A two-dimensional (2D) scalar-tensor gravity theory is used to describe the near-horizon near-extremal behavior of black 3-branes solutions of type IIB string theory. The asymptotic symmetry group of the 2D, asymptotically anti de Sitter (AdS) metric, is generated by a Virasoro algebra. The (non-constant) configuration of the 2D scalar field, which parametrizes the volume of the 3-brane, breaks the conformal symmetry and produces a divergent central charge in the Virasoro algebra. Using a renormalization procedure we find a finite value for the central charge and by means of the Cardy formula, the entropy of the black 3-brane in terms of microstates of the conformal field theory living on the boundary of the 2D spacetime. We find for the entropy as a function of the temperature the power-law behavior $S \propto T^3$. Unfortunately, owing to a scale symmetry of the 2D model the proportionality constant is undetermined. The black 3-brane case is a nice example of how finite temperature effects in higher dimensional AdS/CFT dualities can be described by a AdS$_2$/CFT$_1$ duality endowed with a scalar field that breaks the conformal symmetry and produces a non-vanishing central charge.

I. INTRODUCTION

The brane solutions of type II string theory in ten dimensions play a crucial role in the anti de Sitter/conformal field theory correspondence (AdS/CFT) [1, 2, 3]. Specializing to the AdS$_5$/CFT$_4$ case, the low energy string theory splits into two decoupled pieces, bulk supergravity and the near-horizon limit of the extremal 3-brane. In the near-horizon limit the extremal 3-brane has the AdS$_5 \times S^5$ geometry. On the other hand the same theory has an other low energy limit describing bulk supergravity and super Yang-Mills theory in four dimensions. This fact led Maldacena to the conjecture that $U(N)$ super Yang-Mills in four dimensions is dual to type IIB string theory on AdS$_5 \times S^5$ [1, 3].

Most of the progress about the string/gauge theory duality came from comparison between the two theories at zero temperature, which for the supergravity side means considering the extremal brane. Finite temperature behavior, which is important both for testing the duality and for discussing the thermodynamics of the brane from a microscopical point of view, can be discussed considering the near-extremal brane.

Finite temperature breaks conformal invariance. Moreover, the discussion of the strong ’t Hooft coupling regime of the Yang-Mills theory at finite temperature is problematic. The entropy of the brane calculated using the usual Bekenstein-Hawking formula differs from that calculated for the (weak-coupled) Yang-Mills theory by a proportionality factor [3, 4, 5, 6].

The discrepancy is due to the fact that the gravity description is supposed to be valid at strong ’t Hooft coupling. The difficulty in performing calculations at strong ’t Hooft coupling makes it almost impossible to go beyond a qualitative explanation of the presence of the proportionality factor. In particular, it prevents a quantitative explanation of the Bekenstein-Hawking entropy of the brane in terms of microstates.

Two-dimensional gravity models, in particular dilaton gravity models, have been used with success in several contexts to discuss the near-horizon, near-extremal limit of higher dimensional black hole solutions of string theory [7, 8, 9, 10, 11]. They emerge in a natural way after dimensional reduction of the higher-dimensional gravity theory and give a simple, effective description of the higher dimensional physics. It is therefore tempting to use them to describe the near-horizon, near-extremal limit of black 3-branes. An other feature that makes 2D gravity models very interesting in this context is the relatively simple form that the AdS/CFT correspondence takes in two spacetime dimensions (AdS$_2$/CFT$_1$) [12, 13, 14, 15, 16, 17]. If the near extremal 3-brane admits an effective near horizon description in terms of a 2D gravity model, one can also hope that the (finite temperature) AdS$_5$/CFT$_4$ duality admits an effective description in terms of a AdS$_2$/CFT$_1$ duality.

This paper is devoted to the attempt of giving a two-dimensional description of both the near-horizon, near-extremal regime of black 3-branes of type II string theory and of the AdS$_5$/CFT$_4$ duality. After a brief review of the black 3-brane solutions of type IIB string theory and the related AdS$_5$/CFT$_4$ duality (Sect. II), we proceed performing a dimensional reduction that produces a 2D dilaton gravity model describing the near-horizon, near-extremal behavior of
the black 3-brane (Sect. III). We study the group of asymptotical symmetries of the 2D metric (which has asymptotical AdS behavior), show that it is generated by a Virasoro algebra and discuss its relationship with the conformal group in four dimensions (Sect. IV). The central charge of the Virasoro algebra diverges and we have to use a renormalization procedure to find a finite value for it. Using the Cardy formula we calculate the entropy of the 3-brane by counting states in the boundary conformal field theory. We reproduce the power-law behavior $S \propto T^3$ (Sect. V). Owing to a scale symmetry of the 2D model, which we discuss in detail in Sect. VI and VII, the proportionality constant is undetermined.

II. BLACK 3-BRANES AND THE ADS$_5$/CFT$_4$ DUALITY

Black 3-branes solutions of type IIB string theory in ten dimensions, in particular their near horizon behavior, play a fundamental role in the formulation of the AdS$_5$/CFT$_4$ correspondence. In this section we will briefly review some known facts about these solutions.

Let us consider the low energy effective action for type IIB string theory in ten dimensions. In the Einstein frame the action reads

$$S_{IIB} = \frac{1}{(2\pi)^2}\alpha' g_s^2 \int d^{10}x \sqrt{-g} \left( R - \frac{1}{2} (\nabla \Phi)^2 - \frac{2}{5!} F_5^2 \right),$$  

(1)

where $\alpha'$ is the string tension, $g_s$ is the asymptotical string coupling constant, $\Phi$ is the dilaton and $F_5$ is the field strength for the 4-form potential, $F_5 = dA_4$. We are interested in solutions of the theory that are charged with respect to the Ramond-Ramond (RR) form $A_4$. They are given by (self dual) black 3-branes with a constant dilaton, [18, 19, 20]

$$ds^2 = H^{-1/2} \left( -f dt^2 + \sum_{i=1}^3 dx^i dx^i \right) + H^{1/2} \left( f^{-1} dr^2 + r^2 d\Omega_3^2 \right), \quad e^{-2\Phi} = g_s^{-2},$$  

(2)

where

$$H(r) = 1 + \frac{r_0^4}{r^4}, \quad f(r) = 1 - \frac{r_0^4}{r^4}. $$  

(3)

The solution describing a 3-dimensional electric source is characterized by the field strength

$$F_{tijk} = \epsilon_{ijk} H^{-2} \frac{N}{r^5},$$  

(4)

where $N$ is the RR charge. The dual magnetic configuration can be easily obtained from Eq. (1) applying the Hodge duality. The parameters $r_-$ and $r_0$ are related to the mass per unit of volume and to the RR charge $N$ of the brane (see for instance [3]). The solution [2] describes a black brane with an event horizon at $r = r_0$. The brane becomes extremal for $r_0 = 0$.

We are particularly interested in the near-horizon behavior of the extremal brane solution [2]. It is exactly this regime that is the relevant one for the AdS$_5$/CFT$_4$ correspondence. The near-horizon limit of the extremal black brane has the geometry of AdS$_5 \times S^5$ [1]. This fact led Maldacena to the conjecture that $N = 4 \ U(N)$ super Yang-mills theory in four dimensions is dual to type IIB superstring theory on AdS$_5 \times S^5$. Most of the evidence about the AdS/CFT duality comes from comparison between quantities of the gauge and string theory that are protected by the conformal invariance (and by supersymmetry) of the AdS$_5$ background. Excitations of the 3-brane above extremality in general break conformal invariance and the brane acquires a finite temperature $T > 0$. The near-extremal 3-brane has a natural correspondence with the gauge field theory at finite temperature. The gravity solution describing the 3-brane in the near-extremal, near-horizon regime can be easily obtained from Eq. (2) taking the $r \to 0$ limit, keeping both $r/\alpha'$ and the energy above extremality finite [1, 3]

$$ds^2 = R_0^2 \left\{ u^2 \left[ - \left( 1 - \frac{u_0^4}{u^4} \right) dt^2 + \sum_{i=1}^3 dx^i dx^i \right] + \frac{du^2}{u^2} \left( 1 - \frac{u_0^4}{u^4} \right)^{-1} + d\Omega_5^2 \right\},$$  

(5)

where $R_0^4 = r_-^4 = 4\pi N(\alpha')^2 g_s$, $u_0 = r_0/R_0^2$ and the coordinate $u$ is related with the coordinate $r$ of Eq. (2) by $u = r/R_0^2$. Moreover, in the extremal limit the RR field strength is given by
The temperature of the brane \( T \) is given by

\[
T = \frac{u_0}{\pi}.
\]

(7)

Working in the canonical ensemble, the Bekenstein-Hawking entropy \( S \) and the energy of the excitation above extremality \( E \) can be expressed as a function of the temperature \([4]\),

\[
S = \frac{\pi^2}{2} V N^2 T^3, \quad E = \frac{3}{8} \pi^2 V N^2 T^4,
\]

(8)

where \( V \) is the volume of the 3-brane. In the spirit of the AdS/CFT correspondence the result \([5]\) should be matched by computations of the entropy in the finite-temperature \( U(N) \) Yang-Mills theory. The gauge theory computation yields the result \([4, 5]\)

\[
S_{YM} = \frac{4}{3} S_{brane}.
\]

(9)

The origin of the factor 4/3 is, qualitatively, well understood \([6]\). The gauge theory computation is performed at zero t’Hooft coupling \( N g^2_{YM} \), whereas the gravity description given by the 3-brane is supposed to be valid at strong t’Hooft coupling. A quantitative explanation of the result \([12]\) is much more involved owing to the difficulty of doing calculations for the Yang-Mills theory at strong t’Hooft coupling. In the next section we will try to address the problem using a simplified two-dimensional gravity model.

### III. DIMENSIONAL REDUCTION

The near-horizon, near-extremal solution \([11]\) factorizes as a product of a, asymptotically AdS, five-dimensional (5D) spacetime times a 5-sphere of radius \( R_0 \). We can obtain an effective 2D gravity model describing the near-horizon, near-extremal regime of the 3-brane by introducing a scalar field \( \phi \), which parametrizes the volume of the 3-brane embedded in the 5D spacetime. We perform the dimensional reduction from ten to two dimensions using for the metric the ansatz,

\[
d_{(10)}^2 = d_{(2)}^2 + \phi^{2/3} \sum_{i=1}^{3} dx^i dx^i + R_0^2 d\Omega_5^2,
\]

(10)

Notice that the volume of the 3-brane embedded in the 5D spacetime is given by

\[
V = \phi V.
\]

(11)

The ansatz for the RR field strength and for the dilaton follows directly from Eqs. \([2]\), \([3]\),

\[
\frac{F_5^2}{5!} = \pm \frac{4}{R_0^2}, \quad e^{-2\phi} = g_s^{-1}.
\]

(12)

where the \pm sign refers to magnetic and electric solutions respectively. In the following we will consider only the dimensional reduction of the magnetically charged 3-brane. The dimensional reduced action is obtained using Eqs. \([10]\) and \([12]\) into the ten-dimensional action \([11]\). It has the form of a 2D dilaton gravity model,

\[
S_{(2d)} = \mathcal{K} \int d^2 x \sqrt{-g} \phi \left[ R + \frac{2}{3} (\nabla \phi)^2 + \lambda^2 \right],
\]

(13)

where \( \mathcal{K} \), the inverse of the 2D Newton constant, is given by

\[
\mathcal{K} = \frac{V N^2}{8 \pi^2 R_0^3}, \quad \lambda^2 = \frac{12}{R_0^2}.
\]

(14)
The 2D gravity model is a particular case of general class of models, which have been already investigated in the literature. The black hole solutions of the theory are given by

\[ ds^2 = -\left( b^2 r^2 - \frac{A^2}{b^2 r^2} \right) dt^2 + \left( b^2 r^2 - \frac{A^2}{b^2 r^2} \right)^{-1} dr^2, \quad \phi = \phi_0 (br)^{3}, \]  

(15)

where \( b = 1/R_0 \) and \( A, \phi_0 \) are integration constants.

The 2D black hole solution gives an effective description of the near-extremal near-horizon regime of the 3-brane. The parameters \( A, \phi_0 \) can be easily identified in terms of brane parameters. Changing in Eq. (15) the radial coordinate, \( r = u R_0^2 \) and requiring the black hole solution (15) to match the 2D section of the black brane solution (5), one readily finds

\[ A = u_0^2 R_0^2 = \left( \frac{r_0}{R_0} \right)^2. \]  

(16)

The integration constant \( \phi_0 \) is related with a on-shell scale symmetry of the 2D gravity theory. Rescaling the scalar field \( \phi, \phi \to \mu \phi \) the 2D action (13) changes by an overall factor, \( S_{(2d)} \to \mu S_{(2d)} \). We can use this classical symmetry of the action to change the normalization of the action (13) to \( K = 1/2 \) (the normalization used in Ref. [21]). This fixes the value of \( \phi_0 \),

\[ \phi_0 = \frac{N^2 V}{4\pi^2 R_0^3}. \]  

(17)

Let us now compute the thermodynamical parameters associated with the 2D black hole and show that they reproduce exactly those of the 3 black brane. The ADM mass \( M_{bh} \), temperature \( T_{bh} \) and entropy \( S_{bh} \) are

\[ M_{bh} = \frac{3}{2} \phi_0 b A^2, \quad T_{bh} = \frac{b}{\pi} \sqrt{A}, \quad S_{bh} = 2\pi \phi_0 A^{3/2}. \]  

(18)

Using Eqs (16), (17) and comparing the previous equations with Eqs. (7), (9) one easily finds \( T_{bh} = T_{brane}, \ S_{bh} = S_{brane} \) and \( M_{bh} = E_{brane} \), where \( E_{brane} \) is the energy of the brane above extremality. This result can be easily understood. The thermodynamics of both the 2D black hole and of the black 3-brane is determined by the behavior at the horizon, \( r = (\sqrt{A})/b \). On the other hand this behavior is determined by the 2D sections of the metric, which are the same for the 2D black hole and the 3-brane. The story looks different if we want to give a microscopical interpretation of the thermodynamical behavior or if we want to consider the region where the dual gauge theory is weak coupled. In both cases we need to move away from the horizon and to consider the asymptotical region \( r \to \infty \). The natural framework for doing this is to investigate the asymptotical symmetries of the 2D solution. This will be the subject of the next section.

### IV. ASYMPTOTICAL SYMMETRIES

The group of asymptotical symmetries of the metric have been already investigated in Ref. [21]. Consistently with the fact that the metric is asymptotically AdS, it was found that the asymptotic symmetries are generated by operators \( L_m \) spanning a Virasoro algebra. Unfortunately, the central charge of the Virasoro algebra turns out be divergent. This divergence is essentially due to the asymptotical behavior of the scalar field \( \phi \sim r^3 \). In this paper we will use an alternative approach. We will renormalize the central charge by subtracting the divergent contribution of the background.

Changing the radial coordinate,

\[ b^2 r^2 \to b^2 r^2 + A, \]  

(19)

the solution (15) becomes

\[ ds^2 = -\left( b^2 r^2 + 2A \right) \left( 1 + \frac{A}{b^2 r^2} \right)^{-1} dt^2 + \left( b^2 r^2 + 2A \right)^{-1} dr^2, \quad \phi = \phi_0 \left( (br)^2 + A \right)^{3/2}. \]  

(20)

Expanding the solution near \( r \to \infty \) we get,

\[ g_{tt} = -b^2 r^2 - A + \frac{A^2}{b^2 r^2} + O(r^{-4}), \]
The conformal symmetry is broken by a non-constant value of a scalar field. This breaking is a general feature of 2D AdS solutions endowed with a non-constant scalar field and it is the source of a non-vanishing central charge in the Virasoro algebra [15]. Notice that whereas the leading terms of the Killing vectors are typical of every 2D metric with AdS asymptotic behavior [12, 21], that of the scalar field are not. The non-constant configuration of the scalar field breaks the conformal symmetry of the metric. This breaking is a general feature of 2D AdS solutions endowed with a non-constant scalar field and it is the source of a non-vanishing central charge in the Virasoro algebra [12].

However, if we forget the field \( \phi \) and concentrate ourselves on the metric part of the solution (26), we see that the isometry group of the AdS\(_5\) metric has been promoted by the dimensional reduction to the \( \text{diff}_1 \) group generated by the Killing vectors (28). Of course, this symmetry is broken by the non-constant value of \( \phi \), but as we shall see in the next section the effect of this breaking is the appearance of a non-vanishing central charge in the Virasoro algebra (24). We see here at work a very nice mechanism: the dimensional reduction breaks the full isometry group of AdS\(_5\), but as we shall see in the next section the effect of this breaking is the appearance of a non-vanishing central charge in the Virasoro algebra (24). We see here at work a very nice mechanism: the dimensional reduction allows us to describe the AdS\(_5\)/CFT\(_4\) duality in terms of an AdS\(_2\)/CFT\(_1\) duality in which the conformal symmetry is broken by a non-constant value of a scalar field.

\[
g_{rr} = \frac{1}{b^2 r^2} - \frac{2A}{b^4 r^4} + \frac{4A^2}{b^6 r^6} + \mathcal{O}(r^{-8}), \quad \phi = \phi_0 \left( b^4 r^3 + \frac{3}{2} A b r + \frac{3 A^2}{8} b r + \mathcal{O}(r^{-3}) \right).
\]

The metric is asymptotically AdS. We are led to impose the following \( r \to \infty \) boundary conditions

\[
\begin{align*}
g_{tt} &= -b^2 r^2 + \gamma_{tt} + \frac{\Gamma_{tt}}{b^2 r^2} + \mathcal{O}(r^{-4}), \\
g_{rr} &= \frac{1}{b^2 r^2} + \gamma_{rr} + \frac{\Gamma_{rr}}{b^2 r^2} + \mathcal{O}(r^{-8}), \\
g_{tr} &= \frac{\gamma_{tr}}{b^2 r^2} + \mathcal{O}(r^{-5}), \\
\phi &= \phi_0 \left( \rho(br)^3 + \gamma_{\phi\phi} br + \frac{\Gamma_{\phi\phi}}{br} + \mathcal{O}(r^{-3}) \right)
\end{align*}
\]

Where \( \gamma, \Gamma, \rho \) are deformations of the metric and of the scalar field. The asymptotic symmetry group (ASG) of the metric is given by 2D diffeomorphisms which preserve the boundary conditions (22). The associated Killing vectors are

\[
\chi^t = \epsilon(t) + \frac{\tilde{\epsilon}(t)}{2b^4 r^2} + \mathcal{O}(r^{-5}), \quad \chi^r = -\dot{\epsilon}(t) r + \mathcal{O}(r^{-2}),
\]

where \( \epsilon(t) \) is an arbitrary function of the time \( t \). The leading terms of the Killing vectors (23) are typical of every 2D metric with AdS asymptotic behavior [12, 21].

The generators \( L_k \) of the ASG satisfy the Virasoro algebra,

\[
[L_k, L_l] = (k-l)L_{k+l} + \frac{c}{12} (k^3 - k) \delta_{k+l,0}, \quad k, l = 0, 1, 2, 3, 4
\]

where we allow for a nonvanishing central charge \( c \). The ASG has a natural realization in terms of the conformal group in one dimension (the \( \text{diff}_1 \) group) leading in a natural way to a AdS\(_2\)/CFT\(_1\) duality [12]. Notice that whereas the leading terms of the 2D metric are invariant under the transformations generated by the Killing vectors (28), that of the scalar field are not. The non-constant configuration of the scalar field \( \phi \) breaks the conformal symmetry of the metric. This breaking is a general feature of 2D AdS solutions endowed with a non-constant scalar field and it is the source of a non-vanishing central charge in the Virasoro algebra [12].

Using the boundary conditions (22) and the Killing vectors (23) we can easily find the transformation laws of the boundary fields \( \gamma, \Gamma, \rho \) under the action of the conformal group. In the following we will make use only of the transformation laws for \( \gamma_{rr}, \gamma_{tt}, \rho, \gamma_{\phi\phi} \). They are:

\[
\begin{align*}
\delta \gamma_{rr} &= \epsilon \gamma_{rr} + 2\epsilon \gamma_{rr}, \\
\delta \gamma_{tt} &= \epsilon \gamma_{tt} + 3\epsilon \gamma_{tr} - \frac{\epsilon}{b} (\gamma_{rr} + \gamma_{tt}), \\
\delta \gamma_{tr} &= \epsilon \gamma_{tr} + 3\epsilon \gamma_{tr}, \\
\delta \gamma_{rr} &= \epsilon \gamma_{rr} + \epsilon \gamma_{rr} + \frac{\epsilon}{2b^2}.
\end{align*}
\]

Let us now discuss the relationship between the ASG of the 2D metric (20) and the isometry group of the extremal 3-brane (AdS\(_5\)). The isometry group of AdS\(_5\) is \( \text{SO}(2, 4) \), which is locally isomorphic to the conformal group in four dimensions. The dimensional reduction of the previous section applied to AdS\(_5\) produces the 2D background solution (20) with \( A = 0 \),

\[
d s_6^2 = -(br)^2 dt^2 + (br)^{-2} dr^2, \quad \phi_b = \phi_0 (br)^3
\]

describing AdS\(_2\) endowed with a non-constant field \( \phi \). Owing to the scalar character of \( \phi \), the configuration (20) is not invariant under the full conformal group \( \text{SO}(2, 4) \) but only under its subgroup generated by time-translations. The dimensional reduction breaks the full isometry group of AdS\(_5\). However, if we forget the field \( \phi \) and concentrate ourselves on the metric part of the solution (26) we see that the isometry group of the AdS\(_5\) metric has been promoted by the dimensional reduction to the \( \text{diff}_1 \) group generated by the Killing vectors (23). Of course, this symmetry is broken by the non-constant value of \( \phi \), but as we shall see in the next section the effect of this breaking is the appearance of a non-vanishing central charge in the Virasoro algebra (24). We see here at work a very nice mechanism: the dimensional reduction allows us to describe the AdS\(_5\)/CFT\(_4\) duality in terms of an AdS\(_2\)/CFT\(_1\) duality in which the conformal symmetry is broken by a non-constant value of a scalar field.
V. CENTRAL CHARGE AND ENTROPY

The central charge appearing in the Virasoro algebra \[21\] can be calculated using a canonical realization of the ASG \[12, 21\]. Using the parametrization of the metric,

\[
ds^2 = -N^2 dt^2 + \sigma^2 (dr + N^r dt)^2 ,
\]

one can easily derive the Hamiltonian \( H \) of the theory \[21\]. Boundary term \( J \) (the charges) must be added to \( H \) in order to have well-defined variational derivatives. On the \( r \to \infty \) boundary of 2D AdS spacetime the variation \( \delta J \) is given by \[21\]

\[
\delta J = - \lim_{r \to \infty} \left[ N \left( \sigma^{-1} \delta \phi' - \sigma^{-2} \phi' \delta \sigma - \frac{2}{3} \sigma^{-1} \phi' \delta \phi \right) - N' \sigma^{-1} \delta \phi + N' \left( \Pi_\sigma \delta \phi - \sigma \Pi_\phi \right) \right] ,
\]

where \( \Pi_\sigma \) and \( \Pi_\phi \) are momenta conjugate to \( \sigma \) and \( \phi \) respectively and the prime denotes derivative with respect to \( r \). The central charge in the Virasoro algebra \[21\] can be calculated introducing time-integrated charges \[12\]

\[
\hat{J} = \frac{b}{2\pi} \int_0^{2\pi/b} J dt ,
\]

and using the commutator

\[
\delta_\omega \hat{J}(\epsilon) = \left[ \hat{J}(\epsilon), \hat{J}(\omega) \right] .
\]

If we follow Ref \[21\] and compute the central charge using Eq. \[28\] we get a divergent result. The divergent contribution is due to the cubic divergent term in the scalar field \( \phi \). More physically, it is due to the fact that arbitrary excitations of the \( r = \infty \) boundary have infinite energy \[21\]. In Ref \[21\] we did not try to subtract the divergent part of the charges in order to get a renormalized, finite answer. Here we will use an alternative approach. We will show that the charges \( J \) can be consistently renormalized by subtracting the divergent contribution of the background.

Let us first compute the energy \( M \) of the excitations on the \( r \to \infty \) boundary. This can be done either evaluating \( \hat{J}(\epsilon = 1) \) from Eq. \[28\] or using Mann’s formula for the mass \[22\]. In both cases we find

\[
M = 2b\phi_0 \rho^{4/3} \left\{ \left[ \frac{1}{12b^2} \left( \frac{\dot{\rho}}{\rho} \right)^2 + \frac{\gamma_{\phi\phi}}{\rho} + \frac{3}{4} \gamma_{rr} \right] (br)^2 + \left[ 2 \frac{\Gamma_{\phi\phi}}{\rho} + \frac{3}{4} \left( \Gamma_{rr} - \gamma_{rr}^2 \right) \right] + \mathcal{O}(r^{-2}) \right\} .
\]

For arbitrary deformations \( \gamma, \Gamma, \rho \) \( M \) diverges quadratically as \( r \to \infty \). However, the quadratically divergent term cancels if we consider only on-shell deformations. In fact evaluating the field equations coming from the 2D action \[18\] on the boundary conditions \[22\] one finds that \( \gamma, \Gamma, \rho \) must satisfy the equation \[21\]

\[
\frac{1}{12b^2} \left( \frac{\dot{\rho}}{\rho} \right)^2 + \frac{\gamma_{\phi\phi}}{\rho} + \frac{3}{4} \gamma_{rr} = 0 .
\]

The previous observations indicate the way to keep the energy of the boundary excitations finite: one just needs to consider on-shell deformations. Unfortunately, this is not enough to guarantee the finiteness of the other charges and of the central charge. The cubic asymptotic behavior of the scalar field \( \phi \) is also responsible for the divergence of the other charges. Owing to its scalar character a non-constant configuration of \( \phi \) always breaks the asymptotic symmetry of the metric. This is a general feature of 2D scalar-tensor theories of gravity, which plays an important role for the determination of the central charge in the Virasoro algebra. In these models the origin of the central charge can be traced back to the breaking of the conformal symmetry due to a non-constant value of \( \phi \) \[17\]. As long as the field \( \phi \) depends linearly on \( r \) the central charge remains finite \[21\]. A behavior \( \phi \sim r^h \) with \( h < 1 \) implies a vanishing central charge, whereas for \( h > 1 \) the central charge diverges.

The divergence of the central charge for \( h = 3 \) can be easily cured. The asymptotic AdS background configuration \[20\] is, strictly speaking, not invariant under the action of the ASG generated by the Killing vectors \[28\], owing to
the non-constant value of $\phi$. It is therefore natural to subtract to the charges $J$ defined by Eq. (28) the contribution $J_b$ obtained evaluating $J$ on the background (26). We are therefore led to define renormalized charges $J_R$,

$$J_R = J - J_b.$$  \hfill (33)

Using Eqs. (28), (33) and computing the variations near the classical solution (21), we find after some manipulations

$$\delta J_R(\epsilon) = \epsilon \delta M + \delta J_1,$$  \hfill (34)

where $M$ is given by Eq. (31) and

$$\delta J_1 = \frac{\phi_0}{b} \left( \dot{\epsilon} \dot{\gamma}_{\phi\phi} + \frac{3}{4} \epsilon \dot{\gamma}_{rr} - 3A \epsilon \delta \rho - \frac{3}{2} A \dot{\epsilon} \delta \rho - 3 \dot{b} \dot{\epsilon} \delta \gamma_{tr} \right).$$  \hfill (35)

Considering only on-shell deformations $M$ and $J_R$ are finite. The central charge of the Virasoro algebra is determined by the term $\delta J_1$, through equation (30).

Evaluating Eq. (35) on the classical configuration (21), using Eq. (25), and taking into account that the charges $\hat{J}$ (29) are defined only up to a total time derivative, we find

$$\delta \omega J_1(\epsilon) = \frac{6 \phi_0 A b}{b} (\dot{\omega} \dot{\epsilon} - \dot{\epsilon} \dot{\omega}).$$  \hfill (36)

Expanding in Fourier modes and using Eq. (30) we find the central charge $c$ of the Virasoro algebra

$$c = 144 \phi_0 A.$$  \hfill (37)

The central charge of the Virasoro algebra depends on the mass of the 2D black hole. In fact from Eq. (18) we find

$$c \propto \sqrt{M_{bh}}.$$  

Using the Cardy formula (23),

$$S = 2 \pi \sqrt{c l_0 / 6},$$  \hfill (38)

we can calculate the statistical entropy $S$ associated with the boundary conformal theory. The eigenvalue $l_0$ of the Virasoro operator $L_0$ can be given in terms of the black hole mass $l_0 = M_{bh} / b = (3/2) A^2 \phi_0$. The statistical entropy turns out to be

$$S = 12 \pi \phi_0 A^{3/2}.$$  \hfill (39)

We can now express, by means of Eqs. (17), (18) the entropy either as a function of the temperature (canonical ensemble) or as function of the energy (microcanonical ensemble)

$$S = 3 \pi^2 N^2 V T^3, \quad S = 4 (6)^{1/4} \sqrt{\pi} \left( \frac{N^2 V}{R_0} \right)^{1/4} (E R_0)^{3/4}.$$  \hfill (40)

The statistical entropy associated with the conformal field theory living in the boundary of the 2D spacetime reproduces correctly the power-law behavior $S \propto T^3$ (or $S \propto E^{3/4}$ for the microcanonical ensemble) of the 3-brane (5) and of the $U(N)$ gauge theory (11). This is a highly nontrivial, and to some extend unexpected, result. In previous calculations for a different kind of 2D AdS spacetime we have found the scaling behavior $S \propto T$, which is typical for a 2D CFT (12). Our result indicates that the conformal field theory living on the boundary of 2D AdS spacetimes endowed with a scalar field behaving asymptotically as $\phi \sim \rho^{d-1}$ can reproduce the entropy/temperature thermodynamical relation of conformal field theories in $d \geq 2$ dimensions. The explanation of this peculiar feature can be found in the breaking of the conformal symmetry (the ASG) of the 2D AdS spacetime caused by the nonconstant scalar field $\phi$.

The proportionality factor between $S$ and $T^3$ in Eq. (40) deserves also some surprise. It is not the same as that associated with the horizon of the black 3-brane (or 2D black hole) (5). This was somehow to be expected because the theory on the horizon is strongly coupled whereas that on the $r \to \infty$ region is weakly coupled. For this reason the most natural result for the entropy should have been Eq. (4), i.e the entropy of the Yang-Mills theory at zero t’Hooft coupling. This is not the case because of a factor 6. In the next section we will argue that the 2D theory can only determine the scaling behavior $S \propto T^3$ whereas the proportionality factor is from the 2D point of view essentially undetermined.
VI. SCALE SYMMETRY AND RENORMALIZATION

We have already noted in Sect. III that the 2D action (13) has a classical symmetry \( \phi \rightarrow \mu \phi \) that rescales the 2D Newton constant. More in detail, this scale symmetry appears as a subgroup of the isometry group of AdS\(_2\). It is easy to realize that the scale transformation

\[
\begin{align*}
 r & \rightarrow \mu r, \\
 t & \rightarrow \mu^{-1} t,
\end{align*}
\]

leaves the AdS metric (26) invariant, whereas the scalar \( \phi \) scales as a three-dimensional volume

\[
\phi \rightarrow \mu^3 \phi.
\]

Hence the on-shell symmetry that rescales the 2D Newton constant can be identified as the scale isometry of AdS\(_2\). Again, it is the non constant value of the scalar \( \phi \) which is responsible for the breaking of the isometry group of AdS\(_2\), whose effect is a change of the 2D Newton constant. As a consequence of this symmetry \( \phi_0 \) is a sort of sliding field, which at the classical level remains undetermined. One could argue that this feature does not affect computation of the ratio \( S^{(a)}/S^{(h)} \) between the entropy associated with the horizon and the entropy of the CFT living in the asymptotic region, because one can use for calculating both \( S^{(a)}, S^{(h)} \) the same classical solution. This is not the case because in order to have a finite central charge we have to subtract the divergences. The renormalization procedure of the previous section is equivalent to define a renormalized scalar field \( \phi_R = \phi - \phi_0 \). This introduces the usual ambiguities related with the renormalization scheme. In general we are not allowed anymore to use the same \( \phi_0 \) for computations at the horizon and computations in the asymptotic region.

For the above reasons the numerical factor appearing in Eq. (37) has no physical meaning, it can be changed simply by using an isometric transformation of AdS\(_2\). Consequently, all the thermodynamical relations derived at the end of the previous section are determine up to a unknown dimensionless proportionality factor. This behavior is very similar to what has been observed in 2D cosmological context (24). Investigating the relationship between the Cardy formula and the Friedmann equation for 2D de Sitter gravity, we found a proportionality relation between the central charge and the inverse of the Newton constant. Owing to the scale symmetry of the theory the proportionality factor is undetermined.

Although classically undetermined, \( \phi_0 \), or better the ratio \( \phi^{(a)}_0 / \phi^{(h)}_0 \) between its asymptotic value and that at the horizon, could be determined at the quantum level. For instance one could use the sigma-model formulation of 2D dilaton gravity (27), to compute the running of \( \phi_0 \). As observed in Ref. (21), this is far from being trivial. In the \( r \rightarrow \infty \) asymptotic region the sigma model is weakly coupled and one can use a perturbative approach, but near the horizon the theory becomes strongly coupled so that usual perturbative calculations become useless.

Until now our discussion was based on arguments coming from the 2D gravity theory. These arguments have a nice formulation in terms of the 3-brane and of the AdS\(_{5}\)/CFT\(_4\) correspondence. The scale transformation (41) appears also as isometry of AdS\(_5\) (solution (13) with \( u_0 = 0 \)). In the 5D context the scale transformation (41) is an exact symmetry of the background solution if the coordinates \( x^i \) of the 3-brane transform as \( x^i \rightarrow \mu^{-1} x^i \). This is the so-called ultraviolet/infrared (UV/IR) connection (24). Points near the \( r = 0 \) origin of the AdS spacetime are mapped into the infrared region of the Yang mills theory. In the 2D theory the brane is compactified and its volume is described by the scalar field \( \phi \), which encodes the information about the embedding of the 3-brane in the 5D spacetime. The scale symmetry is broken and the change of the scalar field \( \phi \) under scale transformations can be describe as a change of the parameter \( \phi_0 \), \( \phi_0 \rightarrow \mu^3 \phi_0 \). Thus, running from the UV to the IR in the Yang mills theory is equivalent to running of the 2D Newton constant. In particular, because \( \phi_0 \propto G_N^{-1/2} \), the UV region of Yang Mills theory correspond to the weak-coupled regime of the 2D gravity theory, whereas the IR region is in correspondence with the strong-coupled regime.

There is evidence that also the near-horizon region of the 2D black hole can be described by a CFT (24). Therefore the 3-brane geometry is characterized by two CFT’s. One is generanted by the ASG of the AdS\(_2\) metric and the other is associated with the black hole horizon. If this is true the running of the 2D Newton constant can be interpreted as the running of the central charge when moving from a conformal point to the other. This 2D feature has a counterpart in the large \( N \) behavior of conformal field theories, for which relevant double-trace deformations produce a flow from an ultraviolet to an infrared fixed point (28).

VII. SCALING OF THE THERMODYNAMICAL PARAMETERS

In the previous section we have considered the realization of the scale symmetry for the vacuum AdS\(_2\) solution (20). The vacuum solution is characterized by zero mass, temperature and entropy. Physically, the scale symmetry means that we can change the size of the system without changing the thermodynamical parameters. This is not
true anymore when we consider the solution at finite temperature, i.e. the 2D black hole (15) (corresponding in the ten-dimensional theory to the black brane (9). We expect now the scale symmetry to be broken and that changing the size of the system will affect $M, T, S$. Nonetheless, we will see that the scale symmetry determines the scaling behavior of $M, T, S$. This is very similar to what has been discussed for a class of 2D dilaton gravity model in Ref. 29.

The black hole metric (15) is not invariant under the scale transformation (41). However it can be made invariant if we change in Eq. (15) together with $r, t$ also the parameter $A$, $A \rightarrow \mu A$. From Eqs. (18) it follows immediately the scaling of the thermodynamical parameters

$$ T \rightarrow \mu T, \quad M \rightarrow \mu^4 M, \quad S \rightarrow \mu^3 S. \quad (43) $$

The scale transformation for $\phi, \phi \rightarrow \mu^3 \phi$ gives the scaling law for the volume $V$ of Eq. (11), $V \rightarrow \mu^3 V$, which is the expected one for the volume of a three-dimensional object. Also the entropy $S$ scales extensively as a volume, whereas the mass $M$ is non-extensive.

The previous scaling laws contain all the information necessary to determine up to a proportionality factor the thermodynamical behavior of the system. Writing $V = \mu^3 V'$, we get from Eq. (43) $M(V) = \mu^4 M(V'), \ S(V) = \mu^3 S(V')$. Setting $V' = 1$, trading $V$ for $S$ and using the equation $dM = T dS$ we get the mass/entropy and temperature/entropy relations

$$ M = \alpha S^{4/3}, \quad T = \alpha \frac{4}{3} S^{1/3}, \quad (44) $$

where $\alpha$ is a proportionality constant, which cannot be determined using only scaling arguments. Again, 2D arguments enable us to determine the exponents of the power-law behavior but not the proportionality constant. Owing to the indetermination of $\alpha$, Eq. (44) hold both for the near horizon 3-brane solution (see Eqs (7), (8)) and for the Yang Mills Theory (see Eq. (9). We can also eliminate $\alpha$ from the two Eqs (44)

$$ M = \frac{3}{4} TS. \quad (45) $$

From the 2D point of view this equation can be understood as a violation of the Euler identity $M = TS$. It is a consequence of the non extensivity of the thermodynamical system.

VIII. CONCLUSION

In this paper we have shown that one can use an effective two-dimensional description for the near-extremal near-horizon behavior of the black 3-brane of type IIB string theory. This effective description allows us to formulate the AdS$_5$/CFT$_4$ duality at finite temperature as a AdS$_2$/CFT$_1$ duality with the conformal symmetry broken by nonconstant configuration of the scalar field parametrizing the volume of the 3-brane. The breaking of the symmetry produces a nonvanishing central charge. In particular, this allows for a computation of the entropy of the 3-brane in terms of the conformal field theory living in the boundary of AdS$_2$. Although we could reproduce the entropy/temperature power-law behavior of the 3-brane a dimensionless proportionality constant, related with the 2D Newton constant, remains undetermined. This indetermination seems a general feature of 2D gravity solutions with AdS (or de Sitter) behavior and is due to a scale symmetry of the model. From the 3-brane point of view this scale symmetry is related with the UV/IR connection in the AdS$_5$/CFT$_4$ duality.

The presence of an undetermined dimensioness constant results in a loss of predictive power of the two-dimensional model. In particular, it makes impossible, at least at the level of our present investigation, the determination of the ratio between the Bekenstein-Hawking entropy of the black 3-brane and the entropy of the Yang-Mills theory. Qualitatively this ratio can be explained in the 2D gravity theory in terms of the running of the Newton constant (or equivalently of the central charge) when moving from one to the other conformal point of the geometry. It is very difficult to go beyond this qualitative analysis. The theory is weakly coupled at the one conformal point (the $r \rightarrow \infty$ asymptotical region) but becomes strongly coupled at the other (the horizon).
Acknowledgments

We are very grateful to P. Carta and S. Mignemi for interesting discussions and useful comments.

[1] J. M. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231 [Int. J. Theor. Phys. 38 (1999) 1113] [arXiv:hep-th/9711200].
[2] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253 [arXiv:hep-th/9802150].
[3] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz. Phys. Rept. 323 (2000) 183 [arXiv:hep-th/9905111].
[4] S. S. Gubser, I. R. Klebanov and A. W. Peet, Phys. Rev. D 54 (1996) 3915 [arXiv:hep-th/9602135].
[5] S. S. Gubser, I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. B 534 (1998) 202 [arXiv:hep-th/9805156].
[6] I. R. Klebanov, arXiv:hep-th/0009139.
[7] C. G. Callan, S. B. Giddings, J. A. Harvey and A. Strominger, Phys. Rev. D 45 (1992) 1005 arXiv:hep-th/9111056.
[8] M. Cadoni and S. Mignemi, Nucl. Phys. B 427 (1994) 669 [arXiv:hep-th/9312171].
[9] M. Cadoni, Phys. Rev. D 60 (1999) 084006 [arXiv:hep-th/9904011].
[10] D. Youm, Phys. Rev. D 61 (2000) 044013 [arXiv:hep-th/9910244].
[11] D. Grumiller, W. Kummer and D. V. Vassilevich, Phys. Rept. 369 (2002) 327 [arXiv:hep-th/0204252].
[12] M. Cadoni and S. Mignemi, Phys. Rev. D 59 (1999) 081501 [arXiv:hep-th/9810251].
[13] M. Cadoni and S. Mignemi, Nucl. Phys. B 557 (1999) 165 [arXiv:hep-th/9902040].
[14] M. Caldarelli, G. Catelani and L. Vanzo, JHEP 0010 (2000) 005 [arXiv:hep-th/0008058].
[15] M. Cadoni and S. Mignemi, Phys. Lett. B 490 (2000) 131 [arXiv:hep-th/0002256].
[16] M. Cadoni, P. Carta, D. Klemm and S. Mignemi, Phys. Rev. D 63 (2001) 125021 [arXiv:hep-th/0009185].
[17] M. Astorino, S. Cacciatori, D. Klemm and D. Zanon, Annals Phys. 304 (2003) 128 [arXiv:hep-th/0212096].
[18] G. T. Horowitz and A. Strominger, Nucl. Phys. B 360 (1991) 197.
[19] M. J. Duff and J. X. Lu, Phys. Lett. B 273 (1991) 409.
[20] M. J. Duff, R. R. Khuri and J. X. Lu, Phys. Rept. 259 (1995) 213 [arXiv:hep-th/9412184].
[21] M. Cadoni, P. Carta, M. Cavaglia and S. Mignemi, Phys. Rev. D 65 (2002) 024002 [arXiv:hep-th/0105113].
[22] R. B. Mann, Phys. Rev. D 47 (1993) 4438 [arXiv:hep-th/9206044].
[23] J. L. Cardy, Nucl. Phys. B 270 (1986) 186.
[24] M. Cadoni, P. Carta and S. Mignemi, Nucl. Phys. B 632 (2002) 383 [arXiv:hep-th/0202180].
[25] M. Cavaglia, Phys. Rev. D 59 (1999) 084011 [arXiv:hep-th/9811095].
[26] L. Susskind and E. Witten, arXiv:hep-th/9805114.
[27] S. Carlip, Phys. Rev. Lett. 88 (2002) 241301 [arXiv:gr-qc/0203001].
[28] S. S. Gubser and I. R. Klebanov, Nucl. Phys. B 656 (2003) 23 [arXiv:hep-th/0212138].
[29] M. Cadoni and P. Carta, Phys. Lett. B 522 (2001) 126 [arXiv:hep-th/0107234].