Bogomolov’s inequality and Higgs sheaves on normal varieties in positive characteristic

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Abstract. We prove Bogomolov’s inequality on a normal projective variety in positive characteristic and we use it to show some new restriction theorems and a new boundedness result. We also redefine Higgs sheaves on normal varieties and we prove restriction theorems and Bogomolov-type inequalities for semistable logarithmic Higgs sheaves on some normal varieties in an arbitrary characteristic.

Introduction

Let $X$ be a smooth projective variety of dimension $n$ defined over an algebraically closed field $k$ and let $H$ be an ample line bundle on $X$. Let $\mathcal{E}$ be a coherent torsion free $\mathcal{O}_X$-module of rank $r$ and let us set $\Delta(\mathcal{E}) := 2rc_2(\mathcal{E}) - (r - 1)c_1(\mathcal{E})^2$.

A well-known theorem of Bogomolov says that if $k$ has characteristic zero and $\mathcal{E}$ is slope $H$-semistable, then

$$\int_X \Delta(\mathcal{E}) H^{n-2} \geq 0.$$ 

It was first proven by F. Bogomolov in the surface case. The higher-dimensional case follows from restriction theorems for semistability (e.g., one can use the Mehta–Ramanathan restriction theorem). An analogue of this theorem for slope $H$-stable Higgs bundles was proven by C. Simpson in [35] using analytic methods. Simpson’s paper contains also applications of this result to uniformization and the Miyaoka–Yau inequality in higher dimensions (although only in the non-logarithmic case). Later, T. Mochizuki in [32] generalized this inequality to the logarithmic case (he also used analytic methods). An algebraic proof of Bogomolov’s inequality for Higgs sheaves appeared in [24] and in the logarithmic case in [25]. These papers contained also generalization of these results to positive characteristic.

More recently, in characteristic zero the above results have been generalized in [10] to projective varieties with klt singularities (but not in the logarithmic case). In the mildly singular
logarithmic case one also knows the Miyaoka–Yau inequality (see [16, Chapter 10] and [20] for the 2-dimensional case and [11] for higher dimensions).

One of the main motivations behind this paper is generalization of the above results to positive characteristic and strengthening of the results known in the characteristic zero. We also deal with semistability defined by a collection of nef line bundles instead of one ample line bundle. An importance of considering this generalized situation was first recognized by Y. Miyaoka in [31], who proved Bogomolov’s inequality for torsion free sheaves on normal varieties smooth in codimension 2 in case of collections of ample and one nef line bundles. However, it is not completely clear to the author if the original proof of Mehta–Ramanathan’s restriction theorem works so easily for multipolarizations on normal varieties as claimed in [31, Corollary 3.13]. In case of one ample line bundle on a normal projective variety defined over an algebraically closed field of characteristic zero, restriction theorem for semistable sheaves has been proven by H. Flenner in [7]. However, it seems that his proof cannot be generalized to multipolarizations.

In case of smooth projective varieties the corresponding Bogomolov’s inequality in any characteristic was proven in [21] (however, the proof uses a different, new restriction theorem). Using resolution of singularities one can use this to obtain Mehta–Ramanathan’s restriction theorem for multipolarizations on normal varieties in characteristic zero. In case of Higgs sheaves on smooth projective varieties, restriction theorem and Bogomolov’s inequality for multipolarizations has been proven in [24] and in the logarithmic case in [25].

Our first main result is a strong restriction theorem for multipolarized normal varieties in positive characteristic, analogous to [21, Theorem 5.2 and Corollary 5.4]. One of the problems here is with the definition of Chern classes. Below we use Chern classes of reflexive sheaves defined in [26] (see Section 1.3 for a few basic properties).

Let $X$ be a normal projective variety of dimension $n$ defined over an algebraically closed field $k$ of characteristic $p > 0$. Fix a collection $(L_1, \ldots, L_{n-1})$ of ample line bundles on $X$ (in fact, we usually need weaker assumptions on this collection). Let us set $d = L_1^2 L_2 \cdots L_{n-1}$.

Then we have the following result (see Section 3.1 for the definition of $\beta_r$).

**Theorem 0.1.** Let $\mathcal{E}$ be a coherent reflexive $\mathcal{O}_X$-module of rank $r \geq 2$. Let $m$ be an integer such that

$$m > \left[ \frac{r - 1}{r} \int_X \Delta(\mathcal{E}) L_2 \cdots L_{n-1} + \frac{1}{dr(r-1)} + \frac{(r-1)\beta_r}{dr} \right]$$

and let $H \in |L_1^m|$ be a normal hypersurface. If $\mathcal{E}$ is slope $(L_1, \ldots, L_{n-1})$-stable, then $\mathcal{E}|_H$ is slope $(L_2|_H, \ldots, L_{n-1}|_H)$-stable.

The above theorem implies the following boundedness result.

**Theorem 0.2.** Let us fix some positive integer $r$, integer $\text{ch}_1$ and some real numbers $\text{ch}_2$ and $\mu_{\text{max}}$. Then the set of reflexive coherent $\mathcal{O}_X$-modules $\mathcal{E}$ of rank $r$ with

- $\int_X \text{ch}_1(\mathcal{E}) L_1 \cdots L_{n-1} = \text{ch}_1$,
- $\int_X \text{ch}_2(\mathcal{E}) L_1 \cdots L_{i-1} L_{i+1} \cdots L_{n-1} \geq \text{ch}_2$ for $i = 1, \ldots, n-1$,
- $\mu_{\text{max}}(\mathcal{E}) \leq \mu_{\text{max}}$

is bounded.
In the statement above it is not even clear that the number of Hilbert polynomials of sheaves in the considered family is finite. In case $L_1 = \cdots = L_{n-1}$ the above theorem follows from [21, Theorem 4.4]. If $X$ is smooth, then the above theorem follows from [21, Corollary 5.4]. But it is no longer the case if we consider multipolarizations on normal varieties.

We also prove an analogue of Theorem 0.1 for (semi)stable Higgs sheaves on normal varieties (see Theorem 5.4 for a more precise version).

**Theorem 0.3.** Let $D \subset X$ be a reduced effective Weil divisor and let $(\mathcal{E}, \theta)$ be a reflexive logarithmic Higgs sheaf of rank $r \geq 2$ on $(X, D)$. Let $m_0$ be a non-negative integer such that $T_{X/k}(\log D) \otimes L_1^{\otimes m_0}$ is globally generated. Let $m$ be an integer such that

$$m > \max \left( \frac{r-1}{r} \int_X \Delta(\mathcal{E}) L_2 \cdots L_{n-1} + \frac{1}{dr(r-1)} + \frac{(r-1)\beta_r}{dr}, 2(r-1)m_0^2 \right)$$

and let $H \in |L_1^{\otimes m}|$ be a general divisor. If $(\mathcal{E}, \theta)$ is slope $(L_1, \ldots, L_{n-1})$-stable, then the logarithmic Higgs sheaf $(\mathcal{E}, \theta)|_H$ on $(H, D \cap H)$ is slope $(L_2|_H, \ldots, L_{n-1}|_H)$-stable.

Note that this theorem generalizes [24, Theorem 10] that works for smooth varieties liftable to $W_2(k)$.

Finally, we use the above results to prove the following Bogomolov’s inequality for reflexive Higgs sheaves on mildly singular normal varieties. Note that, unlike previously known results for singular varieties in characteristic zero, our theorem holds for log pairs.

**Theorem 0.4.** Let $D \subset X$ be a reduced effective Weil divisor such that the pair $(X, D)$ is almost liftable to $W_2(k)$ and it has $F$-liftable singularities in codimension 2. Then for any slope $(L_1, \ldots, L_{n-1})$-semistable logarithmic reflexive Higgs sheaf $(\mathcal{E}, \theta)$ of rank $r \leq p$ we have

$$\int_X \Delta(\mathcal{E}) L_2 \cdots L_{n-1} \geq 0.$$
Now let us also fix a reduced effective Weil divisor $D \subset X$ such that the pair $(X, D)$ is log canonical in codimension 2. Theorem 0.3 implies the following result:

**Theorem 0.6.** Let $(\mathcal{E}, \theta)$ be a reflexive logarithmic Higgs sheaf of rank $r \geq 2$ on $(X, D)$. Let $m_0$ be a non-negative integer such that $T^1_{X/k}(\log D) \otimes L_1^{\otimes m_0}$ is globally generated. Let $m$ be an integer such that $m > \max \left( \left\lfloor \frac{r-1}{r} \int_X \Delta(\mathcal{E}) L_2 \ldots L_{n-1} + \frac{1}{d(r-1)} \right\rfloor, 2(r-1)m_0^2 \right)$

and let $H \in |L_1^{\otimes m}|$ be a general divisor. If $(\mathcal{E}, \theta)$ is slope $(L_1, \ldots, L_{n-1})$-stable, then the logarithmic Higgs sheaf $(\mathcal{E}, \theta)|_H$ on $(H, D \cap H)$ is slope $(L_2|_H, \ldots, L_{n-1}|_H)$-stable.

Theorem 0.6 generalizes [10, Theorem 5.22] and [9, Theorem 6.1], which are non-effective. In fact, we prove a stronger version of Theorem 0.6 (see Theorem 7.2) that works for all normal divisors $H$ for which restriction of $(\mathcal{E}, \theta)$ to $H$ gives a logarithmic Higgs sheaf on $(H, D \cap H)$. Similarly, Theorem 0.4 can be used to prove the following inequality generalizing [10, Theorem 6.1].

**Theorem 0.7.** For any slope $(L_1, \ldots, L_{n-1})$-semistable logarithmic reflexive Higgs sheaf $(\mathcal{E}, \theta)$ we have

$$\int_X \Delta(\mathcal{E}) L_2 \ldots L_{n-1} \geq 0.$$
This allows us to obtain stronger results, e.g., Bogomolov’s inequality for reflexive extensions of semistable Higgs sheaves on the regular locus (cf. [10, Theorem 6.1]).

Further applications of the obtained results to non-abelian Hodge theory and Simpson’s correspondence appear in [29].

The structure of the paper is as follows. In the first section we gather some auxiliary results and introduce some notation. In Section 2 we prove a few results on Chern classes of reflexive sheaves on normal varieties in positive characteristic. These results are used in Section 3 to prove generalized versions of Theorems 0.1 and 0.2. In Section 4 we study modules over Lie algebroids and generalized Higgs sheaves on normal varieties. In Section 5 we prove generalized versions of Theorems 0.3 and 0.4. In Section 6 we apply these results to obtain the Miyaoka–Yau inequality for some normal varieties in positive characteristic. In Section 7 we show some applications of the obtained results in characteristic zero, proving Theorems 0.5, 0.6 and 0.7. Section A contains an appendix in which we recall construction of the inverse Cartier transform used in Section 5.

**Notation.** If $f : W \to X$ is a morphism between normal schemes and $\mathcal{E}$ is a coherent reflexive $\mathcal{O}_X$-module, then we set
$$f^{[*]} \mathcal{E} = (f^* \mathcal{E})^{**}.$$ If $f$ is flat, then the canonical map $f^* \mathcal{E} \to f^{[*]} \mathcal{E}$ is an isomorphism.

If $X$ is a normal scheme of characteristic $p$, then we denote by $F_X$ its absolute Frobenius morphism. If $\mathcal{E}$ is a coherent reflexive $\mathcal{O}_X$-module, then for any positive integer $m$ we set
$$F_X^{[m]} \mathcal{E} = (F_X^m)^{[*]} \mathcal{E}.$$

### 1. Preliminaries

#### 1.1. Reflexive sheaves.**

In this subsection $X$ is an integral normal scheme, which is of finite type over a field $k$. By $\operatorname{Ref}(\mathcal{O}_X)$ we denote the category of coherent reflexive $\mathcal{O}_X$-modules. It is a full subcategory of the category $\operatorname{Coh}(\mathcal{O}_X)$ of coherent $\mathcal{O}_X$-modules. The inclusion functor $\operatorname{Ref}(\mathcal{O}_X) \to \operatorname{Coh}(\mathcal{O}_X)$ comes with a left adjoint $(\cdot)^{**} : \operatorname{Coh}(\mathcal{O}_X) \to \operatorname{Ref}(\mathcal{O}_X)$ given by the reflexive hull. The category $\operatorname{Ref}(\mathcal{O}_X)$ comes with an associative and symmetric tensor product $\otimes$ given by
$$\mathcal{E} \otimes \mathcal{F} := (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})^{**}.$$ An open subset $U \subset X$ is called big if its complement $X \setminus U$ has codimension $\geq 2$ in $X$. If we consider $U$ as a subscheme of $X$, then we talk about a big open subscheme. The following well-known lemma can be found in [37, Lemma 0EBJ].

**Lemma 1.1.** Let $j : U \hookrightarrow X$ be a big open subscheme. Then $j_*$ and $j^*$ define adjoint equivalences of categories $\operatorname{Ref}(\mathcal{O}_X)$ and $\operatorname{Ref}(\mathcal{O}_U)$.

Since $X$ is normal, its regular locus $X_{\text{reg}} \subset X$ is a big open subset.

**Lemma 1.2.** Let $f : X \to Y$ be a finite dominant morphism of integral Noetherian normal schemes. If $\mathcal{E}$ is a coherent $\mathcal{O}_Y$-module, then we have a canonical isomorphism
$$f^{[*]}(\mathcal{E}^*) \simeq (f^* \mathcal{E})^*.$$
Proof. We have a natural map
\[ f^*(\mathcal{E}) = f^* \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{O}_Y) \to \mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{E}, f^*\mathcal{O}_Y) = (f^*\mathcal{E})^*. \]
If \( \mathcal{E} \) is torsion free, then there exists a big open subset \( V \subset Y \) such that \( \mathcal{E}|_V \) is finite locally free. Then the above map is an isomorphism on \( U = f^{-1}(V) \). This subset of \( X \) is big because \( f \) is finite and dominant. Since \( (f^*\mathcal{E})^* \) is reflexive, this induces an isomorphism
\[ f^![1](\mathcal{E})^* \simeq (f^*\mathcal{E})^*. \]
If \( \mathcal{E} \) is not torsion free, then the pullback of the quotient map \( \mathcal{E} \to \overline{\mathcal{E}} := \mathcal{E}/\text{Torsion} \) is surjective. Hence the dual map \( (f^*\overline{\mathcal{E}})^* \to (f^*\mathcal{E})^* \) is an isomorphism. But if we apply the lemma to \( \overline{\mathcal{E}} \), then we get \( f^![1](\mathcal{E})^* \simeq (f^*\overline{\mathcal{E}})^* \), which proves the required assertion. \( \square \)

1.2. F-liftable schemes. A Weil divisor on an integral Noetherian scheme \( X \) is a finite formal sum \( \sum a_i D_i \), where \( a_i \in \mathbb{Z} \) and \( D_i \) are prime divisors. However, if all non-zero \( a_i \) are equal to 1, then we can consider the Weil divisor as a reduced closed subscheme of \( X \) of whose irreducible components have codimension one. In this case we talk about a reduced effective Weil divisor.

A log pair \((X, D)\) is a pair consisting of a normal variety \( X \) defined over a perfect field \( k \) and a reduced effective Weil divisor \( D \) on \( X \) (we allow \( D = 0 \)). We say that \((X, D)\) is log smooth if \( X \) is smooth and \( D \) is a normal crossing divisor. In this subsection we assume that \( k \) has positive characteristic \( p \). We also set \( S = \text{Spec} \; k \) and \( \tilde{S} = \text{Spec} \; W_2(k) \).

Definition 1.3. Let \((X, D)\) be a log pair and let us write \( D = \sum D_i \), where \( D_i \) are irreducible. We say that \((X, D)\) is

1. liftable to \( W_2(k) \) if there exists a flat morphism \( \tilde{X} \to \tilde{S} \) and subschemes \( \tilde{D}_i \subset \tilde{X} \) such that \( \tilde{D}_i \to \tilde{S} \) is flat and \((X, D_i) = (\tilde{X} \times_{\tilde{S}} S, \tilde{D}_i \times_{\tilde{S}} S)\) for all \( i \). We write \( \tilde{D} := \sum \tilde{D}_i \).

Then the pair \((\tilde{X}, \tilde{D})\) is called a lifting of \((X, D)\) to \( W_2(k) \).

2. \( F \)-liftable if there exists a lifting \((\tilde{X}, \tilde{D})\) of \((X, D)\) to \( W_2(k) \) and a morphism \( \tilde{F}_X : \tilde{X} \to \tilde{X} \) restricting to \( F_X \) modulo \( p \) such that for each \( D_i \), the image of \( \tilde{F}_X^* I_{\tilde{D}_i} \to \mathcal{O}_{\tilde{X}} \) is contained in \( I_{\tilde{D}_i}^p \). In this case we say that \( \tilde{F}_X \) is compatible with \( \tilde{D} \) and we call the morphism \( \tilde{F}_X \) an \( F \)-lifting of \((X, D)\) (compatible with \((\tilde{X}, \tilde{D})\)).

3. almost liftable to \( W_2(k) \) if there exists a big open subset \( U \subset X \) such that the pair \((U, D_U = D \cap U)\) is liftable to \( W_2(k) \). The corresponding lifting of \((U, D_U)\) is called an almost lifting of \((X, D)\).

4. almost \( F \)-liftable if there exists a big open subset \( U \subset X \) such that the pair \((U, D_U)\) is \( F \)-liftable. The corresponding lifting is called an almost \( F \)-lifting of \((X, D)\).

Remark 1.4. (1) If \( U \subset X \) as in (3)–(4) exists, then we can find a big open subset \( V \subset X \) such that \((V, D_V)\) is log smooth and the corresponding condition is satisfied.

(2) If \((X, D)\) is almost \( F \)-liftable and \( D = \bigcup D_i \), where \( D_i \subset X \) are prime divisors, then \((D_i, (\bigcup_{j \neq i} D_j \cap D_i))\) is also almost \( F \)-liftable. This observation follows from the corresponding fact for \( F \)-liftable log smooth pairs (see [2, Lemma 3.2] for a simple proof).

We also need to introduce some notions of singularities in presence of liftings.
Definition 1.5.  (1) If $(X,D)$ is liftable to $W_2(k)$, we say that it is **locally $F$-liftable** if there exists a lifting $(\bar{X},\bar{D})$ of $(X,D)$ such that every $x \in X$ has an open neighborhood $V \subset X$ for which there exists an $F$-lifting of $(V,D_{\mid V})$ compatible with the lifting induced from $(\bar{X},\bar{D})$.

(2) If $(X,D)$ is almost liftable to $W_2(k)$, we say that it is **locally $F$-liftable** if there exists a big open subset $U \subset X$ and a lifting $(\bar{U},\bar{D}_U)$ of $(U,D_U)$ such that every point of $x \in X$ has an open neighborhood $V \subset X$ for which there exists an $F$-lifting of $(V,D_{\mid V})$ compatible with the lifting induced from $(\bar{U},\bar{D}_U)$.

(3) If $(X,D)$ is almost liftable to $W_2(k)$, we say that $(X,D)$ **has $F$-liftable singularities in codimension** 2 if there exists a closed subset $Z \subset X$ of codimension $\geq 3$ such that $(X \setminus Z,D \setminus Z)$ is locally $F$-liftable.

Remark 1.6.  (1) If $(X,D)$ is log smooth and (almost) liftable to $W_2(k)$, then it is also locally $F$-liftable.

(2) Note that if there exists a big open subset $U \subset X$ such that the pair $(U,D_U = D \cap U)$ is liftable to $W_2(k)$ and locally $F$-liftable, then $(X,D)$ is almost liftable to $W_2(k)$ but it does not need to be locally $F$-liftable.

A characteristic $p$ scheme $X$ is called **$F$-split** if there exists an $\mathcal{O}_X$-linear map

$$\varphi : (F_X)_*\mathcal{O}_X \to \mathcal{O}_X$$

splitting

$$F_X^\#: \mathcal{O}_X \to (F_X)_*\mathcal{O}_X.$$  

If $Y_1,\ldots,Y_s$ are closed subschemes of $X$, then we say that they are compatibly $F$-split by $\varphi$ if $\varphi((F_X)_*I_{Y_j}) \subseteq I_{Y_j}$ for all $j$. For the basic facts about these notions we refer the reader to [4].

We need the following generalization of the second part of [4, Proposition 1.3.11].

Lemma 1.7.  Let $X$ be a smooth variety defined over an algebraically closed field of characteristic $p > 0$. Assume that $\varphi \in H^0(X,\omega_X^{1-p}) \simeq \text{Hom}_{\mathcal{O}_X}((F_X)_*\mathcal{O}_X,\mathcal{O}_X)$ splits $X$. Let $Z(\varphi)$ be the divisor of zeroes of $\varphi$ and let us write $Z(\varphi) = (p-1)D + D'$ for some effective divisors $D$ and $D'$. Then $D$ is reduced and $\varphi$ splits $X$ compatibly with all irreducible components of $D$.

Proof. Let $Y$ be an irreducible component of $D$ and let $x$ be a smooth point of the support of $Z(\varphi)$ that belongs to $Y$. Then we can choose a system of local coordinates $(t_1,\ldots,t_n)$ at $x$ such that the local equation of $Y$ is given by $t_1 = 0$. Note that by assumption the local expansion of $\varphi$ at $x$ is given by

$$t_1^{m(p-1)} g(t_1,\ldots,t_n)(dt_1 \wedge \cdots \wedge dt_n)^{1-p},$$

where $g(t_1,\ldots,t_n)$ is not divisible by $t_1$ and $m \geq 1$ is the multiplicity of $Y$ in $D$. Since $\varphi$ splits $X$, the coefficient of the monomial $(t_1\cdots t_n)^{p-1}$ in $t_1^{m(p-1)} g(t_1,\ldots,t_n)$ is non-zero (see [4, Theorem 1.3.8]). Hence $m = 1$ and the splitting $\varphi$ is compatible with $Y$ at $x$. It follows that $\varphi$ is compatible with $Y$ at smooth points of the support of $Z(\varphi)$. So the required assertion follows from [4, Lemma 1.1.7 (ii)].
Proposition 1.8. Let \((X, D)\) be a log pair.

1. If \(X\) is \(F\)-split compatibly with all irreducible components of \(D\), then \((X, D)\) is liftable to \(W_2(k)\).

2. If \((X, D)\) is almost \(F\)-liftable, then \(X\) is \(F\)-split compatibly with all irreducible components of \(D\).

Proof. In case \(D = 0\) the first part is contained in [24, Proposition 4] and the second one follows from [5, Theorem 2] (see also [2, Section 2]). In general, the first part follows from [3, Lemma 5.2.1]. By [4, Lemma 1.1.7 (ii) and (iii)] to prove the second part it is sufficient to prove that if \((X, D)\) is log smooth and \(F\)-liftable, then irreducible components of \(D\) are compatibly \(F\)-split. Note that the \(F\)-splitting induced by a lifting \(\tilde{F}_X\) that is compatible with \(\tilde{D}\) vanishes to order \((p - 1)\) along \(D\) (see the proof of [2, Lemma 3.1]). So we can conclude by Lemma 1.7.

Remark 1.9. If \((X, D)\) is log smooth, then the fact that \(D\) is compatibly \(F\)-split is claimed in [2, Lemma 3.1] but the proof there contains a gap. The problem is that the Frobenius splitting coming from the lifting of the Frobenius morphism to \(W_2(k)\) does not need to come from \((p - 1)\)-th power of a section of \(H^0(X, \omega_X^{-1})\). See below for an explicit example.

Example 1.10. Let us consider divisor \(D := (x_1 = 0) \subset X := \text{Spec} \ k[x_1, x_2]\), where \(k\) is a perfect field of characteristic \(p > 2\). Let \(\tilde{X} := \text{Spec} \ W_2(k)[x_1, x_2]\) be a lifting of \(X\) to \(W_2(k)\) and let \(\tilde{D} := (x_1 = 0) \subset \tilde{X}\) be a lifting of \(D \subset X\). Let us consider a lifting \(\tilde{F}_X\) of \(F_X\) given by \(x_1 \to x_1^p\) and \(x_2 \to x_2^p + px_2^2\). This lifting is compatible with \(\tilde{D}\). However, it is easy to see that the Frobenius splitting associated to \(\tilde{F}_X\) is given by

\[
\varphi = x_1^{p-1}x_2(x_2^{p-2} + 2)(d x_1 \wedge d x_2)^{1-p} \in H^0(X, \omega_X^{1-p}),
\]

so \(\varphi\) is not a \((p - 1)\)-th power of a section of \(H^0(X, \omega_X^{-1})\). In fact, in this case one cannot find any open subset \(U \subset X\) such that \(\varphi|_U\) is a \((p - 1)\)-th power of a section of \(H^0(U, \omega_U^{-1})\). On \(U = \{x_2(x_2^{p-2} + 2) \neq 0\}\) one can multiply \(\varphi\) by an invertible \(u \in \Gamma(U, \mathcal{O}_U)\) so that

\[
u \cdot \varphi|_U = \psi^{p-1}
\]

for some \(\psi \in H^0(U, \omega_U^{-1})\) and apply [4, Proposition 1.3.11] to this new splitting. This shows that \(u \cdot \varphi|_U\) splits \(U\) compatibly with \(D \cap U\). However, this is not sufficient to apply [4, Lemma 1.1.7 (ii)] to conclude that \(\varphi\) splits \(X\) compatibly with \(D\).

Example 1.11. The following example is motivated by [36, Example 5.1] (note that the argument showing \(F\)-liftable works for \(p > 2\); for \(p = 2\) \(F\)-liftable needs to be proven using [36, Corollary 4.12]).

Let us consider divisor \(D := (x_1x_2(x_1 + x_2) = 0) \subset X := \text{Spec} \ k[x_1, x_2]\), where \(k\) is a perfect field of characteristic \(p > 0\). Then \(\tilde{X} := \text{Spec} \ W_2(k)[x_1, x_2]\) is a lifting of \(X\) to \(W_2(k)\) and it has a natural lifting \(\tilde{F}_X\) of \(F_X\) given by \(x_i \to x_i^p\) for \(i = 1, 2\). Let

\[
\tilde{D} := (x_1x_2(x_1 + x_2) = 0) \subset \tilde{X}
\]

be a lifting of \(D \subset X\). If \(p > 2\), then \(\tilde{F}_X\) induces a compatible lifting \(\tilde{F}_X|_{\tilde{D}} : \tilde{D} \to \tilde{D}\) of \(F_D\). However, \(\tilde{F}_X\) is not compatible with \(\tilde{D}\) as \(\tilde{F}_X^p \tilde{D} = (x_1^p x_2^p(x_1^p + x_2^p) = 0)\) is not equal to
$p\tilde{D} = (x_1^p x_2^p (x_1 + x_2)^p = 0)$. In fact, $X$ is not compatibly $F$-split with $D$ (e.g., because the intersection of $x_1 x_2 = 0$ with $x_1 + x_2 = 0$ is non-reduced, which would give a contradiction with [4, Proposition 1.2.1]). So by Proposition 1.8 $(X, D)$ is not $F$-liftable.

We also need the following logarithmic version of [1, Theorem 3.3.6 (a), (iii)]. The proof is analogous to the one from [1] and we leave it to the reader.

**Lemma 1.12.** Let $(X, D)$ be a log scheme and let $U \subset X$ be a big open subset of $X$. Let $(\tilde{X}, \tilde{D})$ be a $W_2(k)$-lifting of $(X, D)$ and let $F_U$ be an $F$-lifting of $(\tilde{U}, \tilde{D}_U)$, where $\tilde{U} = (U, \Theta_{\tilde{X}|U})$ and $\tilde{D}_U = (D_U, \Theta_{\tilde{D}|D_U})$. Then there exists an $F$-lifting $F_{\tilde{X}} : \tilde{X} \to \tilde{X}$ compatible with $\tilde{D}$.

The following theorem shows that an almost liftable log pair, which is locally almost $F$-liftable is already liftable to $W_2$ and locally $F$-liftable.

**Theorem 1.13.** Let $(X, D)$ be a log pair. Then the following conditions are equivalent:

1. $(X, D)$ is liftable to $W_2(k)$ and it is locally $F$-liftable.
2. $(X, D)$ is almost liftable to $W_2(k)$ and it is locally $F$-liftable.
3. There exists a big open subset $U \subset X$ and a lifting $(\tilde{U}, \tilde{D}_U)$ of $(U, D_U = D \cap U)$ such that every $x \in X$ has an open neighborhood $V \subset X$ for which there exists an almost $F$-lifting of $(V, D_V)$ compatible with the almost lifting induced from $(\tilde{U}, \tilde{D}_U)$.

**Proof.** Implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear. So let us assume (3). Then every point $x \in X$ has an open neighborhood $V \subset X$ and a big open subset $V' \subset V \cap U$ with a compatible $F$-lifting $F_{\tilde{V}} : (\tilde{V}', \tilde{D}') \to (\tilde{V}', \tilde{D}')$, where $\tilde{V}' = (V', \Theta_{\tilde{V}|V'})$ and $\tilde{D}' = (D', \Theta_{\tilde{D}|D'})$. By Lemma 1.12 we can extend $F_{\tilde{V}}$ to $F_{\tilde{V} \cap U}$, where $\tilde{V} \cap U := (V \cap U, \Theta_{\tilde{V}|V \cap U})$. Moreover, $F_{\tilde{V} \cap U}$ is compatible with $\tilde{D}_{(U \cap V)} := (\tilde{V} \cap U, \Theta_{\tilde{D}|(U \cap V)})$. By Proposition 1.8 we know that $\tilde{V}$ is $F$-split compatible with irreducible components of $D_V$ and hence we have a canonical lifting of $(V, D_V)$ to $W_2(k)$. Moreover, this lifting extends lifting $(\tilde{V} \cap U, \tilde{D}_{(U \cap V)})$. So again using Lemma 1.12 we can extend $F_{\tilde{V} \cap U}$ to an $F$-lifting of $(V, D_V)$. This shows (2).

Assertion (1) follows from the fact that for all $x$ the obtained canonical liftings $(\tilde{V}, \tilde{D}_V)$ glue to $(\tilde{U}, \tilde{D}_U)$ giving a locally $F$-liftable lifting of $(X, D)$ to $W_2(k)$. The gluing is possible since an $F$-lifting is uniquely determined up to a canonical isomorphism (this is a log version of [2, Theorem 2.7]).

The above theorem immediately implies the following corollary:

**Corollary 1.14.** If $(X, D)$ is almost liftable to $W_2(k)$ and it has $F$-liftable singularities in codimension 2, then there exists a closed subset $Z \subset X$ of codimension $\geq 3$ such that $(X \setminus Z, D \setminus Z)$ is liftable to $W_2(k)$ and it is locally $F$-liftable.

**Remark 1.15.** Note that it is usually much easier to lift to $W_2(k)$ a big open subset of $X$ than the whole $X$. For example, if $X$ is a smooth projective surface, then any open subset $U \subset X$ is liftable to $W_2(k)$. This follows from the fact that the obstruction to lifting of $U$ to $W_2(k)$ lies in $H^2(T_U)$, which vanishes by Lichtenbaum’s theorem (see [37, Theorem 0G5F]).
Theorem 1.13 says that if $X$ is not liftable to $W_2(k)$, then it is not locally (almost) $F$-liftable with respect to any lifting of $U$.

**Remark 1.16.** If $X$ is $F$-liftable, then it does not need to have rational singularities. In fact, [36, Example 5.2] shows that the cone over an ordinary elliptic curve is $F$-liftable. This singularity is log canonical but not klt. Let us also recall that by [2, Theorem 2.10 (c)] if $X$ is $F$-liftable and $G$ is a linearly reductive group acting on $X$, then the quotient $X \sslash G$ is also $F$-liftable.

Finally, note that by [36, Theorem 4.15] ordinary double points of the form $\cdot x^2_1 \ldots x^2_n \subset \mathbb{A}^n_k$ for $n \geq 5$ in characteristic $p \geq 3$ are $F$-split but they are not (locally) $F$-liftable. These singularities are not only log canonical but even terminal. The hypersurface $(x^2_1 + \ldots + x^2_n = 0) \subset \mathbb{A}^n_k$ is almost liftable to $W_2(k)$ and it has $F$-liftable singularities in codimension 2 (since for $n \geq 4$ it is regular in codimension 2).

**1.3. Intersection theory on normal varieties.** Let $X$ be a normal projective variety of dimension $n$ defined over an algebraically closed field $k$. In the following we write $A_m(X)$ for the group of $m$-cycles modulo rational equivalence on $X$. In particular, $A^1(X) = A_{n-1}(X)$ is the class group of $X$. If $\mathcal{E}$ is a coherent $\mathcal{O}_X$-module of rank $r \geq 1$, the sheaf $\det \mathcal{E} = (\mathcal{E}^r)^{**}$ is reflexive of rank 1 and we can consider the associated class $c_1(\mathcal{E}) \in A^1(X)$ of Weil divisors on $X$.

We say that two line bundles $L$ and $M$ on $X$ are numerically equivalent if for every proper curve $C \subset X$ we have

$$\int_X c_1(L) \cap [C] = \int_X c_1(M) \cap [C].$$

If $L$ is numerically equivalent to $\mathcal{O}_X$, then we say that $L$ is numerically trivial. The group of line bundles modulo numerical equivalence is denoted by $N^1(X)$. This is a torsion free quotient of the Néron--Severi group of $X$. So by the theorem of the base, $N^1(X)$ is a free $\mathbb{Z}$-module of finite rank.

Below we recall some results from [26]. First, let us recall that on $X$ one can define the intersection product

$$A^1(X)^{\times 2} \times N^1(X)^{\times (n-2)} \rightarrow \mathbb{Q}, \quad (D_1, D_2, L_1, \ldots, L_{n-2}) \rightarrow D_1 \cdot D_2 \cdot L_1 \ldots L_{n-2}$$

of two Weil divisors with $(n-2)$ Cartier divisors. This behaves as expected and on a general complete intersection surface it reduces to Mumford’s intersection product. The following result says that one can also define an analogue of the second orbifold Chern character:

**Theorem 1.17.** Assume that $k$ has positive characteristic $p$. For any coherent reflexive $\mathcal{O}_X$-module $\mathcal{E}$ on $X$ there exists a $\mathbb{Z}$-multilinear symmetric form

$$\int_X \mathrm{ch}_2(\mathcal{E}) : N^1(X)^{\times (n-2)} \rightarrow \mathbb{R}.$$
such that:

1. If $E$ is a vector bundle on $X$, then
   $$\int_X \text{ch}_2(E)L_1 \ldots L_{n-2} = \int_X \text{ch}_2(E) \cap c_1(L_1) \cap \cdots \cap c_1(L_{n-2}) \cap [X].$$

2. If $k < K$ is an algebraically closed field extension, then
   $$\int_{X_K} \text{ch}_2(E_K)(L_1)_K \cdots (L_{n-2})_K = \int_X \text{ch}_2(E)L_1 \cdots L_{n-2}.$$  

3. If $n > 2$ and $L_1$ is very ample, then for a very general hypersurface $H \in |L_1|$ we have
   $$\int_X \text{ch}_2(E)L_1 \cdots L_{n-2} = \int_H \text{ch}_2(E|_H)L_2|_H \cdots L_{n-2}|_H.$$  

4. If $X$ is a surface, then
   $$\int_X \text{ch}_2(E) = \lim_{m \to \infty} \frac{\chi(X, F^m_X(E))}{p^{2m}}.$$  

5. $\int_X \text{ch}_2(F^1_X(E))L_1 \cdots L_{n-2} = p^2 \int_X \text{ch}_2(E)L_1 \cdots L_{n-2}.$

By linearity we can also extend the obtained form to $\mathbb{Q}$-line bundles. Once we have the above results, we can easily define real numbers $\int_X c_1^2(E)L_1 \ldots L_{n-2}$, $\int_X c_2(E)L_1 \ldots L_{n-2}$ and $\int_X \Delta(E)L_1 \ldots L_{n-2}$ for any reflexive coherent $\mathcal{O}_X$-module $E$.

1.4. Numerical groups of divisors. Let $X$ be a normal proper $n$-dimensional variety defined over an algebraically closed field $k$. We say that a Weil divisor $D$ is algebraically equivalent to zero if there exists a smooth variety $T$, a Weil divisor $G$ on $X \times_k T$ and $k$-points $t_1, t_2 \in T$ such that $D = G_{t_1} - G_{t_2}$ in $A^1(X)$ (see [8, Definition 10.3]). Then we write $D \sim_{\text{alg}} 0$.

The group of algebraic equivalence classes of Weil divisors on $X$ is denoted by $B^1(X)$. By $\tilde{B}^1(X)$ we denote the quotient of $B^1(X)$ by torsion.

Let us recall that $N^1(X) \simeq \text{Pic}_X / \text{Pic}^0_X$. If $X$ is smooth, then $N^1(X) \simeq \tilde{B}^1(X)$ (see [15, Theorem 9.6.3]). The theorem of the base [13, Théorème 3] implies that $\tilde{B}^1(X)$ is a free $\mathbb{Z}$-module of finite rank.

**Lemma 1.18.** If a Weil divisor $D_1$ is algebraically equivalent to zero, then for every Weil divisor $D_2$ and all line bundles $L_1, \ldots, L_{n-2}$ we have

$$D_1.D_2.L_1 \ldots L_{n-2} = 0.$$  

**Proof.** Let us first assume that $X$ is a surface and let $f : \tilde{X} \to X$ be a resolution of singularities. By assumption there exists a smooth variety $T$, a Weil divisor $G$ on $X \times_k T$ and $k$-points $t_1, t_2 \in T$ such that $D = G_{t_1} - G_{t_2}$ in $A^1(X)$. Let us consider the map

$$g := f \times \text{Id} : \tilde{X} \times_k T \to X \times_k T.$$  

One can use Mumford’s construction of pullback to define $g^*G$ that restricts to $f^*(G_{t_1})$ on $\tilde{X} \times \{t\}$ for every $t \in T(k)$. Then we have $f^*D = (g^*G)_{t_1} - (g^*G)_{t_2}$ in $A^1(\tilde{X})_Q$. This implies that some multiple of $f^*D_1$ is algebraically equivalent to zero and hence

$$f^*D_1.f^*D_2 = D_1.D_2 = 0.$$
In general, by linearity of the intersection product it is sufficient to prove that

\[ D_1.D_2.L_1 \ldots L_{n-2} = 0 \]

assuming that \( L_1, \ldots, L_{n-2} \) are very ample. Let \( S \in |L_1| \cap \cdots \cap |L_{n-2}| \) be a general complete intersection surface. Since cycles algebraically equivalent to zero are preserved by Gysin homomorphisms (see [8, Proposition 10.3]) the restriction \( D_1|_S \) is algebraically equivalent to zero. So we have

\[ D_1.D_2.L_1 \ldots L_{n-2} = D_1|_S.D_2|_S = 0. \]

The above lemma shows that the intersection pairing induces a \( \mathbb{Z} \)-multilinear map

\[ B^1(X) \times B^1(X) \times N^1(X)^{\times(n-2)} \rightarrow \mathbb{Q}. \]

Let us fix a collection (i.e., an order set) \( L = (L_1, L_2, \ldots, L_{n-1}) \) of nef line bundles on \( X \). Assume that \( L_1.L_2 \ldots L_{n-1} \) is numerically non-trivial, i.e., there exists some Weil divisor \( D \) such that \( D.L_1 \ldots L_{n-1} \neq 0 \). We consider a \( \mathbb{Q} \)-valued intersection pairing

\[ \langle \cdot, \cdot \rangle_L : B^1(X) \times B^1(X) \rightarrow \mathbb{Q} \]

given by

\[ \langle D_1, D_2 \rangle_L := D_1.D_2.L_2 \ldots L_{n-1}. \]

We write \( N_L(X) \) for the quotient of \( B^1(X) \) by the radical of this intersection pairing. Then we have the induced intersection pairing \( \langle \cdot, \cdot \rangle_L : N_L(X) \times N_L(X) \rightarrow \mathbb{Q} \), which is non-degenerate. Note that both \( N_L(X) \) and \( \langle \cdot, \cdot \rangle_L \) do not depend on \( L_1 \) but for convenience we preserve this line bundle in the collection \( L \).

The next two lemmas are versions of the Hodge index theorem in our situation.

**Lemma 1.19.** If \( \langle L_1, L_1 \rangle_L > 0 \) and \( \langle D_1, L_1 \rangle_L = 0 \), then \( \langle D_1, D_1 \rangle_L \leq 0 \) with equality if and only if the class \( [D_1] \in N_L(X) \) is zero.

**Proof.** Let us fix some ample line bundle \( A \). Then the \( \mathbb{Q} \)-line bundles \( L_i + \epsilon A \) are ample for \( \epsilon \in \mathbb{Q}_{>0} \). So by [26, Lemma 2.6] we have inequalities

\[
D_1^2.(L_2 + \epsilon A) \ldots (L_{n-1} + \epsilon A) \cdot (L_1 + \epsilon A)^2.(L_2 + \epsilon A) \ldots (L_{n-1} + \epsilon A) \\
\leq (D_1.(L_1 + \epsilon A) \ldots (L_{n-1} + \epsilon A))^2.
\]

Taking the limit when \( \epsilon \rightarrow 0 \), we get \( \langle D_1, D_1 \rangle_L \leq 0 \). Now let us assume that \( \langle D_1, D_1 \rangle_L = 0 \) but \( \langle D_1, D_2 \rangle_L \neq 0 \) for some Weil divisor \( D_2 \). Replacing \( D_2 \) by \( \langle L_1, L_1 \rangle_L D_2 - \langle D_2, L_1 \rangle L_1 \) we can assume that \( \langle D_2, L_1 \rangle_L = 0 \). Therefore we have

\[ 0 \geq (tD_1 + D_2)^2.L_2 \ldots L_{n-1} = 2t \langle D_1, D_2 \rangle_L + \langle D_2, D_2 \rangle_L, \]

which gives a contradiction with some \( t \in \mathbb{Z} \).

The following lemma generalizes [26, Lemma 2.6].

**Lemma 1.20.** The quotient \( N_L(X) \) is a free \( \mathbb{Z} \)-module of finite rank. If \( \text{rk}_\mathbb{Z} N_L(X) = s \), then the intersection pairing \( \langle \cdot, \cdot \rangle_L \) has signature \((1, s - 1)\).
Proof. By definition $N_L(X)$ is torsion free and it is a quotient of $\tilde{B}^1(X)$. So $N_L(X)$ is also a free $\mathbb{Z}$-module of finite rank. If $(L_1, L_1)_L > 0$, then the second assertion follows from Lemma 1.19. In general, there exists some Weil divisor $D$ such that $(D, L_1)_L \neq 0$. Without loss of generality we can assume that $(D, L_1)_L > 0$. Then for every ample Cartier divisor $H$ we have $H^0(X, \mathcal{O}(mH - D)) \neq 0$ for $m \gg 0$. So $mHL_1 \ldots L_{n-1} \geq D.L_1 \ldots L_{n-1} > 0$. Then $M = (L_1 + H)$ is ample and $(M, M)_L > 0$. Since the definition of $N_L(X)$ does not depend on $L_1$, we get the required assertion from the previous case. □

As in [28, Lemma 2.1] the above lemma implies the following result (in fact, the first part follows from the proof of Lemma 1.20).

Corollary 1.21. If $H$ is an ample line bundle, then one has $(H, L_1)_L > 0$. Moreover, if $(D, L_1)_L = 0$ for some Weil divisor $D$, then $(D, D)_L \leq 0$.

2. Some auxiliary results on the Chern character

In this section we prove several results on Chern characters of reflexive sheaves. We assume that $X$ is a normal projective variety of dimension $n$ defined over an algebraically closed field $k$ of characteristic $p > 0$. We also fix a collection $L = (L_1, \ldots , L_{n-1})$ of nef line bundles on $X$ such that $L_1L_2 \ldots L_{n-1}$ is numerically non-trivial.

Lemma 2.1. Let

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2$$

be an exact sequence of reflexive sheaves on $X$ and assume that the cokernel of the last map is supported in codimension $\geq 2$. Then we have

$$\int_X \text{ch}_2(\mathcal{E})L_1 \ldots L_{n-2} \leq \int_X \text{ch}_2(\mathcal{E}_1)L_1 \ldots L_{n-2} + \int_X \text{ch}_2(\mathcal{E}_2)L_1 \ldots L_{n-2}.$$

Moreover, if the last map in the above sequence is surjective and the sequence is locally split in codimension 2, then in the above inequality we have equality.

Proof. Since numerical equivalence classes of nef line bundles are limits of classes of ample $\mathbb{Q}$-line bundles, we can by continuity assume that all $L_i$ are ample $\mathbb{Q}$-line bundles. Passing to their multiples we can also assume that all $L_i$ are very ample line bundles. By Theorem 1.17 (2) and (3), we can assume that the base field $k$ is uncountable and then by restricting to a very general complete intersection surface $S \in |L_1| \cap \cdots \cap |L_{n-2}|$ we can assume that $X$ is a surface.

Let $U$ be a big open subset on which all $\mathcal{E}$, $\mathcal{E}_1$ and $\mathcal{E}_2$ are locally free and let $j : U \hookrightarrow X$ denote the open embedding. Since $X$ is normal, we can also assume that $U$ is contained in the regular locus $X_{\text{reg}}$ of $X$ and the sequence

$$0 \to (\mathcal{E}_1)_U \to \mathcal{E}_U \to (\mathcal{E}_2)_U$$

is exact. Since $F_U^*$ is exact, the sequences

$$0 \to (F_U^*)^*(\mathcal{E}_1)_U \to (F_U^*)^*\mathcal{E}_U \to (F_U^*)^*(\mathcal{E}_2)_U \to 0$$

are exact. Thus, the inequalities

$$\int_X \text{ch}_2(\mathcal{E})L_1 \ldots L_{n-2} \leq \int_X \text{ch}_2(\mathcal{E}_1)L_1 \ldots L_{n-2} + \int_X \text{ch}_2(\mathcal{E}_2)L_1 \ldots L_{n-2}$$

hold. Taking into account that $L_1 \ldots L_{n-2}$ is numerically non-trivial, it follows that $\text{ch}_2(\mathcal{E})L_1 \ldots L_{n-2}$ is numerically non-trivial. Since $\text{ch}_2(\mathcal{E})L_1 \ldots L_{n-2} = \text{ch}_2((\mathcal{E}_1)_U)L_1 \ldots L_{n-2}$, we get the required inequality.

Moreover, if the last map in the above sequence is surjective and the sequence is locally split in codimension 2, then in the above inequality we have equality. □
are exact. This implies that each sequence
\[ 0 \to F_X^m \mathcal{E}_1 \to F_X^m \mathcal{E} \to F_X^m \mathcal{E}_2 \]
is exact and the cokernel of the last map is supported on a finite set of points contained in \( X \setminus U \). So we get inequalities
\[ \chi(X, F_X^m \mathcal{E}) \leq \chi(X, F_X^m \mathcal{E}_1) + \chi(X, F_X^m \mathcal{E}_2). \]
Dividing by \( p^{2m} \), passing to the limit and using Theorem 1.17, (3), we get the required inequality. Equality follows from the fact that the above mentioned sequence becomes exact on the right if the sequence
\[ 0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0 \]
is locally split.

**Lemma 2.2.** Let \( \mathcal{E} \) be a rank \( r \) reflexive coherent \( \mathcal{O}_X \)-module and let
\[ \mathcal{E} = N^0 \supset N^1 \supset \cdots \supset N^s = 0 \]
be a filtration such that all \( N^i / N^{i+1} \) are torsion free. Set \( \mathcal{F}_i := (N^i / N^{i+1})^{**} \), \( r_i := \text{rk} \mathcal{F}_i \), \( \mu_i := \mu_L(\mathcal{F}_i) \) and \( \mathcal{F} := \bigoplus \mathcal{F}_i \). Then the following conditions are satisfied:
1. \( c_1(\mathcal{E}) = c_1(\mathcal{F}) \).
2. \( \int_X Ch_2(\mathcal{E}) L_1 \cdots L_{n-2} \leq \int_X Ch_2(\mathcal{F}) L_1 \cdots L_{n-2} \).
3. \( \int_X \Delta(\mathcal{E}) L_1 \cdots L_{n-2} \geq \int_X \Delta(\mathcal{F}) L_1 \cdots L_{n-2} \).
4. if \( d := L_1^2 L_2 \cdots L_{n-1} > 0 \), then
   \[ \frac{\int_X \Delta(\mathcal{E}) L_2 \cdots L_{n-1}}{r} \geq \sum_i \frac{\int_X \Delta(\mathcal{F}_i) L_2 \cdots L_{n-1}}{r_i} - \frac{1}{rd} \sum_{i < j} r_ir_j(\mu_i - \mu_j)^2, \]
5. if \( \mu_i = \mu_L(\mathcal{E}) \) for all \( i \), then
   \[ \frac{\int_X \Delta(\mathcal{E}) L_2 \cdots L_{n-1}}{r} \geq \sum_i \frac{\int_X \Delta(\mathcal{F}_i) L_2 \cdots L_{n-1}}{r_i}. \]

**Proof.** The first condition is clear as \( c_1(\mathcal{E}) = c_1(\text{Gr}_N(\mathcal{E})) = c_1(\mathcal{F}). \) We prove the second condition by induction on the length \( s \) of the filtration. If \( s > 1 \), then \( N^1 \) is reflexive as \( N^0 / N^1 \) is torsion free. So by Lemma 2.1 we have
\[ \int_X Ch_2(\mathcal{E}) L_1 \cdots L_{n-2} \leq \int_X Ch_2(N^1) L_1 \cdots L_{n-2} + \int_X Ch_2(\mathcal{F}_0) L_1 \cdots L_{n-2}. \]
Lemma 2.1 implies that
\[ \int_X Ch_2(\mathcal{F}) L_1 \cdots L_{n-2} = \sum_i \int_X Ch_2(\mathcal{F}_i) L_1 \cdots L_{n-2}. \]
So applying the induction assumption to the filtration \( N^1 \supset N^2 \supset \cdots \supset N^s = 0 \) of \( N^1 \), we get the required inequality. (3) follows immediately from (1) and (2).
Rewriting the above equality of Chern characters gives
\[
\int_X \Delta(\mathcal{F}_r) L_2 \ldots L_{n-1} = \sum_i \int_X \Delta(\mathcal{F}_i) L_2 \ldots L_{n-1} - \frac{1}{r} \sum_{i<j} r_i r_j \left( \frac{c_1 \mathcal{F}_i}{r_i} - \frac{c_1 \mathcal{F}_j}{r_j} \right)^2 L_2 \ldots L_{n-1}.
\]

But by the Hodge index theorem (see Lemma 1.20) we have
\[
(\mu_i - \mu_j)^2 = \left( \frac{c_1 \mathcal{F}_i}{r_i} - \frac{c_1 \mathcal{F}_j}{r_j} \right)^2 L_1 \ldots L_{n-1} \geq d \cdot \left( \frac{c_1 \mathcal{F}_i}{r_i} - \frac{c_1 \mathcal{F}_j}{r_j} \right)^2 L_2 \ldots L_{n-1}.
\]

So (4) follows from (3). Under assumption (5), Corollary 1.21 implies that
\[
\left( \frac{c_1 \mathcal{F}_i}{r_i} - \frac{c_1 \mathcal{F}_j}{r_j} \right)^2 L_2 \ldots L_{n-1} \leq 0,
\]
so again the inequality follows from (3).

**Lemma 2.3.** Let \( H \in |L_1| \) be a normal variety and let \( \mathcal{T} \) be a rank \( r \) torsion free \( \mathcal{O}_D \)-module and let \( i : H \hookrightarrow X \) be the closed embedding. Let
\[
0 \to \mathcal{G} \to \mathcal{E} \to i_* \mathcal{T} \to 0
\]
be a short exact sequence of coherent \( \mathcal{O}_X \)-modules, where \( \mathcal{E} \) is reflexive. Then
\[
\int_X \Delta(\mathcal{G}) L_2 \ldots L_{n-1} = \int_X \Delta(\mathcal{E}) L_2 \ldots L_{n-1} - \tau(r - \tau) L_1^2 L_2 \ldots L_{n-1} + 2(r c_1(\mathcal{T}) - \tau c_1(i^* \mathcal{E})).i^* L_2 \ldots i^* L_{n-1}.
\]

**Proof.** Note that \( \mathcal{G} \) is a coherent reflexive \( \mathcal{O}_X \)-module. Since both sides of our inequality depend continuously on \( L_2, \ldots, L_{n-1} \) when considered as functions on \( N^1(X)_\mathbb{R} \), and the inequality does not change when we pass to multiples, we can assume that \( L_2, \ldots, L_{n-1} \) are very ample. By Theorem 1.17 we can assume that the base field \( k \) is uncountable and then we can restrict to a very general complete intersection surface in \( |L_2| \cap \cdots \cap |L_{n-1}| \) to reduce the assertion to the surface case. An exact sequence
\[
(F^m_X)^* \mathcal{G} \to (F^m_X)^* \mathcal{E} \to (F^m_X)^* (i_* \mathcal{T}) \to 0
\]
leads to
\[
0 \to F^{[m]}_X \mathcal{G} \to F^{[m]}_X \mathcal{E} \to \mathcal{T}_m \to 0,
\]
where \( \mathcal{T}_m \) is set-theoretically supported on \( H \). Moreover, we have a canonical map
\[
(F^m_X)^* (i_* \mathcal{T}) \to \mathcal{T}_m,
\]
which is an isomorphism on the set where \( F_X \) is flat, i.e., on \( X_{\text{reg}} \). But \( X \) is a surface and \( H \) is a smooth curve, which is also a Cartier divisor. So \( H \) is contained in \( X_{\text{reg}} \) and hence
\[
\mathcal{T}_m \simeq (F^m_X)^* (i_* \mathcal{T}).
\]
This gives
\[
\int_X \text{ch}_2(\mathcal{E}) = \lim_{m \to \infty} \frac{\chi(X, F_X^m \mathcal{E})}{p^{2m}}
= \lim_{m \to \infty} \frac{\chi(X, F_X^m \mathcal{E})}{p^{2m}} + \lim_{m \to \infty} \frac{\chi(X, (F_X^m)^*(i_* \mathcal{T}))}{p^{2m}}
= \int_X \text{ch}_2(\mathcal{E}) + \lim_{m \to \infty} \frac{\chi(X, (F_X^m)^*(i_* \mathcal{T}))}{p^{2m}}.
\]
To compute the last limit, let us consider a resolution of singularities \(f : \widetilde{X} \to X\), which is an isomorphism on \(X_{\text{reg}}\). So we have a closed embedding \(i : H \hookrightarrow \widetilde{X}\) such that \(f \circ i = i\). Then we have a short exact sequence
\[
0 \to \mathcal{F} \to f[*] \mathcal{E} \to i_* \mathcal{T} \to 0,
\]
where \(\mathcal{F}\) is a rank \(r\) vector bundle. Then by the same arguments as above we have
\[
\lim_{m \to \infty} \frac{\chi(X, (F_X^m)^*(i_* \mathcal{T}))}{p^{2m}} = \lim_{m \to \infty} \frac{\chi(\widetilde{X}, (F_{\widetilde{X}}^m)^*(i_* \mathcal{T}))}{p^{2m}}
= \int_{\widetilde{X}} \text{ch}_2(f[*] \mathcal{E}) - \int_{\widetilde{X}} \text{ch}_2(\mathcal{F})
= \int_{\widetilde{X}} \text{ch}_2(i_* \mathcal{T}).
\]
To compute \(\int_{\widetilde{X}} \text{ch}_2(i_* \mathcal{T})\), one can use the Riemann–Roch theorem on \(\widetilde{X}\) and on \(H\) to get
\[
\deg_H \mathcal{T} + \tau \chi(\mathcal{O}_H) = \chi(H, \mathcal{T}) = \chi(X, i_* \mathcal{T}) = \int_{\widetilde{X}} \text{ch}_2(i_* \mathcal{T}) - \frac{1}{2} c_1(i_* \mathcal{T}) K_{\widetilde{X}}
\]
Since \(c_1(i_* \mathcal{T}) = \tau H\) and \(\chi(\mathcal{O}_H) = -\frac{1}{2} H(K_{\widetilde{X}} + H)\) (e.g., because \(K_H = (K_{\widetilde{X}} + H)|_H\) and \(\deg K_H = -2 \chi(\mathcal{O}_H))\), we get
\[
\int_{\widetilde{X}} \text{ch}_2(i_* \mathcal{T}) = \deg_H \mathcal{T} - \frac{\tau}{2} H^2.
\]
Summing up, we get
\[
\int_X \text{ch}_2(\mathcal{E}) = \int_X \text{ch}_2(\mathcal{F}) + \deg_H \mathcal{T} - \frac{\tau}{2} L_1^2.
\]
After rewriting, using \(2r \int_X \text{ch}_2(\mathcal{E}) = c_1^2(\mathcal{E}) - \int_X \Delta(\mathcal{E})\), \(2r \int_X \text{ch}_2(\mathcal{F}) = c_1^2(\mathcal{F}) - \int_X \Delta(\mathcal{F})\) and \(c_1(\mathcal{F}) = c_1(\mathcal{E}) - \tau H\), we get the required formula. \(\square\)

3. Boundedness on normal varieties in positive characteristic

In this section we prove strong restriction theorems for semistable sheaves and we show some boundedness results. In particular, we prove Theorems 0.1 and 0.2.

Let \(X\) be a normal projective variety of dimension \(n\) defined over an algebraically closed field \(k\) of characteristic \(p > 0\) and let \(L = (L_1, \ldots, L_{n-1})\) be a collection of nef line bundles on \(X\) such that the 1-cycle \(L_1 \ldots L_{n-1}\) is numerically non-trivial.

We write \(d\) for the number \(L_1^2 L_2 \ldots L_{n-1}\).

For a divisor \(H \in |L_1|\) we write \(L_H\) for the collection \((L_2|_H, \ldots, L_{n-1}|_H)\).
3.1. Slope semistability and its behavior under pullbacks. For a coherent $\mathcal{O}_X$-module $\mathcal{E}$ of positive rank $r$ we define its slope with respect to the collection $L$ by

$$
\mu_L(\mathcal{E}) := \frac{c_1(\mathcal{E}).L_1 \ldots L_{n-1}}{r}.
$$

Note that by definition $\mu_L(\mathcal{E}^*) = \mu_L(\mathcal{E})$.

We say that a coherent $\mathcal{O}_X$-module $\mathcal{E}$ is slope $L$-semistable if either it is torsion or for every coherent $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{E}$ of positive rank we have $\mu_L(\mathcal{F}) \leq \mu_L(\mathcal{E})$. We say that $\mathcal{E}$ is strongly slope $L$-semistable if for all $m \geq 0$ the Frobenius pullbacks $(F_X^m)^*\mathcal{E}$ are slope $L$-semistable. In this section we usually consider slope semistability with respect to our fixed collection $L$ (unless explicitly stated). So for simplicity of notation we usually ignore dependence of slopes on $L$.

Every coherent $\mathcal{O}_X$-module $\mathcal{E}$ of positive rank admits the Harder–Narasimhan filtration

$$
0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = \mathcal{E}.
$$

It is a unique filtration by coherent $\mathcal{O}_X$-submodules such that $\mathcal{E}_0$ is torsion, quotients $\mathcal{E}^i := \mathcal{E}_i/\mathcal{E}_{i-1}$ are torsion free and slope $L$-semistable for $i = 1, \ldots, s$ and we have $\mu_1 = \mu_L(\mathcal{E}^1) > \cdots > \mu_s = \mu_L(\mathcal{E}^s)$. In the following we write $\mu_{\max,L}(\mathcal{E})$ for $\mu_1$ and $\mu_{\min,L}(\mathcal{E})$ for $\mu_s$. Sometimes we omit $L$ in the notation if it is clear which collection is used.

The proof of [21, Theorem 2.7] works on normal varieties and it gives the following result:

**Theorem 3.1.** Let $\mathcal{E}$ be a coherent $\mathcal{O}_X$-module of positive rank. Then there exists $m_0$ such that for all $m \geq m_0$ all quotients in the Harder–Narasimhan filtration of $(F_X^m)^*\mathcal{E}$ are strongly slope $L$-semistable.

For a coherent reflexive $\mathcal{O}_X$-module $\mathcal{E}$ we set

$$
L_{\max}(\mathcal{E}) = \lim_{m \to \infty} \frac{\mu_{\max}(F_X^m\mathcal{E})}{p^m} \quad \text{and} \quad L_{\min}(\mathcal{E}) = \lim_{k \to \infty} \frac{\mu_{\min}(F_X^m\mathcal{E})}{p^m}.
$$

By Theorem 3.1 the above sequences stabilize and hence $L_{\max}(\mathcal{E})$ and $L_{\min}(\mathcal{E})$ are well defined rational numbers.

Now choose a nef Cartier divisor $A$ on $X$ such that the twisted tangent sheaf $T_X(A)$ is globally generated. Then [21, Corollary 2.5] still holds and it implies the following lemma.

**Lemma 3.2.** For any coherent reflexive $\mathcal{O}_X$-module $\mathcal{E}$ of rank $r$ we have

$$
\max(L_{\max}(\mathcal{E}) - \mu(\mathcal{E}), \mu(\mathcal{E}) - L_{\min}(\mathcal{E})) \leq AL_1 \ldots L_{n-1}.
$$

As in [21] we also set

$$
\beta_r := \left(\frac{r(r-1)}{p-1} AL_1 \ldots L_{n-1}\right)^2.
$$

3.2. Restriction theorem and Bogomolov’s inequality. We define an open cone $K_L^+$ in $N_L(X)$ by

$$
K_L^+ := \{D \in N_L(X) : \langle D, D \rangle_L > 0 \text{ and } \langle D, L_1 \rangle_L \geq 0\}.
$$
As in the smooth case, by Lemma 1.20 this cone is “self-dual” in the following sense:

\[ D \in \mathcal{K}^+_L \text{ if and only if } \langle D, D' \rangle_L > 0 \text{ for all } D' \in \mathcal{K}^+_L \setminus \{0\}. \]

Let us fix a coherent reflexive \( \mathcal{O}_X \)-module \( \mathcal{E} \) of rank \( r > 0 \). Using Lemma 2.2, one can follow the proofs of [21, Theorems 3.1, 3.2, 3.3 and 3.4] to get the following results:

**Theorem 3.3.** Assume that \( L_1 \) is very ample and the restriction of \( \mathcal{E} \) to a general divisor \( H \in |L_1| \) is not slope \( L_H \)-semistable. Let \( r_i \) and \( \mu_i \) denote ranks and slopes (with respect to \( L_H \)) of the Harder–Narasimhan filtration of \( \mathcal{E}|_H \). Then

\[
\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \leq d \cdot \int_X \Delta(\mathcal{E})L_2 \ldots L_{n-1} + 2r^2(L_{\max}(\mathcal{E}) - \mu(\mathcal{E}))(\mu(\mathcal{E}) - L_{\min}(\mathcal{E})).
\]

**Theorem 3.4.** If \( \mathcal{E} \) is strongly slope \( L \)-semistable, then

\[
\int_X \Delta(\mathcal{E})L_2 \ldots L_{n-1} \geq 0.
\]

**Theorem 3.5.** If \( \mathcal{E} \) is slope \( L \)-semistable, then

\[
d \cdot \int_X \Delta(\mathcal{E})L_2 \ldots L_{n-1} + \beta_r \geq 0.
\]

**Theorem 3.6.** If \( d \cdot \int_X \Delta(\mathcal{E})L_2 \ldots L_{n-1} + \beta_r < 0 \), there exists a rank \( 1 \leq r' < r \) saturated reflexive subsheaf \( \mathcal{E}' \subset \mathcal{E} \) such that \( (\frac{c_1(\mathcal{E}')}{r'}) - (\frac{c_1(\mathcal{E})}{r}) \) lies in \( \mathcal{K}^+_L \).

The only difference in proofs with respect to [21] is that one should consider \( F_X^{[m]} \mathcal{E} \) instead of \((F_X^m)^* \mathcal{E}\). Also in the surface case one needs to use [26, Corollary 6.6] instead of using the arguments of [21] that do not work for normal surfaces.

### 3.3. Strong restriction theorems.

In this subsection we assume that \( d > 0 \). As in [21, Theorem 5.1], Lemma 2.2 and Theorems 3.4 and 3.5 imply the following Bogomolov’s inequality for all reflexive sheaves.

**Theorem 3.7.** If \( \mathcal{E} \) is a coherent reflexive \( \mathcal{O}_X \)-module of rank \( r \), then

\[
d \cdot \int_X \Delta(\mathcal{E})L_2 \ldots L_{n-1} + r^2(L_{\max}(\mathcal{E}) - \mu(\mathcal{E}))(\mu(\mathcal{E}) - L_{\min}(\mathcal{E})) \geq 0
\]

and

\[
d \cdot \int_X \Delta(\mathcal{E})L_2 \ldots L_{n-1} + r^2(\mu_{\max}(\mathcal{E}) - \mu(\mathcal{E}))(\mu(\mathcal{E}) - \mu_{\min}(\mathcal{E})) + \beta_r \geq 0.
\]

This immediately implies the following corollary.

**Corollary 3.8.** Let us fix some positive integer \( r \) and some non-negative rational number \( \alpha \). There exists some constant \( C = C(X, L, r, \alpha) \) depending only on \( X, L, r \) and \( \alpha \) such
that for every coherent reflexive $\mathcal{O}_X$-module $\mathcal{E}$ of rank $r$ with $\mu_{\text{max}, L}(\mathcal{E}) - \mu_L(\mathcal{E}) \leq \alpha$ we have

$$\int_X \Delta(\mathcal{E}) L_2 \ldots L_{n-1} \geq C.$$ 

As in [21] we can use Theorem 3.7 to prove strong restriction theorems for reflexive sheaves on normal varieties. We take this opportunity to prove a stronger result that combines [21, Theorem 5.2] with [28, Theorem 3.7].

**Theorem 3.9.** Let $\mathcal{E}$ be a coherent reflexive $\mathcal{O}_X$-module of rank $r \geq 2$. Let us set

$$m_0 = \frac{r - 1}{r} \int_X \Delta(\mathcal{E}) L_2 \ldots L_{n-1} + \frac{1}{dr(r-1)} + \frac{(r-1)\beta_r}{dr}.$$ 

Assume that $\mathcal{E}$ is slope $L$-stable and let $H \in |L^{\otimes m}|$ be an integral normal divisor.

1. If $m > m_0$, then $\mathcal{E}|_H$ is slope $L_H$-stable.
2. If $m \leq m_0$, then

$$\mu_{\text{max}, L_H}(\mathcal{E}|_H) - \mu_{L_H}(\mathcal{E}|_H) \leq \frac{1}{2r} \left( d \int_X \Delta(\mathcal{E}) L_2 \ldots L_{n-1} - \frac{r}{r-1}m + \frac{1}{(r-1)^2} + \beta_r \right).$$

**Proof.** Let $i : H \hookrightarrow X$ denote the closed embedding. Since $\mathcal{E}$ is reflexive, it follows that $i^* \mathcal{E}$ is a torsion free $\mathcal{O}_H$-module. Let $\mathcal{B}$ be a saturated subsheaf of $i^* \mathcal{E}$ of rank $\rho$. Set $\mathcal{T} := (i^* \mathcal{E})/\mathcal{B}$ and let $\mathcal{E}'$ be the kernel of the composition $\mathcal{E} \to i_*(i^* \mathcal{E}) \to i_* \mathcal{T}$. Since $\mathcal{T}$ is a torsion free $\mathcal{O}_H$-module, it follows that $\mathcal{E}'$ is a coherent reflexive $\mathcal{O}_X$-module and we have two short exact sequences

$$0 \to \mathcal{E}' \to \mathcal{E} \to i_* \mathcal{T} \to 0$$

and

$$0 \to \mathcal{E}(-H) \to \mathcal{E}' \to i_* \mathcal{B} \to 0.$$ 

Lemma 2.3 implies that

$$\int_X \Delta(\mathcal{E}') L_2 \ldots L_{n-1} = \int_X \Delta(\mathcal{E}) L_2 \ldots L_{n-1} - \rho(r - \rho)H^2 L_2 \ldots L_{n-1}$$

$$+ 2(r c_1(\mathcal{T}) - (r - \rho) c_1(i^* \mathcal{E})). L_2 H \ldots L_{n-1}|_H.$$ 

Since $\mathcal{E}' \subset \mathcal{E}$ and $\mathcal{E}$ is slope $L$-stable, we have

$$\mu_{\text{max}}(\mathcal{E}') - \mu(\mathcal{E}') = \frac{r}{r} \frac{r - \rho}{r} H L_1 \ldots L_{n-1} + \mu_{\text{max}}(\mathcal{E}') - \mu(\mathcal{E})$$

$$\leq \frac{r - \rho}{r} \cdot md \cdot \frac{1}{r(r - 1)}.$$ 

Similarly, since $\mathcal{E}(-H) \subset \mathcal{E}'$, we have

$$\mu(\mathcal{E}') - \mu_{\text{min}}(\mathcal{E}') = \frac{\rho}{r} H L_1 \ldots L_{n-1} + \mu(\mathcal{E}(-H)) - \mu_{\text{min}}(\mathcal{E}')$$

$$\leq \frac{\rho}{r} \cdot md \cdot \frac{1}{r(r - 1)}.$$
So Theorem 3.7 gives
\[
0 \leq d \cdot \int_X \Delta(\mathcal{E}')L_2 \ldots L_{n-1} + r^2(\mu_{\text{max}}(\mathcal{E}') - \mu(\mathcal{E}'))(\mu(\mathcal{E}') - \mu_{\text{min}}(\mathcal{E}')) + \beta_r
\]
\[
\leq d \cdot \int_X \Delta(\mathcal{E}')L_2 \ldots L_{n-1} - \rho(r - \rho)m^2d^2 - 2r\rho(\mu(\mathcal{E}) - \mu(i^* \mathcal{E}))
\]
\[
+ \left( pm \frac{1}{r - 1} \right) \left( (r - \rho)md - \frac{1}{r - 1} \right) + \beta_r.
\]
If \( \mu(\mathcal{E}) \geq \mu(i^* \mathcal{E}) \), then we get
\[
\frac{2(r - 1)}{d}(\mu(\mathcal{E}) - \mu(i^* \mathcal{E})) + m
\]
\[
\leq \frac{r - 1}{r} \int_X \Delta(\mathcal{E})L_2 \ldots L_{n-1} + \frac{1}{d \rho(r - 1)} + \frac{(r - 1)\beta_r}{d r},
\]
which implies the required assertions. \( \square \)

**Remark 3.10.** The above proof works also for an arbitrary irreducible normal divisor \( D \subset X \), which is nef and Cartier. In this way one gets restriction theorems taking into account the difference of directions of lines given by classes of \( D \) and \( L_1 \) in \( N_L(X) \). We leave the details of proof to the interested reader.

As in [21, Corollary 5.4] the above theorem together with Lemma 2.2 implies the following result:

**Corollary 3.11.** Let \( \mathcal{E} \) be a coherent reflexive \( \mathcal{O}_X \)-module of rank \( r \geq 2 \). Assume that \( \mathcal{E} \) is slope \( L \)-semistable and let \( H \in |L_1^{\mathfrak{m}}| \) be an irreducible normal divisor. Let
\[
0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = \mathcal{E}
\]
be a Jordan–Hölder filtration of \( \mathcal{E} \) and let us assume that all \( (\mathcal{E}_i/\mathcal{E}_{i-1})|_H \) are torsion free. Let \( m_0 \) be defined as in Theorem 3.9.

1. If \( m > [m_0] \), then \( \mathcal{E}|_H \) is slope \( L_H \)-semistable.
2. If \( m \leq [m_0] \), then
\[
\mu_{\text{max},L_H}(\mathcal{E}|_H) \leq \mu(\mathcal{E}) + \frac{1}{2r} \left( d \int_X \Delta(\mathcal{E})L_2 \ldots L_{n-1} - \frac{r}{r - 1} m \right) + \frac{1}{(r - 1)^2} + \beta_r.
\]

**Proof.** Note that all \( \mathcal{E}_i \) are reflexive as they are saturated in \( \mathcal{E} \). Set \( \mathcal{F}_i := (\mathcal{E}_i/\mathcal{E}_{i-1})^{**} \) and \( r_i = \text{rk} \mathcal{F}_i \). By Lemma 2.2 we have
\[
\int_X \frac{\Delta(\mathcal{E})L_2 \ldots L_{n-1}}{r} \geq \sum_i \int_X \frac{\Delta(\mathcal{F}_i)L_2 \ldots L_{n-1}}{r_i}.\]

In the first case either \( r_i = 1 \) or \( r_i \geq 2 \) and
\[
m > \left\lfloor \frac{r_i - 1}{r_i} \int_X \Delta(\mathcal{F}_i)L_2 \ldots L_{n-1} + \frac{1}{d r_i (r_i - 1)} + \frac{(r_i - 1)\beta r_i}{d r_i} \right\rfloor.
\]
Note that in the above inequality we need to worry about the term \( \frac{1}{\kappa r_i(r_i-1)} \), which can be larger than \( \frac{1}{d r_i(r_i-1)} \). However, the difference is compensated by the other terms unless both of them are 0 in which case
\[
\left| \frac{1}{d r_i(r_i-1)} \right| = 0.
\]

Applying Theorem 3.9, we see that \( \mathcal{F}_i|_H \) is stable with the same slope as \( E_j|_H \). As \( (E_i/E_{i-1})|_H \) are torsion free, the sequences
\[
0 \to E_{i-1}|_H \to E_i|_H \to \mathcal{F}_i|_H
\]
are exact. Now a simple induction shows that all \( E_i|_H \) are slope \( L_H \)-semistable.

The second case is completely analogous. We just need to use the fact that
\[
\mu_{\max}(E_i|_H) \leq \max_i \mu_{\max}(\mathcal{F}_i|_H).
\]

Remark 3.12. The above results give also restriction theorems for torsion free sheaves. More precisely, if \( E \) is a slope \( L \)-(semi)stable coherent torsion free \( \mathcal{O}_X \)-module, then the reflexive hull \( E^{**} \) is also slope \( L \)-(semi)stable. So we can apply Theorem 3.9 and Corollary 3.11 to \( E^{**} \). If \( E|_H \) is torsion free, then it is slope \( L_H \)-(semi)stable if (and only if) \( E^{**}|_H \) is slope \( L_H \)-(semi)stable.

3.4. Boundedness. In this subsection we assume that \( n \geq 2 \) and all line bundles in the collection \( L \) are ample. For a collection \( L \) we denote by \( \mathcal{L} \) the collection
\[
(L_1, L_1, \ldots, L_i, \ldots, L_n-1)
\]
(in particular, \( \mathcal{L} = L \)).

For \( \alpha \in \tilde{B}^1(X) \) we write \( \alpha \sim 0 \) if \( \alpha \cdot L_1 \ldots L_{n-1} = 0 \) and
\[
\alpha^2 \cdot L_1 \ldots L_i \ldots L_{n-1} = 0
\]
for \( i = 1, \ldots, n-1 \). Note that the subset \( S := \{ \alpha \in \tilde{B}^1(X) : \alpha \sim 0 \} \) forms a \( \mathbb{Z} \)-submodule of \( \tilde{B}^1(X) \). Indeed, Lemma 1.19 implies that \( \alpha \in S \) if and only if the class of \( \alpha \) in \( N_i(L) \) vanishes for \( i = 1, \ldots, n-1 \). So \( S \) is the intersection of kernels of quotient maps
\[
\tilde{B}^1(X) \to N_i(L).
\]

In the following we set \( C^1(X; L) := \tilde{B}^1(X)/S \). Later we prove a rather surprising fact that the quotient map \( \tilde{B}^1(X) \to C^1(X; L) \) is an isomorphism (see Corollary 3.17).

Lemma 3.13. The quotient \( C^1(X; L) \) is a free \( \mathbb{Z} \)-module of finite rank. Moreover, for a general divisor \( H \in |L_1| \) the Gysin homomorphism \( B^1(X) \to B^1(H) \) induces the homomorphism
\[
C^1(X; L) \to C^1(H; L_H).
\]

Proof. The first assertion follows from the definition and the fact that \( \tilde{B}^1(X) \) is of finite rank. By [8, Proposition 10.3] we have a Gysin homomorphism \( B^1(X) \to B^1(H) \). This induces \( \tilde{B}^1(X) \to \tilde{B}^1(H) \). Let us denote the image of \( \alpha \in \tilde{B}^1(X) \) in \( \tilde{B}^1(H) \) by \( \alpha|_H \). It is suf-
Then the set $A_s$ satisfied:

\[ \text{numbers } [22, \text{Theorem 3.4}]. \]

immediately follow from Kleiman’s criterion. Part of the proof follows the idea of proof of $f$icient to show that if $\alpha \sim 0$ in $\tilde{B}^1(X)$ then $\alpha|_H \sim 0$ in $\tilde{B}^1(H)$. Since $S \subset \tilde{B}^1(X)$ is a finitely generated $\mathbb{Z}$-submodule, it is sufficient to check this condition for finitely many generators of $S$. But for a general divisor $H \in |L_1|$ we have

\[
\alpha^2|_{H} L_2|_H \cdots \widehat{L_i}|_H \cdots L_{n-1}|_H = \alpha_1^2 L_1 \cdots \widehat{L_i} \cdots L_{n-1} = 0
\]

for $i = 1, \ldots, n - 1$. We also have $\alpha|_{H} L_1|_H \cdots L_{n-1}|_H = \alpha L_1 \cdots L_{n-1} = 0$, which proves that the class of $\alpha|_H$ in $C^1(H; \mathcal{L}_H)$ vanishes.

Corollary 3.11 can be used to prove the following boundedness result, which does not immediately follow from Kleiman’s criterion. Part of the proof follows the idea of proof of [22, Theorem 3.4].

**Theorem 3.14.** Let us fix some class $c_1 \in C^1(X; L)$, a positive integer $r$ and some real numbers $c_2$ and $\mu_{\max}$. Let $\mathcal{A}$ be the set of reflexive coherent $\mathcal{O}_X$-modules $\mathcal{E}$ such that

1. $\mathcal{E}$ has rank $r$,
2. the class of $c_1(\mathcal{E})$ in $C^1(X; L)$ is equal to $c_1$,
3. $\int_X c_2(\mathcal{E}) L_1 \cdots \widehat{L_i} \cdots L_{n-1} \leq c_2$ for $i = 1, \ldots, n - 1$,
4. $\mu(\mathcal{E}) \leq \mu_{\max}$.

Then the set $\mathcal{A}$ is bounded.

**Proof.** For $n = 2$ the assertion is well known (see, e.g., [21, Theorem 4.4]), so we can assume that $n \geq 3$. Without loss of generality we can also assume that all $L_i$ are very ample. Note that if $k \subset K$ is an algebraically closed field extension, then the set $\mathcal{A}$ is bounded if and only if the set $\mathcal{A}_K := \{\mathcal{E}_K : \mathcal{E} \in \mathcal{A}\}$ of sheaves on $X_K$ is bounded. This follows from [12, Lemma 1.7.6] and the fact that the Castelnuovo–Mumford regularity of $\mathcal{E}$ (with respect to some fixed very ample line bundle $\mathcal{O}_X(1)$) coincides with the Castelnuovo–Mumford regularity of $\mathcal{E}$ (here we use the fact that $H^i(X_K, \mathcal{E}(j)) = H^i(X, \mathcal{E}(j)) \otimes_k K$). So by Theorem 1.17 (2) and an analogue of [12, Theorem 1.3.7] for slope semistability, we can assume that the base field $k$ is uncountable.

For fixed $\mathcal{E} \in \mathcal{A}$ and for a very general divisor $H \in |L_1|$, the following conditions are satisfied:

1. $\mathcal{E}|_H$ is reflexive as an $\mathcal{O}_H$-module (by [12, Corollary 1.1.14]),
2. for $i = 2, \ldots, n - 1$ we have

\[
\int_H c_2(\mathcal{E}|_H) L_2|_H \cdots \widehat{L_i}|_H \cdots L_{n-1}|_H = \int_X c_2(\mathcal{E}) L_1 \cdots \widehat{L_i} \cdots L_{n-1}.
\]

3. for $i = 2, \ldots, n - 1$ we have

\[
\int_H c_2(\mathcal{E}|_H) L_2|_H \cdots \widehat{L_i}|_H \cdots L_{n-1}|_H = \int_X c_2(\mathcal{E}) L_1 \cdots \widehat{L_i} \cdots L_{n-1} \leq c_2
\]

(by Theorem 1.17),
(4) if \( 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = \mathcal{E} \) is the Harder–Narasimhan filtration of \( \mathcal{E} \), then all \((\mathcal{E}_i / \mathcal{E}_{i-1})|_H\) are torsion free and restriction of any quotient of a Jordan–Hölder filtration of \( \mathcal{F}_i := (\mathcal{E}_i / \mathcal{E}_{i-1})^{**} \) to \( H \) is torsion free as an \( \mathcal{O}_H \)-module (by [12, Corollary 1.1.14]).

Fix a normal hypersurface \( H \in |L_1| \) such that the map \( C^1(X; L) \to C^1(H; L_H) \) induced by the Gysin homomorphism is well defined (see Lemma 3.13). Let us consider the set \( \mathcal{A} \) of reflexive sheaves on normal varieties (see [37, Lemma 0FD8]). Let us fix

\[
\text{Let us set } r_i = \text{rk } \mathcal{F}_i, \quad \text{and } \mu_i := \mu_{L}(\mathcal{F}_i). \text{ By Lemma 2.2 and [21, Lemma 1.4] we have}
\]

\[
\int_X \Delta(\mathcal{F}_i)L_2 \cdots L_{n-1} \leq \sum_j \frac{\int_X \Delta(\mathcal{F}_j) L_2 \cdots L_{n-1}}{r_j} - (s - 1)C
\]

\[
\leq \int_X \Delta(\mathcal{E})L_2 \cdots L_{n-1} + \frac{1}{r} \sum_{i < j} r_i r_j (\mu_i - \mu_j)^2 - (s - 1)C
\]

\[
\leq \frac{2r c_2 - (r - 1)c_1^2}{r} L_2 \cdots L_{n-1}
\]

\[
+ \frac{r}{d} (\mu(\mathcal{E}) - \mu_{\min}(\mathcal{E}))(\mu(\mathcal{E}) - \mu_{\min}(\mathcal{E})) - (s - 1)C.
\]

Since

\[
\mu(\mathcal{E}) - \mu_{\min}(\mathcal{E}) \leq (s - 1)(\mu_{\max}(\mathcal{E}) - \mu(\mathcal{E})) \leq (s - 1) \left( \mu_{\max} - \frac{1}{r} c_1 L_1 \cdots L_{n-1} \right).
\]

Corollary 3.11 and condition (4) imply that for all \( i \) we have \( \mu_{\max, \mathcal{F}_i}(\mathcal{F}_i|_H) \leq C_1 \) for some \( C_1 \) that depends only on \( r, c_1, c_2 \) and \( \mu_{\max} \). Condition (4) implies also that the sequences

\[
0 \to \mathcal{E}_i|_H \to \mathcal{E}_i \to \mathcal{F}_i|_H
\]

are exact, so

\[
\mu_{\max, \mathcal{F}_i}(\mathcal{E}_i|_H) \leq \max_i \mu_{\max, \mathcal{F}_i}(\mathcal{F}_i|_H) \leq C_1.
\]

For any \( \mathcal{E} \in \mathcal{A} \) the class of \( c_1(\mathcal{E}|_H) \) in \( C^1(H; L_H) \) coincides with \( c_1|_H \). This follows from the fact that \( \mathcal{E}|_H \) is torsion free, so also locally free on a big open subset of \( H \). So already the class of \( c_1(\mathcal{E}|_H) \) coincides with \( c_1(\mathcal{E})|_H \in A^1(H) \). By the induction assumption this implies that the set of sheaves \( \{ \mathcal{E}|_H \}_{\mathcal{E} \in \mathcal{A}_H} \) is bounded. To simplify notation we write \( \mathcal{O}_X(1) = L_1 \).

There exists some integers \( a, b \) and \( c \) such that for all \( \mathcal{E} \in \mathcal{A}_H \) the following conditions are satisfied:

1. \( H^i(H, \mathcal{E}|_H(m)) = 0 \) for all \( m \geq a \) and all \( i > 0 \),
2. \( H^1(H, \mathcal{E}|_H(-m)) = 0 \) for all \( m \geq b \),
3. \( \text{h}^1(H, \mathcal{E}|_H(m)) \leq c \) for all \( m \).

The second condition above follows from the well-known Enriques–Severi–Zariski lemma for reflexive sheaves on normal varieties (see [37, Lemma 0FD8]). Let us fix \( \mathcal{E} \in \mathcal{A}_H \). For all
Let us take \( i \geq 2 \). Then for all \( m \geq a \) we have \( H^i(X, \mathcal{E}(m-1)) = H^i(X, \mathcal{E}(m)) \). So by Serre’s vanishing theorem we get \( H^i(X, \mathcal{E}(m-1)) = 0 \) for \( m \geq a \). We also know that

\[
h^1(X, \mathcal{E}(m)) \leq h^1(X, \mathcal{E}(m-1))
\]

for all \( m \geq a \). Let us consider the embedding \( X \hookrightarrow \mathbb{P}^N \) given by the linear system \(|\mathcal{O}_X(1)|\) and let \( \mathcal{H} \subset \mathbb{P}^N \) be the hyperplane defining \( \mathcal{H} \). For any \( m \in \mathbb{Z} \) we have a commutative diagram

\[
\begin{array}{ccc}
H^0(\mathbb{P}^N, \mathcal{E}(m)) & \otimes & H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \\
\beta_1 \downarrow & & \downarrow \beta_2 \\
H^0(\mathcal{H}, \mathcal{E}|_\mathcal{H}(m)) & \otimes & H^0(\mathcal{H}, \mathcal{O}_{\mathcal{H}}(1)) \\
& \alpha_2 & \rightarrow \\
& H^0(\mathcal{H}, \mathcal{E}|_\mathcal{H}(m+1)). & \\
\end{array}
\]

Assume that \( h^1(X, \mathcal{E}(m)) = h^1(X, \mathcal{E}(m-1)) \) for some \( m \geq a + n - 1 \). Then the map \( \beta_1 \) in the above diagram is surjective. Since \( m \geq a + n - 1 \), we also know that \( H^i(\mathcal{H}, \mathcal{E}|_\mathcal{H}(m-i)) = 0 \) for \( i > 0 \) so by the Castelnuovo–Mumford theorem the map \( \alpha_2 \) is also surjective. It follows that \( \beta_2 \) is surjective. This implies that \( h^1(X, \mathcal{E}(m+1)) = h^1(X, \mathcal{E}(m)) \). So by Serre’s vanishing theorem we see that \( h^1(X, \mathcal{E}(m)) = 0 \). This shows that for \( m \geq a + n - 2 \) the sequence \( \{h^1(X, \mathcal{E}(m))\} \) is strictly decreasing until it reaches 0. It follows that \( h^1(X, \mathcal{E}(l)) = 0 \) for \( l \geq h^1(X, \mathcal{E}(a + n - 2)) + a + n - 2 \). Since \( \mathcal{E} \) is reflexive and \( X \) is normal, we know that \( h^1(X, \mathcal{E}(-l)) = 0 \) for \( l \gg 0 \) (here we again use [37, Lemma 0FD8]). So for all \( m \in \mathbb{Z} \) we have

\[
h^1(X, \mathcal{E}(m)) \leq h^1(X, \mathcal{E}(m-1)) + h^1(H, \mathcal{E}|_H(m))
\]

\[
\leq h^1(X, \mathcal{E}(m-2)) + h^1(H, \mathcal{E}|_H(m)) + h^1(H, \mathcal{E}|_H(m-1))
\]

\[
\leq \cdots \leq \sum_{l \geq 0} h^1(H, \mathcal{E}|_H(m-l)) \leq (m + b)c.
\]

In particular, we have \( h^1(X, \mathcal{E}(a + n - 2)) \leq (a + n - 2 + b)c \). Therefore \( h^1(X, \mathcal{E}(m)) = 0 \) for \( m \geq (a + n - 2)(c + 1) + bc \). This shows that there exists a constant \( m_0 \) such that all \( \mathcal{E} \in \mathcal{A}_H \) are \( m \)-regular for all \( m \geq m_0 \) and hence \( \mathcal{A}_H \) is a bounded family (see [12, Lemma 1.7.6]). Since the family of divisors \( H \in |L_1| \) is bounded, this also gives boundedness of the family \( \mathcal{A} \).

**Remark 3.15.** If \( L_1 = \cdots = L_{n-1} \), then the above theorem follows from [21, Theorem 4.4] (see [26, Theorem 6.4]). In general, the problem is that we need restriction theorems for multipolarizations on normal varieties and the method of proof of [21, Theorem 4.4] for singular varieties depends on the projection method that works only if we have one polarization. In case of characteristic zero, restriction theorems needed for multipolarizations follow easily from the results of [21] by passing to the resolution of singularities and using Bertini’s theorem. Unfortunately, this method also does not work for varieties defined over a field of positive characteristic. However, even in characteristic zero Theorem 3.14 is new.

As in [26, Corollary 6.7], Corollary 3.8 and the above theorem imply the following result.
**Corollary 3.16.** Let us fix some positive integer \( r \), integer \( \text{ch}_1 \) and some real numbers \( \text{ch}_2 \) and \( \mu_{\text{max}} \). Let \( \mathcal{B} \) be the set of reflexive coherent \( \mathcal{O}_X \)-modules \( \mathcal{E} \) such that

1. \( \mathcal{E} \) has rank \( r \),
2. \( \int_X \text{ch}_1(\mathcal{E})L_1 \ldots L_{n-1} = \text{ch}_1 \),
3. \( \int_X \text{ch}_2(\mathcal{E})L_1 \ldots \widehat{L}_i \ldots L_{n-1} \geq \text{ch}_2 \) for \( i = 1, \ldots, n-1 \),
4. \( \mu_{\text{max}, L}(\mathcal{E}) \leq \mu_{\text{max}} \).

Then the set \( \mathcal{B} \) is bounded.

**Proof.** By Corollary 3.8 there exists a constant \( C \) such that for all \( \mathcal{E} \in \mathcal{B} \) we have

\[
C \leq \int_X \Delta(\mathcal{E})L_1 \ldots \widehat{L}_i \ldots L_{n-1} = \text{ch}_1(\mathcal{E})^2L_1 \ldots \widehat{L}_i \ldots L_{n-1} - 2r \int_X \text{ch}_2(\mathcal{E})L_1 \ldots \widehat{L}_i \ldots L_{n-1}
\]

for \( i = 1, \ldots, n-1 \). Therefore

\[
c_1(\mathcal{E})^2L_1 \ldots \widehat{L}_i \ldots L_{n-1} \geq C + 2r\text{ch}_2.
\]

Let us write

\[
[c_1(\mathcal{E})] = \alpha_i[L_i] + D_i \in N_i L(X),
\]

where \( \alpha_i = \frac{\text{ch}_1}{L_1 \ldots \widehat{L}_i \ldots L_{n-1}} \). Then \( D_iL_1 \ldots L_{n-1} = 0 \) and

\[
c_1(\mathcal{E})^2L_1 \ldots \widehat{L}_i \ldots L_{n-1} = \alpha_i^2 + D_i^2L_1 \ldots \widehat{L}_i \ldots L_{n-1}.
\]

Therefore

\[
D_i^2L_1 \ldots \widehat{L}_i \ldots L_{n-1} \geq C + 2r\text{ch}_2 - \alpha_i^2L_1 \ldots \widehat{L}_i \ldots L_{n-1}.
\]

But by the Hodge index theorem (see Lemma 1.19) the intersection form is negative definite on \( L_i^{-1} \subset N_i L(X) \), so there are only finitely many possibilities for \( D_i \in N_i L(X) \). Since the canonical map

\[
C^1(X; L) \rightarrow \bigoplus_{i=1}^{n-1} N_i L(X)
\]

is injective, there are also only finitely many possibilities for the classes \([c_1(\mathcal{E})] \in C^1(X; L)\). Now the assertion follows from Theorem 3.14.

The above corollary has some non-trivial implications even in the rank one case:

**Corollary 3.17.** The canonical map \( \tilde{B}^1(X) \rightarrow C^1(X; L) \) is an isomorphism.

**Proof.** Let \( D \) denote a Weil divisor representing the class in the kernel of the map \( B^1(X) \rightarrow C^1(X; L) \). Corollary 3.16 implies that the set \( \{\mathcal{O}_X(mD)\}_{m \in \mathbb{Z}} \) is bounded. So the set \( \{[mD]\}_{m \in \mathbb{Z}} \) of the corresponding classes in \( B^1(X) \) is finite. Therefore \([D] = 0 \in \tilde{B}^1(X)\) by the pigeonhole principle.
Remark 3.18. The above corollary implies that some multiple of a Weil divisor \( D \) on \( X \) is algebraically equivalent to 0 if and only if \( D \cdot L^{n-1} = D^2 \cdot L^{n-2} = 0 \) for some ample line bundle \( L \). This allows to give generalization of [15, Theorem 9.6.3] to rank 1 reflexive sheaves on normal projective varieties.

4. Modules over Lie algebroids and Higgs sheaves

In this section we recall some definitions and prove some simple results on modules over Lie algebroids and generalized Higgs sheaves. This is an already existing and rather classical theory. We simply adapt it to our set-up essentially following [23, Sections 2 and 3] to which we refer for the precise definitions and further references (see also the preprint version of the paper). For simplicity, in the whole section we assume that \( X \) is a scheme of finite type over some fixed base field \( k \) and we deal only with coherent sheaves.

4.1. Basic definitions. We will identify sections of \( T_{X/k} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) \) with \( k \)-derivations of \( \mathcal{O}_X \) without mentioning it. A Lie algebroid on \( X/k \) is a triple \((\mathcal{L}, [\cdot, \cdot]_\mathcal{L}, \alpha)\) consisting of a quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{L} \), a Lie bracket \([\cdot, \cdot]_\mathcal{L}: \mathcal{L} \otimes_k \mathcal{L} \to \mathcal{L}\) and an anchor map \( \alpha: \mathcal{L} \to T_{X/k} \) that is compatible with the Lie algebra structure. In the following we often abuse notation and say that \( \mathcal{L} \) is a Lie algebroid, meaning that it is equipped with the Lie algebroid structure.

A standard example of a Lie algebroid is provided by the tangent sheaf \( T_{X/k} \) with the usual Lie bracket and identity anchor map.

An \( \mathcal{L} \)-module is a pair \((E, \nabla)\) consisting of a coherent \( \mathcal{O}_X \)-module \( E \) and an \( \mathcal{O}_X \)-linear map of left \( \mathcal{O}_X \)-modules \( \nabla: \mathcal{L} \to \mathcal{End}_k E \), which is also a map of sheaves of \( k \)-Lie algebras and satisfies modified Leibniz’s rule using the anchor map. In the case \( \mathcal{L} = T_{X/k} \) with the canonical Lie algebroid structure, an \( \mathcal{L} \)-module is a coherent \( \mathcal{O}_X \)-module with an integrable \( k \)-connection.

A morphism between \( \mathcal{L} \)-modules \((E_1, \nabla_1)\) and \((E_2, \nabla_2)\) is an \( \mathcal{O}_X \)-linear map \( \varphi: E_1 \to E_2 \) commuting with the operators \( \nabla_1 \) and \( \nabla_2 \). We can also define the tensor product of \( \mathcal{L} \)-modules \((E_1, \nabla_1)\) and \((E_2, \nabla_2)\) by giving the \( \mathcal{L} \)-module structure \( \nabla \) on \( E_1 \otimes_{\mathcal{O}_X} E_2 \) by the formula

\[
\nabla(x) = \nabla_1(x) \otimes \text{Id} + \text{Id} \otimes \nabla_2(x).
\]

Similarly, we have the canonical \( \mathcal{L} \)-module structure \( \nabla’ \) on \( \mathcal{H}om_{\mathcal{O}_X}(E_1, E_2) \). It is given by

\[
(\nabla'(x))(\psi) = (\nabla_2(x)) \circ \psi - \psi \circ (\nabla_1(x)).
\]

One can check that the corresponding category \( \mathcal{L} \)-Mod(\( X \)) of \( \mathcal{L} \)-modules is a symmetric monoidal abelian category. The monoidal structure is given by the above tensor product \( \otimes: \mathcal{L} \)-Mod(\( X \)) \times \mathcal{L} \)-Mod(\( X \)) \to \mathcal{L} \)-Mod(\( X \)) and the unit object is given by the \( \mathcal{L} \)-module structure on \( \mathcal{O}_X \) given by the anchor map.

Note that the unit object allows to define the canonical \( \mathcal{L} \)-module structure on the dual \( E^* = \mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X) \) of an \( \mathcal{L} \)-module. Hence we can also consider the reflexive hull of an
\( \mathcal{L} \)-module. As in the case of \( \mathcal{O}_X \)-modules (see Section 1.1) the inclusion functor

\[ \mathcal{L} \text{-Mod}^{\text{ref}}(X) \to \mathcal{L} \text{-Mod}(X) \]

comes with the left adjoint functor \( (\cdot)^{**} : \mathcal{L} \text{-Mod}(X) \to \mathcal{L} \text{-Mod}^{\text{ref}}(X) \) given by the reflexive hull. In particular, we have a natural map \( \mathcal{E} \to \mathcal{E}^{**} \) of \( \mathcal{L} \)-modules coming from the adjoint map to the identity on \( \mathcal{E}^{**} \).

### 4.2. Extensions of modules over Lie algebroids

Assume the underlying \( \mathcal{O}_X \)-module of a Lie algebroid \( \mathcal{L} \) on \( X/k \) is coherent and reflexive. By abuse we call such Lie algebroid reflexive.

Let \( j : U \hookrightarrow X \) be a big open subscheme of \( X \). Let \( (\mathcal{E}, \nabla : \mathcal{L}_U \to \mathcal{E} \text{nd}_k \mathcal{E}) \) be an \( \mathcal{L}_U \)-module. By assumption \( \mathcal{L} = j_* \mathcal{L}_U \), so we can set \( j_*(\mathcal{E}, \nabla) := (j_*\mathcal{E}, j_*\nabla) \), where \( j_* \nabla \) acts as \( \nabla \) on the sections of \( j_*\mathcal{E} \) (which are the same as sections of \( \mathcal{E} \)). In this way we can define the functor \( j_* : \mathcal{L}_U \text{-Mod}(U) \to \mathcal{L} \text{-Mod}(X) \).

An \( \mathcal{L} \)-module \( (\mathcal{E}, \nabla) \) is called reflexive if \( \mathcal{E} \) is a reflexive \( \mathcal{O}_X \)-module. By \( \mathcal{L} \text{-Mod}^{\text{ref}}(X) \) we denote the full subcategory of \( \mathcal{L} \text{-Mod}(X) \), whose objects are reflexive \( \mathcal{L} \)-modules. After restricting to reflexive modules, \( j_* \) and \( j^* \) give adjoint equivalences of categories \( \mathcal{L}_U \text{-Mod}^{\text{ref}}(U) \) and \( \mathcal{L} \text{-Mod}^{\text{ref}}(X) \).

### 4.3. Generalized Higgs sheaves

Let \( \mathcal{L} \) be a quasi-coherent \( \mathcal{O}_X \)-module.

**Definition 4.1.** An \( \mathcal{L} \)-Higgs sheaf \( (\mathcal{E}, \theta : \mathcal{L} \to \mathcal{E} \text{nd}_X \mathcal{E}) \) is an \( \mathcal{L} \)-module for the trivial Lie algebroid structure on \( \mathcal{L} \) given by the zero Lie bracket and the zero anchor map.

The category of \( \mathcal{L} \)-Higgs sheaves on \( X \) is denoted by \( \text{HIG}_X(\mathcal{O}_X) \). By \( \text{HIG}^{\text{ref}}_\mathcal{E}(\mathcal{O}_X) \) we denote the category of reflexive \( \mathcal{L} \)-Higgs sheaves on \( X \).

On any scheme \( X \), if \( \mathcal{E}, \mathcal{F}, \mathcal{G} \) are sheaves of \( \mathcal{O}_X \)-modules, then we have a functorial isomorphism of \( \Gamma(X, \mathcal{O}_X) \)-modules

\[ \text{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G}) \cong \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})). \]

In particular, we have an isomorphism

\[ \alpha : \text{Hom}_{\mathcal{O}_X}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E}, \mathcal{E}) \cong \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{E} \text{nd}_{\mathcal{O}_X}(\mathcal{E})). \]

This shows that we can replace \( \theta \) by an \( \mathcal{O}_X \)-linear map \( \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E} \) that by abuse of notation will also be denoted by \( \theta \).

Note that we have a map \( \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L} \) given by sending \( x \otimes y \) to \( x \otimes y - y \otimes x \).

Since it maps \( x \otimes x \) to \( 0 \), this map factors through the canonical projection \( \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L} \to \bigwedge^2 \mathcal{L} \).

Hence we get the map

\[ \iota : \bigwedge^2 \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L} \]

fitting into an exact sequence

\[ \bigwedge^2 \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L} \to \text{Sym}^2 \mathcal{L} \to 0, \]

where the second map is the canonical projection.

The following lemma explains when an \( \mathcal{O}_X \)-linear map \( \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E} \) gives rise to an \( \mathcal{L} \)-Higgs sheaf.
Lemma 4.2. Let us fix an $O_X$-linear map $\theta : \mathcal{L} \otimes_{O_X} \mathcal{E} \to \mathcal{E}$. Then the following conditions are equivalent:

1. The composition
   \[ \bigwedge^2 \mathcal{L} \otimes_{O_X} \mathcal{E} \xrightarrow{\iota \otimes \text{Id}} \mathcal{L} \otimes_{O_X} \mathcal{L} \otimes_{O_X} \mathcal{E} \xrightarrow{\text{Id} \otimes \theta} \mathcal{L} \otimes_{O_X} \mathcal{E} \xrightarrow{\theta} \mathcal{E} \]
   vanishes.

2. The map $\tilde{\theta} := \alpha(\theta) : \mathcal{L} \to \text{End}_{O_X}(\mathcal{E})$ is a homomorphism of sheaves of Lie rings.

3. The map $\theta$ extends to a $\text{Sym}^\bullet \mathcal{L}$-module structure on $\mathcal{E}$, i.e., there exists an $O_X$-linear map $\widetilde{\theta} : \text{Sym}^\bullet \mathcal{L} \otimes_{O_X} \mathcal{E} \to \mathcal{E}$ such that $\tilde{\theta}|_{O_X} : O_X \otimes_{O_X} \mathcal{E} \to \mathcal{E}$ is the identity, $\tilde{\theta}|_{\mathcal{L}} = \theta$, and the diagram
   \[
   \begin{array}{ccc}
   \text{Sym}^\bullet \mathcal{L} \otimes_{O_X} \mathcal{E} & \xrightarrow{\mu \otimes \text{Id}} & \text{Sym}^\bullet \mathcal{L} \otimes_{O_X} \mathcal{E} \\
   & \searrow{\tilde{\theta}} & \downarrow{\widetilde{\theta}} \\
   \text{Sym}^\bullet \mathcal{L} \otimes_{O_X} \mathcal{E} & & \mathcal{E},
   \end{array}
   \]
   where $\mu$ is the multiplication in $\text{Sym}^\bullet \mathcal{L}$, is commutative.

Proof. Let $x$, $y$ be local sections of $\mathcal{L}$ and $e$ a local section of $\mathcal{E}$. Then the first condition means that
   \[
   \theta(x \otimes \theta(y \otimes e)) = \theta(y \otimes \theta(x \otimes e)),
   \]
   which can be rewritten as $\tilde{\theta}(x)\tilde{\theta}(y) = \tilde{\theta}(y)\tilde{\theta}(x)$, i.e., $[\tilde{\theta}(x), \tilde{\theta}(y)] = 0 = \tilde{\theta}([x, y])$. This shows equivalence of the first two conditions.

If these conditions are satisfied, there exists a homomorphism of sheaves of $O_X$-algebras $\text{Sym}^\bullet \mathcal{L} \to \text{End}_{O_X}(\mathcal{E})$ extending $\alpha(\theta)$. This follows from the definition of $\text{Sym}^\bullet \mathcal{L}$ as the quotient of the tensor algebra of $\mathcal{L}$ by the two-sided ideal generated by local sections of the form $x \otimes y - y \otimes x$.

This map provides $\mathcal{E}$ with a $\text{Sym}^\bullet \mathcal{L}$-module structure. Clearly, if we have such a structure, then also the second condition is satisfied. \qed

Interpretation of a Higgs field as an $O_X$-linear map $\theta : \mathcal{L} \otimes_{O_X} \mathcal{E} \to \mathcal{E}$ is sometimes more convenient. For example, we can use it to introduce the following definition that will play an important role in the paper.

Definition 4.3. A system of $\mathcal{L}$-Hodge sheaves is an $\mathcal{L}$-Higgs sheaf
   \[(\mathcal{E}, \theta : \mathcal{L} \otimes_{O_X} \mathcal{E} \to \mathcal{E})\]
   for which $\mathcal{E}$ splits into a direct sum $\bigoplus \mathcal{E}^i$ so that $\theta$ maps $\mathcal{L} \otimes_{O_X} \mathcal{E}^i$ to $\mathcal{E}^{i-1}$.

This interpretation is also convenient when considering morphisms of $\mathcal{L}$-Higgs sheaves. Giving a morphism of $\mathcal{L}$-Higgs sheaves $\varphi : (\mathcal{E}_1, \theta_1) \to (\mathcal{E}_2, \theta_2)$ is equivalent to giving an $O_X$-linear map $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{L} \otimes_{O_X} \mathcal{E}_1 & \xrightarrow{\theta_1} & \mathcal{E}_1 \\
\downarrow{\text{Id} \otimes \varphi} & & \downarrow{\varphi} \\
\mathcal{L} \otimes_{O_X} \mathcal{E}_2 & \xrightarrow{\theta_2} & \mathcal{E}_2.
\end{array}
\]
4.4. Reflexive Higgs sheaves. In this subsection let us assume that \( \mathcal{L} \) is a coherent \( \mathcal{O}_X \)-module. We set
\[
\Omega^{[m]}_{\mathcal{L}} = \left( \bigwedge^m \mathcal{L} \right)^*.
\]
for \( m \geq 1 \). So in particular we have \( \Omega^{[1]}_{\mathcal{L}} : = \mathcal{L}^* \). We also fix a reflexive coherent \( \mathcal{O}_X \)-module \( \mathcal{E} \).

By [37, Lemma 0AY4] the sheaf \( \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{E}) \) is also reflexive. In particular, if \( X \) is normal, then \( \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{E}) \simeq \mathcal{E} \otimes \Omega^{[1]}_{\mathcal{L}} \).

So in particular we have \( \mathcal{E} \otimes \mathcal{L} \). We also fix a reflexive coherent \( \mathcal{O}_X \)-module \( \mathcal{E} \).

By \( \text{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes \mathcal{L}, \mathcal{E}) \) we also have a canonical isomorphism
\[
\text{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes \mathcal{L}, \mathcal{E}) \simeq \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes \Omega^{[1]}_{\mathcal{L}}).
\]

**Lemma 4.4.** Assume that \( X \) is normal. If \( \theta : \mathcal{E} \otimes \mathcal{L} \to \mathcal{E} \) and \( \hat{\theta} : \mathcal{E} \to \mathcal{E} \otimes \Omega^{[1]}_{\mathcal{L}} \) are \( \mathcal{O}_X \)-linear maps corresponding to each other under the above isomorphism, then \( \mathcal{E} \) is an \( \mathcal{L} \)-Higgs sheaf if and only if the composition
\[
\mathcal{E} \overset{\hat{\theta}}{\longrightarrow} \mathcal{E} \otimes \Omega^{[1]}_{\mathcal{L}} \overset{\theta \otimes \text{Id}}{\longrightarrow} \mathcal{E} \otimes \Omega^{[2]}_{\mathcal{L}}
\]
vanishes.

**Proof.** Note that we have a canonical isomorphism
\[
\text{Hom}_{\mathcal{O}_X}(\bigwedge^2 \mathcal{E} \otimes \mathcal{L}, \mathcal{E}) \simeq \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes \Omega^{[2]}_{\mathcal{L}}).
\]
So the required assertion follows from Lemma 4.2 and the fact that the above defined composition \( \mathcal{E} \to \mathcal{E} \otimes \Omega^{[2]}_{\mathcal{L}} \) corresponds to the composition
\[
\bigwedge^2 \mathcal{E} \otimes \mathcal{L} \overset{\theta \otimes \text{Id}}{\longrightarrow} \mathcal{E} \otimes \mathcal{L} \overset{\text{Id} \otimes \theta}{\longrightarrow} \mathcal{L} \otimes \mathcal{E} \overset{\theta}{\longrightarrow} \mathcal{E}.
\]

We need to change the sign in the last map to make it compatible with de Rham sequences for modules over Lie algebroids. The above lemma and Lemma 1.1 give a different proof of the following corollary (cf. Section 4.2).

**Corollary 4.5.** Assume that \( X \) is normal and \( \mathcal{L} \) is reflexive. If \( j : U \hookrightarrow X \) is a big open subscheme \( X \), then \( j_* \) and \( j^* \) define adjoint equivalences of categories \( \text{HIG}^\text{ref}_L(\mathcal{O}_X) \) and \( \text{HIG}^\text{ref}_L(\mathcal{O}_U) \).

**Corollary 4.6.** If \( X \) is normal and \( \theta : \mathcal{E} \otimes \mathcal{L} \to \mathcal{E} \) is an \( \mathcal{L} \)-Higgs sheaf on \( \mathcal{E} \), then we have a canonical structure of an \( \mathcal{L}^* \*- \text{Higgs sheaf} \) on \( \mathcal{E} \).

**Proof.** The assertion follows immediately from the previous lemma and the remark that \( \Omega^{[1]}_{\mathcal{L}^*} = \Omega^{[1]}_{\mathcal{L}} \) and \( \Omega^{[2]}_{\mathcal{L}^*} = \Omega^{[2]}_{\mathcal{L}} \).

4.5. Pullback of generalized Higgs sheaves. If \( f : X \to Y \) is a morphism of schemes of finite type over \( k \) and \( \mathcal{L} \) is a quasi-coherent \( \mathcal{O}_Y \)-module, then we can easily define the pullback of \( \mathcal{L} \)-Higgs sheaves. Namely, if \( (\mathcal{E}, \theta : \mathcal{E} \otimes \mathcal{L} \to \mathcal{E}) \) is an \( \mathcal{L} \)-Higgs sheaf then it
is easy to see that

$$(f^*\mathcal{E}, f^*\theta : f^*\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{E} = f^*(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{E}) \to f^*\mathcal{E})$$

is an $f^*\mathcal{L}$-Higgs sheaf (for example one can check condition (1) from Lemma 4.2). This defines the pullback functor on the corresponding categories of generalized Higgs sheaves, which is functorial with respect to morphisms between schemes.

4.6. Reflexive pullback for Higgs sheaves under finite morphisms. Let $f : X \to Y$ be a finite dominant morphism of integral normal schemes of finite type over $k$.

4.6.1. Pullback in the separable case. Lemma 1.2 implies that

$$f^{[*]}\mathcal{T}_{Y/k} = (f^*\mathcal{O}_{Y/k})^*.$$ 

So we have a canonical map $T_{X/k} \to f^{[*]}\mathcal{T}_{Y/k}$ dual to $df : f^*\mathcal{O}_{Y/k} \to \mathcal{O}_{X/k}$. Since this map is non-interesting for purely inseparable morphisms, from now on we assume that the induced field extension $k(Y) \to k(X)$ is separable. In this case the map $T_{X/k} \to f^{[*]}\mathcal{T}_{Y/k}$ is injective and it uniquely extends to a homomorphism of sheaves of $\mathcal{O}_X$-algebras

$$\text{Sym}^* T_{X/k} \to \text{Sym}^* f^{[*]}\mathcal{T}_{Y/k}.$$ 

If $(\mathcal{E}, \theta)$ be a Higgs sheaf on $Y$, then $(f^*\mathcal{E}, f^*\theta)$ is an $f^*\mathcal{T}_{X/k}$-Higgs sheaf. Assume that $\mathcal{E}$ is reflexive. Taking reflexivization we get an $f^*\mathcal{T}_{X/k}$-Higgs module structure on $f^{[*]}\mathcal{E}$. By Corollary 4.6 we also have an induced $f^{[*]}\mathcal{T}_{X/k}$-Higgs module structure on $f^{[*]}\mathcal{E}$. Then the homomorphism $\text{Sym}^* T_{X/k} \to \text{Sym}^* f^{[*]}\mathcal{T}_{Y/k}$ provides $f^{[*]}\mathcal{E}$ with a canonical Higgs module structure. This Higgs module will be denoted by $f^{[*]}(\mathcal{E}, \theta) = (f^{[*]}\mathcal{E}, f^{[*]}\theta)$.

One can also describe the above construction explicitly in the following way (this will be useful in the next construction). Namely, let $(\mathcal{E}, \theta : \mathcal{E} \to \mathcal{E} \otimes \mathcal{O}_Y^{[1]})$ be a reflexive Higgs sheaf on $Y$ (see Lemma 4.4). Then there exists a big open subset $V \subset Y$ such that $(\mathcal{E}, \theta)$ is (log) smooth on $V$. Since $f$ is finite, $U = f^{-1}(V)$ is a big open subset of $X$. Let $j : U \to X$ be the corresponding open embedding. Then we can define the map

$$\begin{align*}
(f^*\mathcal{E})_U &= f^*(\mathcal{E}_V) \\
\xrightarrow{f^*\theta} &= f^*(\mathcal{E}_V \otimes_{\mathcal{O}_V} \mathcal{O}_{V/k}) = f^*(\mathcal{E}_V) \otimes_{\mathcal{O}_U} f^*\mathcal{O}_{V/k} \\
\xrightarrow{\text{Id} \otimes df} &= (f^*\mathcal{E})_U \otimes_{\mathcal{O}_U} \mathcal{O}_{U/k}.
\end{align*}$$

This gives the map $T_{U/k} \otimes (f^*\mathcal{E})_U \to (f^*\mathcal{E})_U$, which leads to

$$\begin{align*}
T_{X/k} \otimes_{\mathcal{O}_X} f^{[*]}\mathcal{E} &= j_*((T_{U/k}) \otimes_{\mathcal{O}_X j^*((f^*\mathcal{E})_U))} \\
&\to j_*((f^*\mathcal{E})_U) \\
&\to j_*((f^*\mathcal{E})_U) = f^{[*]}\mathcal{E}.
\end{align*}$$

So we get the induced map $f^{[*]}\theta : f^{[*]}\mathcal{E} \to f^{[*]}\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_{X/k}^{[1]}$. One can easily check that $(f^{[*]}\mathcal{E}, f^{[*]}\theta)$ is a reflexive Higgs sheaf on $X$.

4.6.2. Pullback in the inseparable case. Unfortunately, the above construction is rather useless in case $f$ is purely inseparable as then $df = 0$ and $f^{[*]}\theta$ always vanishes. However, if $f = F_X$ and the big open subset $U$ is $F$-liftable (see Definition 1.3), then we have
an induced map $\xi : F_U^* \Omega \rightarrow \Omega$. Now if in the above construction we replace $df$ by $\xi$, then for any reflexive Higgs sheaf $(E, \theta)$ on $X$ we can define

$$F_X^{[*]} \theta : F_X^{[*]} E \rightarrow F_X^{[*]} E \otimes_{\mathcal{O}_X} \Omega^{[1]}_X.$$ 

This construction is used in the proof of Lemma 5.10. Note that this map depends on the choice of the lifting.

Similar constructions as above work also for log pairs.

### 4.7. Semistability for $L$-modules

Let $X$ be a normal projective variety of dimension $n$ defined over an algebraically closed field $k$. Let $L = (L_1, \ldots, L_{n-1})$ be a collection of nef line bundles on $X$ and let $\mathcal{L}$ be a Lie algebroid on $X/k$ with coherent underlying $\mathcal{O}_X$-module.

**Definition 4.7.** Let $(\mathcal{E}, \nabla)$ be an $\mathcal{L}$-module such that $\mathcal{E}$ is a torsion free $\mathcal{O}_X$-module. We say that $(\mathcal{E}, \nabla)$ is **slope $L$-semistable** if for any $\mathcal{L}$-submodule $(\mathcal{F}, \nabla_\mathcal{F}) \subset (\mathcal{E}, \nabla)$ we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$. We say that $(\mathcal{E}, \nabla)$ is **slope $L$-stable** if for any $\mathcal{L}$-submodule $(\mathcal{F}, \nabla_\mathcal{F}) \subset (\mathcal{E}, \nabla)$ such that $1 \leq \text{rk} \mathcal{F} < \text{rk} \mathcal{E}$ we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$.

In further part of this subsection we consider slope semistability with respect to the fixed $L$ that we omit in the notation.

If $(\mathcal{E}, \theta : \mathcal{L} \otimes \mathcal{O}_X \mathcal{E} \rightarrow \mathcal{E})$ is an $\mathcal{L}$-Higgs sheaf, then it is slope semistable if and only if for every $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{E}$ such that the image of $L \otimes \mathcal{O}_X \mathcal{F} \rightarrow L \otimes \mathcal{O}_X \mathcal{E} \rightarrow \mathcal{E}$ is contained in $\mathcal{F}$, we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ (and similarly for slope stability). This should be compared with [10, Definition 4.13] that considers semistability using so called generically $\theta$-invariant subsheaves. It is easy to see that for reflexive Higgs sheaves in the sense of [10] the obtained notions of semistability coincide.

Let us also remark that if $(\mathcal{E}, \theta)$ is a system of $L$-Hodge sheaves, then we can define notion of semistability using only subsystems of $L$-Hodge sheaves. It is easy to see that this is equivalent to semistability of $(\mathcal{E}, \theta)$ as an $L$-Higgs sheaf. We will use this fact in Section 6.

We have the following general lemma allowing to bound instability of slope semistable $L$-modules. It is a weak form of [24, Lemma 5] but it works also for nef polarizations.

**Lemma 4.8.** Let $A$ be an ample Cartier divisor such that $\mathcal{L}(A)$ is globally generated. Let $(\mathcal{E}, \nabla)$ be an $\mathcal{L}$-module such that $\mathcal{E}$ is torsion free. If $(\mathcal{E}, \nabla)$ is slope semistable then

$$\mu_{\text{max}}(\mathcal{E}) - \mu_{\text{min}}(\mathcal{E}) \leq (r - 1)AL_1 \ldots L_{n-1},$$

where $r$ is the rank of $\mathcal{E}$.

**Proof.** If $\mathcal{F} \subset \mathcal{E}$ is an $\mathcal{O}_X$-submodule, then the $\mathcal{L}$-module structure on $(\mathcal{E}, \nabla)$ induces the $\mathcal{O}_X$-linear map $\mathcal{L} \otimes \mathcal{O}_X \mathcal{F} \rightarrow \mathcal{E}/\mathcal{F}$. If this map vanishes, then $\mathcal{F}$ has a natural structure of an $\mathcal{L}$-submodule of $(\mathcal{E}, \nabla)$.

Let $\mathcal{E}_0 = 0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = \mathcal{E}$ be the Harder–Narasimhan filtration of $\mathcal{E}$ and let us set $\mathcal{E}_i^j := \mathcal{E}_i/\mathcal{E}_{i-1}$ for $i = 1, \ldots, s$. Then we have non-zero $\mathcal{O}_X$-linear maps

$$\mathcal{L} \otimes \mathcal{O}_X \mathcal{E}_i \rightarrow \mathcal{E}/\mathcal{E}_i$$

for $i = 1, \ldots, s - 1$. Since for some $N > 0$ we have a surjective map $\mathcal{O}_X^{\oplus N}(-A) \rightarrow \mathcal{L}$, there
exists for every $i = 1, \ldots, s$ a non-zero map $\mathcal{E}_i(-A) \to \mathcal{E}/\mathcal{E}_i$. So we have

$$\mu(\mathcal{E}^i) - AL_1 \cdots L_{n-1} \leq \mu(\mathcal{E}^{i+1}).$$

Summing these inequalities we get

$$\mu_{\text{max}}(\mathcal{E}) - \mu_{\text{min}}(\mathcal{E}) \leq (s-1)AL_1 \cdots L_{n-1} \leq (r-1)AL_1 \cdots L_{n-1}. \quad \square$$

4.8. Strong restriction theorem for generalized Higgs sheaves. We keep the notation from the previous subsection but we assume that $\mathcal{L}$ has trivial Lie algebroid structure. The same proofs as that of Theorem 3.9 and Corollary 3.11 give the following theorem (cf. [24, Theorem 9] in the smooth case). See Section 4.5 for the definition of pullback used in the statement.

**Theorem 4.9.** Let $(\mathcal{E}, \theta)$ be a reflexive $\mathcal{L}$-Higgs sheaf of rank $r \geq 2$. Let us assume that $d = L_1^2L_2 \cdots L_{n-1} > 0$ and let $m$ be an integer such that

$$m > \left[ \frac{r - 1}{r} \int_X \Delta(\mathcal{E})L_2 \cdots L_{n-1} + \frac{1}{dr(r-1)} + \frac{(r-1)\beta}{dr} \right].$$

Let $H \in |L_1^m|$ be an irreducible normal divisor and let $i : H \hookrightarrow X$ denote the corresponding embedding.

1. If $(\mathcal{E}, \theta)$ is slope $L$-stable, the $i^*\mathcal{L}$-Higgs sheaf $(i^*\mathcal{E}, i^*\theta)$ is slope $L_H$-stable.

2. If $(\mathcal{E}, \theta)$ is slope $L$-semistable and restrictions of all quotients of a Jordan–Hölder filtration of $(\mathcal{E}, \theta)$ to $H$ are torsion free, the $i^*\mathcal{L}$-Higgs sheaf $(i^*\mathcal{E}, i^*\theta)$ is slope $L_H$-semi-stable.

Note that the above theorem should be thought of as a restriction theorem for sheaves with operators and not a genuine restriction theorem for Higgs sheaves (cf. Theorem 5.4).

5. Bogomolov’s inequality for logarithmic Higgs sheaves on singular varieties

This section contains proofs of Theorems 0.3 and 0.4. The main idea is to use the Ogus–Vologodsky correspondence and suitably generalized Higgs–de Rham sequences.

5.1. The Ogus–Vologodsky correspondence on normal varieties. Let $X$ be a normal variety defined over an algebraically closed field $k$ of positive characteristic $p$. Let $D$ be a reduced effective Weil divisor on $X$.

Consider a (reflexive) Lie algebroid $\mathcal{L}$, whose underlying $\mathcal{O}_X$-module is $T_{X/k}(\log D)$ with the anchor map $T_{X/k}(\log D) \hookrightarrow T_{X/k}$ and the Lie bracket induced from $T_{X/k}$. An $\mathcal{L}$-module $(\mathcal{E}, \nabla)$ is called an $\mathcal{O}_X$-module with an integrable logarithmic connection on $(X, D)$. Note that $\mathcal{L}$ carries a restricted Lie algebroid structure (see [23, Section 4]) given by raising logarithmic derivations to the $p$-th power. For any $(\mathcal{E}, \nabla)$ this allows us to define its logarithmic $p$-curvature $F^*_p T_{X/k}(\log D) \to \mathfrak{End}_{\mathcal{O}_X} \mathcal{E}$.

If $(\mathcal{E}, \nabla : T_{X/k}(\log D) \to \mathfrak{End}_k \mathcal{E})$ is a reflexive $\mathcal{O}_X$-module with an integrable logarithmic connection, then we can also define its residues in the following way. For every open subset $U \subset X$ an element $\delta \in (T_{X/k}(\log D))(U)$ can be considered as a logarithmic $k$-derivation
of $\mathcal{O}_U$. Let $J$ be the ideal subsheaf of $\mathcal{O}_U$ generated by the image of $\delta$. Then

$$\mathcal{E}|_U \xrightarrow{\nabla(\delta)} \mathcal{E}|_U \rightarrow (\mathcal{E}|_U)/(J\mathcal{E}|_U)$$

induces an endomorphism $\rho_\delta$ of $(\mathcal{E}|_U)/(J\mathcal{E}|_U)$ called the residue associated to $\delta$. We say that the residues of $(\mathcal{E}, \nabla)$ are nilpotent of order $\leq p$ if for every $U \subset X$ and $\delta \in (T_{X/k}(\log D))(U)$ we have $\rho_\delta^p = 0$.

The category of reflexive $\mathcal{O}_X$-modules with an integrable logarithmic connection on $(X, D)$ with logarithmic $p$-curvature nilpotent of level $\leq (p - 1)$ and with nilpotent residues of order $\leq p$ is denoted by $\text{MIC}^\text{ref}_{\leq (p-1)}(X, D)$.

Similarly, one can consider $T_{X/k}(\log D)$ with the trivial Lie bracket and zero anchor map. Modules over this reflexive Lie algebroid are called logarithmic Higgs sheaves on $(X, D)$. We say that a logarithmic Higgs sheaf $(\mathcal{E}, \theta : T_{X/k}(\log D) \rightarrow \mathcal{E}_{\text{nd}})$ has a nilpotent Higgs field of level $\leq (p - 1)$ if for every open subset $U \subset X$ and every $\delta \in (T_{X/k}(\log D))(U)$ we have $\theta(\delta)^p = 0$, where $\theta(\delta)$ is considered as an $\mathcal{O}_U$-linear endomorphism of $\mathcal{E}|_U$.

The category of reflexive logarithmic Higgs $\mathcal{O}_X$-modules on $(X, D)$ with a nilpotent Higgs field of level $\leq (p - 1)$ is denoted by $\text{HIG}^\text{ref}_{\leq (p-1)}(X, D)$.

The following theorem generalizes the Ogus–Vologodsky correspondence to normal varieties:

**Theorem 5.1.** Let us assume that there exists a big open subset $U \subset X$ such that the pair $(U, D_U = D \cap U)$ is log smooth and liftable to $\mathcal{W}_2(k)$. Let us fix a lifting $(\bar{U}, \bar{D}_U)$ of $(U, D_U)$. Then there exists an equivalence of categories

$$C_{(\bar{U}, \bar{D}_U)} : \text{MIC}^\text{ref}_{\leq (p-1)}(X, D) \rightarrow \text{HIG}^\text{ref}_{\leq (p-1)}(X, D),$$

called a reflexivized Cartier transform.

**Proof.** As remarked in Section 4.2, for any reflexive Lie algebroid $L$ and any big open subset $U \subset X$, we have an equivalence of categories of reflexive $L$-modules on $X$ and reflexive $L_U$-modules on $U$. So the results of Ogus and Vologodsky in the usual case (see [33]) and Schepler in the logarithmic one (see [34]; see also [24, Theorem 2.5] and [17, Appendix]) give the above correspondence on $U$. One needs only to check that extension to $X$ preserves the remaining conditions. For Higgs modules it is clear that having a nilpotent Higgs field of level $\leq (p-1)$ on $U$ gives the same condition on $X$. Similarly, for modules with logarithmic connections checking nilpotency of the logarithmic $p$-curvature on $U$ implies the one on $X$. □

A quasi-inverse to $C_{(\bar{U}, \bar{D}_U)}$ is denoted by $C^{[-1]}_{(\bar{U}, \bar{D}_U)}$ (or simply $C^{[-1]}$) and it is called the reflexivized inverse Cartier transform.

### 5.2. Strong restriction theorem for logarithmic Higgs sheaves

We keep the notation from the previous subsection.

**Definition 5.2.** Let $j : H \hookrightarrow X$ be an effective Cartier divisor, i.e., a closed subscheme of $X$ with the invertible ideal sheaf. We say that $H$ is good for the pair $(X, D)$ if the following conditions are satisfied:

1. $H$ is irreducible and normal,
2. $H$ is not contained in any irreducible component of $D$, 

(3) if $U \subset X$ is the maximal open subset on which $(U, D \cap U)$ is log smooth, then $H \cap U$ is globally generated. Assume also that $H$ and $j$ are good for $(X, D)$.

(4) the pair $(H \cap U, D_H \cap U)$, where $D_H = H \cap D$, is log smooth.

If $H$ is good for $(X, D)$, then we have a canonical map

$$T_H/J_k(\log D_H) \rightarrow (j^*T_X/J_k(\log D))**,$$

obtained by extension of the canonical map $T_H/J_k(\log D_H \cap U) \rightarrow j^*T_X/J_k(\log D)|_{H \cap U}$. In particular, if $H$ is good for $(X, D)$, then Corollary 4.6 shows that any logarithmic Higgs sheaf $(E, \theta)$ on $(X, D)$ gives rise to a reflexive logarithmic Higgs sheaf structure

$$j^*[E, \theta] := (j^*[E], T_{H/J_k}(\log D_H) \otimes \mathcal{O}_H j^*[E] \rightarrow (j^*T_X/J_k(\log D))** \otimes \mathcal{O}_H j^*[E] \rightarrow j^*[E])$$

on $j^*[E] := (j^*[E])**$ over $(H, D_H)$.

**Remark 5.3.** If $A$ is a very ample line bundle, then Bertini’s theorem implies that for all $m \geq 1$ a general hypersurface $H \in |A^{\otimes m}|$ is good for $(X, D)$.

**Theorem 5.4.** Let $(E, \theta)$ be a reflexive logarithmic Higgs sheaf of rank $r \geq 2$ on $(X, D)$. Assume that $L_1$ is ample and let $m_0$ be a non-negative integer such that $T_{X/J_k}(\log D) \otimes L_1^{\otimes m_0}$ is globally generated. Assume also that $d = L_1^2 + \ldots + L_{n-1}^2 > 0$ and let $m$ be an integer such that

$$m > \max \left( \frac{r-1}{r} \int_X \Delta(E)L_2 \ldots L_{n-1} + \frac{1}{\delta(r-1)} + \frac{(r-1)\beta r}{dr} \right) 2(r-1)m_0^2 \right).$$

Let $H \in |L_1^{\otimes m}|$ be good for $(X, D)$ with closed embedding $j : H \hookrightarrow X$.

1. If $(E, \theta)$ is slope $L$-stable, then $j^*[E, \theta]$ on $(H, D_H)$ is slope $L_H$-stable.

2. If $(E, \theta)$ is slope $L$-semistable and restrictions of all quotients of a Jordan–Hölder filtration of $(E, \theta)$ to $H$ are torsion free, then $j^*[E, \theta]$ is slope $L_H$-semistable.

**Proof.** Using Theorem 4.9, one can follow the proof of [24, Theorem 10] to obtain the first part of the theorem.

Now let us remark that if $0 = (E_0, \theta_0) \subset (E_1, \theta_1) \subset \cdots \subset (E_s, \theta_s) = (E, \theta)$ is a Jordan–Hölder filtration of $E$ and $(F_i, \theta_i) := ((E_i, \theta_i)/(E_{i-1}, \theta_{i-1}))**$, then by Lemma 2.2 we have

$$\int_X \Delta(E)L_2 \ldots L_{n-1} \geq \sum_{i} \int_X \Delta(F_i)L_2 \ldots L_{n-1} \frac{m}{r_i},$$

where $r_i = rk F_i$. So by the first part all $j^*[F_i, \theta_i]$ are slope $L_H$-semistable. Since $H$ is good for $(X, D)$, there exists a big open subset $U \subset X$ such that $(U, D \cap U)$ is log smooth and $H \cap U$ is big in $H$. Now a logarithmic Higgs subsheaf destabilizing $j^*[E, \theta]$ would destabilize it on $H \cap U$. This would show that one of the restrictions $j^*[E_i, \theta_i]|_{H \cap U}$ is not slope $L_H$-semistable. But this contradicts the fact that the reflexivization of its extension to $H$ (which is equal to $j^*[F_i, \theta_i]$) is slope $L_H$-stable. \qed
For $r = 2$ the assumptions of this theorem can be slightly relaxed (cf. [24, Theorem 10]). Note that unlike in [24] we do not have any assumptions on lifting on $X$. These assumptions were added in [24] only to avoid the term containing $\beta_r$ so that the results could hold uniformly in all characteristics (including 0). The above result is restricted to the positive characteristic and it was not known in the characteristic zero case even if one assumes that $D = 0$ and $X$ has klt singularities (cf. [10, Theorem 5.22] for a non-effective restriction theorem for general hypersurfaces). The above theorem will be used to obtain a strong restriction theorem for Higgs sheaves in characteristic zero in Section 7.

### 5.3. Deformations to systems of Hodge sheaves

Let $X$ be a normal projective variety defined over an algebraically closed field $k$ and let $\mathcal{L}$ be a Lie algebroid on $X$, which is coherent as an $\mathcal{O}_X$-module.

It is convenient to consider $\mathcal{L}$-modules as modules over the universal enveloping algebra $\Lambda_{\mathcal{L}}$ of differential operators associated to $\mathcal{L}$ (see [23, Section 2.2]). So we consider an $\mathcal{L}$-module as a pair $(E, \nabla)$, where $E$ is a coherent $\mathcal{O}_X$-module and $\nabla : \Lambda_{\mathcal{L}} \otimes_{\mathcal{O}_X} E \to E$ is a $\Lambda_{\mathcal{L}}$-module structure. If the underlying sheaf of an $\mathcal{L}$-module is torsion free as an $\mathcal{O}_X$-module, we say that $(E, \nabla)$ is a torsion free $\mathcal{L}$-module.

If $(E, \nabla)$ is an $\mathcal{L}$-module, then we say that a filtration $E = N^0 \supset N^1 \supset \cdots \supset N^m = 0$ satisfies Griffiths transversality if it is a filtration of $E$ by coherent $\mathcal{O}_X$-submodules and satisfies $\nabla(\Lambda_{\mathcal{L}} \otimes_{\mathcal{O}_X} N^i) \subset N^{i-1}$. For every such filtration the associated graded object

$$\text{Gr}_N(E) := \bigoplus_i N^i/N^{i+1}$$

carries a canonical coherent $\mathcal{L}$-Higgs module structure $\theta : \mathcal{L} \otimes_{\mathcal{O}_X} \text{Gr}_N(E) \to \text{Gr}_N(E)$ defined by $\nabla$. This can be seen by considering the following commutative diagram:

\[
\begin{array}{ccc}
\Lambda_0 \otimes_{\mathcal{O}_X} N^{i+1} & \xrightarrow{\Lambda_0 \otimes_{\mathcal{O}_X} N^i} & \Lambda_0 \otimes_{\mathcal{O}_X} N^i \\
\downarrow & & \downarrow \\
\Lambda_1 \otimes_{\mathcal{O}_X} N^{i+1} & \xrightarrow{\Lambda_1 \otimes_{\mathcal{O}_X} N^i} & \Lambda_1 \otimes_{\mathcal{O}_X} N^i \\
\downarrow & & \downarrow \\
\Lambda_1/\Lambda_0 \otimes_{\mathcal{O}_X} N^{i+1} & \xrightarrow{\Lambda_1/\Lambda_0 \otimes_{\mathcal{O}_X} N^i} & \Lambda_1/\Lambda_0 \otimes_{\mathcal{O}_X} N^i \\
\downarrow & & \downarrow \\
0 & \xrightarrow{0} & 0
\end{array}
\]

where $\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_\mathcal{L}$ is the standard filtration on $\Lambda_{\mathcal{L}}$. By the above diagram, the map $\Lambda_1 \otimes_{\mathcal{O}_X} N^{i+1} \to N^{i-1}/N^i$ is zero and hence the map $\Lambda_1/\Lambda_0 \otimes_{\mathcal{O}_X} N^{i+1} \to N^{i-1}/N^i$ is also zero. So we have an induced map $\Lambda_1/\Lambda_0 \otimes_{\mathcal{O}_X} N^i/N^{i+1} \to N^{i-1}/N^i$. But $\Lambda_1/\Lambda_0 = \mathcal{L}$ and one can easily check that the obtained map gives an $\mathcal{L}$-Higgs module structure on $\text{Gr}_N(E)$. Note also that by construction the obtained pair $(\text{Gr}_N(E), \theta)$ is a system of $\mathcal{L}$-Hodge sheaves on $X$.

In the remainder of this section, in order to define semistability we fix a collection $L = (L_1, \ldots, L_{n-1})$ of nef line bundles on $X$ such that $L_1L_2\ldots L_{n-1}$ is numerically non-trivial.
We say that a Griffiths transverse filtration \( N^\bullet \) on \((\mathcal{E}, \nabla)\) is slope gr-semistable if the associated \( \mathcal{L}\)-Higgs sheaf \((\text{Gr}_N(\mathcal{E}), \theta)\) is (torsion free and) slope semistable. A partial \( \mathcal{L}\)-oper is a triple \((\mathcal{E}, \nabla, N^\bullet)\) consisting of a torsion free coherent \( \mathcal{O}_X\)-module \( \mathcal{E} \) with a \( \Lambda_{\mathcal{L}}\)-module structure \( \nabla \) and a Griffiths transverse filtration \( N^\bullet \), which is slope gr-semistable.

**Remark 5.5.** Note that analogous definitions in [23, Section 5.2] work only for smooth Lie algebroids. The above definitions allow us to deal with general Lie algebroids and they are equivalent to those in [23, Section 5.2] in case of smooth Lie algebroids.

The following theorem can be proven in the same way as [25, Theorem 5.5]. The only difference is that in the proof one needs to consider \( \mathcal{L}\)-modules as \( \nabla\)-modules.

**Theorem 5.6.** If \((\mathcal{E}, \nabla)\) is slope semistable, then there exists a canonically defined slope gr-semistable Griffiths transverse filtration \( S^\bullet \) on \((\mathcal{E}, \nabla)\) providing it with a partial \( \mathcal{L}\)-oper structure. This filtration is preserved by the automorphisms of \((\mathcal{E}, \nabla)\).

The above filtration \( S^\bullet \) is called Simpson’s filtration. Even in the case of a trivial Lie algebroid structure on \( \mathcal{L} \) the above theorem gives a non-trivial corollary:

**Corollary 5.7.** Let \((\mathcal{E}, \theta : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E})\) be a slope semistable \( \mathcal{L}\)-Higgs sheaf. Then there exists a decreasing filtration
\[
\mathcal{E} = N^0 \supset N^1 \supset \cdots \supset N^m = 0
\]
such that \( \theta(\mathcal{L} \otimes_{\mathcal{O}_X} N^i) \subset N^{i-1} \) and the associated graded is a slope semistable system of \( \mathcal{L}\)-Hodge sheaves.

### 5.4. Higgs–de Rham sequences on normal varieties.

Let \( X \) be a normal projective variety defined over an algebraically closed field \( k \) of positive characteristic \( p \). Let \( D \) be a reduced effective Weil divisor on \( X \).

Let us assume that \((X, D)\) is almost liftable to \( W_2(k) \). Then we can find a big open subset \( U \subset X \) such that the pair \((U, D_U = D \cap U)\) is log smooth and liftable to \( W_2(k) \). Let us fix a lifting \((\bar{U}, \bar{D}_U)\).

Let \((\mathcal{E}, \theta : T_{X/k}(\log D) \otimes \mathcal{E} \to \mathcal{E})\) be a reflexive logarithmic Higgs \( \mathcal{O}_X\)-module of rank \( r \leq p \). Let us assume that \((\mathcal{E}, \theta)\) is slope semistable. Then by Corollary 5.7 there exists a canonical filtration \( N^\bullet \) on \( \mathcal{E} \) such that the associated graded \((\bar{\mathcal{E}}_0, \bar{\theta}_0)\) is a slope semistable system of logarithmic Hodge sheaves. Let \((\mathcal{E}_0, \theta_0)\) be the reflexive hull of \((\bar{\mathcal{E}}_0, \bar{\theta}_0)\). By construction, it is a slope semistable reflexive logarithmic system of Hodge sheaves. In particular, since its rank \( r \) is \( \leq p \), it is also a reflexive logarithmic Higgs \( \mathcal{O}_X\)-module with a nilpotent Higgs field of level less or equal to \( p - 1 \). So we can define
\[
(V_0, \nabla_0) := C_{[-1]}^{(\bar{U}, \bar{D}_U)}(\bar{\mathcal{E}}_0, \bar{\theta}_0).
\]
Let \( S^\bullet_0 \) be (decreasing) Simpson’s filtration on \((V_0, \nabla_0)\) and let \((\bar{\mathcal{E}}_1 = \text{Gr}_{S_0}(V_0), \bar{\theta}_1)\) be the associated system of Hodge sheaves. Then we set
\[
(\mathcal{E}_1, \theta_1) := ((\bar{\mathcal{E}}_1)^{**}, \bar{\theta}_1^{**})
\]
and repeat the procedure. In this way we get the following sequence:

\[
\begin{align*}
(\mathcal{E}, \theta) & \xrightarrow{\mathcal{G} \mathcal{L}_{X}^{**}} (V_{0}, \nabla_{0}) \xrightarrow{\mathcal{G} \mathcal{L}_{S_{0}}^{**}} (V_{1}, \nabla_{1}) \xrightarrow{\mathcal{G} \mathcal{L}_{S_{1}}^{**}} \ldots, \\
(\mathcal{E}_{0}, \theta_{0}) & \xrightarrow{\mathcal{G} \mathcal{L}_{X}^{**}} (\mathcal{E}_{1}, \theta_{1}) \xrightarrow{\mathcal{G} \mathcal{L}_{S_{0}}^{**}} (\mathcal{E}_{2}, \theta_{2}) \xrightarrow{\mathcal{G} \mathcal{L}_{S_{1}}^{**}} \ldots,
\end{align*}
\]

in which each logarithmic Higgs sheaf \((\mathcal{E}_{j}, \theta_{j})\) is reflexive rank \(r \leq p\) and slope semistable. We call this sequence the \textit{canonical Higgs–de Rham sequence} of \((\mathcal{E}, \theta)\).

\textbf{Remark 5.8.} Higgs–de Rham sequences were invented by G. Lan, M. Sheng and K. Zuo in [19] and their existence was proven in [19] and [23]. Canonical Higgs–de Rham sequences in the above sense first appeared in the proof of [27, Lemma 3.10]. They are better suited to dealing with normal varieties as one cannot define suitable Chern classes for torsion free sheaves on normal varieties.

\textbf{Remark 5.9.} Although the above construction is very general, it does not seem easy to compare numerical invariants of the sheaves \(\mathcal{E}_{i}\) without some further assumptions on the singularities of the pair \((\mathcal{X}, D)\).

\textbf{5.5. Inverse Cartier transform on log varieties with locally \(F\)-liftable singularities.}

Let \(\mathcal{X}\) be a normal variety defined over an algebraically closed field \(k\) of positive characteristic \(p\). We define the Grothendieck group \(K_{\text{ref}}^{\mathcal{X}}(\mathcal{X})\) of reflexive sheaves on \(\mathcal{X}\) as the free abelian group on the isomorphism classes \([\mathcal{E}]\) of coherent reflexive \(\mathcal{O}_{\mathcal{X}}\)-modules modulo the relations \([\mathcal{E}] 2 = [\mathcal{E}] + [\mathcal{E}] 3\) for each locally split short exact sequence

\[
0 \to \mathcal{E}_{1} \to \mathcal{E}_{2} \to \mathcal{E}_{3} \to 0
\]

of coherent reflexive \(\mathcal{O}_{\mathcal{X}}\)-modules.

For a coherent \(\mathcal{O}_{\mathcal{X}}\)-module \(\mathcal{F}\) we denote by \(\nabla_{\text{can}}^{\mathcal{F}}\) the canonical connection on \(\mathcal{F}_{\mathcal{X}}^{\mathcal{X}}\) given by differentiating along the fibers of the Frobenius morphism.

\textbf{Lemma 5.10.} Let \(D\) be a reduced effective Weil divisor on \(\mathcal{X}\) such that \((\mathcal{X}, D)\) is liftable to \(W_{2}(k)\) and it is locally \(F\)-liftable. If \((\mathcal{E}, \theta) = \bigoplus \mathcal{E}^{i}, \theta\) is a reflexive system of logarithmic Hodge sheaves on \((\mathcal{X}, D)\) and we set \(\mathcal{E}_{j} = \bigoplus_{j=1}^{\infty} \mathcal{E}^{j}\) with induced \(\theta_{j}\), then for every \(j\) we have a short exact sequence

\[
0 \to C^{-1}(\mathcal{E}_{j}, \theta_{j}) \to C^{-1}(\mathcal{E}_{j+1}, \theta_{j+1}) \to (\mathcal{F}_{\mathcal{X}}^{\mathcal{X}}\mathcal{E}^{j+1}, \nabla_{\text{can}}^{\mathcal{E}^{j+1}}) \to 0
\]

of reflexive \(\mathcal{O}_{\mathcal{X}}\)-modules with a logarithmic connection, which is locally split as a sequence of \(\mathcal{O}_{\mathcal{X}}\)-modules. In particular, we have \([C^{-1}]\mathcal{E} = [\mathcal{F}_{\mathcal{X}}^{\mathcal{X}}\mathcal{E}]\) in \(K_{\text{ref}}^{\mathcal{X}}(\mathcal{X})\).

\textbf{Proof.} By construction we have a short exact sequence of Higgs sheaves

\[
0 \to (\mathcal{E}_{j}, \theta_{j}) \to (\mathcal{E}_{j+1}, \theta_{j+1}) \to (\mathcal{E}^{j+1}, 0) \to 0,
\]

which is split as a sequence of \(\mathcal{O}_{\mathcal{X}}\)-modules. Applying \(C^{-1}\) to this sequence we get

\[
0 \to C^{-1}(\mathcal{E}_{j}, \theta_{j}) \to C^{-1}(\mathcal{E}_{j+1}, \theta_{j+1}) \to (\mathcal{F}_{\mathcal{X}}^{\mathcal{X}}\mathcal{E}^{j+1}, \nabla_{\text{can}}^{\mathcal{E}^{j+1}}) \to 0.
\]
because $C^{[-1]}(\mathcal{E}^j+1, 0) = (F_X^{[\ast]}\mathcal{E}^j+1, \nabla^{\text{can}}_{\mathcal{E}^j+1})$. So it is sufficient to show that this sequence is locally split. To do so we fix a point $x \in X$ and an open neighborhood $x \in U \subset X$, which is $F$-liftable. Let $V$ be a big open subset of $U$, which is contained in the log smooth locus of $(X, D)$. The pair $(V, D \cap V)$ has an $F$-lifting $\tilde{F}_V : \tilde{V} \to V$ compatible with the $W_2(k)$-lifting $(\tilde{V}, \tilde{D})$ induced from the given $W_2(k)$-lifting of $(X, D)$. On $V$ we have a short exact sequence of modules with integrable connections

$$0 \to (F_V^{[\ast]}\mathcal{E}_j, \nabla^{\text{can}}_{\mathcal{E}_j}) \to (F_V^{[\ast]}\mathcal{E}_{j+1}, \nabla^{\text{can}}_{\mathcal{E}_{j+1}}) \to (F_V^{[\ast]}\mathcal{E}^j+1, \nabla^{\text{can}}_{\mathcal{E}^j+1}) \to 0,$$

which is split as a sequence of $\mathcal{O}_V$-modules. Extending the above sequence to $U$, we get a short exact sequence of reflexive $T_U$-modules

$$0 \to F_U^{[\ast]}\mathcal{E}_j \to F_U^{[\ast]}\mathcal{E}_{j+1} \to F_U^{[\ast]}\mathcal{E}^j+1 \to 0,$$

which is split as a sequence of $\mathcal{O}_U$-modules. By construction (see Section A)

$$C^{[-1]}(\mathcal{E}_j)|_U \simeq F_U^{[\ast]}\mathcal{E}_j$$

and $C^{[-1]}(\theta_j)|_U$ is obtained by extension of the logarithmic connection $\nabla^{\text{can}}_{\mathcal{E}_j} + \zeta_V(F_V^{[\ast]}\theta_j)$, where $\zeta_V := p^{-1}\tilde{F}_V : F_V^{[\ast]}\Omega_V \to \Omega_V$ (see Section A.1). Since the above isomorphisms are compatible with restrictions to $V$, we see that the sequence

$$0 \to C^{[-1]}(\mathcal{E}_j, \theta_j) \to C^{[-1]}(\mathcal{E}_{j+1}, \theta_{j+1}) \to (F_X^{[\ast]}\mathcal{E}^j+1, \nabla^{\text{can}}_{\mathcal{E}^j+1}) \to 0$$

is split as a sequence of $\mathcal{O}_U$-modules. \hfill \qed

**Corollary 5.11.** Let $X$ be a normal projective variety with a collection $(L_1, \ldots, L_{n-2})$ of nef line bundles. Let $D$ be a reduced effective Weil divisor on $X$ such that $(X, D)$ is almost liftable to $W_2(k)$ and it has $F$-liftable singularities in codimension 2. Then we have

$$\int_X \text{ch}_2(C^{[-1]}\mathcal{E})L_1 \ldots L_{n-2} = p^2 \int_X \text{ch}_2(\mathcal{E})L_1 \ldots L_{n-2}.$$

**Proof.** By Theorem 1.17 we can reduce the assertion to the surface case. Then Theorem 1.13 says that $(X, D)$ satisfies assumptions of Lemma 2.1 and hence we get

$$\int_X \text{ch}_2(C^{[-1]}\mathcal{E})L_1 \ldots L_{n-2} = \sum_j \int_X \text{ch}_2(F_X^{[\ast]}\mathcal{E}^j)L_1 \ldots L_{n-2}$$

$$= p^2 \sum_j \int_X \text{ch}_2(\mathcal{E}^j)L_1 \ldots L_{n-2}$$

$$= p^2 \int_X \text{ch}_2(\mathcal{E})L_1 \ldots L_{n-2}. \hfill \Box$$

### 5.6. Bogomolov’s inequality for Higgs sheaves

In this subsection we give the first version of Bogomolov’s inequality for logarithmic Higgs sheaves on singular varieties. The following theorem generalizes Bogomolov’s inequality for logarithmic Higgs sheaves to singular varieties (see [24, Theorem 8] in case $X$ is smooth and [25, Theorem 3.3] for the log smooth case).
Theorem 5.12. Let \( L = (L_1, \ldots, L_{n-1}) \) be a collection of nef line bundles on \( X \) such that \( L_1 L_2 \ldots L_{n-1} \) is numerically non-trivial. Assume that the pair \((X, D)\) is almost liftable to \( W_2(k) \) and it has \( F \)-liftable singularities in codimension 2. Then for any slope \( L \)-semistable logarithmic reflexive Higgs sheaf \((\mathcal{E}, \theta)\) of rank \( r \leq p \) we have

\[
\int_X \Delta(\mathcal{E}) L_2 \ldots L_{n-1} \geq 0.
\]

Proof. Let 
\[
(\mathcal{E}, \theta : T_{X/k} (\log D) \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E})
\]
be a reflexive logarithmic Higgs \( \mathcal{O}_X \)-module of rank \( r \leq p \). Let us assume that \((\mathcal{E}, \theta)\) is slope semistable. Let

\[
(\mathcal{E}, \theta) \xrightarrow{(\text{Gr}_X)^{**}} (V_0, \nabla_0) \xrightarrow{C[-1]} (\text{Gr}_{S\mathcal{E}})^{**} \xrightarrow{(\text{Gr}_1)^{**}} (V_1, \nabla_1) \xrightarrow{C[-1]} \ldots
\]

be the canonical Higgs–de Rham sequence of \((\mathcal{E}, \theta)\).

By Lemma 4.8 there exists \( \alpha \) such that \( \mu_{\max, L}(\mathcal{E}_m) - \mu_L(\mathcal{E}_m) \leq \alpha \) for all \( m \geq 0 \). So by Corollary 3.8 there exists some constant \( C \) such that for every non-negative integer \( m \) we have

\[
\int_X \Delta(\mathcal{E}_m) L_2 \ldots L_{n-1} \geq C.
\]

Lemma 2.2 implies that

\[
\int_X \Delta(\mathcal{E}) L_2 \ldots L_{n-1} \geq \int_X \Delta(\mathcal{E}_0) L_2 \ldots L_{n-1}
\]

and

\[
\int_X \Delta(V_m) L_2 \ldots L_{n-1} \geq \int_X \Delta(\mathcal{E}_{m+1}) L_2 \ldots L_{n-1}.
\]

By Corollary 5.11 we have

\[
\int_X \Delta(V_m) L_2 \ldots L_{n-1} = p^2 \int_X \Delta(\mathcal{E}_m) L_2 \ldots L_{n-1}.
\]

Therefore

\[
C \leq \int_X \Delta(\mathcal{E}_m) L_2 \ldots L_{n-1} \leq p^{2m} \int_X \Delta(\mathcal{E}) L_2 \ldots L_{n-1}.
\]

Dividing by \( p^{2m} \) and passing with \( m \) to infinity, we get \( \int_X \Delta(\mathcal{E}) L_2 \ldots L_{n-1} \geq 0. \)

Remark 5.13. The above theorem holds also for reflexive sheaves with an integrable logarithmic connection. Indeed, if \((\mathcal{E}, \nabla)\) is a rank \( r \leq p \) slope \( L \)-semistable reflexive sheaf with an integrable logarithmic connection and \( S^\bullet \) is its Simpson’s filtration, then by the above theorem and Lemma 2.2 we have

\[
\int_X \Delta(\mathcal{E}) L_2 \ldots L_{n-1} \geq \int_X \Delta((\text{Gr}_S \mathcal{E})^{**}) L_2 \ldots L_{n-1} \geq 0.
\]
6. The Miyaoka–Yau inequality on singular varieties in positive characteristic

In this section we prove the Miyaoka–Yau inequality on some mildly singular varieties in positive characteristic. The ideas are similar to that from [35] and [25] but we show full proofs to show where they need additional facts related to the use of our Chern classes.

We fix a log pair $(X, D)$ defined over an algebraically closed field of characteristic $p > 0$. We assume that $(X, D)$ is almost liftable to $W_2(k)$ and it has $F$-liftable singularities in codimension 2. Let $n = \dim X$ and let us fix a collection $L = (L_1, \ldots, L_{n-1})$ of nef line bundles on $X$ such that $L_1^2 L_2 \cdots L_{n-1} > 0$. As in Section 3.2 we consider a positive open cone $K_L^+ \subset N_L(X)$.

The proof of the following proposition is essentially the same as that of [25, Proposition 4.1].

**Proposition 6.1.** Let $\mathcal{L}$ be a rank 1 reflexive sheaf contained in $\Omega_{X/k}^{[1]}(\log D)$. Then $c_1(\mathcal{L}) \notin K_L^+$.

**Proof.** Assume that $c_1(\mathcal{L}) \in K_L^+$ and consider a system of logarithmic Hodge sheaves $(\mathcal{E} := \mathcal{E}^1 \oplus \mathcal{E}^0, \theta)$ with $\mathcal{E}^1 = \mathcal{L}, \mathcal{E}^0 = \mathcal{O}_X$ and

$$\theta : \mathcal{E}^1 \to \mathcal{E}^0 \otimes \Omega_{X/k}^{[1]}(\log D) = \Omega_{X/k}^{[1]}(\log D)$$

given by the inclusion. Then $(\mathcal{E}, \theta)$ is slope $L$-stable since the only rank 1 logarithmic sub-system of Hodge sheaves of $(\mathcal{E}, \theta)$ is of the form $(\mathcal{O}_X, 0)$. Therefore by Lemma 2.1 and Theorem 5.12 we have

$$0 = 4 \int_X c_2(\mathcal{E}) L_2 \cdots L_{n-1} \geq \int_X c_1^2(\mathcal{E}) L_2 \cdots L_{n-1} = c_1(\mathcal{L})^2 L_2 \cdots L_{n-1},$$

a contradiction. \qed

Similarly as [25, Theorem 4.4] one can also get the following theorem generalizing the Miyaoka–Yau inequality in the surface case:

**Theorem 6.2.** Let us assume that $p \geq 3$ and let $\mathcal{F} \subset \Omega_{X/k}^{[1]}(\log D)$ be a rank 2 reflexive subsheaf with $c_1(\mathcal{F}) \in K_L^+$. Then

$$3 \int_X c_2(\mathcal{F}) L_2 \cdots L_{n-1} \geq \int_X c_1^2(\mathcal{F}) L_2 \cdots L_{n-1}.$$

**Proof.** Let us consider the system of logarithmic Hodge sheaves $(\mathcal{E} := \mathcal{E}^1 \oplus \mathcal{E}^0, \theta)$ given by $\mathcal{E}^1 = \mathcal{F}, \mathcal{E}^0 = \mathcal{O}_X$ and

$$\theta : \mathcal{E}^1 = \mathcal{E} \to \Omega_{X/k}^{[1]}(\log D) = \mathcal{E}^0 \otimes \Omega_{X/k}^{[1]}(\log D).$$

If $(\mathcal{E}, \theta)$ is slope $L$-semistable, then by Lemma 2.1 and Theorem 5.12 we have

$$3 \int_X c_2(\mathcal{F}) L_2 \cdots L_{n-1} = 3 \int_X c_2(\mathcal{E}) L_2 \cdots L_{n-1} \geq \int_X c_1^2(\mathcal{F}) L_2 \cdots L_{n-1} = \int_X c_1^2(\mathcal{E}) L_2 \cdots L_{n-1}.$$
So we can assume that \((\mathcal{E}, \theta)\) is not slope \(L\)-semistable. Let \((\mathcal{E}', \theta')\) be its maximal destabilizing subsystem of logarithmic Hodge subsheaves. Note that \((\mathcal{E}, \theta)\) contains only one saturated rank 1 system of logarithmic Hodge subsheaves, namely \((\mathcal{O}_X, 0)\). Since this subsystem does not destabilize \((\mathcal{E}, \theta)\), the sheaf \(\mathcal{E}'\) has rank 2. Note that \((\mathcal{E}', \theta')\) is slope \(L\)-stable so

\[
c_1(\mathcal{M}).L_1 \ldots L_{n-1} > 0.
\]

We can decompose \(\mathcal{E}'\) into a direct sum \(\mathcal{O}_X \oplus \mathcal{M}\), where \(\mathcal{M}\) is a saturated rank 1 reflexive sheaf contained in \(\mathcal{F}\). By assumption \((\mathcal{E}', \theta')\) destabilizes \((\mathcal{E}, \theta)\), so

\[
\mu_L(\mathcal{E}') = \frac{c_1(\mathcal{M}).L_1 \ldots L_{n-1}}{2} > \mu_L(\mathcal{E}) = \frac{c_1(\mathcal{F}).L_1 \ldots L_{n-1}}{3}.
\]

Therefore \((3c_1(\mathcal{M}) - 2c_1(\mathcal{F})).L_1 \ldots L_{n-1} > 0\). If \((3c_1(\mathcal{M}) - 2c_1(\mathcal{F})) \in K^+_L\), then

\[
3c_1(\mathcal{M}) = (3c_1(\mathcal{M}) - 2c_1(\mathcal{F})) + 2c_1(\mathcal{F}) \in K^+_L,
\]

which contradicts Proposition 6.1. This shows that

\[
(3c_1(\mathcal{M}) - 2c_1(\mathcal{F}))^2.L_2 \ldots L_{n-1} \leq 0.
\]

Let us set \(\mathcal{L} := (\mathcal{F}/\mathcal{M})^{**}\). Then the sequence

\[
0 \to \mathcal{M} \to \mathcal{F} \to \mathcal{L}
\]

satisfies assumptions of Lemma 2.1 and hence

\[
\int_X \text{ch}_2(\mathcal{F}).L_2 \ldots L_{n-1} \leq \frac{1}{2} c_1(\mathcal{M})^2 . L_2 \ldots L_{n-1} + \frac{1}{2} c_1(\mathcal{L})^2 . L_2 \ldots L_{n-1},
\]

which after rewriting gives

\[
\int_X (3c_2(\mathcal{F}) - c_1(\mathcal{F})^2).L_2 \ldots L_{n-1} + \frac{3}{4} c_1(\mathcal{M})^2 . L_2 \ldots L_{n-1}
\]

\[
\geq -\frac{1}{4} (3c_1(\mathcal{M}) - 2c_1(\mathcal{F}))^2 . L_2 \ldots L_{n-1} \geq 0.
\]

Since \(c_1(\mathcal{M})^2 . L_2 \ldots L_{n-1} \leq 0\) by Proposition 6.1, this implies the required inequality.

\[\square\]

### 7. Applications to characteristic zero

In this section we show a few applications of our results to study varieties defined in characteristic zero. In particular, we prove Theorems 0.5, 0.6 and 0.7.

First we recall the following lemma that follows from Lemma 3.19 in the preprint version of [10] (note that any normal surface with quotient singularities is klt).

**Lemma 7.1.** Let \(X\) be a normal projective surface with at most quotient singularities defined over an algebraically closed field \(k\) of characteristic 0. Let \(\mathcal{E}\) be a coherent reflexive \(\mathcal{O}_X\)-module. Then there exists a normal projective surface \(Y\) and a finite morphism \(\pi : Y \to X\) such that \(\pi([\ast])\mathcal{E}\) is locally free. In this case we have

\[
\int_X \Delta(\mathcal{E}) = \frac{1}{\deg \pi} \int_Y \Delta(\pi([\ast])\mathcal{E}) \quad \text{and} \quad \int_X \text{ch}_2(\mathcal{E}) = \frac{1}{\deg \pi} \int_Y \text{ch}_2(\pi([\ast])\mathcal{E}).
\]
From now on we fix the following notation in this section. Let $X$ be a normal projective variety of dimension $n$ defined over an algebraically closed field $k$ of characteristic $0$. We assume that $X$ has quotient singularities in codimension $2$ and we fix a reduced divisor $D \subset X$ such that the pair $(X, D)$ is log canonical in codimension $2$. For sheaves on such a variety we use Chern classes defined in [26, 5.3]. They coincide with classical Mumford’s $Q$-Chern classes considered in [16, Chapter 10] and in [10, Theorem 3.13] (see [26, Remark 5.9]).

7.1. Strong restriction theorems. Let us fix a collection $L = (L_1, \ldots, L_{n-1})$ of ample line bundles and let us set $d = L_1^2L_2 \cdots L_{n-1}$. The proof of the following theorem is based on a standard spreading out argument.

**Theorem 7.2.** Let $(\mathcal{E}, \theta)$ be a reflexive logarithmic Higgs sheaf of rank $r \geq 2$ on $(X, D)$. Let $m_0$ be a non-negative integer such that $T_{X/k}(\log D) \otimes L_1^{\otimes m_0}$ is globally generated. Let $m$ be an integer such that

$$m > \max \left( \frac{r-1}{r} \int_X \Delta(\mathcal{E})L_2 \cdots L_{n-1} + \frac{1}{r(r-1)d}, 2(r-1)m_0^2 \right).$$

Let $H \in |L_1^{\otimes m}|$ be good for $(X, D)$.

1. If $(\mathcal{E}, \theta)$ is slope $L$-stable, then the logarithmic Higgs sheaf $(\mathcal{E}, \theta)|_H$ on $(H, D \cap H)$ is slope $L_H$-stable.

2. If $(\mathcal{E}, \theta)$ is slope $L$-semistable and restrictions of all quotients of a Jordan–Hölder filtration of $(\mathcal{E}, \theta)$ to $H$ are torsion free, then $(\mathcal{E}, \theta)|_H$ is slope $L_H$-semistable.

**Proof.** Let us write $D$ as a sum $\sum D_i$ of prime divisors. We can find a subring $R \subset k$, which is finitely generated over $\mathbb{Z}$ and there exists a flat projective morphism $\mathcal{X} \to S = \text{Spec } R$ with $D = \sum D_i$, where $D_i \to \mathcal{X}$ are $S$-flat subschemes of $\mathcal{X}$, such that $X \simeq \mathcal{X} \times_S \text{Spec } k$ and $D_i \simeq \mathcal{D}_i \times_S \text{Spec } k$ for all $i$. We can assume that there exist line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_{n-1}$ on $\mathcal{X}$ lifting $L_1, \ldots, L_{n-1}$, a relative logarithmic Higgs sheaf $(\mathcal{E}, \theta : T_{\mathcal{X}/S}(\log D) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{E} \to \mathcal{E})$ lifting $(\mathcal{E}, \theta)$ and a relative effective Cartier divisor $\mathcal{H} \in |\mathcal{L}_1^{\otimes m}|$ lifting $H$ (in particular, $\mathcal{H} \to S$ is flat). Shrinking $S$ if necessary we can assume that the following conditions are satisfied:

1. $S$ is regular,

2. all fibers of $\mathcal{X} \to S$ and $\mathcal{H} \to S$ are geometrically integral and geometrically normal,

3. $\mathcal{L}_1, \ldots, \mathcal{L}_n$ are relatively ample,

4. $T_{\mathcal{X}/S}(\log D) \otimes \mathcal{L}_1^{\otimes m_0}$ is relatively globally generated,

5. for all closed points $s \in S$ the fiber $\mathcal{X}_s$ is liftable modulo $W_2(\kappa(s))$ (see the proof of [24, Theorem 7]),

6. for all geometric points $\bar{s}$ of $S$, $\mathcal{H}_{\bar{s}}$ is good for $(\mathcal{X}_{\bar{s}}, D_{\bar{s}})$,

7. for all geometric points $\bar{s}$ of $S$ the sheaf $\mathcal{E}_{\bar{s}}$ is a coherent reflexive $\mathcal{O}_{\mathcal{X}_{\bar{s}}}$-module,

8. a fixed Jordan–Hölder filtration $\mathcal{E}_{\bullet}$ of $(\mathcal{E}, \theta)$ extends to a filtration $\mathcal{E}_{\bullet}$ of $(\mathcal{E}, \theta)$.

Following the proof of [30, Theorem 4.2] (see also [12, Proposition 2.3.1]), one can see that geometric slope (semi)stability of logarithmic Higgs sheaves is an open condition in flat families. It is sufficient to prove that there exists some geometric point $\bar{s}$ of $S$, the logarithmic...
Higgs sheaf \((\tilde{E}, \tilde{\theta})|_{\mathcal{H}_s}\) on \((X_s, \mathcal{H}_s)\) is slope \((\mathcal{L}_2|_{\mathcal{H}_s}, \ldots, \mathcal{L}_{n-1}|_{\mathcal{H}_s})\)-(semi)stable. Then the restriction of \((\tilde{E}, \tilde{\theta})|_{\mathcal{H}}\) to the fiber of \(\mathcal{H} \to S\) over the generic geometric point of \(S\) is also slope (semi)stable, proving the theorem.

By the above openness of semistability, we can assume that for all geometric points \(\bar{s}\) of \(S\) the logarithmic Higgs sheaf \((\tilde{E}_s, \tilde{\theta}_s)\) is slope (semi)stable. By the same argument the restriction of the filtration \(\tilde{E}_s\) to \(X_s\) gives a Jordan–Hölder filtration of \((\tilde{E}_s, \tilde{\theta}_s)\). Note also that restrictions of quotients of this filtration to \(\mathcal{H}_s\) are torsion free for \(\bar{s}\) over an open subset of \(S\). We need to check that there exists some non-empty open subset \(U \subset S\) such that for all geometric points \(\bar{s}\) over closed points of \(U\) we have

\[
\int_{\mathcal{X}_s} \Delta(\tilde{E}_s) \mathcal{L}_2|_{\mathcal{X}_s} \ldots \mathcal{L}_{n-1}|_{\mathcal{X}_s} = \int_\mathcal{X} \Delta(E)L_2 \ldots L_{n-1}.
\]

This is not obvious as \(\tilde{E}\) is not locally free and Chern numbers of reflexive sheaves do not remain constant in flat families. Using Theorem 1.17 and [26, Theorem 5.8], we can reduce to the surface case. By Lemma 7.1 we can find a normal projective surface \(Y\) and a finite covering \(\pi : Y \to X\) such that \(\pi|_{\mathcal{S}}\mathcal{S}\) is locally free. Then, shrinking \(S\) if necessary, we can find a flat projective morphism \(\tilde{\pi} : \mathcal{Y} \to S\) and a morphism \(\mathcal{Y} \to \mathcal{X}\) lifting \(\pi : Y \to X\). We can also assume that all fibers of \(g : \mathcal{Y} \to S\) are geometrically integral and geometrically normal. Since \(S\) is normal, the schemes \(\mathcal{X}\) and \(\mathcal{Y}\) are also normal. So we can consider \(\tilde{\pi}^{[*]}/ \mathcal{E}\), which is reflexive on \(\mathcal{Y}\). This sheaf is locally free outside of a closed subscheme \(Z \subset \mathcal{Y}\) of codimension \(\geq 2\). Since \(Z\) does not intersect the generic fiber of \(\mathcal{Y} \to S\), it follows that \(\tilde{\pi}^{[*]}/ \mathcal{E}\) is locally free over a non-empty open subset \(S' = S \setminus g(Z) \subset S\). Now let us consider a commutative diagram

\[
\begin{array}{ccc}
\mathcal{Y}_S & \xrightarrow{j_s} & \mathcal{Y} \\
\downarrow \pi_s & & \downarrow \pi \\
\mathcal{X}_S & \xrightarrow{i_s} & \mathcal{X}.
\end{array}
\]

Since \(j_s^{[*]}(\tilde{\pi}^{[*]}/ \mathcal{E})\) is locally free for \(s \in S'\), we have an induced map \(\varphi_s\) that fits into a commutative diagram

\[
\begin{array}{ccc}
\tilde{\pi}_s^{[*]}i_s^{[*]}(\tilde{E}) & \xrightarrow{\simeq} & \tilde{\pi}_s^{[*]}(i_s^{[*]}(\tilde{E})) \\
\downarrow \varphi_s & & \downarrow \varphi_s \\
j_s^{[*]}(\tilde{\pi}^{[*]}/ \mathcal{E}) & \xrightarrow{\simeq} & j_s^{[*]}(\tilde{\pi}^{[*]}/ \mathcal{E}).
\end{array}
\]

Let us set

\[
U := \{x \in X : \tilde{E}_x\text{ is a free }\mathcal{O}_{X,x}\text{-module}\}.
\]

Since \(\varphi_s\) is an isomorphism over a big open subset \(\mathcal{Y}_S \cap \tilde{\pi}^{-1}(U)\) of \(\mathcal{Y}_S\) and \(\tilde{\pi}_s^{[*]}(i_s^{[*]}(\tilde{E}))\) is reflexive, it follows that \(\varphi_s\) is an isomorphism. So by Theorem 1.17 and Lemma 7.1 we have

\[
\int_{\mathcal{X}_s} \Delta(E_x) = \frac{1}{\deg \pi_s} \int_{\mathcal{Y}_s} \Delta(\tilde{\pi}_s^{[*]}E_x) = \frac{1}{\deg \pi} \int_Y \Delta(\tilde{\pi}^{[*]}E) = \int_X \Delta(E)
\]

as claimed. Now the required assertion follows by applying Theorem 5.4 to fibers over geometric points \(\bar{s}\) of \(S\) with large characteristic of the residue field (then \(\beta_\tau(\bar{s}) \to 0\)).

The same argument as above also shows that Theorem 0.1 implies the following strong restriction theorem of Bogomolov’s type.
Theorem 7.3. Let $E$ be a coherent reflexive $\mathcal{O}_X$-module of rank $r \geq 2$. Let $m$ be an integer such that
\[
m > \left[ \frac{r - 1}{r} \int_X \Delta(E)L_2 \cdots L_{n-1} + \frac{1}{r(r-1)d} \right]
\]
and let $H \in |L_1^\otimes m|$ be a normal hypersurface.

1. If $E$ is slope $L$-stable, then $E|_H$ is slope $L_H$-stable.
2. If $E$ is slope $L$-semistable and restrictions of all quotients of a Jordan–Hölder filtration of $E$ to $H$ are torsion free, then $E|_H$ is slope $L_H$-semistable.

Remark 7.4. Although Theorem 0.1 works for any normal varieties in positive characteristic, it does not seem easy to use a similar spreading out argument to obtain even the usual Mehta–Ramanathan theorem for ample multipolarizations on a general normal projective variety in characteristic zero. The problem is that the choice of spreading out depends on $m$ as we need to spread out divisors $H \in |L_1^\otimes m|$. But since Chern numbers of reflexive sheaves are in general not well behaved in families of normal varieties, we cannot choose one $m$ so that Theorem 0.1 works for this fixed $m$ on even one geometric fiber $X_n$. However, using Corollary 3.11 one can show bounds on the maximal destabilizing slope of $E|_H$ on any normal variety in terms of numerical invariants of reductions of $E$.

7.2. Bogomolov’s inequality for logarithmic Higgs sheaves. We will need an analogue of the first part of Lemma 2.1 in the characteristic zero case (the analogue of the second part also holds but we will not need it):

Lemma 7.5. Let $(L_1, \ldots, L_{n-2})$ be a collection of nef line bundles on $X$. Let
\[0 \to E_1 \to E \to E_2\]
be an exact sequence of reflexive sheaves on $X$ and assume the cokernel of the last map is supported in codimension $\geq 2$. Then
\[
\int_X \text{ch}_2(E)L_1 \cdots L_{n-2} \leq \int_X \text{ch}_2(E_1)L_1 \cdots L_{n-2} + \int_X \text{ch}_2(E_2)L_1 \cdots L_{n-2}.
\]

Proof. As in the proof of Lemma 2.1 we first reduce to the surface case. Then by Lemma 7.1 there exists a normal projective surface $Y$ and a finite morphism $\pi : Y \to X$ such that $\pi^{[*]}E, \pi^{[*]}E_1$ and $\pi^{[*]}E_2$ are locally free. As in the proof of Lemma 2.1 we see that the sequence
\[0 \to \pi^{[*]}E_1 \to \pi^{[*]}E \to \pi^{[*]}E_2\]
is exact on $Y$ and the cokernel of the last map is supported on a finite set of points. Therefore
\[
\chi(Y, \pi^{[*]}E) \leq \chi(Y, \pi^{[*]}E_1) + \chi(Y, \pi^{[*]}E_2).
\]
Using the Riemann–Roch theorem for locally free sheaves on normal projective surfaces, this can be rewritten as
\[
\int_Y \text{ch}_2(\pi^{[*]}E) \leq \int_Y \text{ch}_2(\pi^{[*]}E_1) + \int_Y \text{ch}_2(\pi^{[*]}E_2).
\]
Dividing now by the degree of $\pi$, we get the required inequality from the second part of Lemma 7.1. \qed
**Theorem 7.6.** Let \( L = (L_1, \ldots, L_{n-1}) \) be a collection of nef line bundles on \( X \) such that \( L_1^2L_2 \cdot \ldots \cdot L_{n-1} > 0 \). For any slope \( L \)-semistable logarithmic reflexive Higgs sheaf \((\mathcal{E}, \theta)\) we have
\[
\int_X \Delta(\mathcal{E})L_2 \cdot \ldots \cdot L_{n-1} \geq 0.
\]

**Proof.** First assume that \( L_1, \ldots, L_{n-1} \) are ample. Then by the above theorem we can restrict to the surface case. As finite quotients of smooth affine log surface pairs are \( F \)-liftable in large characteristics (cf. [36, Lemma 4.21]), we can apply Theorem 5.12 and an easy spreading out argument.

In general, we reduce to the above case by an argument analogous to that from the proof of [21, 3.6]. Namely, we fix an ample line bundle \( A \) and consider the classes 
\[
L_i(t) = c_1(L_i) + tc_1(A)
\]
in \( N_L(X)_{\mathbb{Q}} \) for \( t \in \mathbb{Q}_{>0} \). These classes are ample and the Harder–Narasimhan filtration of \((\mathcal{E}, \theta)\) with respect to \((L_1(t), \ldots, L_{n-1}(t))\) is independent of \( t \) for small \( t \in \mathbb{Q}_{>0} \). We have an analogue of Lemma 2.2 for normal projective varieties in characteristic zero that have quotient singularities in codimension 2 (this follows from Lemma 7.5 in the same way as Lemma 2.2 follows from Lemma 2.1). Applying this result to the above filtration, using the inequality for ample collections of line bundles and taking the limit as \( t \to 0 \) gives the required inequality. \( \square \)

### A. The inverse Cartier transform after Lan–Sheng–Zuo

In this section we recall the construction of inverse Cartier transform from the Ogus–Vologodsky correspondence [33], following [18]. For simplicity of notation we consider only the non-logarithmic case. The logarithmic case is essentially the same.

#### A.1. Results of Deligne and Illusie

Below we recall the construction from [6] of the canonical splitting of the Cartier operator that is associated to a fixed lifting of Frobenius of a smooth \( F \)-liftable variety.

Let \( k \) be a perfect field of characteristic \( p > 0 \) and set \( S = \text{Spec} \ k \) and \( \widetilde{S} = \text{Spec} \ W_2(k) \).

Let \( X \) be a smooth \( k \)-variety with a fixed lifting \( \widetilde{X}/\widetilde{S} \). Let \( X' \) be the fiber product of \( X \) over the absolute Frobenius morphism of \( S \). Then we have the induced relative Frobenius morphism \( F_{X/S} : X \to X' \). Note that \( X' \) has the natural lifting \( \widetilde{X}' \) to \( \widetilde{S} \) defined as the base change of \( \widetilde{X} \to \widetilde{S} \) via \( \overline{S} \to \widetilde{S} \) coming from \( \sigma_2 : W_2(k) \to W_2(k) \). Let us assume that \( F_{X/S} : X \to X' \) has a lifting \( \widetilde{F}_{X/S} : \widetilde{X} \to \widetilde{X}' \) so that we have a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{F_{X/S}} & \widetilde{X} \\
\downarrow & & \downarrow \widetilde{F}_{X/S} \\
S & \xrightarrow{X'} & \widetilde{S} \\
\end{array}
\]

Since the map \( \widetilde{F}^* : \Omega^{1}_{\widetilde{X}'/\widetilde{S}} \to \Omega^{1}_{\widetilde{X}/\widetilde{S}} \) vanishes after pulling back to \( X \), we can define
\[
\zeta = p^{-1} \widetilde{F}^* : F^{*}_{X/S} \Omega^{1}_{X/S} \to \Omega^{1}_{X/S}.
\]
Since \( d\zeta = 0 \), we can consider \( \zeta \) as the map of sheaves of abelian groups \( F_{X/S}^{*}\Omega^{1}_{X/S} \to Z^{1}_{X/S} \),
where \( Z^{1}_{X/S} \) is the kernel of \( d : \Omega^{1}_{X/S} \to \Omega^{2}_{X/S} \). Its adjoint \( \zeta^{\text{ad}} : \Omega^{1}_{X'/S} \to F_{X'/S,*}Z^{1}_{X/S} \) is \( \mathcal{O}_{X'} \)-linear and it splits the composition

\[
F_{X/S,*}Z^{1}_{X/S} \longrightarrow \mathcal{H}^{1}(F_{X/S,*}\Omega^{*}_{X/S}) \xrightarrow{C_{X/S}} \Omega^{1}_{X'/S}
\]

of the Cartier operator with the canonical projection.

**A.2. The inverse Cartier transform.** Assume \( X \) is smooth and there exists a global lifting \( \bar{X} \) of \( X \) to \( W_{2}(k) \).

Let \( (\mathcal{E}, \theta) \) be a Higgs \( \mathcal{O}_{X'} \)-modules with a nilpotent Higgs field of level \( \leq (p-1) \). We want to construct an \( \mathcal{O}_{X} \)-module \( \mathcal{V} \) with an integrable connection \( \nabla \), whose \( p \)-curvature is nilpotent of level \( \leq (p-1) \). The pair \( (\mathcal{V}, \nabla) \) will be called the inverse Cartier transform of \( (\mathcal{E}, \theta) \) and denoted by \( C_{-1}\mathcal{E}_{/W_{2}(k)}(\mathcal{E}, \theta) \). Let us take a covering \( \{ \bar{U}_{\alpha} \}_{\alpha \in I} \) of \( \bar{X} \) such that for each \( \alpha \in I \) there exists

\[
\bar{F}_{\alpha} : \bar{U}_{\alpha} \to \bar{U}'_{\alpha}
\]
lifting the relative Frobenius morphism \( F_{\alpha} : U_{\alpha} \to U'_{\alpha} \). By the previous subsection, the lifting \( \bar{F}_{\alpha} \) leads to

\[
\zeta_{\alpha} = p^{-1}\bar{F}_{\alpha}^{*} : \Omega^{1}U'_{\alpha/k} \to \Omega^{1}_{U_{\alpha/k}}
\]

Therefore over each \( U_{\alpha} \) we can define \((V_{\alpha}, \nabla_{\alpha})\) by setting

\[
V_{\alpha} := F^{*}_{\alpha}(\mathcal{E}|_{U_{\alpha}}) \quad \text{and} \quad \nabla_{\alpha} := \nabla_{\text{can}} + \zeta_{\alpha}(F^{*}_{\alpha}\theta|_{U'_{\alpha}}).
\]

To glue \((V_{\alpha}, \nabla_{\alpha})\) and \((V_{\beta}, \nabla_{\beta})\) over \( U_{\alpha\beta} = U_{\alpha} \cap U_{\beta} \), one uses the following lemma due to Deligne and Illusie (see [6]):

**Lemma A.1.** There exist \( \mathcal{O}_{U'_{\alpha\beta}} \)-linear maps \( h_{\alpha\beta} : \Omega^{1}_{U'_{\alpha\beta}} \to (F^{*}_{U'_{\alpha\beta}})_{*}\mathcal{O}_{\alpha\beta} \) such that

1. for all \( \alpha, \beta \) we have
   \[
   \zeta^{\text{ad}}_{\alpha} - \zeta^{\text{ad}}_{\beta} = dh_{\alpha\beta},
   \]
2. for all \( \alpha, \beta, \gamma \) we have over \( U_{\alpha\beta\gamma} = U_{\alpha} \cap U_{\beta} \cap U_{\gamma} \),
   \[
   h_{\alpha\beta} + h_{\beta\gamma} = h_{\alpha\gamma}.
   \]

Let \( h'_{\alpha\beta} : F^{*}_{U'_{\alpha\beta}} \Omega^{1}_{U'_{\alpha\beta}} \to \mathcal{O}_{\alpha\beta} \) be adjoint to \( h_{\alpha\beta} \). Now we define gluing maps

\[
g_{\alpha\beta} : V_{\alpha}|_{U_{\alpha\beta}} \to V_{\beta}|_{U_{\alpha\beta}}
\]

using

\[
h'_{\alpha\beta}(F^{*}\theta|_{U_{\alpha\beta}}) : F^{*}\mathcal{E}|_{U_{\alpha\beta}} \to F^{*}\mathcal{E}|_{U_{\alpha\beta}} \otimes F^{*}\Omega^{1}_{U'_{\alpha\beta}} \to F^{*}\mathcal{E}|_{U_{\alpha\beta}}
\]

by setting

\[
g_{\alpha\beta} := \exp(h_{\alpha\beta}(F^{*}\theta|_{U_{\alpha\beta}})) = \sum_{i=0}^{p-1} \frac{(h_{\alpha\beta}(F^{*}\theta|_{U_{\alpha\beta}}))^{i}}{i!}.
\]

Note that the maps \( g_{\alpha\beta} \) allow us to glue \((V_{\alpha}, \nabla_{\alpha})\) and \((V_{\beta}, \nabla_{\beta})\) over \( U_{\alpha\beta} \) to a global object \((\mathcal{V}, \nabla) \in \text{MIC}_{p-1}(X/k)\).
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