Solitary-waves of the Nonpolynomial Schrodinger Equation:
Bright Solitons in Bose-Einstein Condensates

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Abstract

Recently we have derived an effective one-dimensional nonpolynomial Schrödinger equation (NPSE) [Phys. Rev. A 65, 043614 (2002)] that accurately describes atomic Bose-Einstein condensates under transverse harmonic confinement. In this paper we analyze the stability of the solitary-wave solutions of NPSE by means of the Vakhitov-Kolokolov criterion. Stable self-focusing solutions of NPSE are precisely the 3D bright solitons experimentally found in Bose-Einstein condensates [Nature 417, 150 (2002); Science 296, 1290 (2002)]. These self-focusing solutions generalize the ones obtained with the standard nonlinear Schrödinger equation (NLSE) and take into account the formation of instabilities in the transverse direction. We prove that neither cubic nor cubic-quintic approximations of NPSE are able to reproduce the unstable branch of the solitary-wave solutions. Moreover, we show that the analytically found critical point of NPSE is in remarkable good agreement with recent numerical computations of 3D GPE. We also discuss the formation of multiple solitons by a sudden change of the sign of the nonlinear strength. This formation can be interpreted in terms of the modulational instability of the time-dependent macroscopic wave function of the Bose condensate. In
particular, we derive an accurate analytical formula for the number of multiple bright solitons.

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I. INTRODUCTION

A nonlinear Schrödinger equation with nonpolynomial nonlinearity, the so-called non-polynomial Schrödinger equation (NPSE), has been recently introduced and studied [1-3]. This equation has been derived in the context of Bose-Einstein condensation [4,5] from the 3D Gross-Pitaevskii equation [6,7] as a reliable effective 1D equation which describes the axial wavefunction $\psi(x,t)$ of a cigar-shaped condensate confined in the transverse direction $(y,z)$ by a harmonic trapping potential. NPSE has been successfully used to numerically investigate [8] the dynamics of Bose condensate solitary-waves experimentally observed by two groups [9,10]. In this letter we rigorously determine the linear stability domain of the solitary-wave solutions of NPSE by using the Vakhitov-Kolokolov criterion [11,12]. In particular, we find that NPSE admits both stable and unstable stationary solitary-wave solutions.

II. GENERALIZED SCHRÖDINGER EQUATIONS

For the generalized nonlinear Schrödinger equation (GNLSE), given by

$$\left[ i \frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} + F(|\psi|^2) \right] \psi = 0 , \quad (1)$$

where $F(|\psi|^2)$ is the nonlinear term, the linear stability of solitary-waves is determined by the Vakhitov-Kolokolov criterion [11-13]. With the standard position

$$\psi(x,t) = \phi(x) e^{i\omega t} , \quad (2)$$

where $\omega$ is the frequency of the phase and $\phi(x)$ is a real scalar field, from GNLSE one obtains the stationary second-order differential equation

$$\left[ \frac{1}{2} \frac{d^2}{dx^2} + F(\phi^2) \right] \phi = \omega \phi . \quad (3)$$

The first derivative $N'_s(\omega)$ of the number of particles invariant

$$N_s(\omega) = \int dx \phi^2(x) , \quad (4)$$
calculated for the soliton solution $\phi(x)$, allows one to predict the parameter region in $\omega$ where the soliton amplitude can grow or decay exponentially with a nonzero growth rate. The solitons are stable if $N'_s(\omega) > 0$ and unstable if $N'_s(\omega) < 0$ [13].

III. NONPOLYNOMIAL SCHRÖDINGER EQUATION

In the case of the self-focusing nonpolynomial Schrödinger equation (NPSE) [1-3] the nonlinearity of Eq. (1) and Eq. (3) is given by

$$F(\phi^2) = \frac{g\phi^2}{\sqrt{1-g\phi^2}} - \frac{1}{2} \left( \frac{1}{\sqrt{1-g\phi^2}} + \sqrt{1-g\phi^2} \right),$$

(5)

where $g$ is the strength of the attractive interaction between particles in the Bose-Einstein condensate. In particular, the parameter $g$ is related to the s-wave scattering length $a_s$ and to the characteristic length $a_\perp$ of the transverse harmonic confinement by $g = 2|a_s|/a_\perp$ [1-3,8].

Given a GNLSE like Eq. (3) it is quite natural to try a power expansion of $F(\phi^2)$. In the case of NPSE one finds

$$F(\phi^2) = -1 + g\phi^2 + \frac{3}{8}g^2\phi^4 + \ldots.$$  

(6)

The lowest order in $g\phi^2$ gives the familiar one-dimensional self-focusing cubic NLSE. By including the next term of the power expansion one has a cubic-quintic NLSE, which has been investigated in several papers (see for instance [13]). But, as we shall show, the cubic-quintic approximation of NPSE is not able to correctly describe the stability of NPSE solitary waves.

The exact stationary solitary-wave solutions of NPSE can be obtained in the following way. A simple constant of motion of the stationary NPSE is

$$E = \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + V(\phi),$$

(7)

where
\[ V(\phi) = -\omega \phi^2 - \phi^2 \sqrt{1 - g\phi^2} \]  

(8)

is the effective potential energy of the system. In Figure 1 we plot the potential energy \( V(\phi) \) of the full NPSE and its cubic and cubic-quintic approximations. As shown by Figure 1, for \( \omega < 0 \) \( V(\phi) \) is a double-well potential and the two approximations are quite close to the exact curve, apart near \( \phi = \pm 1/\sqrt{g} \), the end-points of the NPSE which are related to the singularity of the term \((1 - g\phi^2)^{1/2}\). By imposing the boundary condition \( \phi \to 0 \) for \( x \to \infty \) one finds \( E = 0 \) and so

\[ \frac{d\phi}{dx} = \sqrt{2\phi^2 \sqrt{1 - g\phi^2} + 2\omega \phi^2} . \]  

(9)

The previous formula gives the integral equation

\[ \int dx = \int d\phi \frac{1}{\sqrt{2\phi^2 \sqrt{1 - g\phi^2} + 2\omega \phi^2}} , \]  

(10)

from which one obtains the solitary-wave solution \( \phi(x) \) written in implicit form (see also [8])

\[ \sqrt{2x} = \frac{1}{\sqrt{1 + \omega}} \text{ arcth } \left[ \sqrt{\frac{1 - g\phi^2 + \omega}{1 + \omega}} \right] - \frac{1}{\sqrt{1 - \omega}} \text{ arctg } \left[ \sqrt{\frac{1 - g\phi^2 + \omega}{1 - \omega}} \right] . \]  

(11)

IV. VAKHITOV-KOLOKOLOV STABILITY CRITERION

The number of particles \( N_s(\omega) \) of Eq. (4) can be calculated by observing that

\[ dx = d\phi \left( \frac{d\phi}{dx} \right)^{-1} , \]  

(12)

from which one gets

\[ N_s(\omega) = \int d\phi \frac{\phi}{\sqrt{2\sqrt{1 - g\phi^2} + 2\omega}} . \]  

(13)

This integral is analytically solvable and one obtains

\[ N_s(\omega) = \frac{2\sqrt{2}}{3g} (1 - 2\omega) \sqrt{1 + \omega} . \]  

(14)
In Figure 2 we plot the function $N_s(\omega)$ for some values of the nonlinear strength $g$. The Vakhitov-Kolokolov criterion [11,12] is based on the study of the sign of the first derivative

$$N'_s(\omega) = -\frac{\sqrt{2} (1 + 2\omega)}{g \sqrt{1 + \omega}}$$

(15)

of the number of particle $N_s(\omega)$. One easily finds that the solitary-wave $\phi(x)$ given by Eq. (11) is linearly stable for $-1 < \omega < -1/2$ and unstable for $-1/2 < \omega < 1/2$. Moreover $N_s(\omega)$ has its maximum value $N_{s,max} = 4/(3g)$ at the critical frequency $\omega = -1/2$.

In Figure 2 we also plot $N_s(\omega)$ obtained from the cubic NLSE and the cubic-quintic NLSE. In the case of cubic NLSE one finds

$$N_s(\omega) = \frac{2\sqrt{2}}{g} \sqrt{1 + \omega},$$

(16)

from which one immediately obtains that the solitary-wave solutions are always stable. In the case of the cubic-quintic NLSE the number of particles is instead given by

$$N_s(\omega) = \frac{2}{g} \left( \frac{\pi}{2} - \arctg \left[ \frac{\sqrt{2}}{2\sqrt{1 + \omega}} \right] \right).$$

(17)

Remarkably also in this case the Vakhitov-Kolokolov stability criterion tells us that there are no unstable solitary-waves. Thus the quintic term in the power expansion of NPSE is not sufficient to determine the stability domain of NPSE.

As previously stated, in the context of Bose-Einstein condensation [1-3] the nonlinear strength $g$ is given by $g = 2|a_s|/a_\perp$. By using this expression one finds that the maximum value of $N_s(\omega)$ for the full NPSE is given by

$$N_{s,max} = \frac{2}{3} \frac{a_\perp}{|a_s|}.$$ 

(18)

This value is very close to the critical number $N_c$ of attractive Bose-condensed atoms under transverse harmonic confinement found in [14] with a full numerical solution of the 3D GPE:

$$N_c = 0.676 \frac{a_\perp}{|a_s|}.$$ 

(19)

Beyond $N_c$ there is the so-called collapse of the Bose-Einstein condensate. The collapse is due to the fact that the solitary-wave solution becomes linearly unstable, i.e. $N_s(\omega) < 0$. 
in our NPSE. Note that this collapse has nothing to do with the singularity of the term \((1 - g\phi^2)^{1/2}\) that is present in NPSE: in fact, also at the collapse the condition \(g\phi^2 < 1\) is satisfied.

V. MODULATIONAL INSTABILITY

In a recent experiment [9] it has been reported the formation of bright solitons in a Bose-Einstein condensate of \(^{7}\text{Li}\) atoms induced by a sudden change in the sign of the scattering length from positive to negative. The formation of these solitons can be explained as due to the modulational instability of the time-dependent wave function of the Bose condensate, driven by imaginary Bogoliubov excitations [15].

Let us consider a Bose condensate with a repulsive inter-atomic interaction \((a_s > 0)\). The condensate is described by Eq. (1) and Eq. (5), setting \(g' = -g\) instead of \(g\) in Eq. (5). Under box axial confinement, the stationary homogeneous wave function is given by

\[
\phi(x) = \sqrt{\frac{N}{L}}
\]

for \(-L/2 < x < L/2\) and zero elsewhere. \(L\) is the size of the confining box and \(N\) the total number of bosons. The Bogoliubov elementary excitations \(\epsilon_k\) of the static Bose condensate \(\phi(x)\) are found by looking for solutions of the form

\[
\psi(x, t) = e^{i\omega t} \left[ \phi(x) + u_k(x) e^{-i\epsilon_k t} + v_k^*(x) e^{i\epsilon_k t} \right],
\]

and keeping terms linear in the complex functions \(u(x)\) and \(v(x)\) (linear-stability analysis). In the quasi-1D limit (defocusing NLSE) one finds

\[
\epsilon_k = \sqrt{\frac{k^2}{2} \left( \frac{k^2}{2} + 2g'n \right)},
\]

where \(\omega = -g'n\), \(n = N/L\) is the axial density and \(g' = 2a_s/a_\perp\).

By suddenly changing the scattering length \(a_s\) to a negative value, the excitations frequencies corresponding to
\( k < k_c = \sqrt{16\pi|g'|n} \)  \hspace{1cm} (23)

become imaginary and, as a result, small perturbations grow exponentially in time. This phenomenon is known as modulational instability [16,17]. It is easy to find that the maximum rate of growth is at \( k_0 = k_c/\sqrt{2} \). The wavelength of this mode is \( \lambda_0 = 2\pi/k_0 \) and the ratio \( L/\lambda_0 \) gives an estimate of the number \( N_{BS} \) of bright solitons which are generated:

\[
N_{BS} = \frac{\sqrt{N|a_s|L}}{\pi a_\perp}.
\]  \hspace{1cm} (24)

The predicted number \( N_{BS} \) of bright solitons is in very good agreement with the numerical results of 3D GPE [18] and in rough agreement with the experimental results [9]. Note that the finite resolution of the imaging process reduces the number of detected solitons [18].

Finally, we observe that Eq. (24) has been obtained by using NLSE. NPSE gives the same results of NLSE in the quasi-1D limit (of experiment [9]) but predicts the formation of a large number of solitons also close to the critical point \( gn = 1 \). The consequences of this prediction will be discussed elsewhere.

**CONCLUSIONS**

In this paper we have shown that the solitary-waves studied in [8] belong to the stable branch \(-1 < \omega < -1/2\) we have here analytically found by using the Vakhitov-Kolokolov criterion. In this branch, growing \( \omega \) the number of particles \( N_s \) increases and for a fixed \( N_s \) the solitary-wave solution is fully determined. Since the study of the 3D model is usually time consuming, it is nice to have a simple 1D model to understand the dynamics of a Bose-Einstein condensate. NPSE has the same numerical complexity of other one-dimensional nonlinear Schrödinger equations but we have shown that it is not well approximated by the cubic or cubic-quintic Schrödinger equations. NPSE reflects correctly the 3D dynamics of Bose condensed solitary-waves and it predicts with considerable accuracy the collapse of the condensate. Moreover, we have shown the NPSE explains remarkably well the formation of multiple bright solitons by a sudden change of the sign of the nonlinear strength from
positive to negative, as a consequence the modulational instability of the time-dependent wave function.
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FIG. 1. Effective potential energy $V(\phi)$ of the NPSE. Exact: full line; cubic approximation: dotted line, cubic-quintic approximation: dashed line. $\omega$ is the phase frequency and $g$ is the nonlinear strength.
FIG. 2. Number of particles $N_s(\omega)$ as a function of the phase frequency $\omega$ for the NPSE. Exact: full line; cubic approximation: dotted line; cubic-quintic approximation: dashed line. $g$ is the nonlinear strength.