The inequalities for convex functions due to Hermite and Hadamard are found to be of great importance, for example, see [5, 14]. According to the inequalities [7, 8],

- if \( u : I \to \mathbb{R} \) is a convex function on the interval \( I \subset \mathbb{R} \) and \( a, b \in I \) with \( b > a \), then
  \[
  u\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b u(y)dy \leq \frac{u(a) + u(b)}{2}.
  \] (1.1)

For a concave function \( u \), the inequalities in (1.1) hold in the reversed direction. We note that Hadamard’s inequality refines the concept of convexity, and it follows from Jensen’s inequality. The classical Hermite-Hadamard inequality yields estimates for the mean value of a continuous convex function \( u : [a, b] \to \mathbb{R} \). The well-known inequalities dealing with the integral mean of a convex function \( u \) are the Hermite-Hadamard inequalities or its weighted versions. They are also known as Hermite-Hadamard-Fejér inequalities.

In [6], Fejér obtained the weighted generalization of Hermite-Hadamard inequality (1.1) as follows.

- Let \( u : [a, b] \to \mathbb{R} \) be a convex function. Then the inequality
  \[
  u\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b u(y)w(y)dy \leq \frac{u(a) + u(b)}{2} \int_a^b w(y)dy
  \] (1.2)
holds for a nonnegative, integrable function \( v : [a, b] \to \mathbb{R} \), which is symmetric to \( \frac{a+b}{2} \).

In [4], Dragomir and Agarwal obtained the following results in connection with the right part of (1.1):

- **Let** \( u : I \subseteq \mathbb{R} \to \mathbb{R} \) **be a differentiable mapping on** \( I, a, b \in I \). If \( |u'| \) is convex on \([a, b]\), then the following inequality holds:

\[
\left| \frac{u(a) + u(b)}{2} - \frac{1}{b-a} \int_a^b u(y)dy \right| \leq \frac{b-a}{8} (|u'(a)| + |u'(b)|). \tag{1.3}
\]

In [13], Pachpatte established two new Hermite-Hadamard type inequalities for products of convex functions as follows:

- **Let** \( u \) and \( w \) **be nonnegative and convex functions on** \([a, b] \subseteq \mathbb{R} \), then

\[
\frac{1}{b-a} \int_a^b u(y)w(y)dy \leq \frac{u(a)w(a) + u(b)w(b)}{3} + \frac{u(a)w(b) + u(b)w(a)}{6} \tag{1.4}
\]

and

\[
2u \left( \frac{a+b}{2} \right) w \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b u(y)w(y)dy \]

\[
+ \frac{u(a)w(a) + u(b)w(b)}{6} + \frac{u(a)w(b) + u(b)w(a)}{3}. \tag{1.5}
\]

Next we present some results on the generalization of aforementioned inequalities.

In [15], Sarikaya et. al. represented Hermite-Hadamard and Dragomir-Agarwal inequalities in fractional integral forms as follows.

- **Let** \( u : [a, b] \to \mathbb{R} \) **be a positive function and** \( u \in L^1([a, b]) \). If \( u \) is a convex function on \([a, b]\), then the following inequalities for fractional integrals hold

\[
u \left( \frac{a+b}{2} \right) \leq \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ I_a^\alpha u(b) + I_b^\alpha u(a) \right] \leq \frac{u(a) + u(b)}{2}
\]

with \( \alpha > 0 \).

- **Let** \( u : [a, b] \to \mathbb{R} \) **be a differentiable mapping on** \((a, b)\). If \( |u'| \) is convex on \([a, b]\), then the following inequality for fractional integrals holds:

\[
\left| \frac{u(a) + u(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ I_a^\alpha u(b) + I_b^\alpha u(a) \right] \right| \leq \frac{b-a}{2(\alpha + 1)} \frac{(1 - 2^{-\alpha}) (|u'(a)| + |u'(b)|)}{2}.
\]

In [11], Işcan obtained the following Hermite-Hadamard-Fejér integral inequalities via fractional integrals:

- **Let** \( u : [a, b] \to \mathbb{R} \) **be convex function with** \( a < b \) and \( u \in L^1([a, b]) \). If \( v : [a, b] \to \mathbb{R} \) is nonnegative, integrable and symmetric to \((a + b)/2\), then the following inequalities for fractional integrals hold

\[
u \left( \frac{a+b}{2} \right) \left[ I_a^\alpha v(b) + I_b^\alpha v(a) \right] \leq \left[ I_a^\alpha (uv)(b) + I_b^\alpha (uv)(a) \right].
\]
Many generalizations and extensions of the Hermite-Hadamard, Hermite-Hadamard-Fejér, Dragomir-Agarwal and Pachpatte type inequalities were obtained for various classes of functions using fractional integrals; see [1, 2, 3, 9, 10, 11, 12, 15, 16, 17] and references therein.

These studies motivated us to consider a new class of functional inequalities for convex functions generalizing the classical Hermite-Hadamard, Hermite-Hadamard-Fejér, Dragomir-Agarwal and Pachpatte inequalities. Here we emphasize that we derive some functional inequalities for the new fractional integral operators with exponential kernel. The difference between our results and the known generalizations is that the above fractional analogues of functional inequalities do not follow from our results. In fact our results are the simplest generalizations of only classical inequalities.

The paper is organized as follows. Section 2 contains some basic concepts related to our proposed study. In Section 3 a Hermite-Hadamard type inequality for a fractional integral with an exponential kernel is proved. The fractional analogue of the Hermite inequality is investigated in Section 4. Section 5 is devoted to the generalization of Dragomir-Agarwal’s inequality. In Section 6 we obtain generalized Pachpatte-type inequalities with fractional integrals in the class of convex functions.

2. Preliminaries

We give some definitions for further use.

**Definition 2.1.** A function $u : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if

$$u(\mu x + (1 - \mu)y) \leq \mu u(x) + (1 - \mu)u(y)$$

for all $x, y \in [a, b]$ and $\mu \in [0, 1]$. We call $u$ a concave function if $(-u)$ is convex.

Now we give some necessary concepts related to the new fractional integral which are used in the sequel.

**Definition 2.2.** Let $f \in L_1(a, b)$. The fractional integrals $I_\alpha^a u(x)$ and $I_\alpha^b u(x)$ of order $\alpha \in (0, 1)$ are defined by

$$I_\alpha^a u(x) = \frac{1}{\alpha} \int_a^x \exp \left( -\frac{1-\alpha}{\alpha} (x-s) \right) u(s)ds, \ x > a \quad (2.1)$$

and

$$I_\alpha^b u(x) = \frac{1}{\alpha} \int_x^b \exp \left( -\frac{1-\alpha}{\alpha} (s-x) \right) u(s)ds, \ x < b \quad (2.2)$$

respectively.

If $\alpha = 1$, then

$$\lim_{\alpha \to 1} I_\alpha^a u(x) = \int_a^x u(s)ds, \ \lim_{\alpha \to 1} I_\alpha^b u(x) = \int_x^b u(s)ds.$$
Moreover, in view of
\[
\lim_{\alpha \to 0} \frac{1}{\alpha} \exp \left( -\frac{1}{\alpha} (x-s) \right) = \delta(x-s),
\]
we deduce that
\[
\lim_{\alpha \to 0} I_\alpha^{[a]} u(x) = u(x), \quad \lim_{\alpha \to 0} I_\alpha^{[b]} u(x) = u(x).
\]

**Definition 2.3.** The left and right Riemann–Liouville fractional integrals \( I_\alpha^{[a]} \) and \( I_\alpha^{[b]} \) of order \( \alpha \in \mathbb{R} \) \( (\alpha > 0) \) are given by
\[
I_\alpha^{[a]} [f] (t) = \frac{1}{\Gamma (\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \, ds, \quad t \in (a, b],
\]
and
\[
I_\alpha^{[b]} [f] (t) = \frac{1}{\Gamma (\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) \, ds, \quad t \in [a, b),
\]
respectively. Here \( \Gamma \) denotes the Euler gamma function.

We henceforth set \( \rho = \frac{1-\alpha}{\alpha} (b-a) \).

### 3. Hermite-Hadamard Type Inequality

**Theorem 3.1.** Let \( u : [a, b] \to \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( u \in L_1(a, b) \). If \( u \) is a convex function on \( [a, b] \), then the following inequalities for fractional integrals (2.1) and (2.2) hold:
\[
u \left( \frac{a+b}{2} \right) \leq \frac{1-\alpha}{2 (1-\exp (-\rho))} \left[ I_\alpha^{[a]} u(b) + I_\alpha^{[b]} u(a) \right] \leq \frac{u(a) + u(b)}{2}.
\]

**Proof.** Since \( u \) is a convex function on \( [a, b] \), we get for \( x \) and \( y \) from \( [a, b] \) with \( \rho = \frac{1}{2} \)
\[
u \left( \frac{x+y}{2} \right) \leq \frac{u(x) + u(y)}{2},
\]
which, for \( x = ta + (1-t)b \), \( y = (1-t)a + tb \), takes the form:
\[
2u \left( \frac{a+b}{2} \right) \leq u(ta + (1-t)b) + u((1-t)a + tb).
\]

Multiplying both sides of (3.3) by \( \exp (-\rho t) \) and then integrating the resulting inequality with respect to \( t \) over \( [0, 1] \), we obtain
\[
\frac{2 (1-\exp (-\rho))}{\rho} u \left( \frac{a+b}{2} \right) \leq \int_0^1 \exp (-\rho t) [u(ta + (1-t)b) + u((1-t)a + tb)] \, dt
\]
\[
= \int_0^1 \exp (-\rho t) u(ta + (1-t)b) \, dt + \int_0^1 \exp (-\rho t) u((1-t)a + tb) \, dt
\]
\[ \begin{align*}
&= \frac{1}{b-a} \int_a^b \exp\left(-\frac{1}{\alpha} (b-s)\right) u(s) ds \\
&\quad + \frac{1}{b-a} \int_a^b \exp\left(-\frac{1}{\alpha} (s-a)\right) u(s) ds \\
&= \frac{\alpha}{b-a} \left[ I_\alpha^a u(b) + I_\alpha^b u(a) \right].
\end{align*} \]

As a result, we get
\[
\frac{2 (1 - \exp(-\rho))}{\rho} u \left( \frac{a + b}{2} \right) \leq \frac{\alpha}{b-a} \left[ I_\alpha^a u(b) + I_\alpha^b u(a) \right].
\]

Thus the first inequality of (3.1) is established.

For the proof of the second inequality in (3.1), we first note that if \( u \) is a convex function, then, for \( t \in [0,1] \), it yields
\[ u(ta + (1-t)b) \leq tu(a) + (1-t)u(b) \]
and
\[ u((1-t)a + tb) \leq (1-t)u(a) + tu(b). \]

By adding the above two inequalities, we have
\[ u(ta + (1-t)b) + u((1-t)a + tb) \leq u(a) + u(b). \quad (3.4) \]

Multiplying both sides of (3.4) by \( \exp(-\rho t) \) and integrating the resulting inequality with respect to \( t \) over \([0,1] \), we obtain
\[
\frac{2 (1 - \exp(-\rho))}{\rho} [u(a) + u(b)] \geq \int_0^1 \exp(-\rho t) u(ta + (1-t)b) dt \\
+ \int_0^1 \exp(-\rho t) u((1-t)a + tb) dt,
\]
that is,
\[
\frac{\alpha}{b-a} \left[ I_\alpha^a u(b) + I_\alpha^b u(a) \right] \leq \frac{2 (1 - \exp(-\rho))}{\rho} [u(a) + u(b)].
\]

Hence the second inequality in (3.1) is proved. This completes the proof of Theorem 3.1. \( \square \)

**Corollary 3.2.** Let \( u : [a,b] \to \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( u \in L_1(a,b) \). If \( u \) is a concave function on \([a,b] \), then the following inequalities for fractional integrals (2.1) and (2.2) hold:
\[
u \left( \frac{a + b}{2} \right) \geq \frac{1 - \alpha}{2 (1 - \exp(-\rho))} \left[ I_\alpha^a u(b) + I_\alpha^b u(a) \right] \geq \frac{u(a) + u(b)}{2}. \]

**Remark 3.3.** For \( \alpha \to 1 \), observe that
\[
\lim_{\alpha \to 1} \frac{1 - \alpha}{2 (1 - \exp(-\rho))} = \frac{1}{2(b-a)}.
\]

Thus, Hermite-Hadamard inequality (1.1) follows from Theorem 3.1 in the limit \( \alpha \to 1 \).
4. HÉRITTE-HADAMARD-FÉJÉR TYPE INEQUALITY

Theorem 4.1. Let \( u : [a, b] \to \mathbb{R} \) be a convex and integrable function with \( a < b \). If \( w : [a, b] \to \mathbb{R} \) is nonnegative, integrable and symmetric with respect to \( \frac{a + b}{2} \), that is, \( w(a + b - x) = w(x) \), then the following inequalities hold

\[
\begin{align*}
u \left( \frac{a + b}{2} \right) \left[ \mathcal{I}_a^\alpha w(b) + \mathcal{I}_b^\alpha w(a) \right] & \leq \left[ \mathcal{I}_a^\alpha (uw)(b) + \mathcal{I}_b^\alpha (uw)(a) \right] \\
& \leq \frac{u(a) + u(b)}{2} \left[ \mathcal{I}_a^\alpha w(b) + \mathcal{I}_b^\alpha w(a) \right].
\end{align*}
\] (4.1)

Proof. Since \( u \) is a convex function on \([a, b]\), we have the inequality (3.3) for all \( t \in [0; 1] \). Multiplying both sides of (3.3) by \( \exp \left( -\rho t \right) w \left( (1 - t)a + tb \right) \), (4.2) and then integrating the resulting inequality with respect to \( t \) over \([0, 1]\), we obtain

\[
\begin{align*}
2u \left( \frac{a + b}{2} \right) \int_0^1 \exp \left( -\rho t \right) w \left( (1 - t)a + tb \right) dt & \\
\leq \int_0^1 \exp \left( -\rho t \right) u \left( ta + (1 - t)b \right) w \left( (1 - t)a + tb \right) dt \\
& + \int_0^1 \exp \left( -\rho t \right) u \left( (1 - t)a + tb \right) w \left( (1 - t)a + tb \right) dt \\
& = \frac{1}{b - a} \int_a^b \exp \left( -\frac{1}{\alpha} (s - a) \right) u(a + b - s) w(s) ds \\
& + \frac{1}{b - a} \int_a^b \exp \left( -\frac{1}{\alpha} (s - a) \right) u(s) w(s) ds \\
& = \frac{1}{b - a} \int_a^b \exp \left( -\frac{1}{\alpha} (b - s) \right) u(s) w(a + b - s) ds \\
& + \frac{\alpha}{b - a} \mathcal{I}_b^\alpha [u(a)w(a)] = \frac{\alpha}{b - a} \left[ \mathcal{I}_a^\alpha [u(a)w(a)] + \mathcal{I}_b^\alpha [u(a)w(a)] \right],
\end{align*}
\]

that is,

\[
\begin{align*}
2u \left( \frac{a + b}{2} \right) \int_0^1 \exp \left( -\rho t \right) w \left( (1 - t)a + tb \right) dt & \\
\leq \frac{\alpha}{b - a} \left[ \mathcal{I}_a^\alpha [u(a)w(a)] + \mathcal{I}_b^\alpha [u(a)w(a)] \right].
\end{align*}
\]

Since \( w \) is symmetric with respect to \( \frac{a + b}{2} \), we have

\[
\mathcal{I}_a^\alpha w(b) = \mathcal{I}_b^\alpha w(a) = \frac{1}{2} \left[ \mathcal{I}_a^\alpha w(b) + \mathcal{I}_b^\alpha w(a) \right].
\]
Therefore, we have
\[ u \left( \frac{a + b}{2} \right) \left[ \mathcal{I}_a^\alpha w(b) + \mathcal{I}_b^\alpha w(a) \right] \leq \mathcal{I}_a^\alpha [w(b) u(b)] + \mathcal{I}_b^\alpha [w(a) u(a)]. \]

This establishes the first inequality of Theorem 4.1

To prove the second inequality in (1.1), we first notice that if \( u \) is a convex function, then, for all \( t \in [0, 1] \), it yields the inequality \( (\mathcal{I}_a^\alpha u(b) + \mathcal{I}_b^\alpha u(a)) \leq \frac{u(a) + u(b)}{2} \left( \mathcal{I}_a^\alpha w(b) + \mathcal{I}_b^\alpha w(a) \right) \).

Corollary 4.2. Let \( u : [a, b] \to \mathbb{R} \) be a concave and integrable function with \( a < b \). If \( w : [a, b] \to \mathbb{R} \) is nonnegative, integrable and symmetric with respect to \( \frac{a + b}{2} \), that is, \( w(a + b - x) = w(x) \), then the following inequalities hold
\[
u \left( \frac{a + b}{2} \right) \left[ \mathcal{I}_a^\alpha w(b) + \mathcal{I}_b^\alpha w(a) \right] \geq \left[ \mathcal{I}_a^\alpha (uw)(b) + \mathcal{I}_b^\alpha (uw)(a) \right] \geq \frac{u(a) + u(b)}{2} \left[ \mathcal{I}_a^\alpha w(b) + \mathcal{I}_b^\alpha w(a) \right].
\]

Remark 4.3. From Theorem 4.1 with \( \alpha \to 1 \), we indeed have Hermite-Hadamard-Fejér inequality (1.2).

5. Dragomir-Agarwal type inequality

Theorem 5.1. Let \( u : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I, a, b \in I \). If \( |u'| \) is convex on \( [a, b] \), then the following inequality involving fractional integrals \( 2.1 \) and \( 2.2 \) holds:
\[
\left| \frac{u(a) + u(b)}{2} - \frac{1 - \alpha}{2 (1 - \exp(-\rho))} \left[ \mathcal{I}_a^\alpha u(b) + \mathcal{I}_b^\alpha u(a) \right] \right| \leq \frac{b - a}{2 \rho} \tanh \left( \frac{\rho}{4} \right) \left( |u'(a)| + |u'(b)| \right). \quad (5.1)
\]

Proof. For \( u' \in L_1(a, b) \), it is easy to find that
\[
\frac{u(a) + u(b)}{2} - \frac{1 - \alpha}{2 (1 - \exp(-\rho))} \left[ \mathcal{I}_a^\alpha u(b) + \mathcal{I}_b^\alpha u(a) \right]
\]
\[ \frac{b - a}{2(1 - \exp(-\rho))} \left\{ \int_0^1 \exp(-\rho t) u'(ta + (1-t)b) \, dt - \int_0^1 \exp(-\rho(1-t)) u'(ta + (1-t)b) \, dt \right\} \] 

(5.2)

Then, using (5.2) and the convexity of \(|u'|\), we obtain

\[ \left| \frac{u(a) + u(b)}{2} - \frac{1 - \alpha}{2(1 - \exp(-\rho))} [T_\alpha^b u(a) + T_\alpha^a u(b)] \right| \]
\[ \leq \frac{b - a}{2} \int_0^1 \frac{|\exp(-\rho t) - \exp(-\rho(1-t))|}{1 - \exp(-\rho)} |u'(ta + (1-t)b)| \, dt \]
\[ \leq \frac{b - a}{2} \int_0^1 \frac{|\exp(-\rho t) - \exp(-\rho(1-t))|}{1 - \exp(-\rho)} t |u'(a)| \, dt \]
\[ + \frac{b - a}{2} \int_0^1 \frac{|\exp(-\rho t) - \exp(-\rho(1-t))|}{1 - \exp(-\rho)} (1-t) |u'(b)| \, dt \]
\[ = \frac{b - a}{2} |u'(a)| \int_0^1 \frac{\exp(-\rho t) - \exp(-\rho(1-t))}{1 - \exp(-\rho)} \, dt \]
\[ + \frac{b - a}{2} |u'(a)| \int_0^1 \frac{\exp(-\rho(1-t)) - \exp(-\rho t)}{1 - \exp(-\rho)} \, dt \]
\[ + \frac{b - a}{2} |u'(b)| \int_0^1 \frac{\exp(-\rho t) - \exp(-\rho(1-t))}{1 - \exp(-\rho)} (1-t) \, dt \]
\[ + \frac{b - a}{2} |u'(b)| \int_0^1 \frac{\exp(-\rho(1-t)) - \exp(-\rho t)}{1 - \exp(-\rho)} (1-t) \, dt \]
\[ = \frac{b - a}{2(1 - \exp(-\rho))} \left[ |u'(a)| (I_1 + I_2) + |u'(b)| (I_3 + I_4) \right]. \]

As a result, we get

\[ \frac{u(a) + u(b)}{2} - \frac{1 - \alpha}{2(1 - \exp(-\rho))} [T_\alpha^b u(a) + T_\alpha^a u(b)] \]
\[ \leq \frac{b - a}{2(1 - \exp(-\rho))} \left[ |u'(a)| (I_1 + I_2) + |u'(b)| (I_3 + I_4) \right], \] 

(5.3)

where

\[ I_1 = \int_0^\frac{1}{2} (\exp(-\rho t) - \exp(-\rho(1-t))) \, dt \]
\[ I_2 = \int_{\frac{1}{2}}^{1} (\exp(-\rho t) - \exp(-\rho(1-t))) t \, dt \]
\[ = \frac{1}{\rho} \left( 1 - \exp \left( -\frac{\rho}{2} \right) + \exp(-\rho) \right) - \frac{1}{\rho^2} (1 - \exp(-\rho)) , \quad (5.5) \]
\[ I_3 = \int_{0}^{\frac{1}{2}} (\exp(-\rho t) - \exp(-\rho(1-t))) (1-t) \, dt \]
\[ = -\frac{\exp \left( -\frac{\rho}{2} \right)}{\rho} + \frac{1}{\rho} (1 + \exp(-\rho)) - \frac{1}{\rho^2} (1 - \exp(-\rho)) \quad (5.6) \]
and
\[ I_4 = \int_{\frac{1}{2}}^{1} (\exp(-\rho t) - \exp(-\rho(1-t))) (1-t) \, dt \]
\[ = -\frac{\exp \left( -\frac{\rho}{2} \right)}{\rho} + \frac{1}{\rho^2} (1 - \exp(-\rho)) . \quad (5.7) \]
Inserting the values of \( I_i \) \( (i = 1, 2, 3, 4) \) given by \( (5.4)-(5.7) \) in \( (5.3) \), we obtain the inequality \( (5.1) \). This completes the proof. \( \square \)

**Corollary 5.2.** Let \( u : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I, a, b \in I \). If \( |u'| \) is concave on \( [a, b] \), then the following inequality holds:

\[
\left| \frac{u(a) + u(b)}{2} - \frac{1 - \alpha}{2(1 - \exp(-\rho))} \left[ I_\alpha^a u(b) + I_\alpha^b u(a) \right] \right| \\
\geq \frac{b-a}{2\rho} \tanh \left( \frac{\rho}{4} \right) (|u'(a)| + |u'(b)|). 
\]

**Remark 5.3.** For \( \alpha \rightarrow 1 \), we find that

\[
\lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{2(1 - \exp(-\rho))} = \frac{1}{2(b-a)}, \\
\lim_{\alpha \rightarrow 1} \frac{b-a}{2\rho} \tanh \left( \frac{\rho}{4} \right) = \frac{b-a}{8} .
\]
Thus we get Dragomir-Agarwal inequality \( (1.3) \) from Theorem \( 5.1 \) when \( \alpha \rightarrow 1 \).

### 6. Pachpatte Type Inequalities

**Theorem 6.1.** Let \( u \) and \( w \) be real-valued, nonnegative and convex functions on \( [a,b] \). Then the following inequalities involving fractional integrals \( (2.1) \) and \( (2.2) \) hold:

\[
\frac{\alpha}{2(b-a)} \left[ I_\alpha^a (u(b)w(b)) + I_\alpha^b (u(a)w(a)) \right]
\]
Multiplying both sides of inequality (6.3) by \( \exp(-\rho) \), we have

\[
\int_0^1 \exp(-\rho \xi) u(\xi a + (1 - \xi) b) w(\xi a + (1 - \xi) b) d\xi \\
= \frac{\alpha}{b - a} \left[ T_a^\alpha (u(b)w(b)) + T_b^\alpha (u(a)w(a)) \right] \\
\leq [u(a)w(a) + u(b)w(b)] \int_0^1 \exp(-\rho \xi) (2\xi^2 - 2\xi + 1) d\xi \\
+ [u(a)w(b) + u(b)w(a)] \int_0^1 \exp(-\rho \xi) 2\xi(1 - \xi) d\xi \\
= [u(a)w(a) + u(b)w(b)] \frac{\rho^2 - 2\rho + 4 - (\rho^2 + 2\rho + 4) \exp(-\rho)}{\rho^3}. \tag{6.1}
\]

**Proof.** Since \( u \) and \( w \) are convex on \([a, b] \), then, for \( \xi \in [0, 1] \), it follows from definition [2.1] that

\[
\begin{align*}
2u \left( \frac{a + b}{2} \right) w \left( \frac{a + b}{2} \right) &\leq \frac{1 - \alpha}{2(1 - \exp(-\rho))} \left[ T_a^\alpha u(b)w(b) + T_b^\alpha u(a)w(a) \right] \\
&+ [u(a)w(a) + u(b)w(b)] \frac{\rho - 2 + \exp(-\rho)(\rho + 2)}{\rho^2(1 - \exp(-\rho))} \\
&+ [u(a)w(b) + u(b)w(a)] \frac{\rho^2 - 2\rho + 4 - (\rho^2 + 2\rho + 4) \exp(-\rho)}{2\rho^2(1 - \exp(-\rho))}. \tag{6.2}
\end{align*}
\]

Consequently, we have

\[
\begin{align*}
&u(\xi a + (1 - \xi) b) w(\xi a + (1 - \xi) b) \leq \xi^2 u(a)w(a) + (1 - \xi)^2 u(b)w(b) \\
&\quad + \xi(1 - \xi) [u(a)w(b) + u(b)w(a)] \\
\text{and} \\
&u((1 - \xi)a + \xi b) w((1 - \xi)a + \xi b) \leq (1 - \xi)^2 u(a)w(a) + \xi^2 u(b)w(b) \\
&\quad + \xi(1 - \xi) [u(a)w(b) + u(b)w(a)].
\end{align*}
\]
\[ + 2 \left[ u(a)w(b) + u(b)w(a) \right] \rho - 2 + \exp(-\rho) (\rho + 2) \frac{\rho^3}{\rho^3}. \]

So
\[
\frac{\alpha}{2(b - a)} \left[ I_a^\alpha(u(b)w(b)) + I_b^\alpha(u(a)w(a)) \right] \leq \left[ u(a)w(a) + u(b)w(b) \right] \frac{\rho^2 - 2\rho + 4 - (\rho^2 + 2\rho + 4) \exp(-\rho)}{2\rho^3}
+ \left[ u(a)w(b) + u(b)w(a) \right] \frac{\rho - 2 + \exp(-\rho) (\rho + 2)}{\rho^3},
\]

which completes the proof of (6.1).

Next we establish the inequality (6.2). Again using convexity of the functions \( u \) and \( v \) on \([a, b]\), we have
\[
u \left( \frac{a + b}{2} \right) w \left( \frac{a + b}{2} \right)\]
\[
= u \left( \frac{\xi a + (1 - \xi) b}{2} + \frac{1 - \xi) a + \xi b}{2} \right) w \left( \frac{\xi a + (1 - \xi) b}{2} + \frac{1 - \xi) a + \xi b}{2} \right)
\leq \left( \frac{u(\xi a + (1 - \xi) b) + u(1 - \xi) a + \xi b}{2} \right) \left( \frac{w((\xi a + (1 - \xi) b) + w(1 - \xi) a + \xi b)}{2} \right)
\leq \frac{u(\xi a + (1 - \xi) b) w(\xi a + (1 - \xi) b)}{4} + \frac{u(1 - \xi) a + \xi b) w((1 - \xi) a + \xi b)}{4}
+ \frac{2\xi^2 - 2\xi + 1}{4} [u(a)w(b) + u(b)w(a)].
\]

Thus
\[
u \left( \frac{a + b}{2} \right) w \left( \frac{a + b}{2} \right)\]
\[
\leq u \left( \frac{\xi a + (1 - \xi) b}{2} w(\xi a + (1 - \xi) b) + \frac{u(1 - \xi) a + \xi b) w((1 - \xi) a + \xi b)}{4} \right) (6.4)
+ \frac{t(1 - \xi)}{2} [u(a)w(a) + u(b)w(b)] + \frac{2\xi^2 - 2\xi + 1}{4} [u(a)w(b) + u(b)w(a)].
\]

Multiplying both sides of (6.4) by \( \exp(-\rho \xi) \) and then integrating the resulting inequality with respect to \( t \in [0, 1] \), we have
\[
\frac{1 - \exp(-\rho \xi)}{\rho} u \left( \frac{a + b}{2} \right) w \left( \frac{a + b}{2} \right)
\leq \int_0^1 \exp(-\rho \xi) \frac{u(\xi a + (1 - \xi) b) w(\xi a + (1 - \xi) b)}{4} d\xi
+ \int_0^1 \exp(-\rho \xi) \frac{u((1 - \xi) a + \xi b) w((1 - \xi) a + \xi b)}{4} d\xi
+ \int_0^1 \exp(-\rho \xi) \frac{\xi(1 - \xi)}{2} [u(a)w(a) + u(b)w(b)] d\xi
\]
Remark 6.3. Using the limiting values

\[
\lim_{\alpha \to 1} \frac{1 - \alpha}{2(1 - \exp(-\rho))} = \frac{1}{2(b - a)}, \quad \lim_{\alpha \to 1} \frac{\rho - 2 + \exp(-\rho)(\rho + 2)}{\rho^3} = \frac{1}{6},
\]

\[
\lim_{\alpha \to 1} \frac{\rho^2 - 2\rho + 4 - (\rho^2 + 2\rho + 4)\exp(-\rho)}{2\rho^2(1 - \exp(-\rho))} = \frac{1}{3},
\]

we obtain Pachpatte inequalities (1.3) and (1.5) from Theorem 6.1 when \(\alpha \to 1\).
DISCUSSIONS AND CONCLUSIONS

We obtained the generalization of the Hermite-Hadamard, Hermite-Hadamard-Fejér, Dragomir-Agarwal and Pachpatte type inequalities for new fractional integral operators with exponential kernel. As an immediate consequence of the results derived in this paper, one can obtain similar inequalities for the following fractional integrals with Mittag-Leffler nonsingular kernel:

\[ \mathcal{J}_a^\alpha u(x) = \frac{1}{\alpha} \int_a^x E_{\alpha,1} \left( -\frac{1-\alpha}{\alpha} (x-s)^\alpha \right) u(s) ds, \quad x > a \]

and

\[ \mathcal{J}_b^\alpha u(x) = \frac{1}{\alpha} \int_x^b E_{\alpha,1} \left( -\frac{1-\alpha}{\alpha} (s-x)^\alpha \right) u(s) ds, \quad x < b \]

for \( f \in L_1(a,b) \) and \( \alpha \in (0,1) \). Here \( E_{\alpha,\mu}(z) \) is the Mittag-Leffler type function:

\[ E_{\alpha,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \mu)}. \]

Moreover, we believe that the present work would serve as a strong motivation for the fellow researchers to enhance/enrich similar known literature on the related topics.

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