A generalization of Ohno’s relation for multiple zeta values

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Abstract

In the present paper, we prove that certain parametrized multiple series satisfy the same relation as Ohno’s relation for multiple zeta values. This result gives us a generalization of Ohno’s relation for multiple zeta values. By virtue of this generalization, we obtain a certain equivalence between the above relation among the parametrized multiple series and a subfamily of the relation. As applications of the above results, we obtain some results on multiple zeta values.

1 Introduction

The multiple zeta value (MZV for short) is defined by the multiple series

\[ \zeta(k) = \zeta(k_1, \ldots, k_n) := \sum_{0 < m_1 < \cdots < m_n} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}, \]

where \( k_1, \ldots, k_n \in \mathbb{Z}_{\geq 1} \) and \( k_n \geq 2 \). The case \( n = 2 \) was studied by L. Euler in [9], and the general case was introduced by M. E. Hoffman [13] and D. Zagier [34]. MZVs are related to several mathematical objects (see, e.g., [4], [6], [11], [23], [33], [34]), and satisfy various relations (see, e.g., [12], [13], [16], [19], [21], [27], [28], [32], [34]).

In the present paper, we deal with the parametrized multiple series

\[ Z(k; \alpha, \beta) = Z(k_1, \ldots, k_n; \alpha, \beta) := \sum_{0 \leq m_1 < \cdots < m_n} (\alpha)_{m_1} \frac{m_n!}{m_1! (\alpha)_{m_1+1} (m_1 + \beta)^{k_1} \cdots (m_{n-1} + \beta)^{k_{n-1}} (m_n + \beta)^{k_n}}, \]

where \( k_1, \ldots, k_n \in \mathbb{Z}_{\geq 1} \) and \( k_n \geq 2 \).
where \( k_1, \ldots, k_n \in \mathbb{Z}_{\geq 1}, k_n \geq 2, \alpha, \beta \in \mathbb{C} \) with \( \text{Re} \alpha > 0, \beta \notin \mathbb{Z}_{\leq 0} \), and \((a)_m\) denotes the Pochhammer symbol defined by
\[
(a)_m = \begin{cases} 
\frac{a(a+1)\cdots(a+m-1)}{m!} & \text{if } m \in \mathbb{Z}_{\geq 1}, \\
1 & \text{if } m = 0.
\end{cases}
\]

By the above definition, we immediately see that \( Z(k;1,1) = \zeta(k) \) and \( Z(k_1;\alpha,\alpha) = \zeta(k_1;\alpha) \), where \( \zeta(s;\alpha) := \sum_{m=0}^{\infty}(m+\alpha)^{-s} \) is the Hurwitz zeta-function. For simplicity, we denote \( Z(k;\alpha,\alpha) \) by \( Z(k;\alpha) \). By the above facts, the parametrized multiple series \( Z(k;\alpha) \) is not only a generalization of MZV but also a multiple version of the Hurwitz zeta-function. In [17], the author found \( Z(k;\alpha) \) and \( Z(k;\alpha,\beta) \) by studying Ochiai’s proof of the sum formula for MZVs ([26]; see also below and Remark 2.5). We note that a multiple series like \( Z(k;\alpha) \) was studied by M. Émery in [7]. (However, as was stated in [7], the results in the first version of [7] follow from some results of others published before [7] (see, e.g., [10, Proposition 2.1]).) We also note that multiple series like \( Z(k;\alpha,\beta) \) appear in the study of the hypergeometric construction of \( \mathbb{Q} \)-linear forms in zeta values (see, e.g., [5, Paragraphe 2.2], [22, Propositions 1 and 2], [35, Lemma 2]). In particular, Krattenthaler and Rivoal’s hypergeometric identity ([22, Proposition 1 (ii)]), which is a non-terminating version of a limiting case of a basic hypergeometric identity of G. E. Andrews, contains a certain relation among the non-strict version of \( Z(k;\alpha) \) (for the details, see [18, Remark 2.7]).

Before we state our results, we recall some definitions. An index \((k_1,\ldots,k_n)\) is called an admissible index if it satisfies that \( k_1,\ldots,k_n \) are positive integers and \( k_n \geq 2 \). The sum \( k_1 + \cdots + k_n \) and the integer \( n \) are called the weight and the depth of the index \((k_1,\ldots,k_n)\), respectively. Any admissible index \((k_1,\ldots,k_n)\) can be expressed as
\[
k := (k_1,\ldots,k_n) = (1,\ldots,1,b_1+2,\ldots,b_{s-1}+2,1,\ldots,1,b_s+2),
\]
where \( a_1,\ldots,a_s,b_1,\ldots,b_s \) are non-negative integers. Under this expression, the dual index of \( k \) is defined by
\[
k' := (k_1',\ldots,k_n') = (1,\ldots,1,a_s+2,\ldots,a_2+2,1,\ldots,1,a_1+2).
\]

By the above definition, it is easy to verify that
\[
k_1 + \cdots + k_n = k_1' + \cdots + k_n' = n + n'.
\]
In the present paper, we prove that the parametrized multiple series $Z(k; \alpha)$ satisfy the same relation as Ohno’s relation for MZVs ([27, Theorem 1]). Namely we shall prove the following.

**Theorem 1.1.** Let $(k_1, \ldots, k_n)$ be an admissible index and $(k'_1, \ldots, k'_n)$ the dual index of $(k_1, \ldots, k_n)$. Then the identity

$$\sum_{l_1 + \cdots + l_n = l, \ l_i \in \mathbb{Z}_{\geq 0}} Z(k_1 + l_1, \ldots, k_n + l_n; \alpha) = \sum_{l_1 + \cdots + l'_n = l, \ l_i \in \mathbb{Z}_{\geq 0}} Z(k'_1 + l_1, \ldots, k'_n + l'_n; \alpha)$$

(1)

holds for any integer $l \geq 0$ and all complex numbers $\alpha$ with $\text{Re} \alpha > 0$.

Taking $\alpha = 1$ in Theorem 1.1, we get Ohno’s relation for MZVs. Therefore Theorem 1.1 is a generalization of Ohno’s relation for MZVs. As Y. Ohno stated in [27], Ohno’s relation for MZVs generalizes simultaneously the duality ([13], [34]) and the sum formula for MZVs ([9], [12], [13], [15], [24]), and contains Hoffman’s relation for MZVs ([13, Theorem 5.1]). In consequence of Theorem 1.1, the same relations as above also hold for $Z(k; \alpha)$.

For example, the following sum formula for $Z(k; \alpha)$, which was proved by the author in [17, Proposition 1], can be derived from Theorem 1.1: the identity

$$\sum_{k_1 + \cdots + k_m = n, \ k_i \in \mathbb{Z}_{\geq 1}, k_m \geq 2} Z(k_1, \ldots, k_m; \alpha) = \zeta(n; \alpha)$$

holds for any integers $m$, $n$ with $0 < m < n$ and all $\alpha \in \mathbb{C}$ with $\text{Re} \alpha > 0$. Indeed, this follows by applying the identity (1) for the index $(k)$, $k \in \mathbb{Z}_{\geq 2}$.

In [17], to prove the above sum formula for $Z(k; \alpha)$, the author used Ochiai’s method of proving the sum formula for MZVs ([26]). In the present paper, we also use Ochiai’s method to prove Proposition 2.8 below, which is equivalent to Theorem 1.1. The sum formula for MZVs was first proved by A. Granville [12] and D. Zagier, independently. Though Ochiai’s proof of the sum formula for MZVs is unpublished, it can be found in [1, pp. 17–20] and [20, pp. 60–61]. Generally speaking, Ochiai’s method is as follows: first, for generating functions of sums of multiple series, we find a multiple integral representation like the Drinfel’d integral; secondly, applying a change of variables to the multiple integral, we get some duality formula for the generating functions; and finally we derive a relation among the multiple series from the above duality formula for the generating functions.
Alternative proofs of Ohno’s relation for MZVs were given by K. Ihara, M. Kaneko and D. Zagier [19], J. Okuda and K. Ueno [30], and G. Kawashima [21] (see also Remark 2.10 below). Further D. M. Bradley proved a \(q\)-analogue of Ohno’s relation for MZVs ([3, Theorem 5]).

By virtue of our generalization of Ohno’s relation for MZVs, we can obtain the following theorem.

**Theorem 1.2.** The following assertion is equivalent to Theorem 1.1:

(A) Let \((k_1, \ldots, k_n)\) be an admissible index and \((k'_1, \ldots, k'_n)\) the dual index of \((k_1, \ldots, k_n)\). Then the identity (1) holds for any “even” integer \(l \geq 0\) and all complex numbers \(\alpha\) with \(\text{Re} \alpha > 0\).

Theorem 1.2 asserts an equivalence between the relation in Theorem 1.1 and a subfamily of the relation. We note that, in the case MZVs, J. Okuda and K. Ueno [30] found a relationship between Ohno’s relation and a subfamily of the relation.

One of our motivations for studying parametrized multiple series is to apply the results to the study of MZVs. In the present paper, as applications of Theorems 1.1 and 1.2, we shall obtain a relation among MZVs (Corollary 2.11 below) and an equivalence of the relation (Corollary 3.3 below). Other applications of the property of parametrized multiple series to the study of MZVs can be found in, e.g., [2], [13, Section 4], [21], [29, Section 3], [30], [31], [36]. (For the reference [13, Section 4], see also Remark 2.10 below.)

We note that the content of Section 2 is an expansion of part of the author’s master’s thesis ([17, Proof of Proposition 3], where the author proved a certain sum formula for \(Z(k; \alpha, \beta)\) (see Remark 2.5 below) by using Ochiai’s method of proving the sum formula for MZVs).

## 2 Proof of Theorem 1.1 and a relation among MZVs

In the present section, we prove Theorem 1.1. First, using Ochiai’s method of proving the sum formula for MZVs, we prove Proposition 2.8 below. Secondly we prove the equivalence between Theorem 1.1 and Proposition 2.8. As an application of Theorem 1.1, we get a relation among MZVs.

In order to prove Theorem 1.1, we prove some properties of \(Z(k; \alpha, \beta)\).
Lemma 2.1. Let $k$ be an admissible index. Then the multiple series $Z(k; \alpha, \beta)$ converges absolutely for $(\alpha, \beta) \in \{(\alpha, \beta) \in \mathbb{C}^2 : \text{Re} \alpha > 0, \beta \notin \mathbb{Z}_{\leq 0}\}$ and uniformly in any compact subset of $\{(\alpha, \beta) \in \mathbb{C}^2 : \text{Re} \alpha > 0, \beta \notin \mathbb{Z}_{\leq 0}\}$.

Proof. We fix any real number $r$ with $0 < r < 1$ and any compact subset $K$ of $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. Let $(\alpha, \beta) \in \{(\alpha, \beta) \in \mathbb{C}^2 : \text{Re} \alpha \geq r, \beta \in K\}$ and let $(k_1, \ldots, k_n)$ be an admissible index. For a given positive integer $m$, we first estimate the finite multiple sum

$$\sum_{0 \leq m_1 < \cdots < m_{n-1} < m} \frac{(\alpha)_{m_1}}{m_1!} \frac{m!}{(\alpha)_{m+1}} \left\{ \prod_{i=1}^{n-1} \frac{1}{(m_i + \beta)^{k_i}} \right\} \frac{1}{(m + \beta)^{k_n-1}}. \tag{2}$$

In the case $n = 1$ in (2), we regard the sum (2) as

$$\frac{1}{(m + \alpha)(m + \beta)^{k_1-1}}.$$

Using Stirling's formula for the gamma function, we get

$$\leq \sum_{0 \leq m_1 < \cdots < m_{n-1} < m} \frac{(r)_{m_1}}{m_1!} \frac{m!}{(r)_{m+1}} \left\{ \prod_{i=1}^{n-1} \frac{1}{|m_i + \beta|^{k_i}} \right\} \frac{1}{|m + \beta|^{k_n-1}} \leq \frac{m!}{(r)_{m+1}|m + \beta|^{k_n-1}} \prod_{i=1}^{n-1} \left( \sum_{m_i=0}^{m-1} \frac{1}{|m_i + \beta|^{k_i}} \right) \ll \frac{1}{m^{1+r}} \left( \sum_{l=1}^{m} \frac{1}{l} \right)^{n-1} \ll \frac{\{\log (m + 1)\}^{n-1}}{m^{1+r}},$$

where the implied constants depend only on $r, (k_1, \ldots, k_n)$ and $K$. Since the series

$$\sum_{m=1}^{\infty} \frac{\{\log (m + 1)\}^{n-1}}{m^{1+r}}$$

converges for $r > 0$, we get the assertion by the Weierstrass $M$-test. \qed
By Lemma 2.1, the multiple series $Z(k; \alpha, \beta)$ is holomorphic in $\{ (\alpha, \beta) \in \mathbb{C}^2 : \text{Re} \alpha > 0, \beta \not\in \mathbb{Z}_{\leq 0} \}$ for any admissible index $k$.

The following multiple integral representation of $Z(k; \alpha, \beta)$, which has a symmetry with respect to the parameters $\alpha$ and $\beta$, plays an essential role for the proof of Theorem 1.1.

**Lemma 2.2.** Let $(k_1, \ldots, k_n)$ be an admissible index. Then the multiple integral representation

$$Z(k_1, \ldots, k_n; \alpha, \beta) = \int \cdots \int_{1 > t_k > \cdots > t_1 > 0} (1 - t_k)^{\alpha - 1} t_k^{-\beta} \left\{ \prod_{i=1}^{k} \omega_i(t_i) \right\} t_1^{\beta - 1} (1 - t_1)^{1 - \alpha} \, dt_k \cdots dt_1 \quad (3)$$

holds for all complex numbers $\alpha$ and $\beta$ with $\text{Re} \alpha > 0$ and $\text{Re} \beta > 0$, where $k$ denotes the weight of the index $(k_1, \ldots, k_n)$ (i.e., $k = k_1 + \cdots + k_n$), and

$$\omega_i(t_i) = \begin{cases} (1 - t_i)^{-1} & \text{if } i \in \{k_1 + \cdots + k_j + 1 : j = 0, 1, \ldots, n - 1 \}, \\ t_i^{-1} & \text{otherwise}. \end{cases}$$

**Proof.** Using the Taylor expansions of $(1 - t_1)^{-\alpha}$ and $(1 - t_i)^{-1}$ for $i \in \{k_1 + \cdots + k_j + 1 : j = 1, \ldots, n - 1 \}$ at the origin and the integration term by term,
we can calculate the right-hand side of (3) as follows:

\[
\int \cdots \int (1-t_k)^{\alpha-1} \sum_{i=1}^{k} \omega_i(t_i) \prod_{i=1}^{k} (1-t_i)^{1-\alpha} \, dt_k \cdots dt_1
\]

\[
= \int \cdots \int (1-t_k)^{\alpha-1} \frac{dt_k \cdots dt_1}{t_k^\beta t_{k-1}^\beta \cdots t_{k+1}^\beta (1-t_{k+1}) t_k \cdots t_2 (1-t_2)^\alpha}
\]

\[
= \int_0^1 \frac{(1-t_k)^{\alpha-1}}{t_k^\beta} \frac{dt_k}{t_{k-1}^\beta} \cdots \frac{dt_{k+1}}{t_{k+1}^\beta} \frac{dt_{k+1}}{1-t_{k+1}} \frac{dt_{k+1}}{t_{k+1}}
\]

\[
\int \frac{dt_2}{t_2} \sum_{l_1=0}^\infty (\alpha)_{l_1} \frac{l_1+\beta}{l_1+1} l_1!
\]

\[
= \sum_{l_1, \ldots, l_n \geq 0} \frac{(\alpha)_{l_1}}{l_1!} \frac{l_1+\beta}{l_1+1} \prod_{i=1}^{k} (1+l_i+n-1+\beta)_{k-1} \frac{1}{(l_1+\cdots+l_n+n-1+\beta)_{k-1}} \int_0^1 (1-t_k)^{\alpha-1} t_k^{l_1+\cdots+l_n+n-1} dt_k
\]

\[
= \sum_{l_1, \ldots, l_n \geq 0} \frac{(\alpha)_{l_1}}{l_1!} \frac{l_1+\beta}{l_1+1} \prod_{i=1}^{k} (1+l_i+n-1+\beta)_{k-1} \frac{1}{(l_1+\cdots+l_n+n-1+\beta)_{k-1}} \frac{\Gamma(\alpha) \Gamma(l_1+\cdots+l_n+n) \Gamma(\alpha+1+\cdots+l_n+n)}{\Gamma(\alpha+1+\cdots+l_n+n)}
\]

\[
= Z(k_1, \ldots, k_n; \alpha, \beta).
\]

The above calculation can be justified by the convergence of the multiple series

\[
Z(k_1, \ldots, k_n; \Re \alpha, \Re \beta), \quad \Re \alpha > 0, \Re \beta > 0.
\]

This completes the proof of Lemma 2.2. \(\square\)

**Remark 2.3.** The multiple integral representation in Lemma 2.2 is a generalization of the one used by H. Ochiai in [26] to prove the sum formula for
MZVs: he used the case $\alpha = 1$, $k_i = 1$ ($i = 1, \ldots, n - 1$) and $k_n = 2$ in Lemma 2.2 (see also [1, pp. 17–20] and [20, pp. 60–61]). We also note that the case $k_i = 1$ ($i = 1, \ldots, n - 1$) and $k_n = 2$ in Lemma 2.2 was used by the author in [17, Proof of Proposition 3] to prove a certain sum formula for $Z(k; \alpha, \beta)$ (see Remark 2.5 below).

Using (3), we can immediately prove the following duality for $Z(k; \alpha, \beta)$.

**Lemma 2.4** (Duality formula for $Z(k; \alpha, \beta)$). Let $k$ be an admissible index and $k'$ the dual index of $k$. Then the identity

$$Z(k; \alpha, \beta) = Z(k'; \beta, \alpha)$$

holds for all complex numbers $\alpha$ and $\beta$ with $\text{Re} \alpha > 0$ and $\text{Re} \beta > 0$.

**Proof.** The proof is the same as that for MZVs in [34, p. 510]. Indeed, the assertion follows from applying the change of variables

$$t_i \mapsto 1 - t_{k-i+1},$$

where $i = 1, \ldots, k$, to the multiple integral on the right-hand side of (3). □

Taking $\alpha = \beta$ in Lemma 2.4, we get the duality formula for $Z(k; \alpha)$.

**Remark 2.5.** Considering the index $(k)$, $k \in \mathbb{Z}_{\geq 2}$, we can derive the following sum formula for $Z(k; \alpha, \beta)$ from Lemma 2.4: the identity

$$\sum_{k_1 + \ldots + k_n = m+n \atop k_i \in \mathbb{Z}_{\geq 1}, k_m \geq 2} Z(k_1, \ldots, k_m; \alpha, \beta) = \sum_{l=0}^{\infty} \frac{1}{(l+\alpha)^m(l+\beta)^n}$$

holds for any integers $m, n \geq 1$ and all $\alpha, \beta \in \mathbb{C}$ with $\text{Re} \alpha > 0$, $\text{Re} \beta > 0$. This sum formula was proved by the author in [17, Proposition 3] by using Ochiai’s method of proving the sum formula for MZVs. We note that the condition $\text{Re} \beta > 0$ in the above sum formula can be changed into $\beta \notin \mathbb{Z}_{\leq 0}$, because both sides are holomorphic in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ as functions of $\beta$.

For simplicity, we put

$$S_l(k; \alpha) = S_l(k_1, \ldots, k_n; \alpha) := \sum_{l_1 + \ldots + l_n = l \atop l_i \in \mathbb{Z}_{\geq 0}} Z(k_1 + l_1, \ldots, k_n + l_n; \alpha)$$

8
and
\[ i_n^{(m)} := i_1^{(m)} + \cdots + i_n^{(m)} \]
for \( m \in \mathbb{Z}_{\geq 1}, n \in \mathbb{Z}_{\geq 0} \) and \( i_1^{(m)}, \ldots, i_n^{(m)} \in \mathbb{Z}_{\geq 0} \). We regard a sum of the form \( x_1 + \cdots + x_n \) as 0 if \( n = 0 \).

By using the notation \( S_l(k; \alpha) \), the identity (1) in Theorem 1.1 can be written as \( S_l(k; \alpha) = S_l(k'; \alpha) \).

The following lemma asserts that the multiple series \( Z(k; \alpha, \beta) \) is a generating function of sums of \( S_l(k; \alpha) \).

**Lemma 2.6.** Let \((k_1, \ldots, k_n)\) be an admissible index and let \( \alpha \) be a complex number with positive real part. Then, for all complex numbers \( \beta \) with \(|\beta - \alpha| < \text{Re} \alpha\), the following two expansions hold:

(i)  
\[
Z(k_1, \ldots, k_n; \beta, \alpha) = \sum_{l=0}^{\infty} (\alpha - \beta)^l 
\times \sum_{i=0}^{l} \sum_{i_1, \ldots, i_n \in \mathbb{Z}_{\geq 0}} S_{l-i}(k_1, 1, \ldots, 1, k_2, \ldots, k_n; \alpha); 
\]

(ii)  
\[
Z(k_1, \ldots, k_n; \alpha, \beta) = \sum_{l=0}^{\infty} (\alpha - \beta)^l 
\times \sum_{i=0}^{l} \sum_{i_1^{(1)} + \cdots + i_n^{(n)} = i} S_{l-i}(k_1 + i_1^{(1)}, \ldots, k_n - 1 + i_n^{(n)}; \alpha). 
\]

**Proof.** We fix any admissible index \((k_1, \ldots, k_n)\) and any complex number \( \alpha_0 \) with \( \text{Re} \alpha_0 > 0 \). Then, expanding \( Z(k_1, \ldots, k_n; \beta, \alpha_0) \) into the Taylor series at \( \beta = \alpha_0 \), we get

\[
Z(k_1, \ldots, k_n; \beta, \alpha_0) = \sum_{l=0}^{\infty} \frac{1}{l!} \frac{d^l}{d \beta^l} Z(k_1, \ldots, k_n; \beta, \alpha_0) \bigg|_{\beta = \alpha_0} (\beta - \alpha_0)^l \quad (4) 
\]
for all $\beta \in \mathbb{C}$ with $|\beta - \alpha_0| < \text{Re} \alpha_0$. By induction on $l$, we obtain

$$\frac{(-1)^l}{l!} \frac{d^l}{d\beta^l} (\beta)_{m_1} \frac{1}{(n_1 + \beta) \cdots (n_l + \beta)}$$

$$= \frac{(\beta)_{m_1}}{(\beta)_{m_n+1}} \sum_{i=0}^l \sum_{i_1 + \cdots + i_{n-1} = i} \sum_{i_1, \ldots, i_{n-1} \in \mathbb{Z}_{\geq 0}} \left\{ \prod_{r=1}^{n} \frac{1}{(m_r + \beta)^{l_r}} \right\}$$

$$\times \sum_{m_1 < m_1 < \cdots < m_{i_1} < m_2} \sum_{m_1 < m_1 < \cdots < m_{i_2} < m_2} \cdots \sum_{m_1 < m_1 < \cdots < m_{i_{n-1}} < m_n}$$

$$\times \sum_{m_1 < m_1 < \cdots < m_{i_{n-1}} < m_n} \frac{1}{(m_{pq} + \beta)^{i_{pq}+1}}$$

for any integer $l \geq 1$ and all integers $m_1, \ldots, m_n$ with $0 \leq m_1 < \cdots < m_n$. Using (4) and (5), we get (i).

Similarly, expanding $Z(k_1, \ldots, k_n; \alpha_0, \beta)$ into the Taylor series at $\beta = \alpha_0$, we get

$$Z(k_1, \ldots, k_n; \alpha_0, \beta) = \sum_{l=0}^{\infty} \frac{1}{l!} \frac{d^l}{d\beta^l} Z(k_1, \ldots, k_n; \alpha_0, \beta) \bigg|_{\beta = \alpha_0} (\beta - \alpha_0)^l$$

for all $\beta \in \mathbb{C}$ with $|\beta - \alpha_0| < \text{Re} \alpha_0$. It is easy to verify that the identity

$$\frac{(-1)^l}{l!} \frac{d^l}{d\beta^l} (m_1 + \beta)^{k_1} \cdots (m_{n-1} + \beta)^{k_{n-1}} (m_n + \beta)^{k_n-1}$$

$$= \sum_{i=0}^l \sum_{i_1^{(k_1)} + \cdots + i_{n-2}^{(k_{n-2})} = i} \sum_{i_1, \ldots, i_{n-1} \in \mathbb{Z}_{\geq 0}} \left\{ \prod_{j=1}^{n} \frac{1}{(m_j + \beta)^{k_j + i_j^{(k_j - 1)} + i_j^{(k_j - 1)}}} \right\}$$

$$\times \frac{1}{(m_n + \beta)^{k_n-1 + i_{n-2}^{(k_{n-2})} + i_{n-1}}}$$

holds for any admissible index $(k_1, \ldots, k_n)$ and any integer $l \geq 0$. Using (6) and (7), we get (ii).
Remark 2.7. The first identity of (5) can be derived from Fu and Lascoux’s $q$-identity ([10, Proposition 2.1]), and was proved by M. Émery in [7, Lemma] by using a different method from ours. We also note that an identity like the first identity of (5) was used by M. E. Hoffman in [13, Proof of Corollary 4.2] and G. Kawashima in [21, Proofs of Propositions 4.7 and 5.2] to prove relations among MZVs: they used the derivative of the binomial coefficient (see also [12, p. 97]). The idea used in the proof of Lemma 2.6 (i) is essentially the same as the one used by M. E. Hoffman and G. Kawashima in the above papers.

Using the above properties of $Z(k; \alpha, \beta)$, we can prove the following relation among $Z(k; \alpha)$.

**Proposition 2.8.** Let $(k_1, \ldots, k_n)$ be an admissible index and $(k'_1, \ldots, k'_n)$ the dual index of $(k_1, \ldots, k_n)$. Then the identity

$$
\sum_{i=0}^{l} \sum_{i_1+\ldots+i_{n-1} = i}^{l} S_{l-i}(k_1, 1, \ldots, 1, k_2, \ldots, k_{n-1}, 1, \ldots, 1, k_n; \alpha) = \sum_{i=0}^{l} \sum_{i_1^{(1)} + \ldots + i_{n'}^{(n')} = i} S_{l-i}(k'_1 + 1, k'_1 + 1, \ldots, k'_1 + 1, k'_{n' - 1} + 1, k'_{n' - 1} + 1, k'_{n' - 2} + 1, \ldots, k'_{n'} + 1; \alpha)
$$

holds for any integer $l \geq 0$ and all complex numbers $\alpha$ with $\text{Re} \alpha > 0$.

**Proof.** By using Lemmas 2.4 and 2.6, the generating functions of both sides of (8) coincide. Therefore we get the assertion.

Theorem 1.1 follows from Proposition 2.8. In fact, these are equivalent.

**Proposition 2.9.** Theorem 1.1 and Proposition 2.8 are equivalent.

**Proof.** Let $(k_1, \ldots, k_n)$ be an admissible index, $(k'_1, \ldots, k'_n)$ the dual index of $(k_1, \ldots, k_n)$, and $i_1, \ldots, i_{n-1}$ non-negative integers. Then we note that, by the definition of dual indices, the dual index of the index

$$(k_1, 1, \ldots, 1, k_2, \ldots, k_{n-1}, 1, \ldots, 1, k_n)$$

...
takes the form
\[(k'_1 + i^{(1)}_{k'_1-1}, \ldots, k'_{n'-1} + i^{(n'-1)}_{k'_{n'-1}-1}, k'_n + i^{(n')}_{k'_n-2})\]
with
\[i^{(1)}_{k'_1-1} + \cdots + i^{(n'-1)}_{k'_{n'-1}-1} + i^{(n')}_{k'_n-2} = i_1 + \cdots + i_{n-1}.
By this fact, it is easy to prove that Theorem 1.1 implies Proposition 2.8.
Conversely we suppose that Proposition 2.8 is true. Then, by the above fact, we can rewrite (8) as
\[S_l(k_1, \ldots, k_n; \alpha) - S_l(k'_1, \ldots, k'_{n'}; \alpha)
= -\sum_{i=1}^{l} \sum_{i_1+\cdots+i_{n-1}=i} \left\{S_{l-i}(k_{i_1, \ldots, i_{n-1}}; \alpha) - S_{l-i}(k'_{i_1, \ldots, i_{n-1}}; \alpha)\right\}, \quad (9)\]
where
\[k_{i_1, \ldots, i_{n-1}} := (k_1, 1, \ldots, 1, k_2, \ldots, k_{n-1}, 1, \ldots, 1, k_n),\]
and \(k'_{i_1, \ldots, i_{n-1}}\) is the dual index of \(k_{i_1, \ldots, i_{n-1}}\). We note that Proposition 2.8 contains the duality formula for \(Z(k; \alpha)\) (i.e., the case \(l = 0\) in Proposition 2.8). Therefore, by using (9) and induction on \(l\), we get Theorem 1.1. This completes the proof of Proposition 2.9.

**Remark 2.10.** Taking \(\alpha = 1\) in Proposition 2.8, we get a relation among MZVs. By using this relation and the same argument as in the proof of Proposition 2.9, we can obtain Ohno’s relation for MZVs. This is an alternative proof of Ohno’s relation for MZVs which follows Ochiai’s proof of the sum formula for MZVs. In this alternative proof, we used the identity \(Z(k; 1, \beta) = Z(k'; \beta, 1)\) (i.e., the case \(\alpha = 1\) in Lemma 2.4). We note that this kind of identity was already used by M. E. Hoffman in [13, Section 4] to prove a relation among MZVs: he used a modification of an identity of L. J. Mordell to prove a special case of the duality formula for MZVs.

As is obvious from the proof of Lemma 2.6, the derivative of \(Z(k; \alpha)\) is a \(Z\)-linear combination of \(Z(k; \alpha)\). Therefore the following corollary to Theorem 1.1 gives us a \(Z\)-linear relation among \(Z(k; \alpha)\).
Corollary 2.11. Let $\alpha_0$ be a complex number with positive real part, $k$ an admissible index, and $k'$ the dual index of $k$. Then the identity
\[
\frac{d^m}{d\alpha^m} S_l(k; \alpha) \bigg|_{\alpha=\alpha_0} = \frac{d^m}{d\alpha^m} S_l(k'; \alpha) \bigg|_{\alpha=\alpha_0}
\]
holds for any integers $l, m \geq 0$.

Using (5) and (7), we obtain
\[
(-1)^m \frac{d^m}{d\alpha^m} S_l(k_1, \ldots, k_n; \alpha)
= \sum_{l_i \in \mathbb{Z}_{\geq 0}} \sum_{l_1+\cdots+l_n=l} S_{l_n}(k_1 + l_1 + i^{(1)}_{k_1+l_1}, 1, \ldots, 1, k_2 + l_2 + i^{(2)}_{k_2+l_2}, \ldots, k_{n-1} + l_{n-1} + i^{(n-1)}_{k_{n-1}+l_{n-1}}, 1, \ldots, 1, k_n + l_n + i^{(n)}_{k_n+l_n}; \alpha)
\]
for any integers $l, m \geq 0$. This gives us an explicit form of the relation in Corollary 2.11.

Taking $\alpha_0 = 1$ in Corollary 2.11, we get a relation among MZVs. In [21], G. Kawashima proved a relation among MZVs which contains Ohno's relation properly (see also [32, Section 1]). Clearly our relation among MZVs also contains Ohno's relation.

Remark 2.12. By Lemma 2.4, we get the identity
\[
\frac{\partial^{m+n}}{\partial \alpha^m \partial \beta^n} Z(k; \alpha, \beta) \bigg|_{\beta=\alpha_0} = \frac{\partial^{m+n}}{\partial \alpha^m \partial \beta^n} Z(k'; \beta, \alpha) \bigg|_{\beta=\alpha_0}
\]
for any integers $m, n \geq 0$ and any $\alpha_0 \in \mathbb{C}$ with $\text{Re} \alpha_0 > 0$, where $k$ is any admissible index, and $k'$ is the dual index of $k$. As is obvious from the proof of Lemma 2.6, the above identity gives us a $Z$-linear relation among $Z(k; \alpha)$, which contains the relation in Proposition 2.8 (i.e., the case $n = 0$).

Remark 2.13. It is well-known that $\zeta(4) = 4\zeta(1, 3)$ (see, e.g., [19, p. 309]). However the functions $\zeta(4; \alpha)$ and $4Z(1, 3; \alpha)$ are not identically equal on the half-plane $\{ \alpha \in \mathbb{C} : \text{Re} \alpha > 0 \}$. Indeed, it is easy to verify that $\zeta(4; 2) \neq 4Z(1, 3; 2)$. On the other hand, by Theorem 1.1, the identities $\zeta(4; \alpha) = Z(2, 2; \alpha) + Z(1, 3; \alpha) = Z(1, 1, 2; \alpha)$ hold for all $\alpha \in \mathbb{C}$ with $\text{Re} \alpha > 0$. This difference probably comes from the absence of suitable expressions of the product $Z(h; \alpha)Z(k; \alpha)$ for $\alpha \neq 1$. In the case $\alpha = 1$ (i.e., MZVs), see [14] and [19].
3 Proof of Theorem 1.2 and an equivalence of a relation among MZVs

In the present section, using Lemma 3.2 below, we prove Theorem 1.2. As an application of Theorem 1.2, we also prove an equivalence for the relation in Corollary 2.11.

For each complex number $\beta$ with $\text{Re} \beta > 0$, we put

$$D(\beta) := \{ \alpha \in \mathbb{C} : |\alpha - \beta| < \text{Re} \beta/2 \}.$$ 

We first prove a lemma.

**Lemma 3.1.** Let $(k_1, \ldots, k_n)$ be an admissible index and let $\beta$ be a complex number with positive real part. Then the series

$$\sum_{l=0}^{\infty} (\alpha - \beta)^l \times \sum_{i=0}^{l} \sum_{i_1+\cdots+i_{n-1}=i} \sum_{i_1, \ldots, i_{n-1} \in \mathbb{Z}_{\geq 0}} S_{l-i}(k_1, 1, \ldots, 1, k_2, \ldots, k_n-1, 1, \ldots, 1, k_n, \alpha)$$

and

$$\sum_{l=0}^{\infty} (\alpha - \beta)^l \times \sum_{i=0}^{l} \sum_{i_1^{(1)}+\cdots+i_{n-2}^{(n)}=i} S_{l-i}(k_1 + i_{k_1-1}^{(1)}, \ldots, k_{n-1} + i_{k_{n-1}-1}^{(n-1)}, k_n + i_{k_n-2}^{(n)}; \alpha)$$

converge absolutely for $\alpha \in D(\beta)$ and uniformly in any compact subset of $D(\beta)$.
Proof. We fix any real number \( r \) with \( \Re \beta/2 < r < 3\Re \beta/2 \). Then we get

\[
\left| \sum_{i=0}^{l} \sum_{i_1 + \cdots + i_{n-1} = i \atop i_1, \ldots, i_{n-1} \in \mathbb{Z}_{\geq 0}} S_{l-i}(k_1, 1, \ldots, 1, k_2, \ldots, k_{n-1}, 1, \ldots, 1, k_n; \alpha) \right| \\
\leq \sum_{i=0}^{l} \sum_{i_1 + \cdots + i_{n-1} = i \atop i_1, \ldots, i_{n-1} \in \mathbb{Z}_{\geq 0}} S_{l-i}(k_1, 1, \ldots, 1, k_2, \ldots, k_{n-1}, 1, \ldots, 1, k_n; r) \\
= \frac{(-1)^l}{l!} \frac{d^l}{dz^l} Z(k_1, \ldots, k_n; z, r) \bigg|_{z=r}
\]

for all \( \alpha \in \mathbb{C} \) with \( \Re \alpha \geq r \) and any integer \( l \geq 0 \). Further, by Cauchy’s theorem, we get

\[
\frac{(-1)^l}{l!} \frac{d^l}{dz^l} Z(k_1, \ldots, k_n; z, r) \bigg|_{z=r} = \frac{(-1)^l}{2\pi \sqrt{-1}} \int_{|z-r|=\rho} \frac{Z(k_1, \ldots, k_n; z, r)}{(z-r)^{l+1}} \, dz \\
\leq \frac{Z(k_1, \ldots, k_n; r-\rho, r)}{\rho^l},
\]

where \( \rho \in \mathbb{R} \) with \( \Re \beta/2 < \rho < r \). Thus we get

\[
|\alpha - \beta|^l \left| \sum_{i=0}^{l} \sum_{i_1 + \cdots + i_{n-1} = i \atop i_1, \ldots, i_{n-1} \in \mathbb{Z}_{\geq 0}} S_{l-i}(k_1, 1, \ldots, 1, k_2, \ldots, k_{n-1}, 1, \ldots, 1, k_n; \alpha) \right| \\
< \left( \frac{\Re \beta}{2\rho} \right)^l Z(k_1, \ldots, k_n; r-\rho, r)
\]

for all \( \alpha \in D(\beta) \cap \{ \alpha \in \mathbb{C} : \Re \alpha \geq r \} \), any integer \( l \geq 0 \) and a fixed \( \rho \in \mathbb{R} \) with \( \Re \beta/2 < \rho < r \). By the above estimate and the Weierstrass M-test, we get the assertion for (10).

By the same argument as above, we can prove the assertion for (11). \( \square \)

By Lemma 3.1, the series (10) and (11) are holomorphic in \( D(\beta) \) as functions of \( \alpha \).

We shall use the following lemma to prove Theorem 1.2.

15
Lemma 3.2. Let \((k_1, \ldots, k_n)\) be an admissible index and let \(\beta\) be a complex number with positive real part. Then, for any integer \(m \geq 0\), the following two identities hold:

\[
\sum_{i=0}^{m} S_{m-i}(k_1 + i_{k_1-1}^{(1)}, \ldots, k_n-1 + i_{k_{n-2}-1}^{(n)}; k_n + i_{k_{n-2}}^{(n)}; \beta)
= (-1)^m \sum_{l=0}^{m} \frac{1}{(m-l)!} \frac{d^{m-l}}{d\alpha^{m-l}} \left\{ \sum_{i=0}^{l} \sum_{i_1 + \cdots + i_{n-1} = i} S_{l-i}(k_1, 1, \ldots, 1, k_2, i_{n-1}) \right\}_{\alpha=\beta}
\]

and

\[
\sum_{i=0}^{m} \sum_{i_1 + \cdots + i_{n-1} = i \atop i_1, \ldots, i_{n-1} \in \mathbb{Z}_{\geq 0}} S_{m-i}(\underbrace{k_1, \ldots, 1}_{i_1}, \ldots, \underbrace{k_{n-1}, 1, \ldots, k_n}_{i_{n-1}}; \beta)
= (-1)^m \sum_{l=0}^{m} \frac{1}{(m-l)!} \frac{d^{m-l}}{d\alpha^{m-l}} \left\{ \sum_{i=0}^{l} \sum_{i_1 + \cdots + i_{n-1} = i} S_{l-i}(k_1 + i_{k_1-1}^{(1)}, \ldots, k_n-1 + i_{k_{n-2}-1}^{(n)}; \beta) \right\}_{\alpha=\beta}
\]

Proof. We fix any admissible index \((k_1, \ldots, k_n)\) and any complex number \(\beta_0\) with \(\operatorname{Re} \beta_0 > 0\). We note that the inequality \(\operatorname{Re} \beta_0 / 2 < \operatorname{Re} \alpha\) holds for all \(\alpha \in D(\beta_0)\). Hence, by Lemma 2.6 (i), the expansion

\[
Z(k_1, \ldots, k_n; \beta_0, \alpha)
= \sum_{l=0}^{\infty} (\alpha - \beta_0)^l
\]

\[
\times \sum_{i=0}^{l} \sum_{i_1 + \cdots + i_{n-1} = i \atop i_1, \ldots, i_{n-1} \in \mathbb{Z}_{\geq 0}} S_{l-i}(k_1, 1, \ldots, 1, k_2, \ldots, k_{n-1}, 1, \ldots, 1, k_n; \alpha)
\]

holds for all \(\alpha \in D(\beta_0)\). Further, by Lemma 3.1, we can differentiate the series on the right-hand side of (14) term by term with respect to \(\alpha\) in \(D(\beta_0)\).
Thus we obtain
\[
\frac{(-1)^m}{m!} \frac{d^m}{d\alpha^m} Z(k_1, \ldots, k_n; \beta_0, \alpha) \bigg|_{\alpha=\beta_0} = (-1)^m \sum_{l=0}^{m} \frac{1}{(m-l)!} \frac{d^{m-l}}{d\alpha^{m-l}} \left\{ \sum_{i=0}^{l} \sum_{i_1 + \cdots + i_{n-1} = i \atop i_1, \ldots, i_{n-1} \in \mathbb{Z}_{\geq 0}} S_{l-i}(k_1, 1, \ldots, 1, k_2, \ldots, k_{n-1}, 1, \ldots, 1, k_n; \alpha) \right\} \bigg|_{\alpha=\beta_0}
\]
for any integer \(m \geq 0\). In the proof of Lemma 2.6 (ii), we proved that the left-hand side of the above identity is equal to that of (12). This completes the proof of (12).

By Lemmas 2.6 and 3.1, and the same argument as above, we can prove (13).

Now we prove Theorem 1.2.

**Proof of Theorem 1.2.** It is trivial that Theorem 1.1 implies Theorem 1.2 (A).

Conversely we suppose that Theorem 1.2 (A) is true. Then it is enough to prove the following assertion: the identity \(S_{m}(k; \alpha) = S_{m}(k'; \alpha)\) holds for any positive “odd” integer \(m\) and all \(\alpha \in \mathbb{C}\) with \(\text{Re} \alpha > 0\), where \(k\) is any admissible index, and \(k'\) is the dual index of \(k\).

Using the identities (12) for \(k = (k_1, \ldots, k_n)\) and (13) for \(k' = (k'_1, \ldots, k'_n)\), and recalling what we noted the form of the dual index of the index
\[
(k_1, 1, \ldots, 1, k_2, \ldots, k_{n-1}, 1, \ldots, 1, k_n)
\]
in the proof of Proposition 2.9, we get

\[
\sum_{i=0}^{m} \sum_{i_1 + \cdots + i_{n-1} = i} \left\{ S_{m-i}(h_{i_1, \ldots, i_{n-1}}; \beta) - S_{m-i}(h_{i_1, \ldots, i_{n-1}}; \beta) \right\} = (-1)^m \sum_{i=0}^{m} \frac{1}{(m-l)!} \left. \left. \frac{d^{m-l}}{d \alpha^{m-l}} \left[ \sum_{i=0}^{l} \sum_{i_1 + \cdots + i_{n-1} = i} \left\{ S_{l-i}(k_{i_1, \ldots, i_{n-1}}; \alpha) - S_{l-i}(k'_{i_1, \ldots, i_{n-1}}; \alpha) \right\} \right] \right|_{\alpha = \beta} \right] \tag{15} \]

for any \( \beta \in \mathbb{C} \) with \( \text{Re}\ \beta > 0 \) and any integer \( m \geq 0 \), where \( k_{i_1, \ldots, i_{n-1}} \) and \( k'_{i_1, \ldots, i_{n-1}} \) are the same notations as in (9),

\[ h_{i_1, \ldots, i_{n-1}} := (k'_{i_1}, 1, \ldots, 1, k'_{2}, \ldots, k'_{n-1}, 1, \ldots, 1, k'_{n}), \]

and \( h'_{i_1, \ldots, i_{n-1}} \) is the dual index of \( h_{i_1, \ldots, i_{n-1}} \). The identity (15) can be rewritten as

\[
\left\{ 1 + (-1)^{m+1} \right\} \left\{ S_m(k; \beta) - S_m(k'; \beta) \right\} = - \sum_{i=1}^{m} \sum_{i_1 + \cdots + i_{n-1} = i} \left\{ S_{m-i}(h'_{i_1, \ldots, i_{n-1}}; \beta) - S_{m-i}(h_{i_1, \ldots, i_{n-1}}; \beta) \right\} + (-1)^m \sum_{i=1}^{m} \sum_{i_1 + \cdots + i_{n-1} = i} \left\{ S_{m-i}(k_{i_1, \ldots, i_{n-1}}; \beta) - S_{m-i}(k'_{i_1, \ldots, i_{n-1}}; \beta) \right\} \tag{16} \]

Further we suppose that \( m \) is a positive odd integer. Then, applying Theorem
1.2 (A) to the right-hand side of (16), we get

\[
2 \left\{ S_m(k; \beta) - S_m(k'; \beta) \right\} = - \sum_{i=0}^{m-1} \sum_{i_1+\ldots+i_{n-1}=m-i \atop i \text{ odd}} \{ S_i(h'_{i_1,\ldots,i_{n-1}}; \beta) - S_i(h_{i_1,\ldots,i_{n-1}}; \beta) \} = - \sum_{i=0}^{m-1} \sum_{i_1+\ldots+i_{n-1}=m-i \atop i \text{ odd}} \{ S_i(k_{i_1,\ldots,i_{n-1}}; \beta) - S_i(k'_{i_1,\ldots,i_{n-1}}; \beta) \} \]

(17)

\[
- \sum_{l=0}^{m-l} \frac{1}{(m-l)!} \cdot \frac{d^{m-l}}{d\alpha^{m-l}} \left[ \sum_{i=0}^{l} \sum_{i_1+\ldots+i_{n-1}=l-i \atop i \text{ odd}} \{ S_i(k_{i_1,\ldots,i_{n-1}}; \beta) - S_i(k'_{i_1,\ldots,i_{n-1}}; \beta) \} \right] \bigg|_{\alpha=\beta}.
\]

Therefore, by using (17) and induction on the positive odd integer \( m \), we get the assertion which we stated at the beginning of this proof. This completes the proof of Theorem 1.2. \( \square \)

As an application of Theorem 1.2, we can prove the following equivalence for the relation in Corollary 2.11.

**Corollary 3.3.** Let \( \alpha_0 \) and \( \alpha_1 \) be complex numbers with positive real parts.

Then the following two assertions are equivalent:

(i) Let \( k \) be an admissible index and \( k' \) the dual index of \( k \). Then the identity

\[
\frac{d^m}{d\alpha^m} S_l(k; \alpha) \bigg|_{\alpha=\alpha_0} = \frac{d^m}{d\alpha^m} S_l(k'; \alpha) \bigg|_{\alpha=\alpha_0}
\]

holds for any “even” integer \( l \geq 0 \) and any integer \( m \geq 0 \).

(ii) Let \( k \) be an admissible index and \( k' \) the dual index of \( k \). Then the identity

\[
\frac{d^m}{d\alpha^m} S_l(k; \alpha) \bigg|_{\alpha=\alpha_1} = \frac{d^m}{d\alpha^m} S_l(k'; \alpha) \bigg|_{\alpha=\alpha_1}
\]

holds for any integers \( l, m \geq 0 \).
Proof. We suppose that (i) is true. Then, expanding $S_l(k; \alpha)$ into the Taylor series at $\alpha = \alpha_0$, we see that the identity $S_l(k; \alpha) = S_l(k'; \alpha)$ holds for any “even” integer $l \geq 0$ and all $\alpha \in \mathbb{C}$ with $|\alpha - \alpha_0| < \Re \alpha_0$. By the uniqueness theorem for holomorphic functions, the above identity holds for all $\alpha \in \mathbb{C}$ with $\Re \alpha > 0$. This is exactly Theorem 1.2 (A). Therefore, by Theorem 1.2, we get Theorem 1.1. Clearly Theorem 1.1 implies (ii).

By the same argument as above, we can prove that (ii) implies (i). \qed

Taking $\alpha_0 = \alpha_1 = 1$ in Corollary 3.3, we get an equivalence between the relation among MZVs which we stated at the end of Section 2 and a subfamily of the relation.

Addendum for the revised version

Using Lemma 3.2, we can prove the following proposition.

**Proposition.** Let $\alpha_0$ be a complex number with positive real part. Then the following two assertions are equivalent:

(i) Let $k$ be an admissible index and $k'$ the dual index of $k$. Then the identity

$$S_l(k; \alpha_0) = S_l(k'; \alpha_0)$$

holds for any integer $l \geq 0$.

(ii) Let $k$ be an admissible index and $k'$ the dual index of $k$. Then the identity

$$\sum_{p+q=l \atop p,q \in \mathbb{Z}_{\geq 0}} \frac{1}{p!} \frac{d^p}{d\alpha^p} S_q(k; \alpha) \bigg|_{\alpha=\alpha_0} = \sum_{p+q=l \atop p,q \in \mathbb{Z}_{\geq 0}} \frac{1}{p!} \frac{d^p}{d\alpha^p} S_q(k'; \alpha) \bigg|_{\alpha=\alpha_0}$$

holds for any integer $l \geq 0$.

**Proof.** As we stated in the proof of Theorem 1.2, using Lemma 3.2, we get (15). We note that the right-hand side of (15) can be rewritten as

$$(-1)^m \sum_{i=0}^{m} \sum_{i_1 + \ldots + i_{n-1} = i \atop i_1,\ldots,i_{n-1} \in \mathbb{Z}_{\geq 0}} \sum_{p+q=m-i \atop p,q \in \mathbb{Z}_{\geq 0}} \frac{1}{p!} \frac{d^p}{d\alpha^p} \left\{ S_q(k_{i_1,\ldots,i_{n-1}}; \alpha) - S_q(k'_{i_1,\ldots,i_{n-1}}; \alpha) \right\} \bigg|_{\alpha=\beta}. $$

20
Hence, by using (15) and the same argument as in the proof of Proposition 2.9, we can prove the assertion.

Taking \( \alpha_0 = 1 \) in the above proposition, we get an equivalence of Ohno’s relation for MZVs.

**Remark 1.** The author proved the cyclic sum formulas for \( Z(k; \alpha) \) and the non-strict version of \( Z(k; \alpha) \) (see [18]).

**Remark 2.** In [17, Proposition 2], by using Ochiai’s method of proving the sum formula for MZVs, the author proved the following identity

\[
\sum_{k_1 + \cdots + k_n = k \atop k_i \in \mathbb{Z}_{\geq 1}, k_n \geq 2} \zeta(k_1, \ldots, k_n; \alpha) = \frac{1}{(k - n - 1)!} \sum_{l=1}^{\infty} \frac{1}{l^n} \frac{\partial^{k-n-1}}{\partial x^{k-n-1}} \left\{ \frac{(1-x)_{l-1}}{(\alpha-x)_l} \right\} \bigg|_{x=0}
\]

for any integers \( k, n \) with \( 0 < n < k \) and all \( \alpha \in \mathbb{C} \) with \( \Re \alpha > 0 \), where \((a)_l\) denotes the Pochhammer symbol and

\[
\zeta(s_1, \ldots, s_n; \alpha) := \sum_{0 \leq m_1 < \cdots < m_n} \frac{1}{(m_1 + \alpha)^{s_1} \cdots (m_n + \alpha)^{s_n}}
\]

is the multiple Hurwitz zeta-function (see also [8] and [25]). Indeed, the identity (18) can be proved by applying the same method as in Section 2 to the generating function of the left-hand side of (18). We note that the identity (18) contains the sum formula for MZVs (i.e., the case \( \alpha = 1 \) in (18)).

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