EHRHART POLYNOMIAL
AND MULTIPLICITY TUTTE POLYNOMIAL

MICHELE D’ADDERIO AND LUCA MOCI

ABSTRACT. We prove that the Ehrhart polynomial of a zonotope is a
specialization of the multiplicity Tutte polynomial introduced in [15].
We derive some formulae for the volume and the number of integer
points of the zonotope.

1. Introduction

Let $P$ be a convex $n$-dimensional polytope having all its vertices in $\Lambda = \mathbb{Z}^n$. For every $q \in \mathbb{N}$, let us denote by $qP$ the dilation of $P$ by a factor $q$.
A celebrated theorem of Ehrhart states that the number $E_P(q)$ of integer
points in $qP$ is a polynomial function in $q$, called the Ehrhart polynomial. A
special case is when $P$ is the zonotope $Z(X)$ generated by a list $X$ of vectors
in $\Lambda$. In [15] we associated to such a list a polynomial $M_X(x, y)$, called the
multiplicity Tutte polynomial. The definition of $M_X(x, y)$ is similar to the
one of the classical Tutte polynomial, but it also encodes some information
on the arithmetics of the list $X$.

In this paper we prove that the multiplicity Tutte polynomial specializes
to the Ehrhart polynomial of the zonotope:

$$E_{Z(X)}(q) = q^n M_X(1 + 1/q, 1).$$

The polynomials $M_X(x, y)$ and $E_P(q)$ have positive coefficients. In [5]
we studied $M_X(x, y)$ in the more general framework of arithmetic matroids,
and we provided an interpretation of its coefficients, which extends Crapo’s
formula [4], and it is inspired by the geometry of generalized toric arrange-
ments.

On the other hand, the meaning of the coefficients of the Ehrhart poly-
nomial is still quite mysterious. The only known facts are that the leading
coefficient is the volume of $P$, the coefficient of the monomial of degree $n - 1$
is half of the volume of the boundary of $P$, and the constant term is 1.
Significant effort has been devoted to the attempt of finding a combinatorial
or geometrical interpretation of the other coefficients. We hope that the
present work can be a step in this direction, by relating the two polynomials.

Furthermore, our results strengthen the connection of the polynomial
$M_X(x, y)$ with the partition function $P_X(\lambda)$. For every $\lambda \in \Lambda$, $P_X(\lambda)$ is

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defined as the number of solutions of the equation
\[ \lambda = \sum_{a \in X} x_a a, \]  
with \( x_a \in \mathbb{Z}_{\geq 0} \) for every \( a \).

(Since we want this number to be finite, the elements of \( X \) are assumed to lie on the same side of some hyperplane). The problem of computing \( P_X(\lambda) \) goes back to Euler, but it is still the object of an intensive research (see for instance [18], [3], [7], [8]; see also [1], [11], [12], [13] for related combinatorial work). This problem is easily seen to be equivalent to the problem of counting integer points in a variable polytope.

In order to study \( P_X(\lambda) \), in [6] Dahmen and Micchelli introduced a space of quasipolynomials \( DM(X) \). It is a theorem that the cone on which \( P_X(\lambda) \) is supported decomposes in regions called big cells, such that on every big cell \( \Omega \), \( P_X(\lambda) \) coincides with a quasipolynomial \( f_{\Omega}(\lambda) \in DM(X) \). In fact \( P_X(\lambda) \) and \( f_{\Omega}(\lambda) \) coincide on a slightly larger region, which is the Minkowsky sum of \( \Omega \) and \( -Z(X) \). In [7] this result is obtained by studying the algebraic Laplace transform of \( P_X(\lambda) \), which is a function defined on the complement of a toric arrangement (see [14], [16], [17]).

The polynomial \( M_X(x,y) \) is related with the partition function in at least three ways. First, many topological and combinatorial invariants of the toric arrangement are specializations of \( M_X(x,y) \). Second, \( DM(X) \) is a graded vector space, whose Hilbert series turns out to be \( M_X(1,y) \). Both these appear in [15]. The third connection is the subject of the present paper: the Ehrhart polynomial of \( Z(X) \) is a specialization of \( M_X(x,y) \). This yields some formulae for the number of integer points in the zonotope and in its interior, which are stated in Corollary 3.4.

2. Notations and recalls

2.1. Ehrhart polynomial. Let \( P \subset \mathbb{R}^n \) be a convex \( n \)-dimensional polytope having all its vertices in a lattice \( \Lambda \). For the sake of concreteness, the reader may assume \( \Lambda = \mathbb{Z}^n \). Let \( P_0 \) be the interior of \( P \). For every \( q \in \mathbb{N} \), let us denote by \( qP \) the dilation of \( P \) by a factor \( q \). Then also \( qP \) is a polytope with vertices in \( \Lambda \), and we denote its interior by \( qP_0 \). We define
\[ E_P(q) = |qP \cap \Lambda| \]
and
\[ I_P(q) = |qP_0 \cap \Lambda|. \]

A beautiful theorem of Ehrhart states that \( E_P(q) \) is a polynomial in \( q \) of degree \( n \) (see [9], [2]). Then \( E_P(q) \) is called the Ehrhart polynomial of \( P \). Also \( I_P(q) \) is a polynomial. Indeed we have
\[ I_P(q) = (-1)^n E_P(-q). \]

This important fact is known as Ehrhart-Macdonald reciprocity (see [2], Theor 4.1).

In this paper, we focus on the case when \( P \) is a zonotope. Namely, we take a finite list \( X \) of vectors in \( \Lambda \). We can assume that \( X \) spans \( \mathbb{R}^n \) as a
vector space. Then
\[ Z(X) = \left\{ \sum_{x \in X} t_x x, 0 \leq t_x \leq 1 \right\} \]
is a convex polytope with integer vertices, called the zonotope of \( X \).
Zonotopes play a crucial role in several areas of mathematics, such as hyperplane arrangements, box splines, and partition functions (see [7]).
To simplify the notation, we set \( E_X(q) = E_{Z(X)}(q) \) and \( I_X(q) = I_{Z(X)}(q) \).

2.2. Multiplicity Tutte polynomial. We take \( X \subseteq \Lambda = \mathbb{Z}^n \) as above.

For every \( A \subseteq X \), let \( r(A) \) be the rank of \( A \), i.e. the dimension of the spanned subspace of \( \mathbb{R}^n \).
The Tutte polynomial of \( X \), defined in [20], is
\[ T_X(x, y) = \sum_{A \subseteq X} (x - 1)^{|A| - r(A)} (y - 1)^{|A| - r(A)}. \]

Following [15], we denote by \( \langle A \rangle_{\mathbb{Z}} \) and \( \langle A \rangle_{\mathbb{R}} \) respectively the sublattice of \( \Lambda \) and the subspace of \( \mathbb{R}^n \) spanned by \( A \). Let us define
\[ \Lambda_A = \Lambda \cap \langle A \rangle_{\mathbb{R}}, \]
the largest sublattice of \( \Lambda \) in which \( \langle A \rangle_{\mathbb{Z}} \) has finite index. We define \( m \) as this index:
\[ m(A) = [\Lambda_A : \langle A \rangle_{\mathbb{Z}}]. \]
Notice that for every \( A \subseteq X \) of maximal rank, \( m(A) \) is equal to the greatest common divisor of the determinants of the bases extracted from \( A \).

We define the multiplicity Tutte polynomial of \( X \) as
\[ M_X(x, y) = \sum_{A \subseteq X} m(A)(x - 1)^{|A| - r(A)} (y - 1)^{|A| - r(A)}. \]

Remark 2.1. We say that the list \( X \) is unimodular if every basis \( B \) extracted from \( X \) spans \( \Lambda \) over \( \mathbb{Z} \) (i.e. \( B \) has determinant \( \pm 1 \)). In this case \( m(A) = 1 \) for every \( A \subseteq X \). Then \( M_X(x, y) = T_X(x, y) \).

3. Theorems
For every \( q \in \mathbb{N} \), let us consider the dilated list
\[ qX = \{ qx, x \in X \} \]
and its multiplicity Tutte polynomial \( M_{qX}(x, y) \). We have

Lemma 3.1.
\[ M_{qX}(x, y) = q^n M_X \left( \frac{x - 1}{q} + 1, y \right). \]

Proof. By definition
\[ M_{qX}(x, y) = \sum_{A \subseteq X} m(qA)(x - 1)^{|A| - r(A)} (y - 1)^{|A| - r(A)}. \]
Since clearly \( m(qA) = q^{r(A)} m(A) \), this sum equals
\[ q^n \sum_{A \subseteq X} m(A) \left( \frac{x - 1}{q} \right)^{n-r(A)} (y - 1)^{|A| - r(A)} = q^n M_X \left( \frac{x - 1}{q} + 1, y \right). \]
Now we prove the main result of this paper.

**Theorem 3.2.**

\[ E_X(q) = q^n M_X(1 + 1/q, 1). \]

We give two different proofs of this theorem.

**Proof.** (I). By [15, Prop. 4.5], the number of integer points in the zonotope \( Z(X) \) is equal to \( M_X(2, 1) \). Since clearly \( qZ(X) = Z(qX) \), by the above Lemma we have that

\[ E_X(q) = M_{qX}(2, 1) = q^n M_X(1 + 1/q, 1). \]

**Proof.** (II). Let \( I(X) \) be the family of all linearly independent sublists of \( X \). By [10, Prop. 2.1] (or [19, Ex. 31]), we have

(2) \[ E_X(q) = \sum_{A \in I(X)} m(A)q^{|A|}. \]

On the other hand we have by definition

\[ M_X(t + 1, 1) = \sum_{A \in I(X)} m(A)t^{n-|A|} \]

since for every \( A \in I(X) \), \( r(A) = |A| \).

Hence this polynomial is obtained from the Ehrhart polynomial by reversing its coefficients:

(3) \[ M_X(t + 1, 1) = t^n E_X(1/t). \]

By setting \( t = 1/q \) we get the claim.

Notice that the second proof does not rely on [15, Prop. 4.5], which we can now obtain as a corollary of the above Theorem.

By the Theorem above and Formula [4] we immediately get the following formula for the number of integer points in the interior of \( qZ(X) \).

**Corollary 3.3.**

\[ I_X(q) = (-q)^n M_X(1 - 1/q, 1). \]

We also get the following results. The first two were proved in [15], while the third is new.

**Corollary 3.4.**

1. The number \( |Z(X) \cap \Lambda| \) of integer points in the zonotope is equal to \( M_X(2, 1) \).
2. The volume \( \text{vol}(Z(X)) \) of the zonotope is equal to \( M_X(1, 1) \).
3. The number \( |Z(X)_0 \cap \Lambda| \) of integer points in the interior of the zonotope is equal to \( M_X(0, 1) \).

**Proof.** The first and the third statement are proved by evaluating at \( q = 1 \) the polynomials \( E_X(q) \) and \( I_X(q) \) respectively. As for the second claim, we recall that \( \text{vol}(Z(X)) \) is equal to the leading coefficient of \( E_X(q) \) (see [2, Cor. 3.20]). But by Formula (3) this is equal to the constant coefficient of \( M_X(1 + t, 1) \), that is \( M_X(1, 1) \).
4. An example

Consider the list in $\mathbb{Z}^2$

$$X = \{(3,0), (0,2), (1,1)\}.$$

Then

$$M_X(x,y) = (x-1)^2 + (3+2+1)(x-1) + (6+3+2) + (y-1) = x^2 + 4x + y + 5,$$

the Ehrhart polynomial is

$$E_X(q) = 11q^2 + 6q + 1$$

and

$$I_X(q) = 11q^2 - 6q + 1.$$

In the picture below we draw the zonotope $Z(X)$ and its dilation $2Z(X)$. The former contains 18 points, the latter 57. The interiors of these polytopes contain 6 and 33 points respectively.

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Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: mdadderio@yahoo.it

Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni, 136, Berlin, Germany
E-mail address: moci@math.tu-berlin.de