On Fractional $q$-Sturm–Liouville Problems

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Abstract

In this paper, we formulate a regular $q$-fractional Sturm–Liouville problem (qF-SLP) which includes the left-sided Riemann–Liouville and the right-sided Caputo $q$-fractional derivatives of the same order $\alpha$, $\alpha \in (0,1)$. The properties of the eigenvalues and the eigenfunctions are investigated. A $q$-fractional version of the Wronskian is defined and its relation to the simplicity of the eigenfunctions is verified. We use the fixed point theorem to introduce a sufficient condition on eigenvalues for the existence and uniqueness of the associated eigenfunctions when $\alpha > 1/2$. These results are a generalization of the integer regular $q$-Sturm–Liouville problem introduced by Annaby and Mansour in [1]. An example for a qFSLP whose eigenfunctions are little $q$-Jacobi polynomials is introduced.

Keywords: Left and right sided Riemann–Liouville and Caputo $q$-derivatives, eigenvalues and eigenfunctions, existence and uniqueness theorem, $q, \alpha$ Wronskian.

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1. Introduction

In the joint paper of Sturm and Liouville [2], they studied the problem

$$- \frac{d}{dx} \left( p \frac{dy}{dx} \right) + r(x)y(x) = \lambda w y(x), \quad x \in [a,b],$$

(1.1)

with certain boundary conditions at $a$ and $b$. Here, the functions $p$, $w$ are positive on $[a,b]$ and $r$ is a real valued function on $[a,b]$. They proved the existence of non-zero solutions (eigenfunctions) only for special values of the parameter $\lambda$ which is called eigenvalues. For a comprehensive study for the contribution of Sturm and Liouville to the theory, see [3]. Recently, many mathematicians were interested in a fractional version of (1.1), i.e. when the derivative is replaced by a fractional derivative like Riemann–Liouville derivative or Caputo derivative, see [4–9]. In [1] Annaby and Mansour introduced a $q$-version of (1.1), i.e.,
when the derivative is replaced by Jackson $q$-derivative. Their results are applied and developed in different aspects, for example, see [10–15]. In this paper, we introduce a $q$-fractional Sturm–Liouville problem. The operator $-\frac{d}{dx} \left( p \frac{d}{dx} \right)$ in (1.1) is self-adjoint operator because the adjoint of the operator $\frac{d}{dx}$ is $-\frac{d}{dx}$ in a certain subspace of $L^2(0, a)$. So, the negative sign in (1.1) exists on purpose. Therefore, the fractional $q$-Sturm–Liouville problem under consideration contains both of the left-sided Caputo $q$-fractional derivative and the right-sided Riemann–Liouville $q$-fractional derivatives because they are adjoint operators in a certain Hilbert space.

This paper is organized as follows. Section 2 is on the $q$-fractional operators and their properties which we need in the sequel. In Section 3, we formulate the fractional $q$-Sturm–Liouville problem, we show that the eigenvalues are real and the eigenfunctions associated to different eigenvalues are orthogonal in a certain Hilbert space. We use the fixed point theory to give a sufficient condition on the parameter $\lambda$ to guarantee the existence and uniqueness of the solution. We also impose a condition on the domain of the problem in order to prove the existence of solution for any $\lambda$. The Wronskian associated with the problem is defined and a relation between its value at zero and the simplicity of eigenfunctions is proved in Section 4. Finally, in Section 5, an example for a qFSLP whose eigenfunctions are little $q$-Jacobi polynomials is introduced. Throughout this paper $q$ is a positive number less than 1. The set of non negative integers is denoted by $\mathbb{N}_0$, and the set of positive integers is denoted by $\mathbb{N}$. For $t > 0$,

$$A_{q,t} := \{tq^n : n \in \mathbb{N}_0\}, \quad A_{q,t}^* := A_{q,t} \cup \{0\},$$

and

$$A_{q,t} := \{\pm tq^n : n \in \mathbb{N}_0\}.$$ When $t = 1$, we simply use $A_q$, $A_{q,1}^*$, and $A_q$ to denote $A_{q,1}$, $A_{q,1}^*$, and $A_{q,1}$, respectively. We follow [16] for the definitions and notations of the $q$-shifted factorial, the $q$-gamma and $q$-beta functions, the basic hypergeometric series, and Jackson $q$-difference operator and integrals. A set $A$ is called a $q$-geometric set if $qx \in A$ whenever $x \in A$. Let $X$ be a $q$-geometric set containing zero. A function $f$ defined on $X$ is called $q$-regular at zero if

$$\lim_{n \to \infty} f(xq^n) = f(0) \quad \text{for all } x \in X.$$ Let $C(X)$ denote the space of all $q$-regular at zero functions defined on $X$ with values in $\mathbb{R}$. $C(X)$ associated with the norm function

$$\|f\| = \sup \{|f(xq^n)| : x \in X, \ n \in \mathbb{N}_0\},$$

is a normed space. The $q$-integration by parts rule [17] is

$$\int_a^b f(x) D_q g(x) = f(x) g(x)|_a^b + \int_a^b D_q f(x) g(qx) \, dq x, \ a, b \in X, \quad (1.2)$$

and $f$, $g$ are $q$-regular at zero functions.
For $p > 0$, and $Y$ is $A_{q,t}$ or $A_{q,t}^*$, the space $L^p_q(Y)$ is the normed space of all functions defined on $Y$ such that

$$\|f\|_p := \left( \int_0^t |f(u)|^p d_q u \right)^{1/p} < \infty.$$  

If $p = 2$, then $L^2_q(Y)$ associated with the inner product

$$\langle f, g \rangle := \int_0^t f(u)g(u) d_q u$$  

(1.3)

is a Hilbert space. By a weighted $L^2_q(Y, w)$ space is the space of all functions $f$ defined on $Y$ such that

$$\int_0^t |f(u)|^2 w(u) d_q u < \infty,$$

where $w$ is a positive function defined on $Y$. $L^2_q(Y, w)$ associated with the inner product

$$\langle f, g \rangle := \int_0^t f(u)g(u)w(u) d_q u$$

is a Hilbert space. The space of all $q$-absolutely functions on $A_{q,t}^*$ is denoted by $AC_q(A_{q,t}^*)$ and defined as the space of all $q$-regular at zero functions $f$ satisfying

$$\sum_{j=0}^{\infty} |f(uq^j) - f(uq^j+1)| \leq K \text{ for all } u \in A_{q,t},$$

and $K$ is a constant depending on the function $f$, c.f. [17, Definition 4.3.1]. I.e.

$$AC_q(A_{q,t}^*) \subseteq C_q(A_{q,t}^*).$$

The space $AC_q^{(n)}(A_{q,t}^*)$ ($n \in \mathbb{N}$) is the space of all functions defined on $X$ such that $f, D_q f, \ldots, D_q^{n-1} f$ are $q$-regular at zero and $D_q^{n-1} f \in AC_q(A_{q,t}^*)$, c.f. [17, Definition 4.3.2]. Also it is proved in [17, Theorem 4.6] that a function $f \in AC_q^{(n)}(A_{q,t}^*)$ if and only if there exists a function $\phi \in L^1_q(A_{q,t}^*)$ such that

$$f(x) = \sum_{k=0}^{n-1} \frac{D_q^k f(0)}{\Gamma_q(k+1)} x^k + \frac{x^{n-1}}{\Gamma_q(n)} \int_0^x (qu/x; q)_{n-1} \phi(u) d_q u, \ x \in A_{q,t}^*.$$  

In particular, $f \in AC(A_{q,t}^*)$ if and only if $f$ is $q$-regular at zero such that $D_q f \in L^1_q(A_{q,t}^*)$. It is worth noting that in [17], all the definitions and results we have just mentioned are defined and proved for functions defined on the interval $[0, a]$ instead of $A_{q,t}^*$.  

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2. Fractional $q$-Calculus

This section includes the definitions and properties of the left sided and right sided Riemann–Liouville $q$-fractional operators which we need in our investigations.

The left sided Riemann–Liouville $q$-fractional operator is defined by

$$I_{q,a}^{\alpha} f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_a^x (qt/x; q)_{\alpha-1} f(t) d_q t. \quad (2.1)$$

This definition is introduced by Agarwal in [18] when $a = 0$ and by Rajković et.al [19] for $a \neq 0$. We define a right sided Riemann–Liouville $q$-fractional operator by

$$I_{q,b}^{\alpha} f(x) = \frac{1}{\Gamma_q(\alpha)} \int_b^x t^{\alpha-1} (qx/t; q)_{\alpha-1} f(t) d_q t. \quad (2.2)$$

One can prove that if $x = bq^m$, $m \in \mathbb{N}_0$, then

$$I_{q,b}^{\alpha} f(x) = b^{\alpha} (1-q)^{\alpha} \sum_{j=0}^{m} q^{\alpha j} (q; q)_{m-j} f(bq^j)$$

$$= b^{\alpha} (1-q)^{\alpha} \frac{(q^\alpha; q)_m}{(q; q)_m} \sum_{j=0}^{m} q^{j \mu} \frac{(q^{-m}; q)_j}{(q^{1-m-a}; q)_j} f(bq^j),$$

where we used [16, Eq. (I.11)]. For example,

$$I_{q,b}^{\alpha} f(x) = b^{\alpha} (1-q)^{\alpha} (q^\alpha; q)_m (q; q)_{m-1} (q^{-m}; q)_{m-1} f(bq^m)$$

$$= \frac{\Gamma_q(\mu + 1)}{\Gamma_q(\mu + \alpha + 1)} b^{\alpha+\mu} \frac{(q^\alpha; q)_m}{(q; q)_m} f(bq^m).$$

The left sided Riemann–Liouville $q$-fractional operator satisfies the semigroup property

$$I_{q,a}^{\alpha} I_{q,b}^{\beta} f(x) = I_{q,a+\beta}^{\alpha+\beta} f(x).$$

The case $a = 0$ is proved in [18] and the case $a > 0$ is proved in [19].

**Theorem 2.1.** The right sided Riemann–Liouville $q$-fractional operator satisfies the semigroup property

$$I_{q,b}^{\alpha} I_{q,b}^{\beta} f(x) = I_{q,b}^{\alpha+\beta} f(x), \quad x \in A^*_q,$$

for any function defined on $A^*_q$ and for any values of $\alpha$ and $\beta$. 

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Proof. Let \( x \in A_{q,b}^* \). Then
\[
I_{q,b}^\alpha - I_{q,b}^\beta f(x) = \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{qx}^b t^{\alpha-1}(qx/t; q)_{\alpha-1} t^{\beta-1}(qt/u; q)_{\beta-1} f(u) \, du \, dq t.
\]
\[
= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{qx}^b u^{\beta-1} f(u) \int_{t=qx}^u t^{\alpha-1}(qx/t; q)_{\alpha-1} (qt/u; q)_{\beta-1} f(u) \, du \, dq u.
\]
But
\[
\int_{qx}^u t^{\alpha-1}(qx/t; q)_{\alpha-1} (qt/u; q)_{\beta-1} \, dq t = u^\alpha \beta_q(\alpha, \beta)(qx/u; q)_{\alpha+\beta-1}.
\]
then substituting from (2.5) into (2.4) yields (2.3).
\]

**Definition 2.2.** Let \( \alpha > 0 \) and \( q\alpha^{-1} = m \). The left and right side Riemann–Liouville fractional \( q \)-derivatives of order \( \alpha \) are defined by
\[
D_{q,a}^\alpha f(x) := D_q^m I_{q,a}^{m-\alpha} f(x), \quad D_{q,a}^\alpha f(x) := \left(\frac{-1}{q}\right)^m D_q^m I_{q,a}^{m-\alpha} f(x),
\]
the left and right sided Caputo fractional \( q \)-derivatives of order \( \alpha \) are defined by
\[
cD_{q,a}^\alpha f(x) := I_{q,a}^{m-\alpha} D_q^m f(x), \quad cD_{q,a}^\alpha f(x) := \left(\frac{-1}{q}\right)^m I_{q,b}^{m-\alpha} D_q^m f(x).
\]
From now on, we shall consider left sided Riemann–Liouville and Caputo fractional \( q \)-derivatives when the lower point \( a = 0 \) and right sided Riemann–Liouville and Caputo fractional \( q \)-derivatives when \( b = a \). According to [17, pp. 124, 148], \( D_{q,a}^\alpha, f(x) \) exists if
\[
f \in L_q^1(A_{q,a}^*) \text{ such that } I_{q,a}^{m-\alpha} f \in AC_q^{(m)}(A_{q,a}^*),
\]
and \( cD_{q,a}^\alpha f \) exists if
\[
f \in AC_q^{(m)}(A_{q,a}^*).
\]
**Lemma 2.3.** Let \( \alpha \in (0,1) \).
(i) If \( f \in L_q^1(A_{q,a}^*) \) such that \( I_{q,0+a}^\alpha f \in AC_q(A_{q,a}^*) \) then
\[
cD_{q,0+a}^\alpha I_{q,0+a}^\alpha f(x) = f(x) - \frac{I_{q,0+a}^\alpha f(0)}{\Gamma_q(1-\alpha)} x^{-\alpha}.
\]
Moreover, if \( f \) is bounded on \( A_{q,a}^* \) then
\[
cD_{q,0+a}^\alpha I_{q,0+a}^\alpha f(x) = f(x).
\]
(ii) For any function \( f \) defined on \( A_{q,a}^* \)
\[
cD_{q,a}^\alpha - I_{q,a}^\alpha f(x) = f(x) - \frac{a^{-\alpha}}{\Gamma_q(1-\alpha)} (qx/a; q)_{-\alpha} \left( I_{q,a}^\alpha f \right) \left( \frac{a}{q} \right).
\]
(iii) If \( f \in L^1_q(A_{q,a}) \) then
\[
D_{q,a}^\alpha I_{q,a}^\alpha f(x) = f(x). 
\]
(2.9)

(iv) For any function \( f \) defined on \( A^*_q \)
\[
D_{q,a}^\alpha f(x) = f(x).
\]
(2.10)

(v) If \( f \in AC_q(A_{q,a}) \) then
\[
I_{q,a}^\alpha D_{q,a}^\alpha f(x) = f(x) - f(0).
\]
(2.11)

(vi) If \( f \) is a function defined on \( A_{q,a}^* \)
\[
D_{q,a}^\alpha I_{q,a}^\alpha f(x) = f(x) - f(0) - \frac{\Gamma_q(1 - \alpha)}{q} \left( I_{q,a}^\alpha f \right) \left( \frac{a}{q} \right) d_q t.
\]
(2.12)

Proof. The proof of (2.6) is a special case of [17, Eq. (5.7)] but note that there is a misprint in the formula (5.7), the summation should start from \( i = 1 \). If \( f \) is bounded on \( A_{q,a} \), then \( I_{q,a}^\alpha D_{q,a}^\alpha f(0) = 0 \), and (2.7) follows at once from (2.6). Now we prove (2.8).c\( D_{q,a}^\alpha I_{q,a}^\alpha f(x) = f(x) - f(0) - \frac{\Gamma_q(1 - \alpha)}{q} \left( I_{q,a}^\alpha f \right) \left( \frac{a}{q} \right) d_q t, \)
where we used \( \frac{\Gamma_q(1 - \alpha)}{q} \left( I_{q,a}^\alpha f \right) \left( \frac{a}{q} \right) = 0 \), and the semigroup property (2.3). Hence, the result follows from the semigroup property (2.3). The proof of (2.10) follows from the fact that
\[
D_{q,a}^\alpha I_{q,a}^\alpha f(x) = f(x) - \frac{1}{q} I_{q,a}^\alpha f(x) = - \frac{1}{q} D_{q,a}^\alpha I_{q,a}^\alpha f(x) = f(x),
\]
where we used the semigroup property (2.3). The proof of (2.11) is a special case of [17, Eq. (5.6)]. The proof of (2.12) is similar to the proof of (2.10) and is omitted. Finally, the proof of (2.13) is a special case of [17, Eq. (5.8)].
Set $X = A_{q,a}$ or $A_{q,a}^*$. Then

$$C(X) \subseteq L_q^2(X) \subseteq L_q^1(X).$$

Moreover, if $f \in C(X)$ then

$$\|f\|_1 \leq \sqrt{a} \|f\|_2 \leq a \|f\|.$$

We have also the following inequalities:

1. If $f \in C(A_{q,a}^*)$ then $I_{q,0}^\alpha f \in C(A_{q,a}^*)$ and

$$\left\| I_{q,0}^\alpha f \right\| \leq \frac{a^\alpha}{\Gamma_q(\alpha + 1)} \|f\|. \quad (2.14)$$

2. If $f \in L_q^1(X)$ then $I_{q,0}^\alpha f \in L_q^1(X)$ and

$$\left\| I_{q,0}^\alpha f \right\|_1 \leq M_{\alpha,1} \|f\|_1, \quad M_{\alpha,1} := \frac{a^\alpha(1 - q)^\alpha}{(1 - q^\alpha)(q; q)_{\infty}}. \quad (2.15)$$

3. If $f \in L_q^2(X)$ then $I_{q,0}^\alpha f \in L_q^2(X)$ and

$$\left\| I_{q,0}^\alpha f \right\|_2 \leq M_{\alpha,2} \|f\|_2, \quad M_{\alpha,2} := \frac{a^\alpha}{\Gamma_q(\alpha)} \sqrt{\frac{(1 - q^2)(q; q)_{\infty}^2}{(1 - q^2\alpha)^2}} \left( \int_0^1 (q\xi; q)_{\alpha-1}^2 d\xi \right)^{1/2}. \quad (2.16)$$

4. If $\alpha > \frac{1}{2}$ and $f \in L_q^2(X)$ then $I_{q,0}^\alpha f \in C(X)$ and

$$\left\| I_{q,0}^\alpha f \right\| \leq \tilde{M}_\alpha \|f\|, \quad \tilde{M}_\alpha := \frac{a^{\alpha - \frac{1}{2}}}{\Gamma_q(\alpha)} \left( \int_0^1 (q\xi; q)_{\alpha-1}^2 d\xi \right)^{1/2}. \quad (2.17)$$

5. Since $\|f\|_2 \leq \sqrt{a} \|f\|$, we conclude that if $f \in C(X)$ then $I_{q,0}^\alpha f \in L_q^2(X)$ and

$$\left\| I_{q,0}^\alpha f \right\|_2 \leq K_\alpha \|f\|, \quad K_\alpha := \sqrt{a} M_{\alpha,2}. \quad (2.18)$$

6. If $f \in C(A_{q,a}^*)$ then $I_{q,a}^\alpha f \in C(A_{q,a}^*)$ and

$$\left\| I_{q,a}^\alpha f \right\| \leq c_{\alpha,0} \|f\|, \quad c_{\alpha,0} := \frac{a^\alpha(1 - q)^\alpha}{(1 - q^\alpha)(q; q)_{\infty}}. \quad (2.19)$$

7. If $f \in L_q^1(X)$ then $I_{q,a}^\alpha f \in L_q^1(X)$ and

$$\left\| I_{q,a}^\alpha f \right\|_1 \leq \begin{cases} \frac{(1 - q)^\alpha a^\alpha}{(1 - q^\alpha)(q; q)_{\infty}} \|f\|_1, & \text{if } \alpha < 1, \\ \frac{(1 - q)^{\alpha-1}q^{\alpha-1}}{(q; q)_{\infty}} \|f\|_1, & \text{if } \alpha \geq 1. \end{cases}$$
8. If $\alpha \neq \frac{1}{2}$ and $f \in L^2_q(X)$ then $I^\alpha_{q,a} f \in L^1_q(X)$ and
\[
\|I^\alpha_{q,a} f\|_2 \leq \begin{cases} 
\frac{(1-q)^{\alpha - \frac{1}{2}}}{\sqrt{1-q^{2\alpha-1}(q; q)_\infty}} \|f\|_2, & \text{if } \alpha < \frac{1}{2}, \\
\frac{(1-q)^\alpha}{(q; q)_\infty \sqrt{(1-q^{2\alpha-1})(1-q^{2\alpha - 1})}} \|f\|_2, & \text{if } \alpha > \frac{1}{2}.
\end{cases}
\]

**Lemma 2.4.** Let $\alpha > 0$. If

(a) $f \in L^1_q(X)$ and $g$ is a bounded function on $A_{q,a}$, or

(b) $\alpha \neq \frac{1}{2}$ and $f$, $g$ are $L^2_q(X)$ functions

then
\[
\int_0^a g(x)I^\alpha_{q,a} f(x) \, dq(x) = \int_0^a f(x)I^\alpha_{q,a} g(x) \, dq(x).
\]

**Proof.** The condition (a) or (b) of the present lemma assures the convergence of the $q$-integrals in (2.19). Since
\[
\int_0^a g(x)I^\alpha_{q,a} f(x) \, dq(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^a g(x)x^{\alpha-1} \int_0^x (qt/x; q)_{\alpha-1} f(t) \, dq(t) \, dq(x),
\]
from the conditions on the functions $f$ and $g$, the double $q$-integral is absolutely convergent, therefore we can interchange the order of the $q$-integrations to obtain
\[
\int_0^a g(x)I^\alpha_{q,a} f(x) \, dq(x) = \int_0^a f(t) \frac{1}{\Gamma_q(\alpha)} \int_0^t x^{\alpha-1} (qt/x; q)_{\alpha-1} g(x) \, dq(x) \, dq(t)
\]
\[
= \int_0^a f(t)I^\alpha_{q,a} g(t) \, dq(t).
\]

**Lemma 2.5.** Let $\alpha \in (0, 1)$.

(a) If $g \in L^1_q(A^*_{q,a})$ such that $I^{1-\alpha}_q g \in AC_q(A^*_{q,a})$, and $D^i_q f \in C(A^*_{q,a})$ ($i = 0, 1$) then
\[
\int_0^a f(x)D^\alpha_{q,a} y(x) \, dq(x) = -f(\frac{x}{q})I^{1-\alpha}_q g(x)|^a_x + \int_0^a g(x)D^\alpha_{q,a} f(x) \, dq(x).
\]

(b) If $f \in AC_q(A^*_{q,a})$, and $g$ is a bounded function on $A^*_{q,a}$ such that $D^\alpha_{q,a} g \in L^1_q(A^*_{q,a})$ then
\[
\int_0^a g(x)D^\alpha_{q,a} f(x) \, dq(x) = \left(I^{1-\alpha}_q g \left(\frac{x}{q}\right) f(x)\right)|^a_x + \int_0^a f(x)D^\alpha_{q,a} g(x) \, dq(x).
\]

**Proof.** The conditions on the functions $f$ and $g$ guarantee the convergence of the $q$-integrals in (2.20) and (2.21), and their proofs follow from Lemma 2.4 and the $q$-integration by parts rule [1.2].
3. Regular Fractional $q$-Sturm Liouville problems

**Definition 3.1.** Let $\alpha \in (0, 1)$. With the notation

$$L_{q,\alpha} y := D_{q,a}^\alpha - p(x)^c D_{q,0}^\alpha y(x) + r(x)y(x),$$

consider the fractional $q$-Sturm–Liouville equation

$$L_{q,\alpha} y(x) - \lambda w_{\alpha}(x)y(x) = 0, \quad x \in A_{q,a}^*, \quad (3.1)$$

where $p(x) \neq 0$ and $w_{\alpha} > 0$ for all $x \in A_{q,a}^*$, $p, r, w_{\alpha}$ are real valued functions defined in $A_{q,a}^*$ and the associated boundary conditions are

$$c_1 y(0) + c_2 \left[I_{q,a}^{1-\alpha} p^c D_{q,0}^\alpha y\right](0) = 0, \quad (3.2)$$

$$d_1 y(a) + d_2 \left[I_{q,a}^{1-\alpha} p^c D_{q,0}^\alpha y\right](a) = 0, \quad (3.3)$$

with $c_1^2 + c_2^2 \neq 0$ and $d_1^2 + d_2^2 \neq 0$.

As in the classical case, the problem of finding the complex numbers $\lambda$'s such that the boundary value problem has a non-trivial solution will be called a regular $q$-fractional Sturm–Liouville problem (regular qFSLP). Such a value $\lambda$, is called an eigenvalue and the corresponding non-trivial solution, the eigenfunction.

In the following, we assume that $0 < \alpha < 1$ and consider the subspace of the vector space $L_q^2(A_{q,a}^*)$ of all $q$-regular at zero functions satisfying the boundary conditions (3.2)–(3.3). Hence $V \subseteq L_q^2(A_{q,a}^*) \cap C(A_{q,a}^*)$ and $V$ associated with the inner product is a Hilbert space.

**Theorem 3.2.** $L_{q,\alpha}$ is a self-adjoint operator on the Hilbert space $V$.

**Proof.** One can prove that for any functions $f, g \in L_q^2(0, a) \cap C(A_{q,a}^*)$, we have

$$\left\langle D_{q,a}^\alpha f, g \right\rangle = -g(x) \left[I_{q,a}^{1-\alpha} p^c D_{q,0}^\alpha y\right](x) \bigg|_0^a + \left\langle f, c D_{q,0}^\alpha y \right\rangle. \quad (3.4)$$

Therefore, for $u, v \in V$

$$\langle L_{q,\alpha} u, v \rangle - \langle u, L_{q,\alpha} v \rangle = \left[u(x) \left(I_{q,a}^{1-\alpha} p^c D_{q,0}^\alpha u\right) \left(\frac{x}{q}\right) - v(x) \left(I_{q,a}^{1-\alpha} p^c D_{q,0}^\alpha u\right) \left(\frac{x}{q}\right) \right]_{x=0}^a + .$$

This yields the Green’s identity

$$\int_0^a \left(u(x)L_{q,\alpha} v(x) - v(x)L_{q,\alpha} u(x)\right) \, d_q x = \left[u(x) \left(I_{q,a}^{1-\alpha} p^c D_{q,0}^\alpha u\right) \left(\frac{x}{q}\right) - v(x) \left(I_{q,a}^{1-\alpha} p^c D_{q,0}^\alpha u\right) \left(\frac{x}{q}\right) \right]_{x=0}^a. \quad (3.5)$$
If \( u \) and \( v \) are in the space \( V \), then they satisfy the boundary condition (3.2) at \( x = 0 \), then
\[
\begin{pmatrix}
  u(0) \\
  v(0)
\end{pmatrix}
\begin{pmatrix}
  (I_1 - \alpha q,a - p c D_{\alpha q,0} + u)(0) \\
  \cdot
\end{pmatrix}
\begin{pmatrix}
  c_1 \\
  c_2
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  0
\end{pmatrix}.
\]

Since \( c_2^2 + c_2^2 \neq 0 \), we should have
\[
\begin{pmatrix}
  u(0) \\
  v(0)
\end{pmatrix}
\begin{pmatrix}
  (I_1 - \alpha q,a - p c D_{\alpha q,0} + u)(0) \\
  \cdot
\end{pmatrix}
\begin{pmatrix}
  c_1 \\
  c_2
\end{pmatrix}
= 0
\]

In the same way we see that if \( u \) and \( v \) satisfy the conditions (3.3) at \( x = a \), then
\[
\begin{pmatrix}
  u(a) \\
  v(a)
\end{pmatrix}
\begin{pmatrix}
  (I_1 - \alpha q,a - p c D_{\alpha q,0} + u)(a) \\
  \cdot
\end{pmatrix}
\begin{pmatrix}
  c_1 \\
  c_2
\end{pmatrix}
= 0
\]

Hence
\[
\langle L_{q,\alpha} u, v \rangle = \langle u, L_{q,\alpha} v \rangle
\]
and the theorem follows.

**Theorem 3.3.** The eigenvalues of the regular qFSLP (3.1)–(3.3) are real.

**Proof.** Assume that \( \lambda \) is an eigenvalue associated with an eigenfunction \( y \). Then we have
\[
L_{q,\alpha} y(x) = \lambda w_\alpha(x) y(x), \quad L_{q,\alpha} \overline{y}(x) = \overline{\lambda} w_\alpha(x) \overline{y(x)}.
\]
Therefore from Green’s identity (3.6)
\[
(\overline{\lambda} - \lambda) \int_0^a w_\alpha(x) |y(x)|^2 d_q x = \int_0^a \left( y(x)L_{q,\alpha} \overline{y(x)} - \overline{y(x)}L_{q,\alpha} y(x) \right) d_q x = 0.
\]
Since \( y \) is a non trivial solution and \( w_\alpha > 0 \), we obtain \( \lambda = \overline{\lambda} \).

**Lemma 3.4.** If \( u \) and \( v \) are eigenfunctions of the regular qFSLP (3.1)–(3.3) associated with different eigenvalues \( \lambda \) and \( \mu \), then \( u \) and \( v \) are orthogonal on the weighted space \( L^2_q(A_{q,a}^*, w_\alpha) \).

**Proof.** Since \( L_{q,\alpha} \) is self adjoint, then substituting with \( L_{q,\alpha} u = \lambda w_\alpha u \) and \( L_{q,\alpha} v = \mu w_\alpha v \) in the identity (3.6) taking into consideration that the eigenvalues are real, we obtain
\[
(\overline{\lambda} - \lambda) \int_0^a u(x)v(x) w_\alpha(x) d_q x = 0.
\]
Since \( \lambda \neq \mu \), then \( \int_0^a u(x)v(x) w_\alpha(x) d_q x = 0 \) and the lemma follows.

In the following we use the fixed point theorem to show that for the regular qFSLP (3.1)–(3.3) if the eigenvalues satisfying certain condition, then the eigenfunctions are unique up to a multiplying constant on the space \( C(A_{q,a}^*) \) for any
We also prove that under a certain constrain on the domain of solutions, for any eigenvalue \( \lambda \), the eigenfunction is unique up to a constant multiplying factor. Since 
\[
(I_{\alpha}^{q,a} - 1)(x) = I_{\alpha}^{q,a} \frac{a^\alpha (qx/a; q)_\alpha}{\Gamma_q(\alpha + 1)}
\]
\[
= \frac{a^\alpha x^\alpha}{\Gamma^2_q(\alpha + 1)} \phi_1(\frac{q^{-\alpha}, q; q^{a+1}, q, x^{a+1}}{a}) =: \phi(x),
\]
for \( |x^{a+1}| < 1 \). The general solution of the equation
\[
D_{\alpha}^{q,a} p(x) D_{\alpha}^{q,a} \phi_0(x) = 0
\]
takes the form
\[
\phi_0(x) = \xi_1 + \xi_2 I_{\alpha}^{q,a} \frac{a^{\alpha-1}(qx/a; q)_{\alpha-1}}{\Gamma_q(\alpha)p(x)} =: \xi_1 + \xi_2 \psi_{\alpha,a}(x). \tag{3.7}
\]

**Lemma 3.5.** Let \( \alpha \in (0, 1) \) and
\[
Y_{\alpha}(x) := r(x)y(x) - \lambda w_\alpha(x)y(x),
\]
\[
\Delta := c_1 d_2 - c_2 d_1 + c_1 d_1 \psi_{\alpha,a}(a). \tag{3.8}
\]
Assume that \( \Delta \neq 0 \), then on the space \( C(A_{\alpha}^{q,a}) \), the regular \( q \)-Sturm–Liouville problem \( (3.1) - (3.3) \) is equivalent to the \( q \)-integral equation
\[
y(x) = -I_{\alpha}^{q,a} \left( \frac{1}{p(\cdot)} I_{\alpha}^{q,a} - Y_{\alpha} \right)(x) + A(x) \int_0^a Y_{\alpha}(x) dq(x)
\]
\[
+ B(x) \left( I_{\alpha}^{q,a} \frac{1}{p(\cdot)} I_{\alpha}^{q,a} - Y_{\alpha} \right)(x) \bigg|_{x=a},
\]
where the coefficients \( A(x) \) and \( B(x) \) are
\[
A(x) := \frac{c_2}{\Delta} [d_2 + d_1 (\psi_{\alpha,a}(a) - \psi_{\alpha,a}(x))]
\]
\[
B(x) := \frac{d_1}{\Delta} [c_1 \psi_{\alpha,a}(x) - c_2],
\]
and the function \( \psi_{\alpha,a} \) is defined in \( (3.7) \).

**Proof.** Using \( (3.8) \), we can rewrite \( (3.1) \) as follows:
\[
D_{\alpha}^{q,a} p(x) D_{\alpha}^{q,a} \left[ y(\cdot) + I_{\alpha}^{q,a} \frac{1}{p(\cdot)} I_{\alpha}^{q,a} - Y_{\alpha} \right](x) = 0.
\]
Thus,
\[
y(x) + I_{\alpha}^{q,a} \left[ \frac{1}{p(\cdot)} I_{\alpha}^{q,a} - Y_{\alpha} \right](x) = \xi_1 + \xi_2 \psi_{\alpha,a}(x).
\]
From the boundary conditions \((3.2)–(3.3)\), we have

\[
\begin{align*}
\xi_1 &= y(0) \\
\xi_2 &= \int_0^a Y_y(x) \, dq_a + \left( I_{\alpha}^{1-\alpha} p^c D_{\alpha}^{\alpha} y \right)(0) \\
\xi_1 + \xi_2 I_{\alpha}^{\alpha} \frac{a^{\alpha-1}(qx/a; q)_{\alpha-1}}{\Gamma(\alpha)p(x)} \bigg|_{x=a} &= y(a) + \left( I_{\alpha}^{\alpha} p^c D_{\alpha}^{\alpha} y \right)(a) \\
\xi_2 &= \left( I_{\alpha}^{1-\alpha} p^c D_{\alpha}^{\alpha} y \right)(\frac{a}{q}).
\end{align*}
\]

This leads to the system of equations

\[
\begin{align*}
c_1 \xi_1 + c_2 \xi_2 &= c_2 X \\
d_1 \xi_1 + (d_2 + d_1 \psi_{\alpha,a}(a)) \xi_2 &= d_1 Z,
\end{align*}
\]

where

\[
X := \int_0^a Y(y)(x) \, dq_a, \quad \text{and} \quad Z := \left( I_{\alpha}^{\alpha} p^c D_{\alpha}^{\alpha} y \right)(a).
\]

Since \(\Delta \neq 0\), the solution for coefficients \(\xi_j, j = 1, 2\), is unique:

\[
\begin{align*}
\xi_1 &= \frac{c_2}{\Delta} (X (d_2 + d_1 \psi_{\alpha,a}(a)) - d_1 Z), \\
\xi_2 &= \frac{d_1}{\Delta} (c_1 Z - c_2 X).
\end{align*}
\]

Let us introduce the notation

\[
\begin{align*}
A &:= \|A(x)\| \\
m_p &:= \inf_{x \in A_{\alpha,\alpha}} |p(x)| \\
B &:= \|B(x)\| \\
M_\phi &:= \|\phi(x)\|.
\end{align*}
\]

\(3.10\)

\(3.11\)

**Theorem 3.6.** Let \(0 < \alpha < 1\). Assume that \(\Delta \neq 0\). Then unique \(q\)-regular at zero function \(y_\lambda\) for the regular \(q\)FSLP \((3.1)\) with the boundary conditions \((3.2)–(3.3)\) corresponding to each eigenvalue obeying

\[
\|r - \lambda w_\alpha\| < \frac{m_p}{M_\phi + B\phi(a) + Aam_p}
\]

exists and such eigenvalue is simple.

**Proof.** One can verify that \((3.1)\) can be interpreted as a fixed point for the mapping \(T : C(A_{\alpha,\alpha}) \to C(A_{\alpha,\alpha})\) defined by

\[
Tf(x) = -I_{\alpha}^{1-\alpha} \left[ \frac{1}{p(\cdot)} I_{\alpha}^{\alpha} Y_f \right](x) + A(x) \int_0^a Y_f(x) \, dq_a + B(x) I_{\alpha}^{1-\alpha} \left[ \frac{1}{p(\cdot)} I_{\alpha}^{\alpha} Y_f(\cdot) \right] \bigg|_{x=a},
\]

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using the estimate
\[ \|Y_g - Y_h\| \leq \|g - h\| \|r - \lambda w_\alpha\|, \]
then
\[
\|Tg - Th\| \leq \left\| I_{q,0}^a \left[ \frac{1}{p(x)} I_{q,a}^a (Y_g - Y_h) \right] \right\| + \|A(x)\| \left\| \int_0^a Y_g(x) - Y_h(x) \, dq_x \right\|
\]
\[
+ \|B(x)\| \left| I_{q,0}^a \left[ \frac{1}{p(\cdot)} I_{q,a}^a (Y_g - Y_h)(\cdot) \right] (x) \right|_{x=a}
\]
\[
\leq \|g - h\| \|r - \lambda w_\alpha\| \left( \frac{\|\phi\|}{m_p} + Aa + \frac{B\phi(a)}{m_p} \right)
\]
(3.13)
\[
= \|g - h\| \|r - \lambda w_\alpha\| L,
\]
where \( L = \|r - \lambda w_\alpha\| \left( \frac{M_\phi}{m_p} + Aa + \frac{B\phi(a)}{m_p} \right) \). Using the assumption of the theorem, we conclude that there is a unique fixed point denoted by \( y_\lambda \in C(A_{q,a}^*) \) satisfies (3.1) or equivalently (3.9) and the boundary conditions (3.2)–(3.3). Therefore, such eigenvalue is simple.

**Theorem 3.7.** Let \( 0 < \alpha < 1 \) and \( k_i \ (i = 0, 1) \) be real numbers. Assume that the functions \( p, r, \) and \( w_\alpha \) are \( C(A_{q,a}^*) \) functions such that \( \inf_{x \in A_{q,a}} p(x) > 0 \). Then, there exists \( m_0 \in \mathbb{N}_0 \) such that the regular qFSLP (3.1) with the initial conditions

\[ y(0) = k_0, \quad \left( I_{q,a}^{1-\alpha} p^\alpha D_q^\alpha y \right) (0) = k_1 \]
(3.14)
has a unique solution in \( C(A_{q,a}^{m_0}) \).

**Proof.** Let \( y_1 \) and \( y_2 \) be two solutions of (3.1) satisfying (3.14). Set \( z := y_1 - y_2 \), hence \( z \) is a solution of (3.1) with the conditions
\[
z(0) = 0, \quad \left( I_{q,a}^{1-\alpha} p^\alpha D_q^\alpha z \right) (0) = 0.
\]
(3.15)
In this case, simple manipulations show that (3.1) can be interpreted as a fixed point for the mapping \( T : C(A_{q,a}^*) \rightarrow C(A_{q,a}^*) \) defined by
\[
Tf(x) = -I_{q,0}^a \left[ \frac{1}{p(\cdot)} I_{q,a}^a Y_f(\cdot) \right] (x)
\]
\[
+ \psi_{\alpha,a}(x) \int_0^a Y_f(x) \, dq_x,
\]
(3.16)
using the estimate
\[ \|Y_g - Y_h\| \leq \|g - h\| \|r - \lambda w_\alpha\|, \]
(3.17)
and the inequality
\[
|\psi_{\alpha,a}(x)| \leq C \frac{x^\alpha}{m_p}, \quad C := \frac{q^{-\alpha}[\alpha]}{(q^{\alpha+1}; q)_\alpha \Gamma_{q}^2(\alpha + 1)},
\]
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for all $x \in A_{q,a}$. Therefore if $x \in A_{q,aq^m}$, $m \in \mathbb{N}$, then
\[ \|T(g - h)\| \leq C \frac{\|r - \lambda w_\alpha\|}{m_p} a^{2\alpha} q^{m\alpha} \|g - h\|. \]

We can choose $m_0 \in \mathbb{N}$ such that
\[ C \frac{\|r - \lambda w_\alpha\|}{m_p} a^{2\alpha} q^{m_0\alpha} < 1. \]

Thus $T : C(A_{q,aq^{m_0}}) \to C(A_{q,aq^{m_0}})$ is a contraction mapping. Hence $z$ is the unique fixed point of (3.16). Therefore, $z \equiv 0$ on $A_{q,aq^{m_0}}$. I.e. $y_1 = y_2$ on $A_{q,aq^{m_0}}$.

Another version of Theorem 3.6 holds if we release the conditions on the functions $r$ and $w_\alpha$ to be only $L^2_q(A_{a,q})$ functions.

**Theorem 3.8.** Let $\frac{1}{2} < \alpha < 1$. Assume that the functions $r$, and $w_\alpha$ are $L^2_q(A_{a,q})$ functions and $p$ is a functions satisfying $\inf_{x \in A_{q,a}} p(x) > 0$. If $\Delta \neq 0$, then unique $q$-regular at zero function $y_\lambda$ for the regular qFSLP (3.1) with the boundary conditions (3.2)–(3.3) corresponding to each eigenvalue obeying
\[ \|r - \lambda w_\alpha\|_2 < \frac{m_p}{(1 + B)a^{2\alpha - \frac{1}{2}}c_\alpha + A\sqrt{\alpha}m_p}, \quad c_\alpha := \frac{(1 - q)^{\alpha - \frac{1}{2}}}{(q; q)_{\infty} \sqrt{1 - q^{2\alpha - 1}}} \] (3.18)
if $\frac{1}{2} < \alpha < 1$, and obeying
\[ \|r - \lambda w_\alpha\|_2 < \frac{m_p}{(1 + B)a^{2\alpha - \frac{1}{2}}\gamma_\alpha + A\sqrt{\alpha}m_p}, \] (3.19)
where
\[ \gamma_\alpha := \frac{\Gamma_q(\alpha + \frac{1}{2})}{(q^\alpha; q)_{\infty} \Gamma_q(2\alpha + \frac{1}{2})} \sqrt{\frac{1 - q}{1 - q^{1 - 2\alpha}}} \]
if $\frac{1}{2} < \alpha < \frac{1}{2}$ exists and such eigenvalue is simple.

**Proof.** Similar to the proof of Theorem 3.6, (3.1) can be interpreted as a fixed point for the mapping $T : C(A_{q,a}) \to C(A_{q,a})$ defined by
\[ Tf(x) = -I^{\alpha}_{q,0^+} \left[ \frac{1}{p(\cdot)} I^{\alpha}_{q,a} Y_f \right] (x) + A(x) \int_0^a Y_f (x) d_q x \]
\[ + B(x) I^{\alpha}_{q,0^+} \left[ \frac{1}{p(\cdot)} I^{\alpha}_{q,a} Y_f (\cdot) \right] (a). \] (3.20)

**Case 1:** $\frac{1}{2} < \alpha < 1$ Using the estimate
\[ \left\| I^{\alpha}_{q,a} (Y_f - Y_\lambda) \right\| \leq \|g - h\| \left\| r - \lambda w_\alpha \right\|_2 \frac{1}{\Gamma_q(\alpha)} \left( \int_{q^2}^{a} t^{2\alpha - 2}(q^2/t; q)_\alpha^{\alpha - 1} d_q t \right)^{1/2}, \] (3.21)
then if $\alpha > \frac{1}{2}$, we obtain
\[
\int_0^a t^{2\alpha - 2}(qx/t; q^2)_{\alpha - 1}^2 \, dt \leq \frac{1}{(q^\alpha; q^\infty)^2} \int_0^a t^{2\alpha - 2} \, dt
\]
\[
= a^{2\alpha - 1} \frac{(1 - q)}{(q^\alpha; q^\infty)^2} \frac{(1 - q^{(2\alpha - 1)(m + 1)})}{1 - q^{2\alpha - 1}} \leq \frac{a^{2\alpha - 1}}{(q^\alpha; q^\infty)^2} \frac{1 - q}{1 - q^{2\alpha - 1}}.
\]
Consequently,
\[
\left| I_{q,a,0}^\alpha \left( \frac{1}{p} I_{q,a,0}^\alpha (Y_g - Y_h) \right) (x) \right|
\]
\[
\leq \|g - h\| \|r - \lambda w_\alpha\|_2 \frac{a^{\alpha - \frac{1}{2}}}{m_p(q^\alpha; q^\infty)} \sqrt{\frac{1 - q}{1 - q^{2\alpha - 1}}} \frac{a^{2\alpha - 1}}{(q^\alpha; q^\infty)^2} \frac{1 - q}{1 - q^{2\alpha - 1}}.
\]
A simple manipulation gives
\[
\left| \int_0^a (Y_g - Y_h) (x) \, dt \right| \leq \|g - h\| \|r - \lambda w_\alpha\|_2 \sqrt{a}. \quad (3.22)
\]
Therefore,
\[
\|Tg - Th\|
\]
\[
\leq \|g - h\| \|r - \lambda w_\alpha\|_2 \left( \frac{(1 + B)}{m_p(q^\alpha; q^\infty)} a^{2\alpha - \frac{1}{2}} (1 - q)^{\alpha - \frac{1}{2}} + A\sqrt{a} \right)
\]
\[
= L_1 \|g - h\|, \quad (3.23)
\]
where $L_1 = \|r - \lambda w_\alpha\|_2 \left( \frac{(1 + B)}{m_p(q^\alpha; q^\infty)} a^{2\alpha - \frac{1}{2}} (1 - q)^{\alpha - \frac{1}{2}} + A\sqrt{a} \right)$. Using the assumption of the theorem, we conclude that there is a unique fixed point denoted by $y_1 \in C(A_{q,a}^\gamma)$ satisfies (3.1) or equivalently (3.9) and the boundary conditions (3.2)–(3.3). Therefore, such eigenvalue is simple.

**Case 2:** $\frac{1}{2} < \alpha < 1$ In this case, we have
\[
\int_0^a t^{2\alpha - 2}(qx/t; q^2)_{\alpha - 1}^2 \, dt \leq \frac{x^{2\alpha - 1}}{(q^\alpha; q^\infty)^2} \frac{1 - q}{1 - q^{1 - 2\alpha}}.
\]
Consequently, if we set $\sigma_\alpha := \sqrt{\frac{1 - q}{1 - q^{1 - 2\alpha}a^{1 - 2\alpha}}}$ then
\[
\left| I_{q,a,0}^\alpha \left( \frac{1}{p} I_{q,a,0}^\alpha (Y_g - Y_h) \right) (x) \right| \leq \sigma_\alpha \|g - h\| \|r - \lambda w_\alpha\|_2 \left( I_{q,a,0}^{\alpha + \frac{1}{2}} \right) (x)
\]
\[
\leq \sigma_\alpha \|g - h\| \|r - \lambda w_\alpha\|_2 \frac{\Gamma_q(\alpha + 2)}{\Gamma_q(2\alpha + 1)} x^{2\alpha - \frac{1}{2}}
\]
\[
\leq \|g - h\| \|r - \lambda w_\alpha\|_2 \gamma_\alpha a^{2\alpha - \frac{1}{2}},
\]
for all $x \in A_{q,a}$. Therefore, using (3.22), we obtain
\[ \|Tg - Th\| \leq \|g - h\| \|r - \lambda w_\alpha\| \left( \frac{1 + B}{m_p} \gamma_\alpha \alpha^{2\alpha - \frac{1}{\alpha}} + A \sqrt{a} \right) \] (3.24)

\[ = \|g - h\| L_2, \]

where $L_2 = \|r - \lambda w_\alpha\| \left( \frac{1 + B}{m_p} \gamma_\alpha \alpha^{2\alpha - \frac{1}{\alpha}} + A \sqrt{a} \right)$. Using the assumption of the theorem, we conclude that there is a unique fixed point denoted by $y_3 \in C(A_{q,a})$ satisfies (3.1) or equivalently (3.9) and the boundary conditions (3.2)–(3.3). Therefore, such eigenvalue is simple.

4. The associated Wronskian

**Definition 4.1.** Let $y_1$, $y_2$ be two functions in $AC_q(A_{q,a})$ and let $0 < \alpha < 1$. Assume that $p \in C(A_{q,a})$ is a positive function. The $q,p,\alpha$ Wronskian of $y_1$ and $y_2$ is denoted by $W_{q,p,\alpha}(y_1, y_2)$ and defined by

\[ W_{q,p,\alpha}(y_1, y_2)(x) = y_1(x) I_{q,a}^{1-\alpha} \left( p^\alpha D_{q,0}^{\alpha} + y_2 \right) (x) - y_2(x) I_{q,a}^{1-\alpha} \left( p^\alpha D_{q,0}^{\alpha} + y_1 \right) (x). \]

**Theorem 4.2.** If $y_1$ and $y_2$ are two solutions of (3.1)–(3.3), then

\[ W_{q,p,\alpha}(y_1, y_2)(0) = W_{q,p,\alpha}(y_1, y_2)(a) \]

**Proof.** Let $y_1$ and $y_2$ be two solutions of (3.1)–(3.3). Then

\[ D_{q,a}^\alpha \left( p^\alpha D_{q,0}^{\alpha} + y_1 \right) (x) + r(x)y_1(x) = \lambda w_\alpha y_1(x), \] (4.1)

\[ D_{q,a}^\alpha \left( p^\alpha D_{q,0}^{\alpha} + y_2 \right) (x) + r(x)y_2(x) = \lambda w_\alpha y_2(x), \] (4.2)

for all $x \in A_{q,a}$. Multiply (4.1) by $y_2$ and (4.2) by $y_1$ and subtracting the two equations. This gives

\[ y_1(x) D_{q,a}^\alpha \left( p^\alpha D_{q,0}^{\alpha} + y_2 \right) (x) - y_2(x) D_{q,a}^\alpha \left( p^\alpha D_{q,0}^{\alpha} + y_1 \right) (x) = 0. \] (4.3)

using that $-\frac{1}{q} D_{q-1} f(x) = D_{q,x} f \left( \frac{x}{q} \right)$, (4.3) can be written as

\[ y_1(x) D_{q,x} \left( I_{q,a}^{1-\alpha} p^\alpha D_{q,0}^{\alpha} + y_2 \right) \left( \frac{x}{q} \right) - y_2(x) D_{q,x} \left( I_{q,a}^{1-\alpha} p^\alpha D_{q,0}^{\alpha} + y_1 \right) \left( \frac{x}{q} \right) = 0. \] (4.4)

Hence,

\[ D_q W_{q,p,\alpha}(y_1, y_2)(x) = D_q y_1(x) \left( I_{q,a}^{1-\alpha} p^\alpha D_{q,0}^{\alpha} + y_2 \right) (x) - D_q y_2(x) \left( I_{q,a}^{1-\alpha} p^\alpha D_{q,0}^{\alpha} + y_1 \right) (x). \] (4.5)
Thus from (2.19)
\[
\int_{0}^{a} D_{q} W_{q,p,\alpha}(y_1, y_2)(x) \, dq \, dx \\
= \int_{0}^{a} \left[ D_{q}y_1(x)I_{q,\alpha}^{1-\alpha} \left( P_{q,0}^{\alpha} y_2 \right)(x) - D_{q}y_2(x)I_{q,\alpha}^{1-\alpha} \left( P_{q,0}^{\alpha} y_1 \right)(x) \right] \, dq \, dx = 0.
\]

Hence,
\[
W_{q,p,\alpha}(y_1, y_2)(0) = W_{q,p,\alpha}(y_1, y_2)(a).
\]

We have the following theorems:

**Theorem 4.3.** Let \( y_1 \) and \( y_2 \) be two functions in \( \mathcal{AC}_q(A_{q,a}^\alpha) \). Then \( y_1 \) and \( y_2 \) are two linearly independent solutions of (3.1) if and only if \( W_{q,p,\alpha}(y_1, y_2)(0) \neq 0 \).

**Proof.** Let \( y_1 \) and \( y_2 \) be two solutions of (3.1) such that \( W_{q,p,\alpha}(y_1, y_1)(0) \neq 0 \).

If
\[
k_1 y_1(x) + k_2 y_2(x) = 0, \quad x \in A_{q,a}^\alpha,
\]

then
\[
k_1 y_1(0) + k_2 y_2(0) = 0,
\]
\[
k_1 \left( I_{q,\alpha}^{1-\alpha} p_{q,0}^{\alpha} y_1 \right)(0) + k_2 \left( I_{q,\alpha}^{1-\alpha} p_{q,0}^{\alpha} y_2 \right)(0) = 0,
\]

but \( W_{q,p,\alpha}(y_1, y_2)(0) \neq 0 \) implies that \( k_1 = k_2 = 0 \). I.e. \( y_1 \) and \( y_2 \) are linearly independent. To prove the necessary condition, we suppose on the contrary that \( y_1 \) and \( y_2 \) are linearly independent solutions and \( W_{q,p,\alpha}(y_1, y_2)(0) = 0 \). Hence there exist constants \( r_1 \) and \( r_2 \) not both zeros such that
\[
r_1 y_1(0) + r_2 y_2(0) = 0,
\]
\[
r_1 \left( I_{q,\alpha}^{1-\alpha} p_{q,0}^{\alpha} y_1 \right)(0) + r_2 \left( I_{q,\alpha}^{1-\alpha} p_{q,0}^{\alpha} y_2 \right)(0) = 0.
\]

Set \( y := r_1 y_1 + r_2 y_2 \). Hence, \( y \) is a solution of (3.1) satisfying the initial conditions
\[
y(0) = \left( I_{q,\alpha}^{1-\alpha} p_{q,0}^{\alpha} y \right)(0).
\]

According to Theorem 3.7 there exists \( m_0 \in \mathbb{N} \) such that (3.1) with (4.7) has a unique solution in \( C(A_{q,a}^{\alpha m_0}) \). Hence, \( y \equiv 0 \). I.e. \( y_1 \) and \( y_2 \) are linearly dependent which is a contradiction. Hence, we should have \( W_{q,p,\alpha}(y_1, y_2)(0) \neq 0 \).

**Theorem 4.4.** The geometric multiplicity of each eigenvalue of the qFSLP (3.1)–(3.3) is 1. I.e. for each eigenvalue, the associated eigenfunction is unique except for a constant multiplier.
constant multiplier of $y$. Consequently, from Theorem 4.3, $y$ under the boundary conditions Lemma 5.1.

5. A discrete spectrum of a qFSLP

In this section, we solve the fractional $q$-Sturm–Liouville problem

$$D_{q,a}^{\mu} (qx; q)_{\beta} y(x) = \lambda x^{-\mu} (qx; q)_{\beta} y(x), \ 0 < \mu < 1, \ x \in \mathbb{A}_q^*,$$  (5.1)

under the boundary conditions

$$y(0) = \left[ I_{q,1}^{1-\mu} (q^{\beta+1}; q)_{\mu} c D_{q,0}^{\mu} y(x) \right] \left( \frac{1}{q} \right) = 0. \quad (5.2)$$

We show that it has a discrete spectrum $\{\phi_n, \lambda_n\}$ where $\phi_n$ is a little $q$-Jacobi polynomial and the eigenvalues $\{\lambda_n\}$ have no finite limit points. To achieve our goal, we need the following preliminaries. The little $q$-Jacobi polynomial is defined by, see [20, P. 92]

$$p_n(x; q^\alpha, q^\beta | q) = 2 \phi_1 \left( q^{-n}, q^{\alpha+\beta+n+1}; q^{\alpha+1}; q, qx \right), \ \alpha > -1, \ \beta > -1.$$  

It satisfies the orthogonality relation

$$\int_0^1 w_{\alpha,\beta} p_n(t; q^\alpha, q^\beta | q) p_m(t; q^\alpha, q^\beta | q) \, dq = C_n(\alpha, \beta) \delta_{n,m}, \quad (5.3)$$

where

$$w_{\alpha,\beta} (t) := t^\alpha (qt; q)_{\beta}, \ \alpha > -1, \ \beta > -1,$$

and

$$C_n(\alpha, \beta) := q^{(\alpha+1)n} \frac{(1-q) (1-q^{\alpha+\beta+1}) (q, q^{\alpha+\beta+2}; q)_\infty (q, q^{\beta+1}; q)_n}{(1-q^{\alpha+\beta+2n}) (q^{\alpha+1}, q^{\beta+1}; q)_\infty (q^{\alpha+1}, q^{\beta+1}; q^2)_n}.$$  

Lemma 5.1.

$$I_{q,0}^{\mu} (\cdot)^\alpha p_n(\cdot; q^\alpha, q^\beta | q) (x) = \frac{\Gamma_q(\alpha + 1)}{\Gamma_q(\mu + \alpha + 1)} x^{\alpha+\mu} p_n(x; q^{\alpha+\mu}, q^{\beta-\mu} | q).$$
Jacobi polynomial, we can prove

\[ p_n(x; q^{\alpha+\mu}, q^{\beta-\mu}|q) = \frac{\Gamma_q(\mu + \alpha + 1)}{\Gamma_q(\alpha + 1)} \int_0^1 t^\alpha (qy; q)_\mu p_n(xy; q^\alpha, q^\beta|q) \, dq y. \]  

(5.4)

Make the substitution \( t = xy \) on the \( q \)-integral of the right hand side of (5.4), we obtain

\[ p_n(x; q^{\alpha+\mu}, q^{\beta-\mu}|q) = \frac{x^{-\alpha-1}\Gamma_q(\mu + \alpha + 1)}{\Gamma_q(\alpha + 1)} \int_0^x t^\alpha (qt/x; q)_\mu p_n(t; q^\alpha, q^\beta|q) \, dq t. \]  

(5.5)

\[ \square \]

**Corollary 5.2.**

\[ cD^\mu_{q,0+} \left( x^{\alpha+\mu} p_n(x; q^{\alpha+\mu}, q^{\beta-\mu}|q) \right) = \begin{cases} \frac{\Gamma_q(\mu + \alpha + 1)}{\Gamma_q(\alpha + 1)} x^\alpha p_n(x; q^\alpha, q^\beta|q), & \alpha > -\mu \\ \frac{1}{\Gamma_q(1 - \mu)} x^{-\mu} [p_n(x; q^\alpha, q^\beta|q) - 1], & \alpha = -\mu. \end{cases} \]

**Proof.** The proof follows form (2.6) and Lemma 5.1. \[ \square \]

**Lemma 5.3.** If \( \alpha, \beta, \) and \( \mu \) are real numbers satisfying

\[ \alpha > -1, \beta > -1, \beta - 1 < \mu < \alpha + 1 \]

\[ I_{q,1-}^\mu (\langle qt; q \rangle s p_m(t; q^\alpha, q^\beta|q) = \Gamma_q(\beta + m + 1) \Gamma_q(\alpha - \mu + 1 + m) \Gamma_q(\alpha + 1) \Gamma_q(\alpha - \mu + m + 1) \Gamma_q(\alpha - \mu) p_m(t; q^{\alpha-\mu}, q^{\beta+\mu}|q). \]

**Proof.** In (5.5) replace \( \alpha \) by \( \alpha - \mu \), \( \beta \) by \( \beta + \mu \) and then substitute in the orthogonality relation (5.3). This gives

\[ C_n(\alpha, \beta) \delta_{n,m} = \frac{\Gamma_q(\alpha + 1)}{\Gamma_q(\mu) \Gamma_q(\alpha - \mu + 1)} \times \int_0^1 t^{\mu-1}(qt; q)_\beta p_n(t; q^\alpha, q^\beta|q) \int_0^t u^{\alpha-\mu}(qu/t; q)_{\mu-1} p_m(u; q^{\alpha-\mu}, q^{\beta+\mu}|q) \, dq u \, dq t. \]  

(5.6)

Changing the order of the \( q \)-integration gives

\[ C_n(\alpha, \beta) \delta_{n,m} = \frac{\Gamma_q(\alpha + 1)}{\Gamma_q(\mu) \Gamma_q(\alpha - \mu + 1)} \int_0^1 t^{\mu-\mu}(qt; q)_\beta p_n(t; q^{\alpha-\mu}, q^{\beta+\mu}|q) F_m(t) \, dq t, \]
In Corollary 5.2, set
\[ F_m(t) = \frac{1}{(qt; q)_{\beta+\mu}} \int_0^1 u^{\mu-1}(qu; q)_\beta(qt/u; q)_{\mu+1}p_m(u; q^\alpha, q^\beta | q)) \, dq.u. \]

Set
\[ G_m(t) := F_m(t) - \frac{\Gamma_q(\mu)\Gamma_q(\alpha - \mu + 1)}{\Gamma_q(\alpha + 1)} p_m(t; q^{\alpha-\mu}, q^{\beta+\mu} | q) \]

Therefore,
\[ \int_0^1 w_{\alpha, \beta}(u)p_n(u; q^{\alpha-\mu}, q^{\beta+\mu} | q)G_m(u) \, dq.u = 0 \]

for all \( n \in \mathbb{N} \). From the completeness of the little \( q \)-Jacobi polynomials \( p_n(x; q^\alpha, q^\beta | q) \) in the weighted space \( L_q^2(A_\alpha, w_{\alpha, \beta}) \), \( w_{\alpha, \beta}(x) = x^n(qx; q)_\beta \) we obtain
\[ F_m(u) = \frac{\Gamma_q(\mu)\Gamma_q(\alpha - \mu + 1)}{\Gamma_q(\alpha + 1)} \frac{c_m(\alpha, \beta)}{c_m(\alpha - \mu, \beta + \mu)} p_m(u; q^{\alpha-\mu}, q^{\beta+\mu} | q). \]

\[ \square \]

**Corollary 5.4.**
\[ D_{q,1-}^\mu ((qt; q)_{\beta+\mu}p_m(t; q^{\alpha-\mu}, q^{\beta+\mu} | q)) = q^{-\mu} \frac{\Gamma_q(\mu + \beta + n + 1)\Gamma_q(\alpha + n + 1)}{\Gamma_q(\beta + 1 + n)\Gamma_q(n + 1)\Gamma_q(\mu + 1)} (qt; q)_\beta p_n(t; q^\alpha, q^\beta | q). \]

*Proof.* The proof follows from (2.9) and Lemma 5.1. \( \square \)

**Theorem 5.5.** For \( 0 < \mu < 1 \), and \( \beta > -1 \), the functions
\[ \phi_n(x) = x^n p_n(x; q^\mu, q^\beta | q), \quad n \in \mathbb{N}_0 \]

are eigenfunctions of the qFSLP (5.1) - (5.2), associated to the eigenvalues
\[ \lambda_n := q^{-\mu n} \frac{\Gamma_q(\mu + \beta + n + 1)\Gamma_q(\alpha + n + 1)}{\Gamma_q(\beta + 1 + n)\Gamma_q(n + 1)\Gamma_q(\mu + 1)} \]
\[ , \quad n \in \mathbb{N}_0. \]

Moreover, the eigenfunctions are unique up to a constant multiplier.

*Proof.* In Corollary 5.2 set \( \alpha = 0 \) and replace \( \beta \) by \( \beta + \mu \). This gives
\[ \mathcal{D}_{q,0+} (x^n p_n(x; q^\mu, q^\beta | q)) = \Gamma_q(\mu + 1)p_n(x; 1, q^{\beta+\mu} | q), \]

(5.7)

In Corollary 5.4 set \( \alpha = \mu \), we obtain
\[ D_{q,1-}^\mu ((qx; q)_\beta p_n(x; 1, q^{\beta+\mu} | q)) = q^{-\mu n} \frac{\Gamma_q(\mu + \beta + n + 1)\Gamma_q(\alpha + n + 1)}{\Gamma_q(\beta + 1 + n)\Gamma_q(n + 1)\Gamma_q(\mu + 1)} (qx; q)_\beta p_n(x; q^\mu, q^\beta | q). \]

(5.8)

Combining (5.7) and (5.8) gives the required result. \( \square \)
6. Conclusion and future work

This paper is the first paper introduces fractional \( q \)-Sturm–Liouville problems. It proves the main properties of the eigenvalues and the eigenfunctions of the fractional Sturm–Liouville problem. We also give a sufficient condition on an eigenvalue to have a unique eigenfunction, and a sufficient condition on the domain is given for the existence and uniqueness of eigenfunction. The discrete spectrum of a qFSLP is given. This work generalizes the study of integer Sturm–Liouville problem introduced by Annaby and Mansour in [1]. It is worth mentioning that a \( q \)-fractional variational calculus is developed and used in [21] to prove that the qFSLp \( (3.1) \) with the boundary condition \( y(0) = y(a) = 0 \) has a countable set of real eigenvalues and associated orthogonal eigenfunctions when \( \alpha > 1/2 \) and a similar study for the fractional Sturm–Liouville problem

\[
^cD_{q,a}^{-p(x)}D_{q,0+a}^{\alpha}y(x) + r(x)y(x) = \lambda w(x)y(x),
\]

is in progress.

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