LATTICE SIZE OF WIDTH ONE LATTICE POLYTOPES IN \( \mathbb{R}^3 \)

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Abstract. Lattice size \( \Delta(P) \) of a lattice polytope \( P \) is an important invariant, which was formally introduced in relation to the problem of bounding the total degree and the bi-degree of the defining equation of an algebraic curve, but appeared implicitly earlier in geometric combinatorics. In this paper, we show that for an empty lattice polytope \( P \subset \mathbb{R}^3 \) there exists a reduced basis of \( \mathbb{Z}^3 \) which computes its lattice size \( \Delta(P) \). This leads to a fast algorithm for computing \( \Delta(P) \) for such \( P \). We also extend this result to another class of lattice width one polytopes \( P \subset \mathbb{R}^3 \). We then provide a counterexample demonstrating that this result does not hold true for an arbitrary lattice polytope \( P \subset \mathbb{R}^3 \) of lattice width one.

1. Introduction

A point in \( \mathbb{R}^d \) is a lattice point if it belongs to \( \mathbb{Z}^d \subset \mathbb{R}^d \). We say that \( P \subset \mathbb{R}^n \) is a lattice polytope if \( P \) is a convex polytope all of whose vertices are lattice points. A segment connecting two lattice points is called a lattice segment. A lattice segment is primitive if its only lattice points are its endpoints.

Recall that a unimodular matrix \( A \in \text{GL}(d, \mathbb{Z}) \) is a square matrix of size \( d \) with integer entries that satisfies \( \det A = \pm 1 \). Further, \( \text{AGL}(d, \mathbb{Z}) \) is the group of affine unimodular transformations \( T: \mathbb{R}^d \to \mathbb{R}^d \) of the form \( T(x) = Ax + v \), where \( A \in \text{GL}(d, \mathbb{Z}) \) and \( v \in \mathbb{Z}^d \). Two lattice polytopes in \( P, Q \subset \mathbb{R}^d \) are lattice-equivalent if \( Q = T(P) \) for some \( T \in \text{AGL}(d, \mathbb{Z}) \). We will write \( AP \) for the image of \( P \) under the map \( T: \mathbb{R}^d \to \mathbb{R}^d \) defined by \( T(x) = Ax \).

Recall that for a lattice polytope \( P \subset \mathbb{R}^d \) the lattice width of \( P \) in the direction of \( h \in \mathbb{R}^d \) is

\[
\text{w}_h(P) = \max_{x \in P} \langle h, x \rangle - \min_{x \in P} \langle h, x \rangle,
\]

where \( \langle h, x \rangle \) denotes the standard dot-product. Then the lattice width \( \text{w}(P) \) of \( P \) is the minimum of \( \text{w}_h(P) \) over all non-zero primitive vectors \( h \in \mathbb{Z}^d \).

Let \( 0 \in \mathbb{R}^d \) denote the origin and let \( \{e_1, \ldots, e_d\} \) be the standard basis of \( \mathbb{R}^d \). Then \( \Delta = \text{conv}\{0, e_1, \ldots, e_d\} \subset \mathbb{R}^d \) is the standard simplex. The central object of this paper is the lattice size. It was formally defined in [6], appeared earlier implicitly in [2, 4, 5, 10, 13], and was further studied in [7, 8, 14]. We now reproduce its definition.

Definition 1.1. The lattice size \( \Delta(P) \) of a lattice polytope \( P \subset \mathbb{R}^d \) with respect to the standard simplex \( \Delta \) is the smallest \( l \geq 0 \) such that \( T(P) \) is contained in the \( l \)-dilate \( l\Delta \) for some \( T \in \text{AGL}(d, \mathbb{Z}) \).

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That is, if we define
\[ l_1(P) = \max_{(x_1, \ldots, x_d) \in P} (x_1 + \cdots + x_d) - \min_{(x_1, \ldots, x_d) \in P} x_1 - \cdots - \min_{(x_1, \ldots, x_d) \in P} x_d, \]
then \( \text{ls}_\Delta(P) = \min \{ l_1(T(P)) \mid T \in \text{AGL}(d, \mathbb{Z}) \} \).

If we replace in Definition 1.1 the standard simplex \( \Delta \) with the unit cube \( \Box = [0, 1]^d \), we obtain the definition of the lattice size with respect to the cube, denoted by \( \text{ls}_\Box(P) \).

In \([7, 8]\) the authors explained an algorithm for computing the lattice size \( \text{ls}_\Delta(P) \) and \( \text{ls}_\Box(P) \) of a plane lattice polygon \( P \). This algorithm is based on a procedure for mapping a polytope inside a small multiple of the unit simplex, introduced by Schicho in \([13]\). In this algorithm, called the “onion skins” algorithm, one passes recursively from \( P \) to the convex hull of the interior lattice points of \( P \). It was then shown in \([7, 8]\) that basis reduction provides a faster way of computing the lattice size \( \text{ls}_\Delta(P) \) and \( \text{ls}_\Box(P) \), which works not only for lattice polygons, but also for plane convex bodies. We next introduce further definitions in order to explain these results.

**Definition 1.2.** For \( P \subset \mathbb{R}^d \), the naive lattice size \( \text{nl}_\Delta(P) \) is the smallest \( l \) such that \( T(P) \subset l \Delta \), where \( T : \mathbb{R}^d \to \mathbb{R}^d \) is a composition of matrix multiplication by a diagonal matrix \( A \) with entries \( \pm 1 \) on the main diagonal, and a lattice translation. Further, for \( d = 2 \), let

\[
\begin{align*}
l_1(P) & := \max_{(x,y) \in P} (x+y) - \min_{(x,y) \in P} x - \min_{(x,y) \in P} y, \\
l_2(P) & := \max_{(x,y) \in P} x + \max_{(x,y) \in P} y - \min_{(x,y) \in P} (x+y), \\
l_3(P) & := \max_{(x,y) \in P} y - \min_{(x,y) \in P} x + \max_{(x,y) \in P} (x-y), \\
l_4(P) & := \max_{(x,y) \in P} x - \min_{(x,y) \in P} y + \max_{(x,y) \in P} (y-x).
\end{align*}
\]

Then in the case of \( d = 2 \) the naive lattice size \( \text{nl}_\Delta(P) \) is the smallest of these four values. For \( P \subset \mathbb{R}^3 \) one can similarly write \( \text{nl}_\Delta(P) \) as the minimum of eight \( l_i(P) \), each corresponding to a vertex of the unit cube.

**Definition 1.3.** We say that a basis \( (h^1, \ldots, h^d) \) of \( \mathbb{Z}^d \) computes \( \text{ls}_\\Delta(P) \) if for matrix \( A \) with rows \( h^1, \ldots, h^d \) we have \( \text{ls}_\Delta(P) = \text{ls}_\Delta(AP) \).

Let \((-P)\) be the reflection of \( P \) in the origin, and define \( K \) to be the polar dual of the Minkowski sum of \( P \) with \((-P)\), that is, \( K := (P + (-P))^\circ \). Then \( K \) is origin-symmetric and convex and it defines a norm on \( \mathbb{R}^d \) by

\[ \|h\|_K = \inf \{ \lambda > 0 \mid h/\lambda \in K \} \].

For details see, for example, \([3]\). We then have

\[ \|h\|_K = \inf \{ \lambda > 0 \mid \langle h, x \rangle \leq \lambda \text{ for all } x \in K^\circ \} = \max_{x \in K^\circ} \langle h, x \rangle = \frac{1}{2} w_h(K^\circ) = w_h(P). \]

In what follows we will often write \( \|h\| \) for the lattice width \( w_h(P) \).

**Definition 1.4.** Let \( P \subset \mathbb{R}^2 \). We say that a basis \( (h^1, h^2) \) of the integer lattice \( \mathbb{Z}^2 \subset \mathbb{R}^2 \) is reduced (with respect to \( P \)) if

\[
(1) \quad \|h_1\| \leq \|h_2\| \text{ and }
\]

\[
(2) \quad \|h_1\| = \|h_2\| \text{ and } \langle h_1, h_2 \rangle = \pm 1.
\]


It was shown in [7] that if the standard basis \((e_1,e_2)\) is reduced with respect to a convex body \(P \subset \mathbb{R}^2\) then \(ls(P) = w_{e_2}(P)\). This result was extended to the lattice size with respect to the unit simplex \(\Delta \subset \mathbb{R}^2\) in [8], where it was shown that if the standard basis \((e_1,e_2)\) is reduced with respect to \(P\) then \(ls(P)\). Since one can find a reduced basis using the generalized Gauss reduction algorithm [9, 11], these results provide a fast way of finding \(ls(P)\) and \(ls(P)\) for plane convex bodies \(P\), which, as explained in [7, 8], outperforms the “onion skins” algorithm of [6].

Since the condition \(\|h_1 + h_2\| \geq \|h_2\|\) is equivalent to requiring that \(\|mh_1 + h_2\| \geq \|h_2\|\) for \(m \in \mathbb{Z}\) (see [7]), the following definition is a natural extension of the previous one.

**Definition 1.5.** Let \(P \subset \mathbb{R}^3\). We say that a basis \((h^1,h^2,h^3)\) of \(\mathbb{Z}^3\) is reduced (with respect to \(P\)) if

\[
\begin{align*}
(1) & \quad \|h_1\| \leq \|h_2\| \leq \|h_3\|; \\
(2) & \quad \|h_1 + h_2\| \geq \|h_2\|; \\
(3) & \quad \|mh_1 + nh_2 + h_3\| \geq \|h_3\| \quad \text{for all } m,n \in \mathbb{Z}.
\end{align*}
\]

It was shown in [7] that if the standard basis \((e_1,e_2,e_3)\) is reduced with respect to a convex body \(P \subset \mathbb{R}^3\), then \(ls(P) = w_{e_3}(P)\). This result does not extend to the lattice size with respect to the standard simplex. An example provided in [8] demonstrates that there exists a lattice polytope \(P \subset \mathbb{R}^3\) such that there is no reduced basis that computes \(ls(P)\).

In order to compute the lattice size \(ls(\Delta)\) of a lattice polytope \(P \subset \mathbb{R}^3\) one can use the following observation from [5]. Let \(l = l_1(P)\) and shift \(P\) so that its barycenter is at the origin. Consider a ball \(B \subset P\) centered at the origin of the largest possible radius \(R\). Denote by \(\|h\|_2\) the Euclidean norm of \(h\). Then if \(\|h\|_2 \geq l\) we have

\[
\|h\| = w_h(P) \geq w_h(B) = R \cdot \|h\|_2 \geq l.
\]

Hence, in order to compute the lattice size \(ls(P)\), one can find all directions \(h\) with \(\|h\|_2 < \frac{l}{R}\), check which ones of them satisfy \(w_h(P) < l\) and then use vectors from this list as rows to build all possible unimodular matrices \(A\). Then \(ls(P)\) is the minimum of \(l_1(AP)\) over all such matrices. We implemented this algorithm, which we call here the Brute Force Algorithm, using Magma, and ran the code on polytopes which are convex hulls of \(n\) random lattice points in \(\mathbb{R}^3\) with absolute values of coordinates bounded by 7, where \(n\) is a random number between 10 and 15. The Brute Force Algorithm is fairly fast: We ran 100 such experiments and they took 4.3 sec on average. The described experimentation is implemented in Experiment1.mgm, [1].

In contrast, when we considered lattice polytopes \(P\) of lattice width one, the algorithm was much slower. We randomly picked two polygons \(P_0\) and \(P_1\) as convex hulls of \(n_0\) and \(n_1\) lattice points in the planes \(x = 0\) and \(x = 1\) correspondingly, with the absolute value of the \(y\) and \(z\) coordinates of these points bounded by 7, and \(n_0\) and \(n_1\) picked randomly between 5 and 8. We then defined \(P\) as the convex hull of \(P_0\) and \(P_1\). We ran 100 such experiments, and they took 97.5 sec on average.

Furthermore, this algorithm takes even longer when \(P \subset \mathbb{R}^3\) is an empty lattice tetrahedron. Recall that a lattice polytope \(P\) is empty if its only lattice points are its vertices. It was shown in [15] that up to lattice equivalence empty lattice tetrahedra \(P \subset \mathbb{R}^3\) are of
the form
\[
P = \begin{bmatrix}
1 & 0 & 0 & p \\
0 & 1 & 0 & q \\
0 & 0 & 1 & 1
\end{bmatrix},
\]
where \( p \) and \( q \) are non-negative integers satisfying \( \gcd(p, q) = 1 \). Note that the columns of this matrix are the vertices of \( P \). (We will be using this notation throughout the paper: The polytope written as a matrix is the convex hull of its column vectors.) It was further shown in [12] that any empty lattice polytope \( P \subset \mathbb{R}^3 \) has lattice width one.

In Experiment 1.mgm in [4] we ran the Brute Force Algorithm on empty lattice tetrahedra \( P \) with \( p \) and \( q \) picked randomly between 0 and 14. The average run time in 25 such experiments was 2907 sec.

In this paper we concentrate our attention on computing the lattice size of lattice polytopes \( P \subset \mathbb{R}^3 \) with \( w(P) = 1 \) and, in particular, of empty lattice polytopes. In Theorem 3.1 we prove that for an empty lattice polytope \( P \subset \mathbb{R}^3 \) there exists a reduced basis that computes its lattice size \( \ell s_\Delta(P) \) and provide an algorithm (Algorithm 3.2) for finding such a basis, which works significantly faster than the Brute Force Algorithm explained above.

In Section 4 we consider another class of lattice polytopes \( P \subset \mathbb{R}^3 \) of lattice width one and show in Theorem 4.3 that there exists a reduced basis that computes their lattice size. It follows that Algorithm 3.2 computes the lattice size in this case as well.

Based on our results, it is reasonable to ask whether the conclusion extends to all lattice polytopes \( P \subset \mathbb{R}^3 \) of lattice width one. In Section 5 we explain an algorithm for testing this question. We ran 10000 experiments and found a lattice width one polytope \( P \), for which there is no reduced basis that computes its lattice size \( \ell s_\Delta(P) \). Interestingly, such counterexamples are quite rare: Only two out of our 10000 experiments resulted in counterexamples.

2.Definitions and First Lemmas

Note that for \( h \in \mathbb{R}^3 \) and \( A \in \text{GL}(3, \mathbb{Z}) \) we have
\[
w_h(AP) = \max_{x \in P} \langle h, Ax \rangle - \min_{x \in P} \langle h, Ax \rangle = \max_{x \in P} \langle A^T h, x \rangle - \min_{x \in P} \langle A^T h, x \rangle = w_{A^T h}(P).
\]
Therefore, if the rows of \( A \) are \( h_1, h_2, h_3 \) then \( w_{e_i}(AP) = w_{h_i}(P) \) for \( i = 1, 2, 3 \).

If \( (e_1, e_2, e_3) \) is the standard basis, then \( l_1(P) \geq w_{e_i}(P) \). We check this for \( i = 1 \):
\[
l_1(P) = \max_{(x,y,z) \in P} (x+y+z) - \min_{(x,y,z) \in P} x - \min_{(x,y,z) \in P} y - \min_{(x,y,z) \in P} z
\]
\[
\geq \max_{(x,y,z) \in P} x + \min_{(x,y,z) \in P} y + \min_{(x,y,z) \in P} z - \min_{(x,y,z) \in P} x - \min_{(x,y,z) \in P} y - \min_{(x,y,z) \in P} z
\]
\[
= \max_{(x,y,z) \in P} x - \min_{(x,y,z) \in P} x = w_{e_1}(P).
\]

Hence, for \( A \in \text{GL}(3, \mathbb{Z}) \) with rows \( h_1, h_2, h_3 \) we get
\[
l_1(AP) \geq l_1(AP) = w_{h_i}(P).
\]

We next prove 3D versions of Lemmas 2.5 and 2.6 from [8].

**Lemma 2.1.** Let \( h_1, h_2, \) and \( h_3 \) be the rows of \( A \in \text{GL}(3, \mathbb{Z}) \). Then
\[
(a) \ l_1(AP) = \max_{x \in P} \langle h_1+h_2+h_3, x \rangle - \min_{x \in P} \langle h_1, x \rangle - \min_{x \in P} \langle h_2, x \rangle - \min_{x \in P} \langle h_3, x \rangle;
\]
(b) $l_1(AP)$ does not depend on the order of the rows in $A$;

(c) $l_1(AP) = l_1 \left( \begin{bmatrix} h_1 \\ h_2 \\ -(h_1 + h_2 + h_3) \end{bmatrix} P \right)$.

Proof. We check (c) using (a)

$$l_1 \left( \begin{bmatrix} h_1 \\ h_2 \\ -(h_1 + h_2 + h_3) \end{bmatrix} P \right) = \max_{x \in B} \langle -h_3, x \rangle - \min_{x \in P} \langle h_1, x \rangle - \min_{x \in P} \langle h_2, x \rangle - \min_{x \in P} \langle -h_1 - h_2 - h_3, x \rangle = l_1(AP).$$

$\Box$

**Lemma 2.2.** Let $l \in \mathbb{R}$ and suppose that for matrix $A \in \text{GL}(3, \mathbb{Z})$ with the rows $h_1$, $h_2$, and $h_3$ we have $l_1(AP) < l$. Then the lattice width of $P$ in the directions $h_1, h_2, h_3, h_1 + h_2, h_1 + h_3, h_2 + h_3$, and $h_1 + h_2 + h_3$ is also less than $l$.

Proof. Denote $l_1 = l_1(P)$. Since $AP \subset l \Delta$, we get $w_{e_1}(AP) \leq w_{e_1}(l \Delta) = l_1$. Hence $w_{h_1}(P) = w_{e_1}(AP) \leq l_1 < l$. The argument for the remaining directions is similar. $\square$

**Definition 2.3.** Let $P \subset \mathbb{R}^3$ be a convex body and denote $\|h\| = w_h(P)$. Then a basis $(h_1, h_2, h_3)$ of $\mathbb{Z}^3$ is Minkowski reduced (with respect to $P$) if

1. $\|h_1\| \leq \|h\|$ for all $h \in \mathbb{Z}^3$;
2. $\|h_2\| \leq \|h\|$ for all $h \in \mathbb{Z}^3$ that are not multiples of $h_1$;
3. $\|h_3\| \leq \|h\|$ for all $h \in \mathbb{Z}^3$ that are not linear combinations of $h_1$ and $h_2$.

Clearly, a Minkowski reduced basis is reduced. It was explained in Theorem 3.3 of [7] how to find a Minkowski reduced basis starting with a reduced basis $(h_1, h_2, h_3)$. For this, consider set $E = \{ ah_1 + bh_2 + ch_3 : |a| = |b| = 1, |c| = 2 \}$ and let $u$ be the direction of the smallest norm in $E$. Then the two vectors of smallest norm among $\{ h_1, h_2, u \}$, written in the order of increasing norm together with $h_3$ form a Minkowski reduced basis. Note that if in a reduced basis $\|h_3\| \geq \|h_1\| + \|h_2\|$ then for any $h \in E$ we have

$$\|h\| \geq 2\|h_3\| - \|h_1\| - \|h_2\| \geq \|h_3\|$$

and hence such a reduced basis is also Minkowski reduced.

**Lemma 2.4.** Let $(h_1, h_2, h_3)$ be a Minkowski reduced basis. Then $\text{ls}_\Delta(P) \geq \|h_3\|.

Proof. Let $A \in \text{GL}(3, \mathbb{Z})$ be such that $\text{ls}_\Delta(P) = l_1(AP)$. The rows $r_1, r_2, r_3$ of $A$ form a basis of $\mathbb{Z}^3$ and hence for one of them, say, $r_3$, we have $\|r_3\| \geq \|h_3\|$. We conclude

$$\text{ls}_\Delta(P) = l_1(AP) \geq w_{e_3}(AP) = w_{r_3}(P) \geq \|h_3\|. \quad \square$$

Let $P \subset \mathbb{R}^3$ be an empty lattice polytope. As we explained above, it was shown in [12] that any such polytope $P$ is of lattice width one and hence, using lattice equivalence, we can assume that $w_{e_1}(P) = 1$. Let $\Pi_0 = P \cap \{ x = 0 \}$ and $\Pi_1 = P \cap \{ x = 1 \}$. Then each of
Π₀ and Π₁ is lattice-equivalent to a point, a primitive segment, the standard simplex, or the unit square \([0,1]^2\). Note the unimodular map

\[
A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]

fixes each point in the plane \(x = 0\) and shifts the plane \(x = 1\) by the vector \((0, a, b)\). Hence, composing such maps with lattice translations, one can use affine unimodular maps to independently shift \(\Pi_0\) and \(\Pi_1\) in the planes \(x = 0\) and \(x = 1\) correspondingly.

Each of the bases \(\Pi_0, \Pi_1\) of an empty lattice polytope \(\Pi\) is either a lattice parallelogram of area 1 or is properly contained in such a parallelogram. We next study some properties of such parallelograms.

**Proposition 2.5.** Let \(\Pi \subset \mathbb{R}^2\) be a lattice parallelogram of area 1. Then one can shift \(\Pi\) so that one of its vertices is at the origin and the entire parallelogram \(\Pi\) is contained in the first quadrant or in the second quadrant.

**Proof.** Shift \(\Pi\) so that its lowest vertex is at the origin. If there are two vertices at the lowest level, shift \(\Pi\) so that one of them is at the origin. Let \((a,b)\) and \((c,d)\) be the two vertices adjacent to the vertex at the origin. If \(\Pi\) is not entirely in the first or in the second quadrant, then we may assume \(a > 0, b \geq 0, c < 0, d \geq 0\). Since \(\Pi\) is of area 1 we have \(1 = ad - bc \geq d + b\), which implies that either \(b = 0, d = 1\) or \(b = 1, d = 0\). (Note that we cannot have \(b = d = 0\).) In the first of these cases we have \(a = 1\) and we can shift \(\Pi\) left by 1 so that one of its vertices is at the origin and \(\Pi\) is contained in the second quadrant. In the second case, we have \(c = -1\) and we shift \(\Pi\) right by 1 unit. The second statement also follows. \(\square\)

If we can shift \(\Pi\) so that one of its vertices is at the origin and \(\Pi\) is contained in the first quadrant, we say that \(\Pi\) is **positively oriented**; if we can shift \(\Pi\) so that one of its vertices is at the origin and \(\Pi\) is contained in the second quadrant, we say that \(\Pi\) is **negatively oriented**. Clearly, the only \(\Pi\) which are both positively and negatively oriented are translations of the unit square \(\text{conv}\{(0,0), (1,0), (0,1), (1,1)\}\).

**Proposition 2.6.** Let \(a, b, c, d\) be non-negative integers and let \(\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 1\). Then

(i) \(a \geq b, c \geq d\) or
(ii) \(a \leq b, c \leq d\) or
(iii) \(\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\).

**Proof.** Suppose that \(a > b\) but \(c < d\). Then \(1 = ad - bc > bd - b(d - 1) = b\), so \(b = 0\) and \(a = d = 1\). Also, \(c \leq d - 1 = 0\) implies \(c = 0\), so \(\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\). The case \(a < b, c > d\) is handled similarly. \(\square\)

**Proposition 2.7.** Let \(\Pi \subset \mathbb{R}^2\) be a positively oriented lattice parallelogram of area 1 with vertices \((0,0), (a,b), (c,d), (a+c, b+d)\), where \(a, b, c, d \geq 0\) and \(\Pi\) is not the unit square
conv\{(0, 0), (1, 0), (0, 1), (1, 1)\}. Then the maximum and the minimum of \(x - y\), \(x + y\), \(x\), and \(y\) are attained at \((0, 0)\) and \((a + c, b + d)\).

**Proof.** The statement about \(x + y\), \(x\), and \(y\) is obvious. By Proposition 2.6, we need to take care of two situations: when \(a \geq b, c \geq d\) and when \(a \leq b, c \leq d\). In the first of these we have \(0 \leq a - b\) and \(0 \leq c - d\), which implies \(a - b \leq (a - b) + (c - d)\) and \(c - d \leq (a - b) + (c - d)\), so the minimum of \(x - y\) is attained at \((0, 0)\), while the maximum is attained at \((a + c, b + d)\). In the second situation the minimum and the maximum are swapped.

**Proposition 2.8.** Let \(\Pi \subset \mathbb{R}^2\) be a negatively oriented lattice parallelogram of area 1 with vertices \((0, 0), (a, b), (c, d), (a + c, b + d)\), where \(b, d \geq 0\), while \(a, c \leq 0\), and \(\Pi\) is not the unit square \(\text{conv}\{(0, 0), (-1, 0), (0, 1), (-1, 1)\}\). Then the maximum and the minimum of \(x, y, x + y,\) and \(x - y\) are attained at \((0, 0)\) and \((a + c, b + d)\).

**Proof.** This follows from Proposition 2.7 by reflecting \(\Pi\) in the \(y\)-axis.

Note that our definition of orientation extends naturally to primitive segments and empty lattice triangles. We say that such a segment or triangle \(\Pi\) is **positively/negatively oriented** if we can shift \(\Pi\) so that one of its vertices is at the origin and \(\Pi\) is contained in the first/second quadrant correspondingly. Clearly, for a segment the orientation coincides with the sign of its slope, with horizontal and vertical segments being both positively and negatively oriented. Unless \(\Pi\) is a shift of

\[
\text{conv}\{(0, 0), (1, 0), (1, 1)\}, \text{ or conv}\{(0, 0), (0, 1), (1, 1)\}, \text{ or conv}\{(0, 0), (1, 0), (0, 1)\},
\]

a positively oriented lattice triangle \(\Pi\) after a shift has vertices \((0, 0), (a, b), (a + c, b + d)\), where \(a, b, c, d \geq 0\) and the maximum and minimum of \(x, y, x - y,\) and \(x + y\) are attained at \((0, 0)\) and \((a + b, c + d)\).

Similarly, unless \(\Pi\) is a shift of \(\text{conv}\{(0, 0), (0, 1), (-1, 1)\}, \text{ or conv}\{(0, 0), (-1, 0), (0, 1)\}, \text{ or conv}\{(0, 0), (1, 0), (0, 1)\}\), a negatively oriented lattice triangle \(\Pi\) after a shift has vertices \((0, 0), (a, b), (a + c, b + d)\), where \(a, c \leq 0\) and \(b, d \geq 0\) and the maximum and the minimum of \(x, y, x - y,\) and \(x + y\) are attained at \((0, 0)\) and \((a + b, c + d)\).

**Definition 2.9.** Let \(\Pi\) be either an empty lattice triangle an empty lattice parallelogram, and assume \(\Pi\) is not contained in a shift of the unit square. Then we say that the **main vertices** are the two vertices where the maximum and minimum of \(x, y, x + y, x - y\) are attained. In the case when \(\Pi\) is a lattice segment, we call both of its vertices **main vertices**.
Note that if we reflect $\Pi$ in the $x$ or $y$-axes, its main vertices are mapped to the main vertices of the image. The observations that we made above will allow us to treat empty lattice parallelograms and triangles as a lattice segments connecting main vertices.

3. Lattice Size of Empty Lattice Polytopes

**Theorem 3.1.** Let $P \subset \mathbb{R}^3$ be an empty lattice polytope. Then there exists a basis which is Minkowski reduced with respect to $P$ and computes the lattice size $ls_\Delta(P)$.

*Proof.* Let the standard basis $(e_1, e_2, e_3)$ be Minkowski reduced with respect to $P$. Since $w(P) = 1$, we have $\|e_1\| = 1$, $\|e_2\| = m$, and $\|e_3\| = n$, where $1 \leq m \leq n$.

If $n = 1$, then we also have $m = 1$, and hence $P$ is contained in the unit cube. Then we have three options: $P$ is lattice-equivalent to the unit simplex, in which case $ls_\Delta(P) = 1$; $P$ is the unit cube, in which case $ls_\Delta(P) = 3$; and anything in between, in which case we have $ls_\Delta(P) = 2$.

In what follows we assume that $n > 1$. Denote $\Pi_0 = P \cap \{x = 0\}$ and $\Pi_1 = P \cap \{x = 1\}$. Then we can assume that each of $\Pi_0$ and $\Pi_1$ is a primitive segment, a lattice triangle of area 1/2, or a lattice parallelogram of area 1, since if one of the $\Pi_i$ is a vertex, we can unimodularly map the other inside the unit square and we would have $n = 1$.

Assume first that $\Pi_0$ and $\Pi_1$ are of opposite orientation, say $\Pi_0$ is positively oriented and $\Pi_1$ is negatively oriented. Additionally, assume that each $\Pi_0$ and $\Pi_1$ has main vertices. Let in $\Pi_0$ the main vertices be $(0, 0, 0)$ and $(0, \alpha, \beta)$, and in $\Pi_1$, let these vertices be $(1, \gamma, 0)$ and $(1, 0, \delta)$, where $\alpha, \gamma \in [0, m]$ and $\beta, \delta \in [0, n]$. Since $\|e_2\| = m$ and $\|e_3\| = n$, and the non-main vertices cannot maximize or minimize $y$ or $z$, we have $\alpha = m$ or $\gamma = m$ and $\beta = n$ or $\delta = n$.

Note that we can swap the two layers $x = 0$ and $x = 1$, swapping $\alpha$ with $\gamma$ and $\beta$ with $\delta$ by applying

$$
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
\delta - \beta & 0 & -1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 & 1 \\
0 & \alpha & \gamma & 0 \\
0 & \beta & 0 & \delta
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & \alpha & \gamma & 0 \\
\beta & 0 & \delta & 0
\end{bmatrix}
$$

Hence we can assume that we have $\delta = n$, which implies that $\gamma > 0$ for otherwise $[(1, \gamma, 0), (1, 0, \delta)]$ would not be primitive.

Since the standard basis is reduced we have $\| - \gamma e_1 + e_2 + e_3 \| \geq n$, which gives

$$
\max\{0, \alpha + \beta, n - \gamma\} - \min\{0, \alpha + \beta, n - \gamma\} \geq n.
$$

(Note that since we are minimizing and maximizing $(y + z)$ on each of the layers here, only the main vertices matter.) Since $\gamma > 0$, this condition implies $\alpha + \beta \geq n$. Hence $\alpha > 0$ since otherwise $\beta = n$ and $[(0, 0, 0), (0, \alpha, \beta)]$ is not primitive.

Suppose first that $\alpha + \beta = n$. Then $l_1(P) = n + 1$. Let’s show that in this case we have $ls_\Delta(P) = n + 1$. For this, we look for integer directions $(a, b, c)$ with $c \geq 0$ such that the lattice width of $P$ in such directions is at most $n$. We have

$$
\max\{0, ba + c\beta, a + b\gamma, a + cn\} - \min\{0, ba + c\beta, a + b\gamma, a + cn\} \leq w_{(a, b, c)}(P) \leq n.
$$

Plugging in $\beta = n - \alpha$ we get $ba + c(n - \alpha) \leq n$ and $cn - b\gamma \leq n$. If $c > 1$ we get $n - b\gamma < cn - b\gamma \leq n$, so $b\gamma > 0$ and hence $b > 0$ since we have $\gamma > 0$. Also,

$$
ba + (n - \alpha) < ba + c(n - \alpha) \leq n
$$

Thus, we have $\max\{0, ba + c\beta, a + b\gamma, a + cn\} - \min\{0, ba + c\beta, a + b\gamma, a + cn\} = n$.
implies \( b\alpha < \alpha \) and hence \( b < 1 \) since \( \alpha > 0 \). We have shown that \( c = 0 \) or 1. This implies that if \( l_1(T(P)) = n \) then the last coordinate of each of the rows of the corresponding unimodular matrix \( A \) has last coordinate equal to 0, 1, or \(-1\). By Lemma 2.2, the same applies to the sum of any two rows of \( A \) as well as to the sum of all three rows. Hence up to permuting the rows, we have two options for the last column of \( A \):

\[
\pm \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \pm \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},
\]

where by Lemma 2.1 we only need to consider the first option. Denote the rows of \( A \) by \( h_1, h_2, \) and \( h_3 \). Since the standard basis is reduced and \( h_3 \) is of the form \( ae_1 + be_2 \pm e_3 \) we have \( w_{h_3}(P) = \|h_3\| \geq n \). Together with \( T(P) \subset n\Delta \), this implies that \((0,0,n) \in T(P)\). We also have \( \|h_1 + h_2 + h_3\| \geq n \) and hence \((0,0,0) \in T(P)\). Hence the entire segment \([0,0,0), (0,0,n)\] is contained in \( T(P) \). Since \( P \) is empty and \( n > 1 \) we conclude that there is no such map \( T \) and hence \( \text{ls}_\Delta(P) = n + 1 \).

We can now assume that \( \alpha + \beta \geq n + 1 \). In this case we have \( l_1(P) = \alpha + \beta \). We also have \( \gamma + \delta = n + \gamma \geq n + 1 \). If we swap the layers \( x = 0 \) and \( x = 1 \), as explained above, we can unimodularly inscribe \( P \) inside \((\gamma + \delta)\Delta \). We now show that \( \text{ls}_\Delta(P) = \min\{\alpha + \beta, \gamma + \delta\} \). Let \((a,b,c)\) be an integer direction with \( c \geq 0 \) such that \( \|(a,b,c)\| < \min\{\alpha + \beta, \gamma + \delta\} \).

Then

\[
\max\{0, b\alpha + c\beta, a + b\gamma, a + cn\} - \min\{0, b\alpha + c\beta, a + b\gamma, a + cn\} < \min\{\alpha + \beta, \gamma + \delta\}.
\]

If \( c > 1 \) we get \( b\alpha + b\beta < b\alpha + c\beta < \alpha + \beta \), so \( b\alpha < \alpha \), and since \( \alpha > 0 \) we conclude \( b \leq 0 \). Using this, we get \( 2n \leq cn \leq cn - b\gamma < n + \gamma \), which implies \( n < \gamma \) and this contradiction proves \( c \leq 1 \). If \( c = 1 \), we have \( b\alpha + \beta < \alpha + \beta \) and hence \( b < 1 \). Also, \( n - b\gamma < n + \gamma \), so \( b > -1 \), and we conclude that \( b = 0 \). We have checked that \( c = 0 \) or 1 and in the case when \( c = 1 \) we also have \( b = 0 \).

As before, if we have \( l_1(T(P)) < \min\{\alpha + \beta, \gamma + \delta\} \) we can assume that the last column of the corresponding unimodular matrix \( A \) is the transpose of \( \begin{bmatrix} 0 & 0 & \pm 1 \end{bmatrix} \). Let the rows of \( A \) be \( h_1, h_2, \) and \( h_3 \). Then the last coordinate of \( h_3, h_1 + h_3, \) and \( h_2 + h_3 \) is 1 so by what we just showed and by Lemma 2.2 the second component of each of these vectors is 0, so the entire second column of \( A \) would consist of all zeroes. We have checked that in this case \( \text{ls}_\Delta(P) = \min\{\alpha + \beta, \gamma + \delta\} \).

We next consider the situation where at least one of the \( \Pi_i \) does not possess main vertices, that is, \( \Pi_i \), up to a lattice shift, is either the unit square or a triangle which is half of the unit square. If this happens for both \( \Pi_0 \) and \( \Pi_1 \), we have \( n = 1 \). Let us assume \( \Pi_1 \) is either the unit square

\[
\text{conv}\{(1,0,0), (1,0,1), (1,1,0), (1,1,1)\}
\]

or a triangle obtained by dropping one of its vertices, and that the main vertices in \( \Pi_0 \) are \((0,\alpha,0)\) and \((0,0,\beta)\). Then since \( n > 1 \) we have \( \beta = n \). We also have \( \alpha = m \). This is clear if \( m \geq 1 \). If \( m = 1 \), we cannot have \( \alpha = 0 \) since then \([(0,\alpha,0),(0,0,\beta)]\) would not be primitive.

We have \( \max\{n, m, n - 1, n + 1\} - \min\{n, m, n - 1, n + 1\} \geq \|(n-1)e_1 + e_2 + e_3\| \geq n \) if \( n > m \) we get \( n + 1 - m \geq n \), which implies \( m = 1 \). If \( n \geq 3 \) we get \( l_1(P) = n \) and hence by Lemma 2.4 we conclude that \( \text{ls}_\Delta(P) = n \). If \( n = 2 \), we have \( l_1(P) = 3 \). Let \((a,b,c)\) be an integer direction such that the lattice width of \( P \) in this direction is at most 2. As
when showing above that \( l_\Delta(P) = n + 1 \) under the assumption that \( \alpha + \beta = n \), here one can easily check that \( |e| \leq 1 \) and there is no unimodular map \( T \) with \( l_1(T(P)) = n \).

It remains to consider the case when \( \Pi_0 \) and \( \Pi_1 \) are of the same orientation. We can then assume that \( \Pi_0 \) and \( \Pi_1 \) are both oriented positively. Shift \( \Pi_0 \) so that its main vertices are \((0,0,0)\) and \((0,\alpha,\beta)\), where \( 0 \leq \alpha \leq m \) and \( 0 \leq \beta \leq n \). Shift \( \Pi_1 \) so that its main vertices are at \((1,\gamma,\delta)\) and \((1,m,n)\), where \( 0 \leq \gamma \leq m \) and \( 0 \leq \delta \leq n \).

We have \( \alpha = m \) or \( \gamma = 0 \) since otherwise we can shift \( \Pi_1 \) in \( x = 1 \) and decrease \( \|e_2\| \). Similarly, we have \( \beta = n \) or \( \delta = 0 \). Switching the layers \( x = 0 \) and \( x = 1 \), if necessary, we can assume that \( \beta = n \).

Since the standard basis is reduced, we have \( \|e_3 - e_2\| \geq n \). By Proposition 2.7 this gives \( \max\{0,n - \alpha,\delta - \gamma\} - \min\{0,n - \alpha,\delta - \gamma\} \geq n \). Since \( n - \alpha \geq 0 \) this implies \( \max\{n - \alpha,\delta - \gamma\} - \min\{0,\delta - \gamma\} \geq n \).

If this gives us \( n - \alpha \geq n \), and then \( \alpha = 0 \), which implies that \( \beta = 1 \) for otherwise \([(0,0,0),(0,\alpha,\beta)]\) is not primitive. Hence \( \Pi_0 = [(0,0,0),(0,0,1)] \) is also negatively oriented and we have considered this case earlier.

If the above gives \( \delta - \gamma \geq n \) then \( \delta = n \) and \( \gamma = 0 \), so \( m = 1 \) and \( \Pi_1 = [(1,0,n),(1,1,1)] \) is negatively oriented, which we have covered before.

Finally, if \( n - \alpha - \delta + \gamma \geq n \) we get \( \alpha + \delta \leq \gamma \). Since \( \alpha = m \) or \( \gamma = 0 \), we have \( \gamma \leq \alpha \) and hence \( \gamma + \delta \leq \alpha + \delta \leq \gamma \), so \( \delta = 0 \) and \( \alpha = \gamma \). If \( \alpha = \gamma = m \), we get \([(1,m,0),(1,m,n)] \subset \Pi_1 \) and hence \( n = 1 \). If \( \alpha = \gamma = 0 \) then \( \Pi_0 = [(0,0,0),(0,0,\beta)] \), so \( \beta = 1 \) and \( \Pi_0 \) is again negatively oriented.

Based on the above argument, we can compute the lattice size of an empty lattice polytope \( P \) using the following algorithm.

**Algorithm 3.2.** Reduce the standard basis with respect to \( P \), using Algorithm 3.11 from [7]. Pass to the corresponding Minkowski reduced basis, as explained in the paragraph following Definition 2.3. For each of the eight matrices

\[
A = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix},
\]

shift the bases \( \Pi_0 \) and \( \Pi_1 \) of \( AP \) in the layers \( x = 0 \) and \( x = 1 \) so that each of them is in the corresponding positive \((y,z)\)-octant and touches the coordinate axes, and call the obtained polytope \( Q \). Then \( l_\Delta(P) \) is the minimum of \( l_1(Q) \) over the eight matrices \( A \).

To confirm that this algorithm outperforms the Brute Force Algorithm we ran 25 examples using Algorithm 3.2 and the Brute Force Algorithm. In all these examples we randomly picked \( a, b, c, \) and \( d \) in the interval from 1 to 14, checked if \( ad - bc = \pm 1 \), and if this was the case, computed the lattice size of

\[
P = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & a & c & a + c \\ 0 & 0 & 1 & 1 & 0 & b & d & b + d \end{bmatrix},
\]

using each of the two algorithms. The average running time for Algorithm 3.2 was 0.198 sec, while for the Brute Force Algorithm it was 1196.697 sec. For the code and further details see Experiment2.mgm, [1].
4. Another class of 3D polytopes of lattice width one

In the next proposition, we are working with the plane case, so now $\Delta \subset \mathbb{R}^2$ is the standard 2-simplex. We show that if one replaces unimodular matrices in the definition of $l_{\Delta}(P)$ with nonsingular integer matrices, then this will define the same object.

**Proposition 4.1.** Let $P \subset \mathbb{R}^2$ be a lattice polygon. Then for any integer nonsingular matrix $A$ of size 2 we have $l_1(AP) \geq l_{\Delta}(P)$.

**Proof.** Let $A = UDV$ be the Smith normal form for $A$. That is, $U$ and $V$ are unimodular matrices and $D$ is an integer diagonal matrix with positive entries on the diagonal. Then, since $U$ and $V$ are unimodular, we have $l_{\Delta}(UDVP) = l_{\Delta}(DVP)$ and $l_{\Delta}(VP) = l_{\Delta}(P)$.

Since $D$ is diagonal with positive integer entries on the diagonal and we can assume that $VP$ contains the origin, we have $VP \subset DVP$ and hence $l_{\Delta}(VP) \leq l_{\Delta}(DVP)$. Hence $l_1(AP) \geq l_{\Delta}(AP) = l_{\Delta}(UDVP) = l_{\Delta}(DVP) \geq l_{\Delta}(VP) = l_{\Delta}(P)$.

\[ \square \]

**Proposition 4.2.** Let $P \subset \mathbb{R}^2$ be a polygon in the $(x, y)$-plane of $\mathbb{R}^3$. Let $l_{\Delta_2}(P) = l$, where $\Delta_2$ is the standard simplex in the $(x, y)$-plane. Then $l_{\Delta}(P) = l$, where $P$ is now considered as a subset of $\mathbb{R}^3$, and $\Delta$ is the standard simplex in $\mathbb{R}^3$.

**Proof.** Suppose that for

\[
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \text{GL}(3, \mathbb{Z})
\]

we have $l_1(AP) = l' < l$ and hence $AP + v \subset l'\Delta$ for some $v = (b_1, b_2, b_3) \in \mathbb{Z}^3$.

Since $A$ is unimodular, at least one of the matrices $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$, and $\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ is nonsingular. We can assume that the determinant of $B := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is nonzero. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection defined by $(x, y, z) \mapsto (x, y)$. We have $\pi(AP + v) \subset \pi(l'\Delta)$ and hence $BP + (b_1, b_2) \subset l'\Delta_2$, where $\Delta_2$ is the standard unit simplex in the $(x, y)$-plane. Since $l_{\Delta_2}(P) = l$ and $l' < l$, this contradicts the result of Proposition 4.1. \[ \square \]

**Theorem 4.3.** Let $P \subset \mathbb{R}^3$ be a convex lattice polytope of lattice width one enclosed between the planes $x = 0$ and $x = 1$. Denote

\[
B_0 = P \cap \{x = 0\} \quad \text{and} \quad B_1 = P \cap \{x = 1\}.
\]

Consider $B_0$ and $B_1$ as subsets of the $(y, z)$-plane by ignoring the $x$-coordinate and suppose that up to a lattice translation $B_1$ is contained in the convex hull of the interior lattice points of $B_0$. Then $l_{\Delta}(P) = l_{\Delta_2}(B_0)$ and there exists a reduced basis that computes $l_{\Delta_2}(B_0)$.

**Proof.** Use a unimodular transformation to shift $B_1$ in the plane $x = 1$ so that its projection onto the plane $x = 0$ along the $x$-axis is contained in the convex hull of the interior lattice points of $B_0$. Next, reduce $(e_2, e_3)$ with respect to $B_1$. We will still refer to the obtained polytope and its bases as $P$, $B_0$ and $B_1$. 

that computes its lattice size \( l_\Delta \) in \([1]\), where we first reduce the standard basis (see \texttt{ComputeLSReducton.mgm}) using Algorithm 3.11 from \([7]\), and then search through reduced bases of the form \((\pm e_1, ae_1 \pm e_2, be_1 + ce_2 \pm e_3)\) to minimize \( l_1(AP) \) over the corresponding

\[
A = \begin{bmatrix}
\pm 1 & 0 & 0 \\
 a & \pm 1 & 0 \\
b & c & \pm 1
\end{bmatrix}
\]

We have checked that the standard basis is reduced with respect to the obtained \( P \).

Denote \( l = l_{\Delta_2}(P) \). By Proposition 4.2 we have

\[
l_{\Delta}(P) \geq l_{\Delta}(B_0) = l_{\Delta_2}(B_0) = l.
\]

Shifting \( P \), if necessary, we can assume that \( B_0 \subset l\Delta \). Since the projection of \( B_1 \) is contained in the convex hull of the interior lattice points of \( B_0 \), we also have \( B_0 \subset l\Delta \) and it follows that \( P \subset l\Delta \). Hence \( l_{\Delta}(P) \leq l \) and we have checked that \( l_{\Delta}(P) = l \). \( \square \)

Note that Algorithm 3.2 computes the lattice size for the lattice polytopes \( P \) of lattice width one described in Theorem 4.3.

5. Computer Experimentation and Counterexample

In light of our results proved in Theorems 3.1 and 4.3, it is natural to ask whether it is true that for any lattice polytope \( P \subset \mathbb{R}^3 \) of lattice width one there exists a reduced basis that computes its lattice size \( l_{\Delta}(P) \).

To test this question, we developed and implemented in MAGMA a procedure (see \texttt{ComputeLSReduction.mgm} in \([1]\)), where we first reduce the standard basis \((e_1, e_2, e_3)\) with respect to \( P \) using Algorithm 3.11 from \([7]\), and then search through reduced bases of the form \((\pm e_1, ae_1 \pm e_2, be_1 + ce_2 \pm e_3)\) to minimize \( l_1(AP) \) over the corresponding

We then compare this minimum with the lattice size \( l_{\Delta}(P) \) computed using the Brute Force Algorithm described in the Introduction. See \texttt{Experiment3.mgm} in \([1]\) for the code related to the described experimentation.

We tested 5,000 polytopes \( P \) whose bases \( P_0 \subset \{x = 0\} \) and \( P_1 \subset \{x = 0\} \) are picked as convex hulls of \( n_0 \) and \( n_1 \) random points \((y, z) \in \mathbb{R}^2\) correspondingly, that satisfy \(|y| \leq 7\) and \(|z| \leq 7\). The numbers of points \( n_0 \) and \( n_1 \) in each of the planes were picked randomly in the interval from 3 to 10. We also tested 5,000 additional examples where we replaced the bound of 7 with 3 and \( n_0 \) and \( n_1 \) were picked randomly between 3 and 7. In all of the 10,000 examples, except two, we found a reduced basis of \( P \) that computes \( l_{\Delta}(P) \). We now explain one of these two examples.
Example 5.1. Let $P = \text{conv}\{(0, 2, 5), (0, -2, 5), (0, 1, -6), (1, -8, 5), (1, 2, -5), (1, -4, -3)\}$. We ran the Brute Force Algorithm (see BruteForceOriginal.mgm) to find that $\text{ls}_\Delta(P) = 13$. One of the matrices that gives this answer is

$$
\begin{bmatrix}
1 & 0 & 0 \\
2 & -1 & 0 \\
7 & 2 & 1
\end{bmatrix}
$$

For this $P$, our procedure, implemented in ComputeLSReduction.mgm, returned the answer of 14. The standard basis in this example is reduced and the norms of the vectors in a Minkowski reduced basis are 1, 10, and 11. Since there is no guarantee that our procedure searches through all possible reduced bases, we next found all the primitive vectors with such norms, and then considered all unimodular matrices $A$ with rows $h_1, h_2$ and $h_3$ that satisfy $\|h_1\| = 1, \|h_2\| = 10, \|h_3\| = 11$. We then minimized $l_1(AP)$ over all such matrices $A$ and confirmed that the minimum is indeed 14 using CheckReducedBruteForce.mgm in [1].

Note that in this example we have $\|h_3\| \geq \|h_1\| + \|h_2\|$ and hence it follows from the comment after Definition 1.5 that for this $P$ any reduced basis is also Minkowski reduced. We conclude that we have searched through all the reduced bases and hence for this $P$ there is no reduced basis that computes its lattice size.

We conclude that one cannot use basis reduction to compute $\text{ls}_\Delta(P)$ for lattice polytopes $P \subset \mathbb{R}^3$ with $w(P) = 1$. According to our experimentation, such examples of lattice polytopes $P$ with $w(P) = 1$ for which there is no reduced basis that computes $\text{ls}_\Delta(P)$ are quite rare: We only found two such $P$ among 10,000 random examples.

While basis reduction is not guaranteed to compute the lattice size, one can compute the lattice size using basis reduction in combination with the Brute Force Algorithm to reduce the running time. For this, one can first reduce the standard basis with respect to $P$ and then run the Brute Force Algorithm. This speeds up the algorithm: In our experimentation, see Experiment4.mgm, [1], the average time of the lattice size computation on 100 random lattice polytopes $P$ with $w(P) = 1$ ran with the Brute Force Algorithm was 103 sec, while the average time on the same polytopes when we first reduced the basis was 34 sec. Here each $P$ is the convex hull of two polygons $P_0$ and $P_1$ in the planes $x = 0$ and $x = 1$ correspondingly, picked as convex hulls of $n_0$ and $n_1$ random points in the $(y, z)$-plane with the magnitude of the components bounded by 7, where each $n_0$ and $n_1$ is picked randomly between 3 and 10.

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