Elliptic Islands on the Elliptical Stadium

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Abstract

We investigate the existence of elliptic islands for a special family of periodic orbits of a two-parameter family of maps $T_{a,h}$, corresponding to the billiard problem on the elliptical stadium.

The hyperbolic character of those orbits were studied in 2 for $1 < a < \sqrt{2}$ and here we look for the elliptical character for every $a > 1$.

We prove that, for $a < \sqrt{2}$, the lower bound for chaos $h = H(a)$ found in 2 is the upper bound of ellipticity for this special family.

For $a > \sqrt{2}$ we prove that there is no upper bound on $h$ for the existence of elliptic islands.

The main results we use are Birkhoff Normal Form and Moser’s Twist Theorem.

1 Introduction

The elliptical stadium is a plane region bounded by a curve $\Gamma$, constructed by joining two half-ellipses, with major axes $a > 1$ and minor axes $b = 1$, by two straight segments of equal length $2h$ (see fig. 1).

![Figure 1: The elliptical stadium.](image)

The billiard on the elliptical stadium consists in the study of the free motion of a point particle inside the stadium, being reflected elastically at the impacts with $\Gamma$. Since the motion is free inside $\Gamma$, it is determined either by two consecutive points of reflection at $\Gamma$ or by the point of reflection and the direction of motion immediately after each collision.

For fixed $a$ and $h$, let $s \in [0, L)$ be the arc length parameter for $\Gamma$ and the direction of motion be given by the angle $\beta$ with the normal to the boundary at the impact point. The billiard defines an invertible map $T_{a,h}$ from the annulus $\mathcal{A} = [0, L) \times (-\pi/2, \pi/2)$ into itself, preserving the measure $d\mu = \cos \beta \, d\beta \, ds$.\footnote{AMS subject classification: 37E40, 70K42}
Since \( \Gamma \) is globally \( C^1 \) but not \( C^2 \), \( T_{a,h} \) is a homeomorphism (see, for instance, [6]) and if \((s_0, \beta_0)\) and \((s_1, \beta_1) = T_{a,h}(s_0, \beta_0) \in \mathcal{A} \) such that \( \Gamma \) is analytic in some neighborhoods of \( s_0 \) and \( s_1 \), then clearly \( T_{a,h} \) is analytic in some neighborhoods of \((s_0, \beta_0)\) and \((s_1, \beta_1)\).

For each \((a, h)\), \((\mathcal{A}, \mu, T_{a,h})\) defines a discrete dynamical system, whose dynamics depends on the values of \( a \) and \( h \). For instance, when \( h = 0 \) we have an ellipse and the billiard is integrable.

When \( h \neq 0 \), two main features appear. If \( a < \sqrt{2} \), Donnay [3] proved that the billiard on the elliptical stadium is chaotic (in the sense of non-vanishing Lyapunov exponents) when \( h \) is sufficiently large. Lower bounds for \( h \) for this behaviour were found by Markarian and ourselves in [5] and by Canale, Markarian and ourselves in [2]. In the present work we show that the lower bound found in [2] is optimal, in the sense that below it we can assure the existence of elliptic islands of positive measure.

In [1] Bunimovich conjectured the existence of a stable periodic orbit, with island of positive measure, for billiards such as the elliptical stadium with \( a > \sqrt{2} \) and \( h \neq 0 \). In this work we make some progress in this direction, proving that there is no upper bound on \( h \) for the existence of elliptic islands if \( a > \sqrt{2} \). So, there is no way to destroy the elliptic islands by just increasing the distance between the half-ellipses.

To prove the existence of elliptic islands we extend the results about a special family of periodic orbits, called pantographic, studied in [2] and find regions on the parameter plane where at least one of its members is elliptic and stable, so, with islands of positive measure in phase space.

### 2 Pantographic orbits: existence and ellipticity

In this section we define the special family of periodic orbits and investigate the existence and ellipticity of its members. This family has already been investigated in [2] for \( a < \sqrt{2} \). Here we extend that work for all \( a > 1 \). We will skip most of the proofs which can be found in the work cited above.

Given \( a, h \) and a positive integer \( n \), an \((n, a, h)\)-pantographic orbit, denoted by \( Pan(n, a, h) \), is a symmetric \((4 + 2n)\)-periodic orbit, with exactly 2 impacts at each half-ellipse, joined by a vertical path, and crossing any vertical line only twice. One example can be seen in figure 2.

![Figure 2: The 10-periodic pantographic orbit \((n = 3)\).](image)

Let the right half-ellipse of the stadium be parametrized by \((x, y) = (a \cos \lambda + h, \sin \lambda)\) and \( P \) be the point marked on figure 2. Using the obvious symmetries (see figure 3), the parameter \( \lambda \) of \( P \) must satisfy:

\[
\tan 2\beta = \frac{a \tan \lambda}{a^2 \tan^2 \lambda - 1} = \frac{h + a \cos \lambda}{n + \sin \lambda} \quad \text{and} \quad \tan \beta = \frac{\cos \lambda}{a \sin \lambda} \quad (1)
\]

where \( \beta > 0 \) is, as defined above, the angle of the trajectory from \( P \), with the normal to the boundary.
The following proposition gives the region of existence of those pantographic orbits in the parameter plane.

**Proposition 1**

- \( Pan(0, a, h) \) and \( Pan(1, a, h) \) exist for every \( a > 1 \) and \( h > 0 \).

- For \( n \geq 2 \), \( Pan(n, a, h) \) exists for every \( 1 < a \leq 2 \) and \( h > 0 \) or for every \( a > 2 \) and \( h > (n - 1) \sqrt{a(a - 2)} \).

**Proof:** Equation (1) can be written as

\[
 n = \frac{a^2 t^2 - 1}{2at} h + \frac{(a^2 - 2)t^2 - 1}{2t\sqrt{1 + t^2}}.
\]

where \( t = \tan \lambda \). As proven in [2], for each integer \( n \geq 0 \), this equation has a unique solution \( t(n, a, h) = \tan \lambda(n, a, h) > 0 \) for every \( a > 1 \) and \( h > 0 \) and \( t(n, a, h) \in \left(\frac{1}{a}, +\infty\right) \).

For \( n = 0 \) and \( n = 1 \) this results implies the existence of the corresponding \( Pan(n, a, h) \) for every \( a > 1 \) and \( h > 0 \).

However, for \( n \geq 2 \) one must also ask that

\[
 \tan 2\beta \geq \frac{a \cos \lambda}{1 + \sin \lambda}
\]

in order to guarantee that the next impact point from \( P \) is on the straight part of the boundary. This is equivalent to

\[
 0 \leq \sin \lambda \leq \frac{1}{a - 1}
\]

(3)
which is always true if $1 < a \leq 2$.

If $a > 2$ we rewrite (3) as $0 \leq t = \tan \lambda \leq \frac{1}{\sqrt{a(a-2)}}$. Since $\frac{\partial h}{\partial t}(n, a, h) < 0$ (which is easily verified from (2)), and $\frac{1}{a} < \frac{1}{\sqrt{a(a-2)}}$ there exists a unique $\overline{h}$ such that $t(n, a, \overline{h}) = \frac{1}{\sqrt{a(a-2)}}$ and if $h > \overline{h}$, $\frac{1}{a} < t(n, a, h) < \frac{1}{\sqrt{a(a-2)}}$. From (3), $\overline{h} = (n-1)\sqrt{a(a-2)}$.

For each fixed $n$, we denote by $U_n$ the open region in the parameter plane where $Pan(n, a, h)$ exists, according to Proposition 4. For $(a, h) \in U_n$, let $s$ be the arc length corresponding to the point $P$ of $Pan(n, a, h)$ and $\beta$ the angle with the normal of the trajectory at this point as before. Then $T_{a, h}^{4+2n} (s, \beta)$ is the ellipse at $(s, \beta)$ and the ellipticity of this orbit is determined by the eigenvalues of $DT_{a, h}^{4+2n}(s, \beta)$.

As shown in [2], we can write $DT_{a, h}^{4+2n}(s, \beta) = (M_1 M_2)^2$ with

$M_j = \frac{1}{cos \beta} \begin{pmatrix} l_j K - cos \beta & l_j \\ K(l_j K - 2 cos \beta) & l_j K - cos \beta \end{pmatrix}$

and where $l_1$ is the length of the trajectory between two impacts with the same half-ellipse, $l_2$ is the length of the trajectory between two impacts with the different half-ellipses and $K$ is the curvature of the ellipse at $s$.

Let

$\Delta_n(a, h) = \left( \frac{l_1 K}{cos \beta} - 1 \right) \left( \frac{l_2 K}{cos \beta} - 1 \right)$.

Since $det(M_1 M_2) = 1$ and $tr(M_1 M_2) = 4\Delta_n(a, h) - 2$, it follows that if $0 < \Delta_n(a, h) < \frac{1}{2} < \Delta_n(a, h) < 1$ then $Pan(n, a, h)$ is elliptic (meaning that the eigenvalues of $DT_{a, h}^{4+2n}(s, \beta)$ are unitary with non zero imaginary part).

The following lemma summarizes some properties of $\Delta_n(a, h)$ and its technical proof has been postponed to the appendix.

**Lemma 2** For every $n \geq 0$, let

$\hat{U}_n = \{(a, h) | 1 < a < \sqrt{2}, h > 0\} \cup \{(a, h) | a \geq \sqrt{2}, h > na\sqrt{a^2 - 2} \equiv h_0^n(a)\} \subset U_n$.

The function $\Delta_n(a, h)$ has the following properties:

1. $\Delta_n(a, h)|_{\hat{U}_n} > 0$
2. $\frac{\partial \Delta_n}{\partial h}|_{\hat{U}_n} > 0$
3. $\lim_{h \to +\infty} \Delta_n(a, h) = +\infty$
4. for $1 < a < \sqrt{2}$, $\lim_{h \to 0} \Delta_n(a, h) = L_n(a) = \left( \frac{2}{a^2} - 1 \right)\left( \frac{2(n+1)}{a^2} - 1 \right) > 0$
   for $a \geq \sqrt{2}$, $\lim_{h \to na\sqrt{a^2 - 2}} \Delta_n(a, h) = 0$.

For each $0 < c \leq 1 + 2n$, let $\alpha_n^c$ be the unique solution of $L_n(a) = c$. We have that $1 < \alpha_n^c < \sqrt{2}$ and if $c < d$ then $\alpha_n^d < \alpha_n^c$.

It follows from the lemma that every level curve $\Delta_n = c$ is given by a graph $h_n^c : (\alpha_n^c, +\infty) \to \mathbb{R}$ such that $\Delta_n(a, h_n^c(a)) = c$ and $\Delta_n(a, h) < c$ if $h < h_n^c(a)$ and $\Delta_n(a, h) > c$ if $h > h_n^c(a)$.

The characterization of the region of ellipticity is then given by:
Figure 4: Region of ellipticity with lines of resonance up to order 4 for $n = 3$

**Proposition 3** For each fixed $n$, the region in the parameter plane where $P_{an}(n,a,h)$ is elliptic is the union of two open adjacent strips, bounded by the graphs of $h^0_n(a), h^{1/2}_n(a), h^1_n(a)$ and by the segments $\{(a,0)|\alpha^1_n < a < \alpha^{1/2}_n\}$ and $\{(a,0)|\alpha^{1/2}_n < a < \sqrt{2}\}$. (see figure [4]).

For each $n$ and $(a,h)$ in the region of ellipticity of $P_{an}(n,a,h)$, let $\mu$ and $\bar{\mu}$ be the eigenvalues of $DT_{a,h}^{4+2n}|_{(s,\beta)}$. $P_{an}(n,a,h)$ has a resonance of order $k$ if $\mu^k = 1$, i.e., $\mu = e^{j\pi k/2n}$, $j = 1, \ldots, k - 1$. It is easy to show that this corresponds, for $k > 1$, to

$$\Delta_n(a,h) = \frac{1}{2} \left( 1 + \cos \frac{j\pi}{k} \right) \equiv c_{jk}.$$  

Clearly $0 < c_{jk} < 1$ and we have the curves of resonance given by the graphs of $h^{c_{jk}}_n : (\alpha_n^{c_{jk}}, +\infty) \rightarrow \mathbb{R}$.

We can then characterize the region of ellipticity with no resonances up to order $k$ by:

**Proposition 4** For a fixed $n$, the region in the parameter plane where $P_{an}(n,a,h)$ is elliptic with no resonances up to order $k$ is a finite union of open disjoint adjacent strips contained in the region of ellipticity. (see figure [4])

### 3 Existence of elliptic islands

In order to establish if the elliptic periodic orbits described in the previous section have invariant curves surrounding them, we will invoke the two classical results:

**Birkhoff Normal Form:** Let $f$ be an area preserving map in $C^l$ ($l \geq 4$) with a fixed point at the origin, with eigenvalues $\mu$ and $\overline{\mu}$, $|\mu| = 1$. If for some integer $q$ in $4 \leq q \leq l + 1$ one has $\mu^k \neq 1$ for $k = 1, 2, \ldots, q$ then there exists a real analytic transformation taking $f$ into the normal form

$$\zeta \rightarrow f(\zeta,\overline{\zeta}) = \mu \zeta e^{i\tau(\zeta\overline{\zeta})} + g(\zeta,\overline{\zeta})$$  

(4)
where $\tau(\zeta^2) = \tau_1|\zeta|^2 + ... + \tau_s|\zeta|^{2s}$, with $s = \left[\frac{q}{2}\right] - 1$, is a real polynomial in $|\zeta|^2$ and $g$ vanishes with its derivatives up to order $q - 1$ at $\zeta = 0$.

**Theorem (Moser, [8], p.56)** If the polynomial $\tau(|\zeta|^2)$ does not vanishes identically, $\zeta = 0$ is a stable fixed point (which means that there are invariant curves surrounding it, and so an elliptic island of positive measure).

For each fixed period $n_0$, we will then investigate the resonances of $T_{a,h}^{4+2n_0}$ and the zeros of the coefficients of its Birkhoff normal form, near the Pantographic orbit.

Let us fix a period $n_0$ and a major axis $a_0 > a_{n_0}^1 = \sqrt{(2 + 2n_0)/(2 + n_0)}$. According to proposition [4], $\text{Pan}(n_0, a_0, h)$ is elliptic with no resonances up to order $q$ if $h$ is in the finite union of open disjoint adjacent intervals denoted $\cup I_j^q$.

Let $\lambda(h), s(h), \beta(h)$ be respectively the parameter, the arc length and the angle with the normal for the point $P$ of $\text{Pan}(n_0, a_0, h)$.

**Lemma 5** For any $q \geq 4$, $\lambda(h), s(h)$ and $\beta(h)$ are analytic functions of $h$ on each $I_j^q$.

**Proof:** The point $P$ will belong to $\text{Pan}(n_0, a_0, h)$ if $t = \tan \lambda$ satisfies equation (8):

$$n_0 = \frac{(a_{n_0}^2 - 1)^2}{2a_{n_0}^2} h + \frac{(a_{n_0}^2 - 2)t^2 - 1}{2t(1 + t^2)}.$$

$A(t) = \frac{2t^2 - 1}{2a_{n_0}^2}$ and $B(t) = \frac{(a_{n_0}^2 - 2)t^2 - 1}{2t(1 + t^2)}$ are analytical functions of $t$ and $A(t) \neq 0$, since $\frac{1}{a_{n_0}^2} < t < \infty$. So $h = h(t) = \frac{n_0 - B(t)}{A(t)}$ is analytic. As this equation has a unique solution for each $h$, the inverse $t = t(h)$ exists and is then locally analytic for every $h \in I_j^q$.

The functions $\lambda(h) = \arctan t(h)$ and the corresponding arc length of the ellipse $s(h) = s(\lambda(h))$ are then analytic. The same is true for $\beta(h) = \beta(\lambda(h))$.

For each fixed $h$, let $f$ be the translation of $T_{a_0,h}^{4+2n_0}$ by $(s(h), \beta(h))$. The map $f$ is clearly area preserving and analytic in $(s, \beta)$ on a neighbourhood of the origin. The eigenvalues of $Df(0,0)$ are the same as those of $DT_{a_0,h}^{4+2n_0}(s(h), \beta(h))$. If $h \in I_j^q$, $f$ can be written in the Birkhoff normal form (18). If one of the Birkhoff coefficients is not zero, $f$ has an elliptic island surrounding $(0,0)$ and, by translation, there is an elliptic island surrounding $\text{Pan}(n_0, a_0, h)$.

**Lemma 6** For $1 \leq m \leq \left[\frac{q}{2}\right] - 1$, $\tau_m(h)$, the $m$-th Birkhoff coefficient of $f$, is an analytic function of $h$ on each $I_j^q$.

**Proof:** For each fixed $h$, $\tau_m(h)$ is a combination of the coefficients of the $(q - 1)$-th jet of $f$ at $(0,0)$ (see, for instance, [4]). The steps giving rise to the calculation of $\tau_m(h)$ (complexification of the space, elimination of unwanted terms) are analytical. So, if the coefficients of the jet $J_{q-1}f(0,0)$ are analytical in $h$, then $\tau_m(h)$ will also be an analytic function of $h$.

Now these coefficients are combinations of the entries of $DT_{a_0,h}^{4+2n_0}$ and their derivatives up to $(q - 1)$-th order with respect to $s$ and $\beta$, calculated at $(s(h), \beta(h))$.

Let $(s, \beta)$ and $(s', \beta')$ be two consecutive impacts of a trajectory with the two different half-ellipses (with $l \geq 0$ impacts with the straight parts between them), or two consecutive impacts of a trajectory with the
same half-ellipse (with $l = 0$). Then $DT^{4+2n}_0(h)$ is a finite product of matrices of the form (see, for instance, [6])

$$
\left(\frac{-1}{\cos \beta'} \begin{array}{cc}
LK - \cos \beta \\
KK' - K' \cos \beta - K \cos \beta' \\
LK' - \cos \beta'
\end{array}\right)
$$

where $K$ stands for the curvature of the ellipse at the impact and $L$ is the total length of the trajectory between the two impacts with the half-ellipses.

Since $\cos \beta \neq 0$ for $\beta$ near $\beta(h)$, all the entries of the matrix above, as well as their derivatives up to any order in $s$ and $\beta$, are analytic functions of $h$. Using lemma [6] we conclude that all the coefficients of $J_{q-1,f(0,0)}$ are analytic in $h$.

If follows that $\tau_m(h)$ is analytic in $h$, leading immediately to the next corollary.

**Corollary 7** On each $I^q_j$ and for $1 \leq m \leq \left\lfloor \frac{q}{2} \right\rfloor - 1$, the set $\{h / \tau_m(h) = 0\}$ is either the entire $I^q_j$ or a discrete set.

In order to prove the existence of islands we use the natural recurrence on the order of the resonances.

We begin by analysing the zeros of $\tau_1$ on the non resonant intervals $I^4_j$. If $\tau_1 = 0$ only on a discrete subset of each $I^4_j$, $Pan(n_0,a_0,h)$ has elliptic island except for a discrete set of values of $h$ (which can be smaller than the union of the discrete subsets of zeros of $\tau_1$ and the values of resonance up to order 4, since on the discrete subsets a non resonant value of higher order may have a non zero Birkhoff coefficient).

If $\tau_1$ is identically zero on one of those intervals, we proceed to the next step, applying the same analysis to the zeros of $\tau_2$.

We continue the recurrence and it will end up in a finite number of steps if for some order of resonance the last Birkhoff coefficient does not vanish identically on a whole non resonant interval. Otherwise, all the Birkhoff coefficients will vanish on at least one open interval, bounded by resonant values of $h$. In this last case, since $\frac{\partial \rho}{\partial h} > 0$, the rotation number $\rho(h)$ of $Pan(n_0,a_0,h)$ is not constant and there exists $h_0$ such that $\rho(h_0)$ is diophantine. As $f$ is analytic, it is conjugate to a rotation and there will be invariant curves [4].

We conclude that:

**Theorem 1** Given $n$ and $a > a_n^1$, there are at least countably many values of $h$ in $\cup I^4_j$ such that $Pan(n,a,h)$ has an elliptic island.

**Remark:** As Moeckel proved in [7], in a generic one-parameter family of area preserving maps with elliptic fixed points, the first Birkhoff coefficient $\tau_1$ varies from $-\infty$ to $+\infty$ as the rotation number varies from $0$ to $1/3$ or from $2/3$ to $1$. So, we do not expect $\tau_1(h)$ to be always different from zero, neither to vanish identically. Furthermore, generically, at a zero of $\tau_1$, a higher Birkhoff coefficient will not vanish. So, although we can handle the case $\tau_s(h_0) = 0$, $\forall s$, we do not expect it to happen in our case.

### 4 Bounds for the existence of islands
### 4.1 The case $a < \sqrt{2}$

As shown in proposition 3 and in [2], if $(a, h)$ is in the region of ellipticity of $Pan(n, a, h)$ then $a \geq \alpha_n^1 = \frac{2n+2}{n+2}$. So, given $a \in (1, \sqrt{2})$, there is only a finite number of periods $4+2n$ such that $Pan(n, a, h)$ can be elliptic. More precisely, $n \leq \frac{2(a^2-1)}{a^2-2}$.

Let $H(a)$ be the maximum of $h_n^1(a)$ for those periods. As proved in [2], $H(a)$ is a lower bound for chaos. By theorem 1, it is also an upper bound for the existence of elliptic islands for the Pantographic family.

### 4.2 The case $a > \sqrt{2}$

On the other hand, if $a > \sqrt{2}$, $Pan(n, a, h)$ can be elliptic for any period $n$. Moreover, for each $n$ and $q$, $\cup I^q_n \subset (h^0_n(a), h^1_n(a))$, with $h^0_n(a) = na\sqrt{a^2-2}$. We also have that $(h^0_n(a), h^1_n(a)) \cap (h^0_{n+1}(a), h^0_{n+1}(a))$ is a non empty open interval. So we can find $h$, $na\sqrt{a^2-2} < h < (n+1)a\sqrt{a^2-2}$ such that $Pan(n, a, h)$ has an elliptic island. This proves the following

**Theorem 2** Given $a > \sqrt{2}$ there is no upper bound on $h$ for the existence of elliptic islands on the elliptical stadium billiard.

However, as can be seen in figure 5, for values of $a$ further away from $\sqrt{2}$, the strips of ellipticity are disjoint. In these gaps all pantographic orbits are hyperbolic, having, thus, no islands.

![Figure 5: Gaps between the strips of ellipticity](image)

Nevertheless, in our simulations other islands appear, obviously corresponding to different periodic orbits. In figure we exemplify this fact for $a = 2$ and $h = 2$, a value located in the gap between the strips of ellipticity for $n = 0$ and $n = 1$ (see figure 5). We show three non-pantographic orbits and their islands and the whole phase space, where we can see many other islands surrounded by what seems to be a chaotic sea. As far as our results indicate and our simulations show, this should be the typical picture for the phase space when $a > \sqrt{2}$.
Figure 6: Phase space for \((a, h) = (2, 2)\)

5 Appendix

5.1 Some properties of \(t(n, a, h)\)

We called 
\[ U_n \]
the open region in the parameter plane where \(Pan(n, a, h)\) exists:
\[ U_0 = U_1 = \{(a, h) / a > 1, h > 0\} \]
\[ U_n = \{(a, h) / 1 < a \leq 2, h > 0\} \cup \{(a, h) / a > 2, h > (n - 1)\sqrt{a(a-2)}\}, \]
for \(n \geq 2\).

The following lemma gives some useful information about \(t(n, a, h)\), the solution of \(f)\) or \(g)\).

**Lemma 8** Given \(n \geq 0\), let \((a, h) \in U_n\) and \(t(n, a, h)\) be the unique solution of \(f)\). Then

- \(\forall n, \) when \(h \to +\infty\), \(t(n, a, h) \to \frac{1}{n}\).
- **For** \(n = 0\)
  - if \(1 < a \leq \sqrt{2}\), when \(h \to 0^+\), \(t(0, a, h) \to +\infty\)
  - if \(a > \sqrt{2}\), when \(h \to 0^+\), \(t(0, a, h) \to \frac{1}{\sqrt{a-2}}\)
- **For** \(n = 1\)
  - if \(1 < a \leq 2\), when \(h \to 0^+\), \(t(1, a, h) \to +\infty\)
  - if \(a > 2\), when \(h \to 0^+\), \(t(1, a, h) \to \frac{1}{\sqrt{a(a-2)}}\)
- **For** \(n \geq 2\)
  - if \(1 < a \leq 2\), when \(h \to 0^+\), \(t(n, a, h) \to +\infty\)
  - if \(a > 2\), when \(h \to (n - 1)\sqrt{a(a-2)}^+\), \(t(n, a, h) \to \frac{1}{\sqrt{a(a-2)}}\)
Proof: Since $\frac{\partial}{\partial n} < 0$ and $\frac{1}{a} < t$, $\lim_{h \to +\infty} t$ exists. From equation (1),

$$at - \frac{1}{at} = 2 \frac{n\sqrt{1+t^2} + t}{h\sqrt{1+t^2} + a} \to 0$$

as $h \to +\infty$ and so $t \to \frac{1}{a}$.

To study the limit as $h \to 0^+$, let us take $x = \frac{1}{a}$. Equation (1) becomes

$$x(2n\sqrt{1 + x^2} + x^2 - (a^2 - 2)) = (a^2 - x^2)\sqrt{1 + x^2} \frac{h}{a}.$$ 

Since $0 < x < \frac{1}{a}$, $x(2n\sqrt{1 + x^2} + x^2 - (a^2 - 2)) > 0$.

Let $\mathcal{P} = \lim_{h \to 0^+} x$. We have that $0 \leq \mathcal{P} \leq a$ and $\mathcal{P}(2n\sqrt{1 + \mathcal{P}^2} + \mathcal{P}^2 - (a^2 - 2)) = 0$.

For $n = 0$, if $1 < a \leq \sqrt{2}$, $\mathcal{P} = 0$ is the unique solution of this equation and $\lim_{h \to 0^+} t(0, a, h) = +\infty$. If $a > \sqrt{2}$ we have a new solution $\mathcal{P} = \sqrt{a^2 - 2}$. But for $0 < x < \sqrt{a^2 - 2}$, $x(x^2 - (a^2 - 2)) < 0$. So $\lim_{h \to 0^+} x = \sqrt{a^2 - 2}$ and $\lim_{h \to 0^+} t(0, a, h) = \frac{1}{\sqrt{a^2 - 2}}$.

For $n = 1$ we have $\mathcal{P}(2\sqrt{1 + \mathcal{P}^2} + \mathcal{P}^2 - (a^2 - 2)) = 0$. If $a^2 - 2 \leq 2$, i.e. $a \leq 2$, the unique solution is $\mathcal{P} = 0$ and $\lim_{h \to 0^+} t(1, a, h) = +\infty$. If $a > 2$ the second solution is $\mathcal{P} = \sqrt{a(a - 2)}$. As above, if $0 < x < \sqrt{a(a - 2)}$, $x(2\sqrt{1 + x^2} + x^2 - (a^2 - 2)) < 0$ and so $\lim_{h \to 0^+} t(1, a, h) = \frac{1}{\sqrt{a(a - 2)}}$.

For $n \geq 2$, we remark first that if $k > l$ then $t(k, a, h) > t(l, a, h)$. So, for $1 < a \leq 2$ $\lim_{h \to 0^+} t(n, a, h) = +\infty$.

For $a > 2$, the limit as $h \to (n-1)\sqrt{a(a - 2)}$ is the unique solution of equation (1) for $h = (n-1)\sqrt{a(a - 2)}$ which is $t = \frac{1}{\sqrt{a(a - 2)}}$.

Remark: When $h \to 0^+$, the elliptical stadium becomes an ellipse. $t \to +\infty$ means that the pantographic orbit goes to the elliptic periodic orbit which corresponds to the minor axis of the ellipse.

Let us call pantographic-like orbits in the elliptical billiard the periodic trajectories that have vertical segments both at left and right extremes. As can be seen in (1), the 4-periodic pantographic-like orbit exists if $a > \sqrt{2}$ and the 6-periodic if $a > 2$, they are parabolic and their position is given, respectively, by $t = \frac{1}{\sqrt{a^2 - 2}}$ and $t = \frac{1}{\sqrt{a(a - 2)}}$. They are the calculated limits of the 4 and 6-periodic pantographic orbits of the elliptical stadium.

5.2 Proof of the lemma 2

For a fixed $n$, let $(a, h) \in U_n$ and $\lambda_n(a, h)$ be the solution of (1), $\beta$ be the angle, with the normal, of the outgoing trajectory at $P = (a \cos \lambda_n + h, \sin \lambda_n)$ and $s$ the corresponding arc length.

Let

$$\Delta_n(a, h) = \left(\frac{l_1 K}{\cos \beta} - 1\right) \left(\frac{l_2 K}{\cos \beta} - 1\right)$$

where $l_1 = 2 \sin \lambda_n$, $l_2 = 2 \sqrt{(h + a \cos \lambda_n)^2 + (n + \sin \lambda_n)^2}$ and $K = a/(a^2 \sin^2 \lambda_n + \cos^2 \lambda_n)^{3/2}$.

Let $\delta_1(a, h) = \left(\frac{l_1 K}{\cos \beta} - 1\right)$ and $\delta_2(a, h) = \left(\frac{l_2 K}{\cos \beta} - 1\right)$.

Lemma 9 For every $n \geq 0$ the function $\delta_1(a, h)$ has the following properties:
1. $\delta_1(a, h) > 0$ for $1 < a < \sqrt{2}$ and $h > 0$.

2. $\lim_{h \to 0^+} \delta_1(a, h) = \frac{a}{2} - 1$ for $1 < a < \sqrt{2}$.

3. $\delta_1(a, n a \sqrt{a^2 - 2}) = 0$ for $a \geq \sqrt{2}$.

4. $\frac{\partial \delta_1}{\partial h} > 0$ for $(a, h) \in U_n$.

**Proof:** We have $\delta_1(a, h) = \frac{h K}{\cos \beta} + 1 = 2 \frac{1 + a^2 t^2}{1 + a^2(1 + t^2)} - 1$ and properties 1 and 2 follow immediately.

If $a \geq \sqrt{2}$, $\delta_1 = 0$ implies $(a^2 - 2)t^2 - 1 = 0$ and $t = \frac{1}{\sqrt{a^2 - 2}}$. From equation (2), $h = na\sqrt{a^2 - 2}$ and property 3 follows.

Since $\frac{\partial \delta_1}{\partial h} = \frac{\partial_1 \partial h}{\partial h}$, as $\frac{\partial_1 \partial}{\partial h} = \frac{4(1-a^2)(1+\alpha^2)}{(1+a^2) \sqrt{2}} < 0$, for $a > 1$, and $\frac{\partial_1}{\partial h} < 0$, $\frac{\partial_2 \delta_1}{\partial h} > 0$.

**Lemma 10** For every $n \geq 0$ the function $\delta_2(a, h)$ has the following properties:

1. $\delta_2(a, h) > \delta_1(a, h) > 0$ for $(a, h) \in U_n$.

2. $\lim_{h \to 0^+} \delta_2(a, h) = 2 \frac{n+1}{a^2} - 1$ for $1 < a < \sqrt{2}$.

3. $\lim_{h \to +\infty} \delta_2(a, h) = +\infty$ for $1 < a$.

4. $\frac{\partial \delta_2}{\partial h} > 0$ for $(a, h) \in U_n$.

**Proof:** Since $l_2 > l_1$, $\delta_2 > \delta_1$.

For $1 < a < \sqrt{2}$, when $h \to 0$, $t(n, a, h) \to +\infty$. So $l_2 \to 2(n+1), K \to 1/a^2$ and $\cos \beta \to 1$ and property 2 follows.

Property 3 is obvious since $l_2 \to \infty$ as $h \to \infty$ and all the other quantities are bounded.

By definition $\delta_2(a, h) = l_2(a, h, t(a, h)) = \frac{K(t(a, h))}{\cos \beta(t(a, h))} - 1$ and $\frac{\partial \delta_2}{\partial t} = \frac{K}{\cos \beta} \frac{\partial l_2}{\partial t} + \frac{\partial}{\partial t} \left( \frac{l_2 \cos \beta}{\cos \beta} \right) \frac{\partial l_2}{\partial \beta}$.

We have that $\frac{\partial l_2}{\partial h} < 0$ and $\frac{\partial l_2}{\partial \beta} > 0$. The curvature $K > 0$ and for $0 < \lambda < \pi/2$, $\frac{\partial K}{\partial x} < 0$. So $\frac{\partial K}{\partial \beta} < 0$. As $0 < \beta < \pi/4$, $\cos \beta > 0$. As $\tan \beta = 1/\alpha$, $\frac{\partial}{\partial t} \cos \beta > 0$. This implies that $\frac{\partial}{\partial t} \left( \frac{K}{\cos \beta} \right) < 0$.

We have that $\frac{1}{l_2} \frac{\partial l_2}{\partial h} = (h + a \cos \lambda)^2 + (n + \sin \lambda)^2$. So $\frac{1}{l_2} \frac{\partial l_2}{\partial h} = 2(h + a \cos \lambda) > 0$ in $U_n$, implying that $\frac{\partial l_2}{\partial h} > 0$. We also have that $\frac{\partial l_2}{\partial h} = -2a \sin \lambda(n + \sin \lambda)(\tan 2\beta - \tan \beta) < 0$ at $0 < \beta < \pi/4$. So $\frac{\partial l_2}{\partial h} = \frac{\partial l_2}{\partial \beta} \frac{\partial \lambda}{\partial \lambda} < 0$, and $\frac{\partial l_2}{\partial \beta} < 0$.

This shows that $\frac{1}{l_2 \cos \beta}$ is the product of two positive decreasing functions of $t$ and so $\frac{\partial}{\partial t} \left( \frac{l_2 \cos \beta}{K} \right) < 0$.

We conclude that $\frac{\partial \delta_2}{\partial h} > 0$.

We have defined $\tilde{U}_n = \{(a, h)/1 < a < \sqrt{2}, h > 0\} \cup \{(a, h)/a \geq \sqrt{2}, h > na\sqrt{a^2 - 2}\} \subset U_n$.

**Lemma 2** For every $n \geq 0$ the function $\Delta_n(a, h)$ has the following properties:
1. $\Delta_n(a, h)|_{\tilde{U}_n} > 0$

2. $\frac{\partial \Delta_n}{\partial h}|_{\tilde{U}_n} > 0$

3. $\lim_{h \to +\infty} \Delta_n(a, h) = +\infty$

4. for $1 < a < \sqrt{2}$, $\lim_{h \to 0} \Delta_n(a, h) = L_n(a) = \left(\frac{2}{a^2} - 1\right)\left(\frac{2(n+1)}{a^2} - 1\right) > 0$

   for $a \geq \sqrt{2}$, $\lim_{h \to 0} \Delta_n(a, h) = 0$.

**Proof:** In $\tilde{U}_n$, $\Delta_n$ is the product of two positive increasing functions of $h$ and properties 1 and 2 follows. Properties 3 and 4 follow immediately from lemmas 9 and 10.

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