State-extended uncertainly relations and tomographic inequalities as quantum system state characteristics

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Abstract

Some inequalities for probability vector are discussed. The probability representation of quantum mechanics where the states are mapped onto probability vectors (either finite or infinite dimensional) called the state tomograms is used. Examples of inequalities for qudit tomograms and a state extended uncertainly relation are considered. Tomographic cumulant related to photon state tomographic probability distributions is introduced and it is used as parameter of the state nongaussianity.

1 Introduction

Recently [1, 2, 3] the tomographic probability representation of quantum states [4, 5] was used to study some uncertainly relations introduced in [6, 7] so called state extended uncertainly relations. These relations were presented in the form of integral inequalities [8] for measurable optical tomograms [9, 10, 11]. The aim of our work is to consider some other inequalities which can be obtained for any probability vectors and to apply the inequalities to tomographic probability distributions describing the quantum states of photons and qudits. We consider recently found new uncertainly relations for arbitrary observables [6, 7] and present the relations in the form of inequalities for measurable optical photon tomograms. Also the simple inequalities available for probability vectors we use to get the inequalities for spin tomographic probability distributions. Another aim is to introduce the cumulant related to optical tomogram as a characteristics of the photon state gaussianity. The paper is organized as follows. In next section 2 we present the state extended uncertainty relations in the form of integral inequalities for the photon state tomograms. In section 3 we remind some properties of probability vectors and discuss the linear maps of such vectors. The inequalities for Shanon entropy associated with a probability distribution considered as probability vector are studied in section 4. New entropic inequalities are obtained for qudit tomograms in section 5. The tomographic cumulant and corresponding integral inequality expressed in terms of optical tomogram of quantum state is suggested to be used as nongaussianity parameter in experiments on homodyne photon detection in section 6. Conclusion and prospectives are presented in section 7.
2 State-extended uncertainly relations

In our previous work [3], some of the state-extended uncertainly relations [6] were presented in tomographic form suitable for experimental check. The state-extended position and momentum uncertainly relations were confirmed in experiments with homodyne photon detection in [8]. Now we consider the other state-extended uncertainly relations; namely, we study the inequality

\[
\left((\Delta A(\psi_1))^2 + \langle \psi_1 | A | \psi_1 \rangle^2 \right) \left((\Delta A(\psi_2))^2 + \langle \psi_2 | A | \psi_2 \rangle^2 \right) \geq |\langle \psi_2 | A^2 | \psi_1 \rangle|^2, \tag{1}
\]

where \( | \psi_1 \rangle \) and \( | \psi_2 \rangle \) are the pure-state vectors, \( A \) is an observable, and \( \Delta A(\psi_1) \) is dispersion of the observable \( A \).

Our aim is to rewrite this inequality in the tomographic form. We use the optical tomographic representation for the one-mode photon state. The tomogram \( w(X, \Theta) \) depends on the homodyne quadrature \( X \) and local oscillator phase \( \Theta \); this tomogram can be measured in the experiments with homodyne photon state detection.

If the observable \( A \) in (1) is an analog of the position operator the inequality in the tomographic form reads

\[
\left[ \int X^2 w_1(X, \mu = 1, \nu = 0) dX \right] \left[ \int X^2 w_2(X, \mu = 1, \nu = 0) dX \right] \geq \frac{1}{2\pi} \int \tilde{w}_1(X, \mu, \nu) w_2(-Y, \mu, \nu) \exp \left( \frac{i}{\nu} (X + Y) \right) dX dY d\mu d\nu, \tag{2}
\]

where \( \tilde{w}_1(X, \mu, \nu) \) is the symplectic tomogram of “the state vector” \( | \varphi_1 \rangle = A^2 | \psi_1 \rangle \) and \( w_2(Y, \mu, \nu) \) is symplectic tomogram of the state \( | \psi_2 \rangle \). If one knows wave function \( \varphi_1(y) = \langle y | \varphi_1 \rangle \) and \( A \) is position operator, the tomogram is

\[
\tilde{w}_1(X, \mu, \nu) = \frac{1}{2\pi |\nu|} \left| \int y^2 \varphi_1(y) \exp \left( \frac{iy \mu}{2\nu} - \frac{iX \nu}{\nu} y \right) dy \right|^2.
\]

For \( \mu = \cos \Theta \) and \( \nu = \sin \Theta \), the symplectic tomogram coincides with the optical tomogram. Inequality (2) can be expressed in terms of optical tomograms of states \( | \psi_1 \rangle \) and \( | \psi_2 \rangle \).

3 Probability distributions and some maps of probability vectors

In this section we study some relations for probability distributions which are considered as probability vectors. We start with example of classical object which can be found in four different states \( a_1, a_2, a_3 \) and \( a_4 \) with probabilities \( p_1, p_2, p_3 \) and \( p_4 \), respectively. The nonnegative numbers \( p_k, k = 1, 2, 3, 4 \) satisfy the condition \( \sum_{k=1}^4 p_k = 1 \). The numbers can be considered as the components \( p_k \) of 4–vector \( \vec{p} \) which can be called the probability vector. Also these four numbers can be considered as coordinates of point on the plane and the domain occupied by all the probability vectors is called simplex. Let us consider linear maps of the probability vectors by means of the following two 4x4–matrices

\[
M^{(1)} = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}; \quad M^{(2)} = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \tag{3}
\]
We get new probability 4-vectors

\[ \tilde{\varphi}^{(1)} = M^{(1)} \tilde{p} = \begin{pmatrix} p_1 + p_2 \\ p_3 + p_4 \\ 0 \\ 0 \end{pmatrix}; \quad \tilde{\varphi}^{(2)} = M^{(2)} \tilde{p} = \begin{pmatrix} p_1 + p_3 \\ p_2 + p_4 \\ 0 \\ 0 \end{pmatrix}. \tag{4} \]

For these vectors the components \( \varphi_1^{(1)}, \varphi_2^{(1)} \) and \( \varphi_1^{(2)}, \varphi_2^{(2)} \) are the nonnegative numbers satisfying the condition \( \varphi_1^{(1)} + \varphi_2^{(1)} = \varphi_1^{(2)} + \varphi_2^{(2)} = 1 \). These pairs of numbers can be considered as probability outcomes in experiments either with two different classical coins or spin \(-1/2\) particles when one measures spin projections \( m = +1/2, -1/2 \) of two spins on two different directions \( \vec{n}_1 \) and \( \vec{n}_2 \). The analogous procedure we used to map the 4-vectors onto the 2-vectors and it was considered as the method of qubit portrait of qudit states to study entanglement phenomenon of qudit states in [12]. All the other matrices providing the result of the map on the probability 4-vector with two components equal to zero can be obtained from the matrix \( M^{(1)} \) by all the permutations of rows and columns.

It is clear that the map of vectors \( \tilde{p} \) realized by permutations of the vector components provides another 4-vector. The map is given by the set of bistochastic matrices \( \tilde{M}_s, s = 1, 2, ..., 24 \), where

\[ \tilde{M}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{5} \]

other 23 matrices can be obtained from \( \tilde{M}_1 \) by all the permutations of the matrix columns and rows. There exist two specific kinds of the linear maps of the probability vectors. One is realized by the bistochastic matrix

\[ \tilde{M}_c = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{6} \]

The matrix projects each probability vector \( \tilde{p} \) onto one vector with all components equal to \( 1/4 \), which are coordinates of the simplex center. Another map is determined by the stochastic matrix \( M_{1}^{\text{par}} \), with all rows excepting the first one containing only zero matrix elements and satisfying the equality

\[ (M_{1}^{\text{par}})^2 = M_{1}^{\text{par}}. \tag{7} \]

The matrix maps all the vectors \( \tilde{p} \) onto one vector which is analog of a ”pure state” of qudit

\[ M_{1}^{\text{par}} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{8} \]

There exist other three matrices which are obtained from the matrix \( M_{1}^{\text{par}} \) by means of the permutations of rows. Thus the maps \( M_{k}^{\text{par}}, k = (1, 2, 3, 4) \) project any vector \( \tilde{p} \) onto vertices of the simplex. There
are stochastic matrices $M^{(3)}$ which provide the maps of 4-vectors onto probability 3-vectors which can be called as qutrit portrait of the state. For example the matrix

$$M_3 = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$  \hspace{1cm} (9)$$

yields the map $(p_1, p_2, p_3, p_4) \rightarrow (p_1 + p_2, p_3, p_4, 0)$. All the other matrices which map one 4-vectors $\vec{p}$ onto probability vectors with one zero component are obtained from the matrix $M^{(3)}$ by all permutations of rows and columns. The qubit portrait map may be realized by other kind of stochastic matrix which has zero elements in two rows like the matrix

$$M_4 = \begin{pmatrix}
1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$  \hspace{1cm} (10)$$

which has the property $M_4 M_4 = M_4^{pur}$. One has the map $(p_1, p_2, p_3, p_4) \rightarrow (p_1 + p_2 + p_3, p_4, 0, 0)$.

Again other matrices of this kind providing qubit portrait of the qudit state are obtained from $M_4$ by all the permutations of rows and columns. All the stochastics matrices $M$ of linear maps of the probability vectors $\vec{p}$ form semigroup. The stochastic matrices providing different portraits of the qudit state form subsemigroup of the set of all the matrices $M$. Analogous construction of the maps of the probability vectors can be presented for any dimension of the linear space $N$.

4 Entropies and information

The probability vectors can be considered as arguments of some functions characterising the degree of randomness in the system. For example Shanon entropy [13] reads

$$H(\vec{p}) = - \sum_{k=1}^{4} p_k \ln p_k \equiv -\vec{p}\ln\vec{p}. \hspace{1cm} (11)$$

For the 2-vectors (qubits) or 3-vectors (qutrits) one has entropies given by (11) where instead of 4-vectors $\vec{p}$ one uses these 2-vectors or 3-vectors. One has the following entropic inequalities. Any map $M$ which acting on the probability vector with 4 nonzero components provides the new vectors with zero components can only decrease entropy, i.e.

$$- M\vec{p}\ln M\vec{p} \leq -\vec{p}\ln\vec{p}. \hspace{1cm} (12)$$

All the 24 permutation matrices like $\tilde{M}_c$ do not decrease the Shannon entropy. The map given by matrix $\tilde{M}_c$ increases the entropy up to maximal value $\ln 4$. One can consider analogous properties for qubits. For 2-vectors the discussed maps are given by four matrices

$$M^{(1)} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \hspace{0.5cm} M^{(2)} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \hspace{0.5cm} M^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \hspace{0.5cm} M^{(4)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
and the matrix $M^{(5)} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

First two matrices decrease the entropy up to zero. The two permutation matrices $M^{(3)}$ and $M^{(4)}$ keep the entropy of qubit unchanged and the bistochastic matrix $M^{(5)}$ creates maximal entropy $\ln 2$.

One can see that among the stochastic matrices $M$ with zeros and unit matrix elements there exists the following ordering. Let us denote in generic case of $N$-dimensional probability vectors such stochastic matrices with $k$ rows containing only zero matrix elements as $M_k^{(N)}$. Then it can be easily proved that the Shannon entropies obey inequalities

$$-\bar{p} \ln \bar{p} \geq -M_1^{(N)} \bar{p} \ln M_1^{(N)} \bar{p} \geq -M_2^{(N)} \bar{p} \ln M_2^{(N)} \bar{p} \geq ... \geq -M_k^{(N)} \bar{p} \ln M_k^{(N)} \bar{p} \geq ... \geq -M_{(N-1)}^{(N)} \bar{p} \ln M_{(N-1)}^{(N)} \bar{p}.$$  \hfill (13)

In particular for probability vector in qutrit case $\bar{p} = (p_1, p_2, p_3)$ one has the inequality for nonnegative numbers $p_k$

$$-p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3 \geq -(p_1 + p_2) \ln (p_1 + p_2) - p_3 \ln p_3. \hfill (14)$$

For any probability $N$-vector $\bar{p}$ one has

$$-\sum_{k=1}^{N} p_k \ln p_k \geq -\sum_{k=3}^{N} p_k \ln p_k - (p_1 + p_2) \ln (p_1 + p_2). \hfill (15)$$

In view of permutation symmetry of the Shannon entropy the decreasing of the entropy appears if one adds any two $\bar{p}$-vector components.

There exists subadditivity condition for a joint probability distribution of composite system with two subsystems.

We present the example of such distribution for two classical coins. The probability 4-vector $\bar{p}$ for such distribution has the following indices

$$p_1 = p_{++}, p_2 = p_{+-}, p_3 = p_{-+}, p_4 = p_{--}. \hfill (16)$$

These indices show that we have probability for two "spin projections" to have parallel or antiparallel directions along $z-$axes. Thus we have entropic inequality obtained by considering Shannon entropies of separate subsystems

$$[-(p_1 + p_2) \ln (p_1 + p_2) - (p_3 + p_4) \ln (p_3 + p_4)]$$

$$+[-(p_1 + p_3) \ln (p_1 + p_3) - (p_2 + p_4) \ln (p_2 + p_4)] \geq -p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3 - p_4 \ln p_4. \hfill (17)$$

It is clear that analogous inequalities can be obtained from this one by any permutation of four numbers 1, 2, 3, 4 though in this case the sense of inequalities for entropy of the subsystems is changed. Analogously one can get inequality

$$[-p_1 \ln p_1 - (p_2 + p_3 + p_4) \ln (p_2 + p_3 + p_4)] +$$

$$[p_2 \ln p_2 - p_3 \ln p_3 - (p_1 + p_4) \ln (p_1 + p_4)] \geq -p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3 - p_4 \ln p_4. \hfill (18)$$
This inequality is equivalent to subadditivity condition for 6–dimensional probability vector \( \vec{p} \) for which first two components are zero and we consider it as probability of qubit-qutrit system with notation for the vector \( \vec{q} \) like
\[
q_1 \equiv q_{+(1)} = 0, \quad q_2 \equiv q_{+(0)} = 0, \quad q_3 \equiv q_{+(-1)} = p_1, \quad q_4 \equiv q_{-(1)} = p_2, \quad q_5 \equiv q_{-(0)} = p_3, \quad q_6 \equiv q_{-(-1)} = p_4.
\]
Qubit has indices \( \pm \) and qutrit has indices \( +1, \ 0, \ -1 \). Thus, calculating Shannon entropies for this joint probability distribution we get inequality (18). It is obvious that this inequality creates other inequalities for all the permutations of numbers 1, 2, 3, 4.

For bipartite system one has the notion of mutual information which equals to difference of left and right sides on the inequalities (18)
\[
I = p_4 \ln p_4 - (p_2 + p_3 + p_4) \ln(p_2 + p_3 + p_4) - (p_1 + p_4) \ln(p_1 + p_4).
\] (19)

This information is known to be nonnegative, i.e. \( I \geq 0 \). One can obtain analogs of the information which is nonnegative applying in above equality all the permutations.

5 Qudit and qubit tomograms

For quantum spin states or for qudits the probability vectors appear being determined by the state density matrix \( \rho \). So one has for a qudit state the state unitary tomogram
\[
w(m,u) = \langle m|u\rho u^\dagger|m \rangle.
\] (20)

This tomogram was introduced in [14]. Here \( m = -j, -j + 1, ..., j - 1, j \), \( j = 0, 1/2, 1, 3/2, ... \) the pure state \( |m \rangle \) satisfies the eigenvalue condition
\[
\hat{J}_z|m \rangle = m|m \rangle
\] (21)
where \( \hat{J}_z \) is spin projection on \( z \)–axes. The matrix \( u \) is \( (2j + 1) \times (2j + 1) \) unitary matrix. If the matrix \( u \) is the matrix of irreducible representation of the group \( SU(2) \) the unitary tomogram \( w(m,u) \) becomes the function \( w(m,\vec{n}) \) where \( \vec{n} \) is unit 3–vector determining the point on Poincare sphere \( S^2 \). Then the tomogram is called spin-tomogram. The unitary and spin tomograms satisfy the nonnegativity condition, i.e. \( w(m,u) \geq 0 \) and normalisation condition
\[
\sum_{m=-j}^{j} w(m,u) = \sum_{m=-j}^{j} w(m,\vec{n}) = 1.
\] (22)

The density matrix can be reconstructed if one knows the tomogram \( w(m,\vec{n}) \) or \( w(m,u) \). In case of two qudits the unitary tomogram of the bipartite system state determined by density matrix \( \rho(1,2) \) is defined as
\[
w(m_1,m_2,u) = \langle m_1m_2|u\rho(1,2)u^\dagger|m_1m_2 \rangle.
\] (23)
Here we have the spin projection $m_k$

$$-j_k \leq m_k \leq j_k, \; k = 1, 2$$

(24)

and the matrix $u$ is $(2j_1 + 1)(2j_2 + 1)x(2j_1 + 1)(2j_2 + 1)$ unitary matrix. In case of $u = u_1 \otimes u_2$ where $u_k$ are $(2j_k + 1)$ unitary matrices which are the matrices of irreducible representations of the $SU(2)xSU(2)$ group the unitary tomogram $w(m_1, m_2, u)$ becomes the spin tomogram $w(m_1, m_2, \vec{n}_1, \vec{n}_2)$ where vectors $\vec{n}_k$ are unit vectors determining the points on two Poincare spheres. If one knows the tomograms $w(m_1, m_2, \vec{n}_1, \vec{n}_2)$ or $w(m_1, m_2, u)$ the density matrix $\rho(1, 2)$ can be reconstructed (see, e.g. [5]). The unitary and spin-tomograms can be represented as probability vectors. For example for qudit state with density matrix $\rho$ which has the nonnegative eigenvalues $\rho_1, \rho_2, ..., \rho_{2^j+1}$ and corresponding normalized eigenvectors $\vec{u}_{01}, \vec{u}_{02}, ..., \vec{u}_{02^j+1}$ the tomographic probability vector $\vec{w}(u)$ with components $w(m, u)$ reads

$$\vec{w}(u) = |uu_0|^2 \vec{\rho}.$$  

(25)

Here $\vec{\rho}$ is column vector with the nonnegative components $\rho_k, \; k = 1, 2, ..., 2j + 1$. The columns of the unitary matrix $u_0$ are the vectors $\vec{u}_{ok}$. The notation $|A|^2$ for any matrix $A$ means $|A|_{jk}^2 = |A_{jk}|^2$. Now we formulate the inequalities for the qudit state tomographic probability vector. Applying the inequalities (13) to the tomographic probability vector $\vec{w}(u)$ of qudit state we get

$$-\vec{w}(u) \ln \vec{w}(u) \geq -M_1^{(2j+1)} \vec{w}(u) \ln M_1^{(2j+1)} \vec{w}(u) \geq -M_2^{(2j+1)} \vec{w}(u) \ln M_2^{(2j+1)} \vec{w}(u)$$

$$\geq \ldots \geq -M_k^{(2j+1)} \vec{w}(u) \ln M_k^{(2j+1)} \vec{w}(u) \geq \ldots \geq -M_{(2j-1)}^{(2j+1)} \vec{w}(u) \ln M_{(2j-1)}^{(2j+1)} \vec{w}(u).$$

(26)

These inequalities take place for any unitary matrix $u$. Also for the matrix $u$ which is the matrix of irreducible representation of the group $SU(2)$ the corresponding inequalities take place for any unit vector $\vec{n}$ determining the point on Poincare sphere. Since the minimum of the Shannon entropy corresponding to the spin-tomographic probability vector $\vec{w}(u)$ for $u = u_0^{-1}$ is equal to von Neuman entropy we get inequality

$$S_{\text{VN}} \geq -M_1^{(2j+1)} \vec{w}(u_0^{-1}) \ln M_1^{(2j+1)} \vec{w}(u_0^{-1}) \geq -M_2^{(2j+1)} \vec{w}(u_0^{-1}) \ln M_2^{(2j+1)} \vec{w}(u_0^{-1})$$

$$\geq \ldots \geq -M_k^{(2j+1)} \vec{w}(u_0^{-1}) \ln M_k^{(2j+1)} \vec{w}(u_0^{-1}) \geq \ldots \geq -M_{(2j-1)}^{(2j+1)} \vec{w}(u_0^{-1}) \ln M_{(2j-1)}^{(2j+1)} \vec{w}(u_0^{-1}).$$

(27)

Thus the von Neuman entropy provides upper bound for all the entropies associated with the portrait tomographic probability vectors taken in the point $u = u_0^{-1}$. It is clear that for pure state the von Neuman entropy equals zero. Since the entropy in the inequalities (26) are nonnegative it means that all these entropies have minimal value equal to zero for $u = u_0^{-1}$. Let us consider expression for information $I$ (19) where we interpret probability vector as tomogram of the qudit state corresponding to $j = 3/2$. Then the nonnegativity of the information $I \geq 0$ gives inequality

$$w(-\frac{3}{2}, u) \ln w(-\frac{3}{2}, u) - [w(\frac{1}{2}, u) + w(-\frac{1}{2}, u) + w(-\frac{3}{2}, u)] \ln[w(\frac{1}{2}, u) + w(-\frac{1}{2}, u)w(-\frac{3}{2}, u)]$$

$$- [w(\frac{3}{2}, u) + w(-\frac{3}{2}, u)] \ln[w(\frac{3}{2}, u) + w(-\frac{3}{2}, u)] \geq 0.$$  

(28)
It means that for the point \( u_0^{-1} \) we have condition of positivity of the "tomographic information" provided the von Neuman entropy is given. The physical meaning of this inequality needs extra clarification. If the probability vector in (19) corresponds to two-qubit state the information nonnegativity gives

\[
\begin{align*}
&\quad w\left(-\frac{1}{2}, -\frac{1}{2}, u\right) \ln w\left(-\frac{1}{2}, -\frac{1}{2}, u\right) - [w\left(\frac{1}{2}, -\frac{1}{2}, u\right) + w\left(-\frac{1}{2}, \frac{1}{2}, u\right) + w\left(-\frac{1}{2}, -\frac{1}{2}, u\right)] \\
&\ln[w\left(\frac{1}{2}, -\frac{1}{2}, u\right) + w\left(-\frac{1}{2}, \frac{1}{2}, u\right) + w\left(-\frac{1}{2}, -\frac{1}{2}, u\right)] \\
&- [w\left(\frac{1}{2}, -\frac{1}{2}, u\right) + w\left(-\frac{1}{2}, \frac{1}{2}, u\right) + w\left(-\frac{1}{2}, -\frac{1}{2}, u\right)] \geq 0.
\end{align*}
\] (29)

For pure two qubit state which is entangled state violating Bell inequality [15] one can consider specific inequality for the "information" which provides some relation for the given probability \( w(m_1, m_2, n_1, n_2) \).

It means that there exists some correlation of the violation of Bell inequality for particular unitary matrix \( u = u_1 \otimes u_2 \) corresponding to directions \( n_1, n_2 \) and the information inequality.

6 Tomographic cumulants

Any probability distribution is characterized by specific numbers, like Shannon entropy, moments of random variables, etc. One of such characteristics is cumulant. For given probability distribution \( W(X) \) of continuous variable \( X \) the cumulants are defined as

\[
g(t) = \ln\langle \exp(tX) \rangle = \sum_{n=2}^{\infty} K_n \frac{t^n}{n!}.
\] (30)

Here

\[
\langle \exp(tX) \rangle = \int W(X) \exp(tX)dx
\] (31)
and cumulants \( K_n \) are coefficients in the series. In probability representation of quantum mechanics the states are described by symplectic tomogram \( M(X, \mu, \nu) \) which is probability distribution of homodyne quadrature \( X \) depending on two real parameters \( \mu \) and \( \nu \). Thus we introduce the tomographic cumulants \( K_n(\mu, \nu) \) which are given by the formula

\[
g(t, \mu, \nu) = \ln \int M(X, \mu, \nu) \exp(tX)dx = \sum_{n=1}^{\infty} \frac{t^n K_n(\mu, \nu)}{n!}.
\] (32)

For optical tomogram \( w(X, \Theta) = M(X, \cos \Theta, \sin \Theta) \) the cumulants are defined by generating function

\[
g(t, \Theta) = \ln \int w(X, \Theta) \exp(tX)dx = \sum_{n=1}^{\infty} \frac{t^n K_n(\Theta)}{n!}.
\] (33)

Let us introduce the function \( C(t, \Theta) \) which we will use as a characteristic of state gaussianity

\[
C(t, \Theta) = \ln \int w(X, \Theta) \exp(tX)dx - t \int X w(X, \Theta)dx - \frac{t^2}{2} \left[ \int X^2 w(X, \Theta)dx - \int X w(X, \Theta)dx \right]^2.
\] (34)

In experiments with homodyne photon detection the optical tomogram \( w(X, \Theta) \) is measured. For gaussian photon states the introduced function must be equal to zero. The deviation of this function of parameter
and local oscillator phase $\Theta$ from zero gives the information on nongaussianity degree of the quantum state. This characteristics can be easily extracted from the experimental homodyne detection data. One can introduce the parameter of nongaussianity

$$C_h = \int_0^\infty \int_0^{2\pi} C(t, \Theta)e^{-t} dt d\Theta.$$ 

For the Gaussian state, it is equal to zero.

7 Conclusion

To resume we point out the main results of our work. We expressed the state-extended uncertainty relations for two states of the photon in tomographic form providing some inequalities which can be checked experimentally. Some new entropic inequalities for spin-tomograms are obtained including inequality for bipartite system and inequality for particle with spin equal $3/2$.

We introduced tomographic cumulant as parameter which can be measured in experiments on homodyne photon detection [8] and it provides the characteristics of degree of nongaussianity of the photon state.

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