CURVATURE OF HIGHER DIRECT IMAGE SHEAVES AND ITS APPLICATION ON NEGATIVE-CURVATURE CRITERION FOR THE WEIL-PETERSSON METRIC

XU WANG

ABSTRACT. We shall show that $q$-semipositivity of the vector bundle $E$ over a Kähler total space $X$ implies the Griffiths-semipositivity of the $q$-th direct image of $O(K_{X/B} \otimes E)$. As an application, we shall give a negative-curvature criterion for the generalized Weil-Petersson metric on the base manifold.

KEYWORDS: Higher direct image, Hodge theory, Chern curvature, $\overline{\partial}$-equation, Weil-Petersson metric, canonically polarized manifold, Calabi-Yau manifold.

CONTENTS

1. Introduction 1
   1.1. Set up 1
   1.2. Main result 3
   1.3. Applications 3
   1.4. List of notations 5
2. Motivation: the product case 5
   2.1. Griffiths-positivity of the bundle of harmonic forms 5
   2.2. Nakano-positivity of the bundle of $\overline{\partial}$-closed forms 8
3. Curvature formula 8
   3.1. Holomorphic vector bundle structure on $\mathcal{H}^{p,q}$ 9
   3.2. Chern connection on $\mathcal{H}^{p,q}$ 10
   3.3. Curvature of $\mathcal{H}^{p,q}$ 12
   3.4. Generalized $\overline{\partial}$-equation associated to the curvature formula 13
   3.5. Curvature of $\mathcal{H}^{n,q}$ 16
4. Curvature of the Weil-Petersson metric 19
5. Acknowledgement 20
References 20

1. INTRODUCTION

1.1. SET UP. Let $\pi : X \to B$ be a proper holomorphic submersion from a complex manifold $X$ to a connected complex manifold $B$. We call $B$ the base manifold of the fibration $\pi$. Assume that $B$ is $m$-dimensional and each fibre $X_t := \pi^{-1}(t)$ of $\pi$ is $n$-dimensional. Let $E$...
be a holomorphic vector bundle over $\mathcal{X}$. Denote by $E_t$ the restriction of $E$ to $X_t$. Let us denote by $K_{X/B}$ the relative canonical bundle on $\mathcal{X}$. Recall that
\begin{equation}
K_{X/B} = K_{\mathcal{X}} - \pi^*(K_B).
\end{equation}
Let us denote by $H^{p,q}(E_t)$ the space of $E_t$-valued $\overline{\partial}$-closed $(p,q)$-type Dolbeault cohomology classes over $X_t$. Let us consider
\begin{equation}
\mathcal{H}^{p,q} := \{H^{p,q}(E_t)\}_{t \in B}.
\end{equation}
It is known that $\mathcal{H}^{p,q}$ has a natural holomorphic vector bundle structure if
\begin{itemize}
\item[A:] The dimension of $H^{p,q}(E_t)$ does not depend on $t \in B$.
\end{itemize}
The usual way to prove this fact is to use the base change theorem. In fact, by the base change theorem, A implies that the $q$-th direct image sheaf
\begin{equation}
\mathcal{E}^{p,q} := R^q\pi_*\mathcal{O}(\wedge^p T_{X/B}^* \otimes E).
\end{equation}
is locally free and the fibres of the holomorphic vector bundle associated to $\mathcal{E}^{p,q}$ are
\begin{equation}
H^q(X_t, \mathcal{O}(\wedge^p T_{X_t}^* \otimes E_t)), \quad t \in B.
\end{equation}
By Dolbeault’s theorem,
\begin{equation}
H^q(X_t, \mathcal{O}(\wedge^p T_{X_t}^* \otimes E_t)) \simeq H^{p,q}(E_t),
\end{equation}
thus there is a holomorphic vector bundle structure on $\mathcal{H}^{p,q}$.

In this paper, we shall use the Newlander-Nirenberg theorem \cite{24} and a theorem of Kodaira-Spencer (see Page 347 in \cite{16}) to construct a holomorphic vector bundle structure on $\mathcal{H}^{p,q}$ directly (see Theorem 3.1 below).

Assume that $\mathcal{X}$ possesses a Kähler form $\omega$, put
\begin{equation}
\omega^j := \omega|_{X_t} > 0, \quad \text{on } X_t.
\end{equation}
Let $h$ be a Hermitian metric on $E$. By the Hodge theory, every cohomology class in $H^{p,q}(E_t)$ has a unique harmonic representative in
\begin{equation}
\mathcal{H}^{p,q}_t := \ker \overline{\partial}^j \cap \ker((\overline{\partial}^j)^*)_t,
\end{equation}
where $\overline{\partial}^j := \overline{\partial}|_{X_t}$ and $(\overline{\partial}^j)^*_t$ is the adjoint of $\overline{\partial}^j$ with respect to $\omega^j$ and $h$. In this paper, we shall identify a cohomology class in $H^{p,q}(E_t)$ with its harmonic representative in $\mathcal{H}^{p,q}_t$. Thus we have
\begin{equation}
\mathcal{H}^{p,q} = \{\mathcal{H}^{p,q}_t\}_{t \in B}.
\end{equation}
The natural $L^2$-inner product on each harmonic space $\mathcal{H}^{p,q}_t$ defines a Hermitian metric on $\mathcal{H}^{p,q}$. Let us denote by $D$ the associated Chern connection on $\mathcal{H}^{p,q}$. Put
\begin{equation}
D = \sum dt^j \otimes D_{t^j} + d\overline{t}^j \otimes \overline{\partial}_{t^j}.
\end{equation}
Then
\begin{equation}
D^2 = \sum dt^j \wedge d\overline{t}^k \otimes \Theta_{jk}, \quad \Theta_{jk} := D_{t^j} \overline{\partial}_{t^k} - \overline{\partial}_{t^j} D_{t^k}.
\end{equation}
Denote by $(\cdot, \cdot)$ and $|| \cdot ||$ the associated inner product and norm on the fibre, $\mathcal{H}^{p,q}_t$, of $\mathcal{H}^{p,q}$ respectively. By definition, $\mathcal{H}^{p,q}$ is semi-positive in the sense of Griffiths if and only if
\begin{equation}
\sum (\Theta_{jk} u, u)_{t^j \overline{t}^k} \xi_{t^j} \overline{\xi}_{t^k} \geq 0, \quad \forall u \in \mathcal{H}^{p,q}_t, \quad \xi \in \mathbb{C}^m.
\end{equation}
Moreover, $\mathcal{H}^{p,q}$ is semi-positive in the sense of Nakano if and only if
\[ \sum_j (\Theta u_j, u_k) \geq 0, \forall u_j \in \mathcal{H}^{p,q}_t, 1 \leq j \leq m. \]

In this paper, we shall give a curvature formula for $\mathcal{H}^{p,q}$ based on the formulas in [3, 4, 7] for $\mathcal{H}^{n,0}$, see [18] and [19] for earlier results and [29, 25, 17, 14, 20, 21] for other generalizations and related results.

1.2. Main result. Denote by $\Theta(E, h)$ (resp. $\Theta(E_t, h)$) the curvature operator of the Chern connection
\[ d^E := \overline{\partial} + \partial^E, \quad (\text{resp. } d^{E_t} := \overline{\partial}^t + \partial^{E_t}), \]
on $(E, h)$ (resp. $(E_t, h)$) respectively. Put
\[ \omega_q := \frac{\omega^q}{q!}, \quad \omega_t^q := \frac{(\omega^t)^q}{q!}. \]

We shall use the following definition:

**Definition 1.1.** $\Theta(E, h)$ is said to be $q$-semipositive on $X$ with respect to $\omega$ if
\[ \omega_q \wedge c_q \{i \Theta(E, h)u, u\} \geq 0, \quad \text{on } X, \]
for every $E$-valued $(n + m - q - 1)$-form $u$ in $X$. Where $c_q = \binom{n+m-q-1}{2}$ is chosen such that
\[ \omega_q+1 \wedge c_q \{u, u\} \geq 0, \]
as a semi-positive volume form on the total space $X$.

Our main theorem is the following:

**Theorem 1.1.** Assume that the total space is Kähler and the dimension of $H^{n,q}(E_t)$ does not depend on $t \in B$. Assume further that $\Theta(E, h)$ is $q$-semipositive with respect to a Kähler form on the total space. Then $\mathcal{H}^{n,q}$ is Griffiths-semipositive.

**Remark:** The following fact is also true:

With the assumptions in the above theorem, assume further that
\[ H^{n,q}_t = \text{ker } \partial^{E_t} \cap \text{ker } (\partial^{E_t})^*, \forall t \in B, \]
then $\mathcal{H}^{n,q}$ is Nakano-semipositive.

If $q = 0$ then (1.15) is always true. If $q \geq 1$, by Siu’s $\partial\overline{\partial}$-Bochner formula (see [27] or [2]), we know that (1.15) is true in case
\[ i \Theta(E_t, h) \wedge \omega_{q-1}^t \equiv 0, \]
on $X_t$ for every $t \in B$.

1.3. Applications. We shall use our main theorem to study the curvature properties of the base manifold $B$. Let us denote by $\kappa$ the Kodaira-Spencer mapping
\[ \kappa : v \mapsto \kappa(v) \in H^{0,1}(T_{X_t}) \simeq H^{n,n-1}(T_{X_t}^*), \forall v \in T_t B, t \in B. \]
We shall introduce the following definition:

**Definition 1.2.** We call the pull back pseudo-metric on $T_B$ defined by
\[ ||v||_{WP} := ||\kappa(v)||_{H^{n,n-1}(T_{X_t}^*)}, \forall v \in T_t B \]
the generalized Weil-Petersson metric on $B$.
Assume that the dimension of $H^{0,1}(T_X)$ does not depend on $t$ in $B$. Then we know that the dimension of the dual space, $H^{n,n-1}(T^*_{X_B})$, of $H^{0,1}(T_X)$ does not depend on $t$ in $B$. Moreover, we shall prove that (see Proposition 1.1)

\[(1.19) \quad \kappa : v \mapsto \kappa(v) \in H^{n,n-1}(T^*_{X_B}), \ \forall \ v \in T_tB,\]

defines a holomorphic bundle map from $T_B$ to the dual bundle of \n
\[(1.20) \quad H^{n,n-1} := \{H^{n,n-1}(T^*_X)\}_{t \in B},\]

Assume that the total space $X$ possesses a Kähler form $\omega$. We shall introduce the following definition:

**Definition 1.3.** The relative cotangent bundle $T^*_{X/B}$ is said to be $(n-1)$-semipositive with respect to $\omega$ if there is a smooth metric, say $h$, on the relative cotangent bundle such that $\Theta(T^*_{X/B}, h)$ is $(n-1)$-semipositive with respect to $\omega$.

By Theorem 1.1, if the relative cotangent bundle is $(n-1)$-semipositive with respect to $\omega$ then $H^{n,n-1}$ is Griffiths-semipositive. Assume further that $\kappa$ is injective. Then $T_B$ is Griffiths-seminegative with respect to the generalized Weil-Petersson metric. Inspired by [23] and [9], we shall introduce the following definition:

**Definition 1.4.** $|| \cdot \||_{WP}$ defines a Griffiths-seminegative singular metric if and only if for every holomorphic vector field $v : t \mapsto v_t$ on the base, $\log ||v||_{WP}$ is plurisubharmonic or equal to $-\infty$ identically, where $||v||_{WP}$ denotes the upper semicontinuous regularization of $t \mapsto ||v_t||_{WP}$.

We shall prove that:

**Theorem 1.2.** Let $\pi$ be a proper holomorphic submersion from a Kähler manifold $(X, \omega)$ to a complex manifold $B$. Assume that the relative cotangent bundle is $(n-1)$-semipositive with respect to $\omega$. Then the associated generalized Weil-Petersson metric defines a Griffiths-seminegative singular metric on $T_B$.

**Remark A:** If the canonical line bundle of each fibre is positive then by Aubin-Yau’s theorem (see [1] and [34]), each fibre possesses a unique Kähler-Einstein metric, which defines a smooth Hermitian metric, say $h$, on the relative cotangent bundle. Then we know that the cotangent bundle of each fibre is $(n-1)$-semipositive. Moreover, if $n = 1$ then it is well known that relative cotangent bundle is 0-semipositive. But for a general canonically polarized family, we don’t know whether the relative cotangent bundle is $(n-1)$-semipositive or not, for related results, say [25]. In [10], we shall introduce another way to study the curvature properties of the base manifold of a canonically polarized family.

**Remark B:** Given a Kähler total space $(X, \omega)$, if the canonical line bundle of each fibre is trivial then by Yau’s theorem [34], we know that there is a smooth function, say $\phi$, on $X$ such that $\omega + i\partial \bar{\partial} \phi$ is Ricci-flat on each fibre. Let us denote by $h$ the smooth Hermitian metric on the relative cotangent bundle defined by $\omega + i\partial \bar{\partial} \phi$. Then we know that the cotangent bundle of each fibre is $(n-1)$-semipositive. But we don’t know whether the relative cotangent bundle is $(n-1)$-semipositive or not, except for some special case, e.g. deformation of torus or other families with flat relative cotangent bundle. In [33], we shall introduce another metric to study the base manifold of a Calabi-Yau family.
1.4. List of notations.

\textbf{Basic notions:}
1. \( \pi : \mathcal{X} \to B \) is a proper holomorphic submersion, \( E \): holomorphic vector bundle on \( \mathcal{X} \);
2. \( X_t := \pi^{-1}(t) \) is the fibre at \( t \), \( E_t := E|_{X_t} \);
3. \( d^E := \overline{\partial} + \partial^E \) is the Chern connection on \( E \), \( d^{E_t} := \overline{\partial} + \partial^{E_t} \) is its restriction;
4. \( \Theta(E, h) := (d^E)^2 \) is Chern curvature of \( E \), \( \Theta(E_t, h) := (d^{E_t})^2 \);
5. \( H^{p,q}(E_t) \): Dolbeault cohomology group on \( X_t \);
6. \( \mathcal{H}^{p,q}_t := \ker \overline{\partial}^t \cap \ker(\overline{\partial})^* \simeq H^{p,q}(E_t) \) is the harmonic space;
7. \( \mathcal{H}^{p,q} := \{ \mathcal{H}^{p,q}_t \}_{t \in B} \simeq \{ H^{p,q}(E_t) \}_{t \in B} \);
8. \( D = \sum d^j \otimes D_{ij} + d^j \otimes \overline{\partial}_j \) is the Chern connection on \( \mathcal{H}^{p,q} \);
9. \( \Theta_{jk} := D_{ij}\overline{\partial}_k - \overline{\partial}_j D_{ij} \) is the curvature operator on \( \mathcal{H}^{p,q} \).

\textbf{Other notations:}
1. \( i_t \): the inclusion mapping \( X_t \hookrightarrow \mathcal{X} \);
2. \( t \): local holomorphic coordinate system on \( B \), \( t^j \): components of \( t \);
3. \( \delta_V \): usual Lie-derivative;
4. \( V_j \): smooth \((1, 0)\)-vector field on \( \mathcal{X} \) such that \( \pi_* V_j = \partial/\partial t^j \), \( L_{V_j} \): Lie-derivative;
5. \( \delta_{V_j} := d^E \delta_{V_j} + \delta_V d^E = [d^E, \delta_{V_j}] \), \( \delta_{V_j} := d^E \delta_{V_j} + \delta_{V_j} d^E = [d^E, \delta_{V_j}] \);
6. \( u : t \to u^t \in \mathcal{H}^{p,q}_t \) is a section of \( \mathcal{H}^{p,q} \);
7. \( \Gamma(\mathcal{H}^{p,q}) \): space of smooth sections of \( \mathcal{H}^{p,q} \);
8. \( u^t \): dual-representative of \( u \) such that \( u^t|_{\mathcal{X}_t} = *u^t \) for every \( t \in B \);
9. \( \mathbb{H} \): orthogonal projection to \( \mathcal{H}^{p,q}_t \);
10. \( \kappa : \partial/\partial t^j \mapsto \kappa(\partial/\partial t^j) \) is the Kodaira-Spencer mapping.

2. Motivation: the product case

2.1. Griffiths-positivity of the bundle of harmonic forms. We shall give an example to show the ideas behind the proof of Theorem 1.1.

Let us consider the following product case: Let \( L \) be a holomorphic line bundle over a compact Kähler manifold \((X, \omega)\). Let \( h \) be a fixed smooth metric on \( L \) and let \( \phi \) be a smooth function on \begin{equation}
\mathcal{X} := X \times \mathbb{B}.
\end{equation}

Put \begin{equation} h^t = he^{-\phi^t}, \quad \phi^t := \phi|_{X \times \{t\}}. \end{equation}
Thus \( \{h^t\}_{t \in B} \) defines a smooth metric, say \( \tilde{h} \), on \( L \times \mathbb{B} \). We shall consider \begin{equation} \mathcal{H}^{n,q} := H^{n,q}(L) \times \mathbb{B} = \{ \mathcal{H}_t^{n,q} \}_{t \in \mathbb{B}}, \end{equation}
where \( \mathbb{B} \) is the unit ball in \( \mathbb{C}^m \) and each \( \mathcal{H}_t^{n,q} \) is the harmonic space with respect to \( \omega \) and \( h^t \). We know that \( \mathcal{H}^{n,q} \) is a trivial vector bundle with non-trivial metric.

For every \( u \in H^{n,q}(L) \) there is an associated holomorphic section \begin{equation} t \mapsto u^t := \mathbb{H}^t(u), \end{equation}
where each \( u \) is a fixed \( \overline{\partial}_X \)-closed representative of \( u \) and \( \mathbb{H}^t \) is the orthogonal projection to the harmonic space \( \mathcal{H}_t^{n,q} \). Now we know that \begin{equation} (u, v) : t \mapsto (u^t, v^t), \quad u, v \in H^{n,q}(L) \end{equation}
is a smooth function on $\mathbb{B}$. Moreover, we have
\begin{equation}
(u, v)_{jk} = (D_{ij} u^t, D_{ik} v^t) - (\Theta_{jk} u^t, v^t).
\end{equation}
By definition, we have
\begin{equation}
(u^t, v^t) = \int_X \{u^t, *v^t\} e^{-\phi^t},
\end{equation}
where $\{\cdot, \cdot\}$ is the pairing associated to $h$. Thus we have
\begin{equation}
(u, v)_j = (L_j u^t, v^t) + \int_X \{u^t, L_j * v^t\} e^{-\phi^t},
\end{equation}
where
\begin{equation}
L_j := \partial/\partial t^j - \phi_j^t, \quad L_{\overline{j}} := \partial/\partial \overline{t}^j.
\end{equation}
Since $(u^t)^* \wedge \omega_q = i^{(n-q)^2} (-1)^{n-q} u^t$, we have
\begin{equation}
L_j^\ast = \ast L_j, \quad \overline{L_j}^\ast = \ast \overline{L_j}, \quad \forall \ 1 \leq j \leq m.
\end{equation}
Now we have
\begin{equation}
(u, v)_j = (L_j u^t, v^t) + (u^t, L_{\overline{j}} v^t).
\end{equation}
Let $v$ be a fixed (does not depend on $t$) $\overline{\partial}_X$-closed representative of $v$. Thus each $v^t - v$ is $\overline{\partial}_X$-exact, which implies that
\begin{equation}
L_{\overline{j}} v^t = L_{\overline{j}} (v^t - v),
\end{equation}
is $\overline{\partial}_X$-exact. Then we have
\begin{equation}
(u^t, L_{\overline{j}} v^t) \equiv 0.
\end{equation}
Hence we get that
\begin{equation}
D_{ij} u^t = \overline{\partial}^t (L_j u^t),
\end{equation}
and
\begin{equation}
(u, v)_{jk} = (L_j u^t, v^t)_{\overline{k}} = (L_{\overline{k}} L_j u^t, v^t) + (L_j u^t, L_{\overline{k}} v^t).
\end{equation}
Notice that
\begin{equation}
L_{\overline{k}} L_j u^t = L_j L_{\overline{k}} u^t - \phi_{j\overline{k}} u^t.
\end{equation}
Then we have
\begin{equation}
(L_{\overline{k}} L_j u^t, v^t) = -(\phi_{j\overline{k}} u^t, v^t) + (L_{\overline{k}} u^t, v^t)_j - (L_{\overline{k}} u^t, L_j v^t).
\end{equation}
Since $(L_{\overline{k}} u^t, v^t) \equiv 0$, thus we have
\begin{equation}
(u, v)_{jk} = -(\phi_{j\overline{k}} u^t, v^t) - (L_{\overline{k}} u^t, L_j v^t) + (L_j u^t, L_{\overline{k}} v^t).
\end{equation}
The following lemma is a crucial step (see Lemma 3.6 for the general case).

**Lemma 2.1.** Put $a^t_j = D_{ij} u^t - L_j u^t$, then each $a^t_j$ is the $L^2$-minimal solution of
\begin{equation}
\overline{\partial}_X a^t_j = -\overline{\partial}_X L_j u^t = \overline{\partial}_X \phi_j \wedge u^t.
\end{equation}
Proof. It suffices to show that $\overline{\partial} X(a^t) \equiv 0$. Since
$$\overline{\partial} X = -* (\partial X - \partial X \phi^t)^*, \ \overline{\partial} X(a^t) = -\overline{\partial} X(L_j u^t),$$
it is enough to show $*L_j u^t$ is $(\partial X - \partial X \phi^t)$-closed. By (2.10), we have
(2.20) $*L_j u^t = L_j *u^t$.
Thus
(2.21) $(\partial X - \partial X \phi^t) * L_j u^t = (\partial X - \partial X \phi^t) L_j *u^t$.
Since each $u^t$ is harmonic, thus
(2.22) $(\partial X - \partial X \phi^t) * u^t \equiv 0$.
Thus it is enough to show that
(2.23) $[\partial X - \partial X \phi^t, L_j] \equiv 0$,
which follows by direct computation. \qed

Fix $u_1, \ldots, u_m \in H^{n,q}(L)$, then we have
(2.24) $\sum (\Theta_{jk} u_j^t, u_k^t) = -||a^t||^2 + \sum (\phi_{jk} u_j^t, u_k^t) + \sum (L_k u_j^t, L_j u_k^t),$
where
(2.25) $a^t := \sum a_j^t, \overline{\partial} X a^t = \sum \overline{\partial} X \phi_j \wedge u_j^t := c^t$
By the classical Bochner-Kodaira-Nakano formula, if
(2.26) $i^{(n-q-1)^2} \omega_q \wedge \{i \Theta(L, h^t) u, u\} > 0$, on $X$,
for every $(n-q-1,0)$-form $u$ that has no zero point in $X$ then we have
(2.27) $||a^t||^2 \leq ([i \Theta(L, h^t), \Lambda_\omega]^{-1} c^t, c^t) = \sum (T_{jk} u_j^t, u_k^t),$
where $\Lambda_\omega$ is the adjoint of $\omega \wedge$ and
(2.28) $T_{jk} := (\overline{\partial} X \phi_k \wedge \cdot)^* [i \Theta(L, h^t), \Lambda_\omega]^{-1} (\overline{\partial} X \phi_j \wedge \cdot)$
Now we have
(2.29) $\sum (\Theta_{jk} u_j^t, u_k^t) = R + \sum (\phi_{jk} u_j^t, u_k^t) + \sum (L_k u_j^t, L_j u_k^t),$
where
(2.30) $R := ([i \Theta(L, h^t), \Lambda_\omega]^{-1} c^t, c^t) - ||a^t||^2 \geq 0$.
We shall use the following lemma (see Lemma 3.10 for the general case):

Lemma 2.2. Assume that (2.26) is true for every $t \in \mathbb{B}$. Then we have
(2.31) $\sum ((\phi_{jk} - T_{jk}) u_j^t, u_k^t) \wedge i^{m^2} dt \wedge d\overline{t} = \pi_* \left(i^{(m+n-1-q)^2} \omega_q \wedge \{i \Theta(L \times \mathbb{B}, \tilde{h}) u^*, u^*\}\right),$
where $u^* := u_j^t \wedge (\partial/\partial \overline{t}) \wedge dt$ and each $u_j^t$ satisfies that
(2.32) $i^*_t (\partial/\partial \overline{t}) (\omega_q \wedge \Theta(L \times \mathbb{B}, \tilde{h}) u_j^t) \equiv 0, \ \ i^*_t u_j^t = *u_j^t$,
on $X$ for every $t \in \mathbb{B}$.
Thus we get the following result (see Theorem 1.1 for the general case):
Theorem 2.3. Assume that
\begin{equation}
\omega_q \wedge i\Theta(L \times B, \tilde{h}) \geq 0, \text{ on } X,
\end{equation}
and
\begin{equation}
\omega_q \wedge i\Theta(L, h^t) > 0, \text{ on } X, \text{ for all } t \in B.
\end{equation}
Then we have
\begin{equation}
\sum (\Theta_{jk} u_j^t, u_k^t) \geq \sum (L_k u_j^t, L_j u_k^t).
\end{equation}
In particular, \( H^{n,q} \) is Griffiths-semipositive.

We shall show in the next section that \( H^{n,q} \) can be seen as a holomorphic quotient bundle of a Nakano-semipositive bundle. But in general, a holomorphic quotient bundle of a Nakano-semipositive bundle is not Nakano-semipositive (see Page 340 in [13] for a counterexample).

2.2. Nakano-positivity of the bundle of \( \overline{\partial} \)-closed forms. Let us denote by \( \text{ker} \overline{\partial} \) (resp. \( \text{Im} \overline{\partial} \)) the space of smooth \( \overline{\partial} \)-closed (resp. \( \overline{\partial} \)-exact) \( L \)-valued \((n, q)\)-forms on \( X \) respectively. Then we have the following trivial bundles:
\begin{equation}
\mathcal{K} := \text{ker} \overline{\partial} \times B, \quad \mathcal{I} := \text{Im} \overline{\partial} \times B.
\end{equation}
But in general the metrics on \( \mathcal{K} \) and \( \mathcal{I} \) defined by \( \{ h^t \}_{t \in B} \) are not trivial. By definition, we know that \( H^{n,q} \) is just the quotient bundle \( \mathcal{K}/\mathcal{I} \). And the metric on \( H^{n,q} \) is just the quotient metric (see [32] for more results).

For every \( u, v \in \text{ker} \overline{\partial} \), we shall write
\begin{equation}
(\Phi^\mathcal{K} t, s) \mapsto \int_X \{ u, *v \} e^{-\phi_t}.
\end{equation}
Let us denote by \( \Theta^\mathcal{K}_{jk} \) the curvature operators on \( \mathcal{K} \). Fix \( u_1, \cdots, u_m \in \text{ker} \overline{\partial} \). Since now
\begin{equation}
L_j u_k \equiv 0, \quad \forall \ 1 \leq j, k \leq m,
\end{equation}
we know that the following theorem is true.

Theorem 2.4. With the assumptions in Theorem 2.3, then we have
\begin{equation}
\sum (\Theta^\mathcal{K}_{jk} u_j^t, u_k^t) = R + \sum ((\phi^t_{jk} - T_{jk}) u_j^t, u_k^t) \geq 0.
\end{equation}
In particular, \( \mathcal{K} \) is Nakano-semipositive.

Remark: One may also study the positivity properties of the bundle of \( \overline{\partial} \)-closed forms for non-trivial fibrations (see [31]).

3. Curvature formula

We shall give a curvature formula for \( H^{p,q} \) in this section.
3.1. Holomorphic vector bundle structure on $\mathcal{H}^{p,q}$. By a theorem of Kodaira-Spencer (see Page 349 in [16]), we know that $\mathcal{H}^{p,q}$ has a smooth complex vector bundle structure if $A$ is true. More precisely, $A$ implies that for every $t_0 \in B$ and every $u^{t_0} \in \mathcal{H}^{p,q}_{t_0}$ (see (1.7) for the definition of the $\overline{\partial}$-harmonic space $\mathcal{H}^{p,q}_{t_0}$), there is a smooth $E$-valued $(p,q)$-form, say $u$, on $X$ such that

\[(3.1)\quad u|_{X_{t_0}} = u^{t_0},\]

and

\[(3.2)\quad u|_{X_t} \in \mathcal{H}^{p,q}_t,\]

for every $t \in B$. Then the smooth vector bundle structure $\mathcal{H}^{p,q}$ can be defined as follows:

**Definition 3.1.** We call $u : t \mapsto u^{t} \in \mathcal{H}^{p,q}_t$ a smooth section of $\mathcal{H}^{p,q}$ if there exists a smooth $E$-valued $(p,q)$-form, say $u$, on $X$ such that $u|_{X_t} = u^{t}, \forall t \in B$. And we call $u$ a representative of $u$. We shall denote by $\Gamma(\mathcal{H}^{p,q})$ the space of smooth sections of $\mathcal{H}^{p,q}$.

By using the Newlander-Nirenberg theorem, we shall prove that:

**Theorem 3.1.** Assume that $\mathcal{H}^{p,q}$ satisfies $A$. Then $D^{0,1} := \sum d\overline{\partial} \otimes \overline{\partial}_t$ defines a holomorphic vector bundle structure on $\mathcal{H}^{p,q}$, where each $\overline{\partial}_t$ is defined by

\[(3.3)\quad \overline{\partial}_t u : t \to \mathbb{H}^t \left( i^{*}_t [\overline{\partial}, \delta_{V_j}] u \right), \quad [\overline{\partial}, \delta_{V_j}] := \overline{\partial}\delta_{V_j} + \delta_{V_j}\overline{\partial}.

Here $u$ is an arbitrary representative of $u \in \Gamma(\mathcal{H}^{p,q})$, $\mathbb{H}^t$ denotes the orthogonal projection to $\mathcal{H}^{p,q}_t$ and $V_j$ is an arbitrary smooth $(1,0)$-vector field on $X$ such that $\pi_* V_j = \partial/\partial t^j$.

**Proof.** First, let us show that $D^{0,1}$ is well defined. Since each $u^{t} \in \mathcal{H}^{p,q}_t$ is harmonic, we know that $i^{*}_t \overline{\partial} u \equiv 0$, thus the definition of $\overline{\partial}_t u$ does not depend on the choice of $V_j$. Thus it suffices to check that $\overline{\partial}_t u$ does not depend on the choice of $u$. Let $u'$ be another representative of $u$ then we have

$$u - u' = \sum dt^j \wedge a_j + \sum dt^k \wedge b_k.$$  

Thus

\[(3.4)\quad i^{*}_t \delta_{V_j} \overline{\partial}(u - u') = -i^{*}_t \overline{\partial}b_j,

which implies that

\[(3.5)\quad \mathbb{H}^t \left( i^{*}_t [\overline{\partial}, \delta_{V_j}] (u - u') \right) = 0.

Thus $D^{0,1}$ is well defined. It is easy to check that

\[(3.6)\quad \overline{\partial}_t (fu) = f \overline{\partial}_t u + f_j u,

where $f$ is an arbitrary smooth function on $B$. By Newlander-Nirenberg’s theorem, it suffices to show that $D^{0,1}$ is integrable, i.e. $(D^{0,1})^2 = 0$. By definition, it is sufficient to show

\[(3.7)\quad \overline{\partial}_t \overline{\partial}_k u = \overline{\partial}_k \overline{\partial}_t u,

for every $u \in \Gamma(\mathcal{H}^{p,q})$ and every $1 \leq j, k \leq m$.

Notice that

\[(3.8)\quad u_j := [\overline{\partial}, \delta_{V_j}] u,
satisfies (see Lemma 3.5 since $i^*_t \overline{\partial} u \equiv 0$)
\begin{equation}
(3.9) \quad \overline{\partial} i^*_t u_j = i^*_t \overline{\partial} u_j = i^*_t [\overline{\partial}, \delta_v] u_j = 0.
\end{equation}
Thus each $i^*_t u_j$ has the following orthogonal decomposition
\begin{equation}
(3.10) \quad i^*_t u_j = (\overline{\partial}_l u)(t) + \overline{\partial}'(\overline{\partial}^*)G^t(i^*_t u_j),
\end{equation}
where $G^t$ is the Green operator on $X_t$. Using Kodaira-Spencer’s theorem again, we know that $G^t(i^*_t u_j)$ depends smoothly on $t$. Thus $(\overline{\partial}^*)G^t(i^*_t u_j)$ depends smoothly on $t$. For each $j$, let us choose a smooth form $v_j$ on $\mathcal{X}$ such that
\[ i^*_t v_j = (\overline{\partial}^*)G^t(i^*_t u_j). \]
By definition, we know that each
\begin{equation}
(3.11) \quad u_j - \overline{\partial} v_j,
\end{equation}
is a representative of $\overline{\partial}_l u$. Thus we have
\begin{equation}
(3.12) \quad (\overline{\partial}_l \overline{\partial}_k - \overline{\partial}_k \overline{\partial}_l) u = \mathbb{H}^l \left( i^*_t [\overline{\partial}, \delta_v] (u_k - \overline{\partial} v_k) - i^*_t [\overline{\partial}, \delta_{\overline{\partial} k}] (u_j - \overline{\partial} v_j) \right)
\end{equation}
\begin{equation}
(3.13) \quad = \mathbb{H}^l i^*_t \left( [\overline{\partial}, \delta_v] [\overline{\partial}, \delta_{\overline{\partial} k}] - [\overline{\partial}, \delta_{\overline{\partial} k}] [\overline{\partial}, \delta_v] \right) u.
\end{equation}
Notice that
\begin{equation}
(3.14) \quad [L_{\overline{\partial} j}, L_{\overline{\partial} k}] = L_{[\overline{\partial} j, \overline{\partial} k]},
\end{equation}
implies that
\begin{equation}
(3.15) \quad [\overline{\partial}, \delta_{\overline{\partial} j}] [\overline{\partial}, \delta_{\overline{\partial} k}] - [\overline{\partial}, \delta_{\overline{\partial} k}] [\overline{\partial}, \delta_{\overline{\partial} j}] = [\overline{\partial}, \delta_{[\overline{\partial} j, \overline{\partial} k]}]
\end{equation}
Since each $u^t$ is harmonic (thus $\overline{\partial}$-closed), we have
\begin{equation}
(3.16) \quad (\overline{\partial}_l \overline{\partial}_k - \overline{\partial}_k \overline{\partial}_l) u = \mathbb{H}^l i^*_t [\overline{\partial}, \delta_{[\overline{\partial} j, \overline{\partial} k]}] u = \mathbb{H}^l i^*_t \delta_{[\overline{\partial} j, \overline{\partial} k]} \overline{\partial} u = \mathbb{H}^l \delta_{[\overline{\partial} j, \overline{\partial} k]}|_{X_t} \overline{\partial} u^t = 0.
\end{equation}
The proof is complete. \hfill \square

3.2. Chern connection on $H^{p,q}$. In this subsection, we shall define the Chern connection on $H^{p,q}$.

Assume that $\mathcal{X}$ possesses a Kähler form, say $\omega$. Then each fibre $X_t$ possesses a Kähler form $\omega^t := \omega|_{X_t}$. Recall that a smooth $k$-form is said to be primitive with respect to $\omega^t$ if $k \leq n$ and $\omega^t_{n-k+1} \wedge \alpha = 0$ on $X_t$. Let $u : t \mapsto u^t \in H^{p,q}_t$ be a smooth section of $H^{p,q}$. By the Lefschetz decomposition theorem, each $u^t$ has a unique decomposition of the form
\begin{equation}
(3.17) \quad u^t := \sum_r \omega^t_r \wedge u^t_r,
\end{equation}
where each $u^t_r$ is a smooth $E_r$-valued primitive $(p - r, q - r)$-form. Let us denote by $*$ the Hodge-Poincaré-de Rham star operator with respect to $\omega^t$. Then we have
\begin{equation}
(3.18) \quad * u^t := \sum_r C_r \omega^t_{n+r-p-q} \wedge u^t_r, \quad C_r = i^{(p+q-2r)^2}(-1)^{p-r}.
\end{equation}
Since $u^t$ depends smoothly on $t$, we know that each $u^t_r$ also depends smoothly on $t$. Thus for each $r$, there exists a smooth $E$-valued $(p - r, q - r)$-form, say $u_r$ on $\mathcal{X}$ such that
\begin{equation}
(3.19) \quad u_r|_{X_t} = u^t_r.
\end{equation}
By Definition 3.1, we know that
\[ \sum_r \omega_r \wedge u_r \]
is a representative of \( u \). We shall use the following definition:

**Definition 3.2.** We call a smooth \( E \)-valued \((n-q,n-p)\)-form \( u^* \) on \( X \) a dual-representative of \( u \in \Gamma(H^{p,q}) \) if
\[ u^* = \sum_r C_r \omega_{n+r-p-q} \wedge u_r. \]

By definition, we know that if \( u \) is a representative of \( u \in \Gamma(H^{p,q}) \) and \( v^* \) is a dual-representative of \( v \in \Gamma(H^{p,q}) \) then
\[ (u, v) = \pi_*\{u, v^*\}, \]
where \( \{\cdot, \cdot\} \) is the canonical sesquilinear pairing (see page 268 in [13]). Now we have (see page 12 in [30])
\[ \frac{\partial}{\partial t_j}(u, v) = \frac{\partial}{\partial t_j}\pi_*\{u, v^*\} \]
(3.23)
\[ = \pi_*\{L_{V_j}\{u, v^*\}\} \]
(3.24)
\[ = \pi_*\{(L_j u, v^*) + \{u, L_j v^*\}\}, \]
where \( V_j \) is an arbitrary smooth \((1,0)\)-vector field on \( X \) such that \( \pi_*V_j = \partial/\partial t_j \) and
\[ L_j := d^E \delta V_j + \delta V_j d^E, \quad L_j := d^E \delta V_j + \delta V_j d^E. \]

Here \( d^E = \partial + \overline{\partial} \) denotes the Chern connection on \( E \). In order to find a good expression of the Chern connection on \( H^{p,q} \), we shall introduce the following definition:

**Definition 3.3.** Assume that \( X \) possesses a Kähler form \( \omega \). A smooth \((1,0)\)-vector field \( V \) on \( X \) is said to be horizontal with respect to \( \omega \) if
\[ i_t^* (\delta \omega) = 0, \]
on \( X_t \) for every \( t \in B \). Moreover, for each \( 1 \leq j \leq m \), we call \( V_j \) the horizontal lift vector field of \( \partial/\partial t_j \) with respect to \( \omega \) if \( V_j \) is horizontal with respect to \( \omega \) and satisfies
\[ \pi_* (V_j) = \partial/\partial t_j. \]

Now we can prove that:

**Proposition 3.2.** Assume that \( X \) possesses a Kähler form \( \omega \) and each \( V_j \) is the horizontal lift vector field of \( \partial/\partial t_j \). Then
\[ \pi_*\{u, L_j v^*\} = \pi_*\{u, \overline{\partial}_j [\delta V_j] v^*\}, \]
for every smooth sections \( u, v \) of \( H^{p,q} \).

**Proof.** For bidegree reason, we have
\[ \pi_*\{u, L_j v^*\} = \pi_*\{u, [\overline{\partial}_j, \delta V_j] v^*\}, \]
Thus it suffices to show that
\[ (\overline{\partial}_j v)(t) = H^t \left( (-1)^{p+q} * i_t^* [\overline{\partial}_j, \delta V_j] v^* \right). \]
By Theorem 3.1 and Definition 3.2 it suffices to check that
\begin{equation}
(3.32) \quad i^*_t[\overline{\partial}, \delta_{\overline{V}_j}], \omega] = 0.
\end{equation}
Since each \( V_j \) is horizontal, the above equality is always true. The proof is complete.

By Proposition 3.2 we have
\begin{equation}
(3.33) \quad \frac{\partial}{\partial t}(u, v) = \pi_\ast\{L_j u, v^*\} + (u, \overline{\partial}_{V_j} v).
\end{equation}
By definition, the \((1,0)\)-part of the Chern connection \( D^{1,0} = \sum dt^j \otimes D_{t^j} \) should satisfy
\begin{equation}
(3.34) \quad \frac{\partial}{\partial t}(u, v) = (D_{t^j} u, v) + (u, \overline{\partial}_{V_j} v).
\end{equation}
Thus we have:

**Proposition 3.3.** Assume that \( X \) possesses a Kähler form \( \omega \) and each \( V_j \) is the horizontal lift vector field of \( \partial/\partial t^j \). Then the \((1,0)\)-part of the Chern connection on \( \mathcal{H}^{p,q} \) satisfies
\begin{equation}
(3.35) \quad D_{t^j} u : t \mapsto H^f(i^*_t[\partial^E, \delta_{V_j}]u),
\end{equation}
where \( u \) is an arbitrary representative of \( u \in \Gamma(\mathcal{H}^{p,q}) \).

### 3.3. Curvature of \( \mathcal{H}^{p,q} \)
In this section, we shall assume that \( X \) possesses a Kähler form, say \( \omega \), and \( \mathcal{H}^{p,q} \) satisfies \( A \). For each \( 1 \leq j \leq m \), we shall denote by \( V_j \) the horizontal lift vector field of \( \partial/\partial t^j \) with respect to \( \omega \).

Let \( u, v \) be two holomorphic sections of \( \mathcal{H}^{p,q} \). By Proposition 3.2 and (3.25), we have
\begin{equation}
(3.36) \quad \frac{\partial}{\partial t}(u, v) = \pi_\ast\{L_j u, v^*\}.
\end{equation}
Thus we have
\begin{align}
(3.37) \quad (u, v)_{j\bar{k}} &= \frac{\partial}{\partial t^k}\pi_\ast\{L_j u, v^*\} = \pi_\ast\{L_{\bar{k}} L_j u, v^*\} \\
(3.38) &= \pi_\ast\{\{L_k L_j u, v^*\} + \{L_j L_k u, L_k v^*\}\} \\
(3.39) &= \pi_\ast\{\{L_k L_j u, v^*\} + \{L_j L_k u, v^*\} + \{L_j u, L_k v^*\}\}.
\end{align}
Since \( u \) is a holomorphic section, by Theorem 3.1 for bidegree reason, we have
\begin{equation}
(3.40) \quad \pi_\ast\{L_k u, v^*\} = 0.
\end{equation}
Thus
\begin{equation}
(3.41) \quad 0 = \frac{\partial}{\partial t^k}\pi_\ast\{L_k u, v^*\} = \pi_\ast\{\{L_j L_k u, v^*\} + \{L_k L_j u, v^*\}\}.
\end{equation}
By (3.39), we have
\begin{equation}
(3.42) \quad (u, v)_{j\bar{k}} = \pi_\ast\{\{L_k L_j u, v^*\} - \{L_k u, L_j v^*\} + \{L_j u, L_k v^*\}\}.
\end{equation}
For the last term, since each \( V_j \) is horizontal, by (3.18), we have
\begin{align}
\pi_\ast\{L_j u, L_k v^*\} &= \pi_\ast\{\{\partial^E, \delta_{V_j}\}u, \{\partial^E, \delta_{V_k}\}v^*\} + \{\overline{\partial}, \delta_{V_j}\}u, \{\overline{\partial}, \delta_{V_k}\}v^*\} \\
&= \{i^*_t[\partial^E, \delta_{V_j}]u, i^*_t[\partial^E, \delta_{V_k}]v\} - (\overline{D}_j X_{t^j} u^\prime, \overline{D}_k X_{t^k} v^\prime).
\end{align}
By the same reason, we have
\begin{equation}
(3.43) \quad \pi_\ast\{L_k u, L_j v^*\} = \{i^*_t[\overline{\partial}, \delta_{V_k}]u, i^*_t[\overline{\partial}, \delta_{V_j}]v\} - (\overline{D}_k X_{t^k} u^\prime, \overline{D}_j X_{t^j} v^\prime).
\end{equation}
By definition of the Chern connection, we have
\[(3.44) \quad (\Theta_{jk} u, v) = (D_{lj} u, D_{lk} v) - (u, v)_{jk}.\]
Put
\[(3.45) \quad a_j^u = D_{lj} u - i^*_t [\partial^E, \delta_{Vj}] u; \quad b_j^u = \overline{\partial V_j}|_{X_t, u} u^t, \]
and
\[(3.46) \quad a_j^v := i^*_t [\overline{\partial}, \delta_{Vj}] v, \quad b_j^v := \overline{\partial V_j}|_{X_t, v} v^t.\]
Then we have:

**Theorem 3.4.** Assume that $\mathcal{X}$ possesses a Kähler form $\omega$ and $\mathcal{H}^{p,q}$ satisfies $A$. Let $u$ and $v$ be holomorphic sections of $\mathcal{H}^{p,q}$. Then we have
\[(3.47) \quad (\Theta_{jk} u, v) = (b_j^u, b_k^v) - (a_j^u, a_k^v) + \pi_* \{ [L_j, L_k]|_u, v^* \} + (a_k^u, a_j^v) - (b_j^u, b_k^v).\]

Remark: For the middle term in the above formula, we shall use
\[(3.48) \quad [L_j, L_k] = [d^E, \delta_{[Vj, V_k]}] + \Theta(E, h)(V_j, V_k).\]
In order to study the other terms, we have to use Hörmander’s $L^2$-theory \[15\] for the generalized $\overline{\partial}$-equation.

3.4. **Generalized $\overline{\partial}$-equation associated to the curvature formula.** We shall use the following lemma:

**Lemma 3.5.** If both $u$ and $u'$ are representatives of $u$ then
\[(3.49) \quad i^*_t L_k(u - u') = 0, \quad i^*_t L_j(u - u') = 0.\]

Proof. By definition, we have
\[(3.50) \quad u - u' = \sum dt^j \land a_j + \sum dE^k \land b_k.\]
Since
\[(3.51) \quad (dE \delta_V + \delta_V dE)(df \land a) = d(V f) \land a + df \land (dE \delta_V + \delta_V dE)a.\]
Apply this formula to $f = t^j, l^k$, $V = V_j, V_k$ and $a = a_j, b_k$, we get (3.49). \[\square\]

Now we can prove:

**Lemma 3.6.** With the notation in Theorem [3.4], we have
\[(3.52) \quad (\overline{\partial}^*)^t a_j^u = (\overline{\partial}^*)^t b_j^v = 0, \quad \forall t \in B, \quad \forall 1 \leq j \leq m.\]
Moreover, if $\mathcal{H}^{p,q}_{t} \subset \ker \partial^E_{t}$ for every $t \in B$ then
\[(3.53) \quad \overline{\partial}^t a_j^u = \partial^E_{t} a_j^u + c_j^u, \quad \partial^E_{t} b_j^v = -\overline{\partial}^t b_j^v - c_j^v,\]
and each $a_j^u$ is the $L^2$-minimal solution of (3.53), where
\[(3.54) \quad c_j^u := \overline{(V_j \cup \Theta(E, h))}|_{X_t} \land u^t, \quad c_j^v := \overline{(V_j \cup \Theta(E, h))}|_{X_t} \land v^t.\]
Assume further that $\mathcal{H}^{p,q}_{t} = \ker \partial^E_{t} \cap \ker (\partial^E_{t})^*$ for every $t \in B$ then each $a_j^v$ is also the $L^2$-minimal solution of (3.53).
Proof. By (3.45), we have
\[(3.55) \ (\overline{\partial}^t)^* a^u_j = -(\overline{\partial}^t)^* (i^*_t [\partial^E, \delta V_j] u) .\]

Since \((\overline{\partial}^t)^* = - * \partial^{E_t *},\) by (3.18), the following equality
\[(3.56) \ i^*_t (\partial^E [\partial^E, \delta V_j] u^*) = 0 .\]
implies \((\overline{\partial}^t)^* a = 0.\) Notice that
\[(3.57) \ \partial^E [\partial^E, \delta V_j] = [\partial^E, \delta V_j] \partial^E ,\]
and
\[(3.58) \ i^*_t \partial^E u^* = \partial^{E_t} * u^t = 0 .\]

Thus (3.56) follows from (3.51). By the same proof, we have \((\overline{\partial}^t)^* b^v_j = 0,\) thus (3.52) is true. Now let us prove (3.53). By (3.45), we have
\[(3.59) \ \overline{\partial} a^u_j - \partial^{E_t} b^v_j = -i^*_t (\overline{\partial} [\partial^E, \delta V_j] u + \partial^E [\bar{\partial}, \delta V_j] u) .\]

Since by our assumption, \(H^{p,q}_t \subset \ker \partial^{E_t},\) thus we have
\[(3.60) \ i^*_t (\partial^E u) = 0, \ i^*_t (\bar{\partial} u) = 0 ,\]
by (3.51), we have
\[(3.61) \ i^*_t [\partial^E, \delta V_j] \bar{\partial} u = i^*_t [\bar{\partial}, \delta V_j] \partial^E u = 0 .\]

Thus
\[(3.62) \ \overline{\partial} a^u_j - \partial^{E_t} b^v_j = -i^*_t (\overline{\partial} [\partial^E, \delta V_j] u + [\partial^E, [\bar{\partial}, \delta V_j]] u) = i^*_t [\delta V_j, [\bar{\partial}, \partial^E]] u = c^v_j .\]

By the same proof, we have
\[\partial^{E_t} a^v_j = - \bar{\partial}^t b^v_j - c^v_j .\]

Thus (3.53) is true. Now let us prove the last part. Since \(v\) is a holomorphic section, by Theorem 3.1, we have that each \(a^v_j\) has no \(\overline{\partial}\)-harmonic part. By our assumption, each \(\overline{\partial}\)-harmonic space is equal to the \(\partial^{E_t}\)-harmonic space, we know that each \(a^v_j\) has no \(\partial^{E_t}\)-harmonic part. Thus it is sufficient to prove that each \(a^v_j\) is \((\partial^{E_t})^*\)-closed. Since \((\partial^{E_t})^* = - * \overline{\partial}^* ,\) it is sufficient to show that
\[(3.63) \ \overline{\partial}^* a^v_j = 0 .\]

By (3.18), we know that
\[(3.64) \ \overline{\partial}^* a^v_j = \overline{\partial} i^*_t [\bar{\partial}, \delta V_j] v^* = i^*_t [\bar{\partial}, \delta V_j] \overline{\partial} v^* .\]

Moreover, by our assumption, each \(v^j\) is also in the \(\partial^{E_t}\)-harmonic space. Thus we have
\[(3.65) \ i^*_t \overline{\partial} v^* = \overline{\partial}^* v^t = 0 ,\]
by (3.51) and (3.64), we know that \(\overline{\partial}^* a^v_j = 0.\) The proof is complete. \(\square\)

By (3.47) and the above lemma, the \(L^2\)-estimates for the generalized \(\overline{\partial}\)-equation (see (3.53)) determine the positivity of \(H^{p,q}\). We shall show how to use the following version of Hörmander’s \(L^2\)-theory \([15]\) to study the curvature of \(H^{p,q}:\)
Theorem 3.7. Let \((E, h)\) be a Hermitian vector bundle over an \(n\)-dimensional compact complex manifold \(X\). Let \(q \geq 0\) be an integer. Assume that \(X\) possesses a Hermitian metric \(\omega\) such that \(\omega_{\text{max}\{q,1\}}\) is \(\partial\bar{\partial}\)-closed. Let \(v\) be a smooth \(\partial\bar{\partial}\)-closed \(E\)-valued \((n, q + 1)\)-form. Assume that

\[
\iota \Theta(E, h) \wedge \omega_q > 0 \text{ on } X, \quad \text{(resp. } \iota \Theta(E, h) \wedge \omega_q \equiv 0 \text{ on } X),
\]

and

\[
I(v) := \inf_{v = \iota \partial \bar{\partial}I} |b|_0^2 + ([\iota \Theta(E, h), \Lambda_\omega]^{-1} c, c)_\omega < \infty, \quad \text{(resp. } I(v) := \inf_{v = \iota \partial \bar{\partial}I} |b|_0^2 < \infty).
\]

Then there exists a smooth \(E\)-valued \((n, q)\)-form \(a\) such that \(\partial a = v\) and

\[
|a|_0^2 \leq I(v).
\]

Proof. Let \(\gamma\) be an arbitrary \(E\)-valued smooth \((n - q - 1, 0)\)-form on \(X\). Put

\[
T = i^{(n-q-1)^2} \{\gamma, \gamma\} \wedge \omega_q, \quad u = \gamma \wedge \omega_{q+1}.
\]

Since \(\partial \omega_q = 0\), one may check that \(i \partial \bar{\partial} T\) can be written as

\[
-2 \text{Re} \langle \partial \bar{\partial} u, u \rangle \omega_n + |\partial \bar{\partial} u|^2 \omega_n + i^{(n-q-1)^2} \{i \Theta(E, h) \gamma, \gamma\} \wedge \omega_q - S,
\]

where

\[
S = i^{(n-q)^2} \{\partial \gamma, \bar{\partial} \gamma\} \wedge \omega_q.
\]

By Lemma 4.2 in Berndtsson’s lecture notes \([6]\), we have

\[
S = (|\omega_{q+1} \wedge \partial \gamma|^2 - |\partial \gamma|^2) \omega_n.
\]

Since \(\omega_{\text{max}\{q,1\}}\) is \(\partial\bar{\partial}\)-closed, we have \(\partial \omega_{q+1} = 0\), thus

\[
S = (\partial \bar{\partial} u|^2 - |(\partial E^*) u|^2) \omega_n.
\]

Notice that \(\int_X i \partial \bar{\partial} T = 0\), thus we have

\[
|\partial \bar{\partial} u|^2 + |\partial \bar{\partial} u|^2 - |(\partial E^*) u|^2 = \int_X i^{(n-q-1)^2} \{i \Theta(E, h) \gamma, \gamma\} \wedge \omega_q
\]

\[
= ([i \Theta(E, h), \Lambda_\omega] u, u),
\]

where \(\Lambda_\omega\) is the adjoint of \(\omega \wedge \cdot\). By Hörmander’s \(L^2\) theory, it suffices to show that

\[
|(v, u)|^2 \leq I(v)(|\partial \bar{\partial} u|^2 + |\partial \bar{\partial} u|^2), \quad \forall \text{ smooth } u,
\]

which follows from (3.72) and (3.73) by the Cauchy-Schwartz inequality. \(\square\)

We shall also use the following generalized version of Theorem 3.7 (see Remark 13.5 in \([11]\) for related results):

Theorem 3.8. Let \((E, h)\) be a Hermitian vector bundle over an \(n\)-dimensional compact complex manifold \(X\). Let \(q \geq 0\) be an integer. Assume that \(X\) possesses a Hermitian metric \(\omega\) such that \(\omega_{\text{max}\{q,1\}}\) is \(\partial\bar{\partial}\)-closed. Let \(v\) be a smooth \(\partial\bar{\partial}\)-closed \(E\)-valued \((n, q + 1)\)-form. Assume that \(i \Theta(E, h) \wedge \omega_q \geq 0\) on \(X\) and

\[
I(v) := \inf_{v = \iota \partial \bar{\partial}I} \left\{ ||b||_\omega^2 + \lim_{\varepsilon \to 0} \left( [i \Theta(E, h), \Lambda_\omega] + \varepsilon \right)^{-1} c, c \right\}_\omega \leq \infty.
\]

Then there exists a smooth \(E\)-valued \((n, q)\)-form \(a\) such that \(\partial a = v\) and

\[
|a|_0^2 \leq I(v).
\]
Proof. By (3.72) and (3.73), for every \( \varepsilon > 0 \), we have
\[
|(v, u)|^2 \leq I(v)(||\overline{\partial} u||^2 + ||\overline{\partial} v||^2 + \varepsilon ||u||^2), \quad \forall \text{ smooth } u,
\]
By Hörmander’s \( L^2 \) theory, there exist \( a_\varepsilon \) and \( \alpha_\varepsilon \) such that
\[
(3.77) \quad \overline{\partial} a_\varepsilon + \sqrt{\varepsilon} \alpha_\varepsilon = v,
\]
and
\[
(3.78) \quad ||a_\varepsilon||^2 + ||\alpha_\varepsilon||^2 \leq I(v).
\]
Thus \( \sqrt{\varepsilon} \alpha_\varepsilon \to 0 \) in the sense of current as \( \varepsilon \to 0 \). By taking a weak limit, say \( a' \), of \( a_\varepsilon \), we know that there exists \( a' \) such that \( \overline{\partial} a' = v \) in the sense of current and
\[
(3.79) \quad ||a'||^2 \leq I(v).
\]
Let \( a \) be the \( L^2 \)-minimal solution. Then \( a \) fits our needs. \( \square \)

3.5. Curvature of \( \mathcal{H}^{n,q} \). Theorem 3.3 implies that there is a good \( L^2 \)-estimate for (3.53) if \( p = n \). We shall show how to use it to prove Theorem 1.1 in this subsection.

Let us study the middle term in (3.47) first. By (3.48), we have
\[
(3.80) \quad \pi_*\{[L_j, L_k]u_j, u_k^*\} = (\Theta(E, E)(V_j, V_k)u_j, u_k) + (\overline{\partial}E_v \delta_{ij}u_j, u_k).
\]
The first term is clear. For the second term, we need the following lemma:

Lemma 3.9. \( \sum(\overline{\partial}E_v \delta_{ij}u_j, u_k) \) can be written as \( A + NAK \), where
\[
(3.81) \quad A := \sum \epsilon^{(n-q)} \pi_*\{c_{jk}(\omega)\omega_{q-1} \wedge i\Theta(E, h)u_j^*, u_k^*\}, \quad c_{jk}(\omega) := \langle V_j, V_k \rangle_{\omega},
\]
and \( NAK := \sum(c_{jk}(\omega)\overline{\partial} v, u_j^*\overline{\partial} u_k^* \rangle \geq 0 \).

We shall prove the above Lemma later. By the above lemma, we have
\[
(3.82) \quad \sum \pi_*\{[L_j, L_k]u_j, u_k^*\} = A + B + NAK,
\]
where
\[
(3.83) \quad B := \sum (\Theta(E, h)(V_j, V_k)u_j, u_k).
\]

Thus by (3.47), put
\[
(3.84) \quad b = \sum b_j^u, \quad a = \sum a_j^u,
\]
than we have (notice that \( b_j^u = 0 \) since \( p = n \))
\[
(3.85) \quad \sum (\Theta_{jk}u_j, u_k) = ||b||^2 - ||a||^2 + A + B + NAK + GRJ,
\]
where
\[
(3.86) \quad GRJ := \sum(a_j^u, a_j^u),
\]
is non-negative if \( u_j = \xi_j u \). Put
\[
(3.87) \quad C_{\varepsilon} := \left( ([i\Theta(E^\varepsilon, h), A_{\omega}^\varepsilon] + \varepsilon)^{-1}c, c \right), \quad c := \sum c_j^u.
\]
Then we have
\[
(3.88) \quad \sum(\Theta_{jk}u_j, u_k) = A + B - C_{\varepsilon} + NAK + GRJ + R_{\varepsilon},
\]
where
\[
(3.89) \quad R_{\varepsilon} := \left( ([i\Theta(E^\varepsilon, h), A_{\omega}^\varepsilon] + \varepsilon)^{-1}c, c \right) + ||b||^2 - ||a||^2.
\]
Proof of Theorem 3.11 By (3.33) and Theorem 3.8 we know that $R_\varepsilon$ is always non-negative. In order to show that $H^{n,q}$ is Griffiths semi-positive, it suffices to prove
\[
\lim_{\varepsilon \to 0} (A + B - C_\varepsilon) \geq 0,
\]
which follows from the following lemma:

**Lemma 3.10.** If $\Theta(E, h)$ is $q$-semipositive with respect to $\omega$ then
\[
A + B + \varepsilon \sum (c_{jk}(\omega)u_j, u_k) - C_\varepsilon \geq 0, \quad \forall \varepsilon > 0.
\]

**Proof.** Put
\[
I_\varepsilon := A + B + \varepsilon \sum (c_{jk}(\omega)u_j, u_k) - C_\varepsilon,
\]
and
\[
T_\varepsilon := \omega_q \wedge i\Theta(E, h) + \varepsilon \omega_{q+1} \otimes Id_E, \quad T_\varepsilon^1 := T_\varepsilon|_{X_t}.
\]
We claim that for each $j$ one may choose a dual representative $u_j^*$ of $u_j$ such that
\[
i_j^* \delta V_j (T_\varepsilon \wedge u_j^*) \equiv 0,
\]
on $X_t$ for every $t \in B$ and
\[
I_\varepsilon \wedge t^m dt \wedge dt = \pi_* (c_q \{ T_\varepsilon \wedge u_j^* \}) \geq 0,
\]
where $c_q := i^{(m+n-q-1)^2}$ and $dt := dt^1 \wedge \cdots \wedge dt^m$.
In fact, (3.93) is equivalent to
\[
\omega_j^t \wedge \omega_{j+1}^t \wedge \varepsilon \omega_{j+1}^t \delta \omega_{j+1}^t \equiv 0.
\]
Notice that
\[
Q_\varepsilon^{-1}(T_\varepsilon^t \wedge i_j^* \delta V_j u_j^* ) = \omega_q^t + \varepsilon \omega_{q+1}^t \delta \omega_{q+1}^t + i_j^* \delta V_j u_j^*;
\]
 Thus there exists $u_j^*$ such that (3.93) is true. Now choose $u_j^*$ that satisfies (3.93), then
\[
c = \sum i^{(n-q)^2+1} t_\varepsilon^t \wedge i_j^* \delta V_j u_j^*,
\]
which implies
\[
Q_\varepsilon^{-1} c = \sum i^{(n-q)^2+1} \omega_q^t \wedge \varepsilon \omega_{q+1}^t \delta \omega_{q+1}^t \equiv 0.
\]
Thus we have
\[
C_\varepsilon = (Q_\varepsilon^{-1} c, c) = \pi_* \left( i^{(n-q-1)^2} \{ T_\varepsilon \wedge \sum \delta V_j u_j^*, \sum \delta V_j u_j^* \} \right).
\]
Recall that
\[
A = \sum i^{(n-q)^2} \pi_* \{ c_{jk}(\omega) \omega_{q-1} \wedge \Theta(E, h) u_j^*, u_k^* \},
\]
and
\[
B = \sum i^{(n-q)^2} \pi_* \{ \omega_q \wedge \Theta(E, h) (V_j, \bar{V}_k) u_j^*, u_k^* \}.
\]
Notice that
\[
i_j^* \left( i \delta V_j \Theta_{\varepsilon} T_\varepsilon \right) = \omega_q^t \wedge \Theta(E, h) (V_j, \bar{V}_k) + i \Theta(E, h) \wedge c_{jk}(\omega) \omega_{q-1}^t + \varepsilon c_{jk}(\omega) \omega_q^t.
\]
Thus
\[
A + B + \varepsilon \sum (c_{jk}(\omega)u_j, u_k) = \sum i^{(n-q)^2} \pi_* \left( i \delta V_j \Theta_{\varepsilon} T_\varepsilon \wedge u_j^*, u_k^* \right).
\]
Notice that
\[
\delta_j \delta_{V_k} \{ T_\epsilon \wedge u_j^*, u_k^* \} = (-1)^{n-q} \{ \delta_j T_\epsilon \wedge u_j^*, \delta_{V_k} u_k^* \} - \{ \delta_{V_k} T_\epsilon \wedge \delta_j u_j^*, \delta_k u_k^* \} + \{ \delta_j \delta_{V_k} T_\epsilon \wedge u_j^*, u_k^* \} + (-1)^{n-q} \{ T_\epsilon \wedge \delta_j u_j^*, \delta_{V_k} u_k^* \}.
\]
By (3.93), we have
\[
(3.104) \quad \pi_* \{ \delta_j T_\epsilon \wedge u_j^*, \delta_{V_k} u_k^* \} = -\pi_* \{ T_\epsilon \wedge \delta_j u_j^*, \delta_{V_k} u_k^* \},
\]
and
\[
(3.105) \quad \pi_* \{ \delta_{V_k} T_\epsilon \wedge \delta_j u_j^*, u_k^* \} = (-1)^{n-q} \pi_* \{ T_\epsilon \wedge \delta_j u_j^*, \delta_{V_k} u_k^* \}.
\]
Thus
\[
(3.106) \quad \pi_* \delta_j \delta_{V_k} \{ T_\epsilon \wedge u_j^*, u_k^* \} = \pi_* \{ \delta_j \delta_{V_k} T_\epsilon \wedge u_j^*, u_k^* \} = (-1)^{n-q} \pi_* \{ T_\epsilon \wedge \delta_j u_j^*, \delta_{V_k} u_k^* \},
\]
which implies that
\[
(3.107) \quad \sum \int \frac{(-1)^{n-q} \pi_* \{ \delta_j \delta_{V_k} \{ T_\epsilon \wedge u_j^*, u_k^* \} \}}{\pi_* \{ T_\epsilon \wedge \delta_j u_j^*, \delta_{V_k} u_k^* \}} = I_\epsilon.
\]
Notice that for bi-degree reason, we have
\[
(3.108) \quad \{ \delta_j \delta_{V_k} \{ T_\epsilon \wedge u_j^*, u_k^* \} \} \wedge dt \wedge \overline{dt} = (-1)^m \{ T_\epsilon \wedge u_j^*, u_k^* \} \wedge \delta_j dt \wedge \overline{dt}.
\]
Thus (3.94) is true. The proof is complete.

**Proof of the remark behind Theorem 1.1** Since \( p = n \), we know that
\[
(3.109) \quad b_j^{u_k} = c_j^{u_k} = 0.
\]
Thus by Lemma 3.6 if \( H_t^{n,q} = \ker \partial E^t \cap \ker(\partial E_i)^* \) for every \( t \in B \) then we have
\[
(3.110) \quad a_j^{u_k} = 0,
\]
which implies that \( G \rho_i = 0 \). Thus \( H^{n,q} \) is Nakano semi-positive.

**Proof of Lemma 3.9** Put
\[
(3.111) \quad I_{jk} := (\partial E_i \delta_{[V_j, \overline{V}_k]} u_j, u_k).
\]
Denote by \( V \) the \((1,0)\)-part of \([V_j, \overline{V}_k]\), then we have
\[
(3.112) \quad I_{jk} = \int_{X_t} \{ \partial E_i \delta_\omega u_j^t, u_k^t \} = (-1)^{n-q} \int_{X_t} \{ \delta_\omega u_j^t, \overline{\delta}^t u_k^t \}.
\]
Since
\[
(3.113) \quad u_j^t = i^{(n-q)} \omega_q^t \wedge \ast u_j^t,
\]
and
\[
(3.114) \quad i^{(n-q)} \omega_q^t \wedge \overline{\delta}^t \ast u_k^t = \overline{\delta}^t u_k^t = 0,
\]
we have
\[
(3.115) \quad I_{jk} = (-1)^{n-q} i^{(n-q)} \int_{X_t} \{ \delta_\omega \omega_q^t \wedge \ast u_j^t, u_k^t \}.
\]
By definition, \( \delta_\omega \omega_q^t \) is the \((q-1, q)\)-part of
\[
(3.116) \quad u_i^t \left( (L_{V_j} \overline{V}_k) \wedge \omega_q \right).
\]
Since
\[
(3.117) \quad (L_{V_j} \overline{V}_k) \wedge \omega_q = L_{V_j} (\overline{V}_k \wedge \omega_q) - \overline{V}_k \wedge L_{V_j} \omega_q,
\]
and
\[(3.118) \quad V_k \cdot \omega_q = -i \sum c_{jk}(\omega)dt^j \wedge \omega_{q-1},\quad L_{V_j}\omega_q = d\left(i \sum c_{jk}(\omega)dt^k \wedge \omega_{q-1}\right).
\]

Thus
\[(3.119) \quad i_t^*((L_{V_j} \tilde{V}_k) \cdot \omega_q) = i_t^*d\left(i c_{jk}(\omega) \wedge \omega_{q-1}\right),\]

and
\[(3.120) \quad \delta_V \omega^t_q = \bar{\partial} \left(i c_{jk}(\omega) \wedge \omega^t_{q-1}\right).
\]

Now \(I_{jk}\) can be written as
\[i^{(n-q+1)^2} \int_{X_t} \{\bar{\partial} (c_{jk}(\omega) \wedge \omega^t_{q-1} \wedge *u_j^t), \bar{\partial} \ast u_k^t\} - \{c_{jk}(\omega) \wedge \omega^t_{q-1} \wedge \bar{\partial} \ast u_j^t, \bar{\partial} \ast u_k^t\}.
\]

By (3.114), each \(\bar{\partial} \ast u_k^t\) is primitive, thus
\[(3.121) \quad - \sum i^{(n-q+1)^2} \int_{X_t} \{c_{jk}(\omega) \wedge \omega^t_{q-1} \wedge \bar{\partial} \ast u_j^t, \bar{\partial} \ast u_k^t\} = Nak.
\]

Now it suffices to show
\[(3.122) \quad A = \sum i^{(n-q+1)^2} \int_{X_t} \{\bar{\partial} (c_{jk}(\omega) \wedge \omega^t_{q-1} \wedge *u_j^t), \bar{\partial} \ast u_k^t\}.
\]

Notice that the right hand side can be written as
\[(3.123) \quad \sum i^{(n-q)^2}(-i) \int_{X_t} \{c_{jk}(\omega) \wedge \omega^t_{q-1} \wedge *u_j^t, \bar{E}_t \partial \ast u_k^t\}.
\]

Since each \(u_k^t\) is harmonic, we have \(\partial E_t \ast u_k^t \equiv 0\) and
\[(3.124) \quad \partial E_t \partial \ast u_k^t = \Theta(E_t, h) \ast u_k^t.
\]

Thus (3.122) follows from (3.123) and (3.124). The proof is complete.

4. Curvature of the Weil-Petersson metric

We shall prove Theorem 1.2 in this section. Let \(\pi : \mathcal{X} \to B\) be a proper holomorphic submersion. Then we have the classical Kodaira-Spencer map
\[(4.1) \quad \kappa : v \mapsto \kappa(v) \in H^{0,1}(T_{X_t}) \simeq H^{n,n-1}(T_{X_t}^*), \quad \forall v \in T_tB, \quad t \in B.
\]

we shall prove that:

**Proposition 4.1.** Assume that the dimension of \(H^{0,1}(T_{X_t})\) does not depend on \(t\). Then
\[(4.2) \quad \kappa : v \mapsto \kappa(v) \in H^{n,n-1}(T_{X_t}^*), \quad \forall v \in T_tB,
\]
defines a holomorphic bundle map from \(T_B\) to the dual bundle of
\[(4.3) \quad \mathcal{H}^{n,n-1} := \{H^{n,n-1}(T_{X_t}^*)\}_{t \in B}.
\]

**Proof.** Let \(v : t \mapsto v^t\) be a holomorphic vector field on \(B\). Let \(V\) be a smooth \((1,0)\)-vector field on the total space such that \(\pi_*V = v\). Then we know that for each \(t\), \((\partial V)|_{X_t}\) defines a representative of \(\kappa(v^t)\). It suffices to prove that
\[(4.4) \quad \pi_*(u \wedge \partial V)
\]
is holomorphic for every holomorphic section \( u \) of \( H^{n,n-1} \), where \( u \) is an arbitrary representative of \( u \). Let us write
\[
\bar{\partial}u = \sum dt^j \wedge a_j + \sum d\bar{t}^k \wedge b_k.
\]
By the proof of Theorem \( 3.1 \), we know that each \( b_k|_{X_t} \) is \( \bar{\partial} \)-closed and the cohomology class of \( b_k|_{X_t} \) does not depend of the choice of \( b_k \). Moreover, if \( u \) is a holomorphic section then the cohomology class of \( b_k|_{X_t} \) is zero. Thus
\[
\pi_*(b_k \wedge \bar{\partial}V) = 0,
\]
and
\[
\bar{\partial}_\pi(u \wedge \bar{\partial}V) = \pi_*(\bar{\partial}u \wedge \bar{\partial}V) = \sum d\bar{t}^k \wedge \pi_*(b_k \wedge \bar{\partial}V) = 0,
\]
which implies that \( \pi_*(u \wedge \bar{\partial}V) \) is holomorphic. The proof is complete. \( \square \)

**Proof of Theorem 1.2**: Let \( v : t \mapsto v^t \) be an arbitrary holomorphic vector field on \( B \). Let us denote by \( ||v||_{WP} \) the upper semicontinuous regularization of
\[
t \mapsto ||v^t||_{WP}.
\]
Then it suffices to prove that \( \log ||v||_{WP} \) is plurisubharmonic or equal to \(-\infty\) identically. Since there is a Zariski open subset, say \( B_0 \), of \( B \) such that the dimension of \( H^{n,n-1}(T^*_X) \) is a constant on \( B_0 \), by Theorem 3.1 we know that \( H^{n,n-1} \) is Griffiths-semipositive on the complement of a proper analytic subset. Thus by Proposition 4.1, we know that \( \log ||v||_{WP} \) is plurisubharmonic or equal to \(-\infty\) identically on the complement of a proper analytic subset. Now it is sufficient to prove that \( ||v||_{WP} \) is locally bounded from above.

Let \( V \) be a smooth \((1,0)\)-vector field on the total space such that \( \pi_* V = v \). By definition, we know that
\[
||v^t||_{WP} \leq ||(\bar{\partial}V)|_{X_t}||_{H^{0,1}(T^*_X)}.
\]
Since
\[
t \mapsto ||(\bar{\partial}V)|_{X_t}||_{H^{0,1}(T^*_X)},
\]
is smooth, we know that \( ||v||_{WP} \) is locally bounded from above. The proof is complete.

5. **Acknowledgement**

I would like to thank Bo Berndtsson for many inspiring discussions (in particular for his suggestion to relate our results to the Weil-Petersson geometry), and for his many useful comments on this paper. Thanks are also given to Bo-Yong Chen and Qing-Chun Ji for their constant support and encouragement.

**References**

[1] T. Aubin, *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, Bull. Sci. Math. **102** (1978), 63–95.

[2] B. Berndtsson, *An eigenvalue estimate for the \( \bar{\partial} \)-Laplacian*, J. Differential Geom. **60** (2002), 295–313.

[3] B. Berndtsson, *Subharmonicity properties of the Bergman kernel and some other functions associated to pseudoconvex domains*, Ann. Inst. Fourier (Grenoble), **56** (2006), 1633–1662.

[4] B. Berndtsson, *Curvature of vector bundles associated to holomorphic fibrations*, Ann. Math. **169** (2009), 531–560.

[5] B. Berndtsson, *Positivity of direct image bundles and convexity on the space of Kähler metrics*, J. Diff. Geom. **81** (2009), 457–482.
[6] B. Berndtsson, *An introduction to things $\bar{\partial}$*, Analytic and algebraic geometry, IAS/Park City Math. Ser., vol. 17, Amer. Math. Soc., Providence, RI, 2010, 7–76.

[7] B. Berndtsson, *Strict and nonstrict positivity of direct image bundles*, Math. Z. 269 (2011), 1201–1218.

[8] B. Berndtsson, *$L^2$-extension of $\bar{\partial}$-closed form*, Illinois J. Math. 56 (2012), 21–31.

[9] B. Berndtsson and M. Paun, *Bergman kernel and the pseudoeffectivity of relative canonical bundles*, Duke Math. J. 145 (2008), 341–378.

[10] B. Berndtsson, M. Paun and X. Wang, *Iterated Kodaira-Spencer map and its applications*, preprint.

[11] J.-P. Demailly, *$L^2$ estimates for the $\bar{\partial}$-bar operator on complex manifolds*. Lecture notes available from the author’s homepage, 1996.

[12] J.-P. Demailly, *Analytic Methods in Algebraic Geometry*. Higher Education Press, Surveys of Modern Mathematics, 2010, 231 pages.

[13] J.-P. Demailly, *Complex analytic and differential geometry*. Book available from the author’s homepage.

[14] T. Geiger, G. Schumacher, *Curvature of higher direct image sheaves*, arXiv:1501.07070v1 [math.AG]

[15] L. Hörmander, *$L^2$-estimates and existence theorems for the $\bar{\partial}$-operator*, Acta Math. 113 (1965), 89–152.

[16] K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures, III. Stability theorems for complex structures*, Ann. Math. 71 (1960), 43–76.

[17] K. Liu and X. Yang, *Curvatures of direct image sheaves of vector bundles and applications*, J. Diff. Geom. 98 (2014), 117–145.

[18] F. Maitani, *Variations of meromorphic differentials under quasiconformal deformations*, J. Math. Kyoto Univ. 24 (1984), 49–66.

[19] F. Maitani, H. Yamaguchi, *Variation of Bergman metrics on Riemann surfaces*, Math. Ann. 330 (2004), 477–489.

[20] C. Mourougane, S. Takayama, *Hodge metrics and positivity of direct images*, J. Reine Angew. Math. 606 (2007), 167–178.

[21] C. Mourougane, S. Takayama, *Hodge metrics and the curvature of higher direct images*, Ann. Sci. Éc. Norm. Supér. 41 (2008), 905–924.

[22] A. M. Nadel, *Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature*, Proc. Nat. Acad. Sci. U.S.A., 86 (1989), 7299–7300 and Annals of Math., 132 (1990), 549–596.

[23] H. Raufi, *Singular hermitian metrics on holomorphic vector bundles*. Arkiv för Matematik, 53 (2015), 359–382.

[24] A. Newlander and L. Nirenberg, *Complex Analytic Coordinates in Almost Complex Manifolds*. Annals of Mathematics, 65 (1957), 391–404.

[25] G. Schumacher, *Positivity of relative canonical bundles and applications*, Invent. Math. 190 (2012), 1–56.

[26] G. Schumacher *Curvature properties for moduli of canonically polarized manifolds: An analogy to moduli of Calabi-Yau manifolds*, Comptes Rendus Mathematique, 352 (2014), 835–840.

[27] Y. T. Siu, *Complex-analyticity of harmonic maps, vanishing and Lefschetz theorems*, J. Differential Geom. 17 (1982), 55–138.

[28] W. K. To and S. K. Yeung, *Finsler Metrics and Kobayashi hyperbolicity of the moduli spaces of canonically polarized manifolds*, Ann. Math. 181 (2015), 547–586.

[29] H. Tsuji, *Variation of Bergman kernels of adjoint line bundles*, arXiv:0511342 [math.CV].

[30] X. Wang, *A curvature formula associated to a family of pseudoconvex domains*, arXiv:1508.00242, to appear in Annales de l’Institut Fourier.

[31] X. Wang, *Subharmonicity properties of the $L^2$-minimal solution of the $\bar{\partial}$-equation*, preprint.

[32] X. Wang, *Relative $\bar{\partial}$-complex and its curvature properties*, preprint.

[33] X. Wang, *Higgs bundle*, preprint.

[34] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I*, Comm. Pure Appl. Math. 31 (1978), 339–411.

**School of Mathematical Sciences, Fudan University, Shanghai, 200433, China**

**Current address:** Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg. SE-412 96 Gothenburg, Sweden

**E-mail address:** wangxu1113@gmail.com

**E-mail address:** xuwa@chalmers.se