Research Article

Terminal Value Problem for Implicit Katugampola Fractional Differential Equations in $b$-Metric Spaces

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1. Introduction and Preliminaries

An interesting extension and unification of fractional derivatives of the type Caputo and the type Caputo-Hadamard is called Katugampola fractional derivative that has been introduced by Katugampola [1, 2]. Some fundamental properties of this operator are presented in [3, 4]. Several results of implicit fractional differential equations have been recently provided (see [4–14] and the references therein). A new class of mixed monotone operators with concavity and applications to fractional differential equations has been considered in [15]. In [16], the authors presented some existence and uniqueness results for a class of terminal value problem for differential equations with Hilfer-Katugampola fractional derivative.

On the other side, a novel extension of $b$-metric was suggested by Czerwik [17, 18]. Although the $b$-metric standard looks very similar to the metric definition, it has a quite different structure and properties. For example, in the $b$-metric topology framework, an open (closed) set is not open (closed). Additionally, the $b$-metric function is not continuous. These weaknesses make this new structure more interesting (see [19–28]).

Throughout the paper, any mentioned set is nonempty. We consider the following type of terminal value problems of Katugampola implicit differential equations of noninteger orders:

\[
\begin{align*}
\left(\rho D^{r}_{0^+} + \vartheta\right)(\tau) &= \kappa(\tau, \vartheta(\tau), (\rho D^{r}_{0^+} + \vartheta)(\tau)), \quad \tau \in I = [0, T], \\
\vartheta(T) &= \vartheta_{T} \in \mathbb{R},
\end{align*}
\]

with $T > 0$ and the function $\kappa : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. Here, $\rho D^{r}_{0^+}$ is the Katugampola fractional derivative of order $r \in (0, 1]$.

Set $C(I) = \{ h : h \text{ real continuous functions on } I = [0, T]\}$. Then, $C(I)$ forms a Banach space with the norm $\|h\|_{\infty} = \sup |h(\tau)|$.

Set $L^{1}(I) = \{ \vartheta : I \to \mathbb{R} | \vartheta \text{ is measurable function and Lebesgue integrable} \}$. Then, $L^{1}(I)$ becomes a Banach space with the norm $\|\vartheta\|_{L^{1}} = \int_{0}^{T} |\vartheta(\tau)| \, d\tau$. 
Set \(C_{r,p}(I) = \{ \theta : (0, T] \to \mathbb{R} | \tau^{(1-r)} \theta(\tau) \in C(I) \} \). Then, it forms a Banach space \(\| \theta \|_{C_r} = \sup_{\tau \in I} \| \tau^{(1-r)} \theta(\tau) \| \). Here, \(C_{r,p}(I)\) is called the weighted space of continuous functions.

**Definition 1** (Katugampola fractional integral) [1]. The Katugampola fractional integrals of order \(r > 0\) and \(p > 0\) of a function \(y \in X_p^p(I)\) are defined by

\[
\rho D_0^r y(\tau) = \int_0^\tau \frac{\theta^{-1} y(s)}{\Gamma(r)} \tau^{(p-1)(r-1)} ds, \quad \tau \in I. \tag{2}
\]

**Definition 2** (Katugampola fractional derivatives) [1, 2]. The generalized fractional derivatives of order \(r > 0\) and \(p > 0\) corresponding to the Katugampola fractional integrals (2) defined for any \(\tau \in I\) by

\[
\rho D_0^r y(\tau) = \left( \left( \tau^{1-p} \frac{d}{d\tau} \right)^{r-n} \left( \tau^{1-p} \frac{d}{d\tau} \right)^{n} \right) \frac{\theta^{-1} y(s)}{\Gamma(r)} \tau^{(p-1)(r-1)} ds, \quad \tau \in I. \tag{3}
\]

where \(n = \lfloor r \rfloor + 1\); if the integrals exist.

**Remark 1** ([1, 2]). As a basic example, we quote for \(r, \rho > 0\) and \(\theta > -\rho\),

\[
\rho D_0^r t^\rho = \frac{\rho^{-1} \Gamma(1 + (\theta/\rho))}{\Gamma(1 - r + (\theta/\rho))} t^{\rho - r}. \tag{4}
\]

Giving in particular,

\[
\rho D_0^r t^\rho = 0, \quad \text{for each } i = 1, 2, \ldots, n. \tag{5}
\]

In fact, for \(r, \rho > 0\) and \(\theta > -\rho\), we have

\[
\rho D_0^r t^\rho = \frac{\rho^{-1} \Gamma(1 + (\theta/\rho))}{\Gamma(1 - r + (\theta/\rho))} t^{\rho - r}, \tag{6}
\]

If we put \(r = (\theta/\rho)\), we obtain from (6):

\[
\rho D_0^r t^\rho = \rho^{-1} \Gamma(r - i + 1) \frac{n!}{(n - i)!} t^{\rho - r}. \tag{7}
\]

So, \(\rho D_0^r t^\rho = 0\), \(\forall r, \rho > 0\).

**Theorem 1** ([2]). Let \(r, \rho, c \in \mathbb{R}\), be such that \(r, \rho > 0\). Then, for any \(\kappa, \omega \in X_p^p(I)\), where \(1 \leq p \leq \infty\), we have

1. Inverse property:

\[
\rho D_0^r \rho D_0^r \kappa(\tau) = \kappa(\tau), \quad \text{for all } r \in (0, 1]. \tag{8}
\]

2. Linearity property: for all \(r \in (0, 1)\), we have

\[
\begin{align*}
\rho D_0^r (\kappa + \omega)(\tau) &= \rho D_0^r \kappa(\tau) + \rho D_0^r \omega(\tau), \\
\rho D_0^r (\kappa + \omega)(\tau) &= \rho I_0^r \kappa(\tau) + \rho I_0^r \omega(\tau). \tag{9}
\end{align*}
\]

**Lemma 1** ([2]). Let \(r, \rho > 0\). If \(\theta \in C(I)\); then the fractional differential equation \(\rho D_0^r \theta(\tau) = 0\), has a unique solution

\[
\theta(\tau) = C_1 t^{\rho(r-1)} + C_2 t^{\rho(r-2)} + \cdots + C_n t^{\rho(r-n)}, \tag{10}
\]

where \(C_i \in \mathbb{R}\) with \(i = 1, 2, \ldots, n\).

**Proof.** Let \(r, \rho > 0\). from Remark 1, we have

\[
\rho D_0^r t^{\rho(r-1)} = 0, \quad \text{for each } i = 1, 2, \ldots, n. \tag{11}
\]

Then, the fractional equation \(\rho D_0^r \theta(\tau) = 0\) has a particular solution as follows:

\[
\theta(\tau) = C_i t^{\rho(r-1)}, \quad C_i \in \mathbb{R}, \text{ for each } i = 1, 2, \ldots, n. \tag{12}
\]

Thus, the general solution of \(\rho D_0^r \theta(\tau) = 0\) is a sum of particular solutions (12), i.e.

\[
\theta(\tau) = C_1 t^{\rho(r-1)} + C_2 t^{\rho(r-2)} + \cdots + C_n t^{\rho(r-n)}, C_i \in \mathbb{R}; (i = 1, 2, \ldots, n). \tag{13}
\]

**Lemma 2.** Let \(r, \rho > 0\). If \(\theta \in C(I)\) and \(0 < r \leq 1\), then

\[
\rho I_0^r \rho D_0^r \theta(\tau) = \theta(\tau) + c t^{\rho(r-1)}, \tag{14}
\]

for some constant \(c \in \mathbb{R}\).

**Proof.** Let \(\rho D_0^r \theta \in C(I)\) be the fractional derivative (3) of order \(0 < r \leq 1\). If we apply the operator \(\rho D_0^r\) to \(\rho I_0^r \rho D_0^r \theta(\tau) - \theta(\tau)\) and use the properties (8) and (9), we get

\[
\rho D_0^r (\rho I_0^r \rho D_0^r \theta(\tau) - \theta(\tau)) = \rho D_0^r \rho I_0^r \rho D_0^r \theta(\tau) - \rho D_0^r \theta(\tau) = \rho D_0^r \rho D_0^r \theta(\tau) = 0. \tag{15}
\]

From the proof of Lemma 1, there exists \(c \in \mathbb{R}\), such that

\[
\rho I_0^r \rho D_0^r \theta(\tau) - \theta(\tau) = c t^{\rho(r-1)}, \tag{16}
\]

which implies (14).
Lemma 3. Let $h \in L^1(I, \mathbb{R})$ and $0 < r \leq 1$ and $\rho > 0$. A function $\vartheta \in C(I)$ forms a solution for

$$
\begin{cases}
(\rho D_{t}^{(r)} \vartheta)(t) = z(t), & t \in I, \\
\vartheta(T) = \vartheta_T,
\end{cases}
$$

(17)

if and only if $\vartheta$ fulfills

$$
\vartheta(t) = (\vartheta_T - \rho^r l^0, z(T)) \left( \frac{T}{t} \right)^{\rho(r-1)} + \frac{\rho^{1-r}}{F(r)} \int_0^t \left( r^\rho - s^\rho \right)^{-1} z(s) ds.
$$

(18)

Proof. Let $r, \rho > 0$. and $0 < r \leq 1$. Suppose that $\vartheta$ satisfies (17). Employing the operator $\rho D_{t}^{(r)}$ to the each side of the equation

$$
(\rho D_{t}^{(r)} \vartheta)(t) = z(t),
$$

(19)

we find

$$
\rho l^0, \rho D_{t}^{(r)} \vartheta(t) = \rho^r l^0, z(t).
$$

(20)

From Lemma 2, we get

$$
\vartheta(t) = \vartheta_T - \rho^r l^0, z(T) = c T^\rho(r-1),
$$

(21)

for some $c \in \mathbb{R}$. If we use the terminal condition $\vartheta(T) = \vartheta_T$ in (21), we find

$$
\vartheta(t) = \vartheta_T - \rho^r l^0, z(T) = c T^\rho(r-1),
$$

(22)

which shows

$$
c = (\rho^r l^0, z(T) - \vartheta_T) T^\rho(1-r).
$$

(23)

Henceforth, we deduce (18). Contrariwise, if $\vartheta$ achieves (18), then $(\rho D_{t}^{(r)} \vartheta)(t) = z(t)$; for $t \in I$ and $\vartheta(t) = \vartheta_T$.

Lemma 4. Contemplate the problem (1), and set $g \in C(I)$, and

$$
\omega(t) = \kappa(t, \vartheta(t), \omega(t)).
$$

We presume $\vartheta$ achieves

$$
\vartheta(t) = (\vartheta_T - \rho^r l^0, \omega(T)) \left( \frac{T}{t} \right)^{\rho(r-1)} + \frac{\rho^{1-r}}{F(r)} \int_0^t \left( r^\rho - s^\rho \right)^{-1} \omega(s) ds.
$$

(24)

Then, $\vartheta$ forms a solution of (1).

Definition 3 [29, 30]. A function $d : S \times S \longrightarrow [0, \infty)$ is called $b$-metric if there is $c \geq 1$ and $d$ fulfills

(i) (bM1) $d(v, \emptyset) = 0$ if and only if $v = \emptyset$

(ii) (bM2) $d(v, \mu) = d(\mu, v)$

(iii) (bM3) $d(\mu, \emptyset) \leq c[d(\mu, v) + d(v, \emptyset)]$

for all $\mu, v, \emptyset \in S$. We say that the tripled $(S, d, c)$ is $b$-metric space (in short, b.m.s.).

Example 1 [29, 30]. Let $d : C(I) \times C(I) \longrightarrow [0, \infty)$ be described as

$$
d(v, \emptyset) = \left\| (v - \emptyset) \right\|^\rho = \sup_{t \in I} \left\| v(t) - \emptyset(t) \right\|^\rho, \quad \text{for all } v, \emptyset \in C(I).
$$

(25)

Ergo, $(C(I), d, 2)$ is $b$-metric space.

Example 2 [29, 30]. Set $S = [0, 1]$ and $d : S \times S \longrightarrow [0, \infty)$ be designated by

$$
d(v, \emptyset) = |v' - \emptyset'|, \quad \text{for all } v, \emptyset \in S.
$$

(26)

Henceforth, $(S, d, r)$ with $r \geq 2$ is $b$-metric space.

We set the following: $\{ \phi : [0, \infty) \rightarrow [0, \infty) | \phi \text{ is continuous, increasing, } \phi(0) = 0 \text{ and } \phi(c \mu) \leq c \phi(\mu) \leq c \mu \text{ for } c > 1 \}$.

For some $c \geq 1$, we set $\mathcal{F} = \{ \lambda : [0, \infty) \rightarrow [0, (1/c^2)] | \lambda \text{ is nondecreasing} \}.

Definition 4 [29, 30]. A self-operator $T$, on a b.m.s. $(S, d, c)$, is called a generalized $a \neq \emptyset -$ Geraghty contraction whenever there exists $a : S \times S \longrightarrow [0, \infty)$, and some $L \geq 0$ such that

$$
D(v, \emptyset) = \sup \left\{ d(v, \emptyset), d(\emptyset, T(\emptyset)), d(v, T(v)), \frac{d(v, T(\emptyset)) + d(\emptyset, T(v))}{2s} \right\},
$$

(27)

$$
N(v, \emptyset) = \min \left\{ d(v, \emptyset), d(\emptyset, T(\emptyset)), d(v, T(v)) \right\},
$$

(28)

we have

$$
a(\mu, v) \varphi(c^2 d(T(\mu), T(v))) \leq \lambda(\varphi(D(\mu, v))) \varphi(D(\mu, v)) + L \varphi(\lambda(\varphi(D(\mu, v))),
$$

(29)

for all $\mu, v, \emptyset \in S$, where $\lambda \in \mathcal{F}, \varphi, \psi \in \Phi$.

Remark 2. In the case when $L = 0$ in Definition 4 and the fact that

$$
d(\mu, v) \leq D(\mu, v), \quad \text{for all } \mu, v \in S,
$$

(30)

the inequality (29) becomes

$$
a(\mu, v) \varphi(c^2 d(T(\mu), T(v))) \leq \lambda(\varphi(d(\mu, v))) \varphi(d(\mu, v))).
$$

(31)

Definition 5 [29, 30]. Set $a : S \times S \longrightarrow [0, \infty)$. An operator $T : S \longrightarrow S$, is $a -$ admissible if

$$
a(\mu, v) \geq 1 \Rightarrow a(T(\mu), T(v)) \geq 1,
$$

(32)

for all $\mu, v \in S$. 
Definition 6 [29, 30]. Let $(S, d, c)$ with $c \geq 1$ be a b.m.s and $\alpha : S \times SR^*_c$.

We say that $S$ is $\alpha$ - regular if for any sequence $\{v_n\}_{n \in \mathbb{N}}$ in $S$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$ and $\alpha(v_n, v_{n+1}) \geq 1$ for each $n$; there exists a subsequence $\{v_{n(k)}\}_{k \in \mathbb{N}}$ of $\{v_n\}_n$ with $\alpha(v_{n(k)}, x) \geq 1$ for all $k$.

Theorem 2 [29, 30]. We presume that a self-operator $T$ over a complete b.m.s. $(S, d, c)$ with $c \geq 1$ forms a generalized $\alpha - \varphi$ - Geragthy contraction. Furthermore,

(i) $T$ is $\alpha -$ admissible with initial value $\alpha(\mu_0, T(\mu_0)) \geq 1$ for some $\mu_0 \in M$ (ii) either $T$ is continuous or $M$ is $\alpha$ - regular

Then $T$ possesses a fixed point. Furthermore, if

(iii) for all fixed points $\mu, \nu$ of $T$, either $\alpha(\mu, \nu) \geq 1$ or $\alpha(\nu, \mu) \geq 1$, then the found fixed point is unique.

This manuscript launches the study of Katugampola implicit fractional differential equations on b.m.s.

2. Main Results

Observe that $(C_{\rho, \varphi}(I), d, 2)$ is a complete b.m.s. with $d : C_{\rho, \varphi}(I) \times C_{\rho, \varphi}(I) \rightarrow [0, \infty)$ described as

$$d(\vartheta, \varphi) = \| (\vartheta - \varphi) \|_C := \sup_{t \in I} \tau^{3(1-\tau)}|\vartheta(t) - \varphi(t)|^2. \quad (33)$$

A function $\vartheta \in C_{\rho, \varphi}(I)$ is called a solution of (1) if it archieves

$$\vartheta(\tau) = (\vartheta_{-\rho}^{T^*}I_{\rho}^{\omega(T)}(T)\frac{\rho(\vartheta, \omega(T))}{\rho(\vartheta, \omega(T))}) + \rho(\vartheta, \omega(T)) \int_0^\tau s^{\rho-1}(\tau^s - s)^{-1}\omega(s)ds, \quad (34)$$

with $\omega(t) = \alpha(\omega(t), \vartheta(t), \varphi(t)) \in C(I)$.

In the sequel, we shall need the following hypotheses:

(H$_1$) There exist $\varphi \in \Phi, \rho : C(I) \times C(I) \rightarrow (0, \infty)$ and $q : I \rightarrow (0, 1)$ so that for each $\vartheta, \varphi, \vartheta_1, \varphi_1 \in C_{\rho, \varphi}(I)$, and $t \in I$,

$$|\alpha(\vartheta, \varphi, \vartheta_1, \varphi_1)| \leq \tau^{\rho(1-t)}|\rho(\vartheta, \varphi)|\vartheta - \vartheta_1| + q(\varphi)\varphi - \varphi_1|. \quad (35)$$

with

$$\left\| \frac{\rho^{1-\tau}}{\rho(T)} \int_0^\tau s^{\rho-1}(\tau^s - s)^{-1}\rho(\vartheta, \omega(T))\frac{\rho(\vartheta, \omega(T))}{\rho(\vartheta, \omega(T))}ds \right\|_C \leq \varphi(\|\vartheta - \varphi\|_C) \quad (36)$$

(H$_2$) There are $\mu_0 \in C_{\rho, \varphi}(I)$ and $\vartheta : C_{\rho, \varphi}(I) \times C_{\rho, \varphi}(I) \rightarrow \mathbb{R}$, so that

$$\sigma \left( \mu_0(t), \left( \vartheta_{-\rho}^{T^*}I_{\rho}^{\omega(T)}(T)\frac{\rho(\vartheta, \omega(T))}{\rho(\vartheta, \omega(T))} \right) + \rho(\vartheta, \omega(T)) \int_0^\tau s^{\rho-1}(\tau^s - s)^{-1}\omega(s)ds \right) \geq 0, \quad (37)$$

with $\sigma \in C(I)$ and $\omega(t) = \alpha(t, \mu_0(t), \varphi(t))$.

(H$_3$) For any $t \in I$, and $\vartheta, \varphi \in C_{\rho, \varphi}(I), \vartheta(\vartheta(t), \varphi(t)) \geq 0$ implies

$$\sigma \left( \rho(T) \int_0^\tau s^{\rho-1}(\tau^s - s)^{-1}\varphi(s)ds + \rho(T) \int_0^\tau s^{\rho-1}(\tau^s - s)^{-1}\omega(s)ds \right) \geq 0. \quad (38)$$

with $\vartheta, \varphi \in C(I)$ so that

$$(H_4) \text{ If } \vartheta_{n+1} \in C(I) \text{ with } \vartheta_n \rightarrow \vartheta \text{ and } \vartheta(\vartheta_n, \vartheta_{n+1}) \geq 1, \text{ then } \vartheta(\vartheta_n, \vartheta) \geq 1. \quad (40)$$

Theorem 3. We presume (H$_1$)-(H$_4$). Then, the problem (1) possesses at least a solution on $I$.

Proof. Take the operator $N : C_{\rho, \varphi}(I) \rightarrow C_{\rho, \varphi}(I)$ into account that is described as

$$(N\vartheta)(t) = (\vartheta_{-\rho}^{T^*}I_{\rho}^{\omega(T)}(T)\frac{\rho(\vartheta, \omega(T))}{\rho(\vartheta, \omega(T))}) + \rho(\vartheta, \omega(T)) \int_0^\tau s^{\rho-1}(\tau^s - s)^{-1}\omega(s)ds, \quad (41)$$

where $\omega \in C(I)$, with $\omega(t) = \alpha(t, \vartheta(t), \varphi(t))$.

On account of Lemma 4, we deduce that solutions of (1) are the fixed points of $N$.

Let $C_{\rho, \varphi}(I) \times C_{\rho, \varphi}(I) \rightarrow (0, \infty)$ be the function defined by

$$\vartheta(\vartheta(t), \varphi(t)) = 1, \quad \text{if } \vartheta(\vartheta(t), \varphi(t)) \geq 0, t \in I, \quad (42)$$

$$\vartheta(\vartheta(t), \varphi(t)) = 0, \quad \text{otherwise.}$$

First, we demonstrate that $N$ form a generalized $\alpha - \varphi$ -Geragthy operator. For any $t \in I$ and each $\vartheta, \varphi \in C(I)$, we derive that

$$\int_0^\tau s^{\rho-1}(\tau^s - s)^{-1}\omega(s)ds \leq \int_0^\tau s^{\rho-1}(\tau^s - s)^{-1}\omega(s)ds, \quad (43)$$
where \( \omega, \varphi \in C(I) \), with
\[
\omega(t) = \kappa(t, \theta(t), \omega(t)), \quad (44)
\]
\[
\varphi(t) = \kappa(t, \nu(t), \varphi(t)). \quad (45)
\]

From \((H_1)\), we have
\[
|\omega(t) - \varphi(t)| = |\kappa(t, \theta(t), \omega(t)) - \kappa(t, \nu(t), \varphi(t))| 
\leq p(\theta, \nu)T^{(r-1)}|\theta(t) - \nu(t)| + q(t)|\omega(t) - \varphi(t)| 
\leq p(\theta, \nu)\left(T^{(r-1)}|\theta(t) - \nu(t)|^2\right)^{1/2} + q(t)|\omega(t) - \varphi(t)|. \quad (46)
\]

Thus,
\[
|\omega(t) - \varphi(t)| \leq \frac{p(\theta, \nu)}{1 - q} \left\| (\theta - \nu)^2 \right\|_C^{1/2}, \quad (47)
\]
where \( q = \sup_{t \in I}|q(t)| \).

Next, we have
\[
|\tau^{(r-1)}(N\theta(t) - \tau^{(r-1)}(N\nu)(t)| 
\leq \tau^{(r-1)}|\theta(t) - \nu(t)| + \int_0^T (T^p - s^p)^{-1} \frac{p(\theta, \nu)}{1 - q} \left\| (\theta - \nu)^2 \right\|_C \, ds 
\leq \tau^{(r-1)}\left(T^p - s^p\right)^{-1} \frac{p(\theta, \nu)}{1 - q} \left\| (\theta - \nu)^2 \right\|_C \, ds. \quad (48)
\]

Thus,
\[
\alpha(\theta, \nu) \left|\tau^{(r-1)}(N\theta(t) - \tau^{(r-1)}(N\nu)(t)\right|^2 
\leq \left\| (\theta - \nu)^2 \right\| \left( \frac{1}{1 - q} \int_0^T (T^p - s^p)^{-1} \frac{p(\theta, \nu)}{1 - q} \, ds \right)^2 
+ \left\| (\theta - \nu)^2 \right\| \left( \frac{1}{1 - q} \int_0^T (T^p - s^p)^{-1} \frac{p(\theta, \nu)}{1 - q} \, ds \right)^2 
\leq \left\| (\theta - \nu)^2 \right\| C. \quad (49)
\]

Hence,
\[
\alpha(\theta, \nu) \varphi(2^3d(N(\theta), N(\nu))) \leq \lambda(\varphi(d(\theta, \nu))\varphi(d(\theta, \nu)). \quad (50)
\]

Accordingly, for any \( t \in I \), we find
\[
\theta(\theta(t), \nu(t)) \geq 0. \quad (52)
\]

This implies from \((H_2)\) that
\[
\theta(Nu(t), Nu(t)) \geq 0, \quad (53)
\]
which gives \( \alpha(N(\theta), NV) \geq 1 \).

Ergo, \( N \) is a \( \alpha \)-admissible.

Now, from \((H_2)\), there exists \( \mu_0 \in C_{r\rho}(I) \) such that
\[
\alpha(\mu_0, N(\mu_0)) \geq 1. \quad (54)
\]

Finally, from \((H_3)\), if \( \mu_n \in N \subset M \) with \( \mu_n \rightarrow \mu \) and \( \alpha(\mu, \mu + 1) \geq 1 \), then
\[
\alpha(\mu, \mu) \geq 1. \quad (55)
\]

Theorem 2 implies that fixed point \( \theta \) of \( N \) forms a solution for \((1)\).

3. An Example

The tripled \( (C_{r\rho}([0,1]), d, 2) \) is a complete b.m.s. with \( d : C_{r\rho}([0,1]) \times C_{r\rho}([0,1]) \rightarrow [0, \infty) \) such that
\[
d(\mu, \theta) = \left\| (\mu - \theta)^2 \right\|_C. \quad (56)
\]

We take the following fractional differential problem into consideration
\[
\begin{cases}
(\rho D^\alpha_0, \mu)(t) = \kappa(t, \mu(t), (\rho D^\alpha_0, \mu)(t)), \quad t \in [0, 1], \\
\mu(1) = 2,
\end{cases} \quad (57)
\]
with
\[
\kappa(t, \mu(t), \theta(t)) = \frac{\tau^{(r-1)}(1 + \sin(\theta(t)))}{4(1 + |\mu(t)|)} + \frac{e^{-t}}{2(1 + |\theta(t)|)}; \quad t \in [0, 1]. \quad (58)
\]

Let \( t \in (0, 1) \), and \( \mu, \theta \in C_{r\rho}([0,1]) \). If \( |\mu(t)| \leq |\theta(t)| \), then
\[
\alpha(\theta, \nu) \geq 1. \quad (51)
\]

where \( \lambda \in \Phi, \varphi \in \Phi, \) with \( \lambda(\tau) = 1/8t \), and \( \varphi(\tau) = \tau \).

So, \( N \) is generalized \( \alpha - \varphi \) - Geraghty operator.

Let \( \theta, \nu \in C_{r\rho}(I) \) such that
\[
\alpha(\theta, \nu) \geq 1. \quad (51)
\]

where \( \omega, \nu \in C(I) \), with
\[
\omega(t) = \kappa(t, \theta(t), \omega(t)), \quad (44)
\]
\[
\varphi(t) = \kappa(t, \nu(t), \varphi(t)). \quad (45)
\]

From \((H_1)\), we have
\[
|\omega(t) - \varphi(t)| = |\kappa(t, \theta(t), \omega(t)) - \kappa(t, \nu(t), \varphi(t))| 
\leq p(\theta, \nu)T^{(r-1)}|\theta(t) - \nu(t)| + q(t)|\omega(t) - \varphi(t)| 
\leq p(\theta, \nu)\left(T^{(r-1)}|\theta(t) - \nu(t)|^2\right)^{1/2} + q(t)|\omega(t) - \varphi(t)|. \quad (46)
\]

Thus,
\[
|\omega(t) - \varphi(t)| \leq \frac{p(\theta, \nu)}{1 - q} \left\| (\theta - \nu)^2 \right\|_C^{1/2}, \quad (47)
\]
where \( q = \sup_{t \in I}|q(t)| \).

Next, we have
\[
|\tau^{(r-1)}(N\theta(t) - \tau^{(r-1)}(N\nu)(t)| 
\leq \tau^{(r-1)}|\theta(t) - \nu(t)| + \int_0^T (T^p - s^p)^{-1} \frac{p(\theta, \nu)}{1 - q} \left\| (\theta - \nu)^2 \right\|_C \, ds 
\leq \tau^{(r-1)}\left(T^p - s^p\right)^{-1} \frac{p(\theta, \nu)}{1 - q} \left\| (\theta - \nu)^2 \right\|_C \, ds. \quad (48)
\]

Thus,
\[
\alpha(\theta, \nu) \left|\tau^{(r-1)}(N\theta(t) - \tau^{(r-1)}(N\nu)(t)\right|^2 
\leq \left\| (\theta - \nu)^2 \right\| \left( \frac{1}{1 - q} \int_0^T (T^p - s^p)^{-1} \frac{p(\theta, \nu)}{1 - q} \, ds \right)^2 
+ \left\| (\theta - \nu)^2 \right\| \left( \frac{1}{1 - q} \int_0^T (T^p - s^p)^{-1} \frac{p(\theta, \nu)}{1 - q} \, ds \right)^2 
\leq \left\| (\theta - \nu)^2 \right\| C. \quad (49)
\]

Hence,
\[
\alpha(\theta, \nu) \varphi(2^3d(N(\theta), N(\nu))) \leq \lambda(\varphi(d(\theta, \nu))\varphi(d(\theta, \nu)). \quad (50)
\]

where \( \lambda \in \Phi, \varphi \in \Phi, \) with \( \lambda(\tau) = 1/8t \), and \( \varphi(\tau) = \tau \).

So, \( N \) is generalized \( \alpha - \varphi \) - Geraghty operator.

Let \( \theta, \nu \in C_{r\rho}(I) \) such that
\[
\alpha(\theta, \nu) \geq 1. \quad (51)
Hypothesis (H_2) is satisfied with \( \mu_0(t) = \mu_0 \). Also, (H_4) holds the definition of the function \( \delta \). So, Theorem 3 yields that problem (57) admits a solution.

### Data Availability

No data is used. No data is available in this work.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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