THE CHARACTER DEGREE RATIO AND COMPOSITION FACTORS OF A FINITE GROUP

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Abstract. For a finite non-abelian group $G$ let $\text{rat}(G)$ denote the largest ratio of degrees of two nonlinear irreducible characters of $G$. We prove that the number of non-abelian composition factors of $G$ is bounded above by $1.8 \ln(\text{rat}(G)) + 1.3$.

1. Introduction

For a finite non-abelian group $G$ let $\text{rat}(G)$ be the largest ratio of degrees of two nonlinear irreducible characters of $G$. That is,

$$\text{rat}(G) = \frac{b(G)}{c(G)},$$

where $b(G)$ denotes the largest degree of an (ordinary) irreducible character of $G$ and $c(G)$ denotes the minimum degree of a nonlinear irreducible character of $G$. This ratio is often referred to as the character degree ratio of $G$. When $G$ is abelian, we adopt a convention that $c(G) = 1$ and $\text{rat}(G) = 1$.

The main result of this note shows that the number of non-abelian composition factors of an arbitrary finite group is logarithmically bounded above by its character degree ratio.

Theorem 1. The number of non-abelian composition factors of a finite group $G$ is bounded above by $1.8 \ln(\text{rat}(G)) + 1.3$.

In [6, Theorem C], I. M. Isaacs has shown that if $G$ is solvable then the derived length of $G$ is bounded by $3 + 4 \log_2(\text{rat}(G))$. Our Theorem 1 can be considered as a nonsolvable version of Isaacs’s result. Theorem 1 also significantly improves a result of J. P. Cossey and the second author [3, Theorem A] that if $S$ a non-abelian composition factor of $G$ different from the simple linear groups $\text{PSL}_2(q)$, then the number of times that $S$ occurs as a composition factor of $G$ is bounded in terms of...
rat(G). We refer the reader to [3, 7, 8, 10] for more discussion on the influence of the character degree ratio and character degrees in general on the structure of finite groups.

Let us now describe some ideas in the proof of the main result. Let \( O_\infty(G) \) denote the solvable radical of \( G \), i.e. \( O_\infty(G) \) is the maximal solvable normal subgroup of \( G \). Then \( O_\infty(G/O_\infty(G)) = 1 \) and the number of non-abelian composition factors of \( G \) equals to that of \( G/O_\infty(G) \). Moreover, as \( c(G/O_\infty(G)) \geq c(G) \) and \( b(G/O_\infty(G)) \leq b(G) \), we have \( \text{rat}(G/O_\infty(G)) \leq \text{rat}(G) \). Therefore in the proof of Theorem 1 we can assume that \( O_\infty(G) = 1 \). It follows that if \( N \) be a minimal normal subgroup of \( G \), then \( N \) is isomorphic to a direct product of copies of a non-abelian simple group \( S \).

One of the key results we need is [3, Theorem 1], which asserts that if \( S \) is a nonabelian simple group not isomorphic to \( \text{PSL}_2(q) \), then \( S \) has two non-principal irreducible characters \( \alpha \) and \( \beta \) which are extendible to \( \text{Aut}(S) \) such that \( \alpha(1)/\beta(1) > |S|^{1/14} \). This allows us to produce two character degrees of \( G \) of large ratio, and to obtain a better bound \( \text{ncf}(G) < 1.8 \ln(\text{rat}(G)) \) in the case \( S \not\cong \text{PSL}_2(q) \). At this point and from now on, we write \( \text{ncf}(G) \) to denote the number of non-abelian composition factors of \( G \).

It turns out that the case \( S \cong \text{PSL}_2(q) \) is exceptional and raises some complications. We first show that the number of copies of \( S \) in \( N \) is bounded above by \( \ln(\text{rat}(G)) + 1.3 \) and this solves Theorem 1 when \( G/N \) is solvable. When \( G/N \) is nonsolvable, we use another key result of J. P. Cossey, Z. Halasi, A. Maróti, and H. N. Nguyen [2, Theorem 6] on upper bound for the product of the orders of the non-abelian composition factors of a finite group \( G \) in terms of its largest character degree.

To end this introduction, we would like to make a couple of remarks. Firstly, though the bound obtained is of the right order of magnitude as shown by characteristically simple groups, we think that the constants 1.8 and 1.3 can be improved and it would be interesting to find the correct bounding constants. Indeed, we know of no finite groups with \( \text{ncf}(G) \geq \ln(\text{rat}(G)) + 1 \). Secondly, since the available proofs of [3, Theorem 1] and [2, Theorem 6] both depend on the classification of finite simple groups, Theorem 1 depends on the classification as well.

2. The simple linear groups \( \text{PSL}_2(q) \)

We start with the following result, which implies Theorem 1 in the case \( S \cong \text{PSL}_2(q) \) and \( G/N \) is solvable.

**Proposition 2.** Let \( S = \text{PSL}_2(q) \) where \( q \geq 5 \) is a prime power. Assume that \( N := S \times \cdots \times S \), a direct product of \( k \) copies of \( S \), is a minimal normal subgroup of \( G \). Then \( k < \ln(\text{rat}(G)) + 1.3 \).

**Proof.** Write \( N = S_1 \times \cdots \times S_k \) where \( S_i \cong S \) for every \( i = 1, 2, \ldots, k \). Let \( T = N_G(S_1) \), so \( |G : T| = k \). Furthermore \( S_1 \) can be considered as a subgroup of \( T/C_G(S_1) \), which in turn is isomorphic to a subgroup of \( \text{Aut}(S_1) \). Consider the so-called Steinberg
character $St_{S_1}$ of degree $q$ of $S_1$. It is well-known that $St_{S_1}$ is extendible to $\text{Aut}(S_1)$ (see [4] for instance). Therefore $St_{S_1}$ is extended to an irreducible character, say $\psi$, of $T/C_G(S_1)$. It follows that $St_{S_1}$ is extended to an irreducible character, say $\chi$, of $T$ whose kernel contains $C_G(S_1)$. Now since $S_2 \times \ldots \times S_k$ is inside $\text{Ker}(\chi)$, we conclude that the character $St_{S_1} \times S_2 \times \ldots \times S_k \in \text{Irr}(S_1)$ is extendible to $T$.

Observe that the stabilizer of $St_{S_1} \times S_2 \times \ldots \times S_k$ normalizes $S_1$, and $St_{S_1} \times S_2 \times \ldots \times S_k$ has $k$ conjugates under the action of $G$. Thus, $T$ must be the stabilizer of $St_{S_1} \times S_2 \times \ldots \times S_k$ in $G$. Applying Clifford’s theorem, we deduce that $St_{S_1}(1)k = kq$ is a character degree of $G$. In particular, $c(G) \leq qk$.

We now show that $b(G) \geq q^k$. Since $\text{Aut}(S)$ stabilizes $St_S$, the product character $St_S \times \ldots \times St_S \in \text{Irr}(N)$ is invariant under $\text{Aut}(N) = \text{Aut}(S)\text{Aut}_k$. Using [9] Lemma 1.3 (see also [11] Lemma 5]), we deduce that $St_S \times \ldots \times St_S$ is extendible to $\text{Aut}(N)$. Since $N$ can be embedded into $G/C_G(N)$ and $G/C_G(N)$ embeds into $\text{Aut}(N)$, it follows that $St_S \times \ldots \times St_S$ is extendible to $G/C_G(N)$ and hence $b(G) \geq St_S(1)k = q^k$.

Combining the bounds for $b(G)$ and $c(G)$ together, we obtain

$$\text{rat}(G) = \frac{b(G)}{c(G)} > \frac{q^k}{kq}.$$  

As it is easy to see that $k < 1.445^k$ for all positive integers $k$, it follows that $\text{rat}(G) > q^k/(q1.445^k) = q^{k-1}/1.445^k$. Therefore we obtain $\ln(\text{rat}(G)) > (k-1)\ln q - k(\ln(1.445)$, which yields

$$k < \frac{\ln(\text{rat}(G))}{\ln q - \ln 1.445} + \frac{\ln q}{\ln q - \ln 1.445}.$$  

It is straightforward to check that $\ln q - \ln 1.445 > 1$ and $\ln q/(\ln q - \ln 1.445) < 1.3$ for every $q \geq 5$. We now obtain the required bound $k < \ln(\text{rat}(G)) + 1.3$.  

As mentioned already, Proposition 2 implies the main result in the case $S \cong \text{PSL}_2(q)$ and $G/N$ is solvable. By making use of the following deep result, we can prove the bound $\text{ncf}(G) < 1.4\ln(\text{rat}(G))$ whenever $S = \text{PSL}_2(q)$ and $G/N$ is non-abelian.

**Theorem 3** (Theorem 6 of [2]). Let $G$ be a finite group. Then the product of the orders of the non-abelian composition factors of $G$ is at most $b(G)^3$.

**Proposition 4.** Let $S = \text{PSL}_2(q)$ where $q \geq 5$ is a prime power. Assume that $N := S \times \ldots \times S$, a direct product of $k$ copies of $S$, is a minimal normal subgroup of $G$ such that $G/N$ is non-abelian. Then $\text{ncf}(G) < 1.4\ln(\text{rat}(G))$.

**Proof.** As in the proof of Proposition 2 the character $St_S \times \ldots \times St_S \in \text{Irr}(N)$ is extended to an irreducible character, say $\chi$, of $G/N$, which can also be viewed as an irreducible character of $G$. By Gallagher’s Theorem [5] Corollary 6.17, there is a bijection $\lambda \leftrightarrow \lambda\chi$ between $\text{Irr}(G/N)$ and the set of irreducible characters of $G$ lying
above \(\text{St}_S \times \cdots \times \text{St}_S\). In particular, by taking \(\lambda\) to be an irreducible character of \(G/N\) of the largest degree, we deduce that
\[
b(G/N)\chi(1) = b(G/N)\text{St}_S(1)^k = b(G/N)q^k
\]
is an irreducible character degree of \(G\). As it is clear that \(c(G/N) \geq c(G)\) and note that \(G/N\) is non-abelian, it follows that
\[
\text{rat}(G) \geq \frac{b(G/N)q^k}{c(G/N)}.
\]
In particular,
\[
(2.1) \quad k \leq \frac{\ln(\text{rat}(G))}{\ln q}.
\]

We now aim to bound \(\text{ncf}(G) - k\) in terms of \(\text{rat}(G)\). Since \(q^k\) is a character degree of \(G\), we have \(c(G) \leq q^k\) and hence
\[
\text{rat}(G) \geq \frac{b(G/N)q^k}{q^k} = b(G/N).
\]
Applying Theorem 3, we deduce that \(\text{rat}(G)^3\) is at least the product of the orders of the non-abelian composition factors of \(G/N\).

Note that the number of non-abelian composition factors of \(G/N\) is \(\text{ncf}(G) - k\) and the order of each non-abelian composition factor is at least 60. It follows that
\[
\text{rat}(G) \geq 60^{(\text{ncf}(G) - k)/3},
\]
which yields
\[
(2.2) \quad \text{ncf}(G) - k \leq \frac{3\ln(\text{rat}(G))}{\ln 60}.
\]

Inequalities (2.1) and (2.2) then imply that
\[
\text{ncf}(G) \leq \frac{\ln(\text{rat}(G))}{\ln q} + \frac{3\ln(\text{rat}(G))}{\ln 60},
\]
and thus the proposition follows. \(\square\)

We have proved Theorem 4 in the case \(S \cong \text{PSL}_2(q)\).

3. Other non-abelian simple groups

We now handle the case \(S \not\cong \text{PSL}_2(q)\).

Proposition 5. Let \(S\) be a non-abelian simple group different from \(\text{PSL}_2(q)\). Assume that \(N := S \times \cdots \times S\) is a minimal normal subgroup of \(G\). Then \(\text{ncf}(G) < 1.8\ln(\text{rat}(G))\).

Our current proof of Proposition 5 is largely based on Theorem 3 and the following result, whose proofs both depend on the classification of finite simple groups.
**Theorem 6** (Theorem 1 of [3]). *Let $S$ be a non-abelian simple group different from $\text{PSL}_2(q)$. Then $S$ has two non-principal irreducible characters $\alpha$ and $\beta$ which are extendible to $\text{Aut}(S)$ such that
\[
\frac{\alpha(1)}{\beta(1)} > |S|^{1/14}.
\]

*Proof of Proposition 5.* By Theorem 6, the simple group $S$ has two non-principal irreducible characters $\alpha$ and $\beta$ which are extendible to $\text{Aut}(S)$ such that
\[
\frac{\alpha(1)}{\beta(1)} > |S|^{1/14}.
\]
The product characters $\alpha_1 := \alpha \times \cdots \times \alpha$ and $\beta_1 := \beta \times \cdots \times \beta$ are then invariant under $\text{Aut}(N) = \text{Aut}(S) \rtimes S_k$. Arguing as before, we see that $\alpha(1)^k$ and $\beta(1)^k$ are irreducible character degrees of $G$.

As in the proof of Proposition 4, one can show that $b(G/N)\alpha(1)^k$ is an irreducible character degree of $G$. Together with the conclusion of the last paragraph, we have
\[
\text{rat}(G) \geq \frac{b(G/N)\alpha(1)^k}{\beta(1)^k}.
\]
As $\alpha(1)/\beta(1) > |S|^{1/14}$, we obtain
\[
\text{rat}(G) > b(G/N)|S|^{k/14},
\]
and hence
\[
\frac{\text{rat}(G)}{|S|^{k/14}} > b(G/N).
\]
Applying Theorem 3 again, we then have that $\text{rat}(G)^3/|S|^{3k/14}$ is at least the product of the orders of non-abelian composition factors of $G/N$.

Since the number of non-abelian composition factors of $G/N$ is precisely equal to $\text{ncf}(G) - \text{ncf}(N) = \text{ncf}(G) - k$, it follows that
\[
\frac{\text{rat}(G)^3}{|S|^{3k/14}} > 60^{\text{ncf}(G)-k}.
\]
Note that the order of any $S \neq \text{PSL}_2(q)$ is at least 2520. We deduce that
\[
\text{rat}(G) > 2520^{\text{ncf}(G)/14},
\]
which in turns implies that
\[
\text{ncf}(G) < 1.8 \ln(\text{rat}(G)),
\]
as claimed.

□
4. Proof of the main result

Proof of Theorem 1. As mentioned in the Introduction, to prove the theorem it suffices to assume that $O_\infty(G) = 1$. Let $N$ be a minimal normal subgroup of $G$. As $G$ has trivial solvable radical, $N$ is direct product of copies of a non-abelian simple group $S$.

When $S = \text{PSL}_2(q)$ and $G/N$ is solvable we have $\text{ncf}(G) < \ln(\text{rat}(G)) + 1.3$ by Proposition 2. Otherwise we have the bound $\text{ncf}(G) < 1.8 \ln(\text{rat}(G))$ by Propositions 4 and 5. The theorem is now completely proved. \hfill \Box

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