Reversible Mapping for Tree Structured Quantum Computation

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Abstract

A hierarchical, reversible mapping between levels of tree structured computation, applicable for structuring the Quantum Computation algorithm for \mathcal{N}P-complete problem is presented. It is proven that confining the state of a quantum computer to a subspace of the available Hilbert space, where states are consistent with the problem constraints, can be done in polynomial time. The proposed mapping, together with the method of the state reduction can be potentially used for solving \mathcal{N}P-complete problems in polynomial time.
Motivation  Expected running time scaling polynomially with input size, displayed by Shor’s factoring algorithm and, to even greater extent, the more recent results give evidence that Quantum Mechanics offers a computation model more powerful than a classical one, yet still physically feasible.

An application this power for simulating nondeterminism associated with \( \mathcal{NP} \)-complete problems had been discouraged by Bennett et al. in \[4\]. However, Theorem 2 of \[4\], which can be interpreted as a “no-go” proof for unstructured attempts to solve \( \mathcal{NP} \)-complete problems in polynomial time, assumes particular quantum computation consisting of unitary evolution of simple input state followed by a single measurement of a state of a system.

We will follow different approach, patterned after error correction, with repeated measurements forcing projections of a system state onto a subspace spanned on eigenvectors of particular eigenvalue of the Hermitian operator representing measurement.

In remaining part of this paper, after choosing a known \( \mathcal{NP} \)-complete problem as our object of study, we will present the deterministic, data independent mapping between the levels of a solution construction tree, thus providing a method for structuring quantum algorithms for \( \mathcal{NP} \)-complete problems.

Sample Problem  Let us consider a \( \mathcal{NP} \)-complete combinatorial optimization problem known as a symmetric Traveling Salesman Problem (TSP) defined as follows:

Given a weighted complete graph \( G = (V, E, w) \), where \( V \) is a set of vertices \( v_k, k = 1..n \), \( E \) is a set of edges, \( \forall v_i, v_k \in V \{v_i, v_k\} \in E \) and a function \( w : E \to \mathbb{Z} \) is symmetric: \( w_{i,k} = w(e_{i,k}) \equiv w(e_{k,i}) \), find a Hamiltonian cycle \( t \) (a closed tour through the graph, visiting each vertex exactly once) with the minimal total weight:

\[
W(t) = w(\{v_{t(n)}, v_{t(1)}\}) + \sum_{k=1}^{n-1} w(\{v_{t(k)}, v_{t(k+1)}\})
\]

Instead of coding each cycle as a list of vertices, we will code it as a set of its edges, since in this case ordering information is coded locally, reducing number of bits modified at a cycle modification. We can represent each edge present in a cycle by a single set bit in a list of all \( n(n-1)/2 \) edges. The ordering of the list is such that the edges are grouped by the higher vertex number present in the edge and, within the group, by the number of the other vertex of the edge: \( (e_{2,1}, e_{3,1}, e_{3,2}, e_{4,1}, \ldots) \). In such a way, incomplete potential solutions i.e. cycles over first \( m \) vertices of a \( n \)-vertex problem have all bits above \( m(m+1)/2 \) reset to 0. We need \( \mathcal{O}(n^2) \) bits to code a \( n \)-vertex cycle, as compared with \( \mathcal{O}(n \log(n)) \) of more common coding as a list of vertices. Such a redundancy can be used for rudimentary error control, but the main motivation is robustness, universality and particular simplicity of unitary operator used for mapping between levels of a construction tree.

From each cycle of \( m \) vertices we can construct \( m \) cycles connecting \( m + 1 \) vertices by breaking it at one of its \( m \) edges, and inserting the vertex \( m + 1 \) at the breaking point. We reset the relevant bit to zero (since the edge \( e_{i,j} \) we break at no longer exists in a resulting cycle ) and replace it by two bits representing edges which connect broken ends of the cycle of a previous level to the newly inserted vertex: \( e_{m+1,j}, e_{m+1,i} \).

Reversible Mapping for TSP  A quantum computation method of producing all the permutations of a given set of elements (all vertices \( v_i \in V \) in a case of TSP) has been
already provided in [3], however, it requires resources scaling exponentially with number of elements in a set.

It is interesting to note that latest results in Quantum Computation are not presented in Quantum Turing Machine formalism. This, in author opinion, reflects the growing belief that if we are ever able to build a quantum computer it will rather be in a form of uniform array of identical, locally interacting elements with quantum degrees of freedom [6,7] than the counterpart of a classical computer. We will follow this approach.

We start with the only tour connecting three vertices represented by a state $|1_2, 1_3, 1_3, 0_3, 0_3, 0\rangle$, where $0$ is used to denote that all the remaining qbits in all remaining groups are $0$. In subsequent equations we will drop the indices for shorter notation, while preserving the same convention for the order of qbits.

Let us denote $l$–th bit of a pure state $|s> \text{ by } s\{l\}$. Clearly, the mapping of the previous paragraph is not unitary, but it can be made so by supplying an ancilla prepared in an equal superposition of all natural unit states $|u^m\rangle$, $u^m\{l\} = \delta_{l,l'}$, with $l, l' = 1..m(m-1)/2$. [3]

Let us assume for a moment that our system is prepared in an equal superposition of all Hamiltonian cycles over first $m$ vertices:

$$|p\rangle = \frac{1}{\sqrt{(m - 1)!/2}} \sum_{k=1}^{m} |p_k^m\rangle. \quad (2)$$

Qbit set to 1 in ancilla $(u^m_l = 1)$ points to the edge in a tour of the input level which should be broken (if present i.e. if $p_k^m\{l\} = 1$) by the insertion of a vertex of the next level. On output both these qbits are reset to 0, while setting two bits representing newly created edges $p_k^{m+1}\{i\} = p_k^{m+1}\{i'\} = 1$. All other bits of the input path remain unchanged, since the edges they represent are common for $p_k^m$ and $p_k^{m+1}$. Partially expanded construction tree is depicted in Fig. [4] with levels labeled by the number of vertices.

Thus defined $U_m$ is an identity operator except for the matrix elements:

$$\langle u^m_l, p^m_k | 0, p^{m+1}_{k'} \rangle = \langle 0, p^{m+1}_{k} | u^m_l, p^m_k \rangle = 1, \quad (3a)$$

$$\langle 0, p^{m+1}_{k'} | 0, p^{m+1}_{k} \rangle = \langle u^m_l, p^m_k | u^m_l, p^m_k \rangle = 0, \quad (3b)$$

for such $k, k'$ that:

$$\begin{align*}
p_k^m\{l\} &= u^m_l\{l\} = 1, \quad p_k^{m+1}\{l\} = 0, \\
p_k^{m+1}\{i\} &= p_k^{m+1}\{i'\} = 1,
\end{align*}$$

where, if $l = 1..m(m-1)/2$ is a position of a qbit representing the edge $e_{l', l''}$, $i, i'$ are the positions of qbits representing the edges $e_{m, l''}, e_{m, l''}$ and all other bits of $p_k^m$ are equal to respective bits of $p_k^{m+1}$ and $0$ denotes that all the bits of ancilla are $0$.

Note that within an adopted convention there is one-to-one correspondence between indices of a given edge and a position of the relevant bit in a list, which can be determined in polynomial time:

In order to find indices $i, k$ of an edge, coded as an $l$-th bit, add the integers $1 + 2 + \ldots$ until the sum is greater than or equal to $l$. Number of terms in this sum plus one equals $i$. Subtracting the last term from the sum and subtracting the result from $l$ yields $k$.

To complete the construction, we will use the following example:
Example 1. We want to construct the unitary transformation, mapping the only cycle over three vertices: $|1110\rangle$ to the superposition of three possible tours connecting four vertices using ancilla:

$$|u_3\rangle = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle)$$ (4)

and acting with unitary transformation $U_m$ in a product of Hilbert spaces of our primary set of qbits $H_p$ and ancilla $H_u$:

$$\frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)|1110\rangle \xrightarrow{U_3} \frac{1}{\sqrt{3}}|000\rangle \times$$

$$\times (|011100\rangle + |101010\rangle + |110001\rangle)$$ (5)

We can construct this mapping in polynomial time for arbitrary $m$, factoring $U_m$ into $m(m-1)/2$ operations $U_m^l$ by iterating over bits of ancilla and, for each $l$ in turn, performing the four bit operation $U_m^l$ on affected qbits of $p$ and a bit of ancilla as defined by relations of Eqs. 3. This construction for the above example is depicted in Fig. 2.

Thus, in order to recursively implement this construction to produce the superposition of all cycles over $n$ vertices we need the time $O(n^3)$. If we do not reuse the ancillae the space requirements will also be $O(n^3)$.

**Measurement** Note that the example describes the only case when the valid tour has all bits set, and so the ancilla has just right number of states to pass all the amplitude to outputs under operation of $U_3$.

In all subsequent levels some portion of the amplitude will remain in the input level. We can correct this by the measurement with projector $M_m = |0_u\rangle\langle 0_u|$ onto state $|0_u\rangle$ of ancilla after using $U_m$. Since all cycles of the output level are entangled with $|0_u\rangle$ the perfect measurement does not destroy the superposition of the output tours, while reducing to 0 the amplitude remaining in the input level tours.

To establish the scaling behavior of the measurement part of the procedure let us consider in detail the application of $U_m$ to the state of our system

$$\frac{1}{\sqrt{\frac{m(m-1)}{2}}} \frac{1}{\sqrt{(m-1)!}} U_m \sum_{l=1}^{m-1/2} \sum_{k=1}^{(m-1)!/2} |u_l^m\rangle |p_k^m\rangle =$$

$$= \frac{2}{\sqrt{(m-1)m!}} \left( \sum_k |0_u, p_k^{m+1}\rangle + \sum_{l,k} |u_l, p_k^m\rangle \right)$$ (6)

where $\tilde{k}$ is indexing all the states $|p_k^{m+1}\rangle$ fulfilling Eqs. 3. They are $m$ qbits set to 1 in each $|p_k^m\rangle$ so they are $m(m-1)!/2 = m!/2$ such states — which can be expected, since it is the number of distinct cycles at the $m + 1$ level. The indices $\tilde{k}, l$ run through the states where the set bit of ancilla corresponds to unset bit of $|p_k^m\rangle$.

The measurement with $M_m$ reduces the state of our primary register to:

$$|p\rangle = \frac{1}{\sqrt{m!/2}} \sum_{k'} |p_{k'}^{m+1}\rangle,$$ (7)
compatible with the *ancilla* in a one dimensional subspace of $\mathcal{H}_u$ defined by the eigenvector belonging to the eigenvalue 1 of the projector. The probability of finding the system in the desired state is:

$$P_m = \frac{m}{m(m-1)/2} = \frac{2}{m-1} \quad (8)$$

If a measurement of any state of ancilla requires time $T$, the measurement with $M_m$ will yield the result in average time $t = T/P_m = (m-1)T/2$.

Thus, total measurement time for the construction of $n$-vertex cycles is $O(n^2)$. To simplify the measurement we can use an auxiliary qubit $|a_m\rangle$ holding the result of a function $\text{AND}(p_m\{l\}, u_l^m\{l\})$ (represented by a square in Fig. 2) calculated for each $|p_m^m, u_l^m\rangle$. As far as the state reduction is concerned, the measurement with $|1_a\rangle\langle 1_a|$ is equivalent to the measurement with $M_m$.

The reversibility of our construction has special meaning, due to the irreversible measurements $M_m$: application of $U_m^\dagger$ to the final state $|0, p_{m+1}^m\rangle$ results in each $p_k^m$ entangled with all $u_l^m$ representing set qbits of the cycle, but nevertheless can be used to push the amplitude back to a single state in $|p\rangle$, while leaving the information of the cycles in (entangled) *ancillae*.

The present construction is not time nor space optimal, but it is very simple — it is not using all the resources postulated for quantum computation so far and confines much of a calculation to isolated space, thus reducing errors. We need only real amplitudes, and our dynamics is truly digital: matrix elements of our unitary operators are only 0 or 1. What is more, the polynomial amplitude errors will lead only to the polynomial increase of expected running time, and changing of phases between the states $|p_k^m\rangle$ of a given level will not affect the computation at all.

In principle we could have calculated the total weight for each path, but the probability of finding a minimum in single measurement will be vanishingly small for large $n$. We could have used *ancillae* better matching the requirements of unitarity and, instead of the postulated measurement, the measurement with a sum of elementary projectors onto all eigenvectors of associated operator (repeating $U_m$ with new ancilla n case that the resulting state of ancilla is not $|0_u\rangle$, much like in quantum error correction).

However, the application of the described mapping and postulated method of state reduction will be very useful for polynomial time algorithms for $\mathcal{NP}$-complete problems, to be presented in a follow-up paper.

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[8] A procedure for preparing this superposition in time polynomial in $m$ can be deduced from the one described in Eqs.6-8 of [5], here we only note that in an array of spins it can be induced by appropriately focused $\pi-$pulse. The state is robust in a sense that decoherence will require energy exchange with the environment.
FIG. 1. Partially expanded solution construction tree for TSP, with Hamiltonian cycles coded by set qbits representing their edges, where levels of a tree are labeled by the number of vertices $m$. $U_m$ is a sought mapping between levels. $0$ is used to denote that all other qbits of a state $|p^m_k\rangle$ are 0. Ancillae, necessary at all levels for unitarity of $U_m$ are not shown.
\[ p^3 \left\{ \begin{array}{c}
|0> \\
|0>
\end{array} \right\} \left\{ \begin{array}{c}
|1> \\
|1>
\end{array} \right\} \left\{ \begin{array}{c}
|1> \\
|1>
\end{array} \right\} \left\{ \begin{array}{c}
\frac{1}{\sqrt{3}} \sum_{l=1}^{3} |u_i^4> \\
\frac{1}{\sqrt{3}} \sum_{l=1}^{3} |u_i^4>
\end{array} \right\} \left\{ \begin{array}{c}
|0> \\
|0>
\end{array} \right\} \left\{ \begin{array}{c}
|0> \\
|0>
\end{array} \right\} \left\{ \begin{array}{c}
|1>
\end{array} \right\} \}

\frac{1}{\sqrt{3}} \sum_{k=1}^{3} |P_k>

FIG. 2. Factoring of \( U_m \) into the bitwise operations in the case of \( m = 3 \). Full circles and crosses denote respectively change from 0 to 1 and from 1 to 0 according to Eqn.3. For each of thus defined 4-bit operations affected bits of \( |p_k> \) (positions of circles) can be determined from the set bit of the ancilla \( u_l \) (a position of a cross) in polynomial time. Dashed line and a squares denote a possible extention of \( U_m \) and an associated auxiliary qbit used to facilitate the measurements, necessary at all other levels of the construction.