Quantum Integrals of Motion for the Heisenberg Spin Chain

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An explicit expression for all the quantum integrals of motion for the isotropic Heisenberg $s = 1/2$ spin chain is presented. The conserved quantities are expressed in terms of a sum over simple polynomials in spin variables. This construction is direct and independent of the transfer matrix formalism. Continuum limits of these integrals in both ferromagnetic and antiferromagnetic sectors are briefly discussed.

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1. Over the years, the study of quantum spin chains has provided many important results both in the theory of magnetism and in mathematical physics. The oldest and still one of the most interesting models of this type is the isotropic (XXX) Heisenberg spin chain, with the Hamiltonian

\[ H = g \sum_{i \in \Lambda} S_i S_{i+1}, \]  

(1)

where \( \Lambda \) is the spin lattice and \( g \) the coupling. The normalization of the spin variables \( S^a_i \) is chosen to be

\[ [S^a_i, S^b_k] = 2i \delta_{jk} \epsilon^{abc} S^c_k, \]  

(2)

(i.e. \( S^a_k \) is a Pauli sigma matrix, acting nontrivially only on the \( k \)-th factor of the tensor product Hilbert space \( \bigotimes_{j \in \Lambda} \mathbb{C}^2 \)). The mathematical structure arising from this innocuous Hamiltonian is astonishingly rich. The key feature accounting for it is quantum integrability, i.e. the existence of a complete set of mutually commuting integrals of motion.

The isotropic Heisenberg chain and its anisotropic generalizations (XXZ, XYZ) are one of the simplest quantum integrable systems and as such ideal laboratories for developing techniques for the study of more complicated models. Moreover, these lattice models have quite interesting continuum limits: the quantum nonlinear Schrödinger (NLS) equation in the ferromagnetic sector \( (g < 0) \) and the quantum Thirring (or equivalently sine-Gordon) model in the antiferromagnetic regime \( (g > 0) \). We can thus hope that results obtained in the lattice case can be transported to these highly nontrivial quantum field theories. One of the outstanding problems in these continuous theories is the explicit construction of the conservation laws. With this in mind, we have initiated a study of the conserved charges in spin chains, looking first at efficient ways of calculating their explicit forms, and then deriving their continuum limits. On the first issue, we end up with remarkably simple and compact expressions for all the conservation laws of the XXX model, valid for a finite chain with periodic boundary conditions and an infinite chain. These are reported below, following a brief review of previously known results. The second issue, which has met with less success, is briefly discussed at the end of this letter.

2. The quantum charges for a spin chain are usually defined by means of the transfer matrix \( T \), which for the XXX model, is a function of a single spectral parameter \( \lambda \). Building on the Lieb’s solution of the two-dimensional classical ice-type models [1], Sutherland [2] showed that \([H, T(\lambda)] = 0\). Independently, Baxter [3], in a more general context, proved
the key property of the transfer matrix: \([T(\lambda), T(\mu)] = 0\). This implies that the logarithmic derivatives of \(T\),
\[
Q_n \equiv 2i \frac{d^{n-1}}{d\lambda^{n-1}} \ln T^{-1}(\lambda_0)T(\lambda)|_{\lambda=\lambda_0},
\]
where \(\lambda_0 = i/2\) in the usual parametrization [4], mutually commute:
\[
[Q_n, Q_m] = 0.
\]
The hamiltonian is related to \(Q_2\) by \(H = gQ_2 + \text{const}\); higher charges correspond to hamiltonians with more neighbors interacting. As shown by Lüscher [5], these charges are local operators, that is, they can be put in the form:
\[
Q_n = \sum_{\{i_1, \ldots, i_{n-1}\}} G^T_{n-1}(i_1, \ldots, i_{n-1}),
\]
where the summation is over ordered subsets \(\{i_1, \ldots, i_{n-1}\}\) of \(\Lambda\), and \(G^T\) is a translationally covariant and totally symmetric function, obeying the locality property:
\[
G^T_n(i_1, \ldots, i_n) = 0, \quad \text{for } |i_n - i_1| \geq n.
\]
Some additional properties of the XXX charges, in particular their completeness, have been proved in [6]. Although the integrals are implicitly known, it is difficult to extract explicit formulae from (3), even by using computer programs for symbolic computations, since the size of the transfer matrix grows exponentially with the length of the chain.

There exists however a shortcut. The boost operator, given by the first moment of the hamiltonian (and which turns out to be the derivative of the logarithm of the Baxter’s corner transfer matrix [7]),
\[
B = \frac{1}{2i} \sum_{j \in \Lambda} j S_j S_{j+1},
\]
has been shown [8] to obey
\[
[B, T(\lambda)] = \frac{\partial}{\partial\lambda} T(\lambda).
\]
This immediately implies that, up to an additive constant,
\[
[B, Q_n] = Q_{n+1}.
\]
$B$ is thus a master-symmetry (see [9]). Eq. (9) provides a convenient way for a recursive calculation of conserved charges for an infinite chain. Using (9) we get, up to additive constants:

$$Q_3 = \sum_{j \in \Lambda} (S_j \times S_{j+1}) \cdot S_{j+2},$$

$$Q_4 = 2 \sum_{j \in \Lambda} (((S_j \times S_{j+1}) \times S_{j+2}) \cdot S_{j+3} + S_j \cdot S_{j+2} - 4Q_2,$$

$$Q_5 = 6 \sum_{j \in \Lambda} (((S_j \times S_{j+1}) \times S_{j+2}) \times S_{j+3}) \cdot S_{j+4} + (S_j \times S_{j+2}) \cdot S_{j+3}$$

$$+ (S_j \times S_{j+1}) \cdot S_{j+3} - 18Q_3.$$  

(10)

Notice that for a finite chain there appear additional boundary terms in (9). Nevertheless, the expressions (10) are also valid in this case, if addition in $\Lambda$ is understood modulo $N$, where $N$ is the number of spins. These expressions suggest a natural pattern for the structure of the quantum charges, which we describe and prove below.

3. Before proceeding further, we need to introduce some notation. A sequence of $n > 2$ spin variables, $C = \{S_{i_1}, \ldots, S_{i_n}\}$, with $i_1 < i_2 < \ldots < i_n$, will be called a cluster of order $n$; if the ordering condition is not met, the sequence will be called a disordered cluster. Clusters of a given order can be further classified by specifying their “holes”, that is the sites between $i_1$ and $i_n$ that are not included in $C$. The number of holes in $C$ is clearly $k = i_n - i_1 + 1 - n$. Obviously, $k = 0$ for a cluster containing only contiguous spins. We denote as $C^{(n,k)}$ the set of all clusters of $\Lambda$ of order $n$ with $k$ holes. For instance, $C^{(3,1)}$ contains $\{S_1, S_2, S_4\}$, $\{S_1, S_3, S_4\}$ and all their translations.

For any sequence of spins $C$, we define $V_m(C)$ as the vector product of the first $m$ spins, with products nested toward the left, e.g.

$$V_1 = S_{i_1},$$

$$V_2 = S_{i_1} \times S_{i_2},$$

$$V_3 = (S_{i_1} \times S_{i_2}) \times S_{i_3},$$

$$\ldots$$

$$V_{m+1} = V_m \times S_{i_m}.$$  

(11)

From these vectors, we construct scalar $n$-linear polynomials in spin variables

$$f_n(C) = V_{n-1} \cdot S_{i_n}.$$  

(12)
In particular, one has
\[ f_0 = f_1 = 0, \quad f_2 = S_{i_1} S_{i_2}, \]
\[ f_3 = (S_{i_1} \times S_{i_2}) \cdot S_{i_3}, \quad f_4 = ((S_{i_1} \times S_{i_2}) \times S_{i_3}) \cdot S_{i_4}. \] (13)

The \( f_n \)'s satisfy an interesting property which is that the dot product can be placed at an arbitrary position, provided that parentheses to its left (right) are nested toward the left (right), e.g:
\[ f_5 = (((S_{i_1} \times S_{i_2}) \times S_{i_3}) \times S_{i_4}) \cdot (S_{i_5} \times S_{i_6}) = S_{i_1} \cdot (S_{i_2} \times (S_{i_3} \times (S_{i_4} \times S_{i_5}))). \] (14)

This is a direct consequence of the familiar vector identity:
\[ (A \times B) \cdot C = A \cdot (B \times C). \] (15)

Finally, we define
\[ F_{n,k} = \sum_{C \in \mathcal{C}(n,k)} f_n(C). \] (16)

The conserved charges can be expressed in a very simple way as linear combinations of the quantities \( F_{n,k} \). It is easily seen that \( F_{2,0} = Q_2 \) and \( F_{3,0} = Q_3 \). For \( n > 3 \) the charges \( Q_n \) obtained from (9) contain terms proportional to lower order charges. It will be more convenient to express the charges in a transformed basis, denoted \( \{ H_n \} \), in which these lower order contributions are stripped off. Our explicit expression for \( H_n \) can be most simply visualized as the sum of the vertices of the tree in fig. 4, with all vertices contributing with unit weight. In particular, one has
\[ H_2 = F_{2,0} = g^{-1} H, \]
\[ H_3 = F_{3,0}, \]
\[ H_4 = F_{4,0} + F_{2,1}, \]
\[ H_5 = F_{5,0} + F_{3,1}, \]
\[ H_6 = F_{6,0} + F_{4,1} + F_{2,2} + F_{2,1}, \]
\[ H_7 = F_{7,0} + F_{5,1} + F_{3,2} + F_{3,1}, \]
\[ H_8 = F_{8,0} + F_{6,1} + F_{4,2} + F_{4,1} + F_{2,3} + 2F_{2,2} + 2F_{2,1}, \]
\[ H_9 = F_{9,0} + F_{7,1} + F_{5,1} + F_{5,2} + F_{3,3} + 2F_{3,2} + 2F_{3,1}, \]
\[ H_{10} = F_{10,0} + F_{8,1} + F_{6,2} + F_{6,1} + F_{4,3} + 2F_{4,2} + 2F_{4,1} + F_{2,4} + 3F_{2,3} + 5F_{2,2} + 5F_{2,1}. \] (17)
Note that the trees describing $H_{2m}$ and $H_{2m+1}$ have identical structures.

The algebraic translation of this construction yields the general expression:

$$H_n = F_{n,0} + \sum_{k=1}^{[n/2]-1} \sum_{\ell=1}^{k} \alpha_{k,\ell} F_{n-2k,\ell},$$

where the square bracket indicates integer part and the coefficients $\alpha_{k,\ell}$ are defined via the recurrence relation:

$$\alpha_{k+1,\ell} = \sum_{m=\ell-1}^{k} \alpha_{k,m},$$

with $\alpha_{1,1} = 1$ and $\alpha_{k,0} = 0$. Notice that $\alpha_{k,1} = \alpha_{k,2}$ for $k \geq 2$. The recurrence relation (19) can be rewritten in the form:

$$\alpha_{k,\ell} = \alpha_{k-1,\ell-1} + \alpha_{k,\ell+1},$$

with the understanding that $\alpha_{k,\ell} = 0$ if $\ell > k$. This is the defining relation for the generalized Catalan numbers, $\alpha_{k,\ell} = C_{2k-\ell-1,\ell}$, with $C_{n,m}$ given by

$$C_{n,m} = \left( \frac{n-1}{p} \right) - \left( \frac{n-1}{p-2} \right),$$

where $\left( \frac{a}{b} \right)$ are the binomial coefficients, with $p = [(n-m+1)/2]$, $m+n$ odd and $m < n+2$. In particular, $\alpha_{k,1} = C_{2k-1,1}$ are the usual Catalan numbers.

For the XXX chain of length $N$ with periodic boundary conditions, the construction above yields $N-1$ charges $\{H_2, \ldots, H_N\}$, which are clearly independent of each other. To complete this set we may take any of the three components of the total spin, $H_1^a = \sum_{j \in \Lambda} S_j^a$. Charges for the infinite XXX chain are similarly given by the sequence $\{H_1^a, H_2, \ldots, H_n, \ldots\}$.

4. Below we sketch our proof that $\{H_n\}$ is a family of conserved charges in involution. (A complete argument will be published elsewhere.) First, we note that since $F_{n,k}$ are invariant under global spin rotation, $[H_1^a, H_n] = 0$. Next we will show that $[H_2, H_n] = 0$, by evaluating directly the commutators:

$$[H_2, f_n(C)] = \sum_{j \in \Lambda} [S_j S_{j+1}, f_n(C)].$$
Remarkably, this commutator contains only terms expressible in terms of the polynomials $f$, namely $f_{n \pm 1}(C')$, where $C'$ can be obtained from $C = \{ S_{i_1}, \ldots, S_{i_n} \}$ using a few simple transformations:

\[
S_{i_1-1}C \equiv \{ S_{i_1-1}, S_{i_1}, \ldots, S_{i_n} \},
\]

\[
CS_{i_n+1} \equiv \{ S_{i_1}, \ldots, S_{i_n}, S_{i_n+1} \},
\]

\[
C_{i_k} \equiv \{ S_{i_1}, \ldots, S_{i_{k-1}}, S_{i_{k+1}}, \ldots, S_{i_n} \},
\]

\[
C_{s_i \rightarrow s_j s_{i+1}} \equiv \{ S_{i_1}, \ldots, S_{i_j-1}, S_{i_k}, S_{i_j}, S_{i_{j+1}}, \ldots, S_{i_n} \},
\]

with the last operation being defined only if $S_{i_k}, S_{i_m}$ are not in $C_{i_j}$. For $n < N$ the calculation gives:

\[
[H_2, f_n(C)] = a_{n+1,k}(C) + b_{n-1,k+1}(C) + d_{n+1,k-1}(C) + e_{n-1,k}(C) + r(C),
\]

where

\[
a_{n+1,k}(C) = -2if_{n+1}(S_{i_1-1}C) + 2if_{n+1}(CS_{i_n+1}),
\]

\[
b_{n-1,k+1}(C) = -4if_{n-1}(C_{i_2})\delta_{i_1+1,i_2} + 4if_{n-1}(C_{i_{n-1}})\delta_{i_{n-1}+1,i_n} + 2i \sum_{j=2}^{n-2} [f_{n-1}(C_{i_j}) - f_{n-1}(C_{i_{j+1}})]\delta_{i_{j+1},i_{j+1}},
\]

\[
d_{n+1,k-1}(C) = 2if_{n+1}(S_{i_1}S_{i_1+1}C_{i_1})(1 - \delta_{i_1,i_2-1}) - 2if_{n+1}(C_{i_n}S_{i_n-1}S_{i_n})(1 - \delta_{i_{n-1},i_{n-1}+1}),
\]

\[
e_{n-1,k}(C) = 4if_{n-1}(C_{i_1})\delta_{i_1+1,i_2} - 4if_{n-1}(C_{i_{n-1}})\delta_{i_{n-1}+1,i_n},
\]

\[
r(C) = -2i \sum_{j=2}^{n-2} f_{n-1}(C_{s_j \rightarrow s_j s_{i+1}})(1 - \delta_{i_{j+1},i_{j+1}}) + 2i \sum_{j=3}^{n-1} f_{n+1}(C_{s_j \rightarrow s_{i-1} s_i})(1 - \delta_{i_{j-1},i_{j-1}}).
\]

The reader is cautioned to distinguish between $S_{i_j+1}$ and $S_{i_j+1}$ in the above formulae. Note that, with the exception of $r(C)$, all of the terms on the left hand-side of (24) involve only ordered clusters. Summing up over all possible clusters, we get:

\[
[H_2, F_{n,k}] = A_{n+1,k} + D_{n+1,k-1} + B_{n-1,k+1},
\]

where $A_{n+1,k} = \sum_c a_{n+1,k}(C)$, $B_{n-1,k+1} = \sum_c b_{n-1,k+1}(C)$, $D_{n+1,k-1} = \sum_c d_{n+1,k-1}(C)$. Due to symmetry $\sum_c e_{n-1,k}(C) = \sum_c r(C) = 0$ (hence contributions from disordered clusters cancel). In the case $n = N$ the calculation yields:

\[
[H_2, F_{N,0}] = B_{N-1,1}.
\]
It follows from (30) and (31), that in order to prove that $H_n$ commute with the hamiltonian, it is sufficient to show that for any $n \leq N$,

(i) $[H_2, F_{n,0}]$ does not contain terms of order $n + 1$,

(ii) $[H_2, F_{n,k} + \sum_{\ell=1}^{k+1} F_{n-2,\ell}]$ does not contain terms of order $n - 1$.

The assertion (i) is immediate. If $n = N$, it follows from (31). If $n < N$, it follows from the fact that, due to translational symmetry, $A_{n+1,0} = 0$. (ii) is equivalent to:

$$B_{n-1,k+1} + \sum_{\ell=1}^{k+1} (A_{n-1,\ell} + D_{n-1,\ell-1}) = 0.$$  \hspace{1cm} (32)

The above sum contains contributions of clusters of order $n - 1$, with hole numbers ranging from 0 to $k + 1$; thus the sum vanishes if and only if the individual contributions vanish, which can be proved by a tedious calculation using (25)-(27).

Having established that the $H_n$’s commute with the hamiltonian, we have yet to show that they commute among themselves. We also want to express the logarithmic derivatives of the transfer matrix in the basis $\{H_n\}$. To this end, we calculate the commutators of $H_n$ ($n \geq 2$) with the boost operator:

$$[B, H_n] = \max(1, \lfloor n/2 \rfloor - 1) \sum_{m=0}^{\lfloor n/2 \rfloor - 1} \beta_m^{(n)} H_{n+1-2m},$$  \hspace{1cm} (33)

where

$$\beta_0^{(n)} = n - 1, \quad \beta_1^{(n > 2)} = 5 - 3n, \quad \beta_1^{(n < 1 < \ell < \lfloor n/2 \rfloor)} = -(n - 2\ell - 1)\alpha_{\ell,1}.$$  \hspace{1cm} (34)

From (33) it is clear that $Q_n$ of even (odd) order $n$ can be expressed as a linear combination of the $H_m$ with even (odd) $m \leq n$:

$$Q_n = \sum_{p=0}^{[n/2]-1} \gamma_p^{(n)} H_{n-2p}.$$  \hspace{1cm} (35)

The coefficients $\gamma$ satisfy the recurrence relation:

$$\gamma_{\ell}^{(n+1)} = \sum_{p, m \geq 0} \gamma_p^{(n)} \beta_m^{(n-2p)},$$  \hspace{1cm} (36)

with $\gamma_p^{(2)} = \delta_{p,0}$. In particular, modulo additive constants,

$$Q_4 = 2H_4 - 4H_2,$$

$$Q_5 = 6H_5 - 18H_3,$$

$$Q_6 = 24H_6 - 96H_4 + 72H_2.$$  \hspace{1cm} (37)
Since $\{Q_n\}$ form a family of conserved charges in involution, (cf. (4)), it then follows from (33) that all of the $H_n$ mutually commute. This can be also proved directly, without using (4), by an inductive argument based on (33) and the Jacobi identity.

5. The ferromagnetic sector of the XXX chain with an arbitrary spin $s$ has a non-relativistic dispersion relation, which can be brought to light by means of the Holstein-Primakoff transformation \(10\). Under this transformation the chain is mapped to a lattice version of the quantum nonlinear Schrödinger (NLS) equation \(11\), whose continuous Hamiltonian is

$$H_{NLS} = \int_{-\infty}^{\infty} (\Psi^+ x \Psi_x + \kappa \Psi^+ \Psi \Psi) dx,$$

with $[\Psi^+ (x), \Psi(y)] = \delta(x - y)$. The coupling constant $\kappa$ is related to the spin $s$ by $\kappa = -2/(s\Delta)$, $\Delta$ being the lattice constant. Finding explicit forms of the conserved charges in this system is an interesting open problem, which has been recently investigated by several authors \(12\). To make contact with our results for the $s = 1/2$ chain, one must take the limit $\kappa \to \infty$ (the impenetrable boson system). In this limit the conserved charges have the form:

$$H_{2n} \to \int_{-\infty}^{\infty} \sum_{m=1}^{n} a_m^{(n)} (\Psi^+)^m(x) \Psi^m(y) \delta^{n-m+1}(x - y) dx dy,$$

$$H_{2n+1} \to \int_{-\infty}^{\infty} \sum_{m=1}^{n} b_m^{(n)} (\Psi^+)^m(x) \Psi_y(y) \Psi^{m-1}(y) \delta^{n-m+1}(x - y) dx dy,$$

where $\delta^k$ denotes a suitably regularized $k$-th power of the Dirac delta function. (An extended discussion of these results will be published elsewhere.) The continuum limit of the $s = 1/2$ XXX charges using the Holstein-Primakoff transformation, does indeed lead to integrals of this type. However, in general the limiting process is ambiguous as far as ordering of operators is concerned (e.g. on the lattice $\Psi_i \Psi^+_j = \Psi^+_j \Psi_i$ if $i \neq j$, but these two expressions have different continuous limits). We have not yet found a prescription which fixes these ambiguities and reproduces the values of the coefficients in (39).

In the antiferromagnetic sector, the continuum limit of the $s = 1/2$ XXX spin chain is a special case of the massive Thirring model \(13\), related to the sine-Gordon model through bosonization \(14\). It can be equivalently described by a level one $SU(2)$ Wess-Zumino-Novikov-Witten (WZNW) model with a marginal perturbation \(15\). In the framework of the latter model, by replacing $S^a_j$ by the the current algebra generators $J^a(x_j)$, and taking the lattice constant $\Delta \to 0$, one gets the holomorphic part of the conserved charges:
$H_2 \rightarrow \int T \, dx, \ H_3 \rightarrow 0$ (this generalizes to all $H_{2m+1}$), $H_4 \rightarrow \int (TT) \, dx$, where $T$ is the Sugawara energy-momentum tensor and the parenthesis denotes the standard normal ordering in conformal field theory. These conserved integrals are common to the quantum KdV equation [16] and, via a Feigen-Fuchs transformation, the quantum sine-Gordon equation. However, calculation of the limits of the higher order charges, needed to probe the value of the WZNW central charge $c = 1$, is again plagued by ordering ambiguities.

6. In this work we have presented a simple and compact expression for the conservation laws of the $s = 1/2$ XXX Heisenberg chain. Our construction of the conserved charges is independent of the transfer matrix formalism and uses only the algebraic relations (30), (31), and (32). It thus provides an alternative and direct way of proving the integrability of the XXX chain. It is an interesting question whether this construction can be generalized to the anisotropic case or to other integrable spin chains.

After the completion of this work, we became aware of [17], which presents an expression for the basis of the space of quantum integrals of motion of the infinite XXX chain, given without proof nor with any indication of how it has been found. This basis is different from ours but we have checked that the two yield equivalent results. However, the logarithmic derivatives of the transfer matrix have not been given in [17].
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Figure Captions

Fig. 1. The tree structure corresponding to $H_n$. The tree stops with the terms $F_{2,\ell}$ ($F_{3,\ell}$) when $n$ is even (odd).
This figure "fig1-1.png" is available in "png" format from:

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