Bounds for discrete moments of Weyl sums and applications

by

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1. Introduction. The recent breakthroughs of Bourgain, Demeter and Guth [7] and Wooley [19, 20] have led to a full proof of the main conjecture in Vinogradov’s Mean Value Theorem (VMVT for short). As one consequence among many, new estimates for Weyl sums are available. With a standard approach, in this article we show that these already lead to strong estimates for moments of Weyl sums (see Theorem 9).

In this context we record an observation that for moments of Weyl sums, a small extra improvement can be made using Montgomery’s so-called alternative derivation [17, §4] incorporating VMVT (see Theorem 8). This additional gain can be exploited in a certain range for the approximating denominator (see (8)) assuming the summation range is long enough. We formulate a conjecture stating what this gain might lead to if further refinements were available (see Conjecture 10).

Then, in Section 4 the Weyl sum moment estimates are used to prove kth derivative tests for discrete moments of exponential sums with smooth functions (Theorems 12 and 13; the above mentioned extra improvement is incorporated in Theorem 12).

The bounds achieved for moments of Weyl sums and exponential sums with smooth functions lead to improvements in some number-theoretic applications, and we present two such applications.

The first one, discussed in Section 5, is the problem of counting integer points close to smooth curves. For this, we use a new approach involving exponential sums such that strong bounds for the counting quantity $\mathcal{R}(f, N, \delta)$ (see Definition 15) can be obtained. The novelty is to perform an efficient
Weyl shift step over a set of indices known to be $H'$-spaced so that the cluster structure of the indices is respected. This yields a saving of an extra factor $H^{-1}$ in the proof of Theorem 17 (see the arguments before (29)).

Compared to some existing bounds, the resulting bound in Theorem 17 is stronger, but is valid only for appropriate functions. This is discussed at the end of Section 5.

The second application, discussed in Section 6, concerns the polynomial large sieve inequality from [11, 12]. In the one-dimensional case we obtain a new improvement of the bound. That new bound comes from the extra improvement in Theorem 8.

1.1. Notations and conventions. Let $k$ denote a fixed positive integer and let $\varepsilon$ be an arbitrary small positive real number that may change its value during calculations. By $s, s_0, s_1 \geq 1$ we denote integers that depend on $k$. In this article, we suppress the dependence of the implicit constants on $k$, $s$ or $\varepsilon$ in our notation, simply writing $\ll$ for $\ll_{k,s,\varepsilon}$. Moreover, we write $f \ll g$ if $f(x) = o(g(x))$, that is, $f(x)/g(x) \to 0$ for $x \to \infty$.

For $\alpha \in \mathbb{R}$ we write $e(\alpha) := \exp(2\pi i \alpha)$ for the complex exponential function, and $||\alpha||$ denotes the distance from $\alpha$ to the nearest integer.

For integers $k, s \geq 1$ and a real number $x > 0$ we use the notation $J_k(x, s)$ for Vinogradov’s integral, that is, the number of solutions to Vinogradov’s system

\begin{align*}
    m_1 + \cdots + m_s &= n_1 + \cdots + n_s, \\
    m_1^2 + \cdots + m_s^2 &= n_1^2 + \cdots + n_s^2, \\
    \vdots \\
    m_1^k + \cdots + m_s^k &= n_1^k + \cdots + n_s^k
\end{align*}

with $1 \leq m_1, \ldots, m_s, n_1, \ldots, n_s \leq x$. In this work, although no use of the integral representation of $J_k(x, s)$ is made, the $\ell_2$-norm of the counting function $r_s(\lambda)$ is used in the proof of Theorem 8.

Given a positive integer $n$ we write $\tau(n)$ for the number of divisors of $n$, and $\tau_3(n)$ denotes the number of ways one can write $n$ as a product of three factors. We will use the well-known estimates $\tau(n) \ll n^{\varepsilon}$ and $\tau_3(n) \ll n^{\varepsilon}$.

The set of real functions with continuous derivatives of order up to $k$ on an interval $I$ is denoted by $C^k(I)$.

1.2. Auxiliaries. We collect some auxiliary results needed as tools in this article.

The following is the well-known sum lemma (see e.g. [18, Lemma 4C] for a proof).
Lemma 1 (Sum lemma). For \( \alpha \in \mathbb{R} \) let \( u, q \) be integers with \((u, q) = 1\), \(0 \leq u \leq q - 1\) and \(|\alpha - u/q| < q^{-2}\). Let \( \beta \in \mathbb{R} \). Then

\[
\sum_{Z \leq h \leq Y} \min(X, \|\alpha h + \beta\|^{-1}) \ll (X + q \log q)((Y - Z)q^{-1} + 1).
\]

For further improvements, we will also make use of the following result from [18, Lemma 9C].

Lemma 2 (Variant of sum lemma). For \( \alpha \in \mathbb{R} \) let \( u, q \) be integers with \((u, q) = 1\), \(0 \leq u \leq q - 1\) and \(|\alpha - u/q| < q^{-1}X^{-1}\). Let \( \beta \in \mathbb{R} \). Then

\[
\sum_{1 \leq j \leq q} \min(X, \|\alpha j + \beta\|^{-1}) \ll \min(X, q\|\beta q\|^{-1}) + q \log q.
\]

Next, we need the following simple bound for the number of curve points close to integer points. This is Lemma 2 in [14]; see also [6, Thm. 5.6], where a proof is provided.

Lemma 3 (Curve points close to integer points). Let \( N \) be a positive integer, and suppose that \( g(x) : [0, N] \to \mathbb{R} \) has a continuous derivative on \((0, N)\). Suppose further that \( 0 < \lambda \leq g'(x) \leq A\lambda \) for all \( x \in (0, N) \). Then

\[
\#\{n \leq N : \|g(n)\| \leq \delta\} \ll (1 + AN)(1 + \delta/\lambda).
\]

We also need the following simple assertion.

Lemma 4. Consider positive real functions \( S, f \) and \( A \). Assume that \( S(x) \ll f(x)S(x) + A(x) \). If \( f(x) \) tends to zero for \( x \to \infty \), then \( S(x) \ll A(x) \).

Proof. If \( C \) denotes the implicit constant, then \( S(x)(1-Cf(x)) \leq CA(x) \). With \( f(x) \leq C/2 \) for all large \( x \) we deduce \( S(x) \leq 2CA(x) \).

Another important ingredient is Vinogradov’s Mean Value Theorem. The theorem is elementary for \( k = 1 \) and \( k = 2 \). For the highly nontrivial cases \( k \geq 3 \) it has been proved by Bourgain, Demeter and Guth [7] for \( k \geq 4 \) and by Wooley [19] for \( k = 3 \), and again by Wooley [20] for \( k \geq 3 \). In our analysis, we will make use of this deep estimate.

Theorem 5 (VMVT). Let \( s, k \geq 1 \) be integers and \( \varepsilon > 0 \). Then \( J_k(x, s) \ll (x^s + x^{2s-k(k+1)/2})x^\varepsilon \).

2. Discrete moments of Weyl sums. It is known that VMVT has the following impact on Weyl sum estimates:

Theorem 6 (Weyl sum estimate). Let \( P \in \mathbb{R}[X] \) be a polynomial of degree \( k \geq 2 \), and for the leading coefficient \( \alpha_k \) of \( P \) let \( u, q \) be integers with
\((u, q) = 1, \ q \geq 1\) and \(|\alpha_k - u/q| < q^{-2}\). Then
\[
\sum_{n \leq x} e(P(n)) \ll x \left( \frac{1}{q} + \frac{1}{x} + \frac{q}{x^k} \right)^{1/(k(k-1))} x^\varepsilon.
\]

A proof can easily be found using Montgomery’s exposition \([17, \S 4]\). Our analysis of this proof yields a generalization of this estimate to discrete moments of Weyl sums. By changing a small aspect, it comes with an extra improvement, stated below as Theorem 8. Compared to this, Theorem 9 below is just a straight-forward generalization of Theorem 6 that stems from Montgomery’s original approach in \([17, \S 4]\).

We give the definition of the discrete moments we will look at.

**Definition 7.** Let \(k \geq 2\) be a fixed integer and consider a fixed polynomial \(P_\alpha \in \mathbb{R}[X]\) of degree \(k\) with \(P_\alpha(0) = 0\), say
\[
P_\alpha(X) = \alpha_k X^k + \alpha_{k-1} X^{k-1} + \cdots + \alpha_1 X
\]
with \(\alpha_1, \ldots, \alpha_k \in \mathbb{R}\). Let \(x > 1\) be a sufficiently large real number and for \(a \in \mathbb{N}\) let
\[
S_a(\alpha) := \sum_{m \leq x} e(a P_\alpha(m))
\]
be a corresponding Weyl sum of \(P_\alpha\). The twist with \(a\) allows us to consider discrete moments of the form
\[
(1) \sum_{a \leq T} |S_{az}(\alpha)|^{2s}
\]
with large real \(T > 1\) and with fixed \(z, s \in \mathbb{N}\).

The role of \(z\) is to control a possible dependence of a further factor in the argument of the exponential. We might think of a small \(z\), or even \(z = 1\).

Sums of the shape \((1)\) occur in numerous applications, like Dirichlet’s divisor problem, counting integer points close to curves, or, as we will see below, the polynomial large sieve inequality (for one variable polynomials). We will restrict ourselves to presenting just the latter two applications.

Our first goal is to give good estimates for the expression in \((1)\) depending on \(x\) and \(T\).

Note that bounds for other moments can then easily be derived by Hölder’s inequality:
\[
(2) \sum_{a \leq T} |S_{az}(\alpha)|^\ell \leq T^{1-\ell/(2s)} \left( \sum_{a \leq T} |S_{az}(\alpha)|^{2s} \right)^{\ell/2s}, \quad 0 < \ell < 2s.
\]
Since our results use different values of \(s\), it is convenient to state bounds for the first moment, which makes the statements easy to compare. Therefore, Theorems 8 and 9 below are stated for the first moment.
3. Improved moment estimate. We estimate Weyl sums along the lines of Montgomery’s so-called alternative derivation in [17 §4.4] and carry it over to discrete moments. This approach yields the following result for Weyl sums as given in Definition 7. We call it the improved moment estimate. The direct approach leading to Theorem 9 yields a bound that is weaker in certain ranges. This is discussed in Subsection 3.1.

**Theorem 8 (Improved moment estimate).** Let

\[ s_0 = (k - 1)(k - 2)/2 + 1 \]

for some \( k \geq 3 \), and let \( u, q \) be integers with \( q \geq 1 \), \((u, q) = 1\) and \(|\alpha_k - u/q| < q^{-2}\). Then

\[ \sum_{a \leq T} |S_{az}(\alpha)| \ll Tx \left( \frac{z x^{k-1}}{q} + \frac{z x^{k-1} \log q}{T} + \frac{1}{x} + \frac{q \log q}{Tx} \right)^{1/(2s_0)} x^\varepsilon. \]

In the bound, we ordered the factors on the right hand side: it starts with the trivial estimate \(Tx\), then we give the improvement factor and then a small additional factor \(x^\varepsilon\).

**Proof of Theorem 8.** We need to introduce some of the notations from [17 §4.4], but writing \(x\) instead of \(N\).

Thus for \( j \in \mathbb{N} \), the \( j \)th power sum of a tuple \( m = (m_1, \ldots, m_s) \in \mathbb{N}^s \) is written as \( s_j(m) := m_1^j + \cdots + m_s^j \), and the difference of two power sums as \( d_j = d_j(u, v) := s_j(u) - s_j(v) \), where \( u_i, v_i \in \{-x, \ldots, x\} \).

Multiplying \( S_{az}(\alpha)^s \) out, sorting the summands according to the value of the power sums with power \( j = 1, \ldots, k - 2 \), and applying Cauchy–Schwarz’s inequality, we get

\[ |S_{az}(\alpha)|^{2s} \leq s^{k-2} x^{(k-1)(k-2)/2} \mathcal{T}(a) \]

with

\[ \mathcal{T}(a) := \sum_{m,n} e(az P(m_1)) \cdots e(az P(m_s)) \]

\[ = \sum_{m,n} e((s_k(m) - s_k(n))az \alpha_k) \]

\[ = \sum_{u,v} e(d_k az \alpha_k + d_{k-1} az \alpha_{k-1}) \sum_{m \in I} e(kd_{k-1} maz \alpha_k) \]

(compare [17, eqs. (34)–(38) in §4.4]). Here, \( m \) runs through an interval \( I = I(u, v, x) \) that contains at most \( x \) successive integers, and we have put \( m = m_1, m_i = m + u_i \) for \( 2 \leq i \leq s \), and \( n_i = m + v_i \) for \( 1 \leq i \leq s \).
Note that the vector $u$ consists of one variable less: it has $s - 1$ components. In this step, all variables $m_i, n_i$ in the Vinogradov system $s_j(m) = s_j(n)$, $j = 1, \ldots, k - 2$, have been translated by $m = m_1$, thus we make use of the translation invariance of the Vinogradov system.

Now let $h = d_{k-1} \leq 2skx^{k-1}$ and sort the tuples $u, v$ by their value for $d_{k-1} = h$. Then the summation of $T(a)$ over $a \leq T$ yields

$$
\sum_{a \leq T} T(a) = \sum_{h \ll x^{k-1}} \sum_{d_{k-1} = h} \sum_{m \in I} \sum_{a \leq T} e(az(\alpha_k d_k + \alpha_k khm + h\alpha_{k-1})),
$$

where the last geometric sum can be estimated by

$$
\ll \min(T, \|\alpha_k zd_k + \alpha_k zkm + zh\alpha_{k-1}\|^{-1}).
$$

In the following, $\sum'$ denotes summation under the condition that $d_j = 0$ for $j = 1, \ldots, k - 2$. Hence

$$
\sum_{a \leq T} T(a) \ll \sum_{h \ll x^{k-1}} \sum_{d_{k-1} = h} \sum_{m \in I'} \min(T, \|\alpha_k zd_k + \alpha_k zkm + zh\alpha_{k-1}\|^{-1})
$$

where we extended the interval $I$ of length at most $x$ to an interval $I'$ of length at most $3x$, in order to remove the dependence on the variables $u, v$ except on $h$, which makes the separation of summation possible. We continue with

$$
\sum_{a \leq T} T(a) \ll \sum_{d \ll x^k} \sum_{h \ll x^{k-1}} \sum_{m \in I'} \min(T, \|\alpha_k zd + \alpha_k zkm + zh\alpha_{k-1}\|^{-1}) \cdot \sum_{d_{k-1} = h, d_k = d} 1;
$$
changing $hm$ to $w$ yields

$$
\sum_{a \leq T} T(a) \ll \sum_{d \ll x^k} \sum_{w \ll x^k} \sum_{h \ll w} \min(T, \|\alpha_k z(d + kw) + zh\alpha_{k-1}\|^{-1}) \cdot \sum_{d_{k-1} = h, d_k = d} 1
$$

(4)
\[
\sum_j \sum_d \sum_{h \ll x^{k-1}} \min(T, \|\alpha_k j + z h \alpha_{k-1}\|^{-1}) \sum_{d_k-1 = h, d_k = d} 1
\]

\[
\sum_{d \ll x^k} \sum_{h \ll x^{k-1}} \left( \sum_j \min(T, \|\alpha_k j + z h \alpha_{k-1}\|^{-1}) \right) \sum_{d_k-1 = h, d_k = d} 1,
\]

and an application of Lemma 7 to the sum in large brackets yields

\[
\sum_{a \leq T} T(a) \ll \sum_{d \ll x^k} \sum_{h \ll x^{k-1}} (T + q \log q) (z x^k / q + 1) \sum_{d_k-1 = h, d_k = d} 1,
\]

assuming that the integer \( q \geq 1 \) is such that there exists an integer \( u \) with \( (u, q) = 1 \) and \( |\alpha_k - u/q| < q^{-2} \).

Now we shall estimate the last sum. For \( \lambda \in \mathbb{Z}^{k-2} \) let

\[
r_{s-1}(\lambda) := \# \{ u : u_2 + \cdots + u_s = \lambda_1, \ldots, u_2^{k-2} + \cdots + u_s^{k-2} = \lambda_{k-2} \}
\]

and similarly

\[
r_s(\lambda) := \# \{ v : v_1 + \cdots + v_s = \lambda_1, \ldots, v_1^{k-2} + \cdots + v_s^{k-2} = \lambda_{k-2} \}.
\]

Then the Cauchy–Schwarz inequality yields

\[
\sum_{u,v} 1 = \sum_{j=0, j=1, \ldots, j=k-2} r_{s-1}(\lambda) r_s(\lambda) \leq \left( \sum_{\lambda} r_{s-1}(\lambda)^2 \right)^{1/2} \left( \sum_{\lambda} r_s(\lambda)^2 \right)^{1/2} = (J_{k-2}(x, s-1) J_{k-2}(x, s))^{1/2},
\]

and so we obtain

\[
\sum_{a \leq T} T(a) \ll (J_{k-2}(x, s-1) J_{k-2}(x, s))^{1/2} \left( \frac{T z x^k}{q} + z x^k \log q + T + q \log q \right)
\]

\[
= T x^k (J_{k-2}(x, s-1) J_{k-2}(x, s))^{1/2} \left( \frac{z}{q} + \frac{z \log q}{T} + \frac{1}{x^k} + \frac{q \log q}{T x^k} \right).
\]

For the desired moment of Weyl sums this yields

\[
\sum_{a \leq T} |S_{az}(\alpha)|^{2s} \ll T x^{(k-1)(k-2)/2+k} (J_{k-2}(x, s-1) J_{k-2}(x, s))^{1/2} \left( \frac{z}{q} + \frac{z \log q}{T} + \frac{1}{x^k} + \frac{q \log q}{T x^k} \right)
\]

\[
= T x^{2s} \left( \frac{J_{k-2}(x, s-1) J_{k-2}(x, s)}{x^{4s-(k-1)(k-2)-2k}} \right)^{1/2} \left( \frac{z}{q} + \frac{z \log q}{T} + \frac{1}{x^k} + \frac{q \log q}{T x^k} \right).
\]
Now that we have VMVT, Theorem 5, at hand, we can apply the best possible bound for the term in large brackets that includes the Vinogradov integrals. Choosing $s = s_0$ with $s_0 = \frac{(k - 1)(k - 2)}{2} + 1$, we have

$$J_{k-2}(x, s - 1)J_{k-2}(x, s) \frac{x^{4s-(k-1)(k-2)-2k}}{x^{2s-(k-1)(k-2)/2}} \ll x^{s-1}x^{2s-(k-1)(k-2)/2}x^{-4s+(k-1)(k-2)+2k}x^\varepsilon = x^{2k-2+\varepsilon}$$

for this value of $s$. (We have $\ll x^{2k-1+\varepsilon}$ when choosing $(k - 1)(k - 2)/2$ for $s$ instead, so the choice $s = s_0$ is optimal.) We arrive at

$$\sum_{a \leq T} |S_{az}(\alpha)|^2s_0 \ll Tx^{2s_0}x^{k-1} \left( \frac{z}{q} + \frac{z \log q}{T} + \frac{1}{x^k} + \frac{q \log q}{Tx^k} \right)x^\varepsilon.$$  

Using Hölder’s inequality [2], we deduce an estimate for the first moment. In this way, we obtain the asserted bound from (5).

Theorem 8 has to be compared with the result obtained by the straightforward approach:

**Theorem 9 (Standard approach estimate).** Let $k \geq 2$, $s_1 = k(k - 1)/2$ and $u, q$ be integers with $q \geq 1$, $(u, q) = 1$ and $|\alpha_k - u/q| < q^{-2}$. Then

$$\sum_{a \leq T} |S_{az}(\alpha)| \ll Tx \left( \frac{z}{q} + \frac{z}{x} + \frac{q}{Tx^k} \right)^{1/(2s_1)} (Txz)^\varepsilon.$$  

Note that with $T = 1$ and $z = 1$, we recover Theorem 6 as a special case.

**Proof of Theorem 9.** Montgomery’s original approach [17, §4.4, p. 81, l. 15] yields

$$\sum_{a \leq T} |S_{az}(\alpha)|^{2s} \ll x^{(k-1)(k-2)/2}x^{-1}J_{k-1}(3x, s) \sum_{a \leq T} \sum_{0 \leq h \leq 2s, x^{k-1}} \min(x, \|akhz\alpha_k\|^{-1}),$$

where the double sum can be estimated using Lemma 1. Together with the substitution $w = akhz$ this yields

$$\ll \sum_{w \leq 2sTz_{kx^{k-1}}} \tau_3(w) \min(x, \|w\alpha_k\|^{-1}) \ll \left( \frac{zTz^k}{q} + zTz^{-1} + q \right) (Txz)^\varepsilon,$$

where there exist integers $u, q$ with $q \geq 1$, $(u, q) = 1$ and $|\alpha_k - u/q| < q^{-2}$. We proceed with

$$\sum_{a \leq T} |S_{az}(\alpha)|^{2s} \ll x^{2sT}z_{k}J_{k-1}(3x, s) \frac{z}{q} + \frac{z}{x} + \frac{q}{Tx^k} (Txz)^\varepsilon.$$
Next, using VMVT (Theorem 5) with the optimal \( s = s_1 = k(k - 1)/2 \) leads to

\[
\sum_{0 < a < T} |S_{a}z(\alpha)|^{2s_1} \ll x^{2s_1} T \left( \frac{z}{q} + \frac{z}{x} + \frac{q}{Tx^k} \right) (Tz)^\epsilon.
\]

Applying Hölder’s inequality (2) to (6) yields the desired first moment as given in the assertion of the theorem.  

3.1. Comparison and conjectural considerations. The expressions in large brackets in Theorems 8 and 9 show the improvements compared to the trivial estimate \( Tx \). They lead to a nontrivial assertion if \( z x^{k-1} \ll q \ll Tx \) in Theorem 8 and if \( z \ll q \ll Tx^k \) in Theorem 9. Let \( s_0 = (k - 1)(k - 2)/2 + 1 \) and \( s_1 = k(k - 1)/2 \).

We compare these improved expressions (supposing \( z \) is small in this comparison) and obtain the following assertions:

(i) In these expressions, we compare the typical dominant terms, \( x^{-1/(2s_0)} \) (for \( z x^k \ll q \ll T \)) with \((z/x)^{1/(2s_1)}\) (for \( x \ll q \ll zTx^{k-1} \)), and we immediately see that Theorem 8 yields a sharper estimate in the intersection range \( z x^k \ll q \ll T \).

(ii) The dominant term \((z x^{k-1}/q)^{1/(2s_0)}\) for \( q \ll \min(z x^k, T) \) in Theorem 8 is sharper than \((z/x)^{1/(2s_1)}\) for \( x \ll q \ll T \) if \( q \gg z^\sigma x^{k-\sigma} \) with

\[
\sigma = \sigma_k := 1 - s_0/s_1 = 2/k - 2/(k(k - 1)).
\]

To summarize, with Theorem 8 we obtain an improvement in the range \( z^\sigma x^{k-\sigma} \ll q \ll \min(z x^k, T) \).

(iii) The dominant term \((z x^{k-1}/T)^{1/(2s_0)}\) for \( T \ll q \ll z x^k \) is sharper than \((z/x)^{1/(2s_1)}\) for \( x \ll q \ll zTx^{k-1} \) if \( T \gg z^\sigma x^{k-\sigma} \). Thus in the range \( T \ll q \ll z x^k \) we obtain an improvement.

(iv) The dominant term \((q/(Tx))^ {1/(2s_0)}\) for \( q \gg \max(T, z x^k) \) is sharper than \((z/x)^{1/(2s_1)}\) for \( x \ll q \ll zTx^{k-1} \) if \( q \ll Tx^\sigma z^{1-\sigma} \). Thus in the range \( \max(T, z x^k) \ll q \ll Tx^\sigma z^{1-\sigma} \), for which \( T \gg z^\sigma x^{k-\sigma} \) necessarily, we obtain an improvement.

To summarize, Theorem 8 yields an improvement only if \( T \gg z^\sigma x^{k-\sigma} \), so the term \( z^\sigma x^{k-\sigma} \) turns out to be a critical value for \( T \) from which on we obtain improvements. Moreover, the above conditions on \( q \) have to hold, that is,

\[
\sigma = \sigma_k := 1 - s_0/s_1 = 2/k - 2/(k(k - 1)).
\]

For any other \( q \), Theorem 9 gives a sharper bound.

An observation is that in (4) we made a very coarse estimate. Heuristically, one would expect that it could be doable with the mean value over \( h \). This would provide a gain of an extra factor \( x^{k-1} \) in the estimate. In this
way, we would save it also in (5) and arrive at the following conjectural bound.

**Conjecture 10.** For \( k \geq 3 \), \( s_0 = (k - 1)(k - 2)/2 + 1 \), and \(|\alpha_k - a/q| < q^{-2}\) for \((a, q) = 1\), we have

\[
\sum_{a \leq T} |S_{az}(\alpha)| \ll T x \left( \frac{z}{q} + \frac{z \log q}{T} + \frac{1}{x^k} + \frac{q \log q}{T x^k} \right)^{1/(2s_0)} (T x z)^\varepsilon.
\]

Compared to Theorem 8, this would lead to an improvement factor \( x^{-k/(2s_0)} \) (around \( x^{-1/k} \)) instead of \( x^{-1/(2s_0)} \), provided that the secondary terms do not matter. It is of interest that we can indeed move towards Conjecture 10 if we assume suitable rational approximations to \( \alpha_k \) and \( \alpha_{k-1} \) as follows.

**Theorem 11** (Second improved moment estimate). Let \( k \geq 2 \), \( s_2 = k(k - 1)/2 + 1 \) and \( u, q \) be integers with \( q \geq 1 \), \((u, q) = 1\) and \(|\alpha_k - u/q| < q^{-1} T^{-1}\). Further, let \( v, w \) be integers with \( 1 \leq w \leq x^{-1} z \) and \(|zq\alpha_{k-1} - v/w| < w^{-2}\). Then

\[
\sum_{a \leq T} |S_{az}(\alpha)| \ll T x \left( \frac{x^{k-1}}{qw} + \frac{x^{k-1}}{T} + \frac{1}{x w} + \frac{q}{T x} \right)^{1/(2s_2)} x^\varepsilon.
\]

**Proof.** We start as in the proof of Theorem 8 but continue (4) with

\[
(9) \quad \sum_{a \leq T} T(a) \ll \sum_{d \ll x^k} \sum_{h \ll x^{k-1}} \left( \sum_{j \ll x z^k} \min(T, ||\alpha_k j + h z \alpha_{k-1}||^{-1}) \right) \sum'_{u, v, \min(d_k - 1 = h, d_k = d)} 1
\]

\[
\ll \left( \sum_{h \ll x^{k-1}} \sum_{j \ll x z^k} \min(T, ||\alpha_k j + z h \alpha_{k-1}||^{-1}) \right) \max_{h_0} \sum'_{d_k - 1 = h_0} 1.
\]

To handle the last sum, letting

\[
\tilde{r}_s(\lambda, h_0) := \# \{ v : v_1 + \cdots + v_s = \lambda_1, \ldots, v_1^{k-2} + \cdots + v_s^{k-2} = \lambda_{k-2}, \]

\[
v_1^{k-2} + \cdots + v_s^{k-2} = \lambda_{k-2} + h_0 \},
\]

we obtain

\[
\sum'_{u, v} 1 = 1 \sum_{\lambda \in \mathbb{Z}^{k-1}} \tilde{r}_{s-1}(\lambda, 0) \tilde{r}_s(\lambda, h_0) \leq \left( \sum_{\lambda} \tilde{r}_{s-1}(\lambda, 0)^2 \right)^{1/2} \left( \sum_{\lambda} \tilde{r}_s(\lambda, h_0)^2 \right)^{1/2}
\]

\[
= (J_{k-1}(x, s - 1)J_{k-1}(x, s))^{1/2},
\]

uniformly in \( h_0 \). Now we turn to the sum over \( j, h \) in (9). For each block \( B = [1 + bq, \ldots, q - 1 + bq] \) of consecutive positive integers with \( b \geq 1 \), we
have
\[
\sum_{j \in B} \min(T, \| j \alpha_k + h z \alpha_{k-1} \|^{-1}) \ll \sum_{1 \leq j \leq q} \min(T, \| j \alpha_k + h z \alpha_{k-1} \|^{-1}),
\]
since \( \min(T, \| j \alpha_k + q \alpha_k + h z \alpha_{k-1} \|^{-1}) \ll \min(T, \| j \alpha_k + h z \alpha_{k-1} \|^{-1}) \) provided \( \| q \alpha_k \| < T^{-1} \). Therefore
\[
\sum_{h \leq x^{k-1}} \sum_{j \leq x^k} \min(T, \| \alpha_k j + h z \alpha_{k-1} \|^{-1}) 
\ll (z x^k / q + 1) \sum_{h \leq x^{k-1}} \sum_{1 \leq j \leq q} \min(T, \| \alpha_k j + h z \alpha_{k-1} \|^{-1})
\ll (z x^k / q + 1) \sum_{h \leq x^{k-1}} (\min(T, q \| h z q \alpha_{k-1} \|^{-1}) + q \log q),
\]
where we applied Lemma 2 in the last step. Writing \( m = h z \) and noting that the number of divisors of \( m \) is \( \ll x^\epsilon \), we continue by using Lemma 1
\[
\ll (z x^k / q + 1) x^\epsilon q (T / q + w \log w) (x^{k-1} z / w + 1) + z x^{2k-1+\epsilon} + x^{k-1+\epsilon} q
\ll x^\epsilon z^2 (x^k + q) (T / q + w) x^{k-1} / w + z x^{2k-1+\epsilon} + x^{k-1+\epsilon} q
\ll x^\epsilon z^2 T x^k \left( \frac{x^{k-1}}{qw} + \frac{x^{k-1}}{T} + \frac{1}{xw} + \frac{q}{Tx} \right),
\]
supposing \( w \leq x^{k-1} z \). This together with (9) tells us that
\[
\sum_{a \leq T} T(a) \ll x^\epsilon x^k T \left( \frac{x^{k-1}}{qw} + \frac{x^{k-1}}{T} + \frac{1}{xw} + \frac{q}{Tx} \right) (J_{k-1}(x, s - 1) J_{k-1}(x, s))^{1/2}.
\]
For the desired moment of Weyl sums this now yields
\[
\sum_{a \leq T} |S_{az}(\alpha)|^{2s} \ll T x^{(k-1)(k-2)/2 + k + \epsilon} (J_{k-1}(x, s - 1) J_{k-1}(x, s))^{1/2} \left( \frac{x^{k-1}}{qw} + \frac{x^{k-1}}{T} + \frac{1}{xw} + \frac{q}{Tx} \right)
\]
\[
= T x^{2s + \epsilon} \left( \frac{J_{k-1}(x, s - 1) J_{k-1}(x, s)}{x^{4s - (k-1)(k-2) - 2k}} \right)^{1/2} \left( \frac{x^{k-1}}{qw} + \frac{x^{k-1}}{T} + \frac{1}{xw} + \frac{q}{Tx} \right).
\]
Again, we need to choose the optimal parameter \( s \) which best fits with the Vinogradov integrals. This is provided by the choice \( s_2 = k(k-1)/2 + 1 \); an application of VMVT (Theorem 5) yields
\[
\frac{J_{k-1}(x, s - 1) J_{k-1}(x, s)}{x^{4s - (k-1)(k-2) - 2k}} \ll x^{s - 1} x^{2s - k(k-1)/2} x^{-4s + (k-1)(k-2) + 2k} x^\epsilon \ll x^\epsilon.
\]
Thus applying Hölder’s inequality, we arrive at the assertion
\[
\sum_{a \leq T} |S_{az}(\alpha)| \ll T x \left( \frac{x^{k-1}}{qw} + \frac{x^{k-1}}{T} + \frac{1}{xw} + \frac{q}{Tx} \right)^{1/(2s_2)} x^\epsilon.
\]
We see that in the setting of Theorem 11 we have improved the term $1/x$ to $1/(xw)$, where $w$ may be taken as large as $x^{k-1}$. This allows a saving of up to $x^{-1/k}$ in the estimate, namely

$$x^{-k/(2s_2)} = x^{-k/(k(k-1)+2)} = x^{-1/(k-1+2/k)} \ll x^{-1/k},$$

assuming best parameter choices for $q$ and $T$ (say $x^k \leq q$ and $T \geq qw$). Like this, we come close to Conjecture 10, but the assumptions on $\alpha_k$ and $\alpha_{k-1}$ are more restrictive.

Theorem 11 also suggests that there might be limitations to Conjecture 10, such as if $\alpha_{k-1}$ is close to 0 mod 1 or has good approximation to a rational $v/w$ with small denominator $w$. In cases like these it seems that we may not estimate the sum in (4) much better than the way we proceeded.

4. Discrete moments of exponential sums. We turn now to discrete moments of general exponential sums with smooth functions $f$. The main idea is to approximate $f$ with a polynomial using Taylor’s theorem and apply the bounds of the previous sections.

We proceed much as in Bordellès’ book [6, §6.6.7], or in Heath-Brown’s recent article [14]. The first result is as follows.

**Theorem 12.** Let $N$ be a large positive integer, and let $f \in C^k((0,3N))$, $k \geq 3$. Suppose that there exist real numbers $\lambda,A$ such that $0 < \lambda \leq f^{(k)}(x) \leq A\lambda$ for all $x \in (0,3N)$. Let $\rho = 1/((k-2)(k-3)+2)$ and $\mu = 1+A\lambda N$. Let $z$ be a positive integer, considered to be small, and let $T$ be a real number with $N^{-k}(zA\lambda)^{-1} \leq T \leq (zA\lambda)^{-1}$. Then

$$\sum_{a \leq T} \sum_{N < m < 2N} e(azf(m)) \ll NT(zA\lambda T)^{\rho/k+\varepsilon} + T(zA\lambda T)^{-1/k}$$

$$+ T\mu z^2(zA\lambda T)^{2/k-2} + \mu z(zA\lambda T)^{1/k-1/2} + \lambda^{-1}.$$  

We note that $\lambda$ as well as $A$ and $z$ may depend on $N$ and $T$. If $A$ and $z$ depend on $k$ only, we may hide $A$ and $z$ in the implicit constant, which leads to a slightly easier expression. If additionally $\mu = 1$, the upper bound simplifies to

$$NT(\lambda T)^{\rho/k} + T(\lambda T)^{-1/k} + T(\lambda T)^{2/k-2} + (\lambda T)^{1/k-1/2} + \lambda^{-1}.$$  

The proof uses an adapted circle method. The first term in the bound (10) respectively (11) comes from the minor arc contribution, the second gives a trivial contribution from a Weyl shift, and the last two terms come from the major arc contribution.

**Proof of Theorem 12.** Let $\mathcal{L}_f$ denote the left hand side of (10). We start with a Weyl shift with $1 \leq H \leq N$. Let $\beta_m = e(azf(m))$ if $N < m < 2N$, 

We obtain
\[
\sum_{N < m < 2N} e(azf(m)) = \sum_{m \in \mathbb{Z}} \beta_{m+h'} = \frac{1}{H} \sum_{h \leq H} \sum_{m \in \mathbb{Z}} \beta_{m+h}
\]
\[
= \frac{1}{H} \sum_{m=N-H+1}^{2N-1} \sum_{h \leq H} \beta_{m+h} = \frac{1}{H} \sum_{m=N+1}^{2N-1} \sum_{h \leq H} e(azf(m+h)) + O(H).
\]

We obtain
\[
(12) \quad \mathcal{L}_f = \frac{1}{H} \sum_{a \leq T} \sum_{N < m < 2N} \left| \sum_{h \leq H} e(azf(m+h)) \right| + O(TH).
\]

Taylor’s theorem yields \( f(m + h) = Q_m(h) + u_m(h) \) with
\[
Q_m(h) = hf^1(m) + h^2 f''(m)/2! + \ldots + h^{k-1} f^{(k-1)}(m)/(k-1)!.\]
Note that \( f^{(k-1)}(m)/(k-1)! \) is the leading coefficient of this polynomial of degree \( k - 1 \) in \( h \), and
\[
u_m(h) = f(m) + \frac{1}{(k-1)!} \int_0^h (h - v)^{k-1} f^{(k)}(m + v) \, dv,
\]
so that \( e(azf(m+h)) = e(azQ_m(h))e(azu_m(h)) \).

We separate the exponential expressions containing \( Q_m \) and \( u_m \) by partial summation; this yields
\[
\mathcal{L}_f \leq S_1 + S_2 + O(TH)
\]
with
\[
S_1 \leq \frac{1}{H} \sum_{N < m < 2N} \sum_{a \leq T} \left| \sum_{h \leq H} e(azQ_m(h)) \right|
\]
and
\[
S_2 \leq \sum_{a \leq T} \frac{2\pi az}{H} \sum_{N < m < 2N} \int_0^H \left| \sum_{h \leq x} e(azQ_m(h)) \right| \cdot |u'_m(x)| \, dx
\]
\[
\ll zH^{k-2} \sum_{N < m < 2N} \int_0^H \sum_{a \leq T} \left| \sum_{h \leq x} e(azQ_m(h)) \right| \, dx \cdot \sup_{v \in (0,H)} |f^{(k)}(m + v)|
\]
\[
\ll zAH^{k-2} \int_0^H \sum_{N < m < 2N} \sum_{a \leq T} \left| \sum_{h \leq x} e(azQ_m(h)) \right| \, dx.
\]

Next, we abbreviate
\[
S_{a,m}(x) := \sum_{h \leq x} e(azQ_m(h)),
\]
sum the bounds for $S_1$ and $S_2$ and arrive at

$$
\mathcal{L}_f \ll TH + \frac{1}{H} \sum_{N<m<2N} \sum_{a \leq T} |S_{a,m}(H)|
$$

$$
+ zA\lambda H^{k-2}T \int_1^H \sum_{N<m<2N} \sum_{a \leq T} |S_{a,m}(x)| \, dx.
$$

For the next argument, fix $x$ with $x \leq H \leq N$ and let $\Delta_0 := z^{-1}T^{-1}H^{1-k}$. Consider $m \in (N, 2N) \cap \mathbb{Z}$ and let

$$
\mathcal{A}_m := \left\{ \alpha \in [0, 1] : \left\| \frac{f^{(k-1)}(m)}{(k-1)!} - \alpha \right\| \leq \Delta_0 \right\}.
$$

Fix an $\alpha \in \mathcal{A}_m$. We replace the leading coefficient in $Q_m(h)$ by $b_{k-1} \in \mathbb{R}$ such that $f^{(k-1)}(m)/(k-1)! - b_{k-1} \in \mathbb{Z}$, so that $|b_{k-1} - \alpha| \leq \Delta_0$. Like this, we look at

$$
f^*_m(h) := hf'(m) + \cdots + h^{k-2}f^{(k-2)}(m)/(k-2)! + b_{k-1}h^{k-1}.
$$

Let $S^*_{a,m}(x) := \sum_{h \leq x} e(azf^*_m(h))$, so that $|S^*_{a,m}(x)| = |S_{a,m}(x)|$ and we are able to work with $S^*_{a,m}(x)$ instead of $S_{a,m}(x)$ in (13). Moreover, let

$$
\tilde{f}_{m,\alpha}(h) := hf'(m) + \cdots + h^{k-2}f^{(k-2)}(m)/(k-2)! + \alpha h^{k-1}
$$

and

$$
\tilde{S}_{a,m}(\alpha, x) := \sum_{h \leq x} e(az\tilde{f}_{m,\alpha}(h)).
$$

Then

$$
\frac{d}{dx}(f^*_m(x) - \tilde{f}_{m,\alpha}(x)) \ll |b_{k-1} - \alpha| x^{k-2} \ll \Delta_0 x^{k-2},
$$

and we conclude by partial summation that

$$
S^*_m(x) \ll |\tilde{S}_{a,m}(\alpha, x)| + az \int_1^x y^{k-2} \Delta_0 |\tilde{S}_{a,m}(\alpha, y)| \, dy.
$$

Our task is reduced to proving good upper bounds for

$$
\mathcal{T}_x := \sum_{N<m<2N} \sum_{a \leq T} |\tilde{S}_{a,m}(\alpha, x)|
$$

with $x \leq H$. For each $m$ in the sum there is an $\alpha \in \mathcal{A}_m$ chosen. We intend to apply Theorem 8. We expect a good result if we assume $T$ to be much greater than $zx^{k-2}$. (Note that $\deg \tilde{S}_{a,m}(\alpha, x) = k - 1$.)

For this purpose, we introduce appropriate major and minor arcs. Let

$$
\mathcal{M} = \bigcup_{q \leq zx^{k-1}} \bigcup_{(u,q)=1} \left[ \frac{u}{q} - \frac{1}{qT}, \frac{u}{q} + \frac{1}{qT} \right].
$$
denote the set of major arcs, and \( m = [0, 1] \setminus M \). Now we distinguish two cases.

Say case \((m)\) occurs if \( m \) is such that there exists a real number \( \alpha \) in \( A_m \cap m \). We then choose such an \( \alpha \) for each such \( m \). By Dirichlet’s approximation theorem, there exist coprime integers \( u \) and \( q \) with \( 1 \leq q \leq T \) such that
\[
\left| \alpha - \frac{u}{q} \right| \leq \frac{1}{qT}.
\]
Since \( \alpha \) is contained in \( m \), we conclude that even \( q \geq x^{k-1} \). A closer look at the improved expression in Theorem 8 yields
\[
\left( \frac{zx^{k-2}}{q} + \frac{zx^{k-2}}{T} + \frac{1}{x} + \frac{q}{Tx} \right)^{\rho} \ll x^{-\rho}
\]
with \( \rho = 1/((k-2)(k-3) + 2) \). Therefore by that theorem,
\[
\sum_{a \leq T} |\tilde{S}_{a,m}(\alpha, x)| \ll T x^{1-\rho + \varepsilon},
\]
hence, summing over these \( m \), we get
\[
(15) \quad T_{x,(m)} := \sum_{N < m < 2N}^{(m)} \sum_{a \leq T} |\tilde{S}_{a,m}(\alpha, x)| \ll NT x^{1-\rho + \varepsilon}.
\]

In the major arc case \((M)\), \( A_m \) is contained completely in a major arc interval. Then we conclude for \( m \) with \( N < m < 2N \) that there exist \( q \leq zx^{k-1} \) and \( (u, q) = 1 \) such that \( \|f^{(k-1)}(m)/(k-1)! - u/q\| < 1/(qT) \). Summing over those \( m \), we obtain
\[
T_{x,(M)} := \sum_{N < m < 2N}^{(MN)} \sum_{a \leq T} |\tilde{S}_{a,m}(\alpha, x)| \ll T x \sum_{q \leq zx^{k-1}} \sum_{(u, q) = 1} \#\{m \in (N, 2N) : \|f^{(k-1)}(m)/(k-1)! - u/q\| < 1/(qT)\}
\]
\[
\ll T x \sum_{q \leq zx^{k-1}} \varphi(q)(1 + A\lambda N)(1 + 1/\lambda qT),
\]
where in the last step we have applied Lemma 3 with \( g(x) := f^{(k-1)}(x)/(k-1)! - u/q \). From now on we use the abbreviation \( \mu = 1 + A\lambda N \). This yields
\[
T_{x,(M)} \ll Tx(\mu z^2 x^{2k-2} + \mu zx^{k-1} \lambda^{-1} T^{-1}),
\]
hence, in case \( M \),
\[
(16) \quad T_{x,(M)} \ll T \mu z^2 x^{2k-1} + \mu zx^k \lambda^{-1}.
\]
Then combining (15) and (16) yields
\[
(17) \quad T_x \ll T_{x,(m)} + T_{x,(M)} \ll NT x^{1-\rho + \varepsilon} + T \mu z^2 x^{2k-1} + \mu zx^k \lambda^{-1}.
\]
Next, from (14) together with (17) we obtain

\[
\sum_{N<m<2N} \sum_{a \leq T} |S^*_a,m(x)| \ll T x \Delta_0 \int_1^x y^{k-2} \mathcal{T}_y \, dy
\]

\[
\ll NT x^{1-\rho+\varepsilon} + T z \Delta_0 \int_1^x y^{k-2} NT y^{1-\rho+\varepsilon} \, dy
\]

\[
+ T \mu z^2 x^{2k-1} + T z \Delta_0 \int_1^x y^{k-2} T \mu z^2 y^{2k-1} \, dy
\]

\[
+ \mu z x^k \lambda^{-1} + T z \Delta_0 \int_1^x y^{k-2} \mu z y^k \lambda^{-1} \, dy
\]

\[
\ll NT x^{1-\rho+\varepsilon} + T \mu z^2 x^{2k-1} + \mu z x^k \lambda^{-1},
\]

where in the last step we have used \( \Delta_0 = z^{-1} T^{-1} H^{1-k} \) and \( x \leq H \).

Therefore, by (13),

\[
\mathcal{L}_f \ll TH + H^{-1} \sum_m \sum_{a \leq T} |S^*_a,m(H)|
\]

\[
+ z A \lambda H^{k-2} T \int_1^H \sum_m \sum_{a \leq T} |S^*_a,m(x)| \, dx
\]

\[
\ll TH + H^{-1} N Th^{1-\rho+\varepsilon} + z A \lambda H^{k-2} T \int_1^H NT x^{1-\rho+\varepsilon} \, dx
\]

\[
+ H^{-1} T \mu z^2 H^{2k-1} + z A \lambda H^{k-2} T \int_1^H T \mu z^2 x^{2k-1} \, dx
\]

\[
+ H^{-1} \mu z H^k \lambda^{-1} + z A \lambda H^{k-2} T \int_1^H \mu z x^{k-1} \lambda^{-1} \, dx
\]

\[
\ll N T h^{-\rho+\varepsilon} + TH + T \mu z^2 H^{2k-2} + \mu z H^{k-1} \lambda^{-1},
\]

where we have chosen \( H = [(z A \lambda T)^{-1/k}] \) in (18). This gives

\[
\mathcal{L}_f \ll NT (z A \lambda T)^{\rho/k+\varepsilon} + T (z A \lambda T)^{-1/k}
\]

\[
+ T \mu z^2 (z A \lambda T)^{2/k-2} + \mu z (z A \lambda T)^{1/k-1} \lambda^{-1}.
\]

As a necessary constraint for \( T \) we get \( N^{-k} \leq z A \lambda T \leq 1 \), since we need \( 1 \leq H = [(z A \lambda T)^{-1/k}] \leq N \). 

**Remark.** We have to discuss in which range of \( T \) Theorem 12 provides a nontrivial upper bound for \( \mathcal{L}_F \). The first two terms of the bound (10) clearly give a nontrivial upper bound \( \ll TN \), and also the third term is \( \ll TN \).
provided that $T \mu z^2 H^{2k-2} \ll TN$, which means
\begin{equation}
T \gg \mu^{k/(2k-2)} z^{1/(k-1)} (A\lambda)^{-1} N^{k/(2-2k)}.
\end{equation}

Moreover, the fourth term is $\ll TN$ provided that
\begin{equation}
T \gg \mu z^{1/(2k-2)} A^{1/(2k-1)} \lambda^{-1} N^{k/(2-2k)}.
\end{equation}
The latter means $T \gg \mu \frac{z}{k/(2k-2)}$, which is stronger than just $T \gg zH^{k-1}$ which was expected in the proof to lead to nontrivial results.

A short calculation shows that the lower bounds (19) and (20) for $T$ are admissible with the constraint $T \leq (zA\lambda)^{-1}$ provided that $z^2 \mu \ll N$ and $z^2 \mu A \ll N^k$. We conclude that, for small $z$, there exists a range for $T$ where a nontrivial bound is achieved.

The lower bounds (19) and (20) for $T$ are quite restrictive, but give the advantage of Theorem 8 over Theorem 9. Using Theorem 9 instead will lead to the slightly weaker bound (21) below since $\tau < \rho$, but provides a larger range for $T$.

**Theorem 13.** Let $N$ be a large positive integer and let $f \in C^k((0,3N))$, $k \geq 3$. Suppose that there exist real numbers $\lambda, A$ such that $0 \leq \lambda \leq f^{(k)}(x) \leq A\lambda$ for all $x \in (0,3N)$. Let $\tau = 1/((k-1)(k-2))$ and $\mu = 1 + A\lambda N$. Let $z$ be a positive integer, considered to be small, and let $T$ be a positive real number with $N^{-k}(zA\lambda)^{-1} \leq T \leq (zA\lambda)^{-1}$. Then
\begin{equation}
\sum_{a \leq T} \left| \sum_{N < m < 2N} e(az f(m)) \right| \ll NT(zA\lambda T)^{\tau/k+\varepsilon} + T(zA\lambda T)^{-1/k} + A\mu T(zA\lambda T)^{-2/k}.
\end{equation}

**Proof.** We proceed as in Theorem 12, but now choose the major arc set to be
\[ \mathcal{M} = \bigcup_{q \leq x} \bigcup_{(u,q) = 1} \left[ \frac{u}{q} - \frac{1}{qT z x^{k-1}}, \frac{u}{q} + \frac{1}{qT z x^{k-1}} \right]. \]
In the minor arc case, we consider $m$ with $x \leq q \leq zx^{k-1} T$ and we are in a position to use Theorem 9 instead, which leads to the slightly weaker estimate
\[ T_{x,(m)} \ll NT x^{1-\tau+\varepsilon}, \]
since $\tau < \rho$, where $\tau = 1/((k-1)(k-2))$. Like this, we estimate the major arc contribution in a better way, namely
\[ T_{x,(\mathcal{M})} := \sum_{N-H < m < 2N} \sum_{a \leq T} |\hat{S}_{a,m}(\alpha, x)| \ll Tx \sum_{q \leq x} \sum_{(u,q) = 1} \# \{ m \in (N-H, 2N) : \| f^{(k-1)}(m)/(k-1)! - u/q \| < 1/qT z x^{k-1} \} \]
\[ \ll Tx \sum_{q \leq x} \phi(q)(1 + A\lambda N)(1 + 1/\lambda q Tzx^{k-1}) \]
\[ \ll Tx(\mu x^2 + \mu x/\lambda Tzx^{k-1}) \ll \mu Tx^3 + \mu(\lambda z)^{-1}x^{3-k}, \]
with \( \mu = 1 + A\lambda N \), again by using Lemma 3. We similarly arrive at
\[ L_f \ll NTH^{-\tau+s} + TH + \mu TH^2 + \mu(\lambda z)^{-1}H^{2-k} \]
if we choose \( H = [(zA\lambda T)^{-1/k}] \) again. Since
\[ \max\{1, (T\lambda z)^{-1}H^{-k}\} = \max\{1, A\} = A, \]
the last two terms are \( \ll A\mu TH^2 \). Again noting that \( 1 \leq H \leq N \) provides the assertion of Theorem 13. 

**Remark.** Again, we give the range for \( T \) where Theorem 13 provides a nontrivial bound for \( L_f \).

We need to inspect the third term in this bound; it is \( \ll TN \) provided that \( A\mu TH^2 \ll TN \), which means
\[ T \gg \mu^{k/2}A^{k/2-1}(z\lambda)^{-1}N^{-k/2}. \]
A short calculation shows that this lower bound for \( T \) is admissible with the constraint \( T \leq (zA\lambda)^{-1} \) provided that \( \mu A \ll N \).

Compared to (19) and (20), the range for \( T \) due to (22) will be much wider in most cases.

We compare our theorems with the direct application of the following recent result of Heath-Brown [14, Thm. 1].

**Theorem 14 (Heath-Brown).** Let \( k \geq 3 \), let \( f : [0, N] \to \mathbb{R} \) be in \( C^k((0, N)) \), and suppose that \( 0 < \lambda \leq f^{(k)}(x) \leq A\lambda \) for all \( x \in (0, N) \). Then
\[ \sum_{n \leq N} e(f(n)) \ll_{A, k, \varepsilon} N^{1+\varepsilon}\left(\lambda^{1/(k(k-1))} + N^{-1/(k(k-1))} + N^{-2/(k(k-1))}\right). \]

In principle, \( L_f \) can be estimated by using Theorem 14, but one then needs the explicit dependence of the implicit constant on \( A \) since the term \( az \) occurs in the argument of the complex exponential function, so that \( A \) in Theorem 14 contains the factor \( az \leq zT \).

Writing down the dependence on \( A \) from the proof in [14] Thm. 1] explicitly, we will have a factor \( A^4 \) occurring in the quantity \( \mathcal{N} \) there. The resulting bound for \( L_f \) will then contain the factor \( A^{4/(2s)} = A^{4/(k(k-1))} \). Thus the main term from this method will provide the extra factor \( T^{4/(k(k-1))} \), which is much larger than \( T^{\rho/k} \) or \( T^{\tau/k} \) from our theorems.
So compared to this, Theorems 12 and 13 give sharper estimates for long Weyl sum averages. When no or short averages are considered, Heath-Brown’s bound is sharper.

Note that the potential improvements depend also on the type of functions considered. For example, if \( f(n) = t \log n \) and \( k \) is large with \( t = N^{1/2} \), then the minor arc contribution from Theorem 12 is around \( tN^{1-(k-1)(k-2)/k^2} \), whereas Heath-Brown’s bound is around \( tN^{1-1/k^2} \). It may be that Theorem 12, as it stands, cannot be improved if uniformity for all functions is desired, but further improvements may be feasible if a suitable type of function is assumed.

A careful combination of the methods from Theorems 8 and 12 together with the idea from the proof of Theorem 11 could lead to further improvements, but this is not discussed in this article.

5. First application: Integer points close to smooth curves.

In this application, we will use Theorem 13. We introduce the following quantity.

**Definition 15.** Let \( N \) be a large positive integer, let \( f \in C^k((0,3N)) \), \( k \geq 3 \) and \( 0 < \delta < 1/4 \). Define

\[
\mathcal{R}(f, N, \delta) := \# \{ n \in [N,2N] \cap \mathbb{Z} : \|f(n)\| < \delta \}.
\]

Thus, we count lattice points in \( \mathbb{Z}^2 \) close to the graph of \( f \). Bordellès [6, Ch. 5] gives an overview of several known nontrivial bounds for this quantity and their applications. In Lemma 3 we already gave a bound for \( \mathcal{R}(f, N, \delta) \) in the case \( k = 1 \); it is also known as the first derivative test for \( \mathcal{R}(f, N, \delta) \).

In what follows, we use a property of the set counted by \( \mathcal{R}(f, N, \delta) \). It is proved in [6, Thm. 5.11].

**Lemma 16.** Suppose that there exist \( \lambda, A > 0 \) such that \( \lambda \leq f^{(k)}(x) \leq A \lambda \leq 1/4 \) for all \( x \in (0,3N) \). In the set \( \{ n \in [N,2N] \cap \mathbb{Z} : \|f(n)\| < \delta \} \) there exists an \( H' \)-spaced subset \( \mathcal{R} \) such that \( \mathcal{R}(f, N, \delta) \leq (k+1)(1 + \#\mathcal{R}) \), where \( H' = (A \lambda)^{-2/(k(k+1))} \).

A set is called \( H' \)-spaced if any two elements differ by more than \( H' > 0 \).

**Theorem 17.** Suppose that there exist \( \lambda, A \geq 0 \) such that \( \lambda \leq f^{(k)}(x) \leq A \lambda \leq c_0 \) for all \( x \in (0,3N) \) and some small constant \( c_0 < 1/4 \), and assume \( (A \lambda)^{-2/(k(k+1))} \leq N \). Let \( \lambda_1 > 0 \) be such that \( |f'(x)| \leq \lambda_1 \) for all \( x \in (0,3N) \). Assume that \( \delta > 0 \) satisfies

\[
(23) \quad A \lambda \ll \delta + \lambda_1 \ll (A \lambda)^{1-2(k-1)/(k(k+1))}.
\]

Then

\[
(24) \quad \mathcal{R}(f, N, \delta) \ll 1 + A(1 + A \lambda N)((\delta + \lambda_1)/A \lambda)^{1/(k-1)}.
\]
Proof. We begin the proof as indicated in [6, Ex. 6.7.4]. Let \( m \in \mathbb{Z} \) with \( N \leq m \leq 2N \) and \( \|f(m)\| < \delta \). Then
\[
\left| \sum_{a=0}^{T-1} e(af(m)) \right|^2 = \sum_{a_1,a_2} e((a_1-a_2)f(m)) = \sum_{a_1,a_2} \Re(e((a_1-a_2)f(m))) \geq T^2/2,
\]
since \( \Re(e(af(m))) \geq \sqrt{2}/2 \) for all \( a \in \mathbb{Z} \) with \( |a| < T \), provided that \( 1 \leq T \leq [1/8\delta] + 1 \).

From Lemma 16 we know that there is an \( H' \)-spaced subset \( \mathcal{R} \) of \( \{ m : N \leq m \leq 2N, \|f(m)\| < \delta \} \) with \( \mathcal{R}(f,N,\delta) \leq (k+1)(1 + \#\mathcal{R}) \), where \( H' = (A\lambda)^{-2/(k(k+1))} \). Hence
\[
\mathcal{R}(f,N,\delta) \ll 1 + \sum_{m \in \mathcal{R}} 1 \leq 1 + \frac{2}{T^2} \sum_{m \in \mathcal{R}} \left| \sum_{a=0}^{T-1} e(af(m)) \right|^2.
\]
Now opening the square and separating the summand for \( a = 0 \) shows
\[
\left| \sum_{a=0}^{T-1} e(af(m)) \right|^2 = \sum_{|a| \leq T-1} (T - |a|)e(af(m)) = T + \sum_{0 < |a| \leq T-1} (T - |a|)e(af(m)) = T + a_m,
\]
say. Clearly \( a_m > 0 \) for large \( T \) and for \( m \in \mathcal{R} \) due to (25). We proceed with
\[
\mathcal{R}(f,N,\delta) \ll 1 + \mathcal{R}(f,N,\delta)T^{-1} + T^{-2} \sum_{m \in \mathcal{R}} a_m
\]
\[
\ll 1 + T^{-2} \sum_{m \in \mathcal{R}} a_m,
\]
by an application of Lemma 4 if we take \( T \gg 1 \) (in fact \( T \geq 4(k+1) \) suffices).

Let \( m \in \mathcal{R} \). Then there exists an integer \( n \) with \( |f(m) - n| < \delta \). Thus, by the mean value theorem, \( |f(m-h) - f(m)| = |f'(t)||h| \leq \lambda_1 H \) for some \( t \in (-h,0) \). We conclude \( |f(m-h) - n| \leq |f(m-h) - f(m)| + |f(m) - n| \leq \lambda_1 H + \delta \), so \( \|f(m-h)\| \leq \lambda_1 H + \delta \) for all \( h \leq H \).

This argument shows that for \( m \in \mathcal{R} \) we have \( a_{m-h} \geq T^2/2 - T \gg T^2 \) by (25) assuming
\[
T \leq \frac{1}{8(\lambda_1 H + \delta)}.
\]
Since \( T^2 \geq a_m \) we deduce that
\[
a_m \ll \frac{1}{H} \sum_{h \leq H} a_{m-h}.
\]
This implies
\[ \sum_{m \in \mathcal{R}} a_m \ll \sum_{m \in \mathcal{R}} \frac{1}{H} \sum_{h \leq H} a_{m-h}. \]

Now we have
\[ \sum_{m \in \mathcal{R}} \frac{1}{H} \sum_{h \leq H} a_{m-h} = \frac{1}{H} \sum_{n \in \mathbb{Z}} a_n, \]
since for each \( m \in \mathcal{R} \), the summands \( a_{m-1}, \ldots, a_{m-h} \) occur in the sum on the right hand side. If \( H \leq H' \), then the elements of \( \mathcal{R} \) are \( H \)-spaced, so each such summand occurs exactly once on both sides of this equality.

This improves upon the Weyl step in the proofs above:
\[ \sum_{m \in \mathcal{R}} a_m \ll H^{-1} \sum_{n \in \mathbb{Z}} \sum_{h \leq H} a_{n+h}, \]
\[ = H^{-2} \sum_{n \in \mathbb{Z}} \sum_{h \leq H} a_{n+h}, \]
\[ = H^{-2} \sum_{n \in \mathbb{Z} \cap [N-2H,2N]} \sum_{h \leq H} a_{n+h}. \]

Note that the number of \( n \in \mathbb{Z} \) in this sum is \( \ll \min\{H \# \mathcal{R}, N\} =: R \), which gives an improvement for a sparse set \( \mathcal{R} \) since then \( H \# \mathcal{R} \ll N \). Further, \( h \) lies in \( \mathcal{H} := [1, H] \cap [h_0-H, h_0-1] \), which is an interval of length at most \( H \).

The extra factor \( H^{-1} \) due to the improved Weyl step will be an advantage in the minor arc analysis.

With the above definition (26) of \( a_m \), we derive
\[ (29) \sum_{m \in \mathcal{R}} a_m \ll TH^{-2} \sum_{a \leq T} \sum_{m \in [N-2H,2N]} \left| \sum_{h \in \mathcal{H}} e(af(m+h)) \right|. \]

The right hand side in (29) can now be handled like \( \mathcal{L}_f \) in the proofs of Theorems 12 and 13 above. For this, in analogy to \( \mathcal{L}_f \) set
\[ \mathcal{L}'_f := H^{-1} \sum_{a \leq T} \sum_{m \in [N-2H,2N]} \left| \sum_{h \in \mathcal{H}} e(af(m+h)) \right|. \]

Thus
\[ (30) \mathcal{R}(f, N, \delta) \ll 1 + T^{-1} H^{-1} \mathcal{L}'_f \]
by (27) and (29). Note that \( \mathcal{L}_f \) differs from \( \mathcal{L}'_f \) only by the summation over \( m \), and \( h \) runs through an interval \( \mathcal{H} \) of length at most \( H \), whose boundary points depend on \( m \).
The estimation of $L'_f$ follows that of $L_f$. The only small change in the above proof of Theorem 13 lies in the minor arc estimate (15), where we are able to replace $N$ by $R$ so that

$$\mathcal{T}'_{x,(m)} \ll RTx^{1-\tau+\epsilon}.$$ 

To verify this, note that

$$T'_x := \sum_{N-2H < m < 2N} \sum_{\exists h_0 \leq 2H: m+h_0 \in \mathcal{R}} |\tilde{S}'_{a,m}(\alpha, x)|$$

involves a restriction of $h$ in the sum

$$\tilde{S}'_{a,m}(\alpha, x) = \sum_{h \in \mathcal{H}, h \leq x} e(a\tilde{f}_{m,\alpha}(h))$$

to $h \in \mathcal{H}$. Still, Theorem 9 can be applied since the condition $h \in \mathcal{H}$ restricts the summation to a set which is an interval. Since this interval has an upper bound of at most $x$, this provides the stated bound for $T'_{x,(m)}$.

Next, for appropriate $H$, choose $T = \left[\frac{1}{H^{k}A\lambda}\right]$. In particular, if $H^{k}A\lambda - 8\lambda_1 H \geq 8\delta$ then (28) is true. As in Theorem 13 we obtain

$$L'_f \ll RTH^{-\tau+\epsilon} + A\mu TH^2.$$ 

We put this bound in (30) above to obtain

$$\mathcal{R}(f, N, \delta) \ll 1 + T^{-1}H^{-1}(RTH^{-\tau+\epsilon} + A\mu TH^2) \ll 1 + \mathcal{R}(f, N, \delta)H^{-\tau+\epsilon} + A\mu H.$$ 

By an application of Lemma 4 we leave out the term on the right hand side containing $\mathcal{R}(f, N, \delta)$ assuming that $H$ is large enough in terms of the implicit constant. This works since $H^{-\tau+\epsilon}$ gets arbitrarily small if $H$ increases.

We arrive at the bound

$$\mathcal{R}(f, N, \delta) \ll 1 + A(1 + A\lambda N)H.$$ 

Now we collect all the assumptions made on $H$. Due to Lemma 16 we need $H \leq (A\lambda)^{-2/(k(k+1))} \leq N$.

Moreover, due to (28) we need $H^{k}A\lambda - 8\lambda_1 H \geq 8\delta$, that is,

$$(31) \quad 8\delta \leq H(H^{k-1}A\lambda - 8\lambda_1),$$

for which $H^{k-1}A\lambda > 8\lambda_1$ has to be true. Let $H = ((8\delta + 8\lambda_1)/A\lambda)^{1/(k-1)}$, so that (31) holds true. Now if

$$((\delta + \lambda_1)/A\lambda)^{1/(k-1)} \ll (A\lambda)^{-2/(k(k+1))},$$

that is,

$$(32) \quad \delta + \lambda_1 \ll (A\lambda)^{1-2(k-1)/(k(k+1))},$$

then we conclude that $H$ is appropriate for the asserted bound in the theorem to be valid for all small $\delta$ such that (32) holds. We further need to assume that
$H$ is larger than some constant, which makes the above step with Lemma 4 work, so we shall also assume $A\lambda \ll \delta + \lambda_1$. This yields the assertion. □

We shall compare the bound in Theorem 17 with the well-known theorem of Huxley and Sargos [15, Thm. 1], which states the following bound for $\mathcal{R}(f, N, \delta)$. The version given here is explicit in $A$ and has been taken from [6, Thm. 5.12] where a proof is provided. The known proofs are geometric and do not depend on any exponential sum technique.

**Theorem 18** (Huxley and Sargos; explicit in $A$). Let $k \geq 3$ be an integer and $f \in C^k([N, 2N])$ be such that there exist $\lambda, A > 0$ with $\lambda \leq |f^{(k)}(x)| \leq A\lambda$ for all $x \in [N, 2N]$. Let $0 < \delta < 1/4$. Then

$$\mathcal{R}(f, N, \delta) \leq N(A\lambda)^{2/(k(k+1))} + N(A\delta)^{2/(k(k-1))} + (\delta/\lambda)^{1/k} + 1.$$  

Here the first term $N(A\lambda)^{2/(k(k+1))}$ dominates if $\delta \leq \lambda^{2/(k+1)}A^{-2/(k+1)}$ and $\delta \ll N^kA^{2/(k+1)}\lambda^{1+2/(k+1)}$, in particular when examining very small $\delta$. The main term $N(A\lambda)^{2/(k(k+1))}$ is commonly called the smoothness term and regarded as very difficult to improve (compare also [6, p. 275]).

A comparison of the term $NA^2\lambda((\delta + \lambda_1)/A\lambda)^{1/(k-1)}$ in the bound of Theorem 17 with the smoothness term shows that under the stated assumptions, Theorem 17 gives a sharper bound. Also note that if we let $\delta \to 0$ in Theorem 17, then we obtain the improved bound $\mathcal{R}(f, N, 0) \ll 1 + A(1 + A\lambda N)(\lambda_1/(A\lambda))^{1/(k-1)}$, compared to Theorem 18 which yields the smoothness term as upper bound for $\mathcal{R}(f, N, 0)$.

The bound of Theorem 17 is quite satisfying, the only obstruction lies in the restrictive assumption

$$\lambda_1 \ll (A\lambda)^{1-2(k-1)/(k(k+1))}$$

on $\lambda_1$ and $A\lambda$. This shows that Theorem 17 is in fact of limited use in applications since this condition is only true for certain functions. Unfortunately, functions like $f(x) = B/x^r$ or $f(x) = (B/x)^{1/r}$ for integers $r \geq 2$, which occur in interesting applications, are not of this kind when $A$ is constant. And if $A$ is so large that (33) holds, the bound of Theorem 17 can be quite weak.

In this context, we state the following theorem of Gorny [10].

**Theorem 19** (Gorny). Let $k \geq 2$ and $f \in C^k([1, 1+3N])$. Let $M, A, \lambda \in \mathbb{R}$ be such that $|f(x)| \leq M$ and $|f^{(k)}(x)| \leq A\lambda$ for all $x \in [1, 1+3N]$. Then for all $x \in [1, 1+3N]$,

$$|f'(x)| \ll MN^{-1} + M^{1-1/k}(A\lambda)^{1/k},$$

with implicit constant that depends on $k$ only.

From this theorem, we conclude that (33) is true for all sufficiently large $N$ provided that $M \leq (A\lambda)^{1-2/(k+1)}$. Therefore, functions on $[1, 1+3N]$
that are much smaller in absolute value compared to the maximum of the absolute value of the \( k \)th derivative are admissible for Theorem 17. This gives a nice criterion for Theorem 17 to hold, but it seems to be hard to find easy examples.

**Remark.** Applying Theorem 12 instead of Theorem 13 in the proof of Theorem 17 would also lead to the vanishing of the minor arc contribution in the bound, but the resulting bound for \( R(f, N, \delta) \) would be weaker due to the larger major arc contribution. Instead, the vanishing trick presented may be used with even large minor arc contributions, probably leading to further refinements.

6. **Second application: The polynomial large sieve inequality (LSI) in the one-dimensional case.** In this section we present an application of Theorem 8 to the polynomial large sieve inequality. This generalization of the classical large sieve inequality to sparse moduli sets, usually given as values of some fixed polynomial, has been intensely studied and has already influenced some other areas of number theory, especially the version with \( k = 2 \) from [2]. To name a few such topics, it has been found useful to find variants of Bombieri–Vinogradov’s theorem [4, 13], new results on primes of polynomial shape [9] or primes in APs to spaced moduli [4], divisibility questions with Fermat quotients [8], and mean value estimates for character sums with applications [1, 3, 5, 16]. Furthermore, the multidimensional polynomial LSI can be used for sieving with high powers as seen in [12] and reaches questions related to the abc-conjecture.

**6.1. Setting.** We start by giving the setting and basic assumptions in the polynomial LSI in the one-dimensional case.

Let \( P \in \mathbb{R}[x] \) be a fixed monic polynomial of degree \( k \geq 2 \) with \( P(0) = 0 \). Assume that \( P \) has only positive values in \([Q, 2Q]\) for each real \( Q \geq 1 \) and let \( M_Q := \max\{P(q) : q \in [Q, 2Q]\} \) be the maximal value for integers \( q \in [Q, 2Q] \). Clearly \( M_Q \ll Q^k \); assume also that \( P(q) \gg Q^k \) for all integers \( q \in [Q, 2Q] \) and some implicit constant that may depend only on \( k \). Let \( N, M \) be integers and \((v_n)_{n \in \mathbb{N}}\) be a sequence of complex numbers.

In the theory of the large polynomial LSI (see [2, 11, 21]) we aim to give upper bounds for the quantity

\[
\Sigma_P := \sum_{q \leq Q} \sum_{1 \leq a \leq P(q)} \left| \sum_{M < n \leq M + N} v_n e\left(\frac{an}{P(q)}\right)\right|^2.
\]

When we put the current form of Weyl’s inequality, Theorem 6, in the machinery of [11, 12], the bound \( \Sigma_P \ll Q^\varepsilon \|v\|^2(Q^{k+1} + A_k(Q, N)) \) is easily
derived with
\[ A_k(Q, N) := NQ^{1-1/(k(k-1))} + N^{1-1/(k(k-1))}Q^{1+1/(k-1)}. \]

The interesting range for \( N \) is \( Q^k \ll N \ll Q^{2k} \), since outside that, it is already known by an application of the standard large sieve inequality that the sharp bound \( \Sigma_P \ll Q^\varepsilon \|v\|^2(Q^{k+1}+N) \) holds true. So we assume without loss of generality that \( N \) lies in this range.

This result already offers an improvement compared to [3] in the range \( Q^k \ll N \ll Q^{2k-2+2/(k(k-1))} \) when \( k \geq 3 \). We will now further improve on that.

6.2. The connection of the polynomial LSI with Weyl sums.

From [11, Lemma 2.1], we know that
\[ \Sigma_P \ll Q^\varepsilon \|v\|^2 \left( \sum_{Q < q \leq 2Q} P(q) + \max_{Q < r \leq 2Q} \max_{1 \leq b < P(r)} \sum_{\gcd(b, P(r)) = 1} \frac{1}{2} \frac{\#F_{b, P(r)}(x)}{x^2} \right). \]

The first term in large brackets is \( \ll QM_Q \) and is admissible. It is not necessary to repeat the definition of \( \#F_{b, P(r)}(x) \) since we will just make use of the upper bound
\[ \#F_{b, P(r)}(x) \ll B^{-1}Q + B^{-1} \sum_{1 \leq a \leq B} |S_a| \]
with \( B^{-1} = 2M_Qx \) and
\[ S_a := \sum_{q \leq 2Q} e \left( \frac{ab}{P(r)}P(q) \right), \]
which has been shown in the deduction of [11 Theorem 1.1].

To consider the integral expression in (34), fix a pair \( b, r \) with \( r \in [Q, 2Q] \), \( 1 \leq b < P(r) \) and \( \gcd(b, P(r)) = 1 \). We substitute \( B^{-1} = 2M_Qx \) and estimate as follows:
\[ \int_{1/N}^{1/2} \frac{\#F_{b, P(r)}(x)}{x^2} dx \ll \int_{1/(4M_Q)}^{N/(2M_Q)} \left( B^{-1}Q + \sum_{1 \leq a \leq B} |S_a| \right) MB dB \]
\[ \ll QM_Q \log N + \sum_{1/(4M_Q)}^{N/(2M_Q)} B^{-1} \sum_{a \leq B} |S_a| dB \]
\[ \ll QM_Q \log N + \sum_{a \leq N/(2M_Q)} |S_a| \int_{a}^{N/(2M_Q)} B^{-1} dB \]
\[ \ll QM_Q \log N + M_Q \sum_{a \leq N/(2M_Q)} |S_a|. \]
Here the last sum is a discrete moment of a Weyl sum with the polynomial \( bP(x)/P(r) \) and leading term \( b/P(r) \) since \( P \) is monic. We can apply Theorem 8 directly with \( P(r) \) as approximating denominator (when \( k \geq 3 \)) to get

\[
\sum_{a \leq N/(2M_Q)} |S_a| \ll \frac{N}{MQ}Q^{1+\varepsilon} \left( \frac{Q^{k-1}}{P(r)} + \frac{Q^{k-1}}{N/\sqrt{M_Q}} + \frac{1}{Q^{2}} + \frac{P(r)}{QN/\sqrt{M_Q}} \right)^{1/(2s_0)}
\]

with

\[
s_0 = (k-1)(k-2)/2 + 1, \quad \text{so that} \quad 2s_0 = k(k-1) - 2k + 4.
\]

Let \( \omega := 1/(2s_0) \). In the large bracket expression, the last summand \( P(r)M_Q/(NQ) \) dominates since \( P(r) \gg Q^k \) and \( 1/Q \ll P(r)M_Q/(NQ) \) for \( N \ll Q^{2k} \). So we continue with

\[
\int_{1/N}^{1/2} \# \mathcal{F}_{b,P(r)}(x) \frac{dx}{x^2} \ll QM_Q \log Q + N^{1-\omega}M_Q^{\omega}Q^{1-\omega+\varepsilon}P(r)^\omega \\
\ll QM_Q \log Q + N^{1-\omega}M_Q^{2\omega}Q^{1-\omega+\varepsilon} \\
\ll Q^{k+1} \log Q + N^{1-\omega}Q^{1+2k\omega-\omega+\varepsilon},
\]

where we have used \( M_Q \ll Q^k \) in the last step.

Compared to the dominating term \( NQ^{1-1/k(k-1)} \) in the previous bound \( A_k(Q,N) \), we get an advantage if \( N^{1-\omega}Q^{1+(2k-1)\omega} \ll NQ^{1-1/k(k-1)} \), which is the case if \( N \geq Q^{2k-2/(k-1)+4/(k(k-1))} \), that is, if \( N \) is close to \( Q^{2k} \), but still in the interesting range \( Q^k \ll N \ll Q^{2k} \). We have therefore shown the following new improved bound for the polynomial LSI.

**Theorem 20.** In the setting of this section, when \( k \geq 3 \),

\[
(35) \quad \Sigma_P \ll Q^{\varepsilon} \|v\|^2(Q^{k+1} + \min\{A_k(Q,N), N^{1-\omega}Q^{1+(2k-1)\omega}\})
\]

with \( \omega = 1/((k-1)(k-2)+2) \).

Theorem 20 offers an improvement compared to known previous results when \( k \geq 4 \). This is because for \( k = 3 \), the additional bound from [3] is still stronger in this case.

It is interesting what we would obtain having Conjecture 10. In this case, we would be able to gain a factor \( Q^{(1-k)\omega} \). Thus, we would arrive at the following result.

**Conjecture 21.** In the setting of this section, when \( k \geq 3 \),

\[
\Sigma_P \ll Q^{\varepsilon} \|v\|^2(Q^{k+1} + \min\{A_k(Q,N), N^{1-\omega}Q^{1+k\omega}\})
\]

with \( \omega = 1/((k-1)(k-2)+2) \).

Note that if we could take \( 1/(k(k-1)) \) for \( \omega \), we would get the expression \( N^{1-1/(k(k-1))}Q^{1+1/(k(k-1))} \), which coincides with the second summand in \( A_k(Q,N) \).
Still, these conjectural bounds are far from Zhao’s conjecture \([21]\) stating that
\[
\sum_P \ll Q^\epsilon \|v\|^2 (Q^{k+1} + N).
\]
Conjecture \([21]\) might be within reach of further refinements of the methods presented in this article.

We remark that an attempt to use Theorem \([11]\) will not give more if much improvement on the coefficient of \(q^{k-1}\) in \(P(q)\), say \(\alpha_{k-1}\), is known. This is because the coefficient of \(q^{k-1}\) in the polynomial \(bP(q)/P(r)\) is then \(\alpha_{k-1}b/P(r)\). By Theorem \([11]\), one then needs to look at the rational approximations to \(\alpha_{k-1}b\). Since \(b\) is supposed to be any coprime residue mod \(P(r)\), there will always be one with small denominator which offers no advantage in Theorem \([11]\).

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