Graphene and the Zermelo Optical Metric of the BTZ Black Hole

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May 25, 2012

Abstract

It is well known that the low energy electron excitations of the curved graphene sheet $\Sigma$ are solutions of the massless Dirac equation on a 2 + 1 dimensional ultra-static metric on $\mathbb{R} \times \Sigma$. An externally applied electric field on the graphene sheet induces a gauge potential which could be mimicked by considering a stationary optical metric of the Zermelo form, which is conformal to the BTZ black hole when the sheet has a constant negative curvature. The Randers form of the metric can model a magnetic field, which is related by a boost to an electric one in the Zermelo frame. We also show that there is fundamental geometric obstacle to obtaining a model that extends all the way to the black hole horizon.
1 Introduction

In a recent paper [1] it has been suggested that a surface of revolution with constant negative curvature (Beltrami Trumpet) made from graphene may exhibit some of the interesting effects which arise in quantum field theory in a curved spacetimes.

Of particular interest in this connection would be effects associated with BTZ black hole metrics [2]. These are solutions of Einstein’s equations in $2 + 1$ dimensions with negative cosmological constant and have been intensively studied in recent years because they also arise in string theory descriptions of black holes and their quantum microstates. The metric of the BTZ black hole is specified by the mass $M$, angular momentum $J$ and the negative cosmological constant $\Lambda \equiv -l^{-2}$. Its metric is of the form:

$$ds^2_{BTZ} = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2 \left( d\phi - \frac{J}{2r^2} dt \right)^2,$$

where

$$\Delta(r) = r^2 - M + \frac{J^2}{4r^2}.$$  \hspace{1cm} (2)

It is well known (see [3] for a review) that on a curved graphene sheet $\Sigma$ there are low energy electronic excitations which satisfy the massless Dirac equation on $\mathbb{R} \times \Sigma$ with respect to the metric

$$ds^2 = -dt^2 + h_{ij} dx^i dx^j,$$

where $h_{ij}$ is the metric induced on the sheet $\Sigma$. The case considered in [1] is a surface of revolution in Euclidean three space $\mathbb{E}^3$ with coordinates $(x(x'), y(x'), z(x'))$.

The metric (3) is an example of an ultra-static metric. That is, it is static, i.e. invariant under both time translations and time reversal. The general static metric takes the local form

$$ds^2 = -V dt^2 + g_{ij} dx^i dx^j.$$ \hspace{1cm} (4)

For a static black hole we have $V > 0$ outside the horizon which is located at $V = 0$. A static metric is ultrastatic if $V = 1$. An ultra-static metric is one for which the gravitational red shifting does not occur, i.e one for which $g_{tt}$ is independent of the spatial coordinates. As a consequence a massive particle may remain at rest in such a spacetime because it suffers no gravitational attraction. Since in the case of graphene, there is no obvious source of red shifting, the assumption made in [1] that $g_{tt} = -1$ appears to be physically well justified. Clearly a black hole metric cannot be ultra-static.

However the massless Dirac equation is invariant under conformal rescalings, that is, in $D$ spacetime dimensions, if

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \tilde{\Psi} = \frac{1}{\Omega^{D-1}} \Psi,$$ \hspace{1cm} (5)

we have

$$\gamma^\mu \nabla_\mu \tilde{\Psi} = \frac{1}{\Omega^{D-1}} \tilde{\gamma}^\mu \nabla_\mu \Psi.$$ \hspace{1cm} (6)

Moreover, any static metric is locally conformally ultra static, that is

$$-V dt^2 + g_{ij} dx^i dx^j = V \left\{ -dt^2 + h_{ij} dx^i dx^j \right\},$$ \hspace{1cm} (7)

where $h_{ij} = \frac{1}{V} g_{ij}$ is called [4] the optical metric of the static metric on the left hand side of (7), classically at least there might appear to be no obstacle to mimicking the effects of horizons on massless fermions using ultra-static metrics for graphene of the form (3), as suggested in [1]. In
quantum field theory however, the choice of vacuum state is not necessarily conformally invariant and so the mimicking the Unruh or Hawking effects is not obviously possible.

However not all time-independent (i.e. stationary) metrics are static. In particular there are rotating black holes in in $2 + 1$ dimensions, including BTZ black holes, which are stationary. Their metric take the general form

$$ds^2 = -V(dt + \omega_i dx^i) + g_{ij} dx^i dx^j.$$ (8)

The effect of the rotation is to introduce an so-called gravito-magnetic field, whose effect on particles resembles that of a magnetic field. Now it is possible to consider graphene subject to an externally applied magnetic field. This would induce a gauge field $A_i$ on the graphene sheet $\Sigma$. From a gravitational point of view such a gauge connection, which breaks time reversal invariance but not conformal invariance, would mimic a stationary rather than a static metric.

Locally any stationary metric may be brought by a conformal rescaling [5] to one of two forms [5]. The Randers form

$$ds^2_R = -(dt + \omega_i dx^i)^2 + a_{ij} dx^i dx^j,$$ (9)

where $a_{ij} = V^{-1} g_{ij}$ and $\omega_i dx^i$ may be thought of as the gravito-magnetic vector potential.

Using a different conformal rescaling and completely the square differently we obtain the Zermelo form

$$ds^2_Z = -dt^2 + h_{ij}(dx^i - W^i dt)(dx^j - W^j dt)$$ (10)

where $h_{ij}$ is called the Zermelo optical metric and $W^i$ the wind vector field. Roughly speaking, and as we shall make more precise later, the effect of the wind is similar to that of an electric field. The relation between these two forms and their physical significance may be found in [5]. Note that the form of the metric we have given for the rotating BTZ black hole (1) is, up to a conformal factor $\Delta$, already in Zermelo form.

The Randers optical form of the conformally rescaled optical metric of the BTZ black hole (1) is considerably more complicated. It may be written as

$$ds^2_R = -(dt + \frac{J}{2\Delta_0} d\phi)^2 + \frac{dr^2}{\Delta_0} + r^2 \frac{\Delta}{\Delta_0} d\phi^2,$$ (11)

where

$$\Delta_0 = r^2 - M,$$ (12)

is the conformal factor. There is an ergosphere at the positive zero of $\Delta_0$, i.e. at $r = lM$, which lies outside the horizon which is located at $r = r_+$, where $r_+$ is the outermost zero of $\Delta$, that is

$$r_+^2 = \frac{1}{2} \left(lM + \sqrt{(lM)^2 - J^2} \right).$$ (13)

The ergoregion leads to complications near the horizon, and so in what follows we shall mainly focus our attention on the Zermelo form.

When considering the Dirac equation, we need to introduce an orthonormal frame or triad of vector fields $e_a$, $a = 0, 1, 2$. The frames $e^R_a$ and $e^Z_a$, which are natural to introduce for the respective Randers and the Zermelo form of the metric, differ and they may be thought of as as boosted with respect to one another. In the Randers case the natural frame has a timelike leg of the form

$$e^R_0 = \frac{\partial}{\partial t}.$$ (14)
In the Zermelo case the timelike leg has the form
\[ e_0^\mu = \partial_{\mu t} + W^i \partial_{\mu x^i} \tag{15} \]

These two differ by the wind term. The result of the boost is that a magnetic vector potential in one frame may appear as an electric vector potential in the other.

In fact, we shall show explicitly later, that, at least in the case of a axial symmetry, the effect of the wind is equivalent to an induced electric connection \( A_\mu \). Thus, as we shall show in detail later in the paper, we could in principle construct a graphene analogue of the conformal geometry of a rotating 2 + 1 -dimensional black hole such as that of BTZ \([2]\). Actually as we shall show later, there is a fundamental geometric obstacle, already encountered in \([1]\) to obtaining a model system which extends all the way to the horizon.

The paper is organized in the following way. In Section 2 we show that only a part of the BTZ optical metric, which lies outside the outer horizon, can be mapped onto the Beltrami Trumpet, an example of the axisymmetric metrics which have an isometric embedding into thee-dimensional Euclidean space \( \mathbb{E}^3 \). In Section 3 we embed the exact static as well as rotating BTZ black hole optical metric into \( \mathbb{E}^3 \). In the latter case the metric is of the Zermelo type. In Section 4 we derive the form of the external electro-magnetic field applied to a graphene. In Section 5 we analyze the Dirac equation for fermions confined to the graphed sheet with the applied elector-magnetic field; the Zermelo axissymmetric metrics model an electric field, while the Randers metrics model a magnetic field in a frame that is boosted relative to the Zerlemo metric frame. Conclusions and possible other applications of optical metrics are given in Section 6.

2 Surfaces of revolution

From now on we shall confine attention to axisymmetric metrics which may be isometrically embedded in Euclidean space \( \mathbb{E}^3 \) as surfaces of revolution. We have
\[ h_{ij}dx^idx^j = d\rho^2 + C^2(\rho)d\phi^2, \quad 0 \leq \phi < 2\pi, \tag{16} \]
so that
\[ C^2(\rho) = x^2 + y^2 = R^2 \tag{17} \]
and
\[ d\rho^2 = dR^2 + dz^2. \tag{18} \]
The Gauss curvature is given by
\[ K = -\frac{C''}{C}. \tag{19} \]

The example introduced in \([1]\) is given by the Beltrami Trumpet
\[ C(\rho) = a \exp\left(-\frac{\rho}{a}\right), \quad \rho \geq 0, \quad \Rightarrow \quad K = -\frac{1}{a^2}. \tag{20} \]

If
\[ w = a\phi + ia \exp\left(\frac{\rho}{a}\right), \tag{21} \]
then
\[ h_{ij}dx^idx^j = \frac{a^2|dw|^2}{(3w)^2}, \tag{22} \]
which would be the standard model of $H^2$ as the upper half complex plane, if $\Im w \geq 0$ but we must quotient by the $\mathbb{Z}$ action $w \rightarrow w + 2\pi a$ and moreover take $\Im w \geq a$. As is well known, the Beltrami Trumpet is obtained by revolving a tractrix curve about the $z$-axis. As one may verify, the tractrix is the locus of one end of a light rope of length $a$ to which is attached to a heavy weight as the other end is dragged along the negative $z$ axis. Initially the rope occupies the interval $0 \leq x \leq a$, $y = 0$, $z = 0$ and there is a half cusp at $(a, 0, 0)$.

This failure to embed all of $H^2/\mathbb{Z}$ is not an artifact of our symmetry assumptions. A theorem of Hilbert implies that we cannot get a non-singular embedding of any complete metric of constant negative curvature into three dimensional Euclidean space. As we see shortly this in turn implies that no non-singular embedding of the optical metric of a 2 + 1 dimensional black hole can never reach the horizon. It can only extend a finite way to the horizon, which is at infinite optical distance.

For vanishing angular momentum ($J = 0$) the BTZ metric (1) takes the form:

$$ds_{BTZ}^2 = -(\frac{r^2}{l^2} - M)dt^2 + \frac{dr^2}{(\frac{r^2}{l^2} - M)} + r^2d\phi^2, \quad 0 \leq \phi < 2\pi.$$ (23)

The associated optical metric is

$$ds_o^2 = -dt^2 + \frac{dr^2}{(\frac{r^2}{l^2} - M)^2} + \frac{r^2}{(\frac{r^2}{l^2} - M)}d\phi^2, \quad \Omega^2 = (\frac{r^2}{l^2} - M).$$ (24)

We set

$$r_+ = l\sqrt{M}, \quad a = 2l, \quad (r - r_+) = \frac{1}{8}r_+ \exp(-\frac{2\rho}{a}),$$ (25)

and expand about the horizon to see that

To lowest order the region $\hat{r} \geq \frac{9}{2}r_+$ of the optical metric of BTZ black hole maps onto the Beltrami spacetime.

### 3 Exact Static and Rotating BTZ Black Holes

In this Section we shall embed the exact BTZ optical geometry into $E^3$. We do that first for the static BTZ optical geometry (24). Thus

$$ds_o^2 = -dt^2 + \frac{dr^2}{(\frac{r^2}{l^2} - a^2)^2} + \frac{L^2r^2}{r^2 - a^2}d\phi^2,$$ (26)

where $a = l\sqrt{M}$. We introduce:

$$C^2 = \frac{l^2r^2}{r^2 - a^2},$$ (27)

and thus

$$dz^2 + dC^2 = \frac{l^4dr^2}{(r^2 - a^2)^2}, \quad \Rightarrow \quad dz^2 = l^2(\frac{r^2 - a^2}{(r^2 - a^2)^2})dr^2.$$ (28)

Thus

$$\left(\frac{dz}{dC}\right)^2 = 1 + \frac{a^2}{C^2} - \frac{C^2}{r^2}.$$ (29)

Clearly the embedding must stop at the radius for which $C = \sqrt{l^2 + a^2}$. This is outside the horizon for which $C^2 \rightarrow \infty$. The radial optical distance $\rho$ is given by

$$d\rho = l^2 \frac{dr}{r^2 - a^2},$$ (30)
Thus
\[ r/a = \coth(a/l\rho), \quad C = l \cosh(a/l\rho), \] (31)
and so the optical metric is
\[ ds^2_o = -dt^2 + d\rho^2 + l^2 \cosh^2(a/l\rho)d\phi^2, \] (32)
and the BTZ metric itself is
\[ ds^2_{\text{BTZ}} = \frac{a^2}{\sinh^2(a/l\rho)} \left\{ -dt^2 + d\rho^2 + l^2 \cosh^2(a/l\rho)d\phi^2 \right\}. \] (33)

Note that the Gaussian curvature of the spatial part of the BTZ optical metric is of constant negative curvature. Note also that the BTZ optical geometry is locally of the form \( \mathbb{R} \times H^2 \) and hence conformally flat.

Assuming that such graphene sheets with negative constant curvature can be made in the Laboratory such BTZ Beltrami Trumpets could also be made.

We now turn to the rotating BTZ black hole which has non-zero angular momentum \( J \). Its metric (1) is conformal to the Zermelo metric (10)
\[ ds^2 = -dt^2 + dr^2 \Delta + r^2 \Delta (d\phi - J^2 r^2 dt)^2. \] (34)
and the conformal factor \( V = \Delta \), specified by (2). The Zermelo metric coefficients are thus of the form:
\[ h_{ij}dx^idx^j = \frac{dr^2}{\Delta^2} + \frac{r^2}{\Delta}d\phi^2, \]
\[ W^i\partial_i = \frac{J}{2r^2}\partial_\phi. \] (35)

In the near horizon limit described above \( h_{ij} \) is of Beltrami Trumpet form.

### 4 Gauge Fields

In this Section we study external electric-magnetic fields applied to a graphene surface of revolution. Such external electromagnetic fields will induce a non trivial ones on the surface of the graphene.

In the magnetic case, we are interested in two cases. One in which the magnetic field in the bulk has constant magnitude \( B_0 \). The second is when the magnetic field normal to the surface \( B_n \) has constant magnitude \( B_c \). Suppose that the bulk gauge field \( A \) and its field-strength \( F \) are given by
\[ A_\mu dx^\mu = A(C)d\phi, \quad \Rightarrow \quad F = \frac{A'}{C}CdC \wedge d\phi = \frac{A'}{C}dx \wedge dy. \] (36)

In the embedding space this will correspond to a magnetic field
\[ B_z = \frac{A'}{C} \] (37)

Thus an applied uniform magnetic field with strength \( B_0 \) has the gauge potential has \( A(C) = B_0 C \). On the surface however since
\[ F = \frac{A'(C)}{C} \frac{dC}{d\rho} d\rho \wedge C d\phi = \frac{A'(C)}{C} \frac{dC}{d\rho} \eta, \] (38)
where $\eta = d\rho \wedge C d\phi$ is the area 2-form on the surface, the normal field strength $B_n$ is

$$B_n = \frac{A'(C)}{C} \frac{dC}{d\rho}.$$  
(39)

Thus a constant field $B_0$ will produce a normal field

$$B_n = B_0 \frac{dC}{d\rho} = B_0 \frac{a}{l} \tanh \left( \frac{a}{l} \rho \right)$$  
(40)

In order to produce a uniform field on the surface we need to set

$$\frac{A'(C)}{C} \frac{dC}{d\rho} = B_c = \text{constant}.$$  
(41)

Thus

$$\frac{dA}{d\rho} = B_c C.$$  
(42)

That is

$$B_z = B_c \left( \frac{dC}{d\rho} \right)^{-1} = \frac{l B_c}{a \sinh \left( \frac{4}{a} \rho \right)}.$$  
(43)

Note that the angle $\psi$ that the meridians make with the vertical direction is given by

$$\sin \psi = \frac{dC}{d\rho},$$  
(44)

These formulae are geometrically rather obvious. in particular we have

$$B_n = B_z \sin \psi.$$  
(45)

An analogous analysis can also be performed for the electric field. In this case we have a bulk vector potential

$$A_\mu dx^\mu = \Phi(R) dt,$$  
(46)

which would, if the graphene sheet were non-conducting, induce a radial electric field on the surface

$$F = \frac{d\Phi(R)}{d\rho} \frac{dC}{d\rho} \wedge dt,$$  
(47)

where $R = C(\rho)$. Thus

$$F = \frac{d\Phi(R)}{dR} \frac{dC}{d\rho} \wedge dt.$$  
(48)

### 5 The Dirac equation

In this Section we study the Dirac equation for fermions confined to the graphene sheet. The Dirac equation is of the form:

$$\left( \gamma^i \nabla_i + \gamma^0 \partial_t \right) \Psi = 0,$$  
(49)

where $(i, j)$ are dyad indices and

$$de^i = -\omega^i_j \wedge e^j,$$  
(50)

and

$$\nabla \Psi = d\psi + \frac{1}{4} \omega_{ij} \gamma^i \gamma^j \Psi.$$  
(51)
For a surface of revolution metric the pseudo-orthonormal one forms are (We use $- + +$ signature):
\[ e^1 = d\rho, \quad e^2 = C(\rho)d\phi, \]  
(52)
and
\[ de^2 = \frac{C'}{C} e^1 \wedge e^2, \quad \Rightarrow \quad \omega_{21} = \frac{C'}{C} e^2. \]  
(53)
Thus
\[ \nabla = d + \gamma^2 \gamma_1 \frac{C'}{2C} e^2, \]  
(54)
and the Dirac equation can be cast in the form:
\[ \gamma^1 \frac{1}{\sqrt{C}} \partial_\rho (\sqrt{C}\Psi) + \gamma^2 \frac{1}{C} \partial_\phi \Psi + \gamma^0 \partial_t \Psi = 0, \]  
(55)
If we set
\[ \sqrt{C}\Psi = \tilde{\Psi}, \gamma^i \gamma^j = \gamma^0 \gamma^i \]  
(56)
we obtain
\[ \tilde{\gamma}^1 \partial_\rho \tilde{\Psi} + \tilde{\gamma}^2 \frac{1}{C} \partial_\phi \tilde{\Psi} - \partial_t \tilde{\Psi} = 0. \]  
(57)
Both $\gamma^3$ and $\gamma^0 \gamma^1 \gamma^2$ commute with (57) and themselves, and have eigenvalues $\pm$. The solution can therefore be cast in terms of eigenvalues of these matrices.

We now turn to the axisymmetric optical Zermelo metric, written in the form:
\[ ds^2 = -dt^2 + d\rho^2 + C^2(\rho)(d\phi - W(\rho)dt)^2. \]  
(58)
Note that the rotating BTZ solution is of this form. A pseudo-orthonormal basis of one-forms is defined as:
\[ e^0 = dt, \quad e^1 = d\rho, \quad e^2 = C(d\phi - Wdt), \]  
(59)
and a dual basis of vector fields by
\[ e_0 = \partial_t + W\partial_\rho, \quad e_1 = \partial_\rho, \quad e_2 = \frac{1}{C}\partial_\phi. \]  
(60)
Note that $e_a \phi = \delta^0_a W(\rho)$ and so the Dreibein $e_a$ is differentially rotating. Using the formulae
\[ de^a = -\omega^a_b \wedge e^b, \quad \omega_{ab} = \eta_{ac}\omega^c_b = \omega_{ca}, \]  
(61)
with $a = 0, 1, 2$ and $\eta_{ab} = \text{diag}(-1, 1, 1)$ we find
\[ \omega_{01} = -\frac{1}{2}CW'e^2, \quad \omega_{02} = -\frac{1}{2}CW'e^1, \quad \omega_{21} = -\frac{1}{2}CW'e^0 + \frac{C'}{C}e^2. \]  
(62)
The massless Dirac equation may be written as
\[ \left( \gamma^a e_a + \frac{1}{4} \gamma^a \omega_{abc} \gamma^b \gamma^c \right) \Psi = 0, \]  
(63)
where $\omega_{abc}$ are what are sometimes called Ricci rotation coefficients
\[ \omega_{bc} = \epsilon^{a}_{bc} \omega_{abc}. \]  
(64)
Thus
\[
\left( \gamma^1 (\partial_\rho + \frac{1}{2} \frac{C'}{C}) + \gamma^2 \frac{1}{C} \partial_\phi + \gamma^0 (\partial_t + W \partial_\phi) + \frac{1}{4} \gamma^0 \gamma^1 \gamma^2 CW' \right) \Psi = 0. \tag{65}
\]
If we choose
\[
\gamma^0 = i \sigma_2, \quad \gamma^1 = \sigma_1, \quad \gamma^2 = \sigma_3. \tag{66}
\]
then \(\gamma^0 \gamma^1 \gamma^2 = 1\) and we get a position dependent “mass-like “ term and a connection term. If \(\Psi \propto e^{-i\omega t + im\phi}\) we have that
\[
- ieA_0 = imW, \quad \Rightarrow eA_0 = -mW, \tag{67}
\]
and the effective electric field in what is actually a rotating frame is
\[
eF = edA = mW' d\rho \wedge dt = mW'e^1 \wedge e^0. \tag{68}
\]
Therefore a stationary Zermelo metric induces in the Dirac equation an effective, position dependent radial electric field. The bulk electric potential \(A_\mu dx^\mu = \Phi(R)dt\) is obtained by setting
\[
\Phi(R) = \frac{m}{e} W(\rho). \tag{69}
\]
From (1) and (2) we have
\[
R^2 = \frac{r^2}{r^2} - M + \frac{J^2}{4r^2}, \tag{70}
\]
and
\[
W = \frac{J}{2r^2}, \quad \Rightarrow \quad r^2 = \frac{J}{2W}. \tag{71}
\]
Elimination leads to the quadratic equation
\[
\frac{1}{l^2} - \frac{1}{R^2} - 2 \frac{WM}{J} + W^2 = 0, \tag{72}
\]
which may be solved to give \(\Phi(R)\).

We could have solved the Dirac equation in the Randers form of the metric (9). We define
\[
e^0 = (dt + \omega d\phi), \quad e^\rho = d\rho, \quad e^\phi = C(\rho)d\phi. \tag{73}
\]
\[
e_0 = \frac{\partial}{\partial t}, \quad e_\rho = \frac{\partial}{\partial \rho}, \quad e_\phi = \frac{1}{C} \left( \frac{\partial}{\partial \phi} - \frac{\omega}{\partial t} \right). \tag{74}
\]
We find
\[
\omega^\phi _\rho = -\frac{\omega'}{2C} e^0 + \frac{C'}{C} e^\phi, \quad \omega^0 _\rho = \frac{\omega'}{2C'} e^\phi, \quad \omega^0 _\phi = -\frac{\omega'}{2C} e^\rho. \tag{75}
\]
Because in this case the roles of \(t\) and \(\phi\) have essentially been interchanged, we now find that there is an effective magnetic vector potential in the Dirac equation. Therefore, as pointed out in the introduction the magnetic vector potential in the Randers frame appears as an electric potential in the Zermelo one.
6 Conclusions

We argue that the curved graphene sheet with negative constant curvature in the externally applied magnetic field could be modeled by considering a stationary optical metric of the Zermelo form which is conformal to the BTZ black hole. In particular, the electric field on the surface of the graphene can be modeled with the wind of the Zermelo metric. On the other hand the Randers metric models a magnetic field, related to the electric one of the Zermelo frame by a boost. Furthermore, we establish that there is a fundamental geometric obstruction to obtain a model that extends all the way to the BTZ black hole horizon. We model the low energy electron excitations of such graphene sheets by studying solutions of the 2 + 1-dimensional massless Dirac equation in the stationary optical metric, conformal to the BTZ black hole.

Related analyses can be applied to any other system described with a two-dimensional curved surface Σ. We should also point out that there are other possible embeddings of a surface with constant negative curvature which are not a surface of revolution but a twisted Beltrami Trumpet. It embeds more of $H^2$ than the Beltrami Trumpet. It would be interesting to further study implications of such more general embeddings.

Acknowledgments: G.W.G thanks the UPenn Department of Physics & Astronomy Department for hospitality. MC is supported by the DoE Grant DOE-EY-76-02- 3071, the NSF RTG DMS Grant 0636606, the Fay R. and Eugene L. Langberg Endowed Chair and the Slovenian Research Agency (ARRS).

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\footnote{Note that there the third author is on [v1], only and that [v1] and [v2] differ considerably.}