RIEMANN SURFACES AND 3-REGULAR GRAPHS

Research Thesis

submitted in partial fulfillment of the
requirements for the degree of
Master of Science in Mathematics

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Submitted to the Senate of
The Technion - Israel Institute of Technology

Tevet, 5761 Haifa January 2001
This research thesis was done under the supervision of Professor Robert Brooks in the Department of Mathematics.

I would like to thank Professor Brooks for lighting my way, caring and loving.

The thesis is dedicated to my parents, sister and brother. Cette thèse est également dédiée à mes grand-mères bien aimées Irène et Sarah.

The generous financial help of the Forchheimer Foundation Fellowship is gratefully acknowledged.
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Abstract

In this thesis, we consider a way to construct a rich family of compact Riemann surfaces in a combinatorial way. Given a 3-regular graph with orientation, we construct a finite-area hyperbolic Riemann surface by gluing triangles together according to the combinatorics of the graph. We then compactify this surface by adding finitely many points.

We discuss this construction by considering a number of examples. In particular, we see that the surface depends in a strong way on the orientation.

We then consider the effect the process of compactification has on the hyperbolic metric of the surface. To that end, we ask when we can change the metric in the horocycle neighbourhoods of the cusps to get a hyperbolic metric on the compactification. In general, the process of compactification can have drastic effects on the hyperbolic structure. For instance, if we compactify the 3-punctured sphere we lose its hyperbolic structure.

We show that when the cusps have lengths bigger than $2\pi$ we can fill in the horocycle neighbourhoods and retain negative curvature.

Furthermore, the condition that the horocycles have length $> 2\pi$ is sharp. We show by examples that there exist curves arbitrarily close to horocycles of length $2\pi$, which cannot be so filled in. Such curves can even be taken to be convex.
List of Symbols

\( \Gamma \) A (3-regular) graph.
\( S^O(\Gamma) \) A non-compact Riemann Surface constructed from \( \Gamma \).
\( S^C(\Gamma) \) A compact Riemann Surface constructed from \( \Gamma \).
\( G \) A group.
\( ISO^+ \) Orientation-Preserving Isometries.
\( \mathbb{D} \) \( \{ z \in \mathbb{C} : |z| < 1 \} \).
\( \mathbb{D}^* \) \( \mathbb{D} \setminus \{0\} \).
\( B_r \) \( \{ z \in \mathbb{D}^* : |z| < r \} \).
\( ds^2_\mathbb{D} \) The complete hyperbolic metric on \( \mathbb{D} \).
\( ds^2_\mathbb{D}^* \) The complete hyperbolic metric on \( \mathbb{D}^* \).
\( \kappa(ds^2) \) The curvature of the Riemannian metric \( ds^2 \).
\( \kappa_g \) Geodesic curvature.
\( \Delta \) The Laplacian.
Introduction

Outline of the thesis

This thesis considers a way to construct Riemann surfaces out of 3-regular graphs. In Chapter 1, we present a method to construct a finite-area Riemann surface out of a 3-regular graph. We then compactify it to get a compact Riemann surface. In sections 1.1–1.6 we describe this construction in detail. In section 1.7 we show that symmetries of the graph are reflected in symmetries of the surface. This allows us to give several examples in section 1.9. In the last section of the chapter we show that the compact Riemann surfaces we construct are dense in the moduli space by considering them as Belyi surfaces.

We would like to learn about the compact Riemann surfaces by studying the geometry and spectral properties of the graphs from which they originated. Early works of Prof. Brooks ([Br1, Br2]) show connections between the graphs and the non-compact Riemann surface. Later results in [Br3] show that under nice conditions (the “large cusps condition”), the complete metrics of constant curvature on the non-compact Riemann surface and its compactification are arbitrarily close, outside of standard cusp neighbourhoods.

It is at this point that we start chapter 2. We would like to know, for example, when we can control the metric on the compact Riemann surface. More precisely, when can we change the hyperbolic metric on the non-compact Riemann surface in neighbourhoods of the cusps to get a hyperbolic metric on the compact Riemann surface? For example, if we compactify the 3-punctured sphere we lose the hyperbolic structure. To that end, we work in a punctured-disk neighbourhood of a cusp, and consider the problem of extension of metrics smoothly across the cusps to obtain metrics of negative curvature. We prove that in a “large cusps” condition we can in fact extend...
the metric from the outside to a smooth metric of negative curvature inside. This is the content of theorem 2.3.1.

The “large cusps” condition is discussed in section 2.2. There, we prove also the classical Shimizu–Leutbecher theorem in a nice way. We study the hyperbolic punctured disk in sections 2.4–2.5, and theorem 2.3.1 is proved in sections 2.6 and 2.7.

In section 2.8 we show that the “large cusps” condition in theorem 2.3.1 is sharp. This leads us in section 2.9 to a very interesting problem: Suppose we have a closed curve bounding a domain $V$ in the unit disk, and we have a conformal Riemannian metric of negative curvature outside of $V$. We ask whether we can extend the metric to $V$ in such a way, that we are left with a conformal Riemannian metric of negative curvature. The Gauss–Bonnet theorem imposes a natural restriction on the extension, but this is not enough. We show that we can find curves which satisfy the Gauss–Bonnet restriction for which we cannot extend the metric. We do this by using the maximum principle for subharmonic functions. Indeed, we show that such curves can be taken to be convex and arbitrarily close to the horocycle of length $2\pi$.

We also consider curves which satisfy the Gauss–Bonnet restriction and for which no metric of negative curvature, conformal or nonconformal, can extend the metric from the outside.

In section 2.10 we show results from [Br3]: If we have large cusps, the hyperbolic metric on the compactification is very close to the original hyperbolic metric, outside of cusp neighbourhoods. This is theorem 2.10.1. The basic idea here is to use the Ahlfors–Schwarz Lemma, which is proved and discussed in appendix A.

Acknowledgements

We would like to thank Curt McMullen for giving us his permission to use his example in section 1.9. We would also like to thank Curt McMullen for discussions with us concerning the thesis, giving us some new ideas to think about.

We would like to thank Mikhail Katz for his fruitful ideas concerning the problem of extension of metrics in section 2.9.

4
Chapter 1

Riemann Surfaces and 3-regular Graphs

1.1 Ideal Triangles

Let us denote by $\mathbb{D}$ the unit disk, equipped with the hyperbolic metric.

**Definition 1.1.1.** An ideal triangle in $\mathbb{D}$ is a triangle which has 3 vertices at infinity.

**Remark:** Any two ideal triangles are isometric, since we may find a Möbius transformation, which takes one onto the other.

Choose a point on each edge of the ideal triangle. The chosen points will be called tick-marks.

**Definition 1.1.2.** A marked ideal triangle is an ideal triangle with a tick-mark on each one of its three sides.

An isomorphism between two marked ideal triangles is an isomorphism between the ideal triangles which preserves the tick-marks. The standard marked ideal triangle (figure 1.1) is any marked ideal triangle which is isometric to the marked ideal triangle whose vertices in the disk model are given by

$$v_1 = 1, \ v_2 = \omega, \ v_3 = \omega^2,$$

and whose tick-marks are

$$t_1 = -(2 - \sqrt{3}), \ t_2 = -(2 - \sqrt{3})\omega, \ t_3 = -(2 - \sqrt{3})\omega^2,$$
where \( \omega = e^{2\pi i/3} \).

Under the isometry which takes the half-plane onto the disk and is given by the map
\[
z \mapsto \frac{z - (\omega + 1)}{z - (\bar{\omega} + 1)}
\]
the ideal triangle of figure 1.1 corresponds to the first ideal triangle from the left in figure 1.2. In the second ideal triangle there, we have drawn also three horocyclic segments, which will be called the *standard horocyclic segments*. Note that each standard horocyclic segment is of length 1.

### 1.2 Some notions in Graph Theory

Throughout our work, the term *graph* will refer to a graph with loops and multiple edges permitted.

**Definition 1.2.1.** A *3-regular graph* is a graph for which there are three edges incident at each vertex (loops are counted twice).

Let \( \Gamma \) be a 3-regular graph. Denote by \( V \) and \( E \) the set of its vertices and the set of its edges respectively.

**Definition 1.2.2.** An orientation at a vertex \( v \) is a cyclic ordering of the edges incident at \( v \).
At each vertex of $\Gamma$ choose an orientation (We will suppress the orientation and will denote the graph with orientation by $\Gamma$). We may regard the orientation as follows: If we take a tour on the graph, at each vertex it tells us which way is left. We define:

**Definition 1.2.3.** A left-hand-turn path in $\Gamma$ is a directed closed path in $\Gamma$ such that if $e_1, e_2$ are successive edges in the path meeting at $v$, then $e_2, e_1$ are successive edges with respect to the orientation at $v$ (see figure 1.3).

Where convenient, we will denote a left-hand-turn path by a cyclic sequence $((v_1, e_1), \ldots, (v_n, e_n))$, where $(v_i, e_i) \in (V, E)$ and $\partial_e = \{v_i, v_{i+1}\}$. In this notation $(v_{i+n}, e_{i+n}) = (v_i, e_i)$, and $(v_i, e_i) \neq (v_j, e_j)$ for $i \not\equiv j \pmod{n}$. 

---

**Figure 1.2:** standard and non-standard marked ideal triangles

**Figure 1.3:** A left-hand-turn path
1.3 Triangulations and 3-regular Graphs

Let $S$ be a closed (topological) surface. Let $\mathcal{T}$ be a triangulation of $S$. Mark a point in the interior of each triangle, and connect points corresponding to adjacent triangles. We get a 3-regular graph, $\Gamma$, drawn on the surface. $\Gamma$ inherits an orientation from the surface’s orientation - at each vertex, $v$, of $\Gamma$ we order the three emanating edges from $v$ counterclockwise. For convenience, we will refer to the vertices and edges of the triangulation as $\mathcal{T}$-vertices and $\mathcal{T}$-edges respectively, while the vertices and edges of $\Gamma$ will be referred as $\Gamma$-vertices and $\Gamma$-edges respectively.

First, we note that any $\Gamma$-edge is crossed by exactly one $\mathcal{T}$-edge, and vice versa. Therefore the $\Gamma$-edges are in one-to-one correspondence with the $\mathcal{T}$-edges. Second, any left-hand-turn path in $\Gamma$ cuts, on its left side, a disk from the surface in which there are no $\Gamma$-vertices, since otherwise one could join any $\Gamma$-vertex inside the disk by a shortest path in $\Gamma$ to the boundary of the disk to get that the boundary would not be a left-hand-turn path. In this disk there is exactly one $\mathcal{T}$-vertex: Indeed, any $\Gamma$-edge which crosses the boundary of the disk must have one of its ends inside the disk so there is at least one $\mathcal{T}$-vertex in this disk. Conversely, take any $\mathcal{T}$-vertex, $p$, in the disk, and let $l$ be the length of the boundary of the disk. If there were less than $l$ $\mathcal{T}$-edges emanating from $p$, one of the triangles surrounding $p$ would contain at least two $\Gamma$-vertices from the boundary of the disk. Therefore, there are $l$ $\mathcal{T}$-edges emanating from $p$, and we are left with “no room” for another $\mathcal{T}$-vertex inside the disk.

We have shown that we have a well-defined bijection: A left-hand-turn path in $\Gamma$ maps to the $\mathcal{T}$-vertex it encloses on its left side. As we will see in the next section, this bijection lets us reconstruct the surface from the graph with orientation. Or, in more fancy language, $\{\text{graphs with orientations}\} \cong \{\text{triangulations of surfaces}\}/\{\text{isotopy}\}$.

1.4 Constructing a Riemann Surface out of a 3-regular Graph

Let $\Gamma$ be a 3-regular graph with orientation. Paste one copy of the standard marked ideal triangle on each vertex of $\Gamma$, in such a way that the three successive edges emanating from the vertex of the graph fit the three geodesic segments $ot_1, ot_2, ot_3$ in figure 1.1, respectively. We glue adjacent sides of
marked triangles, such that the tick-marks fit and the orientations of the corresponding triangles match up. We can begin doing the pasting without leaving the hyperbolic plane, until we get a polygon, \( \Pi \), together with a side pairing (Remark: We may think of the graph \( \Gamma \) as made out of threads, and of each triangle as having a white face and a black face. The threads are glued on the white face of each triangle. We build the polygon such that all the triangles have white face up). Then, we apply the Poincaré Polygon Theorem (see Appendix B) to \( \Pi \): We attach to each side pairing \((s_i, s_j)\) an orientation preserving transformation, \( A_{ij} \), such that \( A_{ij}(s_i) = s_j \), \( A_{ij} \) preserves tick-marks, and \( \Pi \cap A_{ij}(\Pi) = \emptyset \). Denote by \( G \) the subgroup of \( \text{ISO}^+(\mathbb{D}) \) generated by all the side-pairing transformations. By the Poincaré Polygon Theorem, \( G \) is a discrete group of isometries with \( \Pi \) as its fundamental domain. Hence \( \mathbb{H}^2/G \) is a complete hyperbolic Riemann surface, which we will denote by \( \mathcal{S}O(\Gamma) \).

As was shown in the preceding section, the cusps of \( \mathcal{S}O(\Gamma) \) are in bijection with the left-hand-turn paths in \( \Gamma \), and the triangulation of \( \mathcal{S}O(\Gamma) \) obtained by our construction corresponds to \( \Gamma \) in the sense of section 1.3. Finally, take the unique conformal compactification (see section 1.8) of \( \mathcal{S}O(\Gamma) \), to get a closed Riemann surface, \( \mathcal{S}C(\Gamma) \).

### 1.5 A slightly more general construction

Again, let \( \Gamma \) be a 3-regular graph with orientation. We paste a marked ideal triangle on each vertex as before, but now we allow also non-standard marked ideal triangles to be pasted. The marked ideal triangles can be parametrized by three real parameters, \((\alpha_1, \alpha_2, \alpha_3)\), which describe the hyperbolic shift of the tick-marks with respect to the tick-marks \( t_1, t_2, t_3 \) of the standard marked ideal triangle. For instance, the standard marked ideal triangle will be denoted by \((0, 0, 0)\), and the non-standard marked ideal triangle in figure 1.2 is \((0, \ln 2, - \ln 3)\). In the next step, we glue adjacent edges as before, remembering to match orientations and tick-marks.

To each pair of a vertex, \( v \), and an edge \( e \) emanating from \( v \), there corresponds one side of the triangle we paste at \( v \). Therefore we can attach to such a pair a real number \( \alpha(v, e) \) which describes the shift of the tick-mark on the corresponding side.

In order to apply the Poincaré Polygon Theorem all the vertex-cycle transformations must be parabolic. This, in turn, is equivalent to the following
For every left-hand-turn path in \( \Gamma \),
\[
\sum_{k=1}^{n} [\alpha(v_k, e_k) - \alpha(v_k, e_{k-1})] = 0.
\]

1.6 The Genus of \( S^C(\Gamma) \)

In section 1.4 we obtained from a 3-regular graph with orientation, \( \Gamma \), a closed Riemann surface \( S^C(\Gamma) \). Let us calculate the genus of \( S^C(\Gamma) \). To that end, let \( V, E, F \), be the number of vertices, edges and faces, respectively, in the triangulation corresponding to \( \Gamma \). In section 1.3 we have shown that \( V = N_{lht} = \) the number of left-hand-turn paths in \( \Gamma \), \( E = N_e = \) the number of edges in \( \Gamma \), and obviously, \( F = N_v = \) the number of vertices in \( \Gamma \). Since \( \Gamma \) is a 3-regular graph, we also have \( 3F = 2E \). Hence, the Euler characteristic of \( S^C(\Gamma) \) can totally be recovered from \( \Gamma \) by
\[
\chi(S^C(\Gamma)) = V - E + F = N_{lht} - N_v/2.
\]

So, the genus of \( S^C(\Gamma) \) is:
\[
g = (2 - \chi)/2 = 1 + \frac{N_v - 2N_{lht}}{4}.
\]

We could also calculate more geometrically as follows: The Euler characteristic, \( \chi \), of \( S^O(\Gamma) \) is \( 2 - 2g - N_{lht} \). We now use Gauss-Bonnet formula for Riemannian surfaces without boundary:
\[
\iint \kappa \, d(\text{area}) = 2\pi \chi,
\]
where \( \kappa \) is the curvature. In our case \( \kappa \equiv -1 \), and the area of \( S^O(\Gamma) \) is \( N_v \pi \), since the area of the ideal triangle is \( \pi \). Substituting, we get \( -N_v \pi = 2\pi(2 - 2g - N_{lht}) \), which leads again to formula (1.1).

1.7 Automorphisms

Let \( \Gamma_1, \Gamma_2 \) be two graphs with orientation. Let \( \phi : \Gamma_1 \to \Gamma_2 \) be an isomorphism of graphs.
Definition 1.7.1. We say that $\phi$ is orientation-preserving if for any three edges, $\{e_1, e_2, e_3\}$, in $\Gamma_1$ meeting at a vertex $v$, the cyclic ordered triplet $(e_1, e_2, e_3)$ fits the orientation at $v$ iff $(\phi(e_1), \phi(e_2), \phi(e_3))$ fits the orientation at $\phi(v)$.

A similar definition holds for orientation-reversing.

We observe that the left-hand-turn paths are preserved under an isomorphism of graphs with orientation. In particular, the length of a left-hand-turn path is preserved. These simple facts will help us later in finding automorphisms of graphs.

In order to understand the automorphisms of a 3-regular graph with orientation, let us first try to understand the automorphisms of its universal cover, the 3-regular tree, $T_3$: We choose some vertex of this tree, and call it the root. Denote by $\tilde{T}_3$ the 3-regular rooted tree. On a rooted tree we can define the notion of level: level $n$ is the set of all vertices which are at a distance $n$ from the root. A map between two rooted trees, is a graph-map which maps the root to the root. It has the property that it preserves the levels of the tree.

Now, Let $\tilde{T}_3^{\text{or}}$ be a 3-regular rooted tree, with some choice of orientation on its vertices (actually, all the choices are equivalent). We have:

Lemma 1.7.2. The group of automorphisms of $\tilde{T}_3^{\text{or}}$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.

Proof. Since we are dealing with orientation-preserving automorphisms, a generator is determined by the image of a first-level vertex of the tree. \[\Box\]

From the last lemma, we immediately get:

Lemma 1.7.3. An automorphism of the 3-regular tree with orientation, $T_3^{\text{or}}$, is determined by the images of one vertex and an edge emanating from it.

Now, that we understand the automorphism group of the universal cover we can proceed:

Lemma 1.7.4. Let $(v_0, e_0)$ and $(v_1, e_1)$ be two pairs of vertices and edges emanating from them. Then, there exists at most one automorphism $\phi : \Gamma \to \Gamma$, such that $\phi(v_0, e_0) = (v_1, e_1)$.

Proof. Let $\Gamma$ be a 3-regular graph with orientation, and $p : \tilde{\Gamma} \to \Gamma$ be its universal covering map (figure [1.7]). Choose liftings of $(v_i, e_i)$ ($i = 0, 1$), $(\tilde{v}_1, \tilde{e}_1)$ in $\tilde{\Gamma}$. If $\phi_1$ and $\phi_2$ both map $(v_0, e_0)$ to $(v_1, e_1)$ and $\phi_1 \neq \phi_2$ then
they can be lifted to two different automorphisms \( \tilde{\phi}_1, \tilde{\phi}_2 : \tilde{\Gamma} \to \tilde{\Gamma} \), which map \((\tilde{v}_0, \tilde{e}_0)\) to \((\tilde{v}_1, \tilde{e}_1)\). This contradicts the previous lemma.

The situation is very similar to what happens in \( \mathbb{D} \): We recall that an automorphism of the disk, \( \mathbb{D} \) is determined by the image of one point and a rotation. This simple analogy shows:

**Proposition 1.7.5.** Any automorphism of \( \Gamma \) induces an automorphism of \( S^O(\Gamma) \).

### 1.7.1 Extension of Automorphisms

Let \( S \) be a complete hyperbolic Riemann surface of finite area, and \( i : S \hookrightarrow S_c \) be a conformal compactification (see section [L.S]).

**Proposition 1.7.6.** Any conformal automorphism of \( S \) can be extended to a conformal automorphism of \( S_c \).

**Proof.** Take a punctured-disk neighbourhood \( U \) of a cusp \( P \). \( i \circ \phi(U) \) is also a punctured disk. Thus, by Riemann’s theorem on removable singularities, \( P \) is a removable singularity of \( i \circ \phi \). \( \Box \)

### 1.8 Conformal Compactification

Let \( S \) be a Riemann surface, and let \( S_c \) be a compact Riemann surface. If \( \phi : S \to S_c \) is a conformal embedding with dense image, we say that \((S_c, \phi)\) is a conformal compactification of \( S \).

Examples:
Example 1: The Riemann sphere is a conformal compactification of the complex plane.

Example 2: Take $S = \mathbb{H}^2$. By Riemann’s Mapping Theorem $S$ is conformally equivalent to $\mathbb{C} - [1, \infty)$. Hence, the Riemann sphere is a conformal compactification of $\mathbb{H}^2$.

Example 3: Suppose $S$ is a complete hyperbolic Riemann surface of finite area. $S$ has a finite number of cusps (this follows from Gauss-Bonnet theorem or from Shimizu-Leutbecher theorem 2.2.2). We may build a conformal compactification of $S$ as follows: We find pairwise disjoint neighbourhoods of the cusps, such that each neighbourhood is conformally equivalent to a punctured disk, and we fill in the missing point in each punctured disk. The conformal structure on the filled-in disk is unique, by the Uniformization Theorem (see appendix C). Thus, we get a compact Riemann surface in which $S$ is conformally and densely embedded.

Example 4: Any Riemann surface of infinite genus does not admit a conformal compactification.

We will mainly be interested in example 3.

Proposition 1.8.1. In example 3 the conformal compactification is unique.

Proof. Denote by $i : S \hookrightarrow S_{c,1}$ the conformal compactification described above, and suppose that $\phi : S \hookrightarrow S_{c,2}$ is a second conformal compactification. If the cusps of $S$ are $\{p_i\}_{i=1}^n$, then we get a conformal map

$$\phi \circ i^{-1} : S_{c,1} - \{p_i\}_{i=1}^n \hookrightarrow S_{c,2}.$$ 

The same arguments as in the proof of proposition 1.7.6 show that $\phi \circ i^{-1}$ can be extended to the cusps of $S$. So, we obtain that $S_{c,1}$ is conformally equivalent to $S_{c,2}$. \qed

1.9 Examples

In order to give some examples, we first recall that any (anti-)automorphism of the graph $\Gamma$ induces an (anti-)automorphism of $S^0(\Gamma)$, and this, in turn
induces an (anti-)automorphism of $S^C(\Gamma)$ (see section 1.7). In other words, symmetries of the graph $\Gamma$ are reflected in symmetries of $S^C(\Gamma)$.

Uniformize now $S^C(\Gamma)$ by a metric of constant curvature. The geodesic triangulation, $\mathcal{T}$, of $S^O(\Gamma)$ induced from $\Gamma$ maps to some triangulation of $S^C(\Gamma)$ under the conformal embedding $i : S^O(\Gamma) \hookrightarrow S^C(\Gamma)$. We may isotope $i(\mathcal{T})$ to a geodesic triangulation of $S^C(\Gamma)$ keeping the vertices of $\mathcal{T}$ fixed. Let us call this geodesic triangulation $\mathcal{T}^C$. The automorphisms of $S^C(\Gamma)$ which are extensions of automorphisms of $S^O(\Gamma)$ stabilize $\mathcal{T}^C$. Thus, in order to determine the conformal structure of $S^C(\Gamma)$ we can follow the following steps:

1. We determine the topological type of $S^O(\Gamma)$ by counting the left-hand-turn paths in $\Gamma$ and using the genus formula (1.1).
2. We name the angles of the triangles in the geodesic triangulation, $\mathcal{T}^C$, of $S^C(\Gamma)$.
3. For each left-hand-turn path in $\Gamma$ we write the corresponding cusp-equation: The angles around the filled-in cusp amount to $2\pi$.
4. We find the automorphisms and anti-automorphisms of the graph with orientation.
5. We write the constraints on the angles implied by the symmetries found in step 4.

Remark: In general these steps are not enough to determine the conformal structure of $S^C(\Gamma)$, since we do not use the full pasting rule in the construction of $S^O(\Gamma)$, which says that tick-marks go to tick-marks. We have yet to find a good method to take these conditions into account.

We begin by the simplest 3-regular graph, viz. two vertices and three edges. It admits two possible non-isomorphic orientations:

**Example 1.** See figure [1.5]. We paste two triangles together to get a 3-punctured sphere. Then we compactify to get the Riemann sphere (the conformal structure on the sphere is unique).

**Example 2.** See figure [1.6]. We are still working with the graph of example [1], but now we flip the right triangle over. By the genus formula (1.1), we get a once-punctured torus. We want to calculate the conformal class of this torus: The angles around
3-punctured sphere. each cusp of length 2.

Figure 1.5: example no. 1

Once-punctured torus. cusp of length 6.

Figure 1.6: example no. 2
We have the automorphism of order 6:

\[ v_1 \mapsto v_2 \quad e_1 \mapsto e_2 \quad e_2 \mapsto e_3 \quad e_3 \mapsto e_1 \]

which implies (together with its iterates) the equalities

\[ x_1 = y_1 = z_1 = x_2 = y_2 = z_2. \]

Hence, we have that each angle equals \( \pi/3 \), and we obtain the equilateral torus.

In the next three examples we take \( \Gamma \) to be the 1-skeleton of the tetrahedron:

**Example 3.** See figure 1.7. We take the 1-skeleton of the tetrahedron, with orientation induced from its surface. The surface \( S^O(\Gamma) \) is a sphere with 4 cusps at the vertices of a tetrahedron. It is readily seen by the symmetries of the tetrahedron that those vertices may be taken to be at 0, 1, \( \omega, \omega^2 \), where \( \omega \) is a primitive cube root of unity. Obviously, \( S^O(\Gamma) \) is the Riemann sphere.
Example 4. See figure 1.8. We take the 1-skeleton of the tetrahedron with the orientation induced from its surface, and we flip orientation at one vertex. $S^O(\Gamma)$ is a 2-punctured torus. The cusp equations are:

$$x_1 + z_2 + x_4 = 2\pi$$
$$y_1 + z_4 + x_3 + x_2 + z_1 + y_3 + y_4 + y_2 + z_3 = 2\pi.$$

We have the automorphism of order 3 (rotation around $v_3$):

$$
\begin{align*}
 v_1 &\mapsto v_2 & e_1 &\mapsto e_3 \\
 v_2 &\mapsto v_4 & e_2 &\mapsto e_5 \\
 v_3 &\mapsto v_3 & e_3 &\mapsto e_6 \\
 v_4 &\mapsto v_1 & e_4 &\mapsto e_2 \\
 e_5 &\mapsto e_4 & e_6 &\mapsto e_1 
\end{align*}
$$

which implies:

$$
\begin{align*}
 x_1 &= z_2 = x_4, \\
 y_1 &= x_2 = y_4, \\
 z_1 &= y_2 = z_4, \\
 x_3 &= y_3 = z_3.
\end{align*}
$$
We have also an anti-automorphism of order 2:

\begin{align*}
v_1 &\mapsto v_1 & e_1 &\mapsto e_1 \\
v_2 &\mapsto v_4 & e_2 &\mapsto e_4 \\
v_3 &\mapsto v_3 & e_3 &\mapsto e_6 \\
v_4 &\mapsto v_2 & e_4 &\mapsto e_2 \\
e_5 &\mapsto e_5 \\
e_6 &\mapsto e_3
\end{align*}

which implies:

\begin{align*}
y_1 &= z_1, \\
x_2 &= z_4, \\
y_2 &= y_4, \\
z_2 &= x_4, \\
y_3 &= z_3.
\end{align*}

Combining all together, we have:

\begin{align*}
x_1 &= z_2 = x_4 = 2\pi/3, \\
y_1 &= x_2 = y_4 = z_1 = y_2 = z_4 = \alpha, \\
x_3 &= y_3 = z_3 = \beta, \\
6\alpha + 3\beta &= 2\pi.
\end{align*}

Since the triangulation on $S^C(\Gamma)$ is geodesic and our geometry is flat, we have the additional equations:

\begin{equation}
x_i + y_i + z_i = \pi \text{ for } 1 \leq i \leq 4.
\end{equation}

We obtain that triangle 3 is an equilateral triangle. This implies that $S^C(\Gamma)$ is an equilateral torus.

**Example 5.** See figure 1.9. We take the 1-skeleton of the tetrahedron with the orientation induced from its surface, and we flip orientation at two vertices. $S^O(\Gamma)$ is a 2-punctured torus. The cusp equations are:

\begin{align*}
x_1 + z_4 + x_3 + x_2 &= 2\pi, \\
y_1 + z_2 + x_4 + z_1 + y_3 + y_4 + y_2 + z_3 &= 2\pi.
\end{align*}
a tetrahedron with two vertices flipped.

Figure 1.9: example no. 5

We have several anti-automorphisms of order 2:

\[
\begin{align*}
  v_1 &\mapsto v_2 & e_1 &\mapsto e_5 \\
  v_2 &\mapsto v_1 & e_2 &\mapsto e_2 \\
  v_3 &\mapsto v_4 & e_3 &\mapsto e_4 \\
  v_4 &\mapsto v_3 & e_4 &\mapsto e_3 \\
  e_5 &\mapsto e_1 \\
  e_6 &\mapsto e_6
\end{align*}
\]

which implies:

\[
\begin{align*}
  x_1 &= x_2, \\
  y_1 &= z_2, \\
  z_1 &= y_2, \\
  x_3 &= z_4, \\
  y_3 &= y_4, \\
  z_3 &= x_4.
\end{align*}
\]

We have a second reflection:

\[
\begin{align*}
  v_1 &\mapsto v_4 & e_1 &\mapsto e_5 \\
  v_2 &\mapsto v_3 & e_2 &\mapsto e_6 \\
  v_3 &\mapsto v_2 & e_3 &\mapsto e_3 \\
  v_4 &\mapsto v_1 & e_4 &\mapsto e_4 \\
  e_5 &\mapsto e_1 \\
  e_6 &\mapsto e_2
\end{align*}
\]
which implies:

\[ x_1 = z_4, \]
\[ y_1 = y_4, \]
\[ z_1 = x_4, \]
\[ x_2 = x_3, \]
\[ y_2 = z_3, \]
\[ y_3 = z_2. \]

... and a third reflection:

\[ v_1 \mapsto v_3 \quad e_1 \mapsto e_1 \]
\[ v_2 \mapsto v_2 \quad e_2 \mapsto e_3 \]
\[ v_3 \mapsto v_1 \quad e_3 \mapsto e_2 \]
\[ v_4 \mapsto v_4 \quad e_4 \mapsto e_5 \]
\[ e_5 \mapsto e_5 \]
\[ e_6 \mapsto e_4 \]

which implies:

\[ y_1 = z_3, \]
\[ z_1 = y_3, \]
\[ x_1 = x_3. \]

combining all together, we have:

\[ x_1 = x_2 = x_3 = z_4 = \frac{\pi}{2}, \]
\[ y_1 = z_1 = y_2 = z_2 = y_3 = z_3 = x_4 = y_4 = \frac{\pi}{4}. \]

Hence, we have a square torus.

Next, we take \( \Gamma \) to be the 1-skeleton of a cube:

**Example 6.** See figure [1.10]. We take the 1-skeleton of the cube, with orientation induced from its surface. The surface \( S^O(\Gamma) \) is a sphere with 6 cusps at the vertices of an octahedron. It is readily seen by a stereographic projection that the vertices can be taken to be \( 0, 1, i, \infty, -1, -i \). Then, we can map them by a Möbius transformation to \( 1, \omega, \omega^2, -R, -R\omega, -R\omega^2 \) respectively, where \( \omega \) is a primitive cube root of unity and \( R = 2 + \sqrt{3} \). (If we apply another Möbius transformation: \( z \mapsto -z/R \), we get that we could take the vertices to be exactly at the vertices and tick-marks of the standard ideal triangle). \( S^C(\Gamma) \) is the Riemann sphere.
Example 7. We take the 1-skeleton of the cube with orientation induced from its surface, and we flip the orientation at one vertex, say $A$. We calculate $S^C(\Gamma)$ according to ideas we learned from Curt McMullen.

Set $X = S^O(\Gamma)$. $X$ is a torus with four cusps. One of the cusps is surrounded by twelve triangles and the other three are surrounded by four each. We have a $\mathbb{Z}/3\mathbb{Z}$ action on $S^O(\Gamma)$, coming from the order-3 rotation of the cube along its diagonal through $A$. The quotient, $Y$, is an orbifold with two cusps and two ramification points of order 3: One cusp is the image of the three cusps of length 4, the other cusp is the image of the cusp of length 12, and the two ramification points come from the centers of the triangles pasted on the vertices of the rotation axis.

The genus of $Y$ may be calculated by the Riemann–Hurwitz relation:

$$
\chi(X) = 3\chi(Y) - 4,
$$

where the relevant Euler characteristics are:

$$
\chi(X) = 2 - 2g_X - 4,
\chi(Y) = 2 - 2g_Y - 2.
$$

After substituting we get that $g_Y = 0$.

The compactification of $X$, $X^C$, is obtained as a 3-regular covering of the sphere, branched over three points, each of
them, with ramification index 3, which may be taken to be 0, 1 and $\infty$. Hence, a possible equation for $X^C$ in $\mathbb{CP}^2$ is:

$$x^3 = zy(y - z).$$

**Example 8.** (Due to Curt McMullen). See figure 1.11. We take the 1-skeleton of the cube, with orientation at the vertices induced from the surface of the cube. Then we flip orientation at 2 opposite vertices, say $A$ & $B$. What we get is a surface, $X$, of genus 2, with 2 cusps, and 12 ideal triangles around each cusp.

Consider the automorphism of $X$ of order 3 obtained from rotating the cube along the diagonal $AB$. When we mod out $X$ by the action of $\mathbb{Z}/3\mathbb{Z}$, we get an orbifold, $Y$, with two cusps and two points of ramification index 3: The points of
order 3 come from the centers of the triangles at $A$ and $B$. The genus of $Y$ may be calculated by the Riemann-Hurwitz relation:

\[
\begin{align*}
\chi(X) & = 3\chi(Y) - 4, \\
\chi(X) & = 2 - 2g_X - 2 = -4, \\
\chi(Y) & = 2 - 2g_Y - 2,
\end{align*}
\]

from which we get $g_Y = 0$. The triangulation of $X$ descends to a triangulation of $Y$ composed of four cells around each cusp.

Normalize $Y$ so that the ramification points of order 3 are at $y = 0$, and $y = \infty$, and the cusps are at $y = 1$, and $y = y_0$. We now take the 3-regular cover, $Z$, of $Y$ with ramification of order 3 at $y = 0$ and $y = \infty$. $Z$ is a surface (not orbifold) of genus 0, with 6 cusps at the cube roots of 1 and $y_0$. One may verify this by again applying the Riemann-Hurwitz relation:

\[
\begin{align*}
\chi(Z) & = 3\chi(Y) - 4, \\
\chi(Y) & = 2 - 2g_Y - 2 = 0, \\
\chi(Z) & = 2 - 2g_Z - 6.
\end{align*}
\]

The covering map $Z \to Y$ is given by $y = z^3$, and the cellulation of $Y$ lifts to a triangulation of $Z$, with four triangles surrounding each cusp. Thus, it is readily seen that $Z$ can be constructed from a 3-regular graph, $\Gamma$, which corresponds to this triangulation. $Z$ is of genus 0 and has six cusps of length 4 each. By formula (1.1), $\Gamma$ has eight vertices. The only graph with this combinatorics is the 1-skeleton of the cube with orientation induced from the surface of the cube. Therefore, $Z$ is a sphere with six cusps at the vertices of an octahedron.

In example 6 it was shown that the vertices of an octahedron may be taken to be $\{1, \omega, \omega^2, -R, -\omega R, -\omega^2 R\}$, where $R = 2 + \sqrt{3}$ (or $R = 2 - \sqrt{3}$), and $\omega$ is a primitive cube root of unity. From here we conclude that $y_0 = -R^3 = -26$ –
Hence, the conformal compactification of $X$, $X^C$, is the surface obtained as a degree 3-regular cover of $S^2$ with ramification points of order 3 at $\{1, -26 - 15\sqrt{3}, 0, \infty\}$. An equation for $X^C$ in $\mathbb{CP}^2$ may be:

$$x^3 = \frac{zy(y-z)}{y+26z+15\sqrt{3}z}.$$

**Example 9.** ([Br3, BFK]). Set $\Gamma = \text{PSL}(2, \mathbb{Z})$, and let $\rho$ be a finite index torsion free subgroup of $\Gamma$. We have:

**Theorem 1.9.1.** $\mathbb{H}^2/\rho$ can be constructed out of a 3-regular graph.

**Proof.** The classical fundamental domain for $\Gamma$ is given by (see figure 1.12):

$$F = \{z \in \mathbb{H}^2 : |\Re(z)| < 1/2, |z| > 1\}.$$

Here, we take as a fundamental domain for $\Gamma$ (see figure 1.13):

$$G = \{z \in \mathbb{H}^2 : 0 < \Re(z) < 1, |z| > 1, |z-1| > 1\}.$$

Observe that three copies of $G$ around $\omega + 1$ ($\omega = e^{2\pi i/3}$) fit together to give the marked ideal triangle. The equivalence between three such copies is given by an elliptic element of order 3 in $\Gamma$.

A fundamental domain for $\rho$ is composed of copies of $G$, and since $\rho$ is torsion free, the three copies above are not equivalent under $\rho$, and can all be included in a fundamental domain for $\rho$. We have shown:

**Lemma 1.9.2.** A fundamental domain for $\rho$ can be chosen such that it is composed of copies of the marked ideal triangle.

Let $J$ be such a fundamental domain for $\rho$, and mark in each marked ideal triangle the three copies of $G$. Take $\Delta$ to be the three-regular graph which is composed of the finite edges of the copies of $G$, and its vertices are the centers of the marked ideal triangles. The orientation on $\Delta$ is induced from the surface $\mathbb{H}^2/\rho$. It is readily seen that $\mathbb{H}^2/\rho \cong S^O(\Delta)$. □

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Figure 1.12: Fundamental Domain for $PSL(2, \mathbb{Z})$
Figure 1.13: 3 copies of $G$ give the marked ideal triangle
A well known family of finite-index torsion free normal subgroups of $\Gamma$ are the congruence subgroups (see [BFK]):

$$\Gamma(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{k} \right\}.$$ 

Observe that $\Gamma/\Gamma(k) \cong PSL(2, \mathbb{Z}/k\mathbb{Z})$. As we have seen $S(k) = H^2/\Gamma(k)$ all arise from graphs. These graphs, called dual Platonic graphs in [BFK], have a simple algebraic description, and generalize the classical Platonic solids.

Denote the conformal compactification of $S(k)$ by $P(k)$. In [Br3] these surfaces were called the Platonic Surfaces.

### 1.10 Belyi Surfaces

In this section we show that we know how to construct Belyi surfaces out of graphs. By Belyi’s theorem it will follow that the set of constructed surfaces is dense in the moduli space.

Let $S$ be a compact Riemann surface. It is well known that there exists a non-constant meromorphic function on $S$, $\phi : S \to S^2$.

**Definition 1.10.1.** If there exists a branched covering $\phi : S \to S^2$, such that $\phi$ is branched over at most three points, then $S$ is called a Belyi surface.

**Theorem 1.10.2 ([Be]).** $S$ is a Belyi surface if and only if as a curve in $CP^2$ its minimal polynomial lies over some number field.

**Corollary 1.10.3.** For every integer $g \geq 0$ Belyi surfaces are dense in the moduli space of compact Riemann surfaces of genus $g$.

We can characterize Belyi surfaces as follows:

**Lemma 1.10.4.** $S$ is a Belyi surface if and only if we can find finitely many points on $S$, $\{p_1, \ldots, p_k\}$, such that $S - \{p_1, \ldots, p_k\}$ is isomorphic to $H^2/\rho$, where $\rho$ is a finite index torsion free subgroup of $PSL(2, \mathbb{Z})$. 

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Proof. We begin by some basic observations: Let $\Gamma = PSL(2, \mathbb{Z})$, and

$$\Gamma(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{k} \right\}.$$  

$\Gamma(2)$ has fundamental domain $F$, which is composed of 2 ideal triangles glued along a common edge (figure 1.14):

$$F = \{ z \in \mathbb{H}^2 : 0 < \Re(z) < 2, |z - 1/2| > \frac{1}{2}, |z - 3/2| > \frac{1}{2} \}.$$  

From this, we see that the 3-punctured sphere (the punctures may be taken to be at 0, 1, $\infty$) may be realized as $\mathbb{H}^2 / \Gamma(2)$. Moreover, as we observed in example 3, each ideal triangle is composed of three copies of the fundamental domain of $\Gamma$. Therefore, the 3-punctured sphere is a degree-6 branched covering of $\mathbb{H}^2 / \Gamma$.

Let $S$ be a compact Riemann surface. If $S$ is a Belyi surface, then there exists a finite number of points on $S$, $\{p_1, \ldots, p_k\}$ such that $S - \{p_1, \ldots, p_k\}$ is a regular smooth finite degree covering of $\mathbb{H}^2 / \Gamma(2)$. Therefore, $S - \{p_1, \ldots, p_k\}$ may be realized as $\mathbb{H}^2 / \rho$, where $\rho$ is a finite-index subgroup of $\Gamma(2)$, and since $\Gamma(2)$ is a finite-index torsion-free (see BFK) subgroup of $\Gamma$, we get that $\rho$ is a finite-index torsion-free subgroup of $\Gamma$.

Conversely, if we can find finitely many points $\{p_1, \ldots, p_k\}$ on $S$, such that $S - \{p_1, \ldots, p_k\}$ is isomorphic to $\mathbb{H}^2 / \rho$, where $\rho$ is a finite index torsion free
subgroup of $\Gamma$, then $S - \{p_1, \ldots, p_k\}$ is a finite-degree branched covering of $\mathbb{H}^2/\Gamma$, which is a sphere with one cusp and 2 ramification points. Therefore, if we remove the 2 ramification points and their pre-images, we get that $S - \{p_1, \ldots, p_k, \ldots, p_n\}$ is a regular smooth finite-degree covering of the 3-punctured sphere. So, by definition, $S$ is a Belyi surface.

Using the last lemma we can now prove:

**Theorem 1.10.5.** A Riemann surface can be constructed out of a 3-regular graph if and only if it is a Belyi surface.

*Proof.* Lemma 1.10.4 and theorem 1.9.1 show that any Belyi surface can be constructed out of some 3-regular graph with orientation.

Conversely, take a 3-regular graph with orientation, $\Delta$. A fundamental domain, $F$, for $S^O(\Delta)$ is composed of copies of the ideal triangle. By decomposing each ideal triangle into three copies of the fundamental domain for $PSL(2, \mathbb{Z})$, we see that $S^O(\Delta)$ may be realized as $\mathbb{H}^2/\rho$, where $\rho$ is a finite-index torsion free subgroup of $PSL(2, \mathbb{Z})$. $\square$
Chapter 2

Geometric Relations between $S^O(\Gamma)$ and $S^C(\Gamma)$.

2.1 Introduction

Let $\Gamma$ be a 3-regular graph with orientation. In chapter 1 we introduced a way to construct a finite area Riemann surface, $S^O(\Gamma)$, out of $\Gamma$. We denoted its conformal compactification by $S^C(\Gamma)$.

According to the Uniformization Theorem (see appendix C), $S^C(\Gamma)$ admits a complete Riemannian metric of constant curvature (which is compatible with the given conformal structure). In general, this metric might be very different from the metric on $S^O(\Gamma)$. For instance, the 3-punctured sphere admits a hyperbolic structure while its compactification, the Riemann sphere, doesn’t admit a hyperbolic metric at all. The main results of this chapter are theorems 2.3.1 and 2.10.1. These results give conditions under which we do have a hyperbolic metric on $S^C(\Gamma)$ which is close in some sense to the metric on $S^O(\Gamma)$. These conditions are related to the size of the cusps of $S^O(\Gamma)$.

As a last remark, we would like to suggest the reader to bear in mind the Ahlfors–Schwarz Lemma (see appendix A) while reading this chapter.

2.2 Large Cusps

The number of cusps on a complete hyperbolic Riemann surface of finite area, $S$, is finite. This follows from Gauss–Bonnet theorem, which reduces
in that situation to:

$$-\text{area}(S) = 2\pi \chi(S) = 2\pi(2 - 2g - N_c),$$

where $g$ is the genus of $S$, and $N_c$ is the number of cusps. Alternatively, it follows from theorem 2.2.2.

Each cusp of $S$ has a horocyclic neighbourhood, i.e a neighbourhood isometric to $B_r = \{z \in \mathbb{D}^* : |z| < r\}$ for some $0 < r < 1$. The area of $B_r$, $\text{area}(B_r)$, will be a measure for the size of the cusp:

**Definition 2.2.1.** We say that the cusps of $S$ are bigger than $c$ if there exist pairwise disjoint horocyclic neighbourhoods of the cusps $U_1, \ldots, U_l$ such that $\forall i \text{ area}(U_i) > c$.

A beautiful classical theorem is the following:

**Theorem 2.2.2 (Shimizu-Leutbecher).** Let $S$ be a complete hyperbolic Riemann surface. Then, the cusps of $S$ are $\geq 1$.

The classical proof of Shimizu and Leutbecher uses an iteration argument and can be found in [K]. We bring here another proof, based on the decomposition of $S$ into ideal triangles. A different proof in a similar spirit can be found in [Bu].

**Proof.** We can find a locally finite triangulation on $S$, such that each vertex of the triangulation is a cusp of $S$. In each ideal triangle we can mark the “standard horocyclic segments” (see section 1.1 and figure 1.2). Fix one cusp, $P$. Look at the ideal triangles which surround $P$. We pick the triangle, $T$, with the “closest” opposite horocycle to the cusp (in the universal cover, we can describe the situation easily - see figure 1.1). We continue this horocycle segment to close a horocyclic neighbourhood around $P$. We repeat this for all cusps of $S$. It is clear that the horocyclic neighbourhoods constructed in this way are pairwise disjoint, and since each horocycle segment is of length 1, the horocycles are of length at least 1. \hfill \Box

### 2.3 Controlling $S^C(\Gamma)$

Let $S$ be a complete hyperbolic Riemann surface of finite area. Let $S_c$ be its conformal compactification (see section 1.8). The following theorem roughly says that if the cusps of $S$ are large, we have some control on the geometry of $S_c$. 

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**Theorem 2.3.1.** If the cusps of $S$ are bigger than $2\pi$, then $S_c$ admits a conformal Riemannian metric of negative curvature which coincides with the metric on $S$ outside horocyclic cusp neighbourhoods, $\{U_i\}_{i=1}^l$, of area $> 2\pi$.

In order to prove this theorem, we begin with some preliminary calculations in the next two sections.

### 2.4 The Complete Hyperbolic Punctured Disk

Let $D^*$ denote punctured unit disk with the Euclidean conformal structure on it. The map from $\mathbb{H}^2$ onto $D^*$ defined by $z \mapsto e^{2\pi i z}$ realizes $D^*$ as $\mathbb{H}^2/\{z \sim z + 1\}$. From this it is easy to give an explicit expression for the complete hyperbolic metric on $D^*$:

**Lemma 2.4.1.** The complete hyperbolic metric on $D^*$ is given by

$$ds^2_{D^*} = \left(\frac{1}{r \log r}\right)^2 |dz|^2,$$

where $r = |z|$.
Proof. Set \( w = e^{2\pi i z} \). Then,
\[
\begin{align*}
ds^2_{D^*} &= 1/\Im(z)^2 |dz|^2 \\
&= 1/(-\log |w|/2\pi)^2 \left| \frac{dw}{2\pi i w} \right|^2 \\
&= \left( -\frac{1}{|w| \log |w|} \right)^2 |dw|^2.
\end{align*}
\]

Denote by \( B_r \) the horocyclic neighbourhood \( \{ z \in \mathbb{D}^* : |z| < r \} \).

**Lemma 2.4.2.** \( \text{area}(B_r) = -2\pi / \log r \).

**Proof.** By lemma 2.4.1
\[
\text{area}(B_r) = \int_{0}^{2\pi} \int_{0}^{r} -\frac{t}{(t \log t)^2} \, dt \, d\theta = -2\pi \left. \frac{1}{\log t} \right|_{0}^{r} = -2\pi / \log r.
\]

We also have the following interpretation of the total geodesic curvature of a closed curve:

**Lemma 2.4.3.** For a simple closed curve, \( \gamma \), which bounds a domain \( V \) which includes 0, we have:
\[
\oint_{\gamma} \kappa_g \, d(\text{length}) = \text{area}(V).
\]

**Proof.** The Euler characteristic of \( V \) is 0. So, by the Gauss–Bonnet theorem we have:
\[
\oint_{\gamma} \kappa_g \, d(\text{length}) + \iint_{V} \kappa \, d(\text{area}) = 2\pi \chi(V) = 0.
\]
But here \( \kappa = -1 \), and the lemma follows.

Set
\[
\begin{align*}
\lambda_{\mathbb{D}^*} &= -\frac{1}{r \log r}, \\
u_{\mathbb{D}^*} &= \log \lambda_{\mathbb{D}^*} = \log \left( \frac{1}{r \log r} \right).
\end{align*}
\]
Then we have (see figure 2.2):
Lemma 2.4.4. \( u_{D^*} \) satisfies:

(i) \( \lim_{r \to 0} u_{D^*}(r) = \infty \).

(ii) \( \lim_{r \to 1} u_{D^*}(r) = \infty \).

(iii) \( u'_{D^*}(r) < 0 \) for \( r < \frac{1}{\epsilon} \).

(iv) \( u'_{D^*}(r) > 0 \) for \( r > \frac{1}{\epsilon} \).

(v) \( \forall 0 < r < 1, \ u''_{D^*}(r) > 0 \).

Proof. Easy. \qed

2.5 The Curvature of Metrics on \( \mathbb{D}^* \)

Let \( ds^2 \) be a Riemannian metric on \( \mathbb{D}^* \), which is compatible with the conformal structure on \( \mathbb{D}^* \). \( ds^2 \) is given by

\[
    ds^2 = \lambda(x, y)^2(dx^2 + dy^2),
\]

where \( \lambda \) is a positive function.
Lemma 2.5.1. The curvature of $d\!s^2$ is given by
\[
\kappa = -\frac{\Delta \log \lambda}{\lambda^2}. \tag{2.4}
\]

Proof. We calculate according to the following formula for the Gaussian curvature:
\[
\kappa = \frac{\left< \mathbf{K}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right>}{\left< \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right> \left< \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right>}, \tag{2.5}
\]
where
\[
\mathbf{K}(X, Y) = \nabla_X \nabla_Y X - \nabla_X \nabla_Y Y - \nabla_{[X,Y]} \tag{2.6}
\]
and $\nabla$ is the Levi-Civita connection. The Christoffel symbols for the Levi-Civita connection are given by
\[
\Gamma^k_{ij} = \frac{1}{2}g^{km} (g_{im,j} + g_{mj,i} - g_{ji,m}), \tag{2.7}
\]
where $g_{ij}$ is the Riemannian metric tensor. Substituting the metric tensor $d\!s^2$ in this formula, we get
\[
\begin{align*}
\Gamma^x_{xx} &= \frac{\partial \log \lambda}{\partial x}, \quad &\Gamma^x_{yx} &= -\frac{\partial \log \lambda}{\partial y}, \\
\Gamma^x_{xy} &= \Gamma^y_{yx} &= \frac{\partial \log \lambda}{\partial y}, \\
\Gamma^y_{yy} &= -\frac{\partial \log \lambda}{\partial x}, \quad &\Gamma^y_{yy} &= \frac{\partial \log \lambda}{\partial y} \tag{2.8}
\end{align*}
\]
We need also $\nabla \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} = \Gamma^k_{ij} \frac{\partial}{\partial x_k}$ to calculate:
\[
\begin{align*}
\nabla \frac{\partial}{\partial x} \frac{\partial}{\partial y} &= \frac{\partial \log \lambda}{\partial x} \frac{\partial}{\partial x} - \frac{\partial \log \lambda}{\partial y} \frac{\partial}{\partial y}, \\
\nabla \frac{\partial}{\partial x} \frac{\partial}{\partial y} &= \nabla \frac{\partial}{\partial y} \frac{\partial}{\partial x} = \frac{\partial \log \lambda}{\partial y} \frac{\partial}{\partial x} + \frac{\partial \log \lambda}{\partial x} \frac{\partial}{\partial y}, \tag{2.9}
\end{align*}
\]
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\[ \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle \langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \rangle = \lambda^4 \]
\[\left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right] = 0. \] (2.10)

When we substitute equations (2.9) – (2.10) into the curvature formula (2.5), we get the desired result (2.4). \(\square\)

In the special case of a radial metric on \(D^*\), we have

**Corollary 2.5.2.** The curvature of

\[ ds^2 = \lambda^2(r)(dr^2 + r^2d\theta^2) \]

is given by

\[ \kappa = - \frac{(\log \lambda)'' + \frac{1}{r}(\log \lambda)'}{\lambda^2}. \]

**Proof.** The Laplacian in polar coordinates is given by

\[ \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}. \] \(\square\)

### 2.6 A Key Lemma

Recall that \(B_r = \{ z \in D^* : |z| < r \}\).

**Lemma 2.6.1.** If \(\text{area}(B_{r_0}) > 2\pi\), then there exists a Riemannian metric on \(ds^2\) on \(D^*\), such that

i) \(ds^2\) is conformally equivalent to \(ds^2_{D^*}\).

ii) \(ds^2\) coincides with \(ds^2_{D^*}\) outside a circle of Euclidean radius \(r_0\).

iii) \(ds^2\) extends smoothly to a metric on \(D\).

iv) \(ds^2\) has negative curvature.
Proof. By lemma 2.4.2, \( \text{area}(B_{r_0}) > 2\pi \) implies \( r_0 > \frac{1}{\sqrt{e}} \). By lemma 2.4.4 \( u'_\ast(r_0) > 0 \) and \( u''_\ast(r_0) > 0 \). Therefore, there exists a strictly convex monotonically increasing \( C^2 \)-function \( u(r) \) on \([0, 1)\) such that

\[
\forall r > r_0, \quad u(r) = u_\ast(r),
\]

\[
u'(0) = 0.
\]

We can construct \( u(r) \) as follows: First, define a positive continuous function \( w(r) \) such that,

\[
\int_0^{r_0} w(r) \, dr = u'_\ast(r_0),
\]

\[
\forall r \geq r_0, \quad w(r) = u''_\ast(r).
\]

Next, define the function \( v(r) \) by

\[
v(r) = \int_0^r w(s) \, ds.
\]

Observe that \( v(r) \) is a \( C^1 \)-function which coincides with \( u'_\ast(r) \) for \( r \geq r_0 \). Finally, define \( u(r) \) by

\[
u(r) = u_\ast(r_0) + \int_{r_0}^r v(s) \, ds.
\]

\( u \) is a \( C^2 \)-function which satisfies:

\[
\forall r \geq r_0, \quad u(r) = u_\ast(r),
\]

\[
u'(0) = v(0) = 0,
\]

\[
\forall r > 0, \quad u'(r) > 0 \quad (2.17)
\]

\[
u'' = w > 0 \Rightarrow u \text{ is strictly convex}. \quad (2.18)
\]

Set \( \lambda(r) = e^{u(r)} \), and \( ds^2 = \lambda^2(r)(dx^2 + dy^2) \). Then, we have:

a) \( \lambda'(0) = 0 \Rightarrow ds^2 \) is smooth at 0.

b) \( ds^2 \) coincides with \( ds^2_\ast \) for \( r > r_0 \).

c) \( \kappa(ds^2) < 0 \) by corollary 2.5.2.

\[\Box\]
2.7 Proof of Theorem 2.3.1

The key lemma 2.6.1 is the heart of the proof of theorem 2.3.1.

Proof. Let \( \{ U_i \}_{i=1}^l \) be pairwise disjoint horocyclic neighbourhoods of the cusps, \( \{ P_i \}_{i=1}^l \), of areas \( > 2\pi \). According to the key lemma 2.6.1, for each \( 1 \leq i \leq l \) we can alter the Riemannian metric in \( U_i \), without affecting the metric outside of this neighbourhood. Doing so, we are left with a Riemannian metric of negative curvature, which extends smoothly across the cusps. In other words, \( S_c \) admits a Riemannian metric of negative curvature, which coincides with the metric on \( S \) outside the horocyclic cusp-neighbourhoods, \( U_i \)'s.

2.8 A Discussion of Theorem 2.3.1

The best constant in theorem 2.3.1 is \( 2\pi \).

One way to see this is by applying the maximum principle for subharmonic functions: Suppose we have a metric of negative curvature in \( \mathbb{D} \) of the form

\[
ds^2 = e^{2u(x,y)}(dx^2 + dy^2),\]

where \( u \) is smooth and coincides with \( u_{\mathbb{D}^*} \) outside \( \mathcal{B}_{r_0} \). By formula (2.4) the negativity of the curvature is equivalent to \( \Delta u > 0 \). Hence, \( u \) is a subharmonic function. By the maximum principle and Hopf’s lemma the maximum of \( u \) in \( \overline{B_{r_0}} \) is attained on the boundary and \( \frac{\partial u}{\partial r} > 0 \) on the boundary. But since \( \frac{\partial u}{\partial r} \) on the boundary = \( u_{\mathbb{D}^*}(r_0) \), it follows by lemma 2.4.4 that \( r_0 > \frac{1}{e} \).

Finally, by lemma 2.4.2 the last inequality is equivalent to area(\( \mathcal{B}_{r_0} \)) > 2\pi.

A different way to see that \( 2\pi \) is a best constant is by applying the Gauss–Bonnet theorem to the region \( \mathcal{B}_{r_0} \):

\[
\int \int_{\mathcal{B}_{r_0}} \kappa \, d(\text{area}) + \int_{\partial \mathcal{B}_{r_0}} \kappa_g \, d(\text{length}) = 2\pi, \quad (2.19)
\]

where \( \kappa \) is the curvature of the metric, and \( \kappa_g \) is the geodesic curvature of \( \partial \mathcal{B}_{r_0} \). Since \( \kappa < 0 \) we obtain a necessary condition on \( \partial \mathcal{B}_{r_0} \):

\[
\int_{\partial \mathcal{B}_{r_0}} \kappa_g \, d(\text{length}) > 2\pi. \quad (2.20)
\]

By lemma 2.4.3 this is equivalent to

\[
\text{area}(\mathcal{B}_{r_0}) > 2\pi.
\]
2.9 An Extension Problem

The necessary condition (2.20) raises a natural question: Suppose we have a simple closed smooth curve $\gamma$ in $\mathbb{D}^*$, which bounds a domain $V$ which contains 0, such that

$$\oint_{\gamma} \kappa_g \, d(\text{length}) > 2\pi.$$  \hfill (2.21)

Is it sufficient in order to extend the metric outside of $V$ to a Riemannian metric with negative curvature in $V$? A more general form of this question was raised already by M. Gromov in [G], pp. 109–110.

Since the metric outside of $V$ is conformally equivalent to the Euclidean metric, we may write:

$$d_{\text{out}}^2 = \lambda_{\text{out}}^2(x, y)(dx^2 + dy^2),$$

for some positive function $\lambda_{\text{out}}$ defined outside of $V$. We prove:

**Theorem 2.9.1.** If the maximum of $\lambda_{\text{out}}$ on $\gamma$ is attained at a point $z_{\text{max}}$, where the normal derivative $\frac{\partial \lambda_{\text{out}}}{\partial \vec{n}}(z_{\text{max}}) < 0$, then $d_{\text{out}}^2$ cannot be extended to a conformal metric of negative curvature in $V$.

**Proof.** Suppose $d_{\text{out}}^2$ is a Riemannian metric of negative curvature on $\mathbb{D}$, which extends $d_{\text{out}}^2$ and is given by

$$d^2 = \lambda^2(x, y)(dx^2 + dy^2),$$

where $\lambda$ is a smooth positive function defined on $\mathbb{D}$ and extends $\lambda_{\text{out}}$. By formula 2.4

$$\kappa(d^2) < 0 \Rightarrow \Delta \lambda > 0.$$ 

So, $\lambda$ is a subharmonic function. By the maximum principle, the maximum of $\lambda$ in $\bar{V}$ is attained on $\partial V = \gamma$. Hence, it is attained at $z_{\text{max}}$. But, this is not possible, since the normal derivative there is negative. \hfill $\square$

The next theorem shows that there exists curves as in theorem 2.9.1:

**Theorem 2.9.2.** There exists a curve $\gamma$ in $\mathbb{D}^*$, bounding a domain $V$ which includes 0 such that

1) $\oint_{\gamma} \kappa_g \, d(\text{length}) > 2\pi$. 

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\( \gamma \) has the property of theorem 2.9.1 with respect to the complete hyperbolic metric on \( \mathbb{D}^* \).

Proof. We can take \( \gamma \) to be a horocycle with a slit (see figure 2.3): Smooth the curve whose image is given by

\[
\{ R_1 e^{it} : |t| < \theta \} \cup \{ R_2 e^{it} : \theta < |t| < \pi \} \cup \{ re^{it} : R_1 \leq r \leq R_2, |t| = \theta \}.
\]

We choose \( R_2 \) to satisfy \( \frac{1}{e} < R_2 < 1 \). By lemma 2.4.4, we can find \( 0 < R_1 < \frac{1}{e} \) such that \( u_{\mathbb{D}^*}(R_1) > u_{\mathbb{D}^*}(R_2) \). Since by lemma 2.4.2 \( \text{area}(\mathcal{B}_{R_2}) > 2\pi \), we can find \( 0 < \theta < \pi \) such that the area bounded by \( \gamma \) in the complete hyperbolic metric on \( \mathbb{D}^* > 2\pi \). By lemma 2.4.3, \( \gamma \) satisfies ii).

By construction of \( \gamma \) and by lemma 2.4.4, we see that the maximum of \( u_{\mathbb{D}^*} = \log \lambda_{\mathbb{D}^*} \) on \( \gamma \) is attained at a point where the normal derivative is negative. \( \square \)

If we work a little harder, we can show:

**Theorem 2.9.3.** \( \gamma \) in theorem 2.9.2 can be taken to be convex (i.e. \( \kappa_g > 0 \)).

Proof. Take \( \gamma \) to be composed of a horocycle and a geodesic circular arc as in figure 2.4. The total geodesic curvature (taking singular points into account) of \( \gamma \) is given by the area above it (lemma 2.4.3):

\[
\oint_{\gamma} \kappa_g \, d(\text{length}) = \frac{2x}{y} + 2\theta.
\]

We would like to have

\[
\frac{2x}{y} + 2\theta > 2\pi, \tag{2.22}
\]
Figure 2.4: The Extension Problem

and recalling that the map which realizes $\mathbb{D}^*$ as $\mathbb{H}^2/\{z \to z + 1\}$ is given by $z \mapsto e^{2\pi iz}$, we would also like to have

$$u_{\mathbb{D}^*}(e^{-2\pi R(y,\theta)}) > u_{\mathbb{D}^*}(e^{-2\pi y}).$$

(2.23)

We have

$$x(y, \theta) = \frac{1 - 2y \tan \theta}{2},$$

(2.24)

$$R(y, \theta) = \frac{y}{\cos \theta}.$$

(2.25)

Combining (2.22)–(2.25) together, and noting that $u_{\mathbb{D}^*}$ is given by (2.2), we see that we should find $y > 0, 0 < \theta < \pi/2$ such that

$$\frac{1}{y} - 2 \tan \theta + 2\theta > 2\pi,$$

(2.26)

$$\frac{2\pi y}{\cos \theta} - \log \frac{2\pi y}{\cos \theta} > 2\pi y - \log 2\pi y.$$

(2.27)

We may first find a pair $(y_0, \theta_0)$ with equality in (2.26), and then decrease $\theta_0$. So extracting $y$ from (2.26), we obtain that we should find $0 < \theta < \pi/2$ such that

$$\frac{\pi}{\pi + \tan \theta - \theta} > \frac{-\cos \theta \log \cos \theta}{1 - \cos \theta}.$$

(2.28)
The existence of such a $\theta$ can be shown by elementary calculus. 

Ultimately, we can show 

**Theorem 2.9.4.** There exists a convex curve $\gamma$ in $\mathbb{D}^*$, such that 

i) $\oint_{\gamma} \kappa_g \, d(\text{length}) > 2\pi$.

ii) no metric of negative curvature, conformal or nonconformal to the Euclidean metric, can extend the outside metric into the domain bounded by $\gamma$.

**Proof.** We construct $\gamma$ as follows (see figure 2.5): Take any geodesic $\delta$ in $\mathbb{H}^2$, whose end points are in $\mathbb{R}$. Take a symmetric segment $\delta'$ about the “center” of $\delta$ of length, say, $3\pi$. Now, extend the end points of $\delta'$, by horocyclic segments of length $\pi$ each. Then, identify $\mathbb{D}^*$ with $\mathbb{H}^2/\{z \mapsto z + C\}$, where $C$ is the Euclidean distance between the endpoints of the horocyclic segments.

We get a convex curve $\gamma$ in $\mathbb{D}^*$ whose total geodesic curvature (taking singular points into account) $> 2\pi$, by lemma 2.4.3. Suppose we have a Riemannian metric $ds^2$ of negative curvature in the domain bounded by $\gamma$, 

Figure 2.5: No extension
V (\infty \in V). Since the distance between the end points of \delta' < 3\pi, there exists a geodesic \neq \delta' which connects these points. But this is a contradiction to the fact that in a Riemannian surface of negative curvature, there is only one geodesic in each homotopy class.

\[ \Box \]

2.10 A Comparison Theorem

If S denotes a Riemann surface with a complete hyperbolic metric of finite area \(ds^2\), and \(S^C\) its conformal compactification, we regard S as embedded conformally in \(S^C\). The hyperbolic metric on \(S^C\) will be denoted by \(ds^c^2\).

**Theorem 2.10.1** ([Br3]). For every \(\epsilon\), there exists \(L(\epsilon)\) such that, if the cusps of \(S \geq L(\epsilon)\) then

\[
\frac{1}{1 + \epsilon} ds^c^2 \leq ds^2 \leq (1 + \epsilon) ds^c^2.
\]

We will need the following lemma, which shows that under a “large cusps” condition we can fix the complete hyperbolic metric on \(\mathbb{D}^*\) to a smooth metric on \(\mathbb{D}\) without losing control on the curvature:

**Lemma 2.10.2** ([Br3]). For any \(\epsilon > 0\) there exists \(r_\epsilon\) and a smooth metric \(ds^2_\epsilon\) on \(\mathbb{D}\), such that

i) \(\forall r \geq r_\epsilon,\quad ds^2_\epsilon = ds^2_{\mathbb{D}^*}\),

ii) \(-(1 + \epsilon) \leq \kappa(ds^2_\epsilon) \leq \frac{-1}{1 + \epsilon}.

**Proof.** We have that for a metric \(ds^2 = e^{2u}|dz|^2\), on the disk (or the punctured disk) the curvature is given by (see corollary 2.5.2).

\[
\kappa = -\frac{u'' + \frac{1}{r}u'}{e^{2u}}. \tag{2.29}
\]

Set

\[
g(r) \overset{\text{def}}{=} \frac{1}{r} \frac{u'(r)}{e^{2u}}. \tag{2.30}
\]
In terms of \( g(r) \), \( u(r) \) is given by

\[
e^{-2u(r)} = \int_r^1 2sg(s) \, ds.
\]  

(2.31)

Hence the curvature is given in terms of \( g(r) \) by

\[
\kappa = -2g(r) - rg'(r) - \frac{2r^2g(r)^2}{\int_r^1 2sg(s) \, ds}.
\]

From this formula we see that if we do not change \( g(r) \) and \( g'(r) \) too much then the curvature doesn’t change much.

We have already seen that the complete hyperbolic metrics on \( \mathbb{D} \) and on \( \mathbb{D}^* \) are given, respectively, by

\[
\begin{align*}
  u_D &= \log \frac{2}{1 - r^2}, \\
  u_{D^*} &= \log \frac{1}{r \log \frac{1}{r}}.
\end{align*}
\]

One may calculate that (see 2.30)

\[
\begin{align*}
  g_D(r) &= \frac{1 - r^2}{2}, \\
  g_{D^*}(r) &= \log \frac{1}{r} - (\log \frac{1}{r})^2.
\end{align*}
\]

Hence,

\[
\begin{align*}
  \lim_{r \to 1} g_D(r) &= \lim_{r \to 1} g_{D^*}(r) = 0, \\
  \lim_{r \to 1} g_D'(r) &= \lim_{r \to 1} g_{D^*}'(r) = -1
\end{align*}
\]  

(2.32)  

(2.33)

Thus, if \( \delta > 0 \) is small, set \( r_\epsilon \) such that \( \forall r > r_\epsilon \)

\[
\begin{align*}
  |g_D(r) - g_{D^*}(r)| < \delta, \\
  |g_D'(r) - g_{D^*}'(r)| < \delta,
\end{align*}
\]

Then, define the function \( \tilde{g}_\epsilon \) as follows:

\[
\tilde{g}_\epsilon(r) = \begin{cases} 
  g_D(r) - g_D(r_\epsilon) + g_{D^*}(r_\epsilon) & , \quad 0 \leq r < r_\epsilon \\
  g_{D^*}(r) & , \quad r_\epsilon \leq r \leq 1.
\end{cases}
\]

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Finally, smooth $\tilde{g}_\epsilon$ at $r_\epsilon$ to a smooth function, $g_\epsilon(r)$, without altering much the derivative.

For any $0 \leq r \leq 1$, $g_\epsilon(r)$ and its derivative are close to $g_D(r)$ or $g_D^*(r)$ and their derivatives. Therefore, the corresponding curvature $\kappa_\epsilon$ is close to $-1$.

If we define $u_\epsilon$ by formula (2.31), then $u_\epsilon$ is smooth at 0, it coincides with $u_D^*(r)$ for $r > r_\epsilon$ and its curvature is close to -1. \qed

Now we can prove the main theorem (2.10.1).

Proof. We use the following version of the Ahlfors–Schwarz Lemma due to Wolpert (see theorem A.2.2):

**Theorem 2.10.3 ([W]).** Let $ds_1^2, ds_2^2$ be two (conformal) metrics on a compact Riemann surface. If $\kappa(ds_1^2) \leq \kappa(ds_2^2) < 0$, then $ds_1^2 \leq ds_2^2$.

For any $\epsilon > 0$ let $r_\epsilon$ be as in lemma (2.10.2). If the cusps of $S > area(B_{r_\epsilon})$, then $S^C$ admits a metric $ds_1^2$ whose curvature satisfies:

$$-(1 + \epsilon) < \kappa(ds_1^2) < -\frac{1}{1 + \epsilon},$$

and it coincides with the metric on $S$ away from horocyclic neighbourhoods of the cusps. We observe that $\kappa(\alpha ds^2) = \frac{1}{\alpha} \kappa(ds^2)$ (lemma 2.5.1). So, we have:

$$\kappa(\frac{1}{1 + \epsilon} ds^2) < \kappa(ds_1^2) < \kappa((1 + \epsilon) ds^2) < 0.$$

By Wolpert’s theorem we get:

$$\frac{1}{1 + \epsilon} ds^2 \leq ds_1^2 \leq (1 + \epsilon) ds^2.$$

\qed
Appendix A

Ahlfors–Schwarz Lemma

A.1 The Lemma of Schwarz

Ahlfors, in his book [Ah1], proves the lemma of Schwarz:

**Lemma A.1.1.** Let \( f : \mathbb{D} \rightarrow \mathbb{D} \) be an analytic function, with \( f(0) = 0 \). Then

\[
|f(z)| \leq |z| \tag{A.1}
\]

and

\[
|f'(0)| \leq 1. \tag{A.2}
\]

If \( |f(z)| = z \) for some \( z \neq 0 \) or \( f'(0) = 1 \) then \( f \) is a rotation.

The proof is based on the maximum principle.

Ahlfors remarks that the condition \( f(0) = 0 \) should be considered only as a normalization: If we set

\[
g_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha}z}
\]

for \( \alpha \in \mathbb{D} \), then \( g_\alpha \) is an order-2 automorphism of the disk switching 0 and \( \alpha \). For any analytic map \( f : \mathbb{D} \rightarrow \mathbb{D} \) and \( z_1 \in \mathbb{D} \) define

\[
h = g_{f(z_1)} \circ f \circ g_{z_1}.
\]

\( h \) maps \( \mathbb{D} \) into \( \mathbb{D} \) and fixes 0. Applying the Lemma of Schwarz to \( h \) with \( z = g_{z_1}(z_2) \) we obtain:

\[
\forall z_1, z_2 \in \mathbb{D}, \quad \left| \frac{f(z_1) - f(z_2)}{1 - f(z_1)f(z_2)} \right| \leq \frac{|z_1 - z_2|}{|1 - \overline{z_1}z_2|} \tag{A.3}
\]
and in a differential form
\[ \forall z \in \mathbb{D}, \quad \frac{|f'(z)|}{(1-|f(z)|^2)} \leq \frac{1}{1-|z|^2}. \]         \hfill (A.4)

Recalling that the complete hyperbolic Riemannian metric on \( \mathbb{D} \) is given by
\[ ds^2 = \frac{4}{(1-|z|^2)^2}|dz|^2, \]
or in its integrated form:
\[ \tanh \text{dist}(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - \overline{z_1}z_2} \right|, \]
we get a first geometric interpretation of the Lemma of Schwarz:

Lemma A.1.2 (Schwarz–Pick). If \( f : \mathbb{D} \rightarrow \mathbb{D} \) is analytic, then \( f \) is length-decreasing. i.e., for any curve, \( \gamma \),
\[ \text{length}(f(\gamma)) \leq \text{length}(\gamma). \]

If \( f \) preserves the distance between one pair of distinct points, then \( f \) is an isometry.

We have the following important corollary:

Corollary A.1.3. Let \( f : S_1 \rightarrow S_2 \) be an analytic mapping between complete hyperbolic Riemann surfaces. Then \( f \) is length-decreasing.

Proof. The universal covering of \( S_i \) is \( \mathbb{D} \) equipped with complete hyperbolic metric. We may lift \( f \) to a an analytic map \( \tilde{f} : \mathbb{D} \rightarrow \mathbb{D} \), between the universal coverings. \( f \) is length-decreasing iff \( \tilde{f} \) is length-decreasing. But \( \tilde{f} \) is length-decreasing by the previous lemma. \( \square \)

### A.2 Curvature and the Lemma of Schwarz

In [Ah2], Ahlfors shows that the Lemma of Schwarz is related with the notion of curvature.

Lemma A.2.1 ([Ah2]). Let \( S \) be a Riemann surface endowed with a Riemannian metric \( ds^2 \) whose curvature \( \kappa(ds^2) \leq -1 \), and let \( f : \mathbb{D} \rightarrow S \) be analytic. Then \( f \) is length-decreasing.
Proof. We denote the complete hyperbolic metric on $D$ by $d_{s_D}^2$. Consider the pull-back singular metric on $D$, $d_{s_f}^2 = f^*(d^2)$. $d_{s_f}^2$ is conformally equivalent to $d_{s_D}^2$. So, we may write:

$$
\begin{align*}
    d_{s_D}^2 &= \lambda_{s_D}^2 |dz|^2, \\
    d_{s_f}^2 &= \lambda_{s_f}^2 |dz|^2,
\end{align*}
$$

We need to show

$$
\lambda_f \leq \lambda_D \quad (A.5)
$$

(sometimes this is written as $d_{s_f} \leq d_{s_D}$), where the $\lambda$’s are nonnegative functions and $\lambda_D > 0$. The last inequality is trivial at the branch points of $f$.

The curvatures (at nonsingular points) are given by (see 2.5.1):

$$
\kappa_i = -\frac{\Delta \log \lambda_i}{\lambda_i^2}, \quad (A.6)
$$

where $\Delta$ is the Laplacian: $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Hence, if we set $u_i = \log \lambda_i$ we have:

$$
\Delta u_i = -\kappa_i e^{2u_i}. \quad (A.7)
$$

Suppose first that $\frac{\lambda_f}{\lambda_D}$ attains its maximum in $D$ (a maximum point is obviously nonsingular), then $u_f - u_D$ attains its maximum in $D$ and at maximum we have,

$$
\begin{align*}
    \Delta(u_f - u_D) &\leq 0 \quad (A.7) \\
    \Rightarrow \quad -\kappa_f e^{2u_f} + \kappa_D e^{2u_D} &\leq 0 \\
    \Rightarrow \quad e^{2(u_f - u_D)} &\leq \frac{\kappa_D}{\kappa_f} \leq 1 \quad (A.8)
\end{align*}
$$

We got max $\frac{\lambda_f}{\lambda_D} \leq 1$, as we wished.

In the general case, we pick $0 < r < 1$ and approximate $u_f$ by

$$
\lambda_{r,f}(z) = r \lambda_f(rz).
$$

$\lambda_{r,f}$ is finite on $|z| = 1$. Hence $\frac{\lambda_{r,f}}{\lambda_D}$ attains its maximum in $D$. Also, $\kappa_{r,f}(z) = \kappa_f(rz) \leq -1$. So, by (A.8) we get $\lambda_{r,f} \leq \lambda_D$. Letting $r$ tend to 1 we obtain the desired result. \qed
One may ask what happens if we replace $D$ with an arbitrary Riemann surface. For compact Riemann surfaces an answer was given by Wolpert:

**Theorem A.2.2 ([W]).** Let $S_1$ and $S_2$ be two Riemann surfaces equipped with Riemannian metrics, where $S_1$ is compact and

$$
\kappa_2 \leq \kappa_1 < 0.
$$

Then, an analytic $f : S_1 \to S_2$ is length decreasing.

**Proof.** We pull-back the metric on $S_2$ to a metric on $S_1$. The two metrics on $S_1$ are conformal. Therefore their ratio is a well defined nonnegative function on $D$ which attains its maximum by compactness. From here, we proceed as in the proof of Ahlfors–Schwarz Lemma.

Brooks proved the following version of Ahlfors–Schwarz Lemma:

**Theorem A.2.3 ([Br4]).** Let $S_1$ and $S_2$ be two Riemann surfaces equipped with Riemannian metrics and with $D$ as an analytic univeral covering. If $S_1$ is complete and

$$
\sup \kappa_2 \leq \inf \kappa_1 < 0,
$$

then an analytic $f : S_1 \to S_2$ is length decreasing.

**Proof.** We consider the universal coverings, $D_1$ and $D_2$ with the pull-back metrics by the projection maps, and we lift $f$ to $\tilde{f} : D_1 \to D_2$. Then we pull-back the metric on $D_2$ to a metric on $D_1$. The completeness of $S_1$ guarantees that the corresponding metric on $D_1$ blows up on the boundary, and the sup – inf inequality guarantees that the approximation argument in the proof of Ahlfors–Schwarz lemma will work.
Appendix B

The Poincaré Polygon Theorem

Let \( \mathbb{H}^2 \) be the hyperbolic plane. Consider a finite-sided polygon with all vertices at infinity, \( P \), in \( \mathbb{H}^2 \). \( P \) has sides of two kinds: The sides of the first kind are the sides which are not contained in the boundary of \( \mathbb{H}^2 \). The sides of the second kind are the sides, which are contained in the boundary of the hyperbolic plane. Suppose that the number of sides of the first kind is even, and we divide them into pairwise disjoint pairs. To each pair \( \{s_i, s_j\} \) we associate an orientation-preserving transformation, \( A_{ij} \), which maps \( s_i \) onto \( s_j \) in such a way that \( A_{ij}(P) \cap P = \emptyset \). We consider the subgroup, \( G \), of \( ISO^+(\mathbb{H}^2) \cong PSL(2, \mathbb{R}) \), which is generated by the side-pairing transformations.

**Definition B.1.** A vertex-cycle transformation is an element of \( G \) which stabilizes a vertex of the polygon \( P \).

Remark: For any vertex \( v \) of \( P \) \( Stab_G(v) \) is a cyclic group.

**Theorem B.2.** If all the vertex-cycle transformations in \( G \) are parabolic, then \( G \) is a discrete subgroup of \( ISO^+(\mathbb{H}^2) \), with \( P \) as a fundamental domain.

I liked De Rham’s survey and proof of the theorem in the compact and non-compact polygons cases in \([dR]\).
Appendix C

The Uniformization Theorem

The Uniformization Theorem can be formulated as follows:

**Theorem C.1.** Any simply connected Riemann surface is conformally equivalent to one and only one of the following surfaces:

(i) The Riemann sphere.

(ii) The complex plane.

(iii) The unit disk.

A proof can be found in [FK].

**Corollary C.2.** On a Riemann surface there exists a complete Riemannian metric of constant curvature 1, 0, or -1.
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