REFLECTION POSITIVITY ON SPHERES

KARL-HERMANN NEEB AND GESTUR ´OLAFSSON

ABSTRACT. In this article we specialize a construction of a reflection positive Hilbert space due to Dimock and Jaffe–Ritter to the sphere \( S^n \). We determine the resulting Osterwalder–Schrader Hilbert space, a construction that can be viewed as the step from euclidean to relativistic quantum field theory. We show that this process gives rise to an irreducible unitary spherical representation of the orthochronous Lorentz group \( G^c = O_{1,n}(\mathbb{R})^\dagger \) and that the representations thus obtained are the irreducible unitary spherical representations of this group. A key tool is a certain complex domain \( \Xi \), known as the crown of the hyperboloid, containing a half-sphere \( S^n \) and the hyperboloid \( \mathbb{H}^n \) as totally real submanifolds. This domain provides a bridge between those two manifolds when we study unitary representations of \( G^c \) in spaces of holomorphic functions on \( \Xi \). We connect this analysis with the boundary components which are the de Sitter space and a bundle over the space of future pointing lightlike vectors.

Keywords: Reflection positivity, symmetric spaces, dissecting involutions, positive definite kernels, spherical representations

MSC [2010] Primary: 22E70, 4385. Secondary: 43A35, 43A80, 58Z99, 81T08

1. INTRODUCTION

In this article we continue our work on reflection positivity and its connection to representation theory and abstract harmonic analysis, concerning the passage from the compact group \( O_{n+1}(\mathbb{R}) \) to its \( c \)-dual group \( O_{1,n}(\mathbb{R})^\dagger \).

To make this more precise, recall that a symmetric Lie group is a pair \((G, \tau)\), consisting of a Lie group \( G \) with an involutive automorphism \( \tau \). The Lie algebra \( g \) of \( G \) decomposes into \( \tau \)-eigenspaces \( g = h \oplus q \), where \( h = \ker(\tau - 1) \) and \( q = \ker(\tau + 1) \). A Lie group \( G^c \) with Lie algebra \( g^c = h \oplus iq \) is called the Cartan dual, or for short \( c \)-dual group \( G \). Reflection positivity now provides a passage from certain unitary representations of \( G \) to unitary representations of \( G^c \). One considers representations \((U, \mathcal{E})\) of \( G \) on reflection positive Hilbert spaces \((\mathcal{E}, \mathcal{E}_+, \theta)\), i.e., \( \mathcal{E}_+ \subseteq \mathcal{E} \) is a closed subspace and \( \theta \) is a unitary involution for which \( \langle \xi, \xi \rangle_\theta := \langle \xi, \theta \xi \rangle \geq 0 \) for \( \xi \in \mathcal{E}_+ \). We further assume that \( \theta U(g) \theta = U(\tau(g)) \) for \( g \in G \). Then the Hilbert space \( \tilde{\mathcal{E}} \) defined by \( \langle \cdot, \cdot \rangle_\theta \) on \( \mathcal{E}_+ \) is expected to carry a unitary representation \((U^c, \tilde{\mathcal{E}})\) of the \( c \)-dual group \( G^c \) (at least if it is 1-connected). Then we call \((U, \mathcal{E})\) a euclidean realization of \((U^c, \tilde{\mathcal{E}})\). We refer to [NÓ18, §§1.3] for background and details.

There is a natural source of reflection positive Hilbert spaces in Riemannian geometry. We start with a complete Riemannian manifold \( M \) and an involutive isometry \( \sigma : M \to M \) which is dissecting in the sense that the submanifold \( M^\sigma \) of fixed points is of codimension one and its complement \( M_{\pm} \) consists of two connected components satisfying \( \sigma(M_+) = M_- \). Typical examples relevant in our context are:

(a) \( \sigma(x) = (-x_0, x_1, \ldots, x_n) \) on \( \mathbb{R}^{n+1} \), where \( M_+ = \{x \in \mathbb{R}^{n+1} : x_0 > 0\} \) is an open half space.

(b) \( \sigma(x) = (-x_0, x_1, \ldots, x_n) \) on the sphere \( S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\} \subseteq \mathbb{R}^{n+1} \), where \( S^+_n = \{x \in S^n : x_0 > 0\} \) is an open half sphere.

(c) \( \sigma(x) = (x_0, x_1, \ldots, x_{n-1}, -x_n) \) on \( \mathbb{H}^n := \{(x_0, x) \in \mathbb{R}^{n+1} : x_0 > 0, x_0^2 - x^2 = 1\} \) (hyperbolic space), where \( \mathbb{H}^+_n = \{x \in \mathbb{H}^n : x_n > 0\} \).

The research of K.-H. Neeb was partially supported by DFG-grant NE 413/9-1. The research of G. ´Olafsson was partially supported by Simons grant 586106.
Let $\Delta$ be the Laplacian of $M$, considered as a negative selfadjoint operator on $L^2(M, \mu)$, where $\mu$ is the volume measure. For any $m > 0$, we obtain a bounded positive symmetric operator $C_m = (m^2 - \Delta)^{-1}$ on $L^2(M, \mu)$. It defines an inner product on $L^2(M, \mu)$ by

$$
\langle \varphi, \psi \rangle_{-1} = \langle \varphi, C_m \psi \rangle_{L^2} = \int_M \overline{\varphi(m)}(C_m \psi)(m) \, d\mu(m).
$$

The corresponding completion is the Sobolev space $H^{-1}(M)$ and $\sigma$ induces a unitary involution $\sigma_\ast(f) := f \circ \sigma$ on this space. It is shown in [AFGS86, JR08, An13, Di04] that the triple $(\mathcal{E}, \mathcal{E}_+, \theta) := (H^{-1}(M), H^{-1}(M_+), \sigma_\ast)$ is a reflection positive Hilbert space. In [Di04], the space $\tilde{\mathcal{E}}$ is even identified with the subspace $H^{-1}_M \subseteq H^{-1}(M_+)$ consisting of all elements whose support, as distributions on $M_+$, is contained in $M^\sigma$.

In this paper we study the case $M = S^n$, its isometry group $G := O_n(\mathbb{R})$, and the representations of the $c$-dual group $G^c := O_{1,n}(\mathbb{R})^\dagger$ (the orthochronous Lorentz group) on the Hilbert spaces $\tilde{\mathcal{E}}$ corresponding to all values $m > 0$. In particular, we shall see that the above construction provides a euclidean realization of all irreducible spherical unitary representations of $G^c$. In addition, it leads to very natural realizations in spaces of holomorphic functions on a complex manifold $\Xi$ containing $S^n_+$ and $\mathbb{H}^n$ as totally real submanifolds. Parts of our results have already been announced in [NO18].

To obtain these results, we proceed as follows. In Subsection 2.1 we describe how the reflection positivity of $(H^{-1}(M), H^{-1}(M_+), \sigma_\ast)$ leads to a positive definite analytic kernel function $\Psi_m$ on the open subset $M_+ \subseteq M$. This is best understood by first interpreting $H^{-1}(M)$ as a Hilbert space of distributions on $M$, defined by a positive definite distribution kernel $\Phi_m$ on $M \times M$. This kernel is analytic on the complement of the diagonal $\Delta_M \subseteq M \times M$ because it satisfies on this domain the elliptic differential equation $\Delta \Phi_m = m^2 \Phi_m$ in both variables. As a consequence, the kernel $\Psi_m(x, y) = \Phi_m(x, \sigma(y))$ is analytic on $M_+ \times M_+$ and positive definite by reflection positivity (Corollary 2.8 Lemma 2.10). We also make an effort to translate between the two different approaches to the reflection positivity result by Jaffé and Ritter [JR08, An13] and Dimock [Di04]. In Subsection 2.2, all this is specialized to the situation where $M \cong G/K$ is a Riemannian symmetric space. In this context the kernel $\Phi_m$ is represented by a $K$-invariant analytic eigenfunction $\varphi_m$ of $\Delta$ on the complement of the base point (Theorem 2.13). Eventually, we recall in Subsection 2.3 the symmetric space structures on the sphere $S^n_+ \cong O_{n+1}(\mathbb{R})^\dagger/O_n(\mathbb{R}) = G^c/K$ and the hyperboloid $\mathbb{H}^n \cong O_{1,n}(\mathbb{R})^\dagger/O_n(\mathbb{R}) = G^c/K$.

To establish the connection between sphere and hyperbolic space in Section 3, we first note that the complex bilinear form on $\mathbb{C}^{n+1}$ restricts on the subspace $V := \mathbb{R}e_0 + i\mathbb{R}^n$ to the Lorentzian form $\langle (u_0, iu), (v_0, iv) \rangle_V := u_0 v_0 - uv$. Accordingly, we obtain a natural realization of $G^c = O_{1,n}(\mathbb{R})^\dagger$ in $\text{GL}(V) \cap O_{n+1}(\mathbb{C})$. Now the complex submanifold $\Xi := G^c S^n_+$ of the complex sphere $S^n_+$ contains the half sphere $S^n_+$ and the hyperbolic space

$$
\mathbb{H}^n_V = \{ u \in V : u_0 > 0, [u, u]_V = 1 \} \cong O_{1,n}(\mathbb{R})^\dagger/O_n(\mathbb{R})
$$

as totally real submanifolds, so that we may translate between $S^n$ and $\mathbb{H}^n$ by analytic continuation. In the literature on representations of semisimple Lie groups, the manifold $\Xi$ is called the crown of hyperbolic space and it plays the role of a natural “complexification” of the Riemannian symmetric space $G^c/K$ (cf. [AG90, KS04, KS03, KO08] and Theorem 3.30). The boundary of $\Xi$ consists of two $G^c$-orbits. One is de Sitter space

$$dS^n = \Xi \cap iV = i\{ v \in V : [v, v]_V = -1 \},$$

and the other orbit projects onto the homogeneous space

$$\mathbb{L}^n_+ := \{ v = (v_0, iv) \in V : [v, v]_V = 0, v_0 > 0 \}$$

2
of positive light rays in $V$. We also note that, for the future cone
$$V_+ = \{ u = (u_0, iu) \in V : u_0 > 0, [u, u] > 0 \},$$
the corresponding tube $T_{V_+} := V_+ + iV$ intersects $\mathbb{S}^n_+$ precisely in $\Xi$.

In Section 4 we obtain the analytic continuation of the kernels $\Psi_m$ on $\mathbb{S}^n_+$ to $G^c$-invariant kernels on $\Xi$. We call a kernel $\Psi$ on $\Xi \times \Xi$ sesquiholomorphic if it is holomorphic in the first and antiholomorphic in the second argument. Our first main result is Theorem 4.8 asserting that $G^c$-invariant sesquiholomorphic kernels on $\Xi$ are of the form
$$\Psi(z, w) = \alpha_\Psi([z, \sigma_V w]_V), \quad \alpha_\Psi : \mathbb{C} \setminus (-\infty, 1) \to \mathbb{C} \text{ holomorphic},$$
where $\sigma_V$ is the complex conjugation on $V_\mathbb{C}$ fixing $V$ pointwise. To obtain an analytic continuation of $\Psi_m$, we thus have to determine the corresponding function $\alpha_{\Psi_m}$, which occupies the remainder of Section 5. The main results are Theorems 4.12 and 4.19 expressing $\Psi_m$ by the hypergeometric function $2F_1$:
$$\Psi_m(z, w) = \frac{\Gamma\left(\frac{n-1}{2} + \lambda\right) \Gamma\left(\frac{n-1}{2} - \lambda\right)}{\Gamma(n)} 2F_1\left(\frac{n-1}{2}, \frac{n-1}{2} - \lambda; n; \frac{1}{2} (1 - [z, \sigma_V w]_V)\right),$$
where
$$\lambda = \lambda_m := \begin{cases} \sqrt{(n-2)^2 - m^2} & \text{for } 0 \leq m \leq (n-1)/2 \\ \sqrt{m^2 - (n-2)^2} & \text{for } m \geq (n-1)/2. \end{cases}$$

In Section 6 we eventually turn to the representation theoretic consequences of these results. Theorem 5.10 provides the key information by showing that the irreducible positive definite spherical functions $(\varphi_m)_{m > 0}$ on $\mathbb{H}^n_V$ are positive multiples of the functions $\Psi_m(\cdot, e_0)$. In particular, they extend to sesquiholomorphic kernels
$$\Phi^c_m(z, w) := \frac{\Psi_m(z, w)}{\Psi_m(e_0, e_0)}, \quad m > 0, \quad \Phi^c_0(z, w) := 1,$$
on $\Xi \times \Xi$. This entails that the representations $(U^c_m, \tilde{\mathcal{E}})$ of $G^c$ that we obtain from the representations of $G = \text{O}_{n+1}(\mathbb{R})$ on the reflection positive Hilbert spaces $(\mathcal{H}^{-1}(M), \mathcal{H}^{-1}(M_+), \sigma_*)$ are precisely the irreducible spherical representations. It also provides a natural realization of these representations in reproducing kernel Hilbert spaces $\mathcal{H}_{\Phi^c_m} \subseteq \mathcal{O}(\Xi)$ of holomorphic functions on $\Xi$ by $\pi_m(g)f = g_*f$ (Corollary 5.11). As the kernels $(\Phi^c_m)_{m \geq 0}$ are the extreme points in the convex set of sesquiholomorphic positive definite $G^c$-invariant kernels $\Psi$ on $\Xi \times \Xi$ normalized by $\Psi(e_0, e_0) = 1$ (Corollary 5.11), for all such kernels there exists a probability measure $\mu$ on $[0, \infty)$ with
$$\Psi = \int_0^\infty \Phi^c_m \, d\mu(m).$$
We apply this in Section 6 to two natural classes of examples.

As spherical representations of $G^c$ are typically realized in functions on the sphere $\mathbb{S}^{n-1} \cong \mathbb{L}^+_n/\mathbb{R}^+\mathbb{X}$, we show in Theorem 5.13 how this leads to an integral representation of the kernels $\Phi^c_m$ in terms of “plane wave kernels”:
$$\Phi^c_m(z, w) = \int_{\mathbb{S}^{n-1}} \left[\sigma_V(w), (1, u)\right]_V^{\lambda - \frac{n-1}{2}} \left[z, (1, u)\right]_V^{-\lambda + \frac{n-1}{2}} d\mu_{\mathbb{S}^{n-1}}(u),$$
where $\mu_{\mathbb{S}^{n-1}}$ is the $\text{O}_n(\mathbb{R})$-invariant probability measure on $\mathbb{S}^{n-1}$. This in turn leads to a Poisson transform from the realization on $\mathbb{S}^{n-1}$ to holomorphic functions on $\Xi$. Section 6 is rounded off by Subsection 5.3 with a brief discussion of the relations between our kernels with canonical kernels on hyperbolic spaces (cf. [vDH97]). These kernels also extend analytically to a neighborhood of $\mathbb{H}^n_V$ in $\Xi$, but not to all of $\Xi$. 

3
We conclude this paper with Section 5 where we discuss various aspects of our results that have not been pursued in this paper. In Section 6.1, we show that $\Xi$ is holomorphically equivalent to the Lie ball, the bounded symmetric domain $SO_{2,n}(\mathbb{R})/S(O_2(\mathbb{R}) \times O_n(\mathbb{R}))$ associated to the Lie group $SO_{2,n}(\mathbb{R})$. In particular, the action of $G^c$ on $\Xi$ extends to a transitive action of $SO_{2,n}(\mathbb{R})$. This observation can already be found in [KS05, Table III, p.229]. It was used in [GKO03] to construct $SO_{1,n}(\mathbb{R})$-invariant distributions on de Sitter space $dS^n$, realized as a $G^c$-orbit in $\partial \Xi$. From this perspective, the $SO_{2,n}(\mathbb{R})$-invariant positive definite kernels on $\Xi$ are of particular interest, and results on their branching behavior on $G^c$ are briefly described in Theorem 6.1 in terms of the integral decompositions in the sense of (1.1).

In Subsection 6.2 we show that the $G^c$-representations in the Hilbert subspaces $\mathcal{H}_{\Phi_m} \subseteq \mathcal{O}(\Xi)$ have natural boundary value maps into $G^c$-invariant Hilbert spaces of distributions on $dS^n \subseteq \partial \Xi$. Subsection 6.3 briefly discusses analogs of our results concerning spheres $S^n$ for $\mathbb{R}^n$ and $\mathbb{H}^n$, endowed with their natural dissecting involutions. We end in Subsection 6.4 by showing that the spherical representations $(\pi_m, \mathcal{H}_m)_{m \geq 0}$ of $G^c$ all extend naturally to antiunitary representations of the full Lorentz group $O_{1,n}(\mathbb{R})$ in the sense of [NO17].

**Background:** In [NO19], we classify all irreducible symmetric spaces with dissecting involution. It turns out that they are quadrics

$$Q := \{ x \in \mathbb{R}^{p+q} : \beta_{p,q}(x,x) = 1 \},$$

where

$$\beta_{p,q}(x,y) = \sum_{j=1}^{p} x_j y_j - \sum_{j=p+1}^{p+q} x_j y_j$$

and the dissecting involution is given either by

$$\sigma(x_1, \ldots, x_n) = (-x_1, x_2, \ldots, x_n) \quad \text{or by} \quad \sigma(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, -x_n).$$

In particular, the only irreducible $n$-dimensional semisimple Riemannian symmetric spaces with dissecting involutions are the sphere $S^n$ ($q = 0$) and the hyperboloid $\mathbb{H}^n$ ($p = 1$).

**Connections with physics:** The origins of reflection positivity lie in the construction of euclidean quantum field theories by Osterwalder and Schrader [OS73, OS75] (see [Ja08] for a historical discussion). The classical example is $M = \mathbb{R}^{n+1}$ on which $\sigma(x_0, x) = (-x_0, x)$ is interpreted as a time reflection and the passage from $\mathcal{E}$ to $\hat{\mathcal{E}}$ corresponds to the passage from euclidean quantum field theories to relativistic ones. In this process the euclidean inner product on $\mathbb{R}^{n+1}$ changes to the Lorentzian form $\langle x, y \rangle = x_0 y_0 - \langle x, y \rangle$.

Since then various constructions of reflection positive spaces and their representation theoretic context have been studied. We refer to [Ja08, JÓ98] for surveys and to the recent monograph [NÓ18] for more details.

Our work is closely related to the series of articles by J. Bros and his coauthors [BM96, BV97, BEM02b, BM04], although our perspective is different. In these papers the focus is mostly on de Sitter space $dS^n$ and on analytic extensions from there to $\Xi$, whereas we focus on the passage from the sphere $S^n$ to $\mathbb{H}^n$ in $\Xi$.

We now comment briefly on the intersection points. In [BM96], $X_n = dS^n$ is $n$-dimensional de Sitter space and $T_\pm = G^c.S_{\pm}$ are called Lorentz tuboids; clearly $T_+ = \Xi$. Instead of sesquiholomorphic kernels on $\Xi$, the authors study holomorphic kernels on $T_+ \times T_-$. *Perikernels* are distributional solutions of the Klein–Gordon equation $(m^2 - \Box)\Psi = 0$ on de Sitter space which extend holomorphically to $T_\pm$, hence correspond to our kernels $\Psi_m$. [BV96, §4.2]. The Källen–Lehmann representation for generalized free fields [BV96, Prop. 3.3] is a variant of our integral representation (1.1) which applies to all $G^c$-invariant positive definite sesquiholomorphic kernels on $\Xi$. It is formulated in terms of distributional boundary values of the $\Psi_m$ and requires certain growth conditions on the
kernels. The integral representation in terms of de Sitter plane waves in [BV96, §§4.1] is closely related to our formula (1.2). In particular [BV96, Thms. 4.3, 4.4] relate to irreducible spherical representation of the de Sitter group $SO_{1,n}(\mathbb{R})$ and identify the corresponding Casimir eigenvalues, where $\rho$ is called the geometric mass. These results are used in [BEM02b] in the context of quantum field theory on de Sitter space $dS^n$. Propositions 1,2,3 in [BM04, §2.1] appear also in our §3.1. In particular, Proposition 3 determines the set $C_{\Xi}$. Techniques based on [Di04] and reflection positivity concerning the passage from $S^2$ to $dS^2$ have recently been used to construct interacting quantum fields on $dS^2$ in [BJM16].

**Contents**

1. Introduction 1
   Notation 5
2. Reflection positivity related to the resolvent of the Laplacian 7
   2.1. Reflection positivity and the resolvent of the Laplacian 7
   2.2. Symmetric spaces 10
   2.3. The sphere and the hyperboloid as symmetric spaces 11
3. The complex manifold $\Xi$ 12
   3.1. The sphere and the crown of the hyperboloid 12
   3.2. The boundary of $\Xi$ 14
4. Reflection positivity on the sphere $S^n$ 15
   4.1. Invariant positive definite kernels 15
   4.2. Reflection positivity on $S^1$ 20
   4.3. The kernel function corresponding to $(m^2 - \Delta)^{-1}$ 21
   4.4. The constant $\gamma_{n,m}$ 23
5. Reflection positivity on the sphere and representation theory 26
   5.1. The spherical unitary representations of the Lorentz group 26
   5.2. An integral representation of $\varphi_m$ 31
   5.3. Canonical kernels 32
6. Perspectives 33
   6.1. Identification of $\Xi$ with a Lie ball 33
   6.2. Boundary values on the de Sitter space 36
   6.3. Further examples 38
   6.4. Extension to anti-unitary representations 40
References 41

**Notation.** In this section we collect notation that will be used throughout the article.

**Euclidean and Minkowski space:** The standard basis for the euclidean space $E := \mathbb{R}^{n+1}$ and $\mathbb{C}^{n+1}$ is denoted by $e_0, e_1, \ldots, e_n$. Accordingly, we use the notation

$$z = (z_0, z_1, \ldots, z_n) = (z_0, \mathbf{z}) \quad \text{with} \quad z_j \in \mathbb{C}, \mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n.$$

Denote by $zw = \sum_{j=0}^n z_j w_j$ the standard $\mathbb{C}$-bilinear form on $\mathbb{C}^{n+1}$ and write $z^2 = zz$. For $j = 0, \ldots, n$, we write $r_j(z) = z - 2z_je_j$ for the orthogonal reflections in $e_j^0$. Half-sphere and complex sphere are denoted by

$$S^n_\pm := \{x \in S^n : \pm x_0 > 0\} \subset S^n := \{x \in \mathbb{R}^{n+1} : x^2 = 1\} \subset S^n_\mathbb{C} := \{z \in \mathbb{C}^{n+1} : z^2 = 1\}.$$
The Lorentzian bilinear form on $\mathbb{R}^{n+1}$ is denoted $[z, w] = z_0 w_0 - z w$. We write $\mathbb{R}^{1,n} = (\mathbb{R}^{n+1}, [\cdot, \cdot])$ for the $(n + 1)$-dimensional Minkowski space and

$$H^n := \{x \in \mathbb{R}^{1,n}; [x, x] = 1, x_0 > 0\}$$

for the hyperboloid model of $n$-dimensional hyperbolic space.

As we shall see below, it is convenient to realize Minkowski space as the subspace

$$V : = \iota \mathbb{R}^{n+1} = \mathbb{R} e_0 \oplus i \mathbb{R}^n \subseteq \mathbb{C}^n$$

for the hyperboloid model of $n$-dimensional hyperbolic space.

Then

$$[x, y] = x_0 y_0 - x y = (ix)(iy) =: [\iota(x), \iota(y)]_V \quad \text{for} \quad x, y \in \mathbb{C}^{n+1}.$$ 

On $\mathbb{C}^{n+1}$ we consider the conjugations

$$\sigma_E(z_0, \ldots, z_n) := (\overline{z_0}, \ldots, \overline{z_n}) \quad \text{and} \quad \sigma_V(z_0, \ldots, z_n) := (\overline{z_0}, -\overline{z_1}, \ldots, -\overline{z_n})$$

with respect to the real subspaces $E = \mathbb{R}^{n+1}$ and $V$, respectively. These conjugations commute and the holomorphic involution $\sigma_E \sigma_V$ is $-r_0$.

**Groups:** For the matrix Lie groups that will be used in this article we use the notation

- $O_{n+1}(\mathbb{C}) = \{g \in \text{GL}_{n+1}(\mathbb{C}); \forall z, w \in \mathbb{C}^{n+1} (gw)(gw) = zw\}$.
- $O_{n+1}(\mathbb{R}) = \text{GL}_{n+1}(\mathbb{R}) \cap O_{n+1}(\mathbb{C})$.
- $O_{1,n}(\mathbb{R}) = \{g \in \text{GL}_{n+1}(\mathbb{R}); \forall z, w \in \mathbb{C}^{n+1} (gw)(gw) = zw\}$.
- $O_{1,n}(\mathbb{R})^\perp := \{g \in O_{1,n}(\mathbb{R}); g 00 > 0\}$.

The natural action of $G := O_{n+1}(\mathbb{R})$ on $\mathbb{R}^{n+1}$ defines a transitive action of $G$ on $\mathbb{S}^n$. The stabilizer of $e_0$ is $K := G_{e_0} \cong O_n(\mathbb{R})$, so that we obtain a $G$-equivariant diffeomorphism $G/K \to \mathbb{S}^n$, $gK \mapsto g.e_0$, by the orbit map. Similarly, the group $O_{1,n}(\mathbb{R})^\perp$ acts transitively on $H^n = O_{1,n}(\mathbb{R})^\perp.e_0 \cong O_{1,n}(\mathbb{R})^\perp/K$.

The group

$$G^c := \iota O_{1,n}(\mathbb{R})^\perp \iota \subseteq \text{GL}(V)$$

preserves the Lorentzian product $[\cdot, \cdot]_V$. It also contains $K = G_{e_0} = G^c_{e_0}$. The involution $\sigma_V$ commutes with $G^c$, and $\sigma_E g \sigma_F = r_0 g r_0 = \tau(g)$ is the involution on $G^c$ whose fixed point group is $K = G^c_{e_0}$. With

$$G_C := O_{n+1}(\mathbb{C}) \quad \text{and} \quad K_C = G_{e_0} \cong O_{n}(\mathbb{C}),$$

we then have $S^0_C = G_{C,e_0} \cong G^c_{e_0} / K_C$.

We shall also consider the following sets:

- $V_+ := \{v \in V; [v, v]_V > 0, v_0 > 0\}$ (open upper light cone),
- $H^n_0 := \iota H^n = S^n_0 \cap V_+ = G^c.e_0 \cong G^c / K$ (hyperbolic space),
- $S^n_{1,C} := \{z \in S^n_C; \text{Re} z_0 > 0\}$,
- $T_{V_+} := iV + V_+$ (future tube), and
- $\Xi := G^c S^n_+ \subset S^n_{1,C}$, called the *crown* of $H^n_0$.

Note that $V, V_+, T_{V_+}$ and $S^n_{1,C}$ are invariant under the action of $G^c$.

**Distributions:** If $M$ is a manifold, then $C^\infty(M)$ denotes the space of smooth complex valued functions on $M$ and $C^\infty_c(M)$ the subspace of compactly supported smooth functions. This space carries the locally convex topology for which it is the direct limit of the closed subspaces $C^\infty_c(\bar{M})$ of smooth functions supported in the compact subset $\bar{M} \subseteq M$, on which the topology is that of uniform convergence of all derivatives on compact subsets of chart neighborhoods. This turns $C^\infty_c(M)$ into a complete locally convex space. Its conjugate linear dual, i.e., the space of continuous antilinear functionals, is denoted by $C^{-\infty}(M)$. This is the space of distributions on $M$. If $M$ is a Riemannian
manifold, then the volume measure $\mu = \mu_M$ defines a locally finite measure on $M$. This leads to a linear injection $L^1_{\text{loc}}(M) \hookrightarrow C^{-\infty}(M)$ given by

$$\Psi_f(\eta) = \int_M \overline{\eta(x)} f(x) \, d\mu(x), \quad f \in L^1_{\text{loc}}(M), \eta \in C^\infty_c(M).$$

If $\theta : M \to M$ is a diffeomorphism then $\theta_* : C^\infty_c(M) \to C^\infty_c(M)$ is defined by $\theta_* \eta = \eta \circ \theta^{-1}$. One defines $\theta_*$ in a similar way on other function spaces and spaces of distributions. If $\theta$ is an isometry, then $\mu$ is $\theta_*$ invariant, i.e., $\int_M \theta_* \eta \, d\mu = \int_M \eta \, d\mu$ for all $\eta \in C^\infty_c(M)$.

2. Reflection positivity related to the resolvent of the Laplacian

In this section we recall the definition of a reflection positive Hilbert space. We then introduce a construction of Hilbert spaces based on resolvents of the Laplace operator $\Delta$ and refer to $\text{[NÖ18]}$ for more details and background on reflection positivity.

2.1. Reflection positivity and the resolvent of the Laplacian. Most of the material in this subsection can be found in our previous articles $\text{[NÖ14, JÖ98, JÖ00]}$ and the monograph $\text{[NÖ18]}$.

**Definition 2.1.** (Reflection positive Hilbert space) A reflection positive Hilbert space is a triple $(\mathcal{E}, \mathcal{E}_+, \theta)$, where $\mathcal{E}$ is a Hilbert space, $\mathcal{E}_+$ is a closed subspace of $\mathcal{E}$ and $\theta \in \text{U(}\mathcal{E})$ is an involution for which the hermitian form $\langle \xi, \eta \rangle_\theta := (\theta \xi, \eta)$ is positive semi-definite on $\mathcal{E}_+$.

Let $(\mathcal{E}, \mathcal{E}_+, \theta)$ be a reflection positive Hilbert space. We write

$$\mathcal{N} := \{ \eta \in \mathcal{E}_+ : \| \eta \|_\theta = 0 \} = \{ \eta \in \mathcal{E}_+ : (\forall \zeta \in \mathcal{E}_+) \langle \zeta, \eta \rangle_\theta = 0 \} = \mathcal{E}_+ \cap \theta(\mathcal{E}_+)^\perp$$

and $q$ or $\hat{\cdot}$ for the quotient map $\mathcal{E}_+ \to \mathcal{E}_+/\mathcal{N}, \eta \mapsto \hat{\eta} = q(\eta)$. Denote by $\hat{\mathcal{E}}$ the Hilbert space completion of $\mathcal{E}_+/\mathcal{N}$ with respect to the norm $\| \hat{\eta} \|_\theta := \sqrt{\langle \theta \eta, \eta \rangle}$.

The passage from $\mathcal{E}$ to $\hat{\mathcal{E}}$ can be used to construct from certain unitary representations of a Lie group $G$ on $\mathcal{E}$ a unitary representation of another Lie group $G^\circ$ on $\hat{\mathcal{E}}$. We refer to $\text{[AÎNÖ14, NÖ14, NÖ15a, NÖ18, JÖ98, JÖ00]}$ for details.

The geometric setup for the main theme of this article is the following. Let $M$ be a complete Riemannian manifold and $\Delta$ denote the negative Laplace–Beltrami operator on $L^2(M, \mu_M)$, considered as a selfadjoint operator. Here the completeness of $M$ is used for the essential selfadjointness (Str83 Thm. 2.4) and $\mu = \mu_M$ is the Riemannian volume measure. Then $\text{Spec}(\Delta)$ is contained in $(-\infty, 0]$. If $M$ is compact then $\mu(M) < \infty$ and the constants are in contained in $L^2(M)$, so that $0 \in \text{Spec}(\Delta)$.

For each $m > 0$, we have a bounded positive operator $C_m := C := (m^2 - \Delta)^{-1}$ on $L^2(M)$.

**Definition 2.2.** Let $M_+ \subset M$ be an open submanifold, let $\sigma : M \to M$ be an isometric involution, and $D \in C^{-\infty}(M \times M)$. Then

(i) $D$ is called *positive definite* if $D(\varphi \otimes \overline{\varphi}) \geq 0$ for $\varphi \in C^\infty_c(M)$. We then write $\mathcal{H}_D \subseteq C^{-\infty}(M)$ for the Hilbert subspace obtained from the positive semidefinite form on $C^\infty_c(M)$ defined by

$$\langle \varphi, \eta \rangle_D := D(\varphi \otimes \overline{\eta}) \quad \text{for} \quad \varphi, \eta \in C^\infty_c(M).$$

This space embeds naturally into $C^{-\infty}(M)$ in such a way that $\xi \in \mathcal{H}_D$ is mapped to the distribution $\xi_D := \langle \cdot, \xi \rangle_D$. 7
Lemma 2.4. If (M,M_+ ,σ) if D is positive definite and the distribution D^\sigma := D \circ (id,σ) on M_+ \times M_+ is positive definite. We write \mathcal{H}_D \subseteq \mathcal{H}_G for the subspace generated by \xi_D, \xi \in \mathcal{C}_c^\infty(M_+). We denote the extension of \theta_\ast to \mathcal{H}_D by the same symbol. Then D is reflection positive if and only if (\mathcal{H}_D, \mathcal{H}_D, \theta_\ast) is reflection positive.

(iii) If G is a group, G_+ \subset G and \tau : G \to G is an involution, then \varphi : G \to \mathbb{C} is positive definite, resp., reflection positive with respect to G_+ and \tau if the kernel D(x,y) = \varphi(y^{-1}x) is positive definite, resp., reflection positive with respect to (G,G_+ ,\tau).

Remark 2.3. If the distribution D is represented by a continuous function d(x,y) with respect to the measure \mu on M, i.e.,

$$D(\varphi \otimes \overline{\psi}) = \int_{M \times M} \varphi(x)d(x,y)\psi(y)d\mu(x)d\mu(y) \quad \text{for} \quad \varphi, \psi \in C_c^\infty(M),$$

then the positive definiteness of D is equivalent to the positive definiteness of the kernel function d, i.e., to

$$\sum_{i,j=1}^n c_i\overline{c}_j d(x_i,x_j) \geq 0 \quad \text{for} \quad x_1, \ldots, x_n, c_1, \ldots, c_n \in \mathbb{C}.$$

We write diag(M) := \{(x,x) \in M \times M : x \in M\} for the diagonal in M \times M.

Lemma 2.4. The bounded operator C_m on L^2(M) defines a distribution \tilde{C}_m on M \times M by

$$\tilde{C}_m(\varphi \otimes \overline{\eta}) := \langle \varphi, C_m \eta \rangle = \int_M \overline{\varphi(x)}(C_m \eta)(x)d\mu_M(x) \quad \text{for} \quad \varphi, \eta \in C_c^\infty(M).$$

Furthermore, the following assertions hold:

(i) \tilde{C}_m is positive definite, in particular \tilde{C}_m(\varphi \otimes \overline{\eta}) = \overline{\tilde{C}_m(\eta \otimes \overline{\varphi})} for \varphi, \eta \in C_c^\infty(M).

(ii) We have \((m^2 - \Delta)y \tilde{C}_m(x,y) = \delta_M(x,y)\) in sense of distributions on M \times M, where \delta_M(\varphi) = \int_M \varphi(x)d\mu(x).

(iii) On M \times M \setminus \text{diag}(M), the distributional derivatives of \tilde{C}_m satisfy

$$0 = (m^2 - \Delta)y \tilde{C}_m(x,y) = (m^2 - \Delta)x \tilde{C}_m(x,y) = (m^2 - \Delta)(m^2 - \Delta)y \tilde{C}_m(x,y).$$

(iv) On the open subset M \times M \setminus \text{diag}(M), the distribution \tilde{C}_m is represented by an analytic function \Phi_m.

(v) The distribution \tilde{C}_m and the function \Phi_m are invariant under the isometry group Isom(M).

Proof. That \tilde{C}_m defines a distribution on M \times M follows from the continuity of the bounded operator C_m ([1767] Thm. 51.6) and a partition of unity argument to reduce to open subsets of \mathbb{R}^n.

(i) follows directly from the positivity of the operator C_m.

(ii) follows from

$$\tilde{C}_m(\varphi \otimes (m^2 - \Delta)\eta) = \langle \varphi, C_m(m^2 - \Delta)\eta \rangle_{L^2(M)} = \langle \varphi, \overline{\eta} \rangle_{L^2(M)} = \int_M \overline{\varphi(x)}\eta(x)d\mu_M(x) = \delta_M(\varphi \otimes \eta).$$

(iii) and (iv): The first equality follows immediately from (ii), and the second one from (i). The third is an immediate consequence of the first two. As \((m^2 - \Delta)x(m^2 - \Delta)y\) is an elliptic operator on M \times M \setminus \text{diag}(M) and annihilates \tilde{C}_m on this open subset, it follows that \tilde{C}_m is represented on the complement of \text{diag}(M) by an analytic function ([R73] Thm. 8.12]).
(v) Since $\Delta$ commutes with the action of $\text{Isom}(M)$ on $L^2(M)$, the operator $C_m$ also commutes with $\text{Isom}(M)$. This implies that the corresponding distribution $\tilde{C}_m$ on $M \times M$ is invariant under $\text{Isom}(M)$. \hfill \Box

**Definition 2.5.** An isometry $\sigma$ of a connected complete Riemannian manifold $M$ is called **dissecting** if the complement of the fixed point set $M^\sigma$ is not connected. Then $\sigma$ is an involution, the complement of $M^\sigma$ has two connected components $M_{\pm}$ with $\sigma(M_{\pm}) = M_{\mp}$, and each connected component of $M^\sigma$ is of codimension 1 (see [AKLM06 Lemma 2.7]).

Let $M$ be a connected complete Riemannian manifold and let $\sigma: M \to M$ be a dissecting involution. For $m > 0$ and $r \in \mathbb{R}$ let $\mathcal{H}^r(M)$ be the completion of $C^\infty_c(M)$ in the norm

$$\langle \varphi, (-\Delta + m^2)^r \psi \rangle^{1/2}.$$

It is easy to see that this space does not depend on $m$. For $\varphi, \psi \in C^\infty_c(M)$ we have $|\langle \varphi, \psi \rangle_{L^2}| \leq \|\varphi\|_r \|\psi\|_{-r}$, so that the $L^2$-pairing extends to the duality pairing on $\mathcal{H}^r(M) \times \mathcal{H}^{-r}(M)$, realizing $\mathcal{H}^r(M)$ as the dual of $\mathcal{H}^{-r}(M)$. We also note that $(-\Delta + m^2): \mathcal{H}^1(M) \to \mathcal{H}^{-1}(M)$ is unitary.

For a closed subset $A \subset M$, let $\mathcal{H}^r_\lambda(M) = \{\varphi \in \mathcal{H}^r(M): \text{supp}\varphi \subseteq A\}$ and for an open subset $\Omega \subseteq M$, we let $\mathcal{H}^r_\Omega(M)$ be the closed subspace of $\mathcal{H}^r(M)$ generated by $C^\infty(\Omega)$.

The following results can be found in [Di04, Lem. 1/2, Cor. 1/2] but it should be pointed out that the reflection positivity for $(\mathcal{H}^{-1}(M), \mathcal{H}^{-1}_M(M), \theta)$ was also established in [JR08, An13, AFG86]. This follows from Lemma 2.9 below which builds a bridge to Dimock’s context.

**Theorem 2.6.** ([Di04]) Consider the following subspaces $\mathcal{E} := \mathcal{H}^{-1}(M) \subset C^{-\infty}(M)$:

$$\mathcal{E}_0 = \mathcal{H}^{-1}_{M^\sigma}(M), \quad \mathcal{E}_+ = \mathcal{E}_0 \oplus (-\Delta + m^2) \mathcal{H}^0_\lambda(M^+) \quad \text{and} \quad \mathcal{E}_- = \theta \mathcal{E}_+ = \mathcal{E}_0 \oplus (-\Delta + m^2) \mathcal{H}^0_\lambda(M^-),$$

where $\oplus$ stands for orthogonal direct sum. We note that $\mathcal{E}_0 = \mathcal{E}_+ \cap \mathcal{E}_- = \mathcal{E}_+^0$. Then the following assertions hold:

(i) $\mathcal{E}_+ = \mathcal{H}^{-1}_{M^\sigma \cup M^\lambda}(M) = \mathcal{H}^{-1}_{-M^\lambda}(M)$.

(ii) $\mathcal{E} = ((-\Delta + m^2) \mathcal{H}^0_\lambda(M_-)) \oplus \mathcal{E}_+$.

(iii) (The Markov condition) If $\varphi \in \mathcal{E}_+$, then $P_- \varphi = P_0 \varphi$, i.e., $P_+ P_+ = P_0$.

(iv) $\mathcal{E}_0 \neq \{0\}$.

(v) For $\varphi \in \mathcal{E}_+$ and $\psi \in \mathcal{E}_-$ we have $\langle \varphi, \psi \rangle_{-1} = \langle \varphi, (m^2 - \Delta)^{-1} \psi \rangle_{H^{-1} \times H^1} = \langle P_0 \varphi, P_0 \psi \rangle_{-1}$.

From this we immediately get (cf. [NO18 §2.3]):

**Theorem 2.7.** (Reflection Positivity) The triple $(\mathcal{H}^{-1}(M), \mathcal{H}^{-1}_{M^\sigma}(M), \theta)$ is a reflection positive Hilbert space with $\mathcal{N} = (-\Delta + m^2) \mathcal{H}^0_\lambda(M^\pm)$ and $\tilde{\mathcal{E}} \simeq \mathcal{E}_0 = \mathcal{H}^{-1}_{M^\sigma}(M)$.

**Corollary 2.8.** The positive definite distribution $\tilde{C}_m$ on $M \times M$ is reflection positive with respect to $(M, M^+, \sigma)$, i.e., the analytic kernel

$$\Psi_m(x, y) := \Phi_m(x, y) := \Phi_m(x, \sigma(y))$$

on $M^+ \times M^+$ defined in Lemma 2.4 is positive definite.

The following lemma is a bridge between Dimock’s approach and the papers by Jaffe and Ritter:

**Lemma 2.9.** The closed subspace $\mathcal{H}^{-1}_{M^\pm}(M) \subseteq \mathcal{H}^{-1}(M)$ is generated by the subspace $C^\infty(M^\pm)$ of test functions on $M^\pm$, considered as elements of $\mathcal{H}^{-1}(M)$.

**Proof.** Let $\mathcal{K}_{\pm} \subseteq \mathcal{H}^{-1}(M)$ denote the closed subspace generated by $C^\infty(M_{\pm})$.

(a) $\mathcal{K}_+ + \mathcal{K}_-$ is dense in $\mathcal{H}^{-1}(M)$ because the subspace $C^\infty(M_{\pm})$ is dense in $L^2(M_{\pm})$ and $L^2(M) \cong L^2(M_+) \oplus L^2(M_-)$ is dense in $\mathcal{H}^{-1}(M)$.
(b) \( \mathcal{K}_\pm \subseteq \mathcal{H}^{-1}_{M_\pm} \) By Dimock’s Theorem 2.6, \( \mathcal{H}^{-1}_{M_\pm} \) is the orthogonal complement of the closed subspace \( (m^2 - \Delta)\mathcal{H}_0^1(M_\pm) \), hence contains \( \mathcal{K}_\pm \) because \( \varphi_\pm \in C^\infty(M_\pm) \) implies
\[
\langle \varphi_+, (m^2 - \Delta)\varphi_+ \rangle_{L^2} = (m^2 - \Delta)\varphi_+ \rangle_{L^2} = 0.
\]
(c) \( (m^2 - \Delta)\mathcal{H}_0^1(M_\pm) \subseteq \mathcal{K}_\pm \) follows from \( (m^2 - \Delta)C^\infty_c(M_\pm) \subseteq \mathcal{K}_\pm \) and the density of \( C^\infty_c(M_\pm) \) in \( \mathcal{H}_0^1(M_\pm) \).
(d) From (b) and (c) and Dimock’s Theorem, we obtain the orthogonal decomposition
\[
\mathcal{K}_\pm = (\mathcal{K}_\pm \cap \mathcal{H}^{-1}_{M_\pm}(M)) \oplus (m^2 - \Delta)\mathcal{H}_0^1(M_\pm).
\]
Further \( \theta(\mathcal{K}_\pm) = \mathcal{K}_\mp \) and \( \mathcal{E}_0 \subseteq \text{Fix}(\theta) \) imply \( \mathcal{K}_\pm \cap \mathcal{H}^{-1}_{M_\pm}(M) = \mathcal{K}_- \cap \mathcal{H}^{-1}_{M_\pm}(M) \). Therefore
\[
\mathcal{K}_+ \cap \mathcal{K}_- = \mathcal{K}_+ \cap \mathcal{H}^{-1}_{M_\pm}(M) \oplus (m^2 - \Delta)\mathcal{H}_0^1(M_\pm) \oplus (m^2 - \Delta)\mathcal{H}_0^1(M_\pm),
\]
so that (a) and Dimock’s Theorem show that \( \mathcal{K}_+ \cap \mathcal{H}^{-1}_{M_\pm}(M) = \mathcal{H}^{-1}_{M_\pm}(M) \). This completes the proof. \( \square \)

**Lemma 2.10.** Define \( C^\sigma_m : C^\infty_c(M_+) \rightarrow L^2(M_+) \) by \( C^\sigma_m(f) := (\sigma \circ C_m)(f)|_{M_+} = (C_m \sigma)(f)|_{M_+} \). Then, for all \( \eta \in C^\infty_c(M_+) \), we have:
(i) \( (m^2 - \Delta)C^\sigma_m \eta = 0 \) on \( M_+ \) and \( C^\sigma_m \eta \) is analytic on \( M_+ \).
(ii) \( (C^\sigma_m \eta)(x) = \int_{M_+} \Psi_m(x,y)\eta(y) d\mu(y) = \int_{M_+} \Phi_m(x,y)\eta(\Phi_m(y,x)) d\mu(y) \) for \( x \in M_+ \) and \( \Phi_m \) is analytic on \( M_+ \times M_+ \).

**Proof.** As \( \sigma(x) \in M_- \) for \( x \in M_+ \), (i) follows from Lemma 2.4 and the fact that \( -\Delta + m^2 \) is elliptic. It now follows that, for \( \eta \in C^\infty_c(M_+) \), we have
\[
(-\Delta + m^2)C^\sigma_m \eta(x) = \int_{M_+} (-\Delta + m^2)\Psi_m(x,y)\eta(y) d\mu(y) = 0.
\]
Hence \( (-\Delta + m^2)\Psi_m(x,y) = 0 \) for all \( y \in M_+ \). Hence \( y \mapsto \Psi_m(x,y) \) is analytic for all \( x \in M_+ \). As \( -\Delta + m^2 \) is symmetric, it follows that \( \Phi_m(x,y) = \Phi_m(y,x) \). Hence \( \Psi_m(\cdot,x) \) is also analytic on \( M_+ \). \( \square \)

**Remark 2.11.** If \( \eta \in C^\infty_c(M_+) \), then \( \sigma(\text{supp} \eta) \subseteq M_- \) is compact. Lemma 2.10(i) thus shows that \( C^\sigma_m \eta \) is analytic on the open subset \( M \setminus \sigma(\text{supp} \eta) \supseteq M_- \cup M^\sigma \).

### 2.2. Symmetric spaces.

**Definition 2.12.** (a) Let \( M \) be a smooth manifold and \( \mu : M \times M \rightarrow M, (x,y) \mapsto x \cdot y = : s_x(y) \) be a smooth map with the following properties: each \( s_x \) is an involution for which \( x \) is an isolated fixed point and
\[
s_x(y \cdot z) = s_x(y) \cdot s_x(z) \quad \text{for all} \quad x,y \in M, \quad \text{i.e.,} \quad s_x \in \text{Aut}(M,\mu).
\]
Then we call \( (M,\mu) \) a symmetric space.
(b) A morphism of symmetric spaces \( M \) and \( N \) is a smooth map \( \varphi : M \rightarrow N \) such that \( \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) \).
(c) The real line \( \mathbb{R} \) is a symmetric space with respect to \( s_x(y) = 2x - y \) and a geodesic in \( M \) is a smooth morphism \( \gamma : \mathbb{R} \rightarrow M \) of symmetric spaces.

In [Lo69] it is shown that, for every \( v \in T_p(M) \), there is a unique geodesic \( \gamma^v_p : \mathbb{R} \rightarrow M \) with \( \gamma_p^v(0) = p \) and \( (\gamma_p^v)'(0) = v \). The corresponding map
\[
\text{Exp} : TM \rightarrow M, \quad \text{Exp}(v) := \text{Exp}_p(v) := \gamma_p^v(1)
\]
is called the exponential function of \( M \). It satisfies \( \gamma_p^v(t) = \text{Exp}_p(tv) \) for all \( t \in \mathbb{R} \).
As shown in [Lo69], connected symmetric spaces are homogeneous spaces of Lie groups and they arise from the following construction that goes back to É. Cartan, see [He78] for detailed discussion. Let \( G \) be a Lie group and \( \theta : G \to G \) be an involution. Let \( K \subset G \) be a subgroup such that \( (G^\theta)_0 \subset K \subset G^\theta \). Then the homogeneous space \( M := G/K \) is a symmetric space with respect to \( s_gK(xK) := g\theta(g^{-1}x)K \). We write \( m_0 = eK \in M \) for the canonical base point and \( \theta_M \) for the reflection \( s_{m_0}(gK) = \theta(g)(K) \) in the base point \( m_0 \).

Denote by \( d\theta : g \to g \) the derived involution and let

\[
\mathfrak{t} := \{ x \in \mathfrak{g} : d\theta(x) = x \} \quad \text{and} \quad \mathfrak{p} := \{ x \in \mathfrak{g} : d\theta(x) = -x \}.
\]

Then \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \), \( \mathfrak{t} \) is the Lie algebra of \( K \) and \( \mathfrak{p} \) can be identified with the tangent space \( T_{m_0}(M) \).

The isomorphism is given by the tangent map \( T_{M_0}(M) \), where \( q : G \to M, g \mapsto gK \) is the quotient map. We note that if \( x \in \mathfrak{q} \) and \( \varphi \in C^\infty(M) \) then \( T_\tau(q)(x) = \frac{d}{dt}_{|t=0} f(\exp(tx).m_0) \).

Assume that \( K \) is compact. Then there is a \( G \)-invariant Riemannian structure on \( M \) given in the following way: Fix a \( K \)-invariant inner product \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{p} \cong T_{m_0}(M) \), which is possible because \( K \) is compact. Write \( \ell_g : M \to M \) for the diffeomorphism \( \ell_g(x) := g.x \). We then define a \( G \)-invariant metric on \( T(M) \) by

\[
g_{g,m_0}((d\ell_g)_m(u), (d\ell_g)_m(v)) := \langle u, v \rangle.
\]

The so-obtained Riemannian manifold is called a Riemannian symmetric space.

**Theorem 2.13.** Let \( M = G/K \) be a Riemannian symmetric space, \( \sigma : M \to M \) be a dissecting involutive involution, and \( m_0 \in M_+ \) such that there exists an involutive automorphism \( \tau \) of \( G \) with \( \tau(K) = K \) such that

\[
\sigma(g.x) = \tau(g).\sigma(x) \quad \text{for} \quad g \in G, x \in M
\]

holds for an involutive automorphism \( \tau \) of \( G \). Let

\[
\varphi_m(x) := \Phi_m(x, m_0), \quad x \neq m_0 \quad \text{and} \quad \psi_m(x) := \Psi_m(x, m_0) = \varphi_m(\sigma(x)), \quad x \neq \sigma(m_0).
\]

Then the following assertions hold:

1. \( \varphi_m \) is a \( K \)-invariant analytic function on \( M \setminus \{ m_0 \} \) and satisfies the differential equation

\[
\Delta \varphi_m = m^2 \varphi_m.
\]

2. \( \psi_m \) is a \( K \)-invariant analytic function on \( M \setminus \{ \sigma(m_0) \} \) satisfying the differential equation

\[
\Delta \psi_m = m^2 \psi_m.
\]

3. \( (C_m^\sigma\eta)(x) = \int_G \psi_m(\tau(h)^{-1}.x)\eta(h.m_0) \, dh \) for \( \eta \in C^\infty_c(M_+) \) and \( x \in M_+ \).

**Proof.** (1) follows from Lemma 2.4(iv),(v), and (2) from Lemma 2.10. For (3) we observe that the function \( \tilde{\eta}(h) := \eta(h.m_0) \) on \( G \) is supported in \( \{ h \in G : h.m_0 \in M_+ \} \). Furthermore, the singularity of \( \psi_m \) is in \( \sigma(m_0) \), and if \( \tau(h)^{-1}.x = \sigma(m_0) \), then \( x = \sigma(h.m_0) \in M_+ \), so that the singularity is not contained in the support of \( \tilde{\eta} \).

**Remark 2.14.** Theorem 2.13 applies in particular to the dissecting involution on \( \mathbb{S}^n \) defined by the reflection \( \sigma = r_0 \), the base point \( m_0 = e_0 \), and \( \tau(g) = r_0gr_0 \) on \( G = \text{O}_{n+1}(\mathbb{R}) \).

2.3. **The sphere and the hyperboloid as symmetric spaces.** Both the sphere \( \mathbb{S}^n \subset \mathbb{R}^{n+1} \) and the hyperboloid \( \mathbb{H}^n \subset \mathbb{R}^{1,n} \) are Riemannian symmetric spaces.

The tangent bundle of the sphere is given by \( T(\mathbb{S}^n) = \{(u, v) \in \mathbb{S}^n \times \mathbb{R}^{n+1} : v \perp u \} \) with the \( \text{O}_{n+1}(\mathbb{R}) \) action \( g.(u, v) = (gu, gv) \). Geodesics and exponential map are given by

\[
\text{Exp}_p(v) = \cos(\|v\|)p + \sin(\|v\|)\frac{v}{\|v\|} \quad \text{for} \quad 0 \neq v \in T_p(\mathbb{S}^n) \cong p_\perp \cong \mathbb{R}^n
\]

and in particular

\[
\text{Exp}_p(tv) = \cos(t)p + \sin(t)v \quad \text{for} \quad \|v\| = 1.
\]
The exponential function can be dealt with more easily if we use the analytic functions \( C, S : \mathbb{C} \to \mathbb{C} \) defined by

\[
(2.3) \quad C(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^k \quad \text{and} \quad S(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^k
\]

which satisfy

\[
\cos z = C(z^2) \quad \text{and} \quad \frac{\sin z}{z} = S(z^2) \quad \text{for} \quad z \in \mathbb{C}^\times.
\]

We thus obtain

\[
(2.4) \quad \text{Exp}_p(v) = C(v^2)p + S(v^2)v.
\]

On the tangent bundle \( T(\mathbb{H}^n) \simeq \{(u, v) \in \mathbb{H}^n \times \mathbb{R}^{n+1} : [u, v] = 0\} \) of \( \mathbb{H}^n \), the Riemannian structure is given by \( g_u((u, v), (u, v)) = -[v, v] \). This shows that the Lorentz group \( O_{1,n}(\mathbb{R}) \) acts by isometries on \( \mathbb{H}^n \). The stabilizer of \( e_0 \) is again the group \( K \) and \( \mathbb{H}^n \cong O_{1,n}(\mathbb{R})^\uparrow / K \).

### 3. The complex manifold \( \Xi \)

In this section we take a closer look at the subset \( \Xi = G^c.S^+_n \) of the complex sphere \( S^+_n \), defined in the introduction. This domain turns out to be open and thus inherits a complex manifold structure. It contains the half sphere \( S^+_n \) and the hyperbolic space \( \mathbb{H}^n \), as totally real submanifolds \( S^+_n = \Xi^{\sigma_F} \) and \( \mathbb{H}^n = \Xi^{\sigma_V} \). Thus if \( F : \Xi \to \mathbb{C} \) is holomorphic and \( F|_{S^+_n} = 0 \) or \( F|_{\mathbb{H}^n} = 0 \) then \( F = 0 \). We therefore consider \( \Xi \) as a bridge between analytic functions on \( S^+_n \) and \( \mathbb{H}^n \), and we shall study this connection in particular for functions invariant under \( K = O_{1,n}(\mathbb{R}) \). In Subsection 3.1 we explore some elementary properties of \( \Xi \) and calculate the set \( \Xi_\Xi \) of all values of the complex bilinear form \( \langle \cdot, \cdot \rangle \) on \( \Xi \). This permits us later to obtain analytic continuations of \( G^c \)-invariant kernels on \( \mathbb{H}^n \) and from certain kernels on \( S^+_n \). In Subsection 3.2 we show that the boundary of \( \Xi \) in \( S^+_n \) consists of two \( G^c \)-orbits: de Sitter space \( \mathbb{S}^n \) and the space \( \mathbb{L}^n \), the non-zero boundary elements of the positive light cone \( V_+ \). These two boundary orbits are of particular importance for realizations of \( G^c \)-representations on these spaces.

#### 3.1. The sphere and the crown of the hyperboloid

The complex sphere \( S^+_n \) is a complex symmetric space and the reflections are given by the same formula as for the sphere. Its exponential function is given by (2.4). For \( p = e_0 \) and \( \|v\| = 1 \), we obtain in particular

\[
\text{Exp}_{e_0}(itv) = \cosh(t)e_0 + i \sinh(t)v \in i \mathbb{H}^n \subset S^+_n, \quad \text{where} \quad t(x_0, x) = (x_0, ix).
\]

**Lemma 3.1.** Let \( u, v \in V \). Then \( z = u + iv \in S^+_n \) if and only if

\[
[u, u]_V - [v, v]_V = 1 \quad \text{and} \quad [u, v]_V = 0.
\]

**Proof.** A simple calculation shows that \( z^2 = [u, u]_V - [v, v]_V + 2i[u, v]_V \) and the claim follows. \( \square \)

**Proposition 3.2.** The following assertions hold:

(i) \( T_{V_+} \cap \mathbb{R}^{n+1} = \mathbb{R}^{n+1}_+ := \{ (x_0, x) : x_0 > 0 \} \) and \( S^+_n = T_{V_+} \cap S^n \).

(ii) \( \Xi = T_{V_+} \cap S^+_n \subset T_{V_+} \cap S_n^+ \).

(iii) We have \( \sigma_V \Xi = \sigma_F \Xi = \Xi \) and \( \Xi^{\sigma_V} = \Xi \cap V = \mathbb{H}^n \) and \( \Xi^{\sigma_F} = \Xi \cap \mathbb{R}^{n+1} = S^+_n \).

**Proof.** (i) It is clear that \( z = u + iv \in T_{V_+} \cap \mathbb{R}^{n+1} \) if and only if \( u = re_0 \) with \( r > 0 \) and \( iv = (0, v) \) with \( v \in \mathbb{R}^n \). This shows that \( T_{V_+} \cap \mathbb{R}^{n+1} = \mathbb{R}^{n+1}_+ \). Intersecting with the sphere now yields the second assertion.
(ii) By (i) we have $\Xi = G_c^c.S^\mathbb{C}_+ = G_c^c.(T_{\mathbb{V}_+} \cap \mathbb{S}^n) \subseteq T_{\mathbb{V}_+} \cap \mathbb{S}^n_{\mathbb{C}}$. Conversely, let $z = u + iv \in T_{\mathbb{V}_+} \cap \mathbb{S}^n_{\mathbb{C}}$. Then $u_0 > 0$ and, as $G_c$ acts transitively on all level sets $[u, u]_V = r > 0$ in $V_+$, we may assume that $u = re_0$ with $r > 0$. Thus $z = (r + iv_0, v)$ with $v_0 \in \mathbb{R}$ and $v \in \mathbb{R}^n$. As $z \in \mathbb{S}^n_{\mathbb{C}}$, we have

$$1 = z^2 = r^2 - v_0^2 + 2iv_0 + \|v\|^2.$$ 

Hence $v_0 = 0$ and this implies that $z \in \mathbb{S}^n_+ \subset \Xi$. Finally, we note that, if $z \in T_{\mathbb{V}_+}$, then $\text{Re} \ z_0 > 0$ hence $T_{\mathbb{V}_+} \cap \mathbb{S}^n_+ = T_{\mathbb{V}_+} \cap \mathbb{S}^n_{\mathbb{C}+}$.

(iii) follows from (i) and (ii).

**Corollary 3.3.** $\Xi$ is an open subset of $\mathbb{S}^n_{\mathbb{C}}$, hence a complex manifold, and the group $G_c^c$ acts on $\Xi$ by holomorphic maps.

**Proof.** The tube domain $T_{\mathbb{V}_+}$ is open in $\mathbb{C}^{n+1}$ and hence $\Xi$ is an open subset of the complex manifold $\mathbb{S}^n_{\mathbb{C}}$ and hence a complex submanifold. The last statement is clear as this is the restriction of the linear action of $O_{n+1}(\mathbb{C})$ to $G_c^c$. 

**Example 3.4.** Let us take a closer look at the case $n = 1$. We parametrize the complex 1-sphere with the biholomorphic map

$$\zeta: \mathbb{C}^\times \to S^1_{\mathbb{C}} \subseteq \mathbb{C}^2, \quad \zeta(z) = \left(\frac{1}{2}(z + \frac{1}{z}), \frac{1}{2i}(z - \frac{1}{z})\right), \quad \zeta(1) = (1, 0) = e_0,$$

whose inverse is given by $\zeta^{-1}(z_0, z_1) = z_0 + iz_1$. That this map is biholomorphic is most easily seen by writing $z = e^{iw}$ with $\cos(w) = \frac{1}{2}(z + z^{-1})$ and $\sin(w) = \frac{1}{2i}(z - z^{-1})$, which leads to $\zeta(z) = (\cos w, \sin w)$ and in particular to $\zeta(1) = e_0$ and $\zeta(i) = e_1$. The map $\zeta$ intertwines the multiplication action of $\mathbb{T}$ on $\mathbb{C}^\times$ with the action of $G_0 = SO_2(\mathbb{R})$ on $S^1_{\mathbb{R}}$. Further, the coordinate reflections act on $\mathbb{C}$ by $r_0(z) = -z^{-1}$ and $r_1(z) = z^{-1}$. Accordingly, $G_0^c \cong \mathbb{R}^\times$.

The tube domain is given by

$$T_{\mathbb{V}_+} = \{z \in \mathbb{C}^2: \text{Im } z_1 < \text{Re } z_0\}.$$ 

To determine $\zeta^{-1}(\Xi)$, we observe that $\text{Re}(z + z^{-1}) > 0$ if and only if $\text{Re } z > 0$, so that

$$S^1_{+,\mathbb{C}} = \zeta(\mathbb{C}_+), \quad \text{where } \mathbb{C}_+ := \{z \in \mathbb{C}: \text{Re } z > 0\}$$

is the open right half-plane. If $z = x + iy$ with $x > 0$, then

$$|\text{Im } \zeta(z)| = \frac{1}{2} |\text{Re } (z - z^{-1})| = \frac{1}{2} |x\left(1 - \frac{1}{x^2 + y^2}\right)| < \frac{1}{2} x\left(1 + \frac{1}{x^2 + y^2}\right) = \text{Re } \zeta(z)_0$$

holds automatically. By Proposition 3.2, this shows that

$$\Xi = S^1_{+,\mathbb{C}} = \zeta(\mathbb{C}_+).$$

Since we shall need it later, we calculate the set $\mathbb{C}_\Xi$ of values of the kernel $[\cdot, \cdot]_V$ on $\Xi$. We have

$$[\zeta(z), \zeta(w)]_V = \frac{1}{4}(z + \frac{1}{z})(w + \frac{1}{w}) - \frac{1}{4}(z - \frac{1}{z})(w - \frac{1}{w}) = \frac{1}{2}\left(\frac{z}{w} + \frac{w}{z}\right).$$

For $z, w \in \mathbb{C}_+$, the values of $zw^{-1}$ are the products of elements in $\mathbb{C}_+$, which leads to the set $\mathbb{C} \setminus (-\infty, 0]$. For $z \in \mathbb{C}^\times$, the complex number $\frac{1}{2}(z + z^{-1})$ is contained in $(-\infty, -1]$ if and only if $z \in (-\infty, 0)$. This implies that $\mathbb{C}_\Xi = \mathbb{C} \setminus (-\infty, -1]$.

With the precise information on the case $n = 1$, we can determine the set $\mathbb{C}_\Xi$ in general:

**Lemma 3.5.** $\mathbb{C}_\Xi := \{[z, w]_V : z, w \in \Xi\} = \mathbb{C} \setminus (-\infty, -1]$. 


Remark 3.6. Let \( \Omega \). This shows that \( \Theta \) is indeed the crown domain.

Thus (3.2) \( [z,z]_V \geq |x,x]_V > 0 \).

For \( u, v \in \Xi = T_{V_{+}} \cap S_{C}^{n} \), we now have \( u + v \in T_{V_{+}} \) and
\[
[u + v, u + v]_{V} = [u, u]_{V} + [v, v]_{V} + 2[u, v]_{V} = 2(1 + [u, v]_{V}).
\]

If \( [u, v]_{V} \) is real, then so is \( [u + v, u + v]_{V} \), and we obtain from the preceding paragraph that
\[ 0 < [u + v, u + v]_{V} = 2(1 + [u, v]_{V}), \quad i.e., \quad [u, v]_{V} > -1. \]

\( \Xi = G^{c} \cdot \text{Exp}_{e_{0}} \Omega = K \cdot \text{Exp}_{e_{0}}((-\pi/2, \pi/2)e_{n}). \)

Thus
\[
\Xi = G^{c} \cdot \text{Exp}_{e_{0}}(\Omega) = G^{c} \cdot \text{Exp}_{e_{0}}((-\pi/2, \pi/2)e_{n}).
\]

This shows that \( \Xi \) is indeed the crown domain
\[
G_{c} \cdot \text{Exp} \left( \left\{ i \mathbf{x} : \mathbf{x} \in \mathfrak{p}, \| \mathbf{ad} \mathbf{x} \| < \frac{\pi}{2} \right\} \right) K_{C}/K_{C} \subseteq G_{C}/K_{C}
\]

of the Riemannian symmetric space \( G^{c} \cdot e_{0} \cong G^{c}/K \) (cf. [GK03, p. 32]). For more information about the crown see [AG90, KS04, KS04, KO08]. The crown is the maximal \( G^{c} \)-invariant domain for the holomorphic extension of all spherical functions on the Riemannian symmetric space \( G^{c}/K \).

It is shown in [KS05] that in our case the crown is the Lie ball, which can be identified with the Riemannian symmetric space \( \text{SO}_{2,n}(\mathbb{R})_{0}/(\text{SO}_{2}(\mathbb{R}) \times \text{SO}_{n}(\mathbb{R})) \); see also the Section 6.1 for a direct argument.

3.2. The boundary of \( \Xi \). In this subsection we show that the boundary of the crown consists of two types of orbits. We then consider the projection of the orbits onto \( V \) and \( iV \), respectively:

\[
\begin{array}{ccc}
V & \partial \Xi & iV \\
p_{V} & & p_{iV}
\end{array}
\]

Both \( p_{V} \) and \( p_{iV} \) are \( G^{c} \)-equivariant maps. Hence the projection of a \( G^{c} \)-invariant subset is again \( G^{c} \)-invariant in \( V \), resp., \( iV \).

Lemma 3.7. The boundary of \( \Xi \) in \( S_{C}^{n} \) is given by
\[
(3.3) \quad \partial \Xi = \{ u + iv : u, v \in V, [u, u]_{V} = 0, u_{0} \geq 0, [v, v]_{V} = -1, [u, v]_{V} = 0 \}. \]

Proof. Recall from Proposition 3.2 that \( \Xi = T_{V_{+}} \cap S_{C}^{n} \). In view of Lemma 3.1 this means that, for \( u, v \in V \), the element \( z = u + iv \) is contained in \( \Xi \) if and only if \( [u, u]_{V} - [v, v]_{V} = 1, [u, v]_{V} = 0, [u, u]_{V} > 0 \) and \( u_{0} > 0 \). This shows that
\[
\Xi \subseteq \{ u + iv \in V_{C} : [u, u]_{V} - [v, v]_{V} = 1, [u, v]_{V} = 0, [u, u]_{V} \geq 0, \text{ and } u_{0} \geq 0 \}.
\]

To see that we actually have equality, suppose that \( z = u + iv \) is contained in the right hand side but not in \( \Xi \).
If \( u_0 = 0 \), then \([u, u]_V = u_0^2 - u^2 = -u^2 \geq 0\) implies that \( u = 0 \). This shows that \( z = iv \) with \([v, v]_V = -1\). Then there exists an element \( \tilde{u} \in V_\pm \) with \([\tilde{u}, v]_V = 0\), so that \( z_\varepsilon := (1 + \varepsilon^2)^{-1}(\varepsilon \tilde{u} + iv) \in \Xi \) with \( z_\varepsilon \to iv \) for \( \varepsilon \to 0 \).

If \( u_0 > 0 \), then we must have \([u, u]_V = 0 \) and \([v, v]_V = -1\). Acting with a suitable element of \( G^c \), we may assume that \( v_0 = 0 \). For \( \varepsilon > 0 \) small enough we then have \( z_\varepsilon := (1+2\varepsilon u_0+\varepsilon^2)^{-1}(z+\varepsilon e_0) \in \Xi \) with \( z_\varepsilon \to z \) for \( \varepsilon \to 0 \).

If \( u = 0 \), then (3.3) leads to a realization of de Sitter space
\[
dS^n := i[v \in V \mid [v, v]_V = -1] = S^n_\varepsilon \cap iV = \partial \Xi \cap iV = p_1 iV(\partial \Xi) .
\]
The stabilizer \( G^c_\varepsilon \) of the point \( e_n \in dS^n \) in the group \( G^c \) is \( H = tO_{1, n-1}(\mathbb{R})^\perp t \), which is a non-compact symmetric subgroup of \( G^c \) with respect to the involution given by conjugation by \( r_n \). Hence \( dS^n \cong G^c/H \) is a Lorentzian symmetric space.

Before discussing the other boundary orbits, some more notation is needed. Let
\[
M := \begin{cases}
\begin{align*}
1 & \quad 0 & 0 \\
0 & \quad A & 0 \\
0 & \quad 0 & \quad 1
\end{align*}
\end{cases} : i A \in O_{n-1}(\mathbb{R}) \end{equation}
\]
\[
N := \begin{cases}
\begin{align*}
iv & \quad -iv & 0 \\
v & \quad I_{n-1} & -\frac{1}{2} \|v\|^2 \\
\frac{1}{2} \|v\|^2 & \quad t & \quad 1 - \frac{1}{2} \|v\|^2
\end{align*}
\end{cases} : v \in \mathbb{R}^{n-1}
\end{equation}
\]
\[
A := \begin{cases}
\begin{align*}
cosh(t) & \quad 0 & \quad -i \sinh(t) \\
0 & \quad \sinh(t) & \quad 0 \\
i \sinh(t) & \quad 0 & \quad \cosh(t)
\end{align*}
\end{cases} : t \in \mathbb{R}
\end{equation}
\]
The group \( MN \) is the stabilizer of the element \( \xi^0 := e_0 + ie_n \), and
\[
a_t \xi^0 = e^t \xi^0 .
\]
The orbit of \( \xi^0 \) is
\[
(3.8) \quad \mathbb{L}^n_+ := \{v \in (v_0, iv) \in V \mid [v, v]_V = 0, v_0 > 0\} = G^c.\xi^0 \simeq G^c/MN
\]
of non-zero forward lightlike vectors. To see that \( \mathbb{L}^n_+ = G^c.\xi^0 \), let \( v = (v_0, iv) \in \mathbb{L}^n_+ \) and note that \( v_0 \neq 0 \) and \( \|v\| = v_0 > 0 \). As \( K \cong O_n(\mathbb{R}) \) acts transitively on each sphere in \( \mathbb{R}^n \), we may assume that \( v = v_0 \xi_n \), or \( v = v_0 \xi \). Then \( v = v_0(e_0 + ie_n) = a_{log v_0} \xi^0 \) by (3.7).

**Lemma 3.8.** Suppose that \( n \geq 2 \). Let \( \mathcal{O} := G^c.(\xi^0 + e_{n-1}) \). Then the boundary of the crown is the union of two \( G^c \)-orbits
\[
(3.9) \quad \partial \Xi = dS^n \cup \mathcal{O} .
\]
The projection of \( \mathcal{O} \) onto \( V \) is \( \mathbb{L}_+^n = G^c.\xi^0 \) and the projection onto \( iV \) is \( dS^n = G^c.e_{n-1} \).

**Proof.** Assume that \( z = u + iv \in \partial \Xi \setminus dS^n \), so that \( u \neq 0 \) (cf. Lemma 3.7). As \([u, u]_V = 0\), the \( G^c \)-orbit of \( u \) contains \( \xi^0 \), so that we may w.l.o.g. assume that \( u = \xi^0 \). Acting with the subgroup \( K = G^c_{\xi^0} \cong O_n(\mathbb{R}) \), we may further assume that \( v \in \mathbb{R}e_0 - \mathbb{R}ie_n \). Then \([u, v]_V = 0\) implies that \( v = v_0 \xi^0 - aie_n \), and \([v, v]_V = -1\) yields \( a = 1 \). We thus obtain \( z = (1 + v_0)\xi^0 + e_{n-1} \). In the notation of (3.5), we have
\[
n_v(\xi^0 + e_{n-1}) = \xi^0 - iv_{n-1}e_{-1}^0 + e_{n-1} = (1 - iv_{n-1})\xi^0 + e_{n-1} ,
\]
so that we may further assume that \( v_0 = 0 \), which eventually shows that \( z \) is \( G^c \)-conjugate to \( \xi^0 + e_{n-1} \). Now the claim follows from
\[
p_1 V(\mathcal{O}) = G^c.p_1 V(\xi^0 + e_{n-1}) = G^c.e_{n-1} = dS^n .
\]
The tangent space of $dS^n$ at $e_n$ is the $n$-dimensional Minkowski space
\[ T_{e_n}(dS^n) = iV \cap e_n^\perp \cong \mathbb{R}^{1,n-1}. \]

By (2.4), we have
\[ \text{(3.10)} \quad \Exp_{e_0}(z) = S(z^2)z + C(z^2)e_n \quad \text{for} \quad z \in T_{e_0}(dS^n)_\mathbb{C} = \mathbb{C} \oplus \mathbb{C}^{n-1}. \]

We now describe how one can obtain the crown by moving inward from the de Sitter space $dS^n$ (see [KS05] for a discussion of the general case):

**Theorem 3.9.** ($\Xi$ from the perspective of $dS^n$) Let
\[ \Omega_{e_n} := \{ v \in iT_{e_0}(dS^n) = \mathbb{R} \oplus i\mathbb{R}^{n-1} \subseteq V : v_0 > 0, [v,v]_V > 0 \} \]
be the $n$-dimensional forward light cone and
\[ \Omega_{e_n}^\pi := \{ v \in \Omega_{e_n} : [v,v]_V < \pi^2 \}. \]
For $g \in G^c$ and $p := g.e_n$, we put $\Omega^\pi_p := g.\Omega^\pi_{e_n}$. Then we have
\[ \Xi = G^c.\Exp_{e_0}(\Omega^\pi_{e_n}) = \bigcup_{p \in dS^n} \Exp_p(\Omega^\pi_p). \]

**Proof.** In view of the $G^c$-invariance of $\Xi$ and the equivariance of the exponential map of $S^n_\mathbb{C}$, it suffices to verify the first equality. From (3.10) we obtain for $v \in \mathbb{R} \oplus i\mathbb{R}^{n-1} \subseteq iT_{e_0}(dS^n)$ and $\mathbb{R}_+ = (0,\infty)$:
\[ \text{(3.11)} \quad \Exp_{e_0}(v) = S([v,v]_V)v + C([v,v]_V)e_n \]
and this is contained in $TV_+ \cap S^n_\mathbb{C} = \Xi$ if $[v,v]_V \in (0,\pi^2)$. Therefore $\Exp_{e_0}(\Omega_{e_n}) \subseteq \Xi$. If, conversely, $z \in \Xi = G^c.S^n_\mathbb{C}$, then there exists a $t \in (0,\pi)$ such that $z$ is $G^c$-conjugate to $x = (\sin t,0,\ldots,0,\cos t)$. But then $te_0 \in \Omega_{e_n}^\pi$ and (3.11) yields $x = \Exp_{e_0}(te_0)$. This proves the claim. \hfill $\square$

**Remark 3.10.** (a) The proof of Lemma 3.8 shows that the second orbit in (3.9) is a homogeneous space of the form
\[ O \cong G^c/\text{O}_{n-2}(\mathbb{R})\{n_v \in N : v_{n-1} = 0\}, \]
but we will not use that fact in this article. This orbit is not a symmetric space but we have a double fibration
\[ \begin{array}{ccc}
\mathbb{L}_n^+ & \xrightarrow{p_V} & O \\
& & \xleftarrow{p^{\prime}_V} \\
& dS^n & 
\end{array} \]
which might be interesting for harmonic analysis on $dS^n$.

(b) We will return to the space $\mathbb{L}_n^+$ in Section 5.4. We recall that an orbit of a subgroup $gNg^{-1}$, $g \in G^c$, in $\mathbb{H}_V^n$ is called a **horocycle**. The group $G^c$ acts transitively on the set of horocycles which is isomorphic to $G^c/MN \cong \mathbb{L}_n^+$ because $M$ leaves every horocycle invariant. As a subset of $\mathbb{H}_V^n$, the basic horocycle is
\[ h_0 := N.e_0 = \{(1 + \frac{1}{2}||v||^2,iv,\frac{i}{2}||v||^2) : v \in \mathbb{R}^{n-1}\} \cong \mathbb{R}^{n-1}. \]
As $N \cap G^c_{e_0} = \{e\}$, the horocycles are paraboloids diffeomorphic to $N$ and hence to $\mathbb{R}^{n-1}$.

(c) For de Sitter space, the stabilizer group $H = G^c_{e_0}$ is a symmetric subgroup of $G^c$ as pointed out above. But $H$ is not compact and $dS^n$ is not a Riemannian symmetric space, but a pseudo-Riemannian symmetric space. The signature of the metric is $(1,n-1)$. The symmetric space $G^c/H$ is an example of a **non-compactly causal symmetric space**, see [HÖ96]. The involution $r_n$ is dissecting and commutes with the action of $H$. 

16
Example 3.11. In the context of Example 3.4 where we discuss the case \( n = 1 \) by an isomorphism \( \zeta: \mathbb{C}^n \to \mathbb{S}^1_\mathbb{C} \), we have \( \mathbb{L}_+^1 = \emptyset \),
\[
\zeta(i\mathbb{R}^n) = \partial \Xi = dS^1 \subseteq i\mathbb{R} \times \mathbb{R} \quad \text{and} \quad \zeta(\mathbb{R}_+^n) = \mathbb{H}^1_\mathbb{V} \subseteq \mathbb{R} \times i\mathbb{R}.
\]

4. Reflection positivity on the sphere \( \mathbb{S}^n \)

In this section we specialize the results from Section 2.1 to the sphere \( \mathbb{S}^n \) and the dissecting involution \( r_0 \). We start by discussing \( G^c \)-invariant kernels on \( \Xi \) arising by analytic extension from \( G \)-invariant kernels on \( \mathbb{S}^n \) by twisting with \( \sigma \). To connect with our previous work on reflection positive functions on the circle \( \mathbb{S}^1 \) \([\text{NO15b}]\), we then discuss the special case \( n = 1 \). After that we turn to the general case and obtain an explicit expression for the kernel \( \Psi_m \) on \( \Xi \) (Theorem 2.13).

4.1. Invariant positive definite kernels. We say that \( \Psi: \Xi \times \Xi \to \mathbb{C} \) is sesquiholomorphic if \( \Psi \) is holomorphic in the first and antiholomorphic in the second argument. Note that this is equivalent to \( \Psi \) being holomorphic on \( \Xi \times \Xi^\text{op} \), where \( \Xi^\text{op} \) carries the opposite complex structure. We write \( \text{Sesh}(\Xi) \) for the complex linear space of sesquiholomorphic \( G^c \)-invariant kernels on \( \Xi \), and \( \Gamma := \Gamma(\Xi) \subseteq \text{Sesh}(\Xi) \) for the convex cone of positive definite kernels. The Fréchet space of holomorphic functions on \( \Xi \) is denoted by \( \mathcal{O}(\Xi) \). We note that every \( \Psi \in \Gamma \) is hermitian in the sense that \( \Psi(z, w) = \overline{\Psi(w, z)} \).

Lemma 4.1. For \( \Psi \in \text{Sesh}(\Xi) \), the following are equivalent:

1. \( \Psi = 0 \).
2. \( \Psi|_{\Xi \times \Xi^\text{op}} = 0 \).
3. \( \Psi|_{\mathbb{H}^1_\mathbb{V}} = 0 \).
4. \( \Psi|_{\mathbb{S}^+} = 0 \).

For \( \Psi \in \Gamma \), these conditions are further equivalent to

5. \( \Psi(e_0, e_0) = 0 \).

Proof. Obviously (i) implies (ii). The equivalence of (ii), (iii) and (iv) follows from the fact that \( \Xi \) is connected and \( \mathbb{S}^1_\mathbb{C} \) and \( \mathbb{H}^1_\mathbb{V} \) are totally real submanifolds of \( \Psi \).

It is also clear that (iv) implies (v). If, conversely, \( \Psi \) is positive definite and \( \Psi(e_0, e_0) = 0 \), then
\[
|\Psi(z, e_0)|^2 \leq \Psi(z, z)\Psi(e_0, e_0) = 0 \quad \text{for} \quad z \in \Xi,
\]
i.e., \( \Psi|_{\Xi} = 0 \).

If \( \Psi|_{\Xi} = 0 \), then the \( G^c \)-invariance of \( \Psi \) leads to \( \Psi|_{\mathbb{H}^1_\mathbb{V}} = \Psi|_{\mathbb{H}^1_\mathbb{V}} = 0 \) for \( g \in G^c \), \( z \in \Xi \). Hence \( \Psi = 0 \) for every \( z \in \mathbb{H}^1_\mathbb{V} \) and \( \Psi(z, w) = 0 \) for all \( z, w \in \Xi \) and since \( \mathbb{H}^1_\mathbb{V} \) is totally real in \( \Xi \), we obtain \( \Psi(w, z) = 0 \) for all \( z, w \in \Xi \). \( \square \)

Corollary 4.2. Any \( \Psi \in \text{Sesh}(\Xi) \) is uniquely determined by any of the \( K \)-invariant functions \( \Psi|_{\mathbb{H}^1_\mathbb{V}} \) and \( \Psi|_{\mathbb{S}^+} \).

Lemma 4.1(v) shows in particular that the convex subset
\[
\Gamma_1 := \{ \Psi \in \Gamma: \Psi(e_0, e_0) = 1 \}
\]
is a base of the cone \( \Gamma \), i.e., the linear functional \( \Psi \mapsto \Psi(e_0, e_0) \) is strictly positive on \( \Gamma \setminus \{0\} \). We write
\[
\Gamma_e := \text{Ext}(\Gamma_1)
\]
for the set of extreme points of \( \Gamma_1 \) which represent the extremal rays of the cone \( \Gamma \). For \( y \in \Xi \) and \( \Psi \in \Gamma \), we obtain a holomorphic function \( \Psi_y \in \mathcal{O}(\Xi) \) by \( \Psi_y(x) := \Psi(x, y) \) such that
\[
\Psi_y(x) = \overline{\Psi(x, y)} = \Psi(y, x) \quad \text{and} \quad \Psi_y(g \cdot x) = \Psi_y(x) \quad \text{for} \quad x, y \in \Xi, g \in G^c.
\]
Our next goal is to identify the elements in $\Gamma_e$. We start with the following easy geometric lemmas:

**Lemma 4.3.** Two pairs $(x, y), (z, w) \in S^n \times S^n$ are conjugate under $G = O_{n+1}(\mathbb{R})$ if and only if $xy = zw$.

**Remark 4.4.** For $n > 1$, Lemma 4.3 remains true with $SO_{n+1}(\mathbb{R})$ instead of $O_{n+1}(\mathbb{R})$. However, for $n = 1$, one needs an additional invariant to separate the $SO_2(\mathbb{R})$ orbits of pairs, namely the determinant $\det(x, y) := x_0y_1 - x_1y_0$, to determine the oriented angle between $x$ and $y$.

**Lemma 4.5.** Let $\Omega \subseteq S^n \times S^n$ be a subset invariant under the diagonal action of $G = O_{n+1}(\mathbb{R})$. For a function $\Phi: \Omega \to \mathbb{C}$, the following are equivalent:

(i) $\Phi$ is $G$-invariant.

(ii) There exists a $K$-biinvariant function $\varphi_\Phi$ on $G_\Omega := \{g \in G: (g.e_0, e_0) \in \Omega\}$ such that $\Phi(g_1.e_0, g_2.e_0) = \varphi_\Phi(g_2^{-1}g_1)$.

(iii) There exists a function $\alpha_\Phi: \{(x, y) \in \Omega\} \to \mathbb{C}$ such that $\Phi(x, y) = \alpha_\Phi(xy)$ for $(x, y) \in \Omega$.

**Definition 4.6.** An analytic kernel $\Psi$ on $S^n_+ \times S^n_+$ is said to be $(g^c, K)$-invariant if the following conditions hold for all $(x, y) \in S^n_+$:

(i) For $X \in g^c$ we have

$$\left(\mathcal{L}_X^1 F\right)(x, y) = \frac{d}{dt}\bigg|_{t=0} F(\exp(tX).x, y) \quad \text{and} \quad \left(\mathcal{L}_X^2 F\right)(x, y) = \frac{d}{dt}\bigg|_{t=0} F(x, \exp(tX).y)$$

and define $\mathcal{L}_X^j$ for $X \in g_\mathbb{C}$ by complex linear extension. Then

$$\left(\mathcal{L}_X^1 \Psi\right)(x, y) = -\mathcal{L}_X^2 \Psi(x, y) \quad \text{for all } X \in g^c,$$

(ii) For $k \in K$ we have $\Psi(k.x, k.y) = \Psi(x, y)$.

If $\Psi \in \text{Sesq}(\Xi)$ is real-valued and symmetric on $S^n_+ \times S^n_+$, then the kernel $\Psi^*(z, w) := \overline{\Psi(w, z)}$ coincides with $\Psi$ on the totally real submanifold $S^n_+ \times S^n_+$ of $\Xi \times \Xi^{op}$, hence on $\Xi \times \Xi^{op}$. We conclude that $\Psi$ is hermitian.

**Lemma 4.7.** Let $\Psi$ be a $(g^c, K)$-invariant analytic kernel on $S^n_+ \times S^n_+$. Then there exist an analytic complex-valued function $\alpha_\Psi$ on an open neighborhood of $(-1, 1)$, such that

$$\Psi(x, y) = \alpha_\Psi([x, \sigma_\Psi y]) \quad \text{for } x, y \in S^n_+.$$

The kernel $\Psi$ is hermitian if and only if $\alpha_\Psi$ is real valued.

**Proof.** 

**Step 1:** If $y \in S^n_+$, then $\sigma_\Psi y \in S^n_+$. Hence $\Phi(x, y) := \Psi(x, \sigma_\Psi(y))$ is defined on $S^n_+ \times S^n_+$. As $\Psi$ is analytic and $(g^c, K)$-invariant, the kernel $\Phi$ is $(g, K)$-invariant and hence locally $G$-invariant. Transforming pairs of points on $S^n_+$ by differentiable paths in $G$ inside of $S^n_+$ into pairs lying in $S^1 \cong S^n \cap (\mathbb{R}e_0 + \mathbb{R}e_1)$, it follows that there exists a function $\alpha_\Phi: (-1, 1) \to \mathbb{C}$ such that

$$\Phi(x, y) = \alpha_\Phi(xy) \quad \text{for } x, y \in S^n_+.$$

**Step 2:** Now we argue that $\alpha_\Phi$ extends to an analytic function on an open interval containing $(-1, 1)$. The analyticity of $\alpha_\Phi$ on the open interval $(-1, 1)$ immediately follows from the analyticity of $\Phi$. To see what happens in 1, we observe that

$$\Phi(x, e_0) = \alpha_\Phi(x.e_0) = \alpha_\Phi(x_0).$$

For $t$ close to 0, this leads to

$$\Phi(\cos(t)e_0 + \sin(t)e_1, e_0) = \alpha_\Phi(\cos(t)) = \alpha_\Phi(C(t^2))$$

18
with $C$ as in (2.3). Since this function is symmetric in $t$, it follows that $\alpha_\Phi \circ C$ extends to a function which is analytic in a neighborhood of 0. As $C(0) = 1$ and $C'(0) \neq 0$, the Inverse Function Theorem for holomorphic functions shows that $\alpha_\Phi$ extends to an analytic function in a neighborhood of 1.

**Step 3:** For $x, y \in S^1_\C$ we now have

$$\Psi(x, y) = \alpha_\Phi(x_{\sigma_V} y) = \alpha_\Phi([x, \sigma_V y]_V).$$

Hence the statement holds with $\alpha_\Phi := \alpha_\Phi$. As $\sigma_V|_{\mathbb{R}^n} = -r_0 \in K$, we have

$$\alpha_\Phi(x_{\sigma_V} y) = \alpha_\Phi(\sigma_V x y) = \alpha_\Phi(y_{\sigma_V} x).$$

Thus $\Psi(x, y) = \Psi(y, x)$. Hence $\Psi$ is hermitian if and only if $\Psi(y, x) = \Psi(x, y) = \overline{\Psi(y, x)}$ which holds if and only if $\alpha_\Phi$ is real valued.

**Theorem 4.8.** A sesquiholomorphic kernel $\Psi$ on $\Xi \times \Xi$ is $G^c$-invariant if and only if there exists a holomorphic function

$$\alpha_\Phi : \mathbb{C}_\Xi = \mathbb{C} \setminus (-\infty, -1) \rightarrow \mathbb{C} \text{ such that } \Psi(z, w) = \alpha_\Phi([z, \sigma_V w]_V) \text{ for } z, w \in \Xi.$$  

Then $\Psi$ is hermitian if and only if $\alpha_\Phi|_{(-1, 1]}$ is real valued.

**Proof.** By Lemma 3.5 any holomorphic function $\alpha$ on $\mathbb{C}_\Xi$ defines a $G^c$-invariant sesquiholomorphic kernel by $\Psi(z, w) := \alpha([z, \sigma_V w]_V)$.

Suppose, conversely, that $\Psi$ is a $G^c$-invariant sesquiholomorphic kernel on $\Xi$. We have already seen that $\Psi$ is uniquely determined by its restriction to $S^1_\C \times S^1_\C$. This restriction is $(g_c, K)$-invariant if and only if $\Psi$ is $G^c$-invariant. In view of Lemma 4.7 it therefore remains to show that the analytic function $\alpha_\Psi$ extends to the domain $\mathbb{C}_\Xi$. Since $S^1_\C$ embeds naturally in $S^1_\Xi$, it suffices to verify this for $n = 1$.

We recall from Example 3.4 that $S^1_\C \cong \mathbb{C}^\times$, $\Xi \cong \mathbb{C}^+$ is the open right half-plane and

$$G^c \cong \mathbb{R}^\times_+ \times \{\text{id}, \sigma\} \text{ with } \sigma(z) = z^{-1},$$

where $r \in \mathbb{R}^\times_+$ acts by multiplication. Further, $\sigma_V \zeta(z) = \zeta(\overline{z})$ for $z \in \Xi$. If $\Psi$ is an $\mathbb{R}^\times_+$-invariant sesquiholomorphic kernel, we can use the family $(\Psi_w)_{w \in \Xi}$ of holomorphic functions on $\mathbb{C}_\Xi$ to obtain a holomorphic function $F$ defined on the domain $\Xi \cdot \Xi = \mathbb{C} \setminus (-\infty, 0)$ such that $\Psi(z, w) = F(z\overline{w}^{-1})$ holds for $z, w \in \Xi$. The $\sigma$-invariance of $\Psi$ yields for $x, y \in \mathbb{R}^\times_+$:

$$F(x y^{-1}) = \Psi(x, y) = \Psi(x^{-1} y^{-1}) = F(x^{-1} y),$$

and thus $F(z^{-1}) = F(z)$. From Example 3.4 we further recall that

$$[\zeta(z), \sigma_V \zeta(w)] = \frac{1}{2} \left( \frac{w}{\overline{w}} + \frac{\overline{w}}{w} \right) = R(z\overline{w}^{-1}) \text{ for } R(z) := \frac{1}{2}(z + z^{-1}).$$

In a 1-neighborhood the function $\alpha_\Psi$ thus satisfies

$$\alpha_\Psi(R(z)) = \Psi(z, 1) = F(z).$$

Next we observe that the equation $R(z) = w$ has for $w \notin (-\infty, -1]$ the solutions

$$z_{1/2} = w \pm \sqrt{w^2 - 1} \in \mathbb{C} \setminus (-\infty, 0) \text{ satisfying } z_1 z_2 = 1,$$

so that $F(z_1) = F(z_2)$. On $\mathbb{C} \setminus (-\infty, -1]$ we thus obtain by

$$w \mapsto F(w + \sqrt{w^2 - 1}) = F(w - \sqrt{w^2 - 1}) \quad (4.2)$$

a well-defined function. In a neighborhood of 1 it coincides with $\alpha_\Psi$, so that it is holomorphic. Outside of 1, both branches of the square root yield the same holomorphic function when composed with $F$, so that $\alpha_\Psi(w) := F(w \pm \sqrt{w^2 - 1})$ defines a holomorphic function on $\mathbb{C} \setminus (-\infty, -1]$. This completes the proof. $\square$
4.2. Reflection positivity on $S^1$. Let us recall the results from [NÖ15b] where we considered the case $G = M = T_\beta = \mathbb{R}/\beta\mathbb{Z}$, $\beta > 0$, with the involution $r_1(z) = z^{-1}$. Assume that $\beta = 2\pi$. Then $\mathbb{T}_\beta$ is identified with $S^1$ by the map $t + 2\pi\mathbb{Z} \mapsto (\cos t, \sin t)$, mapping 0 to $e_0$ and $\pi/2$ to $e_1$.

We let $\mathbb{T}_{\beta,+} = \{ t + \beta\mathbb{Z} : 0 < t < \beta/2 \}$.

Here $G^c_0 := \mathbb{R}_+^\ast$ act by multiplication, $G \cong G_0 \times \{ \text{id}, r_1 \}$ and $G^c = G^c_0 \times \{ \text{id}, r_1 \}$. We also recall from [31] that $\Xi = \mathbb{C}_+$ is the open right half plane and $\mathbb{H}^1 = \mathbb{R}_+^\ast$.

Here the basic example of reflection positive functions are given by

$$f_\lambda(t) = e^{-t\lambda} + e^{-(\beta-t)\lambda} = 2e^{-\beta\lambda/2} \cosh \left( \frac{\beta}{2} - t \right) \lambda$$

(Definition 2.2(iii)). With $\beta = 2\pi$ this becomes $f_\lambda(t) = 2e^{-\pi\lambda} \cosh((\pi - t)\lambda)$. In general, we have (see [KL81, Thm. 3.3] or [N ´O15b, Thm. 2.4]):

**Theorem 4.9.** A $2\pi$-periodic symmetric continuous function $\varphi : \mathbb{R} \to \mathbb{C}$ is reflection positive with respect to $(\mathbb{T}, \mathbb{T}^+, r_1)$, i.e., the kernels $(\varphi(t+s))_{t,s\in[0,\pi]}$ and $(\varphi(t-s))_{t,s\in\mathbb{R}}$ are positive definite, if and only if there exists a positive measure $\mu$ on $[0,\infty)$ such that

$$\varphi(t) = \int_0^\infty e^{-t\lambda} + e^{(\pi-t)\lambda}d\mu(\lambda) \quad \text{for} \quad 0 \leq t \leq 2\pi.$$

The measure $\mu$ is uniquely determined by $\varphi$.

This fits well into our current discussion. For the $O_2(\mathbb{R})$-invariant kernel $\Phi_m$ on the complement of the diagonal in $S^1 \times S^1$ (Lemma 2.4), we write

$$\tilde{\varphi}_m(t) := \alpha_{\Phi_m}(\cos(t)) = \Phi_m((\cos(t), \sin(t)), e_0)$$

for a function $\alpha_{\Phi_m}$ in $[-1, 1)$ (Lemma 4.5). In this notation the differential equation in Theorem 2.13 becomes

$$\tilde{\varphi}_m''(t) = m^2 \tilde{\varphi}_m(t) \quad \text{for} \quad 0 < t < 2\pi.$$

The solutions are of the form $\tilde{\varphi}_m(t) = ae^{mt} + be^{-mt}$. The invariance under $r_1$ leads to

$$\tilde{\varphi}_m(2\pi - t) = \tilde{\varphi}_m(t), \quad 0 < t < 2\pi,$$

and hence to $a = e^{-2\pi mb}$. Thus

$$\tilde{\varphi}_m(t) = b(e^{-mt} + e^{-2\pi mt}) = 2be^{-\pi m} \cosh((\pi - t)m), \quad 0 < t < 2\pi,$$

which is a multiple of the function $f_m$ from above.

For the twisted kernel $\Psi(x,y) = \Phi(x, \sigma y)$ corresponding to an $O_2(\mathbb{R})$-invariant kernel on the complement of the diagonal of $S^1$, we have for $|t|, |s| < \pi/2$:

$$\Psi((\cos t, \sin t), (\cos s, \sin s)) = \Phi((\cos t, \sin t), (\cos s, \sin s))$$

$$= \Phi((\cos t, \sin t), (\cos(\pi - s), \sin(\pi - s))) = \tilde{\varphi}(t + s + \pi).$$

Therefore the positive definiteness of $\Psi$ is equivalent to the positive definiteness of the kernel $(\tilde{\varphi}(t + s))_{0 < t, s < \pi}$.

As $\cosh(mt) = \int_\mathbb{R} e^{-\lambda t}d\mu_m(t)$ is the Laplace transform of the positive measure $\mu_m = \frac{1}{2}(\delta_m + \delta_{-m})$, the kernel $\cosh(m(t+s))$ is positive definite on $\mathbb{R}$. In particular, the kernel $\Psi_m$ corresponding to the function $\psi_m(t) := \varphi_m(\pi + t)$ is positive definite on $S^1_+ \times S^1_+$.

In complex coordinates $z = e^{it}$ on $S^1 \cong \mathbb{T}$, we have

$$\Psi_m(e^{it}, e^{is}) = \tilde{\varphi}_m(t + s + \pi) = 2be^{-\pi m} \cosh(m(t+s)) = be^{-\pi m}(e^{m(t+s)} + e^{-m(t+s)}).$$

For $z, w \in \mathbb{C}$ with $\text{Re} z, \text{Re} w > 0$, we thus obtain the sesquiholomorphic extension

$$\Psi_m(z, w) = 2be^{-\pi m} \cosh(-mi \log(z/w)) = be^{-\pi m}((z/w)^{im} + (z/w)^{-im}).$$
The hypergeometric function

Remark 4.11. If \( a > 1 \), now on that \( n > 2 \)\n
Lemma 4.10. \( \text{Lemma } 4.5(iii) \). Then \( \text{Fa08 Cor. 9.2.4} \) shows that \( \sum_{l=0}^{\infty} (\text{cosh}(mt))^l = 2 \). The substitution \( s = \sin^2(t/2) = \frac{1}{2}(1 - \cos(t)) \in (0,1) \) transforms \( (4.4) \) into the following differential equation for \( y(s) = \eta_x(t), 0 < s < 1 \) (see \( \text{He84 p.484} \) for a general statement):

\[
(1 - s)y''(s) + (\frac{\pi}{2} - ns)y'(s) - m^2y(s) = 0.
\]

This is a special case of the hypergeometric differential equation

\[
s(1 - s)y''(s) + (c - (a + b + 1)s)y'(s) - aby(s) = 0.
\]

This equation has two linearly independent solutions. In general, one of them is singular at the origin. The other one, the Gauss hypergeometric function, denoted by \( _2F_1(a, b; c; x) \), is regular at \( x = 0 \) and normalized by \( _2F_1(1, 1; 2; 0) = 1 \).

We recall here the definition of \( _2F_1 \) as it will be helpful in the following. For \( a \in \mathbb{C} \) and \( k \in \mathbb{N} \) let \( (a)_0 = 1 \) and \( (a)_k := \prod_{j=0}^{k-1} (a + j) \). Then, for \( a, b, c \in \mathbb{C} \) such that \( c \notin -\mathbb{N}_0 \), we have

\[
_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!} \quad \text{for } |z| < 1.
\]

If \( a \) or \( b \) is a negative integer then \( _2F_1(a, b; c; z) \) is a polynomial.

Remark 4.11. The hypergeometric function \( _2F_1 \) has an analytic continuation to \( \mathbb{C} \setminus [1, \infty) \) (see \( \text{WW96 §14.51} \) or \( \text{L73 p. 297} \)). Clearly \( (4.6) \) implies that \( _2F_1(1, b; c; z) = _2F_1(1, c; z) \) and \( _2F_1(a, b; c; z) > 0 \) for \( 0 \leq z < 1 \) and \( a, b, c > 0 \) or \( b = \bar{a}, c > 0 \).

To simplify the notation, let

\[
\rho := \frac{n - 1}{2} > 0 \quad \text{and} \quad \lambda := \lambda_m := \begin{cases} \sqrt{\rho^2 - m^2} & \text{for } m^2 \leq \rho^2 \\ i\sqrt{m^2 - \rho^2} & \text{for } m^2 \geq \rho^2. \end{cases}
\]

Note that

\[
0 \leq \text{Re } \lambda < \rho \quad \text{and} \quad \lambda^2 = \rho^2 - m^2 \quad \text{for } m > 0.
\]
Then the solution to the differential equation (4.5) which is regular at $s = 0$ is, up to a constant, 
\[ 2F_1(\rho + \lambda, \rho - \lambda; n/2; s) = 2F_1(\rho + \lambda, \rho - \lambda; n/2; \frac{1}{2}(1 - \cos(t))). \]

**Theorem 4.12.** There exists a constant $\gamma_{n,m} > 0$ such that the $G^\circ$-invariant kernel $\Psi_m$ on $\Xi \times \Xi$ from Theorem 2.7 is given by
\[
\Psi_m(x, y) = \gamma_{n,m} \cdot 2F_1(\rho + \lambda, \rho - \lambda; n/2; \frac{1}{2}(1 - [x, \sigma_V(y)]_V)) 
= \gamma_{n,m} \cdot 2F_1\left(\frac{n-1}{2} + \lambda, \frac{n-1}{2} - \lambda; \frac{n}{2}; \frac{1}{2}(1 - [x, \sigma_V(y)]_V)\right).
\]

This kernel is positive definite. Furthermore, $\Psi_m$ extends to a sesquiholomorphic kernel on the set
\[
\{(z, w) \in V_C \times V_C: [z, \sigma_V w]_V \in \mathbb{C} \setminus (-\infty, -1]\}.
\]

Note that
\[
\psi_m(x) = \Psi_m(x, e_0) = \gamma_{n,m} \cdot 2F_1\left(\frac{n-1}{2} + \lambda, \frac{n-1}{2} - \lambda; \frac{n}{2}; \frac{1}{2}(1 - x_0)\right).
\]

**Proof.** We recall first that
\[
\Psi_m(x, y) = \Phi_m(x, \sigma(y)) = \alpha \Phi_m(x \sigma(y))
\]
for a function $\alpha \Phi_m$ on the interval $[-1, 1]$ as $\Phi_m$ is $O_{n+1}(\mathbb{R})$-invariant (Lemma 4.3). We also note that $\sigma(x) = -\sigma_V(x)$ on $S^n$. Thus $\Psi_m(x, y) = \alpha \Phi_m(-[x, \sigma_V(y)]_V)$, which is clearly $(g^\circ, K)$-invariant. Thus $\alpha \Phi_m(s)$ from Lemma 4.7 is given by $\alpha \Phi_m(-s)$. We now apply the above discussion to $\psi_m(x) = \alpha \Phi_m(x_0)$ and note that, for $u \in \mathbb{R}$ and $x = \cos(t)e_0 + \sin(t)u$, we have $x_0 = \cos(t)$ and
\[
\sin^2(t/2) = \frac{1}{2} (1 - \cos(t)) = \frac{1}{2} (1 - x_0) = \frac{1}{2} (1 - [x, \sigma_V(e_0)]).
\]

By the discussion preceding the theorem, there exists a constant $\gamma_{n,m}$ such that
\[
\psi_m(x) = \gamma_{n,m} \cdot 2F_1(\rho + \lambda, \rho - \lambda; n/2; \frac{1}{2}(1 - x_0)).
\]

From Lemma 3.3, we know that, for $z, w \in \Xi$, we have $[z, \sigma_V(w)]_V \in \mathbb{C} \setminus (-\infty, -1]$, so that $\frac{1}{2}(1 - [x, \sigma_V(e_0)]) \in \mathbb{C} \setminus [1, \infty)$. Therefore Remark 4.11 combined with Lemma 4.11 implies that the right hand side of (4.12) extends uniquely to a kernel in $\text{Sesh}(\Xi)$. Hence so does $\Psi_m$ and the extension is given by
\[
\Psi_m(x, y) = \gamma_{n,m} \cdot 2F_1(\rho + \lambda, \rho - \lambda; n/2; \frac{1}{2}(1 - [x, \sigma_V(y)]_V)).
\]

Corollary 2.8 implies that the kernel $\Psi_m|_{S^n \times S^n}$ is positive definite. Now [NO14, Thm. A.1] implies that the kernel $\Psi_m$ is positive definite on $\Xi \times \Xi$. As $\Psi_m|_{S^n \times S^n}$ is non-zero, Lemma 4.1 implies that $\Psi_m(e_0, e_0) > 0$. We thus obtain
\[
0 < \Psi_m(e_0, e_0) = \gamma_{n,m} \cdot 2F_1(\rho + \lambda, \rho - \lambda; n/2; 0) = \gamma_{n,m}.
\]

This finishes the proof. \hfill \Box

**Corollary 4.13.** The $G$-invariant reflection positive kernel $\Phi_m(x, y)$ extends to a sesquiholomorphic kernel on $\{(x, y) \in V_C \times V_C: x \sigma_E(y) \in \mathbb{C} \setminus [1, \infty)\}$ given by
\[
\Phi_m(x, y) = \gamma_{n,m} \cdot 2F_1(\rho + \lambda, \rho - \lambda; n/2; \frac{1}{2}(1 + x \sigma_E(y))) 
= \gamma_{n,m} \cdot 2F_1\left(\frac{n-1}{2} + \lambda, \frac{n-1}{2} - \lambda; \frac{n}{2}; \frac{1}{2}(1 + [x, \sigma_V(y)]_V)\right).
\]
Proof. Let $M := \{(x, y) \in V_C \times V_C : x \sigma_E(y) \in C \setminus [1, \infty)\}$ and observe that $M$ is open and starlike with respect to $(0, 0)$, hence in particular connected. The right hand side defines an element in $\text{Sesh}(M)$. By a similar argument as in the proof of Theorem 2.6 we only have to show that both sides of (4.14) coincide on $S^m_+ \times \{e_0\}$. But this follows from Theorem 4.12 and (4.11) as $-e_0 = \sigma(e_0) = -\sigma_V(e_0)$.

Remark 4.14. (A simplification for $n$ odd) For $n = 1$ we have $\rho = 0$, so that $\lambda = im$ and the differential equation (4.3) becomes $\eta^2 = m^2 \eta \varphi$. Together with the fact that $\eta \varphi$ is even, this leads to (see also [C03, p. 3])

\[
\Psi_m((\cos t, \sin t), e_0) = \gamma_{1, m} \cdot 2 F_1\left(\begin{array}{c} im, -im; \\ 1/2 \end{array}; \frac{1}{2}(1 - \cos t)\right)
\]

(4.15)

\[
= \gamma_{1, m} \cosh(mt), \quad \text{for } 0 < t < \pi.
\]

Even if the following discussion can be made uniform for all $\lambda$, we separate the cases where $m > \rho$ and $0 < m < \rho$. Induction and the identity $(a)_{n+1} = a(a+1)_n$ lead to

\[
d^k \frac{d^k}{dz^k} 2 F_1(a, b; c; z) = \frac{(a)_k(b)_k}{(c)_k} 2 F_1(k + a, k + b; c; z).
\]

(4.16)

We use this relation starting in dimension 1 moving up to odd dimension $n = 2k + 1$. Let $z = \frac{1}{2}(1 - \cos t) = \frac{1}{2}(1 - x_0)$, so that $\frac{dz}{dt} = \frac{1}{\sin t} \frac{dt}{dz}$ for $0 < t < \pi$. We then get for $m > \rho$ with $\lambda = i\sqrt{m^2 - \rho^2}$:

\[
\psi_m(x) = \Psi_m(x, e_0) = \gamma_{n, m} \cdot 2 F_1\left(k + \lambda, k - \lambda; k + \frac{1}{2}; \frac{1}{2}(1 - x_0)\right)
\]

\[
= \gamma_{n, m} \cdot \frac{(1/2)_k}{(1 + \lambda)_k(1 - \lambda)_k} \frac{d^k}{dz^k} 2 F_1\left(\begin{array}{c} \lambda, -\lambda; \\ 1/2 \end{array}; \frac{1}{2}(1 - x_0)\right)
\]

(4.17)

\[
= \gamma_{n, m} \frac{(1/2)_k}{(1 + \lambda)_k(1 - \lambda)_k} \frac{d^k}{dz^k} \cosh(mt)
\]

(4.18)

\[
= \frac{\gamma_{n, m}(n - 2)(n - 4) \cdots 3 \cdot 1}{2^k \prod_{j=0}^{k-1} (j^2 + m^2 - \rho^2)} \left(\frac{1}{\sin(t)} \frac{dt}{dz}\right)^k \cosh(mt).
\]

For $n = 3$, we get the following formula for $S^3$:

\[
\tilde{\psi}_m(t) = \gamma_{3, m} \frac{m}{2} \frac{\sin(mt)}{m^2 - \rho^2}.
\]

For the case $0 < m < \rho$ we have $0 < \lambda < \rho$. The arguments are the same as before, the only change is that now $\lambda$ is real so that we get

\[
\psi_m(x) = \gamma_{n, m} \frac{(n - 2) \cdots 1}{2^k \prod_{j=0}^{k-1} (j^2 + \lambda^2)} \left(\frac{1}{\sin(t)} \frac{dt}{dz}\right)^k \cosh(mt).
\]

This formula is originally due to Takahashi, [T63, p. 326]. For a more general statement see [OP04, Thm.5.1 and Ex. 5.3].

4.4. The constant $\gamma_{n, m}$. In this section we evaluate the constant $\gamma_{n, m}$ explicitly. We need the following facts:

Lemma 4.15. ([Fa08, Prop. 9.1.2]) Let $f \in L^1(S^n)$ be $K$-invariant and $\alpha : [-1, 1] \rightarrow \mathbb{C}$ with $f(x) = \alpha(x_0)$. Then

\[
\int_{S^n} f(x) d\mu(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} \alpha(t) \left(1 - t^2\right)^{\frac{n}{2} - 1} dt
\]

holds for the $O_{n+1}(\mathbb{R})$-invariant probability measure $\mu$ on $S^n$. 

23
Proof. We assume first that \( \lambda \geq \gamma \). For Lemma 4.16.

Lemma 4.17. For \( c, d \in \mathbb{C} \) with \( \text{Re} \, c > 0 \) and \( \text{Re} \, d > 0 \), we have for \( 0 < t < x \):

\[
\int_t^x (x-u)^{c-1}(u-t)^{d-1}2F_1\left(a,b;c;1-\frac{u}{x}\right)du = \frac{(x-t)^{c+d-1}\Gamma(c)\Gamma(d)}{\Gamma(c+d)}2F_1\left(a,b;c+d;1-\frac{t}{x}\right).
\]

Proof. We now have the tools to evaluate the constant \( \gamma \). Finally we also have \((\text{Lo67}, \text{Lem. 1})\):

Lemma 4.18. For \( n > 1 \) and \( \rho = (n-1)/2 \), we have

\[
\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n^2}}\gamma_{n,m} \int_{-1}^1 2F_1\left(\rho+\lambda,\rho-\lambda;\frac{n}{2};(1-t)\right)(1-t^{2\rho}) \frac{dt}{\Gamma}\psi_m(x) = \frac{1}{m^2}.
\]

The claim now follows from Theorem 4.12 and Lemma 4.15.

We now have the tools to evaluate the constant \( \gamma_{n,m} \).

Theorem 4.19. Let \( m > 0 \). Then

\[
\gamma_{n,m} = \frac{\Gamma\left(\frac{n-1}{2} + \lambda\right)\Gamma\left(\frac{n-1}{2} - \lambda\right)}{\Gamma(n)}
\]

For \( \lambda \in i\mathbb{R} \), we obtain in particular

\[
\gamma_{n,m} = \left|\frac{\Gamma\left(\frac{n-1}{2} + \lambda\right)}{\Gamma(n)}\right|^2, \quad \text{and} \quad \gamma_{1,m} = \frac{\pi}{m \sinh(\pi m)} \quad \text{for} \quad n = 1.
\]

Proof. We assume first that \( n > 1 \) so that \( \text{Re}(\rho \pm \lambda) > 0 \). Put \( \gamma := \gamma_{n,m} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n^2}} \) and \( F(s) := \gamma \cdot 2F_1(\rho+\lambda,\rho-\lambda;\frac{n}{2};s) \). By Lemma 4.18 and the change of variables \( u = 1 + t \) we get

\[
\frac{1}{m^2} = \int_{-1}^{1} F((1-t)/2)(1-t)^{\frac{n-1}{2}}(1+t)^{\frac{n-1}{2}} dt
\]

\[
= \int_0^2 (2-u)^{\frac{n-1}{2}}u^{\frac{n-1}{2}}F(1-u/2) du
\]

\[
= \lim_{t \to 0^+} \int_t^2 (2-u)^{\frac{n-1}{2}}(u-t)^{\frac{n-1}{2}}F(1-u/2) du
\]

\[
= 2^{n-1} \frac{\Gamma(n/2)^2}{\Gamma(n)} \lim_{t \to 0^+} 2F_1\left(\frac{n-1}{2} + \lambda, \frac{n-1}{2} - \lambda; n; 1 - \frac{t}{2}\right)
\]

\[
= 2^{n-1} \frac{\Gamma(n/2)^2}{\Gamma(n)} \lim_{t \to 0^+} 2F_1\left(\rho+\lambda, \rho-\lambda; n; 1 - \frac{t}{2}\right).
\]
With \( a = \rho + \lambda, b = \rho - \lambda, \rho = (n - 1) / 2 \) and \( c = n \), we have \( c - a - b = 1 > 0 \) and \( \text{Re}(c - b) = \frac{n + 1}{2} + \text{Re} \lambda > 0 \) (see (4.33)). Hence Lemma 4.16 implies

\[
\lim_{t \to 0^+} 2F_1\left( \rho + \lambda, \rho - \lambda; n; 1 - \frac{t}{2} \right) = \frac{\Gamma(n)}{\Gamma\left( \frac{n + 1}{2} + \lambda \right) \Gamma\left( \frac{n + 1}{2} - \lambda \right)} = \frac{\Gamma(n)}{\Gamma(1 + \rho + \lambda) \Gamma(1 + \rho - \lambda)}
\]

\[
= \frac{\Gamma(n)}{(\rho^2 - \lambda^2) \Gamma(\rho + \lambda) \Gamma(\rho - \lambda)} - \Gamma(\rho + \lambda) \Gamma(\rho - \lambda),
\]

where we have used that \( \text{Re}(\rho \pm \lambda) > 0 \). This further leads to

\[
1 = m^2 2^{n-1} \gamma \left( \frac{n+1}{2} \right) \frac{\Gamma(n)}{\Gamma\left( \frac{n + 1}{2} + \lambda \right) \Gamma\left( \frac{n + 1}{2} - \lambda \right)} = 2^{n-1} \gamma_{n,m} \frac{\Gamma(n)}{\Gamma\left( \frac{n + 1}{2} + \lambda \right) \Gamma\left( \frac{n + 1}{2} - \lambda \right)} \frac{1}{\Gamma(\rho + \lambda) \Gamma(\rho - \lambda)}
\]

This, together with the identity \( 2^{z-1} \Gamma(z) \Gamma(z + 1/2) = \sqrt{\pi} \Gamma(2z) \) (L.73 p. 21]) proves the theorem for \( n > 1 \).

For \( n = 1 \) we first note that the left hand side of (4.20) specializes for \( n = 1 \) to

\[
|\Gamma(im)|^2 = \frac{\pi}{m \sinh(\pi m)} \quad \text{for} \quad m > 0.
\]

On the other hand we have by the discussion after (4.14) and our normalization of the measure on the circle

\[
\frac{1}{m^2} = \frac{\gamma_{1,m}}{2\pi} \int_{-\pi}^{\pi} \cosh(mt) dt = \frac{\gamma_{1,m}}{\pi m} \sinh(\pi m) \quad \text{or} \quad \gamma_{1,m} = \frac{\pi}{m \sinh(\pi m)}.
\]

Remark 4.20. (\( m \to 0 \)) As \( \Delta \) annihilates the constants, its inverse \( \Delta^{-1} \) is not densely defined. However, it is bounded on the hyperplane \( 1^\perp \), where \( 1 \) denotes the constant function. The formula for \( \gamma_{n,m} \) shows that

\[
\lim_{m \to 0} m^2 \gamma_{1,m} = 1 \quad \text{for} \quad n = 1.
\]

For \( n > 1 \) and \( m < \rho \) we have for \( m \to 0 \):

\[
\rho - \lambda = \rho \left( 1 - (1 - m^2 / \rho^2)^{1/2} \right) \sim \rho m^2 \frac{2m^2}{2} = \frac{m^2}{2} \rho \quad \text{and} \quad \rho + \lambda \to 2 \rho.
\]

Therefore

\[
\lim_{m \to 0} m^2 \gamma_{n,m} = \lim_{m \to 0} m^2 \frac{\Gamma(n)}{\Gamma(\rho + \lambda) \Gamma(\rho - \lambda)} = \frac{\Gamma(2\rho)}{\Gamma(2\rho + 1)} \lim_{m \to 0} \frac{m^2}{\rho - \lambda} = \lim_{m \to 0} m^2 \frac{1}{2} \frac{2\rho}{m^2} = 1.
\]

We also have by the power series expression of \( 2F_1 \) for \( \lambda = \rho \):

\[
(4.21) \quad \lim_{m \to 0} 2F_1\left( \rho + \lambda, \rho - \lambda; \frac{n}{2}; z \right) = 2F_1\left( n - 1, 0; \frac{n}{2}; z \right) = 1.
\]

Thus,

\[
(4.22) \quad \lim_{m \to 0} m^2 \Psi_m = \lim_{m \to 0} m^2 \gamma_{n,m} \cdot 2F_1\left( \rho + \lambda, \rho - \lambda; \frac{n}{2}; z \right) = 1.
\]

Let \( \mathcal{Y}_q \) be the space of homogeneous degree \( q \) harmonic polynomials on \( \mathbb{R}^{n+1} \), restricted to the sphere \( S^n \). Then the canonical action \( \delta_q(k)f(x) = f(k^{-1}x) \) defines an irreducible representation of

\[
25
\]
\[ G \text{ and } L^2(S^n) \cong \bigoplus_{q=0}^{\infty} \mathcal{Y}_q \text{ is a Hilbert space direct sum (} \text{[Fa08 Thm. 9.3.2] or } \text{vD09 Ch. 7.3}) \text{. For } \eta \in \mathcal{Y}_q \text{ we have, see [Fa08 Prop. 9.3.5]}: \]
\[ \Delta \eta = -q(q + n - 1)\eta. \]

For \( m > 0 \) we therefore obtain with the orthogonal projection \( p_q : L^2(S^n) \to \mathcal{Y}_q \subseteq L^2(S^n) \) the representation
\[ (-\Delta + m^2)^{-1} = \frac{1}{m^2}p_0 + \sum_{q=1}^{\infty} \frac{1}{q(q+n-1)+m^2 \cdot p_q}. \]

Thus
\[ m^2(-\Delta + m^2)^{-1} = p_0 + \sum_{q=1}^{\infty} \frac{m^2}{q(q+n-1)+m^2} \cdot p_q \xrightarrow{m \to 0} p_0. \]

This fits (4.22) because \( p_0(\varphi) = \int_G \varphi(u) \, d\mu(u) \).

5. Reflection positivity on the sphere and representation theory

In this section we explain how our results from the previous section connect to representation theory. In particular we show that all irreducible unitary spherical representations of \( G^c := O_{1,n}(\mathbb{R})^\dagger \) are obtained by reflection positivity. We use [vD09], in particular Section 7.5, as a standard reference.

5.1. The spherical unitary representations of the Lorentz group. Recall that the group \( G^c \) acts transitively on \( \mathbb{L}_+^n = G^c, \xi^0 \cong G^c/MN \), where \( \xi^0 = e_0 + ie_n \) (cf. Subsection 3.2). We embed the sphere \( S^{n-1} \) into \( \mathbb{L}_+^n \) by \( \xi_u = (1,iu) \), so that \( \xi_{e_n} \) coincides with the element \( \xi^0 \) from above. Then
\[ \mathbb{L}_+^n = \{t \xi_u : t > 0, u \in S^{n-1}\} \cong S^{n-1} \times \mathbb{R}_+. \]

Lemma 5.1. For \( g = \begin{pmatrix} a & iv \\ iw^\top & A \end{pmatrix} \in G^c, a \in \mathbb{R}, v, w \in \mathbb{R}^n, A \in M_{n,n}(\mathbb{R}), \text{ and } u \in S^{n-1} \) define
\[ g.u = \frac{w + u A^\top}{a - vu} \text{ and } j(g,u) = a - vu. \]

Then the following holds for \( g, g_1, g_2 \in G^c \) and \( u \in S^{n-1} \):

(i) \( g.u \in S^{n-1} \) and \( (g,u) \mapsto g.u \) defines an action of \( G^c \) on \( S^{n-1} \).

(ii) \( j(g,u) > 0 \) and \( j(g_1, g_2, u) = j(g_1, g_2, u) j(g_2, u) \).

(iii) \( j(g, u) = [g \xi_u, e_0]v \).

(iv) For \( k \in K, a_t \in A \) and \( \tilde{n} \in N \), we have \( j(k a_t \tilde{n}, e_n) = e_t \).

(v) \( g.(t \xi_u) = t \cdot j(g,u) \xi_{g.u} \).

Proof. A direct calculation show that
\[ g.(t \xi_u) = t g(\xi_u)^\top = t \begin{pmatrix} a - vu \\ i(Au^\top + w^\top) \end{pmatrix}^\top = t \cdot j(g,u) \xi_{g,u}. \]

Here we have used that \( x_0 > 0 \) for every \( (x_0, x) \in \mathbb{L}_+^n \) so that \( j(g,u) > 0 \). This proves (v) which then implies (i) and (ii). Part (iii) follows directly from (5.1) by taking \( t = 1 \), and (iv) follows from (iii) as \( K \) stabilizes \( e_0 \), \( N \) stabilizes \( \xi^0 = \xi_{e_n} \), and \( a_t \xi^0 = e_t \xi^0 \).

With \( \rho = \frac{n-1}{2} \) as before, we define
\[ j_\lambda : G^c \times S^{n-1} \to \mathbb{C}_x, \quad j_\lambda(g,u) := j(g,u)^{-\lambda - \rho}, \]

Then (iii) and (iv) above imply that
\[ j_\lambda(k a_t \tilde{n}, e_n) = e^{-\lambda - \rho}t \quad \text{and} \quad j_\lambda(k,u) = 1 \quad \text{for} \quad k \in K, u \in S^{n-1}. \]
shows that
for all $t > 1$. We obtain a representation, not unitary in general, of $G$ semisimple Lie groups (see [vD09, but we provide the main ideas of the proof to stay self-contained and avoid the structure theory of
Lemma 5.2. The following holds:
(i) If $\varphi \in L^1(S^{n-1})$, then $\int_{S^{n-1}} \varphi(gu) j_\mu(gu) \, d\mu(u) = \int_{S^{n-1}} \varphi(u) d\mu(u)$ holds for the $O_n(\mathbb{R})$-invariant probability measure $\mu$ on $S^{n-1}$.
(ii) We obtain a $G^c$-invariant measure on $L^1_+ = G^c/MN$ by
\begin{align*}
\int_{L_+^c} \varphi(\xi) \, d\xi = \int_0^\infty \int_{S^{n-1}} \varphi(r \xi_u) r^{n-1} \, d\mu(u) \, \frac{dr}{r} \quad \text{for } \varphi \in L^1(L_+^c).
\end{align*}
Proof. (i) follows from Lemma 5.1(v) and [vD09, Prop. 7.5.8], and (ii) from [vD09, p. 114].

For $\lambda \in \mathbb{C}$, let $H_\lambda$ be the space of measurable functions $\varphi : L^1_+ \to \mathbb{C}$ such that $\varphi(t\xi) = t^{-\lambda} \varphi(\xi)$ for all $t > 0$ and $\xi \in L^1_+$, endowed with the Hilbert space structure specified by
\begin{align*}
||\varphi||^2 := \int_K |\varphi(k \xi^0)|^2 \, dk = \int_{S^{n-1}} |\varphi(\xi_u)|^2 \, d\mu(u) < \infty.
\end{align*}
The corresponding scalar product is
\begin{align*}
\langle \varphi, \psi \rangle_{L^2} = \int_{S^{n-1}} \overline{\varphi(\xi_u)} \psi(\xi_u) \, d\mu(u) \quad \text{for } \varphi, \psi \in H_\lambda.
\end{align*}
We obtain a representation, not unitary in general, of $G^c$ on $H_\lambda$ by
\begin{align*}
(\pi_\lambda(\varphi))(\xi_u) = \varphi(g^{-1} \xi_u) = j(g^{-1}, u)^{-\lambda} \varphi(\xi_{g^{-1} u}) = j_\lambda(g^{-1}, u) \varphi(\xi_{g^{-1} u}) \quad \text{for } \varphi \in H_\lambda, g \in G^c \text{ and } \xi \in L^1_+.
\end{align*}
We then have
\begin{align*}
(\pi_\lambda(\varphi))(\xi_u) = \varphi(g^{-1} \xi_u) = j(g^{-1}, u)^{-\lambda} \varphi(\xi_{g^{-1} u}) = j_\lambda(g^{-1}, u) \varphi(\xi_{g^{-1} u}) \quad \text{for } u \in S^{n-1}.
\end{align*}

Lemma 5.3. Let $\lambda \in \mathbb{C}$, $\varphi \in H_\lambda$, $\psi \in H_{-\lambda}$, and $g \in G^c$. Then the following assertions hold:
(i) $||\pi_\lambda(\varphi)||^2 = \int_{S^{n-1}} j(g, u)^2 Re \lambda|\varphi(\xi_u)|^2 \, d\mu(u)$.
(ii) $(\pi_\lambda, H_\lambda)$ is unitary if and only if $\lambda \in i\mathbb{R}$, and then it is irreducible.
(iii) Let $L$ be a compact subset of $G$. Then there exists a constant $C_L > 0$ such that we have $||\pi_\lambda(\varphi)|| \leq C_L ||\varphi||$ for all $\varphi \in L$ and all $\varphi \in H_\lambda$.
(iv) $\langle \pi_-\lambda(\varphi), \pi_\lambda(\varphi) \rangle_{L^2} = \langle \psi, \varphi \rangle_{L^2}$, so that the $L^2$-inner product defines a $\pi_-\lambda \times \pi_\lambda$-invariant hermitian pairing on $H_{-\lambda} \times H_\lambda$.

Proof. These are standard facts about principal series representations of semisimple Lie groups, but we provide the main ideas of the proof to stay self-contained and avoid the structure theory of semisimple Lie groups (see [vD09, §7.5] for details).

Assume first that $\eta, \gamma \in L^2(S^{n-1})$ and $\lambda, \mu \in \mathbb{C}$. The cocycle relation and the fact that $j(\epsilon, u) = 1$ shows that $j(g, u) = j(g^{-1}, g, u)^{-1}$. Hence
\begin{align*}
\int_{S^{n-1}} j_\lambda(g, u) \eta(g, u) j_\mu(g, u) \gamma(g, u) \, d\mu(u) = \int_{S^{n-1}} j(g, u)^{-\lambda-\mu} \eta(g, u) \gamma(g, u) j(g, u)^{-2\mu} \, d\mu(u)
= \int_{S^{n-1}} j(g^{-1}, u)^{\lambda+\mu} \eta(u) \gamma(u) \, d\mu(u),
\end{align*}
where we used Lemma 5.2(i) in the last step.

All parts of Lemma 5.3 except the irreducibility assertion ([vD09, Cor. 7.5.12]), follow from this calculation.

\begin{lemma}
Let $z \in \Xi$ and $\xi \in L^1_+$.
(i) $Re[z, \xi]_V > 0$ and $[z, \xi]_V \in \mathbb{R}$ if $z \in H^1_\gamma$.
\end{lemma}
Lemma 4.15, with
Write

\begin{align*}
\text{(ii)} \quad \overline{z, \xi}_V &= [\sigma V(z), \xi]_V. \\
\text{(iii)} \quad \xi \mapsto [z, \xi]^{-\lambda-\rho} \text{ is in } H_\lambda \text{ for all } z \in \mathbb{C}. \\
\text{(iv)} \quad 1_\lambda(\xi) = [e_0, \xi]^{-\lambda-\rho} \text{ is } K \text{-invariant and } H^K_\lambda \subset C1_\lambda.
\end{align*}

\textbf{Proof.} (i) If \( z = (z_0, iz) \in \mathbb{H}_V^0 \) and \( \xi = (u_0, iw) \) then \( z, \xi \mid V = z_0u_0 - zw \in \mathbb{R} \), so \( z, \xi \mid V \) is real. Now let \( z = u + iv \in \mathbb{C} \). Then \( u \in V_+ \) implies that \( \text{Re}[z, \xi]_V = [u, \xi]_V = u_0 - u_n > 0 \). But then \( \text{Re}[z, g, \xi]_V = \text{Re}[g^{-1}, z, \xi]_V > 0 \) for all \( g \in G^c \) because \( \Xi \) is \( G^c \)-invariant. Now (i) follows.

For (ii) we note that the maps \( \Xi \ni z \mapsto [z, \xi]_V, [\sigma V(z), \xi]_V \in \mathbb{C} \) are holomorphic and agree on \( \mathbb{H}_V^0 \) by (i). Hence they agree on all of \( \Xi \) as \( \mathbb{H}_V^0 \) is totally real in \( \Xi \). (iii) is now obvious and (iv) follows from the definition of \( 1_\lambda \) and the transitivity of the \( K \)-action on \( S^{n-1} \). \( \square \)

\textbf{Definition 5.5.} (a) An irreducible unitary representation \((\pi, H)\) of \( G^c \) is called \textit{spherical} if \( H^K \neq \{0\} \). In that case \( \dim H^K = 1 \) and, for every unit vector \( u \in H^K \), the function \( \varphi_\pi(g) = \langle u, \pi(g)u \rangle \) is the called the corresponding \textit{spherical function}. It is positive definite and \( K \)-biinvariant.

(b) The representations \((\pi_\lambda, H_\lambda)_{\lambda \in i\mathbb{R}}\) are called the \textit{spherical principal series representations}. For these parameters, we write \( \langle \cdot, \cdot \rangle_\lambda := \langle \cdot, \cdot \rangle_{L^2} \) for the scalar product on \( H_\lambda \). Note that \( \pi_\lambda \cong \pi_{-\lambda} \) for \( \lambda \in i\mathbb{R} \) ([vD09, p. 119]).

We now consider the case \( 0 < \lambda < \rho \). Lemma 5.3(iv) suggests the existence of a \( G^c \)-intertwining operator \( A_\lambda : H_\lambda \to H_{-\lambda} \) such that the hermitian form
\begin{align*}
\langle \psi, \varphi \rangle_\lambda := \int_{S^{n-1}} \overline{A_\lambda \psi(\xi_u)} \varphi(\xi_u) \, d\mu(u) &= \langle A_\lambda \psi, \varphi \rangle_{L^2}, \quad 0 < \lambda < \rho,
\end{align*}
is non-degenerate and \( G^c \)-invariant.

\textbf{Lemma 5.6.} Let \( x \in \mathbb{L}_+^n \). Then \( u \mapsto [x, \xi_u]^{-\lambda-\rho}_V \) is integrable on \( S^{n-1} \) if and only if \( \text{Re} \lambda > 0 \) and in that case
\begin{align*}
\int_{S^{n-1}} [x, \xi_u]^{-\lambda-\rho}_V \, d\mu(u) &= \left( \frac{2^{\lambda+\frac{n-3}{2}} \Gamma \left( \frac{n}{2} \right) \Gamma(\lambda)}{\sqrt{\pi} \Gamma(\lambda+\rho)} \right) \overline{[e_0, x]}^{-\lambda-\rho}_V.
\end{align*}

For \( \lambda = \rho \) and \( m = 0 \), we have
\begin{align*}
\frac{2^{\lambda+\frac{n-3}{2}} \Gamma \left( \frac{n}{2} \right) \Gamma(\lambda)}{\sqrt{\pi} \Gamma(\lambda+\rho)} = \frac{2^{n-2} \Gamma \left( \frac{n}{2} \right) \Gamma(\frac{n-1}{2})}{\sqrt{\pi} \Gamma(\lambda-1)} = 1
\end{align*}
by the identity \( 2^{2z-1} \Gamma(z + 1/2) \Gamma(z) = \sqrt{\pi} \Gamma(2z) \) ([L73, p. 21]).

\textbf{Proof.} Write \( x = ka_\lambda \xi_{e_\lambda} \) and \( \lambda = s + ir \). Then
\begin{align*}
[x, \xi_u]_V &= e^{r}[\xi_{e_\lambda}, \xi_{k-1, u}]_V = [e_0, x]_V [\xi_{e_\lambda}, \xi_{k-1, u}]_V.
\end{align*}

We can therefore assume that \( x = \xi_{e_\lambda} \). As \( u \mapsto [\xi_{e_\lambda}, \xi_u]_V \) is \( O_{n-1}(\mathbb{R}) \)-invariant, we get by Lemma 4.15 with \( n \) replaced by \( n-1 \),
\begin{align*}
\int_{S^{n-1}} \overline{[\xi_{e_\lambda}, \xi_u]^{-\lambda-\rho}_V} \, d\mu(u) &= \int_{S^{n-1}} (1 - u_n)^{s-\rho} \, d\mu(u) \\
&= \frac{\Gamma \left( \frac{n}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{n-1}{2} \right)} \int_{-1}^{1} (1 - t)^{s-\rho}(1 - t^2)^{\frac{n-3}{2}} \, dt \\
&= \frac{\Gamma \left( \frac{n}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{n-1}{2} \right)} \int_{-1}^{1} (1 - t)^{s-1}(1 + t)^{\frac{n-3}{2}} \, dt \\
&= \frac{\Gamma \left( \frac{n}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{n-1}{2} \right)} 2^{s+\frac{n-3}{2}} \int_{0}^{1} (1 - u)^{s-1} u^{\frac{n-3}{2}} \, du \quad \text{(substitute } 2u = 1 + t).}
\end{align*}
As $n > 1$, this integral is finite if and only if $s > 0$. The same calculation as above, and analytic continuation in $\lambda$ further show that

$$\int_{S^{n-1}} [k, \xi_{e_n}, \xi_u]^{\lambda-\rho} d\mu(u) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}^{n-1}} \frac{2^{n-1} n!}{\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(\lambda + \rho)} \int_0^1 (1 - u)^{\lambda-1} u^{n-1} du = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}} \frac{2^{n-3} n!}{\Gamma(\lambda + \rho)}$$

where we use [WW96 §12.41] for the last equality.

Lemma 5.7. Let $\mu, \lambda \in \mathbb{C}$, $\varphi \in \mathcal{H}_\lambda$ and $\emptyset \neq L \subset \Xi$ compact. Then there exists a constant $C$ such that for all $z \in L$ and $u \in S^{n-1}$ we have

$$|[z, \xi_u]^{\mu-\rho}_V \varphi(\xi_u)| \leq C|\varphi(\xi_u)|.$$  

In particular, $u \mapsto [z, \xi_u]^{\mu-\rho}_V \varphi(\xi_u)$ is integrable and $z \mapsto \int_{S^{n-1}} [z, \xi_u]^{\mu-\rho}_V \varphi(\xi_u) d\mu(u)$ is holomorphic on $\Xi$.

Proof. The continuity of the function $(z, u) \mapsto [z, \xi_u]^{\mu-\rho}_V \varphi(\xi_u)$ on $\Xi \times S^{n-1}$ implies that it is bounded on the compact subset $L \times S^{n-1}$, and this implies the assertion.

Lemma 5.8. For $\Re \lambda > 0$ and $\varphi \in \mathcal{H}_\lambda \cap C(\mathbb{R}_+^n)$, define

$$(A_\lambda \varphi)(x) := \frac{\sqrt{\pi} \Gamma(\lambda + \rho)}{2^{n-1} n! \Gamma\left(\frac{n}{2}\right) \Gamma(\lambda)} \int_{S^{n-1}} [x, \xi_u]^{\lambda-\rho} \varphi(\xi_u) d\mu(u) \text{ for } x \in \mathbb{R}_+^n.$$  

Then

(i) $A_\lambda \varphi \in \mathcal{H}_{-\lambda}$

(ii) $A_\lambda \pi_\lambda(g) = \pi_{-\lambda}(g) A_\lambda$ on $\mathcal{H}_\lambda \cap C(\mathbb{R}_+^n)$

(iii) $A_\lambda 1_{\lambda} = 1_{-\lambda}$.

Proof. By Lemma 5.6 the integral defining $A_\lambda \varphi(x)$ exists for $x \in \mathbb{R}_+^n$ and the homogeneity requirement follows directly from the definition. The continuity of the function $A_\lambda \varphi$ follows from

$$\int_{S^{n-1}} [k, \xi_{e_n}, \xi_u]^{\lambda-\rho} \varphi(\xi_u) d\mu(u) = \int_{S^{n-1}} [\xi_{e_n}, \xi_{k-1,u}]^{\lambda-\rho} \varphi(\xi_u) d\mu(u) = \int_{S^{n-1}} [\xi_{e_n}, \xi_u]^{\lambda-\rho} \varphi(k, \xi_u) d\mu(u)$$

and the fact that the map $k \mapsto \varphi(k, \cdot)$, $K \mapsto C(S^{n-1})$ is continuous. This shows that $A_\lambda \varphi \in \mathcal{H}_{-\lambda}$. The proof of the intertwining relation (ii) is the same as the proof Lemma 5.3 (i). It uses Lemma 5.2 (i), and that $j(g^{-1}, u) = j(g, g^{-1}.u)^{-1}$. For the constant $c > 0$ in the definition of $A_\lambda$ in Lemma 5.8 we have

$$(A_\lambda \pi_\lambda(g) \varphi)(x) = c \int_{S^{n-1}} [x, \xi_u]^{\lambda-\rho} \varphi(g^{-1}.\xi_u) d\mu(u)$$

$$= c \int_{S^{n-1}} [x, \xi_u]^{\lambda-\rho} j(g^{-1}, u) \varphi(\xi_{g^{-1}.u}) d\mu(u)$$

$$= c \int_{S^{n-1}} [x, \xi_{g,u}]^{\lambda-\rho} j(g, u) \varphi(\xi_u) d\mu(u)$$

$$= c \int_{S^{n-1}} [x, j(g, u) \xi_{g,u}]^{\lambda-\rho} \varphi(\xi_u) d\mu(u)$$

$$= c \int_{S^{n-1}} [x, g \xi_{e_n}]^{\lambda-\rho} \varphi(\xi_u) d\mu(u)$$

$$= c \int_{S^{n-1}} [g^{-1}.x, \xi_u]^{\lambda-\rho} \varphi(\xi_u) d\mu(u)$$

$$= \pi_{-\lambda}(g)(A_\lambda \varphi)(x).$$
Corollary 5.11. Let the notation be as above. Then the following holds:

\[ \varphi_m(x) = \int_{\mathbb{S}^{n-1}} [x,\xi_u]^{-\lambda - \rho} d\mu(u) = 2F_1(\rho + \lambda, \rho - \lambda; \frac{n}{2}; -\sin^2(t/2)) \]

\[ = \frac{(n-1)!}{\Gamma\left(\frac{n}{2} + \lambda\right) \Gamma\left(\frac{n}{2} - \lambda\right)} \Psi_m(x, e_0), \quad x = ka_t, e_0 \in \mathbb{H}^n. \]

Proof. The irreducibility of the representation \((\pi_m, \mathcal{H}_m)\) is \cite[7.5.12]{Dixmier}. That those are all the irreducible unitary spherical representations is \cite[p. 119]{Dixmier} and \cite[Thm. 7.5.9]{Dixmier}. The statements about the spherical functions can be found in \cite[p. 111 and p. 126]{Dixmier}. To translate between our setting and \cite{Dixmier}, we recall from \cite[p. 298]{Wallach} the relation

\[ 2F_1(2a, 2b, a + b + 1, z) = 2F_1(a, b, a + b + 1, 4z(1 - z)) \]

for the hypergeometric functions. For \( a = \frac{\rho + \lambda}{2} \) and \( b = \frac{\rho - \lambda}{2} \) with \( a + b + 1 = \frac{n}{2} \), it leads to

\[ 2F_1(\rho + \lambda, \rho - \lambda; \frac{n}{2}; -\sin^2(t/2)) = 2F_1\left(\frac{\rho + \lambda}{2}, \frac{\rho - \lambda}{2}; \frac{n}{2}; -\sin^2(t)\right) \]

because \( z = -\sin^2(t/2) \) implies

\[ 4z(1 - z) = -4 \sin^2(t/2) \cosh^2(t/2) = -\sinh^2(t). \]

The last equality follows from Theorem 4.12 and the fact that

\[ \frac{1}{2} (1 - [ka_t,e_0, e_0]_{\mathcal{V}}) = \frac{1}{2} (1 - \cosh(t)) = -\sinh^2(t/2). \]

From now on we write

\[ \Phi_m^c(z, w) := \frac{\Psi_m(z, w)}{\Psi_m(e_0, e_0)}, \quad m > 0, \]

for the normalization of the kernel \( \Psi_m \), so that \( \varphi_m(x) = \Phi_m^c(x, e_0) \) is the spherical function on \( \mathbb{H}^n \) corresponding to the spherical representation \((\pi_m, \mathcal{H}_m)\). For \( m = 0 \) we put \( \Phi_0^c = 1 \). For \( m \geq 0 \), we write \( \mathcal{O}_m(\Xi) := \mathcal{H}_m \subset \mathcal{O}(\Xi) \) for the corresponding reproducing kernel Hilbert space. Then left translation \((\rho_{m}(g)F)(z) := F(g^{-1}z)\) defines a unitary representation \((\rho_{m}, \mathcal{H}_m)\) of \( G^c \). By Theorem 5.10 the representation \((\pi_m, \mathcal{O}_m(\Xi))\) is irreducible and isomorphic to \((\pi_m, \mathcal{H}_m)\).

Corollary 5.11. Let the notation be as above. Then the following holds:

(i) The representations \((\rho_{m}, \mathcal{O}_m(\Xi))\), \( m \geq 0 \) are unitary and irreducible and every irreducible spherical representation of \( G^c \) is unitarily equivalent to \((\pi_m, \mathcal{O}_m(\Xi))\) for some \( m \geq 0 \).

(ii) Every irreducible unitary spherical representation of \( G^c \) can be constructed via reflection positivity.

(iii) \( \Gamma_c = \{ \varphi_m : m \geq 0 \}. \)

(iv) For \( \Psi \in \Gamma \), there exists a unique positive Radon measure \( \mu_\Psi \) on \([0, \infty)\) such that

\[ \Psi(z, w) = \int_{[0, \infty)} \Phi_m^c(z, w) d\mu_\Psi(m) \quad \text{for} \quad z, w \in \Xi. \]
The integral converges uniformly on compact subsets of \( \Xi \times \Xi \), resp., as a vector-valued integral in the Fréchet space \( \text{Sesh}(\Xi) \).

Proof. Part (iv) follows from (iii) and [KS05, Thm.5.1], see also [PT99, Thm.1]. As pointed out above, everything else follows from Theorem 5.10. \( \square \)

5.2. An integral representation of \( \varphi_m \). In this subsection we obtain an integral representation of the kernel \( \Phi_m^c \) and to use the Poisson transform to construct a concrete unitary intertwining operator \( \mathcal{P}_m : \mathcal{H}_m \to \mathcal{O}_m(\Xi) \).

For \( \lambda \in \mathbb{C} \), we consider the Poisson kernel 
\[
P_{\lambda} : \Xi \times \mathbb{L}_+^n \to \mathbb{C}, \quad P_{\lambda}(z, \xi) = [z, \xi]^\lambda \overline{\rho}.
\]
This kernel is defined by Lemma 5.4 the functions \( P_{\lambda}(\cdot, \xi) \) are holomorphic on \( \Xi \), \( P_{\lambda,z} := P_{\lambda}(z, \cdot) \in \mathcal{H}_\lambda \), and \( P_{\lambda,0} = 1_\lambda \in \mathcal{H}_\lambda^K \). We define the Poisson transform by
\[
(P_{\lambda}\varphi)(z) = \int_{\mathbb{L}_+^{n-1}} P_{\lambda}(z, \xi_u) \varphi(\xi_u) d\mu(u) \quad \text{for} \quad \varphi \in \mathcal{H}_\lambda.
\]
The existence of the integral follows from Lemma 5.7 and \( P_{\lambda}\varphi \) is holomorphic on \( \Xi \). For \( \lambda = \lambda_m \), \( m \geq 0 \), we will also use the notation \( P_m := P_{\lambda_m} \) and \( \mathcal{P}_m = \mathcal{P}_{\lambda_m} \).

Lemma 5.12. If \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda > 0 \), then \( A_{\lambda} P_{\lambda,z} = P_{-\lambda,z} \) for \( z \in \Xi \).

Proof. By Lemma 5.8(iii), we have \( A_{\lambda} P_{\lambda,0} = A_{\lambda} 1_\lambda = 1 - \lambda = P_{-\lambda,0} \). For \( g \in G^c \), we thus obtain
\[
A_{\lambda} P_{\lambda,g,\xi_e}(\xi) = A_{\lambda}(\pi_{\lambda}(g) P_{\lambda,0})(\xi) = \pi_{-\lambda}(g) A_{\lambda} P_{\lambda,0}(\xi) = P_{-\lambda,0}(g^{-1} \xi) = P_{-\lambda,g,\xi_e}(\xi).
\]
Hence the maps \( w \mapsto A_{\lambda} P_{\lambda,w}(\xi) \) and \( w \mapsto P_{-\lambda,w}(\xi) \) which are both holomorphic on \( \Xi \), coincide on \( \mathbb{H}^0 \). This implies that \( A_{\lambda} P_{\lambda,w} = P_{-\lambda,w} \) for all \( w \in \Xi \). \( \square \)

Theorem 5.13. Let \( m > 0 \) and \( \varphi \in \mathcal{H}_{\lambda_m} \). Then \( \mathcal{P}_m(\varphi) \in \mathcal{O}_m(\Xi) \) and \( \mathcal{P}_m : \mathcal{H}_{\lambda_m} \to \mathcal{O}_m(\Xi) \) is unitary and \( G^c \)-equivariant. We further have
\[
\Phi_m^c(z, w) = \langle P_{m, w}, P_{m, z} \rangle_{\lambda_m} = \int_{\mathbb{S}^{n-1}} [\sigma_V(w, \xi_u)]^{\lambda - \rho} [z, \xi_u]^{-\lambda - \rho} d\mu(u) \quad \text{for} \quad z, w \in \Xi.
\]

Proof. It follows from Lemma 5.7 that all the integrals in question exist and that \( \mathcal{P}_m(\varphi) \in \mathcal{O}(\Xi) \). The same argument shows that the kernels
\[
(z, w) \mapsto \langle P_{m, w}, P_{m, z} \rangle_{\lambda},
\]
\[
(z, w) \mapsto \int_{\mathbb{S}^{n-1}} [\sigma_V(w, \xi_u)]^{\lambda - \rho} [z, \xi_u]^{-\lambda - \rho} d\mu(u)
\]
are sesquiholomorphic. We now show that they coincide. For \( m \geq \rho \) we have \( \lambda \in i \mathbb{R} \) and
\[
\langle P_{m, w}, P_{m, z} \rangle_{\lambda} = \int_{\mathbb{S}^{n-1}} [w, \xi_u]^{-\lambda - \rho} [z, \xi_u]^{\lambda - \rho} d\mu(u) = \int_{\mathbb{S}^{n-1}} [\sigma_V w, \xi_u]^{\lambda - \rho} [z, \xi_u]^{-\lambda - \rho} d\mu(u).
\]
For \( 0 < m < \rho \) we have
\[
\langle P_{m, w}, P_{m, z} \rangle_{\lambda} = \int_{\mathbb{S}^{n-1}} (A_{\lambda} P_m)(w, \xi_u) [z, \xi_u]^{\lambda - \rho} d\mu(u) = \int_{\mathbb{S}^{n-1}} [w, \xi_u]^{\lambda - \rho} [z, \xi_u]^{-\lambda - \rho} d\mu(u)
\]
\[
= \int_{\mathbb{S}^{n-1}} [\sigma_V w, \xi_u]^{\lambda - \rho} [z, \xi_u]^{-\lambda - \rho} d\mu(u)
\]
by Lemmas 5.4 and 5.12 because \( \lambda \) is real. This proves the asserted equality.
For $\lambda \in \mathbb{C}$ and $\varphi \in \mathcal{H}_\lambda$, we find with Lemma 5.3
\[
P_\lambda(\varphi)(z) = \int_{\mathbb{S}^{n-1}} [z, \xi_u]^{\lambda-\rho} \varphi(g^{-1} \cdot \xi_u) \, d\mu(u) = \int_{\mathbb{S}^{n-1}} [\sigma_V z, \xi_u]^{\lambda-\rho} \varphi(g^{-1} \cdot \xi_u) \, d\mu(u)
\]
(5.7)
\[
= \int_{\mathbb{S}^{n-1}} [g^{-1} \cdot z, \xi_u]^{\lambda-\rho} \varphi(\xi_u) \, d\mu(u)
\]
Hence $P_\lambda$ is an intertwining operator.

We now consider the sesquiholomorphic kernel
\[
\Lambda_m(z, w) = \langle P_{m, w}, P_{m, z} \rangle_{\lambda} = \int_{\mathbb{S}^{n-1}} [\sigma_V w, \xi_u]^{\lambda-\rho} [z, \xi_u]^{\lambda-\rho} \, d\mu(u).
\]
This kernel is hermitian because
\[
\Lambda_m(w, z) = \langle P_{m, z}, P_{m, w} \rangle_{\lambda} = \langle P_{m, w}, P_{m, z} \rangle_{\lambda} = \Lambda_m(z, w).
\]
By taking $\varphi = P_{m, z}$ and replacing $z$ in (5.7) by $\sigma_V w$, it follows that $\Lambda_m$ is $G^c$-invariant.

Thus $\Lambda_m \in \text{Sesh}(\Xi)$ and it is positive definite. By Corollary 4.2 the kernel $\Lambda_m$ is determined by the function $\Lambda_{m, e_0} |_{\mathcal{H}^c_{\lambda}}$. But
\[
\Lambda_m(z, e_0) = \Lambda_m(e_0, z) = \int_{\mathbb{S}^{n-1}} [z, e_0]^{\lambda-\rho} \, d\mu(u) = \varphi_m(z)
\]
by Theorem 5.9. Hence $\Lambda_m(z, w) = \Phi_m(z, w)$.

As $(\pi_m, \mathcal{H}_m)$ is irreducible and $\varphi_m \in \mathcal{O}_m(\Xi)$ is cyclic, $P_m$ is a $G^c$-isomorphism. We also have $P_m 1_{\lambda} = \varphi_m$ and
\[
\langle P_m 1_{\lambda}, P_m 1_{\lambda} \rangle = \langle \varphi_m, \varphi_m \rangle = \Phi_m(e_0, e_0) = 1.
\]
As $\langle 1_{\lambda}, 1_{\lambda} \rangle = 1$, it follows that $P_m$ is unitary. □

5.3. Canonical kernels. In this section we discuss the relation between our setting and canonical kernels on hyperboloids. To this end, we identify (only in this section) $V$ with $\mathbb{R}^{n+1}$ and use the standard notation for $\mathbb{H}^n$ etc. The conjugation $\sigma_V$ is then the complex conjugation $z \mapsto \overline{z}$ and $G^c \subset \text{GL}_n(\mathbb{R})$ is the standard realization of $\text{O}^+_1(n, \mathbb{R})$.

The Berezin kernel on the open unit ball $\mathbb{B}^n := \{ x \in \mathbb{R}^n : x^2 < 1 \}$ is defined by
\[
B_{\lambda}(x, y) = \left( \frac{(1-x^2)(1-y^2)}{(1-xy)^2} \right)^{\lambda} \text{ for } \lambda > 0, \quad x^2, y^2 < 1
\]
(see [VDH97, §2]). We consider the diffeomorphism
\[
\Gamma : \mathbb{H}^n \to \mathbb{B}^n, \quad \Gamma(x_0, x) := x_0^{-1} x.
\]
Then $\Gamma^{-1}(x) = (1 - x^2)^{-1/2} (1, x)$ leads to
\[
C_{\lambda}(x, y) := B_{\lambda}(\Gamma(x), \Gamma(y)) = \left( \frac{(x_0^2 - x^2)(y_0^2 - y^2)}{(x_0 y_0 - xy)^2} \right)^{\lambda} = [x, y]^{-2\lambda}
\]
(cf. [VDH97, §3]). This is a $G^c$-invariant kernel on $\mathbb{H}^n$. It extends to a sesquiholomorphic kernel on a neighborhood of $\mathbb{H}^n_\lambda$ in $\Xi$ by the formula
\[
C_{\lambda}(z, w) := [z, \overline{w}]^{-2\lambda}.
\]
This kernel is defined on the $G^c$-invariant open subset $\{(z, w) \in \Xi \times \Xi : \text{Re}[z, \overline{w}] > 0 \}$. As $\{(z, w) : z, w \in \Xi \} = \mathbb{C} \setminus (-\infty, -1]$ (Lemma 5.5), $C_{\lambda}$ can not be extended to all of $\Xi$. To cope with this situation, we shrink $\Xi$ to a suitable $G^c$-invariant domain to which $C_{\lambda}$ extends.
Proposition 5.14. Consider the $G^c$-invariant open submanifold 
\[ \Xi' := \{ z \in \Xi : \beta(z) > 0 \} \quad \text{for} \quad \beta(z) := [z, z] = |z_0|^2 - \|z\|^2. \]
Then $C_\lambda$ defines a $G^c$-invariant sesquiholomorphic kernel on $\Xi'$.

Proof. As the continuous function $\beta$ on $V_\xi$ is $G^c$-invariant, $\Xi'$ is an open $G^c$-invariant submanifold of $\Xi$. Since $\beta(z) = 1$ for $z \in \mathbb{H}^n_0$, it contains $\mathbb{H}^n_0$. With $z_t = \cos(t)c_0 + \sin(t)i\epsilon_n \in \mathbb{S}^n_+$ we have 
\[ \Xi = G^c.\{z_t : |t| < \pi/4\} \supseteq \Xi' = G^c.\{z_t : |t| < \pi/4\}. \]

For $z, w \in \mathbb{C}^n \simeq V_\xi$, we write $(z, w) = z \cdot w = \sum_{j=1}^{n} z_j \overline{w_j}$. Then $\beta(z) = |z_0|^2 - \langle z, z \rangle > 0$ implies $z_0 \neq 0$ and $\overline{z} := z_0^{-1}z$ satisfies $\|\overline{z}\| < 1$. For $\beta(z), \beta(w) > 0$, we thus obtain $|\langle \overline{z}, \overline{w} \rangle| < 1$. For $z = (z_0, z), w = (w_0, w) \in \Xi'$, this leads to 
\[ [z, w] = z_0 \overline{w_0} - \langle z, w \rangle = z_0 \overline{w_0} \left(1 - \langle \overline{z}, \overline{w} \rangle\right). \]

As $\Re z_0 > 0$ and $\Re w_0 > 0$ follows from $\Xi' \subseteq \Xi \subseteq T_{V_+}$ (Proposition 3.2), we see that $C_\lambda$ defines a $G^c$-invariant sesquiholomorphic kernel on $\Xi'$.

On $\mathbb{H}^n$, the corresponding $K$-invariant function is given by 
\[ \psi_\lambda(x) = C_\lambda(x, e_0) = x_0^{-2\lambda}, \quad \text{resp.,} \quad \psi_\lambda(\cosh(t)e_0 + \sinh(t)e_n) = \cosh(t)^{-2\lambda}. \]

By [vDH97] Thm. 1], the kernel $C_\lambda \in \text{Sesh}(\Xi')$ has an integral representation 
\[ C_\lambda = \int_0^\infty \Psi_m \, d\mu_\lambda(m), \]
where $\mu_\lambda$ is a measure on $(0, \infty)$, which is on the interval $[\rho, \infty)$ ($\rho = \frac{n-1}{2}$) equivalent to Lebesgue measure. For $\lambda < \frac{n}{2}$, the measure $\mu_\lambda$ has an additional singular part, given by point measures in the points 
\[ s_j(\lambda) := \rho - 2\lambda - 2j \quad \text{for} \quad j \in \mathbb{N}_0 \quad \text{with} \quad s_j(\lambda) > 0. \]

For the corresponding unitary representation of $G^c$ on the reproducing kernel Hilbert space $\mathcal{H}_{C_\lambda} \subseteq \mathcal{O}(\Xi')$, this means that it decomposes into a direct integral of all spherical principal series representations (corresponding to $\lambda > \rho$) and a direct sum of finitely many spherical complementary series representations, corresponding to the values $s_j(\lambda)$. We refer to [vDH97] and [H99] Prop. 2.7.3, Cor. 4.2.2] for more details, where these kernels are considered as real analytic kernels on $\mathbb{H}^n$, resp., $\mathbb{B}^n$. These results extend to restrictions of minimal holomorphic representations of $\text{SU}_{n,m}(\mathbb{C})$ to $\text{SO}_{n,m}(\mathbb{R})$ ([Se07]) and, more generally, to matrix balls ([Ner99] and Makarevich spaces ([FP05]).

6. Perspectives

6.1. Identification of $\Xi$ with a Lie ball. In this subsection we explain how to identify the crown domain $\Xi$ with the $n$-dimensional Lie ball, i.e., the bounded symmetric domain whose isometry group is locally isomorphic to $\text{SO}_{2,n}(\mathbb{R})$.

On the $(n + 1)$-dimensional tube domain $T_{V_+} = V_+ + iV$, we have a natural transitive action of the group $\text{O}_{2,n+1}(\mathbb{R})^+$ which is obtained by extending the action of this group on the Minkowski space $iV \cong \mathbb{R}^{1,n}$ by rational maps to an action by holomorphic automorphisms of the tube domain $T_{V_+}$ whose Shilov boundary is $iV$ ([FK94] §X.5]). The identity $\Xi = \mathbb{S}^n_+ \cap T_{V_+}$ identifies $\Xi$ with a hypersurface defined by the equation $\ell^2 = 1$ in the tube domain $T_{V_+}$.

The function $\Delta(z) := z^2$ on $V_\xi \cong \mathbb{C}^n$ can be interpreted as the determinant of the complex Jordan algebra 
\[ V_\xi \cong \mathbb{C} \oplus \mathbb{C}^{n-1}, \quad (t, z)(t', z') := (tt' - zz', tz' + t'z), \quad t, t' \in \mathbb{C}, z, z' \in \mathbb{C}^{n-1} \]
whose determinant function is given on $V = \mathbb{R} \oplus i\mathbb{R}^n$ by
\[
\Delta(z_0, iz) = z_0^2 - z_1^2 - \cdots - z_{n-1}^2 = (iz)^2 \quad \text{for} \quad z = (z_0, z) \in \mathbb{R}^{n+1}, \tau(z_0, z) = (z_0, iz)
\]
([FK94] p. 31). On $V_C \cong \mathbb{C}^{n+1}$ we have an involutive rational map
\[
r(z) := \Delta(z)^{-1}z = \frac{1}{z^2}z \quad \text{with} \quad (\mathbb{C}^{n+1} \setminus \Delta^{-1}(0))^r = \mathbb{S}_C^n,
\]
called ray inversion, defined in the complement of the hypersurface $\Delta = 0$. If $\Delta(z) = 1$, then $r$ maps $\mathbb{C}^\times z$ into itself and $\lambda z$ to $\lambda^{-1}z$ for $\lambda \in \mathbb{C}^\times$. Next we observe that
\[
r(z) = \alpha(z^{-1}), \quad \text{where} \quad \alpha(z_0, z) = (z_0, -z) \quad \text{and} \quad (z_0, z)^{-1} = \Delta(z_0, z)^{-1}(z_0, -z)
\]
is Jordan inversion. Since $T_{V_+}$ is invariant under Jordan inversion ([FK94] Thm. X.1.1) and $\alpha$, it is also invariant under the holomorphic involution $r$ and we thus obtain
\[
(T_{V_+})^r = T_{V_+} \cap S_C^n = \Xi.
\]
The Cayley transform $C(z) := (z - e)(z + e)^{-1}$ maps the tube domain $T_{V_+}$ biholomorphically onto the Lie ball
\[
D := \{u + iv \in V_C = \mathbb{C}^{n+1} : \|u\|^2 + \|v\|^2 + 2\sqrt{\|u\|^2\|v\|^2} - (u, v) < 1\},
\]
where we identify $V$ with the euclidean space $\mathbb{R}^{n+1}$, so that $\| \cdot \|$ and $(u, v) = \sum j u_j v_j$ refers to the euclidean scalar product ([FK94] §X.2). From $C(r(z)) = -\alpha(z) = (-z_0, z)$ it follows that
\[
C(\Xi) = D^n := D \cap (\{0\} \times \mathbb{C}^n),
\]
which is an $n$-dimensional Lie ball. Since we also have $C \circ \sigma_V = \sigma_V \circ C$, it follows that
\[
C(\mathbb{H}_0^n) = D_0^n := D^n \cap V = \{(0, ix) : |x|^2 < 1\}
\]
is an open unit ball in an $n$-dimensional euclidean space.

Writing $T_{V_+}$ as $G_1/K_1$ for $G_1 \cong SO_{2n+1}(\mathbb{R})_0$ and the stabilizer $K_1 \cong SO_2(\mathbb{R}) \times SO_{2n+1}(\mathbb{R})$ of the base point $e_0$, we obtain from $r(e_0) = e_0$ an involution $\tau_2$ on $G_1$ defined by $r(g.z) = \tau_2(g)r(z)$ for $z \in T_{V_+}$. For the subgroups $G_1^r \subseteq G_1$ and $K_1^r \subseteq K_1$, we then have
\[
\Xi = (G_1, e_0)^r = G_1^r, e_0 \cong G_1^r/K_1^r.
\]
In particular, $\Xi$ is a Riemannian homogeneous space of the group $G_1^r$.

We now determine the groups $G_1$ and $G_1^r$ more explicitly. On
\[
\widetilde{V} := iV \oplus \mathbb{R}^2 = (i\mathbb{R} \oplus \mathbb{R}^n) \oplus \mathbb{R}^2,
\]
we consider the symmetric bilinear form given by
\[
\beta((v, s, t), (v', s', t')) := vv' - s^2 + t^2
\]
and the projective quadric
\[
Q := \{[\tilde{v}] \in \mathbb{P}(\tilde{V}) : \tilde{v} \in \tilde{V}, \beta(\tilde{v}, \tilde{v}) = 0\}.
\]
The map
\[
\eta : iV \to Q, \quad \eta(v) := \left[ (v, \frac{1}{2}(1 + vv), \frac{1}{2}(1 - vv) \right]
\]
is the conformal completion of $iV$. It is a diffeomorphism onto an open dense subset of $Q$. The natural action of the orthogonal group $O(\tilde{V}, \beta) \cong O_{2, n+1}(\mathbb{R})$ on $Q$ corresponds to the action of the conformal group on the Shilov boundary $iV$ of $T_{V_+}$.

Next we observe that
\[
\eta(r(v)) = \left[ ((vv)^{-1}v, \frac{1}{2}(1 + (vv)^{-1}), \frac{1}{2}(1 - (vv)^{-1}) \right] = \left[ (v, \frac{1}{2}(1 + vv), \frac{1}{2}((vv)^{-1} - 1) \right],
\]
so that
\[
\eta(r(v)) = \tilde{r}\eta(v) \quad \text{for} \quad \tilde{r}(ix_0, x, s, t) = (ix_0, x, s, -t).
\]
Therefore the involution on $O(\tilde{V}, \beta)$ corresponding to $\tau_\nu$ corresponds to $\tau_\nu(g) := \tilde{r}g\tilde{r}$, and thus

$$O(\tilde{V}, \beta)^\nu \cong O_{2,n}(\mathbb{R}) \times O_2(\mathbb{R})$$

because the form $\beta$ on the subspace $\tilde{V}^\nu$ has signature $(n, 2)$. This shows that

(6.3) $$\Xi \cong SO_{2,n}(\mathbb{R})_0/(SO_2(\mathbb{R}) \times SO_n(\mathbb{R}))$$

is the Riemannian symmetric space associated to $SO_{2,n}(\mathbb{R})_0$. In particular, the action of $G^c \cong O_{1,n}(\mathbb{R})^\dagger$ on $\Xi$ extends to a transitive action of the group $SO_{2,n}(\mathbb{R})_0$.

**Connection to highest weight representations and corresponding kernels.** On the tube domain $T_{\nu, \tilde{V}}$, there exists a natural family of sesquiholomorphic kernels, given in terms of the Jordan determinant $\Delta(z) = [z, z]_V$ (see [FK94, §III.1]) by $\Delta\left(\frac{z + \sigma_V w}{2}\right)^{-\nu}$. Concretely, we have

$$\left[\frac{z + \sigma_V(w)}{2}, \frac{z + \sigma_V(w)}{2}\right]_V^{-\nu}$$

([PK94 §XIII.1]). These sesquiholomorphic kernels are obviously invariant under $G^c$, hence restrict to $G^c$-invariant kernels on the complex submanifold $\Xi \subset T_{\nu, \tilde{V}}$ given by

$$Q_\nu(z, w) := \left(1 + \frac{\nu}{2}\right)^{-\nu}.$$ 

On $\mathbb{H}^n$, these kernels correspond to the $K$-invariant function

$$q_\nu(x) = Q_\nu(x, e_0) = \left(1 + \frac{x_0}{2}\right)^{-\nu},$$

respectively

$$q_\nu(cosh(t)e_0 + sinh(t)ie_n) = \left(1 + \frac{\cosh(t)}{2}\right)^{-\nu} = \cosh(t/2)^{-2\nu}.$$ 

The kernel $Q_\nu$ corresponds to the reproducing kernel of a highest weight representation of $SO_{2,n}(\mathbb{R})_0$. Its restriction to the real symmetric bounded domain corresponding to the subgroup $SO_{1,n}(\mathbb{R})_0$ has been studied for instance in [Hi99, ´OØ96, ´O00].

Since $\Xi$ is biholomorphic to an $n$-dimensional Lie ball, hence also to an $n$-dimensional tube domain, it follows from [PK94 Thm. XIII.2.7] that the kernel $Q_\nu$ is positive definite if and only if either $\nu = 0$ or $\nu \geq \frac{d}{2} = \frac{n-2}{2}$ (note that $r = 2$ in our case). We thus obtain elements $Q_\nu \in \text{Sesh}(\Xi)$ for $\nu \geq \frac{d}{2}$, and, in view of Corollary 5.11 it is a natural problem to determine the measure $\mu_\nu = \mu_{Q_\nu}$.

This problem has been solved by H. Seppänen in [Se07b], see also [´O00, O00] for part (i). This case and a different connection to reflection positivity has been discussed in [NO14, JO98, J000]. By [Se07b §5.3] we have:

**Theorem 6.1.** There exists a measure $\mu_\nu$ on $[0, \infty)$ such that:

(i) For $\nu \geq \rho = \frac{n-1}{2}$, the measure $\mu_\nu$ is absolutely continuous with respect to Lebesgue measure on the open interval $(\rho, \infty)$. The representations $\pi_m$ are the unitary principal series representations.

(ii) For $\frac{n-2}{2} < \nu < \frac{n-1}{2}$, we have an additional point mass in a point $m_\nu = \sqrt{\rho^2 - \lambda_\nu}$, where $\lambda_\nu = \rho - \nu$. The corresponding representations corresponds to the complementary series representations.

(iii) For $\nu = \frac{n-2}{2}$, the measure $\mu_\nu$ is a point mass in $m_\nu$.

(iv) For the minimal positive value $\nu_{\text{min}} = \frac{n-2}{2}$, the corresponding representation of $SO_{2,n}(\mathbb{R})_0$ actually restricts to an irreducible representation of $G^c$ belonging to the complementary series.
6.2. Boundary values on the de Sitter space. In the last section we discussed the role of the homogeneous space \( \mathbb{L}^n \) for the identification of the positive definite kernels \( \Phi^c_{g_v} \) and the corresponding unitary representations. Here we briefly relate our work to analysis on the other boundary orbit \( dS^n \) (de Sitter space).

Let \( \mathcal{B}(dS^n) \) be the space of holomorphic functions \( F \) on \( \Xi \) extending to continuous functions on \( \Xi \cup dS^n \). We will also write \( \mathcal{B}(dS^n \times \Xi) \) for the space of sesquiholomorphic kernels \( \Phi: \Xi \times \Xi \rightarrow \mathbb{C} \) extending to a continuous function on \( (dS^n \cup \Xi) \times \Xi \). We then write \( \beta(\varphi)(y, w) = \beta(\varphi(\cdot, w))(y) \).

Similarly we define \( \mathcal{B}(\Xi \times S^n) \) as the space of kernels extending continuously to \( \Xi \times (\Xi \cup dS^n) \). Then \( \beta(\varphi)(z, y), (z, y) \in \Xi \times dS^n \), is well defined.

We use the notation from previous sections:

\[
a_w = \begin{pmatrix} \cosh(w) & 0 & -i \sinh(w) \\ 0 & \mathbf{I}_{n-1} & 0 \\ i \sinh(w) & 0 & \cosh(w) \end{pmatrix}, \quad z_w = a_w.e_0 = \cosh(w)e_0 + i \sinh(w)e_n, \quad w \in \mathbb{C}.
\]

Let \( w = t + ir \in \mathbb{R} + i(-\pi/2, \pi/2) \). Then

\[
\cosh(t - ir) = \frac{1}{2} \left( e^t e^{-ir} + e^{-t} e^{ir} \right) \xrightarrow{r \rightarrow \pi/2} -i \sinh(t)
\]

and

\[
\sinh(t - ir) = \frac{1}{2} \left( e^t e^{-ir} - e^{-t} e^{ir} \right) \xrightarrow{r \rightarrow \pi/2} -i \cosh(t).
\]

Thus

\[
\lim_{r \nearrow \pi/2} a_{t-ir}.e_0 = -i \sinh(t)e_0 + \cosh(t)e_n \in dS^n.
\]

Taking \( t = 0 \) gives:

**Lemma 6.2.** Let \( F \in \mathcal{B}(dS^n) \) and \( g \in G^c \). Then \( \beta(F)(ge_n) = \lim_{r \nearrow \pi/2} F(ga_{-ir}.e_0) \).

The following is well known in general but we give a simple proof suitable for our special situation:

**Lemma 6.3.** For the stabilizer \( H := G^c_{e_n} \), we have \( G^c = HAK = KA^H \).

**Proof.** That \( HAK = KA^H \) follows by taking inverses. Thus we only have to prove that \( G^c = KA^H \).

This assertion is equivalent to \( KA.e_n = dS^n \). Let \( v = (iv_0, v) \in dS^n \). Then \(-v_0 + ||v||^2 = 1\). Let \( t \in \mathbb{R} \) be such that \( v_0 = \sinh(t) \) and \( ||v|| = \cosh(t) \) and let \( k \in K \) be such that \( k.e_n = \frac{1}{\cosh(t)} v \).

Then, as \( a_{-t}.e_n = i \sinh(t)e_0 + \cosh(t)e_n \), we get \( k.a_{-t}.e_n = v \). \( \square \)

**Lemma 6.4.** We have \( \{[z, x]: z \in \Xi, x \in dS^n\} \cap \mathbb{R} = (-1, 1) \). In particular, if \( z \in \Xi \) and \( x \in dS^n \), then \([z, x] \in \mathbb{C} \setminus ((-\infty, -1) \cup [1, \infty))\).

**Proof.** As \( dS^n = G^c.e_n \) and \([\cdot, \cdot] \) is \( G^c \) invariant, we can assume that \( x = e_n \). Write \( z = ga_{-is}.e_0 = g(\cos(s)e_0 + \sin(s)e_n) \) with \( |s| < \pi/2 \), and \( g = ha_{tk} \) with \( h \in H, a_t \in A \) and \( k \in K \). Note that

\[
k.(\cos(s)e_0 + \sin(s)e_n) = \cos(s)e_0 + \sin(s) u \quad \text{for some} \quad u \in e_0^+ \cap S^n.
\]

As \( e_n \) is \( H \)-invariant, we get

\[
[z, e_n]_V = [ha_t(\cos(s)e_0 + \sin(s) u), e_n]_V
= [a_t(\cos(s)e_0 + \sin(s) u), e_n]_V = [\cos(s)e_0 + \sin(s) u, a_{-t}e_n]_V
= [\cos(s)e_0 + \sin(s) u, i \sinh(t) e_0 + \cosh(t)e_n]_V
= u_n \cosh(t) \sin(s) + i \sinh(t) \cos(s).
\]

Thus \( [z, e_n]_V \in \mathbb{R} \) implies \( t = 0 \) because \( \cos(s) > 0 \) for \( |s| < \pi/2 \). Then \( \cosh(t) = 1 \) and \( [z, e_n]_V = u_n \sin s \in (-1, 1) \) because \( |\sin(s)| < 1 \) and \( |u_n| \leq 1 \). That \( u_n \sin(s) \) can take any value in \((-1, 1)\) is clear. This proves the lemma. \( \square \)
Theorem 6.5. Let \( z, w \in \Xi \) and \( x, y \in dS^n \). Then the following assertions hold:

(i) \( \Phi^c_m \in \mathcal{B}(dS^n \times \Xi) \) and
\[
\beta(\Phi^c_m)(x, w) = 2F_1(\rho + \lambda, \rho - \lambda; n/2; (1 - [x, \sigma_V w])/2).
\]

(ii) \( \Phi^c_m \in \mathcal{B}(\Xi \times dS^n) \) and
\[
\beta(\Phi^c_m)(z, y) = 2F_1(\rho + \lambda, \rho - \lambda; n/2; (1 + [z, y])/2).
\]

(iii) If \( \nu \geq \frac{n-2}{2} \), then \( Q_\nu \in \mathcal{B}(dS^n \times \Xi) \cap \mathcal{B}(\Xi \times dS^n) \) and
\[
\beta(Q_\nu)(z, y) = \left(1 - [z, y]\right)^{-\nu} \quad \text{and} \quad \beta(Q_\nu)(x, w) = \left(1 + [x, \sigma_V w]\right)^{-\nu}.
\]

Proof. The kernel \( \Phi^c_m \) extends to a real analytic function on an open subset of \( V_\Xi \times V_\Xi \) containing \( (\Xi \cup dS^n) \times \Xi \) (Theorem 6.4 and Lemma 6.3). It follows that the boundary value exists and is given by the value at the point.

(ii) follows in the same way noting that \( dS^n \subset iV \) so that \( \sigma_V |_{dS^n} = -\text{id} \).

(iii) follows also in the same way as \( Q_\nu \) is continuous on \( \Xi \times \Xi \) and \( \Xi \times \Xi \).

Recall the reproducing kernel Hilbert space \( \mathcal{O}_m(\Xi) \subseteq \mathcal{O}(\Xi) \) with the kernel \( \Phi^c_m \). Then \( \mathcal{O}_m(\Xi) = \mathcal{P}_\lambda \mathcal{H}_\lambda \) (Theorem 5.13). In particular \( \mathcal{O}_m(\Xi) \), with left translation as representation, is isomorphic to \( (\pi_m, \mathcal{H}_m) \). For \( y \in dS^n \), the function \( \eta_y := \beta(\Phi^c_m)(\cdot, y) \) is holomorphic on \( \Xi \). We claim that it does not belong to \( \mathcal{O}_m(\Xi) \). To this end, we first observe that it is invariant under the non-compact stabilizer group \( G_y \), but for irreducible unitary representations of \( G^c \), stabilizer subgroups are compact by the Howe–Moore Theorem [HM79, Thm. 5.1].

We claim that \( \eta_y \) defines a distribution vector, i.e., an element of \( \mathcal{O}_m(\Xi)^{-\infty} \). Let \( \mathcal{O}_m(\Xi)^c \subseteq \mathcal{O}_m(\Xi) \) denote the linear subspace of all functions extending continuously to \( \Xi \cup dS^n \). It is dense because it contains all elements \( \Phi^c_m(\cdot, w) \), \( w \in \Xi \). In particular, it contains \( \varphi_m = \Phi^c_m(\cdot, e_0) \). Since each function \( \Phi^c_m(\cdot, w) \) extends to a smooth function on an open subset containing \( \Xi \cup dS^n \), the subspace \( \mathcal{O}_m(\Xi)^c \) contains also the subspace \( \mathcal{L}_{U(1)} \mathcal{H}_m \) of \( K \)-finite functions in \( \mathcal{O}_m(\Xi) \). The Automatic Continuity Theorem [BD92, Thm. 1] then implies that \( \text{ev}_y : \mathcal{O}_m(\Xi)^c \to \mathbb{C}, f \mapsto \int(f(y)) \) extends to a \( G_y \)-invariant distribution vector. The corresponding holomorphic function on \( \Xi \) is given by
\[
\Xi \to \mathbb{C}, \quad z \mapsto \text{ev}_y(\Phi^c_m(\cdot, z)) = \Phi^c_m(y, z) = \Phi^c_m(z, y) = \beta(\Phi^c_m)(z, y) = \eta_y(z).
\]

According to [GKO03, GKO04], we can define the Hardy space \( H^2(dS^n) \) in the following way. The action of \( \text{SO}_{2,n}(\mathbb{R}) \) on \( \Xi \) can be extended to the open semigroup
\[
S := \{ \gamma \in \text{SO}_{2,n}(\mathbb{C}) : \gamma^{-1}\Xi \subset \Xi \} \neq \emptyset.
\]
We define
\[
H^2(dS^n) = \left\{ \psi \in \mathcal{O}(\Xi) : \sup_{\gamma \in S} \int dS^n |\psi(\gamma^{-1}.x)|^2 dx < \infty \right\}.
\]

Then \( H^2(dS^n) \) is a Hilbert space with norm
\[
\left( \sup_{\gamma \in S} \int dS^n |\psi(\gamma^{-1}.x)|^2 dx \right)^{1/2}
\]
([GKO03 Cor. 5.3]). As a unitary representation of \( G^c \) the Hardy space \( H^2(dS^n) \) is isomorphic to
\[
L^2(H^1_n^\oplus) \simeq \int_{m > \rho} (L, \mathcal{O}_m(\Xi)) d\mu_H(m) \simeq \int_{m > \rho} (\pi_m, \mathcal{H}_m) d\mu_H(m)
\]
where the measure \( \mu_H \) is equivalent to Lebesgue measure on \( (\rho, \infty) \).
The evaluation maps \( ev_z(\varphi) := \varphi(z), z \in \Xi \), are continuous on \( H^2(dS^n) \) and hence given by a positive definite \( SO_{2n}(\mathbb{R}) \)-invariant kernel on \( \Xi \times \Xi \). This kernel is, up to multiplication with a positive function, the kernel \( Q_{n/2} \) (GKO03 Thm. C).

**Theorem 6.6.** Let \( \nu > \frac{n-2}{2} \) and let \( \mu_\nu \) be the measure from Theorem 5.1. Then

\[
Q_\nu(z, y) = \int_0^\infty \Phi_m^c(z, y) d\mu_\nu(m) \quad \text{and} \quad Q_\nu(y, z) = \int_0^\infty \Phi_m^c(y, z) d\mu_\nu(m) \quad \text{for} \quad (z, y) \in \Xi \times dS^n.
\]

Proof. For \( g \in SO_{n+2}(\mathbb{C}) \), let \( g^\dagger = g^{-1} \), where \( g^\dagger \) denotes complex conjugation with respect to \( SO_{2n}(\mathbb{R}) \). Then the semigroup \( S \) from (6.4) is \( g \)-invariant. Furthermore, analytic continuation implies that

\[
Q_\nu(\gamma z, w) = Q_\nu(z, \gamma^2z), \quad \Phi_m^c(\gamma z, w) = \Phi_m^c(z, \gamma^2z) \quad \text{for} \quad \gamma \in S^{-1}, (z, w) \in \Xi \times \Xi.
\]

For \( z \in \Xi \) and \( y \in dS^n \), there exists an element \( \gamma \in S^{-1} \) such that \( \gamma^{-1}z \in \Xi \) and \( \gamma^\dagger y \in \Xi \). Hence

\[
Q_\nu(z, y) = Q_\nu(\gamma^{-1}z, y) = Q_\nu(\gamma^{-1}z, \gamma^2y) = \int_{\mathbb{R}^+} \Phi_m^c(\gamma^{-1}z, \gamma^2y) d\mu_\nu(m)
\]

\[
= \int_{\mathbb{R}^+} \Phi_m^c(z, y) d\mu_\nu(m).
\]

\( \Box \)

6.3. **Further examples.** As the classification shows, there are interesting examples of dissecting involution on non-Riemannian symmetric spaces. We discuss here very briefly some examples. For a classification we refer to [NO19].

**Example 6.7.** (Euclidean space) (cf. [NO15a]) Let \( E := \mathbb{R}^n \) denote euclidean \( n \)-space and consider the euclidean motion group

\[
G := E \times O_n(\mathbb{R}) \quad \text{and} \quad K = O_n(\mathbb{R}).
\]

Then \( E \cong G/K \) is a flat Riemannian symmetric space corresponding to the involution \( (x, g) \mapsto (-x, g) \).

The reflection \( \sigma(x_0, x) = (-x_0, x) \) defines a dissecting reflection such that \( \sigma(g, x) = \tau(g).\sigma(x) \) holds for \( \tau(x, g) := (\sigma(x), \sigma g \sigma) \). We then have

\[
E_+ := \{(x_0, x) : x_0 > 0\} \quad \text{and} \quad E_0 = \{(0, x) : x \in \mathbb{R}^{n-1}\} \cong \mathbb{R}^{n-1}.
\]

For \( m > 0 \), the distribution \( (m^2 - \Delta)^{-1}\delta_0 \) is represented by a rotation invariant analytic function \( \varphi_m \) on \( \mathbb{R}^n \setminus \{0\} \) with a singularity in \( 0 \) (for \( n > 1 \)) ([NO15a]). The corresponding distribution kernel \( \Phi_m(x, y) = \varphi_m(x - y) \) is singular on the diagonal, and the flipped kernel \( \Psi_m(x, y) = \varphi_m(x - \sigma(y)) \) is analytic on \( E_+ \times E_+ \).

The subspace \( E^c := \mathbb{R}i e_0 \oplus \mathbb{R}^{n-1} \) carries the Lorentzian form \((ix_0, x), (iy_0, y)\) = \( x_0y_0 - xy \) with the open forward light cone

\[
E_+^c := \{(x_0, ix) : x_0 > 0, x_0^2 - x^2 > 0\}.
\]

The \( c \)-dual group \( G^c := E^c \times SO_{1,n-1}(\mathbb{R})_{0} \) is the identity component of the corresponding isometry group (the Poincaré group). The corresponding tube domain is

\[
\Xi := G^c E_+ = G^c.(0, \infty)ie_0 = E^c + iE_+^c =: T_{E_+^c}.
\]

One can show that also in this context, the kernel \( \Psi_m \) extends to a sesquiholomorphic \( G^c \)-invariant kernel on \( \Xi \). The boundary values on \( E^c \) of the function \( \psi_m(z) := \Psi_m(z, 0) \) is a distribution \( D_m \), satisfying the Klein–Gordon equation

\[
(m^2 - \Box)D_m = 0.
\]
It is the Fourier transform of the $SO_{1,n-1}(\mathbb{R})_0$-invariant measure on the hyperboloid
\[ \mathcal{O}_m = \{(x_0, \ldots, x_{n-1}) = (x_0, x): x_0^2 - x^2 = m^2, x_0 > 0\}. \]
The corresponding $L^2$-space carries an irreducible unitary representation of $G^c$, cf. Remark 6.11 in [NO15a].

**Example 6.8.** Besides $\mathbb{R}^n$, the preceding discussion also applies to quotients of $\mathbb{R}^n$ by discrete $\sigma$-invariant subgroups $\Gamma$. A particularly interesting case is the torus $\mathbb{T}^n$ with $\sigma(z_1, \ldots, z_n) = (\overline{z}_1, z_2, \ldots, z_n)$ (cf. [Ja08 §VIII]).

**Example 6.9.** (Hyperbolic space) In the $n+1$-dimensional Minkowski space $V := \mathbb{R}^{1,n}$, we consider the hyperbolic space
\[ \mathbb{H}^n = \{(x_0, x): x_0^2 - x^2 = 1, x_0 > 0\} \]
on which the group $G := O_{1,n}(\mathbb{R})^\dagger$ acts, and the dissecting involutive automorphisms $\sigma$ of $\mathbb{H}^n$, defined by the reflection $r_1 \in G$. Then $\mathbb{H}_0^n := (\mathbb{H}^n)^\sigma \cong \mathbb{R}^{n-1}$, and we put
\[ \mathbb{H}_\pm^n := \{x \in \mathbb{H}^n: \pm x_1 > 0\}. \]

We also write
\[ [z, w] := z_0w_0 - zw \]
for the complex bilinear extension to $V_C = \mathbb{C}^{n+1}$, so that
\[ \mathbb{H}_C^n := \{z \in V_C: [z, z] = 1\} \cong \mathbb{S}_C^n \]
is the complex sphere. On the dual space
\[ V^c := \mathbb{R}e_0 \oplus \mathbb{R}ie_1 \oplus \mathbb{R}^{n-1} \subseteq V_C = \mathbb{C}^{n+1}, \]
we have
\[ [(x_0, ix_1, x_2, \ldots, x_n), (y_0, iy_1, y_2, \ldots, y_n)] = x_0y_0 + x_1y_1 - x_2y_2 - \cdots - x_ny_n, \]
and this form is invariant under the action of the connected group
\[ G^c := SO_{2,n-1}(\mathbb{R})_0. \]
The stabilizer of $e_1$ in $G^c$ is the subgroup $H := G^c_{e_1} \cong SO_{1,n-1}(\mathbb{R})_0$, in particular it is connected. Since it acts transitively on $\mathbb{H}_0^n$, we have
\[ \mathbb{H}_\pm^n = H \cdot \text{Exp}_{e_0}(\pm(0, \infty)e_1). \]
Accordingly, we obtain two $G^c$-invariant subsets
\[ \Xi_\pm := G^c \cdot \mathbb{H}_\pm^n = G^c \cdot \text{Exp}_{e_0}(\pm(0, \infty)e_1) \subseteq \mathbb{H}_C^n. \]

To the two non-convex open cones $\Omega_\pm := \{v \in V^c: \pm [v, v] > 0\}$ we associate non-convex tube domains
\[ T_{\Omega_\pm} := V^c \oplus i\Omega_\pm. \]

We claim that
\[ T_{\Omega_+} \cap \mathbb{H}_C^n = \Xi_+ \cup \Xi_- = G^c \cdot \text{Exp}_{e_0}(\mathbb{R}^x e_1) \]
(cf. [BEM02 Lemma 3.2]). In fact, for $t \neq 0$, we have
\[ \cosh(t)e_0 + \sinh(t)e_1 = \cosh(t)e_0 + i \sinh(t)(-ie_1) \in T_{\Omega_+} \]
because $[-ie_1, -ie_1] = [-e_1, e_1] = 1$, so that $G^c \cdot \text{Exp}_{e_0}(\mathbb{R}^x e_1) \subseteq T_{\Omega_+}$ follows from the $G^c$-invariance of $T_{\Omega_+}$. For the converse, let $z = u + iv \in T_{\Omega_+} \cap \mathbb{H}_C^n$. Then
\[ 1 = [z, z] = [u, u] - [v, v] + 2i[u, v] \]
is equivalent to
\[ [u, v] = 0 \quad \text{and} \quad [u, u] = [v, v] + 1 > 1. \]
Therefore \((u, v)\) is the basis of a positive 2-plane in \(V^c\). Hence the \(G^c\)-orbit of this pair contains an element of the form \((\lambda_0, \mu ie_1)\) with \(\lambda = \sqrt{1 + \mu^2}\) and \(\mu \neq 0\). Depending on the orientation of the basis \((u, v)\), we have \(\mu > 0\) or \(\mu < 0\). For \(\mp \mu > 0\) we get \(u + iv \in \Xi_\pm\), and this proves our claim.

We thus obtain a situation very analogous to what we have seen in Subsection 3.1 of the sphere. In the physics literature, the domains \(\Xi_\pm\) are called chiral tubes. They carry an action of \(G^c\) by holomorphic maps, and both contain the \textit{anti de Sitter space}

\[
\text{AdS}^n := \{x \in V^c : [x, x] = 1\} = G^c.e_0 \cong \text{SO}_{2,n-1}(\mathbb{R})_0 / \text{SO}_{1,n-1}(\mathbb{R})_0
\]

in their boundary.

However, Theorem 2.13 does not apply here because our base point \(m_0 = e_0 \in \mathbb{H}^n\) is fixed by the involution \(\sigma\). Hence there is no Riemannian symmetric space in \(\Xi_\pm\), such as the hyperbolic space for the sphere. Here one translates directly between \(\mathbb{H}^n\) and its Lorentzian dual \(\text{AdS}^n\) by first extending a positive definite analytic kernel \(\Psi_m\) on \(\mathbb{H}^n_+ \times \mathbb{H}^n_+\) to a sesquiholomorphic kernel on \(\Xi_+\) and then taking boundary values on \(\text{AdS}^n\). Although from a slightly different perspective, this program is carried out to some extent in [BEM02].

### 6.4. Extension to anti-unitary representations

A \textit{graded group} \((G, \varepsilon)\) is a pair of a group \(G\) and a homomorphism \(\varepsilon : G \to \{\pm 1\}\). A typical example is the full Lorentz group \(O_{1,n}(\mathbb{R})\) with \(\varepsilon\) defined by \(gV_+ = \varepsilon(g)V_+\), for which \(\ker \varepsilon = O_{1,n}(\mathbb{R})^\perp\). Another important example is the group \(\text{AU}(\mathcal{H})\) of all unitary and antiunitary operators on a complex Hilbert space \(\mathcal{H}\) with \(\ker \varepsilon = U(\mathcal{H})\).

We call a homomorphism \(U : G \to \text{AU}(\mathcal{H})\) an \textit{antiunitary representation} if \(\varepsilon_{\text{AU}(\mathcal{H})}(U(g)) = \varepsilon\varphi(g)\) for \(g \in G\) (see [NO17] for more background on these concepts).

In this section we discuss the extension of the representation \((\pi_m, \mathcal{H}_m)_{m \geq 0}\) of \(G^c\) to antiunitary representations of \(O_{1,n}(\mathbb{R})\). Let \(\Psi \in \mathcal{G}\) and denote the corresponding reproducing kernel Hilbert space by \(\mathcal{H}_\Psi \subseteq \mathcal{O}(\Xi)\).

**Lemma 6.10.** Let \(\Psi \in \mathcal{G}\) and \(\sigma : \Xi \to \Xi\) be an antiholomorphic involution extending an isometry of \(\mathbb{H}^n_+\) fixing \(e_0\). We further assume the existence of an involution \(\sigma^G\) on \(G^c\) with

\[
(6.5) \quad \sigma(g.z) = \sigma^G(g).\sigma(z) \quad \text{for} \quad g \in G^c, z \in \Xi.
\]

Then

\[
\Psi(z, w) = \Psi(\sigma(w), \sigma(z)) \quad \text{for} \quad z, w \in \Xi.
\]

**Proof.** We consider the sesquiholomorphic kernel

\[
\tilde{\Psi}(z, w) := \Psi(\sigma(w), \sigma(z)) \quad \text{for} \quad z, w \in \Xi.
\]

The relation (6.5) implies that the kernel \(\tilde{\Psi}\) is \(G^c\)-invariant. Furthermore \(\tilde{\Psi}\) is holomorphic in the first and antiholomorphic in the second argument. Next we observe that

\[
\tilde{\Psi}^*(z, w) = \overline{\tilde{\Psi}(\sigma(z), \sigma(w))} = \overline{\Psi(\sigma(w), \sigma(z))} = \overline{\tilde{\Psi}(z, w)},
\]

so that \(\tilde{\Psi}\) is hermitian, and thus \(\tilde{\Psi} \in \mathcal{G}\).

To show that \(\tilde{\Psi} = \Psi\), by Corollary 4.2 it suffices to show that \(\tilde{\Psi}|_{\mathbb{H}^n_+^\circ} = \Psi|_{\mathbb{H}^n_+^\circ}\), i.e., that \(\psi := \tilde{\Psi}|_{e_0}\) is real-valued on \(\mathbb{H}^n_+\). As \(\psi(\sigma(x)) = \overline{\psi(x)}\) and \(\sigma \in K \cong O_n(\mathbb{R})\), this follows from the \(K\)-invariance of \(\psi\). This completes the proof. \(\square\)

It follows that we can define a conjugation on \(\mathcal{H}_\Psi\) by

\[
J \left( \sum_j c_j\Psi_{w_j} \right) := \sum_j \overline{c_j}\Psi_{\sigma(w_j)}; \quad c_j \in \mathbb{C}, w_j \in \Xi,
\]

resp.,

\[
(J f)(z) := \overline{f(\sigma(z))}, \quad z \in \Xi.
\]
Lemma 6.11. The following assertions hold:

(i) $\mathcal{H}_Y = \{ \varphi \in \mathcal{H}_Y \colon J \varphi = \varphi \}$ is the closure of the real linear span of $\Psi_y, y \in \mathbb{H}_Y^n$.

(ii) $J \pi \psi(g) = \pi \psi(\sigma^G(g))J$ for $g \in G^c$.

(iii) If $\Psi = \Phi_m \in \Gamma_e$ and $J_1 : \mathcal{H}_m \rightarrow \mathcal{H}_m$ is a conjugation satisfying (ii), then $J_1 = \mu J$ for some $\mu \in \mathbb{T}$.

Proof. (i) and (ii) follow directly from the definition and Lemma 6.10.

For (iii) we recall first that the representation of $G^c$ on $\mathcal{H}_m$ is irreducible. If $J_1$ satisfies (ii), then $J_1 J : \mathcal{H}_m \rightarrow \mathcal{H}_m$ is a unitary intertwining operator. By Schur’s Lemma there exists $\lambda \in \mathbb{C}$ such that $J_1 J = \lambda \text{id}$ and now $|\lambda| = 1$ by unitarity. □

Example 6.12. The assumptions of Lemma 6.10 are in particular satisfied with $\sigma^G = \text{id}_G$ and $\sigma = \text{id}_\mathcal{H}|\Xi$. By Lemma 6.11 and the fact that $-1$ commutes with $G^c$, we can extend $\pi \psi$ to an antiunitary representation of $O_{1,n}(\mathbb{R}) \cong G^c \rtimes \{1, \sigma^G\}$ by

$$\pi \psi(-1) := J, \quad (J f)(z) := f(\sigma \psi(z)).$$

For every $X \in g^c$ we then have $J \text{ad} \pi \psi(X) = \text{ad} \pi \psi(X) J$. In particular

$$J \pi \psi(\exp(tX)) = \pi \psi(\exp(tX)) J \quad \text{for} \quad t \in \mathbb{R}.$$

We now assume that $\Psi = \Psi_m$. Let $X \in g^c$ be so that $\theta(X) = -X$ (so $X$ is hyperbolic (a boost)) and $\text{ad} X$ has eigenvalues $\pm \lambda$ and $0$ with $0 < \lambda < 1/4$. Consider the homomorphism

$$\gamma_X : \mathbb{R}^\times \rightarrow O_{1,n}(\mathbb{R}) \cong G^c \rtimes \{1, \sigma^G\}, \quad \gamma_X(e^t) := \exp(tX), \quad \gamma_X(-1) := -1$$

of graded groups. Then $\pi \psi \circ \gamma_X : \mathbb{R}^\times \rightarrow \text{AU}(\mathcal{H})$ is a strongly continuous antiunitary one-parameter group taking on $-1$ the value $J$. The corresponding modular operator is given by

$$\Delta_X := e^{2\pi i \partial \pi \psi(X)},$$

so that the corresponding standard subspace is

$$V_X := \{ v \in D(\Delta_X^{1/2}) = D(e^{i\pi \partial \pi \psi(X)}) \colon J v = e^{i\pi \partial \pi \psi(X)} v \}.$$

If $\| \text{ad} X \| < \frac{1}{2},$ then the elements $\Psi_z, z \in \mathbb{H}_V^n$, are contained in the domain of $\Delta_X$ by [KS05] with

$$\Delta_X^{1/4} \Psi_z = \Psi_{\exp(-i\pi X/2)z} = J \Psi_{\exp(i\pi X/2)z} = \Delta_X^{1/2} \Psi_{\exp(i\pi X/2)z},$$

so that the corresponding standard subspace $V_X$ is generated by the elements $\Psi_{\exp(-i\pi X/2)z}, z \in \mathbb{H}_V^n$. In particular, $V_0 = \mathcal{H}_Y$ is generated by $\Psi_z, z \in \mathbb{H}_V^n$. For the most general statement about analytic extension of coefficient functions we refer to [LPT18, KS05] and the references therein.

References

[AG90] Akhiezer, D. N., and S. G. Gindikin, On Stein extensions of real symmetric spaces, Math. Ann. 286 (1990), 1–12

[AKLM06] Alekseevsky, D., A. Kriegl, M. Losik, and P. W. Michor, Reflection groups on Riemannian manifolds, Annali di Matematica Pura ed Applicata 186 (2006), 25–58

[An13] Anderson, C. C. A., “Defining Physics at Imaginary Time: Reflection Positivity for Certain Riemannian Manifolds,” Thesis, Harvard Univ., 2013

[AGF86] de Angelis, G., D. de Falco, and G. Di Genova, Random fields on Riemannian manifolds: a constructive approach, Comm. Math. Phys. 103 (1986), 297–303

[BJM16] Barata, J. C. A., C. D. Jäkel, and J. Mund, Interacting quantum fields on de Sitter space, arXiv:1607.02265

[BEM02] Bros, J., H. Epstein, and U. Moschella, Towards a general theory of quantized fields on the anti-de Sitter space-time, Comm. Math. Phys. 231:3 (2002), 481–528

[BEM02b] —, Asymptotic symmetry of de Sitter spacetime, Phys. Rev. D (3) 65:8 (2002), 084012, 8 pp

[BM96] Bros, J., and U. Moschella, Two-point functions and quantum fields in de Sitter universe, Rev. Math. Phys. 8:3 (1996), 327–391
Fourier analysis and holomorphic decomposition on the one-sheeted hyperboloid in “Géométrie complexe. II. Aspects contemporains dans les mathématiques et la physique,” 27–58, Hermann Éd. Sci. Arts, Paris, 2004

Bros, J., and G.A. Viano, Connection between the harmonic analysis on the sphere and the harmonic analysis on the one-sheeted hyperboloid: an analytic continuation viewpoint–I, II, Forum Math. 8 (1996), 621–658, 659–722

Bros, J.-L. and P. Delorme, Vectors distributions $H$-invariants pour les séries principales généralisés d’espaces symétriques réductifs et prolongement méromorphe d’intégrales d’Eisenstein, Invent. Math. 109 (1992), 619–664

Cook, J. D., Notes on Hypergeometric Functions, https://www.johndcook.com/HypergeometricFunctions.pdf, 2003

Dimock, J., Markov quantum fields on a manifold, Rev. Math. Phys. 16:2 (2004), 243–255

van Dijk, G., “Introduction to Harmonic Analysis and Generalized Gelfand Pairs,” Studies in Math. 36, de Gruyter, Berlin, 2009

van Dijk, G., and S.C. Hille, Canonical representations related to hyperbolic spaces, J. Funct. Anal. 147 (1997), 109–139

Faraut, J., “Analysis on Lie Groups. An Introduction,” Cambridge Studies in Advanced Mathematics 110, Cambridge University Press, 2008.

Faraut, J., and A. Koranyi, “Analysis on Symmetric Cones,” Oxford Mathematical Monographs, Oxford University Press, 1994

Faraut, J., and M. Pevzner, Berezin kernels and analysis on Makarevich spaces, Indag. Mathes. N. S. 16 (2005), 461–486

Faraut, J., and E.G.F. Thomas, Invariant Hilbert spaces of holomorphic functions, J. Lie Theory 9 (1999), 383–402

Faraut, J., and T. Koornwinder, Positive definite spherical functions on a noncompact, rank one symmetric space. In “Analyse Harmonique sur les Groupes de Lie (Sém., Nancy -Strasbourg 1976–1978), II,” 249–282, LNM 739, Springer, Berlin, 1979

Gindikin, G., B. Krótz and G. Ólafsson, Hardy spaces for non-compactly causal symmetric spaces and the most continuous spectrum, Math. Ann. 327:1 (2003), 25–66

Gindikin, P. E. T. and G. Ólafsson, Unitary representations of Lie groups with reflection symmetry, J. Funct. Anal. 158:1 (1998), 26–88

Hille, S. C., Canonical representations, Thesis, Leiden Univ., June 1999

Hilgert, J., and K.-H. Neeb, “Structure and Geometry of Lie Groups,” Springer Monographs in Mathematics, Springer, New York, 2012

Hilgert, J., and G. Ólafsson, “Causal Symmetric Spaces, Geometry and Harmonic Analysis,” Perspectives in Mathematics 18, Academic Press, 1996

Hildebrandt, J., “Differential Geometry, Lie Groups, and Symmetric Spaces,” Acad. Press, London, 1978

Helgason, S., “Groups and Geometric Analysis,” Academic Press, 1984

Hilgert, J., and G. Ólafsson, “Unitary representations of Lie groups with reflection symmetry,” J. Funct. Anal. 158:1 (1998), 26–88

Jaffe, A., Quantum field theory and relativity, in “Group Representations, Ergodic Theory, and Mathematical Physics. A tribute to George W. Mackey,” Doran, Robert S. (ed.) et al., Providence, RI, American Mathematical Society (AMS), Contemporary Mathematics 449 (2008), 209–245

Jaffe, A., and G. Ritter, Quantum field theory on curved backgrounds. I. The euclidean functional integral, Comm. Math. Phys. 270 (2007), 545–572

Jaffe, A., Quantum field theory on curved backgrounds. II. Spacetime symmetries, arXiv:hep-th/0704.0052v1

Jaffe, A., Reflection positivity and monotonicity, J. Math. Phys. 49 (2008), 052301, 10 pp

Jorgensen, P. E. T. and G. Ólafsson, Unitary representations of Lie groups with reflection symmetry, J. Funct. Anal. 158:1 (1998), 26–88

Jørgensen, P. E. T. and G. Ólafsson, Unitary representations and Osterwalder–Schrader duality, In “The Mathematical Legacy of Harish-Chandra (Baltimore, MD, 1998),” Proc. Sympos. Pure Math. 68, 333–401, Amer. Math. Soc., Providence, RI, 2000

Klein, A. and L. Landau, Periodic Gaussian Osterwalder–Schrader positive processes and the two-sided Markov property on the circle, Pac. J. Math. 94 (1981), 341–367

Krótz, B., and E. Opdam, Analysis on the crown domain, Geom. Funct. Anal. 18:4 (2008), 1326001421
[KS04] Krötz, B and R. J. Stanton, Holomorphic extensions of representations. I. Automorphic functions, Annals of Mathematics, 159 (2004), 641–724

[KS05] —, Holomorphic extensions of representations. II. Geometry and harmonic analysis, Geom. Funct. Anal. 15:1 (2005), 190–245

[L73] Lebedew, N. N., “Spezielle Funktionen und ihre Anwendung”, B. I. AG, Zürich, 1973

[Lo69] Loos, O., “Symmetric spaces I: General theory,” W. A. Benjamin, Inc., New York, Amsterdam, 1969

[Lo67] Love, E. R., Some integral equations involving hypergeometric functions, Proc. Edinburgh Math. Soc. 15 (1967), 169–198

[LP18] Liu, G., A. Parthasarathy, Domains of holomorphy for irreducible admissible uniformly bounded Banach representations of simple Lie groups, Transform. Groups 23 (2018), 755–764

[MNÓ14] Merigon, S., K.-H. Neeb, and G. Ólafsson, Integrability of unitary representations on reproducing kernel spaces, Representation Theory 19 (2015), 24–55

[NÓ14] Neeb, K.-H., G. Ólafsson, Reflection positivity and conformal symmetry, J. Funct. Anal. 266 (2014), 2174–2224

[NÓ15a] —, Reflection positive one-parameter groups and dilations, Complex Analysis and Operator Theory 9:3 (2015), 653–721

[NÓ15b] —, Reflection positivity for the circle group, in “Proceedings of the 30th International Colloquium on Group Theoretical Methods,” Journal of Physics: Conference Series 597 (2015), 012004; arXiv:math.RT.1411.2439

[NÓ17] —, Antisunitary representations and modular theory, in “50th Sophus Seminar”, Eds. K. Grabowska et al; Banach Center Publications 113 (2017), 291–362; arXiv:math:RT.1704.01336

[NÓ18] —, Reflection Positivity— A Representation Theoretic Perspective,” Springer Briefs in Mathematical Physics 32, 2018

[NÓ19] —, Symmetric spaces with dissecting involutions, to appear in Transformation Groups, arXiv:1907.07740

[Ner99] Neretin, Y., Plancherel formula for Berezin deformation of L₂ on Riemannian symmetric spaces, arXiv:math.RT.9911020v1 3 Nov 1999

[Ó00] Ólafsson, G., Analytic continuation in representation theory and harmonic analysis. In: “Global Analysis and Harmonic Analysis,” eds. J. P. Bourguignon, T. Branson, and O. Hijazi, Sém. et Congr., Soc. Math. France 4 (2000), 201–233;

[ÓP04] Ólafsson, G. and A. Pasquale, Paley-Wiener Theorem for the Θ-spherical Transform: The Even Multiplicity case, Journal Math. Pures Appl. 83:7 (2004), 869–927

[ÓO96] Ólafsson, G., and B. Ørsted, Generalization of the Bargmann Transform, in “Workshop on Lie Theory and its Applications in Physics, Clausthal, August 1995”, eds. Dobrev et al, World Scientific, 1996

[OS73] Osterwalder, K., and R. Schrader, Axioms for Euclidean Green’s functions. I, Comm. Math. Phys. 31 (1973), 83–112

[OS75] —, Axioms for Euclidean Green’s functions. II, Comm. Math. Phys. 42 (1975), 281–305

[Ru73] Rudin, W., “Functional Analysis,” McGraw Hill, 1973

[Se07] Seppäläinen, H., Branching laws for minimal holomorphic representations, J. Funct. Anal. 251:1 (2007), 174–209

[Se07b] —, Branching of some holomorphic representations of SO(2, n), J. Lie Theory 17:1 (2007), 191–227

[Str83] Strichartz, R. S., Analysis of the Laplacian on the complete Riemannian manifold, J. Funct. Anal 52:1 (1983), 48–79

[T63] Takahashi, R., Sur les représentations unitaires des groupes de Lorentz généralisés, Bull. Soc. Math. France 91 (1963), 289–433

[T67] Treves, F., “Topological Vector Spaces, Distributions, and Kernels,” Academic Press, New York, 1967

[WW96] Whittaker, E. T., and G. Watson, “A Course of Modern Analysis,” 4th ed, reprint, Cambridge University Press, 1996

DEPARTMENT OF MATHEMATICS, FRIEDRICH-ALEXANDER-UNIVERSITY OF ERLangen-NUREMBERG, CAUERSTRASSE 11, 91058 ERLangen, GERMANY
E-mail address: neeb@math.fau.de

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATon ROUGE, LA 70803, U.S.A.
E-mail address: olafsson@math.lsu.edu