Heterotic strings on homogeneous spaces

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Abstract: We construct heterotic string backgrounds corresponding to families of homogeneous spaces as exact conformal field theories. They contain left cosets of compact groups by their maximal tori supported by NS-NS 2-forms and gauge field fluxes. We give the general formalism and modular-invariant partition functions, then we consider some examples such as $SU(2)/U(1) \sim S^2$ (already described in a previous paper) and the $SU(3)/U(1)^2$ flag space. As an application we construct new supersymmetric string vacua with magnetic fluxes and a linear dilaton.

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1. Motivations and summary

The search for exact string backgrounds has been a major motivation in the field for many years. Gravitational backgrounds with a clear geometric interpretation are even more important since they may provide a handle on quantum gravitational phenomena, black holes and ultimately cosmology – for those which are time-dependent. Wess–Zumino–Witten models provide such a class of solutions, with remarkable properties. The target space is in that case a group manifold and, together with the metric, the Neveu–Schwarz antisymmetric tensor is the only background field. Both of these fields are exactly known to all orders in $\alpha'$. So are the spectrum, partition function, two- and three-point functions, …

Wess–Zumino–Witten models appear in many physical set-ups, as near-horizon geometries of specific brane configurations. The three-sphere is part of the near-horizon geometry of $N_5$ NS5-branes. This is the target space of an $SU(2)_k$ super-wzw model at bosonic level $k = N_5 - 2$. Another celebrated example is that of $\text{AdS}_3$. The latter appears in the NS5-brane/fundamental-string background, together with $S^3$, at equal radius $L = \sqrt{\alpha'N_5}$; it is realized in terms of the $SL(2,\mathbb{R})_{\tilde{k}}$ wzw at level $\tilde{k} = k + 4$. These are important examples because of their role in the study of decoupling limits, little-string theory, holographic dualities etc. The knowledge of exact spectra, amplitudes, … is crucial for better understanding of these issues.

Despite the many assets of wzw models, the major limitation comes from the dimension and signature of their target spaces. When dealing with compact groups, the dimension often exceeds six (e.g. $SU(3)$ is eight-dimensional), while for non-compact groups, $SL(2,\mathbb{R})_{\tilde{k}}$ is the only example with a single time direction.

In order to reduce the dimension of the target space, while keeping two-dimensional conformal invariance and tractability, the usual procedure is the gauging. Gauged wzw models are realized algebraically, at the level of the chiral currents and energy–momentum tensor, by following the gko construction [1]. Alternatively, one can work directly on the action and gauge symmetrically a subgroup $H \subset G$. For $H = U(1)$, the gauged model can even be obtained as an extreme marginal deformation of the original model, driven by a $\int d^2z J\bar{J}$ perturbation, where $J$ and $\bar{J}$ are the currents associated with the $U(1) \subset G$.

Target spaces of gauged wzw models are not usual geometric cosets $G/H$. Firstly, the background fields of gauged wzw receive non-trivial $\alpha'$ corrections*, while geometric cosets can be assigned a well-defined metric. Secondly, the isometry groups are different. For geometric cosets, the isometry group is $G$, while it is $H$ for the target space of the gauged wzw.

Geometric cosets could provide alternative backgrounds, with different properties and new possibilities for accommodating six or less space dimensions, or a single time direction (in the non-compact case). Unfortunately, they have not been systematically analyzed, and were even thought to be, at most, leading-order solutions to the string equations. Although some exact solutions were identified in the past [2, 3, 4], no generic pattern for generalization was known.

*The higher-order $\alpha'$ corrections are trivial for wzw models: they boil down to shifting $k \to k + g^*$ in the classical backgrounds ($g^*$ is the dual Coxeter number of the group $G$).
The issue of geometric cosets as exact backgrounds has been recently revisited in [5]. There, it was shown that $S^2 \equiv SU(2)/U(1)$ and $\text{AdS}_2 \equiv SL(2, \mathbb{R})/U(1)_{\text{space}}$, with magnetic and electric fluxes and no dilaton, can be obtained as extreme marginal deformations of the $SU(2)_k$ and $SL(2, \mathbb{R})_{\tilde{k}}$ wzw models. In this case the background fields are exact up to the usual finite renormalization of the radius ($k \rightarrow k + 2$ and $\tilde{k} \rightarrow \tilde{k} - 2$) and spectra, partition functions, ... are within reach. The marginal deformations are asymmetric because the right current that appears in the bilinear does not belong to the right-moving affine algebra of the group at hand.

Asymmetric marginal deformations apply to any group. The aim of the present paper is to investigate on several interesting generalizations of this method, in the case of compact groups, and make contact with asymmetrically gauged wzw models. We will focus in particular on the SU (3) group. In this case the asymmetric marginal deformation leads to the SU(3)/U(1) geometric coset, with magnetic fluxes and no dilaton. In turn, this coset is identified with the asymmetric gauging of a $U(1)^2$ in the original wzw models.

In the cases under consideration, however, more possibilities exist, which we further exploit. We examine the asymmetric gauging of the full Cartan torus $U(1)^2$. The geometric cosets obtained in this way, can be assigned two different metrics depending on the precise manner the gauging is performed, in combination with the extreme asymmetric marginal deformation. One is Kählerian and consequently no NS form survives: we obtain the flag space $F_3 = SU(3)/U(1)^2$, recognized many years ago [6] to be a leading-order solution, thanks to its Kählerian structure. The other metric is not Kählerian, and the background has both magnetic and NS fluxes. It enters into the construction of non-compact manifolds of $G_2$ holonomy [7].

All our solutions are exact string backgrounds with no dilaton – contrary to the usual symmetrically gauged wzw models. We can determine their spectra as well as their full partition functions.

The paper is organized as follows: first we fix the notation by reviewing some known facts about wzw models and then show how to read the background fields corresponding to an asymmetric marginal deformation of such models. We emphasize in particular the decompactification of the Cartan torus that takes place at the extremal points in moduli space (Sec. 2). This formalism is then used to study the deformation of the $SU (2)$ and $SU (3)$ models (Sec. 3). In the following we introduce a different construction in which the limit deformations are identified to asymmetrically gauged wzw models [8] and the deformation is generalized so to reach the different constant-curvature structures admitted by an asymmetric $G/T$ coset, with particular emphasis on the $SU (3)/U (1)^2$ case (Sec. 4). The next section (Sec. 5) deals with the computation of the one-loop partition functions for the asymmetric deformations leading to geometric cosets. Two different methods are proposed, one using the Kazama–Suzuki decomposition in terms of Hermitian symmetric spaces, the other via the direct deformation of the Cartan lattice of the Lie algebra corresponding to the group. In the final section (Sec. 6) we give an example of application by using these scft’s to construct other supersymmetric exact string backgrounds such as the left-coset analogues of the NS5-branes solutions [9, 10]. They provide new holographic backgrounds of the Little String Theory type [11, 12, 13], and may be dual to non-trivial supersymmet-
ric compactifications on manifolds with singularities. The concluding appendices contain some facts about the geometry of coset spaces, partition functions and characters of affine Lie algebras.

2. Compact coset spaces: general formalism

In this section we will fix the notation by reviewing some well known facts about conformal field theories on group manifolds (wzw models) and give the general formalism for the truly marginal deformations leading to exact cft’s on left coset spaces.

2.1 String theory on group manifolds: a reminder

Let $g$ be the (semi-simple) Lie algebra of the (compact) group $G$ and $\{ T_M \}$ a set of generators that satisfy the usual commutation relations $[T_M, T_N] = \sum \delta^M_N \delta^m_n T_p$ and are normalized with respect to the Killing product $\kappa (T_M, T_N) = - \text{tr} (T_M T_N) = \delta_{MN}$. We can always write $g$ as the direct sum $g = j \oplus k$ where $k$ is the Cartan subalgebra and correspondingly distinguish between the Cartan generators $\{ T_a \}$ and the generators of $j, \{ T_\mu \}$.

The generators are in one-to-one correspondence with the Maurer–Cartan left-invariant one-forms defined by:

$$ J_M = \kappa (T_M, g^{-1} dg) = - \text{tr} (T_M g^{-1} dg) \quad (2.1) $$

where $g$ is the general element of the group $G$. It is a well known fact that the scalar product on $g$ naturally induces a scalar product $\langle \cdot, \cdot \rangle$ on the tangent space $T_g$ to $G$ that can be written by decomposing the induced metric (the so-called Cartan–Killing metric) in terms of the currents as follows:

$$ \langle dg, dg \rangle = \kappa (g^{-1} dg, g^{-1} dg) = \sum \delta_{MN} \kappa (T_M, g^{-1} dg) \kappa (T_N, g^{-1} dg) = \sum \delta_{mn} J^m \otimes J^m \quad (2.2) $$

Now let us consider the affine extension of the Lie algebra $\hat{g}_k$, at level $k$. We have two sets of holomorphic and anti-holomorphic currents of dimension one, naturally related to the Maurer–Cartan right- and left-invariant one-forms:

$$ J_M (z) = - k \kappa (T_M, \partial g g^{-1}) , \quad \bar{J}_M (\bar{z}) = k \kappa (T_M, g^{-1} \bar{\partial} g). \quad (2.3) $$

Each set satisfies the following operator product expansion:

$$ J_M (z) J_N (w) = \frac{k \delta_{MN}}{2 (z - w)^2} + \frac{f_{MN}^P J_P (w)}{z - w} + O ((z - w)^0) \quad (2.4) $$

This chiral algebra contains the Virasoro operator, given by the Sugawara construction:

$$ T(z) = \sum_M \frac{J_M J_M}{k + g^*} \quad (2.5) $$

where $g^*$ is the dual Coxeter number and the corresponding central charge is given by:

$$ c = \frac{k \dim (g)}{k + g^*}. \quad (2.6) $$
An $N = 1$ superconformal extension is obtained by adding $(\dim g)$ free fermions transforming in the adjoint representation:

$$ T(z) = \sum_M : J_M J_M : \frac{k}{k + g^*} + : \psi_M \partial \psi_M : $$  \hspace{1cm} (2.7)

$$ G(z) = \frac{2}{k} \left( \sum_M J_M \psi_M - \frac{i}{3k} \sum_{MNP} f_{MNP} : \psi_M \psi_N \psi_P : \right) $$ \hspace{1cm} (2.8)

An heterotic model is provided by considering a left-moving $N = 1$ current algebra and a right-moving $N = 0$ one. The Lagrangian ($\sigma$-model) description of this model is given by the linear combination of the following WZW-model and the action for free fermions transforming in the adjoint representation:

$$ S = \frac{k}{4\pi} \int_{\partial B} \text{Tr} \left( g^{-1} \, dg \wedge * g^{-1} \, dg \right) + \frac{k}{12\pi} \int_B \text{Tr} \left( g^{-1} \, dg \right)^3 + \frac{1}{2\pi} \int d^2 z \: \psi_M \partial \psi_M $$ \hspace{1cm} (2.9)

(the exterior derivative is here understood as acting on the worldsheet coordinates). The background fields corresponding to this action are the Cartan-Killing metric Eq. (2.2) and the NS-NS two-form field, coming from the WZW term:

$$ H = dB = \text{Tr} (g^{-1} \, dg)^3 = \frac{1}{2} f_{MNP} \mathcal{J}^M \wedge \mathcal{J}^N \wedge \mathcal{J}^P $$ \hspace{1cm} (2.10)

### 2.2 Asymmetric deformations

Truly marginal deformations of WZW models were already studied in [14, 15]. In particular in heterotic strings we can consider a deformation obtained with the following exactly marginal operator $V$ built from the total Cartan currents of $g$ (so that it preserves the local $N = (1,0)$ superconformal symmetry of the theory):

$$ V = \frac{\sqrt{k k_g}}{2\pi} \int d^2 z \: \sum_a H_a \left( J^a(z) - \frac{i}{k} f^a_{MN} : \psi^M \psi^N : \right) \tilde{J}(\bar{z}) $$ \hspace{1cm} (2.11)

(where the set $\{H_a\}$ are the parameters of the deformation and $\tilde{J}(\bar{z})$ is a right moving current of the Cartan subalgebra of the heterotic gauge group at level $k_g$). Such a deformation is always truly marginal since the $J_a$ currents commute.

It is not completely trivial to read off the deformed background fields that correspond to the $S+V$ deformed action. A possible way is a method involving a Kaluza-Klein reduction as in [16]. For simplicity we will consider the bosonic string with vanishing dilaton and just one operator in the Cartan subalgebra $\mathfrak{k}$. The right-moving gauge current $\tilde{J}$ used for the deformation has now a left-moving partner and can hence be bosonized as $\tilde{J} = i \partial \varphi$, $\varphi(z,\bar{z})$ being interpreted as an internal degree of freedom. The sigma-model action is recast as

$$ S = \frac{1}{2\pi} \int d^2 z \: \left( G_{MN} + B_{MN} \right) \partial x^M \partial x^N, $$ \hspace{1cm} (2.12)

where the $x^M, M = 1, \ldots, 4$ embrace the group coordinates $x^\mu, \mu = 1,2,3$ and the internal $x^4 \equiv \varphi$:

$$ x^M = \left( \frac{x^\mu}{x^4} \right). $$ \hspace{1cm} (2.13)
If we split accordingly the background fields, we obtain the following decomposition:

\[ G_{MN} = \begin{pmatrix} G_{\mu\nu} & G_{\phi\phi} A_\mu \\ G_{\phi\phi} A_\mu & G_{\phi\phi} \end{pmatrix}, \quad B_{MN} = \begin{pmatrix} B_{\mu\nu} & B_{\mu4} \\ -B_{\mu4} & 0 \end{pmatrix}, \quad (2.14) \]

and the action becomes:

\[ S = \frac{1}{2\pi} \int d^2 z \left\{ (G_{\mu\nu} + B_{\mu\nu}) \partial x^\mu \tilde{\partial} x^\nu + (G_{\phi\phi} A_\mu + B_{\mu4}) \partial x^\mu \tilde{\partial}\phi \\
+ (G_{\phi\phi} A_\mu - B_{\mu4}) \partial\phi \tilde{\partial} x^\mu + G_{\phi\phi} \partial\phi \tilde{\partial}\phi \right\}. \quad (2.15) \]

We would like to put the previous expression in such a form that space–time gauge invariance,

\[ A_\mu \to A_\mu + \partial_\mu \lambda, \quad (2.16) \]

\[ B_{\mu4} \to B_{\mu4} + \partial_\mu \eta, \quad (2.17) \]

is manifest. This is achieved as follows:

\[ S = \frac{1}{2\pi} \int d^2 z \left\{ (\hat{G}_{\mu\nu} + B_{\mu\nu}) \partial x^\mu \tilde{\partial} x^\nu + B_{\mu4} \left( \partial x^\mu \tilde{\partial}\phi - \partial\phi \tilde{\partial} x^\mu \right) + \\
+ G_{\phi\phi} (\partial\phi + A_\mu \partial x^\mu) \left( \tilde{\partial}\phi + A_\mu \tilde{\partial} x^\mu \right) \right\}, \quad (2.18) \]

where \( \hat{G}_{\mu\nu} \) is the Kaluza–Klein metric

\[ \hat{G}_{\mu\nu} = G_{\mu\nu} - G_{\phi\phi} A_\mu A_\nu. \quad (2.19) \]

We can then make the following identifications:

\[ \hat{G}_{\mu\nu} = \frac{k}{2} \left( J_\mu J_\nu - 2k^2 \tilde{J}_\mu \tilde{J}_\nu \right) \quad (2.20a) \]

\[ B_{\mu\nu} = \frac{k}{2} J_\mu \wedge J_\nu, \quad (2.20b) \]

\[ B_{\mu4} = G_{\phi\phi} A_\mu = h \sqrt{\frac{k g^2}{2}} \tilde{J}_\mu, \quad (2.20c) \]

\[ A_\mu = h \sqrt{\frac{2k}{\epsilon_g}} \tilde{J}_\mu, \quad (2.20d) \]

\[ G_{\phi\phi} = \frac{k g}{2}. \quad (2.20e) \]

Let us now consider separately the background fields we obtained so to give a clear geometric interpretation of the deformation, in particular in correspondence of what we will find to be the maximal value for the deformation parameters \( h_a \).

**The metric.** According to Eq. (2.20d), in terms of the target space metric, the effect of this perturbation amounts to inducing a back-reaction that in the basis of Eq. (2.2) is written as:

\[ \langle dg, dg \rangle_\Pi = \sum_M J_M \otimes J_M - 2 \sum_a h_a J_a \otimes J_a = \sum_M J_M \otimes J_M + \sum_a (1 - 2h_a^2) J_a \otimes J_a \quad (2.21) \]
where we have explicitly separated the Cartan generators. From this form of the deformed metric we see that there is a “natural” maximal value $h_a = 1/\sqrt{2}$ where the contribution of the $J_a \otimes J_a$ term changes its sign and the signature of the metric is thus changed. One could naively think that the maximal value $h_a = 1/\sqrt{2}$ can’t be attained since the this would correspond to a degenerate manifold of lower dimension; what actually happens is that the deformation selects the the maximal torus that decouples in the $h_a = H \to 1/\sqrt{2}$ limit as it was shown in \cite{5} for the $SU(2)$ and $SL(2, \mathbb{R})$ algebras.

To begin, write the general element $g \in \mathcal{G}$ as $g = ht$ where $h \in \mathcal{G}/\mathcal{T}, t \in \mathcal{T}$. Substituting this decomposition in the expression above we find:

$$\langle d(h) , d(ht) \rangle_H = \text{tr} \left( (ht)^{-1} d(ht) (ht)^{-1} d(ht) \right) - \sum_a 2h_a^2 \text{tr} \left( T_a (ht)^{-1} d(ht)^{-1} \right)^2 =$$

$$= \text{tr} \left( h^{-1} dh h^{-1} dh \right) + 2\text{tr} \left( dt t^{-1} h^{-1} dt \right) + \text{tr} \left( t^{-1} dt t^{-1} dt \right) +$$

$$- \sum_a 2h_a^2 \left( \text{tr} \left( T_a t^{-1} h^{-1} dt \right) + \text{tr} \left( T_a t^{-1} dt \right) \right)^2 \quad (2.22)$$

let us introduce a coordinate system $(\gamma_\mu, \psi_a)$ such as the element in $\mathcal{G}/\mathcal{T}$ is parametrized as $h = h(\gamma_\mu)$ and $t$ is written explicitly as:

$$t = \exp \left\{ \sum_a \psi_a T_a \right\} = \prod_a e^{\psi_a T_a} \quad (2.23)$$

it is easy to see that since all the $T_a$ commute $t^{-1} dt = dt t^{-1} = \sum_a T_a d\psi_a$. This allows for more simplifications in the above expression that becomes:

$$\langle d(h) , d(ht) \rangle_H = \text{tr} \left( h^{-1} dh h^{-1} dh \right) + 2 \sum_a \text{tr} \left( T_a h^{-1} dh \right) d\psi_a + \sum_a d\psi_a d\psi_a +$$

$$- \sum_a 2h_a^2 \left( \text{tr} \left( T_a h^{-1} dh \right) + d\psi_a \right)^2 = \text{tr} \left( h^{-1} dh h^{-1} dh \right) - \sum_a 2h_a^2 \left( \text{tr} \left( T_a h^{-1} dh \right) \right)^2 +$$

$$+ 2 \sum_a \left( 1 - 2h_a^2 \right) \text{tr} \left( T_a h^{-1} dh \right) d\psi_a + \sum_a \left( 1 - 2h_a^2 \right) d\psi_a d\psi_a \quad (2.24)$$

if we reparametrise the $\psi_a$ variables as:

$$\hat{\psi}_a = \frac{\dot{\psi}_a}{\sqrt{1 - 2h_a^2}} \quad (2.25)$$

we get a new metric $\langle \cdot , \cdot \rangle_H'$ where we’re free to take the $h_a \to 1/\sqrt{2}$ limit:

$$\langle d(h) , d(ht) \rangle_H' = \text{tr} \left( h^{-1} dh h^{-1} dh \right) - \sum_a 2h_a^2 \left( \text{tr} \left( T_a h^{-1} dh \right) \right)^2 +$$

$$+ 2 \sum_a \sqrt{1 - 2h_a^2} \text{tr} \left( T_a h^{-1} dh \right) d\hat{\psi}_a + \sum_a d\hat{\psi}_a d\hat{\psi}_a \quad (2.26)$$

and get:

$$\langle d(h) , d(ht) \rangle_{1/\sqrt{2}}' = \left[ \text{tr} \left( h^{-1} dh h^{-1} dh \right) - \sum_a \left( \text{tr} \left( T_a h^{-1} dh \right) \right)^2 \right] + \sum_a d\psi_a d\psi_a \quad (2.27)$$
where we can see the sum of the restriction of the Cartan-Killing metric\(^*\) on \(T_hG/T\) and the metric on \(T_t\mathcal{T} = T_tU\) \((1)^\prime\). In other words the coupling terms between the elements \(h \in G/T\) and \(t \in \mathcal{T}\) vanished and the resulting metric \(\langle \cdot , \cdot \rangle'_{1/\sqrt{2}}\) describes the tangent space \(T_{ht}\) to the manifold \(G/T \times \mathcal{T}\).

These homogeneous manifolds enjoy many interesting properties. The best part of them can be interpreted as consequence of the presence of an underlying structure that allows to recast all the geometric problems in Lie algebraic terms (see App. [A] for some constructions). There's however at least one intrinsically geometric property that it is worth to emphasize since it will have many profound implications in the following. All these spaces can be naturally endowed with complex structures by using positive and negative roots as holomorphic and anti-holomorphic generators. Moreover for each space there is not in general only one of these structures (but for the lowest dimensional \(SU(2)\) case) and there always exists one of them which is Kähler \([17]\).

**Other Background fields.** The asymmetric deformation of Eq. (2.11) generates a non-trivial field strength for the gauge field, that from Eq. (2.20d) is found to be:

\[
F^a = \sum_a \sqrt{\frac{2}{k_g h^a}} \mathcal{J}^a = - \sum_a \sqrt{\frac{k}{2 k_g h^a}} f_{\mu \nu}^a J^a \wedge J^\nu
\]

(no summation implied over \(a\)).

On the other hand, the \(B\)-field (2.20b) is not changed, but the physical object is now the 3-form \(H\):

\[
H[3] = dB - \frac{1}{k_g} A^a \wedge dA^a = \frac{1}{2} f_{MNP} \mathcal{J}^M \wedge \mathcal{J}^N \wedge \mathcal{J}^P - 2 \sum_a h^a f_{aNP} \mathcal{J}^a \wedge \mathcal{J}^N \wedge \mathcal{J}^P,
\]

where we have used the Maurer-Cartan structure equations. At the point where the fibration trivializes, \(h^a = 1/\sqrt{2}\), we are left with:

\[
H[3] = \frac{1}{2} f_{\mu \nu \rho} \mathcal{J}^\mu \wedge \mathcal{J}^\nu \wedge \mathcal{J}^\rho.
\]

So only the components of \(H[3]\) “living” in the coset \(G/T\) survive the deformation. They are not affected of course by the rescaling of the coordinates on \(\mathcal{T}\).

**A trivial fibration.** The whole construction can be reinterpreted in terms of fibration as follows. The maximal torus \(\mathcal{T}\) is a closed Lie subgroup of the Lie group \(G\), hence we can see \(G\) as a principal bundle with fiber space \(\mathcal{T}\) and base space \(G/T\) \([18]\).

\[
G \xrightarrow{T} G/T
\]

The effect of the deformation consists then in changing the fiber and the limit value \(h^a = 1/\sqrt{2}\) marks the point where the fibration becomes trivial and it is interpreted in terms of a gauge field whose strength is given by the canonical connection on \(G/T\) \([19]\).

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\(^*\)This always is a left-invariant metric on \(G/H\). A symmetric coset doesn’t admit any other metric. For a more complete discussion see App. [A].
2.3 Equations of motion

In this section we want to explicitly show that the background fields we found on the left coset space are solution to the first order (in $\alpha'$) equations of motion \[20\].

For a vanishing dilaton they read:

\[
\delta c = - R + \frac{k_g}{16} F^a_{\mu\nu} F^{a\mu\nu} \tag{2.32a}
\]

\[
\beta^{(G)}_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\sigma} H^{\rho\sigma}_{\nu} - \frac{k_g}{4} F^a_{\mu\rho} F^a_{\nu\rho} = 0 \tag{2.32b}
\]

\[
\beta^{(B)}_{\nu\rho} = \nabla^{\mu} H_{\mu\nu\rho} = 0 \tag{2.32c}
\]

\[
\beta^{(A)}_\mu = \nabla^\nu F^a_{\mu\nu} - \frac{1}{2} F^{a\nu\rho} H_{\mu\nu\rho} = 0 \tag{2.32d}
\]

after applying the proper normalizations\(^\dagger\) our fields are given by:

\[
g_{\mu\nu} = \frac{k}{2} \delta_{\mu\nu} \tag{2.33a}
\]

\[
F^a_{\mu\nu} = - \sqrt{\frac{2k}{k_g}} f^a_{\mu\nu} \tag{2.33b}
\]

\[
H_{\mu\nu\rho} = - \frac{k}{2} f_{\mu\nu\rho} \tag{2.33c}
\]

• the $\beta^{(B)} = 0$ equation (2.32c) is just the restriction of the same equation for the initial wzw model

• the two terms in the $\beta^{(A)} = 0$ equation (2.32d) vanish separately: the first one because $F$ is closed (or, equivalently because $f^a_{\mu\nu}$ seen as a two form in $G/T$ satisfies the condition stated below Eq. (A.13)); the second because it is proportional to:

\[
\sum_{\nu,\rho \in g/h} f_{\nu\rho} f_{\mu\nu\rho} = \sum_{M,R \in 0} f_{aMR} f_{\mu MR} = 2g^* \delta_{a\mu} = 0 \tag{2.34}
\]

• to solve the $\beta^{(G)} = 0$ equation (2.32b) we need some more work. Using the results in App. [B.1] for a general algebra, we obtain

\[
R_{\mu\nu} = \frac{1}{4} \sum_{\rho,\sigma} f_{\mu\rho\sigma} f_{\nu\rho\sigma} + \sum_{a,\rho} f_{a\mu\rho} f_{a\nu\rho} \tag{2.35}
\]

that is consistent with the result in Eq.(A.12).

If we introduce the orthonormal basis described in (B.1) the Ricci tensor can be explicitly written as:

\[
R_{\mu\nu} = \frac{1}{4} \sum_{\rho,\sigma} f_{\mu\rho\sigma} f_{\nu\rho\sigma} + \sum_{a,\rho} f_{a\mu\rho} f_{a\nu\rho} = \frac{1}{2} g^* \delta_{\mu\nu} + \frac{1}{2} \delta_{\mu\nu} \begin{cases} \left| \alpha_{(\mu+1)/2} \right|^2 & \text{if } \mu \text{ is odd} \\ \left| \alpha_{\mu/2} \right|^2 & \text{if } \mu \text{ is even} \end{cases} \tag{2.36}
\]

\(^\dagger\)Unless explicitly stated we consider $\alpha' = 1$ and the highest root $\psi = 2$
In particular for a simply laced algebra reduces to

\[ R_{\mu\nu} = \frac{g^* + 2}{2} \delta_{\mu\nu} = \frac{g^* + 2}{k} g_{\mu\nu} \]  (2.37)

This result can be read by saying that the metric we obtain on a simply laced algebra is Einstein with the following Ricci scalar:

\[ R = \frac{g^* + 2}{k} (\dim g - \dim \mathfrak{t}) \]  (2.38)

For example in the case of \( G = SU(N) \), \( g^* = N \), \( (\dim g - \dim \mathfrak{t}) = N(N - 1) \) and then

\[ R = \frac{(N + 2) N (N - 1)}{k} \]  (2.39)

3. Some Examples

In this section we will give some explicit examples of our construction. In particular we will consider the deformation leading from the \( SU(2) \) background to the \( SU(2)/U(1) \sim S^2 \) coset (which already appeared in \( \mathfrak{g} \) as part of the \( \text{AdS}_2 \times S^2 \) background) and the superconformal field theory on \( SU(3)/U(1)^2 \). Although our construction is quite general and can in principle be applied to any group there is a limited number of examples giving critical heterotic string theory backgrounds with a clear geometrical meaning. This is just because of dimensional reasons: \( SU(2)/U(1) \) is two-dimensional, \( SU(3)/U(1)^2 \) is 6-dimensional and \( USp(4)/U(1)^2 \) is 8-dimensional; higher groups on the other hand would lead to cosets of dimension greater than 10 (in example \( SU(4)/U(1)^3 \) has dimension \( 15 - 3 = 12 \)). On the other hand these higher-dimensional cosets can be used e.g. to obtain non-trivial compactifications generalizing the constructions of \( \mathfrak{g} \), if the level of these CFTs are kept small.

3.1 The two-sphere CFT

The first deformation that we explicitly consider is the marginal deformation of the \( SU(2) \) WZW model. This was first obtained in \[ \mathfrak{g} \] that we will closely follow. It is anyway worth to stress that in their analysis the authors didn’t study the point of maximal deformation (which was nevertheless identified as a decompactification boundary) that we will here show to correspond to the 2-sphere \( S^2 \sim SU(2)/U(1) \). Exact CFT’s on this background have already obtained in \[ \mathfrak{g} \] and in \[ \mathfrak{g} \]. In particular the technique used in the latter, namely the asymmetric gauging of a \( SU(2) \times U(1) \) WZW model, bears many resemblances to our own.

Consider an heterotic string background containing the \( SU(2) \) group manifold, times some \((1,0)\) superconformal field theory \( \mathcal{M} \). The sigma model action is:

\[
S = kS_{SU(2)}(g) + \frac{1}{2\pi} \int d^2 z \left\{ \sum_{a=1}^{3} \lambda^a \bar{\partial} \lambda^a + \sum_{n=1}^{9} \bar{\chi}^n \partial \bar{\chi}^n \right\} + S(\mathcal{M}),
\]  (3.1)

where \( \lambda^i \) are the left-moving free fermions superpartners of the bosonic \( SU(2) \) currents, \( \bar{\chi}^n \) are the right-moving fermions of the current algebra and \( kS_{SU(2)}(g) \) is the WZW action for
the bosonic $SU(2)$ at level $k$. This theory possesses an explicit $SU(2)_L \times SU(2)_R$ current algebra.

A parametrization of the $SU(2)$ group that is particularly well suited for our purposes is obtained via the so-called Gauss decomposition that we will later generalize to higher groups (see App C). A general element $g(z, \psi) \in SU(2)$ where $z \in \mathbb{C}$ and $\psi \in \mathbb{R}$ can be written as:

$$g(z, \psi) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{f} & 0 \\ 0 & \sqrt{f} \end{pmatrix} \begin{pmatrix} 1 \bar{w} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix}$$

(3.2)

where $w = -z$ and $f = 1 + |z|^2$. In this parametrisation the matrix of invariant one-forms $\Omega = g^{-1}(z, \psi)\, dg(z, \psi)$ appearing in the expression for the Maurer-Cartan one-forms is:

$$\Omega_{11} = \bar{z} \, dz - z \, d\bar{z} + if \, d\psi$$

$$\Omega_{21} = -\Omega_{12}$$

$$\Omega_{22} = -\Omega_{11}$$

(3.3)

(3.4)

(remark that $\Omega$ is traceless and anti-Hermitian since it lives in $\mathfrak{su}(2)$). From $\Omega$ we can easily derive the Cartan–Killing metric on $T_gSU(2)_k$ as:

$$\frac{2}{k} \, ds^2 = \text{tr} \left( \Omega^\dagger \Omega \right) = -\frac{1}{2f^2} \left( \bar{z}^2 \, dz \otimes dz + z^2 \, d\bar{z} \otimes d\bar{z} - 2 \left( 2 + |z|^2 \right) \, dz \otimes d\bar{z} \right) +$$

$$+ \frac{i}{f} (z \, d\bar{z} - \bar{z} \, dz) \otimes d\psi + \frac{1}{2} d\psi \otimes d\psi$$

(3.5)

The left-moving current contains a contribution from the free fermions realizing an $SU(2)_2$ algebra, so that the theory possesses (local) $N = (1, 0)$ superconformal symmetry.

The marginal deformation is obtained by switching on a magnetic field in the $SU(2)$, introducing the following $(1, 0)$-superconformal-symmetry-compatible marginal operator:

$$\delta S = \frac{\sqrt{kk'}}{2\pi} (J^3 + \lambda^+ \lambda^-) \tilde{J}$$

(3.6)

where we have picked one particular current $\tilde{J}$ from the gauge sector, generating a $U(1)$ at level $k_g$. For instance, we can choose the level two current: $\tilde{J} = i\bar{\chi}^1 \chi^2$. As a result the solutions to the deformed $\sigma$-model (2.21), (2.28) and (2.29) read:

$$\frac{1}{k} \, ds^2 = \frac{dz \otimes d\bar{z}}{(1 + |z|^2)^2} + (1 - 2h^2) \frac{1}{f^2} (iz \, d\bar{z} - i\bar{z} \, dz + f \, d\psi) \otimes (iz \, d\bar{z} - i\bar{z} \, dz + f \, d\psi)$$

(3.7)

$$dB = \frac{ik}{2} \, \frac{1}{(1 + |z|^2)^2} \, dz \wedge d\bar{z} \wedge d\psi$$

(3.8)

$$A = \frac{\sqrt{kk'}}{2k_g} \left( -\frac{i}{f} (\bar{z} \, dz - z \, d\bar{z}) + d\psi \right)$$

(3.9)
It can be useful to write explicitly the volume form on the manifold and the Ricci scalar:

\[
\sqrt{\det g} \, dz \wedge \bar{dz} \wedge d\psi = \frac{k \sqrt{k (1 - 2h^2)}}{2 (1 + |z|^2)^2} \, dz \wedge \bar{dz} \wedge d\psi \tag{3.10}
\]

\[
R = \frac{6 + 4h^2}{k} \tag{3.11}
\]

It is quite clear that at \( h = h_{\text{max}} = 1/\sqrt{2} \) something happens as it was already remarked in \[23\]. In general the three-sphere \( SU(2) \) can be seen a non-trivial fibration of \( U(1) \sim S^1 \) as fiber and \( SU(2)/U(1) \sim S^2 \) as base space: the parameterization in (3.7) makes it clear that the effect of the deformation consists in changing the radius of the fiber that naively seems to vanish at \( h_{\text{max}} \). But as we already know the story is a bit different: reparameterising as in Eq. (2.25):

\[
\psi \rightarrow \frac{\hat{\psi}}{\sqrt{1 - 2h^2}} \tag{3.12}
\]

one is free to take the \( h \rightarrow 1/\sqrt{2} \) limit where the background fields assume the following expressions:

\[
\frac{1}{k} ds^2 \bigg|_{h \rightarrow 1/\sqrt{2}} \rightarrow \frac{dz \otimes d\bar{z}}{(1 + |z|^2)^2} + d\hat{\psi} \otimes d\hat{\psi} \tag{3.13}
\]

\[
F \bigg|_{h \rightarrow 1/\sqrt{2}} \rightarrow \sqrt{\frac{k}{4k_g}} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2} \tag{3.14}
\]

\[
H \bigg|_{h \rightarrow 1/\sqrt{2}} \rightarrow 0 \tag{3.15}
\]

Now we can justify our choice of coordinates: the \((z, \bar{z})\) part of the metric that decouples from the \( \psi \) part is nothing else than the Kähler metric for the manifold \( \mathbb{C}P^1 \) (which is isomorphic to \( SU(2)/U(1) \)). In this terms the field strength \( F \) is proportional to the Kähler two-form:

\[
F = i \sqrt{\frac{k}{k_g}} g_{z\bar{z}} \, dz \wedge d\bar{z} \tag{3.16}
\]

This begs for a remark. It is simple to show that cosets of the form \( G/H \) where \( H \) is the maximal torus of \( G \) can always be endowed with a Kähler structure. The natural hope is then for this structure to pop up out of our deformations, thus automatically assuring the \( N = 2 \) world-sheet supersymmetry of the model. Actually this is not the case. The Kähler structure is just one of the possible left-invariant metrics that can be defined on a non-symmetric coset (see App. A) and the obvious generalization of the deformation considered above leads to \( C \)-structures that are not Kähler. From this point of view this first example is an exception because \( SU(2)/U(1) \) is a symmetric coset since \( U(1) \) is not only the maximal torus in \( SU(2) \) but also the maximal subgroup. It is nonetheless possible to define exact an \( \text{CFT} \) on flag spaces but this will require a slightly different construction, that we will introduce in Sec. 4.
We conclude this section observing that the flux of the gauge field on the two-sphere is given by:

\[
Q = \int_{S^2} F = \sqrt{\frac{k}{k_g}} \int d\Omega_2 = \sqrt{\frac{k}{k_g}} 4\pi
\]

(3.17)

However one can argue on general grounds that this flux has to be quantized, e.g. because the two-sphere appears as a factor of the magnetic monopole solution in string theory \cite{25}. This quantization of the magnetic charge is only compatible with levels of the affine \( SU(2) \) algebra satisfying the condition:

\[
\frac{k}{k_g} = p^2, \quad p \in \mathbb{Z}.
\]

(3.18)

### 3.2 The \( SU(3)/U(1) \) flag space

Let us now consider the next example in terms of coset dimensions, \( SU(3)/U(1)^2 \). As a possible application for this construction we may think to associate this manifold to a four-dimensional \((1,0)\) superconformal field theory \( \mathcal{M} \) so to compactify a critical string theory since \( \dim \left[ SU(3)/U(1)^2 \right] = 8 - 2 = 6 \). Our construction gives rise to a whole family of cft’s depending on two parameters (since rank \([SU(3)] = 2\) but as before we are mainly interested to the point of maximal deformation, where the \( U(1)^2 \) torus decouples and we obtain an exact theory on the \( SU(3)/U(1)^2 \) coset. Before giving the explicit expressions for the objects in our construction it is hence useful to remember some properties of this manifold. The first consideration to be made is the fact that \( SU(3)/U(1)^2 \) is an asymmetric coset in the mathematical sense defined in App. \( \Delta \) (as we show below). This allows for the existence of more than one left-invariant Riemann metric. In particular, in this case, if we just consider structures with constant Ricci scalar, we find, together with the restriction of the Cartan-Killing metric on \( SU(3) \), the Kähler metric of the flag space \( S^3 \). The construction we present in this section will lead to the first one of these two metrics. This is known to admit a nearly-Kähler structure and has already appeared in the superstring literature as a basis for a cone of \( G_2 \) holonomy \cite{4}.

A suitable parametrisation for the \( SU(3) \) group is obtained via the Gauss decomposition described in App. \( \Box \). In these terms the general group element is written as:

\[
g(z_1, z_2, z_3, \psi_1, \psi_2) = \begin{pmatrix}
\frac{e^{\psi_1/2}}{\sqrt{f_1}} & -\frac{z_1 + z_2 z_3}{\sqrt{f_1} f_2} e^{(\psi_1 - \psi_2)/2} & -\frac{z_3 - \bar{z}_1 z_2}{\sqrt{f_1} f_2} e^{-i\psi_2/2} \\
\frac{z_1 e^{\psi_1/2}}{\sqrt{f_1}} & \frac{1 + |z_2|^2 - z_3 z_1}{\sqrt{f_1} f_2} e^{(\psi_1 - \psi_2)/2} & \frac{|z_3|^2 - z_1 z_2}{\sqrt{f_1} f_2} e^{-i\psi_2/2} \\
\frac{z_3 e^{\psi_1/2}}{\sqrt{f_1}} & -\frac{z_2 - \bar{z}_1 z_3}{\sqrt{f_1} f_2} e^{(\psi_1 - \psi_2)/2} & \frac{1}{\sqrt{f_2}} e^{-i\psi_2/2}
\end{pmatrix}
\]

(3.19)

where \( z_i \) are three complex parameters, \( \psi_j \) are two real parameters and \( f_1 = 1 + |z_1|^2 + |z_3|^2 \), \( f_2 = 1 + |z_2|^2 + |z_3 - z_1 z_2|^2 \). As for the group, we need also an explicit parameterisation for the \( \mathfrak{su}(3) \) algebra, such as the one provided by the Gell-Mann matrices in Eq. (C.10). It is a well known result that if a Lie algebra is semi-simple (or, equivalently, if its Killing form is negative-definite) then all Cartan subalgebras are conjugated by some inner auto-
morphism*. This leaves us the possibility of choosing any couple of commuting generators, knowing that the final result won’t be influenced by such a choice. In particular, then, we can pick the subalgebra generated by \( \mathfrak{f} = (\lambda_3, \lambda_8) \).

We can now specialize the general expressions given in Sec. 2. The holomorphic currents (2.3) of the bosonic \( SU(3)_c \) corresponding to the two operators in the Cartan are:

\[
\mathcal{J}^3 = -\text{tr} \left( \lambda_3 g (z_\mu, \psi_a)^{-1} \text{d} g (z_\mu, \psi_a) \right) \quad \mathcal{J}^8 = -\text{tr} \left( \lambda_8 g (z_\mu, \psi_a)^{-1} \text{d} g (z_\mu, \psi_a) \right)
\]

that in these coordinates read:

\[
\mathcal{J}^3 = -\frac{i}{\sqrt{2}} \left\{ \frac{\bar{z}_1 + z_3 (\bar{z}_1 \bar{z}_2 + \bar{z}_3)}{2 f_1} d\bar{z}_1 - \frac{\bar{z}_2 (1 + |z_1|^2) - z_1 \bar{z}_3}{2 f_2} d\bar{z}_2 + \left( \frac{\bar{z}_3}{f_1} + \frac{\bar{z}_1 \bar{z}_2 - \bar{z}_3}{2 f_2} \right) d\bar{z}_3 \right\} + \text{c.c.} + \frac{d\psi_1}{\sqrt{2}} - \frac{d\psi_2}{\sqrt{2}}
\]

\[
\mathcal{J}^8 = -i \sqrt{\frac{3}{2}} \left\{ \frac{\bar{z}_1 \bar{z}_2 - \bar{z}_3}{2 f_2} dz_1 + \frac{\bar{z}_2 (1 + |z_1|^2) - z_1 \bar{z}_3}{2 f_2} d\bar{z}_2 + \left( \frac{\bar{z}_3}{f_1} + \frac{\bar{z}_1 \bar{z}_2 - \bar{z}_3}{2 f_2} \right) d\bar{z}_3 \right\} + \text{c.c.} + \frac{1}{2} \sqrt{\frac{3}{2}} d\psi_2
\]

they appear in the expression of the exactly marginal operator (2.11) that we can add to the \( SU(3) \) WZW model action is:

\[
V = \frac{\sqrt{kk'}}{2\pi} H \int d\bar{z}^2 H_3 \left( \mathcal{J}^3 - \frac{i}{\sqrt{2k}} (2 : \psi_2 \psi_1 : + : \psi_5 \psi_4 : + : \psi_7 \psi_6 :) \right) \bar{J}^3 +
\]

\[
+ H_8 \left( \mathcal{J}^8 - \frac{i}{k} \sqrt{\frac{3}{2}} (2 : \psi_5 \psi_4 : + : \psi_7 \psi_6 :) \right) \bar{J}^8
\]

where \( \psi^i \) are the bosonic current superpartners and \( \bar{J} \) are two currents from the gauge sector both generating a \( U(1)_{k_9} \).

Since \( \text{rank } [SU(3)] = 2 \) we have a bidimensional family of deformations parameterised by the two moduli \( H_3 \) and \( H_8 \). The back-reaction on the metric is given by:

\[
ds^2 = g_{\alpha\beta} dz^\alpha \otimes d\bar{z}^\beta + (1 - 2H_3^2) \mathcal{J}^3 \otimes \mathcal{J}^3 + (1 - 2H_8^2) \mathcal{J}^8 \otimes \mathcal{J}^8
\]

where \( g_{\alpha\beta} \) is the restriction of the \( SU(3) \) metric on \( SU(3)/U(1)^2 \). It is worth to remark that for any value of the deformation parameters \( H_3 \) and \( H_8 \) the deformed metric is Einstein with constant Ricci scalar.

With a procedure that has by now become familiar we introduce the following reparametrisation:

\[
\hat{\psi}_1 = \frac{\psi_1}{\sqrt{1 - 2H_2}} \quad \hat{\psi}_2 = \frac{\psi_2}{\sqrt{1 - 2H^2}}
\]

*This is the reason why the study of non-semi-simple Lie algebra deformation constitutes a richer subject. In example the \( SL(2,R) \) group admits for 3 different deformations, leading to 3 different families of exact cft’s with different physics properties. On the other hand the 3 possible deformations in \( SU(3) \) are equivalent.

1In this explicit parameterisation it is straightforward to show that the coset we’re considering is not symmetric. It suffices to pick two generators, say \( \lambda_2 \) and \( \lambda_4 \), and remark that their commutator \( [\lambda_2, \lambda_4] = -1/\sqrt{2} \lambda_6 \) doesn’t live in the Cartan subalgebra.
and take the $h_3 \rightarrow 1/\sqrt{2}, h_8 \rightarrow 1/\sqrt{2}$ limit. The resulting metric is:

$$d s^2 = g_{\alpha\beta} d z^\alpha \otimes d \bar{z}^\beta + \frac{d \hat{\psi}_1 \otimes d \hat{\psi}_1 - d \hat{\psi}_1 \otimes d \hat{\psi}_2 + d \hat{\psi}_2 \otimes d \hat{\psi}_2}{2}$$

(3.26)

that is the metric of the tangent space to the manifold $SU(3)/U(1)^2 \times U(1) \times U(1)$. As shown in App. A the coset metric hence obtained has a $\mathbb{C}$-structure, is Einstein and has constant Ricci scalar $R = 15/k$. The other background fields at the boundary of the moduli space read:

$$F = d J^3 + d J^8$$

$$H_3 = -3\sqrt{2} \left\{ J^1 \wedge (J^4 \wedge J^5 - J^6 \wedge J^7) + \sqrt{3} J^2 \wedge (J^4 \wedge J^5 + J^6 \wedge J^7) \right\}$$

(3.27)

(3.28)

If we consider the supersymmetry properties along the deformation line we can remark the presence of an interesting phenomenon. The initial $SU(3)$ model has $N = 2$ but this symmetry is naively broken to $N = 1$ by the deformation. This is true for any value of the deformation parameter but for the boundary point $h_3^2 = h_8^2 = 1/2$ where the $N = 2$ supersymmetry is restored. Following [26, 22, 27] one can see that a $G/T$ coset admits $N = 2$ supersymmetry if it possesses a complex structure and the corresponding algebra can be decomposed as $j = j_+ \oplus j_-$ such as $[j_+, j_+] = j_+$ and $[j_-, j_-] = j_-$. Explicitly, this latter condition is equivalent (in complex notation) to $f_{ijk} = f_{ijk} = f_{aij} = f_{aij} = 0$. These are easily satisfied by the $SU(3)/U(1)^2$ coset (and actually by any $G/T$ coset) since the commutator of two positive (negative) roots can only be proportional to the positive (negative) root obtained as the sum of the two or vanish, as shown in Eq. (B.3). Having $N = 2$ supersymmetry is equivalent to asking for the presence of two complex structures. The first one is trivially given by considering positive roots as holomorphic and negative roots as anti-holomorphic, the other one by interchanging the role in one out of the three positive/negative couples (the same flip on two couples would give again the same structure and on all the three just takes back to the first structure). The metric is hermitian with respect to both structures since it is $SU(3)$ invariant. It is worth to remark that such background is different from the ones described in [27] because it is not Kähler and can’t be decomposed in terms of Hermitian symmetric spaces.

4. Gauging

In this section we want to give an alternative construction for our deformed models, this time explicitly based on an asymmetric wzw gauging. The existence of such a construction is not surprising at all since our deformations can be seen as a generalization of the ones considered in [28]. In these terms, just like $J\bar{J}$ (symmetric) deformations lead to gauged wzw models, our asymmetric construction leads to asymmetrically gauged wzw models, which were studied in [8].

First of all we will give the explicit construction for the most simple case, the $SU(2)$ model, then introduce a more covariant formalism which will be simpler to generalize to higher groups, in particular for the $SU(3)$ case, whose gauging will lead, this time, to
two different exact models corresponding to the two possible Einstein complex structures admitted by the $SU(3)/U(1)^2$ manifold.

To simplify the formalism we will discuss gauging of bosonic CFTs, and the currents of the gauge sector of the heterotic string are replaced by compact $U(1)$ free bosons. It is obvious that all the results are easily translated into heterotic string constructions.

### 4.1 The $SU(2)/U(1)$ asymmetric gauging

In this section we want to show how the $S^2$ background described in [3] can be directly obtained via an asymmetric gauging of the $SU(2) \times U(1)$ WZW model (a similar construction was first obtained in [2]).

Consider the WZW model for the group manifold $SU(2)_k \times U(1)_{k'}$. A parametrisation for the general element of this group which is nicely suited for our purposes is obtained as follows:

$$
g = \begin{pmatrix}
  z_1 & z_2 & 0 \\
  -\bar{z}_2 & \bar{z}_1 & 0 \\
  0 & 0 & z_3
\end{pmatrix}
\in SU(2) \times U(1) \quad (4.1)
$$

where $g_1$ and $g_2$ correspond to the $SU(2)$ and $U(1)$ parts respectively and $(z_1, z_2, z_3)$ satisfy:

$$
SU(2) \times U(1) = \{(w_1, w_2, w_3) \mid |w_1|^2 + |w_2|^2 = 1, |w_3|^2 = 1\} \subset \mathbb{C}^3 \quad (4.2)
$$

A possible choice of coordinates for the corresponding group manifold is given by the Euler angles:

$$
SU(2) \times U(1) = \left\{(z_1, z_2, z_3) = \left(\cos \frac{\beta}{2} e^{i(\gamma + \alpha)/2}, \sin \frac{\beta}{2} e^{i(\gamma - \alpha)/2}, e^{i\varphi}\right) \mid 0 \leq \beta \leq \pi, 0 \leq \alpha, \beta, \varphi \leq 2\pi\right\} \quad (4.3)
$$

In order to obtain the coset construction leading to the $S^2$ background we define two $U(1) \rightarrow SU(2) \times U(1)$ embeddings as follows:

$$
\begin{align*}
\epsilon_L : U(1) & \rightarrow SU(2) \times U(1) \\
\epsilon_R : U(1) & \rightarrow SU(2) \times U(1)
\end{align*}
\quad (4.4)
$$

$$
e^{i\tau} \mapsto (e^{i\tau}, 0, 1) \quad \epsilon^{i\tau} \mapsto (1, 0, e^{i\tau})
$$

so that in terms of the $z$ variables the action of these embeddings boils down to:

$$
g \mapsto \epsilon_L (e^{i\tau}) g \epsilon_R (e^{i\tau})^{-1} \quad (4.5)
$$

$$
(w_1, w_2, w_3) \mapsto (e^{i\tau} w_1, e^{i\tau} w_2, e^{-i\tau} w_3) \quad (4.6)
$$

This means that we are free to choose a gauge where $w_2$ is real or, in Euler coordinates, where $\gamma = \alpha$, the other angular variables just being redefined. To find the background fields corresponding to this gauge choice one should simply write down the Lagrangian where the symmetries corresponding to the two embeddings in (4.4) are promoted to local symmetries, integrate the gauge fields out and then apply a Kaluza-Klein reduction, much in the same spirit as in [3].
The starting point is the wzw model, written as:

\[ S_{wzw}(g) = \frac{k}{4\pi} \int dz^2 \text{Tr} (g_2^{-1} \partial g_2 g_1^{-1} \partial g_1) + \frac{k'}{4\pi} \int dz^2 \text{Tr} (g_1^{-1} \partial g_1 g_1^{-1} \partial g_1) \]  

(4.7)

Its gauge-invariant generalization is given by:

\[ S = S_{wzw} + \frac{1}{2\pi} \int d^2 z \left[ k A \text{Tr} (t_L \partial g g^{-1}) + k' A \text{Tr} (t_R g^{-1} \partial g) + \sqrt{kk'} A \tilde{A} \right] \]  

(4.8)

where \( A \) and \( \tilde{A} \) are the components of the gauge field, and \( t_L \) and \( t_R \) are the Lie algebra generators corresponding to the embeddings in (4.4), i.e.

\[ t_L = i \begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}, \quad \quad t_R = i \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}, \]  

(4.9)

\( \sigma_3 \) being the usual Pauli matrix. For such an asymmetric coset to be anomaly free, one has the following constraint on the embeddings:

\[ k \text{Tr} (t_L) = k' \text{Tr} (t_R) \Rightarrow k = k' p^2, \quad \text{with} \quad p \in \mathbb{N}. \]  

(4.10)

If we pass to Euler coordinates it is simple to give an explicit expression for the action:

\[ S(\alpha, \beta, \gamma, \varphi) = \frac{1}{2\pi} \int d^2 z \frac{k}{4} \left( \partial \alpha \bar{\partial} \alpha + \partial \beta \bar{\partial} \beta + \partial \gamma \bar{\partial} \gamma + 2 \cos \beta \partial \alpha \bar{\partial} \gamma + \frac{k'}{2} \partial \varphi \bar{\partial} \varphi + \right. \]  

\[ \left. + ik (\partial \alpha + \cos \beta \partial \gamma) \bar{A} + ik' \sqrt{2} \bar{\partial} \varphi A - 2 \sqrt{kk'} A \tilde{A} \right) \]  

(4.11)

This Lagrangian is quadratic in \( A, \tilde{A} \) and the quadratic part is constant so we can integrate these gauge fields out and the resulting Lagrangian is:

\[ S(\alpha, \beta, \gamma, \varphi) = \frac{1}{2\pi} \int d^2 z \left( \partial \alpha \bar{\partial} \alpha + \partial \beta \bar{\partial} \beta + \partial \gamma \bar{\partial} \gamma + 2 \cos \beta \partial \alpha \bar{\partial} \gamma + \frac{k'}{2} \partial \varphi \bar{\partial} \varphi + \right. \]  

\[ \left. + \frac{\sqrt{kk'}}{2} (\partial \alpha + \cos \beta \partial \gamma) \bar{\partial} \varphi \right) \]  

(4.12)

now, since we gauged out the symmetry corresponding to the \( U(1) \) embeddings, this action is redundant. This can very simply be seen by writing the corresponding metric and remarking that it has vanishing determinant:

\[ \det g_{\mu\nu} = \begin{vmatrix} k/4 \\ k/4 \cos \beta \\ \sqrt{2kk'}/4 \cos \beta \end{vmatrix} = 0 \]  

(4.13)

Of course this is equivalent to say that we have a gauge to fix (as we saw above) and this can be chosen by imposing \( \gamma = \alpha \), which leads to the following action:

\[ S(\alpha, \beta, \varphi) = \frac{1}{2\pi} \int d^2 z \left( 2 (1 + \cos \beta) \partial \alpha \bar{\partial} \alpha + \partial \beta \bar{\partial} \beta \right) + \frac{k'}{2} \partial \varphi \bar{\partial} \varphi + \frac{\sqrt{2kk'}}{2} (1 + \cos \beta) \partial \alpha \bar{\partial} \varphi \]  

(4.14)
whence we can read a two dimensional metric by interpreting the $\partial \alpha \bar{\partial} \varphi$ term as a gauge boson and applying the usual Kaluza-Klein reduction. We thus recover the two-sphere we expect.

$$ds^2 = g_{\mu \nu} - G_{\varphi \varphi} A_\mu A_\nu = \frac{k}{4} \left( d\beta^2 + \sin^2 \beta d\alpha^2 \right)$$  \hspace{1cm} (4.15)

supported by a (chromo)magnetic field

$$A = \sqrt{\frac{k}{k'}} (1 + \cos \beta) \, d\alpha$$  \hspace{1cm} (4.16)

As advertised above we now turn to rewrite the above gauging in a more covariant form, simpler to generalize. Since we’re interested in the underlying geometry, we’ll mainly focus on the metric of the spaces we obtain at each step and write these metrics in terms of the Maurer-Cartan currents*.

As we’ve already seen in Eq. (2.2), the metric of the initial group manifold is:

$$ds^2 = \frac{k}{2} \sum \mathcal{J}_i^2 \otimes \mathcal{J}_i^2 + \frac{k'}{2} \mathcal{I} \otimes \mathcal{I}$$  \hspace{1cm} (4.17)

where $\{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}$ are the currents of the $SU(2)$ part and $\mathcal{I}$ the $U(1)$ generator. The effect of the asymmetric gauging amounts - at this level - to adding what we can see as an interaction term between the two groups. This changes the metric to:

$$ds^2 = \frac{k}{2} \sum \mathcal{J}_i^2 \otimes \mathcal{J}_i^2 + \frac{k'}{2} \mathcal{I} \otimes \mathcal{I} + \sqrt{kk'} \mathcal{J}_3 \otimes \mathcal{I}$$  \hspace{1cm} (4.18)

Of course if we choose $\langle \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{I} \rangle$ as a basis we can rewrite the metric in matrix form:

$$g = \frac{1}{2} \begin{pmatrix} \frac{k}{k'} & \sqrt{kk'} \\
\sqrt{kk'} & \frac{k'}{k} \end{pmatrix}$$  \hspace{1cm} (4.19)

where we can see that the gauging of the axial symmetry corresponds to the fact that the sub-matrix relative to the $\{\mathcal{J}_3, \mathcal{I}\}$ generators is singular:

$$\left| \begin{array}{cc} \frac{k}{k'} & \sqrt{kk'} \\
\sqrt{kk'} & \frac{k'}{k} \end{array} \right| = 0$$  \hspace{1cm} (4.20)

explicitly this correspond to:

$$k \mathcal{J}_3 \otimes \mathcal{J}_3 + \sqrt{kk'} \mathcal{J}_3 \otimes \mathcal{I} + \sqrt{kk'} \mathcal{J}_3 \otimes \mathcal{I} + k' \mathcal{I} \otimes \mathcal{I} = (k + k') \hat{\mathcal{J}} \otimes \hat{\mathcal{J}}$$  \hspace{1cm} (4.21)

where

$$\hat{\mathcal{J}} = \sqrt{\frac{k \mathcal{J}_3 + \sqrt{kk'} \mathcal{I}}{\sqrt{k + k'}}}$$  \hspace{1cm} (4.22)

*One of the advantages of just working on the metrics is given by the fact that in each group one can consistently choose holomorphic or anti-holomorphic currents as a basis. In the following we will consider the group in the initial WZW model as being generated by the holomorphic and the dividing group by the anti-holomorphic ones.
is a normalized current. In matrix term this corresponds to projecting the interaction sub-matrix on its non-vanishing normalized eigenvector:

$$\left(\sqrt{\frac{k}{k+k'}} \sqrt{k} \sqrt{k'} \right) \left( \begin{array}{c} k \\ \sqrt{kk'} \\ k' \end{array} \right) \left( \begin{array}{c} \sqrt{\frac{k}{k+k'}} \\ \sqrt{k} \\ \sqrt{k'} \end{array} \right) = k + k'$$

(4.23)

and the resulting metric in the \((\mathcal{J}_1, \mathcal{J}_2, \hat{J})\) basis is:

$$\begin{pmatrix} k \\ k \\ k + k' \end{pmatrix}$$

(4.24)

This manifold \(\mathcal{M}\) (whose metric appears in the action (2.12)) corresponds to a \(S^1\) fibration (the fiber being generated by \(\hat{J}\)) over a \(S^2\) base (generated by \(\langle \mathcal{J}_1, \mathcal{J}_2 \rangle\)).

\[
\begin{array}{c}
S^1 \\
\rightarrow \mathcal{M} \\
\downarrow \\
S^2
\end{array}
\]

(4.25)

It should now appear obvious how to generalize this construction so to include all the points in the moduli space joining the unperturbed and gauged model. The decoupling of the \(U(1)\) symmetry (that has been “gauged away”) is obtained because the back-reaction of the gauge field Eq. (4.12) is such that the interaction sub-matrix is precisely singular. On the other hand we can introduce a parameter that interpolates between the unperturbed and the gauged models so that the interaction matrix now has two non-null eigenvalues, one of which will vanish at the decoupling point.

In practice this is done by adding to the the asymmetrically gauged WZW model an auxiliary \(U(1)\) free boson \(Y\) at radius \(R = (kk')^{1/4}(1/\sqrt{a_0-1})^{1/2}\). This \(U(1)\) is coupled symmetrically to the gauge fields such that the anomaly cancelation condition is still given by (4.10). In particular if we choose the gauge \(Y = 0\), the metric reads:

$$\begin{pmatrix} k \\ \sqrt{2H \sqrt{kk'}} \sqrt{kk'} \\ k' \end{pmatrix}$$

(4.26)

which is exactly the model studied above. For a generic value of \(h^2\) the two eigenvalues are given by:

$$\lambda_{1,2} (k, k', h) = \frac{k + k' \mp \sqrt{k^2 + k'^2 + 2 (4H^2 - 1) kk'}}{2}$$

(4.27)

so we can diagonalize the metric in the \(\langle \mathcal{J}_1, \mathcal{J}_2, \hat{J}, \hat{J}\rangle\) basis (\(\hat{J}\) and \(\hat{J}\) being the two eigenvectors) and finally obtain:

$$g = \begin{pmatrix} k \\ \lambda_1 (k, k', h) \\ \lambda_2 (k, k', h) \end{pmatrix}$$

(4.28)
Of course, in the \( h^2 \to 0 \) limit we get the initial wzw model and in the \( h^2 \to 1/2 \) limit we recover the asymmetrically gauged model Eq. (4.24).

It is important to remark that the construction above can be directly generalized to higher groups with non-abelian subgroups, at least for the asymmetric coset part. This is what we will do in the next section.

4.2 SU (3) /U(1)\(^2\)

To study the SU (3) case we will use the “current” approach, since a direct computation in coordinates would be impractical. As one could expect, the study of SU (3) deformation is quite richer because of the presence of an embedded SU (2) group that can be gauged. Basically this means that we can choose two different deformation patterns that will lead to the two possible Einstein structures that can be defined on the SU (3) /U (1)\(^2\) manifold (see App. A).

4.2.1 Direct gauging.

The first possible choice consists in the obvious generalization of the SU (2) /U (1) construction above, ie simply gauging the \( U (1)^2 \) Cartan torus. Consider the initial SU (3)\(_k\) \( \times U (1)^{k'} \times U (1)^{k''} \) model. In the \( \langle J_1, \ldots, J_8, \mathcal{I}_1, \mathcal{I}_2 \rangle \) base \( \{\mathcal{J}_i\} \) being the SU (3) generators and \( \{\mathcal{I}_k\} \) the 2 \( U (1) \)'s, the initial metric is written as:

\[
g = \begin{pmatrix}
    k \mathbb{I}_{8\times8} & 0 & 0 \\
    0 & k' & 0 \\
    0 & 0 & k''
\end{pmatrix}
\]  

(4.29)

the natural choice for the Cartan torus is given by the usual \( \langle \mathcal{J}_3, \mathcal{J}_8 \rangle \) generators, so we can proceed as before and write the deformed metric as:

\[
g = \begin{pmatrix}
    k \mathbb{I}_{2\times2} & \lambda_1 (k, k', H_3) \\
    \lambda_1 (k, k'', H_3) & k \mathbb{I}_{4\times4} \\
    \lambda_1 (k, k', H_3) & \lambda_2 (k, k', H_3) \\
    \lambda_2 (k, k'', H_3) & k \mathbb{I}_{4\times4}
\end{pmatrix}
\]  

(4.30)

where \( H_3 \) and \( H_8 \) are the deformation parameters and \( \lambda_1 \) and \( \lambda_2 \) are the eigenvalues for the interaction matrices, given in Eq. (4.27). In particular, then, in the \( H_3^2 \to 1/2, H_8^2 \to 1/2 \) limit two eigenvalues vanish, the corresponding directions decouple and we’re left with the following (asymmetrically gauged) model:

\[
g = \begin{pmatrix}
    k \mathbb{I}_{6\times6} & 0 \\
    0 & k + k' \\
    0 & 0 \\
    0 & 0 \\
    0 & 0 \\
    0 & k + k''
\end{pmatrix}
\]  

(4.31)

in the \( \langle \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_4, \mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \sqrt{k} \mathcal{I}_1 + \sqrt{k} \mathcal{I}_3, \sqrt{k'} \mathcal{I}_2 + \sqrt{k'} \mathcal{I}_8 \rangle \) basis that can be seen as a \( U (1)^2 \) fibration over a SU (3) /U (1)\(^2\) base with metric \( \text{diag}(1, 1, 1, 1, 1, 1, 1, 1) \) (in the notation
of App A). This is precisely the same result we obtained in Sec. 3.2 when we read the fibration as a gauge field living on the base.

\[
U(1)^2 \longrightarrow \mathcal{M} \downarrow \quad SU(3)/U(1)^2
\]  

(4.32)

As in the previous example all this construction is valid only if the asymmetrically gauged wzw model is anomaly-free. This will be explained in detail in section 5.

4.2.2 The $F_3$ flag space

Let us now turn to the other possible choice for the $SU(3)$ gauging, namely the one where we take advantage of the $SU(2)$ embedding. Let us then consider the $SU(3)_{k_3} \times SU(2)_{k_2} \times U(1)_{k'} \times U(1)_{k''}$ wzw model whose metric is

\[
g = \begin{pmatrix}
k_3 \mathbb{I}_{8 \times 8} & k_2 \mathbb{I}_{3 \times 3} \\
 & & \left( k' \atop k'' \right)
\end{pmatrix}
\]  

(4.33)

in the $\langle J_1, \ldots, J_8, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, K_1, K_2 \rangle$ basis, where $\langle J_i \rangle$ generate the $SU(3)$, $\langle \mathcal{I}_i \rangle$ generate the $SU(2)$ and $\langle K_i \rangle$ generate the $U(1)^2$.

The first step in this case consists in an asymmetric gauging mixing the $\{J_1, J_2, J_3\}$ and $\{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$ currents respectively. At the gauging point, a whole 3-sphere decouples and we obtain the following metric:

\[
g = \begin{pmatrix}
k_3 \mathbb{I}_{5 \times 5} & (k_2 + k_3) \mathbb{I}_{3 \times 3} \\
 & & \left( k' \atop k'' \right)
\end{pmatrix}
\]  

(4.34)

where we have to remember that in order to have an admissible embedding $k_2 = k_3 = k$. Our result is again – not surprisingly – a $SU(2)$ fibration over a $SU(3)/SU(2)$ base (times the two $U(1)$’s).

\[
SU(2) \longrightarrow \mathcal{M} \downarrow \quad SU(3)/SU(2)
\]  

(4.35)

Of course one could be tempted to give $\mathcal{M}$ the same interpretation as before, namely a $SU(3)/SU(2)$ space supported by a chromo-magnetic $SU(2)$ field (or, even better, gauging an additional $U(1)$, of a $\mathbb{C}P^2$ background with a $SU(2) \times U(1)$ chromo-magnetic field). Actually this is not the case. The main point is the fact that this $SU(3) \times SU(2)$ model is essentially different from the previous ones because the $U(1)$ factors were the result of the bosonisation of the right-moving gauge current which in this way received a (fake) left-moving partner as in Sec. 2.2. This is not possible in the non-abelian case.
since one can’t obtain a $SU(2)$ at arbitrary level $k$ out of the fermions of the theory\(^\dagger\). In other words, the $SU(2)$ factor is in this case truly a constituent of the theory and there is no reason why it should be decoupled or be given a different interpretation from the $SU(3)$ part. This is why the structure obtained by the $SU(2)$ asymmetric gauging is to be considered a 8-dimensional space admitting a $SU(2) \to SU(3)/SU(2)$ fibration structure, or, equivalently, a deformed $SU(3)$ where an embedded $SU(2)$ is at a level double with respect to the other generators.

On the other hand we are still free to gauge away the two $U(1)$ factors just as before. This time we can choose to couple $K_1$ with the $J_8$ factor that was left untouched in the initial $SU(3)$ and $K_2$ with the $J_3 + I_3$ generator. Again we find a two-parameter family of deformations whose metric can be written as:

$$g = \begin{pmatrix}
\begin{array}{c|c}
\; & \mu_1 \\
\hline
\mu_1 & 2k \mathbb{I}_{2 \times 2} \\
\end{array} & \begin{array}{c|c}
\; & \nu_1 \\
\hline

\nu_1 & \begin{array}{c|c}
\; & \mu_2 \\
\hline
\mu_2 & \nu_2 \\
\end{array}
\end{array}\end{pmatrix}$$

(4.36)

where:

$$\mu = \lambda (k, k', h')$$

(4.37)

$$\nu = \lambda (2k, k'', h'')$$

(4.38)

In particular now we can take the decoupling $h' = h'' \to 1/2$ limit where we obtain:

$$g = \begin{pmatrix}
\begin{array}{c|c}
\; & k \mathbb{I}_{4 \times 4} \\
\hline
k \mathbb{I}_{4 \times 4} & 2k \mathbb{I}_{2 \times 2} \\
\end{array} & \begin{array}{c|c}
\; & 2k \\
\hline
2k & k + k' \\
\end{array} & \begin{array}{c|c}
\; & k + k'' \\
\hline
k + k'' & 2k + k'' \\
\end{array}
\end{pmatrix}$$

(4.39)

this structure is once more a $U(1)^2 \to SU(3)/U(1)^2$ fibration but in this case it is perfectly fine to separate the space components from the gauge field ones. So we can read out our final background fields as the Kähler metric on $F_3$ (see App [A]) supported by a $U(1)^2$ (chromo)magnetic field.

To summarize our results we can say that the two Einstein structures that one can define on $SU(3)/U(1)^2$ are both exact string theory backgrounds:

- The first one, obtained as the asymmetric coset $\frac{SU(3) \times U(1)^2}{U(1)^2}$ is supported by an NS-NS field strength and a magnetic field;

- The second, corresponding to the $\frac{SU(3) \times SU(2) \times U(1)^2}{SU(2) \times U(1)^2}$ asymmetric coset is Kähler and hence supported by the (chromo-)magnetic field alone.

\(^\dagger\)This would be of course be possible if we limited ourselves to small values of $k$, but in this case the whole geometric interpretation of the background would be questionable. However for Gepner-like string compactifications this class of models is relevant.
This Kähler structure has been deeply studied both from the mathematical and physical points of view. In particular the Kähler form can be written as in App. C:

\[ K(\gamma_\mu, \bar{\gamma}_\mu) = \log \left[ 1 + |\gamma_1|^2 + |\gamma_3|^2 \right] + \log \left[ 1 + |\gamma_2|^2 + |\gamma_3 - \gamma_1 \gamma_2|^2 \right] \] (4.40)

It is immediate to show that this manifold is Einstein and in particular its Ricci scalar is \( R = 12 \). Being Kähler, \( F_3 \) is torsionless, that means in turn that there is no NS-NS form. Moreover there is no dilaton by construction. The only other field that supports the background comes from the \( U(1)^2 \) fibration. Since the manifold is Kähler it is useful to take advantage of the complex structure and write our background fields in complex formalism. In these terms the metric is written as:

\[ g = \frac{k}{2} \left( J^1 \otimes \bar{J}^1 + J^2 \otimes \bar{J}^2 + 2J^3 \otimes \bar{J}^3 \right) \] (4.41)

where \( J^i \) and \( \bar{J}^\bar{i} \) are the Maurer-Cartan corresponding to positive and negative roots respectively and the field strength is given by:

\[ F^a = \sqrt{\frac{k}{2k_g}} f^a_{\mu\bar{\nu}} C^{\delta\sigma} R_{\sigma\bar{\nu}} \mathcal{J}^\mu \wedge \mathcal{J}^\bar{\nu} \] (4.42)

where \( C \) is the following tensor

\[ C = \sum_\alpha J^\alpha \otimes \bar{J}^\bar{\alpha} \] (4.43)

In Sec. B.2 we show that the metric and (chromo)magnetic field solve the first order in \( \alpha' \) equations of motion.

5. Exact construction: partition functions

In this section we will compute the one-loop partition functions for the various asymmetric deformations leading to geometric cosets. We consider the part of the partition function of the CFT affected by the deformation. We have holomorphic supersymmetric characters and anti-holomorphic bosonic characters of the affine Lie algebra \( \hat{\mathfrak{g}}_k \), times some anti-holomorphic fermionic characters from the gauge sector:

\[ Z[a; b; \chi, \bar{\chi}] = \sum_{\Lambda, \bar{\Lambda}} M^{\Lambda\bar{\Lambda}} \chi^\Lambda(\tau) \left( \frac{\partial \chi'[b]}{\eta(\tau)} \right)^{\text{dim}(\mathfrak{g})/2} \bar{\chi}^{\bar{\Lambda}} \prod_{\ell} \bar{\partial} \left[ h_\ell \right] g_{\ell} \] (5.1)

where \((a, b)\) and \((h_\ell, g_\ell)\) are the spin structures of the (left and right) worldsheet fermions. Useful formulas about characters is provided in appendix D. Starting from the CFTs defined by these partition functions we will perform the magnetic deformation that has been discussed in the previous sections from the geometrical point of view.

\(^1\)To be precise one could define a \( B \) field but this would have to be closed

\(^2\)The dilaton would basically measure the difference between the asymmetric coset volume form and the homogeneous space one as it is shown in [23]
5.1 The SU(3)/U(1)$^2$ flag space CFT

The partition function for the asymmetric deformation of SU(2) has already been given in [3]. We can hence begin with the next non-trivial example of SU(3). In this case we will compare explicitly two possible constructions, the Kazama-Suzuki method and the direct deformation along the Cartan torus to show that they give the two inequivalent metrics on the geometric coset.

5.1.1 The Kazama-Suzuki decomposition of SU(3)

We would like to decompose our wzw model in terms of Kazama-Suzuki (ks) cosets, which are conformal theories with extended $N = 2$ superconformal symmetry [22, 27].

The simplest of those models are the $N = 2$ minimal models that are given by the quotient: $SU(2)_{k-2} \times SO(2)\times U(1)_k$, and their characters come from the branching relation:

$$
\chi^j_{k-2} = \sum_{m \in \mathbb{Z}_{2k}} C^j_m \Theta_{m,k} \eta
$$

(5.2)

For convenience, we write the contribution of the worldsheet fermions in terms of $SO(2n)$ characters, see appendix D.

Similarly it is possible to construct an $N = 2$ coset CFT from $SU(3)$ [22, 27]:

$$
SU(3)_{k-3} \times SO(4) \times SU(2)_{k-2} \times U(1)_3
$$

(5.3)

The characters of this theory are implicitly defined by the branching relation:

$$
\chi^j_{k-3} = \sum_{2j=0}^{k-2} \sum_{m \in \mathbb{Z}_{2k}} C^j_m \chi_{k-2}^{s_2} \Theta_{m,3k} \eta
$$

(5.4)

Therefore combining the two branching relations, we obtain the decomposition of $SU(3)$ in terms of $N = 2$ ks models:

$$
\chi^j_{k-3} = \sum_{j,m,n} C^j_m \Theta_{m,k} \theta_{n,3k} \eta
$$

(5.5)

This decomposition follows the following pattern:

$$
SU(3)_{k-3} \times SO(8) \times SU(2)_{k-2} \times U(1)_3 \times U(1)_k \times SO(2)
$$

(5.6)

and we shall perform the deformation on the left lattice of $SO(2)$. However the deformation will also act on an appropriate sub-lattice of the right-moving gauge sector. The last $SO(2)$ factor corresponds to the fermions which are neutral in the process so they won’t be considered afterwards.

*According to our conventions, the weights of a $U(1)$ at level $k$ are $m^2/4k$, $m \in \mathbb{Z}_{2k}$.
5.1.2 The gauge sector

To construct the model we assume that the gauge sector of the heterotic strings contain an unbroken $SO(6)_1$, whose contribution to the partition function is, written in terms of $SO(6)_1$ free fermionic characters $\Xi_6$, see App. D. Since we decompose the characters of the left-moving sector according to eq. (5.6), a natural choice for the action of the deformation in the right-moving gauge sector is to use a similar Kazama-Suzuki decomposition, but for $k = 3$, in which case the bosonic CFT is trivial:

$$SO(8)_1 \rightarrow \frac{SO(4)_1}{SU(2)_1 \times U(1)_9} \times \frac{SU(2)_1 \times SO(2)_1}{U(1)_3} \times U(1)_1 \times SO(2)_1$$  \hspace{1cm} (5.7)

Since as quoted previously two fermions – the $SO(2)_1$ factor – are neutral it is enough that the gauge sector contains an $SO(6)_1$ subgroup. To achieve this decomposition, first we decompose the $SO(6)_1$ characters according to $SO(4)_1$:

$$\Xi_{\bar{s}} = \sum_{\bar{s}_4, \bar{s}_2 \in Z_4} C_{\bar{s}_6; \bar{s}_4, \bar{s}_2} \Xi_{\bar{s}_4} \Xi_{\bar{s}_2}$$  \hspace{1cm} (5.8)

where the coefficients of the decomposition $SO(6) \rightarrow SO(4) \times SO(2)$ are either zero or one. And then we perform a coset decomposition for the $SO(4)_1$ characters:

$$\Xi_{\bar{s}_4} = \sum_{\ell = 0}^{1} \sum_{u \in Z_{18}} \Xi_{\ell u} \chi_{\ell} \Theta_{u,9} \frac{\eta}{\bar{\eta}}$$  \hspace{1cm} (5.9)

in terms of $SU(2)_1$ characters $\chi^{\ell}$ and $U(1)$ characters $\Theta_{u,9}$. It defines implicitly the coset characters $\Xi_{\ell u}$. Then the $SU(2)_1 \times SO(2)_1$ characters are decomposed as:

$$\chi^{\ell} \Xi_{\bar{s}_2} = \sum_{v \in Z_6} \Xi_{v} \Xi_{\bar{s}_2} \Theta_{v,3} \frac{\eta}{\bar{\eta}}$$  \hspace{1cm} (5.10)

So putting together these branching relations we have the following Kazama-Suzuki decomposition for the free fermions of the gauge sector:

$$\Xi_{\bar{s}_6} = \sum_{\bar{s}_4, \bar{s}_2 \in Z_4} \sum_{\ell = 0, 1} \sum_{u \in Z_{18}} \sum_{v \in Z_6} C_{\bar{s}_6; \bar{s}_4, \bar{s}_2} \Xi_{\bar{s}_4} \Xi_{\bar{s}_2} \Xi_{\ell u} \Xi_{v} \Theta_{u,9} \Theta_{v,3} \frac{\eta}{\bar{\eta}} \frac{\bar{\eta}}{\eta}.$$

5.1.3 The deformation

Now we are in position to perform the asymmetric deformation adding a magnetic field to the model. The deformation acts on the following combination of left and right theta functions:

$$\Theta_{n,3k} \bar{\Theta}_{u,9} \times \Theta_{m,k} \bar{\Theta}_{v,3}.$$

As for the case of $SU(2)$ [5], we have to assume that the level obeys the condition:

$$\sqrt{\frac{k}{3}} = p \in \mathbb{N}.$$  \hspace{1cm} (5.13)
to be able to reach the geometric coset point in the moduli space of CFT. Then we have to perform $O(2, 2, \mathbb{R})$ boosts in the lattices of the $U(1)$’s, mixing the left Cartan lattice of the super-WZW model with the right lattice of the gauge sector. These boosts are parameterized in function of the magnetic fields as:

$$\cosh \Omega_a = \frac{1}{1 - 2\eta_a^2}, \quad a = 1, 2$$  \hspace{1cm} (5.14)

Explicitly we have:

$$\sum_{N_1, N_2 \in \mathbb{Z}} \sum_{f_1, f_2 \in \mathbb{Z}} q^{3k(N_1 + \frac{m}{18})^2} q^{k(N_2 + \frac{N_2}{2k})^2} \times \sum_{f_1, f_2 \in \mathbb{Z}} q^{3(f_1 + \frac{m}{18})^2} q^{3(f_2 + \frac{m}{18})^2}$$

$$\to \sum_{N_1, N_2, f_1, f_2 \in \mathbb{Z}} q^{3[\frac{1}{N_1 + \frac{m}{18}}] \cosh \Omega_1 + (f_1 + \frac{m}{18}) \sinh \Omega_1]^{2}} q^{3\left[[N_2 + \frac{m}{18}] \cosh \Omega_2 + (f_2 + \frac{m}{18}) \sinh \Omega_2\right]}$$

$$\times q^{3\left[(f_1 + \frac{m}{18}) \cosh \Omega_1 + (N_1 + \frac{m}{18}) \sinh \Omega_1\right]^{2}} q^{3\left[(f_2 + \frac{m}{18}) \cosh \Omega_2 + (N_2 + \frac{m}{18}) \sinh \Omega_2\right]^{2}}$$  \hspace{1cm} (5.15)

After an infinite deformation, we get the following constraints on the charges:

$$m = p(18\mu - u), \quad \mu \in \mathbb{Z}_p$$  \hspace{1cm} (5.16a)

$$n = p(6\nu - v), \quad \nu \in \mathbb{Z}_p$$  \hspace{1cm} (5.16b)

and the $U(1)^2$ CFT that has been deformed marginally decouples from the rest and can be safely removed. In conclusion, the infinite deformation gives:

$$Z_{F_3}^{(s_4, s_2; s_6)}(\tau) = \sum_{\Lambda} \sum_{k} \sum_{\mu, \nu \in \mathbb{Z}_p} \sum_{s_4, s_2 \in \mathbb{Z}_4} C[\bar{s}_6; \bar{s}_4, \bar{s}_2]$$

$$\sum_{\ell=0,1} \sum_{u \in \mathbb{Z}_{18}} \sum_{v \in \mathbb{Z}_6} \tilde{C}_j^{(s_4)} \tilde{C}_{p(18\mu - u)}^{(s_2)} \times \tilde{\chi}_{k-3}^{\Lambda} \tilde{\omega}^{\bar{s}_4; \ell u} \tilde{\omega}^{s_2; \ell v}$$  \hspace{1cm} (5.17)

where the sum over $\Lambda, j$ runs over integrable representations, see appendix \[21\]. This is the partition function for the $SU(3)/U(1)^2$ coset space. The fermionic charges in the left and right sectors are summed according to the standard rules of Gepner heterotic constructions \[21\]. The modular properties of this partition function are the same as before the deformation, concerning the $\mathbb{Z}_4$ indices of the worldsheet fermions.

### 5.1.4 Alternative approach: direct abelian coset

Here we would like to take a different approach, by deforming directly the Cartan lattice of $\mathfrak{su}_3$ without decomposing the left CFT in terms of $kS N = 2$ theories. As explained in the App. \[21\], it is possible to perform a generalized (super)parafermionic decomposition of the characters of the $\mathfrak{su}_3$ super-algebra at level $k$ (containing a bosonic algebra at level $k - 3$) w.r.t. the Cartan torus:

$$\chi^\Lambda \left( \frac{a}{b} \right) = \sum_{\lambda \in \mathbb{M}^{* \text{mod } k\mathbb{M}}} \mathcal{P}_\lambda \left[ \frac{a}{b} \right] \Theta_{\lambda, k}$$  \hspace{1cm} (5.18)
where the theta function of the \( \hat{su}_3 \) affine algebra reads, for a generic weight \( \lambda = m_i \lambda_i \) (see app. D):

\[
\Theta_{\lambda,k} = \sum_{\gamma \in \mathbb{M}^+} q^{\frac{k}{2} \kappa(\gamma, \gamma)} = \sum_{N^1, N^2 \in \mathbb{Z}} q^{\frac{k}{2} ||N^1 \alpha_1 + N^2 \alpha_2 + \frac{m_1 \lambda_1 + m_2 \lambda_2}{k}||^2} \tag{5.19}
\]

To obtain an anomaly-free model it is natural to associate this model with an abelian coset decomposition of an \( SU(3)_1 \) current algebra made with free fermions of the gauge sector. Thus if the gauge group contains an \( SU(3)_1 \) unbroken factor their characters can be decomposed as:

\[
\bar{\chi}_{\bar{A}} = \sum_{\bar{\lambda} = \bar{\bar{n}}_i \lambda_i} \bar{\varphi}_{\bar{A}} \Theta_{\bar{\lambda}}. \tag{5.20}
\]

Again we will perform the asymmetric deformation as a boost between the Cartan lattices of the left \( \hat{su}_3 \) algebra at level \( k \) and the right \( \hat{su}_3 \) lattice algebra at level one coming from the gauge sector. So after the infinite deformation we will get the quantization condition \( \sqrt{k} = p \) and the constraint:

\[
\lambda + p \bar{\lambda} = 0 \mod p \mathbb{M} =: p \mathbb{M}, \quad \mu \in \mathbb{M}. \tag{5.21}
\]

So we get a different result compared to the Kazama-Suzuki construction. It is so because the constraints that we get at the critical point force the weight lattice of the \( \hat{su}_3 \) at level \( k \) to be projected onto \( p \) times the \( \hat{su}_3 \) weight lattice at level one of the fermions. This model does not correspond to a Kählerian manifold and should correspond to the \( SU(3) \)-invariant metric on the flag space. Indeed with the KS method we get instead a projection onto \( p \) times a lattice of \( \hat{su}_3 \) at level one which is dual to the orthogonal sublattice defined by \( \alpha_1 \mathbb{Z} + (\alpha_1 + 2 \alpha_2) \mathbb{Z} \) – in other words the lattice obtained with the Gell-Mann Cartan generators. In this case it is possible to decompose the model in KS cosets models with \( N = 2 \) superconformal symmetry.

We have seen in section 4 that, in the gauging approach, one obtains the Kähler metric automatically when one starts from the \( SU(2) \) fibration over \( SU(3)/SU(2) \) rather than from the wzw model \( SU(3) \). It is now very easy to understand why it is the case. Indeed once the \( SU(2) \) has been taken out of the \( SU(3) \), the only \( U(1) \) that can be gauged (or deformed) is the \( U(1) \) orthogonal to the root \( \alpha_1 \) of the \( SU(2) \) subalgebra, thus must be along the \( \alpha_1 + 2 \alpha_2 \) vector. This will allow automatically to decompose the abelian coset into KS Hermitean symmetric spaces, and the model corresponds to the Kählerian metric on the flag space. However, at the level of the effective action, the deformation method of section 3.2 is not sensitive to these two possible CFT realizations of the flag space.

### 5.2 Generalization

The previous construction can be easily generalized to any affine Lie algebra, but the formalism gets a little bit bulky. We will consider separately all the families of simple Lie algebras, since the construction differ significantly. We will mainly focus below on the KS decomposition method.

---

†For the symmetrically gauged wzw models, this has been studied in [31].
5.2.1 $A_n$ algebras

For an $SU(n + 1)$ wzw model we use the following decomposition in terms of $N = 2$ Kazama-Suzuki models:

\[
SU(n + 1)_{k-n-1} \times SO(n^2 + 2n)_1 \rightarrow \\
SU(n+1)_{k-n-1} \times SO(2n)_1 \times SU(n)_{k-n} \times SO(2(n-1))_1 \times \cdots \times SU(2k-2) \times SO(2)_1 \\
\times SO(1)_{n+1} \times U(1)_{n+1} \times U(1)_{n+1} \times \cdots \times U(1)_{n+1}
\]

(5.22)

So the left worldsheet fermions of $SO(n^2 + 2n)_1$ are decomposed into:

\[
SO(n^2 + 2n) \rightarrow SO(2n)_1 \times SO(2(n-1))_1 \times \cdots \times SO(2)_1 \times SO(n)_1
\]

(5.23)

where $n$ fermions are neutral. The Kazama-Suzuki decomposition of the characters reads:

\[
\chi^A \equiv \overline{\psi}_{2n} \overline{\psi}_{2(n-1)} \cdots \overline{\psi}_2 \overline{\psi}_1 = \sum_{\Lambda^1, \Lambda^2, \ldots, j} \sum_{m_1 \in \mathbb{Z}_{n+1}^k} \sum_{m_2 \in \mathbb{Z}_{n-1}^k} \cdots \sum_{m_n \in \mathbb{Z}_k} \Theta_{m_1, n+1} \frac{n+1}{\eta} \Theta_{m_2, n} \frac{n}{\eta} \cdots \Theta_{m_n, k} \frac{k}{\eta}
\]

(5.24)

where the sum on $\Lambda^1, \Lambda^2, \ldots, j$ is taken over integrable representations (see App. 3) of $SU(n), SU(n-1), \ldots, SU(2)$. For the right fermions of the gauge sector the story is the same as for the $SU(3)$ example. We will need $n(n+1)$ free fermions realizing an $SO(n^2 + n)_1$ algebra, in order to use the Kazama-Suzuki decomposition for the $A_n$ model at level $k = n + 1$, such that the bosonic part trivializes:

\[
SO(n^2 + n)_1 \rightarrow \\
\frac{SU(2n)_1}{SU(n+1)_{n+1} \times U(1)_{n+1}^2} \times SU(n+1)_{n+1} \times SU(2(n-1))_1 \times \cdots \times SU(2n-1) \times SO(2n)_1 \\
\times U(1)_{n+1} \times U(1)_{n+1} \times \cdots \times U(1)_{n+1}
\]

(5.25)

So we can write the decomposition in terms of coset characters as:

\[
\overline{\psi}_{n+1}^{\Lambda_1, \ldots, \Lambda_n, \eta} = \sum_{\overline{\psi}_{2n}, \overline{\psi}_{2(n-1)}, \ldots, \overline{\psi}_1} C \left[ \overline{\psi}_{n+1}; \overline{\psi}_{2n}, \overline{\psi}_{2(n-1)}, \ldots, \overline{\psi}_1 \right] \sum_{u_1 \in \mathbb{Z}_{n+1}^k} \sum_{u_2 \in \mathbb{Z}_{n-1}^k} \cdots \sum_{u_n \in \mathbb{Z}_{2n+1}^k} \Theta_{u_1, n+1} \frac{n+1}{\eta} \Theta_{u_2, n} \frac{n}{\eta} \cdots \Theta_{u_n, 2n+1} \frac{k}{\eta}
\]

(5.26)

For the left coset to exist one has to assume the following constraint on the level of the $A_n$ affine algebra:

\[
\sqrt{\frac{k}{n+1}} = p \in \mathbb{Z}
\]

(5.27)
Then the decomposition can be carried out straightforwardly, by mixing the lattices of the holomorphic theta function for the decomposition (5.24) and the decomposition (5.25). We get the following constraints:

\[
\begin{aligned}
    m_1 &= p \left[ n(n+1)^2 \mu_1 - u_1 \right], & \mu_1 &\in \mathbb{Z}_p \\
    m_2 &= p \left[ (n-1)n(n+1) \mu_2 - u_2 \right], & \mu_2 &\in \mathbb{Z}_p \\
    \vdots \\
    m_n &= p \left[ 2(n+1) \mu_n - u_n \right], & \mu_n &\in \mathbb{Z}_p \\
\end{aligned}
\]  

(5.28)

So at the end we can remove the \( U(1)^n \) free CFT contribution and we get the following “partition function” for the \( SU(n+1)/U(1)^n \) left coset, with \( N = 2 \) worldsheet superconformal symmetry:

\[
Z_{F_{n+1}}^{(s_{2n}, \cdots, s_{2}; \bar{s}_{n(n+1)})} (\tau) = \sum_{\Lambda} \sum_{\Lambda^1, \Lambda^2, \cdots, \Lambda^j} \sum_{m_1 \in \mathbb{Z}_{n(n+1)}} \sum_{m_2 \in \mathbb{Z}_{(n-1)n+1}} \cdots \sum_{m_n \in \mathbb{Z}_p} \sum_{\bar{s}_{2n}, \bar{s}_{2(n-1)} \cdots \bar{s}_{2} \in \mathbb{Z}_4} \sum_{\bar{s}_1 \in \mathbb{Z}_{n(n+1)^2}} \sum_{\bar{s}_2 \in \mathbb{Z}_{(n-1)n+1}} \cdots \sum_{\bar{s}_j \in \mathbb{Z}_{(n+1)n+1}} \sum_{\mu_1, \cdots, \mu_n \in \mathbb{Z}_p} \sum_{p \in \mathbb{Z}_{n(n+1)^2}} \sum_{\bar{p} \in \mathbb{Z}_{(n-1)n+1}} \cdots \sum_{\bar{p} \in \mathbb{Z}_{(n+1)n+1}} \sum_{\chi_{\Lambda}^\Lambda_{1, \cdots, \Lambda^{j}, s_{2n}, \cdots, s_{2}}} \chi_{\Lambda_1}^{\Lambda_{1}} \chi_{\Lambda_2}^{\Lambda_{2}} \cdots \chi_{\Lambda_{n}}^{\Lambda_{n}} (5.29)
\]

As in the previous example this characters combination behaves covariantly under modular transformation, i.e. is modular invariant up to the transformation of the fermionic indices \( \{ s_i \} \) and \( \bar{s}_{n(n+1)} \). The modular invariance of the complete heterotic string background will be ensured by an appropriate Gepner construction.

Now let us consider the other simple Lie algebras. For sake of brevity we will only sketch the method, which is quite parallel to the present case.

### 5.2.2 Bₙ algebras

In this case, the relevant Kazama-Suzuki \( N = 2 \) coset model is:

\[
SO(2n+1)_{k-2n+1} \times SO(4n-2)_1 \\
SO(2n-1)_{k-2n+3} \times U(1)_{2k}
\]  

(5.30)

therefore the decomposition in \( N = 2 \) models of the group manifold is:

\[
SO(2n+1)_{k-2n+1} \times SO(n(2n+1))_1 \to \\
\to \frac{SO(2n+1)_{k-2n+1} \times SO(4n-2)_1}{SO(2n-1)_{k-2n+3} \times U(1)_{2k}} \times \frac{SO(2n-1)_{k-2n+3} \times SO(4n-6)_1}{SO(2n-3)_{k-2n+5} \times U(1)_{2k}} \times \\
\times \cdots \times \frac{SO(3)_{k-1} \times SO(2)_1}{U(1)_{2k}} \times SO(n) \times (U(1)_{2k})^n
\]  

(5.31)

So there are no specific constraints on the right fermions of the gauge sector. We only need to pick up \( n \) complex fermions with arbitrary boundary conditions, realizing an \([SO(2)]^n\)
algebra. The level of the $SO(2n + 1)$ has to be quantized as $\sqrt{k} \in \mathbb{N}$. Under this condition the deformation can be carried out straightforwardly.

5.2.3 $C_n$ algebras

We consider here the $k\mathbb{S}$ cosets:

$$\frac{Sp(n)_{k - n - 1} \times SO(n(n + 1))_1}{SU(n)_{2k - n} \times U(1)_{nk}}$$

(5.32)

So apart from the first step the decomposition follows the pattern for $A_n$ algebras:

$$Sp(n)_{k - n - 1} \times SO(n(2n + 1))_1 \rightarrow \frac{Sp(n)_{k - n - 1} \times SO(n(n + 1))_1}{SU(n)_{2k - n} \times U(1)_{nk}} \times \frac{SU(n)_{2k - n} \times SO(2(n - 1))_1}{SU(n - 1)_{2k - n + 1} \times U(1)_{(n - 1)n_k}} \times \cdots \times \frac{SU(2)_{k - 2} \times SO(2)_1}{U(1)_k} \times \frac{SO(n)_1 \times U(1)_{nk})}{SU(n)_{2k - n} \times U(1)_{(n - 1)n_k} \times U(1)_{(n - 2)(n - 1)k} \times \cdots \times U(1)_{2k}}$$

(5.33)

Then one need in the gauge sector an $SO(2n^2)_1$ algebra that will be split according to the purely fermionic Kazama-Suzuki decomposition for $C_n$, together with the quantization condition

$$\sqrt{\frac{k}{n + 1}} \in \mathbb{N}$$

(5.34)

Then the deformation will lead to the flag space partition function.

5.2.4 $D_n$ algebras

We consider here the $k\mathbb{S}$ cosets:

$$\frac{SO(2n)_{k - 2n + 2} \times SO(n(n - 1))_1}{SU(n)_{k - n} \times U(1)_{2nk}}$$

(5.35)

This case is very close to the last one. We have the decomposition:

$$SO(2n)_{k - 2n + 2} \times SO(n(n - 1))_1 \rightarrow \frac{SO(2n)_{k - 2n + 2} \times SO(n(n - 1))_1}{SU(n)_{k - n} \times U(1)_{2nk}} \times \frac{SU(n)_{k - n} \times SO(2(n - 1))_1}{SU(n - 1)_{k - n + 1} \times U(1)_{(n - 1)n_k/2}} \times \cdots \times \frac{SU(2)_{k - 2} \times SO(2)_1}{U(1)_k} \times \frac{SO(n)_1 \times U(1)_{2nk} \times U(1)_{(n - 1)n_k/2} \times U(1)_{(n - 2)(n - 1)k/2} \times \cdots \times U(1)_k}{SU(n)_{k - n} \times U(1)_{(n - 1)n_k} \times U(1)_{(n - 2)(n - 1)k} \times \cdots \times U(1)_{2k}}$$

(5.36)

So the fermions of the gauge sector have to realize an $SO[2n(n - 1)]_1$ algebra, together with the quantization condition

$$\sqrt{\frac{k}{2n - 2}} \in \mathbb{N}$$

(5.37)

‡Of course this algebra may be enhanced in the specific model at hand but this is not necessary. Note also that there is another construction when ones starts with and $SO(2n^2)_1$ algebra in the gauge sector and decompose it in terms of the $B_n$ Kazama-Suzuki model at level $2n - 1$. 

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5.2.5 Exceptional algebras

The two exceptional algebras leading to \(N = 2\) theories – i.e. giving Hermitian symmetric coset spaces – are \(E_6\) and \(E_7\). In the first case, we have the decomposition:

\[
(E_6)_{k-12} \times SO(78)_1 \to (E_6)_{k-12} \times SO(32)_{\frac{1}{6k}} \\
\times SU(5)_{k+4} \times SU(1)_{10(k+8)} \times SU(2)_{k+9} \times SO(2)_{k+8} \times U(1)_{k+8} \\
\times SO(6)_{1} \times U(1)_{6k} \times U(1)_{10(k+3)} \times U(1)_{10(k+8)} \times \cdots \times U(1)_{k+8}
\]

In this case we need fermions in the gauge sector realizing an \(E_6\) algebra at level one, and will lead to the quantization condition \(\sqrt{k/12} \in \mathbb{N}\). In the second case, we have the decomposition:

\[
(E_7)_{k-18} \times SO(133)_1 \to (E_7)_{k-18} \times SO(54)_{\frac{1}{3k}} \\
\times SU(5)_{k+14} \times SU(1)_{10(k+8)} \times SU(2)_{k+9} \times SO(2)_{k+8} \times U(1)_{k+8} \\
\times SO(6)_{1} \times U(1)_{6k} \times U(1)_{10(k+3)} \times U(1)_{10(k+8)} \times \cdots \times U(1)_{k+8}
\]

and we see clearly that the conditions on the level we would get from the first \(U(1)\) at level 3\(k\) and the other ones are generically incompatible. Thus one cannot construct a flag space \(\text{cft}\) for \(E_7\) but only a coset by the maximal torus of the \(E_6\) embedded in \(E_7\).

5.3 Kazama-Suzuki decomposition vs. abelian quotient

In this section we would like to stress the ambiguity in defining an abelian coset of \(wzw\) models. We will consider the \(A_n\) case in the discussion, although it’s pretty much the same for the other classical Lie algebras.

An abelian super-coset \(G \times SO(#g-d)/U(1)^d\), (with \(\hat{g}\) at level \(k-g^*\)) must be supplemented with the definition of the action of the abelian subgroup in \(g\), corresponding to a choice of a particular sub-lattice of \(\Gamma \in \sqrt{k}\mathbb{M}\) (these issues have been discussed in \([30]\) for symmetric supercosets of type II superstrings). In our construction, the left-coset structure will require that, in order to achieve modular invariance, the lattice behaves covariantly as some combination of right-moving fermions of the gauge sector of the heterotic string. It will be possible only if the level of the \(\hat{g}\) affine algebra obeys a special quantization condition. In the \(KS\) construction we define with these right-moving fermions an orthogonal lattice; therefore we have also to choose an orthogonal sub-lattice of the root lattice for the \(wzw\) model in order to make this construction possible.

For the \(A_n\) algebra, the relevant orthogonal basis is written as follows:\(^5\)

\[
\begin{align*}
\nu_1 &= \sqrt{k} \alpha_1 \\
\nu_2 &= \sqrt{k} (\alpha_1 + 2 \alpha_2) \\
\nu_3 &= \sqrt{k} (\alpha_1 + 2 \alpha_2 + 3 \alpha_3) \\
\vdots \\
\nu_n &= \sqrt{k} (\alpha_1 + 2 \alpha_2 + \cdots + n \alpha_n)
\end{align*}
\]

\(^5\)In the case of \(A_2\), we find the Gell-Mann matrices of \(SU(3)\) \([3.10]\).
and is of course a sub-lattice of the complete root lattice. More precisely it corresponds to:

\[
\sqrt{k} \Gamma = \sqrt{k} \bigoplus_{a=1}^{n} a \mathbb{Z} \alpha_a \subset \sqrt{k} \bigoplus_{a=1}^{n} \mathbb{Z} \alpha_a \quad (5.41)
\]

Then the associated theta-functions of \( \hat{su}_{n+1} \) can be written as a product of usual \( \hat{su}_2 \) theta functions:

\[
\Theta^{(\Gamma)}_{\lambda,k} = \prod_{a=1}^{n} \Theta_{m_a, a(a+1)k/2} \quad \text{with} \quad \lambda = \sum m_a \nu_a^* . \quad (5.42)
\]

This choice of orthogonal basis allows actually to decompose the abelian coset into a chain of Kazama-Suzuki models, with enhanced \( N = 2 \) supersymmetry on the worldsheet. Indeed we have to choose the lattice of the \( U(1) \) in \( SU(n+1)/SU(n) \times U(1) \) to be \( \mathbb{Z} \nu_n \), such that it will be orthogonal to the root lattice of \( su_{n-1} \) given by \( \sum_{a=1}^{n-1} \mathbb{Z} \alpha_a \), thus allowing to gauge it.

The left coset corresponding to this choice of abelian subgroup is obtained by a marginal deformation with the operator \( \bigoplus_a (\nu_a, H) \). Its partition function is composed of the coset characters obtained through the branching relation:

\[
\chi^A \prod_{r=1}^{n^2+n} \Theta_{s_r, 2} = \sum_{\lambda = \lambda_a \nu_a^* \in \Gamma^* \mod k \Gamma} C^{A(s_1, \ldots, s_{n^2+n})}_\lambda \prod_{a=1}^{n} \Theta_{\lambda_a, a(a+1)k/2} . \quad (5.43)
\]

On the other hand, the standard \( N = 1 \) abelian coset construction is defined with a full \( \sqrt{k}M \) lattice. The left coset is obtained by a marginal deformation with the operator \( \bigoplus_a (\alpha_a, H) \). The relevant coset characters are given by:

\[
\chi^A \prod_{r=1}^{n^2+n} \Theta_{s_r, 2} = \sum_{\lambda = \epsilon \in M^* \mod k M} C^{A(s_1, \ldots, s_{n^2+n})}_\lambda \Theta_{\lambda,k} . \quad (5.44)
\]

As in the \( A_3 \) case, we can show that the left cosets corresponding to these two classes of models are different. They are in correspondence with the different possible metrics (Kählerian and non-Kählerian) on asymmetric cosets spaces discussed in appendix A.

6. New linear dilaton backgrounds of Heterotic strings

These left-coset superconformal field theories can be used to construct various supersymmetric exact string backgrounds. The first class are generalizations of Gepner models [21] and Kazama-Suzuki constructions [22] using the left cosets as building blocks for the internal SCFT. This has already been considered in [3] for the \( S^2 \) coset but can be extended using the new theories constructed above. In this case there is no geometric interpretation from the sigma model point of view since these theories have no semi-classical limit. Indeed the levels of the cosets are frozen because their central charge must add up to \( c = 9 \) (in the case of four-dimensional compactification). However we expect that they correspond to special points in the moduli spaces of supersymmetric compactifications, generalizing the Gepner points of the CY manifolds.
Another type of models are the left cosets analogues of the NS5-branes solutions \[9, 10\] and of their extensions to more generic supersymmetric vacua with a dilaton background. It was shown in \[13\] that a large class of these linear dilaton theories are dual to singular CY manifolds in the decoupling limit. An extensive review of the different possibilities in various dimensions has been given in \[30\] with all the possible $G/H$ cosets. The left cosets that we constructed allows to extend all these solutions to heterotic strings, with a different geometrical interpretation since our cosets differ from ordinary gauged wzw model. However the superconformal structure of the left sector of our models is exactly the same as for the corresponding gauged wzw – except that the values of the N=2 R-charges that appear in the spectrum are constrained – so we can carry over all the known constructions to the case of the geometric cosets.

In the generic case these constructions involve non-abelian cosets, and as we showed the asymmetric deformations and gaugings apply only to the abelian components. Thus in general we will get mixed models which are gauged wzw models w.r.t. the non-abelian part of $H$ and geometric cosets w.r.t. the abelian components of $H$. Below we will focus on purely abelian examples, i.e. corresponding to geometric cosets. The dual interpretation of these models, in terms of the decoupling limit of some singular compactification manifolds, is not known. Note however that by construction there are about $\sqrt{k}$ times less massless states in our models than in the standard left-right symmetric solutions. Therefore they may correspond to some compactifications with fluxes, for which the number of moduli is reduced. It would be very interesting to investigate this issue further.

**Six-dimensional model.** We consider here the critical superstring background:

\[
\mathbb{R}^{5,1} \times \frac{SL(2,\mathbb{R})_{k+2} \times SO(2)_1}{U(1)_k} \times \left[ U(1)_k \backslash SU(2)_{k-2} \times SO(2)_1 \right]
\]

the second factor being a left coset CFT as discussed in this paper. This is the direct analogue of the five-brane solution, or more precisely of the double scaling limit of NS5-branes on a circle \[12, 13\], in the present case with magnetic flux. This theory has $N = 2$ charges but, in order to achieve spacetime supersymmetry one must project onto odd-integral $N = 2$ charges on the left-moving side, as in the type II construction \[31\]. This can be done in the standard way by orbifolding the left $N = 2$ charges of the two cosets.

**Four-dimensional model.** A simple variation of the six-dimensional theory is given by

\[
\mathbb{R}^{3,1} \times \frac{SL(2,\mathbb{R})_{k/2+2} \times SO(2)_1}{U(1)_{2k}} \times \left[ U(1)_k \backslash SU(2)_{k-2} \times SO(2)_1 \right]
\]

\[\times \left[ U(1)_k \backslash SU(2)_{k-2} \times SO(2)_1 \right]
\]

which is the magnetic analogue of the (double scaling limit of) intersecting five-branes solution. Also here an orbifolding of the left $N = 2$ charges is needed to achieve space-time supersymmetry.
Three-dimensional models: the flagbrane. We can construct the following background of the $G_2$ holonomy type, as in the case of symmetric coset \cite{22}:

$$\mathbb{R}^{2,1} \times \mathbb{R}_Q \times \left[ U(1)_k \times U(1)_{3k} \setminus SU(3)_{k-3} \times SO(6)_1 \right]$$

and the non-trivial part of the metric is

$$ds^2 = -dt^2 + dx^2 + dy^2 + \sum_{k=1}^{\infty} \frac{k}{4r^2} \left[ dr^2 + 4r^2 ds^2(SU(3)/U(1))^2 \right].$$

Without the factor of four it would be a direct analogue of the NS5-brane, being conformal to a cone over the flag space.

Another possibility in three dimensions is to lift the $SL(2,\mathbb{R})/U(1)$ coset to the group manifold $SL(2,\mathbb{R})$. In this case, as for the standard gauged wzw construction \cite{33} we will get the following anti-de Sitter background:

$$SL(2,\mathbb{R})_{k/4+2} \times \left[ U(1)_{3k} \setminus SU(3)_{k-3} \times SO(6)_1 \right]$$

and the left moving sector of this worldsheet cft defines an $N = 3$ superconformal algebra in spacetime.

Two-dimensional model In this case we can construct the background:

$$\mathbb{R}^{1,1} \times \frac{SL(2,\mathbb{R})_{k/4+2} \times SO(2)_1}{U(1)_{4k}} \times \frac{U(1)_{3k} \setminus SU(3)_{k-3} \times SO(6)_1}{U(1)_k}$$

which corresponds in the classification of \cite{33} to a non-compact manifold of $SU(4)$ holonomy once the proper projection is done on the left $N = 2$ charges. This solution can be also be thought as conformal to a cone over the Einstein space $SU(3)/U(1)$. Using the same methods are for the NS5-branes in \cite{31}, we can show that the full solution corresponding to the model (6.6) can be obtained directly as the null super-coset:

$$\frac{SL(2,\mathbb{R})_{k/4} \times U(1) \setminus SU(3)_k}{U(1)_L \times U(1)_R}$$

where the action is along the elliptic generator in the $SL(2,\mathbb{R})$, with a normalization $\text{Tr}[(it^3)^2] = -4$, and along the direction $\alpha_1 + 2\alpha_2$ in the coset space $U(1)\setminus SU(3)$, with a canonical normalization. For $r \to \infty$ the solution asymptotes the cone but when $r \to 0$ the strong coupling region is smoothly capped by the cigar.

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A. Coset space geometry

Coset spaces have been extensively studied in the mathematical literature of the last fifty years. In this appendix we limit ourselves to collect some classical results mainly dealing with the geometric interpretation. In particular we will follow the notations of [34].

Let \( G \) be a semisimple Lie group and \( H \in G \) a subgroup. As in the rest of the paper, upper-case indices \( \{ m,n,o \} \) refer to the whole group (algebra) \( G \), lower-case indices \( \{ m,n,o \} \) to the subgroup (subalgebra) and Greek indices \( \{ \mu,\nu,\omega \} \) to the coset.

It is useful to explicitly write down the commutation relations, separating the generators of \( H \) and \( G/H \):

\[
[T_m, T_n] = f^o_{mn} T_o \\
[T_\mu, T_\nu] = f^o_{\mu\nu} T_o + f^\omega_{\mu\nu} T_\omega
\]

Of course there are no \( f^\omega_{mn} \) terms since \( H \) is a group. \( G/H \) is said to be symmetric if \( f^\omega_{\mu\nu} \equiv 0 \), i.e. if the commutator of any couple of coset elements lives in the dividing subgroup. In this case a classical theorem states that the coset only admits one left-invariant Riemann metric that is obtained as the restriction of the Cartan-Killing metric defined on \( G \) (see eg [19]). This is not the case when \( H \) is the maximal torus (except for the most simple case \( G = SU(2) \)) and the coset manifold accepts different structures.

Any metric (or, more generally, any degree-2 covariant tensor) on \( G/H \) can be put in the form

\[
g = g_{\mu\nu} (x) J^\mu \otimes J^\nu \tag{A.2}
\]

One can show that the \( G \) invariance of \( g \) is equivalent to:

\[
f^\kappa_{\alpha\mu} g_{\kappa\nu} (x) + f^\kappa_{\alpha\nu} g_{\kappa\mu} (x) = 0 \tag{A.3}
\]

and the homogeneity imposes

\[
g_{ij} = \text{constant} \tag{A.4}
\]

Both conditions are easily satisfied by \( g_{\mu\nu} \propto \delta_{\mu\nu} \) (this is the metric on \( G/H \) that we obtained in Eq. (2.27)). The Levi-Civita connection 1-forms \( \omega^\mu_\nu \) of \( g \) are determined by

\[
dg_{\mu\nu} - \omega^\kappa_\mu g_{\kappa\nu} - \omega^\kappa_\nu g_{\kappa\mu} = 0 \tag{A.5a}
\]

\[
dJ^\mu + \omega^\mu_\nu \wedge J^\nu = 0 \tag{A.5b}
\]

and are explicitly written in terms of the structure constants as:

\[
\omega^\mu_\nu = f^\mu_{\alpha\nu} J^\alpha + D^\mu_{\rho\nu} J^\rho \tag{A.6}
\]

where \( D^\mu_{\rho\nu} \) can be separated into its symmetric and antisymmetric parts as follows:

\[
D^\mu_{\rho\nu} = \frac{1}{2} f^\mu_{\rho\nu} + K^\mu_{\rho\nu} \tag{A.7a}
\]

\[
K^\mu_{\rho\nu} = \frac{1}{2} \left( g^{\mu\sigma} f^\omega_{\sigma\rho} g_{\omega\nu} + g^{\mu\sigma} f^\omega_{\sigma\nu} g_{\omega\rho} \right) \tag{A.7b}
\]
We can then derive the curvature 2-form $\Omega = d\omega + \omega \wedge \omega$:

$$\Omega^\mu{}_{\nu} = \left( D^\mu{}_{\sigma\nu} D^\nu{}_{\kappa\rho} - D^\rho{}_{\kappa\nu} D^\mu{}_{\sigma\rho} - f^\mu{}_{\kappa\rho} f^\rho{}_{\alpha\nu} - f^\rho{}_{\kappa\sigma} f^\mu{}_{\rho\nu}\right) \frac{J^\kappa \wedge J^\rho}{2}$$  \hfill (A.8)

the Riemann tensor

$$R^\mu{}_{\nu\kappa\rho} = -f^a{}_{\kappa\rho} f^\mu{}_{a\nu} - \frac{1}{2} f^\rho{}_{\kappa\rho} f^\mu{}_{\nu\rho} + \frac{1}{4} f^\rho{}_{\kappa\nu} f^\mu{}_{\rho\sigma} + \frac{1}{4} f^\rho{}_{\kappa\sigma} f^\mu{}_{\nu\rho} + \frac{1}{2} f^\rho{}_{\kappa\nu} K^\mu{}_{\sigma\rho} + \frac{1}{2} f^\rho{}_{\nu\sigma} K^\mu{}_{\kappa\rho} +$$

$$- \frac{1}{2} f^\mu{}_{\rho\kappa} K^\rho{}_{\sigma\nu} - \frac{1}{2} f^\mu{}_{\rho\sigma} K^\rho{}_{\kappa\nu} - f^\rho{}_{\kappa\sigma} K^\mu{}_{\rho\nu} + K^\rho{}_{\kappa\sigma} K^\mu{}_{\rho\nu} - K^\rho{}_{\kappa\nu} K^\mu{}_{\sigma\rho}$$ \hfill (A.9)

and the Ricci tensor:

$$Ric_{\nu\sigma} = R^\mu{}_{\nu\mu\sigma} = -f^a{}_{\mu\nu} f^\mu{}_{a\nu} - \frac{1}{2} f^\rho{}_{\mu\nu} f^\mu{}_{\rho\nu} + \frac{1}{4} f^\rho{}_{\mu\nu} f^\mu{}_{\rho\sigma} + \frac{1}{2} f^\rho{}_{\mu\nu} K^\mu{}_{\sigma\rho} + \frac{1}{2} f^\mu{}_{\rho\sigma} K^\rho{}_{\mu\nu} +$$

$$- \frac{1}{2} f^\rho{}_{\mu\sigma} K^\mu{}_{\rho\nu} - \frac{1}{2} K^\rho{}_{\mu\nu} K^\mu{}_{\sigma\rho}$$ \hfill (A.10)

In particular, in the case of $g_{\mu\nu} = \delta_{\mu\nu}$ the expressions are greatly simplified because the antisymmetric part $K^\mu{}_{\nu\rho}$ vanishes and then the Riemann and Ricci tensors are respectively given by:

$$R^\mu{}_{\nu\kappa\rho} = -f^a{}_{\kappa\rho} f^\mu{}_{a\nu} - \frac{1}{2} f^\rho{}_{\kappa\rho} f^\mu{}_{\nu\rho} + \frac{1}{4} f^\rho{}_{\kappa\nu} f^\mu{}_{\rho\sigma} + \frac{1}{4} f^\rho{}_{\kappa\sigma} f^\mu{}_{\nu\rho}$$ \hfill (A.11)

$$Ric_{\nu\sigma} = -f^a{}_{\mu\nu} f^\mu{}_{a\nu} - \frac{1}{4} f^\rho{}_{\mu\nu} f^\mu{}_{\rho\nu}$$ \hfill (A.12)

Another fact that we used in the paper about $G/H$ cosets is a construction due to Borel \cite{17,35} of a Kähler structure over $G/T$ where $T$ is the maximal torus. First of all we remark that such a coset can be given a $\mathbb{C}$ structure when associating holomorphic and anti-holomorphic sectors to positive and negative roots respectively. One can then show that the (1,1) form defined as:

$$\omega = \frac{1}{2} \sum_{\alpha > 0} c_{\alpha} J^\alpha \wedge J^{\bar{\alpha}}$$ \hfill (A.13)

is closed if and only if for each subset of roots $\{\alpha, \beta, \gamma\}$ such as $\alpha = \beta + \gamma$, the corresponding real coefficients $c_{\alpha}$ satisfy the condition $c_{\alpha} = c_{\beta} + c_{\gamma}$. Of course this is equivalent to say that the tensor

$$g = \sum_{\alpha > 0} c_{\alpha} J^\alpha \otimes J^{\bar{\alpha}}$$ \hfill (A.14)

is a Kähler metric on $G/T$.

In particular, if we consider the $SU(3)$ group, for the $su(3)$ algebra we can choose the Gell-Mann $\lambda$ matrices \cite{10} as a basis. In this case if we divide by the $U(1) \times U(1)$ subgroup generated by $\langle \lambda_3, \lambda_8 \rangle$, the most general metric satisfying \cite{3} has the form $g = \text{diag} \{a, a, b, b, c, c\}$ in $SU(3)/U(1) \times U(1)$ admits a three parameter family of metrics. Among them, the moduli space lines $a = b = c$ (the metric obtained in Sec. \cite{2} and $a = b = c/2$ (the metric in Sec. \cite{3}) represent Einstein structures (with Ricci scalar $15/a$ and $12/a$ respectively). In both cases the manifold can be endowed with complex structures (positive and negative roots respectively generating the holomorphic and anti-holomorphic sectors) but only the latter admits a Kähler structure (in this way we obtain the so-called flag space $F_3$).
B. Equations of motion

B.1 Explicit derivation of some terms

In this appendix we explicitly derive the expressions for the $F_{\mu\rho}^a F_{\nu}^\rho$ and $H_{\mu\rho\sigma} H_{\nu}^{\rho\sigma}$ terms appearing in the equations of motion (2.32b).

**Gauge field strength.** Consider the term coming from the gauge field strength. First of all we can build an orthonormal basis out of the Weyl-Cartan basis by complexifying the Cartan generators and combining opposite ladder operators as follows:

\[
\begin{align*}
T^a &= \mathcal{I} H^a \\
T^{2\mu-1} &= \frac{|\alpha_\mu|}{2} (E^{\alpha_\mu} - E^{-\alpha_\mu}) \\
T^{2\mu} &= \frac{|\alpha_\mu|}{2} (E^{\alpha_\mu} + E^{-\alpha_\mu})
\end{align*}
\]  
(B.1)

if we write explicitly the $(F^2)_{\mu\nu}$ term as follows:

\[
(F^2)_{\mu\nu} \propto \sum_{m,\omega} f^m_{\mu\omega} f^m_{\pi\omega} = \sum_{m,\omega} \kappa (T^m, [T^\nu, T^\omega]) \kappa (T^m, [T^\pi, T^\omega])
\]  
(B.2)

we can see why rewriting everything this choice of basis simplifies the calculation: the only commutators that will give a non-vanishing result when projected on the Cartan generators are the ones involving opposite ladder operators*, that is $[T^{2\mu-1}, T^{2\mu}]$ which are explicitly given by:

\[
[T^{2\mu-1}, T^{2\mu}] = \frac{|\alpha_\mu|^2}{4} 2 \left[ E^{\alpha_\mu}, E^{-\alpha_\mu} \right] = \alpha_\mu \cdot (\mathcal{I} H)
\]  
(B.4)

this means that:

\[
\begin{align*}
\kappa (T^m, [T^\nu, T^\omega]) &= \alpha_\mu |^m \delta_{\nu+1,\omega} & \text{if } \nu = 2\mu - 1 \\
\kappa (T^m, [T^\nu, T^\omega]) &= -\alpha_\mu |^m \delta_{\nu-1,\omega} & \text{if } \nu = 2\mu
\end{align*}
\]  
(B.5)

putting this back in Eq. (B.2) we find:

\[
\sum_{m,\omega} f^m_{\nu\omega} f^m_{\pi\omega} = \delta_{\nu\pi} \left\{ \begin{array}{ll}
|\alpha_{\nu+1/2}|^2 & \text{if } \nu \text{ is odd} \\
|\alpha_{\nu/2}|^2 & \text{if } \nu \text{ is even}
\end{array} \right.
\]  
(B.6)

if $\mathfrak{g}$ is simply laced then we can fix the normalizations to $|\alpha_\mu|^2 = \psi^2 \equiv 2$ and the above expression is greatly simplified:

\[
\sum_{m,\omega} f^m_{\nu\omega} f^m_{\pi\omega} = 2\delta_{\nu\pi}
\]  
(B.7)

*We remember that the Cartan-Weyl basis is defined by:

\[
\begin{align*}
[H^m, H^n] &= 0 & \text{(B.3a)} \\
[H^m, E^{\alpha_\mu}] &= \alpha_\mu |^m E^{\alpha_\mu} & \text{(B.3b)} \\
[E^{\alpha_\mu}, E^{\alpha_\nu}] &= \begin{cases} 
N_{\mu,\nu} E^{\alpha_\mu + \alpha_\nu} & \text{if } \alpha_\mu + \alpha_\nu \in \Delta \\
\frac{2}{|\alpha_\mu|} \alpha_\mu \cdot H & \text{if } \alpha_\mu = -\alpha_\nu \\
0 & \text{otherwise}
\end{cases} & \text{(B.3c)}
\end{align*}
\]
and by applying the right normalizations (see Eq. (2.33)) we find that for a general algebra:

\[ F_{m \nu \omega} \omega^{\nu} F_{m \pi \omega} = \frac{4}{k g} \delta_{\nu \pi} \begin{cases} \left| \alpha_{(\nu + 1)/2} \right|^2 & \text{if } \nu \text{ is odd} \\ \left| \alpha_{\nu/2} \right|^2 & \text{if } \nu \text{ is even} \end{cases} \]  

(B.8)

and for a simply laced one:

\[ F_{m \nu \omega} \omega^{\nu} F_{m \pi \omega} = \frac{8}{k g} \delta_{\nu \pi} \]  

(B.9)

**NS-NS flux.** From the definition of Casimir of the algebra we easily derive that:

\[ Q = - \sum_{m} \sum_{\omega, \pi} f_{m \nu \omega}^{M} f_{m \nu \pi}^{M} = 2 g^{*} \delta_{\nu \pi} \]  

(B.10)

where \( g^{*} \) is the dual Coxeter number. Limit \( N \) and \( P \) to \( j \) (and call them \( \nu \) and \( \pi \)) and separate the two sums (that span over the entire algebra) into the components over \( j \) and \( k \):

\[ \sum_{m \in k} \left( \sum_{\omega \in j} f_{m \nu \omega}^{m} f_{m \nu \omega}^{m} + \sum_{\omega \in j} f_{m \nu \omega}^{m} f_{m \nu \omega}^{m} \right) + \sum_{\mu \in j} \left( \sum_{\omega \in j} f_{m \nu \omega}^{m} f_{m \nu \omega}^{m} + \sum_{\omega \in j} f_{m \nu \omega}^{m} f_{m \nu \omega}^{m} \right) = -2 g^{*} \delta_{\nu \pi} \]  

(B.11)

now,

- the term with two elements in the Cartan is identically vanishing \( f_{m \nu \omega}^{m} \equiv 0 \) (for two generators in \( k \) always commute)

- the terms with one component in \( k \) can be collected and interpreted as field strengths:

\[ \sum_{m, \omega} f_{m \nu \omega}^{m} f_{m \nu \omega}^{m} + \sum_{\omega, \pi} f_{\nu \omega \pi}^{\mu} f_{\nu \omega \pi}^{\mu} \]  

(B.12)

and at the end of the day

\[ \sum_{\mu, \omega} f_{\nu \mu \omega}^{\mu} f_{\pi \mu \omega}^{\pi} = 2 g^{*} \delta_{\nu \pi} - 2 \sum_{m, \omega} f_{m \nu \omega}^{m} f_{m \nu \omega}^{m} \]  

(B.13)

so that for a general algebra, using (B.6):

\[ \sum_{\mu, \omega} f_{\nu \mu \omega}^{\mu} f_{\pi \mu \omega}^{\pi} = 2 g^{*} \delta_{\nu \pi} - 2 \delta_{\nu \pi} \begin{cases} \left| \alpha_{(\nu + 1)/2} \right|^2 & \text{if } \nu \text{ is odd} \\ \left| \alpha_{\nu/2} \right|^2 & \text{if } \nu \text{ is even} \end{cases} \]  

(B.14)

that reduces in the simply laced case to:

\[ \sum_{\mu, \omega} f_{\nu \mu \omega}^{\mu} f_{\pi \mu \omega}^{\pi} = 2 (g^{*} - 2) \delta_{\nu \pi} \]  

(B.15)

and with the proper normalizations:

\[ H_{\nu \mu \omega} \omega^{\nu} g_{\nu \pi \omega}^{\mu} H_{\nu \pi \omega}^{\mu} = 2 g^{*} \delta_{\nu \pi} - 2 \delta_{\nu \pi} \begin{cases} \left| \alpha_{(\nu + 1)/2} \right|^2 & \text{if } \nu \text{ is odd} \\ \left| \alpha_{\nu/2} \right|^2 & \text{if } \nu \text{ is even} \end{cases} \]  

(B.16)

which reads in the simply laced case:

\[ H_{\nu \mu \omega} \omega^{\nu} g_{\nu \pi \omega}^{\mu} H_{\nu \pi \omega}^{\mu} = 2 (g^{*} - 2) \delta_{\nu \pi} \]  

(B.17)
B.2 Equations of motion for the $F_3$ flag space

To verify that the background fields that we obtained in Sec. 4.2 solve the equations of motion at first order in $\alpha'$ it is convenient to consider the complex structure defined on the $SU(3)/U(1)^2$ coset by considering positive and negative roots as holomorphic and anti-holomorphic generators respectively.

To fix the notation let the two simple roots be:

$$\alpha_1 = [\sqrt{2}, 0] \quad \alpha_2 = [-1/\sqrt{2}, \sqrt{3/2}] \quad (B.18)$$

and the third positive root $\alpha_3 = \alpha_1 + \alpha_2 = [1/\sqrt{2}, \sqrt{3/2}]$. We already know that in the complex formalism the metric is diagonal and the coefficient relative to the non-simple root is given by the sum of the two others as in Eq. (A.14). With the right normalization we have the following metric and Ricci tensor:

$$g_{\mu\bar{\nu}} = k^2 \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad R_{\mu\bar{\nu}} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} \quad (B.19)$$

To write the structure constants we just have to remember the defining relations for the Cartan–Weyl basis Eq. (B.3): it is immediate to see that $f^1_{\mu\bar{\nu}}$ and $f^2_{\mu\bar{\nu}}$ are non-vanishing only if $\alpha_\mu$ and $\alpha_\nu$ are opposite roots (which means in turn that in our complex formalism they are represented by diagonal matrices) and, given the above choice of roots, we have:

$$f^1_{\mu\bar{\nu}} = \begin{pmatrix} \sqrt{2} \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad f^2_{\mu\bar{\nu}} = \begin{pmatrix} 0 \\ \sqrt{3/2} \\ \sqrt{3/2} \end{pmatrix} \quad (B.20)$$

Let us now introduce a new tensor $C$ that in this basis assumes the form of the unit matrix (this is indeed shown to be a tensor in App. A):

$$C_{\mu\bar{\nu}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (B.21)$$

we can use this tensor to define the $U(1)^2$ gauge field that supports the $F_3$ background as$^1$:

$$F^a_{\mu\bar{\nu}} = \sqrt{\frac{k}{2k_g}} f^a_{\mu\bar{\rho}} C^{\bar{\rho}\sigma} R_{\sigma\bar{\nu}} \quad (B.22)$$

The only non-trivial equation of motion is $\beta^G = 0$ (2.32b):

$$\beta^G = R_{\mu\bar{\nu}} - \frac{k_g}{4} F^a_{\mu\bar{\sigma}} g^{\bar{\sigma}\rho} F^a_{\rho\bar{\nu}} \quad (B.23)$$

$^1$One can read this additional term with respect to the expression in Eq. (2.28) as a way to keep track of the fact that the embedded $SU(2)$ subalgebra is at a different level with respect to the remaining currents. Actually this expression can be seen just as a generalisation of the initial one where we were restricting to cosets in which the currents played the rôle of vielbeins, i.e. in this formalism the metric was proportional to the unit matrix.
in our basis all the tensors are diagonal matrices. For this reason it is useful to pass to
matrix notation. Let
\[ G = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \] (B.24)
so that the metric and the Ricci tensor are given by \( g = \frac{k}{2} G \) and \( R = 2G \). In this notation
the above equation reads:
\[ \beta^G = R - \frac{k^2}{4} \sum_{a=1}^{2} \sqrt{\frac{k}{2k_g}} f^a R g^{-1} \sqrt{\frac{k}{2k_g}} f^a R = R - \frac{k}{8} \sum_{a=1}^{2} f^a (2G) \left( \frac{2}{k} G^{-1} \right) f^a R = \\
= R - \frac{1}{2} \sum_{a=1}^{2} f^a f^a R = 0 \] (B.25)
since \( \sum_{a=1}^{2} f^a f^a = 2I_{3 \times 3} \) as one can see by direct inspection.

C. The \( SU(3) \) group: an explicit parametrization

In this section we summarize some known facts about the representation of the \( SU(3) \) group so to get a consistent set of conventions.

To obtain the the Cartan-Weyl basis \( \{ H_a, E^{\alpha_j} \} \) (defined in Eq. (B.3)) for the \( \mathfrak{su}(3) \) algebra we need to choose the positive roots as follows:
\[ \alpha_1 = \left[ \sqrt{2}, 0 \right] \quad \alpha_2 = \left[ -\frac{1}{\sqrt{3}}, \sqrt{3/2} \right] \quad \alpha_3 = \left[ \frac{1}{\sqrt{3}}, \sqrt{3/2} \right] \] (C.1)

\[ \begin{diagram}
\alpha_3
\alpha_2 = \alpha_1 + \alpha_3
\alpha_1
\end{diagram} \]

**Figure 1:** Root system for \( \mathfrak{su}(3) \).

The usual choice for the defining representation is:
\[ H_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad H_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad E_1^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_2^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad E_3^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \] (C.2)

and \( E_j^- = \left( E_j^+ \right)^t \).
A good parametrisation for the $SU(3)$ group can be obtained via the Gauss decomposition: every matrix $g \in SU(3)$ is written as the product:

$$g = b_- d b_+$$  \hspace{1cm} (C.3)

where $b_-$ is a lower triangular matrix with unit diagonal elements, $b_+$ is a upper triangular matrix with unit diagonal elements and $d$ is a diagonal matrix with unit determinant. The element $g$ is written as:

$$g (z_1, z_2, z_3, \psi_1, \psi_2) = \exp \left( z_1 E_1^- + z_2 E_3^- + \left( z_3 - \frac{z_1 z_2}{2} \right) E_2^- \right) \exp (-F_1 H_1 - F_2 H_2)$$

$$\exp \left( \bar{w}_1 E_1^+ + \bar{w}_2 E_3^+ + \left( \bar{w}_3 - \frac{\bar{w}_1 \bar{w}_2}{2} \right) E_2^+ \right) \exp (i \psi_1 H_1 + i \psi_2 H_2)$$  \hspace{1cm} (C.4)

where $z_\mu$ are 3 complex parameters, $\psi_i$ are two real and $F_1$ and $F_2$ are positive real functions of the $z_\mu$'s:

$$\begin{align*}
F_1 & = \log f_1 = \log \left( 1 + |z_1|^2 + |z_3|^2 \right) \\
F_2 & = \log f_2 = \log \left( 1 + |z_2|^2 + |z_3 - z_1 z_2|^2 \right)
\end{align*}$$  \hspace{1cm} (C.5)

By imposing $g (z_\mu, \psi_\mu)$ to be unitary we find that the $w_\mu$'s are complex functions of the $z_\mu$'s:

$$\begin{align*}
w_1 & = -\frac{z_1 \bar{z}_2}{\sqrt{f_1}} \\
w_2 & = \frac{z_1 z_2 (1 + |z_1|^2)}{\sqrt{f_1}} \\
w_3 & = - (z_3 - z_1 z_2) \sqrt{\frac{f_2}{f_1}}
\end{align*}$$  \hspace{1cm} (C.6)

and the defining element $g (z_\mu, \psi_\mu)$ can then be written explicitly as:

$$g (z_1, z_2, z_3, \psi_1, \psi_2) = \begin{pmatrix}
1 & 0 & 0 \\
z_1 & 1 & 0 \\
z_3 & z_2 & 1
\end{pmatrix} \begin{pmatrix}
\frac{1}{\sqrt{f_1}} & 0 & 0 \\
0 & \sqrt{f_1/f_2} & 0 \\
0 & 0 & \sqrt{f_2/f_1}
\end{pmatrix} \begin{pmatrix}
1 & \bar{w}_1 & \bar{w}_3 \\
0 & 1 & \bar{w}_2 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
e^{i \psi_1/2} & 0 & 0 \\
0 & e^{-i (\psi_1 - \psi_2)/2} & 0 \\
0 & 0 & e^{i \psi_2/2}
\end{pmatrix}$$  \hspace{1cm} (C.7)

Now, to build a metric for the tangent space to $SU(3)$ we can define the 1-form $\Omega (z, \psi) = g^{-1} (z, \psi) \ d g (z, \psi)$ and write the Killing-Cartan metric tensor as $g_{KC} = \text{tr} (\Omega^\dagger \Omega) = - \text{tr} (\Omega \Omega)$ where we have used explicitly the property of anti-Hermiticity of $\Omega$ (that lives in the $\mathfrak{su}(3)$ algebra). The explicit calculation is lengthy but straightforward. The main advantage of this parametrisation from our point of view is that it allows for a “natural” embedding of the $SU(3)/U(1)^2$ coset (see e.g. \cite{36} or \cite{37}): in fact in these coordinates the Kähler potential is

$$K (z_\mu, \bar{z}_\mu) = \log (f_1 (z_\mu) f_2 (z_\mu)) = \log \left[ \left( 1 + |z_1|^2 + |z_3|^2 \right) \left( 1 + |z_2|^2 + |z_3 - z_1 z_2|^2 \right) \right]$$  \hspace{1cm} (C.8)

and the coset Kähler metric is hence simply obtained as:

$$g_{\alpha\beta} \ dz^\alpha \otimes \ d\bar{z}^\beta = \frac{\partial^2}{\partial z_{\alpha} \partial \bar{z}_{\beta}} K (z_\mu, \bar{z}_\mu) \ dz^\alpha \otimes \ d\bar{z}^\beta$$  \hspace{1cm} (C.9)
Another commonly used $\mathfrak{su}(3)$ basis is given by the Gell-Mann matrices:

$$
\begin{align*}
\gamma_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\gamma_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\gamma_3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\gamma_4 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\
\gamma_5 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\gamma_6 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, \\
\gamma_7 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}, \\
\gamma_8 &= \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}.
\end{align*}
$$

which presents the advantage of being orthonormal $\kappa(\lambda_i, \lambda_j) = \delta_{ij}$. In this case the Cartan subalgebra is generated by $\mathfrak{k} = \langle \lambda_3, \lambda_8 \rangle$.

D. Characters of affine Lie algebras

In this section we will recall some facts about the partition functions and characters of affine Lie algebras. The characters of an affine Lie algebra $\hat{\mathfrak{g}}$ are the generating functions of the weights multiplicities in a given irreducible representation of highest weight $\Lambda$:

$$
\chi_\Lambda(\tau, \nu, u) = e^{-2\pi k u} \sum_{\hat{\lambda} \in \text{Rep}(\Lambda)} \text{dim} V_\hat{\lambda} \exp \left\{ 2\pi \tau \gamma + \sum_i \nu_i \kappa \left( e_i, \hat{\lambda} \right) \right\}, \quad (D.1)
$$

where $\text{dim} V_\hat{\lambda}$ is the multiplicity of the affine weight $\hat{\lambda} = (\lambda, k, n)$ and $\{ e_i \}$ an orthonormal basis of the root space. In the framework CFT we define slightly different characters, weighted by the conformal dimension of the highest weight of the representation:

$$
\chi^A(\tau, \nu, u) = e^{-2\pi k u} \text{Tr}_{\text{rep}(\Lambda)} \left[ q^{L_0 - \gamma/24} e^{2\pi \kappa(\nu, J)} \right] = e^{2\pi \frac{2(\Lambda, \Lambda + 2\rho)}{2(\kappa + g^*)}} \chi_\Lambda(\tau, \nu, u). \quad (D.2)
$$

where $\rho = \sum_{\alpha > 0} \alpha / 2$ and $g^*$ the dual Coxeter number. To each affine weight $\hat{\lambda}$ we shall assign a theta-function as follows:

$$
\Theta_\hat{\lambda}(\tau, \nu, u) = e^{-2\pi k u} \sum_{\gamma \in M_L + \frac{\hat{\lambda}}{k}} e^{i \pi \kappa(\gamma, \gamma)} e^{2\pi \kappa(\nu, \gamma)} \quad (D.3)
$$

with $M_L$ the the long roots lattice. We can write the affine characters in terms of the theta-function with the Weyl-Kač formula:

$$
\chi^A(\tau, \nu, u) = \frac{\sum_{\lambda \in \mathbb{W}} \epsilon(\omega) \Theta_{\omega(\lambda + \rho)}(\tau, \nu, u)}{\sum_{\omega \in \mathbb{W}} \epsilon(\omega) \Theta_{\omega(\rho)}(\tau, \nu, u)}, \quad (D.4)
$$

$\mathbb{W}$ being the Weyl group of the algebra and $\epsilon(\omega)$ the parity of the element $\omega$.

These affine characters are the building blocks of the modular invariant partition function for the wzw model, since the affine Lie algebra is the largest chiral symmetry of the theory:

$$
Z = \sum_{\Lambda, \bar{\Lambda}} M^{AA} \chi^A(\tau, 0, 0) \bar{\chi}^A(\bar{\tau}, 0, 0) \quad (D.5)
$$
where the sum runs over left and right representations of $\mathfrak{g}$ with highest weight $\Lambda$ and $\bar{\Lambda}$. The representations appearing in this partition function are the integrable ones, which are such that:

$$\text{Rep}(\Lambda) \text{ integrable } \iff \frac{2}{\kappa(\theta, \theta)} [k - \kappa(\Lambda, \theta)] \in \mathbb{N},$$

(D.6)

where $\theta$ is the highest root. The matrix $M^{\Lambda\bar{\Lambda}}$ is such that the partition function of $\mathfrak{g}_k$ is modular invariant; at least, the diagonal $\delta_{\Lambda\bar{\Lambda}}$ exists since the characters form an unitary representation of the modular group.

In the heterotic strings, the worldsheet has a local $N = (1,0)$ local supersymmetry so the left algebra is lifted to a super-affine Lie algebra. However the characters can be decoupled as characters of the bosonic algebra times characters of free fermions:

$$Z_{[a \ b]} = \sum_{\Lambda, \bar{\Lambda}} M^{\Lambda\bar{\Lambda}} \chi^{\Lambda}(\tau) \left( \frac{\vartheta_{[a \ b]}(\tau)}{\eta(\tau)} \right)^{\dim(g)/2} \bar{\chi}^{\bar{\Lambda}}$$

(D.7)

where $(a, b)$ are the spin structures of the worldsheet fermions.

The characters of the affine algebras can be decomposed according to the generalized parafermionic decomposition, by factorizing the abelian subalgebra of the Cartan torus. For example, we can decompose the left supersymmetric $\mathfrak{g}_k$ characters in terms of characters of the supersymmetric coset, given by the following branching relation (see [22]):

$$\chi^\Lambda \left( \frac{\vartheta_{[a \ b]}(\tau)}{\eta(\tau)} \right)^{\dim(\tilde{\mathfrak{g}})/2} = \sum_{\lambda \mod (k+g^*)\mathbb{M}_k} \mathcal{P}_{\lambda}^\Lambda \left[ a \ b \right] \Theta_{\lambda, k+g^*}^{\Lambda} \eta^{\dim(\tilde{\mathfrak{g}})}$$

(D.8)

in terms of the theta-functions associated to $\mathfrak{g}_k$.

D.1 The example of $SU(3)$

In an orthonormal basis, the simple roots of $SU(3)$ are:

$$\alpha_1 = \left( \sqrt{2}, 0 \right), \quad \alpha_2 = \left( -1/\sqrt{2}, 1/\sqrt{3} \right).$$

(D.9)

The dual basis of the fundamental weights, defined by $\left( \lambda^1_f, \alpha_j \right) = \delta^i_j$ is given by:

$$\lambda^1_f = \left( 1/\sqrt{2}, 1/\sqrt{3} \right), \quad \lambda^2_f = \left( 0, \sqrt{2}/\sqrt{3} \right).$$

(D.10)

As they should the simple roots belong to the weight lattice:

$$\alpha_1 = 2\lambda^1_f - \lambda^2_f, \quad \alpha_2 = 2\lambda^2_f - \lambda^1_f.$$  

(D.11)

The theta function of the $\mathfrak{su}_3$ affine algebra reads, for a generic weight $\lambda = m_1\lambda^1_f$:

$$\Theta_{\lambda, k} = \sum_{\gamma \in \mathbb{M}} q^{\frac{1}{2} \parallel \gamma + \lambda \parallel^2} = \sum_{n^1, n^2} q^{\frac{1}{2} \parallel n^1 \alpha_1 + n^2 \alpha_2 + \frac{m_1}{k} \alpha_1 \parallel^2}$$

(D.12)

So the vector appearing in the theta function is:

$$\left\{ \sqrt{k} \left( 2n^1 - n^2 \right) + \frac{m_1}{k} \right\} e_1 + \left\{ \sqrt{k} m^2 + \frac{m_1 + m_2}{3\sqrt{k}} \right\} e_2$$

(D.13)
D.2 Modular transformations

We have the following modular transformations for the theta-functions:

$$\Theta_{\lambda, k} \left( -1/\tau \right) = (-i\tau)^{\dim(t)/2} \begin{vmatrix} M^* \end{vmatrix}^{-1/2} \sum_{\mu \in M^* \mod kM_L} e^{2\pi i(\lambda, \mu)/k} \Theta_{\mu, k} (\tau), \quad (D.14)$$

where $M^*$ is the lattice dual to $M_L$, $|M_L|$ is the size of the basic cell of $M_L$ and for the affine characters:

$$\chi^\Lambda (-1/\tau) = \begin{vmatrix} M^* \end{vmatrix}^{-1/2} e^{i\pi |\Delta_+|} \sum_{\Lambda' \in \Lambda^*} \sum_{w \in W} \epsilon(w) e^{2\pi i(\Lambda + \rho)w(\Lambda' + \rho)} \chi^{\Lambda'} (\tau) \quad (D.15)$$

In this formula, $|\Delta_+|$ is the number of positive roots. From these two formulas we deduce the modular transformation of the characters of the super-coset under $\tau \rightarrow -1/\tau$:

$$C^\Lambda_{\mu} \left[ \begin{array}{c} a \\ b \end{array} \right] (-1/\tau) = e^{i\pi ab \dim(t) |\Delta_+|} \sum_{\mu \in M^* \mod kM_L} e^{2\pi i(\Lambda, \mu)/k} \sum_{\Lambda' \in \Lambda^*} \sum_{w \in W} \epsilon(w) e^{2\pi i(\Lambda + \rho)w(\Lambda' + \rho)} C^\Lambda_{\mu} \left[ \begin{array}{c} b \\ -a \end{array} \right] (\tau) \quad (D.16)$$

D.3 Fermionic characters

For an even number of fermions it is possible to express the characters in terms of representations of the $SO(2n)_1$ affine algebra. The characters are labelled by $s = (0, 1, 2, 3)$ for the trivial, spinor, vector and conjugate spinor representations:

$$\Xi^{0}_{2n} = \frac{1}{2\eta^n} \left[ \theta^{[0]}_0 + \theta^{[0]}_1 \right] \text{ trivial}$$

$$\Xi^{2}_{2n} = \frac{1}{2\eta^n} \left[ \theta^{[0]}_0 - \theta^{[0]}_1 \right] \text{ vector}$$

$$\Xi^{1}_{2n} = \frac{1}{2\eta^n} \left[ \theta^{[1]}_0 + i^{-n}\theta^{[1]}_1 \right] \text{ spinor}$$

$$\Xi^{3}_{2n} = \frac{1}{2\eta^n} \left[ \theta^{[1]}_0 - i^{-n}\theta^{[1]}_1 \right] \text{ conjugate spinor} \quad (D.17)$$

Their modular matrices are:

$$T = e^{-m\pi/12} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & e^{m\pi/4} & 0 \\ 0 & 0 & 0 & e^{m\pi/4} \end{pmatrix} \quad (D.18)$$

and

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & i^{-n} & -i^{-n} \\ 1 & -1 & -i^{-n} & i^{-n} \end{pmatrix} \quad (D.19)$$
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