Ground states of the infinite
q-deformed Heisenberg ferromagnet

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\textbf{Abstract.} We set up a general structure for the analysis of “frustration-free
ground states”, or “zero-energy states”, i.e., states minimizing each term in a
lattice interaction individually. The nesting of the finite volume ground state
spaces is described by a generalized inductive limit of observable algebras. The
limit space of this inductive system has a state space which is canonically iso-
orphic (as a compact convex set) to the set of zero-energy states. We show that
for Heisenberg ferromagnets, and for generalized valence bond solid states, the
limit space is an abelian C*-algebra, and all zero-energy states are translationally
invariant or periodic. For the $q$-deformed spin-1/2 Heisenberg ferromagnet in one
dimension (i.e., the XXZ-chain with $S_q U(2)$-invariant boundary conditions) the
limit space is an extension of the non-commutative algebra of compact operators
by two points, corresponding to the “all spins up” and the “all spins down” states,
respectively. These are the only translationally invariant zero-energy states. The
remaining ones are parametrized by the density matrices on a Hilbert space, and
converge weakly to the “all up” (resp. “all down”) state for shifts to $-\infty$ (resp.
$+\infty$).

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1. Introduction

The determination of all ground states of a given model of statistical mechanics is usually a very difficult task. Even for a finite system an explicit characterization of the ground states is hardly ever possible. Additional problems arise in the passage to the thermodynamic limit, where, on the one hand, an accidental ground state degeneracy of the finite volume models can disappear, or, on the other hand, even if the finite volume models have unique ground states, some low lying states can converge to additional ground states [KT].

In the present paper we study the infinite volume limit of ground states of quantum models of an especially simple kind: these models admit states of the infinite system which restrict to ground states on every finite subregion. For an interaction which is the sum of translates of a fixed finite range operator, such states not just minimize the total energy, but even each term in the interaction. For many (classical or quantum) interactions such states do not exist, a phenomenon also known as “frustration”. Therefore we will call such states “frustration-free ground states”, or “zero-energy states”. In the classical case interactions admitting such ground states are known as $m$-potentials [HS,Sla,Mie,EFS]. Often a lattice interaction which appears to be frustrated allows an equivalent form which is an $m$-potential and, in fact, one has to work hard [Mie] to find examples of “intrinsically frustrated” potentials, for which this is impossible.

For one-dimensional nearest neighbour interactions $H_2, \tilde{H}_2$, equivalence corresponds to perturbations of the form

$$\tilde{H}_2 = H_2 + (X \otimes I - I \otimes X) \quad ,$$

with $X$ an arbitrary one-site observable. Clearly, all terms containing $X$ cancel in the sum over all translates of $\tilde{H}_2$, which represents the formal Hamiltonian of the system. In particular, the two interactions generate the same infinite volume dynamics. That is, for all strictly local observables $A$, the commutators

$$\left[ \sum_{x=-L}^{L} \tilde{H}_{x,x+1}, A \right]$$

are equal to the same local element “[H, A]” for all sufficiently large $L$, and all perturbations $X$. Hence they have the same ground states $\omega$, as defined by the property that

$$\omega(A'[H, A]) \geq 0 \quad ,$$

for all strictly local $A$ (see e.g. Definition 5.3.18 in [BR]). Note also that $H_2$ and $\tilde{H}_2$ have the same expectation in every translationally invariant state, and since only such expectations enter the thermodynamic variational principles, it follows that the two interactions determine the same thermodynamic functions. For a translationally invariant state $\omega$, the minimization of $\omega(H_2)$ is in fact equivalent to (1.2) (see Theorem 6.2.58 in [BR]).
The “zero-energy states” investigated in this paper satisfy a sharper requirement: the expectation of each translate of $H_2$ is equal to its smallest possible value, the lowest eigenvalue of $H_2$. These states are free of “defects” [FP], i.e., there are no sites at which some term in the Hamiltonian can only be minimized by a global state change. We do not require translation invariance. Zero-energy states do satisfy (1.2), and, in fact, proving the zero-energy property is often the best constructive way to show the ground state property [Sha]. But in contrast to the ground state property, the zero-energy property now depends on the boundary term $X$.

An example of this dependence is furnished by the ferromagnetic spin-1/2 XXZ chain, the principal model studied in this paper. The Hamiltonian is

$$H_{XXZ}^L = \sum_{x=1}^{L-1} \left\{ \frac{-q}{2(1 + q^2)} (\sigma_x^1 \sigma_{x+1}^1 + \sigma_x^2 \sigma_{x+1}^2) + \frac{1}{4} (I - \sigma_x^3 \sigma_{x+1}^3) \right\} ,$$

(1.3)

where $\sigma_x^i$, $i = 1, 2, 3$ denotes the Pauli matrices at site $x$, and $q \in (0, 1)$ is a parameter. Up to a factor and a constant this is the Hamiltonian written in [Bax,YY,Joh,AKS] with anisotropy parameter $\Delta = (q + q^{-1})/2 > 1$. (Note however, that relative to some treatments, mainly of the antiferromagnetic regime of the model [Góm,FYF] one has to rotate every other spin by $\pi$ around the 3-axis, and change the sign of $\Delta$ to obtain (1.3)).

The ground state vectors of the basic interaction operator $H_2^{XXZ}$ are the product vectors $|++\rangle$ and $|--\rangle$ (we have chosen constants such that the ground state energy is 0). Hence the only way to achieve minimal energy on a finite chain is to take one of the two product states “all up” or “all down”. Clearly, the corresponding infinite product states, denoted by $\omega^\uparrow$ and $\omega^\downarrow$, are zero-energy states of this interaction.

It is easy to see that by a suitable perturbation (1.1) with $X = \lambda \sigma^1$, one obtains an interaction which does not admit any frustration-free ground states at all. What is more surprising, however, is that with another choice of $X$ we can increase the number of zero-energy states in this model: with

$$X^g = \frac{1 - q^2}{4(1 + q^2)} \sigma^3$$

(1.4)

we get the interaction

$$H_2^g = H_2^{XXZ} + (X^g \otimes I - I \otimes X^g) .$$

(1.5)

This operator is the one-dimensional projection, onto the vector in $\mathbb{C}^2 \otimes \mathbb{C}^2$ which is invariant under the product representation $U \otimes U$ of $q$-deformed SU(2) [Wo1]. In this sense it is the deformation of the ordinary SU(2)-invariant Heisenberg ferromagnetic chain. The modification (1.5) of the XXZ chain has been considered by many authors [AB3Q, PS,MN], although often for imaginary deformation parameter $q$ [HMRS], or in the antiferromagnetic regime [JMMMN]. The finite volume ground states are considered in [ASW]. However, in that paper no attempt is made to determine the geometry of the ground state degeneracy in the infinite system.
It is one of the main aims of this paper to determine the infinite dimensional manifold of zero-energy states of the interaction $H_2^q$. The result is stated in the following Theorem (compare Corollary 17 below, where the mentioned identification of states with the density matrices will also be made concrete). Recall that two states are called *quasi-equivalent*, if they are normal with respect to each other, i.e., each one is given by a density matrix in the representation of the other. Equivalently, each one is approximated in norm by local perturbations of the other.

1. **Theorem.** As a convex set, the set of zero-energy states of the interaction (1.5) is isomorphic to the convex hull of three quasi-equivalence classes:

   (1) the set consisting only of the “all spins up” state $\omega^\uparrow$
   (2) the set consisting only of the “all spins down” state $\omega^\downarrow$
   (3) a set of “kink states”, which is isomorphic to the set of the density matrices on a separable Hilbert space. Each of these states converges in the $w^*$-topology to $\omega^\uparrow$ (resp. $\omega^\downarrow$), when shifted along the chain to right (resp. left) infinity.

The states $\omega^\uparrow$ and $\omega^\downarrow$ are the only translationally invariant zero-energy states. Since each forms a quasi-equivalence class by itself, the ground state problem in its GNS representation is non-degenerate. In contrast, the GNS representation of any kink state contains all other kink states, so that the ground state problem is infinitely degenerate. This GNS representation will be constructed explicitly in Section 4.4. It turns out that in all the states described in Theorem 1 the Hamiltonian in the GNS representation has a non-zero gap above zero (see Section 4.5, or [Na2]). The gap estimate vanishes precisely for $q = 1$. This corresponds to the undeformed Heisenberg ferromagnet, for which the vanishing of the gap is well-known (arbitrarily low excitations are given by the so-called magnon states).

Note that all the states described by Theorem 1 are also ground states of the XXZ interaction. In fact, we get four classes of ground states, because we can add

(4) the set of “anti-kink states”, obtained from kink states (3) by exchanging “$+$” and “$-$”.

Of course, the anti-kink states are zero-energy states of the interaction of the form (1.5) with the opposite sign for the perturbation term. They can also be obtained from the kinks by left/right inversion of the sites on the chain. This exhausts the list of ground states arising as zero-energy states of perturbations of the form (1.5). It is not known, however, whether (1), ..., (4) is also the complete list of ground states (1.2) of the XXZ interaction. In fact, even for the case of the ordinary Heisenberg ferromagnet (the case $q = 1$), for which the zero-energy states are well-known (see Section 3.1), this problem seems to be open (compare the brief discussion at the end of Section 6.2 of [BR]).
A classical example with a somewhat similar ground state structure, two translationally invariant states, and an infinity of non-invariant kink states, was given by Pechersky [Pec]. An even simpler classical model with this structure is given by the $q = 0$ case of the model studied here: an Ising chain, diagonally embedded into a quantum chain, in which only the nearest neighbour configuration “$+ -$” is forbidden. In this case the “kink” states are sharp transitions from “$+$” to “$- $”.

Apart from the treatment of the special model, the aim of this paper is also the development of some general techniques for the computation of zero-energy state spaces. The basic result (Theorem 3) is the isomorphism between the set of zero-energy states and the state space of a “space of zero-energy observables”, which arises in an inductive limit from the finite volume zero-energy observables. The inductive limit construction follows closely the usual construction of the quasi-local observable algebra of the infinite system. It is, however, not a C*-inductive limit, so that the limit space is not automatically a C*-algebra.

We illustrate the general construction with two well-known examples. The first is the Heisenberg ferromagnet (in any dimension, and with arbitrary couplings). Here the zero-energy algebra is isomorphic to the algebra of continuous functions on the 2-sphere. The points of this sphere correspond precisely to the pure zero-energy states, which are hence characterized by the one direction in space, along which all spins are directed. The second example is the class of generalized valence bond solid (GVBS) states [Na1,FNW1,AKLT]. In this case the zero-energy observables form a finite dimensional abelian algebra, which implies that every zero-energy state has a unique decomposition into periodic pure states. Moreover, in each extremal zero-energy state the Hamiltonian has a non-zero spectral gap above the ground state energy. In both these cases the zero-energy states retain some translation symmetry, and hence lie in some simplex of translation invariant states. The zero-energy algebra must therefore be commutative. It is perhaps the most interesting feature of the interaction (1.5) that the set of zero-energy states has the structure of the state space of a non-commutative C*-algebra. This is only possible due to the lack of translation invariance of the kink ground states.

Superficially, the $q$-deformed Heisenberg ferromagnet has many features in common with the undeformed ferromagnet ($q = 1$). For example, the ground state degeneracy for the model with chain length $L$ is $L + 1$. Moreover, the Hamiltonian is reduced by the decomposition of the space according to irreducible representations of deformed SU(2) (denoted by $S_q U(2)$) [Wo1,Dri], and these are again in one-to-one correspondence with the representations of SU(2). This would suggest the same ground state structure for the $q$-deformed and undeformed Heisenberg chains. However, as the Theorem shows, the dimension count is too coarse to admit such conclusions.

Even though the model we consider is characterized in terms of a quantum group symmetry, and even though this “symmetry” is quite useful for obtaining an understanding of the ground states in finite volume, it seems to be of little help in treating the infinite
chain. The fundamental reason for this is that the embeddings used to define the quasi-local algebra are not consistent with the quantum group actions defined for each finite volume. In fact, the quasi-local algebra admits no action of quantum SU(2) which commutes with translations [FNW5]. This difficulties are also seen in the ground state problem: on the intuition derived from classical symmetries one would expect the “symmetry” of the local Hamiltonians to act on the space of zero-energy states. Indeed, this is the case for the action of the only classical subgroup of $S_q U(2)$ (the rotations around the $z$-axis), and this fact is crucially used in our theory to obtain the necessary estimates of operator norms. However, the spectrum of the generator of this subgroup is the set of integers with multiplicity one, and this is inconsistent with an extension of this action to an action of $S_q U(2)$.

The paper is organized as follows: In Section 2 we present the general theory of zero-energy state problems. After stating the problem in Section 2.1, we describe, in Section 2.2, the framework for generalized inductive limits of spaces of observables, and, in particular, the definition of zero-energy observables of the infinite system. The connection with Hilbert space representations of the quasi-local algebra is made in Section 2.3. The general structure is exemplified with the Heisenberg ferromagnet (Section 3.1) and the generalized valence bond solid states (Section 3.2). In Section 4 we study the $q$-deformed Heisenberg ferromagnet. The main result is the identification of the space of zero-energy state observables in Section 4.3 (Theorem 16). The Hilbert space representation relevant for the discussion of zero-energy states is described in Section 4.4, and the spectral gap is estimated in Section 4.5.

2. Zero-energy observables

2.1. Statement of the Problem

In this section we describe abstractly the set of zero-energy states of an infinite quantum lattice system. The basic observable algebra of the system is thus the quasi-local algebra [BR], which is constructed as the C*-inductive limit of local algebras, say $A_\Lambda$, where $\Lambda$ runs over some collection of finite subregions of the lattice. For this section, we will only need that the regions under consideration form a directed set with respect to inclusion, so that the notation $\lim_\Lambda x_\Lambda$ for a net $x_\Lambda$ of numbers makes sense. Readers feeling more comfortable with sequences than with nets may of course consider all statements along a definite sequence of increasing regions, and replace the word “net” by “sequence” throughout. The algebras $A_\Lambda$ will be assumed to be finite dimensional C*-algebras. Typically they are full matrix algebras, i.e., $A_\Lambda$ is the algebra of all linear operators on some finite dimensional Hilbert space $H_\Lambda$. We assume that the algebras are embedded into each other, i.e., for $\Lambda' \subset \Lambda$, there is a unit preserving *-homomorphism

$$i_{\Lambda\Lambda'} : A_{\Lambda'} \to A_\Lambda,$$ (2.1)

satisfying $i_{\Lambda\Lambda'} \circ i_{\Lambda'\Lambda''} = i_{\Lambda\Lambda''}$ whenever $\Lambda \supset \Lambda' \supset \Lambda''$. For matrix algebras this amounts to saying that, up to suitable Hilbert space isomorphisms, $H_{\Lambda'}$ is a tensor factor in $H_\Lambda$, and in the applications we have indeed that $H_\Lambda = H_{\Lambda'} \otimes H_{\Lambda\setminus\Lambda'}$. 

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We briefly recall the construction of the quasi-local algebra, denoted here by \( \mathcal{A}_\infty \), since the space of ground state observables will be defined by a modification of this construction. “(Strictly) local” observables are nets of observables \( A_\Lambda \in \mathcal{A}_\Lambda \) with the property that, for some \( \Lambda' \) and all \( \Lambda \supset \Lambda' \), we have \( A_\Lambda = i_{\Lambda \Lambda'}(A_{\Lambda'}) \). Such nets obviously form a *-algebra with respect to \( \Lambda \)-wise operations, and since the *-homomorphisms \( i_{\Lambda \Lambda'} \) are automatically isometric, the norm of such nets is also unambiguously defined as the norm \( \| A_\Lambda \| \), taken for sufficiently large \( \Lambda \). As usual, the C*-inductive limit space \( \mathcal{A}_\infty \) of the system \( \mathcal{A}_\Lambda, i_{\Lambda \Lambda'} \) is now defined as the completion of the normed space of strictly local observables. The image of a local observable \( A_\Lambda = i_{\Lambda \Lambda'}(A_{\Lambda'}) \) in \( \mathcal{A}_\infty \) will be denoted by \( i_{\infty \Lambda'}(A_{\Lambda'}) \). The restrictions of a state \( \omega \) on \( \mathcal{A}_\infty \) to the local algebras \( \mathcal{A}_\Lambda \) are given by \( \omega_{\Lambda'} = \omega \circ i_{\Lambda \Lambda'} \). Conversely, every net of states \( \omega_{\Lambda} \) on \( \mathcal{A}_\Lambda \) satisfying \( \omega_{\Lambda'} = \omega_{\Lambda} \circ i_{\Lambda \Lambda'} \) determines a unique state on \( \mathcal{A}_\infty \). In the sequel, we will use \( K(\mathcal{A}) \) to denote the state space of a C*-algebra \( \mathcal{A} \). For reasons which will become clear in the next section we will not follow the usual practice of suppressing the maps \( i_{\Lambda \Lambda'} \) in the notation.

Suppose now that in each \( \mathcal{A}_\Lambda \) an interaction Hamiltonian \( H_\Lambda \) is given. We want to study the ground states of these Hamiltonians. Since \( \mathcal{A}_\Lambda \) is finite dimensional this amounts to the determination of the eigenspace \( \mathcal{G}_\Lambda \subset \mathcal{H}_\Lambda \) belonging to the lowest eigenvalue of \( H_\Lambda \). We will denote the projection onto \( \mathcal{G}_\Lambda \) by \( g_\Lambda \). Clearly a state \( \omega_\Lambda \in K(\mathcal{A}_\Lambda) \) is a ground state for \( H_\Lambda \), iff \( \omega_\Lambda(g_\Lambda) = 1 \). Is it possible that all local restriction of a state \( \omega \in K(\mathcal{A}_\Lambda) \) are ground states in this sense? This is the basic type of problem addressed in this article:

**Main Problem.** Determine all states \( \omega \) on the quasi-local algebra \( \mathcal{A}_\infty \) such that

\[
\omega \left( i_{\infty \Lambda}(g_\Lambda) \right) = 1 , \quad \text{for all} \quad \Lambda .
\]

Such states will be called zero-energy states, and their set of will be denoted by \( K_z(\mathcal{A}_\infty) \).

It is immediately clear that \( K_z(\mathcal{A}_\infty) \) is a weak*-closed face in \( K(\mathcal{A}_\infty) \), i.e., all convex components of elements of \( K_z(\mathcal{A}_\infty) \) are again in \( K_z(\mathcal{A}_\infty) \). In analogy to “exposed faces”, i.e., the zero sets of positive continuous affine functionals on a convex set, we call such a face locally exposed. In particular, if \( K_z(\mathcal{A}_\infty) \) consists of a single state, this state will be called locally exposed [FNW1,We2].

What does this mean in typical models? Usually the underlying lattice then has a translation symmetry, which is reflected on the algebraic level by isomorphisms \( \tau_x : \mathcal{A}_\Lambda \rightarrow \mathcal{A}_{\Lambda+x} \). We assume \( H_\Lambda \) to be translationally invariant with “free” boundary conditions:

\[
H_\Lambda = \sum_{x : \Lambda_0 + x \cap \Lambda} i_{\Lambda,\Lambda_0 + x} \tau_x(H_0) , \quad (2.3)
\]

where \( \Lambda_0 \) is some “interaction” region with Hamiltonian \( H_{\Lambda_0} \). (For a nearest neighbour interaction, \( \Lambda_0 \) consists of any site together with its neighbours). Now \( \omega \in K_z(\mathcal{A}_\infty) \)
requires, in the special case \( \Lambda = \Lambda_0 + x \), that

\[
\omega(i_{\infty, \Lambda_0 + x} \tau_x(H_0)) = h_0 \quad \text{for all } x;
\]

where \( h_0 \) is the lowest eigenvalue of \( H_0 \). Conversely, (2.4) implies that every term in the \( \omega \)-expectation of (2.3) is equal to its lowest possible value \( h_0 \). Hence \( \omega_\Lambda \) is a lowest energy state for every \( H_\Lambda \), i.e., \( \omega \in K_z(\mathcal{A}_\infty) \).

For generic interactions one typically finds \( K_z(\mathcal{A}_\infty) = \emptyset \), i.e., most interactions are “frustrated”. In the quantum case this even happens in one dimension [We2]. A fundamental example is the spin-1/2 nearest neighbour Heisenberg antiferromagnet. In that model \( \omega \in K_z(\mathcal{A}_\infty) \) requires the state to be supported by the antisymmetric subspace for any pair of nearest neighbours, and hence on the antisymmetric subspace of every \( \mathcal{H}_\Lambda \), which vanishes whenever \( \Lambda \) contains three or more sites. Hence \( K_z(\mathcal{A}_\infty) = \emptyset \). It is not known whether in this case there is an equivalent finite range interaction allowing some zero-energy state. On the other hand, there are some interesting models with a non-trivial \( K_z(\mathcal{A}_\infty) \). Three paradigms are the Heisenberg ferromagnet on an arbitrary lattice (see Section 3.1), the VBS ground states studied in [AKLT, FNW1, FNW3, Na1, BY] (see Section 3.2), and the deformed Heisenberg chain studied in Section 4 of this paper. The zero-energy state spaces \( K_z(\mathcal{A}_\infty) \) are strikingly different in these three cases.

We close this section with the verification of the claim made in the introduction that zero-energy states are ground states in the sense of the standard definition (1.2). This definition would require of a state \( \omega \), in the notation adopted here, that

\[
\lim_{\Lambda} \omega(X^*[i_{\infty, \Lambda}(H_\Lambda), X]) \geq 0 \quad ,
\]

for all strictly local elements \( X = i_{\infty, \Lambda}(X_\Lambda) \in \mathcal{A}_\infty \). But if \( \omega \in K_z(\mathcal{A}_\infty) \), we get

\[
\omega(X^*[i_{\infty, \Lambda}(H_\Lambda), X]) = \omega(X^*i_{\infty, \Lambda}(H_\Lambda)X) - \omega(X^*X)h_\Lambda \geq 0 \quad ,
\]

where \( h_\Lambda \) is the smallest eigenvalue of \( H_\Lambda \). Hence \( \omega \) is a ground state. The converse certainly fails in general, since \( K_z(\mathcal{A}_\infty) \) may be empty. Even if \( K_z(\mathcal{A}_\infty) \neq \emptyset \), however, it is highly unclear which additional conditions make it true.
2.2. The Inductive Limit

For a finite system $\mathcal{A}_\Lambda$ the space of ground states has a very simple structure: it is simply the state space of the algebra of operator on the lowest eigenvalue eigenspace $\mathcal{G}_\Lambda \subset \mathcal{H}_\Lambda$. It is therefore suggestive to define the space of zero-energy observables for the finite system as

$$B_\Lambda = B(\mathcal{G}_\Lambda) = g_\Lambda \mathcal{A}_\Lambda g_\Lambda,$$  \hspace{1cm} (2.6)

where $B(\mathcal{H})$ denotes the space of bounded operators on a Hilbert space $\mathcal{H}$, and $g_\Lambda$ is the projection from $\mathcal{H}_\Lambda$ onto $\mathcal{G}_\Lambda$ as in the previous subsection. Our task in this subsection is to define an analogue of $B_\Lambda$ for the infinite system, whose state space is canonically isomorphic to $K_z(\mathcal{A}_\infty)$. Since there is no analogue of the projection $g_\Lambda$ in the quasi-local algebra $\mathcal{A}_\infty$, it is clear that equation (2.6) is not suitable for this purpose.

It is useful to go back to the definition of the quasi-local algebra as an inductive limit. As described above, with this definition a quasi-local observable $A_\infty \in \mathcal{A}_\infty$ becomes an equivalence class of Cauchy nets whose members are again nets. This sounds rather involved, but causes no technical complications, because the strictly local observables $\Lambda = i_{\Lambda\Lambda'} A_{\Lambda'}$ are so simple that the $\Lambda$-dependence is often suppressed “by canonical identification”. This would not be permissible if the embedding maps $i_{\Lambda\Lambda'}$ were not isometric, or, even more generally, if the inductive limit relation $i_{\Lambda\Lambda'} \circ i_{\Lambda'\Lambda''} = i_{\Lambda\Lambda''}$ were only satisfied approximately. Such “approximate inductive limits” have become a useful tool in a variety of contexts, e.g. in the statistical mechanics of mean-field systems [DW1, DW2], as a general framework for various thermodynamic limits for intensive observables [We3], or in the formulation of classical ($\hbar \to 0$) limit of quantum mechanics [We4]. The definition of the space of zero-energy observables for an infinite system will be yet another application. In all these cases it turns out to be convenient to reduce the implicit double sequence construction of the quasi-local algebra to a construction involving only nets indexed by $\Lambda$.

We will now briefly review the basic idea of approximate inductive limits, referring to [We3,We4] for more details and proofs. So let $(B_\Lambda, j_{\Lambda\Lambda'})$ be a system consisting of normed spaces $B_\Lambda$ indexed by the elements $\Lambda$ of some directed set, and contractive maps $j_{\Lambda\Lambda'} : B_{\Lambda'} \to B_\Lambda$. In this situation we will call a net $B_\Lambda \in B_\Lambda$ j-convergent, if

$$\lim_{\Lambda} \lim_{\Lambda'} ||B_\Lambda - j_{\Lambda\Lambda'}(B_{\Lambda'})|| = 0 \quad .$$  \hspace{1cm} (2.7)

In particular, null nets with $\lim_{\Lambda} ||B_\Lambda|| = 0$ are convergent. We will say that two convergent nets “have the same limit”, if they differ by a null net. Hence we define the limit space of the system $(B_\Lambda, j_{\Lambda\Lambda'})$ as the space of j-convergent nets modulo the space of null nets. This space will be denoted by $B_\infty$. The limit of a convergent net $B_\Lambda$ is the class of the net in this quotient, and will be denoted by $B_\infty$, or more explicitly, by $j_{\lim_{\Lambda}} B_\Lambda \in B_\infty$. One easily checks that, for j-convergent nets, the net of norms is also convergent, so $B_\infty$ becomes a normed space with

$$||j_{\lim_{\Lambda}} B_\Lambda|| := \lim_{\Lambda} ||B_\Lambda|| \quad .$$  \hspace{1cm} (2.8)
$B_\infty$ is always complete [We3]. Note that if all $B_\Lambda$ are the same normed space, and all $j_{\Lambda\Lambda'}$ are the identity operator on this space, the $j$-convergent nets are just the Cauchy nets, and $B_\infty$ is just the completion of the given normed space.

Nets $B_\Lambda$ with the property that $B_\Lambda = j_{\Lambda\Lambda'} B_{\Lambda'}$, for some $\Lambda'$ and all $\Lambda \supset \Lambda'$, are called **basic nets**. We will always assume that such nets are $j$-convergent, which expresses the asymptotic transitivity of the comparison furnished by the maps $j_{\Lambda\Lambda'}$. This condition will be trivially satisfied in this paper, since we will always have $j_{\Lambda\Lambda'} \circ j_{\Lambda'\Lambda''} = j_{\Lambda\Lambda''}$ for $\Lambda \supset \Lambda' \supset \Lambda''$ (compare (2.16)). We can then define maps $j_{\infty\Lambda'} : B_{\Lambda'} \to B_\infty$ by

$$j_{\infty\Lambda'}(B) = \lim_\Lambda j_{\Lambda\Lambda'}(B), \quad \text{for } B \in B_{\Lambda'}, \quad (2.9)$$

It is easy to see that the elements of the form $j_{\infty\Lambda'} B$ are norm dense in $B_\infty$. The basic nets are also dense as a space of nets: a net $B_\Lambda$ is convergent iff for any $\varepsilon > 0$ there is a basic net $B^\varepsilon$ such that $\lim_\Lambda \|B_\Lambda - B^\varepsilon\| \leq \varepsilon$.

In this paper we are also concerned with the limit of states, i.e., with positive normalized functionals. In order that positivity and normalization make sense in $B_\infty$, we will assume that each $B_\Lambda$ is an order unit space [Nag], e.g. a C*-algebra, and that the $j_{\Lambda\Lambda'}$ preserve both the orderings and the unit elements, in the sense that $j_{\Lambda\Lambda'}(I_{\Lambda'}) = I_\Lambda$, and that $A \geq 0$ implies $j_{\Lambda\Lambda'}(A) \geq 0$. This implies that $B_\infty$ has an ordering, for which the positive cone consists of the limits of all convergent sequences of positive elements, and a unit, namely $I_\infty = \lim_{\Lambda} I_\Lambda$. It can be shown [We3] that thereby $B_\infty$ becomes an order unit space, so that we can define the state space of $B_\infty$ as

$$K(B_\infty) := \{ \omega : B_\infty \to \mathfrak{C} \mid \omega \text{ linear, } A \geq 0 \Rightarrow \omega(A) \geq 0, \ \omega(I_\infty) = 1 \} \quad . \quad (2.10)$$

Associated with the definition of $j$-convergent nets (which is a convergence in norm) there is a notion of weak convergence of states: we say that a net $\omega_\Lambda \in K(A_\Lambda)$ is $j^*$-convergent, if, for any $j$-convergent net $A_\Lambda \in A_\Lambda$, the sequence of numbers $\omega_\Lambda(A_\Lambda)$ is convergent. It is easy to see that in this case a state $\omega_\infty \in K(B_\infty)$ is defined by the formula

$$\omega_\infty(j\lim_\Lambda A_\Lambda) = \lim_\Lambda \omega_\Lambda(A_\Lambda) \quad . \quad (2.11)$$

Every state $\omega \in K(B_\infty)$ is a $j^*$-limit of such a net of states, namely of $\omega_\Lambda = \omega \circ j_{\infty\Lambda}$.

After this excursion to generalized inductive limits we come back to the problem of defining zero-energy observables for an infinite system. It is clear that we must make some assumptions about the subspaces $G_\Lambda \subset H_\Lambda$ or, equivalently, about the projections $g_\Lambda \in A_\Lambda$. Our **standing assumptions** will be

$$g_\Lambda \neq 0 \quad , \quad \text{for all } \Lambda, \text{ and} \quad (2.12)$$

$$g_\Lambda \leq i_{\Lambda\Lambda'}(g_{\Lambda'}) \quad , \quad \text{for all } \Lambda \supset \Lambda' . \quad (2.13)$$

These are dictated by the intention to study situations with $K_z(A_\infty) \neq \emptyset$: the support projections $g_\Lambda^\Lambda$ of the restrictions $\omega_\Lambda = \omega \circ i_{\Lambda\Lambda'}$ of any state $\omega \in K(A_\infty)$ automatically
satisfy these assumptions. Moreover, if \( \omega \in K_z(\mathcal{A}_\infty) \), we get \( \tilde{g}_\Lambda^\omega \leq g_\Lambda \). Hence even if the \( g_\Lambda \) do not satisfy (2.13) initially, we can pass to smaller projections satisfying both assumptions. It is also clear that the two assumptions suffice to guarantee \( K_z(\mathcal{A}_\infty) \neq \emptyset \): because of (2.12), we can find states with \( \omega_\Lambda(g_\Lambda) = 1 \). Then by compactness we can find a weak*-cluster point \( \omega_* \in K(\mathcal{A}_\infty) \) of suitable extensions of these states. From (2.13) it then follows that \( \omega_* \in K_z(\mathcal{A}_\infty) \).

We will only be interested in systems of projections which are non-trivial in the sense that the net \( g_\Lambda \) of projections is not quasi-local. In fact, if \( g_\Lambda \) were \( i \)-convergent, \( i_{\infty\Lambda}g_\Lambda \) would be a norm convergent decreasing sequence of projections, which must be eventually constant, because the norm difference of commuting projections is either 1 or 0. Hence \( g_\Lambda \) would even be strictly local. This is impossible in a statistical mechanics model, since increasing the region always introduces more terms in the Hamiltonian to be minimized and hence more constraints on the state.

We choose the algebras \( \mathcal{B}_\Lambda \) as defined in equation (2.6). The unit in \( \mathcal{B}_\Lambda \) is the projection \( g_\Lambda \in \mathcal{A}_\Lambda \), and for this unit to be different from 0, condition (2.12) must be satisfied. Then, for each \( \Lambda \), the map

\[
\begin{align*}
r_\Lambda : & \mathcal{A}_\Lambda \to \mathcal{B}_\Lambda \\
r_\Lambda(A) := & g_\Lambda A g_\Lambda, \quad \text{for } A \in \mathcal{A}_\Lambda.
\end{align*}
\]

is positive, unit preserving, and surjective. Furthermore, we define, for \( \Lambda \supset \Lambda' \):

\[
\begin{align*}
j_{\Lambda\Lambda'} : & \mathcal{B}_{\Lambda'} \to \mathcal{B}_\Lambda \\
j_{\Lambda\Lambda'}(r_{\Lambda'}A) := & r_\Lambda(i_{\Lambda\Lambda'}(A)), \quad \text{for } A \in \mathcal{A}_{\Lambda'}.
\end{align*}
\]

This is well-defined since, by condition (2.13), \( r_{\Lambda'}(A) = 0 \) implies

\[
r_\Lambda(i_{\Lambda\Lambda'}(A)) = g_\Lambda i_{\Lambda\Lambda'}(A) g_\Lambda = g_\Lambda i_{\Lambda\Lambda'}(g_\Lambda) i_{\Lambda\Lambda'}(A) i_{\Lambda\Lambda'}(g_\Lambda) g_\Lambda = g_\Lambda i_{\Lambda\Lambda'}(g_\Lambda A g_\Lambda) g_\Lambda = g_\Lambda i_{\Lambda\Lambda'}(r_{\Lambda'}(A)) g_\Lambda = 0.
\]

Let \( \Lambda \supset \Lambda' \supset \Lambda'' \). Then \( j_{\Lambda\Lambda'} \circ j_{\Lambda'\Lambda''} \) obviously satisfies the defining equation (2.15) for \( j_{\Lambda\Lambda''} \). Hence we have

\[
j_{\Lambda\Lambda'} \circ j_{\Lambda'\Lambda''} = j_{\Lambda\Lambda''}.
\]

Suppose now that \( \mathcal{A}_\Lambda \) is an \( i \)-convergent net for the system \( (\mathcal{A}_\Lambda, i_{\Lambda\Lambda'}) \) defining the quasi-local algebra, and consider the net \( \mathcal{B}_\Lambda := r_\Lambda(\mathcal{A}_\Lambda) \in \mathcal{B}_\Lambda \). Then

\[
||\mathcal{B}_\Lambda - j_{\Lambda\Lambda'}(\mathcal{B}_{\Lambda'})|| = ||r_\Lambda(\mathcal{A}_\Lambda - i_{\Lambda\Lambda'}(\mathcal{A}_{\Lambda'}))|| \leq ||\mathcal{A}_\Lambda - i_{\Lambda\Lambda'}(\mathcal{A}_{\Lambda'})||
\]
goes to zero in just the way required for \( j \)-convergence of \( \mathcal{B}_\Lambda \). The limit \( \mathcal{B}_\infty \) is not changed, if \( \mathcal{A}_\Lambda \) is modified by a null net. Hence there is a well-defined operator

\[
\begin{align*}
r_\infty : & \mathcal{A}_\infty \to \mathcal{B}_\infty \\
r_\infty(j_{\lim_{\Lambda} A_\Lambda}) := & j_{\lim_{\Lambda} r_\Lambda A_\Lambda}, \quad \text{with}
\end{align*}
\]
for any \(i\)-convergent net \(A_\Lambda \in \mathcal{A}_\Lambda\). The surjectivity of the finite volume maps \(r_\Lambda\) also goes to the limit, as the following Lemma shows.

2 Lemma. \(r_\infty\) is surjective, and maps the unit sphere of \(A_\infty\) onto a dense subset of the unit sphere of \(B_\infty\).

Proof: Consider \(B_\infty = \bar{\lim}_n B_\Lambda\) with \(\|B_\infty\| \leq 1\), and fix some summable sequence \(\varepsilon_n\). Then we can find a sequence of regions \(\Lambda_n \subset \Lambda'_n \subset \Lambda_{n+1} \cdots\) such that

\[
\|B_\Lambda - j_{\Lambda\Lambda_n}(B_{\Lambda_n})\| \leq \varepsilon_n
\]

for all \(\Lambda \subset \Lambda'_n\). Considering \(B_{\Lambda_n}\) as an element of \(\mathcal{A}_{\Lambda_n}\), we can define

\[
A_\Lambda = i_{\Lambda\Lambda_0}(B_{\Lambda_0} + \sum_{n \geq 1} \left[ \sum_{\Lambda_n \subset \Lambda} i_{\Lambda\Lambda_n}(B_{\Lambda_n} - j_{\Lambda\Lambda_n}(B_{\Lambda_{n-1}})) \right] )
\]

This sum converges in norm, uniformly in \(\Lambda\), and we have \(\|A_\Lambda\| \leq \|B_\infty\| + 2\varepsilon_0 + \sum_{n \geq 1} \varepsilon_n\). Moreover, \(A\) is \(i\)-convergent, and \(r_\Lambda(A_\Lambda)\) is a telescoping series evaluating to \(B_{\Lambda_n}\) for the largest \(n\) such that \(\Lambda'_n \subset \Lambda\). Hence \(\|r_\Lambda A_\infty - B_\Lambda\| \to 0\), and \(r_\infty A_\infty = B_\infty\). Moreover, if \(\|B_\infty\| < 1\), we can choose the \(\varepsilon_i\) sufficiently small to make \(\|A_\infty\| \leq 1\).

The adjoint \(r^*_\infty\), which is a weak*-continuous map from the dual \(\mathcal{B}_\infty^*\) of \(\mathcal{B}_\infty\) to \(\mathcal{A}_\infty^*\), is therefore an isometric map, and is precisely the desired isomorphism between the zero-energy states and the states on \(\mathcal{B}_\infty\):

3 Theorem. The map \(r^*_\infty : K(\mathcal{B}_\infty) \to K(\mathcal{A}_\infty) \subset K(\mathcal{A}_\infty)\) is an isomorphism of compact convex sets, where all state spaces are equipped with the weak*-topology.

Proof: We will explicitly construct the inverse of \(r^*_\Lambda\). So let \(\eta \in K(\mathcal{A}_\infty)\). Since \(B_\Lambda = g_{\Lambda}A_\Lambda g_{\Lambda}\) can be considered as a subspace of \(\mathcal{A}_\Lambda\), we can evaluate the restriction \(\eta \circ i_{\infty\Lambda}\) on \(\mathcal{B}_\Lambda\). We define

\[
s_\Lambda : K(\mathcal{A}_\infty) \to K(\mathcal{B}_\Lambda)
\]

\[
s_\Lambda(\eta)(r_\Lambda A) = \eta \circ i_{\infty\Lambda}(A) \quad \text{for all } A \in \mathcal{A}_\Lambda.
\]

We have to verify first that \(s_\Lambda(\eta)\) is a well-defined state. Because \(\eta \circ i_{\infty\Lambda}(g_\Lambda) = 1\), we have that the right hand side \(\eta \circ i_{\infty\Lambda}(A) = \eta(i_{\infty\Lambda}(g_\Lambda A g_\Lambda)) = \eta \circ (i_{\infty\Lambda}(r_\Lambda(A)))\) indeed depends only on \(r_\Lambda(A)\). Positivity is obvious, and normalization follows by putting \(A = I\).

Next, since

\[
s_\Lambda(\eta) \circ j_{\Lambda\Lambda'}(r_\Lambda(A)) = s_\Lambda(\eta) \circ r_\Lambda \circ i_{\Lambda\Lambda'}(A) = \eta \circ i_{\infty\Lambda} \circ i_{\Lambda\Lambda'}(A)
\]

\[
= \eta \circ i_{\infty\Lambda'}(A) = s_{\Lambda'}(\eta)(r_{\Lambda'}(A))
\]

we have \(s_\Lambda(\eta) \circ j_{\Lambda\Lambda'} = s_{\Lambda'}(\eta)\). This readily implies that

\[
s_\infty(\eta) := \lim_{\Lambda} s_\Lambda(\eta)
\]

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exists, and is a state on $B_{\infty}$.

We claim that $s_{\infty}$ is continuous for the weak*-topologies. Let $\eta^\alpha$ is a weak*-convergent net in $K_z(A_{\infty})$, which is to say that, for all $A \in A_{\infty}$, the net $\eta^\alpha(A)$ is convergent. We have to show that, similarly, $s_{\infty}(\eta)(B)$ converges for $B \in B_{\infty}$. It suffices to verify this for the norm dense subset of elements of the form $B = j_{\infty,\Lambda'}(r_{\Lambda'}A)$ with $A \in A_{\Lambda'}$ for some $\Lambda'$. But for these we have

$$s_{\infty}(\eta^\alpha)(B) = \lim_{\Lambda} s_{\Lambda}(\eta^\alpha)(j_{\Lambda\Lambda'} \circ r_{\Lambda'}A)$$

$$= \lim_{\Lambda} s_{\Lambda}(\eta^\alpha)(r_{\Lambda} \circ i_{\Lambda\Lambda'}A)$$

$$= \lim_{\Lambda} \eta^\alpha(i_{\Lambda\Lambda'}A) ,$$

which is convergent by assumption.

It remains to be proven that $s_{\infty}$ and $r_{\infty}^\ast$ are inverses of each other. First, let $\eta \in K_z(A_{\infty})$. Then, for $A \in A_{\Lambda}$,

$$r_{\infty}^\ast \circ s_{\infty}(\eta)(i_{\infty,\Lambda}(A)) = s_{\infty}(\eta)(r_{\infty} \circ i_{\infty,\Lambda}(A))) = \lim_{\Lambda} s_{\Lambda}(\eta)(r_{\Lambda}(A))$$

$$= \eta(i_{\infty,\Lambda}(A)) .$$

Hence $r_{\infty}^\ast s_{\infty}(\eta) = \eta$ on the dense set of elements $i_{\infty,\Lambda}(A)$, and hence everywhere. This proves $r_{\infty}^\ast \circ s_{\infty} = id_{K_z(A_{\infty})}$.

Conversely, let $\omega \in K(B_{\infty})$, and $A \in A_{\Lambda}$. Then

$$s_{\infty} \circ r_{\infty}^\ast(\omega)(j_{\infty,\Lambda} \circ r_{\Lambda}(A)) = s_{\Lambda} \circ r_{\infty}^\ast(\omega)(r_{\Lambda}(A)) = r_{\infty}^\ast(\omega)(i_{\infty,\Lambda}(A))$$

$$= \omega(r_{\infty} \circ i_{\infty,\Lambda}(A)) = \omega(j_{\infty,\Lambda} \circ r_{\Lambda}(A)) .$$

Hence $s_{\infty} \circ r_{\infty}^\ast \omega = \omega$ on a norm dense subset of $B_{\infty}$ and, consequently, $s_{\infty} \circ r_{\infty}^\ast = id_{K(B_{\infty})}$.

The upshot of this Theorem is not so much that $K_z(A_{\infty})$ is identified as the state space of some order unit space $B_{\infty}$. In fact, that is true of any compact convex set [Alf]. It is rather that this space is obtained by a direct construction, which is based on the asymptotic behaviour of the finite volume ground state spaces. One can use this to transfer some of the properties of the finite volume spaces to the limit more easily. A special role in this regard is played by the algebraic product. Although each $B_{\Lambda}$ is a C*-algebra, and the embeddings $j_{\Lambda\Lambda'}$ are completely positive and unit preserving, they are not homomorphisms. Therefore, the limit space does not automatically inherit the product operation. However, it is true in all the examples below that $j_{\Lambda\Lambda'}$ is asymptotically a homomorphism, in a sufficiently strong sense to make $B_{\infty}$ into a C*-algebra, as well. We state this special situation in the following Definition for later reference.

4 Definition. The inductive system $(B_{\Lambda}, j_{\Lambda\Lambda'})$ is said to have the product property, if, for any two $j$-convergent nets $A_{\Lambda}, B_{\Lambda}$, the net defined by $C_{\Lambda} = A_{\Lambda}B_{\Lambda}$ for every $\Lambda$, is also $j$-convergent.
If the product property holds, we can immediately define a product on $B_\infty$ by
\[ A_\infty B_\infty = C_\infty \]
and it is clear that $B_\infty$ thereby becomes a C*-algebra. Of course, this makes the determination of the state space much easier, since much more is known about C*-algebras than about general order unit Banach spaces. We have not found a general way of proving the product property. At least the proofs in the three cases considered below are quite different, and are based on specific properties of each of the models. For the undeformed ferromagnet the proof also yields the commutativity of the product, and consequently $K_z(A_\infty)$ is a simplex. On the other hand, in the deformed case considered in Section 4, the product is well-defined, but non-commutative. In other words, there are not only convex combinations, but “coherent superpositions” of zero-energy states, as well.

With this Theorem we arrive at the following procedure for computing $K_z(A_\infty)$:

1. determine the local ground state spaces $G_\Lambda$ from the given Hamiltonians.
2. Compute the inductive limit space $B_\infty$. Decide the product property.
3. Determine the state space of $B_\infty$.

In the cases considered below (1) is fairly easy. Step (2) is usually the most difficult part. Since the spaces $B_\infty$ arising in these examples are quite simple, (3) is trivial. The main work thus goes into (2).

2.3. Hilbert space representations

For the model considered in Section 4 it is easy to find a representation of the quasi-local algebra with respect to which many of the states in $K_z(A_\infty)$ are obviously normal. (Recall that a state $\omega$ on a C*-algebra $C$ is called normal in a representation $\pi : C \to B(H_\pi)$, if there is a trace class operator $D_\omega$ on $H_\pi$ such that $\omega(C) = \text{tr}(D_\omega \pi(C))$.) In that example it is also true, but much more difficult to show that (with two exceptions) all states $\omega \in K_z(A_\infty)$ are normal in this representation. Working inside just one representation, such questions are impossible to decide, and it was precisely for obtaining such complete characterizations of $K_z(A_\infty)$ that the inductive limit scheme of Section 2.2 was set up. Nevertheless, representations are a useful tool. The aim of this section is to describe briefly how representations of $A_\infty$ generate representations of $B_\infty$, and under which circumstances such representations are faithful.

Let $\pi : A_\infty \to B(H_\pi)$ be a *-representation of the quasi-local algebra. It is convenient to prolong the system of embedding maps into the representation, i.e., we define $i_\pi_\Lambda : A_\Lambda \to B(H_\pi)$ by $i_\pi_\Lambda = \pi \circ i_\infty_\Lambda$. For uniformity of notation $\pi$ is sometimes also written as $i_\pi_\infty$. There is a natural ground state projection in $H_\pi$, namely
\[ g_\pi = \lim_{\Lambda} i_\pi_\Lambda(g_\Lambda) \]
(2.19)
where the limit on the right hand side is in the strong operator topology, and exists, because \( i_{\pi\Lambda}(g_\Lambda) \) is a decreasing net of projections by assumption (2.13). Of course, the limit may be zero.

Now let \( \eta \in \mathcal{B}(\mathcal{H}_\pi)^* \) be a state with \( \eta(g_\pi) = 1 \). Then \( \eta \circ \pi \in K_z(A_\infty) \). The converse, namely that \( \eta \circ \pi \in K_z(A_\infty) \) implies \( \eta(g_\pi) = 1 \), is also true when \( \eta \) is normal, i.e., continuous for the limit on the right hand side of (2.19). As a counterexample for singular \( \eta \), consider any faithful representation \( \pi \) of \( A_\infty \) in which every zero-energy state is singular or, equivalently, \( g_\pi = 0 \). By Hahn-Banach extension of \( \omega \) from \( \pi(A_\infty) \) to \( \mathcal{B}(\mathcal{H}_\pi) \) we can write \( \omega = \eta \circ \pi \), with a necessarily singular state \( \eta \in \mathcal{B}(\mathcal{H}_\infty) \). Then \( \eta(g_\pi) = 0 \), because \( \eta(\pi\Lambda(g_\Lambda)) = \omega(\pi\Lambda(g_\Lambda)) = 1 \), for all finite \( \Lambda \).

We would like to prolong the inductive system of ground state observables into the representation as well. Hence we define,

\[
\begin{align*}
  r_\pi : \mathcal{B}(\mathcal{H}_\pi) &\rightarrow \mathcal{B}(\mathcal{G}_\pi) \\
  r_\pi(A) &= g_\pi Ag_\pi
\end{align*}
\]

and, either for finite \( \Lambda \) or for \( \Lambda = \infty \),

\[
\begin{align*}
  j_{\pi\Lambda} : \mathcal{B}_\Lambda &\rightarrow \mathcal{B}(\mathcal{H}_\pi) \\
  j_{\pi\Lambda}(r_\Lambda A) &= r_\pi(i_{\pi\Lambda}(A)) \quad \text{for } A_\Lambda \in A_\Lambda.
\end{align*}
\]

The salient facts about \( j_{\pi\infty} \) are collected in the following Proposition.

**5 Proposition.**

1. The map \( j_{\pi\infty} \) is well-defined by equation (2.21).

2. If \( B_\Lambda \) is \( j \)-convergent, then

\[
j_{\pi\infty}(B_\infty) = \lim_{\Lambda} i_{\pi\Lambda}(B_\Lambda)
\]

where \( B_\Lambda \) is considered as a subspace of \( A_\Lambda \), and the limit is in the strong operator topology.

3. If the inductive system has the product property, then \( j_{\pi\infty} \) is a homomorphism.

4. When every \( \omega \in K_z(A_\infty) \) is \( \pi \)-normal, \( j_{\pi\infty} \) is isometric.

**Proof:**

1. Suppose that \( A_\Lambda \) is \( i \)-convergent, with \( r_\infty(A_\infty) = 0 \). That is to say, \( \lim_\Lambda \|g_\Lambda A_\Lambda g_\Lambda\| = 0 \). Then, since \( g_\pi \leq i_{\pi\Lambda}(g_\Lambda) \), and \( i_{\pi\Lambda} \) is a homomorphism, we have

\[
r_\pi(i_{\pi\Lambda}(A_\Lambda)) = g_\pi(i_{\pi\Lambda}(g_\Lambda A_\Lambda g_\Lambda))g_\pi \rightarrow 0.
\]

2. It suffices to show this for basic nets of the form \( B_\Lambda = g_\Lambda i_{\Lambda\Lambda'}(A_{\Lambda'})g_\Lambda \), for some fixed \( A_{\Lambda'} \in A_{\Lambda'} \). Then

\[
j_{\pi\infty}(B_\infty) = r_\pi(i_{\pi\infty}i_{\infty\Lambda'}(A_{\Lambda'})) = g_\pi i_{\pi\Lambda'}(A_{\Lambda'}) g_\pi.
\]
On the other hand,

$$i_{\pi}(B_{\Lambda}) = i_{\pi}(g_{\Lambda})i_{\pi}(A_{\Lambda}')i_{\pi}(g_{\Lambda})$$

which converges strongly to the previous expression because, by definition (2.19), $g_{\pi} = s\text{-lim}_{\Lambda} i_{\pi}(g_{\Lambda})$, and because the product is continuous for strong limits.

(3) Let $A_{\Lambda}$, $B_{\Lambda}$, and $C_{\Lambda} = A_{\Lambda}B_{\Lambda}$ be $j$-convergent. Then, by (2),

$$j_{\pi}(C_{\infty}) = s\lim_{\Lambda} i_{\pi}(A_{\Lambda}B_{\Lambda}) = s\lim_{\Lambda} i_{\pi}(A_{\Lambda}) i_{\pi}(B_{\Lambda})$$

$$= s\lim_{\Lambda} i_{\pi}(A_{\Lambda}) s\lim_{\Lambda} i_{\pi}(B_{\Lambda}) = j_{\pi}(A_{\infty}) j_{\pi}(B_{\infty})$$.

(4) Let $A_{\Lambda}$ be $i$-convergent, and $B_{\Lambda} = r_{\Lambda}A_{\Lambda}$, and recall that, up to norm small corrections, every $j$-convergent $B_{\Lambda}$ is of this form. Then

$$\|B_{\infty}\| = \sup_{\Omega} |\Omega(B_{\infty})| = \sup_{\Omega} |(r_{\infty}^{*}\Omega)(A_{\infty})| = \sup_{\omega} |\omega(A_{\infty})|$$

where $\Omega$ runs over the unit sphere of $B_{\infty}^{*}$, and $\omega$ runs over all functionals in the linear hull of $K_{2}(A_{\infty}) \subset A_{\infty}^{*}$ of norm $\leq 1$. Now by assumption all such functionals are represented as $\omega(A_{\infty}) = \text{tr}(D_{\omega}\pi(A_{\infty}))$, with $D_{\omega}$ a linear combination of density matrices supported by $G_{\pi}$, and of trace norm $\leq 1$. For such functionals, the last supremum is equal to $\|g_{\pi}(A_{\infty})g_{\pi}\| = \|j_{\pi}(r_{\infty}A_{\infty})\| = \|j_{\pi}(B_{\infty})\|$. 

Condition (4) is by no means necessary to make $j_{\pi}(\infty)$ isometric. For example, if the product property holds, then it is sufficient that $j_{\pi}(\infty)$ is a faithful representation, whereas (4) requires $j_{\pi}(\infty)$ to be quasi-equivalent to the universal representation. In fact, for our main example, we will construct a natural faithful irreducible representation of $B_{\infty}$ arising from a representation of $A_{\infty}$ (see Section 4.4). Whenever $j_{\pi}(\infty)$ is faithful, we can construct $B_{\infty}$ as $B_{\pi} = g_{\pi}(A_{\infty})g_{\pi}$. This space is then a $C^{*}$-subalgebra of $B(\mathcal{H}_{\pi})$, but not of $\pi(A_{\infty})$. 

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3. Two basic examples

3.1. The Heisenberg Ferromagnet

In this section we consider the zero-energy states of a Heisenberg ferromagnet on an arbitrary connected graph, with arbitrary positive coupling constants, and with arbitrary (not necessarily equal) spins. At each vertex \( x \) of the graph we consider the Hilbert space \( \mathcal{H}_\{x\} = \mathbb{C}^{2s(x) + 1} \) with an action of the spin-\( s(x) \) representation \( \mathcal{D}^{s(x)} \) of SU(2), with \( s(x) > 0 \). The observable algebra \( \mathcal{A}_\{x\} \) is the algebra of operators on \( \mathcal{H}_\{x\} \), i.e., the algebra of \((2s(x) + 1) \times (2s(x) + 1)\)-matrices. The spin operators in \( \mathcal{A}_\{x\} \) will be denoted by \( S^x_\alpha \), \( \alpha = 1, 2, 3 \). For larger regions \( \Lambda \) we set

\[
\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_\{x\} \quad \mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{A}_\{x\} .
\]  

(3.1)

The injections \( i_{\Lambda\Lambda'} \) are given by \( i_{\Lambda\Lambda'}(A) = A \otimes \mathbf{1}_{\Lambda \setminus \Lambda'} \), as usual. By abuse of notation we abbreviate the spin operators at vertex \( x \), considered as observables of the region \( \Lambda \ni x \) again as \( i_{\Lambda,\{x\}}(S^x_\alpha) \equiv S^x_\alpha \).

For any finite, connected subset \( \Lambda \) of lattice vertices, the Hamiltonian of the model is given by

\[
H_\Lambda = - \sum_{x,y \in \Lambda \atop x \sim y} J_{xy} \sum_{\alpha=1}^3 S^x_\alpha S^y_\alpha ,
\]  

(3.2)

where “\( x \sim y \)” means that the vertices \( x \) and \( y \) are connected by an edge, and \( J_{xy} \) are arbitrary strictly positive constants.

The operator \( \sum_{\alpha=1}^3 S^x_\alpha S^y_\alpha \) commutes with \( \mathcal{D}^{s(x)} \otimes \mathcal{D}^{s(y)} \), and on the spin-\( j \) subspace of this representation it is equal to

\[
\frac{1}{2}(j(j+1) - s(x)(s(x) + 1) - s(y)(s(y) + 1)) .
\]  

(3.3)

This expression attains its maximum, namely \( s(x)s(y) \), when \( j = s(x) + s(y) \) is the largest spin in the decomposition of \( \mathcal{D}^{s(x)} \otimes \mathcal{D}^{s(y)} \). Hence we have the lower bound

\[
\langle \varphi | H_\Lambda | \varphi \rangle \geq - \sum_{x,y \in \Lambda \atop x \sim y} J_{xy} s(x)s(y) ,
\]  

(3.4)

for any unit vector \( \varphi \in \mathcal{H}_\Lambda \). Clearly, this becomes an equality iff \( \varphi \) is supported by the maximal spin subspace of \( \mathcal{H}_{\{x,y\}} \) for any edge \( x \sim y \). Thus in order to compute the zero-energy states of \( H_\Lambda \) we need to analyze the intersection of maximal spin subspaces on overlapping tensor factors.
6 Proposition. Let $\Lambda$ be a connected subgraph. Then the lowest eigenvalue of $H_\Lambda$ from (3.2) is given by the right hand side of (3.4), and the corresponding eigenspace $G_\Lambda$ is the irreducible subspace of $\bigotimes_{x \in \Lambda} D^{s(x)}$ for the highest spin, $s_\Lambda = \sum_{x \in \Lambda} s(x)$.

Proof: Recall that the spin-$s$ representation $D^s$ of SU(2) is isomorphic to the subrepresentation of the $2s$-fold tensor power of $D^{1/2}$ on the completely symmetric subspace of $(\mathbb{C}^2)^{\otimes 2s}$. Thus we can replace each site $x$ by a collection $\hat{x}$ of $2s(x)$ sites to each of which is associated a Hilbert space $\Phi^2$ with a spin-$1/2$ representation of SU(2). We are looking for the subspace of vectors $\Phi$ which are (a) invariant the unitary operators exchanging any two sites within the same cluster $\hat{x}$, and (b) belong to the highest spin subspace for the representation belonging to $\hat{x} \cup \hat{y}$ for points $x, y$ connected by an edge. Condition (b) simply means that the $\Phi$ is also symmetric with respect to the exchange of points from $\hat{x}$ and $\hat{y}$. Since the graph is connected, such transpositions generate the whole permutation group of the $2s_\Lambda$ sites, and $\Phi$ belongs to the completely symmetric subspace.

From this proof we can determine $K_z(A_\infty)$, using a Theorem of Hudson and Moody [HM]: it characterizes those states on an infinite system, whose restriction to every finite subsystem is supported by the completely symmetric (Bose) subspace, as the infinite products of pure one-site states, and the integrals over such product states. In our case, the one-site Hilbert space is $\Phi^2$, so the set of pure states on a single site is naturally parametrized by a 2-sphere. Translated back to the language of spin systems with arbitrary spin, we find that the extreme points of $K_z(A_\infty)$ are characterized as those in which “all the spins point in the same direction”. To give a more precise description, let $\chi_x \in H_{\{x\}}$ denote the eigenvector of $S^x_3$ for the largest eigenvalue $s(x)$. Then, for each $\Lambda$, and each unit vector $\vec{e} \in \mathbb{R}^3$, we set

$$\chi_\Lambda = \bigotimes_{x \in \Lambda} \chi_x$$

$$\chi_\Lambda(\vec{e}) = \bigotimes_{x \in \Lambda} D_R^{s(x)} \chi_x,$$

where $R$ is a rotation taking the north pole into $\vec{e}$. Because the sphere is the homogeneous space of SU(2) by the subgroup generated by $S_3$, this vector does not depend on the choice of $R$. Then the restriction of the extremal element of $K_z(A_\infty)$ belonging to $\vec{e} \in S$, restricted to a finite region $\Lambda$ is $B \mapsto \langle \chi_\Lambda(\vec{e}), B_\Lambda \chi_\Lambda(\vec{e}) \rangle$.

As a way of obtaining the zero-energy states this treatment is more or less satisfactory. Some questions remain unclear, however. For example, while we find that $K_z(A_\infty)$ is a simplex, i.e., that every $\omega \in K_z(A_\infty)$ has a unique integral decomposition into extreme points, the nature of this simplex is less clear: is it a Bauer simplex (with closed extreme boundary) or a Poulsen simplex (with dense extreme points), like the set of translationally invariant states on a lattice (see Example 4.3.26 in [BR])? The difference between these two
is precisely that the first kind of simplex is the state space of a commutative C*-algebra, so we are led to consider the space of observables with state space $K_z(A_{\infty})$. This is precisely the space $B_{\infty}$ which the inductive limit construction of Section 2.2 yields naturally. Does it have a natural algebraic structure in this case?

For computing the inductive limit, note that according to Proposition 6, $G_\Lambda$ is the irreducible representation space of SU(2) for spin $s(\Lambda) = \sum_{x \in \Lambda} s(x)$, and all details of the graph or the coupling constants become irrelevant. The embeddings $j_{\Lambda\Lambda'}$ are likewise independent of these details. An explicit formula is the following: let

$$V_{\Lambda\Lambda'} : C^{2s(\Lambda)+1} \to C^{2s(\Lambda')+1} \otimes C^{2(s(\Lambda)-s(\Lambda'))+1}$$

be the intertwining isometry between $D^{s(\Lambda)}$ and $D^{s(\Lambda')} \otimes D^{s(\Lambda)-s(\Lambda')}$, which is unique up to a phase. Then

$$j_{\Lambda\Lambda'}(A) = V_{\Lambda\Lambda'}^*(A \otimes 1) V_{\Lambda\Lambda'} .$$

Thus $j_{\Lambda\Lambda'}$ depends only on the spins $s(\Lambda)$ and $s(\Lambda')$, and describes an inductive limit of the observable algebras on irreducible representations of SU(2) with $s \to \infty$. Since the half integer spin parameter is just the angular momentum in units of $\hbar$, this limit is completely equivalent to the classical limit $\hbar \to 0$ for spins with fixed absolute value of angular momentum [Men].

From this perspective it would seem that the computation of $B_{\infty}$ can be based on asymptotic properties of Clebsch-Gordan coefficients. However, it is more efficient to use the picture set up in the proof of Proposition 6, and to exploit the high permutation symmetry. This symmetry is at the root of the theory of Mean-field systems [RW, DW2, We1]. This is, in fact, also the natural home for the Hudson-Moody Theorem, as well as Størmer’s more general non-commutative analogue of the de Finetti Theorem [Sto]. We briefly review the basic notions. Suppose that to each site $x$ in a finite set $\Lambda$ we associate a Hilbert space $H_{\{x\}}$ of the same dimension. Then on $H_\Lambda = \bigotimes_{x \in \Lambda} H_{\{x\}}$ we have a natural action $\pi \mapsto U_\pi$ of the permutation group of $\Lambda$. The inductive limit underlying mean-field theory is given by the algebras $A_\Lambda$ as above and the embedding maps

$$\text{sym}_{\Lambda\Lambda'} : A_\Lambda' \to A_\Lambda$$

$$\text{sym}_{\Lambda\Lambda'}(A) = \frac{1}{|\Lambda|!} \sum_\pi U_\pi(A \otimes 1_{|\Lambda|-|\Lambda'|}) U_\pi^*, \quad (3.7)$$

i.e., a standard embedding $i_{\Lambda\Lambda'}$, followed by an average over all permutations over the larger set. It then turns out that the limit space of the inductive system $(A_\Lambda, \text{sym}_{\Lambda\Lambda'})$, which we denote by $A_{\text{MF}}$, is isomorphic to the space of continuous functions on the state space $K(A_{\{x\}})$ of the one-site algebra [RW]. One feature which carries over from this general structure is the product property as defined in Definition 4:

7 Proposition. The inductive limit defined by the maps (3.6), has the product property. Moreover, the product induced on $B_{\infty}$ is commutative.
Proof: We realize \( G_\Lambda \) as the completely symmetric subspace of \( \mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathbb{C}^2 \), i.e., as the highest spin subspace of the \(|\Lambda|\)-fold tensor product of the defining representation of \( SU(2) \). Thus
\[
g_\Lambda = \frac{1}{|\Lambda|!} \sum_\pi U_\pi .
\]
Because \( r_\Lambda(U_\pi AU_\pi^*) = g_\Lambda(U_\pi AU_\pi^*)g_\Lambda = g_\Lambda A g_\Lambda = r_\Lambda(A) \), an average over permutations is implicit in \( r_\Lambda \). Therefore, we can write
\[
j_{\Lambda\Lambda'}(A_\Lambda(A)) = \sum_\pi U_\pi .
\]
By a simple norm approximation argument it suffices to prove the Proposition for basic nets \( A_\Lambda, B_\Lambda \), i.e., we can set \( A_\Lambda = \sum_\pi U_\pi \) and \( B_\Lambda = \sum_\pi U_\pi \), for some fixed \( A_1 \in A_{\Lambda_1} \) and \( B_2 \in A_{\Lambda_2} \). Since all symmetrized observables \( \sum_{\Lambda_\Lambda'} X_{\Lambda'} \) commute with \( g_\Lambda \), we have
\[
A_\Lambda B_\Lambda = r_\Lambda \left( (\sum_{\Lambda_1} A_1) (\sum_{\Lambda_2} B_2) \right)
\]
\[
= r_\Lambda \left( \sum_{\Lambda_1 \sqcup \Lambda_2} (A_1 \otimes B_2) \right) + \text{Rest} .
\]
Here \( \Lambda_1 \sqcup \Lambda_2 \) denotes the disjoint union of suitable copies of \( \Lambda_1 \) and \( \Lambda_2 \). The first term in the last expression is well-defined, since under symmetrization the labelling of sites becomes irrelevant. The splitting of the last term is based on the intuition that the product of symmetrized observables is an average over products of copies of \( A_1 \) and \( B_2 \), permuted to localization regions \( \pi_1(\Lambda_1) \) and \( \pi_2(\Lambda_2) \). As \( \Lambda \) becomes large, these localization rarely intersect, and the term written is (up to a small correction in normalization) the collection of terms with \( \pi_1(\Lambda_1) \cap \pi_2(\Lambda_2) = \emptyset \). The precise estimate is given in Lemma IV.1 of [RW]:
\[
\|\text{Rest}\| \leq \frac{|\Lambda_1| \cdot |\Lambda_2|}{|\Lambda|} \cdot \|A_1\| \cdot \|B_2\| \xrightarrow{\Lambda \to \infty} 0 .
\]
Since the leading term is itself a basic net for the inductive system \( (B_\Lambda, j_{\Lambda\Lambda'}) \), we have thus shown (1), and because this term is the same for \( B_\Lambda A_\Lambda \), we have shown (2).

As an abelian C*-algebra, \( B_\infty \) is isomorphic to \( C(S) \) for some compact space \( S \). Its state space, which is topologically isomorphic to \( K_\infty(A_\infty) \) is therefore a Bauer simplex. It is also known from the general mean-field theory that \( S \) can be identified with a set of homogeneous product states [RW], and it is easy to see that only products of pure states have the Bose-Einstein symmetry, which gives the Hudson-Moody Theorem [We1]. We will not describe these connections in detail, but instead use the above identification (3.5) of the elements of \( S \) to summarize the result of this section.

8 Theorem. For the Heisenberg ferromagnet, we have \( B_\infty \cong C(S) \), where \( S \) is the 2-sphere. Under this isomorphism the limit \( B_\infty \) of a \( j \)-convergent net \( B_\Lambda \) is the function defined by
\[
B_\infty(\vec{e}) = \lim_\Lambda \langle \chi_\Lambda(\vec{e}), B_\Lambda \chi_\Lambda(\vec{e}) \rangle .
\]
An interesting modification of this model is the chain with nearest neighbour interaction

\[ H_2 = - \sum_{ij} R_{ij} \sigma^i \otimes \sigma^j, \quad (3.9) \]

where \( R \) is a fixed \( 3 \times 3 \) rotation matrix. Then by a suitable SU(2)-rotation at each site we can map the ground state data of the interaction (3.9) onto those of the Heisenberg chain. The resulting inductive system is also isomorphic, and hence the structure of the set of zero-energy states is also the same. However, as states on \( A_\infty \) the states look quite different: if \( R \) is a rotation by an irrational angle, they will be almost periodic.

### 3.2. Valence bond solid states

We will now look at a construction that yields many examples of locally exposed states, i.e., zero-energy state problems in which the face \( K_z(A_\infty) \) reduces to a single point. The typical example is a state studied by Affleck, Kennedy, Lieb, and Tasaki [AKLT], namely the unique ground state of the Hamiltonian

\[ H = \sum_x \left\{ \frac{1}{2} \vec{S}_x \cdot \vec{S}_{x+1} + \frac{1}{6} (\vec{S}_x \cdot \vec{S}_{x+1})^2 + \frac{1}{3} \right\}, \quad (3.10) \]

where \( \vec{S}_x \) denotes the generators of the irreducible spin-1 representation of SU(2), acting in the one-site algebra at site \( x \). It is an anti-ferromagnetic model in the sense that it is an increasing rather than a decreasing polynomial in the scalar product of neighbouring spins. In fact, the expression in braces is nothing but the projection onto the spin-2 subspace in the decomposition of the tensor product of the two representations at sites \( x \) and \( (x + 1) \). Therefore, for any interval \( \Lambda \subset \mathbb{Z} \), \( G_\Lambda \) is characterized by the property that on any two neighbouring sites the total spin is \( \leq 1 \), whereas for the ferromagnetic ground state only the maximal spin 2 occurs.

In this special model the fastest way to determine the finite volume ground states is the realization of the irreducible spin-\( s \) representation as the space of homogeneous polynomials in two variables of homogeneous degree \( 2s \). In this language it is easy to see [KLT, FNW1] that the finite volume ground state spaces \( G_\Lambda \) are all four-dimensional, and contained in the spin\( \leq 1 \) subspace of \( H_\Lambda \). From an investigation of this model one can abstract the following construction [FNW1, We2], which no longer requires any symmetry group:

**9 Definition.** A **generalized valence bond solid** (VBS) on a spin chain with one-site Hilbert space \( \mathcal{H} \) is given by

1. two auxiliary finite dimensional Hilbert spaces \( \mathcal{K} \) and \( \overline{\mathcal{K}} \),
2. a vector \( \Phi \in \overline{\mathcal{K}} \otimes \mathcal{K} \),
3. a linear operator \( S : \mathcal{K} \otimes \overline{\mathcal{K}} \rightarrow \mathcal{H} \).
Then, for every interval $\Lambda \subset \mathbb{Z}$ of length $L$, the ground state space $\mathcal{G}_\Lambda \subset \mathcal{H}_\Lambda \equiv \mathcal{H}^\otimes L$ is defined as the linear span of the set of vectors of the form

$$
S \otimes S \otimes \cdots \otimes S \left( \chi_L \otimes \varphi \otimes \cdots \otimes \varphi \otimes \chi_R \right),
$$

with $\chi_R \in \mathcal{K}$ and $\chi_L \in \mathcal{K}$ arbitrary. A state $\omega$ whose restriction to each local subalgebra is supported by $\mathcal{G}_\Lambda$ is called a generalized VBS-state.

The point of this construction is that because $\Phi \in \mathcal{K} \otimes \mathcal{K}$ has an expansion into product vectors, the condition (2.13) is automatically satisfied, and, unless $S$ is somehow degenerate, (2.12) also holds. Hence we have an inductive limit of zero-energy observables in the sense of Section 2.2.

A fundamental observation in the theory of VBS states is that there is a transfer matrix like operator, whose spectrum determines the ground state degeneracy, and the decay properties of correlations in the possible zero-energy states. It leads to an alternative expression for VBS states, which was introduced in [FNW1], and studied in a series of papers [FNW2,FNW3,FNW4,FNW5] under the name of C*-finitely correlated states (for an introduction, see also [We2]. A copy of the construction was also made in [KSZ]). The basic objects are the operators

$$
V : \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}
$$

$$
V \chi = (S \otimes \text{id}_{\mathcal{K}})(\chi \otimes \varphi), \quad \text{and}
$$

$$
\mathcal{E} : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})
$$

$$
\mathcal{E}(X) = V^* XV.
$$

For fixed $A \in \mathcal{B}(\mathcal{H})$ we define a map $\mathcal{E}_A : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$ by $\mathcal{E}_A(B) = \mathcal{E}(A \otimes B)$. Then one easily verifies that, for each interval of length $L$, the functionals on $\mathcal{B}(\mathcal{H}^\otimes L)$ of the form

$$
\omega(A_1 \otimes A_2 \otimes \cdots \otimes A_L) = \rho(\mathcal{E}_{A_1} \mathcal{E}_{A_2} \cdots \mathcal{E}_{A_L}(B))
$$

with $B \in \mathcal{B}(\mathcal{K})$, and $\rho$ a linear functional on $\mathcal{B}(\mathcal{K})$, span the same space of functionals on $\mathcal{B}(\mathcal{K})$ as those of the form $A \mapsto \langle \psi_1, A \psi_2 \rangle$ with $\psi_1, \psi_2 \in \mathcal{G}_L$. If $B$ and $\rho$ are positive, then the complete positivity of $\mathcal{E}$ implies that $\omega$ is also positive, and hence, with suitable normalization is a state. Correlation functions in this state are defined by setting $A_2 = \cdots = A_{L-1} = 1$ in (3.12). Thus on the right hand side we get powers of the linear operator $\mathcal{E}_1 : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$. The spectral properties of this transfer operator hence determine the behaviour of correlations. As in the classical Frobenius theory of positive matrices, $\mathcal{E}_1$ has a positive eigenvalue on its spectral radius [AHK], and we say that $\mathcal{E}$ has trivial peripheral spectrum, if this eigenvalue is simple, and all other eigenvalues have strictly smaller modulus. By a simple transformation (see Lemma 2.5 in [FNW1]) one can take the Frobenius eigenvector to be the identity element of the algebra, and $\mathcal{E}(1) = 1$. Then, taking $\rho$ in (3.12) to be $\mathcal{E}$-invariant (i.e., $\rho \circ \mathcal{E} = \rho$), $\omega$ becomes normalized as a state for every chain length $L$, and the states for different $L$ are the restrictions of a
unique translationally invariant state on the quasi-local algebra. By construction, we have \( \omega \in K_z(\mathcal{A}_\infty) \). If the eigenvalue 1 of \( \mathbf{E}_I \) is degenerate, then there are several \( \mathbf{E} \)-invariant states \( \rho \), and hence, in general, many states in \( K_z(\mathcal{A}_\infty) \). Since \( \mathbf{E} \) is an operator on the finite dimensional space \( \mathcal{B}(\mathcal{K}) \), the eigenspace of 1 is finite dimensional, so we expect \( K_z(\mathcal{A}_\infty) \) to be finite dimensional, too. Proof of these intuitive statements can be found in [FNW1]. Emphasis in that paper is on the case of trivial peripheral spectrum. A detailed analysis of the degenerate case was undertaken by Nachtergaele [Na1]. The proof of the following Theorem draws on his results.

10 Theorem. Let \( \mathcal{G}_\Lambda, \Lambda \subseteq \mathbb{Z} \) be the ground state spaces of a valence bond solid. Then

(1) \( K_z(\mathcal{A}_\infty) \) is a finite dimensional simplex, whose extreme points are periodic pure states on \( \mathcal{A}_\infty \).

(2) If \( \mathbf{E} \) has trivial peripheral spectrum, then \( K_z(\mathcal{A}_\infty) \) reduces to a single state.

(3) The inductive system \( (\mathcal{B}_\Lambda, j_{\Lambda\Lambda'}) \) has the product property.

Proof: (1) and (2) were proven in [FNW1]. Let \( \omega^\alpha, \alpha = 1, \ldots, N \) denote the extreme points of \( K_z(\mathcal{A}_\infty) \), and let \( g^\alpha_\Lambda \in \mathcal{A}_\Lambda \) denote the support projection of the restriction of \( \omega^\alpha \) to \( \mathcal{A}_\Lambda \). It follows from [FNW3] that the joint support projection of the \( g^\alpha_\Lambda \) coincides with \( g_\Lambda \) for large enough \( \Lambda \). Then Nachtergaele ([Na1], Lemma 5) proves the estimate

\[
\lim_{\Lambda} \left\| g^\alpha_\Lambda i_{\Lambda\Lambda'}(A) g^\beta_\Lambda - \delta_{\alpha\beta} \omega^\alpha(A) g^\alpha_\Lambda \right\| = 0. \tag{3.13}
\]

Putting \( A = I \) in this relation, we find that \( \tilde{g}_\Lambda = \sum \alpha g^\alpha_\Lambda \) is nearly a projection, in the sense that \( \lim_{\Lambda} \left\| (\tilde{g}_\Lambda)^2 - \tilde{g}_\Lambda \right\| = 0 \). Applying the functional calculus, we find that \( \lim_{\Lambda} \left\| g_\Lambda - \tilde{g}_\Lambda \right\| = 0 \). Hence, for fixed \( \Lambda' \), and \( A \in \mathcal{A}_{\Lambda'} \),

\[
\lim_{\Lambda} \left\| g_\Lambda i_{\Lambda\Lambda'}(A) g_\Lambda - \sum \alpha \omega^\alpha(A) g^\alpha_\Lambda \right\| = 0. \tag{3.16}
\]

Since \( g_\Lambda i_{\Lambda\Lambda'}(A) g_\Lambda = j_{\Lambda\Lambda'} r_{\Lambda'}(A) \) is a generic \( j \)-convergent net, we find that the nets

\[
B_\Lambda = \sum_{\alpha} \omega^\alpha(A) g^\alpha_\Lambda \tag{*}
\]

also approximate every \( j \)-convergent net. Since the different \( \omega^\alpha \) are disjoint, varying \( A \) yields arbitrary coefficients \( \omega^\alpha(A) \). Hence the \( \Lambda \)-wise product of nets of the form \( (*) \) is again of the same form, which proves (3).
4. The infinite q-deformed Heisenberg ferromagnet

4.1. Definition of the model

In this subsection we derive the interaction (1.5) as a quantum group symmetric deformation of the Heisenberg chain. This is helpful, for example, for seeing the ground state degeneracy \((L + 1)\) on the length \(L\) chain without computation. However, deformation arguments are quite misleading with regard to the inductive limit. Hence quantum groups will play no further role, and readers who are not interested in this background can safely skip the rest of this section.

The pair interaction of the spin-1/2 Heisenberg model considered in Section 3.1 is the projection onto the spin-0 subspace in the tensor product of two copies of the defining representation of SU(2). The description of the model considered in this section is exactly the same — with SU(2) replaced by its quantum group deformation \(S_q U(2)\). We briefly recall some basic notions of quantum group theory (following the approach of Woronowicz [Wo1, Wo2]).

The structure of an ordinary (non-quantum) compact group \(G\) can be described completely in terms of the algebra \(C \equiv C(G)\). The topological structure of \(G\) can be reconstructed from \(C\) via the Gelfand isomorphism, whereas the group structure can be encoded in the coproduct \(\Delta : C \to C \otimes C \cong C(G \times G)\), given by \((\Delta F)(g_1, g_2) = F(g_1 g_2)\). Associativity and existence of neutral element and inverses can be reformulated in these terms as well. The key observation leading to the theory of quantum groups is that none of these axioms requires the commutativity of \(C\). Hence, by definition (and modulo some important technical details [Wo2]) a quantum group is a non-commutative \(C^\ast\)-algebra with coproduct \(\Delta\) satisfying all these axioms, except commutativity of \(C\).

Basic notions of group theory are transferred to quantum groups by the same principle. For example, a matrix representation of a quantum group \((C, \Delta)\) is a matrix of finite dimension \(d\), with entries \(u_{ij} \in C\) such that

\[
\Delta(u_{ij}) = \sum_{\ell=1}^{d} v_{i\ell} \otimes v_{\ell j} .
\]

The representation is called unitary, if this matrix is unitary in the \(C^\ast\)-algebra \(\mathcal{M}_d(C) \cong \mathcal{M}_d(C) \otimes C\) of \(d \times d\)-matrices over \(C\). In particular, the defining, or fundamental representation of \(S_q U(2)\) is given by the matrix

\[
u = \begin{pmatrix}
\alpha & -q \gamma^* \\
\gamma & \alpha^*
\end{pmatrix},
\]

where \(\alpha\) and \(\gamma\) are special elements in \(C\), and \(q\), with \(0 < q \leq 1\) is the deformation parameter. This is the defining representation also in the sense that the matrix elements \(\alpha\) and \(\gamma\) generate \(C\) as a \(C^\ast\)-algebra. Unitarity of \(u\) entails the relations

\[
\alpha \alpha^* + q^2 \gamma^* \gamma = \alpha^* \alpha + q^2 \gamma \gamma^* = \alpha^* \alpha + \gamma \gamma^* = I
\]

\[
\alpha \gamma^* - q \gamma^* \alpha = \alpha \gamma - q \gamma \alpha = 0
\]
The coproduct of $S_q U(2)$ is the $^*$-homomorphism defined on the generators by
\[
\Delta \alpha = \alpha \otimes \alpha - q \gamma^* \otimes \gamma \\
\Delta \gamma = \gamma \otimes \alpha + \alpha^* \otimes \gamma .
\]
Of course, for $q = 1$ we obtain again the undeformed SU(2).

The tensor product of matrix representations is defined by
\[
(v \otimes w)_{im,jn} = v_{ij} \ w_{mn} ,
\]
where the product on the right is the product in $C$. It is important to note that the non-commutativity of $C$ introduces an additional asymmetry here, i.e., the tensor product defined as $w_{mn} v_{ij}$ is really different from (4.3). A scalar $k$ matrix $V$ is called an intertwiner between the matrix representations $v$ (of dimension $k$), and $w$ (of dimension $\ell$), if
\[
\sum_n V_{in} v_{nm} = \sum_j w_{ij} V_{jm} .
\]

A subgroup of a quantum group is given by a quotient of $C$ by a $^*$-ideal, say $J$, which is compatible with the coproduct in the sense that $\Delta(J) \subset J \otimes C + C \otimes J$. The quantum group $S_q U(2)$ has a (non-quantum) subgroup corresponding to the rotations around the 3-axis. The corresponding $^*$-ideal is generated by the elements $\gamma$, and one readily verifies that the relation $\gamma = 0$ leaves the abelian algebra $C/J$ generated by a unitary $\alpha$, with coproduct $\Delta \alpha = \alpha \otimes \alpha$.

We use this observation to compute the $S_q U(2)$-invariant interactions for the spin-$1/2$ chain, i.e., the hermitian intertwining operators $h$ between $(u \otimes u)$ and itself. This is a straightforward calculation, which can be simplified considerably by appeal to the known representation theory of $S_q U(2)$ [Wo1]: the irreducible representations are labelled by a half-integer spin parameter, and the decomposition of tensor products yields the same irreducible blocks, i.e., the same Clebsch-Gordan series as SU(2). It follows that the space of self-intertwiners of $(u \otimes u)$ is spanned by the identity and one one-dimensional projection. This projection is the interaction we are looking for; it projects onto the (up to a factor unique) vector $\xi_q$ such that $(u \otimes u)\xi_q = \xi_q$. This is a $C$-valued equation, and by passing to the quotient defined by $\gamma = 0$ we immediately find that $\xi_q$ is of the form $\xi_q = A|+\rangle + B|-\rangle$. Then we have, for example,
\[
\langle + | u \otimes u | \xi_q \rangle = A u_{++} u_{+-} + B u_{+-} u_{++} = -q \{ qA + B \} \gamma \alpha \dagger \langle + | \xi_q \rangle = 0 .
\]
Hence $B = -qA$, which is confirmed in the other components of $\xi_q$. To summarize, we consider the nearest neighbour interaction $H_2^2$, where $H_2^2$ is the one-dimensional projection onto the vector
\[
\xi_q = \frac{1}{\sqrt{1 + q^2}} (q|+\rangle - |-\rangle) \in C^2 \otimes C^2 ,
\]
where $q$ is a real parameter with $0 \leq q < 1$. It is easy to check that the projection $H_2^q$ satisfies Temperley-Lieb relations [Lev,BB,AHY].

A direct characterization of such vectors without using quantum groups is the following: the corresponding pure state has the property that its marginals to the first and second factor (which are mixed states) coincide. With a suitable choice of basis (possibly alternating between odd and even sites) every vector of this description can be written in the standard form (4.5). More general one-dimensional projections also define interactions with many zero-energy states. The analysis of such models can be carried out along similar lines.

4.2. The finite chain

In this subsection we determine the zero-energy states of the Hamiltonian

$$H_\Lambda = \sum_{x=1}^{L-1} i_{\Lambda,\{x,x+1\}} (H_2^q)$$

on a chain of finite length $L$ with free boundary conditions. The nearest neighbour interaction operator $H_2^q$ is as determined in the previous section, namely the one-dimensional projection onto the subspace generated by the “$q$-deformed singlet” $q|+\rangle - |+\rangle$. From the representation theory of $S_q U(2)$ it is obvious that the ground state space will have the same dimension as in the undeformed ($q = 1$) case, namely $L + 1$. However, we will need more detailed information.

Let $\Psi(\sigma_1, \ldots, \sigma_L) = \langle \sigma_1, \ldots, \sigma_L | \Psi \rangle$ denote the components of a ground state vector. Then the condition

$$(\mathbb{I}_k \otimes H_2^q \otimes \mathbb{I}_{L-k-2}) \Psi = 0$$

for some $0 \leq k \leq (L - 2)$ is equivalent to the condition that, for arbitrary signs $\sigma_1, \ldots, \sigma_k, \sigma_{k+3}, \ldots, \sigma_L = \pm$,

$$\Psi(\sigma_1, \ldots, \sigma_k, -, +, \sigma_{k+3}, \ldots, \sigma_L) = q \Psi(\sigma_1, \ldots, \sigma_k, +, -, \sigma_{k+3}, \ldots, \sigma_L)$$

(4.7)

Clearly, this determines each component $\Psi(\sigma_1, \ldots, \sigma_L)$ in terms of any other component with the same number $n_+(\sigma_1, \ldots, \sigma_L)$ of “$+$”-signs. Hence the $(L + 1)$ vectors

$$\Phi_L(n)(\sigma_1, \ldots, \sigma_L) = \delta_{n,n_+(\sigma_1, \ldots, \sigma_L)} \cdot q^{\sum_{x=1}^L x (1 + \sigma_x)/2}$$

(4.8)

are an orthogonal, but not normalized basis of the ground state space $\mathcal{G}_{[1,L]}$. The reason for choosing this particular normalization is that, for $z \in \mathbb{C}$,

$$\Psi_L(z) = \sum_{n=0}^L z^n \Phi_L(n)$$

(4.9)
becomes a product state: we have

\[ \Psi_L(z)(\sigma_1,\ldots,\sigma_L) = \prod_{x=1}^{L} zq^{\sigma_x} \]  

(4.10)

\[ \Psi_L(z) = \bigotimes_{x=1}^{L} \chi(zq^{\sigma_x}) , \]

where, for any \( z \in \mathbb{C} \), \( \chi(z) \in \mathbb{C}^2 \) denotes the vector with components \( \langle - | \chi(z) \rangle = 1 \) and \( \langle + | \chi(z) \rangle = z \). Since \( \Psi_L(z) \) is the generating function for the \( \Phi_L(n) \), it is clear that these product vectors likewise span \( \mathcal{G}_{[1,L]} \). The norms of the vectors \( \Phi_L(n) \) will be of crucial importance. The following Lemma collects some basic formulas and estimates.

11 Lemma. Define

\[ N_L(n) = q^{-n(n+1)/2} \| \Phi_L(n) \| , \quad \text{for } 0 \leq n \leq L < \infty, \text{ and} \]

\[ N_{\infty}(n) = \lim_{L \to \infty} N_L(n) . \]

Then with \( p = \prod_{i=1}^{\infty} \sqrt{1 - q^{2i}} \), and the convention \( \prod_{i=1}^{0} \sqrt{1 - q^{2i}} = 1 \), we have, for \( 0 \leq n \leq L \leq \infty \),

\[ N_L(n) = \frac{\prod_{i=1}^{L} \sqrt{1 - q^{2i}}}{\prod_{i=1}^{n} \sqrt{1 - q^{2i}} \prod_{i=1}^{L-n} \sqrt{1 - q^{2i}}} \]  

(4.11.a)

\[ N_L(n) = N_L(L-n) \]  

(4.11.b)

\[ N_L(0) = 1 \]  

(4.11.c)

\[ N_{\infty}(n) = \prod_{i=1}^{n} (1 - q^2)^{-1/2} \]  

(4.11.d)

\[ p^{-1} = \lim_{n \to \infty, L-n \to \infty} N_L(n) \]  

(4.11.e)

\[ p \leq N_L(n) \leq p^{-2} . \]  

(4.11.f)

Proof: Using the generating function (4.9), and the orthogonality of the \( \Phi_L(n) \), we find, with the abbreviation \( \lambda = |z|^2 \),

\[ \sum_{n=0}^{L} \lambda^n N_L(n)^2 q^{n(n+1)} = \prod_{i=1}^{L} (1 + \lambda q^{2i}) . \]  

(\* )

The expansion of this expression can be found in the literature on \( q \)-factorials [GR], where \( N_L(n)^2 \) appears as a \( q \)-deformed binomial coefficient. This leads to (4.11.a). A direct verification uses (\* ) to obtain the recursion formula

\[ N_{L+1}(n)^2 = N_L(n)^2 + q^{2(L+1)} N_L(n-1)^2 , \]
which is satisfied by (4.11.a). The remaining properties are trivial consequences of (4.11.a). The proof of (4.11.f) uses the estimate. \[ p \leq \prod_{i=1}^{n} \sqrt{1 - q^{2i}} \leq 1. \]

The importance of the estimate (4.11.f) is that it is uniform in \( L \), which is crucial for taking the limit to infinite chain lengths. For taking norm estimates we will need to compute matrix elements in an orthonormal basis, which is obtained by normalizing the vectors \( \Phi_L(n) \):

\[
\tilde{\Phi}_L(n) = q^{-n(n+1)/2} N_L(n)^{-1} \Phi_L(n) .
\]

### 4.3. The Inductive Limit

The regions \( \Lambda \) indexing the net will be subintervals of the integers. Each interval \( \Lambda \subset \mathbb{Z} \) will be characterized by two numbers \( \Lambda_{\pm} \) according to

\[
\Lambda = (\Lambda_{-}, \Lambda_{+}) = \{ i \in \mathbb{Z} | \Lambda_{-} < i \leq \Lambda_{+} \} .
\]

Thus \( \Lambda_{+} - \Lambda_{-} \) is the number of sites in \( \Lambda \). We will use \( \Lambda \to \infty \) as shorthand for \( \Lambda_{-} \to -\infty \) and \( \Lambda_{+} \to +\infty \).

We will identify ground state spaces \( G_{\Lambda} \) for intervals of the same length, so we can write \( \Phi_{\Lambda_{+} - \Lambda_{-}}(n) \in G_{\Lambda} \), with the vectors defined in the previous subsection. Since we know that \( G_L \) is spanned by product vectors, we can immediately write down the isometry \( V_{\Lambda \Lambda'} \) identifying \( G_{\Lambda} \) in the product of ground state spaces for smaller chains:

\[
V_{\Lambda \Lambda'} : \ G_{\Lambda} \longrightarrow \ G_{(\Lambda_{-}, \Lambda_{-}')} \otimes \ G_{\Lambda_{+}'} \otimes \ G_{(\Lambda_{+}', \Lambda_{+})}
\]

\[
V_{\Lambda \Lambda'} \Psi_{\Lambda_{+} - \Lambda_{-}}(z) = \bigotimes_{i=\Lambda_{-}+1}^{\Lambda_{-}'} \chi(zq^i) \otimes \bigotimes_{i=\Lambda_{+}'+1}^{\Lambda_{+}} \chi(zq^i) \otimes \bigotimes_{i=\Lambda_{+}'}^{\Lambda_{+}} \chi(zq^i)
\]

\[
= \Psi_{\Lambda_{-}' - \Lambda_{-}}(z) \otimes \Psi_{\Lambda_{+}' - \Lambda_{+}}(zq^{\Lambda_{-}' - \Lambda_{-}}) \otimes \Psi_{\Lambda_{+} - \Lambda_{+}'}(zq^{\Lambda_{-}' - \Lambda_{-}}).
\]

Then it is straightforward to write down the maps \( j_{\Lambda \Lambda'} \) defined for general ground state spaces \( G_{\Lambda} \) in Section 2:

\[
j_{\Lambda \Lambda'}(A) = V_{\Lambda \Lambda'}^*(I_{(\Lambda_{-}, \Lambda_{-}')} \otimes A \otimes I_{(\Lambda_{+}', \Lambda_{+})}) V_{\Lambda \Lambda'} ,
\]

for \( A \in \mathcal{B}_{\Lambda'} \equiv \mathcal{B}(G_{\Lambda'}) \). Our aim is to study the asymptotic behaviour of such expressions as \( \Lambda \to \infty \), up to terms which become small in norm in this limit.

For developing norm estimates the product vectors in (4.14) are not suitable. Therefore we begin by expressing \( V_{\Lambda \Lambda'} \) in the orthonormal bases (4.12). Inserting the generating
function (4.9) into (4.14), collecting terms of the same order in \( z \) and expressing each \( \Phi_L(n) \) in terms of its normalized counterpart (4.12), we find

\[
V_{\Lambda\Lambda'} \hat{\Phi}_{\Lambda^+ - \Lambda^-}(n') = \sum_{\ell', m', r'} \delta_{n', \ell' + m' + r'} C_{\Lambda\Lambda'}(\Lambda' - \Lambda - \ell', m' + \Lambda', r') \times \\
\times \hat{\Phi}_{\Lambda' - \Lambda^-}(\ell') \hat{\Phi}_{\Lambda^+ - \Lambda'(m')} \hat{\Phi}_{\Lambda^+ - \Lambda_+}(r')
\]

\[
= \sum_{\ell m r} \delta_{n - m, r - \ell} C_{\Lambda\Lambda'}(\ell, m, r) \hat{\Phi}_{\Lambda' - \Lambda^-}(\Lambda' - \Lambda - \ell) \times \\
\times \hat{\Phi}_{\Lambda^+ - \Lambda'_-}(m - \Lambda'_-) \hat{\Phi}_{\Lambda^+ - \Lambda_+}(r), \tag{4.16}
\]

where

\[
C_{\Lambda\Lambda'}(\ell, m, r) = q^{\ell(m - \Lambda'_-) + \ell r + (\Lambda'_- - m)r} \times \\
\times \frac{N_{\Lambda'_- - \Lambda_-}(\ell) N_{\Lambda'_- - \Lambda'_-}(m - \Lambda'_-) N_{\Lambda^+ - \Lambda'_+}(r)}{N_{\Lambda^+ - \Lambda_-}(m + r - \ell - \Lambda_-)}. \tag{4.17}
\]

The parameters \( \ell', m', r' \) in the first sum in (4.16) are the numbers of “+”-signs in the left, middle, and right segment of the interval \( \Lambda \), respectively. However, these parameters are not meaningful in the limit \( \Lambda \to \infty \). Therefore the summation indices in the second sum, and the arguments of \( C_{\Lambda\Lambda'} \) were chosen slightly differently. They are the number \( \ell = \Lambda'_- - \Lambda_- - \ell' \) of “-”-spins on the right, the number \( r = r' \) of “+”-spins on the right, and the label \( m = m' + \Lambda_- \) of the site, where “+” changes to “-” if we pack \( m' \) “+”-signs to the left of \( (\Lambda'_+ - \Lambda'_- - m') \) “-”-signs into the interval \( \Lambda' \). The ranges of these parameters are

\[
0 \leq \ell \leq \Lambda'_- - \Lambda_- \\
\Lambda'_- \leq m \leq \Lambda'_+ \\
0 \leq r \leq \Lambda^+ - \Lambda'_+
\]

\[\tag{4.18}\]

As for the interval \( \Lambda' \) we will use for the whole interval \( \Lambda \) the parameter \( n = n' + \Lambda_ = (\ell' + m' + r') + \Lambda_ = m + r - \ell \).

A convenient basis in \( \mathcal{B}_\Lambda \) is given by the usual matrix units, parametrized as above. For any finite interval \( \Lambda \), and \( \Lambda_ \leq n_1, n_2 \leq \Lambda_+ \), we set

\[
E_\Lambda(n_1, n_2) = \left\langle \hat{\Phi}_{\Lambda^+ - \Lambda_+}(n_1 - \Lambda_-) \right| \left| \hat{\Phi}_{\Lambda^+ - \Lambda_+}(n_2 - \Lambda_-) \right\rangle \tag{4.19}
\]

The operators play a dual role in the sequel. On the one hand, because basic nets are dense, and the \( E_{\Lambda'}(m_1, m_2) \) are a basis in \( \mathcal{B}_{\Lambda'} \), the limits of sequences of the form \( j_{\Lambda\Lambda'}(E_{\Lambda'}(m_1, m_2)) \), with fixed \( \Lambda', m_1, m_2 \) span the limit space \( \mathcal{B}_\infty \). On the other hand, they are convergent nets in their own right (with \( m_1, m_2 \) fixed, and \( \Lambda \to \infty \). In either case we need the matrix elements of the operator \( j_{\Lambda\Lambda'}(E_{\Lambda'}(m_1, m_2)) \), which at the same time can be considered as the matrix elements of the operator \( j_{\Lambda\Lambda'} \) itself. Using (4.16) we find

\[
J_{\Lambda\Lambda'}(n_1, n_1, m_2, n_2) \equiv \left\langle \hat{\Phi}_{\Lambda^+ - \Lambda_+}(n_1 - \Lambda_-) \right| j_{\Lambda\Lambda'}(E_{\Lambda'}(m_1, m_2)) \right| \hat{\Phi}_{\Lambda^+ - \Lambda_+}(n_2 - \Lambda_-) \right\rangle
\]
\[
\sum_{\ell_1 r_1 \leq \ell_2 r_2} \delta_{\ell_1 - \ell_2} \delta_{r_1 - r_2} \times \frac{C_{\Lambda \Lambda'}(\ell_1, m_1, r_1)}{C_{\Lambda \Lambda'}(\ell_2, m_2, r_2)} \]

\[
= \sum_{\ell_1 m_1, r_1} \delta_{n_1 - m_1, n_2 - m_2} \sum_{\ell \geq 1} \frac{C_{\Lambda \Lambda'}(\ell, m_1, r)}{C_{\Lambda \Lambda'}(\ell, m_2, r)} \right)
\]

\[
(4.20)
\]

In general, estimating operator norms is a difficult task. However, in the present case we can utilize a special structure of the matrix elements (4.20): they are non-zero only along a single line parallel to the main diagonal. The simple observation which allows us to compute norms of such operators is stated in the following Lemma.

**12 Lemma.** Let \( I \subset \mathbb{Z} \) be a finite or infinite subset, and \( s \in \mathbb{Z} \). Let \( A \) be an operator in \( \ell^2(I) \) whose matrix elements \( A_{n_1, n_2} \) with respect to the canonical basis vanish unless \( n_1 - n_2 = s \). Then

\[
\| A \| = \sup_{n_1, n_2 \in I} |A_{n_1, n_2}|.
\]

**Proof:** The inequality “\( \geq \)” is trivial. For the converse, we may take \( I = \mathbb{Z} \), by defining matrix elements \( A_{n, m} = 0 \), if \( n \notin I \) or \( m \notin I \). Thus \( A = S \tilde{A} \), with the diagonal operator \( \tilde{A} \), defined by \( \tilde{A}_{n, m} = A_{n+s, m} \), and a shift operator \( S \). Hence \( \| A \| = \| S \tilde{A} \| \leq \| \tilde{A} \| = \sup |\tilde{A}_{n, m}| = \sup |A_{n, m}| \).

Our first key result is the convergence of the matrix units themselves:

**13 Proposition.** Fix \( m_1, m_2 \in \mathbb{Z} \). Then the net \( E_\Lambda(m_1, m_2) \) is \( j \)-convergent. Moreover, the net defined by

\[
F^+_\Lambda = \sum_{m=1}^{\Lambda_+} E_\Lambda(n, n)
\]

is \( j \)-convergent.

**Proof:** (1) If we estimate each \( N_L(n) \) as in (4.11.1), a straightforward estimate for \( C_{\Lambda \Lambda'} \) is

\[
|C_{\Lambda \Lambda'}(\ell, m, r)| \leq p^{-7} q^{(\ell(m-N'_+) + r + (\Lambda'_+ - m)r)}.
\]

For an upper bound on (4.20) consider first the case \( n_1 \geq m_1 \). Then we have \( r \geq 1 \) in the whole sum except, possibly, in the term with \( \ell = r = 0 \). Hence, apart from this term we can estimate the above power of \( q \) by \( q^{\ell+(\Lambda'_+ - m_i)} (i = 1, 2) \) in every term with \( \ell \geq 1 \). The
sum of these bounds is then a simple geometric series. Denoting the term with \( \ell = r = 0 \) by \( J_{\Lambda \Lambda'}^{00}(n_1, m_1, m_2, n_2) \), we obtain the estimate

\[
|J_{\Lambda \Lambda'}(n_1, m_1, m_2, n_2) - J_{\Lambda \Lambda'}^{00}(n_1, m_1, m_2, n_2)| \leq \frac{p^{-14}}{1 - q^2} q^{2\Lambda'_{+} - m_1 - m_2}.
\]

An analogous estimate, with \((\Lambda'_{+} - m_i)\) replaced by \((m_i - \Lambda'_-)\) applies for \(n_1 \leq m_1\). For \(J_{\Lambda \Lambda'}^{00}(n_1, m_1, m_2, n_2)\) we get the explicit expression

\[
J_{\Lambda \Lambda'}^{00}(n_1, m_1, m_2, n_2) = \delta_{n_1, m_1} \delta_{n_2, m_2} \frac{N_{\Lambda'_{+} - \Lambda'_-}(m_1 - \Lambda'_-)}{N_{\Lambda'_{+} - \Lambda'_-}(m_1 - \Lambda'_-)} \frac{N_{\Lambda'_{+} - \Lambda'_-}(m_2 - \Lambda'_-)}{N_{\Lambda'_{+} - \Lambda'_-}(m_2 - \Lambda'_-)},
\]

which converges to \(\delta_{n_1, m_1} \delta_{n_2, m_2} \), as \(\Lambda \to \infty\), and then \(\Lambda' \to \infty\). These are the matrix elements of \(E_{\Lambda}(m_1, m_2)\). Applying Lemma 12, we get

\[
\|j_{\Lambda \Lambda'}(E_{\Lambda}(m_1, m_2)) - E_{\Lambda}(m_1, m_2)\| \leq \frac{p^{-14}}{1 - q^2} \max \{q^{\Lambda'_{+} - m_i}, q^{m_i - \Lambda'_-}\} + \sup_{n_1, n_2} |\delta_{n_1, m_1} \delta_{n_2, m_2} - J_{\Lambda \Lambda'}^{00}(n_1, m_1, m_2, n_2)|
\]

\[\to 0 \quad \text{as} \quad \Lambda \to \infty, \text{ and } \Lambda' \to \infty.\]

(2) By Lemma 12 we have to show that, as \(\Lambda \to \infty\), followed by \(\Lambda' \to \infty\),

\[
\sum_{m \geq 1} J_{\Lambda \Lambda'}(n, m, m, n) \to 0 \quad \text{uniformly for } n \leq 0 \quad \text{and}
\]

\[
\sum_{m \geq 1} J_{\Lambda \Lambda'}(n, m, m, n) \to 1 \quad \text{uniformly for } n \geq 1.
\]

Since \(j_{\Lambda \Lambda'} I_{\Lambda'} = I_{\Lambda}\), the unrestricted sum over \(m\) is equal to 1 for all \(n\), so the second statement is equivalent to the convergence \(\sum_{m \leq 0} J_{\Lambda \Lambda'}(n, m, m, n) \to 0\), uniformly for \(n \geq 1\). By left/right symmetry proving this is completely analogous to proving the first statement, so we will only show the first.

Using again the estimate (*) in (4.20), we get

\[
\left|\sum_{m \geq 1} J_{\Lambda \Lambda'}(n, m, m, n)\right| \leq p^{-14} \sum_{m \geq 1} \sum_{\ell, r \geq 0} \delta_{n-m, r-\ell} q^{2\ell(m-\Lambda'_-) + 2\ell r + 2(\Lambda'_+ - m)r}.
\]

Because \(n \leq 0\), the sum over \(r\) begins at \(r = 0\), and \(\ell = r + (m-n) \geq (r+1)\). Hence, replacing the exponent of \(q\) by the smaller exponent \(2r + 2(m-\Lambda'_-)\), we get the estimate

\[
\left|\sum_{m \geq 1} J_{\Lambda \Lambda'}(n, m, m, n)\right| \leq p^{-14} \sum_{m \geq 1} \sum_{r \geq 0} q^{2r + 2(m-\Lambda'_-)} = p^{-14}(1 - q^2)^{-2}q^{2(1-\Lambda'_-)}.
\]

Clearly, this converges to zero as \(\Lambda'_- \to -\infty\), uniformly in \(\Lambda\) and \(n\).

\section*{14 Lemma.} Fix \(\Lambda'\), and \(\Lambda'_- \leq m_1, m_2 \leq \Lambda'_+\). Then

(1) the limit

\[
J_{\infty \Lambda'}(n_1, m_1, m_2, n_2) = \lim_\Lambda J_{\Lambda \Lambda'}(n_1, m_1, m_2, n_2)
\]

exists uniformly in \(n_1\) and \(n_2\).
We now introduce \( \lim_{d \to \pm \infty} J_{|\Lambda\Lambda'}(d + m_1, m_1, m_2, d + m_2) = \delta_{m_1, m_2} \delta_{m_1, \Lambda_-} \) (4.24). and Lemma 12. Then such that
\[
\lim_{J_{|\Lambda\Lambda'}(d + m_1, m_1, m_2, d + m_2) = \delta_{m_1, m_2} \delta_{m_1, \Lambda_+}.
\]
exist uniformly in \( \Lambda \).

**Proof:** (1) In the proof of Proposition 13 we have found a majorant for the sum (4.20), which is independent of \( n_1, n_2, \) and \( \Lambda \). Hence it suffices to show that each term in (4.20) goes to a limit as \( \Lambda \to \infty \), with fixed \( n_1, n_2 \). Hence the proof is completed by the observation that, using (4.17) with (4.11.e), the limit
\[
\lim_{\Lambda} C_{|\Lambda\Lambda'}(\ell, m, r) \equiv C_{\infty\Lambda'}(\ell, m, r) = q^{\ell(m-\Lambda_-)+\ell r} N_{\infty}(\ell) N_{\Lambda_+} N_{\Lambda_-} N_{\infty}(r).
\]
exists.

(2) As in the proof of Proposition 13 we find, for \( d \geq 0 \):
\[
|J_{|\Lambda\Lambda'}(d + m_1, m_1, m_2, d + m_2)| \leq p^{-14(1-q^2)q^{2(2\Lambda_+-m_1-m_2)d}}.
\]
Hence, unless \( m_1 = m_2 = \Lambda_+ \), this goes to zero uniformly in \( \Lambda \). In the exceptional case the only the first term \( (r = \ell = 0) \) in the sum (4.20) survives, the remainder being bounded by a constant depending only on \( q \), times \( q^{2d} \). That the first term converges to 1 follows again from (4.22). The statement for the limit \( d \to -\infty \) follows analogously. \( \square \)

**15 Proposition.** For any \( j \)-convergent net \( B_\Lambda \in B_\Lambda \), and \( \varepsilon > 0 \), there is a finite linear combination
\[
B_\Lambda^\varepsilon = C \mathbf{1}_\Lambda + C_+ P_+ + \sum_{n_1, n_2} C_{n_1, n_2} E_\Lambda(n_1, n_2),
\]
such that \( \lim_\Lambda \| A_\Lambda - A_\Lambda^\varepsilon \| \leq \varepsilon \).

**Proof:** It suffices to consider basic nets of the form \( B_\Lambda = j_{|\Lambda\Lambda'}(E_\Lambda(m_1, m_2)) \) with fixed \( \Lambda', m_1, m_2 \). Assume first that neither \( m_1 = m_2 = \Lambda_- \) nor \( m_1 = m_2 = \Lambda_+ \). For any \( R \in \mathbb{N} \), let \( P_\Lambda^R \) denote the projection in \( B_\Lambda \) onto the subspace generated by the basis elements with \( |n| \leq R \). Then by choosing \( R \) sufficiently large, the norm difference \( \| B_\Lambda - P_\Lambda^R B_\Lambda P_\Lambda^R \| \) can be made arbitrarily small, uniformly in \( \Lambda \), on account of Lemma 14(2), and Lemma 12. We now introduce
\[
B_\Lambda^R = \sum_{|n| \leq R} J_{|\Lambda\Lambda'}(n_1, m_1, m_2, n_1) E_\Lambda(n_1, n_2).
\]
Then \( \| P_\Lambda^R B_\Lambda P_\Lambda^R - B_\Lambda^R \| \) converges to zero as \( \Lambda \to \infty \), because of the first part of Lemma 14. Moreover, \( B_\Lambda^R \) is a finite linear combination of nets of the form (4.19), and therefore satisfies the conditions of the Proposition for sufficiently large \( R \). In the two exceptional cases,
the same arguments apply after subtraction of either $F_\Lambda^+$ or $(I - F_\Lambda^+)$ from both nets.

16 Theorem. The inductive system (4.15) has the product property. $B_\infty$ is the $C^*$-algebra of operators on $\ell^2(\mathbb{Z})$, generated by the compact operators, the identity, and the operator of multiplication with the characteristic function of $\mathbb{N} \subset \mathbb{Z}$.

Proof : It suffices to show that the nets of the form $B_\Lambda^\times$ as in Proposition 15 have the product property. However, this is evident from the multiplication rule

$$E_\Lambda(m_1,m_2)E_\Lambda(m_3,m_4) = \delta_{m_2,m_3}E_\Lambda(m_1,m_4)$$

for matrix units, which holds for every $\Lambda$. Hence $B_\infty$ is the unique $C^*$-algebra generated by elements $E_\infty(m_1,m_2)$, $F_\infty^+$, and $I$ with just these multiplication rules.

Combining this with Theorem 3 we now obtain the convex structure of $K_\mathbb{Z}(A_\infty)$.

17 Corollary. $K_\mathbb{Z}(A_\infty)$ is isomorphic to the convex hull of the set of density matrices on $\ell^2(\mathbb{Z})$, and two additional points $\omega_+$ and $\omega_-$, interpreted as the “all spins up” and the “all spins down state”.

Note that Theorem 16 gives additional information about the topology of $K_\mathbb{Z}(A_\infty)$, which is not contained in the above description of the convex structure. In fact, there is a different $C^*$-algebra with the same convex set as its state space. Since any $C^*$-algebra $C$ is abstractly reconstructed as the set of $\sigma(C^*,C)$-continuous affine functionals on its state space $K(C)$, this shows that the description of $K_\mathbb{Z}(A_\infty)$ as a mere convex set without topology misses a vital element. The second $C^*$-algebra with the same convex state space is the (C*-algebraic) direct sum of a one-dimensional algebra and the algebra of compact operators with identity adjoined. The projection onto the first summand then produces a continuous affine functional which is 1 on the state, say, $\omega_*$, and vanishes on the non-abelian part. In other words, $\omega_*$ is an exposed state. In contrast to this, neither $\omega_-$ nor $\omega_+$ is exposed in $K(B_\infty) \cong K_\mathbb{Z}(A_\infty)$.

The program for computing $K_\mathbb{Z}(A_\infty)$ can be carried out slightly more easily for the same model on the half chain [Got]. In that case analogous results hold, but only one state at infinity needs to be adjoined.
4.4. Representation in an infinite tensor product

For the determination of the inductive limit in the previous section we did not use any particular representation of the quasi-local algebra. Indeed, we argued in Section 2.3 that this is essential for obtaining a complete characterization of $K_z(A_\infty)$, and not just the subset of states which happen to be normal in the given representation. On the other hand, quantities like the limiting matrix elements $J_{\infty\Lambda'}(n_1, m_1, m_2, n_2)$ appearing in the calculation have a natural interpretation in a Hilbert space associated with the infinite system. In this section we make this connection explicit.

The starting point is that the product vectors $\Psi_L(z)$, which span the ground state space for every chain length $L$, contain mostly “spin up” factors on the left, and “spin down” factors on the right. Therefore, infinite product vectors, which should represent ground states of the infinite chain, should make sense in the incomplete tensor product

$$[\text{Gui}]$$

$$\mathcal{H}_\pi = \bigotimes_{i \in \mathbb{Z}} (\mathcal{F}^2, \eta_i),$$

with the reference vectors

$$\eta_i = \begin{cases} (0) & \text{for } i > 0, \\ (1) & \text{for } i \leq 0. \end{cases}$$

(4.27)

This space carries an irreducible representation $\pi : A_\infty \to \mathcal{B}(\mathcal{H}_\pi)$ of the quasi-local algebra, such that $\pi(A_i)$ for $A_i \in A_{\{i\}}$ acts in the $i^{th}$ tensor factor. Of course, since $A_\infty$ is simple, this representation is faithful.

The two types of vectors we used above for every finite $\Lambda$, namely the product vectors $\Psi$, and the orthogonal vectors $\Phi$, can be embedded into $\mathcal{H}_\pi$ as follows. For $z \in \mathbb{C}$, we define

$$\chi_i(z) = \begin{cases} (1) & = \chi(z) \text{ for } i > 0, \\ (1/z) & = \chi(z) \text{ for } i \leq 0, \end{cases}$$

(4.28)

where $\chi(z)$ is defined as in (4.10). For finite $\Lambda$ we now define the product vectors

$$\tilde{\Psi}_\Lambda(z) = \bigotimes_{i \in \Lambda} \chi_i(zq^i) \otimes \bigotimes_{i \notin \Lambda} \eta_i,$$

(4.29)

and the orthogonal vectors

$$\tilde{\Phi}_\Lambda(n) = \Phi_{\Lambda_+ - \Lambda_-} (n - \Lambda_-) \otimes \bigotimes_{i \notin \Lambda} \eta_i,$$

(4.30)

where $\Lambda_- < n \leq \Lambda_+$. The basic property of these vectors is stated in the following Theorem:
18 Theorem. The limits \( \Phi_\infty(n) = \lim_\Lambda \Phi_\Lambda(n) \) and \( \Psi_\infty(z) = \lim_\Lambda \Psi_\Lambda(z) \) exist in the norm of \( \mathcal{H}_\pi \) for all \( n \in \mathbb{Z} \), and all \( z \in \mathbb{C} \setminus \{ 0 \} \). The \( \Phi_\infty(n) \) are an orthonormal basis of \( \mathcal{G}_\pi \), and, for \( z \in \mathbb{C} \setminus \{ 0 \} \), we have the convergent expansion
\[
\Psi_\infty(z) = p^{-1} \sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2} \Phi_\infty(n) .
\] (4.31)
A zero-energy state \( \omega \in K_\pi(A_\infty) \) is normal in the representation \( \pi \) if and only if it is disjoint from both \( \omega_+ \) and \( \omega_- \), in which case it is given by a unique density matrix supported by \( \mathcal{G}_\pi \).

Proof: The convergence of \( \Psi_\Lambda(z) \) follows from the standard theory of incomplete tensor products [Gui], because
\[
\sum_{i \in \mathbb{Z}} \left\| \eta_i - \chi_i(\zeta q^i) \right\|^2 \leq \infty .
\] (4.32)
Since every \( \Phi_\Lambda(n) \) is a unit vector,
\[
\left\| \Phi_\Lambda(n) - \Phi_\Lambda'(n) \right\|^2 = 2 - 2 \Re \langle \Phi_\Lambda(n), \Phi_\Lambda'(n) \rangle .
\]
The scalar product is computed in \( \mathcal{G}_{(\Lambda_{-},\Lambda'_{-})} \otimes \mathcal{G}_{\Lambda'} \mathcal{G}_{(\Lambda'_{+},\Lambda_{+})} \) into which \( \Phi_\Lambda(n) \) is embedded as in (4.16), and \( \Phi_\Lambda'(n) \) by tensoring with suitable spin up, resp. spin down vectors:
\[
\langle \Phi_\Lambda(n), \Phi_\Lambda'(n) \rangle = \langle V_{\Lambda \Lambda'}, \Phi_{\Lambda_{+}+\Lambda_{-}}(n - \Lambda_{+}), \Phi_{\Lambda'_{+}+\Lambda'_{-}}(n - \Lambda'_{+}) \otimes \Phi_{\Lambda'_{+}+\Lambda'_{-}}(n - \Lambda'_{-}) \otimes \Phi_{\Lambda_{+}+\Lambda_{-}}(0) \rangle = C_{\Lambda \Lambda'}(0, n, 0) = \frac{N_{\Lambda'_{+}+\Lambda'_{-}}(n - \Lambda'_{+})}{N_{\Lambda_{+}+\Lambda_{-}}(n - \Lambda_{+})} .
\]
Since numerator and denominator both converge to \( p \), we have \( \lim_{\Lambda \to \Lambda'} \lim_{\Lambda_{+} \to \Lambda_{+}} C_{\Lambda \Lambda'}(0, n, 0) = 1 \), and \( \Phi_\Lambda(n) \) is a Cauchy net in \( \mathcal{H}_\pi \).

For different \( n \) the vectors \( \Phi_\infty(n) \) are clearly orthogonal. They span \( \mathcal{G}_\pi \), because every vector in \( \mathcal{H}_\pi \) is approximated by vectors differing from the reference vector only in a finite region sites \( \Lambda \), and because for approximating a given vector \( \Phi \in \mathcal{G}_\pi \) we may apply the projection \( \pi_\Lambda(g_\Lambda) \leq g_\pi \) to the approximating vectors without loss.
The expansion formula is obtained from the corresponding formula for finite \( \Lambda \). Assuming \( \Lambda_- \leq 0 \) and using, in succession, equations (4.28), (4.10), (4.9), and (4.12), we get

\[
\bigotimes_{i \in \Lambda} \chi_i(zq^i) = \prod_{i=\Lambda_-+1}^{0} (zq^i)^{-1} \bigotimes_{i \in \Lambda} \chi(zq^i)
\]

\[
= z^{\Lambda_-} q^{(|\Lambda_-|-1)|\Lambda_-|/2} \Psi_{\Lambda_+ - \Lambda_-} (zq^{\Lambda_-})
\]

\[
= z^{\Lambda_-} q^{(\Lambda_-+1)\Lambda_-/2} \sum_{n' = 0}^{\Lambda_+ - \Lambda_-} (zq^{\Lambda_-})^{n'} \Phi_{\Lambda_+ - \Lambda_-} (n')
\]

\[
= \sum_{n = \Lambda_-}^{\Lambda_+} z^n N_{\Lambda_+ - \Lambda_-} (n - \Lambda_-) q^{n(n+1)/2} \tilde{\Phi}_{\Lambda_+ - \Lambda_-} (n - \Lambda_-)
\]

Multiplying with the appropriate tensor product of reference vectors \( \eta_i \), we get

\[
\tilde{\Psi}_\Lambda (z) = \sum_{n = \Lambda_-}^{\Lambda_+} z^n N_{\Lambda_+ - \Lambda_-} (n - \Lambda_-) q^{n(n+1)/2} \tilde{\Phi}_\Lambda (n)
\]

and the expansion for infinite \( \Lambda \) follows, because \( N_{\Lambda_+ - \Lambda_-} (n - \Lambda_-) \rightarrow 1 \). Note that the sum converges both for \( n \rightarrow +\infty \) and \( n \rightarrow -\infty \), and arbitrary \( z \neq 0 \), because of the quadratic dependence of the exponent of \( q \) on \( n \).

The two \( \pi \)-singular states \( \omega_\pm \) are clearly the limits of any \( \pi \)-normal state, for shifts to \( \pm \infty \). Hence all states are of the form \( \eta \circ \pi \) for a (possibly singular) state \( \eta \in K(\mathcal{B}(\mathcal{H})) \). Stating this observation as a property of \( j_{\pi \infty} \), and using the product property, we get

19 Proposition. \( j_{\pi \infty} : \mathcal{B}_\infty \rightarrow \mathcal{B}(\mathcal{G}_\pi) \) is a faithful irreducible representation.
4.5. The spectral gap

In the representation \( \pi \) we define the Hamiltonian \( H_\pi \) as the closure of the operator given by

\[
H_\pi \ i_{\pi,\Lambda'}(A) \varphi = \lim_{\Lambda} \left[ \pi(H_\Lambda), i_{\pi,\Lambda'}(A) \right] \varphi , \text{ for } A \in \mathcal{A}_{\Lambda'}, \text{ and } \varphi \in \mathcal{G}_\pi .
\]

Here the net on the right hand side is eventually constant, because \( A \in \mathcal{A}_{\Lambda'} \) is strictly local, and the interaction is finite range. It is a standard argument \([BR]\) to show that the dynamics exist as an automorphism on the quasi-local algebra and is generated by \( H_\pi \). Hence \( H_\pi \) has a dense set of analytic vectors, and is essentially self-adjoint. For a pure VBS state it was shown that \( H_\pi \) has a spectral gap above its unique ground state. These results were extended to general VBS states by \([Na1]\). On the other hand, for the Heisenberg ferromagnet the well-known magnon (or spin wave) states have arbitrarily small energy, and hence \( H_\pi \) has no gap. Here we will show that even an arbitrarily small deformation generates a gap. The basic technique for obtaining lower estimates on a gap is given in the following Lemma, which was proved in \([FNW1]\) (see the proof of Theorem 6.4 in that paper). Some refinements and generalizations are in \([Na1,Na2]\).

20 Lemma. Consider a one-dimensional translationally invariant nearest neighbour interaction, whose ground state projections \( g_\Lambda \) satisfy assumption (2.13). For \( L \in \mathbb{N} \) let \( \gamma_L \) denote the gap of \( H_{(0,L]} \), i.e., the largest number satisfying

\[
H_{(0,L]} \geq \gamma_L (1 - g_{(0,L]}) .
\] (4.33)

For \( p \in \mathbb{N} \), consider the numbers

\[
\varepsilon(p) = \| (g_{(0,2p]} \otimes \mathbb{1}^\otimes p)(\mathbb{1}^\otimes p \otimes g_{(p,3p]}) - g_{(0,3p]} \| .
\] (4.34)

Then, for \( n \geq 3 \):

\[
\gamma_{n \cdot L} \geq \frac{\gamma_2 L}{2} (1 - 2 \varepsilon(p)) .
\] (4.35)

The quantities appearing in this Lemma are readily computable for small chains. With some assistance for doing long symbolic computations \([Mat]\) we find for our model:

\[
\begin{align*}
\gamma_2 &= 1 \\
\gamma_3 &= (1 - q + q^2)/(1 + q^2) \\
\gamma_4 &= (1 - \sqrt{2}q + q^2)/(1 + q^2) \\
\varepsilon_1 &= q/(1 + q^2) \\
\varepsilon_2 &= q^2/(1 + q^4) .
\end{align*}
\]
This suffices to determine the first two bounds resulting from Lemma 20:

\[
\text{bound}(p = 1) = \frac{(1 - q)^2}{2(1 + q^2)}
\]

\[
\text{bound}(p = 2) = \frac{(1 - q^2)^2 (1 - \sqrt{2q - q^2})}{2 (1 + q^2)(1 + q^4)}
\]

Here the second bound turns out to be only a slight improvement over the first. In any case, both bounds go to zero as \( q \to 1 \), which is also clear from the existence of low lying magnon excitations in the undeformed model.

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