EXPLICIT ESTIMATES FOR SOLUTIONS OF NONLINEAR RADIATION-TYPE PROBLEMS

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Abstract. We establish the existence of weak solutions of a nonlinear radiation-type boundary value problem for elliptic equation on divergence form with discontinuous leading coefficient. Quantitative estimates play a crucial role on the real applications. Our objective is the derivation of explicit expressions of the involved constants in the quantitative estimates, the so-called absolute or universal bounds. The dependence on the leading coefficient and on the size of the spatial domain is precise. This work shows that the expressions of those constants are not so elegant as we might expect.

1. Introduction

Thermal effects on steady-state physical and technological models, whatever they are from mechanical engineering, electrochemistry, biomedical engineering, to mention a few, appear as an additional elliptic equation with a nonlinear radiation-type boundary condition into the coupled PDE system under study [5–8]. These form a boundary value problem constituted by an elliptic quasilinear second order equation in divergence form with the leading coefficient depending on the spatial variable and on the solution itself. The problem of determining radiative effects provides an interesting special case of a conormal derivative boundary value problem for an elliptic divergence structure equation [16]. Here, we deal with the radiation-type condition on a part of the boundary, and on the remaining part the Neumann condition is taken into account. Stationary heat conduction equation with the radiation boundary condition (fourth power law) has been studied in two-dimensional [18] and three-dimensional [19] Lipschitz domains.

In the existence theory, the quantitative estimates of solutions to a linear elliptic equation in divergence form, with bounded and measurable coefficient, play a crucial role. Indeed, they enjoy a large interest in the literature (see for instance [14,17,12,15,20], and the references therein). Most mathematicians have bearing to keep abstract the universal bounds along one whole work. The values of the intervener constants are simply carried out. It is forgotten that their values are crucial on the real applications and/or the numerical analysis (see [14] and the references therein) of the problems under study. Our objective is to fill such gap.

The outline of the present paper is as follows. We begin by stating the problem under study and its functional framework in the next section. The Hilbert case is

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studied in Section 3. We derive $L^q$ (Section 4), $L^\infty$ (Section 5), and $W^{1,q}$ (under $L^1$-data in Section 6) estimates for weak solutions. Finally, a $W^{1,p}$-estimate ($p < n/(n-1)$ for the Green kernel and a $W^{1,q}$-estimate for weak solutions of linear boundary value problem, the so-called mixed Robin-Neumann problem, are obtained in Sections 7 and 8 respectively. Lipschitz domains, discontinuous leading coefficient, and $L^1$-data are the three mathematical shortcomings from the physical models on the real world. It is taken them into account that our results are stated.

2. Statement of the problem

Set $\Omega$ a domain (that is, connected open set) in $\mathbb{R}^n$ ($n \geq 2$) of class $C^{0,1}$, and bounded. Its boundary $\partial \Omega$ is constituted by two disjoint open $(n-1)$-dimensional sets, $\Gamma_N$ and $\Gamma$, such that $\partial \Omega = \overline{\Gamma_N} \cup \overline{\Gamma}$. We consider $\Gamma_N$ over which the Neumann boundary condition is taken into account, and $\Gamma$ over which the radiative effects may occur.

We study the following boundary value problem, in the sense of distributions,

\begin{align}
- \nabla \cdot (A \nabla u) &= f - \nabla \cdot f \quad \text{in } \Omega; \\
(A \nabla u - f) \cdot n + b(u) &= h \quad \text{on } \Gamma; \\
(A \nabla u - f) \cdot n &= g \quad \text{on } \Gamma_N,
\end{align}

where $n$ is the unit outward normal to the boundary $\partial \Omega$. Whenever the $(n \times n)$-matrix of the leading coefficient is $A = aI$, where $a$ is a real function and $I$ denotes the identity matrix, the elliptic equation stands for isotropic materials. Our problem includes the conormal derivative boundary value problem. For that, it is sufficient to consider the situation $\Gamma = \partial \Omega$ (or equivalently $\Gamma_N = \emptyset$). The problem (2)-(3) is the so-called mixed Robin-Neumann problem if the boundary condition (2) is linear, i.e.

\begin{align}
b(u) &= b_* u, \quad \text{for some } b_* > 0.
\end{align}

Set for any $p, \ell \geq 1$

\begin{align}
V_{p,\ell} := \{ v \in W^{1,p}(\Omega) : v \in L^\ell(\Gamma) \}
\end{align}

the Banach space endowed with the norm

\begin{align}
\| v \|_{V_{p,\ell}} := \| v \|_{p,\Omega} + \| \nabla v \|_{p,\Omega} + \| v \|_{\ell,\Gamma}.
\end{align}

For the sake of simplicity, we denote by the same designation $v$ the trace of a function $v \in W^{1,1}(\Omega)$. For $p > 1$, the space $V_{p,\ell}$ is reflexive by arguments given in [9]. Observe that $V_{p,\ell}$ is a Hilbert space equipped with the inner product only if $p = \ell = 2$. The above norm is equivalent to

\begin{align}
\| v \|_{1,p,\ell} := \| \nabla v \|_{p,\Omega} + \| v \|_{\ell,\Gamma},
\end{align}

due to a Poincaré inequality [3] Corollary 3):

\begin{align}
\| v \|_{p,\Omega} \leq P_p \left( \sum_{i=1}^{n} \| \partial_i v \|_{p,\Omega} + |\Gamma|^{1/p-1} \left| \int_{\Gamma} v ds \right| \right).
\end{align}

Here $| \cdot |$ stands for the $(n - 1)$-Lebesgue measure. Throughout this work, the significance of $| \cdot |$ also stands for the Lebesgue measure of a set of $\mathbb{R}^n$. 
By trace theorem,
\[ V_{p,\ell} = W^{1,p}(\Omega), \quad \text{if } 1 \leq \ell < p(n-1)/(n-p); \]
\[ V_{p,\ell} \subset \neq W^{1,p}(\Omega), \quad \text{if } \ell > p(n-1)/(n-p). \]

For \( 1 < q < n \), the best constants of the Sobolev and trace inequalities are, respectively, \([2,21]\)

\[ S_q = \pi^{-1/2} n^{-1/q} \left( \frac{q-1}{n-q} \right)^{1-1/q} \left[ \frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/q)\Gamma(1+n-n/q)} \right]^{1/n}; \]
\[ K_q = \pi^{(1-q)/2} \left( \frac{q-1}{n-q} \right)^{q-1} \left[ \frac{\Gamma \left( \frac{q(n-1)}{2(q-1)} \right)}{\Gamma \left( \frac{n-1}{2(q-1)} \right)} \right]^{(q-1)/(n-1)}, \]

where \( \Gamma \) stands for the Gamma function. For \( 1^* = n/(n-1) \), there exists the limit constant \( S_1 = \pi^{-1/2} n^{-1} \left[ \Gamma(1+n/2) \right]^{1/n} \) \([21]\). Hence, we introduce \( S_{q,\ell} = S_q \max \{1 + P_q2^{(n-1)/q}, P_q\Gamma^{1/q-1/\ell} \} \) and \( K_{q,\ell} = K_q \max \{1 + P_q2^{(n-1)(1-1/q)}, P_q\Gamma^{1/q-1/\ell} \} \) that verify

\[ \|v\|_{nq/(n-q),\Omega} \leq S_{q,\ell} \|v\|_{1,q,\ell}; \]
\[ \|v\|_{(n-1)q/(n-q),\partial\Omega} \leq K_{q,\ell} \|v\|_{1,q,\ell}. \]

**Definition 2.1.** We say that \( u \in V_{p,\ell} \) is a weak solution to (1)-(3), if it verifies

\[ \int_{\Omega} (A \nabla u) \cdot \nabla v dx + \int_{\Gamma} b(u)v ds = \int_{\Omega} f \cdot \nabla v dx + \int_{\Gamma} f v ds + \int_{\Gamma_N} g v ds + \int_{\Gamma} h v ds, \quad \forall v \in V_{p',\ell}, \]

where \( f \in L^p(\Omega), \) \( f \in L^t(\Omega), \) with \( t = pn/(n+p) \) if \( p > n/(n-1) \) and any \( t > 1 \) if \( 1 < p \leq n/(n-1), \) \( g \in L^s(\Gamma_N), \) with \( s = p(n-1)/n \) if \( p > n/(n-1) \) and any \( s > 1 \) if \( 1 < p \leq n/(n-1), \) and \( h \in L^{p/(p-1)}(\Gamma). \)

All terms on the right hand side of (8) have sense, since the following embeddings hold:

\[ W^{1,q}(\Omega) \hookrightarrow C(\Omega) \quad \text{for } q = p' > n, \text{ i.e. } p < n/(n-1); \]
\[ W^{1,q}(\Omega) \hookrightarrow L^{q^*}(\Omega) \quad \text{for } q = p' < n, \text{ i.e. } p > n/(n-1), \]
\[ W^{1,q}(\Omega) \hookrightarrow L^{q_*}(\partial\Omega) \]

with \( q^* = qn/(n-q) \) and \( q_* = q(n-1)/(n-q) \) being the critical Sobolev and trace exponents, respectively, and \( p' \) accounts for the conjugate exponent \( p' = p/(p-1). \)

We observe that \( q^* > 1 \) is arbitrary if \( q = n. \)

**Remark 2.1.** We emphasize that the existence of equivalence between the differential (1)-(3) and variational (8) formulations is only available under sufficiently data. For instance, the Green formula may be applied if \( A \nabla u \in L^p(\Omega) \) and \( \nabla \cdot (A \nabla u) \in L^p(\Omega). \)

Assume
\( A = [A_{ij}]_{i,j=1,\ldots,n} \in [L^\infty(\Omega)]^{n \times n} \) is uniformly elliptic, and uniformly bounded:

\[ \exists a_# > 0, \quad A_{ij}(x)\xi_i\xi_j \geq a_#|\xi|^2, \quad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^n; \]

under the summation convention over repeated indices.

**B:** \( b : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function such that it is strictly monotone with respect to the last variable, and it has the following \((\ell - 1)\)-growthness properties:

\[ \exists b_# > 0, \quad b(x,T)\text{sign}(T) \geq b_#|T|^\ell - 1; \]
\[ \exists b^# > 0, \quad |b(x,T)| \leq b^#|T|^\ell - 1, \]

for a.e. \( x \in \Omega \), and for all \( T \in \mathbb{R} \).

**Remark 2.2.** If \( b(T) = |T|^{\ell - 2}T \), for all \( T \in \mathbb{R} \), the property of strong monotonicity occurs with \( b^# = 2^{(2 - \ell)} \). [3, Lemma 3.3].

### 3. \( V_{2,\ell} \)-Solvability \((\ell \geq 2)\)

We establish the existence and uniqueness of weak solution as well as its quantitative estimate. Although their proof is quite standard, the explicit expression of the bound is unknown, as far as we know.

**Proposition 3.1.** Let \( f \in L^2(\Omega), \ f \in L^t(\Omega), \) with \( t = (2^*)' \), i.e. \( t = 2n/(n + 2) \) if \( n > 2 \) and any \( t > 1 \) if \( n = 2 \), \( g \in L^s(\Gamma_N) \), with \( s = 2(n-1)/n \) if \( n > 2 \) and any \( s > 1 \) if \( n = 2 \). Under the assumptions (\( A \))-\( (B) \), there exists \( u \in V_{2,\ell} \) being a weak solution to (1)-(3), i.e. solving (8) for all \( v \in V_{2,\ell} \). Moreover, the following estimate holds

\[ \frac{a_#}{2} \|\nabla u\|^2_{2,\Omega} + \frac{b_#(\ell - 1)}{\ell} \|u\|_{\ell,\Gamma} \leq \frac{1}{2a_#} \left( \|f\|_{2,\Omega} + \mathcal{F}_n(\|f\|_{t,\Omega}, \|g\|_{s,\Gamma_N}) \right)^2 + \frac{\ell - 1}{b_#^{(\ell - 1)}} \left( \|h\|_{t/(\ell - 1),\Gamma} + \mathcal{H}_n(\|f\|_{t,\Omega}, \|g\|_{s,\Gamma_N}) \right)^{\ell/(\ell - 1)} := A, \]

where \( \mathcal{F}_n(A, B) = \mathcal{H}_n(A, B) = S_{2,\ell}A + K_{2,\ell}B \) if \( n > 2 \), \( \mathcal{F}_2(A, B) = \mathcal{H}_2(\|\Omega\|^{1/\ell}A, \|\Omega\|^{1/(2\ell)}B) \), and \( \mathcal{H}_2(A, B) = S_{2t/(3t - 2),\ell}A + K_{2s/(2s - 1),\ell}B \) if \( t < 2 \). In particular, if \( t = 2 = n \), the estimate (13) holds with \( \mathcal{F}_2(A, B) = \mathcal{H}_2(\|\Omega\|^{1/2}A, \|\Omega\|^{1/(2\ell)}B) \), and \( \mathcal{H}_2(A, B) = S_{1,\ell}\|\Omega\|^{1/2 - 1/\ell}A + K_{2s/(2s - 1),\ell}B \).

**Proof.** The existence and uniqueness of a weak solution \( u \in V_{2,\ell} \) is consequence of the Browder-Minty theorem, since the functional \( T : V_{2,\ell} \to (V_{2,\ell}') \) defined by

\[ T(v) = \int_\Omega (\nabla u) \cdot \nabla v \, dx + \int_{\Gamma} b(u)v \, ds \]

is strictly monotone, continuous, bounded and coercive.

Taking \( v = u \in V_{2,\ell} \) as a test function in (8), using the H"older inequality we obtain

\[ a_# \|\nabla u\|^2_{2,\Omega} + b^# \|u\|_{\ell,\Gamma} \leq \|f\|_{2,\Omega} \|\nabla u\|_{2,\Omega} + + \|h\|_{t/(\ell - 1),\Gamma} \|u\|_{t,\Gamma} + \|f\|_{t,\Omega} \|u\|_{t,\Omega} + \|g\|_{s,\Gamma_N} \|u\|_{s,\Gamma_N}. \]
For \( n > 2 \), making use of (6) and (7) with \( q = 2 \), we get
\[
\frac{a}{2} \| \nabla u \|_{2,2}^2 + \frac{b}{\ell} \| u \|_{\ell,1} \leq \frac{1}{2a} (\| f \|_{2,2} + S_2 \| f \|_{t,1} + K_2 \| g \|_{s,G_N})^2 + \frac{1}{\ell} \| u \|_{(\ell-1),1}(\| h \|_{\ell/(\ell-1),1} + S_2 \| f \|_{t,1} + K_2 \| g \|_{s,G_N})^{\ell/(\ell-1)}.
\]
Therefore, (13) follows.

Consider the case of dimension \( n = 2 \). For \( t, s > 1 \), using the Hölder inequality in (6) with \( q = 2t/(t + 2) \) if \( t \geq 2 \), and in (7) for any \( s > 1 \), we have
\[
\| u \|_{t',1} \leq S_{2s} \| u \|_{1,2s/2s-1} \leq S_{2s} \| u \|_{1,2s-1} \leq K_{2s} \| u \|_{1,2s-1} \leq K_{2s} \| u \|_{\Omega}.
\]
Inserting the above inequalities in (14), it results in (13).

Finally, if \( t > 2 \), we have
\[
\| u \|_{t',1} \leq \| \Omega \|^{1/2-1/t} \| u \|_{2,1} \leq \| \Omega \|^{1/2-1/t} S_{1,t} (\| \Omega \|^{1/2} \| \nabla u \|_{2,1} + \| u \|_{1,1}).
\]
This concludes the proof of Proposition 3.1.

\[\square\]

**Remark 3.1.** Proposition 3.1 remains valid if the assumption \( h \in L^{\ell/(\ell-1)}(\Gamma) \) is replaced by \( h \in L^s(\Gamma) \), with the estimate (13) being rewritten with
\[
\mathcal{A} = \frac{1}{2a} (\| f \|_{2,2} + \mathcal{F}_n(\| f \|_{t,1}, \| g \|_{s,G_N} + \| h \|_{s,G_N}))^2 + \frac{t - 1}{b} \| u \|_{1,1} \mathcal{H}_n(\| f \|_{t,1}, \| g \|_{s,G_N} + \| h \|_{s,G_N})^{\ell}.\]

**Corollary 3.1.** Under the conditions of Proposition 3.1, we have
\[
\| u \|_{2p/(2p-2),1} \leq S_{2p} \| u \|_{1,2p/(2p-2)}(\| \Omega \|^{1/2} + \| \nabla u \|_{2,1})^{1/\ell} + (\| \nabla u \|_{2,1})^{1/\ell},
\]
\[
\| u \|_{2s',1} \leq K_{2s} \| u \|_{1,2s} \| \Omega \|^{1/2} \| \nabla u \|_{2,1} \| u \|_{1,1},
\]
for \( p \geq n \), \( p > n = 2 \), \( s \geq n - 1 > 1 \), and \( s > 1 \) \((n = 2)\).

**Proof.** Making use of (6) with \( q = 2pm/[2p + n(p - 2)] \) if \( p > 2 \), and the Hölder inequality for \( p \geq n \), we obtain
\[
\| u \|_{2p/(2p-2),1} \leq S_{2p} \| u \|_{1,2p/[2p+n(p-2)]} \leq S_{2p} (\| \Omega \|^{1/(n-1)/p} \| \nabla u \|_{2,1} + \| u \|_{1,1}).
\]
Applying (13) in the above inequality, we conclude (16).
Making use of (7) with \( q = \frac{2sn}{2s+(n-1)(s-1)} \) if \( s > 1 \), and the Hölder inequality for \( s \geq n-1 \), we obtain

\[
\|u\|_{2s/(s-1),\partial\Omega} \leq K \frac{2^s}{s+1} \|u\|_{1,(s-1)/2} \leq K_{2s, n/(n-1)(s-1),\ell} \, (|\Omega|^{s-n+1}/(2ns)) \|\nabla u\|_{2,\Omega} + \|u\|_{\ell,r}.
\]

Thus, (17) holds as before.

4. \( L^q \)-estimates \((q < 2(n-1)p/[2(n-1) - p], \ 2 < p < 2(n-1))\)

Section 3 ensures the existence of a weak solution, \( u \in L^{2p/(p-2)}(\Omega) \), to (1)-(3) in accordance with Definition 2.1, and (9) and (11) be fulfilled. If Proposition 4.1.

\[
\text{Proposition 4.1. Let } u \in H^1(\Omega) \text{ be any weak solution to (1)-(3) in accordance with Definition 2.1 and (7) and (12) be fulfilled. If } f \in L^p(\Omega), \ g \in L^{(n-1)p/(n-p)}(\Gamma_N), \text{ and } h \in L^r(\Gamma), \text{ then we have, for every } q < 2(n-1)/[n-2-2(n-1)\delta] := Q, \]

\[
\|u\|_{q,\Omega} + \|u\|_{q,\partial\Omega} \leq K_{q,\delta} \left( B + 2|\Omega| |u| > 1\right) \|v\|_{\frac{n-2}{2(n-1)-\delta}}^{\frac{n-2}{2(n-1)-\delta}},
\]

where \( \delta = \min\{1/2 - 1/p, 1/2 - 1/r\} \), and the positive constants \( K \) and \( B \) are

\[
K_{q,\delta} = 2^{\frac{n-2}{2(n-1)-\delta}} \left( \frac{Q}{Q-q} \right)^{1/q} \left( |\Omega|^{\frac{1}{2} - \frac{1}{q}} + |\partial\Omega|^{\frac{1}{2} - \frac{1}{q}} \right);
\]

\[
B = \left( |\Omega|^{\frac{n-2}{2(n-1)-\delta}} S_{2,2} + K_{2,2} \right) \left( \frac{1}{a_\#} + \frac{1}{\sqrt{a_\#} b_\#} \right) \left( \|f\|_{p,\Omega} + C_{n,p,r} |\Omega|^{\frac{1}{2} - \frac{1}{p}} \right)
\]

\[
+ \left( \frac{1}{b_\#} + \frac{1}{a_\# b_\#} \right) \left( \|h\|_{r,\Gamma} + C_{n,p,r} |\Gamma|^{1/2-1/r-\delta} \right);
\]

\[
C_{n,p,r} = \left( q_{np/(p+n),\Omega} + K_{p',r'} \right) \|g\|_{(n-1)p/(n-1),\Omega,N}, \quad \forall n \geq 2.
\]

\textbf{Proof.} Let \( k \geq k_0 = 1 \). Hence forth we use the notation \( A(k) = \{ x \in \Omega : \ |u(x)| > k \} \), with the set \( A \) being either \( \Omega, \Gamma_N, \Gamma, \partial\Omega \) or \( \Omega \). Choosing \( v = \text{sign}(u) |u|^{-k} =
\]
sign(\(u\)) \(\max\{\|u\| - k, 0\}\) \(\in H^1(\Omega)\) as a test function in (8), then \(\nabla v = \nabla u \in L^2(\Omega(k))\). Since \(|u| > 1\) a.e. on \(\Gamma(k)\), taking (9) and (11) into account, we deduce

\[
a_{\#} \int_{\Omega(k)} |\nabla u|^2 \, dx + b_{\#} \int_{\Gamma(k)} (|u| - k)^2 \, ds \leq \|f\|_{2,\Omega(k)} \|\nabla u\|_{2,\Omega(k)} + \|f\|_{\frac{n-1}{n-p-1}p,\Omega} (|u| - k)^+ \|f\|_{\frac{n-1}{n-p-1}p,\Omega} + \\
+ \|g\|_{\frac{n-1}{n-p-1}p,\Gamma_N} (2|u| - k)^+ \|g\|_{\frac{n-1}{n-p-1}p,\Gamma_N} + \|h\|_{2,\Gamma(k)} \|u| - k\|_{2,\Gamma(k)}.
\]

Using the Hölder inequality, it follows that \((p, r > 2)\)

\[
\|f\|_{2,\Omega(k)} \leq \|f\|_{p,\Omega(k)}^{1/2-1/p} \|\Omega(k)\|^{1/2-1/p} ;
\]

\[
\|h\|_{2,\Gamma(k)} \leq \|h\|_{r,\Gamma(k)}^{1/2-1/r} \|\Gamma(k)\|^{1/2-1/r}.
\]

Making use of (6)-(7) and \((|u| - k)^+ \in V_{p',r'}\) with \(p' < 2 \leq n\) and \(r' < 2\), and the Hölder inequality, we get

\[
\|(u| - k)^+ \|_{\frac{n-1}{n-p-1}p,\Omega} \leq S_{p',r'} \left( \|\Omega(k)\|^{1/2-1/p} \|\nabla u\|_{2,\Omega(k)} + \right.
\]

\[
\left. + \|\Gamma(k)\|^{1/2-1/r} \|u| - k\|_{2,\Gamma(k)} \right);
\]

\[
\|(u| - k)^+ \|_{\frac{n-1}{n-p-1}p,\Gamma_N} \leq K_{p',r'} \left( \|\Omega(k)\|^{1/2-1/p} \|\nabla u\|_{2,\Omega(k)} + \right.
\]

\[
\left. + \|\Gamma(k)\|^{1/2-1/r} \|u| - k\|_{2,\Gamma(k)} \right).
\]

Inserting last four inequalities into (20) we obtain

\[
a_{\#} \|\nabla u\|_{2,\Omega(k)}^2 + b_{\#} \|u| - k\|_{2,\Gamma(k)}^2 \leq \frac{(\|f\|_{p,\Omega} + C_{n,p,p})^2}{a_{\#}} \|\Omega(k)\|^{1-\frac{2}{p}} + \\
+ \frac{(\|h\|_{r,\Gamma} + C_{n,p,r})^2}{b_{\#}} \|\Gamma(k)\|^{1-\frac{2}{r}}, \quad \forall p, r > 2.
\]

It results in

\[
\|(u| - k)^+\|_{1,2,2} \leq \left[ \left( \frac{1}{a_{\#}} + \frac{1}{\sqrt{a_{\#} b_{\#}}} \right) \frac{(\|f\|_{p,\Omega} + C_{n,p,p})\|\Omega(k)\|^{1/2-1/p}}{(\|\Omega(k)\| + |\Gamma(k)|)^{\delta}} + \\
+ \left( \frac{1}{b_{\#}} + \frac{1}{\sqrt{a_{\#} b_{\#}}} \right) \frac{(\|h\|_{r,\Gamma} + C_{n,p,r})|\Gamma(k)|^{1/2-1/r}}{(\|\Omega(k)\| + |\Gamma(k)|)^{\delta}} \right] |\Omega(k)|^{\delta}.
\]

For \(h > k > k_0\), we have

\[
(h - k)|\Omega(h)|^{1/\alpha} \leq \||u| - k|_{\alpha,\Omega(k)} + \||u| - k|_{\alpha,\partial\Omega(k)}, \quad \forall \alpha \geq 1.
\]

Choosing \(\alpha = 2(n-1)/(n-2)\), we use (6) and (7), \((|u| - k)^+ \in W^{1,\alpha/(\alpha+n-1)}(\Omega)\), with \(n\alpha/(\alpha + n - 1) < n\), and the Hölder inequality since \(n\alpha/(\alpha + n - 1) \leq 2\). Thus, we have

\[
\||u| - k|_{\alpha,\Omega(k)} \leq |\Omega|^\frac{1}{\alpha} S_{2,2} \left( \|\nabla u\|_{2,\Omega(k)} + \||u| - k|_{2,\Gamma(k)} \right);
\]

\[
\||u| - k|_{\alpha,\partial\Omega(k)} \leq K_{2,2} \left( \|\nabla u\|_{2,\Omega(k)} + \||u| - k|_{2,\Gamma(k)} \right).
\]
Applying (22)-(23), we find
\[(h - k)|\Omega(h)|^{1/\alpha} \leq B|\Omega(k)|^{\delta}.
\]
Observing that \(\beta = \alpha \delta < 1\) if and only if \(p, r < 2(n-1)\), we may appeal to [20] Lemma 4.1 (iii), deducing
\[h^{\alpha/(1-\beta)}|\Omega(h)| \leq 2^{\alpha/(1-\beta)} \left( B^{\alpha/(1-\beta)} + (2k_0)^{\alpha/(1-\beta)}|\Omega(k)| \right).\]
Considering \(k_0 = 1\), using (18) the claimed estimate (19) holds. \(\square\)

5. \(L^\infty\)-estimates

In this section, we establish some maximum principles, by recourse to the De Giorgi technique [20], and the Moser iteration technique [12, pp. 189-190]. New results are stated that provide \(L^\infty\)-estimates up to the boundary under any space dimension \(n \geq 2\).

5.1. De Giorgi technique.

**Proposition 5.1.** Let \(p > n \geq 2, r > 2(n-1)\), \(u \in H^1(\Omega)\) be any weak solution to (1)-(3) in accordance with Definition 2.1 and (9) and (11) be fulfilled. Under the hypotheses \(f \in L^p(\Omega), \ f \in L^{(n-p)/(p+1)}(\Omega), \ g \in L^{(n-1)p/n}(\Gamma_N), \) and \(h \in L^r(\Gamma),\) we have
\[(24) \quad \text{ess sup}_{\Omega, \partial\Omega} |u| \leq 1 + \begin{cases} 2^{\gamma/(n-1)}(|\Omega| + |\partial\Omega|)^{\gamma-1/2} \mathcal{Z}_n & \text{if } n > 2 \\ 2^{\alpha+1/2} (|\Omega| + |\partial\Omega|)^{\gamma-1/(2\alpha)} \mathcal{Z}_2 & \text{if } n = 2 \end{cases}
\]
where \(\alpha > 1/(2\gamma), \ \gamma = \min\{1/2 - 1/p, (1/2 - 1/r)(n-1)/n\}, \) and \(\mathcal{Z}_n\) is
\[
\mathcal{Z}_n = (S_{2,2} + K_{2,2}) \left[ \left( \frac{1}{a_\#} + \frac{1}{\sqrt{a_\# b_\#}} \right) (\|f\|_{\partial\Omega} + C_{n,p,r})|\Omega|^{1/2 - 1/2\gamma} + \left( \frac{1}{b_\#} + \frac{1}{\sqrt{a_\# b_\#}} \right) (\|h\|_{\partial\Omega} + C_{n,p,r})|\Gamma|^{1/2 - 1/2\gamma} \right] \quad \text{if } n > 2;
\]
\[
\mathcal{Z}_2 = (S_{1,1} + K_{1,1}) \left[ \left( \frac{|\Omega|^{1/(2\alpha)}}{a_\#} + \frac{1}{\sqrt{a_\# b_\#}} \right) (\|f\|_{\partial\Omega} + C_{2,p,r})|\Omega|^{1/2 - 1/2\gamma} + \left( \frac{1}{b_\#} + \frac{|\Omega|^{1/(2\alpha)}}{\sqrt{a_\# b_\#}} \right) (\|h\|_{\partial\Omega} + C_{2,p,r})|\Gamma|^{1/2 - 1/2\gamma} \right],
\]
with \(C_{n,p,r}\) being given in Proposition 4.1.

**Proof.** Arguing as in the proof of Proposition 4.1, (21) holds. Defining \(\Sigma(k) = |\Omega(k)| + |\partial\Omega(k)|^{n/(n-1)}\), we have
\[(25) \quad (h - k)|\Sigma(h)|^{1/\alpha} \leq ||u| - k||_{\alpha,\Omega(k)} + ||u| - k||_{\alpha(n-1)/n,\partial\Omega(k)} := I,
\]
for every \(h > k > k_0 = 1\) and \(\alpha \geq n/(n-1)\).
Next, taking $\alpha = 2n/(n - 2)$ if $n > 2$ and any $\alpha > 1/(2\gamma)$ if $n = 2$, we get $(|u| - k)^+ \in W^{1, n\alpha/(\alpha + n)}(\Omega)$. Thus, we use (6) and (7) with $n\alpha/(\alpha + n) < n$, and the Hölder inequality, obtaining
\[
||u| - k||_{\alpha, \Omega(k)} \leq \frac{S_{n\alpha/(\alpha + n), \alpha + n}}{K_{n\alpha/(\alpha + n), \alpha + n}} (||\Omega|^2||\nabla u||_{2, \Omega(k)} + ||u| - k||_{2, \Gamma(k)}) \Sigma(k)^2;
\]
\[
||u| - k||_{(\alpha(n-1)/n), \partial \Omega(k)} \leq \frac{K_{n\alpha/(\alpha + n), \alpha + n}}{K_{n\alpha/(\alpha + n), \alpha + n}} (||\Omega|^2||\nabla u||_{2, \Omega(k)} + ||u| - k||_{2, \Gamma(k)}) \Sigma(k)^2,
\]
where $z = 0$ if $n > 2$, and $z = 1/(2\alpha)$ if $n = 2$. Let us split these two situations.

**Case $n > 2$:** Applying (21), we obtain
\[
I \leq (S_{2,2} + K_{2,2}) \left[ \left( \frac{1}{a_{\#}} + \frac{1}{\sqrt{a_{\#}b_{\#}}} \right) (||f||_{p,\Omega} + C_{n,p,r})||\Omega(k)||^{1/2 - 1/p} + \right.
\]
\[
+ \left. \left( \frac{1}{b_{\#}} + \frac{1}{\sqrt{a_{\#}b_{\#}}} \right) (||h||_{r,\Gamma} + C_{n,p,r})||\Gamma(k)||^{1/2 - 1/r} \right] \Sigma^{\gamma}.
\]

**Case $n = 2$:** Applying (21), we obtain
\[
I \leq (S_{1,1} + K_{1,1}) \left[ \left( \frac{||\Omega||_{2,\Omega}}{a_{\#}} + \frac{1}{\sqrt{a_{\#}b_{\#}}} \right) (||f||_{p,\Omega} + C_{n,p,r})||\Omega(k)||^{1/2 - 1/p} + \right.
\]
\[
+ \left. \left( \frac{||\Omega||_{1/(2\alpha),\Omega}}{b_{\#}} + \frac{1}{\sqrt{a_{\#}b_{\#}}} \right) (||h||_{r,\Gamma} + C_{n,p,r})||\Gamma(k)||^{1/2 - 1/r} \right] \Sigma^{\gamma + 1/(2\alpha)}.
\]

In both cases, we infer from (25) that
\[
|\Sigma(h)| \leq \left( \frac{Z_n}{h - k} \right)^\alpha |\Sigma(k)|^\beta, \quad \beta = \begin{cases} 
\alpha \gamma & \text{if } n > 2 \\
\alpha \gamma + 1/2 & \text{if } n = 2
\end{cases}
\]
where $\beta > 1$ if and only if $p > n$ and $r > 2(n - 1)$.

By appealing to [20] Lemma 4.1 (i) we conclude
\[
|\Sigma(k_0 + Z_n|\Sigma(k_0)|^{(\beta - 1)/\alpha 2^\beta/(\beta - 1)}| = 0.
\]
This means that the essential supremum does not exceed the well determined constant $k_0 + Z_n|\Sigma(k_0)|^{(\beta - 1)/\alpha 2^\beta/(\beta - 1)}$. This completes the proof of Proposition 5.1.

**Remark 5.1.** In particular, if $f = g = h = 0$ on the corresponding domains and $f \in L^p(\Omega)$ for $p > n$, then
\[
\text{ess sup}_{\Omega, \partial \Omega} |u| \leq 1 + Z_n||f||_{p,\Omega} \begin{cases} 
(||\Omega|| + |\partial \Omega|)^{1/p} \left( \frac{1}{a_{\#}} + \frac{1}{\sqrt{a_{\#}b_{\#}}} \right) & \text{if } n > 2 \\
(||\Omega|| + |\partial \Omega|)^{\alpha - 1/2\alpha} \left( \frac{||\Omega||_{2,\Omega}}{a_{\#}} + \frac{1}{\sqrt{a_{\#}b_{\#}}} \right) & \text{if } n = 2
\end{cases}
\]
for every $\alpha > p/(p - 2)$, with $Z_n = (S_{2,2} + K_{2,2})2^\alpha/2^\alpha$ if $n > 2$, and $Z_2 = (S_{1,1} + K_{1,1})2^\alpha/2^\alpha$. 


5.2. Moser iteration technique.

Proposition 5.2. Let \( p > n \geq 2, \ell \geq 2 \), \( u \in V_{2,\ell} \) be any weak solution to (1)-(3), in accordance with Definition 2.1, under \( f \in L^p(\Omega) \), \( f \in L^{p/2}(\Omega) \), \( g = 0 \) on \( \Gamma_N \), and \( h = 0 \) on \( \Gamma \), and (9) and (11) be fulfilled. Then, \( u \) satisfies

\[
\text{ess sup}_{\Omega \cup \partial \Omega} |u| \leq E_n \chi (\sum_{m \geq 0} m \chi^{-m}) (\sqrt{2E})^{1/(x-1)} \|u\|_{2p/(p-2),\Omega},
\]

with \( E_n = S^{\chi/(x-1)}_n \) and \( \chi = n(p-2)/[p(n-2)] \) if \( n > 2 \), and \( E_2 = S^{\chi/(x-1)}_p \max\{|\Omega|(p-2)/[2p(x-1)], |\Gamma|(p-2)/[2p(x-1)]\} \) for any \( \chi > 1 \), and

\[ \mathcal{E} = \sqrt{\langle |f|^2_{p,\Omega}/a_{\#} + 2\|f\|_{p/2,\Omega} \rangle} / \min\{a_{\#}, b_{\#}\}. \]

Proof. Let \( \beta \geq 1 \), and \( k > 1 \). Defining the truncation operator \( T_k(y) = \min\{y, k\} \), set \( w = T_k(|u|) \in H^1(\Omega) \cap L^\infty(\Omega) \) that satisfies \( w \in L^\infty(\partial \Omega) \). Choosing \( v = \beta^2 \text{sign}(u)w^{2\beta-1}/(2\beta - 1) \) as a test function in (3), then \( \nabla v = \beta^2 u^{2\beta-1} \nabla u \)

in \( \Omega \{u < k\} \), and \( \nabla = 0 \) in \( \Omega \setminus \Omega \{u < k\} \). Thus, applying (9) and (11) we deduce

\[
\int_{\Omega} |\nabla (w^{\beta})|^2 dx + \frac{\beta^2}{2\beta - 1} b_\# \int_{\Gamma} |u|^{\ell-1}|w|^{2\beta-1} ds \leq \int_{\Omega} (A\nabla u) \cdot \nabla v dx + \int_{\Gamma} b(u)v ds \leq \frac{a_{\#}}{2} \|\nabla (w^{\beta})\|_{L_2,\Omega}^2 + \frac{\beta^2}{2a_{\#}} \|f w^{\beta-1}\|_{L_2,\Omega}^2 + \frac{\beta^2}{2\beta - 1} \|f w^{\beta-1}\|_{L_1,\Omega},
\]

using the Hölder inequality.

We may suppose that \( w > 1 \). Otherwise, \( w = |u| \leq 1 < k \). Using the Hölder inequality, we separately compute, for \( p > 2 \),

\[
\int_{\Omega} |f|^2 w^{2\beta-1} dx \leq \|f\|_{L_2,\Omega}^2 \|w^{\beta}\|_{L_q,\Omega}^2, \quad q = 2p/(p-2) > 2; \int_{\Omega} |f| w^{2\beta-1} dx \leq \|f\|_{L_2,\Omega} \|w^{\beta}\|_{L_q,\Omega}^2.
\]

Inserting these two inequalities in (27), and considering that the left hand side absorbs the corresponding term of the right hand side, we obtain

\[
(\|\nabla (w^{\beta})\|_{L_2,\Omega}^2 + \|w^{\beta}\|_{L_2,\Gamma}^2)^{1/2} \leq \beta \mathcal{E} \|w^{\beta}\|_{L_q,\Omega}.
\]

Let us split the proof of estimate (26) into two space dimension dependent cases.

Case \( n > 2 \). Making use of (11), \( w^{\beta} \in W^{1,2}(\Omega) \hookrightarrow L^{q_{\chi}}(\Omega) \), with \( q_{\chi} = 2n/(n-2) \)

i.e. \( (n-2)/[p(n-2)] > 1 \) considering that \( p > n \), and after applying (28), we deduce

\[
\|w\|_{L_{q_{\chi},\Omega}} \leq S_{2,2} \|w^{\beta}\|_{L_{2,2},\Omega} \leq S_{2,2} \sqrt{2} \left( \|\nabla (w^{\beta})\|_{L_2,\Omega}^2 + \|w^{\beta}\|_{L_2,\Gamma}^2 \right)^{1/2} \leq S_{2,2} \sqrt{2} \mathcal{E} \beta \|w^{\beta}\|_{L_{q_{\chi},\Omega}}.
\]
Thus, we may pass to the limit the resulting inequality as $k \to \infty$ by Fatou lemma, obtaining
\[ \|u\|_{q\beta,\Omega} \leq (\beta S_{2,2} \sqrt{2E})^{1/\beta} \|u\|_{q\beta,\Omega}. \]

Taking $\beta = \chi^m > 1$, by induction, we have
\[ (29) \|u\|_{q\chi^N,\Omega} \leq (S_{2,2} \sqrt{2E})^{a_N} \chi^N \|u\|_{q,\Omega}, \quad \forall N \in \mathbb{N}, \]
where
\[ a_N = \sum_{m=0}^{N-1} \chi^{-m} \quad \text{and} \quad b_N = \sum_{m=0}^{N-1} m \chi^{-m}. \]

Therefore, by the definition
\[ \|u\|_{\infty,\Omega} = \lim_{N \to \infty} \|u\|_{q\chi^N,\Omega}, \]
and observing that $\lim_{N \to \infty} a_N$ stands for the geometric series, we find
\[ \text{ess sup}_{\Omega} |u| \leq E_n \chi \left( \sum_{m \geq 0} m \chi^{-m} \right) (\sqrt{2E})^{\chi/(\chi-1)} \|u\|_{2p/(p-2),\Omega}. \]

Next, making use of (7), $w^\beta \in W^{1,2}(\Omega) \hookrightarrow L^{2(n-1)/(n-2)}(\partial\Omega)$, and (28), we deduce
\[ \|w\|_{\beta 2(n-1)/(n-2),\partial\Omega} \leq K_{2,2} \sqrt{2E} \left( \|\nabla (w^\beta)\|_{2,\Omega} + \|w^\beta\|_{2,\Gamma} \right)^{1/2} \leq K_{2,2} \sqrt{2E} \beta \|u\|_{q\beta,\Omega}. \]

Taking $\beta = \chi^m > 1$, and applying (29), we get
\[ \|w\|_{\chi^{m 2(n-1)/(n-2)},\partial\Omega} \leq K_{2,2} \chi^{-m} S_{2,2}^m (\sqrt{2E})^{a_{m+1}} \chi^{b_{m+1}} \|u\|_{q,\Omega}. \]

Thus, we may pass to the limit the above inequality first as $k \to \infty$ by Fatou lemma, and next as $m \to \infty$, concluding
\[ \text{ess sup}_{\partial\Omega} |u| \leq E_n \chi \left( \sum_{m \geq 0} m \chi^{-m} \right) (\sqrt{2E})^{\chi/(\chi-1)} \|u\|_{2p/(p-2),\Omega}, \]
which finishes (26).

Case $n = 2$. Making use of $w^\beta \in W^{1,2}(\Omega) \hookrightarrow W^{1,2q\chi/(q\chi+2)}(\Omega) \hookrightarrow L^{q\chi}(\Omega)$, with $q\chi = 2p\chi/(p-2)$ considering that $p > n = 2$, and next applying (28), we deduce
\[ \|w\|_{q\beta,\Omega} \leq S_{2q\chi/(q\chi+2),2q\chi/(q\chi+2)} \|w^\beta\|_{1,2q\chi/(q\chi+2),2q\chi/(q\chi+2)} \leq S_{2q\chi/(q\chi+2),2q\chi/(q\chi+2)} \left( |\Omega|^{1/(q\chi)} \|\nabla (w^\beta)\|_{2,\Omega} + |\Gamma|^{1/(q\chi)} \|w^\beta\|_{2,\Gamma} \right) \leq S_{2q\chi/(q\chi+2),2q\chi/(q\chi+2)} \max \{|\Omega|^{1/(2q\chi)}, |\Gamma|^{1/(2q\chi)}\} \sqrt{2E} \beta \|u\|_{q\beta,\Omega}. \]

For the boundary bound, we use $w^\beta \in W^{1,2}(\Omega) \hookrightarrow W^{1,2q\chi/(q\chi+1)}(\Omega) \hookrightarrow L^{q\chi}(\partial\Omega)$, deducing
\[ \|w\|_{q\beta,\partial\Omega} \leq K_{2q\chi/(q\chi+1),2q\chi/(q\chi+1)} \max \{|\Omega|^{1/(2q\chi)}, |\Gamma|^{1/(2q\chi)}\} \sqrt{2E} \beta \|u\|_{q\beta,\Omega}. \]

Thus, we may proceed as in the above case, completing the proof of Proposition 5.2.

In the following result stands for the particular case: $f = g = h = 0$. \[ \square \]
Corollary 5.1. Under the conditions of Proposition 5.2, there exists a $L^\infty$-constant, $C_\infty$, to the problem (1)-(3), that is, for $p > n$,

$$\text{ess sup}_{\Omega, \beta \Omega} |u| \leq C_\infty \|f\|_{2,p,\Omega}^{1+\chi/(\chi-1)},$$

with

$$C_\infty = E_n(\sum_{m \geq 0} m \chi^{-m}) \left( \frac{2}{a_\# \min\{a_\#, b_\#\}} \right)^{\frac{2}{\chi-1}} \times$$

$$\times S_{\frac{2pn}{2p+n(n-2)}, \ell} \left( \frac{\|\Omega\|_1^{\frac{1}{2p} + \frac{1}{2}}}{a_\#} + \left( \frac{\ell \|\Omega\|^{1-1/p}}{2a_\# b_\#} \right)^{1/\ell} \right).$$

Proof. It suffices to insert the estimate (16) into (26).

Proposition 5.3. Let $n \geq 2$, $s > n - 1$, $u \in H^1(\Omega)$ be any weak solution to (1)-(3), in accordance with Definition 2.1 and (9) and (11) be fulfilled. If $f = 0$ and $f = 0$ in $\Omega$, $g \in L^s(\Gamma_N)$, and $h \in L^s(\Gamma)$, then

$$\text{ess sup}_{\Omega} |u| \leq G_n(\sum_{m \geq 0} m \chi^{-m})(\sqrt{2G})^{\chi/(\chi-1)} \|u\|_{2s/(s-1), \partial \Omega},$$

with $G_n = K_2^{\chi/(\chi-1)}$, $\chi = (s-1)(n-1)/[s(n-2)]$ if $n > 2$, and $G_2 = K_2^{\chi/(\chi-1)}$

max{\|\Omega\|^{(s-1)/[4s(s-1)]}, \|\Gamma\|^{(s-1)/[4s(s-1)]}} for any $\chi > 1$, and

$$\mathcal{G} = \sqrt{(\|g\|_{s, \Gamma_N} + \|h\|_{s, \Gamma}) / \min\{a_\#, b_\#\}.}$$

Proof. Let $\beta \geq 1$, and $k > 1$. For $s > 1$, and $q = 2s/(s-1)$, proceeding as in the proof of Proposition 5.2 we deduce

$$a_\# \int \Omega |\nabla (w^\beta)|^2 dx + \frac{\beta^2}{2\beta - 1} b_\# \int \Gamma |u|^{\ell-1} |w|^{2\beta-1} ds \leq$$

$$\leq \frac{\beta^2}{2\beta - 1} (\|g w^{2\beta-1}\|_{1, \Gamma_N} + \|h w^{2\beta-1}\|_{1, \Gamma}) \leq$$

$$\leq \beta^2 (\|g\|_{s, \Gamma_N} |w|^{\beta\|}_{q, \Gamma_N} + \|h\|_{s, \Gamma} |w^\beta|^{2}_{q, \Gamma}).$$

Thus, we obtain

$$\left( |\nabla (w^\beta)|_{2, \Omega}^2 + |w^\beta|_{2, \Omega}^2 \right)^{1/2} \leq \beta \mathcal{G} \|w^\beta\|_{q, \partial \Omega}.$$ 

Then, we obtain

$$\|u\|_{q, \chi, \Omega} \leq (\beta \sqrt{2G} M_1)^{1/\beta} \|u\|_{q, \partial \Omega};$$

$$\|u\|_{q, \partial \Omega} \leq (\beta \sqrt{2G} M_2)^{1/\beta} \|u\|_{q, \partial \Omega};$$

Case $n > 2$: Setting $M_1 = S_{2,2}$, and $M_2 = K_{2,2}$, by using (6), and $w^\beta \in W^{1,2}(\Omega) \hookrightarrow L^{n\chi}(\Omega)$, (7), and $w^\beta \in W^{1,2}(\Omega) \hookrightarrow L^{n\chi}(\partial \Omega)$ with $q\chi = 2(n-1)/(n-2)$ i.e.

$\chi = (s-1)(n-1)/[s(n-2)] > 1$ if $s > n-1$.

Case $n = 2$: Setting $M_1 = S_{q,2}/(q+1), 2q\chi/(q+1) \max\{|\Omega|^{1/(q\chi)}, |\Gamma|^{1/(q\chi)}\}$, and $M_2 = K_{2q\chi/(q+1), 2q\chi/(q+1)} \max\{|\Omega|^{1/(2q\chi)}, |\Gamma|^{1/(2q\chi)}\}$, by using (7) with $\beta \in W^{1,2q\chi/(q+1)}(\Omega) \hookrightarrow L^{q\chi}(\Omega)$, and (7) with $\beta \in W^{1,2q\chi/(q+1)}(\Omega) \hookrightarrow L^{q\chi}(\partial \Omega).$
In both cases, following the argument of the proof of Proposition 5.2, we get
\[ \|u\|_{q,\chi^{N+1},\Omega} \leq M_{1}^{\chi^{N}}(\sqrt{2}|G|^{a_{N+1}}M_{2}^{a_{N}}\chi^{b_{N+1}}\|u\|_{q,\partial\Omega}; \]
\[ \|u\|_{q,\chi^{N},\partial\Omega} \leq (\sqrt{2}|G|M_{2}^{a_{N}}\chi^{b_{N}}\|u\|_{q,\partial\Omega}, \quad \forall N \in \mathbb{N}. \]

Therefore, we conclude (30), finishing the proof of Proposition 5.3.

\[ \square \]

Corollary 5.2 (Linear Robin-Neumann problem). Under the conditions of Propositions 5.2 and 5.3, if (4) is assumed then there exists a weak solution, \( u \in H^{1}(\Omega) \), to (7)-(3) in accordance with Definition 2.1, such that

\[ (32) \quad \text{ess sup}_\Omega |u| \leq \Xi_{1} \left( \frac{\|f\|_{p,\Omega}/a_{#} + |f|_{p,\Omega}/2}{\min\{a_{#}, b_{*}\}} \right)^{2(\chi_{1}-1)} \left( |\Omega|^{\frac{p_{n}}{mp_{n}} + 1} \right) + \Xi_{2} \left( \frac{|g|_{s,\Gamma_{N}} + \|h\|_{s,\Gamma}}{\min\{a_{#}, b_{*}\}} \right)^{\frac{\chi_{2}}{2(\chi_{2}-1)}} M_{n}(|g|_{s,\Gamma_{N}} + \|h\|_{s,\Gamma})^{\frac{1}{\min\{a_{#}, b_{*}\}}}, \]

where

\[ \Xi_{1} = E_{n} \chi_{1} \left( \sum_{m=0}^{\infty} m^{\chi_{1}^{m}} \right)^{1/\chi_{1}^{m}} \sqrt{2}^{\chi_{1}/(\chi_{1}-1)} S_{2m+2p+6}_{\rho+p-2} \ell \left( |\Omega|^{\frac{p_{n}}{mp_{n}} + 1} \right); \]

\[ \Xi_{2} = G_{n} \chi_{2} \left( \sum_{m=0}^{\infty} m^{\chi_{2}^{m}} \right)^{1/\chi_{2}^{m}} \sqrt{2}^{\chi_{2}/(\chi_{2}-1)} K_{2m+2s_{1}} \ell \left( |\Omega|^{\frac{s_{1}}{2s_{1}} + 1} \right), \]

with \( E_{n} \) and \( \chi_{1} \) being the constants in accordance with Proposition 5.2, \( G_{n} \) and \( \chi_{2} \) being the constants in accordance with Proposition 5.3, \( L_{n} = 2S_{p_{n}}^{\rho} \) if \( n > 2 \), \( L_{2} = (|\Omega|^{1/2} + 1)S_{2p+6}_{\rho+p-2} \ell \) if \( t < 2 \), \( L_{2} = (|\Omega|^{1/2} + 1)S_{2p+6}_{\rho+p-2} \ell \) if \( t \geq 2 \), \( M_{n} = 2K_{2p_{n}}^{\rho} \) if \( n > 2 \), and \( M_{2} = (|\Omega|^{1/2s} + 1)K_{2s_{1}}^{s_{1}} \ell \).

Proof. From Propositions 3.1 and 5.2 there exists \( u_{1} \in H^{1}(\Omega) \) solving

\[ \int_{\Omega} (A\nabla u_{1}) \cdot \nabla vdx + \int_{\Gamma} u_{1}vds = \int_{\Omega} f \cdot \nabla vdx + \int_{\Gamma} fvdx, \quad \forall v \in H^{1}(\Omega), \]

such that it verifies

\[ (33) \quad \text{ess sup}_\Omega |u_{1}| \leq E_{n} \chi_{1} \left( \sum_{m=0}^{\infty} m^{\chi_{1}^{m}} \right)^{1/\chi_{1}^{m}} (\sqrt{2}|G|)^{\chi_{1}/(\chi_{1}-1)} \|u_{1}\|_{2p/(p-2),\partial\Omega}. \]

From Propositions 3.1 and 5.3, and Remark 3.1, there exists \( u_{2} \in H^{1}(\Omega) \) solving

\[ \int_{\Omega} (A\nabla u_{2}) \cdot \nabla vdx + \int_{\Gamma} u_{2}vds = \int_{\Gamma} gvd + \int_{\Gamma} hvds, \quad \forall v \in H^{1}(\Omega), \]

such that it verifies

\[ (34) \quad \text{ess sup}_\Omega |u_{2}| \leq G_{n} \chi_{2} \left( \sum_{m=0}^{\infty} m^{\chi_{2}^{m}} \right)^{1/\chi_{2}^{m}} (\sqrt{2}|G|)^{\chi_{2}/(\chi_{2}-1)} \|u_{2}\|_{2s_{1}/(s_{1}-1),\partial\Omega}. \]

Then, \( u = u_{1} + u_{2} \in H^{1}(\Omega) \) is the required solution. Moreover, from (33)-(34) gathered with Corollary 3.1 we find (32), with \( L_{n}A = F_{n}(A,0) + H_{n}(A,0), M_{n}(B) = F_{n}(0,B)+H_{n}(0,B), \) and \( F_{n} \) and \( H_{n} \) being the functions in accordance with Proposition 3.1.
6. $V_{q,\ell-1}$-Solvability ($q < n/(n-1), \ell \geq 2$)

The $W^{1,q}$-solvability depends on the data regularity. In the presence of the boundary condition (2), the duality theory is more straightforward than the $L^1$-theory when $L^1$ data are taken into account. In order to determine the explicit constant, the following result of the existence of a solution is based on the duality theory.

First let us recall that, for $q > 1$, the $L^q$-norm may be defined as
\begin{equation}
\|u\|_{q,\Omega} = \sup \left\{ \left| \int_{\Omega} u \cdot g \, dx \right| : \quad g \in L^q(\Omega), \|g\|_{q,\Omega} = 1 \right\},
\end{equation}
for all $u \in L^q(\Omega)$.

**Proposition 6.1.** Let $f = 0$ a.e. in $\Omega$, $f \in L^1(\Omega)$, $g \in L^1(\Gamma_N)$, $h \in L^1(\Gamma)$, (A)-(B) be fulfilled, and $A$ be symmetric. For any $\ell \geq 2$, there exists $u \in V_{q,\ell-1}$ solving (3) for every $1 < q < n/(n-1)$. Moreover, we have the following estimate
\begin{align}
\|\nabla u\|_{q,\Omega} &\leq C_\infty \left( |\Gamma|(1 + b^\#) + \|f\|_{1,\Omega} + \|g\|_{1,\Gamma} + \|h\|_{1,\Gamma} + \right. \\
&\quad + \left. (1 + b^\#) (\|f\|_{1,\Omega} + \|g\|_{1,\Gamma} + \|h\|_{1,\Gamma}) / b^\# \right);
\end{align}
with the constant $C_\infty$ being explicitly given in Corollary 5.1.

**Proof.** For each $m \in \mathbb{N}$, take $f_m = F_m(f) \in L^\infty(\Omega)$, $g_m = F_m(g) \in L^\infty(\Gamma_N)$, $h_m = F_m(h) \in L^\infty(\Gamma)$, with
\begin{equation}
F_m(\tau) = \frac{m\tau}{m + |\tau|}.
\end{equation}
Applying Proposition 3.1, there exists a unique solution $u_m \in V_{2,\ell}$ to the following variational problem
\begin{equation}
\int_{\Omega} (A\nabla u_m) \cdot \nabla v \, dx + \int_{\Gamma} b(u_m) v \, ds = \int_{\Omega} f_m v \, dx + \int_{\Gamma_N} g_m v \, ds + \int_{\Gamma} h_m v \, ds, \quad \forall v \in V_{2,\ell}.
\end{equation}
In particular, (38) holds for all $v \in W^{1,q'}(\Omega)$ for $q' > n$. Defining the truncation operator $T_1(y) = \text{sign}(y) \min\{|y|, 1\}$, let us choose $v = T_1(u_m) \in V_{2,\ell}$ as a test function in (38), obtaining
\begin{equation}
b^\# \left( \int_{|u_m| > 1} |u_m|^{\ell-1} \, ds + \int_{|u_m| \leq 1} |u_m|^{\ell} \, ds \right) \leq \|f\|_{1,\Omega} + \|g\|_{1,\Gamma} + \|h\|_{1,\Gamma}.
\end{equation}
Hence, we conclude that (37) is true for $u_m$.

In order to pass to the limit (38) on $m (m \to \infty)$ let us establish the estimate (36) for $u_m$. Let $w \in V_{2,2}$ be the unique weak solution to the mixed Robin-Neumann problem (1)-(3), under $f = g = h = 0$, in accordance with Proposition 3.1. Since $A$ is symmetric, we infer that
\begin{equation}
\int_{\Omega} (A\nabla u_m) \cdot \nabla w \, dx = \int_{\Omega} (A\nabla w) \cdot \nabla u_m \, dx = \int_{\Omega} f \cdot \nabla u_m \, dx - \int_{\Gamma} w u_m \, ds,
\end{equation}
which gathered with (38) under \( v = w \) reads
\[
\int_\Omega f \cdot \nabla u_m \, dx = \int_{\Gamma} w u_m \, ds - \int_{\Gamma} b(u_m) \, w ds + \int_{\Omega} f_m \, w \, dx + \int_{\Gamma_N} g_m \, w \, ds + \int_{\Gamma} h_m \, w \, ds.
\]

For \( f \in L^{q'}(\Omega) \) with \( q' > n \) such that \( \|f\|_{q',\Omega} = 1 \), Corollary 5.1 guarantees that the existence of a \( L^\infty \)-constant \( C_\infty \) such that \( \|w\|_{\infty,\Omega} + \|w\|_{\infty,\partial\Omega} \leq C_\infty \).

By (35) with \( u = \nabla u_m \), and (40), we obtain
\[
\|\nabla u_m\|_{q,\Omega} \leq C_\infty \left( (|\Gamma|)(1 + b^\#) + (1 + b^\#) \|u_m\|_{c-1,\Gamma} \|u_m|_{>1} + \right.
\]
\[
\left. + \|f\|_{1,\Omega} + \|g\|_{1,\Gamma_N} + \|h\|_{1,\Gamma} \right).
\]

Applying (39), then (36) holds for \( u_m \).

Therefore, the passage to the limit as \( m \) tends to infinity is allowed, concluding the proof of Proposition 6.1.

\[ \square \]

7. Green kernel

In this Section, we establish the existence of the Green kernel altogether some of its properties. Here, we follow the approach introduced in [13] in constructing Green’s function for the Dirichlet problem (see also [17]).

**Definition 7.1.** For each \( x \in \Omega \), we say that \( G = G(x, \cdot) \) is a Green kernel associated to (1)-(3), if it solves, in the distributional sense,
\[
\nabla \cdot (A \nabla G(x, \cdot)) = \delta_x \quad \text{in} \; \Omega;
\]
\[
A \nabla G(x, \cdot) \cdot n + b(G(x, \cdot))\chi_\Gamma = 0 \quad \text{on} \; \partial\Omega,
\]
where \( \delta_x \) is the Dirac delta function at the point \( x \). That is, there is \( q > 1 \) such that \( G \) verifies the variational formulation
\[
\int_\Omega A \nabla G(x, \cdot) \cdot \nabla v \, dy + \int_{\Gamma} b(G(x, \cdot)) v \, ds = v(x), \quad \forall v \in V_{q,\ell}.
\]

**Proposition 7.1.** Let \( n \geq 2 \), \( 1 \leq q < n/(n-1) \), (A)-(B) be fulfilled, and \( A \) be symmetric. Then, for each \( x \in \Omega \) and any \( r > 0 \) such that \( r < \text{dist}(x, \partial\Omega) \), there exists a unique Green function \( G = G(x, \cdot) \in V_{q,\ell-1} \cap H^1(\Omega \setminus B_r(x)) \) according to Definition 7.1 and enjoying the following estimates
\[
\|\nabla G\|_{q,\Omega} \leq C_\infty (1 + (1 + b^\#)(|\Gamma| + 1/b^\#));
\]
\[
\|G\|_{c-1,\Gamma} \leq (|\Gamma| + 1/b^\#)^{1/(\ell-1)},
\]
with the constant \( C_\infty \) being explicitly given in Corollary 5.1. Moreover, \( G(x,y) \geq 0 \) a.e. \( x, y \in \Omega \).

**Proof.** Let \( x \in \Omega \) and \( \rho > 0 \) be such that \( B_\rho(x) \subset \subset \Omega \). Thanks to Proposition 3.1 with \( f = 0 \) a.e. in \( \Omega \), \( g, h = 0 \) a.e. on, respectively, \( \Gamma_N \) and \( \Gamma \), and \( f = \chi_{B_\rho(x)}/|B_\rho(x)| \)
belonging to \( L^{2n/(n+2)}(\Omega) \) if \( n > 2 \), and to \( L^2(\Omega) \) if \( n = 2 \), there exists \( G^\rho = G^\rho(x,\cdot) \in V_{2,\ell} \) being the unique solution to

\[
\int_\Omega A \nabla G^\rho \cdot \nabla v dy + \int_\Gamma b(G^\rho)v ds = \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} v dy,
\]

for all \( v \in V_{2,\ell} \). In particular, if \( \ell = 2 \), (13) reads

\[
\| \nabla G^\rho \|_{2,\Omega} + \| G^\rho \|_{2,\Gamma} \leq \frac{2}{\min\{a_\#, b_\#\}} \times \left\{ \begin{array}{ll} S_2 \omega_n \rho^{1-n/2} & \text{if } n > 2 \\
 \sqrt{n+1} S_{1,2} \rho^{-1} & \text{if } n = 2 \end{array} \right. .
\]

Therefore, for any \( r > 0 \) such that \( B_r(x) \subset \subset \Omega \), there exists \( G \in H^1(\Omega \setminus B_r(x)) \) such that

\[
G^\rho \rightharpoonup G \quad \text{in } H^1(\Omega \setminus B_r(x)) \quad \text{as } \rho \to 0^+.
\]

In order to \( G \) verify

\[
G(x, y) = \lim_{\rho \to 0} G^\rho(x, y), \quad \forall x \in \Omega, \ a.e. \ y \in \Omega, \ x \neq y,
\]

we observe that the \( V_{q,\ell-1} \)-estimates (44)-(45) are true for \( G^\rho \) due to (36)-(37), by applying Proposition 6.1 with \( g, h = 0 \), and \( f = \chi_{B_\rho(x)}/|B_\rho(x)| \in L^1(\Omega) \). Then, we can extract a subsequence of \( G^\rho \), still denoted by \( G^\rho \), weakly converging to \( G \) in \( V_{q,\ell-1} \) as \( \rho \) tends to 0, with \( G \in V_{q,\ell-1} \) solving (43) for all \( v \in W^{1,q}(\Omega) \). A well-known property of passage to the weak limit implies (44)-(45).

In order to prove the nonnegativeness assertion, first calculate

\[
a_\# \int_\Omega |\nabla(G^\rho - |G^\rho|)|^2 dy \leq \frac{2}{|B_\rho(x)|} \left( \int_{B_\rho(x)} G^\rho dy - \int_{B_\rho(x)} |G^\rho| dy \right) \leq 0.
\]

Then, \( G^\rho = |G^\rho| \), and by passing to the limit as \( \rho \) tends to 0, the nonnegativeness claim holds, which completes the proof of Proposition 7.1.

\[\square\]

8. ROBIN-NEUMANN PROBLEM (\( \ell = 2 \))

In the two-dimensional space, Proposition 3.1 leads \( H^1 \) solution for the \( L^p \)-data, with an arbitrary \( p > 1 \). Our concern is then the existence of weak solutions and the derivation of their estimates in the \( n \)-dimensional space: \( n > 2 \).

Proposition 8.1. Let \( f = 0 \) a.e. in \( \Omega \), \( f \in L^t(\Omega) \) with \( t \leq 2n/(n+2) \), \( g \in L^s(\Gamma_N) \) and \( h \in L^s(\Gamma) \) with \( s \leq 2(2n-1)/n \), and \( A \) be a symmetric matrix satisfying the assumption (A). Under the assumption (A) with \( b_\ell = 1 \), there exists \( u \in W^{1,q}(\Omega) \) solving (8) for every \( 1 < q < 2(n-1)p/[2(n-1) - p] \) with \( p = \min\{t, s\} \). Moreover, we have the following estimate

\[
\| \nabla u \|_{q,\Omega} \leq M_q \left( \frac{1}{a_\#} + \frac{1}{\sqrt{a_\#}} \right) (K_{t,q^*} f \|_{t,\Omega} + K_{s,q^*}(\|g\|_{s,\Gamma_N} + \|h\|_{s,\Gamma})) ,
\]
with
\[ M_q = |\Omega|^{(n-2)/[2(n-1)n]}S_{2,2} + K_{2,2} + \\
+ 2|\Omega|^{\frac{1}{2} + \frac{2}{d}} \left( S_{n+1, \frac{n}{n+1}} (|\Omega|^{\frac{1}{2} + \frac{2}{d}} + |\partial \Omega|^{\frac{1}{2} + \frac{2}{d}}) + K_{1,1}(|\Omega|^{\frac{1}{2}} + |\partial \Omega|^{\frac{1}{2}}) \right). \]

**Proof.** For each \( m \in \mathbb{N} \), take the approximations \( f_m, g_m, \) and \( h_m \) as in the proof of Proposition 6.1, and the corresponding unique solution \( u_m \in V_{2,2} \) to the variational problem (38). Moreover, (37) is true for \( u_m (\ell = 2) \).

Let \( w \in V_{2,2} \) be the unique weak solution to the mixed Robin-Neumann problem (1)-(3), under \( f = g = h = 0 \), such that (40) reads
\[
\int_{\Omega} f \cdot \nabla u_m \, dx = \int_{\Omega} f_m w \, dx + \int_{\Gamma_N} g_m \, ds + \int_{\Gamma} h_m \, ds.
\]
Moreover, for \( q' \geq 2 \) (42) reads
\[
\|w\|_{1,2,2} \leq \left( \frac{1}{a^{\#}} + \frac{1}{\sqrt{a^{\#}}} \right) |\Omega|^{1/2 - 1/2} \|f\|_{q',\Omega}.
\]

Observe that
\[
\|w\|_{1,\Omega} + \|w\|_{1,\partial \Omega} \leq \left( S_{\frac{n}{n+1}, \frac{n}{n+1}} (|\Omega|^{1/2 + 1/n} + |\Gamma|^{1/2 + 1/n}) + \\
+ K_{1,1} \left( |\Omega|^{1/2} + |\Gamma|^{1/2} \right) \right) \|w\|_{1,2,2}.
\]

For any \( t \leq 2n/(n+2) \), \( s \leq 2(n-1)/n \), and \( q < 2(n-1)p/[2(n-1) - p] \) with \( p = \min\{t, s\} \), which means \( 2n/(n-2) \leq t' < 2(n-1)q'/[2(n-1) - q'] \), and \( 2(n-1)/(n-2) \leq s' < 2(n-1)q'/[2(n-1) - q'] \) if \( 2 < q' < 2(n-1) \), Proposition 4.1 (with \( \delta = 1/2 - 1/2 - 1 \)) since \( h \equiv 0 \) yields
\[
\|w\|_{q',\Omega} \leq K_{t',\delta} \left( |\Omega|^{\frac{n-2}{(n-1)n}} S_{2,2} + K_{2,2} \right) \left( \frac{1}{a^{\#}} + \frac{1}{\sqrt{a^{\#}}} \right) \|f\|_{q',\Omega} + \\
+ 2(\|w\|_{1,\Omega} + \|w\|_{1,\partial \Omega}) \leq K_{t',\delta} M_q \left( \frac{1}{a^{\#}} + \frac{1}{\sqrt{a^{\#}}} \right) \|f\|_{q',\Omega},
\]
considering that \( (2(n-1) - q')/[2(n-1)q'] < 1 \) is taken into account, and applying (52) accomplished with (51). Analogously
\[
\|w\|_{s',\partial \Omega} \leq K_{s',\delta} M_q \left( \frac{1}{a^{\#}} + \frac{1}{\sqrt{a^{\#}}} \right) \|f\|_{q',\Omega}.
\]

By (35) with \( u = \nabla u_m \), we infer from (50) that (49) holds for \( u_m \). Therefore, by passage to the limit as \( m \) tends to infinity, we conclude the claimed result. \( \square \)

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