Network formation with value heterogeneity: centrality, segregation 
and adverse effects

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Abstract

We investigate formation of economic and social networks where agents may form or cut ties. The novelty is combining a setup where agents are heterogeneous in their talent for generating value in the links they form and value may also accrue from indirect ties. We provide sufficient conditions for assortative matching: agents of greater talent have partners of greater talent. A novel feature is that agents with higher talent are more central in networks. Another novel feature is degree assortativity: partnered agents have a similar number of partners. Two suboptimal network structures are noteworthy. One network displays excess assortativity as high and low talented types fail to connect, and thus inefficient due to payoff externalities despite otherwise obeying the conditions of Becker (1973). In another suboptimal network an agent of low talent becomes excessively central.

Keywords: assortative matching; assortativity; complementarity; cooperative games; network formation; one-sided matching.

JEL classification: C71, C78, D61, D62, D85.

1 Introduction

A classic topic in research on social networks is sorting (or homophily as it sometimes known) which is the tendency to associate more with people similar to one-self. Empirical studies have shown that this sorting by self-similarity is a fundamental pattern across various social networks for characteristics such as ethnicity, interests and socioeconomic status, cf. McPherson, Smith-Lovin, and Cook (2001).

Economic research has been interested in sorting since the pioneering of Becker (1973) on labor and dating markets. This line of research is important as it highlights when sorting is an equilibrium in choice of partners by self-interested agents and that sorting may be inefficient (i.e. everyone could be better under another configuration). However, despite these significant findings there has been no attempt to reconcile Becker (1973)’s framework on sorting with the literature on network formation. This implies that there has been neither an investigation whether sorting is consistent with network externalities (i.e. value from an indirect connection to friends of friends), nor studies of whether there are systematic network structures beyond sorting are formed in this context.

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Methodology and overview  The theoretical framework we explore is one where each agent chooses a number of partners under two core assumptions. First; agents are assumed heterogeneous in their ability to create value in a partnership - this is consistent with research on peer effects cf. Zimmer and Toma (2000), Sacerdote (2001), Falk and Ichino (2006) 1. Second; each agent can only choose a limited number of partners and this induces a discrimination of partners - this assumption of limitation on links is consistent with empirical studies cf. Ugander, Karrer, Backstrom, and Marlow (2011) and Miritello, Lara, and Mord (2013) 2.

In this setting we analyze the structure of networks for robustness in the following sense: no agents can form links and be better off than the current network structure. This setting is probed under two settings: one where indirect connections to friends of friends provides a spillover of value and one where there are no such externalities. When externalities are present it implies a trade-off quality vs quantity: on the one hand there is an incentive to choose more able partners; on the other hand there is an incentive to minimize the distance to all indirect partners.

Although the literature on network formation is vast the literature on strategic link formation by heterogeneous agents has been limited to the two following classes. The first class is algorithmic game theory where the problem of computing the pairwise stable outcomes under two-way link formation without transfers see for example Höfer, Vaz, and Wagner (2014). This first class of has not investigated deeper economic issues of structure and efficiency. The second class of literature builds on the one-way link formation Bala and Goyal (2000). Contrary to our setup one-way link formation requires neither mutual acceptance of links nor does it allow for transfers. The research on one-way link formation has modeled heterogeneity in: survival of links, cf. Haller and Sarangi (2005); costs, cf. Galeotti, Goyal, and Kamphorst (2006), as well as; both value and cost, cf. Goeree, Riedl, and Ule (2009).

The present framework combines two existing models: agent heterogeneity which is line with complementarity as in Becker (1973) and network externalities as in the 'connections model' of Jackson and Wolinsky (1996). In our framework an agent’s talent could refer to productive and non-productive capabilities such as vocational intelligence, social aptitude, physical attractiveness. Note the framework could also model bilateral partnerships of organizations such as corporations, educational institutions.

Below we list the contributions of this paper to the literature on network formation and assortative matching. Some results are derived when only evaluating links formed between a pair of agents which corresponds to pairwise (Nash) equilibrium. Other results require that any coalition of agents can form links between them which corresponds to strong (Nash) equilibrium. The list is ordered logically so that later results build on earlier. The results can be crudely divided into predictions on linking behavior in point (i)-(iii) and the normative aspect in (iv) 3.

(i) We demonstrate conditions for when stable networks have a pattern where more talented agents have a more central network position. Without externalities the relevant measure of centrality is number of links/degree, see Theorem 1. With externalities the measure is decay centrality and the requirements are strong stability and no modularity, see Proposition 1. However, failure of either of these conditions may lead to violation of these monotonic patterns of centrality, see Corollary 1.

(ii) We also provide sufficient conditions for stable networks to exhibit assortative behavior in talent when externalities are absent, see Theorem 2. Moreover, with many agents it is shown

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1Note the empirical research on peer effects should be cautiously interpreted as it is problematic to disentangling peer effects from partner selection is difficult, cf. Manski (1993), Shalizi and Thomas (2011), Angrist (2013).

2Ugander, Karrer, Backstrom, and Marlow (2011) investigates the entire Facebook network and Miritello, Lara, and Mord (2013) for call logs for millions of individuals.

3To the best of the author’s knowledge this is the first demonstration of the novel features.
in Proposition 2 that stratification in talent is perfect - this holds even in the presence of externalities under strong stability. This can be interpreted as that the classic results of Becker (1973) holds under certain circumstances. Nevertheless, the assortative pattern in talent may likewise break down when strong stability is violated as shown in Corollary 2.

(iii) We show if there are no externalities then under the conditions for (i) and (ii) and a regularity condition then stable networks also exhibit degree assortativity (sorting by similarity of number of links), see Theorem 3. This is relevant as degree assortativity is a fundamental phenomenon observed in social networks. This indicates that even when assortative behavior in talent cannot be measured perhaps it may potentially be detected as degree assortativity. Yet, assortative behavior in degree may fail under certain conditions when there are externalities, cf. Corollary 3.

(iv) Under pairwise (Nash) stability we exhibit two generic networks which are inefficient. The first is a stratified network where the more and less talented agents fail to connect, see Example 3 and Proposition 3. The reason is that two agents forming a link fail to internalize for the full benefits from their link. This stratified network may help guide policies as it sheds light on costs of stratification. The other inefficient network is where an agent with least talent is the center of a star. This latter network is important as it highlights a generic failure which may happen for instance within organizations or a group of individuals.

Significance of contributions and literature The following paragraphs compare the results above with the most relevant work in existing literature.

(i) Goeree, Riedl, and Uld (2009) show in two examples with unilateral link formation that the star network with a high value agent at the center can be efficient and Nash stable, while other star networks are also stable. The efficient network emerges more frequently in experiments, and its frequency increases with repetition of the experiment.

(ii) While the literature on two-sided assortative matching (labor and dating markets) has a strong tradition this is not the case for assortative matching in networks (one-sided). The research on assortative networks has been limited to formation of clubs (i.e. groups/coalitions where all agents are linked) under varying production technologies/preferences and non-transferable utility, cf. Farrell and Scotchmer (1988), Kremer (1993), Durlauf and Seshadri (2003), Legros and Newman (2002), Pycia (2012), Baccara and Yariv (2013).

(iii) The most related work is Currarini, Jackson, and Pin (2009) which is also driven by assortative behavior, however, their results rely on differences in the distribution of types (which is not necessary in our setup). Other models from statistical physics also yields the same prediction but have no strategic considerations. The most prominent is the preferential attachment model (agents connect with higher likelihood agents with more links, see Barabasi and Albert (1999)). Note that an analogy to that property is found in Kremer (1993) and Farrell and Scotchmer (1988) where more productive groups tend to be larger.

(iv) Inefficiency under network effects was first shown by Katz and Shapiro (1985) and Farrell and Saloner.
This paper contributes by providing structure to these inefficient network. The first inefficient network structure in (v) highlights a new source of inefficiency in assortative matching besides non-transferable utility. This has implications for policy which are discussed after derivation of Proposition 3 in Section 6. The second network structure in (v) highlights a source of inefficiency akin to the two-sided model in Katz and Shapiro (1985) and Farrell and Saloner (1986) - in their setups the source is a worse producer supplying the consumers cannot coordinate on choosing the better producer and get "locked-in". This corresponds to our setup where the limits to coordinating in reforming links under pairwise stability allows a less talented agent to be excessive central.

Structure of paper The rest of the paper is structured as follows: Section 2 introduces the model; Section 3 explores the relationship between agent talent and network centrality; Section 4 investigates assortative behavior in talent; Section 5 examines assortative behavior in degree; Section 6 examines sources of inefficiencies; Section 7 concludes in a discussion of assumptions. The proofs for the analysis are found along with a few auxiliary results in Appendices A-E. Note that although in the analysis we employ two variations in modeling linking costs the results are valid under both specifications (proven for both cases).

2 Link formation with heterogeneous peers

Let \( N \) constitute a set of \( n \) agents. A collection of agents \( t \) is a subset of the set of agents \( t \subseteq N \). A coalition of agents \( t \) is a subset of \( T \) which is the superset of \( N \) excluding the empty set.

Types Each agent \( i \in N \) is endowed a fixed measure of talent, \( x_i \in X \) where \( X \) is the set of possible levels of talents for agents in \( N \). Let \( \bar{x} = \max X \) and \( x = \min X \). Let the vector of talent that is sorted descending be denoted \( X = (x_1, x_2, ..., x_n) \) where by construction \( x_1 \leq x_2 \leq ... \leq x_{n-1} \leq x_n = \bar{x} \). For \( t \) define \( X(t) \) as the vector of talents ordered descending for agents in \( t \).

Network formation Two agents \( i, j \in N \) may form a link between one another. Links are direct connections between the two and are assumed to require mutual acceptance to form and can be broken by any one of two linked agents. When two agents are linked this is denoted \( ij \in \mu \) where \( \mu \) is the set of links. It follows that any set of links, \( \mu \), constitutes an undirected, unweighted network; this definition is used throughout the paper. Networks are represented in diagrams as where nodes/agents are circles and lines between them are edges. Let the network where every agent is linked to one another be denoted \( \mu^c \); this is referred to as the complete network. The set of all potential networks is denoted \( M = \{\mu | \mu \subseteq \mu^c\} \).

A coalitional move from the network \( \mu \) to another network \( \tilde{\mu} \), consists of a set of added links \( \tilde{\mu} \setminus \mu \), and a set of deleted links, \( \mu \setminus \tilde{\mu} \). A coalitional move is feasible for \( t \) if: (i) links added, \( \tilde{\mu} \setminus \mu \) are formed only between agents who are both members of coalition \( t \); (ii) links destroyed, \( \mu \setminus \tilde{\mu} \) have at least one agent in the link who is a member of the coalition \( t \).

Network measures Define an agent’s neighborhood as the set of agents for which the agent has a link in a given network. The neighborhood for agent \( i \) is a correspondence from the network into a subset of agents denoted \( \nu_i : M \rightharpoonup N \). E.g. for \( i \) in \( \mu \) the neighborhood is \( \nu_i(\mu) = \{i^\prime \in N : ii^\prime \in \mu\} \).

\footnote{Note that we use a different pairwise stability concept than Jackson and Wolinsky (1996) - see end of Section 2 (the model) for a discussion.}
It is useful to have a measure of distance in a network beyond being directly linked. A path between agents \( i_1, i_n \) exists in the network \( \mu \) if there is a subset of links \( \{i_1 i_2, ..., i_{n-1} i_n\} \subseteq \mu \) such that no two agents \( i_n, i_{n'} \) are the same. The (geodesic) distance between \( i \) and \( j \), denoted \( p_{ij} \), is the length of the shortest path connecting \( i \) and \( j \); if no path exists this takes the value \( \infty \).

With the definitions above we employ two centrality measures. The first is degree (centrality) which measures the number of links/neighbors, i.e. the size of the neighborhood: \( k_i = |\nu_i| \). For a subset of agents \( t \) define \( K(t) \) analogue to \( X(t) \) but where the measure is agents’ degree. The other measure of centrality is decay centrality; for a decay factor \( \delta \) it is defined as follows, \( d_k^\delta(\mu) = \sum_{j \neq i} \delta^{p_{ij}(\mu)} \).

**Utility**

The utility accruing to each agent is denoted \( u_i \) and aggregate utility \( U(\mu) = \sum_{i \in N} u_i \).

The inputs to utility are network structure and agent characteristics. An agent’s utility consists of benefits \( b_i \) and costs \( c_i \) of linking:

\[
u_i = b_i - c_i.\tag{1}\]

We use two approaches to model costs of linking which reflect constraints in terms of time and effort needed to form a link. Both approaches provides a framework where each agent can maintain only a limited number of links. The first approach is to impose a degree quota. This quota, \( \kappa \), is an upper bound on the number of links; thus for all agents \( i \in N : k_i \leq \kappa \). Under the quota there are no explicit cost of linking (i.e. \( c_i(\mu) = 0 \)) - only implicit opportunity costs. The second approach is that an agent \( i \)'s costs are determined by its \( k_i(\mu) \): \( c_i(\mu) = c(k_i(\mu)) \). The cost function is \( c : \mathbb{N}_0 \to \mathbb{R} \) is positive, strictly increasing and strictly convex function of the degree (i.e. number of links), \( k_i \).

**Structure of benefits**

The benefits to agent \( i \) is the weighted sum over the value of the agent’s connections. As seen in Equation 2 each pair of agents \( ij \) has two associated components. The first component is their network weight: \( \delta^{p_{ij}(\mu)} \). This weight decreases in network distance, \( p_{ij}(\mu) \) by externality factor \( \delta \in [0, 1] \). Note when \( \delta = 0 \) there are no network externalities. The second component is the individual link value which is the value for \( i \) of being connected to \( j \). The value is denoted \( z_{ij} \) which corresponds to input from the two partnering agents’ talent, i.e. \( z_{ij} = z(x_i, x_j) \).

\[
\nu_i(\mu) = \begin{cases} 
\sum_{j \neq i} \delta^{p_{ij}(\mu)} z_{ij} & \delta > 0, \\
\sum_{j \in \nu_i(\mu)} z_{ij} & \delta = 0.
\end{cases}
\tag{2}
\]

The function \( z \) is assumed to be: twice differentiable, and; taking positive and bounded values.\(^8\)

Let the (total) link value be defined as the value for the pair, i.e. \( Z_{ij} = z_{ij} + z_{ji} \). In our subsequent analysis complementarity in link value is key for deriving results - this consists of two distinct features:

**Definition 1.** The link value function has monotonicity if \( \frac{\partial}{\partial x_i} Z(x_i, x_j) > 0 \) [weak \( \frac{\partial}{\partial x_i} Z(x_i, x_j) \geq 0 \)].

**Definition 2.** The link value function has supermodularity if \( \frac{\partial^2}{\partial x_i \partial x_j} Z(x, \bar{x}) > 0 \) [holds weakly if \( \frac{\partial^2}{\partial x_i \partial x_j} Z(x, \bar{x}) \geq 0 \)]; it has no modularity when \( \frac{\partial^2}{\partial x_i \partial x_j} Z(x, \bar{x}) = 0 \). Note that supermodularity entails,

\[
Z_{ij'} + Z_{jj'} > Z_{ij} + Z_{ij'}, \quad x_i > x_{j'}, x_{ij'} > x_j.
\tag{3}
\]

Note when there are only two types we define a coefficient of modularity, \( \beta = |Z(\bar{x}, \bar{x}) + Z(\bar{x}, \bar{x})|/[2Z(\bar{x}, \bar{x})] \). The interpretation is that \( \beta > 1 \) corresponds to supermodularity and \( \beta = 1 \) corresponds to no modularity.

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\(^8\)This approach mimics the connections model in Jackson and Wolinsky (1990).

\(^9\)The upper bound for each particular link’s utility rules out situations with infinite links in equilibrium.
Transfers and side payments  It is assumed that any two agents may transfer utility between them. Denote the net-transfer in amount of utility from agent \( j \) to agent \( i \) as \( \tau_{ij} \in \mathbb{R} \). The set of net transfers is denoted \( \tau \). As we model the net-transfers it holds that for any \( i, j \in N \) where \( i \neq j \) it holds that \( \tau_{ij} = -\tau_{ji} \). For each agent \( i \) its payoff after transfers, \( s_i \), is defined as:

\[
\begin{align*}
    s_i(\mu, \tau) &= u_i(\mu) + \sum_{j \in \nu_i(\mu)} \tau_{ij} \\
    \sum_{i \in N} s_i(\mu, \tau) &= U(\mu)
\end{align*}
\]  

In addition it is assumed that if either of the two linked agents decide to delete their mutual link then the net-transfer between the two agents is set to zero. Note that if both linked agents participate in a coalitional move which entails that their mutual link is removed then these two agents may choose form a new transfer.

The link formation game  This paper explores a static setup where all decisions are simultaneous. The information is complete and thus every agent has certainty about the type of other players.

Two equilibrium concepts are used. Under either equilibrium concept the focus is restricted to pure strategy outcomes which corresponds to the networks described above. Both concepts of equilibrium rely on blocking coalitions: a coalition \( t \in T \) block the network \( \mu \) with net-transfers \( \tau \) if: (i) there is a feasible coalitional move from network \( \mu \) to network \( \tilde{\mu} \) with \( \tilde{\tau} \); (ii) for all coalition members the coalitional move leaves them with a higher payoff, i.e., \( \forall i \in t : s_i(\tilde{\mu}, \tilde{\tau}) > s_i(\mu, \tau) \). The first equilibrium concept is strong stability which requires that no coalition of any size is blocking cf. Aumann (1959). The set of strongly stable equilibria is denoted \( M^{ss} \). The second equilibrium concept is where no coalition of at most two agents is blocking. This other concept is referred to as pairwise (Nash) stability in this paper. The set of pairwise stable networks is denoted \( M^{ps} \). Note this definition of pairwise stability allows substitution of links (simultaneous deletion and formation) - this is stricter than Jackson and Wolinsky (1996)’s pairwise stability which only allows either deletion or formation of a single link.

The modeling assumptions on transfers have a couple of implications. The first implication is that any strongly stable network is efficient and thus maximizes aggregate utility. The second implication is that the set of strongly stable networks is a subset of pairwise stable networks - thus any result derived under pairwise stability is also true for strong stability. Moreover, when there are no network externalities from indirect connections then it is possible to establish that every pairwise stable network is also strongly stable which is captured in Lemma 1.

In terms of stability concepts, the derived properties which hold for any pairwise (Nash) stability also hold for stricter concepts (it is a relatively weak concept). Propositions 1 and 2 rely on strong stability’s implied efficiency. Although efficiency is a unique property for strong stability (and does not hold for weaker concepts) then strong stability should be seen as a refinement with desirable properties which makes it more likely when it exists. In some circumstances the existence of contracts where an agent may subsidize or penalize another agent’s link formation with alternative agents may imply that strong stability even if contracts were limited to being pairwise specified, cf. Bloch and Jackson (2007).

3 Network centrality and talent

In this section we examine the structure of stable networks when measured by specific measures of network centrality. The measures of centrality are degree and decay centrality, see the model in Section 2. The question posed is the following: is there a relation between network measures of centrality and an individuals’ talent and is this monotone? The short answer is yes, under certain
circumstances talent and connectivity go hand in hand: more talented are more central in the network as is shown below.

**No externalities**  We begin by analyzing the setting where network externalities are absent. In this circumstance it can be shown that an agent’s talent and its number of links are positively related. Thus for a given population of agents this implies more talented agents have weakly more links than their lower talented counterparts. This is captured in the following theorem:

**Theorem 1.** Suppose there are no externalities and monotonicity in link value then for every pairwise stable network it holds that an agent’s degree (weakly) monotonically increasing in the agent’s type in the network:

\[ \forall \mu \in M^{ps}: x_i > x_j \Rightarrow k_i(\mu) \geq k_j(\mu). \]

The intuition behind the above result is that the marginal benefits for forming an additional link is strictly increasing in talent. Thus a higher talented agent has an increased incentive to form links.

**Externalities**  In the following paragraphs we investigate the situation where network externalities affect indirect relations. It turns out that the pattern of monotonic centrality in talent from Theorem 1 may be strengthened to hold for decay centrality. This other centrality measure is more general and equalizes degree centrality in the absence of externalities, i.e. \( \delta = 0 \). Thus the results also hold when there are no network externalities. However, this generalization of the pattern comes with two caveats: (i) there must be no modularity, and; (ii) the pattern only holds for strong stability.

**Proposition 1.** Suppose there are externalities as well as monotonicity and no modularity in link value then for every pairwise stable network it holds that an agent’s \( \delta \)-decay centrality is monotonically increasing in the agent’s type in the network:

\[ \forall \mu \in M^{ss}: x_i > x_j \Rightarrow d^\delta_i(\mu) \geq d^\delta_j(\mu). \]

The result above emerges from the fact that there are only monotone and independent effects of more talent (due to the absence of modularity). The independence of effects entails that the link value can be split into a part for each partner, \( Z(x, \tilde{x}) = \bar{Z}(x) + \tilde{Z}(\tilde{x}) \). It is possible to show that a given agent \( i \)’s total contributions to the network is the product between \( \bar{Z}(x_i) \) and the agent \( i \)’s decay centrality at factor \( \delta \). Under these circumstances monotonicity implies that more talented agents must have weakly higher decay centrality, \( d^\delta_i(\cdot) \), or else efficiency implied by strong stability will be violated.

Note that while the derivation of Proposition 1 relies on strong stability and no modularity the result is robust to change in specifications of externalities from indirect peers. One example is a linear discount \( (p_{ij})^{-1} \) rather than exponential. In this case of linear decaying externalities it would be possible to show an analogue result in terms of closeness centrality which is defined as \( \Sigma_j \neq i (p_{ij})^{-1} \). That is, more talented agents have higher closeness centrality in strongly stable networks.

We continue our analysis by examining the two crucial assumptions for Proposition 1: the absence of supermodularity, and; that strong stability holds. We show that if removing either of these conditions then the monotonic pattern of higher decay centrality ceases to exist.

Showing that the presence of supermodularity may violate the pattern of Proposition 1 is sketched in Example 1 (illustrated in Figure 1.B). When externalities are sufficiently low then the efficient network is the one where the types segregate but low type are more central as there are more of them.
Example 1. There are seven agents - four of low type (1,2,3,4) and three of high type (5,6,7). There is supermodularity; a degree quota of two, and; a level of network externalities \( \delta \). Consider the segregated network \( \mu = \{12, 23, 34, 14\} \cup \{56, 67, 75\} \), and the connected network \( \tilde{\mu} = \{12, 23, 34, 45, 56, 67, 71\} \). If the inequality below holds with ”>” then \( \mu \) is an efficient network; if ”<” then \( \tilde{\mu} \) is an efficient network.

\[
3 \cdot Z(\bar{x}, \bar{x}) + 4 \cdot Z(x, x) \geq 2 \cdot Z(\bar{x}, \bar{x}) + 3 \cdot Z(x, x) + (2 + 4\delta + 6\delta^2) \cdot Z(\bar{x}, \bar{x})
\] (6)

When Inequality (6) holds with ”>” then \( \mu \) is furthermore strongly stable when for every pair of agents \( i, j \in N \); \( i \neq j \) it holds that there are no transfers between them, i.e. \( \tau_{ij} = 0 \). In the network \( \mu \) it holds that \( d_i^s(\mu) > d_j^s(\mu) \) where \( i \in \{1, 2, 3, 4\} \); \( j \in \{5, 6, 7\} \). Note that Inequality (6) holds with ”>” if \( \beta > 1 + 2\delta + 3\delta^2 \) and \( \beta > 1 \) due to supermodularity.

When coalitions can only be formed pairwise (i.e. strong stability is not a requirement) the monotonic pattern may fail completely. The failure is such that the least talented is the most central. In fact the least talented is the center of a network where all other individuals have one link and that link is to the least talented, i.e. a \((n-1)\)-star. An illustration of this of this result where \( n = 3 \) can be found in Example 2 and depicted in Figure 1A.

Example 2. There are three agents in the set of agents, i.e \( N = \{1, 2, 3\} \) where \( x_1 = x \) and \( x = x_2 = x_3 \). Moreover it holds that: there is weak supermodularity and monotonicity holds; network externalities are present (\( \delta > 0 \)); there is a convex cost technology. Denote the 2-star network where agent 1 is the center as \( \mu \), that is \( \mu = \{12, 13\} \). We show below that \( \mu \) may constitute a network which is pairwise stable when there exist transfers \( \tau \) such that Inequalities (7)-(11) are satisfied.

An agent can delete any number of active links. By deleting all links an agent can secure itself a utility of 0 - this is captured by Inequality (7). By deleting one link agent 1 can secure itself utility equivalent to the individual net value of the remaining link - see Inequality (8).

\[
s_i(\mu, \tau) \geq 0, \quad i \in N, \tag{7}
\]

\[
\tau_{1i} \geq \Delta c(1) - z(\bar{x}, \bar{x}), \quad i \in \{2, 3\} \tag{8}
\]

In addition, any coalition of two agents can always delete all current links and form a link between them which is captured in Inequality (9)

\[
s_i(\mu, \tau) + s_j(\mu, \tau) \geq Z_{ij} - 2 \cdot c(1), \quad i, j \in N, i \neq j \tag{9}
\]

(A) Pairwise stable network where the least talented agent, \( i_1 \), is the center of 3-star. This illustrates point that higher centrality of more talented is violated. See Example 2 for details.

(B) Strongly stable network where value of assorting is higher than connecting - see Example 1.

Figure 1: Above networks depict failure of Proposition 1: (A) failure is due to pairwise stability, i.e. lack of strong stability; (B) failure is from presence of supermodularity, i.e. no absence of modularity.
Agents 2 and 3 can also form a link between when at least one of them keep a link to agent 1. Inequality 10 checks for a formation of one new link and one link by a high type is severed with 1. Inequality 11 checks for a formation of one new link when neither 2 or 3 sever their link. This condition is true when the cost of forming an additional link is sufficiently large.

\[
s_2(\mu, \tau) + s_3(\mu, \tau) \geq Z(\bar{x}, \bar{x}) + z(\bar{x}, x) \cdot (1 + \delta) - c(1) - c(2) + \max\{\tau_{21}, \tau_{31}\} \tag{10}
\]

\[
2 \cdot \Delta c(2) \geq (1 - \delta) \cdot Z(\bar{x}, x) \tag{11}
\]

Remark 1. In Example 2 the sufficient conditions for existence of \(\tau\) such that \(\mu\) is stable are linear costs (\(\tilde{c}\) of forming a link) and \(\delta\) being sufficiently high along with the two inequalities:

\[
2 \cdot Z(\bar{x}, x) + Z(\bar{x}, \bar{x}) \geq 4\tilde{c} \quad \text{and} \quad Z(\bar{x}, x) > \tilde{c}.
\]

Example 2 is an illustration of a more general result - monotone increasing centrality in talent may not hold, even for many agents. The necessary conditions for the network in Example 2 are extended in the auxiliary results Appendix B. Lemmas 2 and 3 may be rewritten into the following result:

Corollary 1. The monotone increasing degree and decay centrality in talent may fail under pairwise stability if \(\delta > 0\) and the conditions from either Lemmas 2 or 3 are valid. The latter lemma requires only weak supermodularity, linear costs of linking and many agents.

4 Assortative behavior in talent

This section investigates stratification in talent, i.e. based on the agents’ exogenous given types.

No externalities We begin with the setting where network externalities are absent, i.e., \(\delta = 0\). This setting is the simplest as indirect partners (friends of friends) have no impact. The first result of assortative behavior is to describe the set of partners for a given agent, i.e., the neighborhood:

Theorem 2. If there is monotonicity, supermodularity and no externalities then it follows that for every pairwise stable network it holds that a higher talented agent has partners which weakly dominate in talent when compared partner-by-partner with the partners of a lower talented agent:

\[
\forall \mu \in M^{ps}, \forall j \in N, x_i > x_j : X(\nu_i(\mu)/\{j\})_{l} \geq X(\nu_j(\mu)/\{i\}_{l}), \quad l = 1, \ldots, k_{j}(\mu).
\]

Possible externalities We proceed to a more general context where externalities from indirect connections, i.e. friends of friends, may affect utility. This setting of externalities is used to investigate what pattern of linking is exhibited in the case of many potential partners. The aspect of many partners is the presence of an (asymptotic) infinite number of potential partners in \(N\), i.e. the limit for \(n \to \infty\). In order to avoid describing a set of completely homogenous set of agents we focus on the generic set of agents for which there is asymptotic heterogeneity. This condition of agent heterogeneity is satisfied if for \(N \to \infty\) there are at least two types \(x, \bar{x} \in X\) which has a strictly positive share of \(N_n\). With this definition of heterogeneity we can describe the assortative behavior when there are many agents:

\[\text{Note the technical condition is that } \lim_{n \to \infty} \left(\left|\{i \in N_n\}_{x_i=x} / N\right| \right) > 0. \text{ That condition is satisfied when there is asymptotic variance of types, i.e. } \lim_{n \to \infty} \text{Var}[x_i] > 0.\]
Proposition 2. If there is supermodularity, a degree quota where \( \delta < (\kappa - 1)^{-1} \) and many agents then in every strongly stable network any agent links with agents of the same type almost surely:

\[
\lim_{n \to \infty} \mathbb{P}[x_i = x_j | ij \in \mu_n] = 1, \quad \mu_n \in M_{ss}.
\]

Remark 2. For \( \delta = 0 \) the results in Proposition 2 requires only supermodularity and pairwise (Nash) stability and holds even with a cost function.

The results above demonstrates that the availability of many agents for linking can induce a pattern of perfect discrimination. In the assortative matching literature this pattern is called perfect positive assortative matching. In the literature network science a measure of assortative behavior in networks is the coefficient of assortative mixing which in this case would have a value = 1, cf. Newman (2003). Note that Proposition 2 is only valid in the presence of externalities under strong stability. Note also that Proposition 2 corresponds to the conclusion of Becker (1973) in the marriage market.

One interpretation of Proposition 2 is that for \( \kappa = 2 \) the condition level of externalities is only \( \delta < 1 \). Also for any \( \kappa \) the conditions are met in the limit of \( \delta \to 0 \).

Externalities In the remainder of this section we analyze assortative behavior when externalities are present. The first result is that the assortative behavior previously established may fail to materialize when link formation is pairwise. In this case we have already demonstrated that a network where the least talented can be the center with \( n-1 \) links to all other agents is pairwise stable. This situation with least talented as center entails that the pattern predicted by Theorem 2 is not present:

Corollary 2. Positive assortative behavior in talent as described by Theorem 2 may fail under pairwise stability if \( \delta > 0 \) and the conditions from either Lemmas 2 or 3 are valid. The latter lemma requires only weak supermodularity, linear costs of linking and many agents.

The corollary above demonstrates that despite meeting the conditions of Becker (1973) these conditions are not sufficient for assortative matching in social networks with externalities.

5 Assortative behavior in degree

The question we investigate in this section is whether or not there is assortative behavior in degree, i.e. does more gregarious agents (agents with many partners), tend to associate with agents who are as gregarious as themselves? The simple answer is yes, under certain circumstances elaborated below.

No externalities We begin with an analysis of the setup where externalities are absent. This part of the analysis builds on the two previous sections’ results: firstly, there is a positive relationship between higher talent and (weakly) more partners (Theorem 1), and; secondly, partners are chosen assortative in talent (Theorem 2). By combining these two previously established results it can be shown there will also be degree assortativity. Although the intuition of combining the two previous results is straightforward additional structure needs to be imposed to show the result. Let complete heterogeneity be satisfied if for every agent has a unique level of talent, i.e. \( n = |X| \). The first result on degree assortative behavior is as follows:

\[11\] The coefficient of assortative mixing is equivalent to the Pearson measure of correlation between partner types.
Theorem 3. If there are no externalities, supermodularity and monotonicity in link value, as well as complete heterogeneity then it follows that for every pairwise stable network it holds that an agent with higher degree has partners which weakly dominate in degree when compared partner-by-partner with the partners of a lower talented agent:

$$\forall \mu \in M^{ps}, \forall i, j \in N, k_i(\mu) > k_j(\mu) : \ K(\nu_i(\mu)/\{j\})_l \geq K(\nu_j(\mu)/\{i\})_l, \quad l = 1, ..., k_j(\mu).$$

The above result is identical to Theorem 2 with the exception that the assortative property in this result is degree and this result has an added requirement. The emergence of this assortative pattern is not only a theoretical curiosity - this pattern is an underlying property observed across various empirical social networks, see the literature review.

The feature emerging degree assortativity in Theorem 3 is relevant as it indicates that positive degree assortativity may indicate people sort themselves according to ability.

Externalities In the remainder of this section we investigate the setting where externalities may be present. It has been shown respectively in Section 3 and 4 that when externalities arise also from indirect connections: the monotonic relation between talent and centrality may fail; assortative behavior may break down. As Theorem 3 relies crucially on these results it is clear that the theorem The result of this on assortative behavior in degree above relies crucially on these two failing results - thus it is possible to show that assortative behavior in degree may fail under the same conditions:

Corollary 3. Positive assortative behavior in degree as described by Theorem 3 may fail under pairwise stability if $\delta > 0$ and the conditions from either Lemmas 2 or 3 are valid. The latter lemma requires only weak supermodularity, linear costs of linking and many agents.

The above result implies that Theorem 3 should be interpreted with some caution as it may fail when there are externalities.

6 Sub-optimal networks

This section investigates two classes of network structures which are pairwise stable but inefficient.

Inefficiency from to excessive assortative linking The first class of inefficient networks has an excessive level of stratification. Excessive stratification refers to a setting where there is a group of high and low talented agents which are segregated. However, the two groups of high and low talent could collectively benefit from connecting but they fail to as incentives are not aligned under pairwise network formation. In order to demonstrate the idea we begin with an example below which is represented in Figure 2.

Example 3. Let there be six agents, i.e, $N = \{1, 2, ..., 6\}$. Assume there are two types of talent and three of the agents are of the high type: $x_1 = x_2 = x_3 = x$, and; $x_4 = x_5 = x_6 = \bar{x}$. Furthermore, suppose there is supermodularity and a degree quota of two.

In this example we use two main networks: a network of segregated cliques, $\mu = \{12, 13, 23\} \cup \{45, 46, 56\}$, see Figure 2A; the other network is the segregated network $\mu$ but where two agents of each type has linked across types e.g. $\tilde{\mu} = \{12, 13, 24, 35, 46, 56\}$, see Figure 2C. We show there exists a range $(\tilde{\delta}, \hat{\delta})$ where $\delta > 0$ and $\delta < 1$ such that for every $\delta \in (\tilde{\delta}, \hat{\delta})$ the network $\mu$ satisfies that: (i) the network $\mu$ is inefficient, and; (ii) when net transfers between all linked agents are zero then the network is pairwise stable.
In order to demonstrate pairwise stability of the network we can focus on the case where the agents share the gains of linking equally i.e. \( \tau_{ij} = 0 \) for all \( ij \in \mu \). This implies that all formed links have value and deleting a link always leads to a loss. Thus only coalitional moves where links are formed can be valuable. Under pairwise stability at most one link can be formed. As all links to same types are already formed only links across types are relevant. An example is agents 2, 4 forming a link together: \( \bar{\mu} = \mu \cup \{24\} \setminus \{23, 45\} = \hat{\mu} \setminus \{35\} \), see Figure 2B. For any two agents where \( x_i = \bar{x} \) and \( x_j = \bar{x} \):

\[
\begin{align*}
  u_i(\bar{\mu}) + u_j(\bar{\mu}) &= (1 + \delta) [z(\bar{x}, \bar{x}) + z(x, \bar{x})] + [1 + \delta + \delta^2] \cdot [z(\bar{x}, x) + z(x, \bar{x})], \\
  u_i(\mu) + u_j(\mu) &= 2 \cdot [z(\bar{x}, \bar{x}) + z(x, \bar{x})].
\end{align*}
\]

The condition for when segregating is pairwise stable, i.e. when \( u_i(\mu) + u_j(\mu) > u_i(\bar{\mu}) + u_j(\bar{\mu}) \), can be rewritten into the following expression: \( 1 + \delta + \delta^2 < (1 - \delta) \cdot \beta \).

The total value of the two networks \( \mu \) and \( \bar{\mu} \) is expressed below in the two equations:

\[
\begin{align*}
  U(\bar{\mu}) &= (2 + \delta) \cdot [Z(\bar{x}, \bar{x}) + Z(x, \bar{x})] + [2 + 4\delta + 3\delta^2] \cdot Z(\bar{x}, x) \\
  U(\mu) &= 3 \cdot [Z(\bar{x}, \bar{x}) + Z(x, \bar{x})]
\end{align*}
\]

It follows that connecting is efficient \((U(\mu) < U(\bar{\mu}))\) can be rewritten into: \(1 + 2\delta + \frac{3}{2}\delta^2 > (1 - \delta) \cdot \beta \).

Note the solution to the inequalities for pairwise stability inefficiency of \( \mu \) and pairwise stability of \( \bar{\mu} \) exist when there is supermodularity. The solution can be found in Proposition 3 below.

The example above demonstrates that assortative behavior can be inefficient when there are network effects. The inefficiency stems from a novel source - the pairwise formation of links. The intuition is that under pairwise deviation the two agents do not internalize the total value created for other agents number of indirect links between a high and a low agent. Below this example is re-articulated as a general result for a segregated network of two cliques. This result makes it possible to analytically derive an expression of the two threshold for inefficiency and of pairwise stability as a function of the modularity of \( Z \) and the number of agents, \( n \).

**Proposition 3.** Suppose there is supermodularity, a degree quota, \( \kappa \), two types \( \bar{x}, x \) such that there are exactly \( \kappa + 1 \) of each type then whenever \( M_{s,s}^{\kappa} \neq 0 \) there exists thresholds \( \delta, \bar{\delta} \) such that for any \( \delta \in (\delta, \bar{\delta}) \) the segregated network with no links between agents of the two types which is both pairwise stable and inefficient; the thresholds are:

\[
\begin{align*}
  \delta(\beta) &= \frac{-\left(\frac{\beta}{2} - 1 + \beta\right) + \sqrt{\left(\beta + \frac{\beta}{2} - 1\right)^2 + 2\beta - 1} \cdot \left(\frac{\beta}{2} - 2\right)^2 + 2}{\left(\frac{\beta}{2} - 2\right)^2 + 2}, \\
  \bar{\delta}(\beta) &= \frac{-\left(\frac{\beta}{2} - 2 + \beta\right) + \sqrt{\left(\beta + \frac{\beta}{2} - 2\right)^2 + 2\beta - 1}}{\left(\frac{\beta}{2} - 2\right)^2 + 2},
\end{align*}
\]

(A): Segregated network, \( \mu \). Pairwise stable if \( \delta \leq \bar{\delta}(\beta) \).
(B): Pairwise move by agent 2, 4 who form a link to network \( \bar{\mu} \).
(C): Connected network, \( \hat{\mu} \). Efficient if \( \delta \geq \delta(\beta) \).

**Figure 2:** The above situation depicts the three networks from Example 3.
In Figure 3 the two thresholds from Proposition 3, $\bar{\delta}(\beta), \delta(\beta)$. The plots are made for varying size of cliques where clique size is $\kappa + 1$. It can be seen that the scope for inefficiency, i.e. the gap between $\delta(\beta), \bar{\delta}(\beta)$, increases with the number of agents involved. This makes sense intuitively as the two agents forming the link will fail to account for an increasing number of indirect connections between the two groups. This number of indirect connections at an order of squared in the total number of agents.

The result in Proposition 3 is relevant when various organizations and institutions that sort individuals according to talent. This could be education systems (private and public), professional organizations (firms, government and NGO’s). The sorting only takes account the within group complementarity and not the externalities of benefits from linking across groups to other agents.

Note that although Proposition 3 is derived under the crucial assumption of two peer groups of size being $\frac{n}{2}$ then the existence of segregated network which is pairwise stable but inefficient should hold whenever the number of types in the two groups is greater than $\kappa + 1$.

**Inefficiency from excessive centrality of low talented agent** The other class of inefficient network is where an agent of lesser talent becomes excessive central in the network - the agent of less talent could be replaced by a more talented and collectively the agents would be better off. A sufficient condition for when this failure may occur is monotonicity and no modularity in link value - this corresponds to Proposition 1. Recall that this proposition shows that in all strong stable more talented agents are more decay-central at level $\delta$ as it holds for all efficient networks. This property of monotone decay centrality in talent is referred to as *meritocratic (network) centrality*.

Even if there is supermodularity (violating no modularity) then it may still be that meritocratic centrality has to hold in some circumstances. A setting where meritocratic centrality is necessary but violated is from Lemma 2, the $(n-1)$-star network where the least talented agent is the center. By replacing the least talented in this star network with any higher talented agent then it would increase overall value of the network. The replacement would be such that the two agents inherit the other agent’s links.

Once more Lemma 2 may be invoked to demonstrate the failure of meritocratic centrality. In Lemma 2 there is a failure of efficiency due to meritocratic centrality being satisfied but the less talented is more central than the more talented.

![Figure 3: Visualization of thresholds for connecting from Proposition 3 under variation of clique size.](image-url)
Corollary 4. Efficiency may fail under pairwise stability if \( \delta > 0 \) and the conditions from either Lemmas 2 or 3 are valid. The latter lemma requires only weak supermodularity, linear costs of linking and many agents.

The failure of meritocratic centrality entails that there is a market failure of sorts - the worst candidate among the group has been chosen to be the center. As mentioned earlier this result is analogue to the one in markets with (network) externalities as well as buyers and sellers in Farrell and Saloner (1985) and Katz and Shapiro (1985).

7 Conclusive discussion

The analysis of this paper has some caveats. One severe caveat with our analysis, and stable networks in general, is that these networks may not exist. One such instance is the "Roommate Problem" in Gale and Shapley (1962) which also applies to our setting. Another caveat from the analysis is that under some circumstances there are multiple stable pairwise networks and thus it is not possible to derive any general structure. Finally, the analysis is derived under very strict assumptions - these are assessed below.

A critical assumption is the lack of dynamics - this means that peer effects are confined to direct value. In reality peer effects work over time by disseminating information and exchanging favors. A hope is that this extension can be assessed in a subsequent analysis.

The assumption of binary link formation under mutual accept is critical but seems adequate for modeling friendships and partnerships. Non-binary links (i.e. weighted networks) should lead to many of the same conclusions on sorting and centrality. Modeling non-mutual acceptance can be done where benefits flow both ways (as in Bala and Goyal (2000)) or flows one way. The one-way flows would be akin to directed networks which could be used to describe following behavior on social media networks - the results of monotonic centrality should be robust to this specification but sorting only under additional assumptions.\(^ {12}\)

The assumptions on information structure is likewise critical. Agents know each others type, which is crucial but also an open question in matching. Another assumption is the absence of search frictions which implies all agents may choose to offer linking with any agent. Nevertheless, most results should be robust to introduction of search frictions cf. Shimer and Smith (2000).\(^ {13}\)

There is also a number of assumptions about payoffs. The most crucial assumption is that payoffs are separable for each link. The separability closes the model to a variety of possible peer effect specifications, e.g. utility from groups. Another stylized assumptions is the adaption of Jackson and Wolinsky (1996)'s "connection model" which required externalities from indirect connections to be discounted with geometric decay in network distance. Nevertheless by removing separability and allowing for non-geometric decay many of the results are likely to be robust although more difficult to establish. Two critical assumptions are supermodularity and monotonicity along with perfect transferability. Nevertheless, as mentioned in the introduction, these two assumptions can be replaced by monotonicity in individual link value and perfect non-transferability.\(^ {14}\) As noted in the description of methodology, the monotonicity of individual link value is consistent with research on peer effects. Finally the assumptions on cost of linking implied that effects of effort, such as in the coauthor model of Jackson and Wolinsky (1996) are absent. However, Baccara and Yariv\(^ {12}\) The results of monotonic centrality should hold in this setup too for incoming links.

\(^ {13}\)It should be possible to derive similar conditions to those shown in Shimer and Smith (2000) for the marriage market with random search.

\(^ {14}\)Or the generalized increasing in differences condition which allows for a setting of mix between transferable and non-transferable utility, cf. Legros and Newman (2007).
(2013) shows that for some classes of effort then assortative matching into groups can occur.

References

ANGRIST, J. (2013): “The Perils of Peer Effects,” Working Paper 19774, National Bureau of Economic Research.

AUMANN, R. (1959): “Acceptable points in general cooperative n-person games,” in Contributions to the Theory of Games, Volume IV, ed. by A. W. Tucker, and R. D. Luce, vol. 40 of Annals of Mathematics Studies, pp. 287–324. Princeton University Press.

BACCARA, M., AND L. YARIV (2013): “Homophily in Peer Groups,” American Economic Journal: Microeconomics, 5(3), 69–96.

BALA, V., AND S. GOYAL (2000): “A noncooperative model of network formation,” Econometrica, 68(5), 1181–1229.

BARABASI, A.-L., AND R. ALBERT (1999): “Emergence of Scaling in Random Networks,” Science, 286(5439), 509–512.

BECKER, G. (1973): “A Theory of Marriage: Part I,” The Journal of Political Economy, 81(4), 813–846.

BLOCH, F., AND M. O. JACKSON (2007): “The formation of networks with transfers among players,” Journal of Economic Theory, 133(1), 83–110.

CURRARINI, S., M. JACKSON, AND P. PIN (2009): “An economic model of friendship: Homophily, minorities, and segregation,” Econometrica, 77(4), 1003–1045.

DURLAUFS, S. N., AND A. SESHAIDI (2003): “Is assortative matching efficient?,” Economic Theory, 21(2-3), 475–493.

FALK, A., AND A. ICHINO (2006): “Clean Evidence on Peer Effects,” Journal of Labor Economics, 24(1), 39–57.

FARRELL, J., AND G. SALONER (1985): “Standardization, compatibility, and innovation,” The RAND Journal of Economics, pp. 70–83.  

——— (1986): “Installed Base and Compatibility: Innovation, Product Preannouncements, and Predation,” The American Economic Review, 76(5), 940–955.

FARRELL, J., AND S. SCOTCHMER (1988): “Partnerships,” The Quarterly Journal of Economics, 103(2), 279–297.

GALE, D., AND L. SHAPLEY (1962): “College admissions and the stability of marriage,” American Mathematical Monthly, 69(1), 9–15.

GALEOTTI, A., S. GOYAL, AND J. KAMPHORST (2006): “Network formation with heterogeneous players,” Games and Economic Behavior, 54(2), 353–372.

GOEREE, J. K., A. RIEDL, AND A. ULE (2009): “In search of stars: Network formation among heterogeneous agents,” Games and Economic Behavior, 67(2), 445–466.
Haller, H., and S. Sarangi (2005): “Nash networks with heterogeneous links,” *Mathematical Social Sciences*, 50(2), 181–201.

Hoefer, M., D. Vaz, and L. Wagner (2014): “Hedonic Coalition Formation in Networks,” in *Twenty-Ninth AAAI Conference on Artificial Intelligence*.

Jackson, M., and A. Wolinsky (1996): “A Strategic Model of Social and Economic Networks,” *Journal of Economic Theory*, 71(1), 44 – 74.

Katz, M. L., and C. Shapiro (1985): “Network Externalities, Competition, and Compatibility,” *The American Economic Review*, 75(3), 424–440.

Klaus, B., and M. Walzl (2009): “Stable many-to-many matchings with contracts,” *Journal of Mathematical Economics*, 45(7-8), 422–434.

Konig, M. D., C. J. Tessone, and Y. Zenou (2010): “From assortative to dissortative networks: the role of capacity constraints,” *Advances in Complex Systems*, 13(04), 483–499.

Kremer, M. (1993): “The O-Ring Theory of Economic Development,” *The Quarterly Journal of Economics*, 108(3), 551–575.

Legros, P., and A. F. Newman (2002): “Assortative Matching in a Non-Transferable World,” SSRN Scholarly Paper ID 328460, Social Science Research Network, Rochester, NY.

Legros, P., and A. F. Newman (2007): “Beauty is a beast, frog is a prince: Assortative matching with nontransferabilities,” *Econometrica*, 75(4), 1073–1102.

Manski, C. F. (1993): “Identification of Endogenous Social Effects: The Reflection Problem,” *The Review of Economic Studies*, 60(3), 531–542.

Marmaros, D., and B. Sacerdote (2006): “How Do Friendships Form?,” *The Quarterly Journal of Economics*, 121(1), 79–119.

McPherson, M., L. Smith-Lovin, and J. M. Cook (2001): “Birds of a Feather: Homophily in Social Networks,” *Annual Review of Sociology*, 27(1), 415–444.

Miritello, G., R. Lara, and E. Moro (2013): “Time Allocation in Social Networks: Correlation Between Social Structure and Human Communication Dynamics,” in *Temporal Networks*, ed. by P. Holme, and J. Saramaki, Understanding Complex Systems, pp. 175–190. Springer Berlin Heidelberg.

Newman, M. (2002): “Assortative mixing in networks,” *Physical Review Letters*, 89(20), 208701.

——— (2003): “Mixing patterns in networks,” *Physical Review E*, 67(2), 026126.

Pastor-Satorras, R., A. Vazquez, and A. Vespignani (2001): “Dynamical and Correlation Properties of the Internet,” *Physical Review Letters*, 87(25), 258701.

Pycia, M. (2012): “Stability and preference alignment in matching and coalition formation,” *Econometrica*, 80(1), 323–362.

Sacerdote, B. (2001): “Peer Effects with Random Assignment: Results for Dartmouth Roommates,” *The Quarterly Journal of Economics*, 116(2), 681–704.
Shalizi, C. R., and A. C. Thomas (2011): “Homophily and Contagion Are Generically Confounded in Observational Social Network Studies,” *Sociological Methods & Research*, 40(2), 211–239.

Shimer, R., and L. Smith (2000): “Assortative Matching and Search,” *Econometrica*, 68(2), 343–369.

Ugander, J., B. Karrer, L. Backstrom, and C. Marlow (2011): “The Anatomy of the Facebook Social Graph,” *pre-print*, Arxiv preprint arXiv:1111.4503.

Zimmer, R., and E. Toma (2000): “Peer effects in private and public schools across countries,” *Journal of Policy Analysis and Management*, 19(1), 75–92.
A  No externalities - an auxiliary result

Lemma 1. In the absence of no network externalities then the set of strongly stable networks is equivalent to the set of pairwise stable networks, i.e., $M^{ps}_{\delta=0} = M^{st}_{\delta=0}$.

Proof. By definition it holds that $M^{ps}_{\delta=0} \subseteq M^{st}_{\delta=0}$, thus we need to show that $M^{ps}_{\delta=0} \subseteq M^{st}_{\delta=0}$ to prove the claim. This claim is shown using similar to arguments to Klaus and Walzl (2009)’s Theorem 3.i.

Let $\mu$ with associated contracts $\tau$ be a network which is blocked by a coalition. It will be shown that for every coalition $\tau \in T$ that blocks, within the coalition there is a subset of no more than two members that also wishes to block the network. Let $\tilde{\mu}$ be the alternative network that the blocking coalitions implements through a feasible coalition move and $\tau$ be the transfers associated with $\tilde{\mu}$.

It is always possible to partition the set of deleted links $\mu \setminus \tilde{\mu}$ into two: (i) a subset denoted $\tilde{\mu}$ where for each link $ij$ that can be deleted where one of the two partners can benefit, i.e. it holds that either $z_{ij} + \tau_{ij} - [c_i(\mu) - c_i(\mu \setminus ij)] < 0$ or $z_{ji} + \tau_{ji} - [c_j(\mu) - c_i(\mu \setminus ij)] < 0$; (ii) a subset denoted $\tilde{\mu}$ where for each link $ij$ neither of the previous two inequalities are satisfied.

Suppose that the first partition is non-empty, i.e. $\tilde{\mu} \neq \emptyset$. However, as deleting links can be done by a single agent on its own then the move only takes needs the coalition of that agent to delete the link. Thus any part of a coalitional move that only involves profitably removing links can be performed in parts by a coalition with a single agent - therefore this move is also a pairwise block.

Thus it remains to be shown that the remaining part of coalitional move also can be performed as a pairwise block, i.e. when forming $\tilde{\mu} \setminus \mu$ and deleting $\tilde{\mu}$. This part of the coalitional move must entail forming links as no links can be deleted profitably. The set of formed links $\tilde{\mu} \setminus \mu$ can be partitioned into a number of $[\tilde{\mu} \setminus \mu]$ feasible submoves of adding a single link while deleting links by each of the agents $i$ and $j$ who form a link. The feasibility for each of the partitioned moves is always true when there is a cost function as moves are unrestricted. It is now argued that each of the partitioned moves are feasible when there is a degree quota. If the network $\mu \cup ij$ is feasible then the move of simply adding the link is feasible. If $\mu \cup ij$ is not feasible, then agents $i$ and $j$ can delete at most one link each and if both $\mu$ and $\tilde{\mu}$ are feasible then this also feasible as the degree quota is kept.

For the coalitional move to $\tilde{\mu}$ it must be that at least at least one link among the implemented links $\tilde{\mu} \setminus \mu$ has a strictly positive value that exceeds the loss from deleting at most one link for each of two agents forming the link. This follows as it is known that deleting one or more links cannot add any value and thus must have weakly negative value and that by definition the total value to the blocking coalition must be positive. As every one of the partitioned moves is feasible, it follows that for every coalitional move there are two agents who can form link while potentially destroying current links and both be better off. In other words, for every coalition that blocks, there is a pairwise coalition that blocks.

B  Proofs and auxiliary results for monotonic centrality

Theorem 1: Suppose there are no externalities and monotonicity in link value then for every pairwise stable network it holds that an agent’s degree (weakly) monotonically increasing in the agent’s type in the network:

$$\forall \mu \in M^{ps}: \ x_i > x_j \Rightarrow k_i(\mu) \geq k_j(\mu).$$

Proof. Suppose the claim is false; that is, for some pairwise stable network $\mu \in M^{ps}$ it holds for two agents $i$ and $j$ that $x_i > x_j$ but $k_i(\mu) < k_j(\mu)$. The condition that $k_i(\mu) < k_j(\mu)$ entails there is another agent in $j$’s network which is neither in $i$’s network nor is agent $i$, i.e. $\nu_j(\mu) \setminus (\nu_i(\mu) \cup \{i\}) \neq \emptyset$.

Let $j' \in \nu_j(\mu) \setminus (\nu_i(\mu) \cup \{i\})$. From monotonicity in link value it holds that $Z_{ij'} > Z_{jj'}$ as $x_i > x_j$. Moreover, from the cost technologies (either convex or a degree quota) it must be that
\[
\begin{align*}
c_i(\mu \cup i'j') - c_i(\mu) \leq c_j(\mu) - c_j(\mu \setminus j') & \text{ as } k_i(\mu) < k_j(\mu). \text{ These two facts from benefits and costs entail} \\
& \text{that the value created by forming } i'j' \text{ and deleting } j' \text{ will be equivalent to:} \\
Z_{ij'} - (c_i(\mu \cup i'j') - c_i(\mu)) > Z_{jj'} - (c_j(\mu) - c_j(\mu \setminus j')) 
\end{align*}
\]

As the move to \((\mu \cup i'j') \setminus jj'\) is feasible (and respects the degree quota if there is one), it follows that strong stability is violated as it implies that \(\mu\) is not efficient. From Lemma \[1\] it follows that pairwise stability is also violated if the claim is false.

**Proposition** Suppose there are externalities as well as monotonicity and no modularity in link value then for every pairwise stable network it holds that an agent’s \(\delta\)-decay centrality monotonically increasing in the agent’s type in the network:

\[
\forall \mu \in M^{ss} : x_i > x_j \Rightarrow d_i^\delta(\mu) \geq d_j^\delta(\mu).
\]

Suppose there is monotonicity and no modularity in link value then for every \(\mu \in M^{ss}\) it holds that the decay centrality of an agent \(i\) in \(\mu\), \(d_i^\delta(\mu)\), is (weakly) monotone increasing in type:

\[
\forall \mu \in M^{ss} : x_i > x_j \Rightarrow d_i^\delta(\mu) \geq d_j^\delta(\mu).
\]

**Proof.** The no modularity condition implies that individual link value is independent of own talent and thus separable for any \(\hat{x} \in X\): \(Z(\hat{x}, \hat{x}) = \hat{Z}(\hat{x}) + \hat{Z}(\hat{x})\) where \(\hat{Z}\) is the contribution to the link value for a given level of talent. This entails that we can rewrite the aggregate benefits using Equation 2

\[
\sum_{i \in N} b_i(\mu) = \sum_{i \in N} \sum_{j \neq i} \delta_{ij}^{\mu}(\mu)^{-1} z_{i,j} = \sum_{i \in N} \sum_{j \neq i} \delta_{ij}^{\mu}(\mu)^{-1} Z_{i,j} = \sum_{i \in N} \sum_{j \neq i} \delta_{ij}^{\mu}(\mu)^{-1} \hat{Z}(x_i) = \sum_{i \in N} d_i^\delta(\mu) \cdot \hat{Z}(x_i)
\]

Due to monotonicity in link value it also holds that \(\frac{\partial}{\partial x_i} Z(x_i, x_j') = \frac{\partial}{\partial x_i} \hat{Z}(x_j') > 0\). This entails that a necessary condition for the sum of utilities to be maximal by some network \(\mu\) is that for any two agents \(i', j'\) such that \(x_{i'} > x_{j'}\) it holds that \(d_{i'}^\delta(\mu) > d_{j'}^\delta(\mu)\). The necessity is demonstrated in the following.

Denote an alternative network \(\hat{\mu}\) where \(i'\) and \(j'\) have switched positions: if \(i'\) and \(j'\) are not linked in \(\mu\) then let \(\nu_i(\hat{\mu}) = \nu_{i'}(\mu)\) and \(\nu_{i'}(\hat{\mu}) = \nu_i(\mu)\); else if \(i'\) and \(j'\) are linked in \(\mu\) then let \(\nu_i(\hat{\mu}) = \nu_{i'}(\mu) \cup \{i'/i\}\) and \(\nu_{i'}(\hat{\mu}) = \nu_i(\mu) \cup \{i'/i\}\). Thus it holds that \(d_{i'}^\delta(\hat{\mu}) = d_{i'}^\delta(\mu)\) and \(d_{j'}^\delta(\hat{\mu}) = d_{j'}^\delta(\mu)\). A deviation from \(\mu\) to \(\hat{\mu}\) is possible for the grand coalition.

We will show that the alternative network \(\hat{\mu}\) will generate higher aggregate utility which violates efficiency of \(\mu\) and thus the strong stability. Starting with costs there are two cases of cost technology: when there is quota in links then a deviation to the alternative network \(\hat{\mu}\) is consistent with the degree quota\[13\] and has unchanged costs; when there are convex costs then the move will have no change in aggregate costs as the sum of costs for agents \(i\) and \(j\) is unchanged. However, the benefits will be higher under \(\hat{\mu}\), using that \(i\) and \(i\) switch neighborhoods:

\[
\sum_{i \in N} b_i(\hat{\mu}) > \sum_{i \in N} b_i(\mu) \\
\sum_{i \in N} d_i^\delta(\hat{\mu}) \cdot \hat{Z}(x_i) + d_j^\delta(\hat{\mu}) \cdot \hat{Z}(x_j) > d_i^\delta(\mu) \cdot \hat{Z}(x_i) + d_j^\delta(\mu) \cdot \hat{Z}(x_j) \\
\sum_{i \in N} d_i^\delta(\mu) \cdot [\hat{Z}(x_i) - \hat{Z}(x_j)] > d_i^\delta(\mu) \cdot [\hat{Z}(x_i) - \hat{Z}(x_j)] \\
\sum_{i \in N} d_i^\delta(\mu) > \sum_{i \in N} d_i^\delta(\mu)
\]
Inequality 11 is satisfied; thus it suffices to check Inequalities 7, 9 and 10. Expressions are Inequalities 7, 9, 10 and 11. For large enough restriction ensures the minimal requirements for the transfer and net utility for agent 1 are met linear costs (\(\tilde{c}\) of forming a link) and \(\delta\) being sufficiently high along with the two inequalities:

\[
2 \cdot Z(\bar{x}, x) + Z(\bar{x}, \bar{x}) \geq 4\tilde{c} \quad \text{and} \quad Z(\bar{x}, x) > \tilde{c}.
\]

**Proof.** Linear cost entails \(\Delta c(1) = c(1)\) and \(c(2) = 2c(1)\). Note that the total cost of establishing a link is \(2\tilde{c}\). Throughout we assume that \(s_2 = s_3\) and \(\tau_{23} = 0\) which implies \(\tau_{12} = \tau_{13}\). To shorten notation let \(s_i = s_i(\mu, \tau)\).

We begin with deriving the required transfers and net utility for agent 1 - the relevant Inequalities are \(7, 8, 9\). The two inequalities (for \(i = 2, 3\)) in Inequality \(8\) can be rewritten using that \(\tau_{12} = \tau_{13}\):

\[
\tau_{12} \geq \tilde{c} - z(x, \bar{x})
\]

\[
\tau_{12} + \tau_{13} \geq 2\tilde{c} - 2z(x, \bar{x})
\]

\[
2\tilde{c} - 2z(x, \bar{x}) + s_1 \geq 2\tilde{c} - 2z(x, \bar{x})
\]

\[
s_1 \geq 0
\]

(14)

Thus Inequality \(8\) is irrelevant as it is always satisfied when \(\tau_{12} = \tau_{13}\) and Inequality \(7\) (\(s_1 \geq 0\)) is satisfied.

By adding the two inequalities in Inequality \(9\) where \(i = 1\) and \(j = 2, 3\) together and subtracting Equation 5 the following must hold for the net utility of agent 1:

\[
s_1 \geq 2Z(\bar{x}, x) - 4\tilde{c} - [\delta Z(\bar{x}, \bar{x}) + 2Z(\bar{x}, x) - 4\tilde{c}]
\]

\[
s_1 \geq -\delta Z(\bar{x}, \bar{x})
\]

(15)

We see that Inequality \(15\) is satisfied when Inequality \(7\) holds (\(s_1 \geq 0\)). Note that two inequalities in Inequality \(9\) where \(i = 1\) and \(j = 2, 3\) must still hold.

We now turn to derive to check relevant transfer and net utilities for agent 2 and 3. The relevant expressions are Inequalities \(7, 9, 10\) and \(11\). For large enough \(\delta\) (i.e. \(\delta \to 1\)) then it always holds that Inequality \(11\) is satisfied; thus it suffices to check Inequalities \(7, 9\) and \(10\).

We can use a similar same procedure to see what implications the inequalities have for the net utility of agent \(i \in \{2, 3\}\). Let \(j = \{2, 3\}\backslash\{i\}\). By adding the two inequalities from Inequality \(9\) for the pair 1, \(i\) and the pair \(i, j\), together and subtracting Equation 5 it must hold that:

\[
\min\{s_2, s_3\} \geq Z(\bar{x}, x) + Z(\bar{x}, \bar{x}) - 4\tilde{c} - [\delta Z(\bar{x}, \bar{x}) + 2Z(\bar{x}, x) - 4\tilde{c}]
\]

\[
\min\{s_2, s_3\} \geq (1 - \delta)Z(\bar{x}, \bar{x}) - Z(\bar{x}, x)
\]

(16)

For sufficiently large \(\delta\) Inequality \(16\) becomes irrelevant as in the limit of \(\delta \to 1\) its requirement is \(\min\{s_2, s_3\} \geq -Z(\bar{x}, x)\); this inequality is always satisfied when \(\min\{s_2, s_3\} \geq 0\). Note that it still remains to check all of the Inequality \(9\)’s three inequalities.

In addition, it is possible to rewrite Inequality \(10\) in to Inequality \(17\). Note that Inequality \(17\) is a sufficient condition for Inequality \(17\) for both agent 2 and 3.

\[
s_2 + s_3 \geq Z(\bar{x}, x) + (1 + \delta)z(\bar{x}, x) + \max\{\tau_{21}, \tau_{31}\} - 3\tilde{c}
\]

\[
s_2 + s_3 \geq Z(\bar{x}, x) + (1 + \delta)z(\bar{x}, x) + \max\{s_2, s_3\} - \delta z(\bar{x}, x) - z(\bar{x}, x) - 2\tilde{c}
\]

\[
\min\{s_2, s_3\} \geq (2 - \delta)z(\bar{x}, x) + \delta z(\bar{x}, x) - 2\tilde{c}
\]

(17)

In the remainder of this proof we restrict that \(s_1 = 0\) and thus \(\tau_{21} = \tau_{31} = z(\bar{x}, x) - \tilde{c}\). This restriction ensures the minimal requirements for the transfer and net utility for agent 1 are met (see Inequalities \(7, 8, 9\)). This restriction allows me to assess when the remaining requirements are
satisfied for agents 2 and 3 given that the requirements for agent 1 are minimally satisfied. When \( s_1 = 0 \) then Inequality \( \text{9} \) is satisfied if: \( s_1 = 0 \)

\[
\min\{s_2, s_3\} \geq Z(\bar{x}, x) - 2\bar{c} \quad (18)
\]

It is possible a sufficient condition for the remaining relevant inequalities being satisfied when \( s_2 = s_3 \) and \( s_1 = 0 \). Recall that it is necessary to check Inequalities \( \text{7} \), \( \text{9} \) and \( \text{10} \) for large enough \( \delta \). The sufficient condition for the inequalities is captured in Inequality \( \text{19} \) below. The elements in the set from which the maximal element is chosen derived from respectively: Inequality \( \text{7} \), Inequality \( \text{9} \) for the pair 2,3; Inequality \( \text{11} \) which implies Inequality \( \text{9} \) for the pair 1,2 and the pair 1,3; Inequality \( \text{17} \) which implies Inequality \( \text{10} \) is satisfied.

\[
s_2 + s_3 \geq \max\{0, Z(\bar{x}, \bar{x}) - 2\bar{c}, 2Z(\bar{x}, x) - 4\bar{c}, 2(2 - \delta)z(\bar{x}, \bar{x}) + 2\delta z(\bar{x}, x) - 4\bar{c}\} \quad (19)
\]

Using that \( s_1 = 0 \) it holds that \( s_2 + s_3 = s_1 + s_2 + s_3 \). This can be combined by with substituting Equation \( \text{5} \) into Inequality \( \text{19} \). This implies the following four Inequalities - each inequality correspond to an \( n \)’th element in the set from which the maximal element is chosen.

\[
2Z(\bar{x}, x) + \delta Z(\bar{x}, x) - 4\bar{c} \geq 0 \quad (20)
\]

\[
2Z(\bar{x}, x) + \delta Z(\bar{x}, x) - 4\bar{c} \geq 2Z(\bar{x}, x) - 4\bar{c} \quad (21)
\]

\[
2Z(\bar{x}, x) + \delta Z(\bar{x}, x) - 4\bar{c} \geq 2(2 - \delta)z(\bar{x}, \bar{x}) + 2\delta z(\bar{x}, x) - 4\bar{c} \quad (22)
\]

\[
2Z(\bar{x}, x) + \delta Z(\bar{x}, x) - 4\bar{c} \geq Z(\bar{x}, \bar{x}) - 2\bar{c} \\
2Z(\bar{x}, x) \geq (1 - \delta)Z(\bar{x}, \bar{x}) + 2\bar{c} \quad (23)
\]

Of the above we see that Inequality \( \text{21} \) always holds while Inequality \( \text{22} \) holds when \( \delta \) is large enough. Thus all the above inequalities can be satisfied for large enough \( \delta \) when \( 2Z(\bar{x}, x) + \delta Z(\bar{x}, \bar{x}) \geq 4\bar{c} \) and \( Z(\bar{x}, x) > \bar{c} \).

**Condition 1.**

For \( s_1, s_2, \ldots, s_n \) it is required that the following equation holds:

\[
\Sigma_{i \in N} s_i = U(\mu) = \Sigma_{i \in \{2, \ldots, n\}} Z_{i1} + \Sigma_{i, j \in \{2, \ldots, n\}, i \neq j} \delta Z_{ij} - (n - 1) \cdot c(1) - c(n - 1),
\]

along with the following three set of inequalities,

\[
\forall i \in N : \quad s_i \geq 0, \quad (25)
\]

\[
\forall i \neq 1 : \quad \tau_{i1} \geq \Delta c(n - 2) - z_{1i}, \quad (26)
\]

\[
\forall i, j \in N, i \neq j : \quad s_i + s_j \geq Z_{ij} - 2 \cdot c(1), \quad (27)
\]

and finally the following two set of inequalities must hold for any two \( i, j \in (N \setminus \{1\}), i \neq j \):

\[
s_i + s_j \geq Z_{ij} + z_{1i} + \delta z_{1j} + \delta \Sigma_{l \notin \{1, i, j\}} [z_{il} + \delta z_{lj}] - c(1) - c(2) + \tau_{i1}, \quad (28)
\]

\[
2 \cdot \Delta c(2)x \geq (1 - \delta) \cdot Z_{ij}. \quad (29)
\]
In what follows we demonstrate two auxiliary results: Lemma 2 and Lemma 3. These two results are essential for demonstrating Corollaries 1, 2, 3 and 4. The first auxiliary result rely on Condition 1 above.

**Lemma 2.** Suppose there are at least three agents \((n \geq 3)\), weak supermodularity and monotonicity, a cost function and an interval \([\delta, \hat{\delta}]\) where Condition 1 is satisfied, then for every \(\delta \in [\delta, \hat{\delta}]\) the network \(\mu\) where a least talented agent is the center of a \((n-1)\)-star network \((i.e. \ \mu = \{12, 13, \ldots, 1n\})\) is pairwise stable.

**Proof.** Let \(\mu = \{12, 13, \ldots, 1n\}\) which is a network where the least talented is a center of a \((n-1)\)-star network. The aim is to show that a \((n-1)\)-star network with the least talented as center may be pairwise stable in the following setting: there is weak supermodularity and monotonicity, there are at least three agents \((n \geq 3)\) along with Condition 1.

First we check for deviations where an agent deletes all its links in \(\mu\). This move cannot be profitable when Inequality 25 is satisfied for all individuals. This implies for any agent but agent 1 it is not profitable to delete their single link in \(\mu\). Furthermore for agent 1 deleting any link is not profitable due to due to Inequality 26 being satisfied. But this also implies that deleting any number of links for agent 1 is not beneficial as they are all beneficial at the highest marginal cost.

In addition any two agents can at least secure the value of the link that they can form independently of other links which is ensured by Inequality 27.

Finally agents apart from agent 1 can form a link between themselves while keeping one or both of their links to agent 1. The network \(\mu\) is robust to deviations where one agent deletes a link with 1 but forms a new to another agent that keeps its link with agent 1 if Inequality 28 is satisfied. The network \(\mu\) is robust to deviations where both agents keep their current link with agent 1 and form a new link together if Inequality 29 is satisfied.

**Lemma 3.** For Lemma 2 the requirement of Condition 1 being satisfied may be replaced with linear costs of linking and sufficiently many agents. For large number of agents it holds that:

\[
\begin{align*}
s_i(\mu, \tau) & \geq (1 - \delta^2)Z_{in} - (\delta - \delta^2)z_{ni} + \delta z_{i1} + \Sigma_{i \in \{1, n\}} \delta^2 z_{il} - 2\hat{c}, \quad i \neq 1 \\
s_1(\mu, \tau) & \in [0, (1 - \delta)U(\mu) + \Sigma_{i \in \{2, \ldots, n\}} (\delta - \delta^2)z_{ni} - (1 - \delta^2)Z_{in} + \delta z_{i1}]
\end{align*}
\]

where \(\mu = \{12, \ldots, 1n\}\) and \(\tau\) is consistent with the inequalities above.

**Proof.** Let the setting be as specified by Lemma 2 with the following cost technology. The costs are linear such that each link has some cost, i.e., for any degree \(k \in \mathbb{N}_0\); \(\Delta c(k) = \hat{c}\) which is cost of establishing a link - thus the total costs are \(c(k) = k \cdot \hat{c}\). Note that the total cost of establishing a link then is \(2\hat{c}\) as two agents are required.

Due to weak supermodularity and monotonicity it holds that there is an lower and upper bound on link value for any \(i, j : Z(\bar{x}, \bar{x}) \geq Z_{ij} \geq Z(x, x)\). We will use these two properties along with the condition of many agents to show there exist some threshold where the \((n-1)\)-star network with 1 is pairwise stable.

We can use a procedure similar to the one used in Example 2 to find the net utility of agent \(i \in N\). By adding the \((n-1)\) inequalities from Inequality 27 for \(i\) with all other agents (i.e. in \(N\{i\}\)) together and subtracting Equation 24 the expression in Inequality 30 must hold:

\[
\begin{align*}
\min_{i \in \{2, \ldots, n\}} s_i & \geq Z(\bar{x}, x) + (n - 2)Z(\bar{x}, \bar{x}) - [(n - 1) + \delta(n - 2)^2] \cdot Z(x, x)
\end{align*}
\]

\[\text{Inequality 30 uses the upper bound on the value for the links for agent } i \text{ with others (from Inequality 27) and the lower bound on the total value of the network (from Equation 24). Note that costs in the } n-1 \text{ inequalities are equal to those from Equation 21 - so these have zero net effect.}\]
For $\delta > 0$ it holds that for sufficiently large $n$ then Inequality 31 becomes irrelevant as for large $n$ the right hand side take values below zero for all $i \in N$.

In addition, for any two agents $i, j \neq 1$ it is possible to rewrite Inequality 28 in to the following:

$$s_i + s_j \geq Z_{ij} + z_{i1} + \delta z_{i1} + \Sigma_{i \notin \{1, i, j\}} [\delta z_{ij} + \delta^2 z_{i'1}] - 3\tilde{c} + \tau_{j1}$$

$$s_i + s_j \geq Z_{ij} + \delta z_{i1} + \Sigma_{i \notin \{1, i, j\}} \delta^2 z_{i'1} - 2\tilde{c} - \delta z_{ji} + s_j$$

$$s_i \geq z_{ij} + (1 - \delta) z_{ji} + \delta z_{i1} + \Sigma_{i \notin \{1, i, j\}} \delta^2 z_{i'1} - 2\tilde{c}$$

(31)

The maximum of Inequality 31 for agent $i$ is:

$$s_i \geq \max_{j \notin \{1, i\}} \left[ z_{ij} + (1 - \delta) z_{ji} + \delta z_{i1} + \Sigma_{i \notin \{1, i, j\}} \delta^2 z_{i'1} - 2\tilde{c} \right]$$

$$s_i \geq \max_{j \notin \{1, i\}} \left[ (1 - \delta^2) z_{ij} + (1 - \delta) z_{ji} + \delta z_{i1} + \Sigma_{i \notin \{1, i, j\}} \delta^2 z_{i'1} - 2\tilde{c} \right]$$

$$s_i \geq \max_{j \notin \{1, i\}} \left[ (1 - \delta^2) Z_{ij} - (\delta^2 - \delta^2) z_{ji} + \delta z_{i1} + \Sigma_{i \notin \{1, i, j\}} \delta^2 z_{i'1} - 2\tilde{c} \right]$$

(32)

A sufficient condition for Inequality 32 is that (as $x_n = \bar{x}$ and there is weak supermodularity and monotonicity):

$$s_i \geq (1 - \delta^2) Z_{in} - (\delta - \delta^2) z_{ni} + \delta z_{i1} + \Sigma_{i \notin \{1, i, j\}} \delta^2 z_{i'1} - 2\tilde{c}$$

(33)

For any $\delta > 0$ it holds that as $n \to \infty$ then $\Sigma_{i \notin \{1, i\}} \delta^2 z_{i'1} \to \infty$ for any $i \in \{2, ..., n\}$. Thus, it follows that as $n \to \infty$ the only relevant condition among the above is Inequality 33 as the sum term will dominate all other. This implies that for a large number of agents the requirement on net for agents excluding agent 1 are:

$$\Sigma_{i \in \{2, ..., n\}} s_i \geq \Sigma_{i \in \{2, ..., n\}} [(1 - \delta^2) Z_{in} - (\delta - \delta^2) z_{ni} + \delta z_{i1}] + \Sigma_{i,j \in \{2, ..., n\}, i > j} \delta^2 Z_{ij} - (2n - 2)\tilde{c}$$

(34)

We can check when Inequality 34 is possible to with non-waste-fullness from Equation 24

$$s_1 = \Sigma_{i \in \{1, n\}} s_i - \Sigma_{i \in \{2, ..., n\}} s_i$$

$$\leq \Sigma_{i \in \{2, ..., n\}} Z_{i1} + \Sigma_{i,j \in \{2, ..., n\}, i > j} \delta Z_{ij} - (n - 1) \cdot c(1) - c(n - 1)$$

$$- [\Sigma_{i \in \{2, ..., n\}} [(1 - \delta^2) Z_{in} - (\delta - \delta^2) z_{ni} + \delta z_{i1}] + \Sigma_{i,j \in \{2, ..., n\}, i > j} \delta^2 Z_{ij} - (2n - 2)\tilde{c}]$$

$$= \Sigma_{i \in \{2, ..., n\}} [Z_{i1} + (\delta - \delta^2) z_{ni} - (1 - \delta^2) Z_{in} - \delta z_{i1}] + \Sigma_{i,j \in \{2, ..., n\}, i > j} (\delta - \delta^2) Z_{ij}$$

$$= \Sigma_{i \in \{2, ..., n\}} [Z_{i1} + (\delta^2 - \delta^2) z_{ni} - (1 - \delta^2) Z_{in} + \delta z_{i1}] + \Sigma_{i,j \in \{2, ..., n\}, i > j} (\delta - \delta^2) Z_{ij}$$

$$= (1 - \delta) U(\mu) + \Sigma_{i \in \{2, ..., n\}} [(\delta - \delta^2) z_{ni} - (1 - \delta^2) Z_{in} + \delta z_{i1}]$$

The $n-1$ inequalities represented by Inequality 26 for agents 2, ..., $n$ will hold if $\tau_{ii} = \tilde{c} - z(x, x_i)$ and their aggregate inequality holds (as setting $\tau_{ii} = \tilde{c} - z(x, x_i)$ will maximize the net utility for agent $i$ subject to Inequality 26 holding):

$$\Sigma_{i \neq 1} \tau_{ii} = (n - 1)\tilde{c} - \Sigma_{i \neq 1} z(x, x_i)$$

(35)

$$\updownarrow$$

$$s_1 = 0$$

(36)

By adding the $n-1$ inequalities in Inequality 27 for 1 with 2, 3, ..., $n$ together and subtracting Equation 5 the following must hold for the net utility of agent 1:

$$s_1 \geq -\Sigma_{i,j \in \{2, ..., n\}, i > j} \delta Z_{ij}$$

(37)

We see the condition in Inequality 25 is always stronger than that in Inequality 37, thus Inequality 37 is irrelevant and the only condition for agent 1 is that $s_1 \geq 0$. 

\Box

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C Proofs for assortative behavior in talent

**Theorem 2.** If there is monotonicity, supermodularity and no externalities then it follows that for every pairwise stable network it holds that a higher talented agent has partners which weakly dominate in talent when compared partner-by-partner with the partners of a lower talented agent:

$$\forall \mu \in M^{ps}, \forall i, j \in N, x_i > x_j : \mathcal{X}(\nu_i(\mu)/\{j\})_l \geq \mathcal{X}(\nu_j(\mu)/\{i\})_l, \quad l = 1, \ldots, k_j(\mu).$$

**Proof.** Suppose the claim is false. Let $l'$ be the lowest index for which the condition fail such that for all $l < l'$ it holds that $\mathcal{X}(\nu_i(\mu)/\{j\})_l \geq \mathcal{X}(\nu_j(\mu)/\{i\})_l$ is valid. This implies there are two distinct agents $i', j'$ who have the following properties.

For agent $j'$ it holds that $x_{j'} = \mathcal{X}(\nu_j(\mu))_{l'}$ and $j' \notin \nu_j(\mu) \cup \{i\}$. The argument for why $l' \notin \nu_j(\mu) \cup \{j\}$ is that the number of agents in $\nu_j(\mu)$ with talent $\mathcal{X}(\nu_j(\mu))_{l'}$ or lower is higher agent $i$ than for agent $j$. This follows as the number of agents with talent $\mathcal{X}(\nu_j(\mu))_{l'}$ or lower is $k_i(\mu) - (l' - 1)$ for agent $i$ which exceeds corresponding number of agents in for agent $j$ which is $k_j(\mu) - (l' - 1) - 1$ as from Theorem 1 it is known that $k_i(\mu) \geq k_j(\mu)$ due to $x_i > x_j$.

Thus it holds that $x_i > x_{j'} > x_{j''}$ along with $ij', j'' \notin \mu$. However, this fact implies that there is a violation of strong stability. This follows as agents $i, i', j', j''$ can deviate by destroying $\{ij', j''\}$ and forming $\{ij, j''\}$ and thus increase payoffs due to supermodularity (cf. Equation 3). From Lemma 1 it follows that pairwise stability is also violated if strong stability is violated.

**Proposition 2.** If there is supermodularity, a degree quota where $\delta < (\kappa - 1)^{-1}$ and many agents then in every strongly stable network any agent links with agents of the same type almost surely:

$$\lim_{n \to \infty} \mathbb{P}[x_i = x_j | ij \in \mu_n] = 1, \quad \mu_n \in M^{ss}_n.$$

**Proof.** For the case of no externalities (i.e. $\delta=0$) see the proof for Remark 2 below. In the remainder it is assumed that there are externalities $\delta \in (0, (\kappa - 1)^{-1})$.

It is assumed that the asymptotic variance of $x_i$ is strictly positive. This implies that there is a subset of types, $\tilde{X} \subseteq X$, where $|\tilde{X}| \geq 2$ and for every type $x \in \tilde{X}$ it holds that there is an asymptotic strictly share of the total number of agents of that type, i.e., $\lim_{n \to \infty} (|\{i \in N_n, x_i = x\}|/n) > 0$. This follows as the number of types is finite $|X|$ is finite. Thus it is sufficient to simply analyze the network of agents who are of type $\tilde{X}$. The argument is that for asymptotic infinite agents the measure of types not in $\tilde{X}$ will have measure zero as no agent can have more than $\kappa$ links.

Each agent will almost surely have $\kappa$ links as it is assumed that each link adds positive value and there are asymptotic infinite agents (only a finite number can then not fulfill the degree quota).

Let the highest talented type with asymptotic support of be denoted $x = \max \tilde{X}$. Let the second highest talented type in $\tilde{X}$ be denoted $\tilde{x}$. Denote a given pair of agents of type $x$ as $i, i'$. Fort his pair it is possible to compute: (i) an upper bound for agent $i$ and $i'$ of linking with two other agents of different types, and; (ii) a lower bound of forming a link between agent $i$ and $i'$.

For deriving the lower bound of forming a link between agent $i$ and $i'$ we may use there are infinitely many agents of type $x$. As such it is possible that the neighbors of agent $i$ and agent $i'$ and all their neighbors and neighbors’ neighbors etc. are linked with new agents of type $x$. Moreover, as there are infinitely many agents it is possible to assume that agent $i$ and any of agent $i$’s neighbors and neighbors’ neighbors etc. have the shortest path to $i'$ and any of agent $i$’s neighbors and neighbors’ neighbors via the link that agent $i$ and $i'$ forms. This implies that value of the link is equivalent to the sum of:

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(i) The value of the link itself \( Z(x, x) \).

(ii) The value for \( i \) and \( i' \) connecting with each others’ neighbors and neighbors’ neighbors etc. At each step \( l \) there are \((\kappa - 1)^l\) connections discounted at \( \delta^l \). Note that both \( i \) and \( i' \) connect to each others’ neighbors and neighbors’ neighbors.

(iii) The value of indirectly connecting the neighbors and neighbors’ neighbors etc. of agent \( i \) to the neighbors and neighbors’ neighbors etc. of agent \( i' \) (calculated analogue to the previous step).

Thus the lower bound for agent \( i \) and \( i' \) of forming a link can be arbitrarily close to:

\[
Z(x, x) \cdot \left< 1 + 2 \cdot \sum_{l=1}^{\infty} (\kappa - 1)\delta^l + 2 \cdot \sum_{l'=1}^{\infty} ((\kappa - 1)\delta)^{l'} \cdot \sum_{l=1}^{\infty} (\kappa - 1)\delta^l \right> \\
= Z(x, x) + Z(x, x) \cdot 2 \cdot \frac{(\kappa - 1)\delta}{1 - (\kappa - 1)\delta} + Z(x, x) \cdot 2 \cdot \left< \frac{(\kappa - 1)\delta}{1 - (\kappa - 1)\delta} \right>^2
\] (38)

It is possible to compute the lower bound on the value of a link between two agents of type \( \tilde{x} \) by the same procedure as above. When computing this it is assumed the agents of type \( \tilde{x} \) has neighbors and neighbors’ neighbors etc. that are all of type \( x \). Thus the lower bound is

\[
Z(\tilde{x}, \tilde{x}) + Z(x, \tilde{x}) \cdot 2 \cdot \sum_{l=1}^{\infty} (\kappa - 1)\delta^l + Z(x, x) \cdot 2 \cdot \sum_{l'=1}^{\infty} ((\kappa - 1)\delta)^{l'} \cdot \sum_{l=1}^{\infty} (\kappa - 1)\delta^l \\
= Z(\tilde{x}, \tilde{x}) + Z(x, \tilde{x}) \cdot 2 \cdot \frac{(\kappa - 1)\delta}{1 - (\kappa - 1)\delta} + Z(x, x) \cdot 2 \cdot \left< \frac{(\kappa - 1)\delta}{1 - (\kappa - 1)\delta} \right>^2
\] (39)

As there is monotonicity and supermodularity in link value the upper bound from linking with agents not of type \( x \) can be found by letting agent \( i \) and \( i' \) form a link each with respectively agents \( j \) and \( j' \) of type \( \tilde{x} \) (second highest in \( \tilde{X} \)). Let agents \( j \) and \( j' \) have neighborhood that corresponds to the neighborhoods of agent \( i \) and agent \( i' \) above. Let agent \( i \) and agent \( j \) (and same for agent \( i' \) and agent \( j' \)) and all their neighbors and neighbors’ neighbors etc. are linked with new agents of type \( x \). Moreover, as there are infinitely many agents it is possible to assume that agent \( i \) and any of agent \( i' \)’s neighbors and neighbors’ neighbors etc. have the shortest path to \( j \) and any of agent \( j' \)’s neighbors and neighbors’ neighbors via the link that agent \( i \) and \( j \) forms. Thus it is possible to compute the upper bound of on the value of a link between \( i \) and \( j \):

\[
Z(x, \tilde{x}) + \left< 1 + 2 \cdot \sum_{l=1}^{\infty} (\kappa - 1)\delta^l \right> + Z(x, x) \cdot 2 \cdot \sum_{l'=1}^{\infty} ((\kappa - 1)\delta)^{l'} \cdot \sum_{l=1}^{\infty} (\kappa - 1)\delta^l \\
= Z(x, \tilde{x}) + Z(x, \tilde{x}) \cdot 2 \cdot \frac{(\kappa - 1)\delta}{1 - (\kappa - 1)\delta} + Z(x, x) \cdot 2 \cdot \left< \frac{(\kappa - 1)\delta}{1 - (\kappa - 1)\delta} \right>^2
\] (40)

By letting the lower bound on the value of the links \( ii',jj' \) (sum of Equation 38 and 39) be subtracted by the upper bound on the value of \( ij,i'j' \) (twice the value of sum of Equation 40) to show that the difference is arbitrarily close:

\[
[Z(x, x) + Z(\tilde{x}, \tilde{x}) - 2Z(x, \tilde{x})] + [Z(x, x) + Z(x, \tilde{x}) - 2Z(x, \tilde{x})] \cdot 2 \cdot \frac{(\kappa - 1)\delta}{1 - (\kappa - 1)\delta} + Z(x, x) \cdot 2 \cdot \left< \frac{(\kappa - 1)\delta}{1 - (\kappa - 1)\delta} \right> \\
= [Z(x, x) + Z(\tilde{x}, \tilde{x}) - 2Z(x, \tilde{x})] + [Z(x, x) - Z(x, \tilde{x})] \cdot 2 \cdot \frac{(\kappa - 1)\delta}{1 - (\kappa - 1)\delta}
\]
As there is supermodularity and monotonicity we know the value of the difference in value in the equation above is positive. Thus at most finite agents of the highest type, \( x \), will choose to partner with a lower type as the argument above can be repeated for an infinite number of agents of type \( x \). So it holds that for agents of type \( x \) they will almost surely only partner with other agents of type \( x \).

The argument can be repeated for type \( \hat{x} \) against any other type in \( \hat{X} \). And the argument can be applied repeatedly until type min \( \hat{X} \) can only choose to link with other agents of type min \( \hat{X} \). □

Remark 2: For \( \delta = 0 \) the results in Proposition 2 requires only supermodularity and pairwise (Nash) stability and holds even with a cost function.

**Proof.** It is assumed that the asymptotic variance of \( x_i \) is strictly positive. This implies that there is a subset of types, \( \hat{X} \subseteq X \), where \( |\hat{X}| \geq 2 \) and for every type \( x \in \hat{X} \) it holds that there is an asymptotic strictly share of the total number of agents of that type, i.e., \( \lim_{n \to \infty}(|\{i \in N_n\}_{x_i=x}|/n) > 0 \). By assumption the number of types, \( |X| \), is finite and therefore it is sufficient to only look at types in \( \hat{X} \) and look at their connections. The argument is that for asymptotic infinite agents the measure of types not in \( \hat{X} \) will have measure zero.

When there is a cost function it can be shown that there is an upper bound on how many links each can have. When each type \( x \) only link with itself the maximum number of links is \( \kappa \) which solves \( \kappa = \max\{k \in \mathbb{N} : z(x, x) > 2[c(k) - c(k - 1)]\} \). We refer to this as the implicit degree quota.

Each agent will almost surely have \( \kappa \) links as it is assumed that each link adds positive value and there are asymptotic infinite agents (only a finite number can then not fulfill the degree quota).

As there is an infinite number of agents of any two types \( x \in \hat{X} \) it must be that the degree quota or the implicit degree quota is exceeded by number of agents of type \( x \); i.e., \( |\{i \in N_n\}_{x_i=x}| > \kappa \), of convex costs that \( |\{i \in N_n\}_{x_i=x}| > \kappa \).

Now choose the following two types \( x, \hat{x} \in \hat{X} \) where \( x = \max \hat{X} \) and \( x > \hat{x} \). We will show there can be at most a finite number of links between agents of these two types. Suppose instead there are infinite many links between the two types for some \( \mu \in M^\text{ps} \). Choose any link \( ij \in \mu \) where the talents are \( x_i = x, x_j = \hat{x} \). As there is an infinite number of links between type \( x \) and \( \hat{x} \) then there exist another link \( i'j' \in \mu \) where the talents are \( x_i' = x, x_j' = \hat{x} \) such that \( i'i' \notin \mu \). The existence of a link with these properties follows as \( i \) and \( j \) can at most have \( \kappa \) or \( \kappa \) links each. Thus there are at most \( \kappa^2 \) or \( \kappa^2 \) where either \( i'i' \in \mu \) or \( j'j' \in \mu \). Thus the alternative network \( \mu \cup \{i'i', j'j' \} \setminus \{ij, i'j' \} \) is feasible. By supermodularity this also increases total payoffs which violates efficiency. This in turn violates strong stability. From Lemma 1 it follows that pairwise stability is also violated if strong stability is violated. □

### D Proofs for assortative behavior in degree

**Theorem 3.** If there are no externalities, supermodularity and monotonicity in link value, as well as complete heterogeneity then it follows that for every pairwise stable network it holds that an agent with higher degree has partners which weakly dominate in degree when compared partner-by-partner with the partners of a lower talented agent:

\[ \forall \mu \in M^\text{ps}, \forall i, j \in N, k_i(\mu) > k_j(\mu) : \quad \mathcal{K}(\nu_i(\mu)/\{j\})_l \geq \mathcal{K}(\nu_j(\mu)/\{i\})_l, \quad l = 1, \ldots, k_j(\mu). \]

**Proof.** As there are no externalities, supermodularity and monotonicity in link value then Theorem 1 and 2 must hold. In addition, as agents \( i \) and \( j \) are distinct under complete heterogeneity then either \( x_i > x_j \) or \( x_i' > x_i \). As \( k_i(\mu) \geq k_j(\mu) \) it must be that \( x_i > x_i' \) as the converse would violate Theorem 1.
From Theorem $2$ it is known that if there are two agents such that $x_i > x_{i'}$ then this entails that for $l = 1, ..., k_j(\mu)$: $\mathcal{X}(u_i(\mu)/\{j\})_l \geq \mathcal{X}(u_{j'}(\mu)/\{i\})_l$. This inequality entails there are exactly two possible cases for any index $l = 1, ..., k_j(\mu)$. The first case is that $\mathcal{X}(u_{j'}(\mu)/\{i\})_l = \mathcal{X}(u_j(\mu)/\{i\})_l$. In this case $\mathcal{K}(u_j(\mu)/\{i\})_l = \mathcal{K}(u_j(\mu)/\{i\})_l$ as the agent linked to $i$ and $j$ must be the same due to complete heterogeneity. Else in the other case where $\mathcal{X}(u_i(\mu)/\{j\})_l > \mathcal{X}(u_j(\mu)/\{i\})_l$ then reapplying Theorem entails $1$ in which case it follows that, $\mathcal{K}(u_i(\mu)/\{i\})_l \geq \mathcal{K}(u_j(\mu)/\{i\})_l$. 

□

E Inefficient assortative behavior under network externalities

**Proposition 3.** Suppose there is supermodularity, a degree quota, $\kappa$, two types $\tilde{x}, x$ such that there are exactly $\kappa + 1$ of each type then whenever $M_{ss}^{11} \neq 0$ there exists thresholds $\delta, \tilde{\delta}$ such that for any $\delta \in (\delta, \tilde{\delta})$ the segregated network with no links between agents of the two types which is both pairwise stable and inefficient; the thresholds are:

\[
\delta(\beta) = -\left(\frac{\beta}{2} - 1 + \beta \right) + \sqrt{\left(\beta + \frac{\beta}{2} - 1\right)^2 + 2(\beta - 1) \cdot \left((\frac{\beta}{2} - 2)^2 + 2\right) \cdot \left((\frac{\beta}{2} - 2)^2 + 2\right)},
\]

\[
\tilde{\delta}(\beta) = -\left(\frac{\beta}{2} - 2 + \beta \right) + \sqrt{\left(\beta + \frac{\beta}{2} - 2\right)^2 + 2(\beta - 1)},
\]

where the coefficient of modularity is $\beta = \frac{Z(\tilde{x}, x) + Z(x, \tilde{x})}{2 Z(\tilde{x}, \tilde{x})}$.

**Proof.** This proof use the same approach as in Example $3$. There is supermodularity and there are $2 \cdot (\kappa + 1)$ agents, i.e., $N = \{i_1, i_2, ..., i_{2\kappa+2}\}$. Assume there are two types of talent where each type has half the agents in $N$ where $\kappa + 1$. The other network is connected by two links between the high and low cliques. The other network is similar but has the difference that two agents of each type instead has broken their link and instead linked across the two cliques of types. This connected network is denoted $\mu$.

\[
\mu = \left(\bigcup_{i, i' \in N_x, i \neq i'} \{ii'\} \right) \bigcup \left(\bigcup_{j, j' \in N_x, j \neq j'} \{jj'\} \right)
\]

\[
\tilde{\mu} = \mu \setminus \{ii', jj'\} \cup \{ij, j'i'\} \quad \text{for} \quad i, i' \in N_x, j, j' \in N_{\tilde{x}}
\]

The total value of the two networks are:

\[
\sum_{i \in N} u_i(\mu) = \left[ 2 + 2 \cdot \left(\frac{\beta}{2} - 1\right) \cdot \delta + \left(2 + \left(\frac{\beta}{2} - 2\right)^2\right) \cdot \delta^2 \right] \cdot Z(\tilde{x}, x)
\]

\[
+ \left[ \frac{1}{2} \cdot \left(\frac{\beta}{2} - 1\right) \cdot \left(\frac{\beta}{2} - 2\right) \cdot \left(2 + \left(\frac{\beta}{2} - 2\right)^2\right) \cdot \delta^2 \right] \cdot Z(\tilde{x}, \tilde{x})
\]

\[
\sum_{i \in N} u_i(\tilde{\mu}) = \left[ \frac{1}{2} \cdot \left(\frac{\beta}{2} - 1\right) \cdot \left(\frac{\beta}{2} - 2\right) \cdot \delta \right] \cdot [Z(\tilde{x}, \tilde{x}) + Z(x, \tilde{x})]
\]

The condition for when connecting is efficient, i.e. when $\sum_{i \in N} u_i(\mu) > \sum_{i \in N} u_i(\tilde{\mu})$, can be rewritten into the following expression:

\[
1 + 1 \cdot \left(\frac{\beta}{2} - 1\right) \cdot \delta + \frac{1}{2} \cdot \left(2 + \left(\frac{\beta}{2} - 2\right)^2\right) \cdot \delta^2 > (1 - \delta)\beta
\]

We now turn to check pairwise stability of $\mu$. As in Example $3$ we can assume that where the agents share the gains of linking equally. Under this circumstance all links provide value, thus deletion
of any link leads to a loss. This implies that as in Example 3 we only need to check for one kind of deviation. The only kind of link which may be formed is by one agent \( i \in N_x \) and one agent \( i' \in N_x \) when both \( i \) and \( i' \) delete a current link. Denote a network where agents \( i \) and \( i' \) have deviated from \( \mu \) and formed a link as \( \hat{\mu} \). The value for agent \( i \) and agent \( i' \) for the segregated network and the move towards the pairwise move towards the connected network is due to symmetry of utility:

\[
\begin{align*}
    u_i(\mu) + u_{i'}(\mu) &= (\frac{n}{2} - 2 + \delta) \cdot [z(\bar{x}, \bar{x}) + z(x, x)] + \delta \cdot (\frac{n}{2} - 2 + \delta) \cdot [z(x, x) + z(\bar{x}, x)] \\
    &= \frac{1}{2} \cdot (\frac{n}{2} - 2 + \delta) \cdot [Z(\bar{x}, \bar{x}) + Z(x, x)] + \delta \cdot (\frac{n}{2} - 2 + \delta^2) \cdot Z(x, \bar{x}) \\
    u_i(\hat{\mu}) + u_{i'}(\hat{\mu}) &= (\frac{n}{2} - 1) \cdot [z(\bar{x}, \bar{x}) + z(x, x)] \\
    &= \frac{1}{2} \cdot (\frac{n}{2} - 1) \cdot [Z(\bar{x}, \bar{x}) + Z(x, x)] \\
\end{align*}
\]

The condition for when segregating is pairwise stable, i.e. when \( u_i(\mu) + u_{i'}(\mu) > u_i(\hat{\mu}) + u_{i'}(\hat{\mu}) \), can be rewritten into the following expression:

\[
1 + \left( \frac{n}{2} - 2 \right) \cdot \delta + \delta^2 < (1 - \delta) \beta. \quad (44)
\]

Note the solution to these two inequalities in (43) and (44) above always exist when there is supermodularity as \( \beta > 1 \). By solving the quadratic inequalities the solution to the two unique solutions in \([0,1]\) are:

\[
\delta(\beta) = \frac{-\left( \frac{n}{2} - 1 + \beta \right) + \sqrt{\left( \beta + \frac{n}{2} - 1 \right)^2 + 2(\beta - 1) \cdot (\frac{n}{2} - 2)^2 + 2}}{(\frac{n}{2} - 2)^2 + 2}
\]

\[
\delta(\beta) = \frac{-\left( \frac{n}{2} - 2 + \beta \right) + \sqrt{\left( \beta + \frac{n}{2} - 2 \right)^2 + 2(\beta - 1)}}{(\frac{n}{2} - 2)^2 + 2}
\]