OPTIMAL CONTROL FOR THE COUPLED CHEMOTAXIS-FLUID MODELS IN TWO SPACE DIMENSIONS

YUNFEI YUAN AND CHANGCHUN LIU*

Department of Mathematics, Jilin University
Changchun 130012, China

(Communicated by Bin Liu)

Abstract. This paper deals with a distributed optimal control problem to the coupled chemotaxis-fluid models. We first explore the global-in-time existence and uniqueness of a strong solution. Then, we define the cost functional and establish the existence of Lagrange multipliers. Finally, we derive some extra regularity for the Lagrange multiplier.

1. Introduction. In this paper, we study the coupled chemotaxis-fluid models with the initial-boundary conditions

\begin{align}
\frac{\partial n}{\partial t} + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \nabla c) + \gamma n - \mu n^2, & \text{in } Q = (0,T) \times \Omega, \\
\frac{\partial c}{\partial t} + u \cdot \nabla c &= \Delta c - c + n + f, & \text{in } Q, \\
\frac{\partial u}{\partial t} + u \cdot \nabla u &= \Delta u - \nabla \pi + n \nabla \varphi, & \text{in } Q, \\
\nabla \cdot u &= 0, & \text{in } Q, \\
\frac{\partial n}{\partial \nu} &= \frac{\partial c}{\partial \nu} = 0, & u = 0, & \text{on } (0,T) \times \partial \Omega, \\
n(x,0) = n_0(x), c(x,0) = c_0(x), u(x,0) = u_0(x), & \text{in } \Omega,
\end{align}

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial \Omega$. $\nu$ is the outward normal vector to $\partial \Omega$, and $\gamma$, $\mu$ are positive constants. $n$, $c$ denote the bacterial density, the oxygen concentration, respectively. $u$, $\pi$ are the fluid velocity and the associated pressure. Here, the function $f$ denotes a control that acts on chemical concentration, which lies in a closed convex set $U$. We observe that in the subdomains where $f \geq 0$ we inject oxygen, and conversely where $f \leq 0$ we extract oxygen.

In order to understand the development of system (1.1), let us mention some previous contributions in this direction. Jin [11] dealt with the time periodic problem of (1.1) in spatial dimension $n = 2, 3$. Jin [12] also obtained the existence of large time periodic solution in $\Omega \subset \mathbb{R}^3$ without the term $u \cdot \nabla u$.

* Corresponding author: Changchun Liu.

2020 Mathematics Subject Classification. Primary: 92C17, 49J20; Secondary: 49K20, 35K51.

Key words and phrases. Chemotaxis-fluid models, optimal control, Lagrange multipliers.

This work is supported by the Jilin Scientific and Technological Development Program (no. 20210101466JC).

* Corresponding author: Changchun Liu.
Espejo and Suzuki [6] discussed the chemotaxis-fluid model
\begin{align}
nt + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \nabla c) + n(\gamma - \mu n), \\
c_t + u \cdot \nabla c &= \Delta c - c + n, \\
u_t &= \Delta u - \nabla \pi + n \nabla \varphi, \\
\nabla \cdot u &= 0, \\
\frac{\partial n}{\partial \nu} &= \frac{\partial c}{\partial \nu} = 0, \\
&\text{and } u = 0.
\end{align}
(1.2)
(1.3)
(1.4)
(1.5)
(1.6)
They proved the global existence of weak solution. Tao and Winkler [17] proved the existence of global classical solution and the uniform boundedness. Tao and Winkler [18] also obtained the global classical solution and uniform boundedness under the condition of $\mu > 23$.

The optimal control problems governed by the coupled partial differential equations is important. Colli et al. [4] studied the distributed control problem for a phase-field system of conserved type with a possibly singular potential. Liu and Zhang [14] considered the optimal control of a new mechanochemical model with state constraint. Chen et al. [3] studied the distributed optimal control problem for the coupled Allen-Cahn/Cahn-Hilliard equations. Recently, Guilñón-González et al. [9] studied a bilinear optimal control problem for the chemo-repulsion model with the linear production term. The existence, uniqueness and regularity of strong solutions of this model are deduced. They also derived the first-order optimality conditions by using a Lagrange multipliers theorem. Frigeri et al. [8] studied an optimal control problem for two-dimensional nonlocal Cahn-Hilliard-Navier-Stokes systems with degenerate mobility and singular potential. Some other results can be found in [2, 5, 13, 15, 19].

In this paper, we discuss the optimal control problem for (1.1). We adjust the external source $f$, so that the bacterial density $n$, oxygen concentration $c$ and fluid velocity $u$ are as close as possible to a desired state $n_d$, $c_d$ and $u_d$, and at the final moment $T$ is as close as possible to a desired state $n_{\Omega}$, $c_{\Omega}$ and $u_{\Omega}$. The main difficulties for treating the problem are caused by the nonlinearity of $u \cdot \nabla u$.

Our method is based on fixed point method and Simon’s compactness results. We overcome the above difficulties and derive first-order optimality conditions by using a Lagrange multipliers theorem.

2. Basic estimates of linearized problem. In this section, we will construct the existence and some priori estimates of the linearized problem for the chemotaxis-Navier-Stokes system in a bounded domain $\Omega \subset \mathbb{R}^2$. The proofs in this section will be established for a detailed framework.

In the following lemmas we will state the Gagliardo-Nirenberg interpolation inequality [7].

**Lemma 2.1.** Let $l$ and $k$ be two integers satisfying $0 \leq l < k$. Suppose that $1 \leq q, r \leq \infty$, $p > 0$ and $\frac{1}{r} \leq a \leq 1$ such that
\begin{equation}
\frac{1}{p} - \frac{l}{N} = a \left( \frac{1}{q} - \frac{k}{N} \right) + (1 - a) \frac{1}{r}.
\end{equation}
(2.1)
Then, for any $u \in W^{k,q}(\Omega) \cap L^r(\Omega)$, there exist two positive constants $C_1$ and $C_2$ depending only on $\Omega$, $q$, $k$, $r$ and $N$ such that the following inequality holds
\begin{equation}
\|D^l u\|_{L^r} \leq c_1 \|D^k u\|_{L^r} \|u\|_{L^q}^{1-a} + c_2 \|u\|_{L^r}.
\end{equation}
with the following exception: If $1 < q < \infty$ and $k - l - \frac{N}{q}$ is a non-negative integer, the (2.1) holds only for $a$ satisfying $\frac{1}{k} \leq a < 1$.

The following log-interpolation inequality has been proved by [1].

**Lemma 2.2.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Then for all non-negative $u \in H^1(\Omega)$, there holds
\[
\|u\|_{L^q(\Omega)}^2 \leq \delta \|u\|_{H^1(\Omega)}^2 \|u + 1\|_{L^1(\Omega)} + p(\delta^{-1}) \|u\|_{L^1(\Omega)},
\]
where $\delta$ is any positive number, and $p(\cdot)$ is an increasing function.

We first consider the existence of solutions to the linear problem of system (1.1). Assume functions $u_0 \in H^1(\Omega)$, $\hat{u} \in L^4(0, T; L^4(\Omega))$, $n \in L^2(0, T; L^2(\Omega))$, and consider
\[
\begin{aligned}
u_t - \Delta u + \hat{u} \cdot \nabla u &= -\nabla \pi + \hat{n} \nabla \varphi, & \text{in } Q, \\
\nabla \cdot u &= 0, & \text{in } Q, \\
u(0) &= 0, & \text{on } \partial \Omega, \\
u(x, 0) &= u_0(x), & \text{in } \Omega.
\end{aligned}
\]

By using fixed point method, the existence of solutions can be easily obtained. Therefore, we ignore the process of proof and just give the regularity estimate.

**Lemma 2.3.** Let $u_0 \in H^1(\Omega)$, $\hat{u} \in L^4(0, T; L^4(\Omega))$, $\hat{n} \in L^2(0, T; H^1(\Omega))$, $\nabla \varphi \in L^\infty(Q)$, and $u$ be the solution of the problem (2.2), then $u \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$ and $u_t \in L^2(0, T; L^2(\Omega))$.

**Proof.** Multiplying the first equation of (2.2) by $u$, and integrating it over $\Omega$, we get
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 dx = \int_\Omega \hat{n} \nabla \varphi \cdot u dx + \int_\Omega u^2 dx \\ \leq \|\hat{n}\|_{L^2} \|u\|_{L^2} + \|u\|_{L^2}^2 \\ \leq C(\|\nabla \varphi\|_{L^2} + \|u\|_{L^2}^2).
\]

By Gronwall’s inequality, we have
\[
\|u\|_{L^2}^2 + \int_0^T \|u\|_{L^2}^2 d\tau \leq C \left( \int_0^T \|\nabla \varphi\|_{L^2}^2 d\tau + \|u_0\|_{L^2}^2 \right).
\]

Operating the Helmholtz projection operator $P$ to the first equation of (2.2), we know
\[
u_t + Au + P(\hat{u} \cdot \nabla u) = P(\hat{n} \nabla \varphi),
\]
where $A := -P \Delta$ is called Stokes operator, which is an unbounded self-adjoint positive operator in $L^2$ with compact inverse, for more properties of Stokes operator, we refer to [10]. Note that $\nabla \cdot u = 0$, that is $Pu = u$, $P \Delta u = \Delta u$, $Pu_t = u_t$, $S$. So, in following calculations, we ignore the projection operator $P$. Multiplying this equation by $\Delta u$, and integrating it over $\Omega$, we get
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |
abla u|^2 dx + \int_\Omega |\Delta u|^2 dx + \int_\Omega |
abla u|^2 dx
\]

\[ = \int_{\Omega} P(\hat{u} \nabla u) \Delta u dx - \int_{\Omega} P(\hat{n} \nabla \varphi) \Delta u dx + \int_{\Omega} |\nabla u|^2 dx. \]

For the terms on the right, we have
\[ \int_{\Omega} P(\hat{u} \nabla u) \Delta u dx - \int_{\Omega} P(\hat{n} \nabla \varphi) \Delta u dx + \int_{\Omega} |\nabla u|^2 dx \leq \|\hat{u}\|_{L^4} \|\nabla u\|_{L^4} \|\Delta u\|_{L^2} + \|\hat{n}\|_{L^2} \|\Delta u\|_{L^2} + \|\nabla u\|^2_{L^2} \]
\[ \leq \|\hat{u}\|_{L^4} \|\nabla u\|_{L^2}^{1/2} \|\Delta u\|_{L^2}^{3/2} + \|\hat{n}\|_{L^2} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} + \|\hat{n}\|_{L^2} \|\Delta u\|_{L^2} + \|\nabla u\|^2_{L^2} \]
\[ \leq \frac{1}{2} \|\Delta u\|^2_{L^2} + C (\|\hat{u}\|^4_{L^4} + \|\hat{n}\|^2_{L^4} + 1) \|\nabla u\|^2_{L^2} + \|\hat{n}\|^2_{L^2}. \]

Therefore, we get
\[ \frac{d}{dt} \|\nabla u\|^2_{L^2} + \|\nabla u\|^2_{H^1} \leq C (\|\hat{u}\|^4_{L^4} + \|\hat{n}\|^2_{L^4} + 1) \|\nabla u\|^2_{L^2} + C \|\hat{n}\|^2_{L^2} + C. \]

By Gronwall’s inequality, we derive
\[ \|\nabla u\|^2_{L^2} + \int_0^T \|\nabla u\|^2_{H^1} dt \leq C. \]

Multiplying the first equation of (2.2) by \( u_t \), and combining with above inequality, we have
\[ \int_0^T \int_{\Omega} |u_t|^2 dx dt \leq C. \]

Summing up, we complete the proof. \( \square \)

For the above solution \( u \), we consider the following linear problem
\[
\begin{cases}
  c_t - \Delta c + u \cdot \nabla c + c = \hat{n} + f, & \text{in } Q, \\
  \frac{\partial c}{\partial \nu} = 0, & \text{on } (0, T) \times \partial \Omega, \\
  c(x, 0) = c_0(x), & \text{in } \Omega.
\end{cases}
\tag{2.3}
\]

Along with fixed point method, the existence of solutions can be easily obtained. Thus we omit the proof and only give the regularity estimate.

**Lemma 2.4.** Let \( c_0 \in H^2(\Omega) \), \( \hat{n} \in L^2(0, T; H^1(\Omega)) \), \( f \in L^2(0, T; H^1(\Omega)) \), \( u \) be the solution of the problem (2.2), and \( c \) be the solution of (2.3). Then \( c \in L^\infty((0, T), H^2(\Omega)) \cap L^2((0, T), H^3(\Omega)) \) and \( c_t \in L^2(0, T; L^2(\Omega)) \).

**Proof.** Multiplying the first equation of (2.3) by \( c \), and integrating it over \( \Omega \), we infer from \( \int_{\Omega} c_0(c \cdot \nabla c) = -\frac{1}{2} \int_{\Omega} c^2 \nabla \cdot u dx = 0 \) that
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} c^2 dx + \int_{\Omega} |\nabla c|^2 dx + \int_{\Omega} c^2 dx \leq \|\hat{n}\|_{L^2} \|c\|_{L^2} + \|f\|_{L^2} \|c\|_{L^2}. \]

Therefore, we have
\[ \|c\|^2_{L^2} + \|c\|^2_{H^1} \leq C (\|c_0\|^2_{L^2} + \int_0^T (\|\hat{n}\|^2_{L^2} + \|f\|^2_{L^2}) d\tau). \]

Multiplying the first equation of (2.3) by \(-\Delta c\), and integrating it over \( \Omega \), we get
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla c|^2 dx + \int_{\Omega} |\Delta c|^2 dx + \int_{\Omega} |\nabla c|^2 dx \]
\[ = \int_{\Omega} u \nabla c \Delta c dx - \int_{\Omega} \Delta c \hat{n} dx - \int_{\Omega} \Delta f c dx. \]
Using the Young inequality and the Hölder inequality, we obtain

\[
\int_\Omega u \nabla c \Delta c \, dx - \int_\Omega \Delta c \tilde{u} \, dx - \int_\Omega \Delta c f \, dx \\
\leq \|u\|_{L^4} \|\nabla c\|_{L^4} \|\Delta c\|_{L^2} + \|\tilde{u}\|_{L^6} \|\Delta c\|_{L^2} + \|f\|_{L^2} \|\Delta c\|_{L^2} \\
\leq C \|u\|_{H^1} (\|\nabla c\|_{L^6}^2 + \|\nabla \tilde{u}\|_{L^6}^2 + \|\nabla c\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 + \|\tilde{u}\|_{L^2} \|\Delta c\|_{L^2} + \|f\|_{L^2} \|\Delta c\|_{L^2} \\
= C \|u\|_{H^1} (\|\nabla c\|_{L^6}^2 + \|\nabla \tilde{u}\|_{L^6}^2 + \|\nabla c\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 + \|\tilde{u}\|_{L^2} \|\Delta c\|_{L^2} + \|f\|_{L^2} \|\Delta c\|_{L^2} \\
\leq \frac{1}{2} \|\Delta c\|_{L^2}^2 + C \|u\|_{H^1}^2 \|\nabla c\|_{L^2}^2 + C (\|\tilde{u}\|_{L^2}^2 + \|f\|_{L^2}^2).
\]

Combining this and above inequalities, we conclude

\[
\frac{d}{dt} \|\nabla c\|_{L^2}^2 + \|\nabla c\|_{H^1}^2 \leq C \|u\|_{H^1} \|\nabla c\|_{L^2}^2 + C (\|\tilde{u}\|_{L^2}^2 + \|f\|_{L^2}^2).
\]

We therefore verify that

\[
\|\nabla c\|_{L^2}^2 + \int_0^t \|\nabla c\|_{H^1}^2 \leq C \left( \int_0^t \|\tilde{u}\|_{L^2}^2 \, dt + \int_0^t \|f\|_{L^2}^2 \, dt \right).
\]

Applying \(\nabla\) to the first equation of (2.3), multiplying it by \(\nabla \Delta c\), and integrating over \(\Omega\) give

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\Delta c|^2 \, dx + \int_\Omega |\nabla \Delta c|^2 \, dx + \int_\Omega |\Delta c|^2 \, dx \\
= \int_\Omega \nabla (u \nabla c) \nabla \Delta c \, dx - \int_\Omega \nabla \tilde{u} \nabla \Delta c \, dx - \int_\Omega \nabla f \nabla \Delta c \, dx.
\]

For the terms on the right, we obtain

\[
\int_\Omega \nabla (u \nabla c) \nabla \Delta c \, dx - \int_\Omega \nabla \tilde{u} \nabla \Delta c \, dx - \int_\Omega \nabla f \nabla \Delta c \, dx \\
\leq \|\nabla \Delta c\|_{L^2} \left( \|u\|_{L^4} \|\Delta c\|_{L^4} + \|\nabla u\|_{L^4} \|\nabla c\|_{L^4} \right) + \|\tilde{u}\|_{L^2} \|\nabla \Delta c\|_{L^2} \\
+ \|\nabla f\|_{L^2} \|\nabla \Delta c\|_{L^2} \\
\leq \|\nabla \Delta c\|_{L^2} \left( \|u\|_{L^4} \|\Delta c\|_{L^4} \|\nabla c\|_{L^2}^2 + \|u\|_{L^4} \|\Delta c\|_{L^2} \\
+ \|\nabla u\|_{L^2} \|\Delta c\|_{L^2}^2 + \|\nabla u\|_{L^2} \|\nabla c\|_{L^2} \|\Delta c\|_{L^2} \\
+ \|\nabla u\|_{L^2} \|\nabla c\|_{L^2} \|\nabla \tilde{u}\|_{L^2} \|\nabla \Delta c\|_{L^2} \\
+ \|\nabla f\|_{L^2} \|\nabla \Delta c\|_{L^2} \\
\leq \frac{1}{2} \|\nabla \Delta c\|_{L^2}^2 + C (1 + \|\Delta c\|_{L^2}^2 + \|u\|_{L^4}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla \tilde{u}\|_{L^2}^2 + \|\nabla f\|_{L^2}^2).
\]

Straightforward calculations yield

\[
\|\Delta c\|_{L^2}^2 + \int_0^t \|\Delta c\|_{H^1}^2 \, d\tau \leq C \left( 1 + \int_0^t \|\tilde{u}\|_{H^1}^2 \, d\tau + \int_0^t \|f\|_{H^1}^2 \, d\tau \right).
\]

Multiplying the first equation of (2.3) by \(c_t\), and combining with above inequality, we have

\[
\int_0^T \int_\Omega |c_t|^2 \, dx \, dt \leq C,
\]

and thereby precisely arrive at the conclusion. \(\square\)
With above solutions \( u \) and \( c \) in hand, we deal with the following linear problem.

\[
\begin{cases}
\Delta n + u \cdot \nabla n + n = -\nabla \cdot (n \nabla c) + (1 + \gamma)\bar{n}_+ - \mu \bar{n}_+ n, & \text{in } Q, \\
\frac{\partial n}{\partial \nu} \bigg|_{\partial \Omega} = 0, \\
n(x, 0) = n_0(x), & \text{in } \Omega.
\end{cases}
\tag{2.4}
\]

(2.4)

By a similar argument as the above two problems, the existence of solutions can be easily obtained. Therefore, we only give the regularity estimate.

**Lemma 2.5.** Suppose \( 0 \leq n_0 \in H^1(\Omega), \bar{n} \in L^2(0, T; H^1(\Omega)) \cap L^4(0, T; L^4(\Omega)), \) and \( u, c, n \) are the solutions of the problem (2.2), (2.3) and (2.4), respectively. Then \( n \geq 0, n, \bar{n} \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)) \) and \( n_t \in L^2(0, T; L^2(\Omega)). \)

**Proof.** Firstly, we verify the nonnegativity of \( n \). We examine the set \( A(t) = \{ x : n(x, t) < 0 \} \). Along with (2.4), we get

\[
\frac{d}{dt} \int_{A(t)} n dx - \int_{\partial A(t)} \frac{\partial n}{\partial \nu} ds + \int_{A(t)} n dx = (1 + \gamma) \int_{A(t)} \bar{n}_+ dx - \mu \int_{A(t)} \bar{n}_+ n dx.
\]

Since \( \frac{d}{dt} \geq 0 \) on \( \partial \{ n < 0 \} \), from this we deduce that the right hand side is nonnegative. Integrating this equality on \([0, t]\) gives

\[
\int_{A(t)} n dx dt + \int_0^t \int_{A(t)} n dx dt = 0.
\]

Then, we get \( n \geq 0 \).

Next, multiplying the first equation of (2.4) by \( n \), and integrating it over \( \Omega \), we get

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega n^2 dx + \int_\Omega (n^2 + |\nabla n|^2) dx + \mu \int_\Omega \bar{n}_+ n^2 dx
\]

\[
= \int_\Omega n \nabla c \nabla n dx + (1 + \gamma) \int_\Omega \bar{n}_+ n dx
\]

\[
\leq \|n\|_{L^4}\|\nabla c\|_{L^4}\|\nabla n\|_{L^2} + (1 + \gamma)\|\bar{n}\|_{L^2}\|n\|_{L^2}
\]

\[
\leq C(\|n\|_{L^2}^2\|\nabla n\|_{L^2}^2 + \|n\|_{L^2}\|\nabla n\|_{L^2} + (1 + \gamma)\|\bar{n}\|_{L^2}\|n\|_{L^2}
\]

\[
\leq C(\|n\|_{L^2}^2\|\nabla c\|_{H^2}^2 + \|n\|_{L^2}^2\|\nabla c\|_{H^2}^2 + \|\bar{n}\|_{L^2}^2 + \frac{1}{2}\|n\|_{H^2}^2).
\]

So, we derive that

\[
\|n\|_{L^2}^2 + \int_0^T \|n\|_{H^1}^2 dt \leq C \left( 1 + \int_0^T \|\bar{n}\|_{L^2}^2 dt \right).
\]

Multiplying the first equation of (2.4) by \(-\Delta n\), and integrating it over \( \Omega \), we get

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla n|^2 dx + \int_\Omega |\Delta n|^2 dx + \int_\Omega |\nabla n|^2 dx
\]

\[
= \int_\Omega u \nabla \Delta n dx + \int_\Omega (\nabla \cdot (n \nabla c)) \Delta n - (1 + \gamma)\bar{n}_+ \Delta n + \mu \bar{n}_+ n \Delta n) dx
\]

\[
\leq \|u\|_{L^4}\|\nabla n\|_{L^2}\|\Delta n\|_{L^2} + \|n\|_{L^4}\|\Delta c\|_{L^4}\|\Delta n\|_{L^2} + \|\nabla n\|_{L^4}\|\nabla c\|_{L^4}\|\Delta n\|_{L^2}
\]

\[
+ (1 + \gamma)\|\bar{n}\|_{L^2}\|\Delta n\|_{L^2} + \mu \|n\|_{L^4}\|\bar{n}\|_{L^4}\|\Delta n\|_{L^2}
\]

\[
\leq C\|u\|_{H^1} (\|\nabla n\|_{L^2}^2\|\Delta n\|_{L^2}^2 + \|\nabla n\|_{L^2}\|\Delta n\|_{L^2}^2).
\]
Define a map where the \((1.1)\).

Next, we use fixed point method to prove the local existence of solutions of the \((\hat{u}, \hat{n})\) is bounded in \(Y_u \times Y_n\). Note that the embeddings \(H^2(\Omega) \hookrightarrow H^1(\Omega)\) is compact and interpolating between \(L^\infty(0, T; H^1(\Omega))\) and \(L^2(0, T; H^3(\Omega))\). It is easy to get that \(u\) is bounded in \(L^4(0, T; L^4(\Omega))\) and \(n\) is bounded in \(L^4(0, T; L^4(\Omega)) \cap L^2(0, T; H^1(\Omega))\). Therefore, the operator \(\mathcal{F}: X_u \times X_n \to X_u \times X_n\) is a compact operator.

**Lemma 2.6.** The map \(\mathcal{F}: X_u \times X_n \to X_u \times X_n\) is well defined and compact.

**Proof.** Let \((\hat{n}, \hat{u}) \in X_u \times X_n\), by Lemmas 2.3, 2.4, 2.5 we deduce that \((n, u) = \mathcal{F}(\hat{n}, \hat{u})\) is bounded in \(Y_u \times Y_n\). Note that the embeddings \(H^2(\Omega) \hookrightarrow H^1(\Omega)\) is compact and interpolating between \(L^\infty(0, T; H^1(\Omega))\) and \(L^2(0, T; H^3(\Omega))\). It is easy to get that \(u\) is bounded in \(L^4(0, T; L^4(\Omega))\) and \(n\) is bounded in \(L^4(0, T; L^4(\Omega)) \cap L^2(0, T; H^1(\Omega))\). Therefore, the operator \(\mathcal{F}: X_u \times X_n \to X_u \times X_n\) is a compact operator.
3. Existence and uniqueness of strong solution of system. From Lemma 2.6, \((n, u) \in Y_n \times Y_u\) satisfies pointwisely a.e. in \(Q\) the following problem

\[
\begin{aligned}
    n_t - \Delta n + u \cdot \nabla n + n &= -\nabla \cdot (n \nabla c) \\
    + \alpha (1 + \gamma) n - \mu n^2, & \quad \text{in } Q, \\
    c_t - \Delta c + u \cdot \nabla c + c &= n + \alpha f, & \quad \text{in } Q, \\
    u_t - \Delta u + u \cdot \nabla u &= -\nabla \pi + \alpha n \nabla \varphi, & \quad \text{in } Q, \\
    \nabla \cdot u &= 0, & \quad \text{in } Q, \\
    \frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} &= 0, \quad u = 0, & \quad \text{on } (0, T) \times \partial \Omega, \\
    n(x, 0) &= n_0(x), c(x, 0) = c_0(x), u(x, 0) = u_0(x), & \quad \text{in } \Omega.
\end{aligned}
\] (3.1)

In order to prove the existence of solution, we first give some a priori estimates.

**Lemma 3.1.** Let \((n, c, u)\) be a local solution to (3.1). Then, it holds that

\[
\begin{aligned}
    &\|n\|_{L^1} + \int_0^t (\|n\|_{L^1} + \|n\|_{L^2}) d\tau \leq C, \\
    &\|\nabla u\|_{L^2}^2 + \int_0^t \|\nabla u\|_{H^1}^2 d\tau \leq C, \\
    &\|\nabla c\|_{L^2}^2 + \int_0^t \|\nabla c\|_{H^1}^2 d\tau \leq C.
\end{aligned}
\] (3.2) (3.3) (3.4)

**Proof.** With Lemma 2.5 in hand, we get \(n \geq 0\). Integrating the first equation of (3.1) over \(\Omega\), we see that

\[
\frac{d}{dt} \int_\Omega ndx + \int_\Omega ndx + \mu \int_\Omega n^2 dx = \alpha (1 + \gamma) \int_\Omega ndx \leq \frac{\mu}{2} \int_\Omega n^2 dx + C.
\]

Solving this differential inequality, we obtain that

\[
\|n\|_{L^1} + \int_0^t (\|n\|_{L^1} + \|n\|_{L^2}) d\tau \leq C.
\]

Multiplying the third equation of (3.1) by \(u\), and integrating it over \(\Omega\), we get

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 dx = \alpha \int_\Omega n \nabla \varphi \cdot u dx + \int_\Omega u^2 dx \\
\leq \|n\|_{L^2} \|u\|_{L^2} + \|u\|_{L^2}^2 \leq C (\|n\|_{L^2} + \|u\|_{L^2}^2).
\]

Therefore, we see that

\[
\|u\|_{L^2}^2 + \int_0^t \|u\|_{H^1} d\tau \leq C.
\]

By the Gagliardo-Nirenberg interpolation inequality, we deduce that

\[
\int_0^t \|u\|_{L^4} d\tau \leq C \int_0^t (\|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2) d\tau \\
\leq \|u\|_{L^2}^2 \int_0^t \|\nabla u\|_{L^2}^2 d\tau + \int_0^t \|u\|_{L^2}^2 d\tau \\
\leq C.
\]

Multiplying the third equation of (3.1) by \(\Delta u\), and integrating it over \(\Omega\), we get

\[
\frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 \leq C (\|n\|_{L^2} + \|u\|_{L^2}^2 + 1) \|\nabla u\|_{L^2}^2 + C \|n\|_{L^2}^2 + C.
\]
Thus, we know
\[ \| \nabla u \|^2_{L^2} + \int_0^t \| \nabla u \|^2_{H^1} d\tau \leq C. \]

Multiplying the second equation of (3.1) by \( c \), and integrating it over \( \Omega \), we have
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega c^2 dx + \int_\Omega |\nabla c|^2 dx + \int_\Omega c^2 dx \leq \| n \|_{L^2} \| c \|_{L^2} + \alpha \| f \|_{L^2} \| c \|_{L^2}. \]

Then, we have
\[ \| c \|_{L^2} + \int_0^t \| c \|_{H^1} d\tau \leq C. \]

Multiplying the second equation of (3.1) by \( -\Delta c \), and integrating it over \( \Omega \), we get
\[ \frac{d}{dt} \| \nabla c \|^2_{L^2} + \| \nabla c \|^2_{H^1} \leq C \| u \|^4_{H^1} \| \nabla c \|^2_{L^2} + C(\| n \|^2_{L^2} + \| f \|^2_{L^2}). \]

Further, we have
\[ \| \nabla c \|^2_{L^2} + \int_0^t \| \nabla c \|^2_{H^1} d\tau \leq C. \]

The proof is complete.

**Lemma 3.2.** Let \((n,c,u)\) be a local solution to (3.1). Then, it holds that
\[ \| (n+1) \ln(n+1) \|_{L^1} + \| \nabla c \|^2_{L^2} + \| \nabla c \|^2_{H^1} \leq C. \] (3.5)

**Proof.** We rewrite the first equation of (3.1) as
\[ \frac{d}{dt} (n+1) + u \cdot \nabla (n+1) - \Delta (n+1) = -\nabla \cdot ((n+1) \cdot \nabla c) + \Delta c + \alpha (1 + \gamma) n - \mu n^2. \]

Multiplying the above equation by \( \ln(n+1) \) and integrating the equation, we have
\[ \frac{d}{dt} \int_\Omega (n+1) \ln(n+1) dx + 4 \int_\Omega |\nabla \sqrt{n+1}|^2 dx \leq \int_\Omega \nabla (n+1) \cdot \nabla c dx + \int_\Omega \Delta c \ln(n+1) dx + \alpha (1 + \gamma) \int \Omega \ln(n+1) dx \]
\[ = I_1 + I_2 + I_3. \]

For \( I_1 \), integrating by parts and using Young’s inequality with small \( \delta \), we get
\[ I_1 = -\int_\Omega n \Delta c dx \leq \| n \|_{L^2} \| \Delta c \|_{L^2} \leq \delta \| \Delta c \|^2_{L^2} + C \| n \|^2_{L^2}. \]

For the term \( I_2 \), we have
\[ I_2 = \int_\Omega \Delta c \ln(n+1) dx \leq \delta \| \Delta c \|^2_{L^2} + C \| \ln(n+1) \|^2_{L^2} \]
\[ \leq \delta \| \Delta c \|^2_{L^2} + C \int_\Omega (n+1) \ln(n+1) dx. \]

For the rest term \( I_3 \), straightforward calculations yield
\[ I_3 = \alpha (1 + \gamma) \int_\Omega n \ln(n+1) dx \leq (1 + \gamma) \int_\Omega (n+1) \ln(n+1) dx. \]
Combining $I_1$, $I_2$ with $I_3$, we conduct that
\[
\frac{d}{dt} \int_\Omega (n + 1) \ln(n + 1) dx + 4 \int_\Omega |\nabla \sqrt{n + 1}|^2 dx \\
\leq \delta \|\Delta c\|_{L^2}^2 + C \int_\Omega (n + 1) \ln(n + 1) dx + C\|n\|_{L^2}^2. \tag{3.6}
\]

Multiplying the second equation of (3.1) by $\Delta c$, and integrating it over $\Omega$, we get
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla c|^2 dx + \int_\Omega |\Delta c|^2 dx + \int_\Omega |\nabla c|^2 dx \\
= \int_\Omega u \nabla c \Delta c dx - \int_\Omega \Delta c ndx - \alpha \int_\Omega \Delta f dx.
\]

Straightforward calculations yield
\[
\frac{d}{dt} \|\nabla c\|_{L^2}^2 + \|\nabla c\|_{H^1}^2 \leq C\|\nabla c\|_{L^2}^2 + C(\|n\|_{L^2}^2 + \|f\|_{L^2}^2). \tag{3.7}
\]

Combining (3.6) and (3.7), it follows that
\[
\frac{d}{dt} \int_\Omega (n + 1) \ln(n + 1) dx + \int_\Omega |\Delta c|^2 dx + \int_\Omega |\nabla c|^2 dx \\
\leq C \int_\Omega (n + 1) \ln(n + 1) dx + C\|f\|_{L^2}^2 + \|n\|_{L^2}^2.
\]

Taking $\delta$ small enough, and solving this differential inequality, we obtain that
\[
\|(n + 1) \ln(n + 1)\|_{L^1} + \|\nabla c\|_{L^2}^2 + \|\nabla c\|_{H^1}^2 \leq C.
\]

The proof is complete.

**Lemma 3.3.** Assume $f \in L^2(0,T; H^1(\Omega))$, let $(n, c, u)$ be a local solution to (3.1). Then, it holds that
\[
\|n\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 + \int_0^t \|n\|_{H^1}^2 d\tau + \int_0^t \|\Delta c\|_{H^1}^2 d\tau \leq C. \tag{3.8}
\]

**Proof.** Taking the $L^2$-inner product with $n$ for the first equation of (3.1) implies
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega n^2 dx + \int_\Omega (n^2 + |\nabla n|^2) dx + \mu \int_\Omega n^2 dx \\
= \int_\Omega n \nabla c \nabla n dx + \alpha (1 + \gamma) \int_\Omega n^2 dx \\
= - \frac{1}{2} \int_\Omega n^2 \Delta c dx + \alpha (1 + \gamma) \int_\Omega n^2 dx.
\]

Here, we note that
\[
\left| \int_\Omega n^2 \Delta c dx \right| \leq \|n\|_{L^3}^2 \|\Delta c\|_{L^3} \\
\leq C\|n\|_{L^2}^2 (\|\nabla \Delta c\|_{L^2}^2 \|\nabla c\|_{L^2}^2) \ components + \|\nabla c\|_{L^2}^2 \\
\leq C\|n\|_{L^2}^2 (\|\nabla \Delta c\|_{L^2}^2 + 1).
\]

From Lemma 2.2 and (3.2), it follows that
\[
- \frac{\chi}{2} \int_\Omega n^2 \Delta c dx
\]
As an immediate consequence

\[ \frac{d}{dt} \|n\|_{L^2}^2 + \|u\|_{H^1}^2 \leq \delta \|\nabla c\|_{L^2}^2 + C\delta^2 \|n\|_{H^1}^2 + C\|n\|_{L^2}^2. \quad (3.9) \]

Applying \( \nabla \) to the first equation of (3.1), multiplying it by \( \nabla c \), and integrating over \( \Omega \) give

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\Delta c|^2 \, dx + \int_\Omega |\nabla \Delta c|^2 \, dx + \int_\Omega |\nabla c|^2 \, dx
= \int_\Omega \nabla(u\nabla c) \nabla \Delta c \, dx - \int_\Omega \nabla n \nabla \Delta c \, dx - \int_\Omega \nabla f \nabla \Delta c \, dx = I_4 + I_5.
\]

For \( I_4 \), by using the Gagliardo-Nirenberg interpolation inequality, we get

\[
I_4 = \int_\Omega \nabla(u\nabla c) \nabla \Delta c \, dx
\leq \|\nabla \Delta c\|_{L^2} (\|u\|_{L^4} \|\Delta c\|_{L^4} + \|\nabla u\|_{L^4} \|\nabla c\|_{L^4})
\leq \|\nabla \Delta c\|_{L^2} (\|u\|_{L^4} \|\Delta c\|_{L^4} + \|\nabla u\|_{L^4} \|\nabla c\|_{L^4})
+ \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \|\nabla c\|_{L^2} \|\Delta c\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla c\|_{L^2} \|\Delta c\|_{L^2}
+ \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \|\nabla c\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla c\|_{L^2}
\leq \frac{1}{4} \|\nabla \Delta c\|_{L^2}^2 + C(1 + \|\Delta c\|_{L^2}^2 + \|\Delta u\|_{L^2}^2).
\]

For the term \( I_5 \), we have

\[
I_5 = - \int_\Omega \nabla n \nabla \Delta c \, dx - \int_\Omega \nabla f \nabla \Delta c \, dx
\leq C (\|\nabla n\|_{L^2}^2 + \|\nabla f\|_{L^2}^2) + \frac{1}{4} \|\nabla \Delta c\|_{L^2}^2.
\]

along with \( I_4 \) and \( I_5 \), we conclude

\[
\frac{d}{dt} \|\nabla c\|_{L^2}^2 + \|\nabla \Delta c\|_{L^2}^2 + \|\Delta c\|_{L^2}^2
\leq C(1 + \|\Delta c\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + \|\nabla f\|_{L^2}^2).
\]

Combining (3.9) and (3.10), it follows that

\[
\frac{d}{dt} (\|n\|_{L^2}^2 + \|\Delta c\|_{L^2}^2) + \|\Delta c\|_{L^2}^2 + (1 - C\delta^2) \|n\|_{H^1}^2 + (1 - \delta) \|\nabla c\|_{L^2}^2
\leq C(1 + \|\Delta c\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + \|\nabla f\|_{L^2}^2).
\]

By choosing \( \delta \) small enough and using (3.3) and (3.5), we have

\[
\|n\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 + \int_0^t \|n\|_{H^1}^2 \, dt + \int_0^t \|\Delta c\|_{H^1}^2 \, dt \leq C.
\]

The proof is complete.
Lemma 3.4. Assume \( f \in L^2(0, T; H^1(\Omega)) \), let \( (n, c, u) \) be a local solution to (3.1). Then, it holds that
\[
\|\nabla n\|_2^2 + \int_0^t \|n\|_{H^2}^2 \, dt \leq C. \tag{3.11}
\]

Proof. Taking the \( L^2 \)-inner product with \( -\Delta n \) for the first equation of (3.1) implies
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla n|^2 \, dx + \int_\Omega |\Delta n|^2 \, dx + \int_\Omega |\nabla n|^2 \, dx = \int_\Omega u \nabla n \Delta n \, dx + \int_\Omega \nabla \cdot (n \nabla c) \Delta n \, dx + (1 + \gamma) \int_\Omega |\nabla n|^2 \, dx + \mu \int_\Omega n^2 \Delta n \, dx = I_6 + I_7 + I_8.
\]

For the term \( I_6 \), with the estimate (3.3), we have
\[
I_6 = \int_\Omega u \nabla n \Delta n \, dx = -\frac{1}{2} \int_\Omega \nabla u (\nabla n)^2 \, dx \leq \|\nabla u\|_{L^2} \|\nabla n\|_4^4
\leq \|\nabla u\|_{L^2} (\|\nabla n\|_2^\frac{2}{3} \|\Delta n\|_2^\frac{1}{3} + \|\nabla n\|_{L^2})^2
\leq \delta \|\Delta n\|_2^2 + C \|\nabla n\|_{L^2}^2.
\]

For the term \( I_7 \), taking (3.8) into consideration, we conduct that
\[
I_7 = \int_\Omega \nabla \cdot (n \nabla c) \Delta n \, dx
= \int_\Omega (\nabla n \nabla c + n \Delta c) \Delta n \, dx
\leq \|\Delta n\|_{L^2} (\|\nabla n\|_{L^2} \|\nabla c\|_4 + \|n\|_C \|\Delta c\|_{L^2})
\leq C \|\Delta n\|_{L^2} \left( \|\nabla n\|_{H^\frac{5}{2}} \|\nabla c\|_{H^3} + \|n\|_{H^\frac{3}{2}} \|\Delta c\|_{L^2} \right)
\leq C \|n\|_{H^2} \|n\|_{H^\frac{3}{2}} \|c\|_{H^2} \leq C \|n\|_{H^2}^\frac{5}{2} \|n\|_{H^2}^\frac{1}{2} \|c\|_{H^2} \leq \delta \|n\|_{H^2}^2 + C(\delta) \|n\|_{H^2}^4 \|c\|_{H^2} \leq \delta \|n\|_{H^2}^2 + C.
\]

For the term \( I_8 \), thanks to the nonnegativity of \( n \), we see that
\[
I_8 = (1 + \gamma) \int_\Omega |\nabla n|^2 \, dx + \mu \int_\Omega n^2 \Delta n \, dx
= (1 + \gamma) \int_\Omega |\nabla n|^2 \, dx - 2\mu \int_\Omega |\nabla n|^2 \, dx
\leq (1 + \gamma) \|\nabla n\|_2^2.
\]

Combine the estimates about \( I_6 \), \( I_7 \) and \( I_8 \), it follows that
\[
\frac{d}{dt} \|\nabla n\|_{L^2}^2 + (1 - 4\delta) \|n\|_{H^2}^2 \leq C \|\nabla n\|_{L^2}^2 + C.
\]

By taking \( \delta \) small enough, we get
\[
\|\nabla n\|_{L^2}^2 + \int_0^t \|n\|_{H^2}^2 \, dt \leq C.
\]

Therefore, this proof is complete. \( \Box \)

Lemma 3.5. The operator \( \mathcal{F} : X_u \times X_n \to X_u \times X_n \), is continuous.
Proof. Let \( \{ (\tilde{n}_m, \tilde{u}_m) \}_{m \in \mathbb{N}} \) be a sequence of \( X_u \times X_n \). Then, with Lemmas 2.3, 2.4, and 2.5 in hand, we conduct that \( \{ (n_m, u_m) = \mathcal{F}(\tilde{n}_m, \tilde{u}_m) \}_{m \in \mathbb{N}} \) is bounded in \( Y_u \times Y_n \). Taking the compactness of \( Y_u \times Y_n \) in \( X_u \times X_n \) into consider, we see that \( \mathcal{F} \) is a compact operator, which means there exists a subsequence of \( \{ \mathcal{F}(\tilde{n}_m, \tilde{u}_m) \}_{m \in \mathbb{N}} \), for convenience, still denoted as \( \{ \mathcal{F}(\tilde{n}_m, \tilde{u}_m) \}_{m \in \mathbb{N}} \), and exists an element \( (\hat{n}, \hat{u}) \) in \( Y_u \times Y_n \) such that
\[
\mathcal{F}(\tilde{n}_m, \tilde{u}_m) \to (\hat{n}, \hat{u}) \text{ weakly in } Y_u \times Y_n \text{ and strongly in } X_u \times X_n.
\]
Let \( m \to \infty \) and take the limit, it is clear that \( (n, u) = \mathcal{F}(\tilde{n}_m, \tilde{u}_m) \) and \( (\tilde{n}_m, \tilde{u}_m) = (\hat{n}, \hat{u}) \), this means that \( \mathcal{F}(\tilde{n}_m, \tilde{u}_m) = (\hat{n}, \hat{u}) \). Since uniqueness of limit, the map \( \mathcal{F} \) is continuous.

Theorem 3.1. Let \( u_0 \in H^1(\Omega) \), \( n_0 \in H^1(\Omega) \), \( c_0 \in H^2(\Omega) \) with \( n_0 \geq 0 \) in \( \Omega \), and \( f \in L^2(0, T; H^2(\Omega)) \), then (1.1) exists unique strong solution \((n, c, u)\). Moreover, there exists a positive \( C \) constant such that
\[
\begin{align*}
|n|_{L^\infty(0,T;H^1(\Omega))} + |n|_{L^2(0,T;L^2(\Omega))} + |n_t|_{L^2(0,T;L^2(\Omega))} + |c|_{L^\infty(0,T;H^2(\Omega))} &
+ |c|_{L^2(0,T;H^2(\Omega))} + |c_t|_{L^2(0,T;L^2(\Omega))} + |u|_{L^\infty(0,T;H^1(\Omega))} \\
+ |u|_{L^2(0,T;L^2(\Omega))} + |u_t|_{L^2(0,T;L^2(\Omega))} &
\leq C.
\end{align*}
\]
(3.12)

Proof. From Lemmas 3.1, 3.3 and 3.4, it is easy to verify the existence of solution and (3.11). Therefore, we will prove the uniqueness of the solution in the following part. For convenience, we set \( n = n_1 - n_2 \), \( c = c_1 - c_2 \) and \( u = u_1 - u_2 \), where \((n_i, c_i, u_i)\) is the strong solution of the system, where \( i = 1, 2 \). Thus, we obtain the following system
\[
\begin{align*}
n_t - \Delta n + u_1 \cdot \nabla n + u \nabla n_2 &= -\nabla \cdot (n_1 \nabla c) \\
- \nabla (n \nabla c_2) + \gamma n - \mu n(n_1 + n_2), &\quad in \ (0, T) \times \Omega \equiv Q, \quad (3.13) \\
c_t - \Delta c + u_1 \cdot \nabla c + u \nabla c_2 + c &= n, &\quad in \ (0, T) \times \Omega \equiv Q, \quad (3.14) \\
u_t - \Delta u + u_1 \cdot \nabla u + u \cdot \nabla u_2 &= n \nabla \varphi, &\quad in \ (0, T) \times \Omega \equiv Q, \quad (3.15) \\
\nabla \cdot u &= 0, &\quad in \ (0, T) \times \Omega \equiv Q, \quad (3.16) \\
\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} &= 0, \quad u = 0, &\quad on \ (0, T) \times \partial \Omega, \quad (3.17) \\
u_0(x) = c_0(x) = u_0(x) &= 0, &\quad in \ \Omega. \quad (3.18)
\end{align*}
\]
Taking the \( L^2 \)-inner product with \( n \) for the (3.13) implies
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} n^2 dx + \int_{\Omega} |\nabla n|^2 dx + \int_{\Omega} n_t^2 dx \\
\leq - \int_{\Omega} u \nabla n_2 ndx + \int_{\Omega} n_1 \nabla c \nabla n dx + \int_{\Omega} n \nabla c_2 \nabla n dx + (1 + \gamma) \int_{\Omega} n_t^2 dx \\
= I_9 + I_{10} + I_{11} + I_{12}.
\]
For the term \( I_9 \), due to the estimates (3.3) and (3.8), we have
\[
I_9 = - \int_{\Omega} u \nabla n_2 ndx \leq ||\nabla n_2||_{L^2} ||u||_{L^4} ||n||_{L^4} \\
\leq C ||\nabla n_2||_{L^2} ||u||_{H^1} (||n||_{L^2}^\frac{1}{2} ||\nabla n||_{L^2}^\frac{1}{2} + ||n||_{L^2}) \\
\leq \frac{\delta}{3} ||\nabla n||^{2}_{L^2} + C ||n||^{2}_{L^2}.
\]
For the term $I_{10}$, with the estimate (3.8) and (3.11), we get
\[
I_{10} = \int_{\Omega} n_{1} \nabla c \nabla u \, dx \leq \|\nabla n\|_{L^{2}} \| n_{1} \|_{L^{4}} \| \nabla c \|_{L^{4}} \\
\leq C \|\nabla n\|_{L^{2}} \| n_{1} \|_{H^{1}} \| \nabla c \|_{H^{1}} \\
\leq \frac{\delta}{3} \|\nabla n\|_{L^{2}}^{2} + C.
\]

For the term $I_{11}$,
\[
I_{11} = \int_{\Omega} n \nabla c_{2} \nabla u \, dx \leq \|\nabla n\|_{L^{2}} \| \nabla c_{2} \|_{L^{4}} \| n \|_{H^{1}} \\
\leq \frac{\delta}{3} \|\nabla n\|_{L^{2}}^{2} + C.
\]

With the use of estimates $I_{i} (i = 9, 10, 11, 12)$, we have
\[
\frac{d}{dt} \| n \|_{L^{2}}^{2} + \| n \|_{H^{1}} \leq \delta \|\nabla n\|_{L^{2}}^{2} + C \| n \|_{L^{2}}^{2} + C. \quad (3.19)
\]
Taking the $L^{2}$-inner product with $c$ for the (3.14) implies
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} c^{2} \, dx + \int_{\Omega} |\nabla c|^{2} \, dx + \int_{\Omega} c^{2} \, dx \\
= -\int_{\Omega} u_{1} \nabla c \nabla u \, dx - \int_{\Omega} u \nabla c_{2} \nabla u \, dx + \int_{\Omega} n \nabla u \, dx \\
\leq \|c\|_{L^{2}}^{2} \|\nabla u_{1}\|_{L^{2}} + \|u\|_{L^{2}} \|\nabla c_{2}\|_{L^{4}} \|c\|_{L^{4}} + \|n\|_{L^{2}} \|c\|_{L^{2}} \\
\leq C (\|c\|_{L^{2}}^{2} \|\nabla u_{1}\|_{L^{2}} + \|u\|_{L^{2}} \|\nabla c_{2}\|_{L^{4}} \|c\|_{L^{4}} + \|n\|_{L^{2}} \|c\|_{L^{2}}) \\
\leq \delta \|\nabla c\|_{L^{2}}^{2} + C \|c\|_{L^{2}}^{2}.
\]

Then, we get
\[
\frac{d}{dt} \|c\|_{L^{2}}^{2} + \|c\|_{H^{1}} \leq \delta \|\nabla c\|_{L^{2}}^{2} + C \|c\|_{L^{2}}^{2}. \quad (3.20)
\]
Taking the $L^{2}$-inner product with $c$ for the (3.15) implies
\[
\frac{1}{2} \int_{\Omega} u^{2} \, dx + \int_{\Omega} |\nabla u|^{2} \, dx = \int_{\Omega} n \nabla \varphi u \, dx.
\]
Straightforward calculations yield
\[
\frac{d}{dt} \|n\|_{L^{2}}^{2} + \|n\|_{H^{1}} \leq C (\|u\|_{L^{2}}^{2} + \|n\|_{L^{2}}^{2}). \quad (3.21)
\]
Then, a combination of (3.19), (3.20) and (3.21) yields
\[
\frac{d}{dt} (\|n\|_{L^{2}}^{2} + \|c\|_{L^{2}}^{2} + \|u\|_{L^{2}}^{2}) + (\|n\|_{H^{1}} + \|c\|_{H^{1}} + \|u\|_{H^{1}}) \\
\leq \delta (\|\nabla n\|_{L^{2}}^{2} + \|\nabla c\|_{L^{2}}^{2} + \|\nabla u\|_{L^{2}}^{2}) + (\|n\|_{L^{2}}^{2} + \|c\|_{L^{2}}^{2} + \|u\|_{L^{2}}^{2}) + C.
\]

By choosing $\delta$ small enough, we get
\[
\frac{d}{dt} (\|n\|_{L^{2}}^{2} + \|c\|_{L^{2}}^{2} + \|u\|_{L^{2}}^{2}) \leq C (\|n\|_{L^{2}}^{2} + \|c\|_{L^{2}}^{2} + \|u\|_{L^{2}}^{2}) + C.
\]

Applying Gronwall’s lemma to the resulting differential inequality, we finally obtain the uniqueness of the solution.
4. Existence of an optimal control. In this section, we will prove the existence of the optimal solution of control problem. The method we use for treating this problem was inspired by some ideas of Guillén-González et al [9]. Assume \( U \subset L^2(0, T; H^1(\Omega_d)) \) is a nonempty, closed and convex set, where control domain \( \Omega_c \subset \Omega \), and \( \Omega_d \subset \Omega \) is the observability domain. We adjust the external source \( f \), so that the bacterial density \( n \), oxygen concentration \( c \) and fluid velocity \( u \) are as close as possible to a desired state \( n_d \), \( c_d \) and \( u_d \), and at the final moment \( T \) is as close as possible to a desired state \( n_{T}, c_{T} \) and \( u_{T} \). We consider the optimal control problem as follows

Minimize the cost functional

\[
J(n, c, u, f) = \frac{\beta_1}{2} \| n - n_d \|_{L^2(Q_d)}^2 + \frac{\beta_2}{2} \| c - c_d \|_{L^2(Q_d)}^2 + \frac{\beta_3}{2} \| u - u_d \|_{L^2(Q_d)}^2 \\
\quad + \frac{\beta_4}{2} \| n(T) - n_{T} \|_{L^2(\Omega_d)}^2 + \frac{\beta_5}{2} \| c(T) - c_{T} \|_{L^2(\Omega_d)}^2 \\
\quad + \frac{\beta_6}{2} \| u(T) - u_{T} \|_{L^2(\Omega_d)}^2 + \frac{\beta_7}{2} \| f(x, t) \|_{L^2(Q_c)},
\]

subject to the system (1.1). Moreover, the nonnegative constants \( \beta_i, i = 1, 2, \ldots, 7 \) are given but not all zero, the functions \( n_d, c_d, u_d \) represents the desired states satisfying

\[
n_d \in L^2(Q_d), c_d \in L^2(Q_d), u_d \in L^2(Q_d), \\
n_{T}, c_{T}, u_{T} \in L^2(Q_T), f \in U.
\]

The set of admissible solutions of optimal control problem (4.1) is defined by

\[
S_{ad} = \{ s = (n, c, u, f) \in H : s \text{ is a strong solution of (1.1)} \}.
\]

The function space \( H \) is given by

\[
H = Y_n \times Y_c \times Y_u \times U,
\]

where \( Y_c = L^\infty (0, T; H^2(\Omega)) \cap L^2 (0, T; H^3(\Omega)) \).

Now, we prove the existence of a global optimal control for problem (1.1).

**Theorem 4.1.** Suppose \( f \in U \) is satisfied, and \( n_0 \geq 0 \), then the optimal control problem (4.1) admits a solution \((\bar{n}, \bar{c}, \bar{u}, \bar{f}) \in S_{ad}\).

**Proof.** Along with Theorem 3.1, we conduct that \( S_{ad} \neq \emptyset \), then there exists the minimizing sequence \( \{(n_m, c_m, u_m, f_m)\}_{m \in \mathbb{N}} \in S_{ad} \) such that

\[
\lim_{m \to +\infty} J(n_m, c_m, u_m, f_m) = \inf_{(n,c,u,f)\in S_{ad}} J(n, c, u, f).
\]

According to the definition of \( S_{ad} \), for each \( m \in \mathbb{N} \) there exists \((n_m, c_m, u_m, f_m)\) satisfying

\[
\begin{align*}
\n_m + u_m \cdot \nabla n_m &= \Delta n_m - \nabla \cdot (n_m \cdot \nabla c_m) + \gamma n_m - \mu n^2_m, & \text{in } Q, \\
c_m + u_m \cdot \nabla c_m &= \Delta c_m - c_m + n_m + f_m, & \text{in } Q, \\
u_m + u_m \cdot \nabla u_m &= \Delta u_m - \nabla \pi + n_m \nabla \varphi, & \text{in } Q, \\
\nabla \cdot u_m &= 0, & \text{in } Q, \\
\frac{\partial n_m}{\partial \nu} |_{\partial \Omega} &= \frac{\partial c_m}{\partial \nu} |_{\partial \Omega} = 0, & u_m |_{\partial \Omega} = 0, \\
n_m(0) = n_0, c_m(0) = c_0, u_m(0) = u_0, & \text{in } \Omega.
\end{align*}
\]
the definition of $S$

To this end, we will use a result on existence of Lagrange multipliers in Banach order necessary optimality conditions for a local optimal solution of problem (4.1). The first-order necessary optimality condition.

5. Observing that $J$ is solution of the system (1.1), along with (4.2) implies that

Since $\nabla \cdot (n_m \nabla c_m) = \nabla n_m \cdot \nabla c_m + n_m \Delta c_m$ is bounded in $L^2(0, T; L^2(\Omega))$, then

Recalling that $n_m \nabla c_m \rightarrow \tilde{n} \nabla \tilde{c}$, weakly in $L^\infty(0, T; L^2(\Omega))$.

Therefore, we get that $\chi = \nabla (\tilde{n} \nabla \tilde{c})$. Owing to $(\tilde{n}, \tilde{c}, \tilde{u}, \tilde{f}) \in \mathcal{H}$, we see that $(\tilde{n}, \tilde{c}, \tilde{u}, \tilde{f})$ is solution of the system (1.1), along with (4.2) implies that

On the other hand, we deduce from the weak lower semicontinuity of the cost functional

Therefore, this implies that $(\tilde{n}, \tilde{c}, \tilde{u}, \tilde{f})$ is an optimal pair for problem (1.1). ⊓⊔

5. The first-order necessary optimality condition. In order to derive the first-order necessary optimality conditions for a local optimal solution of problem (4.1).

To this end, we will use a result on existence of Lagrange multipliers in Banach spaces ([20]). First, we discuss the following problem

where $J : X \rightarrow \mathbb{R}$ is a functional, $G : X \rightarrow Y$ is an operator, $X$ and $Y$ are Banach spaces, and nonempty closed convex set $\mathcal{H}$ is subset of $X$ and nonempty closed convex cone $\mathcal{N}$ with vertex at the origin in $Y$.

$A^+$ denotes its polar cone

We consider the following Banach spaces

$X = V_n \times V_c \times V_u \times L^2(0, T; H^1(\Omega_c)),$

$Y = L^2(Q) \times \mathbb{R} \times L^2(0, T; H^1(\Omega)) \times L^2(Q) \times H^2(\Omega) \times H^1(\Omega),$
where
\[ V_n = \{ n \in Y_n : \frac{\partial n}{\partial \nu} \text{ on } (0, T) \times \partial \Omega \}, \]
\[ V_c = \{ n \in Y_c : \frac{\partial c}{\partial \nu} \text{ on } (0, T) \times \partial \Omega \}, \]
\[ V_u = \{ n \in Y_u : u = 0 \text{ on } (0, T) \times \partial \Omega \text{ and } \nabla \cdot u = 0 \text{ in } (0, T) \times \Omega \} \]
and the operator \( G = (G_1, G_2, G_3, G_4, G_5, G_6) : X \to Y \), where
\[ G_1 : X \to L^2(Q), \quad G_2 : X \to L^2(0, T; H^1(\Omega)), \quad G_3 : X \to L^2(Q), \]
\[ G_4 : X \to H^1(\Omega), \quad G_5 : X \to H^2(\Omega), \quad G_6 : X \to H^1(\Omega), \]
which are defined at each point \( s = (n, c, u, f) \in X \) by
\[
\begin{align*}
G_1 &= n_t + u \cdot \nabla n - \Delta n + \nabla \cdot (n \cdot \nabla c) - \gamma n + \mu n^2, \\
G_2 &= c_t + u \cdot \nabla c - \Delta c + c - n - f, \\
G_3 &= u_t + u \cdot \nabla u - \Delta u + \nabla \pi - n \nabla \varphi, \\
G_4 &= n(0) - n_0, \\
G_5 &= c(0) - c_0, \\
G_6 &= u(0) - u_0.
\end{align*}
\] (5.2)

The function spaces are given as follows
\[ \mathcal{H} = V_n \times V_c \times V_u \times \mathcal{U}. \]

We see that \( \mathcal{H} \) is a closed convex subset of \( X \) and \( \mathcal{N} = \{0\} \), and rewrite the optimal control problem
\[
\min J(s) \text{ subject to } s \in S_{ad} = \{ s \in \mathcal{H} : G(s) = 0 \}. \quad (5.3)
\]

Taking the differentiability of \( J \) and \( G \) into consider, it follows that

**Lemma 5.1.** The functional \( J : X \to R \) is Fréchet differentiable and the Fréchet derivative of \( J \) in \( \bar{s} = (\bar{n}, \bar{c}, \bar{u}, \bar{f}) \in X \) in the direction \( r = (\bar{n}, \bar{c}, \bar{u}, \bar{f}) \) is given by
\[
J'(\bar{s})[r] = \beta_1 \int_0^T \int_{\Omega_d} (\bar{n} - n_d) \bar{n} \, dx \, dt + \beta_2 \int_0^T \int_{\Omega_d} (\bar{c} - c_d) \bar{c} \, dx \, dt
+ \beta_3 \int_0^T \int_{\Omega_d} (\bar{u} - u_d) \bar{u} (T) \, dx \, dt + \beta_4 \int_{\Omega_d} (\bar{n}(T) - n_0) \bar{n}(T) \, dx
+ \beta_5 \int_{\Omega_d} (\bar{c}(T) - c_0) \bar{c} (T) \, dx + \beta_6 \int_{\Omega_d} (\bar{u}(T) - u_0) \bar{u} (T) \, dx
+ \beta_7 \int_0^T \int_{\Omega_d} \bar{f} \bar{f} \, dx \, dt. \quad (5.4)
\]

**Lemma 5.2.** The operator \( G : X \to Y \) is continuous-Fréchet differentiable and the Fréchet derivative of \( J \) in \( \bar{s} = (\bar{n}, \bar{c}, \bar{u}, \bar{f}) \in X \) in the direction \( r = (\bar{n}, \bar{c}, \bar{u}, \bar{f}) \) is the linear operator
\[
G'(\bar{s})[r] = (G_1'(\bar{s})[r], G_2'(\bar{s})[r], G_3'(\bar{s})[r], G_4'(\bar{s})[r], G_5'(\bar{s})[r], G_6'(\bar{s})[r]).
\]
defined by
\[
\begin{align*}
G_1'(s)[r] &= \bar{n}_t - \Delta \bar{n} + \bar{u} \cdot \nabla \bar{n} + \bar{u} \nabla \bar{n} + \nabla \cdot (\bar{n} \nabla \bar{c}) \\
&\quad + \nabla (\bar{n} \nabla \bar{c}) - \gamma \bar{n} + 2 \mu \bar{n} \bar{n}, \\
G_2'(s)[r] &= \bar{c}_t - \Delta \bar{c} + \bar{u} \cdot \nabla \bar{c} + \bar{u} \cdot \nabla \bar{c} + \bar{c} - \bar{n} - \bar{f}, \\
G_3'(s)[r] &= \bar{u}_t - \Delta \bar{u} + \bar{u} \cdot \nabla \bar{u} + \bar{u} \cdot \nabla \bar{u} - \bar{n} \nabla \varphi, \\
\n\n\end{align*}
\]

\[
\begin{align*}
\n(0) &= \bar{n}_0, \bar{c}(0) = \bar{c}_0, \bar{u}(0) = \bar{u}_0, \\
\n\end{align*}
\]

\[\text{Lemma 5.3.} \quad \text{Let } \bar{s} = (\bar{n}, \bar{c}, \bar{u}, \bar{f}) \in S_{ad}, \text{ then } \bar{s} \text{ is a regular point.}\]

\text{Proof.} \quad \text{For any fixed } (\bar{n}, \bar{c}, \bar{u}, \bar{f}) \in S_{ad}, \text{ we set } (g_n, g_c, g_u, \bar{n}_0, \bar{c}_0, \bar{u}_0) \in Y. \quad \text{Since } 0 \in C(\bar{f}), \text{ it suffices to show the existence of } (\bar{n}, \bar{c}, \bar{u}) \in \bar{Y}_n \times \bar{Y}_c \times \bar{Y}_u \text{ such that}
\]

\[
\begin{align*}
\bar{n}_t - \Delta \bar{n} + \bar{u} \cdot \nabla \bar{n} + \bar{u} \nabla \bar{n} + \nabla \cdot (\bar{n} \nabla \bar{c}) \\
&\quad + \nabla (\bar{n} \nabla \bar{c}) - \gamma \bar{n} + 2 \mu \bar{n} \bar{n} = g_n, \\
\bar{c}_t - \Delta \bar{c} + \bar{u} \cdot \nabla \bar{c} + \bar{u} \cdot \nabla \bar{c} + \bar{c} - \bar{n} = g_c, \\
\bar{u}_t - \Delta \bar{u} + \bar{u} \cdot \nabla \bar{u} + \bar{u} \cdot \nabla \bar{u} - \bar{n} \nabla \varphi = g_u, \\
\n\n\end{align*}
\]

\[\text{(5.5)}\]

\[
\begin{align*}
\n(0) &= \bar{n}_0, \bar{c}(0) = \bar{c}_0, \bar{u}(0) = \bar{u}_0, \\
\n\end{align*}
\]

Next, we use Leray-Schauder’s fixed point method to prove the existence of solutions of the problem (5.5), the operator \( \bar{T} : (\bar{n}, \bar{u}) \in \bar{X}_n \times \bar{X}_u \rightarrow (\bar{n}, \bar{u}) \in \bar{Y}_n \times \bar{Y}_u \) with \((\bar{n}, \bar{c}, \bar{u})\) solving the decoupled problem:

\[
\begin{align*}
\bar{n}_t - \Delta \bar{n} + \bar{u} \cdot \nabla \bar{n} + \bar{u} \nabla \bar{n} + \nabla \cdot (\bar{n} \nabla \bar{c}) \\
&\quad + \nabla (\bar{n} \nabla \bar{c}) - \gamma \bar{n} + 2 \mu \bar{n} \bar{n} = g_n, \\
\bar{c}_t - \Delta \bar{c} + \bar{u} \cdot \nabla \bar{c} + \bar{u} \cdot \nabla \bar{c} + \bar{c} - \bar{n} = g_c, \\
\bar{u}_t - \Delta \bar{u} + \bar{u} \cdot \nabla \bar{u} + \bar{u} \cdot \nabla \bar{u} - \bar{n} \nabla \varphi = g_u, \\
\n\n\end{align*}
\]

\[\text{(5.6)}\]

The system (5.6) is complemented by the corresponding Neumann boundary and initial conditions. Similar to the proof of Lemmas 2.3, 2.4, 2.5 and 2.6, we conduct that operator \( \bar{T} : \bar{X}_n \times \bar{X}_u \rightarrow \bar{X}_n \times \bar{X}_u \) is well-defined and compact.

Similar to the proof of Theorem 3.1, \((\bar{n}, \bar{u})\) solves the coupled problem \((\bar{n}, \bar{c}, \bar{u}, \bar{f}) \in S_{ad}, \text{ and we set } (g_n, g_c, g_u, \bar{n}_0, \bar{c}_0, \bar{u}_0) \in Y. \quad \text{Since } 0 \in C(\bar{f}), \text{ it suffices to show the existence of } (\bar{n}, \bar{c}, \bar{u}) \in Y_n \times Y_c \times Y_u \text{ such that}
\]

\[
\begin{align*}
\bar{n}_t - \Delta \bar{n} + \bar{n} = -\bar{u} \cdot \nabla \bar{n} - \bar{u} \cdot \nabla \bar{n} - \nabla \cdot (\bar{n} \nabla \bar{c}) \\
&\quad - \nabla (\bar{n} \nabla \bar{c}) + \alpha (\gamma + 1) \bar{n} - 2 \mu \bar{n} \bar{n}, \\
\bar{c}_t - \Delta \bar{c} + \bar{c} = -\bar{u} \cdot \nabla \bar{c} - \bar{u} \cdot \nabla \bar{c} + \alpha \bar{n} + \alpha g_c, \\
\bar{u}_t - \Delta \bar{u} = -\bar{u} \cdot \nabla \bar{u} - \bar{u} \cdot \nabla \bar{u} + \alpha \bar{n} \nabla \varphi + \alpha g_u, \\
\n\n\end{align*}
\]

\[\text{(5.7)}\]

complemented by the corresponding Neumann boundary and initial conditions.

Taking the \( L^2 \)-inner product with \( \bar{u} \) for the third equation of (5.7) implies

\[
\frac{1}{2} \int_{\Omega} \bar{u}_t^2 dx + \int_{\Omega} |\nabla \bar{u}|^2 dx = \alpha \int_{\Omega} \bar{n} \nabla \varphi \bar{u} dx + \alpha \int_{\Omega} \bar{u} g_u dx.
\]
By the Poincaré inequality and Young’s inequality, we have
\[ \frac{d}{dt} \|\tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{H^1}^2 \leq C(\|\tilde{n}\|_{L^2}^2 + \|g_u\|_{L^2}^2) + C\|\tilde{u}\|_{L^2}^2. \] (5.8)

Taking the $L^2$-inner product with $\tilde{c}$ for the second equation of (5.7) implies
\[ \frac{1}{2} \int_{\Omega} \tilde{c}^2 dx + \int_{\Omega} |\nabla \tilde{c}|^2 dx + \int_{\Omega} |\tilde{c}|^2 dx \]
\[ = \int_{\Omega} \tilde{u} \nabla \tilde{c} \tilde{c} dx + \alpha \int_{\Omega} \nabla \tilde{c} \tilde{c} dx + \alpha \int_{\Omega} g_c \tilde{c} dx. \]

With the Poincaré inequality and Young’s inequality in hand, we see that
\[ \frac{d}{dt} \|\tilde{c}\|_{L^2}^2 + \|\tilde{c}\|_{H^1}^2 \leq C(\|\tilde{n}\|_{L^2}^2 + \|g_c\|_{L^2}^2) + C\|\tilde{c}\|_{L^2}^2. \] (5.9)

Taking the $L^2$-inner product with $-\Delta \tilde{c}$ for the second equation of (5.7) implies
\[ \frac{1}{2} \int_{\Omega} |\nabla \tilde{c}|^2 dx + \int_{\Omega} |\Delta \tilde{c}|^2 dx + \int_{\Omega} |\nabla \tilde{c}|^2 dx \]
\[ = \int_{\Omega} \tilde{u} \nabla \Delta \tilde{c} \tilde{c} dx + \int_{\Omega} \tilde{u} \nabla \Delta \tilde{c} \tilde{c} dx - \alpha \int_{\Omega} \nabla \Delta \tilde{c} \tilde{c} dx - \alpha \int_{\Omega} g_c \Delta \tilde{c} dx \]
\[ = \int_{\Omega} \tilde{u} \nabla \Delta \tilde{c} \tilde{c} dx = J_1 + J_2 + J_3. \]

For the term $J_1$
\[ J_1 = \int_{\Omega} \tilde{u} \nabla \Delta \tilde{c} \tilde{c} dx \leq \|\Delta \tilde{c}\|_{L^2} \|\nabla \tilde{c}\|_{L^4} \|\tilde{u}\|_{L^4} \]
\[ \leq \frac{1}{6} \|\Delta \tilde{c}\|_{L^2}^2 + C\|\nabla \tilde{c}\|_{H^1}^2 \|\tilde{u}\|_{H^1}^2. \]

For the term $J_2$, we see that
\[ J_2 = \int_{\Omega} \tilde{u} \nabla \Delta \tilde{c} \tilde{c} dx = -\frac{1}{2} \int_{\Omega} \nabla \tilde{u} |\nabla \tilde{c}|^2 dx \]
\[ \leq \|\nabla \tilde{u}\|_{L^2} \|\nabla \tilde{c}\|_{L^4} \]
\[ \leq \|\nabla \tilde{u}\|_{L^2} (\|\nabla \tilde{c}\|_{L^2}^2 \|\Delta \tilde{c}\|_{L^2}^2 + \|\nabla \tilde{c}\|_{L^2}^2) \]
\[ \leq \frac{1}{6} \|\Delta \tilde{c}\|_{L^2}^2 + C\|\nabla \tilde{c}\|_{L^2}^2. \]

For the term $J_3$, we get
\[ J_3 = -\alpha \int_{\Omega} \tilde{u} \Delta \tilde{c} \tilde{c} dx - \alpha \int_{\Omega} g_c \Delta \tilde{c} \tilde{c} dx \]
\[ \leq \frac{1}{6} \|\Delta \tilde{c}\|_{L^2}^2 + C(\|\tilde{n}\|_{L^2}^2 + \|g_c\|_{L^2}^2). \]

Therefore, combining $J_1$, $J_2$ and $J_3$, we have
\[ \frac{d}{dt} \|\nabla \tilde{c}\|_{L^2}^2 + \|\nabla \tilde{c}\|_{H^1}^2 \leq C\|\nabla \tilde{c}\|_{L^2}^2 + C(\|\tilde{n}\|_{L^2}^2 + \|g_c\|_{L^2}^2). \] (5.10)

Taking the $L^2$-inner product with $\tilde{n}$ for the first equation of (5.7) implies
\[ \frac{d}{dt} \int_{\Omega} \tilde{n}^2 dx + \int_{\Omega} |\nabla \tilde{n}|^2 dx + \int_{\Omega} \tilde{n}^2 dx \]
\[ = -\int_{\Omega} \tilde{u} \nabla \tilde{n} \tilde{n} dx + \int_{\Omega} \nabla \tilde{n} \tilde{n} \tilde{c} dx + \int_{\Omega} \tilde{n} \nabla \tilde{n} \tilde{c} dx + \alpha(\gamma + 1) \int_{\Omega} \tilde{n}^2 dx \]
+ 2\mu \int_{\Omega} \tilde{n} \tilde{n}^2 dx + \alpha \int_{\Omega} \tilde{n} g_n dx
\]
\]

\[
= J_4 + J_5 + J_6 + J_7.
\]

For the term \( J_4 \), by Gagliardo-Nirenberg interpolation inequality, we have

\[
J_4 = -\int_{\Omega} \tilde{u} \nabla \tilde{n} \tilde{n} dx \leq \| \tilde{u} \|_{L^2} \| \nabla \tilde{n} \|_{L^2} \| \tilde{n} \|_{L^1},
\]

\[
\leq C(\| \nabla \tilde{u} \|_{L^2}^2 + \| \tilde{n} \|_{L^2}^2) \| \nabla \tilde{n} \|_{L^2} \| \tilde{n} \|_{H^1}
\]

\[
\leq \delta \| \tilde{n} \|_{H^1}^2 + C \| \nabla \tilde{u} \|_{L^2} \| \tilde{n} \|_{L^2} + C \| \tilde{n} \|_{L^2}^2
\]

\[
\leq \delta \| \tilde{n} \|_{H^1}^2 + \delta \| \nabla \tilde{u} \|_{L^2}^2 + C \| \tilde{n} \|_{L^2}^2.
\]

For the term \( J_5 \),

\[
J_5 = \int_{\Omega} \nabla \tilde{n} \nabla \tilde{c} dx \leq \| \nabla \tilde{n} \|_{L^2} \| \tilde{n} \|_{L^2} \| \nabla \tilde{c} \|_{L^2}
\]

\[
\leq \| \nabla \tilde{n} \|_{L^2} \| \tilde{n} \|_{H^1} (\| \nabla \tilde{c} \|_{L^2}^2 + \| \nabla \tilde{c} \|_{L^2})
\]

\[
\leq \delta \| \tilde{n} \|_{L^2}^2 + \| \nabla \tilde{c} \|_{L^2} \| \Delta \tilde{c} \|_{L^2} + C \| \nabla \tilde{c} \|_{L^2}^2
\]

\[
\leq \delta \| \tilde{n} \|_{L^2}^2 + \| \nabla \tilde{c} \|_{L^2} \| \Delta \tilde{c} \|_{L^2} + C \| \nabla \tilde{c} \|_{L^2}^2.
\]

For the term \( J_6 \),

\[
J_6 = \int_{\Omega} \nabla \tilde{n} \nabla \tilde{c} dx \leq \| \nabla \tilde{n} \|_{L^2} \| \tilde{n} \|_{L^2} \| \Delta \tilde{c} \|_{L^2}
\]

\[
\leq (\| \tilde{n} \|_{L^2}^2 + \| \tilde{n} \|_{L^2}) \| \Delta \tilde{c} \|_{L^2}
\]

\[
\leq \delta \| \tilde{n} \|_{L^2}^2 + C \| \tilde{n} \|_{L^2}^2 + C.
\]

For the term \( J_7 \),

\[
J_7 = \alpha (\gamma + 1) \int_{\Omega} \tilde{n}^2 dx + 2\mu \int_{\Omega} \tilde{n} \tilde{n}^2 dx + \alpha \int_{\Omega} \tilde{n} g_n dx
\]

\[
\leq (\gamma + 1) \| \tilde{n} \|_{L^2}^2 + \| g_n \|_{L^2} \| \tilde{n} \|_{L^2} + \| \tilde{n} \|_{L^2} \| \tilde{n} \|_{L^2}^2
\]

\[
\leq (\gamma + 1) \| \tilde{n} \|_{L^2}^2 + \| g_n \|_{L^2} \| \tilde{n} \|_{L^2} + \| \tilde{n} \|_{L^2} (\| \tilde{n} \|_{L^2}^2 + \| \nabla \tilde{n} \|_{L^2}^2 + \| \tilde{n} \|_{L^2})
\]

\[
\leq \delta \| \nabla \tilde{n} \|_{L^2}^2 + C \| \tilde{n} \|_{L^2}^2 + C \| g_n \|_{L^2}^2.
\]

Therefore, by choosing \( \delta \) small enough, from \( J_4, J_5, J_6 \) and \( J_7 \), it follows that

\[
\frac{d}{dt} \| \tilde{n} \|_{L^2}^2 + \| \tilde{n} \|_{H^1}^2
\]

\[
\leq C(\| \tilde{n} \|_{L^2}^2 + \| \tilde{c} \|_{L^2}^2 + \| \tilde{n} \|_{L^2}^2) + \delta \| \Delta \tilde{c} \|_{L^2} + \delta \| \tilde{n} \|_{L^2}^2 + C \| g_n \|_{L^2}^2.
\]  

(5.11)

By choosing \( \delta \) small enough and combining (5.8)-(5.11), we get

\[
\frac{d}{dt} (\| \tilde{n} \|_{L^2}^2 + \| \tilde{c} \|_{L^2}^2 + \| \tilde{n} \|_{L^2}^2) + \| \tilde{n} \|_{H^1}^2 + \| \tilde{c} \|_{H^1}^2 + \| \tilde{n} \|_{H^1}^2
\]

\[
\leq C(\| g_n \|_{L^2}^2 + \| g_c \|_{L^2}^2 + \| g_n \|_{L^2}^2) + C(\| \tilde{n} \|_{L^2}^2 + \| \tilde{c} \|_{H^1}^2 + \| \tilde{n} \|_{L^2}^2).
\]

Applying Gronwall’s lemma to the resulting differential inequality, we obtain

\[
\| \tilde{n} \|_{L^2}^2 + \| \tilde{c} \|_{H^1}^2 + \| \tilde{n} \|_{L^2}^2 + \int_0^t \| \tilde{n} \|_{H^1}^2 d\tau + \int_0^t \| \tilde{c} \|_{H^2}^2 d\tau + \int_0^t \| \tilde{n} \|_{H^1}^2 d\tau \leq C. \]  

(5.12)
Taking the $L^2$-inner product with $-\Delta \tilde{u}$ for the third equation of (5.7) implies
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \tilde{u}|^2 dx + \int_{\Omega} |\Delta \tilde{u}|^2 dx \\
= \int_{\Omega} \tilde{u} \cdot \nabla \tilde{u} \Delta \tilde{u} dx + \int_{\Omega} \tilde{u} \cdot \nabla \tilde{u} \Delta \tilde{u} dx - \alpha \int_{\Omega} \bar{u} \nabla \varphi \Delta \tilde{u} dx - \alpha \int_{\Omega} g_u \Delta \tilde{u} dx \\
= J_8 + J_9 + J_{10}.
\]
With the use of the Gagliardo-Nirenberg interpolation inequality, we derive
\[
J_8 = \int_{\Omega} \tilde{u} \cdot \nabla \tilde{u} \Delta \tilde{u} dx \leq \|\tilde{u}\|_{L^4} \|\nabla \tilde{u}\|_{L^4} \|\Delta \tilde{u}\|_{L^2} \\
\leq \|\tilde{u}\|_{L^4} \|\nabla \tilde{u}\|_{L^4} \|\Delta \tilde{u}\|_{L^2} \leq \delta \|\Delta \tilde{u}\|_{L^2}^2 + C \|\nabla \tilde{u}\|_{L^2}^2
\]
and
\[
J_9 = \int_{\Omega} \tilde{u} \cdot \nabla \tilde{u} \Delta \tilde{u} dx \leq \|\Delta \tilde{u}\|_{L^2} \|\nabla \tilde{u}\|_{L^4} \|\tilde{u}\|_{L^4} \\
\leq C \|\Delta \tilde{u}\|_{L^2} \|\nabla \tilde{u}\|_{L^4} \|\tilde{u}\|_{L^4} \leq \delta \|\Delta \tilde{u}\|_{L^2}^2 + C \|\nabla \tilde{u}\|_{L^2}^2.
\]
For the term $J_{10}$, we deduce
\[
J_{10} = \alpha \int_{\Omega} \bar{u} \nabla \varphi \Delta \tilde{u} dx - \alpha \int_{\Omega} g_u \Delta \tilde{u} dx \\
\leq \delta \|\Delta \tilde{u}\|_{L^2}^2 + C \|\bar{u}\|_{L^2}^2 + C \|g_u\|_{L^2}^2.
\]
By choosing $\delta$ small enough, with the estimates $J_8$, $J_9$ and $J_{10}$, we have
\[
\frac{d}{dt} \|\nabla \tilde{u}\|_{L^2}^2 + \|\Delta \tilde{u}\|_{L^2}^2 \leq C \|\nabla \tilde{u}\|_{L^2}^2 + C \|g_u\|_{L^2}^2.
\]
Applying $\nabla$ to the first equation of (5.7), multiplying it by $\nabla \Delta \tilde{c}$, and integrating over $\Omega$ give
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\Delta \tilde{c}|^2 dx + \int_{\Omega} |\nabla \Delta \tilde{c}|^2 dx + \int_{\Omega} |\Delta \tilde{c}|^2 dx \\
= - \int_{\Omega} \nabla (\bar{u} \nabla \tilde{c}) \nabla \Delta \tilde{c} dx - \int_{\Omega} \nabla (\bar{u} \nabla \tilde{c}) \nabla \Delta \tilde{c} dx + \alpha \int_{\Omega} \nabla \bar{u} \nabla \Delta \tilde{c} dx \\
+ \alpha \int_{\Omega} \nabla g_c \nabla \Delta \tilde{c} dx \\
= J_{11} + J_{12} + J_{13}.
\]
For the first term $J_{11}$, we have
\[
J_{11} = - \int_{\Omega} \nabla (\bar{u} \nabla \tilde{c}) \nabla \Delta \tilde{c} dx = \int_{\Omega} \nabla \bar{u} \nabla \tilde{c} \nabla \Delta \tilde{c} dx - \int_{\Omega} \bar{u} \Delta \tilde{c} \nabla \Delta \tilde{c} dx \\
\leq \|\nabla \Delta \tilde{c}\|_{L^2} \|\bar{u}\|_{L^4} \|\nabla \tilde{c}\|_{L^4} + \|\nabla \Delta \tilde{c}\|_{L^2} \|\bar{u}\|_{L^4} \|\Delta \tilde{c}\|_{L^4} \\
\leq \|\nabla \Delta \tilde{c}\|_{L^2} \|\nabla \bar{u}\|_{L^2} \|\Delta \tilde{c}\|_{L^2} \|\nabla \tilde{c}\|_{L^2} + \|\nabla \Delta \tilde{c}\|_{L^2} \|\bar{u}\|_{L^4} \|\nabla \tilde{c}\|_{L^2} \|\Delta \tilde{c}\|_{L^2} \\
+ \|\nabla \Delta \tilde{c}\|_{L^2} \|\bar{u}\|_{L^4} \|\nabla \tilde{c}\|_{L^2} \|\Delta \tilde{c}\|_{L^2} + \|\Delta \tilde{c}\|_{L^2} \\
\leq \delta \|\nabla \Delta \tilde{c}\|_{L^2}^2 + C \|\nabla \bar{u}\|_{L^2}^2 + C \|\Delta \tilde{c}\|_{L^2}^2.
Similarly, for the term $J_{12}$,
\[
J_{12} = -\int_{\Omega} \nabla (\tilde{u} \nabla \tilde{c}) \nabla \Delta \tilde{n} d\Omega = -\int_{\Omega} \nabla \tilde{u} \nabla \tilde{c} \nabla \Delta \tilde{n} d\Omega - \int_{\Omega} \tilde{u} \Delta \tilde{c} \nabla \Delta \tilde{n} d\Omega
\]
\[
\leq \| \nabla \Delta \tilde{c} \|_{L^2} \| \nabla \tilde{u} \|_{L^4} \| \nabla \tilde{c} \|_{L^4} + \| \tilde{u} \|_{L^4} \| \Delta \tilde{c} \|_{L^4} \| \nabla \Delta \tilde{n} \|_{L^2}
\]
\[
\leq C \| \nabla \Delta \tilde{c} \|_{L^2} (\| \nabla \tilde{u} \|_{L^2}^2 + \| \nabla \tilde{u} \|_{L^2}) \| \nabla \tilde{c} \|_{L^4} H^1
\]
\[
+ (\| \tilde{u} \|_{L^2}^2 + \| \tilde{u} \|_{L^2}) (\| \Delta \tilde{c} \|_{L^2}^2 + \| \nabla \Delta \tilde{n} \|_{L^2} + \| \tilde{u} \|_{L^2}) \| \nabla \Delta \tilde{c} \|_{L^2}
\]
\[
\leq \delta \| \nabla \Delta \tilde{c} \|_{L^2}^2 + \delta \| \Delta \tilde{n} \|_{L^2}^2 + C \| \nabla \Delta \tilde{c} \|_{L^2}^2 + C \| \nabla \tilde{c} \|_{L^2}^2.
\]

For the rest term $J_{13}$, we see
\[
J_{13} = \alpha \int_{\Omega} \nabla \tilde{n} \nabla \Delta \tilde{n} d\Omega + \alpha \int_{\Omega} \nabla g_c \nabla \Delta \tilde{n} d\Omega
\]
\[
\leq \delta \| \nabla \Delta \tilde{c} \|_{L^2}^2 + C (\| \nabla \tilde{n} \|_{L^2}^2 + \| \nabla \tilde{g_c} \|_{L^2}^2).
\]

By choosing $\delta$ small enough, we get
\[
d \| \Delta \tilde{c} \|_{L^2}^2 + \| \Delta \tilde{c} \|_{H^1}^2
\]
\[
\leq C (\| \nabla \tilde{n} \|_{L^2}^2 + \| \Delta \tilde{c} \|_{L^2}^2 + \| \nabla \tilde{c} \|_{L^2}^2) + C \| \Delta \tilde{u} \|_{L^2}^2 + \| \Delta \tilde{n} \|_{L^2}^2
\]
\[
+ C \| \nabla \Delta \tilde{c} \|_{L^2}^2 + C \| \nabla \tilde{g_c} \|_{L^2}^2.
\]

From (5.13) and (5.14), along with $\delta$ small enough, it follows that
\[
d \| \Delta \tilde{c} \|_{L^2}^2 + \| \Delta \tilde{c} \|_{H^1}^2
\]
\[
\leq C (\| \nabla \tilde{n} \|_{L^2}^2 + \| \Delta \tilde{c} \|_{L^2}^2 + \| \nabla \tilde{c} \|_{L^2}^2) + \| \nabla \Delta \tilde{n} \|_{L^2}^2 + \| \Delta \tilde{n} \|_{L^2}^2 + \| \nabla \Delta \tilde{c} \|_{L^2}^2 + \| \nabla \tilde{g_c} \|_{L^2}^2) + C \| \tilde{g_u} \|_{L^2}^2.
\]

Applying Gronwall’s lemma to the resulting differential inequality, we know
\[
\| \nabla \tilde{n} \|_{L^2}^2 + \| \Delta \tilde{c} \|_{L^2}^2 + \int_{0}^{t} \| \Delta \tilde{n} \|_{L^2}^2 d\tau + \int_{0}^{t} \| \Delta \tilde{c} \|_{H^1}^2 d\tau \leq C.
\]

Taking the $L^2$-inner product with $-\Delta \tilde{n}$ for the first equation of (5.7) implies
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \tilde{n}|^2 dx + \int_{\Omega} |\Delta \tilde{n}|^2 dx + \int_{\Omega} |\nabla \tilde{n}|^2 dx
\]
\[
= - \int_{\Omega} \tilde{u} \cdot \nabla \Delta \tilde{n} \tilde{u} dx - \int_{\Omega} \tilde{u} \cdot \nabla \Delta \tilde{n} \tilde{u} dx - \int_{\Omega} \nabla (\tilde{n} \nabla \tilde{c}) \Delta \tilde{n} dx - \int_{\Omega} \nabla (\tilde{n} \nabla \tilde{c}) \Delta \tilde{n} dx
\]
\[
- \alpha (1 + \gamma) \int_{\Omega} \tilde{n} \Delta \tilde{n} dx + 2 \mu \int_{\Omega} \tilde{n} \Delta \tilde{n} dx - \alpha \int_{\Omega} \tilde{g_u} \Delta \tilde{n} dx
\]
\[
= J_{14} + J_{15} + J_{16} + J_{17} + J_{18}.
\]

With the Gagliardo-Nirenberg interpolation inequality in hand, we can estimate $J_{14}$ as follows
\[
J_{14} = - \int_{\Omega} \tilde{u} \cdot \nabla \Delta \tilde{n} \tilde{u} dx \leq \| \tilde{u} \|_{L^4} \| \nabla \tilde{n} \|_{L^4} \| \Delta \tilde{n} \|_{L^2}
\]
\[
\leq C \| \tilde{u} \|_{H^1} (\| \nabla \tilde{n} \|_{L^2}^2 + \| \nabla \tilde{n} \|_{L^2}) \| \Delta \tilde{n} \|_{L^2}
\]
\[
\leq \delta \| \Delta \tilde{n} \|_{L^2}^2 + C \| \nabla \tilde{n} \|_{L^2}^2.
\]

Similar to above estimates, we see
\[
J_{15} = - \int_{\Omega} \tilde{u} \cdot \nabla \Delta \tilde{n} \tilde{u} dx \leq \| \tilde{u} \|_{L^4} \| \nabla \tilde{n} \|_{L^4} \| \Delta \tilde{n} \|_{L^2}
\]

Similarly, we derive
\[ J_{16} = -\int_{\Omega} \nabla (n \nabla \tilde{c}) \Delta \tilde{n} dx = -\int_{\Omega} \nabla \tilde{n} \nabla \tilde{c} \Delta \tilde{n} dx - \int_{\Omega} \tilde{n} \Delta \tilde{c} \Delta \tilde{n} dx \]
\[ \leq \|\nabla \tilde{n}\|_{L^2} \|\nabla \tilde{c}\|_{L^2} \|\Delta \tilde{n}\|_{L^2} + \|\tilde{n}\|_{L^2} \|\Delta \tilde{c}\|_{L^2} \|\Delta \tilde{n}\|_{L^2} \]
\[ \leq \left( \|\nabla \tilde{n}\|_{L^2} \|\Delta \tilde{n}\|_{L^2} + \|\nabla \tilde{n}\|_{L^2} \|\nabla \tilde{c}\|_{H^1} \|\nabla \tilde{n}\|_{L^2} \right) \]
\[ + \|\tilde{n}\|_{H^1} (\|\Delta \tilde{c}\|_{L^2} + \|\nabla \tilde{c}\|_{L^2} + \|\Delta \tilde{n}\|_{L^2}) \|\Delta \tilde{n}\|_{L^2} \]
\[ \leq \delta \|\Delta \tilde{n}\|_{L^2}^2 + C \|\nabla \Delta \tilde{c}\|_{L^2}^2 + C. \]

and
\[ J_{17} = -\int_{\Omega} \nabla (n \nabla \tilde{c}) \Delta \tilde{n} dx = -\int_{\Omega} \nabla \tilde{n} \nabla \tilde{c} \Delta \tilde{n} dx - \int_{\Omega} \tilde{n} \Delta \tilde{c} \Delta \tilde{n} dx \]
\[ \leq \|\nabla \tilde{n}\|_{L^2} \|\nabla \tilde{c}\|_{L^2} \|\Delta \tilde{n}\|_{L^2} + \|\tilde{n}\|_{L^2} \|\Delta \tilde{c}\|_{L^2} \|\Delta \tilde{n}\|_{L^2} \]
\[ \leq \left( \|\nabla \tilde{n}\|_{L^2} \|\Delta \tilde{n}\|_{L^2} \right) \]
\[ + \|\tilde{n}\|_{H^1} (\|\Delta \tilde{c}\|_{L^2} + \|\nabla \tilde{c}\|_{L^2} + \|\Delta \tilde{n}\|_{L^2}) \|\Delta \tilde{n}\|_{L^2} \]
\[ \leq \delta \|\Delta \tilde{n}\|_{L^2}^2 + C \|\nabla \Delta \tilde{c}\|_{L^2}^2 + C. \]

For the rest terms, we know
\[ J_{18} = -\alpha (1 + \gamma) \int_{\Omega} \tilde{n} \Delta \tilde{n} dx + 2\mu \int_{\Omega} \tilde{n} \Delta \tilde{n} dx - \alpha \int_{\Omega} g_n \Delta \tilde{n} dx \]
\[ \leq (1 + \gamma) \|\tilde{n}\|_{L^2} \|\Delta \tilde{n}\|_{L^2} \]
\[ + 2\mu \|\tilde{n}\|_{L^2} \|\Delta \tilde{n}\|_{L^2} + \|g_n\|_{L^2} \|\Delta \tilde{n}\|_{L^2} \]
\[ \leq (1 + \gamma) \|\tilde{n}\|_{L^2} \|\Delta \tilde{n}\|_{L^2} + C (\|\tilde{n}\|_{L^2} \|\nabla \tilde{n}\|_{L^2} + \|\tilde{n}\|_{L^2} \|\Delta \tilde{n}\|_{L^2} \]
\[ + \|g_n\|_{L^2} \|\Delta \tilde{n}\|_{L^2} \]
\[ \leq \delta \|\Delta \tilde{n}\|_{L^2}^2 + C \|\nabla \tilde{n}\|_{L^2}^2 + C \|g_n\|_{L^2}^2. \]

Therefore, Taking $\delta$ small enough and together with $J_{14} - J_{18}$, we see that
\[ \frac{d}{dt} \|\nabla \tilde{n}\|_{L^2}^2 + \|\nabla \tilde{n}\|_{H^1}^2 \]
\[ \leq C (\|\nabla \tilde{n}\|_{L^2}^2 + \|\nabla \tilde{n}\|_{H^1}^2 + \|\Delta \tilde{c}\|_{L^2}^2 + \|\nabla \Delta \tilde{c}\|_{L^2}^2 + \|\Delta \tilde{n}\|_{L^2}^2) + C. \]

Applying Gronwall’s lemma to the resulting differential inequality, we know
\[ \|\nabla \tilde{n}\|_{L^2}^2 + \int_0^t \|\nabla \tilde{n}\|_{H^1}^2 d\tau \leq C. \]

Therefore, from Leray-Schauder theorem, we derive the existence of solution for (5.5). Along with the regularity of $(\tilde{n}, \tilde{c}, \tilde{u})$, the uniqueness of solution can easily get, so we omit the process. \qed

**Theorem 5.1.** Assume that $\tilde{s} = (\tilde{n}, \tilde{c}, \tilde{u}, \tilde{f}) \in S_{ad}$ be an optimal solution for the control problem (5.3). Then, there exist Lagrange multipliers $(\lambda, \eta, \rho, \xi, \varphi, \omega) \in L^2(Q) \times (L^2(0, T; H^1(\Omega)))' \times L^2(Q) \times (H^1(\Omega))' \times (H^2(\Omega))' \times (H^1(\Omega))'$ such that for all $(\tilde{n}, \tilde{c}, \tilde{u}, \tilde{f}) \in V_n \times V_c \times V_u \times C(f)$ has
\[ \beta_1 \int_0^T \int_{\Omega_d} (\tilde{n} - n_d) \tilde{n} dx dt + \beta_2 \int_0^T \int_{\Omega_d} (\tilde{c} - c_d) \tilde{c} dx dt + \beta_3 \int_0^T \int_{\Omega_d} (\tilde{u} - u_d) \tilde{u} dx dt \]
correspond to the linear system
\[
\begin{align*}
-\lambda_t - \Delta \lambda &= \lambda + 2 \mu \lambda \bar{n} + 2 \mu \lambda \bar{n} - \eta - \nabla \varphi_p \\
-\eta_t - \Delta \eta &= \eta + 2 \mu \lambda \bar{n} + \bar{n} \nabla \lambda - \nabla \lambda \nabla \bar{c} - 2 \mu \lambda \bar{n} - \eta - \nabla \varphi_p \\
-\rho_t - \Delta \rho &= \rho + 2 \mu \lambda \bar{n} + \bar{n} \nabla \lambda - \nabla \lambda \nabla \bar{c} - 2 \mu \lambda \bar{n} - \eta - \nabla \varphi_p \\
&= \beta_1(\bar{n} - n_d), \\
-\eta_t - \Delta \eta &= \eta + 2 \mu \lambda \bar{n} + \bar{n} \nabla \lambda - \nabla \lambda \nabla \bar{c} - 2 \mu \lambda \bar{n} - \eta - \nabla \varphi_p \\
-\rho_t - \Delta \rho &= \rho + 2 \mu \lambda \bar{n} + \bar{n} \nabla \lambda - \nabla \lambda \nabla \bar{c} - 2 \mu \lambda \bar{n} - \eta - \nabla \varphi_p \\
&= \beta_2(\bar{c} - c_d), \\
\end{align*}
\]
subject to the following boundary and final conditions
\[
\begin{aligned}
\nabla \cdot \rho &= 0, & & \text{in } Q, \\
\frac{\partial \lambda}{\partial \nu} &= \frac{\partial \eta}{\partial \nu}, \rho = 0, & & \text{on } (0, T) \times \partial \Omega, \\
\lambda(T) &= \beta_4(\bar{n}(T) - n_\Omega), \eta(T) = \beta_5(\bar{c}(T) - c_\Omega), & & \text{in } \Omega, \\
\rho(T) &= \beta_5(\bar{c}(T) - c_\Omega), & & \text{in } \Omega,
\end{aligned}
\]
and the following identities hold
\[
\int_0^T \int_{\Omega_\delta} (\beta_7 \tilde{f} + \eta)(f - \tilde{f}) \, dx \, dt \geq 0, \forall f \in U. \quad (5.20)
\]

**Proof.** By taking \((\bar{c}, \bar{u}, \tilde{f}) = (0, 0, 0)\) in (5.15), then it follows that the equation (5.16) holds. In light of an analogous argument, and in light of the (5.15), it guarantees that (5.17) and (5.18) hold. On the other hand, let \((\tilde{n}, \tilde{u}, \tilde{f}) = (0, 0, 0)\), as an immediate consequence we obtain
\[
\beta_7 \int_0^T \tilde{f} \, dx \, dt + \int_0^T \tilde{f} \, dt \geq 0, \forall \tilde{f} \in C(\tilde{f}).
\]
By choosing \(\tilde{f} = f - \tilde{f} \in C(\tilde{f})\) for all \(\tilde{f} \in U\), thus we achieve (5.20). \qed

**Theorem 5.2.** Under the assumptions of Theorem 5.1, system (5.19) has a unique weak solution such that
\[
\|\lambda\|_{H^1}^2 + \|\eta\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \int_0^T \|\lambda\|_{H^1}^2 \, d\tau + \int_0^T \|\eta\|_{H^1}^2 \, d\tau + \int_0^T \|\rho\|_{H^1}^2 \, d\tau \leq C.
\]

**Proof.** For convenience, we set \(\bar{\lambda} = \lambda(T - t), \bar{\eta} = \eta(T - t), \bar{\rho} = \rho(T - t)\), in order to simplify notations, we still write \(\lambda, \eta, \rho\) instead of \(\bar{\lambda}, \bar{\eta}, \bar{\rho}\), then the adjoint system (5.19) can be written as follow
\[
\begin{aligned}
\lambda_1 - \Delta \lambda + \bar{u} \cdot \nabla \lambda - \nabla \lambda \nabla \bar{c} - \gamma \lambda + 2\mu \bar{n} - \eta - \nabla \varphi &= \beta_1(\bar{n} - n_d), & & \text{in } Q, \\
\eta_1 - \Delta \eta + \bar{u} \cdot \nabla \eta + \eta + \nabla(\bar{n} \nabla \lambda) &= \beta_2(\bar{c} - c_d), & & \text{in } Q, \\
\rho_1 - \Delta \rho + (\bar{u} \cdot \nabla) \bar{u} + (\rho \cdot \nabla) \bar{u} + \lambda \nabla \bar{n} + \eta \nabla \bar{c} &= \beta_3(\bar{u} - u_d), & & \text{in } Q,
\end{aligned}
\]
subject to the following boundary and final conditions
\[
\begin{aligned}
\nabla \cdot \rho &= 0, & & \text{in } Q, \\
\frac{\partial \lambda}{\partial \nu} &= \frac{\partial \eta}{\partial \nu}, \rho = 0, & & \text{on } (0, T) \times \partial \Omega, \\
\lambda(0) &= \beta_4(\bar{n}(T) - n_\Omega), \eta(0) = \beta_5(\bar{c}(T) - c_\Omega), & & \text{in } \Omega, \\
\rho(0) &= \beta_5(\bar{c}(T) - c_\Omega), & & \text{in } \Omega.
\end{aligned}
\]
Following an analogous reasoning as in the proof of Lemma 5.3, we omit the process and just give a number of a priori estimates as follows.

Taking the \(L^2\)-inner product with \(\lambda\) for the first equation of (5.21) implies
\[
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\Omega} \lambda^2 \, dx + \int_{\Omega} |\nabla \lambda|^2 \, dx + 2\mu \int_{\Omega} \lambda^2 \bar{n} \, dx \\
&\quad = \int_{\Omega} \nabla \lambda \nabla \bar{c} \, dx + \gamma \int_{\Omega} \lambda \varphi \, dx + \int_{\Omega} \lambda \nabla \varphi \, dx + \beta_1 \int_{\Omega} (\bar{n} - n_d) \lambda \, dx \\
&\quad \leq \|\nabla \lambda\|_{L^2} \|\nabla \bar{c}\|_{L^2} + \gamma \|\lambda\|_{L^2}^2 + \|\lambda\|_{L^2} \|\varphi\|_{L^2} + \beta_1 \|\bar{n} - n_d\|_{L^2} \|\lambda\|_{L^2}.
\end{aligned}
\]
\[
\frac{d}{dt} \|\nabla \lambda\|^2_{L^2} + \|\lambda\|^2_{H^1} \leq C(\|\nabla \eta\|^2_{L^2} + \|\eta\|^2_{L^2} + \|\rho\|^2_{L^2}) + C\|\bar{n} - n_d\|^2_{L^2}.
\]

Then, we have
\[
\frac{d}{dt} \|\nabla \lambda\|^2_{L^2} + \|\lambda\|^2_{H^1} \leq C(\|\nabla \eta\|^2_{L^2} + \|\eta\|^2_{L^2} + \|\rho\|^2_{L^2}) + C\|\bar{n} - n_d\|^2_{L^2}. \tag{5.22}
\]

Taking the \(L^2\)-inner product with \(-\Delta \eta\) for the first equation of (5.21) implies
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla \lambda|^2 dx + \int_\Omega |\Delta \lambda|^2 dx
\]
\[
= \int_\Omega \bar{n} \cdot \nabla \lambda \Delta \lambda dx - \int_\Omega \nabla \lambda \nabla \Delta \lambda dx - \gamma \int_\Omega \lambda \Delta \lambda dx + 2\mu \int_\Omega \lambda \bar{n} \Delta \lambda dx
\]
\[
- \int_\Omega \eta \Delta \lambda dx - \int_\Omega \nabla \varphi \Delta \lambda dx + \beta_1 \int_\Omega (\bar{n} - n_d) \Delta \lambda dx
\]
\[
\leq \|\bar{n}\|_{L^4} \|\nabla \lambda\|_{L^4} \|\Delta \lambda\|_{L^2} + \|\nabla \lambda\|_{L^4} \|\nabla \varphi\|_{L^4} \|\Delta \lambda\|_{L^2} + \gamma \|\nabla \lambda\|^2_{L^2}
\]
\[
+ \|\lambda\|_{L^4} \|\nabla \lambda\|_{L^4} \|\Delta \lambda\|_{L^2} - \eta \|\lambda\|_{L^2} \|\Delta \lambda\|_{L^2} + \|\rho\|_{L^2} \|\Delta \lambda\|_{L^2}
\]
\[
+ \beta_1 \|\Delta \lambda\|_{L^2} \|\bar{n} - n_d\|^2_{L^2}
\]
\[
\leq \frac{1}{2} \|\Delta \lambda\|^2_{L^2} + C(\|\nabla \lambda\|^2_{L^2} + \|\eta\|^2_{L^2} + \|\rho\|^2_{L^2}).
\]

Thus, we get
\[
\frac{d}{dt} \|\nabla \lambda\|^2_{L^2} + \|\lambda\|^2_{H^1} \leq C(\|\nabla \eta\|^2_{L^2} + \|\eta\|^2_{L^2} + \|\rho\|^2_{L^2}) + C\|\bar{n} - n_d\|^2_{L^2}. \tag{5.23}
\]

Taking the \(L^2\)-inner product with \(\eta\) for the second equation of (5.21) implies
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \eta^2 dx + \int_\Omega |\nabla \eta|^2 dx + \int_\Omega \eta^2 dx
\]
\[
= \int_\Omega \bar{n} \nabla \lambda \nabla \eta dx + \beta_2 \int_\Omega \eta (\bar{\epsilon} - c_d) dx
\]
\[
\leq \|\bar{n}\|_{L^4} \|\nabla \lambda\|_{L^4} \|\nabla \eta\|_{L^2} + \beta_2 \|\eta\|_{L^2} \|\bar{\epsilon} - c_d\|_{L^2}
\]
\[
\leq \|\bar{n}\|_{H^1} (\|\nabla \lambda\|^2_{L^2} + \|\nabla \varphi\|_{L^2} \|\nabla \eta\|_{L^2} + \beta_2 \|\eta\|_{L^2} \|\bar{\epsilon} - c_d\|_{L^2}
\]
\[
\leq \frac{1}{2} \|\nabla \eta\|^2_{L^2} + \delta \|\Delta \lambda\|^2_{L^2} + C\|\nabla \lambda\|_{L^2} + C\|\eta\|^2_{L^2} + C\|\bar{\epsilon} - c_d\|^2_{L^2}.
\]

As an immediate consequence, we obtain
\[
\frac{d}{dt} \|\eta\|^2_{L^2} + \|\eta\|^2_{H^1} \leq \delta \|\Delta \lambda\|^2_{L^2} + C\|\nabla \lambda\|_{L^2} + C\|\eta\|^2_{L^2} + C\|\bar{\epsilon} - c_d\|^2_{L^2}. \tag{5.24}
\]

Taking the \(L^2\)-inner product with \(\rho\) for the third equation of (5.21) implies
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho^2 dx + \int_\Omega |\nabla \rho|^2 dx
\]
\[
= - \int_\Omega (\rho \cdot \nabla \varphi) \bar{n} \rho dx - \int_\Omega \nabla \bar{n} \rho dx - \int_\Omega \eta \nabla \varphi \rho dx + \beta_3 \int_\Omega (\bar{u} - u_d) \rho dx
\]
\[
\leq \|\rho\|_{L^2} \|\nabla \bar{n}\|_{L^2} \|\rho\|_{L^4} + \lambda \|\nabla \bar{n}\|_{L^2} \|\rho\|_{L^2} + \|\eta\|_{L^2} \|\nabla \varphi\|_{L^4} \|\rho\|_{L^4}
\]
\[
+ \beta_3 \|\bar{u} - u_d\|_{L^2} \rho_{L^2} \rho_{L^4}.
\]
Combining (5.22)-(5.25) and taking referee's valuable suggestions for the revision and improvement of the manuscript.

Acknowledgment. The authors would like to express their deep thanks to the referee’s valuable suggestions for the revision and improvement of the manuscript.

REFERENCES

[1] P. Biler, W. Hebisch and T. Nadzieja, The Debye system: Existence and large time behavior of solutions, *Nonlinear Anal.*, 23 (1994), 1189–1209.
[2] B. Chen and C. Liu, Optimal distributed control of a Allen-Cahn/Cahn-Hilliard system with temperature, *Applied Mathematics and Optimization*, 2021.
[3] B. Chen, H. Li and C. Liu, Optimal distributed control for a coupled phase-field system, *Discrete and Continuous Dynamical Systems Series B*, 23 (2018), 95–116.
[4] P. Colli, G. Gilardi, G. Marinoschi and E. Rocca, Optimal control for a conserved phase field system with a possibly singular potential, *Evol. Equ. Control Theory*, 7 (2018), 95–116.
[5] P. Colli, G. Gilardi, E. Rocca and J. Sprekels, Optimal distributed control of a diffuse interface model of tumor growth, *Nonlinearity*, 30 (2017), 2518–2546.
[6] E. Espejo and T. Suzuki, Reaction terms avoiding aggregation in slow fluids, *Nonlinear Anal. Real World Appl.*, 21 (2015), 110–126.
[7] A. Friedman, *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1969.
[8] S. Frigeri, M. Grasselli and J. Sprekels, Optimal distributed control of two-dimensional nonlocal Cahn-Hilliard-Navier-Stokes systems with degenerate mobility and singular potential, *Appl. Math. Optim.*, 81 (2020), 899–931.
[9] F. Guillén-González, E. Mallea-Zepeda and M. Rodríguez-Bellido, Optimal bilinear control problem related to a chemo-repulsion system in 2D domains, *ESAIM Control Optim. Calc. Var.*, 26 (2020), 21pp.
[10] A. Helmut and T. Yutaka, On Stokes operators with variable viscosity in bounded and unbounded domains, *Math. Ann.*, 344 (2009), 381–429.
[11] C. Jin, Large time periodic solutions to coupled chemotaxis-fluid models, *Z. Angew. Math. Phys.*, 68 (2017), 24pp.
[12] C. Jin, Large time periodic solution to the coupled chemotaxis-Stokes model, *Math. Nachr.*, 300 (2017), 1701–1715.
[13] C. Liu and X. Zhang, Optimal distributed control for a new mechanochemical model in biological patterns, *J. Math. Anal. Appl.*, 478 (2019), 825–863.
[14] C. Liu and X. Zhang, Optimal control of a new mecanochemical model with state constraint, *Math. Methods Appl. Sci.*, 44 (2021), 9237–9263.
[15] S. U. Ryu and A. Yagi, Optimal control of Keller-Segel equations, *J. Math. Anal. Appl.*, 256 (2001), 45–66.
[16] J. Simon, Compact sets in the space $L^p(0, T; B)$, *Ann. Mat. Pura Appl.*, 146 (1987), 65–96.
[17] Y. Tao and M. Winkler, Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system, *Z. Angew. Math. Phys.*, 66 (2015), 2555–2573.
[18] Y. Tao and M. Winkler, Blow-up prevention by quadratic degradation in a two-dimensional Keller-Segel-Navier-Stokes system, *Z. Angew. Math. Phys.*, 67 (2016), 1–23.
[19] X. Zhang, H. Li and C. Liu, Optimal control problem for the Cahn-Hilliard/Allen-Cahn equation with state constraint, *Appl. Math. Optim.*, 82 (2020), 721–754.

Received August 2021; revised September 2021; early access October 2021.

E-mail address: yuanyf19@mails.jlu.edu.cn
E-mail address: liucc@jlu.edu.cn