On the Mass Spectrum of the Two–dimensional $O(3)$ Sigma Model with $\theta$ Term

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Form Factor Perturbation Theory is applied to study the spectrum of the $O(3)$ non–linear sigma model with the topological term in the vicinity of $\theta = \pi$. Its effective action near this value is given by the non–integrable double Sine–Gordon model. Using previous results by Affleck and the explicit expressions of the Form Factors of the exponential operators $e^{\pm i\sqrt{\theta} \phi(x)}$, we show that the spectrum consists of a stable triplet of massive particles for all values of $\theta$ and a singlet state of higher mass. The singlet is a stable particle only in an interval of values of $\theta$ close to $\pi$ whereas it becomes a resonance below a critical value $\theta_c$.

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The $O(3)$ non–linear sigma model is a two–dimensional quantum field theory for a 3-component, unit-vector field $n_\alpha (\alpha = 1, 2, 3 ; n_\alpha^2 = 1)$, with the Euclidean action given by

$$ A\theta = \frac{1}{2f^2} \int d^2x (\partial_\mu n_\alpha)^2 + i\theta T , \quad (1) $$

where $f$ and $\theta$ are dimensionless coupling constants and

$$ T = \frac{1}{8\pi} \int d^2x \, \epsilon^{\alpha\beta\gamma} n_\alpha \partial_\mu n_\beta \partial_\nu n_\gamma \quad (2) $$

is the integer–valued topological term related to the instanton solutions of the model. This model has been the subject of a huge amount of study for its theoretical properties and for the large variety of its application to condensed matter systems [1, 2, 3, 4, 5, 6] (see also [7]). For a generic value of $\theta$, the topological term $T$ breaks both the $Z_2$ invariance $n_\alpha \rightarrow -n_\alpha$ and the parity space symmetry of the action $A\theta$, symmetries which are however restored at $\theta = \pi$. As a matter of fact, the two values $\theta = 0, \pi$ are the only ones for which the action (1) is known to be integrable. The physical properties, however, are completely different in the two cases. At $\theta = 0$, the model consists of a $O(3)$ triplet of massive particles, with an exact $S$–matrix given in [8]. At $\theta = \pi$, the theory is instead massless [9, 10, 11] and corresponds to the Renormalization Group flow between the $c = 2$ CFT and the $SU(2)_1$ Wess–Zumino–Witten (WZW) model at level 1, with central charge $c = 1$. In this case the spectrum of the excitations consists of massless particles which transform according to the $s = 1/2$ representation of $SU(2)$, the so–called spinons. The exact massless scattering amplitudes for the right and left moving doublets are given in [11].

Non–integrable Quantum Field Theories. One may wonder how the spectrum of the theory (1) evolves by moving the coupling $\theta$, in particular how the two doublets of the massless spinons at $\theta = \pi$ are transformed into the triplet of massive states at $\theta = 0$. It is of course difficult, if not impossible, to provide an exact answer to this question since the model is non–integrable for generic values of $\theta$. However, one can gain a significant insight about this question by using the Form Factor Perturbation Theory (FFPT) proposed in Refs. [11, 12]. This method allows one to study with a certain accuracy those non–integrable models obtained as deformation of an integrable quantum field theory. For the theory (1) we have two possibilities, i.e. we can either apply the FFPT in the vicinity of $\theta = 0$ or use it to analyze the non–integrable theory defined near $\theta = \pi$. For reasons that will become clear later, it is simpler to follow the evolution of the particle content starting from the value $\theta = \pi$. Let us discuss then in more details the model in the vicinity of this point.

Double Sine–Gordon Model. At $\theta = \pi$, the $O(3)$ sigma model corresponds to a massless flow from the CFT with $c = 2$ to an infrared fixed point described by the $SU(2)_1$ WZW model. In the vicinity of this point, it is appropriate to use the conformal fields of the WZW model to write an effective action of the sigma model (1). This was done in [12] and the results can be summarized as follows. Near its infrared fixed point, the action $A_\pi$ corresponds to the $SU(2)_1$ WZW model perturbed by the marginally irrelevant perturbation $(Tr g)^2$ (i.e. with $\bar{\gamma} > 0$)

$$ A_{eff}^{\pi} = A_{SU(2)_1} + \bar{\gamma} \int d^2x (Tr g)^2 , \quad (3) $$

where $g$ is the $SU(2)$ matrix field with conformal dimension $\Delta = \bar{\Delta} = \frac{4}{3}$. Beside the symmetry $SU(2)_1$, this action has also a $Z_2$ invariance related to the transformation $g \rightarrow -g$. In terms of the WZW fields, the perturbation that moves the topological term away from the value $\theta = \pi$ is proportional to $Tr g \bar{g}$. This is the only relevant field of the WZW model and, moreover, the only $SU(2)_1$ invariant operator in the theory that breaks parity. Thus, for a generic value of $\theta$ in the vicinity of $\theta = \pi$,
we have an effective action given by the following non–
integrable perturbation of (3)

\[ A^{\text{eff}} = A_{0}^{\text{eff}} + \bar{\eta} \int d^{2}x \text{Tr} \, g \ , \]

(4)

with \( \bar{\eta} \simeq |\theta - \pi| \). At \( \bar{\eta} = 0 \) the massless particles of the theorem are the spinons, which can be also viewed as the fundamental excitations of the IR point [12]. However, the operator \( \text{Tr} \, g \) is non–local with respect to them. As shown in [12] for the case of massive theories, this is the crucial property responsible for the confinement of the particles. The same also happens in the massless cases [14]. Hence, in the presence of \( \text{Tr} \, g \), i.e. as soon as we move away from the point \( \theta = \pi \), the spinons are confined and the model has no longer spin \( 1/2 \) excitations. To recover its actual spectrum near the value \( \theta = \pi \), it is convenient to write Eqs. (4) and (5) in terms of a scalar bosonic field, \( \varphi \), as:

\[ A^{\text{eff}} = \int d^{2}x \left[ \frac{1}{2} (\partial \varphi)^{2} + \eta \cos \sqrt{8\pi} \varphi \right] + \int d^{2}x \text{Tr} \, g \ , \]

(5)

This is a non–integrable quantum field theory which has been studied in details in [12]. In this model, the two periodic interactions play a symmetric role and each of the cosine term can be regarded as a deformation of the integrable theorem defined by the other [16].

Affleck’s Result. Due to the particular values of the cosine frequencies, the quantum field theory (3) presents a series of remarkable peculiarities which, as we are going to show, have far–reaching consequences on its spectrum. The first important peculiarity, noticed by Affleck [14], is the special pattern of the integrable sine–Gordon model at \( \beta^{2} = 2\pi \), obtained for \( \gamma = 0 \) in eq. (5). In fact, the spectrum of this integrable model consists of a soliton \( s \) and an anti–soliton \( \bar{s} \) of mass \( m_{s} \), which are degenerate with a breather state \( b_{1} \). Moreover, all these particles have the same \( S \)-matrix. In addition, there is another breather state \( b_{2} \) of higher mass, given by \( m_{2} = \sqrt{3} \, m \). The excitations can be then organized into a triplet \( (s, \bar{s}, b_{1}) \) of bosonic states of mass \( m_{s} \) and a singlet of mass \( m_{s} = \sqrt{3} \, m_{s} \), explicitly showing the hidden \( SU(2) \) symmetry of the model at this specific point [17, 18]. Their exact \( S \)-matrix can be found in [19] and is not given here.

The above pattern for the particles, in particular the triplet of massive bosonic states, strongly reminds the spectrum of the original \( O(3) \) sigma model at \( \theta = 0 \). However, one may wonder and even doubt whether this was just a fortunate coincidence, that would no longer persist in presence of the second interaction in the action (3). The analysis of the double sine–Gordon model shows, in fact, that an additional cosine term has generally a drastic impact on the spectrum of the sector of the unperturbed theory producing, in particular, their confinement [12]. However, this circumstance does not occur for the theory (3) and this is the second remarkable peculiarity of the action (3). To show that, we apply the Form Factor Perturbation Theory.

Form Factor Perturbation Theory. The Form Factor Perturbation Theory [11] allows one to estimate the variation of the spectrum of an integrable theory, once it has been perturbed by an additional term in the action \( \gamma \int d^{2}x \, \Psi(x) \). At first order in \( \gamma \) one has

\[ \delta m_{s}^{2} \simeq 2 \gamma F_{ii}^{\Psi}(i\pi) \ , \]

(6)

where \( F_{ii}^{\Psi}(\lambda_{1} - \lambda_{2}) = \langle 0 \left| \Psi(0) \right| A_{i}(\lambda_{1}) A_{i}(\lambda_{2}) \rangle \) is the two–particle Form Factor of the operator \( \Psi(x) \) as a function of their rapidities parameterizing the dispersion relation \( E_{i} = m_{i} \cosh \lambda_{i}, \quad p_{i} = m_{i} \sinh \lambda_{i} \). For a generic sine–Gordon model

\[ \mathcal{A} = \int d^{2}x \left[ \frac{1}{2} (\partial \varphi)^{2} + g \cos \beta \varphi \right] \ , \]

(7)

perturbed by another cosine term \( \Psi(x) = \cos \alpha \varphi(x) \), the evaluation of (6) may be however problematic. As shown in [12], the matrix element of \( \cos \alpha \varphi(x) \) on the soliton states has in general a pole at \( \lambda = i\pi \), with a residue ruled by the non–locality index of this operator with respect to the soliton (The presence of this pole signals the confinement of the soliton states in the perturbed theory.). Explicitly

\[ - i \Re s_{\lambda = i\pi} F_{ss}^{\Psi}(\lambda) = [1 - \cos(2\pi \alpha / \beta)] \langle 0 \left| \cos \alpha \varphi(0) \right| 0 \rangle \ . \]

(8)

However, for the double sine–Gordon [10], considered as a deformation of the sine–Gordon model (7) with \( \beta = \sqrt{2\pi} \), we have \( \Psi(x) = \cos \sqrt{8\pi} \varphi(x) \), i.e. \( \alpha / \beta = 2 \), and the matrix element \( F_{ss}^{\Psi}(i\pi) \) is instead finite! Moreover, since this Form Factor is determined by the \( S \)-matrix (which is the same for all the particles of the triplet), we have that all of them get the same mass correction. In other words, the initial triplet identified by Affleck in the theory (3) at \( \gamma = 0 \) is going to stay degenerate even at \( \gamma \neq 0 \), a result which can be proved to hold at any order in the FFPT. It remains, then, to compute its actual correction and to compare it with the mass correction of the second breather. Here we will only present the basic results of this calculation while their complete derivation and the relative discussion will be presented somewhere else [14].

The two–particle form factors of the field \( \Psi(x) = \cos \sqrt{8\pi} \varphi(x) \) on the particles of the triplet and on the higher breather \( b_{2} \), can be computed by an analytic continuation of the matrix elements of the cluster operators of the Sinh–Gordon model [10] (see also [20]). They can be written, up to their vacuum normalization, as
The particles are those which become the triplet of the $O(3)$ sigma model at $\theta = 0$, and their mass should become unbounded moving toward this value, causing the complete decoupling of this particle from the theory. It is then easy to argue, by continuity, that this state corresponds to a stable particle of the theory only in an interval of $\theta$ near $\theta = \pi$. It becomes instead a resonance below a certain critical value $\theta_c$, i.e. above $\gamma_c$ determined by the threshold condition $M_s(\gamma_c) \geq 2M_t(\gamma_c)$. Notice, in particular, that at first order in $\gamma$ this equation does not have solution for $\gamma > 0$, i.e. the singlet is still a stable particle of the theory. The above considerations suggest that the masses should have, qualitatively, the behavior given in Figure 2, with their cusp $M_s(\pi - \theta)^{2/3}$ at $\theta = \pi$ dictated by the anomalous dimension of the operator $\text{Tr} g$ (logarithmic corrections to the power law behavior were considered in Ref. [21]).
Conclusions. The FFPT allowed us to gain new insights on the spectrum of the $O(3)$ non-linear sigma model with $\theta$ term in the vicinity of $\theta = \pi$. As soon as $\theta$ is moved away from this value, the spinons are confined as a consequence of the non-local properties of the associated perturbed operator. The spectrum can be obtained by analyzing the effective action of the model near its $SU(2)_1$ fixed point, given by the double Sine–Gordon model \(^4\). The pattern of triplet massive states identified by looking at the integrable model for $\gamma = 0$ turns out to be robust even in the non–integrable field theory with $\gamma \neq 0$. The singlet state, on the contrary, belongs to the stable part of the spectrum only in an interval of values of $\theta$ close to $\pi$, whereas it becomes a resonance below a critical value $\theta_c$. It should not be difficult to confirm these predictions by a numerical study of the model and to determine correspondingly the critical value $\theta_c$.

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