SOME RESULTS IN A NEW POWER GRAPHS IN FINITE GROUPS

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Abstract. The power graph of a group is the graph whose vertex set is the set of non-trivial elements of group, two elements being adjacent if one is a power of the other. We define a new power graph and study on connectivity, diameter and clique number of this graphs.

1. Introduction.

The directed power graph of a semigroup $S$ was defined by Kelarev and Quinn [2] as the digraph $\Gamma(S)$ with vertex set $S$, in which there is an arc from $x$ to $y$ if and only if $x \neq y$ and $y = x^m$ for some positive integer $m$. Motivated by this, Chakrabarty et al. [1] defined the (undirected) power graph $\Gamma(S)$, in which distinct $x$ and $y$ are joined if one is a power of the other.

Let $L$ be a graph. We denote $V(L)$ and $E(L)$ for vertices and edges of $L$, respectively. For any two vertex $x, y$ of $L$, we denote $d(x, y)$ for the length of smallest path between $x, y$. Also $diam(L) = Max\{d(x, y)|x, y \in V(L)\}$. The clique number of $L$, is denoted by $w(L)$, is the maximum size of complete subgraphs of $L$. The (open) neighborhood $N(a)$ of vertex $a \in V(L)$ is the set of vertices are adjacent to $a$. Also the closed neighborhood of $a$, $N[a]$ is $N(a) \cup \{a\}$. All groups in this paper are finite.

Since in current power graphs the trivial element is adjacent two any element and this property is not very good, we define a new power graph.

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Definition 1.1. Let $G$ be a group and $S$ is a subset of $G$. We define the power graph $\Gamma(G, S)$ with vertex set $S$ and two elements are adjacent if and if only one of them is power of another. Also we denote $\Gamma_1(G)$ for $\Gamma(G, S)$ where $S = G - \{1\}$.

Observation 1.2. In the power graph of a finite group $G$,

(i) $N[a] = N[a^i]$ for any $a \in G$ and $(o(a), i) = 1$.

(ii) $\Gamma(G)$ is complete if and only if $G$ is a cyclic group of prime power order.

2. Connectivity in power graphs

Lemma 2.1. Let $G$ be a finite group. Then $\Gamma_1(G)$ is connected if and only if for any two elements $x, y$ of prime orders where $\langle x \rangle \neq \langle y \rangle$, there exist elements $x = x_0, x_1, \ldots, x_t = y$ such that $o(x_{2i})$ is prime, $o(x_{2i+1}) = o(x_{2i})o(x_{2i+2})$ for $i \in \{0, \ldots, t/2\}$ and, $x_i$ is adjacent to $x_{i+1}$ for $i \in \{0, 1, \ldots, t - 1\}$.

Proof. $\Leftarrow$ Is trivial.

$\Rightarrow$ Assume that $\Gamma_1(G)$ is connected and $x, y$ are two elements of prime orders where $d(x, y)$ is minimum and $x, y$ do not satisfy in conclusion of Lemma. Let $x = x_0 - x_1 - \ldots - x_t = y$ is a smallest path. Consequently $o(x_i) \neq o(x_{i+1})$. Since $o(x)$ is prime then $x_0 = x_s^k$, for some integer $s$. Thus $o(x) < o(x_1)$ and $o(x) \mid o(x_1)$. Assume that $k$ is a greatest integer such that $o(x_0) < o(x_1) < \cdots < o(x_k)$. Also let $m$ is a greatest integer such that $o(x_k) > o(x_{k+1}) > \cdots > o(x_m)$. We have $\langle x_0 \rangle \subseteq \langle x_1 \rangle \subseteq \ldots \subseteq \langle x_k \rangle$ and $\langle x_m \rangle \subseteq \langle x_{m-1} \rangle \subseteq \ldots \subseteq \langle x_k \rangle$. Then $x_0, x_m$ are adjacent to $x_k$ and $k = 1, m = 2$. Since $x_0, x_2 \notin \langle x_1 \rangle$ and $x_0, x_2$ are not adjacent, then $\langle x_2 \rangle$ has element $b$ of prime order distinct to $o(x_0)$. Now $x_0 - bx_0 - b$ satisfies in Lemma and $x = x_0 - bx_0 - b - x_3 - \ldots - x_t = y$ is a path. Consequently $x_3 - \ldots - x_t$ do not satisfy Lemma, a contradiction. ∎
For $n \geq 3$ we consider the generalized Quaternion group as following.

$$Q_n = \langle a, b | a^n = b^2, b^{-1}aba = b^4 = 1 \rangle.$$

**Observation 2.2.** [3, Th 5.4.10] The finite $p$-group has exactly one subgroup of order $p$ if and only if it is a cyclic or a generalized Quaternion group.

**Corollary 2.3.** Let $G$ be a $p$-group. Then $\Gamma_1(G)$ is connected if and only if $G$ is a cyclic or a generalized Quaternion group.

**Proof.** Let $\Gamma_1(G)$ is connected. Since any two element $x, y$ of order $p$ is adjacent if and only if $\langle x \rangle = \langle y \rangle$, then by Lemma 2.1, $G$ has exactly one subgroup of order $p$. Now the proof is completed, by Observation 2.2. □

**Corollary 2.4.** For $p$-group $G$, the number of connectivity complement of $\Gamma_1(G)$ is equal to the number of subgroups of order $p$.

**Proposition 2.5.** If $G, H$ is two nontrivial groups and $(|G|, |H|) = 1$, then $\Gamma_1(G \times H)$ is connected.

**Proof.** Let $p, q$ be two prime factors of $|G \times H|$ and $x, y$ be of order of $p, q$, respectively. We have three cases.

Case 1. $x, y \in G \times \{1\}$. Then, there exists an element $z$ of prime order in $H$. Now $x - x(1, z) - (1, z) - y(1, z) - y$ is a path.

Case 2. $x, y \in \{1\} \times H$. Is similar to Case1.

Case 3. $x \in G \times \{1\}$ and $y \in \{1\} \times H$. Now $x - xy - y$ is a path.

Now the proof is complete by Lemma 2.1. □

**Proposition 2.6.** Let $G, H$ be two groups where $\Gamma_1(G)$ is connected and $|G|$ is not prime power. Then $\Gamma_1(G \times H)$ is connected.
Proof. Let \((a, b)\) be an element of prime order in \(G \times H\) and \(b \neq 1\). We set \(o(a, b) = o(b) = p\). By Lemma 2.1, there exist \(x \in G\) such that \(o(x)\) is prime, \((o(x), p) = 1\) and \(ax = xa\). Now \((a, b) - (xa, b) - (a, 1)\) is a path. Consequently every element of prime order is connected to an element of prime order in \(G \times 1\).

By Lemma 2.1, the proof is complete.

\[\Box\]

**Theorem 2.7.** Let \(G\) be a finite nilpotent group. Then \(\Gamma_1(G)\) is connected if and only if \(G\) is cyclic or generalized Quaternion or not a \(p\)-group for any prime \(p\).

**Proof.** By Corollary 2.3 we assume that \(|G|\) is not a prime power. Thus \(G = P_1 \times P_2 \times \ldots \times P_t\) where \(P_1, ..., P_t\) are all sylow subgroups of \(G\). Consequently by Proposition 2.5, \(\Gamma_1(G)\) is connected.

\[\Box\]

**Example 2.8.** \(\Gamma_1(S_3 \times S_3)\) is not connected but \(\Gamma_1(S_3 \times Z_6)\) is connected where \(S_3\) is the Symmetric group of degree 3 and \(Z_6\) is the cyclic group of order 6.

3. DIAMETER AND CLIQUE NUMBER OF THIS NEW POWER GRAPHS OF FINITE GROUPS

Now we calculate the diameter of this graph. We note that the diameter of old power graph is 1 or 2 trivially.

**Observation 3.1.** Let \(G\) be a group. Then \(diam(\Gamma_1(G)) = 1\) if and only if \(G\) is a cyclic \(p\)-group.

**Proof.** In this case \(\Gamma_1(G)\) is complete and the proof is concluded by Observation 3.1.

\[\Box\]

**Proposition 3.2.** Let \(G\) be a group. Then \(diam(\Gamma_1(G)) = 2\) if and only if one of the following is holds.

(i) \(G\) is a cyclic group of not prime power order.

(ii)\(G\) is a direct product of a cyclic group of odd order to a generalized Quaternion group of order of power of 2.
Proof. ”$\Rightarrow$” We first prove that $G$ is nilpotent. Let $x,y$ are two elements of order $p^n,q^m$ where $p,q$ are two distinct primes. Let $x-a-y$ is a path. If $a \in \langle x \rangle$ then $o(a)|o(x) = p^n$. Since $(p^n,q^m) = 1$, we conclude that $a$ is not adjacent to $y$, a contradiction. Thus $x \in \langle a \rangle$ and respectively $y \in \langle a \rangle$. Consequently $xy = yx$, and then $G$ is nilpotent. Therefore $G = P_1 \times \ldots \times P_t$, where $P_1,\ldots,P_t$ are all sylow subgroups of $G$. Assume that $x,y$ are two elements of order $p_i$ in $P_i$ and $\langle x \rangle \neq \langle y \rangle$. Let $x-a-y$ is a path. By similarly the above we should have $x,y \in \langle a \rangle$, a contradiction. Thus $P_i$ has exactly one subgroup of prime order, and consequently is a cyclic or generalized Quaternion group. Now by the Observation 3.1, the proof is complete.

”$\Leftarrow$” Let $G$ is satisfying in one of those condition. We note that $G$ has exactly one subgroup of any prime factors of $|G|$. Let $G$ be a cyclic group of order of not prime power and $x,y \in G$. Let $x,y$ are not adjacent. If $(o(x),o(y)) = 1$ then $o(xy) = o(x)o(y)$ and $x,y \in \langle xy \rangle$. Therefore $x-xy-y$ is a path. We assume that $p|(o(x),o(y))$ and $a$ is an element of order $p$ in $\langle x \rangle$. $a \in \langle y \rangle$, because $G$ is cyclic. Thus $x-a-y$ is a path. Since $\Gamma_1(G)$ is not complete, $diam(G) = 2$.

(ii) is same to (i). $\Box$

Theorem 3.3. Let $G$ be a nilpotent group that $diam(G) > 2$ then $diam(G) = 4$.

Proof. We have $G = P_1 \times P_2 \times \ldots \times P_t$ where $P_i$ is $p_i$-sylow subgroup of $G$ for $i \in \{1,\ldots,t\}$. Also $G$ is not cyclic of prime power. By Proposition 3.2, $G$ has two cyclic subgroup of prime order $p_i$ for some $i$. Let $x,y$ be two element of order $p_i$ where $\langle x \rangle \neq \langle y \rangle$. Indeed $d(x,y) > 2$. Let $d(x,y) = 3$ and $x-a-b-y$ is a path. we see that $x \in \langle a \rangle$ and $y \in \langle b \rangle$. If $a \in \langle b \rangle$ then $x,y \in \langle b \rangle$, a contradiction. Similarly $b \notin \langle a \rangle$ and so $a$ is not adjacent to $b$, a contradiction. Consequently $d(x,y) \geq 4$ and then $diam(\Gamma_1(G)) \geq 4$. On the other hand, for any two elements $x,y \in G$ we have two cases:
Case 1. \( x, y \in P_i \) for some \( i \). We have \( x - xa - a - ay - y \) is a path where \( a \in P_j \) and \( j \neq i \).

Case 2. \( x, y \notin P_i \), for each \( i \). Then there exist primes integer \( p, q \), such that \( p|o(x), q|o(y)| \) and \( p \neq q \). Let \( a \in \langle x \rangle \) of order \( p \) and \( b \in \langle y \rangle \) of order \( q \). We see that \( x - a - ab - b - y \) is a path.

Consequently \( \text{diam}(\Gamma_1(G)) = 4 \). □

Note that by the Observation 3.1, Proposition 3.2 and Theorem 3.3, the diameters of all finite nilpotent groups were calculated.

**Example 3.4.** (i) \( \text{diam}(Q_n) = 3 \) for odd integer \( n \).

(ii) \( \text{diam}(S_3 \times Z_6) = 4 \).

Now we calculate the clique number of power graph of finite groups.

Let \( n \) be an integer. The set \( A = \{d_1, \ldots, d_t\} \) of divisors of \( n \) is said a \( CD \)-set of \( n \) if \( d_1 > 1 \) and one of any two elements of \( A \) divided by another. The maximal of these set is said an \( MCD \)-set of \( n \). We define \( \text{weight}(A) = \varphi(d_1) + \cdots + \varphi(d_t) \) and \( \text{weight}(n) = \max\{\text{weight}(A) | A \text{ is an } MCD\text{-set of } n\} \).

**Lemma 3.5.** Let \( A = \{d_1, \ldots, d_t\} \), where \( d_1 < \cdots < d_t \). Then \( A \) is an \( MCD \)-set if and only if \( d_1 \) be a prime, \( d_t = n \) and, \( d_{i+1}/d_i \) is a prime for \( i \in \{2, \ldots, t\} \).

**Lemma 3.6.** Let \( G \) be a group and \( K \) is an maximal complete subgraph. Then there exist an maximal cyclic subgroup \( \langle a \rangle \) such that \( V(K) \subseteq \langle a \rangle \). Moreover \( A = \{o(x)|x \in V(K)\} \) is an \( MCD \)-set of \( o(a) \).

**Proof.** Assume that \( a \in V(K) \) and \( o(a) = \max(A) \). Clearly \( V(K) \subseteq \langle a \rangle \), \( \langle a \rangle \) is an maximal cyclic subgroup of \( G \) and \( A \) is a \( CD \)-set of \( o(a) \). Let \( A \) is not an \( MCD \)-set and \( A = \{o(a_1), \ldots, o(a_t)\} \), where \( o(a_1) \leq \cdots \leq o(a_t) \). Then, there exist \( i \) such that \( o(a_i) \neq O(a_{i+1}) \) and \( o(a_{i+1})/o(a_i) \) is not a prime.
Assume that $p|o(a_{i+1})/o(a_i)$ and $b$ be an element of order $po(a_i)$ in $\langle a_{i+1} \rangle$. We see that $V(K) \subseteq N(b)$, a contradiction. \hfill \Box

**Theorem 3.7.** Let $G$ be a finite group. Then $\omega(\Gamma_1(G)) = \omega(\Gamma(G)) - 1 = \max\{\text{weight}(o(a))|a \in G\}$.

**Proof.** Assume that $K, A$ and $a$ be similar to last Lemma. We have $\langle a_i \rangle$ and so $V(K)$ has exactly $\varphi(o(a_i))$ elements of order $o(a_i)$. By Lemma 3.6, the proof is complete. \hfill \Box

**Corollary 3.8.** For the nilpotent group $G$,

$$\omega(\Gamma_1(G)) = \text{weight}(\exp(G))$$

where $\exp(G) = \text{Min}\{n|x^n = 1 \text{ for any } x \in G\}$.

**References**

[1] Ivy Chakrabarty, Shamik Ghosh, M.K. Sen, Undirected power graphs of semigroups, *Semigroup Forum* **78** (2009) 410-426.
[2] A.V. Kelarev, S.J. Quinn, Directed graph and combinatorial properties of semigroups, *J. Algebra* **251** (2002) 16-26.
[3] D. Gorenstein, *Finite groups* (Chelsea Publishing Co., New York, 1980), 2nd edn.
[4] . Mirzargar, A. R. Ashrafi, M. J. Nadjafi-Arani, On the Power Graph of a Finite Group, *Filomat* 26:6 (2012), 1201-1208

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