Rheometric Flows of Concentrated Suspensions of Solid Particles

O. I. Skul’skiy

Institute of Continuous Media Mechanics, Ural Branch, Russian Academy of Sciences, Perm, Russia
e-mail: skul@icmm.ru

Received June 22, 2020; revised July 21, 2020; accepted July 21, 2020

Abstract—Publications on experimental and theoretical studies of the rheological properties of concentrated suspensions of solid particles have been analyzed. According to modern concepts, the rheology of suspensions is considered as a result of contact interaction between their constituent particles due to external forces of formation and destruction of various-type conglomerate structures. A new rheological model of a highly concentrated suspension of solid particles in a Newtonian fluid is proposed, which describes both a continuous and discontinuous growth in the effective viscosity with a uniform increase in shear stress. Exact analytical formulas for the velocity profiles of suspension flows in “cone–plane” and “cylinder–cylinder” rotational viscometers, as well as a slit viscometer, are obtained. The proposed model is modified to take into account the non-Newtonian properties of a dispersion medium, which exhibits pseudoplastic and dilatant properties at low and high strain rates, respectively. The effective viscosity of such a suspension is presented as a sum of the contributions from the non-Newtonian dispersion medium and dispersed-phase solid particles. The rheology of the dispersed phase is described using the Ellis model. The velocity profiles in a pressure-driven flat channel are obtained numerically (by the finite-element method). It is shown that they can take various complex forms, depending on the model parameters.

Keywords: highly concentrated suspensions, rheological model, rheometric flows, analytical solutions, non-Newtonian dispersion medium, numerical solution

DOI: 10.1134/S0021894421070166

1. INTRODUCTION

In the last decade, experimental and theoretical rheology has acquired new data on the rheological behavior of dilatant liquids. Concentrated suspensions of solid particles in a liquid exhibiting dilatancy properties can be of either natural or artificial origin. They are used in the chemical industry, medicine, pharmacology, and production of consumer goods (including food and cosmetics).

Even in channels with a simple geometry, flowing non-Brownian suspensions exhibit some non-Newtonian properties and have nontrivial flow fields. For example, an increase in shear stress leads to a continuous increase in the apparent viscosity (continuous shear thickening, CST), while at high particle concentrations there is a discontinuous increase in this parameter (discontinuous shear thickening, DST) and shear jamming [1–12]. Figure 1 shows typical experimental S-shaped suspension flow curves, demonstrating a stepwise increase in the stress with a smooth increase in the shear velocity.

Theoretical fundamentals of the rheology of suspensions were reported in [2, 10, 11]. Several works devoted to experimental investigation and theoretical simulation of the rheological properties of highly concentrated suspensions have recently been published, which demonstrate a stepwise increase in the stress with a smooth increase in the shear velocity. This non-Newtonian rheology occurs because of the friction interaction between particles [13–18].

Experimental study of the rheological properties of non-Newtonian fluids is generally carried out using modern viscometers with built-in data processing software. Results of rheological tests are presented in the form of dependence of the shear stress on the shear velocity at a known particle concentration.

Phenomenological rheological equations are mainly used for mathematical simulation of flows of concentrated suspensions. The data on the most popular (in scientific literature) equations are listed in Table 1.
The following designations are used in Table 1: \( \eta_{\text{eff}} \), \( \eta_0 \), and \( \eta_{\infty} \) are the effective and characteristic viscosities; \( \sigma \) is the stress; \( \sigma_y \) is the yield stress; \( S \) is the stress intensity; \( \dot{\gamma} \) is the shear velocity; \( \phi \) is the concentration; and \( A \), \( B \), and \( K \) are parameters of the models.

**Table 1.** Most popular phenomenological rheological equations

| Author(s) of equation | Rheological equation | Material under study or its property; reference |
|-----------------------|----------------------|------------------------------------------------|
| Einstein              | \( \eta_{\text{eff}} = 1 + \frac{5}{2} \phi \) \( \frac{\eta_0}{\eta_{\text{eff}} = 1 + \left( \frac{S}{S_0} \right)^{\alpha-1}} \) | Low-concentration suspensions [2, 13–15] |
| Herschel–Bulkley      | \( \sigma = \sigma_y + K \dot{\gamma}^n \) | Concentrated suspensions [2, 10, 11, 23] |
| Ellis                 | \( \eta_{\text{eff}} = \frac{\eta_0}{\eta_{\text{eff}} = 1 + \left( \frac{S}{S_0} \right)^{\alpha-1}} = \exp \left( \frac{5 \phi}{2(1-\phi/\phi_0)} \right) \) | Concentrated suspensions [2, 10, 11] |
| Mooney                | \( \eta_{\text{eff}} = \eta_0 + \frac{\eta_0 - \eta_{\infty}}{1 + (K \dot{\gamma})^n} \) | Concentrated suspensions [2, 10, 11] |
| Cross                 | \( \eta_{\text{eff}} = \eta_{\infty} + \left( \frac{\eta_0 - \eta_{\infty}}{1 + (K \dot{\gamma})^n} \right) \) | Concentrated suspensions [7, 10] |
| Singh et al.          | \( \eta_{\text{eff}} = \eta_0 \exp \left( \frac{A \phi}{1-\phi} \right) \), \( \phi a \left( \frac{S}{S_0} \right)^2 \) \( + \left( \frac{1}{1 + \left( \frac{S}{S_0} \right)^2} \right) \) | Concentrated suspensions, discontinuous increase in viscosity [23] |
| Nakanishi et al.      | \( \eta_{\text{eff}} (\phi) = \eta_0 \exp \left( A \frac{\phi}{1-\phi} \right) \), \( \phi a \left( \frac{S}{S_0} \right)^2 \) \( + \left( \frac{1}{1 + \left( \frac{S}{S_0} \right)^2} \right) \) | Concentrated suspensions, discontinuous increase in viscosity [12, 20] |
| Galindo–Rosales et al. | \( \eta (\dot{\gamma}) = \begin{cases} \eta_{\text{II}} (\dot{\gamma}) = \eta_{\text{c}} + \frac{\eta_0 - \eta_{\text{c}}}{1 + \left[ K_{\text{II}} \left( \frac{\dot{\gamma}}{\dot{\gamma}_{\text{max}}} \right) \right]^{n_{\text{II}}}}, & \dot{\gamma} \leq \dot{\gamma}_{\text{c}}, \\ \eta_{\text{III}} (\dot{\gamma}) = \eta_{\text{max}} + \frac{\eta_{\text{c}} - \eta_{\text{max}}}{1 + \left[ K_{\text{III}} \left( \frac{\dot{\gamma}_{\text{c}} - \dot{\gamma}}{\dot{\gamma}_{\text{max}} - \dot{\gamma}_{\text{c}}} \right) \right]^{n_{\text{III}}}}, & \dot{\gamma}_{\text{c}} < \dot{\gamma} \leq \dot{\gamma}_{\text{max}}, \\ \eta_{\text{III}} (\dot{\gamma}) = \frac{\eta_{\text{max}}}{1 + \left[ K_{\text{III}} \left( \dot{\gamma} - \dot{\gamma}_{\text{max}} \right) \right]^{n_{\text{III}}}}, & \dot{\gamma}_{\text{max}} < \dot{\gamma}. \end{cases} \) | Concentrated suspensions in a non-Newtonian fluid [4, 7, 10] |
A microstructural approach based on the Smoluchowski formalism was developed in [19, 21, 22]. This promising line of research has not yet led to construction of a constitutional equation, which can be used in practical calculations. Below, we will analyze the efficiency of the new rheological model of concentrated suspensions of solid particles in a liquid, which exhibit dilatancy properties, by the example of the main rheometric flows.

2. MODEL OF THE FLOW OF A SUSPENSION OF SOLID PARTICLES IN A NEWTONIAN DISPERSION MEDIUM

In the general case, motion of an incompressible liquid is described by the system of differential equations corresponding to conservation laws:

\[ \rho \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = -\nabla P + \nabla \cdot \mathbf{\tau}, \]

\[ \mathbf{\tau} = 2\eta_{ef} \mathbf{D}, \]

\[ \mathbf{D} = \frac{1}{2} \left( \mathbf{VV} + \mathbf{VV}^T \right), \]

\[ \nabla \cdot \mathbf{V} = 0, \]

where \( \mathbf{V} \) is the velocity vector, \( P \) is the pressure (spherical part of the stress tensor), \( \mathbf{\tau} \) is the deviator part of the stress tensor, \( \mathbf{D} \) is the symmetric part of the strain rate tensor with components \( \gamma_{ij} \), \( \eta_{ef} \) is the effective viscosity, and \( \rho \) is the density.

The system (1)–(4) should be concluded by a rheological equation, determining the properties of a specific material, and the corresponding initial and boundary conditions.

In mechanics of non-Newtonian fluids, nonmonotonicity of flow curves is a nonunique phenomenon. It can be caused by, e.g., a sharp decrease in the viscosity with an increase in temperature [25] or a decrease in the viscosity with an increase in the shear velocity, which induces orientation of macromolecules along the flow [26, 27]. For concentrated suspensions, the nonmonotonicity of flow curves is due to the forces of contact interaction of particles, which grow with an increase in the stress. Based on the experience of simulation of the rheological properties of solutions of polymers with nonmonotonic flow curves, the rheological model of concentrated suspensions of solid particles can be written in the form of a phenomenological equation, which describes both the continuous and discontinuous increase in the shear velocity with a smooth increase in the stress:

\[ \dot{\Gamma} = \frac{S}{\eta_{ef}}, \]

where \( \eta_{ef} = \eta_0 \left( 1 + \left( \frac{S}{S_0} \right)^2 \right) \left( 1 + \varepsilon \left( \frac{S}{S_0} \right)^2 \right) \) and \( S = \frac{1}{\sqrt{2}} (\mathbf{\tau} : \mathbf{\tau}) \) are, as above, the effective viscosity and stress intensity, respectively; \( \Gamma = \frac{1}{\sqrt{2}} (\mathbf{D} : \mathbf{D}) \) is the strain rate intensity; and \( \varepsilon = \frac{m \phi^*}{\phi^*} \) is the relative concentration. The characteristic values of stress \( S_0 \), strain rate \( \Gamma_0 \), viscosity \( \eta_0 = S_0 / \Gamma_0 \), and fitting parameter \( m \) for a specified concentration are determined at two characteristic points of the experimental flow curve: at the point of maximum (\( \varepsilon = 0 \)) and at the inflection point (\( \varepsilon = 1/9 \)) (see Fig. 2).

3. SUSPENSION FLOW IN A CONE–PLANE ROTATIONAL VISCOMETER

Cone–plane viscometers (Fig. 3) are most often used for rheological studies of homogeneous non-Newtonian fluids. The cone opening angle (slit) is generally small, and it is assumed that \( \tan \Psi_0 \equiv \Psi_0 \). As a result, the standard formulas used for processing experimental data approximate with a high degree of accuracy the analytical solution to the problem of liquid flow in a conical slit.

The standard formulas for processing experimental data have the form [12]

\[ \tau = \frac{3M}{2\pi R^3}, \quad \dot{\gamma} = \frac{\Omega}{\Psi_0}, \quad \eta_{ef} = \frac{3M\Psi_0}{2\pi R^3 \Omega}, \]

\[ \Psi_0 \equiv \frac{3M \Omega}{2\pi R^3} \]

\[ \Omega \]

\[ \Psi_0 \]

\[ \frac{3M}{2\pi R^3} \]

\[ \frac{\Omega}{\Psi_0} \]

\[ \eta_{ef} \]

\[ \frac{3M\Psi_0}{2\pi R^3 \Omega} \]

\[ \Psi_0 \equiv \frac{3M \Omega}{2\pi R^3} \]
where $\tau$ is the shear stress, $\dot{\gamma}$ is the shear velocity, $\eta_{ef}$ is the effective viscosity, $M$ is the torque, $\Omega$ is the angular velocity, and $R$ and $\Psi_0$ are the viscometer parameters. These formulas are also valid for dilatant liquids, because the shear velocity and shear stress are constant, while the circumferential velocity is linear with respect to the slit height.

It should be noted that study of the rheology of concentrated suspensions relative to large particles should be carried out on viscometers providing validity of the postulates of mechanics of continua; in other words, the representative suspension volume should be much smaller than the slit.

4. SUSPENSION MOTION IN THE SLIT BETWEEN COAXIAL CYLINDERS

Let the internal cylinder with radius $R_1$ be immobile, while the external cylinder with radius $R_2$ rotates under the applied torque $M$ with angular velocity $\Omega$ (Fig. 4).

In the stationary case of creeping one-dimensional flow ($V_r = V_z = 0, \partial V_\phi / \partial t = 0$) of a non-Newtonian liquid, equilibrium equations (1) in the cylindrical coordinate system have the form

$$\frac{d\tau_{r\phi}}{dr} + 2\frac{\tau_{r\phi}}{r} = 0.$$ \hspace{1cm} (6)

Having integrated (6), we obtain $\tau_{r\phi} = C/r^2$, where the integration constant $C$ is determined from the boundary conditions.

We introduce dimensionless variables: $\xi = \frac{r}{R_1}, k = \frac{R_2}{R_1}, \sigma = \frac{\tau_{r\phi}}{\eta_{0}}, G = \frac{M}{4\pi LR_1^2S_0}, U = \frac{V_\phi}{R_2\Gamma_0}$. The relationship between the strain rate and the stress can be written using the proposed model (5):
Integration of (7) and consideration of the boundary conditions yield formulas of velocity distribution in the slit between the cylinders:

\[ G = \frac{(1 + \varepsilon \sigma^2)}{(1 + \sigma^2)} \sigma, \quad G = \frac{dU}{d\xi} - \frac{U}{\xi}. \]  

(7)

For circumferential velocity,

\[ U(\xi) = \varepsilon \sigma \left( \frac{\xi^2 - 1}{\xi} \right) + (1 - \varepsilon) \xi \arctan \left( \frac{\sigma(\xi^2 - 1)}{\sigma^2 + \xi^2} \right); \]  

(8)

and for angular velocity,

\[ \Omega = \varepsilon \sigma \left( \frac{k^2 - 1}{k^2} \right) + (1 - \varepsilon) k \arctan \left( \frac{\sigma(k^2 - 1)}{\sigma^2 + k^2} \right). \]  

(9)

Formula (9) demonstrates the relationship between the set torque and measured angular velocity of the external cylinder. At \( \varepsilon = 1 \), which corresponds to a Newtonian liquid, it is reduced to the standard form used in viscosimetry [12]:

\[ \eta = \frac{M}{4\pi \Omega R_i^2 L} \left( \frac{k^2 - 1}{k^2} \right). \quad k = \frac{R_2}{R_1}. \]

As an example, Fig. 5 shows the distributions of the circumferential velocity and shear velocity in the slit between the cylinders. An increase in the particle concentration (i.e., a decrease in parameter \( \varepsilon = m \frac{\phi - \phi^*}{\phi^*} \)) leads to a decrease in the velocity and its gradient at the torque set at the external cylinder.

5. POISEUILLE FLOW IN A FLAT CHANNEL

In the case of laminar flow of a liquid in a flat channel under the action of a pressure drop (Fig. 6), we obtain the following linear dependence of the shear stress on the transverse coordinate after integration of the equation of motion \( \frac{dP}{dx} = \frac{d\tau_{xy}}{dy} \) with allowance for the symmetry of the computational domain:

\[ \tau_{xy} = (\Delta P/L) y, \]  

with the maximum value at the walls \( \left| \tau_{xy} \right| = (\Delta P/L) h. \)
Having introduced dimensionless variables \( \zeta = y/h \), \( \sigma = (\Delta P/S_0)(h/L) \), \( G = \dot{\gamma}_{xy}/\Gamma_0 \), and \( W = V_x/(h\Gamma_0) \) and taking into account (5), we can write the analytical solutions as follows:

for the shear velocity,

\[
G = \frac{\sigma\zeta(1 + \varepsilon \sigma^2 \zeta^2)}{1 + \sigma^2 \zeta^2},
\]

(10)

for the profile velocity,

\[
W = \frac{1}{2} \left[ \varepsilon \sigma(1 - \zeta^2) + \frac{(1 - \varepsilon)}{\sigma} \ln \left( \frac{1 + \sigma^2}{1 + \sigma^2 \zeta^2} \right) \right],
\]

(11)

for the specific flow rate,

\[
Q = \int_0^1 W d\zeta = 2 \left[ \frac{\sigma \zeta}{3} + \frac{1}{\sigma} (1 - \varepsilon) - \frac{(1 - \varepsilon)}{\sigma^2} \arctan(\sigma) \right].
\]

(12)

Formulas (10)–(12) can be used for developing a technique of fitting material constants of the rheological equation.

Figure 7 shows the profiles of the effective viscosity, longitudinal velocity, and shear velocity in the cross section of a slit channel at \( G = 10 \) and different values of parameter \( \varepsilon \).
If a dispersed liquid is non-Newtonian, it exhibits pseudoplastic and dilatant properties at low and high strain rates, respectively [4, 7, 24]. The effective viscosity of this suspension can be written as a sum of contributions from the dispersed liquid and dispersed-phase solid particles. The rheology of the dispersed liquid was described using the Ellis model, while the contribution to the effective viscosity from the dispersed phase was taken into account by means of the proposed model:

\[
\eta_{ef} = \frac{\eta_0 + a \left( \frac{S}{S_0} \right)^b}{1 + a \left( \frac{S}{S_0} \right)^b} + \eta_0 \left[ 1 + \left( \frac{S}{S_0} \right)^p \right] \left[ 1 + \varepsilon \left( \frac{S}{S_0} \right)^\varepsilon \right],
\]

(13)

where \(\mu_0\), \(a\), \(b\), and \(p\) are material constants.

In most publications on the rheology of suspensions, experimental data are presented in the form of dependence of the viscosity on the shear velocity \(\dot{\gamma}\). In this case, the proposed model can be applied if one passes from shear velocity to stress via the obvious formula:

\[
\tau = \sqrt{\eta_{ef} \dot{\gamma}},
\]

where \(\tau\) is the stress, \(\eta_{ef}\) is the effective viscosity, and \(\dot{\gamma}\) is the shear rate.\(\dot{\gamma}\). The proposed model was used to calculate the multidimensional flows of a suspension in a flat channel by the Galerkin finite-element method with linearization by the secant method:

\[
0 = \mu_{ef} \frac{\partial^2 \psi}{\partial x^2} + \mu_{ef} \frac{\partial^2 \psi}{\partial y^2} + \mu_{ef} \frac{\partial^2 \psi}{\partial z^2} + \frac{1}{\rho} \left( \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z} \right),
\]

where \(\psi\) is the velocity potential, \(\mu_{ef}\) is the effective viscosity, \(\rho\) is the density, and \(p\) is the pressure.

Typical dependences of the effective viscosity on the stress for suspensions with a non-Newtonian dispersed liquid at different particle concentrations are presented in Fig. 8 for demonstration of the efficiency of the proposed model. Figure 9 shows the experimental dependences of the effective viscosity on the stress for suspensions, which are plotted using the proposed model, and, for comparison, the dependences obtained previously in [24].

Application of the proposed rheological model for calculating multidimensional flows requires the application of numerical methods. For example, in the case of creeping suspension flow in a flat channel, one can use the Galerkin finite-element method with linearization by the secant method:
where are weighting functions, $P_0$ is the distributed load (numerically equal to the pressure at the input), and $\eta_{\text{ef}}$ and $\eta_{\text{ef}^{-1}}$ are the effective viscosities calculated from formula (13) at the current and previous iterations, respectively. Calculations were performed with a grid of 6000 triangular elements. Convergence of the iterative process was estimated based on the effective viscosity, calculated for each element.

**Fig. 8.** Typical dependences of the viscosity on the stress for a suspension with different concentrations $\varepsilon$, calculated within the proposed model with the following parameters: $\mu_0 = 1.34$ Pa s, $\eta_0 = 0.63$ Pa s, $S_0 = 1$ Pa, $a = 40$, $b = 64$, and $p = 1$.

**Fig. 9.** Dependences of the effective viscosity on the stress for the suspension of (1) Aerosil® R816 particles in polypropylene glycol [7] and (2) deposited calcium carbonate particles based on polyethylene glycol [24]: (symbols) experiment and (solid lines) calculation using the proposed model.
ment until the condition $\max \left[ \eta_k - \eta_k^{k-1} \right] < \delta$ is satisfied, where $\delta = 10^{-6}$ is a specified small number. Convergence of the solution required from 70 to 300 iterations, depending on the parameters of the problem.

Figure 10 shows the dimensionless velocity profiles and dependences of the maximum velocity on the pressure drop in a flat rectangular channel with height $h = 0.01$ m and length $L = 0.1$ m for the polyethylene-glycol-based suspension of deposited calcium carbonate particles, which were calculated by the finite-element method [24]. The model with the following material constants was used: $\varepsilon = 0.29$, $\mu_0 = 1.34$ Pa s, $\eta_0 = 0.63$ Pa s, $S_0 = 1$ Pa, $a = 40$, $b = 6$, and $p = 1$. The calculations were performed for different specified pressures at the input.

The results of numerical analysis of the flow of the suspension of particles in a non-Newtonian dispersed liquid by the example of a flat channel showed that, with a smooth increase in the pressure gradient, the effective viscosity first decreases to a minimum value and then increases, whereas the flow velocity behaves oppositely. The velocity profiles are almost parabolic at small pressure gradients and tend to a tapered shape at large gradients.

7. CONCLUSIONS

Mathematical simulation of concentrated suspension flows is of great theoretical and practical importance in view of their wide application in industry and medicine. The proposed phenomenological model allows one to obtain exact analytical expressions for the main rheometric flows, which are necessary to pass from the experimentally measured integral characteristics to the flow curve and determine material constants entering the model. A model in the form of a phenomenological equation, which describes both the continuous and discontinuous increase in the shear velocity with a smooth increase in the stress, was proposed to take into account the non-Newtonian properties of a dispersion medium. It differs from the popular models by introduction of the Ellis law for a dispersed phase. Test numerical calculations of the flat Poiseuille flow showed that the profiles of the cross-sectional velocity and shear velocity may have various complex shapes, depending on the pressure applied at the input and values of the model parameters.

REFERENCES

1. Verdier, C., Rheological properties of living materials. From cells to tissues, J. Theor. Med., 2003, vol. 5, no. 2, pp. 67–91. https://doi.org/10.1080/1027336041001678083

2. Khodakov, G.S., Suspension rheology. The theory of phase flow and its experimental substantiation, Ros. Khim. Zh. (Zh. Ros. khim. ob-va Mendeleeva), 2003, vol. 47, no. 2, pp. 33–43.
3. Guillou, S. and Makhloufi, R., Effect of a shear-thickening rheological behaviour on the friction coefficient in a plane channel flow: A study by direct numerical simulation, *J. Non-Newton. Fluid Mech.*, 2007, vol. 144, pp. 73–86. https://doi.org/10.1016/j.jnnfm.2007.03.008

4. Galindo-Rosales, F.J., Rubio-Hernandez, F.J., and Velazquez-Navarro, J.F., Shear-thickening behavior of Aerosil® R816 nanoparticles suspensions in polar organic liquids, *Rheol. Acta*, 2009, vol. 48, pp. 699–708. https://doi.org/10.1007/s00379-009-0367-7

5. Liu, A.J. and Nagel, S.R., The jamming transition and the marginally jammed solid, *Ann. Rev. Condens. Matter Phys.*, 2010, vol. 1, pp. 347–369. https://doi.org/10.1146/annurev-conmatphys-070909-104045

6. Seth, J.R., Mohan, L., Locatelli-Champagne, C., Cloitre, M., and Bonnecaze, R.T., A micromechanical model to predict the flow of soft particle glasses, *Nat. Mater.*, 2011, vol. 10, pp. 838–843. https://doi.org/10.1038/nmat3119

7. Galindo-Rosales, F.J., Rubio-Hernandez, F.J., and Sevilla, A., An apparent viscosity function for shear thickening fluids, *J. Non-Newton. Fluid Mech.*, 2011, vol. 166, pp. 321–325. https://doi.org/10.1016/j.jnnfm.2011.01.001

8. Boyer, F., Guazzell, E., and Pouliquen, O., Unifying suspension and granular rheology, *Phys. Rev. Lett.*, 2011, vol. 107, p. 188301. https://doi.org/10.1103/PhysRevLett.107.188301

9. Nakanishi, H., Nagahiro, S., and Mitarai, N., Fluid dynamics of dilatant fluids, *Phys. Rev. E*, 2012, vol. 85, p. 011401. https://doi.org/10.1103/PhysRevE.85.011401

10. Fortier, A., *Suspension Mechanics*, Paris: Masson et Cie, 1967.

11. Ur'yev, N.B., *Fiziko-khimicheskie osnovy tekhnologii dispersnykh sistem i materialov* (Physicochemical Foundations of the Technology of Dispersed Systems and Materials), Moscow: Khimiya, 1988.

12. Tanner, R.I., *Engineering Rheology*, Oxford: Oxford Univ. Press, 2000.

13. Brown, E. and Jaeger, H.M., Shear thickening in concentrated suspensions: Phenomenology, mechanisms and relations to jamming, *Rep. Prog. Phys.*, 2014, vol. 77, p. 046602. http://iopscience.iop.org/0034-4885/77/4/046602

14. Denn, M.M. and Morris, J.F., Rheology of non-Brownian suspensions, *Ann. Rev. Chem. Biomol. Eng.*, 2014, vol. 5, pp. 203–228. https://doi.org/10.1146/annurev-chembioeng-060713-040221

15. Ardakani, H.A., Mitsoulis, E., and Hatzikiriakos, S.G., Capillary flow of milk chocolate, *J. Non-Newton. Fluid Mech.*, 2014, vol. 210, pp. 56–65. https://doi.org/10.1016/j.jnnfm.2014.06.001

16. Mari, R., Seto, R., Morris, J.F., and Denn, M.M., Nonmonotonic flow curves of shear thickening suspensions, *Phys. Rev. E*, 2015, vol. 91, p. 052302. https://doi.org/10.1103/PhysRevE.91.052302

17. Pan, Zh., de Cagny, H., Weber, B., and Bonn, D., S-shaped flow curves of shear thickening suspensions: Direct observation of frictional rheology, *Phys. Rev. E*, 2015, vol. 92, p. 032202. https://doi.org/10.1103/PhysRevE.92.032202

18. Ness, C. and Sun, J., Shear thickening regimes of dense non-Brownian suspensions, *Soft Matter*, 2016, vol. 12, pp. 914–924. https://doi.org/10.1039/c5sm02326b

19. Vázquez-Quesada, A. and Ellero, M., Rheology and microstructure of non-colloidal suspensions under shear studied with Smoothed Particle Hydrodynamics, *J. Non-Newton. Fluid Mech.*, 2016, vol. 233, pp. 37–47. https://doi.org/10.1016/j.jnnfm.2015.12.009

20. Nagahiro, S. and Nakanishi, H., Negative pressure in shear thickening bands of a dilatant fluid, *Phys. Rev. E*, 2016, vol. 94, p. 062614. https://doi.org/10.1103/PhysRevE.94.062614

21. Vázquez-Quesada, A., Wagner, N.J., and Ellero, M., Planar channel flow of a discontinuous shear-thickening model fluid: Theory and simulation, *Phys. Fluid.*, 2017, vol. 29, p. 103104. https://doi.org/10.1063/1.4997053
22. Singh, A., Mari, R., Denn, M.M., and Morris, J.F., A constitutive model for simple shear of dense frictional suspensions, *J. Rheol.*, 2018, vol. 62, pp. 457–468.  
https://doi.org/10.1122/1.4999237

23. Singh, A., Pednekar, S., Chun, J., Denn, M.M., and Morris, J.F., From yielding to shear jamming in a cohesive frictional suspension, *Phys. Rev. Lett.*, 2019, vol. 122, p. 098004.  
https://doi.org/10.1103/PhysRevLett.122.098004

24. Egres, R.G. and Wagner, N.J., The rheology and microstructure of acicular precipitated calcium carbonate colloidal suspensions through the shear thickening transition, *J. Rheol.*, 2005, vol. 49, pp. 719–746.  
https://doi.org/10.1122/1.1895800

25. Skulskiy, O.I., Slavnov, E.V., and Shakirov, N.V., The hysteresis phenomenon in nonisothermal channel flow of a non-Newtonian liquid, *J. Non-Newton. Fluid Mech.*, 1999, vol. 81, pp. 17–26.  
https://doi.org/10.1016/S0377-0257(98)00091-3

26. Aristov, S.N. and Skul’skii, O.I., Exact solution of the problem on a six-constant Jeffrey’s model of fluid in a plane channel, *J. Appl. Mech. Tech. Phys.*, 2002, vol. 43, pp. 817–822.  
https://doi.org/10.1023/A:1020752101539

27. Aristov, S.N. and Skul’skii, O.I., Exact solution of the problem of flow of a polymer solution in a plane channel, *J. Eng. Phys. Thermophys.*, 2003, vol. 76, pp. 577–585.  
https://doi.org/10.1023/A:1024768930375  

Translated by A. Sin’kov