UNIQUENESS IN INVERSE ACOUSTIC AND ELEKTRMAGNETIC SCATTERING WITH PHASELESS NEAR-FIELD DATA AT A FIXED FREQUENCY

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ABSTRACT. This paper is concerned with uniqueness results in inverse acoustic and electromagnetic scattering problems with phaseless total-field data at a fixed frequency. We use superpositions of two point sources as the incident fields at a fixed frequency and measure the modulus of the acoustic total-field (called phaseless acoustic near-field data) on two spheres containing the scatterers generated by such incident fields on the two spheres. Based on this idea, we prove that the impenetrable bounded obstacle or the index of refraction of an inhomogeneous medium can be uniquely determined from the phaseless acoustic near-field data at a fixed frequency. Moreover, the idea is also applied to the electromagnetic case, and it is proved that the impenetrable bounded obstacle or the index of refraction of an inhomogeneous medium can be uniquely determined by the phaseless electric near-field data at a fixed frequency, that is, the modulus of the tangential component with the orientations $e_\phi$ and $e_\theta$, respectively, of the electric total-field measured on a sphere enclosing the scatterers and generated by superpositions of two electric dipoles at a fixed frequency located on the measurement sphere and another bigger sphere with the polarization vectors $e_\phi$ and $e_\theta$, respectively. As far as we know, this is the first uniqueness result for three-dimensional inverse electromagnetic scattering with phaseless near-field data.

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1. Introduction. Inverse scattering problems occur in many applications such as radar, remote sensing, geophysics, medical imaging and nondestructive testing. These problems aim at reconstructing the unknown scatterers from the measurement data of the scattered waves. In the past decades, inverse acoustic and electromagnetic scattering problems with phased data have been extensively studied mathematically and numerically. A comprehensive account of these studies can be found in the monographs [10, 14].

In many practical applications, it is much harder to obtain data with accurate phase information compared with just measuring the intensity (or the modulus) of the data, and thus it is often desirable to study inverse scattering with phaseless data (see, e.g., [10, Chapter 8] and the references quoted there). In fact, inverse scattering problems with phaseless data have also been widely studied numerically over the past decades (see, e.g. [2, 3, 10–13, 18, 26, 29, 34, 39, 41, 46–48] and the references quoted there).

Recently, uniqueness and stability results have also been established for inverse scattering with phaseless data (see, e.g. [1, 19, 20, 24, 25, 27, 31–33, 37, 42, 44, 45, 49]). For example, for point source incidence uniqueness results have been established in [24, 25] for inverse potential and acoustic medium scattering with the phaseless near-field data generated by point sources placed on a sphere enclosing the scatterer and measured in a small ball centered at each source position for an interval of frequencies, and in [32] for inverse acoustic medium scattering with the phaseless near-field data measured on an annulus surrounding the scatterer at fixed frequency.

The purpose of this paper is to propose a new approach to establish uniqueness results for inverse acoustic scattering problems with phaseless total-field data at a fixed frequency. Our approach is based on using superpositions of two point sources at a fixed frequency as the incident fields and making use of two spheres, which contain the scatterers, as the locations of such incident fields and the measurement surfaces of the modulus of the acoustic total-field (the sum of the incident field and the scattered field). In fact, many phase retrieval algorithms have been developed for inverse scattering problems with phaseless near-field data measured on two surfaces to ensure the reliability of the near-field phase reconstruction algorithms (see, e.g. [17, 35, 36]). Based on this idea, we prove that the impenetrable bounded obstacle or the index of refraction of the inhomogeneous medium can be uniquely determined from the phaseless total-field data at a fixed frequency. Note that the superposition of two point sources was also used in [37] as the incident field to study uniqueness for phaseless inverse scattering problems. Some related uniqueness results can be found in [40, 50, 51].

The idea is also applied to phaseless inverse electromagnetic scattering which is more complicated than the acoustic case. In this case, the electric total field is a complex vector-valued function, so we need to define the phaseless data used in this paper. In many applications (see, e.g. [5, 34, 38]), the phaseless near-field data are based on the measurement of the modulus of the tangential component of the electric total field on the measurement surface. Further, it has been elaborated in [16] that the measurement data are based on two tangential components of the electric field on the measurement sphere (see [16, p.100]). Therefore, the phaseless near-field data used is the modulus of the tangential component in the orientations $\mathbf{e}_\phi$ and $\mathbf{e}_\theta$, respectively, of the electric total field measured on a sphere enclosing the scatters and generated by superpositions of two electric dipoles at a fixed frequency located on the measurement sphere and another bigger sphere with
the polarizations $e_{\phi}$ and $e_{\theta}$, respectively. Following a similar idea as in the acoustic case, we prove that the impenetrable bounded obstacle or the refractive index of the inhomogeneous medium (under the condition that the magnetic permeability is a positive constant) can be uniquely determined by the phaseless total-field data at a fixed frequency. To the best of our knowledge, this is the first uniqueness result for three-dimensional inverse electromagnetic scattering with phaseless near-field data. It should be mentioned that our uniqueness results in this paper are based on parts of the PhD thesis [43].

The outline of this paper is as follows. The acoustic and electromagnetic scattering models considered in this paper are given in Section 2. Sections 3 and 4 are devoted to the uniqueness results for phaseless inverse acoustic and electromagnetic scattering problems, respectively. Conclusions are given in Section 5.

2. The direct scattering problems. We will introduce the acoustic and electromagnetic scattering models considered in this paper. To this end, assume that $D$ is an open and bounded domain in $\mathbb{R}^3$ with a $C^2$-boundary $\partial D$ such that the exterior $\mathbb{R}^3 \setminus \overline{D}$ is connected. Assume further that $D \subset B_{R_1}$, where $B_{R_1}$ is a ball centered at the origin with radius $R_1 > 0$ large enough.

2.1. The acoustic case. In this paper, we consider the problem of acoustic scattering by an impenetrable obstacle or an inhomogeneous medium in $\mathbb{R}^3$. We need the following fundamental solution to the three-dimensional Helmholtz equation

$$\Delta w + k^2 w = 0 \text{ in } \mathbb{R}^3$$

with $k > 0$:

$$\Phi_k(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad x,y \in \mathbb{R}^3, \ x \neq y.$$ 

For arbitrarily fixed $y \in \mathbb{R}^3 \setminus \overline{D}$ consider the time-harmonic ($e^{-i\omega t}$ time dependence) point source

$$w^i := w^i(x,y) = \Phi_k(x,y), \quad x \in \mathbb{R}^3 \setminus \overline{D},$$

which is incident on the obstacle $D$ from the unbounded part $\mathbb{R}^3 \setminus \overline{D}$, where $k = \omega/c > 0$ is the wave number, $\omega$ and $c$ are the wave frequency and speed in the homogeneous medium in the whole space. Then the problem of scattering of the point source $w^i$ by the impenetrable obstacle $D$ is formulated as the exterior boundary value problem:

(1) $\Delta_x w^s(x,y) + k^2 w^s(x,y) = 0, \quad x \in \mathbb{R}^3 \setminus \overline{D},$

(2) $\mathcal{B}w = 0 \quad \text{on } \partial D,$

(3) $\lim_{r \to \infty} r \left( \frac{\partial w^s}{\partial r} - ikw^s \right) = 0, \quad r = |x|,$

where $w^s$ is the scattered field, $w := w^i + w^s$ is the total field, and (3) is the Sommerfeld radiation condition imposed on the scattered field $w^s$. The boundary condition $\mathcal{B}$ in (2) depends on the physical property of the obstacles $D$:

$$\left\{ \begin{array}{ll}
\mathcal{B}w := w & \text{on } \partial D \quad \text{if } D \text{ is a sound-soft obstacle}, \\
\mathcal{B}w := \partial w/\partial \nu + \eta w & \text{on } \partial D \quad \text{if } D \text{ is an impedance obstacle}, \\
\mathcal{B}w := w & \text{on } \Gamma_D, \ \mathcal{B}w := \partial w/\partial \nu + \eta w & \text{on } \Gamma_I \quad \text{if } D \text{ is a partially coated obstacle},
\end{array} \right.$$ 

where $\nu$ is the unit outward normal to the boundary $\partial D$ and $\eta$ is the impedance function on $\partial D$ satisfying that $\text{Im}[\eta(x)] \geq 0$ for all $x \in \partial D$ or $x \in \Gamma_I$. We assume that $\eta \in C(\partial D)$ or $\eta \in C(\Gamma_I)$, that is, $\eta$ is continuous on $\partial D$ or $\Gamma_I$. When $\eta = 0,$
the impedance boundary condition becomes the Neumann boundary condition (a sound-hard obstacle). For a partially coated obstacle, we assume that the boundary ∂D has a Lipschitz dissection ∂D = ΓD ∪ Π ∪ ΓI, where ΓD and ΓI are disjoint, relatively open subsets of ∂D and having Π as their common boundary in ∂D (see, e.g., [8]).

The problem of scattering of the point source \( w^i \) by an inhomogeneous medium is modeled as follows:

\[
\Delta x w^s(x, y) + k^2 n(x)w^s(x, y) = k^2 (1 - n(x))w^i(x, y), \quad x \in \mathbb{R}^3, \\
\lim_{r \to \infty} r \left( \frac{\partial w^s}{\partial r} - ikw^s \right) = 0, \quad r = |x|, 
\]

where \( w^s \) is the scattered field and \( n \) in (4) is the refractive index characterizing the inhomogeneous medium. We assume that \( n-1 \) has compact support \( \mathcal{D}, n \in L^\infty(D), \) \( \text{Re}[n(x)] \geq n_{\text{min}} > 0 \) for a constant \( n_{\text{min}} \) in \( D \) and \( \text{Im}[n(x)] \geq 0 \) in \( D \).

The existence of a unique (variational) solution to the problems (1)-(3) and (4)-(5) has been proved in [7, 14, 21, 22]. In particular, the scattered-field \( w^s \) has the asymptotic behavior:

\[
w^s(x, y) = \frac{e^{ik|x|}}{|x|} \left\{ w^\infty(\hat{x}, y) + O \left( \frac{1}{|x|} \right) \right\}, \quad |x| \to \infty
\]

uniformly for all observation directions \( \hat{x} = x/|x| \in S^2 \), where \( S^2 \) is the unit sphere in \( \mathbb{R}^3 \) and \( w^\infty(\hat{x}, y) \) is the far-field pattern of \( w^s \) which is an analytic function of \( \hat{x} \in S^2 \) for each \( y \in \mathbb{R}^3 \setminus \mathcal{D} \) (see, e.g., [14, (2.13)]).

In this paper, we also consider the superposition of two point sources

\[
w^j = w^i(x; y_1, y_2) = w^i(x, y_1) + w^i(x, y_2) = \Phi_k(x, y_1) + \Phi_k(x, y_2)
\]

as the incident field, where \( y_1, y_2 \in \mathbb{R}^3 \setminus \mathcal{D} \) are the locations of the two point sources. It then follows by the linear superposition principle that the corresponding scattered field

\[
w^s(x; y_1, y_2) = w^s(x, y_1) + w^s(x, y_2)
\]

and the corresponding total field

\[
w(x; y_1, y_2) = w(x, y_1) + w(x, y_2),
\]

where \( w^s(x, y_j) \) and \( w(x, y_j) \) are the scattered field and the total field corresponding to the incident point source \( w^i(x, y_j) \), respectively, \( j = 1, 2 \).

The inverse acoustic obstacle (or medium) scattering problem we consider in this paper is to reconstruct the obstacle \( D \) and its physical property (or the index of refraction \( n \) of the inhomogeneous medium) from the phaseless total field \( |w(x; y_1, y_2)| \) for \( x, y_1, y_2 \) on some spheres enclosing \( D \) and the inhomogeneous medium.

2.2. The electromagnetic case. In this paper, we consider two electromagnetic scattering models, that is, scattering by an impenetrable obstacle and scattering by an inhomogeneous medium. We will consider the time-harmonic \( e^{-i\omega t} \) time dependence) incident electric dipole located at \( y \in \mathbb{R}^3 \setminus \mathcal{D} \) and described by the matrices \( E^i(x, y) \) and \( H^i(x, y) \) defined by

\[
E^i(x, y)p := \frac{i}{k} \text{curl}_x \text{curl}_x [p \Phi_k(x, y)], \quad H^i(x, y)p := \text{curl}_x [p \Phi_k(x, y)], \quad x \neq y,
\]
for \( x \in \mathbb{R}^3 \), where \( p \in \mathbb{R}^3 \{ (0,0,0) \} \) is the polarization vector, \( k := \omega/c > 0 \) is the wave number, \( \omega \) and \( c := 1/\sqrt{\varepsilon_0 \mu_0} \) are the wave frequency and speed in the homogeneous medium in \( \mathbb{R}^3 \setminus D \), respectively, and \( \varepsilon_0 \) and \( \mu_0 \) are the electric permittivity and the magnetic permeability of the homogeneous medium, respectively. A direct calculation shows that for \( x \neq y \),

\[
E^i(x,y) = \frac{i}{k} \left[ k^2 + \frac{1}{|x-y|^2} \right] I + \frac{i}{k} \nabla_x \nabla_y \Phi_k(x,y)
\]

where \( I \) is a 3 \( \times \) 3 identity matrix, \( \nabla_x \nabla_y := (\partial_x, \partial_y)_{3 \times 3} \), and \( f(x) := (x-y)/|x-y|^2 \) and \( f(r) := 3/r^2 - 3ik/r - k^2 \). Then the problem of scattering of the electric dipole \( E^s \) and \( H^s \) by the impenetrable obstacle \( D \) can be modeled as the exterior boundary value problem:

\[
\begin{align*}
\text{curl}_x E^s - ikH^s &= 0 \quad \text{in } \mathbb{R}^3 \setminus D, \\
\text{curl}_x H^s + ikE^s &= 0 \quad \text{in } \mathbb{R}^3 \setminus D, \\
\mathcal{B} E &= 0 \quad \text{on } \partial D, \\
\lim_{r \to \infty} (H^s \times x - rE^s) &= 0, \quad r = |x|,
\end{align*}
\]

where \( (E^s, H^s) \) is the scattered field, \( E := E^i + E^s \) and \( H := H^i + H^s \) are the electric total field and the magnetic total field, respectively, and (13) is the Silver–Müller radiation condition which holds uniformly for all \( \hat{x} \in \mathbb{S}^2 \) and ensures the uniqueness of the scattered field. The boundary condition \( \mathcal{B} \) in (12) depends on the physical property of the obstacle \( D \), that is, \( \mathcal{B} E := \nu \times E \) on \( \partial D \) (called as the PEC condition) if \( D \) is a perfect conductor, where \( \nu \) is the unit outward normal to the boundary \( \partial D \), \( \mathcal{B} E := \nu \times \text{curl}E - i\lambda(\nu \times E) \times \nu \) on \( \partial D \) if \( D \) is an impedance obstacle, where \( \lambda \) is the impedance function on \( \partial D \), and

\[
\mathcal{B} E := \nu \times E \quad \text{on } \Gamma_D, \quad \mathcal{B} E := \nu \times \text{curl}E - i\lambda(\nu \times E) \times \nu \quad \text{on } \Gamma_I
\]

if \( D \) is a partially coated obstacle, where \( \partial D \) has a Lipschitz dissection \( \partial D = \Gamma_D \cup \Pi \cup \Gamma_I \) with \( \Gamma_D \) and \( \Gamma_I \) being disjoint and relatively open subsets of \( \partial D \) and having \( \Pi \) as their common boundary in \( \partial D \) and \( \lambda \) is the impedance function on \( \Gamma_I \). We assume throughout this paper that \( \lambda \in C(\partial D) \) with \( \lambda(x) \geq 0 \) for all \( x \in \partial D \) or \( \lambda \in C(\Gamma_I) \) with \( \lambda(x) \geq 0 \) for all \( x \in \Gamma_I \).

The problem of scattering of an electric dipole by an inhomogeneous medium is modeled as the medium scattering problem:

\[
\begin{align*}
\text{curl}_x E^s - ikH^s &= 0 \quad \text{in } \mathbb{R}^3, \\
\text{curl}_x H^s + ikn(x)E^s &= ik(1-n(x))E^i \quad \text{in } \mathbb{R}^3, \\
\lim_{r \to \infty} (H^s \times x - rE^s) &= 0, \quad r = |x|,
\end{align*}
\]

where \( (E^s, H^s) \) is the scattered field and \( (E, H) := (E^i, H^i) + (E^s, H^s) \) is the total field. The refractive index \( n(x) \) in (15) is given by

\[
n(x) := \frac{1}{\varepsilon_0} \left( \varepsilon(x) + i \frac{\sigma(x)}{\omega} \right),
\]

where \( \varepsilon(x) \) and \( \sigma(x) \) are the electric permittivity and electric conductivity in \( \mathbb{R}^3 \), respectively. In this paper, we assume the magnetic permeability \( \mu = \mu_0 \) to be a positive constant in the whole space. We assume further that \( n - 1 \) has a compact
support \( D \), \( n \in C^2(\mathbb{R}^3) \) for \( 0 < \gamma < 1 \), \( \text{Re}[n(x)] \geq n_{\text{min}} > 0 \) in \( D \) for a constant \( n_{\text{min}} \) and \( \text{Im}[n(x)] \geq 0 \) in \( D \).

The existence of a unique (variational) solution to the problems (10)–(13) and (14)–(16) has been established in [8, 9, 14]. In particular, it is well known that the electromagnetic scattered field \( E^s \) has the asymptotic behavior:

\[
E^s(x, y)p = \frac{e^{ik|x|}}{|x|} \left\{ E^\infty(\hat{x}, y)p + \mathcal{O}\left( \frac{1}{|x|} \right) \right\}, \quad |x| \to \infty
\]

uniformly for all observation directions \( \hat{x} = x/|x| \in \mathbb{S}^2 \), where \( E^\infty(\hat{x}, y) \) is the electric far-field pattern of \( E^s \) which is an analytic function of \( \hat{x} \) in \( \mathbb{S}^2 \) for each \( y \in \mathbb{R}^3 \setminus \hat{D} \) (see, e.g., [14, (6.23)]). Because of the linearity of the direct scattering problem with respect to the incident field, we can express the scattered waves by matrices \( E^s(x, y) \) and \( H^s(x, y) \), the total waves by matrices \( E(x, y) \) and \( H(x, y) \), and the far-field patterns by \( E^\infty(\hat{x}, y) \) and \( H^\infty(\hat{x}, y) \), respectively.

We will also consider the following superposition of two electric dipoles as the incident field:

\[
E^i = E^i(x, y_1)p_1 + E^i(x, y_2)p_2 = \frac{i}{k} \text{curl}_x \text{curl}_x[p_1 \Phi_k(x, y_1) + p_2 \Phi_k(x, y_2)],
\]

\[
H^i = H^i(x, y_1)p_1 + H^i(x, y_2)p_2 = \text{curl}_x[p_1 \Phi_k(x, y_1) + p_2 \Phi_k(x, y_2)],
\]

where \( x \in \mathbb{R}^3 \), \( y_1, y_2 \in \mathbb{R}^3 \setminus \hat{D}, x \neq y_1, x \neq y_2 \) and \( p_1, p_2 \in \mathbb{R}^3 \setminus \{(0, 0, 0)\} \). For convenience, we define the following incident field:

\[
E^i = E^i(x, y_1, p_1, \tau_1, y_2, p_2, \tau_2) := \tau_1 E^i(x, y_1)p_1 + \tau_2 E^i(x, y_2)p_2 = \frac{i}{k} \text{curl}_x \text{curl}_x[\tau_1 p_1 \Phi_k(x, y_1) + \tau_2 p_2 \Phi_k(x, y_2)],
\]

\[
H^i = H^i(x, y_1, p_1, \tau_1, y_2, p_2, \tau_2) := \tau_1 H^i(x, y_1)p_1 + \tau_2 H^i(x, y_2)p_2 = \text{curl}_x[\tau_1 p_1 \Phi_k(x, y_1) + \tau_2 p_2 \Phi_k(x, y_2)],
\]

with \( x \in \mathbb{R}^3, y_1, y_2 \in \mathbb{R}^3 \setminus \hat{D}, x \neq y_1, x \neq y_2, p_1, p_2 \in \mathbb{R}^3 \setminus \{(0, 0, 0)\} \) and \( \{\tau_1, \tau_2\} \in \{(1, 0), (0, 1), (1, 1)\} \). By the linear superposition principle, the electric scattered field and total field corresponding to the incident field \( E^i(x, y_1, p_1, \tau_1, y_2, p_2, \tau_2) \), \( H^i(x, y_1, p_1, \tau_1, y_2, p_2, \tau_2) \) satisfy

\[
E^s(x, y_1, p_1, \tau_1, y_2, p_2, \tau_2) := \tau_1 E^s(x, y_1)p_1 + \tau_2 E^s(x, y_2)p_2
\]

and

\[
E(x, y_1, p_1, \tau_1, y_2, p_2, \tau_2) := \tau_1 E(x, y_1)p_1 + \tau_2 E(x, y_2)p_2,
\]

where \( E^s(x, y_j)p_j \) and \( E(x, y_j)p_j \) are the electric scattered field and the electric total field corresponding to the incident electric field \( E^i(x, y_j)p_j \), respectively, \( j = 1, 2 \).

Following [16, 34, 38], we measure the modulus of the tangential component of the electric total field on a sphere \( \partial B_r \) centered at the origin with radius \( r > 0 \). To represent the tangential components, we introduce the following spherical coordinate

\[
\begin{align*}
x_1 &= r \sin \theta \cos \phi, \\
x_2 &= r \sin \theta \sin \phi, \\
x_3 &= r \cos \theta,
\end{align*}
\]

with \( x := (x_1, x_2, x_3) \in \mathbb{R}^3 \) and \( (r, \theta, \phi) \in [0, +\infty) \times [0, \pi] \times [0, 2\pi] \). For any \( x \in \partial B_r \), the spherical coordinate gives an one-to-one correspondence between \( x \) and \( (r, \phi, \theta) \).
Here, $\partial B_r := \partial B_r \setminus \{ N_r, S_r \}$ with $N_r := (0,0,r)$ and $S_r := (0,0,-r)$ denoting the north and south poles of $\partial B_r$, respectively. If we define
\[
edphi(x) := (-\sin \phi, \cos \phi, 0), \quad \nedthet(x) := (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta),
\]
then $\nedphi(x)$ and $\nedthet(x)$ are two orthonormal tangential vectors of $\partial B_r$ at $x \in \partial B_r$. Now, we can represent our phaseless measurement data by
\[
|e_m(x) \cdot E(x, y_1, e_n(y_1), \tau_1, y_2, e_l(y_2), \tau_2) |
\]
with $x, y_1, y_2 \in \partial B_r, x \neq y_1, x \neq y_2, m, n, l \in \{ \phi, \theta \}$ and $(\tau_1, \tau_2) \in \{(1,0), (0,1), (1,1) \}$.

The inverse electromagnetic obstacle or medium scattering problem we consider in this paper is to reconstruct the obstacle $D$ and its physical property or the index of refraction $n$ of the inhomogeneous medium from the modulus of the tangential component of the electric total field, $|e_m(x) \cdot E(x, y_1, e_n(y_1), \tau_1, y_2, e_l(y_2), \tau_2)|$, for all $x, y_1, y_2$ in some spheres enclosing $D$ or the inhomogeneous medium, $m, n, l \in \{ \phi, \theta \}$ and $(\tau_1, \tau_2) \in \{(1,0), (0,1), (1,1) \}$. The purpose of this paper is to prove uniqueness results for the above inverse acoustic and electromagnetic scattering problems.

3. Inverse acoustic scattering with phaseless total-field data. This section is devoted to the uniqueness results for inverse acoustic scattering with phaseless total-field data at a fixed frequency measured on two spheres enclosing the scatterers (see Figure 1).

Denote by $w_j^s$ and $w_j$ the scattered field and the total field, respectively, associated with the impenetrable obstacle $D_j$ (or the refractive index $n_j$) and corresponding to the incident field $w^i$, $j = 1, 2$. Let $B_{R_2}$ denote the ball centered at the origin with radius $R_2 > R_1 > 0$ with $\partial B_{R_2}$ denoting the boundary of $B_{R_2}$. By appropriately choosing $R_2 > R_1 > 0$, it can be ensured that $k^2$ is not a Dirichlet eigenvalue of $-\Delta$ in $B_{R_2} \setminus \overline{B_{R_1}}$. Here, $k^2$ is called a Dirichlet eigenvalue of $-\Delta$ in a bounded domain $V$ if the the interior Dirichlet boundary value problem
\[
\begin{align*}
\Delta u + k^2 u &= 0 \quad \text{in } V, \\
u &= 0 \quad \text{on } \partial V
\end{align*}
\]
has a nontrivial solution $u$. The above assumption on $R_1$ and $R_2$ can be easily satisfied since the Dirichlet eigenvalues of $-\Delta$ in a bounded domain are discrete and satisfy the strong monotonicity property [28, Theorem 4.7] (see also the arguments in the proof of [14, Theorem 5.2]). Let $G$ denote the unbounded component of the complement of $D_1 \cup D_2$. Then we have the following global uniqueness results for the phaseless inverse scattering problems.

**Theorem 3.1.** Let $D_1, D_2$ be two bounded domains and let $R_2 > R_1 > 0$ be large enough so that $\overline{D_1 \cup D_2} \subset B_{R_1}$. Assume that $k^2$ is not a Dirichlet eigenvalue of $-\Delta$ in $B_{R_2} \setminus \overline{B_{R_1}}$.

(a) Assume that $D_1$ and $D_2$ are two impenetrable obstacles with boundary conditions $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. If the corresponding total fields satisfy
\[
|w_1(x,y)| = |w_2(x,y)|, \quad \forall (x, y) \in (\partial B_{R_1} \times \partial B_{R_1}) \cup (\partial B_{R_2} \times (\{y_0\} \cup \partial B_{R_2})),
\]
when $x \neq y$ and
\[
|w_1(x,y; y_0)| = |w_2(x,y; y_0)|, \quad \forall (x, y) \in (\partial B_{R_1} \times \partial B_{R_1}) \cup (\partial B_{R_2} \times \partial B_{R_2}),
\]
when $x \neq y, y_0$.
Proof. Since \( w \) is the total field of the obstacle scattering problem (1)-(3) or the medium scattering problem (4)-(5) associated with the point source \( w'(x,y) \). Then, for any fixed \( y_0 \in \partial B_{R_1} \), we have

\[
\begin{align*}
(20) & \quad w(x, y_0) \neq 0, \quad x \in \partial B_{R_1}, \quad x \neq y_0, \\
(21) & \quad w'(x, y_0) \neq 0, \quad x \in \partial B_{R_2}, \\
(22) & \quad w(x, y) \neq 0, \quad (x, y) \in \partial B_{R_2} \times \partial B_{R_2}, \quad x \neq y.
\end{align*}
\]

Proof. Since \( w(x, y) \) is singular at \( x = y_0 \) or \( y \), we know that (20) and (22) are true.

We now prove (21). Assume to the contrary that \( w(x, y_0) \equiv 0 \) for \( x \in \partial B_{R_2} \), that is, \( w'(x, y_0) = -\Phi_k(x, y_0) \) for \( x \in \partial B_{R_2} \). Then, by the uniqueness of the exterior Dirichlet problem it follows that \( w'(x, y_0) = -\Phi_k(x, y_0) \) for all \( x \in \mathbb{R}^3 \setminus \overline{B_{R_2}} \). Since the scattered field \( w'(x, y_0) \) is analytic for \( x \in \mathbb{R}^3 \setminus \overline{D} \) and \( \Phi_k(x, y_0) \) is analytic for \( x \in \mathbb{R}^3 \setminus \{y_0\} \), we have \( w'(x, y_0) = -\Phi_k(x, y_0) \) for all \( x \in \mathbb{R}^3 \setminus (\overline{D} \cup \{y_0\}) \). This is a contradiction since \( \Phi_k(x, y_0) \) has a singularity at \( x = y_0 \in \partial B_{R_1} \) and \( w'(x, y_0) \) is analytic when \( x \) is in a neighbourhood of \( y_0 \). Thus, (21) is true.

Lemma 3.3. Under the assumption of Lemma 3.2, we have the following results.

(i) There exist two open sets \( U_1, U_2 \subset \partial B_{R_3} \) such that \( U_1 \cap U_2 = \emptyset \) and \( w(x, y) \neq 0 \) for all \( (x, y) \in U_1 \times U_2 \).

(ii) There exist two open sets \( U'_1, U'_2 \subset \partial B_{R_2} \) such that \( U'_1 \cap U'_2 = \emptyset \) and \( w(x, y) \neq 0 \) for all \( (x, y) \in U'_1 \times (U'_2 \cup \{y_0\}) \), where \( y_0 \in \partial B_{R_3} \).

Proof. We only prove (ii). The proof of (i) is similar.

By (21) we know that for \( y_0 \in \partial B_{R_2} \) there exists \( x_0 \in \partial B_{R_3} \) such that \( w(x_0, y_0) \neq 0 \). Since \( w(x, y) \) is continuous for \( (x, y) \in \mathbb{R}^3 \setminus \overline{D} \), there exists a neighbourhood \( U' \subset \partial B_{R_3} \) of \( x_0 \) such that \( w(x, y_0) \neq 0 \) for all \( x \in U' \). Further, since \( w(x, y) \) is analytic with respect to \( x \in \partial B_{R_2} \) and \( y \in \partial B_{R_2} \), respectively, when

Figure 1. Acoustic scattering by an obstacle (left) or a medium (right).
$x \neq y$, then it follows from (22) that there exist two points $x_1 \in U'$ and $x_2 \in \partial B_{R_2}$ such that $w(x_1, x_2) \neq 0$ with $x_1 \neq x_2$. Finally, again by the continuity of $w(x, y)$ for $x, y \in \mathbb{R}^3 \setminus \mathcal{D}$ with $x \neq y$, there exists a neighbourhood $U'_1 \subset U'$ of $x_1$ and a neighbourhood $U'_2 \subset \partial B_{R_2}$ of $x_2$ such that $U'_1 \cap U'_2 = \emptyset$ and $w(x, y) \neq 0$ for all $(x, y) \in U'_1 \times U'_2$. Thus, $w(x, y) \neq 0$ for all $(x, y) \in U'_1 \times (U'_2 \cup \{y_0\})$. This completes the proof.

**Proof of Theorem 3.1.** From (8) it is easy to see that (19) is equivalent to the equation

$$|w_1(x, y) + w_1(x, y_0)| = |w_2(x, y) + w_2(x, y_0)|$$

for all $(x, y) \in (\partial B_{R_1} \times \partial B_{R_1}) \cup (\partial B_{R_2} \times \partial B_{R_2})$ with $x \neq y, y_0$. This, together with (18), implies that

$$\text{Re}\{w_1(x, y)w_1(x, y_0)\} = \text{Re}\{w_2(x, y)w_2(x, y_0)\}$$

for all $(x, y) \in (\partial B_{R_1} \times \partial B_{R_1}) \cup (\partial B_{R_2} \times \partial B_{R_2})$ with $x \neq y, y_0$. Define $r_j(x, y) := |w_j(x, y)|$, $j = 1, 2$. Then it follows from (18) that $r_1(x, y) = r_2(x, y) =: r(x, y)$, for all $x \in \partial B_{R_1}$, $y \in \partial B_{R_1} \cup \partial B_{R_2}$ with $x \neq y$, so we can write

$$w_j(x, y) = r(x, y)e^{i\vartheta_j(x, y)}, \quad \forall x, y \in \partial B_{R_1} \cup \partial B_{R_2}, \quad x \neq y, \quad j = 1, 2$$

with real-valued functions $\vartheta_j(x, y), j = 1, 2$.

**Case 1.** (23) holds with $(x, y) \in \partial B_{R_1} \times \partial B_{R_1}$, $x \neq y$.

Since $w_j^*(x, y)$, $j = 1, 2$, are analytic functions of $x \in \partial B_{R_1}$ and $y \in \partial B_{R_1}$, respectively, and $\Phi_k(x, y)$ has a singularity at $x = y$, then, by Lemma 3.3 we can choose two open sets $U_1, U_2 \subset \partial B_{R_1}$ small enough so that $U_1 \cap U_2 = \emptyset$, $r(x, y) \neq 0$ for all $(x, y) \in U_1 \times (U_2 \cup \{y_0\})$, and $\vartheta_j(x, y), j = 1, 2$, are analytic with respect to $x \in U_1$ and $y \in U_2$, respectively.

Now, by (23) we have

$$\cos[\vartheta_1(x, y) - \vartheta_1(x, y_0)] = \cos[\vartheta_2(x, y) - \vartheta_2(x, y_0)]$$

for all $(x, y) \in U_1 \times U_2$. Since $\vartheta_j(x, y), j = 1, 2$, are real-valued analytic functions of $x \in U_1$ and $y \in U_2$, respectively, we have either

$$\vartheta_1(x, y) - \vartheta_1(x, y_0) = \vartheta_2(x, y) - \vartheta_2(x, y_0) + 2q\pi, \quad \forall (x, y) \in U_1 \times U_2$$

or

$$\vartheta_1(x, y) - \vartheta_1(x, y_0) = -[\vartheta_2(x, y) - \vartheta_2(x, y_0)] + 2q\pi, \quad \forall (x, y) \in U_1 \times U_2,$$

where $q \in \mathbb{Z}$.

For the case when (25) holds, we have

$$\vartheta_1(x, y) - \vartheta_2(x, y) = \vartheta_1(x, y_0) - \vartheta_2(x, y_0) + 2q\pi, \quad \forall (x, y) \in U_1 \times U_2.$$  

This implies that $\alpha(x) := \vartheta_1(x, y) - \vartheta_2(x, y) = \vartheta_1(x, y_0) - \vartheta_2(x, y_0) + 2q\pi$ depends only on $x \in U_1$. Then it follows that

$$w_1(x, y) = r(x, y)e^{i\vartheta_1(x, y)} = r(x, y)e^{i\alpha(x) + i\vartheta_2(x, y)} = e^{i\alpha(x)}w_2(x, y)$$

for all $x \in U_1$ and $y \in U_2 \cup \{y_0\}$. By the analyticity of $w_1(x, y) - e^{i\alpha(x)}w_2(x, y)$ in $y \in \partial B_{R_1}$ with $y \neq x$, we get

$$w_1(x, y) = e^{i\alpha(x)}w_2(x, y), \quad \forall x \in U_1, y \in \partial B_{R_1}, \quad x \neq y.$$ 

Changing the variables $x \to y$ and $y \to x$ in (27) gives

$$w_1(y, x) = e^{i\alpha(y)}w_2(y, x), \quad \forall x \in \partial B_{R_1}, y \in U_1, \quad x \neq y.$$
Use (27), (28) and the reciprocity relation that \( w_j(x, y) = w_j(y, x) \) for all \( x, y \in \partial B_{R_1}, j = 1, 2 \) (see [14, Theorem 3.17]) to give
\[
(29) \quad e^{i\alpha(x)}w_2(x, y) = e^{i\alpha(y)}w_2(x, y), \quad \forall x, y \in U_1 \text{ with } x \neq y.
\]

Since \( w_j(x, y) \) has a singularity at \( x = y \), and by (29) and the analyticity of \( w_j(x, y) (j = 1, 2) \) with respect to \( x \in \partial B_{R_1} \) and \( y \in \partial B_{R_1} \), respectively, with \( x \neq y \), it follows that \( e^{i\alpha(x)} = e^{i\alpha(y)} \) for all \( x, y \in U_1 \) with \( x \neq y \). This means that \( e^{i\alpha(x)} \equiv e^{i\alpha} \) for all \( x \in U_1 \), where \( \alpha \) is a real constant. Substituting this formula into (27) gives that \( w_1(x, y) = e^{i\alpha}w_2(x, y) \) for all \( x \in U_1, y \in \partial B_{R_1} \) with \( x \neq y \). Again, by the analyticity of \( w_j(x, y) (j = 1, 2) \) with respect to \( x \in \partial B_{R_1} \) with \( x \neq y \) we have
\[
(30) \quad w_1(x, y) = e^{i\alpha}w_2(x, y), \quad \forall x, y \in \partial B_{R_1} \text{ with } x \neq y,
\]
which gives
\[
(31) \quad w_1^1(x, y) - e^{i\alpha}w_2^1(x, y) = (e^{i\alpha} - 1)\Phi_k(x, y), \quad \forall x, y \in \partial B_{R_1} \text{ with } x \neq y.
\]

Since \( w_1^1(x, y), j = 1, 2 \), are analytic for \( x \in G \) and \( y \in G \), respectively, and \( \Phi_k(x, y) \) has a singularity at \( x = y \), then passing the limit \( y \to x \) in (31) gives that \( e^{i\alpha} = 1 \), so
\[
(32) \quad w_1^1(x, y) = w_2^1(x, y), \quad \forall x, y \in \partial B_{R_1}.
\]

For the case when (26) holds, a similar argument as above gives
\[
(33) \quad w_1(x, y) = e^{i\beta}w_2(x, y), \quad \forall x, y \in \partial B_{R_1} \text{ with } x \neq y
\]
for a real constant \( \beta \), that is,
\[
(34) \quad w_1^1(x, y) = e^{i\beta}w_2^1(x, y) = e^{i\beta}\Phi_k(x, y), \quad \forall x, y \in \partial B_{R_1} \text{ with } x \neq y.
\]

Since \( w_j^1(x, y), j = 1, 2 \), are analytic for \( x \in G \) and \( y \in G \), respectively, \( \text{Re}[\Phi_k(x, y)] \) has a singularity at \( x = y \) and \( \text{Im}[\Phi_k(x, y)] \) is analytic for all \( x, y \in \mathbb{R}^3 \), then \( e^{i\beta} = 1 \). Thus, it follows from (33) that
\[
(35) \quad w_1(x, y) = w_2(x, y), \quad \forall x, y \in \partial B_{R_1} \text{ with } x \neq y.
\]

Case 2. (23) holds with \( (x, y) \in \partial B_{R_2} \times \partial B_{R_2}, x \neq y \).

By a similar argument as in Case 1, it can be obtained that there holds either
\[
(36) \quad w_1(x, y) = w_2(x, y), \quad \forall x \in \partial B_{R_2}, y \in \partial B_{R_2} \cup \{y_0\}
\]
or
\[
(37) \quad w_1(x, y) = w_2(x, y), \quad \forall x \in \partial B_{R_2}, y \in \partial B_{R_2} \cup \{y_0\} \text{ with } x \neq y.
\]

We now prove that both (34) and (36) can not hold simultaneously. Suppose this is not the case. Then define \( v(x) := w_1(x, y_0) - w_2(x, y_0) \) for \( x \in G \) with \( x \neq y_0 \). Since \( \Phi_k(x, y) - \Phi_k(x, y)^* = i\sin(k|x - y|)/(2\pi|x - y|) \) is analytic for all \( x, y \in \mathbb{R}^3 \), then, by the analyticity of \( w_j^*(x, y) (j = 1, 2) \) with respect to \( x \in G \) it follows that \( v \) can be extended as an analytic function of \( x \in G \), denoted by \( v \) again. Further, since \( i\sin(k|x - y|)/(2\pi|x - y|) \) and \( w_j^*(x, y) (j = 1, 2) \) as functions of \( x \) satisfy the Helmholtz equation \( \Delta u + k^2u = 0 \) in \( G \), we have by (34) and (36) that \( v \) satisfies the Dirichlet boundary value problem:
\[
\begin{align*}
\Delta v + k^2v &= 0 & \text{in } B_{R_2} \setminus \overline{B_{R_1}}, \\
v &= 0 & \text{on } \partial B_{R_1} \cup \partial B_{R_2}.
\end{align*}
\]
From the assumption that $k^2$ is not a Dirichlet eigenvalue of $-\Delta$ in $B_{R_2} \setminus \overline{B_{R_1}}$, it is known that $v(x) = 0$ for any $x \in B_{R_2} \setminus \overline{B_{R_1}}$. Thus $w_1(x, y_0) = w_2(x, y_0)$ for all $x \in B_{R_2} \setminus \overline{B_{R_1}}$ with $x \neq y_0$. By the analyticity of $w_2(x, y_0)$ ($j = 1, 2$) with respect to $x \in G$ with $x \neq y_0$, we obtain

$$w_1(x, y_0) = w_2(x, y_0), \ \forall x \in G, \ x \neq y_0,$$

which contradicts to the fact that $w_2(x, y_0) = \Phi_k(x, y_0) + w_2^*(x, y_0), \ j = 1, 2$, satisfy the Sommerfeld radiation condition. We then conclude that both (34) and (36) can not hold simultaneously. This means that at least one of the formulas (34) and (36) does not hold.

If (34) does not hold, then it follows that (32) holds. By the reciprocity relation, the well-posedness of the exterior Dirichlet problem and the analyticity of $w_2^*(x, y) \ (j = 1, 2)$ with respect to $x \in G$ and $y \in G$, respectively, it is easily derived from (32) that

$$w_2^*(x, y) = w_2^*(x, y), \ \forall x, y \in G.$$

Then, by [14, Theorem 2.13] and the mixed reciprocity relation $4\pi w_2^\infty(-d, z) = u_j^*(z, d)$ for all $d \in \mathbb{S}^2$ and $z \in G, \ j = 1, 2$ (see [14, Theorem 3.16]) it is obtained on applying (38) that

$$w_2^\infty(\hat{x}, d) = w_2^\infty(\hat{x}, d), \ \forall \hat{x}, d \in \mathbb{S}^2,$$

where $w_2^\infty$ is the far-field pattern associated with the obstacle $D_j$ (or the refractive index $n_1$) and corresponding to the incident field $u'(x, d) = e^{ikx \cdot d}, \ j = 1, 2$. Similarly, if (36) does not hold, then (35) holds and thus we can also show that (39) holds.

Finally, for the case with two impenetrable obstacles $D_1$ and $D_2$, it follows from (39), [14, Theorem 5.6] and [30, Theorem 3.7] that $D_1 = D_2$ and $\mathcal{B}_1 = \mathcal{B}_2$, while for the case with two refractive indices $n_1$ and $n_2$, we have by (39) and [21, Theorem 6.26] that $n_1 = n_2$. Theorem 3.1 is thus proved. \hfill $\square$

**Remark 1.** (i) Theorem 3.1 (a) remains true for the two-dimensional case, and the proof is similar.

(ii) Theorem 3.1 (b) also holds in two dimensions if the assumption $n_1, n_2 \in L^\infty(\mathbb{R}^3)$ is replaced by the condition that $n_j$ is piecewise in $W^{1, p}(D_j)$ with $p > 2, \ j = 1, 2$. In this case, the proof is similar except that we need Bukhgeim’s result in [6] (see also the theorem in Section 4.1 in [4]) instead of [21, Theorem 6.26] in the proof.

(iii) Theorem 3.1 (b) generalizes the uniqueness results in [24,25,27,32] substantially in the sense that our uniqueness results only need the measurement data of the modulus of the total-field on two spheres enclosing the inhomogeneous medium at a fixed frequency, under no smoothness assumption on the refractive index, instead of the measurement data in a ball for each point source in a sphere for an interval of frequencies as used in [24,25,27] or in an open domain for each point source in another open domain at a fixed frequency as used in [32].

4. **Inverse electromagnetic scattering with phaseless electric total field data.** In this section, we extend the uniqueness results in Section 3 for the acoustic case to the case of inverse electromagnetic scattering problems with phaseless electric total-field data at a fixed frequency. In this case, we consider the measurement of the modulus of the tangential component of the electric total-field on two spheres.
enclosing the scatterers, generated by superpositions of two electric dipoles located also on the two spheres. Denote by $E_j, E'_j, H'_j$ and $H_j$ the electric scattered-field, electric total-field, magnetic scattered-field and magnetic total-field, respectively, associated with the obstacle $D_j$ (or the refractive index $n_j$) and corresponding to the incident electric field $E'$, $j = 1, 2$. Let $B_{R_2}$ denote the ball centered at the origin with radius $R_2 > R_1 > 0$ with $\partial B_{R_2}$ denoting the boundary of $B_{R_2}$ and let $G$ denote the unbounded component of the complement of $D_1 \cup D_2$. Denote by $N_{R_j}$ and $S_{R_j}$ the north and south poles of $\partial B_{R_j}$, respectively, $j = 1, 2$. Define $\partial B_{R_j} := \partial B_{R_j} \setminus \{N_{R_j}, S_{R_j}\}$, $j = 1, 2$. See Figure 2 for the geometry of the electromagnetic scattering problem.

![Figure 2. Electromagnetic scattering by an obstacle (left) or a medium (right).](image)

By choosing appropriate $R_1$ and $R_2$ (see Lemma 4.1), it can be ensured that $k^2$ is not a Maxwell eigenvalue in $B_{R_2} \setminus B_{R_1}$. Here, $k^2$ is called a Maxwell eigenvalue in a bounded domain $V$ if the interior Maxwell problem

\[
\begin{align*}
\operatorname{curl} E - ikH &= 0 \quad \text{in } V \\
\operatorname{curl} H + ikE &= 0 \quad \text{in } V \\
\nu \times E &= 0 \quad \text{on } \partial V
\end{align*}
\]

has a nontrivial solution $(E, H)$.

**Lemma 4.1.** $k^2$ is not a Maxwell eigenvalue in $B_{R_2} \setminus B_{R_1}$ if and only if

\[
\begin{vmatrix}
j_n(kR_1) & y_n(kR_1) \\
j_n(kR_2) & y_n(kR_2)
j_n(kR_1) + kR_1j'_n(kR_1) & y_n(kR_1) + kR_1y'_n(kR_1) \\
j_n(kR_2) + kR_2j'_n(kR_2) & y_n(kR_2) + kR_2y'_n(kR_2)
\end{vmatrix} \neq 0
\]

for all $n = 1, 2, \ldots$, where $j_n$ and $y_n$ are the spherical Bessel functions and spherical Neumann functions of order $n$, respectively.

**Proof.** Assume that $(E, H)$ solves the interior Maxwell problem

\[
\begin{align*}
\operatorname{curl} E - ikH &= 0 \quad \text{in } B_{R_2} \setminus B_{R_1} \\
\operatorname{curl} H + ikE &= 0 \quad \text{in } B_{R_2} \setminus B_{R_1} \\
\nu \times E &= 0 \quad \text{on } \partial B_{R_2} \cup \partial B_{R_1}
\end{align*}
\]
A similar argument as in the proof of [23, Theorems 2.48 and 2.50] gives the following expansion in the spherical vector harmonics of the electric field $E$ in $\overline{B_{R_2} \setminus B_{R_1}}$ as a uniformly convergent series:

$$E(x) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[ a_n^m \text{curl} \{ xj_n(k|x|)y_n^m(\hat{x}) \} + b_n^m \text{curl} \{ xj_n(k|x|)y_n^{-m}(\hat{x}) \} \right]$$

$$+ \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[ c_n^m \text{curl} \{ xy_n(k|x|)y_n^m(\hat{x}) \} + d_n^m \text{curl} \{ xy_n(k|x|)y_n^{-m}(\hat{x}) \} \right],$$

$$x \in \overline{B_{R_2} \setminus B_{R_1}},$$

where $y_n^m$, $m = -n, \ldots, n$, $n = 0, 1, 2, \ldots$, are the spherical harmonics. By [14, (6.71) and (6.72)], we have that for any $x \in \partial B_r$ with $r \in [R_1, R_2]$,

$$\hat{x} \times E(x)$$

$$= \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[ a_n^m j_n(kr) \text{Grad} y_n^m(\hat{x}) + b_n^m \frac{1}{r} \{ j_n(kr) + krj'_n(kr) \} \hat{x} \times \text{Grad} y_n^m(\hat{x}) \right]$$

$$+ \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[ c_n^m y_n(kr) \text{Grad} y_n^m(\hat{x}) + d_n^m \frac{1}{r} \{ y_n(kr) + kry'_n(kr) \} \hat{x} \times \text{Grad} y_n^m(\hat{x}) \right].$$

By the perfectly conducting boundary condition on $\partial B_{R_2} \cup \partial B_{R_1}$ we have

$$\left( \begin{array}{cc} j_n(kR_1) & y_n(kR_1) \\ j_n(kR_2) & y_n(kR_2) \end{array} \right) \left( \begin{array}{c} a_n^m \\ c_n^m \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right),$$

$$\left( \begin{array}{cc} j_n(kR_1) + kR_1j'_n(kR_1) & y_n(kR_1) + kR_1y'_n(kR_1) \\ j_n(kR_2) + kR_2j'_n(kR_2) & y_n(kR_2) + kR_2y'_n(kR_2) \end{array} \right) \left( \begin{array}{c} b_n^m \\ d_n^m \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$$

for all $n = 1, 2, \ldots$, $m = -n, \ldots, n$. By (40) we have $a_n^m = b_n^m = c_n^m = d_n^m = 0$ for all $n = 1, 2, \ldots$, $m = -n, \ldots, n$, and so $k^2$ is not a Maxwell eigenvalue in $\overline{B_{R_2} \setminus B_{R_1}}$.

On the other hand, if

$$\left| \begin{array}{cc} j_n(kR_1) & y_n(kR_1) \\ j_n(kR_2) & y_n(kR_2) \end{array} \right| = 0$$

or

$$\left| \begin{array}{cc} j_n(kR_1) + kR_1j'_n(kR_1) & y_n(kR_1) + kR_1y'_n(kR_1) \\ j_n(kR_2) + kR_2j'_n(kR_2) & y_n(kR_2) + kR_2y'_n(kR_2) \end{array} \right| = 0$$

for some $n \in \mathbb{N}^+$, then (42) or (43) has non-zero solutions. Thus there exists a nontrivial solution to the interior Maxwell problem (41), and so $k^2$ is a Maxwell eigenvalue in $\overline{B_{R_2} \setminus B_{R_1}}$. The proof is thus complete.

We have the following uniqueness results for the phaseless inverse electromagnetic scattering problems.

**Theorem 4.2.** Let $D_1, D_2$ be two bounded domains and let $R_2 > R_1 > 0$ be large enough such that $D_1 \cup D_2 \subset B_{R_1}$ and $k^2$ is not a Maxwell eigenvalue in $\overline{B_{R_2} \setminus B_{R_1}}$.

(a) Assume that $D_1$ and $D_2$ are two impenetrable obstacles with boundary conditions $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. If the corresponding electric total fields satisfy

$$|e_m(x) \cdot E_1(x, y_1, e_\phi(y_1), \tau_1, y_2, e_\theta(y_2), \tau_2)|$$

$$= |e_m(x) \cdot E_2(x, y_1, e_\phi(y_1), \tau_1, y_2, e_\theta(y_2), \tau_2)|$$

for all $x \in \partial D_1$ and $y_1, y_2 \in \partial D_2$, then there exists a unique solution $(\mathbf{E}_1, \mathbf{E}_2)$ to the scattering problem.

(b) Assume that $D_1$ and $D_2$ are two impenetrable obstacles with boundary conditions $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. If the corresponding electric total fields satisfy

$$|\mathbf{E}_1(x, y_1, e_\phi(y_1), \tau_1, y_2, e_\theta(y_2), \tau_2)|$$

$$= |\mathbf{E}_2(x, y_1, e_\phi(y_1), \tau_1, y_2, e_\theta(y_2), \tau_2)|$$

for all $x \in \partial D_1$ and $y_1, y_2 \in \partial D_2$, then there exists a unique solution $(\mathbf{E}_1, \mathbf{E}_2)$ to the scattering problem.
for all \( x, y, 1 \in \partial \hat{B}_R \), with \( x \neq y, x, 1 \in \partial \hat{B}_R \), and \( (\tau_1, \tau_2) \in \{(1, 0), (0, 1), (1, 1)\} \), \( m \in \{\phi, \theta\} \) and
\[
|e_m(x) \cdot E_1(x, y_1, e_n(y_1), \tau_1, y_2, e_l(y_2), \tau_2)|
\]
\[
= |e_m(x) \cdot E_2(x, y_1, e_n(y_1), \tau_1, y_2, e_l(y_2), \tau_2)|
\]
for all \( x, y_1 \in \partial \hat{B}_R \), with \( x \neq y, y_2 \in \partial \hat{B}_R \), \( (\tau_1, \tau_2) \in \{(0, 1), (1, 1)\} \), \( m, n, l \in \{\phi, \theta\} \), then \( D_1 = D_2 \) and \( \mathcal{B}_1 = \mathcal{B}_2 \).

(b) Assume that \( n_1, n_2 \in C^{2, \gamma}(\mathbb{R}^3) \) with \( \gamma > 0 \) are the refractive indices of two inhomogeneous media with \( n_j - 1 \) supported in \( \mathcal{D}_j \) \( (j = 1, 2) \). If the corresponding electric total fields satisfy (44) and (45), then \( n_1 = n_2 \).

Remark 2. Since \( E_j^v(x, y) = [E_j^v(y, x)]^\top \), and by the reciprocity relation \( E_j^v(x, y) = [E_j^v(y, x)]^\top \) for all \( x, y \in G \) (see [14, Theorem 6.32]), \( j = 1, 2 \), we know that (44) with \( m = \phi \) and \( (\tau_1, \tau_2) = (0, 1) \) is equivalent to (44) with \( m = \theta \) and \( (\tau_1, \tau_2) = (1, 0) \).

To prove Theorem 4.2, we need some results on the phaseless total-fields measured on \( \partial \hat{B}_R \).

Lemma 4.3. Assume that the assumptions of Theorem 4.2 are satisfied. If for any fixed \( m \in \{\phi, \theta\} \) there holds
\[
|e_m(x) \cdot E_1(x, y_1, e_\phi(y_1), \tau_1, y_2, e_\theta(y_2), \tau_2)|
\]
\[
= |e_m(x) \cdot E_2(x, y_1, e_\phi(y_1), \tau_1, y_2, e_\theta(y_2), \tau_2)|
\]
for all \( x, y_1, y_2 \in \partial \hat{B}_R \), with \( x \neq y_1, y_2 \), \( (\tau_1, \tau_2) \in \{(1, 0), (0, 1), (1, 1)\} \), then we have either
\[
e_m(x) \cdot E_1(x, y_1)e_\phi(y_1) = e_m(x) \cdot E_2(x, y_1)e_\phi(y_1), \forall x, y_1 \in \partial \hat{B}_R, x \neq y_1,
\]
\[
e_m(x) \cdot E_1(x, y_2)e_\theta(y_2) = e_m(x) \cdot E_2(x, y_2)e_\theta(y_2), \forall x, y_2 \in \partial \hat{B}_R, x \neq y_2
\]
or
\[
e_m(x) \cdot E_1(x, y_1)e_\phi(y_1) = -e_m(x) \cdot E_2(x, y_1)e_\phi(y_1), \forall x, y_1 \in \partial \hat{B}_R, x \neq y_1,
\]
\[
e_m(x) \cdot E_1(x, y_2)e_\theta(y_2) = -e_m(x) \cdot E_2(x, y_2)e_\theta(y_2), \forall x, y_2 \in \partial \hat{B}_R, x \neq y_2
\]
Proof. We only consider the case \( m = \phi \) since the case \( m = \theta \) can be proved similarly.

Using (17) and (46) and arguing similarly as in the proof of Theorem 3.1 give
\[
\text{Re}\{e_\phi(x) \cdot E_1(x, y_1)e_\phi(y_1)\} \times \text{Re}\{e_\phi(x) \cdot E_2(x, y_1)e_\phi(y_1)\}
\]
\[
= \text{Re}\{e_\phi(x) \cdot E_1(x, y_2)e_\phi(y_2)\} \times \text{Re}\{e_\phi(x) \cdot E_2(x, y_2)e_\phi(y_2)\}
\]
for all \( x, y_1, y_2 \in \partial \hat{B}_R, x \neq y_1, y_2 \). For \( x, y \in \partial \hat{B}_R, x \neq y, \) define
\[
r_j^{(\phi)}(x, y) := |e_\phi(x) \cdot E_j(x, y)e_\phi(y)|, r_j^{(\theta)}(x, y) := |e_\phi(x) \cdot E_j(x, y)e_\theta(y)|, j = 1, 2.
\]
It then follows from (46) with \( (\tau_1, \tau_2) = (1, 0) \) and \( (\tau_1, \tau_2) = (0, 1) \) that
\[
r_1^{(\phi)}(x, y) = r_2^{(\phi)}(x, y) = r_1^{(\theta)}(x, y) = r_2^{(\theta)}(x, y) = r^{(\phi)}(x, y), j = 1, 2
\]
for all \( x, y \in \partial \hat{B}_R, x \neq y \). Therefore we can write
\[
e_\phi(x) \cdot E_j(x, y_1)e_\phi(y_1) := r^{(\phi)}(x, y_1)e_j^{(\phi)}(x, y_1), \forall x, y_1 \in \partial \hat{B}_R, x \neq y_1,
\]
\[
e_\phi(x) \cdot E_j(x, y_2)e_\theta(y_2) := r^{(\theta)}(x, y_2)e_j^{(\phi)}(x, y_2), \forall x, y_2 \in \partial \hat{B}_R, x \neq y_2,
\]
where \( e_j^{(\phi)} \), \( j = 1, 2 \) are real-valued functions.
We now prove that $r^{(\phi \phi)}(x, y_1) \neq 0$, $x, y_1 \in \partial B_R$, $x \neq y_1$, and $r^{(\phi \theta)}(x, y_2) \neq 0$, $x, y_2 \in \partial B_R$, $x \neq y_2$. In fact, fix $y_1 \in \partial B_R$, and define the circle $C_{e_\phi(y_1)} := \{ x \in \partial B_R : (x - y_1) \cdot e_\phi(y_1) = 0 \}$, which is the intersection of the sphere $\partial B_R$ with the plane whose normal vector is $e_\phi(y_1)$ at $y_1$. When $x$ tends to $y_1$ along the circle $C_{e_\phi(y_1)}$, we have $x - y_1 \rightarrow 0$ and $e_\phi(x) \cdot e_\phi(y_1) \rightarrow 1$. Thus, by (9) it is known that

$$(52) e_\phi(x) \cdot E^t(x, y_1) e_\phi(y_1) \sim \frac{i}{k} \left[ k^2 + \left( ik - \frac{1}{|x - y_1|} \right) \frac{1}{|x - y_1|} \right] \Phi_k(x, y_1)$$

as $x$ goes to $y_1$ along the circle $C_{e_\phi(y_1)}$. The singularity in (52) implies that $r^{(\phi \phi)}(x, y_1) \neq 0$ for $x, y_1 \in \partial B_R$, with $x \neq y_1$ since $E^t_j(x, y_1)$ is analytic with respect to $x, y_1 \in \partial B_R$, with $x \neq y_1$, respectively ($j = 1, 2$). Further, fix $y_2 \in \partial B_R$, and define the circle

$$C_{e_\phi(y_2) + e_\theta(y_2)} := \{ x \in \partial B_R : (x - y_2) \cdot (e_\phi(y_2) + e_\theta(y_2)) = 0 \}.$$

Then, on letting $x$ tend to $y_2$ along $C_{e_\phi(y_2) + e_\theta(y_2)}$ we have $e_\phi(x) \cdot e_\theta(y_2) \rightarrow 0$ and $e_\phi(x) \cdot \left[ x - y_2 \cdot x - y_2 \right] e_\theta(y_2) \rightarrow c_1$ for a non-zero constant $c_1$. Thus it follows from (9) that

$$(53) e_\phi(x) \cdot E^t(x, y_2) e_\theta(y_2) = \frac{1}{|x - y_2|^2} \Phi_k(x, y_2) [c_2 + o(1)]$$

as $x \rightarrow y_2$ along $C_{e_\phi(y_2) + e_\theta(y_2)}$, where $c_2$ is a non-zero constant. Therefore the singularity in (53) implies that $r^{(\phi \theta)}(x, y_2) \neq 0$ for $x, y_2 \in \partial B_R$, with $x \neq y_2$ since $E^t_j(x, y_2)$ is analytic with respect to $x, y_2 \in \partial B_R$, $x \neq y_2$, respectively ($j = 1, 2$).

Then, similarly as in the proof of Theorem 3.1, we can show that there are three small enough open sets $U_1, U_2 \subset \partial B_R$, such that $U \subset U_1$ and $U_2$ are disjoint, $r^{(\phi \phi)}(x, y_1) \neq 0$ and $r^{(\phi \theta)}(x, y_2) \neq 0$ for all $x \in U$, $y_1 \in U_1$ and $y_2 \in U_2$, and $\theta_j^{(\phi \phi)}(x, y_1)$ and $\theta_j^{(\phi \theta)}(x, y_2)$ are analytic with respect to $x \in U$, $y_1 \in U_1$, $y_2 \in U_2$, respectively, $j = 1, 2$.

Now, by (51) we have

$$(54) \cos[\theta_1^{(\phi \phi)}(x, y_1) - \theta_1^{(\phi \phi)}(x, y_2)] = \cos[\theta_2^{(\phi \phi)}(x, y_1) - \theta_2^{(\phi \phi)}(x, y_2)]$$

for all $(x, y_1, y_2) \in U \times U_1 \times U_2$. Since $\theta_j^{(\phi \phi)}(x, y_1)$ and $\theta_j^{(\phi \theta)}(x, y_2)$ are analytic functions of $x \in U$, $y_1 \in U_1$ and $y_2 \in U_2$, respectively ($j = 1, 2$), we obtain that there holds either

$$(55) \theta_1^{(\phi \phi)}(x, y_1) - \theta_1^{(\phi \theta)}(x, y_2) = \theta_2^{(\phi \phi)}(x, y_1) - \theta_2^{(\phi \theta)}(x, y_2) + 2q\pi$$

or

$$(56) \theta_1^{(\phi \phi)}(x, y_1) - \theta_1^{(\phi \phi)}(x, y_2) = -[\theta_2^{(\phi \phi)}(x, y_1) - \theta_2^{(\phi \theta)}(x, y_2)] + 2q\pi$$

for all $(x, y_1, y_2) \in U \times U_1 \times U_2$, where $q \in \mathbb{Z}$.

For the case when (55) holds, we have

$$\alpha(x) := \theta_1^{(\phi \phi)}(x, y_1) - \theta_2^{(\phi \phi)}(x, y_1) = \theta_1^{(\phi \phi)}(x, y_2) - \theta_2^{(\phi \phi)}(x, y_2) + 2q\pi$$
depends only on $x$, which is a real-valued analytic function in $x \in U$. Thus
\[
n(x) \cdot E_1(x, y_1) e_\phi(y_1) = n(x) e^{i\alpha(x)}(x, y_1) \\
= n(x) e^{i\alpha(x)}(x, y_1) \\
= e^{i\alpha(x)} e_\phi(x) \cdot E_2(x, y_1) e_\phi(y_1),
\]
\[
e_\phi(x) \cdot E_1(x, y_2) e_\phi(y_2) = e^{i\alpha(x)}(x, y_2) e_\phi(x) \cdot E_2(x, y_2) e_\phi(y_2),
\]
for all $(x, y_1, y_2) \in U \times U_1 \times U_2$. By the analyticity of $E_1(x, y) - e^{i\alpha(x)} E_2(x, y)$ in $y \in \partial B_{R_1}$ for $y \neq x$, we obtain
\[
(57) \quad e_\phi(x) \cdot E_1(x, y_1) e_\phi(y_1) = e^{i\alpha(x)} e_\phi(x) \cdot E_2(x, y_1) e_\phi(y_1), \forall x \in U, y_1 \in \partial B_{R_1}, x \neq y_1,
\]
\[
(58) \quad e_\phi(x) \cdot E_1(x, y_2) e_\phi(y_2) = e^{i\alpha(x)} e_\phi(x) \cdot E_2(x, y_2) e_\phi(y_2), \forall x \in U, y_2 \in \partial B_{R_1}, x \neq y_2.
\]
From (57) it follows that
\[
(59) \quad \left[ E_1^*(x, y_1) e_\phi(y_1) - e^{i\alpha(x)} E_2^*(x, y_1) e_\phi(y_1) \right] \\
= e^{i\alpha(x)} - 1] e_\phi(x) \cdot E^*(x, y_1) e_\phi(y_1)
\]
for all $x \in U$ and $y_1 \in \partial B_{R_1}$ with $x \neq y_1$. For arbitrarily fixed $y_1 \in U$, the left-hand side of (59) is analytic in $x \in U$, while, by (52) the right-hand side of (59) is singular when $x$ is close to $y_1$ along the circle $C_{e_\phi(y_1)}$. Therefore, $e^{i\alpha(y_1)} = 1$. Since $y_1 \in U$ is arbitrary, we have $e^{i\alpha(x)} = 1$ for all $x \in U$, and so (57) and (58) become
\[
(60) \quad e_\phi(x) \cdot E_1(x, y_1) e_\phi(y_1) = e_\phi(x) \cdot E_2(x, y_1) e_\phi(y_1), \forall x \in U, y_1 \in \partial B_{R_1}, x \neq y_1,
\]
\[
(61) \quad e_\phi(x) \cdot E_1(x, y_2) e_\phi(y_2) = e_\phi(x) \cdot E_2(x, y_2) e_\phi(y_2), \forall x \in U, y_2 \in \partial B_{R_1}, x \neq y_2.
\]
This, together with the analyticity of $E_1(x, y)$ ($j = 1, 2$) in $x \in \partial B_{R_1}$ with $x \neq y$, gives (47) and (48).

Similarly, for the case when (56) holds, we can deduce
\[
(62) \quad e_\phi(x) \cdot E_1(x, y_1) e_\phi(y_1) = e^{i\beta(x)} e_\phi(x) \cdot E_2(x, y_1) e_\phi(y_1), \forall x \in U, y_1 \in \partial B_{R_1}, x \neq y_1,
\]
\[
(63) \quad e_\phi(x) \cdot E_1(x, y_2) e_\phi(y_2) = e^{i\beta(x)} e_\phi(x) \cdot E_2(x, y_2) e_\phi(y_2), \forall x \in U, y_2 \in \partial B_{R_1}, x \neq y_2,
\]
where $\beta$ is a real-valued analytic function of $x \in U$. From (62) it is easy to derive that
\[
(64) \quad e_\phi(x) \cdot [E_1^*(x, y_1) - e^{i\beta(x)} E_2^*(x, y_1)] e_\phi(y_1) \\
= e_\phi(x) \cdot [e^{i\beta(x)} E^*(x, y_1) - E^*(x, y_1)] e_\phi(y_1)
\]
for all $x \in U, y_1 \in \partial B_{R_1}, x \neq y_1$. For arbitrarily fixed $y_1 \in U$, the left-hand side of (64) is analytic in $x \in U$, but, by (9) and a direct calculation, the right-hand side of (64) has a singularity at $x = y_1$ unless $e^{i\beta(x)} = -1$ for $x \in C_{e_\phi(y_1)}$ near $y_1$. This means that $e^{i\beta(y_1)} = -1$. By the arbitrariness of $y_1 \in U$, we have $e^{i\beta(x)} = -1$ for
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all \( x \in U \), and so
\[
e^{i\beta(x)}E^i(x, y) - E^i(x, y) = -\overline{E^i(x, y)} - E^i(x, y) = (k^2I + \nabla_x\nabla_x)^i_k [\Phi_k(x, y) - \Phi_k(x, y)]
\]
is analytic in \( x \in \mathbb{R}^3 \) and \( y \in \mathbb{R}^3 \), respectively, since \( \Phi_k(x, y) - \Phi_k(x, y) \) is analytic in \( x \in \mathbb{R}^3 \) and \( y \in \mathbb{R}^3 \), respectively. Thus (62) and (63) are reduced to
\[
e\phi(x) \cdot E_1(x, y_1)e\phi(y_1) = -e\phi(x) \cdot \overline{E_2(x, y_1)e\phi(y_1)}, \quad \forall x \in U, y_1 \in \partial B_{R_1}, x \neq y_1,
\]
\[
e\phi(x) \cdot E_1(x, y_2)e\phi(y_2) = -e\phi(x) \cdot \overline{E_2(x, y_2)e\phi(y_2)}, \quad \forall x \in U, y_2 \in \partial B_{R_1}, x \neq y_2.
\]
Both (49) and (50) then follow from the analyticity of \( E_j(x, y) (j = 1, 2) \) in \( x \in \partial B_{R_1} \), for \( x \neq y \). The proof is thus complete.

**Lemma 4.4.** Assume that the assumptions of Theorem 4.2 are satisfied. If for any fixed \( m, n, l \in \{\phi, \theta\} \) there holds
\[
|e_m(x) \cdot E_1(x, y_1)e_n(y_1), \tau_1, y_2, e_l(y_2), \tau_2| = |e_m(x) \cdot E_2(x, y_1)e_n(y_1), \tau_1, y_2, e_l(y_2), \tau_2|
\]
for all \( x, y_1 \in \partial B_{R_1} \) with \( x \neq y_1, y_2 \in \partial B_{R_2} \), \( (\tau_1, \tau_2) \in \{(1, 0), (0, 1), (1, 1)\} \), then we have either
\[
e_m(x) \cdot E_1(x, y_1)e_n(y_1) = e_m(x) \cdot E_2(x, y_1)e_n(y_1), \quad \forall x, y_1 \in \partial B_{R_1} \text{ with } x \neq y_1,
\]
\[
e_m(x) \cdot E_1(x, y_2)e_l(y_2) = e_m(x) \cdot E_2(x, y_2)e_l(y_2), \quad \forall x \in \partial B_{R_1}, y_2 \in \partial B_{R_2}
\]
or
\[
e_m(x) \cdot E_1(x, y_1)e_n(y_1) = -e_m(x) \cdot \overline{E_2(x, y_1)e_n(y_1)}, \forall x, y_1 \in \partial B_{R_1} \text{ with } x \neq y_1,
\]
\[
e_m(x) \cdot E_1(x, y_2)e_l(y_2) = -e_m(x) \cdot \overline{E_2(x, y_2)e_l(y_2)}, \forall x \in \partial B_{R_1}, y_2 \in \partial B_{R_2}.
\]

**Proof.** Since \( |e_m(x) \cdot E_1(x, y_2)e_l(y_2)| \) is analytic in \( x \in \partial B_{R_1} \) and \( y_2 \in \partial B_{R_2} \), respectively, we only need to distinguish between two cases:

- **A:** \( |e_m(x) \cdot E_1(x, y_2)e_l(y_2)| \neq 0, \quad \forall (x, y_2) \in \partial B_{R_1} \times \partial B_{R_2} \),
- **B:** \( |e_m(x) \cdot E_1(x, y_2)e_l(y_2)| = 0, \quad \forall (x, y_2) \in \partial B_{R_1} \times \partial B_{R_2} \).

For the case when \( \text{A} \) holds, by arguing similarly as in the proof of Lemma 4.3 it can be deduced from (68) that we have either
\[
e_m(x) \cdot E_1(x, y_1)e_n(y_1) = e^{i\beta(x)}e_m(x) \cdot E_2(x, y_1)e_n(y_1), \forall x \in U, y_1 \in \partial B_{R_1}, x \neq y_1,
\]
\[
e_m(x) \cdot E_1(x, y_2)e_l(y_2) = e^{i\beta(x)}e_m(x) \cdot E_2(x, y_2)e_l(y_2), \forall x \in U, y_2 \in \partial B_{R_2}
\]
Lemma 4.5.

(e)

(50)

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Proof. We first show that for any fixed

where \( U \) is some small open subset of \( \partial \hat{B}_{R_1} \), and \( \alpha(x) \) and \( \beta(x) \) are real-valued functions of \( x \). By (47) and (49) in Lemma 4.3 it follows easily that \( e^{i\alpha(x)} = 1 \) and \( e^{i\beta(x)} = -1 \). This, together with (73)-(76) and the analyticity of the total fields \( E_j(x,y) \), \( j = 1,2 \), in \( x \) for \( x \neq y \), implies that either (69) and (70) hold or (71) and (72) hold.

For the case when \( B \) holds, it follows from (68) that

Therefore, both (70) and (72) hold. Further, by Lemma 4.3 we have that either (69) or (71) holds. The proof is thus complete.

Using Lemmas 4.3 and 4.4 we can prove the following lemma.

**Lemma 4.5.** Assume that the assumptions of Theorem 4.2 are satisfied. If (46) and (68) hold for all \( m,n,l \in \{ \phi, \theta \} \), then we have

(77)

(78)

To this end, for any fixed \( m \in \{ \phi, \theta \} \) we need to distinguish between the following two cases.

**Case 1.** \( \text{Re} e_m(x) \cdot E_1(x,y_1) e_l(y_1) = 0 \) for all \( x,y_1 \in \partial \hat{B}_{R_1} \) with \( x \neq y_1 \) and for all \( l \in \{ \phi, \theta \} \).

In this case, by Lemma 4.3 it follows that \( \text{Re} e_m(x) \cdot E_2(x,y_1) e_l(y_1) = 0 \) for all \( x,y_1 \in \partial \hat{B}_{R_1} \) with \( x \neq y_1 \) and for all \( l \in \{ \phi, \theta \} \). By Lemma 4.3 again we have (78).

**Case 2.** \( \text{Re} e_m(x) \cdot E_1(x,y_1) e_l(y_1) \neq 0 \) for some \( x,y_1 \in \partial \hat{B}_{R_1} \) with \( x \neq y_1 \), \( l \in \{ \phi, \theta \} \). Here, we only consider the case with \( l = \phi \). The case \( l = \theta \) can be treated similarly.

In this case, by Lemma 4.3 we have that either both (47) and (48) hold or both (49) and (50) hold. We can prove that both (49) and (50) can not hold simultaneously. Suppose this is not the case. Then we have

(79)

This, together with Lemmas 4.3 and 4.4, implies that

(80)
We now show that both (79) and (80) can not hold simultaneously. In fact, by the reciprocity relation $E_j(x, y) = [E_j(y, x)]^\top$ for all $x, y \in G$ ($j = 1, 2$), we deduce from (79) and (80) that
\begin{equation}
\varepsilon_n(y_1) \cdot [E_1(y_1, x)e_m(x)] = -\varepsilon_n(y_1) \cdot [E_2(y_1, x)e_m(x)],
\end{equation}
\begin{equation}
\forall x, y_1 \in \partial B_R, x \neq y_1, \forall n \in \{\phi, \theta\},
\end{equation}
\begin{equation}
\varepsilon_n(y_2) \cdot [E_1(y_2, x)e_m(x)] = -\varepsilon_n(y_2) \cdot [E_2(y_2, x)e_m(x)],
\end{equation}
\begin{equation}
\forall x \in \partial B_R, y_2 \in \partial B_R, \forall n \in \{\phi, \theta\}.
\end{equation}
This, together with the linear combination of $e_\phi(y_j)$ and $e_\theta(y_j)$ ($j = 1, 2$), gives that
\begin{equation}
\nu(y_1) \times [E_1(y_1, x)e_m(x)] = -\nu(y_1) \times [E_2(y_1, x)e_m(x)], \forall x, y_1 \in \partial B_R, x \neq y_1,
\end{equation}
\begin{equation}
\nu(y_2) \times [E_1(y_2, x)e_m(x)] = -\nu(y_2) \times [E_2(y_2, x)e_m(x)], \forall x \in \partial B_R, y_2 \in \partial B_R.
\end{equation}
For any fixed $x \in \partial B_R$, and $m \in \{\phi, \theta\}$, define $\bar{E}(y) := E_1(y, x)e_m(x) + E_2(y, x)e_m(x)$, $y \neq x$. Since $2\text{Re}[E'(y, x)] := E'(y, x) + \bar{E}'(y, x)$ is analyticity for all $x, y \in \mathbb{R}^3$ (see (65)), then, by the analyticity of $E_j(x, y)$ with respect to $y \in G$ ($j = 1, 2$), it follows that $\bar{E}$ can be extended as an analytic function of $y \in G$, which we denote by $\bar{E}$ again. Define $\bar{H}(y) := [1/(ik)]\text{curl}_y\bar{E}(y)$. Then $(\text{Re}[E'(y, x)]e_m(x), \text{Im}[H'(y, x)]e_m(x))$ and $(E_j'(y, x)e_m(x), H_j'(y, x)e_m(x))$ satisfy the Maxwell equations for $x \in G, j = 1, 2$. Thus it follows by (83), (84) and the analyticity of $E_j(x, y)$ in $y \in G$ with $y \neq x$ ($j = 1, 2$) that $(\bar{E}, \bar{H})$ satisfies the interior Maxwell problem
\begin{equation}
\begin{cases}
curl \bar{E} - ik\bar{H} = 0 & \text{in } B_{R_2} \setminus \overline{B_{R_1}}, \\
curl \bar{H} + ik\bar{E} = 0 & \text{in } B_{R_2} \setminus \overline{B_{R_1}}, \\
\nu \times \bar{E} = 0 & \text{on } \partial B_{R_1} \cup \partial B_{R_2}.
\end{cases}
\end{equation}
Since $k^2$ is not a Maxwell eigenvalue in $B_{R_2} \setminus \overline{B_{R_1}}$, then $\bar{E} = 0$ in $B_{R_2} \setminus \overline{B_{R_1}}$. Thus, and by the analyticity of $E_j(y, x)$ in $y \in G$ with $y \neq x$ ($j = 1, 2$), we have $E_1(y, x)e_\phi(x) = -E_2(y, x)e_\phi(x)$ for all $y \in G, y \neq x$. This contradicts to the fact that $E_j(y, x)e_m(x) = E_j'(y, x)e_m(x) + E_j^s(y, x)e_m(x)$, $j = 1, 2$, satisfy the Silver-Müller radiation condition. Therefore, (79) and (80) can not be true simultaneously, which means that both (49) and (50) can not hold simultaneously. This then implies that both (47) and (48) are true, and so (78) holds.

Finally, by (78) and the linear combination of $e_\phi$ and $e_\theta$ we obtain that for arbitrarily fixed $y_1 \in \partial B_{R_1}$ and $n \in \{\phi, \theta\}$,
\begin{equation}
\nu(x) \times [E_n^1(x, y_1)e_n(y_1)] = \nu(x) \times [E_n^2(x, y_1)e_n(y_1)], \forall x \in \partial \bar{B}_{R_1}.
\end{equation}
By the well-posedness of the exterior Maxwell problem in $\mathbb{R}^3 \setminus B_{R_1}$ with the PEC condition on $\partial B_{R_1}$, it is deduced that for arbitrarily fixed $y_1 \in \partial B_{R_1}$,
\begin{equation}
E_n^1(x, y_1)e_n(y_1) = E_n^2(x, y_1)e_n(y_1), \forall x \in \mathbb{R}^3 \setminus B_{R_1}, \forall n \in \{\phi, \theta\}.
\end{equation}
This, together with the reciprocity relation $E_j^s(x, y) = [E_j^s(y, x)]^\top$ for all $x, y \in G$, $j = 1, 2$, implies that for any fixed $x \in \mathbb{R}^3 \setminus B_{R_1}$,
\begin{equation}
\nu(y) \times E_n^s(x, y) = \nu(y) \times E_n^s(y, x), \forall y \in \partial B_{R_1}.
\end{equation}
Again, by the well-posedness of the exterior Maxwell problem in $\mathbb{R}^3 \setminus B_{R_1}$ with the PEC condition on $\partial B_{R_1}$, it is derived that for any fixed $x \in \mathbb{R}^3 \setminus B_{R_1}$,

$$E_1^i(y, x) = E_2^j(y, x), \quad \forall y \in \mathbb{R}^3 \setminus B_{R_1}.$$  

The required result (77) then follows from this, the reciprocity relation and the analyticity of $E_j^1(x, y)$ ($j = 1, 2$) in $x \in G$ and $y \in G$, respectively.

**Proof of Theorem 4.2.** By Lemma 4.5 it follows from (44) and (45) that (77) holds. For $j = 1, 2$, denote by $E_j^\infty(\hat{x}, y)$ the far-field pattern of $E_j^1(x, y)$, $x, y \in G$, and by $E_j^1(x, d)$ and $E_j^\infty(\hat{x}, d)$ the electric scattered field and its far-field pattern associated with the obstacle $D_j$ (or the refractive index $n_j$) and corresponding to the incident electromagnetic plane waves described by the matrices $E^i(x, d)$, $H^i(x, d)$ defined by

$$E^i(x, d) p := \frac{i}{k} \text{curl} \text{curl} pe^{ikx \cdot d} = ik(d \times p) \times de^{ikx \cdot d},$$

$$H^i(x, d) p := \text{curl} pe^{ikx \cdot d} = ikd \times pe^{ikx \cdot d},$$

where $d \in S^2$ and $p \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ denote the incident direction and polarization vector, respectively, and $x \in \mathbb{R}^3$. Then, by (77) in Lemma 4.5 and the mixed reciprocity relation that $4\pi E_j^\infty(-d, x) = [E_j^1(x, d)]^\top$ for all $x \in G$, $d \in S^2$ and $j = 1, 2$ (see [14, Theorem 6.31]), we obtain that $E_j^1(x, d) = E_j^2(\hat{x}, d)$ for all $x \in G$ and all $d \in S^2$ or $E_j^\infty(\hat{x}, d) = E_j^\infty(\hat{x}, d)$ for all $\hat{x}, d \in S^2$. By the uniqueness result for inverse electromagnetic scattering with full far-field data (see [14, Theorem 7.1] and [9, Theorem 3.1] for the obstacle case, and [15, Theorem 4.9] for the inhomogeneous medium case) it follows easily that the uniqueness statements (a) and (b) of Theorem 4.2 are true. The theorem is thus proved.

**5. Conclusions.** This paper proposed a new approach to prove uniqueness results for inverse acoustic and electromagnetic scattering for obstacles and inhomogeneous media with phaseless near-field data at a fixed frequency. The idea is to use superpositions of two point sources at a fixed frequency as the incident fields and, as the phaseless near-field data, to measure the modulus of the acoustic total-field on two spheres enclosing the scatterers generated by such incident fields located on the two spheres, in the acoustic case. For the electromagnetic case, the idea is to utilize superpositions of two electric dipoles at a fixed frequency with the polarization vectors $e_\phi$ and $e_\theta$, respectively, as the incident fields and, as the phaseless near-field data, to measure the modulus of the tangential component with the orientations $e_\phi$ and $e_\theta$, respectively, of the electric total-field on a sphere enclosing the scatterers and generated by such incident fields located on the measurement sphere and another bigger sphere. As far as we know, this is the first uniqueness result for three-dimensional inverse electromagnetic scattering with phaseless near-field data.

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