List Multicoloring of Planar Graphs and Related Classes

Glenn G. Chappell
Department of Computer Science
University of Alaska
Fairbanks, AK 99775-6670
ggchappell@alaska.edu
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Abstract

For positive integers \( a \) and \( b \), a graph \( G \) is \((a:b)\)-choosable if, for each assignment of lists of \( a \) colors to the vertices of \( G \), each vertex can be colored with a set of \( b \) colors from its list so that adjacent vertices are colored with disjoint sets.

We show that for positive integers \( a \) and \( b \), every bipartite planar graph is \((a:b)\)-choosable iff \( ab \geq 3 \). For general planar graphs, we show that if \( \frac{a}{b} < \frac{4}{5} \), then there exists a planar graph that is not \((a:b)\)-choosable, thus improving on a result of X. Zhu, which had \( 4\frac{2}{9} \). Lastly, we show that every \( K_5 \)-minor-free graph is \((a:b)\)-choosable iff \( \frac{a}{b} \geq 5 \). Along the way, we mention some open problems.

1 Introduction

For \( L \) an assignment of lists of colors to the vertices of a (finite, simple) graph \( G \), a \( b \)-fold \( L \)-coloring of \( G \) is a mapping \( \varphi \) that colors each vertex \( v \) of \( G \) with a set \( \varphi(v) \subseteq L(v) \) of \( b \) colors such that adjacent vertices are colored with disjoint sets.

Following Erdős, Rubin, and Taylor [6, p. 155], for positive integers \( a \) and \( b \) we say a graph \( G \) is \((a:b)\)-choosable if, for each assignment \( L \) of colors with \( |L(v)| = a \) for each vertex \( v \), graph \( G \) admits a \( b \)-fold \( L \)-coloring. So the usual notion of \( k \)-choosability is the same as \((k:1)\)-choosability.

We are interested in results of the following form. For some fixed class of graphs, the following are equivalent for positive integers \( a \) and \( b \): (i) every graph in the class is \((a:b)\)-choosable; (ii) \( \frac{a}{b} \geq r \) (where \( r \) is a number that depends on the class of interest).
In Section 2 we consider the class of bipartite planar graphs. Alon & Tarsi [1, Corollary 3.4] showed that every bipartite planar graph is 3-choosable. The following stronger result was proven by Gutner [7, Corollary 1.11] (see also Gutner & Tarsi [9, Corollary 1.11]).

**Theorem 1.1** (Gutner 1992). Let $G$ be a bipartite planar graph. Then $G$ is $(3m : m)$-choosable, for each positive integer $m$. □

Using examples based on a construction of Barát, Joret, & Wood [2, proof of Thm. 1], we improve on Gutner’s result, showing that every bipartite planar graph is $(a : b)$-choosable if and only if $\frac{a}{b} \geq 3$. Thus we have a result in our desired form.

In Section 3 we consider planar graphs in general. Erdős, Rubin, & Taylor [6, p. 153] conjectured that every planar graph is 5-choosable, while there exists a planar graph that is not 4-choosable.

That every planar graph is 5-choosable was proven by Thomassen [13]. Tuza & Voigt [14, Thm. 3.1] generalized Thomassen’s argument to prove the following.

**Theorem 1.2** (Tuza & Voigt 1996). Let $G$ be a planar graph. Then $G$ is $(5m : m)$-choosable, for each positive integer $m$. □

A construction of a planar graph that is not 4-choosable was given by Voigt [15]. Smaller examples are due to Gutner [8, Thm. 1.7] and to Mirzakhani [11]. Using a method similar to that of Gutner, one can construct, for each positive integer $m$, a planar graph that is not $(4m : m)$-choosable; for example, see Zhu [19, Thm. 1].

So if $\frac{a}{b} \geq 5$, then every planar graph is $(a : b)$-choosable, while if $\frac{a}{b} \leq 4$, then there exists a planar graph that is not $(a : b)$-choosable. What about ratios strictly between 4 and 5? Zhu [19, Thm. 1] proved the following.

**Theorem 1.3** (Zhu 2017). Let $a$ and $b$ be positive integers. If $\frac{a}{b} < 4\frac{2}{5}$, then there exists a planar graph that is not $(a : b)$-choosable. □

We improve on Zhu’s result by showing that if $\frac{a}{b} < 4\frac{2}{5} = 4\frac{2}{5}$, then there exists a planar graph that is not $(a : b)$-choosable. However, we do not have a result in our desired form. We speculate on whether such a result might hold.

In Section 4 we consider the larger class of $K_5$-minor-free graphs. Škrekovski [12, Thm. 2.3] generalized Thomassen’s proof of the 5-choosability of planar graphs, showing the following.

**Theorem 1.4** (Škrekovski 1998). Let $G$ be a $K_5$-minor-free graph. Then $G$ is 5-choosable. □

Other proofs of Škrekovski’s result are due to He, Miao, & Shen [10, Thm. 2.1] and to Wood & Linusson [18, Thm. 1].

We prove a result in our desired form that generalizes both the Tuza-Voigt result (Theorem 1.2) and the Škrekovski result (Theorem 1.4): that every $K_5$-minor-free graph is $(a : b)$-choosable if and only if $\frac{a}{b} \geq 5$.

We denote the vertex set of a graph $G$ by $V(G)$. When describing lists of colors, we will generally omit union operators in a union of disjoint sets. For example, $XPT$ means $X \cup P \cup T$, for disjoint sets $X$, $P$, and $T$. 

2
2 Bipartite Planar Graphs

In this section we prove the following theorem.

**Theorem 2.1.** The following are equivalent for positive integers $a$ and $b$.

(i) Every bipartite planar graph is $(a : b)$-choosable.

(ii) $a/b \geq 3$. □

We begin with a lemma giving a list-coloring property of $P_4$, a 4-vertex path (see Figure 1). Later, we will verify the (i) $\implies$ (ii) portion of Theorem 2.1 using examples constructed by pasting together multiple copies of $P_4$.

**Lemma 2.2.** Let $a$ and $b$ be positive integers with $2 \leq a/b < 3$. Let $X$, $Y$, $P$, and $T$ be pairwise disjoint lists of colors, such that $X$, $Y$, and $P$ have size $b$, while $T$ has size $a-2b$. (Note that $a-2b \geq 0$; if $a/b = 2$, then $T = \emptyset$.)

Define a color assignment $L$ for $P_4$, a 4-vertex path, as follows. Label the vertices 1, 2, 3, 4, in order along the path. Let $L(1) = X$, $L(2) = XPT$, $L(3) = YPT$, and $L(4) = Y$, as shown in Figure 1.

Then $P_4$ admits no $b$-fold $L$-coloring.

**Proof.** In a $b$-fold $L$-coloring $\varphi$ of $P_4$, we must have $\varphi(1) = X$ and $\varphi(4) = Y$. So $\varphi(2)$ and $\varphi(3)$ are disjoint subsets of $PT$, each of size $b$, and thus $|\varphi(2) \cup \varphi(3)| = 2b$. However, $|PT| < 2b$, so no such coloring can exist. □

When considering $(a : b)$-choosability, Lemma 2.2 says we can forbid a specific coloring of vertices 1 and 4 of $P_4$ with disjoint sets of colors. Using this idea, we can paste together copies of $P_4$ to construct a bipartite planar graph in which all possible colorings of two vertices are forbidden. We use this idea in the proof of Theorem 2.1.

**Proof of Theorem 2.1.** (ii) $\implies$ (i). This follows from Theorem 1.1.

(i) $\implies$ (ii). Let $a$ and $b$ be positive integers with $a/b < 3$. We construct a bipartite planar graph $G$ such that $G$ is not $(a : b)$-choosable.

Figure 1: $P_4$, a 4-vertex path, with vertices labeled and lists of colors shown, used in Lemma 2.2 and the proof of Theorem 2.1.
Figure 2: Graph $G$ from the (i) $\Rightarrow$ (ii) part of the proof of Theorem 2.1, with vertices 1 and 4 labeled. Each copy of $P_4$ is labeled with a circled number from 1 to $q$.

If $\frac{a}{b} < 2$, then we may let $G = K_2$. Suppose, therefore, that $2 \leq \frac{a}{b} < 3$. Let

$$q = \binom{a}{b} \binom{a-b}{b}.$$

To construct graph $G$, begin with $q$ copies of $P_4$, pictured in Figure 1. Identify all the 1 vertices in these copies, labeling the resulting vertex as 1. Similarly identify all the 4 vertices, labeling the resulting vertex as 4. Add an edge joining vertices 1 and 4. Let $G$ be the resulting graph. See Figure 2 for an illustration of graph $G$. (The construction of $G$ is a variation on a construction of Barát, Joret, & Wood [2, proof of Thm. 1].)

Graph $G$ is bipartite and planar. It remains to show that $G$ is not $(a : b)$-choosable.

Assign vertices 1 and 4 the same list of $a$ colors. The number of ways these two vertices can be colored with disjoint sets of size $b$ is $\binom{a}{b} \binom{a-b}{b} = q$. Create an (arbitrary) correspondence between these colorings and the $q$ copies of $P_4$. For each possible coloring of vertices 1 and 4, assign lists of colors to the 2 and 3 vertices in the corresponding copy of $P_4$ so that Lemma 2.2 allows us to conclude that vertices 1 and 4 cannot be colored in this manner.

The result is an assignment $L$ of lists of colors with $|L(v)| = a$ for all $v \in V(G)$, such that $G$ admits no $b$-fold $L$-coloring, since no coloring is possible for vertices 1 and 4. Thus, $G$ is not $(a : b)$-choosable. □

3 General Planar Graphs

In this section we prove the following theorem.

**Theorem 3.1.** Let $a$ and $b$ be positive integers. If $\frac{a}{b} < \frac{22}{5}$, then there exists a planar graph $G$ such that $G$ is not $(a : b)$-choosable. □

Once again, our proof will use examples constructed by pasting together small graphs. Our construction is somewhat similar to one due to Zhu [19, Lemma 2, proof of Thm. 1]—which, in turn, has similarities with a construction of Gutner [8, proof of Thm. 1.7].

We begin with two lemmas concerning graphs we call $F_1$ and $F_2$, which are pictured in the top of Figure 3 and in Figure 4, respectively.
Lemma 3.2. Let \( a \) and \( b \) be positive integers with \( 4 \leq \frac{a}{b} < \frac{22}{5} \). Let \( X, Y, P, Q, R, \) and \( T \) be pairwise disjoint lists of colors, such that \( X, Y, P, Q, \) and \( R \) have size \( b \), while \( T \) has size \( a - 4b \). (Note that \( a - 4b \geq 0 \); if \( \frac{a}{b} = 4 \), then \( T = \emptyset \).)

Let \( F_1 \) be the graph pictured in the top portion of Figure 3, with vertices labeled \( 1, \ldots, 9 \) as shown. Define a color assignment \( L \) such that
\[
L(1) = X \quad \text{and} \quad L(9) = Y,
\]
while vertices 2 through 8 are assigned lists of colors as shown in the bottom portion of Figure 3.

Then in any \( b \)-fold \( L \)-coloring \( \varphi \) of \( F_1 \), we have
\[
|\varphi(8) \cap T| > \frac{1}{2}|T|; \quad \text{that is, the set with which vertex 8 is colored includes more than half of the elements of } T.
\]

Proof. Suppose that \( \varphi \) is a \( b \)-fold \( L \)-coloring of \( F_1 \). Observe that \( \varphi(v) \cap XY = \emptyset \) for all \( v \in \{2, \ldots, 8\} \).

We begin by proving two claims.

Claim 1. \(|\varphi(2) \cap \varphi(5)| \geq 5b - a\).

Suppose not: \(|\varphi(2) \cap \varphi(5)| < 5b - a\). Then
\[
|\varphi(2) \cup \varphi(5)| = |\varphi(2)| + |\varphi(5)| - |\varphi(2) \cap \varphi(5)|
\]
\[
> b + b - (5b - a)
\]
\[
= a - 3b.
\]

Colors usable on vertices 3 and 4 are those in \( PQRT \): a total of \( 3b + (a - 4b) = a - b \) colors. Removing colors in \( \varphi(2) \cup \varphi(5) \)—more than \( a - 3b \) colors, by the above—the number of
colors still available for vertices 3 and 4 is less than \((a - b) - (a - 3b) = 2b\). But vertices 3 and 4 are adjacent, so \(|\varphi(3) \cup \varphi(4)| = 2b\), a contradiction, and Claim 1 is proven.

**Claim 2.** \(|\varphi(5) \cap \varphi(8)| \geq 5b - a\).

The proof of Claim 2 is much the same as that of Claim 1; only the vertex labels differ.

**Finishing.** Now we prove the conclusion of the lemma, that

\[ |\varphi(8) \cap T| > \frac{1}{2}|T|. \]

Suppose not: \(|\varphi(8) \cap T| \leq \frac{1}{2}|T| = \frac{a - 4b}{2}\). Then

\[
10b - 2a \leq |\varphi(2) \cap \varphi(5)| + |\varphi(5) \cap \varphi(8)| \leq \left(|\varphi(5) \cap P| + |\varphi(5) \cap T|\right) \]

\[
+ \left(|\varphi(5) \cap R| + |\varphi(8) \cap T|\right)
\]

\[
= |\varphi(5)| + |\varphi(8) \cap T| \leq b + \frac{a - 4b}{2} \]

by our supposition

\[
= \frac{a}{2} - b.
\]

Gathering terms in \(10b - 2a \leq \frac{a}{2} - b\), we have:

\[
11b \leq \frac{5}{2}a
\]

\[
< \frac{5}{2} \cdot \frac{22}{5} b = 11b \]

since \(\frac{a}{b} < \frac{22}{5}\).

But \(11b < 11b\) is impossible, and the lemma is proven.  \(\square\)

**Lemma 3.3.** Let \(a\) and \(b\) be positive integers with \(4 \leq \frac{a}{b} < \frac{22}{5}\). Let \(X, Y, P, Q, R,\) and \(T\) be pairwise disjoint lists of colors, such that \(X, Y, P, Q,\) and \(R\) have size \(b\), while \(T\) has size \(a - 4b\). (Note that \(a - 4b \geq 0\); if \(\frac{a}{b} = 4\), then \(T = \emptyset\).)

Let \(F_1\) be the graph pictured in the top portion of Figure 3. Construct graph \(F_2\) as follows: take two copies of \(F_1\), the first with vertices labeled 1, \ldots, 9, as in Figure 3, the second with vertices similarly labeled 1', \ldots, 9'. Identify vertices 1 and 1', labeling the resulting vertex as 1. Similarly identify vertices 9 and 9', labeling the resulting vertex as 9. Lastly, add an edge joining vertices 8 and 8'. The resulting graph \(F_2\) is shown in Figure 4.

Define a color assignment \(L\) such that \(L(1) = X\) and \(L(9) = Y\), while vertices 2 through 8 are assigned lists of colors as shown in the bottom portion of Figure 3. Let vertices 2', \ldots, 8' be similarly assigned colors so that vertex 2' has the same list as vertex 2, vertex 3' has the same list as 3, and so on.

Then graph \(F_2\) admits no \(b\)-fold \(L\)-coloring.
Figure 4: Graph $F_2$ from Lemma 3.3 and the proof of Theorem 3.1.

Proof. Suppose that $\varphi$ is a $b$-fold $L$-coloring of $F_2$. By Lemma 3.2, more than half of the elements of $T$ lie in $\varphi(8)$. Similarly, more than half of the elements of $T$ lie in $\varphi(8')$. So $\varphi(8) \cap \varphi(8') \neq \emptyset$. But since vertices 8 and 8' are adjacent, this is impossible. □

When considering $(a : b)$-choosability, Lemma 3.3 says we can forbid a specific coloring of vertices 1 and 9 of $F_2$ with disjoint sets of colors. Using this idea, we can paste together copies of $F_2$ to construct a planar graph in which all possible colorings of two vertices are forbidden. We use this idea to prove Theorem 3.1.

Proof of Theorem 3.1. Let $a$ and $b$ be positive integers with $\frac{a}{b} < \frac{22}{5}$. We construct a planar graph $G$ such that $G$ is not $(a : b)$-choosable.

If $\frac{a}{b} < 4$, then we may let $G = K_4$. Suppose, therefore, that $4 \leq \frac{a}{b} < \frac{22}{5}$. Let

$$q = \left( \frac{a}{b} \right) \left( \frac{a-b}{b} \right).$$

To construct graph $G$, begin with $q$ copies of graph $F_2$, pictured in Figure 4. Identify all the 1 vertices in these copies, labeling the resulting vertex as 1. Similarly identify all the 9 vertices, labeling the resulting vertex as 9. Add an edge joining vertices 1 and 9. Let $G$ be the resulting graph.

Graph $G$ is planar. It remains to show that $G$ is not $(a : b)$-choosable.

Assign vertices 1 and 9 the same list of $a$ colors. The number of ways these two vertices can be colored with disjoint sets of size $b$ is $\left( \frac{a}{b} \right) \binom{a-b}{b} = q$. Create an (arbitrary) correspondence between these colorings and the $q$ copies of $F_2$. For each possible coloring of vertices 1 and 9, assign lists of colors to the 2, ... , 8 and 2', ... , 8' vertices in the corresponding copy of $F_2$ so that Lemma 3.3 allows us to conclude that vertices 1 and 9 cannot be colored in this manner.

The result is an assignment $L$ of lists of colors with $|L(v)| = a$ for all $v \in V(G)$, such that $G$ admits no $b$-fold $L$-coloring, since no coloring is possible for vertices 1 and 9. Thus, $G$ is not $(a : b)$-choosable. □
We know that if \( \frac{a}{b} \geq 5 \), then every planar graph is \((a : b)\)-choosable (Theorem 1.2), while, if \( \frac{a}{b} < \frac{22}{5} \), then there exists a planar graph that is not \((a : b)\)-choosable (Theorem 3.1). From \( \frac{22}{5} = 4\frac{2}{5} \) to 5 is a sizable gap. Can we close this gap?

**Conjecture 3.4.** There exists a real number \( r \) such that the following are equivalent for positive integers \( a \) and \( b \).

1. Every planar graph is \((a : b)\)-choosable.
2. \( \frac{a}{b} \geq r \). □

If Conjecture 3.4 is true, then \( \frac{22}{5} \leq r \leq 5 \), and it seems likely that \( r \) is either \( \frac{9}{2} \) or 5. Zhu [19, Conjecture 4] conjectured that every planar graph is \((9 : 2)\)-choosable. If Zhu’s conjecture and Conjecture 3.4 both hold, then \( \frac{22}{5} = \frac{44}{10} \leq r \leq \frac{45}{10} = \frac{9}{2} \), and most likely \( r = \frac{9}{2} \).

As we will see in the next section, we are able to “close the gap” for the larger class of \( K_5 \)-minor-free graphs, with \( r = 5 \); see Theorem 4.1.

What about planar graphs with higher connectivity?

A number of examples have been found of planar graphs that are not 4-choosable. The examples due to Voigt [15], Gutner [8], and Mirzakhani [11] all have connectivity 3. Those due to Zhu [19], along with the examples constructed in this work, have connectivity 2; however, by the well known fact that every edge-maximal simple planar graph of order at least 4 is 3-connected, we may add edges to these graphs to obtain 3-connected examples without compromising their properties.

Increasing connectivity to 4 is a different matter. All of the examples mentioned are constructed by pasting together small graphs along edges or triangular faces, which naturally leads to connectivity at most 3. We ask whether examples with greater connectivity exist.

**Question 3.5.** Does there exist a 4-connected planar graph that is not 4-choosable? □

**Question 3.6.** For each positive integer \( m \), does there exist a 4-connected planar graph that is not \((4m : m)\)-choosable? What about ratios greater than 4? □

## 4 \textit{K}_5\textit{-Minor-Free Graphs}

In this section we prove the following theorem.

**Theorem 4.1.** The following are equivalent for positive integers \( a \) and \( b \).

1. Every \( K_5 \)-minor-free graph is \((a : b)\)-choosable.
2. \( \frac{a}{b} \geq 5 \). □
Our proof uses a method similar to the proof of Theorem 1.4 by He, Miao, & Shen [10, Thm. 2.1], along with examples based on a construction of Barát, Joret, & Wood [2, proof of Thm. 1].

We will make use of two previously known results. The first, due to Tuza & Voigt [14, Thm. 3.2], deals with list multicoloring of planar graphs.

**Theorem 4.2** (Tuza & Voigt 1996). Let $m$ be a positive integer. Let $G$ be a plane near triangulation with outer cycle $C$. Suppose that $L$ is a list assignment for $G$, with the following properties.

(i) There are two adjacent vertices $u$ and $v$ of $C$ with disjoint lists of length $m$ each.

(ii) All the other vertices of $C$ have (unrestricted) lists of length $3m$.

(iii) All vertices of $G - C$ have lists of length $5m$.

Then $G$ admits an $m$-fold $L$-coloring. □

The second previously known result is a structure theorem for $K_5$-minor-free graphs proven by Wagner [17] and often called Wagner’s Theorem. As Wagner used terminology very different from ours, we give a statement based on that of Diestel [3, Thm. 8.3.4].

**Theorem 4.3** (Wagner 1937). Let $G$ be an edge-maximal $K_5$-minor-free graph. If $|V(G)| \geq 4$, then $G$ can be constructed recursively, by pasting along triangles and $K_2$s, from plane triangulations and copies of the graph $M_8$—the 8-vertex Möbius ladder or Wagner Graph, shown in Figure 5. □

We begin with the following lemma, which generalizes a lemma of Škrekovski [12, Lemma 2.2] and of He, Miao, & Shen [10, Lemma 2.1] to list multicoloring. Our proof follows similar lines to that of He, Miao, & Shen.

**Lemma 4.4.** Let $m$ be a positive integer. Let $G$ be an edge-maximal $K_5$-minor-free graph, and let $L$ be a list assignment for $G$ such that $|L(v)| \geq 5m$ for each $v \in V(G)$. Suppose that $H$ is a subgraph of $G$ isomorphic to $K_2$ or $K_3$, and $\varphi$ is an $m$-fold $L$-coloring of $H$. Then $\varphi$ can be extended to an $m$-fold $L$-coloring of $G$. 

![Figure 5: Graph $M_8$, the 8-vertex Möbius ladder, also known as the Wagner Graph, used in Theorem 4.3 and the proof of Lemma 4.4.](image)
Figure 6: Graph $N_8 \cong K_{2,2,2}$, the octahedron graph, with vertices labeled and lists of colors shown, used in Lemma 4.5 and the proof of Theorem 4.1.

**Proof.** We proceed by induction on $|V(G)|$. The result is immediate when $|V(G)| \leq 3$. Suppose $|V(G)| \geq 4$.

First suppose $G$ has a separating $K_2$ or $K_3$. We may write $G = G_1 \cup G_2$, where $G_1 \cap G_2$ is $K_2$ or $K_3$. Then $H$ must be contained in either $G_1$ or $G_2$; without loss of generality say $G_1$. Apply the induction hypothesis to $G_1$, with $H$ precolored by $\varphi$—adding edges to $G_1$ as necessary to make it edge-maximal $K_5$-minor-free. Using the resulting multicoloring, precolor $G_1 \cap G_2$ and apply the induction hypothesis to $G_2$—once again adding edges as necessary. Pasting the two resulting multicolorings together gives the required $m$-fold $L$-coloring of $G$.

Now suppose $G \cong M_8$, the Wagner Graph; see Figure 5. Then the vertices of $G - H$ may be colored from their lists in any order, since every vertex in $G$ has degree 3.

We are left with the case in which $G$ is not the Wagner Graph and has no separating $K_2$ or $K_3$. By Theorem 4.3, $G$ is a plane triangulation. We may assume that $H$ is $K_3$; otherwise choose a vertex that lies in a triangle containing $H$, color this vertex using colors in its list that are not used on either vertex of $H$, and add the new vertex to $H$.

Since $H$ is a triangle and is not separating, we can embed $G$ in the plane so that $H$ is the outer face. Denote the vertices of $H$ by $x$, $y$, $z$. Define a new list assignment $L'$ for $G$ as follows. Let $L'(x) = \varphi(x)$, $L'(y) = \varphi(y)$, and $L'(z) = \varphi(x) \cup \varphi(y) \cup \varphi(z)$. For each vertex $v \in V(G) - \{x, y, z\}$, let $L'(v)$ be a $5m$-element subset of $L(v)$. Apply Theorem 4.2 to obtain an $m$-fold $L'$-coloring of $G$ that extends $\varphi$. This is the required $m$-fold $L$-coloring of $G$.  

Next we prove a list-coloring property of the octahedron graph.

**Lemma 4.5.** Let $a$ and $b$ be positive integers with $4 \leq \frac{a}{b} < 5$. Let $X$, $Y$, $Z$, $P$, $Q$, and $T$ be pairwise disjoint lists of colors, such that $X$, $Y$, $Z$, $P$, and $Q$ have size $b$, while $T$ has size $a - 4b$. (Note that $a - 4b \geq 0$; if $\frac{a}{b} = 4$, then $T = \emptyset$.)

Let $N_8 \cong K_{2,2,2}$ be the octahedron graph with list assignment $L$ shown in Figure 6. Then graph $N_8$ admits no $b$-fold $L$-coloring.
Proof. In a $b$-fold $L$-coloring $\varphi$ of $N_8$, we must have $\varphi(1) = X$, $\varphi(2) = Y$, and $\varphi(3) = Z$. So $\varphi(4)$, $\varphi(5)$, and $\varphi(6)$ are pairwise disjoint subsets of $PQT$, each of size $b$, and thus $|\varphi(4) \cup \varphi(5) \cup \varphi(6)| = 3b$. However, $|PQT| < 3b$, so no such coloring can exist. □

When considering $(a : b)$-choosability, Lemma 4.5 says we can forbid a specific coloring of vertices 1, 2, and 3 of the octahedron graph $N_8$ with pairwise disjoint sets of colors. Using this idea, we can paste together copies of $N_8$ to construct a $K_5$-minor-free graph in which all possible colorings of three vertices are forbidden. We use this idea in the proof of Theorem 4.1.

Proof of Theorem 4.1. (ii) $\implies$ (i). Let $a$ and $b$ be positive integers with $\frac{a}{b} \geq 5$. Let $G$ be a $K_5$-minor-free graph. We may assume that $G$ is edge-maximal $K_5$-minor-free; otherwise add edges until it is. Let $L$ be a list assignment of $G$ such that $|L(v)| = a$ for each $v \in V(G)$. It suffices to show that $G$ admits a $b$-fold $L$-coloring.

If $|V(G)| < 2$, then the result is immediate. If $|V(G)| \geq 2$, then let $H$ be the subgraph induced by two (arbitrary) adjacent vertices. Let $m = b$, and let $\varphi$ be an $m$-fold $L$-coloring of $H$. Applying Lemma 4.4, we obtain the required $b$-fold $L$-coloring of $G$.

(i) $\implies$ (ii). Let $a$ and $b$ be positive integers with $\frac{a}{b} < 5$. We construct a $K_5$-minor-free graph $G$ such that $G$ is not $(a : b)$-choosable.

If $\frac{a}{b} < 4$, then we may let $G = K_4$. Suppose, therefore, that $4 \leq \frac{a}{b} < 5$. Let

$$q = \binom{a}{b} \binom{a-b}{b} \binom{a-2b}{b}.$$

To construct graph $G$, begin with $q$ copies of the octahedron graph $N_8$, pictured in Figure 6. In each copy, there is a triangle having vertices labeled 1, 2, 3. Paste along all of these triangles to obtain $G$, so that $G$ has just one vertex labeled 1, and similarly for 2 and 3. See Figure 7 for an illustration of graph $G$. (The construction of graph $G$ is a variation on a construction of Barát, Joret, & Wood [2, proof of Thm. 1].)

Graph $G$ has no $K_5$ minor. It remains to show that $G$ is not $(a : b)$-choosable.

Assign vertices 1, 2, and 3 the same list of $a$ colors. The number of ways these three vertices can be colored with pairwise disjoint sets of size $b$ is $\binom{a}{b} \binom{a-b}{b} \binom{a-2b}{b} = q$. Create an (arbitrary) correspondence between these colorings and the $q$ copies of $N_8$. For each possible coloring of vertices 1, 2, 3, assign lists of colors to the other 3 vertices of the corresponding copy of $N_8$ so that Lemma 4.5 allows us to conclude that vertices 1, 2, 3 cannot be colored in this manner.

The result is an assignment $L$ of lists of colors with $|L(v)| = a$ for all $v \in V(G)$, such that $G$ admits no $b$-fold $L$-coloring, since no coloring is possible for vertices 1, 2, 3. Thus, $G$ is not $(a : b)$-choosable. □

What about forbidding clique minors of orders other than 5? Results similar to Theorem 4.1 can easily be proven for smaller clique minors.

Theorem 4.6. Let $t$ be a positive integer with $2 \leq t \leq 4$. Then the following are equivalent for positive integers $a$ and $b$. 

\begin{enumerate}
\item $t(a:b)$-choosable
\item $t$-free
\item $t$-free
\end{enumerate}
(i) Every $K_t$-minor-free graph is $(a : b)$-choosable.

(ii) $\frac{a}{b} \geq t - 1$. □

Proof. (i) $\Rightarrow$ (ii). Graph $K_{t-1}$ is a $K_t$-minor-free graph that is not $(a : b)$-choosable for any $a, b$ with $\frac{a}{b} < t - 1$.

(ii) $\Rightarrow$ (i). For $2 \leq t \leq 4$, every $K_t$-minor-free graph with order at least 1 has a vertex of degree at most $t - 2$. In particular, a $K_2$-minor-free graph is an edgeless graph, which of course has a vertex of degree at most 0. A $K_3$-minor-free graph is a forest, which must have a vertex of degree at most 1. And it follows from a proof of Dirac [4, p. 87] (see also Duffin [5, Thm. 1, Corollary 4]) that a $K_4$-minor-free graph must have a vertex of degree at most 2.

By a simple inductive argument, then, for $2 \leq t \leq 4$, every $K_t$-minor-free graph is $(a : b)$-choosable for all $a, b$ with $\frac{a}{b} \geq t - 1$ (see Tuza & Voigt [14, Thm. 2.1]). □

Theorem 4.6 is not really new; it simply restates well known ideas. But it is a result in our desired form (see Section 1), and considering it together with Theorem 4.1 is suggestive. We ask whether results of this kind hold for $K_t$-minor-free graphs for larger values of $t$.

**Question 4.7.** Is it true that for each integer $t \geq 2$, there exists a real number $r_t$ such that the following are equivalent for positive integers $a$ and $b$?

(i) Every $K_t$-minor-free graph is $(a : b)$-choosable.

(ii) $\frac{a}{b} \geq r_t$. □
Question 4.7 has an affirmative answer for $2 \leq t \leq 5$, by Theorems 4.1 and 4.6. It remains open for larger values of $t$.

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