MAXIMAL OPERATORS AND FOURIER RESTRICTION ON THE
MOMENT CURVE

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Abstract. We bound certain $r$-maximal restriction operators on the moment curve.

1. Introduction

Let $\gamma(t) = (t, \frac{1}{2}t^2, \ldots, \frac{1}{d}t^d)$, and let $\Gamma$ be the image of this curve for $t \in \mathbb{R}$. Drury [2] proved the Fourier restriction estimate

$$||\hat{f}\||_{L^q(\Gamma)} \leq C_p ||f||_{L^p(\mathbb{R}^d)}$$

for $1 \leq p < \frac{d^2+d+2}{d+2} \quad \text{and} \quad p' = \frac{d(d+1)}{2} q$. In the spirit of [2], we study the $r$-maximal form of this restriction operator: $M_r\hat{f}|_{\Gamma}$, where

$$M_r h(x) = \left( \sup_{s > 0} \int_{B(x,s)} |h'|^{r} \right)^{1/r}$$

and $1 \leq r < \infty$. For $d \geq 3$, we have the following maximal restriction theorem:

Theorem 1.1. For $q = \frac{d+2}{d+1} p' > p$ and $r$ satisfying

$$\begin{cases} r \leq \frac{p'}{d}, & \text{if } 1 \leq p \leq \frac{d^2+2d}{d+2d-2}; \\ r < p' - \frac{d^2+d-2}{2}, & \text{if } \frac{d^2+2d}{d+2d-2} < p < \frac{d^2+d+2}{d+d}. \end{cases}$$

we have the following estimate for every $f \in L^p(\mathbb{R}^d)$:

$$||M_r\hat{f}\||_{L^q(\Gamma)} \leq C_{p,r} ||f||_{L^p(\mathbb{R}^d)}.$$ (1.1)

Müller, Ricci, and Wright [5] introduced maximal restriction theorems to obtain a pointwise interpretation of the restriction operator associated to $C^2$ curves in $\mathbb{R}^2$. After proving bounds for a two-parameter maximal restriction operator, they introduced the operator

$$M_2 h(x) = (M|h|^2)^{1/2}(x)$$

(1.2)

to aid with bounds for a strong maximal restriction operator. Following their logic, for the restriction operator $\mathcal{R}$ associated to the moment curve $\gamma$, the case $r = 2$ in Theorem 1.1 implies:

Corollary 1.2. Let $f \in L^p(\mathbb{R}^d)$ and $1 \leq p < \frac{d^2+d+2}{d+d}$. With respect to arclength measure, almost every $x \in \Gamma$ is a Lebesgue point for $\hat{f}$ and the regularized value of $\hat{f}$ at $x$ coincides with $\mathcal{R} f(x)$.
Later, Vitturi [9] proved similar maximal restriction estimates in the case of the unit sphere in any dimension $d \geq 3$. Ramos [8] improved the known results on spheres in all dimensions, and then in [7] focused on dimensions $d = 2$ and $d = 3$. In particular, he generalized the operator (1.2) to

$$M_r h = (M|h|^{r})^{1/r}(x)$$

for $1 \leq r < \infty$, and Theorem 2 in that paper was a maximal restriction result for this operator on the unit circle for $p < \frac{1}{2}$ and $r \leq 2$. Thus, in the case $d = 2$, Theorem 1.1 is due to Ramos [7], since the arguments that apply to the circle also apply to the parabola.

Kovač [4] took a more general approach, proving maximal and variational restriction estimates using restriction inequalities as a black box. Theorem 1, Remark 2, and Remark 3 in that paper combine with Drury’s [2] restriction estimate to show that

$$||M_2 \hat{f}||_{L^p(R^d)} \leq C_p ||f||_{L^p(R^d)}$$

for $1 \leq p < \frac{d^2 + d + 2}{d^2 + d + 1}$ and $p' = \frac{d(d+1)}{2}q$. Theorem 1.1 extends this range of $p$ to the full Drury range for $M_{2}$ and gives estimates of the form (1.1) for $r > 2$. See also [3] for more on variational restriction theorems.

In the case $d \geq 3$, the first two cases of Theorem 1.1 are distinct. When $r = 2$, we obtain the full range given by Drury: $p < \frac{d^2 + 2d}{d^2 + 2d - 2}$. For $r = \frac{d+2}{2}$, the range of $p$ corresponds to the Christ-Prestini $p < \frac{d^2 + 2d}{d^2 + 2d + 1}$ (see [1] and [d]). Figure 1 illustrates these ranges.

![Figure 1](image.png)

**Figure 1.** Range of $r$ and $p$ for which the $r$-maximal restriction operator is bounded from $L^p$ to $L^q$, where $q = \frac{2}{d+1}p'$.

**Outline of Proof.** The overall structure follows Drury’s induction scheme from [2]. To accommodate this, we prove the superficially stronger result:
Proposition 1.3. Let $2 \leq r < \frac{d+2}{2}$. Denote $M^k_r$ the $k$-fold composition of $M_r$ with itself. Then for each $k \in \mathbb{N}$, $1 \leq p < \frac{d+d^2(r-1)}{d+d^2(r-2)}$, and $p' = \frac{1}{d}(d+1)q$, we have

$$||M^k_r f||_{L^p(\mathbb{R}^d)} \leq C_{p,r} ||f||_{L^p(\mathbb{R}^d)}.$$

Indeed, for $r \geq \frac{d+2}{2}$, we can interpolate the above result with the bound

$$||M^k_r f||_{L^p(\mathbb{R}^d)} \leq ||f||_{L^p(\mathbb{R}^d)},$$

which follows from Hausdorff-Young. For $1 \leq r < 2$, we can apply Hölder’s inequality to see that

$$M^k_r f(x) \leq M_2 f(x).$$

Thus, Theorem 1.1 follows from Proposition 1.3. To prove Proposition 1.3, we first linearize the operator $f \mapsto M^k_r f_{|T}$ (Section 2). Then, in Section 3, we apply the induction hypothesis to prove a mixed-norm estimate for the $d$-fold power of the linear operator from Section 2. We interpolate this estimate with an $L^2$ bound for that same operator that comes from Plancherel. This interpolation allows us to increase the value of $p$, which completes the induction.

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2. Kolmogorov–Seliverstov–Plessner Linearization

We first fix $2 \leq r < \frac{d+2}{2}$. This value of $r$ will remain fixed throughout this section and the next. The first step is to linearize the maximal operator given in Proposition 1.3. The technique here is similar to [7], which built on the techniques of [3], [9], and [8].

Let $\chi_a(x)$ be the $L^1$-normalized characteristic function of the ball of radius $a$; that is,

$$\chi_a(x) = \frac{1}{|B_a|} \chi \left( \frac{x}{a} \right),$$

where $B_a$ is the ball centered at $0$ with radius $a$. Let $\rho_1, \ldots, \rho_k : \mathbb{R}^d \to \mathbb{R}_{>0}$ be measurable and $\eta : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a measurable function such that

$$\int_{B_{\rho_1(x)}(x)} \cdots \int_{B_{\rho_k(y_{k-1})}(y_{k-1})} |\eta(x,y_k)|^r \, dy_k \cdots dy_1 \leq 1$$

for every $x \in \mathbb{R}^d$. Set

$$\Delta^k_x(z_1,\ldots,z_k) = \eta(x,x-z_1-\cdots-z_k)\chi_{\rho_1(x)}(z_1)\chi_{\rho_2(x-z_1)}(z_2)\cdots \chi_{\rho_k(x-z_1-\cdots-z_{k-1})}(z_k).$$

and

$$M_{r,\eta,\rho,k} f(x) = \int_{\mathbb{R}^d} \hat{f}(x-z_1-\cdots-z_k) \Delta^k_x(z_1,\ldots,z_k) \, dz_1 \cdots dz_k,$$

or, equivalently,

$$M_{r,\eta,\rho,k} f(x) = \int_{B_{\rho_1(x)}(x)} \cdots \int_{B_{\rho_k(z_{k-1})}(z_{k-1})} \hat{f}(z_k) \eta(x,z_k) \, dz_k \cdots dz_1.$$
Lemma 2.1. Suppose there is $C > 0$ such that for every $\eta$, $\rho_1, \ldots, \rho_k$ as above, and for all $f$ Schwartz,
\[ ||M_{r, \eta, \rho, k} f||_{L^q(\Gamma)} \leq C ||f||_{L^p(\mathbb{R}^d)}. \]
Then
\[ ||M^k f||_{L^q(\Gamma)} \leq C ||f||_{L^p(\mathbb{R}^d)}. \]
for all Schwartz functions $f$.

Proof. Let $f$ be a Schwartz function, and let
\[ \eta(x, y) = \frac{\bar{\hat{f}}(y) |\hat{f}(y)|^{-2}}{\left( \int_{B_{\rho_1}(x)} \cdots \int_{B_{\rho_k}(y_k)} |\eta(x, y_k)|^{-r} \right)^{\frac{1}{r}}}. \]
Then for any $x \in \mathbb{R}^d$,
\[ \int_{B_{\rho_1}(x)} \cdots \int_{B_{\rho_k}(y_k)} |\eta(x, y_k)|^{-r} dy_k \cdots dy_1 \]
\[ = \frac{\int_{B_{\rho_1}(x)} \cdots \int_{B_{\rho_k}(y_k)} |f(y_k)|^{r(r-1)} dy_k \cdots dy_1}{\int_{B_{\rho_1}(x)} \cdots \int_{B_{\rho_k}(y_k)} |f(z_k)|^{r} dz_k \cdots dz_1}. \]
Since $r(r-1) = r$, the numerator and denominator are equal and hence $\eta$ satisfies (2.1). Moreover, using (2.4), we have
\[ M_{r, \eta, \rho, k} f(x) = \frac{\int_{B_{\rho_1}(x)} \cdots \int_{B_{\rho_k}(y_k)} |\hat{f}(z_k)|^{r} dz_k \cdots dz_1}{\left( \int_{B_{\rho_1}(x)} \cdots \int_{B_{\rho_k}(y_k)} |\hat{f}(z_k)|^{r} dz_k \cdots dz_1 \right)^{\frac{1}{r}}}. \]
Thus, we obtain
\[ M_{r, \eta, \rho, k} f(x) = \left( \int_{B_{\rho_1}(x)} \cdots \int_{B_{\rho_k}(y_k)} |\hat{f}(z_k)|^{r} dz_k \cdots dz_1 \right)^{\frac{1}{r}}. \]
For well-chosen $\rho_1, \ldots, \rho_k$, this can be made arbitrarily close to $M^k f(x)$, so the claim holds. \(\square\)

Hereafter, we will use the form of $M_{r, \eta, \rho, k}$ given in (2.3). As is often the case, it will be more convenient to work with an extension operator rather than the restriction. Given $g: \mathbb{R}^d \to \mathbb{C}$ and $f: \mathbb{R} \to \mathbb{C}$,
\[ \langle M_{r, \eta, \rho, k} g, f \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{g}(\gamma(t) - z_1 - \ldots - z_k) \hat{f}(\gamma(t)) dz_1 \ldots dz_k dt \]
\[ = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-2\pi i \xi(t) \cdot z_1 - \ldots - z_k} \hat{g}(\xi) \hat{f}(\gamma(t)) dz_1 \ldots dz_k d\xi dt. \]
Hence the adjoint is given by
\[ M_{r, \eta, \rho, k}^* f(\xi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-2\pi i \xi(t) \cdot z_1 - \ldots - z_k} \hat{f}(\gamma(t)) dz_1 \ldots dz_k dt \]
\[ = \int_{\mathbb{R}} e^{2\pi i \xi(t) \cdot \overline{\gamma(t)}} \hat{f}(t) dt. \]
Setting $\tilde{\xi}^k = (\xi_1, \ldots, \xi_k)$, we have

$$M^*_r,\eta,\rho,k f(\tilde{\xi}) = \int_{\mathbb{R}} e^{2\pi i \xi(t)} \overline{\mathcal{A}^k}(\tilde{\xi}) f(t) dt.$$ 

Proposition 1.3 now follows from the following lemma:

**Lemma 2.2.** Let $1 \leq p < \frac{d^2 + d + 2(r - 1)}{2(r - 1)}$, $q = \frac{d(d + 1)}{2} p'$, and $2 \leq r < \frac{d + 2}{2}$. There is $C > 0$ such that for $\rho_1, \ldots, \rho_k$ and $\eta$ measurable satisfying \[ (2.5) \] and all Schwartz functions $f$, \[ ||M^*_r,\eta,\rho,k f||_{L^q(\mathbb{R})} \leq C ||f||_{L^p(\Gamma)}. \]

### 3. The Induction Argument

The proof of Lemma 2.2 proceeds by induction. The base case is $p = 1$ and $q = \infty$. Here,

$$|M^*_r,\eta,\rho,k f(\tilde{\xi})| = \left| \int_{\mathbb{R}} e^{2\pi i \xi(t)} \overline{\mathcal{A}^k}(\tilde{\xi}) f(t) dt \right| \leq \int_{\mathbb{R}} \sup_{x \in \mathbb{R}} ||\mathcal{A}^k_x||_{L^1(\mathbb{R})} ||f(t)|| dt.$$

By \[ (2.1) \], $||\mathcal{A}^k_x||_{L^1(\mathbb{R})} \leq 1$ for all $x$. Thus,

$$|M^*_r,\eta,\rho,k f(\tilde{\xi})| \leq \int_{\mathbb{R}} ||f(t)|| dt = ||f||_{L^1(\Gamma)}.$$

This completes the base case. The following lemma, along with a little arithmetic, establishes the claimed range of $p$ and $q$:

**Lemma 3.1.** Assume for every $1 \leq p < p_0 < \frac{d^2 + d + 2(r - 1)}{2(r - 1)}$, $q = \frac{d(d + 1)}{2} p'$, there is $C > 0$ such that \[ (2.5) \] holds for all $k$, all measurable $\rho_1, \ldots, \rho_k: \mathbb{R}^d \to \mathbb{R}_{>0}$, all measurable $\eta: \mathbb{R}^d \times \mathbb{R}^d \to C$ satisfying \[ (2.1) \] for every $x \in \mathbb{R}^d$, and all $f$. Then \[ (2.5) \] holds for all such $\eta, \rho, k,$ and $f$, and for all $p$ satisfying \[ \frac{d}{p} > \frac{2}{(d + 2) p'_0 (r' - 1)} + \frac{d}{(d + 2) p_0}, \]

and \[ q = \frac{d(d + 1)}{2} p'. \]

To prove Lemma 3.1, we adapt Drury’s argument in [2]. Thus, we rewrite the left-hand side of \[ (2.5) \] as

$$||M^*_r,\eta,\rho,k f||_{L^q(\mathbb{R})} = ||(M^*_r,\eta,\rho,k f)^d||_{L^q(\mathbb{R}^d)}^{1/d}.$$

Expanding gives

$$(M^*_r,\eta,\rho,k f)^d(\xi) = \int_{\mathbb{R}^d} e^{2\pi i \xi(x)} \Sigma_{j=1}^d y(x_j) \prod_{j=1}^d \overline{\mathcal{A}^k}(\xi_j) f(x_j) dx.$$

Define

$$TG(\xi) = \int_{\Gamma_1 \cdots \Gamma_d} e^{2\pi i \xi y} \prod_{j=1}^d \overline{\mathcal{A}^k}(\xi_j) G(y) dy,$$

where $x = x(y)$ is uniquely determined by $x_1 < x_2 < \cdots < x_d$ and $y = \Sigma_{j=1}^d y(x_j)$. Let $\nu$ be the Vandermonde determinant,

$$|\nu(x)| = \prod_{1 \leq i < j \leq d} |x_i - x_j|.$$
Then with the choice

\[ G(y) = \prod_{j=1}^{d} f(x_j)|\nu(x)|^{-1}, \]

we have

\[ (3.2) \quad TG(\xi) = \frac{1}{d!}(M_{r,q,p,k}^*)^d(\xi). \]

To apply the induction hypothesis, we will need to work with another change of variables. For \( h = (h_1, h') \) with \( h_1 = 0 < h_2 < \cdots < h_d \), set \( x_j = t + h_j \) and \( y_h(t) = \sum_{j=1}^{d} y(x_j) \). Define the auxiliary function

\[ (3.3) \quad \tilde{G}(t, h) = G(y_h(t)) = \prod_{j=1}^{d} f(t + h_j)|\nu(h)|^{-1}. \]

Finally, fix \( 0 < \epsilon < \frac{d+2}{2} \) - \( r \).

**Lemma 3.2.** For \( T \) defined as in (3.1) and \( r + \epsilon < \frac{d+2}{2} \), we have

\[ ||TG||_{L^{r+\epsilon}} \leq C||\tilde{G}||_{L^{r+\epsilon}(\mathbb{R}^d)} \leq C||\tilde{G}||_{L^{r+\epsilon}(|\nu(h)|)}. \]

**Proof.** For any test function \( H \), by (3.1) we have

\[ |\langle TG, H \rangle| = \left| \int_{\mathbb{R}^d} TG(\xi)\overline{H}(\xi)d\xi \right| = \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2\pi iy} \prod_{j=1}^{d} \hat{A}_{y(x_j)}^{k}(\xi) G(y)\overline{H}(\xi)dyd\xi \right|. \]

Changing the order of integration and applying Plancherel in \( \xi \),

\[ |\langle TG, H \rangle| = \left| \int_{\mathbb{R}^d} G(y) \prod_{j=1}^{d} \hat{A}_{y(x_j)}^{k}(w_1, \ldots, w_k)\overline{H}(y - w_1 - \cdots - w_k)dw_1 \cdots dw_k dy \right|. \]

We can now apply the following lemma, which we will prove shortly.

**Lemma 3.3.** Let \( \hat{A}_{s_j}^{k}(w_1, \ldots, w_k) \) be defined as in (2.2), and let \( \hat{H} \) be a test function. Then for each \( n \in \mathbb{N} \) and \( z_1, \ldots, z_n \),

\[ (3.5) \quad \int_{\mathbb{R}^d} |\hat{A}_{s_j}^{k}(w_1, \ldots, w_k)\overline{H}(y - w_1 - \cdots - w_k)|dw_1 \cdots dw_k \leq M_{r}^{nk}\hat{H}(y). \]

Using this lemma, we see that

\[ |\langle TG, H \rangle| \leq \int_{\mathbb{R}^d} |G(y)|M_{r}^{kd}\hat{H}(y)dy. \]

By Hölder’s inequality, we obtain

\[ |\langle TG, H \rangle| \leq ||G||_{(r+\epsilon)'}||M_{r}^{kd}\hat{H}||_{r+\epsilon}. \]

Since \( r < r + \epsilon \), we have

\[ |\langle TG, H \rangle| \leq ||G||_{(r+\epsilon)'}||\hat{H}||_{r+\epsilon}. \]

Finally, \( r + \epsilon > 2 \), so by Hausdorff-Young,

\[ |\langle TG, H \rangle| \leq ||G||_{(r+\epsilon)'}||H||_{(r+\epsilon)'} \leq 2. \]

Thus, \( TG \) is bounded from \( L^{(r+\epsilon)'} \) to \( L^{r+\epsilon} \), which, along with a change of variables, proves the lemma. \( \square \)
Now we prove Lemma 3.3.

Proof. Fix $k \geq 1$. We proceed by induction. The base case is $n = 1$. In this case, the left-hand side of (3.5) is

$$\int_{\mathbb{R}^d} |\hat{H}(y - w_1 - \ldots - w_k)\eta(z - w_1 - \ldots - w_k) \cdot \chi_{\rho_1}(w_1) \ldots \chi_{\rho_k}(z - w_1 - \ldots - w_{k-1})(w_k)|dw_1 \ldots dw_k.$$ 

Applying Hölder’s inequality, this is bounded by

$$\left(\int_{\mathbb{R}^d} |\hat{H}(y - w_1 - \ldots - w_k)|^{r'}|\chi_{\rho_1}(w_1)| \ldots |\chi_{\rho_k}(z - w_1 - \ldots - w_{k-1})(w_k)|dw_1 \ldots dw_k\right)^{\frac{1}{r'}}$$

$$\cdot \left(\int_{\mathbb{R}^d} |\eta(z, z - w_1 - \ldots - w_k)|^{s'}|\chi_{\rho_1}(w_1)| \ldots |\chi_{\rho_k}(z - w_1 - \ldots - w_{k-1})(w_k)|dw_1 \ldots dw_k\right)^{\frac{1}{s'}}.$$ 

Changing variables in each integral transforms the above into

$$\left(\int_{B_{\rho_1}(z)} \ldots \int_{B_{\rho_k}(z)} |\hat{H}(v_k)| \eta(z, v_k)dv_k \ldots dv_1\right)^{\frac{1}{r'}}$$

$$\cdot \left(\int_{B_{\rho_1}(z)} \ldots \int_{B_{\rho_k}(z)} |\eta(z, v_k)| \eta(z, v_k)dv_k \ldots dv_1\right)^{\frac{1}{s'}}.$$ 

By (2.1), the second term is bounded by 1. Moreover, the first term is bounded by $M^k_\rho \hat{H}(y)$, so the base case is done. Now, assume we have (3.5) for some $n$ and all functions $\hat{H}$. We want to bound

$$\int_{\mathbb{R}^d} |(\hat{A}_{z_1}^k * \ldots * \hat{A}_{z_{n+1}}^k)(w_1, \ldots, w_k)\hat{H}(y - w_1 - \ldots - w_k)|dw_1 \ldots dw_k,$$

with the convolution performed $n+1$ times. Split up the convolution as the convolution of an $n$-fold convolution with $A_{z_{n+1}}$ to rewrite (3.6) as

$$\int_{\mathbb{R}^d} |(\hat{A}_{z_1}^k * \ldots * \hat{A}_{z_{n+1}}^k)(w_1, \ldots, w_k)\hat{H}(y - w_1 - \ldots - w_k)|dw_1 \ldots dw_k.$$ 

Expanding this convolution, we can further rewrite (3.6) as

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(\hat{A}_{z_1}^k * \ldots * \hat{A}_{z_n}^k)(v_1, \ldots, v_k)\hat{A}_{z_{n+1}}^k(w_1 - v_1, \ldots, w_k - v_k)$$

$$\cdot \hat{H}(y - w_1 - \ldots - w_k)|dv_1 \ldots dv_k dw_1 \ldots dw_k.$$ 

With the change of variables $u_j = w_j - v_j$, (3.6) becomes

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(\hat{A}_{z_1}^k * \ldots * \hat{A}_{z_n}^k)(v_1, \ldots, v_k)\hat{A}_{z_{n+1}}^k(u_1, \ldots, u_k)$$

$$\cdot \hat{H}(y - v_1 - \ldots - v_k - u_1 - \ldots - u_k)|dv_1 \ldots dv_k du_1 \ldots du_k.$$ 

By the induction hypothesis, the above is bounded by

$$\int_{\mathbb{R}^d} |\hat{A}_{z_{n+1}}^k(u_1, \ldots, u_k)M_{p}^{nk}\hat{H}(y - v_1 - \ldots - v_k)|dv_1 \ldots dv_k.$$ 

Finally, another application of the induction hypothesis shows that (3.6) is bounded by $M_{p}^{(n+1)k}\hat{H}(y)$. \qed
Lemma 3.4. For $T$ defined as in (3.1), there is a constant $C_{p,r}$ such that

$$||TG||_{L^q} \leq C_{p,r}||\hat{G}||_{L^q_{h,t}([v(h)])}.$$  

Proof. By Minkowski’s inequality for integrals,

$$||TG||_{L^q} = \left\| \int_0^\infty \int_{h_2}^\infty \cdots \int_{h_{d-1}}^\infty \int_\mathbb{R} e^{2\pi i \xi y_n(t)} \prod_{j=1}^d \frac{d}{y(t+h_j)}(\xi^k) \hat{G}(t,h)v(h) \, dh' \right\|_{L^q_t}^q \leq \left\| \int_\mathbb{R} e^{2\pi i \xi y_n(t)} \prod_{j=1}^d \frac{d}{y(t+h_j)}(\xi^k) \hat{G}(t,h) \, v(h) \, dh' \right\|_{L^q_{h,t}([v(h)])}.$$

Define the operator

$$S_h F(\xi) = \int_\mathbb{R} e^{2\pi i \xi y_n(t)} \prod_{j=1}^d \frac{d}{y(t+h_j)}(\xi^k) F(t) \, dt,$$

whose adjoint is given by

$$S_h^* H(t) = \int_\mathbb{R} e^{-2\pi i \xi y_n(t)} \prod_{j=1}^d \frac{d}{y(t+h_j)}(\xi^k) H(\xi) \, d\xi$$

$$= \int_{\mathbb{R}^d} \prod_{j=1}^d \frac{d}{y(t+h_j)}(w_1, \ldots, w_k) \hat{H}(y_n(t) - w_1 - \cdots - w_k) \, dw_1 \cdots dw_k.$$

Using Lemma 3.3 we obtain the bound

$$|S_h^* H(t)| \leq M_{p,r} \hat{H}(y_n(t)).$$

Since each $y_n$ is an affine transformation of the original curve, the induction hypothesis yields

$$||S_h^* H||_{L^{q'}} \leq C_{p,r} ||H||_{L^{q'}}.$$

Hence, we have

$$||S_h F||_{L^q_t} \leq C_{p,r} ||F||_{L^r_t}.$$

Setting $F(t) = \hat{G}(t,h)v(h)$ for each $h$ and integrating in $h'$ finishes the proof. \hfill \Box

By interpolating (3.4) and (3.7), we obtain

$$||TG||_{L^c} \leq C_{a,b,r} ||\hat{G}||_{L^a_{h,t}([v(h)])}$$

for all $(a^{-1}, b^{-1})$ in the triangle with vertices $(1, 1), (1, p_0^{-1})$, and $((r+\varepsilon)^{-1}, ((r+\varepsilon)^{-1})$, with $c$ satisfying

$$\frac{(d+2)(d-1)}{2} a^{-1} + b^{-1} + \frac{d(d+1)}{2} c^{-1} = \frac{d(d+1)}{2}.$$

Expanding out $\hat{G}$ using (3.3), we see that

$$||\hat{G}||_{L^a_{h,t}([v(h)])} = \left( \int_{\mathbb{R}} |v(h)|^{-(a-1)} \left( \int_{\mathbb{R}^{d-1}} |f(t + h_1) \cdots f(t + h_d)|^{p} \, dt \right)^{\frac{d}{p}} \, dh' \right)^{\frac{1}{a}}.$$
As noted in [2], \(v(0, h')^{-1} \in \mathcal{L}_{\mathcal{H}}^{d, \infty}\), so we can apply Hölder’s inequality to obtain
\[
\|\mathcal{G}\|_{\mathcal{L}_{\mathcal{H}}^{p, q}(L_r(h'v))} \leq \|f\|_{\mathcal{L}_r^{d, p, q}}^{d}
\]
for
\[
\begin{align*}
1 &< a < \frac{d+2}{2}, \\
a &\leq b < \frac{2a}{d+2-2a}, \text{ and} \\
\frac{d}{p} &\leq \frac{(d+2)(d-1)}{2} a^{-1} + b^{-1} - \frac{d(d-1)}{2}.
\end{align*}
\]

Plugging (3.2) and (3.9) into (3.8),
\[
\|M_{r,p,k}\|_{L^q} \leq \|f\|_{L^p},
\]
for
\[
\frac{d}{p} = \frac{(d+2)(d-1)}{2} a^{-1} + b^{-1} - \frac{d(d-1)}{2},
\]
where \(q = \frac{d(d+1)}{2} p'\), and \(a\) and \(b\) satisfy (Figure 2):
\[
\begin{align*}
\frac{d}{d+2} < a^{-1} < 1, \\
b^{-1} &\leq a^{-1}, \\
(d+2)a^{-1} - 2b^{-1} &< d, \text{ and} \\
(p_0 - (r+\epsilon')a^{-1} + p_0((r+\epsilon)' - 1)b^{-1} &\geq p_0 - 1.
\end{align*}
\]

Figure 2. Range of \(a\) and \(b\) for which (3.11) holds with \(p\) satisfying (3.11) and \(q = \frac{d(d+1)}{2} p'\).

Since \(r + \epsilon \leq \frac{d+2}{2}\), the point \((a^{-1}, b^{-1}) = \left(\frac{d}{d+2}, \frac{2}{(d+2)p_0((r+\epsilon)' - 1)} + \frac{d}{(d+2)p_0}\right)\) lies on the boundary of this region and satisfies
\[
\frac{(d+2)(d-1)}{2} a^{-1} + b^{-1} - \frac{d(d-1)}{2} < \frac{d}{p_0}.
\]
Taking \((a^{-1}, b^{-1})\) slightly inside of the region and using real interpolation, we obtain
\[
||M_{r,\eta,\rho, k}f||_{L^q} \leq ||f||_{L^p}, \quad q = \frac{d(d + 1)}{2}p', \quad \frac{d}{p} > \frac{2}{(d + 2)p_0'((r + \varepsilon)' - 1)} + \frac{d}{(d + 2)p_0}.
\]
Since this is true for all \(0 < \varepsilon < \frac{d+2}{d} - r\), we have
\[
||M_{r,\eta,\rho, k}f||_{L^q} \leq ||f||_{L^p}, \quad q = \frac{d(d + 1)}{2}p', \quad \frac{d}{p} > \frac{2}{(d + 2)p_0'((r - 1)'^1)} + \frac{d}{(d + 2)p_0},
\]
which proves Lemma 2.2 and hence Theorem 1.1.

4. Bounds on \(r\)

We are not able to show, nor do we believe, that the range of \(r\) is sharp. The following proposition shows that \(r \leq p'\) is necessary in any bound of the form (1.1), which corresponds to \(r \leq \frac{d+2}{d} - \frac{1}{d}\) in the full Drury range. This counterexample in dimension \(d = 2\) is due to Ramos [7].

**Proposition 4.1.** Suppose that for some \(p, q, \) and \(r,\) and all \(f \in L^p(\mathbb{R}^d),\) we have the bound (4.1)
\[
||M_r \hat{f}||_{L^q(\Omega)} \leq C_{p,r} ||f||_{L^p(\mathbb{R}^d)}.
\]
Then \(r \leq p'.\)

*Proof.* For \(0 < t < 1,\) let \(\hat{f}_t = \chi_{[-t,t]} \hat{f},\) and let \(k = 1.\) We first compute \(f_t.\)
\[
f_t(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} \chi_{[-t,t]}(\xi) d\xi = \int_{-t}^t \int_{-t}^t \int_{-t}^t e^{2\pi i x \cdot \xi} d\xi
\]
\[
= \prod_{j=1}^d \frac{e^{2\pi i x_j t} - e^{-2\pi i x_j t}}{2\pi i x_j} = \prod_{j=1}^d \frac{\sin(2\pi x_j t)}{\pi x_j}.
\]
Thus, we have
\[
||f_t||_{L^p(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} \left| \prod_{j=1}^d \frac{\sin(2\pi x_j t)}{\pi x_j} \right|^p dx \right)^\frac{1}{p} = \left( \int_{\mathbb{R}^d} \prod_{j=1}^d \left| \frac{2t \sin(y_j)}{y_j} - \frac{1}{2\pi t} \right|^p dx \right)^\frac{1}{p} = Ct^d.p.
\]
For \(x \in [-1, 1]^d,\) by taking the ball centered at \(x\) with radius 10, we see that
\[
M_r \hat{f}_t(x) \geq t^d.
\]
Hence, we have
\[
||M_r \hat{f}_t||_{L^q(\Omega)} \geq t^d.
\]
Combining these estimates and (4.1) gives
\[
t^\frac{d}{r} \leq t^\frac{d}{p'}.\]
Sending \(t \to 0\) shows that
\[
\frac{d}{r} \geq \frac{d}{p'},
\]
which means that \(r \leq p'.\) \hfill \Box
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