Abstract. A cosmological model with anisotropic dark energy is analyzed. The amount of deviation from isotropy of the equation of state of dark energy, the skewness $\delta$, generates an anisotropization of the large-scale geometry of the Universe, quantifiable by means of the actual shear $\Sigma_0$. Requiring that the level of cosmic anisotropization at the time of decoupling is such to solve the “quadrupole problem” of cosmic microwave background radiation, we find that $|\delta| \sim 10^{-4}$ and $|\Sigma_0| \sim 10^{-5}$, compatible with existing limits derived from the magnitude-redshift data on type Ia supernovae.

Keywords: Dark energy theory

I. INTRODUCTION

The isotropy of the cosmic microwave background (CMB) radiation, first seen by the Cosmic Background Explorer (COBE) satellite [1] and then reinforced by the Wilkinson Microwave Anisotropy Probe (WMAP) data [2], together with the assumption that we are not in any special position in the Universe, underlies the Cosmological Principle, according to which we live in a homogeneous and isotropic universe described by a Robertson-Walker line element.

Tiny deviations from perfect isotropy, at the level of $10^{-5}$, have been also reported by COBE [3] and thereafter confirmed by the high resolution WMAP data. The observed CMB anisotropy spectrum is in impressive agreement with the predictions of the $\Lambda$CDM concordance model, namely the widely accepted standard model of (inflationary) cosmology [4].

Nevertheless, data from 7-yr WMAP observations [5] do not have removed an effect seen at lower significance in the COBE data, that is a lack of power on the CMB quadrupole moment.

Since measurements at the largest angular scales are affected by foreground and other systematic effects, we have to wait for the full results of the PLANCK mission [6] that will check, using a different frequency coverage and scanning strategy than WMAP, the anomalously low quadrupole so far observed.

Surprisingly, however, analyzing the first light sky map released by the PLANCK team, which covers about only the 10% of the sky, Liu and Li [7] have recently claimed that the amplitude of CMB power spectrum on large scales is significantly lower than that reported by the WMAP team.

The “smallness” of CMB quadrupole amplitude, referred to as quadrupole problem, signals a suppression of power at large scales and has been intensively studied [8] mainly because it may signal either a nontrivial topology or a deviation from isotropy of the large scale geometry of the Universe. Indeed, recently enough, it has been shown that a spatially homogeneous but anisotropic cosmological model of Bianchi type I described by a plane-symmetric line element, the ellipsoidal universe [9, 10], allows a better matching of the large-scale CMB anisotropy data.

Various mechanisms could have induced a planar symmetry in the spatial geometry of the universe: topological defects (e.g. cosmic strings, domain walls) [11], a uniform cosmological magnetic field [9, 10], magnetic fields possessing planar symmetry at cosmic scales [12], or a moving [13] dark energy.

In particular, the mechanism proposed by Koivisto and Mota of dark energy with anisotropic equation of state [14, 15] (see also Refs. [16–20]) is very attractive, since cosmic anisotropy originates from the actual dominant component of the Universe and then can be directly tested, for example, by either observations of magnitude and redshift of type Ia supernovae or cosmic parallax effects of distant sources [21].

The aim of this paper is to connect the dark energy anisotropization mechanism to the ellipsoidal universe proposal. As we will see, the deviation from isotropy of the equation of state of dark energy triggers an anisotropization of the Universe which can be well described by an ellipsoidal universe model. The level of such an anisotropization can be then chosen to solve the quadrupole problem.

The plan of the paper is as follows. In section II, we briefly review the ellipsoidal universe proposal. In section III, we introduce a cosmological model with an anisotropic dark energy component and we study its connection with both the ellipsoidal universe model and the quadrupole problem. Finally, we draw our conclusions in section IV.
II. ELLIPSOIDAL UNIVERSE: CMB QUADRUPOLE

Planar symmetry, in a cosmological context, is described by the Taub line element [22]:

\[ ds^2 = dt^2 - a^2(t)(dx^2 + dy^2) - b^2(t)dz^2, \]

where the scale factors \( a \) and \( b \) are normalized as \( a(t_0) = b(t_0) = 1 \) at the present (cosmic) time \( t_0 \).

As shown in Ref. [9], the above metric generates a quadrupole term in the CMB radiation, through the Sachs-Wolfe effect, which adds to that caused by the inflation-produced gravitational potential.

More precisely, let us introduce the CMB temperature anisotropy in terms of spherical harmonics \( Y_{lm} \),

\[ \Delta T(n) = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(n), \]

where \( n \) is a unit direction vector. The coefficients of the expansion, \( a_{lm} \), are related to the angular power spectrum, \( C_l \), through

\[ C_l = \frac{1}{2l+1} \sum_{m=-l}^{l} |a_{lm}|^2, \]

and the quadrupole \((l = 2)\) component is defined as

\[ Q \equiv \sqrt{\frac{3}{4\pi}} C_2. \]

The observed CMB anisotropy map is then a linear superposition of two contributions [22]:

\[ a_{lm} = a_{lm}^A + a_{lm}^I, \]

where \( a_{lm}^A \) comes from the anisotropic spacetime background, while the \( a_{lm}^I \) term is the standard isotropic fluctuation due to inflation-produced gravitational potential at the last scattering surface.

The 7-year WMAP data give, for the quadrupole amplitude, the value [5]

\[ Q_{2,\text{obs}}^2 \simeq 200 \mu K^2, \]

while their best-fit for the isotropic \( \Lambda \text{CDM} \) standard cosmological model gives [5]:

\[ Q_{2,\text{I}}^2 \simeq 1200 \mu K^2. \]

Even taking into account the cosmic variance [24], which dominates theoretical uncertainties at low \( l \),

\[ \sigma_{\text{cosmic}} = \sqrt{\frac{2}{5}} Q_2^2 \simeq 759 \mu K^2, \]

the quadrupole amplitude remains anomalously low.

The quadrupole anisotropy associated to the anisotropic spacetime background has been instead calculated in Ref. [8]:

\[ Q_A = \frac{2}{5\sqrt{3}} T_{\text{cmb}} e_{\text{dec}}^2, \]

where \( T_{\text{cmb}} \simeq 2.73 \) K is the actual (average) CMB temperature and \( e_{\text{dec}} = e(t_{\text{dec}}) \) is the “eccentricity”

\[ e = \begin{cases} \sqrt{1 - (b/a)^2}, & a > b, \\ \sqrt{1 - (a/b)^2}, & a < b, \end{cases} \]

As pointed out in Ref. [15], this is an approximation which is good when the spacetime background anisotropy is small (this will be indeed our case, see section III), and that a rigorous treatment of CMB anisotropies should presuppose the analysis of perturbations in the anisotropic background Eq. [7].
evaluated at the time of decoupling $t_{\text{dec}}$.

The "ellipsoidal universe" proposal could indeed explains the "low" value of the observed quadrupole, Eq. (6), compared to that predicted by the standard cosmological model, Eq. (7). This, however, requires a suitable orientation of the anisotropy associated to planar geometry with respect to the inflation-produced one, in such a way to lower the overall power to a sufficient extent. As shown in Refs. [10, 12] this is attained for eccentricities approximatively given by

$$e_{\text{dec}}^2 \simeq \sqrt{15 \frac{3\sqrt{13} - 5 \text{sgn}(a-b)}{24}} \frac{Q_I}{T_{\text{cmb}}},$$

where sgn $x$ is the sign function (sgn $x = \pm 1$ if $x \gg 0$). In the Appendix we show that the probability that such an orientation occurs by chance is, indeed, not negligible.

In the next Section, we show that an anisotropic dark energy component causes an anisotropization of the Universe described by the line element (1). In particular, the skewness, which parameterizes the deviation from isotropy of the equation of state of dark energy, and the shear, which quantify the level of cosmic anisotropization, will be connected to the amount of eccentricity at decoupling.

**III. ANISOTROPIC DARK ENERGY: SKEWNESS AND SHEAR**

The most general energy-momentum tensor consistent with planar symmetry is

$$T_{\mu \nu} = \text{diag} (\rho, -p^\parallel, -p^\parallel, -p^\perp),$$

where $\rho$ is the energy density, while $p^\parallel$ and $p^\perp$ are the "longitudinal" and "normal" pressures. Taking into account the energy-momentum tensor (12), the Einstein’s equations read

$$3(1 - \Sigma^2)H^2 = 8\pi G\rho,$$

$$3(1 - \Sigma + \Sigma^2)H^2 + [(2 - \Sigma)H] = -8\pi Gp^\parallel,$$

$$3(1 + \Sigma)^2H^2 + 2[(1 + \Sigma)H]^2 = -8\pi Gp^\perp,$$

where a dot denotes a differentiation with respect to the cosmic time. Here, we have introduced, in the usual way, the cosmic shear, $\Sigma$, and the “mean Hubble parameter”, $H$, as

$$\Sigma \equiv (H_a - H)/H, \quad H \equiv \dot{A}/A,$$

where $H_a \equiv \dot{a}/a$ and $A \equiv (a^2b)^{1/3}$ is the “mean expansion parameter”.

Instead of considering Eqs. (15) and (14), we will analyze the equation obtained by subtracting side by side them,

$$3(H\Sigma) + 9H^2\Sigma = 8\pi G(p^\parallel - p^\perp)$$

(from which it is apparent that the source of the shear is proportional to the difference between the longitudinal and normal pressures of the anisotropic fluid), and the equation coming from the conservation of the energy-momentum tensor, $T^\mu_{\nu;\rho} = 0$, which gives

$$\dot{\rho} + 3H \left(\rho + \frac{2p^\parallel + p^\perp}{3}\right) + 2H(p^\parallel - p^\perp)\Sigma = 0.$$

To proceed further, we assume that the anisotropic fluid defined by the energy-momentum tensor (12) is indeed made up of three different components: an isotropic radiation component ($r$), an isotropic dark matter component ($m$), and an anisotropic dark energy (DE) component,

$$\rho = \rho_r + \rho_m + \rho_{\text{DE}},$$

$$p^\parallel = p_r + p_m + p_{\text{DE}},$$

$$p^\perp = p_r + p_m + p_{\text{DE}},$$

with equations of state: $p_r = \rho_r/3$, $p_m = 0$, and

$$p_{\text{DE}} \equiv w\rho_{\text{DE}}, \quad p_{\text{DE}} \equiv w\rho_{\text{DE}}.$$
Moreover, we will assume that the $w^\parallel$ and $w^\perp$ coefficients are constants and that the three components are non-interacting. The latter assumption ensures that each component is separately conserved, so that Eq. (18) gives

$$\rho_r = \rho_r^{(0)} A^{-4}, \quad \rho_m = \rho_m^{(0)} A^{-3},$$

$$\dot{\rho}_{\text{DE}} + [3(1 + w) + 2\delta \Sigma] H \rho_{\text{DE}} = 0,$$  

where from now on an index “0” defines quantities evaluated at the actual time, and we have introduced the “mean equation of state parameter” $w$ and “skewness” $\delta$ as

$$w \equiv (2w^\parallel + w^\perp)/3, \quad \delta \equiv w^\parallel - w^\perp.$$  

(25)

Finally, we assume that $\Sigma$ and $\delta$ are small quantities (as we will verify a posteriori) so that we can neglect the second term in the square brackets of Eq. (24). In this case, Eq. (24) simply gives:

$$\rho_{\text{DE}} = \rho_{\text{DE}}^{(0)} A^{-3(1+w)}.$$  

(26)

Hereafter, we use this approximate result for the evolution of dark energy density.

Introducing the “mean density parameters”

$$\Omega_X \equiv \rho_X^{(0)}/\rho_c^{(0)}, \quad \rho_c^{(0)} \equiv \frac{3H_0^2}{8\pi G},$$

(27)

where $X = r, m, \text{DE}$, and taking into account Eqs. (19)-(27), the shear equation (17) gives

$$\Sigma(A) = \frac{\Sigma_0 + (E - E_0) \delta}{A^3 H/H_0},$$

(28)

where

$$H(A)/H_0 = \sqrt{\Omega_r A^{-4} + \Omega_m A^{-3} + \Omega_{\text{DE}} A^{-3(1+w)}}.$$  

(29)

and we have defined the function

$$E(A) = \Omega_{\text{DE}} \int_0^A \frac{dx}{x^{1+3w} H(x)/H_0}.$$  

(30)

It is worth noting that the density parameters are not all independent, since evaluating Eq. (29) at the present time gives

$$\Omega_r + \Omega_m + \Omega_{\text{DE}} = 1.$$  

(31)

In order to not to spoil the predictions of the standard isotropic model (such as those coming from Big Bang Nucleosynthesis and Large Scale Structure formation), we only consider anisotropic cosmological models which isotropize at early times. Therefore, we impose the isotropization condition:

$$\lim_{A \rightarrow 0} \Sigma(A) = 0.$$  

(32)

Taking into account Eqs. (28)-(30), it is easy to verify that the above condition is satisfied if and only if

$$\Sigma_0 = E_0 \delta,$$  

(33)

where we have taken into account that the dark energy component is subdominant with respect to the radiation one for $A \rightarrow 0$. For completeness, we give the asymptotic expansion of the shear for small values of the expansion parameter:

$$A \ll 1: \quad \Sigma(A) \simeq \frac{\delta}{2 - 3w} \frac{\Omega_{\text{DE}}}{\Omega_r} A^{1 - 3w}.$$  

(34)

Moreover, since dark energy dominates over the other components in the far future, $A \rightarrow \infty$, we get from Eq. (28):

$$\Sigma_\infty \equiv \lim_{A \rightarrow \infty} \Sigma(A) = \frac{2\delta}{3(1 - w)}.$$  

(35)
where we used Eq. (33).

Finally, taking into account Eqs. (28)-(30) and Eqs. (33)-(35), it is possible to show that the following inequality holds:

\[ \max_{A \in (0, \infty)} |\Sigma(A)| = |\Sigma_\infty|. \tag{36} \]

Equations (33)-(35) makes evident the fact that an anisotropy in the equation of state of dark energy (\(\delta\)) generates an anisotropy in the cosmic geometry (\(\Sigma\)), and that the Universe will never isotropize, although its level of anisotropization is very low being \(\delta\) a small quantity.

In Fig. 1, we plot the shear as a function of the expansion parameter for various values of \(w\) and \(\Omega_{\text{DE}}\). Here and in the following we take \(\Omega_r = 0.83 \times 10^{-4} \) [25].

To proceed further, we note that the eccentricity and the shear are connected by the following relation:

\[ e^2 = 6 \text{sgn}(a - b) \int_1^A \frac{dx}{x} \Sigma(x), \tag{37} \]

valid for small eccentricities, \(e \ll 1\), and coming from definitions (10) and (16). The above equation calculated at the time of decoupling, together with Eqs. (28) and (33), implies that \(\text{sgn}(a - b) = -\text{sgn} \delta\) and allow us to write \(\delta\) and \(\Sigma_0\) in the form:

\[ |\delta| = c_\delta(w, \Omega_{\text{DE}}) e_{\text{dec}}^2, \tag{38} \]

\[ |\Sigma_0| = c_\Sigma(w, \Omega_{\text{DE}}) e_{\text{dec}}^2, \tag{39} \]

where

\[ c_\delta \equiv \left[ 6 \int_{A_{\text{dec}}}^{1} \frac{dx}{x^2} \frac{E(x)}{H(x)/H_0} \right]^{-1}, \tag{40} \]

\[ c_\Sigma \equiv E_0 c_\delta. \tag{41} \]

Here, \(A_{\text{dec}} = A(t_{\text{dec}})\) is the mean expansion parameter evaluated at the time of decoupling. In the following we simply assume that \(A_{\text{dec}} = 1/(1 + z_{\text{dec}})\), where \(z_{\text{dec}} \simeq 1090\) [26] is the redshift at decoupling.

In Fig. 2, we plot the skewness and the actual shear as a function of the mean dark energy density, for various values of the mean dark energy equation of state parameter. It is evident from the figure that, for a wide range of values of \(\Omega_{\text{DE}}\) and \(w\), the parameters \(\delta\) and \(\Sigma_0\) are indeed small quantities, as we have previously assumed.
FIG. 2: The skewness $\delta$ (upper panel) and the shear $\Sigma_0$ (lower panel) as a function of the mean dark energy density $\Omega_{DE}$ for different values of the mean dark energy equation of state parameter $w$. The values of $\delta$ and $\Sigma_0$ ensure that the eccentricity at decoupling is just that given by Eqs. (7) and (11).

The smallness of the parameters of anisotropy, $\delta$ and $\Sigma_0$, is a reason to believe that the values of mean parameters $\Omega_{DE}$ and $w$ are very close to the analogous ones for the isotropic standard cosmological model, which according the 7-yr WMAP data are [20]:

$$\Omega_{DE}^{\text{(isotropic)}} = 0.734 \pm 0.029 \text{ (68\% C.L.)},$$
$$w^{\text{(isotropic)}} = -0.980 \pm 0.053 \text{ (68\% C.L.)}.$$  

From the above equations and looking at Fig. 2, we finally get an order of magnitude estimate of $\delta$ and $\Sigma_0$:

$$|\delta| \sim 10^{-4}, \quad |\Sigma_0| \sim 10^{-5}.$$  

The above value of $\delta$ is compatible with the constraint obtained by Koivisto and Mota [14] coming from the analysis of luminosity distance-redshift relation of type Ia supernovae. Instead, no constraint exists on $\Sigma_0$ in the literature.

---

2 The dark energy equation of state parameter $w$ and the skewness $\delta$ introduced in this paper correspond, respectively, to the parameters $w + \gamma$ and $-3\gamma$ defined in Ref. [13]. Translating the results of Ref. [13] to our case, we get $-0.15 \leq \delta \leq 0.21 \text{ (68.3\% C.L.)}$. 

Nevertheless, a full analysis of magnitude-redshift data on type Ia supernovae can put limits both on $\delta$ and $\Sigma_0$ and, indeed, preliminary results \[27\] indicate that
\[
-0.40 \leq \delta \leq 0.13 \quad (1\sigma \text{ C.L.}),
\]
\[
-0.026 \leq \Sigma_0 \leq 0.014 \quad (1\sigma \text{ C.L.}).
\]
This leaves open the possibility of having an ellipsoidal universe \textit{via} an anisotropic dark energy.

\section*{IV. CONCLUSIONS}

The 7-year WMAP observations \[5\] do not have alleviated the so-called quadrupole problem of CMB anisotropy spectrum, and a tension between data and the quadrupole amplitude predicted by the best-fit $\Lambda$CDM concordance model still remains.

On the other hand, an anisotropic cosmological model described by a Bianchi type I metric, the “ellipsoidal universe” \[9, 10\], allows a lower quadrupole and better matches the large-scale CMB anisotropy data.

In the seminal papers \[14, 15\] by Koivisto and Mota, the peculiar features of a universe filled with an anisotropic dark energy fluid were deeply and exhaustively studied.

In this paper, instead, we have investigated the possibility that a dark energy component with anisotropic equation of state could generate an ellipsoidal universe with the right characteristics to explain the low quadrupole in the CMB fluctuations.

The amount of anisotropy in the equation of state of dark energy, the skewness $\delta$, is transferred to the background geometry which gets anisotropized at a level that today is given the actual shear $\Sigma_0$. Imposing that such a level of cosmic anisotropy is exactly that necessary to generate a quadrupole term in the CMB spectrum in such a way to match the “low” value of the observed one, we have obtained the estimates: $|\delta| \sim 10^{-4}$ and $|\Sigma_0| \sim 10^{-5}$.

These values are well within the $1\sigma$ confidence region allowed by magnitude-redshift data of type Ia supernovae \[14, 27\], whose analysis constitutes, up to today, the only test available in the literature constraining cosmic anisotropy of Bianchi type I.

This kind of test when combined with other tests on cosmic anisotropy, such as that coming from the study of CMB polarization spectrum, could put severe limits on the existence of an isotropic dark energy, confirm the ellipsoidal universe proposal or rules it out. All of this is, however, beyond the aim of this paper and will be the subject of future investigations.

\section*{APPENDIX}

The total CMB quadrupole intensity, $Q_{\text{tot}}$, is given by Eqs. (3)-(5) as \[9, 10\]:
\[
Q_{\text{tot}}^2 = Q_A^2 + Q_I^2 - 2fQ_AQ_I^2,
\]
where $Q_I$ and $Q_A$ are respectively given by Eqs. (7) and (9),
\[
f(\theta, \varphi_2, \varphi_3) = \frac{1 + 3 \cos(2\vartheta) + 2\sqrt{6} \sin \vartheta [\sin \vartheta \cos(2\varphi + \varphi_3) - 2 \cos \vartheta \cos(\varphi + \varphi_2)]}{4\sqrt{5}},
\]
and we are considering, for the sake of simplicity, the case $a > b$. In Eq. \[48\], $\vartheta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ are the polar angles defining, in the galactic coordinate system, the direction of the axis of symmetry associated to planar symmetry of ellipsoidal universe model \[9, 10\]. Together with the eccentricity at decoupling, $e_{\text{dec}}$, they completely define the coefficients $a_{2m}^I$. The parameters $\varphi_{2, 3} \in [0, 2\pi]$, instead, are unknown phases (which, roughly speaking, define the “direction” of the inflation-produced quadrupole) that together with $Q_I$ completely define the coefficients $a_{2m}^A$. Explicitly, we have \[9, 10\]:
\[
a_{20}^A = -\frac{\sqrt{\pi}}{6\sqrt{5}} [1 + 3 \cos(2\vartheta)] e_{\text{dec}}^2,
\]
\[
a_{21}^A = -(a_{2,-1}^A)^* = \sqrt{\frac{\pi}{30}} e^{-i\varphi} \sin(2\vartheta) e_{\text{dec}}^2,
\]
\[
a_{22}^A = (a_{2,-2}^A)^* = -\sqrt{\frac{\pi}{30}} e^{-2i\varphi} \sin^2 \vartheta e_{\text{dec}}^2,
\]
\[
Q_{\text{tot}}^2 = Q_A^2 + Q_I^2 - 2fQ_AQ_I^2, \quad (47)
\]
\[
f(\theta, \varphi_2, \varphi_3) = \frac{1 + 3 \cos(2\theta) + 2\sqrt{6} \sin \theta [\sin \theta \cos(2\varphi + \varphi_3) - 2 \cos \theta \cos(\varphi + \varphi_2)]}{4\sqrt{5}}, \quad (48)
\]
FIG. 3: The probability distribution function of $f$ (upper panel), and the probability that the observed quadrupole $Q_{\text{obs}}^2$ is in the 1σ confidence interval of the theoretical total quadrupole $Q_{\text{tot}}^2$, at the varying of the eccentricity at decoupling $e_{\text{dec}}$ (lower panel).

Let us assume that there exists some mechanism able to produce an anisotropization of the Universe, so that $e_{\text{dec}}$ is a parameter fixed by the model. Generally, on the other hand, the direction of the symmetry axis $(\vartheta, \varphi)$ cannot be fixed by the model and should be considered as unknown. The same is true for the direction of the inflation-produced quadrupole $(\phi_2, \phi_3)$. We therefore assume that $\vartheta, \varphi, \phi_2$, and $\phi_3$ are stochastic variables which assume random values in their intervals of existence. In Fig. 3 (see the upper panel), we show the probability distribution function of $f$, $F[f]$, derived by performing a Monte Carlo simulation.

We want now to calculate the probability, whatever are the values of $\vartheta, \varphi, \phi_2$, and $\phi_3$, with $e_{\text{dec}}$ being fixed, that

\begin{align*}
a_{20}^1 &= \sqrt{\frac{\pi}{3}} Q_1, \\
a_{21}^1 &= -(a_{2,-1}^1)^* = \sqrt{\frac{\pi}{3}} e^{i\phi_2} Q_1, \\
a_{22}^1 &= (a_{2,-2}^1)^* = \sqrt{\frac{\pi}{3}} e^{i\phi_3} Q_1.
\end{align*}
the observed quadrupole $Q_{\text{obs}}^2$, given in Eq. (3), is in the $1\sigma$ confidence interval of $Q_{\text{tot}}^2$, namely

$$P(e_{\text{dec}}) \equiv P(e_{\text{dec}} \mid Q_{\text{obs}}^2 \in [Q_{\text{tot}}^2 - \sigma_{\text{tot}}, Q_{\text{tot}}^2 + \sigma_{\text{tot}}]).$$

The uncertainty $\sigma_{\text{tot}}$ on $Q$ is obtained by simply propagating the uncertainty on the inflation-produced quadrupole $Q_1$, i.e. the cosmic variance in Eq. (3), by means of Eq. (47):

$$\sigma_{\text{tot}} = \left| 1 - f \frac{Q_A}{Q_1} \right| \sigma_{\text{cosmic}}.$$

It is easy to see that the condition $Q_{\text{obs}}^2 \in [Q_{\text{tot}}^2 - \sigma_{\text{tot}}, Q_{\text{tot}}^2 + \sigma_{\text{tot}}]$ is equivalent to the condition $f \in [f_-, f_+]$, where

$$f_{\pm}(e_{\text{dec}}) = \frac{1 \pm \sqrt{\frac{2}{3} + Q_1^2 - Q_{\text{obs}}^2}}{(2 + \frac{2}{3}) Q_A Q_1}.$$

We then have that

$$P(e_{\text{dec}}) = \int_{f_-}^{f_+} df F[f]$$

(54)

gives the probability, at fixed $e_{\text{dec}}$, that a cancelation between the quadrupole associated to the anisotropic spacetime background and the inflation-produced one, so to explain the “quadrupole problem”, occurs coincidently (i.e., whatever is the direction of the axis of symmetry). The lower panel of Fig. 3 shows $P(e_{\text{dec}})$ as a function of $e_{\text{dec}}$, and indicates that such a probability is not negligible for $e_{\text{dec}}$ in the range $[0.25, 1.15] \times 10^{-2}$. In particular, $P(e_{\text{dec}}) \approx 29\%$ for $e_{\text{dec}} \approx 0.65 \times 10^{-2}$ given by Eq. (11), which amounts to have $Q_{\text{tot}} \approx Q_{\text{obs}}[3, 10].$
[17] D. C. Rodrigues, Phys. Rev. D 77, 023534 (2008).
[18] O. Akarsu and C. B. Kilinc, arXiv:0807.4867 [gr-qc].
[19] S. Appleby, R. Battye and A. Moss, arXiv:0912.0397 [astro-ph.CO].
[20] R. Battye and A. Moss, Phys. Rev. D 80, 023531 (2009).
[21] C. Quercellini, M. Quartin and L. Amendola, Phys. Rev. Lett. 102, 151302 (2009); C. Quercellini, P. Cabella, L. Amendola, M. Quartin and A. Balbi, Phys. Rev. D 80, 063527 (2009); C. Quercellini, L. Amendola, A. Balbi, P. Cabella and M. Quartin, arXiv:1011.2646 [astro-ph.CO].
[22] A. H. Taub, Annals Math. 53, 472 (1951).
[23] E. F. Bunn and A. Bourdon, Phys. Rev. D 78, 123509 (2008).
[24] R. Durrer, The Cosmic Microwave Background (Cambridge University Press, Cambridge, United Kingdom, 2008).
[25] E. W. Kolb and M. S. Turner, The Early Universe (Addison-Wesley, Redwood City, California, 1990).
[26] E. Komatsu et al., arXiv:1001.4538 [astro-ph.CO].
[27] L. Campanelli, P. Cea, G. L. Fogli and A. Marrone, arXiv:1012.5596 [astro-ph.CO].