OPERATORS PRESERVING INEQUALITIES BETWEEN THE POLYNOMIALS

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Abstract. In this paper, by combining the operators $B$ and $D\alpha$, we investigate the dependence of $B[D\alpha(P(Rz) - \beta P(z))]$ on the maximum modulus of $P(z)$ on $|z| = 1$ for every real or complex numbers $\alpha$ and $\beta$ with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $R > 1$. Our results include not only some known polynomial inequalities as special case, but also the results recently proved by Bidkham and Mezerji as a particular case.

1. Introduction

If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree at most $n$ and $P'(z)$ is its derivatives, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \quad (1.1)$$

and

$$\max_{|z|=R>1} |P'(z)| \leq R^n \max_{|z|=1} |P(z)|. \quad (1.2)$$

Inequality (1.1) is an immediate consequence of S. Bernstein’s inequality on the derivative of a trigonometric polynomial (for reference see [6, 11]), where as inequality (1.2) is a simple deduction from the maximum modulus principle [12, p.346]. In both inequalities (1.1) and (1.2) equality holds only when $P(z)$ is a constant multiple of $z^n$.

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If we restrict ourselves to a class of polynomials having no zero in \(|z| < 1\), then the above inequality can be sharpened. In fact, Erdős conjectured and latter Lax [10] proved that if \(P(z) \neq 0\) in \(|z| \leq 1\), then
\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|.
\]
and
\[
\max_{|z|=R>1} \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|.
\]
Turán [14] proved that, if \(P(z)\) has all its zeros in \(|z| \leq 1\), then
\[
\min_{|z|=1} |P'(z)| \geq n \min_{|z|=1} |P(z)|.
\]
Concerning the minimum modulus of a polynomial \(P(z)\) and its derivative \(P'(z)\), Aziz and Dawood [2] proved that, if \(P(z)\) has all its zeros in \(|z| \leq 1\), then
\[
\min_{|z|=1} |P'(z)| \geq n \min_{|z|=1} |P(z)|.
\]
Let \(\alpha\) be any complex number, the polynomial \(D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)\) denote the polar derivative of the polynomial \(P(z)\) of degree at most \(n\) with respect to \(\alpha\). The polynomial \(D_\alpha P(z)\) is of degree at most \(n - 1\) and it generalizes the ordinary derivative in the sense that
\[
\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).
\]
Aziz [1] extended inequality (1.3) and (1.5) to the polar derivative of a polynomial and proved that if \(P(z)\) is a polynomial of degree \(n\) which does not vanish in \(|z| < 1\), then for every complex number \(\alpha\) with \(|\alpha| \geq 1\),
\[
\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n}{2} \{ |\alpha z^{n-1}| + 1 \} \max_{|z|=1} |P(z)| \text{ for } |z| \geq 1.
\]
Rahman [11, p.538] introduced a class \(B_n\) of operators \(B\) that map \(P \in P_n\) into itself. That is, the operator \(B\) carries \(P \in P_n\) into
\[
B[P(z)] = \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) P'(z) + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!},
\]
where \(\lambda_0, \lambda_1,\text{ and } \lambda_2\) are real or complex numbers such that all the zeros of
\[
u(z) := \lambda_0 + C(n,1)\lambda_1 z + C(n,2)\lambda_2 z^2, \quad C(n,r) = \frac{n!}{r!(n-r)!},
\]
lie in the half plane
\[
|z| \leq \left| z - \frac{n}{2} \right|.
\]
Concerning this operator Shah and Liman [13] proved:
Theorem A. If \( P(z) \in P_n \) and \( P(z) \neq 0 \) in \( |z| > 1 \), then for \( |z| \geq 1 \),
\[
|B[P(z)]| \geq |B[z^n]| \min_{|z|=1} |P(z)|. \tag{1.9}
\]

Theorem B. If \( P(z) \in P_n \) and \( P(z) \neq 0 \) in \( |z| < 1 \), then for \( |z| \geq 1 \),
\[
|B[P(z)]| \leq \frac{1}{2} \left[ \{ |B[z^n]| + |\lambda_\alpha| \} \max_{|z|=1} |P(z)| - \{ |B[z^n]| - |\lambda_\alpha| \} \min_{|z|=1} |P(z)| \right]. \tag{1.10}
\]

Concerning the dependence of \( |P(Rz) - P(z)| \) on \( |P(z)| \) Aziz and Rather [4] proved:

Theorem C. If \( P(z) \) is a polynomial of degree \( n \), then for every real or
complex number \( \beta \) with \( |eta| \leq 1 \) and \( R \geq 1 \),
\[
|P(Rz) - \beta P(z)| \leq |R^n - \beta||z|^n \max_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1. \tag{1.11}
\]

Theorem D. If \( P(z) \) is a polynomial of degree \( n \) which does not vanish in
\( |z| < 1 \), then for every real or complex number \( \beta \) with \( |eta| \leq 1 \) and \( R \geq 1 \),
\[
|P(Rz) - \beta P(z)| \leq \left\{ \frac{|R^n - \beta||z|^n + |1 - \beta|}{2} \right\} \max_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1. \tag{1.12}
\]

Recently Birkhahm and Mezerji [7] have generalised some of the above inequalities by combining \( B \) and \( D_\alpha \) operators and proved the following results:

Theorem E. If \( P(z) \) is a polynomial of degree at most \( n \), having all its zeros
in \( |z| \leq 1 \), then for every complex number \( \alpha \) with \( |\alpha| \geq 1 \),
\[
|B[D_\alpha P(z)]| \geq n|\alpha||B[z^{n-1}]| \min_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1. \tag{1.13}
\]

Theorem F. If \( P(z) \) is a polynomial of degree at most \( n \), having no zero in
\( |z| < 1 \), then for every \( \alpha \) with \( |\alpha| \geq 1 \),
\[
|B[D_\alpha P(z)]| \leq \frac{n}{2} \left\{ |\alpha||B[z^{n-1}]| + |\lambda_\alpha| \right\} \max_{|z|=1} |P(z)|
- \{ |\alpha||B[z^{n-1}]| - |\lambda_\alpha| \} \min_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1. \tag{1.14}
\]

In this paper we combine the different ideas and techniques used above and consider the operator \( B \) and \( D_\alpha \) such that the operator \( B \) carries \( D_\alpha P(z) \) into
\[
B[D_\alpha P(z)] = \lambda_\alpha D_\alpha P(z) + \lambda_1 \left( \frac{mz}{2} \right) D_\alpha P'(z) + \lambda_2 \left( \frac{mz}{2} \right)^2 \frac{D_\alpha P''(z)}{2!},
\]
where $0 \leq m \leq n - 1$ and $\lambda_0, \lambda_1, \lambda_2$ are real or complex numbers such that all zeros of
\[ u(z) := \lambda_0 + C(m, 1)\lambda_1 z + C(m, 2)\lambda_2 z^2, \quad C(m, r) = \frac{m!}{r!(m-r)!}, \tag{1.15} \]
lie in the half plane
\[ |z| \leq \left| z - \frac{m}{2} \right| \]
and obtain compact generalizations of some well-known polynomial inequalities. We first prove the following:

**Theorem 1.1.** If $P(z)$ is a polynomial of degree $n$, then for every real or complex numbers $\alpha, \beta$ with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $R > 1$
\[ |B[D_\alpha(P(Rz) - \beta P(z))]| \leq |\alpha|n|R^n - \beta||B[z^{n-1}]| \max_{|z|=1} |P(z)|, \tag{1.16} \]
for $|z| \geq 1$.

The result is sharp and equality holds in inequality (1.16) for $P(z) = az^n, a \neq 0$.

Substituting for $B[D_\alpha(P(Rz) - \beta P(z))]$, we have for $|z| \geq 1$,
\[
\left| \lambda_0 D_\alpha(P(Rz) - \beta P(z)) + \lambda_1 \left( \frac{mz}{2} \right) D_\alpha(P(Rz) - \beta P(z))' \right. \\
+ \lambda_2 \left( \frac{mz}{2} \right)^2 D_\alpha(P(Rz) - \beta P(z))'' \right| \\
\leq |\alpha|n|R^n - \beta| \lambda_0 z^{n-1} + \lambda_1 \left( \frac{(n-1)z}{2} \right) (n-1)z^{n-2} \\
+ \lambda_2 \left( \frac{(n-1)z}{2} \right)^2 (n-1)(n-2)z^{n-3} \left| \max_{|z|=1} |P(z)| \right|, \tag{1.17} \\
\]
where $0 \leq m \leq n - 1$ and $\lambda_0, \lambda_1$ and $\lambda_2$ are such that all the zeros of $u(z)$ defined by inequality (1.15) lie in the half plane $Re \ z \leq \frac{m}{4}$.

If, we choose $\beta = 0$ and let $R \to 1$ in inequality (1.16) we get the following result:

**Corollary 1.2.** If $P(z)$ is a polynomial of degree $n$, then for every real or complex number $\alpha$ with $|\alpha| \geq 1$,
\[ |B[D_\alpha P(z)]| \leq |\alpha|n|B[z^{n-1}]| \max_{|z|=1} |P(z)| \text{ for } |z| \geq 1. \]
The result is sharp and equality holds for the polynomial $P(z) = az^n$, $a \neq 0$. 

Remark 1.3. If, we choose \( \lambda_1 = 0 = \lambda_2 \) with \( \beta = 0 \) and letting \( R \to 1 \) inequality (1.17) will reduce to
\[
|D_\alpha P(z)| \leq |\alpha|n|z^{n-1}| \max_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1. \tag{1.18}
\]
Dividing both side of inequality (1.18) by \( |\alpha| \) and letting \( |\alpha| \to \infty \), inequality (1.18) will reduce to inequality (1.1).

Choosing \( \lambda_0 = 0 = \lambda_2 \) in inequality (1.17) will give the following result:

Corollary 1.4. If \( P(z) \) is a polynomial of degree \( n \), then for every real or complex numbers \( \alpha, \beta \) with \( |\alpha| \geq 1, |\beta| \leq 1 \) and \( R > 1 \),
\[
\left| \frac{m}{2} D_\alpha (P(Rz) - \beta P(z))' \right| \leq |\alpha|n|R^n - \beta| \left| \left( \frac{(n-1)^2}{2} \right) z^{n-2} \right| \max_{|z|=1} |P(z)|. \tag{1.19}
\]
Dividing both side of inequality (1.19) by \( |\alpha| \) and letting \( |\alpha| \to \infty \), then for \( m = n - 1 \) and for \( \beta = 0 \) and \( R \to 1 \), inequality (1.19) will reduce to,
\[
|P''(z)| \leq n(n-1)|z^{n-2}| \max_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1. \tag{1.20}
\]
The result is best possible and equality holds in inequality (1.20) for \( P(z) = az^n \).

We now prove the theorem which gives the extension of \([13, \text{Lemma (2.3)}]\) to the polar derivative.

Theorem 1.5. If \( P(z) \) is a polynomial of degree \( n \), then for every real or complex numbers \( \alpha, \beta \) with \( |\alpha| \geq 1, |\beta| \leq 1 \) and \( R > 1 \),
\[
|B[D_\alpha(P(Rz) - \beta P(z))]| + |B[D_\alpha(Q(Rz) - \beta Q(z))]| \\
\leq n(|\alpha||R^n - \beta| |B[z^{n-1}]| + |1 - \beta||\lambda_0|) \max_{|z|=1} |P(z)|, \tag{1.21}
\]
for \(|z| \geq 1\), where \( Q(z) = z^n P\left(\frac{1}{z}\right) \).

The result is best possible and the equality holds in inequality (1.21) for \( P(z) = z^n + 1 \). Substituting for \( B[D_\alpha(P(Rz) - \beta P(z))] \) in inequality (1.21), we have for \(|z| \geq 1\),
\[
\left| \lambda_0 D_\alpha(P(Rz) - \beta P(z)) + \lambda_1 \left( \frac{mz}{2} \right) D_\alpha(P(Rz) - \beta P(z))' \right. \\
+ \left. \lambda_2 \left( \frac{mz}{2} \right)^2 D_\alpha(P(Rz) - \beta P(z))'' + \left| \lambda_0 D_\alpha(Q(Rz) - \beta Q(z)) \right. \right. \\
+ \left. \lambda_1 \left( \frac{mz}{2} \right) D_\alpha(Q(Rz) - \beta Q(z))' + \lambda_2 \left( \frac{mz}{2} \right)^2 D_\alpha(Q(Rz) - \beta Q(z))'' \right|
\]
\[
\leq n\left\{ |\alpha| |R^n - \beta| \lambda_0 z^{n-1} + \lambda_1 \left( \frac{(n-1)z}{2} \right) (n-1)z^{n-2} + \lambda_2 \left( \frac{(n-1)z}{2} \right)^2 \frac{(n-1)(n-2)z^{n-3}}{2!} \left| 1 - \beta |\lambda_0| \right| \right\}_{|z|=1} \max |P(z)|, \tag{1.22}
\]

where \( 0 \leq m \leq n - 1 \) and \( \lambda_0, \lambda_1 \) and \( \lambda_2 \) are such that all the zeros of \( u(z) \) defined by inequality (1.15) lie in the half plane \( \text{Re} \ z \leq \frac{m}{2} \).

If, we choose \( \beta = 0 \) and let \( R \to 1 \) in inequality (1.21), we get the following extension of [13, Lemma (2.3)] to polar derivatives.

**Corollary 1.6.** If \( P(z) \) is a polynomial of degree \( n \), then for every real or complex numbers \( \alpha \) with \( |\alpha| \geq 1 \) and for \( |z| \geq 1 \)

\[
|B[D_\alpha P(z)]| + |B[D_\alpha Q(z)]| \leq n(\alpha|B[z^{n-1}]| + |\lambda_0|) \max |P(z)|,
\]

which implies

\[
|B[nP(z) + (\alpha - z)P'(z)]| + |B[nQ(z) + (\alpha - z)Q'(z)]| \leq n(|B[\alpha z^{n-1}]| + |\lambda_0| \max |P(z)|, \]

taking \( \alpha = z \) in the above inequality, we get [13, Lemma (2.3)] that is

\[
|B[P(z)]| + |B[Q(z)]| \leq (|B[z^n]| + |\lambda_0|) \max |P(z)| \text{ for } |z| \geq 1.
\]

Taking \( \lambda_1 = 0 = \lambda_2 \) with \( \beta = 0 \) and letting \( R \to 1 \) in inequality (1.22), we get the following result:

**Corollary 1.7.** If \( P(z) \) is a polynomial of degree \( n \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq 1 \),

\[
|D_\alpha P(z)| + |D_\alpha Q(z)| \leq n\{\alpha|z^{n-1}| + 1\} \max |P(z)| \text{ for } |z| \geq 1. \tag{1.23}
\]

Dividing both sides by \( |\alpha| \) and letting \( |\alpha| \to \infty \), inequality (1.23) will reduce to,

\[
|P'(z)| + |Q'(z)| \leq n|z^{n-1}| \max |P(z)| \text{ for } |z| \geq 1. \tag{1.24}
\]

The result is best possible and equality holds in inequality (1.24) for \( P(z) = z^n + 1 \). The above result is a special case of the result due to Govil and Rahman [8, Inequality (3.2)].

Taking \( \lambda_0 = 0 = \lambda_2 \) with \( \beta = 0 \) and letting \( R \to 1 \) in inequality (1.22), we get the following result:
Corollary 1.8. If \( P(z) \) is a polynomial of degree \( n \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq 1 \),

\[
m\{|D_\alpha P'(z)| + |D_\alpha Q'(z)|\} \leq n|\alpha|(n-1)^2|z|^{n-2}\max_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1. \tag{1.25}
\]

Dividing both sides by \( |\alpha| \) and letting \( |\alpha| \to \infty \), then \( m = n-1 \), inequality (1.25) will reduce to

\[
|P''(z)| + |Q''(z)| \leq n(n-1)|z|^{n-2}\max_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1. \tag{1.26}
\]

The result is best possible and equality holds in inequality (1.26) for \( P(z) = z^n + 1 \).

Next, we prove a result for the class of polynomials not vanishing in a unit disc and obtain compact generalization of inequalities (1.7). Infact we prove:

\[\textbf{Theorem 1.9. If } P(z) \text{ is a polynomial of degree } n \text{ which does not vanish in } |z| < 1, \text{ then for every real or complex numbers } \alpha, \beta \text{ with } |\alpha| \geq 1, |\beta| \leq 1 \text{ and } R > 1,\]

\[
|B[D_\alpha(P(Rz) - \beta P(z))]| \leq \frac{n}{2}\{|\alpha||R^n - \beta||B[z^{n-1}]| + |1 - \beta||\lambda_o|\} \max_{|z|=1} |P(z)|, \tag{1.27}
\]

for \( |z| \geq 1 \).

The result is best possible and equality in inequality (1.27) holds for \( P(z) = z^n + 1 \). Substituting for \( B[D_\alpha(P(Rz) - \beta P(z))] \) in inequality (1.27), we have for \( |z| \geq 1 \),

\[
\left|\lambda_oD_\alpha(P(Rz) - \beta P(z)) + \lambda_1\left(\frac{mz}{2}\right)D_\alpha(P(Rz) - \beta P(z))'\right|
\]

\[
+ \lambda_2\left(\frac{mz}{2}\right)^2\frac{D_\alpha(P(Rz) - \beta P(z))''}{2!} \right| \leq \frac{n}{2}\{|\alpha||R^n - \beta||\lambda_o|z^{n-1} + \lambda_1\left(\frac{(n-1)z}{2}\right)(n-1)|z|^{n-2}
\]

\[
+ \lambda_2\left(\frac{(n-1)z}{2}\right)^2\frac{(n-1)(n-2)z^{n-3}}{2!} + |1 - \beta||\lambda_o|\} \max_{|z|=1} |P(z)|, \tag{1.28}
\]

where \( 0 \leq m \leq n-1 \) and \( \lambda_o, \lambda_1 \) and \( \lambda_2 \) are such that all the zeros of \( u(z) \) defined by inequality (1.15) lie in the half plane \( \Re z \leq \frac{m}{4} \).

Remark 1.10. If we take \( \beta = 0 \) and let \( R \to 1 \), inequality (1.27) will reduce to the following result due to Bidkham and Mezerji [7].
If \( P(z) \) is a polynomial of degree at most \( n \), having no zero in \(|z| \leq 1\), then for every \( \alpha \) with \(|\alpha| \geq 1\),
\[
|B[D_\alpha P(z)]| \leq \frac{n}{2} \{ |\alpha| |B[z^{n-1}]| + |\lambda_0| \} \max_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1.
\]

**Remark 1.11.** If we take \( \lambda_1 = 0 = \lambda_2 \) with \( \beta = 0 \) and letting \( R \to 1 \), inequality (1.28) reduces to inequality (1.7) that is
\[
|D_\alpha P(z)| \leq \frac{n^2}{2} \{ |\alpha z^{n-1}| + 1 \} \max_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1.
\]

On dividing both sides of above inequality by \(|\alpha|\) and letting \(|\alpha| \to \infty\), we get inequality (1.3).

Choosing \( \lambda_0 = 0 = \lambda_2 \) with \( \beta = 0 \) and letting \( R \to 1 \) in inequality (1.28), we get the following result:

**Corollary 1.12.** If \( P(z) \) is a polynomial of degree \( n \) which does not vanish in \(|z| < 1\), then for every real or complex numbers \( \alpha, \beta \) with \(|\alpha| \geq 1\), \(|\beta| \leq 1\) and \( R > 1 \)
\[
|mD_\alpha P'(z)| \leq \frac{n(n-1)^2}{2} |\alpha| |z^{n-2}| \max_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1. \tag{1.29}
\]

Dividing both sides of inequality (1.29) by \(|\alpha|\) and letting \(|\alpha| \to \infty\), then \( m = n - 1 \) and we have
\[
|P''(z)| \leq \frac{n(n-1)}{2} |z^{n-2}| \max_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1. \tag{1.30}
\]

The result is best possible and equality in inequality (1.30) holds for \( P(z) = z^n + 1 \).

We now prove the following interesting result, which provides the compact generalisation of inequality (1.13).

**Theorem 1.13.** If \( P(z) \) is a polynomial of degree \( n \) having all its zeros in \(|z| \leq 1\), then for every real or complex numbers \( \alpha, \beta \) with \(|\alpha| \geq 1\), \(|\beta| \leq 1\) and \( R > 1 \)
\[
|B[D_\alpha (P(Rz) - \beta P(z))]| \geq |\alpha| n |R^n - \beta| |B[z^{n-1}]| \min_{|z|=1} |P(z)|, \tag{1.31}
\]

for \(|z| \geq 1\).

The result is sharp and equality holds in inequality (1.31) for \( P(z) = az^n \). Substituting for \( B[D_\alpha (P(Rz) - \beta P(z))] \), we have for \(|z| \geq 1\),
\[
|\lambda_0 D_\alpha (P(Rz) - \beta P(z)) + \lambda_1 \left( \frac{nz}{2} \right) D_\alpha (P(Rz) - \beta P(z))'| \]
Operators preserving inequalities between the polynomials

\[ + \lambda_2 \left( \frac{mz^2}{2} \right)^2 \frac{D_\alpha (P(Rz) - \beta P(z))''}{2!} \]

\[ \geq |\alpha| |n|R^n - \beta | \lambda_0 z^{n-1} + \lambda_1 \left( \frac{(n-1)z}{2} \right)(n-1)z^{n-2} \] \hspace{1cm} (1.32)

\[ + \lambda_2 \left( \frac{(n-1)z}{2} \right)^2 (n-1)(n-2)z^{n-3} \frac{2!}{2!} \left| \min_{|z|=1} |P(z)| \right| \]

where \( 0 \leq m \leq n - 1 \) and \( \lambda_0, \lambda_1, \lambda_2 \) are such that all the zeros of \( u(z) \) defined by (1.15) lie in the half plane \( \text{Re } z \leq \frac{m}{4} \).

**Remark 1.14.** If we take \( \beta = 0 \) and let \( R \to 1 \), inequality (1.31) will reduce to inequality (1.13).

Taking \( \lambda_1 = 0 = \lambda_2 \) with \( \beta = 0 \) and letting \( R \to 1 \) in inequality (1.32), we will get the following result from which result of Aziz and Dawood [2] follows as a special case.

**Corollary 1.15.** If \( P(z) \) is a polynomial of degree at most \( n \) having all its zeros in \( |z| \leq 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq 1 \),

\[ |D_\alpha P(z)| \geq n|\alpha||z^{n-1}| \min_{|z|=1} |P(z)| \] \hspace{1cm} for \( |z| \geq 1 \). \hspace{1cm} (1.33)

The result is best possible and equality holds in inequality (1.33) for \( P(z) = az^n \). Dividing the inequality (1.33) both sides by \( |\alpha| \) and letting \( |\alpha| \to \infty \), then \( m = n - 1 \), we obtain the inequality (1.5) as a special case.

Choosing \( \lambda_0 = 0 = \lambda_2 \) with \( \beta = 0 \) and letting \( R \to 1 \) in inequality (1.32), we get the following result:

**Corollary 1.16.** If \( P(z) \) is a polynomial of degree at most \( n \), having all its zeros in \( |z| \leq 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq 1 \),

\[ |mD_\alpha P'(z)| \geq n(n-1)^2 |\alpha||z^{n-2}| \min_{|z|=1} |P(z)|. \] \hspace{1cm} (1.34)

Dividing both sides of the inequality (1.34) by \( |\alpha| \) and letting \( |\alpha| \to \infty \), then \( m = n - 1 \) we obtain

\[ |P''(z)| \geq n(n-1)|z^{n-2}| \min_{|z|=1} |P(z)|. \] \hspace{1cm} (1.35)

The result is best possible and the equality holds in inequality (1.35) for \( P(z) = az^n \).

As an improvement of inequality (1.31) and generalisation of inequality (1.10), we prove the following result:
Theorem 1.17. If \( P(z) \) is a polynomial of degree at most \( n \) which does not vanish in \( |z| < 1 \), then for every real or complex numbers \( \alpha, \beta \) with \( |\alpha| \geq 1, |\beta| \leq 1 \) and \( R > 1 \)

\[
|B[D_\alpha(P(Rz) - \beta P(z))]| \leq \frac{n}{2} \left( \left| |\alpha| R^n - \beta |B[z^{n-1}]| + |\lambda_0| |1 - \beta| \right| \max_{|z|=1} |P(z)| 
- \left| |\alpha| R^n - \beta |B[z^{n-1}]| - |\lambda_0| |1 - \beta| \right| \min_{|z|=1} |P(z)| \right) \quad \text{for } |z| \geq 1.
\]

(1.36)

The result is sharp and equality in inequality (1.36) holds for the polynomial having all the zeros on the unit disk. Substituting for \( B[D_\alpha(P(Rz) - P(z))]| \) in inequality (1.36), we have for \( |z| \geq 1, \)

\[
|\lambda_0 D_\alpha(P(Rz) - \beta P(z)) + \lambda_1 \left( \frac{mz}{2} \right) D_\alpha(P(Rz) - \beta P(z))'| 
+ \lambda_2 \left( \frac{mz}{2} \right)^2 D_\alpha(P(Rz) - \beta P(z))''| 
\leq \frac{n}{2} \left( \left| |\alpha| R^n - \beta |\lambda_0 z^{n-1} + \lambda_1 \left( \frac{(n-1)z}{2} \right) (n-1)z^{n-2} 
+ \lambda_2 \left( \frac{(n-1)z}{2} \right)^2 \frac{(n-1)(n-2)z^{n-3}}{2!} \right| + |1 - \beta| |\lambda_0| \right| \max_{|z|=1} |P(z)| 
- \left| |\alpha| R^n - \beta |\lambda_0 z^{n-1} + \lambda_1 \left( \frac{(n-1)z}{2} \right) (n-1)z^{n-2} 
+ \lambda_2 \left( \frac{(n-1)z}{2} \right)^2 \frac{(n-1)(n-2)z^{n-3}}{2!} \right| - |1 - \beta| |\lambda_0| \right| \min_{|z|=1} |P(z)| \right),
\]

(1.37)

where \( 0 \leq m \leq n - 1 \) and \( \lambda_0, \lambda_1 \) and \( \lambda_2 \) are such that all the zeros of \( u(z) \) defined by (1.15) lie in the half place \( Re \ z \leq \frac{m}{4} \).

Remark 1.18. If we take \( \beta = 0 \) and letting \( R \to 1 \), inequality (1.36) will reduce to inequality (1.14).

Remark 1.19. Taking \( \lambda_1 = 0 = \lambda_2 \) with \( \beta = 0 \) and let \( R \to 1 \), inequality (1.37) will reduce to the following result due to Aziz and Shah [5].

If \( P(z) \) is a polynomial of degree \( n \) which does not vanish in \( |z| < 1 \), then for every complex number \( \alpha \) with \( |\alpha| \geq 1, \)

\[
\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n}{2} \left( (|\alpha| + 1) \max_{|z|=1} |P(z)| - (|\alpha| - 1) \min_{|z|=1} |P(z)| \right).
\]
Dividing both sides by $|\alpha|$ and letting $|\alpha| \to \infty$, in the above inequality, it follows that if $P(z) \neq 0$ in $|z| < 1$, then
\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right\}.
\]
The above result is an interesting refinement of Erdös-Lax theorem (inequality (1.3)) and was proved by Aziz and Dawood [2].

If we take $\lambda_o = 0 = \lambda_2$ with $\beta = 0$ and let $R \to 1$ in (1.37), we get the following result:

**Corollary 1.20.** If $P(z)$ is a polynomial of degree at most $n$, having no zero in $|z| \leq 1$, then for every $\alpha$ with $|\alpha| \geq 1$ and $|z| \geq 1$,
\[
|mD_\alpha P(z)| \leq \frac{n(n-1)^2}{2} |\alpha||z^{n-2}| \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right\}.
\] (1.38)

The result is best possible and equality holds in inequality (1.38) for $P(z) = z^n + 1$. Dividing both sides of the inequality (1.38) by $|\alpha|$ and letting $|\alpha| \to \infty$, then $m = n - 1$ and we get
\[
|P''(z)| \leq \frac{n(n-1)^2}{2} |z^{n-2}| \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right\}.
\] (1.39)

2. **Lemmas**

For the proof of above theorems we need the following lemmas. The first lemma follows from [9].

**Lemma 2.1.** If all the zeros of polynomial $P(z)$ of degree $n$ lie in $|z| \leq k$, where $k \leq 1$, then for $|\alpha| \geq k$, the polar derivative $D_\alpha [P(z)]$ of $P(z)$ at the point $\alpha$ also has all its zeros in $|z| \leq k$.

The following lemma which we need is in fact implicit in [11, Lemma 14.5.7, p.540].

**Lemma 2.2.** If all the zeros of the polynomial $P(z)$ of degree $n$ lie in a circle $|z| \leq 1$, then all the zeros of the polynomial $B|P(z)|$ also lie in $|z| \leq 1$.

As an application of Lemmas 2.1 and 2.2 we have the following lemma.

**Lemma 2.3.** If all the zeros of polynomial $P(z)$ of degree $n$ lie in $|z| \leq 1$, then for $|\alpha| \geq 1$, all the zeros of the polynomial $B[D_\alpha P(z)]$ also lie in $|z| \leq 1$.

**Proof.** From Lemma 2.1 for $k = 1$, all the zeros of the polynomial $D_\alpha P(z)$ lie in $|z| \leq 1$ and so from Lemma 2.2 the polynomial $B[D_\alpha P(z)]$ has all its zeros in $|z| \leq 1$. □
The next lemma is due to Aziz and Rather [3].

**Lemma 2.4.** If \( P(z) \) is a polynomial of degree at most \( n \) having all its zeros in \( |z| < k \), where \( k \leq 1 \), then \( |P(Rz)| > |P(z)| \), for \( |z| \geq 1 \) and \( R > 1 \).

**Lemma 2.5.** If \( P(z) \) is a polynomial of degree \( n \) which does not vanish in \( |z| < 1 \), then for every real or complex numbers \( \alpha, \beta \) with \( |\alpha| \geq 1, |\beta| \leq 1 \) and \( R \geq 1 \),

\[
|B[D\alpha(P(Rz) - \beta P(z))]| \leq |B[D\alpha(Q(Rz) - \beta Q(z))]|, \tag{2.1}
\]

for \( |z| \geq 1 \), where \( Q(z) = z^np(\frac{1}{z}) \).

**Proof.** For \( R = 1 \), the result reduces to Bidkham and Mezerji [7, Lemma 4, p.597]. Now we will prove the result for \( R > 1 \). Since all the zeros of \( P(z) \) lie in \( |z| \geq 1 \) and for every real or complex number \( \lambda \) with \( |\lambda| > 1 \), the polynomial \( G(z) = P(z) - \lambda Q(z) \), where \( Q(z) = z^np(\frac{1}{z}) \) has all its zeros in \( |z| \leq 1 \). Applying lemma 4 to the polynomial \( G(z) \) with \( k = 1 \), we get

\[
|G(z)| < |G(Rz)| \quad \text{for } |z| = 1 \text{ and } R > 1.
\]

Since all the zeros \( G(Rz) \) lie in \( |z| \leq \frac{1}{R} < 1 \), therefore for any real or complex number \( \beta \) with \( |\beta| \leq 1 \), the polynomial \( H(z) = G(Rz) - \beta G(z) \), has all its zeros in \( |z| < 1 \), for every \( \lambda \) with \( |\lambda| > 1 \) and \( R > 1 \), by Lemma 2.3 all the zeros of \( B[D\alpha H(z)] \) lie in \( |z| < 1 \). This implies

\[
B[D\alpha(G(Rz) - \beta G(z))]
= B[D\alpha(P(Rz) - \beta P(z))] - \lambda B[D\alpha(Q(Rz) - \beta Q(z))], \tag{2.2}
\]

for \( |z| \geq 1 \) and \( R > 1 \). Inequality (2.2) implies

\[
|B[D\alpha(P(Rz) - \beta P(z))]| \leq |B[D\alpha(Q(Rz) - \beta Q(z))]|, \tag{2.3}
\]

for \( |z| \geq 1 \) and \( R > 1 \). For if it is not true, then there is a point \( z = z_0 \) with \( |z_0| \geq 1 \), such that

\[
|B[D\alpha(P(Rz_0) - \beta P(z_0))]| \geq |B[D\alpha(Q(Rz_0) - \beta Q(z_0))]|, \tag{2.4}
\]

for \( |z| \geq 1 \) and \( R > 1 \). Since all the zeros of \( Q(z) \) lie in \( |z| \leq 1 \), therefore it follows that all the zeros of \( Q(Rz) - \beta Q(z) \), lie in \( |z| \leq 1 \) for every \( \beta \) with \( |\beta| \leq 1 \). Hence \( Q(Rz_0) - \beta Q(z_0) \neq 0 \), for \( |z_0| \geq 1 \). Which implies

\[
B[D\alpha(Q(Rz_0) - \beta Q(z_0))] \neq 0 \quad \text{for } |z| \geq 1 \text{ and } R > 1.
\]

We take

\[
\lambda = \frac{B[D\alpha(P(Rz_0) - P(z_0))]}{B[D\alpha(Q(Rz_0) - Q(z_0))]},
\]

so that \( |\lambda| > 1 \). Which shows that \( B[D\alpha H(z)] \) has a zero in \(|z| \geq 1 \). Which is contradiction to the fact that all the zeros of \( B[D\alpha H(z)] \) lie in \(|z| < 1 \). Thus

\[
|B[D\alpha(P(Rz) - \beta P(z))]| \leq |B[D\alpha(Q(Rz) - \beta Q(z))]|,
\]
for $|z| \geq 1$ and $R \geq 1$. □

3. Proof of Theorems

**Proof of Theorem 1.1.** Let $M = \max_{|z|=1} |P(z)|$, then $|P(z)| \leq M$ for $|z| = 1$. Therefore, by Rouche’s Theorem we have all the zeros of the polynomial $G(z) = P(z) + \lambda z^n M$, lie in $|z| < 1$ for every $\lambda$ with $|\lambda| > 1$. Now from Lemma 2.4, we have

$$|G(z)| < |G(Rz)| \text{ for } |z| = 1 \text{ and } R > 1.$$ 

Since all the zeros of $G(Rz)$ lie in $|z| < \frac{1}{R} < 1$, therefore if $\beta$ is any real or complex number with $|\beta| \leq 1$, we have all the zeros of the polynomial

$$G(Rz) - \beta G(z) = (P(Rz) - \beta P(z)) + \lambda(R^n - \beta)z^n M,$$

also lie in $|z| < 1$ for every $R > 1$ and $|\lambda| > 1$. Therefore by Lemma 2.3, all the zeros of $B[D_\alpha(G(Rz) - \beta G(z))]$, lie in $|z| < 1$ for every $R > 1$ and $|\lambda| > 1$. Which implies

$$B[D_\alpha(G(Rz) - \beta G(z))] = B[D_\alpha(P(Rz) - \beta P(z))] + \lambda n(R^n - \beta)MB[z^{n-1}], \quad \text{(3.1)}$$

for $|z| < 1$ and $R > 1$. Inequality (3.1) implies

$$|B[D_\alpha(P(Rz) - \beta P(z))]|$$

$$\leq |\alpha|n|R^n - \beta||B[z^{n-1}]|M \text{ for } |z| \geq 1 \text{ and } R > 1,$$ \quad \text{(3.2)}

for if this is not true, then there is a point $z = z_o$ with $|z_o| \geq 1$ such that

$$|B[D_\alpha(P(Rz_o) - \beta P(z_o))]| > |\alpha|n|R^n - \beta||B[z_o^{n-1}]|M.$$ 

We take

$$\lambda = -\frac{B[D_\alpha(P(Rz_o) - \beta P(z_o))]}{\alpha n(R^n - \beta)B[z_o^{n-1}]},$$

so that $|\lambda| > 1$, for this choice of $|\lambda|$, we have $B[D_\alpha(G(Rz_o) - \beta G(z_o))] = 0$ for $|z_o| \geq 1$. Which is a contradiction to the fact that all the zeros of $B[D_\alpha(G(Rz) - \beta G(z))]$ lie in $|z| < 1$. Thus

$$|B[D_\alpha(P(Rz) - \beta P(z))]| \leq \alpha n|R^n - \beta||B[z^{n-1}]|\max_{|z|=1} |P(z)|,$$

for $|z| \geq 1$ and $R > 1$. □

**Proof of Theorem 1.5.** Let $M = \max_{|z|=1} |P(z)|$, then $|P(z)| \leq M$ for $|z| = 1$. Now for every real or complex number $\gamma$ with $|\gamma| > 1$, it follows from Rouche’s Theorem, the polynomial $G(z) = P(z) + \gamma M$ does not vanish
in $|z| < 1$. Now applying Lemma 2.4 and 2.5 to the polynomial $G(z)$, we have for every real or complex number $\beta$ with $|\beta| \leq 1$,
\[ |B[D_\alpha(P(Rz) - \beta P(z) + \gamma(1 - \beta)M)]| \leq |B[D_\alpha(Q(Rz) - \beta Q(z) + \gamma(R^n - \beta)z^n M)]|, \tag{3.3} \]
for $|z| \geq 1$ and $R > 1$, where $Q(z) = z^n p(z)$. Inequality (3.3) implies
\[ |B[D_\alpha(P(Rz) - \beta P(z))] + n\gamma(1 - \beta)M\lambda_o| \leq |B[D_\alpha(Q(Rz) - \beta Q(z))] + \alpha n\gamma(R^n - \beta)B[z^{n-1}]M|, \tag{3.4} \]
for $|z| \geq 1$ and $R > 1$. Now choosing the argument of $\gamma$ on the R.H.S of inequality (3.4), such that
\[ |B[D_\alpha(Q(Rz) - \beta Q(z))]| + n\beta M\lambda_o| \leq |B[D_\alpha(Q(Rz) - \beta Q(z))]|, \tag{3.5} \]
for $|z| \geq 1$ and $R > 1$. Therefore we get from inequality (3.4),
\[ |B[D_\alpha(P(Rz) - \beta P(z))]| - |n\beta M\lambda_o| \leq |B[D_\alpha(Q(Rz) - \beta Q(z))]|, \tag{3.6} \]
for $|z| \geq 1$ and $R > 1$. Therefore, inequality (3.6) implies
\[ |B[D_\alpha(P(Rz) - \beta P(z))]| + |B[D_\alpha(Q(Rz) - \beta Q(z))]| \leq |\alpha|n\gamma||R^n - \beta||B[z^{n-1}]M + |B[D_\alpha(Q(Rz) - \beta Q(z))]|, \tag{3.7} \]
for $|z| \geq 1$ and $R > 1$. Letting $|\gamma| \to 1$, in inequality (3.7), we get
\[ |B[D_\alpha(P(Rz) - \beta P(z))]| + |B[D_\alpha(Q(Rz) - \beta Q(z))]| \leq n(|\alpha||R^n - \beta||B[z^{n-1}]M + |1 - \beta||\lambda_o|) \max_{|z|=1} |P(z)|, \]
for $|z| \geq 1$ and $R > 1$. Which proves the theorem. \[ \square \]

**Proof of Theorem 1.9.** We have from Lemma 2.5,
\[ |B[D_\alpha(P(Rz) - \beta P(z))]| \leq |B[D_\alpha(Q(Rz) - \beta Q(z))]|, \]
for $|z| \geq 1$ and $R \geq 1$, where $Q(z) = z^n p(z)$. Also from Theorem 1.5, we have
\[ |B[D_\alpha(P(Rz) - \beta P(z))]| + |B[D_\alpha(Q(Rz) - \beta Q(z))]| \leq n(|\alpha||R^n - \beta||B[z^{n-1}]M + |1 - \beta||\lambda_o|) \max_{|z|=1} |P(z)|,
We take that $z$ does not vanish in $\mathbb{R}$ or complex number for $1$, therefore if $m$ Proof of Theorem 1.17. □

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Which is a contradiction to the fact that all the zeros of $B$ from Lemma 2.3, all the zeros of $n$ of degree $\min$ Proof of Theorem 1.13. If $P(z)$ has a zero on $|z| = 1$, then the result is trivial. So we suppose that $P(z)$ has all its zeros in $|z| < 1$. If $m = \min_{|z|=1} |P(z)|$, then $m > 0$ and $m \leq |P(z)|$ for $|z| = 1$. Therefore, if $\gamma$ is any complex number with $|\gamma| < 1$, we have the polynomial $G(z) = P(z) - \gamma m z^n$ of degree $n$ has all its zeros in $|z| < 1$. Now from Lemma 2.4, we have $|G(z)| < |G(Rz)|$ for $|z| = 1$ and $R > 1$.

Since all the zeros of $G(Rz)$ lie in $|z| < \frac{1}{R} < 1$, therefore for any real or complex number $\beta$ with $|\beta| \leq 1$ and $R > 1$, it follows from Rouche’s Theorem, the polynomial $H(z) = G(Rz) - \beta G(z)$ has all its zeros in $|z| < 1$. Therefore from Lemma 2.3, all the zeros of $B[D_B H(z)]$ lie in $|z| < 1$. This implies

$$B[D_B(G(Rz) - \beta G(z))] = B[D_B(P(Rz) - \beta P(z))] - \alpha n \gamma (R^n - \beta) B[z^{n-1}] m,$$

so that $|\gamma| < 1$. For this choice of $|\gamma|$, we have $B[D_B H(z)] = 0$, for $|z| \geq 1$. Which is a contradiction to the fact that all the zeros of $B[D_B H(z)]$ lie in $|z| < 1$. Thus we have $|B[D_B(\beta P(z))]| \geq |\alpha n |R^n - \beta| |B[z^{n-1}]| \min_{|z|=1} |P(z)|$.

Hence the theorem follows. □

Proof of Theorem 1.17. Since the polynomial $P(z)$ does not vanish in $|z| < 1$, therefore if $m = \min_{|z|=1} |P(z)|$, then $m \leq |P(z)|$ for $|z| \leq 1$. Now for any real or complex number $\lambda$ with $|\lambda| \leq 1$, the polynomial $G(z) = P(z) + \lambda m z^n$ does not vanish in $|z| < 1$. For if this is not true, then there is a point $z = z_o$. for $|z| \geq 1$ and $R > 1$. Which proves the theorem.
with \(|z_0| < 1\), such that \(G(z_0) = P(z_0) + \lambda m z_0^n = 0\). Which implies \(|P(z_0)| = |m \lambda z_0^n| \leq m |z_0|^n < m\), contradicting the fact that \(m \leq |P(z)|\) for \(|z| \leq 1\). Thus \(G(z)\) has no zero in \(|z| < 1\) for every \(\lambda\) with \(|\lambda| \leq 1\). Applying Lemma 2.5 to the polynomial \(G(z)\), we have for \(|\beta| \leq 1\) and \(R > 1\),

\[
|B[D_\alpha(P Rz) - \beta P(z)] + (R^n - \beta) \lambda m z^n| \\
\leq |B[D_\alpha(Q Rz) - \beta Q(z)] + (1 - \beta) \lambda m|,
\]

(3.10)

for \(|z| \geq 1\) and \(R > 1\), where \(Q(z) = z^n p(\frac{1}{z})\). Inequality (3.10) implies

\[
|B[D_\alpha(P Rz) - \beta P(z)] + \alpha(R^n - \beta) \lambda mn R[z^{n-1}]m| \\
\leq |B[D_\alpha(Q Rz) - \beta Q(z)] + \alpha n|R^n - \beta| B[z^{n-1}]m|,
\]

(3.11)

for \(|z| \geq 1\) and \(R > 1\). Choosing \(\lambda\) in inequality (3.11) such that

\[
|B[D_\alpha(P Rz) - \beta P(z)] + \alpha n|R^n - \beta| B[z^{n-1}]m| \\
= |B[D_\alpha(P Rz) - \beta P(z)]| + |\alpha n|R^n - \beta| B[z^{n-1}]m|,
\]

(3.12)

for \(|z| \geq 1\) and \(R > 1\). Inequality (3.12) implies

\[
|B[D_\alpha(P Rz) - \beta P(z)]| + |\alpha n|R^n - \beta| B[z^{n-1}]m| \\
\leq |B[D_\alpha(Q Rz) - \beta Q(z)]| + n|\lambda_0| R^n - \beta| R[z^{n-1}]m|,
\]

(3.13)

for \(|z| \geq 1\) and \(R > 1\). Inequality (3.13) implies

\[
|B[D_\alpha(P Rz) - \beta P(z)]| \\
\leq |B[D_\alpha(Q Rz) - \beta Q(z)]| + n|\lambda_0| R^n - \beta| R[z^{n-1}]m|,
\]

(3.14)

for \(|z| \geq 1\) and \(R > 1\). Letting \(|\lambda| \to 1\), we have for \(|z| \geq 1\) and \(R > 1\)

\[
|2|B[D_\alpha(P Rz) - \beta P(z)]| \\
\leq |B[D_\alpha(P Rz) - \beta P(z)]| + |B[D_\alpha(Q Rz) - \beta Q(z)]| + n|\lambda_0| R^n - \beta| B[z^{n-1}]m|,
\]

(3.15)

for \(|z| \geq 1\) and \(R > 1\). Applying Theorem 1.5, we get from inequality (3.15)

\[
|B[D_\alpha(P Rz) - \beta P(z)]| \\
\leq \frac{n}{2} \left\{ |\alpha|R^n - \beta| B[z^{n-1}]| + |\lambda_0| R^n - \beta| R[z^{n-1}]m| \right\} \max_{|z|=1} |P(z)| \\
- \left\{ |\alpha|R^n - \beta| B[z^{n-1}]| - |\lambda_0| R^n - \beta| R[z^{n-1}]m| \right\} \min_{|z|=1} |P(z)|
\]

(1.36)

for \(|z| \geq 1\).

Hence the Theorem follows. \(\square\)
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