The Vliegenthart-Lekkerkerker relation. The case of the

\textit{Mie}-fluids

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Abstract

The Vliegenthart-Lekkerkerker relation for the second virial coefficient value at the critical temperature found in [G. A. Vliegenthart and H. N. W. Lekkerkerker, J. Chem. Phys. 112 5364 (2000)] is discussed in connection with the scale invariant mean-field approach proposed in [V. L. Kulinskii and L. A. Bulavin, J. Chem. Phys. 133 134101 (2010)]. We study the case of the \textit{Mie}-class potentials which is widely used in simulations of the phase equilibrium of the fluids. It is shown that due to the homogeneity property of the \textit{Mie}-class potentials it is possible to connect the loci of the fluids with these model potentials in different dimensions.

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I. INTRODUCTION

Thermodynamic principle of the corresponding states (PCS) \[\text{[1, 2]}\] is one of the most vivid examples of the unifying nature of the scale invariance. From the microscopic point of view the PCS is based on simple scaling properties of the interaction potential which must be conformal \[\text{[3]}\]. Actually, the microscopic interactions in real substances are more complex and do not conform with the conditions at which the PCS can be derived rigorously from the first principles of statistical mechanics. To expand the range of the applicability of the PCS its simple scaling form was extended to include more parameters which are connected with the most important properties of the interparticle interactions of the simple fluids (see e.g. \[\text{[4]}\]). The locus of the critical point and corresponding compressibility factor \(Z_c = P_c/n_c T_c\), where \(P_c, n_c, T_c\) are the critical pressure, density and the temperature correspondingly, is the core element of the PCS came originally from van der Waals.

The interaction between nearest neighbors dominates in a dense, condensed phase of the molecular system. The liquid-gas critical point (CP) is adjacent to the region of condensed liquid phase. Therefore its locus is determined by the short scale properties of the interactions. They are commonly the distance which corresponds to the minimum of the potential and the depth the potential well \(\varepsilon\). But it is obvious that the very existence of the CP is due to the attraction at long distances \[\text{[5, 6]}\]. For the potential \(\Phi(r)\) of the general type its attractive and repulsive parts are independent. The short-ranged characteristics of the potential weakly depend on the long-range asymptotic behavior of the attractive part. For potentials of specific classes with simple analytical structure, e.g. the conformal potentials, it is possible to relate the short- and the long-ranged characteristics. In such case, as it was noted in \[\text{[7]}\], it is possible to construct the version of the PCS on the basis of a long-range interaction only. The coordinates of the critical point can be related with the long-range properties of the potential. For the potentials of the Lennard-Jones type such scaling approach was proposed in \[\text{[8]}\]. The generalized Lennard-Jones (LJ) potentials or Mie potentials \[\text{[9]}\]:

\[
\Phi(r; m, q) = \varepsilon \Phi_0(q/m) \left( \left( \frac{\sigma}{r} \right)^q - \left( \frac{\sigma}{r} \right)^m \right), \quad q > m.
\] (1)
serves as the example of conformal potentials. Here \( \varepsilon \) is the depth of the potential well, the amplitude coefficient \( \Phi_0(q, m) \) is as following:

\[
\Phi_0(q/m) = \frac{m}{q - m} \left( \frac{q}{m} \right)^{q/(q-m)}.
\] (2)

Below we use common dimensionless units for the temperature \( T \rightarrow T/\varepsilon \) and the density \( n \rightarrow n \sigma^d \), where \( d \) is the dimension. Due to advances in computer simulations there is a great body of results on the liquid-gas equilibrium for the systems with the potentials (1) [10, 11].

Obvious fundamental characteristic which incorporates both repulsive and attractive parts of the potential is the second virial coefficient \( B_2(T) \) [12]:

\[
B_2(T) = 2 \pi \int_0^{\infty} \left( 1 - e^{-\Phi(r)/T} \right) r^2 \, dr.
\] (3)

In Ref. [5] it was demonstrated that for many three dimensional (3D) systems, where particles interact via spherically symmetrical pair interaction \( \Phi(r) \), the relation:

\[
B_2(T_c)/v_0 = -C,
\] (4)

holds, where \( C \) is some constant. It should be noted that \( C \) depends on the number of dimensions \( d \) too. As it follows from [5] the value of \( C \) weakly depends on the details of the potential as far as it belongs to some class of potential functions. In particular the authors of Ref. [5] established that whereas the critical temperature \( T_c \) drops considerably upon narrowing the range of attraction, the combination \( B_2(T_c)/v_0 \) remains practically constant [5]. According to [5] for the generalized potentials \( \Phi(r; 6, q) \), \( C \approx 6.2 \) in three dimensions. These results put the question about the status of (4) as the mean of parametrization the classes of potentials in a sense more general than simple PCS. In such general statement the problem seems rather difficult. Nevertheless the possibility for its simplification may be found in reduction this problem for the continuous systems to their lattice analogues [13].

In this paper we propose the derivation of the Vliegerthart-Lekkerkerker relation for the potentials of Mie-class (1) based on the global isomorphism between the Lennard-Jones fluids and the lattice gas model [8]. The potentials (1) have simple analytical structure and possess homogeneity dependence on the relevant parameters \( (m, q) \) (see Eq. (2)).

The structure of the paper is organized as follows. In Section II we discuss the dependence of the coordinates of the CP on the parameters of the potential based on the results of [8, 14].
In Section III we consider the dependence of the Vliegerthart-Lekkerkerker parameter $C$ on the characteristics of the Mie-class in $d$-dimensional case. The comparison with the known results is given and some predictions are made which can be tested in simulations. The discussion of the obtained results is in conclusive section.

II. THE LOCUS OF THE CRITICAL POINT WITHIN THE GLOBAL ISOMORPHISM APPROACH

It seems that there is no special symmetry property which governs the locus of the critical point in the continuum fluid. Because of the regular structure and the particle-hole symmetry of the configurations the lattice models possess additional symmetry properties which allow to calculate the locus of the critical point exactly without the direct calculation of the thermodynamic potential [15]. For the continuum fluid the particle-hole symmetry is broken and the binodal is asymmetrical. The well-known rectilinear diameter law gives the representation of this asymmetry [1, 16]. Recently the relation between the phase diagrams of the lattice gas and the simple liquid have been proposed in [17]. It is based on the mapping between the fluid part of the phase diagram of the Lennard-Jones (LJ) fluids in the form of the projective transformation:

$$n = n_0 \frac{x}{1 + zt}, \quad T = T_0 \frac{zt}{1 + zt},$$

(5)

where $x, t$ are the density and temperature variables of the lattice gas, $n$ and $T$ are the corresponding quantities for the continuum fluid. Here $n_0$ and $T_0$ are the parameters which will be defined below (see Eq. (8)). The temperature variable $t$ is defined so that its value at the critical point (CP) is $t_c = 1$. Here the parameter $z$ is determined by the correspondence between the loci of critical points of fluid $(n_c, T_c)$ and the lattice gas $x_c = 1/2, t_c = 1$:

$$z = \frac{T_c}{T_0 - T_c}.$$  

(6)

The inverse transformation has the form:

$$x = \frac{n}{1 - T/T_0}, \quad t = \frac{1}{z} \frac{T}{T_0 - T}.$$  

(7)

The main assumption which governs the simple form of the transformation (5) is the validity of the rectilinear diameter law for the density in a broad temperature interval of liquid-gas equilibrium [1]. The inverse form (7) gives the procedure of the symmetrization of
the binodal of the fluid in terms of the variable $x$ which is combination of the density $n$ and the temperature $T$ of the continuum fluid. The idea that the difference between irregularity configuration for continuum fluids and regularity of configurations of lattice models is unimportant for order-disorder transitions was pronounced by K.S. Pitzer in [18]. It was stressed that the main difference is the shape of the holes in real or continuum liquid and that in the lattice gas. This is the reason why the behavior of continuum liquids beyond the fluctuational region is different from that of the lattice gas (Ising model). The global character of the transformation (5) shows that the particle-hole simplified picture is valid not only in near critical region but also far away from it. This rehabilitates the hole theory for expanded liquids [19] which gives the possibility to derive (5) from the microscopic point of view. Yet the transformation Eq. (5) is not exact since its simple form is heavily based on the validity of the law of the rectilinear diameter. In the close vicinity of the critical point there are singular fluctuation corrections to the classical law of the rectilinear diameter [20]. Nevertheless, it is valid for the Lennard-Jones fluids in a broad temperature interval and is widely used to locate the critical point [21]. The numerical data of [22, 23] are consistent with this law and therefore simple form of Eq. (5) is applicable for these systems. The results of [24] for the mapping between Lennard-Jones fluids and the Ising model in 2 and 3 dimensions support the validity of such approximation.

From Eq. (5) the loci of the CP for fluid and lattice gas are connected by simple relations:

$$n_c = \frac{n_*}{2(1+z)}, \quad T_c = T_* \frac{z}{1+z},$$

(8)

where the parameter $n_*$ is given by:

$$n_* = T_* \frac{B'_2(T_*)}{B_3(T_*)},$$

(9)

and $T_*$ is equal to the Boyle temperature in the vdW approximation $T_B^{(vdW)} = a/b$ [8, 24]. Here $b = 4v_0$ is the fourfold of the eigenvolume of the particles $v_0 = \pi \sigma^3/6$ and

$$a = -2\pi \int_{\sigma}^{+\infty} \Phi_{\text{attr}}(r) r^2 dr,$$

(10)

where $\Phi_{\text{attr}}(r)$ is the attractive part of the potential $\Phi(r)$. As has been shown in [14] it is possible to relate the parameter $z$ of the transformation (5) with the exponent of the
attractive part of the potential \( \Phi_{\text{attr}}(r) \simeq -r^{-m} \). It is based on the relation

\[
-\frac{d \ln \left( \frac{T_c}{T^*} \right)}{d \ln \left( \frac{n_c}{n_*} \right)} = \frac{1}{z},
\]

which directly follows from Eq. (8) and the scaling relation

\[
\frac{d \ln \left( \frac{T_c}{T^*} \right)}{d \ln \left( \frac{n_c}{n_*} \right)} = \frac{d \ln \left( \frac{\Phi_{\text{attr}}(n_c^{-1/d})}{T^*} \right)}{d \ln \left( \frac{n_c}{n_*} \right)},
\]

between the critical temperature and that if \( \Phi_{\text{attr}}(r) \simeq -r^{-m} \). This relation generalizes commonly used reasoning that the critical temperature \( T_c \) is of order of the magnitude of the potential well \[7\]. Then in \( d \)-dimensions \( z \) is determined as:

\[
z = \frac{d}{m}.
\]

According to Ref. \[8\] \( C \approx 6.3 \) in 3D for the potentials of the LJ type \( \Phi(r; 6, q) \), \( q \geq 7 \) as well as for \( \Phi(r; m, 2m) \), \( m \geq 6 \). In Ref. \[14\] it was demonstrated that the results of Eq. (8) for the critical temperature are very close to those following from Eq. (11) for the potentials with \( r^{-6} \) asymptotic of the attractive part.

If the interaction does not have the hard core, there is an arbitrariness in the definition of the diameter of the particle \( \sigma \) in Eq. (10). Usually, it is chosen as the root of the equation \( \Phi(\sigma) = 0 \). The locus of the CP determined by Eq. (8) depends on the definition of soft core diameter \( \sigma \) and changes with the change of this scale. Nevertheless, it should be noted that Eq. (11) does not depend on the spatial scale used to determine the size of the molecule like \( \sigma \). This size is introduced from the physical reasonings to distinguish between the short and the long range part of the potential (see e.g. \[25, 26\]).

Note that according to Eq. (11) the quantity \( \Phi_0 \) is the homogeneous function of \( m \) and \( q \). The change of the the scale for \( \sigma \) leads to simple rescaling of the parameter \( T^* \) because it is determined by the long-range power-like asymptotic behavior of the interaction. This points to the connection between (11) and (8). Moreover, Eq. (8) in principle gives the value of the critical density too, while Vliegenthart-Lekkerkerker relation (4) as it is formulated originally does not allow to estimate \( n_c \). Therefore it is interesting to clarify the interrelation between Eq. (8) and Eq. (11). We consider the results of Ref. [3] for the generalized Lennard-Jones potentials as the nontrivial extension of the PCS. The class of PCS consists of the family of potential functions given by Eq. (11). This inference is supported by the computer simulations [10, 11].
Below we elucidate the nature of the Vliegenthart-Lekkerkerker relation (4) and its connection with the global isomorphism approach. We use the results of Ref. [14] to connect the value \( C \) with the parameter \( z \). The latter represents the class of the thermodynamic similarity in accordance with the generalized principle of corresponding states [14].

III. THE NATURE OF THE VLIEGENTHART-LEKKERKERKER RELATION WITHIN THE GLOBAL ISOMORPHISM APPROACH

The relation Eq. (4) looks quite unusual in a sense that it states the relation between the characteristic \( B_2 \) determined in low density region \( n \to 0 \) with \( T_c \). The last is related with the moderate densities of liquid state. In the framework of the approach of Ref. [17] this is naturally explained since the Boyle temperature \( T_B^{(vdW)} = T_\ast \) is connected with the critical one \( T_c \) according to Eq. (8). In previous works [14, 24] we show how the conception of the global isomorphism proposed in Ref. [17] can be applied to formulate the scale invariant mean-field approach for calculation of the locus of the CP.

We are not restricted by the 3D case and consider the case of general dimension \( d > 1 \) so that the second virial coefficient is as following:

\[
B_2(T) = \frac{S_d}{2} \int_0^\infty \left( 1 - e^{-\Phi(r)/T} \right) r^{d-1} dr, \tag{14}
\]

where \( S_d = \pi^{d/2}/\Gamma \left( \frac{d}{2} \right) \) is the unit sphere in \( d \)-dimensional space. This gives us the possibility to establish the dependence of \( C \) on the relevant parameters in \( d \)-dimensional case.

For the temperature we will use the common unit, the absolute value of the minimum \( \varepsilon \) of the potential well. The spatial scale is commonly connected with the parameter \( \sigma \). But in accordance with the scale invariant nature of Eq. (4) one can choose it in appropriate way. We use such freedom for the definition of the spatial scale which separate the intervals of distance where the repulsive and attractive forces dominate correspondingly.

It is easy to derive the estimate for the parameter \( C \) using the high-temperature asymptotic expansion for Eq. (14) (see e.g. [27]):

\[
B_2(T) = b \left( 1 - T_\ast/T \right) + o \left( T_\ast/T \right), \tag{15}
\]

Taking into account that in \( d \) dimensions \( b = 2^{d-1} v_0 \) and using the relation (8), from Eq. (4)
we obtain:

\[ C \approx \frac{2^{d-1}}{z}. \]  \hspace{1cm} (16)

For the LJ potential \( \Phi(r; 6, 12) \) in 3D case for which \( z = 1/2 \) this gives \( C \approx 8 \), for 2D case \( z = 1/3 \), \( C \approx 6 \) correspondingly.

Below we improve the estimate (16) for \( C \) in a case of Mie potentials (1). For these potentials the left hand side of Eq. (14) transforms into:

\[ B_2(T)/v_0 = 2^{d-1} b_0, \]  \hspace{1cm} (17)

where

\[ b_0(T/\Phi_0, m/d, q/m) = \int_0^\infty \left( 1 - \exp \left( \frac{\Phi_0/T}{x^{m/d}} \left( 1 - x^{-(q-m)/d} \right) \right) \right) dx, \]  \hspace{1cm} (18)

and \( x = r^d \) and \( v_0 = S_d/d (\sigma/2)^d \) is the \( d \)-dimensional volume. Notably, the parameter \( b_0 \) depends on its arguments homogeneously. So if the critical temperature \( T_c \) is determined by some constraint for \( B_2(T) \) with the dimension \( d \) fixed, then it is the homogeneous function of the relevant parameters \((m/d, q/m)\). We can say that the last represents the corresponding homogeneity class. The same homogeneous dependence is inherent to the parameter \( T_* \) for the potentials of Mie-type (1):

\[ T_* = -d \int_1^{+\infty} \Phi_{\text{attr}}(r) r^{d-1} dr = \frac{\left( \frac{q}{m} \right)^{\frac{q-m}{m}}}{(m/d-1)(q/m-1)}. \]  \hspace{1cm} (19)

Therefore, \( T_* \) is the same for the potentials with the same ratio \( q/m \). This allows to use simple scaling considerations to connect the critical temperatures for the systems with different potentials belonging to the same homogeneity class. Besides, in view of the isomorphism with the lattice gas we can state that

\[ T_c(d_1)/T_c(d_2) \approx d_1/d_2, \]  \hspace{1cm} (20)

for the potentials of the same homogeneity class \((m/d, q/m)\). Indeed, the expression (8) assumes that the critical temperature of the isomorphic lattice model is set to unit \( t_c = 1 \). So the correct comparison of the critical temperatures for the potentials in different dimensions demands the corresponding scaling since \( t_c \) is proportional to the number of the nearest neighbors, which in its turn proportional to the dimension of the lattice \( d \). Of course, the relation (20) is approximate since the fluctuations are neglected. Nevertheless it is
TABLE I. The critical temperatures for the Mie potentials of the similarity class \((m/d, q/m) = (3, 2)\). It includes \(\Phi(r; 6, 12)\) in \(d = 2\), \(\Phi(r; 9, 18)\) in \(d = 3\) and \(\Phi(r; 12, 24)\) in \(d = 4\).

| \(d\) | \(2, \Phi(r; 6, 12)\) | \(3, \Phi(r; 9, 18)\) | \(4, \Phi(r; 12, 24)\) |
|------|-----------------|-----------------|-----------------|
| \(T_c\) | 0.5 | 0.75 | 1.0 |
| \(T_c^{(\text{num})}\) | 0.515, 0.525 | 0.73, 0.74 | ? |
| \(n_c\) | 0.353 | 0.436 | 0.55 |
| \(n_c^{(\text{num})}\) | 0.355, 0.354 | 0.354, 0.355 | ? |

instructive to check the validity of Eq. (20) and the statement that the critical temperatures for the potentials belonging to the same similarity class can be connected via simple scaling relations. E.g. the potentials \(\Phi(r; 6, 12)\) in \(d = 2\) and \(\Phi(r; 9, 18)\) in \(d = 3\) fall into the same homogeneity class \((2, 2)\). Therefore

\[
T_c(d = 3; 9, 18)/T_c(d = 2; 6, 12) \approx 3/2.
\]

Though the Vliegenthart-Lekkerkerker relation itself does not allow to calculate other parameters of the critical point like the critical density and the pressure, they can be calculated using the methods developed in [14]. The value of the critical density is calculated in accordance with Eq. (8). Let us check (20) considering the potentials \(\Phi(r; m, 2m)\) for \(d = 2, 3, 4\). For these potentials the parameter \(\Phi_0\) takes the constant value \(\Phi_0 = 4\) so that:

\[
B_2(T) = 2^{d-1} b_0(T/4; m/d, 2).
\] (21)

The results of calculations are presented in Tables I-III along with the known numerical estimates. We take the results for the critical temperature of the Lennard-Jones potential \(\Phi(r; 6, 12)\) as known since they can be extracted from the corresponding lattice models (see also [8, 24]).

Using relations (8) and (13) for the case of the LJ potential \(\Phi(r; 6, 12)\) we obtain:

\[
b_0(T_c/4; 2, 2) = 3^{1/4} {}_1F_1\left(-\frac{1}{4}; \frac{1}{2}; \frac{3}{4}\right) \Gamma\left(\frac{3}{4}\right) - 3^{3/4} {}_1F_1\left(\frac{1}{4}; \frac{3}{2}; \frac{3}{4}\right) \Gamma\left(\frac{5}{4}\right) \approx 1.51,
\] (22)

for \(d = 3\). Here \(_1F_1\) is the Kummer confluent hypergeometric function \([30]\). In accordance with Eq. (17) this gives \(C \approx 6.04\). In two-dimensional case for the LJ potential \(\Phi(r; 6, 12)\)
TABLE II. The critical temperatures for the Mie potentials of the similarity class \((m/d, q/m) = (2, 2)\). It includes \(\Phi(r; 4, 8)\) in \(d = 2\), \(\Phi(r; 6, 12)\) in \(d = 3\) and \(\Phi(r; 8, 16)\) in \(d = 4\).

| \(d\) | \(2, \Phi(r; 4, 8)\) | \(3, \Phi(r; 6, 12)\) | \(4, \Phi(r; 8, 16)\) |
|-------|-----------------|-----------------|-----------------|
| \(T_c\) | 0.89 | 1.33 | 1.78 |
| \(T_c^{(\text{num})}\) | ? | 1.313 , | ? |
| \(n_c\) | 0.283 | 0.322 | 0.392 |
| \(n_c^{(\text{num})}\) | ? | 0.316 , | ? |

\[ a \] In \([8]\) we gave another value \(n_c \approx 0.404\). The difference is caused by different integration method and the precision for the calculation of multidimensional integral for \(B_3\) in higher dimensions \((d \geq 4)\). Here we obtained \(n_c \approx 0.408\) and rounded the result up to hundredth.

TABLE III. The critical temperatures for the Mie potentials of the similarity class \((m/d, q/m) = (3/2, 2)\). It includes \(\Phi(r; 3, 6)\) in \(d = 2\), \(\Phi(r; 9/2, 9)\) in \(d = 3\) and \(\Phi(r; 6, 12)\) in \(d = 4\).

| \(d\) | \(2, \Phi(r; 3, 6)\) | \(3, \Phi(r; 9/2, 9)\) | \(4, \Phi(r; 6, 12)\) |
|-------|-----------------|-----------------|-----------------|
| \(T_c\) | 1.6 | 2.4 | 3.2 |
| \(T_c^{(\text{num})}\) | ? | ? | 3.4 , |
| \(n_c\) | 0.333 | 0.353 | 0.41 |
| \(n_c^{(\text{num})}\) | ? | ? | 0.34 , |

\[ a \] In \([8]\) we gave another value \(n_c \approx 0.404\). The difference is caused by different integration method and the precision for the calculation of multidimensional integral for \(B_3\) in higher dimensions \((d \geq 4)\). Here we obtained \(n_c \approx 0.408\) and rounded the result up to hundredth.

the value of the Vliegenthart-Lekkerkerker constant \(C\) is determined by

\[
b_0(T_c/4; 3, 2) = \sqrt{2} F_1 \left( -\frac{1}{6}; \frac{1}{2}, \frac{3}{2}; 2 \right) \Gamma \left( \frac{5}{6} \right) \left( 2 F_1 \left( \frac{1}{3}; \frac{3}{2}, \frac{3}{2}; 2 \right) \Gamma \left( \frac{4}{3} \right) \right) \approx 3.81 , \tag{23}
\]

which leads to \(C \approx 7.6\). Finally, for the homogeneity class \((3/2, 2)\):

\[
b_0(T_c/4; 3/2, 2) = \left( \frac{5}{4} \right)^{1/3} F_1 \left( -\frac{1}{3}; \frac{1}{2}, \frac{5}{16}; \frac{5}{3} \right) \Gamma \left( \frac{5}{3} \right) - \sqrt{5} \Gamma \left( \frac{7}{6} \right) \left( \frac{3}{2}; \frac{5}{16} \right) \left( 2 F_1 \left( \frac{1}{6}; \frac{3}{2}, \frac{5}{16} \right) \right) \approx 1.19 , \tag{24}
\]

with the corresponding value \(C \approx 9.5\). This corrects the high-temperature estimate \([13]\).

IV. CONCLUSIONS

In this paper we derive the Vliegenthart-Lekkerkerker relation for the value of the second virial coefficient at the critical temperature \([3]\) within the approach proposed in \([14]\) for the
potentials of Mie-class. The very existence of such relation has appeared as the direct consequence of the global isomorphism between the fluids with the simple conformal potentials and the lattice models. We extended the applicability of this relation by considering the general case of d dimensions. The estimates for the loci of the critical points of the systems with \( \Phi(r; m, 2m) \)-type potentials are obtained. These estimates can be checked in simulations and it would be interesting to investigate the relations between the critical parameters for the systems with the Mie potentials of the same homogeneity class but with different dimensions. This provides the test for the validity of the proposed approach.

The application of the proposed approach to other types of the potentials with nonalgebraic behavior, like hard spheres plus attractive Yukawa potential, Morse potential or square-well potential is connected with the derivation of the relation between the parameter \( z \) of the transformation (\[1\]) and the relevant integral characteristics of the potential. E.g., the relation (\[2\]) can not be applied directly to the square well potential because of nonanalytic dependence \( \Phi(r) \). Obviously, these characteristics should be related with the symmetry properties of the potential under the scaling of its parameters, namely the effective range of interaction. The results of Ref. \[5\] show that the value of the Vliegenthart-Lekkerkerker parameter \( C \) differs essentially for Lennard-Jones like potentials \( \Phi(r; 6, q) \), \( q \geq 7 \) and square-well potential. In the last case there is strong dependence on the width of the potential well which governs the range of the interaction. The weak dependence of \( C \) on the exponent \( q \) of the repulsive part for \( \Phi(r; 6, q) \) potentials in naturally explained within the global isomorphism approach because of Eq. (\[13\]) and Eq. (\[8\]). The square-well potential is the continuum version of the nearest neighbor interaction in the lattice model. So it is possible to connect the dependence of the critical parameters for the square-well potential and the corresponding results for the regular lattice gas model. These topics will be the subjects of further studies.

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