q-Analogues of two “divergent” Ramanujan-type supercongruences

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Abstract
Guillera and Zudilin proved three “divergent” Ramanujan-type supercongruences by means of the Wilf–Zeilberger algorithmic technique. In this paper we prove q-analogues of two of them via the q-WZ method. Additionally, we give q-analogues of two related congruence of Sun; one is confirmed and the other is conjectural.

Keywords q-Binomial coefficients · Wilf–Zeilberger method · q-WZ method · q-WZ pair · Cyclotomic polynomials

Mathematics Subject Classification Primary: 11B65 · 05A10; Secondary: 05A30

1 Introduction

By using the Wilf–Zeilberger method, Guillera and Zudilin [8] proved the following three supercongruences:

\[
\sum_{k=0}^{p-1} \frac{(\frac{1}{2})^3}{k!^3} (3k + 1) 2^{2k} \equiv p \pmod{p^3} \quad \text{for} \quad p > 2, \quad (1.1)
\]

\[
\sum_{k=0}^{p-1} \frac{(\frac{1}{2})^5}{k!^5} (10k^2 + 6k + 1) 2^{2k} \equiv p^2 \pmod{p^5} \quad \text{for} \quad p > 3, \quad (1.2)
\]

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\[
\sum_{k=0}^{p-1} \frac{(\frac{1}{2})_k^3}{k!^3} (3k + 1)(-1)^k 2^{3k} \equiv p(-1)^{p-1} \mod p^3 \quad \text{for } p > 2. \tag{1.3}
\]

Here, and throughout the paper, the letter \( p \) always denotes a prime, and the Pochhammer symbol \((a)_b\) is used for denoting \( \Gamma(a + b)/\Gamma(a) \) also in the case when \( b \) is not a non-negative integer. In the spirit of [40], the supercongruences (1.1)–(1.3) correspond to divergent Ramanujan-type series for \( 1/\pi \) or \( 1/\pi^2 \), such as
\[
\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{(1)_k^3} (3k + 1) 2^{2k} \equiv \frac{-2i}{\pi}, \quad \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{(1)_k} (3k + 1)(-1)^k 2^{3k} \equiv \frac{1}{\pi} \tag{1.4}
\]
(see [8, (47)]). The summations in (1.4) have to be understood as the analytic continuation of the corresponding hypergeometric series. For instance, the second formula in (1.4) can be written as
\[
\frac{1}{2\pi i} \int_{-\frac{1}{2} - i\infty}^{-\frac{1}{2} + i\infty} \frac{(\frac{1}{2})_s^3}{(1)_s^3} \Gamma(-s)(3s + 1) 2^{3s} ds = \frac{1}{\pi}.
\]

It is worth mentioning that Guillera [7] has given proofs of several divergent hypergeometric formulas for \( 1/\pi \) and \( 1/\pi^2 \) by using a version of the Wilf–Zeilberger method.

In a previous paper, motivated by Zudilin’s work [40], the author [10] utilized the \( q \)-WZ method [24,39] to give the following \( q \)-analogue of a Ramanujan-type supercongruence of Van Hamme [37]:
\[
\sum_{k=0}^{p-1} (-1)^k q^{-k} (4k + 1) (q; q^{-1})_k^3 \equiv [p]q^{(p-1)^2/4} (-1)^{p-1} \mod [p]^3 \quad \text{for } p > 2, \tag{1.5}
\]
where the \( q \)-shifted factorial is defined by \((a; q)_0 = 1\) and \((a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})\) for \( n \geq 1\), while the \( q \)-integer is defined as \([n] = 1 + q + \cdots + q^{n-1}\) (see [6]). We point out that, in this paper, two rational functions in \( q \) are congruent modulo a polynomial \( P(q) \) if the numerator of their difference is congruent to 0 modulo \( P(q) \) in the polynomial ring \( \mathbb{Z}[q] \) while the denominator is relatively prime to \( P(q) \). Note that some other interesting \( q \)-congruences can be found in [1,4,12–16,20,28,35,36].

Although supercongruences have been widely studied by many mathematicians including Beukers [2], Long and Ramakrishna [21], Rodriguez-Villegas [27], Sun [30], Sun [31], Van Hamme [37], and Zudilin [40], there are still many problems on \( q \)-congruences which are worthwhile to investigate. In fact, it is not always easy to give \( q \)-analogues of known congruences, and many congruences might have no \( q \)-analogues. Sometimes in order to prove an ordinary congruence (without \( q \)) we need to establish its \( q \)-analogue [4]. Proving \( q \)-congruences requires a variety of methods, including properties of roots of unity [28], basic hypergeometric series identities.
and the \( q \)-Wilf–Zeilberger (\( q \)-WZ) method \([10,11,36]\). In many cases, the method used to prove an ordinary congruence cannot be generalized directly to prove the corresponding \( q \)-congruence. For instance, no one knows how to extend the \( p \)-adic analysis in \([19]\) to the \( q \)-case. On the other hand, sometimes studying \( q \)-congruences will enable us to find new basic hypergeometric series identities \([9,15]\). Moreover, combinatorial proofs of ordinary congruences are little known. Finding out the \( q \)-analogues will help us to have a better understanding of the original congruences, and the theory of integer partitions might be applied to give the desirable combinatorial interpretations.

This paper is a continuation of \([10]\). We want to give \( q \)-analogues of (1.1) and (1.3) by the \( q \)-WZ method. Recall that the \( n \)-th cyclotomic polynomial \( \Phi_n(q) \) may be defined as

\[
\Phi_n(q) := \prod_{1 \leq k \leq n, \gcd(n,k)=1} (q - \zeta_k),
\]

where \( \zeta \) is an \( n \)-th primitive root of unity. It is well known that \( \Phi_n(q) \) is an irreducible polynomial with integer coefficients and \( \Phi_p(q) = [p] \).

Note that in (1.1)–(1.3) we can sum up to \( p - 1 \), since the \( p \)-adic order of \( \left(\begin{array}{c}1/2 \\ k\end{array}\right) k! \) is 1 for \( p + 1/2 \leq k \leq p - 1 \). Our \( q \)-analogues of (1.1) can be stated as follows.

**Theorem 1.1** Let \( n \) be a positive odd integer. Then

\[
\sum_{k=0}^{n-1} [3k + 1] \frac{(q; q^2)^3 k q^{-(k+1)/2}}{(q; q)^2_k (q^2; q^2)_k} \equiv [n] q^{1/n} \left( \frac{1}{2} \right) \mod [n] \Phi_n(q^2), \tag{1.7}
\]

\[
\sum_{k=0}^{n-1} [3k + 1] \frac{(q; q^2)^3 k q^{-(k+1)/2}}{(q; q)^2_k (q^2; q^2)_k} \equiv [n] q^{1/n} \left( \frac{1}{2} \right) \mod [n] \Phi_n(q^2). \tag{1.8}
\]

It is easy to see that, when \( n = p \) is an odd prime, the congruences (1.7) and (1.8) are equivalent to each other, since

\[
\frac{(q; q^2)^3}{(q; q^2)_k (q^2; q^2)_k} \equiv 0 \mod [p] \text{ for } \frac{p + 1}{2} \leq k \leq p - 1.
\]

But for general \( n \) they are clearly not equivalent.

Let \( n = p^r \) be an odd prime power and \( q = 1 \) in Theorem 1.1. Noticing that \( \Phi_{p^r}(1) = p \) and the denominator of the reduced form of \( \left(\begin{array}{c}1/2 \\ k\end{array}\right)^3 (3k + 1) 2^{2k} \) is relatively prime to \( p \), we obtain the following generalization of (1.1).

**Corollary 1.2** Let \( p > 2 \) and \( r > 0 \). Then

\[
\sum_{k=0}^{p^{r-1}/2} \frac{1}{k!^3} (3k + 1) 2^{2k} \equiv p^r \mod p^{r+2}, \tag{1.9}
\]
\[
p^r - 1 \sum_{k=0}^{\frac{p^r - 1}{2}} \left( \frac{1}{2} \right)_k^3 \left( 3k + 1 \right) 2^{2k} \equiv p^r \pmod{p^r + 2}. \quad (1.10)
\]

We also have two different \(q\)-analogues of (1.3) as follows.

**Theorem 1.3** Let \(n\) be a positive odd integer. Then

\[
\sum_{k=0}^{n-1} (-1)^k [3k + 1] \frac{(q; q^2)_k^3}{(q; q)_k^3} \equiv [n]q^{(\frac{n-1}{2})^2} (-1)^{\frac{n-1}{2}} \pmod{[n]\Phi_n(q)^2}. \quad (1.11)
\]

From (1.11), we can easily deduce the following generalization of (1.3).

**Corollary 1.4** Let \(p > 2\) and \(r > 0\). Then

\[
\sum_{k=0}^{p^r - 1} \left( \frac{1}{2} \right)_k^3 \left( 3k + 1 \right) (-1)^k 2^{3k} \equiv p^r (-1)^{\frac{p^r - 1}{2}} \pmod{p^r + 2} \text{ for } p > 2. \quad (1.12)
\]

**Conjecture 1.5** Let \(n\) be a positive odd integer. Then

\[
\sum_{k=0}^{n-1} (-1)^k [3k + 1] \frac{(q; q^2)_k^3}{(q; q)_k^3} \equiv [n]q^{(\frac{n-1}{2})^2} (-1)^{\frac{n-1}{2}} \pmod{[n]\Phi_n(q)^2}. \quad (1.13)
\]

It should be mentioned that Sun [31, Conjecture 5.1(ii)] discovered refinements of (1.1) and (1.3) modulo \(p^4\), which have been recently confirmed by Mao and Zhang [23] and Chen et al. [5], respectively. Moreover, Sun [31, Conjecture 5.1(i)] also proposed the following conjectural congruences:

\[
\sum_{k=0}^{n} (3k + 1) \binom{2k}{k}^3 (-8)^{n-k} \equiv 0 \pmod{4(2n + 1)\binom{2n}{n}}, \quad (1.14)
\]
\[
\sum_{k=0}^{n} (3k + 1) \binom{2k}{k}^3 16^{n-k} \equiv 0 \pmod{4(2n + 1)\binom{2n}{n}}, \quad (1.15)
\]

which have been recently proved by He [17] and Mao and Zhang [23], respectively. In this paper we shall give a \(q\)-analogue of (1.14) as follows.

**Theorem 1.6** Let \(n\) be a positive integer. Then

\[
\sum_{k=0}^{n} (-1)^k [3k + 1] \binom{2k}{k}^3 (-q; q)_n^3 \equiv 0 \pmod{(1 + q^n)^2[2n + 1]\binom{2n}{n}}, \quad (1.16)
\]
where the $q$-binomial coefficients $\left[ \frac{M}{N} \right]_q$ are defined by

\[
\left[ \frac{M}{N} \right] = \left[ \frac{M}{N} \right]_q = \begin{cases} 
\frac{(q; q)_M}{(q; q)_N(q; q)_{M-N}} & \text{if } 0 \leq N \leq M, \\
0 & \text{otherwise.}
\end{cases}
\]

We also have the following conjecture on a $q$-analogue of (1.15).

**Conjecture 1.7** Let $n$ be a positive integer. Then

\[
\sum_{k=0}^{n} [3k+1] \left[ \frac{2k}{k} \right]^3 \frac{(-q; q)_n^4}{(-q; q)^k_n} q^{-\binom{k+1}{2}} \equiv 0 \pmod{(1 + q^n)^2[2n+1] \left[ \frac{2n}{n} \right]}.
\] (1.17)

Sun and Tauraso (a special case of [33, Theorem 1.3]) proved that

\[
\sum_{k=1}^{p-1} \left( \frac{2k}{k} \right) \frac{1}{k} \equiv 0 \pmod{p^2} \text{ for } p > 3.
\] (1.18)

In this paper we shall give the following $q$-analogue of (1.18).

**Theorem 1.8** Let $n$ be a positive odd integer. Then

\[
\sum_{k=1}^{n-1} \left[ \frac{3k}{2k} \right] \left[ \frac{2k}{k} \right] q^{-\binom{k+1}{2}} \equiv \frac{(n^2-1)(1-q)^2}{24}[n] \pmod{\Phi_n(q)^2}.
\] (1.19)

It is easy to see that when $n = p > 3$ is a prime and $q = 1$, the congruence (1.19) reduces to (1.18). However, we cannot obtain any interesting congruence from (1.19) for the case $q = 1$ and $n = p^r$ with $r > 1$, since when $n = p^r$ the denominator of the reduced form of the left-hand side of (1.19) has factors $\Phi_{p^j}(q)$ for $1 \leq j \leq r - 1$.

The rest of the paper is organized as follows. In Sect. 2 we present auxiliary congruences, some of which are of interest on their own. We shall prove Theorems 1.1 and 1.3 in Sects. 3 and 4, respectively, using the $q$-WZ method. In Sect. 5 we shall prove Theorem 1.6 using the same $q$-WZ pair in Sect. 4. We give a short proof of Theorem 1.8 in Sect. 6. The final section, Sect. 7, provides some remarks on Conjecture 1.7 and proposes more conjectures for further study, including a refinement of the congruence (1.8).
2 Some auxiliary results

In this section we summarize our needs for proving the congruences of Theorems 1.1 and 1.6. Recall that Staver’s identity [29] can be written as

\[ \sum_{k=1}^{N} \left( \frac{2k}{k} \right) \frac{1}{k} = \frac{N+1}{3} \left( \frac{2N+1}{N} \right) \sum_{k=1}^{N} \frac{1}{k^2 \left( \frac{N}{k} \right)^2}. \]

It plays an important part in Guillera and Zudilin’s proof of (1.1), and has also been utilized by Sun [31], Sun and Tauraso [33], and Mao and Zhang [23] to prove certain supercongruences. The following result is a \(q\)-analogue of Staver’s identity.

**Lemma 2.1** Let \( n \) be a positive integer. Then

\[ \sum_{k=1}^{n} \left( \frac{3k}{2k} \right) \left( \frac{2k}{k} \right) q^{-\left( \frac{k}{2} \right)} = [n+1] \left( \frac{2n+1}{n} \right) \sum_{k=1}^{n} \frac{q^{-\left( \frac{n-2k+1}{2} \right)}}{[2k]^2 \left[ \frac{n}{k} \right]^2}. \] (2.1)

**Proof** We use the same technique as Paule [26], who introduced symmetry factors to simplify the proofs of terminating \(q\)-hypergeometric identities. It is easy to see that

\[ \sum_{k=1}^{n} \frac{1-q^{2k-n-1}q^{-\left( \frac{n-2k+1}{2} \right)}}{[2k]^2 \left[ \frac{n}{k} \right]^2} = 0, \]

since the \( k \)-th summand and the \( n+1-k \) summand cancel each other. Hence, the identity (2.1) is equivalent to

\[ \sum_{k=1}^{n} \left( \frac{3k}{2k} \right) \left( \frac{2k}{k} \right) q^{-\left( \frac{k}{2} \right)} = [n+1] \left( \frac{2n+1}{n} \right) \sum_{k=1}^{n} \frac{1+q^{2k-n-1}q^{-\left( \frac{n-2k+1}{2} \right)}}{[2k]^2 \left[ \frac{n}{k} \right]^2}. \] (2.2)

Let

\[ F(n, k) = \frac{[n+1]}{2} \left( \frac{2n+1}{n} \right) \frac{(1+q^{2k-n-1})q^{-\left( \frac{n-2k+1}{2} \right)}}{[2k]^2 \left[ \frac{n}{k} \right]^2}, \]

\[ G(n, k) = -\frac{[3n-2k+5]}{2} \left( \frac{2n+1}{n} \right) \frac{(1+q^{n+1})q^{-\left( \frac{n-2k+3}{2} \right)}}{[2k]^2 \left[ \frac{n+1}{k} \right]^2}. \]

Then we can check that

\[ F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k). \] (2.3)
Namely, the functions \( F(n, k) \) and \( G(n, k) \) form a \( q \)-WZ pair. Summing (2.3) over \( k \) from 1 to \( n \), we obtain

\[
\sum_{k=1}^{n+1} F(n + 1, k) - \sum_{k=1}^{n} F(n, k) = F(n + 1, n + 1) + G(n, n + 1) - G(n, 1) = \frac{[n+2][2n+3](1+q^n)q^{-\binom{n+1}{2}}}{2[2n+2]^2} - \frac{[n+3][2n+1](1+q^{n+1})q^{-\binom{n+1}{2}}}{2[2n+2]^2} + \frac{[3n+3][2n+1](1+q^{n+1})q^{-\binom{n+1}{2}}}{2[2n+2]^2}
\]

\[
= \frac{[2n+2]q^{-\binom{n+1}{2}}}{[n+1]}.
\]

It is clear that the identity (2.2) immediately follows from the above recurrence by induction on \( n \).

We now give a \( q \)-analogue of [8, (13)].

**Lemma 2.2** Let \( n \) be a positive odd integer. Then

\[
\sum_{k=1}^{\frac{n+1}{2}} \binom{3k}{2k} q^{-\binom{k}{2}} \equiv 0 \pmod{\Phi_n(q)}. \tag{2.4}
\]

**Proof** Replacing \( n \) by \( \frac{n+1}{2} \) in (2.1) and noticing that \( \binom{n}{\frac{n+1}{2}} \equiv 0 \pmod{\Phi_n(q)} \) and

\[
\gcd\left(\binom{2n+1}{2k}, \Phi_n(q)\right) = 1 \quad \text{for} \quad 1 \leq k \leq \frac{n-1}{2},
\]

we obtain (2.4).

We need two auxiliary lemmas on properties of \( q \)-factorials.

**Lemma 2.3** Let \( n \) be a positive odd integer, and let \( k \) be a non-negative integer. Then

\[
\frac{(q; q^2)_n(q^{2k+1}; q^2)_{n-1}^2}{(q; q^2)_{n-1}} \equiv 0 \pmod{\lfloor n\rfloor \Phi_n(q)^2}. \tag{2.5}
\]

Moreover, if \( k \leq \frac{n-1}{2} \), then

\[
\frac{(q; q^2)_{(n+1)/2}(q^{2k+1}; q^2)_{(n-1)/2}^2}{(q; q^2)_{(n-1)/2}^3} \equiv 0 \pmod{\lfloor n\rfloor \Phi_n(q)^2}. \tag{2.6}
\]
Proof It is well known that
\[ q^n - 1 = \prod_{d|n} \Phi_d(q). \] (2.7)

Therefore,
\[
(q; q)_{n-1} = (-1)^{n-1} \prod_{d=1}^{n-1} \Phi_d(q)^{\lfloor \frac{n-1}{d} \rfloor},
\]
\[
(q; q^2)_n = \left(\frac{q; q^2}{q^2; q^2} \right)_n = (-1)^n \prod_{d=1}^{n} \Phi_{2d-1}(q)^{\lfloor \frac{2n}{2d-1} \rfloor - \lfloor \frac{n}{2d-1} \rfloor},
\]
\[
(q^{2k+1}; q^2)_n = \left(\frac{q^{2k+1}; q^2}{q^2; q^2} \right)_n = (-1)^n \prod_{d=1}^{n+k} \Phi_{2d-1}(q)^{\lfloor \frac{2n+2k}{2d-1} \rfloor - \lfloor \frac{k}{2d-1} \rfloor - \lfloor \frac{n+1}{2d-1} \rfloor - \lfloor \frac{2k}{2d-1} \rfloor},
\]

where \( \lfloor x \rfloor \) denotes the greatest integer less than or equal to \( x \).

We now suppose that \( t|n \) and \( t > 1 \). Since \( n \) is odd, we know that \( t \) is also odd.

Hence, the exponent of \( \Phi_t(q) \) in \( \frac{(q^{2k+1}; q^2)_n}{(q; q)_{n-1}} \) is
\[
\left\lfloor \frac{2n + 2k}{t} \right\rfloor + \left\lfloor \frac{k}{t} \right\rfloor - \left\lfloor \frac{n + k}{t} \right\rfloor - \left\lfloor \frac{2k}{t} \right\rfloor - \left\lfloor \frac{n - 1}{t} \right\rfloor = \frac{n}{t} - \left\lfloor \frac{n - 1}{t} \right\rfloor = 1.
\]

Since \( [n] = \prod_{t|n, t > 1} \Phi_t(q) \), we obtain
\[
\frac{(q^{2k+1}; q^2)_n}{(q; q)_{n-1}} \equiv 0 \pmod{[n]}.
\] (2.8)

It is easy to see that
\[
\frac{(q^{2k+1}; q^2)_{n-1}}{(q; q)_{n-1}} \equiv 0 \pmod{\Phi_n(q)}.
\] (2.9)

Moreover, using the following inequality
\[
\lfloor 2x + 2y \rfloor + \lfloor y \rfloor \geq \lfloor x \rfloor + \lfloor x + y \rfloor + \lfloor 2y \rfloor
\] (2.10)
(see [3,38] for more such inequalities and related results), we can show that the denominator of the reduced form of (2.9) is a product of even-th cyclotomic polynomials, and is therefore relatively prime to \([n]\). The proof of (2.5) then follows from (2.8) with \( k = 0 \) and (2.9).

Similarly, for any positive odd integer \( t \), the exponent of \( \Phi_t(q) \) in \( \frac{(q^{2k+1}; q^2)_{(n-1)/2}}{(q; q)_{(n-1)/2}} \) is
\[
\left\lfloor \frac{n - 1 + 2k}{t} \right\rfloor + \left\lfloor \frac{k}{t} \right\rfloor - \left\lfloor \frac{n - 1 + 2k}{2t} \right\rfloor - \left\lfloor \frac{2k}{t} \right\rfloor - \left\lfloor \frac{n - 1}{2t} \right\rfloor \geq 0.
\]
by (2.10). Furthermore, if \( k \leq \frac{n-1}{2} \), then the exponent of \( \Phi_n(q) \) in \( \frac{(q^{2k+1}; q^2)_n(q; q^2)_{n-1}}{(q; q)_n(q; q^2)_{n-1/2}} \) is

\[
\left\lfloor \frac{n-1+2k}{n} \right\rfloor + \left\lfloor \frac{k}{n} \right\rfloor - \left\lfloor \frac{n-1+2k}{2n} \right\rfloor - \left\lfloor \frac{2k}{n} \right\rfloor - \left\lfloor \frac{n-1}{2n} \right\rfloor = 1.
\]

The proof of (2.6) then follows from \( (q; q^2)_{n+1/2} = (1-q)[n](q; q^2)_{n-1/2} \). \( \square \)

**Lemma 2.4** Let \( n \) be a positive odd integer. Then for \( k = 1, \ldots, \frac{n-1}{2} \) we have

\[
[n]\left[\frac{2n-2k}{n-1}\right](q; q^2)_n(q; q^2)_{n-k} \equiv 0 \pmod{[n]\Phi_n(q)^2}.
\]

**(Proof)** We can write the left-hand side of (2.11) as

\[
[n]\left[\frac{2n-2k}{n-1}\right]\left[\frac{2n}{n}\right]\left[\frac{2n-2k}{n-k}\right] \frac{1}{(-q; q)_n(-q; q^2)_{n-k}}.
\]

For \( k = 1, \ldots, \frac{n-1}{2} \), we have \( 2n - 2k > n \) and \( n - 2k + 1 < n \). Therefore, by the definition of \( q \)-binomial coefficients, we immediately get

\[
\left[\frac{2n-2k}{n-1}\right] \equiv \left[\frac{2n-2k}{n-k}\right] \equiv 0 \pmod{\Phi_n(q)}.
\]

The proof then follows from the fact that \( [n] \) is relatively prime to \( (-q; q)_n \times (-q; q^2)_{n-k} \). \( \square \)

We need the following result for the proof of Theorem 1.8.

**Lemma 2.5** Let \( n \) be a positive integer. Then

\[
\sum_{k=1}^{n-1} \frac{q^k}{[2k]^2} \equiv \frac{(n^2 - 1)(1-q)^2}{24} \pmod{\Phi_n(q)}.
\]

**(Proof)** The proof is similar to that of [28, Lemma 2]. For the sake of completeness, we provide it here. Since \( \frac{1}{[2k]} = \frac{1-q}{1-q^{2k}} \), the congruence (2.12) is equivalent to

\[
G(q) := \sum_{k=1}^{n-1} \frac{q^k}{(1-q^{2k})^2} \equiv \frac{n^2 - 1}{24} \pmod{\Phi_n(q)}.
\]

Let \( \zeta = e^{\frac{2\pi i}{n}} \) be an \( n \)-th primitive root of unity. By (1.6), to prove (2.13), it suffices to show that

\[
G(\zeta^m) = \frac{n^2 - 1}{24}
\]
for all positive integers \( m \leq n - 1 \) such that \( \gcd(m, n) = 1 \). Since \( n \) is odd, it is easy to see that
\[
G(\zeta^m) = \sum_{k=1}^{n-1} \frac{\zeta^{mk}}{(1 - \zeta^{2mk})^2} = \sum_{k=1}^{n-1} \frac{\zeta^{k}}{(1 - \zeta^{2k})^2} = G(\zeta),
\]
provided that \( \gcd(m, n) = 1 \).

We now define
\[
G(q, z) := \sum_{k=1}^{n-1} \frac{q^k}{(1 - q^{2k}z)^2}.
\]
As
\[
\sum_{k=1}^{n-1} \zeta^{jk} = \begin{cases} n - 1, & \text{if } n \mid j, \\ -1, & \text{if } n \nmid j, \end{cases}
\]
for any complex number \( z \) with \( |z| < 1 \), we have
\[
G(\zeta, z) = \sum_{k=1}^{n-1} \zeta^k \sum_{j=0}^{\infty} \zeta^{2jk}(j + 1)z^j
= \sum_{j=1}^{\infty} j\zeta^{j-1} \sum_{k=1}^{n-1} \zeta^{(2j-1)k} = n \sum_{j=1}^{\infty} \left( jn - \frac{n-1}{2} \right) z^{jn-\frac{n+1}{2}} - \sum_{j=1}^{\infty} jz^{j-1}
= \frac{n^2 z^{\frac{n-1}{2}}}{(1 - z^n)^2} - \frac{n(n-1)z^{\frac{n-1}{2}}}{2(1 - z^n)} - \frac{1}{(1-z)^2}.
\]
(2.14)

Letting \( z \to 1 \) in (2.14) and using L’Hôpital’s rule, we get
\[
G(\zeta) = G(\zeta, 1)
= \lim_{z \to 1} \frac{2n^2 z^{\frac{n-1}{2}}(1-z)^2 - n(n-1)z^{\frac{n-1}{2}}(1-z)^2 - 2(1-z^n)^2 - 1}{2(1-z^n)^2(1-z)^2}
= \frac{n^2 - 1}{24},
\]
as desired.  

\[
3 \text{ Proof of Theorem 1.1}
\]

\textbf{Proof of (1.7).} Define the following two functions:
\[
F(n, k) = [3n + 2k + 1] \frac{(q; q^2)_n (q^{2k+1}; q^n)_n (q^{(n+1)} - (2n+1)k)}{(q; q^2)_n (q^2; q^2)_n}.
\]
\[ G(n, k) = -\frac{(1 + q^{n+2k-1})(q^2; q^2)_n(q^{2k+1}; q^2)_n^{-(-2n-1)k}}{(1 - q)(q^2; q^2)_n^{-(-2n-1)k}}, \]

where we use the convention that \(1/(q^2; q^2)_a = 0\) for any negative integer \(a\). Let \(m\) be a positive odd integer. Then

\[
\sum_{n=0}^{m-1} [3k + 1] \frac{(q^2; q^2)_k^{-(-k+1)}}{(q; q)_k^2(q^2; q^2)_k} F(n, 0) = \sum_{n=0}^{m-1} F(n, 0).
\]

(3.1)

It is easy to check that

\[
F(n, k - 1) - F(n, k) = G(n + 1, k) - G(n, k).
\]

(3.2)

Namely, the functions \(F(n, k)\) and \(G(n, k)\) form a \(q\)-WZ pair. (Usually the \(q\)-hypergeometric functions satisfying (2.3) are called a \(q\)-WZ pair. But it is convenient and reasonable to call the \(q\)-hypergeometric functions satisfying (3.2) a \(q\)-WZ pair too, just as Zudilin [40]). Summing (3.2) over \(n = 0, 1, \ldots, \frac{m-1}{2}\), we obtain

\[
\sum_{n=0}^{m-1} F(n, k - 1) - \sum_{n=0}^{m-1} F(n, k) = G\left(\frac{m + 1}{2}, k\right) - G(0, k)
\]

\[= G\left(\frac{m + 1}{2}, k\right).\]

(3.3)

By (2.6), for \(k = 1, 2, \ldots, \frac{m-1}{2}\), we have

\[
G\left(\frac{m + 1}{2}, k\right) = -\frac{(1 + q^{m-1/2 + 2k})(q^2; q^2)_{(m+1)/2}(q^{2k+1}; q^2)_{(m-1)/2}^2 - q^{m-1/2 - pk}}{(1 - q)(q^2; q^2)_{(m-1)/2}^{-1}(q^2; q)_{(m-1)/2}^{-1}}
\]

\[\equiv 0 \pmod{m\Phi_m(q)^2}.
\]

(3.4)

because \((-q; q)_{(m-1)/2}\) is relatively prime to \([m]\). Substituting (3.4) into (3.3), we see that

\[
\sum_{n=0}^{m-1/2} F(n, 0) \equiv \sum_{n=0}^{m-1/2} F(n, 1) \equiv \sum_{n=0}^{m-1/2} F(n, 2)
\]

\[\equiv \cdots \equiv \sum_{n=0}^{m-1/2} F\left(n, \frac{m-1}{2}\right) \pmod{m\Phi_m(q)^2}.
\]
Hence, modulo \([m] \Phi_m(q)^2\), the sum (3.1) can be replaced by

\[
\sum_{n=0}^{m-1} F \left( n, \frac{m - 1}{2} \right) = \sum_{n=0}^{m-1} \left[ 3n + m \right] \frac{(q; q^2)_n (q^m; q^2)_n q^{-(n+1)/2} - (2n+1)(m-1)}{(q; q)_n^2 (q^2; q^2)_n}
\]

\[
= [m] q^{1-m/2} + \sum_{n=1}^{m-1} \frac{(q; q^2)_n (q^m; q^2)_n q^{-(n+1)/2} - (2n+1)(m-1)}{(q; q)_n^2 (q^2; q^2)_n}
\]

\[
+ q^m \sum_{n=1}^{m-1} \frac{[3n](q; q^2)_n (q^m; q^2)_n q^{-(n+1)/2} - (2n+1)(m-1)}{(q; q)_n^2 (q^2; q^2)_n}. \tag{3.5}
\]

Similar to the proof of Lemma 2.3, we can show that the denominator of the reduced form of the fraction

\[
\frac{(q; q^2)_n (q^m; q^2)_n^2}{(q; q)_n^2 (q^2; q^2)_n}
\]

is relatively prime to \([m]\) (since \(m\) is odd). It is clear that the numerator of this reduced form is divisible by \(\Phi_m(q)^2\) for \(1 \leq n \leq m - 1\). Thus, comparing the expression (3.5) with (3.1), we see that (1.7) with \(n \mapsto m\) is equivalent to

\[
q^m \sum_{n=1}^{m-1} \frac{[3n](q; q^2)_n (q^m; q^2)_n q^{-(n+1)/2} - (2n+1)(m-1)}{(q; q)_n^2 (q^2; q^2)_n} \equiv 0 \pmod{[m] \Phi_m(q)^2}. \tag{3.6}
\]

We can further show that

\[
\frac{[3n](q; q^2)_n (q^m; q^2)_n^2}{(q; q)_n^2 (q^2; q^2)_n} \equiv 0 \pmod{[m]} \quad \text{for} \quad n \leq \frac{m - 1}{2}.
\]

Thus, writing the left-hand side of (3.6) as

\[
[m]^2 \sum_{n=1}^{m-1} \frac{(1 - q)_n^2 (q; q^2)_n (q^m; q^2)_n q^{-(n+1)/2} - (2n+1)(m-1)}{(q; q)_n^2 (q^2; q^2)_n},
\]

and noticing that \([m] \equiv 0 \pmod{\Phi_m(q)}\), we see that (3.6) is equivalent to

\[
\sum_{n=1}^{m-1} \frac{[3n](1 - q)_n^2 (q; q^2)_n (q^m; q^2)_n q^{-(n+1)/2} - (2n+1)(m-1)}{(q; q)_n^2 (q^2; q^2)_n} \equiv 0 \pmod{\Phi_m(q)}. \tag{3.7}
\]
Since \( q^m \equiv 1 \pmod{\Phi_m(q)} \), we can reduce (3.7) to its equivalent form
\[
\sum_{n=1}^{m-1} (1 + q^n + q^{2n}) \frac{(1 - q)(q^2)_n(q^2_n q^{-\frac{n}{2}} - m-1)}{(1 + q^n)(q_2^n)}
\]
\[
= \sum_{n=1}^{m-1} 1 + q^n + q^{2n} \left[ 2n \right] q^{-\frac{n}{2} - \frac{m-1}{2}} \equiv 0 \pmod{\Phi_m(q)},
\]
which is (2.4) with \( n \mapsto m \) differing only by a factor \( q^{-\frac{m-1}{2}} \). \( \square \)

**Proof of (1.8).** Let \( m \) be a positive odd integer again. Then
\[
\sum_{k=0}^{m-1} [3k + 1] \frac{(q; q^2)_k q^{-\frac{k+1}{2}}}{(q; q^2)_k} = \sum_{n=0}^{m-1} F(n, 0). \tag{3.8}
\]

Similarly as before, summing (3.2) over \( n = 0, 1, \ldots, m - 1 \), we obtain
\[
\sum_{n=0}^{m-1} F(n, k - 1) - \sum_{n=0}^{m-1} F(n, k) = G(m, k). \tag{3.9}
\]

By (2.5), for \( k = 1, 2, \ldots, m - 1 \), we have
\[
G(m, k) = \frac{(1 + q^{m+2k-1})(q; q^2)_m(q^{2k+1}, q^2)^2_{m-1} q^{-\frac{m}{2} - (2m-1)k}}{(1 - q)(q; q^2)_{m-1} (-q; q)_{m-1}}
\]
\[
\equiv 0(\text{mod}[m] \Phi_m(q)^2),
\]
since \( \gcd((-q; q)_{m-1}, [m]) = 1 \). Thus, we get
\[
\sum_{n=0}^{m-1} F(n, 0) \equiv \sum_{n=0}^{m-1} F(n, 1) \equiv \cdots \equiv \sum_{n=0}^{m-1} F \left( n, \frac{m-1}{2} \right) (\text{mod } [m] \Phi_m(q)^2).
\]

Therefore, modulo \([m] \Phi_m(q)^2\), the sum (3.8) can be replaced by
\[
\sum_{n=0}^{m-1} F \left( n, \frac{m-1}{2} \right) = \sum_{n=0}^{m-1} [3n + m] \frac{(q; q^2)_n(q^m; q^2)^2_n q^{-\frac{n+1}{2}} - (2n+1)(m-1)}{(q; q^2)_n^2(q^2; q^2)_n}.
\]

Similar to the proof of (1.7), we see that (1.8) with \( n \mapsto m \) is equivalent to
\[
\sum_{n=1}^{m-1} \frac{(1 - q)^2(q; q^2)_n(q^{m+2}; q^2)^2_{n-1} q^{-\frac{n+1}{2}} - (2n+1)(m-1)}{(q; q^2)_n^2(q^2; q^2)_n} \equiv 0 \pmod{\Phi_m(q)},
\]
which immediately follows again from (2.4) with \( n \mapsto m \) (note that in (2.4) we may sum up to \( n - 1 \)).

**Remark** The functions \( F(n, k) \) and \( G(n, k) \) are not easy to find (they always require a preliminary human guess). We note that only one of the functions needs to be guessed and the WZ method produces the other one algorithmically. Since Guillera and Zudilin [8] have given the corresponding WZ pair (the limiting case \( q \to 1 \)), we can find the \( q \)-WZ pair not so difficultly.

### 4 Proof of Theorem 1.3

Let

\[
F(n, k) = (-1)^n [3n - 2k + 1] \frac{2n - 2k}{n} (q; q^2)_n (q; q^2)_{n-k},
\]

\[
G(n, k) = (-1)^{n+1} [n] \frac{2n - 2k}{n-1} (q; q^2)_n (q; q^2)_{n-k} q^{n+1-2k} (q; q^2)_n (q; q^2)_{n-k}.
\]

These functions \( F(n, k) \) and \( G(n, k) \) are \( q \)-analogues of the WZ pair given by He in the proof of [17, Theorem 1.1]. It is easy to check that \( F(n, k) \) and \( G(n, k) \) satisfy

\[
F(n, k - 1) - F(n, k) = G(n + 1, k) - G(n, k). \tag{4.1}
\]

Let \( m \) be a positive odd integer. Since

\[
\binom{2k}{k} = \frac{(q; q^2)_k (-q; q^2)_k}{(q^2; q^2)_k},
\]

we have

\[
\sum_{k=0}^{m-1} (-1)^k [3k + 1] \frac{(q; q^2)^3_k}{(q; q)^3_k} = \sum_{n=0}^{m-1} F(n, 0). \tag{4.2}
\]

Summing (4.1) over \( n = 0, 1, \ldots, m - 1 \), we obtain

\[
\sum_{n=0}^{m-1} F(n, k - 1) - \sum_{n=0}^{m-1} F(n, k) = G(m, k). \tag{4.3}
\]

By Lemma 2.4, for \( k = 1, \ldots, \frac{m-1}{2} \), we have \( G(m, k) \equiv 0 \pmod{[m] \Phi_m(q)^2} \). Therefore, from (4.3) we deduce that

\[
\sum_{n=0}^{m-1} F(n, 0) \equiv \sum_{n=0}^{m-1} F(n, 1) \equiv \cdots \equiv \sum_{n=0}^{m-1} F \left( n, \frac{m-1}{2} \right) \pmod{[m] \Phi_m(q)^2}. \tag{4.4}
\]
Furthermore, since $F(n, k) = 0$ for $n < 2k$, we get
\[
\sum_{n=0}^{m-1} F\left(n, \frac{m-1}{2}\right) = F\left(m-1, \frac{m-1}{2}\right) = [2m-1] \frac{(q; q^2)_{m-1} (q^2; q^2)_{(m-1)/2}}{(q; q)_{m-1} (q^2; q^2)_{(m-1)/2}} = \left[\frac{2m-1}{m-1}\right] \left[\frac{m-1}{2}\right] q^2 (-q; q^2)_{m-1}.
\]
(4.5)

Substituting the following two congruences (see [10, (3.1)] and [20, (1.5)])
\[
\left[\frac{2m-1}{m-1}\right] \equiv (-1)^{m-1} \left(\frac{m}{2}\right) \pmod{\Phi_m(q^2)},
\]
\[
\left[\frac{m-1}{2}\right] q^2 \equiv (-1)^{m-1} q^{\frac{1-m^2}{4}} (-q; q^2)_{m-1} \pmod{\Phi_m(q^2)}
\]
into (4.5), and noticing (4.2) and (4.4), we complete the proof of the theorem with $n \mapsto m$.

**Remark** The $q = 1$ case of our proof gives a new proof of (1.3), which is simpler than Guillera and Zudilin’s original proof. Moreover, summing (4.1) over $n = 0, 1, \ldots, \frac{m-1}{2}$, we can only show that
\[
\sum_{k=0}^{\frac{m-1}{2}} (-1)^k \left[3k + 1\right] \frac{(q; q^2)^3}{(q; q^3)_k} \equiv 0 \pmod{\Phi_m(q^2)}.
\]
Therefore, to confirm Conjecture 1.5, we believe that we need new methods or techniques, though Theorem 1.3 and Conjecture 1.5 are equivalent when $n$ is an odd prime power.

### 5 Proof of Theorem 1.6

Summing (4.1) over $n$ from 0 to $N$, we obtain
\[
\sum_{n=0}^{N} F(n, k-1) - \sum_{n=0}^{N} F(n, k) = G(N+1, k).
\]
(5.1)

Furthermore, summing (5.1) over $k$ from 1 to $N$, we get
\[
\sum_{n=0}^{N} F(n, 0) - \sum_{n=0}^{N} F(n, N) = \sum_{k=1}^{N} G(N+1, k).
\]
(5.2)
Since \( F(n, N) = 0 \) for \( n = 0, 1, \ldots, N \), it follows from (5.2) that

\[
(-q; q)_N^3 \sum_{n=0}^{N} F(n, 0) = (-q; q)_N^3 \sum_{k=1}^{N} G(N + 1, k)
\]

\[
= (-1)^N \sum_{k=1}^{N} [N + 1] \left[ \frac{2N + 2}{N + 1} \right] \left[ \frac{2N - 2k + 2}{N} \right] \left[ \frac{2N - 2k + 2}{N - k + 1} \right] q^{-\frac{2k+2}{N-k+1}}
\]

\times \frac{(-q; q)_N^2 q^{N-2k+2}}{(1 + q^{N+1})(-q; q)_N^2}.
\]

(5.3)

Since \( [N + 1] \left[ \frac{2N+2}{N+1} \right] / (1 + q^{N+1}) = [2N + 1] \left[ \frac{2N}{N} \right] \), it is easy to see that each summand on the right-hand side of (5.3) is divisible by \((1 + q^N) [2N + 1] \left[ \frac{2N}{N} \right] \) (whenever \( k = 1 \) or \( k \geq 2 \)). This proves that the congruence (1.16) holds for \( n = N \).

\[ \square \]

6 Proof of Theorem 1.8

Replacing \( n \) by \( n - 1 \) in (2.1), we obtain

\[
\sum_{k=1}^{n-1} \left[ \frac{3k}{2k^2} \right] k q^{-\frac{k^2}{2k^2}} = [n] \left[ \frac{2n - 1}{n - 1} \right] \sum_{k=1}^{n-1} \frac{q^{-\frac{2k+2}{n-k+1}}}{[2k]^2 \left[ \frac{n-1}{k} \right] q^2}.
\]

(6.1)

It is easy to see that

\[
\left[ \frac{2n - 1}{n - 1} \right] \equiv 1 \pmod{\Phi_n(q)}
\]

and

\[
\left[ \frac{n - 1}{k} \right] \equiv (-1)^k q^{-k^2-k} \pmod{\Phi_n(q)}.
\]

Since \([n] \equiv 0 \pmod{\Phi_n(q)}\) and \(q^n \equiv 1 \pmod{\Phi_n(q)}\), we deduce from (6.1) that

\[
\sum_{k=1}^{n-1} \left[ \frac{3k}{2k^2} \right] k q^{-\frac{k^2}{2k^2}} \equiv [n] \sum_{k=1}^{n-1} \frac{q^k}{[2k]^2} \pmod{\Phi_n(q)^2}.
\]

The proof then follows from (2.12).
7 Concluding remarks and open problems

Numerical calculation suggests the following refinement of (1.8).

**Conjecture 7.1** Let $n$ be a positive odd integer. Then

$$\sum_{k=0}^{n-1} [3k + 1] \frac{(q; q^2)_k^3 q^{-(k+1)}_k}{(q; q^2)_k(q^2; q^2)_k} \equiv [n] q^{\frac{1-n}{2}} + \frac{(n^2 - 1)(1-q)^2}{24} [n] q^{\frac{1-n}{2}} \pmod{[n]\Phi_n(q)^3}.$$  

Note that if $n = p^r$ and $q = 1$, then the above congruence reduces to

$$\sum_{k=0}^{p^r-1} \frac{(\frac{1}{2})_k^3}{k!^3} (3k + 1) 2^{2k} \equiv p^r \pmod{p^{r+3}} \quad \text{for } p > 3. \quad (7.1)$$

A stronger version of (7.1) modulo $p^{r+4}$ was conjectured by Sun [31, Conjecture 5.1(ii)]. Moreover, the $r = 1$ case of (7.1) has been proved by Dian-Wang Hu [18].

Motivated by (1.5), we give the following $q$-analogue of a supercongruence in [31, Conjecture 5.9] (with $a = 1$).

**Conjecture 7.2** Let $n$ be a positive integer with $n \equiv 1 \pmod{4}$. Then

$$\sum_{k=0}^{n-1} [4k + 1] \frac{(q; q^2)_k^3 q^{\frac{2k+1}{k}}}{(q^2; q^2)_k^3} \equiv 0 \pmod{\Phi_n(q)^2}.$$  

Guillera and Zudilin [8] proved (1.3) by using the following WZ pair:

$$F(n, k) = (-1)^n (3n + 2k + 1) \frac{(\frac{1}{2})_n (\frac{1}{2} + k)_n (\frac{1}{2})_k 2^{3n}}{(1)_2^2 (1 + 2k)_n (1)_k},$$

$$G(n, k) = (-1)^n \frac{(\frac{1}{2})_n (\frac{1}{2} + k)_n (\frac{1}{2})_k 2^{3n-2}}{(1)_2^2 (1 + 2k)_n (1)_k}. $$

We also found a $q$-analogue of the above WZ pair as follows:

$$F(n, k) = (-1)^n [3n + 2k + 1] \frac{(q; q^2)_n (q^{2k+1}; q^2)_n (q^2; q^2)_k}{(q^2; q^2)_n (q^{2k+1}; q^2)_n (q^2; q^2)_k},$$

$$G(n, k) = (-1)^n q^{\frac{n+2k-1}{2}} \frac{(q; q^2)_n (q^{2k+1}; q^2)_n (q^2; q^2)_n}{(1-q)(q; q^2)_n (q^{2k+1}; q^2)_n (q^2; q^2)_k}.$$  

Unfortunately, we cannot find and prove the corresponding $q$-analogue of [8, Lemma 4], and so we are unable to use this $q$-WZ pair to prove Theorem 1.3. On the other
hand, from Theorem 1.3 and this \(q\)-WZ pair we can deduce that, for any positive odd integer \(n\), there holds
\[
\sum_{k=1}^{n-1} (-1)^k \frac{(n+3k)\left(-q^{n+1}; q\right)_{k-1}}{[2k]} \binom{2k}{k} \equiv 2 \left(q \binom{n}{2} - \left(-q; q\right)_{n-1}\right) \left(\Phi_n(q)\right)^2, \\
\]
which seems difficult to prove directly. We plan to give details of the proof of this congruence in another paper [11].

Mao and Zhang [23] proved (1.14) by applying the \(x = -\frac{1}{2}\) case of the following identity (see [22, Lemma 3.2]):
\[
\sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{x+k}{2n+1}\right)^2 = \frac{1}{(4n+2)(2n)} \sum_{k=0}^{n} (2x - 3k) \binom{x}{k}^2 \binom{2k}{k}. \\
\tag{7.2}
\]
Therefore, a possible way to prove Conjecture 1.7 is to give a \(q\)-analogue of the curious binomial coefficient identity (7.2).

Swisher [34] has proposed many interesting conjectures on supercongruences that generalize Van Hamme’s original 13 supercongruences [37]. For example, she conjectured that [34, (J.3)]
\[
\sum_{k=0}^{p - 1} \frac{(6k + 1) \left(\frac{1}{2}\right)_k^3}{k!^3 4^k} \equiv (-1)^{\frac{p-1}{2}} p \sum_{k=0}^{p - 1} \frac{(6k + 1) \left(\frac{1}{2}\right)_k^3}{k!^3 4^k} \pmod{p^{4r}} \text{ for } p > 3.
\]

Motivated by Swisher’s conjectures and also the conjectures of Sun [32], we propose the following refinements of (1.9), (1.10), (1.12), and (7.1) (there is also a similar conjecture in [25]).

**Conjecture 7.3** Let \(p\) be an odd prime and \(r\) a positive integer. Then
\[
\sum_{k=0}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} (3k + 1) 2^{2k} \equiv p \sum_{k=0}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} (3k + 1) 2^{2k} \pmod{p^{3r}}, \\
\sum_{k=0}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} (3k + 1) 2^{2k} \equiv p \sum_{k=0}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} (3k + 1) 2^{2k} \pmod{p^{4r-\delta_{p,3}}},
\]
\[
\sum_{k=0}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} (3k + 1) (-1)^k 2^{3k} \equiv p(-1)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} (3k + 1) 2^{2k} \pmod{p^{3r+\delta_{p,3}}}, \\
\sum_{k=0}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} (3k + 1) (-1)^k 2^{3k} \equiv p(-1)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} (3k + 1) 2^{2k} \pmod{p^{3r}},
\]
where \(\delta\) is the Kronecker delta with \(\delta_{i,j} = 1\) if \(i = j\) and \(\delta_{i,j} = 0\) otherwise.
Finally, it would be interesting to find a $q$-analogue of (1.2).

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