Vibrational modes and spectrum of oscillators on a scale-free network

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We study vibrational modes and spectrum of a model system of atoms and springs on a scale-free network in order to understand the nature of excitations with many degrees of freedom on the scale-free network. We assume that the atoms and springs are distributed as nodes and links of a scale-free network, assigning the mass \( M_i \) and the specific oscillation frequency \( \omega_i \) of the \( i \)-th atom and the spring constant \( K_{ij} \) between the \( i \)-th and \( j \)-th atoms.

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I. INTRODUCTION

Recently there has been a notable progress in the study of the so-called scale-free network (SFN) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. In the point of view of network theory, the random network theory (RN) was first invented by Erdős and Rényi[11] and has been applied to many areas of sciences from physics such as Anderson localization [12], percolation [13], and energy landscape [14, 15] to biology such as the Kauffman’s NK model [16]. Recently it was generalized to the small world network (SWN) models [17, 18, 19, 20, 21, 22, 23, 24, 25]. This system is constructed by the following process: Initially we put \( n \) nodes with \( m \) links at the \( i \)-th node and we have assumed that the attachment of new nodes to old nodes existed already in the system with a preferential attachment, seeds for the system. Every time when a new node is added, \( m \) new links are distributed from the node to the existed nodes in the system with a preferential attachment probability \( \Pi_i(k_i) = k_i/ \sum_{k_{ij}=1}^{N} k_i \), where \( k_i \) is the number of links at the \( i \)-th node and we have assumed \( m \leq m_0 \). The development of this model is described by a continuum model \( \frac{dM_i}{d\tau} = m\Pi_i(k_i) = \frac{m \phi_i}{2} \). Then at time \( \tau \) the system consists of \( N(\tau) \) nodes and the \( L(\tau) \) links with \( L(\tau) = \frac{1}{2} \sum_{i=1}^{N(\tau)} k_i \). This model exhibits \( \gamma = 3 \) for the power-law. Thus, it has been concluded that the essential points of why a network grows to a SFN are attributed to the growth of the system and the preferential attachment of new nodes to old nodes existed already in the network.

However, although time evolution of the SFN has been intensively studied regarding nodes and links as metaphysical objects such as agents and relationships in an area of science, it seems that very few physical models putting real meaning on nodes and links in the SFNs have been studied in order to investigate excitations such as vibrations, phonons, and electrons, except diffusion [13] and spins [17] on the SFN and excitations in the RN [11, 12, 13, 14, 15, 16] and SWN [17, 18, 19, 20, 21, 22, 23, 24, 25]. So, we explore to study, as a prototype model, vibrational modes and spectrum of a system of atoms coupled by springs where the atoms and springs are located regarding as nodes and links of a SFN.

II. VIBRATIONAL MODEL

Let us introduce our vibrational model. We first adopt the AB’s SFN model for the construction of a SFN and we regard nodes and links in the SFN as atoms and springs in our physical model. Assuming that \( q_i \) and \( \omega_i \) are the displacement and the specific frequency of the \( i \)-th atom of mass \( M_i \), respectively, we can define the hamiltonian of the system:

\[
H = \sum_{i=1}^{N(\tau)} \left( \frac{M_i q_i^2}{2} + \frac{M_j \omega_i^2 q_j^2}{2} \right) + \sum_{i,j=1, i \neq j}^{N(\tau)} K_{ij} \left( q_i - q_j \right)^2,
\]

where \( \dot{q}_i = \frac{m \omega_i}{2} \) is the velocity of the \( i \)-th atom and \( K_{ij} \) is the spring constant between the \( i \)-th and \( j \)-th atoms with \( K_{ij} = K_{ji} \). Although we will concentrate to study only this atom-spring model in this paper, the generalizations of this model and the applications to other systems are straightforward. We expect that the physical nature of this model shares with those of such models and systems.

We now assume that the time interval \( \Delta \tau \) for the development of link addition process is much larger than
that $\Delta t$ of the physical model such that $|\Delta \tau| \gg |\Delta \tau|$. This guarantees that although the network grows in the course of its development, as long as the network consists of $N(\tau)$ nodes and $L(\tau)$ links, the vibrational model can be simultaneously solved. This means that time evolution the network is adiabatic to the time motion of the atoms and springs. By using the Euler-Lagrange equation, $\frac{\partial}{\partial \tau}(\frac{\partial H}{\partial \dot{\tau}^i}) = \frac{\partial H}{\partial q^i}$, we obtain

$$M_i(q_i + \omega_i^2 q_i) = \sum_{j=1}^{N(\tau)} K_{ji} (q_j - q_i),$$

for $i = 1, \ldots, N(\tau)$. Assuming $q_i(t) = q_i(\omega) e^{-i\omega t}$, Eq. (2) becomes

$$M_i(\omega_i^2 - \omega^2) q_i = \sum_{j=1}^{N(\tau)} K_{ji} (q_j - q_i),$$

for $i = 1, \ldots, N(\tau)$. This is the eigenequation for our system.

Let us assume that all springs are identical for the sake of simplicity such that $K_{ij} = K_0 A_{ij}$, where $K_0$ is the spring constant and $A_{ij}$ is the $ij$-th component of the adjacency matrix $\hat{A}$ for the network geometry. The components of the adjacency matrix are non-negative such that $A_{ij} = 0$ or 1 according to whether or not a link between the $i$-th and $j$-th nodes exist in the network. The link number $k_i(\tau)$ at the $i$-th atom (i.e., the order of the $i$-th node) is given by $k_i(\tau) = \sum_{j=1}^{N(\tau)} A_{ij}$. From this, the last term in Eq. (3) becomes $\sum_{j=1}^{N(\tau)} K_{ji} q_i = K_0 k_i(\tau) q_i$. Hence, in this setting, we obtain

$$\Omega_i q_i = K_0 \sum_{j=1}^{N(\tau)} A_{ji} q_j,$$

for $i = 1, \ldots, N(\tau)$, where

$$\Omega_i \equiv M_i(\omega_i^2 - \omega^2) + K_0 k_i(\tau).$$

### III. GREEN'S FUNCTION FORMALISM

Let us now define the Green’s function by

$$\sum_{j=1}^{N(\tau)} [\Omega_j \delta_{ij} - K_0 A_{ij}] G_{jk} = \delta_{ik},$$

for $i, k = 1, \ldots, N(\tau)$, which is represented by $[\hat{G}_0^{-1} - K_0 \hat{A}] \hat{G} = \hat{1}$ in the matrix representation where $\hat{1}$ is the $N(\tau) \times N(\tau)$ unit matrix and $\hat{G}_0$ is the $N(\tau) \times N(\tau)$ diagonal matrix defined by $\hat{G}_0 = \Omega_i^{-1} \delta_{ij}$. Thus, the Green’s function is formally obtained as $\hat{G}^{-1} = \hat{G}_0^{-1} - K_0 \hat{A}$. Furthermore, we can derive a series expansion of $\hat{G}$ in terms of $\hat{G}_0$ and $\hat{A}$ as

$$\hat{G} = \hat{G}_0 + K_0 \hat{G}_0 \hat{A} \hat{G}_0 + K_0^2 \hat{G}_0 \hat{A} \hat{G}_0 \hat{A} \hat{G}_0 + \cdots = \hat{G}_0 + K_0 \hat{G}_0 \hat{A} \hat{G}_0 = \hat{G}_0 + \hat{T} \hat{G}_0,$$

where $\hat{T}$ is called the $T$-matrix defined as

$$\hat{T} = K_0 \hat{A} + K_0^2 \hat{A} \hat{G}_0 \hat{A} + \cdots \equiv \hat{T}_1 + \hat{T}_2 + \cdots.$$  

We can now derive the following:

$$\langle \hat{T}_n \rangle_{ij} = K_0^n (\hat{A} \hat{G}_0 \hat{A} \cdots \hat{G}_0 \hat{A})_{ij}$$

$$= K_0^n \sum_{j_1,j_2,\ldots,j_{n-1}} \frac{A_{j_1} A_{j_2} \cdots A_{j_{n-1}j}}{\Omega_{j_1} \Omega_{j_2} \cdots \Omega_{j_{n-1}}} \equiv K_0^n \Gamma_{ij}^{(n)} ,$$

and since $K_{ij} = K_{ji}$ (i.e., $A_{ij} = A_{ji}$), we find

$$\langle \hat{T}_n \rangle_{ij} = \langle \hat{T}_n \rangle_{ji}.$$  

From Eqs. (8) and (9), we find $\langle \hat{T} \rangle_{ij} = K_0 \Gamma_{ij}^{(1)} + K_0^2 \Gamma_{ij}^{(2)} + \cdots$.

Let us consider the trace of the Green’s function. We now get

$$\text{Tr} \hat{G} = \text{Tr} (\hat{G}_0 + \hat{T} \hat{G}_0) = \sum_i \left( \frac{1}{\Omega_i} + \frac{\langle \hat{T} \rangle_{ii}}{\Omega_i} \right).$$

We note here that if $M_i(\omega_i^2 - \omega^2) = 0$ such that $\Omega_i = K_0 k_i(\tau)$, then $\langle \hat{T} \rangle_{ii} = K_0 k_i P_i$ with $k_i P_i = k_j P_{ji}$, where $P_{ji}(\tau)$ means the probability that the walker starts at node $i$ at time $t = 0$ and found at node $j$ at time $t = \tau$ in terms of the language of the diffusion theory of Noh and Rieger [see Eqs(2) and (3) in [13]]. As is well-known, the density of states $\rho(\omega)$ is given by

$$\rho(\omega) = -\frac{1}{\pi} \text{Tr} \hat{G}(\omega + i\epsilon).$$

Thus, the poles of the Green’s function produce the spectrum of the system.

### IV. SPECIAL LIMITS

Before going to do the direct calculation of the spectrum of the system, let us consider some limits. (i) The independent atom limit. First, in the case of no springs of $K_0 = 0$, since $\text{Tr} (\hat{G}) = \text{Tr} (\hat{G}_0)$, the poles of the Green’s function are given by $\Omega_j \equiv M_j (\omega_j^2 - \omega^2) = 0$, which trivially provides the discrete spectrum $\omega = \omega_j$ for $j = 1, \ldots, N(\tau)$. This means that the atoms independently vibrate with specific frequencies $\omega_j$.

(ii) The AB limit. Second, in the case of very weak spring constant such as $K_0 \ll 1$, the poles of the Green’s function are obtained as

$$\omega^2 = \omega_j^2 + \frac{K_0}{M_j} k_j(\tau).$$

This means that each atom vibrates with frequency related to the number of links of the atom. Since the distribution of the nodes with $k$ links is given by $P(k) \propto k^{-\gamma}$.
in the SFN, the distribution of the spectrum is given as
\[ P(\omega^2 - \omega_j^2) \propto (\omega^2 - \omega_j^2)^{-\gamma}. \] (14)
Hence, this limit shares with the nature of the AB’s SFN geometry. Therefore, we may call this limit the AB limit.

(iii) The localized mode limit. Third, let us consider the limit of very small mass (\( M_i \ll 1 \)) or very strong spring (\( K_0 \gg 1 \)). In this case, we can ignore the frequency dependence in the eigenvalue of Eq.(4) such as \( \Omega_i \approx K_0 k_i \), which then yields
\[ k_i q_i = \sum_{j=1}^{N(\tau)} A_{ij} q_j, \] (15)
for \( i = 1, \ldots, N(\tau) \). Since we can rewrite the above equation as \( \sum_{j=1}^{N(\tau)} [k_i \delta_{ij} - A_{ij}] q_j = 0 \), non-trivial solutions may exist only when the determinant \( \det[k_i \delta_{ij} - A_{ij}] \) vanishes. This is realized when \( q_j = q_i \) where \( j \) runs the adjacent links around the \( i \)-th atom. In this sense the mode is localized within the adjacent atoms.

V. CALCULATIONS OF THE SPECTRUM

Let us now calculate the spectrum of the system of oscillators in the SFN. This is carried out by directly diagonalizing Eq.(4). For the sake of simplicity, we assume that \( M_i = M_0 = \text{const.} \) and \( \omega_i = \omega_0 = \text{const.} \) and we adopt the AB-model for generating the SFN geometry. We have performed the calculations for the systems up to \( N = 10^4 \).

Fig.1 (a) shows the density of states of the system, where we have calculated for the case of \( m = 2 \) and \( N = 10^4 \) (blue) and the case of \( m = 4 \) and \( N = 10^4 \) (red), respectively, where we have used \( m_0 = 5 \). To obtain the distributions, we have used twenty configurations with different random numbers. The vertical axis means \( \rho(\omega) \sqrt{\langle k \rangle_2} \), while the horizontal axis means \( \omega^2 \), where \( \omega \) is the vibrational frequency of the oscillators and \( \langle k \rangle_2 \) the second order average degree of a node [See Eq.(18)]. The shape of the curve is unique such that there is a peak at \( \omega = \omega_0 \) and the spectral tail exists in the whole range of the spectrum. This tendency means that there is a scale-free nature in the spectrum of the vibrational modes in the system.

Fig.1 (b) shows the tail behavior of the density of states. The density of states is shown in a log-log plot for the cases of the AB-model with \( m = 2 \) and \( N = 10^4 \) (blue circles), with \( m = 4 \) and \( N = 7 \times 10^3 \) (red crosses), and with \( m = 4 \) and \( N = 10^4 \) (red squares), respectively. The red line is a guide for showing \( (\omega^2)^{-3} \). From this we find that the tail behavior of Eq.(14) holds valid for the general cases as well. Therefore, we can conclude that our vibrational model shares common nature with the AB-model of the SFN. This is contrary to the conclusion previously obtained from the calculations of the spectrum of the adjacency matrix \( \tilde{A} \) of the AB-model. There, when the network has the tail behavior of \( P(k) \propto k^{-\gamma} \), the spectral tail for the eigenvalues \( \lambda \) of the adjacency matrix is given by \( \rho(\lambda) \propto \lambda^{-\gamma'} \) where \( \gamma' = 2\gamma - 1 \). Therefore, since the AB-model has \( \gamma = 3 \), we conclude \( \gamma' = 5 \). This is different from our result of \( \gamma' = 3 \). The main reason for this phenomenon is explained as follows: In our vibrational model the \( \Omega_i \) consists of the degree \( k_i \) of the node [see Eq.(4)]. Therefore, as the system grows, so does the magnitude of \( \Omega_i \). This can reduce the contributions of the adjacency matrix \( \tilde{A} \) in the higher terms of the perturbation series.
of Eq. (7). Hence, the spectral behavior is dominated by
the pole of the unperturbed Green’s function \( G_0 \). Thus, we
are led to the same spectral behavior in the AB-limit.

The physical meaning of the above results can be un-
derstood as follows: The main peak in the density of
states is attributed to vibrational modes with frequency \( \omega_0 \). These modes are extremely localized within the least
connected nodes in the SFN such that the total number
of the localized modes provides the height of the peak.
Since the number of nodes is nothing but the number of
degeneracy of the eigenequation, these localized modes
are highly degenerate.

On the other hand, there is the power-law tail of \( \rho(\omega^2) \propto (\omega^2)^{-3} \) as \( \omega \to \infty \). This means that the larger
the frequency of modes the fewer the number of modes.
In other words, as the frequency is increasing, the num-
ber of modes is decreasing by the power-law. As the re-
result, there appears only one mode with the maximum fre-
quency (i.e., the maximum eigenvalue). The mode with
the maximum frequency is extended over the entire sys-
tem of the SFN. This situation means that in the SFN
the lowest frequency modes can be very easily excited,
but it is very hard to excite the maximum energy mode.
Thus, the high frequency modes are very hard to exist
in the system of oscillators coupled in the SFN. This na-
ture is very different from that of the standard systems
of networks such as RN [11] and lattices [13] that there
are a small number of orders of nodes. This is the most
prominent characteristic of our system.

VI. THE MAXIMUM EIGENVALUE

The behavior of the maximum eigenvalue \( \lambda_{\text{max}} \) of the
adjacency matrix \( \hat{A} \) is very important in the network
theory [12, 13]. In the standard networks such as the random networks [2, 11], the maximum eigenvalue
\( \lambda_{\text{max}} \) cannot grow so fast as the network grows [12, 13,
46, 47, 48]. And also, as in solid state physics, networks
in most of physical systems provide the so-called energy
band that is a spectrum with a finite region [12, 13]. This
is due to the topology of the finite coordination number
of atoms in the network of the lattice structure [12, 13]. So,
in order to elucidate the difference between the SFNs and
other networks the growth of the maximum eigenvalue is
an important signature.

As was numerically studied by many authors [1, 2, 3,
43, 44, 45], the maximum eigenvalue \( \lambda_{\text{max}} \) of the adja-
cy matrix \( \hat{A} \) in the AB-model is proportional to
\( \sqrt{k_{\text{max}}} \) such that

\[
\lambda_{\text{max}} \propto \sqrt{k_{\text{max}}}. \tag{16}
\]

Here \( k_{\text{max}} \) means the maximum order of nodes in the
network such that \( k_{\text{max}} = \max_i \{ k_i \} \) (We will use this
notation for later purposes). And the numerical studies
showed that \( k_{\text{max}} \propto \sqrt{N} \). Therefore, we obtain

\[
\lambda_{\text{max}} \propto N^{1/4}. \tag{17}
\]

To see whether or not this is true in an arbitrary SFN
and to know how general it is, very recently, Chung, Lu
and Vu [15] have proved a very general theorem:

**Theorem 1** Suppose that the distribution of degrees of
nodes in a SFN is represented by \( P(k) \propto k^{-\gamma} \). Denote
by \( \langle k \rangle_2 \) the second order average degree of a node. This
is defined by

\[
\langle k \rangle_2 \equiv \frac{\sum_{i=1}^{N} k_i^2}{\sum_{i=1}^{N} k_i} = \frac{\langle k^2 \rangle}{\langle k \rangle}, \tag{18}
\]

where \( \langle k^p \rangle = \frac{1}{N} \sum_{i=1}^{N} k_i^p \) with \( p \) integer. Then, (C1)
if the exponent \( \gamma > 2.5 \), then

\[
\text{const.} \langle k \rangle_2 \leq \lambda_{\text{max}} \leq \text{const.} \sqrt{k_{\text{max}}}. \tag{19}
\]

(C2) If the exponent \( 2 < \gamma < 2.5 \), then

\[
\text{const.} \sqrt{k_{\text{max}}} \leq \lambda_{\text{max}} \leq \text{const.} \langle k \rangle_2. \tag{20}
\]

(C3) And if the exponent \( \gamma = 2.5 \), then a transition hap-
pens.

We note here that in the paper of Chung, Lu and Vu [15]
they used the notation \( \gamma \) for the second order average
degree, instead of our notation \( \langle k \rangle_2 \) for it. Applying the
above theorem to the AB-model of \( \gamma = 3 \), we find that
the AB-model belongs to the first category. Hence, the
theorem explains the numerical results [2, 4, 12, 46, 47].

In spite of such efforts, whether or not the growth of
the maximum eigenvalue of a physical model on the SFN
is not so well-known. This is because the eigenvalues
of the adjacency matrix \( \hat{A} \) is a nontrivial problem. So, in order to see this point, let
us consider the maximum eigenvalue \( \omega_{\text{max}} \) of our vibra-
tional system of oscillators.

We have performed calculations of the maximum eigen-
value (i.e., vibrational mode) \( \omega_{\text{max}} \) in our model of oscil-
lators on the AB-SFN, where \( m = 4 \) and \( N \) is developed
up to \( N = 10^4 \). This is shown in Fig.2. The maximum
eigenvalue \( \omega^2_{\text{max}} \) (circles), the maximum degree of a node
\( k_{\text{max}} \) (triangles), and the second order average degree \( \langle k \rangle_2 \) of a node (+) are shown, respectively. Here we have
obtained the following relation:

\[
\omega^2_{\text{max}} + 2\langle k \rangle_2 \leq \omega^2_{\text{max}} \leq \omega^2_{\text{max}} + 2k_{\text{max}}. \tag{21}
\]

This looks similar to the result of Eq. (19) such that

\[
\omega^2_{\text{max}} \propto \omega^2_{\text{max}} + \sqrt{k_{\text{max}}}. \tag{22}
\]

However, this is not supported by our numerical calcu-
lations. Therefore, as the spectral tail of our vibrational
model is different from that of the AB-model as discussed
in the previous section, so is the growth behavior of the
maximum eigenvalue of our vibrational model. This is
an important character of our physical model with the
AB-SFN.
The maximum eigenvalue $\omega^2$ belongs to the spectrum (a) the AB-model with $m=4$ and (b) the AB-model with $m=2$.
in the network theory. Denote by $\tilde{k}_i = (A_{i1}, \ldots, A_{iN})^t$ the $i$-th column vector of $\hat{A}$. The vector represents the way of links between the $i$-th node and other linked nodes, such that it defines the order $k_i$ of the $i$-th node such that

$$k_i = \sum_j A_{ij}.$$  

(29)

Therefore, let us call $\tilde{k}_i$ vectors the link vectors. Using this representation, we can rewrite the adjacency matrix as $\hat{A} = (\tilde{k}_1, \ldots, \tilde{k}_N)^t = (\tilde{k}_1^t, \ldots, \tilde{k}_N^t)^t$, where $t$ means the transpose. From this, we can derive that

$$\hat{A}^2 = (\tilde{k}_1 \cdot \tilde{k}_i),$$  

(30)

which is a symmetric matrix and nothing but the Gramian matrix between the link vectors, $\tilde{k}_i$, where

$$Tr(\hat{A}^2) = \sum_{i=1}^{N} \tilde{k}_i \cdot \tilde{k}_i = \sum_{i=1}^{N} k_i = 2L(\tau).$$  

(31)

Let us go back to Eq.(4). We now rewrite it as $\Omega \tilde{q} = K_0 \hat{A} \tilde{q}$. Therefore, $\Omega^2 \tilde{q} = K_0^2 \hat{A}^2 \tilde{q}$. Let us now use the Hadamard-Gerschgorin theorem \cite{13} (Appendix A) or the Perron-Frobenius theorem \cite{12} (Appendix B) for $K_0^2 \hat{A}^2$, we can derive an inequality

$$|\Omega^2_i| \leq K_0^2 \sum_{j=1}^{N(\tau)} |(\hat{A}^2)_{ji}| \frac{|q_j|}{q_i}. $$  

(32)

Since $|q_i|/|q_i| \leq 1$ and $(\hat{A}^2)_{ji} = (\hat{A})_{ji} = \tilde{k}_j \cdot \tilde{k}_i$, we can derive $|\Omega^2_i| \leq K_0^2 \sum_{j=1}^{N(\tau)} |(\hat{A}^2)_{ji}| = K_0^2 \sum_{j=1}^{N(\tau)} (\tilde{k}_j \cdot \tilde{k}_i) = K_0^2 \tilde{k}_i \cdot \tilde{k}_i = K_0^2 \tilde{k}_i$.

Then we have

$$\omega^2 - \omega_i^2 - \frac{K_0}{M_i} k_i(\tau) \leq \left( \frac{K_0}{M_i} \right)^2 \tilde{k}_i \cdot \tilde{k}_i, $$  

(34)

Therefore, it then yields a theorem:

**Theorem 6**

$$\omega^2 - \omega_i^2 - \frac{K_0}{M_i} k_i(\tau) \leq \left( \frac{K_0}{M_i} \right)^2 \tilde{k}_i \cdot \tilde{k}_i. $$  

(35)

Thus, there exists at least one atomic site (i.e., node) that satisfies Eq.(35) for all eigenvalues $\omega$. This implies that $\omega^2$ is included within a disk of radius $\frac{K_0}{M_i} \sqrt{\tilde{k}_i \cdot \tilde{k}_i}$ and its center $\omega_i^2 + \frac{K_0}{M_i} k_i(\tau)$. Since $|\omega^2 - \omega_i^2| - \left| \frac{K_0}{M_i} k_i(\tau) \right| \leq \left| \omega^2 - \omega_i^2 - \frac{K_0}{M_i} k_i(\tau) \right|$, we obtain

$$|\omega^2 - \omega_i^2| \leq \left( \frac{K_0}{M_i} \right) \left( k_i(\tau) + \sqrt{\tilde{k}_i \cdot \tilde{k}_i} \right). $$  

(36)

Therefore, for the maximum frequency we obtain

$$|\omega_{max}^2 - \omega_i^2| \leq \max_i \left[ \frac{K_0}{M_i} \left( k_i(\tau) + \sqrt{\tilde{k}_i \cdot \tilde{k}_i} \right) \right]. $$  

(37)

Hence, we obtain

$$\omega_{max}^2 \leq \omega_i^2 + \max_i \left[ \frac{K_0}{M_i} \left( k_i(\tau) + \sqrt{\tilde{k}_i \cdot \tilde{k}_i} \right) \right]. $$  

(37)

Since $k_i \leq \tilde{k}_i$ for all atomic sites, we have

$$\max_i \left[ \frac{K_0}{M_i} \left( k_i(\tau) + \sqrt{\tilde{k}_i \cdot \tilde{k}_i} \right) \right] \leq \max_i \left[ \frac{K_0}{M_i} \left( \sqrt{\tilde{k}_i \cdot \tilde{k}_i} \right) \right].$$

Therefore, the right hand of Eq.(37) is comparable with that of Eq.(26). In this way, Theorems 2-5 work for the SFN systems as well.

**VIII. CONCLUSIONS**

In conclusion, we have studied the system of oscillators connected by springs in the geometry of the AB SFN model. We first presented the Green function formalism for obtaining the spectrum of the vibrational modes of the system. In the case of very weak spring constant, using this formalism we find that the distribution of eigenmodes follows the same type of power-law distribution of degrees of a node in the AB model [see Eq.(13)] such that $P(\omega^2 - \omega_0^2) \propto (\omega^2 - \omega_0^2)^{-\gamma}$ with $\gamma = 3$. In the case of an arbitrary strength of spring constants, we have performed numerical calculations in order to obtain the spectrum of vibrational modes. We have found that even in this case, the distribution of eigenmodes obeys the same type of the power-law distribution of degrees of a node in the AB model as well [see Fig.1]. This is contrary to the distribution of eigenvalues of adjacency matrix in the AB model, where power-law distribution is given by $\rho(\lambda) \propto \lambda^{-\gamma'}$ with $\gamma' = 2\gamma - 1 = 5$.

This is a consequence of our model, where relative displacements between the individual oscillators are included in the Hamiltonian. This Hamiltonian provides the diagonal matrix elements in the eigenequation, which are proportional to the degrees of nodes [see Eq.(4)]. These diagonal elements can be regarded as on-site potentials in the problem. Since the degree of a node develops indefinitely, the on-site potential can be arbitrary large as the system is progressing. Therefore, the eigenvalues are strongly dominated by the magnitude of the diagonal elements of the eigenequation. Thus, the distribution of eigenmodes is affected by that of degrees of nodes such that the distribution of eigenmodes coincides with that of degrees of nodes in the network.

We finally have investigated the asymptotic behavior of the maximum eigenvalue $\omega_{max}$ of the system. We have found numerically that the maximum eigenvalue is bounded as in Eq.(21). From this, as the total number of nodes, $N$, is increasing, the maximum degree of
nodes becomes arbitrarily large. Therefore, the maximum eigenvalue can be arbitrarily large as \( N \to \infty \). This coincides with the result of the maximum eigenvalue of adjacency matrix in the AB model. We have also proved the above numerical results by some mathematical theorems that are proved using the Hadamard-Gerschgorin theorem and the Perron-Frobenius theorem.

Thus we conclude that when we apply a certain physical model to the geometry of a SFN, the physical properties are strongly dominated by the nature of the SFN. In this sense, not only the network geometry of a SFN but also the property of physical models on a SFN are important in the study of the SFN. This direction will be very interesting for further researches.

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APPENDIX A: THE HADAMARD-GERSCHGORIN THEOREM

The following theorem is known as the Hadamard-Gerschgorin theorem in linear algebra \([\text{[1]}\)\]. Consider the following eigenequation:

\[
(\lambda - h_i)q_i = \sum_{j=1, j \neq i}^{N} h_{ij}q_j, \quad (A.1)
\]

for \( i = 1, \ldots, N \). Then, we find

\[
|\lambda - h_i| \leq \sum_{j=1, (\neq i)}^{N} |h_{ij}| |q_j| / |q_i|, \quad (A.2)
\]

for \( i = 1, \ldots, N \). Since always \( |q_i| / |q_i| \leq 1 \), we obtain

\[
|\lambda - h_i| \leq \sum_{j=1, (\neq i)}^{N} |h_{ij}| \equiv B_i, \quad (A.3)
\]

for \( i = 1, \ldots, N \). Now, we find a theorem that there exists at least one site such that the above equation Eq.(A.3) is valid for all \( \lambda \). Eq.(A.3) means that \( \lambda \) is included within a disk of radius \( B_i \) with its center of \( h_i \).

**APPENDIX B: THE PERRON-FROBENIUS THEOREM**

The following theorem is known as the Perron-Frobenius theorem in linear algebra \([\text{[1]}\)\]. Suppose that an \( n \times n \) symmetric matrix \( H \) has all non-negative entries \( h_{ij} \geq 0 \). Then this satisfies an eigenequation \( H|\psi_i\> = \lambda_i|\psi_i\> \). For any positive constants \( c_1, c_2, \ldots, c_n \), the maximum eigenvalue \( \lambda_{\max}(H) \) satisfies

\[
\lambda_{\max}(H) \leq \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} c_j h_{ij} \right\}. \quad (A.4)
\]
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