NOTE ON QUASI-POLARIZED CANONICAL CALABI-YAU THREEFOLDS

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ABSTRACT. Let \((X, L)\) be a quasi-polarized canonical Calabi-Yau threefold. In this note, we show that \(|mL|\) is basepoint free for \(m \geq 4\). Moreover, if the morphism \(\Phi_{|4L|}\) is not birational onto its image and \(h^0(X, L) \geq 2\), then \(L^3 = 1\). As an application, if \(Y\) is an \(n\)-dimensional Fano manifold such that \(-K_Y = (n-3)H\) for some ample divisor \(H\), then \(|mH|\) is basepoint free for \(m \geq 4\) and if the morphism \(\Phi_{|4H|}\) is not birational onto its image, then \(Y\) is either a weighted hypersurface of degree 10 in the weighted projective space \(\mathbb{P}(1, \cdot \cdot \cdot , 1, 2, 5)\) or \(h^0(Y, H) = n-2\).

1. INTRODUCTION

A normal projective complex threefold \(X\) is called a canonical Calabi-Yau threefold if \(\mathcal{O}(K_X) \cong \mathcal{O}_X, h^1(X, \mathcal{O}_X) = 0\) and \(X\) has only canonical singularities. We say that \(X\) is a minimal Calabi-Yau threefold, if, in addition, \(X\) has only Q-factorial terminal singularities. A pair of a normal projective variety \(X\) and a line bundle \(L\) is called a polarized variety if the line bundle \(L\) is ample, and a quasi-polarized variety if the line bundle \(L\) is nef and big. For a given quasi-polarized canonical Calabi-Yau threefold \((X, L)\), the following questions naturally arise.

1.1. Question.
(1) When \(\Phi_{|mL|}\) (the rational map defined by \(|mL|\)) is birational onto its image?
(2) When \(|mL|\) is basepoint free?

These two questions have already been investigated by several mathematicians in various different settings [6, 13, 14] etc. Our first result can be viewed as a generalization of [13, Theorem 1.1] and [14, Theorem 1].

1.2. Theorem. Let \((X, L)\) be a quasi-polarized canonical Calabi-Yau threefold. Then \(|mL|\) is basepoint free when \(m \geq 4\). Moreover, if \(\Phi_{|4L|}\) is not birational onto its image, then either \(L^3 = 1\) or \(h^0(X, L) = 1\).

The estimate is sharp as showed by a general weighted hypersurface of degree 10 in the weighted projective space \(\mathbb{P}(1, 1, 1, 2, 5)\). We remark also that we have always \(h^0(X, L) \geq 1\) by [8, Proposition 4.1] and the morphism \(\Phi_{|5L|}\) is always birational onto its image by [6, Theorem 1.7]. The basepoint freeness of \(|4H|\) is an easy consequence of [12, Theorem 24] and the existence of semi-log canonical member in \(|H|\) (cf. [8, Proposition 4.2]), and for the second part of the theorem, our proof basically goes along the line

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of [14, Theorem 1]. As the first application of Theorem 1.2, we generalize our previous result in [11, Theorem 1.7].

1.3. Corollary. Let $X$ be a weak Fano fourfold with at worst Gorenstein canonical singularities. Then

1. the complete linear system $| - mK_X|$ is basepoint free for $m \geq 4$;
2. the morphism $\Phi|_{-mK_X}$ is birational onto its image for $m \geq 5$.

As above, the estimates in Corollary 1.3 are both optimal as showed by a general weighted hypersurface of degree 10 in the weighted projective space $\mathbb{P}(1,1,1,1,2,5)$. As the second application, in higher dimension, using the existence of good ladder on Fano manifolds with coindex four proved in [11] and the work of Fujita on polarized projective manifold with small $\Delta$-genus and sectional genus (cf. [4]), we derive the following theorem which can also be viewed as a generalization of [13, Theorem 1.1] in higher dimension.

1.4. Theorem. Let $X$ be an $n$-dimensional Fano manifold such that $-K_X = (n-3)H$ for some ample divisor $H$. Then

1. the complete linear system $|mH|$ is basepoint free when $m \geq 4$;
2. the morphism $\Phi|_{mH}$ is birational onto its image when $m \geq 5$.

Moreover, if the morphism $\Phi|_{4H}$ is not birational onto its image, then one of the following holds.

1. $X$ is a weighted hypersurface of degree 10 in the weighted projective space $\mathbb{P}(1,\cdots,1,2,5)$.
2. $h^0(X,H) = n - 2$.

As in dimension 4, the same example given in Theorem 1.4 guarantees that the estimates given in Theorem 1.4 are best possible, and we have always $h^0(X,H) \geq n - 2$ in Theorem 1.4 (cf. [11, Theorem 1.2]). On the other hand, if $X$ is a general weighted complete intersection of type $(6,6)$ in the weighted projective space $\mathbb{P}(1,\cdots,1,2,2,3,3)$ and $H \in |O_X(1)|$, then we have $h^0(X,H) = n - 1$. This leads us to ask the following natural question.

1.5. Question.[4, 2.14][10, Problems 2.4] Is there an example of Fano $n$-fold $X$ such that $-K_X = (n-3)H$ for some ample divisor $H$ and $h^0(X,H) = n - 2$?

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2. PROOF OF THE MAIN RESULTS

Throughout the present paper, we work over the complex numbers and we adopt the standard notation in Kollár-Mori [9], and will freely use them. We start by selecting some results in minimal model program, and we shall use them in the sequel.

2.1. Lemma. Let $(X,L)$ be a quasi-polarized projective variety with at most Gorenstein canonical singularities.

1. There exists a projective variety $Y$ with only $\mathbb{Q}$-factorial terminal singularities and a proper surjective birational morphism $\nu: Y \rightarrow X$ such that $K_Y = \nu^* K_X$. Moreover, in this case, $M := \nu^* L$ gives a quasi-polarization on $Y$. 2
2.2. Definition. Let $X$ be a reduced equi-dimensional algebraic scheme and $B$ an effective \textit{\(\Phi\)} morphism Macaulay singularities. By Lemma 

\begin{proof}
proof of Theorem
\end{proof}

2.3. Definition. Let $(X, L)$ be a $n$-dimensional quasi-polarized projective manifold.

(1) The $\Delta$-genus $\Delta(X, L)$ of $(X, L)$ is defined to be 

(2) The sectional genus $g(X, L)$ of $(X, L)$ is defined to be 

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Recall that canonical singularities are normal rational Cohen-Macaulay singularities. By Lemma 2.1 (2), there exists a proper surjective birational morphism $\mu: X \to Z$ such that $L = \mu^*H$ for some ample line bundle $H$ on $Z$. Moreover, as $\mu_*K_X = K_Z$, we have $O(K_Z) = O_X$. In particular, we get $\mu^*K_Z = K_X$. It follows that $Z$ has only canonical singularities. Thus, $Z$ has only rational singularities and $R^i\mu_*O_X = 0$ for $i > 0$. This implies $h^1(Z, O_Z) \cong h^1(X, O_X) = 0$. As a consequence, $(Z, H)$ is a polarized canonical Calabi-Yau threefold. On the other hand, using the projection formula, we get $\mu_*O_X(mL) = O_Z(mH)$ and $R^i\mu_*O_X(mL) = 0$ for $i > 0$. This implies that the induced morphism $\mu^*: H^0(Z, mH) \to H^0(X, mL)$ is an isomorphism for all $m$. In particular, $|mL|$ is basepoint free if and only if $|mH|$ is basepoint free and $\Phi_{|mL|}$ is birational onto its image if and only if $\Phi_{|mH|}$ is birational onto its image. According to [8, Proposition 4.2], there exists an member $S \in |H|$ such that $S$ is a stable surface with $K_S = H|_S$. By Kawamata-Viehweg vanishing theorem and our assumption, the natural restriction 

$$H^0(Z, mH) \to H^0(S, mH|_S)$$
is surjective for all $m \in \mathbb{Z}$. Thanks to [12, Theorem 24], $|mK_S|$ is basepoint free for all $m \geq 4$. Consequently, $|mH|$ is also basepoint free for all $m \geq 4$.

Next we consider the case when $\Phi_{[4L]}$ is not birational onto its image. By Lemma 2.1 (1), there exists a terminal modification $\nu : Y \to X$ such that $(Y, M)$ is a quasi-polarized minimal Calabi-Yau threefold where $M = v^*L$. As above, we see that $L^3 = M^3$ and the induced morphism $v^* : H^0(X, mL) \to H^0(Y, mM)$ is an isomorphism for all $m$. In particular, $\Phi_{|mL|}$ is birational onto its image if and only if $\Phi_{|mM|}$ is birational onto its image. Thus, after replacing $(X, L)$ by $(Y, M)$, we may assume that $(X, L)$ itself is a quasi-polarized minimal Calabi-Yau threefold. In particular, $X$ is actually factorial by [7, Lemma 5.1]. As mentioned in the introduction, we have always $h^0(X, L) \geq 1$ by [8, Proposition 4.1]. To prove Theorem 1.2, we assume that $h^0(X, L) \geq 2$ and we distinguish two cases according to whether $\dim \Phi_{[L]}(X) = 1$.

1st case. $\dim \Phi_{[L]}(X) \geq 2$. By Hironaka’s resolution theorem, there exists a smooth projective threefold $Y$ and a proper surjective birational morphism $\pi : Y \to X$ and a decomposition

$$\pi^*L = |F| + B$$

such that $|F|$ is basepoint free. Let $T \in |F|$ be a general smooth member. By the proof of [14, Theorem 1], $\Phi_{[(m+1)L]}$ is birational onto its image if $\Phi_{|\pi^*mL| + K_T}$ is birational onto its image. Thus, if $(\pi^*L|_T)^2 \geq 2$, by [16, Theorem 1 (ii)], the complete linear system $|\pi^*mL|_T + K_T|$ is birational onto its image if $m \geq 3$. If $(\pi^*L|_T)^2 = 1$, by the projection formula, we get $L^2 \cdot \pi_*T = 1$ since $T$ is a general member in the movable family $|F|$. Thanks to [14, Lemma 1.1 (4)], we see that $L^3 = 1$.

2nd case. $\dim \Phi_{[L]}(X) = 1$. Since $h^1(X, O_X) = 1$, there exists a smooth projective threefold $Y$ and a proper surjective birational morphism $\mu : Y \to X$ and a decomposition

$$\mu^*L = n|F| + B$$

such that $|F|$ is a free pencil. Let $T$ be a general smooth element in $|F|$. Then $\Phi_{[(m+1)L]}$ is birational onto its image if $\Phi_{|\mu^*mL| + K_T}$ is birational onto its image. Using the same argument as in the 1st case, we obtain $L^3 = 1$ if $\Phi_{[4L]}$ is not birational onto its image. \hfill $\square$

Corollary 1.3 is an immediate consequence of Theorem 1.2 and the existence of good divisor on weak Fano fourfolds established in [8, Theorem 5.2].

Proof of Corollary 1.3. The statement (2) was proved in [11, Theorem 1.7]. By Lemma 2.1 (2), there exists a surjective proper birational map $\mu : X \to Z$ and an ample line bundle $H$ on $Z$ such that $\mu^*H = -K_X$. Moreover, as $\mu_*K_X = K_Z$, it follows that $-K_Z = H$ and $\mu^*K_Z = K_X$. According to [8, Theorem 5.2], there exists a member $Y \in (-K_Z)$ such that $Y$ has only Gorenstein canonical singularities. As a consequence, $(Y, -K_Z|_Y)$ is a polarized canonical Calabi-Yau threefold. Thanks to Kawamata-Viehweg vanishing theorem, the natural restriction map

$$H^0(Z, -mK_Z) \longrightarrow H^0(Y, -mK_Z|_Y)$$

is surjective for all $m \in \mathbb{Z}$. Then, by Theorem 1.4, we see that $|-mK_Z|$ is basepoint free if $m \geq 4$. On the other hand, the same argument as in Theorem 1.2 shows that the induced morphism $\mu^* : H^0(Z, -mK_Z) \to H^0(X, -mK_X)$ is an isomorphism for all $m$. Hence, $|-mK_X|$ is basepoint free for all $m \geq 4$. \hfill $\square$
Next we give the proof of Theorem 1.4.

**Proof of Theorem 1.4.** By [11, Theorem 1.2] and [3, Theorem 1.1], there exists a descending sequence of subvarieties of $X$

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_3$$

such that $X_{i+1} \in |H|_{X_i}$ and $X_i$ has only Gorenstein canonical singularities. Moreover, it is easy to see that $(X_3, H|_{X_3})$ is a polarized canonical Calabi-Yau threefold. Thanks to Theorem 1.2, $|mH|_{X_{n-3}}$ is basepoint free if $m \geq 4$. By Kawamata-Viehweg vanishing theorem, it is easy to see that the natural restriction

$$H^0(X, mH) \to H^0(X_3, mH|_{X_3})$$

is surjective for all $m \in \mathbb{Z}$. Thus $|mH|$ is basepoint free if $m \geq 4$. On the other hand, if $\Phi|_{4H}$ is not birational onto its image, since we can choose all $X_i$ to be general, $\Phi|_{4H|_{X_3}}$ is not birational onto its image (cf. [14, Lemma 1.3]). If $h^0(X, H) \neq n - 2$, by [11, Theorem 1.2], we get $h^0(X, H) \geq n - 1$. As a consequence, we obtain

$$h^0(X_3, H|_{X_3}) = h^0(X, H) - (n - 3) \geq 2.$$  

Then Proposition 1.2 implies $H^n = (H|_{X_3})^3 = 1$. Then, by definition, we have

$$g(X, L) = (K_X \cdot H^{n-1} + (n - 1)H^n)/2 + 1 = H^n + 1 = 2,$$

and

$$\Delta(X, H) = H^n + n - h^0(X, H) \leq n + 1 - (n - 1) = 2.$$  

On the other hand, it is well-known that we have $\Delta(X, H) \geq 0$ with equality if and only if $g(X, L) = 0$ (cf. [5, Theorem 12.1]). This implies that $\Delta(X, H) = 1$ or 2 in our situation. According to [4, Proposition 2.3 and 2.4], $X$ is isomorphic to either a weighted hypersurface of degree 10 in the weighted projective space $\mathbb{P}(1, \cdots, 1, 2, 5)$ or a weighted complete intersection of type $(6, 6)$ in the weighted projective space $\mathbb{P}(1, \cdots, 1, 2, 2, 3, 3)$. However, if $X$ is a weighted complete intersection of type $(6, 6)$ in the weighted projective space $\mathbb{P}(1, \cdots, 1, 2, 2, 3, 3)$, then the group $H^0(X, mH)$ ($m \geq 3$) contains the monomials

$$\{x_1x_0^{m-1}, \cdots, x_{n-2}x_0^{m-1}, x_{n-1}x_0^{m-2}, x_nx_0^{m-2}, x_{n+1}x_0^{m-3}, x_{n+2}x_0^{m-3}\},$$

where $x_i$ are the weighted homogeneous coordinates of $\mathbb{P}(1, \cdots, 1, 2, 2, 3, 3)$ in order. This shows that $\Phi|_{mH}$ ($m \geq 3$) is one-to-one on the non-empty Zariski open subset $\{x_0 \neq 0\} \cap X$ and this case is excluded.

3. **FURTHER DISCUSSIONS**

Let $(X, L)$ be a quasi-polarized canonical Calabi-Yau threefold such that $h^0(X, L) = 1$. Let $(Y, M)$ be the terminal modification of $(X, L)$. Then $Y$ is smooth in codimension two. By Riemann-Roch formula and the projection formula, we obtain

$$\chi(X, mL) = \chi(Y, mM) = \frac{M^3}{6}m^3 + \frac{M \cdot c_2(Y)}{12} + \chi(Y, O_Y).$$
As $h^1(X, \mathcal{O}_X) = 0$, by Serre duality, we get $\chi(Y, \mathcal{O}_Y) = 0$. Thus, using Kawamata-Viehweg vanishing theorem, we obtain

$$1 = h^0(X, L) = h^0(Y, M) = \frac{1}{6}M^3 + \frac{1}{12}M \cdot c_2(Y).$$

Moreover, thanks to [15, Thereom 0.5], we have $M \cdot c_2(X) \geq 0$. It follows that

$$1 \leq L^3 = M^3 \leq 6.$$

On the other hand, a smooth ample divisor $S$ on a Calabi-Yau threefold $X$ (not necessarily simply connected) is a minimal surface of general type. This simple observation yields a bridge between two important classes of algebraic varieties. Moreover, a smooth ample divisor $S$ on a Calabi-Yau threefold is called a rigid ample surface if $h^0(X, \mathcal{O}_X(S)) = 1$. In this case, the geometric genus $p_g(S): = h^0(S, K_S)$ is zero and, by the Lefschetz theorem, the natural map $\pi_1(S) \to \pi_1(X)$ is an isomorphism. Thus, according to theorem 1.2, it may be interesting to ask the following question.

3.1. Question. Is there a simply connected smooth Calabi-Yau threefold $X$ containing a rigid ample surface $S$?

We remark that if we do not require the simple connectedness of $X$, such an example of $(X, S)$ with the quaternion group of order 8 $\pi_1(X) = H_8$ as its fundamental group was constructed by Beauville in [1].

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