On the completeness of impulsive gravitational wave spacetimes

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Abstract
We consider a class of impulsive gravitational wave spacetimes, which generalize impulsive pp-waves. They are of the form $M = N \times \mathbb{R}^2_1$, where $(N, h)$ is a Riemannian manifold of arbitrary dimension and $M$ carries the line element $ds^2 = dh^2 + 2 du dv + f(x) \delta(u) du^2$, with $dh^2$ being the line element of $N$ and $\delta$ the Dirac measure. We prove a completeness result for such spacetimes $M$ with complete Riemannian part $N$.

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1. Introduction
Plane-fronted gravitational waves with parallel rays—pp-waves, for short—are defined by the existence of a covariantly constant null vector field $k$ and are usually associated with the line element in the so-called Brinkmann form

$$ds^2 = 2 du dv + (dx^1)^2 + (dx^2)^2 + H(x^1, x^2, u) du^2$$

on $\mathbb{R}^4$. These spacetimes model gravitational or electromagnetic waves and other forms of null matter and have been extensively studied (see e.g. [GP09, chapter 17] and the literature cited therein). The geodesic null congruence with tangent $k$ is non-expanding, shear-free and twist-free and the latter property implies the existence of a family of two-surfaces perpendicular to $k$ which are interpreted as wave surfaces. Moreover, since $k^{\mu}_{\nu}$ vanishes, they are planar and rays orthogonal to them are parallel.

It should be noted, however, that Brinkmann, who studied these geometries in the context of conformal mappings of Einstein spaces [Bri25], also included a rotational term (rediscovered by Bonner [Bon70] and recently studied further under the name gyraton [Fro07]), as well as allowed for a general wave surface. Including the latter effect, i.e. allowing for a Riemannian manifold of arbitrary dimension as the wave surface, we arrive at the following geometry $(M, g)$: let $(N, h)$ be a connected Riemannian manifold of dimension $n$, set $M = N \times \mathbb{R}^2_1$ and equip $M$ with the line element

$$ds^2 = dh^2 + 2 du dv + H(x, u) du^2,$$
where \(d\ell^2\) denotes the line element of \((N, h)\). Moreover \(u\) and \(v\) are global null-coordinates on the two-dimensional Minkowski space \(\mathbb{R}^2\) and \(H : N \times \mathbb{R} \rightarrow \mathbb{R}\) is a smooth function.

These models have been studied in a series of papers by J. Flores and M. Sanchez in part together with A. Candela [CFS03, FS03, CFS04, FS06] mainly focusing on causality and geodesics. These geometries allow one to shed some light on some of the peculiar causal properties especially of plane waves (i.e. pp-waves (1) with \(H(x^1, x^2, u) = h_{ij}(u)x^ix^j\)), see e.g. [BEE96, chapter 13]. They turn out to be caused by the high degree of symmetries of plane waves and the fact that the wave surfaces of (1) are flat \(\mathbb{R}^2\).

In [CFS03], spacetimes of the form (2) have been called (general) plane-fronted waves (PFW). However, by the geometric interpretation given above and by the analogy with pp-waves it seems more natural to us to call the spacetimes (2) \textit{N-fronted waves with parallel rays} (NPW), which we shall do from now on.

It turns out that the behaviour of \(H\) at spatial infinity, i.e. for ‘large \(x\)’ is decisive for many of the global properties of NPWs. In order to formulate precise statements, we recall that one says that \(H\) behaves \textit{subquadratically at spatial infinity} if there exist a fixed point \(\bar{x} \in N\), continuous functions \(0 \leq R_1, R_2\) and a continuous function \(p < 2\) such that for all \((x, u) \in N \times \mathbb{R}\)

\[
H(x, u) \leq R_1(u)d(x, \bar{x})^{p(u)} + R_2(u).
\]  

(3)

Here \(d\) denotes the Riemannian distance function on \(N\). Similarly, we say that \(H\) behaves at most quadratically, respectively, superquadratically if \(p \leq 2\), respectively, \(p > 2\). In [FS03] it has been shown that the causality of NPWs depends crucially on the exponent \(p\) in (3), with \(p = 2\) being the critical case. In particular, NPWs are causal, but not necessarily distinguishing, they are strongly causal if \(-H\) behaves at most quadratically at spatial infinity and they are globally hyperbolic if \(-H\) is subquadratic and \(N\) is complete. Similarly, the global behaviour of geodesics in NPWs is governed by the behaviour of \(H\) at spatial infinity. From the explicit form of the geodesic equations, it follows ([CFS03, theorem 3.2]) that an NPW is complete if and only if \(N\) is complete and

\[
D^\langle N \rangle_x \dot{x} = \frac{1}{2} \nabla_x H(x, s)
\]

has complete trajectories. Here, \(D^\langle N \rangle_x \) is the induced covariant derivative on \(N\) and \(\nabla_x\) denotes the spatial gradient. Applying classical results on complete vector fields (e.g. [AMR88, theorem 3.7.15]) completeness of \(M\) follows for autonomous \(H\) (i.e. independent of \(u\)) in case \(H\) grows at most quadratic at spatial infinity. Clearly, this implies completeness for at most quadratic sandwich waves, that is, waves with \(H\) compactly supported in \(u\).

In this work, we consider \textit{impulsive} NPWs (INPWs), i.e. we set \(H(x, u) = f(x)\delta(u)\) in (2), where \(\delta(u)\) is the Dirac measure on the hypersurface \(\{u = 0\}\). Impulsive pp-waves (for a summary see [GP09, chapter 20]) have been introduced by Penrose using a ‘scissors-and-paste method’ (e.g. [Pen72]) gluing two halves of Minkowski space along the null hypersurface \(\{u = 0\}\) with a warp. On the other hand, impulsive pp-waves arise as ultrarelativistic limits of Kerr–Newman black holes, with the prototype being the Aichelburg–Sexl geometry [AS71].

The distributional term in the metric of impulsive pp-waves and INPWs makes it a delicate matter to mathematically deal with these spacetimes; for a general account on distributional geometries in GR [SV06]. Therefore, impulsive pp-waves have been treated using the nonlinear distributional geometry [GKOS01] built upon algebras of generalized functions [Col85]. In particular, geodesics in impulsive pp-wave spacetimes have been considered in [Bal97, Ste98] and in [KS99], where an existence and uniqueness result for the geodesic equations has been proved. From a global point of view, these results imply that impulsive pp-waves are geodesically complete.
In this short note, we prove a completeness result for INPWs with complete \( N \). We do so without using any theory of nonlinear distributions leaving a detailed study of INPWs as distributional geometries to a subsequent paper. More precisely, we view INPWs as geometries with a small but finitely extended impulse: let \( \delta_\epsilon \) be some smooth approximation of the Dirac-delta (i.e. \( \delta_\epsilon \to \delta \) weakly as \( \epsilon \to 0 \)) and for fixed \( \epsilon > 0 \) consider the metric
\[
d_{\epsilon}^2 = d_{h}^2 + 2 \, du \, dv + f(x) \delta_\epsilon (u) \, du^2
\]
on \( M \), where \( f \) is an arbitrary smooth function on \( N \). We will show that for any geodesic \( \gamma \) in \( (M, d_{\epsilon}^2) \) there is \( \epsilon_0 \) small enough, such that \( \gamma \) can be defined for all values of an affine parameter provided \( \epsilon \leq \epsilon_0 \). Moreover, the size of \( \epsilon_0 \) for which the geodesic becomes complete can be explicitly estimated in terms of (derivatives of) \( f \) and the initial data of \( \gamma \). Finally, we also show that the globally defined geodesics converge to the geodesics of the background \( N \times \mathbb{R}^2_1 \) which, however, have to be joined with a suitable warp at the shock hypersurface.

2. The geodesic equations for INPWs

In this section, we derive the geodesic equations for INPWs and fix some notation to be used in the remainder of this work. We start by making precise the class of regularizations we use for the Dirac delta. We set \( I := (0, 1] \).

**Definition 2.1.** A net \((\delta_\epsilon)_{\epsilon \in I}\) of smooth functions on \( \mathbb{R} \) is called a strict delta net if it satisfies the following three properties.

1. The supports shrink to zero, \( \text{supp}(\delta_\epsilon) \to \{0\} \) for \( \epsilon \searrow 0 \).
2. The integrals converge to 1, \( \int_{\mathbb{R}} \delta_\epsilon (x) \, dx \to 1 \) for \( \epsilon \searrow 0 \).
3. The \( L^1 \)-norms are uniformly bounded, \( \exists K > 0 : \int_{\mathbb{R}} |\delta_\epsilon (x)| \, dx \leq K \) \( \forall \epsilon \in I \).

Observe that this is a very general class of approximations of \( \delta \). (Although smoothness excludes ‘boxes’, nets arbitrarily close to ‘boxes’ and even discontinuous regularizations are practically included by the fact that \( C^\infty_c \) is dense in \( L^1 \).) Without the loss of generality, we will always assume that \( \text{supp}(\delta_\epsilon) \subseteq (-\epsilon, \epsilon) \) for all \( \epsilon \in I \).

Now let \( M = N \times \mathbb{R}^2_1 \) be an INPW with \( N \) a connected \( n \)-dimensional and complete Riemannian manifold and let \( M \) be endowed with the family of line elements (4), where \( (\delta_\epsilon)_{\epsilon} \) is a strict delta net.

Denoting the Christoffel symbols of the Riemannian manifold \((N, h)\) by \( \Gamma^{(N)}_{ij} \), one obtains the non-vanishing Christoffel symbols for \( M \) with respect to a coordinate system \((x^1, \ldots, x^n)\) of \( N \) and \((u, v)\) null-coordinates of \( \mathbb{R}^2_1 \):
\[
\begin{align*}
\Gamma^k_{ij} & = \Gamma^{(N)}_{ij} \quad \text{for all} \quad 1 \leq i, j, k \leq n, \\
\Gamma^u_{aj} & = \Gamma^u_{ja} = \frac{1}{2} \frac{\partial f}{\partial x^j} \delta_\epsilon \quad \text{for all} \quad 1 \leq j \leq n, \\
\Gamma^v_{au} & = \frac{1}{2} f \delta_\epsilon , \\
\Gamma^k_{au} & = -\frac{1}{2} h^{km} \frac{\partial f}{\partial x^m} \delta_\epsilon .
\end{align*}
\]

Since all Christoffel symbols of the form \( \Gamma^u_{uv} \) vanish, we may use \( u \) as an affine parameter (thereby only excluding geodesics parallel to the shock hypersurface). Hence, the geodesic equations reduce to the following set of \( n + 1 \) equations:
\[
\ddot{v}_\epsilon = -\frac{\partial f}{\partial x^j} (x_\epsilon) \dot{x}_\epsilon^j \delta_\epsilon - \frac{1}{2} f (x_\epsilon) \dot{\delta}_\epsilon ,
\]
(5)
Here $D^{(N)}$ and $\nabla_x$ denote the covariant derivative, respectively, the gradient with respect to $h$. First observe that equation (5) can be integrated once the second equation has been solved. So we have to concentrate on equation (6), which is just the perturbed geodesic equation on $N$ with potential $f$ and the non-autonomous term $\delta \epsilon$. Moreover, since the latter vanishes for $|u| \geq \epsilon$, the $x$-component of the geodesics on $M$ will for large $u$ coincide with the (unperturbed) geodesics on $N$. By completeness of $N$, the question of completeness of $M$ reduces to the question whether all perturbed geodesics on $N$ that enter the regularization strip at $u = -\epsilon$ also leave it at $u = \epsilon$, that is, whether the perturbed geodesics blow up before $u = \epsilon$ or not.

Bearing this in mind, we apply the following procedure to solve the geodesic equation on $M$ as well as to address the problem of geodesic completeness of $M$. We fix $\epsilon > 0$ and impose initial data $x_0 \in N$, $\dot{x}_0 \in T_{x_0}N$ at $u = -1$ 'long before' the shock and then follow the unperturbed Riemannian geodesic on $N$ with this data, i.e. the solution of

\[
D^{(N)}_{\dot{x}} \dot{x} = 0, \quad x(-1) = x_0, \quad \dot{x}(-1) = \dot{x}_0,
\]

which we denote by $x[x_0, \dot{x}_0]$. By completeness of $N$, this geodesic $x[x_0, \dot{x}_0]$ will reach the shock region at $u = -\epsilon$ and until then it will also be a solution of the perturbed geodesic equation (6) with the same data, which we will denote by $x_\epsilon[x_0, \dot{x}_0]$. With this notation, we have $x_\epsilon[x_0, \dot{x}_0] = x[x_0, \dot{x}_0]$ on $]-\infty, -\epsilon]$ and to continue $x_\epsilon[x_0, \dot{x}_0]$ into the shock region $|u| \leq \epsilon$ we consider the initial value problem

\[
(6) \text{ with data } x_\epsilon(-\epsilon) = x[x_0, \dot{x}_0](-\epsilon), \quad \dot{x}_\epsilon(-\epsilon) = \dot{x}[x_0, \dot{x}_0](-\epsilon).
\]

To prove that $x_\epsilon[x_0, \dot{x}_0]$ extends to all values of the parameter $u$, we only have to show that the latter initial value problem possesses a solution denoted by $x_\epsilon[x_0, \dot{x}_0]$ until $u = \epsilon$, since for $u \geq \epsilon$, the right-hand side of (6) vanishes and we are solving the (unperturbed) geodesic equation in the complete manifold $N$. That is, we only have to show that no blow-up occurs within the shock region $|u| \leq \epsilon$, which, in fact, will be done in the following section (at least for $\epsilon$ small enough). In total, we will then have the global perturbed geodesic $x_\epsilon[x_0, \dot{x}_0]$ which we denote by $x_\epsilon[x_0, \dot{x}_0]$. Finally, as observed above, once we have such a solution of the $x$-component of the geodesic in $M$, the equation for $v$ can be integrated to give a solution for all $u \in \mathbb{R}$ and we will use the following notation: for initial conditions $v_0$, $\dot{v}_0 \in \mathbb{R}$ we denote by $v[v_0, \dot{v}_0]$ the straight line $v[v_0, \dot{v}_0](u) = v_0 + \dot{v}_0 (1 + u)$, i.e. a solution of (5) for $u \leq -\epsilon$ and similarly $v_\epsilon[v_0, \dot{v}_0]$ denotes a solution of (5) with $v_\epsilon[v_0, \dot{v}_0](-\epsilon) = v[v_0, \dot{v}_0](-\epsilon)$ and $\dot{v}_\epsilon[v_0, \dot{v}_0](-\epsilon) = \dot{v}_0$.

3. Completeness

We now show that for any geodesic in $M$ we can choose $\epsilon$ sufficiently small such that the geodesic can be extended through the shock. More precisely, we prove that (using the notation introduced above) the initial value problem

\[
D^{(N)}_{\dot{x}} \dot{x} = \frac{1}{2} \nabla_x f(x_\epsilon) \delta \epsilon, \quad x_\epsilon(-\epsilon) = x[x_0, \dot{x}_0](-\epsilon), \quad \dot{x}_\epsilon(-\epsilon) = \dot{x}[x_0, \dot{x}_0](-\epsilon)
\]

has a local solution defined up to $u = \epsilon$, provided $\epsilon$ is small enough.

**Proposition 3.1.** For all $x_0 \in N$, $\dot{x}_0 \in T_{x_0}N$ there exists $\epsilon_0$ such that the initial value problem (8) has a solution $x_\epsilon[x_0, \dot{x}_0]$ defined up to $u = \epsilon$, provided $\epsilon \leq \epsilon_0$. 

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The proof heavily rests on a fixed point argument which we provide in detail in lemma A.2 in the appendix. Here we only observe that this argument indeed provides the assertion of the proposition.

**Proof.** We invoke lemma A.2 with \( b > 0, \ c > 0, \ F_1(y, z)^k := -\Gamma^k_{ij}(y)z^iz^j \) (to express \( D_v^{(N)} \) in local coordinates) and \( F_2(y)^k := \frac{1}{2} h^{ku}(y) \frac{\partial}{\partial y^u}(y) \) which is just \( \frac{1}{2} \nabla_v f \) in coordinates. Clearly, \( F_1 \) and \( F_2 \) are smooth since \( f \) and \( h \) (and hence the Christoffel symbols) are assumed to be smooth. Hence, lemma A.2 guarantees the existence of a solution \( \tilde{x}_{\epsilon} [x_0, \dot{x}_0] \) of (8) until \( u = \alpha - \epsilon \). So choosing \( \epsilon_0 = \frac{\epsilon}{\alpha} \), the solution \( \tilde{x}_\epsilon [x_0, \dot{x}_0] \) exists at least until \( u = \epsilon \), provided \( \epsilon \leq \epsilon_0 \).

Observe that lemma A.2 also implies that the solution \( \tilde{x}_\epsilon [x_0, \dot{x}_0] \) together with its first derivative \( \dot{\tilde{x}}_\epsilon [x_0, \dot{x}_0] \) is uniformly bounded (in \( \epsilon \) on \( [-\epsilon_0, \epsilon_0] \)). Moreover, (A.2) gives an upper bound on \( \epsilon_0 \) in terms of the initial velocity \( \dot{x}_0 \) and of the Christoffel symbols on \( \mathcal{N} \) as well as of \( \nabla_v f \) on a neighborhood of the data \( x_0 \). Next we state our completeness result.

**Theorem 3.2.** For all \( x_0 \in \mathcal{N}, \dot{x}_0 \in T_{x_0} \mathcal{N} \) and all \( v_0 \in \mathbb{R} \) there exists \( \epsilon_0 \) such that the solution \( (x_\epsilon [x_0, \dot{x}_0], v_\epsilon [v_0, \dot{v}_0]) \) of the geodesic equation (5,6) with initial data \( x_\epsilon (-1) = x_0, \dot{x}_\epsilon (-1) = \dot{x}_0, v_\epsilon (-1) = v_0, \dot{v}_\epsilon (-1) = \dot{v}_0 \) is defined for all \( u \in \mathbb{R} \), provided \( \epsilon \leq \epsilon_0 \).

**Proof.** Given \( x_0 \) and \( \dot{x}_0 \), proposition 3.1 provides us with \( \epsilon_0 \) such that the solution of (8) is defined for \( u \in [-\epsilon, \epsilon] \) for all \( \epsilon \leq \epsilon_0 \). In this case, we hence may define \( x_\epsilon [x_0, \dot{x}_0] \) as in (7) for all \( u \in \mathbb{R} \) so it only remains to integrate (5) twice to obtain a globally defined solution \( v_\epsilon \). Hence in total we obtain a unique globally defined geodesic \( \mathcal{R} \ni u \mapsto (x_\epsilon [x_0, \dot{x}_0](u), v_\epsilon [v_0, \dot{v}_0](u)) \).

We point out that \( \alpha \) in the proof of proposition 3.1 and hence \( \epsilon_0 \) for which the geodesic is defined on all \( \mathbb{R} \) depends on the choice of the initial data \( x_0 \) and \( \dot{x}_0 \). Hence, we can not, in general, obtain a global bound \( \epsilon_0 \) such that for fixed \( \epsilon \leq \epsilon_0 \) the manifold \( M \) is geodesically complete. There are, however, two special cases where we actually obtain geodesic completeness of \( M \) for \( \epsilon \) sufficiently small. First assume that \( \mathcal{N} \) is compact. Then we obtain a globally defined \( \epsilon_0 \) since \( x_0 \) varies in a compact set only and upon reparametrization we may achieve that \( |\dot{x}_0| = 1 \). On the other hand, if \( -f \) behaves subquadratically (cf (3)) then by the compactness of the support of \( \delta_\epsilon \), we may apply the results of [FS03] mentioned in the introduction to obtain completeness without even the need to invoke the fixed point argument.

However, one may say that ‘in the limit \( \epsilon \to 0 \)’ we obtain a geodesically complete manifold; hence one may say that INPWs are geodesically complete irrespective of the behaviour of the profile function \( f \). This is in sharp contrast to the case of extended NPWs where completeness depends crucially on the behaviour of \( H \) at ‘spatial infinity’: the role of the \( x \)-asymptotics of \( H \) becomes irrelevant in the impulsive limit.

However, the precise meaning of the completeness statement (i.e. the dependence of \( \epsilon_0 \) on the data) is encoded in the formulation of our theorem above. A more straightforward completeness result for INPWs can be provided using nonlinear distributional geometry [GKOS01, KS02] in the sense of J F Colombeau [Col85], and we will address this topic in a subsequent paper.

4. Limits

In this section, we compute the limits of the global geodesics derived above as \( \epsilon \to 0 \). We start by analysing the \( x \)-component and introduce some more notations in the same spirit as at the
end of section 2. We define the prospective limit of \( x_\epsilon[x_0, \dot{x}_0] \) by pasting together the solution \( x[x_0, \dot{x}_0] \) of the unperturbed equation for \( u < 0 \) with an appropriate solution of the unperturbed equation for \( u > 0 \). To this end, denote by \( \bar{x}[x_0, \dot{x}_0] \) the solution of the (unperturbed) geodesic equation on \( N \) with data \( \bar{x}(0) = x[x_0, \dot{x}_0](0) \) and \( \bar{x}(0) = x[x_0, \dot{x}_0](0) + \frac{1}{2} \nabla_e f(x[x_0, \dot{x}_0](0)) \).

Finally, denote the prospective limit by

\[
y[x_0, \dot{x}_0](u) :=
\begin{cases}
x[x_0, \dot{x}_0](u) & u \leq 0 \\
\bar{x}[x_0, \dot{x}_0](u) & u \geq 0.
\end{cases}
\]

Observe that \( y[x_0, \dot{x}_0] \) is a continuous curve \( \mathbb{R} \to N \) which is piece-wise smooth with a single break point at \( u = 0 \). Moreover, it is not differentiable (in general) as we have

\[
\lim_{u \to 0^-} y[x_0, \dot{x}_0](u) = \bar{x}[x_0, \dot{x}_0](0),
\]

\[
\lim_{u \to 0^+} y[x_0, \dot{x}_0](u) = \bar{x}[x_0, \dot{x}_0](0) + \frac{1}{2} \nabla_e f(x[x_0, \dot{x}_0](0)).
\]

For simplicity, we write \( F_1(y, z) := -\Gamma^{\epsilon}(0)(y) z^i z^j \) and \( F_2^\epsilon = \frac{1}{2} \nabla^k f = \frac{1}{2} h^{ij} \frac{d f}{d s} \) as in the proof of theorem 3.1 and start with an auxiliary result needed throughout the remainder of this section.

**Lemma 4.1.** The global solution \( x_\epsilon[x_0, \dot{x}_0] \) of (6) (defined in (7)) satisfies

\[
x_\epsilon[x_0, \dot{x}_0](\epsilon u) \to y[x_0, \dot{x}_0](0) = x[x_0, \dot{x}_0](0) \text{ uniformly on } [-1, 1] \text{ as } \epsilon \searrow 0.
\]

**Proof.** To keep the notation transparent, we abbreviate \( x[x_0, \dot{x}_0] \) by \( x \) and \( x_\epsilon[x_0, \dot{x}_0] \) by \( x_\epsilon \). We have

\[
\sup_{u \in [-1, 1]} |x_\epsilon(\epsilon u) - x(0)| \leq \sup_{u \in [-1, 1]} |x_\epsilon(\epsilon u) - x(\epsilon u)| + \sup_{u \in [-1, 1]} |x(\epsilon u) - x(0)|.
\]

The second term goes to zero as \( \epsilon \searrow 0 \) since \( x \) is uniformly continuous on compact sets. To estimate the first term, we integrate the differential equations for \( x_\epsilon \) and \( x \) (see also (A.4) in the appendix) to obtain

\[
\int_{-\epsilon}^{\epsilon} \left( \int_{-\epsilon}^{\epsilon} |F_1(x_\epsilon(r), \dot{x}_\epsilon(r)) - F_1(x(r), \dot{x}(r))| dr \right) ds + \int_{-\epsilon}^{\epsilon} \left( \int_{-\epsilon}^{\epsilon} |F_2(x_\epsilon(r))| \delta_\epsilon(r) dr \right) ds
\]

\[
\leq C_1 \epsilon^2 + C_2 \| \delta_\epsilon \|_L^2 \epsilon \leq C \epsilon \to 0 \quad (\epsilon \searrow 0),
\]

where we have used that by lemma A.2, \( x_\epsilon \) and \( \dot{x}_\epsilon \) are bounded independently of \( \epsilon \), and the constants \( C_1 \) and \( C_2 \) contain the \( L^\infty \)-norms of \( F_1 \) and \( F_2 \), respectively, on suitable compact sets. \( \square \)

**Proposition 4.2.** The global solution \( x_\epsilon[x_0, \dot{x}_0] \) of (6) (defined in (7)) satisfies

\[
x_\epsilon[x_0, \dot{x}_0] \to y[x_0, \dot{x}_0] \text{ uniformly on compact subsets of } \mathbb{R},
\]

\[
\dot{x}_\epsilon[x_0, \dot{x}_0] \to \dot{y}[x_0, \dot{x}_0] \text{ uniformly on compact subsets of } \mathbb{R}\setminus\{0\}.
\]

**Proof.** Again we write \( x \) for \( x[x_0, \dot{x}_0] \) and \( x_\epsilon \) for \( x_\epsilon[x_0, \dot{x}_0] \) and similarly \( y \) for \( y[x_0, \dot{x}_0] \) and \( \dot{x}_\epsilon \) for \( \dot{x}_\epsilon[x_0, \dot{x}_0] \). Without loss of generality, we only consider the compact interval \([-1, 1] \). We distinguish three cases: \(-1 \leq u \leq -\epsilon, -\epsilon \leq u \leq \epsilon \) and \( \epsilon \leq u \leq 1 \).

In the first case, \( x_\epsilon \equiv x = y \) on \([-1, -\epsilon] \) (and hence \( \dot{x}_\epsilon \equiv \dot{x} \) on the same interval), since \( x_\epsilon \) and \( x \) solve the same initial value problem. If \( -\epsilon \leq u \leq \epsilon \), then the result for \( x_\epsilon \) follows immediately from lemma 4.1 while for the derivative \( \dot{x}_\epsilon \) there is nothing to prove in this case.
Finally, for \( \epsilon \leq u \leq 1 \) we observe that \( x = \hat{x}_\epsilon \) and \( y = \hat{y} \) solve the same differential equation but now with different initial conditions, namely \( \hat{x}_\epsilon(\epsilon), \hat{x}_\epsilon(\epsilon), \) and \( \hat{x}(\epsilon) \) and \( \hat{x}(\epsilon), \) respectively. By continuous dependence on the initial data, we obtain
\[
\max(\|\hat{x}_\epsilon(u) - \hat{x}(u)\|, |\hat{x}_\epsilon(\epsilon) - \hat{x}(\epsilon)|) \leq \max(\|\hat{x}_\epsilon(\epsilon) - \hat{x}(\epsilon)\|, |\hat{x}_\epsilon(\epsilon) - \hat{x}(\epsilon)|) e^L,
\]
where \( L \) is a Lipschitz constant of \( F_1 \) on the compact image of \([0, 1]\) under \( \hat{x}, \hat{x}_\epsilon, \hat{\hat{x}}, \) and it suffices to estimate the difference of the data. Indeed, we have
\[
|\hat{x}_\epsilon(\epsilon) - \hat{x}(\epsilon)| \leq |\hat{x}_\epsilon(\epsilon) - \hat{x}(0)| + |\hat{x}(0) - \hat{x}(\epsilon)| \rightarrow 0,
\]
since the first term converges to zero by lemma 4.1 and the second by continuity. Similarly, we have
\[
|\hat{x}_\epsilon(\epsilon) - \hat{x}(\epsilon)| \leq |\hat{x}_\epsilon(\epsilon) - \hat{x}(0)| + |\hat{x}(0) - \hat{x}(\epsilon)|,
\]
where again the second term on the right-hand side goes to zero by continuity. To estimate the first term, we plug in the integral representation of \( \hat{x}_\epsilon \) to obtain
\[
|\hat{x}_\epsilon(\epsilon) - \hat{x}(0)| = |\hat{x}_\epsilon(\epsilon) - \hat{x}(0) - F_2(\hat{x}_\epsilon(\epsilon))| \\
\leq |\hat{x}(\epsilon) - \hat{x}(0)| + \int_{-\epsilon}^{\epsilon} |F_1(\hat{x}_\epsilon(s), \hat{x}_\epsilon(s))| ds \\
+ \left| \int_{-\epsilon}^{\epsilon} F_2(\hat{x}_\epsilon(s)) \delta_\epsilon(s) ds - F_2(\hat{x}(0)) \right|.
\]
Now, the first term on the right-hand side vanishes in the limit since \( \hat{x}_\epsilon(-\epsilon) = \hat{x}(-\epsilon) \rightarrow \hat{x}(0). \)
The second term goes to zero again by the uniform boundedness of \( \hat{x}_\epsilon \) and \( \hat{x}_\epsilon. \) To obtain the same conclusion for the third term, we again take into account the uniform boundedness of \( \hat{x}_\epsilon \) and the fact that \( \delta_\epsilon(s) \) is a strict delta net.

Next we turn to the \( v \)-component and recall that \( (u, v) \in \mathbb{R}_1^3 \) and so we may work distributionally.

**Proposition 4.3.** The global solution \( v \) of (5) satisfies
\[
v[v_0, \tilde{v}_0] \rightarrow v[v_0, \tilde{v}_0] - \frac{1}{2} f(x(0)) H - \frac{1}{2} \nabla f'(x(0)) D_J f(x(0)) u_+, \]
where \( u_+(u) = u H(u) \) denotes the so-called kink function and we again have abbreviated \( x[v_0, \tilde{v}_0] \) by \( x. \)

**Proof.** In addition to the abbreviations \( x \) and \( x_\epsilon \) used already above, we write \( v \) for \( v[v_0, \tilde{v}_0] \) and \( u_\epsilon \) for \( u[v_0, \tilde{v}_0]. \) Since we have \( v_\epsilon(u) = v_0 + \tilde{v}_0 \cdot (1 + u) + H \ast H \ast v_\epsilon(u) \) and since convolution is a separately continuous operation, it suffices to calculate the distributional limit of \( v_\epsilon. \) Inserting the integral representation of \( \hat{v}_\epsilon \) into equation (5), we obtain
\[
\tilde{v}_\epsilon(u) = -D_J f(x_\epsilon(u)) \delta_\epsilon(u) \hat{v}_\epsilon(\epsilon) - \frac{1}{2} D_J f(x_\epsilon(u)) \delta_\epsilon(u) \int_{-\epsilon}^{u} F_1(x_\epsilon(s), \hat{x}_\epsilon(s)) ds \\
+ \left\{ \begin{array}{ll}
\frac{1}{2} D_J f(x_\epsilon(u)) \delta_\epsilon(u) \int_{-\epsilon}^{u} F_2(x_\epsilon(s)) \delta_\epsilon(s) ds - \frac{1}{2} f(x_\epsilon(u)) \delta_\epsilon(u) \right\},
\end{array} \right.
\]
It is easily seen that \( I \rightarrow \hat{v}_\epsilon(\epsilon) D_J f(x(0)) \delta \) and that \( IV \rightarrow f(x(0)) \delta \) in \( \mathcal{D}'(\mathbb{R}). \) On the other hand, \( I I \rightarrow 0 \) in \( \mathcal{D}'(\mathbb{R}) \) since for all test functions \( \phi \in \mathcal{D}(\mathbb{R}) \) we have (again using the uniform boundedness of \( x_\epsilon \) and \( \hat{x}_\epsilon \))
\[
\left| \int_{\mathbb{R}} \phi(r) D_J f(x_\epsilon(u)) \delta_\epsilon(u) \int_{-\epsilon}^{u} F_1(x_\epsilon(s), \hat{x}_\epsilon(s)) \, ds \right| \leq 2C\|\phi\|_{\infty}.
\]
Finally, we show that \( (III) \rightarrow \frac{1}{2}D_j f(x(0))F^j_2(x(0))\delta. \) Indeed, we have

\[
\left| \int_x \phi(u)D_j f(x(u))\delta(u) \int_{\epsilon}^{\mu} F^j_2(x(r))\delta(r) \, dr \, du - \frac{1}{2} \phi(0)D_j f(x(0))F^j_2(x(0)) \right|
\]

\[
\leq \left| \int_x \phi(u)\delta(u) \int_{\epsilon}^{\mu} F^j_2(x(r))\delta(r) \, dr \, du \right|
\]

\[
+ \left| \int_x \phi(u)\delta(u) \int_{\epsilon}^{\mu} (F^j_2(x(r)) - F^j_2(x(0)))\delta(r) \, dr \, du \right| \left| D_j f(x(0)) \right|
\]

\[
+ C\|\phi\|_{\infty} \sup_{u[-1,1]} \left| D_j f(x(u)) - D_j f(x(0)) \right|
\]

\[
+ C\int_{\epsilon}^{\mu} \phi(u)\delta(u) \int_{\epsilon}^{\mu} \delta(r) \, dr \, du - \frac{1}{2} \phi(0),
\]

where we have absorbed all constant terms into the ‘generic constant’ \( C. \) Now the first and the second terms converge to zero, again by lemma 4.1. Finally, the integral term in the last line converges to zero by an elementary calculation.

Summing up we have shown that the \( x \)-component of the limit is continuous but has a kink at the shock hypersurface. The \( v \)-component, however, is not even continuous but has a jump at the shock in addition to a kink. The parameters of the kinks and the jump are given in terms of the profile function \( f \) and its derivatives at the point where the geodesic hits the shock hypersurface. So globally the geodesics on \( M \) are given by geodesics on the background \( N \times \mathbb{R}^2 \), which have to be joined suitably at the shock hypersurface.

This result complements the completeness result (theorem 3.2) of section 3. The globally defined geodesics in the complete limiting spacetime are given by suitably gluing together the geodesics of the background spacetime at the shock hypersurface.

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**Appendix**

In this appendix, we detail the fixed point argument used in the proof of our main result. Our argument is built on a slightly sharper version of the Banach fixed point theorem (see [Wei52]). Indeed, the integral operator \( A_\epsilon \) used below to solve the initial value problem is not a contraction on the naturally chosen Banach space \( \mathcal{C} \), and so the Banach fixed point theorem does not apply.

**Theorem A.1** (Weissinger’s fixed point theorem). Let \( X \) be a nonempty closed subset of a Banach space \( (E, \| \cdot \|) \). Moreover, let \( \sum_{n=1}^{\infty} a_n \) be a convergent series of positive real numbers \( (a_n) \), and \( A : X \rightarrow X \) a map with the property that

\[
\|A^n(u) - A^n(v)\| \leq a_n\|u - v\| \quad \forall u, v \in X \forall n \in \mathbb{N}.
\]  

(\( A.1 \))

Then \( A \) has a unique fixed point.
Now we state and prove our main technical result. For brevity, we write $\|F\|_{L,\infty}$ for the $L^\infty$-norm of the function $F$ on the set $I$.

**Lemma A.2.** Let $F_1 \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and $F_2 \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, let $x_0, \tilde{x}_0 \in \mathbb{R}^n$, let $b > 0$, $c > 0$ be given and let $(\delta_n)_n$ be a strict delta net with $L^1$-bound $K > 0$. Define $I_1 := \{x \in \mathbb{R}^n : |x - x_0| \leq b\}$, $I_2 := \{x \in \mathbb{R}^n : |x - \tilde{x}_0| \leq c + K\|F_2\|_{L,\infty}\}$ and $I_3 := I_1 \times I_2$. Moreover, set

$$\alpha := \min\left(1, \frac{b}{|\tilde{x}_0| + \|F_1\|_{L,\infty} + K\|F_2\|_{L,\infty}}, \frac{c}{\|F_1\|_{L,\infty}}\right).$$

(A.2)

Then the initial value problem

$$\dot{x}_t = F_1(x_t, \dot{x}_t) + F_2(x_t)\delta_t, \quad x_t(\epsilon) = x_0, \quad x_t(-\epsilon) = \tilde{x}_0,$$

(A.3)

has a unique solution $x_t$ on $I_3 := [-\epsilon, \alpha - \epsilon]$ with $(x_t(J_3), \dot{x}_t(J_3)) \subseteq I_3$. In particular, both $x_t$ and $\dot{x}_t$ are bounded, uniformly in $\epsilon$.

**Proof.** We consider the closed subset $X_\epsilon := \{x_\epsilon \in C^1(J_\epsilon, \mathbb{R}^n) : x_\epsilon(J_\epsilon) \subseteq I_1, \dot{x}_\epsilon(J_\epsilon) \subseteq I_2\}$ of the Banach space $C^1(J_\epsilon, \mathbb{R}^n)$ with norm $\|x\|_{C^1} := \|x\|_{L,\infty} + \|\dot{x}\|_{L,\infty}$. We define the operator $A_\epsilon$ on $X_\epsilon$ by ($t \in J_\epsilon$):

$$A_\epsilon(x_\epsilon)(t) := x_\epsilon(0) + x_\epsilon(0)(t + \epsilon) + \int_{-\epsilon}^{t} \int_{-\epsilon}^{s} F_1(x_\epsilon(r), \dot{x}_\epsilon(r)) \, dr \, ds + \int_{-\epsilon}^{t} \int_{-\epsilon}^{s} F_2(x_\epsilon(r)) \, dr \, ds.$$

(A.4)

First we show that the operator $A_\epsilon$ maps $X_\epsilon$ to itself. Let $x_\epsilon \in X_\epsilon$ and $t \in J_\epsilon$, then we have for the zero-order derivative of $A_\epsilon(x_\epsilon)$:

$$|A_\epsilon(x_\epsilon)(t) - x_0| \leq |\tilde{x}_0|(t + \epsilon) + \int_{-\epsilon}^{t} \int_{-\epsilon}^{s} |F_1(x_\epsilon(r), \dot{x}_\epsilon(r))| \, dr \, ds$$

$$+ \int_{-\epsilon}^{t} \int_{-\epsilon}^{s} |F_2(x_\epsilon(r))| \, dr \, ds \leq \alpha |\tilde{x}_0| + \alpha^2 \|F_1\|_{L,\infty} + \alpha \|F_2\|_{L,\infty} \|\delta_t\|_{L^1}$$

$$\leq \alpha |\tilde{x}_0| + \|F_1\|_{L,\infty} + K\|F_2\|_{L,\infty} \leq b,$$

and for the first-order derivative:

$$|A_\epsilon'(x_\epsilon)(t) - \tilde{x}_0| \leq \int_{-\epsilon}^{t} |F_1(x_\epsilon(r), \dot{x}_\epsilon(r))| \, dr + \int_{-\epsilon}^{t} |F_2(x_\epsilon(r))| \, dr$$

$$\leq \alpha \|F_1\|_{L,\infty} + \|F_2\|_{L,\infty} \|\delta_t\|_{L^1} \leq c + K\|F_2\|_{L,\infty}.$$

At this point we claim that we can find a sequence of positive real numbers $(a_n)_{n \geq 2}$ such that $\sum_{n=2}^{\infty} a_n < \infty$ and

$$\left\|A_\epsilon^n(x_\epsilon) - A_\epsilon^n(y_\epsilon)\right\|_{C^1(J_\epsilon)} \leq a_n \|x_\epsilon - y_\epsilon\|_{C^1(J_\epsilon)}.$$

So let $n \geq 2$, $x_\epsilon, y_\epsilon \in X_\epsilon$, and $t \in J_\epsilon$. Denoting by $\left[\int_{-\epsilon}^{t} r\right]$ the $n$-times iterated integral, we obtain (with Lip($F_1$, $I_j$) a Lipschitz constant for $F_1$ on $I_j$)

$$|A_\epsilon^n(x_\epsilon)(t) - A_\epsilon^n(y_\epsilon)(t)| \leq \left[2n \int_{-\epsilon}^{t} |F_1(x_\epsilon(r), \dot{x}_\epsilon(r)) - F_1(y_\epsilon(r), \dot{y}_\epsilon(r))| \, dr\right]$$

$$+ \left[2n \int_{-\epsilon}^{t} |F_2(x_\epsilon(r)) - F_2(y_\epsilon(r))| \, dr\right] \, d\alpha^n r$$

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\[ \begin{align*}
\leq \left( \text{Lip}(F_1, I_3) \left[ 2n \int_{-\epsilon}^{\epsilon} d^2 r \right] + \text{Lip}(F_2, I_1) \| \delta \|_{L^1} \left[ (2n-1) \int_{-\epsilon}^{\epsilon} d^2 r \right] \right) \| x - y \|_{C^1(I, L)} \\
\leq \left( \text{Lip}(F_1, I_3) \frac{\alpha^{2n}}{(2n)!} + \text{Lip}(F_2, I_1) \| \delta \|_{L^1} \frac{\alpha^{2n-1}}{(2n-1)!} \right) \| x - y \|_{C^1(I, L)}.
\end{align*} \]

Furthermore, for the derivative of \( A^\mu_\nu \) we find that
\[
\left| \frac{d}{dr} (A^\mu_\nu(x)) (t) - \frac{d}{dr} (A^\mu_\nu(y)) (t) \right| \leq \left( (2n-1) \int_{-\epsilon}^{\epsilon} \right) |f_1(x, r) - f_1(y, r)| |x - y| |d^2 r|
\]
\[
+ \left[ (2n-2) \int_{-\epsilon}^{\epsilon} \right] |f_2(x, r) - f_2(y, r)| |\delta| |d^2 r|
\]
\[
\leq \left( \text{Lip}(F_1, I_3) \frac{\alpha^{2n-1}}{(2n-1)!} + \text{Lip}(F_2, I_1) \| \delta \|_{L^1} \frac{\alpha^{2n-2}}{(2n-2)!} \right) \| x - y \|_{C^1(I, L)}.
\]

Summing up we obtain
\[
\| A^\mu_\nu(x) - A^\mu_\nu(y) \|_{C^1(I, L)} \leq 4 \max (\text{Lip}(F_1, I_3), K \text{Lip}(F_2, I_1)) \frac{\alpha^{2n-2}}{(2n-2)!} \| x - y \|_{C^1(I, L)},
\]
which proves our claim.

Now Weissinger’s fixed point theorem provides us with the existence of a unique solution \( x_\epsilon \in X_\epsilon \).

Finally, since \( x_\epsilon \) and \( \dot{x}_\epsilon \) stay in \( I_1 \) respectively \( I_2 \) (which are defined independently of \( \epsilon \)), \( x_\epsilon \) and \( \dot{x}_\epsilon \) are bounded by \( b \) and \( c + K \| F_2 \|_{L^1, \infty} \), respectively. \( \square \)

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