Maximum Likelihood Estimation of Stochastic Fractional Singular Models

KOMEIL NOSRATI\textsuperscript{1,}, DEREK ABBOTT\textsuperscript{2,} (Fellow, IEEE), AND MASOUD SHAFIEE\textsuperscript{1,} (Senior Member, IEEE)

\textsuperscript{1}School of Electrical Engineering, Amirkabir University of Technology, Tehran 1591634311, Iran
\textsuperscript{2}School of Electrical and Electronic Engineering, The University of Adelaide, Adelaide, SA 5000, Australia

Corresponding author: Komeil Nosrati (nosrati_k@aut.ac.ir)

ABSTRACT Using the non-causal nature of a fractional-order singular (FOS) model, this paper deals with the modification of an estimation algorithm developed by Nosrati and Shafiee, and demonstrates how the derived estimation procedure can be adjusted by additional information related to the future dynamics. The procedure adopts the maximum likelihood (ML) method leading to a 3-block fractional singular Kalman filter (FSKF). In addition to some conditions on existence and uniqueness of solutions for discrete-time linear stochastic FOS models, the estimability analyses are given and an optimal filter is presented. Finally, the performance of the derived filter is verified and validated via numerical simulation on a three machine infinite bus system.

INDEX TERMS Filtering and estimation, maximum likelihood approach, fractional-order singular model, non-causality, machine infinite bus system.

I. INTRODUCTION

Fractional or non-integer calculus steers us to a more general form called fractional-order singular (FOS) models, which are utilized to model various physical systems and scientific processes [2]–[4], and at the same time, share characteristics of both non-integer theories and singular systems. Although there have been some notable studies in stability [5], normalization and stabilization [6], estimator and observer design problems [1], [7], [8], issues have been reported for deterministic FOS models such as control designs for nonlinear and rectangular FOS systems, admissibility conditions and stabilization problems for convex intervals $1 < \alpha < 2$ of fractional order or based on the complex domain that has broader descriptions and more complex behaviors than linear square FOS systems with fractional order $0 < \alpha < 1$. However, there are many remaining challenges and few studies have considered the stochastic terms for applications in estimation and filter design.

In the state estimator problem, the Kalman filter (KF) is well established for optimal state observers [9]. The extension of this algorithm to the discrete time non-integer order model was elaborated in [10], which has been called the fractional KF (FKF). On the other hand, recursive state estimations for discrete-time singular models have been studied, in which different algorithms based on existing methods were derived [11]. That study transformed a singular model to a normal form to apply the classic KF algorithms to estimate the states of the system [12]. Moreover, some normal approaches such as the least-squares (LS), ML and deterministic approaches [13]–[15] were applied to solve the estimation problem without the need for transformation, yielding a singular KF (SKF).

Recently, the problem of filtering for FOS cases has been taken into account to a limited extent. In [12], in an indirect method, a FOS model in its continuous form was first decomposed into normal sub-systems with several transformations, and then, existing KF algorithms were used. Decomposition of the system may result in an important loss of relevant information, and in many cases, with inaccurate estimation of state variables. In order to overcome this problem and to decrease the estimation error, a direct approach is required. Using the data-fitting problem approach, a filtering algorithm directly from the original TI FOS model was derived [1]. That study aimed to formulate the deterministic estimation
algorithm of such a model and introduced the fractional singular Kalman filter (FSKF) algorithm. In both these two studies, some issues such as solvability theorems, future dynamics, singular measurement noise remained open. The research motivation here addresses the noncausal FOS system filtering issue of using ML estimation concepts, which motivate us to derive a 3-block FSKF algorithm and Riccati equation that not only considers the future dynamics, but also performs when the measurement noise is singular, in which the derived algorithm requires only standard matrix inverses instead of pseudo inverses. The main contributions and outcomes of the present work are outlined as follows,

1) Solvability Theorem: The necessary and sufficient conditions on the solvability of discrete FOS models are established in this article. The model is considered to be stochastic with zero-mean white Gaussian measurement and process noise, in the form of Grunwald-Letnikov (GL) difference equations with consistent initial conditions.

2) ML-Based Estimation Algorithm: This article presents a review of the fundamental theories and aspects related to the ML method focused on deducing a valid form to the ML method focused on deducing a valid form.

3) Adjusted Estimation Algorithm: This work studies the adjusted estimate.

4) Modeling and Simulation (A three-machine infinite bus system): The estimation performance of the obtained filter is validated using a numerical simulation on a new FOS model of a three-machine infinite bus power system.

In the following, and in Section 2, we aim to bring some preliminaries on the discrete FOS model in its linear stochastic form, followed by necessary and sufficient conditions on an unique solution to this model. Then in Section 3, we present the ML estimation method that motivates us to take steps towards the filtering issue of the model and a modified version of the filter with respect to the information resulting from the constraints, which include future dynamics. Finally in Section 4, a FOS model of a three-machine infinite bus system is considered as a case study to verify our hypothesis.

**II. PRELIMINARIES**

Consider the following discrete stochastic FOS model

\[
E_{0}^{\Delta^{\alpha}_{k+1}x_{k+1}} = A_{x}x_{k} + w_{k}, \quad (1a)
\]

\[
y_{k} = Cx_{k} + v_{k}, \quad (1b)
\]

where \(E \in \mathbb{R}^{n \times n}\) with \(\text{rank } E < n\), \(A \in \mathbb{R}^{n \times n}\) and \(C \in \mathbb{R}^{p \times n}\) are real constant matrices, \(x_{k} \in \mathbb{R}^{n}\) is the state vector and \(y_{k} \in \mathbb{R}^{p}\) is the output vector, and the sequences \(w_{k} \in \mathbb{R}^{n}\) and \(v_{k} \in \mathbb{R}^{p}\) are zero mean white vectors. Also, \(G_{\Delta}^{\alpha}_{k+1} \Delta_{k+1}^{\alpha} \in \mathbb{R}^{n \times n}\), where \(\alpha_{i} \in \mathbb{R}, i = 1, \ldots, n\) are the fractional orders assigned to \(n\) equations. Furthermore, \(G_{\Delta}^{\alpha}_{k}\) denotes the fractional GL difference defined as

\[
h^{\alpha} \cdot G_{\Delta}^{\alpha}_{k}x_{k} = \sum_{j=0}^{k}(-1)^{j} \gamma_{j}x_{k-j}, \quad \text{in which } h \text{ is the value of duration or the interval of each two samples out of } k \text{ samples, where the derivative is computed, and } \gamma_{j} = \begin{bmatrix} \alpha_{i} \end{bmatrix}_{j} \end{bmatrix} \]
into (3) yields
\[ x_{2,k} = -\sum_{i=0}^{\mu-1} N^i q^{i+1} \left( -N \sum_{j=1}^{k} (-1)^j \gamma_j x_{k-j} + P_2 w_{k-1} \right) \]
\[ = \sum_{i=0}^{\mu-1} N^i q^{i+1} \left( -N \sum_{j=1}^{k} (-1)^j \gamma_j x_{k+j} + \sum_{i=0}^{\mu-1} N^i P_2 w_{k+i} \right). \] (4)

Also, for (2a) one has the solution \( x_{1,k} = \phi_k x_{1,0} + \sum_{j=0}^{k-1} \phi_k \cdot p_{1w,k} \) using iterative methods, where \( x_{k+1} = \phi_k A_{1a} + \sum_{j=1}^{k} (-1)^j \gamma_j x_{k-j+1} \) with \( \phi_0 = I_n \) and \( A_{1a} = A_1 + \alpha I_{n_2} \). Therefore, from these two solutions, one can conclude that there is a solution for the FOS model (1). To show its uniqueness, in equation (2b), let \( N = \text{diag} (N_1, N_2, \ldots, N_q) \) and \( P_2^T = \begin{bmatrix} \hat{P}_1^T \hat{P}_2^T \cdots \hat{P}_q^T \end{bmatrix} \), where \( \hat{P}_i \in \mathbb{R}^{g_i \times g_i} \), \( \sum_i g_i = n_2 \) and \( N_i \in \mathbb{R}^{g_i \times g_i} \), \( i = 1, 2, \ldots, q \) are shift matrices with ones only on the super-diagonal, and zeroes elsewhere. Therefore, the \( i \)th relation of the equation (2b) has the form of
\[ N_1 \hat{G}_{k+1}^a x_{2(k+1)} = x_{2,k} + \hat{P}_w k, \]
where \( x_{2,k} = \begin{bmatrix} x_{2,k}^{(1)} & x_{2,k}^{(2)} & \cdots & x_{2,k}^{(g_i)} \end{bmatrix}^T \) and \( \hat{P}_w = \begin{bmatrix} \hat{P}_1^{(1)} & \hat{P}_2^{(2)} & \cdots & \hat{P}_q^{(g_i)} \end{bmatrix} \), which can be decomposed into the following equations:
\[
\begin{align*}
0 \hat{G}_{k+1}^a x_{2(k+1)} &= x_{2,k} + \hat{P}_1^{(1)} \omega, \\
0 \hat{G}_{k+1}^a x_{2(k+1)} &= x_{2,k} + \hat{P}_2^{(2)} \omega, \\
\vdots \\
0 \hat{G}_{k+1}^a x_{2(k+1)} &= x_{2,k} + \hat{P}_q^{(g_i)} \omega, \\
0 &= x_{2,k} + \hat{P}_q^{(g_i)} \omega. 
\end{align*}
\] (5)

It is obvious that (5) is equivalent to two equations \( 0 = x_{2,k} + \hat{P}_1^{(1)} \omega \) and \( 0 = x_{2,k} + \hat{P}_q^{(g_i)} \omega \). Suppose that equation (2) has two solutions defined by \( x_{1,k}^{(i)} \) and \( x_{1,k}^{(v)} \). From the first solution, one has that system (5) has a unique solution \( x_{2,k} \), \( i = 1, 2, \ldots, q \). As a result, one can conclude that \( x_{2,k} \) is a unique solution of equation (2b). Due to the uniqueness of solution for system (2a), the first part of proof is completed.

Necessity: Let us assume the solution of the FOS model (1) is unique. According to Kronescker’s theorem [16], for any two matrices \( E \) and \( A \), there exist the invertible matrices \( Q, P \) and \( S \) such that \( QAP = \tilde{A} \) and \( QEP = \tilde{E} \) with the matrices \( \tilde{E} \) and \( \tilde{A} \) given by \( \tilde{E} = \text{diag} (0_{n_0 \times n_0}, L_1, L_2, \ldots, L_m, L_1^N, L_2^N, \ldots, L_m^N, I, N) \) and \( \tilde{A} = \text{diag} (0_{n_0 \times n_1}, J_1, J_2, \ldots, J_p, J_1^\infty, J_2^\infty, \ldots, J_p^\infty, A_1, I) \), where \( L_i \in \mathbb{R}^{n \times n_0} \), \( L_i \in \mathbb{R}^{n \times (n_0+1)} \), \( L_i^N \in \mathbb{R}^{n \times n_1} \), \( J_i^\infty \in \mathbb{R}^{n \times n_1} \) with
\[
L_i = \begin{bmatrix} 1 & 0 & 1 & 0 \\
. & . & . & . \\
1 & 0 \\
\end{bmatrix}, \quad J_i = \begin{bmatrix} 0 & 1 & 0 \\
0 & 1 \\
. & . & . \\
0 & 1 \\
\end{bmatrix}
\]
for \( i = 1, 2, \ldots, p \) and \( j = 1, 2, \ldots, q, \) and \( N = \text{diag} (N_1, N_2, \ldots, N_q) \in \mathbb{R}^{n \times n} \), where \( N_i \in \mathbb{R}^{n \times n_1} \), \( I = 1, 2, \ldots, I \) possess the special forms described before. The dimensions of these matrices satisfy the relations \( \sum_{i=1}^{p} k_i = g, \sum_{i=1}^{q} q_i = n, \sum_{i=1}^{q} q_i = n \), \( q_i + \sum_{i=1}^{q} q_i = n \), \( n_0 \), \( n_1 \), and \( n_i \) \( (n_i + k_i) \). Therefore, the FOS model (1) is equivalent to the following form
\[ \tilde{E} \hat{G}_{k+1}^a x_{k+1} = \tilde{A} x_{k} + \tilde{w}_k. \] (6)

where \( \tilde{x}_k = P^{-1} x_k \) and \( \tilde{w}_k = Q w_k \). It is not difficult to show that the systems (1) and (6) have the same solution property. When a solution to (6) is derived, immediately, a solution to FOS model (1) can be obtained. According to the structures of the matrices \( \tilde{E} \) and \( \tilde{A} \), the vectors \( \tilde{x} \) and \( \tilde{w} \) can be partitioned as \( \tilde{x}_k = [x_{k0}^T, x_{k1}^T, \ldots, x_{kn_0}^T]^T \) and \( \tilde{w}_k = [w_{k0}^T, w_{k1}^T, \ldots, w_{kn_0}^T]^T \), where \( x_{k0}^T = \text{diag} ([x_{k01}^T, x_{k02}^T, \ldots, x_{k0n_0}^T]), w_{k0}^T = [w_{k01}^T, w_{k02}^T, \ldots, w_{k0n_0}^T] \) for \( \omega = p, q \). Then, system (6) can be decomposed into the following equations:
\[
\begin{align*}
0_{n_0 \times n_0} \hat{G}_{k+1}^a x_{n_0,k+1} &= w_{n_0,k}, \\
L_1 \hat{G}_{k+1}^a x_{n_1,k+1} &= J_1 x_{n_1,k} + w_{n_1,k}, \\
L_2 \hat{G}_{k+1}^a x_{n_1,k+1} &= J_2 \infty x_{n_1,k} + w_{n_1,k}, \\
N_{n_m} \hat{G}_{k+1}^a x_{n_m,k+1} &= x_{n_m,k} + w_{n_m,k}, \\
0_{n_0 \times n_0} \hat{G}_{k+1}^a x_{n_0,k+1} &= A_1 x_{n_1,k} + w_{h,k},
\end{align*}
\] (7) (8) (9) (10) (11)
for \( i = 1, 2, \ldots, p, j = 1, 2, \ldots, q, \) and \( m = 1, 2, \ldots, I \). Thus, the solution of (6) is equivalent to the solutions of equations (7) to (11) which are elaborated as follows.

(a) For the identical Equation (7), there either does not exist a solution or does an infinite number of solutions for any differentiable function \( x_{n_0,k} \).

(b) In Equation (8), suppose that the system has the dimension of \( (r-1) \times r \). Let us denote the vectors \( x_{n_1} \) and \( w_{n_1} \) by \( z_i \) and \( w_i \), respectively. Then one has \( 0 \hat{G}_{k+1}^a z_{k+1} = z_{j+1,k} + w_{j+1,k} \), for \( j = 1, 2, \ldots, r-1 \), which can be converted into the \( z_{j+1,k} \) \( 0 \hat{G}_{k+1}^a \hat{j}_{k+1} + \hat{w}_{k+1} \), forms, for \( j = 1, 2, \ldots, r-1 \). Clearly, for any given function \( z_{1,k} \), which is sufficiently differentiable, the other variables \( z_{2,k}, \ldots, z_{r-1,k} \) can be obtained easily in turn. As a result, there exist an infinite number of solutions for this type of equations.

(c) In Equation (9), suppose that the system has the dimension of \( (r+1) \times r \). Again denote the vectors \( x_{n_1} \) and \( w_{n_1} \) by \( z_i \) and \( w_i \), respectively. Therefore, the system is equivalent to the equations \( z_{r,k} + w_{r+1,k} = 0, \)
Following from the above points, one can conclude that the necessary and sufficient condition for the solution of system (6) to be unique is the fading of the equations (7) to (9). Accordingly, the linear FOS system (6) has a unique solution if and only if the matrices \( Q \) and \( P \) in the equivalent model (6) of the system (1) take the forms as \( \tilde{E} = QEP = \text{diag}(I, N) \) and \( \tilde{A} = QAP = \text{diag}(\tilde{A}_1, I_{n_2}) \). Based on Lemma 1, this fact immediately gives that the system (1) is regular.

Remark 1: Under regularity condition, the FOS system (1) is of index one and, accordingly, causal, if \( \mu = 1 \). On the contrary, for an index greater than one, we have a non-causal FOS system.

Now, we can take steps to obtain an appropriate dynamical model which interprets the system observations based on the following assumptions:

**Assumption 1:** The discrete stochastic FOS system (1) is regular.

**Assumption 2:** The initial state \( x_0 \) is a random variable as \( x_0 \sim \mathcal{N}(\tilde{x}_0, P_0) \), where \( \tilde{x}_0 \) is mean value and the positive definite (PD) matrix \( P_0 \) is covariance of prior information.

**Assumption 3:** The independent vectors \( w_k \in \mathbb{R}_n \) and \( v_k \in \mathbb{R}_p \) of \( x_0 \) are zero mean white sequences with PD covariance as \( \mathbb{E}\left[\begin{bmatrix} w_k & v_k \end{bmatrix}^T \begin{bmatrix} w_l & v_l \end{bmatrix}\right] = \text{diag}(Q_k, R_k) \delta_{kl} \), where \( \mathbb{E}\{\cdot\} \) denotes the mathematical expectation and \( \delta_{kl} \) is the discrete delta function.

**Assumption 4:** The set of observed signals \( y_i = \{y_i\}_{i=1}^m \) from discrete stochastic FOS regular model (1) are given.

The following section is devoted to the KF derivation for the causal and non-causal system (1) based on ML approach.

### III. ML-BASED FSKF

This section aims to investigate the estimation issue of the FOS model (1) according to ML conception, with respect to the system dynamics and prior information on \( x_0 \) as additional piece of observations. Before deriving an algorithm of estimation based on ML approach, let us first discuss some features of this linear method of estimation.

#### A. ML ESTIMATION TECHNIQUE

Let \( x \in \mathbb{R}^n \) be an unknown vector based on the following measurement vector

\[
y = C_1 x + n_1
\]

where \( y \in \mathbb{R}^p, n_1 = \mathcal{N}(0, N_1), N_1 \in \mathbb{R}^{p \times p} \) and \( C_1 \in \mathbb{R}^{p \times n} \) is a constant matrix. With respect to the estimation of this problem, we aim to bring here some aspects of ML estimation technique.

**Lemma 3:** The recent problem actually includes LS estimation method of Gaussian vectors. In other words, the issue of LS estimate for a vector \( x = \mathcal{N}(m, P) \) relies on the measurement \( z = C_2 x + n_2 \), where \( n_2 \sim \mathcal{N}(0, N_2) \) \( N_2 \in \mathbb{R}^{p \times p} \) and \( C_2 \in \mathbb{R}^{p \times n} \) is a constant matrix, yields the same estimate as the ML problem with

\[
y_{\text{new}} = \begin{bmatrix} m \\ z \end{bmatrix}, \quad C_{1,\text{new}} = \begin{bmatrix} I_n \\ C_2 \end{bmatrix}, \quad N_{1,\text{new}} = \begin{bmatrix} P & 0 \\ 0 & N_2 \end{bmatrix}
\]

where \( y_{\text{new}} \in \mathbb{R}^{(n+p)\times 1}, \quad C_{1,\text{new}} \in \mathbb{R}^{(n+p)\times n} \) and \( N_{1,\text{new}} \in \mathbb{R}^{(n+p)\times (n+p)} \).

**Proof:** Let \( p(y|x) \) denote a probability density function of \( y \) parameterized by \( x \). Since \( n_1 \) is Gaussian, so is \( y \), and as a result, \( p(y|x) = \xi e^{-\frac{1}{2}(y-C_1 x)^T N_1^{-1}(y-C_1 x)} \), where \( \xi \) is a normalization constant. The ML estimate \( \hat{x}_{\text{ML}} \) based on observation (12) satisfies \( p(y|x_{\text{ML}}) \geq p(y|x) \) for all \( x \). Then, \( \hat{x}_{\text{ML}} \) can be obtained by noting that \( \frac{\partial}{\partial x} \ln p(y|x) |_{x=x_{\text{ML}}} = 0 \). Accordingly if \( C_1 \) and \( N_1 \) have full-rank, for the ML problem (12), one can obtain the following solution

\[
\hat{x}_{\text{ML}} = \left(C_1^T N_1^{-1} C_1\right)^{-1} C_1^T N_1^{-1} y.
\]

with the associated error variance

\[
p_{\text{ML}} = \mathbb{E}\left\{(x - \hat{x}_{\text{ML}})^T (x - \hat{x}_{\text{ML}})\right\}
\]

\[
= \mathbb{E}\left\{(x - (C_1^T N_1^{-1} C_1)^{-1} C_1^T N_1^{-1} (C_1 x + n_1))^T (x - (C_1^T N_1^{-1} C_1)^{-1} C_1^T N_1^{-1} (C_1 x + n_1))\right\}
\]

\[
= \left(C_1^T N_1^{-1} C_1\right)^{-1} C_1^T N_1^{-1} n_1 n_1^T \left(C_1^T N_1^{-1} C_1\right)^{-1} C_1^T N_1^{-1}
\]

\[
= \left(C_1^T N_1^{-1} C_1\right)^{-1}.
\]
When an a priori estimate of $x$ exists, we can consider this statistics as an extra observation which takes the form $m = x + r$, where $r \sim \mathcal{N}(0, P)$, $P \in \mathbb{R}^{p \times n}$ is an independent Gaussian vector. By defining a new independent zero-mean Gaussian vector as $n_{1, \text{new}}^T = [r^T \; n_2^T]$, a new ML estimation problem can be defined as (12). Applying the ML estimation technique ((14a) and (14b)) to this problem, one can rewrite the ML estimate as

$$\hat{x}^\text{ML} = \left( \begin{bmatrix} P \\ 0 \end{bmatrix} \begin{bmatrix} P \\ 0 \end{bmatrix} - 1 \begin{bmatrix} I_n \\ C_2 \end{bmatrix} \begin{bmatrix} I_n \\ C_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} I_n \\ C_2 \end{bmatrix}^T \begin{bmatrix} P \\ 0 \end{bmatrix} - 1 \begin{bmatrix} I_n \\ C_2 \end{bmatrix} \begin{bmatrix} I_n \\ C_2 \end{bmatrix} \times \begin{bmatrix} P \end{bmatrix} \begin{bmatrix} P \end{bmatrix} - 1 \begin{bmatrix} m \\ z \end{bmatrix} = \left( P^{-1} + C_2^T C_2^{-1} \right)^{-1} \left( P^{-1} m + C_2^T C_2^{-1} z \right) = \left( P - PC_2^T \left( N_2 + C_2 PC_2^T \right)^{-1} C_2 P \right) \times \left( P^{-1} m + C_2^T C_2^{-1} z \right) = \left( I_n - PC_2^T \left( N_2 + C_2 PC_2^T \right)^{-1} C_2 \right) m + \left( P - PC_2^T \left( N_2 + C_2 PC_2^T \right)^{-1} C_2 P \right) \times C_2^T C_2^{-1} z = \left( I_n - PC_2^T \left( N_2 + C_2 PC_2^T \right)^{-1} C_2 \right) m + \left( I_n - \right)^{-1} \left( I_n - KC_2 \right) \right),$$

which are exactly the Bayesian estimate algorithm (generating a posterior density $p(x|z)$ from the prior density and current measurement, and then updating this density to be the prior density for the next time step) reported in [17].

**Remark 2:** According to Lemma 3, any linear Gaussian Bayesian estimation problem can be transformed into an ML problem by considering a priori statistics of $x$ as an observation.

**Remark 3:** To estimate the components of $x$ completely, the measurement (12) should provide us adequate constraints. To guarantee this condition, the matrix $C_1$ must assure the relation rank($C_1$) = dim($x$) which is always satisfied in the Bayesian problems.

In spite of the explicit recursive formulation for sequential estimation issues, Lemma 3 cannot be feasible for a singular matrix $N_1$. To overcome this problem when $N_1$ is a singular matrix, it is sufficient to consider the ML estimation issue as the quadratic minimization approach [14]. Using the Lagrange multipliers, one can derive a set of equations as

$$\Pi \begin{bmatrix} \lambda^T \\ x^T \end{bmatrix} = \begin{bmatrix} y^T \; 0_{1 \times n} \end{bmatrix}^T \mbox{ of dimension } (p + n), \mbox{ where}$$

$$\Pi = \begin{bmatrix} N_1 & C_1 \\ C_1^T & 0_{n \times n} \end{bmatrix}. \quad (15)$$

To calculate recent equations, the obvious question is about the conditions on invertibility of the matrix $\Pi$. According to [14], for a positive semi-definite matrix and full column rank matrix $C_1$, if $[N_1 \; C_1]$ has full row rank, then $\Pi$ is invertible. Now, the solution to the aforementioned ML problem can be expressed by the following lemma.

**Lemma 4:** Suppose that $\Pi$ is invertible. The ML estimate of and its error covariance based on the measurement vector (12) is given by

$$\hat{x}^\text{ML} = \begin{bmatrix} \hat{x} \; p^\text{ML} \end{bmatrix} = \begin{bmatrix} 0_{n \times p} \\ I_n \end{bmatrix} \Pi^{-1} \begin{bmatrix} y \\ 0_{n \times 1} - I_n \end{bmatrix}. \quad (16)$$

**Proof:** By solving the equation $\Pi \begin{bmatrix} \lambda^T \\ x^T \end{bmatrix} = \begin{bmatrix} y^T \; 0_{1 \times n} \end{bmatrix}$ of dimension $(p + n)$ in terms of $x$, one has the ML estimate of $x$ as $\hat{x}^\text{ML} = \begin{bmatrix} 0_{n \times p} \\ I_n \end{bmatrix} \Pi^{-1} \begin{bmatrix} y \\ 0_{n \times 1} - I_n \end{bmatrix}$. Also, by substituting $\hat{x}^\text{ML}$ into the equation $p^\text{ML} = \mathbb{E} \left( x - \hat{x}^\text{ML} \right) \left( x - \hat{x}^\text{ML} \right)^T$, the ML error covariance is given by

$$\hat{p}^\text{ML} = \mathbb{E} \left\{ \left( x - \begin{bmatrix} 0_{p \times n} \\ I_n \end{bmatrix} \end{bmatrix} \Pi^{-1} \begin{bmatrix} \begin{bmatrix} C_1 \\ 0_n \end{bmatrix} \right)^T x + \begin{bmatrix} n_1 \\ 0_{n \times 1} \end{bmatrix} \right\}.$$
follows by the inverse partitioned matrix lemma [18] estimate of state $x$ and its error covariance can be stated as

$$
K. Nosrati
$$

algorithm can be adopted here. Applying the GL definition dynamic as in (1a), which can be considered as additional mandatory to apply pseudo-inverse algorithms. The derived that just the terms includes the variable $x_k$, which yields the block matrix $C_k^T = [E^T C_k^T]$. Remark 5: According to Remark 3, if the matrix is full column rank, then is estimable. This obtained result coincides with the estimability theorem of state variable of a discrete stochastic FOS model reported in [1].

Remark 4: Let $N_k$ be a PD matrix. The derived ML-based estimate of state $x$ and its error covariance can be stated as follows by the inverse partitioned matrix lemma [18]

$$
\begin{aligned}
\mathbf{z}_{k}^{\text{ML}} &= \left[\begin{array}{c}
0_{p \times n}^T \\
I_n
\end{array}\right] \Pi^{-1} \left[\begin{array}{c}
y \\
0_{n \times 1}
\end{array}\right] - I_n
\end{aligned}
$$

Then, $\mathbf{y}_{k} = \mathbf{C} x_{k} + v_{k}$ (17a)

$$
E x_{k} = A x_{k-1} - \sum_{j=1}^{k} E(-1)^j v_{j} x_{k-j} + w_{k-1} (17b)
$$

where $\tilde{r}_0 \sim \mathcal{N}(0, \tilde{P}_0)$ is a Gaussian independent vector of $w_k$ and $v_k$. Equations (17) provide a set measurements related to the states vector $\mathbf{x}_k = [x_k]_{k=1}^K$. By investigating this set of measurements according to the problem (12), one can deduce that just the terms $E x_k$ and $C x_k$ include the variable $x_k$, which yields the block matrix $C_k^T = [E^T C_k^T]$. By substituting the derived estimation at the prior step as $\tilde{x}_0 = x_0 + \tilde{r}_0$, where $r_0 \sim \mathcal{N}(0, P_0^{ML})$ is an independent Gaussian vector, in these observations, one has the following equations

$$
\begin{aligned}
y_1 &= C x_1 + v_1 \\
(A + E \gamma_1) \tilde{x}_0^{\text{ML}} &= E x_1 + (A + E \gamma_1) r_0 - w_0.
\end{aligned}
$$

By the principle of mathematical induction and applying ML method, we can establish the following theorem to derive our desirable FSKF recursive algorithm.

**Theorem 2:** Consider the full column rank matrix $[E^T C_k^T]^T$, and suppose that the sequence $Y_i$ together with the prior information about $x_0$ are given. The ML-based optimal estimate $\tilde{x}_k^{\text{ML}}$ can be successively derived by use of the algorithm outlined in the Table 1.

**Proof:** From Remark 4, the full column rank matrix $[E^T C_k^T]^T$ ensure that (15) is a full column rank matrix, and then, the state vector $x_k$ is estimable. For $k = 0$, the given observations are as $y_0 = C x_0 + v_0$ and (17c). By using the fact that $x_0$ is estimable and defining the matrices (13) as

$$
\begin{aligned}
y_{\text{new}} &= \left[\begin{array}{c}
\tilde{x}_0 \\
y_0
\end{array}\right], \\
C_{1, \text{new}} &= \left[\begin{array}{c}
I_n \\
C
\end{array}\right], \\
N_{1, \text{new}} &= \left[\begin{array}{c}
P_0 \\
0
\end{array}\right] - \tilde{R}_0,
\end{aligned}
$$

one needs to take Lemma 4 into (17a) and (17c), when $i = 0$, to derive $\tilde{x}_0^{\text{ML}}$ and $P_0^{ML}$ as follows,

$$
\begin{aligned}
\tilde{x}_0^{\text{ML}} &= \left[\begin{array}{c}
0_{n \times (n+p)}^T \\
I_n
\end{array}\right] N_{1, \text{new}} C_{1, \text{new}}^{-1} \left[\begin{array}{c}
0_{n \times (n+p) \times n} \\
I_n
\end{array}\right]^{-1} \\
&= \left[\begin{array}{c}
0_{n \times n} \\
y_0
\end{array}\right] \tilde{P}_0 - P_0^{n \times p} R_0 C I_n^{-1} \\
&= \left[\begin{array}{c}
\tilde{x}_0 \\
y_0
\end{array}\right] 0_{n \times n} \\
&= \left[\begin{array}{c}
\tilde{x}_0 \\
y_0
\end{array}\right] 0_{n \times n} \\
&= \left[\begin{array}{c}
\tilde{x}_0 \\
y_0
\end{array}\right] 0_{n \times n} \\
&= \left[\begin{array}{c}
\tilde{x}_0 \\
y_0
\end{array}\right] 0_{n \times n}
\end{aligned}
$$

In an analogous manner, the observations can be given as

$$
\begin{aligned}
y_1 &= C x_1 + v_1 \\
(A + E \gamma_1) \tilde{x}_0^{\text{ML}} &= E x_1 + (A + E \gamma_1) r_0 - w_0.
\end{aligned}
$$

TABLE 1. ML-based FSKF recursive algorithm.

| Step 0: Initial values |
|------------------------|
| $\mathbf{P}_0^{ML} = \left[\begin{array}{c}
\mathbf{P}_0 \\
\mathbf{Q}_0
\end{array}\right]$ |

| Step k: Prediction |
|-------------------|
| $\mathbf{x}_k = \mathbf{Q}_{k-1} + (A + E \gamma_k) \mathbf{P}_{k-1} + (A + E \gamma_k)^T + \sum_{j=1}^{k} \mathbf{Q}_{j-1} (E \gamma_k)^T$ |

| Step k: Update |
|----------------|
| $\mathbf{P}_k^{ML} = \left[\begin{array}{c}
\mathbf{P}_k \\
\mathbf{Q}_k
\end{array}\right]$ |
To derive $\tilde{x}_{k}^{ML}$ and $P_{k}^{ML}$, it is sufficient to apply Lemma 4 to matrices (13) as

$$y_{new} = \begin{bmatrix} \hat{x}_{1} \\ y_{1} \end{bmatrix}, \quad C_{1,new} = \begin{bmatrix} E \\ C \end{bmatrix}, \quad N_{1,new} = \begin{bmatrix} \tilde{P}_{1} & 0 \\ 0 & R_{1} \end{bmatrix},$$

where $\hat{x}_{1} = (A + EY_{1}) \hat{x}_{0}^{ML}$ and $\tilde{P}_{1} = \mathbb{E} \left[ w_{0,new} w_{0,new}^{T} \right] = Q_{0} + (A + EY_{1}) P_{0}^{ML} (A + EY_{1})^{T}$ with $w_{0,new} = (A + EY_{1})$.

Then, the 3-block FSKF in the second step can be given as

$$\begin{bmatrix} \hat{x}_{k}^{ML} \\ P_{k}^{ML} \end{bmatrix} = \begin{bmatrix} 0_{n\times(n+p)} & I_{n} \\ 0_{n\times1} & -I_{n} \end{bmatrix} \begin{bmatrix} N_{1,new} C_{1,new} \\ C_{1,new} \end{bmatrix}^{-1} \begin{bmatrix} \hat{P}_{1} \\ 0_{n\times1} \end{bmatrix} \begin{bmatrix} 0_{n\times1} & E \\ 0_{n\times1} & R_{1} \end{bmatrix} \begin{bmatrix} E^{T} \\ C^{T} \end{bmatrix} \begin{bmatrix} 0_{n\times1} \\ 0_{n\times1} \end{bmatrix}. $$

Likewise for $k > 1$, if we denote $P_{k-1}^{ML}, \ldots, P_{2}^{ML}$ and $P_{1}^{ML}$ as the error covariance matrices associated with the filtered estimates $\hat{x}_{k-1}^{ML}, \ldots, \hat{x}_{2}^{ML}$ and $\hat{x}_{1}^{ML}$, with initial conditions $\hat{x}_{0}^{ML}$ and $P_{0}^{ML}$ as the prior distribution for $x_{0}$. By defining the prior observations as $\hat{x}_{i}^{ML} = x_{i} + r_{i}, 0 \leq i \leq k$, where $r_{i} \sim \mathcal{N}(0, P_{i}^{ML})$ are Gaussian random independent vectors of $w_{j}$. By substituting these observations into (17b), one can obtain the following observations.

$$(A + EY_{1}) \tilde{x}_{k-1}^{ML} - \sum_{j=2}^{k} E(-1)^{j-1} Y_{j} \tilde{x}_{k-j}^{ML} = E x_{k} + w_{k-1,new},$$

where $w_{k-1,new} = (A + EY_{1}) r_{k-1} - \sum_{j=1}^{k-1} E(-1)^{j-1} Y_{j} r_{k-j} - w_{k-1}$. Therefore, in order to derive the recursive equations of estimation at step $k$, one can apply Lemma 4 to matrices (13) as

$$y_{new} = \begin{bmatrix} \tilde{x}_{k} \\ y_{k} \end{bmatrix}, \quad C_{1,new} = \begin{bmatrix} E \\ C \end{bmatrix}, \quad N_{1,new} = \begin{bmatrix} \tilde{P}_{k} & 0 \\ 0 & R_{k} \end{bmatrix},$$

where $\tilde{x}_{k} = (A + EY_{1}) \tilde{x}_{k-1}^{ML} - E \sum_{j=2}^{k} (-1)^{j-1} Y_{j} \tilde{x}_{k-j}^{ML}$ and $\tilde{P}_{k} = \mathbb{E} \left[ w_{k-1,new} w_{k-1,new}^{T} \right] = Q_{k-1} + (A + EY_{1}) P_{k-1}^{ML} (A + EY_{1})^{T} + \sum_{j=2}^{k} (EY_{j}) P_{j-2}^{ML} (EY_{j})^{T}$. Finally, it is easy to see that the 3-block FSKF $\tilde{x}_{k}^{ML}$ and $P_{k}^{ML}$ can be respectively derived as

$$\begin{bmatrix} \hat{x}_{k}^{ML} \\ P_{k}^{ML} \end{bmatrix} = \begin{bmatrix} 0_{n\times(n+p)} & I_{n} \\ 0_{n\times1} & -I_{n} \end{bmatrix} \begin{bmatrix} N_{1,new} C_{1,new} \\ C_{1,new} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{P}_{k} \\ 0_{n\times1} \end{bmatrix} \begin{bmatrix} 0_{n\times1} & E \\ 0_{n\times1} & R_{k} \end{bmatrix} \begin{bmatrix} E^{T} \\ C^{T} \end{bmatrix} \begin{bmatrix} 0_{n\times1} \\ 0_{n\times1} \end{bmatrix}.$$

Theorem 2 organizes a recursive FSKF algorithm begins from $x_{0}$ and $P_{0}$ as a priori information. Then, at the following steps, the estimate of $\tilde{x}_{k}^{ML}$ can be achieved by using the prediction and the update equations.

**Corollary 1:** Suppose that system (1) has an input term defined by $B_{k} u_{k}$ with $B_{k} \in \mathbb{R}^{m \times n}$ and $u_{k} \in \mathbb{R}^{m}$ in right-hand side of its first equation. Then, Theorem 2 can be stated with a slight modification as follows: Consider the full column rank matrix $[E^{T} C^{T}]$, and let $U_{k} = \{u_{i}\}_{i=0}^{n}$ and $Y_{i}$ be the known input and measurement data, respectively. Now, for the modified FOS model (1), the FSKF algorithm presented in Table 1 remains unchanged, except for $\tilde{x}_{k}$ that can be rewritten as $\tilde{x}_{k} = (A + EY_{1}) \tilde{x}_{k-1}^{ML} + B u_{k-1} - E \sum_{j=2}^{k} (-1)^{j} Y_{j} \tilde{x}_{k-j}^{ML}$.

Proof: In the same analogous as the proof of Theorem, one needs to take the observation (19b) as

$$(A + EY_{1}) \tilde{x}_{k-1}^{ML} - \sum_{j=2}^{k} E(-1)^{j-1} Y_{j} \tilde{x}_{k-j}^{ML} + B u_{k-1} = E x_{k} + w_{k-1,new},$$

for $k > 0$, and apply Lemma 4 to matrices (13) as

$$y_{new} = \begin{bmatrix} \tilde{x}_{k} \\ y_{k} \end{bmatrix}, \quad C_{1,new} = \begin{bmatrix} E \\ C \end{bmatrix}, \quad N_{1,new} = \begin{bmatrix} \tilde{P}_{k} & 0 \\ 0 & R_{k} \end{bmatrix},$$

with $\tilde{x}_{k} = (A + EY_{1}) \tilde{x}_{k-1}^{ML} + B u_{k-1} - E \sum_{j=2}^{k} (-1)^{j} Y_{j} \tilde{x}_{k-j}^{ML}$ and $\tilde{P}_{k}$ same as before.

Remark 6: It is easy to see that the derived 3-block FSKF recursive algorithm based on ML method coincides to that of LS-based approach introduced in [1].

**C. ADJUSTED FSKF ALGORITHM**

According to Remark 1, the FOS model (1) with $\mu > 1$ possess non-causal behavior in its dynamics. This phenomenon is completely alien to normal models. This causes the state vector $x_{i}$ to be subject to constraints due to future, which makes the situation even more complicated. It can be seen that the information provided by this dynamics is not dependent of $i$. The impact of these dynamics can be considered as one observation. By incorporating this observation as additional information, an adjusted estimation algorithm can be derived by modification of the FSKF algorithm as derived from Theorem 2. In the following, we apply the effect of future dynamics in 17b and replace it with an augmented observation given as

$$F x_{k} = \eta_{k},$$

where $\eta_{k} \sim \mathcal{N}(0, \Gamma_{k})$. Therefore, the our issue is to find the matrices $F$ and $\Gamma_{k}$ in terms of the matrices $E$, $A$ and $Q_{k}$. Manipulating 17b, one can derive the following matrix

$$\begin{bmatrix} \tilde{x}_{k} \\ y_{k} \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & 0_{p\times n} \\ 0_{p\times n} & R_{k} \end{bmatrix} \begin{bmatrix} E^{T} \\ C^{T} \end{bmatrix} \begin{bmatrix} 0_{n\times1} \end{bmatrix},$$

$$\begin{bmatrix} \tilde{x}_{k} \\ y_{k} \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & 0_{p\times n} \\ 0_{p\times n} & R_{k} \end{bmatrix} \begin{bmatrix} E^{T} \\ C^{T} \end{bmatrix} \begin{bmatrix} 0_{n\times1} \end{bmatrix}.$$
we suppose out to eliminate the rest of the measurement. To be specific, $x$ be indicated as exogenous variables. Using the elementary
\[
\begin{align*}
\Theta &= \begin{bmatrix}
-W_k & \cdots & \cdots & \cdots \\
W_{k+1} & \cdots & \cdots & \cdots \\
W_{k+2} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
\end{bmatrix}.
\end{align*}
\tag{21}
\]

where $W_{k+n} = w_{k+n} - \sum_{j=n+2}^{k+n+1}(-1)^j E\gamma_j x_{k+n+1-j}$, $n = 0, 1, \cdots$ and

\[
\Theta = \begin{bmatrix}
-(A + E\gamma_1) & E & \cdots & \cdots \\
(-1)^2 E\gamma_2 & -(A + E\gamma_1) & E & \cdots \\
(-1)^3 E\gamma_3 & (-1)^2 E\gamma_2 & -(A + E\gamma_1) & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
\end{bmatrix}.
\]

Equation (21) provides some additional data about the state $x_k$ together with the state vectors $x_i$ for $i > k$, which can be indicated as exogenous variables. Using the elementary row operations, the state vectors $x_i$, $i > k$ can be dropped out to eliminate the rest of the measurement. To be specific, we suppose

\[
\left( \begin{array}{c}
T_i \\
T_i \\
\vdots \\
\end{array} \right) = \begin{bmatrix}
T_0 (A + E\gamma_1) + T_1 E\gamma_2 + \cdots \\
0 \\
0 \\
\vdots \\
\end{bmatrix}
\tag{22}
\]

where $T_i \in \mathbb{R}$, $i = \{0, 1, 2, \ldots\}$. Equation (23) is of the form (20) with $\eta_i = \sum_m T_m w_{i+m}$ and $F = -T_0 (A + E\gamma_1) + \sum_m (-1)^m T_m E\gamma_{i+m+1}$. So the problem is finding a row rank matrix satisfying (22) which can be rewritten as

\[
T(z) \left( zE - (A + E\gamma_1) - \begin{bmatrix} 0 & \cdots & \cdots \end{bmatrix} \right) x_k = \sum_m T_m w_{k+m} - \sum_{m=0}^{\infty} \sum_{i=2+m}^{k+m+1} (-1)^i E\gamma_i x_{k+m+1-i}
\tag{23}
\]

with $m = 0, 1, 2, \ldots$. Finally, from (23) one has $F x_k = \sum_{m=1}^{\infty} T_m w_{k+m} - \sum_{m=0}^{\infty} \sum_{i=2+m}^{k+m+1} (-1)^i E\gamma_i x_{k+m+1-i}$, where $\sum_{m=1}^{\infty} T_m w_{k+m} = T(z)$. Thus, one obtains (20) with

\[
\Gamma_k = \sum_{m} T_m \Omega_{k+m} T_m^T + \sum_{m} \sum_{i=2+m}^{k+m+1} T_m E\gamma_i p_{i+k+m+1-i} (T_m E\gamma_i)^T.
\]

Now, applying the presented method in the previous subsection, we manipulate the optimal estimate $\hat{x}_{ML}^i$ of $x_k$ by the swing equations of the machines $\gamma h_k$, where $h_k \sim \mathcal{N}(0, P_{ML}^i)$ is a Gaussian vector. Applying the future dynamics (20), we can derive the estimate of the state vector $x_k$ by using the information (12) given as

\[
\left[ \begin{array}{c}
\hat{x}_{ML}^i \\
0 \\
\vdots \\
\end{array} \right] = \left[ \begin{array}{c}
I \\
F \\
\vdots \\
\end{array} \right] x_k + \left[ \begin{array}{c}
\eta_k \\
0 \\
\vdots \\
\end{array} \right].
\]

According to Lemma 4 and modified matrices (13) as

\[
\begin{bmatrix}
\hat{x}_{ML}^i \\
0 \\
\vdots \\
\end{bmatrix} = \left[ \begin{array}{c}
C_{1,new} \\
F \\
\vdots \\
\end{array} \right] \begin{bmatrix}
N_{1,new} \\
0 \\
0 \\
\vdots \\
\end{bmatrix}^{-1} \begin{bmatrix}
\eta_{new} \\
0 \\
0 \\
\vdots \\
\end{bmatrix} - I
\]

\[
= \left[ \begin{array}{c}
0 \\
0 \\
\vdots \\
\end{array} \right] \begin{bmatrix}
p_{ML}^k \\
0 \\
\Gamma_k \\
0 \\
\vdots \\
\end{bmatrix}^T \begin{bmatrix}
F \\
0 \\
\vdots \\
\end{bmatrix} - I
\]

As seen, the effect of future dynamics is interpreted as a posteriori modification to the estimate that relies on past dynamics and observations. The obstacle of this method is the use of polynomial matrix manipulations, which makes it numerically intractable. A solution to alleviate this problem can be to convert the estimation problem to where future information does not have any effect on present state estimates, which can be considered as a future research direction.

IV. SIMULATION RESULTS: A THREE MACHINE INFINITE BUS POWER SYSTEM

Here, a three machine infinite bus power system is used to verify the estimation performances of the derived filter in subsection III. As shown in Fig. 1, the dynamic behavior of model is controlled by the swing equations of the machines G1, G2 and G3. The spinning of the generators has considerable inertia where it makes the dynamic behaviors that are dependent on the history. According to the features of
fractional calculus, a fractional order model of the generator system can be considered based on [19]. Also, the fifth node introduces the algebraic behavior [20]. Owing to the conception of fractional calculus and singular theory, the corresponding linear FOS model of the three machines infinite bus system is described as follows,

\[
\begin{align*}
D^\alpha \delta_1 &= \omega_1 \\
D^\alpha \delta_2 &= \omega_2 \\
D^\alpha \delta_3 &= \omega_3 \\
D^\alpha \omega_1 &= \frac{1}{M_1} (P_1 - Y_{12} V_1 V_2 (\delta_1 - \delta_2) - Y_{15} V_1 V_5 (\delta_1 - \delta_5) - D_1 \omega_1) \\
D^\alpha \omega_2 &= \frac{1}{M_2} (P_2 - Y_{21} V_2 V_1 (\delta_2 - \delta_1) - Y_{25} V_2 V_5 (\delta_2 - \delta_5) - D_2 \omega_2) \\
D^\alpha \omega_3 &= \frac{1}{M_3} (P_3 - Y_{34} V_3 V_\infty \delta_3 - Y_{35} V_3 V_5 (\delta_3 - \delta_5) - D_3 \omega_3) \\
0 &= -Y_{51} V_5 V_1 (\delta_5 - \delta_1) - Y_{52} V_5 V_2 (\delta_5 - \delta_2) - Y_{53} V_5 V_3 (\delta_5 - \delta_3) - Y_{54} V_5 V_\infty \delta_5.
\end{align*}
\]

The parameter \( \alpha \) is the fractional order and \( D^\alpha \) refers to GL operator with \( 0 < \alpha < 2 \). Also, \( \delta_1, \delta_2 \) and \( \delta_3 \) are the generator angles, \( \delta_5 \) is the bus angle, and \( \omega_1, \omega_2 \) and \( \omega_3 \) denotes the variation of the generator angles. In addition, \( P_1, P_2 \) and \( P_3 \) are the mechanical power inputs, \( M_1, M_2 \) and \( M_3 \) are the angular momenta, \( Y_{12}, Y_{15}, Y_{25}, Y_{34}, Y_{35} \) and \( Y_{45} \) are the admittances, \( D_1, D_2 \) and \( D_3 \) are the damping factors, and \( V_3, V_5 \) and \( V_\infty \) are the potentials. The nominal values of parameters are chosen as follows,

\[
\begin{align*}
M_1 &= 14 & M_2 &= 26 & M_3 &= 20 \\
D_1 &= 0.057 & D_2 &= 0.15 & D_3 &= 0.11 \\
Y_{12} &= 1 & Y_{15} &= 0.5 & Y_{25} &= 1.2 \\
Y_{34} &= 0.7 & Y_{35} &= 0.8 & Y_{45} &= 1 \\
V_i &= 1 & i &= 1, 2, 3, 5, \infty & \alpha &= 1.
\end{align*}
\]

Assuming that the only available measurements are the generator angles and denoting \( \delta_1, \delta_2, \delta_3, \omega_1, \omega_2, \omega_3 \) and \( \delta_5 \) by the state variables \( x_1, x_2, x_3, x_4, x_5, x_6 \) and \( x_7 \), respectively, and also the parameters \( P_1, P_2 \) and \( P_3 \) by the inputs \( u_1, u_2 \) and \( u_3 \), respectively, the FOS model of the three machines infinite

**FIGURE 1.** A three machine infinite bus system.

**FIGURE 2.** Trajectories of the generator angles, their estimated and absolute error values of system (26).
with respect to Definition 2 in [1], the continuous FOS infinite bus system can be discretized. First, the block matrix $[E \ T \ C]^T$ is full column rank and accordingly (26) is estimable. Then, after discretization and applying the derived ML-based FSKF algorithm, the results including actual value of states, their estimated signals and absolute error values are depicted in Figs. (2), (3) and (4). One can see that the results make sense as the states of (26) are estimated with reasonable accuracy at the right time. Also, the absolute error of each state is depicted beneath the time graph of actual value and its estimation to provide an improved clarity of estimation performance. We observe that these errors between actual values and their estimations are desirable. Note that electric power systems are very complicated dynamical systems due to their intrinsic property of high nonlinearity. In three machine infinite bus system, the variation of generator angles

$E = \text{diag} \{ I_6, 0_{1 \times 6} \}$, $A = \begin{bmatrix} 0_3 & I_3 & 0_{3 \times 1} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0_{1 \times 3} & -3.5 \end{bmatrix}$,

$B = \begin{bmatrix} 0_{3 \times 3} \\ b_{21} \\ 0_{1 \times 3} \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$,

where $a_{22} = -\text{diag} \{ 0.0041, 0.0058, 0.0055 \}$, $b_{21} = \text{diag} \{ 0.07, 0.038, 0.05 \}$, $a_{31} = \begin{bmatrix} 0.5 & 1.2 & 0.8 \end{bmatrix}$ and

$a_{21} = \begin{bmatrix} -0.107 & 0.071 & 0 \\ 0.0038 & -0.0085 & 0 \\ 0 & 0 & -0.075 \end{bmatrix}$, $a_{23} = \begin{bmatrix} 0.0036 \\ 0.0046 \\ 0.004 \end{bmatrix}$.

With respect to Definition 2 in [1], the continuous FOS infinite bus system can be discretized. First, the block matrix $[E \ T \ C]^T$ is full column rank and accordingly (26) is estimable. Then, after discretization and applying the derived ML-based FSKF algorithm, the results including actual value of states, their estimated signals and absolute error values are depicted in Figs. (2), (3) and (4). One can see that the results make sense as the states of (26) are estimated with reasonable accuracy at the right time. Also, the absolute error of each state is depicted beneath the time graph of actual value and its estimation to provide an improved clarity of estimation performance. We observe that these errors between actual values and their estimations are desirable. Note that electric power systems are very complicated dynamical systems due to their intrinsic property of high nonlinearity. In three machine infinite bus system, the variation of generator angles
has complex dynamics with more coupling effects and behavioral interactions than the generator and bus angle. Also, the only available measurements are the generator angles, which have been picked off by the observation matrix \( C \) that converts the system state estimate from the state space to the measurement space and outputs. Here, the matrix \( C \) just select certain states in the form of a linear transformation and projects the three first states and the last one to the measurement unit. That is why the estimated plots for the states \( x_i, i = 4, 5, 6 \) show slightly different behavior than the other four states where they show constant steps during the estimation process. Moreover, the mean square errors (MSEs) of the FSKF algorithm for the state estimation of the three machine infinite bus system are presented in Table 2. Again, we observe that the estimation of each state has enough accuracy.

### V. CONCLUSION

In this study, we have considered an estimation problem using stochastic FOS discrete linear models. First, we showed that regularity is a necessary and sufficient condition on the solvability of these systems. Second, the ML-based optimal filter algorithm and its corresponding error covariance have been derived in a 3-block structure. This possesses a number of advantages in comparison to the existing filtering algorithms. It works when the measurement noise is singular, in which the derived algorithm requires only standard matrix inverses instead of pseudo inverses. In addition, we have demonstrated that how the derived estimation procedure can be adjusted by some additional information related to the future dynamics. Finally, we verified the estimation performance parameters of the proposed filter by a numerical example on a new FOS model of a three-machine infinite bus system, where the estimation algorithm showed a desirable result with enough accuracy. It may be noted that a drawback of the adjusted filter is its implementation feasibility. Due to increasing the matrix dimension, computational complexity may limit some practical implementations. For some classes of problems with large matrices, it has been shown that use of geometric algebra can result in improvement in terms of tractability [21]. This problem together with the investigation of filter stability, continuous-time case studies, etc. can be considered as possible extensions of this work.

### REFERENCES

[1] K. Nosrati and M. Shafiee, “Kalman filtering for discrete-time linear fractional-order singular systems,” *IET Control Theory Appl.*, vol. 12, no. 9, pp. 1254–1266, Jun. 2018.

[2] Y. Wei, P. W. Tse, Z. Yao, and Y. Wang, “The output feedback control synthesis for a class of singular fractional order systems,” *ISA Trans.*, vol. 69, pp. 1–9, Jul. 2017.

[3] K. Nosrati and M. Shafiee, “Dynamic analysis of fractional-order singular Holling type-II predator–prey system,” *Appl. Math. Comput.*, vol. 313, pp. 159–179, Nov. 2017.

[4] K. Nosrati and M. Shafiee, “Fractional-order singular logistic map: Stability, bifurcation and chaos analysis,” *Chaos, Solitons Fractals*, vol. 115, pp. 224–238, Oct. 2018.

[5] X. Zhang and Z. Zhao, “Robust stabilization for rectangular descriptor fractional order interval systems with order \( 0 < \alpha < 1 \),” *Appl. Math. Comput.*, vol. 366, Feb. 2020, Art. no. 124603.

[6] S.-L. Sun and J. Ma, “Optimal filtering and smoothing for discrete-time stochastic singular systems,” *Signal Process.*, vol. 87, no. 1, pp. 189–201, Jan. 2007.

[7] F. P. Komachali, M. Shafiee, and M. Darouach, “Design of unknown input fractional order proportional–integral observer for fractional order singular systems with application to actuator fault diagnosis,” *IET Control Theory Appl.*, vol. 13, no. 14, pp. 2163–2172, Sep. 2019.

[8] K. Nosrati and M. Shafiee, “On the convergence and stability of fractional singular Kalman filter and Riccati equation,” *J. Franklin Inst.*, vol. 357, no. 4, pp. 2263–2281, Mar. 2020.

[9] Z. Gao, “Reduced order Kalman filter for a continuous-time fractional-order system using fractional-order average derivative,” *Appl. Math. Comput.*, vol. 316, pp. 155–166, Jan. 2018.

[10] Z. Gao, “Leader-following consensus conditions for fractional-order descriptor uncertain multi-agent systems with \( 0 < \alpha < 2 \) via output feedback control,” *J. Franklin Inst.*, vol. 357, no. 4, pp. 2263–2281, Mar. 2020.

[11] G. Liu, S. Xu, Y. Wei, Z. Qi, and Z. Zhang, “New insight into reachable set estimation for uncertain singular time-delay systems,” *Appl. Math. Comput.*, vol. 320, pp. 769–780, Mar. 2018.

[12] S.-L. Sun and J. Ma, “Optimal filtering and smoothing for discrete-time stochastic singular systems,” *Signal Process.*, vol. 87, no. 1, pp. 189–201, Jan. 2007.

[13] B. Boulkroune, M. Zasadzinski, and M. Darouach, “Moving horizon state estimation for linear discrete-time singular systems,” *IET Control Theory Appl.*, vol. 4, no. 3, pp. 339–350, Mar. 2010.

[14] R. Nikoukhah, A. S. Willsky, and B. C. Levy, “Kalman filtering and Riccati equations for descriptor systems,” *IEEE Trans. Autom. Control*, vol. 37, no. 9, pp. 1325–1342, 1992.

[15] J. Y. Ishihara, M. H. Terra, and A. F. Bianco, “Recursive linear estimation for general discrete-time descriptor systems,” *Automatica*, vol. 46, no. 4, pp. 761–766, Apr. 2010.

[16] F. R. Gantmache, *The Theory of Matrices II*. New York, NY, USA: Chelsea Publishing Company, 1960.

[17] A. L. Barker, D. E. Brown, and W. N. Martin, “Bayesian estimation and the Kalman filter,” *Comput. Math. with Appl.*, vol. 30, no. 10, pp. 55–77, Nov. 1995.

[18] H. V. Henderson and S. R. Searle, “On deriving the inverse of a sum of matrices,” *SIAM Rev.*, vol. 23, no. 1, pp. 53–60, Jan. 1981.

[19] D. Koenig, “Unknown input proportional multiple-integral observer design for linear descriptor systems: Application to state and fault estimation,” *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 212–217, Feb. 2005.

[20] F. P. Komachali and M. Shafiee, “Sensor fault diagnosis in fractional-order singular systems using unknown input observer,” *Int. J. Syst. Sci.*, vol. 51, no. 1, pp. 116–132, Jan. 2020.

[21] J. M. Chappell, A. Iqbal, and D. Abbott, “N-player quantum games in an EPR setting,” *PLoS ONE*, vol. 7, no. 5, May 2012, Art. no. e36404.
KOMEIL NOSRATI was born in Sari, Iran, in 1983. He received the B.Sc. degree in computer engineering from Iran University of Science and Technology, Tehran, Iran, in 2007, the M.Sc. degree in electrical engineering from Ferdowsi University of Mashhad, Mashhad, Iran, in 2010, and the Ph.D. degree in electrical engineering from Amirkabir University of Technology, Tehran, in 2018, under the supervision of M. Shafiee.

He currently has research collaboration with Amirkabir University of Technology. He is also with the Research and Development Group, Tehran Electrical Distribution Company and incorporated in leading activities around smart grid technologies. He has recently joined with DETCO Group as a Project Manager involved in design, procurement, and configuration and commissioning of automation and telecommunication systems of power plants and substations. He authored or coauthored more than 16 research papers in archival journals, book chapters, and international conference proceedings. His research interests include fractional calculus, singular systems, estimation and filtering, nonlinear systems, chaotic systems, and secure communication.

DEREK ABBOTT (Fellow, IEEE) was born in South Kensington, London, U.K., in 1960. He received the B.Sc. degree (Hons.) in physics from Loughborough University, Leicestershire, U.K., in 1982, and the Ph.D. degree in electrical and electronic engineering from The University of Adelaide, Australia, in 1995, under the supervision of K. Eshraghian and B. R. Davis. His research interests include multidisciplinary physics and electronic engineering applied to complex systems. His research programs span a number of areas of stochastics, game theory, photonics, energy policy, biomedical engineering, and computational neuroscience. He is a fellow of the Institute of Physics, U.K., and an Honorary Fellow of Engineers Australia. He received number of awards, including South Australian Tall Poppy Award for Science, in 2004, the Premier’s SA Great Award in Science and Technology for outstanding contributions to South Australia, in 2004; an Australian Research Council Future Fellowship, in 2012; David Dewhurst Medal, in 2015; Barry Inglis Medal, in 2018; and M. A. Sargent Medal, in 2019. He has served as an Editor and/or Guest Editor for a number of journals, including IEEE Journal of Solid-State Circuits, Journal of Optics B: Quantum and Semiclassical Optics, Microelectronics Journal, Chaos, smart Materials and Structures, Fluctuation and Noise Letters, Proceedings of the IEEE, and IEEE Photonics Journal. He has served on the Editorial Board for Proceedings of the IEEE, from 2009 to 2014. He has been serving on the Editorial Board for IEEE Access, since 2015. Since 2019, he has been serving on the IEEE Publications Services and Products Board (PSPB).

MASOUD SHAFIEE (Senior Member, IEEE) was born in Yazd, Iran, in 1957. He received the B.Sc. degree in mathematics from the College of Management and Mathematics, Karaj, Iran, in 1975, the M.Sc. degree in mathematics and the M.Sc. degree in systems engineering from Wright State University, Dayton, Ohio, in 1980 and 1982, respectively, and the M.Sc. and Ph.D. degrees in electrical engineering from Louisiana State University, Baton Rouge, LA, USA, in 1984 and 1987, respectively.

He is currently a Full Professor with the Department of Electrical Engineering, Amirkabir University of Technology, Tehran, Iran, where he is with SIC Research Institute. He is the author of more than 230 research articles, 13 books (as author), and 11 books (as translator). His research interests include multidimensional systems, singular systems, control of communication networks, information and communications technology (ICT), wavelets, robotics, and automation.

* * *