TRANSVERSAL LIGHTLIKE SUBMANIFOLDS OF
METALLIC SEMI-RIEMANNIAN MANIFOLDS

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ABSTRACT. The Metallic Ratio is fascinating topic that continually generated news ideas. A Riemannian manifold endowed with a Metallic structure will be called a Metallic Riemannian manifold. The main purpose of the present paper is to study the geometry of transversal lightlike submanifolds and radical transversal lightlike submanifolds and of Metallic Semi-Riemannian manifolds. We investigate the geometry of distributions and obtain necessary and sufficient conditions for the induced connection on these manifolds to be metric connection. We also obtain characterization of transversal lightlike submanifolds of Metallic semi-Riemannian manifolds. Finally, we give an example.

1. INTRODUCTION

Lightlike submanifolds are one of the most interesting topics in differential geometry. It is well known that a submanifold of a Riemannian manifold is always a Riemannian one. Contrary to that case, in semi-Riemannian manifolds the induced metric by the semi-Riemann metric on the ambient manifold is not necessarily nondegenerate. Since the induced metric is degenerate on lightlike submanifolds, the tools which are used to investigate the geometry of submanifolds in Riemannian case are not favorable in semi-Riemannian case and so the classical theory can not be used to define any induced object on a lightlike submanifold. The main difficulties arise from the fact that the intersection of the normal bundle and the tangent bundle of a lightlike submanifold is nonzero. In 1996, K.Duggal-A.Bejancu [10] put forward the general theory of lightlike submanifolds of semi-Riemannian manifolds in their book. In order to resolve the difficulties that arise during studying lightlike submanifolds, they introduced a non-degenerate distribution called screen distribution to construct a lightlike transversal vector bundle which does not intersect to its lightlike tangent bundle. It is well-known that a suitable choice of screen distribution gives rises to many substantial results in lightlike geometry. Many authors have studied the geometry of lightlike submanifolds [12, 13, 14, 15, 16, 17, 24, 25, 26].
in different manifolds. For further read we refer [10, 11] and the references therein.

Manifolds with various geometric structures are convenient to study submanifold theory. In recent years, one of the most studied manifold types are Riemannian manifolds with metallic structures. Metallic structures on Riemannian manifolds allow many geometric results to be given on a submanifold.

As a generalization of the Golden mean which contains the Silver mean, the Bronze mean, the Copper mean and the Nickel mean etc., Metallic means family was introduced by V. W. de Spinadel [21] in 2002. The positive solution of the equation given by

\[ x^2 - px - q = 0, \]

for some positive integer \( p \) and \( q \), is called a \((p, q)\)-metallic number [18, 20] which has the form

\[ \sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}. \]

For \( p = q = 1 \) and \( p = 2, q = 1 \), it is well-known that we have the Golden mean \( \phi = 1 + \frac{\sqrt{5}}{2} \) and Silver mean \( \sigma_{2,1} = 1 + \sqrt{2} \), respectively. The metallic mean family plays an important role to establish a relationship between mathematics and architecture. For example golden mean and silver mean can be seen in the sacred art of Egypt, Turkey, India, China and other ancient civilizations [22].

S. I. Goldberg, K. Yano and N. C. Petridis in ([4] and [5]) introduced polynomial structures on manifolds. As some particular cases of polynomial structures C. E. Hretcanu and M. Crasmareanu defined Golden structure [1, 2, 3, 28] and some generalizations of this, called metallic structure [6]. Being inspired by the Metallic mean, the notion of Metallic manifold \( \tilde{N} \) was defined in [6] by a \((1, 1)\)-tensor field \( \tilde{J} \) on \( \tilde{N} \), which satisfies \( \tilde{J}^2 = p\tilde{J} + qI \), where \( I \) is the identity operator on the Lie algebra \( \chi(\tilde{N}) \) of vector fields on \( \tilde{N} \) and \( p, q \) are fixed positive integer numbers. Moreover, if \((\tilde{N}, g)\) is a Riemannian manifold endowed with a metallic structure \( \tilde{J} \) such that the Riemannian metric \( \tilde{g} \) is \( \tilde{J} \)-compatible, i.e., \( \tilde{g}(\tilde{J}V, W) = \tilde{g}(V, \tilde{J}W) \), for any \( V, W \in \chi(\tilde{N}) \), then \((\tilde{g}, \tilde{J})\) is called metallic Riemannian structure and \((\tilde{N}, \tilde{g}, \tilde{J})\) is a Metallic Riemannian manifold. Metallic structure on the ambient Riemannian manifold provides important geometrical results on the submanifolds, since it is an important tool while investigating the geometry of submanifolds. Invariant, anti-invariant, semi-invariant, slant and semi-slant submanifolds of a Metallic Riemannian manifold are studied in
and the authors obtained important characterizations on submanifolds of Metallic Riemannian manifolds.

One of the most important subclasses of Metallic Riemannian manifolds is the Golden Riemannian manifolds. Many authors have studied Golden Riemannian manifolds and their submanifolds in recent years (see [1, 2, 3, 7, 8, 9, 23]). N. Poyraz Önen and E. Yaşar [16] initiated the study of lightlike geometry in Golden semi-Riemannian manifolds, by investigating lightlike hypersurfaces of Golden semi-Riemannian manifolds. B. E. Acet introduced lightlike hypersurfaces in Metallic semi-Riemannian manifolds [32].

Motivated by the studies on submanifolds of Metallic Riemannian manifolds and lightlike submanifolds of semi-Riemannian manifolds, in the present paper we introduce the transversal lightlike submanifolds of a Metallic semi-Riemannian manifold.

Considering given brief above, in this paper, we introduce transversal lightlike submanifolds of Metallic semi-Riemannian manifolds and studied their differential geometry. The paper is organized as follows: In Section 2 is devoted to basic definitions needed for the rest of the paper. In Section 3 and Section 4 , we introduce a Metallic semi-Riemannian manifold along with its subclasses, namely radial transversal and transversal lightlike submanifolds and obtain some characterizations. We investigate the geometry of distributions and find necessary and sufficient conditions for induced connection to be a metric connection. Furthermore, we give an example.

2. PRELIMINARIES

A submanifold $\tilde{N}^m$ immersed in a semi-Riemannian manifold $(\tilde{\mathbf{N}}^{m+k}, \tilde{g})$ is called a lightlike submanifold if it admits a degenerate metric $g$ induced from $\tilde{g}$, whose radical distribution $\text{Rad} T\tilde{N}$ is of rank $r$, where $1 \leq r \leq m$. Then $\text{Rad} T\tilde{N} = T\tilde{N} \cap T\tilde{N}^\perp$, where

$$T\tilde{N}^\perp = \bigcup_{x \in \tilde{N}} \left\{ u \in T_x\tilde{N} \mid \tilde{g}(u, v) = 0, \forall v \in T_x\tilde{N} \right\}.$$  

Let $S(T\tilde{N})$ be a screen distribution which is a semi-Riemannian complementary distribution of $\text{Rad} T\tilde{N}$ in $T\tilde{N}$ i.e., $T\tilde{N} = \text{Rad} T\tilde{N} \perp S(T\tilde{N})$.

We consider a screen transversal vector bundle $S(T\tilde{N}^\perp)$, which is a semi-Riemannian complementary vector bundle of $\text{Rad} T\tilde{N}$ in $T\tilde{N}^\perp$. Since, for any local basis $\{\xi_i\}$ of $\text{Rad} T\tilde{N}$, there exists a lightlike transversal vector bundle $ltr(T\tilde{N})$ locally spanned by $\{N_i\}$. Let $tr(T\tilde{N})$ be
complementary (but not orthogonal) vector bundle to \( T\tilde{N} \) in \( T\tilde{N}^\perp |_{\tilde{N}} \). Then, we have

\[
\begin{align*}
\text{tr}(T\tilde{N}) &= \operatorname{ltr} T\tilde{N} \perp S(T\tilde{N}^\perp), \\
T\tilde{N} |_{\tilde{N}} &= S(T\tilde{N}) \perp [\operatorname{Rad} T\tilde{N} \oplus \operatorname{ltr} T\tilde{N}] \perp S(T\tilde{N}^\perp).
\end{align*}
\]

Although \( S(T\tilde{N}) \) is not unique, it is canonically isomorphic to the factor vector bundle \( T\tilde{N}/\operatorname{Rad} T\tilde{N} \) [10].

The following result is important for this paper.

**Proposition 2.1.** The lightlike second fundamental forms of a lightlike submanifold \( \tilde{N} \) do not depend on \( S(T\tilde{N}), S(T\tilde{N}^\perp) \) and \( \operatorname{ltr} T\tilde{N} \) [10].

We say that a submanifold \((\tilde{N}, g, S(T\tilde{N}), S(T\tilde{N}^\perp))\) of \( \tilde{N} \) is

- Case 1: \( r \)-lightlike if \( r < \min \{ m, k \} \);
- Case 2: Co-isotropic if \( r = k < m; S(T\tilde{N}^\perp) = \{ 0 \} \);
- Case 3: Isotropic if \( r = m = k; S(T\tilde{N}) = \{ 0 \} \);
- Case 4: Totally lightlike if \( r = k = m; S(T\tilde{N}) = \{ 0 \} = S(T\tilde{N}^\perp) \).

The Gauss and Weingarten equations are

\[
\begin{align*}
(2.2) \tilde{\nabla}_W U &= \nabla_W U + h(W, U), \quad \forall W, U \in \Gamma(T\tilde{N}), \\
(2.3) \tilde{\nabla}_W V &= -A_W V + \nabla_W^V V, \quad \forall W, V \in \Gamma(\text{tr}(T\tilde{N})),
\end{align*}
\]

where \( \{ \nabla_W U, A_W V \} \) and \( \{ h(W, U), \nabla_W^V V \} \) belong to \( \Gamma(T\tilde{N}) \) and \( \Gamma(\text{tr}(T\tilde{N})) \), respectively. Here, \( \nabla \) and \( \nabla^t \) denote linear connections on \( \tilde{N} \) and the vector bundle \( \text{tr}(T\tilde{N}) \), respectively. Moreover, we have

\[
\begin{align*}
(2.4) \tilde{\nabla}_W U &= \nabla_W U + h^\ell(W, U) + h^s(W, U), \quad \forall W, U \in \Gamma(T\tilde{N}), \\
(2.5) \tilde{\nabla}_W N &= -A_N W + \nabla_W^N N + D^s(W, N), \quad N \in \Gamma(\operatorname{ltr} T\tilde{N}), \\
(2.6) \tilde{\nabla}_W Z &= -A_Z W + \nabla_W^Z Z + D^\ell(W, Z), \quad Z \in \Gamma(S(T\tilde{N}^\perp)).
\end{align*}
\]

Denote the projection of \( T\tilde{N} \) on \( S(T\tilde{N}) \) by \( P \). Then by using (2.2)- (2.4) and the fact that \( \tilde{\nabla} \) being a metric connection, we obtain

\[
\begin{align*}
(2.7) \tilde{g}(h^s(W, U), Z) + \tilde{g}(U, D^\ell(W, Z)) &= \tilde{g}(A_Z W, U), \\
(2.8) \tilde{g}(D^s(W, N), Z) &= \tilde{g}(N, A_Z W).
\end{align*}
\]

From the decomposition of the tangent bundle of a lightlike submanifold, we have

\[
\begin{align*}
(2.9) \nabla_W PU &= \nabla^*_W PU + h^s(W, PU), \\
(2.10) \nabla_W \xi &= -A^*_\xi W + \nabla^*_W \xi,
\end{align*}
\]
for $W, U \in \Gamma(T\tilde{N})$ and $\xi \in \Gamma(Rad T\tilde{N})$. By using above equations, we obtain

\begin{align}
(2.11) & \quad g\left(h^\ell(W, PU), \xi\right) = g\left(A_\xi W, PU\right), \\
(2.12) & \quad g\left(h^s(W, PU), N\right) = g\left(A_N W, PU\right), \\
(2.13) & \quad g\left(h^\ell(W, \xi), \xi\right) = 0, \quad A_\xi^2 \xi = 0.
\end{align}

In general, the induced connection $\nabla$ on $\tilde{N}$ is not a metric connection. Since $\tilde{\nabla}$ is a metric connection, by using (2.4) we get

\begin{equation}
(2.14) \quad (\nabla_W g)(U, V) = \bar{g}\left(h^\ell(W, U), V\right) + \bar{g}\left(h^\ell(W, V), U\right).
\end{equation}

However, we note that $\nabla^*$ is a metric connection on $S(T\tilde{N})$.

Fix two positive integers $p$ and $q$. The positive solution of the equation

$$x^2 - px - q = 0,$$

is entitled member of metallic means family (18–22). These numbers, denoted by

\begin{equation}
(2.15) \quad \sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2},
\end{equation}

are called $(p, q)$-metallic numbers.

**Definition 2.1.** A polynomial structure on a manifold $\tilde{N}$ is called a Metallic structure if it is determined by an $(1, 1)$-tensor field $\tilde{J}$, which satisfies

\begin{equation}
(2.16) \quad \tilde{J}^2 = p\tilde{J} + qI,
\end{equation}

where $I$ is the identity map on $\tilde{N}$ and $p, q$ are positive integers. Also, if

\begin{equation}
(2.17) \quad \tilde{g}(\tilde{J}W, U) = \tilde{g}(W, \tilde{J}U)
\end{equation}

holds then the semi-Riemannian metric $\tilde{g}$ is called $\tilde{J}$-compatible, for every $U, W \in \Gamma(T\tilde{N})$. In this case $(\tilde{N}, \tilde{g}, \tilde{J})$ is named a Metallic semi-Riemannian manifold. Also a Metallic semi-Riemannian structure $\tilde{J}$ is called a locally Metallic structure if $\tilde{J}$ is parallel with respect to the Levi-Civita connection $\nabla$, that is

\begin{equation}
(2.18) \quad \nabla_W \tilde{J}U = \tilde{J}\nabla_W U
\end{equation}

[1].
If \( J \) be a metallic structure, then (2.17) is equivalent to
\[
(2.19) \quad \tilde{g}(\tilde{J}W, \tilde{J}U) = p\tilde{g}(\tilde{J}W, U) + q\tilde{g}(W, U),
\]
for any \( W, U \in \Gamma(T\tilde{N}) \).

3. **Radical Transversal Lightlike Submanifolds of Metallic semi-Riemannian Manifolds**

In this section, we introduce radical transversal lightlike submanifolds of a Metallic semi-Riemannian manifold.

**Definition 3.1.** Let \((\tilde{N}, g, S(T\tilde{N}), S(T\tilde{N}^{\perp}))\) be a lightlike submanifold of a Metallic semi-Riemannian manifold \((\tilde{N}, \tilde{g}, \tilde{J})\). If the following conditions are satisfied, then the lightlike submanifold \( \tilde{N} \) is called a radical transversal lightlike submanifold:
\[
(3.1) \quad \tilde{J}Rad T\tilde{N} = ltr T\tilde{N},
\]
\[
(3.2) \quad \tilde{J}S(T\tilde{N}) = S(T\tilde{N}).
\]

**Proposition 3.1.** Let \( \tilde{N} \) be a Metallic semi-Riemannian manifold. In this case, there is not 1-radical transversal lightlike submanifold of \( \tilde{N} \).

**Proof.** Let \( \tilde{N} \) be a 1-radical transversal lightlike submanifold. Hence, \( Rad T\tilde{N} = \{ \xi \} \) and \( ltr T\tilde{N} = \{ N \} \). From the equation (2.19), we have
\[
(3.3) \quad \tilde{g}(\tilde{J}\xi, \xi) = \tilde{g}(\xi, \tilde{J}\xi) = 0.
\]
On the other hand, from (3.1), we have
\[
\tilde{g}(\tilde{J}\xi, \xi) = \tilde{g}(\xi, \tilde{J}\xi) = \tilde{g}(N, \xi) = \tilde{g}(\xi, N) = 1,
\]
which contradicts equation (3.3). The proof is completed. \( \Box \)

**Theorem 3.1.** Let \( \tilde{N} \) be a radical transversal lightlike submanifold of a Metallic semi-Riemannian manifold \( \tilde{N} \). In this case, the distribution \( S(T\tilde{N}^{\perp}) \) is invariant with respect to \( \tilde{J} \).

**Proof.** For \( V \in \Gamma(S(T\tilde{N}^{\perp})) \) and \( \xi \in \Gamma(Rad T\tilde{N}) \), from (2.19), we find
\[
\tilde{g}(\tilde{J}V, \xi) = \tilde{g}(V, \tilde{J}\xi) = 0,
\]
which implies that there is not a component in \( ltr T\tilde{N} \) of \( \tilde{J}V \).

Similarly, for \( N \in \Gamma(tr T\tilde{N}) \), from (2.19), we have
\[
(3.4) \quad \tilde{g}(\tilde{J}V, N) = \tilde{g}(V, \tilde{J}N) = \frac{1}{p}\tilde{g}(\tilde{J}V, \tilde{J}N).
\]
From definition of a radical transversal lightlike submanifold, for \( \xi_1 \in \Gamma(Rad T\tilde{N}) \) and \( N_1 \in \Gamma(ltr T\tilde{N}) \), we get
\[
\tilde{J}\xi_1 = N_1.
\]
If we apply \( \tilde{J} \) to the last equation, we can write
\[
p\tilde{J}\xi_1 + q\xi_1 = \tilde{J}N_1,
\]
which implies equation (3.4) equals zero. Namely, we see that, there is no component of \( \tilde{J}V \) in \( Rad T\tilde{N} \).

By a similar way, for \( W \in \Gamma(S(T\tilde{N})) \), we obtain
\[
\tilde{g}(\tilde{J}V, W) = \tilde{g}(V, \tilde{J}W) = 0,
\]
that is, there is no component of \( \tilde{J}V \) in \( S(T\tilde{N}) \). Hence, the proof is completed. \( \square \)

Let \( \tilde{N} \) be a radical transversal lightlike submanifold of Metallic semi-Riemannian manifold \( \tilde{N} \). \( Q \) and \( T \) denote projection morphisms in \( Rad T\tilde{N} \) and \( S(T\tilde{N}) \), respectively. For any \( W \in \Gamma(T\tilde{N}) \), we can write
\[
W = TW + QW,
\]
where \( TW \in \Gamma(S(T\tilde{N})) \) and \( QW \in \Gamma(Rad T\tilde{N}) \). By applying \( \tilde{J} \) to (3.5), we have
\[
\tilde{J}W = \tilde{J}TW + \tilde{J}QW.
\]
Here, if we write \( \tilde{J}TW = SW \) and \( \tilde{J}QW = LW \), then (3.6) becomes
\[
\tilde{J}W = SW + LW,
\]
where, \( SW \in \Gamma(S(T\tilde{N})) \) and \( LW \in ltr T\tilde{N} \).

Assume, \( \tilde{N} \) be a radical transversal submanifold of a locally Metallic semi-Riemannian manifold \( \tilde{N} \). From (2.18), (2.4) and (2.6), we have
\[
\tilde{\nabla}_U (SW + LW) = \tilde{J} \left( \nabla_U W + \check{h}(U, W) + h^s(U, W) \right),
\]
where \( U, W \in \Gamma(T\tilde{N}) \). If we write \( \check{h}(U, W) = K_1\check{h}(U, W) + K_2\check{h}(U, W) \), where \( K_1 \) and \( K_2 \) are projection morphisms of \( \tilde{J}ltr T\tilde{N} \) in \( ltr T\tilde{N} \) and \( Rad T\tilde{N} \), respectively, we find
\[
\left( \begin{array}{c} \nabla_U SW + h^l(U, SW) + h^s(U, SW) \\ -A_{LW}U + \nabla_U LW + D^s(U, LW) \end{array} \right) = \left( \begin{array}{c} S\nabla_U W + L\nabla_U W + \check{h}(U, W) \\ + K_1\check{h}(U, W) + K_2\check{h}(U, W) \end{array} \right).
\]
Thus, by equating the tangent, screen transversal and lightlike transversal parts components, we have

\[ \nabla_U SW - A_{LW} U = S \nabla_U W + K_2 \tilde{J} h^l(U, W), \]
\[ h^S(U, SW) + D^s(U, LW) = \tilde{J} h^s(U, W), \]
\[ h^l(U, SW) + \nabla_U LW = L \nabla_U W + K_1 \tilde{J} h^l(U, W). \]

Therefore we give the following proposition.

**Proposition 3.2.** Let \( \hat{N} \) be a radical transversal lightlike submanifold of a locally Metallic semi-Riemannian manifold \( \hat{N} \). Then, we have

\[ (3.8) \quad \nabla_U SW - A_{LW} U = S \nabla_U W + K_2 \tilde{J} h^l(U, W), \]
\[ (3.9) \quad 0 = h^S(U, SW) + D^s(U, LW) - \tilde{J} h^s(U, W), \]
\[ (3.10) \quad 0 = h^l(U, SW) + \nabla_U LW - L \nabla_U W - K_1 \tilde{J} h^l(U, W), \]

for \( W, U \in \Gamma(T \hat{N}) \).

**Theorem 3.2.** Let \( \hat{N} \) be a radical transversal lightlike submanifold of a locally Metallic semi-Riemannian manifold \( \hat{N} \). Then, the induced connection \( \nabla \) on \( \hat{N} \) is a metric connection if and only if there is no component of \( A_{J \xi} W \) in \( \Gamma(S(T \hat{N})) \), for \( W \in \Gamma(T \hat{N}) \) and \( \xi \in \Gamma(Rad T \hat{N}) \).

**Proof.** Assume that the induced connection \( \nabla \) is a metric connection. In this case, for \( W \in \Gamma(T \hat{N}) \) and \( \xi \in \Gamma(Rad T \hat{N}) \), \( \nabla_\xi W \in \Gamma(Rad T \hat{N}) \). Here, \( U \in \Gamma(S(T \hat{N})) \), we have

\[ g(\nabla_\xi W, U) = \tilde{g}(\tilde{\nabla}_\xi W, U) = 0. \]

If we use \( (2.19) \), we find

\[ 0 = \tilde{g}(\tilde{J} \tilde{\nabla}_\xi W, \tilde{J} U) - p\tilde{g}(\tilde{\nabla}_\xi W, \tilde{J} U), \]

and from \( (2.5) \), we have

\[ \tilde{g}(A_{J \xi} W, \tilde{J} U) = 0, \]

which implies, there is no component of \( A_{J \xi} W \) in \( \Gamma(S(T \hat{N})) \).

Since the converse is obvious, then we omit it. \( \square \)

Now, we shall investigate the conditions for integrability of the distributions involved in the definition of radical transversal lightlike submanifolds.
Theorem 3.3. Let \( \tilde{N} \) be a radical transversal lightlike submanifold of a locally Metallic semi-Riemannian manifold \( \tilde{\mathcal{N}} \). In this case, the screen distribution is integrable if and only if
\[
h^l(U, SW) = h^l(W, SU),
\]
for \( W, U \in \Gamma(S(T\tilde{N})) \).

Proof. For \( W, U \in \Gamma(S(T\tilde{N})) \), if we use equation (3.10) and by interchanging the roles of \( U \) and \( W \), we find
\[
h^l(U, SW) - h^l(W, SU) - K_1 \left( \tilde{J}h^l(U, W) - \tilde{J}h^l(W, U) \right) = L[U, W].
\]
Since \( h^l \) is symmetric, we obtain
\[
h^l(U, SW) - h^l(W, SU) = L[U, W],
\]
which completes the proof. \( \square \)

Theorem 3.4. Let \( \tilde{N} \) be a radical transversal lightlike submanifold of a locally Metallic semi-Riemannian manifold \( \tilde{\mathcal{N}} \). The radical distribution is integrable if and only if
\[
A_{LU}W = A_{LW}U,
\]
for \( V, W \in \Gamma(\text{Rad } T\tilde{N}) \).

Proof. For \( V, W \in \Gamma(\text{Rad } T\tilde{N}) \), if we use equation (3.8), we have
\[
-S\nabla_U W = A_{LW}U + K_2\tilde{J}h^l(U, W),
\]
by virtue of \( SW = 0 \). By changing the roles of \( U \) and \( W \), we find
\[
S(\nabla_W U - \nabla_U W) = A_{LU}W - A_{LW}U + K_2 \left( \tilde{J}h^l(U, W) - \tilde{J}h^l(W, U) \right).
\]
Since \( h^l \) is known to be symmetric, we obtain
\[
S[W, U] = A_{LU}W - A_{LW}U.
\]
Therefore, the proof is completed. \( \square \)

Theorem 3.5. Let \( \tilde{N} \) be a radical transversal lightlike submanifold of a locally Metallic semi-Riemannian manifold \( \tilde{\mathcal{N}} \). Then the radical distribution define as a totally geodesic foliation if and only if
\[
h^*(W, JZ) = ph^*(W, Z),
\]
for \( W \in \Gamma(\text{Rad } T\tilde{N}), Z \in \Gamma(S(T\tilde{N})) \).
Proof. By using the definition of a lightlike submanifold, it is known that the radical distribution defines totally geodesic foliation if and only if

\[ \bar{g}(\nabla W U, Z) = 0, \]

for \( W, U \in \Gamma(\text{Rad} T\dot{N}) \) and \( Z \in S(T\dot{N}) \). Since \( \bar{\nabla} \) is a metric connection, if we use (2.18), (2.19), we have

\[ \bar{g}(\bar{J} U, \bar{\nabla}_W \bar{J} Z) = 0. \]

Then from (2.9), we get

\[ \bar{g}\left(\bar{J} U, h^*(W, \bar{J} Z) - ph^*(W, Z)\right) = 0. \]

Hence, the proof is completed.

\[ \square \]

Theorem 3.6. Let \( \dot{N} \) be a radical transversal lightlike submanifold of a locally Metallic semi-Riemannian manifold \( \dot{N} \). Then the screen distribution defines a totally geodesic foliation if and only if either

\[ h^*(W, \bar{J} U) + K_2 h^l(W, \bar{J} U) = p(h^*(W, U) + K_2 h^l(W, U)), \]

or there is no component of \( \bar{J} N \) in \( ltr T\dot{N} \) for \( W, U \in \Gamma(S(T\dot{N})), N \in \Gamma(ltr T\dot{N}) \).

Proof. Since the screen distribution defines a totally geodesic foliation if and only if

\[ \bar{g}(\nabla W U, N) = 0, \]

for any \( W, U \in \Gamma(S(T\dot{N})), N \in \Gamma(ltr T\dot{N}) \). Here, if we use (2.4), then we have

\[ \bar{g}(\bar{\nabla}_W U, N) = 0. \]

Also from (2.19) and (2.18), we have

\[ \bar{g}(\bar{\nabla}_W \bar{J} U, \bar{J} N) - p\bar{g}(\bar{\nabla}_W U, \bar{J} N) = 0. \]

By using (2.4) and (2.9) in the last equation, we find

\[ \bar{g}(h^*(W, \bar{J} U) + K_2 h^l(W, \bar{J} U), \bar{J} N) - p\bar{g}(h^*(W, U) + K_2 h^l(W, U), \bar{J} N) = 0. \]

Therefore, we conclude.

\[ \square \]

4. TRANSVERSAL LIGHTLIKE SUBMANIFOLDS OF METALLIC SEMI-RIEMANNIAN MANIFOLDS

In this section, we give definition of transversal lightlike submanifolds and investigate the geometry of distributions.
Definition 4.1. Let \((\mathcal{N}, g, S(T\mathcal{N}), S(T\mathcal{N}^\perp))\) be a lightlike submanifold of a Metallic semi-Riemannian manifold \((\mathcal{N}, \bar{g}, \bar{J})\). If the following conditions are satisfied, then the lightlike submanifold \(\mathcal{N}\) is called transversal lightlike submanifold:

\[
\bar{J} \text{Rad } T\mathcal{N} = \text{ltr } T\mathcal{N}, \\
\bar{J}(S(T\mathcal{N})) \subseteq S(T\mathcal{N}^\perp).
\]

We shall denote the orthogonal complement sub bundle to \(\bar{J}(S(T\mathcal{N}))\) in \(S(T\mathcal{N}^\perp)\) by \(\mu\).

Proposition 4.1. Let \(\mathcal{N}\) be a transversal lightlike submanifold of a locally Metallic semi-Riemannian manifold \(\mathcal{N}\). In this case, the distribution \(\mu\) is invariant according to \(\bar{J}\).

Proof. For \(V \in \Gamma(\mu)\), \(x \in \Gamma(\text{Rad } T\mathcal{N})\) and \(N \in \Gamma(\text{ltr } T\mathcal{N})\), from (2.16), (2.17) and (2.19), we have

\[
(4.1) \quad \bar{g}(\bar{J}V, x) = \bar{g}(V, \bar{J}x) = 0,
\]

and

\[
(4.2) \quad \bar{g}(\bar{J}V, N) = \bar{g}(V, \bar{J}N) = 0.
\]

Therefore, there is no component of \(\bar{J}V\) in \(\text{Rad } T\mathcal{N}\) and \(\text{ltr } T\mathcal{N}\).

Similarly, for \(W \in \Gamma(S(T\mathcal{N}))\) and \(V_1 \in \Gamma(S(T\mathcal{N}^\perp))\), we have

\[
(4.3) \quad \bar{g}(\bar{J}V, W) = \bar{g}(V, \bar{J}W) = 0,
\]

and

\[
(4.4) \quad \bar{g}(\bar{J}V, V_1) = \bar{g}(V, \bar{J}V_1) = 0,
\]

which imply that there is no component of \(\bar{J}V\) in \(S(T\mathcal{N})\) and \(\bar{J}(S(T\mathcal{N}))\). From (4.1), (4.2), (4.3) and (4.4), we conclude. \(\square\)

Proposition 4.2. There does not exist a 1-lightlike transversal lightlike submanifold of a locally Metallic semi-Riemannian manifold.

Proof. Assume that \(\mathcal{N}\) is a 1-lightlike transversal lightlike submanifold of a locally Metallic semi-Riemannian manifold \(\mathcal{N}\). In this case, \(\text{Rad } T\mathcal{N} = Sp\{x\}\) and \(\text{ltr } T\mathcal{N} = Sp\{N\}\). From (2.16) and (2.19), we obtain

\[
(4.5) \quad \bar{g}(\bar{J}\xi, \xi) = \bar{g}(\xi, \bar{J}\xi) = 0.
\]

On the other hand, from the fact that \(\bar{J}\text{Rad } T\mathcal{N} = \text{ltr } T\mathcal{N}\), we have \(\bar{J}\xi = N \in \Gamma(\text{ltr } T\mathcal{N})\). So, we find

\[
\bar{g}(\xi, \bar{J}\xi) = \bar{g}(\xi, N) = 1,
\]
which contradicts with (4.5). The proof is completed. □

From Definition 4.1 and Proposition 4.2, we have

**Corollary 4.1.** Let \( \mathcal{N} \) be a transversal lightlike submanifold of a locally Metallic semi-Riemannian manifold \( \mathcal{N} \). Then,

(i): \( \dim(\text{Rad } T\mathcal{N}) \geq 2 \),

(ii): The transversal lightlike submanifold of 3-dimensional is 2-lightlike.

Let \( \mathcal{N} \) be a transversal lightlike submanifold of a locally Metallic semi-Riemannian manifold \( \mathcal{N} \). \( Q \) and \( T \) are projection morphisms in \( \text{Rad } T\mathcal{N} \) and \( S(\mathcal{N}) \), respectively. For any \( W \in \Gamma(T\mathcal{N}) \), we can write (4.6)

\[
W = TW + QW,
\]

where, \( TW \in \Gamma(S(T\mathcal{N})) \) and \( QW \in \Gamma(\text{Rad } T\mathcal{N}) \). If we applied \( \tilde{J} \) to (4.6), we have

(4.7)

\[
\tilde{J}W = \tilde{J}TW + \tilde{J}QW.
\]

By writing \( \tilde{J}TW = KW \) and \( \tilde{J}QW = LW \), the expression (4.7) is

(4.8)

\[
\tilde{J}W = KW + LW.
\]

Here, \( KW \in \Gamma(S(T\mathcal{N}^\perp)) \) and \( LW \in \text{ltr } T\mathcal{N} \). Besides, let \( D \) and \( E \) be projection morphisms in \( \tilde{J}S(T\mathcal{N}) \) and \( \mu \) in \( S(T\mathcal{N}^\perp) \), respectively. For \( V \in \Gamma(S(T\mathcal{N}^\perp)) \), we write

(4.9)

\[
V = DV + EV.
\]

By applying \( \tilde{J} \) to (4.9), we have

(4.10)

\[
\tilde{J}V = \tilde{J}DV + \tilde{J}EV.
\]

If we write \( \tilde{J}DV = BV \) and \( \tilde{J}EV = CV \), expression (4.10) becomes

(4.11)

\[
\tilde{J}V = BV + CV,
\]

where \( BV \in \tilde{J}S(T\mathcal{N}) \oplus S(T\mathcal{N}) \), \( CV \in \Gamma(\mu) \). Since \( \mathcal{N} \) is a locally Metallic semi-Riemannian manifold, then from (2.4), (2.6) and (4.8), we have

(4.12)

\[
\begin{pmatrix}
-A_{KW}U + \nabla^{*}_{U}KW + D^{s}(U, KW) \\
-A_{LW}U + \nabla^{*}_{U}LW + D^{s}(U, LW)
\end{pmatrix} = \begin{pmatrix}
K\nabla_{U}W + L\nabla_{U}W + \tilde{J}h^{s}(U, W) \\
+Bh^{s}(U, W) + Ch^{s}(U, W)
\end{pmatrix},
\]

where \( U, W \in \Gamma(T\mathcal{N}) \). For projection morphisms \( K_{1} \) and \( K_{2} \) of \( \tilde{J}\text{ltr } T\mathcal{N} \) in \( \text{ltr } T\mathcal{N} \) and \( \text{Rad } T\mathcal{N} \) respectively, we write

\[
\tilde{J}h^{s}(U, W) = K_{1}\tilde{J}h^{s}(U, W) + K_{2}\tilde{J}h^{s}(U, W).
\]
Also, for projection morphisms $S_1$ and $S_2$ of $\tilde{J}S(T\tilde{N}^\perp)$ in $\tilde{J}S(T\tilde{N}) \subseteq S(T\tilde{N}^\perp)$ and $S(T\tilde{N})$, we have

$$Bh^s(U, W) = S_1 Bh^s(U, W) + S_1 Bh^s(U, W).$$

Therefore (4.12) can be rewritten as

$$
\begin{pmatrix}
-A_{KW}U + \nabla^s_UKW + D^i(U, KW) \\
-A_{LW}U + \nabla^l_ULW + D^s(U, LW)
\end{pmatrix} = \begin{pmatrix}
K\nabla_UW + L\nabla_UW + K_1 \tilde{J}h^l(U, W) \\
+K_2 \tilde{J}h^l(U, W) + S_1 Bh^s(U, W) \\
+S_2 Bh^s(U, W) + Ch^s(U, W)
\end{pmatrix}.
$$

If we equate the tangent and transversal parts of the above equation, then we get

$$
\begin{align*}
(4.13)\quad &-A_{KW}U - A_{LW}U = K_2 \tilde{J}h^l(U, W) + S_2 Bh^s(U, W), \\
(4.14)\quad &\nabla^s_UKW + D^s(U, LW) = K\nabla_UW + S_1 Bh^s(U, W) + Ch^s(U, W), \\
(4.15)\quad &\nabla^l_ULW = L\nabla_UW + K_1 \tilde{J}h^l(U, W).
\end{align*}
$$

Now we shall investigate the integrable of the distributions on transversal lightlike submanifolds.

**Theorem 4.1.** Let $\tilde{N}$ be a transversal lightlike submanifold of a locally Metallic semi-Riemannian manifold $\tilde{N}$. Then the radical distribution is integrable if and only if

$$D^s(U, LW) = D^s(W, LU),$$

for $V, W \in \Gamma(\text{Rad } T\tilde{N})$.

**Proof.** For $V, W \in \Gamma(\text{Rad } T\tilde{N})$, from equation (4.14), by interchanging roles of $W$ and $U$, we find

$$\nabla^s_UKW - \nabla^s_WKU + D^s(U, LW) - D^s(W, LU) - K(\nabla_UW - \nabla_WU) = 0,$$

since $h^s$ is symmetric. Also, we have $\nabla^s_UKW = \nabla^s_WKU = 0$. Then, we get

$$D^s(U, LW) - D^s(W, LU) = K[U, W],$$

which completes the proof. \hfill \Box

**Theorem 4.2.** Let $\tilde{N}$ be a transversal lightlike submanifold of a locally Metallic semi-Riemannian manifold $\tilde{N}$. Then the screen distribution is integrable if and only if

$$D^i(U, KW) = D^i(W, KU),$$

$W, U \in \Gamma(S(T\tilde{N}))$. 
Proof. From (4.15), the fact that $h^l$ is symmetric and $LW = LU = 0$, we have

$$D^l(U, KW) - D^l(W, KU) = L \{U, W\},$$

by interchanging the roles of $W, U \in \Gamma(S(T \hat{N}))$. Thus, the proof is completed. \qed

Theorem 4.3. Let $\hat{N}$ be a transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $\hat{N}$. Then the screen distribution defines a totally geodesic foliation if and only if $D^l(W, \hat{J}U) = -ph^l(W,U)$, $h^*(W,U) = 0$ and there is no component of $A_{\hat{J}U}W$ in $\text{Rad} T \hat{N}$, for $W, U \in \Gamma(S(T \hat{N}))$, $N \in \Gamma(ltr T \hat{N})$.

Proof. By the definition of a lightlike submanifold, it is known that $S(T \hat{N})$ defines a totally geodesic foliation if and only if

$$\tilde{g}(\nabla_W U, N) = 0,$$

where $W, U \in \Gamma(S(T \hat{N}))$ and $N \in \Gamma(ltr T \hat{N})$. If we use (2.17), (2.18) and (2.19), we find

$$0 = \tilde{g}(\nabla_W \hat{J}U, \hat{J}N) - p\tilde{g}(\nabla_W U, \hat{J}N).$$

Since $\hat{J}U \in \Gamma(S(T \hat{N}))$, from equation (2.4) and (2.6), we have

$$\tilde{g}(-A_{\hat{J}U}W + D^l(W, \hat{J}U), \hat{J}N) - p\tilde{g}(\nabla_W U + h^l(W,U), \hat{J}N) = 0.$$

Then by using (2.9), we obtain

$$\tilde{g}(-A_{\hat{J}U}W + D^l(W, \hat{J}U) - ph^*(W,U) + ph^l(W,U), \hat{J}N) = 0,$$

which completes the proof. \qed

Theorem 4.4. Let $\hat{N}$ be a transversal lightlike submanifold of a locally metallic semi-Riemannian manifold $\hat{N}$. Then the radical distribution defines a totally geodesic foliation if and only if there is no component in $\text{Rad} T \hat{N}$ of $A_{\hat{J}Z}W$, that is, either $K_2\hat{J}h^l(W,Z) = 0$ or $-A_{\hat{J}Z}W = S_2Bh^*(W,Z)$, for $W, U \in \Gamma(\text{Rad} T \hat{N})$, $Z \in \Gamma(S(T \hat{N}))$.

Proof. The radical distribution defines a totally geodesic foliation if and only if

$$\tilde{g}(\nabla_W U, Z) = 0,$$

for $W, U \in \Gamma(\text{Rad} T \hat{N})$ and $Z \in S(T \hat{N})$. From (2.4), we find

$$\tilde{g}(\nabla_W U, Z) = \tilde{g}(\tilde{\nabla}_W U, Z) = 0.$$

Since, $\tilde{\nabla}$ is a metric connection, from (2.17), (2.18) and (2.19), we have

$$0 = -\tilde{g}(\hat{J}U, \tilde{\nabla}_W \hat{J}Z) + p\tilde{g}(\hat{J}U, \tilde{\nabla}_W Z).$$
For $\mathcal{J}Z \in \Gamma(S(T\mathcal{N}^\perp))$, from (2.6) and (2.9), we get
\[
0 = \mathcal{J}g(\mathcal{J}U, A_{\mathcal{J}Z}W) + p\mathcal{J}g(\mathcal{J}U, h^s(W, Z)).
\]
Here, since $\mathcal{J}U \in \Gamma(ltr T\mathcal{N})$, we conclude that either there is no component of $A_{\mathcal{J}Z}W$ in $Rad T\mathcal{N}$ or by changing the roles of $U, W$ and taking $U = Z$, we have $K_2\mathcal{J}h^l(W, Z) = 0$, by virtue of
\[
-A_{\mathcal{J}Z}W = K_2\mathcal{J}h^l(W, Z) + S_2\mathcal{J}h^s(W, Z).
\]

**Theorem 4.5.** Let $\mathcal{N}$ be a transversal lightlike submanifold of a locally Metallic semi-Riemannian manifold $\mathcal{N}$. Then the induced connection on $\mathcal{N}$ is a metric connection if and only if
\[
Q_1\mathcal{J}D^s(W, \mathcal{J}\xi) = pM_1\mathcal{J}h^s(W, \xi),
\]
for $W, U \in \Gamma(T\mathcal{N}), \xi \in \Gamma(Rad T\mathcal{N})$.

**Proof.** For $W, U \in \Gamma(T\mathcal{N})$ and $\xi \in \Gamma(Rad T\mathcal{N})$, we have
\[
\mathcal{N}_W\mathcal{J}\xi = \mathcal{J}\mathcal{N}_W\xi.
\]
From equations (2.4) and (2.5), we write
\[
-A_{\mathcal{J}Z}W + \mathcal{N}_W\mathcal{J}\xi + D^s(W, \mathcal{J}\xi) = \mathcal{J}(\mathcal{N}_W\xi + h^l(W, \xi) + h^s(W, \xi)).
\]
If we apply $\mathcal{J}$ to the above equation and use (2.16), (4.8) and (4.11) we obtain
(4.16)
\[
\begin{pmatrix}
-KA_{\mathcal{J}\xi}W - LA_{\mathcal{J}\xi}W \\
+T_1\mathcal{J}\mathcal{N}_W^l\mathcal{J}\xi + T_2\mathcal{J}\mathcal{N}_W^l\mathcal{J}\xi \\
+Q_1\mathcal{J}D^s(W, \mathcal{J}\xi) + Q_2\mathcal{J}D^s(W, \mathcal{J}\xi)
\end{pmatrix} =
\begin{pmatrix}
p\mathcal{J}\mathcal{N}_W\xi + q\mathcal{N}_W\xi + p\mathcal{J}h^l(W, \xi) + qh^l(W, \xi) \\
+p\mathcal{J}h^s(W, \xi) + qh^s(W, \xi)
\end{pmatrix},
\]
for $V \in \Gamma(S(T\mathcal{N}^\perp))$, where $T_1$ and $T_2$ are projection morphisms of $\mathcal{J}\mathcal{N}_W^l\mathcal{J}\xi$ in $Rad T\mathcal{N}$ and $ltr T\mathcal{N}$, respectively. Then we have
\[
\mathcal{J}\mathcal{N}_W^l\mathcal{J}\xi = T_1\mathcal{J}\mathcal{N}_W^l\mathcal{J}\xi + T_2\mathcal{J}\mathcal{N}_W^l\mathcal{J}\xi.
\]
Also, for projection morphisms $M_1$ and $M_2$ are of $\mathcal{J}h^s(W, \xi)$ in $S(T\mathcal{N})$ and $\mathcal{J}S(T\mathcal{N})$, respectively, then we have
\[
\mathcal{J}h^s(W, \xi) = M_1\mathcal{J}h^s(W, \xi) + M_2\mathcal{J}h^s(W, \xi).
\]
Additionally, we get
\[
\mathcal{J}D^s(W, \mathcal{J}\xi) = Q_1\mathcal{J}D^s(W, \mathcal{J}\xi) + Q_2\mathcal{J}D^s(W, \mathcal{J}\xi).
\]
where $Q_1$ and $Q_2$ are projection morphisms of $\hat{J}D^s(W, \hat{J}\xi)$ in $S(T\hat{N})$ and $S(T\hat{N}^\perp)$, respectively. By equating tangent parts in equation (4.10), we find

$$\frac{1}{q} \left( T_1\hat{J}\nabla_W^i \hat{J}\xi + Q_1 \hat{J}D^s(W, \hat{J}\xi) - pM_1 \hat{J}h^s(W, \xi) - pK_2 \hat{J}h^l(W, \xi) \right) = \nabla_W \xi.$$

Therefore $\nabla_W \xi$ is belong to $\text{Rad } T\hat{N}$ if and only if $Q_1 \hat{J}D^s(W, \hat{J}\xi) = pM_1 \hat{J}h^s(W, \xi)$.

This completes the proof. \hfill \Box

**Example 4.1.** Let $(\hat{N} = \mathbb{R}^5_2, \hat{g}, \hat{J})$ be the 5-dimensional semi-Euclidean space with the semi-Euclidean metric of signature $(-, +, -, +, +)$ and the structure $\hat{J}$ given by

$$\hat{J}(x_1, x_2, x_3, x_4, x_5) = ((p - \sigma)x_1, \sigma x_2, (p - \sigma)x_3, \sigma x_4, \sigma x_5),$$

where $(x_1, x_2, x_3, x_4, x_5)$ is the standard coordinate system of $\mathbb{R}^5_2$. If we take $\sigma = \frac{p + \sqrt{p^2 + 4q^2}}{2}$, then we have

$$\hat{J}^2 = p\hat{J} + qI,$$

which implies $\hat{J}$ is a metallic structure on $\mathbb{R}^5_2$. Hence, $(\hat{N} = \mathbb{R}^5_2, \hat{g}, \hat{J})$ is a Metallic semi-Riemannian manifold. Let $\hat{N}$ be a submanifold in $\hat{N}$ defined by

$$x_2 = 0, \quad x_4 = \sigma x_1 + \sigma x_3.$$

Then, $T\hat{N} = \text{Sp}\{W_1, W_2, W_3\}$, where

$$W_1 = \frac{\partial}{\partial x_1} + \sigma \frac{\partial}{\partial x_4}, \quad W_2 = \frac{\partial}{\partial x_3} + \sigma \frac{\partial}{\partial x_4}, \quad W_3 = \frac{\partial}{\partial x_5}.$$

It is easy to check that $\hat{N}$ is a lightlike submanifold. Therefore,

$$\text{Rad } T\hat{N} = \text{Sp}\{\xi = \sigma W_1 - \sigma W_2 + \sigma \sqrt{2}W_3\},$$

$$\text{ltr } T\hat{N} = \text{Sp}\left\{ N = \frac{1}{2\sigma^2 (2\sigma - p)} \left( (p - \sigma) \frac{\partial}{\partial x_1} - (p - \sigma) \frac{\partial}{\partial x_3} + \sigma \sqrt{2} \frac{\partial}{\partial x_5} \right) \right\},$$

$$S(T\hat{N}) = \text{Sp}\{W_3\},$$

and we have

$$N = \frac{1}{4\sigma^2} \hat{J}\xi,$$

for $p = 0$. That is, $\hat{J}\xi \in \Gamma(\text{ltr } T\hat{N})$ and $\hat{J}W_3 = \sigma W_3 \in S(T\hat{N})$. Therefore, $\hat{N}$ is a radical transversal lightlike submanifold of $(\hat{N} = \mathbb{R}^5_2, \hat{g}, \hat{J})$. 

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