Improved algorithm to determine 3-colorability of graphs with the minimum degree at least 7

Nicholas Crawford, Sogol Jahanbekam, and Katerina Potika

Abstract

Let $G$ be an $n$-vertex graph with the maximum degree $\Delta$ and the minimum degree $\delta$. We give algorithms with complexity $O(1.3158n^{\frac{1}{0.7}}\Delta(G))$ and $O(1.32n^{\frac{1}{0.73}}\Delta(G))$ that determines if $G$ is 3-colorable, when $\delta(G) \geq 8$ and $\delta(G) \geq 7$, respectively.

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1 Introduction

A coloring of the vertices of a graph is proper if adjacent vertices receive different colors. A graph $G$ is $k$-colorable if it has a proper coloring using $k$ colors. The chromatic number of a graph $G$, written as $\chi(G)$, is the smallest integer $k$ such that $G$ is $k$-colorable.

The proper coloring problem is one of the most studied problems in graph theory. To determine the chromatic number of a graph, one should find the smallest integer $k$ for which the graph is $k$-colorable. The $k$-colorability problem, for $k \geq 3$, is one of the classical NP-complete problems [9].

Even approximating the chromatic number has been shown to be a very hard problem. Lund and Yannakakis [8] have shown that there is an $\epsilon$ such that the chromatic number of a general $n$-vertex graph cannot be approximated with ratio $n^\epsilon$ unless $P = NP$.

In 1971, Christofides obtained the first non-trivial algorithm computing the chromatic number of $n$-vertex graphs running in $n!n^{O(1)}$ time [3]. Five years later Lawler [7] used dynamic programming and enumerations of maximal independent sets to improve it to an
algorithm with running time $O^*(2.4423^n)$. Later the running time was improved by Eppstein [4]. The best-known complexity for determining the chromatic number of graphs is due to Björklund, Husfeldt, and Koivisto [2] who used a combination of inclusion-exclusion and dynamic programming to develop a $O(2^n)$ algorithm to determine the chromatic number of $n$-vertex graphs.

The $k$-colorability problem for small values of $k$, like 3 and 4 is also a highly-studied problem that has attracted a lot of attention. Not only this problem has its own importance, but also improving the bounds for small values of $k$ could be used to improve the bound for higher values of $k$ and as a result, improve the complexity of the general coloring problem. The fastest known algorithm deciding if a graph is 3-colorable or not runs in $O(1.3289^n)$ time and is due to Beigel and Eppstein [1]. The fastest known algorithm for 4-colorability runs in $O(1.7272^n)$ and is due to Fomin, Gaspers, and Saurabh [5].

In this paper, we prove the following.

**Theorem 1.** Let $G$ be an $n$-vertex graph with maximum degree $\Delta$ and minimum degree $\delta$, where $\delta(G) \geq 8$. We can determine in $O(1.3158^n - 0.7\Delta)$ time if $G$ is 3-colorable or not.

**Theorem 2.** Let $G$ be an $n$-vertex graph with maximum degree $\Delta$ and minimum degree $\delta$, where $\delta(G) \geq 7$. We can determine in $O(1.32^n - 0.73\Delta)$ time if $G$ is 3-colorable or not.

For smaller minimum degree conditions, results similar to the statements of Theorems 1 and 2 can be proved, but the complexity would increase. For example, the 3-colorability of a graph with minimum degree 6 can be determined in $O(1.368^n - 0.7d(v))$ time. This result is not an improvement compared to that of Beigel and Eppstein [1] however, because $1.368 > 1.3289$.

## 2 Definitions, Notation, and Tools

In this section we define the terms and notation we use to prove Theorems 1 and 2.

For a graph $G$ with vertex set $V(G)$ and edge set $E(G)$, we denote the minimum degree by $\delta(G)$ and the maximum degree by $\Delta(G)$. We suppose all graphs studied in this note are simple. Let $v$ be a vertex in $G$. The degree of $v$ in $G$ is denoted by $d_G(v)$ or simply $d(v)$ (when there is no fear of confusion). The open neighborhood of $v$ in $G$, denoted by $N_G(v)$ (or simply $N(v)$), is the set of neighbors of $v$ in $G$ and $N^2(v)$ denotes the set of vertices in $G$ that are in distance (exactly) 2 from $v$. Therefore $N(v) \cap N^2(v) = \emptyset$. The closed neighborhood of $v$ in $G$, denoted by $N[v]$, is equal to $N(v) \cup \{v\}$. 

Let $A$ be a subset of $V(G)$. The graph $G[A]$ is the induced subgraph of $G$ with vertex set $A$. Let $u$ and $v$ be two vertices of $G$. The graph $G/uv$ is the graph obtained from $G$ after contracting (identifying) the vertices $u$ and $v$ in $G$ and replacing multiple edges by one edge, so that the resulting graph is simple.

Suppose for each vertex $v$ in $V(G)$, there exists a list of colors denoted by $L(v)$. A proper list coloring of $G$ is a choice function that maps every vertex $v$ to a color in the list $L(v)$ in such a way that the coloring is proper. A graph is $k$-choosable if it has a proper list coloring whenever each vertex has a list of size $k$.

A Boolean expression is a logical statement that is either TRUE or FALSE. In computer science, the Boolean satisfiability problem (abbreviated to SAT) is the problem of determining if there exists an interpretation that satisfies a given Boolean expression. The 3-satisfiability problem or 3-SAT problem is a special case of SAT problem, where the Boolean expression can be divided into clauses such that every clause contains three literals.

The constraint satisfiability problem is a satisfiability problem which is not necessarily Boolean. In an $(r,t)$-CSP instance, we are given a collection of $n$ variables, each of which can be given one of up to $r$ different colors and a set of constraints, where each constraint is expressed using $t$ variables, i.e. certain color combinations are forbidden for $t$ variables.

By the above definition 3-SAT is the same as $(2,3)$-CSP. It was proved in [1] that each $(a,b)$-CSP instance is equivalent to a $(b,a)$-CSP instance. Therefore any 3-SAT is equivalent to a $(3,2)$-CSP instance.

The following result was proved by Beigen and Eppstein in [1]. We will apply this theorem in the proof of Theorem 1.

**Theorem 3.** [1] $n$-variable $(3,2)$-CSP instances can be solved in $O(1.3645^n)$ time.

## 3 Proof of Theorem [1]

To prove Theorem [1] we prove the following stronger theorem.

**Theorem 4.** Let $G$ be a graph and $v$ be a vertex in $G$ with the property that all vertices in $V(G) - (N[v] \cup N^2(v))$ have degree at least 8 in $G$, then we can determine in time $O(1.3158^n - 0.7d(v))$ if $G$ is 3-colorable or not.

**Proof.** We apply induction on $n - d(v)$ to prove the assertion. Since $G$ is simple, we have $d(v) \leq n - 1$. Therefore $n - d(v) \geq 1$.

When $n - d(v) = 1$, the graph $G$ has a vertex $v$ of degree $n-1$. In this case $G$ is 3-colorable if and only if $G - v$ is 2-colorable. Since 2-colorability can be determined in polynomial time
(for example using a simple Breadth First Search algorithm we can determine in linear time if the graph is bipartite), the assertion holds in this case.

Let us assume that for any \( n \)-vertex graph \( H \), with a vertex \( v \) of degree \( d(v) \), where \( n - d(v) \leq k \) and \( k \geq 1 \), we can determine if \( H \) is 3-colorable in \( O(1.3158^{n-0.7d(v)}) \) time, given all vertices in \( V(H) - (N[v] \cup N^2(v)) \) have degree at least 8 in \( H \).

We prove that the Theorem holds when the graph \( G \) is an \( n \)-vertex graph having a vertex \( v \) with \( n - d(v) = k + 1 \), where all vertices in \( V(G) - (N[v] \cup N^2[v]) \) have degree at least 8 in \( G \).

If there are three vertices \( u_1, u_2, u_3 \) in \( N(v) \) with \( u_1u_2, u_2u_3 \in E(G) \) (see Figure 1), then \( u_1u_3 \in E(G) \) implies that \( G \) is not 3-colorable, and \( u_1u_3 \notin E(G) \) implies that the vertices \( u_1 \) and \( u_3 \) must get the same colors in any proper 3-coloring of \( G \). As a result, we can identify \( u_1 \) and \( u_3 \) in \( G \) and study the smaller graph. Hence we may suppose that \( G[N(v)] \) has no vertex of degree at least 2.

![Figure 1: When \( G[N(v)] \) has a vertex \( u_2 \) of degree at least 2.](image1)

We consider three cases.

### 3.1 Case 1: When \( d(v) > 0.309n \).

In this case we transfer the problem into a \((3,2)\)-CSP problem with \( n - d(v) - 1 \) vertices. With no loss of generality we may suppose that in any coloring the color of \( v \) is 1. As a result, the vertices in \( N(v) \) must get colors in \( \{2, 3\} \). We create a \((3,2)\)-CSP on \( V(G) - N[v] \) in such a way that \( G \) is 3-colorable if and only if the \((3,2)\)-CSP problem has a solution.

Suppose \( N(v) = \{u_1, \ldots, u_r, w_1, \ldots, w_r, z_1, \ldots, z_t\} \), where \( u_1w_1, \ldots, u_rw_r \) are the only edges with both ends in \( N(v) \). This holds because \( G[N(v)] \) has no vertex of degree at least 2.

![Figure 2: Notation of Case 1.](image2)
If \( u_i \) and \( w_i \) for some integer \( i \), have a common neighbor \( y \) in \( N^2(v) \), then in any proper 3-coloring of \( G \) the vertices \( v \) and \( y \) must get the same color. As a result we can contract \( v \) and \( y \) in \( G \) and study the smaller graph. Hence we may suppose that \( u_i \) and \( w_i \) have no common neighbors in \( N^2(v) \).

Let \( H \) be a graph with \( V(H) = V(G) - N[v] \). We define a (3,2)-CSP on \( H \) as follows.

For vertices \( x, y \in V(H) \), if \( xy \in E(G) \), then we need to avoid patterns 1-1, 2-2, and 3-3 on \( x \) and \( y \), i.e. we need \((x,y) \neq (1,1), (2,2), (3,3)\). If \( x \) and \( y \) have a common neighbor in \( N(v) \) (in \( G \)), then we need to avoid patterns 2-3 and 3-2 on \( x \) and \( y \) (i.e. \((x,y) \neq (2,3), (3,2)\)), since otherwise we cannot extend the coloring on \( V(H) \) to a proper 3-coloring of \( G \). Finally, if \( xu_i, yw_i \in E(G) \), then we need to avoid patterns 2-2 and 3-3 on \( x \) and \( y \) (i.e. \((x,y) \neq (2,2), (3,3)\)), since otherwise we cannot extend the coloring on \( V(H) \) to a proper 3-coloring of \( G \).

By the above construction of the (3,2)-CSP on \( H \), the graph \( G \) is 3-colorable if and only if the (3,2)-CSP on \( H \) has a solution. Note that constructing \( H \) takes a polynomial time process and by Theorem 3 determining if the (3,2)-CSP instance on \( H \) has a solution or not has complexity \( O((1.3645)^{n-d(v)-1}) \). Since \( O((1.3645)^{n-d(v)}) \subseteq O(1.3157^{n-0.7d(v)}) \) for \( d(v) > 0.309n \). Therefore a polynomial factor of \( O(1.3157^{n-0.7d(v)}) \) is a subset of \( O(1.3158^{n-0.7d(v)}) \), as desired.

### 3.2 Case 2. When \( V(G) = N[v] \cup N^2(v) \) and \( d(v) \leq 0.309n \).

In this case with no loss of generality we may suppose that in any coloring the color of \( v \) is 1. As a result, the vertices in \( N(v) \) must get colors in \( \{2, 3\} \). Therefore there are at most \( 2^{d(v)} \) different possibilities for the colors of the vertices in \( N[v] \). Since \( V(G) = N[v] \cup N^2(v) \), all vertices in \( V(G) - N[v] \) have at least one neighbor in \( N[v] \).

Let \( c \) be a proper coloring over \( G[N[v]] \) using colors 2 and 3. As a result, to extend this coloring to a proper coloring of \( G \) each vertex in \( N^2(v) \) must avoid at least one color (the color(s) of its neighbor(s) in \( N[v] \)). Hence each vertex in \( N^2(v) \) has a list of size at most 2, such that \( c \) can be extended to a proper coloring of \( G \) if and only if there exists a proper list coloring on \( N^2(v) \). Note that we can determine in polynomial time if there exists a proper list coloring on the vertices of a graph, when each list has size at most 2 (see [3]).

Since there are at most \( 2^{d(v)} \) proper coloring on \( N(v) \) in which all vertices get colors in \( \{2, 3\} \), we can determine in a polynomial factor of \( 2^{d(v)} \) if \( G \) is 3-colorable or not. Since \( d(v) \leq 0.309n \), we have \( 2^{d(v)} \leq (1.31578)^{n-0.7d(v)} \). Hence \( 2^{d(v)} \subseteq O(1.31578)^{n-0.7d(v)} \), which implies \( poly(n)2^{d(v)} \subseteq O(1.3158)^{n-0.7d(v)} \), as desired.
3.3 Case 3. When $V(G) \neq N[v] \cup N^2(v)$ and $d(v) \leq 0.309n$.

Let $x$ be a vertex in $V(G) - (N[v] \cup N^2(v))$. In any proper 3-coloring of $G$, if it exists, the vertex $x$ either gets the same color as $v$ or $x$ receives a different color than $v$. Therefore it is enough to determine if any of the graphs $G/xv$ and $G \cup xv$ are 3-colorable. Recall that by our hypothesis $d(x) \geq 8$.

Let $H = G/xv$ and $H' = G \cup xv$. The graph $H$ has $n - 1$ vertices. Since $x$ has degree at least 8 in $G$ and since it has no common neighbor with $v$, we have $d_H(v) \geq d_G(v) + 8$. Similarly, we have $n(H') = n(G)$ and $d_{H'}(v) = d_G(v) + 1$. Therefore by the induction hypothesis, we can determine in $O(1.3158^{n-1-0.7(d(v)+8)})$ time if the graph $H$ is 3-colorable and we can determine in $O(1.3158^{n-0.7(d(v)+1)})$ time if the graph $H'$ is 3-colorable. Therefore to determine if $G$ is 3-colorable, we require an algorithm of complexity at most $O(1.3158^{n-0.7d(v)-6.6} + O(1.3158^{n-0.7d(v)-0.7})$.

Note that $1.3158^{n-0.7d(v)-6.6} + 1.3158^{n-0.7d(v)-0.7} < 1.3158^{n-0.7d(v)}$. Therefore the assertion holds.

4 Proof of Theorem 2

The proof of Theorem 2 is very similar to the proof of Theorem 1. To avoid redundancy we skip the parts of the proof that are similar. We prove the following stronger result.

Theorem 5. Let $G$ be a graph and $v$ be a vertex in $G$ with the property that all vertices in $V(G) - (N[v] \cup N^2(v))$ have degree at least 7 in $G$, then we can determine in $O(1.32^{n-0.73d(v)})$ time if $G$ is 3-colorable or not.

Proof. We apply induction on $n - d(v)$. When $n - d(v) = 1$, the graph $G$ has a vertex $v$ of degree $n - 1$. In this case $G$ is 3-colorable if and only if $G - v$ is 2-colorable (can be determined in polynomial time), the assertion holds in this case.

Assume that for any $n$-vertex graph $H$, with a vertex $v$ of degree $d(v)$, where $n - d(v) \leq k$ and $k \geq 1$, we can determine if $H$ is 3-colorable in $O(1.32^{n-0.73d(v)})$ time, given all vertices in $V(H) - (N[v] \cup N^2(v))$ have degree at least 7 in $H$.

We prove that the statement holds when an $n$-vertex graph $G$ has a vertex $v$ with $n - d(v) = k + 1$, where all vertices in $V(G) - (N[v] \cup N^2(v))$ have degree at least 7 in $G$.

Similar to the argument in the proof of Theorem 1 there are no three vertices $u_1, u_2, u_3$ in $N(v)$ with $u_1u_2, u_2u_3 \in E(G)$ (see Figure 1).

We consider the following three cases.
Case 1. When \( d(v) > 0.309n \).
Case 2. When \( V(G) = N[v] \cup N^2(v) \) and \( d(v) \leq 0.309n \).
Case 3. When \( V(G) \neq N[v] \cup N^2(v) \) and \( d(v) \leq 0.309n \).

The proof of Cases 1 and 2 is almost identical to that in the proof of Theorem 4 with the small difference that the base of the complexity \( 1.3158 \) must be replaced by \( 1.32 \) and 1.3157 and 1.31578 in Cases 1 and 2 must be replaced by 1.3199. Hence we move forward to the proof of Case 3, which is also similar to that in the proof of Theorem 4.

Let \( x \) be a vertex in \( V(G) - (N[v] \cup N^2(v)) \). Note that \( G \) is 3-colorable if and only if \( G/xv \) or \( G \cup xv \) is 3-colorable. Therefore it is enough to determine if any of the graphs \( G/xv \) and \( G \cup xv \) is 3-colorable. Recall that by our hypothesis \( d(x) \geq 7 \).

Let \( H = G/xv \) and \( H' = G \cup xv \). The graph \( H \) has \( n-1 \) vertices and \( d_H(v) \geq d_G(v) + 7 \). Similarly, we have \( n(H') = n(G) \) and \( d_{H'}(v) = d_G(v) + 1 \). Hence, by the hypothesis, we can determine in \( O(1.32^{n-1-0.73d(v)+7}) \) time if the graph \( H \) is 3-colorable, and we can determine in \( O(1.32^{n-0.73(d(v)+1)}) \) time if the graph \( H' \) is 3-colorable. All together, to determine if \( G \) is 3-colorable, the algorithm has a complexity of at most \( O(1.32^{n-0.73d(v)+6.11}) + O(1.32^{n-0.73d(v)-0.73}) \).

Since \( 1.32^{n-0.73d(v)-6.11} + 1.32^{n-0.73d(v)-0.73} < 1.32^{n-0.73d(v)} \), the assertion holds.

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