Rainbow connection in some digraphs

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Abstract

An edge-coloured graph $G$ is rainbow connected if any two vertices are connected by a path whose edges have distinct colours. This concept was introduced by Chartrand et al. in \cite{3}, and it was extended to oriented graphs by Dorbec et al. in \cite{5}. In this paper we present some results regarding this extention, mostly for the case of circulant digraphs.

Keywords: arc-coloring; rainbow connected; connectivity

1 Introduction

Given a connected graph $G = (V(G), E(G))$, an edge-coloring of $G$ is called rainbow connected if for every pair of distinct vertices $u, v$ of $G$ there is a $uv$-path all whose edges received different colors. The rainbow connectivity number of $G$ is the minimum number $rc(G)$ such that there is a rainbow connected edge-coloring of $G$ with $rc(G)$ colors. Similarly, an edge-coloring of $G$ is called strong rainbow connected if for every pair $u, v \in V(G)$ there is a $uv$-path of minimal length (a $uv$-geodesic) all whose edges received different colors. The strong rainbow connectivity number of $G$ is the minimum number $src(G)$ such that there is a strong rainbow connected edge-coloring of $G$ with $src(G)$ colors.

The concepts of rainbow connectivity and strong rainbow connectivity of a graph were introduced by Chartrand et al. in \cite{3} and, been the connectivity one fundamental notion in Graph Theory, it is not surprising that several works around these concepts has been done since then (see for instance \cite{2, 4, 6, 7, 8, 9, 10, 11, 12}). For a survey in this topic see (\cite{13}). As a natural extension of this notions is that of

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the rainbow connection and strong rainbow connection in oriented graphs, which was introduced by Dorbec et al. in [5].

Let \( D = (V(D), A(D)) \) be a strong connected digraph and \( \Gamma : A(D) \to \{1, \ldots, k\} \) be an arc-coloring of \( D \). Given \( x, y \in V(D) \), a directed \( xy \)-path \( T \) in \( D \) will be called rainbow if no two arcs of \( T \) receive the same color. \( \Gamma \) will be called rainbow connected if for every pair of vertices \( x, y \in V(D) \) there is a rainbow \( xy \)-path and a rainbow \( yx \)-path. The rainbow connection number of \( D \), denoted as \( rc^*(D) \), is the minimum number \( k \) such that there is a rainbow connected arc-coloring of \( D \) with \( k \) colors. Given a pair of vertices \( x, y \in V(D) \), an \( xy \)-path \( T \) will be called an \( xy \)-geodesic if the length of \( T \) is the distance, \( d_D(x, y) \), from \( x \) to \( y \) in \( D \). An arc-coloring of \( D \) will be called strongly rainbow connected if for every pair of distinct vertices \( x, y \) of \( D \) there is a rainbow \( xy \)-geodesic and a rainbow \( yx \)-geodesic. The strong rainbow connection number of \( D \), denoted as \( src^*(D) \), is the minimum number \( k \) such that there is a strong rainbow connected arc-coloring of \( D \) with \( k \) colors.

In this paper we present some results regarding this problem, mainly for the case of circulant digraphs. For general concepts we may refer the reader to [1].

2 Some remarks and basic results on biorientations of graphs

Let \( D = (V(D), A(D)) \) be a strong connected digraph of order \( n \) and let \( \text{diam}(D) \) be the diameter of \( D \). As we see in [5], it follows that

\[
\text{diam}(D) \leq rc^*(D) \leq src^*(D) \leq n.
\]

Also, it is not hard to see that if \( H \) is a strong spanning subdigraph of \( D \), then \( rc^*(D) \leq rc^*(H) \). However, as in the graph case (see [2]), this is not true for the strong rainbow connection number, as we see in the next lemma.

**Lemma 2.1.** There is a digraph \( D \) and a spanning subdigraph \( H \) of \( D \) such that \( src^*(D) > src^*(H) \).
Proof. Let $H$ be as in Figure 1, where $D$ is obtained from $H$ by adding the arc $a_1a_2$. It is not hard to see that the colouring in Figure 1 is a strong rainbow 6-coloring of $H$, thus $src^*(H) \leq 6$. We will show that $src^*(D) \geq 7$. Suppose there is a strong rainbow 6-coloring $\rho$ of $D$. First notice that, for each $i$ and $j$, the $u_iv_j$-geodesic is unique and contains the arcs $u_iv_i$ and $u_jv_j$, hence there are no two arcs of the type $u_iv_i$ sharing the same colour. Without loss of generality let $\rho(u_iv_i) = i$ for $1 \leq i \leq 4$. By an analogous argument, since $P_i = u_iv_i a_1 a_2 u_4 v_4$ is the only $u_iv_i$-geodesic for $i \leq 3$, and $a_1a_2, a_2u_4 \in A(P_i)$, we can suppose that such arcs have colours 5 and 6, respectively. If we assign any of the six colours to the arc $v_1a_1$, we see that for some $j \geq 2$ the unique $u_1v_j$-geodesic is no rainbow, contradicting the choice of $\rho$. Hence $src^*(G) \geq 7$ and the result follows.

![Figure 1: The digraphs $D$ and $H$ from Lemma 2.1.](image)

Given a pair $v, u \in V(D)$, if the arcs $uv$ and $vu$ are in $D$, then we say that $uv$ and $vu$ are symmetric arcs. When every arc of $D$ is symmetric, $D$ is called a symmetric digraph. Given a graph $G = (V(G), E(G))$, its biorientation is the symmetric digraph $\overrightarrow{G}$ obtained from $G$ by replacing each edge $uv$ of $G$ by the pair of symmetric arcs $uv$ and $vu$. 

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Given a graph \( G \) and a (strong) rainbow connected edge-coloring of \( G \), it is not hard to see that the arc-coloring of \( \vec{G} \), obtained by assigning the color of the edge \( uv \) to both arcs \( uv \) and \( vu \) is a (strong) rainbow connected arc-coloring of \( \vec{G} \). Thus \( rc^*(\vec{G}) \leq rc(G) \) and \( src^*(\vec{G}) \leq src(G) \). Although for some graphs and its biorientations these values coincide (for instance, as we will see, for \( n \geq 4, rc(C_n) = src(C_n) = rc^*(\vec{C_n}) = src^*(\vec{C_n}) \)), for other graphs and its biorientations the difference between those values is unbounded, as we see in the case of the stars, where for each \( n \geq 2, rc(K_{1,n}) = n \) (for each path between terminal vertices we need two colors) and \( rc^*(\vec{K_{1,n}}) = src^*(\vec{K_{1,n}}) = 2 \) (the colouring that assigns color 1 to the in-arcs of the “central” vertex and assigns color 2 to the ex-arcs of the central vertex is a strong rainbow coloring).

**Theorem 2.2.** Let \( D \) be a nontrivial digraph, then

(a) \( src^*(D) = 1 \) if and only if \( rc^*(D) = 1 \) if and only if, for some \( n \geq 2, D = \vec{K_n} \);

(b) \( rc^*(D) = 2 \) if and only if \( src^*(D) = 2 \).

**Proof.** First observe that since \( D \) is nontrivial, \( rc^*(D) \geq 1 \) and therefore if \( src^*(D) = 1 \) then \( rc^*(D) = 1 \). If \( rc^*(D) = 1 \) then \( diam(D) = 1 \) and hence \( D = \vec{K_n} \) for some \( n \geq 2 \). On the other hand, if \( D = \vec{K_n} \) it follows that every 1-colouring of \( D \) is a strong rainbow colouring. Thus \( 1 \geq src^*(D) \geq rc^*(D) \geq 1 \) and (a) follows. For (b), if \( src^*(D) = 2 \), by (a), \( rc^*(D) > 1 \) and hence \( rc^*(D) = 2 \). If \( rc^*(D) = 2, D \) has a 2-rainbow colouring and, by (a) \( D \neq \vec{K_n} \). Therefore for every pair \( u,v \in V(D), with d(u,v) \geq 2, \) exists a \( uv \)-rainbow path of length 2, which is also geodesic. Hence \( src^*(D) = 2 \) and (b) follows. \qed

**Theorem 2.3.** (a) For \( n \geq 2, rc^*(\vec{P_n}) = src^*(\vec{P_n}) = n - 1 \);

(b) For \( n \geq 4, rc^*(\vec{C_n}) = src^*(\vec{C_n}) = \lceil n/2 \rceil \)

(c) Let \( k \geq 2, if \vec{K_{n_1,n_2,...,n_k}} \) is the complete \( k \)-partite digraph where \( n_i \geq 2 \) for some \( i, \) then \( rc^*(\vec{K_{n_1,n_2,...,n_k}}) = src^*(\vec{K_{n_1,n_2,...,n_k}}) = 2. \)
Proof. In [3] it is shown that for every \( n \geq 4 \), \( \text{src}(C_n) = \lceil \frac{n}{2} \rceil \) and for every \( n \geq 2 \), \( \text{src}(P_n) = n - 1 \). Since \( \text{diam}(P_n) = n - 1 \) it follows that \( n - 1 \leq \text{rc}^*(P_n) \leq \text{src}(P_n) \leq \text{src}(P_n) = n - 1 \) and the first part of the theorem follows. In an analogous way, if \( n \) is even, \( \lceil \frac{n}{2} \rceil = \text{diam}(C_n) \leq \text{rc}^*(C_n) \) and since \( \text{rc}^*(C_n) \leq \text{src}(C_n) = \lceil \frac{n}{2} \rceil \), \( \text{rc}^*(C_n) = \text{src}(C_n) = \lceil \frac{n}{2} \rceil \). Let \( n = 2k + 1 \) with \( k \geq 2 \) and let us suppose there is a rainbow \( k \)-colouring \( \rho \) of \( C_n \). Observe that for every \( 0 \leq i \leq n - 1 \), \((v_i, v_i+1, \ldots, v_i+k)\) is the only \( v_iv_{i+k} \)-path of length \( d(v_i, v_i+k) = k \) in \( C_n \) and therefore the \( k \) colours of \( \rho \) occurs in each of such geodesic paths. Thus \( \rho(v_i, v_i+1) = \rho(v_i+k, v_i+k+1) \) for each \( 0 \leq i \leq n - 1 \), which, since \( k \), \( n = 2k + 1 \) implies that all the arcs \( v_iv_{i+k} \) in \( C_n \) receive the same color which is a contradiction. Thus \( \text{rc}^*(C_n) \geq k + 1 = \lceil \frac{n}{2} \rceil \) and (b) follows. For (c), since \( n_i \geq 2 \) for some \( i \), then \( K_{n_1, n_2, \ldots, n_k} \) is not a complete digraph, hence \( \text{rc}^*(K_{n_1, n_2, \ldots, n_k}) \geq 2 \). Let \( V_1, V_2, \ldots, V_k \) be the \( k \)-partition on independent sets of \( V(K_{n_1, n_2, \ldots, n_k}) \), and for each arc \( uv \), with \( u \in V_i \) and \( v \in V_j \), assign color 1 to \( uv \) if \( i < j \) and color 2 if \( i > j \). Since \( \text{diam}(K_{n_1, n_2, \ldots, n_k}) = 2 \), it is not hard to see that this is a strong rainbow connected 2-coloring and therefore \( \text{src}(K_{n_1, n_2, \ldots, n_k}) \leq 2 \).

**Theorem 2.4.** Let \( D \) be a spanning strong connected subdigraph of \( C_n \) with \( k \geq 1 \) asymmetric arcs. Thus

\[
\text{rc}^*(D) = \begin{cases} 
  n-1 & \text{if } k \leq 2; \\
  n & \text{if } k \geq 3.
\end{cases}
\]

Moreover, if \( k \geq 3 \), \( \text{rc}^*(D) = \text{src}^*(D) = n \).

Proof. Let \( V(C_n) = \{v_0, \ldots, v_{n-1}\} \) and suppose \( v_0v_{n-1} \notin A(D) \). Since \( D \) is strong connected the \( v_0v_{n-1} \)-path \( T = (v_0, v_1, \ldots, v_{n-1}) \) is contained in \( D \), thus \( \text{diam}(D) \geq n - 1 \). Therefore, \( n - 1 \leq \text{rc}^*(D) \leq n \). If \( k = 1 \) we see that \( P_n \) is a spanning subdigraph of \( D \), hence \( n - 1 \leq \text{rc}^*(D) \leq \text{rc}^*(P_n) \), which by Theorem 2.3 (a) implies that \( \text{rc}^*(D) = n - 1 \). Let \( k \geq 2 \). If \( v_{n-1}v_0 \notin A(D) \), since \( D \) is strong connected it follows that \( D \) is isomorphic to \( P_n \) which have no asymmetric arcs and thus this is not possible. Therefore \( v_{n-1}v_0 \in A(D) \). If there is a \((n-1)\)-rainbow coloring \( \rho \) of \( D \), since \( v_{n-1}v_0 \in A(D) \), the directed cycle \( C \) induced by \( A(T) \cup v_{n-1}v_0 \) is a
spanning subdigraph of $D$ and therefore there are two arcs $v_i v_{i+1}, v_j v_{j+1} \in A(C)$ such that $\rho(v_i v_{i+1}) = \rho(v_j v_{j+1})$. Since $\rho$ is a rainbow coloring, there is a rainbow $v_i v_{j+1}$-path and a rainbow $v_j v_{i+1}$-path in $D$. Thus the paths $(v_i, v_{i-1}, \ldots, v_{j+2}, v_{j+1})$ and $(v_j, v_{j-1}, \ldots, v_{i+2}, v_{i+1})$ must be contained in $D$ and therefore the number of assymmetric arcs in $D$ is at most 2. Thus, if $k \geq 3$ then $rc^*(D) \geq n$ and hence, $rc^*(D) = n$. Finally, if $k = 2$, let $\rho$ be the $(n-1)$-arc coloring of $D$ which assigns the same color to the assymmetric arcs, and for the remaining $n-2$ pairs of simmetric arcs and the remaining $n-2$ colors, $\rho$ assigns the same color to each pair of simmetric arcs. It is not hard to see that $\rho$ is a rainbow coloring of $D$, thus $rc^*(D) \leq n-1$ and the first part of the theorem follows. The second part is directly from the first part of the theorem and from the fact that $src^*(D) \leq n$. $\square$

As a direct corollary of the previous result we have

**Corollary 2.5.** Let $D$ be a strong connected digraph with $m \geq 3$ arcs. Thus $rc^*(D) = src^*(D) = m$ if and only if $D = \vec{C}_m$.

# 3 Circulant digraphs

For an integer $n \geq 2$ and a set $S \subseteq \{1, 2, \ldots, n-1\}$, the *circulant digraph* $C_n(S)$ is defined as follows: $V(C_n(S)) = \{v_0, v_1, \ldots, v_{n-1}\}$ and

$$A(C_n(S)) = \{v_i v_j : j - i \equiv s \mod n, \ s \in S\},$$

where $a \equiv b$ means: *a congruent with b modulo n*. The elements of $S$ are called *generators*, and an arrow $v_i v_j$, where $j - i \equiv s \mod n$, will be called an *s-jump*. If $s \in S$ we denote by $C(s)$ the spanning subdigraph of $C_n(S)$ induced by all the $s$-jumps. Observe that for every pair of vertices $v_i$ and $v_j$ there is at most one $v_i v_j$-path in $C(s)$. If such $v_i v_j$-path in $C(s)$ exists will be denoted by $v_i C(s) v_j$. From now on the subscripts of the vertices are taken modulo $n$. Given an integer $k \geq 1$, let $[k] = \{1, 2, \ldots, k\}$.

**Theorem 3.1.** If $1 \leq k \leq n-2$, then $rc^*(C_n([k])) = src^*(C_n([k])) = \left\lceil \frac{n}{k} \right\rceil$. 

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Proof. Let \( D = C_n[k] \). The case when \( k = 1 \) is proved in Theorem 2.3. Let \( 2 \leq k \leq n - 2 \), and \( V(D) = \{v_0, \ldots, v_{n-1}\} \). By definition it follows that for every pair \( 0 \leq i \leq j \leq n - 1 \), \( d(v_i, v_j) = d(v_0, v_{j-i}) \) and \( d(v_j, v_i) = d(v_0, v_{i+n-j}) \). Also it is not hard to see that for every \( 0 \leq i \leq n - 1 \), \( d(v_0, v_i) = \lceil \frac{n}{k} \rceil \). From here it follows that \( \text{diam}(D) = \lceil \frac{n-1}{k} \rceil \).

Let \( P = \{V_1, V_2, \ldots, V_{\lceil \frac{n}{k} \rceil}\} \) be a partition of \( V(D) \) such that for each \( i \), with \( 1 \leq i \leq \lfloor \frac{n}{k} \rfloor \), \( V_i = \{v_j : (i-1)k \leq j \leq ik - 1\} \) and, if \( \lceil \frac{n}{k} \rceil \neq \lfloor \frac{n}{k} \rfloor \), \( V_{\lceil \frac{n}{k} \rceil} = \{v_j : k\lfloor \frac{n}{k} \rfloor \leq j \leq n-1\} \).

Claim 1 For every pair \( v_i, v_j \in V(D) \) there is a \( v_i v_j \)-geodesic path \( T \) such that for every \( V_p \in P \), \( |V_p \cap V(T \setminus v_j)| \leq 1 \).

Let \( v_{rk+i}, v_{sk+j} \in V(D) \). If \( r \neq s \) let \( 0 \leq q \leq k - 1 \) and \( t \) be the minimum integer such that \( (r+t)k + i + q \equiv sk + j \) and let

\[
T = (v_{rk+i}, v_{(r+1)k+i}, \ldots, v_{(r+t)k+i}, v_{(r+t)k+i+q})
\]

be a \( v_{rk+i}v_{sk+j} \)-path. Since \( t \) is minimum and \( 0 \leq q \leq k - 1 \) it follows that \( T \) is a \( v_{rk+i}v_{sk+j} \)-geodesic path and, since for every \( V_p \in P \), \( |V_p| \leq k \), hence for every \( V_p \in P \), \( |V_p \cap V(T \setminus v_{sk+j})| \leq 1 \).

If \( r = s \) and \( i \leq j \) it follows that \( v_{rk+i}v_{sk+j} \in A(D) \) and \( T = (v_{rk+i}, v_{sk+j}) \) is a \( v_{rk+i}v_{sk+j} \)-geodesic path with the desired properties. So, let us suppose \( i \geq j + 1 \). Thus

\[
d(v_{rk+i}, v_{sk+j}) = \left\lfloor \frac{n - k(r - s) - (i - j)}{k} \right\rfloor = \left\lfloor \frac{n - (i - j)}{k} \right\rfloor.
\]

Let \( t \) be the maximum integer such that \( (r+t)k + i \leq n - 1 \). If \( v_{(r+t)k+i}v_j \in A(D) \), then

\[
T = (v_{rk+i}, v_{(r+1)k+i}, \ldots, v_{(r+t)k+i}, v_j, v_{k+j}, \ldots v_{sk+j})
\]

is a \( v_{rk+i}v_{sk+j} \)-geodesic path such that for every \( V_p \in P \), \( |V_p \cap V(T \setminus v_{sk+j})| \leq 1 \). If

\[
v_{(r+t)k+i}v_j \notin A(D),
\]

since \( i \geq j + 1 \) and \( t \) is maximum, it follows that \( v_{(r+t)k+i} \in V_{\lfloor \frac{n}{k} \rfloor - 1} \) and \( v_{(r+t)k+i}v_{n-1} \in A(D) \). Therefore

\[
T = (v_{rk+i}, \ldots, v_{(r+t)k+i}, v_{n-1}, v_j, v_{k+j}, \ldots v_{sk+j})
\]
is a $v_{rk+i}v_{sk+j}$-geodesic path such that for every $V_p \in P$, $|V_p \cap V(T \setminus v_{sk+j})| \leq 1$, and the claim follows.

Let $\rho : A(D) \rightarrow \{1, 2, \ldots, \lceil \frac{n}{k} \rceil \}$ be the arc-coloring of $D$ defined as follows: for every $v_iv_j \in A(D)$, $\rho(v_iv_j) = p$ if and only if $i \in V_p$. Given $v_i, v_j \in V(D)$, from Claim 1 we see there is a $v_i v_j$-geodesic path $T$ such that for every $V_i \in P$, $|V_i \cap V(T \setminus v_j)| \leq 1$ which, by definition of $\rho$, is a rainbow path. From here it follows that $\rho$ is a strong rainbow coloring of $D$. Thus, $sr^c(D) \leq \lceil \frac{n}{k} \rceil$, and since $\text{diam}(D) = \lceil \frac{n-1}{k} \rceil$, for every $n$ such that $\lceil \frac{n}{k} \rceil = \lceil \frac{n-1}{k} \rceil$ we have $rc^*(D) = sr^c(D) = \lceil \frac{n}{k} \rceil$. Hence, to end the proof just remain to verify the case $n = kt + 1$. Let suppose there is a $t$-rainbow coloring $\rho$ of $D$, and consider $C_{(k)}$, the spanning subdigraph of $D$ induced by the $k$-jumps. Since $(k, n = kt + 1) = 1$ it follows that $C_{(k)}$ is a cycle, and each $v_iv_{i+tk}$-path in $C_{(k)}$ is the only $v_iv_{i+tk}$-path of length $t$ in $D$. Thus, since $\rho$ is a $t$-rainbow coloring, in every $v_iv_{i+tk}$-path in $C_{(k)}$ most appear the $t$ colors. Therefore, for every $0 \leq i \leq n - 1$, $\rho(v_iv_{i+k}) = \rho(v_{i+kt}v_{i+k(t+1)})$, which, since $(k, n = kt + 1) = 1$, implies that every arc in $C_{(k)}$ receives the same color which is a contradiction. Therefore $rc^*(D) \geq t + 1 = \lceil \frac{n}{k} \rceil$ and since $sr^c(D) \leq \lceil \frac{n}{k} \rceil$, the theorem follows. \qed

Now, we turn our attention on the circulant digraphs with a pair of generators $\{1, k\}$, with $2 \leq k \leq n - 1$. Observe that for every circulant digraph $C_n(\{a_1, a_2\})$, if $(a_1, n) = 1$ and $b \in \mathbb{Z}_n$ is the solution of $a_1x \equiv 1$, then $C_n(\{1, ba_2\}) \cong C_n(\{a_1, a_2\})$. From here, we obtain the following.

**Corollary 3.2.** For $k \geq 1$, $rc^*(C_{2k+1}(1, k + 1)) = sr^c(C_{2k+1}(1, k + 1)) = k + 1$.

**Proof.** By Theorem 3.1 for every $n \geq 4$, $rc^*(C_n([2]) = sr^c(C_n([2]) = \lceil \frac{n}{2} \rceil$. Since $(k + 1, 2k + 1) = 1$ and $2$ is the solution of $(k + 1)x 
\equiv 1$, then $C_{2k+1}(\{1, k + 1\}) \cong C_{2k+1}(\{1, 2\}) = C_{2k+1}(\{2\})$ and the result follows. \qed

Observe that given any circulant digraph $C_n(\{1, k\})$, for every pair $v_i, v_j \in C_n(\{1, k\})$ we have $d(v_i, v_j) = d(v_0, v_{j-i})$ (where $j - i$ is taken modulo $n$). Thus, $\text{diam}(C_n(\{1, k\})) = \max \{d(v_0, v_i) : v_i \in V(C_n(\{1, k\}))\}$.

Given two positive integers $i, k$, let denote as $re(i, k)$ the residue of $i$ modulo $k$. 8
Lemma 3.3. Let $C_n(\{1,k\})$ be a circulant digraph and $V = \{v_0, \ldots, v_{n-1}\}$ the set of vertices. If $n \geq (k-1)\left\lceil \frac{n}{k} \right\rceil$ then for every $v_i \in V$, $d(v_0, v_i) = \left\lfloor \frac{i}{k} \right\rfloor + re(i,k)$.

Moreover $\text{diam}(C_n(\{1,k\})) = \left\lfloor \frac{n-1}{k} \right\rfloor + \max\{re(n-1,k), k-2\}$.

Proof. Let $v_i \in V$, $P = (v_0 = u_0, u_1, \ldots, u_s = v_i)$ be a $v_0 v_i$-geodesic path with a minimum number of $k$-jumps, and suppose in $P$ there are $p$ $k$-jumps and $q$ 1-jumps. Also suppose the first $p$ steps of $P$ are $k$-jumps, and the last $q$ are 1-jumps. Thus $d(v_0, v_i) = p + q$. Since $P$ is geodesic, it follows that $q \leq k - 1$ and therefore $p \geq \left\lceil \frac{n}{k} \right\rceil$. Hence $v_{k\lfloor \frac{i}{k} \rfloor} = u_{\lfloor \frac{i}{k} \rfloor} \in V(P)$ and the subpath

$$Q = (v_{k\lfloor \frac{i}{k} \rfloor} = u_{\lfloor \frac{i}{k} \rfloor}, u_j, \ldots, u_s = v_i)$$

is a $v_{k\lfloor \frac{i}{k} \rfloor} v_i$-geodesic path with $p' = p - \left\lfloor \frac{i}{k} \right\rfloor$ $k$-jumps and $d(v_{k\lfloor \frac{i}{k} \rfloor}, v_i) = p' + q \leq i - k\left\lceil \frac{i}{k} \right\rceil = re(i,k)$. If $p > \left\lceil \frac{i}{k} \right\rceil$ then $q < re(i,k)$ and since $re(i,k) < k$, it follows that $p' \geq \left\lceil \frac{n}{k} \right\rceil$. Therefore, if $m = k\left\lceil \frac{n}{k} \right\rceil - n$, $v_{k\lfloor \frac{i}{k} \rfloor + m} = u_{\lfloor \frac{i}{k} \rfloor + \left\lceil \frac{n}{k} \right\rceil} \in V(Q)$ and the subpath

$$(v_{k\lfloor \frac{i}{k} \rfloor} = u_{\lfloor \frac{i}{k} \rfloor}, u_j, \ldots, u_{\lfloor \frac{i}{k} \rfloor + \left\lceil \frac{n}{k} \right\rceil} = v_{k\lfloor \frac{i}{k} \rfloor + m})$$

is a $v_{k\lfloor \frac{i}{k} \rfloor} v_{k\lfloor \frac{i}{k} \rfloor + m}$-geodesic path of $\left\lceil \frac{n}{k} \right\rceil$ $k$-jumps and $d(v_{k\lfloor \frac{i}{k} \rfloor}, v_{k\lfloor \frac{i}{k} \rfloor + m}) = \left\lceil \frac{n}{k} \right\rceil \leq m$. Since $n \geq (k-1)\left\lceil \frac{n}{k} \right\rceil$ it follows that $\left\lceil \frac{n}{k} \right\rceil \geq k\left\lceil \frac{n}{k} \right\rceil - n = m$ and therefore $d(v_{k\lfloor \frac{i}{k} \rfloor}, v_{k\lfloor \frac{i}{k} \rfloor + m}) = m$. Thus, replacing in $P$ the subpath

$$(v_{k\lfloor \frac{i}{k} \rfloor} = u_{\lfloor \frac{i}{k} \rfloor}, u_j, \ldots, u_{\lfloor \frac{i}{k} \rfloor + \left\lceil \frac{n}{k} \right\rceil} = v_{k\lfloor \frac{i}{k} \rfloor + m})$$

by the subpath

$$(v_{k\lfloor \frac{i}{k} \rfloor}, v_{k\lfloor \frac{i}{k} \rfloor + 1}, \ldots, v_{k\lfloor \frac{i}{k} \rfloor + m})$$

we obtain a $v_0 v_i$-geodesic path with less $k$-jumps than $P$, which is a contradiction. Thus $p = \left\lceil \frac{i}{k} \right\rceil$ and therefore $q = re(i,k)$ which implies that $d(v_0, v_i) = \left\lfloor \frac{i}{k} \right\rfloor + re(i,k)$ and the first part of the result follows. For the second part, first observe that $d(v_0, v_{n-1}) = \left\lceil \frac{n-1}{k} \right\rceil + re(n-1,k)$ and $d(v_0, v_{k\lfloor \frac{n-1}{k} \rfloor - 1}) = \left\lceil \frac{n-1}{k} \right\rceil + k - 2$, thus $\text{diam}(C_n(\{1,k\})) \geq \left\lceil \frac{n-1}{k} \right\rceil + \max\{re(n-1,k), k-2\}$. If there is $v_i \in V$ such that $d(v_0, v_i) > \left\lceil \frac{n-1}{k} \right\rceil + k - 2$, it follows that $n - 1 \geq i \geq k\left\lceil \frac{n-1}{k} \right\rceil$ but then $d(v_0, v_i) \leq d(v_0, v_{n-1}) = \left\lceil \frac{n-1}{k} \right\rceil + re(n-1,k)$ and the result follows. □
Theorem 3.4. For every integer $k \geq 2$

(i) $rc^*(C_{2k}(\{1,k\})) = src^*(C_{2k}(\{1,k\})) = k$.

(ii) $rc^*(C_{2k}(\{1,k+1\})) = src^*(C_{2k}(\{1,k+1\})) = k$.

Proof. Let $V = \{v_0, \ldots, v_{2k-1}\}$ be the set of vertices of $C_{2k}(\{1,k\})$. By Lemma 3.3 we see that $k = \text{diam}(C_{2k}(\{1,k\}))$ and therefore $k \leq rc^*(C_{2k}(\{1,k\}))$. Let $\{V_0, \ldots, V_{k-1}\}$ be a partition of $V$, where $V_r = \{v_r, v_{r+k}\}$ for $0 \leq r \leq k-1$ and define a $k$-colouring $\rho$ such that for every $0 \leq r \leq k-1$, $(u, u') \in \rho^{-1}(r)$ if $u \in V_r$. Let $v_i, v_j \in V$ and suppose $i + q + pk \equiv j$ where $d(v_i, v_j) = p + q$ and $q \leq k-1$.

Observe that, since $q < k$, $v_iC_{(1)}v_{i+q}C_{(k)}v_{i+pk+q}$ is a rainbow $v_iv_j$-path and by Lemma 3.3 is $v_i, v_j$-geodesic. Therefore $src^*(C_{2k}(\{1,k\})) \leq k$ and (i) follows. For (ii), let $V = \{v_0, \ldots, v_{2k-1}\}$ be the set of vertices of $C_{2k}(\{1,k+1\})$ and let $\{V_0, \ldots, V_{k-1}\}$ as before. By Lemma 3.3 it follows that $\text{diam}(C_{2k}(\{1,k+1\})) = k$ which implies $k \leq rc^*(C_{2k}(\{1,k+1\}))$. Now let $\rho$ be a $k$-colouring such that $(u, u') \in \rho^{-1}(r)$ if $u \in V_r$.

Since $N^+(u) = V_{r+1}$ for each $u \in V_r$ (taken $r + 1$ modulo $k$), it follows that every path of length at most $k$ is rainbow, in particular every geodesic path is rainbow. Thus $k \geq src^*(C_{2k}(\{1,k+1\}))$ and (ii) follows.

Theorem 3.5. For every integer $k \geq 3$ we have

$$src^*(C_{(k-1)^2}(\{1,k\})) = rc^*(C_{(k-1)^2}(\{1,k\})) = 2k - 4.$$ 

Proof. By Lemma 3.3 we see that $\text{diam}(C_{(k-1)^2}(\{1,k\})) = 2k - 4$ and therefore $rc^*(C_{(k-1)^2}(\{1,k\})) \geq 2k - 4$. Let $V = \{v_0, \ldots, v_{(k-1)^2-1}\}$ be the set of vertices of $C_{(k-1)^2}(\{1,k\})$ and for each $i$, with $0 \leq i < (k-1)^2$, identify the vertex $v_i$ with the pair $\langle \frac{i}{k-1}, re(i, k-1) \rangle$. Let $\mathcal{V} = \{V_0, \ldots, V_{k-2}\}$ be a partition of $V$, where $V_r = \{(r, s) \mid 0 \leq s \leq k - 2\}$ for $0 \leq r \leq k - 2$, and let $\rho$ be a $(2k - 4)$-colouring defined as follows: For each $r$ with $0 \leq r \leq k - 1$,

1. The arc $\langle (r, s)(r+1, s), 0 \leq s \leq k - 2$, receives color $r$.

2. The arcs $\langle (r, 0)(r, 1) \rangle$ and $\langle (r, k-2)(r+1, 0) \rangle$ receive colour $r$.

3. The arc $\langle (r, s)(r, s+1) \rangle$, with $1 \leq s \leq k - 3$, receives colour $k - 2 + s$. 

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Observe that every path with length at most \( k - 1 \) in \( C_{(k)} \) is rainbow, and, except for those paths of length \( k - 1 \) in \( C_{(1)} \) starting at \( \langle r, 0 \rangle \) (with \( 0 \leq r < k - 1 \)), every path in \( C_{(1)} \) with length at most \( k - 1 \) is rainbow. From the structure of \( \rho \) we see that to prove \( \rho \) is a strong coloring we just need to show that for every \( v \in V_0 \) and every \( w \in V \) there is a rainbow \( vw \)-geodesic path.

Let \( \langle 0, s_0 \rangle \in V_0 \) and \( \langle r, s \rangle \in V_r \). Since \( \langle 0, s_0 \rangle = v_{s_0} \) and \( \langle r, s \rangle = v_{(k-1)+s} \), by Lemma 3.3

\[
d(v_{s_0}, v_{(k-1)+s}) = \left\lfloor \frac{(k-1)r + s - s_0}{k} \right\rfloor + re((k-1)r + s - s_0, k)
\]
(taken \( (k-1)r + s - s_0 \) modulo \( (k-1)^2 \)). Thus, if \( t = \left\lfloor \frac{(k-1)r + s - s_0}{k} \right\rfloor \),

\[
P = \langle 0, s_0 \rangle C_{(k)}(\left\lfloor \frac{s_0 + tk}{k-1} \right\rfloor, re(s_0 + tk, k - 1)) C_{(1)}(r, s)
\]
is a geodesic path. The subpath in \( C_{(k)} \) receives colors \( j \), with \( 0 \leq j \leq \left\lfloor \frac{s_0 + tk}{k-1} \right\rfloor - 1 \leq k - 2 \), and the subpath in \( C_{(1)} \) receives colors \( i \) with \( k-1 \leq i \leq 2k-3 \) or \( i = \left\lfloor \frac{s_0 + tk}{k-1} \right\rfloor \).

Thus, if \( P \) is not rainbow then we have that: the subpath in \( C_{(1)} \) most be of length \( k - 1 \); \( \langle \left\lfloor \frac{s_0 + tk}{k-1} \right\rfloor, re(s_0 + tk, k - 1) \rangle = \langle r - 1, 0 \rangle \) and \( \langle r, s \rangle = \langle r, 0 \rangle \).

If \( r - 1 = 0 \) it follows that \( \langle 0, s_0 \rangle = \langle 0, 0 \rangle \) and the path \( Q \) of \( k \)-jumps \( \langle 0, 0 \rangle C_{(k)} \langle 1, 0 \rangle \) of length \( k - 1 \) is a geodesic rainbow. If \( r - 1 = 1 \), \( \langle 0, s_0 \rangle \langle 1, 0 \rangle \) most be a \( k \)-jump which is not possible. If \( r - 1 \geq 2 \), let \( Q \) be the rainbow geodesic obtained by the concatenation of the paths \( \langle 0, s_0 \rangle C_{(k)} \langle r - 3, k - 2 \rangle \) (which receives colors between 0 and \( r - 4 \)); the arcs \( \langle (r - 3, k - 2), (r - 2, 0) \rangle \) and \( \langle (r - 2, 0), (r - 1, 1) \rangle \) (with colors \( r - 3 \) and \( r - 2 \) respectively); and \( \langle r - 1, 1 \rangle C_{(1)} \langle r, 0 \rangle \) (which receives the colors \( r - 1 \) and \( k - 1, \ldots, 2k - 3 \)). Hence, \( P \) or \( Q \) is a \( \langle 0, s_0 \rangle \langle r, s \rangle \)-geodesic rainbow, and the theorem follows.

\[\square\]

**Theorem 3.6.** If \( n = a_n k \) with \( a_n \geq k - 1 \geq 2 \), then

\[
src(C_n(\{1, k\})) = rc(C_n(\{1, k\})) = a_n + k - 2.
\]

**Proof.** By Lemma 3.3 we see that \( \text{diam}(C_n(\{1, k\})) = a_n + k - 2 \) and then to prove the result just remain to show that \( src(C_n(\{1, k\})) \leq a_n + k - 2 \). Let \( V = \{v_0, \ldots, v_{n-1}\} \)
be the set of vertices of \( C_n(\{1,k\}) \) and, for each \( i \), with \( 0 \leq i < n \), identify the vertex \( v_i \) with the pair \( \langle \frac{i}{n}, re(i, k) \rangle \). Let \( \{V_0, \ldots, V_{an-1}\} \) be a partition of \( V \), where \( V_r = \{\langle r, s \rangle : 0 \leq s < k\} \) for \( 0 \leq r < a_n \), and let \( \rho \) be a \( (a_n + k - 2) \)-colouring defined as follows: For each \( r \), with \( 0 \leq r \leq a_n - 1 \), let

1. The arc \( \langle \langle r, s \rangle (r + 1, s) \rangle \), with \( 0 \leq s < k \), receives color \( r \).

2. If \( r \geq k - 2 \) the arcs \( \langle \langle r, 0 \rangle (r, 1) \rangle \) and \( \langle \langle r, k - 1 \rangle (r + 1, 0) \rangle \) receive color \( r \); and, for each \( 1 \leq j \leq k - 2 \), the arc \( \langle \langle r, j \rangle (r, j + 1) \rangle \) receives color \( a_n - 1 + j \).

3. If \( r \leq k - 3 \) the arc \( \langle \langle r, k - 2 \rangle (r, k - 1 - r) \rangle \) receives color \( r \); for each \( 0 \leq j \leq k - 3 - r \) the arc \( \langle \langle r, j \rangle (r, j + 1) \rangle \) receives color \( a_n + r + j \); for each \( k - 1 - r \leq j \leq k - 2 \) the arc \( \langle \langle r, j \rangle (r, j + 1) \rangle \) receives color \( a_n + j - (k - 1 - r) \); and the arc \( \langle \langle r, k - 1 \rangle (r + 1, 0) \rangle \) receives color \( a_n + r \).

Observe that for every pair \( 1 \leq r, r' < a_n \) the path \( \langle r, s \rangle C(k) \langle r', s' \rangle \) is a rainbow path with colors \( r, r+1, \ldots, r'-1 \) (taken the sequence modulo \( a_n \)). Also every path \( P \) of length at most \( k-1 \) in \( C(1) \) is rainbow. Moreover, if for some \( 0 \leq r < a_n \), \( V(P) \subseteq V_r \) then the colors appearing in \( P \) are contained in \( \{a_n, \ldots, a_n+(k-3)\} \cup \{r\} \); and if \( V(P) \) starts at \( V_r \) and ends at \( V_{r+1} \), the colors of \( P \) are in \( \{a_n, \ldots, a_n+(k-3)\} \cup \{r, r+1\} \).

Let \( \langle r, s \rangle \) and \( \langle r', s' \rangle \) be distinct vertices of \( C_n(\{1,k\}) \). If \( r \neq r' \) it is not hard to see that either \( \langle r, s \rangle C(k) \langle r', s' \rangle C(1) \langle r', s' \rangle \) (if \( s \leq s' \)) or \( \langle r, s \rangle C(k) \langle r' - 1, s \rangle C(1) \langle r', s' \rangle \) (if \( s > s' \)) is a rainbow \( \langle \langle r, s \rangle (r', s') \rangle \)-path. If \( r = r' \) and \( s < s' \) we see that \( \langle r, s \rangle C(1) \langle r, s' \rangle \) is a rainbow path. Let us suppose \( r = r' \) and \( s > s' \). If no arc \( \langle \langle r, t \rangle (r, t + 1) \rangle \), with \( 0 \leq t < s' \), receives color \( r \), the path \( \langle r - 1, s \rangle C(1) \langle r, s' \rangle \) receive only colors in \( \{a_n, \ldots, a_n+(k-3)\} \cup \{r-1\} \), and therefore \( \langle r, s \rangle C(k) \langle r - 1, s \rangle C(1) \langle r, s' \rangle \) is a rainbow path. If some arc \( \langle \langle r, t \rangle (r, t + 1) \rangle \), with \( 0 \leq t < s' \), receives color \( r \), by definition of \( \rho \) most be either \( \langle \langle r, 0 \rangle (r, 1) \rangle \) (if \( r \geq k - 2 \)), or \( \langle \langle r, k - 2 - r \rangle (r, k - 1 - r) \rangle \) (if \( r \leq k - 3 \)). For the first case in the path \( P = \langle r, s \rangle C(k) \langle a_n - 1, s \rangle C(1) \langle 0, s' \rangle C(k) \langle r, s' \rangle \), the \( k \)-jumps receive colors \( \{0, \ldots, r, \ldots, a_n - 2\} \) and, by definition of \( \rho \), the only 1-jump of color 0 is \( \langle \langle 0, k - 2 \rangle (0, k - 1) \rangle \). Thus, since \( s' < s \leq k - 1 \), the colors appearing in \( \langle a_n - 1, s \rangle C(1) \langle 0, s' \rangle \) are contian in \( \{a_n, \ldots, a_n + (k-3)\} \cup \{a_n - 1\} \) and therefore \( P \) is rainbow. For the second case in the path \( P = \langle r, s \rangle C(k) \langle k - 2 - s, s \rangle C(1) \langle k - 1 -
$s, s')C_{(k)}(r, s')$ the $k$-jumps receive colors $\{0, \ldots, k - 3 - s, k - 1 - s, \ldots, a_n - 1\}$ and, since $s > s' > t \geq 0$, $k - 1 - s \leq k - 3$ and therefore the only 1-jump of color $k - 1 - s$ is $(\langle k - 1 - s, s - 1 \rangle \langle k - 2 - s, s \rangle)$. Thus the colors in $\langle k - 2 - s, s \rangle C_{(1)} \langle k - 1 - s, s' \rangle$ are contain in $\{a_n, \ldots, a_n + (k - 3)\} \cup \{k - 2 - s\}$ and $P$ is a rainbow path. In all the cases, from Lemma 3.3 we see that all the paths are geodesic and the result follows.

\[\square\]

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