The effects of a minimal length are investigated within an algebraically extended theory of General Relativity (GR). Former attempts to include a minimal length in GR are first resumed, with a conformal factor of the metric as a consequence. Effective potentials for various black hole masses (as ratios to the minimal length) are deduced. It is found that the existence of a minimal length has, for a small mass black hole, important effects on the effective potential near the event horizon, creating barriers which inhibit that particles can pass the event horizon. Further, a new limit for the minimal mass of a black hole is derived.

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1. Introduction

General Relativity (GR) is a well established theory for describing numerous observations in the solar system\(^1\) and in cosmology, it is one of the best tested theories. Nevertheless, it is not decided if GR is complete to describe phenomena in very strong gravitational fields, as in the vicinity of a black hole. For example, quantization effects are expected to play a role. There are further problems, as the loss of information and the appearance of an event horizon which disconnects an outer observer to the interior of a black hole (though, many do not think that this is a problem).
Therefore, in very strong gravitational fields one can speculate on how possibly to extend GR. Einstein himself intended to modify his theory such that it contains Electrodynamics.\textsuperscript{2,3} In order to do so, he introduced a complex metric. Another approach was followed by M. Born,\textsuperscript{4,5} who complained about the dominant role of coordinates in GR, contrary to Quantum Mechanics where the momentum is treated on an equal footing. M. Born tried to solve the problem by introducing the concept of \textit{complementarity}, modifying the length element such that it is defined in the coordinate-momentum space.

Later, Caianiello and his coworkers retook the ideas of M. Born and wrote a series of papers about a theory called Quantum Geometry\textsuperscript{6–12} (for the formulation in differential geometry see\textsuperscript{13}). In\textsuperscript{14} a pseudo-complex extension of GR (pcGR) was proposed, with many applications,\textsuperscript{15} and in\textsuperscript{16} the most recent review is given including several predictions. M. Born’s theory corresponds to a particular limit in pcGR.\textsuperscript{16} It was shown that deviations to GR, for macroscopic black holes, only appear near the event horizon (not having yet being resolved due to resolution problems). In all applications of pcGR the minimal length was neglected, because its effects where considered to be negligible.

In earlier contributions\textsuperscript{17} (and references therein) the effects of a minimal length on the structure near the event horizon were investigated within GR. The questions, we would like to answer, are: What happens when GR is extended algebraically to pcGR, which includes the effects of a dark energy in the vicinity of a black hole? In what range of the black hole mass the effects of a minimal length are noticeable? In\textsuperscript{17} the effects of the \textit{maximal acceleration}, as a parameter, were studied and not the role of a minimal length and its relation to the mass of the black hole (though, they are related). As we will see the minimal length affects small mass black holes, which may have been created during the Big Bang.

The inclusion of a minimal length in the extended theory is quite involved and we have to recur to approximations, as restricting to the Schwarzschild solution only.

The paper is structured as follows: In Section 2 the theory proposed by E. R. Caianiello is resumed. In Section 3 the effects of the minimal length in pcGR in the Schwarzschild case is discussed and finally in Section 4 the results are interpreted and conclusions are drawn.

\section{2. The theory proposed by E. R. Caianiello}

This model\textsuperscript{6–12,17} interprets the quantization via a curvature of the relativistic eight-dimensional space-time tangent bundle $TM = M_4 + TM_4$, with $M_4$ as the usual flat space-time manifold with the metric $\eta_{\mu\nu}$ (see also\textsuperscript{13}). This description satisfies the Born reciprocity principle\textsuperscript{4,5} and it incorporates the notion that the proper accelerations of massive particles along their worldliness with upper limit $A_m$, referred to as maximal acceleration (MA). Indeed the usual Minkowski line element $d^2s = \eta_{\mu\nu}dx^\mu dx^\nu$ is replaced by the line element in the eight-dimensional...
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space-time tangent bundle $TM$, with $\eta_{\mu\nu}$ substituted by $g_{\mu\nu}$,

$$d^2 w = g_{AB}dX^A dX^B, \quad A, B = 0, \ldots, 7,$$

(1)

where

$$g_{AB} = g_{\mu\nu} \otimes g_{\mu\nu},$$

$$X^A = \left(x^\mu, \frac{c^2}{A_m} \frac{dx^\mu}{ds}\right), \quad \mu = 0, \ldots, 3,$$

(2)

$x^\mu = (ct, \vec{x})$ is the usual space-time four-vector and $\dot{x}^\mu = u_\mu = \frac{dx^\mu}{ds}$ the four-velocity, $c$ is the speed of light in the vacuum. The velocity components are strictly speaking only valid in flat space and it is a consequence of the dispersion relation.

As noted above, E.R. Caianiello substituted the $\eta_{\mu\nu}$ by a general metric $g_{\mu\nu}$, where the relation is approximate.

The line element in the ordinary four-dimensional space-time $M_4$, in which a particles moves when the constraints of a maximal acceleration is present, is rewritten such that the length element has the usual form augmented by a conformal factor. In fact, we can calculate the effective four dimensional metric $\tilde{g}_{\mu\nu}$ on the hypersurface locally embedded in $TM_4$. In this case, the line element becomes

$$d^2 w = (1 + g_{\mu\nu}a_\mu a_\nu) \frac{d^2 s}{A_m^2}$$

(3)

$$d^2 w = \tilde{g}_{\mu\nu} dx^\mu dx^\nu$$

(4)

with $d^2 s = g_{\mu\nu} dx^\mu dx^\nu$ and $\tilde{g}_{\mu\nu} = (1 + g_{\mu\nu}a_\mu a_\nu)g_{\mu\nu} = \sigma^2(r)g_{\mu\nu}$,

(5)

with the four-acceleration $\dot{x}^\mu = a^\mu = c^2 \frac{\dot{x}^\mu}{d^2 s}$ (ds = cd$\tau$, with $\tau$ as the eigentime).

It is quite involved to obtain the exact form for the conformal factor $\sigma^2(r)$ and one recurs to an iterative procedure: In the first step, the equations of motion with $\sigma^2(r) = 1$ are deduced in the usual manner, solving the geodesic equations. These give the components of the 4-acceleration $a_\mu^{(0)}$. Substituting these components into $\sigma^2(r)$ results in

$$\sigma^2_{(1)}(r) = 1 + \frac{g_{\mu\nu}a_\mu^{(0)}a_\nu^{(0)}}{A_m^2}$$

(6)

The index $(i)$ denotes the order of the iteration. We can iterate this procedure by resolving the equations of motion with the $\sigma^2_{(i)}$, obtaining the next iteration $a_\mu^{(i)}$ for the 4-acceleration. As a result we obtain the next iterative expression for $\sigma^2 = \sigma_{(2)}$, namely

...
This iteration procedure can be continued, however, in general it is stop after the second iteration and study the effects on the solutions, the form of the effective potential, etc.

After a second iteration, the modified length element acquires the form

$$d^2w = \left(1 + \frac{g_{\mu\nu}a_{(1)}^\mu a_{(1)}^\nu}{A_m^2}\right)d^2s$$

(8)

$$\tilde{g}_{\mu\nu} = \left(1 + \frac{g_{\mu\nu}a_{(1)}^\mu a_{(1)}^\nu}{A_m^2}\right)g_{\mu\nu}$$

(9)

Limiting values for the accelerations were also derived by several authors on different grounds and applied to many branches of physics such as string theory, cosmology, quantum field theory, black hole physics etc.\textsuperscript{13,18–29} In\textsuperscript{17} this maximal acceleration is obtained as $A_m = \frac{2mc^3}{\hbar}$, where the causal limit was taken into account. Another group starts from an eight dimensional space by introducing pseudo-complex variables and projecting this space to a four dimensional physical space.\textsuperscript{30–33}

Up to here, the notion of a maximal acceleration is restricted to GR, formulated with the coordinates $x^\mu$. The central question of our contribution is: What happens when not only a maximal acceleration is introduced, but also the coordinates are algebraically extended\textsuperscript{34} to pseudo-complex coordinates

$$X^\mu = x^\mu + l y^\mu$$

(10)

with $l^2 = 1.$\textsuperscript{34}

In\textsuperscript{14} just such an extension of the general relativity was proposed, called the pseudo-complex General Relativity (pcGR), with the most recent review given in.\textsuperscript{16} The pseudo-complex extension is proven to be the only consistent algebraic extension of GR,\textsuperscript{34} not having neither ghost nor tachyon solutions. In pseudo-complex General Relativity the space-time coordinates are of the form $X_\mu = x_\mu + \frac{l}{c}u_\mu$, with its pseudo-real part $x^\mu$ and its pseudo-imaginary component, chosen to have the form $y^\mu = \frac{L}{c}u_\mu$ in analogy to.\textsuperscript{18} The component $\frac{L}{c}u_\mu$ is an approximation, strictly speaking valid only in flat space, and can be associated to the components of the tangent vector (four velocity vector) at a given space-time point. The factor $l$ has the unit of a length, which is introduced due to dimensional reasons. The consequence of that is the appearance of a minimal length scale $l$ within the theory which implies a maximal acceleration. As mentioned above, we assume that the minimal length scale is of order of the Planck length ($l_P = \sqrt{\frac{\hbar k}{2\pi}} = 1.616199 \times 10^{-35}m$, $k$ is the gravitational constant and $\hbar$ is the reduced Planck constant).
The line element in pcGR is given by

$$d^2 w = g_{\mu\nu}(X, P) DX^\mu DX^\nu,$$

(11)

where $X^\mu$ and $P_\nu$ are pseudo-complex.

Further assumptions to the metric are applied, namely that $g_{\mu\nu}(X) = g_{\mu\nu}(x)$, i.e. that it does not depend on the four-velocity, and with the constraint that the length element itself is pseudo-real, which is the case when the dispersion relation is satisfied. With this, the length element in pcGR acquires the form

$$d^2 w = g_{\mu\nu}(x) \left[ dx^\mu dx^\nu + \left( \frac{l}{c} \right)^2 du^\mu du^\nu \right]$$

(12)

In this extension the geometry is understood as a consequence of curvature in eight-dimensional phase space, in which the coordinates of the velocity-space manifold are the components of the four-velocity with a dimensional factor $l$ of the order of the Planck length $l_P$. In (12) the length element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is extracted and we use $ds = c d\tau$, with $d\tau$ as the eigentime. We obtain

$$d^2 w = d^2 s \left( 1 - \frac{l^2}{c^4} |a^\mu a^\nu| \right)$$

(13)

$$d^2 w = \tilde{g}_{\mu\nu} dx^\mu dx^\nu$$

$$\tilde{g}_{\mu\nu} = \left( 1 - \frac{l^2}{c^4} |a^\mu a^\nu| \right) g_{\mu\nu}
$$

(14)

with $\tilde{g}_{\mu\nu}$ representing the modified metric and $a^\mu$ the acceleration component $\frac{d^2 x^\mu}{dt^2}$, with $\tau$ as the eigentime.

Therefore, one can rewrite the length element as

$$d^2 w = \sigma^2(\tau) d^2 s$$

with

$$\sigma^2(\tau) = \left( 1 - \frac{l^2}{c^4} |a^\mu a^\nu| \right).$$

(15)

It is clear that the maximal acceleration is included automatically in pseudo-complex general relativity ($A_m = \frac{c^2}{l}$). The new metric $\tilde{g}_{\mu\nu}$ depends on the coordinates $x_\mu$ and on the acceleration field $a_\mu$. In this case, the generalized proper-time interval (13) becomes

$$d^2 w = \left( 1 - \frac{l^2}{c^2} |a|^2 \right) d^2 s,$$

(16)

which restricts the acceleration, as before, to a finite interval with a maximal acceleration $\frac{c^2}{l} = \frac{2mc^2}{\hbar}$. In it was applied to a pc-field theory with the important
feature that it is by construction regularized, due to the appearance of a minimal length scale \( l \). Due to the fact that this minimal length scale is a parameter, it is not affected by a Lorentz transformation and, thus, all symmetries are preserved, which is a huge simplification compared to theories which require a violation of the Lorentz symmetry. In\(^{16,36}\) this fact was used in the hope that it prevents the formation of a black hole. Indeed the formation of a black hole is avoided in this theory, but not due to the minimal length scale but due to the appearance of dark energy which appears naturally within this theory. The same effects appear when it was applied to the Robertson-Walker universe.\(^{15,36}\) In all cases, the variational principle introduced contributions, which can be interpreted as dark energy, acting repulsively such that the formation of an event horizon and a singularity at the center is avoided. This is a most important result, as any proper theory should not contain singularities.

Finally, one has to note that the second term in the conformal factor \( \sigma^2 \) is only a covariant quantity in flat space, where the dispersion relation \( \eta_{\mu\nu} dx^\mu dy^\nu = 0 \), with \( y^\mu = \left( \frac{1}{c} \right) u^\mu \) (\( u^\mu \) is the 4-velocity) is valid.\(^{17}\) In a curved space, one has to solve a complicated equation, which is the result of the constraint that the pseudo-complex length element is real, i.e., the pseudo-imaginary term has to vanish. For more details and first attempts to solve it, please consult.\(^{16}\) This is a price to pay when the iteration procedure is used.

### 3. The Pseudo-complex Schwarzschild geometry

In pcGR it is assumed that a central mass generates, due to quantum effects, a distribution of dark energy in its vicinity, which is based on theoretical grounds using semi-classical Quantum Mechanics.\(^{37}\) The distribution is parametrized as \( B_n/r^n \), with two phenomenological parameters, \( B_n \) and \( n \). The \( B_n = bm^n \) describes the coupling of the central mass to the dark energy and \( n \) its fall-off as a function of the radial distance. This is the simplest ansatz and one easily can add further complicated dependencies in \( r \). In\(^{14} \) \( n \) was set to 2, which results in a metric already excluded by solar system experiments.\(^1\) In\(^{14,38,39} \) the \( n \) was equal to 3 and several observable predictions were made. Using the first observation of gravitational waves,\(^{40}\) in\(^{41,42} \) it is shown that \( n = 3 \) is also excluded, thus \( n = 4 \) is assumed now. In this case the pseudo-complex Schwarzschild solution is

\[
d^2 s = \left( 1 - \frac{2m}{r} + \frac{\Omega}{2r^2} \right) d^2 t - \frac{d^2 r}{\left( 1 - \frac{2m}{r} + \frac{\Omega}{2r^2} \right)} - r^2 (d^2 \theta + \sin^2 \theta d^2 \phi) \tag{17}\]

\[
\Omega(r) = \frac{B_4}{3} r^{-3} \tag{18}\]

The results for \( n = 3 \) are quite similar, changing only details.

In\(^{15} \) the \( B_4 \) parameter was varied within the pc-Kerr solution from zero (GR) to a maximal value, from which on no event horizon exists anymore. The transition
from the existence of an event horizon to its disappearance could be related to a phase transition. However, again the effects of a minimal length were neglected. Here, we also will consider the whole range of $B_4$, i.e., investigating the transition from GR to pcGR, but now with the inclusion of a minimal length.

With the approximation applied, the line element with a minimal length $l$ is given by the relation (16), which is similar in form as given by E. R. Caianiello.

In the following sub-section the factor $\sigma^2(l)$ will be investigated further.

### 3.1. The factor $\sigma^2(r)$

In order to calculate the corrections to the pc-Schwarzschild metric, experienced by a particle and its motion along a geodesic, one must determine the factor $\sigma^2(r)$ of the order of $l^2$. Before that, we will impose some conditions on the factor $\sigma^2(r)$. By using the embedding procedure mentioned above of the order $l^2$, the line element acquires the form (16), which has to be positive definite, i.e.,

$$d^2 w > 0 \iff |a^2| < \frac{c^4}{l^2},$$  \hspace{1cm} (19)

This implies that the acceleration is limited from above by $A_m = c^2/l$, called the maximal acceleration. With this condition, the factor $\sigma^2(r)$ is limited by 0 from below, when the acceleration is maximal, and by 1 from above, when the acceleration is zero:

$$0 \leq \sigma^2(r) \leq 1$$  \hspace{1cm} (20)

As a consequence, the pcGR implies a maximal acceleration, as in GR.$^{6-12}$

In order to deduce the explicit expression of $\sigma^2(r)$ we recur again to an iteration procedure (restricting now to the first iteration), which provides an approximate solution for this factor: In the first iteration, to which we will restrict, the metric with $\sigma^2 = 1$ is taken and the corresponding geodesic equations are derived and solved. To obtain these geodesic equations, we start from the variational integral principle

$$\delta \int \left[ \left(1 - \frac{2m}{r} + \frac{B_4}{6r^4}\right) \left(t_0\right)^2 - \frac{1}{\left(1 - \frac{2m}{r} + \frac{B_4}{6r^4}\right)} \left(\dot{r}\right)^2 - r^2 \left(\dot{\theta}\right)^2 + \sin^2 \theta \left(\dot{\phi}\right)^2 \right] ds^2 = 0,$$  \hspace{1cm} (21)

where the dot refers to the derivative with respect to the variable $s$. The variation leads to equations of motion
\[
ds(s) = r^2 \sin \theta \cos \phi
\]
\[
ds(s) = 0
\]
\[
ds\left[\left(1 - \frac{2m}{r} + \frac{B_4}{6r^4}\right)t\right] = 0
\]
from where the accelerations \(a^{(0)}_\mu\) in this first iteration are obtained.

In this completely relativistic metric of the pc-Schwarzschild, all orbits will be still in the orbital plane defined by

\[
\theta = \frac{\pi}{2}; \quad \dot{\theta} = 0
\]

With this, the equations of motion for the energy and the angular momentum, respectively, are

\[
\left(1 - \frac{2m}{r} + \frac{B_4}{6r^4}\right)t = E
\]
\[
r^2 \phi = L
\]

In addition, we can use \(\frac{ds^2}{ds^2} = 1\), which leads to a third equation, namely

\[
1 = \left(1 - \frac{2m}{r} + \frac{B_4}{6r^4}\right)c^2t^2 - \frac{1}{\left(1 - \frac{2m}{r} + \frac{B_4}{6r^4}\right)r^2 - r^2 \phi^2}
\]

Substituting Eqs. (26) and (27) into (28), we obtain

\[
\frac{\ddot{r}}{r} = E^2 - \left(1 - \frac{2m}{r} + \frac{B_4}{6r^4}\right)\left(\frac{L^2}{r^2} + 1\right)
\]

Substituting the last results into \(\sigma^2 = \left(1 - \frac{\ell^2}{r^2} |a^2|\right)\) leads to \(\sigma_{(1)}^2\) in terms of \(t, \frac{\ddot{t}}{t}, \ddot{r},\) and \(\ddot{\phi}\)

\[
\sigma_{(1)}^2(r) = \left\{1 - \frac{\ell^2}{c^4} \left[1 - \frac{2m}{r} + \frac{B_4}{6r^4}\right] \frac{\ddot{t}}{t} + \frac{1}{\left(1 - \frac{2m}{r} + \frac{B_4}{6r^4}\right)r^2 + r^2 \phi^2}\right\}
\]
with
\[
\ddot{t} = \frac{-E}{c^2 (1 - \frac{2m}{r} + \frac{B^2}{r^2})^2} \left( \frac{2m}{r^2} - \frac{2B^4}{3r^5} \right) \dot{r},
\]
\[
\ddot{r} = \left( -\frac{m}{r^2} + \frac{L^2}{r^3} - \frac{3mL^2}{r^4} + \frac{B^4}{3r^5} + \frac{2B^4L^2}{6r^7} \right),
\]
\[
\ddot{\phi} = -\frac{2L}{r^3} \dot{r},
\]
\[
\ddot{\sigma} = \left[ \frac{E^2}{c^2} \left( 1 - \frac{2m}{r} + \frac{B^4}{6r^4} \right) \left( \frac{L^2}{r^2} + 1 \right) \right] (31)
\]

The $m$ is the mass of the source in units of length, $L$ the angular momentum and $E$ the energy, with $E$ and $L$ being constant of motions, in units of the particle mass $M$. In what follows, we will present some plots of the factor $\sigma^2(r)$, in order to show how $\sigma^2$ varies as a function of the a-dimensional variables.

\[
\rho = \frac{r}{m}, \quad \lambda = \frac{L}{m}, \quad \epsilon = \frac{l}{m}, \quad \alpha = \frac{B}{m^4}. \quad (32)
\]

The dependence of $\sigma^2$ on different parameter values $\epsilon$ ($l$) and $B$ ($\alpha$) is shown in Figs. 1 and 2. As can be seen, there are two effects: i) The deviations to $\sigma^2 = 1$ become noticeable the more $\alpha$ approaches from above $\frac{81}{8}$, where still an event horizon exists, for larger $\alpha$ the deviations are smoothed out. When $\alpha$ is smaller than $\frac{81}{8}$ the deviations increase. In all cases, $\sigma^2$ is strongly lowered below $\frac{3}{m}$, where still an event horizon exists, for larger $\alpha$ the deviations are smoothed out. When $\alpha$ is smaller than $\frac{81}{8}$ the deviations increase. In all cases, $\sigma^2$ is strongly lowered below $\frac{3}{m}$. ii) When $\dfrac{\rho}{m}$ is large, the $\sigma^2$ approaches 1. Note, that $\epsilon = \dfrac{l}{m} = 0.001$ implies a small mass only thousand time larger than the minimal length. For a large mass, for example of the size of a regular star, only small or no effects are seen.
Fig. 2. Dashed line: the factor $\sigma^2(\rho)$ for $\lambda = 0$, $\alpha = 8, \epsilon = 0.1$, $E = 1$. Solid line: the factor $\sigma^2(\rho)$ for $\lambda = 0, \alpha = 81/8, \epsilon = 0.1$ and $E = 1$ and Dotted line: the factor $\sigma^2(\rho)$ for $\lambda = 0, \alpha = 12, \epsilon = 0.1$ and $E = 1$.

To resume, $\sigma^2(\rho = \frac{r}{m}) \rightarrow 1$ as $\rho = \frac{r}{m} \rightarrow \infty$, as it should be because at large $r$ the result of standard GR has to be reproduced and $\sigma^2$ has to acquire the value 1. Taking $\alpha = (81/8)$ and $\alpha = 8$, the factor $\sigma^2(\rho)$ shows a divergence near $\rho = 3/2$ and near $\rho_+ = \frac{(1+\sqrt{5})}{2}, \rho_- = 1$ respectively (see Fig. 1 and 2). In Fig. 1 two lines (dashed and solid) are depicted. The values $\lambda, \alpha$ and $E$ are kept constant, while the dashed line is for $\epsilon = 0.1$ (black hole mass ten times $l$) and the solid line for $\epsilon = 0.001$ (a thousand times $l$). For $\sigma^2$ the behavior is regular and finite for $\alpha = 12$, above the critical value, even when the mass is only ten times $l$, indicating the (see Fig 2) vanishing of event horizons.

A vanishing $d^2w$ also indicates that the effective distance at this point is zero.

3.2. Event Horizons

In order to give an interpretation for the divergences of the factor $\sigma^2(r)$, where it passes zero and tends to $-\infty$ (note, that a negative value of $\sigma^2$ is not allowed, i.e., it corresponds to a non-physical region), it is useful to determine the position of the event horizon(s) (there may be several) for different values of $B_4 = \alpha m^4$. For example, in the GR-Schwarzschild solution, the surface $r = 2m$ is an event horizon. It is also a surface of infinite red-shift. As will be see below, these surfaces are different in the pc-Schwarzschild solution. We begin by investigating the possible positions of the event horizons more explicitly.

A null surface satisfies the equation

$$f(x^0, x^1, x^2, x^3) = 0,$$

with $x^0 = ct, x^1 = r, x^2 = \theta$ and $x^3 = \phi$.

In the case of the Schwarzschild solution in GR the function $f$ for the event horizon is given by

$$f(x^0, x^1, x^2, x^3) = 0$$
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\( f = r - 2m = x^1 - 2m \)

Let us turn to pc-Schwarzschild solution, for which \( g_{\mu\nu} \) is given by

\[
g_{\mu\nu} = \begin{pmatrix}
1 - \frac{2m}{r} + \frac{B_4}{6r^4} & 0 & 0 & 0 \\
0 & 1 - \frac{1}{1 - \frac{2m}{r} + \frac{B_4}{6r^4}} & 0 & 0 \\
0 & 0 & -r^2 & 0 \\
0 & 0 & 0 & -r^2 \sin^2 \theta
\end{pmatrix}
\]  \hspace{1cm} (34)

The \( B_4 \) can, in principle, acquires any value and we will consider the whole range, from 0 to above \( \frac{81}{8} m^4 \).

According to the possible values of \( B_4 = \alpha m^4 \), we can distinguished three cases:

1. When \( 0 \leq \alpha < \frac{81}{8} \), there are two event horizons at \( r_{\pm} \), for example for \( \alpha = 8 \) we get \( r_+ = 1.75211m \) and \( r_- = 1.17221m \), which agrees with.\(^{43}\)
2. For \( \alpha = \frac{81}{8} \) the two event horizons \( r_+ \) and \( r_- \) merge into a single surface at \( r_e = \frac{3m}{2} \) as in.\(^{43}\) This case is mainly taken for the applications of pcGR.\(^{16}\)
3. For \( \alpha > \frac{81}{8} \) there are no event horizons.

For all cases, we investigate the effects within the pc-Schwarzschild geometry. The null hypersurface is determined by \( (1 - \frac{2m}{r} + \frac{B_4}{6r^4}) = 0 \). It is clear that, the factor \( \sigma^2(r) \) diverges at certain points for \( \alpha \leq \frac{81}{8} \) on the two existing event horizons. For example \( \alpha = 8 \), the factor \( \sigma^2(r) \) has two divergences at \( r_+ = 1.75211m \) and \( r_- = 1.17221m \), respectively (see Fig. 2, dashed line). Below and above each singularity, the \( \sigma^2(r) \) function has to go through zero. According to the position of the divergences \( r_{\pm} \) we call these positions, where \( \sigma^2(r) \) has a zero, \( r_{\pm 1} \) and \( r_{\pm 2} \), respectively.

In fact, in the case \( B_4 < \frac{81}{8} m^4 \) (more specifically, the example \( \alpha = 8 \) is shown) the factor \( \sigma^2(r) = 0 \) has five possible values for \( r \) (as explained above), one is near center \( (r = 0) \) \( r_0 \), two \( r_{-1}, r_{-2} \) are below and above the \( r_- \), thus \( r_{-1} < r_- < r_{-2} \) and two \( r_{+1}, r_{+2} \) are below and above the \( r_+ \), i.e. \( r_{+1} < r_+ < r_{+2} \). For \( B_4 = \frac{81}{8} m^4 \) \( (\alpha = \frac{81}{8} \) ) there are three solutions \( r \) where \( \sigma^2(r) = 0 \). One is always near the center \( r = 0 \) and two \( r_{e1}, r_{e2} \) are near \( r_e \), where the index \( e \) refers to the event horizon. For \( B_4 > \frac{81}{8} m^4 \) (more specific \( \alpha = 12 \)), there is one value of \( r \) near the center in which \( \sigma^2(r) = 0 \). Therefore, the factor \( \sigma^2(r) \) is always negative between \( (0, r_0), (r_{-1}, r_{-2}) \) and \( (r_{+1}, r_{+2}) \). As is known, the factor \( \sigma^2(r) = 0 \) represents a particle with a maximal acceleration \( \frac{\dot{r}^2}{r} \), thus, the zeros of the \( \sigma^2(r) \) reveals these positions of maximal acceleration.

3.3. The effective Potential

In order to make a direct comparison with the motion in the pc-Schwarzschild geometry possible, we adopt the same procedure as before. The new action is given
by

\[
S = \int \sigma^2(r) \left\{ \left( 1 - \frac{2m}{r} + \frac{B}{6r^4} \right) t^2 - \left( \frac{1}{1 - \frac{2m}{r} + \frac{B}{6r^4}} \right)^2 - r^2 \left( \theta^2 + \sin^2 \theta \phi \right)^2 \right\} dw
\]

The equation of motions become (\( \theta = \frac{\pi}{2} \))

\[
\sigma^2(r) \left( 1 - \frac{2m}{r} + \frac{B}{6r^4} \right) t = E
\]

\[
\sigma^2(r) r^2 \phi = L
\]

In addition we use \( \frac{d\sigma}{dw}^2 = 1 \), i.e.,

\[
1 = \sigma^2(r) \left\{ \left( 1 - \frac{2m}{r} + \frac{B}{6r^4} \right) \left( \frac{dt}{dw} \right)^2 - \left( \frac{1}{1 - \frac{2m}{r} + \frac{B}{6r^4}} \right)^2 - r^2 \left( \frac{d\theta}{dw} \right)^2 - r^2 \left( \frac{d\phi}{dw} \right)^2 \right\}
\]

Substituting the equations (29) and (30) into equation (31), we get

\[
\frac{1}{(1 - \frac{2m}{r} + \frac{B}{6r^4})} \left( \frac{dr}{dw} \right)^2 = \frac{E^2}{\sigma^4(r) \left( 1 - \frac{2m}{r} + \frac{B}{6r^4} \right)} - \frac{L^2}{\sigma^4(r) r^2} - \frac{1}{\sigma^2(r)}
\]

Solving for \( \left( \frac{dr}{dw} \right)^2 \), we obtain

\[
\left( \frac{dr}{dw} \right)^2 = \left( 1 - \frac{2m}{r} + \frac{B}{6r^4} \right) \left[ \frac{E^2}{\sigma^4(r) \left( 1 - \frac{2m}{r} + \frac{B}{6r^4} \right)} - \frac{L^2}{\sigma^4(r) r^2} - \frac{1}{\sigma^2(r)} \right]
\]

Comparing both sides we obtain for the effective potential

\[
V_{eff}(r) = E^2 - \frac{E^2}{\sigma^4(r)} + \left( 1 - \frac{2m}{r} + \frac{B}{6r^4} \right) \left( \frac{L^2}{\sigma^4(r) r^2} + \frac{1}{\sigma^2(r)} \right)
\]

In a more traditional approach, one writes

\[
\frac{1}{2} \left( \frac{dr}{dw} \right)^2 + P_{eff}(r) = \omega,
\]

where

\[
P_{eff}(r) = \frac{1}{2} \left( V_{eff}^2(r) - 1 \right) \text{ and } \omega = \frac{1}{2} (E^2 - 1)
\]

In order to plot the effective potential, it is preferable to use the a-dimensional variables (32) and the effective potential becomes, with \( \rho = \frac{r}{m} \),
Inspecting (44), one notes that the potential exhibits singularities as soon as $\sigma^2(r)$ approaches zero. On the other hand, when $\sigma^2(r) \to -\infty$ the two last terms tend to zero and the effective potential approaches $E^2$.

For a selected set of parameter values, the effective potential is plotted in the Figs. 3 to 11. In all examples one notes the important feature that for all $\alpha$ and small $\epsilon$ (large masses) the effective potential behaves smoothly, showing a repulsive behavior toward small distances. It shows a minimum between 1 and 1.5 which finally increases toward larger $r$, i.e., it has the typical form of an attractive "molecular" potential.

![Graph](image-url)

**Fig. 3.** Solid line: the effective potential $V^2_{\text{eff}}(\rho)$ for $\lambda = 0, \alpha = 8, \epsilon = 10^{-35} \text{ and } E = 1$. Dashed line: the effective potential $V^2_{\text{eff}}(\rho)$ for $\lambda = 0, \alpha = 8, \epsilon = 0.1 \text{ and } E = 1$. 

\[
V^2_{\text{eff}}(r) = E^2 - \frac{E^2}{\sigma^4(\rho)} + \left(1 - \frac{2}{\rho} + \frac{\alpha}{6\rho^4}\right) \left(\frac{\lambda^2}{\sigma^4(l, \rho)\rho^2} + \frac{1}{\sigma^2(\rho)}\right) \tag{44}
\]
Fig. 4. Solid line: the effective potential $V_{\text{eff}}^2(\rho)$ for $\lambda = 0, \alpha = 81/8, \epsilon = 10^{-35}$ and $E = 1$. Dashed line: the effective potential $V_{\text{eff}}^2(\rho)$ for $\lambda = 0, \alpha = 81/8, \epsilon = 0.1$ and $E = 1$.

Fig. 5. Solid line: the effective potential $V_{\text{eff}}^2(\rho)$ for $\lambda = 0, \alpha = 12, \epsilon = 10^{-35}$ and $E = 1$. Dashed line: the effective potential $V_{\text{eff}}^2(\rho)$ for $\lambda = 0, \alpha = 12, \epsilon = 0.1$ and $E = 1$. 
Fig. 6. Dashed line: the effective potential $V_{eff}^2(\rho)$ for $\lambda = 0, \alpha = 81/8, \epsilon = 0.1, E = 1$. Solid line: the effective potential $V_{eff}^2(\rho)$ for $\lambda = 0, \alpha = 81/8, \epsilon = 0.1$ and $E = 2$ and Dotted line: the effective potential $V_{eff}^2(\rho)$ for $\lambda = 0, \alpha = 81/8, \epsilon = 0.1$, and $E = 4$.

Fig. 7. Dashed line: the effective potential $V_{eff}^2(\rho)$ for $\lambda = 0, \alpha = 12, \epsilon = 0.1$ and $E = 1$. Solid line: the effective potential $V_{eff}^2(\rho)$ for $\lambda = 0, \alpha = 12, \epsilon = 0.01$ and $E = 1$. Dotted line: the effective potential $V_{eff}^2(\rho)$ for $\lambda = 0, \alpha = 12, \epsilon = 0.001$ and $E = 1$. 

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Fig. 8. A zoom of the effective potential near $r/m = 1.5$, with the intention to note the existence of a potential barrier still for rather large black hole masses. For very large masses, e.g. $\epsilon = 10^{-20}$, the barrier seems to have vanished. But effects are present up to $\epsilon = 10^{-18}$ (not shown here). Dashed line: the effective potential $V_{eff}^2(\rho)$ for $\lambda = 0$, $\alpha = 81/8$, $\epsilon = 10^{-8}$ and $E = 1$. Solid line: the effective potential $V_{eff}^2(\rho)$ for $\lambda = 0$, $\alpha = 81/8$, $\epsilon = 10^{-9}$ and $E = 1$. Dotted line: the effective potential $V_{eff}^2(\rho)$ for $\lambda = 0$, $\alpha = 81/8$, $\epsilon = 10^{-20}$ and $E = 1$.

Fig. 9. Dashed line: the effective potential $V_{eff}^2(\rho)$ for $\lambda = 90$, $\alpha = 12$, $\epsilon = 0.1$ and $E = 10$. Solid line: the effective potential $V_{eff}^2(\rho)$ for $\lambda = 90$, $\alpha = 12$, $\epsilon = 0.1$ and $E = 15$. Dotted line: the effective potential $V_{eff}^2(\rho)$ for $\lambda = 90$, $\alpha = 12$, $\epsilon = 0.1$ and $E = 20$. 
Fig. 10. Dashed line: the effective potential $V_{\text{eff}}^2(\rho)$ for $\lambda = 90$, $\alpha = (81/8, \epsilon = 0.000001$ and $E = 10$. Solid line: the effective potential $V_{\text{eff}}^2(\rho)$ for $\lambda = 100, \alpha = 12, \epsilon = 0.000001$ and $E = 10$. For $r \rightarrow 0$ the effective potential approaches $E^2 = 100$.

Fig. 11. Dashed line: the effective potential $V_{\text{eff}}^2(\rho)$ for $\lambda = 90$, $\alpha = 12, \epsilon = 10^{-35}$ and $E = 10$. Solid line: the effective potential $V_{\text{eff}}^2(\rho)$ for $\lambda = 90, \alpha = 8, \epsilon = 10^{-35}$ and $E = 10$. Dotted line: the effective potential $V_{\text{eff}}^2(\rho)$ for $\lambda = 90, \alpha = 81/8, \epsilon = 10^{-35}$ and $E = 10$. The nearly vertical line at small $r/m$ is the potential coming down from large values and approaching $E^2 = 100$ at $r/m = 0$. Due to the poor resolution, the bending of the curve near 0 is not resolved.

The following discussion is easier to understand, noting that the effective potential in (44) has two important terms: the term $-E^2/\sigma^4$ becomes dominant when the $\sigma^2$ tends to zero. In this case, the potential goes to $-\infty$. When $\sigma^2(r)$ tends to $-\infty$, the only term which survives is the first one, namely $E^2$, which the effective potential finally acquires.

In Figs. 3 and 4 we studied the effect of changing $\alpha$ from 8, below the critical limit, to $\alpha = 81/8$, the critical limit, and $\alpha = 12$, above the critical limit. In these figures the solid line represents a macroscopic black hole and it shows a typical
behavior of an attractive, molecular type of potential. In reference to it, in Fig. 3 for $\alpha$ we used the value of 8, in Fig. 4 the value $\frac{81}{8}$ and in Fig. 5 the $\alpha$ is above the critical value. We see that contrary to $\frac{17}{10}$ several potential barriers appear. This is due to the behavior of the $\sigma^2(r)$-function which exhibits several zeros and also the potential exhibits various zeros where $\sigma^2(r)$ tends to $-\infty$. We also see that when the $\alpha$ changes, also the number of singularities change.

Interesting is how the height of the barrier changes with increasing energy $E$ of the particle, see Fig. 6. As in $\frac{17}{10}$ the height of the barrier increases with $E$, indicating that the particle is always inhibited to enter the area below the first position of the barrier. Note, that near the singularity of $\sigma^2(r)$ the effective potential barrier approaches $E^2$ (see the discussion above), thus, the barrier increases with $E^2$.

In Fig. 7 we studied the effect of lowering $\epsilon$ (increasing the mass of the black hole), from $\epsilon = 0.1$ on, which is ten times the minimal value possible. We see, that with increasing mass (smaller $\epsilon$) the barrier structure with its singularities vanish. the effective potential for large black hole masses is plotted in the vicinity of $\rho = \frac{3}{2}$, where the event horizon still exists. We note a sharp barrier, which is the result of introducing a minimal length. Thus, no particle can pass this point, while without the minimal length a particle can fall in. Here we see an important effect in taking into account the presence of a minimal length.

In Fig. 8 a zoom to near $\frac{r}{m} = 1.5$ of the effective potential is depicted, for the critical $\alpha$ value of $\frac{81}{8}$. Without a minimal length, the event horizon is at $\frac{r}{m} = \frac{3}{2}$, where according to our calculations still a repulsive barrier is noted up to at least $\epsilon = 10^{-18}$ ($10^{18}$ times the Planck length), which is already of macroscopic size, compared to the scale of an elementary particle! This barrier is not seen anymore for $\epsilon = 10^{-20}$. That the effective potential approaches zero at $r = \frac{3}{2}m$ for $\alpha = \frac{81}{8}$ and has there a minimum. For $\alpha = 8$, at the position of the event horizons $r_+ = 1.75211m$ and $r_- = 1.17221m$, the potential is zero but does not acquire a minimum there. These observations can be understood from the expression of the effective potential in Eq. 41: The factor $(1 - \frac{2m}{r} + \frac{H}{6\sigma^2})$ is zero at the $r$ positions mentioned and the effective potential reduces to $E^2 - \frac{E^2}{r^2}$. The $\sigma^2(r)$ acquires there the value of 1, thus, the two terms cancel and the effective potential is zero. This is a unique behavior at the values of $\alpha$ mentioned, which disappears when the $\alpha$-value is larger than its limiting value, because then the factor in from of the last term in (41) is not zero anymore.

In Fig. 9 the angular momentum is now different from zero. Increasing the Energy of the particle, with $\alpha = 12$, the main effect is an increase of the barrier, i.e., similar as discussed before. Also, introducing a large angular momentum smooths out the singular structure of the potential.

Finally, in Figs. 10 and 11 nearly macroscopic black holes (very small $\epsilon$) are studied, with a large angular momentum. Increasing $\alpha$, only lifts the potential to higher values, which is due to the dark energy alone, no $l$-dependence is observed. According to (44) the effective potential always tends to $E^2$ at $r = 0$. This is because
the $\sigma^2(r)$-function goes to $-\infty$, which eliminates the last terms in (44). Approaching $r = 0$, the potential comes from very large values, which can be seen as a nearly vertical line in the figures.

3.4. Minimal mass of an accelerating object $A_m = c^2/l$

In\textsuperscript{44} it is argued that there exists a value for the minimal mass of a black hole, equal to the Plank mass. The main argument is that causality has to be maintained and that for smaller mass quantum effects set in, thus the classical theory is not valid any more.\textsuperscript{13,17} In this sub-section we show that due to the minimal length the limit of a minimal black hole mass increases.

The new mass scale is obtained when $m$ is equal to the minimal length, which we will put here as the Planck length ($m = l$), or $m = l = 10^{-35}$ meter.

The Planck mass is determined as

$$M_P = \sqrt{\frac{\hbar c}{k}} = \left( \frac{0.105457 \times 10^{-33} \times 3 \times 10^8}{6.67384 \times 10^{-11}} \right)^{\frac{1}{2}}$$

$$= 0.217726 \times 10^{-7} \text{ kg}$$

Setting $m = l$ implies $\epsilon = 1$, which in turn leads to (using $m = \frac{kM_{\text{object}}}{\epsilon^2}$, where $M_{\text{object}}$ is the mass in kg)

$$\epsilon = \frac{l}{m} = \frac{lc^2}{kM_{\text{object}}}$$

Setting now the limit $\epsilon = 1$ and solving for $M_{\text{object}}$, we obtain for the mass

$$M_{\text{object}} = \frac{lc^2}{k} = \frac{9 \times 10^{16} \times 6.62606 \times 10^{-34}}{6.67384 \times 10^{-11}} = 8.93557 \times 10^{-7} \text{ kg}$$

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The ratio of $M_{\text{object}}$ to $M_P$ is

$$\frac{M_{\text{object}}}{M_P} = 41$$

As a consequence, the minimal mass of an object, corresponding to the minimal length scale, is 41 times larger than the Planck mass, a more stringent limit than the one proposed by S. Hawking.\textsuperscript{44}

4. Discussion and Conclusions

In this contribution we investigated on the effects of a minimal length scale parameter in an extended version of General Relativity, called pseudo-complex General Relativity, which is equivalent to introducing a $r$-dependent mass.\textsuperscript{45} Thus, the pc-GR can be seen as a representative other models which try to extend General
Relativity. In past applications of pcGR the minimal length was ignored, but here we showed that it generates important differences for small black holes, which we will resume in what follows. It is important to stress that the minimal length is treated as a parameter and as a consequence the Lorentz symmetry is maintained. This represents a great advantage to theories treating the minimal length as a physical length, violating the Lorentz symmetry resulting in laborious, involved theories. One may ask if the use of a minimal length as a parameter is physical or not. In spite of this question, even when the theory presented is thought as an effective theory with a minimal length, the consequences discussed in this manuscript can serve as an orientation for more general theories, thus, be very useful.

A minimal length is related to a maximal acceleration. Its effect is cast into a conformal factor $\sigma^2(r)$ of the metric. When the acceleration of a particle’s tends to its maximal value, then the metric correction factor $\sigma^2(r) \rightarrow 0$. At large distances the classical potential of GR is recovered, i.e., $V_{eff}^2(r) \rightarrow 1$ as $r \rightarrow \infty$.

As one main result, we showed that the effects of the minimal length can mainly be noted when the mass is of the order of the Planck mass and a few orders larger than it. For macroscopic black holes also some effect can be seen when $\alpha = \frac{A}{l}$, up to $\epsilon = 10^{-18}$, corresponding to a length of $10^{-15}$fm, which is of the order of the size of a hadron.

But there is also an effect in pcGR due to the coupling of the central mass to the size of the dark energy in the vicinity of a black hole. Depending on the value of $B_4 = bm^4$, we can distinguish three cases, given by particular ranges of the parameter $B_4$, where we only resume the results mainly for zero angular momentum ($\lambda = 0$), though, also some remarks on $\lambda > 0$ are included:

- **case a:** $B_4 < 81 m^4/8$

  The range of $\epsilon = (l/m)$ is divided into two regions, the first is $\epsilon_0 \leq \epsilon \leq 1$, where $\epsilon_0$ marks the value above which no singularity in $\sigma^2$ appears. In the second case ($\epsilon < \epsilon_0$) the conformal factor $\sigma^2$ shows singularities, as can be appreciated in Fig. 1 and 2, where the factor shows three and five points where $\sigma^2$ passes to zero to infinite negative values. The region where $\sigma^2$ is negative is unphysical and has to be excluded. The number and position of these singularities vary with $\epsilon$. The exact value of $\epsilon_0$ cannot be given but only estimated.

  We have two cases to consider. For the first one $\epsilon_0 \leq \epsilon \leq 1$, the effective potential tends to $E^2$ near $r_\pm$ and the center $r = 0$. Moreover, the effective potential diverges when $\sigma^2(\rho) = 0$ in the range $r_0 < r_{-1} < r_{-2} < r_{+1} < r_{+2}$. (The $r_m$ values refer to the position where $\sigma^2(r)$ passes through zero, see the main text.) The divergences appear as a potential barrier near $r_\pm$ and at the center ($r = 0$). The potential barrier also increases with larger values of $E$ and, as a consequence, the incoming massive particle of energy $E$ would never falls into the black hole, i.e., the accretion of mass to the mini-black hole is stopped. Fig. 3 and subsequent figures show that the
effective potentials can not be distinguished from each other at infinity. For \( \lambda \neq 0 \), the effective potential still has the value of \( E^2 \) near the center \( r = 0 \) and to the horizons \( r_+ \) and \( r_- \). There is a critical values of \( E \), which determines the sign of effective potential; it becomes positive when

\[
E^2 < \frac{\lambda^2}{\rho_c^2} (1 - \frac{\rho}{\rho_c} + \frac{\alpha}{6\rho^2})
\]

where \( \rho_c \) is the value of \( \rho \) when coming from \( \rho = 0 \) the effective potential becomes negative (see Figs. 9-11). At a higher value of the critical energy, there are two singularities in which the factor \( \sigma^2(\rho) = 0 \).

The second case is for \( \epsilon \ll 1 \), which corresponds to the classical limit, i.e., the mass, in length units, of the black hole is much bigger than the minimal length. This means that \( \sigma^2(\rho) \longrightarrow 1 \). As consequence, the potential barrier in this case disappears. Due to this, the effects of the minimal length can be neglected.

- **case b: \( B_4 = 81m^4/8 \)**

  In this case, according the value of \( \epsilon = (l/m) \), we can also distinguish two ranges: In the first one, we have \( \epsilon_0 \leq \epsilon \leq 1 \) (here \( \epsilon_0 \) represents the value of \( \epsilon \) for which it makes the transition from the first to the second range). The effective potential has three singularities, one near zero and two degenerate ones at \( \rho_e = \frac{l}{m} = \frac{3}{2} \) (the index \( e \) refers to the event horizon) but the effective potential has the maximum value \( E^2 \) for \( \rho_e = \frac{3m}{2} \) and near the center \( r = 0 \). These singularities act as a potential barrier (see, for example, Fig. 3). When \( \alpha = \frac{81}{8} \) the effective potential for a large mass black hole still shows a repulsive barrier at \( \rho = \frac{3}{2} \), which inhibits the falling in of particles, contrary to the case when no minimal length is taken into account, where the particle can pass the horizon. It is noticeable that at the position of the event horizon \( \rho_e = \frac{3}{2} \) there is still a very large barrier for relative large black hole masses, at least up to \( 10^{18} \) and the penetration of a particle is suppressed. For much larger black hole of \( \epsilon = 10^{-20} \) masses this barrier seems to disappear. An \( \epsilon = 10^{-18} \) corresponds to a length of \( 10^{-15} \) cm, which implies the possibility that effects of the minimal length could be seen there, if \( \alpha = \frac{81}{8} \) and these small effect can be measured. For the second range, we consider \( \epsilon_0 \geq \epsilon \), here \( \epsilon = (l/m) \) is very small (i.e., the black hole is of macroscopic size), this means \( \sigma^2(\rho) \longrightarrow 1 \), in this case the effective potential has no singularities. This implies that there is also no potential barrier. It is clear that the effect of a minimal length is large when \( l \approx m \), i.e., for microscopic black holes.

- **case c: \( B_4 > 81m^4/8 \)**

  For this case, in the range \( \epsilon_0 \leq \epsilon \leq 1 \), the effective potential has also

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a barrier in the potential, see Fig. 3 for $\lambda = 0$. In addition, for $\alpha = 12$ ($B > 81m^4/8$) the effective potential has always a maximum at $E^2$ near $r = 3m/2$ and near $r = 0$.

In addition, for $\lambda \neq 0$ the effective potential has a critical value for the energy, in which the factor $\sigma^2(r) = 0$, for which we can determined the sign of the effective potential: It is positive for $E^2 < \rho_c^2(1 - \frac{2}{\rho_c^2} + \frac{\alpha}{6\rho_c^4})$. For $\epsilon_0 \geq \epsilon$, ($\rho_c = \frac{\epsilon_0}{m}$ is the critical value of $\rho$ where the effective potential becomes negative, when $\rho_c$ is approached from below) the effective potential is regular and finite, which means that the potential barrier disappears in this case, but the event horizon, which appears there, is not the effect of minimal length $l$.

In conclusion, in all cases the effect of minimal length $l$, or maximal acceleration, are noticeable only for small black hole masses, as a potential barrier at the horizons $0 < r_0 < r_- < r_+$ in which the effective potential diverges or has singularities. A "small" black hole can still be relatively large, as $m = 10^{18}l$ corresponds to a black hole mass of about $10^8$kg. The height of the barriers increase with the particle energy. As a consequence, the formation of a larger black hole is stopped. Also the positions of maximal accelerations $r_{\pm 1}$ and $r_{\pm 2}$ were revealed. At these points, the effective potential has a barrier and a particle is prevented to pass this barrier.

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