Lattice propagators and Haldane-Wu fractional statistics

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Abstract – We point out a formal analogy between lattice kinetic propagators and Haldane-Wu fractional statistics. The analogy could be used to compute the partition function of fractional quantum systems by solving a corresponding lattice kinetic equation for classical dissipative flowing systems.

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The analogy between the partition function of a statistical system at equilibrium and the time-propagator of evolution equations has been noted since long, and makes the basis for a number of theoretical insights as well as powerful numerical methods, especially those based on the path integral formalism [1].

Consider a classical evolution problem governed by the first-order Liouville equation:

\[ \partial_t \psi(x; t) = -L \psi(x; t), \]

where \( x \) is a phase-space coordinate and \( \psi \) is a real scalar describing the dynamic state of the system. The formal solution is given by

\[ \psi(x; t) = e^{-L t} \psi(x; 0), \]

where \( \psi(x; 0) \) is the initial datum and the Liouville operator \( L \) is assumed to be non-negative definite on stability accounts.

The stationary regime associated with the Liouville equation, assuming it exists, is described by a time-independent solution \( \psi(x) \), obeying the relation

\[ \{ H, \psi(x) \} = 0, \]

where \( \{ \cdot \} \) denotes the Poisson bracket, and \( H \) is the Hamiltonian of the system. Note that, by definition, the Liouville operator is given by the Poisson bracket of the Hamiltonian, \( i.e., L = \{ H, \cdot \} \).

A simple additive ansatz reads \( \psi(x) = Z^{-1} e^{-\beta H} \), with \( Z \) a normalization factor and \( \beta \) a positive constant.

The formal solution in eq. (2) also invites an interesting analogy with the partition function of the classical statistical system \( Z(\beta) = \langle e^{-\beta H} \rangle \), where \( \beta = 1/k_B T \) is the inverse temperature, and brackets denote integration over the microscopic degrees of freedom of the Hamiltonian. The analogy is highlighted by computing the autocorrelation function \( C(t, \tau) = \langle \psi(x; t + \tau) \psi(x; t) \rangle = C(t) e^{-\beta L \tau} \), where brackets denote averaging over the distribution \( \rho(x; t) = \psi^2(x; t) / \int \psi^2(x; t) dx \) and \( C(t) = \int \langle \psi^2(x; t) \rangle dx \) [2]. The natural identification is \( \beta = \tau \) and \( H = L \), indicating that the long-time behaviour of the autocorrelation associates with low-temperature partition functions [3].

Note that the weight factor \( e^{-\beta H} \) also includes quantum mechanical systems in imaginary time, with the identification \( k_B T = \hbar/\tau \) and \( L = iH/\hbar \) (imaginary time).

As mentioned above, this is a direct consequence of the additivity of the continuum time, which singles out the exponential as the only functional form supporting the group properties \( P_a P_b = P_b P_a = P_{a+b} \) and \( P_a P_{-a} = 1 \), \( P_t \equiv e^{-\beta t} \) being the continuum time-propagator.

In the following, we shall show that even though discrete space-time propagators are generally non-additive, they still support statistical analogies, encompassing not only the three standard statistics, Maxwell-Boltzmann,
Fermi-Dirac and Bose-Einstein, but also fractional statistics associated to exotic anyons [4]. As we shall see, the basic parameter defining the given fractional statistics is the relaxation time of the Liouvillean operator (with dissipative systems in mind, most notably classical fluids).

A large variety of nonlinear field theories, in particular hydrodynamics, can be encoded within a lattice Boltzmann equation (LBE) of the form [5,6]

\[ f_i(\vec{r} + \vec{c}_i \Delta t, t + \Delta t) - f_i(\vec{r}, t) = \frac{\Delta t}{\tau} (f_i^{eq} - f_i(\vec{r}, t)) \]  

(4)

where \( f_i(\vec{r}, t) \) represents the probability of finding a particle at the lattice position \( \vec{r} \) at time \( t \) with discrete velocity \( \vec{c}_i \), \( i = 0, b \).

The left-hand side (LHS) represents the free-streaming from site \( \vec{r} \) to site \( \vec{r}' = \vec{r} + \vec{c}_i \Delta t \) in a time lapse \( \Delta t \), while the right-hand side (RHS) represents the effects of collisions in the form of a relaxation to the local equilibrium \( f_i^{eq} \) on a time scale \( \tau \).

We wish to reiterate that the above formalism is based on the assumption that the system exhibits sufficient universality to i) admit one-body local equilibria, whose space-time dependence is carried solely by slow hydrodynamic modes, ii) relax to such local equilibria according to a constant time scale, (easily generalised to the case of a multi-relaxation matrix).

While such an assumption might set a restriction on the class of admissible many-body Liouvilleans, it is nonetheless sufficiently general to embrace an amazingly broad class of linear and nonlinear non-equilibrium transport phenomena, characterised by a weak departure from local equilibrium, as is typical of the hydrodynamic regime [7].

A similar statement holds for quantum mechanics in imaginary time, with local equilibria replaced by the ground state of the quantum system, a well-known analogy that lies at the heart of the Diffusion Monte Carlo method [8]. Even more interestingly, the lattice kinetic formalism, eq. (4), also extends to the real-time evolution of relativistic and non-relativistic quantum systems, as detailed in [9]. This speaks clearly for the generality of the mathematical framework associated with the lattice kinetic equation (4).

For a classical system, the local equilibrium is given by a Maxwell-Boltzmann distribution in velocity space:

\[ f_i^{eq} = \rho p(v_i), \]

(5)

where \( \rho(v_i) \) is a polynomial truncation of the Gaussian distribution \( e^{-v_i^2/2} \) (for non-quadratic fluids), \( v_i = |\vec{c}_i - \vec{u}|/V_{th} \) and \( V_{th} = \sqrt{k_B T/m} \) is the thermal speed.

Note that \( f_i^{eq} \) depends on space and time only through the slow-conserved hydrodynamic modes, the fluid density, momentum and temperature, namely:

\[ \rho(\vec{r}, t) = m \sum_i f_i(\vec{r}, t), \quad \rho(\vec{r}, t) \vec{u}(\vec{r}, t) = m \sum_i \vec{c}_i f_i(\vec{r}, t), \]

\[ \rho(\vec{r}, t) V^2_{th}(\vec{r}, t) = m \sum_i f_i(\vec{r}, t)(\vec{c}_i - \vec{u})^2/2. \]

Once the set of discrete velocities is chosen so as to ensure sufficient symmetry to recover rotational invariance (isotropy), the above LBE can be shown to reproduce macroscopic hydrodynamics in the large-scale long-time limit of small Knudsen numbers, i.e., molecular mean free-path much smaller than the macroscale in space and time. Under such conditions, a second-order expansion in the Mach number \( u/V_{th} \) of the local equilibria is sufficient to secure compliance with the isothermal Navier-Stokes equations, so that the local equilibrium is a local, quadratic mapping of the actual distribution function.

Symbolically,

\[ f_i^{eq} = M(f_i), \]

where \( M \) is the local quadratic map.

The kinematic viscosity of the lattice fluid is given by

\[ \nu = C_s^2(\tau - \Delta t/2), \]

(6)

where \( C_s \) is the sound speed. Note that, owing to the lattice discreteness, the viscosity of the LB fluid vanishes at a finite value of the relaxation time, namely \( \tau = \Delta t/2 \).

The LB fluid viscosity is often represented in lattice units as follows:

\[ \nu = C_s^2 \frac{\Delta x^2}{\Delta t}(1/2), \]

(7)

where \( \Delta x \) is the lattice spacing and we have set \( \omega \equiv \Delta t/\tau \).

In the above \( C_s^2 \) is the sound speed in lattice units, typically \( 1/3 \) for most LB models. By stability requirements, the LBE scheme operates in the range \( 0 \leq \omega \leq 2 \), corresponding to a finite, non-negative viscosity.

Three distinguished limits are immediately apparent:

1) \( \omega \rightarrow 0 \) (\( \nu \rightarrow \infty \));
2) \( \omega \rightarrow 1 \) (\( \nu \rightarrow \nu_l \));
3) \( \omega \rightarrow 2 \) (\( \nu \rightarrow 0 \)).

where \( \nu_l = C_s^2 \Delta x^2/2\Delta t \) denotes the characteristic lattice viscosity.

We hasten to note that, from a hydrodynamic viewpoint, the divergence of the expression (7) for \( \omega \rightarrow 0 \) is purely formal, since in the continuum limit \( \Delta t/\tau \rightarrow 0 \), the correct value of the fluid viscosity is \( \nu_p = C_s^2 \tau \), according to the expression (6).

This divergence signals the fact that in a regime where the relaxation time \( \tau \) is much larger than the lattice spacing \( \Delta t \), the LB scheme is not supposed to recover any hydrodynamic behavior, simply because collisions are too infrequent to promote collective motion. However, since the present work is by no means restrained to LB as a hydrodynamic solver, the regime of virtually infinite viscosity remains relevant to our discussion.

The interpretation of these distinguished limits becomes particularly informative by recasting the LBE in the following compact form:

\[ \dot{f}_i = f'_i \]

(8)
The normalised lattice fluid viscosity, \(\nu/\nu_1\), as a function of the relaxation parameter \(\omega\). The dual structure around the quick-interaction value \(\omega = 1\) is apparent. For a geometrical interpretation in terms of entropy minimization see [10].

where \(\hat{\mathbb{f}}_i \equiv f_i(\vec{r} + \vec{c}_i \Delta t, t + \Delta t)\) is the space-time shifted (post-streaming) distribution and \(f_i^\infty \equiv (1 - \omega) \hat{f}_i + \omega f_i^0\) denotes the post-collisional distribution.

Case 1 corresponds to no interaction, \(f' = f\), no relaxation to local equilibrium, no fluid, but just a collection of independent particles. Case 2 corresponds to \(f' = f_{eq}\), i.e., the post-collisional state is set to the equilibrium, the quickest path to equilibrium. Case 3 corresponds to maximum change, \(f' = f - 2 f_{eq}\), the strongest interaction, leading to zero diffusivity.

The dictionary vs. standard numerical analysis is as follows:

1. Under-Relaxation (Weak-Interaction): \(0 \leq \omega < 1\);
2. No-Relaxation (Quick-Interaction): \(\omega = 1\);
3. Over-Relaxation (Strong-Interaction): \(1 < \omega \leq 2\).

It is instructive to recast the lattice viscosity in fully symmetric form as

\[
\nu/\nu_1 = \frac{1 - \epsilon}{1 + \epsilon}, \quad (9)
\]

where \(\epsilon \equiv 1 - \omega\) (see fig. 1). From the above, it is seen that the lattice viscosity is dual around \(\epsilon = 0\), i.e., \(\nu(-\epsilon) = 1/\nu(\epsilon)\), a property which evokes the reversibility of group operators.

As we shall show shortly, these limiting cases map onto the three main statistics, Bose, Maxwell and Fermi, respectively.

Before going into these matters, in passing we note that a suitable choice of the local equilibria gives rise to different types of nonlinear PDEs, not just fluids [11], including quantum ones [9]. In this respect, the LBE can be regarded as a functional generator, lifting the nonlinear field theory to a higher-dimensional kinetic space, whereby field theory emerges as a large-scale, long-time, limit projection of the higher-dimensional lattice kinetic theory.

The extra dimensions are exposed by the non-equilibrium component of the distribution which depends not only on the conserved modes but also on their space-time derivatives to all orders (non-local, non-universal terms), associated with increasing powers of the Knudsen number.

To elicit the role of the Knudsen number in probing extra-dimensions of kinetic space, it proves expedient to recast eq. (4) in the following compact integral form:

\[
f_i(\vec{r}, t) = G(k_i, \omega) f_i^\infty(\vec{r}; t), \quad (10)
\]

where the “Green function” reads as

\[
G(k_i, \omega) = \frac{1}{1 + \frac{\omega}{2\tau}(e^{\Delta \omega k} - 1)} = \frac{\omega}{e^{\Delta \omega k} - 1 + \omega}. \quad (11)
\]

In the above, \(k_i \equiv \Delta t \partial_i\), is the lattice Knudsen operator along the \(i\)-th direction associated with the directional covariant derivative \(\partial_i = \partial_t + \vec{c}_i \cdot \nabla\). The continuum limit \(\Delta t/\tau \to 0\), yields \(G(k_i, \omega) \to 1/(1 + \tau \partial_i)\), corresponding to the continuum time semidiscrete Boltzmann-BGK equation:

\[
(1 + \tau \partial_i)f_i = f_i^\infty. \quad (12)
\]

Interestingly, the lattice version can be written exactly in the same form by simply replacing \(\partial\) with its lattice-deformed analogue \(\partial_{\Delta} \equiv (e^{\Delta t \partial} - 1)/\Delta t\). This offers an elegant example of lattice regularization-renormalization, in the sense that LB can be regarded as a coarse-grained Boltzmann equation in discrete phase-space whereby the covariant derivative \(\partial\) is replaced by its lattice-deformed counterpart. The result is the appearance of a negative contribution to the viscosity, as expressed by the factor \(-1/2\) at the right-hand side of eq. (7).

The Green function maps the local equilibrium into the actual distribution by resuming an infinite series in the Knudsen number, each term incorporating higher orders of inhomogeneity and non-locality [12].

Indeed, the Green function exhibits an interesting series expansion \(G(k, \omega) = \sum_{n=0}^{\infty} g_n(\omega)k^n\) where the polynomial coefficients \(g_n(\omega)\) are related to the Bernoulli polynomials through the relation \(z/(e^z - 1) = e^{-z^2} \sum_{n=0}^{\infty} \frac{B_n(z)}{n!} z^n\). The first three are \(g_0 = 1\), \(g_1 = 0\), \(g_2 = 26^2 - 1\), and so on [13], where we have set \(\theta = 1/\omega = \tau/\Delta t\). In other words, the Green function (11) is the generating function of the \(g_n(\omega)\) polynomials, each polynomial carrying the dependence on the Knudsen number at the corresponding order. This is a lattice version of the continuum Green function \((1 + \tau \partial)^{-1}\), whose generating polynomials are \((-\tau)^n\).

It is important to note that the above Green function differs from a standard time-propagator in that, instead of propagating the state from time \(t\) to time \(t + \Delta t\), it generates the actual distribution at time \(t + \Delta t\) starting from the local equilibrium at the time \(t\). In other words, it bootstraps the non-equilibrium component via an all-term application of the space-time gradient (streaming
operator) to the equilibrium distribution at the previous time-step.

It is readily observed that the three distinguished cases \( \omega = 0,1,2 \) correspond to the three major statistics, namely:

1) Bose: \( \omega = 0, \epsilon = -1, G_0 = \omega/(e^k - 1) \);
2) Maxwell: \( \omega = 1, \epsilon = 0, G_1 = e^{-k} \);
3) Fermi: \( \omega = 2, \epsilon = +1, G_2 = 2/(e^k + 1) \).

To be noted that the Bose case corresponds to continuum time, \( \Delta t/\tau \to 0 \), in which case both numerator and denominators of \( G_0 \) vanish, recovering the continuum Green function \( 1/(1 + \tau \partial) \) as a proper limit, exposing the physical Knudsen number \( \tau \partial \) instead of the lattice one, \( \Delta t \partial \), as it should. Interestingly, the generic case \( 0 < \omega < 2 \) also exhibits an approximate statistical parallel with the exotic case of anyons, i.e., particles with fractional statistics [14,15].

To elucidate this point, let us remind that the Haldane-Wu distribution for a fractional excitation of index \( a \), obeys the relation \[ W^a(1 + W)^{1-a} = p(z) \] (12)
where we have set \( p(z) \equiv e^z \) and \( z \equiv (E - \mu)/kT \). In the following, we shall set the chemical potential to \( \mu = 0 \), on the assumption that this does not imply any loss of generality.

The probability distribution function (pdf) of anyons with index \( 0 \leq a \leq 1 \), \( (a = 0 \text{ Bose}, a = 1 \text{ Fermi}) \) is

\[ f_a(z) = \frac{1}{W(z) + a} \] (13)

Since \( W \geq 0 \), \( f_a(z) \leq 1/a \), as should be for particles obeying fractional statistics. From the physical point of view, the exponent \( a \) associates with the fractional charge of quasi-particle excitations in one or two spatial dimensions.

The special case of semions \((a = 1/2)\) can be solved analytically, to deliver \( W_{1/2}(z) = -1 + \sqrt{1 + 4z^2(1)} \), where the minus sign has been excluded on positiveness grounds. The cases \( a = 1/3 \) and \( a = 2/3 \) can also be solved analytically via cubic roots. However, the corresponding expressions are rather cumbersome and shall not be investigated here. In the low-energy limit, \( W \ll 1 \), the expression (12) delivers \( W \sim p^{1/a} \), that is \( f_a(z) \sim \frac{1}{p^{1/a} + a} \). In the high-energy limit, \( W \gg 1 \), the expression (12) yields \( W \sim p \), and more precisely, to first-order accuracy in \( 1/W \), \( W = p - 1 + a \), whence

\[ f_a(z) \sim \frac{1}{p(z) + 2a - 1}. \]

Note that, within this high-energy approximation, the semion pdf with \( a = 1/2 \) matches exactly Maxwell’s distribution.

The corresponding pdfs are reported in fig. 2. The inspection of the graph of the function \( F_a(w) = w^a/(1 + w)^{1-a} \) for different values \( 0 \leq a \leq 1 \) shows that \( F_a(w) \) transits from \( w + 1 \) for \( a = 0 \) to \( w \) for \( a = 1 \), without ever significantly departing from a straight line \( w + k(a) \), with \( k(a) \) a fraction of \( a \), except in the vicinity of \( w = 0 \) (very-low-energy limit).

Based on the above observations, we approximate the anyon pdf as

\[ f_a(z) = \frac{1}{p(z) + s(a)}. \] (14)

where the shift \( s(a) = a + k(a) \) obeys the boundary conditions \( s(0) = -1 \), \( s(1) = 1 \). As a first-order approximation, we take \( s(a) = 2a - 1 \), which, as noted before, is tantamount to equating the semion statistics to the classical one.

The analogy is now apparent, with the following identifications:

\[ -\Delta t \partial \leftrightarrow \beta \mathcal{H}, \]
\[ -\omega = s(a) + 1. \]

The latter simplifies to \( \omega = 2a \) in the semion=Maxwell approximation. The analogy portrayed in this letter is intriguing and maybe even useful.

As mentioned earlier on, the analogy between time-propagators and the partition function permits to compute the latter by calculating the correlation function of the corresponding evolution Liouville problem, and vice-versa. Likewise, the analogy brought up in this paper could be used to compute the partition function of anyon systems.
by solving a corresponding lattice Boltzmann equation with the corresponding value of the relaxation parameter \( \omega \). The relevant observable to this purpose is the Equilibrium-NonEquilibrium (ENE) correlator, namely:

\[
G_f(t; \omega, \beta) = \frac{\langle f^{eq}_{i_x}, f^{eq}_{i_y} \rangle}{\langle f^{eq}_{i_x}, f^{eq}_{i_y} \rangle},
\]

(15)

where brackets denote averaging over configuration space. Here the index \( i = 0, 8 \) corresponds to the standard D2Q9 lattice [18], with one rest particle \( \{c_{0x}, c_{0y}\} = (0, 0) \), four particles with speed 1, \( \{c_{1x}, c_{1y}\} = \{(1, 0), (0, 1), (-1, 0), (0, -1)\} \) and four particles with speed \( \sqrt{2} \), namely \( \{c_{2x}, c_{2y}\} = \{(1, 1), (-1, 1), (-1, -1), (1, -1)\}. \)

As an example, in fig. 3, we report the time-asymptotic ENE correlator of the mainstream \( (f_1) \), cross-flow \( (f_2) \) and diagonal \( (f_3) \) populations, for the case of a two-dimensional channel (Poiseuille) flow, \( u_x(x, y) = U_0 y(1 - y) \), \( u_y(x, y) = 0 \), where \( x \) is the mainstream coordinate and \(-1 \leq y \leq 1\) the normalized cross-flow coordinate. The simulation runs on a \( 128 \times 64 \) grid, corresponding to a resolution \( \Delta x/L = 1/64, L \) being the cross-flow dimension of the channel.

We note that the ENE correlator is well fitted by \( G_{f_1}(\omega) = A \omega^{2(2 - \omega)} \), a distribution which correlates with the lattice fluid viscosity \( \nu \propto \omega^{-1}(2 - \omega) \). Although not obvious \textit{a priori}, this is not surprising, since the ENE correlator is a close relative to the fluid viscosity via the fluctuation-dissipation theorem. Indeed, similar correlators have been recently shown to play a major role in designing a new class of LB models providing enhanced stability via compliance with entropic constraints [19]. The departure of the singular exponent around \( \omega \to 0 \) is also understandable, due to the lack of hydrodynamic content of the LB scheme in this limit. In passing, we note a close resemblance to the ranking distribution of citation records and other statistical indicators of human performance [20].

It should be noted that the ENE correlator may depend on the specific flow which is used to compute it. This is plausible, as it reflects the influence of the boundary conditions on the corresponding partition function, an effect which is hard to capture by analytical methods.

We also remark that, for the sake of simplicity, the LB simulations shown in fig. 4 are based on truncated Maxwell-Boltzmann equilibria, but other choices can be readily accommodated within the LB formalism.

The analogy between time-propagators and the canonical distribution in classical statistical mechanics rests on the exponential function as a very special common cornerstone, securing time and entropy additivity, respectively [21]. In this work, we have shown that in discrete time, where such additivity no longer holds, the analogy still applies, although in the form of a more general fractional Haldane-Wu statistics. The standard Bose-Fermi and Maxwell statistics are recovered as special instances of the Haldane-Wu distribution for the case of strongly interacting systems (Fermi), weakly interacting systems (Bose) and quickly interacting systems (Maxwell, close to semions). The parameter dictating the fractional statistics is the ratio between lattice time-step (advection time scale in discrete space-time) and the collision-relaxation time scale. This might open intriguing prospects for computing the equilibrium partition function of exotic quantum materials with fractional statistics by means of lattice kinetic simulations of classical, dissipative flowing systems.

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