On the lower central factors of groups

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Abstract

A general method for calculating or constructing lower central factors of groups is presented. Relative basic commutators are defined.

1 Introduction

For basic definitions and results the reader may consult any standard book on group theory as for example [9], [10] or [11]. For a group $G$, the lower central series of $G$ is defined by: $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n(G), G]$ where $[A, B]$ denotes the subgroup generated by all commutators $[a, b] = a^{-1}b^{-1}ab$ with $a \in A, b \in B$. The factor group of $G$ by the normal subgroup $H$ will be denoted by $G/H$ or $G/H$ as is convenient.

The $n$th lower central factor of $G$ is the factor group $\frac{\gamma_n(G)}{\gamma_{n+1}(G)}$.

Every group is the factor group of a free group. Suppose then $G$ is presented as $G \cong F/R$ where $F$ is free and $R$ is a normal subgroup of $F$. The following result follows from standard isomorphism theorems. For completeness we include a proof.

**Proposition 1** Suppose $G \cong F/R$. Then $\frac{\gamma_n(G)}{\gamma_{n+1}(G)} \cong \frac{\gamma_n(F)}{R}\frac{\gamma_{n+1}(F)}{R\gamma_{n+1}(F)}$.

**Proof:** Write $\gamma_i$ for $\gamma_i(F)$. From $G \cong F/R$, it follows that $\gamma_i(G) \cong \frac{\gamma_i(R)}{R}$. Hence by standard isomorphism theorems:

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Thus

\[
\frac{\gamma_n(G)}{\gamma_{n+1}(G)} \cong \frac{\gamma_n/\gamma_{n+1}}{(R \cap \gamma_n) \cdot \gamma_{n+1}/\gamma_{n+1}} \\
\cong \frac{\gamma_n/\gamma_{n+1}}{(R \cap \gamma_n)/(R \cap \gamma_n) \cap \gamma_{n+1})} \\
\cong \frac{\gamma_n/\gamma_{n+1}}{(R \cap \gamma_n)/(R \cap \gamma_{n+1})}
\]

QED

Note also that \(\frac{R \cap \gamma_n}{\gamma_{n+1}(G)} \cong \frac{(R \cap \gamma_n) \gamma_{n+1}}{\gamma_{n+1}}\) and this is a free abelian group as indeed it is a subgroup of the free abelian group \(\gamma_n(F)/\gamma_{n+1}(F)\). Thus if \(R \cap \gamma_n\) mod \(\gamma_{n+1}\) is known, the \(n^{th}\) lower central factor of \(G\) is a factor group of the known (see below) free abelian group \(\gamma_n/\gamma_{n+1}\) by the (now known) free abelian group \(R \cap \gamma_n\) mod \(\gamma_{n+1}\).

2 Basic commutators

The structure of \(\frac{\gamma_n(F)}{\gamma_{n+1}(F)}\), the \(n^{th}\) lower central factor of the free group, is well-known as the free abelian group on the basic commutators of weight \(n\).

These basic commutators can be defined as follows:

Let \(F\) be free on a set \(X\). The basic commutators of weight 1 are the elements of \(X\). Suppose then the basic commutators of weight \(n < n\), with \(n \geq 2\), have been defined and ordered. Then a basic commutator of weight \(n\) is a commutator of the form \([b, c]\) where \(b, c\) are basic commutators of weight \(n\) such that weight \(b +\) weight \(c = n\) and if \(b = [d, e]\) where \(d, e\), are basic commutators then \(e \leq c\).

This condition that the second component in a basic commutator must be less than or equal to the next component is a ‘consequence’ of a Jacobi-type identity:

\([x, y, z][y, z, x][z, x, y] \in \gamma_{n+1}(G)\)
where \([x, y, z] \in \gamma_n(G)\). The Jacobi Identity in groups is a “mod \(\gamma_{n+1}(G)\)” identity which is not a direct group identity; this indeed makes group identities and their consequences more complicated in general – unless of course one is only interested in working modulo some term of the lower central series which is often sufficient.

The most general ‘free version’ of a Jacobi/Witt-Hall type identity is
\[
[c, b, a] = [c, b][c, a]^{-1}[b, a, c]^{-1}[b, a]^{-1}[c, b][c, a][c, a, b][b, a]
\]
and this enables \(a\) to be switched to its ‘correct position’ when it is less than \(b\) and \(c\) in some ordering.

Basic commutators were introduced by Philip Hall and Marshall Hall, see \([2], [3], [4], [11]\) and \([10]\). The notes in \([2]\) originated in a series of lectures given by P Hall in 1957 and were available in manuscript form and then reproduced with minor changes and additions by Queen Mary College College Lecture Notes series in 1969. The notes are also available within the collected works of Philip Hall \([3]\).

Basic commutators proved and are still proving very useful in many areas. Computations which involve a ‘collecting process’ of some form follow on from the collecting process introduced in \([2]\) and used in the theory of basic commutators and basic products. The connections between free groups, free Lie Algebras, free associative rings, basic commutators in groups and Lie Algebras, the ideas of basic products (in free associative rings), Hall-Witt Theorems, identities, Baker-Campbell-Hausdorff formulae etc. are presented beautifully and in a very coherent manner in \([2]\) and \([10]\).

The relationship with Fox derivatives is also clear from \([10]\). Basics commutators are used by Gruenberg \([1]\) for his famous result that soluble Engel groups are locally nilpotent. Gruenberg shows that for a finitely generated and soluble group the series of basic commutators eventually grow to ‘Engel-like’ and from this follows the nilpotency of the group. Whole theories of basic commutators are presented in \([12]\) which haven’t received the attention they should and implicitly contain further advancement and potential applications. A general ‘free system’ of free basic-commutator-like generators relating the free structure of terms of the lower central series and related groups is given in \([8]\); this follows from initial work of Ward \([13]\) and \([14]\). They also appear quite frequently in papers on Burnside’s problem and many other areas.

Is it time to ‘go back to basics’?

### 2.1 Factor of a free abelian group by a free abelian group

The basic commutators gives the structure of the free abelian group \(\frac{\gamma_n(F)}{\gamma_{n+1}(F)}\) when \(F\) is a free group and are thus a starting point for the study of lower central factors of any group.

From
\[
\frac{\gamma_n(G)}{\gamma_{n+1}(G)} \cong \frac{\frac{\gamma_n}{\gamma_{n+1}}}{\frac{\gamma_n}{\gamma_{n+1}}}
\]
it is seen the structure of the lower central factors of $G$ is known once the structure of $R_{\gamma_n}$, which is $R \cap \gamma_n \mod \gamma_{n+1}$, is known.

From now on write $\gamma_n G$ for $\gamma_n(G)$.

2.2 Examples of use

2.2.1 Free metabelian example

Consider $R = \gamma_2 \gamma_2 F = F''$, the second derived group of $F$. We wish to know the structure of the lower central factors of $F / F''$. It is relatively easy to show that if given $a \in F''$ then, modulo the $n^{th}$ term of the lower central series of $F$, $a$ is a product of basic commutators of the form $[[...],[...]]$, that is, each basic commutator occurring in an element of $a \in F''$ as the product of basic commutators modulo $\gamma_n F$ has the ‘shape’ of an element of $F''$.

Hence for this $R = F''$, $R \cap \gamma_n F$ modulo $\gamma_{n+1} F$ is simply the free abelian group on the basic commutators of weight $n$ of the form $[[...],[...]]$.

Thus every element in the lower central factors of $F / F''$ can be written uniquely as a product of basic commutators of the form $[[...],[...]]$ - no double brackets - and thus is free abelian on the simple basic commutators, that is basic commutators of the form $[x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n}]$ with $i_1 > i_2 \leq i_3 \leq \ldots \leq i_n$. This is a result due to Magnus (unpublished) – see Hanna Neumann’s book [11] pages 107-109 which has a different and certainly longer proof.

2.2.2 Other examples

This method may also be used on other free groups in a variety. Take $R = \gamma_2 \gamma_3 F$. (The group $F / R$ is the free group in the variety abelian-by-(nilpotent of class $\leq 2$).)

Every element in this $R$ will be a product of basic commutators of the form $[[\geq 3],[\geq 3]]$, that is those that have ‘shape’ in $\gamma_2 \gamma_3 F$; the others then will be a basis for the lower central factors of $\frac{F}{\gamma_2 \gamma_3 F}$. So for example $[[3],[2],\ldots,[2]]$ will be in this basis as well as the simple basic commutators.

In fact this trick works for any $R = \gamma_{n_1} \gamma_{n_2} \gamma_{n_3} \ldots \gamma_{n_m} F$ in a polynilpotent series of $F$ for a sequence $n_1, n_2, n_3, \ldots, n_m$. Included here would, for example, be the derived series in which each $n_i = 2$. The group $F / R$ for this $R$ is known as the free polynilpotent group (relative to the sequence $n_1, n_2, n_3, \ldots, n_m$).

Every element in $R$ is congruent modulo $\gamma_{n+1} F$ to a product of basic commutators whose ‘shape’ is in $R$ and the lower central factors of $F / R$ is the free abelian group on the rest, that is those basic commutators whose ‘shape’ is less than (or not in) $R$. See Martin Ward [12] for much more detail on this and many forms and shapes of basic commutators and basic sequences.
2.3 Further areas of application

The ‘shape’ of a basic commutator has also been exploited in identifications of various intersections in free groups and for groups determined by modules within the free group ring; see for example [5], [6], [7] for solutions to the Fox-type and Lie Dimension problems and other related identities in free groups, where basic-type commutators come into play.

The method also gives a way of identifying $\gamma_{n_1} \gamma_{n_2} \gamma_{n_3} \ldots \gamma_{n_m} F \cap \gamma_w F$ for any integer $w$. For example

$$F'' \cap \gamma_w F = \prod_{i+j=w; i, j \geq 2} [\gamma_i F, \gamma_j F]$$

$$\gamma_2 \gamma_3 F \cap \gamma_w F = \prod_{i+j=w; i, j \geq 3} [\gamma_i F, \gamma_j F]$$

$$\gamma_3 \gamma_2 F \cap \gamma_w F = \prod_{i+j+k=w; i, j, k \geq 2} [\gamma_i F, \gamma_j F, \gamma_k F]$$

or more generally:

$$\gamma_{n_1} \gamma_{n_2} \gamma_{n_3} \ldots \gamma_{n_m} F \cap \gamma_w F = \prod_{i_1 + i_2 + \ldots + i_{n_1} = w} (\gamma_{n_2} \gamma_{n_3} \ldots \gamma_{n_m} F \cap \gamma_{i_1} F), (\gamma_{n_2} \gamma_{n_3} \ldots \gamma_{n_m} F \cap \gamma_{i_2} F), \ldots , (\gamma_{n_2} \gamma_{n_3} \ldots \gamma_{n_m} F \cap \gamma_{i_{n_1}} F)$$

where the product is over all integers $i_1, i_2, \ldots, i_{n_1}$ with $i_1 + i_2 + \ldots + i_{n_1} = w$ and each of $\gamma_{n_2} \gamma_{n_3} \ldots \gamma_{n_m} F \cap \gamma_{i_j} F$ is determined by induction.

3 Relative basic commutators

Suppose $F$ is finitely generated and that $R$ is finitely generated as a normal subgroup. An algorithm may be given for the determination of the structure of $R' \gamma_n F / R \gamma_{n+1} F$ in terms of free generators of $\gamma_n F / \gamma_{n+1} F$; this gives an algorithm for the determination of the the lower central factors of $G$.

The way to do this is to construct basic commutators relative to $R$ in a process now to be defined.

Suppose $F$ is freely generated by $X = \{x_1, x_2, \ldots, x_n\}$ and that $R$ is generated as a normal subgroup by $A = \{r_1, r_2, \ldots, r_m\}$. We assume no element of $A$ occurs in $X$ and order $A \cup X$ by saying the elements of $A$ come after those of $X$. 

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Nielsen transformations were originally constructed to show that every subgroup of a free group is free but has since found applications in many areas.

A **Nielsen Transformation** on a set \((a_i)_{i \in I}\) is one of the following:

1. Exchange two of the \(a_j\).
2. Replace an \(a_j\) by \(a_j^{-1}\).
3. Replace and \(a_j\) by \(a_j a_k, j \neq k\).
4. Carry out substitutions of types 1, 2, 3, repeatedly a finite number of times.

See for example [10], [9] for further details on Nielsen transformations.

A Nielsen transformation transforms a set of free generators onto a set of free generators in such a way that the free group generated by each set is the same.

Let \(b\) be a basic commutator of weight \(\geq 2\) with \(b = [a, c]\) for basic commutators \(a, c\) and if \(a = [d, e]\) then \(e \leq c\). We call \(c\) the second component of \(b\). If Nielsen transformations are applied to a basic commutator we wish to define the second component of such an expression. Suppose then the second components of \(b\) and \(b'\) have been defined. In a transformation of type 1 the second components are the same as the original second components. We define the second component of \(b^{-1}\) to be that of \(b\).

First let us consider \(R \cap \gamma_2 F \cong R \gamma_2 F \gamma_2 F\). Let \(H_1\) denote the subgroup of \(F\) generated by \(A\). Then we may construct a set of free generators \(x_{1,1}, x_{1,2}, \ldots, x_{1,m}\) for \(F\) and a set of free generators \(y_{1,1}, y_{1,2}, \ldots, y_{1,s}\) for \(H\) such that

\[
\begin{align*}
y_{1,i} &\equiv x_{1,i}^{d_{1,i}} \mod \gamma_2 F, & 1 \leq i \leq s_1, & * \\
y_{1,i} &\equiv 1 \mod \gamma_2 F, & s_1 < i \leq s_1, & **
\end{align*}
\]

where \(0 < d_{1,i}\) divides \(d_{1,i+1}\), and \(s_1 \leq m_1\).

This is done in the usual way by performing Nielsen Transformations and reducing an \(n \times m\) matrix of integers to diagonal form. Of course this immediately gives the structure of \(R \cap \gamma_2 F \cong R \gamma_2 F \gamma_2 F\) as the direct product of infinite cyclics and cyclics of orders \(d_{1,i}\).

Note that \(R = H_1^F\), the normal closure of \(H_1\) in \(F\).

Can we extend this to \(R \cap \gamma_2 F \gamma_3 F\) and perhaps higher?

Consider first of all \(R \cap \gamma_2 F \gamma_3 F\).

What we do is construct basic commutators relative to \(R\) of degree two in this case.

We define an \(\overline{R}_2\) - basic to be an element of \([(y_{1,i}, x_{1,j}); i > j] \cup [(x_{1,i}, y_{1,j}); j > i]\) for \(1 \leq i \leq s_1\), and for all \(j, 1 \leq j \leq m_1\).

An \(\overline{R}_2\) - basic is an element in \(R \cap \gamma_2 F\) and is easy to show that the set of \(\overline{R}_2\) - basics is linearly independent modulo \(R \cap \gamma_3 F\).
We haven’t got a full generating set for \( \frac{R \cap \gamma_2 F}{R \cap \gamma_3 F} \) as we have to include some more elements.

Let \( H_2 \) be the group defined by
\[
\{ y_{2i} - \text{basics} \} \cup \{ y_{1i}, \quad s_{11} < i \leq s_{12} \} \cup \{ y_{1i}, x_{1i}, \quad 1 \leq i \leq s_{11} \}.
\]

Then \( H_2 \) is a finitely generated subgroup of \( \gamma_2 F \cap R \). In fact it can be shown that \( H_2 \) will generate \( \frac{R \cap \gamma_2 F}{R \cap \gamma_3 F} \). We also know a free generating set for \( \frac{\gamma_2 F}{\gamma_3 F} \) as the free abelian group on the basic commutators of weight 2.

Apply the diagonalisation process again to \( H_2 \) and \( \frac{\gamma_2 F}{\gamma_3 F} \) will give the structure of \( \frac{R \cap \gamma_2 F}{R \cap \gamma_3 F} \).

Then there exists a set of free generators \( x_{21}, x_{22}, \ldots, x_{2m_2} \) for \( \frac{\gamma_2 F}{\gamma_3 F} \) and a set of generators \( y_{21}, y_{22}, \ldots, y_{2s_{22}} \) for \( H_2 \) such that
\[
y_{2i} \equiv x_{2i}^{d_{2i}} \pmod{\gamma_3 F} \quad \text{for} \quad 1 \leq i \leq s_{21}
\]
\[
y_{2i} \equiv 1 \pmod{\gamma_3 F} \quad \text{for} \quad s_{21} < i \leq s_{22}
\]

Denote by \( R_2 \) the set of all \( \{ y_{2i}/1 \leq i \leq s_{21} \} \). An element of \( R_2 \) will be called an \( R \)-basic of weight 2. Provided \( \frac{R \cap \gamma_2 F}{R \cap \gamma_3 F} \) is generated by \( R_2 \) this will give the structure of \( \frac{\gamma_2 F}{\gamma_3 F} \) and hence the structure of \( \frac{\gamma_3 G}{\gamma_3 G} \).

Of course this process can be continued and we can define a set of \( R \)-basic of weight \( n \) which will be a basis for \( \frac{R \cap \gamma_2 F}{R \cap \gamma_n+1 F} \).

The process is really in a sense replacing a basic commutator which corresponds non-trivially to a free generator modulo \( \gamma_n F \) by this free generator and consequently by any basic commutator which contains this basic commutator as a constituent. The following basis theorem follows from these constructions.

**Theorem 3.1** Every element \( w \) in \( R \) can be written uniquely in the form
\[
w \equiv r_1^{\alpha_1} r_2^{\alpha_2} \ldots r_t^{\alpha_t} \pmod{R \cap \gamma_{n+1} F}
\]
where the \( r_1, r_2, \ldots, r_t \) are the \( R \)-basic commutators of weights \( \leq n \) and \( r_1 < r_2 < \ldots < r_t \) and the \( \alpha_i \) are integers.

**Corollary 3.1** \( R \) generates \( \frac{R \cap \gamma_n F}{R \cap \gamma_{n+1} F} \) freely.

This can be seen by noting that \( R \) is generated by \( \{ r \cup [r, x] \} \) for any \( r \in A \) and any \( x \in F \).

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