Abstract. Two-grid methods with exact solution of the Galerkin coarse-grid system have been well studied by the multigrid community: an elegant identity has been established to characterize the convergence factor of exact two-grid methods. In practice, however, it is often too costly to solve the Galerkin coarse-grid system exactly, especially when its size is large. Instead, without essential loss of convergence speed, one may solve the coarse-grid system approximately. In this paper, we develop a new framework for analyzing the convergence of inexact two-grid methods: two-sided bounds for the energy norm of the error propagation matrix of inexact two-grid methods are presented. In the framework, a restricted smoother involved in the identity for exact two-grid convergence is used to measure how far the actual coarse-grid matrix deviates from the Galerkin one. As an application, we establish a unified convergence theory for multigrid methods.

Key words. multigrid, inexact two-grid methods, convergence factor, two-sided bounds

AMS subject classifications. 65F08, 65F10, 65N55, 15A18

1. Introduction. Multigrid is one of the most efficient methods for solving large-scale linear systems that arise from discretized partial differential equations. It has been shown to be a powerful solver, with linear or near-linear computational complexity, for a large class of linear systems; see, e.g., [19, 12, 29, 30]. Such a desirable property stems from combining two complementary error-reduction processes: smoothing (or relaxation) and coarse-grid correction. In multigrid methods, these two processes will be applied iteratively until a desired residual tolerance is achieved.

Consider solving the linear system

\[ Au = f, \]

where \( A \in \mathbb{R}^{n \times n} \) is symmetric positive definite (SPD), \( u \in \mathbb{R}^n \), and \( f \in \mathbb{R}^n \). Let \( M \) be an \( n \times n \) nonsingular matrix such that \( M + M^T - A \) is SPD. Given an initial guess \( u^{(0)} \in \mathbb{R}^n \), we perform the following smoothing process:

\[ u^{(k+1)} = u^{(k)} + M^{-1}(f - Au^{(k)}) \quad k = 0, 1, \ldots \]

The process (1.2) is typically a simple iterative method, such as the (weighted) Jacobi and Gauss–Seidel iterations. In general, (1.2) is efficient at eliminating high-frequency (i.e., oscillatory) error, while low-frequency (i.e., smooth) error cannot be eliminated effectively; see, e.g., [12, 29]. To further reduce the low-frequency error, a coarse-grid correction strategy is used in multigrid methods. Let \( u^{(f)} \in \mathbb{R}^n \) be an approximation to the exact solution \( u \equiv A^{-1}f \), and let \( P \in \mathbb{R}^{n \times n_c} \) be a prolongation (or interpolation) matrix with full column rank, where \( n_c (< n) \) is the number of coarse variables. Then,
the (exact) coarse-grid correction can be described by
\begin{equation}
\mathbf{u}^{(t+1)} = \mathbf{u}^{(t)} + P A_c^{-1} P^T (\mathbf{f} - A \mathbf{u}^{(t)}),
\end{equation}
where $A_c := P^T A P \in \mathbb{R}^{n_c \times n_c}$ is known as the Galerkin coarse-grid matrix. Define
\begin{equation}
\Pi_A := P A_c^{-1} P^T A.
\end{equation}
From (1.3), we have
\[
\mathbf{u} - \mathbf{u}^{(t+1)} = (I - \Pi_A) (\mathbf{u} - \mathbf{u}^{(t)}).
\]
Since $I - \Pi_A$ is a projection along (or parallel to) $\text{range}(P)$ onto $\text{null}(P^T A)$, it follows that
\[
(I - \Pi_A) e = 0 \quad \forall e \in \text{range}(P).
\]
That is, an efficient coarse-grid correction will be achieved if the coarse space $\text{range}(P)$ can accurately cover the low-frequency error.

Combining (1.2) and (1.3) yields a two-grid procedure, which is the fundamental module of multigrid methods. A symmetric two-grid scheme (i.e., the presmoothing and postsmoothing processes are performed in a symmetric way) for solving (1.1) can be described by Algorithm 1.1. If $B_c$ in Algorithm 1.1 is taken to be $A_c$, then the algorithm is called an exact two-grid method; otherwise, it is called an inexact two-grid method. In particular, if $\mathbf{e} = 0$, then Algorithm 1.1 contains only two smoothing steps, in which case the convergence factor is $1 - \lambda_{\min}(\tilde{M}^{-1} A) < 1$, where $\tilde{M}$ is defined by (2.1b).

Algorithm 1.1 Two-grid method.

1. Presmoothing: $\mathbf{u}^{(1)} \leftarrow \mathbf{u}^{(0)} + M^{-1} (\mathbf{f} - A \mathbf{u}^{(0)})$ \hspace{1cm} $\triangleright$ $M + M^T - A \in \mathbb{R}^{n \times n}$ is SPD
2. Restriction: $\mathbf{r}_c \leftarrow P^T (\mathbf{f} - A \mathbf{u}^{(1)})$ \hspace{1cm} $\triangleright$ $P \in \mathbb{R}^{n \times n_c}$ has full column rank
3. Coarse-grid correction: $\tilde{\mathbf{e}}_c \leftarrow B_c^{-1} \mathbf{r}_c$ \hspace{1cm} $\triangleright$ $B_c \in \mathbb{R}^{n_c \times n_c}$ is SPD
4. Prolongation: $\mathbf{u}^{(2)} \leftarrow \mathbf{u}^{(1)} + P \tilde{\mathbf{e}}_c$
5. Postsmoothing: $\mathbf{u}_{\text{ITG}} \leftarrow \mathbf{u}^{(2)} + M^{-T} (\mathbf{f} - A \mathbf{u}^{(2)})$

The convergence theory of exact two-grid methods has been well developed (see, e.g., [16, 38, 24, 33]), and the convergence factor of exact two-grid methods can be characterized by an elegant identity [32, 16]. In practice, however, it is often too costly to solve the linear system $A_c \mathbf{e}_c = \mathbf{r}_c$ exactly, especially when its size is large. Moreover, the Galerkin coarse-grid matrix may affect the parallel efficiency of algebraic multigrid methods [14]. Instead, without essential loss of convergence speed, one may solve the coarse-grid system approximately (see, e.g., [18, 5, 17, 11]) or find a cheap alternative to $A_c$ (see, e.g., [9, 27, 26, 14, 28, 7]). A typical strategy is to apply Algorithm 1.1 recursively in the correction steps. The resulting multigrid method can be viewed as an inexact two-grid scheme. This enables us to analyze the convergence of multigrid methods via inexact two-grid theory.

In [23], an upper bound for the convergence factor of Algorithm 1.1 was presented, which has been successfully applied to the convergence analysis of W-cycle multigrid methods. Nevertheless, the two-grid estimate in [23] is not sharp in some situations; see Remark 2.5 and Example 3.7. Recently, we established an improved convergence theory for Algorithm 1.1 in [37], the basic idea of which is to measure the accuracy of the coarse solver by using the extreme eigenvalues of $B_c^{-1} A_c$. 
In this paper, we develop a novel framework for analyzing the convergence of Algorithm 1.1: two-sided bounds for the energy norm of the error propagation matrix of Algorithm 1.1 are presented, from which one can easily obtain the identity for exact two-grid convergence. In the framework, the restricted smoother $P_1 M P_0$ is used to measure the deviation $B_c - A_c$, which will be readily available if $B_c$ is a sparsification of $A_c$. Such an idea is completely different from that in [37]. Indeed, it is inspired by an explicit expression for the inexact two-grid preconditioner (see Lemma 3.1). As an application of the framework, we establish a unified convergence theory for multigrid methods, which allows the coarsest-grid system to be solved approximately.

The rest of this paper is organized as follows. In section 2, we introduce some results on the convergence of two-grid methods and a useful tool for eigenvalue analysis. In section 3, we present an analytical framework for the convergence of Algorithm 1.1. Based on the proposed framework, we establish a unified convergence theory for multigrid methods in section 4. In section 5, we give some concluding remarks.

2. Preliminaries. In this section, we review some results on the convergence of two-grid methods and the well-known Weyl’s theorem. For convenience, we list some notation used in the subsequent discussions.

- $I_n$ denotes the $n \times n$ identity matrix (or $I$ when its size is clear from context).
- $\lambda_{\text{min}}(\cdot)$ and $\lambda_{\text{max}}(\cdot)$ stand for the smallest and largest eigenvalues of a matrix, respectively.
- $\Lambda(\cdot)$ denotes the spectrum of a matrix.
- $\rho(\cdot)$ denotes the spectral radius of a matrix.
- $\|\cdot\|_2$ denotes the spectral norm of a matrix.
- $\|\cdot\|_A$ denotes the energy norm induced by an SPD matrix $A \in \mathbb{R}^{n \times n}$; for any $v \in \mathbb{R}^n$, $\|v\|_A = \langle Av, v \rangle^{\frac{1}{2}}$; for any $B \in \mathbb{R}^{n \times n}$, $\|B\|_A = \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{\|Bv\|_A}{\|v\|_A}$.

From (1.2), we have

$$u - u^{(k+1)} = (I - M^{-1}A)(u - u^{(k)}),$$

which leads to

$$\|u - u^{(k)}\|_A \leq \|I - M^{-1}A\|_A \|u - u^{(0)}\|_A.$$  

For any initial guess $u^{(0)} \in \mathbb{R}^n$, if $\|I - M^{-1}A\|_A < 1$, then

$$\lim_{k \to +\infty} \|u - u^{(k)}\|_A = 0.$$  

Such a smoother $M$ (i.e., $\|I - M^{-1}A\|_A < 1$) is said to be $A$-convergent, which is in fact equivalent to the positive definiteness of $M + M^T - A$; see, e.g., [30, Proposition 3.8].

For an $A$-convergent smoother $M$, we define two symmetrized variants:

$$\overline{M} := M(M + M^T - A)^{-1}M^T,$$  

$$\tilde{M} := M^T(M + M^T - A)^{-1}M.$$  

It is easy to verify that

$$I - \overline{M}^{-1}A = (I - M^{-T}A)(I - M^{-1}A),$$  

$$I - \tilde{M}^{-1}A = (I - M^{-1}A)(I - M^{-T}A),$$

from which one can easily deduce that both $\overline{M} - A$ and $\tilde{M} - A$ are symmetric positive semidefinite (SPSD).
The iteration matrix (or error propagation matrix) of Algorithm 1.1 is
\[ E_{ITG} = (I - M^{-T}A)(I - PB^{-1}cP^TA)(I - M^{-1}A), \]
which satisfies
\[ u - u_{ITG} = E_{ITG}(u - u(0)). \]
The iteration matrix \( E_{ITG} \) can be expressed as
\[ E_{ITG} = I - B_{ITG}^{-1}A, \]
where
\[ B_{ITG}^{-1} = \bar{M}^{-1} + (I - M^{-T}A)PB^{-1}cP^TA(I - AM^{-1}). \]

Since \( \bar{M} \) and \( B_c \) are SPD, it follows that \( B_{ITG} \) is an SPD matrix, which is called the inexact two-grid preconditioner. By (2.4), we have
\[ \|E_{ITG}\|_A = \rho(E_{ITG}) = \max \left\{ 1 - \frac{1}{\lambda_{\min}(A^{-1}B_{ITG})}, 1 - \frac{1}{\lambda_{\max}(A^{-1}B_{ITG})} \right\}, \]
which is referred to as the convergence factor of Algorithm 1.1.

In particular, if \( B_c = A_c \), the corresponding iteration matrix is denoted by \( E_{TG} \), which takes the form
\[ E_{TG} = (I - M^{-T}A)(I - \Pi A)(I - M^{-1}A) = I - B_{TG}^{-1}A, \]
where \( \Pi A \) is defined by (1.4) and
\[ B_{TG}^{-1} = \bar{M}^{-1} + (I - M^{-T}A)PAc^{-1}P^TA(I - AM^{-1}). \]
The SPD matrix \( B_{TG} \) is called the exact two-grid preconditioner.

The following theorem provides an elegant identity for \( \|E_{TG}\|_A \) [16, Theorem 4.3], which is a two-level version of the XZ-identity [32, 38].

**Theorem 2.1.** Let \( \tilde{M} \) be defined by (2.1b), and define
\[ \Pi_{\tilde{M}} := \bar{P}(P^T\bar{M}P)^{-1}P^T\tilde{M}. \]
Then, the convergence factor of Algorithm 1.1 with \( B_c = A_c \) can be characterized as
\[ \|E_{TG}\|_A = 1 - \frac{1}{K_{TG}}, \]
where
\[ K_{TG} = \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{\|(I - \Pi_{\tilde{M}})v\|^2_{\tilde{M}}}{\|v\|^2_A}. \]

**Remark 2.2.** It is easy to verify that \( \Pi_{\tilde{M}}^2 = \Pi_{\tilde{M}} \), range(\( \Pi_{\tilde{M}} \)) = range(\( P \)), and \( \Pi_{\tilde{M}} \) is self-adjoint with respect to the inner product \( \langle \cdot, \cdot \rangle_{\tilde{M}} := \langle M\cdot, \cdot \rangle \). That is, \( \Pi_{\tilde{M}} \) is an \( \tilde{M} \)-orthogonal projection onto the coarse space range(\( P \)); see, e.g., [16].
Remark 2.3. From (2.7), we deduce that $A^\frac{1}{2}E_{TG}A^{-\frac{1}{2}}$ is a symmetric matrix with smallest eigenvalue 0. Since

$$A^\frac{1}{2}E_{TG}A^{-\frac{1}{2}} = I - A^\frac{1}{2}B_{TG}^{-1}A^\frac{1}{2},$$

we obtain that $B_{TG} - A$ is SPD and $\lambda_{\text{max}}(B_{TG}^{-1}A) = 1$. Furthermore,

$$\lambda_{\text{min}}(B_{TG}^{-1}A) = 1 - \max\{1, \lambda_{\text{max}}(E_{TG})\} = \frac{1}{K_{TG}}.$$

Hence,

$$K_{TG} = \frac{\lambda_{\text{max}}(B_{TG}^{-1}A)}{\lambda_{\text{min}}(B_{TG}^{-1}A)},$$

that is, $K_{TG}$ is the corresponding condition number when Algorithm 1.1 with $B_c = A_c$ is applied as a preconditioning method.

The identity (2.10) is a powerful tool for analyzing two-grid methods (see, e.g., [16, 33, 10, 36]), which reflects the interplay between smoother and coarse space. The conventional strategy of designing algebraic multigrid methods is to fix a simple smoother (like the Jacobi and Gauss–Seidel types) and then optimize the choice of coarse space. Alternatively, one may fix a coarse space and then optimize the choice of smoother. It is also possible to optimize them simultaneously [33].

Compared with the exact two-grid case, the difficulty of inexact two-grid analysis is increased by the fact that the middle term in (2.3), $I - PB_c^{-1}P^TA$, is no longer a projection. Based on the idea of hierarchical basis [6] and the minimization property of Schur complements (see, e.g., [3, Theorem 3.8]), Notay [23] derived an upper bound for the convergence factor $\|E_{ITG}\|_A$, as described in the following theorem.

**Theorem 2.4.** Under the assumptions of Algorithm 1.1, it holds that

$$\|E_{ITG}\|_A \leq \max\left\{1 - \frac{\min\{1, \lambda_{\text{max}}(B_c^{-1}A_c)\}}{K_{TG}}, \max\{1, \lambda_{\text{max}}(B_c^{-1}A_c)\} - 1\right\},$$

where $K_{TG}$ is given by (2.11).

**Remark 2.5.** If $B_c$ in Algorithm 1.1 is taken to be $\omega I_n$ with $\omega > 0$, then

$$E_{ITG} = (I - M^{-T}A) \left(I - \frac{1}{\omega}PP^TA\right)(I - M^{-1}A).$$

As $\omega \to +\infty$, Algorithm 1.1 reduces to an algorithm containing only the presmoothing and postsmoothing steps, whose convergence factor is

$$\|(I - M^{-T}A)(I - M^{-1}A)\|_A = \lambda_{\text{max}}\left((I - M^{-T}A)(I - M^{-1}A)\right) = \lambda_{\text{max}}\left((I - M^{-1}A)(I - M^{-T}A)\right) = 1 - \lambda_{\text{min}}(\tilde{M}^{-1}A).$$

It is easy to see that the upper bound in (2.13) tends to 1 (as $\omega \to +\infty$), from which one cannot determine whether the limiting algorithm is convergent.

In the next section, we will establish a new convergence theory for Algorithm 1.1, which is based on the following Weyl’s theorem; see, e.g., [20, Theorem 4.3.1].
THEOREM 2.6. Let $H_1$ and $H_2$ be $n \times n$ Hermitian matrices. Assume that the spectra of $H_1$, $H_2$, and $H_1 + H_2$ are $\{\lambda_i(H_1)\}_{i=1}^n$, $\{\lambda_i(H_2)\}_{i=1}^n$, and $\{\lambda_i(H_1 + H_2)\}_{i=1}^n$, respectively, where $\lambda_i(\cdot)$ denotes the $i$th smallest eigenvalue of a matrix. Then, for each $k = 1, \ldots, n$, it holds that

$$
\lambda_{k-j+1}(H_1) + \lambda_j(H_2) \leq \lambda_k(H_1 + H_2) \leq \lambda_{k+\ell}(H_1) + \lambda_{n-\ell}(H_2)
$$

for all $j = 1, \ldots, k$ and $\ell = 0, \ldots, n-k$. In particular, one has

\begin{align*}
(2.1a) \quad & \lambda_{\min}(H_1 + H_2) \geq \lambda_{\min}(H_1) + \lambda_{\min}(H_2), \\
(2.1b) \quad & \lambda_{\min}(H_1 + H_2) \leq \min \{\lambda_{\min}(H_1) + \lambda_{\max}(H_2), \lambda_{\max}(H_1) + \lambda_{\min}(H_2)\}, \\
(2.1c) \quad & \lambda_{\max}(H_1 + H_2) \geq \max \{\lambda_{\max}(H_1) + \lambda_{\min}(H_2), \lambda_{\min}(H_1) + \lambda_{\max}(H_2)\}, \\
(2.1d) \quad & \lambda_{\max}(H_1 + H_2) \leq \lambda_{\max}(H_1) + \lambda_{\max}(H_2).
\end{align*}

3. Convergence of inexact two-grid methods. In this section, we develop a theoretical framework for analyzing the convergence of Algorithm 1.1. The main idea is to measure the deviation $B_c - A_c$ by using the restricted smoother $P^T M P$, which is inspired by the following expression.

LEMMA 3.1. The inexact two-grid preconditioner $B_{ITG}$ can be expressed as

$$
B_{ITG} = A + (I - AM^{-T})\tilde{M}(I - P(P^T \tilde{M}P + B_c - A_c)^{-1}P^T \tilde{M})(I - M^{-1}A).
$$

Proof. In view of (2.1a) and (2.1b), we have

$$
\overline{M}(I - M^{-T}A) = (I - AM^{-T})\tilde{M} \quad \text{and} \quad (I - AM^{-1})\overline{M}(I - M^{-T}A) = \tilde{M} - A.
$$

Using (2.5) and the Sherman–Morrison–Woodbury formula [25, 31, 35], we obtain

$$
B_{ITG} = \overline{M} - (I - AM^{-T})\tilde{M}P(P^T \tilde{M}P + B_c - A_c)^{-1}P^T \tilde{M}(I - M^{-1}A).
$$

Note that

$$
\overline{M} = A + (I - AM^{-T})\tilde{M}(I - M^{-1}A).
$$

The expression (3.1) follows immediately by combining (3.2) and (3.3). \hfill \Box

Remark 3.2. If $B_c = A_c$, we get from (3.1) that

$$
B_{TG} = A + (I - AM^{-T})\tilde{M}(I - \Pi_{\tilde{M}})(I - M^{-1}A),
$$

from which one can readily see that $B_{TG} - A$ is SPSD.

The following lemma gives some technical eigenvalue identities used in the subsequent analysis.

LEMMA 3.3. The extreme eigenvalues of $(A^{-1}\tilde{M} - I)(I - \Pi_{\tilde{M}})$ and $(A^{-1}\tilde{M} - I)\Pi_{\tilde{M}}$ have the following properties:

\begin{align*}
(3.5a) \quad & \lambda_{\min}(\{A^{-1}\tilde{M} - I\}(I - \Pi_{\tilde{M}})) = 0, \\
(3.5b) \quad & \lambda_{\max}(\{A^{-1}\tilde{M} - I\}(I - \Pi_{\tilde{M}})) = K_{TG} - 1, \\
(3.5c) \quad & \lambda_{\min}(\{A^{-1}\tilde{M} - I\)\Pi_{\tilde{M}}) = 0, \\
(3.5d) \quad & \lambda_{\max}(\{A^{-1}\tilde{M} - I\)\Pi_{\tilde{M}}) = \lambda_{\max}(A^{-1}\tilde{M}\Pi_{\tilde{M}}) - 1.
\end{align*}
Proof. Since $A^{-1} - \tilde{M}^{-1}$ is SPD and $\Pi_{\tilde{M}}$ is an $\tilde{M}$-orthogonal projection, it holds that
\[
\lambda((A^{-1} - \tilde{M}^{-1})(I - \Pi_{\tilde{M}})) = \lambda((A^{-1} - \tilde{M}^{-1})\frac{1}{2}\tilde{M}(I - \Pi_{\tilde{M}})(A^{-1} - \tilde{M}^{-1})\frac{1}{2}) \subset [0, +\infty).
\]
Similarly, one has
\[
\lambda((A^{-1} - \tilde{M}^{-1})\Pi_{\tilde{M}}) \subset [0, +\infty) \quad \text{and} \quad \lambda(A^{-1}\tilde{M}\Pi_{\tilde{M}}) \subset [0, +\infty).
\]
Due to the fact that $\Pi_{\tilde{M}}$ is a projection matrix of rank $n_c$, there exists a nonsingular matrix $X \in \mathbb{R}^{n \times n}$ such that
\[
X^{-1}\Pi_{\tilde{M}}X = \begin{pmatrix} I_{n_c} & 0 \\ 0 & 0 \end{pmatrix}.
\]
Let
\[
X^{-1}A^{-1}\tilde{M}X = \begin{pmatrix} \hat{X}_{11} & \hat{X}_{12} \\ \hat{X}_{21} & \hat{X}_{22} \end{pmatrix},
\]
where $\hat{X}_{ij} \in \mathbb{R}^{m_i \times m_j}$ with $m_1 = n_c$ and $m_2 = n - n_c$. Then
\[
X^{-1}(A^{-1} - \tilde{M}^{-1})(I - \Pi_{\tilde{M}})X = \begin{pmatrix} 0 & \hat{X}_{12} \\ 0 & \hat{X}_{22} - I_{n-n_c} \end{pmatrix},
\]
\[
X^{-1}(A^{-1} - \tilde{M}^{-1})\Pi_{\tilde{M}}X = \begin{pmatrix} \hat{X}_{11} - I_{n_c} & 0 \\ \hat{X}_{21} & 0 \end{pmatrix},
\]
\[
X^{-1}A^{-1}\tilde{M}\Pi_{\tilde{M}}X = \begin{pmatrix} \hat{X}_{11} & 0 \\ \hat{X}_{21} & 0 \end{pmatrix},
\]
from which one can obtain the identities (3.5a), (3.5c), and (3.5d).

From (2.12), we have
\[
K_{TG} = \lambda_{\max}(A^{-1}B_{TG}),
\]
which, together with (2.2b) and (3.4), yields
\[
K_{TG} = 1 + \lambda_{\max}(A^{-1}(I - AM^{-T})\tilde{M}(I - M^{-1}A))
\]
\[
= 1 + \lambda_{\max}((I - M^{-1}A)(I - M^{-T}A)A^{-1}\tilde{M}(I - \Pi_{\tilde{M}}))
\]
\[
= 1 + \lambda_{\max}(\lambda_{\max}(A^{-1} - \tilde{M}^{-1})(I - \Pi_{\tilde{M}})),
\]
which leads to the identity (3.5b). This completes the proof.

Define
\[
d_1 := \frac{1}{1 + \lambda_{\max}((PTMP)^{-1}(B_c - A_c))},
\]
\[
d_2 := \frac{1}{1 + \lambda_{\min}((PTMP)^{-1}(B_c - A_c))}.
\]
Recall that $B_c$ is SPD and $PT\tilde{M}P - A_c$ is SPD. We then have
\[
0 < d_1 < \frac{1}{1 - \lambda_{\min}((PTMP)^{-1}A_c)} = \frac{\lambda_{\max}(A_c^{-1}PTMP)}{\lambda_{\max}(A_c^{-1}PTMP) - 1},
\]
\[
0 < d_2 < \frac{1}{1 - \lambda_{\max}((PTMP)^{-1}A_c)} = \frac{\lambda_{\min}(A_c^{-1}PTMP)}{\lambda_{\min}(A_c^{-1}PTMP) - 1}.
\]
We are now in a position to present a new convergence theory for Algorithm 1.1.
Theorem 3.4. Let $d_1$ and $d_2$ be defined by (3.6a) and (3.6b), respectively. Under the assumptions of Algorithm 1.1, $\|E_{ITG}\|_A$ satisfies the following estimates.

(i) If $d_2 \leq 1$, then

$$L_1 \leq \|E_{ITG}\|_A \leq U_1,$$

where

$$L_1 = 1 - \frac{1}{\max \{K_{TG}, \lambda_{\max}(A^{-1}M) - d_2 \lambda_{\max}(A^{-1}M \Pi_{\overline{M}}) + d_2\}},$$

$$U_1 = 1 - \frac{1}{d_1 K_{TG} + (1 - d_1) \lambda_{\max}(A^{-1}M)}.$$

(ii) If $d_1 \leq 1 < d_2 < \frac{\lambda_{\max}(A^{-1}M \Pi_{\overline{M}})}{\lambda_{\max}(A^{-1}M)},$ then

$$L_2 \leq \|E_{ITG}\|_A \leq \max \{U_1, U_2\},$$

where

$$L_2 = 1 - \frac{1}{\max \{\lambda_{\min}(A^{-1}M), d_2 K_{TG} + (1 - d_2) \lambda_{\max}(A^{-1}M)\}},$$

$$U_2 = \frac{1}{(1 - d_2) \lambda_{\max}(A^{-1}M \Pi_{\overline{M}}) + d_2} - 1.$$

(iii) If $1 < d_1 \leq d_2 < \frac{\lambda_{\max}(A^{-1}M \Pi_{\overline{M}})}{\lambda_{\max}(A^{-1}M)},$ then

$$\max \{L_2, L_3\} \leq \|E_{ITG}\|_A \leq U_3,$$

where

$$L_3 = \frac{1}{\min \{\lambda_{\max}(A^{-1}M) - d_1 \lambda_{\max}(A^{-1}M \Pi_{\overline{M}}), (1 - d_1) \lambda_{\min}(A^{-1}M)\} + d_1} - 1,$$

$$U_3 = \max \left\{1 - \frac{1}{K_{TG}}, \frac{1}{(1 - d_2) \lambda_{\max}(A^{-1}M \Pi_{\overline{M}}) + d_2} - 1 \right\}.$$

Proof. By (3.1), we have

$$A^{-1}B_{ITG} = I + (I - M^{-T}A)A^{-1}\tilde{M}(I - P(P^T\tilde{M}P + B_c - A_c)^{-1}P^T\tilde{M})(I - M^{-1}A).$$

Then

$$\lambda(A^{-1}B_{ITG}) = \lambda(I + (I - \tilde{M}^{-T}A)A^{-1}\tilde{M}(I - P(P^T\tilde{M}P + B_c - A_c)^{-1}P^T\tilde{M}))$$

$$= \lambda(I + (A^{-1}\tilde{M} - I)(I - P(P^T\tilde{M}P + B_c - A_c)^{-1}P^T\tilde{M})),$$

which leads to

$$\lambda_{\min}(A^{-1}B_{ITG}) = 1 + \lambda_{\min}((A^{-1}\tilde{M} - I)(I - P(P^T\tilde{M}P + B_c - A_c)^{-1}P^T\tilde{M})).$$

$$\lambda_{\max}(A^{-1}B_{ITG}) = 1 + \lambda_{\max}((A^{-1}\tilde{M} - I)(I - P(P^T\tilde{M}P + B_c - A_c)^{-1}P^T\tilde{M})).$$
Observe that \((A^{-1}\tilde{M} - I)(I - P(P^T\tilde{M}P + B_c - A_c)^{-1}P^T\tilde{M})\) has the same eigenvalues as the symmetric matrix
\[
(A^{-1} - \tilde{M}^{-1})^\mathcal{M}(I - P(P^T\tilde{M}P + B_c - A_c)^{-1}P^T\tilde{M})(A^{-1} - \tilde{M}^{-1})^\mathcal{M}.
\]
Since \(B_c - A_c - (\frac{1}{\sigma_2} - 1)P^T\tilde{M}P\) and \((\frac{1}{\sigma_1} - 1)P^T\tilde{M}P - (B_c - A_c)\) are SPSD, it follows that
\[
\begin{align*}
(3.10a) \quad & 1 + s_2 \leq \lambda_{\min}(A^{-1}B_{\text{ITG}}) \leq 1 + s_1, \\
(3.10b) \quad & 1 + t_2 \leq \lambda_{\max}(A^{-1}B_{\text{ITG}}) \leq 1 + t_1,
\end{align*}
\]
where
\[
\begin{align*}
s_k &= \lambda_{\min}((A^{-1}\tilde{M} - I)(I - d_k\Pi_{\tilde{M}})), \\
t_k &= \lambda_{\max}((A^{-1}\tilde{M} - I)(I - d_k\Pi_{\tilde{M}})).
\end{align*}
\]

Next, we determine the upper bounds for \(s_1\) and \(t_1\), as well as the lower bounds for \(s_2\) and \(t_2\). The remainder of this proof is divided into three parts, which correspond to three cases stated in this theorem.

**Case 1: \(d_2 \leq 1\).** By (2.14b), we have
\[
\begin{align*}
s_1 &= \lambda_{\min}((A^{-1}\tilde{M} - I)(I - d_1\Pi_{\tilde{M}})) \\
&= \lambda_{\min}((A^{-1} - \tilde{M}^{-1})^\mathcal{M}(I - d_1\Pi_{\tilde{M}})(A^{-1} - \tilde{M}^{-1})^\mathcal{M}) \\
&= \lambda_{\min}((A^{-1} - \tilde{M}^{-1})^\mathcal{M}(I - \Pi_{\tilde{M}} + (1 - d_1)\Pi_{\tilde{M}})(A^{-1} - \tilde{M}^{-1})^\mathcal{M}) \\
&\leq \lambda_{\min}((A^{-1} - \tilde{M}^{-1})^\mathcal{M}(I - \Pi_{\tilde{M}})(A^{-1} - \tilde{M}^{-1})^\mathcal{M}) \\
&\quad + (1 - d_1)\lambda_{\max}((A^{-1} - \tilde{M}^{-1})^\mathcal{M}\Pi_{\tilde{M}}(A^{-1} - \tilde{M}^{-1})^\mathcal{M}) \\
&= \lambda_{\min}((A^{-1}\tilde{M} - I)(I - \Pi_{\tilde{M}})) + (1 - d_1)\lambda_{\max}((A^{-1}\tilde{M} - I)\Pi_{\tilde{M}}) \\
&= (1 - d_1)(\lambda_{\max}(A^{-1}\Pi_{\tilde{M}}) - 1),
\end{align*}
\]
where we have used the identities (3.5a) and (3.5d). This suggests that Weyl's theorem (Theorem 2.6) can also be applied to the nonsymmetric matrix \((A^{-1}\tilde{M} - I)(I - d_k\Pi_{\tilde{M}})\). Similarly,
\[
\begin{align*}
s_1 &= \lambda_{\min}((A^{-1}\tilde{M} - I)(I - d_1\Pi_{\tilde{M}})) \\
&\leq \lambda_{\min}(A^{-1}\tilde{M} - I) - d_1\lambda_{\min}((A^{-1}\tilde{M} - I)\Pi_{\tilde{M}}) \\
&= \lambda_{\min}(A^{-1}\tilde{M}) - 1,
\end{align*}
\]
where we have used the fact (3.5c). Hence,
\[
(3.11) \quad s_1 \leq \min \{ (1 - d_1)\lambda_{\max}(A^{-1}\Pi_{\tilde{M}}) + d_1, \lambda_{\min}(A^{-1}\tilde{M}) \} - 1.
\]

Using (2.14a), we obtain
\[
\begin{align*}
s_2 &= \lambda_{\min}((A^{-1}\tilde{M} - I)((1 - d_2)I + d_2(I - \Pi_{\tilde{M}}))) \\
&\geq (1 - d_2)\lambda_{\min}(A^{-1}\tilde{M} - I) + d_2\lambda_{\min}((A^{-1}\tilde{M} - I)(I - \Pi_{\tilde{M}})),
\end{align*}
\]
which, together with (3.5a), yields
\[
(3.12) \quad s_2 \geq (1 - d_2)\lambda_{\min}(A^{-1}\tilde{M}) + d_2 - 1.
\]
By (2.14d), we have
\[
    t_1 = \lambda_{\max}\left( (A^{-1}\tilde{M} - I)((1 - d_1)I + d_1(I - \Pi_{\tilde{M}\tilde{M}})) \right)
    \leq (1 - d_1)\lambda_{\max}(A^{-1}\tilde{M} - I) + d_1\lambda_{\max}((A^{-1}\tilde{M} - I)(I - \Pi_{\tilde{M}\tilde{M}})).
\]

The above inequality, combined with (3.5b), leads to
\[
    t_1 \leq d_1K_{TG} + (1 - d_1)\lambda_{\max}(A^{-1}\tilde{M}) - 1. \tag{3.13}
\]

Applying (2.14c) and (3.5b)–(3.5d), we get that
\[
    t_2 = \lambda_{\max}((A^{-1}\tilde{M} - I)(I - \Pi_{\tilde{M}\tilde{M}}))
    \geq \lambda_{\max}((A^{-1}\tilde{M} - I)(I - \Pi_{\tilde{M}\tilde{M}})) + (1 - d_2)\lambda_{\min}((A^{-1}\tilde{M} - I)\Pi_{\tilde{M}\tilde{M}})
    = K_{TG} - 1
\]
and
\[
    t_2 = \lambda_{\max}((A^{-1}\tilde{M} - I)(I - \Pi_{\tilde{M}\tilde{M}}))
    \geq \lambda_{\max}(A^{-1}\tilde{M} - I) - d_2\lambda_{\max}((A^{-1}\tilde{M} - I)\Pi_{\tilde{M}\tilde{M}})
    = \lambda_{\max}(A^{-1}\tilde{M}) - d_2\lambda_{\max}(A^{-1}\tilde{M}\Pi_{\tilde{M}\tilde{M}}) + d_2 - 1.
\]

Thus,
\[
    t_2 \geq \max\{K_{TG}, \lambda_{\max}(A^{-1}\tilde{M}) - d_2\lambda_{\max}(A^{-1}\tilde{M}\Pi_{\tilde{M}\tilde{M}}) + d_2\} - 1. \tag{3.14}
\]

The estimate (3.7) then follows by combining (2.6), (3.10a), (3.10b), and (3.11)–(3.14).

Case 2: \(d_1 \leq 1 < d_2 < \frac{\lambda_{\max}(A^{-1}\tilde{M}\Pi_{\tilde{M}\tilde{M}})}{\lambda_{\max}(A^{-1}\tilde{M}\Pi_{\tilde{M}\tilde{M}}) - 1}\). Since \(d_1 \leq 1\), the estimates (3.11) and (3.13) still hold. We next determine the lower bounds for \(s_2\) and \(t_2\). By (2.14a), we have
\[
    s_2 = \lambda_{\min}((A^{-1}\tilde{M} - I)(I - \Pi_{\tilde{M}\tilde{M}}) + (1 - d_2)\Pi_{\tilde{M}\tilde{M}}))
    \geq \lambda_{\min}((A^{-1}\tilde{M} - I)(I - \Pi_{\tilde{M}\tilde{M}})) + (1 - d_2)\lambda_{\max}((A^{-1}\tilde{M} - I)\Pi_{\tilde{M}\tilde{M}}),
\]
which, together with (3.5a) and (3.5d), gives
\[
    s_2 \geq (1 - d_2)\lambda_{\max}(A^{-1}\tilde{M}\Pi_{\tilde{M}\tilde{M}}) + d_2 - 1. \tag{3.15}
\]

In light of (2.14c), (3.5b), and (3.5c), we have that
\[
    t_2 = \lambda_{\max}((A^{-1}\tilde{M} - I)(I - \Pi_{\tilde{M}\tilde{M}}))
    \geq \lambda_{\min}(A^{-1}\tilde{M} - I) - d_2\lambda_{\min}((A^{-1}\tilde{M} - I)\Pi_{\tilde{M}\tilde{M}})
    = \lambda_{\min}(A^{-1}\tilde{M}) - 1
\]
and
\[
    t_2 = \lambda_{\max}((A^{-1}\tilde{M} - I)((1 - d_2)I + d_2(I - \Pi_{\tilde{M}\tilde{M}})))
    \geq (1 - d_2)\lambda_{\max}(A^{-1}\tilde{M} - I) + d_2\lambda_{\max}((A^{-1}\tilde{M} - I)(I - \Pi_{\tilde{M}\tilde{M}}))
    = d_2K_{TG} + (1 - d_2)\lambda_{\max}(A^{-1}\tilde{M}) - 1.
\]
Then
\[(3.16) \quad t_2 \geq \max \left\{ \lambda_{\min}(A^{-1} \tilde{M}), d_2 K_{TG} + (1 - d_2) \lambda_{\max}(A^{-1} \tilde{M}) \right\} - 1.\]

Combining (2.6), (3.10a), (3.10b), (3.11), (3.13), (3.15), and (3.16), we can arrive at the estimate (3.8).

Case 3: \(1 < d_1 \leq d_2 < \frac{\lambda_{\max}(A^{-1} \tilde{M} \Pi_{\tilde{M}})}{\lambda_{\max}(A^{-1} \tilde{M} \Pi_{\tilde{M}}) + 1}\). In such a case, the inequalities (3.15) and (3.16) are still valid. We then focus on the upper bounds for \(s_1\) and \(t_1\). By (2.14b), (3.5a), and (3.5d), we have that
\[
\begin{align*}
s_1 &= \lambda_{\min}((A^{-1} \tilde{M} - I)(I - d_1 \Pi_{\tilde{M}})) \\
&\leq \lambda_{\max}(A^{-1} \tilde{M} - I) - d_1 \lambda_{\max}((A^{-1} \tilde{M} - I) \Pi_{\tilde{M}}) \\
&= \lambda_{\max}(A^{-1} \tilde{M}) - d_1 \lambda_{\max}(A^{-1} \tilde{M} \Pi_{\tilde{M}}) + d_1 - 1
\end{align*}
\]
and
\[
\begin{align*}
s_1 &= \lambda_{\min}((A^{-1} \tilde{M} - I)((1 - d_1)I + d_1(1 - \Pi_{\tilde{M}}))) \\
&\leq (1 - d_1)\lambda_{\min}(A^{-1} \tilde{M} - I) + d_1 \lambda_{\min}((A^{-1} \tilde{M} - I)(1 - \Pi_{\tilde{M}})) \\
&= (1 - d_1)\lambda_{\min}(A^{-1} \tilde{M}) + d_1 - 1.
\end{align*}
\]
It follows that
\[(3.17) \quad s_1 \leq \min \left\{ \lambda_{\max}(A^{-1} \tilde{M}) - d_1 \lambda_{\max}(A^{-1} \tilde{M} \Pi_{\tilde{M}}), (1 - d_1)\lambda_{\min}(A^{-1} \tilde{M}) \right\} + d_1 - 1.
\]
Using (2.14d), we obtain
\[
\begin{align*}
t_1 &= \lambda_{\max}((A^{-1} \tilde{M} - I)(I - \Pi_{\tilde{M}} + (1 - d_1)\Pi_{\tilde{M}})) \\
&\leq \lambda_{\max}((A^{-1} \tilde{M} - I)(1 - \Pi_{\tilde{M}})) + (1 - d_1)\lambda_{\min}((A^{-1} \tilde{M} - I)\Pi_{\tilde{M}}),
\end{align*}
\]
which, together with (3.5b) and (3.5c), yields
\[(3.18) \quad t_1 \leq K_{TG} - 1.
\]
In view of (2.6), (3.10a), (3.10b), and (3.15)–(3.18), we conclude that the inequality (3.9) holds.

Remark 3.5. If \(B_c = A_c\), then \(d_1 = d_2 = 1\) and hence
\[
\begin{align*}
\mathcal{L}_1 &= 1 - \frac{1}{\max \left\{ K_{TG}, \lambda_{\max}(A^{-1} \tilde{M}) - \lambda_{\max}(A^{-1} \tilde{M} \Pi_{\tilde{M}}) + 1 \right\}}, \\
\mathcal{B}_1 &= 1 - \frac{1}{K_{TG}}.
\end{align*}
\]
By (2.14d), (3.5b), and (3.5d), we have
\[
K_{TG} + \lambda_{\max}(A^{-1} \tilde{M} \Pi_{\tilde{M}}) - 2 \geq \lambda_{\max}((A^{-1} \tilde{M} - I)(I - \Pi_{\tilde{M}}) + (A^{-1} \tilde{M} - I)\Pi_{\tilde{M}}) \\
&= \lambda_{\max}(A^{-1} \tilde{M}) - 1,
\]
that is,
\[
K_{TG} \geq \lambda_{\max}(A^{-1} \tilde{M}) - \lambda_{\max}(A^{-1} \tilde{M} \Pi_{\tilde{M}}) + 1,
\]
which leads to
\[
\mathcal{L}_1 = 1 - \frac{1}{K_{TG}}.
\]
Thus, the estimate (3.7) will reduce to the identity (2.10) when \(B_c = A_c\).
Remark 3.6. If $B_c = \omega I_n$ with $\omega > 0$, then

$$
\lim_{\omega \to +\infty} d_1 = \lim_{\omega \to +\infty} \frac{1}{1 + \lambda_{\max}((P^T M P)^{-1}(\omega I_n - A_c))} = 0,
$$
\[
\lim_{\omega \to +\infty} d_2 = \lim_{\omega \to +\infty} \frac{1}{1 + \lambda_{\min}((P^T M P)^{-1}(\omega I_n - A_c))} = 0.
\]

It is easy to verify that, as $\omega \to +\infty$, both $L_1$ and $U_1$ tend to $1 - \lambda_{\min}(\widetilde{M}^{-1} A)$, which is exactly the convergence factor of the limiting algorithm. That is, our estimate has fixed the defect of (2.13) indicated in Remark 2.5.

Example 3.7. Let $A$ be partitioned into the two-by-two block form

$$
A = \begin{pmatrix}
A_{ff} & A_{fc} \\
A_{cf} & A_{cc}
\end{pmatrix},
$$

where $A_{ff} \in \mathbb{R}^{n_f \times n_f}$, $A_{fc} \in \mathbb{R}^{n_f \times n_c}$, $A_{cf} = A_{fc}^T$, and $A_{cc} \in \mathbb{R}^{n_c \times n_c}$ ($n_f + n_c = n$). The Cauchy–Bunyakowski–Schwarz (C.B.S.) constant associated with (3.19) (see, e.g., [13, 3]) is defined as

$$
\alpha := \max_{v_f \in \mathbb{R}^{n_f} \setminus \{0\}} \frac{\|v_f^T A_{fc} v_c\|_2}{\sqrt{\|v_f^T A_{ff} v_f\|_2 \|v_c^T A_{cc} v_c\|_2}} = \|A_{ff}^{-\frac{1}{2}} A_{fc} A_{cc}^{-\frac{1}{2}}\|_2.
$$

The positive definiteness of $A$ implies that $\alpha \in [0, 1)$. Take

$$
M = \begin{pmatrix}
A_{ff} & 0 \\
0 & A_{cc}
\end{pmatrix}, \quad P = \begin{pmatrix}
-A_{ff}^{-1} A_{fc} \\
I_n
\end{pmatrix},
$$

and

$$
B_c = P_0^T A P_0 \quad \text{with} \quad P_0 = \begin{pmatrix}
0 \\
I_n
\end{pmatrix} \in \mathbb{R}^{n \times n_c}.
$$

Here, $M$ is an $A$-convergent smoother and $P$ is an ideal interpolation [15, 36]. Then, the iteration matrix $E_{ITG}$ is of the form

$$
E_{ITG} = \begin{pmatrix}
(A_{ff}^{-1} A_{fc} A_{cc}^{-1} A_{cf})^2 & 0 \\
* & A_{cc}^{-1} A_{cf} A_{ff}^{-1} A_{fc}
\end{pmatrix}.
$$

Hence,

$$
\|E_{ITG}\|_A = \rho(E_{ITG}) = \|A_{ff}^{-\frac{1}{2}} A_{fc} A_{cc}^{-\frac{1}{2}}\|_2 = \alpha^2.
$$

It is easy to check that

$$
K_{ITG} = \frac{1}{1 - \alpha^2}, \quad \lambda_{\max}(A^{-1} \widetilde{M}) = \frac{1}{1 - \alpha^2}, \quad d_1 = \frac{1}{1 + \alpha^2}, \quad \text{and} \quad d_2 \leq 1.
$$

An application of (3.7) yields

$$
\|E_{ITG}\|_A = \alpha^2.
$$

On the other hand, since

$$
\lambda_{\min}(B_c^{-1} A_c) = 1 - \alpha^2 \quad \text{and} \quad \lambda_{\max}(B_c^{-1} A_c) \leq 1,
$$
the upper bound in (2.13) gives \( \alpha^2(2 - \alpha^2) \), which is strictly greater than \( \alpha^2 \) (unless \( \alpha = 0 \)). This example shows that the estimate (2.13) is not sharp. Furthermore, the relative error of the bound \( \alpha^2(2 - \alpha^2) \) is

\[
\left| \frac{\alpha^2(2 - \alpha^2) - \alpha^2}{\alpha^2} \right| = 1 - \alpha^2,
\]

which is not tiny if \( \alpha \) is bounded away from 1.

**Remark 3.8.** The C.B.S. constant \( \alpha \) can be expressed as

\[
\alpha = \max_{v_i \in \mathbb{R}^{n_i} \setminus \{0\}} \frac{\mathbf{v}_i^T S_0^T A P_0 \mathbf{v}_c}{\sqrt{\mathbf{v}_i^T S_0^T A S_0 \mathbf{v}_i \cdot \mathbf{v}_c^T P_0^T A P_0 \mathbf{v}_c}},
\]

where

\[
S_0 = \begin{pmatrix} I_{n_i} \\ 0 \end{pmatrix} \in \mathbb{R}^{n \times n_i} \quad \text{and} \quad P_0 = \begin{pmatrix} 0 \\ I_{n_c} \end{pmatrix} \in \mathbb{R}^{n \times n_c}.
\]

As a result, \( \alpha \) can be viewed as the cosine of the abstract angle between \( \text{range}(S_0) \) and \( \text{range}(P_0) \) with respect to \( A \)-inner product, defined by \( \langle \cdot, \cdot \rangle_A := \langle A \cdot, \cdot \rangle \). As pointed out in [22], for finite element matrices associated with the standard nodal basis, \( \alpha \) is generally close to 1, whereas, for hierarchical basis finite element matrices, \( \alpha \) may be nicely bounded away from 1; see, e.g., [21, 2, 1, 8, 4].

As an alternative to \( (P^T \tilde{M})^{-1} \), \( P^T \tilde{M}^{-1} P \) can also be used to analyze the convergence of Algorithm 1.1. It may be easier to compute \( P^T \tilde{M}^{-1} P \) in practice, because the action of \( \tilde{M}^{-1} \) is always available. Indeed, there is a spectral equivalence relation between \( (P^T \tilde{M})^{-1} \) and \( P^T \tilde{M}^{-1} P \) (see, e.g., [15, Lemma 5.2]), as discussed below.

Let \( S \) be an \( n \times (n - n_c) \) matrix, with full column rank, such that \( P^T S = 0 \). This implies that \( (S \ P) \in \mathbb{R}^{n \times n} \) is nonsingular. Let

\[
L_P = \begin{pmatrix} I_{n - n_c} \\ -P^T \tilde{M} S (S^T \tilde{M} S)^{-1} \cdot P^T P \end{pmatrix}.
\]

Then

\[
P^T \tilde{M}^{-1} P = P^T (S \ P)((S \ P)^T \tilde{M} (S \ P))^{-1}(S \ P)^T P
\]

\[
= (0 \ P^T P) \begin{pmatrix} S^T \tilde{M} S & S^T \tilde{M} P \\ P^T \tilde{M} S & P^T \tilde{M} P \end{pmatrix}^{-1} (0 \ P^T P)
\]

\[
= L_P \begin{pmatrix} S^T \tilde{M} S & 0 \\ 0 & P^T \tilde{M} P - P^T \tilde{M} S (S^T \tilde{M} S)^{-1} S^T \tilde{M} P \end{pmatrix}^{-1} L_P
\]

\[
= P^T (P^T \tilde{M} P - P^T \tilde{M} S (S^T \tilde{M} S)^{-1} S^T \tilde{M} P)^{-1} P^T P.
\]

Assume that the Cholesky factorization of \( P^T P \in \mathbb{R}^{n_c \times n_c} \) is

\[
P^T P = U_c^T U_c,
\]

where \( U_c \in \mathbb{R}^{n_c \times n_c} \) is upper triangular. Let

\[
P_c = P U_c^{-1},
\]
which is a normalized prolongation, i.e., $P^T P_f = I_n_c$. Then
\[
(P^T \tilde{M}^{-1} P_f)^{-1} = P^T \tilde{M} P_f - P^T \tilde{M} S (S^T \tilde{M} S)^{-1} S^T \tilde{M} P_f,
\]
which yields
\[
(P^T \tilde{M} P_f)^{-1} (P^T \tilde{M}^{-1} P_f)^{-1} = I_{n_c} - (P^T \tilde{M} P_f)^{-1} P^T \tilde{M} S (S^T \tilde{M} S)^{-1} S^T \tilde{M} P_f.
\]
Thus,
\[
\lambda \left( (P^T \tilde{M} P_f)^{-1} (P^T \tilde{M}^{-1} P_f)^{-1} \right) \subset [1 - \beta^2, 1],
\]
where $\beta \in [0, 1)$ is the C.B.S. constant associated with the matrix
\[
\begin{pmatrix}
S^T \tilde{M} S & S^T \tilde{M} P_f \\
P^T \tilde{M} S & P^T \tilde{M} P_f
\end{pmatrix}.
\]
It follows from (3.20) that
\[
(1 - \beta^2) v_c^T P^T \tilde{M}^{-1} P_f v_c \leq v_c^T (P^T \tilde{M} P_f)^{-1} v_c \leq v_c^T P^T \tilde{M}^{-1} P_f v_c \quad \forall v_c \in \mathbb{R}^{n_c}.
\]
Some approaches to estimating the C.B.S. constant can be found, e.g., in [13, 3, 15, 30].

**Remark 3.9.** We mention that the quantities $K_{TG}$ and $\lambda_{\text{max}}(A^{-1} \tilde{M} \Pi_{\text{MG}})$ involved in Theorem 3.4 will not change if $P$ is replaced by $P_f$.

### 4. Convergence of multigrid methods

In practice, it is often too costly to solve the Galerkin coarse-grid system exactly when its size is relatively large. Instead, without essential loss of convergence speed, one may solve the coarse-grid system approximately. A typical strategy is to apply Algorithm 1.1 recursively in the correction steps. The resulting multigrid algorithm can be treated as an inexact two-grid method. In this section, we establish a unified convergence theory for multigrid methods based on Theorem 3.4.

To describe the multigrid algorithm, we need the following notation and assumptions.

- The algorithm involves $L + 1$ levels with indices $0, \ldots, L$, where 0 corresponds to the coarsest level and $L$ to the finest level.
- $n_k$ denotes the number of unknowns at level $k$ ($n = n_L > n_{L-1} > \cdots > n_0$).
- For each $k = 1, \ldots, L$, $P_k \in \mathbb{R}^{n_k \times n_{k-1}}$ denotes a prolongation matrix from level $k - 1$ to level $k$, and $\text{rank}(P_k) = n_{k-1}$.
- Let $A_L = A$. For each $k = 0, \ldots, L - 1$, $A_k := P_{k+1}^T A_{k+1} P_{k+1}$ denotes the Galerkin coarse-grid matrix at level $k$.
- Let $A_0$ be an $n_0 \times n_0$ matrix such that $A_0 - A_0$ is SPD.
- For each $k = 1, \ldots, L$, $M_k \in \mathbb{R}^{n_k \times n_k}$ denotes a nonsingular smoother at level $k$ with $M_k + M_k^T = A_k$ being SPD.
- $\gamma$ denotes the cycle index involved in the coarse-grid correction steps.

With the above assumptions and an initial guess $u_k^{(0)} \in \mathbb{R}^{n_k}$, the standard multigrid scheme for solving the linear system $A_k u_k = f_k$ (with $f_k \in \mathbb{R}^{n_k}$) can be described by Algorithm 4.1. The symbol MG$^\gamma$ in Algorithm 4.1 means that the multigrid scheme will be carried out $\gamma$ iterations. In particular, $\gamma = 1$ corresponds to the V-cycle and $\gamma = 2$ to the W-cycle.
Algorithm 4.1 Multigrid method at level $k$: $\mathbf{u}_{\text{IMG}} \leftarrow \text{MG}(k, A_k, \mathbf{f}_k, \mathbf{u}^{(0)}_k)$.

1. Presmoothing: $\mathbf{u}^{(1)}_k \leftarrow \mathbf{u}^{(0)}_k + M_k^{-1}(\mathbf{f}_k - A_k \mathbf{u}^{(0)}_k)$
2. Restriction: $\mathbf{r}_{k-1} \leftarrow P^T_k(\mathbf{f}_k - A_k \mathbf{u}^{(1)}_k)$
3. Coarse-grid correction: $\hat{\mathbf{e}}_{k-1} \leftarrow \begin{cases} \hat{A}_0^{-1} \mathbf{r}_0 & \text{if } k = 1, \\ \text{MG}^{\gamma}(k-1, A_{k-1}, \mathbf{r}_{k-1}, \mathbf{0}) & \text{if } k > 1. \end{cases}$
4. Prolongation: $\mathbf{u}^{(2)}_k \leftarrow \mathbf{u}^{(1)}_k + P_k \hat{\mathbf{e}}_{k-1}$
5. Postsmoothing: $\mathbf{u}_{\text{IMG}} \leftarrow \mathbf{u}^{(2)}_k + M_k^{-T}(\mathbf{f}_k - A_k \mathbf{u}^{(2)}_k)$

From Algorithm 4.1, we have

$$\mathbf{u}_k - \mathbf{u}_{\text{IMG}} = E^{(k)}_{\text{IMG}}(\mathbf{u}_k - \mathbf{u}^{(0)}_k),$$

where

$$E^{(k)}_{\text{IMG}} = (I - M_k^{-T} A_k)\left[I - P_k\left(I - (E^{(k-1)}_{\text{IMG}})^{\gamma}\right)A_{k-1}^{-1}P^T_k A_k\right](I - M_k^{-1} A_k).$$

In particular,

$$E^{(1)}_{\text{IMG}} = (I - M_1^{-T} A_1)\left[I - P_1\hat{A}_0^{-1}P^T_1 A_1\right](I - M_1^{-1} A_1).$$

From (4.1), we deduce that

$$A_k^{\frac{1}{2}}E^{(k)}_{\text{IMG}}A_k^{\frac{1}{2}} = N_k^T\left[I - A_k^{\frac{1}{2}}P_k A_{k-1}^{\frac{1}{2}}\left(I - (A_{k-1}^{-1}E^{(k-1)}_{\text{IMG}} A_{k-1}^{-1})^{\gamma}\right)A_{k-1}^{-\frac{1}{2}}P^T_k A_k^{\frac{1}{2}}\right]N_k,$$

where

$$N_k = I - A_k^{\frac{1}{2}}M_k^{-1}A_k^{\frac{1}{2}}.$$ By induction, one can get that $A_k^{\frac{1}{2}}E^{(k)}_{\text{IMG}}A_k^{\frac{1}{2}}$ is symmetric and

$$\lambda(E^{(k)}_{\text{IMG}}) = \lambda\left(A_k^{\frac{1}{2}}E^{(k)}_{\text{IMG}}A_k^{\frac{1}{2}}\right) \subset [0, 1) \quad \forall k = 1, \ldots, L,$$

which lead to

$$\|E^{(k)}_{\text{IMG}}\|_{A_k} < 1.$$ Comparing (4.1) with (2.3), we can observe that Algorithm 4.1 is essentially an inexact two-grid method with $M = M_k$, $A = A_k$, $P = P_k$, and

$$B_c = A_{k-1}\left(I - (E^{(k-1)}_{\text{IMG}})^{\gamma}\right)^{-1}.$$ It is easy to verify that $B_c$ given by (4.2) is SPD.

Define

$$\sigma^{(k)}_{\text{TG}} := \|E^{(k)}_{\text{TG}}\|_{A_k} \quad \text{and} \quad \sigma^{(k)}_{\text{IMG}} := \|E^{(k)}_{\text{IMG}}\|_{A_k},$$

which are the convergence factors of the exact two-grid method and inexact multigrid method at level $k$, respectively. By (4.2), we have

$$B_c - A_c = A_{k-1}\left(I - (E^{(k-1)}_{\text{IMG}})^{\gamma}\right)^{-1} - P_k^T A_k P_k$$

$$= A_{k-1}^{\frac{1}{2}}\left[I - \left(A_{k-1}^{\frac{1}{2}}E^{(k-1)}_{\text{IMG}} A_{k-1}^{\frac{1}{2}}\right)^{\gamma}\right]^{-1} A_{k-1}^{\frac{1}{2}} - A_{k-1}$$

$$= A_{k-1}^{\frac{1}{2}}\left[I - \left(A_{k-1}^{\frac{1}{2}}E^{(k-1)}_{\text{IMG}} A_{k-1}^{\frac{1}{2}}\right)^{\gamma}\right]^{-1} \left(A_{k-1}^{\frac{1}{2}}E^{(k-1)}_{\text{IMG}} A_{k-1}^{\frac{1}{2}}\right)^{\gamma} A_{k-1}^{\frac{1}{2}}.$$
Define 

$$\tilde{M}_k := M_k^T (M_k + M_k^T - A_k)^{-1} M_k.$$ 

For any \(v_{k-1} \in \mathbb{R}^{n_{k-1}} \setminus \{0\}\), it holds that 

$$\frac{v_k^T (B_c - A_c) v_{k-1}}{v_{k-1}^T P_k^T M_k P_k v_{k-1}} = \frac{v_k^T (B_c - A_c) v_{k-1}}{v_{k-1}^T A_{k-1} v_{k-1}}, \frac{v_k^T A_{k-1} v_{k-1}}{v_{k-1}^T P_k^T M_k P_k v_{k-1}} \in \left[0, \lambda_{\text{max}} \left( (P_k^T \tilde{M}_k P_k)^{-1} A_{k-1} \right) \frac{\sigma_{\text{IMG}}^{(k-1)}}{1 - (\sigma_{\text{IMG}}^{(k-1)})^\gamma} \right],$$

where we have used the fact 

$$\lambda \left( A_{k-1} E^{(k-1)}_{\text{IMG}} A_{k-1} \right) \subset \left[0, \sigma_{\text{IMG}}^{(k-1)} \right].$$

Then 

$$\frac{1}{1 + \lambda_{\text{max}} \left( (P_k^T \tilde{M}_k P_k)^{-1} A_{k-1} \right) \frac{\sigma_{\text{IMG}}^{(k-1)}}{1 - (\sigma_{\text{IMG}}^{(k-1)})^\gamma}} \leq d_1 \leq d_2 \leq 1,$$

where \(d_1\) and \(d_2\) are defined by (3.6a) and (3.6b), respectively. Since \(\mathcal{L}_1\) and \(\mathcal{B}_1\) are decreasing functions with respect to \(d_2\) and \(d_1\), respectively, it follows from (3.7) that 

$$\sigma_{\text{IMG}}^{(k)} \geq 1 - \frac{1}{K_{\text{TG}}^{(k)}} = \sigma_{\text{TG}}^{(k)}$$

and 

$$\sigma_{\text{IMG}}^{(k)} \leq 1 - \frac{1 + \lambda_{\text{max}} \left( (P_k^T \tilde{M}_k P_k)^{-1} A_{k-1} \right) \frac{\sigma_{\text{IMG}}^{(k-1)}}{1 - (\sigma_{\text{IMG}}^{(k-1)})^\gamma}}{K_{\text{TG}}^{(k)} + \lambda_{\text{max}} (A_{k-1} \tilde{M}_k) \lambda_{\text{max}} \left( (P_k^T \tilde{M}_k P_k)^{-1} A_{k-1} \right) \frac{\sigma_{\text{IMG}}^{(k-1)}}{1 - (\sigma_{\text{IMG}}^{(k-1)})^\gamma}},$$

where 

$$K_{\text{TG}}^{(k)} = \max_{v_k \in \mathbb{R}^{n_k} \setminus \{0\}} \left\| (I - \Pi_{\tilde{M}_k}) v_k \right\|_2^2 / \left\| v_k \right\|_2^2 \quad \text{with} \quad \Pi_{\tilde{M}_k} = P_k (P_k^T \tilde{M}_k P_k)^{-1} P_k^T \tilde{M}_k.$$

**Remark 4.1.** The estimate (4.3) suggests that a well converged multigrid method entails that the corresponding exact two-grid method has a fast convergence speed.

In what follows, we establish a convergence theory for Algorithm 4.1 based on the estimate (4.4). For brevity, we define 

$$\sigma_L := \max_{1 \leq k \leq L} \sigma_{\text{TG}}^{(k)},$$

$$\tau_L := \max_{1 \leq k \leq L} \lambda_{\text{max}} \left( (P_k^T \tilde{M}_k P_k)^{-1} A_{k-1} \right),$$

$$\varepsilon_L := \min_{1 \leq k \leq L} \lambda_{\text{min}} (\tilde{M}_k^{-1} A_k).$$

In view of (4.5) and (4.8), we have 

$$K_{\text{TG}}^{(k)} = \lambda_{\text{max}} \left( A_k^{-1} \tilde{M}_k (I - \Pi_{\tilde{M}_k}) \right) \leq \lambda_{\text{max}} \left( A_k^{-1} \tilde{M}_k \right) = \frac{1}{\lambda_{\text{min}} (\tilde{M}_k^{-1} A_k)} \leq \frac{1}{\varepsilon_L}.$$
Then

\[ \sigma_{TG}^{(k)} = 1 - \frac{1}{R_{TG}^{(k)}} \leq 1 - \varepsilon_L \quad \forall k = 1, \ldots, L, \]

which, together with (4.6), yields

\[ 0 \leq \sigma_L \leq 1 - \varepsilon_L. \]

Note that the extreme cases \( \sigma_L = 0 \) and \( \sigma_L = 1 - \varepsilon_L \) seldom occur in practice. In the subsequent analysis, we only consider the nontrivial case

(4.9) \[ 0 < \sigma_L < 1 - \varepsilon_L. \]

In addition, we deduce from (4.7) that, for any \( k = 1, \ldots, L, \)

\[ \tau_L \geq \max_{\nu_k \in \text{range}((F_k)^0)} \frac{v_k^T A_k \nu_k}{v_k^T M_k \nu_k} \geq \min_{\nu_k \in \text{range}((F_k)^0)} \frac{v_k^T A_k \nu_k}{v_k^T M_k \nu_k} \geq \min_{\nu_k \in \mathbb{R}^{n_k} \setminus \{0\}} \frac{v_k^T A_k \nu_k}{v_k^T M_k \nu_k}, \]

which, combined with (4.8), yields

(4.10) \[ 0 < \varepsilon_L \leq \tau_L. \]

We first prove a technical lemma, which plays an important role in the convergence analysis of Algorithm 4.1.

**Lemma 4.2.** Let \( \sigma_L, \tau_L, \) and \( \varepsilon_L \) be defined by (4.6), (4.7), and (4.8), respectively. Then, there exists a strictly decreasing sequence \( \{x_\gamma\}_{\gamma=1}^{+\infty} \subset (\sigma_L, 1 - \varepsilon_L) \) with limit \( \sigma_L \) such that \( x_\gamma \) is a root of the equation

\[ \frac{\sigma_L \varepsilon_L (1 - x_\gamma) + \tau_L (1 - \varepsilon_L)(1 - \sigma_L)x_\gamma^\gamma}{\varepsilon_L (1 - x_\gamma) + \tau_L (1 - \sigma_L)x_\gamma^\gamma} - x = 0 \quad (0 < x < 1). \]

**Proof.** Let

\[ F_\gamma(x) = \frac{\sigma_L \varepsilon_L (1 - x_\gamma) + \tau_L (1 - \varepsilon_L)(1 - \sigma_L)x_\gamma^\gamma}{\varepsilon_L (1 - x_\gamma) + \tau_L (1 - \sigma_L)x_\gamma^\gamma} - x. \]

Obviously, \( F_\gamma(x) \) is a continuous function in \((0, 1)\). Direct computations yield

\[ F_\gamma(\sigma_L) = \frac{\tau_L (1 - \sigma_L - \varepsilon_L)(1 - \sigma_L)\sigma_L^\gamma}{\varepsilon_L (1 - \sigma_L^\gamma) + \tau_L (1 - \sigma_L)\sigma_L^\gamma} > 0, \]

\[ F_\gamma(1 - \varepsilon_L) = \frac{\varepsilon_L (1 - \sigma_L - \varepsilon_L)((1 - \varepsilon_L)^\gamma - 1)}{\varepsilon_L - \varepsilon_L(1 - \varepsilon_L)^\gamma + \tau_L(1 - \sigma_L)(1 - \varepsilon_L)^\gamma} < 0. \]

Hence, \( F_\gamma(x) = 0 \) has at least one root in \((\sigma_L, 1 - \varepsilon_L)\).

Let \( x_\gamma \in (\sigma_L, 1 - \varepsilon_L) \) be a root of \( F_\gamma(x) = 0 \). Note that \( F_\gamma(x) + x \) is a strictly increasing function with respect to \( x \). We then have

\[ F_{\gamma+1}(x_\gamma) = F_\gamma\left(x_\gamma^{1+\frac{1}{\gamma}}\right) + x_\gamma^{1+\frac{1}{\gamma}} - x_\gamma < F_\gamma(x_\gamma) = 0. \]

Since \( F_{\gamma+1}(\sigma_L) > 0 \) and \( F_{\gamma+1}(x_\gamma) < 0 \), there exists an \( x_{\gamma+1} \in (\sigma_L, x_\gamma) \) such that \( F_{\gamma+1}(x_{\gamma+1}) = 0 \). Repeating this process, one can obtain a strictly decreasing sequence \( \{x_\gamma\}_{\gamma=1}^{+\infty} \).
Due to $F_\gamma(x_\gamma) = 0$, it follows that
\[ x_\gamma = \frac{\sigma_L\varepsilon_L(1 - x_\gamma)}{\varepsilon_L(1 - x_\gamma)} + \tau_L(1 - \varepsilon_L)(1 - \sigma_L)x_\gamma, \]
which leads to
\[ \lim_{\gamma \to +\infty} x_\gamma = \sigma_L. \]
This completes the proof.

Using (3.7), (4.4), and Lemma 4.2, we can derive the following estimate.

**Theorem 4.3.** Under the assumptions of Algorithm 4.1 and Lemma 4.2, if
\[ \lambda((P_1^T M_1 P_1)^{-1}(A_0 - A_0)) \subset \left[ 0, \frac{\varepsilon_L(x_\gamma - \sigma_L)}{(1 - \sigma_L)(1 - \varepsilon_L - x_\gamma)} \right], \]
then
\[ \sigma_{I_{\text{MG}}}^{(k)} \leq x_\gamma \quad \forall k = 1, \ldots, L. \]

**Proof.** By (3.7) and (4.11), we have
\[ \sigma_{I_{\text{MG}}}^{(1)} \leq 1 - \frac{1 + \frac{\varepsilon_L(x_\gamma - \sigma_L)}{(1 - \sigma_L)(1 - \varepsilon_L - x_\gamma)}}{1 - \sigma_L + \frac{\varepsilon_L(x_\gamma - \sigma_L)}{(1 - \sigma_L)(1 - \varepsilon_L - x_\gamma)}}. \]
From (4.6) and (4.8), we deduce that
\[ \frac{1}{1 - \sigma_{I_{\text{MG}}}^{(1)}} \leq \frac{1}{1 - \sigma_L} \quad \text{and} \quad \lambda_{\text{max}}(A_1^{-1} \tilde{M}_1) = \frac{1}{\lambda_{\text{min}}(M_1^{-1} A_1)} \leq \frac{1}{\varepsilon_L}. \]
Hence,
\[ \sigma_{I_{\text{MG}}}^{(1)} \leq 1 - \frac{1 + \frac{\varepsilon_L(x_\gamma - \sigma_L)}{(1 - \sigma_L)(1 - \varepsilon_L - x_\gamma)}}{1 - \sigma_L + \frac{\varepsilon_L(x_\gamma - \sigma_L)}{(1 - \sigma_L)(1 - \varepsilon_L - x_\gamma)}} = x_\gamma. \]
It is easy to check that
\[ 1 - \frac{1 + \xi}{1 - \sigma_{I_{\text{MG}}}^{(1)}} + \frac{\eta}{1 - \eta} \]
are increasing functions with respect to $\xi \in (0, +\infty)$ and $\eta \in (0, 1)$, respectively. If $\sigma_{I_{\text{MG}}}^{(k-1)} \leq x_\gamma$, we get from (4.4) and (4.6)–(4.8) that
\[ \sigma_{I_{\text{MG}}}^{(k)} \leq 1 - \frac{1 + \max_{1 \leq k \leq L} \lambda_{\text{max}}((P_1^T \tilde{M}_1 P_1)^{-1} A_{k-1}) x_\gamma^2}{1 - \sigma_{I_{\text{MG}}}^{(1)}} \leq 1 - \frac{1 + \tau_L x_\gamma^2}{1 - \sigma_L + \max_{1 \leq k \leq L} \lambda_{\text{max}}((P_1^T \tilde{M}_1 P_1)^{-1} A_{k-1}) x_\gamma^2} \]
\[ = 1 - \frac{1 + \tau_L x_\gamma^2}{1 - \sigma_L + \frac{x_\gamma}{\varepsilon_L(1 - x_\gamma)}} = x_\gamma, \]
\[ \sigma_L \varepsilon_L(1 - x_\gamma) + \tau_L(1 - \varepsilon_L)(1 - \sigma_L)x_\gamma \]
\[ \varepsilon_L(1 - x_\gamma) + \tau_L(1 - \sigma_L)x_\gamma = x_\gamma, \]
where, in the last equality, we have used the fact \( F_\gamma(x_\gamma) = 0 \). The desired result then follows by induction.

Remark 4.4. Observe that a key relation in the proof of Theorem 4.3 is

\[
\sigma^{(1)}_{\text{IMG}} \leq x_\gamma.
\]

The purpose of the condition (4.11) is to validate such an inequality. In fact, we are allowed to replace (4.11) by any condition which can validate (4.13). For example, if

\[
\lambda(\hat{A}_0^{-1} A_0) \subset \left[ \frac{1 - \epsilon_L - x_\gamma}{1 - \epsilon_L - \sigma_L}, 1 \right],
\]

one can show that (4.13) is still valid (see [37, Theorem 4.4]).

In particular, we have the following convergence estimates for the V- and W-cycle multigrid methods.

Corollary 4.5. Let

\[
\mu_L = 1 + \sigma_L - \frac{\tau_L(1 - \epsilon_L)(1 - \sigma_L)}{\epsilon_L}.
\]

Under the assumptions of Theorem 4.3, it holds that, for any \( k = 1, \ldots, L \),

\[
\sigma^{(k)}_{\text{IMG}} \leq \begin{cases} x_1 & \text{if } \gamma = 1, \\ \hat{x}_2 & \text{if } \gamma = 2, \end{cases}
\]

where

\[
x_1 = \frac{2\sigma_L}{\mu_L + \sqrt{\mu_L^2 - 4\sigma_L(1 - \frac{\tau_L}{\epsilon_L}(1 - \sigma_L))}},
\]

\[
\hat{x}_2 = \begin{cases} \frac{2\sigma_L}{1 + \sqrt{1 - 4\sigma_L(1 - \sigma_L) - \frac{\tau_L}{\epsilon_L}(1 - \sigma_L)}} & \text{if } \tau_L(1 - \sigma_L) = \epsilon_L, \\ \frac{\sigma_L \epsilon_L(1 - x_1^2) + \tau_L(1 - \epsilon_L)(1 - \sigma_L)x_1^2}{\epsilon_L(1 - x_1^2) + \tau_L(1 - \sigma_L)x_1^2} & \text{otherwise}. \end{cases}
\]

Proof. Clearly, the equation

\[
F_1(x) = \frac{\sigma_L \epsilon_L(1 - x) + \tau_L(1 - \epsilon_L)(1 - \sigma_L)x}{\epsilon_L(1 - x) + \tau_L(1 - \sigma_L)x} - x = 0 \quad (\sigma_L < x < 1 - \epsilon_L)
\]

has the same roots as

\[
(\epsilon_L - \tau_L(1 - \sigma_L))x^2 + (\tau_L(1 - \epsilon_L)(1 - \sigma_L) - \epsilon_L(1 + \sigma_L))x + \sigma_L \epsilon_L = 0.
\]

If \( \tau_L(1 - \sigma_L) = \epsilon_L \), then the root of \( F_1(x) = 0 \) is \( x_1 = \frac{\sigma_L}{\sigma_L + \epsilon_L} \); otherwise, \( x_1 \) is of the form (4.15). Note that these two cases can be combined together.

Next, we consider the roots of

\[
F_2(x) = \frac{\sigma_L \epsilon_L(1 - x^2) + \tau_L(1 - \epsilon_L)(1 - \sigma_L)x^2}{\epsilon_L(1 - x^2) + \tau_L(1 - \sigma_L)x^2} - x = 0 \quad (\sigma_L < x < x_1).
\]

Due to the fact that \( F_2(x) + x \) is a strictly increasing function, it follows that

\( F_2(x) + x < F_2(x_1) + x_1 \).
Then
\[ F_2(x) < \frac{\sigma_L \varepsilon_L (1 - x_1^2) + \tau_L (1 - \varepsilon_L)(1 - \sigma_L)x_1^2}{\varepsilon_L (1 - x_1^2) + \tau_L (1 - \sigma_L)x_1^2} - x, \]
which yields
\[ F_2 \left( \frac{\sigma_L \varepsilon_L (1 - x_1^2) + \tau_L (1 - \varepsilon_L)(1 - \sigma_L)x_1^2}{\varepsilon_L (1 - x_1^2) + \tau_L (1 - \sigma_L)x_1^2} \right) < 0. \]
Since \( F_2(\sigma_L) > 0 \) and
\[ \sigma_L < \frac{\sigma_L \varepsilon_L (1 - x_1^2) + \tau_L (1 - \varepsilon_L)(1 - \sigma_L)x_1^2}{\varepsilon_L (1 - x_1^2) + \tau_L (1 - \sigma_L)x_1^2} < F_1(x_1) + x_1 = x_1, \]
one can find a root \( x_2 \) satisfying that
\[ x_2 < \frac{\sigma_L \varepsilon_L (1 - x_1^2) + \tau_L (1 - \varepsilon_L)(1 - \sigma_L)x_1^2}{\varepsilon_L (1 - x_1^2) + \tau_L (1 - \sigma_L)x_1^2}. \]
In particular, if \( \tau_L (1 - \sigma_L) = \varepsilon_L \), then
\[ x_2 = \frac{2\sigma_L}{1 + \sqrt{1 - 4\sigma_L(1 - \varepsilon_L)}}. \]
Thus, \( \hat{x}_2 \) given by (4.16) is an upper bound for \( x_2 \). The estimate (4.14) then follows from Theorem 4.3.

Remark 4.6. It is easy to see that \( F_2(x) = 0 \) has the same roots as
\[ (\varepsilon_L - \tau_L (1 - \sigma_L))x^3 + (\tau_L (1 - \varepsilon_L)(1 - \sigma_L) - \sigma_L \varepsilon_L)x^2 - \varepsilon_L x + \sigma_L \varepsilon_L = 0, \]
which is a cubic equation if \( \tau_L (1 - \sigma_L) \neq \varepsilon_L \). For the sake of brevity, we only give an upper bound for \( x_2 \) in Corollary 4.5. Indeed, one can derive the precise expression of \( x_2 \) by using the well-known Cardano’s formula.

5. Conclusions. In this paper, we present a novel framework for analyzing the convergence of inexact two-grid methods, which is inspired by an explicit expression for the inexact two-grid preconditioner. Based on the analytical framework, we establish a unified convergence theory for multigrid methods, which allows the coarsest-grid system to be solved approximately. In the future, we expect to analyze other multilevel methods by using the proposed framework.

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CONVERGENCE ANALYSIS OF INEXACT TWO-GRID METHODS

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