Default Logic and Bounded Treewidth∗

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Abstract
In this paper, we study Reiters propositional default logic when the treewidth of a certain graph representation (semi-incidence graph) of the input theory is bounded. We establish a dynamic programming algorithm on tree decompositions that decides whether a theory has a consistent stable extension or can even be used to enumerate all generating defaults that lead to stable extensions. We show that, for input theories whose semi-incidence graph has bounded treewidth, our algorithm decides whether a theory has a stable extension in linear time and enumerates all characteristic generating defaults with linear delay.

1 Introduction
Reiter’s default logic (DL) is one of the most fundamental formalisms to non-monotonic reasoning where reasoners draw tentative conclusions that can be retracted based on further evidence [16,13]. DL augments classical logic by rules of default assumptions (default rules). Intuitively, a default rule expresses “in the absence of contrary information, assume ...”. Formally, such rule is a triple \( P : J \rightarrow C \) of formulas \( P \), \( J \), and \( C \) expressing “if prerequisite \( P \) can be deduced and justification \( J \) is never violated then assume conclusion \( C \)”.

For an initial set of facts, beliefs supported by default rules are called an extension of this set of facts. The default rules must not lead to an inconsistency if conflicting rules are present, instead such rules should be avoided. If the default rules can be “fired” consistently until a fixed point, the extension is a maximally consistent view (consistent stable extension) with respect to the facts together with the default rules. In DL stable extensions involve the construction of the deductive closure, which can be generated from the conclusions \( C \) of the defaults and the initial facts by means of so-called generating defaults. If a generating default also leads to a consistent stable extension, we call it a characteristic generating default. Even though formulas in DL can be used for various classical logics, we consider only propositional formulas in conjunctive normal form (cnf). Our problems of interest are deciding whether a default theory has a consistent stable extension (Ext), counting the number of characteristic generating defaults (CompSE), and enumerating all generating defaults (EnumSE). The computational complexity of various problems arising in default logic has been subject of several studies. Gottlob [11] established that the problem Ext is \( \Sigma_2^P \)-complete and thus are of high intractability. Beyersdorff et al. considered the complexity of fragments for default logic in the sense of Post’s lattice [1].

Parameterized algorithms [8] have attracted considerable interest in recent years and allow to tackle hard problems by directly exploiting certain structural properties present in input instances (the parameter). When conducting a complexity analysis the computational complexity of the problem is then considered in dependency of the parameter. Problems where the complexity drops and where we have an algorithm that solves the problem in polynomial time when the parameter is fixed are of particular interest. Such problems are called fixed-parameter tractable and fixed-parameter linear, if the runtime bounds are linear in

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the program size. Recently, a parameter for DL has been gained from backdoors \cite{10} and fixed-parameter tractability results have been established for the problem \textsc{Ext}. Another parameter is treewidth, which has been researched extensively \cite{4} and widely applied in knowledge representation and reasoning \cite{5, 9, 12, 17}. Treewidth is the parameter of our interest. A classical way to establish fixed-parameter tractability results when the problem is parameterized by the width of a given TD is to encode the problem in monadic second order logic and apply Courcelle’s theorem \cite{6}. In that way, Meier et al. \cite{14} established that the extension existence problem \textsc{Ext} is fixed-parameter tractable when parameterized by treewidth of a certain graph representation of the default theory (incidence graph). However, in many settings dynamic programming on tree decompositions is more feasible when we are not just interested in the complexity result, but also aim to implement and use the algorithm in a practical setting \cite{2, 3, 5, 9}, respectively.

Contributions. In this paper, we introduce dynamic programming algorithms on tree decompositions to solve the problems \textsc{Ext} and \textsc{EnumSE}. Our algorithms are fixed-parameter linear and fixed-parameter linear for a pre-computation step followed by linear delay for outputting the solutions (Output-FPT~\cite{7}), respectively.

2 Formal Background

We assume familiarity with standard notions in computational complexity, the complexity classes \textit{P} and \textit{NP} as well as the polynomial hierarchy. For more detailed information, we refer to other standard sources.

For parameterized (decision) problems we refer to work by Cygan et al. \cite{8}.

A literal is a propositional variable or its negation and a clause is a finite set of literals. A CNF formula is a finite set of clauses. A (truth) assignment is a mapping \(\theta : X \rightarrow \{0, 1\}\) defined for a set \(X\) of variables. For \(x \in X\) we put \(\neg x := 1 - \theta(x)\). By \(\mathcal{A}(X)\) we denote the set of all truth assignments \(\theta : X \rightarrow \{0, 1\}\).

The truth assignment reduct of a CNF formula \(F\) with respect to \(\theta \in \mathcal{A}(X)\) is the CNF formula \(F_\theta\) obtained from \(F\) by first removing all clauses \(c\) that contain a literal set to 1 by \(\theta\), and second removing from the remaining clauses all literals set to 0 by \(\theta\). \(\theta\) satisfies \(F\) if \(F_\theta = \emptyset\), and \(F\) is satisfiable if it is satisfied by some \(\theta\). Let \(F\) and \(G\) be CNF formulas and \(X = \operatorname{Vars}(F) \cup \operatorname{Vars}(G)\). We write \(F \models G\) if and only if for all assignments \(\theta \in \mathcal{A}(X)\) it holds that all assignments \(\theta\) that satisfy \(F\) also satisfy \(G\). Further, we define the deductive closure of \(F\) as \(\text{Th}(F) := \{G \in \text{CNF} \mid F \models G\}\) and let for formula \(F\) and a family \(\mathcal{M}\) of sets of variables be \(\text{Mod}(\mathcal{M}, F) = \{M \mid M \in \mathcal{M}, M \models F\}\). We denote with \(\text{SAT}\) the problem, given a propositional formula \(F\) asking whether \(F\) is satisfiable. The problem \textsc{Taut} is defined over a given formula \(F\) asking whether \(F\) tautological.

We define for CNF formulas \(P, J, C\) a default rule \(\delta\) as a triple \(\frac{P \leftarrow J}{C}\); \(P\) is called the prerequisite, \(J\) is called the justification, and \(C\) is called the conclusion; we set \(\alpha(d) := P, \beta(d) := J, \text{and } \gamma(d) := C\). We follow the definitions by Reiter \cite{10}. A default theory \(\langle W, D\rangle\) consists of a set \(W\) of propositional CNF formulas (knowledge base) and a set of default rules. A default theory \(D\) is a set of default rules. Let \(\langle W, D\rangle\) be a default theory and \(E\) be a set of formulas. Then, \(\Gamma(E)\) is the smallest set of formulas such that: (i) \(W \subseteq \Gamma(E)\), (ii) \(\Gamma(E) = \text{Th}(\Gamma(E))\), and (iii) for each \(\frac{\alpha : \beta}{\gamma} \in D\) with \(\alpha \in \Gamma(E)\) and \(\neg \beta \notin E\), it holds that \(\gamma \in \Gamma(E)\). \(E\) is a stable extension of \(\langle W, D\rangle\), if \(E = \Gamma(E)\). An extension is inconsistent if it contains \(\emptyset\), otherwise it is called consistent. The definition of stable extensions allows inconsistent stable extensions. However, inconsistent extensions only occur if the set \(W\) is already inconsistent where \(\langle W, D\rangle\) is the theory of interest \cite{13} Corollary 3.60. In consequence, (i) if \(W\) is consistent, then every stable extension of \(\langle W, D\rangle\) is consistent, and (ii) if \(W\) is inconsistent, then \(\langle W, D\rangle\) has a stable extension. For Case (ii) the stable extension consists of all formulas. Thus, we consider only consistent stable extensions. For default theories with consistent \(W\), we can trivially transform every formula in \(W\) into a default rule. Hence, in this paper we generally assume that \(W = \emptyset\) and write a default theory simply as set of default rules. Moreover, we refer by \(\text{SE}(D)\) to the set of all consistent stable extensions of \(D\).

A definition of stable extensions beyond fixed point semantics, which has been introduced by Reiter \cite{10} as well, uses the principle of a stage construction.
Proposition 1 (Stage construction, [10].) Let \( D \) be a default theory and \( E \) be a set of formulas. Then define \( E_0 := \emptyset \) and \( E_{i+1} := \text{Th}(E_i) \cup \{ \gamma(d) \mid \alpha(d) \in E_i, \beta(d) \notin E, d \in D \} \). \( E \) is a stable extension of \( \langle W, D \rangle \) if and only if \( E = \bigcup_{i \in \mathbb{N}} E_i \). The set \( G = \{ d \mid \alpha(d) \in E \land \beta(d) \notin E, d \in D \} \) is called the set of generating defaults. If \( E \) is a stable extension of \( D \), then \( E = \text{Th}(\{ \gamma(d) \mid d \in G \}) \). In that case, we call \( G \) a characteristic generating default.

Example 1. Let the default theories \( D_1 \) and \( D_2 \) be given as

\[
D_1 := \left\{ d_1 = \begin{cases} \{a\} : \{\neg a\} & a \lor b \\ \{\neg b\} & \end{cases} , d_2 = \begin{cases} \{a\} : \{\neg a\} & a \lor b \\ \{\neg b\} & \end{cases} \right\},
\]

\[
D_2 := \left\{ d_1 = \begin{cases} \{c\} : \{a\} & a \lor b \\ \{\neg a\} & \end{cases} , d_2 = \begin{cases} \{c\} : \{\neg a\} & a \lor b \\ \{\neg b\} & \end{cases} , d_3 = \begin{cases} \{c\} & c \\ \{\neg c\} & \end{cases} , d_4 = \begin{cases} \{\neg c\} & c \\ \{c\} & \end{cases} \right\}.
\]

\( D_1 \) has no stable extension, while \( D_2 \) has only one stable extension \( E_1 = \{\neg c\} \).

Given a default theory \( D \) we are interested in the following problems: The extension existence problem (called \( \text{EXT} \)) asks whether \( D \) has a consistent stable extension. \( \text{EXT} \) is \( \Sigma^P_2 \)-complete [11]. The computation problem (called \( \text{COMPSE} \)) asks to output a characteristic generating default of \( D \). The enumerating problem asks to enumerate all characteristic generating defaults of \( D \) (called \( \text{ENUMSE} \)).

Tree Decompositions. Let \( G = (V,E) \) be a graph, \( T = (N,F,n) \) a rooted tree, and \( \chi : N \to 2^V \) a function that maps to each node \( t \in T \) a set of vertices. We call the sets \( \chi(v) \) bags and \( N \) the set of nodes. Then, the pair \( T = (T,\chi) \) is a tree decomposition (TD) of \( G \) if the following conditions hold: (i) for every vertex \( v \in V \) there is a node \( t \in N \) with \( v \in \chi(t) \); (ii) for every edge \( e \in E \) there is a node \( t \in N \) with \( e \subseteq \chi(t) \); and (iii) for any three nodes \( t_1,t_2,t_3 \in N \), if \( t_2 \) lies on the unique path from \( t_1 \) to \( t_3 \), then \( \chi(t_1) \cap \chi(t_3) \subseteq \chi(t_2) \). We call \( \max\{|\chi(t)|-1 \mid t \in N\} \) the width of the TD. The treewidth \( tw(G) \) of the graph \( G \) is the minimum width over all possible TDs of \( G \).

Note that each graph has a trivial TD \( (T,\chi) \) consisting of the tree \( (\{n\},\emptyset,n) \) and the mapping \( \chi(n) = V \). It is well known that the treewidth of a tree is 1, and a graph containing a clique of size \( k \) has at least treewidth \( k-1 \). For arbitrary but fixed \( k \) we can compute a TD of a graph of width that equals its treewidth in time \( 2^{O(k^2)} \cdot |V| \) [12].

Given a TD \( (T,\chi) \) with \( T = (N,\cdots) \), for a node \( t \in N \) we say that type(\( t \)) is leaf if \( t \) has no children; join if \( t \) has children \( t' \) and \( t'' \) with \( t' \neq t'' \) and \( \chi(t) = \chi(t') = \chi(t'') \); int (“introduce”) if \( t \) has a single child \( t' \), \( \chi(t') \subseteq \chi(t) \) and \( |\chi(t)| = |\chi(t')| + 1 \); rem (“removal”) if \( t \) has a single child \( t' \), \( \chi(t) \subseteq \chi(t') \) and \( |\chi(t)| = |\chi(t')| + 1 \). If every node \( t \in N \) has at most two children, type(\( t \) \( \in \{ \text{leaf, join, int, rem} \} \), and bags of leaf nodes and the root are empty, then the TD is called nice.

For every TD, we can compute a nice TD in linear time without increasing the width [14]. In our algorithms we will traverse a TD bottom up, therefore, let post-order(\( T,t \)) be the sequence of nodes in post-order of the induced subtree \( T' = (N',\cdots,t) \) of \( T \) rooted at \( t \).

For the ease of presentation, we also require the notion of labelled tree decompositions (LTDs). Again, let \( G = (V,E) \) be a graph and \( T = (T,\chi) \) be a tree decomposition of \( G \). Further, let \( L \subseteq V \) be a set of labels. We call \( T \) a labelled tree decomposition, if (i) for each node \( t \) of \( T \), we have \( |\chi(t) \cap L| = 1 \), and (ii) whenever for some node \( t \) of \( T \), \( \chi(t) \cap L \neq \emptyset \), then \( \chi(t) \supseteq \{v \mid (v,l) \in E, \{l\} = \chi(t) \cap L \} \) as well. Assume in the following, that we use labelled graphs and nice labelled TDs, unless mentioned otherwise.
Algorithm 1: Algorithm $\mathcal{DP}_A(T)$ for Dynamic Programming on TD $T$ for DL, cf. [9].

**In:** Table algorithm $A$, nice TD $T = (T, \chi)$ with $T = (N, \cdot, n)$ of $G(D)$ according to $A$.

**Out:** Table: maps each TD node $t \in T$ to some computed table $\tau_t$.

```
for iterate $t$ in post-order$(T, n)$ do
    Child-Tabs := \{Tables[t'] | t' is a child of $t$ in $T$\}
    Tables[t] := $A(t, \chi(t), D_t, at_{\leq t}, Child-Tabs)$
```

**Example 2.** Figure 7 (left) depicts a graph $G$ together with a LTD of width 3 of $G$. $G$ has treewidth 3, since there is a clique on the set $\{a, b, d_1, l_{\text{con},d_1}\}$ of vertices. Further, the LTD $T$ in Figure 3 sketches main parts of a nice LTD of $G$ (obvious parts are left out).

**Graph Representations of Default Theories.** In order to use TDs for DL solving, we need dedicated graph representations. These definitions adapt similar concepts from ASP [9]. For a default theory $D$, its semi-incidence graph $P(D)$ is the graph that has the atoms of $D$ as vertices and an edge $a \cdot b$ if there exists a default $d \in D$ and $a, b \in at(d)$. The incidence graph $I(G)$ is the bipartite graph, where the vertices are of atoms of $D$ and defaults $d \in D$, and there is an edge $d \cdot a$ between a default $d \in D$ and a corresponding atom $a \in at(d)$. In order to simplify the presentation of our algorithm, we mainly cover the semi-incidence graph $S(D)$ of $D$, where the vertices are atoms $at(D)$, defaults of $D$ and labels $l_{\text{pre},d}$, $l_{\text{just},d}$ and $l_{\text{con},d}$ for each default $d \in D$. For each default $d \in D$, we have edges $l_{\text{pre},d} \cdot d$, $l_{\text{just},d} \cdot d$ and $l_{\text{con},d} \cdot d$, and a $d$ if atom $a \in at(d)$ occurs in $d$. Moreover, there are edges $a \cdot l_{\text{pre},d}$, $a \cdot l_{\text{just},d}$, or $a \cdot l_{\text{con},d}$ if atom $a$ occurs in the respective formula of $d$. Finally, we have an edge $a \cdot b$ if either $a, b \in at(\alpha(d))$, or $a, b \in at(\beta(d))$, or $a, b \in at(\gamma(d))$. For sake of simplicity of the presentation, which we will see later, we require in addition that in each bag of the decomposition two fresh vertices are present $\varepsilon$ and $bf$, where $\varepsilon$ represents an empty default and $bf$ represents a fail state.

**Example 3.** Recall default theory $D_1$ of Example 7. We observe that graph $G$ in the left part of Figure 7 is the semi-incidence graph of $D_1$.

**Sub-Theories.** Let $T = (T, \chi)$ be a nice LTD of the semi-incidence graph $S(D)$ of a default theory $D$. Further, let $T = (N, \cdot, n)$ and $t \in N$. The bag-defaults are defined as $D_t := D \cap \chi(t)$. Moreover, the bag-default parts $\alpha_t, \beta_t$ and $\gamma_t$ contain the corresponding preconditions $\alpha_t := \alpha(D) \cap \chi(t)$, justifications $\beta_t := \beta(D) \cap \chi(t)$ and conclusions $\gamma_t := \gamma(D) \cap \chi(t)$, respectively. Further, the set $at_{\leq t} := \{a \mid a \in at(D) \cap \chi(t), t' \in \text{post-order}(T, t)\}$ is called atoms below $t$, the default theory below $t$ is defined as $D_{\leq t} := \{d \mid d \in D, t' \in \text{post-order}(T, t)\}$, and the default theory strictly below $t$ is $D_{< t} := D_{\leq t} \setminus D_t$. It holds that $D_{\leq n} = D_{< n} = D$ and $at_{\leq n} = at(D)$. The former definitions naturally extend to the respective bag-default parts, i.e., we also use $\alpha_{\leq t}, \beta_{\leq t},$ and $\gamma_{\leq t}$.

**Example 4.** Intuitively, the LTD of Figure 7 enables us to evaluate $D$ by analyzing sub-theories $\{\{d_1\} \text{ and } \{d_2\}\}$ and combining results agreeing on $a, b$. Indeed, for the given LTD of Figure 7 (right), $D_{\leq t_3} = \{d_1\}$, $D_{\leq t_4} = \{d_2\}$ and $D = D_{\leq t_7} = D_{< t_7} = D_{\leq t_3} \cup D_{\leq t_4}$. Moreover, at $t_7 = at(D)$, $\alpha_{\leq t_3} = \{\alpha(d_1)\}$ and $\gamma_{\leq t_7} = \{\gamma(d_1), \gamma(d_2)\}$.

## 3 Computing Characterizing Generating Defaults

In this section, we present our dynamic programming (DP) algorithms to solve the problems Ext and EnumSE. Our algorithms are inspired by earlier work [9] in answer set programming, but require significant extensions due to much more evolved semantics in DL. Algorithm 1 ($\mathcal{DP}_A$) outlines the basis of our algorithms and sketches the general dynamic programming scheme on TD for computing characterizing generating defaults. Roughly, the algorithm splits the search space for a stage construction (see Proposition 1) based on a given LTD and evaluates the input default theory $D$ in parts. The results are stored in so-called
Algorithm 2: Table algorithm SINC\((t,\chi_t, D_t, \leq t, \text{Child-Tabs})\).

**In:** Bag \(\chi_t\), bag-theory \(D_t\) and child tables Child-Tabs of node \(t\).  
**Out:** Table \(\tau_t\).

1. \(\text{cpy}_\alpha(C, \delta) := \{(\rho, MC, RC)| (\rho, MC, RC) \in C, \rho(d) \neq \delta\}\)
2. \(\text{if type}(t) = \text{leaf} \quad \tau_t := \{(\emptyset, \emptyset), \{\emptyset, \emptyset, \emptyset\}, \emptyset\}\) /* Abbreviations see Footnote 1 */
3. \(\text{else if type}(t) = \text{int}, d \in D_t\) is the introduced default, and \(\tau' \in \text{Child-Tabs}\)
4. \(\text{sub}_{s,M}(C) := \{(\rho, MC \cup MC, RC)| (\rho, MC, RC) \in C, \delta \in s\}\)
5. \(\tau_t := \{(Z_{}\delta_{\alpha}, \text{sub}_{s,M}(C)), \text{sub}_{\{\alpha, \beta\}, s}(C)) \cup \text{sub}_{\{\gamma, \delta\}, M}(\rho)\), \(\langle Z, M, P, C\rangle \in \tau'\}\)
6. \(\text{else if type}(t) = \text{int}, \chi_t \cap MC \in \chi_t\) for \(d \in D_t\) is the introduced label, and \(\tau' \in \text{Child-Tabs}\)
7. \(\text{Upd}_\alpha(C) := \{(\rho, \text{Upd}(\gamma(d)), M, RC) | \rho(d) \neq \gamma\}\)
8. \(\text{Upd}_\alpha(C) := \{(\rho, \text{Upd}(\gamma(d)), M, RC) | \rho(d) \neq \gamma\}\)
9. \(\tau_t := \{(Z, M, \text{Upd}_\alpha(C), \text{Upd}_\beta(C) \cup \text{Upd}_\gamma(C))\), \(\langle Z, M, P, C\rangle \in \tau'\}\)
10. \(\text{else if type}(t) = \text{int}, \alpha \in \chi_t\) for \(d \in D_t\) is the introduced atom, and \(\tau' \in \text{Child-Tabs}\)
11. \(\text{AGuess}(C) := \{(\rho, MC \cup MC^o, R'| R' \in R^o, R \in RC) | (\rho, MC, RC) \in C\}\)
12. \(\tau_t := \{(Z, M, \text{Upd}_\alpha(C), \text{AGuess}(C))\), \(\langle Z, M, P, C\rangle \in \tau'\}\)
13. \(\text{else if type}(t) = \text{rem}, d \notin D_t\) is the removed default, and \(\tau' \in \text{Child-Tabs}\)
14. \(\text{SProj}(\{d \mapsto \alpha, d \mapsto \beta, d \mapsto \gamma\}) = (MC, RC) \in C\)
15. \(\tau_t := \{(Z, M, \text{SProj}(\{d \mapsto \alpha, d \mapsto \beta, d \mapsto \gamma\}) = (MC, RC) \in C\}\)
16. \(\text{else if type}(t) = \text{rem}, a \notin \chi_t\) is the removed atom or label, and \(\tau' \in \text{Child-Tabs}\)
17. \(\text{AProj}(\{d \mapsto \alpha, d \mapsto \beta, d \mapsto \gamma\}) = (MC, RC) \in C\)
18. \(\tau_t := \{(Z, M, \text{AProj}(\{d \mapsto \alpha, d \mapsto \beta, d \mapsto \gamma\}) = (MC, RC) \in C\}\)
19. \(\text{else if type}(t) = \text{join and \(\tau' \neq \tau''\) then}
20. \(\text{else if type}(t) = \text{join and \(\tau' \neq \tau''\) then}
21. \(\text{else if type}(t) = \text{join and \(\tau' \neq \tau''\) then}
22. \(\text{else if type}(t) = \text{join and \(\tau' \neq \tau''\) then}
23. \(\text{else if type}(t) = \text{join and \(\tau' \neq \tau''\) then}
24. \(\text{else if type}(t) = \text{join and \(\tau' \neq \tau''\) then}
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29. \(\text{else if type}(t) = \text{join and \(\tau' \neq \tau''\) then}
30. \(\text{else if type}(t) = \text{join and \(\tau' \neq \tau''\) then}\)

Figure 2: Selected DP tables of SCONS for nice LTD \(T\).
We call $Z$ the witness generating default, since $\gamma(Z)$, is, roughly speaking, part of a potential generating default. In particular, $Z$ witnesses the existence of an extension $Z' \supseteq \gamma(Z)$ for a sub-theory $S$ of $D_{\less}$, i.e., the default theory strictly below $t$. The set $M$ of witness models contains models of $F' := \{C \mid C \in F, F \in \Delta\}$ where $\Delta \subseteq \gamma(D_{\less}) \cup Z'$ is a set of conclusions of the firing defaults of $D_{\less}$. For our assumed witness generating default $Z$, we require a witness proof $P$. The set $P$ consists of tuples of the form $\langle \sigma, B, R, \rangle$, where $\sigma : D_l \cup \{e\} \rightarrow \{\overline{p}, \overline{j}, c\}$, $B \subseteq 2^X$ and $R \subseteq 2^X$ for $X = \chi(t) \cap at(D)$. The function $\sigma$, which we call witness states function, maps each default $d \in D_l$ to a certain decision state. In particular, $\sigma(d) = c$ marks that $d$ should “fire”, in other words, we assume that $d$ is part of a characterizing generating default. Assignment $\sigma(d) = \overline{p}$ or $\sigma(d) = \overline{j}$ marks that $d$ does not fire, because of $\alpha$ or $\beta$, respectively. Note that we require to remember whether any default in $D_{\less}$ has been set to $p$, resulting in $\sigma(e) = \overline{p}$ in such a case (using empty default $\varepsilon$). The set $B$, consists of backfire witness models, containing atom $bf$. Such a model $M \in B$ is not only a model of $F'$, but also of $\beta(d)$ for some default $d \in \sigma^{-1}(\beta) \cup D_{\less}$ and proves that $d$ “backfires”, i.e., $d$ should have fired. In the end, $Z$ proves an extension of sub-theory $S$, where such backfiring defaults are excluded. Hence, $\{d \mid d \in D_{\less}, \sigma(d) = \beta, \exists M \in B : M \models Z', \beta(d)\} \cap S = \emptyset$.

In the end, we want to have $S = D_{\less n}$ for the root $n$. If there is such a tuple $\langle \sigma, \{\{bf\}\}, \cdot \rangle \in P$, this indicates a wrong assignment to $\sigma'(d) = \overline{j}$ for some default $d$ and node in the tree decomposition below. For assignment $\sigma(d) = \overline{p}$ to be correct, it is required that $d$ does not fire, because of $\alpha(d)$ is not a consequence of $Z'$. To be more concrete, additionally we have to find at least one model $M$ such that $M \models Z', \neg \alpha(d)$. Since such a model $M$ is required for each $d \in D_{\less} \setminus \{d \mid \gamma(d) \in Z'\}$, which does not fire because of $\alpha$, the set $R$ of required witness model sets keeps a set of such potential model sets $R \in R$ containing at least one such model $M \in R$ for each such default $d$. Note that $R$ contains and keeps every $R$ of these potential sets, as long as each $M \in R$ also satisfies $Z'$, and otherwise the whole set is excluded. This set $R$ of sets allows us to not distinguish at any TD node $t$ between defaults $d$ with $\sigma(d) = \alpha$, and to therefore still maintain the runtime guarantees. To be more concrete, we have that $\{d \mid d \in D_{\less}, \sigma(d) = \alpha, R = \emptyset\} \cap S = \emptyset$, i.e., if at least one default $d \in D_{\less}$ did not fire because of $\alpha$, we require to have at least one $R \in R$ in order to guarantee that all of these defaults are satisfied (we can not differentiate). Finally, we end up with $S = D_{\less} \setminus \{d \mid d \in D_{\less}, \sigma(d) = \beta, \exists M \in B : M \models Z', \beta(d)\} \cup \{d \mid d \in D_{\less}, \sigma(d) = \alpha, R = \emptyset\}$. To conclude, there is a consistent default $E$ of the default theory $D$ if table $t_n$ for (empty) root $n$ contains $\langle \emptyset, \{\emptyset\}, P \rangle$, where $P$ either contains $\langle \emptyset, \emptyset, \emptyset \rangle$ or $\{\{e \mapsto \alpha\}, \emptyset, \{\{\emptyset\}\}\}$. The main aim of $C$ is to invalidate the subset-minimality of $\gamma(Z)$, and will be covered later.

In the following, we intuitively discuss important cases of Algorithm 2 for Part (i), which consists only of the first three tuple positions (colored red and green) and ignore the remaining parts of the tuple. We call the resulting algorithm $SCONS$. Let $t$ again be a node of a given LTD and $\langle Z, M, P, \cdot \rangle$ a tuple of table $\in \tau'$ for an LTD child node of $t$. Further, we also assume any tuple $\langle \sigma, B, R, \rangle \in P$. When a default $d$ is introduced (type($t$) = int), Line 5 is responsible for making decisions concerning whether $d$ fires, and the reason why it does not fire, otherwise. Lines 6–16 cover nodes with type($t$) = int using labels, “intermediate” node types designed for the ease of presentation, and are explained as follows: In the case of $l_{con, d}$ is the introduced label (Lines 6–9) and if $\sigma(d) = c$, we enforce that each $M \in M$ is also a model of $\gamma(d)$ and only keep sets $R$ in $R$, where each $M \in R$ is a model of $\gamma(d)$. Next, Lines 10–13 mark for defaults $d$, which do not fire because of $\alpha$ ($\sigma(d) = \overline{p}$), the empty default $\varepsilon$ by setting $\sigma(\varepsilon) = \overline{p}$. Additionally, it is enforced for these defaults $d$ that each set $R \in R$ contains any non-backfire model of $M$, which is also a model of $\neg \alpha(d)$. Finally, if $\sigma(d) = \beta$, Lines 14–16 increase $B$ such that it grows by models of $M$ that are also models of $\beta(d)$ and contain the atom $bf$. Next, we cover the case, where an atom $a$ is introduced. Lines 17–19 cover this node type, which increases existing witness set $M$, backfire witness sets $B$, and required witness sets $R$ by which $a$ is set to true. In particular, we have to compute all potential

\[ S_{bf} = \{S : S \in S, bf \in S\}, S^-_e := S \setminus \{e\}, S^+_e := \{S_e^+ \mid S \in S\}, S^-_e := S \cup \{e\}, S^+_e := \{S_e^+ \mid S \in S\}, \emptyset := \{\emptyset\} \text{ and } \] 
\[ S^e := \bigcup_{S \in S, S' \in S(S)} \{S' \cup S\} \text{ and } S^-_e := S' \cup S\} \text{ Further, identity function } \text{ is defined by } f(x) \mapsto x, \text{ and constant function } f_S \text{ is given by } f(\varepsilon) \mapsto S. \]
sets \( R \in \mathcal{R} \) of models, where \( a \) is either set to true or to false. When a default \( d \) gets removed in Lines 20–22, we remove the mapping of \( d \in \sigma \) to any of \( \{ p, j, c \} \) since \( d \) is not considered anymore. The removal of an atom \( a \) (Lines 23–25) is roughly speaking, a kind of projection to the current bag. We remove objects anywhere within the tuples, which contain the atom \( a \) we do not consider anymore. We also use this case for removing a label, since we only copy all the tuples there. Finally, the case where \( \text{type}(t) = \text{join} \) also talks about a second tuple \( u'' \) of a different, second child table \( \tau'' \). Intuitively, both tuples, representing intermediate results of two different branches, have to agree on the witness extension, witness states, and the witness models. Note that for a backfire model \( B \) to endure in \( B \in \mathcal{P} \) of \( \tau_i \), it suffices if \( B \) is a backfire model in one branch \( B \cup \{bf\} \in B' \) and an ordinary model in the other branch \( (M \in \mathcal{M}) \).

**Example 5.** Consider default theory \( D \) from Example 7 and in Figure 3 (left) LTD \( \mathcal{T} = (\cdot, \chi) \) of \( S(D) \) and the tables \( \tau_1, \ldots, \tau_{16} \) illustrating computation results obtained during post-order traversal of \( \mathcal{T} \) by \( \mathcal{DP}_{SCONS} \). Table \( \tau_1 = \{ \{0, \{0\}\}, \{\{0, \{0, 0\}\}\} \} \) as \( \text{type}(t_1) = \text{leaf} \) (see Line 2). Since \( \text{type}(t_2) = \text{int} \) and \( a \) is the introduced atom, we construct table \( \tau_2 \) from \( \tau_1 \) by \( \{ \{\sigma_{1,1}, \mathcal{M}_{2,1}, \mathcal{R}_{1,1}\}\} \), where \( \mathcal{M}_{2,1} \) contains \( \mathcal{M}_{1,1,k} \) for each \( \mathcal{M}_{1,1,k} \) \( (k \leq 1) \) in \( \tau_1 \) (corresponding to a guess on \( a \)). Precisely, \( \mathcal{M}_{2,1} := \{ \{0, \{a\}\}\} \) (L 19). Then, \( t_3 \) introduces default \( d_1 \) fire. In this case, the reason for not firing can be \( \alpha(d_1) \) or \( \beta(d_1) \) (see \( \mathcal{P}_{3,1}, L 5 \)). Otherwise, i.e., default \( d_1 \) fires, we have \( \mathcal{Z}_{1,1} = \{d_1\} \) and \( \mathcal{P}_{3,2} = \{\{d_1 \rightarrow c\}, \{0, 0\}\} \), since \( d_1 \) fires. Node \( t_4 \) introduces label \( \text{lab}_{past_{d_1}} \) and modifies \( \mathcal{P}_{1,1} \). In particular, it chooses among \( \mathcal{M} \) candidates, which might prove that \( d_1 \) backfires (see Line 16). Obviously, since \( \beta(d_1) = b \), \( \mathcal{B}_{1,1,2} = \{\{a, bf\}\} \). In Table \( \tau_6 \), we cover cases, where \( \text{default}(d_1) \) does not fire because of its preconditions \( \alpha(d_1) = \{\} \). In this case (see \( \mathcal{P}_{5,1,1}, L 11 \)), we do not find any model of \( \{0\} \), hence \( \mathcal{R}_{5,1,1} = \emptyset \). Whenever such a set \( \{\sigma, \emptyset\} \in \mathcal{P} \) occurs in some table, we could discard the tuple in case of \( \varepsilon \in \sigma^{-1}(\alpha) \), since at least one choice concerning \( \alpha \) was wrong. Table \( \tau_7 \) concerns about the conclusion \( \gamma(d_1) \) of a default, and, intuitively, updates every model occurring in the table, such that the models satisfy \( \gamma(d_1) \) in case it fires. The right branch of the LTD works similarly to the left branch, and in the end, join node \( t_{16} \), intuitively, just combines witnesses agreeing on its bag content.

Next, we briefly discuss the handling of counter-witnesses, which completes Algorithm SINC. Observe that the handling of counter-witnesses \( C \) is quite similar to the witness proofs \( P \). The tuples \( (\rho, \mathcal{MC}, \mathcal{RC}) \in C \) consist of a counter-witness state \( \rho : D_{\leq 1} \cup \{\varepsilon\} \to \{\{p, j, c\}\} \), counter-witness models \( \mathcal{MC} \subseteq 2^X \), and a set of required counter-witness models \( \mathcal{RC} \subseteq 2^X \) for \( X = \text{at}_{\leq 1} \cup \{bf\} \). Compared to the witness models, \( \mathcal{MC} \) potentially contains backfire counter-witness models per design, since here we do not require to enumerate
all the justifications, why a certain witness does not lead to a (subset-minimal) generating default. Instead, the existence of a certain counter-witness tuple for a witness in a table \( \tau \) proves that the corresponding witness can not be extended to a generating default of \( D_{\leq t} \).

In the following, we provide insights on the correctness of the algorithm \( \mathcal{DP}_{\text{SINC}} \).

**Theorem 1.** Given a default theory \( D \), the algorithm \( \mathcal{DP}_{\text{SINC}} \) correctly solves \( \text{Ext} \).

The correctness proof of these algorithms need to investigate each node type separately. We have to show that a tuple at a node \( t \) guarantees existence of a model for the program \( D_{\leq t} \), proving soundness. Conversely, one can show that each generating default set is indeed evaluated while traversing the TD, which provides completeness. We employ this idea using the notions of (i) partial solutions consisting of partial extensions and the notion of (ii) local partial solutions.

**Definition 1.** Let \( D \) be a default theory, \( T = (T, \chi) \) be an LTD of the semi-incidence graph \( S(D) \) of \( D \), where \( T = (N, \cdot, \cdot) \), and \( t \in N \) be a node. Further, let \( \emptyset \subseteq M \subseteq 2^{M_{\leq t}(\{\chi\})} \), \( \mathcal{R} \subseteq 2^{M_{\leq t}} \), \( \sigma : (D_{\leq t} \cup \{\chi\}) \rightarrow \{\emptyset, \mathcal{R}, \chi\} \), \( E \subseteq \gamma_{\leq t} \), and \( Z := \gamma(\sigma^{-1}(\chi)) \) with \( Z \subseteq E \). The tuple \((\sigma, M, \mathcal{R})\) is a partial extension under \( E \) for \( t \) if the following conditions hold:

1. \( Z \models D_{\leq t} \setminus \{d \in D_{\leq t} \mid \sigma(d) = \emptyset \} \exists M \models M_{\mathcal{BF}} : M \models E, \beta(d) \} \cup \{d \in D_{\leq t} \mid \sigma(d) = \emptyset, \mathcal{R} = \emptyset\} \}, \n2. \sigma(\emptyset) = \mathcal{R} \Leftarrow \sigma^{-1}(\emptyset) \neq \emptyset; \quad \sigma^{-1}(\emptyset) = \emptyset \Rightarrow \sigma(\emptyset) \) undefined, \n3. \( M \) is the largest set such that:
   \[(a) \; M \models Z \quad \text{for every} \; \emptyset \subseteq M \subseteq M_{\mathcal{BF}}, \n(b) \; M \models E, \beta(d) \Rightarrow M_{\mathcal{BF}} = \emptyset \quad \text{for every} \; d \in D_{\leq t} \; \text{s.t.} \; \sigma(d) = \emptyset, \beta(d) \in \beta_{\leq t}, \; M \in \mathcal{M}, \n(c) \exists d \in D_{\leq t} : \sigma(d) = \emptyset, \beta(d) \in \beta_{\leq t}, \; M \models E, \beta(d) \quad \text{for every} \; M \in M_{\mathcal{BF}}, \; \text{and} \n\]

4. \( \mathcal{R} \) is the largest set such that:
   \[(a) \; |\mathcal{R}| \leq |\sigma^{-1}(\alpha)| - 1 \quad \text{for every} \; \emptyset \subseteq \mathcal{R} \subseteq M_{\mathcal{BF}}, \n(b) \exists d \in D_{\leq t} : \sigma(d) = \emptyset, \alpha(d) \in \alpha_{\leq t}, \; M \models Z, \neg \alpha(d) \quad \text{for every} \; M \in \bigcup_{\emptyset \subseteq \mathcal{R} \subseteq M_{\mathcal{BF}}} \mathcal{R}, \; \text{and} \n(c) \exists \emptyset \subseteq \mathcal{R} : M \models Z, \neg \alpha(d) \Leftarrow \sigma(d) = \emptyset \quad \text{for every} \; d \in D_{\leq t} \; \text{s.t.} \; \alpha(d) \in \alpha_{\leq t}, \mathcal{R} \subseteq \mathcal{R} \]

**Definition 2.** Let \( D \) be a default theory, \( T = (T, \chi) \) where \( T = (N, \cdot, \cdot) \) be an LTD of \( S(D) \), and \( t \in N \) be a node. A partial solution for \( t \) is a tuple \((Z, M, \mathcal{P}, \mathcal{C})\) where \( Z \subseteq \gamma_{\leq t} \), and \( \mathcal{P} \) is the largest set of tuples such that each \((\sigma, B, R) \in \mathcal{P} \) is a partial extension under \( Z \) with \( B_{\mathcal{BF}} = B \) and \( Z = \gamma^{-1}(\chi) \). Moreover, \( \mathcal{C} \) is the largest set of tuples such that for each \((\rho, M_{\mathcal{C}}, R_{\mathcal{C}}) \in \mathcal{C} \), we have that \((\rho, M_{\mathcal{C}}, R_{\mathcal{C}}) \) is a partial extension under \( Z \) with \( \rho^{-1}(\chi) \subseteq \gamma^{-1}(\chi) \). Finally, \( \mathcal{M} \subseteq 2^{M_{\leq t}} \) is the largest set with \( M \models \gamma(Z) \) for each \( M \in \mathcal{M} \).

The following lemma establishes correspondence between stable extensions and partial solutions.

**Lemma 1.** Let \( D \) be a default theory, \( T = (T, \chi) \) be an LTD of the semi-incidence graph \( S(D) \), where \( T = (N, \cdot, \cdot) \), and \( \chi(n) = \emptyset \). Then, there exists a stable extension \( Z' \) for \( D \) if and only if there exists a partial solution \( u = (Z, M, \mathcal{P}, \mathcal{C}) \) for root \( n \) with at least one tuple \((\sigma, B, R) \in \mathcal{P} \) where \( B_{\mathcal{BF}} = \emptyset \), and either \( \sigma^{-1}(\emptyset) = \emptyset \), or \( R \neq \emptyset \), where \( \mathcal{C} \) is of the following form: For each \((\rho, M_{\mathcal{C}}, R_{\mathcal{C}}) \in \mathcal{C} \), either \( M_{\mathcal{BF}} \neq \emptyset \), or \( R_{\mathcal{C}} = \emptyset \) and \( \rho^{-1}(\emptyset) \neq \emptyset \).

Next, we need the notion of local partial solutions corresponding to the tuples obtained in Algorithm 2.

**Definition 3.** Let \( D \) be a default theory, \( T = (T, \chi) \) an LTD of the semi-incidence graph \( S(D) \), where \( T = (N, \cdot, \cdot) \), and \( t \in N \) be a node. A tuple \((\sigma, B, R) \) is a local partial solution part of partial solution \((\hat{\sigma}, \hat{B}, \hat{R}) \) for \( t \) if

\[\text{Proofs for statements marked with “\#” can be found in the appendix.}\]
1. \( \sigma = \hat{s} \cap (\chi(t) \cup \{ \varepsilon \}) \times \{ \overline{p}, \overline{j}, c \} \),
2. \( \mathcal{B} = \hat{S}_t \), where \( \mathcal{S}_t := \{ S \cap (\chi(t) \cup \{ b \}) \mid S \in \mathcal{S} \} \), and
3. \( \mathcal{R} = \{ R_t \mid R \in \mathcal{R} \} \).

**Definition 4.** Let \( D \) be a default theory, \( \mathcal{T} = (T, \chi) \) an LTD of the semi-incidence graph \( S(D) \), where \( T = (N, \cdot, n) \), and \( t \in N \) be a node. A tuple \( u = \langle Z, \mathcal{M}, \mathcal{P}, \mathcal{C} \rangle \) is a local partial solution for \( t \) if there exists a partial solution \( \hat{u} = \langle \hat{Z}, \mathcal{M}, \hat{\mathcal{P}}, \hat{\mathcal{C}} \rangle \) for \( t \) such that the following conditions hold: (1) \( Z = \hat{Z} \cap 2^\mathcal{O} \), (2) \( \mathcal{M} = \mathcal{M}_t \), (3) \( \mathcal{P} \) is the smallest set containing local partial solution part \( (\sigma, \mathcal{B}, \mathcal{R}) \) for each \( (\hat{\sigma}, \hat{\mathcal{B}}, \hat{\mathcal{R}}) \in \hat{\mathcal{P}} \), and (4) \( \mathcal{C} \) is the smallest set with local partial solution part \( (\rho, \mathcal{MC}, \mathcal{RC}) \in \mathcal{C} \) for each \( (\hat{\rho}, \hat{\mathcal{MC}}, \hat{\mathcal{RC}}) \in \hat{\mathcal{C}} \). We denote by \( u^t \) the local partial solution for \( t \) given partial solution \( \hat{u} \).

The following proposition provides justification that it suffices to store local partial solutions instead of partial solutions for a node \( t \in N \).

**Lemma 2.** Let \( D \) be a default theory, \( \mathcal{T} = (T, \chi) \) an LTD of \( S(D) \), where \( T = (N, \cdot, n) \), and \( \chi(n) = \emptyset \). Then, there exists a stable extension for \( D \) if and only if there exists a local partial solution of the form \( \langle \emptyset, \{ \emptyset \}, \mathcal{P}, \mathcal{C} \rangle \) for the root \( n \in N \) with at least one tuple of the form \( \langle \sigma, \emptyset, \mathcal{R} \rangle \in \mathcal{P} \) where either \( \sigma^{-1}(\alpha) = \emptyset \), or \( \mathcal{R} \neq \emptyset \). Moreover, for each \( (\rho, \mathcal{MC}, \mathcal{RC}) \in \mathcal{C} \), either \( \mathcal{MC}_{BF} \neq \emptyset \), or \( \rho^{-1}(\alpha) \neq \emptyset \) and \( \mathcal{RC} = \emptyset \).

**Proof.** Since \( \chi(n) = \emptyset \), every partial solution \( n \) is an extension of the local partial solution \( \hat{u} \) for the root \( n \in N \) according to Definition 4. By Lemma 1 we obtain that the lemma is true.

In the following, we abbreviate atoms occurring in bag \( \chi(t) \) by \( at_t \), i.e., \( at_t := \chi(t) \setminus D_t \).

**Proposition 2 (Soundness).** Let \( D \) be a default theory, \( \mathcal{T} = (T, \chi) \) an LTD of \( S(D) \), where \( T = (N, \cdot, \cdot) \), and \( t \in N \) be a node. Given a local partial solution \( u' \) of child table \( \tau' \) (or local partial solution \( u' \) of table \( \tau' \) and local partial solution \( u'' \) of table \( \tau'' \)), each tuple \( u \) of table \( \tau_t \) constructed using table algorithm \( \text{SINC} \) is also a local partial solution.

**Proposition 3 (Completeness).** Let \( D \) be a default theory, \( \mathcal{T} = (T, \chi) \) where \( T = (N, \cdot, \cdot) \) be an LTD of \( S(D) \) and \( t \in N \) be a node. Given a local partial solution \( u' \) of table \( \tau_t \), either \( t \) is a leaf node, or there exists a local partial solution \( u'' \) of child table \( \tau' \) (or local partial solution \( u'' \) of table \( \tau' \) and local partial solution \( u''' \) of table \( \tau''' \)) such that \( u \) can be constructed by \( u' \) (or \( u' \) and \( u'' \), respectively) and using table algorithm \( \text{SINC} \).

**Corollary 1 (Completeness for Enumeration).** Let \( D \) be a default theory, \( \mathcal{T} = (T, \chi) \) where \( T = (N, \cdot, \cdot) \) be an LTD of \( S(D) \) and \( t \in N \) be a node. Given a partial solution \( \hat{u} \) and the corresponding local partial solution \( u = \hat{u}^t \) for table \( \tau_t \), either \( t \) is a leaf node, or there exists a local partial solution \( u' \) of child table \( \tau' \) (or local partial solution \( u' \) of table \( \tau' \) and local partial solution \( u'' \) of table \( \tau'' \)) such that \( u \) can be constructed by \( u' \) (or \( u' \) and \( u'' \), respectively) and using table algorithm \( \text{SINC} \).

**Proof.** The result directly follows from the proof for completeness (see Proposition 3).

Now, we are in the situation to prove Theorem 1.

**Proof of Theorem 1.** We first show soundness. Let \( \mathcal{T} = (T, \chi) \) be the given LTD, where \( T = (N, \cdot, n) \). By Lemma 2 we know that there is a stable extension \( \mathcal{D} \) if and only if there exists a local partial solution for the root \( n \). Note that the tuple is by construction of the form \( \langle \emptyset, \{ \emptyset \}, \mathcal{P}, \mathcal{C} \rangle \), where \( \mathcal{P} \neq \emptyset \) can contain a combination of the following tuples \( \langle \emptyset, \emptyset, \emptyset \rangle, \langle \{ \varepsilon \rightarrow \alpha \}, \emptyset, \{ \emptyset \} \rangle \). For each \( (\rho, \mathcal{MC}, \mathcal{RC}) \in \mathcal{C} \) either \( \mathcal{MC}_{BF} \neq \emptyset \) or \( \rho^{-1}(\alpha) \neq \emptyset \) and \( \mathcal{RC} = \emptyset \). In total, this results in 48 possible tuples, since \( C \subseteq 2^\mathcal{E} \) can contain any combination (16 many) of \( C \), where \( C = \{ \emptyset, \{ \emptyset \}, \emptyset \}, \{ \varepsilon \rightarrow \alpha \}, \{ \emptyset \}, \{ \emptyset \}, \{ \varepsilon \rightarrow \alpha \}, \{ \emptyset, \{ b \} \}, \{ \emptyset \} \), \( \{ \varepsilon \rightarrow \alpha \}, \{ \emptyset, \{ b \} \}, \{ \emptyset \} \).
Hence, we proceed by induction starting from the leaf nodes in order to end up with such a tuple at the root node \( n \). In fact, the tuple \( \langle \emptyset, \{ \emptyset \} \rangle \) is trivially a partial solution for (empty) leaf nodes by Definitions 1 and 2 and also a local partial solution of \( \langle \emptyset, \{ \emptyset \} \rangle \) by Definition 3. We already established the induction step in Proposition 2. Hence, when we reach the root \( n \), when traversing the TD in post-order by Algorithm \( \mathcal{DP}_{\text{SINC}} \), we obtain only valid tuples inbetween and a tuple of the form discussed above in the table of the root \( n \) witnesses an answer set.

Next, we establish completeness by induction starting from the root \( n \). Let therefore, \( \hat{Z} \) be an arbitrary stable extension of \( D \). By Lemma 2 we know that for the root \( n \) there exists a local partial solution of the discussed form \( \langle \emptyset, \{ \emptyset \}, \mathcal{P}, \mathcal{C} \rangle \) for some partial solution \( \langle \hat{Z}, \hat{M}, \hat{\mathcal{P}}, \hat{\mathcal{C}} \rangle \) with \( \gamma(\hat{Z}') = \hat{Z} \). We already established the induction step in Proposition 3. Hence, we obtain some (corresponding) tuples for every node \( t \). Finally, stopping at the leaves \( n \). In consequence, we have shown both soundness and completeness resulting in the fact that Theorem 1 is true.

Theorem 1 states that we can decide the problem \( \text{Cons} \) by means of Algorithm \( \mathcal{DP}_{\text{SINC}} \). The following theorem states its that we obtain fourfold exponential runtime in the treewidth.

**Theorem 2 (**) Given a default theory \( D \), the algorithm \( \mathcal{DP}_{\text{SINC}} \) and runs in time \( \mathcal{O}(4^{2^{k+2}} \cdot \|D\|) \), where \( k := \text{tw}(S(D)) \) is the treewidth of the semi-incidence graph \( S(D) \).

Next, we establish that we can slightly extend \( \mathcal{DP}_{\text{SINC}} \) to enumerate characterizing generating defaults. The algorithm on top of \( \mathcal{DP}_{\text{SINC}} \) is relatively straightforward and presented in Algorithm 3.

**Theorem 3 (**) Given a default theory \( D \), the algorithm \( \mathcal{DP}_{\text{SINC}} \) can be used as a preprocessing step to construct tables from which we can correctly solve the problem \( \text{EnumSE} \).

The correctness proof requires to extend the previous results to establish a one-to-one correspondence when traversing the tree the TD and we can reconstruct each solution as well as we do not get duplicates. Therefore, we require the following three results.

**Observation 1.** Let \( D \) be a default theory, \( T = (T, \gamma) \) where \( T = (N, \cdot, \cdot) \) be an LTD of \( S(D) \) and \( t \in N \) be a node. Then, for each partial solution \( u = (Z, \mathcal{M}, \mathcal{P}, \mathcal{C}) \) for \( t \), \( \mathcal{M}, \mathcal{P} \) and \( \mathcal{C} \) are functional dependent from \( Z \), i.e., for any partial solution \( u' = (Z, \mathcal{M}', \mathcal{P}', \mathcal{C}') \) for \( t \), we have \( u = u' \).

**Lemma 3 (**) Let \( D \) be a default theory, \( T = (T, \gamma) \) with \( T = (N, \cdot, \cdot) \) be an LTD of \( S(D) \), and \( Z \) be a characterizing generating default. Then, there is a unique set of tuples \( S \), containing exactly one tuple per node \( t \in N \) containing only local partial solutions of the unique partial solution for \( Z \).

**Proposition 4 (**) Let \( D \) be a default theory, \( T = (T, \gamma) \) with \( T = (N, \cdot, \cdot) \) be an LTD of \( S(D) \), and \( Z \) be a characterizing generating default. Moreover, let \( S \) be the unique set of tuples, containing exactly one tuple per node \( t \in N \) containing only local partial solutions of the unique partial solution for \( Z \). Given \( S \), and tables of Algorithm \( \text{SINC} \), one can compute in time \( \mathcal{O}(|D|) \) a characterizing generating default \( Z' \) with \( Z' \neq Z \), assuming one can get for a specific tuple \( u \) for node \( t \) its corresponding \(-\)-ordered predecessor tuple set \( \text{orig}_t(u) \) of tuples in the child node(s) of \( t \) in constant time.

4 Conclusion

In this paper, we established dynamic programming algorithms that operate on tree decompositions of the semi-incidence graph of a given default theory. Our algorithms can be used to decide whether the default theory has a stable extension or to enumerate all characterizing generating defaults. Our algorithms run in \( \text{fpt-time} \) and linear delay, respectively.

We believe that our algorithms can be extended to tree decompositions of the incidence graph. However, then we need additional states to handle the situation that prerequisite, justification, and conclusion may...
not occur together in one bag, consequently such an algorithm will likely be very complex. An interesting research question is whether we can improve our runtime bounds. Still it might be worth implementing our algorithms to enumerate characterizing generating defaults for DL, as previous work showed that a relatively bad worst-case runtime may anyways lead to practical useful results [5].

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A Omitted Proofs

Lemma 4. Let $D$ be a default theory, $T = (T, \chi)$ be an LTD of the semi-incidence graph $S(D)$, where $T = \langle \cdot, \cdot, \cdot, \cdot, \cdot \rangle$, and $\chi(n) = \emptyset$. Then, there exists a stable extension $Z'$ for $D$ if and only if there exists a partial solution $u = (Z, M, P, C)$ for root $n$ with at least one tuple $(\sigma, B, R) \in P$ where $B_{BF} = \emptyset$, and either $\sigma^{-1}(\emptyset) = \emptyset$, or $R \neq \emptyset$, where $C$ is of the following form: For each $(\rho, MC, RC) \in C$, either $MC_{BF} \neq \emptyset$, or $RC = \emptyset$ and $\rho^{-1}(\emptyset) \neq \emptyset$.

Proof (Sketch). Given a stable extension $Z'$ of $D$ we construct $u = (Z, M, P, C)$ where we generate every potential function $\sigma : D_{equal} \cup \{z\} \rightarrow \{p, j, c\}$ with $\sigma(d) : = \emptyset$ (only if $Z', \neg(\alpha(d)) \not\subseteq \emptyset$), or $\sigma(d) : = j$ (only if $Z', \beta(d) \not\subseteq \emptyset$) for any $d \in D_{equal}$. Finally, if $\sigma(d)$ has not been set yet (and neither one of the two conditions holds), let $\sigma(d) : = c$. In case of $\sigma^{-1}(\emptyset) \neq \emptyset$, we also set the empty default $\sigma(\emptyset) : = \emptyset$. Observe that since $Z'$ is a stable extension, $\sigma(d) = c \implies \gamma(d) \in Z'$.

For each of these functions $\sigma$, we require $(\sigma, \emptyset, R) \in P$, where $R : = \{R | R \subseteq 2^{st(D)}, |R| \leq |\sigma^{-1}(\emptyset)| - 1, \mbox{ for all } d \in \sigma^{-1}(\emptyset) \setminus \{\emptyset\} \} : \exists M \in R : M \models Z', \neg(\alpha(d))$. Moreover, we define set $Z : = \sigma^{-1}(\emptyset)$ and set $M : = \text{Mod}(\{C \mid C \in Z', Z' \in Z', 2^{st(D)}\})$, in order for $u$ to be a partial solution for $n$ (see Definition[2]). We construct $C$, consisting of partial solutions $(\rho, MC, RC)$ where we use every potential state function $\rho$ with $\rho^{-1}(\emptyset) \subseteq \sigma^{-1}(\emptyset)$. For this, let $E : = \gamma(\rho^{-1}(\emptyset))$.

For the defaults $d$ with $\rho(d) \neq c$, i.e., defaults $d$ that do not fire because of $\alpha(d)$ or $\beta(d)$, we also set their state $\rho(d)$ to $\alpha$ or $\beta$, respectively (analogous to above). Again, in case of $\rho^{-1}(\emptyset) \neq \emptyset$, we also set the empty default $\rho(\emptyset) : = \emptyset$, otherwise $\rho(\emptyset)$ is undefined. Finally, we define set $MC : = \text{Mod}(\{C \mid C \in E_i, E_i \in E\}, 2^{st(D)} \cup \bigcup_{d \in (\rho(d) = j)} \text{Mod}(\{Z_i' \mid C \in Z_i', Z_i' \in Z', 2^{st(D)}\})$ and $RC : = \{R | R \subseteq 2^{st(D)}, |R| \leq |\rho^{-1}(\emptyset)| - 1, \mbox{ for all } d \in \rho^{-1}(\emptyset) \setminus \{\emptyset\} \} : \exists M \in R : M \models E, \neg(\alpha(d))$ according to Definition[1].

For the other direction, Definitions[1] and [2] guarantee that $Z'$ is a stable extension if there exists such a partial solution $u$. In consequence, the lemma holds.

Proposition 5 (Soundness). Let $D$ be a default theory, $T = (T, \chi)$ an LTD of the semi-incidence graph $S(D)$, where $T = \langle N, \cdot, \cdot, \cdot, \cdot \rangle$, and $\chi(t) \in N$ a node. Given a local partial solution $u'$ of child table $\tau'$ (or local partial solution of table $\tau'$ and local partial solution $u''$ of table $\tau''$), each tuple $u$ of table $\tau_i$ constructed using table algorithm SINC is also a local partial solution.

Proof. Let $u'$ be a local partial solution for $t' \in N$ and $u$ a tuple for node $t \in N$ such that $u$ was derived from $u'$ using table algorithm SINC. Hence, node $t'$ is the only child of $t$ and $t$ is either removal or introduce node.

Assume that $t$ is a removal node and $d \in D_{t} \setminus D_{t}$ for some default $d$. Observe that for $u = (Z, M, P, C)$ and $u' = (Z', M', P', C')$, models $M$ and model sets $R$ are equal, i.e., $(\cdot, M, R) \in P \iff (\cdot, M, R) \in P'$ and $(\cdot, M, R) \in C \iff (\cdot, M, R) \in C'$. Since $u'$ is a local partial solution, there exists a partial solution $\bar{u}'$ of $t'$, satisfying the conditions of Definition[1]. Then, $\bar{u}'$ is also a partial solution for node $t$, since it satisfies all conditions of Definitions[1] and [2]. Finally, note that $u = (\bar{u}')^t$ since the projection of $\bar{u}'$ to the bag $\chi(t)$ is $u$ itself. In consequence, the tuple $u$ is a local partial solution.

For $a \in at_{t} \setminus at_{t}$ as well as for introduce nodes, we can analogously check the proposition.

Next, assume that $t$ is a join node. Therefore, let $u'$ and $u''$ be local partial solutions for $t', t'' \in N$, respectively, and $u$ be a tuple for node $t \in N$ such that $u$ can be derived using both $u'$ and $u''$ in accordance with the SINC algorithm. Since $u'$ and $u''$ are local partial solutions, there exists partial solution $\bar{u}' = (Z', M', P', C')$ for node $t'$ and partial solution $\bar{u}'' = (Z'', M'', P'', C'')$ for node $t''$. Using these two partial solutions, we can construct $\bar{u} = (Z' \cup Z'', M' \bowtie M'', P' \bowtie P'', (C' \bowtie C'') \cup (P' \bowtie C'') \cup (C' \bowtie P''))$. Finally, note that $u = (\bar{u})^t$ since the projection of $\bar{u}$ to the bag $\chi(t)$ is $u$ itself. In consequence, the tuple $u$ is a local partial solution.
where for $\bowtie (\cdot, \cdot)$ we refer to Algorithm 2. We define $\bowtie (\cdot, \cdot)$ in accordance with Algorithm 2 as follows:

$$
\text{compat}_f (\hat{R}) := \text{true} \iff |\{ \hat{M} | \hat{M} \in \hat{R}, \hat{M} \cap \chi(t) = f(\hat{M}) \cap \chi(t) \}| = |\hat{R}|
$$

$$
\hat{R}' \bowtie \hat{R}'' := \bigcup_{f : \hat{R}' \rightarrow \hat{R}'' \text{ surjective, } \text{compat}_f (\hat{R}') \hat{M}' \in \hat{R}'} \{ \hat{M}' \cup f(\hat{M}') \}
$$

$$
\hat{R}' \bowtie \hat{R}'' := \bigcup_{\hat{r} \in \hat{R}' \bowtie \hat{r}'' \in \hat{R}''} (\hat{r}' \bowtie \hat{r}'') \cup (\hat{r}'' \bowtie \hat{r}')
$$

$$
\mathcal{MC}' \bowtie_{\mathcal{M}', \mathcal{M}''} \mathcal{MC}'' := (\mathcal{MC}' \bowtie \mathcal{MC}'') \cup [\mathcal{MC}_{BF} \bowtie (\mathcal{MC}_{BF}'' \cup \mathcal{M}'')] \cup [(\mathcal{MC}_{BF} \cup \mathcal{M}') \bowtie \mathcal{MC}_{BF}'']
$$

$$
\hat{C}' \bowtie \hat{C}'' := \{ (\hat{\rho}' \cup \hat{\rho}'', \mathcal{M}' \bowtie_{\hat{\mathcal{M}'}, \hat{\mathcal{M}}''} \mathcal{M}''', \hat{R}\hat{C}' \bowtie \hat{R}\hat{C}'', \hat{\rho}'' \bowtie \hat{\rho}'' (\pi) \cap \chi(t) = \hat{\rho}'' (\hat{\pi}) \cap \chi(t) \text{ forall } \pi \in \{ \hat{p}, \hat{j}, \hat{c} \}) \}
$$

Then, we check all conditions of Definitions 1 and 2 in order to verify that $\hat{u}$ is a partial solution for $t$. Moreover, the projection $\hat{u}'$ of $\hat{u}$ to the bag $\chi(t)$ is exactly $u$ by construction and hence, $u = \hat{u}'$ is a local partial solution.

Since one can provide similar arguments for each node type, we established soundness in terms of the statement of the proposition.

**Proposition 6** (Completeness). Let $D$ be a default theory, $T = (T, \chi)$ where $T = (N, \cdot, \cdot)$ be an LTD of $S(D)$ and $t \in N$ be a node. Given a local partial solution $u$ of table $\tau_t$, either $t$ is a leaf node, or there exists a local partial solution $u'$ of child table $\tau'$ (or local partial solution $u'$ of table $\tau''$) such that $u$ can be constructed by $u'$ (or $u'$ and $u''$, respectively) and using table algorithm $\text{SINC}$.

**Proof.** Let $t \in N$ be a removal node and $d \in D_{\tau'} \setminus D_t$ with child node $t' \in N$. We show that there exists a tuple $u'$ in table $\tau_{t'}$ for node $t'$ such that $u$ can be constructed using $u'$ by $\text{SINC}$ (Algorithm 2). Since $u$ is a local partial solution, there exists a partial solution $\hat{u} = (\hat{Z}, \hat{M}, \hat{P}, \hat{C})$ for node $t$, satisfying the conditions of Definition 2. It is easy to see that $\hat{u}$ is also a partial solution for $t'$ and we define $u' := \hat{u}'$, which is the projection of $\hat{u}$ onto the bag of $t'$. Apparently, the tuple $u'$ is a local partial solution for node $t'$ according to Definition 2. Then, $u$ can be derived using $\text{SINC}$ algorithm and $u'$. By similar arguments, we establish the proposition for $a \in at_{t'} \setminus at_{t}$ and the remaining node types. Hence, the propositions sustains.

**Theorem 2.** Given a default theory $D$, the algorithm $\text{DP}_{\text{SINC}}$ and runs in time $O(4^{2^{k+2}} \cdot |S(D)|)$, where $k := tw(S(D))$ is the treewidth of the semi-incidence graph $S(D)$.

First, we give a proposition on worst-case space requirements in tables for the nodes of our algorithm.

**Proposition 7.** Given a default theory $D$, an LTD $T = (T, \chi)$ of the semi-incidence graph $S(D)$, and a node $t \in N$. Then, there are at most $2^{k+1} \cdot 2^{2^{k+1}} \cdot 4^{3^{k+1} \cdot 2^{k+1} \cdot 2^{k+1}}$ tuples in $\tau_t$ using algorithm $\text{DP}_{\text{SINC}}$ for width $k$ of $T$.

**Proof.** (Sketch) Let $D$ be the given default theory, $T = (T, \chi)$ an LTD of the semi-incidence graph $S(D)$, where $T = (N, \cdot, \cdot)$, and $t \in N$ a node of the TD. Then, by definition of a decomposition of the semi-incidence graph for each node $t \in N$, we have $|\chi(t)| - 1 \leq k$. In consequence, we can have at most $2^{k+1}$ many witness defaults and $2^{2^{k+1}}$ many witnesses models. Each set $\mathcal{P}$ may contain a set of witness proof tuples of the form $(\sigma, \mathcal{B}, \mathcal{R})$, with at most $3^{k+1}$ many witness state $\sigma$ mappings, $2^{k+1}$ many backfire
Algorithm 3: Algorithm $N\overrightarrow{GD}_S(T)$ for computing next characteristic generating default.

| In: TD $T = (T, \cdot)$ with $T = (N, \cdot, n)$, solution tuples $S$, total ordering $\prec$ of orig$(\cdot)$.
| Out: Next solution tuples using $\prec$.
| for iterate $t$ in post-order$(T, n)$ do
| Child-Tabs := \{Tables$t'\mid t'$ is a child of $t$ in $T$\}
| $\hat{t}$ := parent of $t$
| $S[t]$ := direct successor $s' > S[t]$ in orig$_t(S[\hat{t}])$
| if $S[t]$ defined then
| for iterate $t'$ in Child-Tabs do
| for iterate $t''$ in pre-order$(T, t')$ do
| $\hat{t}''$ := parent of $t''$
| $S[t'']$ := $\prec$-smallest element in orig$_{t''}(S[t''])$
| return $S$
| return undefined

Given a default theory $T$, $S(D) = (V, \cdot)$ its semi-incidence graph, and $k$ be the treewidth of $S(D)$. Then, we can compute in time $2^\mathcal{O}(k^n) \cdot |V|$ an LTD of width at most $k$ [1]. We take such a TD and compute in linear time a nice TD [4] $T = (T, \chi)$ be such a nice TD with $T = (N, \cdot, n)$. Since the number of nodes in $N$ is linear in the graph size and since for every node $t \in N$ the tuple $\tau_t$ is bounded by $2^{k+1} \cdot 2^{2k+1} \cdot 4^{(3^{k+1} \cdot 2^{k+1} \cdot 2^{2k+1})}$ according to Proposition [4], we obtain a running time of $\mathcal{O}(4^{2^{2k+1} \cdot |S(D)|})$. Consequently, the theorem sustains.

**Proof of Theorem 3** Let $D$ be a default theory, $S(D) = (V, \cdot)$ its semi-incidence graph, and $k$ be the treewidth of $S(D)$. Then, we can compute in time $2^\mathcal{O}(k^n) \cdot |V|$ an LTD of width at most $k$ [1]. We take such a TD and compute in linear time a nice TD [4] $T = (T, \chi)$ be such a nice TD with $T = (N, \cdot, n)$. Since the number of nodes in $N$ is linear in the graph size and since for every node $t \in N$ the tuple $\tau_t$ is bounded by $2^{k+1} \cdot 2^{2k+1} \cdot 4^{(3^{k+1} \cdot 2^{k+1} \cdot 2^{2k+1})}$ according to Proposition [4], we obtain a running time of $\mathcal{O}(4^{2^{2k+1} \cdot |S(D)|})$. Consequently, the theorem sustains.

**Observation 2.** Let $D$ be a default theory, $T = (T, \chi)$ where $T = (N, \cdot, n)$ be an LTD of $S(D)$ and $t \in N$ be a node. Then, for each partial solution $u = (Z, M, P, C)$ for $t$, $M, P$ and $C$ are functional dependent from $Z$, i.e., for any partial solution $u' = (Z, M', P', C')$ for $t$, we have $u = u'$.

**Proof.** The claim immediately follows from Definition [2].

**Lemma 5.** Let $D$ be a default theory, $T = (T, \chi)$ with $T = (N, \cdot, n)$ be an LTD of $S(D)$, and $Z$ be a characterizing generating default. Then, there is a unique set of tuples $S$, containing exactly one tuple per node $t \in N$ containing only local partial solutions of the unique partial solution for $Z$.

**Proof.** By Observation [1] given $Z$, we can construct one unique partial solution $\hat{u} = (Z, M, P, C)$. Then define the set $S$ by $S := \bigcup_{t \in N} \{u^t\}$. Assume that there is a different set $S' \neq S$ containing also exactly one tuple per node $t \in N$. Then there is at least one node $t \in N$, for which the corresponding tuples $u \in S, u^t \in S'$ differ ($u \neq u'$), since $\hat{u}$ is unique and the computation $u^t$ is defined in a deterministic, functional way (see Definition [4]). Hence, either $\hat{u}^t \neq u$ or $\hat{u}^t \neq u'$, leading to the claim. 

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*Note: The above text is a simplified representation of the content in the image, focusing on the key points and avoiding the technical details and symbols that are not translated clearly into natural text.*
**Proposition 8.** Let $D$ be a default theory, $T = (T, \chi)$ with $T = (N, \cdot, \cdot)$ be an LTD of $S(D)$, and $Z$ be a characterizing generating default. Moreover, let $S$ be the unique set of tuples, containing exactly one tuple per node $t \in N$ and containing only local partial solutions of the unique partial solution for $Z$. Given $S$, and tables of Algorithm $\text{SINC}$, one can compute in time $O(\|D\|)$ a characterizing generating default $Z'$ with $Z' \neq Z$, assuming one can get for a specific tuple $u$ for node $t$ its corresponding $\prec$-ordered predecessor tuple set $\text{orig}(u)$ of tuples in the child node(s) of $t$ in constant time.

**Proof.** Note that with $Z$, it is easy to determine, which element of $S$ belongs to which node $t$ in $T$, hence, we can construct a mapping $S : N \rightarrow S$. With $S$, we can easily apply algorithm $\text{NGD}$, which is given in Algorithm 3 in order to construct a different solution $S'$ in a systematic way with linear time delay, since $T$ is nice.

**Proof of Theorem 3 (Sketch).** First, we construct an LTD $T = (T, \chi)$ with $T = (N, \cdot)$ for graph $S(D)$. Then we run our algorithm $\text{DP}_{\text{SINC}}$ and get tables for each TD node. In order to enumerate all the characterizing generating defaults, we investigate each of these tuple, which lead to a valid characterizing generating default (see proof of Theorem 1). For each of these tuples (if exist), we construct a first solution $S$ (if exist) (as done in Lines 6 to 9 of Algorithm 3) using $\text{orig}(\cdot, \cdot)$, and total order $\prec$. Thereby, we keep track of which tuple in $S$ belongs to which node, resulting in the mapping $S$ (see proof of Proposition 4). Note that $\text{orig}(\cdot, \cdot)$ and $\prec$ can easily be provided by remembering for each tuple an ordered set of predecessor tuple sets during construction (using table algorithm $\text{SINC}$). Now, we call algorithm $\text{NGD}_S(T)$ multiple times, by outputting and passing the result again as argument, until the return value is undefined, enumerating solutions in a systematic way. Using correctness results (by Theorem 1), and completeness result for enumeration by Corollary 1, we obtain only valid solution sets, which directly represent characterizing generating defaults and, in particular, we do not miss a single one. Observe, that we do not get duplicates (see Lemma 3).