Canonical transformations of the extended phase space, Toda lattices and Stäckel family of integrable systems.

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Abstract

We consider compositions of the transformations of the time variable and canonical transformations of the other coordinates, which map completely integrable system into other completely integrable system. Change of the time gives rise to transformations of the integrals of motion and the Lax pairs, transformations of the corresponding spectral curves and $R$-matrices. As an example, we consider canonical transformations of the extended phase space for the Toda lattices and the Stäckel systems.
1 Introduction

It is well known, in classical mechanics any canonical transformation of variables maps a given integrable system into other integrable system. In this paper we consider compositions of the change of the time variable $t$ and canonical transformations of the other coordinates, which map completely integrable system into other completely integrable system.

On the $2n$-dimensional symplectic manifold $M$ (phase space) with coordinates $\{p_j, q_j\}_{j=1}^n$ let us consider hamiltonian system determined by the Hamilton function $H(p, q)$. By definition canonical transformation of the phase space $M$ have to preserve canonical form of the Hamilton equations

$$\dot{q}_i = \frac{\partial H(p, q)}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H(p, q)}{\partial q_i}. \quad (1.1)$$

As sequence, the action integral

$$S = \int \left( \sum_{i=1}^{n} p_i dq_i - H \, dt \right),$$

differential form $\sum_{i=1}^{n} p_i dq_i$ on $M$ and the Poisson brackets on $M$ are invariant with respect to the canonical transformations of the phase space $M$.

At the Hamilton equations (1.1) and at the canonical transformations of $M$ the time $t$ plays a role of parameter. If we want to consider change of the time $t$, we have to add new coordinate $q_{n+1} = t$ with the corresponding momenta $p_{n+1} = H$ to the phase space $M \cup \{t\}$. The resulting $2n + 2$-dimensional space $M_E$ is so-called extended phase space of the hamiltonian system.

Canonical functional $S$ on $M_E$ has the following completely symmetric form

$$S = \int_{r_1}^{r_2} \sum_{i=1}^{n+1} p_i q'_i \, d\tau.$$

On the extended phase space $M_E$ the Jacobi, Euler-Lagrange and Hamilton variational principles $\delta S = 0$ are differed by an additional constraint

$$\mathcal{H}(p_1, \ldots, p_{n+1}; q_1, \ldots, q_{n+1}) = 0. \quad (1.2)$$

Here $\mathcal{H}$ is called generalised Hamilton function \cite{1}. According by (1.2), any hamiltonian system is a conservative system on the extended phase space $M_E$.

As an example, the Hamilton principle with generalised hamiltonian

$$\mathcal{H} = p_{n+1} - E = H - E$$

gives rise to the Hamilton equations on the extended phase space $M_E$

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}, \quad i = 1, \ldots, n + 1. \quad (1.3)$$

On zero-valued energy surface $\mathcal{H} = 0$ these equations are the initial Hamilton equations (1.1) on the phase space $M$ and two additional equations

$$\dot{i} = 1, \quad \dot{E} = \frac{\partial H}{\partial t}. \quad (1.4)$$
Below we shall consider conservative hamiltonian systems at $\partial H/\partial t = 0$ only.

By definition canonical transformations of the extended phase space $\mathcal{M}_E$ preserve canonical form of the Hamilton equations (1.3) or the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H = 0.$$  \hspace{1cm} (1.4)

Invariance of these equations leads to the invariance of the canonical differential form $\sum_{i=1}^{n+1} p_i dq_i$ with respect to canonical transformations of the space $\mathcal{M}_E$. Thus, change of the time variable

$$t \mapsto \tilde{t}, \quad d\tilde{t} = v(p,q) \, dt$$  \hspace{1cm} (1.5)

and transformation of the corresponding momenta $H$

$$H \mapsto \tilde{H} = \frac{H}{v(p,q)}$$  \hspace{1cm} (1.6)

are canonical transformation of the extended phase space. Here $H$ and $\tilde{H}$ are variables of the extended phase space, but simultaneously they may be considered as the functions on the initial phase space $\mathcal{M}$. In general relativity function $v(p,q)$ (1.5) is so-called lapse function, which determines transformation from physical time to coordinate time $\tilde{t}$.

Canonical transformations of the extended phase space $\mathcal{M}_E$ may be defined for any hamiltonian system. However, among all the hamiltonian systems a set of the completely integrable hamiltonian systems attracts a special attention. It is known, this set is invariant by canonical transformations of the initial phase space $\mathcal{M}$. In this paper, we want to discuss the following

**Problem:** *How to construct canonical transformation of the extended phase space $\mathcal{M}_E$, which maps a given integrable system into the other integrable system.*

An exact technical definition of a completely integrable system is provided by the well-known Liouville theorem. Briefly, such system possesses $n$ functionally independent integrals $\{I_j\}_{j=1}^n$ of motion in the involution

$$\{I_j, I_k\} = 0, \quad j, k = 1, \ldots, n, \quad H = I_1.$$  \hspace{1cm}

Thus, to construct canonical transformation of $\mathcal{M}_E$ preserving integrability, we have to explicitly determine new algebra $\tilde{A}_I$ of integrals by using a given algebra $A_I$ of integrals of motion.

For an integrable system we can consider dynamics with respect to the time $t_j$ conjugated to the integral $I_j$. Thus, in generic, we can add to the phase space $\mathcal{M}$ all the integrals of motion $\{I_j\}$ with the conjugated times $\{t_j\}$. The geometry of the $4n$-dimensional extended phase space $\mathcal{M}_I = \mathcal{M} \oplus \{I_j, t_j\}$ may be used to study of the transformations of the integrals of motion and conjugated to them times $t_j$.

By the Liouville theorem we can introduce the action-angle variables $s_1, \ldots, s_n$ and $\varphi_1, \ldots, \varphi_n$ in the vicinity of the common level surface of integrals

$$\mathcal{M}_\alpha = \{z \in \mathcal{M} : I_j(z) = \alpha_j, \ j = 1, \ldots, n.\}$$

Transformation of the time $t$ (1.5) acts on the half of the equations of motion only

$$\frac{d\varphi_j}{dt} = \omega_j(s_1, \ldots, s_n) \mapsto \frac{d\varphi_j}{d\tilde{t}} = \frac{\omega_j(s_1, \ldots, s_n)}{v(t, s, \varphi)},$$
whereas other equations are invariant

\[
\frac{ds_j}{dt} = 0 \quad \mapsto \quad \frac{ds_j}{\tilde{t}} = 0.
\]

If the lapse function \( v(p(s, \varphi), q(s, \varphi)) \) depends on the action variables only

\[
v(s, \varphi) = v(s),
\]

change of the time (1.3) preserves canonical form of the equations of motion in the action-angle variables

\[
\frac{ds_j}{dt} = 0, \quad \frac{d\varphi_j}{dt} = \tilde{\omega}_j(s_1, \ldots, s_n) = \frac{\omega_j(s_1, \ldots, s_n)}{v(s_1, \ldots, s_n)}.
\] (1.7)

Moreover, at \( v = v(s) \) canonical transformations (1.5-1.6) of the extended phase space \( \mathcal{M}_E \) map a given integrable system into other integrable systems. In this case mapping (1.6) determines factorization of the initial Hamiltonian

\[
H(s) = v(s) \cdot \tilde{H}(s)
\] (1.8)

up to canonical transformations of the action-angles variables \( \{s, \varphi\} \).

Thus, by the Liouville theorem we could construct canonical transformations of the extended phase space, which map a given integrable system into other integrable systems. However, it may of course be quite difficult to construct the action-angles variables for a given mechanical system, even if it is known to be completely integrable. Below we shall construct canonical transformation (1.5-1.6) in the physical variables only.

Change of the time preserving integrability induces transformation of all the machinery developed for integrable systems. Let us recall that the key idea which has started the modern age in the study of classical integrable systems is to bring them into Lax form

\[
\{H, L\} = [L, M],
\]

see reviews [13, 10]. The first matrix \( L \) or, more precisely, the coefficients of its characteristic polynomial \( P(\lambda) = \det(L - \lambda) \) determine integrals of motion. Change of the time gives rise to an algebraic transformation of the Lax matrices, a geometric transformation of the corresponding algebraic curves and transformations of the separation of variables methods. By using all the possible algebraic and geometric transformations, we can try to construct canonical transformations of the extended phase space preserving integrability.

Transformations of the time variable and extensions of the phase space have been discussed many times in classical and quantum mechanics. The wittingly incomplete list includes:
- solution of the equations of motion, as example for the Kowalewski top [9];
- the Kowalewski-Painlevé analysis [19];
- study of singularities of solutions equations of motion [14];
- geometrical theory of integrable system [11, 23];
- qualitative theory of dynamical systems [3, 8];
- construction of the bi-Hamiltonian [1] and quasi bi-Hamiltonian systems [24];
- the Birman-Schwinger formalism in quantum mechanics [17].
As a first example, we consider integrable system determined by a natural Hamiltonian

\[ H(p, q) = H_0 + V = \sum g_{ij} p_i p_j + V(q_1, \ldots, q_n), \]

where \( H_0(p) \) and \( V(q) \) are kinetic and potential parts of the Hamilton function. For any given energy \( E \) of the system we can use the Hamiltonian \( \mathcal{H} = H - E \) to represent dynamics, provided that we impose the constraint \( \mathcal{H} = 0 \) \([11, 23]\). The passage to the geometric representation is accomplished by the Jacobi time transformation at

\[ v(q) = E - V(q) \]

such that

\[ dt = \frac{d\tilde{t}}{E - V(q)}, \]

which maps orbits of an energy surface \( \mathcal{H} = 0 \) into geodesics of the Jacobi geometry, i.e. into the orbits of the following Hamiltonian

\[ \tilde{H} = \frac{H_0}{E - V(q)} = \sum \tilde{g}_{ij} p_i p_j, \quad \tilde{g}_{ij} = (E - V)g_{ij}. \]

The new Hamiltonian \( \tilde{H} \) will then give the same equations of motion on the surface \( \tilde{H} = 1 \). Here for a positive-defined kinetic energy \( H_0 \) the term \((E - V)\) is always non-negative in the physically allowed region. A modern discussion of this canonical time transformation may found in \([24]\).

Of course, the geodesic motion may be replaced by other systems. For instance, let us consider two-dimensional oscillator

\[ H = H_0 + a V(q) = (p_1^2 + p_2^2 + b) + a(q_1^2 + q_2^2). \]

In this case free Hamiltonian \( H_0(p, q) \) does not pure kinetic part of \( H \) \([1.9]\). This choice may be considered as a shift of the energy surface \( \mathcal{H} = -b \). However, it is more convenient to consider the free Hamiltonian \( H_0 \) as a sum of the kinetic energy and the constant potential \( U(q) = b \). For this system the Kepler canonical transformation of the time variable \((1.6)\) with the function

\[ v(p, q) = V(q) = q_1^2 + q_2^2 \]

preserves integrability. This change of the time \( t \) has been known by Kepler, the corresponding transformation of the Hamilton function has been studied by Levi-Civita \([12]\) and extended in \([14]\).

After change of the time \((1.6)\) and the point canonical transformation of the other variables \((p_1, q_1, p_2, q_2) \rightarrow (p_x, x, p_y, y)\) the orbits of the oscillator maps into the orbits of the Kepler problem

\[ \tilde{H} = \frac{H_0(p, q)}{v(q)} + a = p_x^2 + p_y^2 + \frac{b}{\sqrt{x^2 + y^2}} + a. \]

Both these systems are degenerate Stäckel systems in plane. We can construct at least four different Lax matrices for them. According to \([27]\), the Kepler change of the time induces the following transformation of the Lax matrices

\[ L(\lambda) \mapsto \tilde{L}(\lambda) = L(\lambda) \begin{pmatrix} 0 & 0 \\ \tilde{H} & 0 \end{pmatrix}. \]
The Lax matrix $L(\lambda)$ associated with the oscillator is defined on the initial phase space $\mathcal{M}$. The Lax matrix $\tilde{L}(\lambda)$ associated with the Kepler problem is defined on the extended phase space $\mathcal{M}_E$. This transformation may be considered as a shift of the initial Lax matrix $L(\lambda)$ on the matrix from the extended phase space $\mathcal{M}_E$. Additional term in (1.12) is the constant matrix with respect to the new time.

Here both Hamiltonians $H$ and $\tilde{H}$ linearly depend on parameters $a$ and $b$. Let the origin integrable Hamiltonian $H(p, q; a_1, \ldots, a_k)$ linearly depends on arbitrary parameters $(a_1, \ldots, a_k)$. Then the function $v(p, q)$ (1.6) may be founded by using additional propositions, for instance, that other integrals of motion are polynomials on parameters $[8]$. In this case the passage to the new time may be considered as a coupling constant metamorphosis acting on the one of the constant $a_j [8]$. The similar additional propositions Kepler used by investigations of the celestial mechanics laws [3].

It is known, in quantum mechanics, the eigenvalue problem of the Hamiltonian operator $H$

$$H(p, q)\Psi = (H_0 + a V + b)\Psi = E\Psi,$$

may be joined with the eigenvalue problem of the charge operator $a$

$$\tilde{H}(p, q)\tilde{\Psi} = (\tilde{H}_0 + (b - E)V^{-1})\tilde{\Psi} = -a\tilde{\Psi}.$$

In quantum mechanics, such duality of the two eigenvalue problems has been used by Fok [3], Schrödinger [21] and many other. In the Birman-Schwinger formalism function $v(q)$ is called a ”sandwich” potential [7]. The canonical transformation of the time variable (1.6) is an analog of this duality. Recall, in quantum mechanics zeroes of the potential $v(p, q) = V(q)$ and signs of the parameters $a$ and $b$ determine compactness or non-compactness of the Sturm operator $\tilde{H}$ in the corresponding space [7]. In the classical mechanics zeroes of the potential $v(p, q) = V(q)$ and signs of the parameters $a$ and $b$ determine behavior of the system with respect to the inversion of the time [3].

Now let us consider an action of the canonical transformations of the extended phase space on the algebra of integrals of motion. Let the phase space of the initial system may be modelled on the dual space of certain Lie algebra $\mathfrak{g}$. It is known, the set of integrals $\{I_j\}$ gives rise to the subalgebra $\mathcal{A}_I$ in the corresponding universal enveloping algebra $U(\mathfrak{g}^*)$. A classical system is called integrable if the commutant $\mathcal{A}_I$ of the Hamiltonian $H$ in $U(\mathfrak{g}^*)$ contains an abelian subalgebra of the necessary rank. Of course, different abelian subalgebras in $U(\mathfrak{g}^*)$ determine different integrable systems on $\mathfrak{g}$*. However, the modern representation theory does not allows us to construct and classify all the possible abelian subalgebras in $U(\mathfrak{g}^*)$.

The search of the canonical transformations (1.6) may be reformulated as the construction of the abelian subalgebra in $U(\mathfrak{g}^*)$ by using information about the known algebra of integrals $\mathcal{A}_I$. For this purpose algebra $\mathcal{A}_I$ is too little, but algebra $U(\mathfrak{g}^*)$ is too much. By using some additional propositions, we want to extend the first algebra (or restrict the second algebra).

Let the integrals of motions be either the fixed highest order polynomials in momenta, either polynomials in parameters, such that

$$I_j = \sum_{i,k} I_{jk}^i p_k^i, \quad \tilde{I}_j = \sum_{i,k} \tilde{I}_{jk}^i a_k^i. \quad (1.13)$$
In the first case instead of origin algebra $\mathcal{A}_I$ we can introduce new algebra $\mathcal{A}^{p,a}_I \supset \mathcal{A}_I$ generated by momenta and corresponding coefficients $I_j^{ik}$ in (1.13). In the second case new algebra $\mathcal{A}^{\tilde{p},a}_I \supset \mathcal{A}_I$ is generated by parameters and coefficients $\tilde{I}_j^{ik}$. Expansions (1.13) give rise to grading of the new algebras $\mathcal{A}^{\tilde{p},a}_I$ by power of momenta or parameters.

By the Liouville theorem initial system is integrable, if the algebras $\mathcal{A}^{p,a}_I$ contain the abelian subalgebras of the necessary rank. If one of the elements $v(p,q) = I_j^{ik}$ or $v(p,q) = \tilde{I}_j^{ik}$ is invertible in a given representation, we can introduce a completed algebra $\mathcal{J}^{p,a}_I = \{v^{-1}, \mathcal{A}^{p,a}_I\}$, such that rank $\mathcal{J}^{p,a}_I \geq \text{rank} \mathcal{A}_I$. The search of the two abelian subalgebras in the algebra $\mathcal{J}^{p,a}_I$ with elements $H$ and $\tilde{H}$ (1.6) is equivalent to construction of the canonical transformation (1.7).

Next we can consider two integrable systems on a common phase space $\mathcal{M}$. By using two algebras of integrals $\mathcal{A}_I$ and $\mathcal{A}_J$ we can construct new algebra $\mathcal{A}_{IJ} = \mathcal{A}_I \oplus \mathcal{A}_J$ or $\mathcal{A}_{IJ} = \mathcal{A}_I \otimes \mathcal{A}_J$. Of course, algebra $\mathcal{A}_{IJ}$ should be much richer than $\mathcal{A}_I$ and $\mathcal{A}_J$, separately. This algebra has two abelian subalgebras, which associate with two initial integrable systems.

If one of the elements $v(p,q) = I_j$ or $v(p,q) = J_j$ is invertible, we can introduce a completed algebra $\mathcal{A}_{IJ} = \{v^{-1}, \mathcal{A}_{IJ}\}$ such that rank $\mathcal{A}_{IJ} \geq \text{rank} \mathcal{A}_I$. Now, existence of the third abelian subalgebra in $\mathcal{A}_{IJ}$ with elements $\tilde{H}$ (1.6) is provided existence of the canonical transformation (1.7).

Obviously, initial algebras $\mathcal{A}_I$ and $\mathcal{A}_J$ contain all the necessary information about the completed algebras $\mathcal{J}^{p,a}_I$ or $\mathcal{J}_{IJ}$. Thus, we have to extract this information only. As an example of the solution of such algebraic problems, we consider integrable systems related by canonical transformations with the Toda lattices and the Stäckel systems.

In the second Section we propose canonical change of the time for the Toda lattices similar to the Kepler change of the time (1.11). Let us rewrite the Hamiltonian of the Toda lattice as a sum of the free Hamiltonian $H_0$ and potential $V_\beta$

$$H = H_0 + a_\beta \cdot V_\beta = \left( \sum_{\alpha \in P, \alpha \neq \beta} a_\alpha \cdot e^{\alpha(q)} \right) + a_\beta e^{\beta(q)}, \quad a_\alpha, \ a_\beta \in \mathbb{R},$$

where $P$ is a system of simple roots and $\beta$ is one of these roots. Here free Hamiltonian $H_0$ is a sum of the kinetic part with the nontrivial potential. Similar to the Kepler transformation (1.11), change of the time (1.5,1.6) preserves integrability at

$$v(q) = V_\beta = e^{\beta(q)} \quad \beta \in P.$$  \hspace{1cm} (1.14)

The corresponding mapping of the Lax matrices looks like (1.12). To construct these new integrable systems we shall use other automorphisms of the $A_1$ subalgebras in the grading simple Lie algebra $g$ associated to the Toda lattice.

In the third Section we consider some pair of the Stäckel systems with a common Stäckel matrix, defined by two families of integrals $\{I_k\}_{k=1}^n$ and $\{J_k\}_{k=1}^n$. In this case ratio of any two integrals $\tilde{H} = I_m/J_m$ (1.9) and the inner product $(I \wedge J)$ of the vectors $I$ and $J$ on $\mathbb{R}^n$ may be used to determine of the third integrable set of integrals of motion. To construct corresponding change of the time (1.6) we explicitly used the Jacobi ideas in the separation of variables method. Recall, Jacobi proposed to construct various classes of completely integrable systems starting from a set of separated one-dimensional problems.
2 The Toda lattices.

Before proceeding further, it is useful to recall some known facts about generalised Toda lattices (all details may be founded in review [18]).

Let $\mathfrak{g}$ be a real, split, simple Lie algebra rank $\mathfrak{g} = n$. Let $K(,)\,$ be its Killing form, let $\mathfrak{a}$ be a split Cartan subalgebra, $\Delta$ the associated root system, $\Delta_+$ the set of positive roots and $P$ the system of simple roots. For $\alpha \in \Delta_+$, let $\mathfrak{g}_\alpha$ be the corresponding root space and $e_\alpha \in \mathfrak{g}_\alpha$ a root vector. It will be convenient to normalize $e_\alpha$ in such a way that $K(e_\alpha, e_{-\alpha}) = 1$.

The root space decomposition, $\mathfrak{g} = \mathfrak{a} + \bigoplus \mathfrak{g}_\alpha$, gives rise a natural grading on $\mathfrak{g}$, the so-called principal grading. Let $\mathfrak{g}_+ = \bigoplus_{\alpha \geq 0} \mathfrak{g}_\alpha$ be a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{g}_-$ be the opposite nilpotent subalgebra. The $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$ is the generalised Gauss decomposition for simple Lie algebra, which induces a dual decomposition $\mathfrak{g}^* = \mathfrak{g}_+^* + \mathfrak{g}_-^*$ and an injection $i = \mathfrak{g}_+^* \hookrightarrow \mathfrak{g}^*$. Let $\mathcal{O}_a \subset \mathfrak{g}_+^*$ denote the coadjoint orbit of some $a \in \mathfrak{g}_+^*$. The image $i(\mathcal{O}_a) \in \mathfrak{g}^*$ has, in general, nothing to do with the coadjoint orbit $\mathcal{O}_{i(a)}$ of $i(a)$ in $\mathfrak{g}^*$. So $i^*$ of the Casimir function $\phi$ on $\mathfrak{g}^*$ can be a complicated, non-constant function on $\mathcal{O}_a$. The Adler-Kostant-Symes theorem claims that any two such functions $i^*\phi_1$ and $i^*\phi_2$ are in involution with respect to the natural symplectic structure on $\mathcal{O}_a$.

The Killing form $K$ allows us to identify $\mathfrak{g}_i^*$ with $\mathfrak{g}_{-i}$, so that

$$\mathfrak{g}_i^* = \oplus_{i \leq 0} \mathfrak{g}_i, \quad \mathfrak{g}_- = \oplus_{i > 0} \mathfrak{g}_i.$$ 

Choose a vector

$$a = \sum_{\alpha \in P} a_\alpha e_{-\alpha}, \quad a_\alpha \in \mathbb{R}$$

and let $\mathcal{O}_a$ be the $\mathfrak{g}_+$ orbit of $a$ in $\mathfrak{a} + \mathfrak{g}_{-1}$. The points of $\mathcal{O}_a$ have the form

$$\xi = p + \sum_{\alpha \in P} c_\alpha a_\alpha e_{-\alpha}, \quad p \in \mathfrak{a}, \quad c_\alpha > 0.$$ 

The orbit $\mathcal{O}_a$ does not really depends on the $a_\alpha$’s but only on their signs.

It is convenient to introduce the variable $q \in \mathfrak{a}$ such that $c_\alpha = \exp \alpha(q)$. Fix a basis $\{h_j\}$ in $\mathfrak{a}$, and let $\{h'_j\}$ be the dual basis with respect to the Killing form $K(h_i, h'_j) = \delta_{ij}$. Set

$$q_j = K(q, h_j), \quad p_j = K(p, h'_j), \quad p, q \in \mathfrak{a}.$$ 

Then

$$\{p_i, p_j\} = \{q_i, q_j\} = 0, \quad \{p_i, q_j\} = \delta_{ij},$$

i.e. the variables $\{p_j, q_j\}$ are canonical. The orbit $\mathcal{O}_a$ is parametrized by the canonical variables as follows

$$\xi = \sum_{i=1}^n p_i h_i + \sum_{\alpha \in P} a_\alpha \cdot \exp \left( \sum_{i=1}^n q_i K(\alpha, h'_i) \right) \cdot e_{-\alpha}.$$ 

Replace the inclusion $i = \mathfrak{g}_+^* \hookrightarrow \mathfrak{g}^*$ by its shifted version $i + e$, where $e$ is an element of $\mathfrak{g}^*$ which annihilates $[\mathfrak{g}_+, \mathfrak{g}_-]$. For the Toda lattice the orbit $\mathcal{O}_a$ is an orbit of $\mathfrak{g}_+$, what we are really interested in the orbits of the full algebra $\mathfrak{g}_R = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. Let us translate $\mathcal{O}_a$ by adding to it a constant vector $e = \sum_{\alpha \in P} e_\alpha$ which may be regarded as a one-point orbit of $\mathfrak{g}_-$. The resulting orbit

$$\mathcal{O}_{ae} = \mathcal{O}_a + e$$

(2.1)
is parametrized by the canonical variables as follows
\[
L = \sum_{i=1}^{n} p_i h_i + \sum_{\alpha \in P} a_\alpha \cdot \exp K(\alpha, q) \cdot e_{-\alpha} + \sum_{\alpha \in P} e_\alpha .
\] (2.2)
Let us consider the simplest Hamiltonian on \( O_{ae} \) generated by the Killing form on \( \mathfrak{g} \)
\[
H(X) = \frac{1}{2} K(X, X) .
\] (2.3)
Its restriction to \( O_{ae} \) is given by
\[
H(p, q) = \frac{1}{2} K(p, p) + \sum_{\alpha \in P} a_\alpha e^{\alpha(q)} .
\] (2.4)
To keep in mind consequent canonical transformation of the time, we choose the second Lax matrix at the following special form
\[
\frac{d}{dt} L = [L, L_-], \quad L_- = - \sum_{\alpha \in P} a_\alpha \cdot \exp K(\alpha, q) \cdot e_{-\alpha} .
\] (2.5)
Any finite-dimensional linear representation of \( \mathfrak{g} \) gives rise to a matrix valued function \( L \) on \( \mathcal{M} \), the coefficients of its characteristic polynomial are integrals of motion.

The behavior of the dynamical system with the Hamilton function (2.4) depends crucially on the signs \( a_\alpha \). The associated hamiltonian flow is complete if and only if \( a_\alpha \geq 0 \). In this case, its solution is reduced to the generalised Gauss decomposition in the corresponding Lie group. Systems with other signs of \( a_\alpha \) are of course also solvable, but their solutions blow up \[18\].

Now let us construct canonical change of the time variable for these Toda lattices. Recall, in a shifted version of the Adler-Kostant-Symes scheme in order to get orbit \( O_{ae} \) in \( \mathcal{M} \simeq \mathfrak{g}_R^* = \mathfrak{g}_+^* \oplus \mathfrak{g}_-^* \) we translate orbit \( O_a \) living in \( \mathfrak{g}_+^* \) by adding to it a constant vector \( e \) from the remaining part of \( \mathcal{M} \). Let us replace the phase space \( \mathcal{M} \) on the extended phase space \( \mathcal{M}_E \). In this case we can also translate the same orbit \( O_a \) in \( \mathfrak{g}_+^* \) by adding to it a constant vector from the remaining part of the whole space \( \mathcal{M}_E \). As above, this vector has to be a character and a constant with respect to the new time. The third condition is that the initial invariant polynomial (2.3) have to generate the coupling constant
\[
K(\bar{L}, \bar{L}) = -b
\]
instead of the Hamiltonian \( \bar{H} \), as for the Kepler problem [27].

**Proposition 1** For each simple root \( \beta \in P \) and for any constant \( b_\beta \in \mathbb{R} \) the following canonical transformation of the extended phase space \( \mathcal{M}_E \)
\[
d\bar{t} = e^{\beta(q)} \cdot dt ,
\] (2.6)
\[
\bar{H}_\beta = e^{-\beta(q)} \cdot \left( H + b_\beta \right)
\]
maps the Toda lattice into the other integrable system. This canonical transformation induces
the following transformation of the Lax matrices

\[ \tilde{L}_\beta = L - \tilde{H}_\beta \cdot \frac{e_\beta}{a_\beta}, \quad \tilde{L}_- = e^{-\beta(q)} \cdot L_- \]  

(2.7)
such that

\[ \{ \tilde{H}, \tilde{L} \} = [\tilde{L}, \tilde{L}_-] . \]  

(2.8)

Here \( H, L \) and \( L_- \) are the Hamiltonian (2.4) and the Lax matrices (2.2, 2.5) for the corre-
sponding Toda lattice.

As for the Toda lattices, restrictions of the invariant function on the orbit (2.7) give rise
to the new set of integrals of motion.

There are \( n = \text{rank} \, g \) functionally independent invariant polynomials on \( g \). Restricted to
the orbit \( O_{ae} \) they remain functionally independent and give rise to integral of motion for the
Toda lattice. In (2.7) we translate \( O_{ae} \) by adding to it a "constant" vector proportional to the
element of the universal enveloping algebra. All the invariant polynomials are invariant with
respect to this transformation. Thus, we can construct \( n \) independent integrals of motion in
the involution for the system with the Hamiltonian (2.6).

The number of the functional independent Hamilton functions \( \tilde{H}_\beta, \beta \in P \) depends on the
symmetries of the associated root system. For the closed Toda lattices associated with the
affine Lie algebras canonical time transformation has the similar form. The associated with
the Lax matrices \( L \) (2.2) and \( \tilde{L} \) (2.7) spectral curves depend on the choice of a representa-
tion of \( g \). Therefore, geometric transformations of the spectral curves will be below considered
on the some examples only.

Of course, canonical transformations (2.6) induce transformations of all the machinery
developed for the Toda lattices. As an example, for the Toda lattices associated to the
classical infinite series of the root systems \( A_n, B_n, C_n \) and \( D_n \) we can introduce another
2 \( \times \) 2 Lax matrices [5, 10]. For brevity, here we restrict ourselves the \( A_n \) root systems only.
In this case, the second Lax pair representation has the following form

\[ T = T_1 T_2 \cdots T_n , \quad \text{where} \quad T_j = \begin{pmatrix} \lambda - p_j & a_j e^{q_j} \\ -e^{-q_j} & 0 \end{pmatrix} , \quad a_j \in \mathbb{R} , \]
such that

\[ \{ H, T_j \} = T_j A_j - A_{j-1} T_j , \quad A_j = \begin{pmatrix} \lambda & a_j e^{q_j} \\ -e^{-q_j} & 0 \end{pmatrix} . \]  

(2.9)

Change of the time (1, 2, 6) associated with the root \( \beta = \varepsilon_j - \varepsilon_{j+1} \) by

\[ v(q) = e^{-\beta(q)} = \exp(q_{j+1} - q_j) \]
gives rise to the following transformation of the Lax matrices

\[ T = T_1 T_2 \cdots T_n \quad \mapsto \quad \tilde{T} = T_1 \cdots T_{j-1} \cdot \left[ T_j T_{j+1} + \begin{pmatrix} H + b & 0 \\ 0 & 0 \end{pmatrix} \right] \cdot T_{j+2} \cdots T_n , \]  

(2.10)

the Lax equations

\[ \{ H, T \} = [T, A_n] \quad \mapsto \quad \{ \tilde{H}, \tilde{T} \} = [\tilde{T}, v(q) A_n] , \]
and the spectral data of the Lax matrices

\[
\det T = \prod_{k=1}^{n} a_k, \quad \text{tr} T = \lambda^n + \lambda^{n-1} P + \lambda^{n-2} \left( \frac{P^2}{2} - H \right) + \ldots ,
\]

\[
\det \tilde{T} = (a_j - \tilde{H}) \cdot \prod_{k \neq j}^{n} a_k , \quad \text{tr} \tilde{T} = \lambda^n + \lambda^{n-1} P + \lambda^{n-2} \left( \frac{P^2}{2} + b \right) + \ldots .
\]

All these transformations look like corresponding transformations for the Stäckel systems [27, 28].

Let us consider some heuristic arguments concerning the Lax matrix transformation (2.8). Recall, constructions of the orbit \( O_{ae} \) and the corresponding classical \( R \)-matrix are closely related to the principal grading of \( \mathfrak{g} \). This grading defines all the possible embedding of the three-dimensional subalgebra \( A_1 \simeq sl(2) \) into \( \mathfrak{g} \).

Let \( \{ e_-, e_+, h \} \) be generators of the Lie algebra \( sl(2) \)

\[
[h, e_-] = e_-, \quad [h, e_+] = -e_+, \quad [e_-, e_+] = 2h, \quad (2.11)
\]

and the element

\[
\Delta = h^2 + \frac{1}{2}(e_-e_+ + e_+e_-) \quad (2.12)
\]

of the universal enveloping algebra be Laplace operator in \( SL(2) \). Let us consider infinite-dimensional irreducible representation \( \mathcal{W} \) of the Lie algebra \( sl(2) \) in the linear space \( V \) such that

\[
\mathcal{W} : \{ e_-, e_+, h \} \rightarrow \{ e, f, h \} \in \text{End}(V).
\]

Let operator \( \psi \) be invertible in \( \text{End}(V) \) and \( \varphi(\Delta) \) be arbitrary function on the value of the Casimir operator \( (2.12) \), then the mapping

\[
e_- \rightarrow e'_- = e_-, \quad h \rightarrow h' = h, \quad e_+ \rightarrow e'_+ = e_+ + e_-^{-1} \cdot \varphi(\Delta) \quad (2.13)
\]

is an outer automorphism of the space of infinite-dimensional representations of \( sl(2) \) in \( V \) [20]. These mappings shift the spectrum of the Laplace operator \( \Delta \) \( (2.12) \) by the rule

\[
\Delta \rightarrow \Delta' = \Delta + \varphi(\Delta).
\]

In particular by \( \varphi(\Delta) = -(\Delta + b) \) one gets \( \Delta \rightarrow \Delta' = -b \). So, instead of the spectrum of the Laplace operator \( \Delta \) on the group \( SL(2) \) one gets the spectrum of the coupling constant \( b \).

Now we turn to the Toda lattices related to the Lie algebra \( \mathfrak{g} \), which contains subalgebras \( A_1 \) associated to the roots \( \beta \in P \). The maps \( (2.13) \) may be closely related to the automorphism \( (2.13) \). Instead of the restriction \( \Delta \) \( (2.12) \) of the Casimir operator \( H(X) \) \( (2.3) \) on \( A_1 \) one have to substitute restriction \( H(p, q) \) \( (2.4) \) of the same Casimir operator on the orbit \( O_{ae} \).

Note, the outer automorphism \( (2.13) \) non-trivial acts on the one nilpotent subalgebra of \( sl(2) \) only. Extension of this mapping on the orbit \( O_{ae} \) non-trivial acts on the constant one-point orbit of \( \mathfrak{g}_- \), too.

Of course, the more justified consideration of the canonical transformation of the time variable requires a more advanced technique of representation theory.
2.1 Examples

Let us consider periodic three-particles Toda lattices associated to the affine Lie algebra $\mathcal{L}(A_3)$. The corresponding Hamiltonian $H$ (2.14) reads as

$$H = \frac{1}{2} \left( p_1^2 + p_2^2 + p_3^2 \right) + a_1 e^{q_1 - q_2} + a_2 e^{q_2 - q_3} + a_3 e^{q_3 - q_1},$$

(2.14)

and the Lax matrix $L(\lambda, \mu)$ (2.2) has the following form

$$L(\lambda, \mu) = \begin{pmatrix} \lambda - p_1 & a_1 e^{q_1 - q_2} & \mu^{-1} \\ 1 & \lambda - p_2 & a_2 e^{q_2 - q_3} \\ \mu \cdot a_3 e^{q_3 - q_1} & 1 & \lambda - p_3 \end{pmatrix}.$$  

The Lax equations (2.5) is linearized on the Jacobian of following algebraic hyperelliptic curve

$$C : \quad \det L(\lambda, \mu) = 0, \quad (a_1 a_2 a_3 \mu + \frac{1}{\mu}) + \lambda^3 + \lambda^2 P + \lambda \left( \frac{P^2}{2} - H \right) + K = 0. \quad (2.15)$$

Here complete momenta of the system $P = p_1 + p_2 + p_3$, Hamiltonian $H$ (2.14) and additional integrals of motion $K$ are polynomials of the first, second and third order in momenta, respectively.

The canonical transformation (2.6) maps the origin Hamiltonian (2.14) into the following Hamiltonian

$$\tilde{H} = e^{q_2 - q_1} \cdot (H + b), \quad b \in \mathbb{R}. \quad (2.16)$$

The associated Lax matrix $\tilde{L}$ (2.7) is given by

$$\tilde{L} = L - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{a_1} \tilde{H} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

(2.17)

Its characteristic polynomial is

$$\tilde{C} : \quad \det \tilde{L}(\lambda, \mu) = 0, \quad (a_1 a_2 a_3 \mu + \frac{1}{a_1} \tilde{H}) + \lambda^3 + \lambda^2 P + \lambda \left( \frac{P^2}{2} + b \right) + \tilde{K} = 0. \quad (2.18)$$

Here, in comparison with (2.15), the linear in $\lambda$ term is proportional to the constant $b$ instead of the Hamilton function $H$. The new Hamiltonian $\tilde{H}$ is related to the second spectral parameter now. For the first time similar transformations of the Lax matrices and of their characteristic polynomials have been observed in [27, 28].

The corresponding $2 \times 2$ Lax matrix for the three-particle Toda lattice is equal to

$$T = T_1 T_2 T_3 = \begin{pmatrix} \lambda - p_1 & a_1 e^{q_1} \\ -e^{-q_1} & 0 \end{pmatrix} \begin{pmatrix} \lambda - p_2 & a_2 e^{q_2} \\ -e^{-q_2} & 0 \end{pmatrix} \begin{pmatrix} \lambda - p_3 & a_3 e^{q_3} \\ -e^{-q_3} & 0 \end{pmatrix}. \quad (2.19)$$

(2.19)

Change of the time maps this matrix into the following matrix

$$\tilde{T} = T_1 \cdot \begin{pmatrix} \lambda - p_2 \left( 1 - a_1^{-1} \tilde{H} \right) & a_2 e^{q_2} \\ -e^{-q_2} & 0 \end{pmatrix} \cdot T_3, \quad (2.20)$$

(2.20)
such that
\[ \det T = a_1a_2a_3, \quad \text{tr} T = \lambda^3 + \lambda^2 P + \lambda \left( \frac{P^2}{2} - H \right) + K, \]

\[ \det \tilde{T} = (a_1 - \tilde{H})a_2a_3, \quad \text{tr} \tilde{T} = \lambda^3 + \lambda^2 P + \lambda \left( \frac{P^2}{2} + b \right) + \tilde{K}. \]

If we put \( a_j = 1 \), the Poisson bracket relations for the initial \( 2 \times 2 \) Lax matrices are closed into the Sklyanin quadratic \( r \)-matrix algebra \([5, 10]\)

\[ \{ \frac{1}{T}(\lambda), \frac{2}{T}(\mu) \} = [r(\lambda - \mu), \frac{1}{T}(\lambda)\frac{2}{T}(\mu)], \quad r(\lambda - \mu) = \frac{\Pi}{\lambda - \mu}. \]

Here the standard notations are introduced:
\[ \frac{1}{T}(\lambda) = T(\lambda) \otimes I, \quad \frac{2}{T}(\mu) = I \otimes T(\mu), \]

and \( \Pi \) is the permutation operator of auxiliary spaces \([5]\). Change of the time \((2.16)\) maps the Sklyanin \( r \)-matrix algebra into the following poly-linear algebra

\[ \{ \frac{1}{T}(\lambda), \frac{2}{T}(\mu) \} = [r(\lambda - \mu), \frac{1}{T}(\lambda)\frac{2}{T}(\mu)] + [r_{12}(\lambda, \mu), \frac{1}{T}(\lambda)] + [r_{21}(\lambda, \mu), \frac{2}{T}(\mu)], \]

with dynamical \( r \)-matrix

\[ r_{12}(\lambda, \mu) = A_3(\lambda) \otimes \sigma T_3(\mu), \quad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \]

\[ r_{21}(\lambda, \mu) = -\Pi r_{12}(\mu, \lambda) \Pi. \]

Here \( T_3(\lambda) \) and \( A_3(\lambda) \) were defined in \((2.9)\).

By using complete momenta \( P \), we can reduce initial 6-dimensional phase space to the 4-dimensional space. After this reduction and after the following point canonical transformation of the coordinates

\[ q_1 = \frac{1}{2}(1 + i\sqrt{3}) \ln x + \frac{1}{2}(1 - i\sqrt{3}) \ln y, \]

\[ p_1 = \frac{1}{2}(1 - \frac{i}{\sqrt{3}})x p_x + \frac{1}{2}(1 + \frac{i}{\sqrt{3}})y p_y, \]

\[ q_2 = \frac{1}{2}(-1 + i\sqrt{3}) \ln x - \frac{1}{2}(1 + i\sqrt{3}) \ln y, \]

\[ p_2 = \frac{1}{2}(-1 - \frac{i}{\sqrt{3}})x p_x - \frac{1}{2}(1 - \frac{i}{\sqrt{3}})y p_y, \]

the new Hamiltonian \( \tilde{H} \) reads as

\[ \tilde{H} = p_x p_y + \frac{b}{xy} + a_2 x^{z_1} y^{z_2} + a_3 x^{z_2} y^{z_1} + a_1, \]
where \( z_j \) are roots of the quadratic equation \((z - z_1)(z - z_2) = z^2 + 3z + 3 = 0\). For the first time the system defined by \( \tilde{H}_1 \) (2.20) has been found in [4].

In conclusion we present integrable systems related to the two-particle Toda lattices associated to affine algebras \( X_2^{(1)} \). After an appropriate point transformation of coordinates, all the Hamilton functions have a common form
\[
\tilde{H} = p_x p_y + b \frac{a}{xy} + a x^{z_1} y^{z_2} + c x^{s_1} y^{s_2} + d, \quad a, b, c, d \in \mathbb{R}
\]
where \( z_{1,2} \) and \( s_{1,2} \) be the roots of the different quadratic equations. Below we show these equations only:

- **A\(_3\)\(_{(1)}\):**
  \[
  z^2 + 3z + 3 = 0 \quad s^2 + 3s + 3 = 0
  \]
- **B\(_2\)\(_{(1)}\), C\(_2\)\(_{(1)}\):**
  \[
  z^2 + 4z + 5 = 0 \quad s^2 + 4s + 5 = 0 \\
  z^2 + 2z + 2 = 0 \quad s^2 + 3s + 5/2 = 0
  \]
- **D\(_2\)\(_{(1)}\):**
  \[
  z^2 + 2z + 2 = 0 \quad (s + 2)^2 = 0
  \]
- **G\(_2\)\(_{(1)}\):**
  \[
  z^2 + 2z + 4 = 0 \quad s^2 + 5s + 7 = 0 \\
  z^2 + 2z + 4 = 0 \quad s^2 + 3s + 3 = 0 \\
  z^2 + 3z + 7/3 = 0 \quad s^2 + 3s + 3 = 0
  \]

The corresponding second integrals of motion \( K \) are polynomials third, fourth and sixth order in momenta. Note, for the algebra \( A_3^{(1)} \) all the three Hamiltonians \( H_\beta, \beta \in P \) are equivalent. Two different Hamilton function (2.6) associated with the algebras \( B_2^{(1)}, C_2^{(1)} \) and \( D_2^{(1)} \). For the \( G_2^{(1)} \) algebra we have three different Hamiltonians (2.9).

## 3 The Stäckel systems

In this Section we propose an interesting extension of the integrable family of the Stäckel systems [22]. As an example, we discuss here a family of the two-dimensional integrable systems in detail. In the two limiting cases, the corresponding systems possess the following Hamilton functions
\[
H = p_x^k p_y^k + \alpha (x y)^{-k+1}, \quad \alpha, k \in \mathbb{R}, \quad (3.1)
\]

where \( \alpha \) and \( k \) are arbitrary parameters. At \( k = 1 \) the first Hamiltonian coincides with the Hamiltonian of the Kepler problem. At \( k = 2 \) the second integrable Hamiltonian has been found by Fokas and Lagerstrom [7]. It is known, both these systems are dual to the some Stäckel systems (see review [8] and references therein). By using similar duality we shall prove integrability of the general systems (5.1).

Let variables \( \{p_j, q_j\}_{j=1}^n \) be coordinates in the phase space \( \mathbb{R}^{2n} \) with the standard Poisson brackets
\[
\{p_j, q_k\} = \delta_{jk}, \quad j, k = 1, \ldots, n.
\]
Let us consider two integrable hamiltonian systems on the common phase space $\mathbb{R}^{2n}$. These systems are defined by the two sets of independent integrals of motion $\{I_j\}_{j=1}^n$ and $\{J_j\}_{j=1}^n$, in the involution

$$\{I_j, I_k\} = 0 \quad \text{and} \quad \{J_j, J_k\} = 0, \quad j, k = 1, \ldots, n.$$ 

By the Liouville theorem for given integrable systems we can introduce two family of the action-angle variables $\{s_j, \varphi_j\}_{j=1}^n$ and $\{\tilde{s}_j, \tilde{\varphi}_j\}_{j=1}^n$ for the each of integrable systems. Integrals of motion depend on the action variables only $I_k = I_k(s_1, \ldots, s_n)$, $J_k = J_k(\tilde{s}_1, \ldots, \tilde{s}_n)$.

According to expansion (1.8), if two a’priori different systems of the action coordinates are related

$$\tilde{s}_j = \tilde{s}_j(s_1, \ldots, s_n) \quad (3.2)$$

then the canonical transformations of the extended phase space at

$$v(p, q) = v(s) = I_k(p, q), \quad \text{or} \quad v(p, q) = v(s) = J_k(p, q)$$

map a given pair of integrable system into the third integrable system.

Thus, we have to verify condition (3.2) and have to construct explicitly the third family of integrals of motion in the initial physical variables. For instance, by using inner product of the two independent vectors of integrals $\mathbf{I}$ and $\mathbf{J}$ in $\mathbb{R}^n$ we can introduce some antisymmetric matrix $\mathbf{K} = (\mathbf{I} \otimes \mathbf{J})$. Any column or row of this matrix defines a set of the $n-1$ independent functions

$$K_{ij} = (\mathbf{I} \otimes \mathbf{J})_{ij} = I_i J_j - I_j J_i, \quad i, j = 1, \ldots, n.$$ 

Under some restriction on the initial integrals $\{I_j\}$ and $\{J_j\}$ we could construct from them a third integrable system on the same phase space.

**Proposition 2** If all the differences of integrals of motion $(I_j - J_j)$ with the common index $j = 1, \ldots, n$ are in the involution

$$\left\{I_j - J_j, I_k - J_k\right\} = 0, \quad j, k = 1, \ldots, n, \quad (3.3)$$

then the ratio of integrals

$$K_m = \frac{I_m}{J_m} \quad (3.4)$$

and $n-1$ functions $K_j$, $j \neq m$

$$K_j = \frac{K_{mj}}{J_m} = \frac{I_m J_j - I_j J_m}{J_m} = K_m J_j - I_j, \quad m \neq j = 1, \ldots, n \quad (3.5)$$

are integrals of motion for new integrable system on the same phase space.

By definition all the new integrals $K_m$ and $K_j$ are functionally independent. Thus, the proof is straightforward calculation of the following Poisson brackets

$$\{K_m, K_j\} = \frac{I_m}{J_m I^2} \left(\{I_m, J_j\} + \{J_m, I_j\}\right) = -\frac{I_m}{J_m^2} \left\{I_m - J_m, I_j - J_j\right\} = 0.$$ 

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and
\[
\{K_j, K_k\} = K_m \{J_j, K_k\} - \{I_j, K_k\} = \\
= J_k (K_m \{J_j, K_m\} - \{I_j, K_m\}) - K_m \left( \{J_j, I_k\} + \{I_j, J_k\} \right) = \\
= J_k \left( \{K_j, K_m\} + K_m \left( I_j - J_j, I_k - J_k \right) \right) = 0, \quad j \neq k \neq m
\]
Thus, canonical transformation (3.4-3.5) of the extended phase space 
\((I, J) \rightarrow K\)
preserves the property of integrability. To apply this transformation we have to find two integrable systems satisfying condition (3.3). Below we prove that the Stäckel integrable systems [22] may be considered as the main example of the systems satisfying condition (3.3).

Let us briefly recall some necessary facts about the Stäckel systems [22, 25]. The non-degenerate \(n \times n\) Stäckel matrix \(S\), whose \(j\) column \(s_{kj}\) depends only on \(q_j\)
\[
\det S \neq 0, \quad \frac{\partial s_{kj}}{\partial q_m} = 0, \quad j \neq m
\]
defines functionally independent integrals of motion \(\{I_k\}_{k=1}^n\)
\[
I_k = \sum_{j=1}^n c_{jk} \left( p_j^2 + U_j (q_j) \right), \quad c_{jk} = \frac{S_{kj}}{\det S},
\]
which are quadratic in momenta. Here \(C = [c_{ik}]\) denotes inverse matrix to \(S\) and \(S_{kj}\) be cofactor of the element \(s_{kj}\).

We can see that in practical circumstances the Stäckel approach is not very useful because it is usually unknown what canonical transformation have to be used to transform a Hamiltonian (3.3) to the natural form \(H = T(p_1, \ldots, p_n) + V(q_1, \ldots, q_n)\). This problem was partially solved for the Stäckel systems with a common potential \(U_j = U, j = 1, \ldots, n\) only [25].

According to [27], if the two Stäckel matrices \(S\) and \(\tilde{S}\) be distinguished the \(m\)-th row only
\[
s_{kj} = \tilde{s}_{kj}, \quad k \neq m,
\]
the corresponding Stäckel systems with a common set of potentials \(U_j\) and with the Hamilton functions \(I_m\) and \(\tilde{I}_m\) are related by canonical change of the time
\[
\tilde{I}_m \leftrightarrow I_m, \quad \tilde{I}_m = \frac{I_m (p, q)}{v(q)}.
\]
where
\[
v(q_1, \ldots, q_n) = \frac{\det \tilde{S}(q_1, \ldots, q_n)}{\det S(q_1, \ldots, q_n)}
\]
The canonical transformation (3.7) connects two Stäckel systems with the different Stäckel matrices and with the common set of potentials \(U_j\).
Let us consider a pair of the Stäckel systems with a common Stäckel matrix $S$ and with the different potentials. Namely, in addition to the system with integrals $\{I_k\}$ (3.6), we introduce the second integrable system with the similar integrals of motion

$$J_k = \sum_{j=1}^{n} c_{jk} \left( p_j^2 + W_j(q_j) \right), \quad k = 1, \ldots, n.$$  

(3.8)

Here even one potential $U_j(q_j)$ has to be functionally independent on the corresponding potential $W_j(q_j)$.

**Proposition 3** Any two integrable systems defined by the same Stäckel matrix $S$ and by the functionally independent potentials $U_j(q_j)$ and $W_j(q_j)$ satisfy the necessary condition (3.3) of the previous proposition. Thus, the ratio of the two Stäckel integrable Hamiltonians defines new integrable system

$$K_m \longleftrightarrow (I_m, J_m), \quad K_m = \frac{I_m}{J_m}. \quad (3.9)$$

It is obvious, all the integrals $I_k$ and $J_k$ are differed by the potential part

$$(I_k - J_k) = \sum_{j=1}^{2} c_{jm} \left[ U_j(q_j) - W_j(q_j) \right]$$

depending on coordinates $\{q_j\}$ only. Thus, systems with a common Stäckel matrix $S$ satisfy condition (3.3).

The Hamilton function (3.9) has the following form

$$H = K_m = \frac{\sum_{j=1}^{n} c_{jm} \left[ p_j^2 + U_j(q_j) \right]}{\sum_{j=1}^{n} c_{jm} \left[ p_j^2 + W_j(q_j) \right]}. \quad (3.10)$$

This Hamiltonian $H$ is a rational function in momenta, but next one can try to use canonical transformations to simplify it. In rare case, one obtains again a natural type Hamilton function as will be shown below.

As for the usual Stäckel system, the common level surface of the new integrals (3.4)

$$M_\alpha = \{ z \in \mathbb{R}^{2n} : K_j(z) = \alpha_j, \ j = 1, \ldots, n \}$$

is diffeomorphic to the real torus. Namely, substituting

$$V_j(q_j) = (1 - \alpha_m)^{-1} \left( U_j(q_j) - \alpha_m W_j(q_j) \right),$$

into the definitions (3.10) and (3.6 3.8) we obtain the following equations

$$\sum_{j=1}^{2} c_{jm} \left[ p_j^2 + V_j(q_j) \right] = 0 = \beta_m,$$

$$\sum_{j=1}^{2} c_{jk} \left[ p_j^2 + V_j(q_j) \right] = -\frac{\alpha_k}{1 - \alpha_m} = \beta_k.$$

(3.11)

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After multiplication of these equations by the Stäckel matrix one immediately gets
\[ p_j^2 = \left( \frac{\partial S}{\partial q_j} \right)^2 = \sum_{k=1}^{n} \beta_k s_{kj}(q_j) - V_j(q_j), \]

where \( S(q_1, \ldots, q_n) \) is a reduced action function. The corresponding Hamilton-Jacobi equation on \( M_\alpha \)
\[ \frac{\partial S}{\partial t} + H(t, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}, q_1, \ldots, q_n) = 0, \quad \Rightarrow \quad c_{jm} \frac{\partial S}{\partial q_j} \frac{\partial S}{\partial q_j} = E, \]

admits the variable separation
\[ S(q_1, \ldots, q_n) = \sum_{j=1}^{n} S_j(q_j), \]

where
\[ S_j(q_j) = \int \sqrt{\sum_{k=1}^{n} \beta_k s_{kj}(\lambda) - V_j(\lambda)} \ dq_j. \]

Thus, coordinates \( q_j(t, \alpha_1, \ldots, \alpha_n) \) are determined from the equations
\[ \sum_{j=1}^{n} \int \frac{s_{kj}(\lambda)d\lambda}{\sqrt{\sum_{k=1}^{n} \beta_k s_{kj}(\lambda) - V_j(\lambda)}} = \delta_k, \quad k = 1, \ldots, n, \]

where the constants \( \alpha_j \) and \( \beta_j \) are related by (3.11).

Thus, the solution of the problem is reduced to solving a sequence of one-dimensional problems, which is the essence of the method of separation of variables. Next, the integration problem for equation of motion is reduced to solution of the inverse Jacobi problem in framework of the algebraic geometry [27].

Recall, the first canonical transformation (3.7) may be related to the ambiguity of the Abel map. So, it would be interesting to investigate the underlying algebro-geometric origin of the construction (3.9).

For some Stäckel systems with uniform potentials \( U_j = U_j \), \( j = 1, \ldots, n \) we can construct the \( 2 \times 2 \) Lax matrices [25]. Of course, transformation of the time (3.7) induces transformation of the Lax matrices. As for the Toda lattices, these transformations may be reduced to the adding a constant with respect to the new time matrix [27, 28] to the initial Lax matrix. Transformations of the Lax matrices by new mapping of the time (3.9) will be published elsewhere.

4 Examples

As we have mentioned before, the Hamiltonian \( H \) (3.10) have a rather unusual expression. However, in some cases suitable canonical transformations can reduce it to a sum of the kinetic energy and the potential energy. Note, necessary choice of such transformations is a generic problem for all the Stäckel systems. Thus, in this Section we present several concrete two-dimensional systems only.

Let us consider polar coordinate system on plane with the usual coordinates \((p_r, r)\) and \((p_\phi, \phi)\) instead of \((p_{1,2}, q_{1,2})\), respectively. We take the first system with the axially symmetric potential
\[ I_1 = p_r^2 + \frac{p_\phi^2}{r^2} - a^2 r^{2k} + b, \quad I_2 = p_\phi, \quad a, b, n \in \mathbb{R}. \]
The second system is associated to a free motion
\[ J_1 = p_r^2 + \frac{p_\phi^2}{r^2}, \quad J_2 = p_\phi. \] (4.13)

These systems belong to the Stäckel family of integrable systems with the following common Stäckel matrix
\[ S = \begin{pmatrix} 1 & 0 \\ -r^{-2} & 1 \end{pmatrix} \]
In this case \( s_{12} = e_{21} = 0 \) and \( I_2 - J_2 = 0 \), it allows us to use the second integrals in the non-Stäckel form (3.6).

The ratio (3.4) of the Hamiltonians (4.12-4.13) may be rewritten in the form (3.1) by using the set of canonical transformations.

Let us begin with the usual transformation of the curvilinear coordinates to the cartesian coordinates
\[ r = \sqrt{uv}, \quad \phi = i \arctan \left( \frac{u - v}{u + v} \right), \]
\[ p_r = -\frac{u p_u + v p_v}{r}, \quad p_\phi = i \left( u p_u - v p_v \right). \] (4.14)

Then, one permute coordinates and momenta \((u \leftrightarrow p_u)\) and \((v \leftrightarrow p_v)\) such that new Hamiltonian (3.10) becomes polynomial in momenta
\[ H = \frac{I_1}{J_1} = -\frac{a^2}{4} \frac{p_u^k p_v^k}{u v} + \frac{b}{4 u v} + 1. \] (4.15)

In conclusion, we have to use the point canonical transformation
\[ p_x = p_u \frac{1}{u^{k+1}}, \quad x = (1 + 1/k) \frac{u^k}{u^{k+1}}, \]
\[ p_y = p_v \frac{1}{v^{k+1}}, \quad y = (1 + 1/k) \frac{v^k}{v^{k+1}}, \]
which converts the Hamiltonian (4.13) into the following form
\[ H = p_x^k p_y^k + \alpha (xy)^{-\frac{k}{k+1}} + \beta, \]
after multiplication on a suitable constant and rescaling parameters.

Note, the Stäckel matrix \( S \) and the set of potentials \( U_j(q_j) \) are determined on the half of the phase space \( \mathbb{R}^{2n} \) and depend on coordinates \( q_j \) only. Thus, we have some freedom related to the different canonical transformations of momenta
\[ (p_1, \ldots, p_n) \rightarrow (\tilde{p}_1, \ldots, \tilde{p}_n), \]
\[ p_i - \tilde{p}_i = 2 \frac{\partial F(q_1, \ldots, q_n)}{\partial q_i}, \quad p_i + \tilde{p}_i = 0. \] (4.16)

Here \( F(q_1, \ldots, q_n) \) is a generating function of the transformations (4.16) depending on coordinates \( \{q_j\} \), which are invariant with respect to transformation (4.16).
If condition (3.3) is invariant under these canonical transformations, we can apply them (4.16) to construct new integrable systems by the rule (3.4). Although no general procedure exists for this, one interesting case is known.

As above, one takes system with the axially symmetric potential (4.12) and system associated with a free motion with integrals

\[ J_1 = \tilde{p}_r^2 + \frac{\tilde{p}_\phi^2}{r^2}, \quad J_2 = \tilde{p}_\phi. \] (4.17)

New momenta \((\tilde{p}_r, \tilde{p}_\phi)\) relate with old ones \((p_r, p_\phi)\) by canonical transformation (4.16)

\[ \tilde{p}_r = p_r - \frac{\partial f(r)}{\partial r} \cos(n \phi) \frac{n}{r}, \quad \tilde{p}_\phi = p_\phi + f(r) \sin(n \phi). \] (4.18)

Here \(f(r)\) be any function on variable \(r\) and \(n\) be arbitrary parameters.

Both these systems belong to the Stäckel family of integrable system associated with a common Stäckel matrix. In this case \(I_2 - J_2 \neq 0\) and condition (3.3) do not invariant by transformation (4.18). Let the second integrals be square root from the usual Stäckel integrals (3.6). If this form of the second integrals are used, the pair of the systems (4.12) and (4.17) satisfies condition (3.3) by

\[ \frac{f'(r)}{f(r)} = \frac{n}{r} \Rightarrow f(r) = cr^n, \quad c \in \mathbb{R}. \]

Let us get over to the cartesian coordinates (4.14) and conjugated momenta

\[ \tilde{p}_r = -up_u + vp_v, \quad \tilde{p}_\phi = i(up_u - vp_v). \] (4.19)

After permutation coordinates and momenta \((u \leftrightarrow p_u)\) and \((v \leftrightarrow p_v)\) (analog of the Fourier transformation in the quantum mechanics), one gets

\[ H = \frac{I_1}{J_1} = \frac{1}{4uv} \left( c^2 p_u^{n-1} p_v^{n-1} - d^2 p_u^k p_v^k - 2 c(v p_u^{n-1} + u p_v^{n-1}) \right) + \frac{b}{4uv} + 1. \] (4.20)

At \(c = 0\) we discuss this system before (4.13). Now, we consider the second limiting case by

\[ a = c, \quad k = n - 1, \]

that simplifies potential part of the Hamiltonian (4.20).

As above, the point canonical transformation (4.16) converts the Hamiltonian (4.20) into the following form

\[ H = p_x^k + p_y^k + \alpha (xy)^{-k+1} + \beta, \]

after multiplying on a suitable constant and rescaling parameters.

Thus, we present a family of two-dimensional integrable systems, which includes the Kepler and Fokas-Lagerstrom potentials simultaneously.

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References

[1] M. Blaszak, S. Rauch-Wojciechowski. *J.Math.Phys.*, v.35, p.1693, 1994.

[2] O.I. Bogoyavlensky. *Methods of qualitative theory of dynamical systems in astrophysics and gas dynamics*. Moscow, Nauka, 1980.

[3] C.L. Charlier. *Die Mechanik des Himmels*, Walter de Gruyer, Berlin and Leipzig, 1927.

[4] J. Drach. *Comptes Rendus*, (Paris), v.200, p.22, 1935.

[5] L.D. Faddeev and L.A. Takhtajan. *Hamiltonian methods in the theory of solitons*. (Springer, Berlin, 1987).

[6] V.A. Fok *Z.Physik*, v.98, p.145, 1935.

[7] A.S. Fokas, P. Lagerstrom. *J.Math.Ann.Appl.*, v.74, p.325, 1980.

[8] J. Hietarinta, B. Grammaticos, B. Dorizzi, and A. Ramani. *Phys. Rev. Lett.*, v.53, p.1707, 1984.

[9] G. Kolosoff. *Math. Ann.*, **56**, 265–272, 1903.

[10] V.B. Kuznetsov, A.V Tsiganov. *Zap.Nauchn.Seminars LOMI*, v.172, p.89, 1989.

[11] C. Lanczos. *The variational principles of mechanics*, Toronto, 1949.

[12] T. Levi-Civita. *Acta Math.*, v.30, p.305, 1906.

[13] C.W. Misner, K.S. Thorne, J.A. Wheeler. *Gravitation*, San Francisco, Freeman, 1973.

[14] J. Moser. *Comm. Pure Appl. Math.*, **23**, 609, 1970.

[15] L. Pars. *Amer.Math.Monthly*, v.56, p.395, 1949.

[16] A.M. Perelomov. *Integrable system of classical mechanics and Lie algebras*, Birkhauser, Basel, 1991.

[17] M. Reed, B. Simon. *Methods of modern mathematical physics*, Acad. Press, New York, London, 1972.

[18] A.G. Reyman and M.A. Semenov-Tian Shansky. Group Theoretical Methods in the theory of Finite-Dimensional Integrable systems. In V.I. Arnold and S.P. Novikov, editors, *Dynamical systems VII*, volume EMS 16, Springer, Berlin, 1993.

[19] A. Ramani, B. Grammaticos, and T. Bountis. *Phys.Rep.*, **180**, 159–245, 1989.

[20] K. Rosquist, G. Puccacco. *J.Phys.A.*, v.28, p.3235, 1995.

[21] E. Schrödinger. *Proc.Roy.Irish.Acad.*, v.46A, p.9 and p.183, 1940, v.47A, p.53, 1941.

[22] P. Stäckel. *Comptes Rendus*, (Paris), v.116, p.485 and p.1284, 1893.

[23] J.L. Synge. *Classical dynamics*, Springer, Berlin, 1960.
[24] G. Tondo, C. Morosi. *Bi-Hamiltonian manifolds, quasi bi-Hamiltonian systems and separation of variables*, Preprint solv-int/9811008, 1998.

[25] A.V. Tsiganov. *J.Math.Phys.*, v.40, p.279, 1999.

[26] A.V. Tsiganov. *Phys.Lett.A.*, v.251, p.354, 1999.

[27] A.V. Tsiganov Duality between integrable Stäckel systems. solv-int/9812001, 1998.

[28] A.V. Tsiganov. The Lax representation for the Holt system. solv-int/9812030, 1998.