Casimir energy in the Fulling–Rindler vacuum

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Abstract

The Casimir energy is evaluated for massless scalar fields under Dirichlet or Neumann boundary conditions, and for the electromagnetic field with perfect conductor boundary conditions on one and two infinite parallel plates moving by uniform proper acceleration through the Fulling–Rindler vacuum in an arbitrary number of spacetime dimension. For the geometry of a single plate both regions of the right Rindler wedge, (i) on the right (RR region) and (ii) on the left (RL region) of the plate are considered. The zeta function technique is used, in combination with contour integral representations. The Casimir energies for separate RR and RL regions contain pole and finite contributions. For an infinitely thin plate taking RR and RL regions together, in odd spatial dimensions the pole parts cancel and the Casimir energy for the whole Rindler wedge is finite. In \(d = 3\) spatial dimensions the total Casimir energy for a single plate is negative for Dirichlet scalar and positive for Neumann scalar and the electromagnetic field. The total Casimir energy for two plates geometry is presented in the form of a sum of the Casimir energies for separate plates plus an additional interference term. The latter is negative for all values of the plates separation for both Dirichlet and Neumann scalars, and for the electromagnetic field.

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1 Introduction

The Casimir effect is a phenomenon common to all systems characterized by fluctuating quantities on which external boundary conditions are imposed. It may have important implications on all scales, from cosmological to subnuclear. The imposition of boundary conditions on a quantum field leads to the modification of the spectrum for the zero–point fluctuations and results in the shift in the vacuum expectation values for physical quantities such as the energy density and stresses. In particular, the confinement of quantum fluctuations causes forces that act on constraining boundaries. The particular features of the resulting vacuum forces depend on the nature of the quantum field, the type of spacetime manifold, the boundary geometries and the specific boundary conditions imposed on the field. Since the original work by Casimir in 1948 [1] many theoretical and experimental works have been done on this problem (see, e.g., [2, 3, 4, 5, 6, 7, 8] and references therein). Many different approaches have been used: mode summation method with combination of the zeta function regularization technique, Green function formalism, multiple scattering expansions, heat-kernel series, etc. An interesting topic in the investigations of the Casimir effect is the dependence of the vacuum characteristics on the type of the vacuum. It is well known that the uniqueness of the vacuum state is lost when we work within the framework of quantum field theory in a general curved spacetime or in non–inertial frames. In particular, the use of general coordinate transformations in quantum field theory in flat spacetime leads to an infinite number of unitary

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inequivalent representations of the commutation relations. Different inequivalent representations will in general give rise to different pictures with different physical implications, in particular to different vacuum states. For instance, the vacuum state for an uniformly accelerated observer, the Fulling–Rindler vacuum [9, 10, 11, 12], turns out to be inequivalent to that for an inertial observer, the familiar Minkowski vacuum (for a mathematical discussion by means of a normal mode analysis see Ref. [13]). Quantum field theory in accelerated systems contains many special features produced by a gravitational field. This fact allows one to avoid some of the difficulties entailed by renormalization in a curved spacetime. In particular, the near horizon geometry of most black holes is well approximated by Rindler and a better understanding of physical effects in this background could serve as a handle to deal with more complicated geometries like Schwarzschild. The Rindler geometry shares most of the qualitative features of black holes and is simple enough to allow detailed analysis. Another motivation for the investigation of quantum effects in the Rindler space is related to the fact that this space is conformally related to the de Sitter space and to the Robertson–Walker space with negative spatial curvature. As a result the expectation values of the energy–momentum tensor for a conformally invariant field and for corresponding conformally transformed boundaries on the de Sitter and Robertson–Walker backgrounds can be derived from the corresponding Rindler counterpart by the standard transformation (see, for instance, [14]).

The problem of vacuum polarization brought about by the presence of an infinite plane boundary moving with uniform acceleration through the Fulling-Rindler vacuum was investigated by Candelas and Deutsch [15] for the conformally coupled 4D Dirichlet and Neumann massless scalar and electromagnetic fields. In this paper only the region of the right Rindler wedge to the right of the barrier is considered. In Ref. [16] we have investigated the Wightman function and the vacuum expectation values of the energy momentum-tensor for the massive scalar field with general curvature coupling parameter, satisfying the Robin boundary conditions on the infinite plane in an arbitrary number of spacetime dimensions and for the electromagnetic field. Unlike Ref. [15] we have considered both regions, including the one between the barrier and Rindler horizon. The vacuum expectation values of the energy-momentum tensors for scalar and electromagnetic fields for the geometry of two parallel plates moving by uniform acceleration are investigated in Ref. [17]. In particular, the vacuum forces acting on the boundaries are evaluated. They are presented as a sum of the interaction and self-action parts. The interaction forces between the plates are always attractive for both scalar and electromagnetic cases. The self-action forces contain well-known surface divergencies and needs further regularization. In Refs. [16, 17] the mode summation method is used in combination with the generalized Abel-Plana summation formula [18]. This allowed us to present the vacuum expectation values in the terms of the purely Rindler and boundary parts. Due to the well known non-integrable surface divergences in the boundary parts, the total Casimir energy cannot be obtained by direct integration of the vacuum energy density and needs an additional regularization. Many regularization techniques are available nowadays and, depending on the specific physical problem under consideration, one of them may be more suitable than the others. In particular, the generalized zeta function method [19, 20] is in general very powerful to give physical meaning to the divergent quantities. There are several examples of the application of this method to the evaluation of the Casimir effect (see, for instance, [7, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]). In this paper, by using the zeta function technique, we consider the Casimir energy for the geometries of a single and two parallel plates, moving by uniform proper acceleration through the Fulling-Rindler vacuum.

The paper is organized as follows. In Sec. 2 the Casimir energy is evaluated for the Dirichlet scalar in the case of a single plate. For the both RR and RL regions we construct integral representations of the related zeta functions and analytically continue them to the physical region. The geometry of two plates in the case of the Dirichlet scalar is investigated in Sec. 3. We show that the corresponding zeta function is a sum of the zeta functions for the separate plates plus an additional interference term which is finite in the physical region. Similar problems for the Neumann scalar are investigated in Sec. 4 and Sec. 5, respectively. Section 6 considers the Casimir energy for the electromagnetic field assuming that the plates are perfect conductors. Section 7 concludes the main results of the paper. In Appendix A the case $d = 1$ is considered separately. Appendix B discusses the relation between the calculated energies.
and the energies measured by an uniformly accelerated observer.

2 Casimir energy for a single Dirichlet plate

Consider a real massless scalar field \( \varphi(x) \) with curvature coupling parameter \( \zeta \) satisfying the field equation

\[
\nabla_\mu \nabla^\mu \varphi + \zeta R \varphi = 0,
\]

(2.1)

with \( R \) being the scalar curvature for a \( d + 1 \)-dimensional background spacetime, \( \nabla_\mu \) is the covariant derivative operator associated with the corresponding metric tensor \( g_{\mu\nu} \). For minimally and conformally coupled scalars one has \( \zeta = 0 \) and \( \zeta = (d - 1)/4d \), respectively. Our main interest in this paper will be the Casimir energy in the Rindler spacetime induced by a single and two parallel plates moving with uniform proper acceleration when the quantum field is prepared in the Fulling-Rindler vacuum (note the difference of our problem from that related to the Unruh effect, where the field is in its Minkowski vacuum state). For this problem the background spacetime is flat and in Eq. (2.1) we have \( R = 0 \). As a result the eigenmodes are independent of the curvature coupling parameter and, hence, the Casimir energy will not depend on this parameter. However, the local characteristics of the vacuum such as energy density and vacuum stresses depend on the parameter \( \zeta \) [16, 17].

In the accelerated frame it is convenient to introduce Rindler coordinates \((\tau, \xi, x)\) which are related to the Minkowski ones, \((t, x^1, x)\) by transformations

\[
t = \xi \sinh \tau, \quad x^1 = \xi \cosh \tau,
\]

(2.2)

where \( x = (x^2, \ldots, x^d) \) denotes the set of coordinates parallel to the plate. In these coordinates the Minkowski line element takes the form

\[
ds^2 = \xi^2 d\tau^2 - d\xi^2 - dx^2,
\]

(2.3)

and a wordline defined by \( \xi, x = \text{const} \) describes an observer with constant proper acceleration \( \xi^{-1} \). Rindler time coordinate \( \tau \) is proportional to the proper time along a family of uniformly accelerated trajectories which fill the Rindler wedge, with the proportionality constant equal to the acceleration.

Let \( \{ \varphi_\alpha(x), \varphi^*_\alpha(x) \} \) be a complete set of positive and negative frequency solutions to the field equation (2.1), where \( \alpha \) denotes a set of quantum numbers. For the geometry under consideration the metric and boundary conditions are static and translational invariant in the hyperplane parallel to the plates. It follows from here that the corresponding part of the eigenfunctions can be taken in the standard plane wave form:

\[
\varphi_\alpha = C_\alpha \phi(\xi) \exp \left[ i \left( kx - \omega \tau \right) \right], \quad \alpha = (k, \omega), \quad k = (k_2, \ldots, k_d).
\]

(2.4)

The equation for \( \phi(\xi) \) is obtained from field equation (2.1) on background of metric (2.3):

\[
\xi^2 \phi''(\xi) + \xi \phi'(\xi) + (\omega^2 - k^2 \xi^2) \phi(\xi) = 0,
\]

(2.5)

where the prime denotes a differentiation with respect to the argument, and \( k = |k| \). The linearly independent solutions to equation (2.5) are the Bessel modified functions \( I_{\omega}(k\xi) \) and \( K_{\omega}(k\xi) \) of the imaginary order. The eigenfrequencies are determined from the boundary conditions imposed on the field on the bounding surfaces.

2.1 Vacuum energy in the RR region

In this section we will consider the vacuum energy for a scalar field satisfying Dirichlet boundary condition on a single plate located at \( \xi = \xi_1 \):

\[
\varphi|_{\xi=\xi_1} = 0.
\]

(2.6)
Figure 1: The \((x_1,t)\) plane with the Rindler coordinates. The heavy line \(\xi = \xi_1\) represent the trajectory of the plate.

We will assume that the plate is situated in the right Rindler wedge \(x^1 > |t|\). The surface \(\xi = \xi_1\) represents the trajectory of the boundary, which therefore has proper accelerations \(\xi_1^{-1}\) (see Fig. 1). This trajectory divides the right Rindler wedge into two regions with \(\xi \geq \xi_1\) and \(\xi \leq \xi_1\). In the following we will refer these regions as RR and RL regions, respectively. First let us consider the vacuum energy in the RR region. In this region, for a complete set of solutions that are of positive frequency with respect to \(\partial/\partial \tau\) and bounded as \(\xi \to \infty\), in Eq. (2.4) one has to take \(\phi(\xi) = K_{i\omega}(k\xi)\). For the Dirichlet scalar the corresponding eigenfrequencies are determined from the equation

\[
K_{i\omega}(k\xi_1) = 0. \tag{2.7}
\]

The positive roots to this equation arranged in the ascending order we will denote by \(\omega = \omega_{1Dn}(k\xi_1)\), \(n = 1, 2, \ldots, \omega_{1Dn} < \omega_{1Dn+1}\). To distinguish in the notations we will use indices \(D\) and \(N\) to denote Dirichlet and Neumann boundary conditions, respectively. Similarly the quantities for the RR and RL regions will be denoted by indices \(R\) and \(L\), respectively. The vacuum energy per unit surface of the plate in the region \(\xi \geq \xi_1\) is given by formula

\[
E^{(R)}_{1D}(\xi_1) = \frac{1}{2} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \sum_{n=1}^{\infty} \omega_{1Dn}(k\xi_1) = \frac{B_d}{\xi_1^{d-1}} \int_0^\infty dx x^{d-2} \sum_{n=1}^{\infty} \omega_{1Dn}(x), \tag{2.8}
\]

where we have integrated over angular coordinates, and

\[
B_d = \frac{1}{(4\pi)^{\frac{d-1}{2}}} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)}. \tag{2.9}
\]

Here and below we will assume that \(d > 1\). The case \(d = 1\) is considered in Appendix A. As it stands, the right-hand side of equation (2.8) clearly diverges and needs some regularization. We regularize it by defining the function

\[
Z_{DR}(s) = \int_0^\infty dx x^{d-2} \zeta^{(R)}_{1D}(s, x) \tag{2.10}
\]

where we have introduced the partial zeta function related to the eigenfrequencies defined by Eq. (2.7):

\[
\zeta^{(R)}_{1D}(s, x) = \sum_{n=1}^{\infty} \omega_{1Dn}^{-s}(x). \tag{2.11}
\]
The Casimir energy is expressed as

\[
E_{1D}^{(R)}(\xi_1) = \frac{B_d}{\xi_1} ZDK(s)|_{s=-1}.
\]  

(2.12)

In accordance with this formula, the computation of the Casimir energy requires the analytic continuation of the zeta function to the value \( s = -1 \).

The starting point of our consideration is the representation of the partial zeta function for the corresponding modes in term of contour integral (for a similar treatment of the zeta function as a contour integral see Refs. [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]):

\[
\zeta_{1D}^{(R)}(s,x) = \frac{1}{2\pi i} \int_C dz \, z^{-s} \frac{\partial}{\partial z} \ln K_{iz}(x),
\]

where \( C \) is a closed counterclockwise contour in the complex \( z \) plane enclosing all zeros \( \omega_{1Dn}(x) \). We assume that this contour is made of a large semicircle (with radius tending to infinity) centered at the origin and placed to its right, plus a straight part overlapping the imaginary axis. When the radius of the semicircle tends to infinity the corresponding contribution into \( \zeta_{1D}^{(R)}(s,x) \) vanishes for \( \text{Re} \, s > 1 \). After parametrizing the integrals over imaginary axis we arrive at the expression

\[
\zeta_{1D}^{(R)}(s,x) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty dz \, z^{-s} \frac{\partial}{\partial z} \ln K_z(x).
\]

(2.14)

For small \( z, \, z \to 0 \), one has \( K_z(x) \approx K_0(x) + (\partial^2 K_z(x)/\partial z^2)_{z=0}(z^2/2) \), and the integral in (2.14) converges at the lower limit for \( \text{Re} \, s < 2 \). Hence, this integral representation of the zeta function is valid for \( 1 < \text{Re} \, s < 2 \). To evaluate the Casimir energy we need the zeta function at \( s = -1 \) and so we have to do an analytic continuation of Eq. (2.14). To this aim we employ the uniform asymptotic expansion of the modified Bessel function \( K_z(x) \) for large values of the order [33]:

\[
K_z(x) = \sqrt{\frac{\pi}{2}} e^{-z\eta(x/z)} K_z^{(D)}(x), \quad K_z^{(D)}(x) \sim \sum_{l=0}^{\infty} (-1)^l \frac{\tau_l(t)}{(x^2 + z^2)^{l/2}},
\]

(2.15)

where

\[
t = \frac{z}{\sqrt{x^2 + z^2}}, \quad \eta(x) = \sqrt{1 + x^2} + \ln \frac{x}{1 + \sqrt{1 + x^2}}, \quad \tau_l(t) = \frac{u_l(t)}{t^l},
\]

(2.16)

and the expressions for the functions \( u_l(t) \) are given in [33]. From these expressions it follows that the coefficients \( \tau_l(t) \) have the structure

\[
\tau_l(t) = \sum_{m=0}^l u_{l,m} t^{2m},
\]

(2.17)

with the numerical coefficients \( u_{l,m} \). From the recurrence relations for the polynomials \( u_l(t) \) (see [33]), the following recurrence formulae can be obtained for the coefficients \( u_{l,m} \):

\[
u_{l+1,m} = \frac{1}{2} u_{l,m} \left[ 2m + l + \frac{1}{4(2m + l + 1)} \right] - \frac{1}{2} u_{l,m-1} \left[ 2m + l - 2 + \frac{5}{4(2m + l + 1)} \right],
\]

(2.18)

where \( m = 0, 1, \ldots, l + 1 \), and \( u_{l,-1} = u_{l+1} = 0, \, u_{00} = 1 \). Now making use the expansion (2.15), the zeta function can be presented in the form

\[
\zeta_{1D}^{(R)}(s,x) = \zeta_{1D}^{(R0)}(s,x) + \zeta_{1D}^{(R1)}(s,x),
\]

(2.19)

where

\[
\zeta_{1D}^{(R0)}(s,x) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty dz \, z^{-s} \frac{\partial}{\partial z} \ln \sqrt{\frac{\pi}{2} \frac{e^{-z\eta(x/z)}}{(x^2 + z^2)^{1/4}}},
\]

(2.20)

\[
\zeta_{1D}^{(R1)}(s,x) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty dz \, z^{-s} \frac{\partial}{\partial z} \ln K_z^{(D)}(x).
\]

(2.21)
Under the conditions \( 1 < \text{Re} s < 2 \), the term (2.20) can be easily evaluated to give
\[
\zeta_{1D}^{(R0)}(s,x) = \frac{(x/2)^{1-s}}{\pi(1-s)} B \left( 1 - s, \frac{s-1}{2} \right) \sin \frac{\pi s}{2} - \frac{x^{-s}}{4},
\]  
(2.22)
with the beta function \( B(x,y) \). Now using the standard dimensional regularization result that the renormalized value of the integrals of the type \( \int_0^\infty dx x^\beta \) is equal to zero (see, e.g., [34]), we conclude that the contribution of the term \( \zeta_{1D}^{(R0)}(s,x) \) into Eq. (2.10) vanishes. This can be seen by another way, considering the case of a scalar field with nonzero mass \( m \) and taking the limit \( m \to 0 \) after the evaluation of the corresponding integrals (for this trick in the calculations of the Casimir energy see, for instance, Refs. [35, 36]). For the massive case in Eq. (2.4) one has \( \phi(x) = K_{1\omega}(\xi \sqrt{k^2 + m^2}) \) and the corresponding formulae for \( \zeta_{1D}^{(R)}(s,x) \) are obtained from those given above in this section by replacement \( x \to \sqrt{x^2 + m^2 \xi^2} \). With this replacement the integral corresponding to the contribution of \( \zeta_{1D}^{(R0)} \) into Eq. (2.10) can be easily evaluated in terms of gamma function and vanishes in the limit \( m \to 0 \) for \( \text{Re} s < d - 1 \). For this reason in the following we will concentrate on the contribution of the second term on the right of Eq. (2.19). The corresponding expression for \( Z_{DK}(s) \) from Eq. (2.10) takes the form
\[
Z_{DK}(s) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty dx x^{d-2-2s} \left( \int_0^\infty dz z^{-s} \frac{\partial}{\partial z} \ln K_z^{(D)}(x) \right).  
\]  
(2.23)
To deal with one infinite limit integral in the following it will be convenient to introduce polar coordinates on the plane \((z,x)\):
\[
z = r \cos \theta, \quad x = r \sin \theta,  
\]  
(2.24)
In these coordinates one has the following expressions
\[
t = \cos \theta, \quad z \eta(x/z) = rg(\theta), \quad K_z^{(D)}(x) \sim \sum_{l=0}^\infty (-1)^l \frac{U_l(\cos \theta)}{r^l}.  
\]  
(2.25)
where we have introduced the notation
\[
g(\theta) = 1 + \cos \theta \ln \frac{\sin \theta}{1 + \cos \theta}.  
\]  
(2.26)
The integral representation (2.23) is well suited for the analytic continuation in \( s \). Following the usual procedure applied in the analogous calculations (see, for instance, [7] and references therein), we subtract and add to the integrand in Eq. (2.23) \( M \) leading terms of the corresponding asymptotic expansion and exactly integrate the asymptotic part. For this let us define the coefficients \( U_l(\cos \theta) \) in accordance with
\[
\ln \left( \sum_{l=0}^\infty (-1)^l \frac{U_l(\cos \theta)}{r^l} \right) = \sum_{l=1}^M (-1)^l \frac{U_l(\cos \theta)}{(1 + r^2)^l/2}.  
\]  
(2.27)
Here on the right we have expanded over \((1 + r^2)^{-1/2}\) to avoid convergence problems in the integral over \( r \) at the lower limit \( r = 0 \). Hence, Eq. (2.23) may be split into the following pieces
\[
Z_{DK}(s) = Z_{DK}^{(as)}(s) + Z_{DK}^{(1)}(s),  
\]  
(2.28)
with
\[
Z_{DK}^{(as)}(s) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty dx x^{d-2-2s} \left( \int_0^\infty dz z^{-s} \frac{\partial}{\partial z} \sum_{l=1}^M (-1)^l \frac{U_l(\cos \theta)}{(1 + r^2)^{l/2}} \right),  
\]  
\(Z_{DK}^{(1)}(s) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty dx x^{d-2-2s} \left( \int_0^\infty dz z^{-s} \frac{\partial}{\partial z} \left( \ln K_z^{(D)}(x) - \sum_{l=1}^M (-1)^l \frac{U_l(\cos \theta)}{(1 + r^2)^{l/2}} \right) \right).  
\]  
(2.29)
The number of terms \( M \) in these formulae is determined by the spatial dimension of the problem under consideration. For \( M \geq d \) the expression \( Z^{(1)}_{DK}(s) \) is finite at \( s = -1 \) and, hence, for our aim it is sufficient to subtract \( M = d \) asymptotic terms. For larger \( M \) the convergence is stronger. This means that with \( M \geq d \) we can directly put \( s = -1 \) in the expression for \( Z^{(1)}_{DK}(s) \) in Eq. (2.30) and perform the integral numerically. For numerical evaluation it is useful to integrate by parts the \( z \)-integral and then to introduce polar coordinates in the \((z,x)\)-plane. This yields

\[
Z^{(1)}_{DK}(-1) = \frac{1}{\pi} \int_0^\infty d r r^{d-1} \int_0^{\pi/2} d \theta \sin^{d-2} \theta \left[ \ln \left( \sqrt{\frac{2r}{\pi}} e^{r\theta} K_r(\theta \sin \theta) \right) - \sum_{l=1}^{M} (-1)^l \frac{U_l(\cos \theta)}{(1 + r^2)^{l/2}} \right].
\] (2.31)

Now we turn to the asymptotic part of \( Z_{DK}(s) \) given by expression (2.29). For this term the analytic continuation can be done analytically because its structure is simple. To evaluate this part we integrate over \( z \) by parts and note that the functions \( U_l(t) \) are polynomials in \( t \):

\[
U_l(t) = \sum_{m=0}^{l} U_{lm} t^{2m},
\] (2.32)

where the numerical coefficients \( U_{lm} \) are related to the coefficients in Eq. (2.17) by expansion (2.27). The first five functions are

\[
\begin{align*}
U_1(t) &= \frac{1}{8} - \frac{5}{24} t^2, & U_2(t) &= \frac{1}{16} - \frac{3}{8} t^2 + \frac{5}{16} t^4, \quad (2.33a) \\
U_3(t) &= \frac{49}{384} - \frac{1793}{1920} t^2 + \frac{221}{128} t^4 - \frac{1105}{172} t^6, \quad (2.33b) \\
U_4(t) &= \frac{121}{128} - \frac{83}{32} t^2 + \frac{551}{64} t^4 - \frac{339}{32} t^6 + \frac{565}{128} t^8, \quad (2.33c) \\
U_5(t) &= \frac{1813}{5120} - \frac{297649}{3840} t^2 + \frac{198755}{1024} t^4 - \frac{136907}{3072} t^6 + \frac{82825}{1024} t^8 - \frac{82825}{3072} t^{10}. \quad (2.33d)
\end{align*}
\]

Taking into account Eq. (2.32) and introducing polar coordinates, for the asymptotic part one finds

\[
Z^{(as)}_{DK}(s) = \frac{s}{\pi} \sin \frac{\pi s}{2} \sum_{l=1}^{M} (-1)^l \sum_{m=0}^{l} U_{lm} \int_0^\infty d r r^{d-s-2} \frac{1}{(1 + r^2)^{l/2}} \int_0^{\pi/2} d \theta \sin^{d-2} \theta \cos^{2m-s-1} \theta.
\] (2.34)

Evaluating the integrals by using the standard formulae (see, for instance, [37]), we arrive at the expression

\[
Z^{(as)}_{DK}(s) = \frac{s}{4\pi} \sin \frac{\pi s}{2} \sum_{l=1}^{M} (-1)^l \sum_{m=0}^{l} U_{lm} B \left( \frac{d-s-1}{2}, \frac{l+s-d+1}{2} \right) B \left( m-s-d+1 \right),
\] (2.35)

where the pole contributions are given explicitly in terms of beta function. In the sum over \( l \), the terms with even \( d-l \geq 0 \) have simple poles at \( s = -1 \) coming from the first beta function. Introducing a new summation variable \( p = (d-l)/2 \), the corresponding residue can be easily found by using the standard formula for the gamma function:

\[
Z^{(as)}_{DK,-1} = \frac{(-1)^d}{\pi d} \sum_{p=0}^{d} (-1)^p \sum_{m=0}^{d-2p} U_{d-2p,m} B \left( m-\frac{s}{2}, \frac{d-1}{2} \right) / B \left( \frac{d}{2} - p, p+1 \right),
\] (2.36)

where

\[
p_d = \begin{cases} (d-1)/2, & \text{for odd } d \\ d/2 - 1, & \text{for even } d \end{cases}.
\] (2.37)
Hence, Laurent-expanding near $s = -1$ we can write

$$Z^{(as)}_{DK}(s) = \frac{Z^{(as)}_{DK,-1}}{s + 1} + Z^{(as)}_{DK,0} + O(s + 1),$$

with

$$Z^{(as)}_{DK,0} = \frac{(-1)^d}{2\pi d} \sum_{p=0}^{d/2} (-1)^p \sum_{m=0}^{d-2p} U_{d-2p,m} \frac{B(m + \frac{d}{2}, d - 1)}{B(d - p, p + 1)}$$

$$\times \left[ \psi(p + 1) - \psi\left(\frac{d}{2}\right) + \psi\left(m + \frac{d}{2}\right) - \psi\left(m + \frac{1}{2}\right) - 2 \right]$$

$$+ \frac{1}{4\pi} \left( \sum_{l=1, l=d-l = \text{odd}}^{d-1} + \sum_{l=d+1}^{M} \right) (-1)^l \sum_{m=0}^{l} U_{ln} B \left(\frac{d}{2}, \frac{l - d}{2}\right) B \left(m + \frac{1}{2}, \frac{d - 1}{2}\right),$$

where $\psi(x) = d\ln\Gamma(x)/dx$ is the digamma function and the second sum in the braces of the third line is present only for $M \geq d + 1$. The first term on the right of Eq. (2.39) with digamma functions comes from the finite part of the Laurent expansion of the summands with even $d - l \geq 0$ in Eq. (2.35). Gathering all contributions together, near $s = -1$ we find

$$Z_{DK}(s) = \frac{Z^{(as)}_{DK,-1}}{s + 1} + Z^{(as)}_{DK,0} + Z^{(1)}_{DK}(-1) + O(s + 1),$$

where the separate terms are defined by formulae (2.31), (2.36), (2.39). Using this result, for the vacuum energy induced by a single plate at $\xi = \xi_1$ in the region $\xi \geq \xi_1$ one receives

$$E^{(R)}_{1D}(\xi_1) = E^{(R)}_{1Dp} + E^{(R)}_{1Df},$$

where for the pole and finite contributions one has

$$E^{(R)}_{1Dp} = \frac{B_d Z^{(as)}_{DK,-1}}{\xi_1^{d-1}(s + 1)}, \quad E^{(R)}_{1Df} = \frac{B_d}{\xi_1^{d-1}} \left[ Z^{(as)}_{DK,0} + Z^{(1)}_{DK}(-1) \right].$$

The numerical results corresponding to the vacuum energy (2.41) with separate parts (2.42) are presented in the first two rows of Table 1 for spatial dimensions $d = 2, 3, 4$.

In Appendix B we have discussed the relation between vacuum energy (2.41) and the energy measured by an uniformly accelerated observer with the proper acceleration $g$. These energies are connected by formula (B.4), where $\mu$ is an arbitrary constant with the dimension of mass. Using the numerical results given in Table 1, the Casimir energy in the RR region for the massless Dirichlet scalar in $d = 3$ measured by an uniformly accelerated observer is presented in the form

$$E^{(R)}_{1D}\left(\xi_1\right) = \frac{g}{\xi_1^2} \left(-0.000620 + \frac{1}{1260\pi^2} \left[ \frac{1}{s + 1} + \ln \left(\frac{\mu}{g}\right) \right] \right),$$

| $d$ | $\xi_1^{-1}E^{(R)}_{1Dp}$ | $\xi_1^{-1}E^{(R)}_{1Df}$ | $\xi_1^{-1}E^{(L)}_{1Dp}$ | $\xi_1^{-1}E^{(L)}_{1Df}$ |
|-----|------------------------|------------------------|------------------------|------------------------|
| 2   | $-\frac{1}{512\pi(s+1)}$ | $0.00185$              | $-\frac{1}{512\pi(s+1)}$ | $-0.00269$             |
| 3   | $-\frac{1}{1260\pi^2(s+1)}$ | $-0.000620$            | $-\frac{1}{1260\pi^2(s+1)}$ | $-0.000695$           |
| 4   | $-\frac{1}{262144\pi^3(s+1)}$ | $0.000225$             | $-\frac{1}{262144\pi^3(s+1)}$ | $-0.000235$           |
where the logarithmic term is a consequence of the divergence and has to be viewed as a remainder of
the renormalization process. The discussion for the role of the normalization scale \( \mu \) in the calculations
of the Casimir energy can be found in Ref. \[21\]. The remained pole term in the Casimir energy is a
characteristic feature for the zeta function regularization method and has been found for many cases of
boundary geometries. Note that a very powerful tool for studying divergence structure of the vacuum
energy is the heat kernel expansion (see, for instance, \[7, 38\] and references therein). In particular, the
coefficient of the logarithmic term is determined by the corresponding boundary coefficient in the heat
kernel asymptotic expansion.

2.2 Vacuum energy in the RL region

Now we turn to the Casimir energy in the region between a single plate and the Rindler horizon corre-
sponding to \( \xi = 0 \). As in previous subsection we will assume that the plate is located at \( \xi = \xi_1 \) and the
field satisfies Dirichlet boundary condition (2.6) on it. In the next section we show that the corresponding
vacuum energy can be determined by the formula

\[
E^{(L)}_{1D}(\xi_1) = \frac{B_d}{s_1^{d-1}} Z_D I(s)|_{s=-1},
\]

where we have defined the function

\[
Z_D = \int_0^\infty dx x^{d-2} \zeta^{(L)}_{1D}(s, x),
\]

with the partial zeta function

\[
\zeta^{(L)}_{1D}(s, x) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty dz z^{-s} \frac{\partial}{\partial z} \ln I_z(x).
\]

Here \( \rho \) is a small number which we will put zero later. Unlike to the case (2.14), here we can not directly
put \( \rho = 0 \), as for values \( s \) with convergence at the upper limit, the \( z \)-integral will diverge at the lower
limit. The analytical continuation for (2.46) as a function on \( s \) can be done by the way similar to that
given above for the region on the right of a single plate. First, we note that for the Bessel modified
function one has the uniform asymptotic expansion \[33\]

\[
I_z(x) = \frac{1}{\sqrt{2\pi}} \frac{e^{\pi(x/z)}}{(x^2 + z^2)^{1/4}} I_z^{(D)}(x), \quad I_z^{(D)}(x) \sim \sum_{l=0}^\infty \frac{\bar{\eta}(t)}{(x^2 + z^2)^{l/2}},
\]

where the functions \( \eta(x), t, \bar{\eta}(t) \) are defined in Eq. (2.16). This allows us to present the partial zeta
function in the form

\[
\zeta^{(L)}_{1D}(s, x) = \zeta^{(L0)}_{1D}(s, x) + \zeta^{(L1)}_{1D}(s, x),
\]

with

\[
\zeta^{(L0)}_{1D}(s, x) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty dz z^{-s} \frac{\partial}{\partial z} \ln \frac{e^{\pi(x/z)}}{\sqrt{2\pi}} (x^2 + z^2)^{1/4},
\]

\[
\zeta^{(L1)}_{1D}(s, x) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty dz z^{-s} \frac{\partial}{\partial z} \ln I_z^{(D)}(x).
\]

For \( 1 < \Re s < 2 \) the \( z \)-integral in the expression for \( \zeta^{(L0)}_{1D}(s, x) \) converges in the limit \( \rho \to 0 \) and in this
limit the evaluation of the integral gives

\[
\zeta^{(L0)}_{1D}(s, x) = \frac{(x/2)^{1-s}}{\pi(1-s)} B \left( 1-s, \frac{s-1}{2} \right) \sin \frac{\pi s}{2} - \frac{x^{-s}}{4}.
\]
As in the case of the RR region the contribution of this term into the vacuum energy vanishes.

Now we consider the part due to Eq. (2.50):
\[
Z_{DI}(s) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty dx x^{d-2} \int_0^\infty dz z^{-s} \frac{\partial}{\partial z} \ln f_z^D(x). \tag{2.52}
\]

By using asymptotic expansion (2.47), we can present it in the form
\[
Z_{DI}(s) = Z_{DI}^{(as)}(s) + Z_{DI}^{(1)}(s), \tag{2.53}
\]
where
\[
Z_{DI}^{(as)}(s) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty dx x^{d-2} \int_0^\infty dz z^{-s} \frac{\partial}{\partial z} \sum_{l=1}^M U_l(x) \ln f_z^D(x). \tag{2.54}
\]

\[
Z_{DI}^{(1)}(s) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty dx x^{d-2} \int_0^\infty dz z^{-s} \frac{\partial}{\partial z} \left( \ln f_z^D(x) - \sum_{l=1}^M U_l(x) \right). \tag{2.55}
\]

In the expression for \(Z_{DI}^{(as)}(s)\) we can directly put \(\rho = 0\), and after introducing polar coordinates and integrating, one finds
\[
Z_{DI}^{(as)}(s) = \frac{s}{4\pi} \sin \frac{\pi s}{2} \sum_{l=1}^M \sum_{m=0}^l U_{lm}B\left(\frac{d-s-1}{2}, \frac{l+s-d+1}{2}\right) B\left(m-s, \frac{d-1}{2}\right). \tag{2.56}
\]

At \(s = -1\) this function has a simple pole with the residue
\[
Z_{DI,-1}^{(as)} = (-1)^d Z_{DK,-1}^{(as)}, \tag{2.57}
\]
where \(Z_{DK,-1}^{(as)}\) is determined by formula (2.36). Now we have the following Laurent expansion
\[
Z_{DI}^{(as)}(s) = \frac{Z_{DI,-1}^{(as)}}{s + 1} + Z_{DI,0}^{(as)} + O(s + 1), \tag{2.58}
\]
where
\[
Z_{DI,0}^{(as)} = \frac{1}{2\pi d} \sum_{p=0}^{d} (-1)^p \sum_{m=0}^{d-2p} U_{d-2p,m} B\left(\frac{d+1}{2} - p, p + 1\right)
\times \left[ \psi(p+1) - \psi\left(\frac{d}{2}\right) + \psi\left(m+\frac{d}{2}\right) - \psi\left(m+\frac{1}{2}\right) - 2 \right]
+ \frac{1}{4\pi} \left( \sum_{l=1, d-l = \text{odd}}^{d-1} + \sum_{l=d+1}^{M} \right) \sum_{m=0}^{l} U_{lm}B\left(\frac{d-l}{2}, \frac{d-1}{2}\right) B\left(m+\frac{1}{2}, \frac{d-1}{2}\right). \tag{2.59}
\]

As regards to \(Z_{DI}^{(1)}(s)\), for \(M \geq d\) the corresponding expression (2.54) is finite at \(s = -1\) and we can directly put \(\rho = 0\). After integrating by parts and introducing polar coordinates we find
\[
Z_{DI}^{(1)}(-1) = \frac{1}{\pi} \int_0^\infty dr \sin^{d-2} \theta \left[ \ln \left( \frac{2\pi r e^{-\gamma(\theta)} I_{r\cos(\theta)}(r \sin(\theta))} {1 + \gamma^2}\right) \right] \sum_{l=1}^M U_l(x) \ln f_z^D(x). \tag{2.60}
\]

Collecting all terms for the expansion near \(s = -1\) one finds
\[
Z_{DI}(s) = \frac{Z_{DI,-1}^{(as)}}{s + 1} + Z_{DI,0}^{(as)} + Z_{DI}^{(1)}(-1) + O(s + 1), \tag{2.61}
\]
with the separate terms defined by (2.57), (2.59), (2.60). Now the vacuum energy in the region \(0 \leq \xi \leq \xi_1\) is the sum of the pole and finite terms

\[
E^{(L)}_{1D}(\xi_1) = E^{(L)}_{1Dp} + E^{(L)}_{1Df},
\]

with

\[
E^{(L)}_{1Dp} = \frac{B_dZ^{(as)}_{DK,-1}}{\xi_1^{d-1}(s+1)}, \quad E^{(L)}_{1Df} = \frac{B_d}{\xi_1^{d-1}} \left[ Z^{(as)}_{DK,0} + Z^{(1)}_{DI,0}(-1) + Z^{(1)}_{DI}(1) \right].
\]

The results of the numerical evaluations for these quantities are given in Table 1 for spatial dimensions \(d = 2, 3, 4\). The corresponding vacuum energy \(E^{(L)}_{1Dg}\) measured by an uniformly accelerated observer is related to the energy (2.62) by formula (B.4). In particular, using the data from Table 1, for the case of spatial dimension \(d = 3\) one finds

\[
E^{(L)}_{1Dg}(\xi_1) = \frac{g}{\xi_1^2} \left[ -0.000695 - \frac{1}{1260\pi^2} \left[ \frac{1}{s+1} + \ln \left( \frac{\mu}{g} \right) \right] \right],
\]

where \(g\) is the proper acceleration of the observer.

### 2.3 Total Casimir energy for a single Dirichlet plate

The total Casimir energy for a single Dirichlet plate can be obtained by summing the RR and RL parts considered above:

\[
E_{1D}(\xi_1) = E^{(R)}_{1D}(\xi_1) + E^{(L)}_{1D}(\xi_1).
\]

As separate summands, it contains pole and finite parts,

\[
E_{1D}(\xi_1) = E_{1Dp} + E_{1Df},
\]

where

\[
E_{1Dp} = \frac{B_dZ^{(as)}_{DK,-1}}{\xi_1^{d-1}(s+1)}, \quad E_{1Df} = \frac{B_d}{\xi_1^{d-1}} \left[ Z^{(as)}_{DK,0} + Z^{(as)}_{DK} + Z^{(1)}_{DI,0}(-1) + Z^{(1)}_{DI}(1) \right].
\]

Note that in odd spatial dimensions the pole part vanishes due to the cancellation of corresponding RR and RL parts and the total Casimir energy is finite. In this case this energy can be presented in the form

\[
E_{1D} = \frac{B_d}{\pi\xi_1^{d-1}} \left\{ \frac{1}{2} \sum_{l=1}^{M_1} B \left( \frac{d}{2}, l - \frac{d}{2} \right) \sum_{m=0}^{2l} U_{2l,m} B \left( m + \frac{1}{2}, \frac{d-1}{2} \right) + \int_0^\infty dr \int_0^{\pi/2} d\theta \sin^{d-2} \theta \right. \\
\times \left[ \ln (2r I_{r,\cos \theta}(r \sin \theta) K_{r,\cos \theta}(r \sin \theta)) - 2 \sum_{l=1}^{M_1} U_{2l}(\cos \theta) \right] \right\}.
\]

For the convergence of the integral in this formula it is sufficient to take \(M_1 > d/2 - 1\). The numerical results for the finite part of the total vacuum energy are given in Table 1. For the total Casimir energy measured by an uniformly accelerated observer with the proper acceleration \(g\), in odd dimensions the divergencies cancel and so do the logarithmic terms. In particular, for \(d = 3\) Dirichlet scalar the total Casimir energy

\[
E_{1Dg} = -\frac{0.00131g}{\xi_1^2}
\]

is negative. This means that the resulting vacuum forces tend to accelerate the Dirichlet plate.
3 Casimir energy for two parallel Dirichlet plates

In this section we consider the scalar vacuum in the region between two plates located at \( \xi = \xi_1, \xi = \xi_2 \), which therefore have proper accelerations \( \xi_1^{-1} \) and \( \xi_2^{-1} \). The problem geometry is depicted in Fig. 2. The scalar field satisfies the Dirichlet boundary conditions on the plates:

\[
\varphi|_{\xi=\xi_1} = \varphi|_{\xi=\xi_2} = 0.
\]

(3.1)

The function \( \phi(\xi) \) in Eq. (2.4) satisfying the boundary condition on the plate \( \xi = \xi_2 \) is in the form

\[
\phi(\xi) = D_{i\omega}(k\xi, k\xi_2) \equiv I_{i\omega}(k\xi_2)K_{i\omega}(k\xi) - I_{i\omega}(k\xi)K_{i\omega}(k\xi_2).
\]

(3.2)

From the boundary conditions on the plate \( \xi = \xi_1 \) we find that the corresponding eigenfrequencies are roots to the equation

\[
D_{i\omega}(k\xi_1, k\xi_2) = 0.
\]

(3.3)

This equation has an infinite set of solutions. We will denote the positive roots by \( \omega = \omega_{D_n}(k\xi_1, k\xi_2) \), and will assume that they are arranged in the ascending order \( \omega_{D_n} < \omega_{D_{n+1}} \). For the vacuum energy in the region \( \xi_1 \leq \xi \leq \xi_2 \) per unit surface of the plates on has

\[
E_D = \frac{1}{2} \int_0^\infty \frac{dk}{(2\pi)^d-1} \sum_{n=1}^\infty \omega_{D_n}(k\xi_1, k\xi_2).
\]

(3.4)

This energy can be written as

\[
E_D = B_d \int_0^\infty dk k^{d-2} \zeta_D(s, k\xi_1, k\xi_2)|_{s=-1},
\]

(3.5)

with the partial zeta function

\[
\zeta_D(s, k\xi_1, k\xi_2) = \sum_{n=1}^\infty \omega_{D_n}^{-s}(k\xi_1, k\xi_2), \quad \text{Re}s > 1.
\]

(3.6)
In order to obtain the Casimir energy, one has to find the analytic continuation of Eq. (3.6) to \( s = -1 \). This is done by an analytic continuation of an adequate contour integration in the complex plane. An immediate consequence of the Cauchy’s formula for the residues of a complex function is the expression

\[
\zeta_{D}(s, k\xi_1, k\xi_2) = \frac{1}{2\pi i} \int_{C} dz \, z^{-s} \frac{\partial}{\partial z} \ln D_{iz}(k\xi_1, k\xi_2),
\]

where the contour \( C \) is the same as in Eq. (2.13) with the additional semicircle \( C_{\rho} \) of small radius \( \rho \) avoiding the origin from the right (in Eq. (3.7) we can directly put \( \rho = 0 \), however this semicircle is needed for the next step, see Eq. (3.8) below). Let us denote by \( \zeta \) the function we have done in the previous section.

As a result the vacuum energy in the region \( \xi_1 \leq \xi \leq \xi_2 \) can be written in the form

\[
\zeta_{D}(s, k\xi_1, k\xi_2) = \zeta_{1D}(s, k\xi_1) + \frac{1}{2\pi i} \sum_{\alpha=+,-} \int_{C_{\rho}} dz \, z^{-s} \frac{\partial}{\partial z} \ln I_{-\alpha z}(k\xi_2)
\] 

\[+ \frac{1}{2\pi i} \sum_{\alpha=+,-} \int_{C_{\rho}} dz \, z^{-s} \frac{\partial}{\partial z} \ln \left[ 1 - \frac{I_{-\alpha z}(k\xi_1)K_{iz}(k\xi_2)}{I_{-\alpha z}(k\xi_2)K_{iz}(k\xi_1)} \right],
\]

\[+ \frac{1}{2\pi i} \int_{C_{\rho}} dz \, z^{-s} \frac{\partial}{\partial z} \ln D_{iz}(k\xi_1, k\xi_2),
\]

where we have used (2.13) to introduce the function \( \zeta_{1D}(s, x) \). The last integral over \( C_{\rho} \) vanishes in the limit \( \rho \to 0 \) for \( k \xi < 2 \) and we will omit it in the following consideration. After parametrizing the integrals over the imaginary axis, we arrive at the formula

\[
\zeta_{D}(s, k\xi_1, k\xi_2) = \zeta_{1D}(s, k\xi_1) + \frac{1}{\pi} \frac{\sin \pi s}{2} \int_{\rho}^{\infty} dz \, z^{-s} \frac{\partial}{\partial z} \ln I_{z}(k\xi_2)
\]

\[+ \frac{1}{\pi} \frac{\sin \pi s}{2} \int_{\rho}^{\infty} dz \, z^{-s} \frac{\partial}{\partial z} \ln \left[ 1 - \frac{I_{z}(k\xi_1)K_{z}(k\xi_2)}{I_{z}(k\xi_2)K_{z}(k\xi_1)} \right].
\]

The last integral on the right of this formula is finite at \( s = -1 \) and vanishes in the limits \( \xi_1 \to 0 \) or \( \xi_2 \to \infty \). It follows from here that the second term on the right corresponds to the zeta function \( \zeta_{1D}(s, k\xi_2) \) for the region on the left of a single plate located at \( \xi = \xi_2 \). The analytic continuation of this function we have done in the previous section.

As a result the vacuum energy in the region \( \xi_1 \leq \xi \leq \xi_2 \) can be written in the form of the sum

\[
E_{D} = E_{1D}^{(R)}(\xi_1) + E_{1D}^{(L)}(\xi_2) + \Delta E_{D}(\xi_1, \xi_2),
\]

where the interference term is given by the formula

\[
\Delta E_{D}(\xi_1, \xi_2) = \frac{B_{d}}{\pi} \int_{0}^{\infty} dk \, k^{d-2} \int_{0}^{\infty} dz \ln \left[ 1 - \frac{I_{z}(k\xi_1)K_{z}(k\xi_2)}{I_{z}(k\xi_2)K_{z}(k\xi_1)} \right].
\]

Note that in the corresponding expression we have partially integrated the \( z \)-integral. The function \( D_{z}(k\xi_1, k\xi_2) \) is positive for \( \xi_1 < \xi_2 \) and the energy (3.11) is always negative, \( \Delta E_{D}(\xi_1, \xi_2) < 0 \). In Fig. 3 we have presented the dependence of the interference part of the Casimir energy on the ratio \( \xi_1/\xi_2 \) for \( d = 3 \). In the case of two plates geometry, to obtain the total Casimir energy, \( E_{D}^{(tot)} \), we need to add to the energy in the region between the plates, given by Eq. (3.10), the energies coming from the regions \( \xi \leq \xi_1 \) and \( \xi \geq \xi_2 \). As a result one receives

\[
E_{D}^{(tot)} = E_{1D}(\xi_1) + E_{1D}(\xi_2) + \Delta E_{D}(\xi_1, \xi_2),
\]

where the interference part is given by formula (3.11). Note that in the limit \( \xi_2 \to \infty \) and for a fixed \( \xi_1 \), the last two terms on the right vanish and we recover the result for a single plate located at \( \xi = \xi_1 \). For
Figure 3: Interference part of the Casimir energy in the region between two Dirichlet plates, $\xi_2^{d-1} \Delta E_D(\xi_1, \xi_2)$, as a function on the ratio $\xi_1/\xi_2$ for $d = 3$.

$d = 3$ Dirichlet scalar all terms on the right of formula (3.12) are negative and, hence, the total Casimir energy for two parallel plates is negative.

In the limit $\xi_1 \to \xi_2$, expression (3.11) is divergent and for small values of $\xi_2/\xi_1 - 1$ the main contribution comes from large values of $z$. Replacing the order of integrations and introducing a new integration variable $x = k/z$, we can replace the Bessel modified functions by their uniform asymptotic expansions for large values of the order. After evaluating the integrals, to the leading order one receives

$$\Delta E_D(\xi_1, \xi_2) \sim -(4\pi)^{-\frac{d+1}{2}} \zeta_R(d + 1) \Gamma \left( \frac{d + 1}{2} \right) \frac{\xi_1}{(\xi_2 - \xi_1)^d},$$

(3.13)

where $\zeta_R(x)$ is the Riemann zeta function. Note that the energy $\Delta E_D$ is related to the corresponding quantity $\Delta E_{Dg}$ measured by an uniformly accelerated observer by formula $\Delta E_{Dg} = g\Delta E_D$, where $g$ is the proper acceleration of the observer (see Appendix B). In the limit $\xi_1, \xi_2 \to \infty$, $\xi_2 - \xi_1 = \text{const}$, the parts in Eq. (3.12) corresponding to the contributions from the single plates vanish, $E_{1D}(\xi_i) \to 0$, $i = 1, 2$, and the interference part remains only. In this case, as it follows from Eq. (3.13), the energy measured by an observer with the acceleration $g \to 0$ such that $g\xi_1 \to 1$, coincides with the standard Casimir energy for two parallel plates in $d + 1$-dimensional Minkowski spacetime [39].

As it has been shown in Ref. [17], the vacuum forces acting on the boundaries can be presented as a sum of the self-action and interaction forces. It can be easily checked that the interaction forces are related to the energy (3.11) by the standard thermodynamical relations

$$\frac{\partial}{\partial \ln \xi_1} \Delta E_D(\xi_1, \xi_2) = - p^{(1)}_{D(\text{int})}(\xi_1, \xi_2) = \frac{B_d}{\pi \xi_1^2} \int_0^\infty dk k^{d-2} \int_0^\infty dz \frac{K_z(k\xi_2)}{K_z(k\xi_1)D_z(k\xi_1, k\xi_2)},$$

(3.14a)

$$\frac{\partial}{\partial \ln \xi_2} \Delta E_D(\xi_1, \xi_2) = - p^{(2)}_{D(\text{int})}(\xi_1, \xi_2) = \frac{B_d}{\pi \xi_2^2} \int_0^\infty dk k^{d-2} \int_0^\infty dz \frac{I_z(k\xi_1)}{I_z(k\xi_2)D_z(k\xi_1, k\xi_2)},$$

(3.14b)

where $p^{(j)}_{D(\text{int})}$, $j = 1, 2$ are the vacuum effective pressures on the plate at $\xi = \xi_j$ induced by the presence of the second plate (interaction force per unit surface).
4 Casimir energy for a single plate with Neumann boundary condition

4.1 RR region

In this section we will consider the vacuum energy for a scalar field satisfying Neumann boundary condition on a single plate located at $\xi = \xi_1$:

$$\frac{\partial \varphi}{\partial \xi} \big|_{\xi = \xi_1} = 0.$$  (4.1)

In the region $\xi \geq \xi_1$, the corresponding eigenfrequencies are roots to the equation

$$K'_{in}(k\xi_1) = 0,$$  (4.2)

where the prime denotes the derivative with respect to the argument of the function. The positive roots to this equation arranged in the ascending order we will denote by $\omega = \omega_{1Nn}(k\xi_1)$, $n = 1, 2, \ldots$, $\omega_{1Nn} < \omega_{1Nn+1}$. The vacuum energy per unit surface of the plate in the region $\xi \geq \xi_1$ is given by formula

$$E_{1N}^{(R)}(\xi_1) = \frac{B_d}{\xi_1^{d-1}} \int_0^\infty dx x^{d-2} \sum_{n=1}^\infty \omega_{1Nn}(x).$$  (4.3)

To regularize this energy we consider the function

$$Z_{NK}(s) = \int_0^\infty dx x^{d-2} \zeta_{1N}^{(R)}(s, x),$$  (4.4)

with the partial zeta function related to the corresponding eigenfrequencies:

$$\zeta_{1N}^{(R)}(s, x) = \sum_{n=1}^\infty \omega_{1Nn}(x).$$  (4.5)

The Casimir energy is obtained by the analytic continuation to $s = -1$:

$$E_{1N}^{(R)}(\xi_1) = \frac{B_d}{\xi_1^{d-1}} Z_{NK}(s)|_{s=-1}.$$  (4.6)

This analytic continuation procedure is similar to that presented above for the Dirichlet case. First, we present the partial zeta function in terms of a contour integral

$$\zeta_{1N}^{(R)}(s, x) = \frac{1}{2\pi i} \int_C dz z^{-s} \frac{\partial}{\partial z} \ln K'_{iz}(x),$$  (4.7)

with the same contour $C$ as in Eq. (2.11). When the radius of the large semicircle of this contour tends to infinity the corresponding contribution into $\zeta_{1N}^{(R)}(s, x)$ vanishes for Re $s > 1$. After parametrizing the integrals over imaginary axis we obtain the following integral representation

$$\zeta_{1N}^{(R)}(s, x) = \frac{1}{\pi} \sin \frac{s}{2} \int_0^\infty dz z^{-s} \frac{\partial}{\partial z} \ln K'_{iz}(x),$$  (4.8)

valid for $1 < \text{Re} s < 2$. For the analytic continuation of Eq. (4.8) to $s = -1$ we use the uniform asymptotic expansion of the function $K'_{iz}(x)$ for large values of the order. We will write this expansion in the form

$$K'_{iz}(x) = -\sqrt{\frac{\pi}{2}} \frac{(x^2 + z^2)^{1/4}}{x} e^{-\eta(x/z)} K^{(N)}_z(x), \quad K^{(N)}_z(x) \sim \sum_{l=0}^\infty (-1)^l \frac{\pi_l(t)}{(x^2 + z^2)^{l/2}},$$  (4.9)
where

\[ \varpi_l(t) = \frac{v_l(t)}{t^l}, \quad (4.10) \]

and the expressions of the functions \( v_l(t) \) can be found in Ref. [33]. From these expressions we can see that the coefficients \( \varpi_l(t) \) are polynomials in \( t \):

\[ \varpi_l(t) = \sum_{m=0}^{l} v_{lm} t^{2m}. \quad (4.11) \]

From the formulae for the functions \( v_l(t) \) (see [33]), the coefficients \( v_{lm} \) can be expressed via the coefficients \( u_{lm} \) in Eq. (2.17) by the relations

\[ v_{l+1,m} = -\frac{1}{2} u_{lm} \left[ 2m + l + 1 - \frac{1}{4(2m + l + 1)} \right] + \frac{1}{2} u_{m-1} \left[ 2m + l - 1 - \frac{5}{4(2m + l + 1)} \right], \quad (4.12) \]

where \( m = 0, 1, \ldots, l + 1 \), and, as has been noted before, \( u_{l,-1} = u_{l,l+1} = 0 \), \( u_{00} = 1 \). Using expansion (4.9), the zeta function may be split into two pieces

\[ \zeta_{1N}^{(R)}(s, x) = \zeta_{1N}^{(R0)}(s, x) + \zeta_{1N}^{(R1)}(s, x), \quad (4.13) \]

with

\[ \zeta_{1N}^{(R0)}(s, x) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty \frac{dz}{z} \frac{z^{s-1}}{\left( \sqrt{2} \left( x^2 + z^2 \right)^{1/4} \right)} \ln \frac{\sqrt{2} \left( x^2 + z^2 \right)^{1/4}}{z} e^{-z(x/z)}, \quad (4.14) \]

\[ \zeta_{1N}^{(R1)}(s, x) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty \frac{dz}{z} \frac{z^{s-1}}{\left( \sqrt{2} \left( x^2 + z^2 \right)^{1/4} \right)} \ln K_z^{(N)}(x). \quad (4.15) \]

Under the conditions \( 1 < \text{Res} < 2 \), evaluating the integral in Eq. (4.14) one finds

\[ \zeta_{1N}^{(R0)}(s, x) = \frac{(x/2)^{1-s}}{\pi (1-s)} B \left( 1-s, \frac{s-1}{2} \right) \sin \frac{\pi s}{2} + \frac{x^{-s}}{4}. \quad (4.16) \]

By the arguments similar to those for the Dirichlet case, we conclude that the contribution of the term \( \zeta_{1N}^{(R0)}(s, x) \) into Eq. (4.4) vanishes. The expression with the second term on the right of Eq. (4.13) takes the form

\[ Z_{NK}(s) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty dx x^{d-2} x^{s-2} \int_0^\infty dz z^{s-1} \frac{\partial}{\partial z} \ln K_z^{(N)}(x). \quad (4.17) \]

Now we subtract and add to the integrand in Eq. (4.17) the corresponding asymptotic expression and exactly integrate the asymptotic part. For this we introduce polar coordinates in the \((z, x)\) plane and the coefficients \( V_l(\cos \theta) \) in accordance with

\[ \ln \left( \sum_{l=0}^\infty (-1)^l \varpi_l(\cos \theta) \right) = \sum_{l=1}^\infty (-1)^l \frac{V_l(\cos \theta)}{(1 + r^2)^{l/2}}. \quad (4.18) \]

This allows us to present (4.17) in the form of the sum

\[ Z_{NK}(s) = Z_{NK}^{(as)}(s) + Z_{NK}^{(1)}(s), \quad (4.19) \]

where

\[ Z_{NK}^{(as)}(s) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty dx x^{d-2} x^{s-2} \int_0^\infty dz z^{s-1} \frac{\partial}{\partial z} \sum_{l=1}^M (-1)^l \frac{V_l(\cos \theta)}{(1 + r^2)^{l/2}}, \quad (4.20) \]

\[ Z_{NK}^{(1)}(s) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty dx x^{d-2} x^{s-2} \int_0^\infty dz z^{s-1} \frac{\partial}{\partial z} \left[ \ln K_z^{(N)}(x) - \sum_{l=1}^M (-1)^l \frac{V_l(\cos \theta)}{(1 + r^2)^{l/2}} \right]. \quad (4.21) \]
For $M \geq d$ the expression $Z_{NK}^{(1)}(s)$ is finite at $s = -1$ and we can perform the integral numerically. For numerical evaluation it is useful to integrate by parts the $z$-integral and to perform the polar coordinates. This yields to the final expression

$$Z_{NK}^{(1)}(-1) = \frac{1}{\pi} \int_0^\infty dr \ r^{d-1} \int_0^{\pi/2} d\theta \sin^{d-2}\theta \times \left[ \ln \left( -\sqrt{\frac{2r}{\pi}} \sin \theta e^{r g(\theta)} K'_p \cos \theta (r \sin \theta) \right) - \sum_{l=1}^M (-1)^l \frac{V_l (\cos \theta)}{(1 + r^2)^{l/2}} \right]. \quad (4.22)$$

Now let us consider the asymptotic part of the function $Z_{NK}(s)$ given by expression (4.20). To evaluate this part note that the functions $V_l(t)$ have the structure

$$V_l(t) = \sum_{m=0}^l V_{lm} t^{2m}, \quad (4.23)$$

where the numerical coefficients $V_{lm}$ are related to the coefficients in Eq. (4.11) by expansion (4.18). The first five functions are

$$V_1(t) = -\frac{3}{8} + \frac{7}{24} t^2, \quad V_2(t) = -\frac{3}{16} + \frac{5}{8} t^2 - \frac{7}{16} t^4, \quad (4.24a)$$

$$V_3(t) = -\frac{45}{128} + \frac{2887}{1920} t^2 - \frac{315}{128} t^4 + \frac{1463}{1152} t^6, \quad (4.24b)$$

$$V_4(t) = -\frac{51}{128} + \frac{129}{32} t^2 - \frac{761}{64} t^4 + \frac{441}{32} t^6 - \frac{707}{128} t^8, \quad (4.24c)$$

$$V_5(t) = -\frac{3879}{5120} + \frac{436931}{35840} t^2 - \frac{26475}{4608} t^4 + \frac{173209}{1536} t^6 - \frac{101395}{1024} t^8 + \frac{495271}{15360} t^{10}. \quad (4.24d)$$

Integrating over $z$ by parts and introducing polar coordinates, for the asymptotic part one finds

$$Z_{NK}^{(as)}(s) = \frac{s}{\pi} \sin \frac{\pi s}{2} \sum_{l=1}^M (-1)^l \sum_{m=0}^l V_{lm} \int_0^\infty dr \ r^{d-s-2} \left( 1 + r^2 \right)^{l/2} \int_0^{\pi/2} d\theta \sin^{d-2}\theta \cos^{2m-1-s}\theta. \quad (4.25)$$

The evaluation of the integrals leads to the following formula

$$Z_{NK}^{(as)}(s) = \frac{s}{4\pi} \sin \frac{\pi s}{2} \sum_{l=1}^M (-1)^l \sum_{m=0}^l V_{lm} B \left( \frac{d-s-1}{2}, \frac{l+s-d+1}{2} \right) \right) B \left( m - \frac{s}{2}, \frac{d-1}{2} \right). \quad (4.26)$$

Due to the first beta function on the right this expression has a simple pole at $s = -1$ with the residue

$$Z_{NK,-1}^{(as)} = \frac{(-1)^d}{\pi d} \sum_{p=0}^{pd} (-1)^p \sum_{m=0}^{d-2p} V_{d-2p,m} B \left( m + 1/2, (d-1)/2 \right) B \left( d/2 - p, p + 1 \right), \quad (4.27)$$

where $p_d$ is defined in Eq. (2.37). Hence, the Laurent expansion of the asymptotic part near $s = -1$ has the form

$$Z_{NK}^{(as)}(s) = \frac{Z_{NK,-1}^{(as)}}{s+1} + Z_{NK,0}^{(as)} + O(s+1), \quad (4.28)$$

with the finite part

$$Z_{NK,0}^{(as)} = \frac{(-1)^d}{2\pi d} \sum_{p=0}^{pd} (-1)^p \sum_{m=0}^{d-2p} V_{d-2p,m} B \left( m + 1/2, \frac{d-1}{2} \right) B \left( \frac{d}{2} - p, p + 1 \right) \times \left[ \psi(p+1) - \psi \left( \frac{d}{2} \right) + \psi \left( m + \frac{d}{2} \right) - \psi \left( m + \frac{1}{2} \right) - \frac{1}{4\pi} \sum_{l=1}^{d-1} \sum_{l=d-l=\text{odd}}^{M} (-1)^l \sum_{m=0}^{l} V_{lm} B \left( \frac{d}{2}, \frac{1-l-d}{2} \right) B \left( m + 1/2, \frac{d-1}{2} \right) \right]. \quad (4.29)$$
Table 2: Pole and finite parts of the Casimir energy for a single Neumann plate.

| $d$ | $\xi_{1}^{-1}E^{(R)}_{1NP}$ | $\xi_{1}^{-1}E^{(R)}_{1Nf}$ | $\xi_{1}^{-1}E^{(L)}_{1NP}$ | $\xi_{1}^{-1}E^{(L)}_{1Nf}$ | $\xi_{1}^{-1}E^{(L)}_{1NP}$ | $\xi_{1}^{-1}E^{(L)}_{1Nf}$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 2   | -0.00874        | -0.000033       | -0.000633       | -0.00937        |                  |                  |
| 3   | 0.00213         | 0.000792        | 0.00292         |                  |                  |                  |
| 4   | -0.000593       | -0.000398       | -0.000194       |                  |                  |                  |

where the second sum in the braces of the third line is present only for $M \geq d + 1$. Now adding the part coming from (4.22), for the Laurent expansion of the function $Z_{NK}(s)$ near $s = -1$ one finds

\[
Z_{NK}(s) = \frac{\text{Z}_{NK,-1}^{(as)}}{s + 1} + Z_{NK,0}^{(as)} + Z_{NK}^{(1)}(-1) + O(s + 1),
\]

(4.30)

where the separate terms are defined by formulae (4.22), (4.27), (4.29). Using this result, for the vacuum energy in the case of Neumann scalar induced by a single plate at $\xi = \xi_1$ in the region $\xi \geq \xi_1$ one receives

\[
E^{(R)}_{1N} (\xi_1) = E^{(R)}_{1NP} + E^{(R)}_{1Nf},
\]

(4.31)

where for the pole and finite contributions we have

\[
E^{(R)}_{1NP} = \frac{B_d \text{Z}_{NK,-1}^{(as)}}{\xi_{1}^{-d-1}(s + 1)}, \quad E^{(R)}_{1Nf} = \frac{B_d}{\xi_{1}^{-d+1}} \left[ Z_{NK,0}^{(as)} + Z_{NK}^{(1)}(-1) \right].
\]

(4.32)

The numerical results corresponding to the vacuum energy (4.31) with separate pole and finite parts are presented in Table 2 for spatial dimensions $d = 2, 3, 4$. Comparing to the data from Table 1, we see that in all these cases the Neumann quantities for the RR region dominate the Dirichlet ones.

By making use the formula given in Appendix B, we can obtain the vacuum energy in the RL region measured by an uniformly accelerated observer. In $d = 3$ spatial dimension for this energy one has

\[
E^{(R)}_{1Ng} (\xi_1) = \frac{g}{\xi_1^2} \left( 0.00213 + \frac{1}{180\pi^2} \left[ \frac{1}{s + 1} + \ln \left( \frac{\mu}{g} \right) \right] \right).
\]

(4.33)

It logarithmically depends on the normalization scale.

### 4.2 Neumann vacuum energy in the RL region

Similar to the case of the RR region, the Casimir energy for the Neumann scalar in the region between a single plate at $\xi = \xi_1$ and the Rindler horizon corresponding to $\xi = 0$ is determined by the formula (see Sec. 5 below)

\[
E^{(L)}_{1N} (\xi_1) = \frac{B_d}{\xi_{1}^{-d+1}} Z_{NI}(s)|_{s=-1},
\]

(4.34)

with the function

\[
Z_{NI}(s) = \int_0^{\infty} dx x^{d-2} \zeta_{1N}^{(L)}(s, x),
\]

(4.35)

and the partial zeta function

\[
\zeta_{1N}^{(L)}(s, x) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int_0^\infty dz z^{-s} \frac{\partial}{\partial z} \ln I'_z(x).
\]

(4.36)

For the analytic continuation of the function $Z_{NI}(s)$ to $s = -1$ we note that for the derivative of the Bessel modified function one has the uniform asymptotic expansion

\[
I'_z(x) = \frac{1}{\sqrt{2\pi}} \frac{(x^2 + z^2)^{1/4}}{x} e^{\mp \eta(x/z)} I_z^{(N)}(x), \quad I_z^{(N)}(x) \sim \sum_{l=0}^{\infty} \frac{\bar{v}_l(t)}{(x^2 + z^2)^{1/2}},
\]

(4.37)
where the functions $\eta(x)$, $t$ are defined in Eq. (2.16). Separating the contributions coming from the different factors in Eq. (4.37), we present the partial zeta function in the form

$$\zeta^{(L)}_{1N}(s, x) = \zeta^{(L0)}_{1N}(s, x) + \zeta^{(L1)}_{1N}(s, x),$$

with

$$\zeta^{(L0)}_{1N}(s, x) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int^{\infty}_{0} dx z^{d-2} \int^{\infty}_{0} dz z^{-s} \frac{\partial}{\partial z} \ln \left( \frac{\sqrt{2\pi}}{x} \right) e^{x \eta(z/x)},$$

$$\zeta^{(L1)}_{1N}(s, x) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int^{\infty}_{0} dz z^{-s} \frac{\partial}{\partial z} \ln I^{(N)}_z(x).$$

For $1 < \text{Res} < 2$ the $z$-integral in the expression for the function $\zeta^{(L0)}_{1N}(s, x)$ is convergent in the limit $\rho \to 0$ and the straightforward evaluation of the integral in this limit gives

$$\zeta^{(L0)}_{1N}(s, x) = -\left( \frac{x}{2} \right)^{1-s} B \left( 1 - s, \frac{s - 1}{2} \right) \sin \frac{\pi s}{2} + \frac{x^{-s}}{4}.$$  \hspace{1cm} (4.41)

As in the case of RR region, the contribution of this term into the renormalized vacuum energy vanishes. Hence, we need to consider the quantity

$$Z_{NI}(s) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int^{\infty}_{0} dx x^{d-2} \int^{\infty}_{0} dz z^{-s} \frac{\partial}{\partial z} \ln I^{(N)}_z(x).$$

With the help of asymptotic expansion (4.37), it can be presented in the form

$$Z_{NI}(s) = Z^{(as)}_{NI}(s) + Z^{(1)}_{NI}(s),$$

where

$$Z^{(as)}_{NI}(s) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int^{\infty}_{0} dx x^{d-2} \int^{\infty}_{0} dz z^{-s} \frac{\partial}{\partial z} \sum_{l=1}^{M} \frac{V_l}{(1 + r^2)^{l/2}},$$

$$Z^{(1)}_{NI}(s) = \frac{1}{\pi} \sin \frac{\pi s}{2} \int^{\infty}_{0} dx x^{d-2} \int^{\infty}_{0} dz z^{-s} \frac{\partial}{\partial z} \left( \ln I^{(N)}_z(x) - \sum_{l=1}^{M} \frac{V_l}{(1 + r^2)^{l/2}} \right).$$

Note that for $M \geq d$ the expression (4.45) for $Z^{(1)}_{NI}(s)$ is finite at $s = -1$ and we can directly put $\rho = 0$. After integrating by parts and introducing polar coordinates we find

$$Z^{(1)}_{NI}(-1) = \frac{1}{\pi} \int^{\pi/2}_{0} dr r^{d-1} \int^{\pi/2}_{0} d\theta \sin^{d-2} \theta \left[ \ln \left( \sqrt{2\pi} e^{-r \theta} \sin \theta I_r^{\cos \theta}(r \sin \theta) \right) - \sum_{l=1}^{M} \frac{V_l}{(1 + r^2)^{l/2}} \right].$$

As regards the asymptotic part, in the expression for $Z^{(as)}_{NI}(s)$ we can directly put $\rho = 0$ and after introducing polar coordinates and integrating one finds

$$Z^{(as)}_{NI}(s) = -\frac{s}{4\pi} \sin \frac{\pi s}{2} \sum_{l=1}^{M} \sum_{m=0}^{l} V_{lm} B \left( \frac{d - s - 1}{2}, \frac{l + s - d + 1}{2} \right) B \left( m - \frac{s}{2}, \frac{d - 1}{2} \right).$$

At $s = -1$ this function has a simple pole with the residue

$$Z^{(as)}_{NI,-1} = (-1)^d Z^{(as)}_{NK,-1},$$

(4.48)
with \( Z_{NK,-1}^{(as)} \) defined by Eq. (4.27). Now, taking into account the contribution coming from (4.45), we have the following Laurent expansion near \( s = -1 \)

\[
Z_{NI}(s) = \frac{Z_{NI,-1}^{(as)}}{s + 1} + Z_{NI,0}^{(as)} + Z_{NI}^{(1)}(-1) + O(s + 1),
\]

where

\[
Z_{NI,0}^{(as)} = \frac{1}{2\pi d} \sum_{p=0}^{p_d} (-1)^p \sum_{m=0}^{d-2p} V_{d-2p,m} B\left(m + \frac{d-1}{2}\right) \frac{1}{B\left(\frac{d}{2} - p, p + 1\right)} \times \left[ \psi(p + 1) - \psi\left(\frac{d}{2}\right) + \psi\left(m + \frac{d}{2}\right) - \psi\left(m + \frac{1}{2}\right) - 2 \right]
\]

\[
+ \frac{1}{4\pi} \left( \sum_{l=1, d-l=\text{odd}}^{d-1} + \sum_{l=d+1}^{M} \right) \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{2\pi} \frac{d\theta_1 d\theta_2 d\theta_3}{2\pi d^{d-1}} \right) \cdot B\left( \frac{d}{2}, \frac{l - d}{2} \right) B\left(m + \frac{1}{2}, \frac{d-1}{2}\right).
\]

(4.50)

The vacuum energy for the Neumann scalar in the region \( 0 \leq \xi \leq \xi_1 \) is presented as a sum of the pole and finite terms

\[
E_{1N}^{(L)}(\xi_1) = E_{1Np}^{(L)} + E_{1Nf}^{(L)},
\]

with

\[
E_{1Np}^{(L)} = \frac{B_d Z_{NK,-1}^{(as)}}{\xi_1^{-d-1}(s + 1)}, \quad E_{1Nf}^{(L)} = \frac{B_d}{\xi_1^{-d-1}} \left[ Z_{NK,0}^{(as)} + Z_{NI,0}^{(as)} + Z_{NI}^{(1)}(-1) \right].
\]

(4.52)

For spatial dimensions \( d = 2, 3, 4 \) the results of the corresponding numerical evaluations are reported in Table 2. For spatial dimension \( d = 3 \), the Casimir energy in the RL region measured by an uniformly accelerated observer is obtained with the help of formula (B.4):

\[
E_{1Ng}^{(L)}(\xi_1) = \frac{g}{\xi_1^2} \left( -0.000792 - \frac{1}{180\pi^2} \left[ \frac{1}{s + 1} + \log \left( \frac{\mu}{g} \right) \right] \right),
\]

(4.53)

where \( g \) is the proper acceleration for the observer.

### 4.3 Total Casimir energy for a single Neumann plate

Summing the contributions from the RR and RL regions we obtain the total Casimir energy for a single Neumann plate:

\[
E_{1N}(\xi_1) = E_{1N}^{(R)}(\xi_1) + E_{1N}^{(L)}(\xi_1).
\]

(4.54)

It can be divided into pole and finite parts,

\[
E_{1N}(\xi_1) = E_{1Np} + E_{1Nf},
\]

(4.55)

where

\[
E_{1Np} = \frac{B_d Z_{NK,-1}^{(as)} (1 + (-1)^d)}{\xi_1^{-d-1}(s + 1)}, \quad E_{1Nf} = \frac{B_d}{\xi_1^{-d-1}} \left[ Z_{NK,0}^{(as)} + Z_{NI,0}^{(as)} + Z_{NI}^{(1)}(-1) + Z_{NI}^{(1)}(-1) \right].
\]

(4.56)

In odd spatial dimensions the pole part vanishes due to the cancellation of corresponding RR and RL parts and the total Casimir energy is finite. In this case this energy can be presented in the form

\[
E_{1N} = \frac{B_d}{\pi s \xi_1^{d-1}} \left\{ \sum_{l=1}^{M_1} B\left(\frac{d}{2}, \frac{l - d}{2}\right) \sum_{m=0}^{2l} V_{2l,m} B\left(m + \frac{1}{2}, \frac{d-1}{2}\right) + \int_{0}^{\infty} dr r^{d-1} \int_{0}^{\pi/2} d\theta \sin^{d-2} \theta \right. \\
\times \left. \left[ \ln \left( -2r \sin^2 \theta \right) I_{\cos \theta}^{(1)}(r \sin \theta) K_{\cos \theta}^{(1)}(r \sin \theta) \right) - 2 \sum_{l=1}^{M_1} V_{2l}(\cos \theta) \right\}. \quad (4.57)
\]
Under the condition $M_1 > d/2 - 1$ the integral in this formula is convergent. The numerical results for the finite part of the total vacuum energy are given in Table 2. In the physically most important case $d = 3$, the total Casimir energy for the Neumann scalar induced by a single plate is positive. For the total Casimir energy measured by an uniformly accelerated observer one obtains

$$E_{1Ng} = \frac{0.00292g}{\xi_1^2},$$

which is finite and independent on the normalization scale. As the energy (4.58) is positive, the corresponding vacuum forces tend to decelerate the plate. Here the situation is opposite compared to the case of the Dirichlet scalar.

5 Casimir energy for two Neumann plates

Consider the scalar vacuum in the region between two plates located at $\xi = \xi_1$, $\xi = \xi_2$ and with the Neumann boundary conditions on them:

$$\frac{\partial\phi}{\partial \xi}|_{\xi=\xi_1} = \frac{\partial\phi}{\partial \xi}|_{\xi=\xi_2} = 0.$$ (5.1)

For the function $\phi(\xi)$ in expression (2.4) one has

$$\phi(\xi) = N_{i\omega}(k\xi_1, k\xi_2) \equiv I'_{i\omega}(k\xi_2)K_{i\omega}(k\xi_1) - I_{i\omega}(k\xi_1)K'_{i\omega}(k\xi_2).$$ (5.2)

From the boundary condition on the plate $\xi = \xi_1$ we obtain that the corresponding eigenfrequencies are solutions to the equation

$$N'_{i\omega}(k\xi_1, k\xi_2) \equiv I'_{i\omega}(k\xi_2)K'_{i\omega}(k\xi_1) - I_{i\omega}(k\xi_1)K'_{i\omega}(k\xi_2) = 0.$$ (5.3)

We denote the positive roots to this equation by $\omega = \omega_{Nn}(k\xi_1, k\xi_2)$, and will assume that they are arranged in the ascending order $\omega_{Nn} < \omega_{Nn+1}$. For the vacuum energy in the region $\xi_1 \leq \xi \leq \xi_2$ per unit surface of the plates on has

$$E_N = \frac{1}{2} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \sum_{n=1}^{\infty} \omega_{Nn}(k\xi_1, k\xi_2).$$ (5.4)

We introduce the partial zeta function as

$$\zeta_N(s, k\xi_1, k\xi_2) = \sum_{n=1}^{\infty} \omega_{Nn}^{-s}(k\xi_1, k\xi_2).$$ (5.5)

In order to obtain the Casimir energy, one has to find the analytic continuation of (5.5) to $s = -1$. Employing the Cauchy’s theorem, zeta function (5.5) may be written under the form of contour integral on the complex plane,

$$\zeta_N(s, k\xi_1, k\xi_2) = \frac{1}{2\pi i} \int_C dz z^{-s} \frac{\partial}{\partial z} \ln N'_{iz}(k\xi_1, k\xi_2),$$ (5.6)

with the same contour $C$ as in Eq. (3.7). The integral on the right of Eq. (5.6) can be presented in the form

$$\zeta_N(s, k\xi_1, k\xi_2) = \zeta_{1N}^{(R)}(s, k\xi_1) + \frac{1}{2\pi i} \sum_{\alpha=+, -} \int_{C^\alpha} dz z^{-s} \frac{\partial}{\partial z} \ln I'_{iz}(k\xi_2)$$

$$+ \frac{1}{2\pi i} \sum_{\alpha=+, -} \int_{C^\alpha} dz z^{-s} \frac{\partial}{\partial z} \ln \left[ 1 - \frac{I'_{iz}(k\xi_1)K'_{iz}(k\xi_2)}{I'_{iz}(k\xi_2)K'_{iz}(k\xi_1)} \right]$$

$$+ \frac{1}{2\pi i} \int_{C^\rho} dz z^{-s} \frac{\partial}{\partial z} \ln N'_{iz}(k\xi_1, k\xi_2),$$ (5.7)
where $C^+$ and $C^-$ are the upper and lower halves of the contour $C$ except the parts coming from the small semicircle $C_\rho$. The last integral over $C_\rho$ vanishes in the limit $\rho \to 0$ for $\text{Re}\, s < 2$ and it can be omitted in the following consideration. After parametrizing the integrals over the imaginary axis we obtain the formula

$$\zeta_N(s, k\xi_1, k\xi_2) = \zeta_1^{(R)}(s, k\xi_1) + \frac{1}{\pi} \sin \frac{\pi s}{2} \int_{\rho}^{\infty} dz \, z^{-s} \frac{\partial}{\partial z} \ln I_z'(k\xi_2)$$

$$+ \frac{1}{\pi} \sin \frac{\pi s}{2} \int_{\rho}^{\infty} dz \, z^{-s} \frac{\partial}{\partial z} \ln \left[ 1 - \frac{I_z'(k\xi_1)K_z'(k\xi_2)}{I_z'(k\xi_2)K_z'(k\xi_1)} \right].$$

(5.8)

The last integral on the right of this formula is finite at $s = -1$ and vanishes in the limits $\xi_1 \to 0$ and $\xi_2 \to \infty$. It follows from here that the second term on the right corresponds to the zeta function $\zeta_1^{(L)}(s, x)$ for the region on the left of a single plate located at $\xi = \xi_2$. The procedure for the analytic continuation of this function we have considered in the previous section.

Taking into account (5.7), for the Neumann vacuum energy in the region $\xi_1 \leq \xi \leq \xi_2$ one finds

$$E_N = E_1^{(R)}(\xi_1) + E_1^{(L)}(\xi_2) + \Delta E_N(\xi_1, \xi_2),$$

(5.9)

with the interference term

$$\Delta E_N(\xi_1, \xi_2) = \frac{Bd}{\pi} \int_0^{\infty} dk \, k^{d-2} \int_0^{\infty} dz \, \ln \left[ 1 - \frac{I_z'(k\xi_1)K_z'(k\xi_2)}{I_z'(k\xi_2)K_z'(k\xi_1)} \right],$$

(5.10)

where we have integrated in parts the $z$-integral. From the inequality $N_z'(k\xi_1, k\xi_2) < 0$ for $\xi_1 < \xi_2$ it follows that $\Delta E_N(\xi_1, \xi_2) < 0$. In Fig. 4 we have presented the dependence of the interference part of the Casimir energy (5.10) on the ratio $\xi_1/\xi_2$ for $d = 3$. As seen from Figures 3 and 4, the interference parts of the Casimir energies for Dirichlet and Neumann scalars are numerically close to each other. This is a consequence of that the subintegrands in formulas (3.11) and (5.10) are numerically close. This can be also seen analytically by using the inequalities for the Bessel modified functions given in Ref. [17].

Interference term (5.10) is finite for all $0 < \xi_1 < \xi_2$, and diverges in the limit $\xi_1 \to \xi_2$. In this limit the main contribution comes from the large values of $z$. Introducing a new integration variable $x = k/z$ and using the uniform asymptotic expansions for the Bessel modified functions, for the interference part in the leading order one obtains the same result as in the Dirichlet case, (3.13). In particular, in the limit $\xi_1, \xi_2 \to \infty$ with fixed $\xi_2 - \xi_1$, for the vacuum energy measured by an uniformly accelerated observer with

Figure 4: Interference part of the Casimir energy in the region between two Neumann plates, $\xi_{2}^{d-1} \Delta E_D(\xi_1, \xi_2)$, as a function on the ratio $\xi_1/\xi_2$ for $d = 3$. 22
the proper acceleration \( g \to 0 \) and \( g \xi_1 \to 1 \), one recovers the standard result for the Casimir plates on the Minkowski bulk.

Adding to the energy in the region between the plates, given by Eq. (5.9), the energies coming from the regions \( \xi \leq \xi_1 \) and \( \xi \geq \xi_2 \), we obtain the total Casimir energy, \( E_{D}^{(\text{tot})} \), for two-plates geometry:

\[
E_{N}^{(\text{tot})} = E_{1N}(\xi_1) + E_{1N}(\xi_2) + \Delta E_{N}(\xi_1, \xi_2),
\] (5.11)

where the interference part is given by formula (5.10). For \( d = 3 \) Neumann scalar the first two terms on the right of Eq. (5.11), corresponding to the single plates contributions, are positive and the third (interference) term is negative. For sufficiently close \( \xi_1 \) and \( \xi_2 \) the last term dominates and the total Casimir energy is negative. For the large separations between the plates the contributions from the single plate parts are dominant and the total energy is positive. Hence, for some intermediate values of the separation the total Casimir energy for the Neumann scalar vanishes.

As in the Dirichlet case, it can be seen that the energy (5.10) is related to the interaction forces between the plates by the relations

\[
\frac{\partial}{\partial \ln \xi_1} \Delta E_N(\xi_1, \xi_2) = p_{N(\text{int})}^{(1)}(\xi_1, \xi_2) = \frac{B_d}{\pi \xi_1^2} \int_0^\infty dk k^{d-2} \int_0^{\infty} dz K_1^2(k\xi_1)(1+z^2/k^2\xi_1^2) K_2^2(k\xi_1) N_1^2(k\xi_1, k\xi_2)
\] (5.12a)

\[
\frac{\partial}{\partial \ln \xi_2} \Delta E_N(\xi_1, \xi_2) = -p_{N(\text{int})}^{(2)}(\xi_1, \xi_2) = -\frac{B_d}{\pi \xi_2^2} \int_0^\infty dk k^{d-2} \int_0^{\infty} dz P_1^2(k\xi_1)(1+z^2/k^2\xi_2^2) P_2^2(k\xi_2) N_2^2(k\xi_1, k\xi_2)
\] (5.12b)

Here \( p_{N(\text{int})}^{(j)} \), \( j = 1, 2 \) are the vacuum effective pressures on the plate at \( \xi = \xi_j \) induced by the presence of the second plate. These quantities are investigated in Ref. [17].

6 Casimir energy for the electromagnetic field

We now turn to the case of the electromagnetic field. We will assume that the plates are perfect conductors with the standard boundary conditions of vanishing of the normal component of the magnetic field and the tangential components of the electric field, evaluated at the local inertial frame in which the conductors are instantaneously at rest. As it has been shown in Ref. [15] for \( d = 3 \) and in Ref. [17] for an arbitrary spatial dimension (on the decomposition of the electromagnetic field in the Rindler coordinates see also Ref. [40]), the corresponding eigenfunctions for the vector potential \( A^\mu \) may be resolved into one transverse magnetic (TM) mode and \( d - 2 \) transverse electric (TE) (with respect to the \( \xi \) direction) modes \( A_{\sigma \alpha}^\mu, \sigma = 0, 1, \ldots, d - 2, \alpha = (k, \omega) \):

\[
A_{1\alpha}^\mu = \left(-\xi \frac{\partial}{\partial \xi}, -i \frac{\omega}{\xi}, 0, \ldots, 0\right) \varphi_{0\alpha}, \quad \sigma = 1, \quad \text{TM mode},
\] (6.1)

\[
A_{\sigma \alpha}^\mu = \epsilon_{\sigma}^\mu \varphi_{\sigma \alpha}, \quad \sigma = 0, 2, \ldots, d - 2, \quad \text{TE modes}.
\] (6.2)

Here the polarization vectors \( \epsilon_{\sigma}^\mu \) obey the following relations:

\[
\epsilon_{\sigma}^\alpha = \epsilon_{\sigma} = 0, \quad \epsilon_{\sigma \mu} \epsilon_{\sigma'}^{\mu'} = -k^2 \delta_{\sigma \sigma'}, \quad \epsilon_{\sigma}^\mu k_{\mu} = 0.
\] (6.3)

From the perfect conductor boundary conditions one has the following conditions for the scalar fields \( \varphi_{\sigma \alpha} \):

\[
\varphi_{\sigma \alpha}(|\xi = \xi_1) = \varphi_{\sigma \alpha}(|\xi = \xi_2) = 0, \quad \sigma = 0, 1, \ldots, d - 2,
\] (6.4)

\[
\frac{\partial \varphi_{1\alpha}}{\partial \xi}(|\xi = \xi_1) = \frac{\partial \varphi_{1\alpha}}{\partial \xi}(|\xi = \xi_2) = 0.
\] (6.5)

As a result the TE/TM modes correspond to the Dirichlet/Neumann scalars and the Casimir energy for the electromagnetic field can be obtained from the scalar results by the formula

\[
E_{em} = (d - 2)E^{(\text{tot})} + E_N.
\] (6.6)
The same relation takes place for the energies \( E_{\text{em}}^{(R)} \), \( E_{\text{em}}^{(L)} \), \( \Delta E_{\text{em}} \), \( E_{\text{em}}^{(\text{tot})} \). In particular, for the total electromagnetic Casimir energy in the geometry of a single plate from numerical results given in Tables 1, 2 one has \( E_{\text{em}} = 0.00160/\xi_1^2 \) in \( d = 3 \). Hence, in this case the electromagnetic forces tend to decelerate the plate.

The total Casimir energy for two plates located at \( \xi = \xi_1 \) and \( \xi = \xi_2 \) can be written in the form

\[
E_{\text{em}}^{(\text{tot})} = E_{\text{em}}^{(L)}(\xi_1) + E_{\text{em}}^{(R)}(\xi_2) + \Delta E_{\text{em}}(\xi_1, \xi_2),
\]

where \( \Delta E_{\text{em}}(\xi_1, \xi_2) < 0 \). Note that the quantity \( \xi_1^{1-1} E_{\text{em}}^{(\text{tot})} \) is a function on the ratio \( \xi_2/\xi_1 \) only. For a given \( \xi_1 \) and for large values of this ratio the main contribution into \( E_{\text{em}}^{(\text{tot})} \) comes from the first term on the right of Eq. (6.7). For \( \xi_2/\xi_1 - 1 \ll 1 \) the last term on the right of this formula dominates. From the numerical results given above it follows that in \( d = 3 \) for a given \( \xi_1 \) the energy \( E_{\text{em}}^{(\text{tot})} \) is positive for large values of \( \xi_2/\xi_1 \) and is negative for \( \xi_2/\xi_1 - 1 \ll 1 \). Hence, for some intermediate value of \( \xi_2/\xi_1 \) this Casimir energy vanishes.

### 7 Conclusion

In quantum field theory the different unitary inequivalent representations of the commutation relations in general give rise to different pictures with different physical implications, in particular to different vacuum states. An interesting issue in the investigations of the Casimir effect is the dependence of the vacuum characteristics on the type of the vacuum. In this paper we have investigated the Casimir energies generated by a single and two parallel plates moving by uniform proper acceleration, assuming that the fields are prepared in the Fulling-Rindler vacuum state. The corresponding vacuum expectation values of the energy–momentum tensor were investigated in Refs. [15, 16] for the geometry of a single plate and in Ref. [17] in the case of two plates. Due to the well known surface divergencies in these expectation values, the total Casimir energy cannot be evaluated by direct integration of the vacuum energy density and needs an additional regularization. In this paper as a regularization method we employ the zeta function technique. We have considered the cases of scalar and electromagnetic fields in an arbitrary number of the spacetime dimensions.

For the scalar case both Dirichlet and Neumann boundary conditions are investigated. In the case of a single plate geometry the right Rindler wedge is divided into two regions, referred as RR and RL regions. By using the Cauchy's theorem on residues, we have constructed an integral representations for the zeta functions in both these regions, which are well suited for the analytic continuation. Subtracting and adding to the integrands leading terms of the corresponding uniform asymptotic expansions, we present the corresponding functions \( Z(s) \) as a sum of two parts. The first one is convergent at \( s = -1 \) and can be evaluated numerically. In the second, asymptotic part the pole contributions are given explicitly in terms of beta function. As a consequence, the Casimir energies for separate RR and RL regions contain pole and finite contributions (see, for example, formulae (2.41) and (2.42) in the case of the Dirichlet scalar in the RR region). The remained pole term is a characteristic feature for the zeta function regularization method and has been found for many other cases of boundary geometries. The coefficient for this term is determined by the corresponding boundary coefficient in the heat kernel asymptotic expansion. For an infinitely thin plate taking RR and RL regions together, in odd spatial dimensions the pole parts cancel and the Casimir energy for the whole Rindler wedge is finite. Note that in this case the total Casimir energy can be directly evaluated by making use formulae (2.68) and (4.57) for the Dirichlet and Neumann scalar fields, respectively. The cancellation of the pole terms coming from oppositely oriented faces of infinitely thin smooth boundaries takes place in very many situations encountered in the literature. It is a simple consequence of the fact that the second fundamental forms are equal and opposite on the two faces of each boundary and, consequently, the value of the corresponding coefficient in the heat kernel expansion summed over the two faces of each boundary vanishes [21]. In even dimensions there is no such a cancellation. The numerical results for the separate pole and finite contributions to the Casimir
energy in RR and RL regions are summarized in Table 1 for the Dirichlet case and in Table 2 for the Neumann boundary condition. For the physically most important case $d = 3$ the total Casimir energy is negative for the Dirichlet scalar and positive for the Neumann scalar. This means that the vacuum forces tend to accelerate the Dirichlet plate and to decelerate the Neumann plate.

In the case of two parallel plates configuration we have derived integral representations for both Dirichlet and Neumann zeta functions. The corresponding Casimir energies are presented as a sum of single plate parts and the interference term. The latter is determined by formula (3.11) for Dirichlet scalar and by formula (5.10) for Neumann scalar and is located in the region between the plates. It is always negative and is related to the corresponding interaction forces by the standard thermodynamical relations (see Eqs. (3.14), (5.12)). For large values of the separation between the plates, the total Casimir energy is dominated by the contributions coming from the single plate parts. For a given $\xi_1$ and small values of the separation the interference part is dominant and the total Casimir energy is negative. For $d = 3$ Dirichlet scalar this is the case for all values of the separation. However, for $d = 3$ Neumann scalar due to the positive single plate energies, the total Casimir energy for two plates is negative for small separations and positive for large separations. Consequently, this energy vanishes for some intermediate value of the ratio $\xi_2/\xi_1$. Letting $\xi_2 \to \infty$ correspond to removing one of the plates and from the formulae for two plates we recover the Casimir energy for a single plate. The case $d = 1$ is considered separately in Appendix A. In this dimension the Casimir energies for Dirichlet and Neumann scalars are the same. They vanish for a single plate (point) geometry and are negative for the two plates case. Note that in $d = 1$ the problem is conformally related to the corresponding problem on background of the Minkowski spacetime.

In Sec. 6 the case of the electromagnetic field is considered with the perfect conductor boundary conditions in the local inertial frame in which the boundaries are instantaneously at rest. The corresponding eigenmodes are superposition of TE and TM modes with Dirichlet and Neumann boundary conditions, respectively. The Casimir energies for the electromagnetic field can be derived from the corresponding scalar results making use formula (6.6). In particular, the total electromagnetic vacuum energy of a single plate in $d = 3$ is positive and the vacuum forces tend to decelerate the plate. For two plates geometry the situation for the electromagnetic field is similar to the Neumann scalar case. The total Casimir energy is negative for small values of the plates separation (the interference part dominates) and is positive for large separations (single plates parts dominate).

In the main part of this paper we have considered the vacuum energy corresponding to the dimensionless coordinate $\tau$ in Eq. (2.3). The corresponding eigenfrequencies are also dimensionless and there is no need to introduce an arbitrary mass scale in the definitions of the related zeta functions. However, this mass scale is necessary if we consider the vacuum energy measured by an uniformly accelerated observer. The relation between these energies is discussed in Appendix B, where we have shown that they are connected by formula (B.4). In this formula the logarithmic term with an arbitrary mass scale $\mu$ has to be viewed as a remainder of the renormalization process (for a discussion see [7, 21]). For infinitely thin boundaries in odd dimensions, in calculations of the total vacuum energy, including the parts from two sides of the boundary, these terms cancel and a unique result emerges.

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A \textit{d = 1 case}

Let us consider the scalar vacuum in the region between two boundaries located at $\xi = \xi_1$ and $\xi = \xi_2$. The normalized eigenfunctions satisfying Dirichlet boundary conditions and the corresponding eigenfrequencies are in the form [17]

\[ \varphi_n^D = \frac{\exp(-i\omega_n\tau)}{\sqrt{\pi n}} \sin\left(\omega_n \ln\left(\frac{\xi_2}{\xi_1}\right)\right), \quad \omega_n = \frac{\pi n}{\ln\left(\frac{\xi_2}{\xi_1}\right)}, \quad n = 1, 2, \ldots \quad (A.1) \]

For the corresponding vacuum energy one finds

\[ E_D = \frac{1}{2} \sum_{n=1}^{\infty} \omega_n = \frac{\pi \zeta_R(-1)}{2 \ln(\xi_2/\xi_1)} = -\frac{\pi}{24 \ln(\xi_2/\xi_1)}, \quad (A.2) \]

where $\zeta_R(x)$ is the Riemann zeta function. The renormalized Casimir energies $E_{1D}^{(R)}$ and $E_{1D}^{(L)}$ for a single boundary are obtained from (A.2) in the limits $\xi_1 \to 0$ and $\xi_2 \to \infty$, respectively, and both these quantities vanish.

For the case of the Neumann boundary conditions the eigenfunctions have the form

\[ \varphi_n^N = \frac{\exp(-i\omega_n\tau)}{\sqrt{\pi n}} \cos\left(\omega_n \ln\left(\frac{\xi_2}{\xi_1}\right)\right), \quad n = 0, 1, 2, \ldots \quad (A.3) \]

with the same eigenfrequencies as in Eq. (A.1). The corresponding Casimir energy is the same as in the Dirichlet case: $E_N = E_D$.

B Relation to the energy measured by an uniformly accelerated observer

In this appendix we consider the relation of the energies evaluated above to the energy measured by an uniformly accelerated observer. The frequency $\omega$ in Eq. (2.4) corresponds to the dimensionless coordinate $\tau$ and, hence, is dimensionless. Proper time $\tau_g$ and the frequency $\omega_g$ measured by an uniformly accelerated observer with the proper acceleration $g$ and word line $(x^1)^2 - t^2 = g^{-2}$, are related to $\tau$ and $\omega$ by formulae $\tau_g = \tau/g$, $\omega_g = \omega g$ (the features of the measurements for time, frequency, and length relative to a Rindler frame as compared to a Minkowski frame are discussed in Ref. [41]). The corresponding vacuum energy has the form

\[ E_g = \frac{1}{2} \int \frac{d^d-1 k}{(2\pi)^{d-1}} \sum_{n=1}^{\infty} \omega_{gn}, \quad (B.1) \]

where $\omega_{gn} = \omega_{gn}(x)$ are the eigenfrequencies measured by the uniformly accelerated observer. As above we introduce the partial zeta function related to these eigenfrequencies and the function $Z_g(s)$ by relations

\[ \zeta_g(s, x) = \sum_{n=1}^{\infty} (\omega_{gn}/\mu)^{-s}, \quad Z_g(s) = \int_0^\infty dx x^{d-2} \zeta_g(s, x). \quad (B.2) \]

Note that we have introduced an arbitrary scale $\mu$ with mass dimension, in order to keep the zeta function dimensionless for all $s$. Now the Casimir energy can be written as

\[ E_g = \frac{\mu B_d}{2\zeta_{d-1}^{|s|=1}} Z_g(s)|_{s=-1}. \quad (B.3) \]

By taking into account the relation between the eigenfrequencies corresponding to $\tau$ and $\tau_g$ coordinates, one obtains the relation between $Z_g(s)$ and the function $Z(s)$ considered above: $Z_g(s) = (\mu/g)^s Z(s)$. 

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Laurent-expanding over $s + 1$ and using the corresponding expansion for the function $Z(s)$ given above, we obtain the following relation between the Casimir energies:

$$E_g = g \left( E + \ln(\mu/g) \right) \frac{B_d Z^{(as)}_{s-1}}{\xi d^{-1}},$$  \hspace{1cm} (B.4)

where the energy $E$ corresponds to the dimensionless coordinate $\tau$. In dependence of the boundary conditions and the region under consideration, here as a pair $(E, Z^{(as)}_{s-1})$ we have to take $(E^{(R)}_{1F}, Z^{(as)}_{FK}, -1)$, $(E^{(L)}_{1F}, Z^{(as)}_{FK}, -1)$, $(E^{(L)}_{1F}, Z^{(as)}_{FK}, -1) + Z^{(as)}_{FK, -1}$, where $F = D, N, em$. Note that in odd spatial dimensions for the total Casimir energy $E_{1Fg}$ of a single plate the scale dependent parts from the RR and RL regions cancel and we obtain a scale independent energy.

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