The Riemann Zeta-Function and Hecke Congruence Subgroups. II

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This is a rework of our old file on an explicit spectral decomposition of the mean value

\[ M_2(g; A) = \int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \left| A\left(\frac{1}{2} + it\right) \right|^2 g(t) dt \]

that has been left unpublished since September 1994, though its summary account is given in [9] (see also [11, Section 4.6]); here

\[ A(s) = \sum_n \alpha_n n^{-s} \]

is a finite Dirichlet series and \( g \) is assumed to be even, regular, real-valued on \( \mathbb{R} \), and of fast decay on a sufficiently wide horizontal strip. At this occasion we shall add greater details as well as a rigorous treatment of the Mellin transform

\[ Z_2(s; A) = \int_1^{\infty} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \left| A\left(\frac{1}{2} + it\right) \right|^2 t^{-s} dt \]

which was scantily touched on in [9].

We shall proceed with an arbitrary \( A \) to a considerable extent but later restrict ourselves to the situation where \( \alpha_n \) is supported by the set of square-free integers. This is solely to avoid certain technical complexities pertaining to Kloosterman sums associated with Hecke congruence subgroups which do not appear particularly worth dealing with thoroughly, for our present principal purpose is to look into the nature of \( Z_2(s; A) \).

Our result on \( Z_2(s; A) \) seems to allow us to have a glimpse of the nature of the plain sixth power moment

\[ M_3(g; 1) = \int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2} + it\right) \right|^6 g(t) dt, \]

although we shall set out only certain ensuing problems which are to be solved before stating anything precisely. In fact, this motivation which was implicit in our original file was similar to that expressed in [4]. Our approach was, however, more explicit, being a natural extension of our treatment of the plain fourth moment \( M_2(g; 1) \) that was later published in [11].

As we noted at a few occasions, the reason of the success with \( M_2(g; 1) \) lies probably in the fact that the Eisenstein series in the framework of \( \text{SL}(2, \mathbb{R}) \) is closely related to the product
of two zeta-values and in that the group is of real rank one, with the observation that the later is reflected in that the integral for $M_2(g; 1)$ is single (as is inferred from the arguments developed in e.g. [2][12]). Extrapolating this, we surmise that a proper formulation of the sixth moment of the zeta-function might be expressed instead in terms of a double integral, since the group $\text{SL}(3, \mathbb{R})$ appears to be closely related to the product of three zeta-values and it is of real rank 2. Nevertheless, we shall consider $M_2(g; A)$, as it stands between the pure fourth and sixth moments and requires less machineries than the plausible direct approach to the sixth moment via the spectral theory of $L^2(\text{PSL}(3, \mathbb{Z})\backslash\text{PSL}(3, \mathbb{R}))$ such as proposed in [11, Section 5.4].

There are at least three ways for us to proceed along. The first is the argument that we took in [7][11], the second is a representation theoretic approach developed in [2], and the third is the one in [12] which is more representation theoretic and in fact generalizes to quite a wide extent. We shall take the first way, as we have indicated above, for it appears to be the most explicit and allow us to exploit best the peculiarity of our problem, i.e., the presence of the square of the zeta-function in place of the first power of an automorphic $L$-function. However, it should be stressed that the methods in [2] and [12] have a definite advantage over that in [7][11]; see Remark 3 in Section 15 below.

**Convention.** We shall assume throughout our discussion that there exist no exceptional eigenvalues for any Hecke congruence subgroup $\Gamma_0(q)$.

Thus all spectral data $\kappa_j$ should be understood to be real and non-negative. With this, we might not appear prudent enough, but actually our discussion of $Z_2(s; A)$ is not essentially affected by the assumption, though we are aware of the possible existence of poles in the interval $\left(\frac{1}{2}, 1\right)$.

**Remark 1.** Readers are warned of a number of notational conflicts, none of which should, however, cause any serious misunderstanding. We remark also that our discussion contains details which must be often excessive for experts; nevertheless, we do this because our old file had been prepared for an abortive series of lectures to be given to beginners, and we want to keep the original style. By the way, there exists as well an abridged version of the file that was to be included in [11] as its sixth chapter, but the plan was put away because of a reason which we can no longer remember.

**Remark 2.** We do not mention any of works on mean values of automorphic $L$-functions done in recent years, notably by D. Goldfeld and his colleagues, some of which in fact come close to our interest on $Z_2(s; A)$. This is solely due to our wish to keep ourselves within the framework of the unpublished file of ours; the necessary updating will be made in our relevant forthcoming works.

In passing, we stress that our work [8] (see also [11, Section 5.3]) on $Z_2(s; 1)$ was done without any knowledge of the existence of A. Good’s work [5] on the Mellin transform of the square of an arbitrary automorphic $L$-function. His argument depends on a clever choice of a Poincaré series, whereas ours exploits fully the peculiarity of the Riemann zeta-function as indicated above and produces results more explicit than his. We add that our reasoning extends beyond Good’s situation. This is a consequence of our latest work [12] lying on the lines developed in [2], [7], and [11].
1. To begin with, we have

\[ M_2(g; A) = \sum_{\alpha, \beta, \gamma} \frac{\alpha \beta \gamma}{c \sqrt{ab}} I_2(g; b/a), \]  

where

\[ I_2(g; b/a) = \int_{-\infty}^{\infty} \left| \zeta(\frac{1}{2} + it) \right|^4 (b/a)^it g(t) dt. \]

To study the latter we introduce

\[ I(u, v, w, z; g; b/a) = -i \int (0) \zeta(u + t) \zeta(v - t) \zeta(w + t) \zeta(z - t) (b/a)^t g(-it) dt \]

with \((a, b) = 1\) and \(\text{Re} \ u, \ldots, \text{Re} \ z > 1\). Shifting the contour to \((\alpha)\) lying in the far right, we have

\[ I(u, v, w, z; g; b/a) = -i \int (\alpha) \cdots dt \]

\[ + 2\pi \left\{ \zeta(u + v - 1) \zeta(v + w - 1) \zeta(z - v + 1) (b/a)^{v-1} g(i(1 - v)) \right. 

\[ + \zeta(u + z - 1) \zeta(w + z - 1) \zeta(v - z + 1) (b/a)^{z-1} g(i(1 - z)) \]

\[ + \zeta(w - u + 1) \zeta(u + v - 1) \zeta(u + z - 1) (b/a)^{1-u} g(i(u - 1)) \]

\[ + \zeta(u - w + 1) \zeta(v + w - 1) \zeta(w + z - 1) (b/a)^{1-w} g(i(w - 1)) \right\}; \]

Thus \( I(u, v, w, z; g; b/a) \) is meromorphic throughout \( \mathbb{C}^4 \). With this, we assume that \(\text{Re} \ u, \ldots, \text{Re} \ z < 1\) and shift the last contour back to the original, getting

\[ I(u, v, w, z; g; b/a) = -i \int (\alpha) \cdots dt \]

\[ + 2\pi \left\{ \zeta(u + v - 1) \zeta(v + w - 1) \zeta(z - v + 1) (b/a)^{v-1} g(i(1 - v)) \right. 

\[ + \zeta(u + z - 1) \zeta(w + z - 1) \zeta(v - z + 1) (b/a)^{z-1} g(i(1 - z)) \]

\[ + \zeta(w - u + 1) \zeta(u + v - 1) \zeta(u + z - 1) (b/a)^{1-u} g(i(u - 1)) \]

\[ + \zeta(u - w + 1) \zeta(v + w - 1) \zeta(w + z - 1) (b/a)^{1-w} g(i(w - 1)) \right\}. \]

In the vicinity of \( p_t = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \), the part in the braces is equal to

\[ \zeta(u + v - 1) \zeta(v + w - 1) \frac{1}{z - v} (1 + c_E(z - v) + \cdots)(b/a)^{v-1} g(i(1 - v)) \]

\[ + \zeta(u + v - 1) \left( 1 + \frac{\zeta}{\zeta'} (u + v - 1)(z - v) + \cdots \right) \]

\[ \times \zeta(v + w - 1) \left( 1 + \frac{\zeta'}{\zeta} (v + w - 1)(z - v) + \cdots \right) \frac{1}{v - z} (1 + c_E(v - z) + \cdots) \]
\[
\times (b/a)^{u-1} \left(1 + (\log b/a)(z - v) + \cdots \right)g(i(1 - v)) \left(1 - i \frac{g'}{g}(i(1 - v))(z - v) + \cdots \right)
\]
\[
+ \frac{1}{w - u} \left(1 + c_E(w - u) + \cdots \right)\zeta(u + v - 1)\zeta(u + z - 1)(b/a)^{1-u}g(i(u - 1))
\]
\[
+ \frac{1}{u - w} \left(1 + c_E(u - w) + \cdots \right)\zeta(u + v - 1) \left(1 + \frac{c'}{\zeta}(u + v - 1)(w - u) + \cdots \right)
\]
\[
\times \zeta(u + z - 1) \left(1 + \frac{c'}{\zeta}(u + z - 1)(v - u) + \cdots \right)
\]
\[
\times (b/a)^{1-u} \left(1 + (\log b/a)(u - w) + \cdots \right)g(i(u - 1)) \left(1 + i \frac{g'}{g}(i(u - 1))(w - u) + \cdots \right)
\]
\[
= \zeta(u + v - 1)\zeta(v + w - 1)(b/a)^{v-1}g(i(1 - v))
\]
\[
\times \left\{2c_E - \frac{c'}{\zeta}(u + v - 1) - \frac{c'}{\zeta}(v + w - 1) - \log b/a + \frac{g'}{g}(i(1 - v)) \right\} + O(|z - v|)
\]
\[
+ \zeta(u + v - 1)\zeta(u + z - 1)(b/a)^{1-u}g(i(u - 1))
\]
\[
\times \left\{2c_E - \frac{c'}{\zeta}(u + v - 1) - \frac{c'}{\zeta}(u + z - 1) + \log b/a + \frac{g'}{g}(i(u - 1)) \right\} + O(|u - w|),
\]
(1.6)

where \(c_E\) is the Euler constant. Hence, in particular, \(I(u, v, w, z; g; b/a)\) is regular in a neighbourhood of \(p_{\frac{1}{2}}\), and we get

\[
I_2(g; b/a) = I\left(p_{\frac{1}{2}}; g; b/a\right)
\]
\[
- \frac{\pi}{2} (b/a)^{-1/2} g \left(\frac{1}{2} i\right) \left\{2c_E - 2 \log 2\pi - \log(b/a) + i \frac{g'}{g}(\frac{1}{2} i) \right\}
\]
\[
- \frac{\pi}{2} (a/b)^{-1/2} g \left(\frac{1}{2} i\right) \left\{2c_E - 2 \log 2\pi - \log(a/b) - i \frac{g'}{g}(\frac{1}{2} i) \right\}.
\]
(1.7)

The last two terms can be regarded as practically negligible.

**2.** On the other hand, we have, in the region of absolute convergence,

\[
I(u, v, w, z; g; b/a) = \sum_{k,l,m,n} \frac{1}{k^{u}l^{v}m^{w}n^{z}} \hat{g} \left(\log \frac{b\ln}{akm}\right)
\]
\[
= \zeta(u + v) \sum_{k,l} \frac{1}{k^{u}l^{v}} \sum_{m,n} \frac{1}{m^{w}n^{z}} \hat{g} \left(\log \frac{b\ln}{akm}\right),
\]
(2.1)

where \(\hat{g}\) is the Fourier transform of \(g\); and \((ak, bl) = (a, l) \cdot (b, k) = c \cdot d\), say; note that \((a, b) = 1\). We have

\[
I(u, v, w, z; g; b/a) = \zeta(u + v) \sum_{c|a,d|b} \frac{1}{c^{v}d^{u}} \sum_{k,l,(k,l)\neq(1,1)} \frac{1}{k^{u}l^{v}} \sum_{m,n} \frac{1}{m^{w}n^{z}} \hat{g} \left(\log \frac{bln/d}{akm/c}\right)
\]
\[ \zeta(u + v) \frac{1}{a^u b^v} \sum_{c \mid a, d \mid b} c^v d^u \sum_{k, l, (k, l) = 1} \frac{1}{k u \ell v} \sum_{m, n} \frac{1}{m^w n^z} \hat{g} \left( \log \frac{d \ln}{ckm} \right) \]

\[ = \zeta(u + v) \frac{1}{a^u b^v} \sum_{c \mid a, d \mid b} c^v d^u \sum_{(c, \ell, d, k) = 1} \frac{1}{k u \ell v} \sum_{m, n} \frac{1}{m^w n^z} \hat{g} \left( \log \frac{d \ln}{ckm} \right) \]

\[ = \zeta(u + v) \frac{1}{a^u b^v} \sum_{c \mid a, d \mid b} c^v d^u J(u, v, w, z; g; d/c), \quad (2.2) \]

say.

Then we apply the dissection:

\[ J(u, v, w, z; g; d/c) = \{ J_0 + J_+ \} (u, v, w, z; g; d/c), \quad (2.3) \]

where \( J_-(u, v, w, z; g; d/c) = J_+(v, u, z, w; g; c/d) \) and

\[ J_0(u, v, w, z; g; d/c) = \hat{g}(0) \sum_{k, l \mid (c, d, l) = 1} \frac{1}{k u \ell v} \sum_{m, n \mid ckm = dln} \frac{1}{m^w n^z}, \quad (2.4) \]

\[ J_+(u, v, w, z; g; d/c) = \sum_{k, l \mid (c, d, l) = 1} \frac{1}{k u \ell v} \sum_{m, n \mid ckm > dln} \frac{1}{m^w n^z} \hat{g} \left( \log \frac{d \ln}{ckm} \right) \cdot \quad (2.5) \]

We have

\[ J_0(u, v, w, z; g; d/c) = \hat{g}(0) \sum_{k, l \mid (c, d, l) = 1} \frac{1}{k u \ell v} \sum_{n} \frac{1}{(d \ln)^w (ckm)^z} \]

\[ = \hat{g}(0)e^{-z} d^{-w} \zeta(w + z) \sum_{k, l \mid (c, d, l) = 1} \frac{1}{k u \ell v} \sum_{v + w} \mu(r) \]

\[ = \hat{g}(0)e^{-z} d^{-w} \zeta(w + z) \sum_{k, l \mid (c, d, l) = 1} \frac{1}{k u \ell v} \sum_{v \mid (c, d, l)} \mu(r) \]

\[ = \hat{g}(0)e^{-z} d^{-w} \zeta(w + z) \sum_{r \mid (c, d, l)} \mu(r) \sum_{r \mid (d, r) | l} \frac{1}{l v + w} \]

\[ = \hat{g}(0)e^{-z} d^{-w} \zeta(w + z) \sum_{r \mid (c, d, l)} \mu(r) ((c, r) / r)^{u + z} ((d, r) / r)^{v + w} \]

\[ = \hat{g}(0)e^{-z} d^{-w} \zeta(w + z) \zeta(u + z) \zeta(v + w) \]

\[ \times \prod_{p \mid cd} \left( 1 - \frac{1}{p^{u + v + w + z}} \right) \prod_{p | c} \left( 1 - \frac{1}{p^{v + w}} \right) \prod_{p | d} \left( 1 - \frac{1}{p^{u + z}} \right), \quad (2.6) \]
where \( p \) denotes a generic prime and the condition \((c, d) = 1\) has been used. The contribution of \( J_0(u, v, w, z; g; d/c) \) to \( I(u, v, w, z; g; b/a) \) is thus equal to

\[
\hat{g}(0) a^{-u} b^{-u} \frac{\zeta(u+v) \zeta(u+z) \zeta(w+v) \zeta(w+z)}{\zeta(u+v+w+z)} \times \left\{ \sum_{c|a} c^{v-z} \prod_{p|c} \frac{1}{1 - \frac{1}{p^{u+v+w+z}}} \right\} \left\{ \sum_{d|b} d^{u-w} \prod_{p|d} \frac{1}{1 - \frac{1}{p^{u+v+w+z}}} \right\}. \tag{2.7}
\]

3. Next, we shall consider the non-diagonal part \( J_+ \). We have

\[
J_+(u, v, w, z; g; d/c) = \sum_{k, l \atop (ck, dl) = 1} \frac{1}{k^{u+l} v} \sum_{f} \sum_{m,n} \frac{1}{m^w n^z} \hat{g} \left( \log \frac{d ln}{ckm} \right)
\]

\[
= \sum_{k, l \atop (ck, dl) = 1} \frac{(ck)^w}{k^{u+l} v} \sum_{f} \sum_{n \equiv -df \mod ck} \frac{1}{(dln+f)^w n^z} \hat{g} \left( \log \frac{d ln}{dln+f} \right)
\]

\[
= (c/d)^w \sum_{k, l \atop (ck, dl) = 1} \frac{1}{k^{u-w} v} \sum_{f} \sum_{n \equiv -df \mod ck} \frac{1}{n^{w+z}} \left( 1 + \frac{f}{dln} \right)^{-w} \hat{g} \left( \log \left( 1 + \frac{f}{dln} \right) \right). \tag{3.1}
\]

We introduce the Mellin transform

\[
g^*(s, w) = \int_0^\infty \hat{g}(\log(1+x)) \frac{x^{s-1}}{(1+x)^w} dx
\]

\[
= \Gamma(s) \int_{-\infty}^{\infty} \frac{\Gamma(w-s+it)}{\Gamma(w+it)} g(t) dt, \tag{3.2}
\]

provided \( \text{Re} w > \text{Re} s > 0 \). Shifting the last contour downward appropriately, we see that \( g^*(s, w)/\Gamma(s) \) is entire in \( s, w \); and an upward shift gives that \( g^*(s, w) \) is of rapid decay in \( s \) as far as \( w \) and \( \text{Re} s \) are bounded (see [11, Lemma 4.1]). In particular, we have

\[
J_+(u, v, w, z; g; d/c) = \frac{(c/d)^w}{2\pi i} \sum_{k, l \atop (ck, dl) = 1} \frac{1}{k^{u-w} v} \sum_{f} \sum_{n \equiv -df \mod ck} \frac{1}{n^{w+z}} \int_{(\eta)} g^*(s, w) \left( \frac{f}{dln} \right)^{-s} ds. \tag{3.3}
\]
with $\eta > 0$, which converges absolutely if

$$\eta > 1, \text{ Re } u > \text{ Re } w + 1, \text{ Re } (v + w) > \eta + 1, \text{ Re } (w + z) > \eta + 1. \quad (3.4)$$

On this condition, we have

$$J_+(u, v, w, z; g; d/c) = \frac{(c/d)^w}{2\pi i} \int_{(\eta)} g^*(s, w) d^s \sum_{k,l} \frac{1}{k^{U-w} l^{v+w-s}} \sum_f \frac{1}{f^s} \sum_{n \equiv df \mod ck} \frac{1}{n^{w+z-s}} ds$$

$$= e^{-z d-w} \int_{(\eta)} g^*(s, w)(cd)^s \sum_{k,l} \frac{1}{k^{u+z-s} l^{v+w-s}} \sum_f \frac{1}{f^s} \zeta \left( w + z - s, -\frac{dlf}{ck} \right) ds, \quad (3.5)$$

where $\zeta(s, \omega)$ is the Hurwitz zeta-function. Classifying $l$ into residue classes mod $ck$, we have

$$J_+(u, v, w, z; g; d/c) = \frac{c^{-z d-w}}{2\pi i} \int_{(\eta)} g^*(s, w)(cd)^s \sum_{(k,d)=1} \frac{1}{k^{u+z-s}}$$

$$\times \sum_f \frac{1}{f^s} \sum_{h=1}^{ck} \sum_{l \equiv h \mod ck} \frac{1}{l^{v+w-s}} \zeta \left( w + z - s, -\frac{dlf}{ck} \right) ds$$

$$= e^{-v-w-z d-w} \int_{(\eta)} g^*(s, w)(c^2 d)^s \sum_{(k,d)=1} \frac{1}{k^{u+v+w+z-2s}}$$

$$\times \sum_f \frac{1}{f^s} \sum_{h=1}^{ck} \zeta \left( v + w - s, \frac{h}{ck} \right) \zeta \left( w + z - s, -\frac{dlf}{ck} \right) ds. \quad (3.6)$$

4. We are going to shift the last contour. To this end we assume that there exists a large $\eta_1$ such that $\eta_1 > \eta + 1$ and

$$\text{ Re } (v + w) < \eta_1, \text{ Re } (w + z) < \eta_1, \text{ Re } (u + v + w + z) > 2(\eta_1 + 1). \quad (4.1)$$

On this and $1 < \text{ Re } s < \eta_1 + \varepsilon$ with a small $\varepsilon > 0$, the sum

$$\sum_{(k,d)=1} \frac{1}{k^{u+v+w+z-2s}} \sum_f \frac{1}{f^s} \sum_{h=1}^{ck} \zeta \left( v + w - s, \frac{h}{ck} \right) \zeta \left( w + z - s, -\frac{dlf}{ck} \right) \quad (4.2)$$

is a meromorphic function of the five complex variables. To see this we note that for any finite $s$

$$\zeta(s, \omega) \ll |s - 1|^{-1} + \omega^{-\text{Re } s} \quad (0 < \omega \leq 1), \quad (4.3)$$
as it follows via an application of partial summation to the Dirichlet series defining \( \zeta(s, \omega) \). Thus (4.2) is, provided neither \( v + w - s \) nor \( w + z - s \) is too close to 1,

\[
\ll \sum_k k^{\Re (2s-u-v-w-z)+1} \left( 1 + k^{\Re (v+w-s)} \right) \left( 1 + k^{\Re (w+z-s)} \right) \\
= \sum_k \left\{ k^{\Re (2s-u-v-w-z)+1} + k^{\Re (s-u-z)+1} + k^{\Re (s-u-v)+1} + k^{\Re (w-u)+1} \right\},
\]

in which we have

\[
\Re (2s - u - v - w - z) + 1 < \Re (2s) - 2(\eta_1 + 1) + 1, \\
\Re (s - u - z) + 1 = \Re (s) - \Re (u + v + w + z) + \Re (v + w) + 1 \\
< \Re (s) - 2(\eta_1 + 1) + \eta_1 + 1, \\
\Re (s - u - v) + 1 = \Re (s) - \Re (u + v + w + z) + \Re (w + z) + 1 \\
< \Re (s) - 2(\eta_1 + 1) + \eta_1 + 1, \\
\Re (w - u) + 1 = \Re (s) - \Re (u + v + w + z) + \Re (w + z) + 1 \\
< \eta_1 + \eta_1 - 2(\eta_1 + 1) + 1;
\]

and the assertion follows.

With this, we shift the contour in (3.6) to \( (\eta_1) \). We encounter poles at \( s = w+z-1, v+w-1 \); we may assume without loss of generality that they do not coincide. Before computing the residues, we note that

\[
\sum_{h=1}^{\infty} \zeta(s, hm/q) = \zeta(s) \sum_{\delta | q} \delta \mu(q/\delta)(\delta/(\delta, m))^{s-1}.
\]

To show this we use the functional equation

\[
\zeta(s, \omega) = 2(2\pi)^{s-1} \Gamma(1-s) \sum_n \sin \left( \frac{1}{2} \pi s + 2\pi n \omega \right) n^{s-1} \quad (\Re s < 0).
\]

Thus, for \( \Re s < 0 \),

\[
\sum_{h=1}^{q} \zeta(s, hm/q) = 2\Gamma(1-s)(2\pi)^{s-1} \sum_n \sum_{h=1}^{q} \sin \left( \frac{1}{2} \pi s + 2\pi \frac{h}{q} mn \right) n^{s-1} \\
= 2\Gamma(1-s)(2\pi)^{s-1} \sin \left( \frac{1}{2} \pi s \right) \sum_n c_q(mn)n^{s-1} \\
= 2\Gamma(1-s)(2\pi)^{s-1} \sin \left( \frac{1}{2} \pi s \right) \sum_n n^{s-1} \sum_{\delta | (q, mn)} \delta \mu(q/\delta) \\
= 2\Gamma(1-s)(2\pi)^{s-1} \sin \left( \frac{1}{2} \pi s \right) \zeta(1-s) \sum_{\delta | q} \delta \mu(q/\delta)(\delta/(\delta, m))^{s-1},
\]
with the Ramanujan sum $c_q$ mod $q$; and (4.6) follows via the functional equation for $\zeta$.

Let us compute the residue at $s = w + z - 1$. This is equal to

$$2\pi i (c^2 d)^{w+z-1} g^*(w + z - 1, w) \sum_{(k,d)=1} \frac{1}{k^{u+v-w-z+2}} \sum_f \frac{1}{f^{w+z-1}}$$

$$\times \sum_{h=1}^{ck} \zeta \left( v - z + 1, \frac{h}{ck} \right)$$

$$= 2\pi i (c^2 d)^{w+z-1} g^*(w + z - 1, w) \zeta(w + z - 1) \zeta(v - z + 1)$$

$$\times \sum_{(k,d)=1} \frac{1}{k^{u+v-w-z+2}} \sum_{\delta|ck} \delta^{-v-z+1} \mu(ck/\delta)$$

$$= 2\pi i c^2 w+v+z-1 d^{w+z-1} g^*(w + z - 1, w) \zeta(w + z - 1) \zeta(v - z + 1)$$

$$\times \sum_{(k,d)=1} \frac{1}{k^{u-w+1}} \prod_{p|ck} \left( 1 - \frac{1}{p^{v-z+1}} \right)$$

$$= 2\pi i c^2 w+v+z-1 d^{w+z-1} g^*(w + z - 1, w) \zeta(w + z - 1) \zeta(v - z + 1)$$

$$\times \prod_{p|cd} \left( 1 + \frac{1}{p^{w+1}} \right) \prod_{p|ck} \left( 1 - \frac{1}{p^{v-z+1}} \right)$$

$$\times \prod_{p|c} \left( 1 - \frac{1}{p^{v-z+1}} \right) \left( 1 - \frac{1}{p^{u-w+1}} \right)^{-1} \left( 1 - \frac{1}{p^{u-w+1}} \right)^{-1} . \tag{4.9}$$

Returning to (3.6), we see that the contribution of the residue to $J_+(u,v,w,z;g;d/c)$ is

$$c^{w-1} d^{z-1} g^*(w + z - 1, w) \zeta(v - z + 1) \zeta(w + z - 1) \zeta(u - w + 1)$$

$$\frac{1}{\zeta(u + v - w - z + 2)}$$

$$\times \prod_{p|c} \left( 1 - \frac{1}{p^{v-z+1}} \right) \prod_{p|d} \left( 1 - \frac{1}{p^{u-w+1}} \right). \tag{4.10}$$

5. The residue at $s = v + w - 1$ is equal to

$$2\pi i (c^2 d)^{v+w-1} g^*(v + w - 1, w) \sum_{(k,d)=1} \frac{1}{k^{u-v-w+z+2}}$$

$$\times \sum_f \frac{1}{f^{v+w-1}} \sum_{h=1}^{ck} \zeta \left( z - v + 1, \frac{dhf}{ck} \right)$$

$$\times \sum_{h=1}^{ck} \zeta \left( z - v + 1, \frac{dhf}{ck} \right).$$
Thus and as before. Here

\[ \sum_f \frac{(f, \delta)^{v-z}}{f^{v+w-1}} = \sum_f \frac{1}{f^{v+w-1}} \sum_{\lambda | (f, \delta)} \frac{\lambda^{v-z}}{p|\lambda} \left( 1 - \frac{1}{p^{v-z}} \right) \prod_{p | \lambda} \left( 1 - \frac{1}{p^{v-z}} \right), \tag{5.2} \]

and

\[ \sum_{\delta | ck} \delta^{z-v+1} \mu(ck/\delta) \sum_f \frac{(f, \delta)^{v-z}}{f^{v+w-1}} = \zeta(v + w - 1) \prod_{\lambda | ck} \frac{1}{\lambda^{w+z-1}} \prod_{p | \lambda} \left( 1 - \frac{1}{p^{v-z}} \right) \prod_{p | (ck)/\lambda} \left( 1 - \frac{1}{p^{z-v+1}} \right). \tag{5.3} \]

Thus

\[ \sum_{(k, d) = 1} \frac{1}{k^{u-v-w+z+2}} \sum_{\delta | ck} \delta^{z-v+1} \mu(ck/\delta) \sum_f \frac{(f, \delta)^{v-z}}{f^{v+w-1}} = c^{z-v+1} \zeta(v + w - 1) \sum_{(k, d) = 1} \frac{1}{k^{u-w+1}} \sum_{\lambda | ck} \frac{1}{\lambda^{z-w+1}} \prod_{p | \lambda} \left( 1 - \frac{1}{p^{v-z}} \right) \prod_{p | (ck)/\lambda} \left( 1 - \frac{1}{p^{z-v+1}} \right) \]

\[ = c^{z-v+1} \zeta(v + w - 1) \sum_{(\lambda, d) = 1} \frac{(c, \lambda)^{u-w+1}}{\lambda^{u+z}} \prod_{p | \lambda} \left( 1 - \frac{1}{p^{v-z}} \right) \]

\[ \times \sum_{(k, d) = 1} \frac{1}{k^{u-w+1}} \prod_{p | (ck)/(c, \lambda)} \left( 1 - \frac{1}{p^{z-v+1}} \right) \]

\[ = c^{z-v+1} \zeta(v + w - 1) \sum_{(\lambda, d) = 1} \frac{(c, \lambda)^{u-w+1}}{\lambda^{u+z}} \prod_{p | \lambda} \left( 1 - \frac{1}{p^{v-z}} \right) \]

\[ \times \prod_{p | (cd)/(c, \lambda)} \left( 1 + \frac{1}{p^{u-w+1}} \left( 1 - \frac{1}{p^{z-v+1}} \right) \left( 1 - \frac{1}{p^{z-v+1}} \right)^{-1} \right) \]

\[ \times \prod_{p | c/(c, \lambda)} \left( 1 - \frac{1}{p^{z-v+1}} \right) \left( 1 - \frac{1}{p^{u-w+1}} \right)^{-1}. \]
where the last product is as in (5).

Now let us turn to 6.

In the last sum we write \( \lambda = \lambda_1 \lambda_2 \) with \( (\lambda_1, c) = 1 \) and \( \lambda_2 | c^\infty \); and we see that the sum is equal to

\[
\frac{\zeta(u + z)}{\zeta(u + v)} \prod_{p|cd} \left( \frac{1 - \frac{1}{p^{u+z}}}{1 - \frac{1}{p^{u+v}}} \right) \times \prod_{p\|c} \left( \sum_{j=0}^{\infty} \frac{p^j}{p^j(u+z)} \left( 1 - \frac{1}{p^{v-z}} \right)^{\xi(j)} \left( \frac{1 - \frac{1}{p^{v-1}}}{1 - \frac{1}{p^{v-w+z+2}}} \right)^{\xi(\beta - \min(\beta, j))} \right),
\]

where \( 1 - \xi \) is the unit measure placed at the origin. One could compute the last sum into a finite expression.

The contribution of the residue at \( s = v + w - 1 \) to \( J_+(u, v, w, z; g; d/c) \) is equal to

\[
e^{w-1}d^{v-1}g^*(v + w - 1, w) \frac{\zeta(z - v + 1)\zeta(v + w - 1)\zeta(u - w + 1)\zeta(u + z)}{\zeta(u + v)\zeta(u - v - w + z + 2)} \times \prod_{p|cd} \left( \frac{1 - \frac{1}{p^{u+z}}}{1 - \frac{1}{p^{u+v}}} \right) \prod_{p|d} \left( \frac{1 - \frac{1}{p^{u-w+1}}}{1 - \frac{1}{p^{u-v-w+z+2}}} \right) \prod_{p\|c} (\cdots),
\]

where the last product is as in (5).

6. Now let us turn to

\[
J_+^*(u, v, w, z; g; d/c) = \frac{e^{-v-w-z}d^{-w}}{2\pi i} \int_{(\eta_1)} g^*(s, w)(e^2d)^s \sum_{(k, d) = 1} \frac{1}{k^{u+v+w+z-2s}} \times \sum_{f} \frac{1}{f^s} \sum_{h=1}^{ck} \zeta\left(v + w - s, \frac{h}{ck}\right) \zeta\left(w + z - s, -\frac{dhf}{ck}\right) ds,
\]

(6.1)
where (4.1) holds. On noting that \( \text{Re}(v + w - s) < 0, \text{Re}(w + z - s) < 0 \), we appeal to the functional equation (4.7). Then the last double sum is equal to

\[
4 \frac{\Gamma(1 + s - v - w)\Gamma(1 + s - w - z)}{(2\pi)^2 + 2s - v - 2w - z} \sum_{f, m, n} m^{v+w-s-1} n^{w+z-1} (fn)^{-s} \times \sum_{h=1 \atop (h, ck) = 1} \frac{ck}{h} \sin \left( \frac{1}{2} \pi (v + w - s) + 2\pi \frac{m}{ck} h \right) \sin \left( \frac{1}{2} \pi (w + z - s) - 2\pi \frac{fn}{ck} dh \right)
\]

\[
= 2 \frac{\Gamma(1 + s - v - w)\Gamma(1 + s - w - z)}{(2\pi)^2 + 2s - v - 2w - z} \sum_{f, m, n} m^{v+w-s-1} n^{w+z-1} (fn)^{-s} \times \{ \cos \left( \frac{1}{2} \pi (v - z) \right) S(m, \overline{d} fn; ck) - \cos \left( \frac{1}{2} \pi (v + 2w + z - 2s) \right) S(m, -\overline{d} fn; ck) \}
\]

\[
= 2 \frac{\Gamma(1 + s - v - w)\Gamma(1 + s - w - z)}{(2\pi)^2 + 2s - v - 2w - z} \sum_{m, n} m^{v+w-s-1} n^{w+z-1} \sigma_{w+z-1}(n) \times \left[ \cos \left( \frac{1}{2} \pi (v - z) \right) S(m, \overline{d} n; ck) - \cos \left( \frac{1}{2} \pi (v + 2w + z - 2s) \right) S(m, -\overline{d} n; ck) \right], \quad (6.2)
\]

where \( S \) is the ordinary Kloosterman sum, and \( \sigma_\tau(n) = \sum_{\lambda | n} \lambda^\tau \).

Thus

\[
J^*_+(u, v, w, z; g; d/c) = \frac{c^u d^\frac{u+v-w+z}{2}(u+v-w+1)}{\pi i (2\pi)^{u-w+1}} \times \sum_{m, n} m^{\frac{1}{2}(v+w-u-z-1)} n^{\frac{1}{2}(u+v+w+z-1)} \sigma_{w+z-1}(n) \sum_{(k, d) = 1} \frac{1}{ck\sqrt{d}}
\]

\[
\times \int_{(q_1)} \left[ \cos \left( \frac{1}{2} \pi (v - z) \right) S(m, \overline{d} n; ck) - \cos \left( \frac{1}{2} \pi (v + 2w + z - 2s) \right) S(m, -\overline{d} n; ck) \right]
\]

\[
\times \Gamma(1 + s - v - w)\Gamma(1 + s - w - z) g^*(s, w) \left( \frac{2\pi \sqrt{mn}}{ck\sqrt{d}} \right)^{u+v+w+z-2s-1} ds. \quad (6.3)
\]

We put

\[
\tilde{g}_+(u, v, w, z; x) = \frac{1}{2\pi i} \cos \left( \frac{1}{2} \pi (v - z) \right)
\]

\[
\times \int_{(q_1)} \Gamma(1 + s - v - w)\Gamma(1 + s - w - z) g^*(s, w)(x/2)^{u+v+w+z-2s-1} ds,
\]

\[
\tilde{g}_-(u, v, w, z; x) = -\frac{1}{2\pi i} \int_{(q_1)} \cos \left( \frac{1}{2} \pi (v + 2w + z - 2s) \right)
\]

\[
\times \Gamma(1 + s - v - w)\Gamma(1 + s - w - z) g^*(s, w)(x/2)^{u+v+w+z-2s-1} ds, \quad (6.4)
\]
and
\[
Y_\pm(u, v, w, z; g ; d/c; m, n) = \sum_{(k,d)=1} \frac{1}{ck\sqrt{d}} S(m, \pm\bar{d}n; ck) \tilde{g}_\pm(u, v, w, z; \frac{4\pi \sqrt{mn}}{ck\sqrt{d}}) \\
= \sum_{(k,d)=1} \frac{1}{ck\sqrt{d}} S(n, \pm\bar{d}m; ck) \tilde{g}_\pm(u, v, w, z; \frac{4\pi \sqrt{mn}}{ck\sqrt{d}}). \tag{6.5}
\]

We have
\[
J_+^*(u, v, w, z; g ; d/c) = [K_+ + K_-] (u, v, w, z; g ; d/c), \tag{6.6}
\]
with
\[
K_\pm(u, v, w, z; g ; d/c) = 2 \frac{e^{ud\frac{1}{2}(u+v-w+z)}}{(2\pi)^{u-w+1}} \times \sum_{m,n} m^{\frac{1}{2}(v+w-u-z-1)} n^{\frac{1}{2}(u+v+w+z-1)} \sigma_{w+z-1}(n) Y_\pm(u, v, w, z; g ; d/c; m, n). \tag{6.7}
\]

7. We need to spectrally decompose the sums \(Y_\pm\). To this end we shall begin with some basic facts about a generic discrete subgroup \(\Gamma\) of \(\text{PSL}(2, \mathbb{R})\) and later proceed to the Kuznetsov sum formula for the Hecke congruence subgroup \(\Gamma_0(q)\).

Thus, let \(\Gamma\) be a discrete subgroup of \(\text{PSL}(2, \mathbb{R})\) which has a fundamental domain of finite volume. We call \(a\) a cusp of \(\Gamma\) if and only if there exists a \(\sigma \in \Gamma\) such that \(\sigma\) is parabolic, i.e., \(\text{Tr}(\sigma) = \pm 2\) and \(\sigma(a) = a \in \mathbb{R} \cup \infty\). Let \(\Gamma_a\) be \(\{\sigma \in \Gamma : \sigma(a) = a\}\), i.e., the stabilizer of \(a\). Then \(\Gamma_a\) is cyclic, so all elements in it are parabolic. Hence, there exits a \(\sigma_a\) such that \(\sigma_a(\infty) = a\) and \(\sigma_a^{-1} \Gamma_a \sigma_a = \Gamma_\infty = [S] \) with \(S = (1, 1)\).

The discussion below depends on the choice of \(\sigma_a\) which is not unique. If \(\sigma_a'\) is another choice, then there exists a \(b\) such that \(\sigma_a' = \sigma_a S_b\). In fact, since \(\sigma_a^{-1} \Gamma_a \sigma_a = \sigma_a'^{-1} \Gamma_a \sigma_a'\), we have \(\sigma_aS \sigma_a^{-1} = \sigma_a' S_b \sigma_a^{-1}\) or \(\sigma_a^{-1} \sigma_a' S_b \sigma_a^{-1} = S \sigma_a^{-1} \sigma_a'\). On the other hand \(\sigma_a^{-1} \sigma_a' (\infty) = \infty\) implies that \(\sigma_a^{-1} \sigma_a' = (a/b, c)\); and \((a/b, c) (1 \pm 1) = (1/1) (a/b, c)\) yields that \(a = \pm c\), that is, \(a = c = 1\) and the assertion follows.

Let \(f\) be a \(\Gamma\)-automorphic form of weight \(2k\), with a positive integer \(k\); namely, for \(\sigma = (a/b, c) \in \Gamma\),
\[
f(\sigma(z)) = (cz + d)^{2k} f(z) \\
= j(\sigma, z)^{2k} f(z). \tag{7.1}
\]

The function \(f(\sigma_a(z))(j(\sigma_a, z))^{-2k}\) is of period 1. In fact,
\[
f(\sigma_a S(z))(j(\sigma_a, S(z)))^{-2k} = f(\sigma_a S \sigma_a^{-1} \sigma_a(z))(j(\sigma_a, S(z)))^{-2k} \\
= f(\sigma_a(z))(j(\sigma_a S \sigma_a^{-1}, \sigma_a(z)))^{2k} (j(\sigma_a, S(z)))^{-2k} \\
= f(\sigma_a(z)) [j(\sigma_a S, z)/j(\sigma_a, z)]^{2k} (j(\sigma_a, S(z)))^{-2k} \\
= f(\sigma_a(z))(j(\sigma_a, z))^{-2k}. \tag{7.2}
\]
Thus, if \(f(\sigma_a(z))\) is regular near \(\infty\), then the function \(f(\sigma_a(\log z/2\pi i))(j(\sigma_a, \log z/2\pi i))^{-2k}\) is single valued and regular on a small disk centered and punctured at the origin. Hence

\[
f(\sigma_a(z))(j(\sigma_a, z))^{-2k} = \sum_n \varrho(n, a) \exp(2\pi inz),
\]

which is called the Fourier expansion of \(f\) around the cusp \(a\).

Note that this expansion depends on the choice of \(\sigma_a\). In fact, if \(\sigma'_a\) is another choice, then \(\sigma'_a = \sigma_a S^b\) with a \(b\). We have \(f(\sigma'_a(z))(j(\sigma'_a, z))^{-2k} = f(\sigma_a(z + b))(j(\sigma_a, z + b))^{-2k}\). That is, \(\varrho(n, a)\) is multiplied by \(\exp(2\pi inb)\).

If \(f\) is regular on the upper half plane \(\mathcal{H} = \{z = x + iy : -\infty < x < \infty, y > 0\}\) and \(\varrho(n, a) = 0\) for any \(n \leq 0\) and any \(a\), then \(f\) is termed a holomorphic cusp-form. Let \(S_k(\Gamma)\) be the space of all cusp-forms of weight \(2k\). Then \(S_k(\Gamma)\) is a finite dimensional Hermitian space with the Petersson inner product

\[
\langle f, g \rangle_k = \int_{\Gamma \backslash \mathcal{H}} f(z)\overline{g(z)}y^{2k}d\mu(z), \quad d\mu(z) = dx dy/y^2.
\]

We let \(\{\psi_{j,k}(z), 1 \leq j \leq \vartheta(k)\}\) stand for an orthonormal base of \(S_k(\Gamma)\).

8. Let \(k \geq 2\). We introduce the Poincaré series

\[
P_m(z, a; k) = \sum_{\gamma \in \Gamma_a \backslash \Gamma} (j(\sigma_a^{-1}\gamma, z))^{-2k} \exp(2\pi im\sigma_a^{-1}\gamma(z)).
\]

This is a holomorphic cusp form of weight \(2k\) for any integer \(m > 0\). We shall confirm this claim, though we skip the convergence issue, which causes no difficulty when \(k \geq 2\).

First, each summand is a function over \(\Gamma_a \backslash \Gamma\). In fact, if \(\Gamma_\gamma = \Gamma_a \gamma'\), then \(\Gamma_\infty \sigma_a^{-1}\gamma = \Gamma_\infty \sigma_a^{-1}\gamma'\) and \(\sigma_a^{-1}\gamma(z) \equiv \sigma_a^{-1}\gamma'(z) \mod 1\) as well as \(j(\sigma_a^{-1}\gamma, z) = j(\sigma_a^{-1}\gamma', z)\). Also the relation \(P_m(\gamma(z), a; k) = (j(\gamma, z))^{2k}P_m(z, a; k)\) is obvious; and \(P_m(z, a; k)\) is regular over \(\mathcal{H}\). Thus, it remains to consider the Fourier expansion at a given cusp \(b\). We have

\[
P_m(\sigma_b(z), a; k)(j(\sigma_b, z))^{-2k}
\]

\[
= \sum_{\gamma \in \Gamma_a \backslash \Gamma} (j(\sigma_b, z))^{-2k}(j(\sigma_a^{-1}\gamma, \sigma_b(z)))^{-2k} \exp(2\pi im\sigma_a^{-1}\gamma \sigma_b(z))
\]

\[
= \sum_{\gamma \in \Gamma_a \backslash \Gamma} (j(\sigma_a^{-1}\gamma \sigma_b, z))^{-2k} \exp(2\pi im\sigma_a^{-1}\gamma \sigma_b(z))
\]

\[
= \sum_{\gamma \in \Gamma_a \backslash \Gamma} \frac{1}{(cz + d)^{2k}} \exp\left(2\pi i m \frac{az + b}{cz + d}\right), \quad \sigma_a^{-1}\gamma \sigma_b = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

If \(c = 0\), then \(\sigma_a^{-1}\gamma \sigma_b(\infty) = \infty\) or \(\gamma(b) = a\), that is, \(a \equiv b \mod \Gamma\) as well as \(\gamma \sigma_b = \sigma_a S^b\). Moreover, if \(\gamma \sigma_b = \sigma_a S^b\), then \(\gamma'(b) = a = \gamma(b)\), that is, \(\gamma^{-1} \in \Gamma_a\) or \(\Gamma_a \gamma = \Gamma_a \gamma'\). Hence

\[
\sum_{\gamma \in \Gamma_a \backslash \Gamma \atop c = 0} = \delta_{a,b} \exp(2\pi im(z + b)).
\]
As to the remaining part, we have
\[ \sum_{\gamma \in \Gamma_a \backslash \Gamma} \sum_{c \neq 0} \frac{1}{(cz + d)^{2k}} \exp \left( 2\pi im \frac{a}{c} - 2\pi im \frac{1}{c(cz + d)} \right). \quad (8.4) \]

We observe that if \( \sigma_a^{-1} \gamma \sigma_b = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) appears in the right side, then \( \sigma_a^{-1} \gamma \sigma_b S_n \sigma_b^{-1} \sigma_b = \left( \begin{array}{cc} a + bn & an \\ c & d + cn \end{array} \right) \) does for all \( n \in \mathbb{Z} \). In fact, \( \gamma \sigma_b S_n \sigma_b^{-1} \in \Gamma \) and thus \( \Gamma_a \gamma \sigma_b S_n \sigma_b^{-1} \) is an element of \( \Gamma_a \backslash \Gamma \). Moreover, if \( \Gamma_a \gamma \sigma_b S_n \sigma_b^{-1} = \Gamma_a \gamma \sigma_b S_m \sigma_b^{-1} \), then \( \sigma_a \Gamma_\infty \sigma_a^{-1} \gamma \sigma_b S_m \sigma_b^{-1} = \sigma_a \Gamma_\infty \sigma_a^{-1} \gamma \sigma_b S_n \sigma_b^{-1} \) or \( \sigma_a^{-1} \gamma \sigma_b S_m = S \sigma_a^{-1} \gamma \sigma_b S_n \). This means that
\[ \left( \begin{array}{cc} a + bn & an \\ c & d + cn \end{array} \right) = \left( \begin{array}{cc} a + cl b + an + (d + cn)l \\ c & d + cn \end{array} \right); \quad (8.5) \]

and we get \( l = 0, m = n \), which confirms our claim. On the other hand, since \( \{ \gamma \sigma_b S_n \sigma_b^{-1} : n \in \mathbb{Z} \} = \gamma \Gamma_b \), we should classify the summands in (8.4) according to the double coset decomposition \( \Gamma_a \backslash \Gamma/\Gamma_b \), which naturally we could have introduced already at (8.2).

We have thus
\[ \sum_{\gamma \in \Gamma_a \backslash \Gamma} \sum_{c \neq 0} \sum_{n} \frac{1}{(c(z + n) + d)^{2k}} \exp \left( 2\pi im \frac{a}{c} - 2\pi im \frac{1}{c(cz + d) + n} \right). \quad (8.6) \]

More explicitly, we have the relation \( \gamma \in \Gamma_a \backslash \Gamma/\Gamma_b \) is equivalent to \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_\infty \sigma_a^{-1} \Gamma_b \). With this one may proceed just in the same way as the case of the full modular group and get
\[ P_m(\sigma_b(z), a; k)(j(\sigma_b, z))^{-2k} = \delta_{a,b} \exp(2\pi i m(z + b)) \]
\[ +2\pi (-1)^k \sum_{n > 0} \left\{ \sum_{c > 0} \frac{1}{c} S(m, n; c; a, b) \left( \frac{n}{m} \right)^{-\frac{1}{2}} J_{2k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right) \right\} \exp(2\pi inz). \quad (8.7) \]

Here
\[ S(m, n; c; a, b) = \sum_{\gamma} \exp(2\pi i (am + dn)/c) \quad (8.8) \]

is a Kloosterman sum associated with \( \Gamma \), where \( \gamma \) runs over the representatives of \( \Gamma_a \backslash \Gamma/\Gamma_b \) with the same \( c \) in the sense remarked after (8.6). The expression (8.8) and the constant \( b \) in (8.7) depend of course on the choice of \( \sigma_a, \sigma_b \).

The last summands are functions on \( \Gamma_a \backslash \Gamma/\Gamma_b \). In fact, let \( \Gamma_a \gamma \Gamma_b = \Gamma_a \gamma' \Gamma_b \). Then \( \sigma_a \Gamma_\infty \sigma_a^{-1} \gamma \sigma_b \Gamma_\infty \sigma_b^{-1} \gamma' \) or \( \Gamma_\infty \sigma_a^{-1} \gamma \sigma_b \Gamma_\infty \sigma_a^{-1} \gamma' \sigma_b \Gamma_\infty \), which means that there exist two integers \( l, l' \) such that \( S^l \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} a' & b' \\ c' & d' \end{array} \right) S^{l'} \). Hence \( c = c' \) and \( a \equiv a', d \equiv d' \mod c \). Also, for each \( c > 0 \) there are at most finitely many double cosets having \( c \) as the lower-left element; otherwise the convergence would be violated.
On the assumption that there exists a $c_0 > 0$ such that for any non-zero integers $m, n$ and any pair of cusps $a, b$

$$\sum_{c} \frac{1}{c^{2k}} |S(m, n; c; a, b)| \ll (mn)^{c_0},$$  

we have

$$P_m(\sigma_\infty, a; k) = 0,$$

implying that $P_m$ is a holomorphic cusp form of weight $2k$.

9. We consider the spectral decomposition

$$\langle P_m(\cdot, a; k), P_n(\cdot, b; k) \rangle_k = \sum_{j=1}^{\vartheta(k)} \langle P_m(\cdot, a; k), \psi_{j,k} \rangle_k \langle P_n(\cdot, b; k), \psi_{j,k} \rangle_k.$$  

The left side is

$$\sum_{\gamma \in \Gamma_b \backslash \Gamma} \int_{\Gamma \backslash \mathcal{H}} P_m(z, a; k) \overline{(j(\sigma_b^{-1} \gamma, z))^{-2k} \exp(2\pi in \sigma_b^{-1} \gamma(z))} y^{2k} d\mu(z)$$

$$= \sum_{\gamma \in \Gamma_b \backslash \Gamma} \int_{\sigma_b^{-1} \gamma(\Gamma \backslash \mathcal{H})} P_m(\gamma^{-1} \sigma_b(z), a; k) \overline{(j(\sigma_b^{-1} \gamma, \gamma^{-1} \sigma_b(z)))^{-2k} \exp(2\pi in \gamma(z))} y^{2k} d\mu(z)$$

$$= \sum_{\gamma \in \Gamma_b \backslash \Gamma} \int_{\sigma_b^{-1} \gamma(\Gamma \backslash \mathcal{H})} P_m(\sigma_b(z), a; k) \overline{(j(\gamma^{-1} \sigma_b, z))^{-2k} \exp(-2\pi in \bar{\gamma(z)})} y^{2k} d\mu(z)$$

$$= \sum_{\gamma \in \Gamma_b \backslash \Gamma} \int_{\sigma_b^{-1} \gamma(\Gamma \backslash \mathcal{H})} P_m(\sigma_b(z), a; k) \overline{(j(\gamma^{-1} \sigma_b, z))^{-2k} \exp(-2\pi in \bar{\gamma(z)})} y^{2k} d\mu(z)$$

$$= \int_{\sigma_b^{-1} \bigcup_{\gamma \in \Gamma_b \backslash \Gamma} \gamma(\Gamma \backslash \mathcal{H})} P_m(\sigma_b(z), a; k) \overline{(j(\gamma^{-1} \sigma_b, z))^{-2k} \exp(-2\pi in \bar{\gamma(z)})} y^{2k} d\mu(z)$$

$$= \int_0^1 \int_0^1 P_m(\sigma_b, a; k) (j(\sigma_b, z))^{-2k} \exp(-2\pi in \bar{\gamma(z)}) y^{2k-2} d\gamma d\mu$$

$$= 2\pi^2 (2k - 1)(4\pi \sqrt{mn})^{1-2k} \left\{ \frac{1}{2\pi} \delta_{a,b} \delta_{m,n} \exp(2\pi in b) \right\}$$

$$+ (-1)^k \sum_{c} \frac{1}{c} S(m, n; c; a, b) J_{2k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right),$$

where we have used that $\sigma_b^{-1} \bigcup_{\gamma \in \Gamma_b \backslash \Gamma} \gamma(\Gamma \backslash \mathcal{H}) = \sigma_b^{-1}(\Gamma_b \backslash \mathcal{H}) = \Gamma_\infty \backslash \mathcal{H}$; in fact, since $\sigma_b \Gamma_\infty \sigma_b^{-1}(\Gamma_b \backslash \mathcal{H}) = \mathcal{H}$, we have $\Gamma_\infty \sigma_b^{-1}(\Gamma_b \backslash \mathcal{H}) = \mathcal{H}$. 

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On the other hand, we have in much the same way

\[
\langle P_m(\cdot, a; k), \psi_{j,k} \rangle_k = \int_0^\infty \int_0^1 \exp(2\pi imz)\overline{\psi_{j,k}(\sigma_a(z))}f(\sigma_a, z)^{-2k} d\mu(z)
= \Gamma(2k-1)(4\pi m)^{1-2k} \Theta_{j,k}(m, a),
\]

where we have put, following (7.3),

\[
\psi_{j,k}(\sigma_a(z))f(\sigma_a, z)^{-2k} = \sum_{n>0} \Theta_{j,k}(n, a) \exp(2\pi inz).
\]

Hence we have obtained the Petersson Formula:

**Lemma 1.** For \( k \geq 2 \)

\[
\frac{1}{2\pi} \frac{\Gamma(2k-1)}{(4\pi \sqrt{mn})^{2k-1}} \sum_{j=1}^{\delta(k)} \Theta_{j,k}(m, a)\Theta_{j,k}(n, b)
= \frac{1}{2\pi} \delta_{a,b} \delta_{m,n} \exp(2\pi inb) + (-1)^k \sum_{c} \frac{1}{c} S(m, n; c; a, b) J_{2k-1} \left( 4\pi \sqrt{mn} \right),
\]

provided \( \Gamma \) satisfies (8.9).

The case \( k = 1 \) can also be treated in much the same way as is done with the full modular group (see [11, pp. 52–54]), excepting that (8.9) should be replaced by the assumption: There exists a constant \( \tau < 2 \) such that for any non-zero integers \( m, n \) and for any pair of cusps \( a, b \)

\[
\sum_{c} \frac{1}{c^2} |S(m, n; c; a, b)| \ll |mn|^{\tau_0}.
\]

On this (9.5) holds for all \( k \geq 1 \).

**10.** We turn to real analytic cusp forms. The procedure is similar to the holomorphic case and also to the full modular situation, and we can be brief.

Let \( f \) be a real analytic cusp form of weight zero with respect to \( \Gamma \) so that \( f(\gamma(z)) = f(z) \) for all \( \gamma \in \Gamma \), and \( \Delta f = \nu f \) with \( \Delta = -y^2(\partial_x^2 + \partial_y^2) \). Since \( f(\sigma_a(z)) \) is of period one, we have the Fourier expansion

\[
f(\sigma_a(z)) = \sum_n \varrho(n, a; y) \exp(2\pi inx).
\]

We require that

\[
\lim_{z \to \infty} f(\sigma_a(z)) = 0 \text{ for any } a, \text{ and } \int_{\Gamma \backslash \mathcal{H}} |f|^2 d\mu(z) < \infty.
\]
We have then
\[ f(\sigma_a(z)) = y^{\frac{1}{2}} \sum_{n \neq 0} g(n, a) K_{i\kappa} (2\pi |n|y) \exp(2\pi inx), \quad \nu = \kappa^2 + \frac{1}{4}. \quad (10.3) \]

One may consider more generally the decomposition of the space \( L^2(\Gamma \backslash G), \; G = \text{PSL}(2, \mathbb{R}) \) into irreducible subspaces and appeal to the theory of representations of the Lie group \( G \). This will allow us to deal with all cusp forms of various weights in a unified fashion. However, here we shall rather follow the argument due to Kuznetsov and others.

Thus, let us introduce the Poincaré series of the Selberg type
\[ U_m(z, a; s) = \sum_{\gamma \in \Gamma_a \backslash \Gamma} \left( \text{Im} \sigma_a^{-1}(z) \right)^s \exp(2\pi im\sigma_a^{-1}(z)), \quad (10.4) \]
and the Eisenstein series \( E(z, a; s) = U_0(z; a; s) \), associated with the cusp \( a \). Arguing as in Section 8, we have the Fourier expansion
\[ U_m(\sigma_b(z), a; s) = \delta_{a,b} y^s \exp(2\pi im(z + b)) + y^{1-s} \sum_n \exp(2\pi inx) \sum_c \frac{1}{c^2 \pi} S(m, n; c; a, b) \times \int_{-\infty}^{\infty} \exp \left( -2\pi iny\xi - \frac{2\pi m}{c^2 y(1 - i\xi)} \right) \frac{d\xi}{(1 + \xi^2)^s}, \quad (10.5) \]
On the assumption (9.6), \( U_m(\sigma_b(z), a; s) \) is regular for \( \text{Re} s > \tau/2 \), and also \( U_m(\sigma_b(z), a; s) \ll y^{1-\text{Re} s} \) as \( y \to \infty \). In particular, \( U_m(z, a; s) \in L^2(\Gamma \backslash \mathcal{H}) \) if \( \text{Re} s > \tau/2 \). Also we have
\[ E(\sigma_b(z), a; s) = \delta_{a,b} y^s + \sqrt{\pi} y^{1-s} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} e_0(s; a, b) + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}} \sum_{n \neq 0} |n|^{s-\frac{1}{2}} e_n(s; a, b) K_{s-\frac{1}{2}} (2\pi |n|y) \exp(2\pi inx), \quad (10.6) \]
with
\[ e_n(s; a, b) = \sum_c \frac{1}{c^2 \pi} S(0, n; a, b). \quad (10.7) \]
It can be shown that \( E(\sigma_b(z), a; s) \) is meromorphic for all \( s \). Moreover, in the case of congruence subgroups, \( E(\sigma_b(z), a; s) \) is regular for \( \text{Re} s \geq \frac{1}{2} \) except for a simple pole at \( s = 1 \).

Let \( \{ \psi_j : j \geq 1 \} \) be a complete orthonormal base of the cuspidal subspace of \( L^2(\Gamma \backslash \mathcal{H}) \) such that \( \Delta \psi_j = \nu_j \psi_j \) with \( \nu_j = \kappa_j^2 + \frac{1}{4} \), and
\[ \psi_j(\sigma_a(z)) = y^{\frac{1}{2}} \sum_{n \neq 0} g_j(n, a) K_{i\kappa_j} (2\pi |n|y) \exp(2\pi inx). \quad (10.8) \]
We put also $\psi_0 \equiv (\text{volume of } \Gamma \backslash \mathcal{H})^{-1/2}$. We suppose that $\Gamma$ is such that no $E(z, a; s)$ has poles in the interval $(\frac{1}{2}, 1)$. Then we have the spectral expansion: For any pair $f, g \in L^2(\Gamma \backslash \mathcal{H})$, it holds that
\[
\langle f, g \rangle = \sum_{j=0}^{\infty} \langle f, \psi_j \rangle \langle g, \psi_j \rangle + \frac{1}{4\pi} \sum_{c} \int_{-\infty}^{\infty} E(r, c; f) \overline{E(r, c; g)} dr,
\]
(10.9)
where $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_0$ and
\[
E(r, c; f) = \int_{\Gamma \backslash \mathcal{H}} f(z) \overline{E(z, c; \tfrac{1}{2} + ir)} \, d\mu(z).
\]
(10.10)

11. We collect here analogues of Bruggeman’s and Kuznetsov’s formulas: On the basic assumption (9.6) we have:

**Lemma 2.** Uniformly for any $n \neq 0$ and $a$,
\[
\sum_{\kappa_j \leq K} \frac{|q_j(n, a)|^2}{\cosh \pi \kappa_j} + \sum_{c} \int_{-K}^{K} |e_n(\tfrac{1}{2} + ir; c, a)|^2 dr \ll K^2 + |n|^{c_1},
\]
(11.1)
where $c_1$ depends on $\tau, c_0$ in (9.6). In particular, we have the bound
\[
|q_j(n, a) | \ll (\kappa_j + |n|^{\frac{1}{2} c_1}) \exp \left( \frac{1}{2} \pi \kappa_j \right).
\]
(11.2)

**Lemma 3.** Let $h(r)$ be even, regular and of fast decay on the strip $|\text{Im} r| < \frac{1}{2} + \eta$ with an $\eta > 0$. Then it holds that for any $m, n > 0$ and $a, b$
\[
\sum_{j=1}^{\infty} \frac{|q_j(m, a)|q_j(\pm n, b)}{\cosh \pi \kappa_j} h(\kappa_j) + \frac{1}{\pi} \sum_{c} \int_{-\infty}^{\infty} (n/m)^{it} e_{m}(\tfrac{1}{2} + ir; c, a) e_{n}(\tfrac{1}{2} + ir; c, b) h(r) dr
\]
\[
= \frac{1}{\pi^2} \delta_{a,b} \delta_{m, \pm n} \exp(2\pi imb) \int_{-\infty}^{\infty} r \tanh(\pi r) h(r) dr + \sum_{c} \frac{1}{\pi} S(m, \pm n; c; a, b) h_{\pm} \left( \frac{4\pi \sqrt{mn}}{c} \right),
\]
(11.3)
where $c$ runs over all inequivalent cusps, and
\[
h_+(x) = \frac{2i}{\pi} \int_{-\infty}^{\infty} \frac{r h(r)}{\cosh \pi r} J_{2ir}(x) dr, \quad h_-(x) = \frac{4}{\pi^2} \int_{-\infty}^{\infty} r h(r) \sinh(\pi r) K_{2ir}(x) dr.
\]
(11.4)

**Lemma 4.** Let $\varphi$ be smooth and of fast decay over the positive real axis. Then we have, for any $m, n > 0$ and $a, b$,
\[
\sum_{c} \frac{1}{\pi} S(m, \pm n; c; a, b) \varphi \left( \frac{4\pi \sqrt{mn}}{c} \right) = \sum_{j=1}^{\infty} \frac{|q_j(m, a)|q_j(\pm n, b)}{\cosh \pi \kappa_j} \varphi_{\pm}(\kappa_j)
\]
\[
+ \frac{1 + 1}{4\pi (4\pi \sqrt{mn})^{2k-1}} \sum_{k=1}^{\infty} \Gamma(2k) \varphi_{\pm} \left( \left( \frac{1}{2} - 2k \right) i \right) \sum_{j=1}^{\infty} q_{j,k}(m, a) q_{j,k}(n, b)
\]
\[
+ \frac{1}{\pi} \sum_{c} \int_{-\infty}^{\infty} (n/m)^{it} e_{m}(\tfrac{1}{2} + ir; c, a) e_{n}(\tfrac{1}{2} + ir; c, b) \varphi_{\pm}(r) dr,
\]
(11.5)
where
\[
\hat{\varphi}_+(r) = \frac{\pi i}{2 \sinh \pi r} \int_0^\infty \{J_{2ir}(x) - J_{-2ir}(x)\} \varphi(x) \frac{dx}{x},
\]
\[
\hat{\varphi}_-(r) = 2 \cosh(\pi r) \int_0^\infty K_{2ir}(x) \varphi(x) \frac{dx}{x}.
\]

(11.6)

12. With this, we shall consider the specialization \(\Gamma = \Gamma_0(q)\). Our discussion overlaps, to a certain extent, with that developed in [3]; however, the present work can be read independently of it. In this section we shall fix a representative set of all cusps inequivalent mod \(\Gamma_0(q)\).

We introduce \(V = \{ (\frac{1}{n} 1) : n \in \mathbb{Z} \}\) the stabilizer of the point 0 in \(\Gamma_0(1)\) and the double coset decomposition
\[
\Gamma_0(1) = \bigcup_a \Gamma_0(q) \gamma_a V,
\]
where the symbol \(a\) is to be regarded temporarily as to be just a label. We begin with a particular \(\gamma_a\), and transform it to a matrix suitable for our purpose. We thus look into the product
\[
\begin{pmatrix}
a & b \\
c q & d
\end{pmatrix}
\begin{pmatrix}
e & f \\
g & h
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
n & 1
\end{pmatrix}
= \begin{pmatrix}
* & * \\
* & k
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
n & 1
\end{pmatrix},
\]
where the middle matrix on the left side corresponds to \(\gamma_a\). It is to be observed that \(g\) is fixed mod \(h\), because of the action of \(V\). We assume that \(h \neq 0\). We have \(k = cfq + dh\), and we claim that this can be made equal to \((q, h)\). In fact \(c(fq/(q, h)) + d(h/(q, h)) = 1\) is soluble in \(c\) and \(d\), for \((fq, h) = (q, h)\); then \(d \equiv h/(q, h)\) mod \(fq/(q, h)\), and \(d\) can be a prime large enough so that \((d, q) = 1\), and thus \((d, cq) = 1\). With such a \(d\) we may choose \(a, b\) to satisfy \(ad - bcq = 1\), which confirms our claim. On the other hand, if \(h = 0\), then it suffices to put \(c = \text{sgn}(f), d = 1\). Thus we may suppose that \(\gamma_a = (\begin{smallmatrix} * & * \\ * & w \end{smallmatrix})\) with \(w|q\); that is, each coset in (12.1) contains elements of this property.

We then apply (12.1) to the point 0, getting
\[
\mathbb{Q} \cup \{\infty\} = \bigcup_a \Gamma_0(q) \gamma_a(0).
\]
This means that \(\{\gamma_a(0) : a\}\), with the current definition of \(a\), is the full set of inequivalent cusps mod \(\Gamma_0(q)\). In fact, that \(\Gamma_0(q) \gamma_{a'}(0) \ni \gamma_{a'}(0)\) implies readily that \(\Gamma_0(q) \gamma_a V = \Gamma_0(q) \gamma_{a'} V\); and the stabilizer in \(\Gamma_0(q)\) of \(\gamma_{a}(0)\) is \(\gamma_{a'} V_{a/w}^{-1}\gamma_{a'}^{-1}\) with \(V_{a'} = \{ (\frac{1}{n} 1) : d|n \}\), provided \(\gamma_a = (\begin{smallmatrix} * & * \\ * & w \end{smallmatrix})\). The labels \(\{a\}\) indeed coincide with their former designation. Also, it should be noted that the element \(w\) is unique to each double coset, which can be proved by considering the relation \(\Gamma_0(q) (\begin{smallmatrix} * & * \\ * & w \end{smallmatrix}) V = \Gamma_0(q) (\begin{smallmatrix} * & * \\ * & w' \end{smallmatrix}) V\) with respect to either mod \(w\) or mod \(w'\), getting \(w|w'\) and \(w'|w\), respectively. Namely, if \(w \neq w'\), then \(\Gamma_0(q) (\begin{smallmatrix} * & * \\ * & w \end{smallmatrix}) V \cap \Gamma_0(q) (\begin{smallmatrix} * & * \\ * & w' \end{smallmatrix}) V = \emptyset\).

Hence, it remains to see when the relation
\[
\Gamma_0(q) (\begin{smallmatrix} e & f \\ g & w \end{smallmatrix}) V = \Gamma_0(q) (\begin{smallmatrix} e' & f' \\ g' & w' \end{smallmatrix}) V
\]
(12.4)
holds, where the two matrices are in $\Gamma_0(1)$ with $w|q$ and $(gg', w) = 1$. Thus, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & w \end{pmatrix} = \begin{pmatrix} e' & f' \\ g' & w \end{pmatrix} \begin{pmatrix} 1 & 1 \\ n & 1 \end{pmatrix} \quad \text{with } q|c$$

$$\iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e' + nf' & f' \\ g' + nw & w \end{pmatrix} \begin{pmatrix} w & -f \\ -g & e \end{pmatrix}$$

$$\iff c = w(g' + nw) - gw = w(g' - g + nw)$$

$$\iff w(g' - g + nw) \equiv 0 \mod q$$

$$\iff g' - g + nw \equiv 0 \mod q/w$$

$$\iff g' \equiv g \mod (w, q/w). \quad (12.5)$$

Hence

$$(1.24) \iff (gg', w) = 1 \text{ and } g \equiv g' \mod (w, q/w). \quad (12.6)$$

Namely, when $\gamma_a$ varies with $w$ fixed, then $g$ and thus $f$ runs over the complete residue classes mod $(w, q/w)$ while satisfying $(u, f) = 1$. If $(u, (w, q/w)) = 1$, then obviously there exists an $f$ such that $u \equiv f \mod (w, q/w)$ and $(w, f) = 1$.

Collecting the above, we have

**Lemma 5.** A complete representative set of cusps inequivalent mod $\Gamma_0(q)$ is given by

$$\left\{ \frac{u}{w} : w|q, (u, w) = 1, u \mod (w, q/w) \right\}, \quad (12.7)$$

whose cardinality is

$$\sum_{w|q} \varphi((w, q/w)). \quad (12.8)$$

13. Let us fix the stabilizers of those cusps given in (12.7). To this end we note first that if $a \neq \infty$ is a cusp of a discrete group $\Gamma$, then

$$\Gamma_a = \Gamma \cap \left\{ \begin{pmatrix} 1 + \nu a & -\nu a^2 \\ \nu & 1 - \nu a \end{pmatrix} : \nu \in \mathbb{R} \right\}. \quad (13.1)$$

In fact, since $(aa + b)/(ca + d) = a$, $a + d = 2$, we see that $a = (1 - d)/c$, and the assertion follows with $c = \nu$. If $a = u/w$ with $w|q$, $(u, w) = 1$, then

$$\Gamma_{u/w} = \Gamma_0(q) \cap \left\{ \begin{pmatrix} 1 + \nu \frac{u}{w} & -\nu \frac{u^2}{w^2} \\ \nu & 1 - \nu \frac{u}{w} \end{pmatrix} : \nu \in \mathbb{R} \right\}, \quad (13.2)$$
and thus $\nu \in \mathbb{Z}, \nu \equiv 0 \mod q, \nu \equiv 0 \mod w^2$; namely

$$\Gamma_{u/w} = \left\{ \left(\begin{array}{cc} 1 + \frac{\nu u}{w} & -\frac{\nu u^2}{w^2} \\ \nu & 1 - \frac{\nu u}{w} \end{array}\right) : \nu \equiv 0 \mod \lfloor w^2, q \rfloor \right\}. \quad (13.3)$$

We write

$$q = vw = (v, w)^2 v^* w^*, \quad v^* = \frac{v}{(v, w)}, \quad w^* = \frac{w}{(v, w)}. \quad (13.4)$$

We put

$$\varpi_{u/w} = \left(\begin{array}{c} u \\ \frac{u \bar{u} - 1}{w} \\ w \bar{u} \end{array}\right), \quad u \bar{u} \equiv 1 \mod w, \quad (13.5)$$

and

$$\tau_{v^*} = \left(\begin{array}{c} \sqrt{v^*} \\ 1 \sqrt{v^*} \end{array}\right). \quad (13.6)$$

Obviously we have $\varpi_{u/w}(\infty) = u/w$. Moreover, we have

$$\varpi_{u/w}^{-1} \left(\begin{array}{cc} 1 + \frac{\nu u}{w} & -\frac{\nu u^2}{w^2} \\ \nu & 1 - \frac{\nu u}{w} \end{array}\right) \varpi_{u/w} = \left(\begin{array}{c} u \\ \frac{u \bar{u} - 1}{w} \\ w \bar{u} \end{array}\right) \left(\begin{array}{cc} 1 + \frac{\nu u}{w} & -\frac{\nu u^2}{w^2} \\ \nu & 1 - \frac{\nu u}{w} \end{array}\right) = \left(\begin{array}{c} 1 - \frac{\nu}{w^2} \\ 1 \end{array}\right) = \tau_{v^*} \left(\begin{array}{c} 1 - \frac{\nu}{v^* w^2} \\ 1 \end{array}\right) \tau_{v^*}^{-1}. \quad (13.7)$$

Hence, on noting that $[w^2, q] = v^* w^2$, we get

$$\Gamma_{u/w} = \varpi_{u/w} \tau_{v^*} \Gamma_{\infty} \tau_{v^*}^{-1} \varpi_{u/w}^{-1}, \quad (13.8)$$

which is equivalent to

$$\Gamma_{u/w} = \varpi_{u/w} \left[ S_{u^*}^v \right] \varpi_{u/w}^{-1}. \quad (13.9)$$

14. In the the special instance where $q = v_i w_i$ with $(v_i, w_i) = 1$, we shall consider the structure of the double coset decomposition $\Gamma_{1/w_1} \backslash \Gamma_0(q) / \Gamma_{1/w_2}$ and associated Kloosterman sums.
To this end we put
\[
\sigma_{1/w_i} = \varpi_{1/w_i} \tau_{v_i} S^{-w_i}_{/v_i} = \varpi_{1/w_i} S^{-w_i}_{/v_i},
\]
(14.1)
where \( w_i \varpi_i \equiv 1 \mod v_i \). The choice of a particular value of \( \varpi_i \) is irrelevant to our discussion of the Kloosterman sums, as we shall show later. Note that
\[
\Gamma_{1/w_i} = \sigma_{1/w_i} \Gamma_\infty \sigma_{1/w_i}^{-1},
\]
as is implied by (13.8).

We shall prove that
\[
S^{\varpi_1} \varpi_{1/w_i}^{-1} \Gamma_0(q) \varpi_{1/w_2} S^{-\varpi_2} = \begin{cases} 
\left( \begin{array}{cc} (v_1, w_2)k & (v_1, v_2)l \\
(w_1, w_2)r & (w_1, v_2)s 
\end{array} \right) \in \text{SL}(2, \mathbb{Z}), k, l, r, s \in \mathbb{Z} 
\end{cases}
\]
(14.3)
(cf. [6, p. 534]; note that there \( q \) is square-free but here not assumed to be so). In fact, we have, by (13.5),
\[
\varpi_{1/w_i} S^{-\varpi_i} = \begin{pmatrix} 1 & -\varpi_i \\
\varpi_i & 1 - \varpi_i \varpi_i \end{pmatrix};
\]
(14.4)
thus for \( \begin{pmatrix} a & b \\
c & d \end{pmatrix} \in \Gamma_0(q) \)
\[
S^{\varpi_1} \varpi_{1/w_i}^{-1} \begin{pmatrix} a & b \\
c & d \end{pmatrix} \varpi_{1/w_2} S^{-\varpi_2} 
\equiv \begin{pmatrix} 0 & * \\
* & 0 \end{pmatrix} \begin{pmatrix} a & b \\
c & d \end{pmatrix} \begin{pmatrix} 0 & * \\
* & 0 \end{pmatrix} = \begin{pmatrix} 0 & * \\
* & 0 \end{pmatrix} \mod (v_1, w_2),
\]
\[
\equiv \begin{pmatrix} 0 & * \\
* & 0 \end{pmatrix} \begin{pmatrix} a & b \\
c & d \end{pmatrix} \begin{pmatrix} 0 & * \\
* & 0 \end{pmatrix} = \begin{pmatrix} 0 & * \\
* & 0 \end{pmatrix} \mod (v_1, v_2),
\]
\[
\equiv \begin{pmatrix} 0 & * \\
* & 0 \end{pmatrix} \begin{pmatrix} a & b \\
c & d \end{pmatrix} \begin{pmatrix} 0 & * \\
* & 0 \end{pmatrix} = \begin{pmatrix} 0 & * \\
* & 0 \end{pmatrix} \mod (w_1, w_2),
\]
\[
\equiv \begin{pmatrix} 0 & * \\
* & 0 \end{pmatrix} \begin{pmatrix} a & b \\
c & d \end{pmatrix} \begin{pmatrix} 0 & * \\
* & 0 \end{pmatrix} = \begin{pmatrix} 0 & * \\
* & 0 \end{pmatrix} \mod (w_1, v_2). \tag{14.5}
\]

On the other hand, we have that
\[
\varpi_{1/w_1} S^{-\varpi_1} \begin{pmatrix} (v_1, w_2)k & (v_1, v_2)l \\
(w_1, w_2)r & (w_1, v_2)s \end{pmatrix} S^{\varpi_2} \varpi_{1/w_2}^{-1} 
\equiv \begin{pmatrix} * & 0 \\
0 & * \end{pmatrix} \begin{pmatrix} * & * \\
* & * \end{pmatrix} \begin{pmatrix} 0 & * \\
* & 0 \end{pmatrix} = \begin{pmatrix} * & 0 \\
0 & * \end{pmatrix} \mod (v_1, w_2),
\]
and that \((v_1, w_2)(v_1, v_2)(w_1, w_2)(w_1, v_2) = q\). This proves (14.3).

Hence, we have

\[
\Gamma_{1/w_1} \backslash \Gamma_0(q)/\Gamma_{1/w_2} = \sigma_{1/w_1}^{-1} \backslash \Gamma_0(q)/\sigma_{1/w_2}^{-1} \backslash \Gamma_0(q)
\]

\[
\Longleftrightarrow \Gamma_\infty \backslash \tau_{w_1}^{-1} S_{w_1}^{-1} \backslash \Gamma_0(q) \backslash \tau_{w_2}^{-1} S_{w_2} \backslash \Gamma_\infty
\]

\[
\Longleftrightarrow \Gamma_\infty \backslash \left\{ \begin{pmatrix} (v_1, w_2)k & (v_1, v_2)l \\ (w_1, w_2)r & (w_1, v_2)s \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \right\} \tau_{w_2}/\Gamma_\infty
\]

\[
\Longleftrightarrow \Gamma_\infty \backslash \left\{ \begin{pmatrix} (v_1, w_2)k\sqrt{v_2/v_1} & (v_1, v_2)l/\sqrt{v_1v_2} \\ (w_1, w_2)r\sqrt{v_1v_2} & (w_1, v_2)s\sqrt{v_1/v_2} \end{pmatrix} \right\} /\Gamma_\infty
\]

\[
\Longleftrightarrow \text{classifying the solutions of } (v_1, w_2)(v_1, v_2)sk - (w_1, w_2)(v_1, v_2)rl = 1
\]

according to \((v_1, w_2)k\sqrt{v_2/v_1}, (w_1, v_2)s\sqrt{v_1/v_2} \mod (w_1, w_2)r\sqrt{v_1v_2};\)

\[
\text{note the remark after (8.6)}
\]

\[
\Longleftrightarrow \text{the moduli of the Kloosterman sums have the form } (w_1, w_2)r\sqrt{v_1v_2}
\]

with \(((v_1, w_2)(v_1, v_2), r) = 1 \text{ and}
\]

\((v_1, w_2)(v_1, v_2)sk \equiv 1 \mod (w_1, w_2)(v_1, v_2)r
\]

\((v_1, w_2)k \mod v_1(w_1, w_2)r \longleftrightarrow k \mod (v_1, v_2)(w_1, w_2)r
\]

\((w_1, v_2)s \mod v_2(w_1, w_2)r \longleftrightarrow s \mod (v_1, v_2)(w_1, w_2)r
\]

\[
\Longleftrightarrow c = (w_1, w_2)r\sqrt{v_1v_2}, ((v_1, w_2)(v_1, v_2), r) = 1,
\]

\[
S(m, n; c; 1/w_1, 1/w_2) = \sum_{s,k \mod (v_1, w_2)(v_1, v_2)r, (v_1, w_2)(v_1, v_2)sk \equiv 1 \mod (v_1, v_2)(w_1, w_2)r} \exp \left( \frac{2\pi i km + ns}{(v_1, v_2)(w_1, w_2)r} \right)
\]

\[
= S((v_1, w_2)m, (w_1, v_2)n; (v_1, v_2)(w_1, w_2)r),
\]
where the last member is an ordinary Kloosterman sum.

It remains to show the irrelevance of the choice of values of \( \overline{w}_j \). In fact, if we replace \( \overline{w}_j \) by \( \overline{w}_j + n v_j \), \( n \in \mathbb{Z} \), then the first equivalence assertion in (14.7) does not change, for we have \( \tau_{v_j}^{-1} S^{nv_j} \tau_{v_j} = S^n \in \Gamma_\infty \).

In particular, we find that if \( q = cd \), \( (c, d) = 1 \), and \( (r, d) = 1 \), then

\[
S(m, n; cr\sqrt{d}; 1/q, 1/c) = S(m, n; cr\sqrt{d}; \infty, 1/c) = S(m, dn; cr),
\]

on the specification (14.1) of \( \sigma_{1/q} \) and \( \sigma_{1/c} \).

15. We still need to see if (9.6) is satisfied by the generic \( \Gamma_0(q) \). Until very recently we had been unable to locate any rigorous treatment of those generalized Kloosterman sums over \( \Gamma_0(q) \) in literature, excepting [9] and [10] where the case with \( q \) square-free is explicitly discussed on the basis of (14.7). With this situation, R. Bruggeman kindly provided us with a treatment [1] of the sums using a partly adelic reasoning; and it is assured that (9.6) indeed holds with any \( \Gamma_0(q) \). Here we shall prove the same with an alternative elementary method; this section can be read independently of [1].

We shall first redefine the Kloosterman sums associated with the two cusps \( u_i/w_i \), \( i = 1, 2 \), which are in the set (12.7), by introducing the convention

\[
\sigma_{u_i/w_i} = \overline{w}_{u_i/w_i} \tau_{v_i^*},
\]

(15.1)

with \( v_i^* \) as in Section 13, which is effective within this section only. Note that when \( u_i = 1 \) this does not coincide with (14.1); when discussing the absolute values of generalized Kloosterman sums, obviously no difference is caused. Also, it is expedient to use the Bruhat decomposition; that is, in the big cell of \( \text{PSL}(2, \mathbb{R}) \) we have

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a/c \\ c & 1 \end{pmatrix} \begin{pmatrix} c & -1/c \\ 1 & d/c \end{pmatrix} = B[a, d; c],
\]

(15.2)
say.

With this, let \( \kappa_q \) be the characteristic function of the set \( \Gamma_0(q) \subset \text{PSL}(2, \mathbb{R}) \). Then Kloosterman sums associated with the two cusps \( u_i/w_i \), \( i = 1, 2 \), have moduli \( c\sqrt{v_1^*v_2^*} \), \( c \in \mathbb{N} \); and under (15.1) we have that

\[
S(m, n; c\sqrt{v_1^*v_2^*}; u_1/w_1, u_2/w_2)
= \sum_{ad \equiv 1 \mod c \atop \begin{array}{c} a \equiv 0 \mod v_1^*c \\ d \equiv 0 \mod v_2^*c \end{array}} \kappa_q \left( \overline{w}_{u_1/w_1} B[a, d; c] \overline{w}_{u_2/w_2}^{-1} \right) \exp \left( 2\pi i \left( \frac{ma}{v_1^*c} + \frac{nd}{v_2^*c} \right) \right),
\]

(15.3)
where \(a, c, d \in \mathbb{Z}\). In fact, by (13.8) we need to consider the double coset decomposition

\[
\Gamma_{\infty} \backslash \tau_{v_1^{-1}} \varpi_{\mu_{1/w_1}} \Gamma_0(q) \varpi_{\mu_{2/w_2}} \tau_{v_2^{-1}} / \Gamma_{\infty} = \Gamma_{\infty} \backslash \left\{ B[a, d; c] : \chi_q(\varpi_{u_{1/w_1}} B[a, d, c] \varpi_{u_{2/w_2}}^{-1}) = 1 \right\} / \Gamma_{\infty}, \tag{15.4}
\]

where \(B[a, d; c] \in \Gamma_0(1)\), since \(\varpi_{u_{1/w_1}} \Gamma_0(q) \varpi_{u_{2/w_2}} \subset \Gamma_0(1)\). The expression (15.3) readily follows. In passing, we note that

\[
|S(m, n; c\sqrt{v_1^*v_2^*}; u_{1/w_1}, u_{2/w_2})| \leq v_1^*v_2^*\varphi(c), \tag{15.5}
\]

for the number of summands on the right of (15.3) is less than or equal to \(v_1^*v_2^*\varphi(c)\). In fact, a unique \(d \mod c\) corresponds to each \(a\), \((a, c) = 1\), or \(v_2^*\) classes \(d \mod v_2^*c\) to each of \(v_1^*\varphi(c)\) classes \(a \mod v_1^*c\) with \((a, c) = 1\).

We remark that \(\chi_q\left(\varpi_{u_{1/w_1}} B[a, d; c] \varpi_{u_{2/w_2}}^{-1}\right)\) is a function over \(a \mod v_1^*c\) and \(d \mod v_2^*c\). To see this, we use the relation

\[
\varpi_{u_{1/w_1}} B[a + a', d + d'; c] \varpi_{u_{2/w_2}}^{-1} = \varpi_{u_{1/w_1}}^{-1} \varpi_{u_{1/w_1}} B[a, d; c] \varpi_{u_{2/w_2}}^{-1} \varpi_{u_{2/w_2}} \varpi_{u_{1/w_1}}^{-1} \varpi_{u_{1/w_1}} B[a, d; c] \varpi_{u_{2/w_2}}^{-1} \left(1 \begin{array}{c} d'/c \\ 1 \end{array}\right) \varpi_{u_{2/w_2}}^{-1} \varpi_{u_{2/w_2}} \varpi_{u_{1/w_1}}^{-1} \varpi_{u_{1/w_1}} B[a, d; c] \varpi_{u_{2/w_2}}^{-1}, \tag{15.6}
\]

and (13.9) gives that

\[
\varpi_{u_{1/w_1}}^{-1} \varpi_{u_{1/w_1}} B[a, d; c] \varpi_{u_{2/w_2}}^{-1} \varpi_{u_{2/w_2}} \varpi_{u_{1/w_1}}^{-1} \varpi_{u_{1/w_1}} B[a, d; c] \varpi_{u_{2/w_2}}^{-1} \in \Gamma_{u_{1/w_1}} \subset \Gamma_0(q),
\]

\[
\varpi_{u_{2/w_2}}^{-1} \varpi_{u_{2/w_2}} \varpi_{u_{1/w_1}}^{-1} \varpi_{u_{1/w_1}} B[a, d; c] \varpi_{u_{2/w_2}}^{-1} \varpi_{u_{2/w_2}} \varpi_{u_{1/w_1}}^{-1} \varpi_{u_{1/w_1}} B[a, d; c] \varpi_{u_{2/w_2}}^{-1} \varpi_{u_{2/w_2}} \varpi_{u_{1/w_1}}^{-1} \varpi_{u_{1/w_1}} B[a, d; c] \varpi_{u_{2/w_2}}^{-1} \varpi_{u_{2/w_2}} \varpi_{u_{1/w_1}}^{-1} \varpi_{u_{1/w_1}} B[a, d; c] \varpi_{u_{2/w_2}}^{-1}, \tag{15.7}
\]

provided \(v_1^*|\left(a'/c\right) \in \mathbb{Z}\), \(v_2^*|\left(d'/c\right) \in \mathbb{Z}\), which proves the assertion.

Next, we shall show that if \(ad \equiv 1 \mod c\), then

\[
\chi_q\left(\varpi_{u_{1/w_1}} B[a, d; c] \varpi_{u_{2/w_2}}^{-1}\right) = \chi_q\left(\varpi_{c^1u_{1/w_1}} B[a, d; c_0] \varpi_{c^1u_{2/w_2}}^{-1}\right), \tag{15.8}
\]

where \(c = c_0c^*\) with \(c_0 = (c, q^\infty)\), and \(c^*\) \equiv 1 \mod q; note that \(c^* u_{i/w_i}\) are cusps of \(\Gamma_0(q)\). In fact, computing the lower-left element of \(\varpi_{u_{1/w_1}} B[a, d; c] \varpi_{u_{2/w_2}}^{-1}\), we see that the value of the left side of (15.8) equals 1 if and only if

\[
\varpi_{u_{2}}(aw_1 + c\overline{m_1}) \equiv w_2 (w_1(ad - 1)/c + d\overline{m_1}) \mod q, \tag{15.9}
\]
and this is equivalent to the congruence
\[ \overline{c^*} w_2 (aw_1 + c_0 \overline{c^*} u_1) \equiv w_2 \left( w_1 (ad - 1) / c_0 + \overline{c^*} u_1 \right) \mod q, \] (15.10)
which immediately implies (15.8).

Hence we have
\[
S(m, n; c \sqrt{v_1^* v_2^*}; u_1 / w_1, u_2 / w_2) = \sum_{ad \equiv 1 \mod c \atop a \mod v_1^* c \atop d \mod v_2^* c} \kappa_q \left( \overline{w c^* u_1 / w_1} B[a, d; c_0] \overline{w c^* u_2 / w_2} \right) \exp \left( 2\pi i \left( \frac{ma}{v_1^* c} + \frac{nd}{v_2^* c} \right) \right). \]
(15.11)
Here we have
\[ \frac{1}{v_i^* c} \equiv \frac{c_i^*}{v_i^* c_0} + \frac{\widetilde{c}_i^* c_0}{c^*} \mod 1, \]
(15.12)
with \( c_i^* \equiv 1 \mod v_i c_0 \), \( v_i \widetilde{c}_i c_0 \equiv 1 \mod c^* \). Inserting this into (15.11), putting \( a \equiv a_0 \mod v_1^* c_0 \), \( a \equiv a^* \mod c^* \), \( d \equiv d_0 \mod v_2^* c_0 \), \( d \equiv d^* \mod c^* \), and further, noting the congruence property of \( \kappa_q \) proved in (15.6)–(15.7), we may write (15.11) as
\[
S(m, n; c \sqrt{v_1^* v_2^*}; u_1 / w_1, u_2 / w_2) = \sum_{ad \equiv 1 \mod c \atop a \mod v_1^* c \atop d \mod v_2^* c} \kappa_q \left( \overline{w c^* u_1 / w_1} B[a_0, d_0; c_0] \overline{w c^* u_2 / w_2} \right) \times \exp \left( 2\pi i \left( \frac{c_1^* ma_0}{v_1^* c_0} + \frac{\widetilde{c}_1^* nd_0}{v_2^* c_0} \right) \right) \times \exp \left( \frac{\tau}{2} \left( \frac{\widetilde{c}_1^* c_0 a^*}{c^*} + \frac{\widetilde{c}_2^* c_0 d^*}{c^*} \right) \right). \]
(15.13)
We have thus obtained the factorization
\[
S(m, n; c \sqrt{v_1^* v_2^*}; u_1 / w_1, u_2 / w_2) = S(c_1^* m, c_2^* n; c_0 \sqrt{v_1^* v_2^*}; c^* u_1 / w_1, c^* u_2 / w_2) S(v_1^* c_0 m, v_2^* c_0 n; c^*), \]
(15.14)
where the last \( S \)-factor is an ordinary Kloosterman sum.

In particular, applying (15.5) and the Weil bound, respectively, to the first and the second factors on the right side of (15.14), we get
\[
|S(m, n; c \sqrt{v_1^* v_2^*}; u_1 / w_1, u_2 / w_2)| \leq v_1^* v_2^* \varphi(c_0)|S(v_1^* c_0 m, v_2^* c_0 n; c^*)| \ll v_1^* v_2^* c_0 ((m, n, c^*) c^*)^{1/2 + \varepsilon}, \]
(15.15)
with the implied constant depending only on \( \varepsilon \). Thus we have, for any \( \xi > \frac{1}{2} \),
\[
\sum_{c \equiv \xi \mod q^\infty} \frac{1}{c^{1/2 + \varepsilon}} |S(m, n; c \sqrt{v_1^* v_2^*}; u_1 / w_1, u_2 / w_2)| \ll (v_1^* v_2^*)^{1 - \frac{1}{2} + \varepsilon} \left( \sum_{c \equiv \xi \mod q^\infty} \frac{1}{c^{-1/2}} \right) \left( \sum_{c \equiv \xi \mod q^\infty} \frac{(m, n, c)^{\xi}}{c^{\xi}} \right), \]
(15.16)
which is finite if $\tau - \xi > 1$. Therefore, we have proved that any $\Gamma_0(q)$ satisfies (9.6) with $\tau > \frac{3}{2}$.

**Remark 3.** The methods in [2] and [12] extend to $M_2(g; A)$ with an arbitrary $A$. Since they are independent of any non-trivial treatment of generalized Kloosterman sums, the above confirmation of (9.6) for generic $\Gamma_0(q)$ could be regarded as redundant, as far as the spectral decomposition of $M_2(g; A)$ is concerned.

**16.** With this, we return to the second line of (6.5). We stress that hereafter we shall again work with the definition (14.1).

In view of (14.8) we have

$$Y_\pm(u, v, w, z; g; d/c; m, n) = \sum_{(k, d)} \frac{1}{ck\sqrt{d}} S(n, \pm m; ck\sqrt{d}; \infty, 1/c) \tilde{g}_\pm(u, v, w, z; 4\pi \sqrt{mn}/ck\sqrt{d}). \quad (16.1)$$

Thus Lemma 4 gives the expansion

$$Y_\pm(u, v, w, z; g; d/c; m, n) = \sum_{j=1}^{\infty} [g]_\pm(\kappa_j; u, v, w, z) \frac{\vartheta_j(n, \infty) \vartheta_j(\pm m, 1/c)}{\cosh \pi \kappa_j}$$

$$+ \frac{1 \pm 1}{4\pi (4\pi \sqrt{mn})^{2k-1}} \sum_{k=1}^{\infty} \Gamma(2k)[g]_+((\frac{1}{2} - 2k) i; u, v, w, z) \sum_{j=1}^{\vartheta(k)} \vartheta_{j,k}(n, \infty) \vartheta_{j,k}(m, 1/c)$$

$$+ \frac{1}{\pi} \sum_{\epsilon} \int_{-\infty}^{\infty} [g]_\pm(r; u, v, w, z)(m/n)^i e_n(\frac{1}{2} + ir; \epsilon, \infty) e_m(\frac{1}{2} + ir; \epsilon, 1/c) dr, \quad (16.2)$$

where

$$[g]_+(r; u, v, w, z) = \frac{\pi i}{2 \sinh \pi r} \int_{0}^{\infty} \{ J_{2ir}(x) - J_{-2ir}(x) \} \tilde{g}_+(u, v, w, z; x) \frac{dx}{x},$$

$$[g]_-(r; u, v, w, z) = 2 \cosh(\pi r) \int_{0}^{\infty} K_{2ir}(x) \tilde{g}_-(u, v, w, z; x) \frac{dx}{x}. \quad (16.3)$$

Further, by (6.6)–(6.7) we have that

$$\frac{(2\pi)^{u-w+1}}{2v^d \frac{1}{2}(u+v-w+z)} J_+^*(u, v, w, z; g; d/c)$$

$$= \sum_{\pm} \sum_{j=1}^{\infty} \frac{[g]_\pm(\kappa_j; u, v, w, z)}{\cosh \pi \kappa_j} \left\{ \sum_m \frac{\vartheta_j(n, \infty) \sigma_{w+z-1}(n)}{n^\frac{1}{2}(u+v+w+z-1)} \right\}$$

$$\times \left\{ \sum_{n} \frac{\vartheta_j(\pm n, 1/c)}{n^\frac{1}{2}(u-v-w+z+1)} \right\}$$
\begin{align*}
&+ \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{(2k-1)!}{(4\pi)^{2k-1}} [g]_{\pm} \left( \left( \frac{1}{2} - 2k \right) i; u, v, w, z \right) \left\{ \sum_{n} \frac{\theta_{j,k}(n, \infty)\sigma_{w+z-1}(n)}{n^{k-\frac{1}{2}}n^{\frac{1}{2}(u+v+w+z-1)}} \right\} \\
&\quad \times \left\{ \sum_{n} \frac{\theta_{j,k}(\pm n, 1/c)}{n^{k-\frac{1}{2}}n^{\frac{1}{2}(u-v-w+z+1)}} \right\} \\
&+ \frac{1}{\pi} \sum_{\pm} \sum_{c} \int_{-\infty}^{\infty} \left[ g \right]_{\pm}(r; u, v, w, z) \left\{ \sum_{m} \frac{e_{n}\left( \frac{1}{2} + ir; c, \infty \right)\sigma_{w+z-1}(n)}{n^{\frac{1}{2}(u+v+w+z-1)+ir}} \right\} \\
&\quad \times \left\{ \sum_{n} \frac{e_{n}\left( \frac{1}{2} + ir; c, 1/c \right)}{n^{\frac{1}{2}(u-v-w+z+1)-ir}} \right\} dr,
\end{align*}

(16.4)

as Lemma 2 and the rapid decay of \( [g]_{\pm}(r; u, v, w, z) \) yield absolute convergence on the right side, provided (4.1) (see [11, Section 4.5]).

17. We need to continue (16.4) to a neighbourhood of the point \((u, v, w, z) = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)\). The continuation of \([g]_{\pm}\) is known already ([11, Section 4.6]), and we are concerned with the nature of \(L\)-functions:

\begin{align*}
L_{j}^{\pm}(s; 1/c) &= \sum_{n} \varrho_{j}(\pm n, 1/c)n^{-s}, \\
D_{j}(s, \alpha) &= \sum_{n} \varrho_{j}(n, \infty)\sigma_{\alpha}(n)n^{-s}, \\
L_{j,k}(s; 1/c) &= \sum_{n} \varrho_{j,k}(n, 1/c)n^{-s-k+\frac{1}{2}}, \\
D_{j,k}(s, \alpha) &= \sum_{n} \varrho_{j,k}(n, \infty)\sigma_{\alpha}(n)n^{-s-k+\frac{1}{2}},
\end{align*}

(17.1)

where the sums converge absolutely if \(\text{Re } s\) is sufficiently large, because of (11.2). We shall especially require uniform bounds for these functions. The Dirichlet series involved in the last integral are to be discussed in detail later, but under the restriction on \(A\) mentioned in the introduction.

In our continuation procedure of the right side of (16.4), we exploit the fact that above \(L\)-functions admit meromorphic continuation to \(\mathbb{C}\) with respect to \(s\), and with respect to \(\alpha\) as well in the second and the fourth \(L\)-functions. To reach (16.4) we appealed to Lemma 4, and hence the bound (9.6) becomes crucial. Moreover, the contribution of the continuous spectrum in (16.4) makes it clear how important for us to have explicit representation of Fourier coefficients of Eisenstein series at each cusp, and this is of course closely related to the structure of generalized Kloosterman sums which is partly discussed in Section 15.

We begin with relations between \(\sigma_{\alpha}\) defined by (14.1) and the two basic involutions \(J : z \mapsto -\overline{z}\), and \(F_{q} : z \mapsto -1/qz\), which satisfy

\begin{align*}
JJ_{0}(q)J^{-1} &= J_{0}(q), \\
F_{q}J_{0}(q)F_{q}^{-1} &= J_{0}(q).
\end{align*}

(17.2)
We have
\[ J\sigma_a = \gamma_1\sigma_{b_1}S^{b_1}, \quad F_q\sigma_a = \gamma_2\sigma_{b_2}S^{b_2}, \quad \gamma_1, \gamma_2 \in \Gamma_0(q), \quad b_1, b_2 \in \mathbb{R}, \] (17.3)
where \( J(a), F_q(a) \) are equivalent to \( b_1, b_2 \), respectively. For instance, the latter identity is due to the fact that the stabilizer of \( b_2 \) is
\[ (\gamma_2^{-1}F_q\sigma_a)\Gamma_\infty(\gamma_2^{-1}F_q\sigma_a)^{-1} = \gamma_2^{-1}F_q\Gamma_a F_q^{-1}\gamma_2 \subset \Gamma_0(q) \] (17.4)
(see the remark made prior to (7.1)).

The reflection operator \( J \) is isometric over \( L^2(\Gamma \backslash \mathcal{H}) \), for \( J(\Gamma \backslash \mathcal{H}) \) is a fundamental domain, and
\[ \|\psi J\|^2 = \int_{\Gamma \backslash \mathcal{H}} |\psi J|^2d\mu = \int_{J(\Gamma \backslash \mathcal{H})} |\psi|^2d\mu = \int_{\Gamma \backslash \mathcal{H}} |\psi|^2d\mu = \|\psi\|^2. \] (17.5)
Besides, we have \( J\Delta = \Delta J \) as well as the first relation in (17.3). Hence \( \psi_j J \) is a cusp form belonging to the same eigenspace as \( \psi_j \), for \( \psi_j J(\sigma_a(z)) = \psi_j(\sigma_{b_1}(z + b_1)) \) converges to 0 as \( z \) tends to \( \infty \). Thus \( J \) can be diagonalized on each eigenspace of \( \Delta \); that is, we may choose an orthonormal base \( \{\psi_j\} \) in such a way that
\[ \psi_j(-z) = \epsilon_j\psi_j(z), \quad \epsilon_j = \pm 1. \] (17.6)

Also, we observe that
\[ J\sigma_{1/c}J\sigma_{1/c}^{-1} = \begin{pmatrix} \sqrt{d} & -f/\sqrt{d} \\ -c\sqrt{d} & (1 + cf)/\sqrt{d} \end{pmatrix} \begin{pmatrix} (1 + cf)/\sqrt{d} & -f/\sqrt{d} \\ -c\sqrt{d} & \sqrt{d} \end{pmatrix} = \begin{pmatrix} 1 + 2cf & -2f \\ -2c(1 + cf) & 1 + 2cf \end{pmatrix} \in \Gamma_0(cd). \] (17.7)
This implies that
\[ \psi_j(J\sigma_{1/c}J(z)) = \psi_j(\sigma_{1/c}(z)) \iff \epsilon_j\psi_j(\sigma_{1/c}(-z)) = \psi_j(\sigma_{1/c}(z)); \] (17.8)
namely
\[ g_j(-n, 1/c) = \epsilon_j g_j(n, 1/c). \] (17.9)
In particular, we have
\[ L^+_j(s; 1/c) = \epsilon_j L^+_j(s; 1/c). \] (17.10)

Next, we consider the action of the Fricke operator \( F_q \). We put \( F = F_{cd} \). Then each \( \psi_j F \) is \( \Gamma_0(cd) \)-invariant, and is a cusp form such that \( \Delta\psi_j F = \nu_j\psi_j F \); in fact it is a unit vector as
\[ \int_{\Gamma \backslash \mathcal{H}} |\psi_j F(z)|^2d\mu(z) = \int_{F\Gamma \backslash \mathcal{H}} |\psi_j(z)|^2d\mu(z) = \int_{\Gamma \backslash \mathcal{H}} |\psi_j(z)|^2d\mu(z) = 1. \] (17.11)
for \( F \Gamma \backslash \mathcal{H} \) is a fundamental domain of \( \Gamma = \Gamma_0(cd) \); moreover, \( \psi_j F(\sigma_a(z)) = \psi_j(\sigma_{b_2}(z + b_2)) \) converges to 0 as \( z \) tends to \( \infty \). Since \( FJ = JF \), we may assume, besides (17.6), that

\[
\psi_j F = \varpi_j \psi_j, \quad \varpi_j = \pm 1. \tag{17.12}
\]

Further, we observe

\[
\begin{align*}
\sigma_{1/c} F^{-1} \sigma_{1/c} F &= \frac{1}{cd} \begin{pmatrix} \sqrt{d} & f/\sqrt{d} \\ c\sqrt{d} & (1+cf)/\sqrt{d} \end{pmatrix} \begin{pmatrix} -1 \\ cd \end{pmatrix} \begin{pmatrix} (1+cf)/\sqrt{d} & -f/\sqrt{d} \\ -c\sqrt{d} & \sqrt{d} \end{pmatrix} \begin{pmatrix} cd \end{pmatrix} \\
&= \frac{1}{cd} \begin{pmatrix} cf\sqrt{d} & -\sqrt{d} \\ c(1+cf)\sqrt{d} & -c\sqrt{d} \end{pmatrix} \begin{pmatrix} -cf\sqrt{d} & -(1+cf)/\sqrt{d} \\ cd\sqrt{d} & c\sqrt{d} \end{pmatrix} \\
&= \frac{1}{cd} \begin{pmatrix} -(cf)^2d - cd^2 & -cf - (cf)^2 - cd \\ c^2f(1+cf)d - (cd)^2 & -c(1+cf)^2 - c^2d \end{pmatrix} \\
&= \begin{pmatrix} -cf^2 - d & -1 - (1+cf)/d \\ -cd(1+ f(1+cf)/d) & c - (1+cf)^2/d \end{pmatrix} \in \Gamma_0(cd). \tag{17.13}
\end{align*}
\]

Hence we have

\[
\psi_j(\sigma_{1/c} F(z)) = \psi_j(F^{-1} \sigma_{1/c}(z)) = \psi_j(\sigma_{1/c}(z)) = \varpi_j \psi_j(\sigma_{1/c}(z)); \tag{17.14}
\]

that is, we have

\[
\psi_j (\sigma_{1/c} (-1/cdz)) = \varpi_j \psi_j (\sigma_{1/c} (z)). \tag{17.15}
\]

18. We may now prove the functional equation for \( L_j(s; 1/c) = L_j^+(s; 1/c) \); note that we have (17.10). We have to discuss two cases separately according as \( \epsilon_j = +1 \) or \(-1\).

The case \( \epsilon_j = +1 \): We have, by (17.9),

\[
\begin{align*}
\int_0^\infty \psi_j \left( \sigma_{1/c} \left( \frac{iy}{\sqrt{cd}} \right) \right) y^{s-\frac{3}{2}} dy &= 2 \int_0^\infty \left( \frac{y}{\sqrt{cd}} \right)^{\frac{1}{2}} \sum_{n > 0} \theta_j(n, 1/c) K_{ik_j} \left( 2\pi \frac{n}{\sqrt{cd}} y \right) y^{s-\frac{3}{2}} dy \\
&= 2^{s-1}(cd)^{-\frac{1}{4}} \left( \frac{2\pi}{\sqrt{cd}} \right)^{-s} \Gamma \left( \frac{1}{2}(s + i\kappa_j) \right) \Gamma \left( \frac{1}{2}(s - i\kappa_j) \right) L_j(s; 1/c). \tag{18.1}
\end{align*}
\]
On the other hand, by (17.15),
\[
\int_0^\infty \psi_j \left( \sigma_{1/c} \left( \frac{iy}{\sqrt{cd}} \right) \right) y^{s-\frac{i}{2}} dy = \int_1^\infty \psi_j \left( \sigma_{1/c} \left( \frac{iy}{\sqrt{cd}} \right) \right) y^{s-\frac{i}{2}} dy + \int_1^\infty \psi_j \left( \sigma_{1/c} \left( \frac{i}{\sqrt{cdy}} \right) \right) y^{1-s-\frac{i}{2}} dy
\]
\[
= \int_1^\infty \left\{ \psi_j \left( \sigma_{1/c} \left( \frac{iy}{\sqrt{cd}} \right) \right) y^{s-\frac{i}{2}} + \bar{\psi}_j \psi_j \left( \sigma_{1/c} \left( \frac{i}{\sqrt{cdy}} \right) \right) y^{1-s-\frac{i}{2}} \right\} dy,
\]
which is entire in \( s \), for \( \psi_j \sigma_{1/c} \) decays exponentially as \( y \) tends to \(+\infty\). Namely, the function \( L_j(s; 1/c) \) is entire, and we have
\[
\left( \frac{\pi}{\sqrt{cd}} \right)^{-s} \Gamma \left( \frac{1}{2}(s + i\kappa_j) \right) \Gamma \left( \frac{1}{2}(s - i\kappa_j) \right) L_j(s; 1/c) = \bar{\omega}_j \left( \frac{\pi}{\sqrt{cd}} \right)^{1-s} \Gamma \left( \frac{1}{2}(-1 + s + i\kappa_j) \right) \Gamma \left( \frac{1}{2}(-1 - s - i\kappa_j) \right) L_j(1-s; 1/c),
\]
(18.3)

By the duplication formula for \( \Gamma \)-function, one may transform this relation into
\[
L_j(s; 1/c) = \frac{\bar{\omega}_j}{\pi} \left( \frac{2\pi}{\sqrt{cd}} \right)^{2s-1} \Gamma(1-s+i\kappa_j)\Gamma(1-s-i\kappa_j)
\]
\[
\times (\cosh \pi \kappa_j - \cos \pi s) L_j(1-s; 1/c).
\]
(18.4)

The case \( \epsilon_j = -1 \): We have
\[
\psi_j(\sigma_{1/c}(z)) = 2i\sqrt{y} \sum_{n>0} \varrho_j(n, 1/c) K_{i\kappa_j}(2\pi ny) \sin(2\pi nx),
\]
(18.5)
We put \( f_j(z) = \partial_z \psi_j(\sigma_{1/c}(z - \tau/d)) \). We have
\[
f_j(z) = 4\pi i \sqrt{y} \sum_{n>0} n\varrho_j(n, 1/c) K_{i\kappa_j}(2\pi ny) \cos(2\pi nx),
\]
(18.6)
which implies that as \( x \to 0 \)
\[
\psi_j(\sigma_{1/c}(z)) = f_j(iy)x + O(x^2)
\]
(18.7)
as well as
\[
\psi_j \left( \sigma_{1/c}(-1/cdz) \right) = \psi_j \left( \sigma_{1/c}(i/cdy - x/cdy^2 + O(x^2)) \right) = -(x/cdy^2)f_j(i/cdy) + O(x^2);
\]
(18.8)
that is,
\[
f_j(i/cdy) = -\bar{\omega}_j cdy^2 f_j(iy).
\]
(18.9)
Hence,
\[
\int_0^\infty f_j \left( \frac{i y}{\sqrt{c d}} \right) y^{s-\frac{1}{2}} dy = \pi i (c d)^{-\frac{1}{4}} \left( \frac{\pi}{\sqrt{c d}} \right)^{-s-1} \Gamma \left( \frac{1}{2} (1 + s + i \kappa_j) \right) \Gamma \left( \frac{1}{2} (1 + s - i \kappa_j) \right) L_j(s; 1/c); \tag{18.10}
\]
and
\[
\int_0^\infty f_j \left( \frac{i y}{\sqrt{c d}} \right) y^{s-\frac{1}{2}} dy = \int_1^\infty \left\{ f_j \left( \frac{i y}{\sqrt{c d}} \right) y^{s-\frac{1}{2}} - \bar{\omega}_j f_j \left( \frac{i y}{\sqrt{c d}} \right) y^{1-s-\frac{1}{2}} \right\} dy. \tag{18.11}
\]
Namely, we have that
\[
L_j(s; 1/c) = -\frac{\bar{\omega}_j}{\pi} \left( \frac{2\pi}{\sqrt{c d}} \right)^{2s-1} \Gamma(1-s+i\kappa_j)\Gamma(1-s-i\kappa_j) \\
\times (\cosh \pi \kappa_j + \cos \pi s) L_j(1-s; 1/c). \tag{18.12}
\]

Lemma 6. The function $L_j(s; 1/c)$ is entire, and it holds that for any $s$
\[
L_j(s; 1/c) = \frac{\bar{\omega}_j}{\pi} \left( \frac{2\pi}{\sqrt{c d}} \right)^{2s-1} \Gamma(1-s+i\kappa_j)\Gamma(1-s-i\kappa_j) \\
\times (\epsilon_j \cosh \pi \kappa_j - \cos \pi s) L_j(1-s; 1/c). \tag{18.13}
\]
We have also
\[
L_j(s; 1/c) \ll (\kappa_j + |s| + 1)^{c_0} \exp \left( \frac{\pi}{2} \pi \kappa_j \right), \tag{18.14}
\]
where the constant $c_0$ depends at most on $\text{Re } s$, and the implied constant on $\text{Re } s$.

The second assertion follows via a convexity argument.

We may omit the discussion on $L_{j,k}$, as it is analogous to $L_j$.

19. We turn to $D_j(s, \alpha)$. There are at least two possible ways for us to take here. One is to exploit the theory of Hecke operators in order to relate $D_j$ with a product of two values of Hecke $L$-functions analogously as we did in the case of $M_2(g; 1)$ in [11]. However, the cusp form $\psi_j$ cannot generally be assumed to be such that the corresponding Hecke series is fully decomposed into an Euler product. This is because those $\rho_j(n, \infty)$ with $n|\langle c d \rangle \infty$ are not well related to eigenvalues of Hecke operators, and thus the corresponding part of $D_j(s, \alpha)$ causes difficulties in the continuation as well as the estimation procedures, which is a serious drawback of the method as far as our present purpose is concerned. One may appeal to the notion of new forms whose Hecke series admits a full Euler product; yet it does not seem to resolve our difficulties. Hence, we shall take the second method which is in fact a special instance of applications of Rankin’s unfolding method (see [11, pp. 181–182]). This causes, however, still a technical difficulty, for it requires us to have an explicit description of the scattering matrix of $\Gamma_0(q)$ and all Fourier
coefficients of Eisenstein series at each cusp (see (24.1) below). This task is highly involved. The note [1] contains, in fact, a discussion of the arithmetical nature of those Fourier coefficients and the result appears to be essentially adequate for our purpose, if we let our reasoning in the later sections be somewhat inexplicit; note that the same can be done by extending (15.14) to a full localization. Under such a circumstance, it may be appropriate for us to make here a compromise by introducing the assumption that $A$ is defined by a sum over square-free integers, as underlined in the introduction. Since we have (14.7), this eases our task considerably, yet it does not seem to restrict the scope of our method. In the future, we shall work out a fuller account of $M_2(d; A)$.

20. Thus, we shall hereafter assume that

$$q = cd$$

is square-free. (20.1)

By Lemma 5 in Section 12, we have now

$$\{\text{inequivalent cusps of } \Gamma_0(q)\} = \left\{ \frac{1}{w} : w|q \right\};$$

and we have (14.7) for any combination of cusps. In particular, for those Hecke congruence groups that are relevant in the sequel, (9.6) and thus Lemmas 2–4 have been verified, without the discussion in Section 15.

To make Lemmas 3–4 more explicit, let us compute the Fourier coefficients of Eisenstein series at each cusp. Thus, by the assertion (14.7),

$$E(\sigma_1/w_2(z), 1/w_1; s)$$

$$= \delta_{w_1,w_2} y^s + \sqrt{\pi} y^{1-s} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_{(v_1,v_2)\cdot (w_1,w_2),r=1} \phi((v_1,v_2)(w_1,w_2)r) (w_1,w_2)r \sqrt{w_1w_2}^{2s}$$

$$+ 2\sqrt{\pi} \frac{s}{\Gamma(s)} \sum_{n \neq 0} \exp(2\pi inx) K_{s-\frac{1}{2}}(2\pi |n|y) |n|^{s-\frac{1}{2}} \sum_{(v_1,v_2)\cdot (w_1,w_2),r=1} \frac{c(v_1,v_2)(w_1,w_2)r(n)}{(w_1,w_2)r \sqrt{w_1w_2}^{2s}},$$

where the last numerator is a Ramanujan sum. We have

$$\sum_{(v_1,v_2)\cdot (w_1,w_2),r=1} \frac{\phi((v_1,v_2)(w_1,w_2)r)}{(w_1,w_2)r \sqrt{w_1w_2}^{2s}}$$

$$= \frac{1}{(w_1,w_2)^{2s}(v_1,v_2)^{s}} \left\{ \sum_{r \cdot ((v_1,v_2),(w_1,w_2)) \rightarrow \infty} \frac{\phi((v_1,v_2)(w_1,w_2)r)}{r^{2s}} \right\} \left\{ \sum_{(r,q) = 1} \frac{\phi(r)}{r^{2s}} \right\}$$

$$= \frac{1}{(w_1,w_2)^{2s}(v_1,v_2)^{s}} \prod_{p \mid (v_1,v_2)} \left( \sum_{j=0}^{\infty} \frac{\phi(p^{j+1})}{p^{2js}} \right) \prod_{p \mid (w_1,w_2)} \left( \sum_{j=0}^{\infty} \frac{\phi(p^{j+1})}{p^{2js}} \right) \left\{ \sum_{(r,q) = 1} \frac{\phi(r)}{r^{2s}} \right\}$$

$$= \frac{\zeta(2s-1)}{\zeta(2s)} \prod_{p \mid (v_1,v_2),(w_1,w_2)} \left( \frac{p-1}{p^{2s}-1} \right) \prod_{p \mid (v_1,w_2),(w_1,v_2)} \left( \frac{p^{s-1-1}}{p^{2s}-1} \right).$$
Next,

$$\sum_{((v_1, w_2), (w_1, v_2), r)} \frac{c(v_1, v_2)(w_1, w_2)r(n)}{((w_1, w_2)r \sqrt{v_1 v_2})^{2s}} = \frac{1}{(w_1, w_2)^{2s}(v_1 v_2)^s} \left\{ \sum_{r|((v_1, v_2)(w_1, w_2))} \frac{c(v_1, v_2)(w_1, w_2)r(n)}{r^{2s}} \right\} \left\{ \sum_{(r, q) = 1} \frac{c_r(n)}{r^{2s}} \right\}.$$

(20.5)

We have

$$\sum_{r|((v_1, v_2)(w_1, w_2))} \frac{c(v_1, v_2)(w_1, w_2)r(n)}{r^{2s}} = \prod_{p|((v_1, v_2)(w_1, w_2))} p^{2s} \left\{ \sum_{j=0}^{\infty} \frac{c(p^j)(n)}{p^{2js}} - 1 \right\}$$

$$=((v_1, v_2)(w_1, w_2))^{2s} \prod_{p|((v_1, v_2)(w_1, w_2))} \left\{ \sigma_{1-2s}(n_p) \left( 1 - \frac{1}{p^{2s}} \right) - 1 \right\}$$

(20.6)

and

$$\sum_{(r, q) = 1} \frac{c_r(n)}{r^{2s}} = \frac{\sigma_{1-2s}(n, \chi_q)}{L(2s, \chi_q)},$$

(20.7)

where \(n_p = (n, p^\infty)\) and \(\chi_q\) is the principal character mod \(q\). Thus,

$$\sum_{((v_1, w_2)(w_1, v_2), r)} \frac{c(v_1, v_2)(w_1, w_2)r(n)}{((w_1, w_2)r \sqrt{v_1 v_2})^{2s}} = \frac{\sigma_{1-2s}(n, \chi_q)}{L(2s, \chi_q)} \left( \frac{(v_1, v_2)}{[v_1, v_2]} \right)^s \prod_{p|((v_1, v_2)(w_1, w_2))} \left\{ \sigma_{1-2s}(n_p) \left( 1 - \frac{1}{p^{2s}} \right) - 1 \right\}.$$

(20.8)

Collecting these assertions, we obtain in particular that

**Lemma 7.** The function \(s(1-s)\Gamma(s)L(2s, \chi_{cd})E(\sigma_{1/w_2}(z), 1/w_1; s)\) is regular for all \(s\), and it is \(\ll y^{\text{Res}} + y^{1-\text{Res}}\) as \(y = \text{Re} z\) tends to infinity, as far as \(s\) remains bounded.

21. Lemma 2 holds safely for \(\Gamma = \Gamma_0(q), \mu(q) \neq 0\), and Lemmas 3 and 4 become as follows (see [10]):

**Lemma 8.** Let \(h(r)\) be even, regular and of fast decay on the strip \(|\text{Im} r| < \frac{1}{2} + \eta\) with an \(\eta > 0\). Then it holds that for any \(m, n > 0\) and \(w_1|q, w_2|q\)

$$\sum_{j=1}^{\infty} \frac{\varrho_j(m, 1/w_1)\varrho_j(\pm n, 1/w_2)}{\cosh \pi \kappa_j} h(\kappa_j)$$
\[ + \frac{1}{\pi} \sum_{q = v, w} \int_{-\infty}^{\infty} \left( \frac{n}{m} \right)^{ir} \frac{\sigma_{2ir}(m; \chi_q) \sigma_{-2ir}(n; \chi_q)}{|L(1 + 2ir, \chi_q)|^2} \left( \frac{v}{v, v_1} \right)^{\frac{1}{2} - ir} \left( \frac{v}{v, v_2} \right)^{\frac{1}{2} + ir} \]

\[ \times \prod_{p|(v, v_1)(w, w_1)} \left\{ \sigma_{2ir}(m_p) \left( 1 - \frac{1}{p^{1 - 2ir}} \right) - 1 \right\} \]

\[ \times \prod_{p|(v, v_2)(w, w_2)} \left\{ \sigma_{-2ir}(n_p) \left( 1 - \frac{1}{p^{1 + 2ir}} \right) - 1 \right\} h(r) dr \]

\[ = \frac{1}{\pi^2} \delta_{w_1, w_2} \delta_{m, \pm n} \exp(2\pi i m b w_1, w_2) \int_{-\infty}^{\infty} r \tanh(\pi r) h(r) dr \]

\[ + \sum_{(r, (v, w_1)(w_1, v_2)) = 1} \frac{1}{(w_1, w_2)r \sqrt{v_1 v_2}} \]

\[ \times S((v_1, w_2)m, \pm (w_1, v_2)n; (v_1, v_2)(w_1, w_2)r) h_{\pm} \left( \frac{4\pi \sqrt{mn}}{(w_1, w_2)r \sqrt{v_1 v_2}} \right), \quad (21.1) \]

with \( h_{\pm} \) as in (11.4).

**Lemma 9.** Let \( \varphi \) be smooth and of fast decay over the positive real axis. Then we have, for any \( m, n > 0 \) and \( w_1|q, w_2|q \),

\[ \sum_{(r, (v, w_1)(w_1, v_2)) = 1} \frac{\varphi_{j}(m, 1/w_1) \varphi_{j}(\pm n, 1/w_2)}{\cosh \pi \kappa_j} \cdot \frac{\varphi_{\pm}(\kappa_j)}{\varphi_{\pm}(\kappa_j)} \]

\[ = \sum_{j=1}^{\infty} \frac{1 \pm \Gamma(2k) \varphi_{\pm}((1/2 - 2k) i)}{4\pi (4\pi \sqrt{mn})^{2k-1}} \sum_{k=1}^{\infty} \varphi_{j,k}(m, 1/w_1) \varphi_{j,k}(n, 1/w_2) \]

\[ \quad + \sum_{k=1}^{\infty} \left( \frac{1}{\pi} \right)^{\frac{1}{2} + k} \frac{1}{(u - w + 1) \sqrt{(u - w + 1) + 1}} \]

\[ \times \prod_{p|(v, v_1)(w, w_1)} \left\{ \sigma_{2ir}(m_p) \left( 1 - \frac{1}{p^{1 - 2ir}} \right) - 1 \right\} \]

\[ \times \prod_{p|(v, v_2)(w, w_2)} \left\{ \sigma_{-2ir}(n_p) \left( 1 - \frac{1}{p^{1 + 2ir}} \right) - 1 \right\} \varphi_{\pm}(r) dr, \quad (21.2) \]

where \( \varphi_{\pm} \) are as in (11.6).

We specialize the last assertion as in (14.8), and have, in place of (16.4),

\[ \frac{(2\pi)^{u-w+1}}{2e^{u}d^{\frac{1}{2}(u+v-w+z)}} J^*_+(u, v, w, z; g; d/c) \]
\[ = \sum_{\pm} \sum_{j=1}^{\infty} \frac{[g]_{\pm}(\kappa_j; u, v, w, z)}{\cosh \pi \kappa_j} \times \left\{ \sum_n \frac{\varrho_j(n, \infty)\sigma_{w+z-1}(n)}{n^{1/2(u+v+w+z-1)}} \right\} \left\{ \sum_n \frac{\varrho_j(\pm n, 1/c)}{n^{1/2(u-v-w+z+1)}} \right\} \]

\[ + 2 \sum_{k=1}^{\infty} \frac{(2k - 1)!}{(4\pi)^{2k}} [g]_+ \left( \left( \frac{1}{2} - 2k \right) i; u, v, w, z \right) \times \left\{ \sum_n \frac{\varrho_j,k(n, \infty)\sigma_{w+z-1}(n)}{n^{k-1/2}(u+v+w+z-1)} \right\} \left\{ \sum_n \frac{\varrho_j,k(n, 1/c)}{n^{k-1/2}(u-v-w+z+1)} \right\} \]

\[ + \frac{1}{\pi d^{1/2} + ir} \sum_{cd = c_1 d_1} \frac{1}{d_1} \int_{-\infty}^{\infty} \frac{(d_1, d)^{1+2ir}}{|L(1 + 2ir, \chi_{cd})|^2} [g]_{\pm}(r; u, v, w, z) \times \left\{ \sum_n \frac{\sigma_{2ir}(n; \chi_{cd})\sigma_{w+z-1}(n)}{n^{1/2(u+v+w+z-1) + ir}} \prod_{p|c_1} \left\{ \sigma_{2ir}(n_p) \left( 1 - \frac{1}{p^{1-2ir}} \right) - 1 \right\} \right\} \]

\[ \times \sum_{n} \frac{\sigma_{-2ir}(n; \chi_{cd})}{n^{1/2(u-v-w+z+1) - ir}} \prod_{p|(c_1, c)(d_1, d)} \left\{ \sigma_{-2ir}(n_p) \left( 1 - \frac{1}{p^{1+2ir}} \right) - 1 \right\} dr, \quad (21.3) \]

with \( q = cd \), where we have used the fact that \( \sigma_{1/cd} \in \Gamma_0(cd) \) and thus \( \varrho_j(n, 1/cd) = \varrho_j(n, \infty) \), \( \varrho_j,k(n, 1/cd) = \varrho_j,k(n, \infty) \).

22. We now deal with the function \( D_j(s, \alpha) \). As remarked in Section 19, we shall employ the unfolding method.

To this end we introduce the scattering matrix \( S \) of \( \Gamma_0(cd) \). We thus write (20.3) as

\[ E(\sigma_{1/w_2}(z), 1/w_1; s) = \delta_{w_1, w_2} y^s + \varphi(s; w_1, w_2) y^{1-s} + \cdots. \quad (22.1) \]

We put

\[ S(s) = \left( \varphi(s; w_1, w_2) \right)_{w_1, w_2 | cd} \quad (22.2) \]

and

\[ E(s) = \left( \begin{array}{ccc} \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right)_{w | cd} \quad (22.3) \]

so that

\[ E(s) = \left( \begin{array}{ccc} 1 \\ \vdots \\ 1 \end{array} \right) y^s + S(s) y^{1-s} + \cdots, \quad (22.4) \]
where the error terms decays exponentially and is \( O(y^{\frac{1}{2}-\varepsilon}) \) as \( y \) tends to infinity and to 0, respectively.

We have the functional equation

\[
\mathbb{E}(z, s) = S(s)\mathbb{E}(z, 1 - s),
\]

provided both sides are finite. To confirm this, we let \( \text{Re} \, s, \text{Im} \, s \) be sufficiently large. Then (22.4) implies in particular that \( \mathbb{E}(z, 1 - s) - S(1 - s)\mathbb{E}(z, s) \) is in an obvious vector extension of \( L^2(\Gamma_0(q)\setminus \mathcal{H}) \). However, this vector function, if not trivial, has the eigenvalue \( s(1 - s) \) against \( \Delta \) the hyperbolic Laplacian. Since \( \Delta \) is self-adjoint, its eigenvalues \( s(1 - s) \) should be real, which is a contradiction, and hence (22.5) holds for all complex \( s \) by analytic continuation as far as \( \mathbb{E}(z, s) \) is finite. Consequently, we have got also

\[
S(s)S(1 - s) = 1.
\]

23. We shall assume \( \varepsilon_j = 1 \) till the end of Section 24.

Let \( E(z, s) \) be the Eisenstein series for \( \Gamma_0(1) \), and put \( E^*(z, s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z, s) \), so that

\[
E^*(z, s) = E^*(z, 1 - s)
\]

and

\[
E^*(z, s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{s-1}\Gamma(1 - s)\zeta(2(1 - s))y^{1-s} + 2\sqrt{y}\sum_{n \neq 0}|n|^{-\frac{s}{2}}\sigma_{1-2s}(n)K_{s-\frac{1}{2}}(2\pi|n|y)\exp(2\pi nx),
\]

which shows that \( s(1 - s)E^*(z, s) \) is regular for all \( s \). We have, on a suitable assumption on \( s, \alpha \) to secure convergence, that

\[
\int_{\Gamma_0(cd)\setminus \mathcal{H}} \overline{\psi_j(z)}E^*(z, \frac{1}{2}(1 - \alpha)) E(z, \infty; s - \frac{1}{2}\alpha) \, d\mu(z) = \int_{\Gamma_\infty\setminus \mathcal{H}} \overline{\psi_j(z)}E^*(z, \frac{1}{2}(1 - \alpha)) \, y^{s-\frac{1}{2}\alpha} \, d\mu(z)
\]

\[
= 4\sum_{n > 0} n^{-\frac{1}{2}\alpha}\sigma_{\alpha}(n)\overline{\vartheta_j(n, \infty)} \int_0^\infty K_{\frac{1}{2}\alpha}(2\pi ny)K_{\kappa_j}(2\pi ny)y^{s-\frac{1}{2}\alpha - 1} \, dy
\]

\[
= \frac{\Gamma(s, \alpha; \kappa_j)}{2\pi^{s-\frac{1}{2}\alpha}\Gamma(s - \frac{1}{2}\alpha)} D_j(s, \alpha),
\]

with

\[
\Gamma(s, \alpha; \kappa) = \Gamma\left(\frac{1}{2}(s + i\kappa)\right) \Gamma\left(\frac{1}{2}(s - i\kappa)\right) \Gamma\left(\frac{1}{2}(s - \alpha + i\kappa)\right) \Gamma\left(\frac{1}{2}(s - \alpha - i\kappa)\right).
\]
On noting this, we consider also the relation

\[
\int_{\mathcal{H}} \overline{\psi_j(z)} E^*(z, \frac{1}{2}(1 - \alpha)) E(z, 1/w; s - \frac{1}{2} \alpha) \, d\mu(z) = \frac{\Gamma(s, \alpha; \kappa_j)}{2\pi^{s - \frac{1}{2} \alpha} \Gamma(s - \frac{1}{2} \alpha)} D_j(s, \alpha; 1/w),
\]

where

\[
D_j(s, \alpha; 1/w) = \sum_{n > 0} \overline{a_j(n, 1/w)} \sigma_\alpha(n, 1/w) n^{-s},
\]

with \(\sigma_\alpha(n, 1/w)\) an analogue of \(\sigma_\alpha(n)\).

By (13.7), we have \(E^*(\sigma_1/w(z), s) = E^*(\tau_v(z), s) = E^*(vz, s)\), and thus

\[
E^*(\sigma_1/w(z), s) = \pi^{-s} \Gamma(s) \zeta(2s)(vy)^s + \pi^{s-1} \Gamma(1-s) \zeta(2(1-s))(vy)^{1-s} + 2\sqrt{\pi} \sum_{n \neq 0} |n|^{-\frac{1}{2}} \sigma_{1-2s}(n, 1/w) K_{s-\frac{1}{2}}(2\pi |n| y) \exp(2\pi ny),
\]

That is, we have

\[
\sigma_\alpha(n, 1/w) = v^{\frac{1}{2}(\alpha + 1)} \sigma_\alpha(n/v),
\]

which vanishes if \(v \nmid n\).

Put

\[
\mathbb{D}_j(s, \alpha) = \begin{pmatrix}
\vdots \\
D_j(s, \alpha; 1/w) \\
\vdots \\
\end{pmatrix}_{w|cd}.
\]

Then we have, by (22.5) and (23.1),

\[
\frac{\Gamma(s, \alpha; \kappa_j)}{\pi^{s - \frac{1}{2} \alpha} \Gamma(s - \frac{1}{2} \alpha)} \mathbb{D}_j(s, \alpha) = \frac{\Gamma(1-s, -\alpha; \kappa_j)}{\pi^{1-s + \frac{1}{2} \alpha} \Gamma(1-s + \frac{1}{2} \alpha)} \mathbb{S}(s - \frac{1}{2} \alpha) \mathbb{D}_j(1-s, -\alpha).
\]

In particular, we get the functional equation

\[
D_j(s, \alpha) = \pi^{2s-\alpha-1} \frac{\Gamma(1-s, -\alpha; \kappa_j)}{\Gamma(s, \alpha; \kappa_j)} \frac{\Gamma(s - \frac{1}{2} \alpha)}{\Gamma(1-s + \frac{1}{2} \alpha)} \times \sum_{w|cd} \varphi(s - \frac{1}{2} \alpha; \infty, 1/w) D_j(1-s, -\alpha; 1/w).
\]

24. We decompose the left side of (23.5) as

\[
\sum_{w|cd} \int_0^1 \int_{y_0}^\infty \overline{\psi_j(\sigma_{1/w_1}(z))} E^*(\sigma_{1/w_1}(z), \frac{1}{2}(1 - \alpha)) E(\sigma_{1/w_1}(z), 1/w; s - \frac{1}{2} \alpha) \, d\mu(z)
\]

\[
+ \int_{(\Gamma_0(cd)\backslash \mathcal{H})_y_0} \overline{\psi_j(z)} E^*(z, \frac{1}{2}(1 - \alpha)) E(z, 1/w; s - \frac{1}{2} \alpha) \, d\mu(z),
\]

(24.1)
Lemma 10. The functions

\[(1 - \alpha^2) \left( s - \frac{1}{2}\alpha \right) \left( 1 - s + \frac{1}{2}\alpha \right) \Gamma(2s - \alpha)L(2s - \alpha, \chi_{cd})D_j(s, \alpha; 1/w) \quad (24.2)\]

of the complex variables \(s\) and \(\alpha\) are all entire over \(\mathbb{C}^2\).

In fact, it suffices to note that the multiple of (24.1) by the factor \( (1 - \alpha^2) \left( s - \frac{1}{2}\alpha \right) \left( 1 - s + \frac{1}{2}\alpha \right) \Gamma(2s - \alpha)L(2s - \alpha, \chi_{cd}) \) is regular in \(s\) and \(\alpha\) by Lemma 7.

On the other hand, we have, by (20.4),

\[
L(2s - \alpha, \chi_{cd})\varphi \left( s - \frac{1}{2}\alpha; \infty, 1/w \right) = \frac{1}{\pi} \varphi(w) \left( \pi \frac{\Gamma(2s - \alpha)}{\Gamma(s - \frac{1}{2}\alpha)} \prod_{\mathfrak{p}|v} \left( p^{s-\frac{1}{2}\alpha} - p^{1-s+\frac{1}{2}\alpha} \right) \right). \quad (24.3)
\]

Inserting this into (23.11), we get

\[
L(2s - \alpha, \chi_{cd})D_j(s, \alpha) = \frac{1}{\pi} \left( \frac{\pi}{cd} \right)^{2s-\alpha} \zeta(2(1-s) + \alpha) \frac{\Gamma(1-s, -\alpha; \kappa_j)}{\Gamma(s, \alpha; \kappa_j)} \times \sum_{w|cd} \varphi(w) \prod_{\mathfrak{p}|v} \left( p^{s-\frac{1}{2}\alpha} - p^{1-s+\frac{1}{2}\alpha} \right) D_j(1-s, -\alpha; 1/w). \quad (24.4)
\]

We then let \(\text{Re} s\) be negative and so large that both \(\zeta(2(1-s) + \alpha)\) and \(D_j(1-s, -\alpha; 1/w)\) are absolutely convergent. In this way we obtain, via Lemma 2, Stirling’s formula, and the convexity argument,

Lemma 11. Provided that \(\text{Re} s\) and \(\alpha\) are bounded, we have

\[
(1 - \alpha^2) \left( s - \frac{1}{2}\alpha \right) \left( 1 - s + \frac{1}{2}\alpha \right) L(2s - \alpha, \chi_{cd})D_j(s, \alpha; 1/w) \ll (\kappa_j + |s| + 1)^\tau \exp \left( \frac{1}{2} \pi \kappa_j \right), \quad (24.5)
\]

where \(\tau\) depends only on \(\text{Re} s\) and \(\text{Re} \alpha\), and the implied constant additionally on \(cd\) too.

25. We still need to deal with the case \(\epsilon_j = -1\). Here we shall have to overcome an additional technical difficulties, because Eisenstein series of non-zero weights naturally come up in our argument (see [11, Section 3.2]).

We introduce

\[
\psi_j^-(z) = y(\partial_x - i\partial_y)\psi_j(z), \quad (25.1)
\]

with our present vector \(\psi_j\) such that \(\psi_j J = -\psi_j\). We have

\[
\psi_j^-(\gamma(z)) = \psi_j^- (z) (j(\gamma, z)/|j(\gamma, z)|)^2, \quad \gamma \in \Gamma. \quad (25.2)
\]
In fact, writing $\xi = \Re \lambda(z)$, $\eta = \Im \lambda(z)$ for a regular function $\lambda$, we have $(\partial_x - i \partial_y)[H(\lambda(z))] = \{(\partial H/\partial \xi)(\partial \xi/\partial x - i \partial \xi/\partial y) + (\partial H/\partial \eta)(\partial \eta/\partial x - i \partial \eta/\partial y)\} = \{(\partial H/\partial \xi - i \partial H/\partial \eta)(\partial \xi/\partial x - i \partial \xi/\partial y)\} = [(\partial \xi - i \partial \eta)H](d\lambda/dz)$ by the Cauchy–Riemann equation applied to $\lambda$. We put $\lambda = \gamma$, $H = \psi_j$, and get $y(\partial_x - i \partial_y)\psi_j(z) = (y/\eta)(d\gamma/dz)\eta(\partial \xi - i \partial \eta)\psi_j(\xi + i \eta)$, which confirms (25.2).

To offset the automorphic factor in (25.2), we introduce

$$E_-(z, 1/w; s) = \sum_{\gamma \in \Gamma_{1/w} \backslash \Gamma} \left( \Im \sigma_{1/w}^{-1}(\gamma)(z) \right)^s \left( j(\sigma_{1/w}^{-1}(\gamma), z)/|j(\sigma_{1/w}^{-1}(\gamma), z)| \right)^{-2}. \tag{25.3}$$

We should note the relation

$$y(\partial_x - i \partial_y)[E(z, 1/w; s)] = -is E_-(z, 1/w; s), \tag{25.4}$$

which can be confirmed by setting $\lambda = \sigma_{1/w}^{-1}$, $H = y^s$ in the above; and more precisely

$$y(\partial_x - i \partial_y)[E(\sigma_{1/w}(z), 1/w; s)] = -is E_-(\sigma_{1/w}(z), 1/w; s) \left( j(\sigma_{1/w}(z), z)/|j(\sigma_{1/w}, z)| \right)^{-2}. \tag{25.5}$$

In particular, we have the functional equation

$$sE_-(z, s) = (1 - s)S(s)E_-(z, 1 - s), \tag{25.6}$$

with

$$E_-(s) = \begin{pmatrix} E_-(z, 1/w; s) \\ \vdots \\ E_-(z, 1/w; s) \end{pmatrix} \tag{25.7}$$

Also, (25.5) implies that

$$\Gamma(s + 1)L(2s, \chi_{cd})E_-(\sigma_{1/w}(z), 1/w; s) \ll y^{Re s} + y^{1-\Re s}, \tag{25.8}$$

as $y$ tends to infinity while $s$ remains bounded, which means that the left side is regular for all $s$, too. This is a counterpart of Lemma 7.

In the region of absolute convergence, we have, by (25.2),

$$\int_{\Gamma \backslash \mathcal{H}} \overline{\psi_j(z)}E^*(z, \frac{1}{2}(1 - \alpha)) E_-(z, 1/w; s - \frac{1}{2}\alpha) \, d\mu(z)$$

$$= \sum_{\gamma \in \Gamma_{1/w} \backslash \Gamma} \int_{\sigma_{1/w}^{-1}(\Gamma \backslash \mathcal{H})} \overline{\psi_j(\sigma_{1/w}(z))} \left( \frac{j(\gamma^{-1}, \sigma_{1/w}(z))}{|j(\gamma^{-1}, \sigma_{1/w}(z))|} \right)^{-2} E^*(\sigma_{1/w}(z), \frac{1}{2}(1 - \alpha))$$

$$\times y^{s-\frac{1}{2}\alpha} \left( \frac{j(\sigma_{1/w}, z)}{|j(\sigma_{1/w}, z)|} \right)^{-2} \, d\mu(z)$$

$$= \sum_{\gamma \in \Gamma_{1/w} \backslash \Gamma} \int_{\sigma_{1/w}^{-1}(\Gamma \backslash \mathcal{H})} \overline{\psi_j(\sigma_{1/w}(z))} \left( \frac{j(\sigma_{1/w}, z)}{|j(\sigma_{1/w}, z)|} \right)^2 E^*(\sigma_{1/w}(z), \frac{1}{2}(1 - \alpha)) y^{s-\frac{1}{2}\alpha} \, d\mu(z)$$

$$= \int_0^\infty \int_0^1 (\partial_x - i \partial_y)[\overline{\psi_j(\sigma_{1/w}(z))}]E^*(\sigma_{1/w}(z), \frac{1}{2}(1 - \alpha)) y^{s-\frac{1}{2}\alpha-1} \, dx \, dy, \tag{25.9}$$
since
\[ \psi_j^-(\sigma_{1/w}(z)) = y(\partial_x - i\partial_y)[\psi_j(\sigma_{1/w}(z))](g(\sigma_{1/w}, z)|j(\sigma_{1/w}, z))^2. \]  (25.10)

We observe then that \( E^*(\sigma_{1/w}(z), \frac{1}{2}(1 - \alpha)) \) is even in \( x \) as (23.7) implies, and \( \partial_y[\psi_j(\sigma_{1/w}(z))] \) is odd by (18.5). Hence (25.9) becomes
\[
\int_{\Gamma \setminus \mathcal{H}} \overline{\psi_j(z)} E^*(z, \frac{1}{2}(1 - \alpha)) E_-(z, 1/w; s - \frac{1}{2} \alpha) d\mu(z) = -i \frac{\Gamma(s + 1, \alpha; \kappa_j)}{\pi^{s - \frac{1}{2} \alpha} \Gamma(s + 1 - \frac{1}{2} \alpha)} D_j(s, \alpha; 1/w), \]  (25.11)

provided absolute convergence holds throughout.

We decompose the left side of (25.11) in just the same way as we did in (24.1), and see, via (25.8), that
\[
(1 - \alpha^2) \Gamma(s + 1 - \frac{1}{2} \alpha) L(2s - \alpha, \chi_{cd}) D_j(s, \alpha; 1/w) \]  (25.12)

are all regular in \( s \) and \( \alpha \). Also, (25.11) gives, via (25.6),
\[
\frac{\Gamma(s + 1, \alpha; \kappa_j)}{\pi^{s - \frac{1}{2} \alpha} \Gamma(s + \frac{1}{2} \alpha)} D_j(s, \alpha) = \frac{\Gamma(2 - s, -\alpha; \kappa_j)}{\pi^{1-s+\frac{1}{2} \alpha} \Gamma(1-s + \frac{1}{2} \alpha)} S(s - \frac{1}{2} \alpha) D_j(1-s, -\alpha), \]  (25.13)

and in particular
\[
D_j(s, \alpha) = \pi^{2s-\alpha-1} \frac{\Gamma(2-s, -\alpha; \kappa_j)}{\Gamma(s+1, \alpha; \kappa_j)} \frac{\Gamma(s - \frac{1}{2} \alpha)}{\Gamma(1-s + \frac{1}{2} \alpha)} \\
\times \sum_{w|cd} \varphi(s - \frac{1}{2} \alpha; \infty, 1/w) D_j(1-s, -\alpha; 1/w). \]  (25.14)

Hence, by (24.3), we have
\[
L(2s - \alpha, \chi_{cd}) D_j(s, \alpha) = \frac{1}{\pi^2} \left( \frac{\pi^2}{cd} \right)^{2s-\alpha} \zeta(2(1-s) + \alpha) \frac{\Gamma(2-s, -\alpha; \kappa_j)}{\Gamma(s+1, \alpha; \kappa_j)} \\
\times \sum_{w|cd} \varphi(w) \prod_p \left( p^{s-\frac{1}{2} \alpha} - p^{1-s+\frac{1}{2} \alpha} \right) D_j(1-s, -\alpha; 1/w). \]  (25.15)

With this, we obtain

**Lemma 12.** With \( \epsilon_j = -1 \) as well, the assertions of Lemmas 10 and 11 hold.

This ends our treatment of \( L_j \) and \( D_j \). We omit the discussion of \( L_{j,k}, D_{j,k} \), for they are analogous.
26. Now we may return to (21.3). Here we shall deal with the first term on the right, the contribution of real analytic cusp forms. Its contribution to \( I(u, v, w, z; g; b/a) \) is, via (2.2), (2.3), (17.10), equal to

\[
\frac{2}{abu(2\pi)^{u-w+1}} \sum_{c|a, d|b} e^{u+v}d^\frac{3}{2}(3u+v-w+z) \sum_j R_j \left( \frac{1}{2}(u + v + w + z - 1), w + z - 1 \right) \times L_j \left( \frac{1}{2}(u - v - w + z + 1); 1/c \right) \frac{([g]_+ + \epsilon_j[g]_-)(\kappa_j; u, v, w, z)}{\cosh \pi \kappa_j}.
\]

(26.1)

with

\[
R_j(s, \alpha) = \zeta(2s - \alpha)D_j(s, \alpha).
\]

(26.2)

By Lemmas 9–12, we see readily that the expression (26.1) is meromorphic over \( \mathbb{C}^4 \), and especially in the vicinity of \( p_\frac{a}{b} \), it is regular; the necessary facts about \([g]_\pm\) is to be given shortly. Hence its value at \( p_\frac{a}{b} \) equals

\[
\frac{1}{\pi \sqrt{ab}} \sum_{c|a, d|b} cd \sum_j R_j \left( \frac{1}{2}, 0 \right) L_j \left( \frac{1}{2}; 1/c \right) \frac{([g]_+ + \epsilon_j[g]_-)(\kappa_j; p_\frac{a}{b})}{\cosh \pi \kappa_j}.
\]

(26.3)

We have another contribution of real analytic cusp forms that comes from \( J_\pm \), which is, however, exactly the same as (26.3).

Let us make the last factor in (26.3) explicit. Thus, comparing (6.4) with [11, (4.3.13)–(4.3.14)], we see that the exchange of variables \( u \) and \( z \) is to be applied to [11, Sections 4.6–4.7] to get corresponding identities. More precisely, we have, under (3.4) and (4.1),

\[
[g]_+(r; u, v, w, z) = \frac{1}{4\pi i} \cos(\frac{1}{2}\pi(v - z)) \int_{(n_1)} \sin(\frac{1}{2}\pi(u + v + w + z - 2s)) \times \Gamma(\frac{1}{2}(u + v + w + z - 1) + ir - s) \Gamma(\frac{1}{2}(u + v + w + z - 1) - ir - s) \times \Gamma(s + 1 - w - z) \Gamma(s + 1 - v - w) g^*(s, w) ds,
\]

(26.4)

\[
[g]_-(r; u, v, w, z) = -\frac{1}{4\pi i} \cosh(\pi r) \int_{(n_1)} \cos(\pi(w + \frac{1}{2}(v + z) - s)) \times \Gamma(\frac{1}{2}(u + v + w + z - 1) + ir - s) \Gamma(\frac{1}{2}(u + v + w + z - 1) - ir - s) \times \Gamma(s + 1 - w - z) \Gamma(s + 1 - v - w) g^*(s, w) ds,
\]

(26.5)

corresponding to [11, (4.4.12)] and [ibid, (4.4.15)], respectively. We then put

\[
\Phi_+(\xi; u, v, w, z; g) = -i(2\pi)^{w-v-2} \cos(\frac{1}{2}\pi(v - z)) \times \int_{-i\infty}^{i\infty} \sin(\frac{1}{2}\pi(u + v + w + z - 2s)) \times \Gamma(\frac{1}{2}(u + v + w + z - 1) + \xi - s) \Gamma(\frac{1}{2}(u + v + w + z - 1) - \xi - s) \times \Gamma(s + 1 - w - z) \Gamma(s + 1 - v - w) g^*(s, w) ds;
\]

(26.6)

\[
\Phi_- (\xi; u, v, w, z; g) = i(2\pi)^{w-v-2} \cos(\pi \xi) \int_{-i\infty}^{i\infty} \cos(\pi(w + \frac{1}{2}(v + z) - s)) \times \Gamma(\frac{1}{2}(u + v + w + z - 1) + \xi - s) \Gamma(\frac{1}{2}(u + v + w + z - 1) - \xi - s) \times \Gamma(s + 1 - w - z) \Gamma(s + 1 - v - w) g^*(s, w) ds;
\]

(26.7)
and

\[ \Xi(\xi; u, v, w, z; g) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Gamma(\xi + \frac{1}{2}(u + v + w + z - 1) - s)}{\Gamma(\xi + \frac{1}{2}(3 - u - v - w - z) + s)} \times \Gamma(s + 1 - w - z) \Gamma(s + 1 - v - w) g^*(s, w) ds. \]  

(26.8)

The paths in (26.6) and (26.7) are such that the poles of the first two gamma-factors and those of the other three factors in each integrand are separated to the right and the left, respectively, by the path, and \( \xi, u, v, w, z \) are assumed to be such that the path can be drawn. The path in (26.8) separates the poles of \( \Gamma(\xi + \frac{1}{2}(u + v + w + z - 1) - s) \) and those of \( \Gamma(s + 1 - w - z) \Gamma(s + 1 - v - w) g^*(s, w) \) to the left and the right of the path, respectively. We have the relations

\[ \Phi_{\pm}(\xi; u, v, w, z; g) = -\frac{(2\pi)^{w-u} \cos\left(\frac{1}{2}\pi(v - z)\right)}{4\sin(\pi\xi)} \times \{ \Xi(\xi; u, v, w, z; g) - \Xi(-\xi; u, v, w, z; g) \}, \]  

(26.9)

\[ \Phi_{-}(\xi; u, v, w, z; g) = \frac{(2\pi)^{w-u}}{4\sin(\pi\xi)} \{ \sin(\pi(\frac{1}{2}(u - w) + \xi)) \Xi(\xi; u, v, w, z; g) \]  

\[ - \sin(\pi(\frac{1}{2}(u - w) - \xi)) \Xi(-\xi; u, v, w, z; g) \}, \]  

(26.10)

provided the left sides are well-defined.

Under (4.1), we can obviously take \((\eta_1)\) as the contours in the last three integrals; and we have, for \( r \in \mathbb{R} \),

\[ [g]_+(r; u, v, w, z) = \frac{1}{2} (2\pi)^{1+u-w} \Phi_{+}(ir; u, v, w, z; g), \]

\[ [g]_- (r; u, v, w, z) = \frac{1}{2} (2\pi)^{1+u-w} \Phi_{-}(ir; u, v, w, z; g). \]  

(26.11)

In particular, we have, after continuation,

\[ [g]_+(r; p_{\frac{1}{2}}) = -\frac{\pi}{4\sin(\pi r)} \left( \Xi(ir; p_{\frac{1}{2}}; g) - \Xi(-ir; p_{\frac{1}{2}}; g) \right), \]

\[ [g]_- (r; p_{\frac{1}{2}}) = \frac{\pi}{4} \left( \Xi(ir; p_{\frac{1}{2}}; g) + \Xi(-ir; p_{\frac{1}{2}}; g) \right), \]  

(26.12)

and

\[ ([g]_+ + \varepsilon_j [g]_-) (r; p_{\frac{1}{2}}) = \frac{\pi}{2} \text{Re} \left\{ \left( \varepsilon_j + \frac{i}{\sinh \pi r} \right) \Xi(ir; p_{\frac{1}{2}}; g) \right\}, \]  

(26.13)

since (3.2) and (26.8) imply \( \Xi(ir; p_{\frac{1}{2}}; g) = \Xi(-ir; p_{\frac{1}{2}}; g) \).

From this, we get immediately

**Lemma 13.** Provided the polynomial \( A \) is supported by the set of square-free integers, the contribution of real analytic cusp forms to \( M_2(g; A) \) is equal to

\[ \sum_{c, d} A(c, d) \xi(c, d; g), \]  

(26.14)
where
\[ A(c, d) = \sum_{(ac, bd) = 1} \frac{\alpha_{acl} \alpha_{bdl}}{abl}, \]

and
\[ C(c, d; g) = \sum_{j} \frac{1}{\cosh \pi \kappa_j} R_j \left( \frac{1}{2}, 0 \right) L_j \left( \frac{1}{2}; 1/c \right) \times \text{Re} \left\{ \left( \epsilon_j + \frac{i}{\sinh \pi \kappa_j} \right) \Xi \left( i\kappa_j; p^{\frac{1}{2}}, g \right) \right\}. \]

The fact that the parity symbol $\epsilon_j$ appears in this way will turn out to be crucial in our later discussion of a certain non-vanishing assertion (Sections 31–36).

The contribution of holomorphic cusp forms is analogous, and we may skip it.

27. We turn to the contribution of continuous spectrum; and we see from (21.3) that we need first to consider the sum
\[ \sum_n \frac{\sigma_{-2ir}(n; \chi_{cd})}{n^s} \prod_{p|(c_1, c)(d_1, d)} \left\{ \sigma_{-2ir}(n_p) \left( 1 - \frac{1}{p^{1+2ir}} \right) - 1 \right\} \]
\[ = \sum_{l| (c_1, c)(d_1, d)} \mu((c_1, c)(d_1, d)/l) \prod_{p|l} \left( 1 - \frac{1}{p^{1+2ir}} \right) \sum_n \sigma_{-2ir}(n; \chi_{cd}) \sigma_{-2ir}(n_l)n^{-s}, \]
with $n_l = (n, l^\infty)$. We have
\[ \sum_n \sigma_{-2ir}(n; \chi_{cd}) \sigma_{-2ir}(n_l)n^{-s} = \left\{ \sum_{(n, l) = 1} \frac{\sigma_{-2ir}(n; \chi_{cd})}{n^s} \right\} \left\{ \sum_{n|l^\infty} \frac{\sigma_{-2ir}(n)}{n^s} \right\} \]
\[ = L(s, \chi_l)L(s + 2ir, \chi_{cd}) \prod_{p|l} \left( 1 - \frac{1}{p^s} \right)^{-1} \left( 1 - \frac{1}{p^{s+2ir}} \right)^{-1} \]
\[ = \zeta(s)L(s + 2ir, \chi_{cd}) \prod_{p|l} \left( 1 - \frac{1}{p^{s+2ir}} \right)^{-1}. \]

Thus
\[ \sum_n \frac{\sigma_{-2ir}(n; \chi_{cd})}{n^s} \prod_{p|(c_1, c)(d_1, d)} \left\{ \sigma_{-2ir}(n_p) \left( 1 - \frac{1}{p^{1+2ir}} \right) - 1 \right\} \]
\[ = \zeta(s)L(s + 2ir, \chi_{cd}) \sum_{l| (c_1, c)(d_1, d)} \mu((c_1, c)(d_1, d)/l) \prod_{p|l} \left( 1 - \frac{1}{p^{s+2ir}} \right)^{-1} \left( 1 - \frac{1}{p^{1+2ir}} \right) \]
\[ = \zeta(s)L(s + 2ir, \chi_{cd}) \prod_{p|(c_1, c)(d_1, d)} \left( 1 - \frac{1}{p^{s+2ir}} \right)^{-1} \left( 1 - \frac{1}{p^{1+2ir}} \right) - 1 \right\}. \]
Next, we need to treat
\[
\sum_n \frac{\sigma_{2ir}(n; \chi_{cd})\sigma_\alpha(n)}{n^s} \prod_{p | c_1} \left\{ \frac{\sigma_{2ir}(n_p)}{n_p} \left( 1 - \frac{1}{p^{1-2ir}} \right) - 1 \right\} = \sum_{l | c_1} \mu(c_1/l) \prod_{p | l} \left( 1 - \frac{1}{p^{1-2ir}} \right) \sum_n \frac{\sigma_{2ir}(n; \chi_{cd})\sigma_\alpha(n)\sigma_{2ir}(n_l)}{n^s}.
\] (27.4)

We have
\[
\sum_n \frac{\sigma_{2ir}(n; \chi_{cd})\sigma_\alpha(n)\sigma_{2ir}(n_l)}{n^s} = \left\{ \sum_{(n,l)=1} \frac{\sigma_{2ir}(n; \chi_{cd})\sigma_\alpha(n)}{n^s} \right\} \left\{ \sum_{n|l} \frac{\sigma_{2ir}(n)\sigma_\alpha(n)}{n^s} \right\}. \tag{27.5}
\]

Analogously to a famous formula of Ramanujan, we have
\[
\sum_{(n,l)=1} \frac{\sigma_{2ir}(n; \chi_{cd})\sigma_\alpha(n)}{n^s} = \frac{L(s, \chi_q)L(s-2ir, \chi_q)L(s-\alpha, \chi_{cd})L(s-2ir-\alpha, \chi_{cd})}{L(2s-2ir-\alpha, \chi_{cd})} \prod_{p | cd/l} \left( 1 - \frac{1}{p^s} \right)^{-1} \left( 1 - \frac{1}{p^{s-\alpha}} \right)^{-1},
\]
\[
\sum_{n|l} \frac{\sigma_{2ir}(n)\sigma_\alpha(n)}{n^s} = \prod_{p | l} \frac{1 - \frac{1}{p^{2s-2ir-\alpha}}}{\left( 1 - \frac{1}{p^s} \right) \left( 1 - \frac{1}{p^{s-2ir}} \right) \left( 1 - \frac{1}{p^s-\alpha} \right) \left( 1 - \frac{1}{p^{s-2ir-\alpha}} \right)}. \tag{27.6}
\]

Thus,
\[
\sum_n \frac{\sigma_{2ir}(n; \chi_{cd})\sigma_\alpha(n)\sigma_{2ir}(n_l)}{n^s} = \frac{\zeta(s)L(s-2ir, \chi_{cd})\zeta(s-\alpha)L(s-2ir-\alpha, \chi_{cd})}{L(2s-2ir-\alpha, \chi_{cd})} \times \prod_{p | l} \frac{1 - \frac{1}{p^{2s-2ir-\alpha}}}{\left( 1 - \frac{1}{p^{s-2ir}} \right) \left( 1 - \frac{1}{p^{s-2ir-\alpha}} \right)}. \tag{27.7}
\]

Hence,
\[
\sum_n \frac{\sigma_{2ir}(n; \chi_{cd})\sigma_\alpha(n)}{n^s} \prod_{p | c_1} \left\{ \frac{\sigma_{2ir}(n_p)}{n_p} \left( 1 - \frac{1}{p^{1-2ir}} \right) - 1 \right\}
\]
\[
\zeta(s)L(s - 2ir, \chi_q)\zeta(s - \alpha)L(s - 2ir - \alpha, \chi_{cd})
\]
\[
\times \prod_{p \mid c_1} \left\{ \left( 1 - \frac{1}{p^{1-2ir}} \right) \left( 1 - \frac{1}{p^{2s-2ir-\alpha}} \right) \right\}^{-1}
\]
\[
= \zeta(s)\zeta(s - 2ir)\zeta(s - \alpha)\zeta(s - 2ir - \alpha) \prod_{p \mid cd} \left( 1 - \frac{1}{p^{2s-2ir-\alpha}} \right)^{-1}
\times \prod_{p \mid d_1} \left( 1 - \frac{1}{p^{s-2ir}} \right) \left( 1 - \frac{1}{p^{s-2ir-\alpha}} \right)
\times \prod_{p \mid c_1} \left\{ \left( 1 - \frac{1}{p^{1-2ir}} \right) \left( 1 - \frac{1}{p^{2s-2ir-\alpha}} \right) - \left( 1 - \frac{1}{p^{s-2ir}} \right) \left( 1 - \frac{1}{p^{s-2ir-\alpha}} \right) \right\} . \tag{27.8}
\]

28. Under the conditions (3.4), (4.1) and by (21.3), (27.3), (27.8), the contribution of the continuous spectrum to \( I \) via \( J_+^* \) is equal to

\[
4(2\pi)^{w-u-2} a^v b^u \int_{-\infty}^{\infty} \frac{Y_{a,b}(ir; u, v, w, z)Z(ir; u, v, w, z)}{\zeta(1+2ir)\zeta(1-2ir)}([g]_- + [g]_-)(r; u, v, w, z) \, dr \tag{28.1}
\]

where

\[
Z(\xi; u, v, w, z) = \zeta \left( \frac{1}{2}(u + v + w + z - 1) + \xi \right) \zeta \left( \frac{1}{2}(u + v + w + z - 1) - \xi \right)
\times \zeta \left( \frac{1}{2}(u + v - w - z + 1) + \xi \right) \zeta \left( \frac{1}{2}(u + v - w - z + 1) - \xi \right)
\times \zeta \left( \frac{1}{2}(u - v - w + z + 1) + \xi \right) \zeta \left( \frac{1}{2}(u - v - w + z + 1) - \xi \right) \tag{28.2}
\]

and

\[
Y_{a,b}(\xi; u, v, w, z) = \sum_{c \mid |a, d|} c^{u+v} d^{\frac{1}{2}(3u+v-w+z-1)-\xi} X_{cd}(\xi; u, v, w, z), \tag{28.3}
\]

with

\[
X_{cd}(\xi; u, v, w, z) = \prod_{p \mid cd} \left\{ \left( 1 - \frac{1}{p^{1+2\xi}} \right) \left( 1 - \frac{1}{p^{1-2\xi}} \right) \left( 1 - \frac{1}{p^{u+v}} \right) \right\}^{-1}
\times \sum_{cd=c_1, d_1} \frac{1}{d_1} \left( \frac{(d_1, d)}{(c_1, c)} \right)^{\frac{1}{2}+\xi}
\times \prod_{p \mid (d_1, c)(c_1, d)} \left( 1 - \frac{1}{p^{\frac{1}{2}(u-v-w+z) + \xi}} \right) \prod_{p \mid (c_1, c)(d_1, d)} \left( \frac{1}{p^{\frac{1}{2}(u-v-w+z) - \xi}} \right) \tag{27.8}
\]

\[
\times \prod_{p \mid d_1} \left( 1 - \frac{1}{p^{\frac{1}{2}(u+v+w+z-1) - \xi}} \right) \left( 1 - \frac{1}{p^{\frac{1}{2}(u+v-w-z) - \xi}} \right)
\]
\[ \times \prod_{p|c_1} \left\{ \left( 1 - \frac{1}{p^{1-2\xi}} \right) \left( 1 - \frac{1}{p^{u+v}} \right) \right. \\
\left. - \left( 1 - \frac{1}{p^{\frac{1}{2}(u+v+w+z-1) - \xi}} \right) \left( 1 - \frac{1}{p^{\frac{1}{2}(u+v-w-z+1) - \xi}} \right) \right\} . \] (28.4)

One may carry out the last sum and transform \( X_{cd} \) and thus \( Y_{ab} \) into a more closed expression that is a product over prime divisors of \( ab \); however, for our aim it does not seem particularly expedient to do so, and we leave (28.3) as it is.

To continue (28.1) to a neighbourhood of \( p_{\frac{1}{2}} \), we need to shift the contour rightward and leftward appropriately as is done in [11, Section 4.7], and there appears a residual contribution, which will be treated in detail later. Here we shall compute, at \( p_{\frac{1}{2}} \), the integral thus continued.

By (28.4), we have, for \( r \in \mathbb{R} \),

\[ X_{cd}(ir; p_{\frac{1}{2}}) \]

\[ = \prod_{p|cd} \left| 1 - \frac{1}{p^{\frac{1}{2}+ir}} \right|^2 \left| 1 - \frac{1}{p^{1+2ir}} \right| \sum_{cd=c_1d_1} \frac{1}{d_1} \left( \frac{(d_1, d)}{(c_1, c)} \right)^{\frac{1}{2}+ir} \prod_{p|d_1} \left( 1 - \frac{1}{p} \right)^2 \\
\times \prod_{p|c_1} \left\{ \left( 1 - \frac{1}{p^{1-2ir}} \right) \left( 1 - \frac{1}{p} \right) - \left( 1 - \frac{1}{p^{\frac{1}{2}-ir}} \right)^2 \right\} \\
= \prod_{p|cd} \left| 1 - \frac{1}{p^{\frac{1}{2}+ir}} \right|^2 \left| 1 - \frac{1}{p^{1+2ir}} \right| \prod_{p|cd} \left( \frac{p, d}{p} \right)^{\frac{1}{2}+ir} \left( 1 - \frac{1}{p} \right)^2 \\
+ \frac{1}{(p, c)^{\frac{1}{2}+ir}} \left\{ \left( 1 - \frac{1}{p^{1-2ir}} \right) \left( 1 - \frac{1}{p} \right) - \left( 1 - \frac{1}{p^{\frac{1}{2}-ir}} \right)^2 \right\} \\
= c^{-\frac{1}{2}-ir} \prod_{p|cd} \left| 1 - \frac{1}{p^{\frac{1}{2}+ir}} \right|^2 \left| 1 - \frac{1}{p^{1+2ir}} \right| \left( 1 - \frac{1}{p} \right)^3 \left( 1 - \frac{1}{p^{\frac{1}{2}-ir}} \right)^3 \\
\text{This implies that} \quad Y_{a,b}(ir; p_{\frac{1}{2}}) \]
Lemma 14. Provided the polynomial $A$ is supported by the set of square-free integers, the contribution of continuous spectrum to $M_2(g; A)$ is equal to

$$\frac{1}{\pi} \sum_{(a,b)=1} \frac{\alpha_{ab} \overline{\alpha_{bl}}}{\varphi(ab)l} \int_{-\infty}^{\infty} \frac{\zeta\left(\frac{1}{2} + ir\right)^6}{\zeta(1+2ir)^2} \prod_{p|ab} \left(4 \left| 1 + \frac{1}{p^{1/2+ir}} \right|^2 - \frac{1}{p} \right) \text{Re} \left\{ \left(1 + \frac{i}{\sinh \pi r}\right) \Xi\left(ir; \frac{1}{2}; g\right) \right\} dr. \quad (28.7)$$

29. We shall give the continuation procedure of (28.1) to a neighbourhood of $p_{1/2}$. This is, however, analogous to that pertaining to the pure fourth moment $M_2(g; 1)$ that is developed in [11, Sections 4.6–4.7]; and we can be brief.

By (26.9)–(26.11), we transform (28.1) into

$$i \frac{(2\pi)^{w-u-1}}{a^v b^u} \int_{(0)} \frac{Z(\xi; u, v, w, z)}{\sin(\pi \xi) \zeta(1+2\xi) \zeta(1-2\xi)} \{Y_{a,b}(\xi; u, v, w, z) + Y_{a,b}(-\xi; u, v, w, z)\}$$

$$\times \{\cos\left(\frac{1}{2} \pi (v-z)\right) - \sin(\pi \left(\frac{1}{2}(u-w)+\xi\right))\} \Xi(\xi; u, v, w, z; g) d\xi; \quad (29.1)$$

and applying the functional equation for $\zeta$ to $\zeta(1-2\xi)$, this becomes

$$2i \frac{(2\pi)^{w-u-2}}{a^v b^u} \int_{(0)} \frac{(2\pi)^2 \Gamma(1-2\xi) Z(\xi; u, v, w, z)}{\zeta(2\xi) \zeta(1+2\xi)} \{Y_{a,b}(\xi; u, v, w, z) + Y_{a,b}(-\xi; u, v, w, z)\}$$

$$\times \{\cos(\frac{1}{2} \pi (v-z)) - \sin(\pi \left(\frac{1}{2}(u-w)+\xi\right))\} \Xi(\xi; u, v, w, z; g) d\xi \quad (29.2)$$

(see [11, (4.6.14)–(4.6.15)]). We shift the last contour to the far right, and we obtain a meromorphic continuation to a domain containing the point $p_{1/2}$; then, restricting ourselves to the vicinity of $p_{1/2}$, we shift the contour back to the imaginary axis. The resulting integral has been considered already in the last section.

The residual contribution of the last procedure takes place when

$$\xi_1 = \frac{1}{2} (u + v + w + z - 3), \quad \xi_2 = \frac{1}{2} (u - v - w + z - 1),$$

$$\xi_3 = \frac{1}{2} (3 - u - v - w - z), \quad \xi_4 = \frac{1}{2} (u + v - w - z - 1). \quad (29.3)$$
(see [11,(4.6.16)] and the bottom lines of [ibid, p. 173]). It should be stressed that this assertion depends on the fact that the singularities, save for those belonging to $Z(\xi;u,v,w,z)$, that we encounter in this procedure are independent of the location of $(u,v,w,z)$; especially those of $Y_{a,b}(\pm\xi;u,v,w,z)$ come only from the first product on the right of (28.4) and are independent of $(u,v,w,z)$.

Remark 4. However, one should note that the set of poles of $Y_{a,b}(\xi;u,v,w,z)$ as a function of $\xi$ cluster at the point $\xi = \pm \frac{1}{2}$ if $a,b$ are allowed to vary arbitrarily. Thus, if the length of the polynomial $A$ increases indefinitely, then the nature of the main term of $M_2(g;A)$ should become subtler.

30. With this, we have essentially finished spectrally decomposing $M_2(g;A)$. Although we have not yet computed the main term explicitly, the above is already quite adequate to analyze the error term in the asymptotic formula for the unweighted mean

$$\int_0^T |\zeta(\frac{1}{2} + it)|^4 |A(\frac{1}{2} + it)|^2 \,dt.$$  \hfill (30.1)

With this in mind, we shall investigate the location of poles of the Mellin transform $Z_2(s;A)$, focusing our attention to the contribution of real analytic cusp forms, for the relevant part of $Z_2(s;A)$ seems to be the most interesting.

Having the assertion of Lemma 13, the argument of [11, Section 5.3] works with $Z_2(s;A)$ as well without any essential change. We find, on the assumption on eigenvalues $\kappa_j^2 + \frac{1}{4}$ made in the introduction, that

Lemma 15. The function $Z_2(s;A)$ is meromorphic over the entire complex plane. It has a pole of the fifth order at $s = 1$; and all other poles are in the half plane $\Re s \leq \frac{1}{2}$. More precisely, $Z_2(s;A)$ has a pole at $\frac{1}{2} + i\kappa$, $\kappa > 0$, if and only if it holds that

$$\sum_{c,d} A(c,d) \sum_{\kappa_j = \kappa \atop \kappa_j^2 + \frac{1}{4} \in \text{Sp}(\Gamma_0(cd))} R_j \left(\frac{1}{2}, 0\right) L_j \left(\frac{1}{2}; 1/c\right) \left(\epsilon_j - \frac{i}{\sinh \pi\kappa}\right) \neq 0. \hfill (30.2)$$

We are going to show that if $A$ is fixed besides a natural condition on its coefficients, then (30.2) holds for infinitely many $\kappa$. To this end we shall establish in the sequel that there are infinitely many $\kappa$ such that

$$\Re(\kappa; A) = \sum_{c,d} A(c,d) \sum_{\kappa_j = \kappa \atop \kappa_j^2 + \frac{1}{4} \in \text{Sp}(\Gamma_0(cd))} \epsilon_j R_j \left(\frac{1}{2}, 0\right) L_j \left(\frac{1}{2}; 1/c\right) \neq 0. \hfill (30.3)$$

Remark 5. As to the possible poles coming from the contribution of the continuous spectrum, one may follow the discussion in [11, p. 211]. In view of (28.7), we may have poles at

$$(2l + 1)\frac{\pi i}{\log p}, \quad l \in \mathbb{Z}, \hfill (30.4)$$
where \( p|ab \) with \( \alpha_a \alpha_b \neq 0 \). Thus it can be asserted, somewhat informally, that as the length of \( A \) tends to infinity the imaginary axis is gradually filled up with poles of \( Z_2(s; A) \).

### 31. To deal with \( R(\kappa; A) \), we adopt the argument of [11, Section 3.3].

Thus, on noting the definitions (17.1) and (26.2), we consider more generally the sum

\[
\mathcal{D}(u, v; h) = \zeta(u + v) \sum_j \varrho_j(-f; 1/c) D_j(u, u - v) \frac{h(\kappa_j)}{\cosh \pi \kappa_j} = \zeta(u + v) \mathcal{D}_1(u, v; h), \tag{31.1}
\]

with an integer \( f > 0 \), where the sum is extended over \( \kappa_j^2 + \frac{1}{4} \in \text{Sp}(\Gamma_0(cd)) \) with a fixed pair \( c, d, \mu(cd) \neq 0 \); also the weight \( h \) is assumed to be an even, entire function such that

\[
h(\pm \frac{1}{2}i) = 0 \tag{31.2}
\]

and

\[
h(r) \ll \exp(-c_0|r|^2), \tag{31.3}
\]

with a certain \( c_0 > 0 \), in any fixed horizontal strip. By Lemmas 10–12, \( \mathcal{D}(u, v; h) \) is meromorphic over \( \mathbb{C}^2 \), and regular in the vicinity of \( \left( \frac{1}{2}, \frac{1}{2} \right) \); in particular, we have

\[
\mathcal{D} \left( \frac{1}{2}, \frac{1}{2}; h \right) = \sum_j \epsilon_j \varrho_j(f; 1/c) R_j \left( \frac{1}{2}, 0 \right) \frac{h(\kappa_j)}{\cosh \pi \kappa_j}. \tag{31.4}
\]

In the region of absolute convergence, we have, by definition,

\[
\mathcal{D}_1(u, v; h) = \sum_m m^{-u} \sigma_{u-v}(m) \sum_j \varrho_j(-f; 1/c) \varrho_j(m; \infty) \frac{h(\kappa_j)}{\cosh \pi \kappa_j}. \tag{31.5}
\]

We apply (21.1) to the inner sum, getting

\[
\mathcal{D}_1(u, v; h) = \mathcal{D}_2(u, v; h) + \mathcal{D}_3(u, v; h) \tag{31.6}
\]

where

\[
\mathcal{D}_2(u, v; h) = \frac{1}{c \sqrt{d}} \sum_m m^{-u} \sigma_{u-v}(m) \sum_{l \equiv 1 \pmod{l, d}} \frac{1}{l} S(m, -df; cl) \psi \left( \frac{4\pi}{cl \sqrt{d}} \sqrt{mf} \right), \tag{31.7}
\]

with

\[
\psi(x) = \frac{4}{\pi^2} \int_{-\infty}^{\infty} r \sinh(\pi r) K_{2ir}(x) h(r) dr, \tag{31.8}
\]
and

\[
L(u + v, \chi_{cd}) D_3(u, v; h) = -\frac{1}{\pi} \sum_{c_1 d_1 = cd} \frac{1}{d_1} \int_{-\infty}^{\infty} \left( \frac{d_1}{\sqrt{d}} \right)^{1+2ir} \frac{f^{ir} \sigma_{-2ir}(f; \chi_{cd})}{|L(1+2ir, \chi_{cd})|^2} \zeta(u + ir) \zeta(u - ir) \zeta(v + ir) \zeta(v - ir) \\
\times \prod_{p | (c_1, c)(d_1, d)} \left( \sigma_{-2ir}(f_p) \left(1 - \frac{1}{p^{1+2ir}}\right) - 1\right) \prod_{p | d_1} \left(1 - \frac{1}{p^{u-ir}}\right) \left(1 - \frac{1}{p^{v-ir}}\right) \prod_{p | c_1} \left(1 - \frac{1}{p^{1-2ir}}\right) \left(1 - \frac{1}{p^{u+v}}\right) - \left(1 - \frac{1}{p^{u-ir}}\right) \left(1 - \frac{1}{p^{v-ir}}\right) \right) h(r) dr \tag{31.9}
\]

in which we have used (27.8) with \( s = u + ir, \alpha = u - v. \)

32. To transform \( D_2 \) we use the formula

\[
\psi(x) = \frac{1}{\pi^2} \int_{(\alpha)} \hat{h}(s) \frac{\cos \pi s}{\cos \pi s} \left(\frac{x}{2}\right)^{-2s} ds, \quad 0 < \alpha < -\frac{1}{2}, \tag{32.1}
\]

where

\[
\hat{h}(s) = \int_{-\infty}^{\infty} r h(r) \frac{\Gamma(s + ir)}{\Gamma(1 - s + ir)} dr \tag{32.2}
\]

(see [11, p. 113]). Moving the last path far down, we see that \( \hat{h} \) is entire. Also we have

\[
\hat{h}(\pm \frac{1}{2}) = 0, \tag{32.3}
\]

and (32.1) is replaced by

\[
\psi(x) = \frac{1}{\pi^2} \int_{(\alpha)} \hat{h}(s) \frac{\cos \pi s}{\cos \pi s} \left(\frac{x}{2}\right)^{-2s} ds, \quad -\frac{3}{2} < \alpha < \frac{3}{2}. \tag{32.4}
\]

The integrand decays exponentially, which facilitate our discussion greatly. We stress that the presence of the factor \( \epsilon_j \) in (30.3) has induced this effect.

Thus in (31.7) we have

\[
\sum_{\substack{l \mid (l, d) = 1}} \frac{1}{l} S(m, -d f; cl) \psi \left( \frac{4\pi}{c l \sqrt{d}} \sqrt{m f} \right),
\]

\[
= \frac{1}{\pi^2} \sum_{\substack{l \mid (l, d) = 1}} \frac{1}{l} S(m, -d f; cl) \int_{(\alpha)} \hat{h}(s) \frac{\cos \pi s}{\cos \pi s} \left(\frac{2\pi}{c l \sqrt{d}} \sqrt{m f}\right)^{-2s} ds, \tag{32.5}
\]

with

\[
-\frac{3}{2} < \alpha < -\frac{1}{4}. \tag{32.6}
\]
The right side of (32.5) converges absolutely. Then we assume that
\[ \Re u, \Re v > 1 - \alpha. \] (32.7)

On this we insert (32.5) into (31.7), and get
\[ \mathcal{D}_2(u, v; h) = \frac{1}{\pi^2 c \sqrt{d}} \sum_{(l, d) = 1} \frac{1}{l} P(u, v; l), \] (32.8)

where
\[ P(u, v; l) = \int_{(\alpha)} \left( \frac{2\pi}{cl \sqrt{d} \sqrt{f}} \right)^{-2s} \frac{\hat{h}(s)}{\cos(\pi s)} \times \left\{ \sum_{a = 1}^{cl} \exp(-2\pi i df a / cl) \sum_{m = 1}^{\infty} \sigma_{u-v}(m) \exp(2\pi im\alpha / cl) m^{-u-s} \right\} ds, \] (32.9)

with \( a\alpha \equiv 1 \) mod \( cl \).

We introduce further a sub-region of (32.7):
\[ 1 - \alpha < \Re u, \Re v < -\beta, \quad -\frac{3}{2} < \beta < \alpha - 1, \quad -\frac{1}{2} < \alpha < -\frac{1}{4}. \] (32.10)

Then we move the path in (32.9) to \((\beta)\). On the assumption \( u \neq v \), we have, by Estermann’s functional equation (see [11, Lemma 3.7]),
\[ P(u, v; l) = -2\pi ic_{cl}(f)(cl)^{1-u-v} \left\{ (2\pi \sqrt{f / d})^{2(u-1)} \hat{h}(1-u)/\cos\pi u \\
+ (2\pi \sqrt{f / d})^{2(v-1)} \hat{h}(1-v)/\cos\pi v \right\} \\
+ 2(2\pi)^{u+v-2}(cl)^{1-u-v} \left\{ \sum_{m = 1}^{\infty} m^{u-1} \sigma_{v-u}(m)c_{cl}(dm + f)\Psi_+(u, v; dm / f; h) \\
+ \sum_{m = 1}^{\infty} m^{u-1} \sigma_{v-u}(m)c_{cl}(dm - f)\Psi_-(u, v; dm / f; h) \right\}, \] (32.11)

where
\[ \Psi_+(u, v; x; h) = -\int_{(\beta)} \Gamma(1-u-s)\Gamma(1-v-s) \cos(\pi (s + \frac{1}{2}(u + v))) \frac{\hat{h}(s)}{\cos\pi s} x^s ds \] (32.12)

and
\[ \Psi_-(u, v; x; h) = \cos\left(\frac{1}{2}\pi(u-v)\right) \int_{(\beta)} \Gamma(1-u-s)\Gamma(1-v-s) \frac{\hat{h}(s)}{\cos(\pi s)} x^s ds. \] (32.13)
We insert (32.11) into (32.8). We get under (32.10) that
\[ L(u + v, \chi_{cd}) D_2(u, v; h) = \{ D^1_2 + D^2_2 + D^3_2 + D^4_2 \} (u, v; h), \]

where

\[ D^1_2 = \frac{2}{\pi i \sqrt{d}} \left\{ \left( \frac{2(1 - u)}{\cos \pi u} \right)^{2(u-1)} \frac{h(1-u)}{\cos \pi u} \zeta(1-u+v) + \left( \frac{2(v-1)}{\cos \pi v} \right) \frac{h(1-v)}{\cos \pi v} \zeta(1-v+u) \right\} \times \sigma_{1-u-v}(f, \chi_{cd}) \prod_{p \mid c} \left( \sigma_{1-u-v}(f_p) \left( 1 - \frac{1}{p^{u+v}} \right) - 1 \right) \]

\[ D^2_2 = 8 \frac{(2\pi)^{u+v-4}}{\sqrt{d}} \sum_m m^{u-1} \sigma_{v-u}(m) \sigma_{1-u-v}(dm + f; \chi_{cd}) \Psi_+(u, v; dm/f; h) \times \prod_{p \mid c} \left( \sigma_{1-u-v}((dm + f)_p) \left( 1 - \frac{1}{p^{u+v}} \right) - 1 \right), \]

\[ D^3_2 = 8 \frac{(2\pi)^{u+v-4}}{\sqrt{d}} \sum_{\substack{m \mid dm \neq f}} m^{u-1} \sigma_{v-u}(m) \sigma_{1-u-v}(dm - f; \chi_{cd}) \Psi_-(u, v; dm/f; h) \times \prod_{p \mid c} \left( \sigma_{1-u-v}((dm - f)_p) \left( 1 - \frac{1}{p^{u+v}} \right) - 1 \right), \]

\[ D^4_2 = 8 \frac{(2\pi)^{u+v-4}}{\sqrt{d}} \frac{c^{u+v}}{(u+v)^4} L(u + v - 1, \chi_d) (f/d)^{u-1} \sigma_{v-u}(f/d) \Psi_-(u, v; 1; h), \]

in which \( D^4_2 \) appears only when \( d \mid f \).

The expansion (33.1) with (33.2) has been proved under the assumption that \( u \neq v \) and (32.10) holds. However, the former can be dropped now; and also \( D^2_2 \) and \( D^3_2 \) converge absolutely if \( 1 + \beta < \text{Re} u, \text{Re} v < -\beta \). In particular, \( L(u + v, \chi_{cd}) D_2(u, v; h) \) is regular at \( (\frac{1}{2}, \frac{1}{2}) \), and there (33.1) holds.

Further, shifting the path in (31.9) upward and downward appropriately, we have the following continuation of \( D_3 \) to the domain \( \text{Re} u, \text{Re} v < 1 \):
\[ L(u + v, \chi_{cd}) D_3(u, v; h) = \{ D^1_3 + D^2_3 + D^3_3 \} (u, v; h), \]

where

\[ D^1_3 = -\frac{1}{\pi} \sum_{c_1d_1 = cd} \frac{1}{d_1} \int_{-\infty}^{\infty} \left( \frac{(d_1, d)}{\sqrt{d}} \right)^{1+2i\epsilon} \frac{f^{ir} \sigma_{-2ir}(f; \chi_{cd})}{\text{Re}(1 + 2i\epsilon, \chi_{cd})^2} \times \zeta(u + ir) \zeta(u - ir) \zeta(v + ir) \zeta(v - ir) \times \prod_{p \mid (c_1, c)(d_1, d)} \left( \sigma_{-2ir}(f_p) \left( 1 - \frac{1}{p^{u+2ir}} \right) - 1 \right) \prod_{p \mid d_1} \left( 1 - \frac{1}{p^{u-ir}} \right) \left( 1 - \frac{1}{p^{v-ir}} \right) \]
\[ \times \prod_{p|c_1} \left\{ \left(1 - \frac{1}{p^{1-2r}} \right) \left(1 - \frac{1}{p^{u+v}} \right) - \left(1 - \frac{1}{p^{u-v}} \right) \left(1 - \frac{1}{p^{v-u}} \right) \right\} h(r)dr, \]

\[ D_3^2 = -2f^{1-u}\sigma_{2(u-1)}(f;\chi_{cd}) \frac{\zeta(u + v - 1)\zeta(v - u + 1)}{L(3 - 2u, \chi_{cd})} h(i(u - 1)) \]

\[ \times \sum_{c_1d_1=cd} \frac{\varphi(c_1)}{c_1^{u+v}d_1} \left( \frac{(d_1, d)}{\sqrt{d}} \right)^{3-2u} \prod_{p|(c_1,c)(d_1,d)} \left( \sigma_{2(u-1)}(f_p) \left(1 - \frac{1}{p^{3-2u}} \right) - 1 \right) \]

\[ \times \prod_{p|d_1} \left(1 - \frac{1}{p^{u+v-1}} \right) \]

\[ - 2f^{1-v}\sigma_{2(v-1)}(f;\chi_{cd}) \frac{\zeta(u + v - 1)\zeta(u - v + 1)}{L(3 - 2v, \chi_{cd})} h(i(v - 1)) \]

\[ \times \sum_{c_1d_1=cd} \frac{\varphi(c_1)}{c_1^{u+v}d_1} \left( \frac{(d_1, d)}{\sqrt{d}} \right)^{3-2v} \prod_{p|(c_1,c)(d_1,d)} \left( \sigma_{2(v-1)}(f_p) \left(1 - \frac{1}{p^{3-2v}} \right) - 1 \right) \]

\[ \times \prod_{p|d_1} \left(1 - \frac{1}{p^{u+v-1}} \right), \]

\[ D_3^3 = -2f^{u-1}\sigma_{2(1-u)}(f;\chi_{cd}) \frac{\zeta(u + v - 1)\zeta(v - u + 1)}{L(3 - 2u, \chi_{cd})} h(i(u - 1)) \prod_{p|cd} \left(1 - \frac{1}{p^{2u-1}} \right)^{-1} \]

\[ \times \sum_{c_1d_1=cd} \frac{\varphi(d_1)}{d_1^{2u-1}} \left( \frac{(d_1, d)}{\sqrt{d}} \right)^{2u-1} \prod_{p|(c_1,c)(d_1,d)} \left( \sigma_{2(1-u)}(f_p) \left(1 - \frac{1}{p^{2u-1}} \right) - 1 \right) \]

\[ \times \prod_{p|d_1} \left(1 - \frac{1}{p^{u+1}} \right) \prod_{p|c_1} \left\{ \left(1 - \frac{1}{p^{3-2u}} \right) \left(1 - \frac{1}{p^{u+v}} \right) - \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{p^{v-u+1}} \right) \right\}, \]

\[ - 2f^{v-1}\sigma_{2(1-v)}(f;\chi_{cd}) \frac{\zeta(u + v - 1)\zeta(u - v + 1)}{L(3 - 2v, \chi_{cd})} h(i(v - 1)) \prod_{p|cd} \left(1 - \frac{1}{p^{2v-1}} \right)^{-1} \]

\[ \times \sum_{c_1d_1=cd} \frac{\varphi(d_1)}{d_1^{2v-1}} \left( \frac{(d_1, d)}{\sqrt{d}} \right)^{2v-1} \prod_{p|(c_1,c)(d_1,d)} \left( \sigma_{2(1-v)}(f_p) \left(1 - \frac{1}{p^{2v-1}} \right) - 1 \right) \]

\[ \times \prod_{p|d_1} \left(1 - \frac{1}{p^{u+1}} \right) \prod_{p|c_1} \left\{ \left(1 - \frac{1}{p^{3-2v}} \right) \left(1 - \frac{1}{p^{u+v}} \right) - \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{p^{v-u+1}} \right) \right\}. \quad (33.4) \]

We see readily that \( D_3^1 \) and \( D_3^2 \) are regular at \((\frac{1}{2}, \frac{1}{2})\). As to \( D_3^3 \), the factors \( \prod_{p|cd}(1 - p^{1-2u})^{-1} \) and \( \prod_{p|cd}(1 - p^{1-2v})^{-1} \) diverge at the point unless \( cd = 1 \); however, \( D_3^3 \) itself must be regular there, for \( L(u + v, \chi_{cd})D_1, L(u + v, \chi_{cd})D_2 \) are regular, and thus \( L(u + v, \chi_{cd})D_3 \) as well.

Hence, from (31.1), (31.6), (33.1), and (33.3), we obtain

\[ D \left( \frac{1}{2}, \frac{1}{2}; h \right) = \frac{cd}{\varphi(cd)} \{ D_2^1 + D_2^2 + D_2^3 + D_2^4 + D_2^1 + D_2^2 + D_2^3 \} \left( \frac{1}{2}, \frac{1}{2}; h \right). \quad (33.5) \]
34. The last equation gives

**Lemma 16.** We have, with the weight $h$ as above,

\[
\sum_j \epsilon_j \varphi_j(f; 1/c) R_j(\frac{1}{2}, 0) \frac{h(k_j)}{\cosh \pi k_j} = \sum_{a=1}^7 \mathcal{H}_a(f; h),
\]

where

\[
\mathcal{H}_1 = \frac{2cd}{\pi^3 i \varphi(cd)} \left\{ (c_E - \log(2\pi \sqrt{f/d})) \left( \hat{h}'(\frac{1}{2}) + \frac{1}{4} \hat{h}''(\frac{1}{2}) \right) \tau(f, \chi_{cd}) f^{-\frac{1}{2}} \prod_{p|c} \left( \tau(f_p) \left( 1 - \frac{1}{p} \right) - 1 \right) \right\},
\]

\[
\mathcal{H}_2 = \frac{c\sqrt{d}}{\pi^3 \varphi(cd)} \sum_{m} m^{-\frac{1}{2}} \tau(m) \tau(dm + f; \chi_{cd}) \Psi_+(dm/f; h) \prod_{p|c} \left( \tau((dm + f)_p) \left( 1 - \frac{1}{p} \right) - 1 \right),
\]

\[
\mathcal{H}_3 = \frac{c\sqrt{d}}{\pi^3 \varphi(cd)} \sum_{dm \neq f} m^{-\frac{1}{2}} \tau(m) \tau(dm - f; \chi_{cd}) \Psi_-(dm/f; h) \prod_{p|c} \left( \tau((dm - f)_p) \left( 1 - \frac{1}{p} \right) - 1 \right),
\]

\[
\mathcal{H}_4 = -\frac{\delta_{d,1}}{2\pi^3} f^{-\frac{1}{2}} \tau(f) \Psi_-(1; h),
\]

\[
\mathcal{H}_a = \frac{cd}{\varphi(cd)} D_3^{3-a} \left( \frac{1}{2}, \frac{1}{2}; h \right), \quad 5 \leq a \leq 7.
\]

Here $\tau$ is the divisor function, $\Psi_{\pm}(x; h) = \Psi_{\pm} \left( \frac{1}{2}, \frac{1}{2}; x; h \right)$, and $\mathcal{H}_4$ vanishes unless $d = 1$.

This is a counterpart of [11, Lemma 3.8], and follows immediately from (31.4), (33.2) and (33.5). We have left $\mathcal{H}_a(f; h)$, $5 \leq a \leq 7$, without computing it explicitly, because it seems better to avoid the highly complicated computation of $D_3^{3-a} \left( \frac{1}{2}, \frac{1}{2}; h \right)$ caused by the two products over $p|cd$ mentioned above; and in fact those $\mathcal{H}_a(f; h)$ will readily turn out to be negligible in our application of (34.1) to be given in the next section.

From [11, pp. 119–121], we quote the following:

\[
(\hat{h})'(\frac{1}{2}) = 2 \int_{-\infty}^{\infty} rh(r) \frac{\Gamma'(\frac{1}{2} + ir)}{\Gamma(\frac{1}{2})} dr, \quad (\hat{h})''(\frac{1}{2}) = 4 \int_{-\infty}^{\infty} rh(r) \left\{ \frac{\Gamma'(\frac{1}{2} + ir)}{\Gamma(\frac{1}{2})} \right\}^2 dr,
\]

\[
\Psi_+(x; h) = 2\pi \int_0^1 \left\{ y(1 - y)(1 + y/x) \right\}^{-\frac{1}{2}} \int_{-\infty}^{\infty} rh(r) \tanh(\pi r) \left\{ \frac{y(1 - y)}{x + y} \right\}^r dr dy.
\]

For $x > 1$

\[
\Psi_-(x; h) = 2\pi i \int_0^1 \left\{ y(1 - y)(1 - y/x) \right\}^{-\frac{1}{2}} \int_{-\infty}^{\infty} \frac{rh(r)}{\cosh(\pi r)} \left\{ \frac{y(1 - y)}{x - y} \right\}^r dr dy.
\]

For $x = 1$

\[
\Psi_-(1; h) = 2\pi^2 \int_{-\infty}^{\infty} rh(r) \frac{\sinh(\pi r)}{(\cosh(\pi r))^2} dr.
\]
For \(0 < x < 1\)

\[
\Psi_-(x; h) = \int_0^\infty \left\{ \int_{(\beta)} x^s (y(y + 1))^{s-1} \frac{\Gamma(\frac{1}{2} - s)^2}{\Gamma(1 - 2s) \cos(\pi s)} ds \right\} \\
\times \left\{ \int_{-\infty}^\infty rh(r) \left(\frac{y}{1+y}\right)^{ir} dr \right\} dy,
\]

\(-\frac{3}{2} < \beta < \frac{1}{2}, \beta \neq -\frac{1}{2}. \tag{34.7}\)

35. We shall continue our discussion, adopting the argument given in [11, pp. 124–130]. Thus we first state the following approximation for \(L_j \left(\frac{1}{2}; 1/c\right)\): Let \(K\) tend to infinity, and assume that

\[
|\kappa_j - K| \leq G \log K \tag{35.1}
\]

with

\[
K^{\frac{1}{2} + \delta} < G < K^{1-\delta}, \quad 0 < \delta < \frac{1}{2}. \tag{35.2}\]

Then we have, for any \(N \geq 1\) and \(\lambda = C \log K\) with a sufficiently large \(C > 0\),

\[
L_j \left(\frac{1}{2}; 1/c\right) = \sum_{f \leq 3K \sqrt{cd}} \varrho_j(f; 1/c)f^{-\frac{1}{2}} \exp\left(-\left(f/(K \sqrt{cd})\right)^\lambda\right) \\
- \sum_{\nu=0}^{N_1} \sum_{f \leq 3K \sqrt{cd}} \varrho_j(f; 1/c)f^{-\frac{1}{2}}U_\nu(f/(K \sqrt{cd}))(1 - (\kappa_j/K)^2)^\nu + O(K^{-\frac{1}{2}N} + K^{-\frac{1}{2}C}), \tag{35.3}
\]

with the implied constant depending only on \(\delta, C,\) and \(N\). Here \(N_1 = [3N/\delta]\) and

\[
U_\nu(x) = \frac{1}{2\pi i \lambda} \int_{(-\lambda)^-} (4\pi^2 x)^w u_\nu(w) \Gamma(w/\lambda) dw, \tag{35.4}
\]

where \(u_\nu(w)\) is a polynomial of degree \(\leq 2N_1\), whose coefficients are independent of \(\kappa_j\) and bounded by a constant depending only on \(\delta\) and \(N\).

In fact, this assertion is a counterpart of [11, Lemma 3.9] and the proof is analogous; the necessary change is only in that we now use \((23.13)\) instead under the assumption \(\varpi_j \epsilon_j = +1\) as \(L \left(\frac{1}{2}; 1/c\right) = 0\) if \(\varpi_j \epsilon_j = -1\).

With this, we now set, in \((34.1)\),

\[
h(r) = \left(r^2 + \frac{1}{4}\right) \left\{ \exp\left(-((r - K)/G)^2\right) + \exp\left(-((r + K)/G)^2\right) \right\}. \tag{35.5}\]

We have

\[
(\hat{h})' \left(\frac{1}{2}\right) = 2i\pi^\frac{3}{2} K^3 G + O(K G^3),
\]

\[
(\hat{h})'' \left(\frac{1}{2}\right) = 8i\pi^\frac{3}{2} K^3 G \log K + O(K G^3 \log K). \tag{35.6}
\]

(see [11, p. 129]).
We have, by (34.1) and (35.4),
\[
\sum_j \epsilon_j R_j \left( \frac{1}{2}, 0 \right) L_j \left( \frac{1}{2}; 1/c \right) \frac{h(\kappa_j)}{\cosh \pi \kappa_j} = \sum_{f \leq \sqrt{c}d} f^{-\frac{1}{2}} \exp\left(-\frac{f}{\sqrt{cd}}\right) \sum_{a=1}^{7} \mathcal{H}_a(f; h)
\]
\[
\quad - \sum_{\nu \leq N_1} \sum_{f \leq \sqrt{c}d} f^{-\frac{1}{2}} U_\nu(f/(\sqrt{cd})) \sum_{a=1}^{7} \mathcal{H}_a(f; h_\nu) + O(1),
\]
(35.7)
where the five terms correspond to those on the right side of (34.1), respectively, with the present $h$ and $h_\nu(r) = h(r)(1 - (r/K)^2)^\nu$. Since we have imposed (35.1)–(35.2), those terms with $\nu \geq 1$ can actually be ignored, and it suffices to consider instead
\[
\sum_{f \leq \sqrt{c}d} f^{-\frac{1}{2}} \exp\left(-\frac{f}{\sqrt{cd}}\right) \sum_{a=1}^{7} \mathcal{H}_a(f; h)
\]
\[
\quad - \sum_{f \leq \sqrt{c}d} f^{-\frac{1}{2}} U_0(f/(\sqrt{cd})) \sum_{a=1}^{7} \mathcal{H}_a(f; h).
\]
(35.8)
The discussion in [11, pp. 128–129] works just fine with our present situation as well; and the contribution of $\mathcal{H}_a$, $a = 2, 3, 4$, turns out to be negligible.

**Remark 6.** However, if the uniformity in the Stufe $\sqrt{c}d$ is required, then this part of our argument should become subtle.

As to $\mathcal{K}_1$, its contribution to (35.8) is equal to
\[
\frac{4cd}{\pi^2 \varphi(cd)} K^3 G(K_1 + K_2) + O(KG^3(\log K)^2),
\]
(35.9)
where we have used (35.6), and
\[
\mathcal{K}_1 = \sum_f \left( c_E - \log(2\pi \sqrt{f/d}) + \log K \right) \exp\left(-\frac{f}{\sqrt{cd}}\right) \tau(f: \chi_{cd})
\]
\[
\quad \times \frac{\tau(f; \chi_{cd})}{f} \prod_{p|c} \left( \tau(f_p) \left( 1 - \frac{1}{p} \right) - 1 \right),
\]
\[
\mathcal{K}_2 = \sum_f \left( c_E - \log(2\pi \sqrt{f/d}) + \log K \right) U_0(f/(\sqrt{cd}))
\]
\[
\quad \times \frac{\tau(f; \chi_{cd})}{f} \prod_{p|c} \left( \tau(f_p) \left( 1 - \frac{1}{p} \right) - 1 \right).
\]
(35.10)
To compute $\mathcal{K}_1$, $\mathcal{K}_2$, let us put
\[
z(s) = \sum_f \frac{\tau(f; \chi_{cd})}{f^{s+1}} \prod_{p|c} \left( \tau(f_p) \left( 1 - \frac{1}{p} \right) - 1 \right).
\]
(35.11)
Then
\[ K_1 = \frac{1}{2\pi i\lambda} \int_{(1)} \left\{ (\log(K\sqrt{d}/2\pi) + c_E)z(s) + \frac{1}{2}z'(s) \right\} (K\sqrt{cd})^s \Gamma(s/\lambda) ds, \]
\[ K_2 = -\frac{1}{2\pi i}\int_{(-1)} \left\{ (\log(K\sqrt{d}/2\pi) + c_E)z(-s) + \frac{1}{2}z'(-s) \right\} (4\pi^2/K\sqrt{cd})^s u_0(w) \Gamma(s/\lambda) ds. \] (35.12)

The latter can be replace by
\[ -\frac{1}{2\pi i\lambda} \int_{(-1)} \left\{ (\log(K\sqrt{d}/2\pi) + c_E)z(-s) + \frac{1}{2}z'(-s) \right\} (4\pi^2/K\sqrt{cd})^s \Gamma(s/\lambda) ds \] (35.13)
with an admissible error (see [11, p. 127] for a description of \( u_0 \)).

We have
\[ z(s) = \zeta(s+1)^2 \prod_{p|d} \left( 1 - \frac{1}{p^{s+1}} \right) \prod_{p|c} \left( 1 - \frac{1}{p^{s+1}} - \frac{2}{p^{s+2}} + \frac{1}{p^{2s+2}} + \frac{1}{p^{2s+3}} \right). \] (35.14)

Hence, we get
\[ K_1, K_2 \sim \frac{1}{3}(\log K)^3 \frac{\varphi(cd)}{cd} \prod_{p|c} \left( 1 - \frac{1}{p^2} \right). \] (35.15)

### 36. It remains for us to deal with \( \mathcal{H}_a, 5 \leq a \leq 7 \).

We have obviously
\[ D_3^1 \left( \frac{1}{2}, \frac{1}{2}; h \right) \ll \tau(f) \int_{-\infty}^{\infty} \left| \zeta \left( \frac{1}{2} + ir \right) \right|^4 h(r) dr \ll \tau(f) K^3 (\log K)^5, \] (36.1)

which can of course be replaced by a better bound, but for our purpose this is sufficient. We see that the contribution of \( \mathcal{H}_5 \) to (35.8) is \( \ll K^2 (\log K)^7 \), which is negligible in view of (35.2), (35.9), and (35.15).

As to \( \mathcal{H}_6 \) and \( \mathcal{H}_7 \), we shall treat the latter only, for the former is analogous and certainly easier than the latter. As we have remarked already, \( D_3^3(u, v; h) \) is regular in the vicinity of \( (\frac{1}{2}, \frac{1}{2}) \). Thus we have
\[ D_3^3 \left( \frac{1}{2}, \frac{1}{2}; h \right) = -\frac{1}{(2\pi)^2} \int_{C_2} \int_{C_1} \frac{D_3^3(u, v; h)}{(u - \frac{1}{2})(v - \frac{1}{2})} dudv, \] (36.2)

where
\[ C_1 : |u - \frac{1}{2}| = \frac{1}{B(2 + \log cd)} , \quad C_2 : |v - \frac{1}{2}| = \frac{1}{2B(2 + \log cd)} . \] (36.3)
with a sufficiently large constant $B$. This integrand is, by the explicit formula for $D_3^3(u,v;h)$ in (33.4),
\[
\ll \exp\left(-\frac{1}{2}(K/G)^2\right),
\]
and $\mathcal{H}_7$ is negligible.

Hence we have obtained

**Lemma 17.** Let $h$ be as in (35.5) with (35.1)-(35.2). Then we have, for any fixed $c,d$ with $\mu(cd) \neq 0$,
\[
\sum_{\kappa_j \in \text{Sp}(\Gamma_0(cd))} \epsilon_j R_j \left(\frac{1}{2}, 0\right) L_j \left(\frac{1}{2}; 1/c\right) \frac{h(\kappa_j)}{\cosh \pi \kappa_j} \sim \frac{8}{3\pi^2} GK^3 (\log K)^3 \prod_{p|c} \left(1 - \frac{1}{p^2}\right).
\]

In particular, if $A$ is fixed, we have
\[
\sum_{\kappa} \mathcal{R}(\kappa; A) \frac{h(\kappa)}{\cosh \pi \kappa} \sim \frac{8}{3\pi^2} GK^3 (\log K)^3 \sum_{c,d} A(c,d) \prod_{p|c} \left(1 - \frac{1}{p^2}\right),
\]
where $\frac{1}{4} + \kappa^2 \in \bigcup_{c,d} \text{Sp}(\Gamma_0(cd))$ with $\mu(cd) \neq 0$.

Therefore we have established

**Theorem.** Provided $\alpha_n > 0$ for square-free $n$ and $\alpha_n = 0$ otherwise, the function $Z_2(s;A)$ has infinitely many simple poles on the line $\text{Re } s = \frac{1}{2}$.

This restriction on the support of $\alpha_n$ will be lifted in our forthcoming work.

Our result suggests that the Mellin transform
\[
Z_3(s;1) = \int_1^\infty |\zeta(\frac{1}{2} + it)|^6 t^{-s} dt
\]
should have the line $\text{Re } s = \frac{1}{2}$ as a natural boundary, for $|\zeta|^6 = |\zeta|^4 |\zeta|^2$ and $|\zeta|^2$ may be replaced by a finite expression similar to $|A|^2$ via the approximate functional equation. The same was speculated also by a few people other than us, but it appears that our theorem is so far the sole explicit evidence supporting this conjectural assertion. At any event, in view of of REMARK 5 above, it appears reasonable for us to maintain that $Z_3(s;1)$ does not continue beyond the imaginary axis.

This entails

**Problems:**

(1) Is the set $\bigcup_{q \geq 1} \text{Sp}(\Gamma_0(q))$ dense in the positive real axis?
(2) Is the set of $\kappa$ satisfying (30.3) dense in the positive real axis?

(3) Is the set of $\kappa$ satisfying (30.3) dense in any half line?

(4) Is the set of $\kappa$ satisfying (30.3) dense in any interval whose left end point is the origin?

Obviously (1) is to be solved first and (2) must be far more difficult than (1). The third, weaker than (2), appears highly plausible in the light of Lemma 17; on the other hand our method does not seem to extend without new twists so as to include the situation of (4), i.e., the detection of low lying poles.

Addendum. Recently C.P. Hughes and M.P. Young (arXiv:0709.2345 [math.NT]) obtained an asymptotic formula for the mean value (30.1) where the length of $A$ is less than $T^\eta$ with any fixed $\eta < 1/11$. They did not employ the spectral theory of Kloosterman sums. Our method should give a better result than theirs, if it is combined with works by N. Watt on this mean value.

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