A POSTERIORI ESTIMATES FOR EULER AND NAVIER-STOKES EQUATIONS

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Abstract. The first two sections of this work review the framework of [6] for approximate solutions of the incompressible Euler or Navier-Stokes (NS) equations on a torus $$\mathbb{T}^d$$, in a Sobolev setting. This approach starts from an approximate solution $$u_a$$ of the Euler/NS Cauchy problem and, analyzing it a posteriori, produces estimates on the interval of existence of the exact solution $$u$$ and on the distance between $$u$$ and $$u_a$$. The next two sections present an application to the Euler Cauchy problem, where $$u_a$$ is a Taylor polynomial in the time variable $$t$$; a special attention is devoted to the case $$d = 3$$, with an initial datum for which Behr, Nečas and Wu have conjectured a finite time blowup [1]. These sections combine the general approach of [6] with the computer algebra methods developed in [9]; choosing the Behr-Nečas-Wu datum, and using for $$u_a$$ a Taylor polynomial of order 52, a rigorous lower bound is derived on the interval of existence of the exact solution $$u$$, and an estimate is obtained for the $$H^3$$ Sobolev distance between $$u(t)$$ and $$u_a(t)$$.

1. Preliminaries.

Throughout this work we fix a space dimension $$d \in \{2, 3, \ldots\}$$; in the application of section 4 we will put $$d = 3$$. For $$a, b$$ in $$\mathbb{R}^d$$ or $$\mathbb{C}^d$$ we put $$a \cdot b := \sum_{r=1}^d a_r b_r$$ and $$|a| := \sqrt{a \cdot a}$$, with $$\bar{a}$$ indicating the complex conjugate.

Let us consider the $$d$$-dimensional torus $$\mathbb{T}^d := (\mathbb{R}/2\pi \mathbb{Z})^d$$; we denote with $$(e_k)_{k \in \mathbb{Z}^d}$$ the Fourier basis made of the functions $$e_k : \mathbb{T}^d \to \mathbb{C}$$, $$e_k(x) := (2\pi)^{-d/2} e^{ik \cdot x}$$. Here and in the sequel, “a vector field on $$\mathbb{T}^{dn}$$ means “an $$\mathbb{R}^d$$-valued distribution on $$\mathbb{T}^{dn}$$ (see, e.g., [5]); we write $$\mathbb{D}'(\mathbb{T}^d) \equiv \mathbb{D}'$$ for the space of such distributions. Any $$v \in \mathbb{D}'$$ has a weakly convergent Fourier expansion $$v = \sum_{k \in \mathbb{Z}^d} v_k e_k$$, with coefficients $$v_k \in \mathbb{C}^d$$ such that $$\overline{v_k} = v_{-k}$$.

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In the sequel $L^p(T^d) \equiv L^p$ denotes the space of $L^p$ vector fields $T^d \rightarrow R^d$. For all $n \in R$ we introduce the Sobolev space of zero mean, divergence free vector fields of order $n$; this is

$$H^m_{20}(T^d) = \{ v \in D' \mid \int_{T^d} v dx = 0, \text{div} v = 0, \sqrt{-\Delta} v \in L^2 \}$$

$$= \{ v \in D' \mid v_0 = 0, k \bullet v_k = 0 \text{ for all } k, \sum_{k \in Z^d \setminus \{0\}} |k|^{2n} |v_k|^2 < +\infty \}$$

(in the above, $\int_{T^d} v dx$ indicates the action of $v$ on the test function $1$ and $\sqrt{-\Delta} v := \sum_{k \in Z^d \setminus \{0\}} |k|^{2n} v_k e_k$). $H^m_{20}$ is a Hilbert space with the inner product and the norm

$$\langle v, w \rangle_n := \langle \sqrt{-\Delta} v, \sqrt{-\Delta} w \rangle_{L^2} = \sum_{k \in Z^d \setminus \{0\}} |k|^{2n} \langle v_k, w_k \rangle, \quad \|v\|_n := \langle v, v \rangle_n ;$$

if $m \leq n$ then $H^m_{20} \subset H^n_{20}$.

1.1. The bilinear map for the Euler or Navier-Stokes (NS) equations. Consider two vector fields $v, w$ on $T^d$ such that $v \in L^2$ and $\partial_r w \in L^2$ for $r = 1, ..., d$; then we have a well defined vector field $v \bullet \partial w$ such that $n \in \mathbb{N}$. We choose a “viscosity coefficient” $\nu \in [0, +\infty)$, and put

$$\mathcal{P}(v, w) := -\mathcal{L}(v \bullet \partial w) .$$

The bilinear map $\mathcal{P} : (v, w) \mapsto \mathcal{P}(v, w)$, which is a main character of the incompressible Euler/NS equations, is known to possess the following properties:

(i) For each $n > d/2$, $\mathcal{P}$ is continuous from $H^m_{20} \times H^{n+1}_{20}$ to $H^n_{20}$; so, there is a constant $K_{nd} \equiv K_n$ such that

$$\|\mathcal{P}(v, w)\|_n \leq K_n \|v\|_n \|w\|_{n+1}$$

for $v \in H^m_{20}$, $w \in H^{n+1}_{20}$.

(ii) For each $n > d/2 + 1$, there is a constant $G_{nd} \equiv G_n$ such that

$$|\langle \mathcal{P}(v, w) |w\rangle_n| \leq G_n \|v\|_n \|w\|_n^2$$

for $v \in H^m_{20}$, $w \in H^{n+1}_{20}$.

The result (ii) is due to Kato, see [3]. In papers [7] [8], (1.4) and (1.5) are called the “basic inequality” and the “Kato inequality”, respectively; in these papers, computable upper and lower bounds are given for the sharp constants appearing therein. From here to the end of this work, $K_n$ and $G_n$ are constants fulfilling the previous inequalities (and not necessarily sharp). From [7] [8] we know that we can take

$$K_3 = 0.323, \quad G_3 = 0.438 \quad \text{if } d = 3 ;$$

these values will be useful in the sequel.

1.2. The Euler/NS Cauchy problem. Let us fix a Sobolev order

$$n \in \left( \frac{d}{2} + 1, +\infty \right) .$$

We choose a “viscosity coefficient” $\nu \in [0, +\infty)$, and put

$$\mathcal{P} := \begin{cases} 1 & \text{if } \nu = 0, \\ 2 & \text{if } \nu > 0. \end{cases}$$

Furthermore, we choose a “forcing”

$$f \in C([0, +\infty), H^n_{20}) .$$
and an initial datum
\begin{equation}
\label{eq:1.10}
 u_0 \in \mathbb{H}_{20}^{n+\nu}.
\end{equation}

**Definition 1.1.** The Cauchy problem for the (incompressible) fluid with viscosity \( \nu \), initial datum \( u_0 \) and forcing \( f \) is the following:
\begin{equation}
\label{eq:1.11}
 \text{Find } u \in C([0,T), \mathbb{H}_{20}^{n+\nu}) \cap C^1([0,T), \mathbb{H}_0^{n}) \text{ such that } \frac{du}{dt} = \nu \Delta u + \mathcal{P}(u, u) + f, \quad u(0) = u_0
\end{equation}
(with \( T \in (0, +\infty) \), depending on \( u \)). As usually, we speak of the “Euler Cauchy problem” if \( \nu = 0 \), and of the “NS Cauchy problem” if \( \nu > 0 \).

It is known [4] that the above Cauchy problem has a unique maximal (i.e., non extendable) solution; any solution is a restriction of the maximal one.

2. Approximate solutions of the Euler/NS Cauchy problem

We consider again the Cauchy problem (1.11), for given \( \nu, \nu, f, u_0 \) as in the previous section. The definitions and the theorem that follow are taken from [6].

**Definition 2.1.** An approximate solution of problem (1.11) is any map \( u_a \in C([0,T_a), \mathbb{H}_{20}^{n+\nu}) \cap C^1([0,T_a), \mathbb{H}_0^{n}) \) (with \( T_a \in (0, +\infty) \)). Given such a function, we stipulate (i) (ii).

(i) The differential error of \( u_a \) is
\begin{equation}
\label{eq:2.1}
 \frac{du_a}{dt} - \nu \Delta u_a - \mathcal{P}(u_a, u_a) - f \in C([0,T_a), \mathbb{H}_0^{n}) ;
\end{equation}
the datum error is
\begin{equation}
\label{eq:2.2}
 u_a(0) - u_0 \in \mathbb{H}_{20}^{n+\nu}.
\end{equation}

(ii) Let \( m \in \mathbb{R} \), \( m \leq n \). A differential error estimator of order \( m \) for \( u_a \) is a function
\begin{equation}
\label{eq:2.3}
 \epsilon_m \in C([0,T_a), [0, +\infty)) \text{ such that } \| (\frac{du_a}{dt} - \nu \Delta u_a - \mathcal{P}(u_a, u_a) - f)(t) \|_m \leq \epsilon_m(t) \text{ for } t \in [0, T_a) .
\end{equation}
Let \( m \in \mathbb{R} \), \( m \leq n + \nu \). A datum error estimator of order \( m \) for \( u_a \) is a real number
\begin{equation}
\label{eq:2.4}
 \delta_m \in [0, +\infty) \text{ such that } \| u_a(0) - u_0 \|_m \leq \delta_m ;
\end{equation}
a growth estimator of order \( m \) for \( u_a \) is a function
\begin{equation}
\label{eq:2.5}
 \mathcal{D}_m \in C([0,T_a), [0, +\infty)) \text{ such that } \| u_a(t) \|_m \leq \mathcal{D}_m(t) \text{ for } t \in [0, T_a) .
\end{equation}
In particular \( \epsilon_m(t) := \| (\frac{du_a}{dt} - \nu \Delta u_a - \mathcal{P}(u_a, u_a) - f)(t) \|_m, \delta_m := \| u_a(0) - u_0 \|_m \) and \( \mathcal{D}_m(t) := \| u_a(t) \|_m \) will be called the tautological estimators of order \( m \) for the differential error, the datum error and the growth of \( u_a \).

From here to the end of the section we consider an approximate solution \( u_a \) of problem (1.11) of domain \([0, T_a)\); this is assumed to possess differential, datum error and growth estimators of orders \( n \) or \( n + 1 \), indicated with \( \epsilon_n, \delta_n, \mathcal{D}_n, \mathcal{D}_{n+1} \).

**Definition 2.2.** Let \( \mathcal{R}_n \in C([0,T_c), [0, +\infty)) \), with \( T_c \in (0, T_a] \). This function is said to fulfill the control inequalities if
\begin{equation}
\label{eq:2.6}
 \frac{d^+ \mathcal{R}_n}{dt} \geq -\nu \mathcal{R}_n + (G_n \mathcal{D}_n + K_n \mathcal{D}_{n+1}) \mathcal{R}_n + G_n \mathcal{R}_n^2 + \epsilon_n \text{ everywhere on } [0, T_c),
\end{equation}
\begin{equation}
\label{eq:2.7}
 \mathcal{R}_n(0) \geq \delta_n .
\end{equation}
In the above \(d^+/dt\) indicates the right, upper Dini derivative: so, for all \(t \in [0, T_c)\), 
\[
(d^+/dt)R_n(t) := \limsup_{h \to 0^+} [R_n(t + h) - R_n(t)]/h.
\]

**Proposition 2.1.** Assume there is a function \(R_n \in C([0, T_c), [0, +\infty))\) fulfilling the control inequalities; consider the maximal solution \(u\) of the Euler/NS Cauchy problem \((1.11)\), and denote its domain with \([0, T)\). Then
\[
T \geq T_c,
\]
(2.8)
\[
\|u(t) - u_a(t)\|_n \leq R_n(t) \quad \text{for } t \in [0, T_c).
\]
(2.9)

**Proof.** (Sketch) One introduces the function \(\|u - u_a\|_n : t \in [0, T) \cap [0, T_a) \mapsto \|u(t) - u_a(t)\|_n\) and shows that \(d^+\|u - u_a\|_n/dt \leq -\nu \|u - u_a\|_n + (G_nD_n + K_nD_{n+1})\|u - u_a\|_n + G_n\|u - u_a\|^2_n + \epsilon_n\); (see Lemma 4.2 of [6], greatly indebted to [2]); moreover, \(\|u(0) - u_a(0)\|_n \leq \delta_n\). From here, from the control inequalities \((2.6)\) \((2.7)\) and from the Čaplygin comparison lemma one infers that \(\|u(t) - u_a(t)\|_n \leq R_n(t)\) for \(t \in [0, T) \cap [0, T_c)\). Finally, it is \(T \geq T_c\); in fact, if it were \(T < T_c\), the previous inequality about \(u, u_a\) and \(R_n\) would imply \(\limsup_{t \to T^-} \|u(t)\|_n < +\infty\), a fact contradicting the maximality assumption for \(u\). See [6] for more details. \(\square\)

Paper [6] presents some applications of the previous proposition, dealing with both the Euler case \(\nu = 0\) and the NS case \(\nu > 0\); a special attention is devoted therein to the approximate solutions \(u_a\) provided by the Galerkin method.

In this work we present an application of Proposition 2.1 to the Euler case \(\nu = 0\), choosing for \(u_a\) a polynomial in the time variable \(t\). In the next section we present this procedure in general, giving the error estimators for approximate solutions of this kind; in the last section we apply the procedure choosing for \(u_0\) the so-called Behr-Nečas-Wu initial datum.

3. **Polynomial approximate solutions for the Euler equations**

Let us recall that \(n \in \{d/2 + 1, +\infty\}\), and consider the Euler Cauchy problem with a datum \(u_0 \in \mathbb{H}^{n+1}_{\Sigma_0}\) and zero external forcing:
\[
(3.1) \quad \text{Find } u \in C([0, T], \mathbb{H}^{n+1}_{\Sigma_0}) \cap C^1([0, T], \mathbb{H}^{n}_{\Sigma_0}) \text{ such that } \frac{du}{dt} = P(u, u), \quad u(0) = u_0.
\]

Let us choose an order \(N \in \{0, 1, 2, \ldots\}\) and consider as an approximate solution for \((3.1)\) a polynomial of degree \(N\) in time, of the form
\[
(3.2) \quad u^N : [0, +\infty) \to \mathbb{H}^{n+1}_{\Sigma_0}, \quad t \mapsto u^N(t) := \sum_{j=0}^{N} u_j t^j \quad (u_j \in \mathbb{H}^{n+1}_{\Sigma_0} \text{ for all } j).
\]

Here \(u_0\) is the initial datum, and \(u_j\) is to be determined for \(j = 1, ..., N\).

**Proposition 3.1.** (i) Let \(u^N\) be as in \((3.2)\). The datum and differential errors of \(u^N\) are
\[
(3.3) \quad u^N(0) - u_0 = 0;
\]
\[
(3.4) \quad \frac{du^N}{dt}(t) - P(u^N, u^N)(t)
\]
\[= \sum_{j=0}^{N-1} [(j+1)u_{j+1} - \sum_{\ell=0}^{j} P(u_{\ell}, u_{j-\ell})] t^j - \sum_{j=N}^{2N} \sum_{\ell=j-N}^{N} P(u_{\ell}, u_{j-\ell}) t^j.
\]
(ii) In particular, assume
\begin{equation}
    u_{j+1} = \frac{1}{j+1} \sum_{\ell=0}^{j} P(u_{\ell}, u_{j-\ell}) \quad \text{for } j = 0, \ldots, N-1 ;
\end{equation}
then
\begin{equation}
    \frac{du^N}{dt}(t) - P(u^N, u^N)(t) = - \sum_{j=N}^{2N} \left[ \sum_{\ell=\ell-j}^{N} P(u_{\ell}, u_{j-\ell}) \right] t^j = O(t^N) \quad \text{for } t \to 0 .
\end{equation}

(iii) If (3.5) is used to define recursively $u_1, \ldots, u_N$, it produces a sequence of elements of $\mathbb{H}^n+1_{\Sigma_0}$ under the condition $u_0 \in \mathbb{H}^n+1_{\Sigma_0}$. More precisely, from $u_0 \in \mathbb{H}^n+1_{\Sigma_0}$ it follows $u_j \in \mathbb{H}^{n+1+N-j}_{\Sigma_0} \subset \mathbb{H}^{n+1}$ for $j = 1, \ldots, N$.

(iv) Let $u_0 \in \mathbb{H}^{n+1+N}_{\Sigma_0}$ and use (3.5) to define $u_j$ for $j = 1, \ldots, N$. Then
\begin{equation}
    \epsilon_n(t) := K_n \sum_{j=N}^{2N} \left[ \sum_{\ell=\ell-j}^{N} \|u_{\ell}\|_n \|u_{j-\ell}\|_{n+1} \right] t^j \quad \text{for } t \in [0, +\infty),
\end{equation}
\begin{equation}
    \epsilon_n(t) := K_n \sum_{j=N}^{2N} \left[ \sum_{\ell=\ell-j}^{N} \|u_{\ell}\|_n \|u_{j-\ell}\|_{n+1} \right] t^j \quad \text{for } t \in [0, +\infty).}
\end{equation}

Proof. (i) (3.3) is obvious; let us prove (3.4). To this purpose, we note that
\begin{equation}
    \frac{du^N}{dt} - P(u^N, u^N) = \frac{d}{dt} \left( \sum_{\ell=0}^{N} u_{\ell} t^\ell \right) - P \left( \sum_{\ell=0}^{N} u_{\ell} t^\ell, \sum_{h=0}^{N} u_h t^h \right)
\end{equation}
\begin{equation}
= \sum_{\ell=1}^{N} \ell u_{\ell} t^{\ell-1} - \sum_{\ell,h=0}^{N} P(u_{\ell}, u_h) t^{\ell+h} = \sum_{j=0}^{N-1} (j+1) u_{j+1} t^j - \sum_{j=0}^{2N} \left[ \sum_{(\ell,h) \in I_{N,j}} P(u_{\ell}, u_h) \right] t^j ,
\end{equation}
\begin{equation}
I_{N,j} := \{(\ell, h) \in \{0, \ldots, N\}^2 \mid \ell + h = j \} .
\end{equation}

One easily checks that
\begin{equation}
    j \in \{0, \ldots, N-1\} \implies I_{N,j} = \{(\ell, j-\ell) \mid \ell \in \{0, \ldots, j\} \} ,
\end{equation}
\begin{equation}
    j \in \{N, \ldots, 2N\} \implies I_{N,j} = \{(\ell, j-\ell) \mid \ell \in \{j-N, \ldots, N\} \} ;
\end{equation}
this readily yields the thesis (3.4).

(ii) Obvious.

(iii) Let $u_0 \in \mathbb{H}^{n+1+N}_{\Sigma_0}$ and define $u_1, \ldots, u_N$ via the recursion relation (3.5). Then
\begin{equation}
    u_1 = P(u_0, u_0) \in \mathbb{H}^{n+1+N}_{\Sigma_0}, \quad u_2 = (1/2)P(u_0, u_1) + (1/2)P(u_1, u_0) \in \mathbb{H}^{n+1+N-1}_{\Sigma_0}, \text{ etc.} .
\end{equation}

(iv) Eq. (3.6) implies $\|(du^N/dt)(t) - P(u^N, u^N)(t))\|_n \leq \sum_{j=N}^{2N} \left[ \sum_{\ell=\ell-j}^{N} \|P(u_{\ell}, u_{j-\ell})\|_n \right] t^j .
\end{equation}
On the other hand Eq. (1.4) gives $\|P(u_{\ell}, u_{j-\ell})\|_n \leq K_n \|u_{\ell}\|_n \|u_{j-\ell}\|_{n+1} \text{, whence the thesis (3.7).}$

$\square$
A special case of the previous framework: the Euler equations on \( \mathbb{T}^3 \), with the \( \text{Behr-Nečas-Wu} \) initial datum.

In this section we consider the Euler Cauchy problem (3.1) with space dimension and Sobolev order

\[
(4.1) \quad d = 3, \quad n = 3;
\]

the initial datum is

\[
(4.2) \quad u_0 := \sum_{k=\pm a, \pm b, \pm c} u_{0k} e_k,
\]

\[
a := (1,1,0), \quad b := (1,0,1), \quad c := (0,1,1);
\]

\[
u_{0,\pm a} := (2\pi)^{3/2}(1,-1,0), \quad u_{0,\pm b} := (2\pi)^{3/2}(1,0,-1), \quad u_{0,\pm c} := (2\pi)^{3/2}(0,1,-1)
\]

(of course, being a Fourier polynomial, \( u_0 \) belongs to \( H^m_{30} \) for each \( m \in \mathbb{R} \)). The above initial datum is considered by Behr, Nečas and Wu in \([1]\); it is analyzed with a similar attitude in \([9]\) and, from a different viewpoint, in \([6]\). In both papers \([1,9]\), attention is fixed on the function \( u^N(t) = \sum_{j=0}^N u_j t^j \) for a rather large value of \( N \), where the \( u_j \)'s are determined for \( j = 1, \ldots, N \) by the recursion relation (3.5). The \( u_j \)'s are Fourier polynomials and can be calculated exactly by computer algebra methods; such computations are performed in \([1]\) for \( N = 35 \), and in \([9]\) up to \( N = 52 \) (using, respectively, the \( C^{+,+} \) and the Python languages).

The Python program of \([9]\) gives exact expressions for the \( u_j \)'s, whose Fourier coefficients are rational (up to factors \( (2\pi)^{3/2} \)); for large \( j \), these expressions are terribly complicated. Here, to give a partial illustration of such Python computations we consider the Fourier components \( u_{52}^k(t) \) for \( k = (1,1,0) \) and \( k = (0,0,2) \), and report the graphs of the functions \( t \mapsto |u_{52}^k(t)| \) for these wave vectors: see Figures 1 and 2.

In both papers \([1,9]\), computations are used to get hints about \( \lim_{N \to +\infty} u^N \), giving the exact solution of the Euler Cauchy problem on the time interval where the limit exists; however the statements of \([1,9]\) rely on the assumption that certain facts on the \( N \to +\infty \) limit can be extrapolated from \( u^{35} \) or \( u^{52} \). In particular \([1]\) makes the conjecture, disputed in \([9]\), that the solution of the Euler Cauchy problem blows up for \( t \to \tau^- \), with \( \tau \simeq 0.32 \).

In the present work we make no conjecture or extrapolation about the \( N \to +\infty \) limit and just consider the function \( u^{52} \) of \([9]\) according to the general framework of approximate solutions and control inequalities. This approach produces:

(i) a rigorous lower bound on the interval of existence of the exact solution \( u \) of the \((d = 3, n = 3)\) Cauchy problem (3.1);
(ii) a bound on \( \|u(t) - u^{52}(t)\|_3 \).

To get these results we regard \( u^{52} \) as an approximate solution of (3.1), using the tautological datum error and growth estimators

\[
(4.3) \quad \delta_3 := 0; \quad \mathcal{D}_3(t) := \|u^{52}(t)\|_3, \quad \mathcal{D}_4(t) := \|u^{52}(t)\|_4 \quad \text{for } t \in [0, +\infty)
\]

(concerning \( \delta_3 \), we recall that \( u^{52}(0) - u_0 = 0 \)). For \( m = 3,4 \) one has \( \mathcal{D}_m(t) = (2\pi)^{3/2}[\sum_{j=0}^{52} d_{mj} t^{2j}]^{1/2} \) where the \( d_{mj} \)'s are rational coefficients; the Python program employed for our work \([9]\) computes exactly these coefficients. For \( m = 3 \) these coefficients are reported in \([9]\), in a 16-digits decimal representation (see Eq. (5.12) of \([9]\), not containing the factor \((2\pi)^{3/2} \) due to a different normalization of the norm.
we can grant that: 

Let us pass to the differential error estimator for \( u^{52} \); we use for it the function \( \epsilon_3 \) defined by (3.8) with \( n = 3 \) and \( K_3 = 0.323 \), see (1.6). \( \epsilon_3 \) is computed exactly by our Python program; again, the explicit expression is too complicated to be reported. (The tautological error estimator \( \epsilon_{3}^{*}(t) := \|(du^{52}/dt)(t) - \mathcal{P}(u^{52}, u^{52}(t))\|_{3} \) is more accurate, but it has an even more complicated expression; its calculation by computer algebra is too expensive.)

For the graph of \( \epsilon_3 \) and some information on its numerical values, see Figure 5 and its caption. With the previous ingredients, we build the following "control Cauchy problem": find \( \mathcal{R}_3 \) such that

\[
\mathcal{R}_3 \in C^{1}([0, T_{c}], \mathbb{R}), \quad \frac{d\mathcal{R}_3}{dt} = (G_{3} \mathcal{D}_3 + K_{3} \mathcal{D}_4)\mathcal{R}_3 + G_{3} \mathcal{R}_3^2 + \epsilon_3, \quad \mathcal{R}_3(0) = 0
\]

(\( G_{3} = 0.438 \), see again (1.6)). This control problem has a unique maximal solution \( \mathcal{R}_3 \), which is strictly increasing and thus positive for \( t \in (0, T_{c}) \). Of course, this \( \mathcal{R}_3 \) fulfils as equalities Eqs. (2.6) (2.7) (with \( \nu = 0 \)).

Once we have \( \mathcal{R}_3 : [0, T_{c}] \to [0, +\infty) \), due to Proposition 2.1 we can grant that:

(i) The maximal solution \( u \) of the \( (n = 3) \) Euler Cauchy problem (3.1) is defined on an interval \([0, T)\) with \( T \geq T_{c} \);

(ii) It is

\[
\|u(t) - u^{52}(t)\|_{3} \leq \mathcal{R}_3(t) \quad \text{for} \ t \in [0, T_{c}].
\]

The function \( \mathcal{R}_3 \) can be determined numerically by a cheap computation using any package for ODEs, e.g. Mathematica (the result is reliable, since (4.4) is the Cauchy problem for a simple ODE in one dimension). This numerical computation indicates that the (maximal) domain of \( \mathcal{R}_3 \) is \([0, T_{c})\), with

\[
T_{c} = 0.242\ldots
\]

After having been extremely small for most of the time between 0 and \( T_{c} \), \( \mathcal{R}_3(t) \) diverges abruptly for \( t \to T_{c}^{-} \); for the graph of this function and some information on its numerical values, see Figure 6 and its caption. Due to (4.6), we can grant that the solution \( u \) of the Euler Cauchy problem (1.11) exists on a time interval of length \( T \geq 0.242 \) (this is four times larger than the lower bound on \( T \) obtained in [6] using a Galerkin approximate solution).

Eq. (4.5) and the previously described behavior of \( \mathcal{R}_3 \) ensure that \( u^{52}(t) \) approximates with extreme precision \( u(t) \) on most of the time interval \([0, T_{c})\). We remark that (4.5) can be used to infer other interesting estimates about \( u - u^{52} \), e.g.,

\[
|u_k(t) - u_k^{52}(t)| \leq \frac{\mathcal{R}_3(t)}{|k|^{3}} \quad \text{for} \ k \in \mathbb{Z}^{3} \setminus \{0\}, \ t \in [0, T_{c}];
\]

this follows from (4.5) and from the elementary inequality \( |v_k| \leq \|v\|_{3}/|k|^{3} \), holding for all \( v \in \mathbb{H}^{3}_{\mathbb{Z}_{0}} \) and \( k \in \mathbb{Z}^{3} \setminus \{0\} \) (recall that \( \|v\|_{3}^{2} = \sum_{k \in \mathbb{Z}^{3} \setminus \{0\}} |k|^{6} |v_k|^{2} \).
Figure 1. Plot of $|u_{1,1,0}^2(t)|$ for $t \in [0, 0.32]$.

Figure 2. Plot of $|u_{0,2}^2(t)|$ for $t \in [0, 0.32]$.

Figure 3. Plot of $D_3(t)$ for $t \in [0, 0.32]$.

Figure 4. Plot of $D_4(t)$ for $t \in [0, 0.32]$.

Figure 5. Plot of $\epsilon_3(t)$ for $t \in [0.20, 0.26]$. One has: $\epsilon_3(t) < 10^{-20}$ for $t \in [0, 0.10]$; $\epsilon_3(t) < 10^{-4}$ for $t \in (0.10, 0.20]$; $\epsilon_3(t) < 10^{-3}$ for $t \in (0.20, 0.21]$; $\epsilon_3(t) < 8.6 \times 10^{-3}$ for $t \in (0.21, 0.22]$; $\epsilon(t) < 0.094$ for $t \in (0.22, 0.23]$; $\epsilon(t) < 0.93$ for $t \in (0.23, 0.24]$, $\epsilon_3(t) < 8.6$ for $t \in (0.24, 0.25]$; $\epsilon_3(t) < 74$ for $t \in (0.25, 0.26]$.

Figure 6. Plot of $R_3(t)$ for $t \in [0.20, 0.24]$. One has: $\Re_3(t) < 2 \times 10^{-6}$ for $t \in [0, 0.20]$; $\Re_3(t) < 1.2 \times 10^{-4}$ for $t \in (0.20, 0.21]$; $\Re_3(t) < 0.013$ for $t \in (0.21, 0.22]$; $\Re_3(t) < 2$ for $t \in (0.22, 0.23]$; $\Re_3(t) < 610$ for $t \in (0.23, 0.24]$. 
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