Multiplicity one theorems for Fourier-Jacobi models

Binyong Sun

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MULTIPLICITY ONE THEOREMS FOR FOURIER-JACOBI MODELS

By BINYONG SUN

Abstract. For every genuine irreducible admissible smooth representation \( \pi \) of the metaplectic group \( \widetilde{Sp}(2n) \) over a \( p \)-adic field, and every smooth oscillator representation \( \omega_\psi \) of \( \widetilde{Sp}(2n) \), we prove that the tensor product \( \pi \otimes \omega_\psi \) is multiplicity free as a smooth representation of the symplectic group \( Sp(2n) \). Similar results are proved for general linear groups and unitary groups.

1. Introduction. Fix a non-archimedean local field \( k \) of characteristic zero. The following multiplicity one theorem for general linear groups, unitary groups and orthogonal groups, which has been expected by J. Bernstein and S. Rallis since the 1980s, was established recently by Aizenbud-Gourevitch-Rallis-Schiffmann in [AGRS10].

THEOREM A. Let \( G \) denote the group \( GL(n) \), \( U(n) \), or \( O(n) \), defined over \( k \), and let \( G' \) denote \( GL(n-1) \), \( U(n-1) \), or \( O(n-1) \), respectively, regarded as a subgroup of \( G \) as usual. Then for any irreducible admissible smooth representation \( \pi \) of \( G \), and \( \pi' \) of \( G' \), one has that

\[
\dim \text{Hom}_{G'}(\pi \otimes \pi', \mathbb{C}) \leq 1.
\]

As will be clear later, the groups \( GL(n) \), \( U(n) \) and \( O(n) \) are automorphism groups of “Hermitian modules”. Therefore, we consider Theorem A the multiplicity one theorem in the “Hermitian case”. It is the first step towards the famous Gross-Prasad Conjecture [GP92, GP94, GR06, GGP].

In [GGP], W. T. Gan, B. Gross, and D. Prasad formulate an analog of the Gross-Prasad Conjecture in the “skew-Hermitian case”. The corresponding multiplicity one theorem, whose proof is the main goal of this paper, is the following:

THEOREM B. Let \( G \) denote the group \( GL(n) \), \( U(n) \), or \( Sp(2n) \), defined over \( k \), and regarded as a subgroup of the symplectic group \( Sp(2n) \) as usual. Let \( \widetilde{G} \) be the double cover of \( G \) induced by the metaplectic cover \( \widetilde{Sp}(2n) \) of \( Sp(2n) \). Denote by \( \omega_\psi \) the smooth oscillator representation of \( \widetilde{Sp}(2n) \) corresponding to a non-trivial character \( \psi \) of \( k \). Then for any irreducible admissible smooth representation \( \pi \) of \( G \), and any genuine irreducible admissible smooth representation \( \pi' \) of \( \widetilde{G} \), one has...
that

$$\dim \text{Hom}_G(\pi' \otimes \omega \otimes \pi, \mathbb{C}) \leq 1.$$ 

Recall that an irreducible admissible smooth representation of $\tilde{G}$ is said to be genuine if it does not descend to a representation of $G$. For symplectic groups, Theorem B is conjectured by D. Prasad in [Pr96, p. 20].

The “Hom”-spaces in Theorem A and Theorem B are extreme cases of their very important generalizations, namely, Bessel models and Fourier-Jacobi models, respectively. For definitions of these models, see [GGP, Part 3], for example. As explained in [GPSR97], uniqueness of Bessel models is the basic starting point to study L-functions for orthogonal groups by the Rankin-Selberg method. Similarly, uniqueness of Fourier-Jacobi models is basic to study L-functions for symplectic groups and metaplectic groups ([GJRS]). The importance of Theorem A and Theorem B lies in the fact that they imply uniqueness of these models in general (cf. [GGP, Part 3], [GJRS, Theorem 4.1], and [JSZ10]). Various special cases of these uniqueness are obtained in the literature (see [Nov76, GPSR87, BFG92, GRS98, BR00, GGP] for example).

In order to prove our main results uniformly, we introduce the following notation. By a commutative involutive algebra (over $k$), we mean a finite product of finite field extensions of $k$, equipped with a $k$-algebra involution on it. Let $(A, \tau)$ be a commutative involutive algebra. Let $E$ be a finitely generated $A$-module. For $\epsilon = \pm 1$, recall that a $k$-bilinear map

$$\langle \cdot, \cdot \rangle_E : E \times E \longrightarrow A$$

is called an $\epsilon$-Hermitian form if it satisfies

$$\langle u, v \rangle_E = \epsilon \langle v, u \rangle_E, \quad \langle au, v \rangle_E = a \langle u, v \rangle_E, \quad a \in A, \ u, v \in E.$$ 

Assume that $E$ is an $\epsilon$-Hermitian $A$-module, namely it is equipped with a non-degenerate $\epsilon$-Hermitian form $\langle \cdot, \cdot \rangle_E$. Denote by $U(E)$ the group of all $A$-module automorphisms of $E$ which preserve the form $\langle \cdot, \cdot \rangle_E$. Depending on $\epsilon = 1$ or $-1$, it is a finite product of general linear groups, unitary groups, and orthogonal or symplectic groups.

View $A^2$ as a standard hyperbolic plane, i.e., it is equipped with the $\epsilon$-Hermitian form $\langle \cdot, \cdot \rangle_{A^2}$ so that both $e_1$ and $e_2$ are isotropic, and that

$$\langle e_1, e_2 \rangle_{A^2} = 1,$$

where $e_1$, $e_2$ is the standard basis of $A^2$. The orthogonal direct sum $E \oplus A^2$ is again an $\epsilon$-Hermitian $A$-module. Note that $U(E)$ is identified with the subgroup of $U(E \oplus A^2)$ fixing both $e_1$ and $e_2$. Denote by $J(E)$ the subgroup of $U(E \oplus A^2)$ fixing $e_1$. 
To be explicit, view $E \oplus A^2$ as the space of column vectors of $A \oplus E \oplus A$, then $J(E)$ consists of all matrices of the form

$$j(x,u,t) := \begin{bmatrix} 1 & -u^\tau & t - \frac{\langle u, u \rangle_E}{2} \\ 0 & x & xu \\ 0 & 0 & 1 \end{bmatrix},$$

where $x \in U(E)$, $u \in E$, $t \in A^{\tau = -\epsilon} := \{ t \in A \mid t^\tau = -\epsilon t \}$, and $u^\tau$ is the map

$$E \rightarrow A, \quad v \mapsto \langle v, u \rangle_E.$$

The unipotent radical of $J(E)$ is

$$H(E) := \{ j(1,u,t) \mid u \in E, t \in A^{\tau = -\epsilon} \} = E \times A^{\tau = -\epsilon},$$

with multiplication

$$(u,t)(u',t') = (u + u', t + t' + \frac{\langle u, u' \rangle_E}{2} - \frac{\langle u', u \rangle_E}{2}).$$

By identifying $j(x,u,t)$ with $(x,(u,t))$, we have

$$J(E) = U(E) \ltimes H(E).$$

By Proposition A.1 of Appendix A, Theorem B is a consequence of the following theorem in the skew-Hermitian case, namely when $\epsilon = -1$.

**Theorem C.** For every irreducible admissible smooth representation $\pi_J$ of $J(E)$, and every irreducible admissible smooth representation $\pi_U$ of $U(E)$, one has that

$$\dim \text{Hom}_{U(E)}(\pi_J \otimes \pi_U, \mathbb{C}) \leq 1.$$

To prove Theorem C by the method of Gelfand-Kazhdan, we extend $U(E)$ to a larger group, which is denoted by $\hat{U}(E)$, and is defined to be the subgroup of $GL(E_k) \times \{ \pm 1 \}$ consists of pairs $(g, \delta)$ such that either

$$\delta = 1 \quad \text{and} \quad \langle gu, gv \rangle_E = \langle u, v \rangle_E, \quad u, v \in E,$$

or

$$\delta = -1 \quad \text{and} \quad \langle gu, gv \rangle_E = \langle v, u \rangle_E, \quad u, v \in E.$$
Here $E_k$ is the underlying $k$-vector space of $E$. Note that for every element $(g, \delta) \in \hat{\mathcal{U}}(E)$, if $\delta = 1$, then $g$ is automatically $A$-linear, and if $\delta = -1$, then $g$ is $\tau$-conjugate linear. This extended group is first introduced implicitly by Moeglin-Vigneras-Waldspurger in [MVW87, Proposition 4.I.2]. It contains $\mathcal{U}(E)$ as a subgroup of index two. Let $\hat{\mathcal{U}}(E)$ act on $H(E)$ as group automorphisms by

$$(g, \delta) \cdot (u, t) := (gu, \delta t).$$

This extends the adjoint action of $\mathcal{U}(E)$ on $H(E)$. The semidirect product

$$\hat{J}(E) := \hat{\mathcal{U}}(E) \ltimes H(E)$$

contains

$$J(E) = \mathcal{U}(E) \ltimes H(E)$$

as a subgroup of index two.

By Corollary B.2 of Appendix B, Theorem C is implied by:

**Theorem D.** Let $f$ be a generalized function on $J(E)$. If it is invariant under the adjoint action of $\mathcal{U}(E)$, i.e.,

$$f(gjg^{-1}) = f(j), \quad \text{for all } g \in \mathcal{U}(E),$$

then

$$f(\hat{g}j\hat{g}^{-1}j^{-1}) = f(j), \quad \text{for all } \hat{g} \in \hat{\mathcal{U}}(E) \setminus \mathcal{U}(E).$$

The usual notion of generalized functions, as well as the idea of the proof of Theorem D, will be explained in the next section.

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2. **Proof of Theorem D.** We first recall some basic notions and facts about distributions and generalized functions. By a t.d. space, we mean a topological space which is Hausdorff, secondly countable, locally compact and totally disconnected. By a t.d. group, we mean a topological group whose underlying topological space is a t.d. space. For a t.d. space $M$, denote by $C_0^\infty(M)$ the space of compactly supported, locally constant (complex valued) functions on $M$. Denote by $D^{-\infty}(M)$ the space of linear functionals on $C_0^\infty(M)$. Such functionals are called distributions on $M$. If $M$ is furthermore a locally analytic $k$-manifold or a t.d. group (see [Sc, Part II] for the notion of locally analytic manifolds), denote by $D_0^\infty(M)$ the space of compactly supported distributions on $M$ which are locally scalar multiples of
Haar measure. In this case, denote by \( C^{-\infty}(M) \) the space of linear functionals on \( D_0^\infty(M) \). Such functionals are called generalized functions on \( M \).

Let \( \varphi : M \to N \) be a continuous map of t.d. spaces. If it is proper, then we define the push forward map

\[ \varphi_* : D^{-\infty}(M) \to D^{-\infty}(N) \]

as usual. Recall that \( \varphi \) is said to be proper if \( \varphi^{-1}(C) \) is compact for every compact subset \( C \) of \( N \). When \( \varphi \) is a closed embedding, \( \varphi_* \) identifies \( D^{-\infty}(M) \) with distributions in \( D^{-\infty}(N) \) which are supported in \( \varphi(M) \). If \( \varphi \) is not proper, then the push forward map is still defined, but only for distributions with compact support.

If \( \varphi : M \to N \) is a submersion of locally analytic k-manifolds, then the push forward map sends \( D_0^\infty(M) \) into \( D_0^\infty(N) \), and its transpose defines the pull back map

\[ \varphi^* : C^{-\infty}(N) \to C^{-\infty}(M) \]

When \( \varphi \) is a surjective submersion, \( \varphi^* \) is injective.

If \( G \) is an (abstract) group acting continuously on a t.d. space \( M \), then for any group homomorphism \( \chi_G : G \to \mathbb{C}^\times \), put

\[ D_{\chi_G}^{-\infty}(M) := \{ \omega \in D^{-\infty}(M) \mid (T_g)_* \omega = \chi_G(g) \omega, \ g \in G \}, \]

where \( T_g : M \to M \) is the map given by the action of \( g \in G \). If furthermore \( M \) is a locally analytic k-manifold, and the action of \( G \) on it is also locally analytic, denote by

\[ C_{\chi_G}^{-\infty}(M) \subset C^{-\infty}(M) \]

the subspace consisting of all \( f \) which are \( \chi_G \)-equivariant, i.e.,

\[ f(g.x) = \chi_G(g)f(x), \quad \text{for all } g \in G, \]

or to be precise,

\[ T_g^*(f) = \chi_G(g)f, \quad \text{for all } g \in G. \]

Now we return to the notation of the Introduction. Recall that \( E \) is an \( \epsilon \)-Hermitian \( A \)-module. Denote by

\[ \chi_E : \hat{U}(E) \to \{ \pm 1 \}, \quad (g, \delta) \mapsto \delta \]

the quadratic character projecting to the second factor. Let \( \hat{U}(E) \) act on \( J(E) \) by

\[ \hat{g}.j := \hat{g}j\chi_E(\hat{g})\hat{g}^{-1}. \]

Then Theorem D is clearly equivalent to:
**Theorem D'.** One has that $C_{\chi_E}^{-\infty}(J(E)) = 0$.

The Lie algebra of (the $k$-linear algebraic group) $U(E)$ is

$$u(E) := \{ x \in \text{End}_A(E) \mid \langle xu, v \rangle_E + \langle u, xv \rangle_E = 0, \ u, v \in E \}.$$  

Let $\hat{U}(E)$ act on $u(E)$ and $E$ by

$$(g, \delta) \cdot x := \delta gxg^{-1}, \ x \in u(E),$$

$$(g, \delta) \cdot u := \delta gu, \ u \in E,$$

and act on $u(E) \times E$ diagonally. The linear version of Theorem D' is:

**Theorem D''.** One has that $C_{\chi_E}^{-\infty}(u(E) \times E) = 0$.

A commutative involutive algebra is said to be simple if it is either a field or a product of two isomorphic fields which are exchanged by the involution. Write

$$(A, \tau) = (A_1, \tau_1) \times (A_2, \tau_2) \times \cdots \times (A_k, \tau_k)$$

as a product of simple commutative involutive algebras. Then

$$E = E_1 \times E_2 \times \cdots \times E_k,$$

where $E_j := A_j \otimes_A E$ is obviously an $\epsilon$-Hermitian $A_j$-module. Note that $E_j$ is free as an $A_j$-module. Put

$$\text{sdim}(E) := \dim_k(E) + \sum_{j=1}^k \max \{ \text{rank}_{A_j}(E_j) - 1, 0 \}.$$  

Denote by $Z(E)$ the center of $U(E)$, and by $\mathcal{U}_E$ the set of unipotent elements in $U(E)$. The following result is proved in Section 3:

**Proposition 2.1.** Assume that for all commutative involutive algebras $A^\circ$ and all $\epsilon$-Hermitian $A^\circ$-modules $E^\circ$,

$$\text{sdim}(E^\circ) < \text{sdim}(E) \quad \text{implies} \quad C_{\chi_{E^\circ}}^{-\infty}(J(E^\circ)) = 0.$$  

Then every $f \in C_{\chi_E}^{-\infty}(J(E))$ is supported in $(Z(E)\mathcal{U}_E) \times H(E)$.

With the idea of linearization by Jaquet-Rallis [JR96] in mind, and based on Proposition 2.1, we prove the following implication in Section 4.

**Proposition 2.2.** Theorem D'' implies Theorem D'.

Now we need to prove Theorem D''. Clearly, the case of simple $(A, \tau)$ implies Theorem D'' in general. Note that Theorem D'' is known when $(A, \tau)$ is simple and $\epsilon = 1$. (This is the linear version of the main results (Theorem 2 and Theorem
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2′) of [AGRS10].) When \( \epsilon = -1 \), \((A, \tau)\) is simple and \( \tau \) is nontrivial, take a nonzero element \( c_A \in A \) such that \( c_A^\tau = -c_A \), then \( (E, c_A \langle , \rangle_E) \) is a \(-\epsilon\)-Hermitian \( A \)-module, and Theorem D′′ reduces to the case of \( \epsilon = 1 \). Therefore it remains to prove Theorem D′′ in the symplectic case, namely, when \( A \) is a field, \( \tau \) is trivial, and \( \epsilon = -1 \).

Denote by \( \mathfrak{z}(E) \) the Lie algebra of \( \mathbb{Z}(E) \). It consists of all elements in \( u(E) \) which are scalar multiplications (by certain elements of \( A \)). Denote by \( \mathcal{N}_E \) the set of all elements in \( u(E) \) which are nilpotent as \( k \)-linear operators on \( E \). We first reduce the problem to the null cone as in Proposition 2.1:

**Proposition 2.3.** Assume that for all commutative involutive algebras \( A^0 \) and all \( \epsilon \)-Hermitian \( A^0 \)-modules \( E^0 \),

\[
\text{sdim}(E^0) < \text{sdim}(E) \implies C_{\chi_{E^0}}^{-\infty}(u(E^0) \times E^0) = 0.
\]

Then every \( f \in C_{\chi_E}^{-\infty}(u(E) \times E) \) is supported in \( (\mathfrak{z}(E) + \mathcal{N}_E) \times E \).

The proof of Proposition 2.3 is similar to that of Proposition 2.1, and is also carried out in Section 3.

Let

\[
\mathcal{N}_E = \mathcal{N}_0 \supset \mathcal{N}_1 \supset \cdots \supset \mathcal{N}_r = \{0\} \supset \mathcal{N}_{r+1} = \emptyset
\]

be a filtration of \( \mathcal{N}_E \) by its closed subsets so that each difference

\[
\mathcal{O}_i := \mathcal{N}_i \setminus \mathcal{N}_{i+1}, \quad 0 \leq i \leq r,
\]

is a \( \tilde{U}(E) \)-orbit (which is also a \( U(E) \)-orbit by [MVW87, Proposition 4.1.2]).

**Proposition 2.4.** Assume that \((A, \tau)\) is simple. Fix \( i = 0, 1, \ldots, r \). If every generalized function in \( C_{\chi_E}^{-\infty}(u(E) \times E) \) is supported in \( (\mathfrak{z}(E) + \mathcal{N}_i) \times E \), then every generalized function in \( C_{\chi_E}^{-\infty}(u(E) \times E) \) is supported in \( (\mathfrak{z}(E) + \mathcal{N}_{i+1}) \times E \).

Recall that the nilpotent orbit \( \mathcal{O}_i \) is said to be distinguished if one (or every) element of it commutes with no non-scalar semisimple element in \( u(E) \) (cf. [CM93, Section 8.2]). Use an uncertainty theorem for distributions with supports (Theorem C.1 of Appendix C), we prove Proposition 2.4 for non-distinguished \( \mathcal{O}_i \) in Section 5. We have to go case by case for the proof of Proposition 2.4 for distinguished \( \mathcal{O}_i \). It is carried out in the symplectic case in Section 6. As explained before, this is the only case which is not done in [AGRS10].

Now we are prepared to prove Theorem D′′ by induction on \( \text{sdim}(E) \). If \( \text{sdim}(E) = 0 \), then \( E = 0 \) and Theorem D′′ is trivial. Assume that \( \text{sdim}(E) > 0 \) and Theorem D′′ is proved when \( \text{sdim}(E) \) is smaller. Without loss of generality, assume that \((A, \tau)\) is simple. By Proposition 2.3, every \( f \in C_{\chi_E}^{-\infty}(u(E) \times E) \) is supported in \( (\mathfrak{z}(E) + \mathcal{N}_E) \times E \), and it has to vanish by Proposition 2.4. This proves Theorem D′′.
3. Proofs of Propositions 2.1 and 2.3. We continue with the notation of the last section. The involution \( \tau \) on \( A \) extends to an anti-involution on \( \text{End}_A(E) \), which is still denoted by \( \tau \), by requiring that
\[
\langle x^\tau u, v \rangle_E = \langle u, xv \rangle_E, \quad u, v \in E.
\]
Recall that an element \( x \in \text{End}_A(E) \) is said to be normal if \( x \) commutes with \( x^\tau \).
For every normal semisimple element \( x \in \text{End}_A(E) \), denote by \( A_x \) the subalgebra of \( \text{End}_A(E) \) generated by \( x, x^\tau \) and scalar multiplications by \( A \). Then \( (A_x, \tau) \) is again a commutative involutive algebra. Write \( E_x := E \), viewed as an \( A_x \)-module.

**Lemma 3.1.** There is a unique \( \epsilon \)-Hermitian form \( \langle \cdot, \cdot \rangle_{E_x} \) on the \( A_x \)-module \( E_x \) such that
\[
\dim_k(A) \text{ tr}_{A_x/k}(\langle u, v \rangle_{E_x}) = \dim_k(A_x) \text{ tr}_{A/k}(\langle u, v \rangle_E), \quad u, v \in E.
\]

**Proof.** The form is determined by requiring that
\[
\dim_k(A) \text{ tr}_{A_x/k}(a \langle u, v \rangle_{E_x}) = \dim_k(A_x) \text{ tr}_{A/k}(\langle au, v \rangle_E), \quad a \in A_x, u, v \in E.
\]
Therefore \( E_x \) is an \( \epsilon \)-Hermitian \( A_x \)-module. We omit the proof of the following elementary lemma.

**Lemma 3.2.** If \( x \in U(E) \setminus Z(E) \) or \( u(E) \setminus \mathfrak{z}(E) \), and \( x \) is semisimple, then \( \text{sdim}(E_x) < \text{sdim}(E) \).

**Proof of Proposition 2.1.** Now we come to the proof of Proposition 2.1. Without loss of generality, assume that \( E \) is faithful as an \( A \)-module. Let \( x \) be a semisimple element in \( U(E) \setminus Z(E) \). Note that \( \hat{U}(E_x) \) is a subgroup of \( \hat{U}(E) \). Recall the action of \( \hat{U}(E) \) on \( J(E) \) (and similarly \( \hat{U}(E_x) \) on \( J(E_x) \)) from (2). The homomorphism
\[
\xi_x : J(E_x) = U(E_x) \ltimes (E_x \times A_x^{\tau = -}\epsilon) \rightarrow J(E) = U(E) \ltimes (E \times A^{\tau = -}\epsilon),
\]

\[
(y, (u, t)) \mapsto (y, (u, \text{ tr}_x(t))
\]

is \( \hat{U}(E_x) \)-intertwining, where \( \text{ tr}_x : A_x \rightarrow A \) is the \( A \)-linear map specified by requiring that
\[
\dim_k(A_x) \text{ tr}_{A_x/k}(\text{ tr}_x(t)) = \dim_k(A) \text{ tr}_{A/k}(t), \quad t \in A_x.
\]
Faithfulness of \( E \) implies that the map \( \text{ tr}_x \) is surjective.
For any
\[
j = (y, h) \in J(E_x) = U(E_x) \ltimes H(E_x),
\]
denote by $J(j)$ the determinant of the $k$-linear map

$$1 - \text{Ad}_{y^{-1}} : u(E)/u(E_x) \longrightarrow u(E)/u(E_x).$$

Note that $\text{Ad}_y$ preserves a non-degenerate $k$-quadratic form on $u(E)/u(E_x)$, which implies that $J$ is $\mathbb{U}(E_x)$-invariant. Put

$$J(E_x)^0 := \{ j \in J(E_x) \mid J(j) \neq 0 \}.$$ 

It contains the set $x\mathcal{U}_{E_x} \ltimes \mathbb{H}(E_x)$.

One easily checks that the map

$$\rho_x : \mathbb{U}(E) \times J(E_x)^0 \longrightarrow J(E),$$

$$(\tilde{g}, j) \mapsto \tilde{g} \cdot (\xi_x(j))$$

is a submersion, and we have a well defined map (cf. [JSZ11, Lemma 2.5])

(6) $$r_x : C_{\chi_E}^{-\infty}(J(E)) \longrightarrow C_{\chi_{E_x}}^{-\infty}(J(E_x)^0),$$

which is specified by the rule

$$\rho_x^*(f) = \chi_E \otimes r_x(f), \quad f \in C_{\chi_E}^{-\infty}(J(E)).$$

Lemma 3.2 and the assumption (5) easily imply the vanishing of the range space of (6) (cf. [JSZ11, Lemma 2.6]). Thus every $f \in C_{\chi_E}^{-\infty}(J(E))$ vanishes on the image of $\rho_x$. As $x$ is arbitrary, we finish the proof of Proposition 2.1. \hfill \Box

**Proof of Proposition 2.3.** Proposition 2.3 is proved in [SZ, Proposition 7.1]. We sketch a proof for completeness. Let $x$ be a semisimple element in $u(E) \setminus \mathcal{Z}(E)$. Recall the action of $\mathbb{U}(E)$ on $u(E) \times E$ (and similarly $\mathbb{U}(E_x)$ on $u(E_x) \times E_x$) from (3). For any $y \in u(E_x)$, denote by $J'(y)$ the determinant of the $k$-linear map

$$[y, \cdot] : u(E)/u(E_x) \longrightarrow u(E)/u(E_x).$$

Then $J'$ is a $\mathbb{U}(E_x)$-invariant function on $u(E_x)$. Put

$$u(E_x)^0 := \{ y \in u(E_x) \mid J'(y) \neq 0 \}.$$ 

It contains $x + \mathcal{N}_{E_x}$. The map

$$\rho'_x : \mathbb{U}(E) \times (u(E_x)^0 \times E_x) \longrightarrow u(E) \times E,$$

$$(\tilde{g}, y, v) \mapsto \tilde{g} \cdot (y, v)$$

is a submersion, and we finish the proof as that of Proposition 2.1. \hfill \Box
4. Proof of Proposition 2.2. This section is devoted to a proof of Proposition 2.2. So assume throughout this section that

\[ C_{X_E^0}^{-\infty}(u(E) \times E) = 0 \]

for all commutative involutive algebras \( A^\circ \) and all \( \epsilon \)-Hermitian \( A^\circ \)-modules \( E^\circ \).

We are aimed to show that \( C_{X_E^0}^{-\infty}(J(E)) = 0 \).

The Lie algebra of \( H(E) \) is

\[ \mathfrak{h}(E) := E \times A^{\tau=-\epsilon} \]

with Lie bracket given by

\[ [(u, t), (u', t')] := (0, \langle u, u' \rangle_E - \langle u', u \rangle_E). \]

The Lie algebra of \( J(E) \) is

\[ \mathfrak{j}(E) := u(E) \ltimes \mathfrak{h}(E), \]

where the semidirect product is defined by the Lie algebra action

\[ x \cdot (u, t) := (xu, 0). \]

Let \( \bar{U}(E) \) act on \( \mathfrak{j}(E) \) by the differential of its action on \( J(E) \), i.e.,

\[ \bar{g} \cdot j := \chi_E(\bar{g}) \text{Ad}_\bar{g}(j), \quad j \in \mathfrak{j}(E). \]

It is easy to see that as a \( \bar{U}(E) \)-space,

\[ \mathfrak{j}(E) = u(E) \times E \times A^{\tau=-\epsilon}, \]

where \( A^{\tau=-\epsilon} \) carries the trivial \( \bar{U}(E) \)-action. Therefore the assumption (7) implies that

\[ C_{X_E^0}^{-\infty}(\mathfrak{j}(E)) = 0. \]

We need the following obvious fact of exponential maps in the theory of linear algebraic groups.

**Lemma 4.1.** The set of unipotent elements in \( J(E) \) is \( U_E \times H(E) \), the set of algebraically nilpotent elements in \( \mathfrak{j}(E) \) is \( N_E \times \mathfrak{h}(E) \), and the exponential map is a \( \bar{U}(E) \)-intertwining homeomorphism from \( N_E \times \mathfrak{h}(E) \) onto \( U_E \times H(E) \).

In all cases that concern us, whenever \( M \) is a locally analytic k-manifold with a locally analytic \( \bar{U}(E) \)-action, there is always a canonical choice (up to a scalar) of a positive smooth invariant measure on \( M \). Therefore the space \( C_{X_E^0}(M) \) is canonically identified with \( D_{X_E^0}(M) \). We will use this observation freely.
Lemma 4.2. One has that $D_{-\infty}^{\infty}(\mathcal{U}_E \ltimes H(E)) = 0$.

Proof. By (8), we have that

$$D_{-\infty}^{\infty}(j(E)) = 0,$$

which implies that

$$D_{-\infty}^{\infty}(N_E \ltimes h(E)) = 0.$$

The lemma then follows from Lemma 4.1.

Recall the following localization principle which is due to Bernstein. See [Be84, section 1.4] or [AGRS10, Corollary 2.1].

Lemma 4.3. Let $\varphi : M \to N$ be a continuous map of t.d. spaces, and let $G$ be a group acting continuously on $M$ preserving the fibers of $\varphi$. Then for any group homomorphism $\chi_G : G \to \mathbb{C}^\times$, the condition

$$D_{-\infty}^{\infty}(\varphi^{-1}(x)) = 0 \quad \text{for all } x \in N$$

implies that

$$D_{-\infty}^{\infty}(M) = 0.$$

We use Lemma 4.2 and the localization principle to prove the following:

Lemma 4.4. One has that $D_{-\infty}^{\infty}(Z(E)\mathcal{U}_E \ltimes H(E)) = 0$.

Proof. Note that $z\mathcal{U}_E \ltimes H(E)$ is $\tilde{U}(E)$-stable for every $z \in Z(E)$, and that (by using the trace map) the map

$$Z(E)\mathcal{U}_E \ltimes H(E) \to Z(E), \quad (zx, h) \mapsto z, \quad (x \in \mathcal{U}_E)$$

is a well defined continuous map. By the localization principle, it suffices to show that

(9) $D_{-\infty}^{\infty}(z\mathcal{U}_E \ltimes H(E)) = 0, \quad \text{for all } z \in Z(E)$.

Given $z \in Z(E)$, denote by $T_z$ the left multiplication by $z$. One easily checks that the diagram

$$\begin{array}{ccc}
\mathcal{U}_E \ltimes H(E) & \xrightarrow{T_z} & z\mathcal{U}_E \ltimes H(E) \\
g \downarrow & & g \downarrow \\
\mathcal{U}_E \ltimes H(E) & \xrightarrow{T_z} & z\mathcal{U}_E \ltimes H(E)
\end{array}$$
commutes for all $g \in U(E)$, and the diagram

$$
\begin{array}{c}
\mathcal{U}_E \times H(E) \xrightarrow{T_z} z\mathcal{U}_E \times H(E) \\
\downarrow \bar{g}z \downarrow \bar{g} \\
\mathcal{U}_E \times H(E) \xrightarrow{T_z} z\mathcal{U}_E \times H(E)
\end{array}
$$

commutes for all $\bar{g} \in \bar{U}(E) \setminus U(E)$, where all vertical arrows are given by the actions of the indicated elements. Therefore (9) is a consequence of Lemma 4.2. □

We now prove Proposition 2.2 by induction on $\text{sdim}(E)$. Assume that we have proven the proposition when $\text{sdim}(E)$ is smaller. Then Proposition 2.1 implies that every $T \in D_{\chi E}^\infty(J(E))$ is supported in $(Z(E)\mathcal{U}_E) \times H(E)$, and then $T = 0$ by Lemma 4.4. This finishes the proof.

5. Proof of Proposition 2.4: non-distinguished orbits. View $u(E)$ as a quadratic space over $k$ under the trace form

$$(10) \quad \langle x, y \rangle_{u(E)} := \text{tr}_{A/k}(\text{tr}(xy)).$$

Then we have an orthogonal decomposition

$$u(E) = \mathfrak{z}(E) \oplus \mathfrak{su}(E),$$

where $\mathfrak{su}(E)$ is the space of trace free elements in $u(E)$. Recall that $O_i = \mathcal{N}_i \setminus \mathcal{N}_{i+1}$ is a nilpotent $\hat{U}(E)$-orbit. It is clearly contained in $\mathfrak{su}(E)$.

**Lemma 5.1.** [SZ, Lemma 6.1] If $O_i$ is non-distinguished and $o \in O_i$, then there is a non-isotropic vector in $\mathfrak{su}(E)$ which is perpendicular to the tangent space $T_o(O_i) \subset \mathfrak{su}(E)$.

**Proof.** By definition, $o$ commutes with a nonzero semisimple element $h \in \mathfrak{su}(E)$. Denote by $a_h$ the center of $\mathfrak{su}(E)^h$ (the centralize of $h$ in $\mathfrak{su}(E)$), which is a nonzero non-degenerate subspace of $\mathfrak{su}(E)$.

Using the fact that every element of $a_h$ commutes with $o$, we see that the tangent space

$$T_o(O_i) = [u(E), o] = [\mathfrak{su}(E), o]$$

is perpendicular to $a_h$. □

Recall the action (3) of $\hat{U}(E)$ on $u(E)$ and $E$. Write $E' := E$ as an $\epsilon$-Hermitian $A$-module, but equipped with the action of $\hat{U}(E)$ given by

$$(g, \delta).u := gu.$$
Define a non-degenerate $\hat{\mathcal{U}}(E)$-invariant bilinear map
\[
\langle \cdot, \cdot \rangle : (u(E) \times E) \times (u(E) \times E') \rightarrow k
\]
by
\[
\langle (x, u), (x', u') \rangle_i := \langle x, x' \rangle_{u(E)} + \text{tr}_{A/k} \langle (u, u')_E \rangle.
\]
Fix a nontrivial character $\psi$ of $k$. As in Appendix C, for every distribution $T \in D^{-\infty}(u(E) \times E)$, define its Fourier transform $\hat{T} \in C^{-\infty}(u(E) \times E')$ by
\[
\hat{T}(\omega) := T(\hat{\omega}), \quad \omega \in D_0^\infty(u(E) \times E'),
\]
where $\hat{\omega} \in C_0^\infty(u(E) \times E)$ is given by
\[
\hat{\omega}(j) := \int_{u(E) \times E'} \psi(\langle j, j' \rangle_i) d\omega(j'), \quad j \in u(E) \times E.
\]

**Lemma 5.2.** Assume that $O_i$ is non-distinguished. Let $T \in D^{-\infty}(u(E) \times E)$. If $T$ is supported in $(\mathfrak{g}(E) \oplus \mathcal{N}_i) \times E$, and its Fourier transform $\hat{T} \in C^{-\infty}(u(E) \times E')$ is supported in the null cone
\[
(11) \quad \{ (z + x, u) \in u(E) \times E' | z \in \mathfrak{g}(E), \quad x \in su(E), \quad \langle x, x \rangle_{u(E)} = 0 \},
\]
then $T$ is supported in $(\mathfrak{g}(E) \oplus \mathcal{N}_{i+1}) \times E$.

**Proof.** This is a direct consequence of Lemma 5.1 and Theorem C.1 of Appendix C. \qed

**Proof of Proposition 2.4 for non-distinguished $O_i$.** Let $T \in D_{X_E}^{-\infty}(u(E) \times E)$. Then by the assumption of Proposition 2.4 it is supported in $(\mathfrak{g}(E) \oplus \mathcal{N}_i) \times E$. It is clear that the Fourier transform maps $D_{X_E}^{-\infty}(u(E) \times E)$ into $C_{X_E}^{-\infty}(u(E) \times E')$. By noting that $-1 \in U(E)$, we find that the space $C_{X_E}^{-\infty}(u(E) \times E')$ is identical to the space $C_{X_E}^{-\infty}(u(E) \times E)$. Apply the assumption to $\hat{T}$, we find that $\hat{T}$ is supported in $(\mathfrak{g}(E) \oplus \mathcal{N}_i) \times E'$, which is contained in the null cone (11). By Lemma 5.2, this proves Proposition 2.4 in the case that $O_i$ is non-distinguished. \qed

### 6. Proof of Proposition 2.4: distinguished orbits.

As explained in Section 2, when $O_i$ is distinguished, we prove Proposition 2.4 only in the symplectic case. So assume that $\epsilon = -1$, $A$ is a field, and $\tau$ is trivial. Then $u(E)$ is a symplectic Lie algebra and $\mathfrak{g}(E) = 0$. For simplicity of notation and without loss of generality, we further assume that $A = k$.

For all $v \in E$, put
\[
\phi_v(u) := \langle u, v \rangle_E v, \quad u \in E.
\]
One easily checks that $\phi_v \in u(E)$. For all $o \in \mathcal{O}_i$, put
\[
E(o) := \{ v \in E \mid \phi_v \in [u(E), o] \}.
\]

**Lemma 6.1.** If every distribution in $D^{-\infty}_{\chi_E}(u(E) \times E)$ is supported in $N_i \times E$, then the support of every distribution in $D^{-\infty}_{\chi_E}(u(E) \times E)$ is contained in
\[
(N_{i+1} \times E) \cup \bigcup_{o \in \mathcal{O}_i} \{ o \} \times E(o).
\]

**Proof.** We follow the method of [AGRS10]. Let $T \in D^{-\infty}_{\chi_E}(u(E) \times E)$ and $(o, v) \in \mathcal{O}_i \times E$ be a point in the support of $T$. It suffices to prove that $v \in E(o)$.

For every $t \in k$, define a homeomorphism 
\[
\eta_t : u(E) \times E \to u(E) \times E,
\]
\[
(x, u) \mapsto (x + t\phi_v, u),
\]
which is checked to be $\check{U}(E)$-intertwining. Therefore
\[
(\eta_t)_* T \in D^{-\infty}_{\chi_E}(u(E) \times E).
\]

Since $(o, v)$ is in the support of $T$, $\eta_t(o, v)$ is in the support of $(\eta_t)_* T$. Therefore the assumption implies that
\[
(12) \quad \eta_t(o, v) = (o + t\phi_v, v) \in N_i \times E.
\]

As $\mathcal{O}_i$ is open in $N_i$, (12) implies that
\[
\phi_v \in T_o(\mathcal{O}_i) = [u(E), o]. \quad \square
\]

Fix an element $e \in \mathcal{O}_i$. Extend it to a standard triple $h, e, f$ in $u(E)$, i.e., the $k$-linear map from $\mathfrak{sl}_2(k)$ to $u(E)$ specified by
\[
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mapsto h, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mapsto e, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mapsto f,
\]
is a Lie algebra homomorphism. Existence of such an extension is known as Jacobson-Morozov Theorem. Using this homomorphism, we view $E$ as an $\mathfrak{sl}_2(k)$-module with an invariant symplectic form. In the remaining part of this section assume that $\mathcal{O}_i$ is distinguished. By the classification of distinguished nilpotent orbits ([CM93, Theorem 8.2.14]), we know that $E$ has an orthogonal decomposition
\[
(13) \quad E = E_1 \oplus E_2 \oplus \cdots \oplus E_s, \quad s \geq 0,
\]
where all $E_j$’s are irreducible $\mathfrak{sl}_2(k)$-submodules, with pairwise different even dimensions. Denote by $E^+$ and $E^-$ the subspaces of $E$ spanned by eigenvectors of
\( h \) with positive and negative eigenvalues, respectively. Then

\[
E = E^+ \oplus E^-
\]
is a complete polarization of \( E \).

**Lemma 6.2.** (cf. [SZ, Lemma 4.1]) One has that \( E(e) = E^+ \).

**Proof.** Recall that \( u(E) \) is a quadratic space over \( k \) under the trace form (10). For every \( v \in E \), we have that \( v \in E(e) \) if and only if

\[
\phi_v \in [u(E), e] \iff \phi_v \perp [u(E), e]^\perp \\
\iff \phi_v \perp u(E)^e \quad \text{(the centralizer of } e \text{ in } u(E)) \\
\iff \langle xv, v \rangle_E = 0 \quad \text{for all } x \in u(E)^e.
\]

Thus if \( v \in E(e) \), then we have

\[
\langle e_j^{2k+1}, v \rangle_E = 0 \quad \text{for all } 1 \leq j \leq s \text{ and } k \geq 0,
\]
where \( e_j \) is the restriction of \( e \) to \( E_j \) under the decomposition (13). Therefore \( v \in E^+ \).

On the other hand, every element \( x \in u(E)^e \) stabilizes \( E^+ \). Therefore \( v \in E^+ \) implies that \( \langle xv, v \rangle_E = 0 \). This finishes the proof. \( \square \)

Fix a Haar measure \( du' \) on \( E' \). For any t.d. space \( M \), we define the partial Fourier transform

\[
\mathcal{F}_E : D^{-\infty}(M \times E) \to D^{-\infty}(M \times E')
\]
by

\[
\mathcal{F}_E(T)(\varphi_M \otimes \varphi') := T(\varphi_M \otimes \hat{\varphi'}), \quad \varphi_M \in C_0^\infty(M), \varphi' \in C_0^\infty(E'),
\]
where \( \hat{\varphi'} \in C_0^\infty(E) \) is given by

\[
\hat{\varphi'}(u) := \int_{E'} \psi(\langle u, u' \rangle_E) \varphi'(u') \, du'.
\]

For every \( o \in \mathcal{O}_i \), write \( E'(o) := E(o) \), viewed as a subset of \( E' \). Lemma 6.2 implies the following

**Lemma 6.3.** Let \( T \in D^{-\infty}(E) \). If \( T \) is supported in \( E(e) \), and \( \mathcal{F}_E(T) \in D^{-\infty}(E') \) is supported in \( E'(e) \), then \( T \) is a scalar multiple of a Haar measure of \( E^+ \).
Proof. Since \( \mathcal{F}_E(T) \) is supported in \( E'(e) = E^+ \), \( T \) is invariant under translations by elements of
\[
\left\{ u \in E \mid \langle u, u' \rangle_E = 0, \ u' \in E'(e) \right\} = E^+.
\]
\( \square \)

Denote by \( \hat{U}(E, e) \) the stabilizer of \( e \in \mathcal{O}_i \) in \( \hat{U}(E) \), and by \( \chi_{E,e} \) the restriction of \( \chi_E \) to \( \hat{U}(E, e) \).

**Lemma 6.4.** Let \( T \in D^{-\infty}_{\chi_{E,e}}(E) \). If \( T \) is supported in \( E(e) \), and \( \mathcal{F}_E(T) \) is supported in \( E'(e) \), then \( T = 0 \).

**Proof.** By Lemma 6.3, \( T \) is a scalar multiple of a Haar measure of \( E^+ \). Note that all eigenvalues of \( h \) on \( E \) are odd integers. Let \( g : E \to E \) be the linear map which is the scalar multiplication by \( (-1)^n \) on the \( h \)-eigenspace with eigenvalue \( 2n + 1, n \in \mathbb{Z} \). It is clear that \( (g, -1) \in \hat{U}(E, e) \), and leaves the Haar measure of \( E^+ \)-invariant. This finishes the proof. \( \square \)

Fix a positive \( \hat{U}(E) \)-invariant measure \( do \) on \( \mathcal{O}_i \) (which always exists), and a Haar measure \( d\hat{g} \) on \( \hat{U}(E) \). Define a submersion
\[
\rho_e : \hat{U}(E) \times E \longrightarrow \mathcal{O}_i \times E;
\]
\[
\hat{g}, v \longmapsto \hat{g} \cdot (e, v),
\]
and define the pull back
\[
\rho_e^* : D^{-\infty}(\mathcal{O}_i \times E) \longrightarrow D^{-\infty}(\hat{U}(E) \times E);
\]
\[
f \cdot do \otimes du \longmapsto \rho_e^*(f) \cdot d\hat{g} \otimes du,
\]
where \( du \) is any Haar measure on \( E \), \( f \in C^{-\infty}(\mathcal{O}_i \times E) \), and \( \rho_e^*(f) \) is the usual pull back of a generalized function. By Frobenius reciprocity (cf. [Be84, Section 1.5]), there is a well defined linear isomorphism
\[
(14) \quad r_e : D_{\chi_{E,e}}^{-\infty}(\mathcal{O}_i \times E) \sim \longrightarrow D_{\chi_{E,e}}^{-\infty}(E),
\]
specified by
\[
\rho_e^*(T) = \chi_{E} \cdot d\hat{g} \otimes r_e(T), \quad T \in D_{\chi_{E}}^{-\infty}(\mathcal{O}_i \times E).
\]
Similarly, by using the action of \( \hat{U}(E) \) on \( \mathcal{O}_i \times E' \), we define a map
\[
\rho'_e^* : D^{-\infty}(\mathcal{O}_i \times E') \longrightarrow D^{-\infty}(\hat{U}(E) \times E'),
\]
and a linear isomorphism
\[
r'_e : D_{\chi_{E,e}}^{-\infty}(\mathcal{O}_i \times E') \sim \longrightarrow D_{\chi_{E,e}}^{-\infty}(E').
\]

The routine verification of the following lemma is left to the reader.
Lemma 6.5. The diagram

\[
\begin{array}{ccc}
D_{\chi(E)}^{-\infty}(O_i \times E) & \xrightarrow{r_e} & D_{\chi(E,e)}^{-\infty}(E) \\
\mathcal{F}_E & \downarrow & \mathcal{F}_E \\
D_{\chi(E)}^{-\infty}(O'_i \times E') & \xrightarrow{r'_e} & D_{\chi(E,e)}^{-\infty}(E')
\end{array}
\]

commutes.

Now we are ready to prove Proposition 2.4 for distinguished $O_i$ (in the symplectic case). Let $T \in D_{\chi_E}^{-\infty}(N_i \times E)$. Then Lemma 6.1 implies that
\[
r_e(T|_{O_i \times E}) \in D_{\chi(E,e)}^{-\infty}(E)
\]
is supported in $E(e)$.

Similarly, since $\mathcal{F}_E(T) \in D_{\chi_E}^{-\infty}(N_i \times E')$, we have
\[
r'_e(\mathcal{F}_E(T)|_{O_i \times E'}) \in D_{\chi(E,e)}^{-\infty}(E')
\]
is supported in $E'(e)$.

Lemma 6.4 and Lemma 6.5 implies that
\[
r_e(T|_{O_i \times E}) = 0,
\]
which implies that $T|_{O_i \times E} = 0$. This finishes the proof.

Appendix A. Some Mackey theory. Let $k$ be a non-archimedean local field of characteristic zero, and let

\[
(A, \tau) = \begin{cases} 
(k \times k, \text{ the nontrivial automorphism}), \\
(\text{a quadratic field extension of } k, \text{ the nontrivial automorphism}), \\
\text{or } \\
(k, \text{ the trivial automorphism}).
\end{cases}
\]

It is a commutative involutive algebra over $k$. Let $\epsilon = -1$, and let $(E, \langle , \rangle_E)$ be an $\epsilon$-Hermitian $A$-module. Then the group $U(E)$ is $GL(n)$, $U(n)$, or $Sp(2n)$, respectively, with $2n = \dim_k(E)$. Write $E_k := E$, viewed as a symplectic $k$-vector space under the form

\[
\langle u, v \rangle_{E_k} := \frac{1}{\dim_k(A)} \text{tr}_{A/k} \left( \langle u, v \rangle_E \right).
\]

Recall the Heisenberg group

\[
H(E) = E \times A^{\tau=\epsilon} = E_k \times k = H(E_k)
\]

with group multiplication

\[
(u, t)(u', t') = (u + u', t + t' + \frac{\langle u, u' \rangle_E}{2} - \frac{\langle u', u \rangle_E}{2}) = (u + u', t + t' + \langle u, u' \rangle_{E_k}).
\]
Denote by $\widetilde{\text{Sp}}(E_k)$ the metaplectic cover of the symplectic group $\text{Sp}(E_k)$. It induces a double cover $\widetilde{U}(E)$ of $U(E) \subset \text{Sp}(E_k)$. For any non-trivial character $\psi$ of $k$, denote by $\omega_\psi$ the corresponding smooth oscillator representation of

$$\text{(15)} \quad \widetilde{\text{Sp}}(E_k) \rtimes H(E_k).$$

Up to isomorphism, this is the only genuine smooth representation which, as a representation of $H(E_k)$, is irreducible and has central character $\psi$. We regard $\omega_\psi$ as a representation of $\widetilde{U}(E) \rtimes H(E)$, as the later is a subgroup of (15).

The following Mackey-theoretic result is known (cf. [AP06, Page 222]). We provide a proof for the sake of completeness.

**Proposition A.1.** With the notation as above, for every genuine irreducible admissible smooth representation $\pi_{\tilde{U}}$ of $\widetilde{U}(E)$, the tensor product $\pi_{\tilde{U}} \otimes \omega_\psi$ is an irreducible admissible smooth representation of $J(E) = U(E) \rtimes H(E)$.

**Proof.** The smoothness and admissibility are clear. We prove that $\pi_{\tilde{U}} \otimes \omega_\psi$ is irreducible as a smooth representation of $U(E) \rtimes H(E)$. The space

$$H_0 := \text{Hom}_{H(E)}(\omega_\psi, \pi_{\tilde{U}} \otimes \omega_\psi)$$

is a smooth representation of $\widetilde{U}(E)$ under the action

$$(\tilde{g} \cdot \phi)(v) := g \cdot \left( \phi(\tilde{g}^{-1} \cdot v) \right),$$

where

$$\tilde{g} \in \widetilde{U}(E), \ \phi \in H_0, \ v \in \omega_\psi,$$

and $g$ is the image of $\tilde{g}$ under the quotient map $\widetilde{U}(E) \to U(E)$. Let $\pi_J$ be a nonzero $U(E) \rtimes H(E)$-subrepresentation of $\pi_{\tilde{U}} \otimes \omega_\psi$, then

$$\text{Hom}_{H(E)}(\omega_\psi, \pi_J)$$

is a nonzero $\widetilde{U}(E)$-subrepresentation of $H_0$. Since the linear map

$$\pi_{\tilde{U}} \longrightarrow H_0, \ v \longmapsto v \otimes (\cdot)$$

is bijective and $\widetilde{U}(E)$-intertwining, $H_0$ is irreducible. Therefore

$$\text{Hom}_{H(E)}(\omega_\psi, \pi_J) = H_0,$$

and consequently, $\pi_J = \pi_{\tilde{U}} \otimes \omega_\psi$. \qed
Appendix B. The Gelfand-Kazhdan criterion for multiplicity one pairs.

Recall the notion of t.d. groups from Section 3. The following result is a form of the Gelfand-Kazhdan criterion for multiplicity one pairs.

**Proposition B.1.** Let $G$ be a t.d. group with a closed subgroup $S$. Let $\sigma$ be a continuous anti-automorphism of $G$ such that $\sigma(S) = S$. Assume that for every generalized function $f$ on $G$ or on $S$, the condition
\[ f(sxs^{-1}) = f(x) \quad \text{for all } s \in S \]
implies that
\[ f(x^\sigma) = f(x). \]
Then for all irreducible admissible smooth representation $\pi_G$ of $G$, and $\pi_S$ of $S$, one has that
\[ \dim \text{Hom}_S(\pi_G \otimes \pi_S, \mathbb{C}) \leq 1. \]

**Proof.** This is proved for real reductive groups in [SZ11, Corollary 2.5]. The same proof works here. We sketch a proof for convenience of the reader.

Denote by $\Delta(S)$ the diagonal subgroup $S$ of $G \times S$. The assumption on $G$ implies that every bi-$\Delta(S)$ invariant generalized function on $G \times S$ is $\sigma \times \sigma$-invariant. Then the usual Gelfand-Kazhdan criterion (cf. [SZ11, Theorem 2.3]) implies that
\[ \dim \text{Hom}_S(\pi_G \otimes \pi_S, \mathbb{C}) \cdot \dim \text{Hom}_S(\pi_G^\vee \otimes \pi_S^\vee, \mathbb{C}) \leq 1. \]

(16)

Here and henceforth, "\(\vee\)" stands for the contragredient of an admissible smooth representation.

Denote by $\sigma'$ the automorphism $g \mapsto \sigma(g^{-1})$. By considering characters of irreducible admissible smooth representations (which are conjugation invariant generalized functions on the groups), the assumption implies that
\[ \pi_G^\vee \cong \pi_G^{\sigma'} \quad \text{and} \quad \pi_S^\vee \cong \pi_S^{\sigma'}. \]

(17)

Here $\pi_G^{\sigma'}$ is the representation of $G$ which has the same underlying space as that of $\pi_G$, and whose action is given by $g \mapsto \pi_G(\sigma'(g))$. The representation $\pi_S^{\sigma'}$ is defined similarly. Therefore the two factors in (16) are equal to each other, and consequently,
\[ \dim \text{Hom}_S(\pi_G \otimes \pi_S, \mathbb{C}) \leq 1. \]

For unimodular groups, we have:

**Corollary B.2.** Let $G$ be a unimodular t.d. group with a unimodular closed subgroup $S$. Let $\sigma$ be a continuous anti-automorphism of $G$ such that $\sigma(S) = S$. 
Assume that for every generalized function \( f \) on \( G \), the condition
\[ f(\sigma s^{-1}) = f(s) \quad \text{for all } s \in S \]
implies that
\[ f(x^\sigma) = f(x). \]
Then for all irreducible admissible smooth representation \( \pi_G \) of \( G \), and \( \pi_S \) of \( S \), one has that
\[ \dim \text{Hom}_S(\pi_G \otimes \pi_S, \mathbb{C}) \leq 1. \]

Proof. When both \( G \) and \( S \) are unimodular, the assumption of the corollary implies the assumption of Proposition B.1. Therefore the corollary is a consequence of Proposition B.1. \( \square \)

Appendix C. An uncertainty theorem for distributions with supports.
Let \( k \) be a non-archimedean local field of characteristic zero. Fix a non-trivial character \( \psi \) of \( k \). Let \( E \) and \( F \) be two finite-dimensional \( k \)-vector spaces which are dual to each other, i.e., a non-degenerate bilinear map
\[ \langle , \rangle : E \times F \rightarrow k \]
is given. The Fourier transform
\[ D_0^\infty(F) \rightarrow C_0^\infty(E) \]
is the linear isomorphism given by
\[ \hat{\omega}(x) := \int_F \psi(\langle x, y \rangle) d\omega(y), \quad x \in E. \]
For every \( T \in D^{-\infty}(E) \), its Fourier transform \( \hat{T} \in C^{-\infty}(F) \) is given by
\[ \hat{T}(\omega) := T(\hat{\omega}), \quad \omega \in D_0^\infty(F). \]
For every subset \( X \) of \( E \), a point \( x \in X \) is said to be regular if there is an open neighborhood \( U \) of \( x \) in \( E \) such that \( U \cap X \) is a closed locally analytic submanifold of \( U \). In this case, the tangent space \( T_x(X) \subset E \) is defined as usual. We define the conormal space to be
\[ N_x^*(X) := \{ v \in F \mid \langle u, v \rangle = 0, \ u \in T_x(X) \}. \]
The uncertainty principle says that a distribution and its Fourier transform cannot be simultaneously arbitrarily concentrated. The purpose of this appendix is to prove the following theorem, which is a form of the uncertainty principle.
**Theorem C.1.** Let $x$ be a regular point in a close subset $X$ of $E$. Let $f : F \to k$ be a polynomial function of degree $d \geq 1$, and denote by $f_d$ its homogeneous component of degree $d$. Let $T \in D^{-\infty}(E)$ be a distribution supported in $X$, with its Fourier transform $\hat{T} \in C^{-\infty}(F)$ supported in the zero locus of $f$. If $f_d$ take nonzero values at some points of $N^*_x(X)$, then $T$ vanishes on some open neighborhood of $x$ in $E$.

**Remark.** The archimedean analog of Theorem C.1 also holds. This is a direct consequence of [JSZ11, Lemma 2.2].

Fix a non-archimedean multiplicative norm $|\cdot|_k : k \to [0, +\infty)$ which defines the topology of $k$. Also fix a non-archimedean norm (multiplicative with respect to $|\cdot|_k$)

$$|\cdot|_F : F \to [0, +\infty),$$

which automatically defines the topology of $F$.

Let $f$ and $f_d$ be as in Theorem C.1, and denote by $Z_f \subset F$ the zero locus of $f$. Write $f_0 := f - f_d$, which is a polynomial function of degree $\leq d - 1$. Then

(18) $$|f_0(y)|_k = o(|y|^d_F), \quad \text{as } |y|_F \to +\infty.$$

**Lemma C.2.** Let $V_1$ and $V_2$ be two compact open subsets of $F$. If $f_d$ has no zero in $V_2$, then

$$Z_f + V_1 \cap \lambda V_2 = \emptyset$$

for all $\lambda \in k^\times$ with $|\lambda|_k$ sufficiently large.

**Proof.** Take a positive number $c$ so that

(19) $$|f_d(y)|_k \geq c|y|^d_F, \quad \text{for all } y \in k^\times V_2.$$

It is easy to see that

(20) $$\max_{v \in V_1} |f_d(y + v) - f_d(y)|_k = o(|y|^d_F), \quad \text{as } |y|_F \to +\infty.$$

If $y \in Z_f$, then

(21) $$|f_d(y + v)|_k \leq \max \{ |f_d(y)|_k, |f_d(y + v) - f_d(y)|_k \}$$

$$= \max \{ |f_0(y)|_k, |f_d(y + v) - f_d(y)|_k \}.$$

The inequalities (18), (20) and (21) implies that

(22) $$|f_d(y)|_k = o(|y|^d_F), \quad \text{as } y \in Z_f + V_1, \text{ and } |y|_F \to +\infty.$$
The lemma then follows by comparing (19) and (22). \qed

Recall the following:

**Definition C.3.** (cf. [He85, Section 2]) A distribution $T \in D^{-\infty}(E)$ is said to be smooth at a point $(x,y) \in E \times F$ if there is a compact open neighborhood $U$ of $x$, and a compact open neighborhood $V$ of $y$ such that the Fourier transform $\hat{1}_UT$ vanishes on $\lambda V$ for all $\lambda \in k^\times$ with $|\lambda|_k$ sufficiently large. Here $1_U$ stands for the characteristic function of $U$. The wave front set of $T$ at $x \in E$ is defined to be

$$WF_x(T) := \{ y \in F \mid T \text{ is not smooth at } (x,y) \}.$$  

Clearly, the wave front set $WF_x(T)$ is closed in $F$ and is stable under multiplications by $k^\times$.

**Lemma C.4.** If the Fourier transform $\hat{T}$ of a distribution $T \in D^{-\infty}(E)$ is supported in $Z_f$, then for every $x \in E$, the wave front set $WF_x(T)$ is contained in the zero locus of $f_d$.

**Proof.** Let $y \in F$ be a vector so that $f_d(y) \neq 0$. We need to show that $T$ is smooth at $(x,y)$. Take an arbitrary compact open neighborhood $U$ of $x$, and an arbitrary compact open neighborhood $V$ of $y$ so that $f_d$ has no zero in $V$. We claim that $\hat{1}_UT$ vanishes on $\lambda V$ for all $\lambda \in k^\times$ with $|\lambda|_k$ sufficiently large. The lemma is a consequence of this claim.

Note that $\hat{1}_UT$ is a finite linear combination of generalized functions of the form

$$(1_{V_1} dy) \ast \hat{T}, \quad V_1 \text{ is a compact open subset of } F.$$  

Here $dy$ is a fixed Haar measure on $F$. The support of the convolution $(1_{V_1} dy) \ast \hat{T}$ is contained in $Z_f + V_1$. Therefore the claim follows from Lemma C.2. \qed

**Lemma C.5.** If a distribution $T \in D^{-\infty}(E)$ is supported in a closed subset $X$ of $E$, and $x \in X$ is a regular point, then the wave front set $WF_x(T)$ is invariant under translations by elements of $N^+_x(X) \subset F$.

**Proof.** This is proved in [Ai, Therorem 4.1.2]. We indicate the main steps.

**Step 1.** When we replace a distribution by a translation of it, the wave front set does not change. Therefore we may assume that $x = 0$.

**Step 2.** Let $\varphi : E \to E$ be a locally analytic diffeomorphism which sends 0 to 0 and induces the identity map on the tangent space at 0. When we replace $T$ by its pushing forward $\varphi_*(T)$, the wave front set $WF_0(T)$ does not change. Therefore we may assume that $X \cap U = E_0 \cap U$, for some subspace $E_0$ of $E$, and some open neighborhood $U$ of 0.
Step 3. When we replace $T$ by a distribution which coincides with $T$ on an open neighborhood of 0, the wave front set $WF_0(T)$ does not change. Therefore we may assume that $X = E_0$.

Step 4. Assume that $T$ is supported in $E_0$. Then $\hat{T}$ is invariant under translations by

$$N^*_x(X) = E_0^* := \{ v \in F \mid \langle u, v \rangle = 0, \ u \in E_0 \},$$

which implies that the same holds for $WF_0(T)$. □

Now we are ready to prove Theorem C.1. It is clear that $T$ vanishes on some open neighborhood of $x$ if and only if $0 \notin WF_x(T)$. If $0 \in WF_x(T)$, then Lemma C.5 implies that $N^*_x(X) \subset WF_x(T)$. Now Lemma C.4 further implies that $N^*_x(X)$ is contained in the zero locus of $f_d$, which contradicts the assumption of the theorem.

REFERENCES

[AP06] J. D. Adler and D. Prasad, On certain multiplicity one theorems, Israel J. Math. 153 (2006), 221–245.

[AI] A. Aizenbud, A partial analog of integrability theorem for distributions on $p$-adic spaces and applications, preprint, http://arxiv.org/abs/0811.2768.

[AG09] A. Aizenbud and D. Gourevitch, Multiplicity one theorem for $(GL_{n+1}(\mathbb{R}), GL_n(\mathbb{R}))$, Selecta Math. (N.S.) 15 (2009), no. 2, 271–294.

[AGRS10] A. Aizenbud, D. Gourevitch, S. Rallis, and G. Schiffmann, Multiplicity one theorems, Ann. of Math. (2) 172 (2010), no. 2, 1407–1434.

[BR00] E. M. Baruch and S. Rallis, A uniqueness theorem of Fourier Jacobi models for representations of $Sp(4)$, J. London Math. Soc. (2) 62 (2000), no. 1, 183–197.

[Be84] J. N. Bernstein, $P$-invariant distributions on $GL(N)$ and the classification of unitary representations of $GL(N)$ (non-Archimedean case), Lie Group Representations, II (College Park, Md., 1982/1983), Lecture Notes in Math., vol. 1041, Springer-Verlag, Berlin, 1984, pp. 50–102.

[BFG92] D. Bump, S. Friedberg, and D. Ginzburg, Whittaker-orthogonal models, functoriality, and the Rankin-Selberg method, Invent. Math. 109 (1992), no. 1, 55–96.

[CM93] D. H. Collingwood and W. M. McGovern, Nilpotent Orbits in Semisimple Lie Algebras, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993.

[GGP] W. T. Gan, B. H. Gross, and D. Prasad, Symplectic local root numbers, central critical $L$-values, and restriction problems in the representation theory of classical groups, Astérisque 346 (2012), 111–170.

[GPSR87] S. Gelbart, I. Piatetski-Shapiro, and S. Rallis, Explicit constructions of automorphic $L$-functions, Lecture Notes in Math., vol. 1254, Springer-Verlag, Berlin, 1987.

[GJRS] D. Ginzburg, D. Jiang, S. Rallis, and D. Soudry, $L$-functions for symplectic groups using Fourier-Jacobi models, Arithmetic Geometry and Automorphic Forms, Adv. Lect. Math. (ALM), vol. 19, Int. Press, Somerville, MA, 2011, pp. 183–207.
[GPSR97] D. Ginzburg, I. Piatetski-Shapiro, and S. Rallis, $L$ functions for the orthogonal group, *Mem. Amer. Math. Soc.* **128** (1997), no. 611, viii+218.

[GRS98] D. Ginzburg, S. Rallis, and D. Soudry, $L$-functions for symplectic groups, *Bull. Soc. Math. France* **126** (1998), no. 2, 181–244.

[GP92] B. H. Gross and D. Prasad, On the decomposition of a representation of $\text{SO}_n$ when restricted to $\text{SO}_{n-1}$, *Canad. J. Math.* **44** (1992), no. 5, 974–1002.

[GP94] ———, On irreducible representations of $\text{SO}_{2n+1} \times \text{SO}_{2m}$, *Canad. J. Math.* **46** (1994), no. 5, 930–950.

[GR06] B. H. Gross and M. Reeder, From Laplace to Langlands via representations of orthogonal groups, *Bull. Amer. Math. Soc. (N.S.)* **43** (2006), no. 2, 163–205.

[He85] D. B. Heifetz, $p$-adic oscillatory integrals and wave front sets, *Pacific J. Math.* **116** (1985), no. 2, 285–305.

[JR96] H. Jacquet and S. Rallis, Uniqueness of linear periods, *Compositio Math.* **102** (1996), no. 1, 65–123.

[JSZ10] D. Jiang, B. Sun, and C.-B. Zhu, Uniqueness of Bessel models: the Archimedean case, *Geom. Funct. Anal.* **20** (2010), no. 3, 690–709.

[JSZ11] ———, Uniqueness of Ginzburg-Rallis models: the Archimedean case, *Trans. Amer. Math. Soc.* **363** (2011), no. 5, 2763–2802.

[MVW87] C. Mœglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe sur un corps $p$-adique*, Lecture Notes in Math., vol. 1291, Springer-Verlag, Berlin, 1987.

[Nov76] M. E. Novodvorsky, New unique models of representations of unitary groups, *Compositio Math.* **33** (1976), no. 3, 289–295.

[Pr96] D. Prasad, Some applications of seesaw duality to branching laws, *Math. Ann.* **304** (1996), no. 1, 1–20.

[Sc] P. Schneider, $p$-adic analysis and Lie groups, preprint, http://wwwmath.uni-muenster.de/u/pschnei/publ/lectnotes/p-adic-analysis.pdf.

[SZ11] B. Sun and C.-B. Zhu, A general form of Gelfand-Kazhdan criterion, *Manuscripta Math.* **136** (2011), no. 1-2, 185–197.

[SZ] ———, Multiplicity one theorems: the Archimedean case, *Ann. of Math. (2)* **175** (2012), no. 1, 23–44.