ON THE DUAL CODES OF SKEW CONSTACYCLIC CODES

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Abstract. Let \( \mathbb{F}_q \) be a finite field with \( q \) elements and denote by \( \theta : \mathbb{F}_q \to \mathbb{F}_q \) an automorphism of \( \mathbb{F}_q \). In this paper, we deal with skew constacyclic codes, that is, linear codes of \( \mathbb{F}_q^n \) which are invariant under the action of a semi-linear map \( \phi_{\alpha,\theta} : \mathbb{F}_q^n \to \mathbb{F}_q^n \), defined by \( \phi_{\alpha,\theta}(a_0, a_1, \ldots, a_{n-2}, a_{n-1}) := (\alpha \theta(a_{n-1}), \theta(a_0), \ldots, \theta(a_{n-2})) \) for some \( \alpha \in \mathbb{F}_q \setminus \{0\} \) and \( n \geq 2 \). In particular, we study some algebraic and geometric properties of their dual codes and we give some consequences and research results on 1-generator skew quasi-twisted codes and on MDS skew constacyclic codes.

1. Introduction

Let \( \mathbb{F}_q \) be a field with \( q \) elements. A linear code \( C \) of length \( n \) and dimension \( k \), called an \([n,k]_q\)-code, is a \( k \)-dimensional subspace of \( \mathbb{F}_q^n \). Moreover, an \([n,k]_q\)-code \( C \) with minimum Hamming distance \( d := d(C) \) is denoted as an \([n,k,d]_q\)-code. A fundamental problem in coding theory is to optimize one of the parameters \( n, k \) or \( d \) of a linear code, given the other two. If \( d_q(n,k) \) denotes the largest value of \( d \) for which an \([n,k,d]_q\)-code exists, we call an \([n,k,d_q(n,k)]_q\)-code simply a code with best known linear code (BKLC) parameters. It is well known that a large number of new linear codes achieving the best known bounds \( d_q(n,k) \), in particular over small fields, have been constructed as cyclic, constacyclic or quasi-cyclic codes, including the corresponding ones in the non-commutative case (e.g., see [1], [6] and [20]). In this paper, by extending some results of [18] to the non-commutative case, we study some algebraic and geometric properties of skew constacyclic codes (Definition 2.1) and their duals, the latest ones being strongly related to the minimum Hamming distance \( d \) which is useful in error-correcting codes and for some decoding algorithms. After then, as applications of the previous results, we give some immediate consequences on 1-generator skew quasi-twisted codes and on Maximum Distance Separable (MDS) skew constacyclic codes. Finally, new constructions of some linear codes are provided.
codes, 1-generator skew quasi-cyclic codes and some MDS skew constacyclic codes with the best known parameters for small values of $q$ (Tables 1, 2 and 3) are obtained in the same spirit of [1] and [19] by applying the previous theoretical results and computer programs written in MAGMA [5].

The paper is organized as follows. After some basic notions and remarks, in Section 2 we recall the main properties of skew constacyclic codes and we reprove in an easy way some results which will be useful in the next sections. As a consequence of these results, in Section 3 we show some geometric properties about generator and parity check matrices of skew constacyclic codes (Theorems 3.1 and 3.2), whose columns are composed of orbits of points in projective spaces via the action of semi-linear maps, in line with a work of T. Maruta [18] in the commutative case. As an application of these facts, inspired by the works [2], [1] and [20] respectively, in Section 4.1 we study the main properties of 1-generator skew quasi-twisted codes and we show a method for lengthening skew constacyclic codes of small length to construct in an easy way new examples of linear codes over small fields that meet the parameters of some best known linear codes (Table 1). Moreover, we extend to the non-commutative case the main result in [20] about 1-generator quasi-cyclic codes (Theorem 4.3) and as search results, we give also some new examples of 1-generator skew quasi-cyclic codes with the best known distance $d$ for $q = 4$ and $k = 5$ (Table 2). Finally, in Section 4.2 the existence and the construction of some MDS skew constacyclic codes (Table 3) are completely characterized by simple algebraic conditions (Theorem 4.5 and Corollary 4).

2. Notation and background material

Along all this paper, we will use the following notation.

Let $F_q$ be a finite field with $q$ elements, where $q = p^r$ for some prime $p$ and $r \in \mathbb{Z}_{\geq 1}$. Define $F^*_q := F_q \setminus \{0\}$ and take $\alpha \in F^*_q$. Let $\theta$ be an automorphism of $F_q$, that is, $\theta(z) := z^{p^t}$ for any $z \in F_q$, where $t$ is an integer such that $1 \leq t \leq r$.

Let us recall here the definition of the main codes we will treat in the next sections.

Definition 2.1 ([6],[9],[11]). A linear code $C \subseteq F^n_q$ is called a skew $(\alpha,\theta)$-cyclic code if $C$ is invariant under the semi-linear map

$$\phi_{\alpha,\theta} : (c_0, c_1, \ldots, c_{n-1}) \mapsto (\alpha \theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})).$$

Moreover, a skew $(\alpha, id)$-cyclic code is called simply an $\alpha$-constacyclic code and, for some fixed $\theta$, we will call a linear code $C \subseteq F^n_q$ a skew constacyclic code if $C$ is a skew $(\alpha, \theta)$-cyclic code for some $\alpha \in F^*_q$.

Remark 1. According to [11, §2], in our situation we have $\phi_{\alpha,\theta} = A \circ \Theta$, where

$$\Theta((c_0, \ldots, c_{n-1})) := (\theta(c_0), \ldots, \theta(c_{n-1}))$$

and $A(\vec{v}) := \vec{v}A$ for every $\vec{v} \in F^n_q$ with

$$A := \begin{pmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
\alpha & 0 & \cdots & 0
\end{pmatrix}.$$

Let us show here some algebraic and geometric properties of the dual codes of skew constacyclic codes.
First of all, with the purpose of giving an algebraic structure to skew \((\alpha, \theta)\)-cyclic codes, one defines a ring structure on the set
\[
R := \mathbb{F}_q[x; \theta] := \{a_0 x^n + \ldots + a_1 x + a_0 \mid a_i \in \mathbb{F}_q \text{ and } s \in \mathbb{Z}_{\geq 0}\},
\]
where the addition is defined to be the usual addition of polynomials and the multiplication is defined by the basic rule \(xa = \theta(x)a\) for any \(a \in \mathbb{F}_q\), and extended to all elements of \(R\) by associativity and distributivity.

Consider now the following one-to-one correspondence:
\[
\pi : \mathbb{F}_q^n \longrightarrow R/R(x^n - \alpha)
\]
\[
(a_0, a_1, \ldots, a_{n-1}) \mapsto a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}.
\]
Note that \(\pi\) is an \(\mathbb{F}_q\)-linear isomorphism of left \(\mathbb{F}_q\)-modules. So, we can identify \(\mathbb{F}_q^n\) with \(R/R(x^n - \alpha)\) and any vector \(\bar{a} = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{F}_q^n\) with the polynomial class \(\pi(\bar{a}) := \sum_{i=0}^{n-1} a_i x^i \in R/R(x^n - \alpha)\) via \(\pi\).

Let \(m\) be the order of \(\theta\). If either \(m \not| n\) or \(\alpha \not\in \mathbb{F}_q^\theta := \{a \in \mathbb{F}_q : \theta(a) = a\}\), we know that \(R/R(x^n - \alpha)\) is not a ring and we can not argue about its ideals, as in the commutative case (e.g., see [15, §2.2]). On the other hand, when \(\alpha = 1\) and \(m \mid n\), \(R/R(x^n - \alpha)\) becomes a ring and one can construct a one-to-one correspondence between skew cyclic codes (skew \((1, \theta)\)-cyclic codes) and the ideals of \(R/R(x^n - 1)\) (see, e.g., [6] and [8] for many results on this topic). Anyway, without any condition on \(m\) and \(\alpha\), the set \(R/R(x^n - \alpha)\) can be always considered as a left \(\mathbb{F}_q\)-module, or a left \(R\)-module.

The next two results give an equivalent definition of skew constacyclic codes and some of their well-known properties, respectively.

**Theorem 2.2** (see, e.g., [8] and [22]). A nonempty subset \(C \subset \mathbb{F}_q^n\) is a skew \((\alpha, \theta)\)-cyclic code if and only if \(\pi(C)\) is a left \(R\)-submodule of the left \(R\)-module \(R/R(x^n - \alpha)\).

**Theorem 2.3** (see, e.g., [6], [9] and [11]). Let \(\pi(C)\) be a left \(R\)-submodule of \(R/R(x^n - \alpha)\), i.e. \(C\) is a skew \((\alpha, \theta)\)-cyclic code of \(\mathbb{F}_q^n\). Then there exists a unique monic polynomial \(g(x)\) of minimal degree in \(R\), called the generator polynomial of \(C\), such that

(a) \(g(x)\) is a right divisor of \(x^n - \alpha\);
(b) \(\pi(C) = Rg(x)/R(x^n - \alpha) =: Rg(x)\);
(c) every \(c(x) \in \pi(C)\) can be written uniquely as \(c(x) = f(x)g(x) \in R/R(x^n - \alpha)\), where \(f(x) \in R\) has degree less than or equal to \(n - \deg g(x)\). Moreover, the dimension of \(C\) is equal to \(n - \deg g(x)\);
(d) if \(g(x) := \sum_{i=0}^{k} g_i x^i\), then \(C\) has a generator matrix \(G\) given by

\[
G = \begin{pmatrix}
g_0 & g_1 & \cdots & g_{k-1} & 0 & 0 & \cdots & 0 \\
0 & 0 & \theta(g_0) & \theta(g_1) & \cdots & \theta(g_{k-1}) & 0 & \cdots & 0 \\
0 & 0 & \cdots & \theta(g_{k-1}) & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \theta^p g_{k-1}(g_0) & \theta^p g_{k-1}(g_1) & \cdots & \theta^p g_{k-1}(g_{k-1}) & \cdots & \theta^p g_{k-1}(g_{k})
\end{pmatrix}.
\]

For skew \((\alpha, \theta)\)-cyclic codes of \(\mathbb{F}_q^n\) with \(\theta(z) := z^{p^r}\), where \(q = p^r\) for some prime \(p\), \(r \in \mathbb{Z}_{\geq 2}\) and an integer \(t\) such that \(1 \leq t \leq r - 1\), type the following MAGMA Program to construct by the command SD\((n, a)\) the generator polynomials of all skew \((\alpha, \theta)\)-cyclic codes of \(\mathbb{F}_q^n\):

\[
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\]
Program 1.

\[ \begin{align*}
p &:= \ldots \quad \text{r} := \ldots \quad \text{t} := \ldots \quad \text{F}_{\omega} := \text{GF}(p^r) \seteq \text{TwistedPolynomials}(\text{F}_{q} := p^t) \seteq \text{function}(n,a) \\
\text{P} &:= [0] \\
\text{for} \ i \ \text{in} \ [1..n-2] \ \text{do} \\
\text{P} &:= \text{P} \ \text{cat} \ [0] \\
\text{end for} \\
\text{T} &:= [-a] \ \text{cat} \ \text{P} \ \text{cat} \ [1] \\
\text{f} &:= \text{R}!\text{T} \\
\text{dd} &:= [] \\
\text{E} &:= [x : x \ \text{in} \ \text{F} \ | \ x \ \text{ne} \ 0] \\
\text{S} &:= \text{CartesianProduct}(\text{E}, \text{CartesianPower}(\text{F}, n-1)) \\
\text{for} \ s \ \text{in} \ \text{S} \ \text{do} \\
\text{ll} &:= [s[1]] \ \text{cat} \ [p : p \ \text{in} \ s[2]] \\
\text{if} \ \text{LeadingCoefficient}(\text{R}!\text{ll}) \ \text{eq} \ 1 \ \text{then} \\
\text{q,r} &:= \text{Quotrem}(\text{f}, \text{R}!\text{ll}) \\
\text{if} \ \text{r} \ \text{eq} \ \text{R}![0] \ \text{then} \\
\text{R}!\text{ll} \ \text{dd} &:= \text{dd} \ \text{cat} \ \text{R}!\text{ll} \\
\text{end if} \\
\text{end if} \\
\text{end for} \\
\text{return} \ \text{dd} \\
\text{end function;}
\end{align*} \]

Remark 2. In [17, §2], A. Leroy gives an algorithm to find right divisors of \( x^n - \alpha \in \text{F}_q[x; \theta] \) by showing that factorizations of polynomials in \( \text{F}_q[x; \theta'] \), where \( \theta' \) is the Frobenius automorphism, are translated into factorizations in the usual polynomial ring \( \text{F}_q[x] \). For further MAGMA Programs about the research of right divisors of \( x^n - \alpha \) and the construction of skew \( (\alpha, \theta) \)-cyclic codes, we refer also to [22] and [23].

Example 1. In \( \text{F}_{4}^{14} \) with \( \theta(z) := z^2 \) for any \( z \in \text{F}_4 \), by using the command \( \text{SD}(14,1) \) of Program 1 we can see that there are 603 different nontrivial right divisors of \( x^{14} - 1 \), i.e. 603 different nontrivial skew \( (1, \theta) \)-cyclic codes of \( \text{F}_{4}^{14} \) instead of 25 different nontrivial cyclic codes of \( \text{F}_{4}^{14} \) in the commutative case.

The following results deal with the (Euclidean) dual code \( C^\perp \) of a code \( C \subseteq \text{F}_q^n \), i.e. the set of words which are orthogonal to the code’s words relatively to the Euclidean scalar product. In particular, the next theorem was first presented and proved in very different forms in [7, §4], [8, Theorem 8], [9, Theorem 1] and [13, Theorem 6.1], and here we give a very short and simple proof of it.

Theorem 2.4. Let \( C \subseteq \text{F}_q^n \) be a linear code and take \( \alpha \in \text{F}_q^* \). Then

\( C \) is a skew \( (\alpha, \theta) \)-cyclic code \( \iff \) \( C^\perp \) is a skew \( (\alpha^{-1}, \theta) \)-cyclic code.

Proof. Assume that \( C \) is a skew \( (\alpha, \theta) \)-cyclic code. By [22, Proposition 25] and Remark 1 we deduce that \( C^\perp \) is invariant under the semi-linear map \( T: \text{F}_q^n \to \text{F}_q^n \) given by \( T(c_0, \ldots, c_{n-1}) := (\theta^{-1}(c_1), \ldots, \theta^{-1}(c_{n-1}), \theta^{-1}(\alpha) \theta^{-1}(c_0)) \), i.e. \( TC^\perp = C^\perp \). Since \( T \) is invertible, then we have also \( T^{-1}C^\perp = C^\perp \). Note that

\[ T^{-1}(d_0, \ldots, d_{n-1}) = (\alpha^{-1}\theta(d_{n-1}), \theta(d_0), \ldots, \theta(d_{n-2})) . \]

Thus \( T^{-1} = \phi_{\alpha^{-1}, \theta} \) and by Definition 2.1 we conclude that \( C^\perp \) is a skew \( (\alpha^{-1}, \theta) \)-cyclic code. Finally, having in mind that \( (C^\perp)^\perp = C \), the converse is immediate. \( \square \)
Corollary 1 (see also §§5, 6 of [7], Proposition 13 of [8] and Proposition 1 of [10]).
Let \( \mathcal{C} \) be a skew \((\alpha, \theta)\)-cyclic code. If \( \{0\} \neq \mathcal{C} \subseteq \mathcal{C}^\perp \) (or \( \{0\} \neq \mathcal{C}^\perp \subseteq \mathcal{C} \)), then \( \alpha = \pm 1 \) and \( \dim \mathcal{C} \leq \frac{n}{2} \leq \dim \mathcal{C}^\perp \) (or \( \dim \mathcal{C}^\perp \leq \frac{n}{2} \leq \dim \mathcal{C} \)). In particular, if \( \mathcal{C} = \mathcal{C}^\perp \) then \( \alpha = \pm 1 \), \( n \) is even and \( \dim \mathcal{C} = \frac{n}{2} \).

Proof. Since \( \mathcal{C} = (\mathcal{C}^\perp)^\perp \), without loss of generality, we can assume that \( \mathcal{C} \subseteq \mathcal{C}^\perp \).
First of all, observe that \( \alpha \) are particular orbits under the action of a semi-linear map associated to the code. More precisely, we will see first that the columns of a parity check matrix of a skew constacyclic code can be considered as points in a projective space which is in line with [18], in the non-commutative case we obtain the following two results.

Finally, by Theorems 2.3 and 2.4, let \( g(x), h(x) \) be the generator polynomials of \( \mathcal{C} \) and \( \mathcal{C}^\perp \), respectively. Thus for some \( h_1(x), h_2(x) \in R \) we have
\[
h_1(x)h(x) = x^n - \alpha^{-1} = x^n - \alpha + (\alpha - \alpha^{-1}) = h_2(x)g(x) + (\alpha - \alpha^{-1}) \, .
\]
Since \( \mathcal{C} \subseteq \mathcal{C}^\perp \), we see also that there exists \( q(x) \in R \) such that \( g(x) = q(x)h(x) \). Thus \( (h_1(x) - h_2(x)q(x))h(x) = \alpha - \alpha^{-1} \in \mathbb{F}_q \). If \( \alpha - \alpha^{-1} \neq 0 \) then \( \mathcal{C}^\perp = \mathbb{F}_q^n \), but this gives the contradiction \( \mathcal{C} = (\mathcal{C}^\perp)^\perp = (\mathbb{F}_q^n)^\perp = \{0\} \). So, we get \( \alpha = \alpha^{-1} \) and this implies \( \alpha^2 = 1 \), i.e. \( \alpha = \pm 1 \).

With the next result (see also [8, Theorem 8] and [13, Theorem 6.1]), one can write directly the generator polynomial of the dual code of any skew constacyclic code.

Proposition 1. Let \( \mathcal{C} \subseteq \mathbb{F}_q^n \) be a skew \((\alpha, \theta)\)-cyclic code generated by \( g(x) = g_0 + g_1 x + \cdots + g_{n-k-1} x^{n-k-1} + x^{n-k} \) with \( \deg g(x) := n - k > 0 \). Then the dual code \( \mathcal{C}^\perp \) is generated by
\[
h(x) := \theta^k(h_0^{-1}) \left[ \sum_{i=1}^{k} \theta^i(h_{k-i}) x^i + 1 \right],
\]
where \( h(x) = h_0 + \cdots + h_{k-1} x^{k-1} + x^k \) is such that \( x^n - \theta^{-k}(\alpha) = g(x)h(x) \).

Proof. From Theorem 2.4 it follows that the dual code \( \mathcal{C}^\perp \) is a skew \((\alpha^{-1}, \theta)\)-cyclic code. Then by Theorem 2.3 we know that there exists a unique monic polynomial \( h(x) \) of minimal degree in \( R \) which is the generator polynomial of \( \mathcal{C}^\perp \). So by [8, Theorem 8] we see that there exist \( h(x) \in R \) and \( c \in \mathbb{F}_q^n \) such that
\[
x^n - c = g(x)h(x)
\]
and \( h(x) = \theta^k(h_0^{-1})g^1(x) \), where \( g^1(x) := \sum_{i=0}^{k} \theta^i(h_{k-i}) x^i \) and \( h(x) = h_0 + \cdots + h_{k-1} x^{k-1} + x^k \). Since \( g(x) \in R \) is monic and \( x^n - \alpha = t(x)g(x) \) for some monic \( t(x) \in R \), by [9, Lemma 2] we obtain also that
\[
x^n - \theta^{-k}(\alpha) = g(x)s(x)
\]
for some \( s(x) \in R \). Hence \( c - \theta^{-k}(\alpha) = g(x)(s(x) - h(x)) \). Since \( \deg g(x) > 0 \), this shows that \( c = \theta^{-k}(\alpha) \).

3. Generator and Parity Check Matrices

We will show here a geometric property of the dual codes of skew constacyclic codes. More precisely, we will see first that the columns of a parity check matrix of a skew constacyclic code can be considered as points in a projective space which are particular orbits under the action of a semi-linear map associated to the code.

Denote by \( GL(k, q) \) the set of \( k \times k \) invertible matrices defined over \( \mathbb{F}_q \). Therefore, in line with [18], in the non-commutative case we obtain the following two results.
Moreover, writing $\tau(\vec{v}) := \Theta(\vec{v})T$ for every $\vec{v} \in \mathbb{F}_q^k$.

Proof. Let $(P_1, P_2, \ldots, P_n)$ be a parity check matrix of $C$, where $P_i \in \mathbb{F}_q^k$, and suppose that $C$ is a skew $(\alpha, \theta)$-cyclic code. Then there exists $S \in GL(k, q)$ such that $S[\alpha^{-1} \Theta(P_n), \ldots, \Theta(P_1)] = [P_1, P_2, \ldots, P_n]$. This matrix equality gives

$$
P_1 = \alpha^{-1} \Theta(P_n)S, \quad P_2 = \Theta(P_1)S, \quad P_3 = \Theta(P_2)S = \Theta^2(P_1)S_0S, \quad \ldots
$$

and $\alpha \vec{u} = \alpha P_1 = \Theta(P_n)S = \tau(P_n) = \tau^n(\vec{u})$. Conversely, assume that $C$ has a parity check matrix $H$ of the form

$$\begin{bmatrix}
\vec{u}, \tau(\vec{u}), \tau^2(\vec{u}), \ldots, \tau^{n-1}(\vec{u})
\end{bmatrix}
$$

such that $\tau^n(\vec{u}) = \alpha \vec{u}$, where $\tau(\vec{v}) := \Theta(\vec{v})T$ for every $\vec{v} \in \mathbb{F}_q^k$ and $\vec{h}_j \in \mathbb{F}_q^n$ for $j = 1, \ldots, k$. Since

$$
\tau^n(\vec{u}) = \alpha^{-1} \tau^{n-1}(\vec{u}) = \alpha^{-1} \left( \Theta \left( \tau^{n-1}(\vec{u}) \right) \right),
$$

$$
\tau^i(\vec{u}) = \tau T \Theta(\vec{u}) \quad \text{and} \quad \tau^i \tau(\vec{u}) = \tau T \Theta(\tau^{i-1}(\vec{u})) \quad \text{for } i = 2, \ldots, n - 1,
$$

we have

$$
H = \begin{bmatrix}
\vec{u}, \ldots, \tau^{n-1}(\vec{u})
\end{bmatrix} = T [\vec{u}, \ldots, \tau^{n-1}(\vec{u})]_\theta A = T H A
$$

for any $s \in \mathbb{Z}_{\geq 0}$, where $A$ is an $n \times n$ matrix as in Remark 1 with $\alpha^{-1}$ instead of $\alpha$. Moreover, writing $\phi := \phi_{\alpha^{-1}, \theta} = A \circ \Theta$, note that

$$
H = T^t T_{\theta} \ldots T_{\theta^s} \left( H_{\theta^{s+1}} A_{\theta^s} \ldots A_{\theta} A \right) = T^t T_{\theta} \ldots T_{\theta^s} \left( \phi^{s+1} \begin{bmatrix}
\vec{h}_1 \\
\vdots \\
\vec{h}_k
\end{bmatrix}
\right).
$$
Let $\psi{j}(\tilde{h}_i) \in C^\perp$ for every $i = 1, \ldots, k$ and for any $j \in \mathbb{Z}_{\geq 0}$. Hence $C^\perp$ is a skew $(\alpha^{-1}, \theta)$-cyclic code and we conclude by Theorem 2.4.

**Theorem 3.2.** Let $g(x) = a_0 + a_1 x + \ldots + a_{k-1} x^{k-1} + x^k$ be a monic skew polynomial of degree $k$ in $\mathbb{F}_q[x; \theta]$ that divides on the right $x^n - \alpha$, where $\alpha \in \mathbb{F}_q^*$. Then $C \subseteq \mathbb{F}_q^n$ is a skew $(\alpha, \theta)$-cyclic $[n, n-k]_q$-code with generator polynomial $g(x)$ if and only if $C$ has a parity check matrix

$$\left[ t^P, t^\tau(P), t^\tau^2(P), \ldots, t^\tau^{n-1}(P) \right],$$

where $P = (1,0,\ldots,0) \in \mathbb{F}_q^k$, $\tau(\tilde{v}) := \Theta(\tilde{v}) T_g(x)$ for every $\tilde{v} \in \mathbb{F}_q^n$ and $T_g(x)$ is the companion matrix of $g(x)$, i.e.

$$T_g(x) := \begin{pmatrix}
0 & 1 & \cdots & 0 \\
-1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
-a_0 & -a_1 & \cdots & -a_{k-1}
\end{pmatrix}.$$  
\[ (2) \]

**Proof.** Consider the linear map $\pi'$ defined by

$$\pi' : \mathbb{F}_q^k \rightarrow R/Rg,$$

$$(c_0, c_1, \ldots, c_{k-1}) \mapsto c_0 + c_1 x + \cdots + c_{k-1} x^{k-1}.$$  
\[ (2) \]

Since $P = (1,0,\ldots,0)$, note that $\pi'(\tau^i(P)) = x^i \pi'(P) = x^i$ for all $i \in \mathbb{Z}_{\geq 0}$. Thus, we have

$$\pi'(a_0 P + \cdots + \tau^k(P)) = a_0 \pi'(P) + \cdots + a_{k-1} \pi'(\tau^{k-1}(P)) + \pi'(\tau^k(P))$$

$$= a_0(1) + a_1(x) + \cdots + a_{k-1}(x^{k-1}) + (x^k)$$

$$= g(x) = 0 \in R/Rg(x),$$

i.e. $a_0 P + a_1 \tau(P) + \cdots + a_{k-1} \tau^{k-1}(P) + \tau^k(P) = (0,\ldots,0) \in \mathbb{F}_q^k$. This shows that $\pi^{-1}(g(x)) := (a_0, \ldots, a_{k-1}, 1, 0, \ldots, 0)$ is in the kernel $K$ of $H$, where $H$ is the matrix $[ t^P, t^\tau(P), t^\tau^2(P), \ldots, t^\tau^{n-1}(P) ]$ given in the statement and $\pi$ is the linear map defined by (1). Moreover, as $g(x)$ is a right divisor of $x^n - \alpha$, we deduce that

$$\pi'(\tau^P(x) - \alpha P) = \pi'(\tau(x^P)) - \alpha \pi'(P) = x^n - \alpha = 0 \in R/Rg(x),$$

and this implies that $\tau^P(x) - \alpha P = (0,\ldots,0) \in \mathbb{F}_q^k$, i.e. $\tau^P(x) = \alpha P$. By Theorem 3.1 we deduce that $K$ is a skew $(\alpha, \theta)$-cyclic code, hence by Theorem 2.2 $\pi(K)$ is a left $R$-module which contains $g(x)$. This means that $\pi^{-1}(Rg(x)) \subseteq K$. Since both spaces have dimension $n - k$, we see that $K$ is indeed the skew $(\alpha, \theta)$-cyclic code whose generator polynomial is $g(x)$.

As an immediate consequence of Theorems 2.4, 3.1, 3.2 and Proposition 1, we have also the following characterization of a generator matrix of a skew constacyclic code.

**Corollary 2.** Let $C$ be a linear $[n, k]_q$-code and $\alpha \in \mathbb{F}_q^*$. Then

(a) $C$ is a skew $(\alpha, \theta)$-cyclic code if and only if for some $\tilde{a} \in \mathbb{F}_q^k$ and $T \in GL(k, q)$, $C$ has a generator matrix of the form

$$[ t^{\tilde{a}}, t^\tau(\tilde{a}), t^\tau^2(\tilde{a}), \ldots, t^\tau^{n-1}(\tilde{a}) ]$$

such that $\tau^n(\tilde{a}) = \alpha^{-1} \tilde{a}$, where $\tau(\tilde{v}) := \Theta(\tilde{v}) T$ for every $\tilde{v} \in \mathbb{F}_q^n$.  
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(b) $C$ is a skew $(\alpha, \theta)$-cyclic code with generator polynomial $g(x) = g_0 + g_1 x + \cdots + g_{n-k-1} x^{n-k-1} + x^{n-k}$ if and only if $C$ has a generator matrix of the form

\[
[\vec{v}, \tau(\vec{b}), \tau^2(\vec{b}), \ldots, \tau^{n-1}(\vec{b})],
\]

where $\vec{b} = (1, 0, \ldots, 0)$ and $\tau(\vec{v}) := \Theta(\vec{v}) T_h(x)$ for every $\vec{v} \in \mathbb{F}_q^n$ with $h(x)$ given by Proposition 1.

**Remark 3.** Comparing the two matrices in Corollary 2 (a) and (b), since they differ by an invertible matrix, it follows that the first $k$ vectors $\vec{a}, \tau(\vec{a}), \ldots, \tau^{k-1}(\vec{a})$ generate $\mathbb{F}_q^k$ since they are linearly independent. The same occurs for the first $k$ vectors $\vec{a}, \tau(\vec{b}), \ldots, \tau^{k-1}(\vec{b})$ of matrices as in Theorem 3.1, by comparing these matrices with those in Theorem 3.2 for the same linear code.

We will give now a criterion in Proposition 2 to determine whether two skew $(\alpha, \theta)$-cyclic codes with some parameters are in fact the same linear code.

**Definition 3.3.** A linear code $C \subseteq \mathbb{F}_q^n$ with a parity check matrix of the form

\[
[P, \tau(P), \tau^2(P), \ldots, \tau^{n-1}(P)]
\]

with $P \in \mathbb{F}_q^k$, $\tau = T \circ \Theta$, $T \in GL(k, q)$ and such that $\tau^n(P) = \alpha P$ for some $\alpha \in \mathbb{F}_q^*$ is called a code $C$ defined by $(\tau, P, n)$. Moreover, one can define the following set:

$\Gamma_q^n := \{ (\tau, P, n) \mid \text{the code defined by } (\tau, P, n) \text{ is a skew } (\alpha, \theta)\text{-cyclic } [n, n-k]_q\text{-code}. \}$

Note that a code $C$ defined by $(\tau, P, n)$ is a skew $(\alpha, \theta)$-cyclic $[n, n-k]_q$-code if and only if the first $k$ columns of the parity check matrix $[P, \tau(P), \tau^2(P), \ldots, \tau^{n-1}(P)]$ are linearly independent. The next result shows when two codes $C_i$ defined by $(\tau_i, P_i, n) \in \Gamma_q^n$ for $i = 1, 2$ are the same skew $(\alpha, \theta)$-cyclic code in $\mathbb{F}_q^n$.

**Proposition 2.** Let $C_i$ be the code defined by $(\tau_i, P_i, n) \in \Gamma_q^n$ for $i = 1, 2$. Then, $C_1 = C_2$ if and only if there exists $S \in GL(k, q)$ such that $\tau_2 = \hat{S} \circ \tau_1 \circ \hat{S}^{-1}$ and $S(P_1) = P_2$, where $\hat{S}(\vec{v}) := \vec{v} S$ for every $\vec{v} \in \mathbb{F}_q^k$.

**Proof.** “$\Rightarrow$” Since $C_1 = C_2$, there exists $S \in GL(k, q)$ such that

\[
[P_2, \tau_2(P_2), \tau_2^2(P_2), \ldots, \tau_2^{n-1}(P_2)] = S[P_1, \tau_1(P_1), \tau_1^2(P_1), \ldots, \tau_1^{n-1}(P_1)]
\]

From the first columns, we deduce that $S(P_1) = P_2$, i.e. $\hat{S}^{-1}(P_2) = P_1$. Furthermore, $\tau_2^i(P_2) = \tau_1^i(P_1) S = \hat{S}(\tau_1^i(P_1)) = \hat{S}(\tau_1^i(\hat{S}^{-1}(P_2))) = \hat{S} \circ \tau_1^i \circ \hat{S}^{-1}(P_2)$ for $i = 1, \ldots, n-1$, and $\tau_2^i(P_2) = \alpha P_j$ for $j = 1, 2$.

Since the set $\{P_2, \tau_2(P_2), \ldots, \tau_2^{n-1}(P_2)\}$ generates $\mathbb{F}_q^k$, we see that every vector $\vec{v} \in \mathbb{F}_q^n$ can be written as $\vec{v} = \sum_{i=0}^{n-1} \lambda_i \tau_2^i(P_2)$. So we have

\[
\tau_2(\vec{v}) = \sum_{i=0}^{n-1} \theta(\lambda_i) \tau_2^{i+1}(P_2) = \sum_{i=0}^{n-2} \theta(\lambda_i) \hat{S} \circ \tau_1^{i+1} \circ \hat{S}^{-1}(P_2) + \alpha \theta(\lambda_i) P_2,
\]

\[
\hat{S} \circ \tau_1 \circ \hat{S}^{-1}(\vec{v}) = \sum_{i=0}^{n-1} \theta(\lambda_i) \hat{S} \circ \tau_1 \circ \hat{S}^{-1} \circ \tau_2^i(P_2) = \sum_{i=0}^{n-1} \theta(\lambda_i) \hat{S} \circ \tau_1^{i+1} \circ \hat{S}^{-1}(P_2)
\]

\[
= \sum_{i=0}^{n-2} \theta(\lambda_i) \hat{S} \circ \tau_1^{i+1} \circ \hat{S}^{-1}(P_2) + \alpha \theta(\lambda_i) \hat{S}(P_1),
\]

\[\]

\[\]

\[\]

\[\]

\[\]
i.e. \( \tau_2(\vec{v}) = \hat{S} \circ \tau_1 \circ \hat{S}^{-1}(\vec{v}) \) for any vector \( \vec{v} \in \mathbb{F}_q^k \). Hence \( \tau_2 = \hat{S} \circ \tau_1 \circ \hat{S}^{-1} \).

\( \Leftarrow \) Since there is a matrix \( S \in GL(k, q) \) such that \( \tau_1 = \hat{S}^{-1} \circ \tau_2 \circ \hat{S} \) and \( P_1 = \hat{S}^{-1}(P_2) \), we see that

\[
[P_2, \ i\tau_2(P_2), \ldots, \ i\tau_2^{n-1}(P_2)] = [(P_1S), \ i(\tau_1(P_1)S), \ldots, \ i(\tau_1^{n-1}(P_1)S)].
\]

By Theorem 3.1, the matrices defined by \( (\tau_1, P_1, n) \) and \( (\tau_2, P_2, n) \) are parity check matrices of skew \((\alpha, \theta)\)-cyclic codes which differ by an invertible matrix \( iS \), i.e. they correspond to the same linear code. Hence \( C_1 = C_2 \).

Finally, let \( C_i \) be two codes defined by \( (\tau_i, P_i, n) \in \Gamma_k^n \) for \( i = 1, 2 \). Denote by \( m_{\tau_i} \) the minimal polynomial of the semi-linear map \( \tau_i \), that is, the monic polynomial \( m_{\tau_i} \) of minimal degree such that \( m_{\tau_i}(\tau_1) = 0 \) for \( i = 1, 2 \) (see [23, Proposition 3.2]).

Let us conclude this section by showing here with an example that in the non-commutative case, i.e. when \( \theta \neq id \), we can not obtain a similar result as Theorem 4 in [18]. More precisely, if \( C_1 = C_2 \) then it follows easily that \( m_{\tau_1} = m_{\tau_2} \), since \( \tau_2 = \hat{S} \circ \tau_1 \circ \hat{S}^{-1} \) for some \( S \in GL(k, q) \) by Proposition 2, but the converse of this statement is not true in general, as the following counterexample shows.

**Example 2.** Consider \( q = 4 \), \( g_1(x) = 1 + ax + x^2 + x^3 \) and \( g_2(x) = 1 + a^2 x + x^2 + x^3 \) in \( R \), where \( \theta(\neq id) \) is the Frobenius automorphism and \( a \) is a root of \( y^2 + y + 1 \in \mathbb{F}_2[y] \).

Let \( C_i \) be the skew \((1, \theta)\)-cyclic \([14, 11]_4\)-codes with generator polynomial \( g_i(x) \) for \( i = 1, 2 \). Since \( g_1(x) \neq g_2(x) \) in \( R/R(x^{14} - 1) \), we have \( C_1 \neq C_2 \). On the other hand, we get

\[
\tau_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & a & 1 \end{pmatrix} \circ \Theta \quad , \quad \tau_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & a^2 & 1 \end{pmatrix} \circ \Theta
\]

whose minimal polynomials are \( m_{\tau_1}(x) = x^6 + x^2 + 1 \) and \( m_{\tau_2}(x) = x^6 + x^2 + 1 \) respectively (see [23] for their construction).

This shows clearly that in general the condition \( C_1 = C_2 \) is not equivalent to \( m_{\tau_1} = m_{\tau_2} \), as it happens when \( \theta = id \) (e.g., see Theorem 4 in [18]).

4. Some remarks and applications

In this last section, we will consider two main consequences of some results obtained in Section 3. Let us recall here that \( q = p^r \) for some prime \( p \) and \( r \in \mathbb{Z}_{\geq 1} \) and that \( \theta \) denotes an automorphism of \( \mathbb{F}_q \) given by \( \theta(z) := z^{p^t} \) for any \( z \in \mathbb{F}_q \), where \( t \) is an integer such that \( 1 \leq t \leq r - 1 \).

4.1. On 1-generator skew quasi-twisted codes. The main result of this section (Theorem 4.3) is a consequence of the previous results and it has been motivated principally by the fact that the class of 1-generator skew quasi-cyclic codes generalizes the class of the 1-generator quasi-cyclic codes by obtaining new examples of linear codes with BKLC parameters (see [1]). This suggests that better codes may be found in this new class, or simply by lengthening skew constacyclic codes.

First of all, let us note here that 1-generator skew quasi-twisted (SQT) codes can be easily defined from the notion of 1-generator QT codes (see, [20, §1]). So, inspired by [1], [2] and [20], we give first the following definition of skew quasi-twisted codes.
Definition 4.1. Take \( \alpha \in \mathbb{F}_q^* \) and let \( \theta \) be an automorphism of \( \mathbb{F}_q \). Denote by 
\( \hat{R}_{\alpha,\theta} := R/R(x^N - \alpha) \) the polynomial ring \( R \) modulo \( x^N - \alpha \). For \( m \in \mathbb{Z}_{\geq 1} \) and 
\[ g = (g_1(x), g_2(x), \ldots, g_m(x)) \in \hat{R}_m^{\alpha,\theta}, \]
the set 
\[ \mathcal{C}_g = \{ (r(x)g_1(x), r(x)g_2(x), \ldots, r(x)g_m(x)) \mid r(x) \in \hat{R}_{\alpha,\theta} \} \]
is called the 1-generator skew quasi-twisted (SQT) code of length \( mN \) and index \( m \) with generator \( g \). Moreover, if \( \alpha = 1 \) then \( \mathcal{C}_g \) is called the 1-generator skew quasi-cyclic (SQC) code of length \( mN \) and index \( m \) with generator \( g \).

First of all, observe that by using similar arguments as in [2, Theorem 6], we can easily deduce the following two results.

Theorem 4.2. Let \( \mathcal{C} \) be a 1-generator SQT code of length \( mN \) and index \( m \) over \( \mathbb{F}_q \) generated by 
\[ g = (g_1(x), g_2(x), \ldots, g_m(x)) \in \hat{R}_m^{\alpha,\theta}, \]
with a parity check matrix 
\[ \Pi = \left[\begin{array}{cccc} x\in \mathbb{F}_q^* & x & \cdots & x \end{array}\right], \]
where 
\[ \mathcal{C}_g = \{ (r(x)g_1(x), r(x)g_2(x), \ldots, r(x)g_m(x)) \mid r(x) \in \hat{R}_{\alpha,\theta} \} \]
is the 1-generator skew quasi-twisted (SQT) code of length \( mN \) and index \( m \) with generator \( g \).

Corollary 3. Let \( \mathcal{C} \) be a 1-generator SQT code of length \( mN \) and index \( m \) over \( \mathbb{F}_q \) generated by 
\[ g = (g(x), g(x)p_1(x), \ldots, g(x)p_{m-1}(x)) \in \hat{R}_m^{\alpha,\theta}, \]
where \( g(x) \) is a monic divisor of \( x^N - \alpha \) with \( N \) a multiple of the order of \( \theta \) and \( \alpha \in \mathbb{F}_q^* \). Then \( \mathcal{C} \) is an \( \mathbb{F}_q \)-free code with rank \( N - \deg g(x) \).

Finally, from Theorem 3.2 we know that an \([N, N - k]_q\)-code is a skew \((\alpha, \theta)\)-cyclic code with generator polynomial \( g(x) \) if and only if \( \mathcal{C} \) is a linear code with a parity check matrix 
\[ [g]^N := [P_1, \gamma(P_1), \gamma^2(P_1), \ldots, \gamma^{N-1}(P_1)], \]
where \( P = (1, 0, \ldots, 0) \in \mathbb{F}_q^k \), \( \gamma(v) := \Theta(v)T_{g(x)} \) for every \( v \in \mathbb{F}_q^k \) and \( T_{g(x)} \) is the companion matrix of \( g(x) \).

Let \( g := x^k - \sum_{i=0}^{k-1} a_i x^i \in R \) be a right divisor of the polynomial \( x^N - \alpha \in R \) for some \( \alpha \in \mathbb{F}_q^* \) and denote by \( \mathcal{T} : \mathbb{F}_q^k \to \mathbb{F}_q^{k-1} \) the map defined by \( \gamma \), i.e. \( \mathcal{T}(\bar{v}) := [\gamma(\bar{v})] \) for every \( \bar{v} \in \mathbb{F}_q^k \). We can say that \( \mathcal{T} \) is defined by \( g \). Note that the columns of \([g]^N = [a_0, a_1, \ldots, a_{k-1}]^N\) can be considered as points in \( \mathbb{F}_q^{k-1} \) of an orbit of \( \mathcal{T} \). Conversely, we can obtain similarly as above a skew \((\alpha, \theta)\)-cyclic \([N, N - k]_q\)-code from an orbit of length \( N \) of the map \( \mathcal{T} \).

So, consider \( m \) orbits \( \mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_m \) of \( \mathcal{T} \) of length \( n_i \) and starting points \( P_i \in \mathcal{O}_i \) for \( i = 1, \ldots, m \), respectively. For simplicity, take \( P_1 = (1, 0, \ldots, 0) \in \mathbb{F}_q^k \). Thus \( n_1 = N \) and denote by \([g]^N + P_2^{n_2} + \cdots + P_m^{n_m} \) the matrix 
\[ [P_1, \gamma(P_1), \gamma^2(P_1), \ldots, \gamma^{N-1}(P_1), P_2, \gamma(P_2), \gamma^2(P_2), \ldots, \gamma^{n_2-1}(P_2), \cdots, P_m, \gamma(P_m), \gamma^2(P_m), \ldots, \gamma^{n_m-1}(P_m) \].
Note that the matrix \([g]^N + P_2^{n_2} + \cdots + P_m^{n_m} \) can be interpreted as a generator matrix of a linear \([N + n_2 + \cdots + n_m, k]_q\)-code. This fact gives a simple method to concatenating skew constacyclic codes to obtain new constructions of linear codes that meet the parameters of some best known linear codes (BKLC) for small values of \( N \) and \( q \) (compare the examples in Table 1 with the corresponding codes in http://www.codetables.de/, http://www.win.tue.nl/~aeb/ and the database of Magma in [5]).
Example 3. Consider $F_4 = \{0, 1, a, a^2\}$, where $a$ is a root of $x^2 + x + 1 \in F_2[x]$. Then the polynomial $g(x) = x^6 - (1 + ax + x^2 + a^2x^3 + x^4 + a^2x^5) \in F_4[x; \theta]$ divides $x^5 - a$ ($N = 7, \alpha = a$), where $\theta(z) = z^4$ for every $z \in F_4$ ($p' = 2$). Consider the points $P_1 := [1 : 0 : 0 : 0 : 1], P_2 := [a : a : 0 : 0 : a : 1 : 1] \in \mathbb{P}^5(F_4)$. Then, under $\mathcal{T}$, the orbit of $P_1$ is of length 7 and that of $P_2$ is of length 14. So, we get a linear $[21, 6, 12]_4$-code whose generator matrix is

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & a & 1 & a^2 & 1 & a^2 & 1 & a^2 & a & a^2 & a^2 & a \\
0 & 1 & 0 & 0 & 0 & 0 & a^2 & a & 0 & a^2 & 1 & a^2 & 0 & 1 & 0 & a^2 & a^2 & a^2 & a^2 & a \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & a & a^2 & 1 & 1 & 0 & 1 & a^2 & a^2 & a^2 & 1 & a & a \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & a & 1 & 0 & a & 1 & a^2 & 0 & a^2 & a^2 & a & a \\
0 & 0 & 0 & 0 & 0 & 1 & a^2 & a & a & 1 & a & a & 1 & 1 & a^2 & a^2 & a & a & a & a^2 & a^2
\end{pmatrix}
$$

and for simplicity of notation, the above matrix can be written simply as $[g]^7 + P^4_1 = [1a^21a^21a^2]^7 + [aa0a11]^4$ (see the first case for $q = 4$ and $k = 6$ in Table 1).

Example 4. Consider $F_8 = \{0, 1, w, w^2, \ldots, w^6\}$. Then the polynomial $g(x) = x^3 - (1 + w^6x + w^4x^2) \in F_8[x; \theta]$ divides $x^4 - w$ ($N = 4, \alpha = w$), where $\theta(z) = z^4$ for every $z \in F_8$ ($p' = 2$). Moreover, we can see that the points $P_1 := [1 : 0 : 0 : w], P_2 := [w^4 : w : 1], P_3 := [w^5 : 1 : 0], P_4 := [1 : 1 : 1] \in \mathbb{P}^3(F_8)$ have orbits of length 4, 12, 12, 6, respectively. So, we get a linear $[34, 3, 28]_8$-code whose generator matrix can be written as

$$[g]^4 + P^4_2 + P^4_3 + P^4_4 = [1w^6w^4]^4 + [w^6w1]^12 + [w^5w10]^12 + [111]^8$$

which reaches the best known distance for a linear $[34, 3]_8$-code (see the first case for $q = 8$ in Table 1).

Example 5. Consider $F_9 = \{0, 1, \beta, \beta^2, \ldots, \beta^7\}$, where $\beta$ is a root of $x^2 + 2x + 2 \in F_3[x]$. We set that $g(x) = x^4 + (1 + \beta^7x + \beta^3x^2 + \beta^3x^3) = x^4 - (\beta^4 + \beta^3x + x^2 + \beta^7x^3) \in F_9[x; \theta]$ divides $x^5 - \beta$ ($N = 5, \alpha = \beta$), with $\theta(z) = z^3$ for every $z \in F_9$ ($p' = 3$). Let $\mathcal{T}$ be the semi-linear map defined by $\theta$ and $g(x)$. Consider the following three points $P_1$, $P_2$ and $P_3$ of $\mathbb{P}^3(F_9)$:

$P_1 = [1 : 0 : 0 : 0], P_2 = [1 : \beta^5 : \beta^7 : 1], P_3 = [\beta^3 : \beta^6 : \beta^3 : 1] .

Then, under $\mathcal{T}$, the orbit of $P_1$ is of length 5 and the orbits of $P_2$ and $P_3$ are both of length 10, and the matrix

$[g]^5 + P^3_2 + P^3_3 = [\beta^4\beta^3 \beta^7]^5 + [1\beta^5\beta^7]^10 + [\beta^3\beta^6\beta^3]^10 $

generates a linear $[25, 4, 19]_9$-code (see the first case for $q = 9$ in Table 1).

On the other hand, when $\alpha = 1, N$ is a multiple of the order of $\theta$, $n_1 = n_2 = \ldots = n_m = N$ and $P_1 \in (\mathbb{F}_q^\theta)^k$ for $i = 1, \ldots, m$, the matrix $[g^N] + P^N_2 + \cdots + P^N_m$ becomes a generator matrix of a 1-generator SQC code of length $mN$ and index $m$ with generator polynomial $g$ given by the following result which generalizes [20, Theorem 2.3] (see for instance Table 2 for some examples of 1-generator SQC codes for $q = 4$).

Theorem 4.3. With the same notation as above, if $P_i \in (\mathbb{F}_q^\theta)^k$ for $i = 1, \ldots, m$, then $[g^N] + P^N_2 + \cdots + P^N_m$ generates a 1-generator SQC $[mN, k]_q$-code of length $mN$ and index $m$ with generator

$$g = (h^*(x), h^*(x)b_2(x^{-1}), \ldots, h^*(x)b_m(x^{-1})) \in R^m,$
where
\[ h^*(x) := \theta^{N-k}(h_0)^{-1} \left[ \sum_{i=1}^{N-k} \theta^i (h_{N-k+1}) x^i + 1 \right] \]
with \( h(x) = h_0 + \cdots + h_{N-k-1} x^{N-k} + x^{N-k} \) such that \( x^N - \theta^{N-k}(\alpha) = g(x)h(x) \) and \( b_i(x) \) is the polynomial given by \((1, x, \ldots, x^{k-1})P_i\) for \( 2 \leq i \leq m \).

**Proof.** Define \( H := [g^N] + P_2^N + \cdots + P_m^N \) and use the same notation as above. Note that
\[ H = [P_1, \tau(P_1), \tau^2(P_1), \ldots, \tau^{N-1}(P_1), P_2, \tau(P_2), \tau^2(P_2), \ldots, \tau^{N-1}(P_2), \ldots, P_m, \tau(P_m), \tau^2(P_m), \ldots, \tau^{N-1}(P_m)] \]
with \( P_i = e_i := (1, 0, \ldots, 0) \in (\mathbb{F}_q^2)^k \) and \( \tau^N(P_i) = P_i \) for \( i = 1, \ldots, m \). By putting
\[ H_i := [P_i, \tau(P_i), \tau^2(P_i), \ldots, \tau^{N-1}(P_i)] \]
for all \( i = 1, \ldots, m \), we simply have \( H = [H_1|H_2|\cdots|H_m] \).

**Claim 1.** \( \tau^h(P_i) = P_i(\tau)(\tau^h(P_i)) \quad \forall h = 0, \ldots, N-1 \).

Note that \( \tau^{i-1}(P_i) = e^*_j \in \mathbb{F}_q^k \) for all \( j = 1, \ldots, k \), where \( e^*_j \) is the \( j \)-th canonical vector of \( \mathbb{F}_q^k \). Therefore, by putting \( P_i := (\lambda_{i,0}, \ldots, \lambda_{i,k-1}) \in (\mathbb{F}_q^k)^k \) for \( i = 1, \ldots, m \), we have
\[ P_i = \sum_{j=0}^{k-1} \lambda_{ij} \tau^j(P_i) = \left( \sum_{j=0}^{k-1} \lambda_{ij} \right) \cdot (P_i) =: P_i(\tau)(P_i), \]
where \( P_i(z) := \sum_{j=0}^{k-1} \lambda_{ij} z^j \in \mathbb{F}_q^k[z] \). Thus
\[ \tau^h(P_i) = \sum_{j=0}^{k-1} \lambda_{ij} \tau^{h+j}(P_i) = P_i(\tau)(\tau^h(P_i)) \quad \forall h = 0, \ldots, N-1. \quad Q.E.D. \]

Define \( \tau^s \cdot [Q_0, Q_2, \ldots, Q_{N-1}] := [\tau^s(Q_0), \tau^s(Q_2), \ldots, \tau^s(Q_{N-1})] \) for any \( s \in \mathbb{Z}_{\geq 0} \).

**Claim 2.** \( H_i = P_i(\tau) \cdot H_1 \quad \forall i = 1, \ldots, m. \)

For \( i = 1, \ldots, m \), by Claim 1 we see that
\[
H_i = [P_i, \tau(P_i), \tau^2(P_i), \ldots, \tau^{N-1}(P_i)] \\
= [P_i(\tau)(P_i), P_i(\tau)(\tau(P_i)), \ldots, P_i(\tau)(\tau^{N-1}(P_i))] \\
=: P_i(\tau) \cdot [P_i, \tau(P_i), \tau^2(P_i), \ldots, \tau^{N-1}(P_i)] = P_i(\tau) \cdot H_1. \\
Q.E.D.
\]

Let \( H_i^* \) be a parity check matrix of the code \( \mathcal{C} \) such that \( \pi(\mathcal{C}) = Rg(x)/R(x^{N-1}) \), written as
\[ H_i^* = \begin{pmatrix}
 h^*(x) \\
 xh^*(x) \\
 \vdots \\
 x^{k-1}h^*(x)
\end{pmatrix} = [J \mid \hat{H}_i^*]
\]
with \( \det(J) \neq 0 \), where from Proposition 1 it follows that
\[ h^*(x) := \theta^{N-k}(h_0)^{-1} \left[ \sum_{i=1}^{N-k} \theta^i (h_{N-k+1}) x^i + 1 \right] \]
with \( h(x) = h_0 + \cdots + h_{N-k-1}x^{n-k-1} + x^{N-k} \) such that \( x^N - \theta^{k-N}(x) = g(x)h(x) \).

**Claim 3.** \( H_i = J^{-1}H_i^* \quad \forall i = 1, \ldots, m, \) where \( H_i^* := \begin{pmatrix} h^*(x)P_1(x^{-1}) \\ x(h^*(x)P_1(x^{-1})) \\ \vdots \\ x^{k-1}(h^*(x)P_1(x^{-1})) \end{pmatrix} \).

By hypothesis, \( H_1 \) is the parity check matrix of \( C \) in the standard form which can be written as \( H_1 = [I_k | \tilde{H}_1] \), where \( I_k \) is the \( k \times k \) identity matrix. In particular, this implies that \( H_1 = J^{-1}H_1^* \). Now, let \( A \) be a matrix with \( N \) columns and for any \( h \in \mathbb{Z}_{\geq 0} \) define the bilinear map

\[
\circ : \mathbb{F}_q[x^{-1}] \times \text{Mat}(k, N; \mathbb{F}_q) \to \text{Mat}(k, N; \mathbb{F}_q)
\]

\[
\forall \lambda, \mu \in \mathbb{F}_q, \forall k, h \in \mathbb{Z}_{\geq 0} \text{ and } \forall A, B \in \text{Mat}(k, N; \mathbb{F}_q) \text{ as follows:}
\]

\[
x^{-h} \circ A := A \left( \begin{array}{c} 0 \\ I_N^t \end{array} \right)^h = A \left( \begin{array}{ccc} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{array} \right)^h
\]

\[
(\lambda x^{-h} + \mu x^{-k}) \circ A = \lambda(x^{-h} \circ A) + \mu(x^{-k} \circ A),
\]

\[
x^{-h} \circ (\lambda A + \mu B) = \lambda(x^{-h} \circ A) + \mu(x^{-h} \circ A).
\]

Moreover, observe that \( \lambda x^{-h} \circ (CD) = C(\lambda x^{-h} \circ D) \) for every \( \lambda \in \mathbb{F}_q \), \( C \in \text{Mat}(k, k; \mathbb{F}_q) \) and \( D \in \text{Mat}(k, N; \mathbb{F}_q) \). Furthermore, if \( A = \begin{pmatrix} a_0(x) \\ \vdots \\ a_{k-1}(x) \end{pmatrix} \equiv \pi(x)R/R(x^N-1) \) and \( \pi(\tilde{a}_j) = a_j(x) \) for \( j = 0, \ldots, k-1 \), then we have \( x^{-1} \circ A \equiv \begin{pmatrix} a_0(x)x^{-1} \\ \vdots \\ a_{k-1}(x)x^{-1} \end{pmatrix} \), where \( p(x)x^{-1} = p_1x + p_2x^2 + \cdots + p_{N-1}x^{N-2} + p_0x^{N-1} \) if \( p(x) = p_0 + p_1x + \cdots + p_{N-1}x^{N-1} \), since \( x^N = 1 \) and \( x^{-1} = x^{N-1} \). Hence we get \( (\lambda x^{-h} + \mu x^{-k}) \circ A \equiv \begin{pmatrix} \lambda a_0(x)x^{-h} \\ \vdots \\ \lambda a_{k-1}(x)x^{-h} \end{pmatrix} \) and

\[
x^{-1} \circ H_1 = x^{-1} \circ [\mathcal{P}_1, \mathcal{P}_1^2, \ldots, \mathcal{P}_1^{N-1}]
\]

\[
= [\mathcal{P}_1, \mathcal{P}_1^2, \ldots, \mathcal{P}_1^{N-1}]
\]

\[
= \mathcal{P}_1 \cdot H_1
\]

\[
x^{-2} \circ H_1 = [\mathcal{P}_1^2, \ldots, \mathcal{P}_1^{N-1}]
\]

\[
= \mathcal{P}_1 \cdot H_1
\]

\[
\vdots
\]

\[
x^{-h} \circ H_1 = \tau^h \cdot H_1.
\]
Then we deduce that $q(\tau) \cdot H_1 = q(x^{-1}) \circ H_1$ for any polynomial $q(t) \in \mathbb{F}_q[t]$, where $(x^{-1})^h := x^{-h}$. Keeping in mind that $P_i(z) \in \mathbb{F}_q[z]$, by Claim 2 we have for $i = 1, ..., m$

$$H_i = P_i(\tau) \cdot H_1 = P_i(x^{-1}) \circ (J^{-1}H_1^*) = J^{-1}(P_i(x^{-1}) \circ H_1^*)$$

$$= J^{-1} \begin{pmatrix} P_i(x^{-1}) \circ \left( \begin{array}{c} h^*(x) \\ xh^*(x) \\ \vdots \\ x^{k-1}h^*(x) \end{array} \right) \end{pmatrix} = J^{-1} \begin{pmatrix} (h^*(x))P_i(x^{-1}) \\ (xh^*(x))P_i(x^{-1}) \\ \vdots \\ (x^{k-1}h^*(x))P_i(x^{-1}) \end{pmatrix}$$

$$= J^{-1} \begin{pmatrix} h^*(x)P_i(x^{-1}) \\ xh^*(x)P_i(x^{-1}) \\ \vdots \\ x^{k-1}(h^*(x))P_i(x^{-1}) \end{pmatrix} =: J^{-1}H_i^*.$$  

Q.E.D.

So, by Claim 3 we have

$$H = [H_1 | H_2 | \cdots | H_m] = [J^{-1}H_1^* | J^{-1}H_2^* | \cdots | J^{-1}H_m^*] = J^{-1}[H_1^* | H_2^* | \cdots | H_m^*],$$

where $H_i^* := \left( \begin{array}{c} h^*(x)P_i(x^{-1}) \\ xh^*(x)P_i(x^{-1}) \\ \vdots \\ x^{k-1}(h^*(x))P_i(x^{-1}) \end{array} \right)$ for $i = 1, ..., m$. Finally, by Corollary 3 we can conclude that $H = [H_1 | H_2 | \cdots | H_m]$ is a generator matrix of a 1-generator SQC $[mN, k]_q$-code of length $mN$ and index $m$ with generator polynomial

$$g = (h^*(x), h^*(x)P_2(x^{-1}), \ldots, h^*(x)P_m(x^{-1})) \in \hat{R}^m,$$

where $h^*(x)$ is given by Proposition 1. \qed

4.2. MDS skew $(\alpha, \theta)$-cyclic codes. First of all, let us observe that in the commutative case, from [18, Theorem 6] it is known that there exists a MDS $\alpha$-constacyclic $[n, k]$-code over $\mathbb{F}_q$ with $(n, q) \neq 1$ and $2 \leq k \leq n - 2$, if and only if $n = p$.

Let us show here that in the non-commutative case, there exist MDS skew constacyclic $[n, k]_q$-codes with $(n, q) \neq 1$ and $2 \leq k \leq n - 2$ also when $n \neq p$.

Example 6. Over $\mathbb{F}_4 = \mathbb{F}_2[\alpha]$ with $\alpha^2 + \alpha + 1 = 0$ and $\theta(z) = z^2$, consider the skew $(\alpha, \theta)$-cyclic $[4, 2]_4$-code $C$ generated by $g(x) = \alpha x^2 + \alpha^2 x + \alpha^3$. Then $C$ is a MDS code with parity check matrix

$$\begin{pmatrix} \alpha^2 & \alpha^2 & \alpha & 0 \\ 0 & \alpha & \alpha & \alpha^2 \end{pmatrix}.$$

Example 7. Over $\mathbb{F}_9 = \mathbb{F}_3[\omega]$ with $\omega^3 + \omega + 2 = 0$ and $\theta(z) = z^2$, consider the skew $(1, \theta)$-cyclic $[6, 4]_9$-code $C$ generated by $g(x) = \omega^5 x^2 + \omega^7 x + \omega^7$. Then $C$ is a MDS code with parity check matrix

$$\begin{pmatrix} \omega^7 & \omega^7 & \omega^5 & 0 & 0 & 0 \\ 0 & \omega^5 & \omega^5 & \omega^7 & 0 & 0 \\ 0 & 0 & \omega^7 & \omega^7 & \omega^5 & 0 \\ 0 & 0 & 0 & \omega^5 & \omega^3 & \omega^7 \end{pmatrix}.$$
New constructions of some linear codes with BKLC parameters.

Some 1-generator $	au$ codes with

$$\begin{align*}
\text{Generator Matrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\
\end{pmatrix} \quad \alpha, \theta \\
N &= 5 \\
\alpha &= 2 \\
p^3 &= 2
\end{align*}$$

Table 1. New constructions of some linear codes with BKLC parameters.

By Theorem 3.1, we recall that the parity check matrix of a MDS skew $(\alpha, \theta)$-cyclic $[n, k, d]_q$-code has the form $[^t P, ^t \tau(P), ^t \tau^2(P), \ldots, ^t \tau^{n-1}(P)]$, where $P \in \mathbb{F}_q^k$, $\tau(\vec{v}) := \Theta(\vec{v})T$ for every $\vec{v} \in \mathbb{F}_q^k$ with $T \in GL(k, q)$ and $\tau^n(P) = \alpha P$ with $n \leq \text{ord}(\tau)$.

Therefore, under certain conditions on $q$, $k$, and $n$, the existence of MDS skew $(\alpha, \theta)$-cyclic $[n, k, d]_q$-codes is strictly related to some algebraic conditions, as it is shown in the following results.
Proposition 3. Assume that \( q > 2, \ k \geq 4 \) and
\[
\begin{align*}
q + k - \frac{\sqrt{q} + 5}{4} &< n, \quad \text{for } q \text{ odd} \\
q + k - \frac{2\sqrt{q} + 7}{4} &< n, \quad \text{for } q \text{ even}.
\end{align*}
\]
If there exists a MDS skew \((\alpha, \theta)\)-cyclic \([n, n-k]_q\)-code, then \( n \leq q + 1 \).

Proof. Let \( C \subseteq \mathbb{F}_q^n \) be a MDS skew \((\alpha, \theta)\)-cyclic \([n, n-k]_q\)-code. By Theorem 3.1, a parity check matrix of \( C \) has the form \([ t^1 P, t^\tau (P), t^\tau^2 (P), \ldots, t^\tau^{n-1} (P) \] , where \( P \in \mathbb{F}_q^k \) and \( \tau = T \circ \Theta \) for some \( T \in GL(k, q) \) with \( \tau(v) := \Theta(v) T \) for every \( v \in \mathbb{F}_q^k \) and such that \( \tau^n(P) = \alpha P \). Note that the set
\[
K := \{ |\tau^i (P)| : i = 0, \ldots, n-1 \} \subseteq \mathbb{P}^{k-1} (\mathbb{F}_q)
\]
defines an \( n \)-arc in the finite projective space \( \mathbb{P}^{k-1} (\mathbb{F}_q) \). Thus, by \([3, 4, 16, 21, 24]\) and the hypothesis
\[
\begin{align*}
q - \frac{1}{3} \sqrt{q} + (k-1) - \frac{1}{3} &< n, \quad q \text{ odd} \\
q - \frac{1}{2} \sqrt{q} + (k-1) - \frac{3}{4} &< n, \quad q \text{ even}.
\end{align*}
\]
we deduce that \( K \) lies on a unique rational normal curve. Hence \( n = |K| \leq q + 1 \). \( \square \)

Let us note here that for some \( a \in \mathbb{F}_q^* \), a skew monic polynomial \( F(x) \in \mathbb{F}_q[x; \theta] \) could not divide \( x^m - a \) on the right for every \( m \in \mathbb{Z}_{\geq 0} \). Nevertheless, in \([22, \S3.2.1]\) it was shown that any skew polynomial \( F(x) \) as above with a regular constant term always divides the polynomial \( x^m - 1 \) on the left for some \( m \in \mathbb{Z}_{>0} \). Observe that the proof of this fact works well also for right divisions with only slight modifications.

So, in line with the classical definition of subexponent of a polynomial in \( \mathbb{F}_q[x] \) (see, e.g., \([14, \text{pp. 6–7}]\)), let us give here a similar definition for the non-commutative ring \( \mathbb{F}_q[x; \theta] \) (see also \([12, \text{Definitions 2.1(a) and 3.1}]\)).

Definition 4.4. Let \( F(x) \) be a skew monic polynomial in \( \mathbb{F}_q[x; \theta] \). If \( F(0) \neq 0 \), then \( F(x) \) has right exponent \( e(F) \) if \( e(F) \) is the smallest positive integer such that \( F(x) \) is a right divisor of \( x^{e(F)} - a \) for some \( a \in \mathbb{F}_q^* \). Similarly, one can define the notion of left exponent of \( F(x) \).

The following result is an algebraic characterization of the existence of some MDS skew \((\alpha, \theta)\)-cyclic codes with certain parameters.

Theorem 4.5. Assume that \( q > 2, \ k \geq 4 \) and
\[
\begin{align*}
q + k - \frac{\sqrt{q} + 5}{4} &< n \leq q + 1, \quad \text{for } q \text{ odd} \\
q + k - \frac{2\sqrt{q} + 7}{4} &< n \leq q + 1, \quad \text{for } q \text{ even}.
\end{align*}
\]
Then, there exists a MDS skew \((\alpha, \theta)\)-cyclic \([n, n-k]_q\)-code if and only if there exists a polynomial \( x^2 + ax + b \in \mathbb{F}_q[x; \theta] \) with \( b \neq 0 \) such that \( e(x^2 + ax + b) = n \).

Proof. Let \( C \subseteq \mathbb{F}_q^n \) be a MDS skew \((\alpha, \theta)\)-cyclic \([n, n-k]_q\)-code. By the same notation as in Proposition 3, through a projectivity given by \( A \in GL(k, q) \), we can send the points of \( K \) onto the canonical rational normal curve which is the image of the Venorese map \( \nu_k : \mathbb{P}^1 (\mathbb{F}_q) \rightarrow \mathbb{P}^{k-1} (\mathbb{F}_q) \) given by
\[
\nu_k (x_0 : x_1) = [x_0^{k-1} : x_0^{k-2} x_1 : \cdots : x_0 x_1^{k-2} : x_1^{k-1}].
\]
By taking \( \tau' := A \circ \tau \circ A^{-1} \) given by \( \tau'(Q) := A \circ \tau \circ A^{-1}(Q) = A \circ \tau(QA^{-1}) = \tau(QA^{-1})A \) for every \( Q \in \mathbb{F}_q^k \), we can define the set

\[ K' := \{ [\tau^i(P)A] : i = 0, \ldots, n-1 \} = \{ ([\tau')^i(P)A] : i = 0, \ldots, n-1 \} \subseteq \mathbb{P}^{k-1}(\mathbb{F}_q), \]

where \( \tau' = M' \circ \Theta \) for some \( M' \in GL(k, q) \). Now, consider the following commutative diagram

\[ \begin{array}{ccc}
\mathbb{P}^{k-1}(\mathbb{F}_q) & \xrightarrow{\tau':= M' \circ \Theta} & \mathbb{P}^{k-1}(\mathbb{F}_q) \\
\nu_k \downarrow & & \uparrow \nu_k \\
\mathbb{P}^1(\mathbb{F}_q) & \xrightarrow{\sigma:= M \circ \Theta} & \mathbb{P}^1(\mathbb{F}_q)
\end{array} \]

where \( M \in GL(2, q) \) is such that \( \nu_k \circ \sigma = \tau' \circ \nu_k \). In this way, we can identify the elements of \( K' \) with the elements of

\[ \{ ([\sigma^i(\bar{p})]) = [(M \circ \Theta)^i(\bar{p})] : i = 0, \ldots, n-1 \} \subseteq \mathbb{P}^1(\mathbb{F}_q), \]

where \( [\bar{p}] := \nu_k^{-1}([PA]) \). Consider a projectivity of \( \mathbb{P}^1(\mathbb{F}_q) \) given by \( B \in GL(2, q) \) such that \((1, 0)B = \bar{p}\). Let \( M'' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) be a \( 2 \times 2 \) matrix with \( \beta \neq 0 \) such that the following diagram commutes,

\[ \begin{array}{ccc}
\mathbb{P}^1(\mathbb{F}_q) & \xrightarrow{\sigma:= M \circ \Theta} & \mathbb{P}^1(\mathbb{F}_q) \\
\nu_k \downarrow & & \uparrow \nu_k \\
\mathbb{P}^1(\mathbb{F}_q) & \xrightarrow{\sigma'':= M'' \circ \Theta} & \mathbb{P}^1(\mathbb{F}_q)
\end{array} \]

where \( [(1, 0)]B = [\bar{p}] \). Define \( C := \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \) and observe that \( C^{-1} = \beta^{-1} \begin{pmatrix} \beta & 0 \\ -\alpha & 1 \end{pmatrix} \).

Then

\[ C^{-1} \circ (M'' \circ \Theta) \circ C = \begin{pmatrix} 1 & 0 \\ \theta(\alpha) & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha \beta^{-1} & 1 \end{pmatrix} \circ \Theta = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \circ \Theta \]

for some \( a, b \in \mathbb{F}_q \) with \( b \neq 0 \). This gives the further commutative diagram

\[ \begin{array}{ccc}
\mathbb{P}^1(\mathbb{F}_q) & \xrightarrow{\sigma'':= M'' \circ \Theta} & \mathbb{P}^1(\mathbb{F}_q) \\
C \downarrow & & \uparrow C \\
\mathbb{P}^1(\mathbb{F}_q) & \xrightarrow{A \circ \Theta} & \mathbb{P}^1(\mathbb{F}_q)
\end{array} \]

where \( A := \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \) and \( [(1, 0)]C = [(1, 0)] \). Combining the previous commutative diagrams, we obtain the following commutative diagram

\[ \begin{array}{ccc}
\mathbb{P}^{k-1}(\mathbb{F}_q) & \xrightarrow{\tau':= M' \circ \Theta} & \mathbb{P}^{k-1}(\mathbb{F}_q) \\
\nu_k \circ B \circ C \downarrow & & \uparrow \nu_k \circ B \circ C \\
\mathbb{P}^1(\mathbb{F}_q) & \xrightarrow{A \circ \Theta} & \mathbb{P}^1(\mathbb{F}_q)
\end{array} \]

such that the elements of \( K' \) can be identified with the elements of

\[ \{ [(A \circ \Theta)^i(1, 0)] : i = 0, \ldots, n-1 \} \subseteq \mathbb{P}^1(\mathbb{F}_q). \]
Let $f(x) := x^2 + ax + b$ be the characteristic polynomial of $A$ and define the bijective map

$$\pi: \mathbb{F}_q^2 \longrightarrow \mathbb{F}_q[x; \theta]/(f(x)) =: R$$

defined by $\pi(\vec{v}) := v_0 + v_1 x$, where $\vec{v} := (v_0, v_1)$.

Observe that $\pi((A \circ \Theta)^j(\vec{v})) = x^j \cdot \pi(\vec{v}) = x^j (v_0 + v_1 x)$ for any $j \in \mathbb{Z}_{\geq 0}$ and that $\pi$ induces the bijective map

$$\hat{\pi}: (\mathbb{F}_q^2 \setminus \{(0, 0)\}) / \mathbb{F}_q^* \longrightarrow (R \setminus \{0\}) / \mathbb{F}_q^*$$
given by $\hat{\pi}((v_0, v_1)) := [\pi(v_0, v_1)]$. So, $\hat{\pi}((A \circ \Theta)^j(\vec{v})) = [x^j (v_0 + v_1 x)]$ for any $j \in \mathbb{Z}_{\geq 0}$.

Moreover, since $(\mathbb{F}_q^2 \setminus \{(0, 0)\}) / \mathbb{F}_q^* \cong \mathbb{F}_q^*$, for all $i = 1, \ldots, n - 1$, we have

$$[(1, 0)] \neq [(A \circ \Theta)^i(1, 0)] \Rightarrow \hat{\pi}[(1, 0)] \neq \hat{\pi}[(A \circ \Theta)^i(1, 0)] \Rightarrow [1] \neq [x^i]$$

$$(\forall \lambda \in \mathbb{F}_q^*) \Rightarrow \lambda \neq x^i$$

$$(\forall \lambda \in \mathbb{F}_q^*) \Rightarrow x^i - \lambda \neq 0 \text{ mod } f(x)$$

i.e., there exists $f(x) = x^2 + ax + b \in \mathbb{F}_q[x; \theta]$ with $b \neq 0$ such that $x^i - \lambda \neq 0 \text{ mod } f(x)$ for all $\lambda \in \mathbb{F}_q^*$ and $i = 1, \ldots, n - 1$. Since $[1] = [x^n]$, we conclude that there exists $x^2 + ax + b \in \mathbb{F}_q[x; \theta]$ with $b \neq 0$ such that $e(x^2 + ax + b) = n$.

Looking closely the above proof, by construction the converse of the statement becomes at this point easy to prove.

Finally, observe that the above result provides a method to construct via Veronese embeddings some MDS skew $(\alpha, \theta)$-cyclic $[n, n - k]_q$-code.

**Corollary 4.** If a polynomial $x^2 + ax + b \in \mathbb{F}_q[x; \theta]$ with $b \neq 0$ has right exponent $e(x^2 + ax + b) = n - q + 1$, then for any integer $h$ such that $1 \leq h \leq n - 1$ there exists a MDS skew $(\alpha, \theta)$-cyclic $[n, h]_q$-code for some $\alpha \in \mathbb{F}_q^*$.

**Proof.** Let $k$ be an integer such that $2 \leq k \leq n - 1$ and define $A := \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$.

Note that there exists a matrix $M \in GL(k, q)$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{F}_q^{k-1} & \xrightarrow{M \circ \Theta} & \mathbb{F}_q^{k-1} \\
\uparrow \nu_k & & \uparrow \nu_k \\
\mathbb{F}_q^1 & \xrightarrow{\lambda \circ \Theta} & \mathbb{F}_q^1
\end{array}$$

where $\nu_k[x_0 : x_1] = [x_0^{k-1} : x_0^{k-2} x_1 : \cdots : x_0 x_1^{k-2} : x_1^{k-1}]$. Thus the set

$$\{([A \circ \Theta]^i(1, 0)) : i = 0, \ldots, n - 1\} \subseteq \mathbb{F}_q^1$$

provides a set $\{(M \circ \Theta)^i(1, 0, \ldots, 0) : i = 0, \ldots, n - 1\} \subseteq \mathbb{F}_q^{k-1}$ of points lying on a rational normal curve in $\mathbb{P}_q^{k-1}$. This gives a matrix of type

$$[\tau^0(P), \tau^1(P), \tau^2(P), \ldots, \tau^{n-1}(P)],$$

where $P = (1, 0, \ldots, 0) \in \mathbb{F}_q^k$ and $\tau := M \circ \Theta$ is such that $\tau^n(P) = \beta P$ for some $\beta \in \mathbb{F}_q^*$. We conclude by Theorems 2.4, 3.1 and Corollary 2. \qed
When $\theta$ is the Frobenius automorphism of $\mathbb{F}_q$, in [12, Theorem 3.1] the authors show that the right and left exponents of a skew polynomial $g(x)$ in $\mathbb{F}_q[x; \theta]$ are equal and they provide an algebraic method to compute the right exponent of $g(x)$. On the other hand, the following MAGMA program defines a command EXP($p^t$, [b, a, 1]) which determines the right exponent of any skew polynomial $x^2 + ax + b \in \mathbb{F}_q[x; \theta]$ with $b \neq 0$, when $\theta(z) = z^{p^t}$.

**Program 2.**

```magma
Exp:=function(qq,g,t)
    R<x>:=TwistedPolynomials(F:q:=qq);
    f:=R!g; n:=Degree(f)-1;
    repeat n:=n+1; v:=[t];
        for i in [1..n-1] do
            v:=v cat [0];
        end for;
        s:=v cat [1]; g:=R!s; _,r:=Quotrem(g,f);
    until r eq R![0] or n ge c+2;
    if r eq R![0] then
        return n;
    end if;
    if n ge c+2 then
        return c^Degree(f);
    end if;
    end function;

EXP:=function(aa,h)
    BB:=[x : x in F | x ne 0]; C:={};
    for j in BB do
        C:= C join {Exp(aa,h,j)};
    end for;
    return Minimum(C);
end function;
```

**Example 8.** Let $\mathbb{F}_q := \{0, 1, w, w^2, \ldots, w^{q-2}\}$ be a field with $q$ elements. By using Program 2 and Corollary 4, we can construct, for example, the Table 3 and deduce the existence of some MDS skew $(\alpha, \theta)$-cyclic $[n, k]$-codes whose parameters are not met in the commutative case by any MDS $\alpha$-constacyclic code (compare this table with Tables 1, 2 and 3 in [19]).

For instance, consider the first case $q = 8$ in Table 3 and, for simplicity, let $k = 3$. When $\theta = id$, the only possible exponents less or equal to $q + 1 = 9$ of polynomial of type $x^2 + ax + b$ with $b \neq 0$ are 2, 3, 7 and 9. On the other hand, Table 3 shows that one can construct a MDS skew $(1, \theta)$-cyclic $[6, 3]_8$-code. Indeed, by using the same notations as in the proof of Corollary 4, we have

$$A := \begin{pmatrix}
0 & 1 \\
-w^3 & -1
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
w^3 & 1
\end{pmatrix} \quad \text{and} \quad \nu_3[x_0 : x_1] = [x_0^2 : x_0x_1 : x_1^2].$$

Thus we obtain the set

$$\{(1, 0), (0, 1), ([w^3, 1]), ([w^3, w^2]), ([1, w^3]), ([1, 1])\} \subset \mathbb{P}^1(\mathbb{F}_8)$$
Table 3. Existence of some MDS skew $(\alpha, \theta)$-cyclic $[n,k]_q$-code.

| $q$ | $n$ | $k$ | $\alpha$ | $p^t$ for $\theta$ | Polynomial $x^2 + ax + b$ |
|-----|-----|-----|----------|-------------------|----------------------|
| 8   | 6   | 3   | 1, 1     | 2                 | $x^2 + x + w^3$       |
| 9   | 6   | 3   | 1, $w^4$ | 3                 | $x^2 + x + w^3$       |
| 16  | 8   | 3   | 1, 1, 1  | 2                 | $x^2 + x + w$         |
| 16  | 12  | 3   | 1, 1, 1  | 2                 | $x^2 + wx + w$        |
| 25  | 10  | 3   | $w^{12}, w^6, 1$ | 5             | $x^2 + x + w^3$       |
| 32  | 10  | 3   | 1, 1, 1  | 2                 | $x^2 + x + w$         |
| 32  | 15  | 3   | 1, 1, 1  | 2                 | $x^2 + x + w^7$       |
| 49  | 14  | 3   | $w^{32}, w^{24}, w^{16}$ | 7         | $x^2 + x + w^2$       |
| 64  | 12  | 3   | 1, 1, 1  | 2                 | $x^2 + x + w$         |
| 64  | 18  | 3   | 1, 1, 1  | 2                 | $x^2 + x + w^3$       |

which gives via $\nu_3$ the following set of points in $\mathbb{P}^2(\mathbb{F}_8)$:

$\mathcal{K} := \{(1,0,0), (0,0,1), (w^6,w^3,1), (w^6,w^5,w^4), (1,w^3,w^6), (1,1,1)\}$.

By $\mathcal{K} \subset \mathbb{P}^2(\mathbb{F}_8)$ one can construct the following matrix

$$H = \begin{bmatrix}
1 & 0 & w^6 & w^6 & 1 & 1 \\
0 & 0 & w^3 & w^5 & w^3 & 1 \\
0 & 1 & 1 & w^4 & w^6 & 1
\end{bmatrix}$$

which can be interpreted as a parity check matrix of a MDS skew $(1,\theta)$-cyclic $[6,3]_8$-code whose generator matrix $G$ in standard form is

$$G = \begin{bmatrix}
1 & 0 & 0 & w^5 & w & w^6 \\
0 & 1 & 0 & w^2 & w^2 & w^4 \\
0 & 0 & 1 & w^5 & w^2 & w^5
\end{bmatrix}.$$  

Similarly, one can construct all the MDS skew $(\alpha, \theta)$-cyclic $[n,k]_q$-codes of Table 3.

5. Conclusion

In this paper, we consider skew constacyclic codes $\mathcal{C}$ (called also skew $(\alpha, \theta)$-cyclic codes) and some algebraic and geometric properties of their dual codes. After proving again in an easy way that the dual code of $\mathcal{C}$ is a skew $(\alpha^{-1}, \theta)$-cyclic code with an explicit generator polynomial, we show that the columns of the generator and the parity check matrices of $\mathcal{C}$ are orbits of points in projective spaces via semilinear maps. Two main applications of this property are given. The first application consists of some results on 1-generator skew quasi-twisted codes which prove that a suitable concatenation of skew $(\alpha, \theta)$-cyclic codes gives in fact a 1-generator skew quasi-cyclic code. The second one shows that under certain conditions on the parameters $n, k$ and $q$ of a code, the existence of MDS skew $(\alpha, \theta)$-cyclic $[n,k]_q$-codes is strictly related to some algebraic conditions which are explicitly determined.

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