Weak Gravitational lensing by phantom black holes and phantom wormholes using the Gauss-Bonnet theorem

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In this paper, we study the deflection of light by a class of phantom black hole and wormhole solutions in the weak limit approximation. More specifically, in the first part of this work we study the deflection of light by Garfinkle-Horowitz-Ströminger black hole and Einstein-Maxwell anti-dilaton black hole using the optical geometry and the Gauss-Bonnet theorem. Our calculation show that gravitational lensing is affected by the phantom scalar field (phantom dilaton). In the second part of this work, we explore the deflection of light by a class of asymptotically flat phantom wormholes. In particular we have used three types of wormholes: wormhole with a bounded/unbounded mass function, and a wormhole with a vanishing redshift function. We show that the particular chose of the shape function and mass function plays a crucial role in the final expression for the deflection angle of light. Finally, in the third part of this paper we verify our findings with the help of standard geodesics equations.

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Contents

I. Introduction

II. Phantom Black Holes

A. Garfinkle-Horowitz-Ströminger dilaton black hole

B. Einstein Maxwell anti-dilaton black hole

III. Gauss-Bonnet Theorem and Weak Gravitational Lensing by GHS Black Hole

IV. Gauss-Bonnet Theorem and Weak Gravitational Lensing by EMaD Black Hole

V. Light deflection by Phantom Wormholes

A. Wormholes with a bounded mass function

B. Wormholes with an unbounded mass function

C. Wormholes with vanishing redshift function

VI. Geodesics

A. Deflection angle by Garfinkle-Horowitz-Ströminger black hole

B. Deflection angle by Einstein-Maxwell anti-dilaton black hole

C. Deflection angle by phantom wormholes

VII. Conclusions

Acknowledgments

References

I. INTRODUCTION

Presently, the general relativity is not the only viable theory of gravitation. One of the most promising avenue realizing the unified theories is the string theory, which in the low-energy limit reduces to the Einstein-Maxwell dilaton gravity. Moreover, the most important questions nowadays is devoted to the existence of the dark energy (DA). The natural questions arises whether local manifestations of DE at astrophysical scale can be observed, as considerable efforts are made to study the nature of DE. Different effective models of dark energy have been proposed in literature [1, 2]. In some of them the possibility of describing DE by phantom fields is considered.

The gravitational lensing can be used for a confirmation of the generalized gravity theories. In most of the cases of interest connected with phenomena realizing on large scales, it is possible to assume that the gravitational field is weak, hence the angle of deflection of the light rays caused by a spherically symmetric body...
with mass $M$ can be approximated by: $\hat{\alpha} \approx 4GM/bc^2$, where $b$ is the impact parameter. But the question of the influence of the phantom black hole charge on the light deflection angle in the weak field regime still remains unclear.

As we have already mentioned, the weak gravitational lensing could provide examination in the asymptotically flat space-time region of different kinds black holes. Therefore, following the method of Gibbons and Werner (GW) in the present work we wish to study gravitational lensing in the weak field limit due to a static, spherically symmetric, charge, dilaton Garfinkle-Horowitz-Strominger (GHS) black hole [3] in the heterotic string theory with the aim of investigating the influence of the dilaton field in the behaviour of the light bending angle in the weak deflection limit regime by employing the Gauss-Bonnet theorem (GBT) to the optical geometry. Nowadays, GBT has been applied in numerous studies [4–27].

Moreover, one of the aims of the current paper is to study the effect of phantom scalar field (phantom dilaton) and of the phantom black hole charge on gravitational lensing and in particular to the angle of deflection of the light rays. Calculating the light deflection angle caused by black holes in the weak deflection limit we study the possible manifestation of dark energy at a distances much larger than the black hole gravitational radius. For this purpose we model DE with phantom dilaton and compare the calculated light deflection angle in the geometries of the standard Einstein-Maxwell black hole and Einstein-Maxwell-anti-dilaton (EMaD) black hole which has a phantom dilaton applying the GBT to the optical geometry.

It has been recently shown that, by using the optical geometry, we can calculate the Gaussian optical curvature $\mathcal{K}$ to find the asymptotic bending angle which, in the case of asymptotically flat spacetimes, can be calculated as follows:

$$\hat{\alpha} = -\int \int_{D_\infty} \mathcal{K} dS,$$

which gives exact results for bending angle. It is quite remarkable that one can compute the deflection angle by integrating over a domain outside the impact parameter.

In this paper our main aim is to obtain the deflection angle by phantom black holes [28, 29], and phantom wormholes [30] in the weak gravitational field approximation using the GBT.

This paper is organized as follows. In Section II we briefly review the phantom black holes such as GHS Black Hole and EMaD Black Hole. In section III, we calculate the deflection angle by GHS Black Hole using the GBT. In section IV, we calculate the weak gravitational lensing for the EMaD Black Hole. In Section V, we shall calculate the deflection angle by three particular wormhole solutions. In Section VI we verify our findings with the help of geodesics approach. Finally, in Section VII, we comment on our results.

\section{PHANTOM BLACK HOLES}

\subsection{Garfinkle-Horowitz-Ströminger dilaton black hole}

The line element of the static, spherically-symmetric Garfinkle-Horowitz-Ströminger black hole solution of the Einstein Maxwell scalar field equations obtained in low energy string theory is given [3, 28] by

$$ds^2 = -F(r)dt^2 + F(r)^{-1}dr^2 + H(r)(d\theta^2 + \sin^2 \theta d\phi^2).$$

with functions $F(r) = (1 - \frac{r}{r_c}) (1 - \frac{r}{r_+})$ and $H(r) = r^2 (1 - \frac{r}{r_+})^{1-\gamma}$, where the parameter $\gamma = (1 - \alpha^2)/(1 + \alpha^2)$ has been introduced for convenience. It varies in the interval $[0, 1]$ for $\alpha \in (-\infty, \infty)$, so the stronger coupling corresponds to lower values of $\gamma$. The corresponding solutions for the dilaton and the Maxwell field are

$$e^{2\alpha \varphi} = (1 - \frac{r}{r_-})^{1-\gamma}, \quad F = \frac{Q}{r_-} dt \wedge dr.$$  \hspace{1cm} (3)

It is noted that the ADM mass $M$ and the charge $Q$ in terms of the inner Coughy horizon $r_-$ and the event horizon $r_+$ are

$$2M = r_+ + \gamma r_-, \quad 2Q^2 = (1 + \gamma)r_+ r_-,$$  \hspace{1cm} (4)

whence the black hole horizons could be expressed respectively as follows

$$r_+ = M \left[1 + \sqrt{1 - \frac{2\gamma}{1 + \gamma} \left(\frac{Q}{M}\right)^2}\right],$$  \hspace{1cm} (5)

$$r_- = \frac{M}{\gamma} \left[1 - \sqrt{1 - \frac{2\gamma}{1 + \gamma} \left(\frac{Q}{M}\right)^2}\right].$$  \hspace{1cm} (6)

The metric functions $F(r)$ and $H(r)$ in term of the ADM mass $M$ and the charge $Q$ than respectively are

$$F(r) = \left[1 - \frac{M}{r} \left(1 + \sqrt{1 - \frac{2\gamma}{1 + \gamma} \left(\frac{Q^2}{M^2}\right)}\right)\right] \times \left[1 - \frac{M}{r\gamma} \left(1 - \sqrt{1 - \frac{2\gamma}{1 + \gamma} \left(\frac{Q^2}{M^2}\right)}\right)\right]^\gamma,$$  \hspace{1cm} (7)

$$H(r) = r^2 \left[1 - \frac{M}{r\gamma} \left(1 - \sqrt{1 - \frac{2\gamma}{1 + \gamma} \left(\frac{Q^2}{M^2}\right)}\right)\right]^{1-\gamma}. $$  \hspace{1cm} (8)

The two black hole horizons $r_-$ and $r_+$ merge at

$$\left(\frac{Q}{M}\right)^2 = \left(\frac{Q}{M}\right)_{\text{crit}}^2 = \frac{2}{1 + \gamma}. $$  \hspace{1cm} (9)
and the solution reduces to the Janis-Newman-Winicour solution, describing a naked singularity. In this case, at \( r_− = 2M/\gamma \) a singularity is reached and \( \gamma \in [0,1] \). In the particular case \( \gamma = 0 \), the solution coincides with static, spherically symmetric Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GMGHS) charged solution investigated in weak deflection limit regime from Keeton and Petters in [40]. In the limit case \( \gamma \rightarrow 1 \) the solution restores the Reissner-Nordstrom black hole. In the particular case of absence of electrical charge \( Q = 0 \), the Schwarzschild black hole is recovered with an event horizon \( r_+ = 2M \). In this work we will restrict our considerations to weak gravitational lensing of black holes.

### B. Einstein Maxwell anti-dilaton black hole

The line element of the Einstein Maxwell anti-dilaton black hole is given by [28]

\[
ds^2 = -F(r)dt^2 + F(r)^{-1}dr^2 + H(r)(d\theta^2 + \sin^2 \theta d\phi^2),
\]

with functions \( F(r) = \left(1 - \frac{r}{r_+}\right)\left(1 - \frac{r}{r_-}\right)^{1/\gamma} \) and \( H(r) = r^2 \left(1 - \frac{r}{r_-}\right)^{1-1/\gamma} \). The solutions for the dilaton and the Maxwell field are

\[
e^{2\alpha \varphi} = \left(1 - \frac{r}{r_+}\right)^{1-1/\gamma}, \quad F = \frac{Q}{r^2} dt \wedge dr.
\]

The ADM mass \( M \) and the charge \( Q \) can be expressed by \( r_+ \) and \( r_- \) in the following way

\[
2M = r_+ + \frac{1}{\gamma} r_- , \quad 2Q^2 = \frac{(1 + \gamma)}{\gamma} r_+ r_- ,
\]

whence the black hole horizons represented in terms of the ADM mass \( M \) and the charge \( Q \) are

\[
2M = r_+ + \gamma r_- , \quad 2Q^2 = (1 + \gamma) r_+ r_- ,
\]

whence the black hole horizons could be expressed respectively as follows

\[
r_+ = M \left[1 + \sqrt{1 - \frac{2}{1 + \gamma} \left( \frac{Q}{M} \right)^2} \right], \quad (14)
\]

\[
r_- = \gamma M \left[1 - \sqrt{1 - \frac{2}{1 + \gamma} \left( \frac{Q}{M} \right)^2} \right]. \quad (15)
\]

One can show the metric function \( F(r) \) and \( H(r) \) in terms of the ADM mass \( M \) and the charge \( Q \) respectively are

\[
F(r) = \left[1 - \frac{M}{r} \left(1 + \sqrt{1 - \frac{2}{1 + \gamma} \left( \frac{Q^2}{(1 + \gamma)M^2}\right)}\right)\right]
\]

\[
\times \left[1 - \frac{M\gamma}{r} \left(1 - \sqrt{1 - \frac{2}{1 + \gamma} \left( \frac{Q^2}{(1 + \gamma)M^2}\right)}\right)^{1/\gamma}\right]. \quad (16)
\]

\[
H(r) = r^2 \left[1 - \frac{M\gamma}{r} \left(1 - \sqrt{1 - \frac{2}{1 + \gamma} \left( \frac{Q^2\gamma}{(1 + \gamma)M^2}\right)}\right)^{1-1/\gamma}\right]. \quad (17)
\]

The condition for both horizons to be real is

\[
\left( \frac{Q}{M} \right)^2 \leq \left( \frac{Q}{M} \right)_{\text{crit}}^2 = \frac{1 + \gamma}{2}. \quad (18)
\]

In the limit case \( \gamma \rightarrow 1 \) the Reissner-Nordström black hole is restored. For \( r_- = 0 \), when the black hole charge \( Q \) vanish or \( \gamma \rightarrow 0 \) the Schwarzschild black hole is restored.

### III. GAUSS-BONNET THEOREM AND WEAK GRAVITATIONAL LENSING BY GHS BLACK HOLE

In this section we study the weak gravitational lensing using the GHS black hole with the GBT. First we use the null geodesic \( ds^2 = 0 \) where with the deflection angle of light in the equatorial plane \( \theta = \pi/2 \), we obtain the optical metric of GHS BH as follows:

\[
dt^2 = \frac{1}{F(r)^2} dr^2 + H(r) \, d\varphi^2. \quad (19)
\]

Afterwards, we make the transformation to tortoise coordinates with \( r^* \), and find the \( f(r^*) \)

\[
dr^* = \frac{1}{F(r)} dr , \quad f^2(r^*) = \frac{H(r)}{F(r)}. \quad (20)
\]

The metric of GMGHS reduces to

\[
dr^2 = \tilde{g}_{ab} dx^a dx^b = dr^2 + f^2(r^*) d\varphi^2. \quad (a, b = r, \varphi), \quad (21)
\]

Then we obtain the Gaussian optical curvature \( K \):

\[
K = -\frac{1}{f(r^*)} \frac{d^2 f(r^*)}{dr^{*2}} = -\frac{1}{f(r^*)} \left[ \frac{d}{dr^*} \left( \frac{d}{dr^*} \right) \frac{df}{dr^*} + \left( \frac{df}{dr^*} \right)^2 \frac{d^2 f}{dr^{*2}} \right]. \quad (22)
\]

Since we are interested in the weak limit, we can approximate the optical Gaussian curvature as

\[
K \approx -\frac{\gamma r_-}{r^3} - \frac{r_+}{r^3} + \frac{3r_+^2}{4r^4} + \frac{9\gamma r_- r_+}{2r^4} + \frac{r_+^2 (8\gamma^2 - 6\gamma + 1)}{4r^4}. \quad (23)
\]

Now, we calculate the deflection angle using the Gaussian optical curvature. For this purpose, we select a non-singular region \( \mathcal{D}_R \) with boundary \( \partial \mathcal{D}_R = \gamma \bar{\beta} \cup C_R \).
Note that this region allows the GBT to be stated as follows:
\[
\int_{\mathcal{D}_R} \mathcal{K} \, dS + \int_{\partial \mathcal{D}_R} \kappa \, dt + \sum_i \theta_i = 2\pi \chi(\mathcal{D}_R),
\]
(24)
in which \( \kappa \) gives the geodesic curvature, \( K \) stands for the Gaussian optical curvature, while \( \theta_i \) is the exterior angle at the \( i \)th vertex. We can choose a non-singular domain outside of the light ray with the Euler characteristic number \( \chi(\mathcal{D}_R) = 1 \). In order to find the deflection angle, let us first compute the geodesic curvature using the following relation
\[
\kappa = \hat{g} \left( \nabla \gamma \tilde{\gamma}, \tilde{\gamma} \right)
\]
(25)
forgether with the unit speed condition \( \hat{g}(\gamma, \gamma) = 1 \), where \( \gamma \) gives the unit acceleration vector. If we let \( R \to \infty \), our two jump angles \( (\theta_\gamma, \theta_\varphi) \) become \( \pi/2 \), or in other words, the sum of jump angles to the source \( \mathcal{S} \), and observer \( \mathcal{O} \), satisfies \( \theta_\mathcal{O} + \theta_\mathcal{S} \to \pi \). Hence we can write GBT as
\[
\int_{\mathcal{D}_R} \mathcal{K} \, dS + \int_{\partial \mathcal{D}_R} \kappa \, dt + \int_{\mathcal{D}_R} \mathcal{K} \, dS + \int_0^{\pi+\hat{\alpha}} d\phi = \pi.
\]
(26)
Let us now compute the geodesic curvature \( \kappa \). To do so, we first point out that \( \kappa(\gamma_\hat{g}) = 0 \), since \( \gamma_\hat{g} \) is a geodesic. We are left with the following
\[
\kappa(C_R) = \left| \nabla \mathcal{C}_R \hat{C}_R \right|,
\]
(27)
where we choose \( C_R := r(\varphi) = R = \text{const.} \). The radial part is evaluated as
\[
\left( \nabla \mathcal{C}_R \hat{C}_R \right)^r = \mathcal{C}_R^r \left( \partial_r \mathcal{C}_R^r + \hat{F} \mathcal{C}_R^{r \varphi} \mathcal{C}_R^r \right)^2.
\]
(28)
From the last equation, it obvious that the first term vanishes, while the second term is calculated using Eq. (7) and the unit speed condition. For the geodesic curvature we find
\[
\lim_{R \to \infty} \kappa(C_R) = \lim_{R \to \infty} \left| \nabla \mathcal{C}_R \hat{C}_R \right|, \to 1/R.
\]
(29)
On the other hand, for very large radial distance yields
\[
\lim_{R \to \infty} dt \to \left( R - \frac{\left( \gamma - 1 \right) r \gamma}{2\gamma} \right) d\varphi.
\]
(30)
If we combine the last two equations, we find
\[
\kappa(C_R) dt = d\varphi. \quad \text{It is convenient to choose the deflection line as } r = b / \sin \varphi, \text{in that case, the deflection angle from Eq. (30) can be recast in the following from}
\]
\[
\hat{\alpha} = - \int_0^{\pi} \int_{\mathcal{K}} \mathcal{K} \, dS.
\]
(31)
If we substitute Eq. (34) into the last equation and solve it. Note that we use the following relation \( \sin \theta = \frac{d\varphi}{d\theta} \approx dr^* / dr \), valid in the limit as \( R \to \infty \). One can easily solve this integral in the leading order terms to find the following result
\[
\hat{\alpha} \approx 2 \frac{r_+}{b} + 2 \frac{\gamma r_-}{b\gamma}.
\]
(32)

IV. GAUSS-BONNET THEOREM AND WEAK
GRAVITATIONAL LENSING BY EMAD BLACK HOLE

In this section, we calculate the deflection angle of 
EMaD BH in weak field approximation using the GBT.
First we use the null geodesic \( ds^2 = 0 \) where with 
the deflection angle of light in the equatorial plane \( \theta = \pi/2 \),
we obtain the optical metric of EMaD black hole as 
follows:
\[
dt^2 = \frac{1}{F(r)^2} dr^2 + \frac{H(r)}{F(r)} d\varphi^2,
\]
(33)
with functions \( F(r) = \left( 1 - \frac{r_+}{r} \right) \left( 1 - \frac{r_-}{r} \right)^{\frac{1}{\gamma}} \) and \( H(r) = \frac{r^2 \left( 1 - \frac{r_-}{r} \right)^{1-1/\gamma}}{4r^\gamma}. \)
For this purpose, similarly to previous section, first we obtain the Gaussian optical curvature \( \mathcal{K} \). Since we are interested in the weak limit, we can approximate the optical Gaussian curvature as
\[
\mathcal{K} \approx - \frac{r_+}{r^3} - \frac{r_-}{r^3} + \frac{3r_+^2}{4r^4} + \frac{9r_- r_+}{2r^4\gamma} + \frac{\left( \gamma^2 - 6\gamma + 8 \right) r_-^2}{4r^4\gamma^2}.
\]
(34)
For the geodesic curvature we find
\[
\lim_{R \to \infty} \kappa(C_R) = \lim_{R \to \infty} \left| \nabla \mathcal{C}_R \hat{C}_R \right|, \to 1/R.
\]
(35)
On the other hand, for very large radial distance yields
\[
\lim_{R \to \infty} dt \to \left( R - \frac{\left( \gamma - 1 \right) r \gamma}{2\gamma} \right) d\varphi.
\]
(36)
If we combine the last two equations, we find \( \kappa(C_R) dt = d\varphi \). It is convenient to choose the deflection line as \( r = b / \sin \varphi \), in that case, the deflection angle from Eq. (30) can be recast in the following from
\[
\hat{\alpha} = - \int_0^{\pi} \int_{\mathcal{K}} \mathcal{K} \, dS.
\]
(37)
If we substitute Eq. (34) into the last equation, and solve the integral in the leading order terms, we obtain the following result for deflection angle:
\[
\hat{\alpha} \approx 2 \frac{r_+}{b} + 2 \frac{r_-}{b\gamma}.
\]
(38)
V. LIGHT DEFLECTION BY PHANTOM WORMHOLES

A. Wormholes with a bounded mass function

Recently a class of wormhole solutions with a phantom energy equation of state parameter $\omega < -1$ was reported in Ref. [30]. As a first example we shall examine a particular wormhole solution with a bounded mass function with the metric written as follows [30]

$$ds^2 = -\left(1 + \frac{ar_0}{r}\right)^{1 - \frac{\alpha}{2}}dt^2 + \frac{dr^2}{1 - \frac{ar_0}{r}(\frac{a}{r} + 1 - a)} + r^2d\Omega^2,$$

where the parameter $a$ belongs to the interval $-1 < a < 0$. Note that the wormhole throat which connects two asymptotic regions is located at $r_0$, with the following condition $b(r_0) = r_0$. The mass function with a finite value is given by

$$m(r) = \frac{ar_0}{2}\left(\frac{r_0}{r} - 1\right).$$

Let is write the wormhole optical metric in the $(r, \varphi)$ plane which takes the form

$$dt^2 = \frac{dr^2}{\left(1 + \frac{ar_0}{r}\right)^{1 - \frac{\alpha}{2}}\left(1 - \frac{ar_0}{r} + 1 - a\right)} + \left(1 + \frac{ar_0}{r}\right)^{1 - \frac{\alpha}{2}}r^2d\varphi^2.$$

The Gaussian optical curvature on the other hand is approximated as follows

$$\mathcal{K} \simeq \frac{(7 - 14a)r_0^2 + 4r(a - 1)r_0}{4r^3}.$$  

In addition the geodesic curvature reads

$$\lim_{R \to \infty} \kappa(C_R) = \lim_{R \to \infty} \left|\nabla_{\dot{C}_R} \dot{C}_R\right|, \quad \rightarrow \frac{1}{R}.$$  

On the other hand, for very large radial distance we have

$$\lim_{R \to \infty} dt \rightarrow Rd\varphi.$$  

To put it another way, our optical-wormhole metric is asymptotically Euclidean satisfying the condition $\kappa(C_R)dt/d\varphi = 1$. Moreover if we substitute the expression for the Gaussian optical curvature in Eq. (26) the deflection angle is recast as follows

$$\hat{\alpha} = -\int_0^{\pi} \int_{\pi/2}^{\pi} \frac{(7 - 14a)r_0^2 + 4r(a - 1)r_0}{4r^3} drd\varphi.$$  

As a result, up to the first oder approximation we find the following result for the deflection angle:

$$\hat{\alpha} \simeq \frac{2r_0}{b}(1 - a) - \frac{\pi r_0^2}{16b^2}(7 - 14a).$$  

B. Wormholes with an unbounded mass function

Our second example is a wormhole solution with unbounded mass function with the following spacetime metric [30]

$$ds^2 = -\left(1 + \frac{1 - a}{\sqrt{r/r_0}}\right)^2 dt^2 + \frac{dr^2}{1 - \frac{a}{r/r_0} - \frac{r^2}{r/r_0}} + r^2d\Omega^2.$$  

With the mass function given as

$$m(r) = \frac{a}{2}\left(\sqrt{r/r_0} - r_0\right).$$  

It is noted that, the above expression is positive throughout the spacetime, but unbounded as $r \to \infty$. Furthermore the parameter $a$ lies in the range $0 < a < 2$. The corresponding optical metric in the $(r, \varphi)$ reads

$$dt^2 = \frac{dr^2}{\left(1 + \frac{1-a}{\sqrt{r/r_0}}\right)^2 \left(1 - \frac{a}{\sqrt{r/r_0} - 1 - a}\right)^2} + \left(1 + \frac{1-a}{\sqrt{r/r_0}}\right)^2 r^2d\varphi^2,$$

with the Gaussian optical curvature

$$\mathcal{K} \simeq -\frac{r_0}{2r^3} + \frac{\sqrt{r_0}}{4r^{5/2}} - \frac{3r_0^{3/2}}{2r^{7/2}} - \frac{3r_0^2}{4r^4} + \Xi(r, r_0) a.$$  

in which

$$\Xi(r, r_0) = -\frac{3r_0}{8r^3} - \frac{\sqrt{r_0}}{2r^{5/2}} + \frac{21r_0^{3/2}}{8r^{7/2}} + \frac{9r_0^2}{4r^4}.$$  

The geodesic curvature reveals that

$$\lim_{R \to \infty} \kappa(C_R) = \lim_{R \to \infty} \left|\nabla_{\dot{C}_R} \dot{C}_R\right|, \quad \rightarrow \frac{1}{R},$$  

together with

$$\lim_{R \to \infty} dt \rightarrow Rd\varphi.$$  

From these results it is possible to show $\kappa(C_R)dt = d\varphi$. It is a straightforward analysis to check that the light ray equation is governed by equation $r = b/\sin \varphi$. The GBT implies

$$\hat{\alpha} = -\int_0^{\pi} \int_{\pi/2}^{\pi} \left(-\frac{r_0}{2r^3} + \frac{\sqrt{r_0}}{4r^{5/2}} - \frac{3r_0^{3/2}}{2r^{7/2}} - \frac{3r_0^2}{4r^4} + \Xi a\right) rdrd\varphi.$$  

Solving the integral we obtain the following expression

$$\hat{\alpha} \simeq -\frac{\sqrt{\pi}r_0}{2\sqrt{b}1\Gamma\left(\frac{3}{4}\right)}(1 - 2a) + \frac{r_0}{4b}\left(4 + 3a\right)$$  

$$+ \frac{\sqrt{\pi}r_0^{3/2}}{6b^{3/2}1\Gamma\left(\frac{3}{4}\right)}(6 - 212a) + \frac{\pi r_0^2}{16b^2}(3 - 9a).$$
C. Wormholes with vanishing redshift function

Our last example, is a rather simple wormhole solution with a vanishing redshift function. The spacetime metric in that case is given by [30]

\[ ds^2 = -dt^2 + \frac{dr^2}{1 - (r_0/r)^{1-\alpha}} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \]

(56)

in which \( 0 < \alpha < 1 \). The optical metric takes the form

\[ dt^2 = \frac{dr^2}{1 - (r_0/r)^{1-\alpha}} + r^2 d\phi^2, \]

(57)

and

\[ \mathcal{H} = -\frac{(1-\alpha)r_0}{2r^3} \left( 1 - \frac{r_0}{r} \right)^{-\alpha}. \]

(58)

To simplify the problem, we approximate is result as

\[ \mathcal{H} \simeq -\frac{(1-\alpha)r_0}{2r^3} - \frac{(1-\alpha)\alpha r_0^2}{2r^4}. \]

(59)

It is a straightforward to see that

\[ \lim_{R \to \infty} \kappa(C_R) = \lim_{R \to \infty} \left| \nabla C_R \hat{C}_R \right| \]

\[ \to \frac{1}{R}, \]

(60)

thus the optical geometry is asymptotically Euclidean

\[ \lim_{R \to \infty} \frac{\kappa(C_R)dt}{d\phi} = 1. \]

(61)

Utilizing the GBT it follows

\[ \hat{a} = -\int_0^\infty \int_0^{\pi/2} \left( \frac{(1-\alpha)r_0}{2r^3} - \frac{(1-\alpha)\alpha r_0^2}{2r^4} \right) r dr d\phi. \]

(62)

Finally evaluating the above integral we find

\[ \hat{a} \simeq \frac{(1-\alpha)r_0}{b} + \frac{\pi \alpha r_0^2}{8 b^2} + \mathcal{O}(r_0^3, \alpha^2). \]

(63)

VI. GEODESICS

In this section we apply the standard geodesics approach to show that one can obtain the same results (32) and (38) for the deflection angles in the weak field regime. The geodesic equations can be derived using the Euler-Lagrange formalism [31] with the Lagrangian

\[ 2\mathcal{L} = -F(r)\dot{t}^2 + F(r)^{-1}\dot{r}^2 + H(r)\dot{\phi}^2 \]

(64)

for geodesics in the \( \theta = \pi/2 \) hypersurface of the considered black hole or wormhole geometry. Here, the dot represents the derivative with respect to the affine parameter \( \lambda \), i.e. \( \dot{q} = dq/d\lambda \). Since the metric coefficients do not depend neither on \( \phi \) nor \( t \), than the geodesic equations corresponding to these "cyclic" coordinates will be associated to two integrals of motion according to the equations \( \dot{\Pi}_q - \partial \mathcal{L}/\partial \dot{q} = 0 \) leading to

\[ \dot{\Pi}_r = 0, \quad \dot{\Pi}_\phi = 0. \]

(65)

Here, \( \Pi_q = \partial \mathcal{L}/\partial \dot{q} \) are the conjugate momenta to the spacetime coordinates \( q \) and for the Lagrangian (64) are given by

\[ \Pi_t = -F(r)\dot{t} \equiv -E, \]

(66)

\[ \Pi_r = F(r)^{-1}\dot{r} \]

(67)

\[ \Pi_\phi = H(r)\dot{\phi} \equiv L, \]

(68)

where \( E \) and \( L \) are the first integrals of motion for a test particle, corresponding to the energy at infinity and the angular momentum, respectively. Therefore, the Hamiltonian for a test particle with rest mass \( m \) in the static and spherically symmetric spacetimes under consideration is given by

\[ \mathcal{H} = \Pi_t \dot{t} + \Pi_\phi \dot{\phi} + \Pi_r \dot{r} - \mathcal{L}, \]

(69)

\[ 2\mathcal{H} = -E\dot{t} + L\dot{\phi} + F(r)^{-1}\dot{r}^2 = -m^2. \]

(70)

Considering only massless particles and taking into account the condition \( m = 0 \), for photons we obtain the following equations of motion

\[ \dot{i} = \frac{E}{F(r)}, \]

(71)

\[ \dot{\phi} = \frac{L}{H(r)}, \]

(72)

\[ \dot{r}^2 = E^2 - L^2 \frac{F(r)}{H(r)}. \]

(73)

Finding the set of values of the integrals of motion \( E \) and \( L \), for which \( \dot{r} = 0 \), we can determine a precise relation between the light ray impact parameter \( b = L/E \) and the closest approach distance \( r_0 \)

\[ b(r_0) = \sqrt{\frac{H(r_0)}{F(r_0)}}. \]

(74)

Having the expressions (72) and (73) we find the azimuthal shift of the photon as a function of the coordinate distance

\[ \frac{d\phi}{dr} = \pm \frac{1}{H(r)} \left\{ \frac{1}{b^2} - \frac{F(r)}{H(r)} \right\}^{-1/2}, \]

(75)

where the sign in front of square root is negative along the motion of the photon from the source to the minimum distance \( r_0 \) and the sign is positive for the motion of the photon to the source, afterwards.
Let’s assume the light source and observer are placed in the asymptotically flat region of the spacetime. Then, according to the symmetry in the phase of approach and departure we can write the whole light deflection angle via integration of Eq. (75) from \( r_0 \) to infinity

\[
\hat{\alpha}(r_0) = 2 \int_{r_0}^{\infty} \left| \frac{d\phi}{dr} \right| dr - \pi. \tag{76}
\]

### A. Deflection angle by Garfinkle-Horowitz-Ströminger black hole

The relation between the impact parameter \( b \) and the distance of closest approach of the light ray \( r_0 \) in the spacetime of the Garfinkle-Horowitz-Ströminger dilaton black hole is given by

\[
b(r_0) = r_0^{1+\gamma} \sqrt{\frac{(r_0 - r_-)^{1-2\gamma}}{(r_0 - r_+)}}, \tag{77}
\]

We expect for small deviations of the light rays the distance of a closest approach \( r_0 \) to be of the same order as the impact parameter, under assumptions \( r_0 \gg r_+ \) and \( r_0 \gg r_- \). Therefore, for the solution of Eq. (73) we suggest that

\[
r_0 \simeq b \left\{ 1 + \sum_{i,j=1}^{2} \epsilon_{r_+} \epsilon_{r_-} \epsilon_{r_+} \epsilon_{r_-} + O(\epsilon^3) \right\}, \tag{78}
\]

where we are introducing two independent expansion parameters in terms of the black hole horizons

\[
\epsilon_{r_+} = \frac{r_+}{b}, \quad \epsilon_{r_-} = \frac{r_-}{b}. \tag{79}
\]

Calculations of the deflection angle, according to Eq. (76), show that the zero order term \( O(1) \) evaluated on \( \pi \) is shortening. As opposed to that the first order terms proportional to \( O(\epsilon_{r_+}) \) and \( O(\epsilon_{r_-}) \) respectively, give the most significant contributions \( 2\epsilon_{r_+} \) and \( 2\epsilon_{r_-} \) to the light deflection angle, as can be seen from the next ex-

Here the coefficients \( c_{r_+r_-} \) are real numbers, and the summation is over all possible combinations of epsilon powers \( i, j \) up to and including second order terms. Notice, that the approximation (78) is not longer valid in limit \( \gamma \to -1 \), when the both black hole horizons diverge.

Solving the equation \( \dot{r}(r_0) = 0 \) we find for the closest approach distance

\[
r_0 \simeq b \left\{ 1 - \frac{r_+}{2b} + \frac{(1 - 2\gamma)r_-}{2b} + \frac{(1 - 2\gamma)r_+ r_-}{4b^2} - \frac{3r_+^2}{8b^2} + \frac{(1 - 4\gamma^2)r_-^2}{8b^2} + O(\epsilon^3) \right\}. \tag{80}
\]

The azimuthal shift of the photon follows from the Eqs. (72) and (73) and is given by

\[
\frac{d\phi}{dr} = \frac{(r - r_-)^{\gamma-1}}{r^{1+\gamma}} \left\{ \frac{1}{b^2} - \frac{(r - r_+)}{r^{2(1+\gamma)}(r - r_-)^{1-2\gamma}} \right\}^{-1/2}. \tag{81}
\]

Assuming \( \epsilon_{r_+} = r_+/r_0 \) and \( \epsilon_{r_-} = r_-/r_0 \) are small enough quantities, we can consider a Taylor expansion series of Eq. (81) up to and including first order terms of the new expansion parameters \( \epsilon_{r_+} \) and \( \epsilon_{r_-} \). To make integration afterwards easier, we introduce new variable \( u = r_0/r \), in terms of which

\[
\frac{d\phi}{du} \simeq \frac{1}{\sqrt{1 - u^2}} + \frac{1}{2} \left[ \frac{1 + u + u^2}{\sqrt{1 - u(1 + u)^{3/2}}} \right] \epsilon_{r_+} - \frac{1}{2} \left[ \frac{1 - u(1 + u) - 2\gamma}{\sqrt{1 - u(1 + u)^{3/2}}} \right] \epsilon_{r_-} + \frac{3}{8} \left( \frac{(u^2 + u + 1)^2}{\sqrt{1 - u(1 + u)^{3/2}}} \right) \epsilon_{r_+}^2 - \frac{1}{4} \left[ \frac{(1 - 2\gamma) - 2\gamma u + (1 - 6\gamma)u^2 - 2u^3 - u^4}{\sqrt{1 - u(1 + u)^{3/2}}} \right] \epsilon_{r_+} \epsilon_{r_-} - \frac{1}{8} \left[ \frac{1 - 4\gamma^2 - (12\gamma - 6)u - (8\gamma^2 + 1) u^2 - 6u^3 - 3u^4}{\sqrt{1 - u^2(1 + u)^2}} \right] \epsilon_{r_-}^2 + O(\epsilon^3). \tag{82}
\]

Pression

\[
\hat{\alpha}(r_0) \simeq \frac{2r_+}{r_0} + \frac{2\gamma r_-}{r_0} - \left( \frac{3\pi}{8} - 1 + \frac{3(2 - \pi)\gamma}{2} \right) \frac{r_+ r_-}{r_0^2} + \left( \frac{15\pi}{16} - 1 \right) \frac{r_+}{r_0^2} - \left( \frac{\pi}{16} - \gamma + (2 - \pi)\gamma^2 \right) \frac{r_-}{r_0^2} + O(\epsilon^3). \tag{83}
\]
Hence, taking into account the power series (80), we can represent the light deflection angle in terms of the photon integrals of motion $L$ and $E$, via the impact parameter $b$, up to and including the second orders of $\epsilon$. Therefore, the bending of light ray by GHS black hole leads to the following components of the deflection angle

$$\hat{\alpha}(b) \simeq \frac{2r_+}{b} + \frac{2\gamma r_-}{b} - \frac{3\pi(1-4\gamma)r_+r_-}{8b^2} + \frac{15\pi r_+^2}{16b^2} - \pi \left(\frac{1}{16} - \frac{\gamma^2}{2}\right) \frac{r_-^2}{b^2} + O(\epsilon^3).$$  \hspace{1cm} (84)

Up to the first formal order in the expansion parameter the approximate expression for the light deflection angle (84) coincides with the result (32) found by GBT.

B. Deflection angle by Einstein-Maxwell anti-dilaton black hole

In the case of Einstein-Maxwell anti-dilaton black hole the impact parameter of the photon $b$ and the distance of closest approach $r_0$ obey the following relation

$$b(r_0) = r_0^{1+\frac{1}{2}} \sqrt{(r_0-r_-)\frac{1-\frac{3}{2}}{(r_0-r_+)}}.$$  \hspace{1cm} (85)

We expect in the weak deflection limit the distance of the closest approach $r_0$ to be of the same order as the impact parameter $b$. Therefore, under assumptions $r_0 \gg r_+$ and $r_0 \gg r_-$, we are taking advantage of the two independent expansion parameters

$$\epsilon_{r_+} = \frac{r_+}{b}, \quad \epsilon_{r_-} = \frac{r_-}{b},$$  \hspace{1cm} (86)

which are real quantities for $(Q/M)^2 \leq (Q/M)_{\text{crit}}^2 = (1+\gamma)/2$.

Repeating the same procedure as in the previous case of GHS dilaton black hole, we are solving the equation $\hat{r}(r_0) = 0$ and obtain for the closest approach distance the following expression

$$r_0 \simeq b \left\{ 1 - \frac{r_+}{2b} + \frac{(\gamma - 2)r_-}{2\gamma b} + \frac{(\gamma - 2)r_+r_-}{4\gamma b^2} - \frac{3r_+^2}{b^2} + \frac{(\gamma - 2)(\gamma + 2)r_-^2}{8\gamma^2 b^2} + O(\epsilon^3) \right\}.$$  \hspace{1cm} (87)

The azimuthal shift of the photon as a function of the coordinate distance is given by

$$\frac{d\phi}{dr} = (r-r_-)^{\frac{1}{2}} - \frac{1}{r^{1+\frac{1}{2}}} \left[ \frac{1}{b^2} - \frac{(r-r_+)}{r^{2+\frac{1}{2}}(r-r_-)^{1-\frac{3}{2}}} \right]^{1/2}.$$  \hspace{1cm} (88)

Introducing two independent expansion parameters $\epsilon_{r_+} = r_+/r_0$ and $\epsilon_{r_-} = r_-/r_0$, we can perform Taylor series expansion of Eq. (88) up to and including second order terms, which in terms of the variable $u = r_0/r$ is given by

$$\frac{d\phi}{du} \simeq \frac{1}{\sqrt{1-u^2}} + \frac{1}{2} \left[ \frac{1+u+u^2}{\sqrt{1-u(1+u)^{3/2}}} \right] \epsilon_{r_+} + \frac{1}{2} \left[ \frac{2-\gamma(1-u-u^2)}{\sqrt{1-u(1+u)^{3/2}}} \right] \epsilon_{r_-} + \frac{3}{8} \left[ \frac{(1+u+u^2)^2}{\sqrt{1-u(1+u)^{3/2}}} \right] \epsilon_{r_+}^2 + \epsilon_{r_-}^2 + O(\epsilon^3).$$  \hspace{1cm} (89)

Therefore, the bending angle of the light ray, defined by Eq. (76), up to and including the second orders of the expansion parameters $\epsilon$, is given by

$$\hat{\alpha}(r_0) \simeq \frac{2r_+}{r_0} + \frac{2r_-}{\gamma b} + \frac{(\pi(4-\gamma) - 8(3-\gamma))r_+r_-}{8\gamma r_0^2} + \left(\frac{15\pi}{16} \right) \frac{r_-^2}{r_0^2} - \left(\frac{\pi}{16} - \frac{\pi + \gamma - 2}{\gamma^2} \right) \frac{r_-^2}{r_0^2} + O(\epsilon^3).$$  \hspace{1cm} (90)

Here, similar to the procedure described in the previous paragraph, we consider $\epsilon_r \equiv \frac{b}{r_0} \epsilon_r$, and take into account the power series (80). After completing approximation procedure, we obtain the light deflection angle

$$\hat{\alpha}(b) \simeq \frac{2r_+}{b} + \frac{2r_-}{\gamma b} - \frac{3\pi(4-\gamma)}{8} \left(1 - \frac{4}{\gamma} \right) \frac{r_+r_-}{b^2} + \frac{15\pi r_+^2}{16b^2} + \pi \left(\frac{1}{\gamma^2} - \frac{1}{16} \right) \frac{r_-^2}{b^2} + O(\epsilon^3).$$  \hspace{1cm} (91)

The most significant contributions to the light deflection angle in that case are given by the first order terms $2\epsilon_{r_+}$ and $2\epsilon_{r_-}/\gamma$, proportional to $O(\epsilon_{r_+})$ and $O(\epsilon_{r_-})$, respectively.

As expected, up to the given formal order in the expansion parameters the deflection angle in the weak limit approximation is found to be the same result (38)
found by GBT.

C. Deflection angle by phantom wormholes

In the case of the wormhole solution with a bounded mass function (39), the impact parameter of the photon $b$ and the distance of closest approach $r_m$ obey the following relation

$$b(r_m) = r_m \left(1 - \frac{a r_0}{r_m}\right)^{\frac{1}{2}(\frac{1}{2} - 1)}.$$  \hspace{1cm} (92)

Since in the weak field limit the light deflection angle is small, the distance of the closest approach $r_m$ is of the same order as the impact parameter $b$. Therefore, under assumption the wormhole throat $r_0 \gg r_m$, we introduce the expansion parameter

$$\epsilon_r = \frac{r_0}{b}. \hspace{1cm} (93)$$

Solving the radial equation of motion $\dot{r}(r_m) = 0$ we find for the closest approach distance

$$r_m \simeq b \left\{1 - \frac{1 - (1 - a) r_0}{2b} - \frac{(1 - a)(1 - 3a) r_0^2}{8b^2} - \frac{(1 - a)(1 - 2a)(1 - 3a) r_0^3}{12b^3} + \mathcal{O}(\epsilon^4)\right\}. \hspace{1cm} (94)$$

One can easily solve the integral (76) in the second order terms to find the following result

$$\dot{\alpha}(b) \simeq \frac{2(1 - a) r_0}{b} + \frac{\pi(11 - 22a + 15a^2) r_0^2}{16b^2} + \mathcal{O}(\epsilon_r^3), \hspace{1cm} (97)$$

where we considered $\epsilon_r = \frac{r_0}{r_m} \epsilon_r$, as we took into account the power series (94).

Up to the first formal order in the expansion parameter the approximate expression for the light deflection angle (97) coincides with the result (46) found by GBT.

In the case of the wormhole with unbounded mass function (47) one can show that the impact parameter of the light ray is given by

$$b(r_m) = \frac{r_m}{1 + (1 - a) \sqrt{\frac{r_0}{r_m}}}. \hspace{1cm} (98)$$

The azimuthal shift of the photon follows from the Eqs. (72) and (73) and is given by

$$\frac{d\phi}{dr} \simeq \frac{1}{r^2} \left[\frac{1}{b^2} - \frac{1}{r^2} \left(1 - \frac{a r_0}{r}\right)^{1 - \frac{3}{2}}\right]^{-1/2}. \hspace{1cm} (95)$$

In order to obtain the light deflection angle in weak deflection limit, we perform a Taylor series expansion of (95) up to and including second order terms in the powers of new small expansion parameter $\epsilon_r = r_0/r_m$. Therefore, the azimuthal shift function in terms of the variable $u = r_m/r$ is given by

$$\frac{d\phi}{du} \simeq \frac{1}{\sqrt{1 - u^2}} + \frac{1}{2} \left[\left(1 - a\right) \left(u^2 + u + 1\right) + \mathcal{O}(\epsilon_r^3)\right]. \hspace{1cm} (96)$$

One can easily solve the integral (76) in the second order terms to find the following result

$$\dot{\alpha}(b) \simeq \frac{2(1 - a) r_0}{b} + \frac{\pi(11 - 22a + 15a^2) r_0^2}{16b^2} + \mathcal{O}(\epsilon_r^3), \hspace{1cm} (97)$$

where we considered $\epsilon_r = \frac{r_0}{r_m} \epsilon_r$, as we took into account the power series (94).

Considering a light ray with adistance of closest approach $r_m \gg r_0$, where $r_0$ is the wormhole throat, one can show that the light deflection angle in the weak deflection limit is given by

$$\dot{\alpha}(b) \simeq -\frac{\sqrt{\pi} \Gamma \left(\frac{3}{2}\right)}{2 \Gamma \left(\frac{5}{2}\right)} \sqrt{\frac{r_0}{b}} (1 - 2a) + \frac{(3 - 6a) r_0}{b} + \mathcal{O}(\epsilon_r^3). \hspace{1cm} (99)$$

Here, we have chosen $\epsilon_r = \sqrt{r_0/b}$ for a power series expansion parameter and we have also assumed $a \ll 1$.

Comparison of the result with that obtained by GBT (55) shows that the both expressions for the deflection angle match up to the leading term proportional to $\mathcal{O}(\epsilon_r)$. Moreover, one can notice that the light deflection angle by wormhole with an unbounded mass function is negative, in the limit $r_0/b \to 0$, for small enough $a$. 
In the last case concerning a wormhole with vanishing redshift function (56) one can show that the impact parameter $b$ and the closest approach distance of the light ray $r_m$ coincide

$$b(r_m) = r_m. \tag{100}$$

Following the same procedure applied in the previous cases one can show that the approximate light deflection angle is given by

$$\hat{\alpha}(b) \simeq (1 - \alpha) \frac{r_0}{b} + \frac{\alpha \ln 2}{b} + \frac{3\pi r_0^2}{16b^2} (1 + (1 - \ln 4)\alpha) + \mathcal{O}(\epsilon r_0^3, \alpha^2), \tag{101}$$

where first we have adopt $\epsilon r_0 = r_0/b$ for the series expansion parameter, as we assume that the wormhole throat $r_0$ and the impact parameter $b$ of the photon obey the relation $r_0 \ll b$. Afterwards, we perform a power series expansion of the obtained result, assuming that the metric parameter $\alpha$ is small quantity.

Up to the first formal order in the expansion parameter the approximate expression for the light deflection angle (101) coincides with the result (63) found by GBT.

\section*{VII. CONCLUSIONS}

In this paper, first we have calculated the deflection angle by GHS BH and EMaD BH in weak field approximation. We applied the GBT to the corresponding optical geometries that provide us to calculate the deflection angle by integrating over a domain outside the impact parameter. Notice an interesting fact: the effect of the bending of light ray is a global effect. The deflection angle by GHS BH and then by EMaD BH are calculated as follows:

$$\hat{\alpha}_{\text{GHS}} \simeq 2 \frac{r_+}{b} + 2 \frac{\gamma r_-}{b}, \tag{102}$$

and

$$\hat{\alpha}_{\text{EMaD}} \simeq 2 \frac{r_+}{b} + 2 \frac{r_-}{b\gamma}. \tag{103}$$

The difference between the deflection angles are plotted in Fig. 1. As the impact parameter decreases, the deflection grows more and more.

In the case of GHS BH and EMaD BH, it is interesting to point out that, up to the first order terms, the deflection angles are not affected at all by the electric charge $Q$ which is encoded in $r_{\pm}$. In other words, the effect of electric charge becomes significant if we take into account higher order terms.

Next, to elucidate observational differences between the deflection angle by the black hole and wormhole phenomena, the deflection angle by phantom wormholes such as wormholes with a bounded mass function, wormholes with an unbounded mass function and wormholes with vanishing redshift function are calculated by using the GBT as follows:

$$\hat{\alpha}_{\text{pw1}} \simeq 2 \frac{r_0}{b} (1 - \alpha) + \mathcal{O}(r_0^2, \alpha^2), \tag{104}$$

$$\hat{\alpha}_{\text{pw2}} \simeq -\frac{\sqrt{\pi}}{2} \frac{\Gamma \left(\frac{3}{4}\right)}{\Gamma \left(\frac{5}{4}\right)} \frac{r_0^2}{b^2} (1 - 2\alpha) + \mathcal{O}(r_0^2, \alpha^2) \tag{105}$$

and

$$\hat{\alpha}_{\text{pw3}} \simeq \frac{(1 - \alpha) r_0}{b} + \mathcal{O}(r_0^2, \alpha^2). \tag{106}$$

We see that besides the effect of the wormhole geometry on the banding of light encoded in $r_0$, the parameters $\alpha$ and $a$ also affects the value of the deflection angle in an interesting way. Neglecting the effect of $\alpha$ and $a$, we see that the geometric contribution depends on the particular chose of the mass function. For instance,
in the case of wormhole with a bounded mass function we find a magnitude of the banding angle twice as \( r_0/b \). Moreover, in the case of wormhole with an unbounded mass function the magnitude of the deflection angle can be negative at a distance far enough from the wormhole throat, where the weak field approximation is valid. Finally we should point out that, in current analyses in the present paper the agreement is exact up to the first order terms, thus going to higher order terms the agreement breaks down. The main reason for this is the straight line approximation involved in the integration domain. In principle, this inconsistency can be resolved by choosing a more precise relation for the light ray trajectory in the integration domain. Nevertheless these phantom wormholes are interesting from a theoretical point of view because of arising in general relativity.

Afterwards, we checked these results using the geodesics method and confirmed them. In Fig. 1 we have seen that, the approximated deflection angles by GHS and EMaD black holes (83) and (90) calculated by the geodesics method have a perfect coincidence with the numerically calculated deflection angles in the weak field limit \( r_0/b \to 0 \). Notice, that the approximation breaks down when the minimal distance of the closest approach \( r_m \) converge to the photon sphere radius of the black holes. Moreover, the calculated approximate expressions for the deflection angles describe correctly the slightly repulsive effect on the light rays deviation by the EMaD black hole versus GHS black hole, as the numerical calculations shown.

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