Boundary Lipschitz Regularity of Solutions for Semilinear Elliptic Equations in Divergence Form

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Abstract In this paper, we consider the pointwise boundary Lipschitz regularity of solutions for the semilinear elliptic equations in divergence form mainly under some weaker assumptions on nonhomogeneous term and the boundary. If the domain satisfies $C^{1, \text{Dini}}$ condition at a boundary point, and the nonhomogeneous term satisfies Dini continuity condition and Lipschitz Newtonian potential condition, then the solution is Lipschitz continuous at this point. Furthermore, we generalize this result to Reifenberg $C^{1, \text{Dini}}$ domains.

Keywords Boundary Lipschitz regularity, semilinear elliptic equation, Dini condition, Reifenberg domain

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1 Introduction

In this paper, we will investigate the boundary Lipschitz regularity of solutions for the following semilinear elliptic equation in divergence form:

$$\begin{cases} 
\Delta u = \text{div} \mathbf{F}(x, u) & \text{in } \Omega, \\
 u = g & \text{on } \partial \Omega,
\end{cases}$$

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^n$. 

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1) Corresponding author
There is a complete regularity theory for the classical Poisson equation
\[ \Delta u = f. \]  

(1.2)

For the interior regularity of solutions, as we all know, \( u \) is \( C^{2,\alpha} \) for some \( 0 < \alpha < 1 \) when \( f \) is \( C^\alpha \), and \( u \) is \( C^2 \) when \( f \) is Dini continuous, see [2, 6]. Moreover, if \( f \in L^\infty \), then any weak solution \( u \) satisfies \( u \in W^{2,p}_{loc} \) for any \( 1 \leq p < +\infty \), and consequently \( u \in C^{1,\alpha} \) for \( 0 < \alpha < 1 \) but not for \( \alpha = 1 \). So it is clear that \( f \in L^\infty \) or even \( f \) continuous is not strong enough to assure the \( C^{1,1} \)-regularity. Recently research activity has thus focused on identifying conditions on \( f \) which ensure \( W^{2,\infty}_{loc} \) or \( C^{1,1}_{loc} \) regularity of \( u \). In [1], Andersson, Lindgren and Shahgholian showed that the sharp condition to get the \( C^{1,1} \) regularity of \( u \) is that \( f \ast N \) is \( C^{1,1} \) which is slightly weaker than the Dini condition, where \( N \) is the Newtonian potential and \( \ast \) denotes the convolution.

For the \( C^{1,1} \) regularity of solutions for the semilinear elliptic equation
\[ \Delta u = f(x, u) \quad \text{in } B_1, \]

which are derived from the obstacle problem, Shahgholian in [13] proved \( u \) is \( C^{1,1} \) if \( f(x, u) \) is Lipschitz continuous in \( x \), uniformly in \( u \), and \( \partial_u f \geq -C \). Recently Indrei, Minne and Nurbekyan in [5] obtained the same result under weaker assumptions that \( f(x, u) \) is Dini continuous in \( u \), uniformly in \( x \), and it has a \( C^{1,1} \) Newtonian potential in \( x \), uniformly in \( u \).

With respect to the boundary regularity of solutions, there has been extensive study in the past two decades which is closely related to the regularity of the boundary, such as sphere condition and \( C^{1,1,\text{Dini}} \) condition. It is well-known that the solution of (1.2) is \( C^\alpha \) up to the boundary when \( \partial \Omega \) is Lipschitz. Trudinger in [14, 15] proved the boundary Lipschitz regularity when the boundary satisfies uniform exterior sphere condition. Li and Wang in [7, 8] studied the following Dirichlet problem,
\[
\begin{align*}
- a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} &= f(x) & \text{in } \Omega, \\
u &= g & \text{on } \partial \Omega.
\end{align*}
\]

(1.3)

They proved that the solution is differentiable at any point on the boundary if the domain is convex. Li and Zhang in [9] got the boundary differentiability of (1.3) when \( g = 0 \) under the \( \gamma \)-convexity domain condition which is strictly weaker than the convexity condition. On the other hand, the Dini continuity was a current topic for the regularity theory. In 2011, Ma and Wang [11] considered the following equation,
\[
\begin{align*}
F(D^2 u(x), x) &= f(x) & \text{in } \Omega, \\
u &= g & \text{on } \partial \Omega.
\end{align*}
\]

(1.4)

They showed the pointwise \( C^1 \) estimates up to the boundary under the Dini conditions including that \( \partial \Omega \) is \( C^{1,\text{Dini}} \) (see Definition 1.1) and the boundary value \( g \) is \( C^{1,\text{Dini}} \). In [3], Huang, Li and Wang proved that the solution of (1.3) is Lipschitz if the boundary satisfies exterior \( C^{1,\text{Dini}} \) condition and the solution is differentiable if the boundary is exterior \( C^{1,\text{Dini}} \) and punctually \( C^1 \). Furthermore, in [4] they extended their results to Reifenberg \( C^{1,\text{Dini}} \) domain which is more general than the classical \( C^{1,\text{Dini}} \) domain.
However, there are few results on the boundary behavior of solutions for semilinear elliptic equations in the divergence form. In this paper, we study the pointwise boundary Lipschitz regularity of solutions for the semilinear elliptic equation in divergence form under some weaker assumptions on \( F(x,u) \) and the boundary. For convenience we give some notations and definitions of Dini condition.

**Notations:**
- \( |x| := \sqrt{\sum_{i=1}^{n} x_i^2} \): the Euclidean norm of \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \).
- \( |x'| := \sqrt{\sum_{i=1}^{n-1} x_i^2} \): the Euclidean norm of \( x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \).
- \( B_r(x_0) := \{ x \in \mathbb{R}^n : |x - x_0| < r \} \).
- \( B_r := \{ x \in \mathbb{R}^n : |x| < r \} \).
- \( \Omega_r := B_r \cap \Omega \).
- \( T_r := B_r \cap \{ x_n = 0 \} = \{ (x', 0) \in \mathbb{R}^n : |x'| < r \} \).
- \( \bar{\boldsymbol{a}} \cdot \bar{b} \): the standard inner product of \( \bar{a}, \bar{b} \in \mathbb{R}^n \).
- \( \{ e_i \}_{i=1}^n \): the standard basis of \( \mathbb{R}^n \).

**Definition 1.1** Let \( x_0 \in \partial \Omega \). We say that \( \partial \Omega \) is \( C^{1, \text{Dini}} \) at \( x_0 \), if there exists a unit vector \( \bar{n} \) and a positive constant \( r_0 \), a Dini modulus of continuity \( \omega(r) \) satisfying \( \int_0^{r_0} \frac{\omega(r)}{r} dr < \infty \) such that for any \( 0 < r \leq r_0 \),

\[
B_r(x_0) \cap \{ x \in \mathbb{R}^n : (x - x_0) \cdot \bar{n} > r \omega(r) \} \subset B_r(x_0) \cap \Omega \\
\subset B_r(x_0) \cap \{ x \in \mathbb{R}^n : (x - x_0) \cdot \bar{n} > -r \omega(r) \}.
\]

**Remark 1.2** Any modulus of continuity \( \omega(t) \) is non-decreasing, subadditive, continuous and satisfies \( \omega(0) = 0 \) (see [16]). Hence any modulus of continuity \( \omega(t) \) satisfies

\[
\frac{\omega(r)}{r} \leq 2 \frac{\omega(h)}{h}, \quad 0 < h < r. \tag{1.5}
\]

**Definition 1.3** (Reifenberg \( C^{1, \text{D ini}} \) condition) Let \( x_0 \in \partial \Omega \). We say that \( \Omega \) satisfies the \((r_0, \omega)\)-Reifenberg \( C^{1, \text{Dini}} \) condition at \( x_0 \) if there exists a positive constant \( r_0 \) and a Dini modulus of continuity \( \omega(r) \) satisfying \( \int_0^{r_0} \frac{\omega(r)}{r} dr < \infty \) such that for any \( 0 < r \leq r_0 \), there exists a unit vector \( \bar{n}_r \in \mathbb{R}^n \) such that

\[
B_r(x_0) \cap \{ x \in \mathbb{R}^n : (x - x_0) \cdot \bar{n}_r > r \omega(r) \} \subset B_r(x_0) \cap \Omega \\
\subset B_r(x_0) \cap \{ x \in \mathbb{R}^n : (x - x_0) \cdot \bar{n}_r > -r \omega(r) \}.
\]

**Remark 1.4** Definitions 1.1 and 1.3 mean that for any \( 0 < r \leq r_0 \), \( B_r(x_0) \cap \partial \Omega \) lies between two parallel hyperplanes perpendicular to \( \bar{n} \) or \( \bar{n}_r \) with the distance of \( 2r \omega(r) \). As for the Dini conditions on the domain, we refer the reader to the papers [3, 4, 11].

The following lemma can be found in [4].

**Lemma 1.5** If \( \Omega \) satisfies \((r_0, \omega)\)-Reifenberg \( C^{1, \text{Dini}} \) condition, then there exists a bounded nonnegative function \( S(\theta) \geq 1 \) such that \( |\bar{n}_r - \bar{n}_{r_0}| \leq S(\theta) \omega(r) \) for each \( 0 < \theta < 1 \) and \( 0 < r \leq r_0 \). Furthermore, for a fixed positive constant \( 0 < \lambda < 1 \), \( \{ \bar{n}_{r_0/\lambda} \}_{r_0}^{\infty} \) is a Cauchy sequence. We can set \( \lim_{i \to \infty} \bar{n}_{r_0/\lambda} = \bar{n}_* \).

**Definition 1.6** Let \( x_0 \in \partial \Omega \). The boundary value \( g \) is said to be \( C^{1, \text{Dini}} \) at \( x_0 \) with respect to a function \( v_{x_0}(x) \), if there exists a constant vector \( \bar{a} \), a positive constant \( r_0 \) and a Dini modulus...
of continuity $\sigma(r)$ satisfying $\int_0^{r_0} \frac{\sigma(r)}{r} \, dr < \infty$ such that for any $0 < r \leq r_0$ and $x \in \partial \Omega \cap B_r(x_0)$,
\[ |g(x) - v_{x_0}(x) - (g(x_0) - v_{x_0}(x_0)) - \bar{a} \cdot (x - x_0)| \leq r \sigma(r). \]

Next we propose the following assumptions on $\overrightarrow{F}(x, u)$.

**Assumption 1** $\overrightarrow{F}(x, t) \in L^\infty(\mathbb{B}_d \times \mathbb{R})$ where $d$ is large enough. Moreover $\overrightarrow{F}(x, t)$ is Dini continuous in $t$ with continuity modulus $\omega_1(r)$, uniformly in $x$, i.e.,
\[ |\overrightarrow{F}(x, t_2) - \overrightarrow{F}(x, t_1)| \leq \omega_1(|t_2 - t_1|), \]
and $\int_0^{t_0} \frac{\omega_1(t)}{t} \, dt < \infty$, for some $t_0 > 0$.

**Assumption 2** For every boundary point $x_0$ and each $t \in \mathbb{R}$, there exists a function $v_{x_0}(\cdot, t)$ in $B_1(x_0)$ satisfying
\[ \Delta v_{x_0}(\cdot, t) = \text{div} \overrightarrow{F}(\cdot, t) \text{ in } B_1(x_0). \]
Furthermore, $v_{x_0}(\cdot, t)$ is a Lipschitz function which is uniform in $x_0$ and $t$ with Lipschitz constant $T$.

**Remark 1.7** (1) We can always assume that $\Omega \subset B_{\frac{d}{4}}$.

(2) In the sequel, for $g(x) \in L^\infty(\partial \Omega)$ is the boundary value of (1.1), then for every boundary point $x_0$, we let $v_{x_0}(x)$ solve
\[ \Delta v_{x_0}(x) = \text{div} \overrightarrow{F}(x, g(x)) \text{ in } B_1(x_0). \]

Our main results are the following two theorems.

**Theorem 1.8** Let $x_0 \in \partial \Omega$. $\overrightarrow{F}(x, t)$ satisfies Assumption 1 and 2. If $\partial \Omega$ is $C^{1, \text{Dini}}$ at $x_0$ and $g$ is $C^{1, \text{Dini}}$ at $x_0$ with respect to $v_{x_0}(x)$, then the solution of (1.1) is Lipschitz continuous at $x_0$, i.e.,
\[ |u(x) - u(x_0)| \leq C|x - x_0|, \quad \forall x \in B_1 \cap \Omega, \]
where $C = C(n, \Omega, T, \|u\|_{L^\infty(\Omega)}, \omega, \sigma, \omega_1)$.

**Theorem 1.9** Let $x_0 \in \partial \Omega$. $\overrightarrow{F}(x, t)$ satisfies Assumption 1 and 2. If $\partial \Omega$ satisfies Reifenberg $C^{1, \text{Dini}}$ condition at $x_0$ and $g$ is $C^{1, \text{Dini}}$ at $x_0$ with respect to $v_{x_0}(x)$, then the solution of (1.1) is Lipschitz continuous at $x_0$.

To prove these two theorems, the main method is iterative scheme. In fact we can approximate $u$ by $v$ defined by Assumption 2 and to show this approximation can be improved from $B_1$ to $B_\lambda$. The key point of our proof is that the main part of $u$ is a Lipschitz function $v$ and a linear function. We organize the paper as follows. In Section 2 we will give some necessary lemmas. Next we prove the Lipschitz regularity under $C^{1, \text{Dini}}$ condition in Section 3. Finally, we extend $C^{1, \text{Dini}}$ condition to Reifenberg $C^{1, \text{Dini}}$ condition and give a proof of Theorem 1.9 in Section 4.

## 2 Preliminary Tools

In this section we will give a general approximation lemma of the following divergence form elliptic equation:
\[ \Delta u = \text{div} \overrightarrow{F} \text{ in } \Omega, \quad (2.1) \]
where $\Omega$ is a bounded domain. After proving the approximation lemma, we will give a key lemma which will be used repeatedly in Section 3 and 4. We mainly assume that the boundary lies between two parallel hyperplanes with a very small distance.

**Lemma 2.1**

Let $0 \in \partial \Omega$, $B_1 \cap \{x \in \mathbb{R}^n : x_n > \varepsilon\} \subset B_1 \cap \Omega \subset B_1 \cap \{x \in \mathbb{R}^n : x_n > -\varepsilon\}$ for some $0 < \varepsilon < \frac{1}{4}$. For any $\overline{F}(x) \in L^\infty(B_1 \cap \Omega)$, $g \in L^\infty(B_1 \cap \partial \Omega)$, if $u$ is a weak solution of

$$
\begin{cases}
\Delta u = \text{div} \overline{F} & \text{in } B_1 \cap \Omega, \\
 u = g & \text{on } B_1 \cap \partial \Omega,
\end{cases}
$$

then there exists a universal constant $C_0$ and a harmonic function $h$ defined in $B_{\frac{1}{4}}$ which is odd with respect to $x_n$ satisfying

$$
\|h\|_{L^\infty(B_{\frac{1}{4}})} \leq (1 + 2C_0\varepsilon)\|u\|_{L^\infty(B_1 \cap \Omega)}
$$

such that

$$
\|u - h\|_{L^\infty(B_{\frac{1}{4}} \cap \Omega)} \leq \|g\|_{L^\infty(B_1 \cap \partial \Omega)} + 5C_0\varepsilon\|u\|_{L^\infty(B_1 \cap \Omega)} + C\|\overline{F}\|_{L^\infty(B_1 \cap \Omega)},
$$

where $C$ is a constant depending only on $n$ and $\Omega$.

In order to prove this lemma, we need the following estimate which can be found in many basic books of elliptic partial differential equations, such as Chapter 8 in [2].

**Theorem 2.2**

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. We suppose that $\overline{F} \in L^q(\Omega)$, for some $q > n$. Then if $u$ is a $W^{1,2}(\Omega)$ solution of (2.1), then we have

$$
\|u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\partial \Omega)} + C\|\overline{F}\|_{L^q(\Omega)},
$$

where $C = C(n, q, |\Omega|)$.

Now we give the proof of Lemma 2.1.

**Proof**

Let $v$ solve the following problem

$$
\begin{cases}
\Delta v = 0 & \text{in } B_1 \cap \Omega, \\
v = u & \text{on } \partial B_1 \cap \Omega, \\
v = 0 & \text{on } B_1 \cap \partial \Omega.
\end{cases}
$$

(2.3)

Then by the maximum principle, we have

$$
|v| \leq \|u\|_{L^\infty(B_1 \cap \Omega)} \text{ in } B_1 \cap \Omega.
$$

Next we denote $\|u\|_{L^\infty(B_1 \cap \Omega)}$ by $\mu$. Combining with (2.2) and (2.3) we have

$$
\begin{cases}
\Delta(u - v) = \text{div} \overline{F} & \text{in } B_1 \cap \Omega, \\
u - v = 0 & \text{on } \partial B_1 \cap \Omega, \\
u - v = g & \text{on } B_1 \cap \partial \Omega.
\end{cases}
$$

(2.4)

By the previous theorem we obtain

$$
\|u - v\|_{L^\infty(B_{\frac{1}{2}} \cap \Omega)} \leq \|g\|_{L^\infty(B_1 \cap \partial \Omega)} + C\|\overline{F}\|_{L^\infty(B_1 \cap \Omega)}.
$$
Let \( \Gamma \) be defined for \( x \in \mathbb{R}^n \setminus \{0\} \) by
\[
\Gamma(x) = \Gamma(|x|) = \begin{cases} 
-\frac{1}{2\pi} \ln |x|, & n = 2, \\
\frac{1}{(n-2)\omega_n}|x|^{2-n}, & n \geq 3,
\end{cases}
\]
where \( \omega_n \) is the surface area of the unit sphere in \( \mathbb{R}^n \). This function \( \Gamma \) is usually called the fundamental solution of the Laplace operator. By a simple calculation, we have \( \Delta \Gamma = 0 \) in \( \mathbb{R}^n \setminus \{0\} \). Then for any \( y = (y',0) \in T_{\frac{1}{4}} \), we consider a function
\[
l(x) = \frac{\Gamma(x-(y',-{\frac{1}{4}}-\epsilon)) - \Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4}) - \Gamma(\frac{1}{4})}\mu.
\]
Clearly \( l(x) \) is harmonic between \( B_{\frac{1}{4}}(y',-{\frac{1}{4}}-\epsilon) \) and \( B_{\frac{3}{4}}(y',-{\frac{1}{4}}-\epsilon) \) and
\[
\begin{align*}
l(x) &= 0 & \text{on } \partial B_{\frac{1}{4}}(y',-{\frac{1}{4}}-\epsilon), \\
0 < l(x) < \mu & \text{between } B_{\frac{1}{4}}(y',-{\frac{1}{4}}-\epsilon) \text{ and } B_{\frac{3}{4}}(y',-{\frac{1}{4}}-\epsilon), \\
l(x) &= \mu & \text{on } \partial B_{\frac{3}{4}}(y',-{\frac{1}{4}}-\epsilon).
\end{align*}
\]
From (2.3) and (2.5) we get \( v - l \) satisfies
\[
\begin{align*}
\Delta(v - l) &= 0 & \text{in } B_{\frac{3}{4}}(y',-{\frac{1}{4}}-\epsilon) \cap \Omega, \\
v - l &\leq 0 & \text{on } \partial B_{\frac{1}{4}}(y',-{\frac{1}{4}}-\epsilon) \cap \Omega, \\
v - l &\leq 0 & \text{on } B_{\frac{3}{4}}(y',-{\frac{1}{4}}-\epsilon) \cap \partial \Omega.
\end{align*}
\]
Applying the maximum principle again, it yields that
\[v \leq l \text{ in } B_{\frac{3}{4}}(y',-{\frac{1}{4}}-\epsilon) \cap \Omega.
\]
Similarly, repeating the above process for \( v+l \), it’s easy to see that \( v \geq -l \) in \( B_{\frac{3}{4}}(y',-{\frac{1}{4}}-\epsilon) \cap \Omega \), then
\[|v| \leq l \text{ in } B_{\frac{3}{4}}(y',-{\frac{1}{4}}-\epsilon) \cap \Omega.
\]
Furthermore for arbitrary \( x_1 \in \partial B_{\frac{3}{4}}(y',-{\frac{1}{4}}-\epsilon) \), in the radial direction, we have
\[
l(x) - l(x_1) \leq C_0 \mu \text{ between } B_{\frac{1}{4}}(y',-{\frac{1}{4}}-\epsilon) \text{ and } B_{\frac{3}{4}}(y',-{\frac{1}{4}}-\epsilon),
\]
then
\[
l(x) \leq C_0 \mu \text{ dist}\left(x, \partial B_{\frac{3}{4}}(y',-{\frac{1}{4}}-\epsilon)\right).
\]
In particular along the \( x_n \)-direction, we get \( l(x) \leq C_0 \mu (x_n + \epsilon) \). Since \( y \) can be chosen in \( T_{\frac{1}{4}} \) arbitrarily, then
\[|v(x)| \leq l(x) \leq C_0 \mu (x_n + \epsilon) \text{ in } \overline{B_{\frac{3}{4}}} \cap \Omega. \quad (2.6)
\]
Then (2.3) and (2.6) imply that \( v \) satisfies the following conditions,

\[
\begin{cases}
\Delta v = 0 & \text{in } B_{\frac{1}{4}} \cap \Omega, \\
|v(x)| \leq C_0 \mu(x_n + \varepsilon) & \text{in } \overline{B_{\frac{1}{4}}} \cap \Omega, \\
v = 0 & \text{on } B_{\frac{1}{4}} \cap \partial \Omega.
\end{cases}
\]

Now it’s time to find the harmonic function. We take \( h \) be a harmonic function defined in \( B_{\frac{1}{4}} \), which is odd with respect to \( x_n \) and satisfies the following conditions,

\[
\begin{cases}
\Delta h = 0 & \text{in } B_{\frac{1}{4}}, \\
h = 0 & \text{on } T_{\frac{1}{4}}, \\
h = v & \text{on } \partial B_{\frac{1}{4}} \cap \{x \in \mathbb{R}^n : x_n \geq \varepsilon\}, \\
h = 2C_0 \mu \varepsilon & \text{on } \partial B_{\frac{1}{4}} \cap \{x \in \mathbb{R}^n : 0 < x_n < \varepsilon\}.
\end{cases}
\]

Applying the maximum principle to \( h \), we get

\[
|h(x)| \leq C_0 \mu(x_n + \varepsilon) + 2C_0 \mu \varepsilon \quad \text{in } B_{\frac{1}{4}},
\]

\[
\|h\|_{L^\infty(B_{\frac{1}{4}})} \leq (1 + 2C_0 \varepsilon) \mu.
\]

Next we consider \( v - h \) in \( B_{\frac{1}{4}} \cap \Omega \) to obtain

\[
\begin{cases}
\Delta (v - h) = 0 & \text{in } B_{\frac{1}{4}} \cap \Omega, \\
v - h = 0 & \text{on } \partial B_{\frac{1}{4}} \cap \{x_n \geq \varepsilon\}, \\
-4C_0 \mu \varepsilon \leq v - h \leq 0 & \text{on } \partial B_{\frac{1}{4}} \cap \{0 < x_n < \varepsilon\}, \\
-4C_0 \mu \varepsilon \leq v - h \leq 4C_0 \mu \varepsilon & \text{on } B_{\frac{1}{4}} \cap \partial \Omega, \\
-C_0 \mu \varepsilon \leq v - h \leq C_0 \mu \varepsilon & \text{on } T_{\frac{1}{4}} \cap \Omega.
\end{cases}
\]

Using the maximum principle again we obtain

\[
|v - h| \leq 4C_0 \mu \varepsilon \quad \text{in } B_{\frac{1}{4}} \cap \Omega.
\]

Since \( h \) is odd respect to \( x_n \) and \( B_1 \cap \Omega \subset B_1 \cap \{x \in \mathbb{R}^n : x_n > -\varepsilon\} \) for some \( 0 < \varepsilon < \frac{1}{4} \), it’s easy to get \( |h| \leq 4C_0 \mu \varepsilon \) in \( B_{\frac{1}{4}} \cap \Omega \). Combining with (2.6) we get

\[
|v - h| \leq 5C_0 \mu \varepsilon \quad \text{in } B_{\frac{1}{4}} \cap \Omega.
\]

From above two inequalities we get

\[
\|v - h\|_{L^\infty(B_{\frac{1}{4}} \cap \Omega)} \leq 5C_0 \mu \varepsilon.
\]

Then from (2.4) and (2.8), we can get the following desired result by the triangle inequality,

\[
\|u - h\|_{L^\infty(B_{\frac{1}{4}} \cap \Omega)} \leq 5C_0 \mu \varepsilon + \|g\|_{L^\infty(B_{\frac{1}{4}} \cap \partial \Omega)} + C\|F\|_{L^\infty(B_{\frac{1}{4}} \cap \Omega)}.
\]

**Remark 2.3** \( x_n \) can be regarded as \( x \cdot \vec{e}_n \). Therefore in the above lemma, \( x_n \) can be replaced by \( x \cdot \vec{n} \) for arbitrary unit vector \( \vec{n} \).

Then we give a more general form of Lemma 2.1.
Lemma 2.4  \( 0 \in \partial \Omega, \ B_1 \cap \{x \in \mathbb{R}^n : x \cdot \vec{n} > \varepsilon\} \subset B_1 \cap \Omega \subset B_1 \cap \{x \in \mathbb{R}^n : x \cdot \vec{n} > -\varepsilon\} \) for some \( 0 < \varepsilon < \frac{1}{4} \). For any \( \overrightarrow{F}(x) \in L^\infty(B_1 \cap \Omega), \ g \in L^\infty(B_1 \cap \partial \Omega) \), if \( u \) is a weak solution of
\[
\begin{cases}
\Delta u = \text{div} \overrightarrow{F} & \text{in } B_1 \cap \Omega, \\
u = g & \text{on } B_1 \cap \partial \Omega,
\end{cases}
\]
then there exists a universal constant \( C_0 \) and a harmonic function \( h \) defined in \( B_\frac{1}{4} \) which is odd with respect to \( x \cdot \vec{n} \) satisfying
\[
\|h\|_{L^\infty(B_\frac{1}{4})} \leq (1 + 2C_0\varepsilon)\|u\|_{L^\infty(B_1 \cap \Omega)}
\]
such that
\[
\|u - h\|_{L^\infty(B_\frac{1}{4} \cap \Omega)} \leq \|g\|_{L^\infty(B_1 \cap \partial \Omega)} + 5C_0\varepsilon\|u\|_{L^\infty(B_1 \cap \Omega)} + C\|\overrightarrow{F}\|_{L^\infty(B_1 \cap \Omega)}
\]
where \( C \) is a constant depending only on \( n \) and \( \Omega \).

Lemma 2.5 (Key Lemma) Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). Assume that \( 0 \in \partial \Omega \) and \( B_1 \cap \{x \in \mathbb{R}^n : x_n > \varepsilon\} \subset B_1 \cap \Omega \subset B_1 \cap \{x \in \mathbb{R}^n : x_n > -\varepsilon\} \) for some \( 0 < \varepsilon < \frac{1}{4} \). Then there exists \( \lambda > 0 \) and universal constants \( \tilde{C}, \tilde{C}_1, \tilde{C}_2 > 0 \) such that for any functions \( \overrightarrow{F}(x,u) \in L^\infty(B_1 \cap \Omega), \ g \in L^\infty(B_1 \cap \partial \Omega) \), the solution of
\[
\begin{cases}
\Delta u = \text{div} \overrightarrow{F}(x,u) & \text{in } B_1 \cap \Omega, \\
u = g & \text{on } B_1 \cap \partial \Omega,
\end{cases}
\]
and solution of
\[
\Delta v = \text{div} \overrightarrow{F}(x,0) \text{ in } B_1,
\]
there exists a constant \( K \) such that
\[
\|u - v - Kx_n\|_{L^\infty(B_\lambda \cap \Omega)} \leq \|g - v\|_{L^\infty(B_1 \cap \partial \Omega)} + \tilde{C}_1(\lambda^2 + \varepsilon)\|u - v\|_{L^\infty(B_1 \cap \Omega)} + \tilde{C}_2\|\overrightarrow{F}(x,u) - \overrightarrow{F}(x,0)\|_{L^\infty(B_1 \cap \Omega)}
\]
and
\[
|K| \leq \tilde{C}\|u - v\|_{L^\infty(B_1 \cap \Omega)}.
\]

Proof  By the definition of \( u \) and \( v \) we get
\[
\begin{cases}
\Delta(u - v) = \text{div}(\overrightarrow{F}(x,u) - \overrightarrow{F}(x,0)) & \text{in } B_1 \cap \Omega, \\
u - v = g - v & \text{on } B_1 \cap \partial \Omega.
\end{cases}
\]
Then by Lemma 2.1, there exists a universal constant \( C_0 \) and a harmonic function \( h \) defined in \( B_\frac{1}{4} \) which is odd with respect to \( x_n \) satisfying
\[
\|h\|_{L^\infty(B_\frac{1}{4})} \leq (1 + 2C_0\varepsilon)\|u - v\|_{L^\infty(B_1 \cap \Omega)}
\]
such that
\[
\|u - v - h\|_{L^\infty(B_\frac{1}{4} \cap \Omega)} \leq \|g - v\|_{L^\infty(B_1 \cap \partial \Omega)} + 5C_0\varepsilon\|u - v\|_{L^\infty(B_1 \cap \Omega)} + \tilde{C}\|\overrightarrow{F}(x,u) - \overrightarrow{F}(x,0)\|_{L^\infty(B_1 \cap \Omega)}.
\]
Take $L$ to be the first order Taylor polynomial of $h$ at 0, i.e., $L(x) = Dh(0) \cdot x + h(0)$. Since $h = 0$ on $B_\frac{1}{8} \cap \{x_n = 0\}$, then $L(x) = Kx_n$, where $|K| = |Dh(0)|$. Note that $h$ is a harmonic function which is odd with respect to $x_n$ in $B_{\frac{1}{8}}$, according to the property of harmonic function, when $|x| \leq \frac{1}{8}$,

$$|D^2 h(x)| + |Dh(x)| \leq A \|h\|_{L^\infty(B_{\frac{1}{8}})} \leq A(1 + 2C_0\varepsilon)\|u - v\|_{L^\infty(B_1 \cap \Omega)},$$

where $A$ is a constant depending only on $n$. Then there exists $\xi \in B_{\frac{1}{8}}$ such that for $|x| \leq \frac{1}{8}$,

$$|h(x) - L(x)| \leq \frac{1}{2}|D^2 h(\xi)||x|^2.$$  \hfill (2.10)

Finally, combining with (2.9) and (2.10), by taking $0 < \lambda < \frac{1}{8}$, we have

$$\|u - v - Kx_n\|_{L^\infty(B_\lambda \cap \Omega)} \leq \|u - v - h\|_{L^\infty(B_\lambda \cap \Omega)} + \|h - L\|_{L^\infty(B_\lambda \cap \Omega)}$$

$$\leq \|g - v\|_{L^\infty(B_1 \cap \partial \Omega)} + 5C_0\varepsilon\|u - v\|_{L^\infty(B_1 \cap \Omega)}$$

$$+ C\|ar{F}(x, u) - \bar{F}(x, 0)\|_{L^\infty(B_1 \cap \Omega)}$$

$$+ \frac{1}{8}\lambda^2 A(1 + 2C_0\varepsilon)\|u - v\|_{L^\infty(B_1 \cap \Omega)}$$

$$\leq \|g - v\|_{L^\infty(B_1 \cap \partial \Omega)} + C_1(\lambda^2 + \varepsilon)\|u - v\|_{L^\infty(B_1 \cap \Omega)}$$

$$+ C_2\|ar{F}(x, u) - \bar{F}(x, 0)\|_{L^\infty(B_1 \cap \Omega)}.$$}

where $C_1 = \max\{5C_0, \frac{1}{8}A(1 + 2C_0)\}$, $C_2 = C$.

**Remark 2.6** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Assume that $0 \in \partial \Omega$ and $B_1 \cap \{x \in \mathbb{R}^n : x_n > \varepsilon\} \subset B_1 \cap \Omega \subset B_1 \cap \{x \in \mathbb{R}^n : x_n > -\varepsilon\}$ for some $0 < \varepsilon < \frac{1}{8}$. Then there exists $\lambda > 0$ and universal constants $\tilde{C}, C_1, C_2 > 0$ such that for any functions $\bar{F}(x) \in L^\infty(B_1 \cap \Omega)$, $g \in L^\infty(B_1 \cap \partial \Omega)$, the solution of

$$\begin{aligned}
\Delta u &= \text{div} \bar{F}(x) & \text{in } B_1 \cap \Omega, \\
\quad u &= g & \text{on } B_1 \cap \partial \Omega,
\end{aligned}$$

there exists a constant $K$ such that

$$\|u - Kx_n\|_{L^\infty(B_\lambda \cap \Omega)} \leq \|g\|_{L^\infty(B_1 \cap \partial \Omega)} + C_1(\lambda^2 + \varepsilon)\|u\|_{L^\infty(B_1 \cap \Omega)} + C_2\|ar{F}(x)\|_{L^\infty(B_1 \cap \Omega)}$$

and

$$|K| \leq \tilde{C}\|u\|_{L^\infty(B_1 \cap \Omega)}.$$
Lemma 3.1

if \( \partial \Omega \) is \( C^{1, \text{Dini}} \) at \( x_0 \), then for any \( 0 < r \leq r_0 \), \( B_r(x_0) \cap \partial \Omega \subset B_r(x_0) \cap \{ |x_n - x_{0,n}| \leq r \omega(r) \} \).

Without loss of generality, we can take \( x_0 = 0 \), denote \( v_0 \) by \( v \) and assume that

\[
\begin{align*}
 u(0) &= g(0) = 0, \quad v(0) = 0, \quad r_0 = 1, \\
 \omega(1) &\leq \lambda, \quad \int_0^1 \frac{\omega(r)}{r} dr \leq 1, \quad \int_0^1 \frac{\sigma(r)}{r} dr \leq 1, \quad \int_0^1 \frac{\omega_1(r)}{r} dr \leq 1,
\end{align*}
\]

where \( \lambda \) will be determined in Lemma 2.5 and 3.2. Besides, We can also assume \( \vec{a} = 0 \) in definition 1.6, if not, we can consider \( \vec{u} := u - g(0) - \vec{a} \cdot x \), which satisfies the same equation.

Based on Lemma 2.5 and Remark 2.6, the following lemma is an iteration result.

Lemma 3.1 There exist sequences \( \{N_i\}_{i=0}^\infty \) and nonnegative sequences \( \{M_i\}_{i=0}^\infty \), with \( N_0 = 0, M_0 = \|u - v\|_{L^\infty(\Omega_1)} \), and for \( i = 0, 1, 2, \ldots \),

\[
M_{i+1} = \|g - v - N_i x_n\|_{L^\infty(B_{\lambda^i} \cap \partial \Omega)} + C_1(\lambda^2 + \omega(\lambda^i)) \|u - v - N_i x_n\|_{L^\infty(\Omega_{\lambda^i})} + C_2 \lambda^i \|\tilde{F}(x, u) - \tilde{F}(x, 0)\|_{L^\infty(\Omega_{\lambda^i})},
\]

such that

\[
\|u - v - N_i x_n\|_{L^\infty(\Omega_{\lambda^i})} \leq M_i. \tag{3.1}
\]

Proof. We prove this lemma inductively by using Remark 2.6 repeatedly.

When \( i = 0 \), since \( N_0 = 0 \) and \( M_0 = \|u - v\|_{L^\infty(\Omega_1)} \), it’s easy to see

\[
\|u - v - N_0 x_n\|_{L^\infty(\Omega_1)} = M_0.
\]

When \( i = 1 \), by Definition 1.1, we have \( B_{1} \cap \partial \Omega \subset \{ |x_n| \leq \omega(1) \} \). Therefore by Lemma 2.5, there exists \( |N_1| \leq \tilde{C}\|u - v\|_{L^\infty(B_1 \cap \partial \Omega)} \) such that

\[
\|u - v - N_1 x_n\|_{L^\infty(B_{\lambda^1} \cap \partial \Omega)} \leq \|g - v\|_{L^\infty(B_1 \cap \partial \Omega)} + C_1(\lambda^2 + \omega(1)) \|u - v\|_{L^\infty(B_1 \cap \partial \Omega)} + C_2 \lambda \|\tilde{F}(x, u) - \tilde{F}(x, 0)\|_{L^\infty(B_1 \cap \partial \Omega)} \triangleq M_1,
\]

and

\[
|N_1 - N_0| \leq \tilde{C}\|u - v\|_{L^\infty(B_1 \cap \partial \Omega)}.
\]

Next we assume that the conclusion is true for \( i \). We consider the equation

\[
\begin{align*}
 \Delta(u - v - N_i x_n) &= \text{div}(\tilde{F}(x, u) - \tilde{F}(x, 0)) \quad \text{in} \, \Omega_{\lambda^i}, \\
 u - v - N_i x_n &= g - v - N_i x_n \quad \text{on} \, B_{\lambda^i} \cap \partial \Omega.
\end{align*}
\]

For \( z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n \) we set

\[
\begin{align*}
 \tilde{u}(z) &= \frac{u(\lambda^i z) - v(\lambda^i z) - N_i \lambda^i z_n}{\lambda^i}, \\
 \tilde{g}(z) &= \frac{g(\lambda^i z) - v(\lambda^i z) - N_i \lambda^i z_n}{\lambda^i}, \\
 \tilde{f}(z) &= \tilde{F}(\lambda^i z, u(\lambda^i z)) - \tilde{F}(\lambda^i z, 0).
\end{align*}
\]
Lemma 3.2

\[ \sum \]

Then \( \tilde{u}(z) \) is a solution of

\[
\begin{align*}
\Delta \tilde{u}(z) &= \text{div} \tilde{f}(z) \quad \text{in } B_1 \cap \tilde{\Omega}, \\
\tilde{u}(z) &= \tilde{g}(z) \quad \text{on } B_1 \cap \partial \tilde{\Omega},
\end{align*}
\]

where \( \tilde{\Omega} = \{ z : \lambda' z \in \Omega \} \). Therefore \( B_1 \cap \partial \tilde{\Omega} \subset B_1 \cap \{ |z_n| \leq \omega(\lambda') \} \). Then by Remark 2.6, there exists a constant \( K \) such that

\[
\| \tilde{u} - Kz_n \|_{L^\infty(B_1 \cap \tilde{\Omega})} \leq \| \tilde{g} \|_{L^\infty(B_1 \cap \partial \tilde{\Omega})} + C_1(\lambda^2 + \omega(\lambda'))\| \tilde{u} \|_{L^\infty(B_1 \cap \tilde{\Omega})} + C_2\| \tilde{f} \|_{L^\infty(B_1 \cap \tilde{\Omega})},
\]

where \( |K| \leq \tilde{C}\| \tilde{u} \|_{L^\infty(B_1 \cap \tilde{\Omega})} = \tilde{C}\| u - v - N_i x_n \|_{L^\infty(\Omega_{\lambda_i})} \). Let \( N_{i+1} = N_i + K \), scaling back, then we get

\[
\| u - v - N_{i+1} x_n \|_{L^\infty(\Omega_{\lambda_{i+1}})} \leq \| g - v - N_i x_n \|_{L^\infty(B_1 \cap \partial \Omega)} + C_1(\lambda^2 + \omega(\lambda'))\| u - v - N_i x_n \|_{L^\infty(\Omega_{\lambda_i})} + C_2\lambda^i \| \mathbf{f}(x, u) - \mathbf{f}(x, 0) \|_{L^\infty(\Omega_{\lambda_i})} = M_{i+1},
\]

and

\[
|N_{i+1} - N_i| = |K| \leq \frac{\tilde{C}}{\lambda^i}\| u - v - N_i x_n \|_{L^\infty(\Omega_{\lambda_i})}.
\]

This completes the proof of Lemma 3.1.

The following three lemmas are similar to [3] and [4].

**Lemma 3.2** \( \sum_{i=0}^{\infty} \frac{M_i}{\lambda^i} < \infty \) and \( \lim_{i \to \infty} N_i \) exists. We set

\[
\lim_{i \to \infty} N_i = \tau.
\]

**Proof** We assume \( T \) is the Lipschitz constant respect to \( v \), then

\[
\| v \|_{L^\infty(B_1 \cap \partial \Omega)} = \| v - v(0) \|_{L^\infty(B_1 \cap \partial \Omega)} \leq T\lambda^i.
\]

For \( k \geq 0 \), we suppose \( P_k = \sum_{i=0}^{k} \frac{M_i}{\lambda^i} \). By Lemma 3.1, noting that \( N_0 = 0 \), \( M_0 = \| u - v \|_{L^\infty(\Omega_i)} \), then for any \( k \geq 0 \), we have

\[
N_{k+1} \leq N_k + \frac{Gk}{\lambda^k} \leq \tilde{C}P_k, \quad |N_{k+1}| \leq |N_k| + \frac{Gk}{\lambda^k} \leq \tilde{C}P_k, \quad M_{k+1} \leq \lambda^k \sigma(\lambda^k) + |N_k| \lambda^k \omega(\lambda^k) + C_1(\lambda^2 + \omega(\lambda^k))M_k + C_2\lambda^k \omega_1(\| u \|_{L^\infty(\Omega_{\lambda_k})}), \quad (3.2)
\]

where Definitions 1.1, 1.6 and Assumption 1 are used. Then

\[
\frac{M_{k+1}}{\lambda^{k+1}} \leq \frac{1}{\lambda}(\sigma(\lambda^k) + |N_k| \omega(\lambda^k)) + \frac{C_1(\lambda^2 + \omega(\lambda^k))}{\lambda} \left( \frac{M_k}{\lambda^k} \right) + \frac{C_2}{\lambda} \omega_1(\| u \|_{L^\infty(\Omega_{\lambda_k})}). \quad (3.3)
\]

Recalling the property of the modulus of continuity (see (1.5)) we have

\[
\omega_1(\| u \|_{L^\infty(\Omega_{\lambda_k})}) \leq \omega_1(\| u - v - N_k x_n \|_{L^\infty(\Omega_{\lambda_k})} + \| v \|_{L^\infty(\Omega_{\lambda_k})} + \| N_k x_n \|_{L^\infty(\Omega_{\lambda_k})}) \leq \omega_1(M_k + T\lambda^k + |N_k| \lambda^k) \leq 2 \left( \frac{M_k}{\lambda^k} + T + |N_k| \right) \omega_1(\lambda^k).
\]

By substituting the above inequality and (3.2) into (3.3), we obtain

\[
\frac{M_{k+1}}{\lambda^{k+1}} \leq \frac{1}{\lambda}(\sigma(\lambda^k) + \tilde{C} \omega(\lambda^k)P_{k-1}) + \frac{C_1(\lambda^2 + \omega(\lambda^k))}{\lambda} \left( \frac{M_k}{\lambda^k} \right) + 2\frac{C_2}{\lambda} \left( |N_k| \lambda^k + M_k + T \right) \omega_1(\lambda^k)
\]
\[
\leq \frac{1}{\lambda} \sigma(\lambda^k) + \frac{\tilde{C} + C_1}{\lambda} \omega(\lambda^k) P_k + C_1 \lambda \left( \frac{M_k}{\lambda^k} \right) + \frac{2C_2(\tilde{C} + 1)}{\lambda} \omega_1(\lambda^k)(P_k + T).
\]

We take \( \lambda \) small enough firstly to make \( C_1 \lambda \leq \frac{1}{4} \), then we take \( k_0 \) large enough (then fixed) such that
\[
\sum_{i=k_0}^{\infty} \frac{\tilde{C} + C_1}{\lambda} \omega(\lambda^i) \leq \frac{\tilde{C} + C_1}{\lambda \ln \frac{1}{\lambda}} \int_0^{\lambda^{k_0-1}} \frac{\omega(r)}{r} dr \leq \frac{1}{4},
\]
\[
\sum_{i=k_0}^{\infty} \frac{2C_2(\tilde{C} + 1)}{\lambda} \omega_1(\lambda^i) \leq \frac{2C_2(\tilde{C} + 1)}{\lambda \ln \frac{1}{\lambda}} \int_0^{\lambda^{k_0-1}} \frac{\omega_1(r)}{r} dr \leq \frac{1}{4}.
\]

For such \( k_0(\geq 1) \), we have
\[
\sum_{i=k_0}^{\infty} \sigma(\lambda^i) \leq \frac{1}{\lambda \ln \frac{1}{\lambda}} \int_0^{\lambda} \frac{\sigma(r)}{r} dr \leq \frac{1}{\ln \frac{1}{\lambda}}.
\]

Therefore for each \( k \geq k_0 \), we have
\[
P_{k+1} - P_k = \sum_{i=k_0}^{k} \frac{M_i+1}{\lambda^{i+1}}
\]
\[
\leq \sum_{i=k_0}^{k} \frac{1}{\lambda} \sigma(\lambda^i) + \sum_{i=k_0}^{k} \frac{\tilde{C} + C_1}{\lambda} \omega(\lambda^i) P_i + \sum_{i=k_0}^{k} C_1 \lambda \left( \frac{M_i}{\lambda^i} \right)
\]
\[
+ \sum_{i=k_0}^{k} \frac{2C_2(\tilde{C} + 1)}{\lambda} \omega_1(\lambda^i)(P_i + T)
\]
\[
\leq \frac{1}{\lambda \ln \frac{1}{\lambda}} + P_{k+1} \left( \frac{1}{4} + \sum_{i=k_0}^{k} \frac{\tilde{C} + C_1}{\lambda} \omega(\lambda^i) \right) + (P_{k+1} + T) \sum_{i=k_0}^{k} \frac{2C_2(\tilde{C} + 1)}{\lambda} \omega_1(\lambda^i)
\]
\[
\leq \frac{1}{\lambda \ln \frac{1}{\lambda}} + \frac{3}{4} P_{k+1} + \frac{1}{4} T.
\]

Then for all \( k \geq k_0 \),
\[
P_{k+1} \leq \frac{4}{\lambda \ln \frac{1}{\lambda}} + T + 4P_{k_0}.
\]

Therefore \( \{P_k\}_{k=0}^{\infty} \) is bounded. We already proved \( \sum_{i=0}^{\infty} \frac{M_i}{\lambda^i} \) is convergent and \( \{N_i\}_{i=0}^{\infty} \) is bounded.

Furthermore, by (3.2) and the definition of \( P_i \) it’s easy to see
\[
N_{i+1} - N_i \leq \frac{\tilde{C} M_i}{\lambda^i} = \tilde{C} P_i - \tilde{C} P_{i-1}, \quad \text{for } i \geq 1,
\]
and
\[
N_{i+1} - \tilde{C} P_i \leq N_i - \tilde{C} P_{i-1}, \quad \text{for } i \geq 1.
\]

So \( \{N_i - \tilde{C} P_{i-1}\}_{i=1}^{\infty} \) is a bounded and non-increasing sequence and \( \lim_{i \to +\infty} (N_i - \tilde{C} P_{i-1}) \) exists. In conclusion \( \lim_{i \to +\infty} N_i \) exists and we set \( \tau := \lim_{i \to +\infty} N_i \). The proof is finished.

**Lemma 3.3** \( \lim_{i \to +\infty} \frac{M_i}{\lambda^i} = 0 \).

**Proof** The proof is straightforward from Lemma 3.2 since \( \sum_{i=0}^{\infty} \frac{M_i}{\lambda^i} \) is convergent.
Lemma 3.4 For each \( i = 0, 1, 2, \ldots \), there exists \( B_i \) such that \( \lim_{i \to \infty} B_i = 0 \) and that
\[
\|u - v - \tau x_n\|_{L^\infty(\Omega, \lambda)} \leq B_i \lambda^i.
\]
Proof For any \( i \geq 0 \) we have
\[
\|u - v - \tau x_n\|_{L^\infty(\Omega, \lambda)} \leq \|u - v - N_i x_n\|_{L^\infty(\Omega, \lambda)} + \|N_i x_n - \tau x_n\|_{L^\infty(\Omega, \lambda)}.
\]
Using (3.1) we get
\[
\|u - v - \tau x_n\|_{L^\infty(\Omega, \lambda)} \leq M_i + \lambda^i |N_i - \tau|.
\]
We set \( B_i = \frac{M_i}{\lambda^i} + |N_i - \tau| \), then
\[
\|u - v - \tau x_n\|_{L^\infty(\Omega, \lambda)} \leq B_i \lambda^i.
\]
At the same time, by Lemma 3.2 and 3.3 we have
\[
\lim_{i \to \infty} B_i = 0.
\]
The proof is completed.

Proof of Theorem 1.8 From above four lemmas we already show that \( u - v \) is differentiable at 0. Since \( v \) is a Lipschitz function, it’s clear that \( u \) is Lipschitz at 0.

4 Boundary Lipschitz Regularity under Reifenberg \( C^{1, \text{Dini}} \) Condition

In this section, we will generalize the results in Section 3 to Reifenberg \( C^{1, \text{Dini}} \) domain. The main difficulty is that the unit normal vectors are changing in different scales. But by Lemma 1.5, we notice that the difference of the unit vectors are controlled by Dini modulus of continuity and the unit vectors are convergent. The proof of Theorem 1.9 is similar to Theorem 1.8. We also use the following four lemmas to prove Theorem 1.9. For convenience, we only prove the boundary Lipschitz regularity at 0. Without loss of generality, we can take \( x_0 = 0 \). Denote \( v_0 \) by \( v \) and assume that
\[
\begin{align*}
  u(0) &= g(0) = 0, \quad v(0) = 0, \quad \bar{a} = 0, \quad r_0 = 1, \\
  \omega(1) &\leq \lambda, \quad \int_0^1 \frac{\omega(r)}{r} \, dr \leq 1, \quad \int_0^1 \frac{\sigma(r)}{r} \, dr \leq 1, \quad \int_0^1 \frac{\omega_1(r)}{r} \, dr \leq 1,
\end{align*}
\]
where \( \lambda \) will be determined in Lemma 2.5 and Lemma 4.2. In the following, we denote \( \vec{n}_\lambda \) by \( \vec{n}_i \).

Lemma 4.1 There exist sequences \( \{N_i\}_{i=0}^\infty \) and nonnegative sequences \( \{M_i\}_{i=0}^\infty \), with \( N_0 = 0 \), \( M_0 = \|u - v\|_{L^\infty(\Omega)} \), and for \( i = 0, 1, 2, \ldots \),
\[
M_{i+1} = \left\| \frac{g - v - N_i x \cdot \vec{n}_i}{L^\infty(\Omega, \lambda_i)} \right\|_{L^\infty(\Omega, \lambda_i)}
+ \|N_{i+1} x \cdot (\vec{n}_{i+1} - \vec{n}_i)\|_{L^\infty(\Omega, \lambda_i)}
+ C_1 (\lambda^2 + \omega(\lambda^i)) \|u - v - N_i x \cdot \vec{n}_i\|_{L^\infty(\Omega, \lambda_i)}
+ C_2 \lambda^i \|\tilde{F}(x, u) - \tilde{F}(x, 0)\|_{L^\infty(\Omega, \lambda_i)},
\]
\[
|N_{i+1} - N_i| \leq \frac{C}{\lambda^i} \|u - v - N_i x \cdot \vec{n}_i\|_{L^\infty(\Omega, \lambda_i)}.
\]
such that
\[
\|u - v - N_i x \cdot \vec{n}_i\|_{L^\infty(\Omega, \lambda_i)} \leq M_i.
\]
Proof We prove this lemma inductively by using Lemma 2.5, Remark 2.6 and 2.7 repeatedly.

When \( i = 0 \), since \( N_0 = 0 \) and \( M_0 = \| u - v \|_{L^\infty(\Omega_i)} \), it's easy to see

\[
\| u - v - N_0 x \cdot \vec{n}_0 \|_{L^\infty(\Omega_i)} = M_0.
\]

When \( i = 1 \), by Definition 1.3, we have \( B_1 \cap \partial \Omega \subset \{ | x \cdot \vec{n}_0 | \leq \omega(1) \} \). Therefore by Lemma 2.5 and Remark 2.7, there exists \( | N_1 | < C \| u - v \|_{L^\infty(B_1 \cap \Omega)} \) such that

\[
\| u - v - N_1 x \cdot \vec{n}_1 \|_{L^\infty(B_1 \cap \Omega)} \leq \| g - v \|_{L^\infty(B_1 \cap \partial \Omega)} + C_1 (\lambda^2 + \omega(1)) \| u - v \|_{L^\infty(B_1 \cap \Omega)} + C_2 \| \vec{F}(x, u) - \vec{F}(x, 0) \|_{L^\infty(B_i \cap \Omega)} \frac{M_1}{\lambda^i}.
\]

Then we have

\[
\| u - v - N_1 x \cdot \vec{n}_1 \|_{L^\infty(B_1 \cap \Omega)} \leq \| g - v \|_{L^\infty(B_1 \cap \partial \Omega)} + \| N_1 x \cdot (\vec{n}_1 - \vec{n}_0) \|_{L^\infty(B_1 \cap \Omega)} + C_1 (\lambda^2 + \omega(1)) \| u - v \|_{L^\infty(B_1 \cap \Omega)} + C_2 \| \vec{F}(x, u) - \vec{F}(x, 0) \|_{L^\infty(B_i \cap \Omega)} = M_1
\]

and

\[
| N_1 - N_0 | < C \| u - v \|_{L^\infty(B_1 \cap \Omega)}.
\]

Next we assume that the conclusion is true for \( i \). We consider the equation

\[
\begin{align*}
\Delta (u - v - N_i x \cdot \vec{n}_i) &= \text{div}(\vec{F}(x, u) - \vec{F}(x, 0)) \quad \text{in } \Omega_i, \\
u - v - N_i x \cdot \vec{n}_i &= g - v - N_i x \cdot \vec{n}_i \quad \text{on } B_i \cap \partial \Omega.
\end{align*}
\]

For \( z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n \) we set

\[
\begin{align*}
\tilde{u}(z) &= \frac{u(\lambda^i z) - v(\lambda^i z)}{\lambda^i} - N_i \lambda^i z \cdot \vec{n}_i, \\
\tilde{g}(z) &= \frac{g(\lambda^i z) - v(\lambda^i z)}{\lambda^i} - N_i \lambda^i z \cdot \vec{n}_i, \\
\tilde{f}(z) &= \vec{F}(\lambda^i z, u(\lambda^i z)) - \vec{F}(\lambda^i z, 0).
\end{align*}
\]

Then \( \tilde{u}(z) \) is a solution of

\[
\begin{align*}
\Delta \tilde{u}(z) &= \text{div} \tilde{f}(z) \quad \text{in } B_1 \cap \tilde{\Omega}, \\
\tilde{u}(z) &= \tilde{g}(z) \quad \text{on } B_1 \cap \partial \tilde{\Omega},
\end{align*}
\]

where \( \tilde{\Omega} = \{ z : \lambda^i z \in \Omega \} \). Therefore \( B_1 \cap \partial \tilde{\Omega} \subset B_1 \cap \{ | z \cdot \vec{n}_i | \leq \omega(1) \} \). Then by Remark 2.6, there exists a constant \( K \) such that

\[
\| \tilde{u} - K z \cdot \vec{n}_i \|_{L^\infty(B_1 \cap \tilde{\Omega})} \leq \| \tilde{g} \|_{L^\infty(B_1 \cap \partial \tilde{\Omega})} + C_1 (\lambda^2 + \omega(\lambda^i)) \| \tilde{u} \|_{L^\infty(B_1 \cap \tilde{\Omega})} + C_2 \| \tilde{f} \|_{L^\infty(B_1 \cap \tilde{\Omega})},
\]

where \( | K | \leq C \| \tilde{u} \|_{L^\infty(B_1 \cap \tilde{\Omega})} = \frac{C}{\lambda^i} \| u - v - N_i x \cdot \vec{n}_i \|_{L^\infty(\Omega_i)} \). Let \( N_{i+1} = N_i + K \), scaling back, we get

\[
\| u - v - N_{i+1} x \cdot \vec{n}_{i+1} \|_{L^\infty(\Omega_{i+1})} \leq \| g - v - N_i x \cdot \vec{n}_i \|_{L^\infty(B_{\lambda^i} \cap \partial \Omega)} + \| N_{i+1} x \cdot (\vec{n}_{i+1} - \vec{n}_i) \|_{L^\infty(\Omega_{i+1})} + C_1 (\lambda^2 + \omega(\lambda^i)) \| u - v - N_i x \cdot \vec{n}_i \|_{L^\infty(\Omega_{i+1})} + C_2 \lambda^i \| \vec{F}(x, u) - \vec{F}(x, 0) \|_{L^\infty(\Omega_{i+1})} = M_{i+1}
\]

\( | N_{i+1} - N_i | < C \| u - v \|_{L^\infty(B_1 \cap \partial \Omega)} \).
Lemma 4.2 \( \sum_{i=0}^{\infty} \frac{M_i}{X^i} < \infty \) and \( \lim_{i \to \infty} N_i \) exists. We set

\[
\lim_{i \to \infty} N_i = \tau.
\]

Proof We assume that \( T \) is the Lipschitz constant respect to \( v \), then

\[
\|v\|_{L^\infty(B_{\lambda i} \cap \partial \Omega)} = \|v - v(0)\|_{L^\infty(B_{\lambda i} \cap \partial \Omega)} \leq T \lambda^i.
\]

For \( k \geq 0 \), we suppose \( P_k = \sum_{i=0}^{k} \frac{N_i}{X^i} \). By Lemma 4.1, noting that \( N_0 = 0 \) and \( M_0 = \|u - v\|_{L^\infty(\Omega_i)} \), then for any \( k \geq 0 \) we have

\[
N_{k+1} \leq N_k + \tilde{C} \frac{M_k}{\lambda^k} \leq \tilde{C} P_k, \quad |N_{k+1}| \leq |N_k| + \tilde{C} \frac{M_k}{\lambda^k} \leq \tilde{C} P_k, \tag{4.2}
\]

\[
M_{k+1} \leq \lambda^k \sigma(\lambda^k) + |N_k| \lambda^k \omega(\lambda^k) + |N_{k-1}| \lambda^k S(\lambda) \omega(\lambda^k) + C_1 (\lambda^2 + \omega(\lambda^k)) M_k + C_2 \lambda^k \omega_1(\|u\|_{L^\infty(\Omega_{\lambda k})}),
\]

where Definitions 1.3, 1.6, Lemma 1.5 and Assumption 1 are used. Then

\[
\frac{M_{k+1}}{\lambda^{k+1}} \leq \frac{1}{\lambda} \lambda \sigma(\lambda^k) + |N_k| \omega(\lambda^k) + |N_{k-1}| S(\lambda) \omega(\lambda^k)
\]

\[
+ C_1 (\lambda^2 + \omega(\lambda^k)) \left( \frac{M_k}{\lambda^k} \right) + C_2 \lambda^k \omega_1(\|u\|_{L^\infty(\Omega_{\lambda k})}). \tag{4.3}
\]

Recalling the property of the modulus of continuity (see (1.5)) we have

\[
\omega_1(\|u\|_{L^\infty(\Omega_{\lambda k})}) \leq \omega_1(\|u - v - N_k x \cdot \vec{n}_k\|_{L^\infty(\Omega_{\lambda k})} + \|u\|_{L^\infty(\Omega_{\lambda k})})
\]

\[
\leq \omega_1(M_k + T \lambda^k + |N_k| \lambda^k)
\]

\[
\leq 2 \left( \frac{M_k}{\lambda^k} + T + |N_k| \right) \omega_1(\lambda^k).
\]

By substituting the above inequality and (4.2) into (4.3), we obtain

\[
\frac{M_{k+1}}{\lambda^{k+1}} \leq \frac{1}{\lambda} \lambda \sigma(\lambda^k) + \frac{\tilde{C}}{\lambda} \omega(\lambda^k) (P_{k-1} + S(\lambda) P_k)
\]

\[
+ C_1 (\lambda^2 + \omega(\lambda^k)) \left( \frac{M_k}{\lambda^k} \right) + 2C_2 \lambda \left( |N_k| + \frac{M_k}{\lambda^k} + T \right) \omega_1(\lambda^k)
\]

\[
\leq \frac{1}{\lambda} \lambda \sigma(\lambda^k) + \frac{\tilde{C}}{\lambda} + C_1 + C_2 \frac{S(\lambda)}{\lambda} \omega(\lambda^k) P_k
\]

\[
+ C_1 \lambda \left( \frac{M_k}{\lambda^k} \right) + 2C_2 (\tilde{C} + 1) \frac{1}{\lambda} \omega_1(\lambda^k) (P_k + T).
\]

The remaining proof is the same as Lemma 3.2, then we get \( \sum_{i=0}^{\infty} \frac{M_i}{X^i} < \infty \) and \( \lim_{i \to \infty} N_i \) exists. \( \square \)

Lemma 4.3 \( \lim_{i \to +\infty} \frac{M_i}{X^i} = 0. \)

Lemma 4.4 For each \( i = 0, 1, 2, \ldots \), there exists \( B_i \) such that \( \lim_{i \to \infty} B_i = 0 \) and that

\[
\|u - v - \tau x \cdot \vec{n}_*\|_{L^\infty(\Omega_{\lambda i})} \leq B_i \lambda^i,
\]

where \( \vec{n}_* \) is the limit of \( \{\vec{n}_{\lambda_i}\}_{i=0}^{\infty} \) in Lemma 1.5.
Proof For any \(i \geq 0\) we have
\[
\|u - v - \tau x \cdot \vec{n}_*\|_{L^\infty(\Omega_{\lambda_i})} \leq \|u - v - N_i x \cdot \vec{n}_i\|_{L^\infty(\Omega_{\lambda_i})} + \|N_i x \cdot \vec{n}_i - N_i x \cdot \vec{n}_*\|_{L^\infty(\Omega_{\lambda_i})} \\
+ \|N_i x \cdot \vec{n}_* - \tau x \cdot \vec{n}_*\|_{L^\infty(\Omega_{\lambda_i})}.
\]
Using (4.1) we get
\[
\|u - v - \tau x \cdot \vec{n}_*\|_{L^\infty(\Omega_{\lambda_i})} \leq M_i + |N_i| \lambda^i \|\vec{n}_i - \vec{n}_*\| + \lambda^i |N_i - \tau|.
\]
We set \(B_i = \frac{M_i}{\lambda^i} + |N_i| \|\vec{n}_i - \vec{n}_*\| + |N_i - \tau|\), then
\[
\|u - v - \tau x \cdot \vec{n}_*\|_{L^\infty(\Omega_{\lambda_i})} \leq B_i \lambda^i.
\]
At the same time, by Lemmas 4.2, 4.3 and Lemma 1.5 we have
\[
\lim_{i \to \infty} B_i = 0.
\]
The proof is completed.

Proof of Theorem 1.9 From the above four lemmas we already show that \(u - v\) is differentiable at 0. Since \(v\) is a Lipschitz function, it’s clear that \(u\) is Lipschitz at 0.

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