Doubly nonlinear parabolic equations for a general class of Forchheimer gas flows in porous media

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Abstract

This paper is focused on the generalized Forchheimer flows of compressible fluids in porous media. The gravity effect and other general nonlinear forms of the source term and boundary flux are integrated into the model. We derive a doubly nonlinear parabolic equation for the so-called pseudo-pressure, and study its initial value problem subject to a general nonlinear Robin boundary condition. The growth rates in the source term and the boundary condition are arbitrarily large. The maximum of the solution, for positive time, is estimated in terms of certain Lebesgue norms of the initial and boundary data. The gradient estimates are obtained under a theoretical condition which, indeed, is relevant to the fluid flows in applications. In dealing with the complexity and generality of the equation and boundary condition, suitable trace theorems and Sobolev’s inequalities are utilized, and a well-adapted Moser’s iteration is implemented.

Keywords: Forchheimer flows, nonlinear Robin condition, Moser iteration, compressible fluids, porous media, doubly nonlinear equation

Mathematics Subject Classification numbers: 76S05, 35B45, 35K20, 35K55, 35K65, 35K67
1. Introduction

We consider fluid flows in porous media with pressure $p$, density $\rho$, velocity $v$, and absolute viscosity $\mu$. The medium has permeability $k > 0$ and porosity $\phi \in (0, 1)$. For hydrodynamics in porous media, the following Darcy’s equation is usually used as a default law

$$-\nabla p = \frac{\mu}{k} v. \tag{1.1}$$

However, even Darcy himself [7] noted that there were deviations from the linear equation (1.1). For instance, in the case the Reynolds number is large or fluids in fractured media, (1.1) becomes inaccurate in describing the fluid dynamics. Many works have been devoted to developing alternative nonlinear models to Darcy’s law, see e.g. [3]. Forchheimer, in [12, 13], established the following three models:

$$-\nabla p = a v + b |v| v, \tag{1.2}$$

$$-\nabla p = a v + b |v| v + c |v|^2 v, \tag{1.3}$$

$$-\nabla p = a v + d |v|^{m-1} v, \quad \text{for some real number } m \in (1, 2). \tag{1.4}$$

The numbers $a, b, c, d$ above are empirical positive constants. The equations (1.2)–(1.4) are usually referred to as Forchheimer’s two-term, three-term and power laws, respectively. For more models and discussions, see [3, 33, 34, 38, 43] and references therein.

From the mathematical point of view, the Darcy flows, under the umbrella of ‘porous medium equations’, have been analyzed intensively since 1960s, see e.g. [41] and a large number of references cited there. In contrast, the mathematics of Forchheimer flows and their variations has attracted much less attention. Moreover, the existing papers on this topic mainly treat incompressible fluids, leaving compressible ones barely studied, see [38] and references therein. The current paper aims to explore Forchheimer flows of compressible fluids using analytical techniques from partial differential equation (PDE) theory.

The Forchheimer equations (1.2)–(1.4) are extended to more general form

$$-\nabla p = \sum_{i=0}^{N} a_i |v|^\alpha_i v, \tag{1.5}$$

where $a_i$’s are positive constants, and $\alpha_i$’s are nonnegative exponents. Equation (1.5) is called the generalized Forchheimer equation. It is used to unify the models (1.2)–(1.4), and as a framework for interpretation of different experimental or field data. Single-phase flows of this type are analyzed in [2, 4, 5, 19–22, 25] for slightly compressible fluids. For incompressible fluids, see [15, 16, 36, 38] and references therein, and, also, see [23, 24] for the study of two-phase flows. For their numerical analysis, see e.g. [11, 17, 26, 29, 35].

For compressible fluids, especially gases, the dependence of coefficients $a_i$’s on the density $\rho$ is essential. By using dimension analysis, Muskat [33] and then Ward [43] proposed the following equation for both laminar and turbulent flows in porous media:

$$-\nabla p = f(v^\alpha k^{\alpha-1} \rho^{\alpha-1} \mu^{\alpha-1}), \quad \text{where } f \text{ is a function of one variable.} \tag{1.6}$$

Using this, Ward [43] established from experimental data that

$$-\nabla p = \frac{\mu}{|k|} v + c_F \frac{\rho}{\sqrt{|k|}} |v| v \quad \text{with } c_F > 0. \tag{1.7}$$
This model is widely accepted as the standard form of the Forchheimer’s two term-law.

Based on the arguments by Muskat and Ward, we proposed in [6] the following adaptation for (1.5)

\[-\nabla p = \sum_{i=0}^{N} a_i \rho^{\alpha_i} |v|^{\alpha_i} v, \quad (1.8)\]

where \(N \geq 1, \alpha_0 = 0 < \alpha_1 < \ldots < \alpha_N\) are real numbers, the coefficients \(a_0, \ldots, a_N\) are positive. This equation covers the two-term case (1.7), and the focus is the dependence on the density, but not viscosity and permeability.

Equation (1.8), however, does not take into account the gravity. Because of the nonlinear density-dependence of the model, any addition of new density terms may complicate the analysis as we will see below. Nonetheless, the gravity can be integrated into (1.8) by replacing \((-\nabla p)\) with \((-\nabla p + \rho \vec{g})\), where \(\vec{g}\) is the constant gravitational field. Therefore, we consider

\[\sum_{i=0}^{N} a_i \rho^{\alpha_i} |v|^{\alpha_i} v = -\nabla p + \rho \vec{g}, \quad (1.9)\]

Denote by \(g : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) a generalized polynomial with positive coefficients defined by

\[g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + \cdots + a_N s^{\alpha_N} \quad \text{for} \quad s \geq 0, \quad (1.10)\]

with \(a_0, a_1, \ldots, a_N > 0\). Then (1.9) can be rewritten as

\[g(\rho |v|) v = -\nabla p + \rho \vec{g}. \quad (1.11)\]

With this rule, we now follow the method in [6] to derive the basic PDE and its corresponding boundary condition. Multiplying both sides of (1.11) by \(\rho\), we obtain

\[g(|\rho v|) \rho v = -\rho \nabla p + \rho^2 \vec{g}. \quad (1.12)\]

Solving for \(\rho v\) from (1.12) gives

\[\rho v = -K(|\rho \nabla p - \rho^2 \vec{g}|)(\rho \nabla p - \rho^2 \vec{g}), \quad (1.13)\]

where the function \(K : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) is defined for \(\xi \geq 0\) by

\[K(\xi) = \frac{1}{g(s(\xi))}, \quad (1.14)\]

with \(s(\xi) = s\) being the unique non-negative solution of \(sg(s) = \xi\).

Relation (1.13) will be combined with other equations of fluid mechanics. The first is the continuity equation

\[\phi \rho_t + \text{div}(\rho v) = F, \quad (1.15)\]

where the porosity \(\phi\) is a constant in \((0, 1)\), the source term \(F\) counts for the rate of net mass production or loss due to any source and/or sink in the media.

From (1.13) and (1.15) follows

\[\phi \rho_t = \text{div} \left( K(|\rho \nabla p - \rho^2 \vec{g}|)(\rho \nabla p - \rho^2 \vec{g}) \right) + F. \quad (1.16)\]

We consider below scenarios of isentropic gas flows, ideal gases and slightly compressible fluids.
Isentropic gas flows

In this case
\[ p = \bar{c} \rho^{\gamma} \quad \text{for some constants } \bar{c}, \gamma > 0. \]  
(1.17)

Here, \( \gamma \) is the specific heat ratio. Note that \( \rho \nabla p = \nabla (\bar{c} \rho^{\gamma+1}/(\gamma + 1)) \). Hence by letting
\[ u = \frac{\bar{c} \rho^{\gamma+1}}{\gamma + 1} = \frac{\gamma p^{1/\gamma}}{\bar{c} \rho^{\gamma+1}}, \]  
(1.18)

we rewrite (1.16) as
\[ \phi e^{1/2}(u^\lambda)_t = \nabla \cdot (K(|\nabla u - cu^\ell \vec{g}|)(\nabla u - cu^\ell \vec{g})) + F, \]  
(1.19)

where
\[ \lambda = \frac{1}{\gamma + 1} \in (0, 1), \quad \ell = 2\lambda \quad \text{and} \quad c = \left( \frac{\gamma + 1}{\bar{c} \gamma} \right)^{\ell}. \]  
(1.20)

The new quantity \( u \) in (1.18) is essentially a pseudo-pressure.

Ideal gases

The equation of state is
\[ p = \bar{c} \rho^{\gamma} \quad \text{for some constant } \bar{c} > 0. \]  
(1.21)

We can consider (1.21) as a special case of (1.17) with \( \gamma = 1 \), and derive the same equation (1.19) with \( \lambda = 1/2 \). In this case, the pseudo-pressure \( u \sim p^2 \) which is used commonly in engineering problems.

Although we mainly focus on gases, we also present here, in a unified way, the slightly compressible fluids which are important in petroleum engineering.

Slightly compressible fluids

The equation of state is
\[ p = \bar{c} \rho^{\gamma} \quad \text{for some constant } \bar{c} > 0. \]  
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We can consider (1.21) as a special case of (1.17) with \( \gamma = 1 \), and derive the same equation (1.19) with \( \lambda = 1/2 \). In this case, the pseudo-pressure \( u \sim p^2 \) which is used commonly in engineering problems.

Although we mainly focus on gases, we also present here, in a unified way, the slightly compressible fluids which are important in petroleum engineering.

\[ \frac{1}{\rho} \frac{d\rho}{dp} = \frac{1}{\kappa} = \text{const.} > 0. \]

Then \( \rho \nabla p = \kappa \nabla \rho \), and by letting
\[ u = \kappa \rho, \quad \ell = \frac{1}{\kappa} \quad \text{and} \quad c = 1/\kappa^2, \]  
(1.22)

we obtain the same equation (1.19) with \( \lambda = 1 \).

For all three cases, by scaling the time variable, we can always assume the multiplying factor on the left-hand side of (1.19) to be 1. In summary, we have derived
\[ (u^\lambda)_t = \nabla \cdot (K(|\nabla u - cu^\ell \vec{g}|)(\nabla u - cu^\ell \vec{g})) + F \]  
(1.23)

with constants \( \lambda \in (0, 1], \ell = 2\lambda, c > 0 \), and function \( F(x, t) \) being rescaled appropriately.

Boundary condition

We will study the problem in a bounded domain \( U \) with outward normal vector \( \vec{\nu} \) on the boundary. We consider the volumetric flux condition
This gives $\rho v \cdot \nu = \psi \rho$, hence, together with (1.13) and (1.18) or (1.22) yields
\[
-K(\nabla u - cu^\ell \xi)(\nabla u - cu^\ell \xi) \cdot \nu = c^{1/2} \psi u^\lambda.
\] (1.24)

General formulation and the initial boundary value problem (IBVP)

Let $n = 2, 3, \ldots$, and $U$ be an open, bounded, connected subset of $\mathbb{R}^n$ with $C^2$ boundary $\Gamma = \partial U$. In applications, $n = 2$ or $3$, but we treat all $n \geq 2$ in this paper. Although problem (1.23) and (1.24) is our motivation, in this mathematical investigation, we consider a more general class of equations and boundary conditions, namely,
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \nabla \cdot \left( K(|\nabla u + Z(u)|)(\nabla u + Z(u)) \right) + f(x, t, u) \quad \text{on} \ U \times (0, \infty), \\
u(x, 0) &= u_0(x) \quad \text{on} \ U, \\
K(|\nabla u + Z(u)|)(\nabla u + Z(u)) \cdot \nu &= B(x, t, u) \quad \text{on} \ \Gamma \times (0, \infty),
\end{aligned}
\] (1.25)

where the unknown function $u(x, t)$ is non-negative on $U \times (0, \infty)$, the initial data $u_0(x) \geq 0$ on $U$ is given, $Z(u)$ is a function from $[0, \infty)$ to $\mathbb{R}^n$, $B(x, t, u)$ is a function from $\Gamma \times [0, \infty) \times [0, \infty)$ to $\mathbb{R}$, and $f(x, t, u)$ is a function from $U \times [0, \infty) \times [0, \infty)$ to $\mathbb{R}$.

Concerning the requirement $u \geq 0$, it comes naturally from the term $u^\lambda$ with exponent $\lambda$ real. For the original model (1.23), $u$ is defined by (1.18) or (1.22), and hence, is non-negative.

In (1.25) the source term $f(x, t, u)$ now can depend on $u$, and the boundary term $B(x, t, u)$ can be more general than $-c^{1/2} \psi(x, t)u^\lambda$ in (1.24). For our analysis, they still need some growth conditions.

**Assumption (A1).** Throughout this paper, we assume that functions $Z(u) : [0, \infty) \to \mathbb{R}^n$, $B(x, t, u) : \Gamma \times [0, \infty) \times [0, \infty) \to \mathbb{R}$ and $f(x, t, u) : U \times [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfy
\[
|Z(u)| \leq d_0 u^\ell_z, \quad (1.26)
\]
\[
B(x, t, u) \leq \varphi_1(x, t) + \varphi_2(x, t)u^{\ell_b}, \quad (1.27)
\]
\[
f(x, t, u) \leq f_1(x, t) + f_2(x, t)u^{\ell_f}, \quad (1.28)
\]
with constants $d_0, \ell_z > 0$, $\ell_f, \ell_b \geq 0$, and functions $\varphi_1, \varphi_2, f_1, f_2 \geq 0$.

Note that the growths with respect to $u$ in assumption (A1) are arbitrary. This is different from other existing papers when the exponents $\ell_z, \ell_f, \ell_b$ are restricted to suit certain Sobolev embedding or trace theorems. Also, we do not require the absolute values of $B(\cdot)$ and $f(\cdot)$ in (1.27) and (1.28) because our solutions are considered to be non-negative.

The problem (1.25) is significantly more general than the one in our previous work [6] when
\[
Z(u) = 0, \quad f(x, t, u) = 0, \quad B(x, t, u) = \varphi(x, t)u^\lambda. \quad (1.29)
\]
It can arise from other complex, nonlinear models of fluid flows in porous media. Indeed, a similar PDE with $p$-Laplacian structure was derived in [9] for water vapor. Here, we showed that it comes naturally from the Forchheimer equations and is formulated for the pseudo-pressure $u$ instead. Due to more complicated flows (1.11), the resulting function $K(\xi)$ is non-homogeneous compared to the homogeneous one in [9].
Regarding the PDE in (1.25), it is doubly nonlinear in both $u$ and $\nabla u$. (For the theory of equations of this type, see the monograph [27], review paper [28] and e.g. [1, 30, 31, 40, 42]. See also the book [10] for essential results and techniques for degenerate parabolic equations.) Moreover, our equation contains lower order terms of arbitrary growths in $u$. Therefore, it is not clear whether $L^\infty$-estimates are possible. In addition, the boundary condition is time-dependent, of nonlinear Robin type, and also has arbitrary growth rate in $u$. Thus, the boundary contribution is not trivial and, thanks to the nonlinearity, we cannot shift the solution by subtracting the boundary data. Global maximum estimates for nonlinear PDEs in (1.25) with such boundary conditions have not been dealt with in literature. For vanishing Dirichlet boundary condition, the work [9] uses maximum principle which is not applicable to our problem with nonlinear Robin boundary condition. Also, both related papers [9, 40] impose the $L^\infty$-requirement for the initial data. In contrast, we use Moser’s iteration [32], hence can deal with more complex equation and boundary condition. Furthermore, we only require the initial data belonging to a certain $L^\alpha$-space with finite $\alpha > 0$, and derive the $L^\infty$-estimates of the solution for positive time. This type of result is established in [37, section 5.2] for equations of certain types different from ours, and with the vanishing Dirichlet boundary condition. While the estimates in the current paper are for the entire domain $U$, we recall that the interior $L^\infty$-estimates are dealt with in our previous work [6] for the problem (1.25) and (1.29). In this regard, the local $L^\gamma - L^\infty$ estimates are also obtained in [14] for, again, a different class of doubly nonlinear parabolic equations. We note that, although ours are a priori estimates, they are crucial in establishing, via regularization and approximation, the existence results (see [27]).

The paper is organized as follows. In section 2, we recall needed trace theorem and Poincaré–Sobolev inequalities. In particular, the inequalities in corollary 2.2 are formulated to suit the nonlinear diffusion, the general nonlinearity in the source term and boundary condition in problem (1.25). In section 3, we establish the $L^\alpha$-estimates of a solution $u(x, t)$ of (1.25) for any $\alpha \in (0, \infty)$ in terms of the initial and boundary data, see theorem 3.2. In section 4, we derive an estimate for spatially global $L^\infty$-norm of $u(x, t)$ in theorem 4.6. By adapting Moser’s iteration, we bound the global $L^\infty$-norm of $u(x, t)$ by (4.36) in theorem 4.4 which is ‘quasi-homogeneous’ in its $L^\beta_x$-norm for some $\beta > 0$. This global estimate is an improvement over the generally known ones for degenerate/doubly degenerate equations, and extends a previous local version in [39] and our own in [6], see remark 4.5. From the technical point of view, our implementation of Moser’s method is rather involved with the sequences of exponents in the iterations being carefully constructed based on the nonlinearity of the boundary condition and the source term, see (4.34), (4.37) and (4.40). Nonetheless, the current proof already simplifies those in [6]. In section 5, we establish $L^\beta$-a-estimates for the gradient of positive solutions in theorem 5.3. Even for this simple norm, it is non-trivial due to the arbitrary growth in the nonlinear Robin boundary condition. These gradient estimates are obtained under condition (5.8). This mathematical requirement turns out to be naturally satisfied for the original problem (1.23) and the known gases such as those in the data book [44], see remark 5.1. It modestly shows the relevance of our mathematical analysis. It is also worthy to note that while the estimates established in [6] use the $L^\infty(\Gamma)$-norm for the boundary data $\varphi(x, t)$ in (1.29), we use, in the current paper, only the $L^\beta(\Gamma)$-norms of $\varphi_1$ and $\varphi_2$ for some finite number $\beta$. Such improvements, in fact, require much more complicated calculations in the proofs and our ability to deal with a general power of $u$ on the boundary.
2. Basic inequalities

First, we recall elementary inequalities that will be used frequently. Let $x, y \geq 0$. Then
\[
(x + y)^p \leq x^p + y^p \quad \text{for all } 0 < p \leq 1,
\]
\[
(x + y)^p \leq 2^{p-1}(x^p + y^p) \quad \text{for all } p \geq 1,
\]
\[
x^\beta \leq x^\alpha + x^\gamma \quad \text{for all } 0 \leq \alpha \leq \beta \leq \gamma,
\]
particularly,
\[
x^\beta \leq 1 + x^\gamma \quad \text{for all } 0 \leq \beta \leq \gamma.
\]

Next, we recall particular Poincaré–Sobolev inequality and trace theorem. The classical trace theorem: If a function $u(x)$ belongs to $W^{1,1}(U)$ then
\[
\int_U |u|^\alpha \, d\sigma \leq c_1 \int_U |u|^p \, dx + c_2 \int_U |\nabla u|^p \, dx,
\]
where constants $c_1, c_2 > 0$ depend on $U$.

**Lemma 2.1 (Lemma 2.1).** In the following statements, $u(x)$ is a function defined on $U$.

(i) If $\alpha \geq s \geq 0$, $\alpha \geq 1$, and $p > 1$, then for any $|u|^\alpha \in W^{1,1}(U)$ and $\varepsilon > 0$ one has
\[
\int_U |u|^\alpha \, d\sigma \leq \varepsilon \int_U |u|^\alpha - \alpha \theta \varepsilon^{-s} \int_U |\nabla u|^p \, dx + c_3 \int_U |u|^p \, dx + (c_4 \alpha)^{\frac{\theta}{1-s}} \int_U |u|^{\frac{\theta}{1-s}} \, dx.
\]

(ii) If $n > p > 1$, $r > 0$, $\alpha \geq s \geq 0$, $\alpha \geq \frac{n-1}{p-1}$, and $\alpha > \frac{n+1}{p-1}$, then for any $\varepsilon > 0$ one has
\[
\int_U |u|^\alpha \, dx \leq \varepsilon \int_U |u|^{\alpha - \alpha \theta \varepsilon^{-s} \int_U |\nabla u|^p \, dx + c_3 \int_U |u|^p \, dx + (c_4 \alpha)^{\frac{\theta}{1-s}} \int_U |u|^{\frac{\theta}{1-s}} \, dx + c_5 \int_U |u|^s \, dx
\]
for all $|u|^m \in W^{1,1}(U)$, where
\[
m = \frac{\alpha - s + p}{p}, \quad \theta = \frac{r}{(\alpha p/n) + p - s}, \quad \mu = \frac{r + \theta(s - p)}{1 - \theta},
\]
and positive constants $c_3 = c_3(U,p)$ and $c_4 = c_4(U,p)$ depend on $U,p$, but not on $u(x)$, $\alpha, s$.

The following remarks on lemma 2.1 are in order.

(a) We can calculate $r$ in terms of $\theta$ by the second formula in (2.8), and rewrite the power $\alpha + \mu$ in (2.7) as
\[
\alpha + \mu = \alpha + \left[ \theta \left( \frac{\alpha p}{n} + (p-s) \right) + \theta(s-p) \right] \frac{1}{1-\theta} = \alpha \left( 1 + \frac{p}{n} \cdot \frac{\theta}{1-\theta} \right).
\]

Assume $p \leq s$, then $m \leq \alpha$ and we can rewrite (2.7) as
\[
\int_U |u|^\alpha \, dx \leq \varepsilon \int_U |u|^{\alpha - \alpha \theta \varepsilon^{-s} \int_U |\nabla u|^p \, dx + c_3 \int_U |u|^p \, dx + (c_4 \alpha)^{\frac{\theta}{1-s}} \int_U |u|^{\frac{\theta}{1-s}} \, dx + c_5 \int_U |u|^s \, dx + D_1 \int_U |u|^{\alpha + \frac{\theta}{1-s}} \, dx + c_5 \int_U |u|^s \, dx.
\]
where
\[ D_1 = [2^{\alpha-s+p}c_3^p |U|^{(1/(p-1))} + r + \beta}]^\theta, \quad D_2 = [2^{\alpha-s+p}(c_3^p)^\theta + \beta]^\theta. \]

(b) For any \( r \geq 0, \alpha \geq s \geq 0, \alpha + r \geq 1 \), applying (2.6) for \( \alpha \to \alpha + r \) and \( s \to s + r \) yields
\[ \int_U |u|^{n+\beta} \leq \epsilon \int_U |u|^{n+\beta} |\nabla u| dx + c_1 \int_U |u|^{n+\beta} + (c_2(\alpha + r))^{\frac{n}{p-1}} \int_U |u|^{n+\beta} dx, \tag{2.11} \]
where
\[ \epsilon = \frac{r + s - p}{p - 1} = \frac{pr + s - p}{p - 1}. \]

In our particular case, we have the following corollary.

**Corollary 2.2.** Assume \( 1 > a > \delta > 0, \alpha \geq 2 - \delta \), and \( |u|^{n} \in W^{1,1}(U) \). Let \( r > 0 \).

(i) For any \( \epsilon > 0 \) one has
\[ \int_U |u|^{\alpha+\beta} dx \leq \epsilon \int_U |u|^{\alpha-2+\delta} |\nabla u|^{2-a} dx + c_1 \int_U |u|^{\alpha+\beta} + (c_2(\alpha + r))^{\frac{n}{p-1}} \int_U |u|^{\alpha+\beta} dx. \tag{2.12} \]
(ii) If \( \alpha > \frac{n(r+a-\delta)}{2-a} \), then for any \( \epsilon > 0 \) one has
\[ \int_U |u|^{\alpha+\beta} dx \leq \epsilon \int_U |u|^{\alpha-2+\delta} |\nabla u|^{2-a} dx + D_1 \|u\|_{L^n}^{2-a} + D_2 \epsilon \int_U |u|^{\alpha+\beta} dx, \tag{2.13} \]
where
\[ \epsilon = \frac{r + s - p}{p - 1} = \frac{pr + s - p}{p - 1}. \]

**Proof.** Let \( p = 2 - a \) and \( s = 2 - \delta \), noticing that \( p < s \). Then inequality (2.12) follows (2.11), and inequality (2.13) follows (2.10). \( \square \)

We also recall a particular multiplicative parabolic Sobolev inequality.

**Lemma 2.3 ([6], lemma 2.3).** Assume \( 1 > a > \delta > 0, |u|^{n} \in L^{\infty}(0, T; W^{1,1}(U)) \), and
\[ \alpha \geq 2 - \delta \quad \text{and} \quad \alpha > \alpha_* = \frac{n(a - \delta)}{2 - a}. \tag{2.14} \]

If \( T > 0 \), then
\[ \left( \int_0^T \int_U |u|^n dx \right)^{\frac{n}{n-a}} \leq (c_3^{2-a} |U|^{\alpha(2-a) + \beta}) \left( \int_0^T \int_U |u|^{\alpha+\delta-a} dx dr + \int_0^T \int_U |u|^{\alpha+\delta-a} |\nabla u|^{2-a} dx dr \right)^{\frac{n}{n-a}} \]
\[ \times \sup_{t \in [0, T]} \left( \int_U |u(x, t)|^{n} dx \right)^{\frac{n-a}{n-a+\delta}}. \tag{2.15} \]
where \( c_5 \geq 1 \) is independent of \( \alpha \) and \( T \), and
\[
\hat{\theta} = \frac{\theta}{\alpha \theta + 1}, \quad \kappa = \frac{\kappa(\alpha)}{\alpha} = 1 + \left( \frac{2 - a}{n} \right) - \frac{a - \delta}{\alpha}, \quad \hat{\theta} = 1 + (a - \delta) \left( \frac{1}{\alpha} - \frac{1}{\alpha} \right).
\]

(2.16)

### 3. Estimates of the Lebesgue norms

From now on, we fix a function \( g(s) \) in (1.10) and (1.11). Denote
\[
a = \frac{\alpha N}{\alpha N + 1} \in (0, 1).
\]
(3.1)

This number will be used in our calculations throughout.

The function \( K(\xi) \) defined by (1.14) has the following properties: it is decreasing in \( \xi \), maps \([0, \infty)\) onto \((0, 1)\) and
\[
d_1 \left( \frac{1 + \xi}{(1 + \xi)^a} \right) \leq K(\xi) \leq \frac{d_2}{(1 + \xi)^a},
\]
(3.2)
\[
d_3 (\xi^{2-a} - 1) \leq K(\xi) \xi^2 \leq d_2 \xi^{2-a},
\]
(3.3)

where \( d_1, d_2, d_3 \) are positive constants depending on \( \alpha_i \)’s and \( a_i \)’s of the function \( g(s) \), see [2].

Hereafter, \( u(x, t) \geq 0 \) is a solution of IBVP (1.25).

Define \( \delta = 1 - \lambda \in [0, 1) \) throughout the paper, we assume
\[
a > \delta.
\]
(3.4)

This assumption is to avoid too many possible cases for our estimates. The case \( a \leq \delta \) can be treated similarly and, in fact, is easier to deal with.

Then \( \alpha_* = n(a - \delta)/(2 - a) \) in (2.14) is a fixed positive number. In order to describe our results, we introduce some constants and quantities. Define
\[
\kappa_B = 1 + \left( \frac{1 - a}{n} \right) \quad \text{and} \quad \kappa_f = 1 + \left( \frac{2 - a}{n} \right).
\]
(3.5)

Let \( p_1, p_2, p_3, p_4 \) be fixed numbers such that
\[
1 < p_1, p_2 < \kappa_B \quad \text{and} \quad 1 < p_3, p_4 < \kappa_f.
\]
(3.6)

For \( i = 1, 2, 3, 4 \) let \( q_i \) be the conjugate exponent of \( p_i \), that is, \( 1/p_i + 1/q_i = 1 \). Then (3.6) is equivalent to
\[
1 + \frac{n}{n - a} < q_1, q_2 < \infty \quad \text{and} \quad 1 + \frac{n}{2 - a} < q_3, q_4 < \infty.
\]
(3.7)

We also denote by \( \eta_0 \) a positive number defined by
\[
\eta_0 = \max \left\{ q_1, q_2, q_3, q_4, \frac{n(\ell - 1)}{\ell - 1}, \frac{p_1}{\kappa_B - p_1}, \frac{p_2}{\kappa_B - p_2}, \frac{p_3}{\kappa_f - p_3}, \frac{p_4}{\kappa_f - p_4} \right\}.
\]
(3.8)
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**Norm notation**

Throughout the paper, we make use of the brief notation \( \|f(t)\|_{L^q(U)} \) for \( \|f(\cdot,t)\|_{L^q(U)} \), and \( \|\varphi(t)\|_{L^q(\Gamma)} \) for \( \|\varphi(\cdot,t)\|_{L^q(\Gamma)} \).

In this paper, we are interested in obtaining a priori estimates for a (non-negative) solution \( u(x,t) \) of problem (1.25). Particularly in this section, we focus on estimating its \( L^\alpha \)-norm on \( U \), for any given \( \alpha > 0 \) and \( t > 0 \). We start with a differential inequality for \( \|u(t)\|_{L^\alpha(U)}^{\alpha} \).

A key quantity in expressing our estimates is

\[ \Upsilon(t) = \|\varphi_1(t)\|_{L^q(\Gamma)}^{\nu_1} + \|\varphi_2(t)\|_{L^q(\Gamma)}^{\nu_2} + \|f_1(t)\|_{L^q(U)}^{\nu_1} + \|f_2(t)\|_{L^q(U)}^{\nu_2} \text{ for } t > 0. \]

**Lemma 3.1.** Assume

\[ \alpha \geq \max \left\{ 2, \frac{na}{1-a} \right\} \text{ and } \alpha > \eta_0. \]

For \( t > 0 \), one has

\[ \frac{d}{dt} \int_U u(x,t)^\alpha dx + \int_U |\nabla u(x,t)|^{2-a} u(x,t)^{\alpha-\lambda-1} dx \leq C_0 \cdot \left( \|u(t)\|_{L^q(U)}^{\nu_1} + \|u(t)\|_{L^q(U)}^{\nu_2} + \Upsilon(t) \right), \]

where \( C_0 = C_0(\alpha) > 0 \),

\[ \nu_1 = \nu_1(\alpha) \overset{\text{def}}{=} \alpha - \lambda - 1 \text{ and } \nu_2(\alpha) \overset{\text{def}}{=} \alpha \left( 1 + \frac{2-a}{n} \cdot \frac{\theta}{1-\theta} \right) \]

with

\[ \theta = \theta_\alpha \overset{\text{def}}{=} \frac{\nu_3 - \alpha}{\alpha(2-a)/n + \delta - a} \text{ for } \nu_3 = \nu_3(\alpha) \text{ defined by (3.24) below.} \]

**Proof.** Multiplying the first equation in (1.25) by \( u^{a-\lambda} \), integrating over domain \( U \), and using integration by parts we have

\[ \frac{\lambda}{\alpha} \frac{d}{dt} \int_U u^\alpha dx = \int_U \nabla \cdot (K(|\nabla u + Z(u)|)(\nabla u + Z(u))) u^{a-\lambda} dx + \int_U f u^{a-\lambda} dx \]

\[ = -(\alpha - \lambda) \int_U K(|\nabla u + Z(u)|)(\nabla u + Z(u)) \cdot \nabla u u^{a-\lambda-1} dx \]

\[ + \int_U K(|\nabla u + Z(u)|)(\nabla u + Z(u)) \cdot \nabla u u^{a-\lambda} dx + \int_U f u^{a-\lambda} dx. \]

For the first integral on the right-hand side, we write

\[ (\nabla u + Z(u)) \cdot \nabla u = |\nabla u + Z(u)|^2 - (\nabla u + Z(u)) \cdot Z(u). \]

For the boundary integral, we use the boundary condition in (1.25). These result in

\[ \frac{\lambda}{\alpha} \frac{d}{dt} \int_U u^\alpha dx = -(\alpha - \lambda) \int_U K(|\nabla u + Z(u)|)|\nabla u + Z(u)|^{2a-\lambda-1} dx \]

\[ + (\alpha - \lambda) \int_U K(|\nabla u + Z(u)|)(\nabla u + Z(u)) \cdot Z(u) u^{a-\lambda-1} dx + \int_{\Gamma} f u^{a-\lambda} ds. \]
Using relations (1.26), (3.2) and (3.3) for the first two integrals on the right-hand side, as well as (1.27) and (1.28) for the last two, we have
\[
\frac{\lambda}{\alpha} \frac{d}{dt} \int_U u^\alpha \, dx \leq - (\alpha - \lambda) d_3 \int_U (|\nabla u + Z(u)|^2 - 1) u^{\alpha - \lambda - 1} \, dx \\
+ (\alpha - \lambda) d_2 d_0 \int_U (|\nabla u + Z(u)| + 1)^{1 - a} u^{\alpha - \lambda - 1 + \ell_2} \, dx \\
+ \int_\Gamma u^{\alpha - \lambda} (\varphi_1 + \varphi_2 u^{\ell_2}) \, d\sigma + \int_U (f_1 + f_2 u^{\ell_2}) u^{\alpha - \lambda} \, dx.
\]

Using (2.1), we estimate
\[
(|\nabla u + Z(u)| + 1)^{1 - a} \leq |\nabla u + Z(u)|^{1 - a} + 1.
\]
Hence, we obtain
\[
\frac{\lambda}{\alpha} \frac{d}{dt} \int_U u^\alpha \, dx \leq - (\alpha - \lambda) d_1 + (\alpha - \lambda) d_3 d_2 + (\alpha - \lambda) d_2 d_0 + (\alpha - \lambda) d_2 d_0 d_4 + I_5 + I_6,
\]
where
\[
I_1 = \int_U |\nabla u + Z(u)|^{2 - a} u^{\alpha - \lambda - 1} \, dx, \quad I_2 = \int_U u^{\alpha - \lambda - 1} \, dx, \\
I_3 = \int_U |\nabla u + Z(u)|^{1 - a} u^{\alpha - \lambda - 1 + \ell_2} \, dx, \quad I_4 = \int_U u^{\alpha - \lambda - 1 + \ell_2} \, dx, \\
I_5 = \int_\Gamma (u^{\alpha - \lambda} \varphi_1 + u^{\alpha - \lambda + \ell_2} \varphi_2) \, d\sigma, \quad I_6 = \int_U (u^{\alpha - \lambda} f_1 + u^{\alpha - \lambda + \ell_2} f_2) \, dx.
\]

In calculations below, the positive constants \(\bar{C}, C, C_\varepsilon, C_\varepsilon'\) are generic, with \(\bar{C}\) independent of \(\alpha\), while \(C\) depending on \(\alpha\), and \(C_\varepsilon, C_\varepsilon'\) depending on \(\alpha\) and \(\varepsilon\).

- Applying Young’s inequality to the integrand of \(I_3\) with powers \(\frac{2-a}{1-a}\) and \(2-a\) gives
  \[
  I_5 \leq \frac{d_5}{2d_2 d_0} \int_U |\nabla u + Z(u)|^{2 - a} u^{\alpha - \lambda - 1} \, dx + \bar{C} \int_U u^{(2-a)+\alpha-\lambda-1} \, dx = \frac{d_5}{2d_2 d_0} I_1 + \bar{C} I_7, \quad (3.12)
  \]

  where
  \[
  I_7 = \int_U u^{\mu_1} \, dx \quad \text{with} \quad \mu_1 = \ell_2 (2-a) + \alpha - \lambda - 1. \quad (3.13)
  \]

  Note also, by (2.3), that
  \[
  I_4 \leq I_5 + I_7.
  \]

  Then
  \[
  \frac{\lambda}{\alpha} \frac{d}{dt} \int_U u^\alpha \, dx \leq - (\alpha - \lambda) \frac{d_3}{2} I_1 + \bar{C} \alpha (I_2 + I_7) + I_5 + I_6. \quad (3.14)
  \]
• To estimate $I_1$ we use inequality
  
  \[ |x|^{2-a} \leq 2^{1-a} (|x-y|^{2-a} + |y|^{2-a}) \]  
  which gives \( |x-y|^{2-a} \geq 2^{a-1} |x|^{2-a} - |y|^{2-a} \) \( \forall x, y \in \mathbb{R}^n \).

  Together with (1.26), it gives
  
  \[
  I_1 \geq \int_U \left( 2^{a-1} |\nabla u|^2 - |Z(u)|^2 \right) dx
  \geq 2^{a-1} \int_U |\nabla u|^{2-a} u^{\alpha-\lambda-1} dx - 2^{a-1} \int_{\Gamma} u^{\alpha-\lambda-1} + I_7(2-a) dx
  \geq 1/2 \int_U |\nabla u|^{2-a} u^{\alpha-\lambda-1} dx - \frac{\lambda}{\alpha} \int_U u^{\alpha-\lambda-1} dx.
  \]  
  (3.15)

  This and (3.14) imply
  
  \[
  \frac{\lambda}{\alpha} \frac{d}{dt} \int_U u^{\alpha} dx + (\alpha - \lambda) \frac{d}{dt} \left( \int_U |\nabla u|^{2-a} u^{\alpha-\lambda-1} dx \right) \leq C_0 (I_7 + I_5) + I_5 + I_6.
  \]  
  (3.16)

• For $I_5$, using Young’s inequality, we have
  
  \[
  I_5 \leq \int_{\Gamma} u^{\mu_2} d\sigma + \int_{\Gamma} u^{\mu_3} d\sigma + \|\varphi_1\|^2_{L^q(\Gamma)} + \|\varphi_2\|^2_{L^q(\Gamma)},
  \]  
  (3.17)

  where
  
  \[
  \mu_2 = p_1(\alpha - \lambda) \quad \text{and} \quad \mu_3 = p_2(\alpha - \lambda + \ell_3).
  \]  
  (3.18)

  For the first two integrals on the right-hand side, applying (2.12) to $\alpha + r = \mu_2$ and $\alpha + r = \mu_3$ gives
  
  \[
  \int_{\Gamma} u^{\mu_2} d\sigma + \int_{\Gamma} u^{\mu_3} d\sigma \leq \varepsilon \int_U |u|^{\alpha-\lambda-1} |\nabla u|^{2-a} dx + c_1 \int_U (u^{\mu_2} + u^{\mu_3}) dx + C_5 \int_U (u^{\mu_4} + u^{\mu_5}) dx,
  \]  
  (3.19)

  where $\varepsilon > 0$ is arbitrary,
  
  \[
  \mu_4 = \alpha + \frac{2-a\alpha}{1-a} = \alpha + \frac{(2-a)((\alpha-1)\alpha - \alpha + 1) + a - \delta}{1-a} > \alpha,
  \]  
  \[
  \mu_5 = \alpha + \frac{2-a\alpha}{1-a} = \alpha + \frac{(2-a)((\alpha-1)\alpha + \alpha + 1) - \alpha + \delta}{1-a} > \alpha.
  \]  
  (3.20)

  (The validity of (3.19) will be verified later.) Then
  
  \[
  I_5 \leq \varepsilon \int_U |u|^{\alpha-\lambda-1} |\nabla u|^{2-a} dx + C_5 \int_U \sum_{i=2}^5 u^{\mu_i} dx + \|\varphi_1\|^2_{L^q(\Gamma)} + \|\varphi_2\|^2_{L^q(\Gamma)}.
  \]  
  (3.21)

• For $I_6$, using Young’s inequalities we have
  
  \[
  I_6 \leq \int_U (u^{\mu_4} + u^{\mu_5}) dx + \|f_1\|^2_{L^2} + \|f_2\|^2_{L^2}.
  \]  
  (3.22)
where
\[ \mu_6 = p_2(\alpha - \lambda) \quad \text{and} \quad \mu_7 = p_4(\alpha - \lambda + \ell_f). \] (3.23)

Note for \( 1 \leq i \leq 7 \) that \( \nu_1 \leq \nu_i \leq \nu_3 \), where, referring to (3.13), (3.20) and (3.23),
\[ \nu_3 = \max \{ \mu_i : 1 \leq i \leq 7 \}. \] (3.24)

Hence by using (2.3), \( u^{\nu_i} \leq u^{\alpha} + u^{\nu_3} \) for all \( 1 \leq i \leq 7 \). Combining this with (3.16), (3.21), and (3.22) yields
\[ \frac{\lambda}{\alpha} \frac{d}{dt} \int u^{\alpha} dx + \left[ (\alpha - \lambda \frac{d_1}{4} - \varepsilon) \int |\nabla u|^{2-a} u^{\alpha - \lambda - 1} dx \right] \leq C \int (u^{\alpha} + u^{\nu_3}) dx + \Upsilon(t). \] (3.25)

Using Hölder’s inequality we have
\[ \int u^{\alpha} dx \leq C \left( \int u^{\nu_3} dx \right)^{\frac{\alpha}{\nu_3}} = C \| u \|_{L^{\nu_3}}^{\frac{\alpha}{\nu_3}}. \] (3.26)

Applying (2.13) to \( \alpha + r = \nu_3 \) gives
\[ C \int u^{\nu_3} dx \leq \varepsilon \int |u|^{a-\lambda-1} |\nabla u|^{2-a} dx + C \| u \|_{L^{\nu_3}}^{\nu_3} \| u \|_{L^{\nu_3}}^{\frac{\alpha}{\nu_3}}. \] (3.27)

(Again, we will verify the validity of (3.27) later.) Combining (3.25) with (3.26) and (3.27), also using inequality (2.3) with powers \( \nu_1 \leq \nu_3 \leq \nu_2 \) in dealing with \( \| u \|_{L^2} \), we have
\[ \frac{\lambda}{\alpha} \frac{d}{dt} \int u^{\alpha} dx + \left[ (\alpha - \lambda \frac{d_1}{4} - 2\varepsilon) \int |\nabla u|^{2-a} u^{\alpha - \lambda - 1} dx \right] \leq C \| u \|_{L^{\nu_3}}^{\nu_3} \| u \|_{L^{\nu_3}}^{\frac{\alpha}{\nu_3}} + \Upsilon(t). \] (3.28)

Choosing \( \varepsilon \) sufficiently small in (3.25), we obtain (3.11).

It remains to check conditions for inequalities (3.19) and (3.27) to hold. Inequality (3.19) is valid under conditions \( \mu_2, \mu_3 > \alpha \) and \( \alpha > 2 - \delta \), which, thanks to \( \alpha \geq 2 \), is equivalent to
\[ \alpha > \max \{ q_1 \lambda, q_2 (\lambda - \ell_b) \}. \] (3.29)

The conditions for (3.27) are \( \nu_3 > \alpha, \alpha \geq 2 - a \), and
\[ \alpha > \frac{n(r + a - \delta)}{2 - a} = \frac{n(\nu_3 - \alpha + a - \delta)}{2 - a}. \] (3.30)

Since \( \nu_3 \geq \mu_2, \mu_3 \), the condition \( \nu_3 > \alpha \) is met if \( \alpha \) satisfies (3.29).

Based on definition (3.24) of \( \nu_3 \) for \( i = 1, \ldots, 7 \), when \( \nu_3 = \mu_i \) the corresponding condition (3.30) becomes
\[ \alpha > \frac{p_2(-\lambda + \ell_b) + a - \delta}{\kappa_f - p_2} \quad \text{if} \quad \nu_3 = \mu_3. \] (3.33)
\[ \alpha > -p_1\lambda + (a - \delta) \]

\[ \kappa_B - p_1 \]

if \( \nu_3 = \mu_4 \),

(3.34)

\[ \alpha > \frac{n}{2-a} \left( \frac{(2-a)((p_2-1)\alpha + p_2(-\lambda + \ell_B)) + a - \delta}{1-a} \right) \]

\[ = \frac{n}{1-a} \left( (p_2-1)\alpha + p_2(-\lambda + \ell_B) + a - \delta \right) \]

equivalently, \( \alpha > \frac{p_2(-\lambda + \ell_B) + a - \delta}{\kappa_B - p_2} \)

if \( \nu_3 = \mu_5 \),

(3.35)

\[ \alpha > \frac{-p_3\lambda + a - \delta}{\kappa_f - p_3} \]

if \( \nu_3 = \mu_6 \),

(3.36)

\[ \alpha > \frac{p_4(-\lambda + \ell_f) + a - \delta}{\kappa_f - p_4} \]

if \( \nu_3 = \mu_7 \).

(3.37)

In summary, the conditions (3.29) and (3.31)–(3.37) are equivalent to \( \alpha > \eta_0 \), which is our choice of \( \alpha \). The proof is complete. \( \square \)

We then obtain the estimates in terms of initial and boundary data in the next theorem.

**Theorem 3.2.** Let \( \alpha, C_0, \theta \) be as in lemma 3.1 and \( \Upsilon(t) \) be defined by (3.9). If \( T > 0 \) satisfies

\[ \int_0^T (1 + \Upsilon(t))dt < C_1 \cdot \left( 1 + \int_U u_0^\alpha(x)dx \right)^{-\nu_4}, \]

(3.38)

where

\[ \nu_4 = \nu_4(\alpha) \overset{\text{def}}{=} \frac{(2-a)\theta}{n(1-\theta)} \quad \text{and} \quad C_1 = C_1(\alpha) \overset{\text{def}}{=} \frac{1}{4C_0\nu_4}, \]

(3.39)

then for all \( t \in [0, T] \)

\[ \int_U u^\alpha(x,t)dx \leq \left\{ 1 + \int_U u_0^\alpha(x)dx \right\}^{-\nu_4} - \frac{1}{C_1} \int_0^t (1 + \Upsilon(\tau))d\tau \right\}^{-\frac{\nu_4}{2}}. \]

(3.40)

In particular, if \( T > 0 \) satisfies

\[ \int_0^T (1 + \Upsilon(t))dt < C_1 \cdot (1 - 2^{-\nu_4}) \left( 1 + \int_U u_0^\alpha(x)dx \right)^{-\nu_4}, \]

(3.41)

then

\[ \int_U u^\alpha(x,t)dx \leq 1 + 2 \int_U u_0^\alpha(x)dx \quad \text{for all} \quad t \in [0, T], \]

(3.42)

and

\[ \int_0^T \int_U |\nabla u|^{2-\alpha-a-\lambda-1}dxdt \leq 2^{\nu_4+1}(1 + 1/\nu_4) \left( 1 + \int_U u_0^\alpha(x)dx \right). \]

(3.43)
Proof. In (3.11), by applying Young’s inequality to the right-hand side we find that
\[
\frac{d}{dt} \int_U u^\alpha \, dx + \int_U |\nabla u|^{2-\alpha} u^{\alpha-\lambda-1} \, dx \leq 2C_0 \cdot (1 + \|u(t)||^\nu_x + \Upsilon(t)) \leq 4C_0 \cdot (1 + \Upsilon(t)) \left(1 + \int u^\alpha \, dx\right)^{\nu_x/2}.
\]  
(3.44)

Let \( V(t) = 1 + \int_U u^\alpha(x,t) \, dx \). It follows from (3.44) that
\[
V'(t) \leq 4C_0 \cdot (1 + \Upsilon(t))V(t)^{\nu_x/2} = 4C_0 \cdot (1 + \Upsilon(t))V(t)^{\nu_x+1}.
\]

Solving this differential inequality, under condition (3.38) yields the estimate (3.40) for all \( t \in [0,T] \).

Now, let \( T > 0 \) satisfy (3.41). Then we have from (3.40) that
\[
V(t) \leq 2V(0) \quad \forall t \in [0,T],
\]
and hence, estimate (3.42) follows.

Next, integrating (3.44) in time from 0 to \( T \) and using (3.45) and (3.41) we have
\[
\int_0^T \int_U |\nabla u(x,t)|^{2-\alpha} u(x,t)^{\alpha-\lambda-1} \, dx \, dr \leq \int_0^T \int_U u_0^\alpha(x) \, dx + 4C_0 \int_0^T \left(1 + \int_U u^\alpha(x,t) \, dx\right)^{\nu_x+1} \left(1 + \Upsilon(t)\right) \, dt \leq \int_0^T u_0^\alpha(\cdot) \, dx + 4C_0 \cdot 2^{\nu_x+1} \left(1 + \int_0^T u_0^\alpha(\cdot) \, dx\right)^{\nu_x+1} \left(1 + \Upsilon(t)\right) \, dt
\]
\leq \int_0^T u_0^\alpha(\cdot) \, dx + 4C_0 C_1 \cdot 2^{\nu_x+1} (1 - 2^{-\nu_x}) \left(1 + \int_0^T u_0^\alpha(\cdot) \, dx\right).
\]
(3.46)

Note that
\[
4C_0 C_1 \cdot 2^{\nu_x+1} (1 - 2^{-\nu_x}) = \frac{2^{\nu_x+1}(1 - 2^{-\nu_x})}{\nu_x} \leq \frac{2^{\nu_x+1}}{\nu_x}.
\]
Thus we obtain (3.43) from (3.46).  

\[\square\]

4. Maximum estimates

In this section, we use Moser’s iteration to estimate the \( L^\infty \)-norm of the solution \( u(x,t) \).

Denote \( Q_T = U \times (0,T) \) and \( \Gamma_T = \Gamma \times (0,T) \). For \( i = 1, 2, 3, 4 \), let \( q_i \) and \( p_i \) be as in section 3.

Assume (3.6). Let
\[
\kappa_+ = \max \left\{ \frac{(2-a)p_1 - 1}{1-a}, \frac{(2-a)p_2 - 1}{1-a}, p_3, p_4 \right\},
\]
(4.1)
\[
z_1 = \frac{p_1(2-a)\lambda - a + \delta}{(2-a)p_1 - 1}, \quad z_2 = \frac{p_2(2-a)(\lambda - \ell_b) - a + \delta}{(2-a)p_2 - 1},
\]

Lemma 4.1. Given $\tilde{\kappa} > \kappa_*$, suppose

$$\alpha > \max\{2, \alpha_*, \eta_1\} \quad \text{and} \quad \alpha \geq \frac{\eta_2}{\tilde{\kappa} - \kappa_*}. \quad (4.2)$$

If $T > T_2 > T_1 \geq 0$ then

$$\sup_{t \in [T_1, T]} \int_U u^\alpha(x, t)dx + \int_{T_1}^T \int_U |\nabla u(x, t)|^{2-\delta}u(x, t)^{\alpha-\lambda-1}dxdt \leq c_6(1 + T) \left(1 + \frac{1}{T_2 - T_1}\right) \alpha^2 M_0 \left(\|u\|_{L^\infty(U \times (T, T))}^{\nu_5} + \|u\|_{L^\infty(U \times (T, T))}^{\nu_6}\right), \quad (4.3)$$

where $c_6 \geq 1$ is independent of $\alpha, \tilde{\kappa}, T, T_1$ and $T_2$.

The positive powers $\nu_5$ and $\nu_6$ are defined by

$$\nu_5 = \alpha - h_1 \quad \text{and} \quad \nu_6 = \alpha + h_2 \quad (4.5)$$

with

$$h_1 = \max\{\lambda + 1, \eta_1\} > 0,$$

$$h_2 = \max\{0, \ell_2(2 - a) - \lambda - 1, \ell_f - \lambda, \ell_B - \lambda, -z_1, -z_2\} \geq 0.$$

Proof. Let $\xi = \xi(t)$ be a $C^1$-function on $[0, T]$ with $\xi(0) = 0$ and $0 \leq \xi(t) \leq 1$. In calculations below, the generic positive constant $C$ is independent of $\alpha, \tilde{\kappa}, T, T_1$ and $T_2$.

Multiply the PDE in (1.25) by test function $u^{\alpha-\lambda}\xi^2$ and integrating the resulting equation over $U$ give

$$\frac{\lambda}{\alpha} \frac{d}{dt} \int_U u^\alpha \xi^2 dx - \frac{\lambda}{\alpha} \int_U 2u^\alpha \xi \xi' dx = \int_U \nabla \cdot \left(K(|\nabla u + Z(u)|)(\nabla u + Z(u))\right) u^{\alpha-\lambda} \xi^2 dx + \int_U f u^{\alpha-\lambda} \xi^2 dx.$$

Since $\xi(t)$ is independent of $x$, same estimates as in section 3 give the following version of (3.16)

$$\frac{\lambda}{\alpha} \frac{d}{dt} \int_U u^\alpha \xi^2 dx - \frac{\lambda}{\alpha} \int_U 2u^\alpha \xi \xi' dx \leq \frac{C}{4} \int_U |\nabla u|^2 u^{\alpha-\lambda-1} \xi^2 dx \leq C \alpha \int_U (u^\alpha + u^{\nu_5}) \xi^2 dx + \frac{\lambda}{\alpha} \int_U f u^{\alpha-\lambda} \xi^2 dx + \int_U \int_U (f_1 + f_2 u^{\nu_5}) u^{\alpha-\lambda} \xi^2 dx + \frac{\lambda}{\alpha} \int_U 2u^\alpha \xi' dx.$$

Integrating the previous inequality in time and applying Hölder’s inequality give
\[
\frac{\lambda}{\alpha} \sup_{r \in [0, T]} \int_{Q_r} u^\alpha(x, t)\xi^2(t)dx + \frac{(\alpha - \lambda)d_3}{4} \int \int_{Q_T} |\nabla u|^{2-\alpha} u^{\alpha-\lambda-1}\xi^2dxdt \\
\leq C\alpha \int \int_{Q_T} (u_{i1} + u_{i1})\xi^2dxdt + \frac{\lambda}{\alpha} \int \int_{Q_T} 2u^{\alpha}\xi^2dxdt + \sum_{i=1}^{4} I_i,
\]

where
\[
I_i = \left( \int \int_{Q_T} u^{\alpha}\xi^2dxdt \right)^{1/p_i} F_i \quad \text{with} \quad F_i = \left( \int \int_{Q_T} \varphi_i^\alpha\xi^2dxdt \right)^{1/q_i}, \quad i = 1, 2,
\]
\[
I_3 = \left( \int \int_{Q_T} u^{\alpha}\xi^2dxdt \right)^{1/p_3} F_3 \quad \text{with} \quad F_3 = \left( \int \int_{Q_T} \varphi_i^\alpha\xi^2dxdt \right)^{1/q_3},
\]
\[
I_4 = \left( \int \int_{Q_T} u^{\alpha}\xi^2dxdt \right)^{1/p_4} F_4 \quad \text{with} \quad F_4 = \left( \int \int_{Q_T} \varphi_i^\alpha\xi^2dxdt \right)^{1/q_4},
\]

with
\[
\alpha_1 = p_1(\alpha - \lambda) > p_1 > 1, \quad \alpha_2 = p_2(\alpha - \lambda + \ell_3) > p_2(1 + \ell_3) > 1,
\]
\[
\alpha_3 = p_3(\alpha - \lambda) > p_3 > 1, \quad \alpha_4 = p_4(\alpha - \lambda + \ell_4) > p_4(1 + \ell_4) > 1.
\]

For \(I_1\), using the trace theorem (2.5), we have
\[
\int \int_{Q_T} \varphi_i^\alpha\xi^2dxdt \leq c_1 \int \int_{Q_T} u^{\alpha}\xi^2dxdt + c_2\alpha_1 \int \int_{Q_T} u^{\alpha-1}|\nabla u|\xi^2dxdt.
\]

In the last integral, writing \(u^{\alpha-1}|\nabla u|\) as a product of \(u^{\frac{2-\lambda-1}{\alpha}}|\nabla u|\) and \(u^{\alpha-\frac{2-\alpha+\delta}{1-a}}\), and applying Hölder’s inequality with powers \(2 - a\) and \(2 - a)/(1 - a)\), we obtain
\[
\int \int_{Q_T} u^{\alpha}\xi^2dxdt \leq c_1 \int \int_{Q_T} u^{\alpha}\xi^2dxdt + c_2\alpha_1 \left( \int \int_{Q_T} u^{\alpha-\lambda-1}|\nabla u|^{2-a}\xi^2dxdt \right)^{\frac{1}{1-a}}
\times \left( \int \int_{Q_T} u^{\alpha-\lambda}|\nabla u|^{2-\alpha}\xi^2dxdt \right)^{\frac{1}{1-a}}, \quad (4.7)
\]
where
\[
m_1 = \frac{(2 - a)\alpha_1 - \alpha + a - \delta}{1-a} = \frac{((2 - a)p_1 - 1)\alpha - p_1(2 - a)\lambda + a - \delta}{1-a} > 0.
\]

The last inequality comes from the fact \(\alpha > \eta_1\) in condition (4.2).

Denote \(X(\alpha) = \int \int_{Q_T} u^\alpha(x, t)\xi(t)dxdt\). We will use the fact \(0 \leq \xi^2 \leq \xi \leq 1\). By (4.7) and Young’s inequality
\[
I_1 \leq c_1^{1/p_1} X(\alpha_1)^{1/p_1} F_1 + (c_2\alpha_1)^{1/p_1} \left( \int \int_{Q_T} u^{\alpha-\lambda-1}|\nabla u|^{2-a}dxdt \right)^{\frac{1}{m_1}} X(m_1)^{\frac{1}{m_1}} F_1
\]
\[
\leq c_1^{1/p_1} X(\alpha_1)^{1/p_1} F_1 + \varepsilon \int \int_{Q_T} u^{\alpha-\lambda-1}|\nabla u|^{2-a}dxdt + \varepsilon^{-1}(c_2\alpha_1)^{2-a} \left( \int \int_{Q_T} u^{\alpha-\lambda}|\nabla u|^{2-\alpha}dxdt \right)^{\frac{1}{m_1}}
\times X(m_1)^{\frac{1}{m_1}} F_1^{\frac{1-a}{(2-a)m_1}}, \quad (4.8)
\]
Similarly,
\[ I_2 \leq c_3 X(\alpha_2)^{1/p} F_2 + \varepsilon \int_{Q_T} u^{\alpha - \lambda - 1} |\nabla u|^2 \, dx \, dt + \left[ \varepsilon^{-1} (c_4 \alpha_2)^{2 - \alpha} \right]^{1/(2 - \alpha)} \]
\[ \times X(m_2)^{1/(2 - \alpha)} F_2^{p/(2 - \alpha)}, \]
(4.9)
where
\[ m_2 = \frac{(2 - a) \alpha_2 - \alpha + a - \delta}{1 - a} = \frac{(2 - a) p_2 - 1 + p_2 (2 - a) (\ell_2 - \lambda) + a - \delta}{1 - a} > 0. \]

We now choose \( \xi(t) \) such that
\[ \xi(t) = 0 \quad \text{for} \quad 0 \leq t \leq T_1, \quad \xi(t) = 1 \quad \text{for} \quad T_2 \leq t \leq T, \quad \text{and} \quad 0 \leq \xi'(t) \leq \frac{2}{T_2 - T_1} \quad \text{for} \quad 0 \leq t \leq T. \]
(4.10)

Then
\[ \frac{\lambda}{\alpha} \int_{Q_T} 2 u^\alpha \xi' \, dx \, dt \leq \frac{C}{T_2 - T_1} X(\alpha). \]
(4.11)

Therefore, with \( \varepsilon = (\alpha - \lambda) d_3 / 16 \), it follows (4.6) and the above estimates (4.8), (4.9) and (4.11) that
\[ \frac{\lambda}{\alpha} \sup_{t \in [0, T]} \int_{U} u^\alpha (x, t) \xi^2(t) \, dx + \frac{(\alpha - \lambda) d_3}{8} \int_{U} \int_{Q_T} |\nabla u|^{2 - a} u^{\alpha - \lambda - 1} \xi^2 \, dx \, dt \leq C \alpha (X(\mu_1) + X(\mu_1)) + \frac{C}{T_2 - T_1} \sum_{j=1}^{4} X(\alpha_1)^{1/p} F_1 + \frac{C}{T_2 - T_1} \sum_{j=1}^{2} \left( \frac{\alpha_2^{2 - a}}{\alpha - \lambda} \right)^{\frac{1}{p/(2 - \alpha)}} X(m_j)^{1/(2 - \alpha)} F_2^{p/(2 - \alpha)}. \]
(4.12)

We bound the left-hand side of (4.12) from below by
\[ I_0 \overset{\text{def}}{=} \frac{\lambda}{\alpha} \sup_{t \in [0, T]} \int_{U} u^\alpha (x, t) \xi^2(t) \, dx + \frac{d_3}{8} \int_{Q_T} |\nabla u(x, t)|^{2 - a} u(x, t)^{\alpha - \lambda - 1} \xi^2(t) \, dx \, dt. \]

We now bound the right-hand side of (4.12). We will estimate many \( X(\cdot) \)-terms by using
\[ J_0 \overset{\text{def}}{=} ||u||_{L^{\infty}(U \times (T_1, T))}. \]

Claim. Under condition (4.2), one has
\[ 0 < \mu_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, m_1, m_2 \leq \kappa. \]
(4.13)

We accept this claim at the moment. By Hölder’s inequality if \( 0 < \beta \leq \kappa \alpha \) then
\[ X(\beta) \leq \int_{T_1}^{T} \int_{U} u^\beta \, dx \, dt \leq J_0^\beta |Q_T|^{1 - \frac{\beta}{\alpha}} \leq J_0^\beta (1 + |Q_T|). \]
(4.14)

Note for \( j = 1, 2 \) that
\[ \frac{1 - a}{p_j (2 - a) - 1} < 1 \quad \text{and} \quad \frac{p_j (2 - a)}{p_j (2 - a) - 1} < \frac{2 - a}{1 - a}. \]
(4.15)
Using also $\alpha - \lambda \geq \alpha/2$ and $\alpha_1, \alpha_2 \leq (p_1 + p_2)(1 + \ell_B)\alpha$, then for $j = 1, 2$, 
\[
\left(\frac{\alpha_2^{2-a}}{\alpha - \lambda}\right)^{1/(2-a-1)} \leq \left(2[(p_1 + p_2)(1 + \ell_B)]^{2-a} \alpha^{1-a}\right)^{1/(2-a-1)} \leq C_0.
\]

Thus, we have from (4.12) and (4.14) that
\[
I_0 \leq C(1 + |Q_T|) \left(\alpha F_0^{p_1} + \alpha F_0^{p_2} + \sum_{i=1}^{4} \frac{\alpha}{F_0^{p_i} F_i} + \alpha \sum_{j=1}^{2} \alpha_0^{\alpha_j} |F_j|_{s_j^{\alpha_j}}\right).
\]

Calculating the powers of $J_0$ gives
\[
\frac{\alpha_1}{p_1} = \frac{\alpha_2}{p_2} = \alpha - \lambda, \quad \frac{\alpha_4}{p_4} = \alpha - \lambda + \ell_f, \quad \frac{m_1(1-a)}{p_1(2-a)-1} = \alpha + \frac{p_1(2-a)\lambda + a - \delta}{p_1(2-a) - 1} = \alpha + z_1,
\]
\[
\frac{m_2(1-a)}{p_2(2-a)-1} = \alpha + \frac{p_2(2-a)(\ell_B - \lambda) + a - \delta}{p_2(2-a) - 1} = \alpha + z_2.
\]

Hence,
\[
I_0 \leq C(1 + |Q_T|) \alpha \left(1 + \frac{1}{T_2 - T_1}\right) F \cdot J,
\]

where
\[
F = 1 + F_1 + F_2 + \frac{p_1(2-a)}{p_1(2-a)-1} + \frac{p_2(2-a)}{p_2(2-a)-1} + F_3 + F_4,
\]
\[
J = J_0^{\alpha + \ell_f(2-a) - \lambda - 1} + J_0^{\lambda - 1} + J_0^{\alpha} + J_0^{\alpha + \ell_f - \lambda} + J_0^{\alpha + \ell_f - \lambda} + J_0^{\alpha - z_1} + J_0^{\alpha - z_2}.
\]

Using the second inequality in (4.15) and inequality (2.4), we simply estimate
\[
F \leq 5(1 + F_1^{\frac{\lambda - 1}{2}} + F_2^{\frac{\lambda - 1}{2}} + F_3 + F_4) \leq 5M_0.
\]

Also, by using (2.3),
\[
J \leq 8(J_0^{-h_1} + J_0^{h_1}).
\]

Therefore
\[
I_0 \leq C(1 + |Q_T|) \alpha \left(1 + \frac{1}{T_2 - T_1}\right) M_0(J_0^{-h_1} + J_0^{h_1}). \tag{4.16}
\]

By (4.16) and definition of $I_0$ we have
\[
\sup_{t \in [0,T]} \int_0^1 u^\alpha(x,t)\xi^2(t)dx + \int \int |\nabla u(x,t)|^{2-a} u(x,t)^{\alpha + \lambda - 1}\xi^2(t)dxdt \leq C(1 + |Q_T|) \alpha \left(1 + \frac{1}{T_2 - T_1}\right) M_0(J_0^{-h_1} + J_0^{h_1}).
\]
This estimate, together with property (4.10) of function $\xi(t)$, and the fact $1 + |Q_T| \leqslant (1 + |U|)(1 + T)$, implies (4.3).

Finally, we verify the claim (4.13). Dividing (4.13) by $\alpha$ gives an equivalent statement

$$
1 + \frac{\ell_2(2 - a) - \lambda - 1}{\alpha} \cdot \rho_1 - \frac{\lambda}{\alpha}, \quad p_1 + p_2 - \frac{\lambda}{\alpha}, \quad p_3 - \frac{\lambda}{\alpha}, \quad p_4 - \frac{\lambda}{\alpha}, \quad \frac{(2 - a)p_1 - 1}{1 - a} \cdot \frac{(2 - a)p_2 - 1}{1 - a} \leqslant \frac{\ell_2^0}{\alpha(1 - a)} - a - \delta
$$

Expanding, we have

$$
\frac{(2 - a)p_1 - 1}{1 - a} \cdot \frac{(2 - a)p_2 - 1}{1 - a} \leqslant \frac{\ell_2^0}{\alpha(1 - a)} - \frac{\delta}{\alpha(1 - a)} \cdot \frac{(2 - a)p_1 \cdot (2 - a)p_2 - 1}{1 - a} \leqslant \frac{\ell_2^0}{\alpha(1 - a)} - a - \delta.
$$

Note for $i = 1, 2$ that $p_i \leqslant \frac{(2 - a)p - 1}{1 - a} - 1$. Then by definition (4.1) of $\kappa*$,

$$
1, p_1, p_2, p_3, p_4 \leqslant \frac{(2 - a)p - 1}{1 - a}, \quad \frac{(2 - a)p - 1}{1 - a} \leqslant \kappa*.
$$

Hence, we can bound the left-hand side of (4.17) by

$$
\kappa* + \max \left\{ \frac{\ell_2(2 - a) - \lambda - 1}{\alpha} \cdot \rho_1 - \frac{\lambda}{\alpha}, \frac{\ell_2^0}{\alpha}, \frac{(2 - a)p_1 \cdot (2 - a)p_2 - 1}{1 - a} \cdot \frac{\ell_2^0}{\alpha(1 - a)} \cdot \frac{(2 - a)p_1 \cdot (2 - a)p_2 - 1}{1 - a} \right\} \leqslant \kappa* + \frac{\eta_1}{\alpha}.
$$

Since $\kappa > \kappa*$, we choose a sufficient condition for (4.17) to be $\eta_1/\alpha \leqslant \kappa - \kappa*$, which is satisfied by (4.2). The proof is complete. \qed

**Lemma 4.2.** Let $\kappa$, $\nu$, $\nu_5$ and $\mathcal{M}_0$ be as in lemma 4.1. If $T > T_2 > T_1 \geqslant 0$ then

$$
\|u\|^\nu_{\mathcal{L}^{\nu}}(U \times (T_1, T)) \leqslant A_{\alpha}^{\frac{1}{\nu}} \left( \|u\|^\nu_{\mathcal{L}^{\nu}}(U \times (T_1, T)) + \|u\|^\nu_{\mathcal{L}^{\nu}}(U \times (T_1, T)) \right)^{\frac{1}{\nu}},
$$

where $\nu$ is defined by (2.16),

$$
\nu = \nu_5(\alpha) \equiv \frac{\alpha_5 \nu}{\alpha + \delta - a},
$$

and

$$
A_\alpha = c_7(1 + T)^2 \left( 1 + \frac{1}{T_2 - T_1} \right)^2 \alpha^\delta a \mathcal{M}_0^2
$$

with $c_7 \geqslant 1$ independent of $\alpha$, $\kappa$, $T$, $T_1$ and $T_2$.

**Proof.** We follow the proof of proposition 6.2 in [6]. We use Sobolev inequality (2.15) in lemma 2.3:

$$
\|u\|^\nu_{\mathcal{L}^{\nu}}(U \times (T_1, T)) \leqslant \hat{C}^{\frac{\alpha^\delta}{\alpha}} \left\{ \tilde{I}^{\frac{\hat{\theta} - \gamma}{\alpha}} \cdot \sup_{r \in [T_1, T]} \left( \int_U |u(x, t)|^{\alpha} dx \right)^{\frac{1 - \tilde{\theta}}{\alpha}} \right\}^{\frac{1}{\nu}},
$$

where $\hat{C} = c_3 \alpha^\delta a$ with $c_3 \geqslant 1$, the numbers $\hat{\theta}$ and $\kappa$ are defined in (2.16) and

$$
I = \left[ \int_{T_2}^T \int_U |u|^{\alpha + \delta - a} dx dr + \int_{T_2}^T \int_U |u|^{\alpha - \lambda - 1} \|\nabla u\|^{2 - a} dx dr \right]^{\frac{\alpha}{\alpha - \lambda}}.
$$
Note that
\[
\frac{\alpha}{\alpha + \delta - a} \leq 2 \quad \text{and} \quad 2^{\frac{\alpha}{\alpha + \delta - a}} - 1 \leq 2. \tag{4.23}
\]

Then applying inequality (2.2), we find that
\[
I \leq 2 \left( \int_{T_2}^{T} \int_{U} |u|^{\alpha + \delta - a} \, dx \, dt \right)^{\frac{\alpha}{\alpha + \delta - a}} + 2 \left( \int_{T_2}^{T} \int_{U} |u|^{\alpha - \lambda - 1} |\nabla u|^{2 - a} \, dx \, dt \right)^{\frac{\alpha}{2 - a}}. \tag{4.24}
\]

Applying Hölder’s inequality to the first integral on the right-hand side of (4.24) with conjugate exponents \(\frac{\alpha}{\alpha + \delta - a}\) and \(\frac{\alpha}{\alpha - \lambda - 1}\), we get
\[
I \leq 2|Q_r|^{\left(1 - \frac{\alpha}{\alpha + \delta - a}\right)} \left( \int_{T_2}^{T} \int_{U} |u|^{\alpha} \, dx \, dt \right)^{\frac{\alpha}{\alpha}} + 2 \left( \int_{T_2}^{T} \int_{U} |u|^{\alpha - \lambda - 1} |\nabla u|^{2 - a} \, dx \, dt \right)^{\frac{\alpha}{2 - a}}. \tag{4.25}
\]

Next, we use (4.3) to estimate the second term on the right-hand side of (4.25). We denote
\[
S = \|u\|_{L^{\alpha}(U \times (T_1, T))} \quad \text{and} \quad \mathcal{M} = c_6 (1 + T) \left( 1 + \frac{1}{T_2 - T_1} \right) \alpha^2 \mathcal{M}_0. \tag{4.26}
\]

where \(c_6\) is the positive constant in (4.3). Then combining (4.22) and (4.25) yields
\[
\|u\|_{L^{\alpha}(U \times (T_1, T))} \leq \hat{C} \frac{\alpha}{\alpha - \lambda} \left\{ \left( 2(1 + |Q_r|)^{\frac{\alpha}{\alpha + \delta - a}} S^{\alpha} + 2(\mathcal{M}(S^{\alpha} + S^{\alpha}))(S^{\alpha} + S^{\alpha}) \right)^{\frac{\alpha}{\alpha + \delta - a}} \right\}^{\frac{1}{2}} \leq \hat{C} \frac{\alpha}{\alpha - \lambda} \left( 2(1 + |Q_r|)^{\frac{\alpha}{\alpha + \delta - a}} S^{\alpha} + 2(\mathcal{M}(S^{\alpha} + S^{\alpha}))(S^{\alpha} + S^{\alpha}) \right)^{\frac{1}{2}}.
\]

Since \(\nu_5 < \alpha < \nu_6\), we use (2.3) to estimate \(S^{\alpha} \leq S^{\nu_5} + S^{\nu_6}\). Also, by (2.2) and (4.23),
\[
(S^{\nu_5} + S^{\nu_6})^{\frac{\alpha}{\alpha - \lambda}} \leq 2(S^{\frac{\nu_5}{\nu_5 + \nu_6}} + S^{\frac{\nu_6}{\nu_5 + \nu_6}}).
\]

Thus, we have
\[
\|u\|_{L^{\alpha}(U \times (T_1, T))} \leq \hat{C} \frac{\alpha}{\alpha - \lambda} \left( 2(1 + |Q_r|)^{\frac{\alpha}{\alpha + \delta - a}} S^{\alpha} + 4\mathcal{M}^{\frac{\alpha}{\alpha - \lambda}} (S^{\frac{\nu_5}{\nu_5 + \nu_6}} + S^{\frac{\nu_6}{\nu_5 + \nu_6}}) \right)^{\frac{1}{2}}.
\]

Using \(S^{\nu_5}, S^{\frac{\nu_5}{\nu_5 + \nu_6}} \leq S^{\nu_5} + S^{\frac{\nu_5}{\nu_5 + \nu_6}}\), we obtain
\[
\|u\|_{L^{\alpha}(U \times (T_1, T))} \leq \hat{C} \frac{\alpha}{\alpha - \lambda} \left( 8 \left( 1 + |Q_r| \right)^{\frac{\alpha}{\alpha + \delta - a}} + \mathcal{M}^{\frac{\alpha}{\alpha - \lambda}} (S^{\nu_5} + S^{\frac{\nu_5}{\nu_5 + \nu_6}}) \right)^{\frac{1}{2}}. \tag{4.27}
\]

Note also that \(\hat{C}^{1/2} \leq \hat{C}\), then we obtain from (4.27) that
\[
\|u\|_{L^{\alpha}(U \times (T_1, T))} \leq \mathcal{M}^{\frac{1}{2}} \left( \|u\|_{L^{\alpha}(U \times (T_1, T))}^{\nu_5} + \|u\|_{L^{\alpha}(U \times (T_1, T))}^{\frac{\nu_5}{\nu_5 + \nu_6}} \right)^{\frac{1}{2}}, \tag{4.28}
\]
where
\[ M_1 = 8c_5\alpha^{2-a}[(1 + |Q_T|)^2 + M + M^{\frac{a^2}{2}}]. \] (4.29)

We estimate \( M_1 \). Note that \( M_0, M, M_1 \geq 1 \). Recall from (4.23) that the last power of \( M \) in (4.29) is less than or equal to 2. Then
\[ M_1 \leq 8c_5\alpha^{2-a} \left( (1 + |U|)^2 (1 + T)^2 + 2M^2 \right) \]
\[ \leq 8c_5\alpha^{2-a} \left( (1 + |U|)^2 (1 + T)^2 + 2\left(c_6\alpha^2 \left(1 + \frac{1}{T_2 - T_1}\right)(1 + T)M_0\right)^2\right) \]
\[ \leq 8c_5(1 + |U|)^2 + 2c_6^2\alpha^{6-a} \left(1 + \frac{1}{T_2 - T_1}\right)^2 (1 + T)^2M_0^2. \] (4.30)

Hence we obtain (4.19) from (4.28) and (4.30).

Next, we apply Moser’s iteration. For that we recall the following lemma on finding the upper bounds of certain numeric sequences.

**Lemma 4.3 ([6], lemma A.2).** Let \( y_j \geq 0, \kappa_j > 0, s_j \geq r_j > 0 \) and \( \omega_j \geq 1 \) for all \( j \geq 0 \).

Suppose there is \( A \geq 1 \) such that
\[ y_j + 1 \leq A \omega_j \kappa_j (y_j r_j + y_j s_j) \] \( \kappa_j \forall j \geq 0. \)

Denote \( \beta_j = r_j / \kappa_j \) and \( \gamma_j = s_j / \kappa_j \). Assume
\[ \bar{\alpha} = \sum_{j=0}^{\infty} \frac{\omega_j}{\kappa_j} < \infty \] and the products \( \prod_{j=0}^{\infty} \beta_j \prod_{j=0}^{\infty} \gamma_j \) converge to positive numbers \( \bar{\beta}, \bar{\gamma} \), resp.

Then
\[ y_j \leq (2A)^{G_j} \max \left\{ \gamma_0^{\gamma_0 \ldots \gamma_{j-1}}, y_0^{\beta_0 \ldots \beta_{j-1}} \right\} \] \( \forall j \geq 1, \) (4.31)

where \( G_j = \max \{1, \gamma_m \gamma_{m+1} \ldots \gamma_n : 1 \leq m \leq n < j\} \). Consequently,
\[ \lim_{j \to \infty} \sup y_j \leq (2A)^G \max \{y_0, y_0^3\}, \] where \( G = \lim_{j \to \infty} G_j. \) (4.32)

For iterations below, we will require
\[ \kappa_2^2 < \kappa_f, \]
which is equivalent to
\[ p_1, p_2 < \frac{(1-a)\sqrt{\kappa_f} + 1}{2 - a} \] and \( p_3, p_4 < \sqrt{\kappa_f}. \) (4.33)

Therefore, we assume (4.33) for the remainder of this section. We now fix \( \tilde{\kappa} \) such that
\[ \kappa_* < \tilde{\kappa} < \sqrt{\kappa_f}. \] (4.34)

(For example, we can simply take \( \tilde{\kappa} = \sqrt{[\kappa_*^2 + \kappa_f]/2} \) or \( \tilde{\kappa} = (\kappa_* + \sqrt{\kappa_f})/2 \).)
Theorem 4.4. Let \( \alpha_0 \) be a positive number such that
\[
\alpha_0 > \max\{2, \alpha_*, \eta_1\} \quad \text{and} \quad \alpha_0 \geq \max\left\{ \frac{\eta_2}{K - \kappa_*}, \frac{a - \delta}{\kappa_0 - \kappa^2} \right\}.
\] (4.35)

Then there are positive constants \( C, \bar{\mu}, \bar{\nu}, \omega, \omega_2, \omega_3 \) such that if \( T > 0 \) and \( \sigma \in (0, 1) \) then
\[
\|u\|_{L^\infty(U \times \sigmaT, T)} \leq C\left( 1 + \frac{1}{T^2} \right)^\omega \left( 1 + T \right)^\omega \mathcal{M}_0^\omega \max\left\{ \|u\|_{L^\infty(U \times (0,T))}^A, \|u\|_{L^\infty(U \times (0,T))}^B \right\},
\] (4.36)
where \( \mathcal{M}_0 \) is defined by (4.4).

Proof. We set up to iterate inequality (4.19) in lemma 4.2. Note that \( \alpha = \alpha_0 \) satisfies (4.2). Define for \( j \geq 0 \),
\[
\beta_j = \bar{\kappa}^j \alpha_0.
\] (4.37)

Since \( \bar{\kappa} > 1 \), the sequence \( \{\beta_j\}_{j=0}^\infty \) is increasing, and hence, so is the sequence \( \{\kappa(\beta_j)\}_{j=0}^\infty \). We then have from the last fact and the choice of \( \alpha_0 \) that
\[
\kappa(\beta_j) \geq \kappa(\alpha_0) = \kappa_0 - \frac{a - \delta}{\alpha_0} \geq \bar{\kappa}^2.
\] (4.38)

For \( j \geq 0 \), let \( t_j = \sigma T(1 - \frac{1}{j}) \). For \( j \geq 0 \), applying (4.19) of lemma 4.2 with \( \alpha = \beta_j \), \( T_2 = t_{j+1} \) and \( T_1 = t_j \), we have
\[
\|u\|_{L^\kappa(\beta_j, \beta_j)(U \times (t_j, T_j))} \leq \lambda_{\beta_j}^\frac{1}{\kappa} \left( \|u\|_{L^{\kappa, \beta_j}(U \times (t_j, T_j))}^\kappa + \|u\|_{L^{\kappa, \beta_j}(U \times (t_j, T_j))}^\mu \right)^{\frac{1}{\kappa}}
\]
\[
= \lambda_{\beta_j}^\frac{1}{\kappa} \left( \|u\|_{L^{\kappa, \beta_j+1}(U \times (t_j, T_j))}^\kappa + \|u\|_{L^{\kappa, \beta_j+1}(U \times (t_j, T_j))}^\mu \right)^{\frac{1}{\kappa}},
\] (4.39)
where, see formulas (4.5) and (4.20),
\[
\tilde{r}_j = \nu \kappa(\beta_j) \quad \text{and} \quad \tilde{s}_j = \nu \alpha_0.
\] (4.40)

Note from (4.38) that
\[
\kappa(\beta_j) \beta_j \geq \bar{\kappa}^2 \beta_j = \beta_{j+2}.
\]

Define for \( j \geq 0 \) that \( Q_j = U \times (t_j, T) \) and \( Y_j = \|u\|_{L^{\kappa, \beta_j}(Q_j)} \). By Hölder’s inequality
\[
Y_{j+1} = \|u\|_{L^{\kappa, \beta_{j+2}}(Q_{j+1})} \leq \|u\|_{L^{\kappa, \beta_{j+2}}(Q_{j+1})} \leq (1 + |Q_0|)^\frac{1}{\kappa} \|u\|_{L^{\kappa, \beta_{j+2}}(Q_{j+1})}.
\] (4.41)

Combining inequalities (4.39) and (4.41) gives
\[
Y_{j+1} \leq \tilde{\lambda}_{\beta_j} \left( Y_j^\frac{1}{\kappa} + Y_j^\frac{1}{\mu} \right)^{\frac{1}{\kappa}} \forall j \geq 0, \quad \text{where} \quad \tilde{\lambda}_j = (1 + |Q_0|)A_{\beta_j}.
\] (4.42)

Now we estimate \( \tilde{\lambda}_j \), Note that \( |Q_0| = |Q_j| \leq (1 + |U|)(1 + T) \). From (4.21) and (4.42), we have
\[ \tilde{A}_j \leq c_7 (1 + |U|) \beta_j^{a-a} \left( 1 + \frac{2^{j+1}}{\sigma T} \right)^2 (1 + T)^3 M_0^2 \]
\[
\leq c_7 (1 + |U|) (\tilde{K}^{1/\alpha_0})^{a-a} 4^{j+1} \left( 1 + \frac{1}{\sigma T} \right)^2 (1 + T)^3 M_0^2 \leq A_{T, \sigma, \alpha_0}^{j+1},
\]

where
\[ A_{T, \sigma, \alpha_0} = \max \left\{ 4\tilde{K}^{1/\alpha_0}, 4c_7 (1 + |U|) \alpha_0^{a-a} (1 + \frac{1}{\sigma T})^2 (1 + T)^3 M_0^2 \right\} \geq 1. \quad (4.43) \]

Hence
\[ Y_{j+1} \leq A_{T, \sigma, \alpha_0}^{j+1} (Y_{j}^{\tilde{\beta}} + \tilde{Y}_{j}^{\tilde{\beta}})^{\tilde{\nu}.} \quad (4.44) \]

We have
\[ \sum_{j=0}^{\infty} j + 1 / \beta_j = \frac{1}{\alpha_0} \sum_{j=0}^{\infty} j + 1 / \tilde{K} < \infty. \]

Note that
\[ 1 \geq \tilde{r}_j / \beta_j = 1 - \frac{h_1}{\tilde{K}^{1/\alpha_0}} \quad \text{and} \quad 1 \leq \tilde{s}_j / \beta_j = \frac{\beta_j + h_2}{\beta_j + \delta - a} = 1 + \frac{h_2 + a - \delta}{\tilde{K}^{1/\alpha_0} + \delta - a}. \quad (4.45) \]

Then it is elementary to show that the products
\[ \tilde{\mu} = \prod_{j=0}^{\infty} \tilde{r}_j / \beta_j \quad \text{and} \quad \tilde{\nu} = \prod_{j=0}^{\infty} \tilde{s}_j / \beta_j \quad (4.46) \]
are positive numbers. By (4.44) and lemma 4.3, we obtain
\[ \limsup_{j \to \infty} Y_j \leq (2A_{T, \sigma, \alpha_0})^{\omega} \max \{ Y_0^{\tilde{\mu}}, \tilde{Y}_0^{\tilde{\nu}} \}, \quad (4.47) \]

where \( \omega = \mathcal{G} \sum_{j=0}^{\infty} \frac{j+1}{\beta_0} \) with \( \mathcal{G} = \prod_{k=1}^{\infty} (\tilde{s}_k / \beta_k) \in (0, \infty). \)

Note that \( Y_0 = \| u \|_{L^{\infty}([0, T])} \) and, by (4.43),
\[ (2A_{T, \sigma, \alpha_0})^{\omega} \leq C \left( 1 + \frac{1}{\sigma T} \right)^{\omega_1} (1 + T)^{\omega_2} M_0^{\omega_3}, \]
where \( \omega_1 = 2\omega, \omega_2 = 3\omega \) and \( \omega_3 = 2\omega. \) Then estimate (4.36) follows (4.47). \( \square \)

**Remark 4.5.** Estimate (4.36) can be rewritten as
\[ \| u \|_{L^\infty(U \times (\sigma T, T))} \leq C_{\sigma, T, \beta, \omega_1} \left( \| u \|_{L^{\infty}(U \times (0, T))} + \| u \|_{L^{\infty}(U \times (0, T))} \right). \]

The bound on the right-hand side is quasi-homogeneous in \( \| u \|_{L^{\infty}(U \times (0, T))} \), hence, relevant for both small and large values of \( \| u \|_{L^{\infty}(U \times (0, T))}. \) This is different from the commonly obtained estimates when a positive constant is added to the inequality’s right-hand side. This
Combining theorems 3.2 and 4.4 gives the following particular estimates of in terms of initial and boundary data.

**Theorem 4.6.** Let \( \alpha_0 \) satisfy

\[
\alpha_0 > \max \left\{ 2, \alpha_*, \eta, \frac{\eta_0}{\kappa} \right\} \quad \text{and} \quad \alpha_0 \geq \max \left\{ \frac{na}{(1 - a)\kappa}, \frac{\eta_2}{\kappa - \kappa_*}, \frac{a - \delta}{\kappa_* - \kappa^2} \right\}. \tag{4.48}
\]

Let \( \bar{\mu}, \bar{\nu}, \omega_1, \omega_2, \omega_3 \) be the same constants as in theorem 4.4, and denote \( \beta_1 = \bar{\kappa} \alpha_0 \).

(i) If \( T > 0 \) satisfies (3.38) for \( \alpha = \beta_1 \), then for \( 0 < \varepsilon < \min \{1, T\} \), one has

\[
\|u\|_{L^\infty(U \times (\varepsilon, T))} \leq C \varepsilon^{-\omega_1}(1 + T)^{\omega_2} M_{\alpha_0}^{\omega_3} \cdot \max \left\{ \left( \int_0^T \mathcal{V}(t) dt \right)^{\frac{\beta_1}{1 - \beta_1}}, \left( \int_0^T \mathcal{V}(t) dt \right)^{\frac{\beta_1}{1 + \beta_1}} \right\}, \tag{4.49}
\]

where \( M_0 \) is defined by (4.4), and for \( 0 \leq t \leq T \)

\[
\mathcal{V}(t) = \left( 1 + \int_U u_0(x)^{\beta_1} dx \right)^{-\nu_4(\beta_1)} = \frac{1}{C_1(\beta_1)} \int_0^t \mathcal{Y}(\tau) d\tau \quad \tag{4.50}
\]

with \( \nu_4(\beta_1) \) defined as in (3.39), and \( \mathcal{Y}(t) \) in (3.9).

(ii) If \( T > 0 \) satisfies (3.41) for \( \alpha = \beta_1 \), then for \( 0 < \varepsilon < \min \{1, T\} \), one has

\[
\|u\|_{L^\infty(U \times (\varepsilon, T))} \leq C_2 \varepsilon^{-\omega_1}(1 + T)^{\omega_2} (1 + \|u_0(x)\|_{L^\beta(U)})^{\bar{\nu}} M_{\alpha_0}^{\omega_3}. \tag{4.51}
\]

Above \( C \) is a positive constant independent of \( T \) and \( \varepsilon \).

**Proof.** Note that \( \alpha_0 \) satisfies (4.35), and \( \alpha = \beta_1 \) satisfies (3.10). The constant \( C > 0 \) in this proof is generic.

(i) Applying estimate (4.36) to \( \sigma T = \varepsilon \) we have

\[
\|u\|_{L^\infty(U \times (\varepsilon, T))} \leq C \varepsilon^{-\omega_1}(1 + T)^{\omega_2} M_{\alpha_0}^{\omega_3} \cdot \max \left\{ \left( \int_0^T \int_U |u|^{\beta_1} dx dt \right)^{\frac{\beta_1}{1 - \beta_1}}, \left( \int_0^T \int_U |u|^{\beta_1} dx dt \right)^{\frac{\beta_1}{1 + \beta_1}} \right\}. \tag{4.52}
\]

By (3.40), we have

\[
\int_U u^{\beta_1}(x,t) dx \leq \mathcal{V}(t). \quad \tag{4.53}
\]

Then (4.52) and (4.53) yield (4.49).

(ii) If \( T \) satisfies (3.41) with \( \alpha = \beta_1 \), then by (3.42), we have

\[
\int_U u^{\beta_1}(x,t) dx \leq 2 \left( 1 + \int_U u_0^{\beta_1}(x) dx \right) \quad \text{for all} \quad t \in [0, T]. \tag{4.54}
\]

Combining (4.52) with (4.54), and noticing that \( \bar{\mu} < \bar{\nu} \), we obtain
\[ \|u\|_{L^\infty(U \times (\varepsilon, T))] \leq C e^{-\omega_1 (1 + T)^{\omega_2} M_0^{\omega_3}} \times \max \left\{ \left( 2 \int_0^T \left( 1 + \int_U u_0^{\beta_1}(x) \, dx \right) \, dt \right)^{\frac{1}{\beta_1}}, \left( 2 \int_0^T \left( 1 + \int_U u_0^{\beta_2}(x) \, dx \right) \, dt \right)^{\frac{1}{\beta_2}} \right\} \leq C e^{-\omega_1 (1 + T)^{\omega_2} M_0^{\omega_3} \cdot (1 + T)^{\frac{1}{\beta_1}}} \left( 1 + \int_U u_0^{\beta_2}(x) \, dx \right)^{\frac{1}{\beta_2}}, \]

and estimate (4.51) follows.

5. Gradient estimates

In theorem 3.2 we have an estimate for \( \int_0^T \int_U |\nabla u(x,t)|^{2\alpha} u(x,t)\alpha^{-\lambda-1} \, dx \, dt \). Note that the double integral is in both spatial and time variables, and it is not yet a direct estimate for the gradient. In this section, we focus on estimating \( \int_U |\nabla u(x,t)|^{2\alpha} \, dx \) for \( t > 0 \).

We assume, for the remaining of the section, that \( u(x,t) \geq 0 \) on \( U \times [0, \infty) \) is a solution of (1.25), positive on \( U \times (0, \infty) \).

In connection with properties (1.26) and (1.27) in assumption (A1), it is natural to make the following assumptions.

**Assumption (A2).**

(i) The function \( Z(u) \) satisfies
\[ |Z'(u)| \leq d u^{d-1} \quad \forall u \in (0, \infty), \tag{5.1} \]
for some constant \( d > 0 \).

(ii) There are non-negative functions \( \varphi_3(x,t) \) and \( \varphi_4(x,t) \) defined on \( \Gamma \times (0, \infty) \) such that
\[ \left| \frac{\partial B(x,t,u)}{\partial t} \right| \leq \varphi_3(x,t) + \varphi_4(x,t) u^d \quad \forall (x,t,u) \in \Gamma \times [0, \infty) \times [0, \infty). \tag{5.2} \]

(iii) We also assume a slightly stronger version of (1.27) and (1.28), namely,
\[ |B(x,t,u)| \leq \varphi_1(x,t) + \varphi_2(x,t) u^d \quad \forall (x,t,u) \in \Gamma \times [0, \infty) \times [0, \infty), \tag{5.3} \]
\[ |f(x,t,u)| \leq f_1(x,t) + f_2(x,t) u^d \quad \forall (x,t,u) \in U \times [0, \infty) \times [0, \infty). \tag{5.4} \]

To deal with the boundary condition, we define for any \( x \in \Gamma, t \geq 0, u \geq 0 \)
\[ Q(x,t,u) = \int_0^t B(x,t,v) \, dv. \tag{5.5} \]

Then by assumptions (A1) and (A2), we have for any \( x \in \Gamma, t \geq 0, u \geq 0 \) that
\[ |Q(x,t,u)| \leq \varphi_1(x,t) u + \frac{1}{\ell_B + 1} \varphi_2(x,t) u^{d+1} \tag{5.6} \]
and
\[ \left| \frac{\partial Q(x,t,u)}{\partial t} \right| \leq \varphi_3(x,t) u + \frac{1}{\ell_B + 1} \varphi_4(x,t) u^{d+1}. \tag{5.7} \]
Finally, we impose a technical condition on exponents $\ell_Z$ and $\lambda$.

**Assumption (A3).**

$$2\ell_Z > \lambda + 1. \quad (5.8)$$

**Remark 5.1.** For our original problem (1.23), $\ell_Z = 2\lambda$, then condition (5.8) becomes $\lambda > 1/3$. For slightly compressible fluids, $\lambda = 1$. For ideal gases, $\lambda = 1/2$. For isentropic gas flows, from the data in [44] or section III of [8], all values of the specific heat ratio $\gamma$ belong to the interval $(1, 2)$, therefore $\lambda = 1/(1 + \gamma)$, see (1.20), satisfies $1/3 < \lambda < 1/2$. Thus (5.8) is naturally met in all cases.

For the gradient estimates, same as in [2, 18], we will make use of the following function

$$H(\xi) = \int_{0}^{\xi} K(\sqrt{s}) ds \quad \text{for} \quad \xi \geq 0.$$

The function $H(\xi)$ satisfies

$$K(\xi) \xi^2 \leq H(\xi) \leq 2K(\xi) \xi^2, \quad (5.9)$$

and hence, as a consequence of (3.3) and (5.9),

$$d_3(\xi^{2-a} - 1) \leq H(\xi) \leq 2d_3 \xi^{2-a}. \quad (5.10)$$

Based on the structure of the PDE in (1.25), we define an intermediate quantity

$$I(t) = \int_{U} H\left( |\nabla u(x, t) + Z(u(x, t))| \right) dx \quad \text{for} \quad t \geq 0,$$

and, for initial values,

$$Z_0 = \int_{U} \phi_{0}^{\beta+1}(x) dx + Z(0) + \int_{\Gamma} \left( \varphi_1(x, 0) u_0(x) + \varphi_2(x, 0) u_0^{\beta+1}(x) \right) d\sigma.$$

First, we estimate $I(t)$ in terms of certain Lebesgue norms of $u(x, t)$.

**Proposition 5.2.** For $t > 0$,

$$I(t) \leq 2Z_0 + C \left\{ t + 1 + \int_{U} u^{\eta_1}(x, t) dx + \int_{0}^{t} \int_{U} u^{\eta_2}(x, \tau) dx d\tau + N_1(t) \right\} + \int_{0}^{t} N_2(\tau) d\tau,$$

where $C$ is a positive constant,

$$\eta_1 = \max \left\{ \frac{2-a}{1-a} \left( 2\ell_B + 1 \right), (2-a)\ell_Z \right\}, \quad \eta_2 = \max \left\{ \frac{(2-a)(2\ell_B + 1)}{1-a}, \frac{2\ell_Z(2-a)}{a}, (2\ell_f + 1) \right\},$$

and

$$N_1(t) = \int_{\Gamma} \left( \varphi_{1}^{\beta}(x, t) + \varphi_{2}^{\beta}(x, t) \right) d\sigma,$$

$$N_2(t) = N_1(t) + \int_{\Gamma} \left( \varphi_{2}^{\beta}(x, t) + \varphi_{2}^{\beta}(x, t) \right) d\sigma + \int_{U} \left( f_1^{\nu}(x, t) + f_1^{\nu}(x, t) \right) dx.$$
with
\[ \eta_5 = \frac{2(1 + \ell_B)}{1 + 2\ell_B}, \quad \eta_6 = \max \left\{ \frac{2(1 + \ell_f)}{1 + 2\ell_f}, \frac{4(2\ell_f + 1 - \lambda)}{4\ell_f + 1 - \lambda} \right\}. \]

**Proof.** The proof is divided into three steps below. As usual, constants \( C, C_\varepsilon \) are positive and generic.

**Step 1.** Multiplying the PDE in (1.25) by \( u \), integrating over \( U \) and using integration by parts, we have
\[
\frac{\lambda}{\lambda + 1} \frac{d}{dt} \int_U u^{\lambda+1} dx = \int_U \nabla \cdot (K(|\nabla u + Z(u)|)(\nabla u + Z(u))) u dx + \int_U f u dx
\]
\[
= - \int_U K(|\nabla u + Z(u)|)(\nabla u + Z(u)) \cdot \nabla u dx + \int_U B u dx + \int_U f u dx
\]
\[
= - \int_U K(|\nabla u + Z(u)|)|\nabla u + Z(u)|^2 dx
\]
\[
+ \int_U K(|\nabla u + Z(u)|)(\nabla u + Z(u)) \cdot Z(u) dx + \int_U B u dx + \int_U f u dx.
\]

On the right-hand side of the last equation, using relation (5.9) for the first integral, relations (3.2) and (1.26) for the second integral, (1.27) for the third, and (1.28) for the last, we have
\[
\frac{\lambda}{\lambda + 1} \frac{d}{dt} \int_U u^{\lambda+1} dx \leq - \frac{1}{2} \int_U H(|\nabla u + Z(u)|) dx + C \int_U |\nabla u + Z(u)|^{1-a} u^{\ell_f} dx
\]
\[
+ \int_U (\varphi_1 + \varphi_2 u^s) u dx + \int_U (f_1 + f_2 u^s) u dx
\]
\[
\overset{\text{def}}{=} - \frac{1}{2} \mathcal{I} + J_1 + J_2 + J_3. \quad (5.12)
\]

In the following, \( \varepsilon \in (0, 1) \).

- For \( J_1 \), using Young’s inequality with powers \( \frac{2-a}{1-a} \) and \( 2-a \), and then relation (5.10),
\[
J_1 \leq \varepsilon \mathcal{I} + C_\varepsilon + C_\varepsilon \int_U u^{(2-a)} dx. \quad (5.13)
\]

- For \( J_2 \), using Young’s inequality, we have
\[
J_2 \leq 2 \left\| u^{1+\ell_f} \right\|_{L^2(\Gamma)}^2 + N_1(t).
\]

Next, applying inequality (2.6) for \( \alpha = s = 2(\ell_B + 1), \ p = 2-a \) to the boundary integral \( \left\| u^{1+\ell_f} \right\|_{L^2(\Gamma)}^2 \), we have
\[
J_2 \leq \varepsilon \int_U |\nabla u|^{2-a} dx + C \left\| u \right\|_{L^{2(\ell_B+1)}}^{2(\ell_B+1)} + C_\varepsilon \left\| u \right\|_{L^{2s_1}}^{s_1} + N_1(t)
\]
with
\[
s_1 = 2(\ell_B + 1) + \frac{2\ell_B + a}{1-a} = \frac{(2-a)(2\ell_B + 1)}{1-a}.
\]

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By (2.2) and then relations (1.26) and (5.10),
\[ \int_U |\nabla u|^{2-a} \, dx \leq 2 \int_U |\nabla u + Z(u)|^{2-a} \, dx + 2 \int_U |Z(u)|^{2-a} \, dx \]
\[ \leq \frac{2}{d^3} \mathcal{I} + C + C \int_U u^{(2-a)\ell_2} \, dx. \]

Thus
\[ J_2 \leq \frac{2\varepsilon}{d^3} \mathcal{I} + C + C \int_U u^{(2-a)\ell_2} \, dx + C \int_U u^{2(\ell_2+1)} \, dx + C_\varepsilon \int_U u^4 \, dx + N_1(t). \]

Since \((2-a)\ell_2, 2(\ell_B + 1), s_1 \leq \eta_3\), by using Young’s inequality, we have
\[ J_2 \leq \frac{2\varepsilon}{d^3} \mathcal{I} + C + C \int_U u^{\eta_1} \, dx + N_1(t). \quad (5.14) \]

• For \(J_3\), using Young’s inequality
\[ J_3 \leq 2 \int_U u^{2(\ell_2+1)} \, dx + \int_U \left( f_1^{2(\ell_2+1)} + f_2^3 \right) \, dx. \quad (5.15) \]

Combining (5.12)–(5.15) gives
\[ \frac{\lambda}{\lambda + 1} \frac{d}{dt} \int_U u^{\lambda+1} \, dx + \left( \frac{1}{2} - \varepsilon - \frac{2\varepsilon}{d^3} \right) \mathcal{I} \leq C \left( 1 + \int_U (u^{(2-a)} + u^{2(\ell_2+1)}) \, dx \right) + N_3(t), \]

where
\[ N_3(t) = N_1(t) + \int_U \left( f_1^{2(\ell_2+1)} (x,t) + f_2^3 (x,t) \right) \, dx. \]

Taking \(\varepsilon\) sufficiently small and using Young’s inequality show that
\[ \frac{d}{dt} \int_U u^{\lambda+1} \, dx + \mathcal{I} \leq C + C \int_U u^s \, dx + CN_3(t), \quad (5.16) \]

where
\[ s_2 = \max \{2(\ell_2 + 1), \eta_3\}. \]

**Step 2.** Multiplying both sides of the PDE in (1.25) by \(u_t\), integrating over \(U\) and using the boundary condition, one has
\[ \lambda \int_U u^{\lambda-1} u_t^2 \, dx + \int_U K(|\nabla u + Z(u)|)(\nabla u + Z(u)) \cdot \nabla u_t \, dx = \int_{\Gamma} Bu_t \, d\sigma + \int_U f_t \, dx. \]

Then by definition of \(H(\xi)\), we have
\[ \lambda \int_U u^{\lambda-1} u_t^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_U H(|\nabla u + Z(u)|) \, dx \]
\[ = \lambda \int_U u^{\lambda-1} u_t^2 \, dx + \int_U K(|\nabla u + Z(u)|)(\nabla u + Z(u)) \cdot (\nabla u + Z(u)) \, dx \]
\[ = \int_U K(|\nabla u + Z(u)|)(\nabla u + Z(u)) \cdot (Z(u)) \, dx + \int_{\Gamma} Bu_t \, d\sigma + \int_U f_t \, dx. \]
Estimating $K(|\nabla u + Z(u)|)$ by (3.2), we obtain
\[
\begin{align*}
\lambda \int_U u^{\lambda - 1} u_t^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_U H(|\nabla u + Z(u)|) \, dx & \leq C \int_U |\nabla u + Z(u)|^{1-\alpha} |\nabla u| \, dx \\
+ \int_G B u_t \, d\sigma + \int_U f u \, dx.
\end{align*}
\tag{5.17}
\]

- We bound the first term of the right hand side in (5.17) by (5.1) and then Young’s inequality,
\[
C \int_U |\nabla u + Z(u)|^{1-\alpha} |\nabla u| \, dx \leq C \int_U |\nabla u + Z(u)|^{1-\alpha} u^2 \, dx
\leq \varepsilon \int_U |\nabla u + Z(u)|^2 \, dx + C_\varepsilon \int_U u^{(\lambda - 1)\eta (\xi - 1)} |u_t|^{2-\alpha} \, dx.
\]

By (5.10) and then using Young’s inequality one more time for the last term, we have
\[
C \int_U |\nabla u + Z(u)|^{1-\alpha} |\nabla u| \, dx \leq \frac{\varepsilon}{d_3} I + C_\varepsilon \int_U u^{\lambda - 1} u_t^2 \, dx + C_\varepsilon \int_U u^{\lambda - 1} u_t^2 \, dx.
\tag{5.18}
\]

- For the middle term on the right-hand side of (5.17), using definition (5.5) we have
\[
B(x, t, u(x, t)) u_t(x, t) = \frac{d}{dt} Q(x, t, u(x, t)) - \frac{\partial Q}{\partial t}(x, t, u(x, t)).
\]

Thus (5.2) yields
\[
\int_G B u_t \, d\sigma = \frac{d}{dt} \int_G Q(x, t, u) \, d\sigma + \frac{\partial Q}{\partial t}(x, t, u) \, d\sigma \leq \frac{d}{dt} \int_G Q(x, t, u) \, d\sigma
+ C \int_G (\varphi_3 u + \varphi_4 u^{4+1}) \, d\sigma.
\tag{5.19}
\]

For the last term of (5.17), similar to estimate (5.14) we have
\[
C \int_G (\varphi_3 u + \varphi_4 u^{4+1}) \, d\sigma \leq \frac{2\varepsilon}{d_3} I + C_\varepsilon + C_\varepsilon \int_U u^2 \, dx + C \int_G (\varphi_3^2 + \varphi_4^2) \, d\sigma.
\tag{5.20}
\]

- For the last term of (5.17), using Cauchy’s inequality, (5.4) and Young’s inequality, we obtain
\[
\int_U f u \, dx \leq \frac{\lambda}{4} \int_U u^{\lambda - 1} u_t^2 \, dx + C \int_U f^2 u^\lambda \, dx
\leq \frac{\lambda}{4} \int_U u^{\lambda - 1} u_t^2 \, dx + C \int_U (f_1^2 + f_2^2 u^{2\xi}) u^\lambda \, dx
\leq \frac{\lambda}{4} \int_U u^{\lambda - 1} u_t^2 \, dx + C \int_U u^{2\xi + 1 - \lambda} \, dx + C \int_U \left( \frac{4(2\xi + 1 - \lambda)}{2\xi + 1 - \lambda} f_1^2 + f_2^2 \right) \, dx.
\tag{5.21}
\]

Then combining (5.17)–(5.21), we obtain
\[
\frac{\lambda}{2} \int_U u^{\lambda - 1} u_t^2 \, dx + \frac{1}{2} \frac{d}{dt} I - \frac{d}{dt} \int_G Q(x, t, u) \, d\sigma
\leq \frac{3\varepsilon}{d_3} I + C_\varepsilon + C_\varepsilon \int_U u^{\frac{\lambda - 1}{2\xi + 1 - \lambda}} + u^{2\xi + 1 - \lambda}) \, dx + C N_4(t).
\tag{5.22}
\]
where

\[ N_4(t) = \int_{\Gamma} \left( \varphi_1^T(x, t) + \varphi_2^T(x, t) \right) \, d\sigma + \int_{U} \left( \int_{\Gamma} \frac{4(2\ell_f+1-\lambda)}{4\eta_f+1-\lambda} (x, t) + f_2^T(x, t) \right) \, dx. \]  

(5.23)

**Step 3.** Let \( t \in (0, T) \). Note by assumption (5.8) that the power \((2-a)(2\ell_f - \lambda - 1)/a\) in (5.22) is positive. Summing (5.16) and (5.22) with sufficiently small \( \varepsilon \), and using Young’s inequalities, we have

\[
\frac{d}{dt} \left( \int_{U} u^{\lambda+1} \, dx + \frac{1}{2} \mathcal{I} - \int_{\Gamma} Q(x, t, u) \, d\sigma \right) \leq C + C \int_{U} u^{\eta} \, dx + C(N_3(t) + N_4(t))
\]  

(5.24)

with

\[
s_3 = \max \left\{ s_2, \frac{(2-a)(2\ell_f - \lambda - 1)}{a}, 2(2\ell_f + 1 - \lambda) \right\}.
\]

For simplicity, we bound \( s_3 \leq \eta_f \), and \( u^{\eta} \leq 1 + u^{\eta_1} \). Also, the powers \( \frac{2(1+\varepsilon)}{1+2\varepsilon} \) and \( \frac{d(2\ell_f+1-\lambda)}{4\eta_f+1-\lambda} \) of \( f_1 \) in \( N_3(t) \) and \( N_4(t) \) are less than or equal to \( \eta_f \), then by Young’s inequality

\[ N_3(t) + N_4(t) \leq C(1 + N_2(t)). \]

Therefore, (5.24) yields

\[
\frac{d}{dt} \left( \int_{U} u^{\lambda+1} \, dx + \frac{1}{2} \mathcal{I} - \int_{\Gamma} Q(x, t, u) \, d\sigma \right) \leq C + C \int_{U} u^{\eta} \, dx + CN_2(t).
\]

(5.25)

Integrating (5.25) in time from 0 to \( t \) gives

\[
\int_{U} u^{\lambda+1}(x,t) \, dx + \frac{1}{2} \mathcal{I}(t) \leq \int_{U} u_0(\lambda+1) \, dx + \mathcal{I}(0) - \int_{\Gamma} Q(x, 0, u_0(x)) \, d\sigma + \int_{\Gamma} Q(x, t, u) \, d\sigma + \frac{1}{2} \mathcal{I}(t) \leq C + C \int_{0}^{t} \int_{U} u^{\eta} \, dx \, d\tau + C \int_{0}^{t} N_2(\tau) \, d\tau.
\]

(5.26)

We will neglect the first term on the left-hand side of (5.26). On the right-hand side, by (5.6) we have

\[
\int_{\Gamma} |Q(x, 0, u_0(x))| \, d\sigma \leq \int_{\Gamma} (\varphi_1(x, 0)u_0(x) + \varphi_2(x, 0)u_0 u^{\eta+1}(x)) \, d\sigma,
\]

while for \( t > 0 \), using also estimate (5.14) we obtain

\[
\int_{\Gamma} Q(x, t, u) \, d\sigma \leq \int_{\Gamma} (\varphi_1 u + \varphi_2 u^{\eta+1}) \, d\sigma \leq \frac{2\varepsilon}{d_3} \mathcal{I} + C \varepsilon + C \int_{U} u^{\eta} \, dx + CN_1(t).
\]

(5.27)

Again, choosing \( \varepsilon \) sufficiently small, we derive from (5.26) and (5.27) that

\[
\frac{1}{2} \mathcal{I}(t) \leq Z_0 + C(t + 1) + C \int_{0}^{t} \int_{U} u^{\eta} \, dx \, d\tau + C \int_{0}^{t} N_2(\tau) \, d\tau + C \int_{U} u^{\eta} \, dx + CN_1(t),
\]

and, hence, estimate (5.11) follows.

Finally, we combine the estimate in proposition 5.2 with those in theorem 3.2 to obtain bounds for the gradient.
Theorem 5.3. Let
\[ \eta_T = \max \left\{ \frac{na}{1 - a}, \eta_0 + 1, \eta_4 \right\} \text{ and } \eta_T = \nu_T(\eta_T) \text{ defined as in (3.39).} \]

(i) If \( T > 0 \) satisfies (3.38) for \( \alpha = \eta_T \), then for all \( t \in (0, T] \)
\[
\int_U |\nabla u(x, t)|^{2-a} dx \leq C \left( Z_0 + (t + 1)(1 + \mathcal{V}_u(t)) + N_1(t) + \int_0^t N_2(\tau) d\tau \right), \tag{5.28}
\]
where \( N_1(t), N_2(t) \) are defined in proposition 5.2, and
\[
\mathcal{V}_u(t) = \left\{ \left( 1 + \int_U u_0(x)^{\eta_T} dx \right)^{-\eta_T} - \frac{1}{C_1(\eta)} \int_0^t (1 + \Upsilon(\tau)) d\tau \right\}^{-\frac{1}{\eta_T}}
\]
with \( \Upsilon(t) \) being defined by (3.9).

(ii) Moreover, if \( T > 0 \) satisfies (3.41) for \( \alpha = \eta_T \) then for all \( t \in (0, T] \)
\[
\int_U |\nabla u(x, t)|^{2-a} dx \leq C \left\{ Z_0 + (t + 1) \left( 1 + \int_U u_0^\eta(x) dx \right) + N_1(t) + \int_0^t N_2(\tau) d\tau \right\}. \tag{5.29}
\]

Above, \( C \) is a positive constant.

Proof. Thanks to (2.2), (5.10) and (5.11), we have
\[
\int_U |\nabla u|^{2-a} dx \leq 2 \int_U (|\nabla u + Z(u)|^{2-a} + |Z(u)|^{2-a}) dx \leq CI(t) + C + \int_U u^{\eta T(2-a)} dx
\leq C \left\{ Z_0 + (t + 1) + \int_U (u^{\eta T} + u^{\eta T(2-a)}) dx + \int_0^t \int_U u(x, \tau)^{\eta T} dx d\tau + N_1(t) + \int_0^t N_2(\tau) d\tau \right\}.
\]
Using the fact \( \eta_T, \ell_T(2 - a), \eta_4 \leq \eta_T \) and (2.4) to bound \( u^{\eta T}, u^{\eta T(2-a)}, u^{\eta T} \leq 1 + u^{\eta T} \), we obtain
\[
\int_U |\nabla u|^{2-a} dx \leq C \left\{ Z_0 + (t + 1) + \int_U u(x, \tau)^{\eta T} dx + \int_0^t \int_U u(x, \tau)^{\eta T} dx d\tau + N_1(t) + \int_0^t N_2(\tau) d\tau \right\}. \tag{5.30}
\]

(i) Note that \( \alpha = \eta_T \) satisfies (3.10). Let \( T > 0 \) satisfy (3.38) for \( \alpha = \eta_T \). Now using (3.40) with \( \alpha = \eta_T \), we have
\[
\int_U u(x, t)^{\eta T} dx \leq \mathcal{V}_u(t). \tag{5.31}
\]
By (5.31) and the fact \( \mathcal{V}_u(t) \) is increasing in \( t \), we have
\[
\int_U |\nabla u|^{2-a} dx \leq C \left\{ Z_0 + (t + 1) + (t + 1) \mathcal{V}_u(t) + N_1(t) + \int_0^t N_2(\tau) d\tau \right\}, \tag{5.32}
\]
and, therefore, obtain (5.28).

(ii) Now, assume \( T > 0 \) satisfies (3.41) for \( \alpha = \eta_T \). By using (3.42) instead of (3.40), we can replace \( \mathcal{V}_u(t) \) with \( 2(1 + \int_U u_0^\eta(x) dx) \) in (5.31) and (5.32), and consequently, estimate (5.29) follows.
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