The Proof that Maxwell Equations with the 3D E and B are not Covariant upon the Lorentz Transformations but upon the Standard Transformations. The New Lorentz Invariant Field Equations

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In this paper the Lorentz transformations (LT) and the standard transformations (ST) of the usual Maxwell equations (ME) with the three-dimensional (3D) vectors of the electric and magnetic fields, E and B respectively, are examined using both the geometric algebra and tensor formalisms. Different 4D algebraic objects are used to represent the usual observer dependent and the new observer independent electric and magnetic fields. It is found that the ST of the ME differ from their LT and consequently that the ME with the 3D E and B are not covariant upon the LT but upon the ST. The obtained results do not depend on the character of the 4D algebraic objects used to represent the electric and magnetic fields. The Lorentz invariant field equations are presented with 1-vectors \( E \) and \( B \), bivectors \( E_{Hv} \) and \( B_{Hv} \) and the abstract tensors, the 4-vectors \( E^a \) and \( B^a \). All these quantities are defined without reference frames, i.e., as absolute quantities. When some basis has been introduced, they are represented as coordinate-based geometric quantities comprising both components and a basis. It is explicitly shown that this geometric approach agrees with experiments, e.g., the Faraday disk, in all relatively moving inertial frames of reference, which is not the case with the usual approach with the 3D E and B and their ST.

KEY WORDS: standard and Lorentz transformations of Maxwell equations.

1. INTRODUCTION

Recently it is shown in the tensor formalism\(^{(1)}\) and the geometric (Clifford) algebra formalism\(^{(2)}\) that the standard transformations (ST)\(^{(3,4)}\) (see also the well-known textbooks, e.g. Refs. 5,6) of the three-dimensional (3D) vectors of the electric and magnetic fields, E and B respectively, drastically differ
from the correct Lorentz transformations (LT) of the corresponding 4D algebraic objects representing the electric and magnetic fields. The fundamental difference is that in the ST, e.g., the components of the transformed 3D $E'_{st}$ are expressed by the mixture of components of the 3D $E$ and $B$, and similarly for $B'_{st}$. However, the correct LT always transform the 4D algebraic object representing the electric field only to the electric field, and similarly for the magnetic field. The results from Refs. 1, 2 are used here to investigate the LT and the ST of the usual Maxwell equations (ME) with the 3D $E$ and $B$. Different 4D algebraic objects are used to represent the standard observer dependent and the new observer independent electric and magnetic fields. First the electric and magnetic fields are represented by the observer dependent 1-vectors $E_f$ and $B_f$ defined in the $\gamma_0$ - frame. The usual ME in the component form are derived in Sec. 2.1. The LT of the ME are considered in Sec. 2.3. It is explicitly shown in Sec. 2.3., using the correct LT of $E_f$ and $B_f$, that the Lorentz transformed ME are not of the same form as the original ones. This proves that, contrary to the general opinion, the usual ME are not Lorentz covariant equations. In Sec. 2.4. the ST of the usual ME are considered taking into account the ST of the components of the 3D $E$ and $B$. It is proved that both the ST of the 3D $E$ and $B$ and the ST of the usual ME have nothing in common with the correct LT. The new Lorentz invariant field equations are constructed in Sec. 2.6. in which the electric and magnetic fields are represented by the 1-vectors $E$ and $B$ that are defined without reference frames. The whole consideration is briefly repeated in the same sections using the observer dependent bivectors $E_H$ and $B_H$ defined in the $\gamma_0$ - frame and the coordinate-free bivectors $E_{ Hv}$ and $B_{ Hv}$. In the geometric algebra formalism the active LT are used. This geometric approach is compared with the usual manner in which the ME with the 3D $E$ and $B$ are obtained from the covariant ME with $F^{\mu \nu}$, Sec. 2.2., and from the Lorentz transformed $F'^{\mu \nu}$, Sec. 2.5.. In Sec. 3. the whole consideration is performed in the tensor formalism using the coordinate-free 4-vectors $E^a$ and $B^a$ and the observer dependent 4-vectors $E^a_f$ and $B^a_f$ defined in the $\gamma_0$ - frame. In the tensor formalism the passive LT are used. All quantities in the Lorentz invariant field equations, with 1-vectors $E$ and $B$, bivectors $E_{ Hv}$ and $B_{ Hv}$ and the abstract 4-vectors $E^a$ and $B^a$ are geometric, coordinate-free quantities, i.e., the absolute quantities (AQs). They are defined without reference frames, or, when some basis has been introduced, they are represented as coordinate-based geometric quantities (CBGQs) comprising both components and a basis. All such equations are completely equivalent to the
field equations with $F$ (given, e.g. in Refs. 7-9 and discussed in detail in Ref. 10) or with $F_{ab}$ (already presented, e.g., in Ref. 11). It can be concluded from the consideration presented in all mentioned sections that the difference between the ST and the LT of the ME does not depend on the character of the 4D algebraic objects used to represent the electric and magnetic fields. The comparison with experiments is given in Sec. 4. and it shows that this geometric approach agrees with experiments, e.g., the Faraday disk, in all relatively moving inertial frames of reference, which is not the case with the usual approach with the 3D $E$ and $B$ and their ST. (The comparison of the geometric approach to special relativity (SR) and of the standard formulation of SR with experiments that test SR is also given in detail in Ref. 12.) The summary and conclusions are presented in Sec. 5. (We note that the great part of the consideration exposed in this paper is also presented in Ref. 13.)

2. THE PROOF OF THE DIFFERENCE BETWEEN THE LT AND THE ST OF THE ME USING THE GEOMETRIC ALGEBRA APPROACH

For the usual formulation of electrodynamics with the Clifford multivectors, see, e.g., Refs. 7-9. In Refs. 7-9 the electromagnetic field is represented by a bivector-valued function $F = F(x)$ on the spacetime. The source of the field is the electromagnetic current $j$ which is a 1-vector field and the gradient operator $\partial$ is also 1-vector. A single field equation for $F$ is first given by M. Riesz\(^{(14)}\) as

$$\partial F = j/\varepsilon_0 c, \quad \partial \cdot F + \partial \wedge F = j/\varepsilon_0 c.$$  \hspace{1cm} (1)

The trivector part is identically zero in the absence of magnetic charge. The geometric (Clifford) product is written by simply juxtaposing multivectors $AB$. The dot “·” and wedge “∧” in (1) denote the inner and outer products respectively. All quantities in (1) are AQs. Thence they are independent of the reference frame and the chosen system of coordinates in that frame. Consequently the equation (1) is a Lorentz invariant field equation. In the geometric algebra formalism (as in the tensor formalism as well) one mainly deals either with 4D AQs, e.g., the Clifford multivector $F$ (the abstract tensor $F_{ab}$) or, when some basis has been introduced, with CBGQs that comprise both components and a basis. The SR that exclusively deals with AQs or,
equivalently, with CBGQs, can be called the invariant SR.\textsuperscript{(11,12,10,15)} The reason for this name is that upon the passive LT any 4D CBGQ remains unchanged. The invariance of some 4D CBGQ upon the passive LT reflects the fact that such mathematical, invariant, geometric 4D quantity represents the same physical object for relatively moving observers. It is taken in the invariant SR that such 4D geometric quantities are well-defined not only mathematically but also experimentally, as measurable quantities with real physical meaning. Thus they have an independent physical reality.

In the usual geometric algebra formalism, e.g., Refs. 7, 8, 9, instead of to work only with such observer independent quantities one introduces (in order to get a more familiar form for (11)) a space-time split and the relative vectors in the $\gamma_0$ - frame, i.e., a particular time-like direction $\gamma_0$ is singled out. $\gamma_0$ is tangent to the world line of an observer at rest in the $\gamma_0$ - frame.

(The generators of the spacetime algebra are four basis vectors $\gamma_\mu, \mu = 0...3$, satisfying $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = diag(+---)$. They form the standard basis $\{\gamma_\mu\}$. This basis is a right-handed orthonormal frame of vectors in the Minkowski spacetime $M^4$ with $\gamma_0$ in the forward light cone. The $\gamma_k$ ($k = 1, 2, 3$) are spacelike vectors. The $\gamma_\mu$ generate by multiplication a complete basis for spacetime algebra: $1, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu \gamma_5, \gamma_5$ (16 independent elements). $\gamma_5$ is the pseudoscalar for the frame $\{\gamma_\mu\}$. It is worth noting that the standard basis corresponds, in fact, to the specific system of coordinates, i.e., to Einstein’s system of coordinates. In the Einstein system of coordinates the Einstein synchronization\textsuperscript{(4)} of distant clocks and Cartesian space coordinates $x^i$ are used in the chosen inertial frame of reference. However different systems of coordinates of an inertial frame of reference are allowed and they are all equivalent in the description of physical phenomena. For example, in Ref. 11 two very different, but completely equivalent systems of coordinates, the Einstein system of coordinates and “radio” (“r”) system of coordinates, are exposed and exploited throughout the paper. In this paper, for the sake of brevity and of clearness of the whole exposition, we shall work only with the standard basis $\{\gamma_\mu\}$, but remembering that the approach with 4D quantities that are defined without reference frames holds for any choice of the basis.)

The bivector field $F$ is decomposed in the $\gamma_0$ - frame into electric and magnetic parts using different algebraic objects to represent these fields. The explicit appearance of $\gamma_0$ in these expressions implies that the space-time split is observer dependent and thus all quantities obtained by the space-time split in the $\gamma_0$ - frame are observer dependent quantities. In Refs. 7,8 the observer
independent $F$ field from (1) is expressed in terms of observer dependent quantities, i.e., as the sum of a relative vector $E_H$ and a relative bivector $\gamma_5 B_H$

\[
F = E_H + c\gamma_5 B_H, \quad E_H = (F \cdot \gamma_0)\gamma_0 = (1/2)(F - \gamma_0 F\gamma_0), \\
\gamma_5 B_H = (1/c)(F \wedge \gamma_0)\gamma_0 = (1/2c)(F + \gamma_0 F\gamma_0).
\]  

(The subscript $H$ is for “Hestenes.”) Both $E_H$ and $B_H$ are, in fact, bivectors. Similarly in Ref. 9 $F$ is decomposed in terms of observer dependent quantities, 1-vector $E_J$ and a bivector $B_J$ (the subscript $J$ is for “Jancewicz”) as $F = \gamma_0 \wedge E_J - cB_J$, where $E_J = F \cdot \gamma_0$ and $B_J = -(1/c)(F \wedge \gamma_0)\gamma_0$. The $F$ field can be also decomposed in terms of other algebraic objects; the observer dependent electric and magnetic parts of $F$ are represented with 1-vectors that are denoted as $E_f$ and $B_f$ (see also Refs. 2, 15). The physical description with 1-vectors $E_f$ and $B_f$ is simpler but completely equivalent to the description with the bivectors $E_H$, $B_H$, Refs. 7,8, or with 1-vector $E_f$ and a bivector $B_f$, Ref. 9. Such decomposition of $F$ is not only simpler but also much closer to the classical representation of the electric and magnetic fields by the 3D vectors $E$ and $B$ than those used in Refs. 7, 8, 9. Thus

\[
F = E_f \wedge \gamma_0 + c(\gamma_5 B_f) \cdot \gamma_0, \\
E_f = F \cdot \gamma_0, \quad B_f = -(1/c)\gamma_5(F \wedge \gamma_0).
\]  

Having at our disposal different decompositions of $F$ into observer dependent quantities we proceed to present the difference between the ST and the LT of the ME using the decomposition (3) and only briefly the decomposition (2). We shall not deal with the decomposition of $F$ into $E_J$ and $B_J$ from Ref. 9 since both the procedure and the results are completely the same as with the decompositions (3) and (2).

2.1. The Field Equations in the $\gamma_0$ - Frame. The Maxwell Equations

When (3) is introduced into the field equation for $F$, Eq. (1), we find

\[
\partial[(F \cdot \gamma_0) \wedge \gamma_0 + (F \wedge \gamma_0) \cdot \gamma_0] = j/\varepsilon_0 c \\
\partial(E_f \wedge \gamma_0 + c(\gamma_5 B_f) \cdot \gamma_0) = j/\varepsilon_0 c.
\]  


The equations (4) can be now written as coordinate-based geometric equations (CBGEs) in the standard basis \{\gamma_\mu\} and the second equation becomes
\[
\partial_\alpha [\delta^{\alpha\beta}_{\mu\nu} E^\nu_f (\gamma_0)^\nu + c\varepsilon^{\alpha\beta\mu\nu} (\gamma_0)^\mu B_{f,\nu}] - (j^\beta/c\varepsilon_0)\gamma_0 = 0,
\]

where \(\gamma_0 = (\gamma_0)^\mu\gamma_\mu\) with \((\gamma_0)^\mu = (1, 0, 0, 0)\) and

\[
E_f = E^\mu_f \gamma_\mu = 0\gamma_0 + F^{0\gamma_0},
B_f = B^\mu_f \gamma_\mu = 0\gamma_0 + (-1/2c)\varepsilon^{0kli} F_{kl}.\]

Thence the components of \(E_f\) and \(B_f\) in the \{\gamma_\mu\} basis are

\[
E^i_f = F^{i0}, \quad B^i_f = (-1/2c)\varepsilon^{0kli} F_{kl}.\]

The relation (7) is nothing else than the standard identification of the components \(F^{\alpha\beta}\) with the components of the 3D vectors \(E\) and \(B\), see, e.g., Refs. 1,2. (It is worth noting that Einstein’s fundamental work\(^\text{[16]}\) is the earliest reference on covariant electrodynamics and on the identification of some components of \(F^{\alpha\beta}\) with the components of the 3D \(E\) and \(B\).) We see that in the \(\gamma_0\)-frame \(E_f\) and \(B_f\) do not have the temporal components \(E^0_f = B^0_f = 0\).

Thus \(E_f\) and \(B_f\) actually refer to the 3D subspace orthogonal to the specific timelike direction \(\gamma_0\). Notice that we can select a particular, but otherwise arbitrary, inertial frame of reference as the \(\gamma_0\)-frame, to which we shall refer as the frame of our “fiducial” observers (for this name see Ref. 17). The subscript “\(f\)” in the above relations stands for “fiducial” and denotes the explicit dependence of these quantities on the \(\gamma_0\)-, i.e., “fiducial” - observer.

Using that \(E^0_f = B^0_f = 0\) and \((\gamma_0)^\mu = (1, 0, 0, 0)\) the equation (5) becomes

\[
(\partial_k E^k_f - j^0/c\varepsilon_0)\gamma_0 + (-\partial_0 E^i_f + c\varepsilon^{ijk0} \partial_j B_{fk} - j^i/c\varepsilon_0)\gamma_i +
(-c\partial_k B^k_f)\gamma_5\gamma_0 + (c\partial_0 B^i_f + \varepsilon^{ijk0} \partial_j E_{fk})\gamma_5\gamma_i = 0.
\]

The first part (with \(\gamma_0\)) in Eq. (8) is from the 1-vector part of Eq. (1), i.e., Eq. (2), whereas the second one (with \(\gamma_5\gamma_0\)) is from the trivector (pseudovector) part of Eq. (1), i.e., Eq. (3). Both parts in Eq. (8) are written as CBGEs in the standard basis \{\gamma_\mu\} and cannot be further simplified as geometric equations. In the first part (with \(\gamma_0\)) in Eq. (8) one recognizes \textit{two Maxwell equations} in the \textit{component form}, the Gauss law for the electric field (the
first bracket, with $\gamma_0$ and the Ampère-Maxwell law (the second bracket, with $\gamma_i$). Similarly from the second part (with $\gamma_5\gamma_\alpha$) in Eq. (8) we recognize the component form of another two Maxwell equations, the Gauss law for the magnetic field (with $\gamma_5\gamma_0$) and Faraday’s law (with $\gamma_5\gamma_i$).

The whole procedure can be repeated using the decomposition of $F$, Eq. (2), into the bivectors $E_H, B_H$ as in Refs. 7,8. We shall quote only the results (the complete derivation is given in Ref. 13). When the decomposition (2) is substituted into Eq. (1) we find

$$\partial(E_H + c\gamma_5 B_H) = j/\varepsilon_0 c.$$  \hfill (9)

All quantities in Eq. (9) can be written as CBGQs in the standard basis $\{\gamma_\mu\}$ (see also Refs. 2, 13). Thus $E_H = F^{0i}\gamma_i \land \gamma_0, B_H = (1/2c)e^{kil}F_{kl}\gamma_i \land \gamma_0$. Both bivectors $E_H$ and $B_H$ are parallel to $\gamma_0$, that is, it holds that $E_H \land \gamma_0 = B_H \land \gamma_0 = 0$. When written in terms of components (e.g., $(E_H)^{\mu\nu} = \gamma^\nu \cdot (\gamma^\mu \cdot E_H) = (\gamma^\nu \land \gamma^\mu) \cdot E_H$) one finds that $E_H = (E_H)^{00}\gamma_0 \land \gamma_0 = E^i\gamma_i \land \gamma_0, B_H = (B_H)^{i0}\gamma_i \land \gamma_0 = B^i\gamma_i \land \gamma_0$. Thus it holds that $(E_H)^{ij} = (B_H)^{ij} = 0$. Multiplying Eq. (9) by $\gamma_0$ and using the above expressions for $E_H, B_H$ we write the resulting equation as a CBGE

$$(\partial_k E^k - j^0/c\varepsilon_0) + (\partial_0 E^i - e\varepsilon^{ijk}0 \partial_j B_k + j^i/c\varepsilon_0)(\gamma_i \land \gamma_0) + (c\partial_k B^k)\gamma_5 + (c\partial_0 B^i + \varepsilon^{ijk0}\partial_j E_k)\gamma_5(\gamma_i \land \gamma_0) = 0.$$  \hfill (10)

The equation (10) is exactly the same as the equations obtained in the geometric algebra formalism, e.g., the equations (8.5) and (8.6a-8.6d) in the first of Ref. 7, now written as a CBGE. Eq. (11) encodes all four ME in the component form in the same way as it happens with the equation (8). It is worth noting that this step, the multiplication of Eq. (9) by $\gamma_0$, in order to get the usual ME, is unnecessary in the formulation with 1-vectors $E_f$ and $B_f$. This shows that the approach with 1-vectors $E_f$ and $B_f$ is simpler than the approach with bivectors $E_H$ and $B_H$ and also it is much closer to the classical formulation of electromagnetism with the 3D vectors $E$ and $B$.

2.2. The Comparison of the usual Covariant Approach and the Geometric Approach, I

Let us now examine the difference between the usual covariant approach, e.g., Refs. 5,6, and the above geometric approach. The covariant approach
deals with the *component form* (implicitly taken in the standard basis \(\{\gamma_\mu\}\)) of the ME with \(F^{\alpha\beta}\) and its dual \(\ast F^{\alpha\beta}\)

\[
\partial_\alpha F^{\alpha\beta} = j^\beta / \varepsilon_0 c, \quad \partial_\alpha \ast F^{\alpha\beta} = 0,
\]

where \(\ast F^{\alpha\beta} = (1/2) \varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}\). (Almost always in the usual covariant approaches to SR one considers *only the components* of the geometric quantities taken in the \(\{\gamma_\mu\}\) basis and thus not the whole tensor. However the components are coordinate quantities and they do not contain the whole information about the physical quantity.) In order to get the component form of the ME with the 3D \(E\) and \(B\)

\[
\partial_k E_k - j^0 / c\varepsilon_0 = 0, \quad -\partial_0 E_i + c\varepsilon_{ijk} \partial_j B_k - j^i / c\varepsilon_0 = 0,
\]

\[
\partial_k B_k = 0, \quad c\partial_0 B_i + \varepsilon_{ijk} \partial_j E_k = 0
\]

from Eq. (11) one simply makes the *identification of six independent components* of \(F^{\mu\nu}\) with three components \(E_i\) and three components \(B_i\)

\[
E_i = F^{i0}, \quad B_i = (1/2c)\varepsilon_{ikl} F_{lk}.
\]

(The components of the 3D fields \(E\) and \(B\) are written with lowered (generic) subscripts, since they are not the spatial components of the 4D quantities. This refers to the third-rank antisymmetric \(\varepsilon\) tensor too. The super- and subscripts are used only on the components of the 4D quantities.) Then the 3D \(E\) and \(B\), as *geometric quantities in the 3D space*, are constructed from these six independent components of \(F^{\mu\nu}\) and the *unit 3D vectors* \(i, j, k\), e.g., \(E = F^{10}i + F^{20}j + F^{30}k\). The usual ME with the 3D \(E\) and \(B\) are obtained from Eq. (12) and so constructed 3D \(E\) and \(B\) as

\[
\nabla E(r, t) = \rho(r, t) / \varepsilon_0, \quad \nabla \times E(r, t) = -\partial B(r, t) / \partial t
\]

\[
\nabla B(r, t) = 0, \quad \nabla \times B(r, t) = (1/\varepsilon_0 c^2) j(r, t) + (1/c^2) \partial E(r, t) / \partial t.
\]

Such usual procedure has a number of disadvantages. They are:

i) The covariant ME (11) are written in the component form and these components are taken in the Einstein system of coordinates, whereas the field equation (1) is written with AQs, i.e., it is independent of the reference frame and of the chosen system of coordinates in that frame. When Eq. (11) is written as a CBGE in the \(\gamma_0\) - frame with the \(\{\gamma_\mu\}\) basis and when *only* the components are taken then Eq. (11) becomes Eq. (11).
ii) It is considered by the identification (13) that $E_i$ and $B_i$ are the primary quantities for the whole electromagnetism and that the components $F^{\alpha\beta}$ are derived from and determined with $E_i$ and $B_i$. But the components $F^{\alpha\beta}$ are determined as the solutions of the field equations (11) for the given sources and, in principle, they are not in any obvious relation with $E_i$ and $B_i$, which are the solutions of Eq. (12). It is shown in Ref. 10 that the whole electromagnetism can be formulated exclusively by the well-defined geometric 4D quantity, the Faraday bivector $F$, without even mentioning the 3D $E$ and $B$ or the 4D electromagnetic potentials (which are gauge dependent). Thus $F$ is the primary quantity and not the 3D $E$ and $B$, or the potentials.

iii) The simple identification (13) of the components $E_i$ and $B_i$ with the components of $F^{\alpha\beta}$ is not a permissible tensor operation; permissible tensor operations with components of tensors produce components of new tensors, for example: a) multiplication by a scalar field b) addition of components of two tensors c) contraction on a pair of indices, ... .

iv) Such identification of the components of the 3D $E$ and $B$ with components of $F^{\mu\nu}$ is dependent on the chosen system of coordinates. In the usual covariant approaches the standard basis $\{\gamma^\mu\}$ is implicitly assumed. However the identification (13) is meaningless, e.g., in the “t” system of coordinates, the $\{r_\mu\}$ basis, in which only the Einstein synchronization is replaced by an asymmetric synchronization, the “radio” synchronization.

Then $F^{10}_{r} = F^{10} + F^{12} + F^{13}$, which means that by the relation (13) $E_{1r} = F^{10}_{r}$ the component $E_{1r}$ in the $\{r_\mu\}$ basis is expressed as the combination of $E_i$ and $B_i$ components from the $\{\gamma^\mu\}$ basis, $E_{1r} = E_1 - B_3 + B_2$, see Ref. 11.

v) $E_i$ and $B_i$ in Eq. (12) are the components of vectors defined on the 3D space while $F^{\alpha\beta}$ are the components of tensor defined on the 4D spacetime. Thence when forming the geometric quantities the components of the 4D quantity would need to be multiplied with the unit vectors $\gamma_i$ from the 4D spacetime and not with the unit vectors $i, j, k$ from the 3D space.

On the other hand in the above geometric approach the mapping between $F$ and 1-vectors $E_f, B_f$, or bivectors $E_H, B_H$, given by the equations (3) and (2) respectively, is performed by a correct mathematical procedure and all quantities are defined on the same 4D spacetime. Instead of Eq. (12) that contains a combination of quantities (components) from the 4D spacetime $(\partial_\mu, j^\mu)$ and from the 3D space $(E_i, B_i, \varepsilon_{ijk})$, we have the CBGEs (3) and (10) in the geometric approach, which contain only components $E^\mu_f, B^\mu_f$ and $(E_H)^{\mu\nu}, (B_H)^{\mu\nu}$ of the well-defined 4D quantities $E_f, B_f$, and $E_H, B_H$. Similarly instead of the usual ME (14) with geometric quantities from the 3D
space \( \mathbf{E} \) and \( \mathbf{B} \) we have the ME (1) and (2) with geometric quantities from the 4D spacetime \( E_f, B_f, \) and \( E_H, B_H \). However it has to be noted that the decompositions (3) and (2) still have some disadvantages. In Eqs. (3) and (2) the observer independent 4D quantity \( F \) is decomposed into the observer dependent 4D quantities \( E_f, B_f, \) or \( E_H, B_H \) by using the space-time split in the \( \gamma_0 \) - frame. The space-time split in another \( \gamma'_0 \) - frame is not obtained by the LT from that one in the \( \gamma_0 \) - frame. This problem will be discussed in the subsequent sections and in Secs. 2.6. and 3. we shall present the new decompositions of \( F \) without using the space-time split.

2.3. The LT of the Maxwell Equations

Let us now apply the active LT upon Eq. (8), or Eq. (5). We write Eq. (8), or Eq. (5), in the form

\[
a^\alpha \gamma_\alpha + b^\alpha (\gamma_5 \gamma_\alpha) = 0. \tag{15}
\]

The coefficients \( a^\alpha \) and \( b^\alpha \) are clear from Eq. (8), or Eq. (5); they are the usual ME in the component form. In the Clifford algebra formalism, e.g., Refs. 7-9, the LT are considered as active transformations; the components of, e.g., some 1-vector relative to a given inertial frame of reference (with the standard basis \( \{ \gamma_\mu \} \)) are transformed into the components of a new 1-vector relative to the same frame (the basis \( \{ \gamma_\mu \} \) is not changed). Furthermore the LT are described with rotors \( R, R \tilde{R} = 1 \), in the usual way as \( p \to p' = Rp \tilde{R} = p^\mu \gamma_\mu \). To an observer in the \( \{ \gamma_\mu \} \) frame the vector \( p' \) appears the same as the vector \( p \) appears to an observer in the \( \{ \gamma'_\mu \} \) frame. For boosts in the direction \( \gamma_1 \) the rotor \( R \) is given by the relation

\[
R = (1 + \gamma + \gamma \beta \gamma_0 \gamma_1)/(2(1 + \gamma))^{1/2}, \tag{16}
\]

\( \beta \) is the scalar velocity in units of \( c \), \( \gamma = (1 - \beta^2)^{-1/2} \). Then the LT of Eq. (4) are given as

\[
R\{\partial \left[ (F \cdot \gamma_0) \wedge \gamma_0 + (F \wedge \gamma_0) \cdot \gamma_0 \right] - j/\varepsilon_0 c \} \tilde{R} = 0,
\]

\[
R\{\partial [E_f \wedge \gamma_0 + c(\gamma_5 B_f) \cdot \gamma_0] - j/\varepsilon_0 c \} \tilde{R} = 0, \tag{17}
\]

where \( R \) is given by Eq. (16). (A coordinate-free form of the LT is also given in the Clifford algebra formalism in Ref. 15 and in the tensor formalism in Ref. 11, see also Ref. 18. The form presented in Ref. 15 does not need to
use rotors but, of course, it can be expressed by rotors as well.) Then the LT of the usual ME (15) are

\[ R\{a^\alpha \gamma_\alpha + b^\alpha (\gamma_5 \gamma_\alpha)\} \tilde{R} = 0. \]  

(18)

Performing the LT we find the explicit expression for Eq. (18) as

\[ \gamma_0 (\gamma a^0 - \beta \gamma a^1) + \gamma_1 (\gamma a^1 - \beta \gamma a^0) + \gamma_2 a^2 + \gamma_3 a^3 + \gamma_5 \gamma_0 (\gamma b^0 - \beta \gamma b^1) + \gamma_5 \gamma_1 (\gamma b^1 - \beta \gamma b^0) + \gamma_5 \gamma_2 b^2 + \gamma_5 \gamma_3 b^3 = 0. \]  

(19)

It can be simply written as

\[ a'^{\alpha} \gamma_\alpha + b'^{\alpha} (\gamma_5 \gamma_\alpha) = 0, \]  

(20)

where, e.g., \( a'^{0} = \gamma a^0 - \beta \gamma a^1 \) and, as it is said, \( a^{\alpha} \) and \( b^{\alpha} \) are the usual ME in the component form given in Eq. (8), or Eq. (5). This result, Eq. (19), i.e., Eq. (20), is exactly the usual result for the active LT of a 1-vector and of a pseudovector. It is important to note that, e.g., the Gauss law for the electric field \( a^0 \) does not transform by the LT again to the Gauss law but to \( a'^{0} \), which is a combination of the Gauss law and a part of the Ampère-Maxwell law (\( a^1 \)).

The second equation in (17) can be expressed in terms of Lorentz transformed derivatives and Lorentz transformed 1-vectors \( E_f \) and \( B_f \) as

\[ \partial'[E'_f \wedge (v'/c) + c(\gamma_5 B'_f) \cdot (v'/c)] - j'/\varepsilon_0 c = 0, \]  

(21)

where \( \partial' = R \partial \tilde{R}, \ v'/c = R \gamma_0 \tilde{R} = \gamma \gamma_0 - \beta \gamma \gamma_1 \) and (see also Ref. 2) the Lorentz transformed \( E'_f \) is

\[ E'_f = R (F \cdot \gamma_0) \tilde{R} = R E_f \tilde{R} = R (F^{\mu_0} \gamma_\mu) \tilde{R} = E'_f^\mu \gamma_\mu = \]

\[ = - \beta \gamma E'_f^1 \gamma_0 + \gamma E'_f^1 \gamma_1 + E'_f^2 \gamma_2 + E'_f^3 \gamma_3, \]  

(22)

what is the usual form for the active LT of the 1-vector \( E_f \). Similarly we find that \( B'_f \) is

\[ B'_f = R \left[ -(1/c) \gamma_5 (F \wedge \gamma_0) \right] \tilde{R} = RB_f \tilde{R} = R \left[ (-1/2c) \varepsilon_0^{0kli} F_{kl} \gamma_i \right] \tilde{R} = \]

\[ = B'_f^\mu \gamma_\mu = - \beta \gamma B'_f^1 \gamma_0 + \gamma B'_f^1 \gamma_1 + B'_f^2 \gamma_2 + B'_f^3 \gamma_3. \]  

(23)

It is worth noting that \( E'_f \) and \( B'_f \) are no longer orthogonal to \( \gamma_0 \), i.e., they have the temporal components \( \neq 0 \). Furthermore the components \( E'_f^\mu \) (\( B'_f^\mu \))
transform upon the active LT again to the components \( E_f^\mu \) (\( B_f^\mu \)) as seen from Eqs. \((22)\) and \((23)\); there is no mixing of components. When Eq. \((21)\) is written in an expanded form as a CBGE in the standard basis \( \{ \gamma_\mu \} \) it takes the form of Eq. \((20)\) but now the coefficients \( a^\alpha \) are written by means of the Lorentz transformed components \( \partial_k, E_f^k \) and \( B_f^k \) (for simplicity only the term \( a^0 \gamma_0 \) is presented)

\[
a^0 \gamma_0 = \{ [\gamma(\partial_k E_f^k) - j^0/c\varepsilon_0] + \beta\gamma[\partial_1 E_f^0 + c(\partial_2 B_f^3 - \partial_3 B_f^2)] \} \gamma_0. \tag{24}
\]

and it substantially differs in form from the term \( a^0 \gamma_0 = (\partial_k E_f^k - j^0/c\varepsilon_0) \gamma_0 \) in Eq. \((8)\). As explained above the coefficient \( a^0 \) is the Gauss law for the electric field written in the component form. It is clear from Eq. \((24)\) that the LT do not transform the Gauss law into the “primed” Gauss law but into quite different law Eq. \((23)\); \( a^0 \) contains the time component \( E_f^0 \) (while \( E_f^0 = 0 \)), and also the new “Gauss law” includes the derivatives of the magnetic field. The same situation happens with other Lorentz transformed terms, which explicitly shows that the Lorentz transformed ME (\(21\) with \(21\)) are not of the same form as the original ones Eq. \((8)\). This is a fundamental result which reveals that, contrary to the previous derivations, e.g., Refs. 4,16,5-9, and contrary to the general opinion, the usual ME are not Lorentz covariant equations. The physical consequences of this achievement will be very important and they will be carefully examined.

Again as in Sec 2.1. we give only the results for the case when \( E_H, B_H \) are used (all details are given in Ref. 13.) The relation \((10)\) can be written in the form \( a^0 + a^i (\gamma_i \wedge \gamma_0) + b^0 \gamma_5 + b^i \gamma_5 (\gamma_i \wedge \gamma_0) = 0 \). The coefficients \( a^0, a^i \) and \( b^0, b^i \) are clear from Eq. \((10)\); they are the usual ME in the component form. As it is said the usual ME \((10)\) are obtained multiplying Eq. \((9)\) by \( \gamma_0 \). The LT of the resulting equation (after multiplication by \( \gamma_0 \)) are

\[
R\{ \gamma_0 [\partial(E_H + c\gamma_5 B_H) - j'/\varepsilon_0 c]\} \tilde{R} = 0. \tag{25}
\]

Then after applying the LT upon Eq. \((10)\) we find \( a^0 + R[a^i (\gamma_i \wedge \gamma_0)] \tilde{R} + b^0 \gamma_5 + R[b^i \gamma_5 (\gamma_i \wedge \gamma_0)] \tilde{R} = 0 \), where, e.g.,

\[
R[a^i (\gamma_i \wedge \gamma_0)] \tilde{R} = a^i (\gamma_i \wedge \gamma_0) + \gamma[a^2 (\gamma_2 \wedge \gamma_0) + a^3 (\gamma_3 \wedge \gamma_0)] - \beta\gamma[a^2 (\gamma_2 \wedge \gamma_1) + a^3 (\gamma_3 \wedge \gamma_1)], \tag{26}
\]

see Ref. 13. This result is the usual result for the active LT of a multivector from Eq. \((10)\). The equation \((25)\) can be expressed in terms of Lorentz transformed derivatives and Lorentz transformed \( E_H \) and \( B_H \) as

\[
(v'/c)[\partial'(E_H' + c\gamma_5 B_H') - j'/\varepsilon_0 c] = 0,
\]
where \( v'/c = R\gamma_0 \tilde{R}, \) \( \partial' = R\partial \tilde{R}, \) and the Lorentz transformed bivectors are \( \mathbf{E}'_H \) and \( \mathbf{B}'_H. \) This \( \mathbf{E}'_H \) is

\[
\mathbf{E}'_H = R[(F \cdot \gamma_0)\gamma_0] \tilde{R} = R\mathbf{E}_H \tilde{R} = E^1 \gamma_1 \land \gamma_0 + \gamma(E^2 \gamma_2 \land \gamma_0 + E^3 \gamma_3 \land \gamma_0) - \beta \gamma \gamma(E^2 \gamma_2 \land \gamma_1 + E^3 \gamma_3 \land \gamma_1),
\]

(27)

where \( E^i = F^{i0} \) and it is similarly obtained for \( \mathbf{B}'_H, \) see Refs. 2, 13. \( \mathbf{E}'_H, \) Eq. (27) (and also \( \mathbf{B}'_H) \) are the familiar forms for the active LT of bivectors, here \( \mathbf{E}_H \) and \( \mathbf{B}_H. \) It is worth noting that \( \mathbf{E}'_H \) and \( \mathbf{B}'_H, \) in contrast to \( \mathbf{E}_H \) and \( \mathbf{B}_H, \) are not parallel to \( \gamma_0, \) i.e., it does not hold that \( \mathbf{E}'_H \land \gamma_0 = \mathbf{B}'_H \land \gamma_0 = 0 \) and thus there are \( (\mathbf{E}'_H)^{ij} \neq 0 \) and \( (\mathbf{B}'_H)^{ij} \neq 0. \) Further, as it happens for \( E_f \) and \( B_f, \) see Eqs. (22) and (23), the components \( (\mathbf{E}_H)^{\mu\nu} ((\mathbf{B}_H)^{\mu\nu}) \) transform upon the active LT again to the components \( (\mathbf{E}'_H)^{\mu\nu} ((\mathbf{B}'_H)^{\mu\nu}); \) there is no mixing of components. Thus by the active LT \( \mathbf{E}_H \) transforms to \( \mathbf{E}'_H \) and \( \mathbf{B}_H \) to \( \mathbf{B}'_H. \) Actually, as we said, this is the way in which every bivector transforms upon the active LT. Then Eq. (26) can be written as a CBGE in the standard basis \( \{\gamma_\mu\}, \) but for simplicity we only quote the scalar term \( a^0 \)

\[
a^0 = -\beta \gamma \partial'_0 (\mathbf{E}'_H)^{10} + \gamma [\partial'_k (\mathbf{E}'_H)^{k0}] + \beta \gamma [\partial'_2 (\mathbf{E}'_H)^{21} + \partial'_3 (\mathbf{E}'_H)^{31}] - (\gamma j^0 - \beta j^1)/\varepsilon_0 c
\]

(28)

Comparing \( a^0, \) Eq. (28), with \( a^0 \) from the usual ME (10) \( a^0 = \partial_k (\mathbf{E}_H)^{k0} - j^0/\varepsilon_0, \) we again see, as with \( E_f \) and \( B_f, \) that \( a^0 \) substantially differs in form from the term \( a^0 \) in Eq. (10). The same situation happens with other transformed terms, which shows that the Lorentz transformed ME, (10) with (28), are not of the same form as the original ones, Eq. (10). This is a fundamental result which once again reveals that, contrary to the previous derivations, e.g., Refs. 4, 16, 5-9, and contrary to the generally accepted belief, the usual ME are not Lorentz covariant equations.

2.4. The ST of the Maxwell equations

In contrast to the correct active LT of \( E_f, \) Eq. (22), and \( B_f, \) Eq. (23), it is wrongly assumed in the usual derivations of the the ST for \( E'_{st} \) and \( B'_{st} \) (the subscript \( st \) is for standard) that the quantities obtained by the active LT of \( E_f \) and \( B_f \) are again in the 3D subspace of the \( \gamma_0 \)-observer, see also Ref. 2. This means that it is wrongly assumed in all usual derivations, e.g.,
in the Clifford algebra formalism\(^{(7,8,9)}\) (and in the tensor formalism\(^{(16,5,6)}\) as well), that one can again perform the same identification of the transformed components \(F'_{\mu\nu}\) with the components of the 3D \(E'\) and \(B'\) as in Eq. \((7)\). Thus it is taken in Refs. 7, 8, 9 that for the transformed \(E'\) and \(B'\) again hold \(E'_{\mu0} = B'_{\mu0} = 0\) as for \(E_f\) and \(B_f\)

\[
E'_{\mu} = (RF \tilde{R}) \cdot \gamma_0 = F' \cdot \gamma_0 = F'^{i0} \gamma_i = E'^{\mu}_{\nu} \gamma_i = E_f^1 \gamma_1 + (\gamma E_f^2 - \beta \gamma c B_f^2) \gamma_2 + (\gamma E_f^3 + \beta \gamma c B_f^2) \gamma_3, \tag{29}
\]

where \(F' = RF \tilde{R}\), and similarly for \(B'_{\mu}\)

\[
B'_{\mu} = -(1/c) \gamma_5 (F' \wedge \gamma_0) = -(1/2c) \varepsilon^{0kl} F'_{kl} \gamma_i = B'^{\mu}_{\nu} \gamma_i = B_f^1 \gamma_1 + (\gamma B_f^2 + \beta \gamma E_f^3/c) \gamma_2 + (\gamma B_f^3 - \beta \gamma E_f^2/c) \gamma_3. \tag{30}
\]

From the relativistically incorrect transformations \((29)\) and \((30)\) one simply finds the transformations of the spatial components \(E'_{\mu}\) and \(B'_{\mu}\)

\[
E'^{\mu}_{\nu} = F'^{i0}, \quad B'^{\mu}_{\nu} = (1/2c) \varepsilon^{0kl} F'_{kl}. \tag{31}
\]

As can be seen from Eqs. \((29)\) and \((30)\), i.e., from Eq. \((31)\), the transformations for \(E'_{\mu}\) and \(B'_{\mu}\) are exactly the ST of components of the 3D vectors \(E\) and \(B\) that are quoted in almost every textbook and paper on relativistic electrodynamics. Notice that, in contrast to the active LT \((22)\) and \((23)\), according to the ST \((29)\), i.e., \((31)\), the transformed components \(E'^{\mu}_{\nu}\) are expressed by the mixture of components \(E_f^i\) and \(B_f^i\), and Eq. \((30)\) shows that the same holds for \(B'^{\mu}_{\nu}\). In all previous treatments of SR, e.g., Refs. 7-9 (and Refs. 4,5,6,16) the transformations for \(E_{\mu}\) and \(B_{\mu}\) are considered to be the LT of the 3D electric and magnetic fields. However the above analysis, and Refs. 1,2 as well, show that the transformations for \(E'^{\mu}_{\nu}\) and \(B'^{\mu}_{\nu}\), Eq. \((31)\), are derived from the relativistically incorrect transformations \((29)\) and \((30)\), which are not the LT; the LT are given by the relations \((22)\) and \((23)\).

It is also argued in all previous works, starting in the year 1905 with Einstein’s fundamental paper on SR\(^{(4)}\) that the usual ME with the 3D \(E\) and \(B\) are Lorentz covariant equations. The relation \((21)\) together with Eq. \((24)\) shows that it is not true; the Lorentz transformed ME are not of the same form as the original ones. Here we explicitly show that in the usual derivations the ME remain unchanged in form not upon the LT but upon some transformations which, strictly speaking, have nothing to do with the LT of the equation \((4)\), i.e., of the ME \((8)\). The difference between the
Lorentz transformed ME, given by Eq. (17) or finally by Eq. (21) with Eq. (24) (or by Eq. (19)) and the equations (given below) obtained by applying the ST is the same as the difference between the LT of $E_f$ ($B_f$) given by Eqs. (22) (or 23) and their ST given by Eqs. (29) (30). Thus the ST of the equation are

$$\{[(RF\tilde{R}) \cdot \gamma_0] \wedge \gamma_0 + [(RF\tilde{R}) \wedge \gamma_0] \cdot \gamma_0\} - (Rj\tilde{R})/\varepsilon_0 c = 0, \quad \partial'\{E'_{\text{st}} \wedge \gamma_0 + c(\gamma_5 B'_{\text{st}}) \cdot \gamma_0\} - j'/\varepsilon_0 c = 0, \quad (32)$$

where $E'_{\text{st}}$ and $B'_{\text{st}}$ are determined by Eqs. (29) and (30). Notice that, in contrast to the correct LT (17) or (21), $\gamma_0$ is not transformed in Eq. (32).

When this second equation in (32) is written as a CBGE in the standard basis $\{\gamma_\mu\}$ it becomes

$$\partial'_k E'_{\text{st}}^{jk} - j^0/c\varepsilon_0 \gamma_0 + (\partial'_k E'_{\text{st}}^i + c\varepsilon^{ijk0} \partial'_i B'_{\text{st},k} - j^i/c\varepsilon_0)\gamma_0 +$$

$$-c\partial'_k B'_{\text{st}}^{jk} \gamma_5 \gamma_0 + (c\partial'_0 B'_{\text{st}}^i + \varepsilon^{ijk0} \partial'_j E'_{\text{st}}^{ik}) \gamma_5 \gamma_0 = 0. \quad (33)$$

The equation (33) is of the same form as the original ME (8) but the electric and magnetic fields are not transformed by the LT than by the ST. Therefore, as can be seen from Eq. (32) (together with Eqs. (29) and (30), $\gamma_0$ is not transformed in Eq. (32). When this second equation in (32) is written as a CBGE in the standard basis $\{\gamma_\mu\}$ it becomes

$$\partial'_k E'_{\text{st}}^{jk} - j^0/c\varepsilon_0 \gamma_0 + (\partial'_k E'_{\text{st}}^i + c\varepsilon^{ijk0} \partial'_i B'_{\text{st},k} - j^i/c\varepsilon_0)\gamma_0 +$$

$$-c\partial'_k B'_{\text{st}}^{jk} \gamma_5 \gamma_0 + (c\partial'_0 B'_{\text{st}}^i + \varepsilon^{ijk0} \partial'_j E'_{\text{st}}^{ik}) \gamma_5 \gamma_0 = 0. \quad (33)$$

Let us discuss the ST in the formulation with $E'_H$ and $B'_H$. As can be easily shown, see also Ref. 2, the ST for $E'_{H,\text{st}}$ and $B'_{H,\text{st}}$ are derived wrongly assuming that the quantities obtained by the active LT of $E_H$ and $B_H$ are again parallel to $\gamma_0$, i.e., that again holds $E'_H \wedge \gamma_0 = B'_H \wedge \gamma_0 = 0$ and consequently that $(E'_{H,\text{st}})^{ij} = (B'_{H,\text{st}})^{ij} = 0$. Hence, in contrast to the correct LT of $E_H$ (Eq. (27)) (and $B_H$), it is taken in the usual derivations (Ref. 7, Space-Time Algebra (eq. (18.22)), New Foundations for Classical Mechanics (Ch. 9 eqs. (3.51a,b)), Ref. 8 (Ch. 7.1.2 eq. (7.33))) that

$$E'_{H,\text{st}} = (F' \cdot \gamma_0)\gamma_0 = (E'_{H,\text{st}})^{0i} \gamma_i \wedge \gamma_0 = E'_{H,\text{st}}^i \gamma_i \wedge \gamma_0 =$$

$$E^1 \gamma_1 \wedge \gamma_0 + (\gamma E^2 - \beta \gamma c B^3) \gamma_2 \wedge \gamma_0 + (\gamma E^3 + \beta \gamma c B^2) \gamma_3 \wedge \gamma_0. \quad (34)$$

where $F' = RF\tilde{R}$, and similarly for $B'_{H,\text{st}}$, see Ref. 2. The relation (33) (and that one for $B'_{H,\text{st}}$) immediately gives the familiar expressions for the ST of the 3D vectors $E$ and $B$. Now, in contrast to the correct LT of $E_H$
(Eq. (27)) (and $B_H$), the components of the transformed $E'_{H,st}$ are expressed by the mixture of components $E^i$ and $B^i$, and the same holds for $B'_{H,st}$. The ST of Eq. (9) (after multiplication by $\gamma_0$) are given as

$$\gamma_0[\partial' (E'_{H,st} + c\gamma_5 B'_{H,st}) - j'/\varepsilon_0 c] = 0,$$

where $E'_{H,st}$ is determined by Eq. (34) (and similarly for $B'_{H,st}$). Notice again that, in contrast to the correct LT (25) or (26), $\gamma_0$ is not transformed in Eq. (35), as it is not transformed in the ST $E'_{H,st}$, Eq. (34) (and $B'_{H,st}$).

When Eq. (35) is written as a CBGE in the standard basis $\{\gamma_\mu\}$ it becomes

$$\partial'_k E'_{k,st} - \gamma_0 c/\varepsilon_0 c + \gamma_0 \gamma_5,$$

$$\partial'_0 E'_{i,st} - c\varepsilon_0\partial'_j B'_{st,k} + j'/c\varepsilon_0 (\gamma_i \wedge \gamma_0) + (c\partial'_k B'_{st})\gamma_5 + (c\partial'_0 B'_{st} + c\varepsilon_0\partial'_j E'_{st,k})\gamma_5 (\gamma_i \wedge \gamma_0) = 0.$$

This equation is of the same form as the original ME (10) but the bivectors $E_H$ and $B_H$ representing the electric and magnetic fields are not transformed by the LT than by the ST. A s seen from Eq. (35) this equation is not the LT of the original ME (10); the LT of the ME (10) is the equation (26) with (28).

2.5. The Comparison of the usual Covariant Approach and the Geometric Approach, II

In the usual covariant approach, e.g., Refs. 5,6, one transforms by the passive LT the covariant ME (11) and finds $\partial'_\alpha F'^{\alpha\beta} = j'^\beta/\varepsilon_0 c$, $\partial'_\alpha \wedge F'^{\alpha\beta} = 0$. (Upon the passive LT the set of components, e.g., $j^\mu$ from the $S$ frame transform to $j'^\mu$ in the relatively moving inertial frame of reference $S'$, $j'^\mu = L^\mu_{\nu} j^\nu$, where (for the boost in the $\gamma_1$ direction) $L^0_0 = L^1_1 = \gamma_1$, $L^0_1 = L^1_0 = -\beta \gamma$, $L^2_2 = L^3_3 = 1$ and all other components are zero.) Then the same identification as in Eq. (13) is assumed to hold for the transformed components $E'_i$ and $B'_i$

$$E'_i = F'^{i0}, \quad B'_i = (1/2c)\varepsilon_{ikl} F'^{kl},$$

e.g., $F'^{20} = \gamma F^{20} - \beta \gamma F^{21}$, which yields (by Eqs. (13) and (30)) that $E'_2 = \gamma E_2 - \beta \gamma cB_3$, see Jackson’s book (5) Sec. 11.10. Thus in the usual covariant approach the components $F'^{\alpha\beta}$ are transformed by the passive LT into $F'^{\alpha\beta}$ and then it is simply argued that six independent components of $F'^{\alpha\beta}$ are the “Lorentz transformed” components $E'_i$ and $B'_i$. The identification (36) reveals an additional disadvantage in the usual covariant approach that is not mentioned in Sec. 2.2.. It is
vi) It is not possible to speak about the LT of some components of \( F^{03} \) as in Eq. (36); the LT always transform the whole geometric 4D quantity and not some components. Further, by the same procedure as in Sec. 2.2., one finds the “transformed” equations of the same form as Eqs. (12) and (14), but with primed quantities replacing the unprimed ones, e.g., \( \partial_k E_k' - j^0/c\varepsilon_0 = 0 \), and

\[
\nabla' E'(r', t') = \rho'(r', t')/\varepsilon_0, \ldots, \tag{37}
\]

where, e.g., the 3D vector \( E' \) is again obtained multiplying the components \( F_{g0} \) by the unit 3D vectors \( i', j', k' \), \( E' = F_{10}i' + F_{20}j' + F_{30}k' \). However the meaning of the 3D vectors \( \hat{y}', \hat{j}', \hat{k}' \) is undefined; they are not obtained by any transformation, particularly not by the LT from the 3D vectors \( \hat{i}, \hat{j}, \hat{k} \). Obviously such procedure has the same disadvantages as those discussed in Sec. 2.2 including the new one, vi). The components \( E'_i, B'_i \) and the 3D fields are all ill-defined in the 4D spacetime. On the other hand the meaning of all quantities in the above geometric approach is very clear; they are all well-defined in the 4D spacetime. Moreover, the difference between the LT and the ST of the 4D quantities representing the electric and magnetic fields is clearly seen; in the LT always the whole 4D geometric quantity is transformed as, e.g., in Eqs. (22) and (23), whereas in the ST only a part of the whole 4D geometric quantity is transformed as, e.g., in Eqs. (29) and (30). Nevertheless the usual procedure, the identifications (13) and (36) and the derivation of the “transformed” equations (37) is considered for almost hundred years as relativistically correct procedure. It is argued in every paper and textbook on the relativistic electrodynamics (without exception as I am aware) that the equations (14) are Lorentz covariant equations, i.e., that the LT of the equations (14) are the equations (37). Our discussion explicitly shows that in the 4D spacetime the usual procedure is not justified either mathematically or physically.

2.6. Lorentz Invariant Field Equations with 1-Vectors \( E, B \) and Bivectors \( E_{Hv}, B_{Hv} \)

Let us now remove the disadvantage mentioned at the end of Sec. 2.2. that still exists in all Clifford algebra approaches to the electromagnetism. Instead of decomposing \( F \) into the observer dependent \( E_f \) and \( B_f \) in the \( \gamma_0 \) - frame, as in Eq. (3), we present the decomposition of \( F \) into the AQs, 1-vectors of the electric \( E \) and magnetic \( B \) fields that are defined without
reference frames, see also Ref. 15. We define

\begin{align*}
F &= (1/c)E \wedge v + (IB) \cdot v, \\
E &= (1/c)F \cdot v, \\
IB &= (1/c^2)F \wedge v, \\
B &= -(1/c^2)I(F \wedge v),
\end{align*}

(38)

where \(I\) is the unit pseudoscalar. (\(I\) is defined algebraically without introducing any reference frame, as in Ref. 19 Sec. 1.2.) It holds that \(E \cdot v = B \cdot v = 0\) (since \(F\) is skew-symmetric). \(v\) in Eq. (38) can be interpreted as the velocity (1-vector) of a family of observers who measures \(E\) and \(B\) fields. The velocity \(v\) and all other quantities entering into Eq. (38) are defined without reference frames. Thus the relations (38) hold for any observer. However it has to be emphasized that Eq. (38) is not a physical definition of \(E\) and \(B\); the physical definition has to be given in terms of the Lorentz force and Newton's second law as, e.g., in Ref. 15. The relations (38) actually establish the equivalence of the formulation of electrodynamics with the field bivector \(F\), see Ref. 10, and the formulation with 1-vectors of the electric \(E\) and magnetic \(B\) fields. Both formulations, with \(F\) and \(E, B\) fields, are equivalent formulations, but every of them is a complete, consistent and self-contained formulation. When Eq. (38) is inserted into the field equation for \(F\), Eq. (1), then Eq. (1) becomes the field equation for \(E, B\) fields

\[
\partial[E \wedge (v/c) + (IB) \cdot v] = j/\varepsilon_0 c.
\]

(39)

In contrast to the field equation (4), that holds only for the \(\gamma_0\)-observer, the field equation (39) holds for any observer; the quantities entering into Eq. (39) are all AQs. The equation (39) is physically completely equivalent to the field equation for \(F\) (1). In some basis \(\{e_\mu\}\) the field equation (39) can be written as a CBGE

\[
[\partial_\alpha (\delta^{\alpha\beta}_{\mu\nu}E^\mu v^\nu + \varepsilon^{\alpha\beta\mu\nu}v_\mu cB_\nu) - (j^\beta/\varepsilon_0)]e_\beta + \\
\partial_\alpha (\delta^{\alpha\beta}_{\mu\nu}v^\mu cB^\nu + \varepsilon^{\alpha\beta\mu\nu}v_\mu E_\nu)e_\delta e_\beta = 0,
\]

(40)

where \(E^\alpha\) and \(B^\alpha\) are the basis components of the electric and magnetic 1-vectors \(E\) and \(B\), \(\delta^{\alpha\beta}_{\mu\nu} = \delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu\) and \(e_5\) is the pseudoscalar for the frame \(\{e_\mu\}\). The first part in Eq. (40) (it contains sources) emerges from \(\partial \cdot F = j/\varepsilon_0 c\) and the second one (the source-free part) is obtained from \(\partial \wedge F = 0\), see also Ref. 15. Instead of working with the observer independent field equation in the \(F\)-formulation, Eq. (1), one can equivalently use the
E, B - formulation with the field equation (39), or in the \{e_\mu\} basis Eq. (40). (The complete E, B formulation of relativistic electrodynamics will be reported elsewhere.) Furthermore one can completely forget the manner in which the equation with E and B is obtained, i.e., the field equation with \( F \), and consider the equation with E and B, Eq. (39), which is defined without reference frames, or the corresponding CBGE (40), as the primary and fundamental equations for the whole classical electromagnetism. In such a correct relativistic formulation of electromagnetism the field equation with 1- vectors E and B, Eq. (39), takes over the role of the usual ME with the 3D E and B, i.e., of the ME (8). We note that the equivalent formulation of electrodynamics with tensors \( E^a \) and \( B^a \) is reported in Refs. 11, 20, whereas the component form in the Einstein system of coordinates is given in Refs. 17, 21 and Ref. 22.

Let us now take that in Eq. (40) the standard basis \( \{\gamma_\mu\} \) is used instead of some general basis \( \{e_\mu\} \). Then Eq. (40) can be written as \( C^\beta \gamma_\beta + D^\beta \gamma_5 \gamma_\beta = 0 \), where \( C^\beta = \partial_\alpha (\delta^{\alpha\beta} v_\mu E^\nu - \epsilon^{\alpha\beta\mu\nu} v_\mu c B^\nu) - j^\beta / \varepsilon_0 \) and \( D^\beta = \partial_\alpha (\delta^{\alpha\beta} \mu\nu E^\nu + \epsilon^{\alpha\beta\mu\nu} v_\mu E^\nu) \). When the active LT are applied to Eq. (40) with the \( \{\gamma_\mu\} \) basis the equation remains of the same form but with primed quantities replacing the unprimed ones (of course the basis is unchanged). This can be immediately seen since the equation (40) is written in a manifestly covariant form. Thus the Lorentz transformed Eq. (40) is

\[
R(C^\beta \gamma_\beta + D^\beta \gamma_5 \gamma_\beta) \tilde{R} = 0,
\]
\[
C'^\beta \gamma_\beta + D'^\beta \gamma_5 \gamma_\beta = 0,
\]

(41)

where, e.g., \( C'^\beta = \partial'_\alpha (\delta^{\alpha\beta} v_\mu E^\nu - \epsilon^{\alpha\beta\mu\nu} v_\mu c B^\nu) - j'^\beta / \varepsilon_0 \). Obviously such a formulation of electromagnetism with the fundamental equation (39) or (40) is a relativistically correct formulation.

What is the relation between the relativistically correct field equation (39) or (40) and the usual ME (8)? From the above discussion and from Sec. 2.1. one concludes that if in Eq. (39) we specify the velocity \( v \) of the observers who measure E and B fields to be \( v = c \gamma_0 \), then the equation (39) becomes the equation (1). Further choosing the standard basis \( \{\gamma_\mu\} \) in the \( \gamma_0 \) - frame, in which \( v = c \gamma_0 \), or in the components \( v^a = (c, 0, 0, 0) \), then in that \( \gamma_0 \) - frame E and B become \( E_f \) and \( B_f \) and they do not have temporal components, \( E_f^0 = B_f^0 = 0 \). The CBGE (40) becomes the usual ME (8). Thus the usual Clifford algebra treatments of electromagnetism (7,8,9) with the space-time split in the \( \gamma_0 \) - frame and the usual ME (8) are simply
obtained from our observer independent formulation with field equation (39) or (40) choosing that \( v = c\gamma_0 \) and choosing the standard basis \( \{\gamma_\mu\} \). We see that the correspondence principle is simply satisfied in this formulation with \( E \) and \( B \) fields; all results obtained in the previous treatments from the usual ME with the 3D \( E \) and \( B \) remain valid in the formulation with the 1-vectors \( E \) and \( B \) if physical phenomena are considered only in one inertial frame of reference. Namely the selected inertial frame of reference can be chosen to be the \( \gamma_0 \) - frame with the \( \{\gamma_\mu\} \) basis. Then there, as explained above, the CBGE (40) can be reduced to the equations containing only the components, the four ME in the component form, the ME (8). Thus for observers who are at rest in the \( \gamma_0 \) - frame \( (v = c\gamma_0) \) the components of the 3D \( E \) and \( B \) can be simply replaced by the space components of the 1-vectors \( E \) and \( B \) in the \( \{\gamma_\mu\} \) basis. We remark that just such observers are usually considered in the conventional formulation with the 3D \( E \) and \( B \). The dependence of the field equations (40) on \( v \) reflects the arbitrariness in the selection of the \( \gamma_0 \) - frame but at the same time it makes the equations (40) independent of that choice. The \( \gamma_0 \) - frame can be selected at our disposal, which proves that we don’t have a kind of the “preferred” frame theory. All experimental results that are obtained in one inertial frame of reference can be equally well explained by our geometric formulation of the electromagnetism with the 1-vectors \( E \) and \( B \) as they are explained by the usual ME with the 3D \( E \) and \( B \).

However there is a fundamental difference between the standard approach with the 3D \( E \) and \( B \) and the approach with the 4D AQs \( E \) and \( B \). It is considered in all standard treatments that the equation (33) is the LT of the original ME (8). But, as shown here, the equation (33) is not the LT of the original ME (8); the LT of the ME (8) are the equations (19) (i.e., (20) with (24), or (21)). The ME (8) are obtained from our field equation (40) putting \( v = c\gamma_0 \) and choosing the standard basis \( \{\gamma_\mu\} \). In the same way the equations (41), which are the LT of the equations (40), become the LT of the ME (8), that is, the equations (19) (or (20) with (24), or (21)), when in Eq. (41) it is taken that \( v' \), \( \partial' \), \( E' \) and \( B' \) are the LT of \( v = c\gamma_0 \), \( \partial \), \( E_f \) and \( B_f \), that is, \( v' = R(c\gamma_0)\tilde{R}, \partial' = R\partial\tilde{R}, E' = RE_f\tilde{R} = E'_f, B' = RB_f\tilde{R} = B'_f \). We recall from Sec. 2.3. that to an observer in the \( \{\gamma_\mu\} \) frame the vector \( p' \) \( (p' = Rp\tilde{R} = p^{\mu}\gamma_\mu) \) appears the same as the vector \( p \) \( (p = p^{\mu}\gamma_\mu) \) appears to an observer in the \( \{\gamma'_\mu\} \) frame. This, together with the preceding discussion, show that the usual ME with the 3D \( E \) and \( B \), i.e., the equation (8) and the equation (33) obtained by the ST from (8), cannot be used for the explanation.
of any experiment that tests SR, i.e., in which relatively moving observers have to compare their data obtained by measurements on the same physical object. In contrast to the description of the electromagnetism with the 3D E and B, the description with the 4D fields E and B, i.e., with the equations (40) and (41), is correct not only in the γ₀ - frame with the standard basis \{γ_µ\} but in all other relatively moving frames and it holds for any permissible choice of coordinates, i.e., basis \{e_µ\}. We see that the relativistically correct fields E and B and the new field equations (39) and (40) do not have the same physical interpretation as the usual 3D fields E and B and the usual 3D ME (8) except in the γ₀ - frame with the \{γ_µ\} basis in which E₀ = B₀ = 0. This consideration completely defines the relation between our approach with 4D E and B and all previous approaches.

As explained in the preceding sections the observer independent F field is decomposed in Eq. (2), see Refs. 7, 8, in terms of observer dependent quantities, i.e., as the sum of a relative vector \(E_H\) and a relative bivector \(γ_5 B_H\), by making the space-time split in the γ₀ - frame. But, here we present the new decomposition of F into the AQs, the bivectors \(E_{HV}\) and \(B_{HV}\), which are independent of the chosen reference frame and of the chosen system of coordinates in it. We define

\[
F = E_{Hv} + cIB_{Hv}, \quad E_{Hv} = (1/c^2)(F \cdot v) \wedge v \\
B_{Hv} = -(1/c^3)I[(F \wedge v) \cdot v], \quad IB_{Hv} = (1/c^3)(F \wedge v) \cdot v
\]

(42)

(The subscript \(HV\) is for “Hestenes” with \(v\) and not, as usual, Refs. 7,8, with \(γ_0\).) Obviously Eq. (42) holds for any observer. When we use Eq. (42) in the field equation for \(F\) (1), and after multiplication by \(v/c\) (instead of by \(γ₀\)), the equation (1) becomes

\[
(v/c)\{\nabla(E_{Hv} + cIB_{Hv}) - j/ε_0 c\} = 0.
\]

(43)

In contrast to the field equation (9) that holds only for the \(γ₀\)-observer, the field equation (43) holds for any observer; the quantities entering into Eq. (43) are the AQs. The equation (43) is physically completely equivalent to the field equation for \(F\) (1), i.e., to the field equation with 1- vectors E and B (39). (The equation (9) corresponds to the equation (4), whereas Eq. (43) corresponds to Eq. (39).) The field equation (43) can be written as a CBGE, and it looks much more complicated than the equation (40) with 1- vectors E and B. We write it (for better comparison) as two equations; the first one
will yield the scalar and bivector parts of Eq. (10) when \( v/c = \gamma_0 \). It is

\[
\frac{1}{c} v_\beta \partial_\alpha (E_{Hv})^{\alpha\beta} + \left[ (1/2c) v^\alpha \partial_\alpha (E_{Hv})^{\beta\sigma} - (1/2) \epsilon^{\mu\nu\alpha\sigma} v^\beta \partial_\alpha (B_{Hv})_{\mu\nu} \right] \gamma_\beta \wedge \gamma_\sigma \\
= \frac{1}{\varepsilon_0 c^2} (v_\alpha j^\alpha + v^\beta j^\sigma \gamma_\beta \wedge \gamma_\sigma).
\]

The second equation will yield the pseudoscalar and pseudobivector parts of Eq. (10) when \( v/c = \gamma_0 \) and it is

\[
v_\beta \partial_\alpha (B_{Hv})^{\alpha\beta \gamma_5} + (1/2) v^\alpha \partial_\alpha (B_{Hv})^{\mu\nu \gamma_5} (\gamma_\mu \wedge \gamma_\nu) + (v_\beta \partial_\alpha - v^\alpha \partial_\beta) (E_{Hv})_{\alpha\nu} \gamma_\beta \wedge \gamma_\nu = 0.
\]

In the \( \{\gamma_\mu\} \) basis \( I = \gamma_5 \). The equation (44) is with sources and it emerges from \( \partial \cdot \vec{F} = j/\varepsilon_0 c \), while Eq. (45) is the source-free equation and it emerges from \( \partial \wedge \vec{F} = 0 \). Comparing Eqs. (44) and (45) in the \( E_{Hv}, B_{Hv} \) formulation with the corresponding parts in Eq. (10) with 1-vectors \( E \) and \( B \) we see that the formulation with \( E \) and \( B \) is much simpler and more elegant than the formulation with bivectors \( E_{Hv} \) and \( B_{Hv} \); the physical content is completely equivalent.

The equations (44) and (45) are written in a manifestly covariant form. This means that when the active LT are applied upon such Eqs. (44) and (45) the equations remain of the same form but with primed quantities replacing the unprimed ones (of course the basis is unchanged).

The whole discussion with 1-vectors \( E \) and \( B \) about the correspondence principle applies in the same measure to the formulation with bivectors \( E_{Hv} \) and \( B_{Hv} \). The only difference is the simplicity of the formulation with 1-vectors \( E \) and \( B \).

The same conclusions hold for the formulation with 1-vector \( E_J \) and a bivector \( B_J \) from Ref. 9, but for the sake of brevity that formulation will not be considered here.

### 3. The Proof of the Difference Between the ST and the LT of the ME Using the Tensor Formalism with 4-Vectors \( E^a \) and \( B^a \)

The same proof and the whole consideration as with 1-vectors \( E \) and \( B \) can be given in the tensor formalism as well (it is presented in detail in Ref. 13). The important parts of this issue are already treated in Refs. 11, 15, 1. Therefore we only quote the main results. Now we start with Lorentz invariant field equations with \( v \) and with the decomposition of \( F^{ab} \) into the
AQs $E^a$ and $B^a$ since in the tensor formalism such field equations and the
decomposition are already in use, Refs. 23, 24.

The electromagnetic field tensor $F^{ab}$ is defined as an AQ; it is an abstract
tensor. Latin indices $a,b,c,\ldots$ are to be read according to the abstract index
notation, as in Refs. 23, 24 and Refs. 11, 12, 20. As already said in the
invariant SR that uses 4D AQs in the tensor formalism, Refs. 11, 12, 20,
1, and in the Clifford algebra formalism, Refs. 10, 15, 2, any permissible
system of coordinates, not necessary the Einstein system of coordinates, i.e.,
the standard basis $\{\gamma_\mu\}$, can be used on an equal footing. However, for
simplicity, we shall only deal with the standard basis $\{\gamma_\mu\}$. In the tensor
formalism $\gamma_\mu$ denote the basis 4-vectors forming the standard basis $\{\gamma_\mu\}$.

In the abstract index notation the field equations with $F^{ab}$ are given as

\[
(-g)^{-1/2}\partial_a((-g)^{1/2}F^{ab}) = \frac{j_b}{\varepsilon_0 c}, \quad \varepsilon^{abcd}\partial_b F_{cd} = 0,
\]

where $g$ is the determinant of the metric tensor $g_{ab}$ and $\partial_a$ is an ordinary
derivative operator. Now there are two field equations whereas in the geo-
metric algebra formalism they are united in only one field equation. When
written in the $\{\gamma_\mu\}$ basis as CBGEs the relations (46)
become

\[
\partial_\alpha F^{\alpha\beta\gamma\delta} = (1/\varepsilon_0 c)j^\beta, \quad \partial_\alpha * F^{\alpha\beta\gamma\delta} = 0.
\]

Instead of Eq. (38) from Sec. 2.6. we have the decomposition of $F^{ab}$ into
the AQs, the 4-vectors $E^a$ and $B^a$

\[
F^{ab} = (1/c)\delta^{ab}_{cd}E^c v^d + \varepsilon^{abcd}v_c B^d,
\]

\[
E^a = (1/c)F^{ab} v^b, \quad B^a = (1/2c^2)\varepsilon^{abcd}v_c F_{bd}.
\]

Inserting Eq. (48) into Eq. (46) we find the Lorentz invariant field equations
with $E^a$ and $B^a$ that correspond to Eq. (39) from Sec. 2.6. When these
equations are written as CBGEs in the $\{\gamma_\mu\}$ basis they become

\[
\partial_\alpha (\delta^{\alpha\beta}_{\mu\nu}E^\mu v^\nu + \varepsilon^{\alpha\beta\mu\nu}v_\mu cB_\nu)\gamma_\beta = (j^\beta/\varepsilon_0)\gamma_\beta
\]

\[
\partial_\alpha (\delta^{\alpha\beta}_{\mu\nu}v^\mu cB^\nu + \varepsilon^{\alpha\beta\mu\nu}v_\mu E_\nu)\gamma_\beta = 0.
\]

The equations (49) correspond to Eq. (40) from Sec. 2.6. (when written
in the standard basis $\{\gamma_\mu\}$). It is clear from the form of the equations (49)
(with some general $v^\mu$) that they are invariant upon the passive LT. Namely
in a relatively moving frame $S'$ all quantities in (49) will be replaced with

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the primed quantities that are obtained by the passive LT (of course, \( \delta^{\alpha \beta}_{\mu \nu} \) and \( \varepsilon^{\alpha \beta \mu \nu} \) are unchanged). All the primed quantities (components and the basis) are obtained from the corresponding unprimed quantities through the LT. The components of any 4D CBGQ transform by the LT, while the basis vectors \( \gamma_{\mu} \) transform by the inverse LT, thus leaving the whole 4D CBGQ invariant upon the passive LT. The invariance of some 4D CBGQ upon the passive LT reflects the fact that such 4D quantity represents the same physical object for relatively moving observers. Due to the invariance of every CBGQ upon the passive LT the field equations with primed quantities, thus in \( S' \), are exactly equal to the corresponding equations in \( S \), given by Eq. (49). Thus the equations (49) are not only covariant but also the Lorentz invariant field equations. The principle of relativity is automatically included in such formulation.

The usual ME are simply obtained from eq. (49) specifying that \( v^{\alpha} = c(\gamma_{0})^{\alpha} \), i.e., choosing the rest frame of “fiducial” observers, the \( \gamma_{0} \) - frame with the \{\( \gamma_{\mu} \)\} basis. Then from Eq. (49) we first find the ME exactly corresponding to Eq. (5) from Sec. 2.1. and further the component form of the usual ME corresponding to Eq. (8) (but now there are two equations)

\[
(\partial_{k} E^{k}_{j} - j^{0}/c\varepsilon_{0})\gamma_{0} + (-\partial_{0} E^{i}_{j} + c\varepsilon^{ijk0}\partial_{j} B_{fk} - j^{i}/c\varepsilon_{0})\gamma_{i} = 0
\]

\[
(-c\partial_{k} B^{k}_{j})\gamma_{0} + (c\partial_{0} B^{i}_{j} + \varepsilon^{ijk0}\partial_{j} E_{fk})\gamma_{i} = 0.
\]

As in Sec. 2.1., in the \( \gamma_{0} \) - frame with the \{\( \gamma_{\mu} \)\} basis, \( E^{0}_{j} = B^{0}_{j} = 0 \), and the relations (7) hold also here \( E^{i}_{j} = F^{i0}_{j} \), \( B^{i}_{j} = (-1/2c)\varepsilon^{0kl}F_{kl} \) (the standard identification), since \( E^{\mu}_{j} = F^{\mu \nu}(\gamma_{0})_{\nu} \), \( B^{\mu}_{j} = (1/c)(F^{*})^{\mu \nu}(\gamma_{0})_{\nu} \). The equations (50) (and (49) as well) can be written as \( a^{a}\gamma_{\alpha} = 0 \) and \( b^{a}\gamma_{\alpha} = 0 \). The coefficients \( a^{a} \) and \( b^{a} \) are clear from the first and second equation respectively in Eq. (50); they are the usual ME in the component form.

Let us now apply the passive LT to the ME (50). Upon the passive LT the sets of components \( E^{\mu}_{j} \) and \( B^{\mu}_{j} \) and the basis \{\( \gamma_{\mu} \)\} of the \( \gamma_{0} \) - frame (the \( S \) frame) transform to \( E'^{\mu}_{j} \) and \( B'^{\mu}_{j} \) and the new basis \{\( \gamma'_{\mu} \)\} in the relatively moving inertial frame of reference \( S' \), e.g., \( E'^{\nu}_{\delta} = L_{\delta}^{\nu} E^{\delta}_{\nu} \) and \( \gamma'_{\mu} = (L^{-1})^{\delta}_{\mu} \gamma_{\delta} \) (the components \( L'_{\delta} \) are quoted in Sec. 2.5). For the boost in the \( \gamma_{1} \) direction the Lorentz transformed sets of components \( E'^{\mu}_{j} \) and \( B'^{\mu}_{j} \) are given as

\[
E'^{\mu}_{j} = (1/c)F'^{\mu \nu}_{\nu} v'_{\nu}, \quad E'^{\mu}_{j} = (-\beta \gamma E^{1}, \gamma E^{1}, E^{2}, E^{3}),
\]

\[
B'^{\mu}_{j} = (1/c^{2})(F^{*})^{\mu \nu}_{\nu} v'_{\nu}, \quad B'^{\mu}_{j} = (-\beta \gamma B^{1}, \gamma B^{1}, B^{2}, B^{3}).
\]
where \( v' = (c\gamma, c\beta\gamma, 0, 0) \), and \( \gamma' \), as the LT of \((\gamma_0)\nu, \gamma_{\nu}/c = L_{\delta}^{\nu}(\gamma_0)^{\delta} \), is not in the time direction in \( S' \), i.e., it is not \((\gamma_0)^{\nu} \). Note that \( E^\mu_f \) and \( B^\mu_f \) have the temporal components as well. Further the components \( E^\mu_f \) (\( B^\mu_f \)) in \( S \) transform upon the LT again to the components \( E^\mu_{f'} \) (\( B^\mu_{f'} \)) in \( S' \); there is no mixing of components. Actually this is the way in which every well-defined 4-vector (the components) transforms upon the LT. The relations \( \gamma_{\mu} \) are given in Ref. 1 and they correspond to relations \( 22 \) and \( 23 \) from Sec. 2.3. The AQ, e.g., an abstract tensor \( E^\alpha \), can be represented by CBGQs in \( S \) and \( S' \) as \( E^\mu_{f}\gamma_{\mu} \) and \( E^\mu_{f'}\gamma_{\mu}' \) and, of course, it must hold that, e.g., \( E^\alpha = E^\mu_{f}\gamma_{\mu} = E^\mu_{f'}\gamma_{\mu}' \). Then the equations \( 50 \) transform to

\[
ad^{\alpha}_{\gamma_{\delta}} = 0, \quad b^{\alpha}_{\gamma_{\delta}} = 0,
\]

and it holds, as for any 4-vector (a geometric quantity), that \( a^{\alpha}_{\gamma_{\delta}} = a^{\alpha}_{\gamma_{\delta}} \), and \( b^{\alpha}_{\gamma_{\delta}} = b^{\alpha}_{\gamma_{\delta}} \); the coefficients transform by the LT, e.g. \( a^{0} = \gamma a^{0} - \beta a^{1} \), while the basis 4-vectors transform by the inverse LT, e.g., \( \gamma_0 = \gamma \gamma_0 + \beta \gamma_1 \). Of course \((\gamma_0)^{\nu} \) transforms to \( \gamma_{\nu}/c \), \( E^\mu_{f}, B^\mu_{f} \) are given by Eq. \( 11 \). When the coefficients \( a^{\alpha} \) and \( b^{\alpha} \) are written in terms of the primed quantities (from the \( S' \) frame) then we find the same expressions as in Sec. 2.3., e.g., the expression \( 24 \) is obtained for \( a^{0} \), and, of course, \( a^{0} \) is completely different in form than the coefficient \( a^{\alpha}_{\gamma_{\delta}} = (\partial_{k}E_{f}^{k} - j^{0}/c\varepsilon_{0}) \) in Eq. \( 50 \). Thus these Lorentz transformed ME exactly correspond to the equation \( 20 \) with Eq. \( 24 \) from Sec. 2.3.. We again see that the usual ME are not Lorentz covariant equations.

As shown above upon the LT \((\gamma_0)^{\nu} \) transforms to \( v_{\nu}^{'}/c = L^{\nu}_{\delta}(\gamma_0)^{\delta} \), which is not \( (\gamma_0)^{\nu} \), i.e., it is not in the time direction in \( S' \). However it is implicitly assumed in all usual treatments, e.g., Ref. 5 and Ref. 6 eqs. (3.5) and (3.24), that in \( S' \) one can again make the identification of six independent components of \( F'^{\mu\nu} \) with three components \( E^{i}_{f}, E^i_{f} = F'^{i0}, \) and three components \( B_{f}, B_{f} = (1/2c)\varepsilon_{ijk}F'_{ik} \), Eq. \( 36 \), see Secs. 2.2. and 2.5. This means that standard treatments assume that upon the passive LT the set of components \((\gamma_0)^{\nu} = (1, 0, 0, 0) \) from \( S \) transforms to \((\gamma_0')^{\nu} = (1, 0, 0, 0) \) \((\gamma_0')^{\nu} \) are the components of the unit 4-vector \( \text{in the time direction in } S' \), and consequently that, as shown in Ref. 1, \( E^\nu_{f} \) and \( B^\nu_{f} \) transform to \( E^\nu_{st} \) and \( B^\nu_{st} \) in \( S' \) as

\[
E^\nu_{st} = F'^{\mu\nu}(\gamma_0')^{\nu}, \quad E^\nu_{st} = (0, E^1, E^2, E^3), \quad B^\nu_{st} = (B^1, B^2, B^3),
\]

(53)
The temporal components of $E_{\mu}^{'st.}$ and $B_{\mu}^{'st.}$ in $S'$ are again zero as are the temporal components of $E_{\mu}^{'f}$ and $B_{\mu}^{'f}$ in $S$. This fact clearly shows that the transformations (53) are not the LT of some well-defined 4D quantities; the LT cannot transform the unit 4-vector in the time direction in one frame $S$ to the unit 4-vector in the time direction in another relatively moving frame $S'$. Obviously $E_{\mu}^{'st.}$ and $B_{\mu}^{'st.}$ are completely different quantities than $E_{\mu}^{'f}$ and $B_{\mu}^{'f}$, Eq. (51), that are obtained by the correct LT. We can easily check that $E_{\mu}^{'st.}\gamma_{\mu}' \neq E_{\mu}^{'f}\gamma_{\mu}$, and $B_{\mu}^{'st.}\gamma_{\mu}' \neq B_{\mu}^{'f}\gamma_{\mu}$. This means that, e.g., $E_{\mu}^{'st.}\gamma_{\mu}$ and $E_{\mu}^{'f}\gamma_{\mu}$ are not the same quantity for observers in $S$ and $S'$. As far as relativity is concerned the quantities, e.g., $E_{\mu}^{'f}\gamma_{\mu}$ and $E_{\mu}^{'st.}\gamma_{\mu}$, are not related to one another. The observers in $S$ and $S'$ are not looking at the same physical object but at two different objects; every observer makes measurement on its own object and such measurements are not related by the LT. From the relativistically incorrect transformations (53) one simply derives the transformations of the spatial components $E_{\mu}^{'st.}$ and $B_{\mu}^{'st.}$, the relations (31) from Sec. 2.4., which are exactly the ST of components of the 3D $\mathbf{E}$ and $\mathbf{B}$. According to the ST the transformed components $E_{\mu}^{'st.}$ and $B_{\mu}^{'st.}$, are expressed by the mixture of components $E_{\mu}^{'f}$ and $B_{\mu}^{'f}$. This completely differs from the correct LT (51). The transformations (53) and the transformations for $E_{\mu}^{'f}$ and $B_{\mu}^{'f}$ (31) are typical examples of the “apparent” transformations that are first discussed in Refs. 25 and 26. The “apparent” transformations of the spatial distances (the Lorentz contraction) and the temporal distances (the dilatation of time) are elaborated in detail in Refs. 11, 12, see also Ref. 22. It is explicitly shown in Ref. 12 that the true agreement with experiments that test SR exists when the theory deals with well-defined 4D quantities, i.e., the quantities that are invariant upon the passive LT. However new experiments that test SR are continuously published in leading physical journals, e.g., Ref. 27, and in these papers the dilatation of time and the Lorentz contraction are still considered as fundamental relativistic effects. (These experiments will be discussed in detail elsewhere.)

Let us now perform the ST of the ME (50) supposing that $E_{\mu}^{'f}$ and $B_{\mu}^{'f}$ in $S$ are transformed into $E_{\mu}^{'st.}$ and $B_{\mu}^{'st.}$ in $S'$ according to Eq. (53). They are

\begin{align}
\left(\partial_{k}E_{\mu}^{nk} - j^{0}/c\varepsilon_{0}\right)\gamma_{0}' + \left(-\partial_{0}E_{\mu}^{ni} + c\varepsilon_{ijk}\partial_{j}B_{\mu}^{nk} - j^{n}/c\varepsilon_{0}\right)\gamma_{i}' = 0, \\
\left(-c\partial_{k}B_{\mu}^{nk}\right)\gamma_{0}' + \left(c\partial_{0}B_{\mu}^{ni} + \varepsilon_{ijk}\partial_{j}E_{\mu}^{nk}\right)\gamma_{i}' = 0. 
\end{align}

(54)

These equations are of the same form as the original ME (50), but $E_{\mu}^{'f}$ and $B_{\mu}^{'f}$ from $S$ are not transformed by the LT than by the ST (53) into $E_{\mu}^{'st.}$ and $B_{\mu}^{'st.}$ 26
in $S'$. Thence the equations (54) are not the correct LT, but relativistically incorrect transformations of the original ME (50); the LT of the ME (50) are the equations (52) with $a^0$ as in Eq. (24), where the Lorentz transformed $E_f^\mu$ and $B_f^\mu$ are given by the relations (51).

4. SHORT COMPARISON WITH EXPERIMENTS.

**FARADAY DISK**

Let us now briefly discuss, as an example, the Faraday disk, using both the conventional formulation of electromagnetism with the 3D $E$ and $B$ and their ST and this new formulation, the invariant relativistic electrodynamics, with geometric 4D quantities. The comparison will be made in the tensor formalism from Sec. 3, since it is better known for physicists than the geometric algebra formalism. A conducting disk is turning about a thin axle passing through the center at a right angle to the disk and parallel to a uniform magnetic field $B$. The circuit is made by connecting one end of the resistor to the axle (the spatial point $A$) and the other end to a sliding contact touching the external circumference (the spatial point $C$). The disk of radius $R$ is rotating with angular velocity $\omega$. (For the description and the picture of the Faraday disk see, e.g., Ref. 28 Chap. 18 or the recent paper.) Let us determine the electromotive force (emf) in two inertial frames of reference, the laboratory frame $S$ in which the disk is rotating and the frame $S'$ instantaneously co-moving with a point on the external circumference (say $C$, taken at some moment $t$, e.g., $t = 0$). The $x'$ axis is along the 3-velocity $V$ of the point $C$ at $t$ and it is parallel to the $x$ axis. Actually all axes in $S'$ are parallel to the corresponding axes in $S$. The $y'$ axis is along the radius, i.e., along the segment $AC$.

First we calculate the emf using the standard formulation. In the $S$ frame

$$emf = \oint (F_L/q) \cdot dl = \int_{AC} (F_{L,y}/q) dy = \omega R^2 B/2,$$

where $F_L$ is the usual form for the 3D Lorentz force $F_L = qE + qU \times B$, $E = 0$ in $S$, $B$ is along the $+z$ axis, $qU \times B$ is the magnetic part of the Lorentz force seen by the charges co-moving with the disk along the segment $AC$. The integral along the segment $AC$ is taken at the same moment $t$. In the $S'$ frame the usual treatments suppose that the Lorentz force becomes $F'_L = qE'_{st} + qU' \times B'_{st}$, where the components of the 3D $E'_{st}$ and $B'_{st}$ are determined by the ST (53). Thus it is argued in the conventional formulation
that in $S'$ the charges experiences the fields $E_{st}' = \gamma_V \beta_V \times c B$ and $B_{st}' = \gamma_V B$, where $V = \omega R$, $\beta_V = (V/c)i$ and $\gamma_V = (1 - \beta_V^2)^{-1/2}$. Then only the $y'$ component of the force $F'_L$ remains and it is

$$F'_{L,y} = -qcB\beta_U/\gamma_V(1 - \beta_U\beta_V). \quad (56)$$

Notice that the same relation can be obtained from the definition of the 4-force (the components) $K^\mu = (\gamma_U F \cdot U, \gamma_U F)$ and its LT. This gives $\gamma_U^2 F'^2 = \gamma_U F^2$ whence the same $F'_{L,y}$ is obtained. (This happens here accidentally since $F'_{L,y}$ is calculated along the $y$ axis and $E = 0$ in $S$. Generally the expression $F'_{L,y} = qE_{st}' + qU' \times B_{st}'$ and the expression obtained from the LT of the 4-force will not give the same result.) In $S'$ the velocity (in units of $c$) $\beta'_U$ of some point on the segment $AC$ is $\beta'_U = (\beta_U - \beta_V)/(1 - \beta_U\beta_V)$ and the corresponding $\gamma'_U$ is $\gamma'_U = \gamma_V \gamma_U (1 - \beta_U\beta_V)$. The emf is again given by the integral of $F'_{L,y}/q$ over the common $y, y'$ axis (along the segment $AC$, $dl' = dl$) taken again at the same moment of time, $t' = 0$ ($y$ axis is orthogonal to the relative velocity $V$)

$$emf' = \int_{AC} (F'_{L,y}/q) dy. \quad (57)$$

It is clear from the expression for the emf in $S$, Eq. (55), and the corresponding one for the emf in $S'$, Eq. (57), together with Eq. (56) that these electromotive forces, in general, are not equal. Really

$$emf' = (c^2 B/\omega \gamma_V \beta_V^2)[\beta_V^2 - \ln(1 + \beta_V^2)], \quad (58)$$

thus $emf' \neq emf$. Only in the limit $\beta_U, \beta_V \ll 1$ $emf' \approx emf$. This result explicitly shows that the standard formulation is not relativistically correct formulation.

Let us now consider the same example in the invariant relativistic electrodynamics. In the tensor formalism the invariant Lorentz force $K^a$ is investigated in Ref. 11 Sec. 6.1. In terms of $F^{ab}$ it is $K^a = (q/c)F^{ab}u_b$, where $u^b$ is the 4-velocity of a charge $q$. In the general case of an arbitrary spacetime and when $u^a$ is different from $v^a$ (the 4-velocity of an observer who measures $E^a$ and $B^a$), i.e. when the charge and the observer have distinct world lines, $K^a$ can be written in terms of $E^a$ and $B^a$ as a sum of the $v^a$ - orthogonal component, $K_\perp^a$, and $v^a$ - parallel component, $K_\parallel^a$, $K^a = K_\perp^a + K_\parallel^a$. $K_\perp^a$ is

$$K_\perp^a = (q/c^2) \left[ (v^b u_b) E^a + c \varepsilon_{bc}^a u^b B^c \right] \quad (59)$$
and $\varepsilon_{abc} \equiv \varepsilon_{dabc} v^d$ is the totally skew-symmetric Levi-Civita pseudotensor induced on the hypersurface orthogonal to $v^a$, while

$$K^a_\parallel = \left(\frac{q}{c^2}\right) \left[\left(E^b u_b\right) v^a\right].$$

(60)

Speaking in terms of the prerelativistic notions one can say that $K^a_\parallel$, Eq. (59), plays the role of the usual Lorentz force lying on the 3D hypersurface orthogonal to $v^a$, while $K^a_\perp$, Eq. (60), is related to the work done by the field on the charge. However in the invariant SR only both components together, that is, $K^a$, does have definite physical meaning and $K^a$ defines the Lorentz force both in the theory and in experiments. Of course $K^a_\parallel, K^a_\perp$ and $K^a$ are all 4D quantities defined without reference frames, the AQs, and the decomposition of $K^a$ is an observer independent decomposition. Then we define the emf also as an invariant 4D quantity, the Lorentz scalar,

$$emf = \int_\Gamma \left(\frac{K^a}{q}\right) dl_a,$$

(61)

where $dl_a$ is the infinitesimal spacetime length and $\Gamma$ is the spacetime curve. Let the observers are at rest in the $S$ frame, $\nu^\mu = (c, 0, 0, 0)$ whence $E^0 = B^0 = 0$; the $S$ frame is the rest frame of “fiducial” observers, the $\gamma_0$ - frame with the $\{\gamma_\mu\}$ basis. Thus the components of the 4-vectors in the $\{\gamma_\mu\}$ basis are $E^\mu = (0, 0, 0, 0), B^\mu = (0, 0, 0, B), u^\mu = (\gamma_\mu c, \gamma_\mu U = \gamma_\mu \omega y, 0, 0), dl^\mu = (0, 0, dl^2 = dy, 0)$. Thence $K^a_\parallel = 0, K^a_\perp = K^1_\perp = K^3_\perp = 0, K^2_\perp = \gamma_\mu q U B$. When all quantities in Eq. (61) are written as CBGQs in the $S$ frame with the $\{\gamma_\mu\}$ basis we find

$$emf = (c^2 B/\omega) \left\{1 - \left[1 - (\omega R/c)^2\right]^{1/2}\right\},$$

(62)

which for $\omega R/c \ll 1$ becomes the usual expression $emf = \omega R^2 B/2$ as in Eq. (55). Since the expression (61) is independent of the chosen reference frame and of the chosen system of coordinates in it we shall get the same result, Eq. (62), in the relatively moving $S'$ frame as well;

$$emf = \int_{\Gamma(in\ S)} (K^\mu/q) dl_\mu = \int_{\Gamma(in\ S')} (K'^\mu/q) dl'_\mu$$

$$= (c^2 B/\omega) \left\{1 - \left[1 - (\omega R/c)^2\right]^{1/2}\right\}.$$  

(63)

This can be checked directly performing the LT of all 4-vectors as CBGQs from $S$ to $S'$ including the transformation of $\nu^\mu\gamma_\mu$. Obviously the approach...
with Lorentz invariant 4D quantities gives the relativistically correct answer in an enough simple and transparent way. From the viewpoint of the geometric approach the agreement with the usual approach exists only in the frame of the “fiducial” observers and when $V \ll c$.

5. SUMMARY AND CONCLUSIONS

The covariance of the ME is considered to be a cornerstone of the modern relativistic field theories, both classical and quantum. Einstein$^{(4)}$ derived the ST of the 3D $E$ and $B$ assuming that the ME with $E$ and $B$ must have the same form in all relatively moving inertial frames of reference. In Einstein’s formulation of SR$^{(4)}$ the principle of relativity is a fundamental postulate that is supposed to hold for all physical laws including those expressed by 3D quantities, e.g., the ME with the 3D $E$ and $B$. This derivation is discussed in detail in Ref. 11. The results presented in this paper substantially change generally accepted opinion about the covariance of the ME exactly proving in the geometric algebra and tensor formalisms that the usual ME$^{(8)}$, or $^{(10)}$, or $^{(50)}$ change their form upon the LT (see Eq. (21) with Eq. (24), or Eq. (20) with Eq. (28), or Eq. (52) with $a^0$ from Eq. (24)). It is also proved that the ST of the ME (see Eqs. (32) and (33), or Eq. (35), or Eq. (54)), which leave unchanged the form of the ME, actually have nothing in common with the LT of the usual ME. The difference between the LT of the ME, e.g., Eq. (21) with Eq. (24), and their ST, e.g., Eqs. (32) and (33), is essentially the same as it is the difference between the LT of the electric and magnetic fields (see Eqs. (22) and (23), or Eq. (27), or Eq. (51)) and their ST (see Eqs. (29) and (30), or Eq. (34), or Eq. (53)). This last difference is proved in detail in Refs. 1, 2 and that proof is only briefly repeated in this paper. All this together reveals that, contrary to the generally accepted opinion, the principle of relativity does not hold for physical laws expressed by 3D quantities (a fundamental achievement). A 3D quantity cannot correctly transform upon the LT and thus it does not have an independent physical reality in the 4D spacetime; it is not the same quantity for relatively moving observers in the 4D spacetime (see also, e.g., Figs. 3. and 4. in Ref. 11, and Ref. 12). Since the usual ME change their form upon the LT they cannot describe in a relativistically correct manner the experiments that test SR, i.e., the experiments in which relatively moving observers measure the same 4D physical quantity. Therefore the new field equations with geometric 4D quantities are constructed in geometric algebra formalism with 1-vectors $E$.
and $B$ (Eqs. (39) and (40)), and with bivectors $E_{Hv}$ and $B_{Hv}$ (Eqs. (43) and (44) with (45)), and also in the tensor formalism with 4-vectors $E^a$ and $B^a$ (Eq. (49)); the Lorentz invariant field equations in the tensor formalism are already presented in Refs. 11, 20. All quantities in these geometric equations are independent of the chosen reference frame and of the chosen system of coordinates in it. When the $\gamma_0$ - frame with the $\{\gamma_\mu\}$ basis is chosen, in which the observers who measure the electric and magnetic fields are at rest, then all mentioned geometric equations become the usual ME. This result explicitly shows that the correspondence principle is naturally satisfied in the invariant SR. However, as seen here, the description with 4D geometric quantities is correct not only in the $\gamma_0$ - frame with the $\{\gamma_\mu\}$ basis but in all other relatively moving frames and it holds for any permissible choice of coordinates. We conclude from the results of this paper that geometric 4D quantities, defined without reference frames, i.e., the AQs, or as CBGQs, have an independent physical reality and the relativistically correct physical laws are expressed in terms of such quantities. The principle of relativity is automatically satisfied with such quantities whereas in the standard formulation of SR it is postulated outside the mathematical formulation of the theory. We see that the role of the principle of relativity is substantially different in the Einstein formulation of SR and in the invariant SR. The results of this paper clearly support the latter one. Furthermore we note that all observer independent quantities, i.e., the AQs, introduced here and the field equations written in terms of them hold in the same form both in the flat and curved spacetimes. The results obtained in this paper will have important and numerous consequences in all relativistic field theories, classical and quantum. Some of them will be soon examined.

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REFERENCES

1. T. Ivezić, Found. Phys. 33, 1339 (2003); hep-th/0302188
2. T. Ivezić, physics/0304085.
3. H.A. Lorentz, Proceedings of the Academy of Sciences of Amsterdam,
6 (1904), in W. Perrett and G.B. Jeffery, in *The Principle of Relativity* (Dover, New York).
4. A. Einstein, *Ann. Physik.* **17**, 891 (1905), tr. by W. Perrett and G.B. Jeffery, in *The Principle of Relativity* (Dover, New York).
5. J.D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1977) 2nd edn.; L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, (Pergamon, Oxford, 1979) 4th edn.;
6. C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1970).
7. D. Hestenes, *Space-Time Algebra* (Gordon and Breach, New York, 1966); *Space-Time Calculus*; available at: [http://modelingnts.la.asu.edu/evolution.html](http://modelingnts.la.asu.edu/evolution.html); *New Foundations for Classical Mechanics* (Kluwer Academic Publishers, Dordrecht, 1999) 2nd. edn.; *Am. J Phys.* **71**, 691 (2003).
8. C. Doran, and A. Lasenby, *Geometric algebra for physicists* (Cambridge University Press, Cambridge, 2003).
9. B. Jancewicz, *Multivectors and Clifford Algebra in Electrodynamics* (World Scientific, Singapore, 1989).
10. T. Ivezić, physics/0305092.
11. T. Ivezić, *Found. Phys.* **31**, 1139 (2001).
12. T. Ivezić, *Found. Phys. Lett.* **15**, 27 (2002); physics/0103026 physics/0101091.
13. T. Ivezić, physics/0311043.
14. M. Riesz, *Clifford Numbers and Spinors*, Lecture Series No. 38, The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland (1958).
15. T. Ivezić, [hep-th/0207250](http://arxiv.org/abs/hep-th/0207250) [hep-ph/0205277](http://arxiv.org/abs/hep-ph/0205277).
16. A. Einstein, *Ann. Physik* **49**, 769 (1916), tr. by W. Perrett and G.B. Jeffery, in *The Principle of Relativity* (Dover, New York).
17. H.N. Núñez Yépez, A.L. Salas Brito, and C.A. Vargas, *Revista Mexicana de Física* **34**, 636 (1988).
18. T. Matolesi, *Spacetime without Reference Frames* (Akadémiai Kiadó, Budapest, 1993).
19. D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus* (Reidel, Dordrecht, 1984).
20. T. Ivezić, *Annales de la Fondation Louis de Broglie* **27**, 287 (2002).
21. S. Esposito, *Found. Phys.* **28**, 231 (1998).
22. T. Ivezić, *Found. Phys. Lett.* **12**, 105 (1999); *Found. Phys. Lett.* **12**, 507 (1999).
23. R.M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).
24. M. Ludvigsen, *General Relativity, A Geometric Approach* (Cambridge University Press, Cambridge, 1999); S. Sonego and M.A. Abramowicz, *J. Math. Phys.* **39**, 3158 (1998); D.A. T. Vanzella, G.E.A. Matsas, H.W. Crater, *Am. J. Phys.* **64**, 1075 (1996).
25. F. Rohrlich, *Nuovo Cimento B* **45**, 76 (1966).
26. A. Gamba, *Am. J. Phys.* **35**, 83 (1967).
27. G. Saathoff et al., *Phys. Rev. Lett.* **91**, 190403 (2003); H. Müller et al., *Phys. Rev. Lett.* **91**, 020401 (2003); P. Wolf et al., *Phys. Rev. Lett.* **90**, 060402 (2003).
28. W.K.H. Panofsky and M. Phillips, *Classical electricity and magnetism*, 2nd edn. (Addison-Wesley, Reading, Mass., 1962).
29. L. Nieves, M. Rodriguez, G. Spavieri and E. Tonni, *Nuovo Cimento B* **116**, 585 (2001).