Percolation on interacting networks

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Most networks of interest do not live in isolation. Instead they form components of larger systems in which multiple networks with distinct topologies coexist and where elements distributed amongst different networks may interact directly. Here we develop a mathematical framework based on generating functions for analyzing a system of \( l \) interacting networks given the connectivity within and between networks. We derive exact expressions for the percolation threshold describing the onset of large-scale connectivity in the system of networks and each network individually. These general expressions apply to networks with arbitrary degree distributions and we explicitly evaluate them for \( l = 2 \) interacting networks with a few choices of degree distributions. We show that the percolation threshold in an individual network can be significantly lowered once “hidden” connections to other networks are considered. We show applications of the framework to two real-world systems involving communications networks and socio-technical congruence in software systems.

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In the past decade there has been a significant advance in understanding the structure and function of networks. Mathematical models of networks are now widely used to describe a broad range of complex systems, from spread of disease on networks of human contacts to interactions amongst proteins [1, 2, 3]. However, current methods deal almost exclusively with individual networks treated as isolated systems. In reality an individual network is often just one component in a much larger complex system; a system that can bring together multiple networks with distinct topologies and functions. For instance, a pathogen spreads on a network of human contacts abetted by global and regional transportation networks. Likewise, email and e-commerce networks rely on the Internet which in turn relies on the electric grid. In biological systems, activated genes give rise to proteins some of which go back to the genetic level and activate or inhibit other genes. Results obtained in the context of a single isolated network can change dramatically once interactions with other networks are incorporated.

Consider a system formed by two interacting networks, \( \alpha \) and \( \beta \). Fig 1(a) Network \( \alpha \) could be a human contact network for one geographic region and network \( \beta \) that for a separated region. When viewed as individual systems, only small clusters of connected nodes exist, hence, a disease spreading in either network should stay contained within clusters. In reality, a disease can hop from \( \alpha \) to \( \beta \), for instance, by an infected person flying on a airplane, spread in the \( \beta \) network and eventually hop back to the \( \alpha \) network into new clusters, causing an epidemic outbreak. Next consider interacting networks that contain completely different types of nodes. Network \( \alpha \) can be a social network, such as an email communication network of software developers, while network \( \beta \) can be a technological network, such as the network of calls between functions in software code. Here, bi-partite edges connect developers on \( \alpha \) to code they author on \( \beta \).

An important step towards modeling interacting networks was introduced with the layered network framework of [4]. Yet, the networks in the distinct layers must be composed of the identical nodes (modeling essentially physical connectivity and logical connectivity or flow). Herein we consider systems of \( l \geq 2 \) distinct interacting networks and calculate explicitly how the connectivity within and between networks determines the onset of large scale connectivity in the system and in each network individually. Our mathematical formulation has some overlap with recent works calculating connectivity properties in a single network accounting for a diversity of node attributes [5, 6] or interactions between modules within a network [7, 8]. Here we present our formalism and also applications to real-world systems of interacting networks coming from telecommunications and software.

The onset of large-scale connectivity (i.e., the percolation threshold) corresponding to the emergence of a giant connected component in an isolated network has been

![FIG. 1: a) Two networks \( \alpha \) and \( \beta \). Nodes interact directly with other nodes in their immediate network, yet also with nodes in the second network. b) An illustration of the remaining edges incident to a node in a network \( \mu \) reached by following a random edge between networks \( \nu \) and \( \mu \).](image-url)
studied extensively, first for random networks with Poisson degree distributions [9] and later for random networks with arbitrary degree distributions [10]. Similar results were then derived using generating functions [11, 12], the approach we employ herein. Generating functions, similar to the network configuration model [10, 13], evaluate the ensemble of all possible random networks consistent with a specified degree distribution, \( \{ p_k \} \), and are most accurate in the sparse regime where networks are approximately tree-like. Thus in the regime before the emergence of the giant component, generating functions can be used to calculate the distribution of component sizes. In the supercritical regime they can be used to calculate the distribution in sizes of components that are not part of the giant component.

For our purposes, a system with \( t \geq 2 \) interacting networks is described by a set of degree distributions. Each individual network \( \mu \) is characterized by a multi-degree distribution, \( \{ p_{k_1,k_2,...,k_t}^{\mu} \} \), where \( p_{k_1,k_2,...,k_t}^{\mu} \) is the fraction of all nodes in network \( \mu \) that have \( k_1 \) edges to nodes in network 1, \( k_2 \) edges to nodes in network 2, etc. The multi-degree distribution for each network may be written in the form of a generating function:

\[
G_\mu(x_1, ..., x_t) = \sum_{k_1, ..., k_t = 0}^{\infty} p_{k_1,...,k_t}^{\mu} x_1^{k_1} \cdots x_t^{k_t}. \tag{1}
\]

To simplify notation in what follows, we now define two \( l \)-tuple’s, \( x = (x_1, ..., x_l) \) and \( 1 = (1, 1, ..., 1) \).

Our interest is in calculating the distribution of component sizes, where a component is a set of nodes connected to one another either directly or indirectly by traversing a path along edges. Clearly such components can be composed of nodes distributed among the \( l \) different networks, and our formulation allows us to calculate the distribution of such system-wide components, yet also to refine the focus and calculate the contribution coming from nodes contained in only one of the \( l \) networks.

We begin by deriving the distribution of connectivity for a node at the end of a randomly chosen edge.

Consider selecting uniformly at random an edge falling between a node in network \( \nu \) and a node in network \( \mu \) (i.e., a \( \nu-\mu \) edge). The \( \mu \) node attached to the edge is \( k_\nu \) times more likely to have \( \nu \)-degree \( k_\nu \) than degree 1. We can also account for the remaining local connectivity, to nodes in other networks as shown in Fig. 1(b). In single isolated networks remaining connectivity is called the excess degree of a node [11]. Let \( p_{k_1,...,k_t}^{\mu} \) denote the probability of following a randomly chosen \( \nu-\mu \) edge to a node with excess degree as shown in Fig. 1(b) (which has total \( \nu \)-degree of \( k_\nu + 1 \)). Then \( p_{k_1,...,k_t}^{\mu} \propto (k_\nu + 1)p_{k_1,(k_\nu+1)\cdots,k_t}^{\mu} \) and the generating function for the distribution, \( \{ p_{k_1,...,k_t}^{\mu} \} \) is,

\[
G_{\mu\nu}(x) = \sum_{k_1, ..., k_t = 0}^{\infty} p_{k_1,...,k_t}^{\mu\nu} x_1^{k_1} \cdots x_t^{k_t} = \sum_{k_1, ..., k_t = 0}^{\infty} \sum_{j_1, ..., j_t = 0}^{\infty} (k_\nu + 1)p_{k_1,\cdots,(k_\nu+1)\cdots,j_t}^{\mu} x_1^{k_1} \cdots x_t^{k_t} = \left( \sum_{j_1, ..., j_t = 0}^{\infty} \frac{\partial}{\partial x_{\nu,j_1}} \right) \sum_{k_1, ..., k_t = 0}^{\infty} p_{k_1,\cdots,k_t}^{\mu} x_1^{k_1} \cdots x_t^{k_t} = G_{\mu\nu}(x) = \left( \frac{\partial}{\partial x_{\nu,j_1}} \right) G_{\mu}(x) \tag{2}
\]

where \( G_{\mu\nu}(x) \) denotes the first derivative of \( G_{\mu}(x) \) with respect to \( x_\nu \) and the denominator is a normalization constant so that \( G_{\mu\nu}(1) = 1 \). Also note that \( G_{\mu\nu}(1) = k_\nu \) is the average \( \nu \)-degree for a node in network \( \mu \).

The distribution of second nearest neighbors for that \( \mu \) node via the \( \nu \) layer is calculated by using Eq. 2 as the argument to Eq. 1 namely \( G_{\mu}(1,1, ..., G_{\nu}(x)) \) with the argument to Eq. 1 namely \( G_{\mu}(1,1, ..., G_{\nu}(x)) \) with \( x_1 = 1, x_2 \neq \mu, ..., x_l \). Comparing this distribution calculated via generating functions to that found in real-world interacting networks can reveal interesting statistical features. Returning to the software example, we have a network of email communication between developers, a network of relations between code, and bipartite edges connecting developers to the code they edit. We would expect that the real system does not resemble a random network, but instead reflects a structure conducive to project development. For instance, if two developers edit the same code we would like for them to directly communicate via email and thus be first neighbors. In a sparse random network these developers would typically be second neighbors, connected indirectly via the code they both edit.

We analyze the evolution of the Apache 2.0 Open Source Software project from mid-2000 thru 2004, with data aggregated over three month windows. From this we extracted the multi-degree distribution of the system for each time-shot, which we then plug into our generating functions to calculate the expected distribution of second neighbors found by following first a developer-to-code edge then a code-to-developer edge. We then compare this distribution to the real distribution of such devel-

FIG. 2: Comparison over time of the distribution of the number of developers connected indirectly via co-editing code in the Apache project with the distribution expected in a random network with the same multi-degree distribution. Vertical lines mark the first generally available release in 2002, and a significant deviation from random in 2003, when the communication network shrinks and the project seems to become more efficiently organized.
We recognize the form of this equation from Eq. 2 thus

$$H_{\mu\nu}(x) = x_\mu G_{\mu\nu} [H_{1\mu}(x), \ldots, H_{1\nu}(x)]. \quad (5)$$

We now consider starting from a randomly chosen $\mu$-node, rather than a random $\nu$-$\mu$ edge. A topology such as one from Fig. 3 exists a the end of each edge incident to the $\mu$-node. The generating function for the probability distribution of component sizes is,

$$H_{\mu}(x) = x_\mu G_{\mu} [H_{1\mu}(x), \ldots, H_{1\mu}(x)]. \quad (6)$$

While in theory it is possible to solve Eq. 5 for $H_{\mu}(x)$ and use that solution in Eq. 6 to solve for $H_{\mu}(x)$, in practice, even for the case of a single isolated network, as noted in 11, the equations are typically quite difficult to solve. Yet, Eq. 6 allows calculation of average component size. A component may include multiple node flavors, but we can distinguish between the average number of each type. For example, the average number of $\nu$-nodes in the component of a randomly chosen $\mu$-node is

$$\langle s_\mu \rangle_\nu = \left. \frac{\partial}{\partial x_\nu} H_{\mu}(x) \right|_{x=1} = \delta_{\mu\nu} G_{\mu} [H_{1\mu}(1), \ldots, H_{1\mu}(1)]$$

$$+ \sum_{\lambda=1}^I G_{\mu\lambda} [H_{1\mu}(1), \ldots, H_{1\mu}(1)] H_{\lambda\mu}(1)$$

$$= \delta_{\mu\nu} + \sum_{\lambda=1}^I G_{\mu\lambda}(1) H_{\lambda\mu}(1) \quad (7)$$

Intuitively Eq. 7 is reasonable because $H_{\lambda\mu}(1)$ represents the average number of $\nu$-nodes in the component found by following a $\mu$-$\lambda$ edge towards a $\lambda$-node, and the expected number of $\mu$-$\lambda$ edges incident to an initial $\mu$-node is $G_{\mu\lambda}(1)$ (recall, $G_{\mu\lambda}(1) = \Sigma_\mu$). The product of the two terms summed over all $\lambda$ networks produces the number of $\nu$-nodes in a component connected to a randomly chosen $\mu$-node, $\langle s_\mu \rangle_\nu$.

The preceding results regarding components hold in the sub-critical regime where no giant connected component exists. Once a giant component emerges, generating functions allow us to calculate properties of components not belonging to it. The giant component will span multiple networks and calculating its size requires accounting for the contribution from each network. Let $S_\mu$ be the fraction of $\mu$-nodes belonging to the giant component. The probability that a randomly chosen $\mu$-node is not part of the giant component must then satisfy the following equation,

$$1 - S_\mu = \sum_{k_1, \ldots, k_l=0}^{\infty} q_{k_1, \ldots, k_l} u_{k_1}^{i_1} \cdots u_{k_l}^{i_l} = G_{\mu}(u_{1\mu}, \ldots, u_{l\mu}), \quad (8)$$

where $u_{\nu\mu}$ is the probability that an $\mu$-$\nu$ edge is not part of the giant component. In addition, for all $\mu, \nu \in l$, $u_{\nu\mu}$ must satisfy,

$$u_{\nu\mu} = G_{\nu\mu}(u_{1\nu}, \ldots, u_{l\nu}), \quad (9)$$

FIG. 3: A diagrammatical representation of the topological constraints placed on the generating function $H_{\mu\nu}(x)$ for the distribution of sizes of components reachable by following a randomly chosen $\nu$-$\mu$ edge. The labels attached to each edge indicate type or flavor of the edge and summation notation indicates that we are summing over all possible flavors.
derived using the same self-consistency arguments that resulted in Eq. [5].

Though all the equations above hold for a system of \( l \geq 2 \) interacting networks, we now give a concrete example for \( l = 2 \), with the networks indexed as \( \alpha \) and \( \beta \). Consider first the simplest of systems, where the internal connectivity of \( \alpha \) and \( \beta \) each has a distinct Poisson degree distribution, and the inter-network connectivity is described by a third Poisson degree distribution, for instance, \( p_{\text{inter}}^{\beta} = \frac{\left(k_\alpha^{\beta}\right)^{\kappa_\beta} e^{-k_\alpha^{\beta}}/\kappa_\beta!}{\left(k_\beta^{\beta}\right)^{\kappa_\beta} e^{-k_\beta^{\beta}}/\kappa_\beta!} \).

(Recall \( \overline{k}_\mu \) denotes the average \( \nu \)-degree for a node in network \( \mu \).) Then, from Eq. [7]

\[
G_\alpha(x_\alpha, x_\beta) = e^{k_\alpha^{\beta}(x_\alpha - 1)} e^{k_\beta^{\beta}(x_\beta - 1)},
\]

\[
G_\beta(x_\alpha, x_\beta) = e^{k_\alpha^{\beta}(x_\alpha - 1)} e^{k_\beta^{\beta}(x_\beta - 1)}.
\]

Using Eq. [7] the average number of \( \alpha \)-nodes in a component reachable from a randomly chosen \( \alpha \)-node is,

\[
\langle s_\alpha \rangle = 1 + \frac{\overline{k}_\alpha^{\alpha} + \overline{k}_\alpha^{\beta} \overline{k}_\beta^{\beta} - \overline{k}_\alpha^{\beta} \overline{k}_\beta^{\beta}}{(1 - \overline{k}_\alpha^{\alpha})(1 - \overline{k}_\beta^{\beta}) - \overline{k}_\alpha^{\beta} \overline{k}_\beta^{\beta}}.
\]

The average component size diverges for \( (1 - \overline{k}_\alpha^{\alpha})(1 - \overline{k}_\beta^{\beta}) = \overline{k}_\alpha^{\beta} \overline{k}_\beta^{\beta} \); the point at which the giant component emerges. (Ref. [8] recently presented an alternate method for deriving similar percolation thresholds and connectivity properties, but in a single network with multiple interacting communities.)

Note, following Eq. [7] we can show \( \langle s_\beta \rangle_\alpha, \langle s_\beta \rangle_\beta, \) and \( \langle s_\beta \rangle_\beta \) also all diverge at this point, marking when a giant component emerges in each network and throughout the system. Further simplifying, by assuming the two interacting networks have the same degree distribution, \( \overline{k}_\alpha^{\beta} = \overline{k}_\beta^{\beta} = \overline{k}_\text{inter} \) and \( \overline{k}_\beta^{\alpha} = \overline{k}_\beta^{\beta} = \overline{k}_\text{inter} \), then the giant component emerges when, \( \overline{k}_\text{inter} + \overline{k}_\text{inter} = 1 \), recovering the standard result for a single network (which, by definition, has \( \overline{k}_\text{inter} = 0 \)) that emergence occurs for \( \overline{k}_\text{inter} = 1 \).

Once the giant component emerges the \( u_{\text{inter}} \) which satisfy Eq. [8] are \( u_{\alpha\alpha} = u_{\alpha\beta} = 1 - S_\alpha \) and \( u_{\beta\beta} = u_{\beta\alpha} = 1 - S_\beta \), while \( S_\alpha \) and \( S_\beta \), respectively, the number of \( \alpha \)-nodes and \( \beta \)-nodes in the giant component of the system, satisfy

\[
S_\alpha = 1 - e^{-(\overline{k}_\alpha^{\alpha} S_\alpha + \overline{k}_\alpha^{\beta} S_\beta)}
\]

\[
S_\beta = 1 - e^{-(\overline{k}_\beta^{\alpha} S_\alpha + \overline{k}_\beta^{\beta} S_\beta)}.
\]

To observe the change in connectivity of one network precipitated by an increase in connectivity of a second network attached to the first, we simulated a system of two interacting networks and fixed \( \overline{k}_\alpha^{\alpha}, \overline{k}_\beta^{\beta}, \) and \( \overline{k}_\beta^{\alpha} \) while varying \( \overline{k}_\beta^{\beta} \) from 0 to 5 (Fig. 4). As \( \overline{k}_\beta^{\beta} \) increases the \( \beta \)-network becomes a single connected component (the traditional behavior for a single network) and \( S_\beta \to 1 \). However, the connectivity of \( \alpha \) remains limited. It can be shown that as \( \overline{k}_\beta^{\beta} \) increases \( S_\alpha \to \frac{1}{\overline{k}_\alpha^{\beta}} W \left[ -\overline{k}_\alpha^{\beta} e^{-\overline{k}_\alpha^{\beta} / \overline{k}_\alpha^{\beta}} \right] + 1 \) (dashed horizontal line in Fig. 4), where \( W \) is the Lambert \( W \) function, also known as the product log.

We next consider more complex degree distributions, where \( \alpha \) is still described by a Poisson distribution, but the internal connectivity of \( \beta \) is described by a power-law distribution with an exponential cutoff. While power-law degree distributions have attracted considerable attention as a model for node degree distributions in many types of networks [10], a power-law with an exponential cutoff may be a better model for real-world networks [17]. Here \( p_{\text{inter}}^{\beta} = \frac{\left(k_\beta^{\beta}\right)^{\kappa_\beta} e^{-k_\beta^{\beta}}/\kappa_\beta!}{\left(k_\beta^{\beta}\right)^{\kappa_\beta} e^{-k_\beta^{\beta}}/\kappa_\beta!} \) where \( \text{Li}_n(x) \) is the \( n \)th polylogarithm of \( x \) and serves as a normalizing factor for the distribution. Thus, we can write our basic generating function for network \( \beta \),

\[
G_\beta(x_\alpha, x_\beta) = e^{k_\alpha^{\beta}(x_\alpha - 1)} \frac{\text{Li}_n(x_\beta e^{-1/\kappa})}{\text{Li}_n(x_\beta e^{-1/\kappa})}.
\]

The generating function for \( \alpha \) is still given by Eq. [10]. We simulate the impact on the connectivity of the \( \alpha \)-network as the exponential cutoff and hence the average degree of network \( \beta \) increases, inset of Fig. 4. Again
the dashed curves are the analytic results obtained by solving Eqns. [8] and [9]. The solid red line is the behavior for the \( \beta \) network considered in isolation, showing that even the percolation threshold for \( \beta \) is lowered through connectivity with network \( \alpha \).

Finally we consider an application of connectivity to communications networks, building on the increasing interest in using Bluetooth connectivity between individuals to transmit data [18]. For instance, rather than downloading a webpage (such as the CNN homepage) by connecting to the Internet, a copy could be obtained from a close-by individual already in possession of this data. We construct prototypical networks of local Bluetooth connectivity between individuals from raw data of Bluetooth sightings by 41 attendees at the 25th IEEE International Conference on Computer Communications (INFOCOM) [19]. We initially partition the raw data into discrete 20 minute windows and consider that a communication edge exists between any two devices so long as they are within contact for at least 120 seconds. Each network has approximately a Poisson degree distribution of connectivity. We choose two arbitrary 20 minute snapshots as proxies for two distinct networks, \( \alpha \) and \( \beta \), representing, for instance, two separate rooms at the conference. We calculate how adding long-range connections between \( \alpha \) and \( \beta \) (for instance via text messages or email) enhances overall connectivity in the system. In other words, we calculate how many long-range connections would be needed between two isolated local Bluetooth networks to create the desired large scale connectivity, potentially allowing many users to share information. Figure 3 shows the size of the giant component obtained via numerical simulations using the real data (points) and the analytic calculations obtained via generating functions (dashed line). The analytic calculations slightly overestimate connectivity, yet there is remarkable agreement with empirical data even though the actually networks are quite small.

In summary, we have introduced a formalism for calculating connectivity properties in a system of \( l \) interacting networks. We demonstrate the extreme lowering of the percolation threshold possible once interactions with other networks are taken into account. This framework for calculating connectivity and statistics of interacting networks should be broadly applicable, and we show potential applications to software and communications systems.

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[1] S. N. Dorogovtsev and J. F. F. Mendes, *Advances in Physics*, **51**, 1079-1187 (2002).
[2] M. E. J. Newman, *SIAM Review* **45**, 167 (2003).
[3] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez and D.-U. Hwang, *Physics Reports*, **424**, 175-308 (2006).
[4] M. Kurant and P. Thiran, *Phys. Rev. Lett.* **96**, 138701 (2006).
[5] B. Bollobás, S. Janson and O. Riordan, *Random Structures and Algorithms* **31**, 3-122 (2007).
[6] A. Allard, P-A Noël, L. J. Dubé and B. Pourbohloul, *Phys. Rev. E* **79**, 036113 (2009).
[7] S. N. Dorogovtsev, J. F. F. Mendes, A. N. Samukhin, and A. Y. Zyuzin, *Phys. Rev. E* **78**, 056106 (2008).
[8] M. Ostilli and J. F. F. Mendes, arXiv:0812.0008 (2008).
[9] P. Erdős and A. Rényi, *Publicationes Mathematicae* **6**, 290 (1959).
[10] M. Molloy and B. Reed, *Random Structures and Algorithms* **6**, 161 (1995).
[11] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, *Phys. Rev. E* **64**, 026118 (2001).
[12] D. S. Callaway, M. E. J. Newman, S. H. Strogatz, and D. J. Watts, *Phys. Rev. Lett.* **85**, 5468 (2000).
[13] B. Bollobás, *European Journal of Combinatorics* **1**, 311 (1980).
[14] J. Lin, *IEEE Trans. Information Theory*, **37** (1) 145-151 (1991).
[15] [http://www.apacheweek.com/features/ap2](http://www.apacheweek.com/features/ap2).
[16] A.-L. Barabási and R. Albert, *Science* **286**, 510-512 (1999).
[17] A. Clauset, C. R. Shalizi, and M. E. J. Newman, *SIAM Review*, in-press (2009). (arXiv:0706.1062).
[18] S. Ioannisidis, A. Chaintreau, and L. Massoulié. *Proceedings of IEEE INFOCOM*, 2009.
[19] J. Scott, R. Gass, J. Crowcroft, P. Hu, C. Diot and A. Chaintreau, [http://crawdad.cs.dartmouth.edu/cambridge/haggle](http://crawdad.cs.dartmouth.edu/cambridge/haggle).