GLOBAL EXISTENCE AND OPTIMAL DECAY RATES OF SOLUTIONS FOR COMPRESSIBLE HALL-MHD EQUATIONS

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Abstract. In this paper, we are concerned with global existence and optimal decay rates of solutions for the compressible Hall-MHD equations in dimension three. First, we prove the global existence of strong solutions by the standard energy method under the condition that the initial data are close to the constant equilibrium state in $H^2$-framework. Second, optimal decay rates of strong solutions in $L^2$-norm are obtained if the initial data belong to $L^1$ additionally. Finally, we apply Fourier splitting method by Schonbek [Arch. Rational Mech. Anal. 88 (1985)] to establish optimal decay rates for higher order spatial derivatives of classical solutions in $H^3$-framework, which improves the work of Fan et al. [Nonlinear Anal. Real World Appl. 22 (2015)].

1. Introduction. The application of Hall-magnetohydrodynamics system (in short, Hall-MHD) covers a very wide range of physical objects, for example, magnetic reconnection in space plasmas, star formulation, neutron stars, and geodynamo, refer to [13, 17, 29, 2, 24, 21] and the references therein. Recently, Achenitogaray et al. [1] derived the Hall-MHD equations from the two-fluid Euler-Maxwell system for electrons and ions through a set of scaling limits or from the kinetic equations by taking macroscopic quantities in the equations under some closure assumptions. They also established the global existence of weak solutions with periodic boundary condition. In this paper, we investigate the following compressible Hall-MHD equations in three-dimensional whole space $\mathbb{R}^3$ (see [1]):

\[
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \nu)\nabla \text{div} u + \nabla P(\rho) &= (\text{curl} B) \times B, \\
B_t - \text{curl}(u \times B) + \text{curl} \left[ \frac{(\text{curl} B) \times B}{\rho} \right] &= \Delta B, \quad \text{div} B = 0,
\end{aligned}
\]

where the functions $\rho, u, \text{ and } B$ represent density, velocity, and magnetic field respectively. The pressure $P(\rho)$ is a smooth function in a neighborhood of 1 with

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$P'(1) = 1$. The constants $\mu$ and $\nu$ denote the viscosity coefficients of the flow and satisfy physical condition as follows

$$\mu > 0, \ 2\mu + 3\nu \geq 0.$$  

To complete the system (1), the initial data are given by

$$\begin{align*}
\left. (\rho, u, B) \right|_{t=0} = (\rho_0(x), u_0(x), B_0(x)).
\end{align*}$$

Furthermore, as the space variable tends to infinity, we assume

$$\lim_{|x| \to \infty} (\rho_0 - 1, u_0, B_0)(x) = 0.$$  

Obviously, the compressible Hall-MHD equations transform into the well-known compressible MHD equations when the Hall effect term $\text{curl} \left( \frac{(\text{curl} B) \times B}{\rho} \right)$ is neglected.

When the density is a constant function, Chae et al.\cite{Chae2004} proved local existence of smooth solutions for large data and global smooth solutions for small data in three-dimensional whole space. They also showed a Liouville theorem for the stationary solutions. Chae and Lee \cite{Chae2006} established an optimal blow-up criterion for classical solutions and proved two global-in-time existence results of classical solutions for small initial data, the smallness conditions of which are given by the suitable Sobolev and Besov norms respectively. Later, Fan et al.\cite{Fan2012} also established some new regularity criteria, which were also built for density-dependent incompressible Hall-MHD equations with positive initial density by Fan and Ozawa \cite{Fan2014}. On one hand, Maicon and Lucas \cite{Maicon2015} proved a stability theorem for global large solutions under a suitable integrable hypothesis and constructed a special large solution by assuming the condition of curl-free magnetic fields. On the other hand, Fan et al. \cite{Fan2016} established the global well-posedness of the axisymmetric solutions. Recently, Chae and Schonbek \cite{Chae2017} established temporal decay estimates for weak solutions and obtained algebraic time decay for higher order Sobolev norms of small initial data solutions as follows

$$\|\nabla^k (\rho - 1, u, B)\|_{L^2} \leq C(1 + t)^{-\frac{3+2k}{2}}, \quad k \in \mathbb{N}$$

for all $t \geq T^* (T^* \text{ is a positive constant})$. Furthermore, Weng \cite{Weng2018} extended this result by providing upper and lower bounds on the decay of higher order derivatives. For the compressible Hall-MHD equations (1), Fan et al.\cite{Fan2017} proved the local existence of strong solutions with positive initial density and global small solutions(classical solutions) with small initial perturbation. They also established optimal time decay rate for classical solutions as follows

$$\| (\rho - 1, u, B) \|_{L^2} \leq C(1 + t)^{-\frac{3}{4}}.$$  

Here, they required the initial perturbation was small in $H^3$-norm and bounded in $L^1$-norm.

Recently, the study of decay rates for the MHD equations has aroused many researchers’ interest. First of all, under the $H^3$-framework, Li and Yu \cite{Li2019} and Chen and Tan \cite{Chen2020} not only established the global existence of classical solutions, but also obtained the time decay rates for the three-dimensional compressible MHD equations by assuming the initial data belong to $L^1$ and $L^q(q \in [1, \frac{6}{5}])$ respectively. More precisely, Chen and Tan \cite{Chen2020} built the temporal decay rates

$$\|\nabla^k (\rho - 1, u, B)\|_{H^{3-k}} \leq C(1 + t)^{-\frac{3}{2} \left( \frac{1}{q} - \frac{1}{2} \right) - \frac{k}{2}}, \quad k = 0, 1,$$  

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which has been established by Li and Yu [19] for the case $q = 1$. Motivated by the work of Guo and Wang [16], Tan and Wang [25] established the optimal time decay rates for the higher order spatial derivatives of solutions if the initial perturbation belong to $H^N \cap H^{-s}(N \geq 3, s \in [0, \frac{3}{2}])$. More precisely, they built the following time decay rates

$$
\|\nabla^k (\rho - 1, u, B) (t)\|_{H^{N-k}} \leq C (1 + t)^{-\frac{k + 2s}{4}}
$$

where $k = 0, 1, ..., N - 1$. In the spirit of the work [14], we (see [15]) established the time decay rates for all $t \geq T^*$ ($T^*$ is a positive constant) as follows

$$
\|\nabla^k (\rho - 1) (t)\|_{H^{3-k}} + \|\nabla^k u (t)\|_{H^{3-k}} \leq C (1 + t)^{-\frac{3 + 2k}{4}},
$$

$$
\|\nabla^m B (t)\|_{H^{3-m}} \leq C (1 + t)^{-\frac{3 + 2m}{4}},
$$

(6)

where $k = 0, 1, 2$, and $m = 0, 1, 2, 3$. It is easy to see that the decay rates (6) is better than decay rates (5) since (6) provides faster time decay rates for the higher order spatial derivatives of solutions.

In this paper, we hope to establish the global existence and temporal decay rates of solutions for the compressible Hall-MHD equations (1)-(3). First of all, we construct the global existence of strong solutions by the standard energy method under the condition that the initial data are close to the constant equilibrium state $(1, 0, 0)$ in $H^2$-norm. Second, if the initial data in $L^1$-norm are finite additionally, the optimal time decay rates of strong solutions are established by the method of Green function. Precisely, we obtain the following decay rates for all $t \geq 0$,

$$
\|\nabla^k (\rho - 1) (t)\|_{H^{2-k}} + \|\nabla^k u (t)\|_{H^{2-k}} + \|\nabla^k B (t)\|_{H^{2-k}} \leq C (1 + t)^{-\frac{3 + 2k}{4}},
$$

where $k = 0, 1$. This framework of time convergence rates has been applied to other compressible models, refer to [26, 18, 27, 28]. Although magnetic field equations are nonlinear parabolic equations, we hope to establish optimal time decay rates for the second order spatial derivatives of magnetic field under the condition of small initial perturbation. In order to achieve this goal, we move the nonlinear terms to the right hand side of (1.3) and deal with the nonlinear terms as external force with the property on fast time decay rates. Then, the application of Fourier splitting method by Schonbek [23] helps us to establish optimal time decay rate for the second order spatial derivatives of magnetic field as follows

$$
\|\nabla^2 B (t)\|_{L^2} \leq C (1 + t)^{-\frac{3}{4}}.
$$

Finally, one focuses on establishing optimal time decay rates for higher order spatial derivatives of classical solutions to the compressible Hall-MHD equations. Moreover, we prove that the global classical solution $(\rho, u, B)$ of Cauchy problem (1)-(3) has the temporal decay rates (6). Obviously, these decay rates improve the results (4) by Fan et al. [9] since we build faster decay rates for higher order spatial derivatives of classical solutions.

Notation. In this paper, we use $H^s (\mathbb{R}^3) (s \in \mathbb{R})$ to denote the usual Sobolev spaces with norm $\| \cdot \|_{H^s}$ and $L^p (\mathbb{R}^3) (1 \leq p \leq \infty)$ to denote the usual $L^p$ spaces with norm $\| \cdot \|_{L^p}$. The symbol $\nabla^l$ with an integer $l \geq 0$ stands for the usual any spatial derivatives of order $l$. For example, we define

$$
\nabla^l v = \{ \partial_{x_i}^k v_i | \alpha = k, \ i = 1, 2, 3 \}, \ v = (v_1, v_2, v_3).
$$

We also denote $\mathcal{F} (f) := \hat{f}$. The notation $a \lesssim b$ means that $a \leq Cb$ for a universal constant $C > 0$ independent of time $t$. The notation $a \approx b$ means $a \lesssim b$ and
First of all, we establish the global existence and optimal decay rates of strong solutions for the compressible Hall-MHD equations (1)-(3).

**Theorem 1.1.** Assume that the initial data \((ρ₀ - 1, u₀, B₀) ∈ H^2\) and there exists a small constant \(δ₀ > 0\) such that

\[
\|\rho₀ - 1, u₀, B₀\|_{H^2} ≤ δ₀,
\]

then the problem (1)-(3) admits a unique global strong solution \((ρ, u, B)\) satisfying for all \(t ≥ 0\),

\[
\|ρ(t) - 1, u(t), B(t)\|_{H^2} + \int₀^t (\|∇ρ(s)\|_{H^2} + ∥∇(u, B)(s)∥_{H^2}) ds ≤ C\|ρ₀ - 1, u₀, B₀\|_{H^2}^2.
\]

Furthermore, if \(\|ρ₀ - 1, u₀, B₀\|_{L^1}\) is finite additionally, then the global strong solution \((ρ, u, B)\) has following decay rates for all \(t ≥ 0\),

\[
\|∇^k ρ(t)\|_{H^{2-k}} + \|∇^k u(t)\|_{H^{2-k}} ≤ C(1 + t)^{-\frac{3+2k}{4}},
\]

\[
\|∇^m B(t)\|_{H^{2-m}} ≤ C(1 + t)^{-\frac{3+2m}{4}},
\]

where \(k = 0, 1\), and \(m = 0, 1, 2\).

**Remark 1.** For any \(2 ≤ p ≤ 6\), by virtue of Theorem 1.1 and the Sobolev interpolation inequality, we obtain time decay rates as follows

\[
\|ρ(t)\|_{L^p} + \|u(t)\|_{L^p} ≤ C(1 + t)^{-\frac{3}{4}(1 - \frac{1}{p})},
\]

\[
\|∇^k B(t)\|_{L^p} ≤ C(1 + t)^{-\frac{3}{4}(1 - \frac{1}{p})-\frac{k}{4}},
\]

where \(k = 0, 1\). Furthermore, in the same manner, we also have

\[
\|ρ(t)\|_{L^∞} + \|u(t)\|_{L^∞} ≤ C(1 + t)^{-\frac{3}{4}},
\]

\[
\|B(t)\|_{L^∞} ≤ C(1 + t)^{-\frac{3}{4}}.
\]

Second, one also builds temporal decay rates for the time derivatives of global strong solutions.

**Theorem 1.2.** Under all the assumptions in Theorem 1.1, the global strong solution \((ρ, u, B)\) of Cauchy problem (1)-(3) has the decay rates

\[
\|ρ(t)\|_{H^1} + \|u(t)\|_{L^2} ≤ C(1 + t)^{-\frac{3}{4}},
\]

\[
\|B(t)\|_{L^2} ≤ C(1 + t)^{-\frac{3}{4}}
\]

for all \(t ≥ 0\).

Furthermore, one establishes optimal decay rates for the higher order spatial derivatives of classical solutions to the compressible Hall-MHD equations.

**Theorem 1.3.** Assume that the initial data \((ρ₀ - 1, u₀, B₀) ∈ H^3 ∩ L^1\) and there exists a small constant \(ε₀ > 0\) such that

\[
\|ρ₀ - 1, u₀, B₀\|_{H^3} ≤ ε₀,
\]

then the global classical solution \((ρ, u, B)\) of the problem (1)-(3) has the decay rates

\[
\|∇^k ρ(t)\|_{H^{3-k}} + \|∇^k u(t)\|_{H^{3-k}} ≤ C(1 + t)^{-\frac{3+2k}{4}},
\]

\[
\|∇^m B(t)\|_{H^{3-m}} ≤ C(1 + t)^{-\frac{3+2m}{4}},
\]

for all \(t ≥ 0\).
where $k = 0, 1, 2$, and $m = 0, 1, 2, 3$.

**Remark 2.** Compared with the decay rates of linearized systems (62) stated in Proposition 1, (9) provides optimal decay rates of solutions and its spatial derivatives (except for the third order spatial derivatives of density and velocity) in $L^2$-norm to the nonlinear problem (1)-(3). Here the decay rate of solutions to nonlinear system is optimal in the sense that it coincides with the rate of solutions to the linearized systems.

**Remark 3.** By virtue of the Sobolev inequality and the results (9) in Theorem 1.3, the global classical solution $(\rho, u, B)$ has the decay rates

$$
\| (\rho - 1)(t) \|_{L^p} + \| u(t) \|_{L^p} \leq C(1 + t)^{-\frac{3}{2}(1 - \frac{1}{p})},
$$

$$
\| \nabla^k B(t) \|_{L^p} \leq C(1 + t)^{-\frac{3}{2}(1 - \frac{1}{p}) - \frac{k}{2}},
$$

where $k = 0, 1$, and $p \in [2, \infty]$. Hence, the rate of classical solution $(\rho, u, B)$ converging to the equilibrium state $(1, 0, 0)$ in $L^\infty$-norm is $(1 + t)^{-\frac{3}{2}}$.

**Remark 4.** It is easy to see that (9) provides faster decay rates for higher order spatial derivatives of global classical solutions than (4). Hence, the results in Theorem 1.3 improve the work of Fan et al. [9].

**Remark 5.** Although we only established the time decay rates under the $H^3$-framework in Theorem 1.3, the method here can be applied to the $H^N(N \geq 3)$-framework just following the idea as Gao et al. [14]. Hence, if $(\rho_0 - 1, u_0, B_0) \in H^N \cap L^1(N \geq 3)$, then the global solution $(\rho, u, B)$ has the time decay rates

$$
\| \nabla^k (\rho - 1)(t) \|_{H^{N-k}} + \| \nabla^k u(t) \|_{H^{N-k}} \leq C(1 + t)^{-\frac{3+2k}{2}},
$$

$$
\| \nabla^m B(t) \|_{H^{N-m}} \leq C(1 + t)^{-\frac{3+2m}{2}},
$$

where $k = 0, 1, \ldots, N - 1$, and $m = 0, 1, 2, \ldots, N$.

Finally, one builds decay rates for the mixed space-time derivatives of global classical solutions.

**Theorem 1.4.** Under all the assumptions in Theorem 1.3, the global classical solution $(\rho, u, B)$ of the problem (1)-(3) satisfies the decay rates

$$
\| \nabla^k \rho_t(t) \|_{H^{2-k}} + \| \nabla^k u_t(t) \|_{L^2} \leq C(1 + t)^{-\frac{3+2k}{2}},
$$

$$
\| \nabla^k B_t(t) \|_{L^2} \leq C(1 + t)^{-\frac{3+2k}{2}},
$$

where $k = 0, 1$.

The rest of this paper is organized as follows. In section 2, one establishes some energy estimates that will play an essential role for us to construct the global existence of strong solutions. Then, the estimates will be closed by the standard continuity argument and the global existence of strong solutions follows immediately. Furthermore, one builds the time decay rates by taking the method of Green function and establishes optimal decay rates for the second order spatial derivatives of magnetic field. Finally, we also investigate decay rates for the time derivatives of density, velocity and magnetic field. In section 3, one establishes the optimal decay rates for the higher order spatial derivatives of global classical solutions and mixed space-time derivatives of solutions.
2. Proof of Theorem 1.1 and Theorem 1.2. In this section, we will establish global existence and optimal time decay rates of strong solutions for the compressible Hall-MHD equations. Indeed, computing directly, it is easy to deduce
\[(\text{curl}B) \times B = (B \cdot \nabla)B - \frac{1}{2} \nabla(|B|^2),\]
and
\[\text{curl}(u \times B) = u(\text{div}B) - (u \cdot \nabla)B + (B \cdot \nabla)u - B(\text{div}u).\]
Then, denoting \( \varrho = \rho - 1 \), we rewrite (1) in the perturbation form as
\[
\begin{cases}
\varrho_t + \text{div}u = S_1, \\
u_t - \mu \Delta u - (\mu + \nu)\nabla \text{div}u + \nabla \varrho = S_2, \\
B_t - \Delta B = S_3, \quad \text{div}B = 0,
\end{cases}
\]
where the function \( S_i(i = 1, 2, 3) \) is defined as
\[
\begin{aligned}
S_1 &= -\varrho \text{div}u - u \cdot \nabla \varrho, \\
S_2 &= -u \cdot \nabla u - h(\varrho)[\mu \Delta u + (\mu + \nu)\nabla \text{div}u] - f(\varrho)\nabla \varrho + g(\varrho)\left[B \cdot \nabla B - \frac{1}{2} \nabla(|B|^2)\right], \\
S_3 &= -u \cdot \nabla B + B \cdot \nabla u - B \text{div}u - \text{curl}\left[g(\varrho)\left(B \cdot \nabla B - \frac{1}{2} \nabla(|B|^2)\right)\right].
\end{aligned}
\]
Here the nonlinear functions of \( \varrho \) are defined by
\[
h(\varrho) = \frac{\varrho}{\varrho + 1}, \quad f(\varrho) = \frac{\varrho'(\varrho + 1)}{\varrho + 1} - 1, \quad g(\varrho) = \frac{1}{\varrho + 1}.
\]
The initial data are given as
\[
(\varrho, u, B)(x, t)|_{t=0} = (\varrho_0, u_0, B_0)(x) \to (0, 0, 0) \quad \text{as} \quad |x| \to \infty.
\]
2.1. Energy estimates. First of all, suppose there exists a small positive constant \( \delta \) satisfying following estimate
\[
\|(\varrho, u, B)(t)\|_{H^2} := \|\varrho(t)\|_{H^2} + \|u(t)\|_{H^2} + \|B(t)\|_{H^2} \leq \delta,
\]
which, together with Sobolev inequality, yields directly
\[
\frac{1}{2} \leq \varrho + 1 \leq \frac{3}{2}.
\]
Hence, it is easy to deduce immediately
\[
|f(\varrho)| + h(\varrho) \leq C|\varrho| \quad \text{and} \quad |g^{(k-1)}(\varrho)| + |h^{(k)}(\varrho)| + |f^{(k)}(\varrho)| \leq C \quad \text{for any} \quad k \geq 1,
\]
which will be used frequently to derive a priori estimates.

We state the classical Sobolev interpolation of the Gagliardo-Nirenberg inequality, refer to [22].

Lemma 2.1. Let \( 0 \leq m, \alpha \leq l \) and the function \( f \in C_c^\infty(\mathbb{R}^3) \), then we have
\[
\|\nabla^\alpha f\|_{L^p} \lesssim \|\nabla^m f\|_{L^{p_0}}^{\theta} \|\nabla^l f\|_{L^\infty}^{\theta},
\]
where \( 0 \leq \theta \leq 1 \) and \( \alpha \) satisfy
\[
\frac{1}{p} - \frac{\alpha}{3} = \left(\frac{1}{2} - \frac{m}{3}\right)(1 - \theta) + \left(\frac{1}{2} - \frac{l}{3}\right)\theta.
\]
First of all, one derives following energy estimates.
Lemma 2.2. Under the condition (14), then for $k = 0, 1$, we have

$$\frac{d}{dt} \| \nabla^k (\rho, u, B) \|_{L^2}^2 + C \| \nabla^{k+1} (u, B) \|_{L^2}^2 \lesssim \delta \| \nabla^{k+1} \rho \|_{L^2}^2. \tag{17}$$

Proof. Taking $k$-th spatial derivatives to (10)$_1$ and (10)$_2$ respectively, multiplying the resulting identities by $\nabla^k \rho$ and $\nabla^k u$ respectively and integrating over $\mathbb{R}^3$ by parts, it is easy to obtain

$$\frac{1}{2} \frac{d}{dt} \int (|\nabla_k \rho|^2 + |\nabla_k u|^2) dx + \int (\mu |\nabla^{k+1} u|^2 + (\mu + \nu) |\nabla^k \text{div} u|^2) dx$$

$$= \int \nabla^k S_1 \cdot \nabla^k \rho \ dt + \int \nabla^k S_2 \cdot \nabla^k u \ dt. \tag{18}$$

Taking $k$-th spatial derivatives to (10)$_3$, multiplying the resulting identity by $\nabla^k B$ and integrating over $\mathbb{R}^3$ by parts, we have

$$\frac{1}{2} \frac{d}{dt} \int |\nabla^k B|^2 dx + \int |\nabla^{k+1} B|^2 dx = \int \nabla^k S_3 \cdot \nabla^k B \ dt. \tag{19}$$

Adding (18) to (19), it follows immediately

$$\frac{1}{2} \frac{d}{dt} \int (|\nabla_k \rho|^2 + |\nabla_k u|^2 + |\nabla^k B|^2) dx + \int (\mu |\nabla^{k+1} u|^2 + (\mu + \nu) |\nabla^k \text{div} u|^2 + |\nabla^{k+1} B|^2) dx$$

$$= \int \nabla^k S_1 \cdot \nabla^k \rho \ dt + \int \nabla^k S_2 \cdot \nabla^k u \ dt + \int \nabla^k S_3 \cdot \nabla^k B \ dt. \tag{20}$$

For the case $k = 0$, then the differential identity (20) has the following form

$$\frac{1}{2} \frac{d}{dt} \int (|\rho|^2 + |u|^2 + |B|^2) dx + \int (\mu |\nabla u|^2 + (\mu + \nu) |\nabla \text{div} u|^2 + |\nabla B|^2) dx$$

$$= \int S_1 \cdot \rho \ dt + \int S_2 \cdot u \ dt + \int S_3 \cdot B \ dt = I_1 + I_2 + I_3. \tag{21}$$

Applying (14), Hölder, Sobolev and Cauchy inequalities, it is easy to deduce

$$I_1 \lesssim \| \rho \|_{L^6} \| \nabla \text{div} u \|_{L^6} + \| \rho \|_{L^6} \| \nabla \rho \|_{L^6} \| u \|_{L^6}$$

$$\lesssim \| \rho \|_{H^2} \| \nabla u \|_{L^2} \| \nabla \rho \|_{L^2} + \| \rho \|_{H^2} \| \nabla \rho \|_{L^2} \| \nabla u \|_{L^2}$$

$$\lesssim \delta (\| \nabla \rho \|_{L^2}^2 + \| \nabla u \|_{L^2}^2). \tag{22}$$

Integrating by parts and applying (15), Hölder, Sobolev and Cauchy inequalities, one arrives at

$$- \int h(\rho) (\mu \Delta u + (\mu + \nu) \nabla \text{div} u) u dx$$

$$\approx \int (h'(\rho) \nabla \rho \cdot u + h(\rho) \nabla u) \nabla u dx$$

$$\lesssim \| \nabla \rho \|_{L^2} \| u \|_{L^6} \| \nabla u \|_{L^3} + \| \rho \|_{L^6} \| \nabla u \|_{L^2}$$

$$\lesssim (\| \rho \|_{H^2} + \| \nabla u \|_{H^1}) (\| \nabla \rho \|_{L^2}^2 + \| \nabla u \|_{L^2}^2)$$

$$\lesssim \delta (\| \nabla \rho \|_{L^2}^2 + \| \nabla u \|_{L^2}^2). \tag{23}$$
Hence, with the help of (15), (23), H"older, Sobolev and Cauchy inequalities, we deduce
\[
\begin{align*}
I_2 \lesssim & \left( \|u\|_{L^3} \|\nabla u\|_{L^2} + \|\varphi\|_{L^3} \|\nabla \varphi\|_{L^2} + \|g(\varphi)\|_{L^\infty} \|B\|_{L^3} \|\nabla B\|_{L^2} \right) \|u\|_{L^6} \\
& + \delta (\|\nabla \varphi\|^2_{L^2} + \|\nabla u\|^2_{L^2}) \\
\lesssim & \left( \|u\|_{H^1} \|\nabla u\|_{L^2} + \|\varphi\|_{H^1} \|\nabla \varphi\|_{L^2} + \|B\|_{H^1} \|\nabla B\|_{L^2} \right) \|\nabla u\|_{L^2} \\
& + \delta (\|\nabla \varphi\|^2_{L^2} + \|\nabla u\|^2_{L^2}) \\
\lesssim & \delta (\|\nabla \varphi\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \|\nabla B\|^2_{L^2}).
\end{align*}
\]
(24)

Integrating by part and applying (15), H"older and Sobolev inequalities, we get
\[
\begin{align*}
- \int \text{curl} [g(\varphi)(B \cdot \nabla B)] B dx \\
= - \int g(\varphi)(B \cdot \nabla B) \text{curl} B dx \\
\lesssim & \|g(\varphi)\|_{L^\infty} \|B\|_{L^\infty} \|\nabla B\|_{L^2} \|\text{curl} B\|_{L^2} \\
\lesssim & \|B\|_{H^2} \|\nabla B\|^2_{L^2}.
\end{align*}
\]
(25)

Hence, with the help of (25), H"older, Sobolev and Cauchy inequalities, it is easy to deduce
\[
\begin{align*}
I_3 \lesssim & \left( \|u\|_{L^3} \|\nabla B\|_{L^2} + \|B\|_{L^3} \|\nabla u\|_{L^2} \right) \|\nabla B\|_{L^6} + \|B\|_{H^2} \|\nabla B\|^2_{L^2} \\
\lesssim & \left( \|u\|_{H^1} + \|B\|_{H^1} \right) (\|\nabla u\|^2_{L^2} + \|\nabla B\|^2_{L^2}) + \delta \|\nabla B\|^2_{L^2} \\
\lesssim & \delta (\|\nabla u\|^2_{L^2} + \|\nabla B\|^2_{L^2}).
\end{align*}
\]
(26)

Substituting (22), (24) and (26) into (21) and applying the smallness of \(\delta\), we find
\[
\begin{align*}
\frac{d}{dt} & \int (|\varphi|^2 + |u|^2 + |B|^2) dx + \int (\mu |\nabla u|^2 + |\nabla B|^2) dx \\
\lesssim & \delta \|\nabla \varphi\|^2_{L^2}.
\end{align*}
\]
(27)

For the case \(k = 1\), then the differential identity (20) has the following form
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} & \int (|\nabla \varphi|^2 + |\nabla u|^2 + |\nabla B|^2) dx + \int (\mu |\nabla^2 u|^2 + (\mu + \nu) \text{div} u |\nabla^2 B|^2) dx \\
= & \int \nabla S_1 \cdot \nabla \varphi dx + \int \nabla S_2 \cdot \nabla u dx + \int \nabla S_3 \cdot \nabla B dx = I_{11} + I_{12} + I_{13}.
\end{align*}
\]
(28)

Applying H"older and Cauchy inequalities, we obtain
\[
\begin{align*}
I_{11} \leq & \left( \|\varphi\|_{L^3} \|\text{div} u\|_{L^6} + \|u\|_{L^3} \|\nabla \varphi\|_{L^6} \right) \|\nabla^2 \varphi\|_{L^2} \\
\lesssim & \left( \|\varphi\|_{H^1} + \|u\|_{H^1} \right) \left( \|\nabla^2 \varphi\|^2_{L^2} + \|\nabla^2 u\|^2_{L^2} \right) \\
\lesssim & \delta (\|\nabla^2 \varphi\|^2_{L^2} + \|\nabla^2 u\|^2_{L^2}).
\end{align*}
\]
(29)

Similarly, it is easy to deduce
\[
\begin{align*}
I_{12} \leq & \left( \|u\|_{L^3} \|\nabla u\|_{L^6} + \|h(\varphi)\|_{L^6} \|\nabla^2 u\|_{L^2} \right) \|\nabla^2 u\|_{L^2} \\
& + \left( \|f(\varphi)\|_{L^3} \|\nabla \varphi\|_{L^6} + \|g(\varphi)\|_{L^6} \|B\|_{L^3} \|\nabla B\|_{L^2} \right) \|\nabla^2 \varphi\|_{L^2} \\
\lesssim & \left( \|\varphi\|_{H^2} + \|u\|_{H^1} + \|B\|_{H^1} \right) \left( \|\nabla^2 \varphi\|^2_{L^2} + \|\nabla^2 u\|^2_{L^2} + \|\nabla^2 B\|^2_{L^2} \right) \\
\lesssim & \delta (\|\nabla^2 \varphi\|^2_{L^2} + \|\nabla^2 u\|^2_{L^2} + \|\nabla^2 B\|^2_{L^2}).
\end{align*}
\]
(30)
Integrating by part and applying (15), Hölder and Sobolev inequalities, we have

\[- \int \nabla \operatorname{curl}[g(\varrho)(B \cdot \nabla B)] \nabla B \, dx\]

\[= - \int \nabla [g(\varrho)(B \cdot \nabla B)] \nabla \operatorname{curl} B \, dx\]

\[\lesssim (\|\nabla g(\varrho)\|_{L^1} \|B\|_{L^6} \|\nabla B\|_{L^6} + \|g(\varrho)\|_{L^\infty} \|\nabla B\|_{L^3} \|\nabla B\|_{L^3}) \|\nabla \operatorname{curl} B\|_{L^2}\]

\[+ \|g(\varrho)\|_{L^\infty} \|B\|_{L^\infty} \|\nabla^2 B\|_{L^2} \|\nabla \operatorname{curl} B\|_{L^2}\]

\[\lesssim (\|\nabla^2 \varrho\|_{L^2} \|\nabla B\|_{L^2} + \|\nabla B\|_{L^3} + \|B\|_{H^2}) \|\nabla^2 B\|_{L^2}^2\].\] (31)

Applying (31), Hölder, Sobolev and Cauchy inequalities, we find

\[H_3 \lesssim (\|u\|_{L^2} \|\nabla B\|_{L^6} \|B\|_{L^6} \|\nabla u\|_{L^6}) \|\nabla^2 B\|_{L^2}\]

\[+ (\|\nabla^2 \varrho\|_{L^2} \|\nabla B\|_{L^2} + \|\nabla B\|_{L^3} + \|B\|_{H^2}) \|\nabla^2 B\|_{L^2}^2\]

\[\lesssim \delta (\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 B\|_{L^2}^2).\] (32)

Substituting (29), (30) and (32) into (28), then we get

\[\frac{d}{dt} \int (\|\nabla \varrho\|^2 + \|\nabla u\|^2 + \|\nabla B\|^2) \, dx + \int (\mu \|\nabla^2 u\|^2 + \|\nabla^2 B\|^2) \, dx \lesssim \delta \|\nabla^2 \varrho\|_{L^2}^2,\]

which, together with (27), completes the proof of the lemma.

Next, one derives the second type of energy estimates involving the higher order spatial derivatives of density and velocity.

**Lemma 2.3.** Under the condition (14), then we have

\[\frac{d}{dt} \|\nabla^2 (\varrho, u, B)\|_{L^2}^2 + C \|\nabla^3 (u, B)\|_{L^2}^2 \lesssim \delta \|\nabla^2 \varrho\|_{L^2}^2.\] (33)

**Proof.** Taking $k = 2$ specially in (20), we deduce immediately

\[\frac{1}{10} \frac{d}{dt} \int (\|\nabla^2 \varrho\|^2 + \|\nabla^2 u\|^2 + \|\nabla^2 B\|^2) \, dx + \int (\mu \|\nabla^3 u\|^2 + (\mu + \nu) \|\nabla^2 \operatorname{div} u\|^2 + \|\nabla^3 B\|^2) \, dx\]

\[= \int \nabla^2 S_1 \cdot \nabla^2 \varrho \, dx + \int \nabla^2 S_2 \cdot \nabla^2 u \, dx + \int \nabla^2 S_3 \cdot \nabla^2 B \, dx.\] (34)

Applying Hölder, Sobolev and Cauchy inequalities, it is easy to obtain

\[- \int \nabla^2 (\varrho \operatorname{div} u) \nabla^2 \varrho \, dx\]

\[= - \int (\nabla^2 \varrho \operatorname{div} u + 2 \varrho \nabla \varrho \operatorname{div} u + \varrho \nabla^2 \varrho \operatorname{div} u) \nabla^2 \varrho \, dx\]

\[\lesssim (\|\nabla \varrho\|_{L^\infty} \|\nabla^2 \varrho\|_{L^2} + \|\nabla \varrho\|_{L^2} \|\nabla^2 u\|_{L^6} + \|\varrho\|_{L^\infty} \|\nabla^3 u\|_{L^2}) \|\nabla^2 \varrho\|_{L^2}\]

\[\lesssim (\|\nabla^2 \varrho\|_{L^2} \|\nabla^3 \varrho\|_{L^2} + \|\nabla \varrho\|_{H^1} \|\nabla^3 u\|_{L^2} + \|\varrho\|_{H^2} \|\nabla^3 u\|_{L^2}) \|\nabla^2 \varrho\|_{L^2}\]

\[\lesssim (\|\nabla^2 \varrho\|_{L^2} \|\nabla^3 \varrho\|_{L^2} + \|\nabla \varrho\|_{H^1} \|\nabla^3 u\|_{L^2} + \|\varrho\|_{H^2} \|\nabla^3 u\|_{L^2})\]

\[\lesssim \delta (\|\nabla^2 \varrho\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2).\]
Integrating by part and applying Hölder, Sobolev and Cauchy inequalities, we find

\[- \int \nabla^2(u \cdot \nabla \theta) \nabla^2 \theta \, dx\]

\[= \int \left[ -(\nabla^2 u \nabla \theta + 2 \nabla u \nabla^2 \theta) \nabla^2 \theta + \frac{1}{2} |\nabla^2 \theta|^2 \div u \right] \, dx\]

\[\lesssim \left( \| \nabla^2 u \|_{L^6} \| \nabla \theta \|_{L^3} + \| \nabla u \|_{L^\infty} \| \nabla^2 \theta \|_{L^2} \right) \| \nabla^2 \theta \|_{L^2} \]

\[\lesssim \| \nabla \theta \|_{H^1} \| \nabla^2 \theta \|_{L^2} \| \nabla^3 u \|_{L^2} + \| \nabla^2 u \|_{L^2} \| \nabla^3 u \|_{L^2} \]

\[\lesssim (\| \nabla^2 u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2) (\| \nabla^2 \theta \|_{L^2}^2 + \| \nabla^3 u \|_{L^2}^2)\]

\[\lesssim \delta (\| \nabla^2 \theta \|_{L^2}^2 + \| \nabla^3 u \|_{L^2}^2).\]

The combination of (35) and (36) gives rise to

\[\int \nabla^2 S_1 \cdot \nabla^2 \theta \, dx \lesssim \delta (\| \nabla^2 \theta \|_{L^2}^2 + \| \nabla^3 u \|_{L^2}^2).\]

Now, we give the estimate for the second term on the right hand side of (34). By virtue of Hölder and Sobolev inequalities, we have

\[\int \nabla (u \cdot \nabla u) \nabla^3 u \, dx\]

\[= \int (\nabla u \nabla u + u \nabla^2 u) \nabla^3 u \, dx\]

\[\lesssim \| u \|_{L^3} \| \nabla u \|_{L^3} \| \nabla^3 u \|_{L^2} + \| u \|_{L^6} \| \nabla^2 u \|_{L^6} \| \nabla^3 u \|_{L^2}\]

\[\lesssim \| u \|_{L^2}^{\frac{3}{2}} \| \nabla^3 u \|_{L^2} \| \nabla u \|_{L^2}^{\frac{3}{2}} \| \nabla^3 u \|_{L^2} \| \nabla^2 u \|_{L^2} \| \nabla^3 u \|_{L^2}\]

\[\lesssim \delta \| \nabla^3 u \|_{L^2}^2.\]

In view of (15), Hölder and Sobolev inequalities, we get

\[\int \nabla (h(\theta)(\mu \Delta u + (\mu + \nu) \nabla \div u)) \nabla^3 u \, dx\]

\[\lesssim (\| \nabla h(\theta) \|_{L^3} \| \nabla^2 u \|_{L^6} + \| h(\theta) \|_{L^\infty} \| \nabla^3 u \|_{L^2}) \| \nabla^3 u \|_{L^2}\]

\[\lesssim (\| \nabla \theta \|_{H^1} \| \nabla^3 u \|_{L^2} + \| \theta \|_{H^2} \| \nabla^3 u \|_{L^2}) \| \nabla^3 u \|_{L^2}\]

\[\lesssim \delta \| \nabla^3 u \|_{L^2}^2.\]

and

\[\int \nabla (f(\theta) \nabla \theta) \nabla^3 u \, dx\]

\[\lesssim (\| \nabla \theta \|_{L^4}^2 + \| f(\theta) \|_{L^\infty} \| \nabla^2 \theta \|_{L^2}) \| \nabla^3 u \|_{L^2}\]

\[\lesssim (\| \nabla \theta \|_{L^4}^2 \| \nabla^2 \theta \|_{L^2} + \| \theta \|_{H^2} \| \nabla^2 \theta \|_{L^2}) \| \nabla^3 u \|_{L^2}\]

\[\lesssim (\| \nabla \theta \|_{L^4}^2 \| \nabla^2 \theta \|_{L^2} + \| \theta \|_{H^2} \| \nabla^2 \theta \|_{L^2}) \| \nabla^3 u \|_{L^2}\]

\[\lesssim \delta (\| \nabla^2 \theta \|_{L^2}^2 + \| \nabla^3 u \|_{L^2}^2).\]

Similarly, it is easy to deduce

\[\int \left[ g(\theta)(B \cdot \nabla B - \nabla \left( \frac{1}{2} |B|^2 \right)) \right] \nabla^3 u \, dx\]

\[\lesssim (\| g(\theta) \|_{L^6} \| B \cdot \nabla B \|_{L^6} + \| g(\theta) \|_{L^\infty} \| \nabla B \|_{L^3} \| \nabla B \|_{L^6}) \| \nabla^3 u \|_{L^2}\]
\begin{align*}
+ \|g(\varphi)\|_{L^\infty} \|B\|_{L^3} \|
abla^2 B\|_{L^6} \|\nabla^3 u\|_{L^2} \\
\lesssim (\|\nabla \varphi\|_{L^6} \|\nabla B\|_{L^3} + \|B\|_{L^6} \|\nabla \varphi\|_{L^3} \|\nabla^2 B\|_{L^6} \|\nabla^3 B\|_{L^6} \|\nabla^3 u\|_{L^2}) \\
+ \|B\|_{H^1} \|
abla^3 B\|_{L^2} \|\nabla^3 u\|_{L^2} \\
\lesssim \delta(\|\nabla^2 \varphi\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 + \|\nabla^3 B\|_{L^2}^2).
\end{align*}

By virtue of the estimates (38)-(41), we obtain immediately
\begin{equation}
\int \nabla^2 S_2 \cdot \nabla^2 u \, dx \lesssim \delta(\|\nabla^2 \varphi\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 + \|\nabla^3 B\|_{L^2}^2).
\end{equation}

Integrating by part and applying Hölder and Sobolev inequalities, one arrives at
\begin{align*}
\int \nabla^2 (u \cdot \nabla B) \nabla^2 B \, dx \\
\lesssim (\|u\|_{L^3} \|\nabla B\|_{L^6} + \|u\|_{L^3} \|\nabla^2 B\|_{L^6}) \|\nabla^3 B\|_{L^2} \\
\lesssim (\|u\|_{L^2}^\frac{1}{2} \|\nabla^3 u\|_{L^2} \|\nabla B\|_{L^2} \|\nabla^3 B\|_{L^2} + \|u\|_{H^1} \|\nabla^3 B\|_{L^2}) \|\nabla^3 B\|_{L^2} \\
\lesssim \delta(\|\nabla^3 u\|_{L^2}^2 + \|\nabla^3 B\|_{L^2}^2).
\end{align*}

Similarly, it is easy to deduce
\begin{align*}
\int \nabla^2 (B \cdot \nabla u + B \text{div} u) \nabla^2 B \, dx \\
\lesssim (\|u\|_{L^3} \|\nabla B\|_{L^6} + \|B\|_{L^3} \|\nabla^2 u\|_{L^6}) \|\nabla^3 B\|_{L^2} \\
\lesssim (\|u\|_{L^2}^\frac{1}{2} \|\nabla^3 u\|_{L^2} \|\nabla B\|_{L^2} \|\nabla^3 B\|_{L^2} + \|B\|_{H^1} \|\nabla^3 u\|_{L^2}) \|\nabla^3 B\|_{L^2} \\
\lesssim \delta(\|\nabla^3 u\|_{L^2}^2 + \|\nabla^3 B\|_{L^2}^2).
\end{align*}

Integrating by part and applying (15), Hölder and Sobolev inequalities, we find
\begin{align*}
- \int \nabla^2 \text{curl} \left[ g(\varphi) \left( B \cdot \nabla B - \nabla \left( \frac{1}{2} |B|^2 \right) \right) \right] \nabla^2 B \, dx \\
= - \int \nabla^2 \left[ g(\varphi) \left( B \cdot \nabla B - \nabla \left( \frac{1}{2} |B|^2 \right) \right) \right] \nabla^2 \text{curl} B \, dx \\
\lesssim (\|\nabla^2 g(\varphi)\|_{L^2} \|B\|_{L^\infty} \|\nabla B\|_{L^\infty} + \|\nabla^2 g(\varphi)\|_{L^6} \|\nabla B\|_{L^6} \|\nabla B\|_{L^3}) \|\nabla^2 \text{curl} B\|_{L^2} \\
+ (\|\nabla^2 g(\varphi)\|_{L^6} \|B\|_{L^6} \|\nabla^2 B\|_{L^6} + \|g(\varphi)\|_{L^\infty} \|\nabla B\|_{L^3} \|\nabla^2 B\|_{L^6}) \|\nabla^2 \text{curl} B\|_{L^2} \\
+ \|g(\varphi)\|_{L^\infty} \|B\|_{L^\infty} \|\nabla^3 B\|_{L^2} \|\nabla^2 \text{curl} B\|_{L^2} \\
\lesssim (\|\nabla \varphi\|_{L^2}^2 + \|\nabla^2 \varphi\|_{L^2} \|B\|_{L^\infty} \|\nabla B\|_{L^\infty} + \|\nabla \varphi\|_{L^6} \|\nabla B\|_{L^6} \|\nabla B\|_{L^6}) \|\nabla^3 B\|_{L^2} \\
+ (\|\nabla \varphi\|_{L^6} \|B\|_{L^6} \|\nabla^2 B\|_{L^6} + \|\nabla \varphi\|_{L^6} \|\nabla B\|_{L^3} \|\nabla^2 B\|_{L^6} + \|B\|_{H^1} \|\nabla^3 B\|_{L^2}) \|\nabla^3 B\|_{L^2} \\
\lesssim (\|\nabla \varphi\|_{L^2}^2 \|\nabla^2 \varphi\|_{L^2}^2 + \|\nabla^2 \varphi\|_{L^2} \|B\|_{H^2} \|\nabla^2 B\|_{L^2} \|\nabla^3 B\|_{L^2} \\
+ \|\nabla B\|_{H^2} \|\nabla^2 \varphi\|_{L^2} \|\nabla^3 B\|_{L^2} + \|\nabla B\|_{H^1} \|\nabla^3 B\|_{L^2}) \\
\lesssim \delta(\|\nabla^2 \varphi\|_{L^2}^2 + \|\nabla^3 B\|_{L^2}^2).
\end{align*}

In view of the estimates (43)-(45), it is easy to deduce
\begin{equation}
\int \nabla^2 S_3 \cdot \nabla^2 B \, dx \lesssim \delta(\|\nabla^2 \varphi\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 + \|\nabla^3 B\|_{L^2}^2).
\end{equation}
Substituting (37), (42) and (46) into (34), then we have
\[ \frac{d}{dt} \int (|\nabla^2 \varrho|^2 + |\nabla^2 u|^2 + |\nabla^2 B|^2) \, dx + \int (\mu |\nabla^3 u|^2 + |\nabla^3 B|^2) \, dx \lesssim \delta \| \nabla^2 \varrho \|_{L^2}^2, \]
which completes the proof of the lemma.

Finally, we use the equations (10) to recover the dissipation estimate for \( \varrho \).

**Lemma 2.4.** Under the condition (14), then for \( k = 0, 1 \), we have
\[ \frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \varrho \, dx + C \| \nabla^{k+1} \varrho \|_{L^2}^2 \lesssim \| \nabla^{k+1} u \|_{L^2}^2 + \| \nabla^{k+2} u \|_{L^2}^2 + \| \nabla^{k+2} B \|_{L^2}^2. \] (47)

**Proof.** Taking \( k \)-th spatial derivatives to the second equation of (10), multiplying by \( \nabla^{k+1} \varrho \) and integrating over \( \mathbb{R}^3 \), then we obtain
\[ \int \nabla^k u_t \cdot \nabla^{k+1} \varrho \, dx + \int |\nabla^{k+1} \varrho|^2 \, dx = \int \nabla^k [\mu \Delta u + (\mu + \nu) \nabla \text{div} u] \nabla^{k+1} \varrho \, dx + \int \nabla^k S_2 \cdot \nabla^{k+1} \varrho \, dx. \] (48)

In order to deal with the term \( \int \nabla^k u_t \cdot \nabla^{k+1} \varrho \, dx \), one turns the time derivatives of velocity to the density. Then, applying the mass equation (10)_1, one deduces immediately
\[ \int \nabla^k u_t \cdot \nabla^{k+1} \varrho \, dx \\
= \frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \varrho \, dx - \int \nabla^k u \cdot \nabla^{k+1} \varrho_t \, dx \\
= \frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \varrho \, dx + \int \nabla^k u \cdot \nabla^{k+1} (\text{div} u + \text{div}(\varrho u)) \, dx \\
= \frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \varrho \, dx - \int \nabla^k \text{div} u \cdot \nabla^k (\text{div} u + \text{div}(\varrho u)) \, dx \\
= \frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \varrho \, dx - \int |\nabla^k \text{div} u|^2 \, dx - \int \nabla^k \text{div} u \cdot \nabla^k \text{div}(\varrho u) \, dx. \] (49)

Substituting (49) into (48), it is easy to deduce
\[ \frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \varrho \, dx + \int |\nabla^{k+1} \varrho|^2 \, dx \\
= \int |\nabla^k \text{div} u|^2 \, dx + \int \nabla^k \text{div} u \cdot \nabla^k \text{div}(\varrho u) \, dx + \int \nabla^k S_2 \cdot \nabla^{k+1} \varrho \, dx \\
+ \int \nabla^k [\mu \Delta u + (\mu + \nu) \nabla \text{div} u] \nabla^{k+1} \varrho \, dx. \] (50)

For the case \( k = 0 \), then applying Hölder, Sobolev and Cauchy inequalities, we obtain
\[ \int \text{div} u \cdot \text{div}(\varrho u) \, dx \lesssim \| \varrho \|_{L^\infty} \| \nabla u \|_{L^2}^2 + \| u \|_{L^3} \| \text{div} u \|_{L^6} \| \nabla \varrho \|_{L^2} \]
\[ \lesssim (\| \varrho \|_{H^2} + \| u \|_{H^3}) (\| \nabla \varrho \|_{L^2}^2 + \| \nabla^2 u \|_{L^2}^2) \]
\[ \lesssim \delta (\| \nabla \varrho \|_{L^2}^2 + \| \nabla^2 u \|_{L^2}^2). \] (51)
By virtue of (15) and Hölder inequality, it is easy to deduce\[
\int S_2 \cdot \nabla \varrho dx \lesssim (\|\varrho\|_{L^\infty} \|\nabla u\|_{L^6} + \|g\|_{L^\infty} \|\nabla^2 u\|_{L^2}) \|\nabla \varrho\|_{L^2} \\
+ (\|\varrho\|_{L^\infty} \|\nabla \varrho\|_{L^2} + \|g(\varrho)\|_{L^\infty} \|B\|_{L^3} \|\nabla B\|_{L^6}) \|\nabla \varrho\|_{L^2} \\
\lesssim \delta (\|\nabla \varrho\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 B\|_{L^2}^2),\tag{52}
\]
and\[
\int [\mu \Delta u + (\mu + \nu) \nabla \text{div} u] \nabla \varrho dx \lesssim \|\nabla^2 u\|_{L^2}^2 + \varepsilon \|\nabla \varrho\|_{L^2}^2. \tag{53}
\]
The combination of (51), (52) and (53) helps us complete the proof of (47) for the case of \( k = 0 \). As for the case \( k = 1 \), applying Hölder, Sobolev and Cauchy inequalities, we get\[
\int \nabla \text{div} u \cdot \nabla \text{div}(\varrho u) dx \\
\lesssim (\|\nabla \varrho\|_{L^3} \|\text{div} u\|_{L^6} + \|\varrho\|_{L^\infty} \|\nabla \text{div} u\|_{L^2}) \|\nabla^2 u\|_{L^2} \\
+ (\|\nabla \varrho\|_{L^3} \|\nabla u\|_{L^6} + \|\varrho\|_{L^\infty} \|\nabla^2 \varrho\|_{L^2}) \|\nabla^2 u\|_{L^2} \\
\lesssim \delta (\|\nabla^2 \varrho\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2).\tag{54}
\]
With the help of Hölder inequality and Lemma 2.3, we find\[
\int \nabla S_2 \cdot \nabla^2 \varrho dx \lesssim \delta (\|\nabla^2 \varrho\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 + \|\nabla^3 B\|_{L^2}^2), \tag{55}
\]
and\[
\int \nabla [\mu \Delta u + (\mu + \nu) \nabla \text{div} u] \nabla^2 \varrho dx \lesssim \|\nabla^3 u\|_{L^2}^2 + \varepsilon \|\nabla^2 \varrho\|_{L^2}^2. \tag{56}
\]
The combination of (54), (55) and (56) gives rise to the proof of (47) for the case of \( k = 1 \). \( \Box \)

2.2. Global existence of strong solutions. In this subsection, one combines the energy estimates that have been derived in the previous section to prove the global existence of strong solutions in Theorem 1.1. Summing up (17) from \( k = l \) (\( l = 0, 1 \)) to \( k = 1 \), then we obtain\[
\frac{d}{dt} \|\nabla^l (\varrho, u, B)\|_{H^{2-l}}^2 + C \|\nabla^l (\nabla u, \nabla B)\|_{H^{2-l}}^2 \lesssim \delta \|\nabla^{l+1} \varrho\|_{H^{1-l}}^2,
\]
which, together with (33), gives\[
\frac{d}{dt} \|\nabla^l (\varrho, u, B)\|_{H^{2-l}}^2 + C \|\nabla^{l+1} (u, B)\|_{H^{2-l-1}}^2 \leq \delta C_1 \|\nabla^{l+1} \varrho\|_{H^{1-l-1}}^2. \tag{57}
\]
On the other hand, summing (47) from \( k = l \) (\( l = 0, 1 \)) to \( k = 1 \), we obtain immediately\[
\frac{d}{dt} \sum_{l \leq k \leq 1} \int \nabla^k u \cdot \nabla^{k+1} \varrho dx + C_2 \|\nabla^{l+1} \varrho\|_{H^{1-l-1}}^2 \leq C \left( \|\nabla^{l+1} u\|_{H^{2-l-1}}^2 + \|\nabla^{l+1} B\|_{H^{1-l-1}}^2 \right). \tag{58}
\]
Multiplying (58) by \( 2\delta C_1/C_2 \) and adding the resulting inequality to (57), then one arrives at\[
\frac{d}{dt} \|\nabla^{l+1} \varrho\|_{H^{1-l}}^2 + C_3 \left( \|\nabla^{l+1} \varrho\|_{H^{1-l}}^2 + \|\nabla^{l+1} (u, B)\|_{H^{2-l}}^2 \right) \leq 0. \tag{59}
\]
where $E^l_t(t)(l = 0, 1)$ is defined as

$$E^l_t(t) = \|\nabla^l(\varrho, u, B)\|_{H^{l+2}}^2 + \frac{2\delta C_1}{C_2} \sum_{l \leq k \leq 1} \int \nabla^k u \cdot \nabla^{k+1} \varrho dx.$$ 

By virtue of the smallness of $\delta$ in (59), integrating over $[0, t]$ and applying the equivalent relation (60), we obtain

$$ C_4^{-1} \|\nabla^l(\varrho, u, B)\|_{H^{l+2}}^2 \leq E^l_t(t) \leq C_4 \|\nabla^l(\varrho, u, B)\|_{H^{l+2}}^2. \tag{60} $$

Choosing $l = 0$ in (59), integrating over $[0, t]$ and applying the equivalent relation (60), we obtain

$$ \|(\varrho, u, B)(t)\|_{H^2}^2 \leq C\|(\varrho_0, u_0, B_0)\|_{H^2}^2. $$

Then, by the standard continuity argument (see Theorem 7.1 on page 100 in [20]), one closes the estimate (14). Thus, we extend the local strong solutions to be global one and the uniqueness of global strong solutions is guaranteed by the uniqueness of local solutions that has been prove by Fan et al. [9]. Therefore, choosing $l = 0$ in (59), integrating over $[0, t]$ and applying the equivalent relation (60), it is easy to deduce

$$ \|(\varrho, u, B)(t)\|_{H^2}^2 + \int_0^t \left( \|\nabla \varrho(\tau)\|_{H^2}^2 + \|\nabla \nabla B(\tau)\|_{H^2}^2 \right) d\tau \leq C\|(\varrho_0, u_0, B_0)\|_{H^2}^2, \tag{61} $$

which completes the proof of the global existence of strong solutions.

2.3. Decay rates of strong solution. In this section, we establish optimal decay rates for the compressible Hall-MHD equations (1)-(3). If the initial perturbation belongs to $L^1$ additionally, one applies the method of Green function to establish optimal time decay rates for the global strong solutions. Furthermore, the application of Fourier splitting method by Schonbek [23] helps us to build optimal time decay rates for the second order spatial derivatives of magnetic field.

First of all, let us to consider following linearized systems

$$\begin{align*}
&\begin{cases}
\varrho_t + \text{div} u = 0, \\
u_t - \mu \Delta u - (\mu + \nu) \text{div} u + \nabla \varrho = 0, \\
B_t - \Delta B = 0,
\end{cases} \tag{62}
\end{align*}$$

with the initial data

$$ (\varrho, u, B)(x, t)|_{t=0} = (\varrho_0, u_0, B_0)(x) \to (0, 0, 0) \quad \text{as} \quad |x| \to \infty. \tag{63} $$

Obviously, the solution $(\varrho, u, B)$ of the linearized systems (62)-(63) can be expressed as

$$(\varrho, u, B)^{tr} = G(t) * (\varrho_0, u_0, B_0)^{tr}, t \geq 0. \tag{64}$$

Here $G(t) := G(x, t)$ is the Green matrix of the systems (62) and the exact expression of the Fourier transform $\hat{G}(\xi, t)$ of Green function $G(x, t)$ is

$$\hat{G}(\xi, t) = \begin{bmatrix}
\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} & \frac{-i \xi e^{\lambda_+ t} + e^{\lambda_- t}}{\lambda_+ - \lambda_-} + e^{\lambda_- t} \left( I_{3 \times 3} - \frac{\xi \xi^t}{|\xi|^2} \right) & 0 \\
\frac{-i \xi e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} & \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} \xi^t + e^{\lambda_- t} \left( I_{3 \times 3} - \frac{\xi \xi^t}{|\xi|^2} \right) & 0 \\
0 & 0 & e^{\lambda_1 t} I_{3 \times 3}
\end{bmatrix}$$
Proof. Taking for \( k = 0, 1 \) specially in (59), we find directly

\[
\lambda_0 = -\mu|\xi|^2, \quad \lambda_1 = -|\xi|^2,
\]

\[
\lambda_+ = -\left( \frac{1}{2} + \nu \right) |\xi|^2 + \frac{i}{2} \sqrt{|\xi|^2 - \left( \frac{1}{2} + \nu \right)^2} |\xi|^4,
\]

\[
\lambda_- = -\left( \frac{1}{2} + \nu \right) |\xi|^2 - \frac{i}{2} \sqrt{|\xi|^2 - \left( \frac{1}{2} + \nu \right)^2} |\xi|^4.
\]

Since the systems (62) is an independent coupling of the classical linearized Navier-Stokes equations and heat equation, the representation of Green function \( \hat{G}(\xi, t) \) is easy to verify. Furthermore, one also has the following decay rates for the systems (62)-(63), refer to [8].

**Proposition 1.** Assume that \((\varrho, u, B)\) is the solution of the linearized systems (62)-(63) with the initial data \((\varrho_0, u_0, B_0) \in L^1 \cap H^2\), then

\[
\| \nabla^k \varrho \|_{L^2}^2 \leq C \left( \| (\varrho_0, u_0) \|_{L^1}^2 + \| \nabla^k (\varrho_0, u_0) \|_{L^2}^2 \right) (1 + t)^{-\frac{3}{2}-k},
\]

\[
\| \nabla^k u \|_{L^2}^2 \leq C \left( \| (\varrho_0, u_0) \|_{L^1}^2 + \| \nabla^k (\varrho_0, u_0) \|_{L^2}^2 \right) (1 + t)^{-\frac{3}{2}-k},
\]

\[
\| \nabla^k B \|_{L^2}^2 \leq C \left( \| B_0 \|_{L^1}^2 + \| \nabla^k B_0 \|_{L^2}^2 \right) (1 + t)^{-\frac{3}{2}-k}
\]

for \( 0 \leq k \leq 2 \).

In the sequel, we want to verify some estimates that play an important role for us to derive decay rates for the compressible Hall-MHD equations (10)-(13). Precisely, by computing directly, it is easy to check that

\[
\|(S_1, S_2, S_3)\|_{L^1} \lesssim \delta (\| \nabla \varrho \|_{L^2} + \| \nabla u \|_{H^1} + \| \nabla B \|_{H^1}),
\]

\[
\|(S_1, S_2, S_3)\|_{L^2} \lesssim \delta (\| \nabla \varrho \|_{L^2} + \| \nabla u \|_{H^1} + \| \nabla B \|_{H^1}),
\]

\[
\| \nabla (S_1, S_2, S_3) \|_{L^2} \lesssim \delta (\| \nabla^2 \varrho \|_{L^2} + \| \nabla^2 u \|_{L^2} + \| \nabla^2 B \|_{L^2})
\]

\[
+ \| \nabla (\varrho, B) \|_{H^1} \| \nabla^2 (u, B) \|_{H^1}.
\]

(65)

Now, one establishes the decay rates for the compressible Hall-MHD equations (10)-(13).

**Lemma 2.5.** Under the assumptions of Theorem 1.1, the global strong solution \((\varrho, u, B)\) of problem (10)-(13) has the temporal decay rates

\[
\| \nabla^k \varrho(t) \|_{H^{2-k}}^2 + \| \nabla^k u(t) \|_{H^{2-k}}^2 + \| \nabla^k B(t) \|_{H^{2-k}}^2 \leq C (1 + t)^{-\frac{3}{2}-k}
\]

(66)

for \( k = 0, 1 \).

**Proof.** Taking \( l = 1 \) specially in (59), we find directly

\[
\frac{d}{dt} \mathcal{E}_1^2(t) + C_3 \left( \| \nabla^2 \varrho \|_{L^2}^2 + \| \nabla^2 u \|_{H^1}^2 + \| \nabla^2 B \|_{H^1}^2 \right) \leq 0,
\]

(67)

where \( \mathcal{E}_1^2(t) \) is defined as

\[
\mathcal{E}_1^2(t) = \| \nabla \varrho \|_{H^1}^2 + \| \nabla u \|_{H^1}^2 + \| \nabla B \|_{H^1}^2 + \frac{2C_1 \delta}{C_2} \int \nabla u \cdot \nabla \varrho dx.
\]

With the help of Cauchy inequality and smallness of \( \delta \), it is easy to deduce

\[
C_4^{-1} \| \nabla (\varrho, u, B) \|_{H^1}^2 \leq \mathcal{E}_1^2(t) \leq C_4 \| \nabla (\varrho, u, B) \|_{H^1}.
\]
Adding $\|\nabla (\rho, u, B)\|_{L^2}^2$ to both hand sides of (67) and applying the equivalent relation (68), then we have
\[
\frac{d}{dt} E_1^2(t) + CE_1^2(t) \leq \|\nabla (\rho, u, B)\|_{L^2}^2,
\]
which, by integration over $[0, t]$, yields directly
\[
E_1^2(t) \leq E_1^2(0)e^{-Ct} + \int_0^t e^{-C(t-\tau)}\|\nabla (\rho, u, B)(\tau)\|_{L^2}^2 d\tau.
\] (69)

In order to derive the time decay rate for $E_1^2(t)$, one needs to control the term $\|\nabla (\rho, u, B)\|_{L^2}^2$. In fact, by Duhamel principle, one can represent the solutions for the problem (10)-(13) as
\[
(r, u, B)_{t^r}(t) = G(t) * (r_0, u_0, B_0)_{t^r} + \int_0^t G(t-s) * (S_1, S_2, S_3)_{t^r}(s) ds.
\] (70)

Denoting
\[
E(t) = \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{5}{4}} (\|\nabla \rho(\tau)\|_{H^1}^2 + \|\nabla u(\tau)\|_{H^1}^2 + \|\nabla B(\tau)\|_{H^1}^2),
\]
which, together with (70), (65) and Proposition 1, yields directly
\[
\|\nabla (\rho, u, B)(t)\|_{L^2}^2 \leq C(1 + t)^{-\frac{5}{4}} + C\int_0^t \left( \|\nabla (S_1, S_2, S_3)(\tau)\|_{L^1} + \|\nabla (S_1, S_2, S_3)(\tau)\|_{L^2} \right) (1 + t - \tau)^{-\frac{5}{4}} d\tau
\]
\[
\leq C(1 + t)^{-\frac{5}{4}} + C\int_0^t \delta \left( \|\nabla \rho(\tau)\|_{H^1} + \|\nabla u(\tau)\|_{H^1} + \|\nabla B(\tau)\|_{H^1} \right) (1 + t - \tau)^{-\frac{5}{4}} d\tau
\]
\[
\leq C(1 + t)^{-\frac{5}{4}} + C\delta \sqrt{E(t)} \int_0^t (1 + t - \tau)^{-\frac{5}{4}} (1 + \tau)^{-\frac{5}{4}} d\tau
\]
\[
\leq C(1 + t)^{-\frac{5}{4}} + C\delta \sqrt{E(t)} (1 + t)^{-\frac{5}{4}} \leq (1 + t)^{-\frac{5}{4}} (1 + \delta \sqrt{E(t)}),
\]
where we have used the basic fact
\[
\int_0^t (1 + t - \tau)^{-r}(1 + \tau)^{-r} d\tau
\]
\[
= \int_0^t \left( (1 + t - \tau)^{-r} - (1 + \tau)^{-r} d\tau \right)
\]
\[
\leq \left( 1 + \frac{t}{2} \right)^{-r} \int_0^t (1 + \tau)^{-r} d\tau + \left( 1 + \frac{t}{2} \right)^{-r} \int_\frac{t}{2}^t (1 + t - \tau)^{-r} d\tau
\]
\[
\leq (1 + t)^{-r},
\]
for $r = \frac{5}{2}$ and $r = \frac{5}{4}$ respectively. Thus, we have the estimate
\[
\|\nabla (\rho, u, B)(t)\|_{L^2}^2 \leq C(1 + t)^{-\frac{5}{4}} (1 + \delta E(t)).
\] (71)
Inserting (71) into (69), it follows immediately
\[ E_1^2(t) \leq E_1^2(0)e^{-Ct} + C \int_0^t e^{-C(t-\tau)}(1+\tau)^{-\frac{3}{2}}(1+\delta E(\tau))d\tau \]
\[ \leq E_1^2(0)e^{-Ct} + C(1+\delta E(t)) \int_0^t e^{-C(t-\tau)}(1+\tau)^{-\frac{3}{2}}d\tau \]
\[ \leq E_1^2(0)e^{-Ct} + C(1+\delta E(t))(1+t)^{-\frac{3}{2}} \]
\[ \leq C(1+\delta E(t))(1+t)^{-\frac{3}{2}}, \]
where one has used the basic fact
\[ \int_0^t e^{-C(t-\tau)}(1+\tau)^{-\frac{3}{2}}d\tau \]
\[ = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t e^{-C(t-\tau)}(1+\tau)^{-\frac{3}{2}}d\tau \]
\[ \leq e^{-\frac{t}{2}} \int_0^{\frac{t}{2}} (1+\tau)^{-\frac{3}{2}}d\tau + \left(1 + \frac{t}{2}\right)^{-\frac{5}{2}} \int_{\frac{t}{2}}^t e^{-C(t-\tau)}d\tau \]
\[ \leq C(1+t)^{-\frac{3}{2}}. \]
Hence, by virtue of the definition of \( E(t) \) and (72), it follows immediately
\[ E(t) \leq C(1+\delta E(t)), \]
which, in view of the smallness of \( \delta \), yields
\[ E(t) \leq C. \]

Therefore, we have the following time decay rates
\[ \|\nabla g(t)\|^2_{L^4} + \|\nabla u(t)\|^2_{H^1} + \|\nabla B(t)\|^2_{H^1} \leq C(1+t)^{-\frac{3}{2}}, \]
\[ (73) \]
On the other hand, by (70), (65), (73) and Proposition 1, it is easy to deduce
\[ \|g, u, B\|^2_{L^2} \]
\[ \leq C(1+t)^{-\frac{3}{2}} + C \int_0^t (\|(S_1, S_2, S_3)\|^2_{L^4} + \|(S_1, S_2, S_3)\|^2_{L^2}) (1+t-\tau)^{-\frac{3}{2}}d\tau \]
\[ \leq C(1+t)^{-\frac{3}{2}} + C \int_0^t \delta (\|\nabla g(\tau)\|^2_{L^2} + \|\nabla u(\tau)\|^2_{H^1} + \|\nabla B(\tau)\|^2_{H^1}) (1+t-\tau)^{-\frac{3}{2}}d\tau \]
\[ \leq C(1+t)^{-\frac{3}{2}} + C \int_0^t (1+t-\tau)^{-\frac{3}{2}} (1+\tau)^{-\frac{3}{2}}d\tau \]
\[ \leq C(1+t)^{-\frac{3}{2}}, \]
where we have used the basic fact
\[ \int_0^t (1+t-\tau)^{-\frac{3}{2}} (1+\tau)^{-\frac{3}{2}}d\tau \leq C(1+t)^{-\frac{3}{2}}. \]
Hence, we have the following decay rate
\[ \|g(t)\|^2_{L^2} + \|u(t)\|^2_{L^2} + \|B(t)\|^2_{L^2} \leq C(1+t)^{-\frac{3}{2}}, \]
\[ (74) \]
Therefore, the combination of (73) and (74) completes the proof of the lemma. □

Finally, one establishes optimal decay rates for the second order derivatives of magnetic field.
Lemma 2.6. Under the assumptions of Theorem 1.1, then the magnetic field has the following decay rate

\[ \| \nabla^2 B(t) \|_{L^2} \leq C(1 + t)^{-\frac{7}{4}}. \]  

(75)

Proof. Taking \( k = 2 \) in (19), it follows immediately

\[ \frac{1}{2} \frac{d}{dt} \int |\nabla^2 B|^2 dx + \int |\nabla^3 B|^2 dx = \int \nabla^2 S_3 \cdot \nabla^2 B dx. \]  

(76)

The application of Hölder, Sobolev and Cauchy inequalities yields that

\[ \int \nabla^2 (B \cdot \nabla u + B\text{div}u) \nabla^2 B \ dx \leq (\|\nabla u\|_{L^3} \|\nabla B\|_{L^6} + \|B\|_{L^\infty} \|\nabla^2 u\|_{L^2}) \|\nabla^3 B\|_{L^2} \]  

(77)

\[ \leq (\|\nabla u\|_{H^1} \|\nabla^2 B\|_{L^2} + \|\nabla B\|_{H^1} \|\nabla^2 u\|_{L^2}) \|\nabla^3 B\|_{L^2} \]

\[ \leq \|\nabla(u,B)\|_{H^1}^2 \|\nabla^2(u,B)\|_{L^2}^2 + \delta \|\nabla^3 B\|_{L^2}^2. \]

It follows from (43) and (45) that

\[ \int \nabla^2 (u \cdot \nabla B) \nabla^2 B \ dx \leq \|\nabla u\|_{H^1}^2 \|\nabla^2 B\|_{L^2}^2 + \delta \|\nabla^3 B\|_{L^2}^2, \]  

(78)

and

\[ \int - \nabla^2 \text{curl} \left[ g(\varphi) \left( B \cdot \nabla B - \nabla \left( \frac{1}{2} |B|^2 \right) \right) \right] \nabla^2 B \ dx \leq \|\nabla^2 \varphi\|_{L^2}^2 \|\nabla B\|_{H^1}^2 \]  

(79)

Substituting (77)-(79) into (76) and applying the decay rates (66), then we obtain

\[ \frac{d}{dt} \int |\nabla^2 B|^2 dx + \int |\nabla^3 B|^2 dx \leq \|\nabla(u,B)\|_{H^1}^2 \|\nabla^2(u,B)\|_{L^2}^2 + \|\nabla^2 \varphi\|_{L^2}^2 \|\nabla B\|_{H^1}^2 \]  

\[ \leq (1 + t)^{-\frac{7}{2}} (1 + t)^{-\frac{5}{2}} + (1 + t)^{-\frac{5}{2}} (1 + t)^{-\frac{5}{2}} \]

\[ \leq (1 + t)^{-5}. \]  

(80)

For some constant \( R \) defined below, denoting the time sphere (see [23])

\[ S_0 := \left\{ \xi \in \mathbb{R}^3 \mid |\xi| \leq \left( \frac{R}{1 + t} \right)^{\frac{1}{2}} \right\}, \]

it follows immediately

\[ \int_{\mathbb{R}^3} |\nabla^3 B|^2 dx \geq \int_{\mathbb{R}^3/S_0} |\xi|^6 |\hat{B}|^2 d\xi \]

\[ \geq \frac{R}{1 + t} \int_{\mathbb{R}^3/S_0} |\xi|^4 |\hat{B}|^2 d\xi \]

\[ \geq \frac{R}{1 + t} \int_{\mathbb{R}^3} |\xi|^4 |\hat{B}|^2 d\xi - \left( \frac{R}{1 + t} \right)^2 \int_{S_0} |\xi|^2 |\hat{B}|^2 d\xi, \]

or equivalently

\[ \int_{\mathbb{R}^3} |\nabla^3 B|^2 dx \geq \frac{R}{1 + t} \int_{\mathbb{R}^3} |\nabla^2 B|^2 dx - \left( \frac{R}{1 + t} \right)^2 \int_{\mathbb{R}^3} |\nabla B|^2 dx. \]  

(81)
The combination of (80), (81) and (66) yields directly
\[ \frac{d}{dt} \int |\nabla^2 B|^2 dx + \frac{4}{1+t} \int |\nabla^2 B|^2 dx \leq \frac{16}{(1+t)^2} \int |\nabla B|^2 dx + C(1+t)^{-5} \]
\[ \lesssim (1+t)^{-2}(1+t)^{-\frac{3}{2}} + (1+t)^{-5} \]
\[ \leq C(1+t)^{-\frac{7}{2}}, \]
where we have chosen \( R = 4 \) in (81). Multiplying (82) by \((1+t)^4\), we obtain
\[ \frac{d}{dt} \left[(1+t)^4 \|\nabla^2 B\|_{L^2}^2\right] \leq C(1+t)^{-\frac{1}{2}}. \] (83)

Integrating (83) over \([0, t]\), then we have the following decay rate
\[ \|\nabla^2 B(t)\|_{L^2}^2 \leq C(1+t)^{-\frac{7}{2}}. \]

Therefore, we complete the proof of the lemma.

Proof of Theorem 1.1. With the help of (61), Lemma 2.5 and Lemma 2.6, we complete the proof of Theorem 1.1.

2.4. Proof of Theorem 1.2. In this subsection, one establishes the decay rates for the time derivatives of strong solutions.

Lemma 2.7. Under the assumptions of Theorem 1.1, the global strong solution \((\rho, u, B)\) of problem (10)-(13) satisfies
\[ \|\varrho_t(t)\|_{H^1} + \|u_t(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}}, \]
\[ \|B_t(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}}. \] (84)

Proof. By virtue of the equation (10) \(1\) and decay rates (7), we find
\[ \|\varrho_t\|_{L^2} = \|\text{div} u + \text{div} u + \rho \nabla \varrho + \nabla \varrho + S_2\|_{L^2} \]
\[ \lesssim \|\text{div} u\|_{L^2} + \|
abla \varrho\|_{L^2} \|\nabla u\|_{L^2} + \|\varrho\|_{L^\infty} \|\text{div} u\|_{L^6} \]
\[ \lesssim \|
abla^2 u\|_{L^2} + \|\varrho, u\|_{H^2} \|
abla^2 (\varrho, u)\|_{L^2} \]
\[ \leq C(1+t)^{-\frac{3}{4}}. \] (85)

Similarly, it follows immediately
\[ \|\nabla \varrho_t\|_{L^2} = \|\nabla \text{div} u + \nabla \varrho \nabla u + \rho \nabla \varrho + \nabla \varrho + \nabla^2 \varrho\|_{L^2} \]
\[ \lesssim \|
abla \text{div} u\|_{L^2} + \|
abla \varrho\|_{L^2} \|
abla u\|_{L^2} + \|\varrho, u\|_{L^\infty} \|
abla^2 (\varrho, u)\|_{L^2} \]
\[ \lesssim \|
abla^2 u\|_{L^2} + \|\varrho, u\|_{H^2} \|
abla^2 (\varrho, u)\|_{L^2} \]
\[ \leq C(1+t)^{-\frac{3}{4}}. \] (86)

In view of the equation (10) \(2\), decay rates (7) and estimate (65), we have
\[ \|u_t\|_{L^2} = \|\mu \Delta u + (\mu + \nu) \nabla \text{div} u - \nabla \varrho + S_2\|_{L^2} \]
\[ \lesssim \|
abla^2 u\|_{L^2} + \|
abla \varrho\|_{L^2} + \delta(\|\nabla \varrho\|_{L^2} + \|
abla (u, B)\|_{H^1}) \]
\[ \lesssim (1+t)^{-\frac{3}{4}} + (1+t)^{-\frac{7}{4}} \]
\[ \leq C(1+t)^{-\frac{3}{4}}. \] (87)
By virtue of (10), (7), Hölder and Sobolev inequalities, we obtain
\[
\|B_t\|_{L^2} = \left\|\Delta B - u \cdot \nabla B + B \cdot \nabla u - B \text{div} u - \text{curl} \left[ g(\varrho) \left( B \cdot \nabla B - \frac{1}{2} \nabla (|B|^2) \right) \right] \right\|_{L^2} \\
\lesssim \|\Delta B\|_{L^2} + \|u\|_{L^3} \|\nabla B\|_{L^6} + \|\nabla u\|_{L^6} + \|\nabla g(\varrho)\|_{L^2} \|B\|_{L^6} \|\nabla B\|_{L^6} \\
+ \|g(\varrho)\|_{L\infty} \|\nabla B\|_{L^3} \|\nabla B\|_{L^6}^2 + \|\varrho g\|_{L^2} \|\nabla B\|_{H^1} \|\nabla B\|_{H^2} \|\nabla^2 B\|_{L^2} \\
\lesssim \|\nabla^2 B\|_{L^2} + \|(u, B)\|_{H^1} \|\nabla^2 (u, B)\|_{L^2} + \|\nabla \varrho\|_{L^2} \|\nabla B\|_{H^1} \|\nabla B\|_{H^2} \|\nabla^2 B\|_{L^2} \\
\lesssim (1 + t)^{-\frac{3}{4}} + (1 + t)^{-\frac{5}{4}} (1 + t)^{-\frac{7}{4}} + (1 + t)^{-\frac{3}{4}} (1 + t)^{-\frac{3}{4}} \\
\leq C(1 + t)^{-\frac{3}{4}}. 
\]

In view of the decay rates (85)-(88), we complete the proof of the lemma. \(\square\)

**Proof of Theorem 1.2.** With the help of Lemma 2.7, we complete the proof of Theorem 1.2. \(\square\)

### 3. Proof of Theorem 1.3 and Theorem 1.4.

In this section, one first establishes optimal time decay rates for the higher order spatial derivatives of global classical solutions under the condition of small initial perturbation in \(H^1\)-norm and finite initial perturbation in \(L^1\)-norm. Furthermore, one also studies the decay rates for the mixed space-time derivatives of global classical solutions.

First of all, Fan et al. (see (3.2) on Page 430 in [9]) have established following estimate
\[
\|(\varrho, u, B)(t)\|_{H^3} \leq C \|(\varrho_0, u_0, B_0)\|_{H^3} \leq C \varepsilon_0. \tag{89}
\]

Thus, the inequality (15) also holds under the condition of (8).

#### 3.1. Proof of Theorem 1.3.

Just following the idea as Lemma 2.5, it is easy to establish optimal decay rates for the global classical solutions. For the sake of brevity, we only state the results in the following lemma.

**Lemma 3.1.** Under the assumptions of Theorem 1.3, the global classical solution \((\varrho, u, B)\) of problem (10)-(13) satisfies for all \(t \geq 0\),
\[
\|\nabla^k \varrho(t)\|_{H^{3-k}}^2 + \|\nabla^k u(t)\|_{H^{3-k}}^2 + \|\nabla^k B(t)\|_{H^{3-k}}^2 \leq C(1 + t)^{-\frac{3}{4} - k}, \tag{90}
\]
where \(k = 0, 1\).

Next, one establishes optimal time decay rates for the second order spatial derivatives of magnetic field and enhance the time decay rates for the third order spatial derivatives of magnetic field.

**Lemma 3.2.** Under the assumptions of Theorem 1.3, then the magnetic field has following decay rate for all \(t \geq 0\),
\[
\|\nabla^2 B(t)\|_{H^1} \leq C(1 + t)^{-\frac{3}{4}}. \tag{91}
\]

**Proof.** Taking \(k = 3\) in (19), it follows immediately
\[
\frac{1}{2} \frac{d}{dt} \int |\nabla^3 B|^2 dx + \int |\nabla^4 B|^2 dx \\
= \int \nabla^3 \left[ -u \cdot \nabla B + B \cdot \nabla u - B \text{div} u - \text{curl} \left( g(\varrho) \left( B \cdot \nabla B - \frac{1}{2} \nabla (|B|^2) \right) \right) \right] \nabla^3 B dx \\
= III_1 + III_2 + III_3 + III_4. \tag{92}
\]
By virtue of (89), Hölder and Sobolev inequalities, we find

\[
III_1 = \int (\nabla^2 u \nabla B + 2 \nabla u \nabla^2 B + u \nabla^3 B) \nabla^4 B dx
\]

\[
\lesssim (\|\nabla^2 u\|_{L^1}\|\nabla^2 B\|_{L^\infty} + \|\nabla u\|_{L^2}\|\nabla^2 B\|_{L^6} + \|u\|_{L^3}\|\nabla^3 B\|_{L^6}) \|\nabla^4 B\|_{L^2} (93)
\]

\[
\lesssim \|\nabla^2 u\|_{L^1}^2 \|\nabla^2 B\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \|\nabla^3 B\|_{L^2}^2 + (\varepsilon + \varepsilon_0) \|\nabla^4 B\|_{L^2}^2
\]

\[
\lesssim \|\nabla^2 u\|_{H^2}^2 \|\nabla^2 B\|_{H^1}^2 + (\varepsilon + \varepsilon_0) \|\nabla^4 B\|_{L^2}^2.
\]

In view of the Sobolev and Cauchy inequalities, it is easy to deduce

\[
III_2 = -\int (\nabla^2 B \nabla u + 2 \nabla^3 u \nabla^2 B + B \nabla^3 u) \nabla^4 B dx
\]

\[
\lesssim (\|\nabla^3 B\|_{L^6} \|\nabla^2 u\|_{L^2} + \|\nabla^2 B\|_{L^6} \|\nabla^2 u\|_{L^2} + \|B\|_{L^\infty} \|\nabla^3 u\|_{L^2}) \|\nabla^4 B\|_{L^2} (94)
\]

\[
\lesssim \|\nabla^2 u\|_{H^1}^2 \|\nabla^3 u\|_{H^1}^2 + \|\nabla^2 B\|_{H^1}^2 \|\nabla^3 u\|_{L^2}^2 + \|\nabla^4 B\|_{H^2}^2.
\]

In the same manner, we get

\[
III_3 \lesssim \|\nabla^2 B\|_{H^1}^2 \|\nabla^3 u\|_{H^1}^2 + \|\nabla^2 B\|_{H^1}^2 \|\nabla^3 u\|_{L^2}^2 + \varepsilon \|\nabla^4 B\|_{L^2}^2. (95)
\]

Applying (15), (89), Hölder, Sobolev and Cauchy inequalities, it is easy to deduce

\[
III_4 \lesssim (\|\nabla^3 g(\varrho)\|_{L^6} \|\nabla B\|_{L^\infty} \|\nabla^2 B\|_{L^\infty} + \|\nabla^2 g(\varrho)\|_{L^6} \|\nabla B\|_{L^6} \|\nabla^2 B\|_{L^6} \|\nabla^3 B\|_{L^6}) \|\nabla^4 B\|_{L^2}
\]

\[
+ (\|\nabla^3 g(\varrho)\|_{L^6} \|\nabla B\|_{L^6} \|\nabla^2 B\|_{L^6} + \|\nabla^2 g(\varrho)\|_{L^6} \|\nabla B\|_{L^6} \|\nabla^2 B\|_{L^6} \|\nabla^4 B\|_{L^2})
\]

\[
+ (\|\nabla^3 g(\varrho)\|_{L^6} \|\nabla B\|_{L^6} \|\nabla^2 B\|_{L^6} + \|\nabla g(\varrho)\|_{L^\infty} \|\nabla B\|_{L^3} \|\nabla^3 B\|_{L^6} \|\nabla^4 B\|_{L^2})
\]

\[
\lesssim \|\nabla B\|_{H^1}^2 \|\nabla^2 B\|_{H^1}^2 + \|\nabla^2 B\|_{L^2}^2 \|\nabla^2 B\|_{H^1}^2 + \|\nabla B\|_{H^1}^2 \|\nabla^3 B\|_{L^2}^2
\]

\[
+ (\varepsilon + \varepsilon_0) \|\nabla^4 B\|_{L^2}^2.
\]

(96)

Substituting (93)-(96) into (92) and applying the smallness of \(\varepsilon\) and \(\varepsilon_0\), we find

\[
\frac{d}{dt} \int |\nabla^3 B|^2 dx + \int |\nabla^4 B|^2 dx \lesssim \|\nabla (u, B)\|_{H^2}^2 \|\nabla^2 (u, B)\|_{H^1}^2, (97)
\]

which, together with the time decay rates (90), yields directly

\[
\frac{d}{dt} \int |\nabla^3 B|^2 dx + \int |\nabla^4 B|^2 dx \lesssim (1 + t)^{-5}. (98)
\]

Similar to (81), it is easy to deduce

\[
\int |\nabla^4 B|^2 dx \geq \frac{5}{1 + t} \int |\nabla^3 B|^2 dx - \left(\frac{5}{1 + t}\right)^2 \int |\nabla^2 B|^2 dx. (99)
\]

The combination of (90), (98) and (99) gives

\[
\frac{d}{dt} \int |\nabla^3 B|^2 dx + \frac{5}{1 + t} \int |\nabla^3 B|^2 dx \lesssim \frac{25}{(1 + t)^2} \int |\nabla^2 B|^2 dx + (1 + t)^{-5}
\]

\[
\lesssim (1 + t)^{-2} (1 + t)^{-\frac{5}{2}} + (1 + t)^{-5}
\]

\[
\lesssim (1 + t)^{-2},
\]
which, together with (82), yields immediately
\[
\frac{d}{dt} \int (\nabla^2 B)^2 + |\nabla^3 B|^2) dx + \frac{4}{1 + t} \int (\nabla^2 B)^2 + |\nabla^3 B|^2) dx \lesssim (1 + t)^{-\frac{5}{2}}. \tag{100}
\]
Multiplying (100) by \((1 + t)^4\), then we obtain
\[
\frac{d}{dt} \left[ (1 + t)^4 \|\nabla^2 B\|_{H^1}^2 \right] \leq C (1 + t)^{-5} + \varepsilon_0 \|\nabla^3 \varrho\|_{L^2}^2,
\]
which, integrating over \([0, t]\), gives
\[
\|\nabla^2 B(t)\|_{H^1}^2 \leq C (1 + t)^{-\frac{7}{2}}.
\]
Therefore, we complete the proof of the lemma.

In order to establish optimal decay rate for the third order spatial derivatives of magnetic field, we need to improve the decay rate for the second and third order spatial derivatives of velocity. Indeed, following the idea as the compressible MHD equations (see [15]), it is easy to check that the decay rates (90) holds on for \(k = 2\).

For the convenience of readers, we also introduce the method to improve the decay rates for the second order spatial derivatives of density and velocity here.

**Lemma 3.3.** Under the assumptions of Theorem 1.3, the global classical solution \((\varrho, u, B)\) of Cauchy problem (10)-(13) has
\[
\frac{d}{dt} \|\nabla^2 (\varrho, u)\|_{H^1}^2 + \mu \|\nabla^3 u\|_{H^1}^2 \leq C_5 \left[ (1 + t)^{-5} + \varepsilon_0 \|\nabla^3 \varrho\|_{L^2}^2 \right]. \tag{101}
\]

**Proof.** Taking \(k = 2\) specially in (18), then we get
\[
\frac{1}{2} \frac{d}{dt} \int (|\nabla^2 \varrho|^2 + |\nabla^2 u|^2) dx + \int (\mu |\nabla^3 u|^2 + (\mu + \nu) |\nabla^2 \text{div} u|^2) dx = \int \nabla^2 S_1 \cdot \nabla^2 \varrho dx + \int \nabla^2 S_2 \cdot \nabla^2 u dx. \tag{102}
\]
Integrating by part and applying (90), Hölder, Sobolev and Cauchy inequalities, we obtain
\[
\int \nabla^2 S_1 \cdot \nabla^2 \varrho dx = \int \nabla (\varrho \text{div} u + u \cdot \nabla \varrho) \cdot \nabla^3 \varrho dx \tag{103}
\]
\[
\lesssim (\|\nabla \varrho\|_{L^3} \|\nabla u\|_{L^6} + |\nabla^2 u|_{L^3} \|\varrho\|_{L^6} + |\nabla^2 \varrho|_{L^3} \|\text{div} u\|_{L^6}) \|\nabla^3 \varrho\|_{L^2} \lesssim (1 + t)^{-5} + \varepsilon \|\nabla^3 \varrho\|_{L^2}^2.
\]
From the estimates (38) and (39), it is easy to deduce
\[
\int \nabla^2 [-u \cdot \nabla u - h(\varrho) |\mu \Delta u + (\mu + \nu) \nabla \text{div} u|] \nabla^2 u dx \lesssim \varepsilon_0 \|\nabla^3 u\|_{L^2}^2. \tag{104}
\]
Integrating by part and applying (15), (90), Hölder and Sobolev inequalities, we find
\[
\int \nabla^2 [-f(\varrho) \nabla \varrho] \nabla^2 u dx \lesssim (\|f(\varrho)\|_{L^\infty} \|\nabla^2 \varrho\|_{L^2} + |\nabla f(\varrho)|_{L^3} \|\nabla \varrho\|_{L^6}) \|\nabla^3 u\|_{L^2}.
\]
Applying the Hölder and Sobolev inequalities, it is easy to deduce
\[
\|\nabla \varrho\|_{L^\infty} + \|\nabla \varrho\|_{L^3} \|\nabla^2 \varrho\|_{L^2} \|\nabla^3 u\|_{L^2} 
\lesssim \|\nabla \varrho\|_{L^3}^2 \|\nabla^2 \varrho\|_{L^2}^2 + \varepsilon \|\nabla^3 u\|_{L^2}^2
\]
\[
\lesssim (1 + t)^{-5} + \varepsilon \|\nabla^3 u\|_{L^2}^2.
\]  
(105)

In the same manner, it is easy to deduce
\[
\int \nabla^2 \left[ g(\varrho) \left( B \cdot \nabla B - \frac{1}{2} \nabla (|B|^2) \right) \right] \nabla^2 u \, dx
\approx \int \left( \nabla g(\varrho) B \nabla B + g(\varrho) \nabla B \nabla B + g(\varrho) B \nabla^2 B \right) \nabla^3 u \, dx
\lesssim (\|\nabla \varrho\|_{L^6} \|B\|_{L^6} \|\nabla B\|_{L^6} + \|g(\varrho)\|_{L^\infty} \|\nabla B\|_{L^3} \|\nabla B\|_{L^2}) \|\nabla^3 u\|_{L^2}
+ \|g(\varrho)\|_{L^\infty} \|B\|_{L^6} \|\nabla^2 B\|_{L^2} \|\nabla^3 u\|_{L^2}
\lesssim \|\nabla^2 \varrho\|_{L^2}^2 \|\nabla B\|_{L^2}^3 \|\nabla^2 B\|_{L^2}^2 + \|\nabla B\|_{L^2} \|\nabla^2 B\|_{L^2}^3
+ \|\nabla B\|_{L^2} \|\nabla^2 B\|_{L^2} \|\nabla^3 u\|_{L^2}
\lesssim (1 + t)^{-5} + \varepsilon \|\nabla^3 u\|_{L^2}^2.
\]
(106)

In view of the estimates (104) – (106), it is easy to deduce
\[
\int \nabla^2 S_2 \cdot \nabla^2 u \, dx \lesssim (1 + t)^{-5} + \varepsilon_0 \|\nabla^3 u\|_{L^2}^2.
\]  
(107)

Substituting (103) and (107) into (102) and applying the smallness of \(\varepsilon\) and \(\varepsilon_0\), we obtain
\[
\frac{d}{dt} \int (|\nabla^2 \varrho|^2 + |\nabla^2 u|^2) \, dx + \mu \int |\nabla^3 u|^2 \, dx \lesssim (1 + t)^{-5} + \varepsilon \|\nabla^3 \varrho\|_{L^2}^2.
\]  
(108)

Taking \(k = 3\) in (18) specially, then we have
\[
\frac{1}{2} \frac{d}{dt} \int (|\nabla^3 \varrho|^2 + |\nabla^3 u|^2) \, dx + \int (\mu |\nabla^4 u|^2 + (\mu + \nu) |\nabla^5 \text{div} u|^2) \, dx
= \int \nabla^3 (-\varrho \text{div} u - u \cdot \nabla \varrho) \nabla^4 \varrho \, dx + \int \nabla^3 (-u \cdot \nabla u) \nabla^3 u \, dx
+ \int \nabla^3 \left[ h(\varrho)(\mu \Delta u + (\mu + \nu) \nabla \text{div} u) \right] \nabla^3 u \, dx
+ \int \nabla^3 \left[ -f(\varrho) \nabla \varrho + g(\varrho) \left( B \cdot \nabla B - \frac{1}{2} \nabla (|B|^2) \right) \right] \nabla^3 u \, dx
= IV_1 + IV_2 + IV_3 + IV_4 + IV_5 + IV_6 + IV_7.
\]  

Applying the Hölder and Sobolev inequalities, it is easy to deduce
\[
IV_1 \lesssim (\|\nabla^3 \varrho\|_{L^2} \|\nabla u\|_{L^\infty} + \|\nabla^2 \varrho\|_{L^2} \|\nabla^3 u\|_{L^1}) \|\nabla^3 \varrho\|_{L^2}
+ (\|\nabla \varrho\|_{L^3} \|\nabla^3 u\|_{L^6} + \|\varrho\|_{L^\infty} \|\nabla^3 u\|_{L^2}) \|\nabla^3 \varrho\|_{L^2}
\lesssim \varepsilon_0 (\|\nabla^3 \varrho\|_{L^2}^2 + \|\nabla^4 u\|_{L^2}^2).
\]  
(110)

Similarly, it is easy to check that
\[
IV_2 \lesssim \varepsilon_0 (\|\nabla^3 \varrho\|_{L^2}^2 + \|\nabla^4 u\|_{L^2}^2).
\]  
(111)
Integrating by part and applying decay rates (90), Hölder and Sobolev inequalities, one arrives at

\[ IV_3 = \int \nabla^2(u \cdot \nabla u) \nabla^4 u \, dx \]

\[ \lesssim (\| \nabla u \|_{L^3} \| \nabla^2 u \|_{L^6} + \| u \|_{L^3} \| \nabla^3 u \|_{L^6}) \| \nabla^4 u \|_{L^2} \]

\[ \lesssim \| \nabla u \|_{L^2}^2 \| \nabla^3 u \|_{L^2}^2 (\varepsilon + \varepsilon_0) \| \nabla^4 u \|_{L^2}^2 \]

\[ \lesssim (1 + t)^{-5} + (\varepsilon + \varepsilon_0) \| \nabla^4 u \|_{L^2}^2. \]

In view of (15), (90), Hölder and Sobolev inequalities, we obtain

\[ IV_4 \approx \int \nabla^2(h(q) \nabla^3 u) \nabla^4 u \, dx \]

\[ = \int (\nabla^2 h(q) \nabla^2 u + 2 \nabla h(q) \nabla^3 u + h(q) \nabla^4 u) \nabla^4 u \, dx \]

\[ \lesssim (\| \nabla q \|_{L^6} \| \nabla^2 u \|_{L^6} + \| \nabla^2 q \|_{L^6} \| \nabla^3 u \|_{L^6}) \| \nabla^4 u \|_{L^2} \]

\[ + (\| \nabla q \|_{L^6} \| \nabla^3 u \|_{L^6} + \| h(q) \| \| \nabla^4 u \|_{L^2}) \| \nabla^4 u \|_{L^2} \]

\[ \lesssim \| \nabla q \|_{H^1}^2 \| \nabla^3 u \|_{L^2}^2 + (\varepsilon + \varepsilon_0) \| \nabla^4 u \|_{L^2}^2 \]

\[ \lesssim (1 + t)^{-5} + (\varepsilon + \varepsilon_0) \| \nabla^4 u \|_{L^2}^2. \]

Similarly, it is easy to deduce immediately

\[ IV_5 = \int (f(q) \nabla^3 q + 2 \nabla f(q) \nabla^2 q + \nabla^2 f(q) \nabla q) \nabla^4 u \, dx \]

\[ \lesssim (\| f(q) \|_{L^\infty} \| \nabla^3 q \|_{L^2} + \| \nabla f(q) \|_{L^\infty} \| \nabla^2 q \|_{L^6} + \| \nabla^2 f(q) \|_{L^3} \| \nabla q \|_{L^6}) \| \nabla^4 u \|_{L^2} \]

\[ \lesssim \| \nabla q \|_{H^1} \| \nabla^3 q \|_{L^2}^2 + \| \nabla^2 q \|_{L^6}^2 + \| \nabla^2 q \|_{L^3}^2 \| \nabla^2 q \|_{L^6} + \varepsilon \| \nabla^4 u \|_{L^2}^2 \]

\[ \lesssim (1 + t)^{-5} + \varepsilon \| \nabla^4 u \|_{L^2}^2. \]

Integrating by part and applying (15), (90), Hölder and Sobolev inequalities, we get

\[ IV_6 = -\int (\nabla^2 g(q) B \nabla B + 2 \nabla g(q) \nabla (B \nabla B) + g(q) \nabla^2 (B \nabla B)) \nabla^4 u \, dx \]

\[ \lesssim (\| \nabla^2 g(q) \|_{L^6} \| B \|_{L^6} \| \nabla B \|_{L^6} + \| \nabla g(q) \|_{L^6} \| \nabla B \|_{L^6}^2) \| \nabla^4 u \|_{L^2} \]

\[ + (\| \nabla g(q) \|_{L^6} \| B \|_{L^6} \| \nabla^2 B \|_{L^6} + \| g(q) \|_{L^6} \| \nabla B \|_{L^6} \| \nabla^2 B \|_{L^6}) \| \nabla^4 u \|_{L^2} \]

\[ + \| g(q) \|_{L^6} \| B \|_{L^6} \| \nabla^3 B \|_{L^3} \| \nabla^4 u \|_{L^2} \]

\[ \lesssim \| \nabla B \|_{L^2}^2 \| \nabla^2 B \|_{L^2}^2 + \| \nabla^2 B \|_{L^2} \| \nabla^2 B \|_{L^2} \| \nabla^4 u \|_{L^2} + \varepsilon \| \nabla^4 u \|_{L^2} \]

\[ \lesssim \| \nabla B \|_{H^1} \| \nabla^2 B \|_{L^2}^2 + \varepsilon \| \nabla^4 u \|_{L^2} \]

\[ \lesssim (1 + t)^{-5} + \varepsilon \| \nabla^4 u \|_{L^2} \]

In the same manner, it is easy to deduce

\[ IV_7 \lesssim (1 + t)^{-5} + \varepsilon \| \nabla^4 u \|_{L^2}^2. \]

Substituting (110)-(116) into (109), we find

\[ \frac{d}{dt} \int (|\nabla^3 q|^2 + |\nabla^3 u|^2)^2 \, dx + \mu \int |\nabla^4 u|^2 \, dx \lesssim (1 + t)^{-5} + \varepsilon_0 \|\nabla^3 q\|_{L^2}^2, \]

which, together with (108), completes the proof of the lemma.

Next, we establish an inequality to recover the dissipation estimate for \( \varrho \).
Lemma 3.4. Under the assumptions in Theorem 1.3, the global classical solution \((\varrho, u, B)\) of Cauchy problem (10)-(13) satisfies
\[
\frac{d}{dt} \int \nabla^2 u \cdot \nabla^3 \varrho dx + C_0 \int |\nabla^3 \varrho|^2 dx \leq C_T [(1 + t)^{-5} + \|\nabla^3 u\|_{L^2}^2]. \tag{117}
\]

Proof. Taking \(\nabla^2\) operator on both hand sides of (10)\(_1\), multiplying by \(\nabla^3 \varrho\) and integrating over \(\mathbb{R}^3\), then we have
\[
\int (\nabla^2 u_t \cdot \nabla^3 \varrho + \nabla^3 \varrho dx = \int [\mu \Delta \nabla^2 u + (\mu + \nu) \nabla^3 \text{div} u + \nabla^2 S_2] \nabla^3 \varrho dx. \tag{118}
\]
In order to deal with the term \(\int \nabla^2 u_t \cdot \nabla^3 \varrho dx\), we turn the time derivatives of velocity to density and apply the transport equation (10)\(_1\). More precisely, we get
\[
\frac{d}{dt} \int \nabla^2 u_t \cdot \nabla^3 \varrho dx
\]
\[
= \frac{d}{dt} \int \nabla^2 u \cdot \nabla^3 \varrho dx - \int \nabla^2 u \cdot \nabla^3 \varrho_t dx \tag{119}
\]
\[
= \frac{d}{dt} \int \nabla^2 u \cdot \nabla^3 \varrho dx + \int \nabla^2 \text{div} u \cdot \nabla^3 \varrho dx
\]
\[
= \frac{d}{dt} \int \nabla^2 u \cdot \nabla^3 \varrho dx - \int \nabla^2 \text{div} u \cdot \nabla^3 (\text{div} \varrho + \varrho \text{div} u + \varrho) dx.
\]
The combination of (119) and (118) yields directly
\[
\frac{d}{dt} \int \nabla^2 u \cdot \nabla^3 \varrho dx + \int |\nabla^3 \varrho|^2 dx
\]
\[
= \int \nabla^2 \text{div} u |^2 dx + \int \nabla^2 \text{div} u \cdot \nabla^3 (\text{div} \varrho + \varrho \text{div} u + \varrho) dx + \int \nabla^2 S_2 \cdot \nabla^3 \varrho dx \tag{120}
\]
\[
+ \int [\mu \Delta \nabla^2 u + (\mu + \nu) \nabla^3 \text{div} u] \cdot \nabla^3 \varrho dx.
\]
With the help of time decay rates (90), Hölder and Sobolev inequalities, we obtain
\[
\int \nabla^2 \text{div} u \cdot \nabla^3 (\text{div} \varrho + \varrho \text{div} u + \varrho) dx
\]
\[
= - \int \nabla^3 \text{div} u \cdot \nabla (\text{div} \varrho + \varrho \text{div} u + \varrho) dx
\]
\[
\lesssim (\|\nabla \varrho\|_{L^3} \|\nabla u\|_{L^6} + ||\varrho||_{L^6} \|\nabla^2 u\|_{L^3} + \|u\|_{L^6} \|\nabla^2 \varrho\|_{L^3}) \|\nabla^4 u\|_{L^2} \tag{121}
\]
\[
\lesssim ||\nabla \varrho||_{H^1}^2 ||\nabla^2 u||_{H^2}^2 + ||\nabla u||_{L^2}^2 ||\nabla^3 \varrho||_{H^1}^2 + \epsilon ||\nabla^4 u||_{L^2}^2
\]
\[
\lesssim (1 + t)^{-5} + \|\nabla^4 u\|_{L^2}^2.
\]
On the other hand, just following the idea as (112)-(115), we have
\[
\int \nabla^2 S_2 \cdot \nabla^3 \varrho dx \lesssim (1 + t)^{-5} + ||\nabla^4 u||_{L^2}^2 + \epsilon ||\nabla^3 \varrho||_{L^2}^2 \tag{122}
\]
and
\[
\int [\mu \Delta \nabla^2 u + (\mu + \nu) \nabla^3 \text{div} u] \cdot \nabla^3 \varrho dx \lesssim ||\nabla^4 u||_{L^2}^2 + \epsilon ||\nabla^3 \varrho||_{L^2}^2. \tag{123}
\]
Plugging (121)-(123) into (120), we complete the proof of lemma.

Furthermore, one establishes optimal decay rates for the second order spatial derivatives of density and velocity.
Lemma 3.5. Under the assumptions in Theorem 1.3, then the density and velocity have following decay rate

$$\|\nabla^2 \varrho(t)\|_{H^1} + \|\nabla^2 u(t)\|_{H^1} \leq C(1 + t)^{-\frac{3}{2}}$$  \hspace{1cm} (124)

for all $t \geq T^*(T^* \text{ is a constant defined below}).$

Proof. Multiplying $(117)$ by $\frac{2C_{5,0}}{C_6}$ and adding to $(101)$, then we have

$$\frac{d}{dt} \mathcal{E}_2^3(t) + C_8 \int (|\nabla^3 \varrho|^2 + |\nabla^3 u|^2 + |\nabla^4 u|^2) dx \leq C_9 (1 + t)^{-5},$$

where $\mathcal{E}_2^3(t)$ is defined as

$$\mathcal{E}_2^3(t) = \|\nabla^2 \varrho\|_{H^1}^2 + \|\nabla^2 u\|_{H^1}^2 + \frac{2C_{5,0}}{C_6} \int \nabla^2 u \cdot \nabla^3 \varrho dx.$$  \hspace{1cm} (126)

By virtue of the smallness of $\varepsilon_0$, it is easy to deduce immediately

$$C_{10}^{-1} \|\nabla^2 (\varrho, u)\|_{H^1}^2 \leq \mathcal{E}_2^3(t) \leq C_{10} \|\nabla^2 (\varrho, u)\|_{H^1}^2.$$  \hspace{1cm} (127)

It follows directly from (125) that

$$\frac{d}{dt} \mathcal{E}_2^3(t) + \frac{C_8}{2} \int (|\nabla^3 \varrho|^2 + |\nabla^3 u|^2 + |\nabla^4 u|^2) dx \leq C_9 (1 + t)^{-5}.$$  \hspace{1cm} (128)

In the same manner as $(81)$, we get

$$\int |\nabla^3 \varrho|^2 dx \geq \frac{R}{1 + t} \int |\nabla^2 \varrho|^2 dx - \left(\frac{R}{1 + t}\right)^2 \int |\nabla \varrho|^2 dx,$$  \hspace{1cm} (129)

and

$$\|\nabla^3 u\|_{H^1}^2 \geq \frac{R}{1 + t} \|\nabla^2 u\|_{H^1}^2 - \left(\frac{R}{1 + t}\right)^2 \|\nabla u\|_{H^1}^2.$$  \hspace{1cm} (130)

Plugging $(129)$ and $(130)$ into $(128)$, it follows directly

$$\frac{d}{dt} \mathcal{E}_2^3(t) + \frac{C_8}{2} \left[ \frac{R}{1 + t} \int (|\nabla^2 \varrho|^2 + |\nabla^2 u|^2 + |\nabla^3 u|^2) dx + \int |\nabla^3 \varrho|^2 dx \right]$$

$$\lesssim \left(\frac{R}{1 + t}\right)^2 \int (|\nabla \varrho|^2 + |\nabla u|^2 + |\nabla^2 u|^2) dx + (1 + t)^{-5}$$

$$\lesssim (1 + t)^{-2}(1 + t)^{-\frac{3}{2}} + (1 + t)^{-5} \lesssim (1 + t)^{-\frac{5}{2}}.$$  \hspace{1cm} (131)

For some large time $t \geq R - 1$, we find

$$\frac{R}{1 + t} \leq 1,$$

which implies

$$\frac{R}{1 + t} \int |\nabla^3 \varrho|^2 dx \leq \int |\nabla^3 \varrho|^2 dx.$$  \hspace{1cm} (132)

Combining $(131)$ with $(132)$, it is easy to deduce

$$\frac{d}{dt} \mathcal{E}_2^3(t) + \frac{C_8R}{2(1 + t)} \|\nabla^2 (\varrho, u)\|_{H^1}^2 \lesssim (1 + t)^{-\frac{9}{2}},$$

which, together with the equivalent relation $(127)$, yields

$$\frac{d}{dt} \mathcal{E}_2^3(t) + \frac{C_8R}{2C_{10}(1 + t)} \mathcal{E}_2^3(t) \lesssim (1 + t)^{-\frac{9}{2}}.$$  \hspace{1cm} (133)
If choosing \( R = \frac{8C_{10}}{C_8} \) in (133), it is easy to deduce

\[
\frac{d}{dt} \mathcal{E}^3_3(t) + \frac{4}{1+t} \mathcal{E}^3_3(t) \lesssim (1+t)^{-\frac{3}{2}}
\]  
(134)

for all \( t \geq T^* := \frac{8C_{10}}{C_8} - 1 \). Multiplying (134) by \((1+t)^4\), then we have

\[
\frac{d}{dt} [(1+t)^4 \mathcal{E}^3_3(t)] \lesssim (1+t)^{-\frac{3}{2}}.
\]  
(135)

Integrating (135) over \([0,t]\), then we get

\[
\mathcal{E}^3_3(t) \lesssim (1+t)^{-\frac{5}{2}},
\]

which, together with the equivalent relation (127), gives directly

\[
\|\nabla^2 \varrho(t)\|_{L^2} + \|\nabla^2 u(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{2}}.
\]

Therefore, we complete the proof of the lemma.

Finally, one establishes optimal decay rate for the third order spatial derivatives of magnetic field.

**Lemma 3.6.** Under the assumption of Theorem 1.3, then the magnetic field has following decay rate for all \( t \geq T^* \),

\[
\|\nabla^3 B(t)\|_{L^2} \leq C(1+t)^{-\frac{11}{2}}.
\]  
(136)

**Proof.** Applying (97), (91) and (124), it is easy to check that

\[
\frac{d}{dt} \int |\nabla^3 B|^2 dx + \int |\nabla^4 B|^2 dx \\
\lesssim \|\nabla (u, B)\|_{H^1}^2 \|\nabla^2 (u, B)\|_{H^1}^2 \\
\lesssim (1+t)^{-\frac{3}{2}} (1+t)^{-\frac{7}{2}} \\
\lesssim (1+t)^{-6},
\]

which, together with (99), gives directly

\[
\frac{d}{dt} \int |\nabla^3 B|^2 dx + \frac{5}{1+t} \int |\nabla^3 B|^2 dx \\
\lesssim (1+t)^{-2} \|\nabla^2 B\|_{L^2}^2 + (1+t)^{-6} \\
\lesssim (1+t)^{-2} (1+t)^{-\frac{7}{2}} + (1+t)^{-6} \\
\lesssim (1+t)^{-\frac{11}{2}}.
\]  
(137)

Multiplying (137) by \((1+t)^5\) and integrating the resulting inequality over \([0,t]\), we get

\[
\|\nabla^3 B(t)\|_{L^2}^2 \lesssim (1+t)^{-\frac{5}{2}}.
\]

Therefore, we complete the proof of the lemma.

**Proof of Theorem 1.3.** With the help of Lemma 3.1, Lemma 3.2, Lemma 3.5 and Lemma 3.6, we complete the proof of Theorem 1.3.
3.2. Proof of Theorem 1.4. In this section, one establishes the decay rates for the mixed space-time derivatives of global classical solutions.

Lemma 3.7. Under the assumptions in Theorem 1.3, the global classical solution \((\varrho, u, B)\) of Cauchy problem (10)-(13) has the time decay rates
\[
\|\nabla^k \varrho(t)\|_{H^{2-k}} + \|\nabla^k u(t)\|_{L^2} \leq C(1 + t)^{-\frac{5+2k}{2}}, \\
\|\nabla^k B(t)\|_{L^2} \leq C(1 + t)^{-\frac{7+2k}{4}},
\]
where \(k = 0, 1\).

Proof. First of all, applying the estimate (86) and decay rates (9), it is easy to deduce
\[
\|\nabla \varrho(t)\|_{L^2} \lesssim (1 + t)^{-\frac{7}{2}}. \tag{138}
\]
In view of the equation (10), decay rates (9), Hölder and Sobolev inequalities, we get
\[
\|\nabla^2 \varrho(t)\|_{L^2}^2 = \| - \nabla^2 \text{div} u - \nabla^2 (\varrho \text{div} u + u \cdot \nabla \varrho)\|_{L^2}^2 \\
\lesssim \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 (\varrho, u)\|_{L^2} \|\nabla (\varrho, u)\|_{L^2} \tag{139}
\]
Combining (138)-(139) with (85), it is easy to check that
\[
\|\nabla^k \varrho(t)\|_{H^{2-k}}^2 \leq (1 + t)^{-\frac{5+2k}{2}}, \tag{140}
\]
where \(k = 0, 1\). Secondly, in view of the equation (10), (65) and Hölder inequality, we obtain
\[
\|\nabla u(t)\|_{L^2}^2 = \|\mu \Delta \nabla u + (\mu + \nu) \nabla^2 \text{div} u - \nabla^2 \varrho + \nabla S_2\|_{L^2}^2 \\
\lesssim \|\nabla^3 u\|_{L^2}^2 + \|\nabla^2 \varrho\|_{L^2}^2 + \|\nabla (\varrho, B)\|_{H^1} \|\nabla^2 (\varrho, B)\|_{H^1} \tag{141}
\]
which, together with (87), yields directly
\[
\|\nabla^k u(t)\|_{L^2}^2 \leq C(1 + t)^{-\frac{5+2k}{2}}, \tag{141}
\]
where \(k = 0, 1\). Finally, it follows from (10), (77)-(79), Hölder and Sobolev inequalities that
\[
\|\nabla B(t)\|_{L^2}^2 = \|\nabla \Delta B + \nabla S_2\|_{L^2}^2 \\
\lesssim \|\nabla^3 B\|_{L^2}^2 + \|\nabla (u, B)\|_{H^1} \|\nabla^2 (u, B)\|_{L^2} + \|\nabla^2 \varrho\|_{L^2} \|\nabla B\|_{H^1} \tag{142}
\]
which, together with (88), gives directly
\[
\|\nabla^k B(t)\|_{L^2}^2 \leq C(1 + t)^{-\frac{7+2k}{4}}, \tag{142}
\]
where \(k = 0, 1\). Combining (140), (141) with (142), then we complete the proof of the lemma. \qed

Proof of Theorem 1.4. With the help of Lemma 3.7, we complete the proof of Theorem 1.4. \qed
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