Current quark mass and g-2 of muon and $ee^+ \to \pi^+\pi^-$

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Abstract

Based on a phenomenologically successful effective chiral theory of pseudoscalar, vector, and axial-vector mesons the dependences of $\rho - \omega$ mixing and the vertex $\omega\pi\pi$ on $m_d - m_u$ are found. Using the new data of $ee^+ \to \pi^-\pi^+$, $m_d - m_u$ has been determined to be 4.24MeV. The form factor of pion agrees with data in both space- and time-like regions. The branching ratio of $\tau \to \pi\pi\nu$ is computed to be 25.14% which agrees with data. CVC is satisfied. The values of g-2 of muon from $\pi\pi$ channel are calculated in the range of $q^2 < 1.3^2 GeV^2$.

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In particle physics $ee^+ \rightarrow \pi^- \pi^+$ is a very important process. Recently, measurements of the cross section of $ee^+ \rightarrow \pi^- \pi^+$ in the center of mass energy region from 0.61 to 0.96 GeV with a 0.6% systematic uncertainty have been reported by CMD-2 group[1]. In Ref.[1] two different models[2, 3] have been used to analyze the data. In this paper we exploit another model, effective chiral theory of pseudoscalar, vector, and axial-vector mesons[4] to study $ee^+ \rightarrow \pi^- \pi^+$. The contribution of $\pi \pi$ channel to g-2 of muon has been calculated[5]. To get good accuracy the form factor of pion in both time-like, space-like, and $B(\tau \rightarrow \pi \pi \nu)$ have to be taken into account[5]. However, in Ref.[6] by using CVC it is obtained $B(\tau \rightarrow \pi \pi \nu) = (24.25 \pm 0.77)\%$. The data[7] is $(25.40 \pm 0.14)\%$. The seperation of the isovector form factor of pion from $ee \rightarrow \pi \pi$ depends on the treatment of $\omega \rightarrow \pi \pi$. In this paper a different mechanism for $\omega \rightarrow \pi \pi$ is provided. The amplitude of $\omega \rightarrow \pi^- \pi^+$ is obtained from the effective theory[4]. The mass difference of u and d quarks is determined by fitting the data[1]. The contribution of $\pi \pi$ channel to g-2 of muon is calculated in the range of $q^2 < 1.3^2 GeV^2$. In these calculations both space-, time-like regions and $\tau$ decays are taken into account.

Using the knowledge of current algebra and nonlinear $\sigma$ model, the Lagrangian of the effective chiral theory of pseudoscalar, vector, and axial-vector mesons is constructed as[4] (taking two flavors as an example)

$$
\mathcal{L} = \bar{\psi}(x)(i \gamma \cdot \partial + \gamma \cdot v + \gamma \cdot \alpha \gamma_5 - mu(x))\psi(x) - \bar{\psi}(x)M\psi(x)
+ \frac{1}{2}m_0^2(\rho^\mu_\rho_{\mu i} + \omega^\mu \omega_{\mu} + a^\mu_i a_{\mu i} + f^\mu f_{\mu})
$$

(1)

where $a_\mu = \tau_i a^i_\mu + f_\mu$, $v_\mu = \tau_i \rho^i_\mu + \omega_\mu$, and $u = exp\{i \gamma_5 (\tau_i \pi_i + \eta)\}$, and $M$ is the current quark matrix. Since mesons are bound states solutions of $QCD$ they are not independent.
degrees of freedom. Therefore, there are no kinetic terms for meson fields. The kinetic terms of meson fields are generated from quark loops. m is a parameter, the constituent quark mass, which related to quark condensate[4]. The theory has dynamical chiral symmetry. In the limit, \( m_q \rightarrow 0 \), the theory is explicitly chiral symmetric.

After integrating out the quark fields the effective Lagrangian of mesons is obtained and can be found in Ref.[4]. Normalizing the kinetic terms of pion, \( \rho \), and \( \omega \) fields, the fields of physical mesons are defined

\[
\pi \rightarrow \frac{2}{f_\pi} \pi, \quad \rho \rightarrow \frac{1}{g} \rho, \quad \omega \rightarrow \frac{1}{g} \omega.
\]

\( f_\pi \) is the pion decay constant and g is the universal coupling constant. They are defined as

\[
f_\pi^2 = F^2 (1 - \frac{2c}{g})^{-1}, \quad F^2 = \frac{N_C}{(4\pi)^2} m^2 \int d^4 k \frac{1}{(k^2 + m^2)^2},
\]

\[
g^2 = \frac{F^2}{6m^2}, \quad c = \frac{f_\pi^2}{2gm_\rho^2}.
\]

\( f_\pi \) and g are two inputs. To define \( F^2 \) a cut-off has to be introduced[4].

Feynman diagrams at tree level are at \( O(N_C) \) and loop diagrams are at higher order in \( N_C \) expansion. The theory is phenomenologically successful[8].

The Vector Meson Dominance(VMD) plays major role in studying the process \( ee^+ \rightarrow \pi^- \pi^+ \). VMD is a natural result of this theory and it has been derived as[4]

\[
\frac{1}{2} eg \left\{- \frac{1}{2} F^{\mu\nu} (\partial_\mu \rho_\nu - \partial_\nu \rho_\mu) + A^\mu j_\mu^0 \right\},
\]

\[
\frac{1}{6} eg \left\{- \frac{1}{2} F^{\mu\nu} (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) + A^\mu j_\mu^\omega \right\},
\]

\[- \frac{1}{3\sqrt{2}} eg \left\{- \frac{1}{2} F^{\mu\nu} (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) + A^\mu j_\mu^\phi \right\},
\]

\( (2) \)
The current \( j_0^0 \) is derived from the vertex of \( \rho \pi \pi \) \cite{1}

\[
\mathcal{L}_{\rho \pi \pi} = \frac{2}{g} f_{\rho \pi \pi}(q^2) \epsilon_{ijk} \rho^i_\mu \pi^j_\nu \partial_\mu \pi_k
\]  

(3)

by substituting \( \rho^0_\mu \rightarrow \frac{1}{2} e g A_\mu \) into this equation. In the same way we obtain

\[
\mathcal{L}_{\gamma \pi \pi} = i e A_\mu f_{\rho \pi \pi}(q^2) (\pi^+ \partial_\mu \pi^- - \pi^- \partial_\mu \pi^+).
\]  

(4)

where

\[
f_{\rho \pi \pi}(q^2) = 1 + \frac{q^2}{2 \pi^2 f_{\pi}^2} [(1 - \frac{2c}{g})^2 - 4 \pi^2 c^2].
\]  

(5)

\( q \) is the momentum of \( \rho \) meson. \( f_{\rho \pi \pi}(q^2) \) is an intrinsic form factor which is the physical effect of quark loops. Similarly, \( j_\omega^\omega \) and \( j_\phi^\phi \) can be defined\cite{3} too. We have used VMD and this effective theory to study the form factors of pion and kaons\cite{9}, in which the \( \rho - \omega \) mixing is taken to be a constant. In this paper the amplitude of \( \omega \rightarrow \pi \pi \) is determined by this effective theory\cite{4} completely.

\( \omega \rightarrow \pi^- \pi^+ \) contributes to \( ee^+ \rightarrow \pi^- \pi^+ \). The study of \( \omega \rightarrow \rho \) mixing has a long history. In this paper a different approach is taken to determine the contribution of \( \omega \rightarrow \pi \pi \) to \( ee \rightarrow \pi \pi \). Based on the Lagrangian of mesons derived from Eq.(1) \( \omega \rightarrow \rho \) mixing is found

\[
\mathcal{L}_{\rho \omega} = \left\{ - \frac{1}{4 \pi^2 g^2 m} (m_d - m_u) + \frac{1}{24} e^2 g^2 \right\} (\partial_\mu \rho_\nu - \partial_\nu \rho_\mu)(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu).
\]  

(6)

Eq.(6) has been presented in Refs.\cite{4, 10}. However, there is direct coupling between \( \omega \) and \( \pi \pi \) which is derived from Eq.(1) too

\[
\mathcal{L}_{\omega \pi \pi} = \frac{2i}{g} f_{\omega \pi \pi}(q^2) \omega_\mu (\pi^+ \partial_\mu \pi^- - \pi^- \partial_\mu \pi^+) + \frac{i}{6} e^2 g f_{\rho \pi \pi}(q^2) \omega_\mu (\pi^+ \partial_\mu \pi^- - \pi^- \partial_\mu \pi^+),
\]  

(7)
where
\[
f_{\omega\pi\pi}(q^2) = -\frac{g^2}{2\pi^2 f^2_\pi} \left( 1 - \frac{4c}{g} \right) \frac{1}{m} (m_d - m_u).
\]

Taking Eqs.(2,3,5,6,7) into account(Fig.1), the cross section of \(ee^+ \rightarrow \pi^- \pi^+\) is obtained
\[
\sigma = \frac{\pi\alpha^2}{3} \frac{1}{q^2} (1 - \frac{4m^2_{\pi^+}}{q^2}) \left| F_\pi(q^2) \right|^2.
\]
(8)

where \(F_\pi(q^2)\) is the pion form factor
\[
F_\pi(q^2) = f_{\rho\pi\pi}(q^2) \{1 - \frac{q^2 - m^2_\rho + i\sqrt{q^2} \Gamma_\rho(q^2)}{3 q^2 - m^2_\omega + i\sqrt{q^2} \Gamma_\omega(q^2)}\},
\]
(9)

\[
b(q^2) = -\frac{m_d - m_u}{m} f_{\omega\pi\pi}(q^2) - \frac{(eg)^2}{12} - \frac{m_d - m_u}{2\pi^2 g^2 m} \frac{(eg)^2}{12} \frac{q^2}{q^2 - m^2_\rho + i\sqrt{q^2} \Gamma_\rho(q^2)}.
\]

Where \(b(q^2)\) is the effect of \(\omega \rightarrow \pi\pi\). In Eq.(9), the first term is from Fig.1(1), the second term from Fig.1(3), the last two terms are from \(\rho - \omega\) mixing(Fig.1(2,4,5)). \(f_\pi = 0.186 GeV\) is taken in Ref.[4]. The decay width of \(\rho\) meson is calculated to be
\[
\Gamma_\rho(q^2) = \Gamma_{\rho^0 \rightarrow \pi^+ \pi^-}(q^2) + \Gamma_{\rho^0 \rightarrow K\bar{K}}(q^2),
\]
\[
\Gamma_{\rho^0 \rightarrow \pi^+ \pi^-}(q^2) = \frac{f^2_{\rho\pi\pi}(q^2) \sqrt{q^2}}{12\pi g^2} (1 - \frac{4m^2_{\pi^+}}{q^2})^\frac{3}{2} \theta(q^2 > 4m^2_{\pi^+}),
\]
\[
\Gamma_{\rho^0 \rightarrow K\bar{K}}(q^2) = \frac{f^2_{\rho\pi\pi}(q^2) \sqrt{q^2}}{48\pi g^2} (1 - \frac{4m^2_{K^0}}{q^2})^\frac{3}{2} \theta(q^2 > 4m^2_{K^0})
\]
\[
+ \frac{f^2_{\rho\pi\pi}(q^2) \sqrt{q^2}}{48\pi g^2} (1 - \frac{4m^2_{K^0}}{q^2})^\frac{3}{2} \theta(q^2 > 4m^2_{K^0}),
\]
(10)

when \(q^2 > 4m^2_K\) the \(K\bar{K}\) channel is open. There are other channels, however, in the range of \(\sqrt{q^2} < 1.3 GeV\) the contribution of other channels is negligible. Because \(\omega\) is a narrow resonance \(\Gamma_\omega\) is taken to be a constant. Following Ref.[1] we take
\[
m_\omega = (782.71 \pm 0.08) MeV,
\]
\[
\Gamma_\omega = (8.68 \pm 0.24) MeV.
\]
Comparing with the two models of Ref.[1], there are two new effects:

1. Direct coupling of $\omega\pi\pi$ (not through $\rho - \omega$ mixing),

2. the $\rho - \omega$ mixing depends on $q^2$.

In Ref.[4] $m_\rho = 0.77GeV$ and $g = 0.39$ are taken. By fitting the new CMD2[1] experiment data, we obtain

$$m_\rho = 777.4 \pm 0.5 MeV, \quad g = 0.3938 \pm 0.002,$$

$$m_d - m_u = 4.24 \pm 0.32 MeV. \quad (11)$$

The value of $m_d - m_u$ agrees with the one presented in Ref.[11]. The fit is shown in Fig.2. The comparison between the theory and experimental data of pion for $m$ factor in both time-like (new and old) and space-like regions ($-1.1 GeV < \sqrt{q^2} < 1.3 GeV$)(Fig.3). Theory agrees well with data.

The form factor of pion in space-like region is obtained by taking $\Gamma_\rho$ and $\Gamma_\omega$ to be zero. The radius of pion is expressed as

$$<r_\pi^2> = \frac{6}{m_\rho^2} + \frac{3}{2\pi^2 f_\pi^2} \{(1 - \frac{2c}{g})^2 - 4\pi^2 e^2\} + \frac{e^2 g^2}{6m_\omega^2} = 0.451 fm^2. \quad (12)$$

The data[12] is $(0.439 \pm 0.03) fm^2$.

Using Eqs.(6,7), we obtain

$$\Gamma(\omega \to \pi^+\pi^-) = |b(m_\omega^2)|^2 \Gamma_\rho(m_\omega^2)$$

$$B(\omega \to \pi\pi) = (0.97 \pm 0.19)\%. \quad (13)$$

The results of this paper and Ref.[1] are listed in Tab.1. In this paper $m_\rho$ and $g$ are inputs and
Table 1: Results

| Parameter                  | GS model[2]     | HLS model[3]    | This paper     |
|----------------------------|-----------------|-----------------|----------------|
| $M_\rho (MeV)$             | 776.09 $\pm$ 0.64 $\pm$ 0.50 | 775.23 $\pm$ 0.61 $\pm$ 0.50 | 777.4 $\pm$ 0.50 |
| $\Gamma_\rho (MeV)$       | 144.46 $\pm$ 1.33 $\pm$ 0.80 | 143.88 $\pm$ 1.44 $\pm$ 0.80 | 147.37 $\pm$ 0.40 |
| $\Gamma(\rho \to e^+e^-)$ KeV | 6.86 $\pm$ 0.11 $\pm$ 0.05 | 6.84 $\pm$ 0.12 $\pm$ 0.05 | 6.73 $\pm$ 0.02 |
| $B(\omega \to \pi^+\pi^-)$ (%) | 1.33 $\pm$ 0.24 $\pm$ 0.05 | 1.43 $\pm$ 0.24 $\pm$ 0.05 | 0.97 $\pm$ 0.19 |
| $\chi^2/\nu$              | 0.92            | 0.94            | 1.06           |

all other quantities listed in Tab.1 are theoretical values. Tab.1 shows that the values of $m_\rho$, $\Gamma_\rho$, and $\Gamma(\rho \to ee)$ obtained in this paper are consistent with the values obtained by other two models. However, the central value of the branching ratio of $\omega \to \pi\pi$ obtained in this paper is smaller than others. The reason of this difference is how to treat the contribution of $\omega \to \pi\pi$.

In Ref.[1] a mixing $|\omega >= |\omega_0 > +\epsilon |\rho_0 >$ has been used, where $\epsilon$ is a constant. In this paper the $\rho - \omega$ mixing obtained (Eq.6) strongly depends on $q^2$, and besides $\rho - \omega$ mixing there is direct channel( Eq.7). It is quit clear here that the mechanism of $\rho - \omega$ mixing is the same as VMD(2).

Because of CVC the decay of $\tau \to \pi^+\pi^-\nu_\tau$ is determined by the isovector component of the pion form factor

$$\frac{d\Gamma}{dq^2} = \frac{G^2}{(2\pi)^3} \frac{|V_{ud}|^2}{48m_\tau^3} (m_\tau^2 + 2q^2)(m_\tau^2 - q^2)^2 \left(1 - \frac{4m_\pi^2}{q^2}\right)^3 f_{\rho\pi\pi}^2(q^2) \frac{m_\rho^4 + q^2 \Gamma_\rho^2(q^2)}{(q^2 - m_\rho^2)^2 + q^2 \Gamma_\rho^2(q^2)}. \quad (14)$$

The branching ratio is computed to be

$$B(\tau \to \pi^0\pi^-\nu) = (25.14 \pm 0.31)\%. \quad (15)$$
The data is $(25.40 \pm 0.14)\%$\cite{7}. Theory agrees with data. CVC is satisfied. The reason of the satisfaction of CVC in this paper is the new amplitude of $\omega \to \pi\pi$(9). We predict a small $B(\omega \to \pi\pi)$ and obtain correct $B(\tau \to \pi\pi\nu)$.

Now we are ready to calculate the contribution of $\pi\pi$ channel to g-2 of muon. The results are

$$a^{(2)}(h.v.p) = \int_{4m_d^2}^{\infty} dt \frac{1}{4\pi^3} \sigma(t) k(t),$$

$$k(t) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)t/m_u^2},$$

$$a^{(2)}(2\pi, t < 0.8 GeV^2) = (4697 \pm 34) \times 10^{-11},$$

$$a^{(2)}(2\pi, t < 1.3^2 GeV^2) = (4971 \pm 47) \times 10^{-11},$$

$$a^{(2)}(2\pi, (0.61)^2 < t < (0.96)^2 GeV^2) = (3721 \pm 40) \times 10^{-11},$$

where $t = q^2$.

It is necessary to point out that the results are based on an effective chiral theory. However, in the energy region of Ref.\cite{3} our result agrees with Ref.\cite{3} within their errors, $(3681 \pm 26 \pm 22) \times 10^{-11}$. Theoretical results fit the form factors in both space-like and time-like regions and $\tau$ decays. CVC is satisfied. On the other hand, this effective theory has been applied to study many other physical processes. Theory agrees with data well. We will present the results of many other channels in the near future.

In summary, the CMD2 new data has been used to determine the value of $m_d - m_u$. A smaller $B(\omega \to \pi\pi)$ is obtained. The form factor of pion agrees with data in both space-like and time-like regions. Theoretical $B(\tau \to \pi\pi\nu)$ agrees with data and CVC is satisfied. The contribution of $\pi\pi$ channel to g-2 of muon at $\sqrt{q^2} < 1.3 GeV$ has been calculated. In this study all vertices are derived up to the fourth order in derivatives. They fit the data up to $1.3^2 GeV^2$. 

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To fit the data at higher energies we need to add terms at higher order in derivatives and hard gluon effects. We will present the study in the near future.

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Figure Captions

FIG. 1. Feynman Diagrams of $e^+e^- \rightarrow \pi^+\pi^-$

FIG. 2. Charged Pion form factor fitting by using our theoretical formula and the data are from Ref. [13]. $X = \sqrt{q^2}$

FIG. 3. Charged Pion form factor in time-like region and space-like region. Data are from Refs. [1, 12, 14, 13, 15]. $X = \sqrt{q^2}$ when $q^2 > 0$ and $X = -\sqrt{-q^2}$ when $q^2 < 0$
FIG. 2.
FIG. 3.