ON A CLASS OF MATRIX PENCILS EQUIVALENT TO A GIVEN MATRIX POLYNOMIAL

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Abstract. We say that an \( m \times m \) matrix polynomial \( P(x) = \sum_{i=0}^{n} P_i x^i \) is equivalent to an \( m q \times m q \) matrix polynomial \( A(x) \), and write \( A(x) \equiv P(x) \), if there exist \( m q \times m q \) matrix polynomials \( E(x), F(x) \) such that \( \det E(x) \) and \( \det F(x) \) are nonzero constants and \( E(x)A(x)F(x) = I_{m(q-1)} \oplus P(x) \). Given \( P(x) \) of degree \( n \) we provide an \( m q \times m q \) matrix polynomial \( A(x) \) such that: \( A(x) \equiv P(x), A^\#(x) \equiv P^\#(x) \), where \( P^\#(x) = x^n P(x^{-1}) \) is the reversed polynomial of \( P(x) \); \( A(x) \) has the form \( A(x) = D(x) + [I_m, \ldots, I_m][W_1(x), \ldots, W_q(x)] \), where \( D(x) \) is a diagonal matrix defined by \( D(x) = \text{diag}(b_1(x), I_m, \ldots, b_q(x), I_m) \), the polynomials \( b_1(x), \ldots, b_q(x) \) are any co-prime monic polynomials of degree \( d_1, \ldots, d_q \), respectively, while \( W_1(x), \ldots, W_q(x) \) are matrix polynomials of degree less than \( d_1 + \cdots + d_q = n \) and \( s \) is a constant which makes \( b_i(x)P_m + sI_m \) nonsingular modulo \( b_i(x), i = 1, \ldots, q - 1 \). An explicit expression of the eigenvectors of \( A(x) \) as functions of the eigenvalues is proven. For \( b_i(x) = (x - \beta_i)I_m \), \( i = 1, \ldots, n \), the matrix polynomial \( A(x) \) is a linear pencil of the form diagonal plus low-rank. Numerical experiments show that for suitable choices of \( \beta_1, \ldots, \beta_n \) obtained by means of the generalized Pellet theorem and the use of tropical roots, the eigenvalue problem for \( A(x) \) is much better conditioned than the eigenvalue problem for \( P(x) \).

Keywords: Matrix polynomials, matrix pencils, linearizations, companion matrix, tropical roots

1. Introduction. A standard way to deal with an \( m \times m \) matrix polynomial \( P(x) = \sum_{i=0}^{n} P_i x^i \) is to convert it to a linear pencil, that is to a linear matrix polynomial of the form \( L(x) = Ax - B \) where \( A \) and \( B \) are \( mn \times mn \) matrices such that \( \det P(x) = \det L(x) \). This process, known as linearization, has been introduced in [15].

In certain cases, like for matrix polynomials modeling Non-Skip-Free stochastic processes [4], it is more convenient to reduce the matrix polynomial to a quadratic polynomial of the form \( Ax^2 + Bx + C \), where \( A, B, C \) are matrices of suitable size [4]. The process that we obtain this way is referred to as quadratization. If \( P(x) \) is a matrix power series, like in \( M/G/1 \) Markov chains [25, 24], the quadratization of \( P(x) \) can be obtained with block coefficients of infinite size [27]. In this framework, the quadratic form is desirable since it is better suited for an effective solution of the stochastic model; in fact it corresponds to a QBD process for which there exist efficient solution algorithms [4, 20]. In other situations it is preferable to reduce the matrix polynomial \( P(x) \) of degree \( n \) to a matrix polynomial of lower degree \( \ell \). This process is called \( \ell \)-ification in [13].

Techniques for linearizing a matrix polynomial have been widely investigated. Different companion forms of a matrix polynomial have been introduced and analyzed, see for instance [2] [22] [12] and the literature cited therein, a wide literature exists on matrix polynomials with contribution of many authors [1] [9] [10] [11] [12] [15] [19] [17] [18] [28] [30], motivated both by the theoretical interest of this subject and by the many applications that matrix polynomials have [1] [20] [21] [25] [24] [29]. Techniques for reducing a matrix polynomial, or a matrix power series into quadratic form, possibly with coefficients of infinite size, have been investigated in [27] [4]. Reducing a matrix polynomial to a polynomial of degree \( \ell \) is analyzed in [13].

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Denote $\mathbb{C}[x]^{m \times m}$ the set of $m \times m$ matrix polynomials over the complex field $\mathbb{C}$. If $P(x) = \sum_{i=0}^{n} P_i x^i \in \mathbb{C}[x]^{m \times m}$ and $P_n \neq 0$ we say that $P(x)$ has degree $n$. If $\det P(x)$ is not identically constant we say that $P(x)$ is regular. Throughout the paper we assume that $P(x)$ is a regular polynomial of degree $n$. The following definition is useful in our framework.

**Definition 1.1.** Let $P(x) \in \mathbb{C}[x]^{m \times m}$ be a matrix polynomial of degree $n$. Let $q$ be an integer such that $0 < q \leq n$. We say that a matrix polynomial $Q(x) \in \mathbb{C}[x]^{m \times m}$ is equivalent to $P(x)$, and we write $P(x) \approx Q(x)$ if there exist two matrix polynomials $E(x), F(x) \in \mathbb{C}[x]^{m \times m}$ such that $\det E(x)$ and $\det F(x)$ are nonzero constants, that is $E(x)$ and $F(x)$ are unimodular, and

$$E(x)Q(x)F(x) = \begin{bmatrix} I_{m(q-1)} & 0 \\ 0 & P(x) \end{bmatrix} =: I_{m(q-1)} \oplus P(x).$$

Denote $P^\#(x) = x^n P(x^{-1})$ the reversed polynomial obtained by reverting the order of the coefficients. We say that the polynomials $P(x)$ and $Q(x)$ are strongly equivalent if $P(x) \approx Q(x)$ and $P^\#(x) \approx Q^\#(x)$. If the degree of $Q(x)$ is 1 and $P(x) \approx Q(x)$ we say that $Q(x)$ is a linearization of $P(x)$. Similarly, we say that $Q(x)$ is a strong linearization if $Q(x)$ is strongly equivalent to $P(x)$ and $\deg Q(x) = 1$. If $Q(x)$ has degree $\ell$, we use the terms $\ell$-ification and strong $\ell$-ification.

It is clear from the definition that $P(x) \approx Q(x)$ implies $\det P(x) = \kappa \det Q(x)$ where $\kappa$ is some nonzero constant, but the converse is not generally true. The equivalence property is actually stronger because it preserves also the eigenstructure of the matrix polynomial, and not only the eigenvalues. For a more in-depth view of this subject see [13].

In the literature, a number of different linearizations have been proposed. The most known are probably the Frobenius and the Fiedler linearizations [23]. One of them is, for example,

$$xA - B = x \begin{bmatrix} I_m & & & \\ & \ddots & & \\ & & I_m & \\ P_n & & & I_m \end{bmatrix} - \begin{bmatrix} I_m & -P_0 \\ & \ddots & -P_1 \\ & & \ddots & -P_{n-1} \\ & & & I_m \end{bmatrix},$$

where $I_m$ denotes the identity matrix of size $m$.

**1.1. New contribution.** In this paper we provide a general way to transform a given $m \times m$ matrix polynomial $P(x) = \sum_{i=0}^{n} P_i x^i$ of degree $n$ into a strongly equivalent matrix polynomial $A(x)$ of lower degree $\ell$ and larger size endowed with a strong structure. The technique relies on representing $P(x)$ with respect to a basis of matrix polynomials of the form $C_i(x) = \prod_{j=1, j \neq i}^{q} B_i(x)$, $i = 1, \ldots, q$, where $B_i(x)$ are matrix polynomials of degree $d_i$, for $i = 1, \ldots, q$, pairwise commute and are right coprime. That is, $B_i B_j = B_j B_i$ for any pair $i, j$, and the equation $B_i \alpha + B_j \beta = I_m$ has a solution $\alpha = \alpha_{i,j}, \beta = \beta_{i,j}$ for any $i, j$, where $\alpha_{i,j}$ and $\beta_{i,j}$ are matrix polynomials.

According to the choice of the basis we arrive at different $\ell$-ifications $A(x)$, where $\ell \geq [n/q]$ is determined by the degree of the $B_i(x)$, represented as a $q \times q$ block diagonal matrix with $m \times m$ blocks plus a matrix of rank at most $m$. An example of
\( \ell \)-ification \( A(x) \) is given by

\[
A(x) = D(x) + (e \otimes I_m)[W_1, \ldots, W_q],
\]

\[
D(x) = \text{diag}(B_1(x), \ldots, B_q(x)), \quad e = (1, \ldots, 1)^t \in \mathbb{R}^q,
\]

\[
B_i(x) = b_i(x)I_m, \quad \text{for } i = 1, \ldots, q - 1,
\]

\[
B_q(x) = b_q(x)P_n + sI_m, \quad d_q = \deg b_q(x),
\]

\[
W_i(x) \in \mathbb{C}[x]^{m \times m}, \quad \deg W_i(x) < \deg B_i(x),
\]

where \( b_1(x), \ldots, b_q(x) \) are pairwise co-prime monic polynomials of degree \( d_1, \ldots, d_q \), respectively, such that \( n = d_1 + \cdots + d_q \), and \( s \) is such that \( \lambda b_q(\xi) + s \neq 0 \) for any eigenvalue \( \lambda \) of \( P_n \) and for any root \( \xi \) of \( b_i(x) \) for \( i = 1, \ldots, q - 1 \). The matrix polynomial \( A(x) \) has degree \( \ell = \max\{d_1, \ldots, d_q\} \geq \left\lfloor \frac{n}{q} \right\rfloor \) and size \( mq \times mq \).

If \( b_i(x) = x - \beta_i \) are linear polynomials then \( \deg A(x) = 1 \) and the above equivalence turns into a strong linearization, moreover the eigenvalues of \( P(x) \) can be viewed as the generalized eigenvalues of the matrix pencil

\[
x \begin{bmatrix}
I_m & & \\
& \ddots & \\
& & I_m
\end{bmatrix}
- \begin{bmatrix}
\beta_1 I_m & & \\
& \ddots & \\
& & \beta_{n-1} I_m
\end{bmatrix}
+ \begin{bmatrix}
I_m & & \\
& I_m & \\
& & I_m
\end{bmatrix}
\begin{bmatrix}
W_1 & & \\
& \ddots & \\
& & W_q
\end{bmatrix},
\]

where

\[
W_i = \begin{cases} 
\frac{P(\beta_i)}{\prod_{j \neq i} (\beta_i - \beta_j)}((\beta_i - \beta_n)P_n + sI_m)^{-1} & \text{for } i < n, \\
\frac{P(\beta_i)}{\prod_{j < i} (\beta_i - \beta_j)} - sI_m - s \sum_{j = 1}^{n-1} \frac{W_j}{\beta_i - \beta_j} & \text{otherwise}.
\end{cases}
\]

If \( P(x) \) is a scalar polynomial then \( \det A(x) = \prod_{i=1}^n (x - \beta_i)(\sum_{j=1}^n \frac{W_j}{x - \beta_j} + 1) \) so that the eigenvalue problem can be rephrased in terms of the secular equation \( \sum_{j=1}^n \frac{W_j}{x - \beta_j} + 1 = 0 \). Motivated by this fact, we will refer to this linearization as \textit{secular linearization} and to \( A(x) \) as \textit{secular companion form} of \( P(x) \) or \textit{secular \( \ell \)-ification}.

Observe that this kind of linearization relies on the representation of \( P(x) - \prod_{i=1}^n B_i(x) \) in the Lagrange basis formed by \( C_i(x) = \prod_{j=1, j \neq i}^n B_j(x) \), \( i = 1, \ldots, n \) which is different from the linearization given in [2] where the pencil \( A(x) \) has an arrowhead structure. Unlike the linearization of [2], our linearization does not introduce eigenvalues to the infinity. The secular companion matrix that we obtain can be reduced to a block upper Hessenberg form where only the first block row, the diagonal and the subdiagonal blocks are generally nonzero.

This secular linearization has some advantages with respect to the Frobenius linearization [1,1]. We show that with the linearization obtained by choosing \( \beta_i = \omega_n^i \), where \( \omega_n^i \) is a principal \( n \)th root of 1, our companion form is unitarily similar to the block Frobenius matrix associated with \( P(x) \). By choosing \( \beta_i = \alpha \omega_n^i \), we obtain a matrix unitarily similar to the scaled Frobenius matrix. With these choices, the eigenvalues of the secular companion have the same condition number as the eigenvalues of the (scaled) Frobenius matrix.

This observation leads to better choices of the nodes \( \beta_i \) performed according to the magnitude of the eigenvalues of \( P(x) \). In fact, by using the information provided by the tropical roots in the sense of [6], we may compute at a low cost particular values of the nodes \( \beta_i \) which greatly improve the condition number of the eigenvalues. From
an experimental analysis we find that in most cases the conditioning of the eigenvalues of the linearization obtained this way is lower by several orders of magnitude with respect to the conditioning of the eigenvalues of the Frobenius matrix even if it is scaled with the optimal parameter.

Our experiments are based on some randomly generated polynomials and on some problems taken from the repository NLEVP [3].

We believe that the information about the tropical roots, used in [14] for providing better numerically conditioned problems, can be more effectively used with our ℓ-ification. This analysis is part of our future work.

Another advantage of this representation is that any matrix in the form “diagonal plus low rank” can be reduced to Hessenberg form $H$ by means of Givens rotation with a low number of arithmetic operations provided that the diagonal is real. Moreover, the function $p(x) = \det(xI - H)$ as well as the Newton correction $p(x)/p'(x)$ can be computed in $O(nm^2)$ operations [7]. This fact can be used to implement the Aberth iteration in $O(n^2m^3)$ ops instead of $O(nm^4 + n^2m^2)$ of [3]. This complexity bound seems optimal in the sense that for each one of the $mn$ eigenvalues all the $m^2(n + 1)$ data are used at least once. These results are still work in place and will be part of our future work [7].

As a side result, we derive a block companion form of the matrix polynomial $A(x)$ with null blocks in the strictly block upper triangular part except for the entries in the first block row. Moreover, we provide an explicit version of right and left eigenvectors of $A(x)$ in the general case.

The paper is organized as follows. In Section 2 we provide the reduction of any matrix polynomial $P(x)$ to the equivalent form

$$A(x) = D(x) + (e \otimes I_m)[W_1(x), \ldots, W_q(x)],$$

that is, the ℓ-ification of $P(x)$. In Section 2.2 we show that $P(x)$ is strongly equivalent to $A(x)$ in the sense of Definition 1.1. In Section 3 we provide the explicit form of left and right eigenvectors of $A(x)$. In Section 4 we present the results of some numerical experiments.

2. A diagonal plus low rank ℓ-ification. Here we recall a known companion-like matrix for scalar polynomials represented as a diagonal plus a rank-one matrix, provide a more general formulation and then extend it to the case of matrix polynomials.

Let $p(x) = \sum_{i=0}^{n} p_i x^i$ be a polynomial of degree $n$ with complex coefficients, assume $p(x)$ monic, i.e., $p_n = 1$, consider a set of pairwise different complex numbers $\beta_1, \ldots, \beta_n$ and set $e = (1, \ldots, 1)^t$. Then it holds that [16]

$$p(x) = \det(xI - D + ew^t),$$

$$D = \text{diag}(\beta_1, \ldots, \beta_n), \quad w = (w_i), \quad w_i = \frac{p(\beta_i)}{\prod_{j \neq i} (\beta_i - \beta_j)}. \quad (2.1)$$

Now consider a monic polynomial $b(x)$ of degree $n$ factored as $b(x) = \prod_{i=1}^{q} b_i(x)$, where $b_i(x), i = 1, \ldots, q$ are monic polynomials of degree $d_i$ which are co-prime, that is, $\gcd(b_i(x), b_j(x)) = 1$ for $i \neq j$, where $\gcd$ denotes the monic greatest common divisor. Recall that given a pair $u(x), v(x)$ of polynomials there exist unique polynomials $s(x), r(x)$ such that $\deg s(x) < \deg v(x), \deg r(x) < \deg u(x)$, and $u(x)s(x) + v(x)r(x) = \gcd(u(x), v(x))$. From this property it follows that if $u(x)$ and $v(x)$ are
co-prime, there exists \( s(x) \) such that \( s(x)u(x) \equiv 1 \mod v(x) \). This polynomial can be viewed as the reciprocal of \( u(x) \) modulo \( v(x) \). Here and hereafter we denote \( u(x) \mod v(x) \) the remainder of the division of \( u(x) \) by \( v(x) \).

This way, we may uniquely represent any polynomial of degree \( n \) in terms of the generalized Lagrange polynomials \( c_i(x) = b(x)/b_i(x), \ i = 1, \ldots, q \) as follows.

**Lemma 2.1.** Let \( b_i(x), \ i = 1, \ldots, q \) be co-prime monic polynomials such that \( \deg b_i(x) = d_i \) and \( b(x) = \prod_{i=1}^{q} b_i(x) \) has degree \( n \). Define \( c_i(x) = b(x)/b_i(x) \). Then there exist polynomials \( s_i(x) \) such that \( s_i(x)c_i(x) = 1 \mod b_i(x) \), moreover, any monic polynomial \( p(x) \) of degree \( n \) can be uniquely written as

\[
p(x) = b(x) + \sum_{i=1}^{q} w_i(x)c_i(x),
\]

\[
w_i(x) = p(x)s_i(x) \mod b_i(x), \quad i = 1, \ldots, q,
\]

where \( \deg w_i(x) < d_i \).

**Proof.** Since \( \gcd(b_i(x), b_j(x)) = 1 \) for \( i \neq j \) then \( b_i(x) \) and \( c_i(x) = b(x)/b_i(x) \) are co-prime. Therefore there exists \( s_i(x) = 1/c_i(x) \mod b_i(x) \). Moreover, setting \( w_i(x) = p(x)s_i(x) \mod b_i(x) \) for \( i = 1, \ldots, q \), it turns out that the equation \( p(x) = b(x) + \sum_{i=1}^{q} w_i(x)c_i(x) \) is satisfied modulo \( b(x) \) for \( i = 1, \ldots, q \). For the primality of \( b_1(x), \ldots, b_q(x) \), this means that the polynomial \( \psi(x) := p(x) - b(x) - \sum_{i=1}^{q} w_i(x)c_i(x) \) is a multiple of \( \prod_{i=1}^{q} b_i(x) \) which has degree \( n \). Since \( \psi(x) \) has degree at most \( n - 1 \) it follows that \( \psi(x) = 0 \). That is \eqref{2.2} provides a representation of \( p(x) \). This representation is unique since another representation, say, given by \( \tilde{w}_i(x), \ i = 1, \ldots, q \), would be such that \( \sum_{i=1}^{q} (\tilde{w}_i(x) - w_i(x))c_i(x) = 0 \), whence \( (\tilde{w}_i(x) - w_i(x))c_i(x) \equiv 0 \mod b_i(x) \). That is, for the co-primality of \( b_i(x) \) and \( c_i(x) \), the polynomial \( \tilde{w}_i(x) - w_i(x) \) would be of \( b_i(x) \). The property \( \deg(b_i(x)) < \deg(\tilde{w}_i(x) - w_i(x)) \) implies that \( \tilde{w}_i(x) - w_i(x) = 0 \).

The polynomial \( p(x) \) in Lemma 2.1 can be represented by means of the determinant of a (not necessarily linear) matrix polynomial as expressed by the following result which provides a generalization of \eqref{2.1.1}

**Theorem 2.2.** Under the assumptions of Lemma 2.1 we have

\[
p(x) = \det A(x), \quad A(x) = D(x) + e[w_1(x), \ldots, w_q(x)]
\]

for \( D = \text{diag}(b_1(x), \ldots, b_q(x)) \) and \( e = [1, \ldots, 1]^t \).

**Proof.** Formally, one has \( A(x) = D(x)(I + D(x)^{-1}e[w_1(x), \ldots, w_q(x)]) \) so that

\[
\det A(x) = \det D(x) \det(I + D(x)^{-1}e[w_1(x), \ldots, w_q(x)])
\]

\[
= b(x)(1 + [w_1(x), \ldots, w_q(x)]D(x)^{-1}e),
\]

where \( b(x) = \prod_{i=1}^{q} b_i(x) \). Whence, we find that \( \det A(x) = b(x) + \sum_{i=1}^{q} w_i(x)c_i(x) = p(x) \), where the latter equality holds in view of Lemma 2.1.

Observe that for \( d_i = 1 \) the above result reduces to \eqref{2.1.1} where \( w_i \) are constant polynomials. From the computational point of view, the polynomials \( w_i(x) \) are obtained by performing a polynomial division since \( w_i(x) \) is the remainder of the division of \( p(x)s_i(x) \) by \( b_i(x) \).

### 2.1 The case of matrix polynomials

Now we are ready to generalize the result of Theorem 2.2 to the case of matrix polynomials \( P(x) = \sum_{i=0}^{n} P_i x^i \) where
$P_n \neq 0$ and rank $P_n = k \leq m$. We rely on the Chinese remainder theorem that here we rephrase in terms of matrix polynomials.

**Lemma 2.3.** Let $b_i(x)$, $i = 1, \ldots, q$ be co-prime polynomials of degree $d_1, \ldots, d_q$, respectively such that $\sum_{i=1}^{q} d_i = n$. If $P_1(x)$, $P_2(x)$ are matrix polynomials of degree at most $n - 1$ then $P_1(x) = P_2(x)$ if and only if $P_1(x) - P_2(x) \equiv 0 \pmod{b_i(x)}$, for $i = 1, \ldots, q$.

**Proof.** The implication $P_1(x) - P_2(x) = 0 \Rightarrow P_1(x) - P_2(x) \equiv 0 \pmod{b_i(x)}$ is trivial. Conversely, if $P_1(x) - P_2(x) \equiv 0 \pmod{b_i(x)}$ for every $b_i$ then the entries of $P_1(x) - P_2(x)$ are multiples of $\prod_{i=1}^{q} b_i(x)$ for the co-primality of the polynomials $b_i(x)$.

But this implies that $P_1(x) - P_2(x) = 0$ since the degree of $P_1(x) - P_2(x)$ is at most $n - 1$ while $\prod_{i=1}^{q} b_i(x)$ has degree $n$. $\square$

Now we can extend Lemma 2.3 to the case of matrix polynomials.

**Theorem 2.4.** Let $P(x) = \sum_{i=0}^{n} x^i P_i$ be an $m \times m$ matrix polynomial over an algebraically closed field. Let $b_i(x)$ for $i = 1, \ldots, q$ be co-prime monic polynomials of degree $d_i$ such that $\sum_{i=1}^{q} d_i = n$. Define $B_i(x) = b_i(x)I_m$ for $i = 1, \ldots, q - 1$ and $B_q(x) = b_q(x)P_0 + sI_m$ where $s$ is a constant such that $\lambda b_q(\xi) + s \neq 0$ for any eigenvalue $\lambda$ of $P_n$ and for any root $\xi$ of $b_i(x)$, for $i = 1, \ldots, q - 1$. Set $C_i(x) = \prod_{j \neq i} B_j(x)$. Then there exists unique the decomposition

$$P(x) = B(x) + \sum_{i=1}^{q} W_i C_i(x), \quad B(x) = \prod_{i=1}^{q} B_i(x)$$

where $W_i(x)$ are matrix polynomials of degree less than $d_i$ for $i = 1, \ldots, q$ defined by

$$W_i(x) = \frac{P(x)}{\prod_{j=1, j \neq i}^{q} b_j(x)} (b_q(x)P_n + sI_m)^{-1} \pmod{b_i(x)}, \quad i = 1, \ldots, q - 1$$

$$W_q(x) = \frac{1}{\prod_{j=1}^{q-1} b_j(x)} P(x) - sI_m - \sum_{j=1}^{q-1} \frac{W_j(x)}{b_j(x)} \pmod{b_q(x)}$$

**Proof.** We show that there exist matrix polynomials $W_i(x)$ of degree less than $d_i$ such that $P(x) - B(x) = \sum_{i=1}^{q} W_i(x)C_i(x) \pmod{b_i(x)}$ for $i = 1, \ldots, q$. Then we apply Lemma 2.3 with $P_1(x) = P(x) - B(x)$ that by construction has degree at most $n - 1$, and with $P_2(x) = \sum_{i=1}^{q} W_i(x)C_i(x)$, and conclude that $P(x) = B(x) + \sum_{i=1}^{q} W_i(x)C_i(x)$. Since for $i = 1, \ldots, q - 1$ the polynomial $b_i(x)$ divides every entry of $B(x)$ and of $C_j(x)$ for $j \neq i$, we find that $P(x) = W_i(x)C_i(x) \pmod{b_i(x)}$, $i = 1, \ldots, q$. Moreover, for $i < q$ we have $C_i(x) = \left( \prod_{j \neq i, j < q} b_j(x)I_m \right) (b_q(x)P_n + sI_m)$. The first term is invertible modulo $b_i(x)$ since by assumption $b_i(x)$ is co-prime with $b_j$ for every $j \neq i$. We need to prove that the matrix on the right is invertible modulo $b_i(x)$, that is, its eigenvalues $\mu$ are invertible modulo $b_i(x)$. Now, since the eigenvalues of $b_i(x)P_n + sI_m$ have the form $\mu = b_i(x)\lambda + s$, where $\lambda$ is a generic eigenvalue of $P_n$, it is enough to ensure that for every root $\xi$ of $b_i(x)$ the value $\lambda b_i(\xi) + s$ is different from 0 for $i = 1, \ldots, q - 1$. This is guaranteed by hypothesis, and so we obtain the explicit formula for $W_i(x)$, $i = 1, \ldots, q - 1$ given by (2.4). It remains to find an explicit expression for $W_q(x)$. We have $W_q(x)C_q(x) = P(x) - \sum_{j=1}^{q-1} W_j(x)C_j(x)$, where the right-hand side is made by known polynomial. This way, taking the latter expression modulo $b_i(x)$ we can compute $W_q(x)$ since $C_q(x) = \prod_{j=1}^{q-1} b_j(x)I_m$ is invertible modulo $b_i(x)$ in view of the co-primality of the polynomials $b_1(x), \ldots, b_q(x)$. This way we get the expression of $W_q$ in (2.4). $\square$
In the case where \( b_i(x), i = 1, \ldots, q \) are linear polynomials we have the following

**Corollary 2.5.** If \( b_i(x) = x - \beta_i, i = 1, \ldots, q \), then

\[
W_i = \frac{P(\beta_i)}{\prod_{j=1, j\neq i}^{n} (\beta_i - \beta_j)} ((\beta_i - \beta_n)P_n + sI_m)^{-1}, \quad i = 1, \ldots, n - 1,
\]

\[
W_n = \frac{P(\beta_n)}{\prod_{j=1}^{n} (\beta_n - \beta_j)} - sI_m - s \sum_{j=1}^{n-1} \frac{W_j}{\beta_n - \beta_j}.
\]

Moreover, if \( P(x) \) is monic then with \( s = 0 \) the expression for \( W_i \) turns simply into \( W_i = P(\beta_i)/\prod_{j=1, j\neq i}^{n} (\beta_i - \beta_j) \), for \( i = 1, \ldots, n \).

**Proof.** It follows from Theorem 2.4 and from the property \( v(x) \mod x - \beta = v(\beta) \) valid for any polynomial \( v(x) \).

From the representation of \( P(x) \) given in (2.3) we immediately deduce the following generalization of Theorem 2.2.

**Theorem 2.6.** Under the assumptions of Theorem 2.4, we have

\[
\det P(x) = \det A(x), \quad A(x) = D(x) + (e \otimes I_m) [W_1(x), \ldots, W_q(x)],
\]

where \( D(x) = \text{diag}(B_1(x), \ldots, B_q(x)). \)

**Proof.** Rewrite \( A(x) \) as \( A(x) = D(x)(I_{mq} + D(x)^{-1}(e \otimes I_m) [W_1, \ldots, W_q]) \). Then

\[
\det A(x) = \det D(x) \det(I_m + [W_1, \ldots, W_q]D(x)^{-1}(e \otimes I_m))
= \det B(x) \det(I_m + \sum_{i=1}^{q} W_iB_i(x)^{-1}) = \det((I_m + \sum_{i=1}^{q} W_iB_i(x)^{-1})B(x))
= \det(B(x) + \sum_{i=1}^{q} W_iC_i(x)) = \det P(x)
\]

where the last equality follows in view of Theorem 2.4.

Given \( n \) and \( q \leq n \), let \( \ell = \lfloor \frac{n}{q} \rfloor \). We may choose polynomials \( b_i(x) \) of degree \( d_i \) in between \( \ell - 1 \) and \( \ell \) such that \( \sum_{i=1}^{q} d_i = n \). This way we have \( \deg c_i(x) \leq n - \ell + 1 \) so that \( A(x) \) is an \( mq \times mq \) matrix polynomial of degree \( \ell \). For instance, if \( \ell = 2 \) we obtain a quadratization of \( P(x) \).

We can prove that \( A(x) \) is an \( \ell \)-ification of \( P(x) \) in the sense that there exist matrix polynomials \( E(x), F(x) \) such that \( E(x)A(x)F(x) = \text{diag}(I_{m_1}, \ldots, I_{m_p}, P(x)) \) where \( E(x) \) and \( F(x) \) are unimodular matrix polynomials. We will see this in the next section under slightly more general assumptions.

**2.2. Strong \( \ell \)-ification.** The following technical Lemma is needed to prove the next Theorem 2.8.

**Lemma 2.7.** Let \( B_1, B_2 \in \mathbb{C}[x]^{m \times m} \) be regular and such that \( B_1B_2 = B_2B_1 \). Assume that \( B_1 \) and \( B_2 \) are right co-prime, that is, there exist \( \alpha, \beta \in \mathbb{C}[x]^{m \times m} \) such that \( B_1\alpha + B_2\beta = I_m \). Then the \( 2 \times 2 \) block-matrix polynomial \( F(x) = \begin{bmatrix} \alpha B_2 \\ -\beta B_1 \end{bmatrix} \) is unimodular.

**Proof.** It is enough to prove that \( G(x) = \begin{bmatrix} B_2 & \alpha \\ B_1 \alpha & -\beta \end{bmatrix} \) is unimodular. Since \( B_2 \) is regular, from the decomposition

\[
\begin{pmatrix} I_m & 0 \\ B_1 & -B_2 \end{pmatrix} \begin{pmatrix} B_2 & \alpha \\ B_1 \alpha & -\beta \end{pmatrix} = \begin{pmatrix} B_2 & \alpha \\ 0 & I_m \end{pmatrix}.
\]
we deduce that \( \det G(x) = (-1)^n \), that is, \( \det F(x) = 1. \) \( \Box \)

We prove a stronger and more general result on \( \ell \)-ifications expressed by the following

**Theorem 2.8.** Let \( P(x) = \sum_{i=0}^n P_i x^i \), \( B_1(x), \ldots, B_q(x), \) and \( W_1(x), \ldots, W_q(x) \) be polynomials in \( \mathbb{C}[x]^{m \times m} \). Let \( C_i(x) = \prod_{j \neq i} B_i(x) \) and suppose that the following conditions hold:

1. \( P(x) = \prod_{i=1}^n B_i(x) + \sum_{i=1}^n W_i(x) C_i(x) ; \)
2. the polynomials \( B_i(x) \) are regular, commute, i.e., \( B_i(x)B_j(x) - B_j(x)B_i(x) = 0 \) for any \( i, j \), and are pairwise right co-prime;
3. \( \deg W_i(x) < \deg B_i(x) \) for every \( i = 1, \ldots, q. \)

Then the matrix polynomial \( A(x) \) defined as

\[
A(x) = D(x) + (e \otimes I_m)[W_1(x), \ldots, W_q(x)], \quad D(x) = \text{diag}(B_1(x), \ldots, B_q(x))
\]

is equivalent to \( P(x) \), i.e., there exist unimodular \( q \times q \) matrix polynomials \( E(x), F(x) \) such that \( E(x)A(x)F(x) = I_{m(q-1)} \otimes P(x) \).

**Proof.** Define \( E_0(x) \) the following (constant) matrix:

\[
E_0(x) = \begin{bmatrix}
I_m & -I_m & -I_m \\
- & I_m & -I_m \\
& & \ddots & \ddots & \ddots \\
& & & I_m & -I_m \\
& & & & & I_m
\end{bmatrix}.
\]

A direct inspection shows that

\[
E_0(x)A(x) = \begin{bmatrix}
B_1(x) & -B_2(x) \\
B_2(x) & -B_3(x) \\
& & \ddots & \ddots & \ddots \\
W_1 & \cdots & B_{q-1}(x) & -B_q(x) \\
& & & W_{q-1} & B_q(x) + W_q
\end{bmatrix}.
\]

Using the fact that the polynomials \( B_i(x) \) are right co-prime, we transform the latter matrix into block diagonal form. We start by cleaning \( B_1(x) \). Since \( B_1(x), B_2(x) \) are right co-prime, there exist polynomials \( \alpha(x), \beta(x) \) such that \( B_1(x)\alpha(x) + B_2(x)\beta(x) = I_m \). For the sake of brevity, from now on we write \( \alpha, \beta \) and \( B_i \) in place of \( \alpha(x), \beta(x) \) and \( B_i(x) \). Observe that the matrix

\[
F_1(x) = \begin{bmatrix}
\alpha & B_2 \\
-\beta & B_1
\end{bmatrix} \otimes I_{m(q-2)}.
\]

is unimodular in view of Lemma 2.7, moreover

\[
E_0(x)A(x)F_1(x) = \begin{bmatrix}
I_m & -B_2\beta & -B_3 & & \\
& B_1 B_2 & -B_3 & & \\
& & \ddots & \ddots & \ddots \\
W_1 \alpha - W_2 \beta & W_1 B_2 + W_2 B_1 & \cdots & W_{m-1} & B_q + W_q
\end{bmatrix}.
\]

Using row operations we transform to zero all the elements in the first column of this matrix (by just adding multiples of the first row to the others). That is, there exists
prime polynomials. Then the reversed polynomials $B_i$ reduction process described in Theorem 2.8. Following that procedure proves that $A$ This matrix polynomial is already in the same form of $A$. Moreover, observe that $\alpha_i W_i(x) = x^{\alpha_i} p(x^{-1}) + x^{\alpha_i} q(x^{-1}) = p(x) + x^k q(x).$

This lemma is important in our case since $A(x)$ has a decomposition of this kind. In fact, for every $i$ we have $\deg(W_i(x)) = \deg(B_i(x)) - k_i$ for some positive $k_i$. Moreover, observe that $(r(x)s(x))# = r#(x)s#(x).$ \hfill (2.5)

Lemma 2.9. Let $p(x), q(x)$ be two polynomials of degree $n$ and $n-k$, respectively. Then we have $(p+q)#(x) = p#(x) + x^k q#(x).$

Proof. By definition one has $(p+q)#(x) = x^n p(x^{-1}) + x^k q(x^{-1}) = p#(x) + x^k q#(x).$

Now, we can prove that the polynomial equivalence that we have just presented is actually a strong equivalence. To accomplish this task we will show that the reversed matrix polynomial of $A(x) = D(x) + (e \otimes I_m) W$, $W = [W_1, \ldots, W_q]$, has the same structure of $A(x)$ itself. We need a couple of lemmas.

Lemma 2.10. Let $B_1(x), \ldots, B_n(x) \in \mathbb{C}[x]^{m \times m}$ be pairwise commuting right co-prime polynomials. Then the reversed polynomials $B_1^#(x), \ldots, B_n^#(x)$ are also right co-prime.

Proof. Our hypothesis implies that there exists $\alpha, \beta$ such that $I_m = B_1 \alpha + B_2 \beta$. Taking the reverse on both sides of the latter equation yields:

$I_m = I^#_m = (B_1 \alpha + B_2 \beta)^# = B_1^\# \alpha^# + B_2^\# \beta^#,$

where we have applied Lemma 2.9 with $k = 0$ since the two polynomials $B_1 \alpha$ and $B_2 \beta$ must have the same degree, together with (2.5).

Now we have all the ingredients to prove the following

Theorem 2.11. The linearization of Theorem 2.8 is a strong linearization.

Proof. Consider $A^#(x)$. By virtue of Lemma 2.9 we obtain that

$A^#(x) = \text{diag}(B_1^#(x), \ldots, B_n^#(x)) + (e \otimes I_m)[x^{k_1} W_1^#(x), \ldots, x^{k_q} W_q^#(x)].$

This matrix polynomial is already in the same form of $A(x)$. So we may apply the reduction process described in Theorem 2.8. Following that procedure proves that $A^#(x)$ is equivalent to the polynomial

$U(x) = \prod_{i=1}^q B_i^#(x) + \sum_{i=1}^q x^{k_i} W_i^#(x) \prod_{j \neq i} B_j^#(x)$

\hfill (2.9)
Combining this with Lemma 2.9 we obtain that
\[ U(x) = \left( \prod_{i=1}^{q} B_i(x) \right) ^\# + \sum_{i=1}^{q} x^{k_i} \left( W_i(x) \prod_{j \neq i} B_j(x) \right) ^\# . \]

Combining this with Lemma 2.9 we obtain that
\[ U(x) = \left( \prod_{i=1}^{q} B_i(x) + \sum_{i=1}^{q} W_i(x) \prod_{j \neq i} B_j(x) \right) ^\# = P^\#(x). \]

2.3. A special case. Consider the case where \( P(x) \) is monic, that is \( P_n = I \), and \( B_i(x) \) are diagonal matrix polynomials. Choose \( B_i(x) = \text{diag}(d_i^{(1)}(x), \ldots, d_i^{(\ell_i)}(x)) =: D_i(x) \) monic polynomials such that the corresponding diagonal entries of \( D_i(x) \) and \( D_j(x) \) are pairwise prime for any \( i \neq j \) so that Assumption 2 of Theorem 2.8 is satisfied. Let us prove that there exist matrix polynomials \( W_i(x) \) such that \( \deg W_i(x) < \deg D_i(x) \) and
\[ P(x) = \prod_{i=1}^{q} D_i(x) + \sum_{i=1}^{q} W_i(x) C_i(x), \quad C_i(x) = \prod_{j=1, j \neq i}^{q} D_j(x), \]
so that Theorem 2.8 can be applied. Observe that the terms of degree \( x^n \) in (2.6) cancel out, therefore equating the coefficients of \( x^1 \) in (2.6) for \( i = 0, \ldots, n-1 \) provides a linear system of \( m^2 n \) equations in \( m^2 n \) unknowns. Equating the \( j \)-th columns of both sides of (2.6) modulo \( d_j^{(i)}(x) \) yields
\[ P(x) e_j \mod d_j^{(i)}(x) = \prod_{s=1, s \neq i}^{n} d_j^{(s)} W_i(x) e_j \mod d_j^{(i)}(x), \quad i = 1, \ldots, m. \]

The above equation allows one to compute the coefficients of the polynomials of degree at most \( \deg D_j(x) - 1 \) in the \( j \)-th column of \( W_j(x) \) by means of the Chinese remainder theorem.

3. Additional properties. In this section we provide an explicit expression of right and left eigenvectors of the matrix polynomial \( A(x) \) together with a block companion form for \( A(x) \).

3.1. Eigenvectors. Observe that since \( E(x) A(x) F(x) = I_{m(q-1)} \oplus P(x) \) then for a given eigenvalue \( \lambda \) the condition \( P(\lambda) v = 0 \) implies that \( (I_{m-1} \oplus P(\lambda))(0_{m(q-1)} \oplus v) = 0 \) that is, \( A(\lambda) F(\lambda)(0_{m(q-1)} \oplus v) = 0 \). In view of this remark, we may prove the following

**Lemma 3.1.** Let \( P(x) \) be a matrix polynomial and \( A(x) \) its secular \( \ell \)-ification defined in Theorem 2.8. Then if \( \lambda \) is an eigenvalue for \( P(x) \) and \( v \) its corresponding right eigenvector, i.e. such that \( P(\lambda) v = 0 \) then
\[ v_A = \begin{bmatrix} \prod_{j \neq 1} B_j(\lambda) v \\ \vdots \\ \prod_{j \neq q} B_j(\lambda) v \end{bmatrix} \]
is a right eigenvector for $A(x)$ corresponding to $\lambda$. If $B_t(x) = b_t(x)I_m$ where $t = (t_i)$ is the vector in $\mathbb{C}^n$ defined by $t_i = \prod_{j \neq i} b_j(\lambda)$.

Proof. We have already observed that $F(\lambda)(0_{q-1} \oplus v)$, where $P(\lambda)v = 0$ is eigenvector of $A(x)$ corresponding to $\lambda$. Recall that, in view of Theorem 2.8 we have

$$F(x) = F_1(x) \ldots F_{q-1}(x), \quad F_j(x) = I_{m(j-1)} \oplus \begin{bmatrix} \alpha_i & B_{i+1} \\ -\beta_i & \prod_{j \leq i} B_j \end{bmatrix} \oplus I_m(q-j-1),$$

so that it remains to provide an expression for $v_A = F_1(\lambda) \ldots F_{q-1}(\lambda)(0_{m(q-1)} \oplus v)$. Since $0_{m(q-1)} \oplus v$ has only the last block component different from zero, we find that

$$F_{q-1}(\lambda) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B_q(\lambda)v \prod_{j < q} B_j(\lambda)v \end{bmatrix}.$$ 

Multiplying by $F_{q-2}$ both sides of the above equation, it turns out that the block in position $q-1$ is multiplied by $\prod_{j < q-1} B_j(\lambda)$ and so we obtain the block $\prod_{j \neq q} B_j(\lambda)v$ in position $q-1$. Continuing this way we arrive at the sought form. □

A similar result can be proven for left eigenvectors. In this case, it may be difficult to proceed by explicitly computing the matrix $E(x)$, since its factors are not immediately available from the proof of Theorem 2.8. Instead, we will first analyze the scalar case and then proceed by analogy to guess what the eigenvectors are. We will then give an a-posteriori proof of this characterization.

**Lemma 3.2.** Let $A = D + cx^t$ where $D$ is a diagonal matrix, $c$ is the vector with all the components equal to $1$, and $w = (w_i)_{i=1,n}$. For the left and right eigenvectors $u^t$ and $v$, respectively, corresponding to the eigenvalue $\lambda$ such that $Av = \lambda v$, $u^t A = \lambda u^t$, we have $v = \left( \frac{1}{\lambda-d_i} \right)_{i=1,n}$, $u = \left( \frac{w_i}{\lambda - d_i} \right)_{i=1,n}$.

Proof. It follows by direct inspection. □

**Remark 3.3.** Observe that in the scalar case, dividing the expression of $v_A$ given in Lemma 2.3 by the scalar constant $\prod_{i=1}^n b_i(\lambda)$ provides the expression of $v$ given in the above lemma.

In the spirit of this remark we may try to guess the structure of the left eigenvectors of $A(x)$. It is quite straightforward to analyze the case where the $B_i(x)$ can be written as $b_i(x)I_m$ where $b_i(x)$ are scalar polynomials whilst the general case is more involved. To give a better understanding of the statement in this simple (yet important) case, we shall prove the following Lemma first. A generalization will be given in next Lemma 3.5.

**Lemma 3.4.** Let $A(x)$ be the secular companion form for $P(x)$ defined in Theorem 2.8. Then if $u$ is a left eigenvector for $P(x)$ relative to an eigenvalue $\lambda$, i.e., $u^tP(\lambda) = 0$, and $B_t(x) = b_t(x)I_m$ where $b_t(x) \in \mathbb{C}[x]$ then

$$u^t_A = \left[ u^t W_1 \prod_{j \neq 1} B_j(\lambda), \ldots, u^t W_q \prod_{j \neq q} B_j(\lambda) \right]$$

is a left eigenvector for $A(x)$ relative to $\lambda$. 

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Proof. We prove this statement by direct verification. We have that

$$u_A^t(D(\lambda) + EW^t) = \begin{bmatrix} u_1 \prod_{i=1}^n b_i(\lambda) & \vdots & u_n \prod_{i=1}^n b_i(\lambda) \end{bmatrix}^t + u^t \left( \sum_{i=1}^n \prod_{j \neq i} b_i(\lambda) W_i \right) \begin{bmatrix} W_1^t & \vdots & W_n^t \end{bmatrix} = 0.$$

We can now rewrite this vector in the following way:

$$u_A^t(D(\lambda) + EW^t) = u^t \left( \begin{bmatrix} \prod_{j=1}^n b_j(\lambda) I_m + \sum_{j=1}^n W_j \prod_{j \neq k} b_k(\lambda) \end{bmatrix} \right)^t = u^t P(\lambda) \begin{bmatrix} W_1^t & \vdots & W_n^t \end{bmatrix} = 0$$

that concludes the proof. □

We can now handle the case of more general $B_i(x)$. Suppose that

$$B_i(x) = b_i(x)I_m, \text{ for } i < q,$$

$$B_q(x) = (b_q(x)P_n + sI_m), \quad \text{deg } b_q(x) < d_q,$$

and let $W_i$ be such that $P(x) = \sum_{i=1}^n W_i \prod_{j \neq i} B_j(x) + \prod_{i=1}^n B_i(x)$. Then we have the following:

**Lemma 3.5.** If $P(x)$, $B_i(x)$ and $W_i(x)$ are defined in Theorem 2.8, then for every eigenvalue $\lambda$ of $P(x)$ such that $u^t P(\lambda) = 0$ the vector

$$u_A = \begin{bmatrix} W_1^t \prod_{j \neq i} B_j(x) y \end{bmatrix}_{i=1,...,q}, \quad \hat{B}_j(x) = \begin{cases} B_j(x) \quad \text{if } j < q \\ W_q^t B_q(x) W_q^{-t} \quad \text{otherwise} \end{cases}$$

is a left eigenvector for $A(x) = D(x) + (e \otimes I_m)W^t$ relative to the eigenvalue $\lambda$, where $y$ is such that $\hat{P}(\lambda)y = 0$, with

$$\hat{P}_i(x) = \prod_{i=1}^n \hat{B}_i(x) + \sum_{i=1}^n W_i^t \prod_{j \neq i} \hat{B}_j(x).$$

**Proof.** We need to prove that $u_A^t A(\lambda) = 0$. That is equivalent to verifying that $A'(\lambda)u_A = 0$, i.e., that $u_A$ is a right eigenvector for the transposed matrix polynomial $A'(x)$ relative to the eigenvalue $\lambda$. We will prove this fact by relying on Lemma 3.1 transforming $A'(x)$ to the required structure. Consider the block diagonal matrix $D_W = \text{diag}(W_1^t, \ldots, W_q^t)$. We have that

$$\hat{A}(x) = D_W A'(x) D_W^{-1} = \text{diag}([W_1^t B_1'(x)^t W_i^{-t}]_{i=1,...,q}) + (e \otimes I_m) \begin{bmatrix} W_1^t \\ \vdots \\ W_q^t \end{bmatrix}^t.$$

If we put $\hat{B}_i(x) = W_i^t B_i'(x) W_i^{-t}$ we have that $\hat{A}(x)$ is the linearization of the polynomial $\hat{P}(x) = \prod_{i=1}^n \hat{B}_i(x) + \sum_{i=1}^n W_i^t \prod_{j \neq i} \hat{B}_j(x)$. Moreover, we can easily check that $\hat{B}_i(x) = B_i'(x)$ for every $i < q$, since the $B_i(x)$ are scalar multiples of $I_m$. This implies that the right eigenvectors of $\hat{A}(x)$ are of the form

$$y_A = \begin{bmatrix} W_1^t \prod_{j \neq 1} \hat{B}_j(\lambda) y \\ \vdots \\ W_q^t \prod_{j \neq q} \hat{B}_j(\lambda) y \end{bmatrix}, \quad \hat{P}(\lambda)y = 0.$$
Recalling that \( \tilde{B}_i(x) = B_i'(x) \) for every \( i < q \) and \( \tilde{B}_q(x) = W_q B_q W_q^{-1} \) we can conclude that

\[
\tilde{P}(x) = \prod_{i=1}^{n} \tilde{B}_i(x) + \sum_{i=1}^{n} W_i^t \prod_{j \neq i} \tilde{B}_j(x).
\]

that is precisely the result that we wanted. \( \Box \)

**Remark 3.6.** Lemma 3.3 suggests another way of characterizing the left eigenvectors of \( A(x) \). The left eigenvectors of \( A(x) \) can be obtained by the right ones of \( \tilde{P}(x) \). By choosing a different scaling (with \( W^{-1}_q W_i \) instead of simply \( W_i \)) we obtain the left eigenvectors of \( A(x) \) are of the form

\[
y_A = \begin{bmatrix}
W_1^t W_q^{-1} \prod_{j \neq 1} B_j(\lambda) y \\
\vdots \\
W_q^t W_q^{-1} \prod_{j \neq q-1} B_j(\lambda) y \\
\end{bmatrix}, \quad \tilde{Q}(\lambda) y = 0.
\]

where

\[
Q(x) = \prod_{i=1}^{n} B_i'(x) + \sum_{i=1}^{n} \tilde{W}_i^t \prod_{j \neq i} B_j'(x) \quad \tilde{W}_i = W_q^{-1} W_i W_q.
\]

In the simpler case where the \( B_i(x) \) are of the form \( b_i(x)I_m \) it can be shown that \( \tilde{P}(x) \) and \( Q(x) \) coincide with \( P'(x) \) and with a scaled version of \( P'(x) \), respectively, so we obtain Lemma 3.3 again.

### 3.2. Block companion form.

Consider the block bidiagonal matrix \( L \) having \( I_m \) on the block diagonal and \( -I_m \) on the block subdiagonal. It is immediate to verify that \( L(e \otimes I_m) = e_1 \otimes I_m \), where \( e_1 = (1, 0, \ldots, 0)^t \). This way, for the matrix \( H(x) = LA(x) \) it holds that

\[
H(x) = \begin{bmatrix}
B_1(x) + W_1(x) & W_2(x) & \ldots & W_{q-1}(x) & W_q(x) \\
-B_1(x) & B_2(x) & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & -B_{q-1}(x) & B_q(x) \\
& & & & -B_{q-1}(x) & B_q(x)
\end{bmatrix}
\]

### 4. Numerical issues.

Let \( \omega_n \) be a principal \( n \)th root of the unity, define \( \Omega_n = \frac{1}{\sqrt{n}} (\omega_n^j)_{j=1,n} \) the Fourier matrix such that \( \Omega_n^* \Omega_n = I_n \) and observe that \( \Omega_n e = e_n \), where \( e = (1, \ldots, 1)^t \), \( e_n = (0, \ldots, 0, 1)^t \). Assume for simplicity \( P_n = I_m \). For the linearization obtained with \( \beta_i = \omega_n^j \), \( i = 1, \ldots, n \), we have

\[
A(x) = xI_m - \text{diag}(\omega_n^j I_m, \omega_n^{2j} I_m, \ldots, \omega_n^{nj} I_m) + (e \otimes I_m)[W_1, \ldots, W_n]
\]

with \( W_i = \frac{1}{n} \omega_n^j P(\omega_n^i) \). It is easy to verify that the pencil \( (\Omega_n^* \otimes I_m)A(x)(\Omega_n \otimes I_m) \) has the form

\[
xI_m - F, \quad F = (C \otimes I_m) - (e_n \otimes I_m)[P_0 + I_m, P_1, \ldots, P_{n-1}]
\]

where \( C = (c_{i,j}) \) is the unit circulant matrix defined by \( c_{i,j} = (\delta_{i,j+1 \mod n}) \). That is, \( F \) is the block Frobenius matrix associated with the matrix polynomial \( P(x) \).
This shows that our linearization includes the companion Frobenius matrix with a specific choice of the nodes. In particular, since $\Omega_n$ is unitary, the condition number of the eigenvalues of $A(x)$ coincides with the condition number of the eigenvalues of $F$. Observe also that if we choose $\beta_i = \alpha \omega_i^n$ with $\alpha \neq 0$, then $\Omega_n A(x) \Omega_n^* = xI - D_{\alpha}^{-1}FD_{\alpha}$ for $D_{\alpha} = \text{diag}(1, \alpha, \ldots, \alpha^{n-1})$. That is, we obtain the scaled Frobenius pencil.

Here, we present some numerical experiments to show that in many interesting cases a careful choice of the $B_i(x)$ can lead to linearizations (or $\ell$-ifications) where the eigenvalues are much better conditioned than in the original problem.

The code used to generate these examples can be downloaded from [http://numpi.dm.unipi.it/software/secular-linearization/](http://numpi.dm.unipi.it/software/secular-linearization/)

4.1. Scalar polynomials. As a first example, consider a monic scalar polynomial $p(x) = \sum_{i=0}^{n} p_i x^i$ where the coefficients $p_i$ have unbalanced moduli. In this case, we generate $p_i$ using the MATLAB command $p = \exp(12 \times \text{randn}(1,n+1))$; $p(n+1)=1$;

![Figure 4.1. Conditioning of different linearizations of a degree 50 scalar polynomial with random unbalanced coefficients.](image)

Then we build our linearization by means of the function $\text{seccomp}(b,p)$ that takes a vector $b$ together with the coefficients of the polynomial and generates the linearization $A(x)$ where $B_i(x) = x - \beta_i$ for $\beta_i = b(i)$. Finally, we measure the conditioning of the eigenvalues of $A(x)$ by means of the Matlab function $\text{condeig()}$.

We have considered three different linearizations:
- The Frobenius linearization obtained by $\text{compan}(p)$;
- the secular linearization obtained by taking as $\beta_i$ perturbed values of the roots;
- the secular linearization with nodes given by the tropical roots of the polynomial multiplied by unit complex numbers.

The results are displayed in Figure 4.1. On can see that in the first case the condition numbers of the eigenvalues are much different from each other and can be as large as $10^{13}$ for the worst conditioned eigenvalue. In the second case the condition number of all the eigenvalues is close to 1, while in the third linearization the condition numbers are much smaller than those of the Frobenius linearization and have an almost uniform distribution.

These experimental results are a direct verification of a conditioning result of [8, Sect. 5.2] that is at the basis of the $\text{secsolve}$ algorithm presented in that paper.
These tests are implemented in the function files `Example1.m` and `Experiment1.m` included in the Matlab source code for the experiments. These properties similarly hold in the matrix case.

4.2. The matrix case. Consider now a matrix polynomial $P(x) = \sum_{i=0}^{n} P_i x^i$. As in the previous case, we start by considering monic matrix polynomials. As a first example, consider the case where the coefficients $P_i$ have unbalanced norms. Here is the Matlab code that we have used to generate this test:

```matlab
n = 5; m = 64;
P = {};
for i = 1 : n
    P{i} = exp(12 * randn) * randn(m);
end
P{n+1} = eye(m);
```

We can give reasonable estimates to the modulus of the eigenvalues using the Pellet theorem or the tropical roots. See [14, 26], for some insight on these tools.

The same examples given in the scalar case have been replicated for matrix polynomials relying on the Matlab script published on the website reported above by issuing the following commands:

```matlab
P = Example2();
Experiment2(P);
```

![Figure 4.2. Conditioning of different linearization for some matrix polynomials with random coefficients having unbalanced norms.](image)

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We have considered three linearizations: the standard Frobenius companion matrix, and two versions of our secular linearizations. In the first version the nodes $\beta_i$ are the mean of the moduli of set of eigenvalues with close moduli multiplied by unitary complex numbers. In the second, the values of $\beta_i$ are obtained by the Pellet estimates delivered by the tropical roots.

In Figure 4.2 we report the conditioning of the eigenvalues, measured with Matlab’s `condeig`.

It is interesting to note that the conditioning of the secular linearization is, in every case, not exceeding $10^2$. Moreover it can be observed that no improvement is obtained on the conditioning of the eigenvalues that are already well-conditioned. In contrast, there is a clear improvement on the ill-conditioned ones. In this particular case, this class of linearizations seems to give an almost uniform bound to the condition number of all the eigenvalues.

Further examples come from the NLEVP collection of [3]. We have selected some problems that exhibit bad conditioning.

As a first example we consider the problem `orr_sommerfeld`. Using the tropical roots we can find some values inside the unique annulus that is identified by the Pellet theorem. In this example the values obtained only give a partial picture of the eigenvalues distribution. The Pellet theorem gives about $1.65e^{-4}$ and $5.34$ as lower and upper bound to the moduli of the eigenvalues, but the tropical roots are rather small and near to the lower bound. More precisely, the tropical roots are $1.4e^{-3}$ and $1.7e^{-4}$ with multiplicities 3 and 1, respectively.

This leads to a linearization $A(x)$ that is well-conditioned for the smaller eigenvalues but with a higher conditioning on the eigenvalues of bigger modulus as can be seen in Figure 4.3 on the left (the eigenvalues are ordered in nonincreasing order with respect to their modulus). It can be seen, though, that coupling the tropical roots with the standard Pellet theorem and altering the $\beta_i$ by adding a value slightly smaller than the upper bound (in this example we have chosen 5 but the result is not very sensitive to this choice) leads to a much better result that is reported in Figure 4.3 on the right. In the right figure we have used $b = [1.7e^{-4}, 1.4e^{-3}, -1.4e^{-3}, 5]$. This seems to justify that there exists a link between the quality of the approximations obtained through the tropical roots and the conditioning properties of the secular linearization.

We analyzed another example problem from the NLEVP collection that is called...
Figure 4.4. Conditioning of the eigenvalue problem for three different linearizations on the planar_waveguide problem.

planar_waveguide. The results are shown in Figure 4.2. This problem is a PEP of degree 4 with two tropical roots approximately equal to 127.9 and 1.24. Again, it can be seen that for the eigenvalues of smaller modulus (that will be near the tropical root 1.24) the Frobenius linearization and the secular one behave in the same way, whilst for the bigger ones the secular linearization has some advantage in the conditioning. This may be justified by the fact that the Frobenius linearization is similar to a secular linearization on the roots of the unity.

Note that in this case the information obtained by the tropical roots seems more accurate than in the orr_sommerfeld case, so the secular linearization built using the tropical roots and the one built using the block-mean of the eigenvalues behave approximately in the same way.

As a last example, we have tried to find the eigenvalues of a matrix polynomial defined by integer coefficients. We have used polyeig and our secular linearization (using the tropical roots as $b_i$) and the QZ method. We have chosen the polynomial

$$P(x) = P_{11}x^{11} + P_{9}x^9 + P_2x^2 + P_0$$

where

$$P_{11} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad P_9 = 10^8 \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 3 & 1 & 1 \end{bmatrix}, \quad P_2 = 10^8 P_{11}^t, \quad P_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

In this case the tropical roots are good estimates of the blocks of eigenvalues of the matrix polynomial. We obtain the tropical roots $1.2664 \cdot 10^4$, $0.9347$ and $1.1786 \cdot 10^{-4}$ with multiplicities 2, 7 and 2, respectively. We have computed the eigenvalues with a higher precision and we have compared them with the results of polyeig and of eig applied to the secular linearization. Here, the secular linearization has been computed with the standard floating point arithmetic. As shown in Figure 4.5 we have achieved much better accuracy with the latter choice. The secular linearization has achieved a relative error of the order of the machine precision on all the eigenvalues except the smaller block (with modulus about $10^{-4}$). In that case the relative error is about
The accuracy of the computed eigenvalues using polyeig and the secular linearization with the $b_i$ obtained through the computation of the tropical roots.

$10^{-12}$ but the absolute error is, again, of the order of the machine precision.

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