Non-commutative tomography: A tool for data analysis and signal processing

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Outline

- Integral transforms: linear and bilinear
- Wavelet-type, quasi-distributions and tomograms: Examples and relations
- Tomograms and the conformal group operators
- The time-frequency tomogram: Applications:
  - 1. Denoising and component separation
  - 2. Plasma reflectometry
- Signal-adapted tomography
Tomographic data analysis. General setting

- Integral transforms

Consider signals $f(t)$ as vectors in a dense nuclear subspace $\mathcal{N}$ of a Hilbert space $\mathcal{H}$ with dual space $\mathcal{N}^*$. Let $U(\alpha)$ be a family of operators defined on $\mathcal{N}$. (In many cases $U(\alpha)$ generates a unitary group $U(\alpha) = e^{iB(\alpha)}$.)

Three types of transforms

1. Wavelet-type transform
   $$W(h)f(\alpha) = h U(\alpha) j f i,$$
Tomographic data analysis. General setting

- **Integral transforms**
- *Linear transforms*: Fourier, Wavelets, Hilbert, ...
Tomographic data analysis. General setting

- **Integral transforms**
- *Linear transforms*: Fourier, Wavelets, Hilbert, ...
- *Bilinear transforms*: Wigner-Ville, Bertrand, Tomograms

Consider signals \( f(t) \) as vectors \( j^* f \in \text{dense nuclear subspace} \) of a Hilbert space \( H \) with dual space \( H' \). A family of operators \( U(\alpha) \) is defined on \( H \). (In many cases \( U(\alpha) \) generates a unitary group \( U(\alpha) = e^{iB(\alpha)} \)).

Three types of transforms

Let \( h \in N \) be a reference vector such that the linear span of \( f U(\alpha) h \in N \) is dense in \( N \). In the set \( f U(\alpha) h \), a complete set of vectors can be chosen to serve as a basis.

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W(h) f(\alpha) = h U(\alpha) j^* f
\]
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Consider signals $f(t)$ as vectors $| f \rangle \in$ dense nuclear subspace $\mathcal{N}$ of a Hilbert space $\mathcal{H}$ with dual space $\mathcal{N}^*$

- $\{U(\alpha) : \alpha \in I \}$ a family of operators defined on $\mathcal{N}^*$. (In many cases $U(\alpha)$ generates a unitary group $U(\alpha) = e^{iB(\alpha)}$)
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**Three types of transforms**

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- **1 - Wavelet-type transform**

\[ W_f^{(h)}(\alpha) = \langle U(\alpha)h \mid f \rangle, \]
Tomographic data analysis. General setting

2 - Quasi-distribution

\[ Q_f(\alpha) = \langle U(\alpha) f \mid f \rangle. \]
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\[ Q_f(\alpha) = \langle U(\alpha) f | f \rangle. \]

If \( U(\alpha) \) is a unitary operator there is a self-adjoint operator \( B(\alpha) \)

\[ W_f^{(h)}(\alpha) = \langle h | e^{iB(\alpha)} | f \rangle \]

\[ Q_f^{(B)}(\alpha) = \langle f | e^{iB(\alpha)} | f \rangle \]
2 - Quasi-distribution

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If \( U(\alpha) \) is a unitary operator there is a self-adjoint operator \( B(\alpha) \)

\[ W_f^{(h)}(\alpha) = \langle h \mid e^{iB(\alpha)} \mid f \rangle \]

\[ Q_f^{(B)}(\alpha) = \langle f \mid e^{iB(\alpha)} \mid f \rangle \]

3 - Tomographic transform or tomogram

\[ M_f^{(B)}(X) = \langle f \mid \delta(B(\alpha) - X) \mid f \rangle \]

\( \delta(B(\alpha) - X) = |X\rangle \langle X| = \) projector on the eigenvector of \( B(\alpha) \) with eigenvalue \( X \)
Examples for wavelet-type and quasi-distributions

- **Fourier transform**: is $W_f^{(h)}(\alpha)$ if $U(\alpha)$ is unitary generated by $B_F(\vec{\alpha}) = \alpha_1 t + i\alpha_2 \frac{d}{dt}$ and $h$ is a (generalized) eigenvector of the time-translation operator.
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- **Ambiguity function**: $Q_f(\alpha)$ for the same $B_F(\vec{\alpha})$. 
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- **Ambiguity function**: $Q_f(\alpha)$ for the same $B_F(\alpha)$

- **Wigner–Ville transform**: $Q_f(\alpha)$ for the same $B_F(\alpha)$ plus the parity operator

$$B^{(WV)}(\alpha_1, \alpha_2) = -i2\alpha_1 \frac{d}{dt} - 2\alpha_2 t + \pi \left( t^2 - \frac{d^2}{dt^2} - 1 \right) \frac{\pi}{2}.$$
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- **Fourier transform**: is $W_f^{(h)}(\alpha)$ if $U(\alpha)$ is unitary generated by $B_F(\overrightarrow{\alpha}) = \alpha_1 t + i\alpha_2 \frac{d}{dt}$ and $h$ is a (generalized) eigenvector of the time-translation operator.

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- **Wavelet transform**: $W_f^{(h)}(\alpha)$ for $B_W(\overrightarrow{\alpha}) = \alpha_1 D + i\alpha_2 \frac{d}{dt}$, $D$ being the dilation operator

$$D = -\frac{1}{2} \left(it \frac{d}{dt} + i \frac{d}{dt} t\right).$$
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- **Wavelet transform**: $W_{f}^{(h)}(\alpha)$ for $B_{W}(\vec{\alpha}) = \alpha_{1} D + i \alpha_{2} \frac{d}{dt}$, $D$ being the dilation operator $D = -\frac{1}{2} \left(it \frac{d}{dt} + i \frac{d}{dt} t\right)$.

- **Bertrand transform**: $Q_{f}(\alpha)$ for $B_{W}$.
The tomographic transform (tomogram)

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- \( M_f^{(B)}(\alpha) \) is positive and may be interpreted as a probability distribution. Benefits from the properties of the bilinear transforms, without interpretation ambiguities.
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- \( M^{(B)}_f (\alpha) \) is positive and may be interpreted as a probability distribution. Benefits from the properties of the bilinear transforms, without interpretation ambiguities.

- For normalized \( |f \rangle \),
  \[ \langle f \mid f \rangle = 1 \]
  the tomogram is normalized

  \[ \int M^{(B)}_f (X) \, dX = 1 \]

  It is a probability distribution for the random variable \( X \) corresponding to the observable defined by the operator \( B(\alpha) \).
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  \[ \int M_f^{(B)}(X) \, dX = 1 \]
  It is a probability distribution for the random variable \( X \) corresponding to the observable defined by the operator \( B(\alpha) \)
- The tomogram is a homogeneous function
  \[ M_f^{(B/p)}(X) = |p| M_f^{(B)}(pX) \]
Relations between the three types of transforms

\[ M_f^{(B)}(X) = \frac{1}{2\pi} \int Q_f^{(kB)}(\alpha) e^{-ikX} dk \]
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\[ Q_f^{(B)}(\alpha) = W_f^{(f)}(\alpha), \]

\[ W_f^{(h)}(\alpha) = \frac{1}{4} \int e^{iX} \left[ M_f^{(B)}(X) - iM_f^{(B)}(X) \
- M_f^{(B)}(X) + iM_f^{(B)}(X) \right] dX, \]

with

\[ | f_1 \rangle = | h \rangle + | f \rangle; \quad | f_3 \rangle = | h \rangle - | f \rangle; \]
\[ | f_2 \rangle = | h \rangle + i | f \rangle; \quad | f_4 \rangle = | h \rangle - i | f \rangle. \]
The conformal group

- The generators of the conformal group

\[ \omega_k = i \frac{\partial}{\partial t_k} \]
\[ D = i \left( t \cdot \nabla + \frac{d}{2} \right) \]
\[ R_{j,k} = i \left( t_j \frac{\partial}{\partial t_k} - t_k \frac{\partial}{\partial t_j} \right) \]
\[ K_j = i \left( t_j^2 \frac{\partial}{\partial t_j} + t_j \right) \]
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For \( d = 1 \)

in \( \mathbb{R} \)

\[ \omega = i \frac{d}{dt} \]
\[ D = i \left( t \frac{d}{dt} + \frac{1}{2} \right) \]
\[ K = i \left( t^2 \frac{d}{dt} + t \right) \]
Tomograms associated to the conformal group

- Time-frequency tomogram

\[ B_1 = \mu t + iv \frac{d}{dt} \]
Tomograms associated to the conformal group

- **Time-frequency tomogram**
  \[ B_1 = \mu t + iv \frac{d}{dt} \]

- **Time-scale**
  \[ B_2 = \mu t + iv \left( t \frac{d}{dt} + \frac{1}{2} \right) \]
Tomograms associated to the conformal group

- **Time-frequency tomogram**

\[ B_1 = \mu t + i\nu \frac{d}{dt} \]

- **Time-scale**

\[ B_2 = \mu t + i\nu \left( t \frac{d}{dt} + \frac{1}{2} \right) \]

- **Frequency-scale**

\[ B_3 = i\mu \frac{d}{dt} + i\nu \left( t \frac{d}{dt} + \frac{1}{2} \right) \]
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  \[ B_3 = i\mu \frac{d}{dt} + iv \left( t \frac{d}{dt} + \frac{1}{2} \right) \]

- **Time-conformal**

  \[ B_4 = \mu t + iv \left( t^2 \frac{d}{dt} + t \right) \]
Tomograms associated to the conformal group

- General construction of the tomograms: Let

\[ \int dY \left| Y \right\rangle \left\langle Y \right| = 1 \]

be a decomposition of the unit, with generalized eigenvectors of the operator \( B \). Then

\[ M(\alpha, X) = \int dY \left\langle f \left| \delta \left( B(\alpha) - X \right) \right| Y \right\rangle \left\langle Y \right| \left| f \right\rangle = \left| \left\langle X \left| f \right\rangle \right| ^2 \]
Tomograms associated to the conformal group

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- Therefore the construction of the tomograms reduces to the calculation of the generalized eigenvectors of each \( B \) operator
Tomograms associated to the conformal group

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- Therefore the construction of the tomograms reduces to the calculation of the generalized eigenvectors of each \( B \) operator

\[ B_1 \psi_1 (\mu, \nu, t, X) = X \psi_1 (\mu, \nu, t, X) \]

\[ \psi_1 (\mu, \nu, t, X) = \exp i \left( \frac{\mu t^2}{2\nu} - \frac{tX}{\nu} \right) \]

\[ \int dt \psi_1^*(\mu, \nu, t, X) \psi_1 (\mu, \nu, t, X') = 2\pi \nu \delta (X - X') \]
Tomograms associated to the conformal group

- $B_2 \psi_2 (\mu, \nu, t, X) = X \psi_2 (\mu, \nu, t, X)$

\[
\psi_2 (\mu, \nu, t, X) = \frac{1}{\sqrt{|t|}} \exp \left( i \left( \frac{\mu t}{\nu} - \frac{X}{\nu} \log |t| \right) \right)
\]

\[
\int dt \psi_2^* (\mu, \nu, t, X) \psi_2 (\mu, \nu, t, X') = 4\pi \nu \delta (X - X')
\]
Tomograms associated to the conformal group

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\[
\psi_2 (\mu, \nu, t, X) = \frac{1}{\sqrt{|t|}} \exp \left( i \left( \frac{\mu t}{v} - \frac{X}{v} \log |t| \right) \right)
\]

\[
\int dt \psi_2^* (\mu, \nu, t, X) \psi_2 (\mu, \nu, t, X') = 4\pi \nu \delta (X - X')
\]

- \( B_3 \psi_3 (\mu, \nu, \omega, X) = X \psi_3 (\mu, \nu, \omega, X) \)

\[
\psi_3 (\mu, \nu, t, X) = \exp (-i) \left( \frac{\mu \omega}{v} - \frac{X}{v} \log |\omega| \right)
\]

\[
\int d\omega \psi_3^* (\mu, \nu, \omega, X) \psi_3 (\mu, \nu, \omega, X') = 2\pi \nu \delta (X - X')
\]
Tomograms associated to the conformal group

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  \]

  \[
  \int d\omega \psi_3^* (\mu, \nu, \omega, X) \psi_3 (\mu, \nu, \omega, X') = 2\pi \nu \delta (X - X')
  \]

- $B_4 \psi_4 (\mu, \nu, t, X) = X \psi_4 (\mu, \nu, t, X)$

  \[
  \psi_4 (\mu, \nu, t, X) = \frac{1}{|t|} \exp \left( i \left( \frac{X}{\nu t} + \frac{\mu}{\nu} \log |t| \right) \right)
  \]

  \[
  \int dt \psi_4^* (\mu, \nu, t, s) \psi_4 (\mu, \nu, t, s') = 2\pi \nu \delta (s - s')
  \]
Tomograms associated to the conformal group

\( \mu = 0 \)
Tomograms associated to the conformal group

- Time-frequency tomogram

\[ M_1 (\mu, \nu, X) = \frac{1}{2 \pi |\nu|} \left| \int \exp \left[ i \frac{\mu t^2}{2 \nu} - \frac{i t X}{\nu} \right] f(t) \, dt \right|^2 \]
Tomograms associated to the conformal group

- Time-frequency tomogram
  
  \[ M_1(\mu, \nu, X) = \frac{1}{2\pi|\nu|} \left| \int \exp \left[ \frac{i\mu t^2}{2\nu} - \frac{itX}{\nu} \right] f(t) \, dt \right|^2 \]

- Time-scale tomogram
  
  \[ M_2(\mu, \nu, X) = \frac{1}{2\pi|\nu|} \left| \int dt \, \frac{f(t)}{\sqrt{|t|}} e^{i(\frac{\mu}{\nu} t - \frac{X}{\nu} \log |t|)} \right|^2 \]
Tomograms associated to the conformal group

- **Time-frequency tomogram**
  \[
  M_1(\mu, \nu, X) = \frac{1}{2\pi|\nu|} \left| \int \exp \left[ \frac{i\mu t^2}{2\nu} - \frac{itX}{\nu} \right] f(t) \, dt \right|^2
  \]

- **Time-scale tomogram**
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  M_2(\mu, \nu, X) = \frac{1}{2\pi|\nu|} \left| \int dt \frac{f(t)}{\sqrt{|t|}} e^{i \left( \frac{\mu}{\nu} t - \frac{X}{\nu} \log |t| \right)} \right|^2
  \]

- **Frequency-scale tomogram**
  \[
  M_3(\mu, \nu, X) = \frac{1}{2\pi|\nu|} \left| \int d\omega \frac{f(\omega)}{\sqrt{|\omega|}} e^{-i \left( \frac{\mu}{\nu} \omega - \frac{X}{\nu} \log |\omega| \right)} \right|^2
  \]

- \[f(\omega) = \text{Fourier transform of } f(t)\]
Tomograms associated to the conformal group

- Time-frequency tomogram

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- Time-scale tomogram

\[ M_2(\mu, \nu, X) = \frac{1}{2\pi|\nu|} \left| \int dt \frac{f(t)}{\sqrt{|t|}} e^{i\left(\frac{\mu}{\nu} t - \frac{X}{\nu} \log |t|\right)} \right|^2 \]

- Frequency-scale tomogram

\[ M_3(\mu, \nu, X) = \frac{1}{2\pi|\nu|} \left| \int d\omega \frac{f(\omega)}{\sqrt{|\omega|}} e^{-i\left(\frac{\mu}{\nu} \omega - \frac{X}{\nu} \log |\omega|\right)} \right|^2 \]

\[ f(\omega) = \text{Fourier transform of } f(t) \]

- Time-conformal tomogram

\[ M_4(\mu, \nu, X) = \frac{1}{2\pi|\nu|} \left| \int dt \frac{f(t)}{|t|} e^{i\left(\frac{X}{\nu t} + \frac{\mu}{\nu} \log |t|\right)} \right|^2 \]
Basis functions of the tomograms in the time-frequency plane

Time-frequency

![Graph showing basis functions of tomograms in the time-frequency plane]

- Time (samples): 0, 500, 1000, 1500
- Frequency (samples): -1000, -500, 0, 500, 1000, -30, -20, -10, 0, 10

Colors range from -30 to 10 on the color scale.
Basis functions of the tomograms in the time-frequency plane

Time-scale

![Graph showing basis functions in the time-frequency plane with time in samples ranging from 0 to 1500 and frequency in samples ranging from -1000 to 1000. The color scale indicates intensity with values ranging from -40 to 0.](image-url)
Basis functions of the tomograms in the time-frequency plane

Time-scale
Basis functions of the tomograms in the time-frequency plane

Time-conformal
Applications: Component decomposition

- Most natural and man-made signals are nonstationary and have a multicomponent structure.
  Examples: Bat echolocation, whale sounds, radar, sonar, etc.

\[ M(\theta, X) = \int f(t) \psi_{\theta, X}(t) \, dt = \langle f, \psi \rangle_j^2 \]

\[ \psi_{\theta, X}(t) = \frac{1}{pT} \exp(i \cos \theta_2 \sin \theta t + iX \sin \theta t) \]

\( \mu = \cos \theta, \nu = \sin \theta \).
Applications: Component decomposition

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  Examples: Bat echolocation, whale sounds, radar, sonar, etc.
- The concept of signal component is not uniquely defined. The notion of *component* depends as much on the observer as on the observed object. When we speak about a component of a signal we are in fact referring to a particular feature of the signal that we want to emphasize.

\[
M(\theta, X) = \int f(t) \psi_{\theta, X}(t) \, dt = \langle f, \psi_{\theta, X} \rangle
\]

with \( \psi_{\theta, X}(t) = \frac{1}{\sqrt{T}} \exp \left( \frac{i}{2} \cos \theta t \sin \theta t + iX \sin \theta t \right) \mu = \cos \theta, \nu = \sin \theta \).
Applications: Component decomposition

- Most natural and man-made signals are nonstationary and have a multicomponent structure.
  Examples: Bat echolocation, whale sounds, radar, sonar, etc.
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- One possibility: Separation of components using its behavior in the time-frequency plane. Consider the finite-time tomogram

\[
M(\theta, X) = \left| \int f(t) \psi_{\theta,X}(t) \, dt \right|^2 = |< f, \psi >|^2
\]

with

\[
\psi_{\theta,X}(t) = \frac{1}{\sqrt{T}} \exp \left( -\frac{i \cos \theta}{2 \sin \theta} t^2 + \frac{i X}{\sin \theta} t \right)
\]

\[
\mu = \cos \theta, \quad \nu = \sin \theta.
\]
Component decomposition

- $\theta$ is a parameter that interpolates between the time and the frequency operators, running from 0 to $\pi/2$ whereas $X$ is allowed to be any real number.
Component decomposition

- $\theta$ is a parameter that interpolates between the time and the frequency operators, running from 0 to $\pi/2$ whereas $X$ is allowed to be any real number.

- For all different $\theta$’s the $U(\theta)$ are unitarily equivalent operators, hence all the tomograms share the same information. The component separation technique is based on the search for an intermediate value of $\theta$ where a good compromise might be found between time localization and frequency information.
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- First select a subset \( X_n \) in such a way that the corresponding family \( \{ \psi_{\theta,X_n}(t) \} \) is orthogonal and normalized,

\[
\langle \psi_{\theta,X_n} \psi_{\theta,X_m} \rangle = \delta_{m,n}
\]

This is possible by taking the sequence

\[
X_n = X_0 + \frac{2n\pi}{T} \sin \theta
\]

where \( X_0 \) is freely chosen (in general we take \( X_0 = 0 \)).
We then consider the projections of the signal $f(t)$

$$c^\theta_{X_n}(f) = \langle f, \psi_{\theta,X_n} \rangle$$

which are used for the signal processing.
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Denoising consists in eliminating the $c^{\theta}_{X_n}(f)$ such that

$$
|c^{\theta}_{X_n}(f)|^2 \leq \epsilon
$$

for some threshold $\epsilon$
Component decomposition and denoising

- We then consider the projections of the signal $f(t)$

$$c_{X_n}^\theta(f) = \langle f, \psi_{\theta,X_n} \rangle$$

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for some threshold $\epsilon$.

- Multi-component analysis is done by selecting subsets $\mathcal{F}_k$ of the $X_n$ and reconstructing partial signals ($k$-components) by restricting the sum to

$$f_k(t) = \sum_{n \in \mathcal{F}_k} c_{X_n}^\theta(f) \psi_{\theta,X_n}(t)$$

for each $k$. 
Component decomposition. Examples

\[ y(t) = y_1(t) + y_2(t) + y_3(t) + b(t) \]

\[ y_1(t) = \exp(i25t), \quad t \in [0, 20] \]

\[ y_2(t) = \exp(i75t), \quad t \in [0, 5] \]

\[ y_3(t) = \exp(i75t), \quad t \in [10, 20] \]
Component decomposition. Examples

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Real part of the time signal
Component decomposition

The tomogram
Component decomposition. Examples

- Separation at $\theta = \frac{\pi}{5}$
Component decomposition. Examples

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Component decomposition. Examples

- Reconstruction of the $y_2(t)$
Component decomposition. Examples

- Reconstruction of the $y_2(t)$

- and $y_3(t)$ components
Component decomposition. Examples

- Sum \( y(t) = y_0(t) + y_R(t) + b(t) \) of an “incident” \( y_0(t) \) and a “deformed reflected” chirp \( y_R(t) \) delayed by 3s with white noise added.

\[
y_0(t) = e^{i\Phi_0(t)} \quad y_R(t) = e^{i\Phi_R(t)}
\]

\( \Phi_0(t) = a_0 t^2 + b_0 t \) and

\( \Phi_R(t) = a_R (t - t_R)^2 + b_R (t - t_R) + 10(t - t_R)^{3/2} \).
Component decomposition. Examples

- Sum $y(t) = y_0(t) + y_R(t) + b(t)$ of an “incident” $y_0(t)$ and a “deformed reflected” chirp $y_R(t)$ delayed by 3s with white noise added.

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![Graph of real(y) vs. time]
**Component decomposition. Examples**

- Comparison of the phase derivatives $\frac{d}{dt} \Phi_0(t)$ and $\frac{d}{dt} \Phi_R(t)$. Except for the three first seconds, the spectrum of the signals $y_0(t)$ and $y_R(t)$ is almost the same.
Component decomposition. Examples

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Component decomposition. Examples

- Frequency representation

![Frequency representation graph](image-url)
Component decomposition. Examples

- Tomogram of the chirps signal
Component decomposition. Examples

Separable spectrum at $\theta = \frac{\pi}{5}$
The phase derivative
Component decomposition. Reflectometry
Component decomposition. Reflectometry

- Reflectometry signal
Component decomposition. Examples
Component decomposition. Examples

Spectrogram
Component decomposition. Examples

Oversampled spectrogram

![Oversampled spectrogram image with labels for First wall, Plasma, and Porthole.]
Component decomposition. Examples

- Tomogram of the reflectometry signal
Component decomposition. Examples

- "Spectrum" at $\theta = \pi - \frac{\pi}{5}$
Component decomposition. Examples

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Amplitude (real part)

First wall

Plasma

Porthole

Frequency (GHz)
Component decomposition. Examples

Spectrogram of the plasma component
In $B(\mu, \nu) = \mu t + \nu S$, choose an operator $S$, specially tuned to the features of the signal that one wants to extract.

At particular values of $(\mu, \nu)$ noise effects may cancel out. Separates the information of very small signals from large noise and obtain reliable information on the temporal structure of the signal. A signal-adapted filtering technique.

To construct $S$ consider a set of $N$–dimensional time sequences $\{\vec{x}_1, \cdots, \vec{x}_k\}$, typical of the signal one wants to detect. (May be considered as the code words that later one wishes to detect in a noisy signal).

Form the $k \times N$ matrix $U \in \mathcal{M}_{k \times N}$.

$$
U = \begin{pmatrix}
x_1(1\Delta t) & x_1(2\Delta t) & \cdots & x_1(N\Delta t) \\
\vdots & \vdots & \ddots & \vdots \\
x_k(1\Delta t) & x_k(2\Delta t) & \cdots & x_k(N\Delta t)
\end{pmatrix}
$$

with $k < N$ typically.
Construct the square matrices $A = U^T U \in \mathcal{M}_{N \times N}$ and $B = U U^T \in \mathcal{M}_{k \times k}$.

Diagonalization of $A$ provides $k$ non-zero eigenvalues $(\alpha_1, \cdots, \alpha_j)$ and its corresponding orthogonal $N$-dimensional eigenvectors $(\Phi_1, \cdots, \Phi_k)$, $\Phi_j \in \mathbb{R}^N$. Correspondingly, the diagonalization of $B$ would provide the same $k$ eigenvalues and eigenvectors $(\Psi_1, \cdots, \Psi_k)$ with $\Psi_j \in \mathbb{R}^k$. If needed one may obtain, by the Gram-Schmidt method, the remaining $N - k$ eigenvectors to span $\mathbb{R}^N$, which in this context are associated to the eigenvalue zero.

The linear operator $S$ constructed from the set of typical signals is

$$S = \sum_{i=1}^{k} \alpha_i \Phi_i \Phi_i^T$$

where $S \in \mathcal{M}_{N \times N}$. 
Signal-adapted tomography

- For the tomogram consider an operator $B(\mu, \nu)$ of the form

$$B(\mu, \nu) = \mu t + \nu S = \mu \begin{pmatrix} 1\Delta t \\ 2\Delta t \\ \vdots \\ N\Delta t \end{pmatrix} + \nu \sum_{i=1}^{k} \alpha_i \Phi_i \Phi_i^t$$

where $B \in \mathcal{M}_{N \times N}$.

- The eigenvectors of each $B(\mu, \nu)$ are the columns of the matrix that diagonalizes it. From the projections of the signal on these eigenvectors one constructs a tomogram adapted to the operator pair $(t, S)$. 
Tomography with an adapted operator pair: An example

Typical data: a set of 40 random signals with pulses of duration $\Delta t = 10$ and intensities $+1$ or $-1$. The total length of the signal is 200 time units.
Tomography with an adapted operator pair: An example

Eigenbasis of \( B(\theta) = t \cos \theta + S \sin \theta \) is used to project the signal. Tomogram for 20 different values of \( \theta \) at intervals \( \Delta \theta = \pi/40 \)

Right lower plot is projection on eigenvectors 185 to 200 at \( \theta_{19} = 19\pi/40 \).
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