The Reduced properties and applications of the Yangian algebras

Li-Jun TIAN and Yan-Ling JIN
Department of Physics, Shanghai University, Shanghai, 200444, China
Shanghai Key Lab for Astrophysics, Shanghai, 200234, China

The reduced properties and applications of Yangian $Y(\mathfrak{sl}(2))$ and $Y(\mathfrak{su}(3))$ algebras are discussed. By taking a special constraint, the representation of $Y(\mathfrak{su}(3))$ can be divided into three $3 \times 3$ blocks diagonal based on Gellmann matrices. The reduced Yangian $Y(\mathfrak{sl}(2))$ and $Y(\mathfrak{su}(3))$ are applied to the bi-qubit system and the mixed light pseudoscalar meson state respectively, and are both able to make the final states disentangled after acting on the initial state by the transition operator, composed of the generators of Yangian.

Keywords: $Y(\mathfrak{sl}(2))$; $Y(\mathfrak{su}(3))$; reduction; entanglement degree
PACS number(s): 02.20.-a, 03.65.-w

I. INTRODUCTION

Quantum entanglement, as the nonlocal correlation among different quantum systems, is of crucial importance in quantum computation\cite{1}, quantum teleportation\cite{2}, dense coding\cite{3} and quantum key distribution\cite{4}. However, in real systems, the deterioration of the coherence or even the decoherence and disentanglement due to the interaction with an environment, which is recognized as a main obstacle to realize quantum computing\cite{5} and quantum information processing (QIP)\cite{6}, have to be taken into account in the research in the field of quantum information. Earlier studies had indicated that entanglement decays exponentially\cite{7,8,9} until T. Yu suggested that entanglement decays completely in finite time and called for concerted effect to research entanglement sudden death\cite{10}, for example, different systems are considered\cite{11,12,13}, realization in experiment is given\cite{14,15} and \cite{16} provides theoretical guidance to practical application of controlling entanglement.

In the last decades, Yangian associated with simple Lie algebras have been systematically studied in both mathematics and physics and have many applications through spin operators and quantum fields\cite{16-21}. There has been a remarkable success in studying the long-ranged interaction models by means of various approaches\cite{22,23}, in which the Haldane-Shastry model was regarded as the representative of the spin chain $\mathfrak{su}(n)$ with long-range interaction\cite{26,27}. Recently, Yangian $Y(\mathfrak{su}(3))$ algebra has been demonstrated to be able to realize the hadronic decay channels of light pseudoscalar mesons and predict the unknown particle $X$ in the decay channel $K^0_L \rightarrow \pi^0\pi^0X$\cite{28}. Moreover, the influence of transition operators composed of the generators of Yangian $Y(\mathfrak{sl}(2))$ and $Y(\mathfrak{su}(3))$ on the entanglement degrees of two-qubit system and the mixed light pseudoscalar meson states are discussed respectively\cite{33,34,35}.

In this letter, we will follow the similar method in \cite{36}, where the block-diagonal form of Yangian $Y(\mathfrak{sl}(2))$ algebra is given by taking a special constraint, to make the $Y(\mathfrak{su}(3))$ algebra reduced. And also, an example is presented to compare the effect of the transition operators of the general Yangian algebras with reduced ones on the entanglement degrees. Results show that the generators of the reduced Yangian algebras can make the final states disentangled while the general ones can’t under the same condition.

II. THE REDUCED $Y(\mathfrak{sl}(2))$ ALGEBRA IN THE BI-QUBIT SYSTEM

A. The Reduced $Y(\mathfrak{sl}(2))$ Algebra

The Yangian $Y(\mathfrak{sl}(2))$ is generated by the generators $\{I_{\alpha}, J_{\alpha}\}$ with the commutation relation\cite{37}:

$[I_{\alpha}, I_{\beta}] = i\epsilon_{\alpha\beta\gamma}I_{\gamma}, \quad [I_{\alpha}, J_{\beta}] = i\epsilon_{\alpha\beta\gamma}J_{\gamma} \quad (\alpha, \beta, \gamma = 1, 2, 3),$

where the $\{I_{\alpha}\}$ form a simple Lie algebra $\mathfrak{sl}(2)$ characterized by $\epsilon_{\alpha\beta\gamma}$ and

$[J_{\pm}, [J_{\beta}, J_{\pm}]] = \frac{\hbar^2}{4}I_{\pm}(J_{\pm}I_{3} - I_{\pm}J_{3}),$

where $\hbar$ is the deformation parameter and the notations $I_{\pm} = I_{1} \pm i I_{2}$ and $J_{\pm} = J_{1} \pm i J_{2}$.

Now let us consider a bi-spin system, in this case the $Y(\mathfrak{sl}(2))$ generators take the form of\cite{38}

$I = S = S_{1} + S_{2},$

$J = \frac{\mu}{\mu + \nu}S_{1} + \frac{\nu}{\mu + \nu}S_{2} + \frac{i \lambda}{\mu + \nu}S_{1} \times S_{2}$.
where $S_1$, $S_2$ are the spin-$\frac{1}{2}$ operators and $\mu$, $\nu$ and $\lambda$ are arbitrary parameters. In this case $I$ is the total spin operator satisfying $[I^a_i, I^b_j] = i\epsilon_{abc}I^c_k\delta_{ij}, (i, j = 1, 2)$.

Then direct calculation shows that
\[
J^2 = \frac{1}{(\mu + \nu)^2}(\mu^2 + \nu^2 - \frac{1}{2} \lambda^2)
+ (2\mu\nu + \frac{1}{2} \lambda^2)S^1_1 S^2_2, \tag{3}
\]
and
\[
[J_a, J_b] = i\epsilon_{abc}(\mu + \nu)^2[-(\mu\nu + \frac{1}{2} \lambda^2)J_c
+ (\mu + \nu)^2J_c]. \tag{4}
\]

With a special constraint relation $\frac{\mu\nu}{\lambda^2} = -\frac{1}{4} \lambda^2$, we can get $J^2 = \frac{1}{4}$, $[J_a, J_b] = i\epsilon_{abc}J_c$. Similarity transformations of the generators can be made by the use of the matrix $\tau$ who takes the form of
\[
\tau = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \nu & -\frac{1}{2} \lambda & 0 \\
0 & \frac{1}{2} \lambda & \nu & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. \tag{5}
\]

After the similar transformations, the generators become
\[
Y^+ = \tau^{-1}J^+\tau = \begin{pmatrix}
\xi^+ & 0 \\
0 & \xi^{-1} \sigma^+
\end{pmatrix},
\]
\[
Y^- = \tau^{-1}J^-\tau = \begin{pmatrix}
\xi^{-1} \sigma^- & 0 \\
0 & \xi^+ 
\end{pmatrix},
\]
\[
Y^3 = \tau^{-1}J^3\tau = \frac{1}{2} \begin{pmatrix}
\sigma^3 & 0 \\
0 & \sigma^3
\end{pmatrix}, \tag{6}
\]
where $\xi = \nu - \frac{1}{2} \lambda$ and $\sigma$ are pauli matrices. $\{Y^a\}$ reduce to Lie algebra and $4 \times 4$ matrix is essentially the 4-dimension representation of sl(2) algebra, so it is marked as the reduced $Y(sl(2))$ algebra in this case.

\section*{B. The Reduced $Y(sl(2))$ Algebra to the Entanglement degree of Bi-qubit System}

For an arbitrary two-qubit pure state $|\Phi\rangle = \alpha |00\rangle + \beta |11\rangle$, where $\alpha, \beta, \gamma, \delta$ are the normalized complex amplitudes, the concurrence (the entanglement of formation) C is given by
\[
C = 2|\alpha \delta - \beta \gamma| \quad \text{and} \quad 0 \leq C \leq 1. \tag{7}
\]

For a maximally entangled states (MES) $C = 1$, we can of course construct another general state as the initial state
\[
|\phi\rangle = \frac{1}{\sqrt{2}}[\alpha(|00\rangle + |11\rangle) + \beta(|01\rangle + |10\rangle)] \tag{8}
\]
where $|\alpha|^2 + |\beta|^2 = 1$.

The concurrence $C$ of the initial state is
\[
C = |\alpha^2 - \beta^2|. \tag{9}
\]

If the transition operator is
\[
P = a(J^3 + 2s_1^3 s_2^3) \tag{10}
\]
which is composed of general Yangian $Y(sl(2))$ generators. $a$ is an arbitrary parameter.

We will get the final state $|\phi'\rangle$
\[
|\phi'\rangle = \frac{a}{\sqrt{2}}[-\frac{\mu - \frac{1}{2} \beta}{\mu + \nu}/0 |01\rangle - \frac{\nu + \frac{1}{2} \beta}{\mu + \nu}/10 |11\rangle. \tag{11}
\]

The normalizing condition is
\[
a^2|\alpha^2 - \beta^2| = 2. \tag{12}
\]

The concurrence of the final state $|\phi'\rangle$ is
\[
C'_\phi = \left|\frac{(\mu - \frac{1}{2} \beta)(\nu + \frac{1}{2} \beta)}{(\mu + \nu)^2} |\alpha^2 - \beta^2| \right|. \tag{13}
\]

If we add an restrictive condition $\mu = -\nu = \frac{1}{2}$, namely, it satisfies $\mu\nu = -\frac{\lambda^2}{4}$, so that we can get the final state $|\phi''\rangle$ acted by reduced Yangian:
\[
|\phi''\rangle = \frac{1}{\sqrt{2}}a|11\rangle. \tag{14}
\]

The normalizing condition is
\[
a^2|\alpha|^2 = 2. \tag{15}
\]

The concurrence of the final state $|\phi''\rangle$ is
\[
C''_\phi = 0. \tag{16}
\]

Comparing Eq.(13) with Eq.(16), it’s easy to see that the reduced Yangian $Y(sl(2))$ can make the final state disentangled while the general one can’t.

\section*{III. THE REDUCED $Y(sl(3))$ ALGEBRA IN THE MIXED LIGHT PSEUDOSCALAR MESON STATES}

\subsection*{A. The Reduced $Y(sl(3))$ Algebra}

As is known that the subalgebra of $Y(sl(3))$ is Lie algebra $su(3)$ which we have been familiar with the $su(3)$ symmetry for elementary particles. For $su(3)$ generators is defined by
\[
[F^a, F^b] = i\epsilon_{abc}F^c \tag{17}
\]
where $a, b, c = 1, 2, \cdots, 8$ and the structure constants $f_{abc}$ are antisymmetric for any two indices:

$$f_{123} = 1, \quad f_{458} = f_{672} = \sqrt{3} \over 2,$$

$$f_{147} = f_{246} = f_{257} = f_{345} = -f_{156} = -f_{367} = 1 \over 2. \quad (18)$$

The 3-dimensional representation of $su(3)$ is formed by the well-known Gell-Mann matrices, i.e.,

$$\Lambda^a = 2F^a, \quad \{\Lambda^a, a = 1, 2, \cdots, 8\}$$

$$[\Lambda^a, \Lambda^b] = 2if_{abc}\Lambda^c, \quad (19)$$

which eight Hermitian traceless matrices are the extension of Pauli matrices:

$$\Lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \Lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \Lambda^8 = \sqrt{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (20)$$

The antisymmetric structure constants possess the properties:

$$f_{ijk} = \frac{1}{4i}Tr([\lambda_i, \lambda_j][\lambda_k]), \quad (21)$$

$$Tr(\lambda_i\lambda_j) = 2\delta_{ij}, \quad (22)$$

According to the Jacobi identity $[F_i, [F_j, F_k]] + cyclic(i, j, k) = 0$, we can get

$$f_{ijm}f_{kmn} + f_{kim}f_{jmn} + f_{jkm}f_{imn} = 0. \quad (23)$$

Using $Tr([\lambda_i, \lambda_j][\lambda_k, \lambda_n] + [\lambda_i, \lambda_n][\lambda_j, \lambda_k]) = 0$, we have

$$f_{ijm}d_{kmn} + f_{kim}d_{jmn} + f_{jkm}d_{imn} = 0. \quad (24)$$

There are two Casimir operators in Lie algebra $su(3)$:

$$C_1 = \sum_{i=1}^{8} F_i^2 = \frac{8}{3} \sum_{ijk} f_{ijk}F_iF_jF_k, \quad (25)$$

$$C_2 = \sum_{ijk} d_{ijk}F_iF_jF_k = C_1(2C_1 - 11 \over 6). \quad (26)$$

Introducing the shift operators

$$I^\pm = F^\pm i F^2, \quad U^\pm = F^6 \pm i F^7, \quad V^\pm = F^4 \mp i F^5,$$

$$I^3 = F^3, \quad Y = 2 \sqrt{3} F^8 = I^8, \quad (Y = hypercharge), \quad (27)$$

the commutation relations $[\Lambda^a, \Lambda^b]$ can be rewritten in the form:

$$[I^3, I^\pm] = \pm I^\pm \quad [I^+, I^-] = 2I^3, \quad [I^3, I^a] = (\alpha = \pm, 3), \quad [I^3, U^\pm] = 2U^\pm, \quad [I^3, V^\pm] = \mp 2V^\pm,$$

$$[I^3, F^\pm] = iF^\pm \quad [I^3, Y] = \mp 2Y, \quad [I^3, I^\pm] = \pm I^\pm, \quad (28)$$

$$[U^\pm, V^\pm] = \pm I^\pm \quad [U^\pm, U^-] = 2U^3, \quad [V^\pm, V^-] = 2V^3. \quad (30)$$

Eqs. (30) show that $\{U^a, a = \pm, 3\}$ have the similar commutation relations as the isospin $\{I^a, a = \pm, 3\}$. So, in general, $U$ and $V$ are called the $U$-spin and $V$-spin. The charge operator $Q$ is given by the isospin $I^3$ and hypercharge $Y$ as follows

$$Q = I^3 + 1 \over 2 Y. \quad (31)$$

And it is easy to check that

$$[U^a, Q] = 0 \quad (a = \pm, 3). \quad (32)$$

But there is no the same property between $I$-spin and $V$-spin.

For two particles, we define the operators of $Y(sl(3))$ as follows:

$$I^a = \sum_i F_i^a, \quad J^a = \mu F_1^a + \nu F_2^a + \overline{i \lambda} f_{abc} \sum_{ij} \omega_{ij} F_i^b F_j^c \quad (i, j = 1, 2). \quad (33)$$

Here

$$\omega_{ij} = \left\{ \begin{array}{ll} 1 & i > j \\ -1 & i < j \\ 0 & i = j \end{array} \right. \quad (34)$$

$$\omega_{ij} = -\omega_{ji}$$
and $\mu, \nu, \lambda$ are parameters or casimir operators. 
\{ $F_i^a, a = 1, 2, \cdots, 8$ \} form a local $su(3)$ on the $i$ site, and they obey the commutation relation

$$[F_i^a, F_j^b] = i f_{abc} \delta_{ij} F_i^c,$$  \hspace{1cm} (35)

the index $i$ here represents different sites. Substituting \{ $\bar{I}, \bar{J}$ \} into the Yangian commutation relations shown in the Eqs.(??) and (??), then we can verify the set \{ $I, J$ \} satisfy $Y(sl(3))$ sufficiently. Eq.(33) plays an important role in explaining the physical meaning of the representation theory of Chari-Pressly \[41\] through more calculation \[42\].

Introducing the notations

$$\bar{I}^\pm = J^1 \pm i J^2, \quad \bar{U}^\pm = J^6 \pm i J^7, \quad \bar{V}^\pm = J^4 \pm i J^5,$$

$$\bar{F}^3 = J^3, \quad \bar{F}^8 = \frac{2}{\sqrt{3}} J^8,$$ \hspace{1cm} (36)

and from Eqs. (33) and (36) the $Y(sl(3))$ can be sufficiently realized by

$$I^\pm = \sum_i I_i^\pm, \quad U^\pm = \sum_i U_i^\pm,$$

$$V^\pm = \sum_i V_i^\pm, \quad I^3 = \sum_i I_i^3, \quad I^8 = \sum_i I_i^8,$$

$$\bar{I}^\pm = \mu I_1^\pm + \nu I_2^\pm \pm \lambda \sum_{i \neq j} \omega_{ij} (I_i^3 I_j^3 + \frac{1}{2} U_i^\mp V_j^\pm),$$

$$\bar{U}^\pm = \mu U_1^\pm + \nu U_2^\pm \pm \lambda \sum_{i \neq j} \omega_{ij} [U_i^\pm (I_j^3 - \frac{3}{2} Y_j)],$$

$$\bar{V}^\pm = \mu V_1^\pm + \nu V_2^\pm \pm \lambda \sum_{i \neq j} \omega_{ij} [V_i^\pm (I_j^3 + \frac{3}{2} Y_j)]$$

$$+ U_i^\mp I_j^3],$$

$$\bar{F}^3 = \mu I_1^3 + \nu I_2^3 - \lambda \sum_{i \neq j} \omega_{ij} [I_i^3 I_j^3 - \frac{1}{2} (U_i^+ U_j^- - V_i^+ V_j^-)],$$

$$\bar{F}^8 = \mu I_1^8 + \nu I_2^8 - \lambda \sum_{i \neq j} \omega_{ij} (U_i^+ U_j^- - V_i^+ V_j^-).$$

Using Eq.(28), under the condition of $\mu \nu = -\frac{1}{2} \lambda$ we get

$$[\bar{I}^\pm, \bar{U}^\mp] = 2(\mu + \nu) \bar{I}^3,$$

$$[\bar{I}^3, \bar{I}^\pm] = \pm (\mu + \nu) \bar{I}^3 \quad [\bar{I}^3, \bar{F}^8] = 0 \quad [\bar{F}^3, \bar{F}^8] = 0$$

$$[\bar{I}^3, \bar{U}^\pm] = \pm (\mu + \nu) \bar{I}^3 \quad [\bar{I}^3, \bar{V}^\pm] = \pm (\mu + \nu) \bar{V}^3,$$

$$[\bar{I}^3, \bar{V}^\pm] = \pm (\mu + \nu) \bar{I}^3 \quad \bar{U}^\pm, \bar{V}^\pm, \bar{F}^3 = \bar{F}^8 = 0,$$

$$[\bar{V}^\pm, \bar{I}^\pm] = \pm \bar{V}^\mp \quad [\bar{V}^\pm, \bar{U}^\pm] = \pm \bar{V}^\mp \quad [\bar{V}^\pm, \bar{F}^8] = 0,$$

$$[\bar{V}^\pm, \bar{V}^\pm] = \mp \bar{V}^\mp \quad [\bar{V}^\pm, \bar{F}^8] = 0,$$

$$[\bar{I}^\pm, \bar{U}^\pm] = \pm \bar{V}^\mp \quad [\bar{I}^\pm, \bar{V}^\pm] = \mp \bar{V}^\mp \quad [\bar{I}^\pm, \bar{F}^8] = 0,$$

$$[\bar{F}^3, \bar{F}^8] = 0 \quad [\bar{F}^3, \bar{F}^8] = 0 \quad [\bar{F}^3, \bar{F}^8] = 0$$

If we introduce the notations

$$\bar{U}^3 = -\frac{1}{2} \bar{F}^3 + \frac{3}{4} \bar{F}^8, \quad \bar{V}^3 = -\frac{1}{2} \bar{F}^3 - \frac{3}{4} \bar{F}^8,$$  \hspace{1cm} (39)

then

$$[\bar{U}^+, \bar{U}^-] = 2(\mu + \nu) \bar{U}^3, \quad [\bar{V}^+, \bar{V}^-] = 2(\mu + \nu) \bar{V}^3.$$

With the help of Eq.(37) and the condition of (??), direct calculation shows

$$(J)^2 = (J^1)^2 + (J^2)^2 + (J^3)^2 + (J^4)^2 + (J^5)^2 + (J^6)^2 + (J^7)^2 + (J^8)^2 + (J^9)^2 + (J^{10})^2$$

$$+(J^1)^2 + (J^2)^2 + (J^3)^2 + (J^4)^2 + (J^5)^2 + (J^6)^2 + (J^7)^2 + (J^8)^2 + (J^9)^2 + (J^{10})^2$$

$$+ \frac{1}{3} (\mu + \nu)^2$$  \hspace{1cm} (40)

It means that if we set

$$Y^n = \frac{1}{\mu + \nu} I^n, \quad (a = 1, 2, \cdots, 8)$$  \hspace{1cm} (41)

in the terms of the notations

$$\bar{I}^\pm = Y^1 \pm i Y^2, \quad \bar{U}^\pm = Y^6 \pm i Y^7, \quad \bar{V}^\pm = Y^4 \pm i Y^5,$$

$$\bar{F}^3 = Y^3, \quad \bar{F}^8 = \frac{2}{\sqrt{3}} Y^8,$$  \hspace{1cm} (42)

we have

$$\langle Y \rangle^2 = \frac{1}{3}$$  \hspace{1cm} (43)

In the following we get the commutation relations

$$[\bar{I}^3, \bar{I}^\pm] = 2 \bar{I}^3 \quad [\bar{I}^3, \bar{F}^8] = [\bar{I}^\pm, \bar{F}^8] = 0$$

$$[\bar{I}^3, \bar{U}^\pm] = \pm \bar{U}^\pm \quad [\bar{I}^3, \bar{V}^\pm] = \pm \bar{V}^\pm \quad [\bar{I}^3, \bar{F}^8] = 0$$

$$[\bar{I}^3, \bar{F}^8] = \pm \bar{F}^3 \quad [\bar{I}^3, \bar{F}^8] = \pm \bar{F}^8 \quad [\bar{I}^3, \bar{F}^8] = 0$$

$$[\bar{I}^3, \bar{F}^8] = \pm \bar{F}^3 \quad [\bar{I}^3, \bar{F}^8] = \pm \bar{F}^8 \quad [\bar{I}^3, \bar{F}^8] = 0$$

where $\bar{U}^3 = -\frac{1}{2} \bar{F}^3 + \frac{3}{4} \bar{F}^8$ and $\bar{V}^3 = -\frac{1}{2} \bar{F}^3 - \frac{3}{4} \bar{F}^8$.

It is similar with the commutation relations of the $I$-spin, $U$-spin, and $V$-spin, namely the Equation (28). By the discussion we have the result: a general realization of $Y(sl(3))$ is given by equations

$$\sum_i F_i^a = \frac{\mu}{\mu + \nu} I_1^a + \frac{\nu}{\mu + \nu} I_2^a + \frac{\mu + \nu}{2(\mu + \nu)} f_{abc} \sum_{i \neq j} \omega_{ij} I_i^b I_j^c \quad (i, j = 1, 2).$$  \hspace{1cm} (45)
Then if we take the condition $\mu \nu = -\frac{\lambda^2}{4}$ and the fundamental representation of local $su(3)$ is held, they have the same commutation relations between the two generators of Yangian, namely $Y(su(3))$ algebra has reduced to two sets of $su(3)$ algebras.

In fact, the fundamental representation of local $su(3)$ is given by

$$I^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad U^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$V^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad I^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix}, \quad \text{(46)}$$

and $I^- = (I^+)^*, \quad U^- = (U^+)^*, \quad V^- = (V^+)^*$.

By using Eqs. (37), (41) and (46), we get

$$\tilde{I}^+ = \frac{1}{\mu + \nu} \left[ \mu I_1^+ + \nu I_2^+ + \lambda (I_1^+ I_2^3 + \frac{1}{2} U_1^- V_2^-) - \lambda (I_2^1 I_2^3 + \frac{1}{2} V_1^- U_2^-) \right]$$

$$\tilde{I}^- = \frac{1}{\mu + \nu} \left[ \mu I_1^- + \nu I_2^- - \lambda (I_1^- I_2^3 + \frac{1}{2} U_1^+ V_2^+) + \lambda (I_2^1 I_2^3 + \frac{1}{2} V_1^+ U_2^+) \right]$$

$$\tilde{U}^+ = \frac{1}{\mu + \nu} \left[ \mu U_1^+ + \nu U_2^+ + \frac{1}{2} \left( U_1^- (I_2^3 - \frac{3}{2} Y_2) + I_2^- V_2^- - U_2^- (I_2^3 - \frac{3}{2} Y_1) I_2^- V_1^- \right) \right]$$

$$\tilde{U}^- = \frac{1}{\mu + \nu} \left[ \mu U_1^- + \nu U_2^- + \frac{1}{2} \left( U_1^+ (I_2^3 - \frac{3}{2} Y_2) + I_2^+ V_2^+ - U_2^+ (I_2^3 - \frac{3}{2} Y_1) I_2^+ V_1^+ \right) \right]$$

$$\tilde{V}^+ = \frac{1}{\mu + \nu} \left[ \mu V_1^+ + \nu V_2^+ + \frac{1}{2} \left( V_1^- (I_2^3 + \frac{3}{2} Y_2) + U_1^- V_2^- - (I_2^3 + \frac{3}{2} Y_1) V_2^- + I_2^- U_2^- \right) \right]$$

$$\tilde{V}^- = \frac{1}{\mu + \nu} \left[ \mu V_1^- + \nu V_2^- + \frac{1}{2} \left( V_1^+ (I_2^3 + \frac{3}{2} Y_2) + U_1^+ V_2^+ - (I_2^3 + \frac{3}{2} Y_1) V_2^+ + I_2^+ U_2^- \right) \right]$$

$$\tilde{F}^3 = \frac{1}{\mu + \nu} \left[ \mu F_1^3 + \nu F_2^3 + \frac{1}{2} \left( F_1^3 + \frac{1}{2} (U_1^- U_2^- - I_1^- I_2^-) V_1^- V_2^- \right) \right]$$

$$\tilde{F}^8 = \frac{1}{\mu + \nu} \left[ \mu F_1^8 + \nu F_2^8 - \frac{\lambda}{2} (U_1^+ U_2^- - V_1^- V_2^+) \right] \quad \text{(47)}$$

with $E$ a unite matrix and its solutions are

$$x_1 = 0, \quad x_{2,3} = \pm \frac{1}{2},$$

$$x_{4,5} = \pm \frac{1}{2(\mu + \nu)} \sqrt{\mu^2 - 2\mu \nu + \nu^2 - \lambda^2},$$

$$x_{6,7} = \frac{1}{4} \pm \frac{1}{4(\mu + \nu)} \sqrt{\mu^2 - 2\mu \nu + \nu^2 - \lambda^2},$$

$$x_{8,9} = -\frac{1}{4} \pm \frac{1}{4(\mu + \nu)} \sqrt{\mu^2 - 2\mu \nu + \nu^2 - \lambda^2} \quad \text{(48)}$$

Taking $\mu \nu = -\frac{\lambda^2}{4}$, then we can get

$$x_{1,2,3} = 0, \quad x_{4,5,6} = \frac{1}{2}, \quad x_{7,8,9} = -\frac{1}{2} \quad \text{(49)}$$

If we take the similar matrix as

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \nu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \nu & 0 & \nu & 0 \\ 0 & 0 & 0 & 0 & \nu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \nu & 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{(50)}$$

Setting $x$ the eigenvalue of $\tilde{F}^3$, then $|x E - \tilde{F}^3| = 0$ and its inverse matrix $A^{-1}$ can be obtained easily, thus...
we get

\[
(\tilde{P}_3)' = A^{-1}\tilde{P}_4 A = \begin{pmatrix}
I^3 & 0 & 0 \\
0 & I^3 & 0 \\
0 & 0 & I^3
\end{pmatrix}
\]

\[
(\tilde{P}_8)' = A^{-1}\tilde{P}_8 A = \begin{pmatrix}
I^8 & 0 & 0 \\
0 & I^8 & 0 \\
0 & 0 & I^8
\end{pmatrix}
\]

\[
(\tilde{I}^+)' = A^{-1}\tilde{I}^+ A = \begin{pmatrix}
\alpha I^+ & 0 & 0 \\
0 & \alpha^{-1} I^+ & 0 \\
0 & 0 & I^+
\end{pmatrix}
\]

\[
(\tilde{I}^-)' = A^{-1}\tilde{I}^- A = \begin{pmatrix}
\alpha^{-1} I^- & 0 & 0 \\
0 & \alpha I^- & 0 \\
0 & 0 & I^-
\end{pmatrix}
\]

\[
(\tilde{U}^+)' = A^{-1}\tilde{U}^+ A = \begin{pmatrix}
U^+ & 0 & 0 \\
0 & \alpha U^+ & 0 \\
0 & 0 & \alpha^{-1} U^+
\end{pmatrix}
\]

\[
(\tilde{U}^-)' = A^{-1}\tilde{U}^- A = \begin{pmatrix}
U^- & 0 & 0 \\
0 & \alpha^{-1} U^- & 0 \\
0 & 0 & \alpha U^-
\end{pmatrix}
\]

\[
(\tilde{V}^+)' = A^{-1}\tilde{V}^+ A = \begin{pmatrix}
\alpha^{-1} V^+ & 0 & 0 \\
0 & V^+ & 0 \\
0 & 0 & \alpha V^+
\end{pmatrix}
\]

\[
(\tilde{V}^-)' = A^{-1}\tilde{V}^- A = \begin{pmatrix}
\alpha V^- & 0 & 0 \\
0 & V^- & 0 \\
0 & 0 & \alpha^{-1} V^-
\end{pmatrix}
\]

Eq. (43) will get the same result, that is, the Yangian algebra we discussed hides a u(1) algebra.

If the transition operator is

\[
P = \eta_1 \tilde{V}^+ + \eta_2 \tilde{V}^-
\]

which is composed of general Yangian generators.

We will get the final state |φ’⟩

\[
|φ'⟩ = \frac{1}{\sqrt{3}}[(\mu + \nu)\eta_2 \alpha_1 + (\mu + \lambda)\eta_2 \alpha_2 + (\mu + \nu)\eta_1 (\alpha_1 + \alpha_2)]|η^0⟩ + \frac{1}{\sqrt{2}}[(\mu + \nu)\eta_2 \alpha_1 + (\mu + \lambda)\eta_2 \alpha_2]|π^0⟩
\]

\[
+ \frac{1}{\sqrt{6}}[2(\mu + \nu)\eta_1 (\alpha_1 + \alpha_2) - (\mu + \nu)\eta_2 \alpha_1 - (\mu + \lambda)\eta_2 \alpha_2]|η^0⟩.
\]

The normalizing condition is

\[
[(\mu + \nu)\eta_2 \alpha_1 + (\mu + \lambda)\eta_2 \alpha_2]^2 + (\mu + \nu)^2 \eta_1^2 (\alpha_1 + \alpha_2)^2 = 1
\]

The degree of entanglement of the final state |φ’⟩ is

\[
C_φ' = -[(\mu + \nu)\eta_2 \alpha_1 + (\mu + \lambda)\eta_2 \alpha_2]^2 \log_3[(\mu + \nu)\eta_2 \alpha_1 + (\mu + \lambda)\eta_2 \alpha_2]^2 - (\mu + \nu)^2 \eta_1^2 (\alpha_1 + \alpha_2)^2 \log_3(\mu + \nu)^2 \eta_1^2 (\alpha_1 + \alpha_2)^2.
\]

If we add an restrictive condition \(\mu = -\nu = \frac{\lambda}{2}\), namely, it satisfies \(\mu\nu = -\frac{\lambda^2}{4}\). So that we can get the final state |φ’’⟩ acted by reduced Yangian:

\[
|φ''⟩ = \frac{3}{2} \lambda \eta_2 \alpha_2 \left( \frac{\sqrt[3]{3}}{\sqrt{2}} |η^0⟩ + \sqrt[2]{\eta^0} - \frac{1}{\sqrt{6}} |η^0⟩ \right)
\]

The normalizing condition is

\[
\frac{9}{4} \lambda^2 \eta_2^2 \alpha_2^2 = 1.
\]

The degree of entanglement of the final state |φ’’⟩ is

\[
C_φ'' = 0.
\]

We can obtain the same conclusion that the reduced Yangian Y(su(3)) can make the final state disentangled while the general one can’t from Eq. (57) and Eq. (60).

IV. CONCLUSION

In this paper, we have discussed the reduced properties and applications of Yangian Y(sl(2)) and Y(su(3)). As same as Y(sl(2)), Y(su(3)) algebra has been reduced into two sets of su(3) algebras, moreover the structure is the same as su(2) case, i.e. those formed by the generators \(\textbf{Y}\) of Yangian are all constructed by the consequent generators of su(3). Moreover, we have compared the influence of the transition
operators of general Yangian algebras with reduced ones on the entanglement degrees and found that the reduced ones can make the initial states disentangled while the general ones can not both for $Y(sl(2))$ and $Y(su(3))$.

Now a question put forward: can we use this method to $su(n)$? To our knowledge, this problem has not been discussed in this thesis. But we can surmise that for $Y(su(n))$ the matrices of the generators $Y$ can be written as $n$ pieces of $n \times n$ matrices, furthermore each pieces is formed by the consequently generators of $su(n)$. Consequently, it is of interesting area how to generalize the idea in $Y(su(n))$, which makes the system contact with physical application.

V. ACKNOWLEDGEMENTS

This work is in part supported by the NSF of China under Grant No. 10775092 and No.10875026, Shanghai Leading Academic Discipline Project (Project number S30105) and Shanghai Research Foundation No.07d222020.