) with the subsets \( f(\phi^k_\Delta), f(\psi^k_\Delta) \) and \( f(\theta^k_\Delta) \) subsets of \( \Gamma^q \). We conclude that the type \( q(x) \) can be coded in the sorts \( \Gamma \cup \Gamma^q \), as every definable subset \( D \) in some power of \( \Gamma \) admits a code in \( \Gamma^q \).

**Theorem 5.10.** Let \( K \) be a valued field of equicharacteristic zero, residue field algebraically closed and value group with bounded regular rank. Then \( K \) admits weak elimination of imaginaries in the language \( L_G \), where the stabilizer sorts are added.

**Proof.** By Theorem 5.10, \( K \) admits weak elimination of imaginaries down to the sorts \( G \cup \Gamma^q \), where \( G \) are the stabilizer sorts. In fact, Hrushovki’s criteria requires us to verify the following two conditions:

1. the density of definable types, this is Theorem 5.3, and
2. the coding of definable types, this is Theorem 5.9.

By Corollary 2.3 the value group \( \Gamma \) is stably embedded. By Theorem 2.10, the ordered abelian group with bounded regular rank \( \Gamma \) admits weak elimination of imaginaries once the quotient sorts \( \{ \Gamma/\Delta \mid \Delta \in RJ(\Gamma) \} \cup \{ \Gamma/\Delta + n\Gamma \mid \Delta \in RJ(\Gamma), n \in \mathbb{N}_{>2} \} \) are added. We conclude that \( K \) admits weak elimination of imaginaries down to the stabilizer sorts \( G \), as desired.  

\section{Elimination of imaginaries for henselian valued field with dp-minimal value group}

Let \((K, v)\) be a henselian valued field of equicharacteristic zero, residue field algebraically closed and dp-minimal value group. Let \( I = \{ I_S \mid S \in S_{dp} \} \) the family of definable \( O \) submodules of \( K \), where \( S_{dp} \) is the complete family of end-segments introduced in Corollary 2.25. This is a complete family of \( O \)-submodules, each of them definable over the empty-set. We view \( K \) as a multisorted structure in the language \( \tilde{L} \) extending the language \( L_{val} \) described in Subsection 2, where:

1. the value group is equipped with the language \( L_{dp} \) described in Subsection 2.
2. For each finite tuple \((I_1, \ldots, I_n) \in I \) we add a sort interpreted as the quotient \( B_n(K)/Stab(I_1, \ldots, I_n) \) and projection maps \( \rho_{(I_1, \ldots, I_n)} = \text{Id} \to B_n(K)/Stab(I_1, \ldots, I_n) \) as in Definition 4.4.

We refer to these sorts as the **stabilizer sorts** and we denote their union \( \mathcal{G} = K \cup k \cup \Gamma \cup \{ \Gamma/\Delta \mid \Delta \in RJ(\Gamma) \} \cup \{ B_n(K)/\text{Stab}(I_1, \ldots, I_n) \mid (I_1, \ldots, I_n) \in I^n \} \).

This section is devoted to prove the following statement:

**Theorem 6.1.** Let \( K \) be a henselian valued field of equicharacteristic zero, residue field algebraically closed and dp-minimal value group. Then \( K \) eliminates imaginaries in the language \( \tilde{L} \), where the stabilizer sorts are added.

**Definition 6.2.** We say that a multi-sorted first order theory \( T \) codes finite sets if for every model \( M \models T \), and every set \( S \) of a finitely many elements in \( M \), the code \( 'S' \) is interdefinable with a tuple of elements in \( M \).
The following is a folklore fact.

**Fact 6.3.** Let $T$ be a complete multi-sorted theory. If $T$ has weak elimination of imaginaries and codes finite sets then $T$ eliminates imaginaries.

In view of Theorem 5.10, it is only left to show that any finite set can be coded in $\mathcal{G}$.

**Definition 6.4.**
1. An equivalence relation $E$ on a set $X$ is said to be proper if it has at least two different equivalence classes. It is said to be trivial if for any $x, y \in X$ we have $E(x, y)$ if and only if $x = y$.

2. A finite set $F$ is primitive over $A$ if there is no proper non-trivial $(F^\ast \cup A)$-definable equivalence relation on $F$. If $F$ is primitive over $\emptyset$ we just say that it is primitive.

To code finite sets we need numerous smaller results. This section is organized as follows:

1. Subsection 6.1: we analyze the stable and stably embedded multi-sorted structure $\text{VS}_{k,C}$, consisting of the $k$-vector spaces red$(s)$, where $s$ is some $\mathcal{O}$-lattice definable over $C$. We prove that this structure has elimination of imaginaries.

2. Subsection 6.2: we start by introducing the notion of germ of a definable function over a definable type $p$. We prove that germs can be coded in the stabilizer sorts.

3. Subsection 6.3: later we show that any $\mathcal{O}$-submodule $M \subseteq K^n$ is interdefinable with its projection into the last coordinate and the germ of the function describing each of the fibers. Later, we show that the same statement holds for torsors.

4. Subsection 6.4: we prove several results regarding the coding of the one-dimensional case.

5. Subsection 6.5: we carry a simultaneous induction to prove that any primitive finite set $F$ of torsors can be coded in the stabilizer sorts and any definable function $f : F \to \mathcal{G}$ admits a code in the stabilizer sorts.

6. Subsection 6.6: we combine all the statements to prove that finite sets can be coded in the stabilizer sorts and present the complete elimination of imaginaries down to the stabilizer sorts.

### 6.1 The multi-sorted structure of $k$-vector spaces

By Corollary 2.3 the residue field $k$ is stably embedded and it is a strongly minimal structure, because it is an algebraically closed field. This enables us to construct, over any base set of parameters $C$, a part of the structure that naturally inherits stability-theoretic properties from the residue field. Given an $\mathcal{O}$-lattice $s \subseteq K^n$ we have red$(s) = s/Ms$ is a $k$-vector space, so we define:

**Definition 6.5.** For any parameter set $C$, we let $\text{VS}_{k,C}$ be the many-sorted structure whose sorts are the $k$-vector spaces red$(s)$ where $s \subseteq K^n$ is an $\mathcal{O}$-lattice of rank $n$ definable over $C$. Each sort red$(s)$ is equipped with its $k$-vector space structure. In addition, $\text{VS}_{k,C}$ has any $C$-definable relation on products of the sorts.

**Definition 6.6.** A definable set $D$ is said to be internal to the residue field if there is a finite set of parameters $F \subseteq \mathcal{G}$ such that $D \subseteq \text{dcl}^{eq}(kF)$.

Each of the structures red$(s)$ is internal to the residue field, and the parameters needed to witness the internality lie in red$(s)$, so in particular each of the $k$-vector spaces red$(s)$ is stably embedded. The entire multi-sorted structure $\text{VS}_{k,C}$ is also stably embedded and stable, and in this subsection we will prove that it eliminates imaginaries.

**Notation 6.7.** We recall that given an $\mathcal{O}$-submodule $M$ of $K$, we denote the end-segment induced by $M$ as $S_M = \{v(x) \mid x \in M\}$

We start by characterizing the 1-modules that are internal to the residue field.

**Lemma 6.8.** Let $K \models T$ and $M = A/B$ be a non-trivial 1-definable module. Then exactly one of the following statements hold:

1. $M$ is internal to the residue field. This holds if $M = a\mathcal{O}/a\mathcal{M}$, or
2. there is some definable surjection from $M$ onto an infinite definable convex subset $I \subseteq \Gamma$.

Proof. Let $M$ be a 1-definable module. Consider the case that is a quotient of the form $aO/aM$. The definable $O$-isomorphism $g : O/M \to aO/aM$ defined by sending $x + M \to ax + aM$, witnesses the internality of $M$ over the residue field $k = O/M$. Suppose now that $M = A/B$ where $S_A \setminus S_B$ is an infinite convex set of $\Gamma$. And define the function $g : M \setminus \{B\} \to \Gamma$ that send $b + B \mapsto v(b)$.

Firstly, we prove that $g$ is a well defined map. Let $b, b' \in A$ such that $b + B = b' + B \neq B$. Hence $b - b' \in B$ and therefore $v(b - b') \in S_B$. Because $b \notin B$ then $v(b) \notin S_B$, thus $v(b) < v(b - b')$. We have $v(b') = v((b - b') + b) = v(b)$. On the other hand, the image of $g$ contains an infinite convex subset of $\Gamma$. In fact, for any $\delta \in S_A \setminus S_B$ we can find an element $x_\delta \notin B$ such that $v(x_\delta) = \delta$, and therefore $g(x_\delta + B) = \delta$. \qed

We recall some definitions from [26] to show that $VS_{k,C}$ eliminates imaginaries.

Definition 6.9. Let $t$ be a theory of fields (possibly with additional structure). A $t$-linear structure $A$ is a structure with sort $k$ for a model of $t$, and additional sorts $(V_i \mid i \in I)$ denoting finite-dimensional vector spaces. Each $V_i$ has (at least) a $k$-vector space structure, and $\dim V_i < \infty$. We assume that:

1. $k$ is stably embedded,
2. the induced structure on $k$ is precisely given by $t$,
3. The $V_i$ are closed under tensor products and duals.

Moreover, we say it is flagged if for any finite dimensional vector space $V$ there is a filtration $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V$ by subspaces, with $\dim V_i = i$ and $V_i$ is one of the distinguished sorts.

The following is [26, Lemma 5.2].

Lemma 6.10. If $k$ is an algebraically closed field and $A$ is a flagged $k$-linear structure, then $A$ admits elimination of imaginaries.

Notation 6.11. Given $A$ an $O$-module. Let $MA = \{xa \mid x \in M, a \in A\}$ we denote as $\text{red}(A)$ the quotient $O$-module $A/MA$.

We observe that $\text{red}(A) = A/MA$ is canonically isomorphic to $A \otimes_O k$.

Fact 6.12. Let $A \subseteq K^n$ and $B \subseteq K^m$ be $O$-lattices. Then $\text{red}(A) \otimes_k \text{red}(B)$ is canonically identified with $\text{red}(A \otimes_O B)$.

Proof. This is a straightforward computation and it is left to the reader. \qed

Remark 6.13. Given $A \subseteq K^n$ and $B \subseteq K^m$ $O$-lattices, there is some $O$-module $C \subseteq K^{mn}$ such that $A \otimes_O B$ is canonically identified with $C$. This isomorphism induces as well a one to one correspondence between $\text{red}(A \otimes_O B)$ and $\text{red}(C)$.

Proof. Given $K^n$ and $K^m$ two vector spaces, the tensor product $K^n \otimes K^m$ is a $K$ vector space whose basis is $\{e_i \otimes e_j \mid i \leq n, j \leq m\}$ and it is canonically identified with $K^{mn}$, via a linear map $\phi$ that extends the bijection between the basis sending $e_i \otimes e_j$ to $e_{ij}$. Given $A \subseteq K^n$ and $B \subseteq K^m$ $O$-lattices, then $A \otimes B$ is an $O$-lattice of $K^n \otimes K^m$ and we denote as $C = \phi(A \otimes_O B)$. This map induces as well an identification between $\text{red}(A \otimes_O B)$ and $\text{red}(C)$ such that the following map commutes:

\[
\begin{array}{ccc}
A \otimes O B & \overset{\phi}{\longrightarrow} & C \\
\downarrow \text{red} & & \downarrow \text{red} \\
\text{red}(A \otimes_O \text{red}(B)) & \overset{i}{\longrightarrow} & \text{red}(C)
\end{array}
\]

\qed

Fact 6.14. Let $A \subseteq K^n$ and $B \subseteq K^m$ be $O$-lattices. Then there is a isomorphism $\phi : \text{red}(\text{Hom}_O(A, B)) \to \text{Hom}_k(\text{red}(A), \text{red}(B))$, where for any $f \in \text{Hom}_O(A, B)$ and $a \in A$:

\[
\phi(f + \mathcal{M}\text{Hom}_O(A, B)) : \begin{cases} 
\text{red}(A) & \to \text{red}(B) \\
(a + \mathcal{M}A) & \mapsto f(a) + \mathcal{M}B.
\end{cases}
\]
Proof. This is a straightforward computation and it is left to the reader. □

Remark 6.15. Given an \( \mathcal{O} \)-lattice \( A \subseteq K^n \), then \( \text{Hom}_{\mathcal{O}}(A, \mathcal{O}) \) can be canonically identified with some \( \mathcal{O} \)-submodule \( C \) of \( K^n \). So there is a correspondence between \( \text{red}(\text{Hom}_{\mathcal{O}}(A, \mathcal{O})) \) and \( \text{red}(C) \).

Proof. Let \( A \) be an \( \mathcal{O} \)-lattice of \( K^n \). By linear algebra \( K^n \) can be identified with its dual space \( (K^n)^* \). Let

\[
A^* = \{ T \in (K^n)^* \mid \text{for all } a \in A, T(a) \in \mathcal{O} \}.
\]

\( A^* \) is canonically identified with \( \text{Hom}_{\mathcal{O}}(A, \mathcal{O}) \) via the map that sends a transformation \( T \) to \( T \upharpoonright_A \). Also \( A^* \) is isomorphic to some \( \mathcal{O} \)-submodule \( C \) of \( K^n \), as there is a canonical isomorphism between \( K^n \) and its dual space. So we have a definable \( \mathcal{O} \)-isomorphism \( \phi \) between \( \text{Hom}_{\mathcal{O}}(A, \mathcal{O}) \) and \( C \), and this correspondence induces an identification \( \hat{\phi} \) between \( \text{red}(\text{Hom}_{\mathcal{O}}(A, \mathcal{O})) \) and \( \text{red}(C) \) making the following diagram commute:

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{O}}(A, \mathcal{O}) & \xrightarrow{\phi} & C \\
\downarrow\text{red} & & \downarrow\text{red} \\
\text{red}(\text{Hom}_{\mathcal{O}}(A, \mathcal{O})) & \xrightarrow{\hat{\phi}} & \text{red}(C)
\end{array}
\]

Remark 6.16. Let \( A \subseteq K^n \) be an \( \mathcal{O} \)-lattice. There is a sequence of \( \mathcal{O} \)-lattices \( < A_i \mid i \leq n > \) such that \( < \text{red}(A_i) \mid i \leq n > \) is a flag of \( \text{red}(A) \).

Proof. We proceed by induction on \( n \), the base case is trivial. Let \( A \subseteq K^{n+1} \), and \( \pi_{n+1} : K^{n+1} \to K \) be the projection into the last coordinate. Let \( B \subseteq K^n \) be the \( \mathcal{O} \)-lattice such that \( \ker(\pi_{n+1}) = B \times \{0\} = A \cap (K^n \times \{0\}) \). By Corollary 3.2 \( B \) is a direct summand of \( A \), so we have the exact splitting sequence \( 0 \to B \to A \to \pi_{n+1}^{-1}(A) \to 0 \).

As a consequence, \( 0 \to MB \to MA \to M\pi_{n+1}(A) \to 0 \) and \( 0 \to \text{red}(B) \to \text{red}(A) \to \text{red}(\pi_{n+1}(A)) \to 0 \) are both split exact sequences. By the induction hypothesis, there is a sequence \( \{0\} \leq A_1 \leq \cdots \leq A_n = B \) such that \( < \text{red}(A_i) \mid i \leq n > \) is a flag of \( \text{red}(B) \) such that \( \dim(\text{red}(A_i)) = i \). Let \( A_{n+1} = A \), the sequence \( < A_i \mid i \leq n+1 > \) satisfies the required conditions. □

Theorem 6.17. Let \( C \subseteq K^{eq} \), then \( \text{VS}_{k,C} \) has elimination of imaginaries.

Proof. The sorts \( \text{red}(s) \) where \( s \) is a \( \mathcal{O} \)-lattice of \( K^n \) and \( \text{def}_{eq}(C) \)-definable form the multi-sorted \( \text{VS}_{k,C} \). Each \( \text{red}(s) \) carries a \( k \)-vector space structure. \( \text{VS}_{k,C} \) is closed under tensor operation by Remark 6.13 and Fact 6.12. It is closed under duals by Remark 6.15 and Fact 6.14, by Remark 6.16 it each sort \( \text{red}(s) \) where \( s \) is an \( \mathcal{O} \)-lattice admits a complete filtration. Therefore, \( \text{VS}_{k,C} \) is a flagged \( k \)-linear structure, so the statement is immediate consequence of 6.10. □

6.2 Germs of functions

In this subsection we show how to code the germ of a definable function \( f \) over a definable type \( p(x) \) in the stabilizer sorts.

Definition 6.18. Let \( T \) be a complete first order theory and \( M \models T \). Let \( B \subseteq M \) and \( p \) be a \( B \)-definable type whose solution set is \( P \). Let \( f \) be an \( M \)-definable function whose domain contains \( P \). Suppose that \( f = f_c \) is defined by the formula \( \phi(x,y,c) \) (so \( f_c(x) = y \)). We say that \( f_c \) and \( f_{c'} \) have the same germ on \( P \) if the formula \( f_c(x) = f_{c'}(x) \) lies in \( p \). By the definability of \( p \) the equivalence relation \( E_{\phi}(c,c') \) that states \( f_c \) and \( f_{c'} \) have the same germ on \( P \) is definable over \( B \). The germ of \( f_c \) on \( P \) is defined to be the class of \( c \) under the equivalence relation \( E_{\phi}(y,z) \), which is an element in \( M^{eq} \). We denote by \( \text{germ}(f,p) \) the code for this equivalence class.

Definition 6.19. Let \( p \) be a global type definable over \( B \) and let \( C \) a set of parameters. We say that a realization \( a \) of \( p \) is sufficiently generic over \( BC \) if \( a \models p \upharpoonright_{BC} \).

We start proving some results that will be required to show how to code the germs of a definable function \( f \) over a definable type \( p \) in the stabilizer sorts.
Definition 6.20. Let \( U \subseteq K \) be a 1-torsor, let
\[
\Sigma^\text{gen}_U(x) = \{ x \in U \} \cup \{ x \notin B \mid B \not\subseteq U \text{ is a proper subtorsor of } U \}.
\]
We refer to this type as the generic type of \( U \), and we note that it is \('U'\)-definable.

Proposition 6.21. Let \( I \in \mathcal{I} \) and \( U = y + zI \) be a 1-torsor. Then any completion \( p(x) \) of the generic type of \( U \) is \('U'\)-definable.

Proof. By the quantifier elimination, any complete extension of the generic type of \( U \) is determined by the congruence restrictions that it satisfies. Let \( p(x) \) be any completion of the generic type \( \Sigma^\text{gen}_U(x) \), and let \( a \) be any realization of \( p(x) \). For each \( \Delta \in RJ(\Gamma) \), \( n \in \mathbb{N} \) let \( c \in \Omega^\Delta \) be such that \( \nu_\Delta(a) - \rho_\Delta(c) \in n(\Gamma/\Delta) \). Then, for any \( \beta \in \Gamma/\Delta \) and \( k \in \mathbb{Z} \) we have:
\[
\nu_\Delta(x) + \beta + k_\Delta \in n(\Gamma/\Delta)\text{ if and only if } \beta + k_\Delta - \rho_\Delta(c) \in n(\Gamma/\Delta).
\]
Likewise, the set \( \{ \beta \in \Gamma/\Delta \mid \nu_\Delta(x) = \beta \} \) is either empty or there is some \( \beta_0 \in \Gamma/\Delta \) such that \( \nu_\Delta(a) = \beta_0 \in \text{def}('U') \). By the quantifier elimination, we can conclude that \( p(x) \) is \('U'\)-definable, as required. \( \square \)

Proposition 6.22. Let \( (I_1, \ldots, I_n) \in \mathcal{I}^n \), for every \( \mathcal{O} \)-module \( M \) of type \( (I_1, \ldots, I_n) \) we can find a type \( p_M(\bar{x}_1, \ldots, \bar{x}_n) \in S_{n \times n}(K) \) such that:
1. \( p_M(x) \) is definable over \( \mathcal{M} \),
2. A realization of \( p_M(x) \) is a matrix representation of \( M \). This is if \( (\bar{d}_1, \ldots, \bar{d}_n) \vDash p_M(x) \) then \([\bar{d}_1, \ldots, \bar{d}_n]\) is a representation matrix for \( M \).

Proof. Let \( \mathcal{M} \) be the monster model.

Step 1: We define a partial type \( \Sigma_{(I_1, \ldots, I_n)} \) satisfying condition (1) and (2) for the canonical module \( C_{(I_1, \ldots, I_n)} \). Such a type is left-invariant under the action of \( \text{Stab}(I_1, \ldots, I_n) \).

We consider the set \( \mathcal{J} = \{ (i, j) \mid 1 \leq i, j \leq n \} \), and we equip it with a linear order defined as:
\[
(i, j) < (i', j') \text{ if and only if } j < j' \text{ or } j = j' \land i' > i.
\]
And we consider an enumeration of \( \mathcal{J} = \{ v_1, \ldots, v_{n^2} \} \) such that \( v_1 < v_2 < \cdots < v_{n^2} \). By Proposition 4.10
\[
\text{Stab}_{(I_1, \ldots, I_n)} = \{ ((a_{i,j})_{1 \leq i, j \leq n} \in B_n(K) \mid a_{ii} \in \mathcal{O}^\times_{\Delta_{\beta_{I_i}}} \land a_{ij} \in \text{Col}(I_i, I_j) \text{ for each } 1 \leq i < j \leq n \}.
\]
Hence, for each \( 1 \leq m \leq n^2 \) let:
\[
p_{v_m}(x) = \begin{cases} 
tp(0) & \text{if } v_m = (i, j) \text{ where } 1 \leq j < i \leq n, \\
\Sigma^\text{gen}_{\Delta_{\beta_{I_i}}} (x) & \text{if } v_m = (i, i) \text{ for some } 1 \leq i \leq n, \\
\Sigma^\text{gen}_{\text{Col}(I_i, I_j)}(x) & \text{if } v_m = (i, j) \text{ where } 1 \leq i < j \leq n.
\end{cases}
\]
Consider the partial definable type \( \Sigma_{C_{(I_1, \ldots, I_n)}} = p_{v_{m^2}} \otimes \cdots \otimes p_{v_1} \). Given a realization of this type \( (b_{v_{m^2}}, \ldots, b_{v_1}) \vDash \Sigma_{C_{(I_1, \ldots, I_n)}} \) let
\[
B = \begin{bmatrix} 
b_{v_n} & b_{v_{n-1}} & \cdots & b_{v_2} \\
\vdots & \vdots & \ddots & \vdots \\
b_{v_3} & b_{v_{n+1}} & \cdots & b_{v_{(n-1)^2}+1}
\end{bmatrix}
\]
By construction \( B \) is an upper triangular matrix such that \( (B)_{i,j} \in \text{Col}(I_i, I_j) \) for \( 1 \leq i < j \leq n \) and \( (B)_{ii} \in \mathcal{O}^\times_{\Delta_{\beta_{I_i}}} \), thus its column vectors constitute a basis for the canonical module. To check left invariance, it is sufficient to take \( A \in \text{Stab}_{(I_1, \ldots, I_n)}(\mathcal{M}) \) and argue that for each \( 1 \leq m \leq n^2 \) the element \( (AB)_{v_m} \) is a realization of generic type \( p_{v_m} \) sufficiently generic over \( \mathcal{M} \cup \{ (AB)_{v_k} \mid k < m \} \). Suppose that \( v_m = (i, j) \), then \( (AB)_{v_m} = (AB)_{ij} = \sum_{k=1}^{j} a_{ik}b_{kj} = a_{ii}b_{ij} + \cdots + a_{ij}b_{jj} \).
In the fixed enumeration we guarantee that $b_{ij}$ is chosen sufficiently generic over $\mathfrak{M} \cup \{b_{kj} \mid i \leq k \leq j\}$. Thus, for each $i \leq k \leq j$, $a_{ik}b_{kj} \in Col(I_1, I_j)$, we have that $v(a_{ik}b_{kj}) = v((AB)_{ij})$ and $(AB)_{ij} \models p_{\text{ev}} \cup \mathfrak{M} \cup \{b_{kj} \mid i \leq k \leq j\}$.

**Step 2:** for any $\mathcal{O}$-module $M \subseteq K^n$ of type $(I_1, \ldots, I_n)$ there is an $\langle \mathcal{M} \rangle$-definable type $p_M$, such that any realization of $p_M$ is a representation matrix for $M$.

Let $T = \mathfrak{M} \to \mathfrak{N}$ be a linear transformation whose representation matrix is upper triangular and $T$ sends the canonical module $C_{(I_1, \ldots, I_n)}$ to $M$. And let $\Sigma_M = T(\Sigma_{C_{(I_1, \ldots, I_n)}})$, its definition is independent from the choice of $T$, because given two linear transformations with upper triangular representation matrices $[T]$ and $[T']$ which send the canonical $\mathcal{O}$-module of type $(I_1, \ldots, I_n)$ to $M$, we have that $[T']^{-1}[T] \in \text{Stab}(I_1, \ldots, I_n)$ and the type $\Sigma_{C_{(I_1, \ldots, I_n)}}$ is left invariant under the action of such group. Thus, $\Sigma_M$ is $\langle \mathcal{M} \rangle$-definable and given $B \models \Sigma_M$, the type $\text{tp}(B/\mathfrak{M})$ is still $\langle \mathcal{M} \rangle$-definable by Proposition 6.21.

**Theorem 6.23.** Let $X$ be a definable subset of $K^n$ and let $p(x) \models x \in X$ be a global type definable over $\langle \mathcal{X} \rangle$.

Let $\mathcal{X} = \mathcal{G}$ be a definable function. Then the $p$-germ of $f$ is coded in $\mathcal{G}$ over $\langle \mathcal{X} \rangle$.

**Proof.** We first assume that $f : X \to B_n(K)/\text{Stab}(I_1, \ldots, I_n)$. Let $B = \text{dcl}(\text{germ}(f, p), \langle \mathcal{X} \rangle) = \text{dcl}(\text{germ}(f, p), \langle \mathcal{X} \rangle) \cap \mathcal{G}$. Suppose that $f$ is $c$-definable, and let $q = \text{tp}(c/B)$ and $\mathcal{Q}$ its set of realizations. Fix some $c' \in \mathcal{Q}$. We denote by $f'$ the function obtained by replacing the parameter $c$ by $c'$ in the formula defining $f$. Let $M$ be a small model containing $Bcc'$.

**Step 1:** For any realization $a \models p(x) \models M$ we have $f(a) = f'(a)$.

Let $a$ be a realization of $p(x) \models M$. Let $u_f(a)(y) = \text{definable type over } f(a)$ given by Proposition 6.22. Given any realization $d = (d_1, \ldots, d_n) = u_f(a)(y)$, $[d_1, \ldots, d_n]$ is a representation matrix for the module $f(a)$. In particular, $f(a) = \rho((I_1, \ldots, I_n)) \in \text{dcl}(\text{definable type over } f(a))$.

Let $\mathcal{X} = \text{tp}(f(c)/B)$ we can find an automorphism $\sigma \in \text{Aut}(\mathfrak{M}/B)$ sending $c$ to $c'$. Then $u_{f'(a)} = \sigma(u_f(a))$, which is a definable type over $f'(a)$. Let $d'$ be a realization of $u_{f'(a)} \models M$. Let $r(x, y) = \text{tp}(a, d'/M)$, then $\sigma(r(x, y)) = r'(x, y) = r(x, y)$ by the $B$-invariance of $r(x, y)$, so $\text{tp}(a, d/M) = \text{tp}(a, d'/M)$. Since $f(a) \in \text{definable type over } f'(a)$ and $f'(a) \in \text{definable type over } f(a)$, we must have that $\text{tp}(a, f(a)/M) = \text{tp}(a, f'(a)/M)$ and since $f$ and $f'$ are both definable over $M$ this implies that $f(a) = f'(a)$.

**Step 2:** The germ $(f, p)$ is coded in the stabilizer sorts $\mathcal{G}$ over $\langle \mathcal{X} \rangle$.

Firstly, note that for any $a \models p(x) \models Bcc'$ it is the case that $f(a) = f'(a)$. In fact, by Step 1 $f(x) = f'(x) \in \text{tp}(a/M)$ and $f(x) = f'(x)$ is a formula in $\text{tp}(a/Bcc')$. Then $f$ and $f'$ both have the same $p$-germ. Since $p(x)$ is definable over $B = \text{dcl}(\mathcal{G})$ the equivalence relation $E$ stating that $f$ and $f'$ have both the same $p$-germ is $B$-definable. Since for any realization $a \models p(x) \models Bcc'$ it is the case that $f(a) = f'(a)$, the class $E(c, c')$ is $B$-invariant, therefore germ $(f, p)$ is definable over $B = \text{dcl}(\mathcal{G})$.

We continue arguing that the statement for $f : X \to B_n(K)/\text{Stab}(I_1, \ldots, I_n)$ is sufficient to conclude the entire result. For each $\Delta \in \mathcal{R}(\mathcal{G})$ there is a canonical isomorphism $\Gamma/\Delta \cong K^{\times}/\mathcal{O}_\Delta$, where $\mathcal{O}_\Delta$ is the valuation ring of the coarsened valuation $v_\Delta$ induced by $\Delta$. The functions whose image lie in $\Gamma/\Delta$ are being considered in the previous case, because $\text{Stab}(\mathcal{O}_\Delta) = \mathcal{O}_\Delta^\times$. Lastly, by Proposition 3.5 any definable function $f : X \to k = \mathcal{O}/M$ can be seen as a function whose image lies in $B_2(K)/\text{Stab}(\mathcal{O}, M)$.

**6.3 Some useful lemmas**

In this subsection we prove several lemmas that will be required to show the coding of finite sets.

**Definition 6.24.** Let $U \subseteq K$ be a 1-torsor, the we say that:

1. it is closed if it is a translate of a submodule of $K$ of the form $a\mathcal{O}$. In this case we define $S_U = v(a) + \mathcal{O}$,

2. it is open if it is either $K$ or a translate of a submodule of the form $aI$ for some $a \in K$, where $I \subseteq \mathcal{I}\cap \mathcal{O}$. In this case we define $S_U = v(a) + I$.

**Notation 6.25.** Through the following lemmas for each $A \subseteq K$ a torsor, we will denote as $p_A(x)$ some $\langle A \rangle$-definable type. Its existence is guaranteed by Proposition 6.21.

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Lemma 6.26. Let $F = \{B_1, \ldots, B_n\}$ be a primitive finite set of 1-torsors. Let $W = \{\{x_1, \ldots, x_n\} \mid x_i \in B_i\}$, and $W^* = \{\{x_1, \ldots, x_n\} \mid x_i \in W\}$ Then there is a $W^*$-definable type $q$ concentrated on $W^*$.

Furthermore, given $b^*$ a realization of $q$ sufficiently generic over a set of parameters $C$, if we take $B$ the finite set coded by $b^*$, then if $b \in B$ is the element that belongs to $B_i$ then it is generic in $B_i$, i.e. it is a relaxation of $\Sigma^\text{gen}_{B_i}(x)$ sufficiently generic over $C$.

Proof. Suppose that each 1-torsor $B_i = c_i + b_i I_i$ for some $I_i \in I$. By transitivity all the balls are of the same type $I \in I$ and for all $1 \leq i, j \leq n$ we have that $v(b_i) = v(b_j)$. Hence, we may assume that each $B_i$ is of the form $c_i + b I$ for some fixed $c_i, b \in K$ and $I \in I$. We argue by cases:

1. Case 1: All the 1-torsors $B_i$ are closed.

For each $i \leq n$, let $p_B(x)$ be the $B_i$-definable type given by Proposition 6.21. Define $r(x_1, \ldots, x_n) = p_{B_i}(x_1) \otimes \cdots \otimes p_{B_i}(x_n)$, which is definable over $'W' = \{\{B_1, \ldots, B_n\}\}$. Let $(a_1, \ldots, a_n) \models r(x_1, \ldots, x_n)$ and let $q = \text{tp}'(\{a_1, \ldots, a_n\})/\text{def}(M)$. This type is well defined independently of the choice of the order, because each $p_{B_i}$ is generically stable, thus it commutes with any definable type by [MIP][Proposition 2.33]. The type $q$ is $W^*$-definable and it is a realization of $\Sigma^\text{gen}(x)$ sufficiently generic over $C$.

2. Case 2: All the 1-torsors $B_i$ are open, i.e. $I \in I \{\emptyset\}$.

Let $S_M = v(b) + S_I = \{v(b) + v(x) \mid x \in I\}$, this is a definable end-segment of $\Gamma$ with no minimal element. Let $r(y)$ be the $S_M$-definable type given by Fact 2.24, extending the partial generic type $\Sigma^{\text{gen}}_{S_M}(y)$. Fix elements $a = \{a_1, \ldots, a_n\} \in W(\text{def})$ and $\delta \models r(y)$, we define $\text{Ca}(a, \delta) = \{C_1(a), \ldots, C_n(a)\}$, where each $C_i(a)$ is the closed ball around $a_i$ of radius $\delta$. Let $q^\delta_a$ be the symmetrized generic type of $C_1(a) \times \cdots \times C_n(a)$ as described in the first case, i.e. we take $\text{tp}'(\{b_1, \ldots, b_n\}/\text{def}(M))$ where $(b_1, \ldots, b_n)$ is a realization of the generically stable type $p_{C_1(a)} \otimes \cdots \otimes p_{C_n(a)}$. Let $q^\delta$ be the definable global type satisfying that $d \models q^\delta$ if and only if there is some $\delta \models r(y)$ and $d \models q^\delta_{\delta}$.

Claim: The type $q^\delta$ does not depend on the choice of $\delta$.

Let $\delta' = \{a_1, \ldots, a_n\} \in W(\text{def})$ and $\delta \models r(y)$. For each $1 \leq n$, $a_i, a'_i \in B_i$ meaning that $a_i - a'_i \in b I$ i.e. $v(a_i - a'_i) \in S_M = v(b) + S_I$ and note that $v(a_i - a'_i) \in \Gamma(M)$.

By construction, $\delta \in S_M$ and $\delta < v(a_i - a'_i)$, thus the closed ball of radius $\delta$ concentrated on $a_i$ is the same closed ball of radius $\delta$ concentrated on $a'_i$. As the set of closed balls $\text{Ca}(a, \delta) = \text{Ca}(a', \delta$) we must have that $q^\delta_a = q^\delta_{\delta'}$, and since this holds for any $\delta \models r(y)$ we conclude that $q^\delta$ does not depend on the choice of $\delta$ and we simply denote it as $q$. This type $q$ is $W^*$-definable and it is a realization of the type $q$.

In both cases the second part of the statement follows directly by the construction of the type $q$.

Notation 6.27. Let $M \subseteq K^n$ be a non-trivial definable $\text{O}$-module and let $Z = \bar{d} + M$ be a torsor. Let $\pi_n = K^n \to K$ be the projection to the last coordinate. Consider the function that describes the fiber in $Z$ of each element at the projection, this is $h_Z(x) = \{y \in K^{n-1} \mid (y, x) \in Z\}$.

Fact 6.28. Let $M$ be a $\text{O}$-submodule of $K^n$ and let $Z = \bar{d} + M$ be a torsor. Then for any $x, z \in \pi_n(Z)$ we have that $h_Z(x) + h_Z(y) = h_Z(x + y)$.

Proof. This is a straightforward computation and it is left to the reader.

Lemma 6.29. Let $n \geq 2$ be a natural number and $M \subseteq K^n$ be a definable $\text{O}$-submodule. Then $'M'$ is interdefinable with $(\pi_n(M), \text{germ}(h_M, \pi_n(M)))$.

Proof. Let $M_1$ and $M_2$ be $\text{O}$-modules of the same type. Suppose that $A = \pi_n(M_1) = \pi_n(M_2)$, and $\text{germ}(h_{M_1}, p_A) = \text{germ}(h_{M_2}, p_A)$. We must show that $M_1$ and $M_2$ are the same $\text{O}$-module.

Claim: Given $y \in A$ there are realizations $c, d$ of the type $p_A(x)$ sufficiently generic over $'M_1', 'M_2'$ such that $y = c - d$.

Proof. Let $c$ be a realization of the type $p_A(x)$ sufficiently generic over $'M_1', 'M_2'$ and $d = c - y$. As $\Sigma^\text{gen}_A(x) \subseteq p_A(x)$, $c \in A$ and $c \notin U$ for any proper subtorsor $U \subseteq A$. Note that $d = \Sigma_A(x)$, in fact $A$ is an $\text{O}$-submodule of $K$ so $d \in A$. And if there is a subtorsor $U \not\subseteq A$ such that $d \notin U$, then $c \in y + U \not\subseteq A$ contradicting that $c \models \Sigma_A^\text{gen}(x)$. For any $\Delta \in R(\Gamma)$, element $z \in K$ and realization $a \models \Sigma^\text{gen}_A(x)$ we have $v_\Delta(z - a) = v_\Delta(a)$. Thus for any $n \in \mathbb{N}$ and $\beta \in \Gamma/\Delta$:

$$v_\Delta(z - a) - \beta \in n(\Gamma/\Delta)$$ if and only if $v_\Delta(a) - \beta \in n(\Gamma/\Delta)$. 

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We conclude that $d$ and $c$ must satisfy the same congruence and coset formulas, because $c$ is generic, $y \in A$ and $v_\Delta(c) = v_\Delta(c - y) = v_\Delta(d)$. Thus $d$ is a realization of $p_A(x)$ sufficiently generic over $\langle M_1^\wedge, M_2^\wedge \rangle$.

Let $y \in A$ take realizations $c, d$ of the type $p_A(x)$ sufficiently generic over $\langle M_1^\wedge, M_2^\wedge \rangle$ such that $y = c - d$ given by the previous Claim. As $\text{germ}(h_{M_1}, p_A) = \text{germ}(h_{M_2}, p_A)$, we have that $h_{M_1}(c) = h_{M_2}(c)$ and $h_{M_1}(d) = h_{M_2}(d)$. By Fact 6.28, $h_{M_1}(y) = h_{M_1}(c) - h_{M_1}(d) = h_{M_2}(c) - h_{M_2}(d) = h_{M_2}(y)$. Consequently, $M_1 = M_2$ as desired.

**Corollary 6.30.** Let $n \geq 2$ be a natural number and $N \subseteq K^n$ be a definable $O$-submodule. Let $Z = \tilde{b} + N$ be a torsor, then $\langle Z \rangle$ is interdefinable with $\langle \langle \pi_n(Z), \text{germ}(h_Z, p_{\pi_n}(Z)) \rangle \rangle$.

**Proof.** We first show that $\langle Z \rangle$ is interdefinable with $\langle \langle \pi_n(Z), \langle N \rangle, \text{germ}(h_Z, p_{\pi_n}(Z)) \rangle \rangle$. Let $Z_1 = \tilde{b}_1 + N$ and $Z_2 = \tilde{b}_2 + N$ torsors, and suppose that $A = \pi_n(Z_1) = \pi_n(Z_2)$. Let $c$ be a realization of the type $p_A(x)$ sufficiently generic over $\langle Z_1 \rangle, \langle Z_2 \rangle$, then $h_{Z_1}(c) = h_{Z_2}(c)$. If $Z_1 = Z_2$, then they must be disjoint because they are different cosets of $N$. If $h_{Z_1}(c) = h_{Z_2}(c)$ then $Z_1 \cap Z_2 = \emptyset$, so $Z_1 = Z_2$.

We continue showing that $N$ is definable over $\langle \langle \pi_n(Z), \text{germ}(h_Z, p_{\pi_n}(Z)) \rangle \rangle$. By Lemma 6.32 it is sufficient to show $\langle \langle \pi_n(N), \text{germ}(h_N, p_{\pi_n(N)}) \rangle \rangle \in dcl^\forall(\langle \langle \pi_n(Z), \text{germ}(h_Z, p_{\pi_n(Z)}) \rangle \rangle)$. We first observe that $\pi_n(N) = \{ y - y' \mid y, y' \in \pi_n(Z) \}$, so $\pi_n(N) \in dcl^\forall(\langle \pi_n(Z) \rangle)$. Let $c$ be a realization of $p_{\pi_n(N)}(x)$ sufficiently generic over $\langle Z_1 \rangle, \langle Z_2 \rangle$, and let $d$ be a realization of $p_{\pi_n(Z)}(x)$ sufficiently generic over $\langle Z_1 \rangle, \langle Z_2 \rangle$. Then $d + c$ are independent generic elements of $\pi_n(Z)$, and $h_N(c) = h_Z(d + c) - h_Z(d)$, by Fact 6.28 $\text{germ}(h_{\pi_n(N)}, p_{\pi_n(N)}) \in dcl^\forall(\langle \pi_n(Z) \rangle), \text{germ}(h_Z, p_{\pi_n(Z)}))$, as desired.

### 6.4 Some coding lemmas

**Lemma 6.31.** Let $A$ be a definable $O$-lattice in $K^n$ and $U \in K^n/A$ be a torsor. Let $B$ be the $O$-lattice in $K^{n+1}$ that is interdefinable with $U$ (given by Proposition 3.5). Then there is a $U$-definable injection:

$$f = \begin{cases} \text{red}(U) & \to \text{red}(B) \\ b + MA & \to (1, b) + MB. \end{cases}$$

**Proof.** We recall how the construction of $B$ was achieved. Given any $\tilde{d} \in U$, we can represent $B = \left[ \begin{array}{c} 1 \\ \tilde{d} \end{array} \right] O + A_2$, where $A_2 = \{0\} \times A$. This definition is independent from the choice of $\tilde{d}$. We consider the $U$-definable injection $\phi = U \to B$ that sends each element $\tilde{b}$ to $\left[ \begin{array}{c} 1 \\ \tilde{b} \end{array} \right]$. The interpretable sets $\text{red}(U)$ and $\text{red}(B) = B/MB$ are both $U$-definable. It follows by a standard computation that for any $b, b' \in U$ $b - b' \in MA$ if and only if $\left[ \begin{array}{c} 1 \\ b - b' \end{array} \right] \in MB$. This shows that the map $f$ is a $U$-definable injection.

**Lemma 6.32.** Let $F$ be a primitive finite set of 1-torsors, then $F$ can be coded in $G$.

**Proof.** If $|F| = 0$ or $|F| = 1$ the statement clearly follows, so we suppose that $|F| > 1$. By primitivity all the torsors in $F$ are translates of the same $O$-submodule of $K$. Indeed, there are some $b \in K$ and $I \subseteq \mathcal{I}$ that for any $t \in F$ there is some $a_i \in K$ satisfying $t = a_i + I$. Moreover, there is some $\delta \in v(b)$ such that for any two different torsors $t, t' \in F$ if $x \in t$ and $y \in t'$ then $v(x - y) = \delta$. Let $T = \bigcup_{t \in F} t$. We define

$$J_F = \{ Q(x) \in K[x] \mid v(Q(x)) \text{ has degree at most } |F| \text{ and for all } x \in T, v(Q(x)) = v(b) + (|F| - 1)\delta + S_I \}.$$  

**Step 1:** $J_F$ is interdefinable with $F'$.

Observe that $J_F$ is definable over $F'$, because $v(b), \langle T', \delta \rangle$ lie in $dcl^\forall(F')$. Hence, it is sufficient to prove that we can recover $F'$ from $J_F$. For this we will show that given a monic polynomial $Q(x)$, we have that $Q(x) \in J_F$ if and only if $Q(x)$ has $|F|$-different roots in $K$, each with multiplicity one $Q(x) \in J_F$ and each of the roots lies in exactly one of the torsors $t \in F$. Let $Q(x)$ be a monic polynomial with exactly $|F|$-different roots $\{\beta_1, \ldots, \beta_{|F|}\}$ in $K$, and suppose that each of them lies in a unique torsor $t \in F$. For any element $x \in T = \bigcup_{t \in F} t$, there is some $t \in T$ and $i \leq |F|$ such that $x, \beta_i \in t$ so $v(x - \beta_i) = v(b) + S_I$. For any other index
\[ j \neq i, \text{ we have that } x \text{ and } \beta_j \text{ lie in different torsors, thus } v(x - \beta_j) = \delta. \] Summarizing we have:
\[ v(Q(x)) = v\left( \prod_{k \in |F|} (x - \beta_k) \right) = v(x - \beta_i) + \sum_{j \neq i} v(x - \beta_j) \in v(b) + (|F| - 1)\delta + S_I. \]
Consequently, \( Q(x) \in J_F \). For the converse, let \( Q(x) \in J_F \) be a monic polynomial with exactly \(|F|\)-different roots in \( K \) say \( B = \{ \beta_1, \ldots, \beta_{|F|}\} \). We aim to prove that for each torsor \( t \in F \), there is exactly one of these roots in \( t \).

We first show that given any torsor \( t \in F \), there is some root \( \beta \in B \) such that for all elements \( x \in t \), \( v(x - \beta) > \delta \).

We argue by contradiction, so let \( t \in F \) and assume that there is no root \( \beta \in B \) such that \( v(x - \beta) > \delta \) for all \( x \in t \). Then for each element \( x \in t \) we have:
\[ v(Q(x)) = v\left( \prod_{i \in |F|} (x - \beta_i) \right) = \sum_{i \in |F|} v(x - \beta_i) \leq |F|\delta. \]

In this case we would have that \( Q(x) \notin J_F \), because \( |F|\delta \neq v(b) + (|F| - 1)\delta + S_I \) since \( \delta \neq v(b) + s_I \).

Therefore, after relabeling we have \( F = \{ t_i \mid i \leq |F| \} \) and for any \( x \in t_i \), \( v(x - \beta_i) > \delta \). In particular, if \( x \in t_i \) and \( j \neq i \) then \( v(x - \beta_j) = \delta \). So for any \( x \in t_i \):
\[ v(Q(x)) = v\left( \prod_{k \in |F|} (x - \beta_k) \right) = v(x - \beta_i) + \sum_{j \neq i} v(x - \beta_j) = v(x - \beta_i) + (|F| - 1)\delta. \]

We must have that \( v(x - \beta_i) \in v(b) + S_I \), because \( Q(x) \in J_F \). Consequently \( \beta_i \in t_i \), as desired.

**Step 2:** \( F \) admits a code in the geometric sorts.

By the first step \( F \) is interdefinable with \( J_F \). The latter one is an \( O \)-module, so by Lemma 5.5 it admits a code in the stabilizer sorts \( G \).

**Lemma 6.33.** Let \( F \) be a primitive finite set of 1-torsors such that \( |F| > 1 \). There are \( F \)-definable \( O \)-lattice \( s \subseteq K^2 \) and an \( F \)-definable injective map \( g : F \to VS_{k,s} \).

**Proof.** Let \( F \) be a primitive finite set of 1-torsors. By primitivity, there is some \( d \in K \) and \( I \in \mathcal{I} \) such that for any \( t \in F \) there is some \( a_t \in K \) satisfying \( t = a_t + dI \). Moreover, there is some \( \delta \in \Gamma \setminus \{ v(d) + S_I \} \) such that for any pair of different torsors \( t, t' \in F \), and \( x \in t, y \in t' \) we have \( v(x - y) = \delta \). Let \( T = \bigcup t \), and take elements \( c \in T \) and \( b \in K \) such that \( v(b) = \delta \), and consider \( U = c + bO \). Then \( U \) is the smallest closed 1-torsor that contains all the elements of \( F \). Note that \( U \) is definable over \( F \). Let \( h \) be the map sending each element of \( F \) to the unique element that contains it in \( red(U) \). By construction, such a map must be injective and it is \( F \)-definable. Let \( s \) be the \( O \)-lattice in \( K^2 \), whose code is interdefinable with \( U \). By Lemma 6.31 there is a \( s^3 \)-definable injection \( f = \text{red}(U) \to \text{red}(s) \). Let \( g = f \circ h \), the composition map \( g = F \to VS_{k,s} \) satisfies the required conditions.

**Lemma 6.34.** Let \( F \) be a finite set of 1-torsors and let \( f : F \to G \) be a definable function. Suppose that \( F \) is primitive over \( F \), then:

1. for any set of parameters \( C \) if \( f(F) \subseteq VS_{k,C} \) then \( f \) is coded in \( G \) over \( C \),
2. if \( f(F) \subseteq K \) then \( f \) is coded in \( G \),
3. if \( f(F) \) is a finite set of 1-torsors of the same type \( I \in \mathcal{I} \). Then \( f \) is coded in \( G \).

**Proof.** In all the three cases, by primitivity of \( F \) over \( F \), \( f \) is either constant or injective. If it is constant and equal to \( c \), the tuple \( (F, c) \) is a code for \( f \). By Lemma 6.32 \( F \) admits a code in \( G \), so \( (F, c) \) lies in the stabilizer sorts. In the following arguments we assume that \( f \) is an injective function and that \( |F| \geq 2 \).

1. By Lemma 6.32 \( F^* \in G \). Let \( s \) be the \( O \)-lattice of \( K^2 \) and \( g : F \to VS_{k,C^*} \) the injective map given by Lemma 6.33. Both \( s \) and \( g \) are \( F \)-definable. Let \( F^* = g(F) \), by Theorem 6.17 the map \( f \circ g^{-1} = F^* \subseteq VS_{k,C^*} \to VS_{k,C^*} \), can be coded in \( G \). Hence, the tuple \( (f \circ g^{-1}, d) \) is a code in \( G \) of \( f \) over \( C \), as \( g \) is a \( F \)-definable bijection.

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2. Let $D = f(F) \subseteq K$, this is a finite set in the main field so it can be coded by a tuple $d$ of elements in $K$. By primitivity there is some $\delta \in \Gamma$ such that for any pair of different elements $x, y \in D$ $v(x - y) = \delta$. Let $b \in K$ be such that $v(b) = \delta$, take $x \in D$ and let $U = x + b\mathcal{O}$ be the closed 1-torsor. The elements of $D$ all lie in different classes of $\text{red}(U)$ and let $g = D \to \text{red}(U)$ be the definable map sending each element $x \in D$ to the unique element in $\text{red}(U)$ that contains $x$. Both $U$ and $g$ are $\langle D \rangle$-definable, and therefore $d$-definable. By Proposition 3.5, there is an $\mathcal{O}$-lattice $s \subseteq K^2$, whose code interdefinable with $\langle U \rangle$. Let $h = \text{red}(U) \to \text{red}(s)$ the $\langle U \rangle$- definable injective map given by Lemma 6.31. Both $U$ and $h$ are $d$-definable. By (1) of this statement the function $h \circ g \circ f : F \to VS_{k, r,s}$ can be coded in $\mathcal{G}$. Since $f$ and $h \circ g \circ f$ are interdefinable over $d$, the statement follows.

3. Let $D = f(F)$ then $D$ must be a primitive set of 1-torsors because $F$ is primitive over $\langle f \rangle$. By Lemma 6.32, we may assume $\langle D \rangle$ is a tuple in the stabilizer sorts. Let $s \subseteq K^2$ and $g = D \to \text{red}(s) \subseteq VS_{k, r,s}$, the injective map given by Lemma 6.31. Both $s$ and $g$ are $\langle D \rangle$-definable, thus they are $d$-definable. By part (1) of this statement the composition $g \circ f$ can be coded in $\mathcal{G}$, as $g$ is a $d$-definable bijection the tuple $(\langle g \circ f \rangle, d)$ is a code of $f$ over the empty set.

\[ \square \]

### 6.5 Coding primitive finite sets of $\mathcal{G}$

We start by recalling some terminology from previous sections for sake of clarity.

**Notation 6.35.** Let $M \subseteq K^n$ be an $\mathcal{O}$-module, and $(I_1, \ldots, I_n) \in \mathcal{I}$ such that $M \cong \oplus_{i \leq n} I_i$. For any torsor $Z = d + M \in K^n/M$ we say that $Z$ is of type $(I_1, \ldots, I_n)$ and it has complexity $n$. We denote by $\pi_n = K^n \to K$ the projection to the last coordinate and for a torsor $Z = d + M \in K^n/M$ we write as $A_Z = \pi_n(Z)$. We recall as well the notation introduced in 6.27 for the function that describes the fiber in $Z$ of each element at the projection, this is $h_Z(x) = \{ y \in K^{n-1} | (y, x) \in Z \}.$

We start by considering the following observation, that states that finite sets of modules can be coded in the stabilizer sorts.

**Observation 6.36.** Let $F$ be a finite set of codes of $\mathcal{O}$-submodules of $K$. Then:

1. $F$ admits a code in the stabilizer sorts $\mathcal{G}$.
2. Any definable function $f : F \to \mathcal{G}$ admits a code in the stabilizer sorts $\mathcal{G}$.

**Proof.** Any finite set of $\mathcal{O}$-submodules is already coded in $\mathcal{G}$ as there is a definable order on $F$. In fact, if $M$ and $N$ are $\mathcal{O}$-submodules of $K$ then either $M \subseteq N$ or $N \subseteq M$. For the second part of the statement, let $< M_i | i \leq m >$ be an enumeration of $F$ such that $M_1 \subseteq \cdots \subseteq M_m$. Let $c_i = f(M'_i) \in \mathcal{G}$, then $\langle f \rangle$ is interdefinable with the sequence of elements $(\langle M'_i, c_i \rangle)_{1 \leq i \leq m},$ as desired.

We continue proving that primitive finite subsets of torsors can be coded in the stabilizer sorts.

**Theorem 6.37.** Let $n \geq 2$ and $F$ be a primitive finite set of torsors of complexity $n$. Let $S$ a finite set of 1-torsors. Then:

1. $(1)_n$ $F$ can be coded in $\mathcal{G}$.
2. $(2)_n$ Let $f : S \to F$ be a definable injective function and assume that $S$ is a primitive set over $\langle f \rangle$, then $f$ admits a code in $\mathcal{G}$.

**Proof.** We argue by induction on the complexity of the torsors in $F$ to prove both statements $I_n$ and $II_n$. The base case follows by Lemma 6.33 and Lemma 6.34 (3). We suppose that both $I_n$ and $II_n$ hold, and prove $I_{n+1}$ and $II_{n+1}$. By primitivity of $F$ the projections to the last coordinate are either all equal or all different. We argue by cases:

1. **Case 1:** All the projections are equal, i.e. $A = A_Z$ for all $Z \in F$.
Proof. First, we prove $I_{n+1}$. For each $x \in A$, the set of fibers $\{h_Z(x) \mid Z \in F\}$ is a primitive finite set of torsors of lower complexity, so by the induction hypothesis $I_n$ it admits a code in the stabilizer sorts. By compactness we can uniformize such codes, and we can define the function $g : A \to \mathcal{G}$ by sending $x \mapsto \{h_Z(x) \mid Z \in F\}$.

We note that $g$ is \(^F\)-definable. Let $p_A(x)$ be the definable type over \(^A\) given by Proposition 6.21. By Theorem 6.23 the germ of $g$ over $p_A$ can be coded in $\mathcal{G}$ over \(^A\). By Corollary 6.30 for any $Z \in F$ the code \(^Z\) is interdefinable with the tuple \(^{\{A\}, \text{germ}(h_Z,p_A)}\), then \(^F\) is interdefinable with \(^{\{A\}, \{\text{germ}(h_Z,p_A) \mid Z \in F\}}\).

Claim: \(^{\{\text{germ}(h_Z,p_A) \mid Z \in F\}}\) is interdefinable with \(^{\{A\}}\).

Proof. We first prove that \(^{\{\text{germ}(h_Z,p_A) \mid Z \in F\}}\) is interdefinable with \(^{\{A\}}\). Let $\sigma \in \text{Aut}(\mathcal{G})/^{\{\text{germ}(h_Z,p_A) \mid Z \in F\}}$, \(^{\{A\}}\), we want to show that $\sigma(\text{germ}(h_Z,p_A)) = \text{germ}(\sigma(g),p_A) = \text{germ}(g,p_A)$. Let $B$ the set of all the parameters required to define all the objects that have been mentioned so far. It is therefore sufficient to argue that for any realization $c$ of $p_A(x)$ sufficiently generic over $B$ we have $\sigma(g)(c) = g(c)$, where $\sigma(g) : A \to \mathcal{G}$ is the function given by sending $x \mapsto \{h_{\sigma(Z)}(x) \mid Z \in F\}$.

Note that $\sigma(\{\text{germ}(h_Z,p_A) \mid Z \in F\}) = \{\text{germ}(\sigma(Z),p_A) \mid Z \in F\} = \{\text{germ}(h_Z,p_A) \mid Z \in F\}$, because $\sigma(\{\text{germ}(h_Z,p_A) \mid Z \in F\}) = \{\text{germ}(h_Z,p_A) \mid Z \in F\}$. As a result, for any realization $c$ of $p_A(x)$ sufficiently generic over $B$ we must have that \(^{\{h_Z(c) \mid Z \in F\}}\) = \(^{\{\text{germ}(h_Z,p_A) \mid Z \in F\}}\). Therefore, we conclude that $\sigma(\{\text{germ}(h_Z,p_A) \mid Z \in F\}) = \{\text{germ}(h_Z,p_A) \mid Z \in F\}$, as desired.

For the converse, let $\sigma \in \text{Aut}(\mathcal{G})/^{\{A\}, \text{germ}(g,p_A)}$, we want to show that $\sigma(\{\text{germ}(h_Z,p_A) \mid Z \in F\}) = \{\text{germ}(h_Z,p_A) \mid Z \in F\}$. Let $c$ be a realization of $p_A(x)$ sufficiently generic over $B$ by hypothesis $g(c) = \sigma(g)(c)$. Then:

$g(c) = \{h_Z(c) \mid Z \in F\} = \{h_{\sigma(Z)}(c) \mid Z \in F\} = \sigma(g(c))$.

Therefore, for each $Z \in F$ there is some $Z' \in F$ such that $h_Z(c) = h_{\sigma(Z')}(c)$ and this implies that $\text{germ}(h_Z,p_A) = \text{germ}(h_{\sigma(Z')},p_A)$. Thus

$\sigma(\{\text{germ}(h_Z,p_A) \mid Z \in F\}) = \{\text{germ}(\sigma(Z),p_A) \mid Z \in F\} = \{\text{germ}(h_Z,p_A) \mid Z \in F\}$. We conclude that $\sigma(\{\text{germ}(h_Z,p_A) \mid Z \in F\}) = \{\text{germ}(h_Z,p_A) \mid Z \in F\}$, as desired.

Hence, $F$ is coded by the tuple \(^{\{A\}, \text{germ}(g,p_A)}\) which is a sequence of elements in $\mathcal{G}$. We continue proving $I_{n+1}$. Let $f = S \to F$ be a definable injective map. For each $x \in A$ we define the function $g_x : S \to \mathcal{G}$ by sending $t \mapsto h_{f(t)}(x)$.

This is the function that sends each torsor $t \in S$ to the fiber at $x$ of the torsor $f(t) \in F$. [See Figure 1.]

**Figure 1**

```
|         |         |         |
|---------|---------|---------|
| h_{f(t)}(x) | h_{f(t')}(x) | h_{f(t'')}(x) |
|         |         |         |
| f(t)   | f(t')  | f(t'') |
| A      | A      | A      |
| t      | t'     | t''    |
```

By the induction hypothesis $I_n$ for each $x \in A$, the function $g_x$ can be coded in $\mathcal{G}$ because its range is of lower complexity. By compactness we can uniformize such codes, so we can define the function $r : A \to \mathcal{G}$ by sending $x \mapsto g_x$.

By Theorem 6.23 the germ of $r$ over $p_A$ can be coded in $\mathcal{G}$ over \(^A\). By Lemma 6.32 the set $S$ admits a code \(^S\) in the stabilizer sorts.

Claim: \(^f\) is interdefinable with \(^{\{A\}, \text{germ}(r,p_A)}\), and the later is a sequence of elements in $\mathcal{G}$. It is clear that \(^{\{A\}, \text{germ}(r,p_A)}\) \epsilon \text{dcl}(f). We want to show that \(^f\) \epsilon \text{dcl}(f)\(^{\{A\}}\).

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Let $\sigma \in \text{Aut}(\mathfrak{M}, A, S, \text{germ}(r, p_A))$. By Corollary 6.30 for each torsor $Z \in F = \{f(t) \mid t \in S\}$, the code $\langle Z \rangle$ is being identified with the tuple $\langle A', \text{germ}(h_Z, p_A) \rangle$. Thus, the function $f$ is interdefinable over $\langle A' \rangle$ with the function: $f': S \to G$ that sends $t \mapsto \text{germ}(h_{f(t)}, p_A)$. So, it is sufficient to argue that $\sigma(\langle f' \rangle) = \langle f' \rangle$. Let $B$ be the set of parameters required to define all the objects that have been mentioned so far. For any realization $c$ of $p_A(x)$ sufficiently generic over $B$ we must have that $r(c) = \sigma(r)(c)$. Because $\text{germ}(r, p_A) = \sigma(\text{germ}(r, p_A)) = \sigma(\text{germ}(r), p_A)$. By definition, $r(c) = \langle g_c \rangle$ and $\sigma(r)(c) = \langle \sigma(g)_c \rangle$, where $\sigma(g)_c: S \to G$ is the function that sends $t \mapsto h_{\sigma(f(t))}(c)$. For any torsor $t \in S$ there must be a unique element $t' \in S$ such that $\sigma(t') = t$ and $h_{\sigma(f(t))}(c) = h_{\sigma(f)}(c(t'))$, as $g_c = \sigma(g)_c$. The later implies that $\text{germ}(h_{f(t)}, p_A) = \text{germ}(h_{\sigma(f)}(c(t')), p_A)$. We conclude that $\sigma(t', \text{germ}(h_{f(t)}, p_A)) = (t, \text{germ}(h_{f(t)}, p_A))$ meaning that $\sigma$ is acting as a bijection among the elements in the graph of $f'$. Therefore, $\sigma(\langle f' \rangle) = \langle f' \rangle$, as desired. 

2. Case 2: All the projections are different i.e. $A_Z \neq A_{Z'}$ for all $Z \neq Z' \in F$.

Proof. We first show the statement for $I_{n+1}$. To simplify the notation fix some enumeration of the projections $\{A_Z \mid Z \in F\}$ say $\{A_1, \ldots, A_n\}$. Let $W = \{\{x_1, \ldots, x_n\} \mid x_i \in A_i\}$, such set is independent from the choice of the enumeration. Each set $\{x_1, \ldots, x_n\} \in W$ admits a code in the home sort $K$, because fields uniformly code finite sets. We denote by $W^* = \{\langle x_1, \ldots, x_n \rangle \mid \{x_1, \ldots, x_n\} \in W\}$, i.e. the set of all these codes. For each $x^* \in W^*$, we define the function $f_{x^*}: S \to K$ that sends $A_Z \to x_Z$, where $x_Z$ is the unique element in the set coded by $x^*$ that belongs to $A_Z$. Let $l_{x^*}: S \to G$ the function given by sending $A_Z \to h_Z(f_{x^*}(A_Z))$.

This map sends the projection $A_Z$ to the code of the fiber in the module $Z$ at the point $x_Z$, which is the unique point in the set coded by $x^*$ that belongs to $A_Z$ [See Figure 2].

**Figure 2**

For each $x^* \in W^*$ the functions $f_{x^*}$ and $l_{x^*}$ can be coded in $G$, this follows by Lemma 6.34 (2) and the induction hypothesis $I_{n-1}$. By compactness we can uniformize all such codes, so we can define the function $g: W^* \to G$ by sending $x^* \to \langle f_{x^*}^-, l_{x^*}^- \rangle$.

By Lemma 6.26 there is some $\langle W^* \rangle$-definable type $q(x^*) \vdash x^* \in W^*$. The second part of this Lemma also guarantees that given $d'$ a generic realization of $q$ over a set of parameters $B$, if we take $Y$ the set coded by $d'$ and $b$ is the element in $Y$ that belongs to $A_Z$ then $b$ is a realization of the type $p_{A_Z}(x)$ which is also generic over $B$. By Theorem 6.23 the germ of $g$ over $q$ can be coded in the stabilizer sorts $G$ over $\langle W^* \rangle \equiv \text{dcl}_{\mathfrak{M}}(\langle S \rangle)$. By Lemma 6.33 we may assume $\langle S \rangle \in \mathcal{G}$.

Claim: $(\text{germ}(g,q), \langle S \rangle) \in \mathcal{G}$ is interdefinable with $\langle F \rangle$.

It is clear that $(\text{germ}(g,q), \langle S \rangle) \in \text{dcl}_{\mathfrak{M}}(\langle F \rangle)$. For the converse, let $\sigma \in \text{Aut}(\mathfrak{M}/\text{germ}(g,q), \langle S \rangle)$ want to show that $\sigma(F) = F$. By Corollary 6.30 the code of each torsor $Z \in F$ is interdefinable with the pair $(A_Z, \text{germ}(h_Z, p_A))$. Hence it is sufficient to argue that:

$$\sigma(\{ (A_Z, \text{germ}(h_Z, p_A)) \mid Z \in F \}) = \{ (A_Z, \text{germ}(h_Z, p_A)) \mid Z \in F \}.$$ 

We have that $\sigma(\langle W^* \rangle) = \langle W^* \rangle$ because $\sigma(S) = S$. Therefore $\sigma(\text{germ}(g,q)) = \text{germ}(\sigma(g), q) = \text{germ}(g,q)$. Let $B$ be the set of parameters required to define all the objects that have been mentioned so far. For
any realization $d^*$ of the type $q$ sufficiently generic over $B$ we have $g(d^*) = \sigma(g)(d^*)$, where $\sigma(g)$ is
the function sending an element $x^*$ in $W^*$ to the tuple $(\sigma((f)_{x^*}), \sigma((l)_{x^*}))$. As a result, $\langle f_{d^*}, l_{d^*} \rangle = (\langle \sigma(f)_{d^*} \rangle, \langle \sigma((l)_{d^*}) \rangle)$. Let $D = \{d_t \mid t \in T\}$ be the set of elements coded by $d^*$. The action of $\sigma$ is just permuting the elements of the graph $f_{d^*}$, because $\langle f, l \rangle = (\langle \sigma(f) \rangle, \langle \sigma(l) \rangle)$.

The function $f_{d^*} : S \rightarrow K$ sends a 1-torsor $t$ to the unique element $d_t \in D$ such that $d_t \in t$, then $\sigma$ is sending the pair $(t, d_t)$ to $(\sigma(t), d_{\sigma(t)})$, where $d_{\sigma(t)}$ is a realization $p_{\sigma(t)}(x)$ sufficiently generic over $B$. By assumption, we also have that $\langle l_{d_t} \rangle = \langle \sigma(l)_{d^*} \rangle$, thus the action of $\sigma$ is a bijection among the elements of the graph of $l_{d^*}$. Consequently, for any $t \in T$ there is some unique $t' \in S$ such that $\sigma((t, h_{B}(d_t))) = (\sigma(t), h_{c}(Z)(d_{\sigma(t)})) = (t', h_{c}(Z)(d_{\sigma(t)}))$. Thus $\sigma(Z)$ is a torsor whose projection is $t \in S$, and $d_t \in t$ is a realization of the type $p_t(x)$ sufficiently generic over $B$. As a result, $(\sigma(t, germ(h_{Z}, p_t))) = (t', germ(h_{c}(Z), p_{t'}))$. We conclude that:

$$\sigma\{(A_{Z}, germ(h_{Z}, p_{A_{Z}})) \mid Z \in F\} = \sigma\{(t, germ(h_{Z}, p_t)) \mid t \in S\}$$

$$= \{(t', germ(h_{c}(Z), p_{t'})) \mid t' \in S\} = \{(A_{Z}, germ(h_{c}(Z), p_{A_{Z}})) \mid Z \in F\},$$

as desired.

We continue proving $II_{n+1}$. Let $f : S \rightarrow F$ be a definable injective function where $S$ is a finite set of
1-torsors primitive over $F$. We consider the definable function that sends each torsor $t \in S$ to the code
of the projection into the last coordinate of the torsor $f(t) \in F$, more explicitly:

$$\pi_{n+1} \circ f : \begin{cases} S & \rightarrow \mathcal{G} \\ t & \mapsto \pi_{n+1}(f(t)) \end{cases}.$$

By Lemma (3) 6.34, $\pi \circ f$ can be coded in $\mathcal{G}$, and by $I_{n+1}$ the finite set $F$ is coded by a tuple in $\mathcal{G}$. It is sufficient to show the following claim:

Claim: $\langle f \rangle$ is interdefinable with the tuple $(\langle \pi \rangle, \langle F \rangle)$, which is a tuple in the stabilizer sorts.

Clearly $(\langle \pi \rangle, \langle F \rangle) \in \text{dcl}(\langle \pi \rangle)$. Note that $\langle S' \rangle \in \text{dcl}(\langle \pi \rangle)$ because $S$ is the domain of the given function, we can describe the function $f : S \rightarrow F$ by sending $t \mapsto \langle Z \rangle$, where $Z_t$ is the unique torsor in $F$ such that $\langle \pi_{n+1}(Z) \rangle = \langle \pi \circ f \rangle(t)$. We conclude that $\langle f \rangle \in \text{dcl}(\langle \pi \rangle)$. As a consequence, $f$ is coded in $\mathcal{G}$ by the tuple $(\langle \pi \rangle, \langle F \rangle)$.

$\square$

Theorem 6.38. Let $F$ be a primitive finite subset of $\mathcal{G}$, then $F$ admits a code in $\mathcal{G}$.

Proof. Let $F$ be a finite set of elements from the stabilizer sorts. By primitivity all the elements of $F$ lie in the same sort. If $F$ is either contained in the main field or the residue field, then $F$ is coded by a tuple of elements in the same field, because fields code uniformly finite sets. If $F$ is a finite set of $\mathcal{O}$-modules of the same type $I \in \mathcal{I}$ the statement follows immediately by Observation 6.36. If $F \in B_n(K)/\text{Stab}(t_1, \ldots, t_n)$ for some $n \geq 2$, by Theorem 6.37 $F$ admits a code in $\mathcal{G}$. (Indeed, $\mathcal{O}$-modules are in particular torsors).

$\square$

Proposition 6.39. Let $F$ and $P$ be a finite sets of torsors and $f : F \rightarrow P$ be a definable function. Suppose that $F$ is primitive over $\langle F \rangle$ then $f$ can be coded in the stabilizer sorts $\mathcal{G}$.

Proof. We proceed by induction on the complexity of the torsors in $F$. The base case follows directly by Theorem 6.37 ($II_m$ for all $m \in \mathbb{N}$). We assume the statement for any set of torsors $F$ with complexity $n$ and we prove it for complexity $n + 1$. By primitivity $f$ is either constant or injective, if it constant and equal to some $c$ then $\langle f \rangle$ is interdefinable with the tuple $(\langle F \rangle, c)$. The code of $F$ can be found in $\mathcal{G}$ by Theorem 6.37 part 1, and $c$ is interdefinable with an element in $\mathcal{G}$ by Proposition 3.5. Thus we may assume $f$ to be injective. By primitivity all the projections into the last coordinate are either equal or all distinct. We argue by cases:

1. Case 1: All the projections are equal and let $A = A_{f}$ for all $Z \in F$.

For each $x \in A$, let $I_x = \langle \langle h_{Z}(x) \rangle \mid Z \in F \rangle$ which describes the set of fibers at $x$. We define $B = \{x \in A \mid |I_x| = |F|\}$ which is an $\langle F \rangle$-definable set. For each $y \in B$ we consider the map $g_y : I_y \rightarrow P$ defined by sending $h_{Z}(y) \mapsto f(Z)$, which is the function that sends each fiber to the image of the torsor under $f$. By the induction hypothesis we can find a code $\langle g_y \rangle$ in $\mathcal{G}$, and by compactness we can uniformize such codes. Therefore we can define the function: $r : B \rightarrow \mathcal{G}$ by sending $y \mapsto \langle g_y \rangle$.

Let $p_{A}(x)$ be the generic type definable over $\langle A \rangle$ given by Proposition 6.21. By Corollary 6.30 we must
have that \( p_A(x) \vdash x \in B \). In fact, if we fix a realization of the generic type \( c \) of \( p_A(x) \) sufficiently generic over \( \{ Z \mid Z \in F \} \), if we take \( Z \neq Z' \in F \) then the fibers \( h_Z(c) \) and \( h_{Z'}(c) \) must be different. By Theorem 6.23 the germ of \( r \) over \( p_A(x) \) can be coded in \( G \) over \( A' \).

Claim: 'f' is interdefinable with \( \langle \text{germ}(r, p_A), F' \rangle \) which is a tuple in the stabilizer sorts \( G \).

Clearly \( \langle \text{germ}(r, p_A), F' \rangle \in \text{dcl}^q(\langle f' \rangle) \). We will argue that for any automorphism \( \sigma \in \text{Aut}(\mathcal{M} F \text{'}, \text{germ}(r, p_A)) \) we have \( \sigma(\langle f' \rangle) = \langle f' \rangle \). As each torsor \( Z \in F \) is being identified with the tuple \( \langle A', \text{germ}(h_Z, p_A) \rangle \), 'A' \( \in \text{dcl}^q(\langle F' \rangle) \) then it is sufficient to argue that:

\[
\sigma(\langle \text{germ}(h_Z, p_A), f(Z) \rangle \mid Z \in F) = \langle \text{germ}(h_Z, p_A), f(Z) \rangle \mid Z \in F\).
\]

For any \( Z \in F \) there is a unique torsor \( Z' \in F \) such that \( \sigma(Z') = Z \), because \( \sigma(\langle f' \rangle) = \langle f' \rangle \). Let \( D \) be the set of parameters required to define all the objects that have been mentioned so far. For any realization \( c \) of the type \( p_A(x) \) sufficiently generic over \( D \) we have \( r(c) = \sigma(r)(c) \), because \( \sigma(\text{germ}(r, p_A)) = \text{germ}(\sigma(r), p_A) \). Consequently \( r(c) = \sigma(g) = \sigma(g) = \sigma(\sigma(c)) = \sigma(g) \). In particular, \( h_{\sigma(Z)}(c) = h_{Z}(c) \) which implies that \( \text{germ}(h_{\sigma(Z)}, p_A) = \text{germ}(h_Z, p_A) \). In addition, \( \sigma(f)(\sigma(Z')) = \sigma(g)(\sigma(c)) = g(c)(h_Z(c)) = f(Z) \). Therefore,

\[
\sigma(\langle \text{germ}(h_Z, p_A), f(Z) \rangle \mid Z \in F) = \langle \text{germ}(h_Z, p_A), f(Z) \rangle \mid Z \in F, \text{as desired.}
\]

2. Case 2: All the projections are different.

By Theorem 6.37 (Part 1) we can find a code in the stabilizer sorts for \( F \), and \( F' \) \( \in \text{dcl}^q(\langle f' \rangle) \) as it is the domain of this function. We can define the set \( S = \{ A_Z \mid Z \in F \} \) and define the function \( g : S \to F \) by sending \( A_Z \to Z \), where \( Z \) is the unique torsor in \( F \) satisfying that \( \pi_0(Z) = A_Z \). Clearly \( g \) is a \( \langle f' \rangle \)-definable bijection. We consider the map \( f \circ g : S \to P \) that sends \( A_Z \to f(Z) \). By the second part of Theorem 6.37, the function \( f \circ g \) admits a code in the stabilizer sorts.

Claim: 'f' is interdefinable with the tuple \( \langle f \circ g', F' \rangle \) which is a tuple in the stabilizer sorts.

It is clear that \( \langle f \circ g', F' \rangle \in \text{dcl}^q(\langle f' \rangle) \). For the converse note that \( S \) is definable over \( \langle f \circ g \rangle \) as it is its domain. As \( F \) is given, we can define the function \( \pi : F \to S \) that sends \( Z \to A_Z \). This is the map that sends each torsor to its projection into the last coordinate. We observe that \( f = (f \circ g) \circ \pi \), in fact \( f(Z) = (f \circ g)(A_Z) \). So \( \langle f' \rangle \in \text{dcl}^q(\langle f \circ g', F' \rangle) \).

\[\square\]

**Theorem 6.40.** Let \( F \subseteq G \) be a finite set of size at least 2. Let \( f : F \to G \) be a definable function, and we suppose that \( F \) is primitive over \( f' \), then \( f \) admits a code in the stabilizer sorts.

**Proof.** Let \( F \subseteq G \), by primitivity all the elements of \( F \) lie in the same sort. By Observation 6.36, \( f \) can be coded in the stabilizer sorts \( G \) if the elements of \( F \) are codes of \( O \)-submodules of \( K \). If \( F \subseteq K \), the residue field, then \( F \) is interdefinable with the code of a finite set of 1-torsors of type \( M \) and the statement follows by Proposition 6.39. If \( F \subseteq B_n(K)/\text{Stab}(I_1, \ldots, I_n) \) for some \( n \geq 2 \), the statement follows by Proposition 6.39, because \( O \)-modules are torsors. It is therefore left to consider the case where \( F \subseteq K \). We may assume that \( \langle F' \rangle \) is a tuple of elements in the main field, as fields code finite sets. Let \( U \) be the smallest closed torsor that contains all the elements of \( F \), this is a \( \langle F' \rangle \)-definable set. Let \( g \) the function that sends each element \( x \in F \) to the unique class of \( \text{red}(U) \) that contains such element. Let \( s \) be the \( O \)-lattice whose code is interdefinable with \( \langle U' \rangle \), and let \( h = \text{red}(U) \to \text{red}(s) \) be the map given by Lemma 6.31. Let \( D = h \circ g(F) \), which is an \( \langle F' \rangle \)-definable finite subset of \( \text{red}(s) \). By Proposition 6.39, the composition \( f \circ g^{-1} \circ h^{-1} = D \to G \) can be coded in the stabilizer sorts \( G \). As \( h \circ g = F \to D \) is a \( \langle F' \rangle \)-definable bijection, then \( f \) is interdefinable with the tuple \( \langle F', f \circ g^{-1} \circ h^{-1} \rangle \) which is a sequence of elements in \( G \).

\[\square\]

**6.6 Putting everything together**

We split the proof of coding finite sets into two main steps, first we prove that a primitive finite set \( F \subseteq G \) can be coded in the stabilizer sorts. Later, we show that coding of primitive sets is sufficient to find a code for any finite set \( F \).
Proposition 6.41. For any \( r > 0 \) let \( F \subseteq \mathcal{G}' \) be a primitive finite set. Then \( F \) can be coded in \( \mathcal{G} \).

Proof. Let \( \pi_i = \mathcal{G}' \rightarrow \mathcal{G} \) be the projection into the \( i \)-th coordinate. By primitivity of \( F \) each projection \( \pi_i \) is either constant or injective. As \( |F| > 1 \) there must be an index \( 1 \leq i_0 \leq r \) such that \( \pi_{i_0} \) is injective and \( F_0 = \pi_{i_0}(F) \) is a primitive finite subset of \( \mathcal{G} \). By Theorem 6.38 we can find a code \( 'F_0' \) in \( \mathcal{G} \). For each other index \( i \neq i_0 \), by Theorem 6.40 we have that \( \pi_i \circ \pi_{i_0} = F_0 \rightarrow \mathcal{G} \) can be coded in the stabilizer sorts. Then \( 'F' \) is interdefinable with the tuple \( ('F_0', (\pi_i \circ \pi_{i_0})_{i \neq i_0}) \) which is a tuple in the stabilizer sorts, as required.

Theorem 6.42. Let \( r > 0 \) and \( F \) be a finite subset of \( \mathcal{G}' \). Then \( F \) can be coded in \( \mathcal{G} \).

Proof. Let \( r > 0 \) and \( F \) be a finite set of \( \mathcal{G}' \), we proceed by induction on \( |F| \). If \( |F| = 1 \) the statement follows trivially. For the inductive step, we assume that any finite set of size \( m \) can be coded in the stabilizer sorts. Assume the statement for finite sets of size smaller or equal than \( m \), we aim to prove it for \( |F| = m + 1 \). If \( F \) is primitive the statement follows directly by Proposition 6.41. So suppose that \( F \) is not primitive, that means that we can find a non trivial equivalence \( E \) relation definable over \( 'F' \), and let \( C_1, \ldots, C_l \) be such classes. For each \( i \leq l, |C_i| < m + 1 \) then by the induction hypothesis we can find a code \( c_i \in \mathcal{G} \). We can find a code \( c \) in the stabilizer sorts of the set \( \{c_1, \ldots, c_l\} \), because \( l < m + 1 \). The code \( 'F' \) is interdefinable with \( c \).

We conclude this section with our main theorem.

Theorem 6.43. Let \( K \) be a henselian valued field of equicharacteristic zero, residue field algebraically closed and dp-minimal value group. Then \( K \) eliminates imaginaries in the language \( \mathcal{L} \), where the stabilizer sorts are added.

Proof. By Theorem 5.10, \( K \) has weak elimination of imaginaries down to the stabilizer sorts. By Fact 6.3 it is sufficient to show that finite sets can be coded, this is guaranteed by Theorem 6.42.

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