BOSONIC LAPLACIANS, THE GENERALIZED MAXWELL OPERATORS AND THEIR PROPERTIES

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Abstract. In this paper, we study boundary value problems involving Bosonic Laplacians in the upper-half space and the unit ball. Poisson kernels in the upper-half space and the unit ball are constructed, which give us solutions to the Dirichlet problems with $L^p$ boundary data, $1 \leq p \leq \infty$. We also prove the uniqueness of solutions to the Dirichlet problem with continuous data for the generalized Maxwell equations and provide several analogs of properties of harmonic functions, such as Cauchy’s estimates, mean-value property, Liouville’s theorem, Harnack’s inequality, representation formula, etc. for null solutions of the generalized Maxwell equations.

Key words: Bosonic Laplacians, The generalized Maxwell operators, Dirichlet problem, Mean-value property, Maximum principle, Liouville’s theorem, Harnack’s inequality, Cauchy’s estimates.

2000 Mathematics Subject Classification: 32A50, 35J25, 35B50, 35B53, 35C15

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1. Introduction

Boundary value problems are extremely important as they model a large amount of phenomena and applications, such as solid mechanics, heat transfer, fluid mechanics, acoustic diffusion, etc. Among the earliest boundary value problems to be investigated are the Dirichlet problems of Laplace’s equation and Poisson’s equation, which are frequently studied in many branches of physics, for instance, electrostatics, gravitation and fluid dynamics. Thus, the study of boundary value problems of Laplace’s equation is considered as one of the most important topics in classical harmonic analysis, more details can be found in [3, 29]. Here we investigate a second order differential operator in Euclidean space whose properties are very similar to those of the Laplacian by applying certain aspects of Clifford analysis.

Clifford analysis is considered as not only a higher dimensional function theory offering a generalization of complex analysis but also a refinement of classical harmonic analysis. Clifford analysis is centered around the study of the Dirac operator and monogenic functions (null solutions of the Dirac operator). In this setting, Dirichlet problems have also been studied by many researchers. For instance, solutions to the Dirichlet problems was studied with quaternionic analysis in [20]. Delanghe and Qian [8] studied half Dirichlet problems for monogenic functions on a cylindrical domain. These solutions were obtained with projections in the Hodge decomposition of quaternion-valued $L^2$ space and a Teodorescu transform. In [14], the Dirichlet boundary value problem for monogenic functions was studied on a cylindrical domain. Delanghe and Qian [8] studied half Dirichlet problems for monogenic functions in the upper-half space and the unit ball by introducing two idempotent functions. This technique was also applied in [24] to study Dirichlet problems involving the iterated Dirac operator in the unit ball. More results on this topic can be found in [20, 21, 22, 23, 25].

The study in the higher spin theory in the framework of Clifford analysis is a rather new topic and it received a considerable amount of attention during the last decades. In this setting, we study operators and function theory on functions taking values in arbitrary irreducible representations of the spin group, which are usually realized as the spaces of homogeneous harmonic (monogenic) polynomials. The first investigation on this topic via Clifford analysis is a systematic study of a class of generalized Rarita-Schwinger operators given by Bureš et. al. [5]. They generalized the classical Rarita-Schwinger equations for spin $3/2$ fields to the case of functions taking values in irreducible representations of the spin group of weight $k + 1/2$. In [13], the authors reinvestigated the Rarita-Schwinger operator with an analytic approach, which provides fundamental solutions, conformal invariance property and some integral formulas for the Rarita-Schwinger operator. Stein and Weiss [28] pointed out that the Dirac operator could also be constructed as a projection to a unique component of a tensor product of two irreducible representations of the spin group. From then on, differential operators constructed with this technique were named as Stein-Weiss gradients. In [12], the authors showed that the Rarita-Schwinger operators could be reconstructed as Stein-Weiss gradients as well. Eelbode et. al. [16] and De Bie et. al. [7] discovered the generalized Maxwell operators and Bosonic Laplacians (also called the higher spin Laplace operators), which act on functions taking values in the spaces of complex-valued homogeneous harmonic polynomials. These second-order conformally invariant differential operators are considered as generalizations of Laplacian with respect to the conformal invariance property in the higher spin theory in Clifford analysis. In [13], the authors constructed other higher-order conformally invariant differential operators in
the higher spin setting with a fundamental solution approach. In [17, 18, 30], conformally invariant operators acting on functions taking values in more general irreducible representation spaces, called simplicial harmonic (monogenic) polynomials, of the spin group were studied. Further, Clerc and Ørsted [6] used a representation-theoretic framework to show the relations between these conformally invariant differential operators and Knapp-Stein intertwining operators.

In [10, 11], the authors started an investigation of integral formulas for Bosonic Laplacian and the higher-order fermionic operators. In these papers, the Borel-Pompeiu formula, Cauchy’s integral formula and Green’s integral formula for the conformally invariant differential operators in the higher spin theory, were provided for the first time. These results are of interest and motivate a study of boundary value problems involving the aforementioned operators which have not been well understood yet.

In this paper, we carry out an investigation of boundary value problems involving Bosonic Laplacians and the generalized Maxwell operators. The intricate form of these operators, together with the interaction of the two variables and the rotation action on the second variable, highly complicates the analysis and leads to the invalidity of some classical techniques. The contribution of the paper is the construction of Poisson kernels in the upper-half space and the unit ball, which leads to the existence of solutions to Dirichlet problems with $L^p$ boundary data, $1 \leq p \leq \infty$. The paper also shows the counterpart of important results regarding Laplace operator such as the mean-value property, strong maximum principle, Liouville’s theorem, Harnack’s inequality and representation formula. These results form the basis for further study of PDEs involving Bosonic Laplacians.

Acknowledgements. Chao Ding and Phuoc-Tai Nguyen are supported by Czech Science Foundation, project GJ19-14413Y.

2. Preliminaries

Let $\{e_1, \cdots, e_m\}$ be a standard orthonormal basis for the $m$-dimensional Euclidean space $\mathbb{R}^m$. The (real) Clifford algebra $\mathcal{C}l_m$ is generated by $\mathbb{R}^m$ with the relationship

$$e_i e_j + e_j e_i = -2 \delta_{ij}, \quad 1 \leq i, j \leq m.$$  

An arbitrary element of the basis of the Clifford algebra can be written as $e_A = e_{j_1} \cdots e_{j_r}$, where $A = \{j_1, \cdots, j_r\} \subseteq \{1, 2, \cdots, m\}$ and $1 \leq j_1 < j_2 < \cdots < j_r \leq m$. Hence for any element $a \in \mathcal{C}l_m$, we have $a = \sum_A a_A e_A$, where $a_A \in \mathbb{R}$, in particular, $e_\emptyset = 1$. The $m$-dimensional Euclidean space $\mathbb{R}^m$ is embedded into $\mathcal{C}l_m$ with the following mapping

$$\mathbb{R}^m \rightarrow \mathcal{C}l_m,$$

$$x = (x_1, \cdots, x_m) \rightarrow \sum_{j=1}^m x_j e_j.$$  

For $x \in \mathbb{R}^m$, one can easily obtain that $|x|^2 = \sum_{j=1}^m x_j^2 = -x^2$. The complex Clifford algebra $\mathcal{C}l_m(\mathbb{C})$ is defined by $\mathcal{C}l_m(\mathbb{C}) = \mathcal{C}l_m \otimes \mathbb{C}$. For $a = \sum_A a_A e_A \in \mathcal{C}l_m$, we define the reversion of $a$ as

$$\tilde{a} = \sum_A (-1)^{|A|(|A|-1)/2} a_A e_A,$$
where $|A|$ is the cardinality of $A$. In particular, $e_{j_1} \cdots e_{j_r} = e_{j_r} \cdots e_{j_1}$. Also $\tilde{a}b = \tilde{b}\tilde{a}$ for $a, b \in \mathcal{C}_m$. 

Suppose $U \subset \mathbb{R}^m$ is a domain, a diffeomorphism $\phi : U \rightarrow \mathbb{R}^m$ is said to be conformal if, for each $x \in U$ and each $u, v \in TU_x$, the angle between $u$ and $v$ is preserved under the corresponding differential at $x$, $d\phi_x$. For $m \geq 3$, a theorem of Liouville tells us that the only conformal transformations are Möbius transformations. Ahlfors and Vahlen showed that any Möbius transformation on $\mathbb{R}^m \cup \{\infty\}$ can be expressed as $y = (ax + b)(cx + d)^{-1}$ with $a, b, c, d \in \mathcal{C}_m$ satisfying the following conditions [2]:

1. $a, b, c, d$ are all products of vectors in $\mathbb{R}^m$.
2. $\tilde{a}b, \tilde{c}d, \tilde{b}c, \tilde{d}a \in \mathbb{R}^m$.
3. $a\tilde{d} - b\tilde{c} = \pm 1$.

The third condition is usually called the normalization condition of the Möbius transformation. Since $y = (ax + b)(cx + d)^{-1} = ac^{-1} + (b - ac^{-1})d(cx + d)^{-1}$, a conformal transformation can be decomposed as compositions of translation, dilation, reflection and inversion. This gives an Iwasawa decomposition for Möbius transformations. See [20] for more details.

Now suppose $a \in S^{-1} \subset \mathbb{R}^m$ and $x \in \mathbb{R}^m$. If we consider $axa$, we may decompose $x = x_{a||} + x_{a\perp}$, where $x_{a||}$ is the projection of $x$ onto $a$ and $x_{a\perp}$ is the remainder part of $x$ perpendicular to $a$. Hence $x_{a||}$ is a scalar multiple of $a$ and we have $axa = ax_{a||}a + ax_{a\perp}a = -x_{a||}a + x_{a\perp}$.

So the action $axa$ describes a reflection of $x$ in the direction of $a$, i.e., $axa = O_a x$, where $O_a \in O(m)$. More details can be found in, for instance, [9].

The classical Dirac operator is defined as $D_x = \sum_{j=1}^m \partial_j e_j$, which factorizes the Laplace operator $\Delta_x = -D^2_x$. Let $H_k$ $(1 \leq k \leq n)$ be the space of real-valued homogeneous harmonic polynomials of degree $k$ in $m$-dimensional Euclidean space. Now consider a function $f(x, u) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m, H_k)$, i.e., for a fixed $x \in \mathbb{R}^m$, $f(x, u) \in H_k$ with respect to $u \in \mathbb{R}^m$. Recall that Bosonic Laplacian [10] is defined as

$$D_k : C^\infty(\mathbb{R}^m \times \mathbb{R}^m, H_k) \rightarrow C^\infty(\mathbb{R}^m \times \mathbb{R}^m, H_k),$$

$$D_k = \Delta_x - \frac{4}{m + 2k - 2} \langle u, D_x \rangle \langle D_u, D_x \rangle + \frac{4|u|^2 \langle D_u, D_x \rangle^2}{(m + 2k - 2)(m + 2k - 4)}, \quad (2.1)$$

where $\langle \ , \ \rangle$ is the standard inner product in $\mathbb{R}^m$. In particular, when $k = 1$, it is the generalized Maxwell operator and it reduces to the classical Maxwell equations given in terms of Faraday-tensor when $m = 4, k = 1$ with signature $(-, +, +, +)$. More details can be found in [10].

3. Dirichlet problems of Bosonic Laplacians

In this section, we will investigate Dirichlet problem involving Bosonic Laplacian in the upper-half space and the unit ball with different boundary data.
3.1. Poisson kernel in the upper-half space. We start this subsection by introducing some technical lemmas for the complex-valued homogeneous harmonic polynomials.

**Lemma 3.1** (Lemma 6, [15]). Let $2 < m \in \mathbb{N}$ and $1 \leq k \in \mathbb{N}$. Suppose $h_k : \mathbb{R}^m \rightarrow Cl_m(\mathbb{C})$ is a harmonic polynomial homogeneous of degree $k$. For any $u \in S^{m-1}$, there holds

$$
\frac{m + 2k - 2}{(m - 2)\omega_m} \int_{S^{m-1}} h_k(xux) dS(x) = c_k h_k(u),
$$

where $\omega_m$ is the surface area of the unit sphere $S^{m-1}$.

For $x \in \mathbb{R}^m$, we write $x = (x', y)$ with $x' = (x_1, \ldots, x_{m-1}) \in \mathbb{R}^{m-1}$ and $y > 0$. Now, we claim that we have a similar result for the homogeneous harmonic polynomials in the upper-half space, which is critical to our later proof of solving the Dirichlet problems.

**Lemma 3.2.** Suppose that $f_k(u) \in \mathcal{H}_k$ with $u \in \mathbb{R}^m$, then there holds

$$
c_{m,k} \int_{\mathbb{R}^{m-1}} \frac{y}{|x|^m} f_k\left(\frac{xux}{|x|^2}\right) dx' = f_k(u),
$$

where $c_{m,k} = \frac{2(m + 2k - 2)}{(m - 2)\omega_m}$.

**Proof.** Since $f_k(u) \in \mathcal{H}_k$, we see that $f_k\left(\frac{xux}{|x|^2}\right)$ is bounded for each fixed $u \in \mathbb{R}^m$, $x' \in \mathbb{R}^{m-1}$, and $\frac{y}{|x|^m} \in L^1(\mathbb{R}^{m-1}, dx')$ (see [3, Chapter 7]). Further, if we let $x = yz$, we derive that the integral

$$
\int_{\mathbb{R}^{m-1}} \frac{y}{|x|^m} f_k\left(\frac{xux}{|x|^2}\right) dx' 
$$

does not depend on the value of $y$. Hence, we can assume that

$$
\int_{\mathbb{R}^{m-1}} \frac{y}{|x|^m} f_k\left(\frac{xux}{|x|^2}\right) dx' = \frac{P(u)}{c_{m,k}},
$$

where $P(u)$ is a function of $u$ and $c_{m,k}$ is a non-zero constant only depending on $m, k$, and it will be determined later on. On the one hand, we have

$$
\int_0^\infty \frac{1}{1 + y^2} dy \int_{\mathbb{R}^{m-1}} \frac{y}{|x|^m} f_k\left(\frac{xux}{|x|^2}\right) dx' = \frac{P(u)}{c_{m,k}} \int_0^\infty \frac{1}{1 + y^2} dy = \frac{\pi}{2c_{m,k}} P(u). \tag{3.1}
$$

On the other hand, if we denote $x = r\zeta$, where $\zeta = (\zeta_1, \cdots, \zeta_m) \in S^{m-1}$, then we have

$$
\int_0^\infty \frac{1}{1 + y^2} dy \int_{\mathbb{R}^{m-1}} \frac{y}{|x|^m} f_k\left(\frac{xux}{|x|^2}\right) dx' = \int_{\mathbb{R}_+^m} \frac{y}{1 + y^2} |x|^{-m} f_k\left(\frac{xux}{|x|^2}\right) dx
$$

$$
= \int_{S^{m-1}} \int_0^\infty \frac{\zeta_m}{1 + (r\zeta_m)^2} dS_f(\zeta) dS(\zeta) = \frac{\pi}{2} \int_{S^{m-1}} f_k(\zeta u\zeta) dS(\zeta). \tag{3.2}
$$
Further, we notice that $\zeta u \zeta$ is invariant under $\zeta \rightarrow -\zeta$. Therefore, one has
\[
\int_{S^{m-1}} f_k(\zeta u \zeta) dS(\zeta) = \frac{1}{2} \int_{S^{m-1}} f_k(\zeta u \zeta) dS(\zeta).
\]
From (3.1), (3.2) and Lemma 3.1, we have
\[
\pi^2 c_{m,k} \mathcal{P}(u) = \frac{\pi}{4} \cdot \left( m - 2 \right) \omega_m + 2\frac{k - 2}{m - 2} f_k(u).
\]
Therefore, we have $\mathcal{P}(u) = f_k(u)$ and $c_{m,k} = 2\left( m + 2k - 2 \right) \omega_m$. This completes the proof. □

Let $Z_k(u, v)$ be the reproducing kernel of the spherical harmonics (see [3]) in the sense that
\[
f(v) = \int_{S^{m-1}} Z_k(u, v) f(u) dS(u), \text{ for all } f(v) \in \mathcal{H}_k.
\]
Notice that Bosonic Laplacian $\mathcal{D}_k$ given in (2.1) is a second-order differential operator with respect to $x$, then $yZ_k(u, v)$ is a trivial null solution of $\mathcal{D}_k$. Further, in [7, 13], it shows that $\mathcal{D}_k$ is a second-order conformally invariant differential operator. In particular, it is conformally invariant under the following special conformal transformation
\[
K : C^\infty(\mathbb{R}^m \times \mathbb{R}^m, \mathcal{H}_k) \rightarrow C^\infty(\mathbb{R}^m \times \mathbb{R}^m, \mathcal{H}_k)
\]
\[
f(x, u) \rightarrow K[f](x, u) := \left| x \right|^{2-m} f\left( \frac{x}{|x|^2}, \frac{x u x}{|x|^2} \right).
\]
We apply $K$ to $yZ_k(u, v)$ to obtain a non-trivial null solution $\frac{y}{|x|^m} Z_k\left( \frac{x u x}{|x|^2}, v \right)$ of $\mathcal{D}_k$.

Now, let $t' \in \mathbb{R}^{m-1}$, $t = (t', 0) = (t_1, \cdots, t_{m-1}, 0)$, $x \in \mathbb{R}^m$, $u \in S^{m-1}$, $v \in \mathbb{B}^m$, and set
\[
P_{\mathcal{H}}(x, t, u, v) = c_{m,k} \frac{y}{|x-t|^m} Z_k\left( \frac{(x-t)u(x-t)}{|x-t|^2}, v \right).
\]
(3.3)
The function $P_{\mathcal{H}}$ is called the Poisson kernel of Bosonic Laplacian in the upper-half space. Notice that, in accordance to Lemma 3.2, we have
\[
\int_{\mathbb{R}^{m-1}} P_{\mathcal{H}}(x, t, u, v) dt' = Z_k(u, v), \quad x \in \mathbb{R}^m, \quad u \in S^{m-1}, \quad v \in \mathbb{B}^m.
\]
Now, we introduce an approximation property for the Poisson kernel.

**Proposition 3.3.** Let $a \in \mathbb{R}^{m-1}$, $x \in \mathbb{R}^m$, $v \in \mathbb{B}^m$ and $\delta > 0$, then we have
\[
\lim_{x \rightarrow a} \int_{|t'-a| > \delta} \int_{S^{m-1}} P_{\mathcal{H}}(x, t, u, v) dS(u) dt' = 0.
\]
Proof. Notice that
\[
\begin{align*}
\int_{|x-a| > \delta} P_H(x, t, u, v) dS(u) dt' & \leq \int_{|x-a| > \delta} \left| c_{m,k} \frac{y}{|x-t|^{m-1}} Z_k \left( \frac{(x-t)u(x-t)}{|x-t|^2}, v \right) \right| dS(u) dt' \\
& \leq c_{m,k} \omega_m \dim \mathcal{H}_k \int_{|x-a| > \delta} \frac{|y|}{|x-t|^{m-1}} dt',
\end{align*}
\]
where we have used the estimate \(|Z_k(u, v)| \leq \dim \mathcal{H}_k\) for \(u, v \in \mathbb{S}^{m-1}\) (see [9, Proposition 5.27]). By in [4, Lemma 1.3.5 (c)], we obtain (3.4). \(\Box\)

For \(1 \leq p < \infty\), let \(L^p(\mathbb{R}^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)\) be the space of Borel measurable functions \(f\) on \(\mathbb{R}^{m-1} \times \mathbb{B}^m\) for which
\[
\|f\|_{L^p(\mathbb{R}^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)} = \left( \int_{\mathbb{R}^{m-1} \times \mathbb{B}^m} |f(x', u)|^p dS(u) dx' \right)^{1/p} < +\infty.
\]
\(L^\infty(\mathbb{R}^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)\) consists of the Borel measurable functions \(f\) on \(\mathbb{R}^{m-1} \times \mathbb{B}^m\) for which \(\|f\|_\infty < +\infty\), where \(\| \cdot \|_\infty\) stands for the essential supremum norm on \(\mathbb{R}^{m-1} \times \mathbb{B}^m\). One might notice that, when we define the norm of \(L^p\), the integration of \(u\) is over \(\mathbb{S}^{m-1}\) instead of \(\mathbb{B}^m\), although \(u \in \mathbb{B}^m\). This is because \(f(x', u)\) is a homogeneous harmonic polynomial of degree \(k\) in \(u\), so the norm defined on \(\mathbb{S}^{m-1}\) is the same as the norm defined on \(\mathbb{B}^m\) up to a multiplicative constant.

One can easily see that
\[
\|f\|_{L^p(\mathbb{R}^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)} = (m + kp)^{-1} \|f\|_{L^p(\mathbb{R}^{m-1} \times \mathbb{S}^{m-1}, \mathcal{H}_k)}.
\]
(3.6)
The Poisson integral of \(f \in L^p(\mathbb{R}^{m-1} \times \mathbb{B}^m, \mathcal{H}_k), p \in [1, \infty]\), is given by
\[
P_H[f](x, v) = \int_{\mathbb{R}^{m-1} \times \mathbb{S}^{m-1}} P_H(x, t, u, v) f(t', u) dS(u) dt', \quad x \in \mathbb{R}^m_+, \quad v \in \mathbb{B}^m.
\]
(3.7)
By the boundedness of \(Z_k(u, v)\), the fact that \(\frac{y}{|x-t|^{m-1}} \in L^p(\mathbb{R}^{m-1}), 1/p + 1/q = 1\), and Hölder’s inequality, we observe that \(P_H[f]\) is well-defined for every \(x \in \mathbb{R}^m_+, \ v \in \mathbb{B}^m\).

3.2. **Dirichlet problem in the upper-half space.** Now, we claim that the Poisson integral given in (3.7) solves the following Dirichlet problem in \(\mathbb{R}^m_+\).

**Theorem 3.4** (Dirichlet problem in \(\mathbb{R}^m_+\) with continuous and bounded data). Suppose \(f \in C(\mathbb{R}^{m-1} \times \mathbb{B}^m, \mathcal{H}_k) \cap L^\infty(\mathbb{R}^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)\). Define \(g\) on \(\mathbb{R}^m_+ \times \mathbb{B}^m\) by
\[
g(x, v) = \begin{cases} P_H[f](x, v), & \text{if } x \in \mathbb{R}^m_+, \quad v \in \mathbb{B}^m, \\ f(x', v), & \text{if } x' \in \mathbb{R}^{m-1}, \quad v \in \mathbb{B}^m. \end{cases}
\]
Then \(g\) is continuous on \(\mathbb{R}^m_+\) with respect to \(x\), \(\mathcal{D}kg = 0\) in \(\mathbb{R}^m_+ \times \mathbb{B}^m\) and
\[
\|g\|_{L^\infty(\mathbb{R}^m_+ \times \mathbb{B}^m)} \leq a_{m,k} \|f\|_{L^\infty(\mathbb{R}^{m-1} \times \mathbb{B}^m)},
\]
(3.8)
where $a_{m,k}$ is a positive constant only depending on $m$ and $k$.

Proof. First, from the expression of $P_H(x, t, u, v)$ given in (3.3), if we consider a fixed derivative of $P_H(x, t, u, v)$ with respect to $y$, $x_j$, $j = 1, \cdots, m - 1$ or $u_j$, $j = 1, \cdots, m$, and assume $V$ is a compact subset in $\mathbb{R}^m$ then for any $(x', y) \in V$ and $u \in S^{m-1}$, the supremum of the derivative as a function of $t$ and $v$, is integrable. This is because of the fact that $P_H(x, t, u, v)$ is nonsingular when $x$ is in a compact set in $\mathbb{R}^m$. Hence, we can differentiate under the integral sign to obtain that $D_t P_H[f] = 0$.

Next we will show that $g(x, v)$ is continuous in $\mathbb{R}^m$ with respect to $x$. Let $a \in \mathbb{R}^{m-1}$ and $v \in \mathbb{R}^m$. By Lemma 3.2 we have

$$f(a, v) = \int_{S^{m-1}} Z_k(u, v) f(a, u) dS(u)$$

$$= c_{m,k} \int_{S^{m-1}} \int_{\mathbb{R}^m} \frac{y}{|x - t|^m} Z_k \left( \frac{(x - t)u(x - t)}{|x - t|^2}, v \right) dt' f(a, u) dS(u)$$

$$= \int_{\mathbb{R}^m} \int_{S^{m-1}} P_H(x, t, u, v) f(a, u) dS(u) dt'.$$

Let $\delta > 0$. From (3.9) and the estimate $|Z_k(u, v)| \leq \dim \mathcal{H}_k$, we obtain

$$|g(x, v) - f(a, v)| = |P_H[f](x, v) - f(a, v)|$$

$$\leq \int_{\mathbb{R}^m} \int_{S^{m-1}} |P_H(x, t, u, v)(f(t', u) - f(a, u))| dS(u) dt'$$

$$\leq c_{m,k} \dim \mathcal{H}_k \cdot \int_{\mathbb{R}^m} \int_{S^{m-1}} \frac{y}{|x - t|^m} \cdot |f(t', u) - f(a, u)| dS(u) dt'$$

$$= c_{m,k} \dim \mathcal{H}_k \cdot \int_{|t' - a| < \delta} \int_{S^{m-1}} \frac{y}{|x - t|^m} \cdot |f(t', u) - f(a, u)| dS(u) dt'$$

$$+ c_{m,k} \dim \mathcal{H}_k \cdot \int_{|t' - a| > \delta} \int_{S^{m-1}} \frac{y}{|x - t|^m} \cdot |f(t', u) - f(a, u)| dS(u) dt'.$$

Take arbitrarily $\epsilon > 0$, we can choose $\delta > 0$ small enough such that the first integral is smaller than $\epsilon$ due to the facts that $f(t', u)$ is continuous at $a$ and $\int_{\mathbb{R}^m} \frac{y}{|x - t|^m} dt'$ is bounded. The second integral above approaching zero when $x \to a$ can be immediately obtained from Proposition 3.3. This completes the proof of the continuity.
Finally, we will prove (3.8). For any $x \in \mathbb{R}^m$ and $v \in \mathbb{B}^m$, we have

$$
|g(x,v)| = \left| \int_{\mathbb{R}^{m-1}} \int_{\mathbb{S}^{m-1}} P_H(x,t,u,v)f(t',u)dS(u)dt' \right| \\
\leq c_{m,k} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{S}^{m-1}} \left| Z_k \left( \frac{(x-t)u(x-t)}{|x-t|^2} \right) f(t',u) \right| dS(u)dt' \\
\leq c_{m,k} \dim \mathcal{H}_k \cdot \omega_m \|f\|_{L^\infty(\mathbb{R}^{m-1} \times \mathbb{B}^m)} \int_{\mathbb{R}^{m-1}} \frac{y}{|x-t|^m} dt' \\
= c_{m,k} \dim \mathcal{H}_k \cdot \omega_m \pi \frac{m}{2} \Gamma \left( \frac{m}{2} \right)^{-1} \|f\|_{L^\infty(\mathbb{R}^{m-1} \times \mathbb{B}^m)} \\
=: a_{m,k} \|f\|_{L^\infty(\mathbb{R}^{m-1} \times \mathbb{B}^m)}.
$$

The last second equality comes from [4, Lemma 1.3.5]. Therefore, we obtain (3.8). □

We now extend the result to the Dirichlet problem with $L^p$ data.

**Theorem 3.5** (Dirichlet problem with $L^p$ data). Assume that $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)$. Let $g(x,v) = P_H[f](x,v) = P_H[f](x',y,v)$, and for $y > 0$, we set $g_y(x',v) = P_H[f](x',y,v)$.

1. There holds $\mathcal{D}_k g = 0$ on $\mathbb{R}^m_+ \times \mathbb{B}^m$.

2. There exists a positive constant $c'_{m,k}$ depending on $m$ and $k$ such that, for any $y > 0$, there holds

$$
\|g_y\|_{L^p(\mathbb{R}^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)} \leq c'_{m,k} \|f\|_{L^p(\mathbb{R}^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)}.
$$

3. There holds $\|g_y - f\|_{L^p(\mathbb{R}^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)} \rightarrow 0$ as $y \rightarrow 0$.

**Proof.** First, for each fixed $t = (t',0)$ with $t' \in \mathbb{R}^{m-1}$, we have $\mathcal{D}_k P_H(x,t,u,v) = 0$. Further, from the expression that $P_H(x,t,u,v)$ in (3.3), if we consider a fixed derivative of $P_H(x,t,u,v)$ with respect to $y$, $x_j$, $j = 1, \cdots, m - 1$ or $u_j$, $j = 1, \cdots, m$, and assume $V$ is a compact subset in $\mathbb{R}^m_+$, then for any $(x',y) \in V$ and $u \in \mathbb{S}^{m-1}$, the supremum of the derivative as a function of $t$ and $u$, is in $L^p(\mathbb{R}^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)$, where $1/p + 1/q = 1$. This is because of the fact that $P_H(x,t,u,v)$ is nonsingular when $x$ is in a compact set in $\mathbb{R}^m_+$. This allows us to differentiate under the integral sign to conclude that $\mathcal{D}_k P_H[f] = 0$. 


Next, we prove statement (2). By Minkowski integral inequality, we have

\[
\|g_{y}\|_{L^{p}(\mathbb{R}^{m-1} \times \mathbb{R}^{m}, \mathcal{H}_{k})} = c_{m,k} \left( \int_{\mathbb{R}^{m-1} \times \mathbb{S}^{m-1}} \int_{\mathbb{R}^{m-1} \times \mathbb{S}^{m-1}} \left| \int_{\mathbb{R}^{m-1} \times \mathbb{S}^{m-1}} P_{H}(x, t, u, v) f(t', u) dS(v) dt' \right|^{p} \frac{1}{p} dS(v) dx' \right)^{\frac{1}{p}}
\]

\[
\leq c_{m,k} \dim \mathcal{H}_{k} \int_{\mathbb{R}^{m-1} \times \mathbb{S}^{m-1}} \left( \int_{\mathbb{R}^{m-1} \times \mathbb{S}^{m-1}} \left| f(x - \eta, u) \right|^{p} dS(v) dx' \right)^{\frac{1}{p}} dS(u) d\eta'
\]

\[
= c_{m,k} \omega_{m} \dim \mathcal{H}_{k} \int_{\mathbb{R}^{m-1}} \left( \int_{\mathbb{S}^{m-1}} \left| f(x - \eta, u) \right|^{p} dS(v) dx' \right)^{\frac{1}{p}} dS(u) d\eta'
\]

\[
\leq c'_{m,k} \dim \mathcal{H}_{k} \int_{\mathbb{R}^{m-1}} \left( \int_{\mathbb{S}^{m-1}} \left| f(x - \eta, u) \right|^{p} dS(v) dx' dS(u) \right)^{\frac{1}{p}},
\]

where \( \eta = (x' - t', y) =: (\eta', y) \). This yields statement (2).

To prove statement (3), we denote \( f_{\eta}(x, u) = f(x - \eta, u) \). Let \( \epsilon > 0 \) and in accordance to the continuity of the translation operator in the \( L^{p} \) norm with \( p < \infty \), we can choose \( \delta > 0 \) so that \( \|f_{\eta} - f\|_{L^{p}(\mathbb{R}^{m-1} \times \mathbb{R}^{m}, \mathcal{H}_{k})} \leq \epsilon \), when \( |\eta| < \delta \). Then by Minkowski integral inequality and the estimate \( |P_{H}(\eta, 0, u, v)| \leq \dim \mathcal{H}_{k} \frac{y}{|\eta|^{m}} \), we obtain

\[
\|g_{y} - f\|_{L^{p}(\mathbb{R}^{m-1} \times \mathbb{R}^{m}, \mathcal{H}_{k})} = \left( \int_{\mathbb{R}^{m-1} \times \mathbb{S}^{m-1}} \int_{\mathbb{R}^{m-1} \times \mathbb{S}^{m-1}} \left( \int_{\mathbb{R}^{m-1} \times \mathbb{S}^{m-1}} P_{H}(\eta, 0, u, v)(f(x - \eta, u) - f(x', u)) dS(u) d\eta' \right|^{p} \frac{1}{p} dS(v) dx' \right)^{\frac{1}{p}}
\]

\[
\leq \int_{\mathbb{R}^{m-1} \times \mathbb{S}^{m-1}} \int_{\mathbb{R}^{m-1} \times \mathbb{S}^{m-1}} \left( \int_{\mathbb{R}^{m-1} \times \mathbb{S}^{m-1}} \left| P_{H}(\eta, 0, u, v)(f(x - \eta, u) - f(x', u)) \right|^{p} \frac{1}{p} dS(u) d\eta' \right)^{\frac{1}{p}}
\]

\[
\leq \dim \mathcal{H}_{k} \int_{\mathbb{R}^{m-1} \times \mathbb{S}^{m-1}} \int_{\mathbb{R}^{m-1} \times \mathbb{S}^{m-1}} \left( \int_{\mathbb{R}^{m-1} \times \mathbb{S}^{m-1}} \left| f(x - \eta, u) - f(x', u) \right|^{p} \frac{1}{p} dS(u) d\eta' \right)^{\frac{1}{p}}
\]

\[
\leq a'_{m,k} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{S}^{m-1}} \left( \int_{\mathbb{R}^{m-1}} \left| f(x - \eta, u) - f(x', u) \right|^{p} \frac{1}{p} dS(u) d\eta' \right)^{\frac{1}{p}}
\]

\[
= a'_{m,k} \int_{\mathbb{R}^{m-1}} \left( \int_{\mathbb{S}^{m-1}} \left| f_{\eta} - f \right|_{L^{p}(\mathbb{R}^{m-1} \times \mathbb{R}^{m}, \mathcal{H}_{k})} d\eta' \right)^{\frac{1}{p}}.
\]
This implies
\[
\|g_y - f\|_{L^p(\mathbb{R}^{m-1} \times \mathbb{R}^m, \mathcal{H}_k)} = a_{m,k}' \left( \int_{|\eta| < \delta} \frac{y}{|\eta|^m} \|f_\eta - f\|_{L^p(\mathbb{R}^{m-1} \times \mathbb{R}^m, \mathcal{H}_k)} \, d\eta' + \int_{|\eta| > \delta} \frac{y}{|\eta|^m} \|f_\eta - f\|_{L^p(\mathbb{R}^{m-1} \times \mathbb{R}^m, \mathcal{H}_k)} \, d\eta' \right) \\
\leq a_{m,k}' \left( \epsilon \int_{|\eta| < \delta} \frac{y}{|\eta|^m} \, d\eta' + 2 \|f\|_{L^p(\mathbb{R}^{m-1} \times \mathbb{R}^m, \mathcal{H}_k)} \int_{|\eta| > \delta} \frac{y}{|\eta|^m} \, d\eta' \right).
\]
By a similar argument as in Theorem 3.4, we derive that the above two integrals approach zero when \( y \) goes to zero. This completes the proof of the last statement. \( \square \)

3.3. Dirichlet problem in the unit ball. In this subsection, we will derive the Poisson kernel for Bosonic Laplacian in the unit ball by applying a particular Cayley transform, which is a Möbius transformation mapping the unit ball to the upper-half space. We need the following technical lemma.

**Lemma 3.6.** Let \( \mathbf{x}, \zeta \in \mathbb{R}^m \) and \( \phi(\mathbf{x}) = (a\mathbf{x} + b)(c\mathbf{x} + d)^{-1} \) be a Möbius transformation. Then, one has

1. \( |\phi(\mathbf{x}) - \phi(\zeta)| = |c\mathbf{x} + d|^{-1}|\mathbf{x} - \zeta||c\zeta + d|^{-1} \),
2. \( Z_k \left( \frac{\phi(\mathbf{x}) - \phi(\zeta)}{|\phi(\mathbf{x}) - \phi(\zeta)|^2}, \frac{\mathbf{v}}{|\mathbf{v}|^2} \right) = Z_k \left( \frac{(\mathbf{x} - \zeta)\omega(\mathbf{x} - \zeta)}{|\mathbf{x} - \zeta|^2}, \frac{\mathbf{v}}{|\mathbf{v}|^2} \right), \)

where
\[
\omega = \frac{(c\zeta + d)\mathbf{u}(c\zeta + d)}{|c\zeta + d|^2} \quad \text{and} \quad \mathbf{v} = \frac{(c\mathbf{x} + d)\mathbf{u}(c\mathbf{x} + d)}{|c\mathbf{x} + d|^2}.
\]

**Proof.** The proof is similar to that of Theorem 8 in [15] and Theorem 5.1 in [27]. We briefly recall it here. The idea is to prove the identities are true for a translation, a dilation, a rotation and an inversion, separately. Then, it is also true for an arbitrary Möbius transformation in accordance to the Iwasawa decomposition.

**Translation.** In this case, \( \phi(\mathbf{x}) = \mathbf{x} + b \), in other words, \( a = d = 1, c = 0 \). Statements 1 and 2 are obviously true.

**Dilation.** In this case \( \phi(\mathbf{x}) = a\mathbf{x} \), i.e., \( b = c = 0, d = 1 \). Since our Möbius transformation is normalized \( (a\tilde{a} - b\tilde{c} = \pm 1) \), we have \( a = \pm 1 \). One can observe that the statements are also true in this case.

**Reflection.** In this case, \( \phi(\mathbf{x}) = a\mathbf{x}a^{-1} \), so \( b = c = 0 \). Since \( a\tilde{a} - b\tilde{c} = \pm 1 \), one has \( a\tilde{a} = \pm 1 \), which means that \( |a| = 1 \). Therefore, one can easily observe that the statements are true as well.

**Inversion.** In this case, \( \phi(\mathbf{x}) = \mathbf{x}^{-1} \), i.e., \( a = d = 0, b = c = 1 \). We see that
\[
\mathbf{x}^{-1} - \zeta^{-1} = -\zeta^{-1}(\mathbf{x} - \zeta)\mathbf{x}^{-1} = -\mathbf{x}^{-1}(\mathbf{x} - \zeta)\zeta^{-1}
\]
\[
= \frac{-\mathbf{x}}{|\mathbf{x}|^2}(\mathbf{x} - \zeta)\frac{\zeta}{|\zeta|^2} = -\frac{\zeta}{|\zeta|^2}(\mathbf{x} - \zeta)\frac{\mathbf{x}}{|\mathbf{x}|^2}.
\]
One can observe that the statement 1 is true. For the statement 2, one has

\[Z_k \left( \frac{x(x - \zeta)\zeta u \tilde{\zeta}(x - \zeta)\tilde{x}}{|x|^2 |x - \zeta|^2 \zeta^2}, v \right) = Z_k \left( \frac{(x - \zeta)\zeta u \tilde{\zeta}(x - \zeta)}{|x - \zeta|^2 \zeta^2}, xv\tilde{x} \right),\]

which is derived from the fact that the reproducing kernel of spherical harmonics is invariant under reflection, which can be observed from the explicit expression of the reproducing kernel given in Theorem 5.38 in [3]. Therefore, the statement 2 is also true under inversion. Hence, both statements are correct in accordance to the Iwasawa decomposition. □

Let \( \mathbb{B}^m \) be the open unit ball in \( \mathbb{R}^m \) and \( \varphi \) is the Cayley transform given as follows.

\[\varphi : \mathbb{B}^m \rightarrow \mathbb{R}^m, \]
\[x \mapsto z = -\frac{1}{2}(x + e_m)(e_m x + 1)^{-1}. \tag{3.10}\]

In particular, if \( \zeta \in \mathbb{S}^{m-1} \) then \( \varphi(\zeta) \in \mathbb{R}^{m-1} \). Let \( x, v \in \mathbb{B}^m \) and

\[P_B[h](x, v) := \frac{c_{m,k}}{2} \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} \frac{1 - |x|^2}{|x - \zeta|^2} Z_k \left( \frac{(x - \zeta)\omega(x - \zeta)}{|x - \zeta|^2}, v \right) h(\zeta, u) dS(u) dS(\zeta), \tag{3.11}\]

where

\[\omega = \frac{(e_m \zeta + 1)u(e_m \zeta + 1)}{|e_m \zeta + 1|^2} \quad \text{and} \quad \nu = \frac{(e_m x + 1)v(e_m x + 1)}{|e_m x + 1|^2}, \tag{3.12}\]

and \( c_{m,k} \) is given in Lemma 3.2. Here, we remind the reader that \( \omega \) and \( \nu \) are actually obtained by rotations of \( u \) and \( v \), respectively, see more details in [19] Theorem 6.3. We claim that

**Theorem 3.7** (Dirichlet problem in \( \mathbb{B}^m \) with continuous data). Suppose \( h \in C(\mathbb{S}^{m-1} \times \mathbb{B}^m, \mathcal{H}_k) \). Define \( h^* \) on \( \mathbb{B}^m \times \mathbb{B}^m \) by

\[h^*(x, v) = \begin{cases} 
P_B[h](x, v), & \text{if } x \in \mathbb{B}^m, \quad v \in \mathbb{B}^m \\
h(x, v), & \text{if } x \in \mathbb{S}^{m-1}, \quad v \in \mathbb{B}^m. \end{cases}\]

Then \( h^* \) is continuous on \( \bar{\mathbb{B}}^m \) with respect to \( x \), \( \mathcal{D}_h h^* = 0 \) in \( \mathbb{B}^m \times \mathbb{B}^m \) and

\[\|h^*\|_{L^\infty(\mathbb{B}^m \times \mathbb{B}^m)} \leq a'_{m,k} \|h\|_{L^\infty(\mathbb{S}^{m-1} \times \mathbb{B}^m)}, \tag{3.13}\]

where \( a'_{m,k} \) is a positive constant only depending on \( m \) and \( k \).

**Proof.** We will prove this theorem by showing that it can derived from Theorem 3.4 after applying the Cayley transform \( \varphi \). Recall that, in the upper-half space case, the function

\[P_H[f](z, v) = \int_{\mathbb{R}^{m-1}} \int_{\mathbb{S}^{m-1}} P_H(z, t, u, v)f(t', u)dS(u)dt', \quad z \in \mathbb{R}^m_+, \quad v \in \mathbb{B}^m, \]

solves the Dirichlet problem of Bosonic Laplacian, where \( z = (z', z_m) \in \mathbb{R}^m_+ \). By plugging \( \varphi(x) = z \) and \( \varphi(\zeta) = t \) as defined in (3.10), where \( x \in \mathbb{B}^m \) and \( \zeta \in \mathbb{S}^{m-1} \), into \( P_H[f](z, v) \)
and by taking into account that \( z_m = \frac{1 - |x|^2}{2|e_m x + 1|^2} \), we obtain

\[
P_B[f](z, v) = c_{m,k} \int_{S^{m-1}} \int_{S^{m-1}} \frac{z_m}{|x - t|^m} Z_k \left( \frac{(x - t)u(x - t)}{|x - t|^2}, v \right) f(t', u) dS(u) dt'
\]

\[
= \frac{c_{m,k}}{2} \int_{S^{m-1}} \int_{S^{m-1}} \frac{(1 - |x|^2)|e_m x + 1|^{-2}}{|e_m x + 1 - \zeta|^m} |e_m \zeta + 1|^{-m} Z_k \left( \frac{(x - \zeta)\omega(x - \zeta)}{|x - \zeta|^2}, \nu \right)
\]

\[
\cdot |e_m \zeta + 1|^{2-m} f(\zeta, u) dS(u) dS(\zeta),
\]

where \( J(\varphi, \zeta) \) is the Jacobian. It can be checked that \( J(\varphi, \zeta) = |e_m \zeta + 1|^{-2m+2} \), which implies

\[
P_B[f](\varphi(x), v) = \frac{c_{m,k}}{2} |e_m x + 1|^{m-2} \int_{S^{m-1}} \int_{S^{m-1}} \frac{1 - |x|^2}{|x - \zeta|^m} Z_k \left( \frac{(x - \zeta)\omega(x - \zeta)}{|x - \zeta|^2}, \nu \right)
\]

\[
\cdot |e_m \zeta + 1|^{2-m} f(\varphi(\zeta), u) dS(u) dS(\zeta).
\]

In other words, one has

\[
|e_m x + 1|^{2-m} P_B[f](\varphi(x), v) = \frac{c_{m,k}}{2} \int_{S^{m-1}} \int_{S^{m-1}} \frac{1 - |x|^2}{|x - \zeta|^m} Z_k \left( \frac{(x - \zeta)\omega(x - \zeta)}{|x - \zeta|^2}, \nu \right)
\]

\[
\cdot |e_m \zeta + 1|^{2-m} f(\varphi(\zeta), u) dS(u) dS(\zeta). (3.14)
\]

If we let \( h(\zeta, v) = |e_m \zeta + 1|^{2-m} f(\varphi(\zeta), v) \), we obtain from the definition of \( P_B \) that

\[
P_B[h](x, v) = \frac{c_{m,k}}{2} \int_{S^{m-1}} \int_{S^{m-1}} \frac{1 - |x|^2}{|x - \zeta|^m} Z_k \left( \frac{(x - \zeta)\omega(x - \zeta)}{|x - \zeta|^2}, \nu \right) h(\zeta, u) dS(u) dS(\zeta). (3.15)
\]

From (3.14) and (3.15), we deduce that \( P_B[h](x, v) = |e_m x + 1|^{2-m} P_B[f](\varphi(x), v) \). Since \( D_k P_B[f] = 0 \) and \( D_k \) is conformally invariant \([7,13]\), it follows that

\[
D_k P_B[h](x, v) = D_k |e_m x + 1|^{2-m} P_B[f](\varphi(x), v) = 0.
\]

With the boundedness of \( Z_k(u, v) \) and noticing that \( \frac{1 - |x|^2}{|x - \zeta|^m} \) is the Poisson kernel of the Laplacian in the unit ball, we derive (3.13) by using a similar argument used in the upper-half space case. This completes the proof of Theorem 3.7. \( \square \)

From Theorem 3.7, we can denote the Poisson kernel of Bosonic Laplacian in the unit ball by

\[
P_B(x, \zeta, u, v) = \frac{c_{m,k}}{2} \frac{1 - |x|^2}{|x - \zeta|^m} Z_k \left( \frac{(x - \zeta)\omega(x - \zeta)}{|x - \zeta|^2}, \nu \right), \quad x, v \in \mathbb{B}^m, \zeta, u \in S^{m-1}.
\]

Notice that although this kernel also consists of the Poisson kernel of Laplacian in the unit ball and the reproducing kernel \( Z_k \), it also involves two different actions on \( u \) and \( v \), which are caused by the Cayley transform applied to \( x \) and \( \zeta \).
Now, we deal with the $L^p$ data with techniques from [3]. Here, for $1 \leq p < \infty$, $L^p(S^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)$ consists of the Borel measurable functions $f$ on $S^{m-1} \times \mathbb{B}^m$, for which
\[
\|f\|_{L^p(S^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)} = \left( \int_{S^{m-1}} \int_{\mathbb{B}^m} |f(x,u)|^p dS(x)dS(u) \right)^{1/p} < +\infty.
\]
$L^\infty(S^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)$ consists of the Borel measurable functions $f$ on $S^{m-1} \times \mathbb{B}^m$ for which $\|f\|_\infty < +\infty$, where $\| \cdot \|_\infty$ stands for the essential supremum norm on $S^{m-1} \times \mathbb{B}^m$. We will use $\| \cdot \|_p$ to represent $\| \cdot \|_{L^p(S^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)}$ in the rest of this section when there is no confusion. For $f \in L^p(\mathbb{B}^m \times \mathbb{B}^m, \mathcal{H}_k)$, we define $f_r(\eta, u) = f(r\eta, u)$ with $\eta \in S^{m-1}$, $u \in \mathbb{B}^m$, and $0 \leq r < 1$. Now, we claim that

**Theorem 3.8.** Let $1 \leq p \leq \infty$, $f \in C(S^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)$ and $g = P_B[f]$. Then, for any $0 < r < 1$, we have $\|g_r\|_p \leq b_{m,k} \|f\|_p$, where $b_{m,k}$ is a constant depending on $m$ and $k$.

**Proof.** We firstly prove the result for $1 \leq p < \infty$. $Z_k(u, v)$ is bounded, we have
\[
\|g_r\|_p^p = \int_{S^{m-1}} \int_{\mathbb{B}^m} |g_r(\eta, v)|^p dS(v)dS(\eta)
= \int_{S^{m-1}} \int_{\mathbb{B}^m} \int_{S^{m-1}} \int_{\mathbb{B}^m} \frac{1 - r^2}{|r\eta - \zeta|^m} Z_k\left(\frac{(r\eta - \zeta)\omega(r\eta - \zeta)}{|r\eta - \zeta|^2}, \nu\right) f(\zeta, u)dS(u)dS(\zeta) dS(v)dS(\eta)
\leq b_{m,k} \int_{S^{m-1}} \int_{\mathbb{B}^m} \int_{S^{m-1}} \int_{\mathbb{B}^m} \frac{1 - r^2}{|r\eta - \zeta|^m} |f(\zeta, u)|dS(u)dS(\zeta)dS(\eta).
\]
Recall that
\[
\int_{S^{m-1}} \frac{1 - r^2}{|r\eta - \zeta|^m} dS(\zeta) = 1,
\]
then according to the above estimate and Jensen’s inequality, we have
\[
\|g_r\|_p^p \leq b_{m,k} \int_{S^{m-1}} \int_{\mathbb{B}^m} \int_{S^{m-1}} \int_{\mathbb{B}^m} \frac{1 - r^2}{|r\eta - \zeta|^m} |f(\zeta, u)|dS(u)dS(\zeta)dS(\eta)
= b_{m,k} \int_{S^{m-1}} \int_{\mathbb{B}^m} \int_{S^{m-1}} \int_{\mathbb{B}^m} \frac{1 - r^2}{|r\eta - \zeta|^m} dS(\eta) |f(\zeta, u)|dS(u)dS(\zeta)
= b_{m,k} \int_{S^{m-1}} \int_{\mathbb{B}^m} |f(\zeta, u)|dS(u)dS(\zeta) = b_{m,k} \|f\|_p.
\]
In the above calculation, we have used the fact that $|r\zeta - \eta| = |r\eta - \zeta|$, $\zeta, \eta \in S^{m-1}$ and Fubini’s theorem.
For $p = \infty$, it is easy to observe that

$$|g_r(\zeta, v)| = \left| \int_{S^{m-1}} \int_{S^{m-1}} \frac{1 - r^2}{|r \eta - \zeta|^m} Z_k \left( \frac{(r \eta - \zeta)\omega(r \eta - \zeta)}{|r \eta - \zeta|^2} , v \right) f(\zeta, u) dS(u) dS(\zeta) \right|$$

$$\leq b'_{m, k} \|f\|_\infty \int_{S^{m-1}} \frac{1 - r^2}{|r \eta - \zeta|^m} dS(\zeta) = b'_{m, k} \|f\|_\infty,$$

which completes the proof of the theorem. □

An immediate consequence of the theorem above is stated as follows.

**Proposition 3.9.** Let $1 \leq p \leq \infty$. Assume that $D_k f = 0$ on $B^m \times B^m$ and $0 \leq r < 1$. Then $\|f_r\|_p \leq b_{m, k} \|f_s\|_p$.

**Proof.** From Theorem 3.8, we have $\|f_r\|_p = \|P_B[f_s]\|_p \leq b_{m, k} \|f_s\|_p$. □

Theorem 3.8 also implies a $L^p$-convergence of $f_r$ stated as follows.

**Theorem 3.10.** Assume $1 \leq p < \infty$. Let $f \in L^p(S^{m-1} \times B^m, \mathcal{H}_k)$ and $g = P_B[f]$. Then

$$\lim_{r \to 1} \|g_r - f\|_p = 0. \quad (3.16)$$

**Proof.** Fix $\epsilon > 0$ and choose $h \in C(S^{m-1} \times B^m, \mathcal{H}_k)$ such that $\|f - h\|_p \leq \epsilon$. Let $g'_r = P_B[h]$, then we have

$$\|g_r - f\|_p \leq \|g_r - g'_r\|_p + \|g'_r - h\|_p + \|h - f\|_p.$$  

Notice that $g_r - g'_r = P_B[f - h]$, which yields

$$\|g_r - g'_r\|_p \leq b_{m, k} \|f - h\|_p < b_{m, k} \epsilon$$

by Theorem 3.8. Therefore, one has

$$\|g_r - f\|_p \leq \|g'_r - h\|_p + (b_{m, k} + 1) \epsilon.$$  

Since $h \in C(S^{m-1} \times B^m, \mathcal{H}_k)$, Theorem 3.7 tells us that $\|g'_r - h\|_p \to 0$ when $r \to 1$. Hence,  

$$\lim_{r \to 1} \|g_r - f\|_p \leq (b_{m, k} + 1) \epsilon.$$  

Since $\epsilon$ is arbitrary and $b_{m, k}$ depends only on $m$ and $k$, we conclude (3.16). □

With the $L^p$-convergence of $f_r$, we have the result of Dirichlet problem of Bosonic Laplacian in the unit ball with $L^p$ data as follows.

**Theorem 3.11** (Dirichlet problem in $B^m$ with $L^p$ data). Assume $1 \leq p < \infty$ and $h(x, u) \in L^p(S^{m-1} \times B^m, \mathcal{H}_k)$. Define $h^*$ on $B^m \times B^m$ by

$$h^*(x, v) = \begin{cases} P_B[h](x, v), & \text{if } x \in B^m, \ v \in B^m, \\ h(x, v), & \text{if } x \in S^{m-1}, \ v \in B^m. \end{cases}$$

Then the following hold

1. $D_k h^* = 0$ in $B^m \times B^m$,
2. $\|h^*_r\|_p \leq b'_{m, k} \|h\|_p$, where $b'_{m, k}$ is a positive constant depending on $m$ and $k$,
3. $\lim_{r \to 1} \|h^*_r - h\|_p = 0.$
4. Existence and uniqueness for the Dirichlet problem

In this section, we will show the existence and uniqueness of continuous and bounded solutions to the Dirichlet problem involving the generalized Maxwell operator \( k = 1 \) in the upper-half space. This implies the existence and uniqueness result in the unit ball due to the Cayley transform \( \varphi \). The strategy is the following: firstly reformulate the Dirichlet problem as a partial differential system with Dirichlet conditions. Then, we show that an estimate theorem of elliptic systems in \( \mathbb{H} \) is applicable to the Dirichlet problem for this equation system. This estimate helps us to obtain the uniqueness property of the Dirichlet problem for the system.

4.1. The elliptic system form of the generalized Maxwell equation. Recall that, Bosonic Laplacian given in (2.1) with \( k = 1 \), also known as the generalized Maxwell operator, denoted by \( D_1 \), has the following form [16]:

\[
D_1 = \Delta_x - \frac{4}{m}(u,D_x)(D_u,D_x) : C^2(\mathbb{R}^m \times \mathbb{R}^m, \mathcal{H}_1) \rightarrow C(\mathbb{R}^m \times \mathbb{R}^m, \mathcal{H}_1).
\]

We remind the reader that the third term in (2.1) disappears because the function space \( f \) given in (3.10) is the following

\[
\phi^{-1} : \mathbb{R}^m_+ \rightarrow \mathbb{B}^m,
\]

\[
x \mapsto z = \phi^{-1}(x) = (x - e_m)(-e_m x + 1)^{-1}.
\]

Further, in accordance to the intertwining operators of \( D_1 \) given in Theorem 1 in [11], with the conformal transformation \( z = \phi^{-1}(x) \), one has that

\[
D_{1,z} f(z, \omega) = |-e_m x + 1|^{-2-m} D_{1,x,u} - e_m x + 1 |^2-m f(\phi^{-1}(x), u),
\]

where \( f(z, \omega) \in C^2(\mathbb{B}^m \times \mathbb{B}^m, \mathcal{H}_1) \), \( D_{1,x,u} \) stands for the operator \( D_1 \) with respect to \( x \) and \( u \) and

\[
\omega = \frac{(-e_m x + 1) u (-e_m x + 1)}{|-e_m x + 1|^2}.
\]

This relation means that the uniqueness for the following Dirichlet problem in the unit ball

\[
\begin{cases}
D_{1,z} f(z, \omega) = 0, \quad z, \omega \in \mathbb{B}^m, \\
f(z, \omega) = 0, \quad z \in \mathbb{S}^{m-1}, \quad \omega \in \mathbb{B}^m,
\end{cases}
\]

is the same as the uniqueness for the following Dirichlet problem in the upper-half space

\[
\begin{cases}
D_{1,x,u} - e_m x + 1 |^{2-m} f(\phi^{-1}(x), u) = 0, \quad x \in \mathbb{R}^m_+, u \in \mathbb{B}^m, \\
f(x, u) = 0, \quad x \in \mathbb{R}^{m-1}, \quad u \in \mathbb{B}^m.
\end{cases}
\]

Moreover, since \( f(z, \omega) \in C(\mathbb{B}^m \times \mathbb{B}^m, \mathcal{H}_1) \), one can observe that \( f \) is bounded on \( \mathbb{B}^m \times \mathbb{B}^m \). Hence, one can see that \( g(x, u) := |-e_m x + 1|^{2-m} f(\phi^{-1}(x), u) \) is also bounded on \( \mathbb{R}^m_+ \times \mathbb{B}^m \). Therefore, we only need to consider the uniqueness of continuous and bounded solutions of Dirichlet problem in the upper-half space.

For \( g(x, u) \in C^2(\mathbb{R}^m_+ \times \mathbb{B}^m, \mathcal{H}_1) \cap C(\mathbb{B}^m \times \mathbb{B}^m, \mathcal{H}_1) \), we can write

\[
g(x, u) = \sum_{j=1}^{m} u_j g_j(x),
\]
with \( g_j(x) \in C^2(\mathbb{R}_+^m, \mathbb{R}) \cap C(\overline{\mathbb{R}_+^m}, \mathbb{R}) \) and \( g_j \) is bounded in \( \mathbb{R}_+^m \). The equation
\[
D_1 g(x, u) = \left( \Delta_x - \frac{4}{m} (u, D_x)(D_u, D_x) \right) g(x, u) = 0
\]
can be rewritten as
\[
\Delta_x g(x, u) - \frac{4}{m} \sum_{s,j,k=1}^m u_s \partial_{x_s} \partial_{x_j} u_k g_k(x) = \sum_{s=1}^m u_s \Delta_x g_s(x) - \frac{4}{m} \sum_{s,j=1}^m u_s \partial_{x_s} \partial_{x_j} g_j(x) = 0.
\]
This tells us
\[
\sum_{s=1}^m u_s \left( \Delta_x g_s(x) - \frac{4}{m} \sum_{j=1}^m \partial_{x_s} \partial_{x_j} g_j(x) \right) = 0.
\]
From the linear independence of \( \{u_s\}_{s=1}^m \), the equation above becomes
\[
\Delta_x g_s(x) - \frac{4}{m} \sum_{j=1}^m \partial_{x_s} \partial_{x_j} g_j(x) = 0, \quad 1 \leq s \leq m.
\]
Therefore, when \( k = 1 \), the uniqueness of solutions to the Dirichlet problem in the upper-half space given in Theorem \( \text{[34]} \) can be reformulated as follows. We are looking for continuous and bounded solutions \( g(x, u) = \sum_{j=1}^m u_j g_j(x) \in C^2(\mathbb{R}_+^m \times \mathbb{R}^m, \mathcal{H}_1) \) of
\[
\begin{align*}
\Delta_x g_s(x) - &\frac{4}{m} \sum_{j=1}^m \partial_{x_s} \partial_{x_j} g_j(x) = 0, \quad 1 \leq s \leq m, x \in \mathbb{R}_+^m, \\
g_j(x) = 0, \quad x \in \mathbb{R}^{m-1}, \quad j = 1, \ldots, m.
\end{align*}
\]
In the Dirichlet problem above we assume that \( x_m = y \) for convenience. This reformulation of the generalized Maxwell operator can also be found in \( \text{[16]} \).

4.2. An estimate of solutions of elliptic systems. In this section, we will introduce an estimate of solutions of elliptic systems satisfying certain conditions in terms of the boundary data and the bound of the solutions. This estimate helps us to determine uniqueness of the Dirichlet problem mentioned above. For more details, we refer the reader to \( \text{[1]} \).

In order to have this article self-contained, we firstly introduce some notations. Here, we consider functions \( g(x) = g(x', y) \), where \( x \in \mathbb{R}_+^m \) and
\[
\partial_i = \partial / \partial x_i, \quad \partial = (\partial_1, \ldots, \partial_m),
\]
where we let \( x_m = y \) above for convenience. Here, we consider the system
\[
\begin{align*}
\sum_{j=1}^N l_{ij}(x, \partial) g_j(x', y) &= \phi_i(x', y), \quad i = 1, \ldots, N, \quad y > 0, \\
\sum_{j=1}^N B_{hj}(x, \partial) g_j(x', y) &= \varphi_h(x'), \quad h = 1, \ldots, n,
\end{align*}
\]
where $l_{ij}(x, \partial)$ stand for homogeneous partial differential operators with coefficients depending on $x$ over $\mathbb{R}^m_+$ and $B_{h_j}(\partial)$ are differential operators with constant coefficients.

Further, for $l_{ij}(x, \partial)$, we assume that there exist integers $s_1, \ldots, s_N, t_1, \ldots, t_N$ such that $\text{deg}(l_{ij}) = s_i + t_j$, $i, j = 1, \ldots, N$. Without loss of generality, we can also assume that $s_i \leq 0$, $t_j \geq 0$. Let $L(x, \zeta) = (l_{ij}(x, \zeta))$ be the matrix obtained by replacing $\partial_j$ by $\zeta_j$ in $l_{ij}(x, \partial)$ and we assume that $n = \frac{1}{2}\text{deg}_{\zeta} \det L(x, \zeta) > 0$. We also suppose that the operator $L$ is uniformly elliptic in the sense that there exists a positive constant $c$ such that

$$c^{-1}|\zeta|^{2m} \leq |\det L(x, \zeta)| \leq c|\zeta|^{2m}$$

for every $\zeta = (\zeta_1, \ldots, \zeta_m)$ and every $x$ in $\mathbb{R}^m_+$. For $B_{h_j}(\partial)$, we also assume that there exist integers $r_1, \ldots, r_n$, such that $\text{deg}_{x} B_{h_j}(\zeta) = r_h + t_j$. Let $l_0 = \max(0, r_h)$ and $l$ is an integer $\geq l_0$.

Let $s$ be a positive integer, the set $C^{s+\lambda}(\mathbb{R}^m_+)$ is the subset of $C^s(\mathbb{R}^m_+)$ consisting of functions which, together with their first $s$-derivatives, satisfy a uniform Hölder condition with exponent $\lambda \in (0, 1)$. The analogous subset of $C^s(\mathbb{R}^m_+)$ we call $C^{s+\lambda}(\mathbb{R}^m_+)$. If $g \in C^{s+\lambda}(\mathbb{R}^m_+)$, we define

$$[g]_{s+\lambda, \mathbb{R}^m_+} = \sup_{x, \bar{x} \in \mathbb{R}^m_+, |\alpha| = s} \frac{|D^\alpha g(x) - D^\alpha g(\bar{x})|}{|x - \bar{x}|^\lambda},$$

where $D^\alpha$ stands for the partial derivative with respect to the multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\alpha_j \in \mathbb{N}$ and $|\alpha| = \sum_j \alpha_j$. We have the following estimate for solutions of system (4.1).

**Theorem 4.1** (Theorem 8.3, [1]). Let $g_j(x', y) \in C^{l_0+t_j+\lambda}(\mathbb{R}^m_+)$, $0 < \lambda < 1$, satisfy (11) in the upper-half space. Assume that for a fixed integer $l \geq l_0$ both the quantities

$$K_0 = \sum_{i=1}^N \sum_{h=1}^n |\phi_i|_{l-s_i+\lambda} + \sum_{h=1}^n |\varphi_h|_{l-r_h+\lambda}$$

and

$$M_0 = \lim_{R \to \infty} \sum_{j=1}^N R^{-(l+t_j+\lambda)} \max_{\Sigma_R} |g_j|$$

are finite, where $\Sigma_R := \{x \in \mathbb{R}^m_+ : \|x\| < R\}$. Then $g_j \in C^{l-t_j+\lambda}(\mathbb{R}^m_+)$, and

$$[g_j]_{l+t_j+\lambda} \leq \text{const.}(K_0 + M_0).$$

### 4.3. Existence and uniqueness for the Dirichlet problem.

The Poisson integral given in Theorem 3.4 already showed the existence of a continuous and bounded solution of the Dirichlet problem in the upper-half space, here we only show the uniqueness of the continuous and bounded solution.

**Proposition 4.2.** Assume $m > 4$. If $g = (g_1, \ldots, g_m)$ with $g_j \in C^2(\mathbb{R}^m_+) \cap C(\mathbb{R}^m_+)$ is a bounded solution of

$$\begin{cases}
\Delta_x g_s(x) - \frac{4}{m} \sum_{j=1}^m \partial_{x_j} \partial_{x_j} g_j(x) = 0, & 1 \leq s \leq m, \ x \in \mathbb{R}^m_+, \\
\quad g_j(x) = 0, & x \in \mathbb{R}^{m-1}, \ j = 1, \ldots, m,
\end{cases}$$

then $g = 0$. 

Proof. The proof is actually a straightforward application of Theorem 4.1 after we write out $h_{ij}$, $B_{ij}$ explicitly and verify that all assumptions in Theorem 4.1 are satisfied. Firstly, it is easy to find out that for (4.2)
\[
(l_{ij})_{m \times m} = \begin{pmatrix}
\Delta - \frac{4}{m} \partial_1^2 & -\frac{4}{m} \partial_1 \partial_2 & \cdots & -\frac{4}{m} \partial_1 \partial_m \\
-\frac{4}{m} \partial_1 \partial_1 & \Delta - \frac{4}{m} \partial_2^2 & \cdots & -\frac{4}{m} \partial_2 \partial_m \\
& \ddots & \ddots & \cdots \\
& & -\frac{4}{m} \partial_m \partial_1 & -\frac{4}{m} \partial_m \partial_2 & \cdots & \Delta - \frac{4}{m} \partial_m^2
\end{pmatrix}.
\]
Therefore, one obtains that $\deg l_{ij} = 2$ so that we can let $s_i = 0$ and $t_j = 2$ such that $\deg h_{ij} = s_i + t_j$ for $i, j = 1, \cdots, m$. Further, we can immediately have
\[
L(x, \zeta) = \begin{pmatrix}
|\zeta|^2 - \frac{4}{m} \zeta_1^2 & -\frac{4}{m} \zeta_1 \zeta_2 & \cdots & -\frac{4}{m} \zeta_1 \zeta_m \\
-\frac{4}{m} \zeta_2 \zeta_1 & |\zeta|^2 - \frac{4}{m} \zeta_2^2 & \cdots & -\frac{4}{m} \zeta_2 \zeta_m \\
& \ddots & \ddots & \cdots \\
& & -\frac{4}{m} \zeta_m \zeta_1 & -\frac{4}{m} \zeta_m \zeta_2 & \cdots & |\zeta|^2 - \frac{4}{m} \zeta_m^2
\end{pmatrix},
\]
so that one can have
\[
\det \, L(x, \zeta) = \det \left( |\zeta|^2 I - \frac{4}{m} \zeta^T \zeta \right) = \det \left( |\zeta|^2 I - \frac{4}{m} \zeta \zeta^T \right) = (1 - \frac{4}{m}) \det |\zeta|^2 I = (1 - \frac{4}{m})|\zeta|^{2m},
\]
where $\zeta = (\zeta_1, \cdots, \zeta_m)$. Hence, $n = \frac{1}{2} \deg \zeta \det L(x, \zeta) = m > 0$ when $m > 4$. We also have that
\[
\frac{1}{6} |\zeta|^{2m} \leq | \det \, L(x, \zeta) | \leq 6 |\zeta|^{2m}.
\]
To obtain the boundary conditions in (4.2), from (4.1), we observe that $(B_{ij})_{m \times m} = E_{hh}$, which is the matrix with 1 in the $(h, h)$ entry and zero elsewhere. Hence, we have $\deg B_{ij} = 0$, so that we can choose $r_h = -2$ to satisfy $\deg B_{ij} = r_h + t_j$ for $h, j = 1, \cdots, m$. Therefore, $l_0 = \max(0, r_h) = 0$. Now, we can apply Theorem 4.1 to (4.2). Indeed, Theorem 4.1 tells us that for $g_j \in C^{2+\alpha}(\mathbb{R}^m_T)$ satisfying (4.2) with $0 < \alpha < 1$, then we have $g_j \in C^{l+\alpha}(\mathbb{R}^m_T)$ with $l \geq 0$ and $|g_j|_{l+\alpha} \leq \text{const.} (K_0 + M_0)$, where $K_0$ and $M_0$ are given in Theorem 4.1. However, we have $\varphi_h = \phi_i = 0$, $i, h = 1, \cdots, m$ and $g_j$ are bounded on $\mathbb{R}^m_T$ in (4.2), which immediately tell us that $K_0 = M_0 = 0$. Hence, we have $|g_j|_{l+2+\lambda} = 0$ for $l \geq 0$ and $0 < \alpha < 1$. In particular, if we let $l = 0$, we have
\[
[g_j]_{2+\lambda} = \sup_{x, x' \in \mathbb{R}^m_T, |\alpha| = 2} \frac{|D^\alpha g(x) - D^\alpha g(x')|}{|x - x'|^{\lambda}} = 0.
\]
Thus one can easily observe that $g_j$ must be of a polynomial with degree $< 2$, which contradicts the assumption that $g_j$ are bounded on $\mathbb{R}^m_T$ unless $g_j = 0$. This completes the proof of the uniqueness. \[\square\]

Remark 4.3. One can observe that if we replace the boundedness condition on $g_j$ by $g_j = O(|x|^{-2})$, when $x \to \infty$, then solutions of system (4.2) is unique up to a polynomial $h(x, u) \in C^2(\mathbb{R}^m_T \times \mathbb{R}^m, \mathcal{H}_1)$, which satisfies that $\deg \, h \leq s$ and $h = 0$ when $x \in \mathbb{R}^{m-1}$. 
By the argument presented at the beginning of the section 4.1, one can also have the uniqueness of continuous solutions over the closure of the unit ball of Dirichlet problem given in Theorem 3.7. This observation is crucial in the following sense.

(1) It implies that the Poisson integral is the only continuous extension from the unit sphere to the interior of the unit ball, which also satisfies the generalized Maxwell equation \( D_1 f = 0 \) and the Dirichlet boundary condition.

(2) Since \( l \geq 0 \) in the proof above is arbitrary, this also implies that any \( C^2 \) solutions of Theorem 3.7 is infinitely differentiable, which is an analog of the smoothness property of harmonic functions.

(3) The uniqueness also allows us to plug zero into the Poisson integral to obtain a mean-value property, which immediately provides us analogs of some results of Laplacian. This will be discussed in the next section.

In the proof of Proposition 4.2, the assumption \( m > 4 \) is imposed to ensure that \( \frac{1}{2} \deg \xi \det L(x, \xi) = m > 0 \). Therefore, we assume that \( m > 4 \) in the rest of this paper.

Recall that Theorem 3.7 implies that the Poisson integral (3.11) is a continuous extension from a continuous function on \( S^{m-1} \times \mathbb{B}^m \) to a solution of the generalized Maxwell operator in \( \mathbb{B}^m \times \mathbb{B}^m \). The uniqueness in the unit ball case gives a Poisson integral formula for continuous null solutions of \( D_1 \).

**Proposition 4.4 (Poisson integral formula).** Let \( f \in C^2(\mathbb{B}^m \times \mathbb{B}^m, \mathcal{H}_1) \cap C(\mathbb{B}^m \times \mathbb{B}^m, \mathcal{H}_1) \) and \( D_1 f = 0 \) in \( \mathbb{B}^m \times \mathbb{B}^m \). Then \( f = P_B[f|_{x=\mathbb{S}^{m-1}}] \) in \( \mathbb{B}^m \times \mathbb{B}^m \).

5. Properties of null solutions of the generalized Maxwell operator

In this section, we will use the uniqueness of continuous extension of Theorem 3.7 to obtain several analogs of results of harmonic functions for null solutions of the generalized Maxwell operator \( (k=1) \) when \( m > 4 \). One can notice that all the results (except solving Poisson’s equation) obtained in this section do not depend on \( k \). In other words, one can easily generalize these results to arbitrary \( k \) cases once one has the uniqueness of solutions to the Dirichlet problem.

5.1. Mean value property, maximum principle and Harnack’s inequality. Since the Poisson integral formula for a harmonic function \( f \) is given by

\[
f(x) = \int_{\mathbb{S}^{m-1}} \frac{1 - |x|^2}{|x - \zeta|^2} f(\zeta)dS(\zeta), \quad x \in \mathbb{B}^m,
\]

one can obtain the mean-value property of \( f \) over the unit sphere by letting \( x = 0 \). This motivates us that choosing \( x = 0 \) in the Poisson integral given in Theorem 3.7 should gives us a mean-value property for null solutions of the generalized Maxwell operator \( D_1 \) as well. It is worth pointing out that the mean-value property discussed below is with respect to the variable \( x \), but \( f(x, u) \) also has mean-value property with respect to \( u \), since \( f \) is harmonic with respect to \( u \) in accordance to the definition of the function space \( C^2(\mathbb{B}^m \times \mathbb{B}^m, \mathcal{H}_1) \).
Theorem 5.1 (Mean-value property). Let $f \in C^2(\mathbb{R}^m \times \mathbb{R}^m, \mathcal{H}_1) \cap C(\overline{\mathbb{R}^m} \times \mathbb{R}^m, \mathcal{H}_1)$ and $\mathcal{D}_1 f = 0$ on $\mathbb{R}^m \times \mathbb{R}^m$. There holds

$$f(0, v) = \frac{c_m}{2} \int_{S^{m-1}} f(\zeta, \frac{\langle \xi - e_m, v(\xi - e_m) \rangle}{|\xi - e_m|^2}) dS(\zeta), \quad \forall v \in \mathbb{R}^m,$$

where $c_m = \frac{2m}{(m - 2)\omega_m}$ given in Lemma 3.2 with $k = 1$.

**Proof.** Theorem 3.1 and the uniqueness property tell us that

$$f(x, v) = \frac{c_m}{2} \int_{S^{m-1}} \int_{S^{m-1}} \frac{1 - |x|^2}{|x - \zeta|^2} Z_1\left(\frac{\langle x - \zeta, \omega - \zeta \rangle}{|x - \zeta|^2}, v\right) f(\zeta, u) dS(u) dS(\zeta),$$

where $x, v \in \mathbb{R}^m$. With $\omega, v$ given in (3.12), let $x = 0$, then we have $v = v$ and

$$f(0, v) = \frac{c_m}{2} \int_{S^{m-1}} \int_{S^{m-1}} Z_1(\zeta, \omega v, v) f(\zeta, u) dS(u) dS(\zeta)$$

$$= \frac{c_m}{2} \int_{S^{m-1}} \int_{S^{m-1}} Z_1\left(\frac{\zeta(e_m + 1)u(e_m + 1)\zeta}{|e_m + 1|^2}, v\right) f(\zeta, u) dS(u) dS(\zeta)$$

$$= \frac{c_m}{2} \int_{S^{m-1}} \int_{S^{m-1}} Z_1\left(\frac{\zeta - e_m}{|\zeta - e_m|^2}, v\right) f(\zeta, u) dS(u) dS(\zeta)$$

$$= \frac{c_m}{2} \int_{S^{m-1}} \int_{S^{m-1}} Z_1\left(\frac{u(e_m v(e_m - e_m))}{|\zeta - e_m|^2}, v\right) f(\zeta, u) dS(u) dS(\zeta)$$

$$= \frac{c_m}{2} \int_{S^{m-1}} f\left(\zeta, \frac{\zeta(e_m + 1)u(e_m + 1)\zeta}{|e_m + 1|^2}, v\right) dS(\zeta),$$

where we have used the facts that $Z_1(a + au, v) = Z_1(u, av)$ for $a \in \mathbb{R}^m$ and that $Z_1$ is the reproducing kernel of $\mathcal{H}_1$ in the last two steps. \qed

One can immediately obtain a mean-value property on the sphere of an arbitrary ball $B(a, r)$ by applying a translation and a dilation to the previous theorem.

**Proposition 5.2.** Let $f \in C^2(B(a, r) \times \mathbb{R}^m, \mathcal{H}_1) \cap C(\overline{B(a, r)} \times \mathbb{R}^m, \mathcal{H}_1)$ and $\mathcal{D}_1 f = 0$ on $B(a, r) \times \mathbb{R}^m$. We have

$$f(a, v) = \frac{c_m}{2} \int_{S^{m-1}} f\left(a + r\zeta, \frac{\zeta(e_m + 1)u(e_m + 1)\zeta}{|e_m + 1|^2}, v\right) dS(\zeta), \quad \forall v \in \mathbb{R}^m.$$

Further, we can also obtain a volume version of the mean-value property as follows.

**Proposition 5.3** (Volume version: mean-value property). Let $f \in C^2(B(a, r) \times \mathbb{R}^m, \mathcal{H}_1) \cap C(\overline{B(a, r)} \times \mathbb{R}^m, \mathcal{H}_1)$ and $\mathcal{D}_1 f = 0$ on $B(a, r) \times \mathbb{R}^m$. We have

$$f(a, v) = \frac{m}{(m - 2)V(B(a, r))} \int_{B(a, r)} f\left(x, \frac{\eta v}{|\eta|^2}\right) dx, \quad \forall v \in \mathbb{R}^m,$$
where \( \eta = \frac{x-a}{|x-a|} - e_m \) and \( V(B(a,r)) \) denotes the volume of the ball \( B(a,r) \).

**Proof.** Without loss of generality, we let \( B(a,r) = B^m \). We calculate the integral on the right side by changing to spherical coordinates. Let \( x \in B^m \) and \( y = \frac{x}{|x|} \in S^{m-1} \), then

\[
\begin{align*}
\frac{m}{(m-2)V(B^m)} \int_{B^m} f \left( x, \frac{\eta \eta \eta}{|\eta|^2} \right) dx & = \frac{mc_m}{2} \int_0^1 \int_{S^{m-1}} t^{m-1} f \left( ty, \frac{(y-e_m)v(y-e_m)}{|y-e_m|^2} \right) dS(y)dt \\
& = mf(0,v) \int_0^1 t^{m-1} dt = f(0,v),
\end{align*}
\]

where in the last second equality we have used Proposition 5.2. The proof is complete. \( \Box \)

An important consequence of the above mean-value property is the following strong maximum principle.

**Theorem 5.4 (Strong maximum Principle).** Suppose \( \Omega \subset \mathbb{R}^m \) is connected and \( f \in C^2(\Omega \times B^m, H) \) is real valued and \( \mathcal{D}_1 f = 0 \) on \( \Omega \times \mathbb{B}^m \). Then, if \( f \) obtains its minimum or maximum at an interior point of \( \Omega \times \mathbb{B}^m \), then \( f = 0 \).

**Proof.** We only prove the case that \( f \) obtains its minimum at an interior point of \( \Omega \times \mathbb{B}^m \), since the other case can be obtained by applying the same argument to \(-f\). Suppose that \( f \) obtains its minimum at a point \((a,v_0)\in \Omega \times \mathbb{B}^m\). Since \( f(x,v) \) is harmonic with respect to \( v \), for each fixed \( x \), the strong minimum principle for harmonic functions implies that \( f(a,v) = f(a,v_0) \) for every \( v \in \mathbb{B}^m \). However, we notice that \( f(x,v) \in H_1 \), which is a degree-1 homogeneous harmonic polynomial in \( v \), for each fixed \( x \in \Omega \). This implies that \( f(a,v) = f(a,v_0) = 0 \) for all \( v \in \mathbb{B}^m \). Hence, \( f(x,v) \) is a non-negative function on \( \Omega \times \mathbb{B}^m \).

Let \( r_1 \) be sufficiently small such that \( B(a,r_1) \subset \Omega \). By Proposition 5.3, we have

\[
f(a,v_0) = \frac{m}{m-2} \frac{1}{V(B(a,r_1))} \int_{B(a,r_1)} f \left( x, \frac{\eta v_0 \eta}{|\eta|^2} \right) dx.
\]

Since \( f \) is non-negative on \( \Omega \times \mathbb{B}^m \), we have \( f \left( x, \frac{\eta v_0 \eta}{|\eta|^2} \right) = f(a,v_0) = 0 \) for every \( x \in B(a,r_1) \), where \( \eta \) is given in Proposition 5.3. Notice that \( \frac{\eta v_0 \eta}{|\eta|^2} \) is a rotation of \( v_0 \), i.e.,

\[
\frac{|\eta v_0 \eta|}{|\eta|^2} = |v_0|,
\]

the previous observation also implies that for each fixed \( x \in B(a,r_1) \), \( f \) obtains its minimum at interior points \( \left( x, \frac{\eta v_0 \eta}{|\eta|^2} \right) \). Hence, the strong minimum principle for harmonic functions with respect to \( v \) gives us that \( f(x,v) = f(a,v_0) = 0 \) for all \( x \in B(a,r_1) \) and \( v \in \mathbb{B}^m \).

Let \((x,u) \in \Omega \times \mathbb{B}^m \) be an arbitrary point, since \( \Omega \) is connected, there exists a path \( \gamma \) in the interior of \( \Omega \times \mathbb{B}^m \) connecting \((a,v_0)\) and \((x,u)\), and \( \gamma \) can be covered by finitely many open subsets of \( \Omega \times \mathbb{B}^m \), denoted by \( \{B(a_j,s_j) \times B(v_j,l_j)\}_{j=1}^N \). Hence, one can apply
similar argument as above to \( f \) on these open subsets to obtain that \( f = 0 \) on these subsets, in particular, \( f(\mathbf{x}, \mathbf{u}) = 0 \) as well. This completes the proof. \( \square \)

One can also obtain an analog of Harnack’s inequality as follows.

**Theorem 5.5** (Harnack’s inequality). Suppose \( \Omega \subset \mathbb{R}^m \) is connected and \( K_1 \times K_2 \) is a compact subset of \( \Omega \times \mathbb{R}^m \). Then, there exists a constant \( c \) only depending on \( m, d_{m-1}(K_1) \) and \( d_{m-1}(K_2) \) such that for any non-negative function \( f \in C^2(\Omega \times \mathbb{R}^m, \mathcal{H}_i) \) satisfying \( \nabla_1 f = 0 \) on \( \Omega \times \mathbb{R}^m \), one has

\[
  f(\mathbf{a}, \mathbf{u}) \leq c f(\mathbf{y}, \mathbf{v}),
\]

for all \( \mathbf{a}, \mathbf{y} \in K_1 \) and \( \mathbf{u}, \mathbf{v} \in K_2 \), where \( d_{m-1}(K_1) \) stands for the distance between \( K_1 \) and \( \mathbb{S}^{m-1} \).

**Proof.** Let \( r = \frac{d_{m-1}(K_1)}{4} \), without loss of generality, we assume that \( K_2 = \overline{B(0, r')} \) where \( 0 < r' < 1 \). Suppose \( (\mathbf{a}, \mathbf{u}) \) is an arbitrary point in \( K_1 \times K_2 \). From the volume version of mean-value property, we have

\[
  f(\mathbf{a}, \mathbf{u}) = \frac{a_m}{V(B(\mathbf{a}, r))} \int_{B(\mathbf{a}, r)} f(x, \frac{\eta a \mathbf{u} \eta a}{|\eta a|^2}) \, dx,
\]

where \( \eta a = \frac{x - a}{|x - a|} - e_m \) is given in Proposition 5.3 and \( a_m \) is a constant only depending on \( m \). Let \( \mathbf{b} \) be an arbitrary point in \( B(\mathbf{a}, r) \), and it is easy to observe that \( B(\mathbf{a}, r) \subset B(\mathbf{b}, 2r) \subset \Omega \). Further, \( f \) is a positive function on \( \Omega \times \mathbb{R}^m \), we have

\[
  f(\mathbf{a}, \mathbf{u}) \leq \frac{a_m}{V(B(\mathbf{a}, r))} \int_{B(\mathbf{b}, 2r)} f(x, \frac{\eta a \mathbf{u} \eta a}{|\eta a|^2}) \, dx. \quad (5.1)
\]

Recall that \( \frac{\eta a \mathbf{u} \eta a}{|\eta a|^2} \) is a rotation of \( \mathbf{u} \), we have that \( f \left( x, \frac{\eta a \mathbf{u} \eta a}{|\eta a|^2} \right) \) is also a harmonic function with respect to \( \mathbf{u} \). Hence, in accordance to the Harnack’s inequality of harmonic functions, there exists a constant \( c_1 \) depending on \( m, d_{m-1}(K_2) \), such that

\[
  f \left( x, \frac{\eta a \mathbf{u} \eta a}{|\eta a|^2} \right) \leq c_1 f \left( x, \frac{\eta a \mathbf{u}_0 \eta a}{|\eta a|^2} \right),
\]

where \( \mathbf{u}_0 \) is an arbitrary point in \( K_2 \), and also \( \frac{\eta a \mathbf{u} \eta a}{|\eta a|^2}, \frac{\eta a \mathbf{u}_0 \eta a}{|\eta a|^2} \in K_2 \) with \( \eta a = \frac{x - \mathbf{b}}{|x - \mathbf{b}|} - e_m \).

Substituting into (5.1), we have

\[
  f(\mathbf{a}, \mathbf{u}) \leq \frac{a_m c_1}{V(B(\mathbf{a}, r))} \int_{B(\mathbf{b}, 2r)} f \left( x, \frac{\eta a \mathbf{u}_0 \eta a}{|\eta a|^2} \right) \, dx = \frac{a_m c_1 V(B(\mathbf{b}, 2r))}{V(B(\mathbf{a}, r))} f(\mathbf{b}, \mathbf{u}_0),
\]

where \( a_m K_2 \) is a constant depending on \( m \) and \( d_{m-1}(K_2) \). Now, let \( (\mathbf{y}, \mathbf{v}) \) be an arbitrary point in \( K_1 \times K_2 \) and since \( \Omega \) is connected, there exists a path \( \gamma \) connecting \( \mathbf{a} \) and \( \mathbf{y} \). Since \( K_1 \) is compact, there exist finitely many balls \( \{B(\mathbf{a}_j, r)\}^N_{j=1} \) to cover \( \gamma \), where the number \( N \) depends on \( r \). Applying the argument above to each ball, one obtains that

\[
  f(\mathbf{a}, \mathbf{u}) \leq (a_m K_2)^N f(\mathbf{y}, \mathbf{v}) =: c f(\mathbf{y}, \mathbf{v}),
\]

where \( c \) depends on \( m, d_{m-1}(K_1), d_{m-1}(K_2) \), which completes the proof. \( \square \)
Remark 5.6. One might notice that for the strong maximum principle, when we conclude that \( f \) is non-negative on \( \Omega \times \mathbb{B}^m \), we can immediately obtain that \( f = 0 \) on \( \Omega \times \mathbb{B}^m \). Indeed, \( f \) is a degree-1 homogeneous polynomial in \( v \), which implies that \( f(x, -v) = -f(x, v) \), actually this also happens for any odd \( k \). Hence, such a non-negative function on \( \Omega \times \mathbb{B}^m \) must be zero. Similarly, the Harnack’s inequality also becomes trivial in this case, i.e., \( f = 0 \) on \( \Omega \times \mathbb{B}^m \) in accordance to the assumptions. However, our proofs above for the strong maximum principle and the Harnack’s inequality do not depend on \( k = 1 \), and we believe the results (except solving Poisson’s equation) obtained in this section are also true for any even \( k \) once we have the uniqueness of solutions to the Dirichlet problem, which will be investigated in an upcoming paper.

5.2. Cauchy’s estimates and Liouville-type theorem. Let \( \alpha = (\alpha_1, \cdots, \alpha_m) \) be a multi-index, where \( \alpha_j, j = 1, \cdots, m \) are nonnegative integers. The partial differentiation is defined as \( D_\alpha^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_m}^{\alpha_m} \). It is easy to see that for each \( \zeta, u \in \mathbb{S}^{m-1} \), the function

\[
P_B(x, \zeta, u, v) = \frac{1}{|x - \zeta|^m} \int Z_1 \left( \frac{(x - \zeta)\omega(x - \zeta)}{|x - \zeta|^2}, \nu \right),
\]

is infinitely differentiable on \( \mathbb{B}^m \) with respect to \( x \), where \( x, v \in \mathbb{B}^m \) and \( \omega, \nu \) are given in (3.12). Further, since there are no singular points occur for \( P_B(x, \zeta, u, v) \) when \( x \in \mathbb{B}^m \) and \( \zeta \in \mathbb{S}^{m-1} \), one observes that \( D^\beta D_\alpha^\alpha P_B(x, \zeta, u, v) \) are integrable over \( \mathbb{B}^m \times \mathbb{B}^m \), where \( \alpha, \beta \) are multi-indices.

Recall that if \( f \in C^2(\mathbb{B}^m \times \mathbb{B}^m, \mathcal{H}_1) \cap C(\overline{\mathbb{B}^m} \times \overline{\mathbb{B}^m}, \mathcal{H}_1) \) and \( D_1 f = 0 \), then Theorem 3.7 tells us that

\[
f(x, v) = \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} P_B(x, \zeta, u, v) f(\zeta, u) dS(u) dS(\zeta), \quad x, v \in \mathbb{B}^m.
\]

Differentiating under the integral sign, we can see that \( f(x, v) \) is infinitely differentiable with respect to \( x, v \) on \( \mathbb{B}^m \). This is an analog of the smoothness property of harmonic functions. Further, we have

\[
D^\beta D_\alpha^\alpha f(x, v) = \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} f(\zeta, u) D^\beta D_\alpha^\alpha P_B(x, \zeta, u, v) dS(u) dS(\zeta).
\]

We can derive an analog of Cauchy’s estimates as follows.

Theorem 5.7 (Cauchy’s estimates). Let \( \alpha, \beta \) be multi-indices. Assume \( f \in C^2(\mathbb{B}^m, \mathcal{H}_1) \) and \( D_1 f = 0 \) on \( \Omega \times \mathbb{B}^m \). Then there exists a constant \( c_{\alpha, m} \) such that

\[
|D^\beta D_\alpha^\alpha f(a, v_0)| \leq \frac{c_{\alpha, m}}{|\alpha| |\beta|} \left\| f \right\|_{L^\infty(B(a, r_1) \times B(v_0, r_2), \mathcal{H}_1)},
\]

for any \( B(a, r_1) \subseteq \Omega \) and \( B(v_0, r_2) \subseteq \mathbb{B}^m \).

Remark 5.8. In the theorem above, one can notice that when \( |\beta| > 1 \), the left side of Cauchy’s estimates becomes zero since \( f(x, v) \) is a degree one polynomial with respect to \( v \).
Proof. Without loss of generality, we may assume \( a = 0 \). Firstly, we consider the case \( \Omega = \mathbb{B}^m \) and we let \( \| f \|_{L^\infty(\mathbb{B}^m \times \mathbb{B}^m, H_1)} = M' \), then we have, for \( v_0 \in \mathbb{B}^m \),

\[
|D^\beta_v D^\alpha_x f(0, v_0)| = \left| \int_{S^{m-1}} \int_{S^{m-1}} f(\zeta, u) D^\beta_v D^\alpha_x P_B(0, \zeta, u, v_0) dS(u) dS(\zeta) \right|
\]

\[
\leq M' \int_{S^{m-1}} \int_{S^{m-1}} |D^\beta_v D^\alpha_x P_B(0, \zeta, u, v_0)| dS(u) dS(\zeta)
\]

\[
= M' \int_{S^{m-1}} \int_{S^{m-1}} \left| D^\alpha_x |_{v=v_0} \left| \frac{1 - |x|^2}{|x - \zeta|^m} \right| Z_1 \left( \frac{1 - |x|^2}{|x - \zeta|^2}, \nu \right) \right| dS(u) dS(\zeta).
\]

Notice that there are no singular points of the function

\[
\frac{1 - |x|^2}{|x - \zeta|^m} Z_1 \left( \frac{1 - |x|^2}{|x - \zeta|^2}, \nu \right),
\]

which is a homogeneous polynomial with respect to \( v \), and (see page 104 in \[3\] with \( k = 1 \)) \( Z_1(u, v) = m(u, v) \), where \( u \in S^{m-1}, \ v \in \mathbb{B}^m \). These imply that

\[
\left| D^\beta_v |_{v=v_0} D^\alpha_x \left| \frac{1 - |x|^2}{|x - \zeta|^m} \right| Z_1 \left( \frac{1 - |x|^2}{|x - \zeta|^2}, \nu \right) \right|
\]

is bounded when \( x = 0, \ u \in S^{m-1}, \ \zeta \in S^{m-1} \) and \( v = v_0 \). Therefore, there exists a constant \( c_{\alpha, m} \), which only depends on \( \alpha \) and \( m \), such that

\[
\int_{S^{m-1}} \left| D^\beta_v |_{v=v_0} D^\alpha_x \left| \frac{1 - |x|^2}{|x - \zeta|^m} \right| Z_1 \left( \frac{1 - |x|^2}{|x - \zeta|^2}, \nu \right) \right| dS(u) dS(\zeta) \leq c_{\alpha, m}.
\]

Hence, we have

\[
|D^\beta_v D^\alpha_x f(0, v_0)| \leq c_{\alpha, m} M'.
\]

Now, suppose \( D_1 f = 0 \) and \( \| f \|_{L^\infty(B(0, r_1) \times B(v_0, r_2), H_1)} = M \). Then we apply the argument above to \( f(r_1 x, r_2 v + v_0) \) with respect to \( x, \ v \in \mathbb{B}^m \), we obtain

\[
|D^\beta_v D^\alpha_x f(0, v_0)| \leq c_{\alpha, m} \frac{M}{r_1^{m+|\beta|} r_2^{m+|\alpha|}}.
\]

One can also obtain a Cauchy’s estimates with a \( L^1 \) norm as follows.

**Proposition 5.9.** Let \( \alpha, \beta \) be multi-indices. Suppose \( f \in C^2(\Omega \times \mathbb{B}^m, H_1), \ D_1 f = 0 \) in \( \Omega \times \mathbb{B}^m \). Then there exists a constant \( c_{\alpha, m} \) such that for any \( B(a, r_1) \subseteq \Omega \) and \( v_0 \in \mathbb{B}^m \), there holds

\[
|D^\beta_v D^\alpha_x f(a, v_0)| \leq c_{\alpha, m} \frac{\| f \|_{L^1(B(a, r_1) \times B(0, 1 - \frac{1}{4} d_{S^{m-1}}(v_0)), H_1)}}{r_1^{m+|\alpha|} d_{S^{m-1}}(v_0)^{m+|\beta|}},
\]

where \( d_{S^{m-1}}(v_0) \) stands for the distance from \( v_0 \in \mathbb{B}^m \) to the unit sphere \( S^{m-1} \) and

\[
\| f \|_{L^1(B(a, r_1) \times B(0, 1 - \frac{1}{4} d_{S^{m-1}}(v_0)), H_1)} := \int_{B(a, r_1)} \int_{B(0, 1 - \frac{1}{4} d_{S^{m-1}}(v_0))} |f(x, v)| dv dx.
\]
Proof. Let \(B(a, r_1) \subseteq \Omega \) and \(v_0 \in \mathbb{B}^m\). Without loss of generality, we may assume that \(a = 0\). Put \(M = \|f\|_{L^\infty(B(0, r_1/2) \times B(v_0, r_2/2), \mathcal{H}_1)} < \infty\), where \(r_2 = d_{\mathbb{B}^{m-1}}(v_0)\). Then we apply the argument in the theorem above to \(f(r_1 x/2, r_2 v/2 + v_0)\) with respect to \(x, v \in \mathbb{B}^m\), we obtain
\[
|D^\alpha_x D^\alpha_v f(0, v_0)| \leq \frac{c_{\alpha, m} M}{r_1^\alpha r_2^\alpha}.
\]
(5.2)

Next, we assume that \(f\) obtains its maximum value \(M\) over \(\overline{B(0, r_1/2) \times B(v_0, r_2/4)}\) at the point \((x_1, v_1)\), then we can see that \(B(x_1, r_1/2) \subseteq B(0, r_1)\), \(B(v_1, r_2/2) \subseteq B(v_0, 3r_2/4)\). Therefore, in accordance to the mean-value property, we have
\[
M = f(x_1, v_1) = \frac{m}{(m - 2)V(B(x_1, r_1/2))} \int_{B(x_1, r_1/2)} f\left(x, \frac{\eta v_1 \eta}{|\eta|^2}\right) dx.
\]
Notice that if we let \(\frac{\eta v_1 \eta}{|\eta|^2} = v_2\), which means that \(v_2\) is obtained from \(v_1\) by a rotation, then \(f(x, v_2) \in \mathcal{H}_1\) with respect to \(v_2 \in \mathbb{B}^m\). Further, one can notice that \(B(v_2, r_2/2) \subseteq B(0, 1 - \frac{1}{4}r_2)\). Hence, one can use the mean-value property of harmonic functions to obtain that
\[
|f(x, v_2)| = \frac{1}{V(B(v_2, r_2/2))} \left| \int_{B(v_2, r_2/2)} f(x, v) dv \right| \leq c_1 r_2^{-m} \int_{B(0, 1 - \frac{1}{4}r_2)} |f(x, v)| dv,
\]
where \(c_1\) is a positive constant only depending on \(m\). Therefore, we have
\[
M = f(x_1, v_1) \leq c' r_2^{-m} V(B(x_1, r_1/2))^{-1} \int_{B(x_1, r_1/2)} \int_{B(0, 1 - \frac{1}{4}r_2)} |f(x, v)| dv dx
\]
\[
\leq c'' r_1^{-m} r_2^{-m} \|f\|_{L^1(B(0, r_1) \times B(0, 1 - \frac{1}{4}r_2), \mathcal{H}_1)},
\]
where the last inequality comes from the fact that \(B(x_1, r_1/2) \subseteq B(0, r_1)\) and \(c', c''\) are positive constants only depending on \(m\). Combining with (5.2) completes the proof. \(\square\)

Remark 5.10. In this proposition, one might notice that the domain of \(v\) in the \(L^1\) norm is \(B(0, 1 - \frac{1}{4}d_{\mathbb{B}^{m-1}}(v_0))\) instead of \(B(v_0, r_2)\), which is the form of the domain of \(a\). This is because of the difficulty caused by \(v_2\) in the proof above. Actually, \(v_2\) is obtained by a rotation of \(v_1\), and it can be anywhere on the sphere with radius \(|v_1|\). Hence, it is impossible to cover \(v_2\) with a ball \(B(v_0, r_2)\) by a similar argument applied for \(a\). Also, the \(L^1\) norm defined here is equivalent to the norm defined in (5.5) up to a constant with similar argument given in (5.6). This constant only depends on \(m\) considering \(3/4 \leq 1 - \frac{1}{4}r_2 \leq 1\), since \(0 \leq r_2 \leq 1\).

One can apply the Cauchy’s estimates to obtain an analog of Liouville’s Theorem for null solutions of the generalized Maxwell operator as follows.

Theorem 5.11 (Liouville-type Theorem). Suppose \(f \in C^2(\mathbb{R}^m \times \mathbb{B}^m, \mathcal{H}_1) \cap L^\infty(\mathbb{R}^m \times \mathbb{B}^m, \mathcal{H}_1)\) and \(D_1 f = 0\) on \(\mathbb{R}^m \times \mathbb{B}^m\). Then \(f = f(v) \in \mathcal{H}_1\).

Proof. Suppose \(f\) is bounded by \(M\) on \(\mathbb{R}^m \times \mathbb{B}^m\). Let \(B(a, R)\) be an arbitrary ball, then from the proof of the Cauchy’s estimates above, one can immediately obtain that
\[
|\nabla_x f(a, v)| \leq \frac{c_m M}{R},
\]
since $M$ does not depend on $R$, we let $R \to \infty$, which gives us that $|\nabla_x f(a,v)| = 0$. Hence, $f = f(v) \in \mathcal{H}_1$.

An immediate consequence of the Liouville-type Theorem given above is the following.

**Proposition 5.12.** Let $1 \leq l \in \mathbb{N}$. Suppose $f \in C^2(\mathbb{R}^m \times \mathbb{B}^m, \mathcal{H}_1)$, $D_1 f = 0$ on $\mathbb{R}^m \times \mathbb{B}^m$ and

$$\|f\|_{L^1(B(0,R) \times \mathbb{B}^m, \mathcal{H}_1)} = o(R^{l+m}) \quad \text{as} \quad R \to \infty.$$  

Then $f$ is a polynomial of $x$ with degree less than $l$.

**Proof.** Let $(a,v)$ be an arbitrary point in $\mathbb{R}^m \times \mathbb{B}^m$, and $B(v,r_2) \subset \mathbb{B}^m$. Now, we denote $|a| = r_1$, then one can see that $B(a,R) \subset B(0, R + 2r_1)$. In accordance to the Cauchy’s estimates, we have, for any multi-index $\alpha$,

$$|D^\alpha_x f(a,v)| \leq \frac{c_{\alpha,m} \|f\|_{L^1(B(a,R) \times \mathbb{B}^m, \mathcal{H}_1)}}{R^{m+|\alpha|} r_1^m} \leq \frac{c_{\alpha,m} \|f\|_{L^1(B(0,R+2r_1) \times \mathbb{B}^m, \mathcal{H}_1)}}{R^{m+|\alpha|} r_2^m},$$

$$= c_{\alpha,m} o((R + 2r_1)^{l+m}) \quad \text{as} \quad R \to \infty.$$  

Notice that $r_1, r_2$ are fixed, and $c_{\alpha,m}$ only depends on $\alpha, m$. Therefore if we let $|\alpha| = l$ and $R \to \infty$, we have $|D^\alpha_x f(a,v)| = 0$ for any $|\alpha| = l$. Further, $(a,v)$ is arbitrary in the domain, this implies that $|D^\alpha_x f(x,v)| = 0$ for any $|\alpha| = l$ on $\mathbb{R}^m \times \mathbb{B}^m$. Hence, $f$ is a polynomial of $x$ with degree less than $l$.

The next result provides the convergence property.

**Proposition 5.13.** Suppose that there is a sequence of functions $\{f_j(x,v)\} \subset C^2(\Omega \times \mathbb{B}^m, \mathcal{H}_1)$ such that $D_1 f_j = 0$ for any $1 \leq j \in \mathbb{N}$. Further, assume that $\{\|f_j\|_{L^1(K \times \mathbb{B}^m, \mathcal{H}_1)}\}_{j=1}^\infty$ is uniformly bounded, where $K$ is an arbitrary compact subset of $\Omega$. Then, there exists a subsequence $\{f_{j_k}\}$ and a function $f(x,v) \in C^2(\Omega \times \mathbb{B}^m, \mathcal{H}_1)$ such that for each multi-index $\alpha$, $\{D^\alpha_x f_{j_k}\}$ converges uniformly to $D^\alpha_x f$ on $K_1 \times K_2$ for an arbitrary compact subset $K_1 \times K_2$ of $\Omega \times \mathbb{B}^m$. Moreover, $f \in C^2(\Omega \times \mathbb{B}^m, \mathcal{H}_1)$ and $D_1 f = 0$ in $\Omega \times \mathbb{B}^m$.

**Proof.** Here, we only need to show that there exists a subsequence $\{f_{j_k}\}$ and a function $f(x,v) \in C^2(\Omega \times \mathbb{B}^m, \mathcal{H}_1)$ such that for each multi-index $\alpha$, $\{D^\alpha_x f_j\}$ converges uniformly to $D^\alpha_x f$ on $K_1 \times K_2$ for each compact subset $K_1 \times K_2$ of $\Omega \times \mathbb{B}^m$ and $D_1 f = 0$ on $K_1 \times K_2$.

Let $a$ be an arbitrary point in $\Omega$, and $B(a, 2r_1) \subset \Omega$, $B(0, 2r_2) \subset \mathbb{B}^m$. We choose $K = \overline{B(a, 2r_2)}$, $K_1 = \overline{B(a, r_1)}$, $K_2 = B(0, r_2)$. According to the assumption, $\{\|f_j\|_{L^1(K \times \mathbb{B}^m, \mathcal{H}_1)}\}$ is uniformly bounded by some $M > 0$. Then, in accordance to the Cauchy’s estimates given in Theorem 5.7, we have that for all $(x,v) \in K_1 \times K_2$,

$$|D^\alpha_x f_j(x,v)| \leq \frac{c_{\alpha,m} M}{r_1^{m+|\alpha|} r_2^m}.$$  

By choosing $|\alpha| = 0, 1$ above respectively, one can see that $\{f_j\}$ is uniformly bounded and uniformly equicontinuous in $K_1 \times K_2$, Arzelà-Ascoli Theorem tells us that there exists a subsequence $\{f_{j_k}\}$ converges uniformly to a function $f \in C^2(K_1 \times K_2, \mathcal{H}_1)$ in $K_1 \times K_2$. Since $a$, $K_1$, $K_2$ are arbitrary, we conclude that $f \in C^2(\Omega \times \mathbb{B}^m, \mathcal{H}_1)$. 


Further, $D_1 f_{j_k} = 0$ tells us that
\[
f_{j_k}(x, v) = \frac{c_m}{2} \int \int_{S^{m-1} \times S^{m-1}} \frac{1 - |x|^2}{|x - \zeta|^m} Z_1 \left( \frac{(x - \zeta)\omega(x - \zeta)}{|x - \zeta|^2}, \nu \right) f_{j_k}(\zeta, u) dS(u) dS(\zeta),\]
for every integer $j_k$, and $x, v \in \mathbb{B}^m$, $\omega, \nu$ are defined in (3.12). Since $f_{j_k}$ and the Poisson kernel are both uniformly bounded in every compact subset $K_1 \times K_2 \subset \Omega \times \mathbb{B}^m$, we can take the limit on both sides to obtain
\[
f(x, v) = \frac{c_m}{2} \int \int_{S^{m-1} \times S^{m-1}} \frac{1 - |x|^2}{|x - \zeta|^m} Z_1 \left( \frac{(x - \zeta)\omega(x - \zeta)}{|x - \zeta|^2}, \nu \right) f(\zeta, u) dS(u) dS(\zeta),\]
for every $x \in K_1$, $v \in K_2$, hence, $D_1 f = 0$ over $K_1 \times K_2$. For a multi-index $\alpha$, we have
\[
D_\alpha^m f_{j_k}(x, v) = \frac{c_m}{2} \int \int D_\alpha^m \left[ \frac{1 - |x|^2}{|x - \zeta|^m} Z_1 \left( \frac{(x - \zeta)\omega(x - \zeta)}{|x - \zeta|^2}, \nu \right) \right] f_{j_k}(\zeta, u) dS(u) dS(\zeta).\]
Since $Z_1(u, v)$ is a degree-1 homogeneous polynomial in $u$ and $v$, one can observe that for every compact subset $K_1 \times K_2 \subset \Omega \times \mathbb{B}^m$,
\[
D_\alpha^m \left[ \frac{1 - |x|^2}{|x - \zeta|^m} Z_1 \left( \frac{(x - \zeta)\omega(x - \zeta)}{|x - \zeta|^2}, \nu \right) \right]
\]
is uniformly bounded in $(x, \zeta, u, v) \in K_1 \times S^{m-1} \times S^{m-1} \times K_2$. Therefore, we have
\[
\lim_{j_k \to \infty} D_\alpha^m f_{j_k}(x, v) = \lim_{j_k \to \infty} \frac{c_m}{2} \int \int D_\alpha^m \left[ \frac{1 - |x|^2}{|x - \zeta|^m} Z_1 \left( \frac{(x - \zeta)\omega(x - \zeta)}{|x - \zeta|^2}, \nu \right) \right] f_{j_k}(\zeta, u) dS(u) dS(\zeta) = D_\alpha^m f(x, v),
\]
which completes the proof.

5.3. Poisson’s equation and representation formula. Recall that the fundamental solution of $D_1$ in $\mathbb{R}^m \times \mathbb{B}^m$ is given by [7]
\[
H_1(x, y, u, v) = \frac{1}{(4 - m)|y - x|^{2-m}} Z_1 \left( \frac{(y - x)u(y - x)}{|y - x|^2}, v \right) = \frac{m}{(4 - m)|y - x|^{2-m}} \frac{(y - x)u(y - x)}{|y - x|^2},
\]
where $Z_1(u, v)$ is the reproducing kernel of degree-1 homogeneous harmonic polynomials. Let $f \in C^2_b(\mathbb{R}^m \times \mathbb{B}^m, H_1)$, that is $f \in C^2(\mathbb{R}^m \times \mathbb{B}^m, H_1)$ and $f$ has compact support with respect to $x$, and set
\[
\Phi(y, v) = \int \int_{\mathbb{R}^m \times S^{m-1}} H_1(x, y, u, v) f(x, u) dS(u) dx.
\]
From the homogeneity of $|x|$ in the integrand and the fact that $f$ has compact support with respect to $x$, one can easily see that $\Phi$ is well-defined.

**Theorem 5.14** (Solving Poisson’s equation). Let $\Phi$ be defined as in (5.3). Then, we have
(1) $\Phi \in C^2(\mathbb{R}^m \times \mathbb{B}^m, \mathcal{H}_1)$,
(2) $D_1 \Phi = f$ in $\mathbb{R}^m \times \mathbb{B}^m$.

Proof. 1. Firstly, we notice that $Z_1 \left( \frac{xux}{|x|^2}, v \right)$ is harmonic with respect to $v$, so $\Phi$ is harmonic with respect to $v$. Further, we have

$$
\Phi(y, v) = \int \int_{\mathbb{R}^m \mathbb{S}^{m-1}} H_1(x, y, u, v) f(x, u) dS(u) dx
$$

$$
= \int \int_{\mathbb{R}^m \mathbb{S}^{m-1}} H_1(x, 0, u, v) f(y - x, u) dS(u) dx,
$$

hence,

$$
\frac{\Phi(y + he_j, v) - \Phi(y, v)}{h} = \int \int_{\mathbb{R}^m \mathbb{S}^{m-1}} H_1(x, 0, u, v) \frac{f(y - x + he_j, u) - f(y - x, u)}{h} dS(u) dx,
$$

where $h \neq 0$ and $e_j = (0, \ldots, 1, \ldots, 0)$ with 1 in the $j^{th}$ spot. Notice that $f \in C^2_{c}(\mathbb{R}^m \times \mathbb{B}^m, \mathcal{H}_1)$, which tells us that

$$
\frac{f(y - x + he_j, u) - f(y - x, u)}{h} \to f_{y_j}(y - x, u), \ h \to 0
$$

uniformly. Hence, we have

$$
\Phi_{y_j}(y, v) = \int \int_{\mathbb{R}^m \mathbb{S}^{m-1}} H_1(x, 0, u, v) f_{y_j}(y - x, u) dS(u) dx.
$$

A similar argument can be applied to the second derivatives, which implies $\Phi \in C^2(\mathbb{R}^m \times \mathbb{B}^m, \mathcal{H}_1)$.

2. For $\epsilon > 0$, let $B_\epsilon := \{x \in \mathbb{R}^m : |x| < \epsilon\}$ and $B_\epsilon^c := \mathbb{R}^m \backslash B_\epsilon$. Then, we have

$$
D_{1,y,v} \int \int_{\mathbb{R}^m \mathbb{S}^{m-1}} H_1(x, 0, u, v) f(y - x, u) dS(u) dx
$$

$$
= D_{1,y,v} \int \int_{B_\epsilon \mathbb{S}^{m-1}} H_1(x, 0, u, v) f(y - x, u) dS(u) dx
$$

$$
+ D_{1,y,v} \int \int_{B_\epsilon^c \mathbb{S}^{m-1}} H_1(x, 0, u, v) f(y - x, u) dS(u) dx,
$$

where $D_{1,y,v}$ stands for $D_1$ with respect to $y$ and $v$. From the homogeneity of $|x|^{2-m}$ in $H_1$, if one changes the coordinates of $x$ to spherical coordinates, one can see that the singularity at $|x|^{2-m}$ disappears and the first integral above goes to zero when $\epsilon \to 0$. Hence, we only need to calculate the second integral above. Recall that

$$
D_{1,y,v} = \Delta_y - \frac{4}{m} \langle v, D_y \rangle \langle D_v, D_y \rangle,
$$
hence, we have

\[
\int \int_{B^c \subset S^{m-1}} H_1(x, 0, u, v)D_1 y, v f(y - x, u)dS(u)dx \\
= \int \int_{B^c \subset S^{m-1}} H_1(x, 0, u, v)\Delta_x f(y - x, u)dS(u)dx \\
- \frac{4}{m} \int \int_{B^c \subset S^{m-1}} H_1(x, 0, u, v)v\langle D_x, D_v \rangle f(y - x, u)dS(u)dx \\
:= I_\epsilon - \frac{4}{m} K_\epsilon.
\]

Here, we remind the reader that in the expression of $K_\epsilon$ above and later in this section, $D_v$ always acts on $H_1(x, 0, u, v)$. Next, with the awareness of the fact that $f$ has compact support with respect to $x$ in $\mathbb{R}^m$, we apply integration by parts to $I_\epsilon$, $K_\epsilon$, respectively. On the one hand,

\[
I_\epsilon = \int \int_{B^c \subset S^{m-1}} H_1(x, 0, u, v)\Delta_x f(y - x, u)dS(u)dx \\
= \int \int_{B^c \subset S^{m-1}} \Delta_x H_1(x, 0, u, v)f(y - x, u)dS(u)dx \\
+ \int \int_{\partial B_s \subset S^{m-1}} H_1(x, 0, u, v)\frac{\partial f}{\partial n_x}(y - x, u)dS(u)dS(x) \\
- \int \int_{\partial B_s \subset S^{m-1}} \frac{\partial H_1}{\partial n(x)}(x, 0, u, v)f(y - x, u)dS(u)dS(x),
\]

where $n(x) := (n_1(x), \ldots, n_m(x)) = -x/\epsilon$ is the outward normal vector of $B^c_s$ and $dS(x)$ is the area element on $\partial B_s$. Notice that the homogeneity of $|x|^{2-m}$ is $2 - m$, this implies that

\[
\lim_{\epsilon \to 0} \int_{\partial B_s \subset S^{m-1}} H_1(x, 0, u, v)\frac{\partial f}{\partial n(x)}(y - x, u)dS(u)dS(x) = 0.
\]

Further, notice that

\[
\frac{\partial}{\partial n(x)} = -\sum_{i=1}^m x_i\frac{\partial}{\partial x_i} = \frac{E_x}{\epsilon},
\]

where $E_x$ is the Euler operator and one can easily check that $E_x H_1(x, 0, u, v) = \deg_x H_1(x, 0, u, v) = (2 - m)H_1(x, 0, u, v)$. Therefore, we have

\[
\lim_{\epsilon \to 0} I_\epsilon = \lim_{\epsilon \to 0} \int \int_{B^c \subset S^{m-1}} \Delta_x H_1(x, 0, u, v)f(y - x, u)dS(u)dx \\
- \frac{m - 2}{\epsilon} \lim_{\epsilon \to 0} \int \int_{\partial B_s \subset S^{m-1}} H_1(x, 0, u, v)f(y - x, u)dS(u)dS(x). \tag{5.5}
\]
On the other hand,

\[
K_\epsilon = \int \int_{B_\epsilon \cap \mathbb{S}^{m-1}} H_1(x, 0, u, v)\langle v, D_x \rangle \langle D_v, D_x \rangle f(y - x, u) dS(u) dx
\]

\[
= \sum_{i,j=1}^m \int \int_{B_\epsilon \cap \mathbb{S}^{m-1}} (v_i \partial_{v_j} H_1(x, 0, u, v)) (\partial_{x_i} \partial_{x_j} f(y - x, u)) dS(u) dx
\]

\[
= \int \int_{B_\epsilon \cap \mathbb{S}^{m-1}} \left[ \langle v, D_x \rangle \langle D_v, D_x \rangle H_1(x, 0, u, v) \right] f(y - x, u) dS(u) dx
\]

\[
+ \sum_{i,j=1}^m \int \int_{\partial B_\epsilon \cap \mathbb{S}^{m-1}} (n_i(x) v_i \partial_{v_j} H_1(x, 0, u, v)) (\partial_{x_j} f(y - x, u)) dS(u) dS(x)
\]

\[
- \sum_{i,j=1}^m \int \int_{\partial B_\epsilon \cap \mathbb{S}^{m-1}} (n_j(x) v_i \partial_{v_i} \partial_{x_i} H_1(x, 0, u, v)) f(y - x, u) dS(u) dS(x).
\]

Similar argument as applied to \( I_\epsilon \), one has

\[
\lim_{\epsilon \to 0} K_\epsilon = \lim_{\epsilon \to 0} \int \int_{B_\epsilon \cap \mathbb{S}^{m-1}} \left[ \langle v, D_x \rangle \langle D_v, D_x \rangle H_1(x, 0, u, v) \right] f(y - x, u) dS(u) dx
\]

\[
- \lim_{\epsilon \to 0} \sum_{i,j=1}^m \int \int_{\partial B_\epsilon \cap \mathbb{S}^{m-1}} (n_j(x) v_i \partial_{v_i} \partial_{x_i} H_1(x, 0, u, v)) f(y - x, u) dS(u) dS(x).
\]

Here, we notice that we applied integration by parts twice in the calculation above, since \( f \in C^2_\epsilon(\mathbb{R}^m \times \mathbb{B}^m, H_1) \), we should obtain the same result if we switch the order of two integration by parts. Switching the order of the integration by parts above gives us that

\[
\lim_{\epsilon \to 0} K_\epsilon = \lim_{\epsilon \to 0} \int \int_{B_\epsilon \cap \mathbb{S}^{m-1}} \langle v, D_x \rangle \langle D_v, D_x \rangle H_1(x, 0, u, v) f(y - x, u) dS(u) dx
\]

\[
- \lim_{\epsilon \to 0} \sum_{i,j=1}^m \int \int_{\partial B_\epsilon \cap \mathbb{S}^{m-1}} (n_i(x) v_i \partial_{v_i} \partial_{x_i} H_1(x, 0, u, v)) f(y - x, u) dS(u) dS(x).
\]

Therefore, we have

\[
\lim_{\epsilon \to 0} \sum_{i,j=1}^m \int \int_{\partial B_\epsilon \cap \mathbb{S}^{m-1}} (n_j(x) v_i \partial_{v_i} \partial_{x_i} H_1(x, 0, u, v)) f(y - x, u) dS(u) dS(x)
\]

\[
= \lim_{\epsilon \to 0} \sum_{i,j=1}^m \int \int_{\partial B_\epsilon \cap \mathbb{S}^{m-1}} (n_i(x) v_i \partial_{v_i} \partial_{x_i} H_1(x, 0, u, v)) f(y - x, u) dS(u) dS(x). \quad (5.6)
\]
Next, we explicitly calculate the differentiation of $H_1$ in the two integrals above. We omit the constant in $H_1$ below in the calculation for convenience.

$$
\sum_{i,j=1}^{m} n_j(x)v_i \partial_{v_i} \partial_{x_i} H_1(x, 0, u, v) = -\epsilon^{-1} \sum_{i,j=1}^{m} x_j v_i \partial_{v_i} \partial_{x_i} |x|^{2-m} \left( \frac{\langle u, x \rangle}{|x|^2}, v \right)
$$

$$
= -\epsilon^{-1} \sum_{i,j=1}^{m} x_j v_i \partial_{x_i} \left( |x|^{2-m} \left( u_j - 2 \frac{\langle x, u \rangle x_j}{|x|^2} \right) \right)
$$

$$
= -\epsilon^{-1} \sum_{i,j=1}^{m} x_j v_i \left[ (2 - m)x_j |x|^{2-m} \left( u_j - 2 \frac{\langle x, u \rangle x_j}{|x|^2} \right) \right]
$$

$$
= -2|x|^{2-m} \left( \frac{u_j x_j + \langle x, u \rangle \delta_{ij}}{|x|^2} - 2 \frac{\langle x, u \rangle x_j^2}{|x|^4} \right)
$$

$$
= -m \epsilon^{-1-m} \langle x, u \rangle \langle x, v \rangle + 2 \epsilon^{-m} \langle u, v \rangle.
$$

Similarly,

$$
\sum_{i,j=1}^{m} n_i(x)v_i \partial_{v_i} \partial_{x_i} H_1(x, 0, u, v) = -\epsilon^{-1} \sum_{i,j=1}^{m} x_i v_i \partial_{v_i} \partial_{x_i} |x|^{2-m} \left( \frac{\langle u, x \rangle}{|x|^2}, v \right)
$$

$$
= -\epsilon^{-1} \langle x, v \rangle \sum_{j=1}^{m} \partial_{x_j} \left( |x|^{2-m} \left( u_j - 2 \frac{\langle x, u \rangle x_j}{|x|^2} \right) \right)
$$

$$
= -\epsilon^{-1} \langle x, v \rangle \sum_{j=1}^{m} \left[ (2 - m)x_j |x|^{-m} \left( u_j - 2 \frac{\langle x, u \rangle x_j}{|x|^2} \right) \right]
$$

$$
= -2|x|^{2-m} \left( \frac{u_j x_j + \langle x, u \rangle \delta_{ij}}{|x|^2} - 2 \frac{\langle x, u \rangle x_j^2}{|x|^4} \right)
$$

$$
= m \epsilon^{-1-m} \langle x, u \rangle \langle x, v \rangle.
$$

Plugging into (5.6), we immediately have

$$
m \lim_{\epsilon \to 0} \int_{\partial B, S^{m-1}} \int \epsilon^{-1-m} \langle x, u \rangle \langle x, v \rangle f(y - x, u) dS(u) dS(x)
$$

$$
= \lim_{\epsilon \to 0} \int_{\partial B, S^{m-1}} \int \epsilon^{-m} \langle u, v \rangle f(y - x, u) dS(u) dS(x), \quad \text{(5.7)}
$$

and with the constant in $H_1$, we also have

$$
\lim_{\epsilon \to 0} K_\epsilon = \lim_{\epsilon \to 0} \int_{B^c, S^{m-1}} \int \langle v, D_x \rangle \langle D_v, D_x \rangle H_1(x, 0, u, v) f(y - x, u) dS(u) dS(x)
$$

$$
= \frac{m}{(4 - m) \omega_m} \lim_{\epsilon \to 0} \int_{\partial B, S^{m-1}} \int \epsilon^{-m} \langle u, v \rangle f(y - x, u) dS(u) dS(x). \quad \text{(5.8)}
$$
Plugging (5.5) and (5.8) into (5.4), we have

\[
\lim_{\epsilon \to 0} \int_{B^c \mathbb{S}^{m-1}} \int_{B^c \mathbb{S}^{m-1}} H_1(x, 0, u, v) D_{1,y,v} f(y - x, u) dS(u) dx = \lim_{\epsilon \to 0} \int_{B^c \mathbb{S}^{m-1}} \int_{B^c \mathbb{S}^{m-1}} D_{1,x,v} H_1(x, 0, u, v) f(y - x, u) dS(u) dS(x)
\]

\[
-\frac{m - 2}{\epsilon} \lim_{\epsilon \to 0} \int_{\partial B_c \mathbb{S}^{m-1}} \int_{\partial B_c \mathbb{S}^{m-1}} H_1(x, 0, u, v) f(y - x, u) dS(u) dS(x)
\]

\[
+ \frac{4}{(4 - m) \omega_m} \lim_{\epsilon \to 0} \int_{\partial B_c \mathbb{S}^{m-1}} \int_{\partial B_c \mathbb{S}^{m-1}} \epsilon^{-m} \langle u, v \rangle f(y - x, u) dS(u) dS(x)
\]

\[
= \lim_{\epsilon \to 0} \int_{\partial B_c \mathbb{S}^{m-1}} \int_{\partial B_c \mathbb{S}^{m-1}} \epsilon^{-m} \langle u, v \rangle f(y - x, u) dS(u) dS(x)
\]

The calculation above used (5.7) and the fact that

\[
D_{1,x,v} H_1(x, 0, u, v) = D_{1,x,v} \frac{1}{(4 - m) \omega_m} |x|^{2-m} Z_1 \left( \frac{x u x}{|x|^2}, v \right) = D_{1,x,v} \frac{1}{(4 - m) \omega_m} |x|^{2-m} Z_1 \left( u, \frac{x v x}{|x|^2} \right) = 0
\]

for \( x \in B^c \). Further, since \( Z_1(u, v) = m(u, v) \) is the reproducing kernel of \( \mathcal{H}_1 \), we have

\[
\lim_{\epsilon \to 0} \int_{B^c \mathbb{S}^{m-1}} \int_{B^c \mathbb{S}^{m-1}} H_1(x, 0, u, v) D_{1,y,v} f(y - x, u) dS(u) dx = \frac{1}{\omega_m} \int_{\partial B_c \mathbb{S}^{m-1}} \int_{\partial B_c \mathbb{S}^{m-1}} \epsilon^{-m} Z_1(u, v) f(y - x, u) dS(u) dS(x)
\]

\[
= \frac{1}{\omega_m} \int_{\partial B_c \mathbb{S}^{m-1}} \int_{\partial B_c \mathbb{S}^{m-1}} \epsilon^{-m} f(y - x, v) dS(x) = f(y, v),
\]

which completes the proof. \( \square \)

With Theorem 5.14 and the Liouville-type Theorem, one can immediately have a representation formula as follows.
Theorem 5.15 (Representation formula). Assume $m > 4$ and $f \in C^2_c(\mathbb{R}^m \times \mathbb{B}^m, \mathcal{H}_1)$. Then any bounded solution of $D_1 g = f$ in $\mathbb{R}^m \times \mathbb{B}^m$ has the form
\[
g(y,v) = \int_{\mathbb{R}^m} \int_{S^{m-1}} H_1(x,y,u,v)f(x,u)dS(u)dx + h(v), \quad y \in \mathbb{R}^m, v \in \mathbb{B}^m,
\]
where $h \in \mathcal{H}_1$.

Proof. Notice that $H_1(x,y,u,v) \to 0$ when $y \to 0$ for $m > 4$, hence,
\[
\Phi(y,v) := \int_{\mathbb{R}^m} \int_{S^{m-1}} H_1(x,y,u,v)f(x,u)dS(u)dx
\]
is a bounded solution in $C^2(\mathbb{R}^m \times \mathbb{B}^m, \mathcal{H}_1)$ for $D_1 g = f$ in $\mathbb{R}^m \times \mathbb{B}^m$. If $h$ is another bounded solution in $\mathbb{R}^m \times \mathbb{B}^m$, then we have a bounded solution $\Phi - h$ for $D_1 g = 0$ in $\mathbb{R}^m \times \mathbb{B}^m$. In accordance to the Liouville-type Theorem, $\Phi - g = h(v)$ in $\mathbb{R}^m \times \mathbb{B}^m$, where $h(v) \in \mathcal{H}_1$, which completes the proof. □

Conclusion

In this paper, an investigation of boundary value problems for generalizations involving Laplacian, named as the generalized Maxwell operator and Bosonic Laplacian, in the higher spin setting is conducted for the first time. Solutions to Dirichlet problems of Bosonic Laplacian in the upper-half space and the unit ball are solved with continuous or $L^p$ data. Uniqueness of solutions of Dirichlet problem of the generalized Maxwell operator is proved by a Hölder estimate for solutions to elliptic systems. This provides us important analogs of some fundamental results, such as mean-value property, maximum principle, Cauchy’s estimates, Liouville’s theorem, Harnack’s inequality and the representation formula of harmonic functions for null solutions of the generalized Maxwell operator. We expect these results are also true for Bosonic Laplacian, however, they can not be derived by the techniques applied here. This will be investigated in an upcoming paper. These results also lead to further investigations on analogs of other important results of harmonic functions. For instance, Böcher’s theorem, harmonic Hardy spaces, sub(sup)-harmonic functions, etc.

References

[1] S. Agmon, A. Douglis, L. Nirenberg, Estimates Near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions II, Communications on Pure and Applied Mathematics, Vol. XVII, 1964, pp. 35-92.
[2] L.V. Ahlfors, Möbius transformations in $\mathbb{R}^n$ expressed through $2 \times 2$ matrices of Clifford numbers, Complex Variables, Vol. 5, 1986, pp. 215-224.
[3] S. Axler, P. Bourdon, W. Ramey, Harmonic function theory, second edition, Graduate Texts in Mathematics, Springer, New York, 2001.
[4] R. Bañuelos, C.N. Moore, Probabilistic Behavior of Harmonic Functions, Birkhäuser, 1991.
[5] J. Bureš, F. Sommen, V. Souček, P. Van Lancker, Rarita-Schwinger Type Operators in Clifford Analysis, J. Funct. Anal. Vol. 185, No. 2, 2001, pp. 425-455.
[6] J.L. Clerc, B. Ørsted, Conformal covariance for the powers of the Dirac operator, https://arxiv.org/abs/1409.4983v1.
[7] H. De Bie, D. Eelbode, M. Roels, The Higher Spin Laplace Operator, Potential Analysis, Vol. 47, Issue 2, 2017, pp. 123-149.
[8] R. Delanghe, T. Qian, Half Dirichlet Problems and Decompositions of Poisson Kernels, Advances in Applied Clifford Algebras, Vol. 17, 2007, pp. 383-393.
[9] R. Delanghe, F. Sommen, V. Souček, *Clifford Algebra and Spinor-Valued Functions, A Function Theory for the Dirac Operator*, Springer Netherlands, 1992.

[10] C. Ding, *Integral Formulas for Higher Order Conformally Invariant Fermionic Operators*, Advances in Applied Clifford Algebras, 29, 37 (2019) doi:10.1007/s00006-019-0953-4.

[11] C. Ding, J. Ryan, *Some Properties of Bosonic Laplacian*, Transactions of the American Mathematical Society, Vol. 371, Issue 5, 2019, pp. 3375-3395.

[12] C. Ding, J. Ryan, *On Some Conformally Invariant Operators in Euclidean Space*, Clifford Analysis and Related Topics, in Honor of Paul A.M. Dirac, CART 2014, Tallahassee, Florida, December 15-17, pp. 53-72.

[13] C. Ding, R. Walter, J. Ryan, *Construction of Arbitrary Order Conformally Invariant Operators in Higher Spin Spaces*, The Journal of Geometric Analysis, Vol. 27, Issue 3, 2017, pp. 2418-2452.

[14] D.C. Dinh, *Dirichlet boundary value problem for monogenic functions in Clifford analysis*, Complex Variables and Elliptic Equations, An International Journal, Vol. 59, Issue 9, 2014, pp. 1201-1213.

[15] C.F. Dunkl, J. Li, J. Ryan, P. Van Lancker, *Some Rarita-Schwinger type operators*, Computational Methods and Function Theory, Vol. 13, Issue 3, 2013, pp. 397-424.

[16] D. Eelbode, M. Roels, *Generalised Maxwell equations in higher dimensions*, Complex Analysis and Operator Theory, Vol. 10, Issue 2, 2016, pp. 267-293.

[17] D. Eelbode, T. Raeymaekers, P. Van Lancker, *On the fundamental solution for higher spin Dirac operators*, Journal of Mathematical Analysis and Applications, Vol. 405, Issue 2, 2013, 555-564.

[18] D. Eelbode, T. Raeymaekers, *Polynomial solutions for arbitrary higher spin Dirac operators*, Experimental Mathematics, Vol. 24, Issue 3, 2015, 339-354.

[19] J. Gilbert, M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambridge University Press, Cambridge, 1991.

[20] K. Gürlebeck, K. Habetha, W. Sprößig, *Holomorphic Functions in the Plane and n-dimensional Space*, Birkhäuser Basel, 2008.

[21] K. Gürlebeck, K. Habetha, W. Sprößig, *Application of Holomorphic Functions in Two and Higher Dimensions*, Birkhäuser Basel, 2016.

[22] K. Gürlebeck, W. Sprößig, *Quaternionic Analysis and Elliptic Boundary Value Problems*, Birkhäuser Basel, 1989.

[23] S. Huang, Y. Qiao, G. Wen, *Real and Complex Clifford Analysis*, Advances in Complex Analysis and Its Applications, Vol. 5, Springer US, 2006.

[24] M. Ku, U. Kähler, P. Cerejeiras, *Dirichlet-type problems for the iterated Dirac operator on the unit ball in Clifford analysis*, Advances in Hypercomplex Analysis. Springer INdAM Series, vol 1. Springer, Milano, 2013.

[25] A. McIntosh, *Clifford algebras and the higher-dimensional Cauchy integral, Approximation Theory and Function Spaces*, Banach Center Publications, 22(1989), pp. 253-267.

[26] I. Porteous, *Clifford algebra and the classical groups*, Cambridge University Press, Cambridge, 1995.

[27] J. Ryan, *Dirac Operators, Conformal Transformations and Aspects of Classical Harmonic Analysis*, Journal of Lie Theory, Vol. 8, 1998, pp. 67-82.

[28] E. Stein, G. Weiss, *Generalization of the Cauchy-Riemann equations and representations of the rotation group*, Amer. J. Math. 90 (1968), pp. 163-196.

[29] E.M. Stein, G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, (PMS-32). Princeton, New Jersey: Princeton University Press, 1971.

[30] P. Van Lancker, F. Sommen, D. Constales, *Models for irreducible representations of Spin(m)*, Advances in Applied Clifford Algebras, 11(S1), 2001, 271-289.

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