Local existence and uniqueness of solutions for non stationary compressible viscoelastic fluid of Oldroyd type

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Abstract

This work is devoted to the study of a compressible viscoelastic fluids satisfying the Oldroyd-B model in a regular bounded domain. We prove the local existence of solutions and uniqueness of flows by a classical fixed point argument.

1 Introduction

In this paper, we study the local existence of solutions for compressible viscoelastic fluid flows in the case of the Oldroyd-B model in a regular bounded domain in $\mathbb{R}^3$. We also show the uniqueness of solutions. We prove the existence by using the classical method based on the Schauder fixed-point theorem. Valli, in [8], show the local existence in the case of the Navier-Stokes equations. The case of the Oldroyd model for incompressible fluid is studied by Guillopé and Saut in [3]. Talhouk shows the existence and the uniqueness for Jeffreys model’s in [6].

This paper is organized as follows. Section 2 is devoted to the modeling of the problem and to the definition of well-prepared initial conditions. The principal notation and results are detailed in Section 3. The local existence of regular solutions is given in Section 4.

2 The Modeling

2.1 Unsteady Flows of Compressible Viscoelastic Fluids

Consider unsteady flows of viscoelastic fluids in a bounded domain $\Omega^*$ of $\mathbb{R}^3$ with a regular boundary $\Gamma^*$. The system, obtained from the laws of conservation of momentum, and of mass, and from the constitutive equation of the fluid, reads as follows [4]: in $Q^* = (0,T^*) \times \Omega^*$,

$$\begin{align*}
\rho^* \left( \frac{\partial u^*}{\partial t^*} + (u^* \cdot \nabla^*) u^* \right) &= \rho^* f^* + \text{div}^* (\tau^* - p^* I), \\
\rho^* \frac{\partial u^*}{\partial t^*} + \text{div}^* (\rho^* u^*) &= 0, \\
\tau^* + \lambda \frac{D_a \tau^*}{D_t^*} &= 2\eta \left( D^* + \mu \frac{D_a D^*}{D_t^*} \right). 
\end{align*}$$

(2.1)
The *-variables are the dimensional ones in the domain of the flow $\Omega^*$, and $T^* > 0$ is a dimensional time. The unknowns are the velocities $\mathbf{u}^*$, the density $\rho^*$, and the symmetric tensor of constraints $\tau^*$. $\eta$ is the total viscosity of the fluid, $\lambda > 0$ is the relaxation time, and $\mu$ is the retardation time ($0 < \mu < \lambda$).

\[
\frac{D_a\tau^*}{Dt^*} = \left( \frac{\partial}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) \right) \tau^* + \tau^*W^* - W^*\tau^* - a(D^*\tau^* + \tau^*D^*),
\]

where $W^* = W^*[\mathbf{u}^*] = \frac{1}{2}(\nabla^*\mathbf{u}^* - \nabla^*\tau^*\mathbf{u}^*)$ and $D^* = D^*[\mathbf{u}^*] = \frac{1}{2}(\nabla^*\mathbf{u}^* + \nabla^*\tau^*\mathbf{u}^*)$ are, respectively, the rate of rotation and the rate of deformation tensors. $a$ is a real parameter in $[-1, 1]$.

System (2.1) is completed by a condition on the boundary,

\[
\mathbf{u}^* = 0 \text{ on } \Sigma_{T^*}^* = (0, T^*) \times \Gamma^*,
\]

and by the initial data

\[
\mathbf{u}^*(0, \cdot) = \mathbf{u}_0^*, \quad \rho^*(0, \cdot) = \rho_0^*, \quad \tau^*(0, \cdot) = \tau_0^*, \quad \text{in } \Omega^*.
\]

We split $\tau^*$ into two parts: the Newtonian one $\tau_s^*$ related to the solvent, and the polymeric one $\tau_p^*$. We may write

\[
\tau^* = \tau_s^* + \tau_p^* = 2\eta_s D^* + \tau_e^*,
\]

where $\tau_e^* = \tau_p^* - \left( \frac{2\mu}{3} \text{div}^* \mathbf{u}^* \right) I$, and $I$ is the identity tensor. $\eta_s = \eta \mu / \lambda$ and $\xi_s$ are the solvent viscosity and the group viscosity, respectively. Since we are interested in a model for weakly compressible fluids, we suppose that $\xi_s = 0$. From the third equation in (2.1), we can deduce that $\tau_e^*$ satisfies the equation

\[
\tau_e^* + \lambda \frac{D_a\tau_e^*}{Dt^*} = 2\eta_e D^*,
\]

where $\eta_e = \eta - \eta_s$ is called the polymer viscosity. $\eta_s$ and $\eta_e$ are two non-negative numbers.

Therefore, under the assumption $\xi_s = 0$, System (2.1) is equivalent to the system in $Q_{T^*}^*$,

\[
\begin{align*}
\rho^* \left( \frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) \mathbf{u}^* \right) & = \rho^* \mathbf{f}^* + \eta_s (\Delta^* \mathbf{u}^* + \nabla^* \text{div}^* \mathbf{u}^*) - \nabla^* p^* + \text{div}^* \tau^*, \\
\frac{\partial \rho^*}{\partial t^*} + \text{div}^*(\rho^* \mathbf{u}^*) & = 0, \\
\tau^* + \lambda \frac{D_a\tau^*}{Dt^*} & = 2\eta_e D^*[\mathbf{u}^*],
\end{align*}
\]

where we have denoted $\tau_e^*$ by $\tau^*$ to simplify the notation.

### 2.2 Well-Prepared Initial Conditions

We first define the Mach number $\varepsilon$ as being the ratio of the typical velocity of the fluid $U_0$ to the speed of sound $\left( \frac{dp^*}{d\rho^*}(p_0^*) \right)^{1/2}$ in the same fluid at the same state. We divide the density

\[
\frac{d\rho^*}{d\rho^*}(p_0^*) = 1/
\]

...
\[ \rho^* = \rho^{*\varepsilon} \] into two parts: a constant one \( \overline{\rho}_0^* \), independent of \( \varepsilon \), and a remainder, which is small for small \( \varepsilon \)'s, say

\[ \rho^{*\varepsilon} = \overline{\rho}_0^* + \mathcal{O}(\varepsilon^2) = \overline{\rho}_0^* + \varepsilon^2 \sigma^{*\varepsilon}. \]

We also suppose that the initial conditions \( \rho_0^{*\varepsilon}, u_0^{*\varepsilon} \) and \( \tau_0^{*\varepsilon} \) are well-prepared, which means that they take a similar form, say

\[
\begin{align*}
\rho_0^{*\varepsilon} &= \overline{\rho}_0^* + \mathcal{O}(\varepsilon^2) = \overline{\rho}_0^* + \varepsilon^2 \sigma_0^{*\varepsilon}, \\
u_0^{*\varepsilon} &= v_0^* + v_0^{*\varepsilon}, \text{ with } \text{div} v_0^* = 0, \\
\tau_0^{*\varepsilon} &= S^0_s + S_0^{*\varepsilon},
\end{align*}
\]

where \( v_0^* \) and \( S^0_s \) are, respectively, a vector and a symmetric tensor, both independent of \( \varepsilon \).

We assume

\[ m^* = \min_{\overline{\Omega}} \rho_0^* > 0 \quad \text{and} \quad \mathfrak{M}^* = \max_{\overline{\Omega}} \rho_0^*. \]

Assuming that \( p^* = p^*(\rho^*) \) is regular, say class \( C^3 \) at least, we remark

\[ \frac{dp^*}{d\rho^*}(\overline{\rho}_0^* + \varepsilon^2 \sigma^*) - \frac{dp^*}{d\rho^*}(\overline{\rho}_0^*) = \varepsilon^2 \int_0^1 \frac{dp^*}{d\rho^*}(\overline{\rho}_0^* + s\varepsilon^2 \sigma^*) \, ds. \]

We introduce the function \( w^* \), defined by \( w^*(\sigma^*) = \frac{dp^*}{d\rho^*}(\overline{\rho}_0^* + \varepsilon^2 \sigma^*) - \frac{dp^*}{d\rho^*}(\overline{\rho}_0^*) \), and remark that \( w^* \) depends on \( \varepsilon \), satisfies \( w^*(0) = 0 \), and is of class \( C^2 \) at least.

Replacing \( \rho^* \) by its value in the first equation \( (2.2) \), one infers

\[
\begin{align*}
(\overline{\rho}_0^* + \varepsilon^2 \sigma^*) \left( \frac{\partial u^*}{\partial t^*} + (u^* \cdot \nabla^*) u^* \right) + \varepsilon^2 \frac{dp^*}{d\rho^*}(\overline{\rho}_0^* + \varepsilon^2 \sigma^*) \nabla^* \sigma^* &= (\overline{\rho}_0^* + \varepsilon^2 \sigma^*) f^* + \eta_s (\Delta^* u^* + \nabla^* \text{div}^* u^*) + \text{div}^* \tau^*. \\
\end{align*}
\]

We can also rewrite this equality, by taking into account the definitions of \( w^*(\sigma^*) \) and of the Mach number \( \varepsilon \), in the form

\[
\begin{align*}
(\overline{\rho}_0^* + \varepsilon^2 \sigma^*) \left( \frac{\partial u^*}{\partial t^*} + (u^* \cdot \nabla^*) u^* \right) + (U_0)^2 \nabla^* \sigma^* &= (\overline{\rho}_0^* + \varepsilon^2 \sigma^*) f^* + \eta_s (\Delta^* u^* + \nabla^* \text{div}^* u^*) + \text{div}^* \tau^* - \varepsilon^2 w^*(\sigma^*) \nabla^* \sigma^*. \\
\end{align*}
\]

From the second equation in \( (2.2) \) we easily deduce

\[ \varepsilon^2 \frac{\partial \sigma^*}{\partial t^*} + \overline{\rho}_0^* \text{div}^* u^* + \varepsilon^2 \text{div}^* (\sigma^* u^*) = 0. \]

Finally, System \( (2.2) \) can be written as follows, in \( Q^\tau^* \),

\[
\begin{cases}
(\overline{\rho}_0^* + \varepsilon^2 \sigma^*) \left( \frac{\partial u^*}{\partial t^*} + (u^* \cdot \nabla^*) u^* \right) + (U_0)^2 \nabla^* \sigma^* = (\overline{\rho}_0^* + \varepsilon^2 \sigma^*) f^* + \text{div}^* \tau^* + \eta_s (\Delta^* u^* + \nabla^* \text{div}^* u^*) - \varepsilon^2 w^*(\sigma^*) \nabla^* \sigma^*, \\
\varepsilon^2 \frac{\partial \sigma^*}{\partial t^*} + \overline{\rho}_0^* \text{div}^* u^* + \varepsilon^2 \text{div}^* (\sigma^* u^*) = 0, \\
\tau^* + \lambda \frac{D_{\tau^*}}{D_{\tau^*}} = 2\eta_s D^*[u^*].
\end{cases}
\]
2.3 Dimensionless Variables

We introduce the dimensionless variables,

\[ x^* = L_0 x, \quad t^* = \frac{L_0}{U_0} t, \quad \rho^* = a_0 \rho, \quad w^*(\sigma^*) = (U_0)^2 w(\sigma), \]

\[ u^* = U_0 u, \quad \sigma^* = a_0 \sigma, \quad \tau^* = T_0 \tau, \quad p^*(\rho^*) = T_0 p(\rho), \quad f^* = \frac{(U_0)^2}{L_0} f, \]

where \( L_0 \) represents a typical length of the flow. The real numbers \( a_0 = \frac{\eta}{U_0 L_0} \) and \( T_0 = \frac{\eta U_0}{L_0} \) characterize the density and the stress tensor of the fluid. \( \Omega \) denotes the non-dimensional domain of the flow, with boundary \( \Gamma \), and \( T > 0 \) a non-dimensional time.

We introduce three non-dimensional numbers: a number \( \alpha \) similar to the Reynolds number for incompressible flows, the Weissenberg number \( W_e \), and a number \( \omega \) relative to the viscosities of the fluid,

\[ \alpha = \frac{\tilde{\tau}_o}{a_0} = \frac{\tilde{\rho}_o U_0 L_0}{\eta}, \quad W_e = \frac{\lambda U_0}{L_0}, \quad \omega = 1 - \frac{\eta_s}{\eta}. \]

We also define

\[ w(\sigma) = \alpha \left\{ \frac{dp}{d\rho} (\alpha + \varepsilon^2 \sigma) - \frac{dp}{d\rho} (\alpha) \right\}. \]

In dimensionless variables, System (2.3) takes the form, in \( Q_T = (0, T) \times \Omega \),

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{d}{dt}(u^* + (u^* \cdot \nabla) u^*) + \frac{1}{\alpha + \varepsilon^2 \sigma} \Delta \sigma &= f^* + \frac{1 - \omega}{\alpha + \varepsilon^2 \sigma} (\Delta u^* + \nabla \div u^*) + \frac{\div \tau}{\alpha + \varepsilon^2 \sigma} \\
\sigma^* + \varepsilon^2 \div u^* + \div (\sigma u^*) &= 0,
\end{array} \right.
\end{aligned}
\]

\[ \tau + \text{We} \{ \tau^* + (u^* \cdot \nabla) \tau + g(\nabla u^*, \tau) \} = 2\omega D[u], \]

with the notation \( u' = \frac{\partial u}{\partial t}, \sigma' = \frac{\partial \sigma}{\partial t} \) and \( \tau' = \frac{\partial \tau}{\partial t} \), and

\[ g(\nabla u^*, \tau) = \tau W[u] - W[u] \tau - a \left( D[u] \tau + \tau D[u] \right). \]

Introducing the differential operator \( A = -(\Delta + \nabla \div) \) we may rewrite System (2.4) as follows, in \( Q_T \),

\[
\begin{aligned}
\left\{ \begin{array}{l}
\alpha \left[ (u' + (u \cdot \nabla) u) + (1 - \omega) A u + \nabla \sigma \right] &= F(u, \sigma, \tau) + \div \tau, \\
\sigma' + (u \cdot \nabla) \sigma + \sigma \div u &= -\varepsilon^2 \alpha \div u, \\
\tau + \text{We} \left[ \tau^* + (u \cdot \nabla) \tau + g(\nabla u, \tau) \right] &= 2\omega D[u],
\end{array} \right.
\end{aligned}
\]

with

\[ F(u, \sigma, \tau) = \alpha f + \frac{(1 - \omega)\varepsilon^2 \sigma}{\alpha + \varepsilon^2 \sigma} A u + \frac{\varepsilon^2 (\sigma - w(\sigma))}{\alpha + \varepsilon^2 \sigma} \nabla \sigma - \frac{\varepsilon^2 \sigma}{\alpha + \varepsilon^2 \sigma} \div \tau. \]
System (2.5) is completed by an homogeneous condition on the boundary,

\[ u = 0 \text{ on } \Sigma_T = (0,T) \times \Gamma, \]  

and by three initial conditions,

\[ u(0,\cdot) = u_0, \quad \sigma(0,\cdot) = \sigma_0, \quad \tau(0,\cdot) = \tau_0, \text{ in } \Omega. \]  

We also assume the followings,

\[ 0 < m_1 = \frac{m^*}{a_0} \leq \alpha + \varepsilon^2 \sigma_0 \leq \mathcal{M}_1 = \frac{\mathcal{M}^*}{a_0}, \text{ in } \Omega, \]

where \( m_1 \) and \( \mathcal{M}_1 \) are some given constants.

### 3 The Notation and Main Results

#### 3.1 Notation

\( \Omega \) is a bounded domain in \( \mathbb{R}^3 \), with a regular boundary \( \Gamma \), and \( n \) denotes the unit outward-pointing normal vector to \( \Gamma \). For \( x = (x^1, x^2, x^3) \in \mathbb{R}^3 \), we denote by \( |x| \) its Euclidean norm.

We will use the following spaces: the Lebesgue spaces \( L^p(\Omega) \), \( 1 \leq p \leq +\infty \), with norms \( ||\cdot||_L^p \) (except for the \( L^2(\Omega) \)-norm, which is denoted by \( ||\cdot||_2 \)); the Sobolev space \( H^k(\Omega) \), \( k \in \mathbb{N}^* \), with norm \( ||\cdot||_{L^k} \) and inner product \( (\cdot, \cdot)_k \); the vector spaces \( L^p(\Omega) \) and \( H^k(\Omega) \) of vector-valued or tensor-valued functions with components in \( L^2(\Omega) \) and \( H^k(\Omega) \) respectively, their norms being denoted in the same way as above. We will also use the homogeneous Sobolev space \( H^1_0(\Omega) \) and its dual \( H^{-1}(\Omega) \).

If \( I \) is an interval of \( \mathbb{R}_+ \) and \( k \in \mathbb{N}, C(I; H^k(\Omega)) \) is the space of vector- or tensor-valued functions which are continuous on \( I \) with values in \( H^k(\Omega) \). The norm, in this space, is denoted by \( ||\cdot||_{C,k} \). \( C_b(I; H^k(\Omega)) \) is the space of functions of \( C(I; H^k(\Omega)) \) which are bounded on \( I \).

The space \( L^p(I; H^k(\Omega)) \), \( 1 \leq p \leq +\infty \), and \( k \in \mathbb{N} \), consists of \( p \)-integrable functions on \( I \) with values in \( H^k(\Omega) \). For \( 1 \leq p \leq +\infty \), \( k \in \mathbb{N} \) and \( 0 < T \leq \infty \), the norm in \( L^p((0,T), H^k(\Omega)) \) is denoted by \( ||\cdot||_{L^p,k,T} \). \( L^2_{\text{loc}}(\mathbb{R}_+; H^k(\Omega)) \) is the set of functions which are in \( L^2(I; H^k(\Omega)) \) for all bounded interval \( I \) in \( \mathbb{R}_+ \).

The letters \( C, c_i \), or \( c_i^2, i, j = 1, 2, \ldots, \) will denote constants taking different values, but not depending on \( \varepsilon \). \( C_0 \) will be a constant, taking different values, and depending only on \( \Omega \). \((2.1)_n \) denotes the \( n \)-th equation of System (2.1).

#### 3.2 The Main Result

Recall the problem under study:

\[
\begin{cases}
\alpha \left( \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) + (1-\omega)Au + \nabla \sigma = F(u, \sigma, \tau) + \text{div} \ \tau, \\
\sigma' + (u \cdot \nabla) \sigma + \sigma \text{div} u = -\varepsilon^2 \alpha \text{div} u, \\
\tau + \text{We} \left( \tau' + (u \cdot \nabla) \tau + g(\nabla u, \tau) \right) = 2\omega \mathbf{D}[u], \text{ in } Q_T, \\
u(0, \cdot) = u_0, \quad \sigma(0, \cdot) = \sigma_0, \quad \tau(0, \cdot) = \tau_0, \quad \text{in } \Omega, \\
u = 0, \quad \text{on } \Sigma_T.
\end{cases}
\]  

\[ (3.1) \]
where $F$ is defined by (2.6).

**Theorem 3.1.** (Existence of a local solution) Assume $\Omega \subset \mathbb{R}^3$ is a domain of class $C^3$. Let $m_1$ and $M_1$ be two real constants such that $0 < m_1 \leq M_1$. Assume

$f \in L^2_{\text{loc}}(\mathbb{R}^+; H^1(\Omega))$, with $f' \in L^2_{\text{loc}}(\mathbb{R}^+; H^{-1}(\Omega))$, $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, $\tau_0 \in H^2(\Omega)$,

$\sigma_0 \in H^2(\Omega)$, with $\int_{\Omega} \sigma_0(x)dx = 0$, and $0 < m_1 \leq \alpha + \varepsilon^2 \sigma_0 \leq M_1$ in $\Omega$.

Then there exists a time $T_1 > 0$ and a solution $(u, \sigma, \tau)$ of Problem (2.5)-(2.8) in $Q_{T_1} = (0, T_1) \times \Omega$, satisfying

$u \in L^2(0, T_1; H^3(\Omega)) \cap C([0, T_1]; H^1(\Omega) \cap H^1_0(\Omega))$,

$u' \in L^2(0, T_1; H^2(\Omega)) \cap C([0, T_1]; L^2(\Omega))$,

$(\tau, \sigma) \in C([0, T_1]; H^2(\Omega) \times H^2(\Omega))$, $(\tau', \sigma') \in C([0, T_1]; H^1(\Omega) \times H^1(\Omega))$,

with

$\int_{\Omega} \sigma(\cdot, x)dx = 0$, in $[0, T_1]$, and $\frac{m_1}{2} \leq \alpha + \varepsilon^2 \sigma \leq 2M_1$, in $\overline{Q}_{T_1}$.

**Theorem 3.2.** (Uniqueness of a local solution) There exist a unique solution of Problem (2.5)-(2.8), given in Theorem 3.1.

To show that the local solution found in Theorem 3.1 exists for all times under certain regularity and smallness conditions on the data, we also assume that the function $w \in C^2(\mathbb{R})$ has the following properties: for all $h \in L^2(0, T; H^2(\Omega))$,

$\|(w(h))'\| \leq C \|h'\|$, $\|w(h)\| \leq C \|h\|$, $\|w(h)\|_k \leq C \|h\|_k$, $k = 1, 2$,

for some constant $C$ depending on $\Omega$ and $w$.

**Remark 3.3.** There are several examples of functions $p = p(\rho)$, for which $w$ satisfies the conditions above. Let us quote the case where the pressure is given by the linear state law $p(\rho) = \frac{1}{\varepsilon^2}(\rho - \alpha)$, as well as the case of isothermal compressible perfect fluids, where $p(\rho) = (C_s)^2 \rho$, and $C_s$ is the velocity of sound in the fluid.

### 4 Existence and Uniqueness of Local Solutions

We prove Theorem 3.1 by using the classical method based on the Schauder fixed-point theorem. To do that in our case, we study three linear problems: the first one has the velocity $u$ as unknown, and the next ones are two transport equations for the density $\sigma$ and for the stress tensor $\tau$ respectively. The parameter $\varepsilon$ is fixed in the interval $(0, 1]$. 


Let \( w, \pi \) and \( \psi \) a given vector, function and the symmetric tensor of constraints respectively. Let \( T \) a positive real number, \( Q_T = \Omega \times [0, T] \) and \( \Sigma_T = \partial \Omega \times [0, T] \). Consider the linear problem,

\[
\begin{aligned}
\alpha u' + (1 - \omega)Au &= \tilde{F}, \\
\sigma' + (w.\nabla)\sigma + \sigma \text{div } w &= \tilde{G}, \\
\tau + \text{We}(\tau' + (w.\nabla)\tau + g(\nabla w, \tau)) &= 2\omega D[w], \quad \text{in } Q_T, \\
u(0, x) &= u_0(x), \\
\sigma(0, x) &= \sigma_0(x), \\
\tau(0, x) &= \tau_0(x), \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \Sigma_T,
\end{aligned}
\]  

(4.1)

with

\[
\tilde{F} = F(w, \pi, \psi) - \alpha w.\nabla w - \nabla \pi + \text{div } \psi, \\
\tilde{G} = -\varepsilon^{-2} \text{div } w.
\]

and

\[
\frac{m_1}{2} \leq \alpha + \varepsilon^2 \pi \leq 2m_1, \quad \text{in } Q_T.
\]

(4.4)

4.1 Linear problem concerning the velocity \( u \)

Consider the linear problem concerning the velocity \( u \),

\[
\begin{aligned}
\alpha u' + (1 - \omega)Au &= \tilde{F}, \quad \text{in } Q_T, \\
u(0, x) &= u_0(x), \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \Sigma_T,
\end{aligned}
\]

(4.5)

where \( Au = -(D(u + \nabla u)), \tilde{F} \) and \( u_0 \) are given and \( 0 < T \leq +\infty \).

The first Lemma concerns the existence of a unique solution of (4.5). By classical result of Agmon-Dougls-Nirenberg \([1]\), \( A = -\Delta - \nabla \text{div} \) is a strongly elliptic operator, and generates an analytic semigroup in \( L^2(\Omega) \) with domain \( D(A) = H^2(\Omega) \cap H^1_0(\Omega) \) (we can see for instance \([3]\)).

Lemma 4.1. Let \( \Omega \subset \mathbb{R}^3 \) of class \( C^2 \), \( \tilde{F} \in L^2(0, T; L^2(\Omega)) \) and \( u_0 \in H^1_0(\Omega) \). Then there exists a unique solution of problem (4.5)

\[
\begin{aligned}
u &\in L^2(0, T; H^2(\Omega)) \cap C([0, T]; H^1_0(\Omega)), \\
u' &\in L^2(0, T; L^2(\Omega)).
\end{aligned}
\]

Moreover, this solution satisfies the estimate

\[
\frac{\alpha}{2}||u'||^2_{L^2(0,T,L^2(\Omega))} + \frac{(1-\omega)^2}{2}||Au||^2_{L^2(0,T,L^2(\Omega))} + (1-\omega)||D u||^2_{L^\infty(0,T,L^2(\Omega))} + (1-\omega)||\text{div } u||^2_{L^\infty(0,T,L^2(\Omega))} \leq 4(1-\omega)||D u_0||^2 + ||\tilde{F}||^2_{L^2(0,T,L^2(\Omega))}.
\]

(4.6)

Proof.

By classical result of Agmon-Dougls-Nirenberg \([1]\), \( A = -\Delta - \nabla \text{div} \) is a strongly elliptic
operator, and generates an analytic semigroup in $L^2(\Omega)$ with domain $D(A) = H^1(\Omega) \cap H^2_0(\Omega)$ (we can see for instance [8]).

We start by showing the estimate \([4.7]\). Multiply \([4.5]\), in $L^2(\Omega)$ by $u' + \alpha(1 - \omega)Au$, then

$$
\int_{\Omega} |u'|^2 + 2(1 - \omega) \int_{\Omega} u' \cdot Au + (1 - \omega)^2 \int_{\Omega} |Au|^2
= \int_{\Omega} \mathfrak{F} \cdot u' + (1 - \omega) \int_{\Omega} \mathfrak{F} \cdot Au.
$$

Integrate by parts the second term, we obtain

$$
||u'||^2 + (1 - \omega) \frac{d}{dt} \left(||Du||^2 + ||\text{div} \, u||^2\right) + (1 - \omega)^2 ||Au||^2 \leq ||\mathfrak{F}|| \cdot ||u'||^2
+ \frac{(1 - \omega)}{4} ||\mathfrak{F}|| \cdot ||Au||^2.
$$

On the other hand, using Young's inequality on the two terms right, we get

$$
||\mathfrak{F}|| \cdot ||u'|| \leq \frac{1}{2} ||\mathfrak{F}||^2 + \frac{1}{2} ||u'||^2,
$$

$$
(1 - \omega) ||\mathfrak{F}|| \cdot ||Au|| \leq \frac{1}{2} ||\mathfrak{F}||^2 + \frac{(1 - \omega)^2}{2} ||Au||^2.
$$

Integrate over $[0, T]$ and use the inequality

$$
||\text{div} \, u_0|| \leq 3 ||Du_0||,
$$

then we get \([4.6]\).

The second Lemma give some stronger estimates.

**Lemma 4.2** ([8, 6]). Under the conditions of Lemma \([4.7]\) and if $\partial \Omega \in C^3$, $\mathfrak{F} \in L^2(0, T; H^1(\Omega))$, $\mathfrak{F}' \in L^2(0, T; H^{-1}(\Omega))$ and $u_0 \in H^1(\Omega) \cap H^2_0(\Omega)$. Then the solution $u$ of problem \([4.5]\) given by Lemma \([4.7]\) is such that

$$
u \in \begin{cases}
L^2(0, T; H^4(\Omega)) \cap C([0, T]; H^2(\Omega) \cap H^3_0(\Omega)), \\
u' \in \begin{cases}L^2(0, T; H^1(\Omega)) \cap C([0, T]; L^2(\Omega).)
\end{cases}
\end{cases}
$$

and there exists a constant $C_1$, depend only in $\Omega$, $T$, $\alpha$ and $\omega$, such that one has the estimate

$$
||u||_{L^2(0, T; H^4(\Omega))} + ||u||_{L^\infty(0, T; H^2(\Omega) \cap H^3_0(\Omega))} + ||u'||_{L^2(0, T, H^1(\Omega))} + ||u'||_{L^\infty(0, T, L^2(\Omega))}
\leq C_1 \{||Au_0||^2 + ||\mathfrak{F}||^2 + ||\mathfrak{F}'||^2 + ||\mathfrak{F}||_{L^2(0, T; H^1(\Omega))}^2 + ||\mathfrak{F}'||_{L^2(0, T; H^{-1}(\Omega))}^2\}.
$$

**Proof.**

Derive in terms of $t$ the equation \([4.5]\), then we obtain

$$
u'' + \alpha(1 - \omega)Au' = \mathfrak{F}', \quad \text{in } Q_T,
$$

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and \( u'_{|\Omega}(t) = 0 \) for all \( t \in [0, T] \). Let \( v = u' \), then \( v \) verify the system

\[
\begin{cases}
  v' + \alpha(1 - \omega)Av = \tilde{\gamma}', \\
  v(0) = v_0 = \tilde{\gamma}(0) - \alpha(1 - \omega)Au_0, \\
  v = 0,
\end{cases}
\]

in \( Q_T \), \( \Omega \), and \( \Sigma_T \). Multiply by \( v \) the equation (4.1.10) and integrate on \( \Omega \). It comes

\[
\int_{\Omega} v' \cdot v + \alpha(1 - \omega) \int_{\Omega} Av \cdot v = \langle \tilde{\gamma}', v \rangle_{H^{-1}, H^1_0}.
\]

After calculation, we obtain

\[
\frac{1}{2} \frac{d}{dt} \|v\|^2 + \frac{3\alpha(1 - \omega)}{4} \|Dv\|^2 + \alpha(1 - \omega) \|\text{div} v\|^2 \leq \frac{1}{\alpha(1 - \omega)} \|\tilde{\gamma}'\|^2_{H^{-1}}.
\]

Integrate on \( [0, T] \) and replace \( v \) and \( v_0 \) by their values

\[
\frac{1}{2} \|v(t)\|^2_{L^2(\Omega)} + \frac{3\alpha(1 - \omega)}{4} \|Dv(t)\|^2_{L^2(\Omega)} + \alpha(1 - \omega) \|\text{div} v(t)\|^2_{L^2(\Omega)} \\
\leq \frac{1}{\alpha(1 - \omega)} \|\tilde{\gamma}'\|^2_{L^2(\Omega)} + \frac{1}{2} \left( \|\tilde{\gamma}(0)\|^2 + \alpha(1 - \omega) \|Au_0\|^2 \right).
\]

Finally, inequality (4.7) follows from inequality (4.6) and (4.9).

### 4.2 Resolution of the Transport Problems

We consider the following two linear transport problems,

\[
\begin{cases}
  \sigma' + (w \cdot \nabla)\sigma + \sigma \text{div} w = \varepsilon^2 \text{div} \sigma, & \text{in } Q_T, \\
  \sigma(0, \cdot) = \sigma_0, & \text{in } \Omega,
\end{cases}
\]

and

\[
\begin{cases}
  \tau' + \text{We} \left( \tau' + (w \cdot \nabla)\tau + \text{g}(\nabla w, \tau) \right) = 2\omega D[w], & \text{in } Q_T, \\
  \tau(0, \cdot) = \tau_0, & \text{in } \Omega,
\end{cases}
\]

where \( \sigma_0 \) and \( \tau_0 \) are, respectively, some given function and symmetric tensor defined in \( \Omega \). The existence of solutions to this problems follows from the classical method of characteristics. (see for example [3, 6, 8]). The lemmas below give some estimates of the solutions of these problems.

**Lemma 4.3 (8).** Let \( \Gamma \in C^1 \), \( w \in L^1(0, T; H^1(\Omega)) \), \( w, n = 0 \) on \( \Sigma_T \), and \( \sigma_0 \in H^2(\Omega) \), with \( \int_\Omega \sigma_0 dx = 0 \). Then there exists a unique solution \( \sigma \in C([0, T]; H^2(\Omega)) \) of (4.10) such that

\[
\int_\Omega \sigma(\cdot, x) dx = 0 \text{ in } [0, T],
\]

and satisfying the following estimate

\[
\|\sigma\|_{L^1(0, T; H^2(\Omega))} \leq (\|\sigma_0\| + \alpha \varepsilon^2) \exp \left( C_\alpha \|w\|_{L^1(0, T; H^2(\Omega))} \right),
\]
for some positive constant $C_0$ depending on $\Omega$.

If, in addition, $w \in C([0,T]; H^2(\Omega))$, then $\sigma' \in C([0,T]; H^1(\Omega))$ satisfies

$$||\sigma'||_{L^\infty(0,T,H^1(\Omega))} \leq C_0 \|||w||_{L^\infty(0,T,H^2(\Omega))} (||\sigma|| + \alpha \varepsilon^2) \exp \left( C_0 ||w||_{L^1(0,T,H^3(\Omega))} \right).$$

**Lemma 4.4** (If). Let $\Omega \subset \mathbb{R}^3$ be a domain of class $C^3$, $w \in L^1(0,T; H^1(\Omega) \cap H^3_0(\Omega))$ and $\tau_0 \in H^2(\Omega)$. Then there exists a unique solution $\tau \in C([0,T]; H^2(\Omega))$ of (4.11), such that

$$||\tau||_{L^\infty(0,T,H^2(\Omega))} \leq \left( ||\tau_0||^2 + \frac{2\omega}{C_0 We} \right) \exp \left( C_0 ||w||_{L^1(0,T,H^3(\Omega))} \right),$$

for some positive constant $C_0$ depending on $\Omega$.

If, in addition, $w \in C([0,T]; H^2(\Omega) \cap H^3_0(\Omega))$, then $\tau' \in C([0,T]; H^1(\Omega))$ satisfies

$$||\tau'||_{L^\infty(0,T,H^1(\Omega))} \leq C_0 \left( ||w||_{L^\infty(0,T,H^2(\Omega))} + \frac{1}{C_0 We} \right) \left( ||\tau_0|| + \frac{2\omega}{C_0 We} \right) \exp \left( C_0 ||w||_{L^1(0,T,H^3(\Omega))} \right).$$

**4.3 Proof of Theorem 3.1**

We are now in a position to prove the local existence of a solution to problem (3.1). We apply the Theorem of fixed-point of Schauder.

Take $T > 0$, $\mathcal{B}_1, \mathcal{B}_2 > 0$, and define

$$\mathcal{R}_T = \{ (w, \pi, \psi), \]

$$w \in C([0,T]; H^2(\Omega) \cap H^3_0(\Omega)) \cap L^\infty(0,T,H^3(\Omega)), w' \in C([0,T]; L^2(\Omega)) \cap L^2(0,T,H^1(\Omega)) \]

$$\pi \in L^\infty(0,T,H^2(\Omega)), \pi' \in L^\infty(0,T,H^1(\Omega)), \]

$$\psi \in L^\infty(0,T,H^2(\Omega)), \psi' \in L^\infty(0,T,H^1(\Omega)), \]

$$w(0) = u_0, \pi(0) = \sigma_0, \psi(0) = \tau_0 \in \Omega, w = 0 \text{ in } \Sigma_T, \]

$$||w||_{L^\infty(0,T,H^2(\Omega) \cap H^3_0(\Omega))} + ||w'||_{L^2(0,T,H^3(\Omega))} + ||w'||_{L^\infty(0,T,L^2(\Omega))} + ||\pi'||_{L^\infty(0,T,H^2(\Omega))} \leq \mathcal{B}_1, \]

$$||\pi||_{L^\infty(0,T,H^2(\Omega))} + ||\pi'||_{L^\infty(0,T,H^1(\Omega))} + ||\psi'||_{L^\infty(0,T,H^1(\Omega))} \leq \mathcal{B}_2, \]

$$\frac{m_2}{2} \leq \alpha + \varepsilon^2 \pi(t,x) \leq 2\mathcal{M}_1, \text{ in } \Omega_T}. \]

Choose $\mathcal{B}_1$ such that

$$\mathcal{B}_1 > \max \{ C_4 ||Au_0||^2, ||\sigma_0||, ||\tau_0|| \}, \quad (4.12)$$

then $(u_0, \sigma_0, \tau_0) \in \mathcal{R}_T$. In fact, $w$ is a solution of problem

$$\left\{ \begin{array}{l}
 w(\cdot) \in H^1(\Omega),
 w' + (1 - \omega)Aw = 0, \text{ p.p. in } \mathbb{R}_+,
 w(0) = u_0, \text{ in } \Omega,
 w = 0, \text{ on } \Sigma_T.
 \end{array} \right. \quad (4.13)$$

Using estimate (4.7), there exists a constant $C_4$ such that

$$||w'||_{L^2(0,T,H^3(\Omega))} + ||w'||_{L^\infty(0,T,H^2(\Omega) \cap H^3_0(\Omega))} + ||w||_{L^2(0,T,H^1_0(\Omega))} + ||w||_{L^\infty(0,T,L^2(\Omega))} \leq C_4 ||Au_0||^2. \quad (4.14)$$

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Thus the choose of \( \mathcal{B}_1 \) in (4.12) is enough for prove that \( \mathcal{R}_T \) is non empty for each \( T > 0 \).

Define now the application mapping \( \mathfrak{R} \) in this way

\[
\mathfrak{R} : \mathcal{R}_T \rightarrow \mathcal{X}_T = C([0, T]; \mathcal{H}^1_0(\Omega)) \times C([0, T]; \mathcal{H}^1(\Omega)) \times C([0, T]; \mathcal{H}^1(\Omega))
\]

\[
(\mathbf{w}, \pi, \psi) \rightarrow (\mathbf{u}, \sigma, \tau)
\]

where \( \mathbf{u}, \sigma \) and \( \tau \) are solution of (4.5), (4.10) and (4.11), respectively, with

\[
\mathfrak{R} = \alpha f + (1 - \omega) \frac{\varepsilon^2 \pi}{\alpha + \varepsilon^2 \pi} \mathbf{A} \mathbf{w} + \frac{\varepsilon^2}{\alpha + \varepsilon^2 \pi} (\pi - w(\pi)) \nabla \pi - \alpha (\mathbf{w}, \nabla) \mathbf{w} - \nabla \pi + \text{div} \psi,
\]

\[
\mathcal{G} = -\varepsilon^2 \alpha \text{div} \mathbf{w}.
\]

If we take

\[
\mathcal{B}_1 > \max \left\{ C_4 \| \mathbf{A} \mathbf{u}_0 \|^2, \varepsilon \sqrt{T} \left( \| \sigma_0 \|_2 + \| \tau_0 \|_2 + 1 + \frac{2\omega}{C_3 \text{We}} \right), C_2 (2C_5 + 1) \| \mathbf{A} \mathbf{u}_0 \|^2 + C_5 \| \mathbf{A} \mathbf{u}_0 \|^4 + 3 \left( 2(1 + \| \mathbf{w} \|^2) \| \sigma_0 \|_2^2 + \| \tau_0 \|_2^2 \right) + 3 \| \mathbf{f}(0) \|^2 + 3 \| \mathbf{f} \|^2_{L^2(0, T, H^1(\Omega))} + 3 \| \mathbf{f}' \|^2_{L^2(0, T, H^1(\Omega))} \right\},
\]

and

\[
\mathcal{B}_2 > \varepsilon \sqrt{T} \left\{ C_6 \left( \| \sigma_0 \|_2 + \| \tau_0 \|_2 + 1 + \frac{2\omega}{C_3 \text{We}} \right) + \frac{1}{\text{We}} \left( \| \tau_0 \|_2 + \frac{2\omega}{C_3 \text{We}} \right) \right\},
\]

and for all \( T \) small enough such that

\[
T \leq T^* = \min \left( \frac{\mathcal{B}_1}{2C_2 (4C_4 (1 + \| \mathbf{w} \|^2) \mathcal{B}_1^2 + 3 \mathcal{B}_2^2)} \right),
\]

we have \( \mathfrak{R}(\mathcal{R}_T) \subset \mathcal{R}_T \).

We now use Schauder fixed point theorem. The mapping \( \mathfrak{R} \) is defined from convex, bounded and no empty set \( \mathcal{R}_T \) into \( \mathcal{X}_T \). To finish, we need to show the continuity of \( \mathfrak{R} \) in \( \mathcal{X}_T \).

**Lemma 4.5.** To show the continuity of \( \mathfrak{R} \) in \( \mathcal{X}_T \), it is enough to show the continuity of \( \mathfrak{R} \) in

\[
\mathfrak{Y}_T = C([0, T]; \mathcal{L}^2(\Omega)) \times C([0, T]; \mathcal{L}^2(\Omega)) \times C([0, T]; \mathcal{L}^2(\Omega)).
\]

**Proof.** Let \( \left( \mathbf{w}_n, \pi_n, \psi_n \right) \) be a sequence of \( \mathfrak{Y}_T \) and tends to \( \left( \mathbf{w}, \pi, \psi \right) \), such that:

\[
\left( \mathbf{u}_n, \sigma_n, \tau_n \right) = \mathfrak{R}(\mathbf{w}_n, \pi_n, \psi_n) \quad \text{and} \quad \left( \mathbf{u}, \sigma, \tau \right) = \mathfrak{R}(\mathbf{w}, \pi, \psi).
\]

Suppose that \( \mathfrak{R} \) is continuous in \( \mathfrak{Y}_T \), then the sequence \( \left( \mathfrak{R}(\mathbf{w}_n, \pi_n, \psi_n) \right) \) tends to \( \mathfrak{R}(\mathbf{w}, \pi, \psi) \) in \( \mathfrak{Y}_T \), i.e.

\[
\lim_{n \to \infty} \| \left( \mathbf{u}_n, \sigma_n, \tau_n \right) - \left( \mathbf{u}, \sigma, \tau \right) \|_{\mathfrak{Y}_T} = 0.
\]

\( \mathfrak{R}_T \) is a compact set in \( \mathcal{X}_T \) (see for instance [5]). Using (4.17), we can extract of \( \left( \mathbf{u}_n, \sigma_n, \tau_n \right) \) a subsequence converges in \( \mathcal{X}_T \) to the unique accumulation point \( \left( \mathbf{u}, \sigma, \tau \right) \). Then the sequence \( \left( \mathbf{u}_n, \sigma_n, \tau_n \right) \rightarrow \mathfrak{R}(\mathbf{w}_n, \pi_n, \psi_n) \) converges to \( \left( \mathbf{u}, \sigma, \tau \right) = \mathfrak{R}(\mathbf{w}, \pi, \psi) \) in \( \mathcal{X}_T \). This proved the continuity of \( \mathfrak{R} \) in \( \mathcal{X}_T \). \( \square \)
Lemma 4.6. \( \mathcal{R} \) is continuous in \( \mathcal{Y}_T \).

Proof. Let \( (w_n, \pi_n, \psi_n) \) be a sequence of \( \mathcal{R}_T \) and tends to \( (w, \pi, \psi) \), such that:

\[
(u_n, \sigma_n, \tau_n) = \mathcal{R}(w_n, \pi_n, \psi_n) \quad \text{and} \quad (u, \sigma, \tau) = \mathcal{R}(w, \pi, \psi).
\]

Consider two systems. The first is:

\[
\begin{aligned}
\alpha u'_n + (1 - \omega)Au_n &= \mathcal{F}_n, \\
\sigma'_n + (w_n, \nabla)\sigma_n + \sigma_n \text{div } w_n &= \mathcal{G}_n, \\
\tau_n + \text{We}\{\tau'_n + (w_n, \nabla)\tau_n + g(\nabla w_n, \tau_n)\} &= 2\omega\mathcal{D}[w_n], \quad \text{in } Q_T, \\
u_n(0, x) &= u_0(x), \\
\sigma_n(0, x) &= \sigma_0(x), \\
\tau_n(0, x) &= \tau_0(x), \quad \text{in } \Omega, \\
u_n &= 0, \quad \text{on } \Sigma_T,
\end{aligned}
\]

with

\[
\mathcal{F}_n = F(w_n, \pi_n, \psi_n) - \alpha(w_n, \nabla)w_n - \nabla \pi_n + \text{div } \psi_n, \\
\mathcal{G}_n = -\varepsilon^{-2}\text{div } w_n.
\]

And, the second is:

\[
\begin{aligned}
\alpha u'_n + (1 - \omega)Au_n &= \mathcal{F}, \\
\sigma'_n + (w, \nabla)\sigma + \text{div } w &= \mathcal{G}, \\
\tau + \text{We}\{\tau' + (w, \nabla)\tau + g(\nabla w, \tau)\} &= 2\omega\mathcal{D}[w], \quad \text{in } Q_T, \\
u(0, x) &= u_0(x), \\
\sigma(0, x) &= \sigma_0(x), \\
\tau(0, x) &= \tau_0(x), \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \Sigma_T,
\end{aligned}
\]

with

\[
\mathcal{F} = F(w, \pi, \psi) - \alpha(w, \nabla)w - \nabla \pi + \text{div } \psi, \\
\mathcal{G} = -\varepsilon^{-2}\text{div } w.
\]

Let \( v_n = u_n - u, \ q_n = \sigma_n - \sigma \) and \( S_n = \pi_n - \pi \). Using (4.18) and (4.19), we obtain, in \( Q_T \):

\[
\begin{aligned}
\alpha v'_n + (1 - \omega)Av_n &= \mathcal{F}_1, \\
q'_n + (w_n, \nabla)q_n + q_n \text{div } w_n &= \mathcal{G}_1 - ((w_n - w), \nabla)\sigma - \text{div } (w_n - w), \\
S_n + \text{We}\{S'_n + (w_n, \nabla)S_n + g(\nabla w_n, S_n)\} &= \mathcal{H}_1 - \text{We}\{((w_n - w), \nabla)\tau + g(\nabla(w_n - w), \tau)\},
\end{aligned}
\]

with the boundary conditions:

\[
\begin{aligned}
v_n(0, x) &= 0, \\
q_n(0, x) &= 0, \\
S_n(0, x) &= 0, \quad \text{in } \Omega, \\
v_n &= 0, \quad \text{on } \Sigma_T,
\end{aligned}
\]
such that:

\[
\mathfrak{F}_1 = F(w_n, \pi_n, \psi_n) - F(w, \pi, \psi) - \alpha \left( (w_n \cdot \nabla)w_n - (w \cdot \nabla)w \right) - \nabla(\pi_n - \pi) + \text{div}(\psi_n - \psi),
\]

\[
\mathcal{G}_1 = -\varepsilon^{-2}\text{div}(w_n - w),
\]

\[
\mathcal{H}_1 = 2\omega D[w_n - w].
\]

First, multiply the equation (4.20) by \(v_n\), and integrate over \(\Omega\). We get:

\[
\frac{\alpha}{2} \frac{d}{dt} ||v_n||^2 + (1 - \omega) \left( ||\nabla v_n||^2 + ||\text{div} v_n||^2 \right) \leq ||\pi_n - \pi||^2 + ||\psi_n - \psi||^2 + 4 ||v_n||^2
\]

\[
+ \alpha^2 ||(w_n \cdot \nabla)w_n - (w \cdot \nabla)w||^2 + ||F(w_n, \pi_n, \psi_n) - F(w, \pi, \psi)||^2
\]

We now estimate the term \(||F(w_n, \pi_n, \psi_n) - F(w, \pi, \psi)||^2\) on the right hand of (4.22). Using the two inequalities:

\[
\frac{m_1}{2} \leq \alpha + \varepsilon^2 \pi(t, x) \leq 2M, \quad \text{and} \quad \frac{m_1}{2} \leq \alpha + \varepsilon^2 \pi_n(t, x) \leq 2M,
\]

we obtain:

\[
||F(w_n, \pi_n, \psi_n) - F(w, \pi, \psi)||^2 \leq C_\varepsilon m_1 \varepsilon^4 \left[ (1 - \omega)^2 ||\pi_n A w_n - \pi A w||^2 + ||\pi_n \nabla \pi_n - \pi \nabla \pi||^2 
\]

\[
+ ||(w_n \cdot \nabla)w_n - (w \cdot \nabla)w||^2 \right] + ||\pi_n \nabla \pi_n - \pi \nabla \pi||^2 + ||\pi_n \text{div} S_n - \pi \text{div} S||^2
\]

Then, (4.22) satisfies:

\[
\frac{\alpha}{2} \frac{d}{dt} ||v_n||^2 + (1 - \omega) \left( ||\nabla v_n||^2 + ||\text{div} v_n||^2 \right) \leq C_\varepsilon \ell_n + 4 ||v_n||^2,
\]

with

\[
\ell_n = ||\pi_n - \pi||^2 + ||\psi_n - \psi||^2 + \alpha ||(w_n \cdot \nabla)w_n - (w \cdot \nabla)w||^2 + (1 - \omega)^2 ||\pi_n A w_n - \pi A w||^2
\]

\[
+ ||\pi_n \nabla \pi_n - \pi \nabla \pi||^2 + ||(w(\pi_n) \nabla \pi_n - w(\pi) \nabla \pi||^2 + ||\pi_n \text{div} S_n - \pi \text{div} S||^2.
\]

Second, multiply the equation (1.20) by \(\varepsilon^2 q_n\) and integrate over \(\Omega\). This yields:

\[
\varepsilon^2 \frac{d}{dt} ||q_n||^2 \leq (1 + C_\varepsilon ||\sigma||_2) ||w_n - w||^2 + (1 + C_{10} ||w_n||_3) ||q_n||^2
\]

\[
\leq (1 + C_\varepsilon ||\sigma||_2) ||w_n - w||^2 + j_n ||q_n||^2,
\]

with \(j_n = 1 + C_{10} ||w_n||_3\).

Finally, multiply the equation (1.20) by \(S_n / 2\omega\), we obtain

\[
\frac{\text{We}}{2\omega} \frac{d}{dt} ||S_n||^2 \leq \left( 1 + \frac{\text{We} C_{11}}{2\omega} ||\pi||_2 \right) ||w_n - w||^2 + \left( 1 + \frac{1}{2\omega} + \frac{\text{We} C_{12}}{2\omega} ||w_n||_3 \right) ||S_n||^2
\]

\[
\leq \left( 1 + \frac{\text{We} C_{11}}{2\omega} ||\pi||_2 \right) ||w_n - w||^2 + k_n ||S_n||^2,
\]

with \(k_n = 1 + \frac{1}{2\omega} + \frac{\text{We} C_{12}}{2\omega} ||w_n||_3\).
The functions \( j_n \) and \( k_n \), are positive and, because of the class of solutions we consider, \( j_n \) and \( k_n \) belong to \( L^1(0,T) \). Therefore, by using (4.23), (4.24), and (4.25), we deduce from Gronwall’s lemma that:

\[
\|v_n\|^2 \leq \frac{C_8}{\alpha} \int_0^t \exp \left( \frac{-4s}{\alpha} \right) \ell_n(s) ds, 
\]  
(4.26)

\[
\|q_n\|^2 \leq \frac{1}{\varepsilon^2} (\text{dir} 0 + C_9 \mathcal{B}_1) \int_s^t \exp \left( \int_0^s j_n(r) dr \right) \|w_n(s) - w(s)\|^2 ds, 
\]  
(4.27)

\[
\|S_n\|^2 \leq \left( \frac{2\omega}{Wc} + C_{11} \mathcal{B}_1 \right) \int_s^t \exp \left( \int_0^s k_n(r) dr \right) \|w_n(s) - w(s)\|^2 ds. 
\]  
(4.28)

The sequence \((w_n, \pi_n, \psi_n)\) of \( \mathcal{R}_T \) tends to \((w, \pi, \psi)\), and using (4.26), (4.27) and (4.28), we obtain \( v_n, q_n \) and \( S_n \) tend to zero in \( \mathcal{Y}_T \). This means that the sequence \( ((u_n, \sigma_n, \tau_n))_n = (\tilde{u}(w_n, \pi_n, \psi_n))_n \) tends to \((u, \sigma, \tau) = \tilde{u}(w, \pi, \psi)\) and \( \tilde{u} \) is continuous in \( \mathcal{Y}_T \).

**4.4 Proof of Theorem 3.2**

We take, as usual, the difference of two solutions \((u_1, \sigma_1, \tau_1)\) and \((u_2, \sigma_2, \tau_2)\) belonging to the class specified in the theorem 3.2. The vector function \( u = u_1 - u_2 \), the scalar function \( \sigma = \sigma_1 - \sigma_2 \) and the tensor function \( \tau = \tau_1 - \tau_2 \) satisfy the following system:

\[
\begin{cases} 
\alpha \left[ u' + (u, \nabla)u_1 + (u, \nabla)u_2 \right] + (1 - \omega)Au = \tilde{g}_2 - \nabla \sigma + \text{div} \tau, \\
\sigma' + (u_1, \nabla)\sigma + (u, \nabla)\sigma_2 + \sigma \text{div} u_1 + \sigma_2 \text{div} u = -\varepsilon^2 \text{div} u, \\
\tau + \text{We} \{\tau' + (u, \nabla)\tau + (u, \nabla)\tau_2 + g(\nabla u_1, \tau) + g(\nabla u, \tau_2)\} = 2\omega D[u], 
\end{cases}
\]  
(4.29)

with the boundary conditions:

\[
\begin{cases} 
u(0, x) = 0, \\
\sigma(0, x) = 0, \\
\tau(0, x) = 0, \text{ in } \Omega, \\
u = 0, \text{ on } \Sigma_T,
\end{cases}
\]  
(4.30)

such that:

\[
\tilde{g}_2 = F(u_1, \sigma_1, \tau_1) - F(u_2, \sigma_2, \tau_2).
\]

Multiply (4.29), (4.29), and (4.29) by \( u, \varepsilon^2 \sigma / \alpha, \) and \( \tau/(2\omega) \), respectively, and integrate over \( \Omega \). Summing the three obtained equations, one obtains

\[
\frac{1}{2} \frac{d}{dt} \left( \alpha \|u\|^2 + \frac{\varepsilon^2}{\alpha} ||\sigma||^2 + \frac{We}{2\omega} ||\tau||^2 \right) + (1 - \omega) \left( ||\nabla u||^2 + ||\text{div} u||^2 \right) + \frac{1}{2\omega} ||\tau||^2 \leq \alpha C_{12} \left[ ||u||^2 ||u||^2 + ||u_1||^2 ||u_2||^2 \right] + \frac{\varepsilon^2}{\alpha} C_{12} \left[ ||\sigma|| ||u|| ||u|| ||\nabla u|| + ||\sigma_1|| ||\nabla u|| + ||\sigma_2|| ||\nabla u|| \right] 
\]  
(4.31)

\[
+ ||\sigma|| ||\sigma|| ||\text{div} u|| + ||\sigma_1|| ||\text{div} u|| + ||\tau_1|| ||\sigma|| ||\nabla u|| + ||\tau_2|| ||\tau|| ||\nabla u|| \right] + \frac{\varepsilon^2}{\alpha} C_{12} \left[ ||u||^2 ||\sigma|| ||\nabla u|| + ||\nabla u|| ||\sigma|| ||\nabla u|| \right] + \frac{We}{2\omega} C_{12} \left[ ||u|| ||\tau||^2 + ||\nabla u|| ||\tau|| ||\nabla u|| \right].
\]
For $\delta > 0$, (4.31) can be written as:

$$\frac{1}{2} \frac{d}{dt} \left( \alpha ||u||^2 + \frac{\varepsilon^2}{\alpha} ||\sigma||^2 + \frac{We}{2\omega} ||\tau||^2 \right) + (1 - \omega) \left( ||\nabla u||^2 + ||\text{div} u||^2 \right) + \frac{1}{2\omega} ||\tau||^2 \leq C_{12} \left( ||u_1|| + ||u_2|| + \frac{(C_{12})^2}{2\delta} ||\sigma||^2 \right) \alpha ||u||^2$$

$$+ \frac{\delta}{2} \left[ \left( \frac{5\varepsilon^2}{\alpha} + \frac{We}{2\omega} \right) ||\nabla u||^2 + \frac{2\varepsilon^2}{\alpha} ||\text{div} u||^2 \right]$$

$$+ \frac{(C_{12})^2}{2\delta} \left( ||u_1||^3 + ||\sigma||^2 + 2 ||\sigma_2||^2 + ||\tau_1||^2 \right) + C_{12} ||u_1||_3 \frac{\varepsilon^2}{\alpha} ||\sigma||^2$$

$$+ \frac{(C_{12})^2}{2\delta} ||\sigma_2||^2 + C_{12} ||u_1||_3 \frac{We}{2\omega} ||\tau||^2.$$  \hspace{1cm} (4.32)

From (4.32), we then deduce that solutions $(u, \sigma, \tau)$ of (4.29) satisfy the following energy inequality:

$$\frac{1}{2} \frac{d}{dt} \left( \alpha ||u||^2 + \frac{\varepsilon^2}{\alpha} ||\sigma||^2 + \frac{We}{2\omega} ||\tau||^2 \right) + (1 - \omega) \left( 1 - \omega \right) \left( 1 - \frac{\delta \varepsilon^2}{\alpha(1 - \omega)} \right) ||\nabla u||^2$$

$$+ (1 - \omega) \left( 1 - \frac{\delta \varepsilon^2}{\alpha(1 - \omega)} \right) ||\text{div} u||^2 + \frac{1}{2\omega} ||\tau||^2 \leq \chi_\delta \left[ \alpha ||u||^2 + \frac{\varepsilon^2}{\alpha} ||\sigma||^2 + \frac{We}{2\omega} ||\tau||^2 \right],$$  \hspace{1cm} (4.33)

with

$$\chi_\delta = C_{12} \left( ||u_1|| + ||u_2|| + ||u_1||_3 \right) + \frac{(C_{12})^2}{2\delta} \left( ||u_1||^3 + ||\sigma||^2 + 2 ||\sigma_2||^2 + ||\tau_1||^2 \right).$$  \hspace{1cm} (4.34)

The function $\chi_\delta$, defined in (4.34), is positive. Moreover, because of the class of solutions we consider, $\chi_\delta$ belongs to $L^1(0,T)$. Therefore, choosing $\delta > 0$ small enough, we deduce from Gronwall’s lemma that $u = 0$, $\sigma = 0$ and $\tau = 0$ in $Q_T$, and that consequently $u_1 = u_2$, $\sigma_1 = \sigma_2$, $\tau_1 = \tau_2$ in $Q_T$ and the system (4.29) has a unique solution.

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