Perturbative Corrections to Heavy Quark-Diquark Symmetry Predictions for Doubly Heavy Baryon Hyperfine Splittings

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Abstract

Doubly heavy baryons ($QQq$) and singly heavy antimesons ($\bar{Q}q$) are related by the heavy quark-diquark (HQDQ) symmetry because in the $m_Q \to \infty$ limit, the light degrees of freedom in both the hadrons are expected to be in identical configurations. Hyperfine splittings of the ground states in both systems are nonvanishing at $O(1/m_Q)$ in the heavy quark mass expansion and HQDQ symmetry relates the hyperfine splittings in the two sectors. It was expected that corrections to this prediction would scale as $O(1/m_Q^2)$. In this paper, working within the framework of Non-Relativistic QCD (NRQCD), we point out the existence of an operator that couples four heavy quark fields to the chromomagnetic field with a coefficient that is enhanced by a factor from Coulomb exchange. This operator gives a correction to doubly heavy baryon hyperfine splittings that scales as $1/m_Q^2 \times \alpha_s/r$, where $r$ is the separation between the heavy quarks in the diquark. This correction can be calculated analytically in the the extreme heavy quark limit in which the potential between the quarks in the diquark is Coulombic. In this limit, the correction is $O(\alpha_s^2/m_Q)$ and comes with a small coefficient. For values of $\alpha_s$ relevant to doubly charm and doubly bottom systems, the correction to the hyperfine splittings in doubly heavy baryons is only a few percent or smaller.

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I. INTRODUCTION

The first doubly charm baryon $\Xi_{cc}^{++}$ with mass $3621.40 \pm 0.72 \pm 0.27 \pm 0.14$ MeV was recently observed by the LHCb collaboration in the exclusive decay modes, $\Xi_{cc}^{++} \rightarrow \Lambda_c^+ K^+ \pi^+ \pi^+$ and $\Xi_{cc}^{++} \rightarrow \Xi_c^+ \pi^+$ \cite{1,2}. Even though the SELEX collaboration \cite{3,4} had earlier reported the observation of doubly charmed baryons years ago, those observations were not confirmed by other experiments such as FOCUS \cite{6}, Belle \cite{7,8}, and BaBar \cite{9}. The large isospin violation implied by the recent LHCb results also cast doubt on the validity of the SELEX results. The recent experimental observation of the $\Xi_{cc}^{++}$ baryon has greatly revived the interest in the physics of doubly heavy baryons. This includes the experimental efforts to search for the other doubly charm and bottom baryons such as $\Xi_{cc}^+$ \cite{10} as well as recent theoretical studies regarding the lifetimes, production rates, and decay rates of the double heavy baryons \cite{11-15}.

An interesting idea regarding the physics of doubly heavy baryons is that of heavy quark-diquark symmetry (HQDQ) which relates the physics of doubly heavy baryons ($QQq$) to the heavy antimesons ($\bar{Q}q$). The appropriate theory for dealing with heavy mesons is the heavy quark effective field theory (HQET) \cite{16-18}, whereas the appropriate theory for dealing with doubly heavy baryons is nonrelativistic quantum chromodynamics (NRQCD) \cite{19,20}. In the limit of large heavy quark mass $m_Q$, the two heavy quarks $QQ$ in the doubly heavy baryons experience an attractive Coulomb force and the ground state of the two heavy quarks is a tightly bound spin-1 diquark in the $\bar{3}$ color representation. The size of the diquark is small, $r \sim (1/m_Qv)^{-1} \ll \Lambda_{QCD}^{-1}$, where $v$ is the relative velocity of the two heavy quarks in the diquark. This implies that the diquark can be considered as a point source of color charge in the $\bar{3}$ representation that looks the same to the light degrees of freedom as a singly heavy antiquark, up to corrections that are suppressed by inverse powers of heavy quark mass $m_Q$.

The light degrees of freedom in the heavy antimeson also orbit a point source of color charge in the $\bar{3}$ representation. Therefore, the two heavy hadrons have identical configurations for the light degrees of freedom in the $m_Q \rightarrow \infty$ limit. The HQDQ symmetry also relates the double heavy tetraquarks to singly heavy baryons and the chiral lagrangians incorporating this symmetry have been derived in Refs. \cite{21,22}.

One of the implications of the HQDQ symmetry is the relation between the hyperfine mass splittings of the doubly heavy baryons and heavy antimesons. The chromomagnetic interactions of the diquark and quark are responsible for the hyperfine splittings in the doubly heavy baryons and antimesons. The effective Lagrangian describing the chromomagnetic coupling of diquarks at $O(1/m_Q)$ was derived in Ref. \cite{23,24} in the framework of NRQCD. The ground state of the heavy antimeson consists of a spin-0 meson, $P$, and a spin-1 meson, $P^*$. The ground state of the
doubly heavy baryon consists of spin-1/2 baryon, \( \Xi \), and spin-3/2, baryon \( \Xi^* \). These states are degenerate due to heavy quark spin symmetry that breaks at \( \mathcal{O}(1/m_Q) \) due to spin dependent chromomagnetic interactions. The heavy quark-diquark symmetry implies the relation between the hyperfine splittings to be \[ m_{\Xi^*} - m_\Xi = \frac{3}{4} (m_{P^*} - m_P) \] (1)

The purpose of this paper is to study higher order corrections to this prediction. Since the hyperfine splittings themselves are \( \mathcal{O}(1/m_Q) \) one might expect leading corrections to scale as \( 1/m_Q^2 \). What we will see below is that there is a higher dimension operator that scales as \( 1/m_Q^2 \times \alpha_s/r \), where \( r \) is the typical separation between the quarks within the diquark. Since \( 1/r \sim m_Q v \), this operator gives a correction to Eq. (1) of relative order \( \alpha_s v \). In the extreme limit, where the quarks within the diquark are bound by Coulombic gluon exchange, \( v \) is proportional to \( \alpha_s \) and this \( \mathcal{O}(\alpha_s^2) \) correction is computed below. While this correction turns out to be only of order a percent or less for values of \( \alpha_s \) relevant to doubly charm and bottom baryons, we find the unexpected scaling with \( m_Q \) to be interesting. A similar correction to the HQDQ symmetry due to the finite size of diquark was calculated in Ref. [26]. The finite size effects were due to operators coupling the light quarks and the diquarks that contributes to the mass of the double heavy baryon. The correction to the HQDQ was also estimated to be small in Ref. [26]. In the body of this paper we review the effective action for heavy diquark fields, introduce the operator and compute its effect on the prediction for doubly heavy baryon hyperfine splittings. We then give our conclusions. In an Appendix, we derive the form of the operator by matching the full QCD diagrams for \( QQg \rightarrow QQ \) scattering onto NRQCD to \( \mathcal{O}(1/m_Q^2) \).

II. EFFECTIVE ACTION FOR COMPOSITE DIQUARK FIELDS

The effective action for the heavy composite diquark fields with the lowest order heavy quark spin symmetry violating chromomagnetic interaction was derived by Fleming and Mehen in Ref. [23] and Brambilla, Vairo, and Rosch in Ref. [24] in the framework of NRQCD. The leading order chromomagnetic couplings of diquarks gives \( \mathcal{O}(1/m_Q) \) corrections to the heavy quark spin symmetry and is responsible for the hyperfine splittings in the ground state of doubly heavy baryons. In this section, we use the formalism in Ref. [23] to include the correction to the chromomagnetic coupling of diquark fields from diagrams that contribute to the effective action at higher order in NRQCD power counting.

The NRQCD Lagrangian relevant for constructing the effective action for composite diquarks
\[ \mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \sum_p \psi_p^\dagger \left( i D^0 - \frac{(p - i D)^2}{2 m_Q} + \frac{g}{2 m_Q} \sigma \cdot B \right) \psi_p \]

\[ - \frac{1}{2} \sum_{p, q} \frac{g_s^2}{(p - q)^2} \psi_{-q}^\dagger T_A \psi_p \psi_{-q}^\dagger T^A \psi_p + \ldots, \]

where \( \psi_p \) represents the quark field with a three vector label \( p \), \( B \) is the chromomagnetic field, and the ellipsis represents the higher order corrections as well as terms including soft gluons. The color and spin Fierz identities in Eqs. (3) and (4) introduces four terms but two of them vanish due to Fermi statistics. The diquark fields in Eq. (7) are in the \( 6 \) representation of the Dirac \( \beta \)-function and \( 3 \) representation in color space:

\[ \delta_{\alpha \beta} \delta_{\gamma \delta} = -\frac{1}{2} (\sigma^i \epsilon)_{\alpha \beta} (\epsilon \sigma^i)_{\gamma \delta} + \frac{1}{2} \epsilon_{\alpha \beta} \epsilon_{\gamma \delta}, \]

\[ T_{ij} r_{jk} = -\frac{2}{3} \sum_m \frac{1}{2} \epsilon_{mij} \epsilon_{mlk} + \frac{1}{3} \sum_{(mn)} d_{ij}^{(mn)} d_{kl}^{(mn)}, \]

where the Greek letters refer to spin indices, the Roman letters refer to color indices, \( \sigma^i \) denotes the Pauli matrices, \( \epsilon = i \sigma^2 \) is an anti-symmetric \( 2 \times 2 \) matrix, and \( d_{ij}^{(mn)} \) are symmetric matrices in color space:

\[ d_{ij}^{(mn)} = \begin{cases} \frac{(\delta_i^m \delta_j^n + \delta_i^n \delta_j^m)}{\sqrt{2}} & m \neq n \\ \delta_i^m \delta_j^n & m = n \end{cases} \]
in color space, and have spin-1 and spin-0, respectively. We define the following composite
diquark operators

\[ T^i_r = \sum_p e^{ip} r \frac{1}{2} e^{ijk} (\psi_p)_{j} \sigma(\psi_p)_{k} , \]  

(8)

\[ \Sigma^{(mn)}_r = \sum_p e^{ip} r \frac{1}{\sqrt{2}} d^{(mn)}_{ij} (\psi_p)_{i} \epsilon^{T}(\psi_p)_{j} , \]  

(9)

where \( T^i_r \) is a spin-1 vector field and \( \Sigma^{(mn)}_r \) is a spin-0 scalar field.

The composite diquark fields, \( T^i_r \) and \( \Sigma^{(mn)}_r \), enter the theory by using the Hubbard-
Stratonovich transformation, which cancels the quartic interaction terms in heavy quark fields
in favor of interaction terms between the diquark fields and the two heavy quark fields:

\[
\Delta \mathcal{L} = \frac{1}{2} \int d^3 r V^{(3)}(r) \left( T^i_r - \sum_q e^{-iq \cdot r} \frac{1}{2} (\psi^i_q)_{j} \sigma(\psi^i_{-q})_{k} \right) 
\times \left( T^i_r - \sum_p e^{ip} r \frac{1}{2} \epsilon_{ilm}(\psi_p)_{l} \epsilon(\psi_p)_{m} \right) 
+ \frac{1}{2} \int d^3 r V^{(6)}(r) \left( \Sigma^{(mn)}_r \right) 
\times \left( \Sigma^{(mn)}_r - \sum_p e^{ip} r \frac{1}{\sqrt{2}} d^{(mn)}_{ij} (\psi_p)_{i} \epsilon^{T}(\psi_p)_{j} \right). 
\]  

(10)

The NRQCD Lagrangian after using the Hubbard-Stratonovich transformation reduces to

\[
\mathcal{L} + \Delta \mathcal{L} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + \sum_p \psi^i_p \left( iD^0 - \frac{(p - iD)^2}{2m_Q} + \frac{g}{2m_Q} \sigma \cdot B \right) \psi_p 
+ \frac{1}{2} \int d^3 r V^{(3)}(r) \left( T^i_r T^i_r - T^i_r T^i_r \right) 
\times \left( \Sigma^{(mn)}_r \right) 
+ \frac{1}{2} \int d^3 r V^{(6)}(r) \left( \Sigma^{(mn)}_r \right) 
\times \left( \Sigma^{(mn)}_r - \sum_p e^{ip} r \frac{1}{\sqrt{2}} d^{(mn)}_{ij} (\psi_p)_{i} \epsilon^{T}(\psi_p)_{j} \right). 
\]  

(11)

The Feynman rules describing the interaction of diquarks with two heavy quarks corresponding
to the above Lagrangian are shown in Fig[1].

The \( \sigma \cdot B \) term in the NRQCD Lagrangian in Eq. (2) is the chromomagnetic interaction for
heavy quarks. This is the lowest order heavy quark spin symmetry violating term that gives \( \mathcal{O}(1/m_Q) \) corrections to the heavy quark spin symmetry and is responsible for the hyperfine
FIG. 1: Feynman rules for the coupling of the composite diquark fields to quarks.

splittings in the ground state of heavy mesons. The chromomagnetic coupling for the heavy
diquark field $T^i_r$ was derived in Ref. [23] by considering the two one-loop diagrams shown in
Fig. 2 which contributes at $O(v^2)$ to the effective action in the NRQCD power counting. The
effective Lagrangian for the diquark field $T^i_r$ which gives $O(1/m_Q)$ corrections to the heavy
quark spin symmetry and is responsible for the hyperfine splittings in the ground state of
doubly heavy baryons is

$$L_{\sigma, B} = i \frac{g}{2m_Q} \int d^3r \ T^i_r \cdot B^c \ T^c_j \times T^j_r.$$  (13)

Other $O(v^2)$ couplings of the diquark field, $T^i_r$, which do not violate the heavy quark spin
symmetry can be found in Ref. [27]. The composite diquark field, $\Sigma^{(mn)}_r$, is a scalar and

therefore does not have a chromomagnetic coupling. Other chromomagnetic couplings are
possible if one considers diquarks composed of two different heavy quarks.

The effective Lagrangian for the diquark field, $T^i_r$, in Eq. (13) can have corrections from terms
that contribute at $O(v^3)$ and higher to the effective action in the NRQCD power counting. The
leading corrections to the chromomagnetic coupling of diquarks come from a two-loop diagram
shown in Fig. 3. The two-loop diagram contributes at $O(v^4)$ to the effective action and gives
$O(1/m_Q^2)$ corrections to the heavy quark spin symmetry. It arises from an effective five point
contact interaction shown in Fig. 4, which is obtained after matching tree-level scattering of
two heavy quarks and gluon in QCD and NRQCD. The Lagrangian for the effective operator in Fig. 4 is given by Eq. (A12) and a detailed derivation is shown in Appendix A.

In order to evaluate the correction to the chromomagnctic coupling of a diquark from the two-loop diagram in Fig. 3, we consider the external diquark fields $T_r^i$ and $T_{r'}^i$ to be at rest and have energy $E$ and $E'$ respectively. The external diquarks have spin indices $k$ and $l$ and color indices $a$ and $b$ respectively. The usoft gluon has polarization index $m$ and color index $c$.

Using the Feynman rules for the diquark-quark interaction in Fig. 1 and the effective four-quark contact vertex in Eq. (A12), the two-loop diagram in Fig. 3 evaluates to

$$i\Sigma = -\frac{g^3}{6m_Q^2} \epsilon_{klm} T_{ba} \int \frac{d^4l}{(2\pi)^3} \int \frac{d^4l'}{(2\pi)^3} \frac{e^{-il \cdot r} V^{(3)}(r)}{E - l^2/m_Q + ie} \frac{e^{il' \cdot r'} V^{(3)}(r')}{{E'} - l'^2/m_Q + ie} \frac{B_m^c}{|l - l'|^2}. \tag{14}$$

The effective Lagrangian describing the leading correction to the chromomagnetic coupling of diquark field $T_r$ in Eq. (13) is

$$\mathcal{L}'_{r,B} = \int d^4r \int d^4r' T_r^l \Sigma T_{r'}, \tag{15}$$

where $i\Sigma$ is given in Eq. (14) and the color and spin indices of the diquark field, $T_r$, have been suppressed. This Lagrangian can be easily interpreted using the notation of Ref. [23], where
the diquark field $T_r$ are thought of as vectors in a Hilbert space spanned by the position space eigenkets $|r⟩$:

$$\mathcal{L}_{\sigma,B} = i \frac{g^3}{6m_Q^2} \epsilon_{klm} \tilde{T}_m B^c \langle T| \hat{V}^{(3)} \frac{1}{E' - H_0} \frac{1}{4\pi r} \frac{1}{E - H_0} \hat{V}^{(3)} |T⟩,$$

where we define the diquark field $T_r \equiv \langle r|T⟩$, the potential operator $\langle r'|\hat{V}^{(3)}|r⟩ \equiv V^{(3)} \delta^3(r - r')$, the momentum eigenstates $\langle r|L⟩ = e^{-i\mathbf{l} \cdot \mathbf{r}}$, and a free Hamiltonian $H_0|L⟩ = \mathbf{l}^2/m_Q|L⟩$. Using the formalism developed in Ref. [23], the potential operator $\hat{V}^{(3)}$ in Eq. (16) cancels against the factors of $E - H_0$ in the denominator after using the equation of motion for the diquark field $T_r$. Therefore, the effective Lagrangian describing the leading correction to the chromomagnetic coupling of diquarks in Eq. (13) is

$$\mathcal{L}_{\sigma,B} = i \frac{g^2}{2m_Q} \frac{\alpha_s}{3m_Q} \int d^4r \ T_i^{\dagger} \cdot \frac{1}{r} \mathbf{B} \cdot \tilde{T}_{ij} \times T_r^j.$$

The effective Lagrangian in above equation contributes at $\mathcal{O}(v^4)$ to the effective action of diquarks because the chromomagnetic field $B$ scales as $\mathcal{O}(v^4)$, the diquark field $T_r$ scales as $\mathcal{O}(v^3)$, the strong coupling constant $\alpha_s$ scales as $\mathcal{O}(v)$, the position vector $r$ scales as $\mathcal{O}(v^{-1})$, and the integration measure $d^4x$ scales as $\mathcal{O}(v^{-5})$ in the NRQCD power counting that was developed in Ref. [20].

The mesons $P$ and $P^*$ and the baryons $\Xi$ and $\Xi^*$ are both degenerate in the absence of the chromomagnetic interactions that is suppressed by an inverse power of $m_Q$. The chromomagnetic coupling of heavy quarks in Eq. (2) and the chromomagnetic coupling of heavy diquarks in Eqs. (13) and (17) lead to mass splittings of heavy mesons and doubly heavy baryons in ground state. Since the ground state of diquarks in the double heavy baryons is an $s$-wave ($l = 0$), the spatial wavefunction of the diquark can be approximated by a hydrogen-like spatial wavefunction in the limit of extremely large $m_Q$:

$$\phi(r) = \left(\frac{1}{\pi a_0^3}\right)^{1/2} e^{-r/a_0},$$

where $a_0$ is the Bohr radius given by

$$a_0 = \frac{3}{\alpha_s m_Q}.$$

The identical configurations of light degrees of freedom in the heavy $\bar{Q}q$ meson and the heavy $QQq$ baryons implies that the hyperfine mass splittings in the heavy antimeson and the doubly heavy baryon are related by the heavy quark-diquark symmetry. The hyperfine splittings depends on the matrix elements of the chromomagnetic couplings of quarks and diquarks in Eqs. (2), (13), and (17). The matrix element of the $\mathcal{O}(1/m_Q^2)$ chromomagnetic
coupling of diquark in Eq. (17) depends on \( \langle 1/r \rangle \):

\[
\langle 1/r \rangle = \int d^3r \frac{|\phi(r)|^2}{r} = \frac{1}{a_0}.
\]  

This gives a correction to the matrix element of the chromomagnetic operator in Eq. (13) that appears only in the doubly heavy baryon mass splitting as there is no analogue correction in the heavy meson sector. Therefore, the relation between the hyperfine splittings in Eq. (1) is modified to

\[
m_{P^*} - m_P = \frac{4}{3} (m_{\Xi^*} - m_{\Xi}) \left( 1 + \frac{\alpha_s}{3m_Q} \langle 1/r \rangle \right),
\]

where \( \alpha_s^2/9 \) is the correction to the hyperfine splitting from the two-loop diagram in Fig. 3. Of the two powers of \( \alpha_s \) appearing in Eq. (21), one arises from matching QCD onto NRQCD, so naturally lives at the scale \( m_Q \), while the other appears in the evaluation of the matrix element, \( \langle 1/r \rangle \), and naturally lives at the scale \( m_Qv \). It would be interesting to compute the anomalous dimension of the operator in Eq. (17) in order to sum logarithms of ratios of these two scales, but that is beyond the scope of this work.

For numerical purposes, we will use the value of \( \alpha_s \) evaluated at the scale \( m_Qv \). If the doubly heavy baryons are composed of charm quarks, the value of the strong coupling constant is \( \alpha_s(m_cv) \approx 0.52 \), which implies the correction to the hyperfine splitting is \( 3 \times 10^{-2} \). If the doubly heavy baryons are composed of bottom quarks, the value of strong coupling constant is \( \alpha_s(m_bv) \approx 0.35 \), which implies the correction to the hyperfine splitting is \( 1.4 \times 10^{-2} \). Therefore, we conclude the heavy quark diquark symmetry prediction receives very small correction at \( \mathcal{O}(v^2) \), at least as \( m_Q \to \infty \).

**III. CONCLUSION**

The ground state mass hyperfine splittings in the double heavy baryons and singly heavy antimesons are related by the heavy quark-diquark symmetry (HQDQ). The hyperfine splittings are due to the \( \mathcal{O}(1/m_Q) \) chromomagnetic couplings of the diquark and quark and the leading prediction for the splittings is given by Eq. (1). In this paper, we compute the leading correction to the hyperfine splitting of the double heavy baryons in the framework of NRQCD. We point out an effective five-point contact operator that couples the four heavy quark fields with the chromomagnetic field with a coefficient that is enhanced by the Coulomb exchange. Naively, one would expect the leading correction to the chromomagnetic coupling of the diquark to scale as \( 1/m_Q^2 \), instead we find that the correction from the effective operator scales as \( 1/m_Q^2 \times \alpha_s/r \),
where $r$ is the separation between the heavy quarks in the diquark. The Lagrangian describing the leading correction to the chromomagnetic coupling of diquark is given by Eq. (17).

We estimate the correction to the ground state mass hyperfine splitting in the doubly heavy baryons due to the next leading order Lagrangian in Eq. (17). We find that in the $m_Q \to \infty$ limit, when the two quarks within the diquark are bound by strong Coulombic interaction, the leading correction to the hyperfine splitting of double heavy baryons is of $\mathcal{O}(\alpha_s^2/m_Q)$ with a small coefficient as shown in Eq. (21). For values of $\alpha_s$ relevant to doubly charm and doubly bottom systems, we find that the correction to the hyperfine splitting in doubly heavy baryons is $3 \times 10^{-2}$ for doubly charm baryons and $1.4 \times 10^{-2}$ for doubly bottom baryons. While the scaling $\mathcal{O}(\alpha_s^2/m_Q)$ is unexpected, the size of the correction to the hyperfine splittings is very small.

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Appendix A: Effective five-point contact operator

In this Appendix, we derive the Lagrangian for the effective contact operator shown in Fig. 4. The effective five-point contact operator with four heavy quarks and one gluon in Fig. 4 gives an $\mathcal{O}(1/m_Q^2 \times \alpha_s/r)$ correction to the chromomagnetic coupling of diquark field $T_r$. This effective operator is obtained after matching the low-energy tree diagrams for $QQ \to QQg$ in full QCD theory onto NRQCD. In QCD, the diagrams for $QQ \to QQg$ are shown in Fig. 5. The two tree-diagrams at the top of Fig. 5, where the external gluon is attached to the external quarks, match onto two distinct types of NRQCD diagrams. One diagram is the tree diagram shown in Fig. 6, in which the gluon couples to an external quark via the chromomagnetic interaction and there is a virtual nonrelativistic quark. The other diagram is the contact interaction in Fig. 4. The bottom two diagrams in Fig. 5, where the external gluon is attached to the exchanged gluon via the three-gluon vertex, could in principle also contribute. However, the bottom two diagrams in Fig. 5 have a vanishing color factor when the incoming and outgoing diquarks are both in the $\bar{3}$ representation.

In Fig. 5, the incoming heavy quarks have four-momenta $p_1 = (E_1, P_1)$ and $p_2 = (E_2, P_2)$ and color indices $i$ and $j$ respectively. The outgoing heavy quarks have four momenta given by
FIG. 5: Full QCD diagrams for $QQ \rightarrow QQg$. There are six other diagrams in QCD similar to the upper two diagrams, in which the external gluon is attached to each of the external quarks.

FIG. 6: One of the two types of NRQCD required to reproduce the full QCD result at low energy. The other is the contact interaction shown in Fig. [4]

$p_3 = (E_3, P_3)$ and $p_4 = (E_4, P_4)$ and color indices $r$ and $l$ respectively. The external gluon have four-momenta $q$ and color index $c$. In full QCD, the contribution to the low-energy scattering amplitude from the upper two diagrams of Fig. [5] is

$$iA = ig^3 T^a_{rk'} T^{c}_{kj} T^a_{lj} \varepsilon_m \left[ \bar{u}_i (p_4) \gamma_\mu u_j (p_2) \right] \left[ \bar{u}_r (p_3) \gamma_\mu (k + m_Q) \gamma^m u_i (p_1) \right] \frac{1}{(k^2 - m_Q^2 + i\epsilon) (p^2 + i\epsilon)} - (p_3 \leftrightarrow p_4, r \leftrightarrow l),$$

(A1)

where $p = p_4 - p_2$ is the four-momenta of the intermediate gluon in the first term, $k = p_3 + p$ is the four-momenta of the intermediate fermion in the first term, $\varepsilon^m$ is the polarization 4-vector of the incoming gluon and $m_Q$ is the mass of the heavy quark. In the above expression we have explicitly shown the color indices and suppressed the spinor indices. Using the equation
of motion for the external states, the amplitude in Eq. (A1) can be written as

$$iA = ig^3 T_{r'k'}^a T_{k'i}^c T_{lj}^e \varepsilon_m \left[ \bar{u}(p_4) \gamma_\mu u_j(p_2) \right] \left[ \bar{u}_r(p_3) \left( 2p_3^\mu + \gamma_\mu \hat{p} \right) \gamma^m u_i(p_1) \right] \Bigg( k^2 - m_Q^2 + i\epsilon \Bigg) \left( p^2 + i\epsilon \right) - (p_3 \leftrightarrow p_4, r \leftrightarrow l).$$

(A2)

Using the identity for the product of three gamma matrices, the terms in the second square bracket in the numerator of the above equation can rewritten as

$$\bar{u}(p_3) \left( 2p_3^\mu \gamma^m \varepsilon_m + p^\mu \gamma^m \varepsilon_m - \varepsilon^\mu p_m \gamma^m + p^m \varepsilon_m \gamma^\mu - i\epsilon^\mu\varepsilon_m \gamma_\mu \gamma_5 \varepsilon_m \gamma^5 \right) u(p_1).$$

(A3)

In the above expression, $p_3 \sim \mathcal{O}(m_Q)$ and $p = p_4 - p_2 \sim \mathcal{O}(m_Q v)$, thus, the leading contribution comes only from the first term in the parenthesis. The low-energy scattering amplitude for $QQ \rightarrow QQg$ can be obtained from QCD by doing the nonrelativistic expansion of the amplitude in Eq. (A1) in powers of 3-momenta. Using the Gordon identity

$$\bar{u}(p) \gamma^\mu u(q) = \bar{u}(p) \left[ \frac{(p + q)^\mu}{2m} + i\sigma^{\mu\nu} \frac{(p - q)_\nu}{2m} \right] u(q),$$

(A4)

and then taking the nonrelativistic limit of the Dirac spinors

$$u_i(P) = \begin{pmatrix} \xi_i \\ 0 \end{pmatrix},$$

(A5)

where $\xi_i$ is a two-spinor, and $i$ denotes the color index, we find that the leading term with a spin-dependent interaction in the nonrelativistic expansion of the numerator in Eq. (A2) is

$$2p_3^0 \left[ \bar{u}(p_4) \gamma_0 u(p_2) \right] \left[ \bar{u}(p_3) \gamma_0 u(p_1) \right] = -i\xi_{4,i} \xi_{2,j} \xi_{3,r} \sigma \cdot (q \times \varepsilon) \; \xi_{1,i} + \cdots,$$

(A6)

where $\varepsilon$ is the three-space polarization vector and the ellipsis represents terms that are suppressed by powers of $v$ or do not explicitly break the heavy quark spin symmetry.

In the nonrelativistic limit, the fermion propagator in the denominator of the amplitude in Eq. (A2) is given by

$$\frac{1}{k^2 - m_Q^2 + i\epsilon} = \frac{1}{2m_Q} \left[ \frac{1}{k_0 - E_k + i\epsilon} - \frac{1}{2m_Q} \right] + \mathcal{O}(P^2/m_Q^4),$$

(A7)

where $k = P_3 + P_4 - P_2$ and

$$k_0 - E_k = \frac{P_3^2}{2m_Q} + \frac{P_4^2}{2m_Q} - \frac{P_2^2}{2m_Q} - \frac{(P_3 + P_4 - P_2)^2}{2m_Q}.$$

(A8)

Similarly, the gluon propagator can also be expanded in powers of 3-momenta as

$$\frac{1}{(p_4 - p_2)^2 + i\epsilon} = \frac{1}{(P_4 - P_2)^2} \left( 1 + \mathcal{O}(P^2/m_Q^2) \right),$$

(A9)
Using the nonrelativistic expansion of the numerator in Eq. (A6) and the nonrelativistic expansion of the fermion and gluon propagators in Eq. (A7) and (A9), the low-energy scattering amplitude in Eq. (A2) is

\[
iA = ig^3 T^a_{rt} T^c_{r't'} T^a_{i'j'} \left[ \xi_1^i \xi_2^j \xi_3^k \xi_4^l \right] \frac{1}{k_0 - E_k + i\epsilon} \frac{1}{2m_Q} \frac{1}{(P_4 - P_2)^2} \]

\[- (P_3 \leftrightarrow P_4, r \leftrightarrow l) + \cdots,
\]

where \( \mathbf{k} = \mathbf{P}_3 + \mathbf{P}_4 - \mathbf{P}_2 \) and \( k_0 - E_k \) is given by Eq. (A8). In the above expression we have suppressed both the color and spin indices. The low-energy amplitude in Eq. (A10) has a pole where the intermediate fermion propagator goes on-shell. This pole will be reproduced in the effective theory by the contribution to the scattering amplitude from Fig. 6. If the vertex with the external gluon line is the chromomagnetic coupling of the heavy quark, then the contribution to the scattering amplitude from Fig. 6 is

\[
iA = ig^3 T^a_{rt} T^c_{r't'} T^a_{i'j'} \left[ \xi_1^i \xi_2^j \xi_3^k \xi_4^l \right] \frac{1}{k_0 - E_k + i\epsilon} \frac{1}{2m_Q} \frac{1}{(P_4 - P_2)^2} \]

\[- (\xi_4 \leftrightarrow \xi_3, P_4 \leftrightarrow P_3, r \leftrightarrow l),
\]

\( k = \mathbf{P}_3 + \mathbf{P}_4 - \mathbf{P}_2 \) and \( k_0 - E_k \) is given by Eq. (A8). In the above expression we have suppressed both the color and spin indices. On comparing Eqs. (A10) and (A11), we see that the pole in the low-energy scattering amplitude in Eq. (A10) from the intermediate fermion propagator is cancelled exactly by the corresponding contribution in NRQCD. After cancelling the pole, we are left with a term which is reproduced by the effective four-quark and gluon contact interaction shown in Fig. 4. The five-point operator with four heavy quarks and one gluon receives contributions from six other diagrams in QCD and three other diagrams in NRQCD where the external gluon is attached to each of the external quarks. After taking into consideration all these diagrams, the Lagrangian for the five-point contact interaction with four heavy quarks and one gluon in Fig. 4 is

\[
\mathcal{L}_{eff} = -\frac{g^3}{2} \frac{1}{2m_Q} \sum_{P_1, P_2, P_3, P_4} \psi_1^\dagger P_4 T^a \psi_2 P_2 T^c \psi_3 P_3 (T^a T^c + T^c T^a) \frac{\sigma \cdot B^c}{2m_Q} \psi_4 P_4 \frac{1}{(P_4 - P_2)^2}
\]

(A12)

The Feynman rule for the contact vertex in Fig. 4 is given by

\[- i \frac{g^3}{4m_Q^2} T^a \otimes (T^a T^c + T^c T^a) \frac{\sigma \cdot B^c}{(P_4 - P_2)^2}.
\]

(A13)

[1] R. Aaij et al. [LHCb Collaboration], Phys. Rev. Lett. 119, no. 11, 112001 (2017) arXiv:1707.01621 [hep-ex].
[2] R. Aaij et al. [LHCb Collaboration], Phys. Rev. Lett. 121, no. 16, 162002 (2018) arXiv:1807.01919 [hep-ex].

[3] M. Mattson et al. [SELEX Collaboration], Phys. Rev. Lett. 89, 112001 (2002). arXiv:hep-ex/0208014.

[4] M. A. Moinester et al. [SELEX Collaboration], Czech. J. Phys. 53, B201 (2003). arXiv:hep-ex/0212029.

[5] A. Ocherashvili et al. [SELEX Collaboration], Phys. Lett. B 628, 18 (2005) doi:10.1016/j.physletb.2005.09.043 hep-ex/0406033.

[6] S.P. Ratti, Nucl. Phys. Proc. Suppl. 115, 33 (2003).

[7] R. Chistov et al. [Belle Collaboration], Phys. Rev. Lett. 97, 162001 (2006) hep-ex/0606051.

[8] Y. Kato et al. [Belle Collaboration], Phys. Rev. D 89, no. 5, 052003 (2014) arXiv:1312.1026 [hep-ex]].

[9] B. Aubert et al. [BaBar Collaboration], Phys. Rev. D 74, 011103 (2006) hep-ex/0605075.

[10] M. T. Traill [LHCb Collaboration], PoS Hadron 2017, 067 (2018). doi:10.22323/1.310.0067

[11] H. Y. Cheng and Y. L. Shi, Phys. Rev. D 98, 113005 (2018) doi:10.1103/PhysRevD.98.113005 arXiv:1809.08102 [hep-ph].

[12] A. V. Berezhnoy, A. K. Likhoded and A. V. Luchinsky, Phys. Rev. D 98, 113004 (2018) doi:10.1103/PhysRevD.98.113004 arXiv:1809.10058 [hep-ph]].

[13] H. Y. Cheng and F. Xu, Phys. Rev. D 99, 073006 (2019) doi:10.1103/PhysRevD.99.073006 arXiv:1903.08148 [hep-ph]].

[14] V. V. Kiselev and A. K. Likhoded, Phys. Usp. 45, 455 (2002) [Usp. Fiz. Nauk 172, 497 (2002)].

[15] A. K. Ridgway and M. B. Wise, Phys. Lett. B 793, 181 (2019). 10.1016/j.physletb.2019.04.020 arXiv:1902.04582 [hep-ph]]

[16] H. Georgi and M. B. Wise, Phys. Lett. B 243, 279 (1990). doi:10.1016/0370-2693(90)90851-V

[17] A. F. Falk, H. Georgi, B. Grinstein and M. B. Wise, Nucl. Phys. B 343, 1 (1990). doi:10.1016/0550-3213(90)90591-Z

[18] E. Eichten and B. R. Hill, Phys. Lett. B 234, 511 (1990). doi:10.1016/0370-2693(90)90249-O

[19] G. T. Bodwin, E. Braaten and G. P. Lepage, Phys. Rev. D 51, 1125 (1995) [Erratum-ibid. D 55, 5853 (1997)]. arXiv:hep-ph/9407339.

[20] M. E. Luke, A. V. Manohar and I. Z. Rothstein, Phys. Rev. D 61, 074025 (2000). arXiv:hep-ph/9910209.

[21] J. Hu and T. Mehen, Phys. Rev. D 73, 054003 (2006) doi:10.1103/PhysRevD.73.054003 hep-ph/0511321.

[22] T. Mehen, Phys. Rev. D 96, no. 9, 094028 (2017) doi:10.1103/PhysRevD.96.094028
[23] S. Fleming and T. Mehen, Phys. Rev. D 73, 034502 (2006).

[24] N. Brambilla, A. Vairo and T. Rosch, Phys. Rev. D 72, 034021 (2005). [arXiv:hep-ph/0506065].

[25] M. J. Savage and M. B. Wise, Phys. Lett. B 248, 177 (1990). doi:10.1016/0370-2693(90)90035-5

[26] H. Ann and M. B. Wise, Phys. Lett. B 788, 131 (2016). doi:10.1016/j.physletb.2018.11.004 [arXiv:1809.02139]

[27] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Nucl. Phys. B 566, 275 (2000). [arXiv:hep-ph/9907240].