A Characteristic of Similarities by Use of Steinhaus’ Problem on Partition of Triangles

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Abstract
H. Steinhaus [7] has asked whether inside each acute triangle there is a point from which perpendiculars to the sides divide the triangle into three parts of equal areas. In this paper we present a new characteristic of similarities by use of Steinhaus’ Problem on partition of a triangle.

Keywords: Möbius transformation; similarity; Steinhaus’ problem.

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1. Introduction

A Möbius transformation $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a mapping of the form $f(z) = \frac{az + b}{cz + d}$ satisfying $ad - bc \neq 0$, where $a, b, c, d \in \mathbb{C}$. Notice that

$$f(\infty) = \lim_{z \to +\infty} f(z) = \frac{a}{c} \quad \text{and} \quad f\left(- \frac{d}{c}\right) = \infty.$$ 

It is well known that the set of all Möbius transformations is a group with respect to the composition and that Möbius transformations have many beautiful properties. Some of these properties are as follows:

- Any Möbius transformation has at most two fixed points in $\mathbb{C} \cup \{\infty\}$.
- The cross-ratio $[z_1, z_2, z_3, z_4]$ of any four complex numbers, which is defined by

$$[z_1, z_2, z_3, z_4] = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3},$$

is invariant under Möbius transformations, that is

$$[z_1, z_2, z_3, z_4] = [f(z_1), f(z_2), f(z_3), f(z_4)].$$

- Möbius transformations are conformal and continuous.
- Möbius transformations map circles to circles, where straight lines are considered to be circles through $\infty$.

Translations, rotations about origin, stretch transformations (complex dilations), inversions and similarities are most familiar Möbius transformations, which are defined by $f(z) = z + b$, $g(z) = e^{i\theta}z$, $h(z) = az$ ($a \neq 0$), $j(z) = \frac{1}{z}$, $m(z) = az + b$, respectively. It is well known that any Möbius transformation can be written as a composition of translations, complex dilations and inversions. In the literature there are many characterizations of Möbius transformations by use of some geometric objects such as Apollonius points of triangles [2], Apollonius quadrilaterals [3], Apollonius pentagons [1], Apollonius hexagons [4] and others. The aim of this paper is to present

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If $P$ is a solution of Steinhaus’ problem for an acute triangle $ABC$, then there exist corresponding points $K, L, M$ on $AB, BC$ and $CA$, respectively, such that $AB \perp PK, BC \perp PL, CA \perp PM$ satisfying
\[
\text{Area}(AKPM) = \text{Area}(BLPK) = \text{Area}(CMPL) = \frac{\text{Area}(ABC)}{3}.
\]

A new characterization of similarities by use of Steinhaus’ problem on partition of triangles. H. Steinhaus [7] has asked whether inside each acute triangle there is a point from which perpendiculars to the sides decide the triangle into three parts of equal areas, see Fig. 1. For the solution of this problem, we refer the reader to [8].

**Example 1.1.** Let $ABC$ be an arbitrary equilateral triangle in the Euclidean plane and let $L$ be its center. Then
\[
\text{Area}(AELD) = \text{Area}(BFLE) = \text{Area}(CDLF) = \frac{\text{Area}(ABC)}{3}
\]
holds, where $D, E, F$ are the midpoints of the sides $AC, AB$ and $BC$, respectively.

### 2. Main Results

**Lemma 2.1.** Let $ABC$ be an equilateral triangle in the Euclidean plane and let $L$ be its center. Denote the midpoints of the sides $AC, AB, BC$ by $D, E, F$ respectively. Then $AL \perp DE$.

The proof is clear, so we omit it.

Throughout the paper we denote by $X'$ the image of $X$ under $f$, by $AB$ the geodesic segment between points $A$ and $B$, by $|AB|$ the distance between points $A$ and $B$, by $ABC$ the triangle with three ordered vertices $A, B$ and $C$, and by $\angle BAC$ the angle between $AB$ and $AC$. Unless otherwise stated, we consider $w = f(z)$ as a nonconstant meromorphic function of a complex variable $z$ in the plane $|z| < +\infty$.

Now we consider Property $S$.

**Property $S$:** Suppose that $w = f(z)$ is an analytic and univalent mapping in a nonempty domain $R$ of the complex plane. Let $ABC$ be an arbitrary triangle contained in $R$. If $L$ is a solution of Steinhaus’ problem for $ABC$, (that is there exist corresponding points $D, E, F$ on the sides $AC, AB, BC$ respectively, such that $LD \perp AC, LE \perp AB, LF \perp BC$ satisfying
\[
\text{Area}(AELD) = \text{Area}(BFLE) = \text{Area}(CDLF) = \frac{\text{Area}(ABC)}{3},
\]
then $L'$ is a solution of $A'B'C'$ (that is the points $D', E', F'$ are on the sides $A'C', A'B', B'C'$, respectively, such that $L'D' \perp A'C', L'E' \perp A'B', L'F' \perp B'C'$ satisfying
\[
\text{Area}(A'E'L'D') = \text{Area}(B'F'L'E') = \text{Area}(C'D'L'F') = \frac{\text{Area}(A'B'C')}{3}.
\]

**Lemma 2.2.** If $w = f(z)$ is analytic and univalent in a nonempty domain $R$, then $f'(z) \neq 0$ in $R$, see [6].

**Lemma 2.3.** Let $w = f(z)$ satisfy Property $S$. If $l_1$ and $l_2$ are two lines meeting perpendicularly, then $f(l_1)$ meets $f(l_2)$ perpendicularly.
Proof. Let $l_1$ and $l_2$ be two lines meeting at a point, say $B$, perpendicularly. Let $A$ be a point on $l_1$ and let $C$ be a point on $l_2$ such that $\angle A C B = \frac{\pi}{2}$, $\angle C B A = \frac{\pi}{2}$, $\angle B A C = \frac{\pi}{3}$. It is enough to prove that $C' B' \perp A' B'$. Denote the reflection of $C$ with respect to $A B$ by $D$ and denote the reflection of $B$ with respect to $A C$ by $E$. Let $F$ be the symmetry of $C$ with respect to $E$. Hence we construct an equilateral triangle $F C D$. Clearly $A$ is the center of $F C D$. Since $A$ is the solution of Steinhaus' problem for $F C D$, that is

$$Area(BCAE) = Area(BDGA) = Area(GFEA) = \frac{Area(FCD)}{3},$$

where $G$ is the midpoint of the side $DF$. By Property $S$, we get

$$Area(C' B' A' E') = Area(B' D' G' A') = Area(G' F' E' A') = \frac{Area(F'C'D')}{3},$$

which implies that $C' B' \perp A' B'$. Therefore $f(l_1)$ meets $f(l_2)$ perpendicularly. \hfill $\square$

**Theorem 2.1.** $w = f(z)$ has Property $S$ if and only if $w = f(z)$ is a similarity.

**Proof.** Let $f$ be a similarity defined by

$$f(z) = az + b$$

satisfying $a, b \in \mathbb{C}, a \neq 0$ and let $ABC$ be an acute angled triangle. Clearly

$$|A'B'| = |a||AB|, \quad |A'C'| = |a||AC|, \quad |CB'| = |a||CB|.$$

By the side-side-side theorem, we get

$$Area(A'B'C') = |a|^2 Area(ABC).$$

Let $L$ be a solution of Steinhaus' problem for $ABC$. Then one can easily see that there exist three points $D, E, F$ on the sides $AC, AB$ and $BC$, respectively such that

$$Area(AELD) = Area(BFLE) = Area(FCDL) = \frac{Area(ABC)}{3}.$$

Since $f$ preserves the measures of the angles of triangles and preserves the collinearity property of points, we get

$$Area(AELD) = Area(ALD) + Area(ALE) = \frac{Area(A'L'D')}{|a|^2} + \frac{Area(A'L'E')}{|a|^2},$$

$$Area(BFLE) = Area(BFL) + Area(BEL) = \frac{Area(B'F'L')}{|a|^2} + \frac{Area(B'E'L')}{|a|^2},$$

$$Area(FCDL) = Area(CLF) + Area(CLD) = \frac{Area(C'L'F')}{|a|^2} + \frac{Area(C'L'D')}{|a|^2},$$

which implies that $f$ has Property $S$.

Now assume that $w = f(z)$ has Property $S$. Because of the fact that $w = f(z)$ is analytic and univalent in the domain $R$, by Lemma 2.2,

$$f'(z) \neq 0 \quad (2.1)$$

holds in $R$. If $x$ is an arbitrarily fixed point of $R$, then by (2.1) we get

$$f'(x) \neq 0. \quad (2.2)$$

Let $L$ be the point represented by $x$. Because of $L \in R$, there exists a positive real number $\epsilon$ such that $V(L, \epsilon)$ is contained in $R$, where $V(L, \epsilon)$ is $\epsilon$-closed circular neighborhood of $L$. Throughout the proof let $ABC$ denote an arbitrary equilateral triangle which is contained in $V(L, \epsilon)$ and whose center is at $L$. Since $ABC$ is an equilateral triangle contained in $V(L, \epsilon)$, we can represent the points $A, B, C$ by complex numbers

$$A = x + y, \quad B = x + wy, \quad C = x + w^2 y,$$
where \( w = \frac{-1 + \sqrt{3}}{2} \) and \(|y| \leq \epsilon\). Then the midpoints of the sides \( AC, AB \) and \( BC \) are

\[
D = x + \frac{w^2 + 1}{2} y, \quad E = x + \frac{w + 1}{2} y, \quad F = x + \frac{w^2 + w}{2} y,
\]

respectively. Since \( w = f(z) \) is univalent in \( R \), the points \( A', B', C', D', E', F', L' \) are different points. Clearly, there exists some sufficiently small \( \delta \in \mathbb{R}^+ \) satisfying \( \delta \leq \epsilon \) such that \( A', B', C' \) are not collinear on the \( w \)-plane for all \( y \) satisfying \( 0 < |y| \leq \epsilon \) by (2.2) and by the property of analytic functions, see [5]. By hypothesis, \( A', B', C' \) are not collinear and \( L' \) is a solution of Steinhaus’ Problem for \( A'B'C' \), that is

\[
\text{Area}(A'E'L'D') = \text{Area}(B'F'L'E') = \text{Area}(F'C'D'L') = \frac{\text{Area}(A'B'C')}{3},
\]

where

\[
A' = f(x + y), \quad B' = f(x + wy), \quad C' = f(x + w^2 y),
\]

\[
D' = f(x + \frac{w^2 + 1}{2} y), \quad E' = f(x + \frac{w + 1}{2} y), \quad F' = f(x + \frac{w^2 + w}{2} y).
\]

Since

\[
\text{Area}(A'E'L'D') = \text{Area}(B'F'L'E'),
\]

it follows that

\[
\frac{1}{2}|A'L'||D'E'| \sin \alpha = \frac{1}{2}|B'L'||F'E'| \sin \beta,
\]

where \( \alpha \) is the measure of the angle between \( A'L' \) and \( D'E' \), and \( \beta \) is the measure of the angle between \( B'L' \) and \( F'E' \). By Lemma 2.1, we get that \( AL \perp DE \) and \( BL \perp EF \). Since \( f \) preserves right angles by Lemma 2.3, we get \( \alpha = \beta = \frac{\pi}{2} \). Then by (2.2), we obtain

\[
|A'L'||D'E'| = |B'L'||F'E'|,
\]

which implies

\[
\left| (f(x + y) - f(x))(f(x + \frac{w + 1}{2} y) - f(x + \frac{w^2 + 1}{2} y)) \right| = \left| (f(x + wy) - f(x))(f(x + \frac{w^2 + w}{2} y) - f(x + \frac{w + 1}{2} y)) \right|
\]

and this yields

\[
\left| \frac{(f(x + y) - f(x))(f(x + \frac{w + 1}{2} y) - f(x + \frac{w^2 + 1}{2} y))}{(f(x + wy) - f(x))(f(x + \frac{w^2 + w}{2} y) - f(x + \frac{w + 1}{2} y))} \right| = 1.
\]

If we set

\[
g(y) = \frac{(f(x + y) - f(x))(f(x + \frac{w + 1}{2} y) - f(x + \frac{w^2 + 1}{2} y))}{(f(x + wy) - f(x))(f(x + \frac{w^2 + w}{2} y) - f(x + \frac{w + 1}{2} y))}
\]

then we get \(|g(y)| = 1\) in the punctured closed disk \( 0 < |y| \leq \delta \). Since the numerator and the denominator of \( g(y) \) are analytic functions for all \( y \) satisfying \( 0 < |y| \leq \delta \) and since, by the fact that \( w = f(z) \) is univalent in \( R \), the denominator of \( g(y) \) never vanishes in \( 0 < |y| \leq \delta \), \( g(y) \) is analytic in \( 0 < |y| \leq \delta \). Next we prove that \( g(y) \) is also analytic at \( y = 0 \). As \( y \to 0 \), by L’Hôpital’s rule and by the fact that \( f'(x) \neq 0 \), we obtain

\[
\frac{f(x + y) - f(x)}{f(x + wy) - f(x)} \to \frac{f'(x)}{w f'(x)} = \frac{1}{w}
\]

and

\[
\frac{f(x + \frac{w + 1}{2} y) - f(x + \frac{w^2 + 1}{2} y)}{f(x + \frac{w^2 + w}{2} y) - f(x + \frac{w + 1}{2} y)} \to \frac{-w}{1 + w}
\]

holds. Hence, for \( y \to 0 \), we immediately get

\[
g(y) \to \frac{1}{w}, \quad \frac{-w}{1 + w} = -\frac{1}{w + 1}.
\]
If we define
\[ g(0) = \frac{-1}{w+1} \]
and by Riemann’s theorem on removable singularities, the function \( g(y) \) is analytic at \( y = 0 \). Furthermore, since \( g(0) = \frac{-1}{w+1} \) holds, the equality \( |g(y)| = 1 \) still holds at \( y = 0 \). Therefore \( g(y) \) is analytic in the closed disk \( |y| \leq \delta \) and that \( |g(y)| = 1 \) holds for all \( y \) with \( |y| \leq \delta \). By the maximum modulus principle for analytic functions we obtain
\[ g(y) = K \]
in \( |y| \leq \delta \), where \( K \) is a complex constant with modulus 1. Setting \( y = 0 \) in \( g(y) = K \) and using \( g(0) = \frac{-1}{w+1} \), we get
\[ K = \frac{-1}{w+1}. \]
Thus we get
\[ (w+1)(f(x+y) - f(x))(f(x + \frac{w+1}{2}y) - f(x + \frac{w^2+1}{2}y)) + (f(x+wy) - f(x))(f(x + \frac{w^2+w}{2}y) - f(x + \frac{w+1}{2}y)) = 0 \]
(2.4)
Choosing both sides of (2.4) three times with respect to \( y \) and setting \( y = 0 \), we get
\[ f'(x)f''(x) = 0. \]
Since \( f'(x) \neq 0 \), we obtain that
\[ f''(x) = 0, \]
which implies that \( f \) must be a similarity, that is it must be of the form
\[ f(z) = az + b \]
for some \( a, b \in \mathbb{C} \) with \( a \neq 0 \).

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