A New Type of Conjugate Gradient Technique for Solving Fuzzy Nonlinear Algebraic Equations

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Abstract. In this paper, we suggest a new method for the numerical solution of fuzzy nonlinear equations in parametric form using a new Conjugate Gradient Technique. Table of the numerical solution is given to show the efficiency of the proposed method and which is compared with classical algorithms such as (Fletcher and Reeves (FR), Polak and Ribiere (PRP), and Fletcher (CD)) techniques.

1. Introduction

Recently, Systems of simultaneous nonlinear equations have been played a major role in various areas of science such as mathematics, statistic, engineering social sciences, robotics, and medicines. The idea of fuzzy numbers and mathematic activity with these numbers were first presented and researched by[1][2][3][4][5]. One of the significant utilization of fuzzy number-crunching is nonlinear systems whose parameters are all or partially represented by fuzzy numbers [6-8]. Standard analytical methods like Buckley and Qu method [9][10], cannot be suitable for resolving the equations such as

i- $cx^5 + bx^4 + cx^3 + dx - e = f,$

ii- $x - \sin(x) = g,$

Where $x, a, b, c, d, e, f$ and $g$ are fuzzy numbers. In this way, we need to build up the numerical techniques to find the roots of such equations. Here, we think about these conditions, as a rule, as $F(x) = c$.

whose parameters are all or partially represented by fuzzy numbers, [11] research the performance of Newton’s scheme for getting the solve of the fuzzy nonlinear equations and reached out to the systems of fuzzy nonlinear equations by [12]. Newton’s technique converges quickly if the initial point is picked near the solution point. The primary downside of Newton's strategy is figuring the Hessian matrix in each epoch. Probably the easiest variation of Newton’s technique was considered by [13] for solution the double fuzzy nonlinear equations. Another type of Newton’s technique known as Levenberg-Marquardt alteration was used to fathom fuzzy nonlinear equations by[14]. Authors in [15] applied also the Broyden's technique to solve the fuzzy nonlinear equations. Every one of these techniques is Newton-like which requires the calculation and storage of either Hessian matrix or approximate Hessian
matrix at every iterative of iterations. Newly, a diagonal update method for solving fuzzy nonlinear equations was proposed by [14]. A gradient based technique by [16] was applied to get the optimal value of variables of fuzzy nonlinear equations. This technique is simple and requires no Hessian matrix assessment during calculations. Be that as it may, method convergence is linear and very slow toward the optimal solution [17]. The steepest descent technique is additionally affected by ill-conditioning [18].

In this work, we developed a new conjugate gradient coefficient and applied it to solve fuzzy nonlinear equations. The conjugate gradient technique is known to be easy and high proficient in taking care of optimization problem. The plan in this work is to convert the parametric form of a fuzzy nonlinear Algebraic equation into an unconstrained optimization problem before applying the new conjugate gradient technique to get the optimal solution.

2. Preliminaries

Definition 1. A fuzzy number is a fuzzy set like $u: \mathbb{R} \rightarrow [0,1]$ which satisfies [19][20],

1. $u$ is upper semi continuous,
2. $u(x) = 0$ outside some interval $[c,d]$,
3. There are real numbers $a, b$ such that $c \leq a \leq b \leq d$ and
   i. $u(x)$ is monotonic increasing on $[c,a]$,
   ii. $u(x)$ is monotonic decreasing on $[b,d]$,
   iii. $u(x) = 1, a \leq x \leq b$.

The set of all these fuzzy numbers is denoted by $E$. An equivalent parametric is also given in [21] as follows.

Definition 2. A fuzzy number $u$ in parametric form is a pair $(u_l, u_r)$ of function $u_l(r), u_r(r), 0 \leq r \leq 1$, which satisfies the following requirements:

1. $u_l(r)$ is a bounded monotonic increasing left continuous function,
2. $u_r(r)$ is a bounded monotonic decreasing left continuous function,
3. $u_l(r) \leq u_r(r), 0 \leq r \leq 1$.

A popular fuzzy number is the trapezoidal fuzzy number $u = (x_0, y_0, \sigma, \beta)$ with interval defuzzifier $[x_0, y_0]$ and left fuzziness $\sigma$ and right fuzziness $\beta$ where the membership function is

$$
u(x) = \begin{cases} 
\frac{1}{\sigma}(x - x_0 + \sigma), & x_0 - \sigma \leq x \leq x_0, \\
1, & x \in [x_0, y_0], \\
\frac{1}{\beta}(y_0 - x + \beta), & y_0 \leq x \leq y_0 + \beta, \\
0, & \text{otherwise}.
\end{cases}$$

Its parametric form is

$$u_l(r) = x_0 - \sigma + r\sigma, u_r(r) = y_0 + \beta + r\beta.$$ 

Let $TF(\mathbb{R})$ be the set of all trapezoidal fuzzy numbers. The addition and scalar multiplication of fuzzy numbers are defined by the extension principle and can be equivalently represented as follows.

For arbitrary $u = (u_l, u_r), v = (v_l, v_r)$ and $k > 0$ we define addition $(u + v)$ and multiplication by scalar $k$ as

$$
(u + v)(r) = u_l(r) + v_l(r), \quad (u + v)(r) = u_r(r) + v_r(r), \quad (ku)(r) = ku_l(r), \quad (ku)(r) = ku_r(r).
$$

(1)
3. New conjugate gradient method for solving fuzzy nonlinear equations

In this section we will show some of the conjugate gradient methods and then suggest a new algorithm for conjugate gradient algorithm for solving fuzzy nonlinear equations

\[ x_{k+1} = x_k + \alpha_k d_k, \ k \geq 1 \]  

(3)

Where \(\alpha_k\) step-size that satisfy the standard wolfe conditions

\[ f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k \]  

(4)

\[ d_k^T g(x_k + \alpha_k d_k) \geq \sigma d_k^T g_k \]  

(5)

or strong wolfe conditions

\[ f(x_k + \alpha_k d_k) \leq f(x) + \delta \alpha_k g_k^T d_k \]  

(6)

\[ |d_k^T g(x_k + \alpha_k d_k)| \leq -\sigma d_k^T g_k \]  

(7)

\[ d_{k+1} = \begin{cases} -g_1, & k = 1 \\ -g_{k+1} + \beta_k d_k, & k \geq 1 \end{cases} \]  

(8)

The Fletcher and Reeves (FR) [22], Fletcher (CD) [23], Polak and Ribiere (PRP) [24] and \(\beta\) is scalar.

\[ \beta^{FR} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2}, \text{ see [22]} \]

\[ \beta^{CD} = \frac{-\|g_{k+1}\|^2}{\|g_k\|^2}, \text{ see [23]} \]

\[ \beta^{PRP} = \frac{g_{k+1}^T y_k}{\|g_k\|^2}, \text{ see [24]} \]

Where \(g_k = \nabla f(x_k)\), and let \(y_k = g_{k+1} - g_k\) \(y_k = g_{k+1} - g_k\).

Now we suggest a new of conjugate gradient algorithm for solving fuzzy nonlinear equations depend basically on norm-1 instead of norm-2 in Fletcher-Reeves (FR) algorithm so we get new formula

\[ d_{k+1} = -g_{k+1} + \beta_k^{NEW} d_k \]  

(9)

\[ \beta_k^{NEW} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2} \]  

(10)

4. The Descent property for a new Conjugate Gradient Method (CGM)

In this part, we have to show the descent property for our proposed new conjugate gradient method, denoted by \(\beta_k^{NEW}\).

Theorem (1)

Consider the search direction given by equation (9) and (10) if \(\alpha_k\) computed by strong wolfe condition (6) and (7) then \(g_{k+1}^T d_{k+1} < 0\).

\[ d_{k+1} = -g_{k+1} + \beta_k^{NEW} d_k \]  

, for all \(k \geq 1\)

Proof:-

The proof is by using mathematical inducement

1- If \(k = 1\) then \(g_1^T d_2 < 0\), \(d_2 = -g_1 \rightarrow < 0\).

2- Let the relation \(g_k^T d_k < -c|g_k|\) for all \(k\) where \(0 < c < 1\).

3- We prove that the relation is true when \(k = k + 1\) by multiplying the equation (9) in \(g_{k+1}\) we obtain

\[ d_{k+1} = -g_{k+1} + \beta_k^{NEW} d_k \]

\[ \beta_k^{NEW} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2} \]

Multiply above equation by \(g_{k+1}\)

Using the following inequality \(\|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2\) We get
\[ g_{k+1}^T d_{k+1} = -\|g_{k+1}\|^2 + \frac{\|g_{k+1}\|^2}{\|g_k\|^2} g_k^T g_{k+1} d_k \leq -\|g_{k+1}\|^2 + \frac{\|g_{k+1}\|^2}{\|g_k\|^2} |g_k^T d_k| \]

\[ \frac{g_{k+1}^T d_{k+1}}{\|g_{k+1}\|^2} \leq -1 + \frac{|g_k^T d_k|}{\|g_k\|^2} \]

By strong wolfe conditions \( g_k^T d_k \leq -c|g_k| \)

\[ \frac{g_{k+1}^T d_{k+1}}{\|g_{k+1}\|^2} \leq -1 + \frac{\|g_k\|^2}{\|g_k\|^2} \leq -1 - \sigma c \]

\[ \therefore g_{k+1}^T d_{k+1} \leq -(1 + \sigma c)|g_{k+1}| \]

5. Global convergence study

We will display that CG method with \( \beta_k^{NEW} \) convergences globally. We need some assumption for the convergence of the proposed new algorithm.

Assumption (1)

1. Assume \( f \) is bound below in the level set \( S = \{ x \in \mathbb{R}^n : f(x) \leq f(x_0) \} \); In some Initial point.

2. \( f \) is continuously differentiable and its gradient is Lipchitz continuous, that is there exist \( L > 0 \) such that [25]:

\[ \| g(x) - g(y) \| \leq L \| x - y \| \forall x, y \in N . \]

On the other hand, under Assumption (1), it is clear that there exist positive constants \( B \) such

\[ \|x\| \leq B, \forall x \in S \]

\[ \| \nabla f(x) \| \leq \bar{F}, \forall x \in S \]

Lemma (1)

Suppose that Assumption (1) and equation (11) hold. Take into consideration any conjugate gradient method in from (3) and (8), where \( d_k \) is a descent direction and \( \alpha_k \) is obtained by the strong wolfe conditions if

\[ \sum_{k>1} \frac{1}{\| d_{k+1} \|^2} \geq \infty \]

Then we have

\[ \liminf_{k \to \infty} \| g_k \| = 0 \]

More details can be found in[26][27][28][29][30][31].

Theorem (2)

Assume that Assumption (1) and equation (11) and the descent condition hold. Consider a conjugate gradient scheme in the form

\[ d_{k+1} = -g_{k+1} + \beta_k^{NEW} d_k \]

where \( \alpha_k \) is computed from strong Wolfe line search condition for more details see[32][33][34], If the objective function is uniformly on set \( S \), then \( \lim_{n \to \infty} (\inf \| g_k \|) = 0 \).

Proof
\[ d_{k+1} = -g_{k+1} + \beta_k^{NEW} d_k \]
\[ \beta_k^{NEW} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2} \]
\[ \|d_{k+1}\| = -g_{k+1} + \beta_k^{NEW} \|d_k\| \]
\[ \|d_{k+1}\| = -g_{k+1} + \frac{\|g_{k+1}\|^2}{\|g_k\|^2} \|d_k\| \]
\[ \|d_{k+1}\| = -g_{k+1} + \frac{\sqrt{n} \|g_{k+1}\|^2}{\|g_k\|^2} \|d_k\| \]
\[ \|d_{k+1}\| \leq \|g_{k+1}\| + \sqrt{n} \|g_{k+1}\|^2 \|d_k\| \]
\[ \|d_{k+1}\| \leq (1 + \frac{\sqrt{n} \|g_{k+1}\|}{\|g_k\|^2}) \|d_k\| \|g_{k+1}\| \]

Let \( \omega = (1 + \frac{\sqrt{n} \|g_{k+1}\|}{\|g_k\|^2}) \|d_k\| \)
\[ \sum_{k=1}^{\infty} \frac{1}{\|d_{k+1}\|^2} \geq \frac{1}{\omega^2} \sum_{k=1}^{\infty} 1 = \infty \]

Therefore, we have contradiction
\[ \lim_{n \to \infty} (\inf \|g_k\|) = 0. \]

### 6. Numerical Examples

In this part, we present the numerical solution of some examples using the new proposed method for CG method for fuzzy nonlinear equations. All computations are carried out on MATLAB 9.0 using a double precision computer. We use the stop criterion \( \|g_{k+1}\| < 10^{-6} \), and employ the new method to solve fuzzy nonlinear equations. Then, we compare with the other methods (Fletcher and Reeves (FR), Polak and Ribiere (PRP), and Fletcher (CD)).

**Example 1:** Consider the fuzzy nonlinear equation [9]
\[(3,4,5)x^2 + (1,2,3)x = (1,2,3)\]
Without any loss of generality, assume that \( x \) is positive, and then the parametric form of this equation is as follows:
\[
\begin{align*}
(3 + r)x^2(r) + (1 + r)x(r) - (1 + r) &= 0, \\
(5 - r)x^2(r) + (3 - r)x(r) - (3 - r) &= 0.
\end{align*}
\]
The above system needs initial values as follows. For \( r = 1 \)
\[
\begin{align*}
4x^2(1) + 2x(1) - 2 &= 0, \\
4x^2(1) + 2\bar{x}(1) - 2 &= 0,
\end{align*}
\]
For \( r = 0 \)
\[
\begin{align*}
3x^2(0) + \bar{x}(0) - 1 &= 0, \\
5x^2(0) + \bar{x}(0) - 3 &= 0
\end{align*}
\]
With initial values
\[ x_0 = (\bar{x}(0), \bar{x}(1), x(1), \bar{x}(0)) = (0.434, 0.5, 0.5, 0.681) \]
Example 2: Consider the fuzzy nonlinear equation \[9\]
\[(4,6,8)x^2 + (2,3,4)x - (8,12,16) = (5,6,7)\]
Without any loss of generality, assume that \(x\) is positive, and then the parametric form of this equation is as follows:
\[
\begin{align*}
(4 + 2r)x^2 + (2 + r)x - (3 + 3r) &= 0, \\
(8 - 2r)x^2 + (4 - r)x - (9 - 3r) &= 0.
\end{align*}
\]
The above system needs initial values as follows. For \(r = 1\)
\[
\begin{align*}
6x^2(1) + 3x(1) - 6 &= 0, \\
6\overline{x}(1) + 3\overline{x}(1) - 6 &= 0,
\end{align*}
\]
For \(r = 0\)
\[
\begin{align*}
4x^2(0) + 2x(0) - 3 &= 0, \\
8\overline{x}(0) + 4\overline{x}(0) - 9 &= 0.
\end{align*}
\]
With initial values
\[
x_0 = (x(0), \underline{x}(1), \overline{x}(1), \med{x}(0)) = (0.651, 0.7808, 0.7808, 0.8397)
\]

Example 3: Consider the fuzzy nonlinear equation \[9\]
\[(1,2,3)x^3 + (2,3,4)x^2 + (3,4,5) = (5,8,13)\]
Without any loss of generality, assume that \(x\) is positive, and then the parametric form of this equation is as follows:
\[
\begin{align*}
(1 + r)x^3 + (2 + r)x^2 + (2 + 2r) &= 0, \\
(3 - r)x^3 + (4 - r)x^2 + (8 - 4r) &= 0.
\end{align*}
\]
The above system needs initial values as follows. For \(r = 1\)
\[
\begin{align*}
2x^3(1) + 3x^2(1) - 4 &= 0, \\
2\overline{x}(1) + 3\overline{x}(1) - 4 &= 0,
\end{align*}
\]
For \(r = 0\)
\[
\begin{align*}
x^3(0) + 2x^2(0) - 2 &= 0, \\
3\overline{x}(0) + 4\overline{x}(0) - 8 &= 0.
\end{align*}
\]
With initial values
\[
x_0 = (x(0), \underline{x}(1), \overline{x}(1), \med{x}(0)) = (0.76, 0.91, 0.91, 1.06).
\]
| Examples | FR | PRP | CD | NEW |
|----------|----|-----|----|-----|
|          | iter | x\_optimal | f\_optimal | iter | x\_optimal | f\_optimal | iter | x\_optimal | f\_optimal | iter | x\_optimal | f\_optimal |
| 1        | 8   | 0.4343 | 5.2505e-014 | 6    | 0.4343 | 1.2314e-016 | 8    | 0.4343 | 8.1709e-014 | 6    | 0.4343 | 1.7016e-011 |
|          |     | 0.5000 |            |      | 0.5000 |            |      | 0.5000 |            |      | 0.5000 |            |
|          |     | 0.5000 |            |      | 0.5307 |            |      | 0.5307 |            |      | 0.5307 |            |
| 2        | 12  | 0.6514 | 2.3998e-010 | 12   | 0.6514 | 7.9887e-012 | 14   | 0.6514 | 5.9506e-010 | 9    | 0.6514 | 1.9513e-011 |
|          |     | 0.7808 |            |      | 0.7808 |            |      | 0.7808 |            |      | 0.7808 |            |
|          |     | 0.7808 |            |      | 0.8397 |            |      | 0.8397 |            |      | 0.8397 |            |
| 3        | 19  | 0.8393 | 2.6011e-011 | 16   | 0.8393 | 4.6595e-012 | 135  | 0.8393 | 1.4978e-008 | 13   | 0.8393 | 1.6756e-013 |
|          |     | 0.9108 |            |      | 0.9108 |            |      | 0.9108 |            |      | 0.9108 |            |
|          |     | 0.9108 |            |      | 1.0564 |            |      | 1.0564 |            |      | 1.0564 |            |
7. Conclusions
We presented in this research a new type conjugate gradient technique for solving fuzzy nonlinear equations, and the proposed algorithm has shown high efficiency in solving these problems with the least number of iterations and higher accuracy in reaching the approximate solution of the function.

Acknowledgements
I would like to express my sincere gratitude and appreciation to my supervisors Prof. Dr. Khalil K. Abbo for this valuable suggestion, encouragement and invaluable remark during writing this paper.

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