Optimal Regularity for $\overline{\partial}b$ on CR manifolds
by Moulay Youssef Barkatou

Abstract. In this paper a new explicit integral formula is derived for solutions of the tangential Cauchy-Riemann equations on CR $q$-concave manifolds and optimal estimates in the Lipschitz norms are obtained.

0. Introduction

The aim of this paper is to prove the following theorem:

THEOREM 0.1 Let $M$ be a $q$-concave CR generic submanifold (cf. sect 1.2) of codimension $k$ and of class $C^{2+\ell}$ (resp. $C^{3+\ell}$) in $\mathbb{C}^n$ ($\ell \geq 0$) and $z_0$ a point in $M$. Then there exist an open neighborhood $M_0 \subseteq M$ of $z_0$ and kernels $R_r(\zeta, z)$, for $r = 0, \ldots, q-1, n-k-q, \ldots, n-k$, with the following properties,

(i) For every domain $\Omega \subset M_0$ with piecewise $C^1$ boundary and every $C^1(0, r)$-form $f$ on $\Omega$ ($0 \leq r \leq q-1$ or $n-k-q+1 \leq r \leq n-k$), we have

$$f = \overline{\partial}b \int_{\Omega} f \wedge R_{r-1} - \int_{\Omega} \overline{\partial}b f \wedge R_r + \int_{\partial \Omega} f \wedge R_r$$

on $\Omega$.

(ii) For every open set $\Omega \subset M_0$ the integral operator $\int_{\Omega} \cdot \wedge R_r$ is a bounded linear operator from $C^{2+\ell}_0(\Omega) \cap L^\infty(\Omega)$ to $C^{2+\ell}_{0,r+1}(\Omega)$ for $r \geq n-k-q$ (resp. $r \leq q-1$).

Theorem 0.1 has the following corollary

COROLLARY 0.2 Let $M$ be a $1$-concave CR generic $C^{3+\ell}$-submanifold of a complex manifold. Let $T$ be a distribution of order $\ell$ on $M$. If $\overline{\partial}b T$ is defined by a $C^{\ell}(0,1)$-form on $M$ then $T$ is defined by a $C^{\ell+\frac{1}{2}}$-function on $M$.

For a proof of corollary 0.2, we refer to the proof of theorem 4.1.6 in [5]. The interest of this Corollary lies in the fact that under the hypothesis of 1-concavity the tangential Cauchy-Riemann equation for $(0,1)$-currents cannot be solved locally (see [3]).

Theorem 0.1 and Corollary 0.2 essentially improve the results of Airapetjan and Henkin [12], [1], [2] and also of the author in [5] where homotopy formulas were obtained with less explicit kernels giving almost optimal but not optimal estimates.

The study of the tangential Cauchy-Riemann equations with the use of explicit integral formulas was initiated by Henkin [12]. For further references and results on CR manifolds we refer the reader to the survey of Henkin [13], the memoir of Trèves [23] and the book of Boggess [2].

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It is known that a fundamental solution for the $\overline{\partial}_b$ operator on certain hypersurfaces (see Henkin [14], Harvey-Polking [13], Boggess-Shaw [8], Fischer-Leiterer [10]) can be constructed as the jump of two kernels, obtained by applying to the usual Bochner-Martinelli-Koppelman kernel (BMK kernel) in $\mathbb{C}^n$, a solution operator for $\overline{\partial}$, once on the left and once on the right hand side of the hypersurface.

Solutions for such equations can be given by applying the generalized Koppelman (cf. section 1.3) to the BMK section and the barrier functions (cf. section 1.4) of the hypersurface as was done in [14], [11], [8] and [7] or by using a homotopy operator for $\overline{\partial}$ of Grauert-Lieb-Henkin type as was achieved in [10].

Inspired by the definition of a hyperfunction of several variables, the present author generalized in [3] the construction of Fischer-Leiterer [10] to higher codimensional CR submanifolds by solving with estimates up to the boundary some $\overline{\partial}$ equations on certain wedges attached to such manifolds with the use of $\overline{\partial}$ homotopy operators from [17] and [18].

In this paper we shall show that such equations can also be solved up to some error terms by using the Koppelman lemma (see (2.2)) and the key idea in this work is to "deform" via this lemma those terms into ones with vanishing coefficients for some bidegrees (see lemmas 2.2 and 2.3), the strict $q$-convexity plays here an important role.

We shall give two fundamental solutions to the tangential Cauchy-Riemann complex. The first one (cf. sect 2.1) does not yield sharp estimates for the solutions of $\overline{\partial}_b$ (when $k > 1$) but is a "necessary" step to construct the second one (cf. sect 2.2) corresponding to kernels $R_r$. To derive the latter fundamental solution from the former, we shall use an idea of Henkin [14].

In [9] B.Fischer proved Theorem 0.1 and Corollary 0.2 for hypersurfaces by using a version of the first fundamental solution which was suggested to him by I.Lieb and J.Michel.

Recently, Polyakov [21] proved sharp estimates for global solutions of $\overline{\partial}_b$ on $q$-concave CR manifolds, in Lipschitz spaces of Stein [22].

**Polyakov’s theorem.** Let $M$ be a $q$-concave CR generic $C^4$-submanifold in $\mathbb{C}^n$ with $q \geq 2$ and let $M'$ be a relatively compact open subset of $M$. Then for any $r = 1, \ldots, q-1$ there exist linear operators

$$R_r : L^s_{(0,r)}(M) \to \Gamma^{s,1}_{(0,r-1)}(M) \quad \text{and} \quad H_r : L^s_{(0,r)}(M) \to L^s_{(0,r)}(M)$$

such that for any $s \in [1, \infty]$ $R_r$ is bounded and $H_r$ is compact and such that for any differential form $f \in C^\infty_{(0,r)}(M)$ the following equality:

$$f(z) = \overline{\partial}_b R_r(f)(z) + R_{r+1}(\overline{\partial}_b f)(z) + H_r(f)(z)$$

holds for $z \in M'$.

Our method is quite different from that of Polyakov, and it is not clear how one can get an analogous result to Corollary 0.2 from Polyakov’s theorem.

This paper is organized as follows. In section 1.2 we give the definition of a $q$-concave CR manifold and we define the $\overline{\partial}_b$ operator. In section 1.3 we recall the generalized
Koppelman lemma which plays a key role in the construction of our kernels. In section 1.4 we recall the construction of a barrier function and a Leray map for a hypersurface at a point where the Levi form has some positive eigenvalues. In section 1.5 we state some elementary facts from Algebraic Topology, which we shall use later. Section 2.1 is devoted to the construction of our first fundamental solution. In section 2.2 we construct our second fundamental solution and in section 3 we prove estimates for our kernels.

1. Preliminaries and notations

1.1. Let $X$ be a complex manifold, and $M$ a real submanifold of $X$.

Let $f$ be a differential form of degree $m$ defined on a domain $D \subseteq M$. Then we denote by $\|f(z)\|_{z \in D}$, the Riemannian norm of $f$ at $z$ (cf. [16], section 0.4), and we set

$$\|f\|_{0,D} = \sup_{z \in D} \|f(z)\|$$

and

$$\|f\|_{\alpha,D} = \|f\|_{0,D} + \sup_{z \neq \zeta, z, \zeta \in D} \frac{\|f(z) - f(\zeta)\|}{|\zeta - z|^\alpha}$$

for $0 < \alpha < 1$.

If $0 < \alpha < 1$, then a form on $D$ is called $\alpha$-Hölder continuous on $D$ if

$$\|f\|_{\alpha,K} \leq \infty.$$

for all compact sets $K \subseteq D$.

If $\ell$ is a non-negative integer and $0 < \alpha < 1$, then we say $f$ is a $C^{\ell+\alpha}$ form on $D$ if $f$ is of class $C^{\ell}$ and all derivatives of order $\leq \ell$ of $f$ are $\alpha$-Hölder continuous on $D$. $L^\infty(D)$ denotes the space of all bounded forms on $D$.

Throughout this paper $C$ will denote a positive constant which is independent of the variables and the functions. The constant $C$ used in different places may have different values there.

1.2. Let $M$ be a real submanifold of class $C^2$ in $\mathbb{C}^n$ defined by

$$M = \{z \in \Omega; \rho_1(z) = \cdots = \rho_k(z) = 0\} \quad 1 \leq k \leq n \quad (1.1)$$

where $\Omega$ is an open subset of $\mathbb{C}^n$ and the functions $\rho_\nu$, $1 \leq \nu \leq k$, are real-valued functions of class $C^2$ on $\Omega$ with the property $d\rho_1(z) \wedge \cdots \wedge d\rho_k(z) \neq 0$ for each $z \in M$.

We denote by $T_z^\mathbb{C}(M)$ the complex tangent space to $M$ at the point $z \in M$ i.e.,

$$T_z^\mathbb{C}(M) = \{\zeta \in \mathbb{C}^n / \sum_{j=1}^n \frac{\partial \rho_\nu}{\partial z_j}(z)\zeta_j = 0, \nu = 1, \ldots, k\}.$$
We have \( \dim \mathcal{T}_z^k(M) \geq n - k \). The submanifold \( M \) is called a Cauchy-Riemann manifold (CR-manifold) if the number \( \dim \mathcal{T}_z^k(M) \) does not depend on the point \( z \in M \). \( M \) is said to be CR generic if for every \( z \in M \), \( \dim \mathcal{T}_z^k(M) = n - k \); this is equivalent to:

\[
\overline{\partial} \rho_1 \wedge \overline{\partial} \rho_2 \wedge \cdots \wedge \overline{\partial} \rho_k \neq 0 \text{ on } M. \tag{1.2}
\]

If \( M \) is CR generic, we call \( M \) \( q \)-concave, \( 0 \leq q \leq \frac{n - k}{2} \), if for each \( z \in M \) and every \( x \in \mathbb{R}^k \setminus \{0\} \) the following hermitian form on \( T_z^k(M) \)

\[
\sum_{\alpha, \beta} \frac{\partial^2 \rho_x}{\partial z_\alpha \partial \bar{z}_\beta} (z) \zeta_\alpha \bar{\zeta}_\beta, \text{ where } \rho_x = x_1 \rho_1 + \cdots x_k \rho_k
\]

has at least \( q \) negative eigenvalues.

If \( M \) is CR generic then we denote by \( \mathcal{C}^s_{p,r}(M) \) the space of differential forms of type \( (p,r) \) on \( M \) which are of class \( C^s \). Here, two forms \( f \) and \( g \) in \( \mathcal{C}^s_{p,r}(M) \) are considered to be equal if and only if for each form \( \varphi \in \mathcal{C}^\infty_{n-p,n-k-r}(\Omega) \) of compact support, we have

\[
\int_M f \wedge \varphi = \int_M g \wedge \varphi.
\]

We denote by \( \mathcal{L}^{(s)}_{p,r}(M) \) the dual space to \( \mathcal{C}^s_{n-p,n-k-r}(M) \).

We define the tangential Cauchy-Riemann operator on forms in \( \mathcal{L}^{(s)}_{0,r}(M) \) as follows. If \( u \in \mathcal{C}^s_{0,r}(M) \), \( s \geq 1 \), then \( u \) can be extended to a smooth form \( \tilde{u} \in \mathcal{C}^s_0(\Omega) \) and we may set

\[
\overline{\partial}_h u := \overline{\partial} \tilde{u} |_M
\]

It follows from the condition for equality of forms on \( M \) that this definition does not depend on the choice of the extended form \( \tilde{u} \). In general, for for forms \( u \in \mathcal{L}^{(s)}_{0,r-1}(M) \) and \( f \in \mathcal{L}^{(s)}_{0,r}(M) \), by definition,

\[
\overline{\partial}_h u = f
\]

will mean that for each form \( \varphi \in \mathcal{C}^\infty_{n-p,n-k-r}(\Omega) \) of compact support we have

\[
\int_M f \wedge \varphi = (-1)^r \int_M u \wedge \overline{\partial} \varphi.
\]

1.3. The generalized Koppelman lemma. In this section we recall a formal identity (the generalized Koppelman lemma) which is essential for the construction of our kernels. The exterior calculus we use here was developed by Harvey and Polking in [17].

Let \( V \) be an open set of \( \mathbb{C}^n \times \mathbb{C}^n \). Suppose \( G : V \to \mathbb{C}^n \) is a \( C^1 \) map. We write

\[
G(\zeta, z) = (g_1(\zeta, z), \ldots, g_n(\zeta, z))
\]

and we use the following notation

\[
G(\zeta, z)(\zeta - z) = \sum_{j=1}^n g_j(\zeta, z)(\zeta_j - z_j)
\]
\[ G(\zeta, z).d(\zeta - z) = \sum_{j=1}^{n} g_j(\zeta, z)d(\zeta_j - z_j) \]

\[ \overline{\partial}_{\zeta, z} G(\zeta, z).d(\zeta - z) = \sum_{j=1}^{n} \overline{\partial}_{\zeta, z} g_j(\zeta, z)d(\zeta_j - z_j) \]

where \( \overline{\partial}_{\zeta, z} = \overline{\partial}_{\zeta} + \overline{\partial}_{z} \).

We define the Cauchy-Fantappie form \( \omega^G \) by

\[ \omega^G = \frac{G(\zeta, z).d(\zeta - z)}{G(\zeta, z).d(\zeta - z)} \]

on the set where \( G(\zeta, z), (\zeta - z) \neq 0 \).

Given \( m \) such maps, \( G^j, 1 \leq j \leq m \), we define the kernel

\[ \Omega(G^1, \ldots, G^m) = \omega^{G^1} \wedge \ldots \wedge \omega^{G^m} \wedge \sum_{\alpha_1 + \ldots + \alpha_m = n-m} (\overline{\partial}_{\zeta, z} \omega^{G^1})^{\alpha_1} \wedge \ldots \wedge (\overline{\partial}_{\zeta, z} \omega^{G^m})^{\alpha_m} \]

on the set where all the denominators are nonzero.

**Lemme 1.1.** (The generalized Koppelmann lemma)

\[ \overline{\partial}_{\zeta, z} \Omega(G^1, \ldots, G^m) = \sum_{j=1}^{m} (-1)^j \Omega(G^1, \ldots, \hat{G}^j, \ldots, G^m) \]

on the set where the denominators are nonzero, the symbol \( \hat{G}^j \) means that the term \( G^j \) is deleted.

The following lemma is useful for the estimation of the kernel defined above.

**Lemme 1.2.** For \( k \geq 0 \)

\[ \omega^G \wedge (\overline{\partial}_{\zeta, z} \omega^{G^j})^k = \frac{G(\zeta, z).d(\zeta - z)}{G(\zeta, z).d(\zeta - z)}} \wedge \left( \frac{\overline{\partial}_{\zeta, z} G.d(\zeta - z)}{G(\zeta, z).d(\zeta - z)}} \right)^k. \]

For a proof of these two lemmas we refer the reader to [11] or [9].

**Remark.** When \( G = \overline{\zeta - z} \), we see from Lemma 1.2 that \( \Omega(G) \) is the classical Martinelli-Bochner Koppelman kernel in \( \mathbb{C}^n \).
1.4. Barrier function. In this section, we shall construct a barrier function for a hypersurface at a point where the Levi form has some positive eigenvalues.

For a detailed proof of what follows we refer the reader to sect. 3 in [17].

Let $H$ be an oriented real hypersurface of class $C^2$ in $\mathbb{C}^n$ defined by

$$H = \{ z \in \Omega ; \rho(z) = 0 \}$$

where $\Omega$ is an open subset of $\mathbb{C}^n$ and $\rho$ is a real-valued function of class $C^2$ on $\Omega$ with $d \rho(z) \neq 0$ for each $z \in H$.

Denote by $F(\cdot, \zeta)$ the Levi polynomial of $\rho$ at a point $\zeta \in \Omega$, i.e.

$$F(z, \zeta) = 2 \sum_{j=1}^{n} \frac{\partial \rho(\zeta_j - z_j)}{\partial \zeta_j} (\zeta_j - z_j) - \sum_{j,k=1}^{n} \frac{\partial^2 \rho(\zeta)}{\partial \zeta_j \partial \zeta_k} (\zeta_j - z_j)(\zeta_k - z_k)$$

$\zeta \in \Omega, z \in \mathbb{C}^n$.

Let $z^0 \in H$ and $T$ be the largest vector subspace of $\mathbb{C}^n$ such that the Levi form of $\rho$ at $z^0$ is positive definite on $T$. Set $\dim T = d$ and suppose $d \geq 1$.

Denote by $P$ the orthogonal projection from $\mathbb{C}^n$ onto $T$, and set $Q = I - P$. Then it follows from Taylor’s theorem that there exist a number $R$ and two positive constants $A$ and $\alpha$ such that the following holds:

$$\text{Re} \ F(z, \zeta) \geq \rho(\zeta) - \rho(z) + \alpha |\zeta - z|^2 - A |Q(\zeta - z)|^2$$

for $|z^0 - \zeta| \leq R$ and $|z^0 - z| \leq R$.

Since $\rho$ is of class $C^2$ on $\Omega$, We can find $C^\infty$ functions $a^{kj}(k, j = 1, \ldots, n)$ on a neighborhood $U$ of $z^0$ such that

$$\left| a^{kj}(\zeta) - \frac{\partial^2 \rho(\zeta)}{\partial \zeta_k \partial \zeta_j} \right| \leq \frac{\alpha}{2n^2}$$

for all $\zeta \in U$. And then we have

$$\left| \sum_{k,j=1}^{n} \left( a^{kj}(\zeta) - \frac{\partial^2 \rho(\zeta)}{\partial \zeta_k \partial \zeta_j} \right) t_k t_j \right| \leq \frac{\alpha}{2} |t|^2$$

for all $\zeta \in U$ and $t \in \mathbb{C}^n$. Set

$$\bar{F}(\zeta, z) = 2 \sum_{j=1}^{n} \frac{\partial \rho(\zeta_j - z_j)}{\partial \zeta_j} (\zeta_j - z_j) - \sum_{k,j=1}^{n} a^{kj}(\zeta)(\zeta_k - z_k)(\zeta_j - z_j)$$

for $(z, \zeta) \in \mathbb{C}^n \times U$. Then it follows from (1.3) that

$$\text{Re} \ \bar{F}(\zeta, z) \geq \rho(\zeta) - \rho(z) + \frac{\alpha}{2} |\zeta - z|^2 - A |Q(\zeta - z)|^2$$

for $|z^0 - \zeta| \leq R$ and $|z^0 - z| \leq R$.

Denote by $Q_{kj}$ the entries of the matrix $Q$ i.e

$$Q = (Q_{kj})_{k,j=1}^{n} \quad (k = \text{column index}).$$
We set for \((z, \zeta) \in \mathbb{C}^n \times U\)
\[
\begin{align*}
g_j(\zeta, z) &= 2 \frac{\partial \rho(\zeta)}{\partial \zeta_j} - \sum_{k=1}^n a^{kj}(\zeta)(\zeta_k - z_k) + A \sum_{k=1}^n Q^{kj}(\zeta_k - z_k) \\
G(\zeta, z) &= (g_1(\zeta, z), \ldots, g_n(\zeta, z)) \\
\Phi(\zeta, z) &= G(\zeta, z). (\zeta - z).
\end{align*}
\]
Since \(Q\) is an orthogonal projection, then we have
\[
\Phi(\zeta, z) = \tilde{F}(\zeta, z) + A |Q(\zeta - z)|^2
\]
hence it follows from (1.4) that
\[
\text{Re} \Phi(\zeta, z) \geq \rho(\zeta) - \rho(z) + \alpha \frac{|\zeta - z|^2}{2}
\]
for \(|z^0 - \zeta| \leq R\) and \(|z^0 - z| \leq R\).

\(G\) is called a \textbf{Leray map} and \(\Phi\) is called a \textbf{barrier function} of \(H\) (or \(\rho\)) at \(z^0\).

**Definition.** A map \(f\) defined on some complex manifold \(X\) will be called \(k\)-holomorphic if, for each point \(\xi \in X\), there exist holomorphic coordinates \(h_1, \ldots, h_k\) in a neighborhood of \(\xi\) such that \(f\) is holomorphic with respect to \(h_1, \ldots, h_k\).

**Lemma 1.3.** For every fixed \(\zeta \in U\), the map \(G(\zeta, z)\) and the function \(\Phi\), defined above, are \(d\)-holomorphic in \(z \in \mathbb{C}^n\).

**Proof.** Choose complex linear coordinates \(h_1, \ldots, h_n\) on \(\mathbb{C}^n\) with
\[
\{z \in \mathbb{C}^n : Q(z) = 0\} = \{z \in \mathbb{C}^n : h_{d+1}(z) = \cdots = h_n(z) = 0\}.
\]
Then the map \(\mathbb{C}^n \ni z \rightarrow Q(\zeta - z)\) is independent of \(h_1, \ldots, h_d\). This implies that \(G(\zeta, \cdot)\) is complex linear with respect to \(h_1, \ldots, h_d\), and \(\Phi(\zeta, \cdot)\) is quadratic complex polynomial with respect to \(h_1, \ldots, h_d\).

**1.5. Some Algebraic Topology.** Let \(N\) be a positive integer. Then we call \(p\)-simplex, \(1 \leq p \leq N\), every collection of \(p\) linearly independent vectors in \(\mathbb{R}^N\).

We define \(S_p\) as the set of all finite formal linear combinations, with integer coefficients, of \(p\)-simplices.

Let \(\sigma = [a_1, \ldots, a_p]\) be a \(p\)-simplex, then we set
\[
\partial_j \sigma = [a_1, \ldots, \hat{a}_j, \ldots, a_p]
\]
for \(1 \leq j \leq p\) and
\[
\partial \sigma = \sum_{j=1}^p (-1)^j \partial_j \sigma
\]
(this definition holds also for any collection of \(p\) vectors). If \(1 \leq j_1 \leq p \ldots 1 \leq j_r \leq p - r\), we define
\[
\partial_{j_r \ldots j_1} \sigma = \partial_{j_r} (\partial_{j_{r-1} \ldots j_1} \sigma)
\]
where $\partial^1_j \sigma = \partial_j \sigma$.

All of these operations can be extended by linearity to $S_p$.

If $\sigma$ is a $p$-simplex defined as above then we define the barycenter of $\sigma$ by

$$b(\sigma) = \frac{1}{p} \sum_{j=1}^{p} a_j.$$ 

Now we define the first barycentric subdivision of $\sigma$ by the following

$$sd(\sigma) = (-1)^{p+1} \sum_{j_1, \ldots, j_p} (-1)^{j_1 + \cdots + j_p} \left[ b(\sigma), b(\partial_{j_1} \sigma), \ldots, b(\partial_{j_p} \sigma) \right].$$

By linearity we can also define the first barycentric Subdivision of any element of $S_p$.

It is easy to see that

**Lemma 1.4.** If $\sigma$ is an element of $S_p$, then

$$sd(\partial \sigma) = \partial sd(\sigma).$$

The barycentric subdivision of higher order of an element $\sigma$ of $S_p$ is defined as follows, we set for $m \geq 2$

$$sd^m(\sigma) = sd(sd^{m-1}(\sigma)).$$

$sd^0(\sigma)$ and $sd^1(\sigma)$ are defined respectively as $\sigma$ and $sd(\sigma)$.

The following lemma is basic in Algebraic Topology.

**Lemma 1.5.** Given a simplex $\sigma$, and given $\epsilon > 0$, there is an $m$ such that each simplex of $sd^m \sigma$ has diameter less than $\epsilon$.

For a proof of this lemma, see for example [21].

Let $\sigma = [\nu_1, \ldots, \nu_p]$ and $\tau = [\mu_1, \ldots, \mu_r]$. We shall adopt the following notations

$$[\sigma, \tau] = [\sigma, \mu_1, \ldots, \mu_r] = [\nu_1, \ldots, \nu_p, \tau] = [\nu_1, \ldots, \nu_p, \mu_1, \ldots, \mu_r].$$

Now let $\sigma$ be a $p$-simplex, $p \geq 2$, set

$$T(\sigma) = \left[ b(\sigma), \sigma \right] + \sum_{\ell=1}^{p-2} \sum_{j_1, \ldots, j_\ell} (-1)^{j_1 + \cdots + j_\ell} \left[ b(\sigma), b(\partial_{j_1} \sigma), \ldots, b(\partial_{j_\ell} \sigma), \partial_{j_1, \ldots, j_\ell} \sigma \right]$$

and extend $T$ by linearity to $S_p$.

If $\tau$ is an element of $S_1$ then we set

$$T(\tau) = 0$$
Proposition 1.6 If \( \sigma \) is an element of \( S_p \), \( p \geq 2 \), then
\[
\partial T(\sigma) + T(\partial \sigma) = sd(\sigma) - \sigma.
\]

This proposition follows by a straightforward computation.

2. Fundamental solutions for \( \overline{\partial}_b \)

In this section, we shall construct two fundamental solutions for the tangential Cauchy-Riemann Complex. The second solution will be derived from the first and will yield optimal Hölder estimates for \( \overline{\partial}_b \).

Let us begin by some notations.

2.0. Notations. Throughout this section \( M \) will denote a \( q \)-concave CR generic \( C^2 \) submanifold of codimension \( k \) in \( \mathbb{C}^n \).

\( \mathcal{I} \) is the set of all subsets \( I \subseteq \{ \pm 1, \ldots, \pm k \} \) such that \(|i| \neq |j|\) for all \( i, j \in I \) with \( i \neq j \).

For \( I \in \mathcal{I} \), \( |I| \) denotes the number of elements in \( I \).

We set
\[
\Delta_{1, \ldots, |I|} = \{(\lambda_1, \ldots, \lambda_{|I|}) \in (\mathbb{R}^+)^{|I|} \mid \sum_{j=1}^{|I|} \lambda_j = 1\}
\]

\( \mathcal{I}(\ell), 1 \leq \ell \leq k \), is the set of all \( I \in \mathcal{I} \) with \(|I| = \ell \).

\( \mathcal{T}(\ell), 1 \leq \ell \leq k \), is the set of all \( I \in \mathcal{I}(\ell) \) of the form \( I = \{j_1, \ldots, j_\ell\} \) with \( |j_\nu| = \nu \) for \( \nu = 1, \ldots, \ell \).

If \( I \in \mathcal{I} \) and \( \nu \in \{1, \ldots, |I|\} \), then \( I_\nu \) is the element with number \( \nu \) in \( I \) after ordering \( I \) by modulus. We set \( I(\nu) = I \setminus \{I_\nu\} \).

If \( I \in \mathcal{I} \), then
\[
\text{sgn} I := \begin{cases} 
1 & \text{if the number of negative elements in } I \text{ is even} \\
-1 & \text{if the number of negative elements in } I \text{ is odd}
\end{cases}
\]

2.1. First fundamental solution for \( \overline{\partial}_b \). In this section we shall construct our first fundamental solution for the tangential Cauchy-Riemann complex.

Let \( z^0 \in M \), \( U \subseteq \mathbb{C}^n \) be a neighborhood of \( z^0 \) and \( \hat{\rho}_1, \ldots, \hat{\rho}_k : U \to \mathbb{R} \) be functions of class \( C^2 \) such that
\[
M \cap U = \{\hat{\rho}_1 = \cdots = \hat{\rho}_k = 0\} \text{ and } \partial \hat{\rho}_1(z^0) \wedge \cdots \wedge \partial \hat{\rho}_k(z^0) \neq 0.
\]

Since \( M \) is \( q \)-concave, it follows from lemma 3.1.1 in \[1\] that we can find a constant \( C > 0 \) such that the functions
\[
\rho_j := \hat{\rho}_j + C \sum_{\nu=1}^{k} \hat{\rho}_\nu^2 \quad (j = 1, \ldots, k)
\]
\[
\rho_j := -\hat{\rho}_j + C \sum_{\nu=1}^{k} \hat{\rho}_\nu^2 \quad (j = -1, \ldots, -k)
\]
have the following property: for each \( I \in \mathcal{I} \) and every \( \lambda \in \Delta_{I} \) the Levi form of \( \lambda_1 \rho_{I_1} + \cdots + \lambda_{|I|} \rho_{I_{|I|}} \) at \( z^0 \) has at least \( q + k \) positive eigenvalues.

Let \((e_1, \ldots, e_k)\) be the canonical basis of \( \mathbb{R}^k \), set \( e_{-j} := -e_j \) for every \( 1 \leq j \leq k \).

Let \((j_1, \ldots, j_k)\) be in \( \mathcal{I}'(k) \), set

\[
\Delta_I = \left\{ \sum_{i=1}^{k} \lambda_i e_{j_i}, \text{ with } \lambda_i \geq 0, \text{ all } i, \text{ and } \sum_{i=1}^{k} \lambda_i = 1 \right\},
\]

and for each \( a = \sum_{i=1}^{k} \lambda_i e_{j_i} \), let \( G_a \) and \( \Phi_a \) be respectively the Leray map and the barrier function at \( z_0 \) corresponding to \( \rho_a = \lambda_1 \rho_{j_1} + \cdots + \lambda_k \rho_{j_k} \) (see sect. 1.4).

We call \( \rho_a \) (resp. \( \phi_a \)) the defining function (resp. the barrier function) of \( M \) in the direction \( a \).

Let \( \sigma = [a_1, \ldots, a_p] \), \( p \geq 1 \), be a collection of \( p \) vectors, where \( a_i \in \bigcup_{I \in \mathcal{I}'(k)} \Delta_I \), for every \( 1 \leq i \leq k \).

Then we define

\[
\tilde{\Omega}[\sigma] := \Omega(G_{a_1}, \ldots, G_{a_p})
\]

(cf sect 1.3), and for every \( 0 \leq s \leq n \) and every \( 0 \leq r \leq n - p \), we define \( \tilde{\Omega}_{s,r}[\sigma] \) as the piece of \( \tilde{\Omega}[\sigma] \) which is of type \((s, r)\) in \( z \).

If we denote by \( S'_p \) the set of all finite formal linear combinations of such collections, with integer coefficients, and we extend \( \tilde{\Omega} \) by linearity to \( S'_p \); then the generalized Koppelman lemma implies

**Lemma 2.1** For every \( \tau \in S'_p \), we have

\[
\partial_z \tilde{\Omega}[\tau] = \tilde{\Omega}[\partial \tau]
\]

outside the singularities.

Let \( \sigma = [e_{j_1}, \ldots, e_{j_l}] \) be in \( \mathcal{I}'(l) \), \( 1 \leq l \leq k \) and \( \sigma_I = [e_{j_1}, \ldots, e_{j_l}] \). Then by continuity of the Levi form, by lemma 1.3 and lemma 1.5, we can find a positive integer \( m \) independant of \( I \) and \( l \) such that for every simplexe \( \tau = [a^1, \ldots, a^l] \) in \( sd^m(\sigma_I) \), the Leray maps of \( G_{a^1}, \ldots, G_{a^l} \) are \( q + k \)-holomorphic in the same directions with respect to the variable \( z \in \mathbb{C}^n \). Therefore we have the following lemma.

**Lemma 2.2** There is a positive integer \( m \) such that for every \( I \in \mathcal{I}'(l) \), \( 1 \leq l \leq k \), any \( s \geq 0 \) and every \( r \geq n - k - q + 1 \)

\[
(i) \quad \tilde{\Omega}_{s,r}(sd^m(\sigma_I)) = 0
\]

\[
(ii) \quad \partial_z \tilde{\Omega}_{s,r-1}(sd^m(\sigma_I)) = 0
\]

on the set where all the denominators are non-zero.

**Proof.** this follows by linearity from the fact that \( \Omega(G_{a^1}, \ldots, G_{a^l}) = 0 \), for any \([a^1, \ldots, a^l]\) in \( sd^m(\sigma_I) \). The last statement is easy to prove, looking at the definition of \( \Omega \) (see sect. 1.3), because \( G_{a^1}, \ldots, G_{a^l} \) are \( q + k \)-holomorphic in the same directions with respect to the variable \( z \in \mathbb{C}^n \).
By the same arguments, we have $\hat{\Omega}_{s,r}(\partial (sd^n(\sigma_I))) = \hat{\Omega}_{s,r}(sd^n(\partial \sigma_I)) = 0$, for all $r \geq n - k - q + 1$, and from lemma 2.1, we have

$$\tilde{\partial}_{z} \hat{\Omega}_{s,r-1}(sd^n(\sigma_I)) = -\tilde{\partial}_{z} \hat{\Omega}_{s,r}(sd^n(\sigma_I)) + \hat{\Omega}_{s,r}(\partial (sd^n(\sigma_I)))$$

which implies the statement (ii). \qed

Now let $D$ be a neighborhood of $z^0$ such that for every $1 \leq l \leq k$, all $0 \leq i \leq m$ and every vertex $a$ in $sd^l(\sigma_I)$, the barrier function $\Phi_a$ satisfies an inequality such (1.5), for $\zeta, z \in D$. Set

$$M_0 := M \cap D,$$

and for $I \in \mathcal{I}$

$$D_I := \{\rho_{I_1} < 0\} \cap \cdots \cap \{\rho_{I_{|I|}} < 0\} \cap D,$$

$$D_I^+ := \{\rho_{I_1} > 0\} \cap \cdots \cap \{\rho_{I_{|I|}} > 0\} \cap D,$$

$$S_I := \{\rho_{I_1} = \cdots = \rho_{I_{|I|}} = 0\} \cap D,$$

$$S^+_I := D^+_I \cap \mathcal{D}^+_I \text{ for } I \in \mathcal{I} \text{ and } |I| \geq 2.$$

We oriente these manifolds as follows:

- $D_I$ and $D_I^+$ as $\mathbb{C}^n$ for all $I \in \mathcal{I}$
- $S^+_I$ as $D^+_I$ for $j = \pm 1, \ldots, \pm k$
- $S_I$ as $\partial S^+_I$ for all $I \in \mathcal{I}$ such that $|I| \geq 2$
- $M_0$ as $S_I$ where $I = \{1, \ldots, k\}$.

Fix $1 \leq l \leq k$ and $I \in \mathcal{I}(l)$. Let $B = (\zeta_1 - z_1, \ldots, \zeta_n - z_n)$ and define

$$\hat{\Omega}_B[\tau] := \Omega(B, G_{\nu^1}, \ldots, G_{\nu^p}) \quad (2.1)$$

for any $\tau = [\nu^1, \ldots, \nu^p]$ in $S^+_p$, $p \geq 1$. Extend this operation, by linearity, to all elements of $S^+_p$.

Now by applying lemma 1.1 we get

$$\tilde{\partial}_{\zeta,z} \hat{\Omega}_B[\sigma_I] = -\hat{\Omega}[\sigma_I] - \hat{\Omega}_B[\partial \sigma_I] \quad (2.2)$$

(where $\hat{\Omega}_B[\partial \sigma_I] := \Omega(B)$ if $|I| = 1$) for $z \in \mathcal{D}_I$ and $\zeta \in \mathcal{D}_I^-$, with $\zeta \neq z$.

Let $|I| \geq 2$ and $T$ be defined as in sect. 1.5 and $m$ an integer such that lemma 2.2 holds.

By applying lemma 1.4, lemma 2.1, proposition 1.6, we obtain

$$\tilde{\partial}_{\zeta,z} \sum_{i=0}^{m-1} \hat{\Omega}[T(sd^i(\sigma_I))] = -\hat{\Omega}[\sigma_I] - \sum_{i=0}^{m-1} \hat{\Omega}[T(sd^i(\partial \sigma_I))] + \hat{\Omega}[sd^m(\sigma_I)] \quad (2.3)$$

for $z \in \mathcal{D}_I$ and $\zeta \in \mathcal{D}_I^-$, with $\zeta \neq z$. 
Now define for $|I| \geq 2$

$$K^I(\zeta, z) = \tilde{\Omega}_B[\sigma_I](\zeta, z) - \sum_{i=0}^{m-1} \tilde{\Omega}[T(sd^i(\sigma_I))](\zeta, z),$$

$$B^I(\zeta, z) = \sum_{v=1}^{|I|} (-1)^{v+1} K^I(\nu) = -\tilde{\Omega}_B[\partial \sigma_I](\zeta, z) + \sum_{i=0}^{m-1} \tilde{\Omega}[T(sd^i(\partial \sigma_I))](\zeta, z) \quad (2.4)$$

and set for $|I| = 1$

$$K^I(\zeta, z) = \tilde{\Omega}_B[\sigma_I](\zeta, z),$$

and

$$B^I(\zeta, z) = \Omega(B)(\zeta, z) \quad \text{(the Martinelli-Bochner-Koppelman kernel).} \quad (2.5)$$

Then we have the following

**Lemma 2.3**

(i) For $z \in \overline{D_I}$ and $\zeta \in \overline{D_I}$, with $\zeta \neq z$, we have

$$\overline{\partial}_{\zeta,z} K^I = B^I - \tilde{\Omega}[sd^m(\sigma_I)]$$

(ii) There exist a constant $C > 0$ and a finite family $\{\gamma_1, \ldots, \gamma_L\}$ of linearly independent families $\gamma_i = [\gamma_{i,1}, \ldots, \gamma_{i,|I|}]$ in $\Delta_I$ such that

$$\|K^I(\zeta, z)\| \leq C \sum_{i=1}^L \frac{1}{\prod_{j=1}^{|I|} |\Phi_{\gamma_i}(\zeta, z)| |\zeta - z|^{2n-2|I|-1}} \leq C \varepsilon (1 + |\theta \varepsilon|)^{|I|} \quad (2.6)$$

Proof. (i) is a consequence of (2.1), (2.3) and (2.4). The estimate in (ii) is easy to see from the definition of $\Omega$, by using lemma 1.2 and inequality (1.5) (cf. the proof of Lemma 2.6). \hfill \Box

The following lemma shows that the kernel $K^I$ (resp. $B^I$) has locally integrable coefficients on $S_I$ (resp. $S_I^+$) in both variables $\zeta$ and $z$.

**Lemma 2.4**

(i) Let $I \in \mathcal{I}$ and $(\gamma_1, \ldots, \gamma^{|I|})$ be a family of linearly independent vectors in $\mathbb{R}^{|I|}$ and $z \in \overline{D_I}$, then there exists $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and for all $j \in \{\pm 1, \ldots, \pm k\} \setminus I$, $|j| \leq |I|$

$$\int_{\zeta \in S_I} d\lambda(\zeta) \prod_{i=1}^{|I|} |\Phi_{\gamma_i}(\zeta, z)||\zeta - z|^{2n-2|I|-1} \leq C \varepsilon (1 + |\theta \varepsilon|)^{|I|} \quad (2.6)$$

$$\int_{\zeta \in S_I^+} d\lambda(\zeta) \prod_{i=1}^{|I|} |\Phi_{\gamma_i}(\zeta, z)||\zeta - z|^{2n-2|I|-1} \leq C \varepsilon (1 + |\theta \varepsilon|)^{|I|} \quad (2.7)$$
Remark. If is bounded by is continuous from for (2

where inequality (cf. (1.5)) is sufficiently small (cf. lemma 2.3 in [6]). Thus taking into account the following Definition 2.5 lemma 2.3 and estimates (2

continuous :

Proof. Since M is CR generic and for all the above estimates hold if we integrate with respect to instead of .

(ii) Let , , and .

All of the above estimates hold if we integrate with respect to instead of .

Definition 2.5 (i) Let and . It follows from lemma 2.3 and estimates (2.6), (2.7), that the following operators are well defined and continuous :

where

(ii) Let it follows from lemma 2.3 and estimate (2.8) that the operator defined by :

is continuous from into .

Remark. If and then and therefore

thus from the definition of we obtain

(2.9)
LEMMA 2.6 Let $I \in I$, $n - k - q + 1 \leq r \leq n - k$ and $f \in C^1_{0, r}(D)$ with compact support on $D$. Then the following equality holds in the sense of currents:

$$\overline{\partial}K_{0, r-1}^I f + (-1)^{|I|+1} K_s^I \overline{\partial}f = (-1)^r(\overline{\partial}B_{0, r-1}^I f + (-1)^{|I|} \overline{\partial}B_{0, r}^I \overline{\partial}f)$$  (2.10)
on $D_I$.

Proof. The following identity is true from lemma 2.3(i) and Stokes’ theorem: if $n - k - q + 1 \leq s \leq n - k$, $g \in C^1_{0, s+1}(D)$ with compact support on $D$ and if $z \in D_I$ then:

$$\hat{K}_{0, s}^I g(z) = \int_{S^+_I} \overline{\partial}g \wedge K_{0,r}^I(\cdot,z) + (-1)^{s+1} \overline{\partial}B_{0, s}^I g(z) + (-1)^{|I|} \int_{S^+_I} \overline{\partial}g \wedge K_{0,r-1}^I(\cdot,z)$$  (2.11)

Since the forms $\hat{K}_{0, r-1}^I, \hat{K}_{0, r-1}^I, \hat{B}_{0, r-1}^I f$ and $\hat{B}_{0, r}^I \overline{\partial}f$ are continuous on $D_I$, it is sufficient to prove (2.10) on $D_I$ where these forms are smooth.

By setting $s = r$ and $g = \overline{\partial}f$ in (2.11) and using lemma 2.2 (i), we obtain

$$\hat{K}_{0, r}^I \overline{\partial}f(z) = (-1)^r \hat{B}_{0, r}^I \overline{\partial}f(z) + (-1)^{|I|} \int_{S^+_I} \overline{\partial}f \wedge K_{0,r-1}^I(\cdot,z)$$  (2.12)

for all $z \in D_I$.

If we set now $s = r - 1$ and $g = f$ in (2.11), then we get

$$\hat{K}_{0, r-1}^I f(z) = \int_{S^+_I} \overline{\partial}f \wedge K_{0,r-1}^I(\cdot,z) + (-1)^r \hat{B}_{0, r-1}^I f(z)$$

$$+ (-1)^{|I|} \int_{S^+_I} f \wedge K_{0,r-2}^I(\cdot,z) + (-1)^{r-1} \int_{S^+_I} f \wedge \hat{\Omega}_{0, r-1}(sd^m \sigma_I)(\cdot,z),$$

and then by lemma 2.2 (ii)

$$\overline{\partial}\hat{K}_{0, r-1}^I f(z) = \overline{\partial} \int_{S^+_I} \overline{\partial}f \wedge K_{0,r-1}^I(\cdot,z) + (-1)^r \overline{\partial}B_{0, r-1}^I f(z)$$  (2.13)

for all $z \in D_I$.

The lemma now follows from (2.12) and (2.13) .

Now define

$$K(\zeta, z) := \sum_{I \in I^*(k)} (\sgn I)K^I(\zeta, z)$$  (2.14)

for $\zeta, z \in M_0$ with $\zeta \neq z$, and denote by $K_{s,r}$ the piece of $K$ which is of type $(s, r)$ in the variable $z$.

From lemma 2.4, we see that the kernel $K$ has locally integrable coefficients in both variables $\zeta$ and $z$.

Now by applying (2.10) $k$ times , taking into account (2.9) and using the classical Martinelli-Bochner-Koppelman formula (see [3] or [6] for technical details) we obtain the following integral representation
Theorem 2.7. Let \( \Omega \subset M_0 \) of piecewise \( C^1 \) boundary and \( f \) a \((0, r)\) \( C^1 \) form on \( \overline{\Omega} \) with \( n - k - q + 1 \leq r \leq n - k \), then
\[
(-1)^{r(k+1)} f(z) = \int_{\partial \overline{\Omega}} f(\zeta) \wedge K_{0,r}(\zeta, z) - \int_{\Omega} \overline{\partial} f(\zeta) \wedge K_{0,r}(\zeta, z) + \int_{\Omega} f(\zeta) \wedge K_{0,r-1}(\zeta, z). 
\]

By a duality argument we obtain

Corollary 2.8. Let \( \Omega \subset M_0 \) of piecewise \( C^1 \) boundary and \( f \) a \((0, r)\)-form on \( \overline{\Omega} \) with \( 0 \leq r \leq q - 1 \), then we have
\[
(-1)^{r(k+1)} f(\zeta) = \int_{\partial \overline{\Omega}} f(\zeta) \wedge K_{n,n-k-1-r}(\zeta, z) - \int_{\Omega} \overline{\partial} f(\zeta) \wedge K_{n,n-k-1-r}(\zeta, z) + \int_{\Omega} f(\zeta) \wedge K_{n,n-k-r}(\zeta, z).
\]

We say that \( K \) is a fundamental solution for \( \overline{\partial} \) on \( M_0 \).

2.2. Second fundamental solution for \( \overline{\partial} \). In this section, we shall construct our second fundamental solution for the tangential Cauchy-Riemann complex on \( M_0 \). This fundamental solution will be derived from the first one, by using an idea of Henkin [14] (cf. [11]).

Let \( m \) be as in Lemma 2.2 and \( \nu^* \in \bigcup_{I \in \mathcal{I}'(k)} \Delta_I \) such that

\( (*) \) \quad For any \( k \)-simplex \( \tau \) in \( sd^m(\sigma_I) \), each collection of \( k \) elements in \( [\nu^*, \tau] \) is a \( k \)-simplex.

Remark. The choice of such \( \nu^* \) is very important for our optimal estimates.

We adopt the following notation
\[
[\nu^*, \sum_i c_i \sigma_i] = \sum_i c_i [\nu^*, \sigma_i]
\]
for any element \( \sum_i c_i \sigma_i \) in \( S'_p \).

Set
\[
E(\zeta, z) = \sum_{I \in \mathcal{I}'(k)} \text{sgn} I \left( \Omega_B[\nu^*, \sigma_I] + \sum_{i=0}^{m-1} \Omega[\nu^*, T(sd^i(\sigma_I))] \right)
\]
and
\[
R(\zeta, z) = \sum_{I \in \mathcal{I}'(k)} \text{sgn} I \Omega[\nu^*, sd^m(\sigma_I)].
\]
Since
\[ \sum_{I \in \mathcal{I}(k)} (\text{sgn} I) \partial\sigma_I = 0, \]
then by applying lemma 1.1, proposition 1.6 and (2.2), we obtain
\[ \overline{\partial}_{\zeta,z} E(\zeta, z) = K(\zeta, z) - R(\zeta, z) \] (2.17)
for \( \zeta, z \in M_0 \) with \( \zeta \neq z \).
Now we claim that \( R \) is a fundamental solution for \( \overline{\partial}_b \) on \( M_0 \), this means that Theorem 2.7 holds also for the kernel \( R \). To prove it, following Henkin [14], all we have to do is to show that the singularity of \( E \) is mild enough so that the identity (2.17) holds on all \( M_0 \times M_0 \) in the sense of distributions. For once this is done, our claim follows by applying \( \overline{\partial}_{\zeta,z} \) to both sides of (2.17) and then using Theorem 2.7.

The proof of the first part of Theorem 0.1 will be then complete by setting

**Definition 2.9**

\[ \mathcal{R}_r(\zeta, z) := \begin{cases} (-1)^{r(k+1)} R_{0,r}(\zeta, z) & \text{if } n - k - q \leq r \leq n - k \\ (-1)^{r(k+1)} R_{n,n-k-1-r}(z, \zeta) & \text{if } 0 \leq r \leq q - 1. \end{cases} \]

Now to realize our program, we follow the proof of Theorem 1, chap.21 in [7].
First we need the following lemma

**Lemma 2.10.** Given \( W \subset \subset M_0 \), there is a positive constant \( C \) such that for each \( \epsilon > 0 \) and \( z \in W \), we have

(i) \[ \int_{\zeta \in M_0 \atop |\zeta - z| \leq \epsilon} \| K(\zeta, z) \| \, d\lambda(\zeta) \leq C \epsilon(1 + |\delta\epsilon|)^k \]

(ii) \[ \int_{\zeta \in M_0 \atop |\zeta - z| \leq \epsilon} \| R(\zeta, z) \| \, d\lambda(\zeta) \leq C \epsilon \]

(iii) \[ \int_{\zeta \in M_0 \atop |\zeta - z| \leq \epsilon} \| E(\zeta, z) \| \, d\lambda(\zeta) \leq C \epsilon^2(1 + |\delta\epsilon|)^k. \]

(iv) All of the above inequalities hold if we integrate with respect to \( z \) instead of \( \zeta \).

Let us assume the lemma for the moment and show that equation (2.17) holds on all \( M_0 \times M_0 \).
For \( \epsilon > 0 \), choose a smooth function \( \chi_\epsilon \) on \( M_0 \times M_0 \) with the following properties

\[ \chi_\epsilon(\zeta, z) = \begin{cases} 1 & \text{if } |\zeta - z| \geq \epsilon \\ 0 & \text{if } |\zeta - z| \leq \frac{\epsilon}{2}. \end{cases} \]

and for any first-order derivative \( \mathcal{D} \),

\[ |\mathcal{D}\{\chi_\epsilon\}| \leq \frac{C}{\epsilon} \] (2.18)
where $C$ is a positive constant that is independent of $\epsilon$.

Since $\chi_\epsilon$ vanishes near the diagonal of $M_0 \times M_0$, we have from (2.17)

$$
\overline{\partial}_{\zeta,z} \{\chi_\epsilon E\} = (\overline{\partial}_{\zeta,z} \chi_\epsilon) \wedge E + \chi_\epsilon (K - R)
$$

(2.19) on $M_0 \times M_0$. From Lemma 2.10, we have

$$
\chi_\epsilon K \to K, \chi_\epsilon R \to R, \chi_\epsilon E \to E
$$

and

$$
(\overline{\partial}_{\zeta,z} \chi_\epsilon) \wedge E \to 0
$$

as $\epsilon \to 0$, in the sense of currents. So we obtain the desired result by letting $\epsilon \to 0$ in the equation (2.19).

**Proof of Lemma 2.10.** Looking at the definitions of the kernels $K$, $E$ and $R$ (cf. (2.14), . . . , (2.16)) and taking into account Lemma 1.2, we see that we have to estimate the following typical term

$$
\mathcal{N}(\zeta, z) = \prod_{i=1}^{k} (\Phi_{a_i}(\zeta, z))^{r_i} (\Phi_{a_0}(\zeta, z))^{r_0} (\Phi_{a_{k+1}}(\zeta, z))^{r_{k+1}} |\zeta - z|^{2s}
$$

(2.20) where $a^1, \ldots, a^k$ are linearly independent, $a^0 = \sum_{i=1}^{k} x_i a^i$, $a^{k+1} = \sum_{i=1}^{k} y_i a^i$,

$$
r_i \geq 1, \text{ all } 1 \leq i \leq k; \quad s, r_0, r_{k+1} \geq 0 \quad \text{and} \quad
$$

$$
s + \sum_{i=0}^{k+1} r_i = n.
$$

For the kernel $K$, we have $r_{k+1} = 0$ and either $r_0 = 0$, $s \geq 1$ and the function $\mathcal{N}$ involves coefficients of the differential form

$$
\left( G_{a^1}.d(\zeta - z) \right) \wedge \cdots \wedge \left( G_{a^k}.d(\zeta - z) \right) \wedge \left( (\zeta - z).d(\zeta - z) \right)
$$

or $s = 0$, $r_0 \geq 1$ and the function $\mathcal{N}$ contains the coefficients of the term

$$
\left( G_{a^1}.d(\zeta - z) \right) \wedge \cdots \wedge \left( G_{a^k}.d(\zeta - z) \right) \wedge \left( G_{a^0}.d(\zeta - z) \right)
$$

Since

$$
G_{a^0}(\zeta, z) = \sum_{i=1}^{k} x_i G_{a^i} + \mathcal{O}(|\zeta - z|)
$$

we obtain in both cases

$$
|\mathcal{N}(\zeta, z)| \leq C|\zeta - z|
$$

(2.21) Since $M$ is $CR$ generic and $a^1, \ldots, a^k$ are linearly independent,
In this section we shall prove

\[ \text{End of proof of Theorem 0.1} \]

Thus the proof of (i), (ii), (iii) in Lemma 2.10 is complete. (iv) follows in the same way. \( \square \)

3. End of proof of Theorem 0.1

In this section we shall prove \( C^{\ell+\frac{1}{2}} \) -estimates. We first prove \( C^{\ell} \) -estimates and then we derive \( C^{\ell+\frac{1}{2}} \) -estimates by using a kind of integration by parts argument (see [5] and [3]).
3.1. $C^{\frac{k}{2}}$-Estimates. Suppose $M$ of class $C^2$. Recall from the previous section that the coefficients of the kernel $R(\zeta, z)$ have the form

$$\frac{\mathcal{N}(\zeta, z)}{\prod_{i=1}^{k+1}(\Phi_{i}(\zeta, z))^{r_i}}$$

where $a^1, \ldots, a^{k+1}$ are vectors in $\mathbb{R}^k$ such that every subset of $k$ elements in $\{a^1, \ldots, a^{k+1}\}$ is a family of linearly independent vectors (condition (*)), the estimate (2.21) holds for $\mathcal{N}$ and

$$r_i \geq 1, \text{ all } 1 \leq i \leq k + 1; \sum_{i=0}^{k+1} r_i = n.$$ 

We have

$$\int_{\zeta \in M_0} \|R(\zeta, z^1) - R(\zeta, z^2)\| \, d\lambda(\zeta) \leq J_1(z^1, z^2) + J_2(z^1, z^2)$$

where

$$J_1(z^1, z^2) := \int_{\zeta \in M_0 \mid \|\zeta - z^1\| \leq \|\zeta - z^2\|^{\frac{1}{2}}} (\|R(\zeta, z^1)\| + \|R(\zeta, z^2)\|) \, d\lambda(\zeta)$$

and

$$J_2(z^1, z^2) := \int_{\zeta \in M_0 \mid \|\zeta - z^1\| \geq \|\zeta - z^2\|^{\frac{1}{2}}} \|R(\zeta, z^1) - R(\zeta, z^2)\| \, d\lambda(\zeta)$$

It follows from lemma 2.10 (ii) that

$$J_1(z^1, z^2) \leq C|z^1 - z^2|^\frac{1}{2}.$$ 

Since $\mathcal{N}(\zeta, z)$ is smooth in $z$, it is not difficult to see by the same arguments as in the proof of Lemma 2.10 that

$$J_2(z^1, z^2) \leq C|z^1 - z^2|^\frac{1}{2} \int_{X \in \mathbb{R}^{2n-k}} \frac{dX}{(|X_1| + |X|^2)^{2+\frac{1}{2}} \prod_{j=2}^{k} (|X_1| + |X|^2)^{1+\frac{1}{j}} |X|^{2n-2k-3}}$$

$$\leq C|z^1 - z^2|^\frac{1}{2}.$$ 

Thus

$$\int_{\zeta \in M_0} \|R(\zeta, z^1) - R(\zeta, z^2)\| \, d\lambda(\zeta) \leq C|z^1 - z^2|^\frac{1}{2}. \tag{2.23}$$

Analogously we can show that

$$\int_{z \in M_0} \|R(z^1, z) - R(z^2, z)\| \, d\lambda(z) \leq C|z^1 - z^2|^\frac{1}{2}. \tag{2.24}$$

under the hypothesis that $M$ is of class $C^3$. This is because $R(\zeta, z)$ involves second-order derivatives in $\zeta$ of the defining functions of $M$. 
3.2. $C^{\ell+2}$-Estimates. We assume that $M$ is of class $C^{\ell+2}$ ($\ell \geq 1$).

Let $a^1, \ldots, a^k$ be linearly independent vectors in $\bigcup_{I \in \mathcal{I}(k)} \Delta_I$ and $a^{k+1} = \sum_{i=1}^k y_i a^i$ with $y_i \neq 0$, all $1 \leq i \leq k$ (this means that every collection of $k$ vectors in $\{a^1, \ldots, a^{k+1}\}$ is a family of linearly independent vectors).

Denote by $\tilde{\rho}_i$ (resp. $\phi_i$) the defining function (resp. the barrier function) of $M$ in the direction $a^i$ for $1 \leq i \leq k+1$. $E^j(j \geq 0)$ will denote a smooth differential form on $M \times M$ vanishing of order $j$ for $\zeta = z$. It is clear that

$$\phi_{k+1} = \sum_{i=1}^k y_i \phi_i + E^2$$

(2.25)

We need the following lemma.

**Lemma 3.1** There exist $Y_{\zeta}^1, \ldots, Y_{\zeta}^k$, tangential vector fields to $M$ such that for every $\zeta \in M_0$ and every $1 \leq i, j \leq k$,

$$Y_{\zeta}^i \phi_j(\zeta, \zeta) = \delta_{ij},$$

where $\delta_{ij}$ is Kronecker’s symbol.

**Proof.** Since $M$ is CR generic and $a^1, \ldots, a^k$ are linearly independent, we have

$$\partial \tilde{\rho}_1 \wedge \cdots \wedge \partial \tilde{\rho}_k \neq 0 \text{ on } M_0.$$

Then the matrix

$$A = \begin{pmatrix}
< \partial \tilde{\rho}_1(\zeta), \partial \tilde{\rho}_1(\zeta)> & \cdots & < \partial \tilde{\rho}_k(\zeta), \partial \tilde{\rho}_1(\zeta)>
\vdots & \ddots & \vdots \\
< \partial \tilde{\rho}_1(\zeta), \partial \tilde{\rho}_k(\zeta)> & \cdots & < \partial \tilde{\rho}_k(\zeta), \partial \tilde{\rho}_k(\zeta)>
\end{pmatrix}$$

is invertible for all $\zeta \in M_0$ (here $< \ldots >$ denotes the Hermitian inner product), and there exist $\nu_1, \ldots, \nu_k \in \{1, \ldots, n\}$ such that the matrix

$$B = \begin{pmatrix}
\frac{\partial \tilde{\rho}_1}{\partial \zeta_{\nu_1}}(\zeta) & \cdots & \frac{\partial \tilde{\rho}_k}{\partial \zeta_{\nu_1}}(\zeta)
\vdots & \ddots & \vdots \\
\frac{\partial \tilde{\rho}_1}{\partial \zeta_{\nu_k}}(\zeta) & \cdots & \frac{\partial \tilde{\rho}_k}{\partial \zeta_{\nu_k}}(\zeta)
\end{pmatrix}$$

is also invertible for all $\zeta \in M_0$.

Set

$$Y_{\zeta}^i = \frac{1}{2} \sum_{j=1}^k \alpha_{ij}(\zeta) \sum_{\nu=1}^n \frac{\partial \tilde{\rho}_j}{\partial \zeta_{\nu}} \frac{\partial}{\partial \zeta_{\nu}} - \frac{1}{2} \sum_{j=1}^k \beta_{ij}(\zeta) \frac{\partial}{\partial \zeta_{\nu_j}}$$

where $[\alpha_{ij}(\zeta)] = A^{-1}$ and $[\beta_{ij}(\zeta)] = B^{-1}$.

Now it is easy to check that
Let us introduce the following class of kernels for $\delta \geq 0$,

$$
\mathcal{L}_\delta = \frac{\mathcal{E}_j}{\prod_{i=1}^{k+1} (\phi_i + \delta)^{r_i}},
$$

where

$$
2n - 1 - 2 \sum_{i=1}^{k+1} r_i + j \geq 0
$$

and

$$
r_i \geq 1 \text{ for all } 1 \leq i \leq k + 1
$$

**Remark 3.2.** Notice that the kernel $R$ is a finite sum of kernels of type $\mathcal{L}_0$, and estimate (2.23) with estimate (2.24) hold, independently of $\delta$, for kernels $\mathcal{L}_{\delta}$.

If we denote by $X^z$ a tangential vector field to $M$ in $z$-variable and $X^\zeta$ the corresponding operator in $\zeta$-coordinates, then we have the following

**Lemma 3.3** Let $\delta > 0$, then we have

$$
X^z \mathcal{L}_\delta = -X^\zeta \mathcal{L}_\delta + \sum_{i=1}^{k+1} \frac{(X^z + X^\zeta)\phi_i}{Y_i^\zeta \phi_i} Y_i^\zeta (\mathcal{L}_\delta) + S_\delta.
$$

where $S_\delta$ is a finite sum of kernels of type $\mathcal{L}_\delta$.

**Proof.** It is not difficult to see that the following facts are true:

(i) $(X^z + X^\zeta)\mathcal{E}^j$ is of type $\mathcal{E}^j$.

(ii) $(X^z + X^\zeta)\phi_i$ is of type $\mathcal{E}^1$.

(iii) $|Y_i^\zeta \phi_i(\zeta, z)| \geq C$ for $|\zeta - z| \leq \epsilon, \epsilon > 0$, and $1 \leq i \leq k$ (see Lemma 3.1)

(iv) If $i \neq j$ then $Y_i^\zeta \phi_j$ is of type $\mathcal{E}^1$ (cf. Lemma 3.1).

(v) $Y_i^\zeta \phi_{k+1} - y_i Y_i^\zeta \phi_i$ is of type $\mathcal{E}^1$ for $1 \leq i \leq k$ (see (2.25), (iv), Lemma 3.1)

(vi) $(X^z + X^\zeta)\phi_{k+1} - \sum_{i=1}^{k} y_i (X^z + X^\zeta)\phi_i$ is of type $\mathcal{E}^2$ (see (2.25) and (i)).

(vii) $(X^z + X^\zeta)(\frac{1}{\phi_i^1}) = \frac{(X^z + X^\zeta)\phi_i}{Y_i^\zeta \phi_i} Y_i^\zeta (\frac{1}{\phi_i^1})$.

The lemma follows now by a straightforward computation.

Now let $\Omega \subset M_0$ and $f \in L^\infty(\Omega) \cap \mathcal{C}_c^1(\Omega)$. Let $z_1 \in \Omega$ and $\chi$ a smooth compactly supported function on $\Omega$ such that

$$
\chi(\zeta) = \begin{cases} 
0 & \text{if } |\zeta - z_1| \geq \frac{\epsilon}{4} \\
1 & \text{if } |\zeta - z_1| \leq \frac{\epsilon}{4}.
\end{cases}
$$
where $\epsilon$ is chosen so that (see Lemma 3.1)

$$|Y_i^\xi \phi_i(\zeta, z)| \geq C$$

for $|\zeta - z| \leq \epsilon$ and all $1 \leq i \leq k$.

Set $K := \{z \in \Omega/|z - z_1| \leq \tfrac{\epsilon}{4}\}$.

We write

$$\int_\Omega f(\zeta) \wedge R(\zeta, z) = \int_\Omega \chi(\zeta) f(\zeta) \wedge R(\zeta, z) + \int_\Omega (1 - \chi(\zeta)) f(\zeta) \wedge R(\zeta, z).$$

Let $J_1(f)$ denote the first integral in the right-hand side and $J_2(f)$ the second one. Since $R(\zeta, z)$ is of class $C^\infty$ in $z$ for $\zeta \neq z$ then $J_2(f)$ is of class $C^\infty$ on $K$.

By Remark 3.2 to estimate $J_1(f)$, it is enough to do so for $\int_\Omega \chi f \wedge \mathcal{L}_0(\cdot, z)$.

We have

$$\int_\Omega \chi f \wedge \mathcal{L}_0(\cdot, z) = \lim_{\delta \to 0} \int_\Omega \chi f \wedge \mathcal{L}_\delta(\cdot, z).$$

By Lemma 3.3, we obtain from Stokes’ theorem

$$X^z \int_\Omega \chi f \wedge \mathcal{L}_\delta(\cdot, z) = \pm \int_\Omega X^\xi(\chi f) \wedge \mathcal{L}_\delta(\cdot, z)$$

$$\pm \sum_{i=1}^k \int_\Omega Y_i^\xi(\chi f) \wedge \left(\frac{(X^z + X^\xi)\phi_i}{Y_i^\xi \phi_i}\right) \mathcal{L}_\delta(\cdot, z) + \int_\Omega \chi f \wedge S_\delta(\cdot, z).$$

where $S_\delta$ is a finite sum of kernels of type $\mathcal{L}_\delta$.

Now, if we apply $r \leq \ell$ derivatives, we can write

$$X_1^z \cdots X_r^z \int_\Omega \chi f \wedge \mathcal{L}_\delta(\cdot, z)$$

as a sum of terms

$$\int_\Omega \tilde{X}_1^\xi \cdots \tilde{X}_j^\xi (\chi f) \wedge \mathcal{L}_\delta(\cdot, z)$$

with $0 \leq j \leq r$.

Since

$$\|\int_\Omega \tilde{X}_1^\xi \cdots \tilde{X}_j^\xi (\chi f) \wedge \mathcal{L}_\delta(\cdot, z)\|_{C^j} \leq C\|\mathcal{L}_\delta\|_{C^j}$$

for $0 \leq j \leq \ell$, independently of $\delta$ (see Remark 3.2), we conclude that

$$\int_\Omega \chi f \wedge \mathcal{L}_0(\cdot, z)$$

is of class $\mathcal{C}^{\ell + \frac{j}{2}}$ on $\Omega$.

Thus $J_1(f)$ is of class $\mathcal{C}^{\ell + \frac{j}{2}}$ on $\Omega$, and therefore

$$\int_\Omega f(\zeta) \wedge R(\zeta, z)$$

is of class $\mathcal{C}^{\ell + \frac{j}{2}}$ on $K$. 


By noticing that $Y^i_z \Phi_j = -Y^i_z \Phi_j + \mathcal{E}^1$ for $1 \leq i, j \leq k$ one can show in the same way
\[
\int_{\Omega} f(z) \wedge R(\zeta, z) \text{ is of class } \mathcal{C}^{\ell+\frac{1}{2}} \text{ on } K
\]
provided $M$ is of class $\mathcal{C}^{\ell+3}$ (see (2.24)). This completes the proof of the second part of Theorem 0.1 (cf. Definition 2.9).

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Institut Fourier
UFR de Mathématiques
UMR 5582 CNRS
B.P 74 38402 Saint Martin d’Hères France

e-mail: ybarkat@ujf-grenoble.fr