On the null surface formalism
– Formulation in three dimensions and gauge freedom –

Masayuki Tanimoto†

Yukawa Institute for Theoretical Physics, Kyoto University,
Kyoto 606-01, Japan

Abstract
The null surface formalism of GR in three dimensions is presented, and the gauge freedom thereof, which is not just diffeomorphism, is discussed briefly.

1 Introduction
Recently, Frittelli, Kozameh, and Newman [1, 2, 3] presented an unconventional version of GR, where the variable is not any field such as the metric or connection, but 2-parameters family of foliations
\[ Z(x^a; \zeta, \bar{\zeta}) = \text{const.} \]  
(1)
of a manifold, where \( x^a \) is an arbitrary coordinate system, and \( \zeta \) and \( \bar{\zeta} \) are parameters. In this formalism, called the null surface formalism (NSF), part of main equations, called the metricity condition, is obtained from the requirement that the hypersurfaces \( Z(x^a; \zeta, \bar{\zeta}) = \text{const.} \) be null with respect to a metric. This condition not only says the existence of a corresponding metric on the manifold, but also contains information of null geodesics. (For an attempt toward quantization, see Refs.[4, 5].)

We shall in this article present the three-dimensional version of NSF. An establishment of NSF in dimensions other than four was nontrivial at the point the original formulation was presented, since the NSF is not a conventional “field” theory. Our success in three dimensions may suggest that NSF can establish in any dimensions equal to or higher than three. Moreover, the simplicity in three dimensions makes the algebraic structure of the NSF transparent. We use our formalism to discuss “gauge” freedom of NSF, which is also rather unconventional.

2 Null foliations and the intrinsic coordinates
Consider a Lorentzian metric \( g^{ab}(x^a) \) on a three dimensional manifold \( M \), and 1-parameter family of null foliations of \( M \);
\[ Z(x^a, \zeta) = \text{const.}, \]  
(2)
where \( x^a \) is an arbitrary coordinate system, and \( \zeta \) is a real parameter.

As is emphasized in Refs.[1, 2], a family of null foliations defines a particular coordinate system \( \theta^i \), called the intrinsic coordinates, which is given by successive derivatives of \( Z \) with respect to the parameter \( \zeta \):
\[ \theta^0 \equiv u \equiv Z(x^a, \zeta), \]
\[ \theta^1 \equiv \omega \equiv \partial Z(x^a, \zeta), \]
\[ \theta^2 \equiv R \equiv \partial^2 Z(x^a, \zeta), \]  
(3)
where
\[ \partial \equiv \frac{\partial}{\partial \zeta}. \]  
(4)
We may write as
\[ \theta^i = \partial^i Z(x^a, \zeta). \]  
(5)
For later convenience, we define the “Jacobian”
\[ \theta^i_a \equiv \theta^i, a, \quad \theta^i_a \theta^j_a \equiv \delta^i_j. \] (6)

We also define
\[ \Lambda \equiv \partial^3 Z(\theta^i, \zeta). \] (7)

We can find the components of the metric with respect to \( \theta^i \)
\[ g^{ij} = g^{ab} \partial Z, a \partial Z, b, \] (8)
by successively operating \( \partial \) on the null condition of \( Z(x^a, \zeta) \)
\[ g^{00} = g^{ab}(x^a)Z, a Z, b = 0. \] (9)

For example, the first derivative of Eq.(9),
\[ g^{ab} \partial Z, a Z, b = 0, \] immediately implies
\[ g^{01} = 0. \]

If we define
\[ \Omega^2 \equiv g^{02}, \] (10)
which is independent of \( Z(x^a, \zeta) \), the final result is
\[ g^{ij} = \Omega^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & -(1/3)\Lambda_2 & -(1/3)\partial(\Lambda_2) + (1/9)(\Lambda_2)^2 + \Lambda, 1 \\ 1 & -1 & -1 \end{pmatrix}. \] (11)

Here, the operation of \( \partial \) on a function \( f(\theta^i, \zeta) \) is well-defined;
\[ \partial f(\theta^i, \zeta) = \partial' f + (\partial \theta^i) f, i = \partial' f + \omega f, 0 + R f, 1 + \Lambda f, 2, \] (12)
where \( \partial' \) is the differential operator for fixed \( \theta^i \).

With Eq.(11), we can check that \( l^a \equiv (\partial/\partial R)^a \) is a null geodesic generator;
\[ l^a l_a = 0, \quad l^b \nabla_b l^a = -2 \frac{\Omega_2 l^a}{\Omega}. \] (13)

3 The metricity conditions and Einstein’s equation

As in [2], the metricity conditions can be expressed as
\[ \partial g^{ab}(x^a) = 0 \]
\[ \iff \partial (g^{ij}(\theta^i, \zeta) \theta^a \theta^b) = 0 \]
\[ \iff \partial g^{ij}(\theta^i, \zeta) + g^{km}(\theta^i, \zeta) \partial(\theta^k \theta^m \theta^b) \theta^a \theta^j = 0. \] (14)

That is, we demand that the \( \zeta \)-dependence of \( g^{ij}(\theta^i, \zeta) \) can be “absorbed” into a coordinate transformation of the form [2]. It is important to note that \( \theta^i \) themselves are regarded as \( \zeta \)-dependent, so \( g^{ij}(\theta^i, \zeta) = g^{ij}(\theta^i(\zeta), \zeta) \). This implies that even though \( g^{ij} \) has no explicit dependence of \( \zeta \), the metricity conditions are nontrivial.

To proceed, we define matrix \( T^{ij} \) by
\[ \partial \theta^i_a \equiv T^{ij}_a \theta^j_a. \] (15)

With this, Eq.(14) can be written as
\[ \partial g^{ij} - T^{ik} g^{kj} - T^{ij} k g^{ki} = 0. \] (16)

We can easily have the explicit form of \( T^{ij} \). For example, noting \( \theta^i_a = \theta^i, a \), we have for \( i = 0 \) for Eq.15
\[ \partial Z, a = T^0 j \theta^j, a \]
\[ \iff \theta^1, a = T^0 j \theta^j, a, \] (17)
so that
\[ T^0_j = \delta^j_0. \]
Similarly, we have for other components
\[ T^1_j = \delta^j_1, \quad T^2_j = \Lambda_j. \]
We have six conditions Eq.(16) (under Eqs.(18) and (19)), and moreover we impose the null condition \( g^{00} = 0 \) and define Eq.(10). We therefore have eight inputs, so eight outputs must follow. Six of them are given by the components of \( g^{ij} \), Eq.(11). The rest is the metricity conditions we searched for, which is the following two consistency conditions;
\[ \Lambda_2 = 3\Omega^{-1}\partial\Omega, \]
and
\[ -\frac{4}{9}(\Lambda_2)^3 + 2\Lambda_2\partial(\Lambda_2) - 2\Lambda_1\Lambda_2 - \partial^2(\Lambda_2) + 3\partial(\Lambda_1) - 6\Lambda_0 = 0. \]
If we find a solution of Eqs.(20) and (21) for \( \Lambda \) and \( \Omega \), we obtain a metric (11), and simultaneously, have essentially all information of null geodesics thereof (cf. Eq.(13)).

Next, we consider Einstein’s equation with cosmological constant \( \lambda \),
\[ G^{ab} = \lambda g^{ab} \]
where \( G^{ab} \) is the Einstein tensor.
We cast the gradient of \( Z \) to Eq.(22);
\[ G^{ab}Z_{,a}Z_{,b} = \lambda g^{ab}Z_{,a}Z_{,b} = 0 \]
\[ \Leftrightarrow G^{00} = 0 \]
\[ \Leftrightarrow G_{22} = 0. \]
From a straightforward calculation, \( G_{22} = \Omega^{-1}\Omega_{,22}. \) We thus have
\[ \Omega^{-1}\Omega_{,22} = 0. \]
If we operate \( \partial \) on (the first row of) Eq.(23), we obtain
\[ G^{11} + G^{02} = 0, \]
but this does not mean the two components of Einstein’s equation for \((ij) = (11) \) and \((02)\), given by
\[ G^{11} = -\lambda\Omega^2 \]
and
\[ G^{02} = \lambda\Omega^2. \]
Therefore, we need impose one of the two equations independently of Eq.(23) (or Eq.(24)).

\[ \text{We take Eq.(27), which is, after some calculations, found to be} \]
\[ -\frac{1}{36}\Omega^2((\Lambda_{,22})^2 + 6\Lambda_{,221}) + \frac{1}{9}(\Omega_{,2})^2(-3\partial(\Lambda_2) + (\Lambda_2)^2 + 9\Lambda_1) \]
\[ + \frac{1}{18}\Omega_{,2}(3\partial(\Lambda_{,22}) + \Lambda_{,2}\Lambda_{,2}) + \frac{2}{3}\Omega_{,2}(3\Omega_{,0} - \Lambda_{,2}\Omega_{,1}) \]
\[ + \frac{1}{6}\Omega(\Lambda_{,22}\Omega_{,1} + 2\Lambda_{,2}\Omega_{,12} + 6\Omega_{,11} - 6\Omega_{,02} - (\Omega_{,1})^2 = \lambda. \]
We have used, as well as Eq.(24), identity
\[ \partial(\Lambda_{,2})_{,2} = \partial(\Lambda_{,22}) + \Lambda_{,12} + \Lambda_2\Lambda_{,22}, \]
obtained from more general identity for an arbitrary function $\phi$

$$\partial(\phi_2) - (\partial \phi)_2 = -\phi_1 - \Lambda_2 \phi_2.$$  

(30)

We can check that any other components of Einstein’s equation follow from the successive differentiations with respect to $\zeta$ of Eq. (24). Thus, we need only two component equation, Eqs. (24) and (28), rather than six components.

Our fundamental equations are therefore the four coupled nonlinear partial differential equations (20), (21), (24), and (28). The following fact may be noteworthy at this point. First, we can immediately solve Eq. (24), and find that $\Omega$ is a polynomial of variable $R$ at most of first order. Then, it is also easy to see from Eq. (20) that $\Lambda$ is a polynomial of variable $R$ at most of third order. Thus, both $\Lambda$ and $\Omega$ are polynomials of $R$ in any “gauge”, discussed in the next section.

4 ”Gauge” degrees of freedom

For a fixed spacetime, we can consider many varieties of 1-parameter family of null foliations, and which, in general, correspond to different $\Lambda$ and $\Omega$. Therefore, even in the present formalism there exists a sort of “gauge” freedom.

More precisely, we can categorize this “gauge” into two parts, one is essential deformations of way of taking the 1-parameter family of null foliations, and the other is just reparametrizations of the 1-parameter family of null foliations.

We, in this section, discuss two simple cases of reparametrizations.

4.1 Reparametrization of $\zeta$

Consider a 1-parameter family of foliations $u = \text{const.} = Z(x^a, \zeta)$. Reparametrization of the form

$$\zeta \to f(\zeta),$$

(31)

where $f$ is a real function, apparently preserves the original foliation. Hence, this is a sort of gauge.

Under the transformation (31), we have

$$Z(x^a, \zeta) \to \tilde{Z}(x^a, \zeta) = Z(x^a, f(\zeta)).$$

(32)

We denote the new intrinsic coordinates and $\Lambda$ obtained from $\tilde{Z}$ as ($\tilde{u}, \tilde{\omega}, \tilde{R}$) and $\tilde{\Lambda}$, respectively. Operating $\partial$ successively on $\tilde{Z}$, we can easily have

$$\tilde{u} = u, \quad \tilde{\omega} = \omega f'(\zeta),$$

$$\tilde{R} = R f'(\zeta)^2 + \omega f''(\zeta),$$

$$\tilde{\Lambda} = \Lambda f'(\zeta)^3 + 3 R f'(\zeta) f''(\zeta) + \omega f'''(\zeta).$$

(33)

Thus, we have obtained the transformation rule for $\Lambda$. The rule for $\Omega$ can also be obtained, if we think of Eq. (33) as a coordinate transformation $\theta^i \to \tilde{\theta}^i$.

From the Jacobian

$$\tilde{g}^{i,k} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f'(\zeta) & 0 \\ 0 & f''(\zeta) & f'(\zeta)^2 \end{pmatrix},$$

(34)

we have

$$\tilde{g}^{ij} = g^{kl} \tilde{g}^i_{,k} \tilde{g}^j_{,l} = f'(\zeta)^2 \Omega^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & * \\ 1 & * & ** \end{pmatrix},$$

(35)

where

$$* \equiv -\frac{f''(\zeta)}{f'(\zeta)} - \frac{1}{3} f'(\zeta) \Lambda_2.$$  

(36)
and
\[ ** ≡ - \left( \frac{f''(\zeta)}{f'(\zeta)} \right)^2 - \frac{2}{3} f''(\zeta) \Lambda_2 + f'(\zeta)^2 \left( -\frac{1}{3} \partial \Lambda_2 + \Lambda_1 + \frac{1}{9}(\Lambda_2)^2 \right). \] (37)

Comparing with Eq.(11), we find
\[ \Omega \rightarrow \tilde{\Omega} = f'(\zeta) \Omega. \] (38)
That is, \( \Omega \) so transforms as \( \omega \). (We can check that \( * \) and \( ** \) are certainly of the form of Eq.(11), if written in terms of the tilded variables.)

### 4.2 Reparametrization of \( u \)

Consider a foliation
\[ \text{const.} = Z(x^a). \] (39)
Then, the foliation
\[ \text{const.} = \phi(Z(x^a)), \] (40)
where \( \phi \) is a real function, is geometrically the same foliation as the original.

We thus consider the transformation
\[ Z(x^a, \zeta) \rightarrow \tilde{Z}(x^a, \zeta) = \phi(Z(x^a, \zeta)). \] (41)

The following procedure to obtain the transformation rule for \( \Lambda \) and \( \Omega \) is exactly the same as before. We denote the new intrinsic coordinates and \( \Lambda \) obtained from \( \tilde{Z} \) as \((\tilde{u}, \tilde{\omega}, \tilde{R})\) and \( \tilde{\Lambda} \), respectively. Operating \( \partial \) successively on \( Z \), we have
\[ \tilde{u} = \phi(u), \quad \tilde{\omega} = \phi'(u) \omega, \]
\[ \tilde{R} = \phi''(u) \omega^2 + \phi'(u) R, \]
\[ \tilde{\Lambda} = \phi'''(u) \omega^3 + 3 \phi''(u) \omega R + \phi'(u) \Lambda. \] (42)

The rule for \( \Omega \) can also be obtained by observing the coordinate transformation \( \theta^i \rightarrow \tilde{\theta}^i \). We find
\[ \Omega \rightarrow \tilde{\Omega} = \phi'(u) \Omega. \] (43)

Again, \( \Omega \) transforms like \( \omega \).

### 5 An example

As an example of solution, we consider the Minkowski space, which is the only (local) vacuum solution of Einstein’s equation in three dimensions.

The metric is, in the standard coordinates, given by
\[ ds^2 = dt^2 - dx^2 - dy^2. \] (44)

From the 1-parameter family of (or “\( S^1 \)’s worth” of) null vector fields given by
\[ l^a = \left( \frac{\partial}{\partial t} \right)^a - \cos \zeta \left( \frac{\partial}{\partial x} \right)^a - \sin \zeta \left( \frac{\partial}{\partial y} \right)^a, \] (45)
we can easily have a 1-parameter family of null foliations;
\[ u = Z(x^a, \zeta) = l_ax^a = t + x \cos \zeta + y \sin \zeta. \] (46)

Operating \( \partial^a \), we have
\[ \omega = \partial Z = -x \sin \zeta + y \cos \zeta, \]
\[ R = \partial^2 Z = -x \cos \zeta - y \sin \zeta, \]
\[ \Lambda = \partial^3 Z = x \sin \zeta - y \cos \zeta. \] (47)
We, thus, have
\[ \Lambda = -\omega. \quad (48) \]
As for \( \Omega \), like in the previous section, thinking of Eqs.(46) and (47) as a coordinate transformation, we have
\[ \Omega = 1, \quad (49) \]
since the metric in terms of the intrinsic coordinates is found to be
\[ ds^2 = du^2 + 2du dR - d\omega^2. \quad (50) \]
We can easily check that Eqs.(48) and (49) satisfy the fundamental equations (20), (21), (24), and (28).

6 Conclusion
We have formulated the three dimensional version of NSF. It inherits all the properties of the original one, but is, of course, much simpler. We have explicitly discussed two simple classes of gauge with our simple version of NSF. As we have seen, gauge in NSF is not just diffeomorphism of the manifold — it presumably corresponds to a subclass of the diffeomorphisms, but is unclear up to now.

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