Prime Labeling of Rough Approximations for Some Special Graphs

S. Dhanalakshmi* and N. Parvathi
Department of Mathematics, SRM University, Chennai – 603203, Tamil Nadu, India; dhanadhana1980@gmail.com, parvathi.n@ktr.srmuniv.ac.in

Abstract

Background/Objectives: Rough set theory proposed a new mathematical approach to vagueness or imperfect knowledge. It is the learning of approximations of concepts represented by lower and upper approximations which is being attracted by many researchers. The current study is a combination of rough sets approximation and graph labeling. Methods/Analysis: Many researchers have studied prime graph. Here we combine rough approximations with prime labeling under the name of H-prime labeling on graph G. Findings: The current work is to prove that the induced sub graph obtained by the upper approximation of any sub graph H of a friendship graph $F_n$, bistar graph $B_{n,n}$ and splitting graph of a star graph ‘$S’ graph admits prime labeling. The various applications between rough sets and graph labeling are chemical classification, decision analysis, knowledge acquisition, machine learning, job assignment etc.

Keywords: Bistar Graph, Friendship Graph, Lower and Upper Approximation, Prime Labeling, Rough Set, Splitting Graph

1. Introduction

Combination of rough set theory with the graph theory is new research direction of mathematics. Graph theory is one of the dynamic areas of mathematics. For notations and terms in graph theory we follow Harary¹. The concept of graph labeling² was introduced by Rosa in 1967. It is applied in many areas³ like coding theory, X-ray crystallography, radar, a astronomy, circuit design, communication network addressing, data base management. The applications of rough set concept to explore various aspects of graph theory and vice versa is a fruitful area of research. Fuzzy and Rough approximations operations were discussed⁴ for some graphs. Rough set properties were studied⁵ on graph theory concepts and minimum spanning tree was constructed⁶ for a tree having rough weights. Lower and upper approximation admits graceful labeling⁷ were studied for some classes of graphs. Even graceful labeling⁷ was discussed for the induced subgraph of upper approximations of star and path graphs. The current work is also interpretation of graph labeling with rough approximations. In particular we investigate upper approximations of some special graphs admit prime labeling.

2. Preliminaries

Definition 2.1: Let $G = (V(G), E(G))$ be a graph. A bijection $f: V \rightarrow \{1,2,3,\ldots, |V|\}$ is called prime labeling if for each edge $e = uv$, gcd $(f(u),f(v)) = 1$. A graph which admits prime labeling is called a prime graph.

Definition 2.2: Let $H$ be a sub graph of any graph $G$ then the neighborhood of $v$ is:

$N(v) = \{v\} \cup \{u \in V(G) : u \in E(G)\}$

Definition 2.3: For any sub graph $H$ of graph $G$ then we define

- The lower approximation operation as:
  $L(V(H)) = \{v \in V(H) / N(v) \subseteq V(H)\}$

- The upper approximation operation as:
  $U(V(H)) = \cup \{N(v) / v \in V(H)\}$

*Author for correspondence
3. Lower and Upper Approximation H-Prime Labeling on G

Based on the definition of lower and upper H – graceful labeling we here introduce the following definitions:

**Definition 3.1:** Lower approximation H – prime labeling on G :
The induced sub graph obtained by the lower approximation operation of any sub graph H of G has prime labeling then we can say the sub graph H is lower approximation H – prime labeling on G. 

**Definition 3.2:** Upper approximation H – prime labeling on G:
The induced sub graph obtained by the upper approximation operation of any sub graph H of G has prime labeling then we can say the sub graph H is upper approximation H – prime labeling on G.

4. Main Results

- Friendship graph F_n

**Theorem 4.1:** Any sub graph of a friendship graph has upper approximation H – prime labeling on G.

**Proof:** Let the friendship graph be G having n copies of cycle C_3, i.e. V(G) = {v, v_1, v_2, v_3, …, v_2n} and H be a sub graph of G. Based on the sub graph of G we have some cases of upper approximation and its labeling pattern which is given below:

**Case (1):** Consider the sub graph H as the apex vertex ‘v’ i.e. v ∈ V(H)
Then the upper approximation of V(H) is:
U(V(H)) = {v, v_1, v_2, v_3, …, v_2n}
Hence the prime labeling of upper approximation of V(H) is as follows:
f: U(V(H)) → \{1, 2, 3, …, |V(H)|\}
given by f(v) = 1
f(v_i) = i + 1 for 1 ≤ i ≤ n
Hence the upper approximation of the apex vertex admits prime labeling.

**Case (2):** Consider the sub graph H as the apex vertex ‘v’ and any vertex v_i of G
Then the upper approximation of V(H) is
U(V(H)) = {v, v_1, v_2, v_3, …, v_2n}
Hence the labeling pattern of the above approximation is same as that of case (1).

**Case (3):** Consider the sub graph H as the vertices of C_n (except the apex vertex), i.e., {v_1, v_2, v_3, …, v_n} ∈ V(H)
Then the upper approximation of V(H) is
U(V(H)) = {v_1, v_2, v_3, …, v_n}
Then the labeling pattern of above upper approximation of V(H) is same as that of Case (1).

4.1 Illustration

Prime labeling of upper approximation of any sub graph of F_n shown in the following Figure 1.

- Bistar Graph B_{n,n}

**Theorem 4.2:** Any sub graph of a bistar graph B_{n,n} has upper approximation H – prime labeling on G.

**Proof:** Consider the bistar graph be G with two copies of k_{1,n}, Let v_1, v_2, …, v_n and u_1, u_2, …, u_n be the corresponding vertices of each copy of K_{1,n} with apex vertex v and u. Based on the sub graph of G.

We have some cases of upper approximation and its labeling pattern which is given below:

**Case (1):** Consider the sub graph H as the apex vertex ‘v’ i.e. v ∈ V(H)
Then the upper approximation of V(H) is:
U(V(H)) = {v_1, v_2, v_3, …, v_n}
Now the prime labeling of upper approximation of V(H) is:
Case (2): Consider the subgraph $H$ as the apex vertex 'u' ie., $u \in V(H)$.

Then the upper approximation of $V(H)$ is:

$$U(V(H)) = \{u, u_1, u_2, \ldots, u_n, v\}$$

Now the prime labeling of upper approximation of $V(H)$ is as follows:

$$f: U(V(H)) \rightarrow \{1, 2, 3, \ldots, |V(H)|\}$$

given by $f(v) = 1$

$$f(u) = 2$$

$$f(v_i) = 2i+2 \text{ for } 1 \leq i \leq n$$

Hence the upper approximation of the apex vertex admits prime labeling.

Case (3): Consider the subgraph $H$ as the pendant vertices corresponding to the apex vertex 'v' ie., $\{v, v_1, v_2, \ldots, v_n\} \in V(H)$.

Then the upper approximation of $V(H)$ is:

$$U(V(H)) = \{v, v_1, v_2, \ldots, v_n\}$$

The induced subgraph of above upper approximation is a star graph which admits prime labeling.

Case (4): Consider the subgraph $H$ as the pendant vertices corresponding to the apex vertex 'u' ie., $\{u, u_1, u_2, \ldots, u_n\} \in V(H)$.

Then the upper approximation of $V(H)$ is:

$$U(V(H)) = \{u, u_1, u_2, \ldots, u_n\}$$

The induced subgraph of above upper approximation is a star graph which admits prime labeling.

Case (5): Consider the subgraph $H$ as the pendant vertices corresponding to the apex vertex 'v' and the apex vertex $u$ ie., $\{v, v_1, v_2, \ldots, v_n, u\} \in V(H)$.

Then the upper approximation of $V(H)$ is:

$$U(V(H)) = \{v, v_1, v_2, \ldots, v_n, u\}$$

Then the labeling pattern of above upper approximation of $V(H)$ is same as that of Case (1).

Case (6): Consider the subgraph $H$ as the pendant vertices corresponding to the apex vertex 'u' and the apex vertex $v$ ie., $\{u, u_1, \ldots, u_n, v\} \in V(H)$.

Then the upper approximation of $V(H)$ is:

$$U(V(H)) = \{v, u, u_1, u_2, \ldots, u_n\}$$

Then the labeling pattern of above upper approximation of $V(H)$ is same as that of Case (2).

Case (7): Consider the subgraph $H$ as the apex vertex $u,v$ and its corresponding pendant vertices $u_i, v_i$. Then the upper approximation of $V(H)$ is:

$$U(V(H)) = \{v, u, u_1, u_2, \ldots, u_n\}$$

Then the prime labeling of upper approximation of $V(H)$ is as follows:

$$f: U(V(H)) \rightarrow \{1, 2, 3, \ldots, |V(H)|\}$$

given by $f(v) = 1$

$$f(u) = 2$$

$$f(v_i) = 2i+2 \text{ for } 1 \leq i \leq n$$

$$f(u_i) = 2i+1 \text{ for } 1 \leq i \leq n$$

Now we have a bijection $f: U(V(H)) \rightarrow \{1, 2, 3, \ldots, |V(H)|\}$ such that $e = \{u, v\}$

$E$ and $\text{GCD} \{f(u), f(v)\} = 1$.

Hence the upper approximation of the above subgraph $H$ admits prime labeling on $G$.

4.2 Illustration

Prime labeling of upper approximation of any sub graph of $B_{n,n}$ shown in the following Figure 2.

Figure 2.
• Splitting Graph of a Star Graph $s'\text{'}$

**Theorem 4.3:** Any sub graph of a splitting graph of a star graph $s'\text{'}$ has upper approximation $H$ – prime labeling.

**Proof:** Let $G$ be the splitting graph of $K_{1,n}$ with $v$ as the apex vertex and $v'$ be the newly added vertices with $K_{1,n}$ to form $G$. Based on the sub graph of $G$ we have the following cases of upper approximation and its labeling pattern which is given below:

**Case (1):** Consider the sub graph $H$ as the apex vertex $v'$ ie. $v \in V(H)$
Then the upper approximation of $V(H)$ is:
$$U(V(H)) = \{ v, v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n \}$$

Now the prime labeling of upper approximation of $V(H)$ is as follows:
$$f: \{1,2,3,\ldots,|V(H)|\} \rightarrow U(V(H))$$
given by $f(v) = 1$
$$f(v_i) = 2i+2 \text{ for } 1 \leq i \leq n$$
$$f(v'_i) = 2i+1 \text{ for } 1 \leq i \leq n$$

Now we have a bijection $f: U(V(H)) \rightarrow \{1,2,3,\ldots,|V(H)|\}$ such that $e \in E$ and $\text{GCD}\{f(u),f(v)\} = 1$.
Hence the upper approximation of the apex vertex admits prime labeling.

**Case (2):** Consider the sub graph $H$ as the vertex $v'$, ie., $v' \in V(H)$
Then the upper approximation of $V(H)$ is:
$$U(V(H)) = \{ v, v'_1, v'_2, \ldots, v'_n \}$$

Then the prime labeling of upper approximation of $V(H)$ is as follows:
$$f: \{1,2,3,\ldots,|V'(H)|\} \rightarrow U(V(H))$$
given by $f(v') = 1$
$$f(v'_i) = i+1 \text{ for } 1 \leq i \leq n$$

Hence the upper approximation of the apex vertex admits prime labeling.

**Case (3):** Consider the sub graph $H$ as the pendent vertices corresponding to the vertex $v$ ie., $\{v_1',v_2',\ldots,v_n'\} \in V(H)$
Then the upper approximation of $V(H)$ is:
$$U(V(H)) = \{ v, v_1', v_2', \ldots, v_n' \}$$

The induced sub graph of above upper approximation is a star graph which admits prime labeling.

**Case (4):** Consider the sub graph $H$ as the vertices $\{v',v_1,v_2,\ldots,v_n\} \in V(H)$
Then the upper approximation of $V(H)$ is:
$$U(V(H)) = \{ v, v', v_1, v_2, \ldots, v_n \}$$
Now the prime labeling of upper approximation of $V(H)$ is as follows:
$$f: \{1,2,3,\ldots,|V'(H)|\} \rightarrow U(V(H))$$
given by $f(v) = 1$
$$f(v') = 2$$
$$f(v'_i) = 2i+1 \text{ for } 1 \leq i \leq n$$
Hence the upper approximation of the above sub graph $H$ admits prime labeling.

**Case (5):** Consider the sub graph $H$ as the vertices $\{v_1',v_2',\ldots,v_n',v\} \in V(H)$
Then the upper approximation of $V(H)$ is:
$$U(V(H)) = \{ v, v_1', v_2', \ldots, v_n', v \}$$
Now the prime labeling of upper approximation of $V(H)$ is as follows:
f: \( U(V(H)) \rightarrow \{1,2,3,...,|V(H)|\} \)
given by \( f(v) = 1 \)
\( f(v') = 2 \)
\( f(v_i) = 2i+1 \) for \( 1 \leq i \leq n \)
\( f(v'_i) = 2i+2 \) for \( 1 \leq i \leq n \)

Hence the upper approximation of the above sub graph \( H \) admits prime labeling.

### 4.3 Illustration
Prime labeling of upper approximation of any sub graph of \( S' \) shown in the following Figure 3.

### 4.4 Remarks
1. For all the above three graphs, the lower and upper approximation of \( V(G) \) admits prime labeling.
2. Lower and upper approximation of empty set is empty. If \( K \) is a empty set then \( L(V(K)) = \phi = U(V(K)) \)

### 5. Conclusion and Future Work
In this paper friendship graph, bi star graph and splitting graph of a star graph have taken for discussion and proved that the induced sub graph of the upper approximations of any sub graph of graph \( G \) admits prime labeling. Also we have concluded that the lower approximation admits prime labeling for \( V(G) \). In future, we can proceed with some more types of labeling and its lower and upper approximation for all sub graph of different types of graph.

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