BROWNIAN MOTIONS ON METRIC GRAPHS WITH NON-LOCAL BOUNDARY CONDITIONS II: CONSTRUCTION

FLORIAN WERNER

ABSTRACT. A pathwise construction of discontinuous Brownian motions on metric graphs is given for every possible set of non-local Feller–Wentzell boundary conditions. This construction is achieved by locally decomposing the metric graphs into star graphs, establishing local solutions on these partial graphs, pasting the solutions together, introducing non-local jumps, and verifying the generator of the resulting process.

1. INTRODUCTION

This article is the final part in a series of works in which we achieve a classification and pathwise construction of Brownian motions on metric graphs. In [6], we defined Brownian motions on metric graphs in accordance with previous works of Itô and McKean [1] and Kostrykin, Potthoff and Schrader [3], that is, as right continuous, strong Markov processes which behave on every edge of the graph like the standard one-dimensional Brownian motion. There, we showed that the generator $A = \frac{1}{2} \Delta$ of every Brownian motion on a metric graph $G$ satisfies at each vertex point $v \in \mathcal{V}$ a non-local Feller–Wentzell boundary condition

$$
\forall f \in \mathcal{D}(A) : p^v_1 f(v) - \sum_{l \in \mathcal{L}(v)} p^v_{2,l} f'_l(v) + \frac{p^v_{3}}{2} f''(v) - \int_{G \setminus \{v\}} (f(g) - f(v)) p^v_4(dg) = 0
$$

for some constants $p^v_1 \geq 0$, $p^v_{2,l} \geq 0$ for each edge $l \in \mathcal{L}(v)$ emanating from $v$, $p^v_3 \geq 0$ and a measure $p^v_4$ on $G \setminus \{v\}$, normalized by

$$
p^v_1 + \sum_{l \in \mathcal{L}(v)} p^v_{2,l} + p^v_3 + \int_{G \setminus \{v\}} (1 - e^{-d(v,g)}) p^v_4(dg) = 1,
$$

and $p^v_4$ being an infinite measure if $\sum_{l \in \mathcal{L}(v)} p^v_{2,l} + p^v_3 = 0$.

After having developed the necessary process transformations of concatenations and process revivals in [8], collected the characteristic properties of Brownian motions on metric graphs in [6] and constructed all Brownian motions on star graphs in [7], we are now in the position to give a complete pathwise construction of Brownian motions on any metric graph for every admissible set of Feller–Wentzell boundary data, thus proving the following existence theorem:
Theorem 1.1. Let $G = (V, E, I, \partial, \rho)$ be a metric graph, and for every $v \in V$ let constants $p_v^1 \geq 0$, $p_v^{1, l} \geq 0$ for each $l \in L(v)$, $p_v^3 \geq 0$ and a measure $p_v^4$ on $G \setminus \{v\}$ be given with
\[
p_v^1 + \sum_{l \in L(v)} p_v^{1, l} + p_v^3 + \int_{G \setminus \{v\}} (1 - e^{-d(v, g)}) p_v^4(dg) = 1,
\]
and
\[
p_v^4(G \setminus \{v\}) = +\infty, \quad \text{if} \quad \sum_{l \in L(v)} p_v^{1, l} + p_v^3 = 0.
\]
Then there exists a Brownian motion $X$ on $G$ which is continuous inside all edges, such that its generator satisfies
\[
\mathcal{D}(A) \subseteq \left\{ f \in C^2_0(G) : \forall v \in V : \right\}
\]
\[
p_v^1 f(v) - \sum_{l \in L(v)} p_v^{1, l} f_l(v) + \frac{p_v^3}{2} f''(v) - \int_{G \setminus \{v\}} (f(g) - f(v)) p_v^4(dg) = 0.
\]

2. Construction Approach

The construction proceeds as follows: We will begin with Brownian motions on star graphs which implement the corresponding “local” boundary conditions (including “small jumps”) at their respective vertices. When the process is started on one of these star graphs and approaches (or jumps to) the vicinity of another vertex, it is killed and revived on the relevant subgraph with the help of concatenation techniques. That way, we obtain a Brownian motion on a general metric graph by successive pastings of partial Brownian motions on star graphs. The accurate construction approach will be laid out in the following.

Technically, we will not start with star graphs, but with the complete metric graph which we then decompose into subgraphs. This approach is necessary, as the subgraphs (that is, at some level, star graphs) must be chosen appropriately in order to construct the correct complete graph at the end, and the topology of the full graph is required for the pathwise construction and the specification of the Feller–Wentzell data.

Let $G = (V, E, I, \partial, \rho)$ be a metric graph having at least two vertices. We will break $G$ up by decomposing the set of vertices into $V = V^{-1} \sqcup V^{+1}$ and defining two “subgraphs” $\tilde{G}^j$, $j \in \{-1, +1\}$, which possess the respective vertices $V^j$ as well as all of the original edges (with their combinatorial structure) not incident with the other vertices $V^{-j}$. As internal edges $i$ which are incident with vertices of both subgraphs are lost, we need to replace them by new external “shadow” edges $e_i^{-1}$, $e_i^{+1}$ on the respective subgraphs, see the upper graph of figure 1.

By iteratively decomposing the subgraphs further up to the level of star graphs, we are able to apply our results of [7] and introduce Brownian motions on $\tilde{G}^{-1}$ and $\tilde{G}^{+1}$ with the desired boundary behavior at their vertices. In order to paste the two processes—and thus the two graphs—together, we need to cut out the excrecent parts of the external “shadow” edges by removing them from the subgraphs.

---

1As in the previous works, we will assume any metric graph discussed here to have no loops (see [6, Section A.2, Remark 3.1]). Furthermore, we restrict our attention to metric graphs with finite sets of edges and vertices. A short introduction to metric graphs can be found in [6, Appendix A].
Figure 1. Decomposition and gluing of metric graphs: The metric graph $\mathcal{G}$ of [6, Figure 1] is decomposed into two “subgraphs” $\mathcal{G}^{-1}$ and $\mathcal{G}^{+1}$ with vertices $\mathcal{V}^{-1} = \{v_1, v_2, v_3\}$ and $\mathcal{V}^{+1} = \{v_4, v_5, v_6\}$. The internal edges $i$ which are incident with vertices of both subgraphs are replaced by new external edges $e_i^{-1}, e_i^{+1}$ on the respective subgraphs. By performing the transformations explained in section 2, subsets of these “subgraphs” are mapped to the subsets $\mathcal{G}^{-1}, \mathcal{G}^{+1}$ of the graph $\mathcal{G}$.

and killing the partial Brownian motions whenever they hit the removed locations. The remaining parts of these external edges need to be reoriented where necessary (as vertices are always initial points of external edges) and then are mapped to the original internal edges in order to get proper subgraphs $\mathcal{G}^{-1}$ and $\mathcal{G}^{+1}$ of the original graph $\mathcal{G}$, see the lower graph of figure 1.

The resulting Brownian motions on $\mathcal{G}^{-1}$ and $\mathcal{G}^{+1}$ can now be pasted together with the help of the alternating copies technique established in [8, Section 3], namely by reviving the subprocesses at the other subgraph whenever they leave the remaining part of one of their shadow vertices (and thus are killed).

This construction approach will cause two main technical difficulties, which will prescribe the order of applied transformations: Firstly, the “global” jumps, that is jumps to other vertices or subgraphs, can only be implemented once the gluing is complete, as their jump destinations do not exist for the original Brownian motions on the subgraphs. They will be implemented by an instant return process with an appropriate revival measure. Moreover, the implementation of the killing portions $(p_i^v, v \in \mathcal{V})$ via jumps to the cemetery must be postponed until the gluing procedure and the introduction of the global jumps is complete. The reason is that, as just mentioned, both procedures will apply the technique of identical/alternating copies,
which is based on reviving the process and would therefore cancel any killing effect beforehand.

The above-mentioned restrictions and interactions of these techniques lead to some rather unwieldy “workarounds” in the upcoming complete construction. We are giving an overview of the construction steps now, the mathematical justifications will follow in sections 3–6.

Assume that we are given a metric graph \( G = (\mathcal{V}, \mathcal{I}, \mathcal{E}, \partial, \rho) \) and boundary weights

\[
(p_1^v, (p_2^v)_{i \in \mathcal{L}(v)}, p_3^v, p_4^v)_{v \in \mathcal{V}}
\]

which satisfy the conditions of Feller’s theorem [6, Theorem 1.1].

As we cannot introduce the distant jumps yet, we choose for each \( v \in \mathcal{V} \) a distance \( \delta^v > 0 \) such that \( \delta^v \) is smaller than the lengths of all edges emanating from \( v \), and define the restricted jump measure

\[
q_4^v := p_4^v \left( \cdot \cap B_{\delta^v}(v) \right)
\]
on the ball $B_{\delta^v}(v)$ around $v$ with radius $\delta^v$, and the “extended” killing parameter
\[ q_1^v := p_1^v + p_4^v(\mathcal{C}B_{\delta^v}(v)). \]

We are going to construct the complete Brownian motion with the just given boundary weights iteratively. That is, we decompose the metric graph into two subgraphs $\tilde{\mathcal{G}}^{-1}$ and $\tilde{\mathcal{G}}^{+1}$ as explained above, and assume that there exist two Brownian motions $\tilde{X}^{-1}, \tilde{X}^{+1}$ thereon which implement the boundary conditions
\[ (q_1^v, (p_2^{v,j})_{j \in \tilde{\mathcal{E}}(v)}, p_3^v, q_4^v) \in \mathcal{V}, \quad j \in \{-1, +1\}, \]
where we set the reflection parameters for the adjoined edges to $p_2^{v,j} = p_2^{v,j}$.

As the gluing procedure only works for processes with no additional killing effects at the vertices, we further adjoin for every vertex $v \in \mathcal{V}$ an absorbing “fake” cemetery point $\square^v$ to the respective subgraph $\tilde{\mathcal{G}}^j$, and assimilate the killing parameter into the jump measure by reviving the subprocesses at $\square^v$ whenever they die at $v$, see figure 2A. Then the new processes possess the boundary conditions
\[ (0, (p_2^{v,j})_{j \in \mathcal{E}(v)}, p_3^v, q_4^v) \in \mathcal{V}, \quad j \in \{-1, +1\}. \]

Next, we glue both processes together and obtain a process on the complete graph $\mathcal{G}$, as illustrated in figure 2B, with boundary conditions
\[ (0, (p_2^{v,j})_{j \in \mathcal{L}(v)}, p_3^v, q_4^v) \in \mathcal{V}. \]

In order to introduce the global jumps, we split the jump to $\square^v$, with original weight $q_1^v = p_1^v + p_4^v(\mathcal{C}B_{\delta^v}(v))$, into killing with weight $p_1^v$ and non-local jumps relative to the measure $p_4^v(\cdot \cap \mathcal{C}B_{\delta^v}(v))$. To this end, we need to kill the process again: By mapping the absorbing points $\{\square^v, v \in \mathcal{V}\}$ to the “real” cemetery $\Delta$, see figure 2C, we obtain a newly killed process with boundary conditions
\[ (q_1^v, (p_2^{v,j})_{j \in \mathcal{L}(v)}, p_3^v, q_4^v) \in \mathcal{V}. \]

We adjoin another absorbing “fake” cemetery point $\square$ and construct the next process as instant revival process with revival distribution $\left(p_1^v + p_4^v(\cdot \cap \mathcal{C}B_{\delta^v}(v))\right)/q_1^v$. This process now implements jumps relative to the measure $p_1^v \cdot \Delta + p_4^v(\cdot \cap \mathcal{C}B_{\delta^v}(v))$, which adds to the already existing jump measure $q_4^v = p_4^v\left(\cdot \cap \mathcal{C}B_{\delta^v}(v)\right)$. Thus, this process satisfies the boundary conditions
\[ (0, (p_2^{v,j})_{j \in \mathcal{L}(v)}, p_3^v, p_1^v \cdot \Delta + p_4^v) \in \mathcal{V}. \]

Finally, we transform the jumps to $\square$ into killing by mapping $\square$ to $\Delta$, and obtain the complete boundary condition
\[ (p_1^v, (p_2^{v,j})_{j \in \mathcal{L}(v)}, p_3^v, p_4^v) \in \mathcal{V}. \]

As seen above, we need to perform many process transformations in the complete construction, while keeping track of the resulting boundary conditions. In order to keep our results comprehensible, we first analyze the two main components—killing on an absorbing set and introduction of jumps via the instant revival process—together with their effects on the generator separately in the next two sections.
3. Killing a Brownian Motion on an Absorbing Set

In this section, we examine how killing a Brownian motion on an absorbing set \( F \) affects the boundary conditions of its generator. It will turn out that the jump portion which originally led to \( F \) is just transformed into the killing portion, as any jump to \( F \) is now immediately triggering the killing.

We implement the killing transformation by mapping the absorbing set \( F \) to \( \Delta \), that is, we consider the process \( \psi(X) \) for the map

\[
\psi: G \to G \setminus F, \quad x \mapsto \begin{cases} x, & x \in G \setminus F, \\ \Delta, & x \in F. \end{cases}
\]

(3.1)

It has been shown in Appendix A that the transformed process \( \psi(X) \) is a right process if \( X \) is a right process and \( F \) is an isolated and absorbing set for \( X \).

We are able to obtain the following set of necessary boundary conditions by directly computing the generator of the transformed process:

**Lemma 3.1.** Let \( X \) be a Brownian motion on \( G \) with generator

\[
\mathcal{D}(A^X) \subseteq \{ f \in C^2_0(G) : \forall v \in V : \\
\quad c_1^v f(v) - \sum_{l \in \mathcal{L}(v)} c_2^{v,l} f_l(v) + \frac{c_3^v}{2} f''(v) - \int_{G \setminus \{v\}} (f(v) - f(v)) c_4^v(dg) = 0 \},
\]

and \( F \subseteq G \) be an isolated, absorbing set for \( X \). Let \( Y := \psi(X) \) be the process on \( G \setminus F \) resulting from killing \( X \) on \( F \), with \( \psi \) as given in equation (3.1). Then the domain of the generator of \( Y \) satisfies

\[
\mathcal{D}(A^Y) \subseteq \{ f \in C^2_0(G \setminus F) : \forall v \in V \setminus F : \\
\quad (c_1^v + c_4^v(F)) f(v) - \sum_{l \in \mathcal{L}(v)} c_2^{v,l} f_l(v) + \frac{c_3^v}{2} f''(v) - \int_{G \setminus (F \cup \{v\})} (f(v) - f(v)) c_4^v(dg) = 0 \}.
\]

**Proof.** For all \( f \in \mathcal{D}(A^Y) \), we have for \( g \in G \setminus F \)

\[
A^X(f \circ \psi)(g) = \lim_{t \downarrow 0} \frac{E_g(f \circ \psi(X_t)) - f \circ \psi(g)}{t} = \lim_{t \downarrow 0} \frac{E_g(f(Y_t)) - f(g)}{t},
\]

which exists and is equal to \( A^Y f(g) \). On the other hand, if \( g \in F \), then \( X_t \in F \) holds for all \( t \geq 0 \), \( \mathbb{P}_g \)-a.s., because \( F \) is absorbing for \( X \), and it follows that

\[
A^X(f \circ \psi)(g) = \lim_{t \downarrow 0} \frac{E_g(f \circ \psi(X_t)) - f \circ \psi(g)}{t} = \lim_{t \downarrow 0} \frac{E_g(f(\Delta)) - f(\Delta)}{t} = 0.
\]

Thus, we have \( f \circ \psi \in \mathcal{D}(A^X) \) for all \( f \in \mathcal{D}(A^Y) \), and \( A^X(f \circ \psi) = A^Y f \mathbb{1}_F \) in this case.
So, if \( f \in \mathcal{D}(A^Y) \), then \( f \circ \psi \) fulfills the boundary condition for \( X \), that is
\[
0 = c_1^v f(\psi(v)) - \sum_{t \in \mathcal{L}(v)} c_2^{v,t} f_t'(\psi(v)) + \frac{c_3}{2} f''(\psi(v))
- \int_{\mathcal{G} \setminus \{v\}} (f(\psi(g)) - f(\psi(v))) c_4^v(dg)
= c_1^v f(v) - \sum_{t \in \mathcal{L}(v)} c_2^{v,t} f_t'(v) + \frac{c_3}{2} f''(v)
- \int_{\mathcal{G} \setminus (F \cup \{v\})} (f(g) - f(v)) c_4^v(dg) + f(v) c_4^v(F)
\]
for all \( v \in \mathcal{V} \setminus F \), where we used \( f(\psi(g)) = f(\Delta) = 0 \) for all \( g \in F \). \( \square \)

In general, this proof does not provide us with the Feller–Wentzell data of the killed process, as we are only able to directly compare the Feller–Wentzell data with the boundary data of the generator in the star graph case (cf. [6, Lemma 4.1]). Therefore, we need to derive it manually by checking its definitions given in Feller’s theorem [6, Theorem 1.2]:

**Lemma 3.2.** Let \( X \) be a Brownian motion on \( \mathcal{G} \) with Feller–Wentzell data
\[
(c_1^{v,\Delta}, c_1^{v,\infty}, (c_2^{v,t})_{t \in \mathcal{L}(v)}, c_3^v, c_4^v)_{v \in \mathcal{V}},
\]
and \( F \subseteq \mathcal{G} \) be an isolated, absorbing set for \( X \). Let \( Y := \psi(X) \) be the process on \( \mathcal{G} \setminus F \) resulting from killing \( X \) on \( F \), with \( \psi \) as given in equation (3.1). If \( \mathcal{G} \setminus F \) is a metric graph and \( Y \) is a Brownian motion on \( \mathcal{G} \setminus F \), then the Feller–Wentzell data of \( Y \) reads
\[
(c_1^{v,\Delta} + c_4^v(F), c_1^{v,\infty}, (c_2^{v,t})_{t \in \mathcal{L}(v)}, c_3^v, c_4^v(\cdot \cap F^c))_{v \in \mathcal{V} \setminus F}.
\]

**Proof.** We are using the notations of [6, Theorem 1.2], and indicate the corresponding process in the superscript of the variables. Fix \( v \in \mathcal{V} \setminus F \). The processes’ exit behaviors totally coincide, except if \( X \) exits from a small neighborhood of \( v \) by jumping into \( F \) (then \( Y \) jumps to \( \Delta \)). Thus, \( \mathbb{E}_v(\tau^X_\varepsilon) = \mathbb{E}_v(\tau^Y_\varepsilon) \) holds for all sufficiently small \( \varepsilon > 0 \), and the exit distributions read
\[
\mathbb{P}_v(Y_{\tau^X_\varepsilon} \in dg \cap (\mathcal{G} \setminus F)) = \mathbb{P}_v(X_{\tau^X_\varepsilon} \in dg \cap (\mathcal{G} \setminus F))
\]
\[
\mathbb{P}_v(Y_{\tau^Y_\varepsilon} = \Delta) = \mathbb{P}_v(X_{\tau^X_\varepsilon} \in \{\Delta\} \cup F).
\]
Therefore, we have \( \nu_{\varepsilon,v}^X = \nu_{\varepsilon,v}^Y (\cdot \cap (\mathcal{G} \setminus F)) \) and, as \( d(v,f) = +\infty \) for all \( f \in F \),
\[
\int_{F} (1 - e^{-d(v,g)}) \nu_{\varepsilon,v}^X(dg) = \nu_{\varepsilon,v}^X(F) = \frac{\mathbb{P}_v(X_{\tau^X_\varepsilon} \in F)}{\mathbb{E}_v(\tau^X_\varepsilon)}.
\]
It follows that
\[
K_{\varepsilon,v}^Y = 1 + \frac{\mathbb{P}_v(Y_{\tau^Y_\varepsilon} = \Delta)}{\mathbb{E}_v(\tau^Y_\varepsilon)} + \int_{\mathcal{G} \setminus \{v\}} (1 - e^{-d(v,g)}) \nu_{\varepsilon,v}^Y(dg)
= 1 + \frac{\mathbb{P}_v(X_{\tau^X_\varepsilon} = \Delta)}{\mathbb{E}_v(\tau^X_\varepsilon)} + \int_{(\mathcal{G} \setminus \{v\}) \cup F} (1 - e^{-d(v,g)}) \nu_{\varepsilon,v}^X(dg)
= K_{\varepsilon,v}^X.
As $F$ is isolated, we get $\mathbb{P}^{Y,v} = \mathbb{P}^{X,v}(\cdot \cap (G \setminus \{v\} \setminus F))$, and conclude that

$$c^Y_{1,\Delta} = \lim_{\varepsilon \to 0} \left( \frac{\mathbb{P}_v(X_{\tau^X_v} = \Delta)}{\mathbb{E}_v(\tau^X_v) K^X_v} + \frac{\mathbb{P}_v(X_{\tau^X_v} \in F)}{\mathbb{E}_v(\tau^X_v) K^X_v} \right)$$

$$= c^X_{1,\Delta} + \mathbb{P}^{X,v}(F)$$

as well as $c^Y_{1,\infty} = c^Y_{1,\infty}$, $c^Y_{2,l} = c^X_{2,l}$ for each $l \in \mathcal{L}(v)$, $c^Y_{3} = c^X_{3}$, and $c^Y_{4} = c^X_{4}(\cdot \cap (G \setminus F))$. \hfill \Box

**Remark 3.3.** We will apply Lemma 3.2 in the following context: Let $X$ be a Brownian motion on $G$ and $F$ be an isolated and absorbing set for $X$, such that for its first entry time $H_F := \inf \{ t \geq 0 : X_t \in F \}$ and $H_X$ as given in [6, Definition 2.1],

$$H_X < H_F \quad \mathbb{P}_g\text{-a.s.}$$

holds true for all $g \in \mathcal{G} \setminus F$.

It then follows from Theorem A.1 that the killed process $Y = \psi(X)$ is a right process, and therefore strongly Markovian. If $G \setminus F$ is a metric graph, then, as $H_Y = H_X$ and $Y_t = X_t$ for all $t \leq H_X < H_F$, the properties of [6, Theorem 2.5] follow for $Y$ from the respective ones of $X$. Thus, $Y$ is a Brownian motion on $G \setminus F$, and Lemma 3.2 can be applied in order to deduce the Feller–Wentzell data of $Y$.

In particular, the condition above is satisfied if $F$ can only be reached from $\mathcal{G} \setminus F$ via jumps from vertices, which, as $F$ is isolated and thus has positive distance from any vertex $v \in \mathcal{V} \setminus F$, cannot happen immediately due to the normality of the process.

4. INTRODUCTION OF NON-LOCAL JUMPS

We will introduce the “global” jumps, namely jumps to other subgraphs, with the help of the technique of instant revivals as established in [8, Theorem 1.7]. In order to prepare this approach, we examine the effect of this method on the Feller–Wentzell data. Similar results were already attained in the examinations concerning Brownian motions on star graphs (see [6, Lemma 4.2, Lemma 4.3]).

The next lemma shows that, as expected, the killing weight will be transformed to an additional jump portion with distribution given by the revival kernel. It also clarifies that this technique can only be used for the implementation of finite jump measures.

**Lemma 4.1.** Let $X$ be a Brownian motion on $G$ with Feller–Wentzell data

$$(c^v_{1,\Delta}, c^v_{1,\infty}, (c^v_{2,l})_{l \in \mathcal{L}(v)}, c^v_{3}, c^v_{4})_{v \in \mathcal{V}},$$

lifetime $\zeta^X$, and exit times $\tau^X_v := \inf \{ t \geq 0 : d(X_t, X_0) > \varepsilon \}$ for $\varepsilon > 0$. If $c^v_{1,\Delta} > 0$, consider the instant revival process $Y$, constructed from $X$ with the revival kernel

$$k(v, \cdot) = \kappa^v, \quad v \in \mathcal{V},$$

for some probability measure $\kappa^v$ on $G$, and $k(g, \cdot) = \varepsilon_g$ for all $g \notin \mathcal{V}$. Suppose that for every $v \in \mathcal{V}$ there exists $\delta > 0$ such that

(i) $\kappa^v(B_\delta(v)) = 0$, and

(ii) for all $\varepsilon < \delta$, $X_{\tau^X_v} \in B_\delta(v)$ holds $\mathbb{P}_v^X$-a.s. on $\{ \tau^X_v < \zeta^X \}$. 

...
Then \( Y \) is a Brownian motion on \( \mathcal{G} \). For all \( v \in \mathcal{V} \), the generator \( A^v \) of \( Y \) satisfies for every \( f \in \mathcal{D}(A^v) \)
\[
c_1^v \infty f(v) - \sum_{i \in \mathcal{E}(v)} c_2^v f_i(v) + c_3^v Af(v) - \int_{\mathcal{E}(v)} (f(g) - f(v)) (c_4^v + c_1^v \Delta^v)(dg) = 0.
\]
If additionally \( d(v, x) = +\infty \) holds for every \( x \in \text{supp} \kappa^v \), then the Feller–Wentzell data of \( Y \) at \( v \) reads
\[
(0, c_1^v \infty, (c_2^v)_i \in \mathcal{E}(v), c_3^v, c_4^v + c_1^v \Delta^v).
\]

**Proof.** By [8, Theorem 1.7], \( Y \) is a right process and thus strongly Markovian. As \( Y_t = X_t \) holds a.s. for all \( t \leq H_X = H_Y \), \( Y \) is a Brownian motion on \( \mathcal{G} \).

Fix \( v \in \mathcal{V} \). We are going to examine the components evolving in the generator of the process \( Y \) and compare them to the respective ones of \( X \). The components in Feller’s theorem [6, Theorem 1.2] for the process \( X \) at the vertex \( v \) will be named \( c_1^X, \nu^X, K^X \), etc., instead of \( c_1^v, \nu^v, K^v \). The proof will be based on the following two main principles:

- Due to assumption (i), \( \nu^v \) and \( X \) are equivalent in a neighborhood of \( v \), more precisely: There exists \( \delta > 0 \) (e.g. being the minimum of \( \delta \) in assumption (i) and the minimal length of all edges incident with \( v \)) such that
  \[
  \forall \epsilon \leq \delta : \quad E_1^y (\tau^y) = E_0^x (\tau^x),
  \]
  and for all \( n \in \mathbb{N} \), \( f_1, \ldots, f_n \in \mathcal{B}(\mathcal{G}) \), \( 0 = t_1 < \ldots < t_n \),
  \[
  \mathbb{P}_v^Y (f_1(Y_{t_1}) \cdots f_1(Y_{t_n}); t_n < \tau_\delta^Y) = \mathbb{P}_v^X (f_1(X_{t_1}) \cdots f_1(X_{t_n}); t_n < \tau_\delta^X).
  \]
  In particular, we have for all \( \epsilon < \delta \), \( A \in \mathcal{B}(\mathcal{G}) \):
  \[
  \mathbb{P}_v^Y (Y_{\tau^y} \in A | Y_{\tau^y} \in B_\delta(v)) = \mathbb{P}_v^X (X_{\tau^x} \in A | X_{\tau^x} \in B_\delta(v)).
  \]

- Due to assumption (ii), the process \( X \) only has jumps from \( v \) into \( B_\delta(v) \) or to \( \Delta \), that is,
  \[
  \forall \epsilon < \delta : \quad \mathbb{P}_v^X (X_{\tau^x} \in B_\delta(v) \cup \{\Delta\}) = 1.
  \]
  Therefore, \( Y \) only can jump into \( \mathbb{C}B_\delta(v) \) if the underlying process \( X \) is killed and revived again, which yields
  \[
  \mathbb{P}_v^Y (Y_{\tau^y} \in \mathbb{C}B_\delta(v)) = \mathbb{P}_v^X (X_{\tau^x} = \Delta),
  \]
  and the jump distribution is given by the reviving kernel
  \[
  \mathbb{P}_v^Y (Y_{\tau^y} \in A | Y_{\tau^y} \in \mathbb{C}B_\delta(v)) = \kappa^y(A), \quad A \in \mathcal{B}(\mathcal{G}).
  \]
  Furthermore, the revived process \( Y \) is not able to die at all, yielding
  \[
  \mathbb{P}_v^Y (Y_{\tau^y} = \Delta) = 0.
  \]
Let \( f \in \mathcal{D}(A^v) \) and fix \( v \in \mathcal{V} \). The vertex \( v \) cannot be a trap for \( Y \), as otherwise \( v \)
would either be a trap for \( X \), which is impossible by \( c_1^v > 0 \), or \( Y \) would be revived at \( v \) when \( X \) dies there, which contradicts assumption (i). Thus, Dynkin’s formula yields
\[
Af(v) = \lim_{\epsilon \downarrow 0} \frac{E_1^y (f(Y_{\tau^y})) - f(v)}{E_1^y (\tau^y)}.
\]
We are going to reiterate the steps in the proof of Feller’s theorem [6, Theorem 1.2] for the process $Y$, but we will be using the normalization factor $K^X_\varepsilon$ of $X$ instead of $K^Y_\varepsilon$. This will not pose any problems because $K^X_\varepsilon \geq K^Y_\varepsilon$ holds true, which is seen as follows: With the scaled exit distributions from $\overline{CB_\varepsilon(v)}$

$$\nu^Y_\varepsilon(A) = \frac{\mathbb{P}^Y_v(Y_\tau^Y \in A)}{\mathbb{E}_v^Y(\tau^Y_\varepsilon)}, \quad \nu^X_\varepsilon(A) = \frac{\mathbb{P}^X_v(X_\tau^X \in A)}{\mathbb{E}_v^X(\tau^X_\varepsilon)}, \quad A \in \mathcal{B}(\mathcal{G}\setminus\{v\}),$$

for $Y$ and $X$, assumption (i) asserts that for all sufficiently small $\varepsilon > 0$,

$$K^X_\varepsilon = 1 + \frac{\mathbb{P}^X_v(X_\tau^X = \Delta)}{\mathbb{E}_v^X(\tau^X_\varepsilon)} + \int_{\mathcal{G}\setminus\{v\}} (1 - e^{-d(v,g)}) \nu^X_\varepsilon(dg)$$

$$= 1 + \frac{\mathbb{P}^Y_v(Y_\tau^Y \in \overline{CB_\varepsilon(v)})}{\mathbb{E}_v^Y(\tau^Y_\varepsilon)} + \int_{\overline{B_\varepsilon(v)}\setminus\{v\}} (1 - e^{-d(v,g)}) \nu^Y_\varepsilon(dg).$$

As $\mathbb{P}^Y_v(Y_\tau^Y = \Delta) = 0$ and

$$(4.1) \quad \frac{\mathbb{P}^Y_v(Y_\tau^Y \in \overline{CB_\varepsilon(v)})}{\mathbb{E}_v^Y(\tau^Y_\varepsilon)} = \int_{\overline{CB_\varepsilon(v)}} 1 \nu^Y_\varepsilon(dg) \geq \int_{\overline{CB_\varepsilon(v)}} (1 - e^{-d(v,g)}) \nu^Y_\varepsilon(dg),$$

we get

$$K^X_\varepsilon \geq 1 + \frac{\mathbb{P}^Y_v(Y_\tau^Y = \Delta)}{\mathbb{E}_v^Y(\tau^Y_\varepsilon)} + \int_{\mathcal{G}\setminus\{v\}} (1 - e^{-d(v,g)}) \nu^Y_\varepsilon(dg)$$

$$= K^Y_\varepsilon.$$

Thus, by following the proof of Feller’s theorem (see [6, Section 3]), we get

$$\lim_{\varepsilon \searrow 0} \left( f(v) \frac{\mathbb{P}^Y_v(Y_\tau^Y = \Delta)}{\mathbb{E}_v^Y(\tau^Y_\varepsilon)} \frac{1}{K^X_\varepsilon} + Af(v) \frac{1}{K^X_\varepsilon} \int_{\mathcal{G}\setminus\{v\}} (f(g) - f(v)) \nu^Y_\varepsilon(dg) \right) = 0.$$

However, it is $\mathbb{P}^Y_v(Y_\tau^Y = \Delta) = 0$, and the exit distributions of $Y$ decompose into

$$\nu^Y_\varepsilon(A) = \frac{\mathbb{P}^Y_v(Y_\tau^Y \in A \cap B_\varepsilon(v))}{\mathbb{E}_v^Y(\tau^Y_\varepsilon)} + \frac{\mathbb{P}^Y_v(Y_\tau^Y \in A \cap B_\varepsilon(v)^\complement)}{\mathbb{E}_v^Y(\tau^Y_\varepsilon)},$$

with

$$\frac{\mathbb{P}^Y_v(Y_\tau^Y \in A \cap B_\varepsilon(v))}{\mathbb{E}_v^Y(\tau^Y_\varepsilon)} = \frac{\mathbb{P}^Y_v(Y_\tau^Y \in A | Y_\tau^Y \in B_\varepsilon(v))}{\mathbb{E}_v^Y(\tau^Y_\varepsilon)} \frac{1}{\mathbb{P}^Y_v(Y_\tau^Y \in B_\varepsilon(v))} = \frac{\mathbb{P}^X_v(X_\tau^X \in A | X_\tau^X \in B_\varepsilon(v))}{\mathbb{E}_v^X(\tau^X_\varepsilon)} \frac{1}{\mathbb{P}^X_v(X_\tau^X \in B_\varepsilon(v))} = \nu^X_\varepsilon(A),$$

and

$$\frac{\mathbb{P}^Y_v(Y_\tau^Y \in A \cap B_\varepsilon(v)^\complement)}{\mathbb{E}_v^Y(\tau^Y_\varepsilon)} = \frac{\mathbb{P}^Y_v(Y_\tau^Y \in A | Y_\tau^Y \in B_\varepsilon(v)^\complement)}{\mathbb{E}_v^Y(\tau^Y_\varepsilon)} \frac{\mathbb{P}^Y_v(Y_\tau^Y \in B_\varepsilon(v)^\complement)}{\mathbb{E}_v^X(\tau^X_\varepsilon)} = \kappa^v(A) \frac{\mathbb{P}^X_v(X_\tau^X = \Delta)}{\mathbb{E}_v^X(\tau^X_\varepsilon)}.$$
Therefore, we have

\[
\lim_{\varepsilon \downarrow 0} \left( A f(v) \frac{1}{K}\nu_\varepsilon^X(dy) - \int_{\mathcal{G}\setminus\{v\}} (f(y) - f(v)) \nu_\varepsilon^X(dy) \right) = 0,
\]

and knowing that \( \frac{1}{K_{\varepsilon}} \nu_\varepsilon^X(dy) \) converge along the same sequence \((\varepsilon_n, n \in \mathbb{N})\) given by Feller’s theorem \([6, \text{Theorem 1.2}]\) for \(X\), we conclude that

\[
c^1_v \varepsilon f(v) - \sum_{i \in E(v)} c_{2,i} f_i'(v) + c_{3,v} A f(v) - \int_{G\setminus\{v\}} (f(y) - f(v)) c^v_4(dy) \\
- c^\Delta_v \int_{\mathcal{L}_v} (f(y) - f(v)) \kappa^v(dy) = 0.
\]

In case every point in the support of \(\kappa^v\) has distance \(+\infty\) from \(v\), equation (4.1) shows that \(K^v_{\varepsilon} = K^v_{\varepsilon} \) holds true, and therefore the above set of boundary conditions at \(v\) for \(Y\) coincides with the Feller–Wentzell data of \(Y\) at \(v\).

The reader may notice that the resulting boundary data for \(Y\) given in Lemma 4.1 might not satisfy the normalization condition of the Feller–Wentzell data, as given in \([6, \text{Theorem 1.2}]\), in case the support of \(\kappa^v\) does not have infinite distance from \(v\).

**Remark 4.2.** We observe in Lemma 4.1 that the revival of a process upon its death with a revival distribution \(\kappa\) only transforms the “real” killing parameter \(c_1^\Delta\) into an additional jump part \(c_1^\Delta\kappa\), while leaving the artificial killing portion \(c_1^\infty\) intact. The main explanation is that \(c_1^\infty\) does not represent the effect of “killing” in the sense of proper jumps to the cemetery point \(\Delta\). It is rather caused by an explosion of the process, triggered by ever-growing jumps when the process approaches a vertex point, and this effect is not transformed by the revival technique.

In the Brownian context, we do not expect any effects which would contribute to \(c_1^\infty\), and we indeed showed in \([6, \text{Theorem 1.4}]\) that \(c_1^\infty\) vanishes for all Brownian motions on star graphs. As these processes will form the building blocks of the Brownian motions on the general metric graph, the Feller–Wentzell data of all processes considered here will satisfy

\[
\forall v \in V : c_v^{1,\infty} = 0.
\]

5. **Gluing the Graphs Together**

We are going to discuss the main construction method, namely the pasting of the subgraphs and their Brownian motions thereon. As already disclosed in section 2, this technique will compromise several steps:

5.1. **Decomposition of the Graph \(\mathcal{G}\) into \(\mathcal{G}^{-1}, \mathcal{G}^1\).** Let \(\mathcal{G} = (V, E, \mathcal{I}, \mathcal{I}_S, \rho)\) be a metric graph. We partition \(\mathcal{G}\) into two graphs by choosing two disjoint, non-empty sets \(V^{-1}\) and \(V^1\) with \(V = V^{-1} \cup V^1\), and decompose the set of edges into

\[
E = E^{-1} \cup E^1, \quad \mathcal{I} = \mathcal{I}^{-1} \cup \mathcal{I}^1 \cup \mathcal{I}_S, \quad I^1 := \{i \in I : \partial_-(i) \in V^1, \partial_+(i) \in V^1\}.
\]

5.2. **Pasting of the Graphs.** We define \(\mathcal{G}^{-1} \cup \mathcal{G}^1\) as the graph obtained by pasting the two graphs \(\mathcal{G}^{-1}\) and \(\mathcal{G}^1\) along the set of edges \(E\), subject to the following conditions:

\[
E = E^{-1} \cup E^1, \quad \mathcal{I} = \mathcal{I}^{-1} \cup \mathcal{I}^1 \cup \mathcal{I}_S, \quad I^1 := \{i \in I : \partial_-(i) \in V^1, \partial_+(i) \in V^1\}.
\]
As most of the following construction will be performed for both partial graphs in parallel, we will always assume that \( j \in \{-1, +1\} \) when nothing else is said.

We define the metric graphs \( \tilde{G}^{-1}, \tilde{G}^{+1} \) by

\[
\tilde{G}^j := (\mathcal{V}^j, \mathcal{E}^j \cup \mathcal{E}^j_s, \mathcal{I}^j, \partial^j, \rho^j),
\]
equipped with additional external “shadow” edges

\[
\mathcal{E}_s^j := \{ e_i^j, i \in \mathcal{I}_s \}, \quad \text{with} \quad \forall i \in \mathcal{I}_s : \quad e_i^j \notin \mathcal{E} \cup \mathcal{E}_s^{-j} \cup \mathcal{I},
\]
where the combinatorial structure and edge lengths of the original graph are naturally transferred to \( \tilde{G}^{-1}, \tilde{G}^{+1} \) by setting

\[
\partial^j|_{\mathcal{E}_s \cup \mathcal{I}^j} := \partial|_{\mathcal{E}_s \cup \mathcal{I}^j}, \quad \partial^j(e_i^j) := \begin{cases} \partial^j(i), & i \in \mathcal{I}_s, \\ \partial^j(i), & i \in \mathcal{I}_s^{-j}, \end{cases},
\]

\[
\rho^j|_{\mathcal{E}_s \cup \mathcal{I}^j} := \rho|_{\mathcal{E}_s \cup \mathcal{I}^j}, \quad \rho^j|_{e_i^j} := +\infty.
\]

For later use, we also define the “shadow length” of an external “shadow” edge by

\[
\rho_s(e_i^j) := \rho(i), \quad e_i^j \in \mathcal{E}_s^{-1} \cup \mathcal{E}_s^{+1}.
\]

The excrescent parts of the shadow edges, which will be removed in the following development before gluing both subgraphs together, are named

\[
\tilde{G}_s^j := \bigcup_{e \in \mathcal{E}_s^j} \left( \{ e \} \times [\rho_s(e), +\infty) \right).
\]

### 5.2. Introducing the Brownian Motion \( \tilde{X}^j \) on \( \tilde{G}^j \)

Let \( \tilde{X}^{-1}, \tilde{X}^{+1} \) be Brownian motions on \( \tilde{G}^{-1}, \tilde{G}^{+1} \) respectively, which admit the hypotheses of right processes, feature infinite lifetimes, have the Feller–Wentzell data

\[
(0, 0, (p_2^{v, i})_{i \in \mathcal{E}(v)}, p_3^v, p_4^v)_{v \in \mathcal{V}^j},
\]
are continuous inside every edge (cf. [7, Theorem 4.3]), and satisfy for all \( v \in \mathcal{V}^j \)

\[
(5.1) \quad \forall \varepsilon < \delta : \quad P^v_i \left( \tilde{X}^j_t \notin \tilde{G}^j_i \right) = 0,
\]
with \( \delta := \min \{ \rho_i, i \in \mathcal{I}_s \} \) and \( \tilde{G}^j_i := \inf \{ t \geq 0 : d(\tilde{X}^j_t, \tilde{X}^j_0) > \varepsilon \} \).

By gluing the graphs \( \tilde{G}^{-1} \) and \( \tilde{G}^{+1} \) (and thus the Brownian motions \( \tilde{X}^{-1} \) and \( \tilde{X}^{+1} \) thereon) together, we are going to show the following main result of this section:

**Theorem 5.1.** There exists a Brownian motion \( X \) on \( G \) with Feller–Wentzell data

\[
(c_1^v, (c_2^{v, i})_{i \in \mathcal{E}(v)}, c_3^v, c_4^v)_{v \in \mathcal{V}},
\]
such that for each \( v \in \mathcal{V} \), it holds that \( c_1^v = 0, c_2^v = p_3^v, c_3^v = p_4^v \circ (\psi^j)^{-1} \), with \( \psi^j \) being defined by equation (5.2), and

\[
i \in \mathcal{I}(v) : \quad c_2^{v, i} = \begin{cases} p_2^{v, i}, & i \in \mathcal{I}^{-1}(v) \cup \mathcal{I}^{+1}(v), \\ p_2^{v, e_i^j}, & i \in \mathcal{I}_s(v), \text{ with } j \in \{-1, +1\} \text{ such that } v \in \mathcal{V}^j,
\end{cases}
\]

\[
e \in \mathcal{E}(v) : \quad c_2^{e} = p_2^{v, e}.
\]

We construct this process \( X \) explicitly via alternating copies of transformed processes \( X^{-1}, X^{+1} \) of \( \tilde{X}^{-1}, \tilde{X}^{+1} \). Before that, we need to kill the original processes \( \tilde{X}^{-1} \) and \( \tilde{X}^{+1} \) on the excrescent shadow edges and reorientate the remaining parts in order to comply with the direction of the original internal edges of \( G \).
5.3. **Defining $\tilde{X}^j$ by Killing $\tilde{X}^j$ on $\tilde{G}^j$**. Consider the first entry time into $\tilde{G}^j_0$ of the prototype Brownian motion $\bar{X}^j$ on $\bar{G}^j$,

$$\tilde{T}^j := \inf \{ t \geq 0 : \tilde{X}^j_t \in \tilde{G}^j_0 \}. $$

We define $\tilde{X}^j$ to be the process obtained by killing $\bar{X}^j$ at the terminal time $\tilde{T}^j$,

$$\tilde{X}^j_t := \begin{cases} \bar{X}^j_t, & t < \tilde{T}^j, \\ \Delta, & t \geq \tilde{T}^j, \end{cases}$$

on the topological subspace $\tilde{G}^j$ of $\bar{G}^j$ given by

$$\tilde{G}^j := \tilde{G}^j \setminus \tilde{G}^j_s = \mathcal{V}^j \cup \bigcup_{l \in \mathcal{E} \cup \mathcal{Z}} \{(l) \times (0, \rho_l)\} \cup \bigcup_{e \in \mathcal{E}_l} \{(e) \times (0, \rho_e(e))\}. $$

**Lemma 5.2.** $\tilde{X}^j$ is a right process on $\tilde{G}^j$ with lifetime $\tilde{T}^j$.

**Proof.** $\tilde{X}^j$ is a right process with infinite lifetime. By employing [4, Corollary 12.24], it suffices to observe that $\tilde{T}^j$ is the debut of the closed, thus nearly optional set $\tilde{G}^j_s$, and the regular set of the killing time $\tilde{T}^j$ reads

$$F := \{ g \in \tilde{G}^j : \mathbb{P}^g(\tilde{T}^j = 0) = 1 \} = \tilde{G}^j_s,$$

as $\tilde{X}^j$ is a right continuous, normal process and $\tilde{G}^j_s$ is closed. \hfill $\square$

We would like to point out that the just introduced processes $\tilde{X}^j$ are not Brownian motions on a metric graph in the sense of [6, Definitions 2.1, A.1, A.2] anymore, as $\tilde{G}^j$ is not a metric graph. Thus, we will not be able to apply any results on Brownian motions for $\tilde{X}^j$ in the upcoming development.

5.4. **Letting $X^j$ be the Mapping of $\tilde{X}^j$ to the Subspace $G^j \subseteq G$**. We need to fit the subspaces $\tilde{G}^j$ of $\bar{G}^j$ to the corresponding subspaces of $G$. To this end, we introduce the topological subspaces $G^{-1}, G^+1$ of $G$ by

$$\tilde{G}^j := \mathcal{V}^j \cup \bigcup_{l \in \mathcal{E} \cup \mathcal{Z} \cup \mathcal{L}} \{(l) \times (0, \rho_l)\},$$

and consider the mapping $\psi^j : \tilde{G}^j \to G^j$ defined by

$$\forall i \in \mathcal{I}, x \in (0, \rho_i) : \psi^j((e_i^j, x)) := \begin{cases} (i, x), & i \in I^j_s, \\ (i, \rho_i - x), & i \in I^{j+} \setminus I^j_s, \end{cases}$$

$$\psi^j = \text{id} \text{ otherwise}.$$  \hspace{1cm} (5.2)

Clearly, $\psi^j$ is a bijective mapping, with its inverse $(\psi^j)^{-1} =: \varphi^j : G^j \to \tilde{G}^j$ being given by

$$\forall i \in \mathcal{I}, x \in (0, \rho_i) : \varphi^j((i, x)) := \begin{cases} (e^j_i, x), & i \in I^j_s, \\ (e^j_i, \rho_i - x), & i \in I^{j+} \setminus I^j_s, \end{cases}$$

$$\varphi^j = \text{id} \text{ otherwise}.$$ 

Furthermore, $\psi^j$ is a continuous mapping, as it is continuous inside every edge and its preimages of balls with sufficiently small radius around vertices $v \in \mathcal{V}^j$ coincide with the corresponding balls of $\tilde{G}^j$. 
\( \tilde{X}^j \) is a right process on \( \tilde{G}^j \), \( \psi^j \) is a bijective and measurable map from \( \tilde{G}^j \) onto \( G^j \), and \( t \mapsto \psi^j(\tilde{X}_t^j) \) is right continuous (as \( \psi^j \) is continuous and \( t \mapsto \tilde{X}_t^j \) is right continuous). Thus, the following result is a direct consequence of [4, Corollary (13.7)]:

**Lemma 5.3.** The process \( X^j := \psi^j(\tilde{X}^j) \), resulting from the state space mapping of \( \tilde{X}^j \) by \( \psi^j \), is a right process on \( \psi^j(\tilde{G}^j) = G^j \) with lifetime \( \zeta^j = \tilde{\zeta}^j = \tilde{\tau}^j \).

5.5. **Constructing \( X \) as Alternating Copies Process of \( X^{-1}, X^{+1} \).** We apply the technique of [8] to define the process \( X \) obtained by forming alternating copies of \( X^{-1} \) and \( X^{+1} \) via the transfer kernels \( K^{-1} \) and \( K^{+1} \), given by

\[
\begin{align*}
K^{-1} & := \sum_{i \in I^j} \varepsilon_{\partial_+(i)}(i) \mathbb{I}_{\{i\}}(\pi^1(\tilde{X}^{-1}_i_{-}^{j-}))) + \sum_{i \in I^j} \varepsilon_{\partial_-(i)}(i) \mathbb{I}_{\{i\}}(\pi^1(\tilde{X}^{-1}_i_{+}^{j-}))), \\
K^{+1} & := \sum_{i \in I^j} \varepsilon_{\partial_+(i)}(i) \mathbb{I}_{\{i\}}(\pi^1(\tilde{X}^{1}_i_{+}^{j+}))) + \sum_{i \in I^j} \varepsilon_{\partial_-(i)}(i) \mathbb{I}_{\{i\}}(\pi^1(\tilde{X}^{1}_i_{-}^{j+}))).
\end{align*}
\]

(5.3)

That is, the transfer kernels implement the following rules for \( j \in \{-1, +1\} \):

(i) \( X \) is revived as \( X^{+1} \) at \( v = \partial_-(i) \), if \( X^{-1} \) dies on \( i \in I^j \);

(ii) \( X \) is revived as \( X^{-1} \) at \( v = \partial_+(i) \), if \( X^{+1} \) dies on \( i \in I^{j} \).

For later use, we give the following combined formula of the above definitions for the transfer kernels \( K^j, \ j \in \{-1, +1\} \):

\[
K^j = k^j(i) := \begin{cases} 
\varepsilon_{\partial_+(i)}, & i \in I^j, \\
\varepsilon_{\partial_-(i)}, & i \in I^{j\prime}, 
\end{cases} \text{ for } i := \pi^1(\tilde{X}^j_{\zeta-}).
\]

(5.4)

**Lemma 5.4.** \( K^j \) is a transfer kernel from \( X^j \) to \( E^{-j} \).

**Proof.** With probability 1, the process \( \tilde{X}^j \) cannot realize \( \tilde{\tau}^j \) through a direct jump from any vertex \( v \in V^j \). Otherwise, this would imply \( \mathbb{P}^j(\tilde{X}^j_{\tau^j} \in \tilde{G}^j) > 0 \), as \( \tilde{\tau}^j \geq \tilde{\tau}^j \) holds for \( \varepsilon < \delta \), contradicting our fundamental assumption \( (5.1) \). Furthermore, \( \tilde{X}^j \) is continuous on every edge, so \( \tilde{X}^j_{\tau^j} \) exists and is equal to \( \tilde{X}^j_{\tau^j} \). Thus,

\[
\tilde{X}^j_{\zeta^j} = \lim_{t \uparrow \zeta^j} \tilde{X}^j_t = \lim_{t \uparrow \tilde{\tau}^j} \tilde{X}^j_t = \tilde{X}^j_{\tilde{\tau}^j},
\]

exists in \( \{(e, \rho_s(e)), e \in E^j \} \), and

\[
\pi^1(\tilde{X}^j_{\zeta^j}) = \pi^1(\psi^j(\tilde{X}^j_{\zeta^j})) = \pi^1(\psi^j(\tilde{X}^j_{\tilde{\tau}^j})).
\]

(5.5)

exists in \( I^j \). Therefore, \( \pi^1(\tilde{X}^j_{\zeta^j}) \in \mathcal{F}^j_{\{\zeta^j\}} \), so \( K^j \) is indeed a probability kernel \( K \) from \( (\Omega^j, \mathcal{F}^j_{\{\zeta^j\}}) \) to \( (E^{-j}, \mathcal{E}^{-j}) \), that is, a transfer kernel. \qed

Let

- \( \tau_{-1}^j \) be the first entry time of \( X^{-1} \) into \( G^{-1} \setminus G^{+1} \), \( \zeta^{-1} \) the lifetime of \( X^{-1} \),
- \( \tau_{+1}^j \) be the first entry time of \( X^{+1} \) into \( G^{+1} \setminus G^{-1} \), \( \zeta^{+1} \) the lifetime of \( X^{+1} \).

Then, according to [8, Theorem 1.6], \( X \) is a right process on \( G = G^{-1} \cup G^{+1} \) in case the following conditions hold true for all \( g \in G^{-1} \cap G^{+1}, f \in b \mathcal{B}(G), k^{-1} \in b \mathcal{B}(G^{-1}), h^{+1} \in b \mathcal{B}(G^{+1}) \):

(i) \( \mathbb{E}^{-1}_g \left( \int_0^{\tau_{-1}^j} e^{-\alpha t} f(X_{-1}^j) \, dt \right) = \mathbb{E}^{+1}_g \left( \int_0^{\tau_{+1}^j} e^{-\alpha t} f(X_{+1}^j) \, dt \right) \);
(ii) \( \mathbb{E}_g^{-1}(e^{-\alpha \tau_{-1}^{-1}} h^{-1}(X_{-1}^{-1}); \tau_{-1}^{-1} < \zeta^{-1}) = \mathbb{E}_g^{+1}(e^{-\alpha \zeta^{+1}} K^{+1} h^{-1}; \zeta^{+1} < \tau_{+1}^{+1}) \).

By using the definition of \( X \) and employing that the latter is a Brownian motion on \((\mathbb{I}, \tau_{-1}^{-1})\), we get

\[
\tau_{j}^{j} = \inf \{ t \geq 0 : X_{t}^{j} \in G \setminus G^{-j} \} = \inf \{ t \geq 0 : \psi^{-1} \circ \psi^{-1}(X_{t}^{j}) \in G \setminus G^{-j} \} = \inf \{ t \geq 0 : X_{t}^{j} \in G \setminus G^{-j} \}.
\]

The process \( X^{j} \) was constructed by killing \( X^{j} \) at \( \tau_{j}^{T} \). Thus, by introducing the first exit times of \( X^{j} \) from the shadow edges

\[
\overline{\tau}_{j}^{T} := \inf \{ t \geq 0 : X_{t}^{j} \in G \setminus G^{-j} \} = \inf \{ t \geq 0 : X_{t}^{j} \notin \bigcup_{i \in \mathbb{I}} \{ (e_{i}^{j}) \times (0, \infty) \} \},
\]

we obtain the relation

\[
\tau_{j}^{j} \wedge \zeta^{j} = \overline{\tau}_{j}^{T} \wedge \overline{\tau}_{j}^{T}.
\]

Turning to the actual proof of (i) and (ii), let \( g \in G^{-1} \cap G^{+1} \), that is, \( g = (i, x) \) for some \( i \in \mathbb{I}_x \), \( x \in (0, \rho_i) \). Choose \( j \in \{-1, +1\} \) such that \( i \in \mathbb{I}_x^{j} \). By tracing \( X^{j} \) back to \( \tilde{X}^{j} \) and employing that the latter is a Brownian motion on \( \tilde{G}^{j} \), [6, Lemma 2.4 and Corollary 2.10] yield

\[
\mathbb{E}^{-j}_{x_{i}} \left( \int_0^{\tau_{j}^{T}} e^{-\alpha t} f(X_{t}^{j}) \, dt \right) = \mathbb{E}^{-j}_{x_{i}} \left( \int_0^{\tau_{j}^{T}} e^{-\alpha t} f(\psi^{-1}(\tilde{X}_{t}^{j}, \tilde{T}_{j}^{T})) \, dt \right)
\]

and analogously

\[
\mathbb{E}^{j}_{x_{i}} \left( \int_0^{\tau_{j}^{T}} e^{-\alpha t} f(X_{t}^{j}) \, dt \right) = \mathbb{E}^{j}_{x_{i}} \left( \int_0^{\tau_{j}^{T}} e^{-\alpha t} f(\psi^{j}(\tilde{X}_{t}^{j})) \, dt \right)
\]

and

\[
\mathbb{E}^{j}_{x_{i}} \left( \int_0^{\tau_{j}^{T}} e^{-\alpha t} f(i, \rho(i) - B_{t}) \, dt \right) = \mathbb{E}^{j}_{x_{i}} \left( \int_0^{\tau_{j}^{T}} e^{-\alpha t} f(i, B_{t}) \, dt \right).
\]

By the spatial homogeneity and reflection invariance of the one-dimensional Brownian motion \( B \), we have

\[
\mathbb{E}^{j}_{x_{i}} \left( \int_0^{\tau_{j}^{T}} e^{-\alpha t} f(i, \rho(i) - B_{t}) \, dt \right) = \mathbb{E}^{j}_{x_{i}} \left( \int_0^{\tau_{j}^{T}} e^{-\alpha t} f(i, B_{t}) \, dt \right),
\]

which proves the equality of (5.7) and (5.8), and thus concludes (i).
Coming to (ii), we will prove both assertions simultaneously, as they only differ in the initial process. Let \( j \in \{-1, +1\} \). We start by reducing the first expectation to \( \bar{X}^j \), and obtain with the help of equation (5.6) the identity

\[
E^j_g \left( e^{-\alpha \tau^{-j}} h^{-j} (X_{\tau^{-j}}^j) ; \tau^{-j} < \zeta^{-j} \right) = E^j_{(\psi^{-j})^{-1}(g)} \left( e^{-\alpha \bar{\tau}^{-j}} h^{-j} (\psi^{-j}(\bar{X}_{\bar{\tau}^{-j}}^j)) ; \bar{\tau}^{-j} < \bar{\tau}^{-j} \right),
\]

where \((\psi^{-j})^{-1}(g) = (e^{-j}, \rho_i - x)\) or \((\psi^{-j})^{-1}(g) = (e^{-j}, x)\) depending on whether \( i \in I^j_s\) or \( i \in I^j_t\). For all that follows, we define for any \( g \in \mathcal{G}^j \) the first hitting time \( \bar{H}^j_g \) of the set \( \{g\} \) by the process \( \bar{X}^j \). By the continuity of \( \bar{X}^j \) inside the edges, we see that \( \mathbb{P}^j_{(e^j,s)} \), a.s. for any \( i \in I_s \), the relation

\[
\bar{\tau}^j = \bar{H}^j_i \quad \text{on } \{ \bar{\tau}^j < \bar{\tau}^j \} = \{ \bar{H}^j_i < \bar{H}^j_{(e^j,s)} \}
\]

holds true with \( v := \partial(e^j,s) \), so we have

\[
\psi^{-j}(\bar{X}_{\bar{\tau}^{-j}}^j) = \partial(e^{-j}) = \begin{cases} 
\partial_+(i), & i \in I^j_t, \\
\partial_-(i), & i \in I^j_s.
\end{cases}
\]

Therefore, we get

\[
E^j_g \left( e^{-\alpha \tau^{-j}} h^{-j} (X_{\tau^{-j}}^j) ; \tau^{-j} < \zeta^{-j} \right) = E^j_{(\psi^{-j})^{-1}(g)} \left( e^{-\alpha \bar{\tau}^{-j}} h^{-j} (\psi^{-j}(\bar{X}_{\bar{\tau}^{-j}}^j)) ; \bar{\tau}^{-j} < \bar{\tau}^{-j} \right) = \begin{cases} 
\bar{H}^{-j}_v h^{-j} (\partial_+(i)) ; \bar{H}^{-j}_v < \bar{H}^{-j}_{(e^{-j}, \rho_i(e^{-j}))}, & i \in I^j_t, \\
\bar{H}^{-j}_v h^{-j} (\partial_-(i)) ; \bar{H}^{-j}_v < \bar{H}^{-j}_{(e^{-j}, \rho_i(e^{-j}))}, & i \in I^j_s.
\end{cases}
\]

But \( \bar{X}_{\bar{\tau}^{-j}}^j \) is a Brownian motion on \( \mathcal{G}^{-j} \), so [6, Corollary 2.10, Remark 2.9] together with \( \rho_i(e^{-j}) = \rho(i) \) yield

\[
E^j_g \left( e^{-\alpha \tau^{-j}} h^{-j} (X_{\tau^{-j}}^j) ; \tau^{-j} < \zeta^{-j} \right) = \begin{cases} 
h^{-j} (\partial_+(i)) E^j_{(\psi^{-j})^{-1}(g)} (e^{-\alpha \tau_0} ; \tau_0 < \tau_{\rho(i)}), & i \in I^j_t, \\
h^{-j} (\partial_-(i)) E^j_{(\psi^{-j})^{-1}(g)} (e^{-\alpha \tau_0} ; \tau_0 < \tau_{\rho(i)}), & i \in I^j_s.
\end{cases}
\]

Next, we employ the same techniques as above in order to compute the right-hand sides of (ii). Equations (5.5) and (5.6) give

\[
E^j_{(\psi^{-j})^{-1}(g)} \left( e^{-\alpha \bar{\tau}^{-j}} k^j (\pi^1(\psi^{-j}(\bar{X}_{\bar{\tau}^j}^j))) h^{-j} ; \bar{\tau}^{-j} < \bar{\tau}^j \right).
\]

We observe that

\[
\bar{\tau}^j = \bar{H}^j_{(e^j,s)} \quad \text{on } \{ \bar{\tau}^j < \bar{\tau}^j \} = \{ \bar{H}^j_{(e^j,s)} < \bar{H}^j_i \},
\]

as \( \bar{\tau}^j \leq \bar{\tau}^j \) in case \( \bar{\tau}^j = \bar{H}^j_{(e^j,s)} \) for some other \( k \neq i \). Thus, we have

\[
\pi^1(\psi^{-j}(\bar{X}_{\bar{\tau}^j}^j)) = \pi^1(\psi^{-j}(e^j, \rho_i(e^j))) \quad \mathbb{P}_{(e^j,s)} \text{-a.s. on } \{ \bar{\tau}^j < \bar{\tau}^j \},
\]
and because $\psi^j$ maps $e_i^j$ to $i$, the definition of the transfer kernel $K^j$, which was summarized in equation (5.4), gives

$$K^j = k^j\left(\pi^1(\psi^j((e_i^j, \rho_s(e_i^j)))) = \begin{cases} \varepsilon_{\partial_+(i)}, & i \in I_s^j, \\ \varepsilon_{\partial_-(i)}, & i \in I_s^{-j}. \end{cases} \right.$$ 

This results in

$$\mathbb{E}_g^j\left(e^{-\alpha \zeta^j} K^j h^{-j}; \zeta^j < j \right) = \mathbb{E}_g^j\left(e^{-\alpha \tau^j} k^j\left(\pi^1(\psi^j(\bar{X}_g^j))\right) h^{-j}; \tau^j < \bar{\tau}^j \right)$$

\begin{align*}
(5.10) & \quad \mathbb{E}_g^j\left(e^{-\alpha \bar{R}^j_{(e_i^j, x)}(e_i^j, \rho_s(e_i^j))} h^{-j}(\partial_+(i)) \bar{R}^j_{(e_i^j, \rho_s(e_i^j))} < \bar{R}^j_{(e_i^j, x)}, \right. \\
& \quad \mathbb{E}_g^j\left(e^{-\alpha \bar{R}^j_{(e_i^j, \rho(i) - x)}(e_i^j, \rho_s(e_i^j))} h^{-j}(\partial_-(i)) \bar{R}^j_{(e_i^j, \rho_s(e_i^j))} < \bar{R}^j_{(e_i^j, \rho(i) - x)} \right), \\
& \quad i \in I_s^j, \quad i \in I_s^{-j}.
\end{align*}

Now, the first passage time formulas for the one-dimensional Brownian motion $B$ (cf. [2, Section 1.7]) give

$$\mathbb{E}_x^B(e^{-\alpha \tau^i; \tau^i < \tau^i}) = \frac{\sinh(\sqrt{2\alpha} x)}{\sinh(\sqrt{2\alpha} \tau^i)} = \mathbb{E}_x^B(e^{-\alpha \tau^i; \tau^i < \tau^i}),$$

$$\mathbb{E}_x^B(e^{-\alpha \tau^i; \tau^i < \tau^i}) = \frac{\sinh(\sqrt{2\alpha} (\tau^i - x))}{\sinh(\sqrt{2\alpha} \tau^i)} = \mathbb{E}_x^B(e^{-\alpha \tau^i; \tau^i < \tau^i}).$$

A comparison of the equations (5.9) and (5.10) then proves the equalities in (ii).

We have shown that the conditions of [8, Theorem 1.6] are fulfilled and thus have proved:

**Lemma 5.5.** The process $X$ which is obtained by forming alternating copies of $X^{-1}$ and $X^{+1}$ via the transfer kernels $K^{-1}$ and $K^{+1}$, as defined by equation (5.3), is a right process on $G^{-1} \cup G^{+1} = G$.

### 5.6. Proving that $X$ is a Brownian Motion on $G$

As just seen, $X$ is a right process and therefore a strong Markov process on $G$. In regard to [6, Theorem 2.5], it suffices to analyze the stopped resolvent and the exit behavior from any edge in order to show that $X$ is indeed a Brownian motion on $G$:

**Lemma 5.6.** $X$ is a Brownian motion on $G$.

**Proof.** For mutual edges $i \in I_s$, we choose $j \in \{-1, +1\}$ such that $i \in I_s^j$. Then we have $X_t = X_i^j$ for all $t < \tau^j$ and $X_{R^j} \in \partial(i)$, $\mathbb{P}_{(i,x)}$-a.s., for the first revival time $R^j = \inf \{ t \geq 0 : X_t \in G^{-j} \setminus G^j \}$. Therefore, $H_X = \tau^j \wedge R^j$ holds true, and with equation (5.8) we get

\begin{align*}
\mathbb{E}_{(i,x)}\left(\int_0^{H_X} e^{-\alpha t} f(X_t) \, dt\right) &= \mathbb{E}_{(i,x)}^j\left(\int_0^{\tau^j \wedge \rho_s(e_i^j)} e^{-\alpha t} f(X_t^j) \, dt\right) \\
&= \mathbb{E}_x^B\left(\int_0^{\tau^i \wedge \tau^i} e^{-\alpha t} f(i, B_t) \, dt\right) \\
&= \mathbb{E}_x^B\left(\int_0^{H_B} e^{-\alpha t} f(i, B_t) \, dt\right).
\end{align*}
For non-mutual edges \( l \notin \mathcal{I}_s \), on the other hand, choose \( j \in \{-1,+1\} \) such that \((l,x) \in \tilde{G}_i^j \). Then \( X^j_l = \tilde{X}^j_l \) holds for all \( t < \tau^j_l \), \( \mathbb{P}_{(l,x)} \)-a.s., and as \( \tilde{X}^j \) is itself a Brownian motion on \( \tilde{G}_i^j \), the above identity follows immediately.

Coming to the exit distribution from an edge, the identity
\[
\mathbb{P}_{(l,x)} \circ (H_{X^j}, X_{H_{X^j}})^{-1} = \mathbb{P}_x^B \circ (H_B, (l, B_{H_B}))^{-1}
\]
follows for edges \( l \notin \mathcal{I}_s \) from the corresponding property of \( \tilde{X}^{-1} \) or \( \tilde{X}^{+1} \) by [6, Theorem 2.5]. In case \( i \in \mathcal{I}_s \), we choose \( j \in \{-1,+1\} \) with \( i \in \mathcal{I}_s^j \). By employing equations (5.9), (5.10) and \( H_X = \tau^j_l \wedge R^1 \mathbb{P}_{(l,x)} \)-a.s., we get for all \( \alpha > 0 \), \( h \in b\mathcal{B}(\mathcal{G}) \)
\[
\mathbb{E}_{(i,x)}(e^{-\alpha H_{X^j}} h(X_{H_{X^j}})) = \mathbb{E}_{(i,x)}^j(e^{-\alpha \tau^j_l h(X^j_l)}; \tau^j_l < \zeta^j_l) + \mathbb{E}_{(i,x)}^j(e^{-\alpha \zeta^j_l} K^j g; \zeta^j_l < \tau^j_l)
\]
\[
= \mathbb{E}_x^h(e^{-\alpha \tau^j_l} h(\partial^j_l(i)); \tau^j_l < \tau^j_{\rho(i)}) + \mathbb{E}_x^h(e^{-\alpha \tau^j_l} h(\partial^j_l(i)); \tau^j_l < \tau^j_{\rho(i)})
\]
which results in
\[
\mathbb{P}_{(i,x)} \circ (H_{X^j}, X_{H_{X^j}})^{-1} = \mathbb{P}_x^B \circ (H_B, (i, B_{H_B}))^{-1}. \quad \square
\]

5.7. Computing the Feller–Wentzell Data of \( X \). The Feller–Wentzell data of \( X \), as given in [6, Theorem 1.2], is derived from its exit distributions from any arbitrarily small neighborhood of each vertex. \( X \) is constructed via alternating copies of \( X^{-1} \) and \( X^{+1} \), so we first need to analyze their respective exit behavior.

To this end, we consider the exit times of \( X^j \)
\[
\tau^j_{\varepsilon} := \inf \{ t \geq 0 : d(X^j_t, X^j_0) > \varepsilon \}
\]

and the exit distributions \( X^j_{\tau^j_{\varepsilon}} \) for all small \( \varepsilon > 0 \). As we only have information on \( \tilde{X}^j \), we need to trace back the required data to these original processes. Fix \( v \in \mathcal{V} \) and choose \( j \in \{-1,+1\} \) such that \( v \in \mathcal{V}^j \), and let
\[
\tilde{\tau}^j_{\varepsilon} := \inf \{ t \geq 0 : d(\tilde{X}^j_t, \tilde{X}^j_0) > \varepsilon \}.
\]

Using the definition of \( X^j \) and the isometric property of \( \psi^j \), we get for all \( \varepsilon > 0 \)
\[
\tau^j_{\varepsilon} = \inf \{ t \geq 0 : d(\psi^j(X^j_t), \psi^j(X^j_0)) > \varepsilon \}
\]
\[
= \inf \{ t \geq 0 : d(\tilde{X}^j_t, \tilde{X}^j_0) > \varepsilon \}
\]
\[
= : \tilde{\tau}^j_{\varepsilon}.
\]

By its definition, \( \tilde{X}^j_t \) holds for all \( t < \tilde{\tau}^j_{\varepsilon} \), and as \( \tilde{\mathcal{C}} B_v(v) \subset \tilde{G}^j_v \), we obtain
\[
\forall \varepsilon < \delta : \quad \tilde{\tau}^j_{\varepsilon} \leq \tilde{\tau}^j_{\varepsilon} \quad \mathbb{P}_v \text{-a.s.}
\]

More precisely, we even get
\[
\forall \varepsilon < \delta : \quad \tilde{\tau}^j_{\varepsilon} < \tilde{\tau}^j_{\varepsilon} \quad \text{if} \quad \tilde{\tau}^j_{\varepsilon} \neq +\infty, \quad \mathbb{P}_v \text{-a.s.},
\]

because
\[
\mathbb{P}_v^j(\tilde{\tau}^j_{\varepsilon} = \tilde{\tau}^j_{\varepsilon}, \tilde{\tau}^j_{\varepsilon} < +\infty) = \mathbb{P}_v^j(\tilde{\tau}^j_{\varepsilon} = \tilde{\tau}^j_{\varepsilon}, \tilde{X}^j_{\tilde{\tau}^j_{\varepsilon}} \in \tilde{G}^j_v, \tilde{\tau}^j_{\varepsilon} < +\infty)
\]
\[
= \mathbb{P}_v^j(\tilde{\tau}^j_{\varepsilon} = \tilde{\tau}^j_{\varepsilon}, \tilde{X}^j_{\tilde{\tau}^j_{\varepsilon}} \in \tilde{G}^j_v, \tilde{\tau}^j_{\varepsilon} < +\infty)
\]
\[
\leq \mathbb{P}_v^j(\tilde{X}^j_{\tilde{\tau}^j_{\varepsilon}} \in \tilde{G}^j_v) = 0.
\]
Therefore, we see that for all $\varepsilon < \delta$,
\[
\bar{\tau}_\varepsilon = \inf \{ t \in [0, \bar{\tau}_\varepsilon) : d(\hat{X}_t^j, \hat{X}_0^j) > \varepsilon \} \wedge \bar{\tau}_\varepsilon
\]
\[
= \inf \{ t \in [0, \bar{\tau}_\varepsilon) : d(\bar{X}_t^j, \bar{X}_0^j) > \varepsilon \} \wedge \bar{\tau}_\varepsilon
\]
\[
= \inf \{ t \geq 0 : d(\bar{X}_t^j, \bar{X}_0^j) > \varepsilon \},
\]
where we used that $\hat{X}^j$ is a subprocess of $\bar{X}^j$ with lifetime $\bar{\tau}_\varepsilon$, that is
\[
d(\hat{X}_t^j, \hat{X}_0^j) = d(\Delta, \bar{X}_0^j) = +\infty > \varepsilon.
\]
We have thus shown:

**Lemma 5.7.** Let $v \in \mathcal{V}^j$. For all $\varepsilon < \delta$, it holds $\mathbb{P}_v$-a.s. that
\[
\tau_\varepsilon^j = \bar{\tau}_\varepsilon = \bar{\tau}_\varepsilon^j,
\]
and
\[
\bar{\tau}_\varepsilon^j < \bar{\tau}_\varepsilon^j, \quad \text{if} \quad \bar{\tau}_\varepsilon^j < +\infty.
\]

**Corollary 5.8.** For all $v \in \mathcal{V}^j$, $\varepsilon < \delta$, the exit distribution of $X^j$ is given by
\[
X^j_{\tau_\varepsilon^j} = \begin{cases} 
\psi^j(\hat{X}^j_{\tau_\varepsilon^j}), & \bar{\tau}_\varepsilon^j < +\infty, \\
\Delta, & \bar{\tau}_\varepsilon^j = +\infty.
\end{cases}
\]

We are ready to compute the Feller–Wentzell data of $X$. By Lemmas 5.7 and 5.3, we have for all $\varepsilon < \delta$
\[
\tau_\varepsilon^j = \bar{\tau}_\varepsilon < \bar{\tau}_\varepsilon^j = \zeta^j \quad \text{on} \quad \{ \zeta^j < +\infty \},
\]
so $\tau_\varepsilon^j < \zeta^j$ a.s. holds. On the other hand, $X_t = X_t^j$ holds for all $t < R^j = \zeta^j$ (more formally, $X_t^j(\omega^j) = X_t((\omega_1, \omega_2, \ldots))$ with $i = 1$ if $j = -1$, and $i = 2$ if $j = +1$) by the construction of $X$, yielding
\[
\mathbb{P}_v \circ (\tau_\varepsilon, X_{\tau_\varepsilon})^{-1} = \mathbb{P}_v^j \circ (\tau_\varepsilon^j, X_{\tau_\varepsilon^j})^{-1}.
\]
Thus, if $v$ is not a trap, then $\bar{\tau}_\varepsilon^j < +\infty$ holds $\mathbb{P}_v$-a.s. for all sufficiently small $\varepsilon > 0$ (see [6, Lemma B.1]), and therefore $\tau_\varepsilon < +\infty$ holds $\mathbb{P}_v$-a.s. as well. By using the notations of [6, Theorem 1.2] and backtracking $X$ to $\hat{X}^j$, we compute for $\varepsilon < \delta$, for all $A \in \mathcal{B}(\mathcal{G} \setminus \{v\})$:
\[
\nu^v_\varepsilon(A) = \frac{\mathbb{P}_v(X_{\tau_\varepsilon} \in A)}{\mathbb{E}_v(\tau_\varepsilon)} = \frac{\mathbb{P}_v^j(X_{\tau_\varepsilon^j} \in A)}{\mathbb{E}_v^j(\tau_\varepsilon^j)} = \frac{\mathbb{P}_v^j(\psi^j(\hat{X}^j_{\tau_\varepsilon^j}) \in A)}{\mathbb{E}_v^j(\bar{\tau}_\varepsilon^j)} = \tilde{\nu}^j_v(\psi^j)^{-1}(A),
\]
where we naturally extend, here and in all that follows, the mapping $\psi^j : \hat{G}^j \to \mathcal{G}^j$ to $\psi^j : \hat{G}^j \to \mathcal{G}$. This gives
\[
K_\varepsilon^v = 1 + \frac{\mathbb{P}_v(X_{\tau_\varepsilon} = \Delta)}{\mathbb{E}_v(\tau_\varepsilon)} + \int_{\mathcal{G} \setminus \{v\}} (1 - e^{-d(v, g)}) \nu^v_\varepsilon(dg)
\]
\[
= 1 + \frac{\mathbb{P}_v^j(\hat{X}^j_{\tau_\varepsilon^j} = \Delta)}{\mathbb{E}_v^j(\bar{\tau}_\varepsilon^j)} + \int_{\hat{G} \setminus \{v\}} (1 - e^{-d(v, \psi^j(g)))}) \tilde{\nu}^j_v(dg)
\]
\[
= 1 + \frac{\mathbb{P}_v^j(\hat{X}^j_{\tau_\varepsilon^j} = \Delta)}{\mathbb{E}_v^j(\bar{\tau}_\varepsilon^j)} + \int_{\hat{G} \setminus \{v\}} (1 - e^{-d(v, g)}) \tilde{\nu}^j_v(dg)
\]
\[
= \bar{K}_\varepsilon^j,v,
\]
because $\psi^j$ is an isometry with $\psi^j(v) = v$, and as $\tilde{\nu}^v_e(\tilde{G}^j \setminus \tilde{G}^j) = 0$ holds due to the assumption (5.1). Renormalization yields, again because $\psi$ is an isometry,

$$\forall A \in \mathcal{B}(\tilde{G} \setminus \{v\}) : \ \mu^v_e(A) = \int_A \left(1 - e^{-d(v, \rho)}\right) \frac{\nu^v_e(dg)}{K^v_e}$$

(5.11)

$$= \int_{\psi^{-1}(A)} \left(1 - e^{-d(v, \psi(g))}\right) \frac{\tilde{\nu}^v_e(dg)}{K^v_e}$$

$$= \tilde{\mu}^j_e((\psi^j)^{-1}(A)).$$

Next, introduce the topological subspaces $\tilde{G}^j \setminus \{v\}$ of $\tilde{G}^j \setminus \{v\}$ and $\tilde{G} \setminus \{v\}$ of $\tilde{G} \setminus \{v\}$, and consider the continuous extension of $\psi^j : \tilde{G}^j \to \tilde{G}^j$ to $\tilde{\psi}^j : \tilde{G}^j \setminus \{v\} \to \tilde{G} \setminus \{v\}$. Continuity of $\tilde{\psi}^j$ dictates that the new points $\tilde{G}^j \setminus \{v\} \setminus \tilde{G}^j$ are mapped to

$$i \in I^j(v) : \ \tilde{\psi}^j((e_i, 0+)) = \lim_{x \to 0} \psi^j((e_i, x)) = (i, 0+),$$

$$\tilde{\psi}^j((e_i, \rho_i)) = \lim_{x \to \|\rho(i)\|} \psi^j((e_i, x)) = (i, \rho_i),$$

(5.12)

$$i \in I^{-j}(v) : \ \tilde{\psi}^j((e_i, 0+)) = \lim_{x \to 0} \psi^j((e_i, x)) = (i, \rho_i),$$

$$\tilde{\psi}^j((e_i, \rho_i)) = \lim_{x \to \|\rho(i)\|} \psi^j((e_i, x)) = (i, 0+),$$

and analogously

$$i \in I^j(v) : \ \tilde{\psi}^j((i, 0+)) = (i, 0+), \text{ if } v = \partial_-(i),$$

$$\tilde{\psi}^j((i, \rho_i)) = (i, \rho_i), \text{ if } v = \partial_+(i),$$

(5.13)

$$e \in E^j(v) : \ \tilde{\psi}^j((e, 0+)) = (e, 0+),$$

$$e \in E^j(v) : \ \tilde{\psi}^j((e, +\infty)) = (e, +\infty).$$

Proceeding in the course of the proof of [6, Theorem 1.2] for $\tilde{X}^j$, we extend the measures $\tilde{\nu}^v_e$ to measures $\tilde{\mu}^v_e$ on $\tilde{G} \setminus \{v\}$ by

$$\tilde{\mu}^v_e(A) := \tilde{\mu}^v_e(A \cap (\tilde{G}^j \setminus \{v\})), \quad A \in \mathcal{B}(\tilde{G} \setminus \{v\}),$$

and choose a sequence of positive numbers $(\varepsilon_n, n \in \mathbb{N})$ converging to zero, such that $(\tilde{\mu}^v_{\varepsilon_n}, n \in \mathbb{N})$ converges weakly to a measure $\tilde{\mu}^v$. When also extending the measures $\mu^v_e$ to measures $\mu^v_e$ on $\tilde{G} \setminus \{v\}$, we obtain with equation (5.11)

$$\forall A \in \mathcal{B}(\tilde{G} \setminus \{v\}) : \ \mu^v_e(A) = \mu^v_e(A \cap (\tilde{G} \setminus \{v\}))$$

$$= \tilde{\mu}^v_e((\psi^j)^{-1}(A \cap (\tilde{G} \setminus \{v\})))$$

$$= \tilde{\mu}^v_e((\tilde{\psi}^j)^{-1}(A \cap (\tilde{G}^j \setminus \{v\})))$$

$$= \tilde{\mu}^v_e((\tilde{\psi}^j)^{-1}(A) \cap (\tilde{G} \setminus \{v\}))$$

$$= \tilde{\mu}^v_e \circ (\tilde{\psi}^j)^{-1}(A).$$

By the continuous mapping theorem, $(\tilde{\mu}^v_{\varepsilon_n}, n \in \mathbb{N})$ converges weakly to the measure $\tilde{\mu}^v = \tilde{\mu}^j \circ (\tilde{\psi}^j)^{-1}$ on $\tilde{G} \setminus \{v\}$. We summarize all of our results up to this point:
Lemma 5.9. Let \( v \in \mathcal{V}^j \), and \( K^v, \mu^v, \bar{\mu}^v \) and \( \bar{\tilde{K}}^v, \tilde{\mu}^v, \bar{\tilde{\mu}}^v \) be defined as in [6, Theorem 1.2] for the Brownian motions \( X, \bar{X}^j \) respectively. Then,

(i) \( K^v = \bar{\tilde{K}}^v \) for all \( \varepsilon < \delta \),
(ii) \( \mu^v = \tilde{\mu}^v \circ (\tilde{\upsilon}^j)^{-1} \) for all \( \varepsilon < \delta \),
(iii) \( (\bar{\mu}^v_{n}, n \in \mathbb{N}) \) converges weakly along the same sequence \( (\varepsilon_{n}, n \in \mathbb{N}) \) of positive numbers for which \( (\bar{\tilde{\mu}}^v_{n}, n \in \mathbb{N}) \) converges weakly to \( \bar{\tilde{\mu}}^v \), and the limit of \( (\bar{\mu}^v_{n}, n \in \mathbb{N}) \) is

\[
\bar{\mu}^v = \bar{\tilde{\mu}}^v \circ (\tilde{\upsilon}^j)^{-1}.
\]

We are now ready to compute the Feller–Wentzell data of the glued process \( X \), thus completing the proof of Theorem 5.1:

Proof of Theorem 5.1. We have already proved in Lemma 5.6 that \( X \) is a Brownian motion on \( G \). It remains to compute the Feller–Wentzell data of \( X \) by employing Lemma 5.9. To this end, let \( v \in \mathcal{V} \) and choose \( j \in \{-1, 1\} \) such that \( v \in \mathcal{V}^j \).

The killing parameters are given by

\[
e_1^{v, \Delta} = \lim_{n \to \infty} \frac{P_{v}(X_{\tau_{v}} = \Delta)}{E_{v}(\tau_{v})K_{v}}, \quad e_1^{v, \infty} = \lim_{n \to \infty} \frac{P_{v}(\bar{X}^{j}_{\bar{\mu}^{v}_{n}} = \Delta)}{E_{v}(\tau_{v})\bar{\tilde{K}}^{v}}, \quad p_{1}^{v, \Delta},
\]

and thus vanish, as \( p_{1}^{v} = p_{1}^{v, \Delta} + p_{1}^{v, \infty} = 0 \) holds by assumption.

The reflection parameters are defined as

\[
e_2^{v, l} = \begin{cases} \bar{\mu}^v \left\{ (l, 0+) \right\}, & l \in \mathcal{E}(v), \\ \bar{\mu}^v \left\{ (l, 0+) \right\}, & l \in \mathcal{I}(v), v = \partial_-(l), \\ \bar{\mu}^v \left\{ (l, \rho_l) \right\}, & l \in \mathcal{I}(v), v = \partial_+(l). \end{cases}
\]

For \( e \in \mathcal{E}(v) \), the relation \( (\tilde{\upsilon}^j)^{-1}(\left\{ (e, 0+) \right\}) = (e, 0+) \) immediately yields \( c_2^{v, e} = p_2^{v, e} \).

For \( i \in \mathcal{I}(v) \), we need to distinguish some cases, using equations (5.12) and (5.13): For \( i \in \mathcal{I}(v) \) with \( v = \partial_-(i) \), that is if \( i \in \mathcal{I}(v) \cup \mathcal{I}(v) \), we have

\[
c_2^{v, i} = \bar{\mu}^v \left\{ (i, 0+) \right\} = \begin{cases} \bar{\mu}^v \left\{ (i, 0+) \right\} = p_2^{v, i}, & i \in \mathcal{I}(v), \\ \bar{\mu}^v \left\{ (i, \rho_i) \right\} = p_2^{v, i}, & i \in \mathcal{I}(v). \end{cases}
\]

while for \( i \in \mathcal{I}(v) \) with \( v = \partial_+(i) \), that is if \( i \in \mathcal{I}(v) \cup \mathcal{I}(v) \), we have

\[
c_2^{v, i} = \bar{\mu}^v \left\{ (i, \rho_i) \right\} = \begin{cases} \bar{\mu}^v \left\{ (i, \rho_i) \right\} = p_2^{v, i}, & i \in \mathcal{I}(v), \\ \bar{\mu}^v \left\{ (i, \rho_i) \right\} = p_2^{v, i}, & i \in \mathcal{I}(v). \end{cases}
\]

The diffusion parameter is given by

\[
e_3^v = \lim_{n \to \infty} \frac{1}{K_{v}} = \lim_{n \to \infty} \frac{1}{\bar{\tilde{K}}_{v}} = p_{3}^{v}.
\]
For all $A \in \mathcal{A}(\mathcal{G}\setminus\{v\})$, the jump distribution is computed by
\[
c_i^v(A) = \int_A \frac{1}{1 - e^{-d(v,g)}} \overline{\mu}^v(g) \, dg
= \int_{(\psi^v)^{-1}(A)} \frac{1}{1 - e^{-d(v,\psi^v(g))}} \overline{\mu}_i^v(g) \, dg
= p_i^v \circ (\psi^v)^{-1}(A),
\]
as $\overline{\mu}$ is an extension from $\psi^v : \tilde{\mathcal{G}} \to \mathcal{G}$ and an isometry. \hfill \square

6. Completing the Construction

We are ready to carry out the construction that was laid out in section 2.

**Theorem 6.1.** Let $\mathcal{G} = (V, E, I, \partial, \rho)$ be a metric graph, and for every $v \in V$ let constants $p_{v,l}^l \geq 0$ for each $l \in \mathcal{L}(v)$, $p_{v}^v \geq 0$ and a measure $p_i^v$ on $\mathcal{G}\setminus\{v\}$ be given, satisfying
\[
\sum_{l \in \mathcal{L}(v)} p_{v,l}^l + p_{v}^v + \int_{\mathcal{G}\setminus\{v\}} (1 - e^{-d(v,g)}) p_i^v(g) \, dg = 1,
\]
and
\[
p_i^v(\mathcal{G}\setminus\{v\}) = +\infty, \quad \text{if} \quad \sum_{l \in \mathcal{L}(v)} p_{v,l}^l + p_{v}^v = 0,
\]
as well as $p_i^v(\partial \mathcal{G}(v)) = 0$ for some $\delta \in (0, \min_{l \in \mathcal{L}} \rho_l)$. Then there exists a Brownian motion $X$ on $\mathcal{G}$ which has infinite lifetime, is continuous inside all edges, satisfies $X_{\tau_v} \in \mathcal{B}_\delta(v)$ $\mathbb{P}_v$-a.s. for all $\varepsilon < \delta$, $v \in V$, and admits the Feller–Wentzell data
\[
(0, (p_{v,l}^l)_{l \in \mathcal{L}(v)}, p_i^v, p_i^v \circ \psi^v)_{v \in V}.
\]

**Proof.** We proceed via an induction over the count $n := |V|$ of vertices. If $n = 1$, then $\mathcal{G}$ is a star graph, so the construction given in [7] together with [7, Theorem 4.33], [6, Lemma 4.1] and [7, Theorem 4.3] yield the result.

Assume now that such Brownian motions exist for all metric graphs with less than $n$ vertices. Let $\mathcal{G}$ be a metric graph with $n$ vertices $V = \{v_1, \ldots, v_n\}$ and boundary data as given in the theorem. We decompose the graph into $\tilde{\mathcal{G}}^{-1}$ and $\tilde{\mathcal{G}}^{+1}$, as done in section 5, for $V^{-1} = \{v_1, \ldots, v_{n-1}\}$ and $V^{+1} = \{v_n\}$. Then the conditions of the theorem are satisfied for these graphs $\tilde{\mathcal{G}}^{-1}$, $\tilde{\mathcal{G}}^{+1}$ with $n-1$ vertices, one vertex respectively, and corresponding boundary data $(p_{v,l}^l \geq 0, l \in \tilde{\mathcal{L}}^j(v))$, $p_{v}^v \geq 0$ and $p_i^v \circ \psi^j$ (as $\psi^j$ is an isometry, this data satisfies the normalization requirements). Therefore, there exist Brownian motions $\tilde{X}^j$ on $\tilde{\mathcal{G}}^j$ with infinite lifetime which are continuous inside all edges, satisfy $\tilde{X}^j_{\tau_v} \in \mathcal{B}_\delta(v)$ $\mathbb{P}^j_v$-a.s. for all $v \in V^j$ and admit the Feller–Wentzell data
\[
(0, (p_{v,l}^l)_{l \in \tilde{\mathcal{L}}^j(v)}, p_i^v, p_i^v \circ \psi^j)_{v \in V^j}
\]
with $p_{v,l}^l := p_{v,l}^j$ for $i \in \tilde{\mathcal{L}}^j(v)$, $v \in V^j$. We then follow the construction of section 5 in order to glue $\tilde{X}^{-1}$ and $\tilde{X}^{+1}$ together, and Theorem 5.1 concludes the proof. \hfill \square
Theorem 6.2. Let $\mathcal{G} = (V, E, I, \partial, \rho)$ be a metric graph, and for every $v \in V$ let constants $p_{1}^{v} \geq 0$, $p_{2}^{v,l} \geq 0$ for each $l \in L(v)$, $p_{3}^{v} \geq 0$ and a measure $p_{4}^{v}$ on $\mathcal{G} \setminus \{v\}$ be given with
\[ p_{1}^{v} + \sum_{l \in L(v)} p_{2}^{v,l} + p_{3}^{v} + \int_{\mathcal{G} \setminus \{v\}} (1 - e^{-d(v,g)}) p_{4}^{v}(dg) = 1, \]
and
\[ p_{4}^{v}(\mathcal{G} \setminus \{v\}) = +\infty, \quad \text{if} \quad \sum_{l \in L(v)} p_{2}^{v,l} + p_{3}^{v} = 0, \]
as well as $p_{4}^{v}(\overline{\{v\}}) = 0$ for some $\delta \in \left(0, \min_{l \in L} \rho_{l}\right)$. Then there exists a Brownian motion $X$ on $\mathcal{G} \cup \overline{\{v\}}$, $v \in V$, with $\overline{\{v\}}, v \in V$ being an isolated, absorbing set for $X$, such that $X$ has infinite lifetime, is continuous inside all edges, satisfies $X_{t_{v}} \in B_{\delta}(v) \cup \{\overline{v}\}$ $P_{v}$-a.s. for all $\varepsilon < \delta$, $v \in V$, and has the Feller–Wentzell data
\[ (0, (p_{2}^{v,l})_{l \in L(v)}, p_{3}^{v} + p_{4}^{v} \in \{\overline{v}\})_{v \in \mathcal{V}}. \]

Proof. This proof proceeds analogously to the proof of Theorem 6.1, except that we need to adjoin the isolated points $\overline{v}$, $v \in V$, to the partial processes and revive these processes there before gluing the partial graphs together.

If $|\mathcal{V}| = 1$, then $\mathcal{G}$ is a star graph, and the construction of [7] (again with [7, Theorem 4.33], [6, Lemma 4.1], and [7, Theorem 4.3]) gives a Brownian motion on $\mathcal{G}$ with the needed properties and Feller–Wentzell data
\[ \left(p_{1}^{v}, (p_{2}^{v,l})_{l \in L(v)}, p_{3}^{v}, p_{4}^{v}\right). \]
By concatenating it with the constant process on $\overline{\{v\}}$ with the technique of [8], we revive this Brownian motion on a new, isolated, absorbing point $\overline{v}$. Then a computation along the lines of Lemma 4.1 yields that the revived process is a Brownian motion on $\mathcal{G} \cup \overline{\{v\}}$ with its Feller–Wentzell data at $v$ being given by
\[ (0, (p_{2}^{v,l})_{l \in L(v)}, p_{3}^{v} + p_{4}^{v} \in \overline{v}). \]

Now let $V = \{v_{1}, \ldots, v_{n}\}$, and assume that the assertion of the theorem holds for any graph with less than $n$ vertices. We decompose the graph $\mathcal{G}$ into $\overline{\mathcal{G}}^{-1}$ and $\overline{\mathcal{G}}^{+1}$, as done in section 5, for $\mathcal{V}^{-1} = \{v_{1}, \ldots, v_{n-1}\}$ and $\mathcal{V}^{+1} = \{v_{n}\}$. By assumption, there exist Brownian motions $\overline{X}^{j}$ on $\overline{G}^{j} \cup \overline{\{\overline{v}, v \in V^{j}\}}$ with the needed path properties and Feller–Wentzell data
\[ (0, (p_{2}^{v,l})_{l \in L(v)}, p_{3}^{v} + p_{4}^{v} \in \overline{v}). \]
with $p_{v}^{v,i} := p_{2}^{v,i}$ for $i \in I_{v}(v)$, $v \in V^{j}$. We then again follow the construction of section 5 to glue $\overline{\mathcal{G}}^{-1}$ and $\overline{\mathcal{G}}^{+1}$ together, and Theorem 5.1 yields the result. \hfill $\Box$

In order to complete the proof of the existence theorem for Brownian motions on metric graphs with non-local boundary conditions, it remains to implement the “global” jumps:

Proof of Theorem 1.1. Let $\delta > 0$ with $\delta < \min_{l \in L} \rho_{l}$, and define for every $v \in V$
\[ q_{1}^{v} := p_{1}^{v} + p_{4}^{v} (\overline{\mathcal{C}}B_{\delta}(\delta)), \quad q_{4}^{v} := p_{4}^{v} |_{B_{v}(\delta)}, \]
The introduction of the normalizing factor
\[ c_0^u := \left( q_1^u + \sum_{l \in \mathcal{L}(v)} p_2^v, l + p_3^v + \int_{\mathcal{G}\setminus\{v\}} (1 - e^{-d(v,g)}) q_4^v (dg) \right)^{-1} \]
enables us to employ Theorem 6.2 in order to construct a Brownian motion \( X^1 \) on \( \mathcal{G}\cup\{\square^v, v \in \mathcal{V}\} \) which has infinite lifetime, is continuous inside all edges, satisfies \( X_{t_\varepsilon} \in B_\delta(v) \cup \{\square^v\} \) \( \mathbb{P}_v \)-a.s. for all \( \varepsilon < \delta, v \in \mathcal{V} \), and has the Feller–Wentzell data
\[ (c_0^v (0, (p_2^v, l)_{l \in \mathcal{L}(v)}, p_3^v, q_4^v + q_1^v \in \mathcal{V})_{v \in \mathcal{V}}. \]

As \( X^1 \) has infinite lifetime, we can use an application of the concatenation techniques (see [8] and [4, Proposition 14.20]) to adjoin a new, isolated, absorbing point \( \square \) on the isolated, absorbing set
\[ \mathcal{G} \cup \{\square^v, v \in \mathcal{V}\} \cup \{\square\} \]
which has infinite lifetime, is continuous inside all edges, satisfies
\[ \mathcal{G} \cup \{\square^v, v \in \mathcal{V}\} \cup \{\square\} \]
and additional Feller–Wentzell data \( (0, 0, 1, 0) \) at the new vertex \( \square \).

Let \( X^3 \) be the right process on \( \mathcal{G} \cup \{\square\} \) which results from killing \( X^2 \) on the absorbing set \( \{\square^v, v \in \mathcal{V}\} \) (see Appendix A). As \( X^3 \) is strongly Markovian and \( X^3_t = X^2_t \) for all \( t \leq H_v \), \( X^3 \) is a Brownian motion on \( \mathcal{G} \cup \{\square\} \), and Lemma 3.2 asserts that the Feller–Wentzell data of \( X^3 \) reads
\[ (c_0^v, (p_2^v, l)_{l \in \mathcal{L}(v)}, p_3^v, q_4^v + q_1^v \in \mathcal{V})_{v \in \mathcal{V}}. \]

Now construct \( X^4 \) as the revived process obtained from \( X^3 \) by the identical copies method with revival distributions
\[ \kappa^u := (q_1^u)^{-1} (p_1^v \in \square + p_4^v \vert_{B_\delta(v)}) , \quad v \in \mathcal{V}. \]

Then by Lemma 4.1, \( X^4 \) is a Brownian motion on \( \mathcal{G} \cup \{\square\} \), and its generator satisfies
\[ \mathcal{D}(A) \subseteq \left\{ f \in C_0^5(\mathcal{G} \cup \{\square\}) : \forall v \in \mathcal{V} : \right. \]
\[ \left. - \sum_{l \in \mathcal{L}(v)} c_0^v p_2^v, l f'(v) + c_0^v p_3^v f''(v) \right. \]
\[ \left. - \int_{(\mathcal{G}\setminus\{v\}) \cup \{\square\}} (f(g) - f(v)) c_0^v (p_4^v \vert_{B_\delta(v)} + p_5^v \in \square + p_4^v \vert_{B_\delta(v)})(dg) = 0 \right\} \]
\[ = \left\{ f \in C_0^5(\mathcal{G} \cup \{\square\}) : \forall v \in \mathcal{V} : \right. \]
\[ \left. - \sum_{l \in \mathcal{L}(v)} p_2^v, l f'(v) + p_3^v f''(v) \right. \]
\[ \left. - \int_{(\mathcal{G}\setminus\{v\}) \cup \{\square\}} (f(g) - f(v)) (p_4^v + p_5^v \in \square)(dg) = 0 \right. \].

Finally, employ once more the transformation of Appendix A in order to kill \( X^4 \) on the isolated, absorbing set \( \{\square^v, v \in \mathcal{V}\} \) and obtain the Brownian motion \( X^5 \) on \( \mathcal{G} \). Lemma 3.1 asserts that the domain of its generator satisfies
\[ \mathcal{D}(A) \subseteq \left\{ f \in C_0^5(\mathcal{G}) : \forall v \in \mathcal{V} : \right. \]
\[ \left. p_1^v f(v) - \sum_{l \in \mathcal{L}(v)} p_2^v, l f'(v) + p_3^v f''(v) - \int_{\mathcal{G}\setminus\{v\}} (f(g) - f(v)) p_4^v (dg) = 0 \right\}. \]
Appendix A. Killing on an Absorbing Set

We present an easy technique to kill a right process on an absorbing set (see [4, Definition 12.27]), which will be used in the main construction of this article. For the role of the cemetery point ∆ and the lifetime conventions in the context of right processes, the reader may consult [4, Section 11].

Let $\tilde{E} = E_\Delta \cup F$ be the topological union of two disjoint Radon spaces $E_\Delta$ and $F$, and consider a right process $X$ on $\tilde{E}$, with $F$ being an absorbing set for $X$. We kill the process $X$ on this absorbing set $F$ by mapping $F$ to $\Delta$ with

$$\psi: \tilde{E} \to E_\Delta, \ x \mapsto \psi(x) := \begin{cases} x, & x \in E_\Delta, \\ \Delta, & x \in F. \end{cases}$$

By checking the consistency conditions for state space transformations of right processes (see [4, Section 13] and [8, Section 3.1]), we show:

**Theorem A.1.** $\psi(X)$ is a right process on $E_\Delta$.

**Proof.** The transformation $\psi$ is clearly surjective and measurable, as

$$\forall B \in \mathcal{E}_\Delta: \ \psi^{-1}(B) = \begin{cases} B, & \Delta \notin B, \\ B \cup F, & \Delta \in B. \end{cases}$$

Let $H_F$ be the first entry time of $X$ into $F$. We have $X_t \in F$ for all $t \geq H_F$ a.s., as the strong Markov property at $H_F$ yields

$$\mathbb{P}(X_{H_F + t} \in F \text{ for all } t \geq 0) = \mathbb{E}(\mathbb{P}_{X_{H_F}}(X_t \in F \text{ for all } t \geq 0)) = 1.$$ 

Furthermore, it is evident that $X_t \notin F$ for all $t < H_F$, so the transformed process

$$t \mapsto \psi(X_t) = \begin{cases} X_t, & t < H_F, \\ \Delta, & t \geq H_F \end{cases}$$

is a.s. right continuous.

For all $f \in b\mathcal{E}_{\Delta}$, $x \in \tilde{E}$, we have for the semigroup $(T_t, t \geq 0)$ of $X$:

$$T_t(f \circ \psi)(x) = \mathbb{E}_x(f \circ \psi(X_t)) = \mathbb{E}_x(f(X_t); t < H_F) + f(\Delta) \mathbb{P}_x(t \geq H_F) = g \circ \psi(x),$$

with $g \in b\mathcal{E}_{\Delta}$ being defined by

$$g(x) := \begin{cases} \mathbb{E}_x(f(X_t); t < H_F) + f(\Delta) \mathbb{P}_x(t \geq H_F), & x \in E, \\ f(\Delta), & x = \Delta, \end{cases}$$

as $H_F = 0$ holds $\mathbb{P}_x$-a.s. for all $x \in F$. \qed

**Acknowledgements**

The main parts of this paper were developed during the author’s Ph.D. thesis [5] supervised by Prof. Jürgen Potthoff, whose constant support the author gratefully acknowledges.
References

1. Kiyoshi Itô and Henry P. McKean, *Brownian motions on a half line*, Illinois J. Math. **7** (1963), 181–231.

2. **Diffusion processes and their sample paths**, 2 ed., Grundlehren Math. Wiss. 125, Springer, 1974.

3. Vadim Kostrykin, Jürgen Potthoff, and Robert Schrader, *Brownian motions on metric graphs*, J. Math. Phys. **53** (2012), 095206.

4. Michael Sharpe, *General theory of Markov processes*, Pure Appl. Math. 133, Academic Press, 1988.

5. Florian Werner, *Brownian motions on metric graphs*, Ph.D. thesis, University of Mannheim, 2016.

6. **Brownian motions on metric graphs with non-local boundary conditions I: characterization**, ArXiv e-prints (2018), 1805.06709.

7. **Brownian motions on star graphs with non-local boundary conditions**, ArXiv e-prints (2018), 1803.07027.

8. **Concatenation and pasting of right processes**, ArXiv e-prints (2018), 1801.02595.