Structure of wavefunction for interacting bosons in mean-field with random $k$-body interactions

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Abstract

Wavefunction structure is analyzed for interacting many-boson systems using Hamiltonian $H$, which is a sum of one-body $h(1)$ and an embedded GOE of $k$-body interaction $V(k)$ with strength $\lambda$. For sufficiently large $\lambda$, the conditional $q$-normal density describes Gaussian to semi-circle transition in strength functions as body rank $k$ of the interaction increases. This interpolating form describes the fidelity decay after $k$-body interaction quench very well. Also, obtained is the smooth form for the number of principal components, which is a measure of chaos in finite interacting many-particle systems and it describes embedded ensemble results well in chaotic domain for all $k$ values.
I. INTRODUCTION

Wishart’s historical paper [1] on multivariate statistics, gave birth to the field of Random Matrix Theory (RMT). Later, Wigner introduced RMT in the field of Physics, while addressing compound nucleus resonances [2] and the tripartite classification of classical random matrices (Gaussian Orthogonal Ensemble (GOE), Gaussian Unitary Ensemble (GUE) and Gaussian Symplectic Ensemble (GSE)) was given by Dyson [3]. Due to the Universality of RMT [4], accompanied with a great deal of mathematical work done over the years, RMT is now not only limited to the fields of Science [5–11], but also has emerged as a multidisciplinary research area with numerous applications in fields like finance and econophysics [12, 13].

Constituents of isolated quantum systems interact via few-body interactions whereas the classical random matrix ensembles (and in particular the Gaussian orthogonal ensemble (GOE)) take into account many-body interactions. This motivated French and co-workers to introduce random matrix model accounting for few-body interactions, called *Embedded ensemble* (EE) of $k$-body interaction, EGOE($k$) [14, 15]. With two-body interaction (i.e. $k = 2$), and in the presence of mean-field one-body part they are called EGOE(1+2). It is now well established that these EGOE(1+2) models are paradigmatic models to study the dynamical transition from integrability to chaos in isolated finite interacting many-body quantum systems [5, 8, 16, 17]. These models were initially analyzed for isolated finite interacting spin-less systems (both fermion [16, 18–21] as well as boson (denoted by BE- GOE(1+2)) [22–27]), as they are generic models for finite isolated interacting many-particle systems. Moreover, EGOE(1+2) models with spin degree of freedom for isolated interacting fermion and boson systems have also been analyzed in detail. See [5] and references therein. In these investigations, the main focus was on one- plus two-body part of the interaction as inter-particle interaction is known to be only one-body and two-body in nature. However, it is seen that the higher body interactions $k > 2$ play an important role in strongly interacting quantum systems [28, 29], nuclear physics [30], quantum black holes [10, 31] and wormholes [32] with SYK model and also in quantum transport in disordered networks connected by many-body interactions [33–35]. Therefore, it is necessary to extend the analysis of EE to higher $k$-body interactions in order to understand these problems. They are represented by EGOE($k$) (or BEGOE($k$)) for fermion (or boson) systems.
From the previous studies, it is known that with EGOE\((k)\), the eigenvalue density for a system of \(m\) fermions/bosons in \(N\) single particle (sp) states changes from Gaussian form to semi-circle as \(k\) changes from 2 to \(m\) \([5, 9, 18, 36]\). Very recently, \(q\)-Hermite polynomials have been employed to study spectral densities of the so-called SYK model \([37, 38]\) and quantum spin glasses \([39]\), along with studying the strength functions (also known as local density of states (LDOS)) and fidelity decay (also known as survival or return probability) in EE, both for fermion as well as boson systems \([36]\). Formulas for parameter \(q\) in terms of \(m\), \(N\) and \(k\) are derived for fermionic and bosonic EE in \([36]\) which explain the Gaussian to semi-circle transition in spectral densities and strength functions and fidelity decay in many-body quantum systems as a function of rank of interactions. Recently, the lower-order bivariate reduced moments of the transition strengths are examined for the action of a transition operator on the eigenstates generated by EGOE\((k)\) and it is shown that the ensemble averaged distribution of transition strengths follows a bivariate \(q\)-normal distribution \([40]\). Also using the bivariate \(q\)-normal form, a formula for the chaos measure, number of principal components (NPC), in the transition strengths from a state is presented in \([40]\).

Going beyond this, in the present paper, structure of wavefunction, for interacting boson systems using one-body plus an embedded GOE of \(k\)-body interactions (BEGOE\((1+k)\)), has been analyzed by studying strength functions. It is very crucial to study the strength functions in detail, as they carry information that is essential to study structure of wavefunction in isolated finite interacting quantum systems. The chaos measures like NPC, information entropy, fidelity decay etc. can be studied by examining the general features of the strength functions \([5]\). These measures play a very important role in studying quantum many-body chaos and thermalization in isolated finite interacting many-body quantum systems. It is important to note that for BEGOE\((1+2)\) the strength functions become asymmetrical in energy as one moves away from the center \([41]\). Very recently, the analytical formula for the lowest four moments of the strength functions for fermion systems are derived in \([42]\) and it is shown that the conditional \(q\)-normal density \(f_{CqN}\) (defined ahead) can be used to represent strength functions. A similar behavior is expected for boson-systems. Here, in the first analysis, it is established numerically that the strength functions of BEGOE\((1+k)\) for boson systems can be represented by the conditional distribution \(f_{CqN}\) of bivariate \(q\)-normal distribution \(f_{biv-qN}\) in the chaotic domain. Further, the variation of widths of the strength functions, in terms of the correlation coefficient \(ζ\) (defined ahead), is studied numerically.
as a function of $k$-body interaction strength $\lambda$. Then, this interpolating form $f_{CqN}$ is used to describe fidelity decay in BEGOE(1+$k$) after random $k$-body interaction quench. In the second analysis, a simple two parameter analytical formula for NPC as a function of energy, valid in chaotic domain, is derived using $f_{CqN}$ form for the strength functions and theoretical estimates are compared with numerical embedded ensemble results of BEGOE(1+$k$) for all $k$ values. These are the main results of this paper and they are important as the higher body interactions $k > 2$ play an important role in strongly interacting quantum systems [28, 29], quantum black holes [10, 31, 32] and also in quantum transport [33–35]. Also nuclear interactions are now known to have some small 3-body and 4-body parts of interaction. This analysis can be eventually useful to understand further more about transition strengths generated due to $k$-body interactions as these are important for many purposes including that they are observable in many situations [5, 9, 40].

The paper is organized as follows. In Section II, we briefly introduce BEGOE(1+$k$) and the formula for parameter $q$, defining $q$-Hermite polynomials for BEGOE($k$), for the sake of completeness (even though it is clearly given in [9, 36]) along with results for the variation of $q$ parameter as a function of $\lambda$. In Section III, Gaussian to semi-circle transition in the strength functions is described using the conditional $q$-normal form and the variation of width of strength functions is studied by varying $\lambda$ in BEGOE(1+$k$). In Section IV, this interpolating form for the strength functions is used to describe the fidelity decay after random $k$-body interaction quench in BEGOE(1+$k$). Further, two parameter ($\zeta$ and $q$) analytical formula for NPC, valid in chaotic domain, is derived as a function of energy for $k$-body interaction and they are tested with numerical embedded ensemble BEGOE(1+$k$) results in Section V. Also, results for information entropy, $S_{\text{info}}$, are compared with the integral forms involving the conditional $q$-normal density for the strength functions. Finally, Section VI gives conclusions.

II. PRELIMINARIES

A. Embedded bosonic ensembles - BEGOE(1+$k$)

Consider $m$ spinless bosons distributed in $N$ degenerate sp states interacting via $k$-body ($1 \leq k \leq m$) interactions. Distributing these $m$ bosons in all possible ways in $N$ sp states
generates many-particle basis of dimension \( d = \binom{N+m-1}{m} \). The \( k \)-body random Hamiltonian \( V(k) \) is defined as,

\[
V(k) = \sum_{k_a,k_b} V_{k_a,k_b} B^\dagger(k_a)B(k_b) .
\]

Here, operators \( B^\dagger(k_a) \) and \( B(k_b) \) are \( k \)-boson creation and annihilation operators. They obey the boson commutation relations. \( V_{k_a,k_b} \) are the symmetrized matrix elements of \( V(k) \) in the \( k \)-particle space with the matrix dimension being \( d_k = \binom{N+k-1}{k} \). They are chosen to be randomly distributed independent Gaussian variables with zero mean and unit variance, in other words, \( k \)-body Hamiltonian is chosen to be a GOE. \( \text{BEGOE}(k) \) is generated by action of \( V(k) \) on the many-particle basis states. Due to \( k \)-body nature of interactions, there will be zero matrix elements in the many-particle Hamiltonian matrix, unlike a GOE. By construction, we have a GOE for the case \( k = m \). For further details about these ensembles, their extensions and applications, see [5, 43, 44] and references therein.

In realistic systems, bosons also experience mean-field generated by presence of other bosons in the system and hence, it is more appropriate to model these systems by \( \text{BEGOE}(1+k) \) defined by,

\[
H = h(1) + \lambda V(k)
\]

Here, the one-body operator \( h(1) = \sum_{i=1}^{N} \epsilon_i n_i \) is described by fixed sp energies \( \epsilon_i \); \( n_i \) is the number operator for the \( i \)th sp state. The parameter \( \lambda \) represents the strength of the \( k \)-body (\( 2 \leq k \leq m \)) interaction and it is measured in units of the average mean spacing of the sp energies defining \( h(1) \). In this analysis, we have employed fixed sp energies \( \epsilon_i = i + 1/i \) in defining the mean-field Hamiltonian \( h(1) \).

A very significant property of \( \text{EE}(k) \) (and \( \text{EE}(1+k) \)) is that in general they exhibit Gaussian to semi-circle transition in the eigenvalue density as \( k \) increases from 1 to \( m \) [21]. This result is now well established from many numerical calculations and analytical proofs obtained via lower order moments [5, 9, 24, 43, 45, 46]. Very recently, it is shown that generating function for \( q \)-Hermite polynomials describes this transition in spectral densities and in strength functions using \( k \)-body EGOE and their Unitary variants EGUE, both for fermion and boson systems, as a function of rank of interaction \( k \) [36]. In the next sub-section, we will briefly describe \( q \)-Hermite polynomials, \( q \)-normal distribution along with the bivariate \( q \)-normal distribution and give the formula for parameter \( q \).
B. \textit{q-Hermite polynomials and conditional $q$-normal distribution}

The $q$-Hermite polynomials were first introduced by L. J. Rogers in Mathematics. Consider $q$ numbers $[n]_q$ defined as $[n]_q = (1 - q)^{-1}(1 - q^n)$. Then, $[n]_{q-1} = n$, and $[n]_q! = \prod_{j=1}^{n}[j]_q$ with $[0]_q! = 1$. Now, $q$-Hermite polynomials $H_n(x|q)$ are defined by the recursion relation \[47,\]

\[xH_n(x|q) = H_{n+1}(x|q) + [n]_q H_{n-1}(x|q)\] (3)

with $H_0(x|q) = 1$ and $H_{-1}(x|q) = 0$. Note that for $q = 1$, the $q$-Hermite polynomials reduce to normal Hermite polynomials (related to Gaussian) and for $q = 0$ they will reduce to Chebyshev polynomials (related to semi-circle). Importantly, $q$-Hermite polynomials are orthogonal within the limits $\pm 2/\sqrt{1-q}$, with the $q$-normal distribution $f_{qN}(x|q)$ as the weight function defined by \[40,\]

\[f_{qN}(x|q) = \frac{\sqrt{1-q}}{2\pi \sqrt{4-(1-q)x^2}} \prod_{i=0}^{\infty} (1-q^{i+1})[(1+q^i)^2 - (1-q)q^i x^2]. \] (4)

Here, $-2/\sqrt{1-q} \leq x \leq 2/\sqrt{1-q}$ and $q \in [0,1]$. Note that $\int_{s(q)} f_{qN}(x|q) \, dx = 1$ over the range $s(q) = (-2/\sqrt{1-q}, 2/\sqrt{1-q})$. It is seen that in the limit $q \to 1$, $f_{qN}(x|q)$ will take Gaussian form and in the limit $q = 0$ semi-circle form. Now the bivariate $q$-normal distribution $f_{biv-qN}(x,y|\zeta, q)$ is defined as follows \[40, 48,\]

\[f_{biv-qN}(x,y|\zeta, q) = f_{qN}(x|q) f_{CqN}(y|x; \zeta, q)\] (5)

\[= f_{qN}(y|q) f_{CqN}(x|y; \zeta, q)\]

where $\zeta$ is the bivariate correlation coefficient and the conditional $q$-normal densities, $f_{CqN}$ can be given as,

\[f_{CqN}(x|y; \zeta, q) = f_{qN}(x|q) \prod_{i=0}^{\infty} \frac{(1-\zeta^2 q^i)}{h(x,y|\zeta, q)}; \]

\[f_{CqN}(y|x; \zeta, q) = f_{qN}(y|q) \prod_{i=0}^{\infty} \frac{(1-\zeta^2 q^i)}{h(x,y|\zeta, q)}; \] (6)

\[h(x,y|\zeta, q) = (1-\zeta^2 q^{2i})^2 - (1-q)\zeta q^i(1+\zeta^2 q^{2i})xy + (1-q)\zeta^2(x^2 + y^2)q^{2i}. \]
The $f_{CqN}$ and $f_{biv-qN}$ are normalized to 1 over the range $s(q)$, which can be inferred from the following property,

$$\int_{s(q)} H_n(x|q)f_{CqN}(x|y; \zeta, q) \, dx = \zeta^n H_n(y|q).$$

(7)

Recently, it is established via the lower order reduced bivariate moments of $f_{biv-qN}$ that the transition strength densities generated by EGOE and EGUE random matrix ensembles follow bivariate $q$-normal form [40]. From the above equations, the first four moments of the $f_{CqN}$ are given as,

- Centroid = $\zeta y$,
- Variance = $1 - \zeta^2$,
- Skewness, $\gamma_1 = -\frac{\zeta(1-q)y}{\sqrt{1-\zeta^2}}$,
- Excess, $\gamma_2 = (q-1) + \frac{\zeta^2(1-q)^2y^2 + \zeta^2(1-q^2)}{(1-\zeta^2)}$.

(8)

Very recently, first four moments of strength function for $k$-body fermionic embedded ensemble are derived in [42] and it is shown that they are essentially same as that of $f_{CqN}$. Therefore, in general, one can analyze the structure of wavefunction in quantum many-systems with $k$-body interactions using the conditional $q$-normal distribution.

C. Formula for $q$-parameter

Histograms in Figure 1(a) represent ensemble averaged state density obtained for a 100 member BEGOE(1+$k$) ensemble with $m = 10$ bosons distributed in $N = 4$ sp states and the body rank of interaction changing from $k = 2$ to 10. In these calculations, the eigenvalue spectrum for each member of the ensemble is first zero centered ($\epsilon_H$ is centroid) and scaled to unit width ($\sigma_H$ is width) and then the histograms are constructed. The results clearly display transition in the spectral density from Gaussian to semi-circle form as $k$ changes from 2 to $m = 10$. With $E$ as zero centered and using $x = E/\sigma_H$, the numerical results are compared with the normalized state density $\rho(E) = d \, f_{qN}(x|q)$ with $\epsilon_H - \frac{2\sigma_H}{\sqrt{1-q}} \leq E \leq \epsilon_H + \frac{2\sigma_H}{\sqrt{1-q}}$. Here the value for parameter $q$ is computed using the formula given in terms of $(N, m, k)$.
which can be obtained by comparing reduced fourth moment of $q$-normal distribution given by Eq.(4) with that of BEGOE(k). The formula for $q$ is given by,

$$q_{V(k)} \sim \left( \frac{N + m - 1}{m} \right)^{-1} \sum_{\nu=0}^{\nu_{\text{max}}} \frac{X(N, m, k, \nu) \ d(g_{\nu})}{[\Lambda_{0}^{0}(N, m, k)]^{2}};$$

$$X(N, m, k, \nu) = \Lambda^{\nu}(N, m, m - k) \Lambda^{\nu}(N, m, k);$$

$$\Lambda^{\nu}(N, m, r) = \binom{m - \nu}{r} \binom{N + m + \nu - 1}{r},$$

$$d(g_{\nu}) = \left( \frac{N + \nu - 1}{\nu} \right)^{2} - \left( \frac{N + \nu - 2}{\nu - 1} \right)^{2}.$$ 

Although, the above equation is given for BEGOE(k), it can be applicable to BEGOE(1+k) with sufficiently large $\lambda$ in Eq.(2), as the k-body part of the interaction is expected to dominate over one-body part. The numerical results in Figure 1 are in excellent agreement with the theory. Therefore, $f_{qN}(x|q)$ with $q$ given by Eq. (9), describes the eigenvalue density for EE(k) \[36, 45\]. For $\lambda = 0$ in Eq.(2), i.e. with one-body part $h(1)$ only, the analytical formula for $q$, based on trace propagation method \[49\], can be given as,

$$q_{h(1)} = \langle h(1)^{4} \rangle_{m}^{m} - 2,$$

$$= \left\{ \frac{3(m-1)N(1+N)(1+m+N)}{m(2+N)(3+N)(m+N)} - 2 \right\}$$

$$+ \frac{m^{2} + (N + m)^{2} + (N + 2m)^{2}}{m(N + m)} \frac{\sum_{i=1}^{N} \epsilon_{i}^{4}}{(\sum_{i=1}^{N} \epsilon_{i}^{2})^{2}}.$$

Here, $\langle h(1)^{4} \rangle_{m}$ is the reduced fourth moment of one-body part and $\epsilon_{i}$ is the traceless sp energies of i’th state. For $(m = 10, N = 5)$ example with $H = h(1)$, uniform sp energies $\epsilon_{i} = i$ gives $q = 0.71$, while with sp energies used in the present study, $\epsilon_{i} = i + 1/i$ gives $q = 0.68$. It can be clearly seen that in the dense limit \[25\] $(m \to \infty, N \to \infty$ and $m/N \to \infty)$, the second term in Eq.(10) approaches zero and the first term leads to $q \to 1$. Figure 1(b) shows the variation of parameter $q$ as the strength of interaction $\lambda$ varies in BEGOE(1+k) for given body rank $k$. Here, the ensemble averaged value of $q$ is computed for a system of 100 member BEGOE(1+k) ensemble with $m = 10$ bosons in $N = 5$ sp states. It is seen from the results that in the weak interaction domain (with small $\lambda$), $q$ decreases with increasing $k$ with $k \leq 6$. While for $k > 6$ it increases again with $k$. This is due the
fact that for a $k$-body operator, the $m$-particle scaler average is related to $k$-particle scaler average via through binomial coefficient $\binom{m}{k}$. Further, as in the strong interaction domain, $k$-body part dominates over one-body part i.e. BEGOE($1+k$) result reduces to BEGOE($k$), $q$ approaches constant value close to given by Eq.(9). It can be seen from results that the ensemble averaged values are in excellent agreement with the expected values both in weak and strong interaction domain.

Now, we will examine the shape of the strength functions $F_\xi(E)$ in the strong interaction domain as the body rank of interaction $k$ in BEGOE($1+k$) is changed. Recently, using numerical examples, both for fermions and bosons, it is shown that the strength functions $F_\xi(E)$ can be well represented by the $q$-normal $f_{qN}(x|q)$ form for $\xi$ states at the center of the spectrum for all $k$ values in $V(k)$ [36]. However, the strength functions become asymmetrical in $E$ for $\xi \neq 0$ [41]. This feature can be generated by $f_{CqN}$ which can not be incorporated using $f_{qN}$. Here, conditional $q$-normal density function, $f_{CqN}$ (Eq.(6)), is considered for describing the Gaussian to semi-circle transition in strength functions shown that $f_{CqN}$ represents strength functions to a good approximation. More importantly, this can be useful in studying statistical properties like fidelity decay, NPC, information entropy etc.

III. STRENGTH FUNCTION

Given $m$-particle basis state $|\kappa\rangle$, the diagonal matrix elements of $m$-particle Hamiltonian $H$ are denoted as energy $\xi_\kappa$, so that $\xi_\kappa = \langle \kappa | H | \kappa \rangle$. The diagonalization of the full matrix $H$ gives the eigenstates $|E_i\rangle$ with eigenvalues $E_i$, where $|\kappa\rangle = \sum_i C_\kappa^i |E_i\rangle$. The strength function that corresponds to the state $|\kappa\rangle$ is defined as $F_{\xi_\kappa}(E) = \sum_i |C_\kappa^i|^2 \delta(E - E_i)$. In the present study, we take the $|\kappa\rangle$ states to be the eigenstates of $h(1)$. In order to get an ensemble average form of the strength functions, the eigenvalues $E_i$ are scaled to have zero centroid and unit variance for the eigenvalue distribution. The $\kappa$-energies, $\xi_\kappa$, are also scaled similarly. Now, for each member, all $|C_\kappa^i|^2$ are summed over the basis states $\kappa$ with energy $\xi$ in the energy window $\xi \pm \Delta$. Then, the ensemble averaged $F_{\xi}(E)$ vs. $E$ are constructed as histograms by applying the normalization condition $\int_{s(q)} F_{\xi}(E) \, dE = 1$. In Figure 2, histograms represent ensemble averaged $F_{\xi}(E)$ results for body rank $k$ values in BEGOE($1+k$) with $m = 10$ bosons in $N = 5$ sp states. The strength function plots are
FIG. 1. (a) Histograms represent the state density vs. normalized energy $E$ results of the spectra of a 100 member BEGOE(1+$k$) ensemble with $m = 10$ bosons in $N = 4$ sp states for different $k$ values. The strength of interaction $\lambda = 0.5$ is chosen and in the plots $\int \rho(E) dE = d$. Ensemble averaged state density histogram is compared with $q$-normal distribution (continuous black curves) given by $f_{qN}(x|q)$ with the corresponding $q$ values given by Eq. (9). (b) Ensemble averaged $q$ vs. $\lambda$ for a 100 member BEGOE(1+$k$) ensemble with $m = 10$ bosons in $N = 5$ sp states for different $k$ values. The horizontal black mark on left $q$-axis indicates $q$ estimate for $H = h(1)$, while the colored marks on right $q$-axis represent the $q$ values, given by Eq. (9), for corresponding $k$-body rank with $H = V(k)$. See text for more details.
obtained for \( \xi = 0.0, \pm 1.0 \) and \( \pm 2.0 \). The value of \( k \)-body interaction strength \( \lambda \) is chosen close to the region of thermalization \([27, 45]\). The histograms, representing BEGOE(1+\( k \)) results of strength functions, are compared with the conditional \( q \)-normal density function as given by,

\[
F_\xi(E) = f_{CqN}(x = E | y = \xi; \zeta, q).
\]  

Here, the parameter \( \zeta \) (the correlation coefficient between full Hamiltonian \( H \) and the diagonal part \( H_{\text{diag}} \) of the full Hamiltonian) is given by,

\[
\zeta = \sqrt{1 - \frac{\sigma^2_{H_{\text{off-diag}}}}{\sigma^2_H}}. \tag{12}
\]

In the above equation, \( \sigma^2_H \) and \( \sigma^2_{H_{\text{off-diag}}} \) are variances of the eigenvalue distribution using full Hamiltonian and by taking all diagonal matrix elements as zero, respectively. The smooth curves in Figure 2 are obtained via Eq.(11). Here, ensemble averaged \( \zeta \) and \( q \) value given by Eq.(9) are used. In the strong interaction domain \( \zeta \rightarrow 0 \) and then \( q \) value in \( f_{CqN}(x = E | y = \xi; \zeta, q) \) can be given by Eq.(9) \([42]\). The results clearly show very good agreement between the numerical histograms and continuous black curves for all body rank \( k \). The \( F_\xi(E) \) results for \( \xi = 0 \) are given in Figure 2(a) which clearly show that the strength functions are symmetric and also exhibit a transition from Gaussian form to semi-circle as \( k \) changes from 2 to \( m = 10 \) in BEGOE(1+\( k \)). The smooth form given by Eq.(11) using the conditional \( q \)-normal density function interpolates this transition very well. Going further, \( F_\xi(E) \) results for \( \xi \neq 0 \) are shown in Figures 2(b) and 2(c). One can see that \( F_\xi(E) \) results are asymmetrical about \( E \) as demonstrated earlier in \([41]\). Also, \( F_\xi(E) \) are skewed more in the positive direction for \( \xi > 0 \) and skewed more in the negative direction for \( \xi < 0 \) and their centroids vary linearly with \( \xi \). We also calculated the first four moments (centroid, variance, skewness (\( \gamma_1 \)) and excess (\( \gamma_2 \))) of the strength function results shown in Figure 2 for the body rank \( k \) going from 2 to \( m = 10 \). Figure3 represents results for centroid, \( \gamma_1 \) and \( \gamma_2 \) for various values of \( \xi \). The variance of the strength functions (independent of \( \xi \)) is simply related to correlation coefficient, \( \sigma_F^2 = 1 - \zeta^2 \). Therefore, for further discussion about \( \sigma_F^2 \), see results of \( \zeta^2 \) (Figure4) ahead. The results shown in Figs.?? and 3 consistent with the analytical results obtained in \([42]\) and they clearly show that the strength functions can be represented by \( f_{CqN} \) with two parameters, \( q \) and \( \zeta \), for sufficiently large \( \lambda \).

Going further, as \( \lambda \) decreases in BEGOE(1+\( k \)), strength functions will take the Breit-Wigner(BW) form for fixed \( k \). One can observe the BW to Gaussian to semi-circle transition
in strength functions by changing both $\lambda$ and $k$. Therefore, it is possible to have a shape intermediate to BW and semi-circle for some values of $\lambda$ and $k$. This detail can be explored by computing $\zeta^2$ as it is related to width of the strength functions $\sigma_F$. We have calculated ensemble averaged $\zeta^2$ as a function of $k$-body interaction strength $\lambda$ for a 100 member BEGOE(1+$k$) ensemble with $(m = 10, N = 5)$. The results for various values of body rank $k$ are presented in Figure 4(a). It is clear from the results that $\zeta^2$ is close to 1 for all $k$ when interaction $\lambda$ is just switched on in BEGOE(1+$k$) and as $\lambda$ increases, $\zeta^2$ goes on decreasing smoothly. The rate of decrease is faster with up to $k = 6$ and then it increases again for $k > 6$. Similar behavior is observed in the results of $q$ as well. In the past, an analytical formula for $\zeta^2$, as a function of $\lambda$, was derived for $k = 2$ in [26] for bosons. However, it will be more complex to derive the same for BEGOE(1+$k$). Following [26], a simple formula for $\zeta^2$ for BEGOE($k$), with all sp energies as degenerate, can be written as

$$\zeta^2 = \frac{4}{\binom{N+k-1}{k} + 1}. \quad (13)$$

Here, $\binom{N+k-1}{k}$ is the dimensionality of $k$-particle space. From the above equation, it can be immediately seen that $\zeta^2$ is independent of $m$ and for $k$ large $k$, $\zeta^2 \to 4/(N^2 + N + 2)$. Thus, as $N \to \infty$ and $k \to \infty$, $\zeta^2 \to 0$ giving $\sigma_F \to 1$ which is consistent with the result obtained in [26]. We have compared $\zeta$ values calculated for sufficiently strong $\lambda$ with that of obtained using Eq.(13) and results are shown in Figure 4(b). One can see that the match between ensemble averaged results and estimates given by Eq.(13) is very good.

In the study of thermalization and relaxation dynamics of an isolated finite quantum system after a random interaction quench, the strength functions $F_\xi(E)$ play a major role. In the next section, the interpolating form $f_{CqN}$ for the strength function, given by Eq. (11), is used to study fidelity decay in BEGOE(1+$k$).

IV. FIDELITY DECAY AFTER AN INTERACTION QUENCH

Fidelity decay or return probability of a quantum system after a sudden quench is an important quantity in the study of relaxation of a complex (chaotic) system to an equilibrium state. Let’s say the system is prepared in one of the eigenstates ($\psi(0) = |\kappa\rangle$) of the mean-field Hamiltonian $H = h(1)$. With the quench at $t = 0$ by $\lambda V(k)$, the system evolves unitarily with respect to $H \to h(1) + \lambda V(k)$ and the state changes after time $t$ to $\psi(t) = |\kappa(t)\rangle =$
FIG. 2. Strength function vs. normalized energy $E$ for a system of $m = 10$ bosons in $N = 5$ sp states for different $k$ values in BEGOE(1+$k$) ensemble. An ensemble of 250 members is used for each $k$. Strength function plots are obtained for (a) $\xi = 0$ (Purple histogram), (b) $\xi = -1.0$ (Blue histogram) and 1.0 (Red histogram) and (c) $\xi = -2.0$ (Blue histogram) and 2.0 (Red histogram).

In the plots $\int F_\xi(E)dE = 1$. The continuous black curves are due to fitting with $f_{CqN}$ given by Eq. (11) using $q$ and $\xi$ values obtained by Eq. (9) and Eq. (12), respectively. See text for more details.
FIG. 3. Ensemble averaged (a) Centroid, (b) $\gamma_1$ and (c) $\gamma_2$ as a function of body rank $k$ for the strength function results presented in Fig. 2. Results are shown for various values of $\xi$.

Moreover, the probability to find the system in its initial unperturbed state after time $t$, called fidelity decay, is given by,

$$W_0(t) = |\langle \psi(t) | \psi(0) \rangle|^2$$

$$= \left| \sum_E \left[ C_k^E \right]^2 \exp(-iEt) \right|^2$$

$$= \int \mathcal{S}(q) F_\xi(E) \exp(-iEt) dE. \quad (14)$$

Therefore, $W_0(t)$ is the Fourier transform of the strength function given by Eq. (11); this is valid for times not very short or very long. In the thermalization region, the form of $F_\xi(E)$ is Gaussian for $k = 2$ while it is semi-circle for $k = m$. These two extreme situations are recently studied, both analytically and numerically, in [51] and [52–54] respectively. The formula for $W_0(t)$ can be given in terms of $\sigma_\kappa$, width of $\lambda V(k)$ scaled by $\sigma_H$. It is important to note that $\sigma_\kappa$ is connected to correlation coefficient $\zeta$. Clearly, following the results of the previous section, $f_{CqN}$ can be used to obtain $W_0(t)$ generated by BEGOE(1+k). As
FIG. 4. (a) Ensemble averaged $\zeta^2$ as a function of interaction strength $\lambda$ in BEGOE$(1+k)$ ensemble with $N = 5$, $m = 10$ example, is shown for different $k$ values. (b) Ensemble averaged $\zeta$ vs. $k$. Black dots represent ensemble averaged $\zeta$ results using $\lambda = 1$ in BEGOE$(1+k)$ and red dashed curve is due to Eq.(13). The $\zeta$ values are obtained at $k = 2, 3, ..., 10$ and the red dashed curve is drawn just to guide the eye.

analytical formula for the Fourier transform of Eq.(11) is not available, $W_0(t)$ is evaluated by computing the Fourier transform of Eq.(11) numerically. Figure 5 shows results for $W_0(t)$ (red solid circles) for a 100 member BEGOE$(1+k)$ ensemble with $m = 10$ and $N = 5$ for various $k$ values and they are compared with numerical Fourier transform (black smooth curves) of Eq.(11). It is clear from the results that the interpolating form for the strength functions represented by conditional $q$-normal distribution describes fidelity decay after random interaction quench very well. Let us add that fidelity decay is correlated with entropy production and its saturation with time \[51–54\]. This can be investigated using $f_{CqN}$ for higher rank of the interaction as well and this will be addressed in future.

Going beyond the above results, the behavior of fidelity decay over a long time is of great interest as it is expected that $W_0(t)$ surely demonstrates a power-law behavior i.e. $W_0(t) \propto t^{-\gamma}$ with $\gamma \geq 2$ implying thermalization, no matter how fast the decay may initially be. This problem was addressed in \[55\] assuming BW and Gaussian forms for strength functions and also using spin models. The Gaussian domain gives $\gamma = 2$. As shown in \[55\], the power-law behavior appears due to the fact that the energy spectrum is bounded from both the ends. This condition is essentially satisfied by $f_{CqN}$. Therefore, it is interesting to study long-time behavior of fidelity decay for EE using $f_{CqN}$. This is for future.
FIG. 5. Fidelity decay \( W_0(t) \) as a function of time for a 100 member BEGOE(1+k) ensemble with \( N = 5 \) and \( m = 10 \) represented by the red solid circles; the \( \psi(0) \) here corresponds to middle states of \( h(1) \) spectrum. The black smooth curves are obtained by taking numerical Fourier transform of the strength functions represented by Eq.\((11)\).

In the next section, we will analyze the wavefunction structure given by NPC and \( S^{\text{info}} \). Importantly, these quantities for a wavefunctions can be written as integrals over all \( \xi \) energies involving strength functions. Very recently, an integral formula for NPC in the transition strengths from a state as a function of energy for fermionic EGOE(\( k \)) using the bivariate \( q \)-normal form is presented in [40]. In the past, the smooth forms, for NPC and \( S^{\text{info}} \), were derived in terms of energy and correlation coefficient \( \zeta \) for \( k = 2 \) in [56]. It is interesting to check whether these forms are also applicable to \( k \)-body force in these ensembles with \( k > 2 \).

V. NPC AND INFORMATION ENTROPY

The NPC in wavefunction characterizes various layers of chaos in interacting particle systems [20, 57, 58] and for a system like atomic nuclei, NPC for transition strengths is a measure of fluctuations in transition strength sums [40]. For an eigenstate \( \ket{E_i} \) spread over the basis states \( \ket{\kappa} \), with energies \( \xi_\kappa = \langle \kappa | H | \kappa \rangle \), NPC (also known as inverse participation ratio) is defined as,

\[
\text{NPC}(E) = \left\{ \sum_\kappa |C_\kappa|^4 \right\}^{-1}
\]  \hspace{1cm} (15)
NPC essentially gives the number of basis states $|\kappa\rangle$ that constitute an eigenstate with energy $E$. The GOE value for NPC is $d/3$. NPC can be studied by examining the general features of the strength functions $F_\xi(E)$. The smooth forms for NPC($E$) can be written as [56],

$$\text{NPC}(E) = \frac{d}{3} \left\{ \int d\xi \frac{\rho^H_\kappa(\xi)[F_\xi(E)]^2}{[\rho^H(E)]^2} \right\}^{-1}. \tag{16}$$

Taking $E$ and $\xi$ as zero centered and scaled by corresponding widths, one can write above equation in terms of $f_{qN}$ and $f_{CqN}$ [40, 42],

$$\text{NPC}(E) = \frac{d}{3} \left\{ \int d\xi \frac{f_{qN}(\xi|q)f_{CqN}(E|\xi; \zeta, q)^2}{f_{qN}(E|q)} \right\}^{-1}. \tag{17}$$

In the above equation, the integral is over the range $S(q)$. In general, $q$’s in the above equation need not be same [40, 42]. However, in the strong interaction domain, as $\zeta \to 0$, one can approximate $\gamma_2 = q - 1$ in Eq.(8). Then, the formula for $q$ given by Eq.(9) is valid for $f_{CqN}$. This is well verified numerically in Section II. Also, the results of $\gamma_2$ in Figure 3(c) corroborate this claim. With this, it is possible to simplify above using Eqs.(6) and (7) and a simple two parameter formula, valid in chaotic domain, for NPC can be written as,

$$\text{NPC}(E) = \frac{d}{3} \left( \sum_{n=0}^{\infty} \frac{\zeta^{2n}}{n!} H_n^2(E|q) \right)^{-1}. \tag{18}$$

It is easy to see from above formula that NPC($E$) approaches GOE value $d/3$ as $\zeta \to 0$. Also for $q \to 1$, $f_{qN}$ and $f_{CqN}$ in Eq.(17) reduce to Gaussian and then Eq.(18) gives results obtained in [56]. We have tested this formula with numerical ensemble averaged BEGOE($1+k$) results. Figure 6, shows results for ensemble averaged NPC vs. normalized energy, for a 100 member BEGOE($1+k$) with $m = 10$ bosons in $N = 5$ sp states for different values of rank of interaction $k$. The ensemble averaged NPC values are shown with red solid circles and continuous lines are obtained using the theoretical expression given by Eq. (18).

It is clearly seen from the results that for given $k$: (i) for small value of $\lambda$, where the one-body part of the interaction is dominating, the numerical NPC values are zero and the theoretical curve is far away from the numerical results indicating that the wavefunctions are completely localized (the bottom panels in Figure 6); (ii) with further increase in $\lambda$, the theoretical estimate for NPC in the chaotic domain is much above the ensemble averaged curve indicating that the chaos has not yet set in; (iii) However, with sufficiently large $\lambda$, we see that the ensemble averaged curve is matching with the theoretical estimate given by
Eq. (18), indicating that system is in chaotic domain corresponding to the thermalization region given by $\zeta^2 \sim 0.5$ [27] and the strength functions $F_\xi(E)$ are well represented by Eq.(11). Again with further increase in $\lambda$ (the top panels in Figure 6), the match between the theoretical chaotic domain estimate and the ensemble averaged values is very well in the bulk part of the spectrum ($|E| < 2$) for all values of $k$ with deviations near the spectrum tails. Hence, in the chaotic domain, the energy variation of NPC($E$) using Eq. (18) is essentially given by parameters $\zeta$ and $q$. It is also seen from the results that for NPC there is a transition from Gaussian form to the GOE result as the body rank $k$ of interaction in BEGOE($1+k$) increases. One can also see that the thermalization sets in faster as the body rank $k$ of the interaction increases.

Another statistical quantity normally considered is the information entropy defined by $S_{\text{info}}(E) = -\sum_\kappa p_\kappa^i \ln p_\kappa^i = -\sum_\kappa |C_\kappa^i|^2 \ln |C_\kappa^i|^2$, here $p_\kappa^i$ is the probability of basis state $\kappa$ in the eigenstate at energy $E_i$. The localization length, $l_H$ is related to $S_{\text{info}}(E)$ by $l_H(E) = \exp\{S_{\text{info}}(E/\zeta,q)/0.48d\}$. Then the corresponding embedded ensemble expression for $l_H$ involving $F_\xi(E)$, for the chaotic domain can be written as[56],

$$l_H(E) = -\int d\xi \frac{F_\xi(E)}{\rho^H(E)} \ln \left\{ \frac{F_\xi(E)}{\rho^H(E)} \right\}.$$  

(19)

Replacing $\rho^H(\xi)$ and $\rho^H(E)$ by $f_{qN}$ and $F_\xi(E)$ by $f_{CqN}$ will give

$$l_H(E) = -\int d\xi \frac{f_{CqN}(E;\xi,\zeta,q)}{f_{qN}(E;q)} \ln \left\{ \frac{f_{CqN}(E;\xi,\zeta,q)}{f_{qN}(E;q)} \right\}.$$  

(20)

At present, to simplify Eq.(20) for $l_H$ is an open problem. Therefore, we evaluate Eq.(20) numerically and results are compared with ensemble averaged numerical results of BEGOE($1+k$).

Figure 7, shows results for ensemble averaged $l_H$ vs. normalized energy $E$ for a 100 member BEGOE($1+k$) with $m = 10$ bosons in $N = 5$ sp states for different values of $k$. Here, we choose $k$-body interaction strength $\lambda = 1$ so that the system will be in thermalization region. Numerical embedded ensemble results (red solid circles) are compared with theoretical estimates (black curves) obtained using Eq. (20). The $\zeta$ values are shown in the figure. A very good agreement between numerical results and smooth form is obtained for all values of $k$ in the bulk of the spectrum with small deviations near the spectrum tails. Hence, in the chaotic domain, the energy variation of $l_H(E)$, with Eq. (20), is essentially given by conditional $q$ forms for the strength functions.
 Ensemble averaged NPC as a function of normalized energy $\hat{E}$ for a 100 member BEGOE(1+$k$) with $m = 10$ interacting bosons in $N = 5$ sp states for different values of $k$. Ensemble averaged BEGOE(1+$k$) results are represented by solid circles while continuous curves correspond to the theoretical estimates in the chaotic domain obtained using Eq. (18). The ensemble averaged $\zeta$ and $q$ values are also given in the figure. GOE estimate is represented by dotted line in each graph.

VI. CONCLUSIONS

In the present work, we have analyzed wavefunction structure of the many-body bosonic systems by modeling the Hamiltonian of these complex systems using BEGOE(1+$k$). The main results of this work establish that the $q$-Hermite polynomials play an important role in embedded ensembles in explaining the dependence of spectral density, strength functions, fidelity decay and also the measures of chaos, namely NPC and $S^{\text{info}}$, on the rank of interac-
FIG. 7. Ensemble averaged localization lengths $l_H$ vs. normalized energy $E$ for a 100 member BEGOE($1+k$) with $m = 10$ interacting bosons in $N = 5$ sp states for different $k$ values. Here, $\lambda = 1$ chosen for all $k$. Ensemble averaged BEGOE($1+k$) results (red solid circles) are compared with the smooth forms obtained via Eq. (20) involving parameters $\zeta$ and $q$. The ensemble averaged $\zeta$ values are given in the figure and Eq. (9) is used for $q$ values. Dotted lines in each graph represent GOE estimate.

From the $q$ vs $\lambda$ plots, one can infer that the shape of state density varies intermediate between Gaussian to semi-circle as the $k$-body interaction strength in BEGOE($1+k$) is increased. This is important as the nuclear interactions are now known to have some small 3-body and 4-body parts. The strength functions $F_{\xi}(E)$ (defined with respect to $h(1)$ basis), studied numerically in the strong interaction domain, change gradually from Gaussian form to semi-circle as the body rank of interaction $k$ changes from 2 to $m$. The function $f_{CqN}$ with parameters $q$ and $\zeta$, given by Eq. (11), interpolating these two forms is shown to describe the intermediate region very well. This form is also used to study the fidelity decay for any $k$. As with $f_{CqN}$ form the energy spectrum is bounded from both the ends, it is interesting to analyze power-law behavior of fidelity decay for very long time using embedded ensembles with $k$-body forces following the work in [55]. Also the formula for $\zeta^2$ valid in the strong interaction domain is obtained and compared with embedded ensemble numerical results. $\zeta^2$ vs $\lambda$ results clearly show that the thermalization sets in faster as body rank $k$ increases. Furthermore, wavefunction structure given by the BEGOE($1+k$) ensemble is studied by deriving two parameter formula for NPC. It is shown that the smooth form (with $q$ and $\zeta$ parameters) obtained using $f_{qN}$ and $f_{CqN}$ describes the numerical embedded ensemble calculations very well. With these results, it is possible to analyze the thermal-
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