SOME REMARKS ON STABILITY OF CONES FOR THE ONE-PHASE FREE BOUNDARY PROBLEM

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Abstract. We show that stable cones for the one-phase free boundary problem are hyperplanes in dimension 4. As a corollary, both one and two-phase energy minimizing hypersurfaces are smooth in dimension 4.

1 Introduction

We investigate stable homogeneous solutions

$$u : \Omega \to \mathbb{R}, \quad \Omega \subset \mathbb{R}^n,$$

to the one-phase free boundary problem

$$\triangle u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{and} \quad |\nabla u| = 1 \quad \text{on } \partial \Omega \setminus \{0\}. \quad (1.1)$$

Here $\Omega$ is a cone ($\Omega = r\Omega$ for all $r > 0$) equal to the interior of its closure and with smooth cross-section; the function $u$ is homogeneous of degree one and positive in $\Omega$.

We are interested in solutions $u$ that are stable with respect to the Alt–Caffarelli (see [AC81]) energy functional,

$$E(u, B) = \int_B (|\nabla u|^2 + \chi_{\{u>0\}}) \, dx, \quad (1.2)$$

with respect to compact domain deformations that do not contain the origin. Explicitly, the stability we require is that for any smooth vector field $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ with $0 \notin \text{supp } \Psi \subset B_R$ we have

$$\frac{d^2}{dt^2} E(u(x + t\Psi(x)), B_R) \geq 0 \quad \text{at } t = 0. \quad (1.3)$$

There is a vast literature concerning the one-phase free boundary problem; see, for example, the book by Caffarelli and Salsa [CS05]. Many results in the regularity theory of the free boundary $\partial\{u > 0\}$ parallel the corresponding statements in the regularity theory of minimal surfaces, see [Caf87, Caf89, DJ11, Wei99].

Our main result is the following.
Theorem 1.1. The only stable homogeneous solutions in dimension $n \leq 4$ are the one-dimensional solutions $u = (x \cdot \nu)^+$ for unit vectors $\nu$.

For dimension $n = 3$ this result was obtained by Caffarelli, Jerison and Kenig in [CJK04], and they conjectured that it remains true up to dimension $n \leq 6$. On the other hand De Silva and Jerison provided in [DJ09] an example of a nontrivial minimal solution in dimension $n = 7$.

The main consequence of Theorem 1.1 is that it implies the smoothness of the free boundary for minimizers in both the one-phase and two-phase problem in dimension $n \leq 4$. Moreover, by the dimension reduction arguments of Weiss [Wei99], we obtain the following regularity result.

Corollary 1.2. Let $v$ be a minimizer of the energy functional

$$J(v) := \int_{B_1} \left( |\nabla v|^2 + Q_+(x)\chi_{\{v>0\}} + Q_-(x)\chi_{\{v\leq 0\}} \right) dx$$

with $Q_{\pm}$ smooth functions satisfying $Q_+ - Q_- > 0$. Then the free boundary

$$F(v) := \partial\{v > 0\} \cap B_1$$

is a smooth hypersurface except possibly on a closed singular set $\Sigma \subset F(v)$ of Hausdorff dimension $n - 5$, and

$$(v^+_{\nu})^2 - (v^-_{\nu})^2 = Q_+ - Q_- \text{ on } F(v)\setminus \Sigma.$$
2.1 Normals for second derivatives at the boundary. Fix a point 
\[ x_0 \in \partial \Omega \setminus \{0\} \]
and choose a system of coordinates at \( x_0 \) such that 
\[ e_n = \nu_{x_0} \]
the interior normal at \( x_0 \)
and \( \partial \Omega \) is given locally by the graph of a function 
\[ \Omega = \{ x_n > g(x') \}, \quad \text{with} \quad \nabla_{x'} g(x'_0) = 0, \quad D^2_{x'} g(x'_0) \]
diagonal. By differentiating \( u(x', g(x')) = 0 \) in the \( i, j \) directions, \( i, j < n \), we obtain
\[ u_i = 0, \quad u_{ij} = -u_n g_{ij} = -g_{ij} \]
at \( x_0 \), (2.1)
where subscripts indicate partial derivatives. Differentiating \(|\nabla u|^2(x', g(x')) = 1\), we obtain
\[ \sum_{k=1}^{n} u_k u_{ki} = 0, \quad \sum_{k=1}^{n} (u_k u_{kij} + u_{ki} u_{kj}) = -u_n u_{nn} g_{ij} \]
for \( i, j < n \). In conclusion, applying \( u_n = 1 \) and (2.1) at \( x_0 \) we have,
\[ u_{in} = 0 \quad \text{at} \quad x_0, \quad i < n. \]
Consequently, \( D^2 u \) is diagonal at \( x_0 \), and
\[ u_{ijn} = 0 \quad \text{if} \quad i, j < n \quad \text{and} \quad i \neq j, \]
\[ u_{iin} = u_{nn} u_{ii} - u_{ii}^2 \quad \text{for each} \quad i < n, \]
\[ u_{nnn} = \sum_{k=1}^{n} u_{kk}^2, \]
where the last equation follows from the sum over \( i \) of the previous one and \( \triangle u = \triangle u_n = 0 \).

2.2 The linearized equation. A smooth function \( v : \overline{\Omega} \to \mathbb{R} \) solves the linearized equation for a solution \( u \) if
\[ \left\{ \begin{array}{l} \triangle v = 0 \quad \text{in} \quad \Omega, \\ v_\nu = u_{\nu \nu} v \quad \text{on} \quad \partial \Omega \setminus \{0\}. \end{array} \right. \]
(2.3)
Notice that from \( \triangle u = 0 \) and (2.1) it follows that
\[ -u_{\nu \nu} = H \]
where \( H \) denotes the mean curvature of \( \partial \Omega \) oriented towards the complement of \( \Omega \). Thus the second equation in (2.3) can be rewritten as
\[ v_\nu + H v = 0 \quad \text{on} \quad \partial \Omega. \]
In the case when $\Omega$ is a cone different from a half-space, it easily follows that $H > 0$.

Indeed, $|\nabla u|^2/2$ is a subharmonic function homogeneous of degree 0, and its maximum occurs on the boundary. Then either $|\nabla u|^2/2$ is constant or by the Hopf lemma its normal derivative on $\partial \Omega$, which equals $-H$, is negative.

The linearized equation (2.3) is obtained by requiring that $(u + \epsilon v)^+$ solves the original equation up to an error of order $O(\epsilon^2)$ (here we think that $u$ and $v$ are extended smoothly in a neighborhood of $\partial \Omega$). Thus the function $v$ above represents the infinitesimal vertical distance between the graph of a perturbed solution and the graph of the original solution $u$ of (1.1).

We deduce (2.3) briefly. The interior condition for $v$ is obvious. For the boundary condition we see that the free boundary of $(u + \epsilon v)^+$ lies within $O(\epsilon^2)$ of the surface $\Gamma_\epsilon$ obtained as

$$x \in \Gamma_0 := \partial \Omega \mapsto x_\epsilon \in \Gamma_\epsilon, \quad x_\epsilon := x - \epsilon v(x) \nu_x.$$ 

Thus

$$\nabla (u + \epsilon v)(x_\epsilon) = \nabla u(x) - D^2 u(x)(x_\epsilon - x) + \epsilon \nabla v(x_\epsilon) + O(\epsilon^2)$$

$$= \nu - \epsilon \nu (D^2 u) + \epsilon \nabla v(x) + O(\epsilon^2)$$

and

$$|\nabla (u + \epsilon v)(x_\epsilon)|^2 = 1 + 2\epsilon(v_\nu - v u_{\nu\nu}) + O(\epsilon^2),$$

which gives the second condition in (2.3).

Clearly the directional derivatives $v = e \cdot \nabla u$ solve the linearized equation, since they arise from translation of the solution $u$. The boundary equation can also be seen directly in the coordinates of Section 2 for which $D^2 u$ is diagonal at $x_0$: $v = e \cdot \nu$ and $v_\nu = v_n = (e \cdot \nu) u_{nn} = u_{\nu\nu}\nu$.

### 2.3 Criteria for stability and instability.

Let $u$ be a homogeneous one-phase free boundary solution $u$ as in (1.1) supported on the cone $\Omega$. Consider the annulus

$$U = \{ x \in \mathbb{R}^n : 0 < c_1 < |x| < c_2 \}.$$ 

The main lemma of [CJK04] says that the stability (1.3) under perturbations in $U$ implies that for all smooth functions $f$ supported in $U$,

$$\int_{\partial \Omega} H f^2 d\sigma \leq \int_{\Omega} |\nabla f|^2 dx.$$

We will deduce from (2.4) a criterion for instability in the form we will need, that is, expressed in terms of subsolutions.
We say that \( v \) is a subsolution to the linearized equation (2.3) in \( \Omega \cap U \) if
\[
\begin{cases}
\triangle v \geq 0 & \text{in } \Omega \cap U, \\
v_{\nu} + Hv \geq 0 & \text{on } U \cap \partial \Omega,
\end{cases}
\] (2.5)
with
\[ v \geq 0 \quad \text{on } \Omega \cap U, \quad v = 0 \quad \text{on } \Omega \cap \partial U. \]

It follows from integration by parts that if there is a strict subsolution \( v \) as in (2.5), then \( u \) is unstable. Indeed,
\[
\int_{\Omega} |\nabla v|^2 \, dx = -\int_{\Omega} v \triangle v \, dx - \int_{\partial \Omega} vv_{\nu} \, d\sigma \leq -\int_{\partial \Omega} vv_{\nu} \, d\sigma \leq \int_{\partial \Omega} Hv^2 \, d\sigma.
\] (2.6)
If either inequality in (2.6) is strict, then \( u \) is unstable in \( U \).

We prove Theorem 1.1 by constructing an explicit subsolution \( v \) to (2.5) which depends on the second derivatives of \( u \). The function \( v \) is a product of spherical and radial parts. Denote by \( \Omega_S \) the intersection of \( \Omega \) with the unit sphere and write \( \triangle_S \) for the Laplacian on the sphere. The following result is implicit in [CJK04], but not stated or used directly there.

**Proposition 2.1.** Suppose there is a nonnegative function \( \varphi \) defined on \( \bar{\Omega}_S \) that is a strict subsolution to
\[
\begin{cases}
\triangle_S \varphi \geq \lambda \varphi & \text{in } \Omega_S, \\
\varphi_{\nu} + H \varphi \geq 0 & \text{on } \partial \Omega_S,
\end{cases}
\]
and suppose that the constant \( \lambda \) satisfies
\[ \lambda \geq \frac{(n-2)^2}{4}. \]
Then \( u \) is unstable in the sense that (1.3) fails for some perturbation \( \Psi \) in a sufficiently large annulus.

**Proof.** Define \( \Lambda \) by
\[
-\Lambda := \inf_{\psi} \frac{\int_{\Omega_S} |\nabla \psi|^2 - \int_{\partial \Omega_S} H \psi^2}{\int_{\Omega_S} \psi^2}.
\] (2.7)
As in (2.6), an integration by parts and the assumption that \( \varphi \) is a strict subsolution yields
\[
\int_{\Omega_S} |\nabla \varphi|^2 - \int_{\partial \Omega_S} H \varphi^2 < -\lambda \int_{\Omega_S} \varphi^2,
\]
so that \( \Lambda > \lambda \). It is well known that the minimizer \( \bar{\psi} \) of (2.7) exists and satisfies \( \bar{\psi} \in C^\infty(\bar{\Omega}_S) \), \( \bar{\psi} > 0 \) in \( \Omega_S \), and
\[
\triangle_S \bar{\psi} = \Lambda \bar{\psi} \quad \text{on } \Omega_S; \quad \bar{\psi}_{\nu} + H \bar{\psi} = 0 \quad \text{on } \partial \Omega_S.
\]
Extend $\bar{\psi}$ to be homogeneous of degree 0 on $\Omega$ and define $v := f(r)\bar{\psi}; r = |x|$. Then

$$\Delta v = (f'' + (n-1)f'/r + \Lambda f/r^2)\bar{\psi}. $$

On the other hand, it is straightforward to check that if $f$ satisfies the constant coefficients ODE

$$f'' + \alpha f'/r + \beta f/r^2 = 0,$$

then $f$ oscillates around 0 if and only if

$$4\beta > (\alpha - 1)^2.$$

Let $\alpha = n - 1$. Since $\Lambda > \lambda = (n-2)^2/4$, we may choose $\beta$ so that

$$\Lambda > \beta > (\alpha - 1)^2/4 = (n-2)^2/4,$$

and let $\mathcal{U}$ be the annular region between two consecutive zeros of $f$ where $f$ is positive. Then $v > 0$ on $\Omega \cap \mathcal{U}$, and

$$\Delta v = (\Lambda - \beta)f\bar{\psi}/r^2 > 0 \text{ in } \Omega \cap \mathcal{U}.$$

Moreover, since $f$ is radial, $v_\nu + Hv = 0$ on $\partial\Omega$ and $v = 0$ on $\Omega \cap \partial\mathcal{U}$ because $f = 0$ on $\partial\mathcal{U}$. Therefore $v$ is a strict subsolution for (2.5), and $u$ is unstable. \qed

It remains to find the function $\varphi$. It will turn out that $\varphi$ is constructed using functions that are homogeneous of degree $-\mu \neq 0$, so we will rewrite Proposition 2.1 as follows.

**Proposition 2.2.** If there exists $\bar{v} \geq 0$, homogeneous of degree $-\mu$ on $\Omega$, that is a strict subsolution for the following problem

$$\begin{cases}
\Delta \bar{v} \geq \gamma \bar{v}/|x|^2 & \text{in } \Omega, \\
\bar{v}_\nu + H \bar{v} \geq 0 & \text{on } \partial\Omega \setminus \{0\},
\end{cases} \quad (2.8)$$

and the constant $\gamma$ satisfies

$$\gamma \geq \left(\frac{n}{2} - 1 - \mu\right)^2, \quad (2.9)$$

then $u$ is unstable.

Note that (2.8) is equivalent to

$$\begin{cases}
\Delta (\log \bar{v}) + |\nabla (\log \bar{v})|^2 \geq \gamma/|x|^2 & \text{in } \Omega \cap \{\bar{v} > 0\}, \\
\frac{1}{|x|}(\log \bar{v})_\nu \geq -1 & \text{on } \partial\Omega \cap \{\bar{v} > 0\}. \quad (2.10)
\end{cases}$$
Proof. The function $\bar{v}$ satisfies
\[ \Delta_S \bar{v} \geq (\gamma + \mu(n - 2 - \mu)) \bar{v} \text{ on } \Omega_S \]
and the condition
\[ \gamma + \mu(n - 2 - \mu) \geq (n - 2)^2 / 4 \]
is the same as (2.9). \qed

Although we do not need this in the sequel, we remark that the sufficient conditions stated here for instability are also necessary, as shown in the following proposition.

**Proposition 2.3.** The following are equivalent.

(a) The stability inequality (2.4) holds for all $f \in C_0^\infty(U)$.

(b) There exists $\bar{f}$ satisfying $\bar{f} > 0$ and $\Delta \bar{f} = 0$ in $\Omega \cap U$ and the boundary condition
\[ \bar{f}_\nu + H \bar{f} = 0 \text{ on } U \cap \partial \Omega. \]

(c) There are no nonnegative strict subsolutions as in (2.5), on any annulus $U' \subset U$, that is, no $v \geq 0$ strict subsolutions to
\[ \Delta v \geq 0 \text{ in } \Omega \cap U'; \quad v_\nu + H v \geq 0 \text{ on } U' \cap \partial \Omega; \quad v = 0 \text{ on } \Omega \cap \partial U'. \]

**Proof.** To prove that a) implies b), note that the minimizer $\bar{f}$ to
\[ \inf_{f \in C_0^\infty(U)} \int_\Omega |\nabla f|^2 - \int_{\partial \Omega} Hf^2 \]
satisfies the required properties. To prove that b) implies c), observe that if $v$ existed, then $\Delta (v/\bar{f}) \geq 0$ on $\Omega \cap U'$ and $(v/\bar{f})_\nu \geq 0$ on $\partial(\Omega \cap U')$ so that $v/\bar{f}$ is constant. But $v$ cannot be a multiple of $\bar{f}$.

Finally, we prove that c) implies a) by establishing the contrapositive. Suppose that a) is false. Then there exists a slightly smaller annulus $U' \subset\subset U$ for which
\[ -\delta := \inf_{f \in C_0^\infty(U')} \frac{\int_\Omega |\nabla f|^2 - \int_{\partial \Omega} Hf^2}{\int_\Omega f^2} < 0. \]
The minimizer $g$ is a nonnegative strict subsolution in $\Omega \cap U'$. The strictness follows from $\Delta g = \delta g > 0$. This shows that c) does not hold. \qed

Proposition 2.3 says in particular that the stability of a solution $u$ in a region is equivalent to the existence of a positive solution to the linearized equation in the same region. In fact, in non-variational elliptic problems this characterization can be taken as the definition of stability. Typically in such non-variational problems, when such a positive solution exists, then, in a neighborhood of the graph of $u$, the space can be foliated by perturbed solutions. By contrast, the existence of a strict subsolution is essentially equivalent to saying that solutions to the linearized equation must change sign and corresponds in the nonlinear setting to the case when the graph of $u$ and the graph of “nearby” perturbed solutions “cross each other.”
3 The Case $w = \|D^2 u\|$ 

In this section we show that 

$$\bar{v} = w^\alpha$$

satisfies an inequality of the type (2.8) where $w := \|D^2 u\|$, that is 

$$w^2 := \|D^2 u\|^2 = \sum_{i,j=1}^n u_{ij}^2.$$

3.1 The interior inequality. First we obtain an inequality for harmonic functions which is similar to Simons’s inequality for minimal surfaces.

**Proposition 3.1.** Assume $u$ is harmonic and homogeneous of degree 1. Then 

$$w \triangle w \geq \frac{2}{n-1} |\nabla w|^2 + 2 \frac{n-2}{n-1} \frac{w^2}{|x|^2},$$

in the set $\{w > 0\}$.

**Proof.** We have 

$$ww_k = \sum_{i,j=1}^n u_{ij} u_{ijk} \quad \text{for each } k = 1, \ldots, n,$$

and 

$$w \triangle w + |\nabla w|^2 = \sum_{i,j,k=1}^n (u_{ijk}^2 + u_{ij} u_{ijk k}) = \sum_{i,j,k=1}^n u_{ijk}^2.$$ 

(3.1)

Since $u$ is homogeneous of degree one, the radial direction $x/|x|$ is an eigenvector for $D^2 u$. We choose a system of coordinates such that $e_1$ points in the radial direction at $x$. Then 

$$u_{1i} = 0 \quad \text{for each } i = 1, \ldots, n,$$

and since $u_{ij}$ are homogeneous of degree $-1$ we obtain 

$$u_{11j} = -\frac{u_{ij}}{|x|}, \quad u_{11i} = 0.$$ 

Choosing the remaining coordinates $e_j$, $j \geq 2$ so that $D^2 u$ is diagonal at $x$, we have 

$$w^2 = \sum_{i=1}^n u_{ii}^2, \quad w_k = \sum_{i=1}^n \frac{u_{ii}}{w} u_{iik}.$$ 

Thus by the Cauchy–Schwarz inequality, for each $k$, 

$$w_k^2 \leq \sum_i \left( \frac{u_{ii}}{w} \right)^2 \sum_i u_{iik}^2 = \sum_i u_{iik}^2.$$
Then
\[ \sum_{i,j,k} u_{ijk}^2 = \sum_{i,k} u_{iik}^2 + \sum_{i \neq j,k} u_{ijk}^2 \geq |\nabla w|^2 + 2 \sum_{i \neq k} u_{iik}^2. \] (3.2)

Next we estimate for each \( k \) the sum in the last term above. If \( k = 1 \), then
\[ \sum_{i \neq 1} u_{ii1}^2 = \sum_{i \neq 1} \left( \frac{u_{ii}}{|x|} \right)^2 = \frac{w^2}{|x|^2}. \] (3.3)

If \( k \neq 1 \), then we use \( \Delta u_k = 0 \) and \( u_{11k} = 0 \) and obtain
\[ \sum_{i \neq k} u_{iik} = -u_{kkk} \implies (n-2) \sum_{i \neq k} u_{iik}^2 \geq u_{kkk}^2, \quad k \neq 1, \text{fixed}. \]

Now, substituting
\[ u_{iik}^2 = \frac{1}{n-1} u_{iik}^2 + \frac{n-2}{n-1} u_{iik}^2, \]
we find that
\[ \sum_{i \neq k} u_{iik}^2 \geq \frac{1}{n-1} \sum_{i \neq k} u_{iik}^2 + \frac{1}{n-1} u_{kkk}^2 = \frac{1}{n-1} \sum_i u_{iik}^2 \geq \frac{1}{n-1} w_k^2, \quad k \neq 1, \text{fixed}. \]

Using (3.2)–(3.4) in (3.1), we have
\[ w \triangle w = \sum_{i,j,k} u_{ijk}^2 - |\nabla w|^2 \geq 2 \sum_{i \neq k} u_{iik}^2 \geq 2 \frac{w^2}{|x|^2} + \frac{1}{n-1} \sum_{k=2}^n w_k^2 \]
and remarking that
\[ w_1 = -\frac{w}{|x|}, \]
since \( w \) is homogeneous of degree \(-1\), the inequality of Proposition 3.1 is established.

\[ \square \]

**Corollary 3.2.** The function \( \bar{v} = w^\alpha \), which is homogeneous of degree \(-\alpha\), satisfies
\[ \Delta \bar{v} \geq \alpha(\alpha + 1) \frac{\bar{v}}{|x|^2} \quad \text{for all} \quad \alpha \geq 1 - \frac{2}{n-1}, \] (3.5)

**Proof.** The conclusion of Proposition 3.1 can be written as
\[ \Delta (\log w) \geq \left( \frac{2}{n-1} - 1 \right) |\nabla (\log w)|^2 + 2 \frac{n-2}{n-1} \frac{1}{|x|^2}, \]
or
\[ \Delta (\alpha \log w) + |\nabla (\alpha \log w)|^2 \geq \alpha \left( \frac{2}{n-1} - 1 + \alpha \right) |\nabla (\log w)|^2 + 2\alpha \frac{n-2}{n-1} \frac{1}{|x|^2}. \]
Using

\[ |\nabla(\log w)| \geq \frac{w^2}{w^2} = \frac{1}{|x|^2}, \]

and \( \alpha \geq 1 - 2/(n - 1) \), we see that

\[
\triangle(\alpha \log w) + |\nabla(\alpha \log w)|^2 \geq \alpha \left( \frac{2}{n - 1} - 1 + \alpha \frac{2n - 2}{n - 1} \right) \frac{1}{|x|^2} = \alpha(\alpha + 1) \frac{1}{|x|^2}.
\]

It follows that (3.5) holds on the set \( \bar{v} > 0 \). On the other hand, at points of \( \{\bar{v} = 0\} \cap \Omega \), \( \Delta \bar{v} \geq 0 \) holds in the sense of viscosity since \( \bar{v} \geq 0 \).

3.2 The boundary inequality. We have

\[
w^2 = \sum_{i,j} u_{ij}^2 \implies w w_n = \sum_{i,j} u_{ij} u_{ijn}.
\]

Fix a point \( x_0 \) on \( \partial\Omega \setminus \{0\} \) and choose a system of coordinates as in Section 2, i.e. such that \( e_n = \nu_{x_0} \), \( D^2 u(x_0) \) is diagonal, and \( e_1 \) coincides with the radial direction \( x_0/|x_0| \). We recall (2.2)

\[
u_{iin} = u_{nn} u_{ii} - u_{ii}^2 \quad \text{for all } i < n, \]

\[u_{nnn} = w^2.
\]

Thus, using \( u_{nn} = -H \),

\[w w_n = u_{nn} w^2 + \sum_{i<n} (u_{nn} u_{ii}^2 - u_{ii}^3)
\]

\[= u_{nn} w^2 + \sum_{k=1}^n (u_{nn} u_{kk}^2 - u_{kk}^3)
\]

\[= -2H w^2 - \sum_{k=1}^n u_{kk}^3.
\]

Therefore,

\[
\frac{1}{H} (\log w)_n = - \left( 2 + \frac{1}{H w^2} \sum_{k=1}^n u_{kk}^3 \right) \geq -L
\]

where

\[
L := \max_{\partial \Omega} \left( 2 + \frac{1}{H w^2} \sum_{k=1}^n u_{kk}^3 \right)
\]
and we see that the function $\bar{v} = u^\alpha$ satisfies
\[ \frac{1}{H} (\log \bar{v})_\nu \geq -1 \quad \text{if} \quad \alpha \leq \frac{1}{L}. \] (3.7)

From (3.5), (3.7) and Proposition 2.2 we see that $u$ is unstable if there exists $\alpha$ such that
\[ \alpha \geq 1 - \frac{2}{n-1}, \quad \alpha \leq \frac{1}{L}, \quad \text{and} \quad \alpha(\alpha + 1) > \left( \frac{n}{2} - 1 - \alpha \right)^2. \]

Notice that the second lower bound on $\alpha$ guarantees the first lower bound since the second lower bound is equivalent to
\[ \alpha > \frac{(n-2)^2}{4(n-1)} \geq 1 - \frac{2}{n-1}. \]

We summarize these results in the next proposition.

**Proposition 3.3.** Let $u$ be a solution to (1.1) which is not one-dimensional. Then $u$ is unstable if
\[ \frac{(n-2)^2}{4(n-1)} < \frac{1}{L}, \] (3.8)
with $L$ given by (3.6). Moreover, $u$ is unstable also in case of equality in (3.8) provided that equality does not hold at all points in (3.5), (3.7).

**Corollary 3.4.** If $u$ is a stable solution to (1.1) in dimension $n = 3$ then $u$ is one-dimensional.

**Proof.** When $n = 3$ the left side of (3.8) is $1/8$, while $L = 2$ since in our coordinate system $u_{11} = 0$ and $u_{22} = -u_{33}$. \[ \square \]

Unfortunately (3.8) need not be true in dimensions $4 \leq n \leq 6$. To see this we express $L$ at $x_0 \in \partial\Omega$ in terms of the relative sizes of the $n-2$ nonvanishing curvatures of $\partial\Omega$ at that point. Let $\kappa_\ell$, $\ell = 2, \ldots, n-1$, denote the curvatures of $\partial\Omega$ with respect to the outer normal, $(\kappa_1 = 0$ since $e_1$ is the radial direction). Define
\[ \mu_\ell := \frac{\kappa_\ell}{H} \quad \Rightarrow \quad u_{\ell\ell} = \mu_\ell H, \quad u_{nn} = -H. \]

Recall that $H = \sum_{\ell<n} \kappa_\ell > 0$. Since $\Delta u = 0$,
\[ \sum_{\ell=2}^{n-1} \mu_\ell = 1. \]

Thus an upper bound for $L$ in (3.6) is given by
\[ L \leq L^* := \sup \left\{ 2 + \frac{\sum \mu_\ell^3 - 1}{1 + \sum \mu_\ell^2} : \sum \mu_\ell = 1 \right\} \quad \text{(sums from} \ell = 2, \ldots, n-1). \] (3.9)
It is not hard to show that when \( n = 4 \), \( L^* = 7/2 \), whereas the left hand side in (3.8) is \( 1/3 \). Moreover, if \( n \geq 5 \), then \( L^* = \infty \).

We remark however that condition (3.8) gives the sharp result in the case when all curvatures are equal, i.e. the axis symmetric case. Then \( L = L^* = (n - 1)/(n - 2) \) and (3.8) holds for \( n \leq 5 \). When \( n = 6 \) we have equality in (3.8), but in this case, the equality in (3.5) is strict. Indeed, choosing
\[
\alpha = \frac{1}{L} = \frac{4}{5} > \frac{3}{5} = 1 - \frac{2}{n-1},
\]
the computation in the proof of Corollary 3.2 shows that
\[
\triangle (\alpha \log w) + |\nabla (\alpha \log w)|^2 \geq \alpha (\alpha + 1) \frac{1}{|x|^2} + \alpha \left( \frac{2}{n-1} - 1 + \alpha \right) \frac{w_n^2}{w^2}.
\]

The last term is positive on \( \partial \Omega \) because \( w_n/w = -HL < 0 \).

Finally we point out the main difference with the minimal surface theory. Proposition 3.3 requires an exponent \( \alpha \) satisfying
\[
(n - 2)^2/4(n - 1) \leq \alpha \leq 1/L.
\]
The lower bound is essentially maximized when \( D^2u \) has only one negative eigenvalue and the remaining ones are positive and equal (as in the axis symmetric case). On the other hand, the upper bound is minimized (that is, \( L \to \infty \), \( n \geq 5 \)) when, on the boundary \( \partial \Omega \), one tangential eigenvalue is positive and the remaining ones are negative. In other words, the constraints on \( \alpha \) that come from the interior inequality and boundary inequalities are individually nearly optimal but they are achieved for different configurations. This is one way to understand why (3.8) is not sufficient to prove instability in the conjectured optimal range, that is, for \( n \leq 6 \).

Our computation is somewhat consistent with the findings of Hong in [Hon15] where he studied the stability of Lawson-type cones for (1.1) in low dimensions. It turns out that in dimension \( n = 7 \) there are in fact two different stable cones corresponding precisely to the two situations described above.

### 4 The Case \( w^2 = \sum_{\lambda_k > 0} \lambda_k^2 + a \sum_{\lambda_k < 0} \lambda_k^2 \)

In this section we proceed as in Section 3 for a different choice of \( w \) i.e.
\[
w^2 := \sum_{\lambda_k > 0} \lambda_k^2 + a \sum_{\lambda_k < 0} \lambda_k^2, \tag{4.1}
\]
for some constant \( a > 0 \). Here \( \lambda_k \) represent the eigenvalues of \( D^2u \).

When \( a = 1 \) then \( w \) coincides with the function considered in Section 3. We show that when \( a = 4 \) and \( n = 4 \), the interior inequality remains the same as in Section 3, however the boundary inequality improves from \( L \leq 7/2 \) to \( L \leq 3 \) and allows us to prove Theorem 1.1.
4.1 Functions of the eigenvalues. Assume

\[ F(D^2 u) = f(\lambda_1, \ldots, \lambda_n), \]

with \( f \in C^1 \) a symmetric function of its arguments. We choose a system of coordinates at a point \( x \in \Omega \) such that

\[ D^2 u = \text{diag}(\lambda_1, \ldots, \lambda_n), \]

and we use the following orthonormal basis in the space of symmetric matrices

\[ e_{ii} := e_i \otimes e_i, \quad e_{ij} := \frac{1}{\sqrt{2}}(e_i \otimes e_j + e_j \otimes e_i) \quad \text{for } i < j. \]

Then one can check that

\[ F_{e_{ii}}(D^2 u) = f_{\lambda_i} \quad \text{and} \quad F_{e_{ij}}(D^2 u) = 0. \]  

(4.2)

Moreover, if \( f \in C^2 \) then

\[ F_{e_{ii}, e_{kk}}(D^2 u) = f_{\lambda_i, \lambda_k}, \]

\[ F_{e_{ij}, e_{ij}}(D^2 u) = \begin{cases} \frac{f_{\lambda_i} - f_{\lambda_j}}{\lambda_i - \lambda_j} & \text{if } \lambda_i \neq \lambda_j, \\ f_{\lambda_i, \lambda_i} - f_{\lambda_j, \lambda_j} & \text{if } \lambda_i = \lambda_j, \quad i \neq j, \\ 0 & \text{if } e_{ij} \neq e_{kl}, \ i < j. \end{cases} \]

These can be checked from the fact that the eigenvalues of the matrix

\[ \begin{pmatrix} \lambda_1 & \epsilon \\ \epsilon & \lambda_2 \end{pmatrix} \]

are

\[ \lambda_1 + \frac{\epsilon^2}{\lambda_1 - \lambda_2} + O(\epsilon^3) \quad \text{and} \quad \lambda_2 + \frac{\epsilon^2}{\lambda_2 - \lambda_1} + O(\epsilon^3) \quad \text{if } \lambda_1 \neq \lambda_2 \]

or

\[ \lambda_1 + \epsilon, \quad \lambda_2 - \epsilon \quad \text{if } \lambda_1 = \lambda_2. \]

4.2 The interior inequality. We show that the function \( w \) defined in (4.1) satisfies the same differential inequality as in Proposition 3.1. Rather surprisingly we can prove a more general statement: any convex, symmetric, homogeneous of degree one function of the eigenvalues satisfies the same conclusion as Proposition 3.1.
Theorem 4.1. Assume $\triangle u = 0$ and let

$$w = F(D^2 u) := f(\lambda_1, \ldots, \lambda_n),$$

with $f$ a convex, symmetric, homogeneous of degree one function. Then

$$w \triangle w \geq \frac{2}{n} |\nabla w|^2.$$

Moreover, if $u$ is homogeneous of degree 1, the inequality can be improved to

$$w \triangle w \geq \frac{2}{n-1} |\nabla w|^2 + \frac{n-2}{n-1} w^2 |x|^2.$$

(The inequalities above are understood in the viscosity sense.)

We remark that the hypotheses on $f$ easily imply $f \geq 0$. Notice that the first inequality is equivalent to $w^{1-\frac{2}{n}}$ is subharmonic, or in the case $n = 2$ that $\log w$ is subharmonic.

Proof. We assume that $f$ is smooth in $\mathbb{R}^n \setminus \{0\}$. Then the general case easily follows by approximation. Also, it suffices to show the inequality in the set $\{w > 0\}$ since it is obvious in $\{w = 0\}$.

Fix a point $x$ with $D^2 u(x) \neq 0$, and we choose a system of coordinates at $x$ such that

$$D^2 u = \text{diag}(\lambda_1, \ldots, \lambda_n).$$

First we show that

$$(f_{\lambda_i} - f_{\lambda_j})(\lambda_i - \lambda_j) \geq 0,$$  (4.3)

and the inequality is strict if $f$ is strictly convex and $\lambda_i \neq \lambda_j$.

Indeed, let $Z_0 := (\lambda_1, \ldots, \lambda_n)$ and let $Z_1$ denote the vector obtained from $Z_0$ after interchanging $\lambda_i$ with $\lambda_j$. Using the symmetry and convexity of $f$ we obtain

$$0 = f(Z_1) - f(Z_0) \geq \nabla f(Z_0) \cdot (Z_1 - Z_0),$$

and this gives our claim (4.3).

In view of Section 4.1, for each $k$ we have

$$w_k = f_{\lambda_i} u_{iik},$$  (4.4)

and

$$w_{kk} = \sum_i f_{\lambda_i} u_{iikk} + \sum_{i,j} f_{\lambda_i, \lambda_j} u_{iik} u_{jjk} + 2 \sum_{i<j} F_{\epsilon_{i,j}, e_{i,j}} u_{ijk}^2.$$  

Summing over $k$ and using that $f$ is convex and $\triangle u_{ii} = 0$ we find

$$\triangle w \geq 2 \sum_{i<j} \sum_k F_{\epsilon_{i,j}, e_{i,j}} u_{ijk}^2.$$
Notice that all such terms are nonnegative since, by (4.3), $F_{e_{ij}, e_{ij}} \geq 0$. We keep only the terms for which either $i = k$ or $j = k$ and obtain

$$\Delta w \geq 2 \sum_{i \neq j} F_{e_{ij}, e_{ij}} u_{iij}^2,$$

where $i, j$ run over $\{1, \ldots, n\}$ with $i \neq j$.

In order to obtain our inequality it suffices to show that

$$\sum_{i \neq k} F_{e_{ik}, e_{ik}} u_{iik}^2 \geq \frac{1}{n} \frac{w_k^2}{w}$$

for each fixed $k$. (4.5)

From (4.4) and $\Delta u_k = 0$ we find

$$w_k = \sum_{i \neq k} (f_{\lambda_i} - f_{\lambda_k}) u_{iik} \quad (k \text{ fixed}).$$

Notice that from Section 4.1 and the symmetry of $f$ we have

$$f_{\lambda_i} - f_{\lambda_k} = (\lambda_i - \lambda_k) F_{e_{ik}, e_{ik}}.$$

Hence by the Cauchy–Schwarz inequality and (4.3) we obtain

$$w_k^2 \leq \left( \sum_{i \neq k} F_{e_{ik}, e_{ik}} u_{iik}^2 \right) \left( \sum_{i \neq k} (\lambda_i - \lambda_k)(f_{\lambda_i} - f_{\lambda_k}) \right).$$

Thus, in order to prove (4.5) it suffices to show that

$$\sum_{i \neq k} (\lambda_i - \lambda_k)(f_{\lambda_i} - f_{\lambda_k}) \leq nf$$

for each fixed $k$. (4.6)

Indeed, using that $\sum \lambda_i = 0$, $\sum \lambda_i f_{\lambda_i} = f$, and summing over both $i$ and $j$, we obtain the identity

$$\sum_{i < j} (\lambda_i - \lambda_j)(f_{\lambda_i} - f_{\lambda_j}) = \sum_i f_{\lambda_i} \sum_{j \neq i} (\lambda_i - \lambda_j) = \sum_i f_{\lambda_i} n \lambda_i = nf.$$  

Our claim (4.6) follows since, by (4.3), the left hand side in (4.6) is bounded above by the left hand side of (4.7).

Remark. From the equality above and (4.3) we see that, if $f$ is strictly convex in a neighborhood of $Z_0 = (\lambda_1, \ldots, \lambda_n)$, we have equality in (4.6) only when all $\lambda_i$ with $i \neq k$ are equal. In other words, the coefficient of $1/n$ on $w_k^2/w$ in (4.5) can be replaced by $1/n + \epsilon$ in a neighborhood of $x$, if $\lambda_i \neq \lambda_j$ for some $i, j \neq k$ and $f$ is strictly convex near $Z_0$ in the 2-dimensional plane generated by the $\lambda_i, \lambda_j$ directions. Here $\epsilon > 0$ depends on $(\lambda_i - \lambda_j)(f_{\lambda_i} - f_{\lambda_j})$. 
We conclude with the case when \( u \) is homogeneous of degree 1 and show that the inequalities above can be improved. We assume that at the point \( x \), the \( e_1 \) direction represents the radial direction \( x/|x| \), thus

\[
\lambda_1 = 0, \quad u_{ij} = -\frac{u_{ij}}{|x|}.
\]

Let \( k \neq 1 \). Then, the coefficient of \( 1/n \) in (4.5) can be replaced by \( 1/(n-1) \). Indeed, \( u_{11k} = 0, \lambda_1 = 0 \), thus the index \( i = 1 \) can be ignored in the computations above, and we reduce the problem to \( n-1 \) variables.

When \( k = 1 \) the left hand side of (4.6) equals \( f \) since

\[
\sum_{i \neq 1} (\lambda_i - \lambda_1)(f_{\lambda_i} - f_{\lambda_1}) = \sum_{i \neq 1} \lambda_i f_{\lambda_i} = f,
\]

where we used \( \lambda_1 = 0, \sum \lambda_i = 0 \). This shows that the coefficient of \( w^2_1/w^2 \) in (4.5) can be replaced by 1. Since \( w \) is homogeneous of degree \(-1\) we also have \( w_1 = -w/|x| \), and we obtain

\[
w\Delta w \geq \frac{2}{n-1} \sum_{k=2}^n w^2_k + 2w^2_1 = \frac{2}{n-1} |\nabla w|^2 + 2\frac{n-2}{n-1} w^2/|x|^2. \tag*{□}
\]

### 4.3 The boundary inequality.

We show that the function \( w \) defined in (4.1), when \( a = 4, n = 4 \) satisfies

\[
\frac{1}{H} (\log w)_{\nu} \geq -3. \tag{4.8}
\]

Notice that \( w \in C^{1,1} \) in the set \( \{w \neq 0\} \).

Let \( x_0 \in \partial \Omega \setminus \{0\} \) and we choose a system of coordinates as before i.e. with \( D^2 u(x_0) \) diagonal, \( e_n = v_{x_0} \) and \( e_1 = x_0/|x_0| \). Denote by \( i \) and \( s \) the indices for which \( \lambda_i > 0 \) and \( \lambda_s < 0 \), respectively. From (4.4), (2.2) and \( \lambda_n = -H \), we have

\[
w_n = \sum_i \frac{\lambda_i}{w} u_{ii} + a \sum_s \frac{\lambda_s}{w} u_{ssn}
\]

\[
= \sum_i \frac{\lambda_i}{w} (-H\lambda_i - \lambda_i^2) + a \sum_{s \neq n} \frac{\lambda_s}{w} (-H\lambda_s - \lambda_s^2) + a \frac{\lambda_n}{w} \sum_{k=1}^n \lambda_k^2
\]

\[
= -\frac{H}{w} \left( \sum_i \lambda_i^2 + a \sum_{s \neq n} \lambda_s^2 + a \lambda_n^2 \right) - \frac{1}{w} \left( \sum_i \lambda_i^3 + a \sum_{s \neq n} \lambda_s^3 + a\lambda_n^3 \right) - \frac{aH}{w} \sum_{k=1}^n \lambda_k^2.
\]

Multiplying by \(-1/Hw\), we write this in more compact form as

\[
-\frac{w_n}{Hw} = 1 + \frac{\sum \lambda^3_i + a \sum \lambda^3_s + aH \sum \lambda_k^2}{Hw^2}.
\]
where the \( k \) is summed over all indices and, as before, \( i \) is summed over indices for which \( \lambda_i > 0 \), and \( s \) is summed over indices for which \( \lambda_s < 0 \).

Since \( \lambda_1 = 0 \) and \( \lambda_4 < 0 \), we distinguish two cases depending whether \( \lambda_2 \) and \( \lambda_3 \) are both positive or have opposite signs.

**Case 1:** \( \lambda_2 > 0, \lambda_3 \leq 0 \).

Let 
\[
\mu := -\frac{\lambda_3}{H} \quad \text{thus} \quad \frac{\lambda_2}{H} = \mu + 1, \quad \text{and} \quad \mu \geq 0.
\]

We need to show that 
\[
\frac{(1 + \mu)^3 - a\mu^3 + a((1 + \mu)^2 + \mu^2)}{(1 + \mu)^2 + a\mu^2 + a} \leq 2.
\]
This is equivalent to 
\[
(\mu - 1)(a(\mu^2 + \mu - 1) - (\mu + 1)^2) \geq 0,
\]
or, since \( a = 4 \),
\[
(\mu - 1)^2(3\mu + 5) \geq 0,
\]
which is obvious since \( \mu \geq 0 \).

**Case 2:** \( \lambda_2 > 0, \lambda_3 > 0 \).

Let 
\[
\mu := \frac{\lambda_2}{H} \quad \text{thus} \quad \frac{\lambda_3}{H} = 1 - \mu, \quad \text{and} \quad \mu \in (0, 1).
\]

We need to show that 
\[
\frac{\mu^3 + (1 - \mu)^3 + 4(\mu^2 + (1 - \mu)^2)}{\mu^2 + (1 - \mu)^2 + 4} \leq 2.
\]
This is obvious since the numerator is bounded above by 5 thus the fraction is bounded by \( 5/4 < 2 \).

In conclusion (4.8) is proved and equality at a point holds only when 
\[
\lambda_2 > 0, \quad \lambda_3 = \lambda_4 < 0. \tag{4.9}
\]

**Proof of Theorem 1.1.** Let \( n = 4 \) and set 
\[
\bar{v} := w^{\frac{1}{3}}
\]
with \( w \) as in (4.1) and \( a = 4 \). Assume that \( w \) is not identically 0, i.e. \( u \) is not a one-dimensional solution.

By Theorem 4.1 and (4.8) it follows as in Section 3 that \( \bar{v} \) satisfies (2.8). In order to prove that \( u \) is not stable it remains to show that \( \bar{v} \) is a strict subsolution.
We fix a point $x_0 \in \partial \Omega$. If equality holds in (4.8) then, by (4.9), $\lambda_2 \neq \lambda_3$ at $x_0$. Then, from the remark in the proof of Theorem 4.1, it follows that the differential inequality can be improved by adding a term $\epsilon w^2_n$ to the right hand side. Since $w_n = -3Hw < 0$, we find that at $x_0$ we have strict inequality in Theorem 4.1 which in turn gives that $\tilde{v}$ is a strict subsolution for the interior problem in a neighborhood of $x_0$.

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