Extended quantum field theory, index theory and the parity anomaly

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Abstract

We use techniques from functorial quantum field theory to provide a geometric description of the parity anomaly in fermionic systems coupled to background gauge and gravitational fields on odd-dimensional spacetimes. We give an explicit construction of a geometric cobordism bicategory which incorporates general background fields in a stack, and together with the theory of symmetric monoidal bicategories we use it to provide the concrete forms of invertible extended quantum field theories which capture anomalies in both the path integral and Hamiltonian frameworks. Specialising this situation by using the extension of the Atiyah-Patodi-Singer index theorem to manifolds with corners due to Loya and Melrose, we obtain a new Hamiltonian perspective on the parity anomaly. We compute explicitly the 2-cocycle of the projective representation of the gauge symmetry on the quantum state space, which is defined in a parity-symmetric way by suitably augmenting the standard chiral fermionic Fock spaces with Lagrangian subspaces of zero modes of the Dirac Hamiltonian that naturally appear in the index theorem. We describe the significance of our constructions for the bulk-boundary correspondence in a large class of time-reversal invariant gauge-gravity symmetry-protected topological phases of quantum matter with gapless charged boundary fermions, including the standard topological insulator in $3 + 1$ dimensions.
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1 Introduction and summary

The parity anomaly in field theories of fermions coupled to gauge and gravitational backgrounds in odd dimensions was discovered over 30 years ago [Red84, NSS83, AGDPM85], and has found renewed interest recently because of its relevance to certain topological states of quantum matter [Wit16a, SW16]. The purpose of this paper is to re-examine the parity anomaly from the perspective of functorial quantum field theory, and in particular to elucidate its appearance in the Hamiltonian framework (see e.g. [CL87]) which has been largely unexplored. We begin with some preliminary physics background to help motivate the problem we study.

1.1 Anomalies and symmetry-protected topological phases

In recent years considerable progress has been made in condensed matter physics towards understanding the distinct possible quantum phases of matter with an energy gap through their universal long-wavelength properties, and the ensuing interplay between global symmetries and topological degrees of freedom. In many instances the effective low-energy (long-range) continuum theory of a lattice Hamiltonian model can be formulated as a relativistic field theory, which for a gapped phase can be usually reduced to a topological quantum field theory that describes the ground states and their response to external sources; such a gapped phase is known as a ‘topological phase’ of matter. The classic example of this is the integer quantum Hall state and its effective description as a three-dimensional Chern-Simons gauge theory.

A gapped phase $\Psi$ is ‘short-range entangled’ [CGW10], or ‘invertible’ [Fre14, FH16], if there exists a gapped phase $\Psi^{-1}$ such that $\Psi \otimes \Psi^{-1}$ can be deformed to a trivial product state by an adiabatic transformation of the Hamiltonian without closing the bulk energy gap. The macroscopic properties of such gapped states are described by particular kinds of topological quantum field theories which are also called ‘invertible’ with respect to the tensor product of vector spaces [Fre14]; they have the property that their Hilbert space of quantum states is one-dimensional and all propagators are invertible, in contrast to most quantum field theories. The correspondence between invertible topological field theories and short-range entangled phases of matter is discussed in [FH16].

Some gapped systems are non-trivial as a consequence of intrinsic topological order or of protection by a global symmetry group $G$. A short-range entangled state is ‘$G$-symmetry-protected’ if it can be deformed to a trivial product state by a $G$-noninvariant adiabatic transformation [CGW10, CGLW13]. The gapped bulk system is then characterised by gapless boundary states, such as the chiral quantum Hall edge states, which exhibit gauge or gravitational anomalies; conversely, a $d - 1$-dimensional system whose ground state topological order is anomalous can only exist as the boundary of a $d$-dimensional topological phase. While the boundary quantum field theory on its own suffers from anomalies, the symmetry-protected boundary states are described by considering the anomalous theory ‘relative’ to the higher-dimensional bulk theory, where it becomes a non-anomalous quantum field theory under the ‘bulk-boundary correspondence’ [RML12, RZ12] in which the boundary states undertake anomaly inflow from the bulk field.

\footnote{Here ‘$\otimes$’ denotes a ‘stacking’ operation combining gapped phases together which turns them into a commutative monoid with identity element the trivial phase. The short-range entangled phases form the abelian group of units in this monoid. In the corresponding topological quantum field theory, it is the local tensor product induced by that on the state space.}
theory [CH85, FS86]. The standard examples are provided by topological insulators which are protected by fermion number conservation and time-reversal symmetry \((G = U(1) \times \mathbb{Z}_2)\) [HK10, QZ11]. The correspondence between topological field theories and symmetry-protected topological phases of matter is discussed in e.g. [Wen13, KT17, GK16, Wit16a].

In the situations just described, a field theory with anomaly is well-defined as a theory living on the boundary of a quantum field theory in one higher dimension which is invertible. The modern perspective on quantum field theories is that they should not be simply considered on a fixed spacetime manifold, but rather on a collection of manifolds which gives powerful constraints on their physical quantities; in applications to condensed matter systems involving fermions alone the manifolds in question should be spin\(^c\) manifolds [SW16]. For example, an anomaly which arises due to the introduction of a gauge-noninvariant regularization may be resolved on any given spacetime manifold, but it may not be possible to do this in a way consistent with the natural cutting and gluing constructions of manifolds. This is, for example, the case for the global anomaly at the boundary of a 3 + 1-dimensional topological superconductor [Wit16a].

The purpose of the present paper is to cast the topological field theory formulation of the parity anomaly and the 3 + 1-dimensional topological insulator, which was outlined in [Wit16a], into a more general and rigorous mathematical framework using the language of functorial quantum field theory, and to interpret the path integral description of the parity anomaly, together with its cancellation via the bulk-boundary correspondence, in a Hamiltonian framework more natural for condensed matter physics applications. Before outlining precisely what we do, let us first informally review some of the main mathematical background.

1.2 Anomalies in functorial quantum field theory

A natural framework for making the constructions discussed in Section 1.1 mathematically precise is through functorial field theories. The idea is that a \(d\)-dimensional quantum field theory should assign to a \(d\)-dimensional manifold \(M\) a complex number \(Z(M)\), its partition function. Heuristically, this number is given by a path integral of an action functional over the space of dynamical field configurations on \(M\); thus far there is no mathematically well-defined theory of such path integration in general. Functorial quantum field theory is an axiomatic approach to quantum field theory which formalises the properties expected from path integrals. A quantum field theory should not only assign complex numbers to \(d\)-dimensional manifolds, but also a Hilbert space of quantum states \(Z(\Sigma)\) to every \(d - 1\)-dimensional manifold \(\Sigma\). Moreover, the theory should assign a time-evolution operator (propagator) \(Z(\Sigma \times [t_0, t_1])\) to a cylinder over \(\Sigma\), satisfying \(Z(\Sigma \times [t_1, t_2]) \circ Z(\Sigma \times [t_0, t_1]) = Z(\Sigma \times [t_0, t_2])\). More generally, we assign an operator \(Z(M) : Z(\Sigma_-) \to Z(\Sigma_+)\) to every manifold \(M\) with a decomposition of its boundary \(\partial M = \Sigma_- \sqcup \Sigma_+\) satisfying an analogous composition law under gluing.

To make this precise, one generalises Atiyah’s definition of topological quantum field theories [Ati88] and Segal’s definition of two-dimensional conformal field theories [Seg88] to define a functorial quantum field theory as a symmetric monoidal functor\(^2\)

\[
Z : \text{Cob}_{d,d-1} \longrightarrow \text{Hilb}_\mathbb{C} ,
\]

---

\(^2\)This definition also allows for non-unitary or reflection positive, its Euclidean analogue, theories. See [PH16] for an implementation of reflection positivity, which would be the relevant concept for the theories considered in this paper. However, for simplicity we will not consider reflection positivity in this paper.
where $\text{Cob}_{d,d-1}^F$ is a category modelling physical spacetimes, and $\text{Hilb}_C$ is the category of complex Hilbert spaces and linear maps. Loosely speaking, the category $\text{Cob}_{d,d-1}^F$ contains compact $d-1$-dimensional manifolds as objects, $d$-dimensional cobordisms as morphisms, and a further class of morphisms corresponding to diffeomorphisms compatible with the background fields $F$ which represent the physical symmetries of the theory.

If $Z$ is an invertible field theory, described as a functor $Z : \text{Cob}_{d,d-1}^F \rightarrow \text{Hilb}_C$, then the partition function $Z$ of a $d-1$-dimensional field theory with anomaly $Z$ evaluated on a $d-1$-dimensional manifold $M$ takes values in the one-dimensional vector space $Z(M)$, instead of $\mathbb{C}$. We can pick a non-canonical isomorphism $Z(M) \cong \mathbb{C}$ to identify the partition function with a complex number. Furthermore, the group of symmetries acts (non-trivially) on $Z(M)$ describing the breaking of symmetries in the quantum field theory; see Section 2.1 for details.

To extend this description to the Hamiltonian formalism incorporating the quantum state spaces of a field theory $A$, we instead seek a functor $A$ that assigns linear categories to $d-2$-dimensional manifolds $\Sigma$ such that the state space $A(\Sigma)$ is an object of $A(\Sigma)$. In other words, $A$ should be an extended quantum field theory, i.e. a symmetric monoidal 2-functor $A : \text{Cob}_{d,d-1,d-2}^F \rightarrow 2\text{Vect}_C$ appropriately categorifying [1.1]. There are different possible higher replacements of the category of Hilbert spaces, but for simplicity we restrict ourselves to Kapranov-Voevodsky 2-vector spaces [KV94] in this paper, ignoring the Hilbert space structure altogether.

In an analogous way to the partition function, we want to be able to identify the state space of a quantum field theory with anomaly in a non-canonical way with a vector space, i.e. there should be an equivalence of categories $A(\Sigma) \cong \text{Vect}_C$. We enforce the existence of such an equivalence by requiring that $A$ is an invertible extended field theory, i.e. it is invertible with respect to the Deligne tensor product. We can then define a quantum field theory with anomaly in a precise manner as a natural symmetric monoidal 2-transformation $A : \mathbf{1} \Rightarrow \text{tr}A$ between a trivial field theory $\mathbf{1}$ and a certain truncation of $A$; this definition is detailed in Section 3.3. In this formalism one can in principle compute the 2-cocycles of the projective representation of the gauge group characterising the anomalous action on the quantum states [Mon15]. This description of anomalies in terms of relative field theories [PT14] is closely related to the twisted quantum field theories of [ST11, JFS17], the difference being that the twist they use does not have to come from a full field theory.

The extended quantum field theories for some physically relevant anomalies are more or less explicitly known. Some noteworthy examples are:

- Dai-Freed theories describing chiral anomalies in odd dimensions $d$ have been sketched in [Mon15]. They are an extended version of the field theories constructed in [DF95].
- Wess-Zumino theories describing the anomaly in self-dual field theories have been constructed in [Mon15].
- The theory describing the anomaly of the worldvolume theory of M5-branes has been constructed as an unextended quantum field theory in [Mon17].
The anomaly field theory corresponding to supersymmetric quantum mechanics is described in [Fre14].

This paper adds a further example to this list by giving a precise construction of an extended quantum field theory in any even dimension $d$ which encodes the parity anomaly in odd spacetime dimension. We shall now give an overview of our constructions and findings.

1.3 Summary of results

One of the technical difficulties related to extended functorial field theories is the construction of the higher cobordism category equipped with additional structure. For cobordisms with tangential structure there exist a $(\infty, n)$-categorical definition [Lur09, DS15]. A categorical version with arbitrary background fields taking into account families of manifolds has been defined by Stolz and Teichner [ST11]. A bicategory of cobordisms equipped with elements of topological stacks is constructed in [SP11].

One of the main technical accomplishments of this paper is the explicit construction of a geometric cobordism bicategory $\text{Cob}_{d,d-1,d-2}^F$ which includes arbitrary background fields in the form of a general stack $F$ (Section 3.2); although this is only a slight generalisation of previous constructions, it is still technically quite complicated, and its explicit form makes all of our statements precise. Building on this bicategory, we then use the theory of symmetric monoidal bicategories following [SP11] and the ideas of [Mon15] to work out the concrete form of the anomalous quantum field theories sketched in Section 1.2; this is described in Section 3.3 and is the first detailed description of anomalies in extended quantum field theory using the framework of symmetric monoidal bicategories, as far as we are aware. The relation to projective representations of the gauge group in [Mon15], and its extension to projective groupoid representations following [FV15], is explained in Section 3.4.

The central part of this paper is concerned with the construction of a concrete example of this general formalism describing the parity anomaly in odd spacetime dimensions. As the parity anomaly is related to an index in one higher dimension [NS83, AGDPM85, Wit16a], this suggests that quantum field theories with parity anomaly should take values in an extended field theory constructed from index theory; this naturally fits in with the classification of topological insulators and superconductors using index theory and K-theory, see [Ert17] for a recent exposition of this. We build such a theory using the index theory for manifolds with corners developed in [LM02, Loy04], which extends the well-known Atiyah-Patodi-Singer index theorem [APS75] to manifolds with corners of codimension 2. Our construction produces an extended quantum field theory $A^\zeta_{\text{parity}}$ depending on a complex parameter $\zeta \in \mathbb{C}^\times$ in any even spacetime dimension $d$; for $\zeta = -1$ this theory describes the parity anomaly in odd spacetime dimension. The details are contained in Section 4.

To exemplify how our constructions fit into the usual treatments of the parity anomaly from the path integral perspective, we first consider in Section 2 the simpler construction of an ordinary (unextended) quantum field theory $Z^\zeta_{\text{parity}}$ based on a geometric cobordism category $\text{Cob}_{d,d-1}^F$ and the usual Atiyah-Patodi-Singer index theorem for even-dimensional manifolds with boundary. We show that the definition of the partition function $Z^\zeta_{\text{parity}}(M)$ transforms under

\[^3\text{In [FT14] the more general situation of simplicial sheaves is considered.}\]
a gauge transformation $\phi : M \to M$ by multiplication with a 1-cocycle of the gauge group given by $\zeta$ to a power determined by the index of the Dirac operator on the corresponding mapping cylinder $\mathcal{M}(\phi)$. This is precisely the same gauge anomaly at $\zeta = -1$ that arises from the spectral flow of edge states under adiabatic evolution signalling the presence of the global parity anomaly [Red84, AGDP85, Wit16a], which is a result of the sign ambiguity in the definition of the fermion path integral in odd spacetime dimension. We further illustrate how the bulk-boundary correspondence in this case [Wit16a] is captured by the full quantum field theory $\mathcal{Z}^\zeta_{\text{parity}}$.

A key feature of the Hamiltonian formalism defined by our construction of the extended quantum field theory $\mathcal{A}^\zeta_{\text{parity}}$ is that the index of a Dirac operator on a manifold with corners depends on the choice of a unitary self-adjoint chirality-odd endomorphism of the kernel of the induced Dirac operator on all corners, whose positive eigenspace defines a Lagrangian subspace of the kernel with respect to its natural symplectic structure. We assemble all possible choices into a linear category $\mathcal{A}^\zeta_{\text{parity}}(\Sigma)$ assigned to $d-2$-dimensional manifolds $\Sigma$ by $\mathcal{A}^\zeta_{\text{parity}}$. The index theorem for manifolds with corners splits into a sum of a bulk integral over the Atiyah-Singer curvature form and boundary contributions depending on the endomorphisms. We use these boundary terms to define the theory $\mathcal{A}^\zeta_{\text{parity}}$ on 1-morphisms, i.e. on $d-1$-dimensional manifolds $M$; the general idea is to use categorical limits to treat all possible boundary conditions at the same time. The index theorem then induces a natural transformation between linear functors, defining the theory $\mathcal{A}^\zeta_{\text{parity}}$ on 2-morphisms, i.e. on $d$-dimensional manifolds.

A crucial ingredient in the construction of the invertible extended field theory $\mathcal{A}^\zeta_{\text{parity}}$ in Section 4.2 is a natural linear map

$$\Phi_{T_0,T_1}(M_0,M_1) : \mathcal{A}^\zeta_{\text{parity}}(M_1) \circ \mathcal{A}^\zeta_{\text{parity}}(M_0)(T_0) \to \mathcal{A}^\zeta_{\text{parity}}(M_1 \circ M_0)(T_0)$$

for every pair of 1-morphisms $M_0 : \Sigma_0 \to \Sigma_1$ and $M_1 : \Sigma_1 \to \Sigma_2$ with corresponding endomorphisms $T_i$ on the corner manifolds $\Sigma_i$; it forms the components of a natural linear isomorphism $\Phi$ which is associative. A lot of information about the parity anomaly is contained in this map: The construction of $\mathcal{A}^\zeta_{\text{parity}}$ allows us to fix endomorphisms for concrete calculations and still have a theory which is independent of these choices. Viewing a field theory with parity anomaly as a theory $\mathcal{A}^\zeta_{\text{parity}}$ relative to $\mathcal{A}^\zeta_{\text{parity}}$ in the sense explained before, we then get a vector space of quantum states $\mathcal{A}^\zeta_{\text{parity}}(\Sigma)$ for every $d-2$-dimensional manifold $\Sigma$; the group of gauge transformations $\text{Sym}(\Sigma)$ only acts projectively on this space. Denoting this projective representation by $\rho$, for any pair of gauge transformations $\phi_1, \phi_2 : \Sigma \to \Sigma$ one finds

$$\rho(\phi_2) \circ \rho(\phi_1) = \Phi_{T_1,T_2}(\mathcal{M}(\phi_1),\mathcal{M}(\phi_2)) \rho(\phi_2 \circ \phi_1) ,$$

where $\mathcal{M}(\phi_i)$ is the mapping cylinder of $\phi_i$. Using results of [Loy05, LW96], we can calculate the corresponding 2-cocycle $\alpha_{\phi_1,\phi_2}$ appearing in the conventional Hamiltonian description of anomalies [Fad84, FS85, Mic85] in terms of the action of gauge transformations on Lagrangian subspaces of the kernel of the Dirac Hamiltonian on $\Sigma$; the explicit expression can be found in (4.24).

This explicit description of the projective representation of the gauge group due to the parity anomaly is new. The only earlier Hamiltonian description of the global parity anomaly that we are aware of is the argument of [CL87] for the case of fermions coupled to a particular background $SU(2)$ gauge field in $2 + 1$-dimensions. There the second quantized Fock space is constructed in
the usual way from the polarisation of the first quantized Hilbert space into spaces spanned by 
the eigenspinors of the one-particle Dirac Hamiltonian on $\Sigma$ with positive and negative energies. 
When the Dirac Hamiltonian has zero modes, a sign ambiguity arises in the identification of 
Fock spaces with a constant space, which is a result of a spectral flow; whence the fermion 
Fock space only carries a representation of a $\mathbb{Z}_2$-central extension of the gauge group. In our 
general approach, we are able to give a more in-depth description which also lends a physical 
interpretation to the Lagrangian subspaces occurring in the index theorem: While the standard 
Atiyah-Patodi-Singer index theorem is crucial for computing the parity anomaly in the path 
integral and its cancellation via the bulk-boundary correspondence, the extra boundary terms 
that appear in the index theorem on manifolds with corners enable the definition of a second 
quantized Fock space of the quantum field theory at $\zeta = -1$ which is compatible with parity 
symmetry by suitably extending the standard polarisation by Lagrangian subspaces of the kernel 
of the Dirac Hamiltonian on $\Sigma$. As before, the sign ambiguities arise from the definition of $A_{\text{parity}}^{(-1)}$ 
as a natural symmetric monoidal 2-transformation, and the gauge anomaly computed by the 
spectral flow is now completely encoded in the 2-cocycles $\alpha_{\phi_1, \phi_2}$, which cancel between the bulk 
and boundary theories by a mechanism similar to that of the partition function; for details see Section 4.3.

Finally, although the present paper deals exclusively with systems of Dirac fermions and the 
parity anomaly in odd-dimensional spacetimes, the method we develop for the concrete con-
struction of our extended field theory can be used in other contexts to build invertible extended 
field theories from invariants of manifolds with corners. For example, our techniques could be 
applied to primitive homotopy quantum field theories, and to Dai-Freed theories which are 
related to $\eta$-invariants; such a formalism would be based on the Dai-Freed theorem [DF95] rather 
than the Atiyah-Patodi-Singer index theorem and would enable constructions with chiral or Majorana fermions and unoriented manifolds, which are applicable to other symmetry-protected 
states of quantum matter such as topological superconductors [Wit16a] as well as to anomalies 
in M-theory [Wit16b]. Another application involves repeating our constructions with Dirac operators 
replaced by signature operators, which would lead to an extended quantum field theory 
describing anomalies in Reshetikhin-Turaev theories based on modular tensor categories [Tur10]; 
these theories should also have applications to anomalies in M-theory along the lines considered 
in [Sat11].

1.4 Outline

The outline of this paper is as follows.

In Section 2, as a warm-up we construct the theory $A_{\text{parity}}^{\zeta}$ as a quantum field theory $Z_{\text{parity}}^{\zeta}$, 
i.e. as a symmetric monoidal functor, and describe how it captures the parity anomaly at the 
level of path integrals. Following [Wit16a], we provide some explicit examples related to the 
standard topological insulator in 3 + 1-dimensions and other topological phases of matter.

In Section 3, we present the general description of anomalies in the framework of functorial 
quantum field theory using symmetric monoidal bicategories. In particular, in Section 3.2, we 
introduce the geometric cobordism bicategory $\text{Cob}_{d, d-1, d-2}$ with arbitrary background fields $\mathcal{F}$ in quite some detail.

The heart of this paper is Section 4, where we explicitly build an extended quantum field
theory describing the parity anomaly using index theory on manifolds with corners. In particular, in Section 4.3 we give the first detailed account of the parity anomaly in the Hamiltonian framework, which further elucidates the physical meaning of some technical ingredients that go into the index theorem.

Two appendices at the end of the paper include some additional technical background. In Appendix A we collect some facts about manifolds with corners and b-geometry which are essential for this paper. In Appendix B we review definitions connected to symmetric monoidal bicategories, mostly in order to fix notation and conventions.

1.5 Notation

Here we summarise our notation and conventions for the convenience of the reader.

Throughout this paper we use the notation $M^{d,i}$ for manifolds with corners, where $d \in \mathbb{N}$ is the dimension of $M^{d,i}$ and $i$ is the codimension of its corners; for closed manifolds we abbreviate $M^{d,0}$ by $M^d$.

The smooth sections of a vector bundle $E$ over a manifold $M$ are denoted by $\Gamma(E)$. The (twisted) Dirac operator on a manifold $M$ equipped with a spin structure and a principal bundle with connection is denoted by $\mathcal{D}_M$. We denote the corresponding twisted spinor bundle by $S_M$.

By $\mathcal{F}$ we always denote a stack on manifolds of a fixed dimension.

We use calligraphic letters $\mathcal{C}, \mathcal{G}, \mathcal{B}, \ldots$ to denote generic categories, groupoids and bicategories. Strict bicategories are called 2-categories. We denote by $\text{Obj}(\mathcal{C})$ the class of objects of $\mathcal{C}$. (1-)morphisms and 2-morphisms are denoted by $\to$ and $\Rightarrow$, respectively; natural transformations are 2-morphisms in the 2-category of small categories. Modifications, which are 3-morphisms in the 3-category of bicategories, are represented by $\Rightarrow$. We denote by $\text{Hom}_\mathcal{C}$ the set of morphisms in a small category $\mathcal{C}$, and by $\text{Hom}_\mathcal{B}$ the category of 1-morphisms in a bicategory $\mathcal{B}$. We write $s, t$ for the maps from (1-)morphisms to their source and target objects, respectively. The symbol $\circ$ denotes composition of (1-)morphisms and vertical composition of 2-morphisms, while $\bullet$ denotes horizontal composition of 2-morphisms.

Functors and 2-functors are denoted by calligraphic letters $\mathcal{F}, \mathcal{G}, \ldots$.

For concrete categories and bicategories we use sans serif letters. We will frequently encounter the following categories and bicategories:

- $\text{Vect}_\mathcal{C}$: The symmetric monoidal category of finite-dimensional complex vector spaces.
- $\text{2Vect}_\mathcal{C}$: The symmetric monoidal bicategory of 2-vector spaces (see Example B.11).
- $\text{Hilb}_\mathcal{C}$: The symmetric monoidal category of complex Hilbert spaces.
- $\text{Grpd}$: The 2-category of small groupoids.
- $\text{Cob}_{d,d-1}^\mathcal{F}$: The symmetric monoidal category of $d$-dimensional geometric cobordisms with background fields $\mathcal{F}$ (see Section 2.1).
- $\text{Cob}_{d,d-1,d-2}^\mathcal{F}$: The symmetric monoidal bicategory of $d$-dimensional geometric cobordisms with background fields $\mathcal{F}$ (see Section 3.2).
2 Quantum field theory and the parity anomaly

The parity anomaly appears in certain field theories with time-reversal (or space-reflection) symmetry involving fermions coupled to gauge fields and gravity in spacetimes of odd dimension $2n - 1$ if, after quantisation, there is no consistent way to make the path integral real. The phase ambiguity appears in a controlled manner and can be understood by regarding the original quantum field theory as living on the boundary of another quantum field theory defined in $d = 2n$ dimensions with the same global symmetry in the bulk: We say that the anomalous field theory in $d - 1$ dimensions takes values in a non-anomalous quantum field theory in $d$ dimensions, since the phase ambiguity of the boundary theory is cancelled by the phase of the bulk system. As a warmup, in this section we explain a simple functorial perspective that captures the relation between the parity anomaly in $2n - 1$ dimensions and (unextended) quantum field theories in $d = 2n$ dimensions, based on the index theory of the Dirac operators which feature in field theories with Dirac fermions. This categorical formalism captures the anomaly only at the level of partition functions and ignores the action on the Hilbert space of quantum states; the latter will be incorporated in subsequent sections by extending the underlying source and target categories to bicategories.

2.1 Geometric cobordisms and quantum field theories

We begin by explaining the formalism of functorial quantum field theories that we shall use in this paper, and the relation between invertible field theories and anomalies. Let us fix a gauge group $G$ with Lie algebra $\mathfrak{g}$, and a finite-dimensional unitary representation $\rho: G \to \text{Aut}(V)$ of $G$. We define a geometric source category $\text{Cob}_{d,d-1}^\mathcal{F}$ whose objects are closed smooth $d - 1$-dimensional manifolds $M^{d-1}$ equipped with a Riemannian metric $g_{M^{d-1}}$, an orientation, a spin structure and a principal $G$-bundle $\pi_{M^{d-1}}: P_{M^{d-1}} \to M^{d-1}$ with connection $A_{M^{d-1}} \in \Gamma(T^* P_{M^{d-1}} \otimes \mathfrak{g})$; this specifies the background field content $\mathcal{F}$ and we call the resulting objects ‘$\mathcal{F}$-manifolds’.

We think of an object $M^{d-1}$ as sitting in the germ of $d$-dimensional manifolds of the form $M^{d-1} \times (-\epsilon, \epsilon)$ with all structures extended as products.

A diffeomorphism of $\mathcal{F}$-manifolds $M_1^{d-1}$ and $M_2^{d-1}$, or an ‘$\mathcal{F}$-diffeomorphism’, consists of an orientation-preserving smooth isometry $\phi: M_1^{d-1} \to M_2^{d-1}$ of the underlying manifolds, together with bundle isomorphisms from the spinor bundle and principal $G$-bundle on $M_1^{d-1}$ to the pullbacks along $\phi$ of the corresponding bundles over $M_2^{d-1}$ that preserves the Levi-Civita connection on the spinor bundles, and the connections $A_{M_1^{d-1}}$ and $A_{M_2^{d-1}}$.

There are then two types of morphisms in $\text{Cob}_{d,d-1}^\mathcal{F}$. The first type of morphisms are given by equivalence classes of compact $d$-dimensional manifolds $M^{d,1}$ endowed with $\mathcal{F}$-fields up to $\mathcal{F}$-diffeomorphisms preserving collars, a decomposition of their boundary $\partial M^{d,1} = \partial_- M^{d,1} \sqcup \partial_+ M^{d,1}$, and collars $M^{d,1}_\pm$ near the boundary components $\partial_{\pm} M^{d,1}$ such that the $\mathcal{F}$-fields are of product structure on $M^{d,1}_\pm$. Such a manifold $M^{d,1}$ describes a morphism from $M_1^{d-1}$ to $M_2^{d-1}$ if it comes with diffeomorphisms of $\mathcal{F}$-manifolds $\varphi_-: M_1^{d-1} \times [0, \epsilon_1) \to M_2^{d-1}$ and $\varphi_+: M_2^{d-1} \times (-\epsilon_2, 0] \to M_1^{d,1}$ for fixed real numbers $\epsilon_i > 0$. We refer to these geometric cobordisms as ‘regular morphisms’. Composition is defined by gluing along boundaries using the collars and their trivialisations $\varphi_\pm$ as described in Appendix A.2 (see Figure 1); note that the smooth structure on glued manifolds depends on the choice of collars [Koc04, Section 1.3]. This composition is strictly associative.

10
\[
M^{d-1}\times[0, \epsilon_1] \quad M^{d-1}_2\times(-\epsilon_2, 0] \quad M^{d-1}_3\times(-\epsilon_3, 0] \quad M^{d-1}_4\times[0, \epsilon_1] \quad M^{d-1}_3\times(-\epsilon_4, 0]
\]

Figure 1: Illustration of two composable regular morphisms in \(\text{Cob}^{\mathcal{F}}_{d,d-1}\) (on the left) and their composition (on the right).

The second type of morphisms are diffeomorphisms of \(\mathcal{F}\)-manifolds \(\phi: M^{d-1}_1 \to M^{d-1}_2\); they incorporate symmetries of the background fields including internal symmetries such as gauge symmetries. We may regard these morphisms as zero length limits of mapping cylinders, and hence we refer to them as ‘limit morphisms’. Composition of limit morphisms is given by concatenation of \(\mathcal{F}\)-diffeomorphisms. The composition of a limit morphism \(\phi: M^{d-1}_1 \to M^{d-1}_2\) with a regular morphism \(M^{d,1}\) is given by precomposition with the extension of \(\phi\) to \(\phi'\): \(M^{d-1}_1 \times [0, \epsilon_1) \to M^{d-1}_2 \times [0, \epsilon_1)\); this composition affects only the map \(\varphi_-\). The composition of a regular morphism with a limit morphism is defined in a similar way, affecting only \(\varphi_+\).

The disjoint union of manifolds makes \(\text{Cob}^{\mathcal{F}}_{d,d-1}\) into a symmetric monoidal category with monoidal unit 1 given by the empty manifold \(\emptyset\).

**Definition 2.1.** A \(d\)-dimensional functorial quantum field theory (or quantum field theory for short) with background fields \(\mathcal{F}\) is a symmetric monoidal functor

\[
Z: \text{Cob}^{\mathcal{F}}_{d,d-1} \to \text{Hilb}_C,
\]

where \(\text{Hilb}_C\) is the symmetric monoidal category of complex Hilbert spaces and linear maps under tensor product.

**Remark 2.2.** It is unclear to us to which extent physical examples of quantum field theories fit into this framework. See [Seg11] for a discussion of this definition. However, it is enough to capture the field theories related to anomalies, which suffice for this paper.

The simplest example of a quantum field theory is the trivial theory \(1: \text{Cob}^{\mathcal{F}}_{d,d-1} \to \text{Hilb}_C\) sending every object to the monoidal unit \(C\) in \(\text{Hilb}_C\) and every morphism to the identity map on \(C\). Given two quantum field theories \(Z_1: \text{Cob}^{\mathcal{F}}_{d,d-1} \to \text{Hilb}_C\) and \(Z_2: \text{Cob}^{\mathcal{F}}_{d,d-1} \to \text{Hilb}_C\), their tensor product \(Z_1 \otimes Z_2\) can be defined locally. We ignore technical subtleties related to the tensor product of infinite-dimensional Hilbert spaces by tacitly assuming that all vector spaces are finite-dimensional; see [Mon15] for a discussion of how this assumption fits in with the infinite-dimensional state spaces that typically appear in quantum field theory. Using the tensor product one can define the class of quantum field theories relevant for the description of anomalies.

**Definition 2.3.** A quantum field theory \(Z\) is invertible if there exists a quantum field theory
\(Z^{-1}\) and a natural symmetric monoidal isomorphism from \(Z \otimes Z^{-1}\) to \(1\).

The modern perspective on anomalous field theories in \(d-1\) dimensions is that they are “valued” in invertible quantum field theories in \(d\) dimensions [Pre14]. We can understand this point of view at the level of the partition function. This requires restriction to the subcategory \(\text{trCob}_{d,d-1}\) of \(\text{Cob}_{d,d-1}\) containing only invertible morphisms, which are precisely the limit morphisms; we call \(\text{trCob}_{d,d-1}\) the ‘truncation’ of the category \(\text{Cob}_{d,d-1}\). We denote by \(\text{tr} Z\) the restriction of the functor \(Z\) to \(\text{trCob}_{d,d-1}\).

**Definition 2.4.** A partition function of an invertible quantum field theory \(Z\) is a natural symmetric monoidal transformation

\[
Z: 1 \Longrightarrow \text{tr} Z.
\]

Unpacking this definition, we get for every closed \(d-1\)-dimensional manifold \(M^{d-1}\) a linear map \(Z(M^{d-1}): C = 1(M^{d-1}) \to Z(M^{d-1})\), such that the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{Z(M^{d-1})} & Z(M^{d-1}) \\
\downarrow \text{id} & & \downarrow Z(\phi) \\
C & \xrightarrow{Z(\phi(M^{d-1}))} & Z(\phi(M^{d-1}))
\end{array}
\]

commutes for all limit morphisms \(\phi\). Since \(Z\) is an invertible field theory, \(Z(M^{d-1})\) is a one-dimensional vector space and so isomorphic to \(C\), though not necessarily in a canonical way. This translates into an ambiguity in the definition of the partition function as a complex number, which is the simplest manifestation of an anomaly.

**Remark 2.5.** Following [Pre14, Mon15], consider a general stack \(\mathcal{F}\) and an arbitrary invertible quantum field theory \(\mathcal{L}: \text{Cob}_{d,d-1}^{\mathcal{F}} \to \text{Hilb}_C\) (see Section 3 below for precise definitions). Fixing a \(d-1\)-dimensional closed manifold \(M^{d-1}\), for every choice \(f \in \mathcal{F}(M^{d-1})\) we get a one-dimensional vector space \(\mathcal{L}(M^{d-1}, f)\). If we assume that \(\mathcal{F}(M^{d-1})\) is a manifold, then it makes sense to require that \(\mathcal{L}(M^{d-1}, \cdot) \to \mathcal{F}(M^{d-1})\) is a line bundle. In this case the partition function gives rise to a section of this line bundle. This heuristic reasoning reproduces the more common geometric picture of anomalies in terms of the absence of canonical trivialisations of line bundles over the parameter space of the field theory (see e.g. [Nas91]).

Even when it is well-defined, the partition function may fail to be invariant under a limit automorphism \(\phi\) by a linear isomorphism \(Z(\phi)\) of \(Z(M^{d-1})\) that can define a non-trivial \(C\)-valued 1-cocycle of the group of limit automorphisms of \(M^{d-1}\), which is precisely the case of an anomalous symmetry. In this paper we are interested in the more general case of anomalous symmetries encoded by non-trivial \(C\)-valued 1-cocycles of the groupoid \(\text{trCob}_{d,d-1}^{\mathcal{F}}\): A partition function \(Z: 1 \Rightarrow \text{tr} Z\) induces, after picking non-canonical isomorphisms, a groupoid homomorphism \(\chi^Z: \text{trCob}_{d,d-1}^{\mathcal{F}} \to C / \!/ C^\times\).

From a physical viewpoint it is natural to require that these quantum field theories are local, which leads to fully extended field theories. These can be classified in the case of topological quantum field theories by fully dualizable objects in the symmetric monoidal target \((\infty,d)-\)
category \[\text{Lur09}\]. If the theory is moreover invertible then a classification using cobordism spectra and stable homotopy theory is possible \([\text{FH16}]\). However, in subsequent sections we will extend the quantum field theory describing anomalies only up to codimension 2; this has the advantage that all concepts are well-defined and all technical difficulties can be handled rather explicitly.

### 2.2 The quantum field theory \(Z^\zeta_{\text{parity}}\)

Let us now construct a \(d\)-dimensional quantum field theory \(Z^\zeta_{\text{parity}}\) encoding the parity anomaly in \(d - 1\) dimensions. The theory \(Z^\zeta_{\text{parity}}\): \(\text{Cob}_{d,d-1}^\mathcal{F} \to \text{Hilb}_\mathbb{C}\) assigns the one-dimensional vector space \(\mathbb{C}\) to every object \(M^{d-1}\):

\[
Z^\zeta_{\text{parity}}(M^{d-1}) = \mathbb{C}.
\]

The background field content \(\mathcal{F}\) together with the representation \(\rho_G\) defines a Dirac operator \(\mathcal{D}_{M^{d,1}}\) on every manifold \(M^{d,1}\) corresponding to a regular morphism in \(\text{Cob}_{d,d-1}^\mathcal{F}\). Since \(d\) is even, the twisted spinor bundle \(S_{M^{d,1}} = S_{M^{d,1}}^+ \oplus S_{M^{d,1}}^-\) splits into bundles \(S_{M^{d,1}}^\pm\) of positive and negative chirality spinors; the Dirac operator is odd with respect to this \(\mathbb{Z}_2\)-grading. On a closed manifold \(M^d\), the chiral Dirac operator \(\mathcal{D}_{M^d}^+ : H^1(S^+_{M^d}) \to L^2(S^-_{M^d})\) is a first order elliptic differential operator, where \(H^1(S^+_{M^d})\) is the first Sobolev space of sections of \(S^+_{M^d}\); the integration is with respect to the Hermitian structure on \(S_{M^d}\) induced by the metric and the unitary representation \(\rho_G\). Every elliptic operator acting on sections of a vector bundle of finite rank over a closed manifold \(M^d\) is Fredholm, so that we can define a map \(Z^\zeta_{\text{parity}}(M^d) : \mathbb{C} \to \mathbb{C}\) by \(z \mapsto \zeta \text{ind}(\mathcal{D}_{M^d}^+) \cdot z\), where the index of a Fredholm operator \(D\) is defined by

\[
\text{ind}(D) = \dim \ker(D) - \dim \text{coker}(D)
\]

and \(\zeta \in \mathbb{C}^\times\) is a non-zero complex parameter. We would like to extend this map to manifolds with boundary, but the Dirac operator on a manifold with boundary is never Fredholm. A standard solution to this problem is to attach infinite cylindrical ends to the boundary of \(M^{d,1}\).

We define

\[
\hat{M}^{d,1} = M^{d,1} \sqcup_{\partial M^{d,1}} \left( \partial_- M^{d,1} \times (-\infty, 0] \sqcup \partial_+ M^{d,1} \times [0, \infty) \right),
\]

where we use the identification of the collars \(M^{d,1}_\pm\), which are part of the data of a regular morphism, with open cylinders to glue as discussed in Appendix \(A.2\). We extend all of the background field content \(\mathcal{F}\) as products to \(\hat{M}^{d,1}\). The structure of the regular morphisms in \(\text{Cob}_{d,d-1}^\mathcal{F}\) makes it natural to attach inward and outward pointing cylinders to the incoming and outgoing boundary, respectively, contrary to what is normally done in the index theory literature; this will be crucial for compatibility with composition later on. It is further natural, again in contrast to what is normally done in index theory, to glue in the cylinders along the identification of the collars with cylinders; this means that the gluing could “twist” bundles. Alternatively, we could first attach a mapping cylinder for the identification and then an infinite cylinder.

---

4This is equivalent to the introduction of Atiyah-Patodi-Singer spectral boundary conditions on the spinors \([\text{APS75}]\). We use the method of cylindrical ends here, since it can be generalised to manifolds with corners and gives a natural cancellation of certain terms later on.
The Dirac operator $\hat{\mathcal{D}}^+_{M^d,1} : H^1(\hat{S}^+_{M^d,1}) \to L^2(\hat{S}^-_{M^d,1})$ is Fredholm if and only if the kernel of the induced Dirac operator on the boundary of $M^d,1$ is trivial. If the kernel is non-trivial, then we have to regularize the index in an appropriate way, which corresponds physically to introducing small masses for the massless fermions on $M^d,1$. This is done precisely by picking, for every connected component $\partial M^d,i$ of the boundary, a small number $\alpha_i$ with $0 < \alpha_i < \delta_i$, where $\delta_i$ is the smallest magnitude $|\lambda_i|$ of the non-zero eigenvalues $\lambda_i$ of the induced Dirac operator on $\partial M^d,i$. Now we can attach weights $e^{\alpha_i s_i}$ to the integration measure on the cylindrical ends, where $s_i$ is the coordinate on the cylinder over $\partial M^d,i$. Denoting the corresponding weighted Sobolev spaces by $e^{\alpha_i s}H^1(\hat{S}^+_{M^d,1})$ and $e^{\alpha_i s}L^2(\hat{S}^-_{M^d,1})$, we then have

Theorem 2.6. ([Mel93, Theorem 5.60]) The Dirac operator $\hat{\mathcal{D}}^+_{M^d,1} : e^{\alpha_i s}H^1(\hat{S}^+_{M^d,1}) \to e^{\alpha_i s}L^2(\hat{S}^-_{M^d,1})$ is Fredholm and its index is independent of the masses $\alpha_i$.

Having at hand a well-defined notion of an index for manifolds with boundaries, we can now define

$$Z^\zeta_{\text{parity}}(M^d,1) : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \zeta^{\text{ind}}(\hat{\mathcal{D}}^+_{M^d,1}) \cdot z.$$ 

The index can be computed by means of the Atiyah-Patodi-Singer index theorem [APS75], which gives a concrete formula for the index if the attaching of cylindrical ends is taken along identity maps:

$$\text{ind}(\hat{\mathcal{D}}^+_{M^d,1}) = \int_{M^d,1} K_{AS} - \frac{1}{2} \left( \eta(\hat{\mathcal{D}}_{\partial M^d,1}) + \dim \ker(\hat{\mathcal{D}}_{\partial - M^d,1}) - \dim \ker(\hat{\mathcal{D}}_{\partial + M^d,1}) \right), \quad (2.7)$$

where the Atiyah-Singer density

$$K_{AS} = \text{ch}(P_{M^d,1}) \wedge \hat{A}(TM^d,1)\big|_d$$

is the homogeneous differential form of top degree in $\Omega^d(M^d,1)$ occurring in the exterior product of the Chern character of the bundle $P_{M^d,1}$ with the $\hat{A}$-genus of the tangent bundle $TM^d,1$. The $\eta$-invariant of the Dirac operator on a closed manifold $M^{d-1}$ of odd dimension calculates the number of positive eigenvalues minus the number of negative eigenvalues of $\hat{\mathcal{D}}_{M^{d-1}}$, and is defined by

$$\eta(\hat{\mathcal{D}}_{M^{d-1}}) = \lim_{s \to 0} \sum_{\lambda \in \text{spec}(\hat{\mathcal{D}}_{M^{d-1}}) \atop \lambda \neq 0} \frac{\text{sign}(\lambda)}{|\lambda|^s}.$$ 

The limit here should be understood as the value of the analytic continuation of the meromorphic function $\sum_{\lambda \neq 0} \frac{\text{sign}(\lambda)}{|\lambda|^s}$ at $s = 0$; the regularity of this value is proven in [APS75]. The sign difference between the dimensions of the kernels in (2.7) comes from the fact that we attach cylinders with opposite orientation to the incoming and outgoing boundary; this corresponds to a negative sign for the numbers $\alpha_i$ on the outgoing boundary $\partial_+ M^{d,1}$ in the version of the

5To be more precise, we have to first attach a mapping cylinder before we can apply [Mel93, Theorem 5.60].
Atiyah-Patodi-Singer index theorem given in [Mel93]. The \( \eta \)-invariant can be reformulated as an integral over the trace of the corresponding heat kernel operator as
\[
\eta\left(\mathcal{D}_{M^{d-1}}\right) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \text{Tr} \left( \mathcal{D}_{M^{d-1}} e^{-t \mathcal{D}^2_{M^{d-1}}} \right) \, dt .
\] (2.8)

We assign to a limit morphism \( \phi \) the value of \( Z^\xi_{\text{parity}} \) on a corresponding mapping cylinder; in order for \( Z^\xi_{\text{parity}} \) to be well-defined, this construction must then be independent of the length of the mapping cylinder. We prove this as part of Theorem 2.9.

\( Z^\xi_{\text{parity}} \): \( \text{Cob}_{d,d-1} \rightarrow \text{Hilb}_\mathbb{C} \) is an invertible quantum field theory.

Proof. The value of \( Z^\xi_{\text{parity}} \) on a mapping cylinder is independent of its length, since the manifolds constructed by attaching cylindrical ends are \( \mathcal{F} \)-diffeomorphic. This proves that \( Z^\xi_{\text{parity}} \) is well-defined on limit morphisms \( \phi \).

If we cut a manifold \( M^{d,1} \) along a hypersurface \( H \) into two pieces \( M^{d,1}_1 \) and \( M^{d,1}_2 \), then from the Atiyah-Patodi-Singer index theorem (2.7) we get
\[
\text{ind}\left(\mathcal{D}^\xi_{M^{d,1}}\right) = \text{ind}\left(\mathcal{D}^\xi_{M^{d,1}_1}\right) + \text{ind}\left(\mathcal{D}^\xi_{M^{d,1}_2}\right) ,
\]
since the integration is additive and \( \eta\left(\mathcal{D}_{M^{d-1}}\right) = -\eta\left(\mathcal{D}_{-M^{d-1}}\right) \), where \( -M^{d-1} \) is the manifold \( M^{d-1} \) with the opposite orientation. The contributions from the boundary along which the cutting takes place cancel in (2.7) because of the opposite signs of the dimensions of the kernel of the boundary Dirac operator for incoming and outgoing boundaries.

For regular morphisms \( M^{d,1} : M^{d-1}_1 \rightarrow M^{d-1}_2 \) and \( N^{d,1} : M^{d-1}_2 \rightarrow M^{d-1}_3 \), we can cut the manifold \( N^{d,1} \circ M^{d,1} \) into \( M^{d,1} \) and \( N^{d,1} \), with half of the collar around \( M^{d-1}_2 \) removed and mapping cylinders corresponding to the identification attached. This uses the description of the gluing process in terms of mapping cylinders (see Appendix A.2), but as mentioned earlier the index of such pieces is the same as the index corresponding to a manifold where the attachment is twisted by the identification of the collars with cylinders. This implies
\[
Z^\xi_{\text{parity}}\left(N^{d,1} \circ M^{d,1}\right) = Z^\xi_{\text{parity}}\left(M^{d,1}\right) \cdot Z^\xi_{\text{parity}}\left(N^{d,1}\right) .
\]
This proves that \( Z^\xi_{\text{parity}} \) is a functor, which is furthermore symmetric monoidal since all our constructions are multiplicative under disjoint unions. The inverse functor is \( \left(Z^\xi_{\text{parity}}\right)^{-1} = Z^\xi_{\text{parity}}^{-1} \). \( \square \)

Remark 2.10. It may seem unnatural for \( Z^\xi_{\text{parity}} \) to assign the one-dimensional vector space \( \mathbb{C} \) to every closed \( d-1 \)-dimensional manifold \( M^{d-1} \). Rather one would expect a complex line generated by all boundary conditions via an inverse limit construction as for example in [FQ93]. Assigning \( \mathbb{C} \) to every closed manifold is only possible due to the presence of canonical APS-boundary conditions related to the \( L^2 \)-condition on the non-compact manifolds \( \hat{M}^{d,1} \).

2.3 Partition functions and symmetry-protected topological phases

We turn our attention now to the partition function for a quantum field theory describing the parity anomaly. According to Definition 2.3, it is a natural symmetric monoidal transformation
\footnote{For this we need a cylindrical neighbourhood of \( H \) on which all of the field content \( \mathcal{F} \) is of product form.}
$Z^\zeta_{\text{parity}} : 1 \Rightarrow \text{tr} Z^\zeta_{\text{parity}}$. This yields, for every closed $d - 1$-dimensional manifold $M^{d - 1}$, a linear map $Z^\zeta_{\text{parity}}(M^{d - 1}) : \mathbb{C} \to Z^\zeta_{\text{parity}}(M^{d - 1}) = \mathbb{C}$. A linear map $Z^\zeta_{\text{parity}}(M^{d - 1}) : \mathbb{C} \to \mathbb{C}$ can be canonically identified with a complex number $Z^\zeta_{\text{parity}}(M^{d - 1}) \in \mathbb{C}$. Now there is no ambiguity in the definition of the partition function as a complex number. The essence of the parity anomaly, like most anomalies associated with the breaking of a classical symmetry in quantum field theory, is the lack of invariance of $Z^\zeta_{\text{parity}}$ under limit morphisms $\phi$: The naturalness of the partition function implies that it transforms under gauge transformations $\phi$ by multiplication with a 1-cocycle $Z^\zeta_{\text{parity}}(\phi) \in \mathbb{C}^\times$; note that in the present context ‘gauge transformations’ also refer to isometries and isomorphisms of the spinor bundle $S_{M^{d - 1}}$. Since $Z^\zeta_{\text{parity}}$ depends only on topological data, this multiplication is given by

$$Z^\zeta_{\text{parity}}(\phi) = \zeta^{\text{ind}(\mathcal{D}^+_{M^{d - 1}, \phi})} \quad (2.11)$$

where $\Re(M^{d - 1}, \phi)$ is the corresponding mapping torus constructed by identifying the boundary components of $M^{d - 1} \times [0, 1]$ using $\phi$.

Example 2.12. We shall now illustrate how the functorial formalism of this section connects with the more conventional treatments of the parity anomaly in the physics literature, following [Wit16a] (see also [SW16]): indeed, what mathematicians call ‘invertible quantum field theories’ are known as ‘short-range entangled topological phases’ to physicists. A partition function with parity anomaly can be defined by fixing its value on a representative for every gauge equivalence class, where the definition of the determinant requires a suitable regularization. Formally, this is the absolute value of the contribution to the path integral measure from a massless Dirac fermion in the configuration space of the field theory, and changing the sign every time an eigenvalue of the Dirac operator crosses through zero. It is well-known that this spectral flow can be calculated by the index of the Dirac operator on the corresponding mapping cylinder. This physical intuition is formalised by the definition above for $\zeta = -1$: The phase ambiguity is determined by requiring the partition function to define a natural symmetric monoidal transformation.

For an arbitrary chosen background $(A_{M^{d - 1}}, g_{M^{d - 1}})$ in every gauge equivalence class, where the definition of the determinant requires a suitable regularization. Formally, this is the absolute value of the contribution to the path integral measure from a massless Dirac fermion in $d - 1$ dimensions coupled to a background $(A_{M^{d - 1}}, g_{M^{d - 1}})$. There is an ambiguity in defining the phase of $Z^\zeta_{\text{parity}}(M^{d - 1})$. Time-reversal (or space-reflection) symmetry forces $Z^\zeta_{\text{parity}}(M^{d - 1})$ to be real. Here we chose the phase to make the partition function positive at the fixed representative.

From a physical perspective, having set the phase of the partition function at a fixed background $(A_{M^{d - 1}}, g_{M^{d - 1}})$ we can calculate the phase at a gauge equivalent configuration $\phi(A_{M^{d - 1}}, g_{M^{d - 1}})$ by following the path

$$(1 - t) (A_{M^{d - 1}}, g_{M^{d - 1}}) + t \phi(A_{M^{d - 1}}, g_{M^{d - 1}}), \quad t \in [0, 1] \quad (2.13)$$

in the configuration space of the field theory, and changing the sign every time an eigenvalue of the Dirac operator crosses through zero. It is well-known that this spectral flow can be calculated by the index of the Dirac operator on the corresponding mapping cylinder. This physical intuition is formalised by the definition above for $\zeta = -1$: The phase ambiguity is determined by requiring the partition function to define a natural symmetric monoidal transformation.

We can preserve gauge invariance by using Pauli-Villars regularization [Wit16a], leading to the gauge invariant partition function

$$Z_{\text{parity}}(M^{d - 1}) = |\det(\mathcal{D}_{M^{d - 1}})| (-1)^\eta(\phi_{M^{d - 1}})/2.$$
The global parity anomaly is due to the fact that the fermion path integral is in general not a real number, whereas classical orientation-reversal (or ‘parity’) symmetry, which acts by complex conjugation on path integrals, would imply that the path integral is real.

The formula (2.11) for the anomalous phase $Z_{\text{parity}}^{-1}(\phi)$ now immediately suggests a way to cancel the parity anomaly: We combine bulk and boundary degrees of freedom by introducing for the bulk fields the action

$$S_{\text{bulk}}(M^{d,1}) = i \pi \int_{M^{d,1}} K_{\text{AS}},$$

where $M^{d,1}$ is a regular morphism from $\emptyset$ to $M^{d-1}$, i.e. $\partial M^{d,1} = M^{d-1}$. Then after integrating out the boundary fermion fields, the contribution to the path integral measure for the combined system is given by

$$Z_{\text{comb}}(M^{d,1}) = |\det(\mathcal{D}_{M^{d-1}})| (-1)^{\eta(\mathcal{D}_{M^{d-1}})} e^{-S_{\text{bulk}}(M^{d,1})}$$

$$= |\det(\mathcal{D}_{M^{d-1}})| \exp \left( \frac{i \pi}{2} \eta(\mathcal{D}_{M^{d-1}}) - i \pi \int_{M^{d,1}} K_{\text{AS}} \right)$$

$$= |\det(\mathcal{D}_{M^{d-1}})| (-1)^{\text{ind}(\mathcal{D}_{M^{d,1}})},$$

where in the last line we used the Atiyah-Patodi-Singer index formula (2.7). This expression is real. Thus the combined bulk-boundary system is invariant under orientation-reversal and gauge transformations, since now its path integral is real, due to ‘anomaly inflow’ from the bulk to the boundary. In particular, the non-anomalous partition function of the combined system

$$Z_{\text{comb}}(M^{d,1}) = Z_{\text{parity}}^{-1}(M^{d,1}) |\det(\mathcal{D}_{M^{d-1}})|$$

is defined in the full $d$-dimensional quantum field theory $Z_{\text{parity}}^{-1}$, rather than just the truncation $\text{tr} Z_{\text{parity}}^{-1}$ in which the original partition function $Z_{\text{parity}}^{-1}$ lives. Looking at this from a different perspective, we see that the existence of an effective long wavelength action (2.14) for the bulk gauge and gravitational fields implies the existence of gapless charged boundary fermions with an anomaly cancelling the anomaly of the bulk quantum field theory under orientation-reversing transformations.

This example provides a simple model for the general feature of some topological states of matter: Symmetry-protected topological phases in $d$ dimensions are related to global anomalies in $d-1$ dimensions. In the simplest case $d = 2$, the quantum mechanical time-reversal anomaly on the $0+1$-dimensional boundary is encoded by the $1+1$-dimensional symmetry-protected topological phase in the bulk whose topological response action (2.14) evaluates to $i \pi \Phi$, where $\Phi$ is the magnetic flux of the background gauge field through $M^{2,1}$. This sets the two-dimensional $\theta$-angle equal to $\pi$, and the action reduces to the Wilson loop of the gauge field $A_{M^1}$ on $\partial M^{2,1} = M^1$.

For the $d = 4$ example of the time-reversal (or space-reflection) invariant $3+1$-dimensional fermionic topological insulator with $2+1$-dimensional boundary [Wit16a], the integral of the Atiyah-Singer index density $K_{\text{AS}}$ in four dimensions yields the sum of the instanton number $I$ of the background gauge field and a gravitational contribution related to the signature $\sigma$ of the fourmanifold $M^{4,1}$ [Nas91]. For the cancellation of the parity anomaly we had to introduce the term $i \pi I$ in the action, which is the anticipated statement that the $\theta$-angle parameterising the axionic
response action is equal to $\pi$ inside a topological insulator. The bulk-boundary correspondence discussed above then resembles the well-known situation from three-dimensional Chern-Simons theory, to which the bulk theory reduces on $\partial M^{4,1} = M^3$ [NS83, AGDPM85].

The present formalism generalises this perspective to systematically construct quantum field theories with global parity symmetry that characterise gapless charged fermionic boundary states of certain symmetry-protected topological phases of matter in all higher even dimensions $d \geq 6$. Indeed, the anomaly of a quantum field theory in $d = 2n$ dimensions involving an action that integrates the Atiyah-Singer index density $K_{AS}$ reduces on the boundary $\partial M^{d,1} = M^{d-1}$ to coupled combinations of gauge and gravitational Chern-Simons type terms. The bulk action (2.14) will now also involve couplings between gauge and gravitational degrees of freedom through intricate combinations of Chern and Pontryagin classes, such that the bulk symmetry-protected topological phase completely captures the parity anomaly of the boundary theory. Some examples of such mixed gauge-gravity phases can be found e.g. in [WGW15].

3 Anomalies and projective representations

When a quantum field theory has a global symmetry which is non-anomalous, the symmetry group acts on the Hilbert space of quantum states. When the global symmetry is anomalous, the group instead acts projectively on the state space, or equivalently a non-trivial central extension acts linearly. Such central extensions correspond to group 2-cocycles which also specify the class of a gerbe on the classifying space of the symmetry group. To see this effect in our framework, it is necessary to extend the functorial quantum field theories defined in Section 2.1 in order to capture the action of the anomaly on quantum states. In this section we develop a general framework of extended field theories which will encode anomalies in this way, following [Fre14, Mon15, FV15]. In this formalism the same group 2-cocycle characterising the projective representation in the $d - 1$-dimensional boundary field theory also specifies an invariant of the bulk $d$-dimensional quantum field theory in which the anomaly is encoded, so that such cocycles also describe invariants of certain topological phases.

3.1 Invariant background fields

Physical fields should be local, i.e. they form a sheaf on the category of manifolds, and can have (higher) internal symmetries such as gauge symmetries. We thus incorporate all data of fields such as bundles with connections, spin structures and metrics into a stack $\mathcal{F}: \text{Man}_d^{\text{op}} \to \text{Grpd}$ on the category $\text{Man}_d$ of $d$-dimensional manifolds with corners and local diffeomorphisms; we regard $\text{Man}_d$ as a 2-category with only trivial 2-morphisms, and $\text{Grpd}$ denotes the 2-category of small groupoids, functors and natural isomorphisms. One should think of elements of $\mathcal{F}(M)$ as the collection of classical background fields on $M$, which in particular satisfies the sheaf condition, i.e. for every open cover $\{U_a\}$ of a manifold $M$, the diagram

$$
\mathcal{F}(M) \longrightarrow \prod_a \mathcal{F}(U_a) \Rightarrow \prod_{a,b} \mathcal{F}(U_a \cap U_b)
$$

is a weak/homotopy equalizer diagram in $\text{Grpd}$. Using stacks we avoid problems associated to the fact that pullbacks of certain background fields are only functorial up to canonical isomorphism. We can include some geometrical structures such as metrics by considering the corresponding
set as a groupoid with only identity morphisms. With this in mind it would be more general to work with ∞-stacks, but for our purpose stacks are enough.

**Remark 3.1.** We implicitly pick, for every surjective submersion \( \pi: Y \to M \), weak adjoint inverses to the canonical map \( \mathcal{F}(M) \to \text{Desc}_\mathcal{F}(Y) \) where \( \text{Desc}_\mathcal{F}(Y) \) is the category of descent data associated to \( \pi \). For every refinement

\[
\begin{array}{c}
Y_1 \\
\downarrow^{\pi_1} \\
M \\
\downarrow \\
Y_2 \\
\downarrow^{\pi_2}
\end{array}
\]

we get a natural functor \( f^*: \text{Desc}_\mathcal{F}(Y_2) \to \text{Desc}_\mathcal{F}(Y_1) \) for which we pick a weak adjoint inverse. The adjointness condition is essential for ensuring naturality of constructions using descent properties.

To generalize the constructions of Appendix [A.2] we need the following notion.

**Definition 3.2.** Let \( \mathcal{F} \) be a stack, \( M \) a manifold, and \( \mathcal{I}(M) \) a groupoid which consists of a collection of open subsets of \( M \) including \( M \) and a collection of diffeomorphisms as morphisms.

(a) An **invariant structure** with respect to \( \mathcal{I}(M) \) for an element \( f \in \mathcal{F}(M) \) is a natural 2-transformation

\[
\begin{array}{ccc}
\mathcal{I}(M)^{\text{op}} & \xrightarrow{f} & \text{Grpd} \\
\downarrow & & \downarrow \\
1 & & 1
\end{array}
\]

from the constant 2-functor sending every object to the groupoid 1 with one object and one morphism, such that the natural transformation is induced by \( f \) on objects (see the following remark for an explanation). Here we regard \( \mathcal{I}(M) \) as a 2-category with trivial 2-morphisms\(^7\). We call an element \( f \in \mathcal{F}(M) \) together with the choice of an invariant structure an invariant element.

(b) A morphism \( \Theta: f \to f' \) between invariant elements \( f, f' \in \mathcal{F}(M) \) is invariant under \( \mathcal{I}(M) \) if it induces a modification between the natural 2-transformations corresponding to \( f \) and \( f' \).

**Remark 3.3.** Let us spell out in detail what we mean by saying that \( f \in \mathcal{F}(M) \) induces a natural 2-transformation on objects. A map \( f_U: 1 \to \mathcal{F}(U) \) is an element of \( \mathcal{F}(U) \). We set \( f_U = f_{|U} \) for all \( U \in \text{Obj}(\mathcal{I}(M)) \). To equip this with the structure of a natural 2-transformation we have to

---

\(^7\)This is the same as a higher fixed point for the groupoid action corresponding to \( \mathcal{F} \), as discussed for example in [HSV17].
This is the same thing as morphisms \( f_{U_1U_2} : t^* f_{U_1} \to f_{U_2} \) for every morphism \( t : U_2 \to U_1 \) of \( \mathcal{F}(M) \) which have to satisfy the coherence conditions (B.6) and (B.7):

\[
\begin{align*}
 f_{U_2U_3}(t_2) \circ t_2^* f_{U_1U_2}(t_1) &= f_{U_1U_3}(t_1 \circ t_2) \circ \Phi_{\mathcal{F}(U_3),\mathcal{F}(U_2),\mathcal{F}(U_1)}(t_1 \times t_2), \\
 f_{UU}(id_U) \circ \Phi_{\mathcal{F}(U)}(id_\gamma) &= id_{f_U}
\end{align*}
\]

for morphisms \( t_2 : U_3 \to U_2 \) and \( t_1 : U_2 \to U_1 \). For a sheaf considered as a stack, the maps \( f_{U_1U_2}(t) \) must be identity maps and we reproduce, for example, the definition of an invariant function.

**Example 3.6.** Let \( \mathcal{F} = \text{Bun}_G \) be the stack of principal \( G \)-bundles, and let \( M \) be a manifold equipped with an action \( \rho : \Gamma \to \text{Diff}(M) \) of a group \( \Gamma \) by diffeomorphisms of \( M \). We can encode the action into a groupoid \( \mathcal{G}(M) \) as in Definition 3.2 with one object \( M \) and morphisms \( \{ \rho(\gamma) \mid \gamma \in \Gamma \} \). A \( G \)-bundle \( P \) which is invariant under \( \mathcal{G}(M) \) comes with gauge transformations \( \Theta_\gamma : \rho(\gamma)^* P \to P \) satisfying (3.4) and (3.5). This is just a \( \Gamma \)-equivariant \( G \)-bundle. An invariant morphism between two \( \Gamma \)-equivariant \( G \)-bundles is then a \( \Gamma \)-equivariant gauge transformation.

**Definition 3.7.** Let \( \Sigma \) be a \( d - 1 \)-dimensional manifold (with boundary). For every (not necessarily open) interval \( I \subset \mathbb{R} \), we say that an element \( f \in \mathcal{F}(\Sigma \times I) \) is constant along \( I \) if it is invariant under translations in the direction along \( I \), i.e. invariant with respect to the groupoid with open subsets of \( \Sigma \times I \) as objects and translations along \( I \) as morphisms.

**Remark 3.8.** We employ a similar definition for manifolds of the form \( Y \times I_1 \times \cdots \times I_n \).

### 3.2 Geometric cobordism bicategories

Inspired by [SPT11] and the sketch of [Mon15, Appendix A], we introduce a bicategory of manifolds equipped with geometric fields. For the definition of a Dirac operator, a metric on the underlying manifold is crucial, whence we cannot assume that the field content is topological. This leads to technical problems in defining 2-morphisms. We make the assumption that the field content is constant near gluing boundaries and use a specific choice of collars to get around these problems.

We define a bicategory \( \text{Cob}^\mathcal{F}_{d,d-1,d-2} \) with objects given by quadruples

\[
(M^{d-2}, f^{d-2}, \epsilon_1, \epsilon_2)
\]

consisting of a closed \( d - 2 \)-dimensional manifold \( M^{d-2} \) with \( n \) connected components \( M^{d-2}_i \), \( n \)-tuples \( \epsilon_1, \epsilon_2 \in \mathbb{R}_{>0}^n \), and an element \( f^{d-2} \in \mathcal{F}(M^{d-2} \times (-\epsilon_1, \epsilon_1) \times (-\epsilon_2, \epsilon_2)) \) which is constant along \( (-\epsilon_1, \epsilon_1) \times (-\epsilon_2, \epsilon_2) \). Here we introduced the notation

\[
M^{d-2} \times (-\epsilon_1, \epsilon_1) \times (-\epsilon_2, \epsilon_2) = \bigcup_{i=1}^n M^{d-2}_i \times (-\epsilon_{1,i}, \epsilon_{1,i}) \times (-\epsilon_{2,i}, \epsilon_{2,i})
\]
which we will continue to use throughout this section.

There are two different kinds of 1-morphisms in \( \text{Cob}_{d,d-1}^F \):

(a) Regular 1-morphisms

\[
M^{d-1,1} : (M_{d-2}^{d-2}, f_{d-2}^{d-2}, \epsilon_{-1}, \epsilon_{-2}) \rightarrow (M_{d}^{d-2}, f_{+}^{d-2}, \epsilon_{+1}, \epsilon_{+2})
\]

consist of 7-tuples

\[
(M^{d-1,1}, \varphi_{-1}^{d-1}, \varphi_{+1}^{d-1}, f_{-1}^{d-1}, \Theta_{d-1}^{d-1}, \Theta_{d+1}^{d-1}, \epsilon),
\]

where \( M^{d-1,1} \) is a \( d-1 \)-dimensional manifold with boundary and \( n \) connected components together with a decomposition of a collar of its boundary into \( N_{d-1}^{d-1} \) and \( N_{+}^{d-1} \), \( \varphi_{-1}^{d-1} : M_{d-2}^{d-2} \times [0, \epsilon_{-1}) \rightarrow N_{d-1}^{d-1} \) and \( \varphi_{+1}^{d-1} : M_{d-2}^{d-2} \times (-\epsilon_{+2}, 0) \rightarrow N_{d-1}^{d-1} \) are diffeomorphisms, \( \epsilon \in \mathbb{R}_{>0} \), \( f_{d-1} \in \mathcal{F}(M^{d-1,1} \times (-\epsilon, \epsilon)) \) is constant along \( (-\epsilon, \epsilon) \) and \( \Theta_{d-1}^{d-1} : f_{+}^{d-2} \rightarrow \varphi_{+}^{d-1} f_{d-1}^{d-1} \) are constant morphisms. Here we use \( \Theta_{d-1}^{d-1} \) to implicitly define the structure of a constant object on \( N_{d-1}^{d-1} \).

(b) Limit 1-morphisms consist of diffeomorphisms \( \phi : M^{d-2} \rightarrow M_{d}^{d-2} \) together with a morphism \( \Theta : f_{d-2}^{d-2} \rightarrow \phi^{*} f_{d-1}^{d-2} \) which is constant along \( (-\epsilon_{1}, \epsilon_{1}) \times (-\epsilon_{2}, \epsilon_{2}) \).\(^8\) We refer to diffeomorphisms of this form as ‘\( \mathcal{F} \)-diffeomorphisms’.

For the composition of regular 1-morphisms we glue the underlying manifolds using their collars, and define the composed field content using covers \( U_{1} \times (-\epsilon, \epsilon), U_{2} \times (-\epsilon, \epsilon) \) and \( U_{3} \times (-\epsilon, \epsilon) \) constructed from the cover in \( \text{Appendix A.5} \), and the descent property of the stack \( \mathcal{F} \). Note that we can use the interpretation in terms of mapping cylinders also in this general situation. Composition of limit 1-morphisms is given by composition of \( \mathcal{F} \)-diffeomorphisms. The composition of a limit 1-morphism with a regular 1-morphism is given by changing the identification of the collars and \( \Theta_{d-1}^{d-1} \) using the limit 1-morphism.

There are also two different kinds of 2-morphisms in \( \text{Cob}_{d,d-1}^F \):

(a) Regular 2-morphisms (see Figure 2)

\[
M^{d,2} : (M_{d-1}^{d-1,1}, \varphi_{-1}^{d-1}, \varphi_{+1}^{d-1}, f_{-1}^{d-1}, \Theta_{d-1}^{d-1}, \Theta_{d+1}^{d-1}, \epsilon_{1}) \rightarrow (M_{d}^{d-1,1}, \varphi_{-1}^{d-1}, \varphi_{+1}^{d-1}, f_{-1}^{d-1}, \Theta_{d-1}^{d-1}, \Theta_{d+1}^{d-1}, \epsilon_{2})
\]

with \( M_{d-1}^{d-1,1} : (M_{d-2}^{d-2}, f_{d-2}^{d-2}, \epsilon_{-1}, \epsilon_{-2}) \rightarrow (M_{d}^{d-2}, f_{+}^{d-2}, \epsilon_{+1}, \epsilon_{+2}) \) regular 1-morphisms for \( i = 1, 2 \), consist of equivalence classes of 6-tuples

\[
(M^{d,2}, f^{d}, \varphi_{-}^{d}, \varphi_{+}^{d}, \Theta_{-}^{d}, \Theta_{+}^{d})
\]

where \( M^{d,2} \) is a \( d \)-dimensional (2)-manifold (see Appendix A.1) whose corners are equipped with a decomposition of a collar of the 0-boundary into \( N_{d}^{d} \) and \( N_{d}^{d} \) such that the closure of \( N_{d}^{d} \) contains the 1-boundary, \( f^{d} \) is an element of \( \mathcal{F}(M^{d,2}), \varphi_{-}^{d} : M_{d-1}^{d-1,1} \times [0, \epsilon_{-1}) \rightarrow N_{d}^{d} \) and \( \varphi_{+}^{d} : M_{d-1}^{d-1,1} \times (-\epsilon_{+2}, 0) \rightarrow N_{d}^{d} \) are diffeomorphisms, and \( \Theta_{-}^{d} : f_{-1}^{d-1} \rightarrow \varphi_{-}^{d} f^{d} \) and

\(^8\)For this statement to make sense we require that \( \epsilon \) is compatible with \( \epsilon_{\pm 2} \) on the boundary.

\(^9\)For this to make sense we require that all \( n \)-tuples \( \epsilon \) are equal.
\[ \Theta^d : \varepsilon^{d-1}_2 \rightarrow \varphi^d \ast f^d \] are constant morphisms. All of these structures have to be compatible, in the sense that the diagram commutes, where \( \iota_{\pm} \) are inclusions. We change the sign of the coordinates corresponding to both intervals in the lower embedding. This induces a diagram of functors in groupoids and we require that all morphisms \( \Theta \) are compatible with this diagram. Note that the collars of the 1-morphisms induce collars for the 1-boundaries which agree by (3.9). Two such 6-tuples are equivalent if they are \( \mathcal{F} \)-diffeomorphic relative to half of the collars.

(2b) Limit 2-morphisms consist of pairs \((\phi, \Theta)\), where \( \phi : M_1^{d-1,1} \rightarrow M_2^{d-1,1} \) is a diffeomorphism relative to collars together with a morphism \( \Theta : f_1^{d-1} \rightarrow f_2^{d-1} \). There are no non-trivial 2-morphisms between limit 1-morphisms.

We define horizontal and vertical composition of 2-morphisms as follows:
(Ha) Horizontal composition of regular 2-morphisms is given by gluing along 1-boundaries.

(Hb) Horizontal composition of limit 2-morphisms is defined by “gluing together” diffeomorphisms and the descent condition for morphisms in the stack \( \mathcal{F} \). This uses the open cover defined in (A.5).

(Hc) Horizontal composition of a limit 2-morphism with a regular 2-morphism is defined by the attachment of a mapping cylinder to the 1-boundary.

(Va) Vertical composition of limit 2-morphisms is given by composition of diffeomorphisms, pullback and composition of morphisms in the stack \( \mathcal{F} \).

(Vb) Vertical composition of regular 2-morphisms is a little bit more complicated. Simple gluing of \( M_{1}^{d,2} \) and \( M_{2}^{d,2} \) along a common 1-morphism does not give a 2-morphism again, since the resulting 1-boundaries are “too long”. In the context of topological field theories a solution to this problem [SP11] consists in picking once and for all a diffeomorphism \([0,2] \to [0,1]\). We are unable to use this trick here, since the stack we consider in this paper contains a metric. Instead, we will use collars to circumvent this problem. Given two regular 2-morphisms \( M_{1}^{d,2} : M_{1}^{d-1,1} \Rightarrow M_{2}^{d-1,1} \) and \( M_{2}^{d,2} : M_{2}^{d-1,1} \Rightarrow M_{3}^{d-1,1} \), we define

\[
\tilde{M}_{1}^{d,2} = M_{1}^{d,2} \setminus \varphi_{1+}^{d}(M_{2}^{d-1,1} \times (-\xi,0)) , \\
\tilde{M}_{2}^{d,2} = M_{2}^{d,2} \setminus \varphi_{2-}^{d}(M_{2}^{d-1,1} \times [0,\xi]) .
\]

We define the vertical composition \( M_{2}^{d,2} \circ M_{1}^{d,2} \) to be the manifold resulting from gluing \( \tilde{M}_{1}^{d,2} \) and \( \tilde{M}_{2}^{d,2} \) along \( M_{2}^{d-1,1} \). We have to equip this manifold with appropriate collars: Write \( N_{1}^{d} = N_{1}^{d} \cap \varphi_{1+}^{d}(M_{2}^{d-1,1} \times (-\xi,0)) \), where \( N_{1}^{d} \) is the incoming collar of \( M_{1}^{d,2} \). We set

\[
C = (\varphi_{2-}^{d} \circ (id \times + \epsilon) \circ (\varphi_{1+}^{d})^{-1})(N_{1}^{d}) .
\]

We can glue \( C \) to the remainder of the collar of \( M_{1}^{d-1,1} \) to get a new collar; this is only possible because we assumed that the corresponding elements of \( \mathcal{F} \) are constant along the collars. It is possible to define a new collar for \( M_{3}^{d-1,1} \) in the same way.

(Vc) Vertical composition of a limit 2-morphism with a regular 2-morphism is defined by changing the identification of collars as in Section 2.1

This completes the definition of the geometric cobordism bicategory \( \text{Cob}_{d,d-1,d-2}^{F} \). The disjoint union of manifolds makes \( \text{Cob}_{d,d-1,d-2}^{F} \) into a symmetric monoidal bicategory.

### 3.3 Anomalies and extended quantum field theories

We will now give a general description of anomalies in the framework of functorial quantum field theory. The point of view we take in this paper is that anomalies in \( d-1 \) dimensions can be described by invertible extended field theories in \( d \) dimensions [Fre14, PV15, Mon15]. This is naturally formulated in the language of symmetric monoidal bicategories (or \((\infty,d)\)-categories, see [PV15]). The most important concepts for the following treatment are summarized in Appendix B, a detailed introduction can be found in [SP11 Chapter 2].
Definition 3.10. A $d$-dimensional extended functorial quantum field theory with background fields $\mathcal{F}$ (or extended quantum field theory for short) is a symmetric monoidal 2-functor

$$\mathcal{A}: \text{Cob}_{d,d-1,d-2} \to \text{2Vect}_\mathbb{C}$$

from a geometric cobordism bicategory to the 2-category of 2-vector spaces.

Remark 3.11. The definition of the symmetric monoidal bicategory $\text{2Vect}_\mathbb{C}$ is given in Example B.11. Although the 2-category of 2-Hilbert spaces would be more natural for some physical applications, we choose here to work with $\text{2Vect}_\mathbb{C}$ since it reduces some of the technical complexity while still capturing all essential features.

Furthermore, the KV 2-vector spaces used in this paper are categorifications of finite dimensional vector spaces. They are enough for our purpose. However, more complicated extended quantum field theories require more elaborate target 2-categories corresponding to infinite dimensional 2-vector spaces.

The Deligne product $\boxtimes$ induces a tensor product for suitable extended field theories.

Definition 3.12. An extended quantum field theory $\mathcal{A}$ is invertible if there exists an extended quantum field theory $\mathcal{A}^{-1}$ and a natural symmetric monoidal 2-isomorphism from $\mathcal{A} \boxtimes \mathcal{A}^{-1}$ to the trivial theory $1: \text{Cob}_{d,d-1,d-2} \to \text{2Vect}_\mathbb{C}$ sending every object to the monoidal unit $\text{Vect}_\mathbb{C}$ of $\text{2Vect}_\mathbb{C}$, every 1-morphism to the identity functor on $\text{Vect}_\mathbb{C}$, and every 2-morphism to the identity natural transformation.

Defining the truncation of $\text{Cob}_{d,d-1,d-2}$ to be the sub-bicategory $\text{trCob}_{d,d-1,d-2}$ containing only invertible 2-morphisms, and $\text{tr}\mathcal{A}$ the restriction of the 2-functor $\mathcal{A}$ to $\text{trCob}_{d,d-1,d-2}$, we can now give a precise definition of a quantum field theory with anomaly.

Definition 3.13. An anomalous quantum field theory with anomaly described by an invertible extended quantum field theory $\mathcal{A}: \text{Cob}_{d,d-1,d-2} \to \text{2Vect}_\mathbb{C}$ is a natural symmetric monoidal 2-transformation

$$\mathcal{A}: 1 \Rightarrow \text{tr}\mathcal{A}.$$ 

We call $\mathcal{A}$ the anomaly quantum field theory describing the anomaly of $\mathcal{A}$.

This definition is a special case of the relative quantum field theories of [PTT14]. These anomaly field theories are often topological field theories. For some applications, such as to two-dimensional rational conformal field theories or to six-dimensional $(2,0)$ superconformal field theories, it is necessary to consider also non-invertible quantum field theories to capture the feature that the partition function is valued in a vector space of dimension $> 1$ [Mon15].

We can recover the anomalous partition function of Definition 2.4 by restricting $\mathcal{A}$ to a functor

$$\mathcal{Z} := \mathcal{A}|_\emptyset : \text{Cob}_{d,d-1} \cong \text{End}_{\text{Cob}_{d,d-1,d-2}}(\emptyset) \to \text{End}_{\text{2Vect}_\mathbb{C}}(\text{Vect}_\mathbb{C}) \cong \text{Vect}_\mathbb{C}$$

and $\mathcal{A}$ to a natural transformation $\mathcal{Z} := \mathcal{A}|_\emptyset : 1 \Rightarrow \text{tr}\mathcal{A}|_\emptyset$. 24
Unpacking Definition \[\text{B.13}\] we get for every closed \(d-2\)-dimensional manifold \(M^{d-2}\) a \(\mathbb{C}\)-linear functor \(A(M^{d-2}) : \text{Vect}_\mathbb{C} = 1(M^{d-2}) \to A(M^{d-2})\), which can be (non-canonically) identified with a complex vector space in \(\text{Vect}_\mathbb{C}\), and for all pairs \((M_{d-2}^-, M_{d-2}^+)\) a natural transformation

\[
\begin{align*}
\text{Hom}_{\text{Cob}_{d,d-1,d-2}}(M_{d-2}^-, M_{d-2}^+) & \xrightarrow{1} \text{Hom}_{\text{Vect}_\mathbb{C}}(\text{Vect}_\mathbb{C}, \text{Vect}_\mathbb{C}) \\
M & \xrightarrow{\Phi} \\
A & \xrightarrow{A} A(M_{d-2}^+) \xrightarrow{} A(M_{d-2}^-) \\
\text{Hom}_{\text{Vect}_\mathbb{C}}(A(M_{d-2}^-), A(M_{d-2}^+)) & \xrightarrow{A(M_{d-2}^-)} \text{Hom}_{\text{Vect}_\mathbb{C}}(\text{Vect}_\mathbb{C}, A(M_{d-2}^+))
\end{align*}
\]

This consists of a natural transformation

\[A(M^{d-1,1}) : A(M^{d-1,1}) \circ A(M_{d-2}^+) \Rightarrow A(M_{d-2}^-)\]

for every 1-morphism \(M^{d-1,1} : M_{d-2}^+ \to M_{d-2}^-\). The definition further includes a modification \(\Pi_A\) consisting of natural isomorphisms

\[\Pi_A(M_{d-2}^-, M_{d-2}^+) : \chi_A \circ A(M_{d-2}^-) \bowtie A(M_{d-2}^+) \Rightarrow A(M_{d-2}^- \sqcup M_{d-2}^+) \circ \lambda_{\text{Cob}_{d,d-1,d-2}}\]

and a natural isomorphism

\[M_A^{-1} : A(\emptyset) \Rightarrow \iota_A\]

All of these structures have to satisfy appropriate compatibility conditions, which we summarize in

**Proposition 3.14.** For every anomalous quantum field theory \(A\) with anomaly \(\mathcal{A}\), there are identities

\[\begin{align*}
A(M_{d-1,1}^2) \circ A(M_{d-1,1}^1) &= A(M_{d-1,1}^2 \circ M_{d-1,1}^1) \circ \Phi_A(A(M_{d-1,1}^2) \circ A(M_{d-1,1}^1)) , \\
A(id_{M_{d-2}}) \circ \Phi_A(A(id_{M_{d-2}})) &= id_{A(M_{d-2})} , \\
A(M_{d-1,1}^1) &= A(M_{d-1,1}^2) \circ (A(f) \bullet id_{A(M_{d-2})}) ,
\end{align*}\]

for some 2-isomorphism \(f : M_{d-1,1}^1 \Rightarrow M_{d-1,1}^2\), together with the following commutative diagrams wherein we suppress obvious structure 2-morphisms and identity 2-morphisms:

\[
\begin{align*}
&\xrightarrow{A(M_{d-1,1}^2, M_{d-2}^1)} A(M_{d-1,1}^1 \sqcup M_{d-2}^1) \circ \chi_A \circ A(M_{d-2}^1) \bowtie A(M_{d-2}^2) & \xrightarrow{B(A(M_{d-2}^2), A(M_{d-2}^2))} A(M_{d-1,1}^2 \sqcup M_{d-2}^2) \\
\chi_A \circ A(M_{d-2}^1) & \bowtie A(M_{d-2}^2) & \xrightarrow{B(A(M_{d-2}^2), A(M_{d-2}^2))} A(M_{d-2}^1 \sqcup M_{d-2}^2)
\end{align*}
\]

(3.18)
the vector spaces and linear maps defined in this way form a quantum field theory.

An anomalous quantum field theory with trivial anomaly $A: 1 \Rightarrow 1$ is a $d - 1$-dimensional quantum field theory in the sense of Definition 2.1. We can canonically identify the functor $A(M^{d-2}): \text{Vect}_\mathbb{C} \to \text{Vect}_\mathbb{C}$ with the vector space $A(M^{d-2})(\mathbb{C})$ and the natural transformation $A(M^{d-1}): \text{id}_{\text{Vect}} \circ A(M^{d-2}) \Rightarrow A(M^{d-1})$ with a linear map $A(M^{d-1}): A(M^{d-2})(\mathbb{C}) \to A(M^{d-1})(\mathbb{C})$. The compatibility conditions summarised by Proposition 3.14 then imply that the vector spaces and linear maps defined in this way form a quantum field theory.
3.4 Projective anomaly actions

Following [Mon15, EV15] we describe how the extended quantum field theory encodes the projective action on the state space of an anomalous field theory $A$ with anomaly $\chi$. We fix an object $M^{d-2} \in \text{Obj}(\text{Cob}_d^{\mathcal{F}, d-1,d-2})$. The limit 1-automorphisms of $M^{d-2}$ form the group of physical symmetries $\text{Sym}(M^{d-2})$. Every $\phi \in \text{Sym}(M^{d-2})$ gives rise to a $\mathbb{C}$-linear functor $A(\phi) : A(M^{d-2}) \to A(M^{d-2})$. Choosing a non-canonical equivalence $\chi : A(M^{d-2}) \to \text{Vect}_\mathbb{C}$ identifies $A(\phi)$ as a functor which takes the tensor product with a one-dimensional vector space $L_{\chi,\phi}$. The structure of $A$ defines an isomorphism

$$\alpha_{\chi,\phi_1,\phi_2} : L_{\chi,\phi_1} \otimes L_{\chi,\phi_2} \to L_{\chi,\phi_2 \circ \phi_1}.$$ 

If we furthermore pick an isomorphism $\varphi : L_{\chi,\phi} \to \mathbb{C}$ for every $\phi \in \text{Sym}(M^{d-2})$ we get a family of linear isomorphisms

$$\alpha_{\chi,\varphi,\phi_1,\phi_2} : \mathbb{C} \to \mathbb{C}.$$ (3.24)

We will show later on using abstract arguments that this is a 2-cocycle for the group $\text{Sym}(M^{d-2})$ whose group cohomology class is independent of the chosen equivalence $\chi$ and isomorphisms $\varphi$; a concrete proof can be found in [Mon15].

This cocycle describes the projective action of $\text{Sym}(M^{d-2})$ on the space of quantum states of the theory $A(M^{d-2})$ as follows: Let $A : 1 \Rightarrow \text{tr}A$ be an anomalous quantum field theory with anomaly $A$. We use the equivalence $\chi$ chosen above to identify $A(M^{d-2}) : \text{Vect}_\mathbb{C} \to A(M^{d-2})$ with a vector space $A_\chi(M^{d-2})$. From $A$ we get a natural transformation $A(\phi) : A(\phi) \circ A(M^{d-2}) \Rightarrow A(M^{d-2})$, which by horizontal composition with the identity natural transformation of $\chi$ induces a linear map

$$A_\chi(\phi) : A_\chi(M^{d-2}) \otimes L_{\chi,\phi} \to A_\chi(M^{d-2}).$$

By precomposing with the isomorphisms $\varphi^{-1}$ we get a projective representation

$$\rho_\varphi : \text{Sym}(M^{d-2}) \to \text{End}_\mathbb{C}(A_\chi(M^{d-2})).$$

We cannot say anything about the structure of this projective representation in general, but we can describe the failure of the composition law explicitly in terms of $A$:

$$\rho_\varphi(\phi_2) \circ \rho_\varphi(\phi_1) = \alpha_{\chi,\varphi,\phi_1,\phi_2} \rho_\varphi(\phi_2 \circ \phi_1).$$

We say that the quantum field theory $A$ is anomaly-free on $M^{d-2}$ if there is a choice of $\chi$ and $\varphi$ such that the corresponding projective representation $\rho_\varphi$ reduces to an honest representation. This is only possible if the corresponding cohomology class of $\alpha_{\chi,\varphi,\phi_1,\phi_2}$ is trivial.

More generally, we can build a projective representation of the groupoid $\text{SymCob}_d^{\mathcal{F}, d-1,d-2}$ of symmetries having the same objects as $\text{Cob}_d^{\mathcal{F}, d-1,d-2}$ and all limit 1-morphisms as morphisms. For this, we first need to recall the notion of a groupoid 2-cocycle.

**Definition 3.25.** A 2-cocycle of a groupoid $\mathcal{G}$ with values in $\text{Vect}_\mathbb{C}$ is a 2-functor $\alpha : \mathcal{G} \to \text{BLine}_\mathbb{C}$, where we consider $\mathcal{G}$ as a 2-category with trivial 2-morphisms and $\text{BLine}_\mathbb{C}$ is the 2-category with one object, and the symmetric monoidal category $\text{Line}_\mathbb{C}$ of complex lines and linear isomorphisms as endomorphisms.
Let us spell out explicitly some details of this definition:

(a) We can pick an equivalence between $\text{Line}_\mathbb{C}$ and $\mathcal{B}_2 \mathbb{C}^\times$ by choosing for every complex line $L$ an isomorphism $\chi: L \to \mathbb{C}$; the inverse of this equivalence is the embedding of $\mathbb{C} \to \mathbb{C}^\times$ into $\text{Line}_\mathbb{C}$. This induces a 2-functor $\alpha: \mathcal{G} \to \mathcal{B}_2 \mathbb{C}^\times$. Writing out Definition B.2 we get for every pair $(g, g') \in \text{Hom}_\mathcal{G}(G_1, G_2) \times \text{Hom}_\mathcal{G}(G_2, G_3)$ a non-zero complex number $\alpha_{g,g'}$ such that

$$\alpha_{g_3 \circ g_2, g_1} \alpha_{g_3, g_2} = \alpha_{g_3, g_2} \alpha_{g_2, g_1},$$

for all composable morphisms $g_1, g_2, g_3$, and

$$\alpha_{\text{id}_t(g), g} = \alpha_{\text{id}_t(g), \text{id}_t(g)} = \alpha_{g, \text{id}_t(g)}.$$

Note that the 2-morphism $\alpha_1: \alpha(\text{id}) \Rightarrow \text{id}$ is completely fixed by the coherence condition (B.4) and takes the value $\alpha_{\text{id}, \text{id}}^{-1}$.

(b) The data contained in a natural 2-transformation $\sigma: \alpha \Rightarrow \alpha'$ between two 2-cocycles is given by a collection $\sigma_g \in \mathbb{C}^\times$ for all morphisms $g$ in $\mathcal{G}$ such that

$$\sigma_{g_2 \circ g_1} \alpha'_{g_2, g_1} = \alpha_{g_2, g_1} \sigma_{g_1} \sigma_{g_2}$$

for all composable morphisms $g_1, g_2$. This is the coherence condition (B.6) which also implies (B.7). We see that natural 2-transformations restrict to the usual coboundaries on endomorphisms of an object. This immediately implies that the 2-cocycles (3.24) are well-defined up to coboundaries. To see this we pick two different 2-equivalences $\chi_1, \chi_2: \text{BLine}_\mathbb{C} \to \mathcal{B}_2 \mathbb{C}^\times$ which both have the embedding $i: \mathcal{B}_2 \mathbb{C}^\times \to \text{BLine}_\mathbb{C}$ as inverse. We then get a chain of natural 2-transformations

$$\chi_1 \Rightarrow \chi_2 \circ i \circ \chi_1 \Rightarrow \chi_2$$

which implies that the 2-cocycles are independent of the choice of $\chi$ up to coboundary.

(c) The data contained in a modification $\theta: \sigma \Rightarrow \sigma'$ between two natural 2-transformations is an assignment of an element $\theta_G \in \mathbb{C}^\times$ to every $G \in \text{Obj}(\mathcal{G})$ such that

$$\theta_{t(g)} \sigma_g = \sigma'_g \theta_{s(g)},$$

which is the condition (B.9).

Having defined 2-cocycles for groupoids we can now define projective representations (see e.g. [Wil08 Section 2.3.1]).

**Definition 3.27.** A projective representation $\rho$ of a groupoid $\mathcal{G}$ twisted by a 2-cocycle $\alpha: \mathcal{G} \to \mathcal{B}_2 \mathbb{C}^\times$ consists of the following data:

(a) A complex vector space $V_G$ for all $G \in \text{Obj}(\mathcal{G})$.

(b) A linear map $\rho(g): V_{s(g)} \to V_{t(g)}$ for each morphism $g$ of $\mathcal{G}$ such that

$$\rho(g_2) \circ \rho(g_1) = \alpha_{g_2, g_1} \rho(g_2 \circ g_1)$$

for all composable morphisms $g_1, g_2$. 


In this definition we work with cocycles valued in $B^2 \mathbb{C}^\times$. A similar but slightly more complicated definition using cocycles with target $B \mathbb{L} \mathbb{I} \mathbb{N} \mathbb{E}_\mathbb{C}$ can be deduced from

**Proposition 3.28.** A projective groupoid representation with $2$-cocycle $\alpha : \mathcal{G} \to B^2 \mathbb{C}^\times \subset 2\text{Vect}_\mathbb{C}$ is the same as a natural $2$-transformation $\mathbf{1} \Rightarrow \alpha$, where $\alpha$ is considered as a $2$-functor to $2\text{Vect}_\mathbb{C}$.

**Proof.** This follows immediately from Definition 3.5. □

**Remark 3.29.** We can use Proposition 3.28 to define intertwiners between projective representations as modifications between the corresponding natural $2$-transformations.

To apply this general formalism to the anomalous field theories at hand, we introduce the Picard $2$-groupoid $\text{Pic}_2(B)$ of a monoidal $2$-category $B$ consisting of the objects of $B$ which are invertible with respect to the monoidal product, and invertible $1$-morphisms and $2$-morphisms; there is a canonical embedding $\text{Pic}_2(B) \to B$. An extended quantum field theory $A$ is invertible if and only if it factors uniquely through $A : \text{Cob}_{d,d-1,d-2} \to \text{Pic}_2(2\text{Vect}_\mathbb{C}) \hookrightarrow 2\text{Vect}_\mathbb{C}$.

We can pick an equivalence of $2$-categories $\text{Pic}_2(2\text{Vect}_\mathbb{C}) \to B \mathbb{L} \mathbb{I} \mathbb{N} \mathbb{E}_\mathbb{C}$ by choosing a non-canonical equivalence between every invertible $2$-vector space and $\text{Vect}_\mathbb{C}$; an inverse to this equivalence is given by the embedding $B \mathbb{L} \mathbb{I} \mathbb{N} \mathbb{E}_\mathbb{C} \to \text{Pic}_2(2\text{Vect}_\mathbb{C})$. The invertibility of the anomaly quantum field theory $A$ and this equivalence induces a $2$-cocycle of the symmetry groupoid with values in $\text{Vect}_\mathbb{C}$:

$$\alpha^A : \text{SymCob}_d^\mathcal{G} \longrightarrow B \mathbb{L} \mathbb{I} \mathbb{N} \mathbb{E}_\mathbb{C}.$$ 

The same argument as that used in Remark 3.26(b) shows that this cocycle is independent of the choice of equivalence $\text{Pic}_2(2\text{Vect}_\mathbb{C}) \to B \mathbb{L} \mathbb{I} \mathbb{N} \mathbb{E}_\mathbb{C}$ up to coboundary. Combining these facts with Proposition 3.28 we can then infer

**Proposition 3.30.** Every anomalous quantum field theory $A : \mathbf{1} \Rightarrow \text{tr}A$ induces a projective representation of the symmetry groupoid $\text{SymCob}_d^\mathcal{G}$. The $2$-cocycle $\alpha^A$ corresponding to this representation is unique up to coboundary.

We have seen in Proposition 3.28 that natural $2$-transformations $\mathbf{1} \Rightarrow \alpha$ are the same things as projective representations of groupoids, so it should come as no surprise that these cocycles appear in the description of anomalies. The interesting prospect is that we can extend these cocycles to invertible extended field theories. This allows us to calculate quantities related to anomalies using the machinery of extended quantum field theories. Furthermore, we can couple such a theory to a bulk theory cancelling the anomaly as in Example 2.12. It is not clear that every anomaly admits such an extension, but all anomalies should give a projective representation of the symmetry groupoid.

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10This is the same thing as a higher fixed point for the representation $\alpha$ of $\mathcal{G}$.
4 Extended quantum field theory and the parity anomaly

In this section we extend the quantum field theory $\mathcal{Z}_{\text{par}}^C$ constructed in Section 2.2 to an invertible extended quantum field theory $\mathcal{A}_{\text{par}}^C$ describing the parity anomaly of an anomalous field theory $\mathcal{A}_{\text{par}}^C: \mathbf{1} \Rightarrow \text{tr} \mathcal{A}_{\text{par}}^C$. This leads us naturally to the index theorem for manifolds with corners. We use the version of $\text{LM02}$ involving b-geometry; a non-technical introduction to the topic can be found in $\text{Loy04}$. An alternative approach can be found in $\text{Bun09}$. The most important concepts from b-geometry for the ensuing formalism are summarised in Appendix A.4; a detailed introduction can be found in $\text{Mel93}$, see also $\text{Sat11}$ for a more physics oriented introduction.

We fix the background field content introduced in Section 2.1 into the stack

$$\mathcal{F} = \text{Bun}_C^\Sigma \times \text{Met} \times \text{Spin} \times \text{Or}$$

for the geometric cobordism bicategory $\text{Cob}_{d,d-1,d-2}$ constructed in Section 3.2 together with a finite-dimensional unitary representation $\rho_G$ of the gauge group $G$ which specifies the matter field content. For technical reasons we restrict ourselves to the sub-bicategory of $\text{Cob}_{d,d-1,d-2}$ containing only objects with vanishing index for the Dirac operator on every connected component; this imposes conditions on the topology of each manifold $M^{d-2}$. This condition is a requirement for the existence of a well-defined index theory on manifolds with corners $\text{LM02}$. Similar restrictions also appear in $\text{Bun09}$. We further require that all structures on the collars are of product form. By a slight abuse of notation, we continue to call this bicategory $\text{Cob}_{d,d-1,d-2}$.

4.1 Index theory

We have seen in Section 2 that it is helpful to attach mapping cylinders to a manifold encoding the data of the identification of the boundary components with lower-dimensional objects. In the extended case we also need mapping boxes at the corners. Let $Y_i, i = 1, 2, 3, 4$ be four closed manifolds equipped with $\mathcal{F}$-fields of product form $f_i \in \mathcal{F}(Y_i \times (-\epsilon_1, \epsilon_1)^2)$, and a diagram of $\mathcal{F}$-diffeomorphisms $\varphi_{ij}$:

$$\begin{array}{ccc}
Y_1 & \xrightarrow{\varphi_{12}} & Y_2 \\
\downarrow{\varphi_{13}} & & \downarrow{\varphi_{24}} \\
Y_3 & \xrightarrow{\varphi_{34}} & Y_4
\end{array}$$

Then the mapping box $\mathcal{M}(Y, \varphi)$ of length $\epsilon$ corresponding to this data is constructed by gluing $Y_1 \times [0, \frac{3}{2} \epsilon]^2, Y_2 \times (\frac{1}{4} \epsilon, \epsilon] \times [0, \frac{3}{4} \epsilon), Y_3 \times [0, \frac{3}{4} \epsilon) \times (\frac{1}{4} \epsilon, \epsilon] and Y_4 \times (\frac{1}{4} \epsilon, \epsilon] ^2 along \varphi_{ij}$. Using descent we can construct an element $f \in \mathcal{F}(\mathcal{M}(Y, \varphi))$.

Given a regular 2-morphism $M^{d,2}$ from $M_1^{d-1,1} : M_{-}^{d-2} \to M_{+}^{d-2}$ to $M_2^{d-1,1} : M_{-}^{d-2} \to M_{+}^{d-2}$ in $\text{Cob}_{d,d-1,d-2}$, by definition it comes with collars $N_1^d \cong M_1^{d-1,1} \times [0, \epsilon_1]$ and $N_2^d \cong M_2^{d-1,1} \times (-\epsilon_1, 0)$. We first attach mapping cylinders of a fixed length $\epsilon \in \mathbb{R}_{>0}$ to $M_1^{d-1,1}, M_2^{d-1,1}$ and the 0-boundary. In a second step we attach mapping boxes of length $\epsilon$ to the corners of $M^{d,2}$. We denote this new manifold by $M^{d,2}$ (see Figure 3). For this to be well-defined we need compatibility of all collars involved. The new manifold has four distinct boundaries which we denote by $M_1^{d-1,1}, M_2^{d-1,1}, C(M_{-}^{d-2}) = M_{-}^{d-2} \times [-\epsilon - \frac{1}{2} \epsilon_1, \epsilon + \frac{1}{2} \epsilon_1]$ and $C(M_{+}^{d-2}) = M_{+}^{d-2} \times [-\epsilon - \frac{1}{2} \epsilon_1, \epsilon + \frac{1}{2} \epsilon_1]$. 

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We can now attach cylindrical ends to \( \tilde{M}'_{d,2} \). For this, we first define
\[
\tilde{M}'_{d,2} = M'_{d,2} \sqcup \partial M'_{d,2}
\]
where we use the collars to glue the manifolds and extend all fields as products. Then \( \tilde{M}'_{d,2} \) is a non-compact manifold with corners. Further gluing (see Figure 3) produces
\[
\tilde{M}'_{d,2} = \tilde{M}'_{d,2} \sqcup \partial \tilde{M}'_{d,2}
\]
with all structures extended as products. As in the case of manifolds with boundaries, the Dirac operator \( \mathcal{D}_{\tilde{M}'_{d,2}} \) is not Fredholm in general, and one can prove analogously that \( \mathcal{D}_{\tilde{M}'_{d,2}} \) is Fredholm if and only if the induced Dirac operators on the corners and boundaries are invertible [LM02].

When the kernel of the corner Dirac operator is non-trivial, we have to add a mass perturbation [Loy04]. The induced twisted spinor bundle over \( Y = M_{d}^{-2} \sqcup -M_{d}^{-2} \) decomposes into spinors of positive and negative chirality. We pick a unitary self-adjoint isomorphism \( T_i : \ker(\mathcal{D}_{Y_i}) \to \ker(\mathcal{D}_{Y_i}) \), for every connected component \( Y_i \) of the corner \( Y \), which is odd with respect to the \( \mathbb{Z}_2 \)-grading of the spinor bundle; this is possible since the index of \( \mathcal{D}_Y \) is 0 by assumption. We define
\[
T_\pm = \bigoplus_{i=1}^n T_{\pm,i} \quad \text{and} \quad T = T_- \oplus T_+ ,
\]
where \( T_{\pm,i} : \ker(\mathcal{D}_{M_{d,2}^\pm}) \to \ker(\mathcal{D}_{M_{d,2}^\pm}) \). Now the operator \( \mathcal{D}_Y - T \) is invertible. This suggests extending \( T \) to an operator \( \hat{T} \) on \( \tilde{M}'_{d,2} \) such that the massive Dirac operator \( \mathcal{D}_{\tilde{M}'_{d,2}} - \hat{T} \) is Fredholm on weighted Sobolev spaces. A concrete construction of \( \hat{T} \) can be found in [LM02, Section 2.3], from which it is clear that \( \hat{T} \) is independent of the length \( \epsilon \) of the attached

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\[ \text{Figure 3: Illustration of the construction of } \tilde{M}'_{d,2} \text{ near } M_{d}^{-2}. \]
mapping cylinders and boxes. When we choose for every boundary component a small mass \(\alpha_i\) as in Section 2, then
\[
\mathcal{D}_{M^{d-2}}^+ - \hat{T}^+ : e^{\alpha s} H^1(\hat{S}_{M^{d-2}}^+) \to e^{\alpha s} L^2(\hat{S}_{M^{d-2}}^-)
\]
is a Fredholm operator on weighted Sobolev spaces \([LM02\text{, Theorem 2.6}]\). We restrict ourselves to a description of the corresponding index theorem on manifolds which are of the form \(M^{d,2}\) for a regular 2-morphism \(M^{d,2}\) in \(\text{Cob}_{d-1,d-2}\); the more general version can be found in \([LM02\text{, Theorem 6.13}]\).

To define the \(\eta\)-invariant on a manifold \(M^{d-1,1}\) with boundary we proceed as in Section 2 and define \(M^{d-1,1}\) by attaching cylindrical ends to \(M^{d-1,1}\). In general, the Dirac operator \(\mathcal{D}_{M^{d-1,1}}\) has a continuous spectrum, so we have to use the expression (2.8) to define the \(\eta\)-invariant as an integral
\[
b\eta(\mathcal{D}_{M^{d-1,1}}) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} b\text{Tr}(\mathcal{D}_{M^{d-1,1}} e^{-t \mathcal{D}_{M^{d-1,1}}^2}) \, dt,
\]
where we have to replace the usual trace by the b-geometric trace (see Appendix A.4) because its argument is not a trace-class operator on \(M^{d-1,1}\) in general. There are other ways of defining \(\eta\)-invariants for manifolds with boundaries using appropriate boundary conditions \([DF95, LW96, M"{u}l94]\). The \(b\eta\)-invariant agrees with the canonical boundary conditions on spinors at the infinite ends of the cylinders induced by scattering Lagrangian subspaces, which we describe below.

There is a further contribution to the index theorem coming from the corners. We define for \(M^{d-1,1}_i, j = 1, 2\) the scattering Lagrangian subspace
\[
\Lambda_{C_i} = \left\{ \lim_{s_+ \to \infty} \Psi(y-, s_-) \oplus \lim_{s_+ \to \infty} \Psi(y+, s_+) \mid \Psi \in \mathcal{C}^\infty(\hat{S}_{M^{d-1,1}}) \cap \ker(\mathcal{D}_{M^{d-1,1}}) \text{ bounded} \right\}
\]
where \(s_- \in (-\infty, 0]\), \(s_+ \in [0, \infty)\) and \(y_\pm \in M^{d-2}_\pm\). The set \(\Lambda_{C_i} \subset \ker(\mathcal{D}_Y)\) is a Lagrangian subspace of \(\ker(\mathcal{D}_Y)\) with respect to the symplectic form \(\omega(\cdot, \cdot) = \langle \Gamma \cdot, \cdot \rangle_{L^2}\) \([M"{u}l94]\), where \(\Gamma\) is the chirality operator on the spinor bundle over the corners, i.e. \(\Gamma\) acts by scalar multiplication with \(\pm i\) on \(S_\pm^\mp\). We define an odd unitary self-adjoint isomorphism \(C_i\) of \(\ker(\mathcal{D}_Y)\), called the scattering matrix of \(\mathcal{D}_{M^{d-1,1}}\), by setting \(C_i = \text{id} \mid \Lambda_{C_i}\) and \(C_i = -\text{id} \mid \Lambda_{C_i}^\perp\). We denote by \(\Lambda_T \subset \ker(\mathcal{D}_Y)\) the \(+1\)-eigenspace of \(T\); there is a one-to-one correspondence between Lagrangian subspaces of \(\ker(\mathcal{D}_Y)\) and boundary conditions \(T \in \text{End}_\mathbb{C}(\ker(\mathcal{D}_Y))\). Following \([LW96, Bun95]\), we introduce the ‘exterior angle’ between Lagrangian subspaces by the spectral formula
\[
\mu(\Lambda_T, \Lambda_{C_i}) = -\frac{1}{\pi} \sum_{\theta \in \text{spec}(\mathcal{D}_Y)} e^{i \theta} \theta \in \text{spec}(\mathcal{D}_Y),
\]
where the grading is with respect to the \(\mathbb{Z}_2\)-grading of the twisted spinor bundle over the corners.

With this notation, we can now formulate the index theorem for manifolds of the form \(M^{d,2}\) as
Theorem 4.2. If $M^{d,2}$ is a regular 2-morphism in $\text{Cob}_{d-1,d-2}$, then

\[
\text{ind}(\mathcal{D}_M^{d,2} - \mathcal{T}^+) = \int_{M^{d,2}} K_{\text{AS}} - \frac{1}{2} \left( b \eta(\mathcal{D}_M^{d-1,1}) + b \eta(\mathcal{D}_M^{d-1,1}) \right) + \dim \ker(\mathcal{D}_M^{d-1,1}) - \dim \ker(\mathcal{D}_M^{d-1,1}) + \dim(\Lambda_T \cap \Lambda_C) - \dim(\Lambda_T \cap \Lambda_C) + \mu(\Lambda_T, \Lambda_C) \right).
\]

Remark 4.4. The extra corner contributions in the last line of (4.3) to the usual (b-geometric) Atiyah-Patodi-Singer formula (2.7) can be understood as follows. Let $M^{d-1,1}$ be a regular 1-morphism in the bicategory $\text{Cob}_{d-1,d-2}$. Then for every $T \in \text{End}_C(\ker(\mathcal{D}_M^{d-1,1}))$ we can relate the spectral data of the massive Dirac operator on $M^{d-1,1}$ to their massless counterparts as

\[
b \eta(\mathcal{D}_M^{d-1,1} - \mathcal{T}) = b \eta(\mathcal{D}_M^{d-1,1}) + \mu(\Lambda_T, \Lambda_C),
\]

\[
\dim \ker(\mathcal{D}_M^{d-1,1} - \mathcal{T}) = \dim \ker(\mathcal{D}_M^{d-1,1}) + \dim(\Lambda_T \cap \Lambda_C),
\]

where $\Lambda_C$ is the scattering Lagrangian subspace for $M^{d-1,1}$.

Remark 4.5. We describe the relation between $b \eta$-invariants and $\eta$-invariants with boundary conditions [Loy05]. We denote by $\Pi_+$ the projection onto the space spanned by the positive eigenspinors of $\mathcal{D}_\partial M^{d-1,1}$, and by $\Pi_T$ the projection onto the positive eigenspace $\Lambda_T$. This allows us to define a Dirac operator $\mathcal{D}_T$, which coincides with $\mathcal{D}_\partial M^{d-1,1}$, on the domain

\[
\{ \Psi \in H^1(\hat{\mathcal{S}}_{M^{d-1,1}}) \mid (\Pi_+ + \Pi_T)\Psi \big|_{\partial M^{d-1,1}} = 0 \}.
\]

The operator $\mathcal{D}_T$ is self-adjoint and elliptic for all $T$. It is shown in [Loy05] Theorem 1.2 that

\[
\eta(\mathcal{D}_T) = b \eta(\mathcal{D}_M^{d-1,1}) + \mu(\Lambda_T, \Lambda_C),
\]

so that we can combine the $b \eta$-invariant and the exterior angle $\mu$ in (4.3) into an $\eta$-invariant for a Dirac operator with suitable boundary conditions induced by the Lagrangian subspace $\Lambda_T$.

Proof of Theorem 4.2. From the general index theorem for manifolds with corners [LM02 Theorem 6.13] we get

\[
\text{ind}(\mathcal{D}_M^{d,2} - \mathcal{T}^+) = \int_{M^{d,2}} K_{\text{AS}} - \frac{1}{2} \left( \frac{1}{2} \left( b \eta(\mathcal{D}_M^{d-1,1}) + b \eta(\mathcal{D}_M^{d-1,1}) + b \eta(\mathcal{D}_C(M^{d-2})) + b \eta(\mathcal{D}_C(M^{d-2})) \right) + \dim \ker(\mathcal{D}_M^{d-1,1}) + \dim(\Lambda_T \cap \Lambda_C) - \dim \ker(\mathcal{D}_M^{d-1,1}) + \dim \ker(\mathcal{D}_M^{d-1,1}) + \dim(\Lambda_T \cap \Lambda_C) + \mu(\Lambda_T, \Lambda_C) + \mu(\Lambda_T, \Lambda_C) + \mu(\Lambda_T, \Lambda_C) + \mu(\Lambda_T, \Lambda_C)
\]

where $\Lambda_C$ are the scattering Lagrangian subspaces for $C(M^{d-2})$, respectively. We can calculate the contributions from the boundaries $C(M^{d-2})$ explicitly and show that they all vanish.

Attaching infinite cylindrical ends to $C(M^{d-2})$ leads to the manifolds $M^{d-2}_\pm \times (-\infty, \infty)$. The Dirac operator on the manifold $M^{d-2}_\pm \times (-\infty, \infty)$ of odd dimension $d-1$ is given by (A.6):

\[
\mathcal{D}_{M^{d-2}_\pm \times (-\infty, \infty)} = \sigma_t(\mathcal{D}_{M^{d-2}_\pm} + \partial_t),
\]

(4.7)
where \( t \in (-\infty, \infty) \). We are interested in the dimension of the space of harmonic spinors \( \Psi(y_\pm, t) \). By elliptic regularity there exists a basis of smooth sections. Multiplying (4.7) with \( \sigma_t^{-1} \), we get
\[
\left( \mathcal{D}_{M^d_{\pm}} + \partial_t \right) \Psi(y_\pm, t) = 0 .
\]
Using separation of variables \( \Psi(y_\pm, t) = \psi(y_\pm) \alpha(t) \), this equation reduces to a pair of equations
\[
\mathcal{D}_{M^d_{\pm}} \psi(y_\pm) = \lambda \psi(y_\pm) \quad \text{and} \quad \frac{d\alpha(t)}{dt} = -\lambda \alpha(t) ,
\]
for an arbitrary constant \( \lambda \) which must be real since \( \mathcal{D}_{M^d_{\pm}} \) is an elliptic operator. The second equation has solution (up to a constant) \( \alpha(t) = e^{-\lambda t} \), and we finally see that there are no non-zero square-integrable spinors \( \Psi(y_\pm, t) \) with eigenvalue 0. Hence the contributions from the terms \( \dim \ker(\mathcal{D}_{\mathcal{C}(M^d_{\pm})}) \) are 0.

A solution of (4.8) is bounded if and only if \( \lambda = 0 \), and so the scattering Lagrangian subspace takes the form
\[
\Lambda_{C_{\pm}} = \Delta(\ker(\mathcal{D}_{M^d_{\pm}})) = \{ \psi \oplus \psi \mid \psi \in \ker(\mathcal{D}_{M^d_{\pm}}) \} \subset \ker(\mathcal{D}_{M^d_{\pm}}) \oplus \ker(\mathcal{D}_{-M^d_{\pm}}) .
\]
This implies that
\[
\dim(\Lambda_{-i\Gamma T_\pm} \cap \Lambda_{C_{\pm}}) = 0 ,
\]
since the chirality operator \( \Gamma \) on the outgoing and ingoing boundaries differs by a sign while \( T_\pm \) is the same over both boundaries.

Finally, by Remark 4.5 \( \text{b} \eta(\mathcal{D}_{\mathcal{C}(M^d_{\pm})}) + \mu(\Lambda_{-i\Gamma T_\pm}, \Lambda_{C_{\pm}}) \) is the \( \eta \)-invariant on a cylinder with identical boundary conditions at both ends, which vanishes by [LW96, Theorem 2.1]. \( \square \)

For later use we derive here a formula for the index of a 2-morphism under cutting. For this, we first have to study the behaviour of the various quantities in the index formula (4.3) under orientation-reversal.

**Lemma 4.9.** Let \( M^{d-1,1} \) be a regular 1-morphism in \( \text{Cob}_{d,d-1,d-2}^\mathbb{F} \) with fixed boundary condition
\( T \in \text{End}_C(\ker(\mathcal{D}_{\partial M^{d-1,1}})) \) as above. If we reverse the orientation of \( M^{d-1,1} \), then \( T \) still defines a suitable boundary condition of \( \mathcal{D}_{\partial(-M^{d-1,1})} \) and
\[
\dim \ker(\mathcal{D}_{M^{d-1,1}}) = \dim \ker(\mathcal{D}_{-M^{d-1,1}}) \quad \text{and} \quad \text{b}\eta(\mathcal{D}_{M^{d-1,1}}) = \text{b}\eta(\mathcal{D}_{-M^{d-1,1}}) ,
\]
\[
\dim(\Lambda_T \cap \Lambda_C) = \dim(\Lambda_T \cap \Lambda_{-C}) \quad \text{and} \quad \mu(\Lambda_T, \Lambda_C) = -\mu(\Lambda_T, \Lambda_{-C}) ,
\]
where \( \Lambda_{-C} \) is the scattering Lagrangian subspace for \( -M^{d-1,1} \).

**Proof.** There is an equality \( \mathcal{D}_{M^{d-1,1}} = -\mathcal{D}_{-M^{d-1,1}} \) of operators acting on sections of the underlying twisted spinor bundle \( \tilde{S}_{M^{d-1,1}} \), which implies the first two equations. The Lagrangian subspaces \( \Lambda_C \) and \( \Lambda_T \) are independent of the orientation, which implies the third equation.

We can interpret the exterior angle \( \mu(\Lambda_T, \Lambda_C) \) as the \( \eta \)-invariant of a cylinder with boundary conditions induced by \( \Lambda_T \) and \( \Lambda_C \) [LW96]. Reversing the orientation of this cylinder corresponds to \( \mu(\Lambda_T, \Lambda_{-C}) \). The last equation then follows from the fact that the \( \eta \)-invariant changes sign under orientation-reversal. \( \square \)
Proposition 4.10. The index is additive under vertical composition of regular 2-morphisms in \( \text{Cob}_{d,d-1,d-2}^{\mathbb{F}} \) if we choose identical boundary conditions on the corners.

Proof. The contributions from the gluing boundary cancel each other by Lemma 4.9. We still have to show that

\[
\int_{(M^{d,2}_2 \circ M^{d,2}_1)} K_{\text{AS}} = \int_{M^{d,2}_1} K_{\text{AS}} + \int_{M^{d,2}_2} K_{\text{AS}}.
\]

This is not completely obvious, since the vertical composition also involves deleting half of the collars of the gluing boundary. However, from (4.3) and the construction of \( \hat{M}^{d,2} \) it is clear that \( \int_{M^{d,2}} K_{\text{AS}} \) is independent of the length of the collars. Using the description of gluing in terms of mapping cylinders we can cover \( (M^{d,2}_2 \circ M^{d,2}_1)' \) by \( M^{d,2}_1 \) and \( M^{d,2}_2 \), where \( M^{d,2}_i \) is the manifold \( M^{d,2}_i \) with \( \frac{3}{4} \) of the collar corresponding to the gluing boundary removed. \( \square \)

4.2 The extended quantum field theory \( \mathcal{A}^\xi_{\text{parity}} \)

We shall now proceed to extend the quantum field theory \( \mathcal{Z}^\xi_{\text{parity}} \) to an anomaly quantum field theory \( \mathcal{A}^\xi_{\text{parity}} : \text{Cob}_{d,d-1,d-2}^{\mathbb{F}} \to \mathbf{2Vect}_\mathbb{C} \) describing the parity anomaly in \( d-1 \) dimensions. Following Section 2, we would like to define something like \( \mathcal{A}^\xi_{\text{parity}}(M^{d,2}) = \zeta^{\text{ind}(\mathcal{D}^{\mathbb{F}}_{M^{d,2}})} \) for a fixed \( \zeta \in \mathbb{C}^\times \) and every regular 2-morphism \( M^{d,2} \) of \( \text{Cob}_{d,d-1,d-2}^{\mathbb{F}} \). The problem with this definition is that the index may depend on our choice of \( T \in \text{End}_\mathbb{C}(\ker(\mathcal{D}_{Y})) \). The resolution is to include the data about the choice of \( T \) into our extended quantum field theory.

We do this by combining, for each object \( M^{d-2} \) of \( \text{Cob}_{d,d-1,d-2}^{\mathbb{F}} \), all possible boundary conditions \( T \) into a category \( \mathcal{A}^\xi_{\text{parity}}(M^{d-2}) \) in the following way: Let \( T(M^{d-2}) \) be the category with one object for every odd self-adjoint unitary \( T \in \text{End}_\mathbb{C}(\ker(\mathcal{D}_{M^{d-2}})) \) and one morphism between every pair of objects. The category \( \mathcal{A}^\xi_{\text{parity}}(M^{d-2}) \) is then defined to be the finite completion of the \( \mathbb{C} \)-linearisation \( \mathbb{C}T(M^{d-2}) = \{ \mathbb{C}T \mid T \in T(M^{d-2}) \} \) of the category \( T(M^{d-2}) \). A concrete model is given, for example, by the functor category

\[
\mathcal{A}^\xi_{\text{parity}}(M^{d-2}) \cong \left[ T(M^{d-2}), \mathbf{Vect}_\mathbb{C} \right],
\]

(4.11)

of pre-cosheaves of complex vector spaces on \( T(M^{d-2}) \).

We can construct a \( \mathbb{C} \)-linear functor \( \mathcal{A}^\xi_{\text{parity}}(M^{d-1,1}) : \mathcal{A}^\xi_{\text{parity}}(M^{d-2}) \to \mathcal{A}^\xi_{\text{parity}}(M^{d-2}) \) for a regular 1-morphism \( M^{d-2}_{-} \times [0, \epsilon_-] \xrightarrow{\zeta} M^{d-1,1}_{-} \xrightarrow{\zeta} M^{d-2}_{+} \times (-\epsilon_+, 0] \) in \( \text{Cob}_{d,d-1,d-2}^{\mathbb{F}} \) by using the corresponding boundary and corner contributions to the index formula (4.3) to build a complex line encoding all possible boundary conditions on \( M^{d-2} \). For generators \( T_{\pm} \) of \( \mathcal{A}^\xi_{\text{parity}}(M^{d-2}) \), we define

\[
I_{M^{d-1,1}}(T_{-}, T_{+}) = -\frac{1}{2} \left( b\eta(\mathcal{D}_{M^{d-1,1}}) - \dim \ker(\mathcal{D}_{M^{d-1,1}}) - \dim(\Lambda_{T_- \oplus T_+} \cap \Lambda C) + \mu(\Lambda_{T_- \oplus T_+} \cap \Lambda C) \right) \,.
\]

We will sometimes drop the subscript \( M^{d-1,1} \) when it is obvious from the context. We fix a generator \( T_{-} \) of \( \mathcal{A}^\xi_{\text{parity}}(M^{d-2}) \). Then we get a morphism in \( \mathcal{A}^\xi_{\text{parity}}(M^{d-2}) \) from generator \( T_{+,i} \) to generator \( T_{+,j} \) in \( T(M^{d-2}_{+}) \) by multiplying the special morphism between \( T_{+} \) and \( T_{+} \) with

\[
\zeta I_{M^{d-1,1}}(T_{-}, T_{+}) - I_{M^{d-1,1}}(T_{-}, T_{+}) \,.
\]
These morphisms fit into a family of diagrams $\mathcal{J}(M^{d-1,1})(T_-): \mathbf{T}(M^d_{d-2}) \to \mathcal{A}^\zeta_{\text{parity}}(M^d_{d-2})$. We finally define the complex line

$$\mathcal{A}^\zeta_{\text{parity}}(M^{d-1,1})(T_-) = \lim_{\leftarrow} \mathcal{J}(M^{d-1,1})(T_-).$$

To be precise, we have to invoke the axiom of choice to pick a representative for the limit. We define $\mathcal{A}^\zeta_{\text{parity}}(M^{d-1,1})(f)$ for a morphism $f: T_- \to T'_-$ in $\mathbf{T}(M^d_{d-2})$ to be the unique map induced by the matrix of compatible morphisms

$$\left( \zeta I_{M^{d-1,1}}(T'_-,T_{+,i}) - I_{A^{d-1,1}}(T_-,T_{+,i}) \right)$$

via the functoriality of limits. We can summarise our definition of $\mathcal{A}^\zeta_{\text{parity}}$ on regular 1-morphisms by the commutative diagram

$$
\begin{array}{ccc}
\mathcal{J}(M^{d-1,1}) & \to & [\mathbf{T}(M^d_{d-2}), \mathcal{A}^\zeta_{\text{parity}}(M^d_{d-2})] \\
\mathcal{J}_{\text{cont}}(M^{d-1,1}) & \to & \lim_{\leftarrow} \\
\mathcal{A}^\zeta_{\text{parity}}(M^d_{d-2}) & \to & \mathcal{A}^\zeta_{\text{parity}}(M^d_{d-2}) \\
\mathbf{T}(M^d_{d-2}) & \to &
\end{array}
$$

(4.13)

where $\mathcal{J}_{\text{cont}}(M^{d-1,1})$ is the continuous extension of $\mathcal{J}(M^{d-1,1})$, which is unique up to a canonical natural isomorphism.

For a regular 2-morphism $M^{d,2} : M^{d-1,1} \Rightarrow M^{d-1,1}$, we define a family of morphisms in $\mathcal{A}^\zeta_{\text{parity}}(M^d_{d-2})$ by

$$\xi(M^{d,2})_{T_-,T_{+,i}} = \zeta^{\text{ind}(\mathcal{D}_{M^{d,2}}^{+} \circ T_{+}^{+} \circ T_{-,i}^{+})} \cdot \text{id}_{T_{+,i}} : T_{+,i} \to T_{+,i}.$$  

(4.14)

**Proposition 4.15.** The family of morphisms (4.14) induces a natural transformation

$$\mathcal{A}^\zeta_{\text{parity}}(M^{d,2}) = \text{id}_{\lim} \bullet \xi_{\text{cont}}(M^{d,2}) : \mathcal{A}^\zeta_{\text{parity}}(M^{d-1,1}) \Rightarrow \mathcal{A}^\zeta_{\text{parity}}(M^{d-1,1})$$

as indicated in the diagram

$$
\begin{array}{ccc}
\mathcal{J}(M^{d-1,1}) & \to & [\mathbf{T}(M^d_{d-2}), \mathcal{A}^\zeta_{\text{parity}}(M^d_{d-2})] \\
\mathcal{J}(M^d_{d-1,1}) & \to & \lim_{\leftarrow} \\
\mathcal{A}^\zeta_{\text{parity}}(M^d_{d-2}) & \to & \mathcal{A}^\zeta_{\text{parity}}(M^d_{d-2}) \\
\mathbf{T}(M^d_{d-2}) & \to &
\end{array}
$$

(4.16)
where $\xi_{\text{cont}}(M^{d,2})$ is the unique extension of $\xi(M^{d,2})$ to a natural transformation between the continuous extensions of $\mathcal{J}(M^{d-1,1}_1)$ and $\mathcal{J}(M^{d-1,1}_2)$.

**Proof.** The only thing we need to show is that $\xi(M^{d,2})$ is actually a natural transformation. First of all, we have to prove that we really do get a diagram from the definition (4.14). For this, we have to prove that the diagram

\[
\begin{array}{ccc}
T_{+,i} & \xrightarrow{\zeta(M^{d-1,1}_1,T_-,T_+,i,T_+,j)\ind(\Phi^+_{M^{d,2}}-T_-+T_+,i)} & T_{+,i} \\
\downarrow & & \downarrow \\
T_{+,j} & \xrightarrow{\zeta(M^{d-1,1}_2,T_-,T_+,i,T_+,j)\ind(\Phi^+_{M^{d,2}}-T_-+T_+,j)} & T_{+,j}
\end{array}
\]  
(4.17)

commutes, where we defined

\[
I(M^{d-1,1}_1,T_-,T'_-,T_+,i,T_+,j) = I_{M^{d-1,1}}(T'_-,T_+,j) - I_{M^{d-1,1}}(T_-,T_+,i)
\]

This is an immediate consequence of the index theorem (4.3); note that this only works because the incoming and outgoing boundaries contribute with different signs. Thus the diagram (4.17) induces a morphism $\mathcal{A}_\text{parity}^\zeta(M^{d,2})T_- : \mathcal{A}_\text{parity}^\zeta(M^{d-1,1}_1)(T_-) \to \mathcal{A}_\text{parity}^\zeta(M^{d-1,1}_2)(T_-)$. To show that these morphisms fit into a natural transformation it is enough to show that the diagram

\[
\begin{array}{ccc}
T_{+,i} & \xrightarrow{\zeta(M^{d-1,1}_1,T_-,T_+,i,T_+,j)\ind(\Phi^+_{M^{d,2}}-T_-+T_+,i)} & T_{+,i} \\
\downarrow & & \downarrow \\
T_{+,i} & \xrightarrow{\zeta(M^{d-1,1}_2,T_-,T_+,i,T_+,i)\ind(\Phi^+_{M^{d,2}}-T_-+T_+,i)} & T_{+,i}
\end{array}
\]

commutes. This follows again immediately from the index theorem. \qed

We define the theory $\mathcal{A}_\text{parity}^\zeta$ on limit 1-morphisms as the functor corresponding to a mapping cylinder. From the definition of $\mathcal{A}_\text{parity}^\zeta(M^{d-1,1})$ it is clear that this is independent of the length of the mapping cylinder, since only the behaviour at infinity is important. On limit 2-morphisms we define the theory to be the value of $\mathcal{A}_\text{parity}^\zeta$ on a mapping cylinder of length $\epsilon$. This completes the definition of $\mathcal{A}_\text{parity}^\zeta$.

Before demonstrating that $\mathcal{A}_\text{parity}^\zeta$ is an extended quantum field theory, we explicitly calculate the functors corresponding to limit 1-morphisms.

**Proposition 4.18.** Let $\phi : M^{d-2} \to M^{d-2}_\pm$ be a limit 1-morphism in $\text{Cob}^{\mathbb{Z}}_{d,d-1,d-2}$ with mapping cylinder $\mathfrak{M}(\phi)$, and let $T_\pm$ be fixed objects in $\mathcal{T}(M^{d-2}_\pm)$. Then

\[
I_{\mathfrak{M}(\phi)}(T_-,T_+) = -\frac{1}{2} \left( \dim(\Lambda_{T_-} \cap \Lambda_{\phi^*T_+}) + \mu(\Lambda_{T_-}, \Lambda_{\phi^*T_+}) \right).
\]  
(4.19)
Proof. By Remark 4.5 we get a term in $I_{\mathfrak{M}(\phi)}(T_-, T_+)$ corresponding to the $\eta$-invariant with boundary conditions induced by the Lagrangian subspaces $\Lambda_{T_{\pm}}$. There is a diffeomorphism induced by $\phi^{-1}$ and id from the mapping cylinder of $\phi$ with length 1 to the cylinder $M^{-2} \times [0, 1]$. The boundary conditions change to new boundary conditions induced by $\Lambda_{T_-}$ and $\Lambda_{\phi^*T_+}$. The $\eta$-invariant for this situation was calculated in [LW96, Theorem 2.1], from which we get

$$\eta(\Phi_{T_-, \phi^*T_+}) = \mu(\Lambda_{T_-}, \Lambda_{\phi^*T_+}).$$

We can extend the diffeomorphism induced by $\phi^{-1}$ and id above to manifolds with cylindrical ends attached. The expression (4.19) then follows from similar arguments to those used in the proof of Theorem 4.2.

$$\Box$$

Theorem 4.20. $\mathcal{A}_\text{parity}^C: \text{Cob}_{d-1, d-2}^F \to 2\text{Vect}_C$ is an invertible extended quantum field theory.

Proof. We construct a family of natural isomorphisms $\Phi_{M^{d-2}}: \text{id} \Rightarrow \mathcal{A}_\text{parity}^C(\text{id}_{M^{d-2}})$. For this, it is enough to construct a natural isomorphism $\Phi': \mathcal{C} \Rightarrow \mathcal{I}(M^{d-2} \times [0, 1])$, where $\mathcal{C}$ is the functor sending an object $T$ of $\mathcal{T}(M^{d-2})$ to the constant diagram on $T$. The collection of special morphisms from $T_i$ to $T$ multiplied with $\zeta^{I_{M^{d-2} \times [0, 1]}(T, T_i)}$ induces such an isomorphism:

$$\begin{array}{c}
T_i \\
\downarrow \zeta^{I(T,T_i)} \\
T_j \\
\downarrow \zeta^{I(T,T_j)} \\
\end{array}
\begin{array}{c}
T \\
\zeta^{I(T,T_j)-I(T,T_i)} \\
\end{array}$$

The naturality with respect to morphisms in $\mathcal{T}(M^{d-2})$ follows immediately from the commuting diagram

$$\begin{array}{c}
T_1 \\
\downarrow \zeta^{I(T_1,T_i)} \\
T_i \\
\downarrow \zeta^{I(T_2,T_i)} \\
T_2 \\
\end{array}$$

$$\begin{array}{c}
T_i \\
\downarrow \zeta^{I(T_1,T_i)-I(T_2,T_i)} \\
T_i \\
\end{array}$$

For the composition of regular 1-morphisms we have to construct natural $\mathbb{C}$-linear isomorphisms

$$\Phi_{M^{d-1,1}, M^{d-1,1}}: \mathcal{A}_\text{parity}^C(M^{d-1,1}) \circ \mathcal{A}_\text{parity}^C(M^{d-1,1}) \Rightarrow \mathcal{A}_\text{parity}^C(M^{d-1,1} \circ M^{d-1,1}).$$

$$\text{(4.21)}$$
Using (4.13) we get the diagram

\[
\begin{array}{ccc}
\mathcal{J}(M_2^{-1,1} \circ M_1^{-1,1}) & \xrightarrow{\mathcal{J}(M_1^{d-2})} & \mathcal{J}(M_1^{d-2}) \\
\mathcal{T}(M_1^{d-2}), \mathcal{A}_{\text{parity}}^{\mathcal{C}}(M_1^{d-2}) & \xrightarrow{\lim} & \mathcal{T}(M_2^{d-2}), \mathcal{A}_{\text{parity}}^{\mathcal{C}}(M_2^{d-2}) \\
\lim & & \\
\mathcal{A}_{\text{parity}}^{\mathcal{C}}(M_0^{-1,2}) & \xrightarrow{\mathcal{J}(M_1^{d-1,1})} & \mathcal{A}_{\text{parity}}^{\mathcal{C}}(M_1^{d-1,1}) \\
\end{array}
\]

with \( M_1^{d-2} = M_2^{d-2} \). The lower part of this diagram is commutative up to a canonical natural isomorphism coming from the universal property of the limit, since by definition \( \mathcal{J} \) is continuous; this isomorphism depends on the concrete realisation of the limit that we pick. Our goal is to now define a natural isomorphism

\[
\Phi': \lim \circ \mathcal{J} \circ \mathcal{J}(M_1^{d-1,1}) \Rightarrow \mathcal{J}(M_2^{-1,1} \circ M_1^{d-1,1})
\]

which then induces the natural isomorphism (4.21).

We begin by evaluating \( \mathcal{J} \circ \mathcal{J}(M_1^{d-1,1}) \) on a fixed object \( T_- \) of \( \mathcal{T}(M_1^{d-2}) \) to get the family of diagrams

\[
\mathcal{J} \circ \mathcal{J}(M_1^{d-1,1})(T_-): \mathcal{T}(M_1^{d-2}) \rightarrow [\mathcal{T}(M_2^{d-2}), \mathcal{A}_{\text{parity}}^{\mathcal{C}}(M_2^{d-2})]
\]

defined by

\[
\begin{array}{ccc}
T_1 & \xrightarrow{\mathcal{J} \circ \mathcal{J}(M_1^{d-1,1})(T_-)} & T_2 \\
\mathcal{T}(M_2^{d-2}) & \xrightarrow{\mathcal{A}_{\text{parity}}^{\mathcal{C}}(M_2^{d-2})} & \mathcal{T}(M_2^{d-2}) \\
(\mathcal{J} \circ \mathcal{J}(M_1^{d-1,1})(T_-)[T_1], (f: T_2 \rightarrow T_2')) & \xrightarrow{\mathcal{J} \circ \mathcal{J}(M_1^{d-1,1})(T_-)[T_2]} & (\mathcal{J} \circ \mathcal{J}(M_1^{d-1,1})(T_-)[T_2']) \\
\end{array}
\]

Since the category \( \mathcal{A}_{\text{parity}}^{\mathcal{C}}(M_2^{d-2}) \) is complete, we can calculate the limit of this diagram object-wise. For this, we fix an object \( T_2 \) of \( \mathcal{T}(M_2^{d-2}) \). A realisation for the limit is then given by the cone

\[
\mathcal{J}(M_2^{-1,1} \circ M_1^{-1,1})(T_-)[T_2] = T_2
\]
This cone is universal, since all maps involved are isomorphisms, and so by the universal property of limits we get an isomorphism

$$\Phi^{-1}_{T_1,T_2} : J(M_{d-1,1}^2 \circ M_1^{d-1,1})(T_-)[T_2] \to \lim \overset{\leftarrow}{\mathcal{J}} \circ \mathcal{J}^* \circ \mathcal{J}(M_{1}^{d-1,1})(T_-)[T_2].$$

To show that this construction is natural in $T_2$ it is enough to observe that the diagram

$$\begin{array}{ccc}
\mathcal{J}(M_{1}^{d-1,1})(T_-)[T_2] = T_2 & \xrightarrow{\zeta^I(T_-^2-T_1^2, T_-^2-T_2^2)} & \mathcal{J}(M_{1}^{d-1,1})(T_-)[T_2'] = T_2' \\
\zeta^I(T_-^1, T_1^2-T_2^2) & \downarrow & \zeta^I(T_-^1, T_1^2-T_2^2) \\
T_2[T_1] & \xrightarrow{\zeta^I(T_1^2-T_2^2)} & T_2'[T_1] \\
\end{array}$$

It follows that this construction is also natural in $T_-$. This completes the construction of the natural isomorphism $\Phi$.

In order for $\Phi$ to equip $\mathcal{A}^\zeta_{\text{parity}}$ with the structure of a 2-functor, we need to check naturality with respect to 2-morphisms and associativity. We start with the compatibility with 2-morphisms. Using the index theorem, this follows from the calculation

$$\log_\zeta \mathcal{A}_{\text{parity}}^\zeta(M_{2}^{d,2} \bullet M_{1}^{d,2})(T_-, T_+).$$

$$= \int_{(M_{2}^{d,2} \bullet M_{1}^{d,2})} K_{\text{AS}} + I_{\partial_+ M_{2}^{d,2} \bullet M_{1}^{d,2}}(T_-, T_+) - I_{\partial_- M_{2}^{d,2} \bullet M_{1}^{d,2}}(T_-, T_+)$$

$$= \int_{M_{1}^{d,2}} K_{\text{AS}} + \int_{M_{2}^{d,2}} K_{\text{AS}} + I_{\partial_+ M_{2}^{d,2} \bullet M_{1}^{d,2}}(T_-, T_+) - I_{\partial_- M_{2}^{d,2} \bullet M_{1}^{d,2}}(T_-, T_+)$$

$$+ (I_{\partial_+ M_{2}^{d,2}} (T_-, T_1) + I_{\partial_+ M_{2}^{d,2}} (T_1, T_+)) - (I_{\partial_+ M_{2}^{d,2}} (T_-, T_1) + I_{\partial_+ M_{2}^{d,2}} (T_1, T_+))$$

$$+ (I_{\partial_- M_{2}^{d,2}} (T_-, T_1) + I_{\partial_- M_{2}^{d,2}} (T_1, T_+)) - (I_{\partial_- M_{2}^{d,2}} (T_-, T_1) + I_{\partial_- M_{2}^{d,2}} (T_1, T_+))$$

$$= \log_\zeta \mathcal{A}_{\text{parity}}(M_{2}^{d,2})(T_-, T_1) + \log_\zeta \mathcal{A}_{\text{parity}}(M_{2}^{d,2})(T_1, T_+),$$

for all objects $T$ of $\mathcal{T}(\partial_+ \partial_+ M_{1}^{d,2}).$
It remains to demonstrate compatibility with associativity: \( \Phi \circ (\Phi \bullet \text{id}) = \Phi \circ (\text{id} \bullet \Phi) \), i.e. the coherence condition (B.3). For this, we fix three composable \( d-1 \)-dimensional manifolds \( M_i^{d-1,1} \), \( i = 1, 2, 3 \) with incoming boundaries \( M_i^{d-2} \) and outgoing boundaries \( M_i^{d-2} \). By the naturality of all constructions, it is enough to check the relation for fixed objects \( T \) of \( T(M_i^{d-2}) \), \( T_1 \) of \( T(M_i^{d-2}) = T(M_i^{d-2}) \), \( T_2 \) of \( T(M_i^{d-2}) = T(M_i^{d-2}) \), and \( T_+ \) of \( T(M_i^{d-2}) \). This follows immediately from the commutative diagram

We finally have to check the coherence condition (B.4). We fix a regular 1-morphism \( M_{d-1,1} : M_{d-2} \to M_{d-2} \). The situation can be represented diagramatically by

where here we abbreviate the identity regular 1-morphism \( M_{d-2} \times [0,1] \) by \( \text{id} \). We have to show that the composition of the natural transformations in the lower part of this diagram is the
identity. We can do this by showing that the composition in the upper part is the identity. We evaluate the resulting natural transformation at a fixed object $T_-$ of $\mathcal{T}(M^{d-2})$. By naturality we can also fix an object $T_+$ of $\mathcal{T}(M^{d-2})$. Then the composition gives

$$\left( T_+ \xrightarrow{c_{\text{id}(T_+, T_+)}^{-1}} T_+ \xrightarrow{c_{M^{d-2}I(T_-, T_+)}^{-1}} T_+ \right) = (T_+ \xrightarrow{\text{id}} T_+), \quad (4.22)$$

which proves the condition (B.4). The coherence condition for $A^C_{\text{parity}}(\text{id}) \circ A^C_{\text{parity}}(M^{d-1,1})$ can be proven in the same way.

Next we come to the vertical composition of regular 2-morphisms. It is enough to show that the composition is given by multiplication for fixed objects $T_{\pm}$ of $A^C_{\text{parity}}(M^{d-2})$. This follows immediately from Proposition 4.10 by an argument similar to the one used in the proof of Theorem 2.9. The conditions for limit 1-morphisms and limit 2-morphisms follow now from their representations as mapping cylinders.

Now we check compatibility with the monoidal structure. There are canonical $\mathbb{C}$-linear equivalences given on objects by

$$\chi^{-1}_{M^{d-2}, M'^{d-2}} : A^C_{\text{parity}}(M^{d-2} \uplus M'^{d-2}) \longrightarrow A^C_{\text{parity}}(M^{d-2}) \boxtimes A^C_{\text{parity}}(M'^{d-2})$$

sending $(T, T') \in \text{End}_\mathbb{C}(\ker(\mathcal{D}_{M^{d-2}})) \oplus \text{End}_\mathbb{C}(\ker(\mathcal{D}_{M'^{d-2}})) \cong \text{End}_\mathbb{C}(\ker(\mathcal{D}_{M^{d-2} \uplus M'^{d-2}}))$ to $T \boxtimes T'$, and

$$\ell^{-1} : A^C_{\text{parity}}(\emptyset) \longrightarrow \text{Vect}_\mathbb{C}$$

sending $0 \in \{0\} = \text{End}_\mathbb{C}(\ker(\mathcal{D}_{\emptyset}))$ to $\mathbb{C}$. All further structures required for $A^C_{\text{parity}}$ to be a symmetric monoidal 2-functor are trivial. It is straightforward if tedious to check that all diagrams in the definition of a symmetric monoidal 2-functor commute, but we shall not write them out explicitly. Finally, it is straightforward to see that $A^C_{\text{parity}}$ factors through the Picard 2-groupoid $\text{Pic}_2(2\text{Vect}_\mathbb{C})$, and hence $A^C_{\text{parity}}$ is invertible.

Remark 4.23. The proof of Theorem 4.20 is more or less independent of the concrete form of $I_{M^{d-1,1}(T_-, T_+)}$ and the index theorem. It only uses additivity under vertical composition and the decomposition

$$\text{ind}(\mathcal{D}_{M^{d,2}}^{T_+, T_-} + \mathcal{D}_{T_+}^{T_-}) = \int_{M^{d,2}} K_{AS} + I_{\partial_+ M^{d,2}(T_-, T_+)} - I_{\partial_- M^{d,2}(T_-, T_+)} \text{,}$$

into a local part and a global part depending solely on boundary conditions. Hence it should be possible to apply this or a similar construction to a large class of invariants depending on boundary conditions. A particularly interesting example would involve $\eta$-invariants on odd-dimensional manifolds with corners, which should be related to chiral anomalies in even dimensions and extend Dai-Freed theories [DF95].

4.3 Projective representations and symmetry-protected topological phases

A quantum field theory with parity anomaly is now regarded as a theory relative to $A^C_{\text{parity}}$ as described in Section 3.3 i.e. a natural symmetric monoidal 2-transformation $A^C_{\text{parity}} : 1 \Rightarrow \text{tr}A^C_{\text{parity}}$. The concrete description of the extended quantum field theory $A^C_{\text{parity}}$ given in the proof of Theorem 4.20 allows us to calculate the corresponding groupoid 2-cocycle along the
lines discussed in Section 3.4, this information about the parity anomaly is contained in the isomorphism (4.21). We choose a \( \mathbb{C} \)-linear equivalence of categories \( \chi: A^\zeta_{\text{parity}}(M^{d-2}) \to \text{Vect}_\mathbb{C} \) sending all objects \( T \) of \( T(M^{d-2}) \) to \( \mathbb{C} \) and all morphisms \( f \) of \( T(M^{d-2}) \) to \( \text{id}_\mathbb{C} \); a weak inverse is given by picking a particular object \( T_{M^{d-2}} \) in \( T(M^{d-2}) \) and mapping \( \mathbb{C} \) to \( T_{M^{d-2}} \). The functor \( A^\zeta_{\text{parity}}(\phi) \) corresponding to a limit 1-morphism \( \phi: M^{d-2}_1 \to M^{d-2}_2 \) in the symmetry groupoid \( \text{SymCob}_{d-1,d-2}^\phi \) is given by taking the tensor product with the complex line

\[
\begin{array}{c}
L_{\chi,\phi} = \lim_{T(M^{d-2}_2)} \left( \mathbb{C} T_i \xrightarrow{\zeta I_{\mathfrak{M}(\phi)}(T_{M^{d-2}_1} \to T_{M^{d-2}_1}) - I_{\mathfrak{M}(\phi)}(T_{M^{d-2}_1} \to T_{M^{d-2}_1})} \mathbb{C} T_j \right),
\end{array}
\]

where \( \mathfrak{M}(\phi) \) is the mapping cylinder of \( \phi \). A choice of an object \( T_{M^{d-2}} \) of \( T(M^{d-2}_1) \) defines an isomorphism \( \varphi_{M^{d-2} \to} L_{\chi,\phi} \to \mathbb{C} \), which for simplicity we pick to be the same boundary mass perturbation as chosen for the weak inverse above. The groupoid cocycle evaluated at \( \phi_1: M^{d-2}_1 \to M^{d-2}_2 \) and \( \phi_2: M^{d-2}_2 \to M^{d-2}_3 \) corresponding to this choice is then given by

\[
A^\zeta_{\text{parity}} \alpha_{\phi_1,\phi_2} = \zeta I_{\mathfrak{M}(\phi_2 \circ \phi_1)}(T_{M^{d-2}_1} \to T_{M^{d-2}_1}) - I_{\mathfrak{M}(\phi_1)}(T_{M^{d-2}_1} \to T_{M^{d-2}_1}) - I_{\mathfrak{M}(\phi_2)}(T_{M^{d-2}_2} \to T_{M^{d-2}_2}),
\]

We can evaluate this expression explicitly by using (4.19) to get

\[
\log_{\zeta} A^\zeta_{\text{parity}} = -\frac{1}{2} \left( \dim \left( \Lambda T_{M^{d-2}_1} \cap \Lambda \phi_1 \phi_2^* T_{M^{d-2}_3} \right) + \mu \left( \Lambda T_{M^{d-2}_1}, \Lambda \phi_1 \phi_2^* T_{M^{d-2}_3} \right) \right.
- \dim \left( \Lambda T_{M^{d-2}_1} \cap \Lambda \phi_1^* T_{M^{d-2}_2} \right) - \mu \left( \Lambda T_{M^{d-2}_1}, \Lambda \phi_1^* T_{M^{d-2}_2} \right)
- \dim \left( \Lambda T_{M^{d-2}_2} \cap \Lambda \phi_2^* T_{M^{d-2}_3} \right) - \mu \left( \Lambda T_{M^{d-2}_2}, \Lambda \phi_2^* T_{M^{d-2}_3} \right) \right).
\]

(4.24)

To calculate the part of the 2-cocycle involving identity 1-morphisms we can use (4.22) to get

\[
A^\zeta_{\text{parity}} \alpha_{\phi,\text{id}_{M^{d-2}}} = A^\zeta_{\text{parity}} \alpha_{\text{id}_{M^{d-2}}, \phi} = \zeta I_{M^{d-2}_1 \times [0,1)}(T_{M^{d-2}_1} \to T_{M^{d-2}_1}) = \zeta^{-\frac{1}{4}} \dim \ker(\partial_{M^{d-2}}),
\]

where the last equality follows from (4.19). From a physical point of view it is natural to assume this to be equal to 1, since the identity limit morphism should still be a non-anomalous symmetry of every quantum field theory. We can achieve this by normalising our anomaly quantum field theory \( A^\zeta_{\text{parity}} \) to the theory \( \tilde{A}^\zeta_{\text{parity}} \) obtained by redefining

\[
\tilde{I}_{M^{d-1,1}}(T_-, T_+) = I_{M^{d-1,1}}(T_-, T_+) + \frac{1}{8} \left( \dim \ker(\partial_{\partial_{M^{d-1,1}}} - \partial_{M^{d-1,1}}) + \dim \ker(\partial_{\partial_{M^{d-1,1}}} - \partial_{M^{d-1,1}}) \right).
\]

The proof of Theorem 4.20 then carries through verbatim with \( I_{M^{d-1,1}} \) replaced by \( \tilde{I}_{M^{d-1,1}} \) everywhere.

**Example 4.25.** We conclude by illustrating how to extend Example 2.12 to the anomaly quantum field theory \( A^{(-1)}_{\text{parity}} \), glossing over many technical details. To construct the second quantized Fock space of a quantum field theory of fermions coupled to a background gauge field on a Riemannian manifold \( M^{d-2} \), one needs a polarization

\[
H = H^+ \oplus H^-
\]
of the one-particle Hilbert space $H$ of wavefunctions, which we take to be the sections of the twisted spinor bundle $S_{M^{d-2}}$. If the Dirac Hamiltonian $\mathcal{D}_{M^{d-2}}$ has no zero modes, then there exists a canonical polarization given by taking $H^+ = H^{>0}$ (resp. $H^- = H^{<0}$) to be the space spanned by the positive (resp. negative) energy eigenspinors. Given such a polarization we can define

$$A^{(-1)}_{\text{parity}}(M^{d-2}) = \wedge H^+ \otimes \wedge (H^-)^*,$$

where $\wedge H$ denotes the exterior algebra generated by the vector space $H$. Now time-reversal (or orientation-reversal) symmetry acts by interchanging $H^+$ and $H^-$, and there is no problem extending this symmetry to the Fock space $A^{(-1)}_{\text{parity}}(M^{d-2})$.

In the case that $\ker(\mathcal{D}_{M^{d-2}})$ is non-trivial, as is the case for fermionic gapped quantum phases of matter, one could try to declare all zero modes to belong to $H^{>0}$ or $H^{<0}$ and use the corresponding polarization to define a Fock space. We cannot apply this method of quantization, since it breaks time-reversal symmetry. Therefore we are forced to use a different polarization compatible with orientation-reversal symmetry. There is no canonical choice for such a polarization, but rather a natural family parameterized by Lagrangian subspaces $\Lambda_T \subset \ker(\mathcal{D}_{M^{d-2}})$:

$$H^+(\Lambda_T) = H^{>0} \oplus \Lambda_T \quad \text{and} \quad H^-(\Lambda_T) = H^{<0} \oplus \Gamma \Lambda_T.$$

Since orientation reversion acts proportionally to the chirality operator $\Gamma$ on spinors, these polarizations are compatible with the symmetry. We then get a family of Fock spaces

$$A^{(-1)}_{\text{parity}}(M^{d-2}, T) = \wedge H^+(\Lambda_T) \otimes \wedge H^-(\Lambda_T)^* = \wedge H^{>0} \otimes \wedge (H^{<0})^* \otimes F(M^{d-2}, T),$$

where the essential part for our discussion is encoded in the finite-dimensional vector space

$$F(M^{d-2}, T) = \wedge \Lambda_T \otimes \wedge (\Gamma \Lambda_T)^*.$$ 

Fixing an ordered basis for every $\Lambda_T$, these finite-dimensional vector spaces fit into a $\mathbf{Vect}_\mathbb{C}$-valued pre-cosheaf $A^{(-1)}_{\text{parity}}(M^{d-2})$ on $T(M^{d-2})$, where we assign to a morphism $T_1 \to T_2$ the linear map induced by sending the fixed basis of $\Lambda_{T_1}$ to the basis of $\Lambda_{T_2}$. By (4.11) this is an element of $A^{(-1)}_{\text{parity}}(M^{d-2})$, or equivalently a $\mathbb{C}$-linear functor

$$A^{(-1)}_{\text{parity}}(M^{d-2}) : \mathbf{Vect}_\mathbb{C} \to A^{(-1)}_{\text{parity}}(M^{d-2}).$$

We sketch how these pre-cosheaves fit into a natural symmetric monoidal 2-transformation, realising an anomalous quantum field theory $A^{(-1)}_{\text{parity}}$ with parity anomaly according to Definition 3.13. For a 1-morphism $M^{d-1,1} : M_{-}^{d-2} \to M_{+}^{d-2}$ we have to construct a natural transformation $A^{(-1)}_{\text{parity}}(M^{d-1,1}) : A^{(-1)}_{\text{parity}}(M^{d-1,1}) \circ A^{(-1)}_{\text{parity}}(M^{d-2}) \Rightarrow A^{(-1)}_{\text{parity}}(M^{d-2})$. The left-hand side is given by the pre-cosheaf

$$\mathbf{T}(M_{+}^{d-2}) \to \mathbf{Vect}_\mathbb{C},$$

$$T \mapsto \lim_{\leftarrow \mathbf{T}(M_{-}^{d-2})} \left( \cdots \to F(M_{-}^{d-2}, T_{-}) \xrightarrow{(-1)^{I(T_{+}',T_{-})-I(T_{-}',T_{+})}} F(M_{-}^{d-2}, T_{+}') \to \cdots \right).$$

This implies that constructing a natural transformation is the same as defining a family of compatible linear maps

$$A^{(-1)}_{\text{parity}}(M^{d-1,1})_{T_{-},T_{+}} : F(M_{-}^{d-2}, T_{-}) \to F(M_{+}^{d-2}, T_{+}).$$

These should

\footnote{This requires replacing the limit by a colimit, which is possible since it is taken over a groupoid inside $\mathbf{Vect}_\mathbb{C}$.}
again be given by an appropriate regularization of path integrals. As before we assume that these maps are well-defined up to a sign. To fix the sign we have to consistently fix reference background fields on all 1-morphisms. This is possible, for example, by using a connection on the universal bundle and pullbacks along classifying maps. Again we can fix the sign at these reference fields to be positive. Using a spectral flow similar to (2.13) with boundary conditions $T_-$ and $T_+$, we can fix the sign for all other field configurations. Assuming that this spectral flow can be calculated by the index with appropriate boundary conditions, we see that these sign ambiguities satisfy the coherence conditions encoded by $A_{\text{parity}}^{(-1)}$, i.e. they define a natural symmetric monoidal 2-transformation. This demonstrates in which sense a field theory with parity anomaly takes values in $A_{\text{parity}}^{(-1)}$.

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A Manifolds with corners

In this appendix we collect information about manifolds with corners necessary for our constructions, following [SP11, Chapter 3.1] for the most part. We also give a short introduction to the concepts of geometry on such manifolds which are used in the main text, following [MP98, Mel93, Loy98].

A.1 Basic definitions

Roughly speaking, a manifold of dimension $d$ is a topological space which locally looks like open subsets of $\mathbb{R}^d$. The idea behind manifolds with corners of codimension 2 is to replace $\mathbb{R}^d$ by $\mathbb{R}^{d-2} \times \mathbb{R}^2_{\geq 0}$; we denote by $\text{pr}_{\mathbb{R}^2_{\geq 0}} : \mathbb{R}^{d-2} \times \mathbb{R}^2_{\geq 0} \to \mathbb{R}^2_{\geq 0}$ the projection. A chart for a subset $U$ of a topological space $X$ is then a homeomorphism $\varphi : U \to V \subset \mathbb{R}^{d-2} \times \mathbb{R}^2_{\geq 0}$. Two charts $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ are compatible if $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is a diffeomorphism. A map between subsets of $\mathbb{R}^{d-2} \times \mathbb{R}^2_{\geq 0}$ is smooth if there exists an extension to open subsets of $\mathbb{R}^d$ which is smooth. As for manifolds, a collection of charts covering $X$ is called an atlas. An atlas is maximal if it contains all compatible charts.

Definition A.1. A manifold with corners of codimension 2 is a second countable Hausdorff space $M$ together with a maximal atlas.

Remark A.2. Closed manifolds and manifolds with boundary are in particular manifolds with corners.
We define the tangent space $T_x M$ at a point $x \in M$ as the space of derivatives on the real-valued functions $C^\infty(M)$ at $x$. We define embeddings in the same way as for manifolds without corners. We are now able to introduce an essential concept used throughout the main text.

**Definition A.3.** A collar for a submanifold $Y \subset \partial M$ is a diffeomorphism $\varphi: U_Y \to Y \times [0, \epsilon)$ for some fixed $\epsilon > 0$ and a neighbourhood $U_Y$ of $Y$.

Note that there are situations in which no collars exist.

Given $x \in M$ we define the index of $x$ to be the number of coordinates of $(\text{pr}_{R^2_0} \circ \varphi)(x)$ equal to 0 for a chart $\varphi$. Clearly $\text{index}(x) \in \{0, 1, 2\}$, and this definition does not depend on the choice of chart. The corners of $M$ are the collection of all points of index 2. A connected face of $M$ is the closure of a maximal connected subset of points of index 1. A manifold with corners is a manifold with faces if each $x \in M$ belongs to exactly $\text{index}(x)$ connected faces. In this case we define a face of $M$ to be a disjoint union of connected faces, which is a manifold with boundary. A boundary defining function for a face $H_i$ is a function $\rho_i \in C^\infty(M)$ such that $\rho_i(x) \geq 0$ and $\rho_i(x) = 0$ if and only if $x \in H_i$.

**Definition A.4.** A $\langle 2 \rangle$-manifold is a manifold $M$ with faces together with two faces $\partial_0 M$ and $\partial_1 M$ such that $\partial M = \partial_0 M \cup \partial_1 M$ and $\partial_0 M \cap \partial_1 M$ are the corners of $M$.

Denote by $[1]$ the category corresponding to the ordered set $\{0, 1\}$. A $\langle 2 \rangle$-manifold $M$ then defines a diagram $M: [1]^2 \to \text{Man}_c$ of shape $[1]^2$ in the category $\text{Man}_c$ of manifolds with corners and smooth embeddings:

\[
\begin{array}{c}
\partial_0 M \\
\downarrow \\
M \\
\downarrow \\
\partial_1 M \\
\downarrow \\
\partial_0 M \cap \partial_1 M \\
\end{array}
\]

### A.2 Gluing principal bundles

Given principal bundles with connections on manifolds $M_1$ and $M_2$, gluing them along a common boundary $\Sigma$ requires some care. Naively one could try to cover the glued manifold $M$ by $\{M_1, M_2\}$ and use the descent property for the stack of principal bundles with connection. However, this does not work as $M_1$ and $M_2$ are not open subsets of $M$. A way out is to deform $M_1$ and $M_2$ into open subsets of $M$ by cutting out a collar near their common boundary. But there is no canonical choice for such a collar. We need in this case a third open set interpolating between the two manifolds. In general there is no canonical choice for such an interpolation, whence we should consider it together with the collar as part of the gluing data.$^{13}$

We give an explicit construction for this rather complicated gluing procedure of principal bundles with connections over smooth manifolds and specify the additional information needed. We fix principal bundles with connection $\pi_1: P_1 \to M_1$ and $\pi_2: P_2 \to M_2$ over ori-$^{13}$For topological stacks all choices are equivalent.
ent smooth $d$-dimensional compact manifolds $M_1$ and $M_2$. We assume that there are neighbourhoods $M_{1,+} \subset M_1$ and $M_{2,-} \subset M_2$ of parts of the boundaries, and require orientation-preserving diffeomorphisms $\varphi_1: M_{1,+} \to \Sigma \times (-\epsilon_1, 0]$ and $\varphi_2: M_{2,-} \to \Sigma \times [0, \epsilon_2)$ for fixed $\epsilon_i > 0$. Then $\varphi_i$ for $i = 1, 2$ induce projections $p_i: M_{i,+} \to \varphi_i^{-1}(\Sigma \times \{0\})$. We further assume that $P_i$ is of product structure over $M_{i,+}$, i.e. $P_i|_{M_{i,+}} = p_i^*P_1|_{\varphi_1^{-1}(\Sigma \times \{0\})}$. We fix a third bundle with connection $\pi_3^*: P_3' \to \Sigma$, defining a bundle $\pi_3: P_3 \to \Sigma \times (-\epsilon_1, \epsilon_2)$ via pullback along the projection onto $\Sigma \times \{0\}$. We choose connection-preserving gauge transformations $\psi_i^*: P_3' \to (\varphi_i^{-1})^*(P_1|_{\varphi_i^{-1}(\Sigma \times \{0\})})$. These gauge transformations induce gauge transformations over that part of $\Sigma \times (-\epsilon_1, \epsilon_2)$ where both bundles are defined, which are constant along the fibres.

We now glue $M_1$ and $M_2$ along $\Sigma$ as usual to get a manifold

$$M = M_1 \sqcup_{\varphi_2^{-1}|_{\Sigma \times (-\epsilon_1, 0]} \circ \varphi_1^{-1}(\Sigma \times \{0\})} M_2,$$

where the collars $M_{1,+}$ and $M_{2,-}$ are needed to define a unique smooth structure on $M$. We cover $M$ with the three open sets

$$U_1 = M_1 \setminus \varphi_1^{-1}(\Sigma \times [-\frac{\epsilon_1}{4}, 0]),$$
$$U_2 = M_2 \setminus \varphi_2^{-1}(\Sigma \times [0, \frac{\epsilon_2}{4}]),$$
$$U_3 = (\varphi_1^{-1} \sqcup \varphi_2^{-1})(\Sigma \times (-\frac{\epsilon_1}{2}, \frac{\epsilon_2}{2})).$$

The structures fixed so far induce connection-preserving gauge transformations $\psi_1: P_1|_{U_1 \cap U_3} \to (\varphi_1)^*(P_3)|_{U_1 \cap U_3}$ and $\psi_2: P_2|_{U_2 \cap U_3} \to (\varphi_2)^*(P_3)|_{U_2 \cap U_3}$. We use the descent property of the stack of principal bundles to define a bundle over $M$. More concretely we define

$$P_M = P_1|_{U_1} \sqcup P_2|_{U_2} \sqcup (\varphi_1 \sqcup \varphi_2)^*(P_3) / \sim,$$

where $p \sim \varphi_1(p)$ for all $p \in U_1 \cap U_3$ and $p \sim \varphi_2(p)$ for all $p \in U_2 \cap U_3$. Using local trivialisations for all bundles involved, or the fact that principal $G$-bundles with connection form a stack $\text{Bun}_G$, it is easy to see that $P_M$ is a principal bundle with connection over $M$. This gluing construction depends on the choice of collars $\varphi_i$, the definition of $U_i$ and the trivialisations $\psi_i$. Different choices for the collars and the open cover lead to isomorphic bundles with connection, since we assume that the bundles are of product form on the collars.

We give another point of view on this construction using ‘mapping cylinders’. Given two principal bundles with connection $\pi_1: P_1 \to \Sigma_1$ and $\pi_2: P_2 \to \Sigma_2$, a diffeomorphism $f: \Sigma_1 \to \Sigma_2$, and a connection-preserving gauge transformation $\psi: P_1 \to f^*P_2$, the mapping cylinder of length $\epsilon$ is defined as

$$\mathcal{M}(f, \psi) := \Sigma_1 \times [0, \frac{3\epsilon}{4}] \sqcup \Sigma_2 \times (\frac{\epsilon}{4}, \epsilon] / \sim,$$

where $(x, t) \sim (f(x), t)$ for all $(x, t) \in \Sigma_1 \times (\frac{\epsilon}{4}, \frac{3\epsilon}{4})$, together with the principal bundle with connection over $\mathcal{M}(f, \psi)$ given by

$$P_{\mathcal{M}(f, \psi)} := P_1 \times [0, \frac{3\epsilon}{4}] \sqcup P_2 \times (\frac{\epsilon}{4}, \epsilon] / \sim,$$

where $(p, t) \sim ((f^{-1})^*\psi(p), t)$ for all $(p, t) \in P_1 \times (\frac{\epsilon}{4}, \frac{3\epsilon}{4})$. Now we see that the gluing happens by removing half of the collars $M_{1,+}$ and $M_{2,-}$, and attaching mapping cylinders $\mathcal{M}(\varphi_1, \psi_1^*)$ and $\mathcal{M}(\varphi_2^{-1}, \psi_2')$ of appropriate length. This point of view is crucial in the proof of Theorem 2.9.
The important properties we used in this construction are the stack property of \( \text{Bun}_{\mathcal{F}} \) and the notion of a bundle of product structure. In Section 3.2 we use a similar construction to build a bicategory of cobordisms equipped with elements of an arbitrary stack \( \mathcal{F} \).

We also need to glue metrics, which can be done using the open cover \([A.5]\). For this, we assume that there are metrics \( g_1 \in \Gamma(\text{Sym}^2(T^*M_1)), g_2 \in \Gamma(\text{Sym}^2(T^*M_2)) \) and \( g'_3 \in \Gamma(\text{Sym}^2(T^*\Sigma)) \). We equip \( \Sigma \times (-\epsilon_1,\epsilon_2) \) with the metric \( g_3 = g'_3 + dt \otimes dt \). Now it is sensible to assume that \( \varphi_1 \) and \( \varphi_2 \) are isometries. We can define a metric over \( U_3 \) as \( (\varphi_1 \cup \varphi_2)^*(g_3)|_{U_3} \). Then all metrics agree on the intersections, since we assumed that \( \varphi_1 \) and \( \varphi_2 \) are isometries. This defines a metric on \( M \), since sections of \( \text{Sym}^2(T^*M) \) form a sheaf over \( M \).

### A.3 Dirac operators on spin manifolds with boundary

Given a spin structure on a \( d \)-dimensional oriented Riemannian manifold \( M \) with boundary \( \partial M \), i.e. a double cover of the frame bundle \( P_{\text{SO}(d)}(M) \) by a principal \( \text{Spin}(d) \)-bundle \( P_{\text{Spin}(d)}(M) \rightarrow M \), we can include the frame bundle \( P_{\text{SO}(d-1)}(\partial M) \rightarrow P_{\text{SO}(d)}(M) \) by adding the inward pointing normal vector to an orthonormal frame of \( \partial M \). The pullback of the double cover \( P_{\text{Spin}(d)}(M) \) along this inclusion gives a spin structure on \( \partial M \).

Assume from now on that all structures are of product form near the boundary. To describe the relation between the Dirac operator on the boundary and on the bulk manifold we use the embedding of Clifford bundles

\[
\text{Cl}_{d-1}(\partial M) \longrightarrow \text{Cl}_d(M), \quad T_x(\partial M) \ni v \longmapsto v n_x,
\]

where \( n \) is the inward pointing normal vector field corresponding to the boundary. This gives \( S_M|_{\partial M} \) the structure of a Clifford bundle over \( \partial M \). For the relation to the spinor bundle over the boundary we need to distinguish between even and odd dimensions.

If the dimension \( d \) of \( M \) is odd then we can identify \( S_M|_{\partial M} \) with the spinor bundle over \( \partial M \). In this case the Dirac operator can be described in a neighbourhood of \( \partial M \) by

\[
\text{D}_M = n \cdot (\text{D}_{\partial M} + \partial n).
\]

On the other hand, if the dimension \( d \) of \( M \) is even then the spinor bundle \( S_M = S^+_M \oplus S^-_M \) decomposes into spinors of positive and negative chirality. The Clifford action of \( \text{Cl}_{d-1}(\partial M) \) leaves this decomposition invariant and we can identify the spinor bundle over \( \partial M \) with the pullback of the positive spinor bundle \( S^+_M|_{\partial M} \). As the Clifford action of \( \text{Cl}_{d-1}(\partial M) \) commutes with the chirality operator \( \Gamma \), an identification with the negative spinor bundle is possible as well. Near the boundary the Dirac operator is given by

\[
\text{D}_M = n \cdot \begin{pmatrix}
\text{D}_{\partial M} + \partial n & 0 \\
0 & \Gamma_{|S^+_M} \text{D}_{\partial M} \Gamma_{|S^-_M} + \partial n
\end{pmatrix}.
\]

### A.4 b-geometry

b-geometry (for ‘boundary geometry’) is concerned with the study of geometric structures on manifolds with corners which can be singular at the boundary. We fix a \( d \)-dimensional \((2)\)-manifold \( M \) and an ordering of its hypersurfaces \( \{H_1, \ldots, H_k\} \). The central objects in b-geometry are b-vector fields. These are vector fields which are tangent to all boundary hypersurfaces. We denote by \( \text{Vect}_b(M) \) the projective \( C^\infty(M) \)-module of b-vector fields. Then
A b-metric is closed under the Lie bracket of vector fields. By the Serre-Swan theorem, the b-vector fields are naturally sections of the b-tangent bundle with fibres

$$b^\mathcal{I}_x M := \text{Vect}_b(M) \setminus \mathcal{I}_x (M) \cdot \text{Vect}_b(M),$$

where \(\mathcal{I}_x (M) = \{ f \in C^\infty(M) \mid f(x) = 0 \}\) is the ideal of functions vanishing at \(x \in M\). This allows us to define arbitrary b-tensors as in classical differential geometry. The inclusion \(\text{Vect}_b(M) \hookrightarrow \text{Vect}(M)\) induces a natural vector bundle map \(\alpha_b : b^\mathcal{T} M \to TM\).

The structures introduced so far can be summarized by saying that \((\mathcal{B}, b^\mathcal{I}_x M)\) is an example of a manifold with Lie structure at infinity [ALN04].

Using a set of boundary defining functions \(x_i\), the Lie algebra \(\text{Vect}_b(M)\) is locally spanned near a point \(x \in H_i\) of index 1 by \(\{ x_i, \partial x_i, \partial h_1, \ldots, \partial h_{d-1} \}\), where \(\{ h_l \}_{l=1}^{d-1}\) is a local coordinate system for \(H_i\). In a neighbourhood of \(x \in H_i \cap H_j\), \(i \neq j\), we can form a basis given by \(\{ x_i, \partial x_i, x_j \partial x_j, \partial y_1, \ldots, \partial y_{d-2} \}\), where \(\{ y_l \}_{l=1}^{d-2}\) is a local coordinate system on \(Y_{ij} = H_i \cap H_j\). The dual basis for the b-cotangent bundle \(b^\mathcal{T}^* M\) is denoted by \(\{ \frac{dx_i}{x_i}, \frac{dx_j}{x_j}, dy_1, \ldots, dy_{d-2} \}\).

A b-metric \(g\) is now simply a metric on the vector bundle \(b^\mathcal{T} M\) over \(M\). This defines an ordinary metric in the interior of \(M\). The general expression in local coordinates near a corner point is

$$g = \sum_{i,j=0,1} a_{ij} \frac{dx_i}{x_i} \otimes \frac{dx_j}{x_j} + 2 \sum_{i=0,1} \sum_{j=1}^{d-2} b_{ij} \frac{dx_i}{x_i} \otimes dy_j + \sum_{i,j=1}^{d-2} c_{ij} dy_i \otimes dy_j .$$

A b-metric \(g\) is exact if there exists a set of boundary defining functions \(x_i\) such that it takes the form

$$g = \begin{cases} \frac{dx_i}{x_i} \otimes \frac{dx_i}{x_i} + h_{H_i} & \text{near } H_i , \\ \frac{dx_i}{x_i} \otimes \frac{dx_i}{x_i} + \frac{dx_j}{x_j} \otimes \frac{dx_j}{x_j} + h_{H_i \cap H_j} & \text{near } H_i \cap H_j , \end{cases}$$

where \(h_Y\) denotes a metric on \(Y\).

We will now describe the relation between \((2)\)-manifolds with exact b-metrics and the index theory on manifolds with corners considered in the main text. To define the index we attach infinite cylindrical ends \(H_i \times (-\infty,0]\) to the boundary hypersurfaces and \(Y_{ij} \times (-\infty,0]\) to the corners. The coordinate transformation \(x_i = e^{t_i}\) for \(t_i \in (-\infty,0]\) maps this non-compact manifold to the interior of a manifold \(X_i\) with corners. The product metric on the cylindrical ends induces a b-metric on \(X_i\), since \(dt_i \otimes dt_i = \frac{dx_i}{x_i} \otimes \frac{dx_i}{x_i}\). For this reason one can view the study of manifolds with exact b-metrics as the study of manifolds with cylindrical ends.

A b-differential operator is an element of the universal enveloping algebra of \(\text{Vect}_b(M)\), the collection of which act naturally on \(C^\infty(M)\). A b-differential operator \(D \in \text{Diff}_k^b(M, E_1, E_2)\) of order \(k\) between two vector bundles \(E_1\) and \(E_2\) over \(M\) is a smooth fibre-preserving map, which in any local trivialisations of \(E_1\) and \(E_2\) is given by a matrix of linear combinations of products of up to \(k\) b-vector fields. Most concepts from differential geometry such as connections, symbols and characteristic classes can be generalized to the b-geometry setting.

Since exact b-metrics are singular at the boundary it is necessary to define a renormalised b-integral. Heuristically, the problem stems from the fact that the integral \(\int_0^1 \frac{dx}{x}\) is divergent. The cure for this is to multiply with \(x^2\) for \(\text{Re}(z) > 0\).
Lemma A.7. ([Loy04, Lemma 4.1]) Let $M$ be a manifold with corners and an exact b-metric $g$. Then for all $f \in C^\infty(M)$ and $z \in \mathbb{C}$ with $\text{Re}(z) > 0$, the integral

$$F(f, z) := \int_M x^z f \, dg$$

exists and extends to a meromorphic function $F(f, z)$ of $z \in \mathbb{C}$.

Definition A.8. Let $M$ be a manifold with corners and an exact b-metric $g$. The b-integral of a function $f \in C^\infty(M)$ is

$$\int_M f \, dg = \text{Reg}_{z=0} F(f, z).$$

(A.9)

This allows us to define the b-trace of a pseudo-differential operator $D$ in terms of its kernel $D(x, y)$ as

$$\text{bTr}(D) = \int_M \text{tr}(D(x, x)) \, dg(x),$$

where the trace $\text{tr}$ is over the fibres of the vector bundle on which $D$ acts.

B Bicategories

As it is central to the treatment of this paper, in this appendix we provide a fairly detailed account of symmetric monoidal bicategories, following [Lei98, SP11] for the most part.

B.1 Basic definitions

We introduce the basic concepts from the theory of bicategories following [Lei98].

Definition B.1. A bicategory $\mathcal{B}$ consists of the following data:

(a) A class $\text{Obj}(\mathcal{B})$ of objects.

(b) A category $\text{Hom}_{\mathcal{B}}(A, B)$ for all $A, B \in \text{Obj}(\mathcal{B})$, whose objects $f : A \to B$ we call 1-morphisms and whose morphisms $f \Rightarrow g$ we call 2-morphisms.

(c) Composition functors

$$\circ_{ABC} : \text{Hom}_{\mathcal{B}}(B, C) \times \text{Hom}_{\mathcal{B}}(A, B) \to \text{Hom}_{\mathcal{B}}(A, C)$$

for all $A, B, C \in \text{Obj}(\mathcal{B})$.

(d) Identity functors

$$\text{Id}_A : 1 = \star \parallel \{\text{id}_A\} \to \text{Hom}_{\mathcal{B}}(A, A)$$

for all $A \in \text{Obj}(\mathcal{B})$.

(e) Natural associator isomorphisms

$$a_{A,B,C,D} : \circ_{ACD} \circ (\text{id}_{\text{Hom}_{\mathcal{B}}(C,D)} \times \circ_{ABC}) \Rightarrow \circ_{ABD} \circ (\circ_{BCD} \times \text{id}_{\text{Hom}_{\mathcal{B}}(A,B)})$$

for all $A, B, C, D \in \text{Obj}(\mathcal{B})$, expressing associativity of the composition.
(f) Natural right and left unitor isomorphisms

\[ r_A: \circ_{AAB} \circ (\text{id}_{\text{Hom}_B(A,B)} \times \text{id}_A) \Rightarrow \text{id}_{\text{Hom}_B(A,B)} \]

and

\[ l_A: \circ_{AAB} \circ (\text{id}_B \times \text{id}_{\text{Hom}_B(A,B)}) \Rightarrow \text{id}_{\text{Hom}_B(A,B)} \]

for all \( A, B \in \text{Obj}(\mathcal{B}) \).

These data are required to satisfy the following coherence axioms:

(C1) The pentagon diagram

```
  (((k \circ h) \circ g) \circ f) \Rightarrow (k \circ (h \circ g)) \circ f
```

commutes for all composable 1-morphisms \( k, h, g \) and \( f \), where \( \bullet \) denotes the horizontal composition of natural transformations.

(C2) The triangle diagram

```
  (g \circ \text{id}) \circ f \Rightarrow g \circ (\text{id} \circ f)
```

commutes for all composable 1-morphisms \( f \) and \( g \).

There are different definitions for functors between bicategories corresponding to different levels of strictness. For our purposes the following definition is suitable.

**Definition B.2.** A 2-functor \( \mathcal{F}: \mathcal{B} \to \mathcal{B}' \) between two bicategories \( \mathcal{B} \) and \( \mathcal{B}' \) consists of the following data:

(a) A map \( \mathcal{F}: \text{Obj}(\mathcal{B}) \to \text{Obj}(\mathcal{B}') \).

(b) A functor \( \mathcal{F}_{AB}: \text{Hom}_\mathcal{B}(A,B) \to \text{Hom}_\mathcal{B}'(\mathcal{F}(A),\mathcal{F}(B)) \) for all \( A, B \in \text{Obj}(\mathcal{B}) \).

(c) A natural isomorphism \( \Phi_{ABC} \) given by

\[ \text{Hom}_\mathcal{B}(B,C) \times \text{Hom}_\mathcal{B}(A,B) \xrightarrow{\circ} \text{Hom}_\mathcal{B}(A,C) \]

\[ \text{Hom}_\mathcal{B}(\mathcal{F}(B),\mathcal{F}(C)) \times \text{Hom}_\mathcal{B}'(\mathcal{F}(A),\mathcal{F}(B)) \]

for all \( A, B, C \in \text{Obj}(\mathcal{B}) \).
(d) A natural isomorphism $\Phi_A$ given by

\[
\begin{array}{ccc}
1 & \xrightarrow{\text{Id}_A} & \text{Hom}_\mathcal{B}(A, A) \\
\downarrow & & \downarrow \Phi_A \\
1 & \xrightarrow{\text{Id}_F(A)} & \text{Hom}_\mathcal{B}(F(A), F(A))
\end{array}
\]

for all $A \in \text{Obj}(\mathcal{B})$.

These data are required to satisfy the following coherence axioms:

(C1) The diagram

\[
\begin{array}{cc}
(F(h) \circ' F(g)) \circ' F(f) & \xrightarrow{\Phi \circ' \text{Id}} F((h \circ g) \circ f) \\
\downarrow z' & \\
F(h) \circ' (F(g) \circ' F(f)) & \xrightarrow{\text{Id}' \circ' \Phi} F(h) \circ' F(g \circ f) & \xrightarrow{\Phi} F(h \circ (g \circ f))
\end{array}
\]

commutes for all composable 1-morphisms.

(C2) The diagram

\[
\begin{array}{ccc}
F(f) \circ' \text{Id}_F(A) & \xrightarrow{\text{Id}' \circ' \Phi} F(f) \circ' F(\text{Id}_A) & \xrightarrow{\Phi} F(f \circ \text{Id}_A) \\
\downarrow F(r) & \swarrow F(r) & \searrow
\end{array}
\]

commutes for all composable 1-morphisms.

(C3) A diagram analogous to (B.4) for the left unitors $l$ and $l'$ commutes.

Again there are different ways to define natural transformations between 2-functors. The following definition is suitable for our purposes.

**Definition** B.5. Given two 2-functors $F, G : \mathcal{B} \to \mathcal{B}'$, a **natural 2-transformation** $\sigma : F \Rightarrow G$ consists of the following data:

(a) A 1-morphism $\sigma_A : F(A) \to G(A)$ for all $A \in \text{Obj}(\mathcal{B})$.

(b) A natural transformation $\sigma_{AB}$ given by

\[
\begin{array}{cccc}
\text{Hom}_\mathcal{B}(A, B) & \xrightarrow{\sigma_{AB}} & \text{Hom}_\mathcal{B}(F(A), F(B)) \\
\downarrow G_{AB} & & \downarrow \sigma_B \circ' F_{AB} \\
\text{Hom}_\mathcal{B}(G(A), G(B)) & \xrightarrow{\sigma_A} & \text{Hom}_\mathcal{B}(F(A), G(B))
\end{array}
\]

for all $A, B \in \text{Obj}(\mathcal{B})$. In particular, these natural transformations comprise families of 2-morphisms $\sigma_f : G_{AB}(f) \circ' \sigma_A \Rightarrow \sigma_B \circ' F_{AB}(f)$ for all 1-morphisms $f : A \to B$ in $\mathcal{B}$.

\[\text{Here we use } \ast \text{ to denote pullbacks and pushforwards in the usual way.}\]
These data are required to satisfy the following coherence axioms:

(C1) The diagram

\[
\begin{align*}
&\Phi \circ \text{id} \\
\downarrow &\Phi \circ \text{id} \\
\mathcal{G}(g \circ f) \circ \sigma_A &\xrightarrow{\alpha'} \mathcal{G}(g) \circ (\mathcal{G}(f) \circ \sigma_A) \\
\downarrow &\Phi \circ \text{id} \\
\mathcal{G}(g \circ f) \circ \sigma_A &\xrightarrow{(\mathcal{G}(g) \circ \sigma_B) \circ \mathcal{F}(f)} \\
\mathcal{G}(g \circ f) \circ \sigma_A &\xrightarrow{\Phi \circ \text{id}} \mathcal{G}(g) \circ (\sigma_B \circ \mathcal{F}(f))
\end{align*}
\]

commutes for all 1-morphisms \(f: A \to B\) and \(g: B \to C\) in \(\mathcal{B}\).

(C2) The diagram

\[
\begin{align*}
&\text{id} \circ \sigma_A \\
\downarrow &\Phi \circ \text{id} \\
\mathcal{G}(\text{id}_A) \circ \sigma_A &\xrightarrow{\alpha} \sigma_A \circ \text{id} \\
\downarrow &\Phi \circ \text{id} \\
\mathcal{G}(\text{id}_A) \circ \sigma_A &\xrightarrow{(\mathcal{G}(\text{id}) \circ \mathcal{F}(f))} \sigma_A \circ \mathcal{F}(\text{id}_A)
\end{align*}
\]

commutes for all \(A \in \text{Obj}(\mathcal{B})\).

Note that we do not require the natural transformation \(\sigma_{AB}\) to be invertible, whence its direction matters. There is an alternative definition using the opposite direction.

Since there is an additional layer of structure for bicategories, we are able to relate two natural 2-transformations to each other. There is only one way to do this, since there are no higher morphisms.

Definition B.8. Given two natural 2-transformations \(\sigma, \tau: \mathcal{F} \Rightarrow \mathcal{G}\), a modification \(\Gamma: \sigma \Rightarrow \tau\) consists of a 2-morphism \(\Gamma_A: \sigma_A \Rightarrow \tau_A\) for each \(A \in \text{Obj}(\mathcal{B})\) such that the diagram

\[
\begin{align*}
&\mathcal{G}(f) \circ \sigma_A \\
\downarrow &\Phi \circ \text{id} \\
\mathcal{G}(f) \circ \sigma_A &\xrightarrow{\Phi \circ \text{id}} \mathcal{G}(f) \circ \tau_A \\
\downarrow &\Phi \circ \text{id} \\
\mathcal{G}(f) \circ \sigma_A &\xrightarrow{(\mathcal{G}(\tau) \circ \mathcal{F}(f))} \mathcal{G}(f) \circ \tau_A
\end{align*}
\]

commutes for all 1-morphisms \(f: A \to B\) in \(\mathcal{B}\).

B.2 Symmetric monoidal bicategories

We now describe how to introduce symmetric monoidal structures on bicategories following [SP11].

Definition B.10. A symmetric monoidal bicategory consists of a bicategory \(\mathcal{B}\) together with the following data:

(a) A monoidal unit \(1 \in \text{Obj}(\mathcal{B})\).

(b) A 2-functor \(\otimes: \mathcal{B} \times \mathcal{B} \to \mathcal{B}\).

(c) Equivalence natural 2-transformations\(^{15}\) \(\alpha: \otimes \circ (\text{id} \times \otimes) \Rightarrow \otimes \circ (\otimes \times \text{id})\), \(\lambda: 1 \otimes - \Rightarrow \text{id}\)

\(^{15}\)Here ‘equivalence’ means the natural 2-transformations in question have weak inverses.
ρ: id ⇒ · ⊗ 1. We pick adjoint inverses which are part of the data and denoted them by *, leaving the adjunction data implicit, and for every equivalence natural 2-transformation we pick an adjoint weak inverse without writing them out explicitly.

(d) An equivalence natural 2-transformation β: a ⊗ b ⇒ b ⊗ a.

(e) The four invertible modifications

\[ \begin{align*}
\otimes \circ (\otimes \times \otimes) & \xrightarrow{\alpha} \otimes \circ (\otimes \times \otimes) \\
\otimes \circ (\otimes \times \text{id}) \circ (\otimes \times \text{id} \times \text{id}) & \xrightarrow{\alpha \otimes \text{id}} \otimes \circ (\otimes \times \otimes) \circ (\text{id} \times \text{id} \times \otimes) \\
\otimes \circ (\otimes \times \text{id}) \circ (\text{id} \times \otimes \times \text{id}) & \xrightarrow{\alpha} \otimes \circ (\otimes \times \otimes) \circ (\text{id} \times \otimes \times \text{id}) \\
\otimes \circ (\text{id} \times (1 \otimes \cdot)) & \xrightarrow{\alpha} \otimes \circ ((\cdot \otimes 1) \times \text{id}) \\
\otimes \circ ((1 \otimes \cdot) \times \text{id}) & \xrightarrow{\alpha} \otimes \circ ((1 \otimes \cdot) \times \text{id}) \\
\otimes & \xrightarrow{\text{id} \otimes \lambda} \otimes \\
\otimes & \xrightarrow{\Lambda \otimes \text{id}} \otimes
\end{align*} \]

\[ \begin{align*}
\otimes \circ (\text{id} \times (1 \otimes \cdot)) & \xrightarrow{\alpha} \otimes \circ ((\cdot \otimes 1) \times \text{id}) \\
\otimes \circ ((1 \otimes \cdot) \times \text{id}) & \xrightarrow{\Lambda \otimes \text{id}} \otimes
\end{align*} \]

and

\[ \begin{align*}
\otimes & \xrightarrow{\text{id} \otimes \rho} \otimes \circ (\text{id} \times (\cdot \otimes 1)) \\
\otimes & \xrightarrow{\rho} \otimes \circ (\text{id} \times (\cdot \otimes 1)) \\
\otimes & \xrightarrow{\alpha} \otimes \circ (\text{id} \times (\cdot \otimes 1))
\end{align*} \]

(f) Further invertible modifications

\[ \begin{align*}
(a \otimes (b \otimes c)) & \xrightarrow{\beta} (b \otimes c) \otimes a \\
(a \otimes b) \otimes c & \xrightarrow{\alpha} (b \otimes c) \otimes a \\
(b \otimes a) \otimes c & \xrightarrow{\alpha} (b \otimes c) \otimes a \\
(b \otimes a) \otimes c & \xrightarrow{\beta \otimes \text{id}} (b \otimes (c \otimes a)) \\
(b \otimes a) \otimes c & \xrightarrow{\id \otimes \beta} (b \otimes (c \otimes a))
\end{align*} \]

and

\[ \begin{align*}
(a \otimes (b \otimes c)) & \xrightarrow{\beta} (c \otimes (a \otimes b)) \\
(a \otimes (b \otimes c)) & \xrightarrow{\alpha} (c \otimes (a \otimes b)) \\
(b \otimes (a \otimes c)) & \xrightarrow{\alpha} (c \otimes (a \otimes b)) \\
(b \otimes (a \otimes c)) & \xrightarrow{\beta \otimes \id} (a \otimes (c \otimes b)) \\
(a \otimes (b \otimes c)) & \xrightarrow{\alpha \otimes (\beta \otimes \id) \alpha} (a \otimes (c \otimes b)) \\
(b \otimes (a \otimes c)) & \xrightarrow{\beta \otimes \id} (a \otimes (c \otimes b))
\end{align*} \]
(g) An invertible modification

\[
\begin{array}{c}
a \otimes b \\
\downarrow \Sigma \\
b \otimes a
\end{array}
\xrightarrow{\beta}
\begin{array}{c}
a \otimes b \\
\downarrow \Sigma \\
b \otimes a
\end{array}
\]

These data are required to satisfy a long list of coherence diagrams, see [SP11, Appendix C] for details.

**Example B.11.** A Kapranov-Voevodsky 2-vector space [KV94] is a \(\mathbb{C}\)-linear semi-simple additive category \(\mathcal{V}\) with finitely many isomorphism classes of simple objects; in particular, a 2-vector space is also an abelian category. There is a 2-category \(2\text{Vect}_{\mathbb{C}}\) of 2-vector spaces, \(\mathbb{C}\)-linear functors and natural transformations. Given two 2-vector spaces \(\mathcal{V}_1\) and \(\mathcal{V}_2\) we can define their tensor product \(\mathcal{V}_1 \boxtimes \mathcal{V}_2\) [BK01, Definition 1.15] to be the category with objects given by finite formal sums

\[
\bigoplus_{i=1}^{n} V_{1i} \boxtimes V_{2i} ,
\]

with \(V_{1i} \in \text{Obj}(\mathcal{V}_1)\) and \(V_{2i} \in \text{Obj}(\mathcal{V}_2)\). The space of morphisms is given by

\[
\text{Hom}_{\mathcal{V}_1 \boxtimes \mathcal{V}_2} \left( \bigoplus_{i=1}^{n} V_{1i} \boxtimes V_{2i}, \bigoplus_{j=1}^{m} V'_{1j} \boxtimes V'_{2j} \right) = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \text{Hom}_{\mathcal{V}_1}(V_{1i}, V'_{1j}) \otimes_{\mathbb{C}} \text{Hom}_{\mathcal{V}_2}(V_{2i}, V'_{2j}) .
\]

This tensor product coincides with the Deligne product of abelian categories. It furthermore satisfies the universal property with respect to bilinear functors that one would expect from a tensor product. We can also take tensor products of \(\mathbb{C}\)-linear functors and of natural transformations. Then the 2-category \(2\text{Vect}_{\mathbb{C}}\) with \(\boxtimes\) is a symmetric monoidal bicategory with monoidal unit 1 given by the category of finite-dimensional vector spaces \(\text{Vect}_{\mathbb{C}}\).

**Definition B.12.** A symmetric monoidal 2-functor between two symmetric monoidal bicategories \(\mathcal{B}\) and \(\mathcal{B}'\) consists of a 2-functor \(\mathcal{H}: \mathcal{B} \to \mathcal{B}'\) of the underlying bicategories together with the following data:

(a) Equivalence natural 2-transformations \[\chi: \boxtimes' \circ (\mathcal{H}(\cdot) \times \mathcal{H}(\cdot)) \Rightarrow \mathcal{H} \otimes \] and \(\iota: 1' \Rightarrow \mathcal{H}(1)\), where here we consider 1 as a 2-functor from the bicategory with one object, one 1-morphism and one 2-morphism to \(\mathcal{B}\).

(b) The three invertible modifications

\[
\begin{array}{ccc}
\mathcal{H}(a) \otimes' (\mathcal{H}(b) \otimes' \mathcal{H}(c)) & \xrightarrow{id \otimes' \chi} & \mathcal{H}(a) \otimes' \mathcal{H}(b \otimes c) \\
\mathcal{H}(a) \otimes' (\mathcal{H}(b) \otimes' \mathcal{H}(c)) & \xrightarrow{\chi \otimes' \iota} & \mathcal{H}(a \otimes (b \otimes c)) \\
\mathcal{H}(a \otimes b) \otimes' \mathcal{H}(c) & \xrightarrow{\alpha} & \mathcal{H}(a \otimes (b \otimes c)) \\
\mathcal{H}(a \otimes b) \otimes' \mathcal{H}(c) & \xrightarrow{\chi} & \mathcal{H}(a \otimes (b \otimes c))
\end{array}
\]

\[\text{We fix again adjoint inverses and the adjunction data.}\]
We finally come to the central concept in this paper. In contrast to the definition given in [SP11], we require the appearing modifications to be invertible. However, the 2-morphisms corresponding to the underlying natural transformations are not invertible in our definition, so our definition is also weaker than the definition given in [SP11].

Definition B.13. A natural symmetric monoidal 2-transformation between symmetric monoidal 2-functors $\mathcal{H}, \mathcal{K} : \mathcal{B} \to \mathcal{B}'$ consists of a natural 2-transformation $\theta : \mathcal{H} \Rightarrow \mathcal{K}$ of the underlying 2-functors together with invertible modifications

\[
\begin{array}{c}
\mathcal{H}(a \otimes b) \\
\mathcal{H}(a) \otimes' \mathcal{H}(b) \\
\mathcal{K}(a) \otimes' \mathcal{K}(b)
\end{array}
\]

which satisfy the following coherence conditions expressed as equalities between 2-morphisms.
(omitting tensor product symbols on objects and 1-morphisms to streamline the notation):
In (B.14), the unlabelled 2-morphisms in the first diagram are constructed from naturality of $\alpha^*$ and 2-functoriality of $\otimes$, while the unlabelled 2-morphism in the second diagram is induced by the equivalence $\alpha^* \circ \alpha \Rightarrow \text{id}$.

**Definition B.18.** A symmetric monoidal modification between two symmetric monoidal 2-transformations $\theta, \theta': \mathcal{H} \Rightarrow \mathcal{K}$ consists of a modification $m: \theta \Rightarrow \theta'$ of the underlying natural 2-
transformations satisfying

\[
\begin{array}{c}
\mathcal{H}(a) \otimes \mathcal{H}(b) \\
\downarrow \theta \otimes' \theta' \\
\mathcal{K}(a) \otimes' \mathcal{K}(b)
\end{array}
\xrightarrow{x_{\mathcal{H}}} \begin{array}{c}
\mathcal{H}(a \otimes b) \\
\downarrow \phi \\
\mathcal{K}(a \otimes b) \end{array}
\quad = \quad \begin{array}{c}
\mathcal{H}(a) \otimes' \mathcal{H}(b) \\
\downarrow \theta \otimes' \theta' \\
\mathcal{K}(a \otimes' b)
\end{array}
\xrightarrow{x_{\mathcal{K}}} \begin{array}{c}
\mathcal{H}(a \otimes b) \\
\downarrow \phi' \\
\mathcal{K}(a \otimes b) \end{array}
\]

and

\[
\begin{array}{c}
\mathcal{H}(1) \\
\downarrow \theta \otimes' \theta' \\
\mathcal{K}(1)
\end{array}
\xrightarrow{\iota_{\mathcal{H}}} \begin{array}{c}
\mathcal{H}(1) \\
\downarrow \theta' \\
\mathcal{K}(1)
\end{array}
\quad = \quad \begin{array}{c}
\mathcal{H}(1) \\
\downarrow \theta' \\
\mathcal{K}(1)
\end{array}
\xrightarrow{\iota_{\mathcal{K}}} \begin{array}{c}
\mathcal{H}(1) \\
\downarrow \theta' \\
\mathcal{K}(1)
\end{array}
\]

References

[AGDPM85] L. Alvarez-Gaumé, S. Della Pietra, and G. W. Moore. Anomalies and odd dimensions. *Ann. Phys.*, 163:288–317, 1985.

[ALN04] B. Ammann, R. Lauter, and V. Nistor. On the geometry of Riemannian manifolds with a Lie structure at infinity. *Int. J. Math. Sci.*, 2004(1–4):161–193, 2004, [arXiv:math.DG/0201202](arXiv:math.DG/0201202).

[APS75] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and Riemannian geometry I. *Math. Proc. Cambridge Philos. Soc.*, 77(1):43–69, 1975.

[Ati88] M. F. Atiyah. Topological quantum field theory. *Publ. Math. IHÉS*, 68:175–186, 1988.

[BK01] B. Bakalov and A. A. Kirillov, Jr. *Lectures on Tensor Categories and Modular Functors*. American Mathematical Society, 2001.

[Bun95] U. Bunke. On the gluing problem for the $\eta$-invariant. *J. Diff. Geom.*, 41(2):397–448, 1995.

[Bun09] U. Bunke. Index theory, eta forms, and Deligne cohomology. *Memoirs of the AMS*, 198, no. 928, 2009.

[CGLW13] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen. Symmetry-protected topological orders and the group cohomology of their symmetry group. *Phys. Rev. B*, 87(15):155114, 2013, [arXiv:1106.4772](arXiv:1106.4772 [cond-mat.str-el]).

[CGW10] X. Chen, Z.-C. Gu, and X.-G. Wen. Local unitary transformation, long-range quantum entanglement, wavefunction renormalization, and topological order. *Phys. Rev. B*, 82:155138, 2010, [arXiv:1004.3835](arXiv:1004.3835 [cond-mat.str-el]).

[CH85] C. G. Callan, Jr. and J. A. Harvey. Anomalies and fermion zero modes on strings and domain walls. *Nucl. Phys. B*, 250:427–436, 1985.
[CL87] L. N. Chang and Y. Liang. Topological anomalies: Explicit examples. *Comm. Math. Phys.*, 108:139–152, 1987.

[DS15] D. Calaque and C. I. Scheimbauer. A note on the $(\infty, n)$-category of cobordisms. *Preprint*, 2015, [arXiv:1509.08906 [math.AT]]

[DF95] X. Dai and D. S. Freed. $\eta$-invariants and determinant lines. *C. R. Acad. Sci. Paris Sér. I Math.*, 320(5):585–591, 1995, [arXiv:hep-th/9405012]

[Ert17] Ü. Ertem. Index of Dirac operators and classification of topological insulators. *Preprint*, 2017, [arXiv:1709.01778 [math-ph]]

[Fad84] L. D. Faddeev. Operator anomaly for the Gauss law. *Phys. Lett. B*, 145:81–84, 1984.

[FH16] D. S. Freed and M. J. Hopkins. Reflection positivity and invertible topological phases. *Preprint*, 2016, [arXiv:1604.06527 [hep-th]]

[FQ93] D. S. Freed and F. Quinn. Chern-Simons Theory with Finite Gauge Group. *Comm. Math. Phys.*, 156, 435-472, 1993, [arXiv:hep-th/9111004]

[Fre14] D. S. Freed. Anomalies and invertible field theories. *Proc. Symp. Pure Math.*, 88:25–46, 2014, [arXiv:1404.7224 [hep-th]].

[FS85] L. D. Faddeev and S. L. Shatashvili. Algebraic and Hamiltonian methods in the theory of nonabelian anomalies. *Theor. Math. Phys.*, 60:770–778, 1985.

[FS86] L. D. Faddeev and S. L. Shatashvili. Realization of the Schwinger term in the Gauss law and the possibility of correct quantization of a theory with anomalies. *Phys. Lett. B*, 167:225–228, 1986.

[FT14] D. S. Freed and C. Teleman. Relative quantum field theory. *Comm. Math. Phys.*, 326(2):459–476, 2014, [arXiv:1212.1692 [hep-th]].

[FV15] D. Fiorenza and A. Valentino. Boundary conditions for topological quantum field theories, anomalies and projective modular functors. *Comm. Math. Phys.*, 338(3):1043–1074, 2015, [arXiv:1409.5723 [math.QA]].

[GK16] D. Gaiotto and A. Kapustin. Spin TQFTs and fermionic phases of matter. *Int. J. Mod. Phys. A*, 31(28n29):1645044, 2016, [arXiv:1505.05856 [cond-mat.str-el]].

[HK10] M. Z. Hasan and C. L. Kane. Topological insulators. *Rev. Mod. Phys.*, 82(4):3045–3067, 2010, [arXiv:1002.3895 [cond-mat.str-el]].

[HSV17] J. Hesse, C. Schweigert, and A. Valentino. Frobenius algebras and homotopy fixed points of group actions on bicategories. *Theory Appl. Categ.*, 32(18):652–681, 2017, [arXiv:1607.05148 [math.QA]].

[JFS17] T. Johnson-Freyd and C. Scheimbauer. (Op)lax natural transformations, twisted quantum field theories, and “even higher” Morita categories. *Adv. Math.*, 307:147–223, 2017, [arXiv:1502.06526 [math.CT]].
[Koc04] J. Kock. *Frobenius Algebras and 2D Topological Quantum Field Theories*. Cambridge University Press, 2004.

[KT17] A. Kapustin and A. Turzillo. Equivariant topological quantum field theory and symmetry-protected topological phases. *J. High Energy Phys.*, 03:006, 2017, arXiv:1504.01830 [cond-mat.str-el]

[KV94] M. Kapranov and V. Voevodsky. Braided monoidal 2-categories and Manin-Schechtman higher braid groups. *J. Pure Appl. Algebra*, 92:241–267, 1994.

[Lei98] T. Leinster. Basic bicategories. *Preprint*, 1998, arXiv:math.CT/9810017

[LM02] P. Loya and R. B. Melrose. Fredholm perturbations of Dirac operators on manifolds with corners. *Preprint*, 2002.

[Loy98] P. Loya. On the b-pseudodifferential calculus on manifolds with corners. *Ph.D. Thesis*, 1998.

[Loy04] P. Loya. Index theory of Dirac operators on manifolds with corners up to codimension two. *Operator Theory: Adv. Appl.*, 151:131–166, 2004.

[Loy05] P. Loya. Dirac operators, boundary value problems, and the b-calculus. *Contemp. Math.*, 366:241–280, 2005.

[Lur09] J. Lurie. On the classification of topological field theories. *Current Devel. Math.*, 2008:129–280, 2009, arXiv:0905.0465 [math.CT]

[LW96] M. Lesch and K. P. Wojciechowski. On the $\eta$-invariant of generalized Atiyah-Patodi-Singer boundary value problems. *Illinois J. Math.*, 40(1):30–46, 1996.

[Mel93] R. B. Melrose. *The Atiyah-Patodi-Singer Index Theorem*. A.K. Peters, Wellesley, 1993.

[Mic85] J. Mickelsson. Chiral anomalies in even and odd dimensions. *Comm. Math. Phys.*, 97:361–370, 1985.

[Mon15] S. Monnier. Hamiltonian anomalies from extended field theories. *Comm. Math. Phys.*, 338(3):1327–1361, 2015, arXiv:1410.7442 [hep-th]

[Mon17] S. Monnier. The anomaly field theories of six-dimensional (2,0) superconformal theories. *Preprint*, 2017, arXiv:1706.01903 [hep-th]

[MP98] R. Mazzeo and P. Piazza. Dirac operators, heat kernels and microlocal analysis. II. Analytic surgery. *Rend. Mat. Appl. (7)*, 18(2):221–288, 1998, arXiv:math.DG/9807040

[Mül94] W. Müller. $\eta$-invariants and manifolds with boundary. *J. Diff. Geom.*, 40(2):311–377, 1994.

[Nas91] C. Nash. *Differential Topology and Quantum Field Theory*. Academic Press, London, 1991.
[NS83] A. J. Niemi and G. W. Semenoff. Axial anomaly induced fermion fractionization and effective gauge theory actions in odd-dimensional spacetimes. *Phys. Rev. Lett.*, 51:2077–2080, 1983.

[QZ11] X.-L. Qi and S.-C. Zhang. Topological insulators and superconductors. *Rev. Mod. Phys.*, 83(4):1057–1110, 2011.

[Red84] A. N. Redlich. Parity violation and gauge non-invariance of the effective gauge field action in three dimensions. *Phys. Rev. D*, 29:2366–2374, 1984.

[RML12] S. Ryu, J. E. Moore, and A. W. W. Ludwig. Electromagnetic and gravitational responses and anomalies in topological insulators and superconductors. *Phys. Rev. B*, 85:045104, 2012, [arXiv:1010.0936 [cond-mat.str-el]]

[RZ12] S. Ryu and S.-C. Zhang. Interacting topological phases and modular invariance. *Phys. Rev. B*, 85:245132, 2012, [arXiv:1202.4484 [cond-mat.str-el]]

[Sat11] H. Sati. Corners in M-theory. *J. Phys. A*, 44:255402, 2011, [arXiv:1101.2793 [hep-th]]

[Seg88] G. B. Segal. The definition of conformal field theory. In *Differential Geometrical Methods in Theoretical Physics*, pages 165–171. Springer Netherlands, 1988.

[Seg11] G. B. Segal. Three roles of quantum field theory. *Felix Klein Lectures*, 2011 [http://www.mpim-bonn.mpg.de/node/3372/abstracts]

[SP11] C. J. Schommer-Pries. The classification of two-dimensional extended topological field theories. *Ph.D. Thesis*, 2011, [arXiv:1112.1000 [math.AT]]

[ST11] S. Stolz and P. Teichner. Supersymmetric field theories and generalized cohomology. *Proc. Symp. Pure Math.*, 83:279–340, 2011, [arXiv:1108.0189 [math.AT]]

[SW16] N. Seiberg and E. Witten. Gapped boundary phases of topological insulators via weak coupling. *Prog. Theor. Exp. Phys.*, 2016(12):12C101, 2016, [arXiv:1602.04251 [cond-mat.str-el]]

[Tur10] V. G. Turaev. *Quantum Invariants of Knots and 3-Manifolds*. De Gruyter, New York, 2010.

[Wen13] X.-G. Wen. Classifying gauge anomalies through symmetry-protected trivial orders and classifying gravitational anomalies through topological orders. *Phys. Rev. D*, 88(4):045013, 2013, [arXiv:1303.1803 [hep-th]]

[WGW15] J. C. Wang, Z.-C. Gu, and X.-G. Wen. Field theory representation of gauge-gravity symmetry-protected topological invariants, group cohomology and beyond. *Phys. Rev. Lett.*, 114(3):031601, 2015, [arXiv:1405.7689 [cond-mat.str-el]]

[Wil08] S. Willerton. The twisted Drinfeld double of a finite group via gerbes and finite groupoids. *Algebr. Geom. Topol.*, 8(3):1419–1457, 2008, [arXiv:math.QA/0503266]

[Wit16a] E. Witten. Fermion path integrals and topological phases. *Rev. Mod. Phys.*, 88(3):035001, 2016, [arXiv:1508.04715 [cond-mat.mes-hall]]
E. Witten. The “parity” anomaly on an unorientable manifold. *Phys. Rev. B*, 94(19):195150, 2016, [arXiv:1605.02391 [hep-th]]