On using convolutions with exponential distributions for solving a Kolmogorov backward equation

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Abstract. We propose a numerical method to solve 3-dimensional partial differential equations with variable coefficients, which appear in applications focused on studying random walks in the presence of an absorbing barrier. We restrict ourselves to the case of the Kolmogorov backward equation, which most commonly arises in mathematical finance. A probabilistic interpretation of the problem we use allows to apply Carr’s randomization for dimensionality reduction and employ the Wiener-Hopf factorization to solve the arising 1-dimensional problems. The key idea of the approach proposed is to realize the expected present value operators, which appear in the factorization identity, as convolutions with certain exponential distributions. We generalize our method for the case of exponentially distributed jumps presence using the Russian splitting method. The Wiener-Hopf factors related to the jump component of the operator again involve a convolution with exponential kernels. The numerical iterative scheme suggested to evaluate Wiener-Hopf operators is much less computationally expensive than a standard Fast Fourier transform-based approach.

1. Introduction
Diffusion equations are used to describe a vast class of processes in physics, natural sciences and economics. In the years since Dynkin, Feinman, and Katz published their fundamental works, it has been well known that the solution to a diffusion equation in a given domain can be interpreted as an expected value of a function of the Brownian motion at the first exit point from the domain. It is also well known that the Brownian motion itself can in some cases be efficiently used to model a number of physics, social and financial processes.

Driven by applications where the concept of jumps is vital (see, for example, [11]), the idea was later generalized for the case of Lévy processes. This approach has established itself as a valuable and important instrument in mathematical finance, where certain drawbacks of the Black-Scholes model [4] became more and more pronounced. To cover the discrepancies between Brownian motion-based approximations and historical asset prices, a vast of alternative models emerged, an overview of which can be found in [11, 19] and other related sources.

In recent years there has been a dramatic increase in interest in models where there is more than one source of randomness, where Lévy models, either purely gaussian or with discontinuous trajectories are combined together.

A subset of such models where the market volatility or variance is governed by a random process is referred to as “stochastic volatility models”. Perhaps the most widely used model of
this class is the Heston [16] model. It has closed-form solutions for certain basic, but commonly
encountered problems based on Kolmogorov backward equation, where no absorbing barriers
are taken into account.

A more general case of the Heston model, which allows normally distributed jumps similar to
[24] for process trajectories, is known as the Bates model [3]. The former can be later generalized
from into a form where jump process may have a different distribution. As an example we can
mention the double-exponential Kou model introduced in [23].

Solving problems in the presence of an absorbing barrier is more complicated, due to the fact
that the solution depends on the whole trajectory of a process considered. As a rule, there are
no known analytical methods and the only option is to use the methods of numerical analysis.

With a suitable initial and boundary conditions, the Kolmogorov backward equation can be
solved in Heston and Bates models. There is a number of methods to obtain the solution. They
can be roughly divided into several wide branches.

First and foremost, there is a series of Monte-Carlo methods. Among them we may mention
[1], as it provides relatively good speed and accuracy away from a barrier. The most significant
drawback of Monte-Carlo methods is low computational performance, which comes from a huge
number of simulations it takes to achieve good accuracy. It is also important to find a good
unbiased source of randomness for computations.

The second branch of methods uses finite-difference schemes (see, for example, [9, 28]).
A comprehensive review on such methods was presented in [7]. They commonly require
a sufficiently dense grid to achieve good computational accuracy, which also makes them
computationally expensive. To improve convergence speed, some authors also apply operator
splitting technique (see, for example [18, 19]).

Methods based on approximating trees can be considered as implicit-explicit schemes. For
example, a method proposed in [27] has good accuracy and robustness for parameter values, but
the number of nodes in the tree it uses, grows quadratically with the number of time steps.

It is possible, in certain cases, to develop semi-analytical methods based on analyzing an
asymptotic behaviour for short time intervals (see [25]). This approach is reported to have a
good potential for generalization, although being computationally demanding. A wide class of
methods (see [14]) relies on Fourier transform.

There are also hybrid methods where variance process approximation is constructed (for
example, [2, 10]). An idea behind this approach is to reduce dimensions of the original problem
from 3 to 1 and solve a system of partial differential equations. It can be shown (see [20]) that
the method of lines, which is commonly used to justify this approach, is equivalent to time
randomization introduced by P. Carr in [8]. To solve the system either finite-difference methods
(see [7]) or the Wiener-Hopf factorization (see [20, 22]) can be applied.

In the present article, we propose a new hybrid method based on interpreting estimated
present value (EPV) operators as convolutions. The method uses the Wiener-Hopf factorization
and operator splitting method.

2. Materials and methods
2.1. Problem setup
Let us consider a 3-dimensional partial differential equation for function \( u = u(x,t) \), where
\( x = (x_1,x_2) \in \mathbb{R}^2, \ t \in \mathbb{R} \):

\[
\left( \frac{\partial}{\partial t} + L \right) u = 0, \quad \text{with} \quad L = \sum_i \mu_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j},
\]

(1)
indices \( i = 1, 2, j = 1, 2 \); functions \( \mu(x) = (\mu_1(x), \mu_2(x)) \) and

\[
\sigma(x) = \begin{pmatrix}
\sigma_{11}(x) & \sigma_{12}(x) \\
\sigma_{21}(x) & \sigma_{22}(x)
\end{pmatrix}
\]

satisfies conditions of Theorem 5.2.1 [26]. Let us recall that \( \mu : \mathbb{R}^2 \to \mathbb{R}^2 \) and \( \sigma : \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2 \) (or sometimes \( \frac{1}{2}(\sigma \sigma^T) \)) are being referred to as drift and diffusion coefficients, respectively. An equation (1), known as the Kolmogorov backward equation (or diffusion equation), describes an evolution of a 2-dimensional random process and arises in various applications.

Let \( T > 0 \) be a time moment, \( H > 0 \) - an absorbing barrier, \( g(x) : \mathbb{R} \to \mathbb{R}^{\geq 0} \) - some suitable function, which decays rapidly on infinity. Consider an equation (1) with the terminal and boundary conditions defined as follows:

\[
\begin{aligned}
&\left. \left( \frac{\partial}{\partial t} + L \right) u \right|_{x_1 > H, t < T} = 0, \\
&u(x_1, x_2, T) = g(x_1), \quad x_1 > H, \\
&u(x_1, x_2, t) = 0, \quad x_1 \leq H, t \leq T.
\end{aligned}
\]

A review on uniqueness and existence conditions for a class of problems with Kolmogorov equation can be found in [13, 26], and, for less general case, in [9, 17].

Although general analytical solution is not known, one can develop numerical methods based on the exact form of \( \mu_1(x) \) and \( \sigma(x) \). An approach we propose here draws on a probabilistic interpretation of the equation. Let \( B_1(t), B_2(t) \) be independent Wiener processes (Brownian motions). It can be shown (see. [26]) that the function \( u \) allows a representation as a conditional expectation of the event of \( X_t \) exit from a domain \( x_1 \leq H \), where \( X_t = (X_1(t), X_2(t)) \) is defined by a system of differential equations in the Itô form:

\[
\begin{aligned}
&dX_1(t) = \mu_1 dt + \sigma_{11} dB_1(t) + \sigma_{12} dB_2(t), \\
&dX_2(t) = \mu_2 dt + \sigma_{21} dB_1(t) + \sigma_{22} dB_2(t).
\end{aligned}
\]

The solutions of (3) are referred to as Itô diffusions, as they can be used to describe a dynamics of a particle in a fluid flow.

An important case of the problem (1) appears in mathematical finance. Let us define positive constants \( \kappa_V, \theta_V, \sigma_V; \rho \in (-1, 1) \). Let \( \hat{\rho} = \sqrt{1 - \rho^2} \) and set \( \sigma_{22} = 0 \). Let us denote \( X_1(t) \) as \( S_t \) and \( X_2(t) \) as \( V_t \). Assuming \( \mu_1 = 0, \mu_2 = \kappa_V(\theta_V - V) \) and

\[
\sigma = \begin{pmatrix}
\rho \sigma S_t V_t^{-1} & \hat{\rho} \sigma S_t V_t^{-1} \\
\sigma V_t & 0
\end{pmatrix},
\]

from (3) we derive a system:

\[
\begin{aligned}
&dS_t = \sqrt{V_t} S_t (\rho dB_1(t) + \hat{\rho} dB_2(t)), \\
&dV_t = \kappa_V(\theta_V - V_t) dt + \sigma \sqrt{V_t} dB_1(t).
\end{aligned}
\]

For a jump-diffusional case the equation for \( S \) also has a jump component, which at this point we assume zero. A review on existence and uniqueness conditions for this case can be found in [11]. An infinitesimal operator \( L \) from (1), in terms of \( u = u(S, v, t) \) (where \( x_1 \) is replaced with \( S \) for convenience, and \( x_2 \) is replaced with \( v \)), can be represented as:

\[
L = \frac{1}{2} S^2 v \frac{\partial^2}{\partial S^2} + \rho \sigma_V v S \frac{\partial^2}{\partial S \partial v} + \frac{1}{2} \sigma_V^2 v^2 \frac{\partial^2}{\partial v^2} + \kappa_V(\theta_V - v) \frac{\partial}{\partial v}.
\]
In financial terms, the process $S_t$ describes the dynamics of an asset price, whereas $V_t$ (which is the Cox-Ingersoll-Ross (CIR [12]) process) defines its variation. The parameter $\kappa_V$ denotes a mean reversion rate of $V_t$ to a value $\theta_V$, and the parameter $\sigma_V$ is a volatility of variation. The processes $(\rho dB_1(t) + \dot{\rho} dB_2(t))$ and $B_1(t)$ are Brownian motions with correlation $\rho$. Also, $S_t$ is considered a martingale under a suitable risk-neutral measure.

The problem (2) may be, therefore, rewritten in the following terms:

$$
\begin{aligned}
&\left( \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2_v \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial v} \rho \sigma_v \frac{\partial}{\partial y} + \frac{\partial}{\partial v} \left( \mu(y,v) \frac{\partial}{\partial y} + \mu_V(v) \frac{\partial}{\partial v} \right) \right) u = 0, \\
&u(S, v, T) = g(S), \quad S > H, v > 0, t < T, \\
&u(S, v, t) = 0, \quad S \leq H, v > 0, 0 \leq t \leq T.
\end{aligned}
$$

(6)

At the earliest possible moment in time, $T = 0$, the solution for (6) may be written as an expectation in the following form:

$$
u(S, v, 0) = E[1_{(T, \infty)}(T_H) \cdot g(S_T)|S_0 = S, V_0 = v],
$$

(7)

where $T_H$ is the moment when $S_t$ first enters $(0, H]$.

From a financial viewpoint, $u(S, v, 0)$ may be interpreted as a price of a barrier put option with a payoff function $g(S) = \max\{(K - S), 0\}$, where $K > 0$. This type of option become worthless if the price of an underlying asset falls below a barrier $H$. Such contingent claims have both theoretical and practical value, as it is possible to evaluate economical risks associated with crossing certain price levels in their terms. The price of such contracts relies upon the whole trajectory of an underlying asset.

### 2.2. Process substitution

Finding a numerical solution of (6) in its original form is difficult, in particular, due to the fact that the generator $L$ has partial derivatives. This effect may be avoided by using a suitable process substitution, for example: $Y_t = \ln \left( \frac{S_t}{V_t} \right) - \frac{\rho}{\sigma_v} V_t$. It its terms $S_t = H \exp(Y_t + \frac{\rho}{\sigma_v} V_t)$. The operator $L$ in this case has the following simplified form:

$$
L = \frac{1}{2} \rho^2_v \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial y} \mu_V(v) \frac{\partial}{\partial y} + \mu_V(v) \frac{\partial}{\partial v},
$$

where

$$
\mu_V(v) = -\frac{1}{2} v - \frac{\rho}{\sigma_v} \kappa_V(\theta_V - v) \quad \text{and} \quad \mu_V(v) = \kappa_V(\theta_V - v)
$$

are drift coefficients calculated for $v$. The equations for processes can now be written as follows:

$$
\begin{aligned}
&dY_t = \mu_Y(V_t)dt + \dot{\rho} \sqrt{V_t} dB_2(t), \\
&dV_t = \mu_V(V_t)dt + \sigma_V \sqrt{V_t} dB_1(t).
\end{aligned}
$$

(8)

Unfortunately, making this substitution inevitably changes the behaviour of the absorbing barrier. Putting $f(y, v, t) = u \left( H \exp(y + \frac{\rho}{\sigma_v} v), v, t \right)$, $g(y) = g(H e^{y + \frac{\rho}{\sigma_v} v})$ and noting that the condition $S_t > H$ now has a form $Y_t + \frac{\rho}{\sigma_v} V_t > 0$, one can observe that the problem (6) changes respectively:

$$
\begin{aligned}
&\left( \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2_v \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial y} \mu_V(v) \frac{\partial}{\partial y} + \mu_V(v) \frac{\partial}{\partial v} \left( \frac{\partial}{\partial y} + \frac{\partial}{\partial v} \right) \right) f = 0, \\
&f(y, v, T) = g(y), \quad y + \frac{\rho}{\sigma_v} v > 0, v > 0, 0 < t < T, \\
&f(y, v, t) = 0, \quad y + \frac{\rho}{\sigma_v} v \leq 0, v > 0, 0 \leq t \leq T.
\end{aligned}
$$

(9)
Then $T_H$ can be interpreted as the first time the process $Y_t + \frac{\rho}{\sigma} V_t$ enters $(-\infty, 0]$:

$$T_H = \inf\{t : Y_t + \frac{\rho}{\sigma} V_t \leq 0\},$$

and, therefore, the expectation (7) can be rewritten in terms of $f$ as:

$$f(y, v, 0) = E^{y, v}[1_{(T, +\infty)}(T_H)g(Y_T)].$$

(10)

Let us recall that we use the notion $E^{y, v}[\cdot]$ in a sense that the respective conditional expectation is calculated with $Y_0 = y, V_0 = v$.

2.3. Randomization and approximation

Let us choose a large $N \in \mathbb{N}$ and define $\Delta t = T/N$. Let us then put $q > 0$, $r > 0$, and denote a random exponentially distributed variable with an average value of $\Delta t$ as $T_q$. Now we may use an approach introduced in [22] to the model we consider in the present paper.

Let us briefly recall its idea. Using a Carr randomization technique, which has been justified and extended in [5], we can observe that the calculation of expectation (10) comes down to an approach introduced in [22] to the model we consider in the present paper.

For the sake of brevity we will denote each of them as $T_q$. An approximation for each of $f_n(y, v)$ can be written with respect to a Markov chain, which resides on a recombining binomial tree $V_{n,k}$:

$$V_{n,k} = (\sqrt{V_0 + \frac{\sigma V}{2}(2k-n)\sqrt{\Delta t}})^2 \cdot 1_{(0, +\infty)}(\sqrt{V_0 + \frac{\sigma V}{2}(2k-n)\sqrt{\Delta t}}),$$

(11)

where $n = 0, 1, \ldots, N$, $k = 0, 1, \ldots, n$, and transitions from a node with indexes $(n, k)$ are only allowed to nodes with indexes $(n+1, k_d)$ and $(n+1, k_d)$, and happen with probabilities $p_u$ and $p_d$ respectively. The values $k_d$, $k_u$ and $p_u$, $p_d$ are calculated using the method described in [2] based on parameters of $V_t$. With all that, the calculation of approximate values of $f_n(y, v)$ can be reduced to iterative calculation of a sequence of expectations $f_{n,k}(y) := f_n(y, V_{n,k})$ associated with the nodes of the recombining binomial tree, in a form:

$$f_{n,k}(y) = 1_{(0, +\infty)}(y + \frac{\rho}{\sigma} V_{n,k}) \cdot (p_u f_{n,k}^u(y) + p_d f_{n,k}^d(y)),\quad(12)$$

where

$$f_{n,k}^u(y) = E^y[1_{(0, +\infty)}(Y_{T_q}^{n+1,k_u} + \frac{\rho}{\sigma} V_{n+1,k_u}) \cdot f_{n+1}(y + Y_{T_q}^{n+1,k_u}, V_{n+1,k_u})],$$

(13)

and $Y_{T_q}^{n,k}$ is a value of $Y_{T_q}$ from (8) when $V_{T_q} = V_{n,k}$; and the value of $f_{n,k}^d(y)$ is calculated analogously.

2.4. Finding the solution in terms of EPV operators $\mathcal{E}^\pm$

An operator

$$\mathcal{E}_q F(x) = qE^x[\int_0^{+\infty} e^{-qt} F(X_t)dt] = E[F(x + X_{T_q})],$$

(14)

where $X_t$ is an Ito diffusion, is referred to as the Expected Present Value (EPV) operator (see, for example, [21]). Let us define a supremum process $X_t$ as

$$X_t = \sup_{0 \leq s \leq t} X_t,$$
and an infimum process – as

\[ X_t = \inf_{0 \leq s \leq t} X_s, \]

respectively. We can define operators \( \mathcal{E}_q^\pm F(x) \) for them analogously as:

\[
\mathcal{E}_q^+ F(x) = qE^x \left[ \int_0^{+\infty} e^{-\beta u} F(x + \mathcal{X}_t) du \right] = E[F(x + \bar{X}_t)],
\]

\[
\mathcal{E}_q^- F(x) = qE^x \left[ \int_0^{+\infty} e^{-\beta u} F(x + \mathcal{X}_t) du \right] = E[F(x + X_{T_q})].
\]

Using a property \( \mathcal{X}_t \sim X_t - X_t \), one can (see, for example, [6]) calculate the value of \( f_{n,k}^u(y) \) as:

\[
f_{n,k}^u(y) = q\mathcal{E}_q \left( 1_{(0, +\infty)}(y + \frac{\rho}{\sigma} V_{n+1,k_n}) \cdot \mathcal{E}_q^+ f_{n+1,k_n}(y) \right),
\]

and use an analogous formula for \( f_{n,k}^d(y) \). Let us note that \( \mathcal{E}_q^\pm \) can be interpreted as pseudodifferential operators (PDO). Typically, one can implement PDO numerically by means of the Fast Fourier Transform, which require \( O(n \ln n) \) operations (see e.g. [21, 20]).

For the processes considered there also an interpretation of \( \mathcal{E}_q \) and \( \mathcal{E}_q^\pm \) as convolutions exist:

\[
\mathcal{E}_q F(x) = \int_{-\infty}^{+\infty} F(x + u) P_q^u(du), \quad \mathcal{E}_q^\pm F(x) = \int_{-\infty}^{+\infty} F(x + u) P_q^{\pm u}(du),
\]

where \( P_q^u(-\infty, 0) = 0, \quad P_q^u(0, +\infty) = 0 \).

If \( X_t \) is a Lévy process, and its characteristic exponent is a function \( \psi(\xi) \), defined by the Levy-Khintchine formula, then the characteristic functions for the distributions \( P_q^u(du) \) are \( \phi^\pm(\xi) \), such that

\[
\phi^+(\xi) \cdot \phi^-(\xi) = q(q + \psi(\xi))^{-1}. \tag{16}
\]

There are also exist constants \( \omega_- < 0 < \omega_+ \) (see details in [20]) such that \( \phi^+(\xi) \) admits an analytical continuation into half-plane \( \Im \xi > \omega_- \) and \( \phi^-(\xi) \) admits an analytical continuation into \( \Im \xi < \omega_+ \).

The expression (16) is a special case of the Wiener-Hopf factorization for a symbol of a PDO. Let us also note that here \( q(q + \psi(\xi))^{-1} \) is a symbol of EPV operator \( \mathcal{E}_q \), and \( \phi^\pm(\xi) \) are respective symbols of \( \mathcal{E}_q^\pm \).

In the present case, the characteristic function of the process \( Y_{t,n,k}^j, \quad t < T_q \), has a form

\[
\psi(\xi) = \frac{\sigma_{n,k}^2}{2} \xi^2 - i\gamma_{n,k} \xi, \tag{17}
\]

where \( \sigma_{n,k} = \hat{\rho} \sqrt{\gamma_{n,k}}, \quad \gamma_{n,k} = \mu_Y(V_{n,k}) \). It easily can be shown that:

\[
\phi^+(\xi) = \frac{\beta_q^+}{\beta_q^+ - i\xi}, \quad \phi^-(\xi) = \frac{-\beta_q^-}{-\beta_q^- + i\xi}, \tag{18}
\]

where

\[
\beta_q^+ = \frac{-\gamma_{n,k} + \sqrt{\gamma_{n,k}^2 + 2\sigma_{n,k}^2 q}}{\sigma_{n,k}^2}, \quad \beta_q^- = \frac{-\gamma_{n,k} - \sqrt{\gamma_{n,k}^2 + 2\sigma_{n,k}^2 q}}{\sigma_{n,k}^2}. \tag{19}
\]

The formulae for the respective exponential distributions \( P^\pm(du) \) would be as follows:

\[
P_q^-(du) = -\beta_q^- e^{-\beta_q^- u} 1_{(-\infty, 0)}(u) du, \quad P_q^+(du) = \beta_q^+ e^{-\beta_q^+ u} 1_{(0, +\infty)}(u) du.
\]
Let us define a reasonably dense grid of equally spaced points \( y_k \) with a distance \( h \) between them: \( y_k = y^* + yh, \ k = 0, 1, \ldots, M \), where \( M \in \mathbb{N} \) is large. Let us then denote, for brevity, \( F_n(y) := f_{n,k}^u(y) \), \( F_n+1(y) := f_{n+1,ku}^u(y) \) and consider the following equation:

\[
E^+ F_{n+1}(y_k) = \int_0^{+\infty} \beta_q^+ e^{-\beta_q^+ u} F_{n+1}(y_k + u) du.
\]

As \( \lim_{y \to +\infty} F_n(y) = \lim_{y \to +\infty} g(y) = 0 \), we restrict the integration area with a finite \( y^* > 0 \). It is straightforward that if \( y_k + u < y^* \), then \( u < y^* - y_k \). Therefore, we get:

\[
E^+ F_{n+1}(y_k) \approx \int_0^{y^*-y_k} \beta_q^+ e^{-\beta_q^+ u} F_{n+1}(y_k + u) du.
\]

Now let us make a substitution: \( w = u + y_k \). Then \( u = w - y_k, \ u = 0 \to w = y_k, \ u = y^* - y_k \to w = y^* \):

\[
E^+ F_{n+1}(y_k) \approx \int_{y_k}^{y^*} \beta_q^+ e^{-\beta_q^+ (w-y_k)} F_{n+1}(w) dw = e^{\beta_q^+ y} \int_{y_k}^{y^*} \beta_q^+ e^{-\beta_q^+ w} F_{n+1}(w) dw.
\]

From a boundary condition in (13) it follows that if \( y_k - h < -\frac{\rho}{\sigma V} V_{n+1,ku} \), then \( F_n+1(y_k-h) = 0 \). For \( y_k - h > -\frac{\rho}{\sigma V} V_{n+1,ku} \), \( F_{n+1}(y_k-h) := F_{n+1}(y_k-h) \) may be calculated as:

\[
E^+ F_{n+1}(y_k-1) = e^{\beta_q^+ (y_k-h)} \left( \int_{y_k}^{y^*} \beta_q^+ e^{-\beta_q^+ w} F_{n+1}(w) dw + \int_{y_k-h}^{y_k} \beta_q^+ e^{-\beta_q^+ w} F_{n+1}(w) dw \right) = e^{-\beta_q^+ h} E^+ F_{n+1}(y_k) + e^{-\beta_q^+ h} \int_{y_k-h}^{y_k} \beta_q^+ e^{-\beta_q^+ (w-y_k)} F_{n+1}(w) dw.
\]

Using the simplest trapezoid approximation, we can get the following formula:

\[
E^+ F_{n+1}(y_k-1) \approx e^{-\beta_q^+ h} E^+ F_{n+1}(y_k) + \frac{h}{2} \beta_q^+ \left( F_{n+1}(y_k-1) + e^{-\beta_q^+ h} F_{n+1}(y_k) \right).
\]

To use Simpson formula and improve accuracy we may consider two neighboring values:

\[
E^+ F_{n+1}(y_k-2) = e^{-2\beta_q^+ h} E^+ F_{n+1}(y_k) + e^{-2\beta_q^+ h} \int_{y_k-2h}^{y_k} \beta_q^+ e^{-\beta_q^+ (w-y_k)} F_{n+1}(w) dw.
\]

\[
E^+ F_{n+1}(y_k-2) \approx e^{-2\beta_q^+ h} E^+ F_{n+1}(y_k) + \frac{h}{3} \beta_q^+ \left( F_{n+1}(y_k-2) + 4 e^{-\beta_q^+ h} F_{n+1}(y_k-1) + e^{-2\beta_q^+ h} F_{n+1}(y_k) \right).
\]

Let us now put \( z = \frac{\rho}{\sigma V} V_{n+1,ku} \) and consider an expression:

\[
qE^- \left( 1_{(0,+\infty)}(y_k+z) \cdot E^+ F_{n+1}(y_k) \right) = q \int_{-\infty}^0 -\beta_q^- e^{-\beta_q^- u} 1_{(0,+\infty)}(u+(y_k+z)) \cdot E^+ F_{n+1}(u+y_k) du.
\]

Directly applying the condition of the indicator-function, we obtain:

\[
qE^- \left( 1_{(0,+\infty)}(y_k+z) \cdot E^+ F_{n+1}(y_k) \right) = q \int_{-(z+y_k)}^0 -\beta_q^- e^{-\beta_q^- u} \cdot E^+ F_{n+1}(u+y_k) du.
\]
We can make a substitution \( w = u + y_k; u = w - y_k \) again and write it down as:

\[
q \int_{-z}^{y_k} -\beta_q \ e^{-\beta_q \ (w-y_k)} \mathcal{E}^+ F_{n+1}(w)dw = q e^{\beta_q \ y_k} \int_{-z}^{y_k} -\beta_q \ e^{-\beta_q \ w} \mathcal{E}^+ F_{n+1}(w)dw.
\]

Then, for \( y_{k+1} = y_k + h \), using the same reasoning as in the previous case, we finally get:

\[
q \mathcal{E}^{-} (1_{(0, +\infty)}(y_{k+1} + z) \cdot \mathcal{E}^+ F_{n+1}(y_{k+1})) =
q e^{\beta_q \ (y_k+h)} \left( \int_{-z}^{y_k} -\beta_q \ e^{-\beta_q \ w} \mathcal{E}^+ F_{n+1}(w)dw + \int_{y_k}^{y_k+h} -\beta_q \ e^{-\beta_q \ w} \mathcal{E}^+ F_{n+1}(w)dw \right) \approx
\approx q \left( e^{\beta_q \ h} \mathcal{E}^{-} 1_{(-\infty, 0)}(y_k + z) \cdot \mathcal{E}^+ F_{n+1}(y_k) + \frac{h}{2} \cdot (-\beta_q) (\mathcal{E}^+ F_{n+1}(y_{k+1}) + e^{\beta_q \ h} \mathcal{E}^+ F_{n+1}(y_{k})) \right).
\]

A similar modification for the Simpson formula case gives us:

\[
q \mathcal{E}^{-} (1_{(0, +\infty)}(y_{k+2} + z) \cdot \mathcal{E}^+ F_{n+1}(y_{k+2})) =
q e^{\beta_q \ (y_k+2h)} \left( \int_{-z}^{y_k} -\beta_q \ e^{-\beta_q \ w} \mathcal{E}^+ F_{n+1}(w)dw + \int_{y_k}^{y_k+2h} -\beta_q \ e^{-\beta_q \ w} \mathcal{E}^+ F_{n+1}(w)dw \right) \approx
\approx q \left( e^{2\beta_q \ h} \mathcal{E}^{-} 1_{(-\infty, 0)}(y_k + z) \cdot \mathcal{E}^+ F_{n+1}(y_k) + \frac{h}{3} \cdot (-\beta_q) (e^{2\beta_q \ h} \mathcal{E}^+ F_{n+1}(y_{k+2}) + e^{\beta_q \ h} \mathcal{E}^+ F_{n+1}(y_{k+1}) + \mathcal{E}^+ F_{n+1}(y_{k})) \right).
\]

Using the procedure above to solve the arising 1-dimensional problems, we obtain an approximate solution for (2). The scheme presented in this subsection is an iterative method for calculating integrals, where already calculated values of \( \mathcal{E}^\pm F_{n+1}(y_k) \) are used to obtain values for the neighboring \( y_k \). The main advantage of this method is that we only need \( O(n) \) operations to calculate the integrals, in contrast with \( O(n \log n) \) operations that the FFT-based procedure, proposed in [22], required.

2.5. Operator splitting for the case of exponentially distributed jumps presence

The method proposed can be naturally generalized for the Bates-like case, where jump component corresponds to Kou [23] model. It arises in applications where the discontinuous behavior of a random walk is essential, both positive and negative jumps are allowed, and jumps are considered exponentially distributed.

Let us recall (see, for example [19]) that in Kou model the Lévy density is defined by the equation:

\[
\pi(dx) = \lambda [p \lambda_+ e^{-\lambda_+ x} 1_{x \geq 0} + (1 - p) \lambda_- e^{-\lambda_- x} 1_{x < 0}]dx,
\]

where parameters \( \lambda_+ > 1, \lambda > 0, \lambda_- < 0 \) sets the respective jumps intensities, \( p > 0 \) denotes the probability of positive jumps. For model calibration there is an alternative parametrization, where \( c_+ := p \lambda_+ \) and \( c_- := p \lambda_- \). We will use it for the rest of the section, as it is more compact.

In this model, instead of (17) we end up with the following characteristic exponent:

\[
\psi_{n,k}(\xi) = \frac{\sigma_{n,k}^2}{2} \xi^2 - i \gamma_{n,k} \xi + \frac{ic_+ \xi}{i \xi - \lambda_+} + \frac{ic_- \xi}{i \xi - \lambda_-},
\]

(24)
where coefficients $\sigma_{n,k}$ and $\gamma_{n,k}$ are calculated so that the jump-diffusion process is a martingale. It can also be used to derive an analytic form of factorization. Let us recall that $\psi(\xi)$ is a symbol of a respective pseudo-differential operator, which we denote here as $L$. We can use a technique known as the Strang splitting or Russian splitting (see e.g. [19]), and decompose $L$ into two parts: a diffusion part, $D$, with a symbol:

$$\frac{\sigma_{n,k}}{2} \xi^2 - i\gamma_{n,k}\xi,$$

and a jump-related part, $J$, with a symbol:

$$\frac{ic_+\xi}{i\xi - \lambda_+} + \frac{ic_-\xi}{i\xi - \lambda_-},$$

so that $Lf = [D + J]f$. The factorization process of $q(q + D)^{-1}$ was described in Sect. 2.4.

As for operator $q(q + J)^{-1}$, we can derive the relative functions $\phi^\pm_J(\xi)$ as follows.

$$\phi^+_J(\xi) \cdot \phi^-_J(\xi) = \frac{\xi}{q + i\xi c_+ + i\xi c_-} \frac{q + i\xi c_+ + i\xi c_-}{(q + c_+ + c_-)(i\xi - \beta^+_{q,J})(i\xi - \beta^-_{q,J})},$$

where $\beta^+_{q,J}$ and $\beta^-_{q,J}$ are the roots of the following equation in $i\xi$:

$$(q + c_+ + c_-)(i\xi)^2 - \left(\lambda_+(q + c_-) + \lambda_-(q + c_+)\right) i\xi + q \lambda_+ \lambda_- = 0,$$

which appears in a denominator by regrouping the expression:

$$q(i\xi - \lambda_+)(i\xi - \lambda_-) + i\xi c_+(i\xi - \lambda_-) + i\xi c_-(i\xi - \lambda_+).$$

As $q\lambda_+ \lambda_- < 0$, it follows that $i\beta^+_{q,J}$ and $i\beta^-_{q,J}$ to be located at the upper and the lower half-plane, respectively. A straightforward formula for $\beta^\pm_J$ will be:

$$\beta^\pm_J = \frac{\lambda_+(q + c_-) + \lambda_-(q + c_+)}{2(q + c_+ + c_-)} \pm \sqrt{\left(\lambda_-(q + c_+) + \lambda_+(q + c_-)\right)^2 - 4(q + c_+ + c_-)q \lambda_+ \lambda_-}.$$

Taking into account that $\phi^\pm_J(0) = 1$, we obtain:

$$\phi^+_J(\xi) = \frac{\beta^+_J}{\lambda_+} \cdot \frac{i\xi - \lambda_+}{i\xi - \beta^+_{q,J}} = \frac{\beta^+_J}{\lambda_+} + \frac{\lambda_+ - \beta^+_J}{i\xi - \beta^+_{q,J}} \cdot \frac{-\beta^+_J}{i\xi - \beta^+_{q,J}},$$

$$\phi^-_J(\xi) = \frac{\beta^-_J}{\lambda_-} \cdot \frac{i\xi - \lambda_-}{i\xi - \beta^-_{q,J}} = \frac{\beta^-_J}{\lambda_-} + \frac{\lambda_- - \beta^-_J}{i\xi - \beta^-_{q,J}} \cdot \frac{-\beta^-_J}{i\xi - \beta^-_{q,J}}.$$

The expressions $\frac{-\beta^+_J}{i\xi - \beta^+_{q,J}}$ and $\frac{-\beta^-_J}{i\xi - \beta^-_{q,J}}$ are characteristic functions of exponential distributions, so we can write the related Wiener-Hopf operators $E^\pm_J$ in terms of convolutions.

$$E^+_J F(x) = \int_0^{+\infty} F(x + u) \left(\frac{\lambda_+ - \beta^+_J}{\lambda_+} \cdot \beta^+_J e^{-\beta^+_J u}\right) du + \frac{\beta^+_J}{\lambda_+} F(x),$$

$$E^-_J F(x) = \int_{-\infty}^0 F(x + u) \left(\frac{\lambda_- - \beta^-_J}{\lambda_-} \cdot \beta^-_J e^{-\beta^-_J u}\right) du + \frac{\beta^-_J}{\lambda_-} F(x).$$
To combine the results together we use the Strang splitting scheme (see the formula (5.27) from [19]). Notice that the expectation (13) is an approximate 0-time solution to the problem

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( \frac{\partial}{\partial t} + L \right)v(y,t) = 0, \\
v(y,t) = 0, \\
v(y, t + \Delta t) = f_{n+1}(y, V_{n+1,k_u}),
\end{array} \right. & \quad y > -\frac{\rho}{\sigma_y} V_{n+1,k_u}, \quad t < \Delta t, \\
y \leq -\frac{\rho}{\sigma_y} V_{n+1,k_u}, \quad t \leq \Delta t,
\end{align*}
\]

where \( L = D + J \) is an infinitesimal generator of \( Y_t^{n+1,k_u} \). The splitting method involves the following steps.

The step 1.

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( \frac{\partial}{\partial t} + D \right)v_1(y,t) = 0, \\
v_1(y,t) = 0, \\
v_1(y, t + \Delta t/2) = f_{n+1}(y, V_{n+1,k_u}),
\end{array} \right. & \quad y > -\frac{\rho}{\sigma_y} V_{n+1,k_u}, \quad t < \Delta t, \\
y \leq -\frac{\rho}{\sigma_y} V_{n+1,k_u}, \quad t \leq \Delta t,
\end{align*}
\]

The step 2.

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( \frac{\partial}{\partial t} + J \right)v_2(y,t) = 0, \\
v_2(y,t) = 0, \\
v_2(y, t + \Delta t) = v_1(y,0),
\end{array} \right. & \quad y > -\frac{\rho}{\sigma_y} V_{n+1,k_u}, \\
y \leq -\frac{\rho}{\sigma_y} V_{n+1,k_u},
\end{align*}
\]

The step 3.

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( \frac{\partial}{\partial t} + D \right)v_3(y,t) = 0, \\
v_3(y,t) = 0, \\
v_3(y, t + \Delta t/2) = v_2(y,0),
\end{array} \right. & \quad y > -\frac{\rho}{\sigma_y} V_{n+1,k_u}, \quad t < \Delta t, \\
y \leq -\frac{\rho}{\sigma_y} V_{n+1,k_u}, \quad t \leq \Delta t,
\end{align*}
\]

The final solution \( v_0(y,0) \) is approximate for \( v(y,0) \) as well as for the expectation (13). Notice that problems (34)–(36) can be approximately represented as expectation (13) with a related diffusion or jump process provided that \( q = (\Delta t/2)^{-1} \) at the steps 1 and 3, whereas for the step 2 it should be \( q = (\Delta t)^{-1} \). The Strang splitting-based numerical solution scheme can have an error of \( O(\Delta t)^2 \). Following the scheme we can solve the equations using Wiener-Hopf method (see (15)).

3. Results and discussions

As we are calculating the approximating functions values using their known values on the nearest grid elements, we can reduce the amount of computations from \( O(n \ln n) \), which are necessary for the technique based on Fourier transform proposed in [22], to \( O(n) \). Due to the fact that the numerical solution for a 3-dimensional equation we consider involves a significant number of iterations, which requires the use of this technique (more precisely, we need 2 for each node of a tree where the Markov chain resides, except for the very last layer), the method proposed can give a notable advantage in terms of convergence speed and computational performance.

To achieve good accuracy on some coefficients values, it becomes necessary to use high order integration routines, with a suitable modification of an approximate formula, and construct a sufficiently dense grid for a spatial variable.

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