Observable Dirac electron in accelerated frames

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We present a new quantum algebraic description of an electron localized in space-time. Positions in space and time, mass and Clifford generators are defined as quantum operators. Commutation relations and relativistic shifts under frame transformations are determined within a unique algebraic framework. Redshifts, i.e. shifts under transformations to uniformly accelerated frames, are evaluated and found to differ from the expressions of classical relativity.

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I. INTRODUCTION

Well-known difficulties affect Quantum Field Theory at its interface with mechanical, inertial or gravitational effects. In particular, the problem of quantum systems in accelerated frames is quite poorly mastered when compared with quantum theory on one hand and classical relativity on the other hand. Prospects for improved tests of the effect of gravitation on atomic clocks however make this question inescapable. We shall show below that the standard theoretical framework, despite its well-known deficiencies with respect to this problem, may be reformulated to allow for the definition of observables describing localization of a quantum system in space-time. Here we shall consider the exemplary quantum system constituted by a single electron.

Special relativity has led to a profound revision of our conception of space-time. Basic elements of the new conception are events localized in space and time. Time definition and clock synchronization correspond to dating of events such as, respectively, ticks of a clock and emission or detection of light pulses $^1$. Positions in space-time of such events are physical observables which differ from coordinate parameters on a space-time map. Consequently, relativistic shifts of observables under frame transformations are related to relativistic symmetries and are distinct from mere changes of coordinate mapping.

These points which are already delicate in classical relativity still raise more acute difficulties in a quantum context. To illustrate these difficulties, let us consider the simple case of an electron. Clearly, its position in space is described by 3 quantum observables conjugated to momentum components. Lorentz transformations of these positions cannot be properly understood unless a position in time is also defined as a quantum observable $^2$. But such a time operator is commonly considered to be unavailable in the standard quantum formalism where time remains a classical parameter used to describe evolution. More generally, the definition of space-time observables fulfilling quantum and relativistic requirements has to face the problem that the changes of coordinate mapping of relativity have the status of gauge transformations in quantum theory and, as a result, that they cannot affect physical observables $^3,4$.

The answer to these problems, imported from relativistic theories, is to associate the definition of observables as well as the evaluation of their shifts with the algebra of symmetries. In this respect, Einstein’s synchronization or localization procedures are exemplary questions by their direct relations with the symmetry properties of electromagnetic fields. Relativistic effects on space-time positions are well-known to reflect invariance of the laws of physics under Lorentz frame transformations. Moreover, Einstein’s localization relies on Maxwell equations which are invariant also under dilatations and conformal transformations to uniformly accelerated frames $^5,6$.

These symmetry principles still hold in a quantum context $^8$. It is then possible to define observables associated with position in space-time of events and to evaluate their relativistic shifts. For electromagnetic fields, an event may be defined as the intersection of two pulses and its position in space-time may be written from the generators of Poincaré transformations and of dilatations. These position observables are found to be conjugated to momentum operators while simultaneously obeying explicitly Lorentz invariance. A time observable is found besides space ones and the shifts of these observables under Lorentz transformations conform to expectations of classical relativity $^9$.

Furthermore, redshifts, i.e. relativistic shifts under transformations to uniformly accelerated frames may be evaluated in the same manner from conformal symmetry. Localization observables are defined for field states which correspond to a non vanishing mass since they contain

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photons propagating in different directions. Remarkably, this mass observable experiences a redshift which reproduces exactly the effect of the gravitational potential arising in accelerated frames according to Einstein equivalence principle [11–13]. Hence, conformal symmetry forces the mass unit to scale as the inverse of the space-time unit and therefore corresponds to preservation of the quantum constant $\bar{\hbar}$ under frame transformations [14, 15]. These relations have been derived for electromagnetic quantum systems. They may certainly be expected to have a more universal character. As an important example, an electron-positron pair is a system which may decay into a pair of photons. The position of the decay reduces to the point of coincidence of the two emitted photons. Meanwhile, the mass is conserved in the annihilation process. It should therefore be expected that the redshift law consistent with Einstein equivalence principle is valid not only for the post-decay electromagnetic state but also for the pre-decay electron-positron pair. But to clear up this question, one must also be able to define localization observables for massive quantum systems such as electrons. Dirac theory of electrons [14] is certainly insufficient for that purpose since the coordinate parameters used to write quantum fields associated with electrons cannot be considered as localization observables.

For massless field theories such as electromagnetism, positions have been built on Poincaré and dilatation symmetry and redshift laws on the full conformal symmetry. The same achievements cannot be extended for electron as long as the latter is described by field theories, like Dirac theory, which violate dilatation or conformal symmetry by treating electron mass as a classical constant. But modern descriptions of electron consider that its mass is generated through an interaction with Higgs fields [15]. A quantum representation of electron mass is in fact inescapable since it is, at least partly, generated by electromagnetic self-energy. It must therefore possess intrinsic quantum fluctuations [16, 17]. Being a quantum operator rather than a classical parameter, it should vary under frame transformations. Standard forms of coupling to the Higgs field obey conformal invariance [17], so that the mass redshift may be expected to follow the same law as for electromagnetic fields and, therefore, to fit Einstein equivalence principle.

The aim of the present paper is to build up a new quantum description of electrons obeying conformal symmetry and fulfilling the expectation of the preceding paragraph. We consider that electrons are described by a conformally invariant field theory which we do not specify in more detail. Using general properties of conformal algebra and a few specific assumptions, we write down localization observables for electrons and deduce their quantum and relativistic properties. The specific assumptions are drawn from the phenomenology of electrons which are fermions with a spin $\frac{1}{2}$. They are written in a purely quantum algebraic manner and thus present analogies with Connes’ non-commutative geometry [18]. These analogies will be discussed as well as differences between the two approaches.

II. HERMITIAN LOCALIZATION OBSERVABLES

Localization observables will be built upon the algebra of symmetries in the manner already used for electromagnetic fields [19, 20].

Firstly this algebra contains Poincaré algebra characterized by the following relations

\[ (P_\mu, P_\nu) = 0 \]
\[ (J_{\mu\nu}, J_{\rho\sigma}) = \eta_{\mu\nu} P_{\rho\sigma} - \eta_{\mu\rho} P_{\nu\sigma} - \eta_{\mu\sigma} P_{\nu\rho} \]

\[ \eta_{\mu\nu} = \text{diag} (1, -1, -1, -1) \]

Meanwhile they represent the relativistic shifts of observables under translations and rotations. The Minkowski tensor is used throughout the paper for manipulating indices.

We then assume that the symmetry algebra contains a generator $D$ which generates relativistic shifts under global dilatations according to the conformal weight of observables

\[ (D, P_\mu) = P_\mu \]
\[ (D, J_{\mu\nu}) = 0 \]

Equation (2) constitutes a key assumption of our approach to the problem of localization observables. It is the source of important differences with Dirac theory of electrons although, as we shall see later on, it leads to close analogies with the latter. To illustrate these differences, we consider the mass observable $M$ defined in accordance with the standard relativistic relation

\[ M^2 = P^2 \equiv P_\mu P_\mu \]

as a Lorentz scalar with the same conformal weight as momenta

\[ (P_\mu, M) = (J_{\mu\nu}, M) = 0 \]
\[ (D, M) = M \]

These relations determine the mass $M$ up to an ambiguity which will be cleared up later on. At the moment, it is clear that any classical treatment of $M$ would lead to a vanishing commutator with $D$ and, therefore, to a
contradiction with (3). On the contrary, the commutator $(D, M)$ written in (4) is necessary in any framework where mass has its proper conformal dimension.

Localization observables representing positions in space-time may be built on Poincaré and dilatation generators. First, spin observables are introduced in a relativistic framework through the Pauli-Lubanski vector

$$W_\mu = - \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma$$

(5)

$$(P_\mu, W_\rho) = 0 \quad (J_{\mu\nu}, W_\rho) = \eta_{\rho\nu} W_\mu - \eta_{\rho\mu} W_\nu$$

The dot symbol denotes a symmetrised product for non commuting observables.

$$\epsilon_{\mu\nu\sigma\rho}$$ is the completely antisymmetric Lorentz tensor

$$\epsilon_{0123} = -\epsilon^{0123} = +1$$

$$\epsilon_{\mu\nu\sigma\rho} = - \epsilon_{\mu\sigma\nu\rho} = - \epsilon_{\nu\mu\sigma\rho}$$

Commutators between components of the spin vector define a spin tensor

$$S_{\mu\nu} = \frac{(W_\mu, W_\nu)}{M^2} = \epsilon_{\mu\nu\rho\sigma} \frac{W^\rho P^\sigma}{M^2}$$

(6)

Spin observables are transverse with respect to momentum

$$P^\mu S_{\mu\nu} = P_\rho W^\rho = 0$$

(7)

The square modulus of the Lorentz vector $W^\mu$ is a Lorentz scalar that we can write under its standard form in terms of a spin number $s$ taking integer or half-integer values

$$W^2 / M^2 = -\hbar^2 s (s + 1)$$

(8)

The spin number will be fixed to the value $\frac{1}{2}$ in the following. Throughout the paper, the velocity of light is set to unity while the Planck constant $\hbar$ is kept as the characteristic scale of quantum effects.

We are then able to define position observables as the quantities $X_\mu$ solving the following equations

$$J_{\mu\nu} = P_\mu \cdot X_\nu - P_\nu \cdot X_\mu + S_{\mu\nu} \quad D = P^\mu \cdot X_\mu$$

(9)

The dot symbol denotes a symmetrised product for non commuting observables

$$A \cdot B = \frac{AB + BA}{2}$$

Position observables are then obtained as

$$X_\mu = \frac{P_\mu}{M^2} \cdot D + \frac{P^\nu}{M^2} J_{\mu\nu}$$

(10)

They are shifted under translations, dilatation and rotations exactly as ordinary coordinate parameters are shifted under the corresponding transformations in classical relativity

$$x_\mu = X_\mu - i\gamma \frac{W_\mu}{M^2} = X_\mu - \frac{P^\nu}{M^2} s_{\nu\mu}$$

(11)

$$J_{\mu\nu} = P_\mu \cdot x_\nu - P_\nu \cdot x_\mu + s_{\mu\nu} \quad D = P^\mu \cdot x_\mu$$

(12)

Note that relations (7,10,12) are well defined only when the squared mass $M^2$ differs from 0. For electromagnetic states, such a condition was revealing that a localized event may be defined only with photons propagating in at least two different directions [19,20]. This problem does not arise for an electron which has a non vanishing mass.

Relation (12) clearly indicates that the conceptions of space-time inherited from classical relativity have to be revised for quantum objects. Electron cannot be considered as a sizeless point but rather as a fuzzy spot with a size of the order of Compton wavelength.

### III. CANONICAL LOCALIZATION VARIABLES

In this context, it is a remarkable and useful property that an auxiliary set of variables may be defined which possesses algebraic properties of canonical variables.

To obtain these observables, we first consider the involutive duality correspondence

$$\tilde{S}_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} S^{\rho\sigma} \quad S^{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} \tilde{S}^{\rho\sigma}$$

(13)

This definition is such that

$$\tilde{S}_{\mu\nu} = i \frac{P_\mu W_\nu - P_\nu W_\mu}{M^2}$$

(14)

We then introduce a self-dual representation of spin

$$s_{\mu\nu} = S_{\mu\nu} + \gamma \tilde{S}_{\mu\nu}$$

$$\tilde{s}_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} S^{\rho\sigma} = \gamma s_{\mu\nu}$$

(15)

where $\gamma$ is a dimensionless Lorentz scalar with a square equal to unity

$$(P_\mu, \gamma) = (J_{\mu\nu}, \gamma) = (D, \gamma) = 0 \quad \gamma^2 = 1$$

(16)

As a consequence of self-duality (13), $\gamma$ is the orientation of the canonical spin tensor $s_{\mu\nu}$ with the two eigenvalues $\pm 1$ associated with the right/left components. It plays the same role here as $\gamma_5$ in Dirac theory [22].

A new definition of position variables is associated with the self-dual spin tensor (13)

$$x_\mu = X_\mu - i\gamma \frac{W_\mu}{M^2} = X_\mu - \frac{P^\nu}{M^2} s_{\nu\mu}$$

(17)
Poincaré and dilatation generators take the same form in terms of both sets of variables \((\gamma^\mu)\) and \((\gamma^\mu)\). The variables \(x_\mu\) and \(s_\mu\) are quantum operators as \(X_\mu\) and \(S_\mu\) but they obey canonical commutation relations
\[
(P_\mu, x_\nu) = -\eta_{\mu\nu}, \quad (x_\mu, x_\nu) = 0 \\
(s_\mu s_\nu + \eta_{\mu\nu} s_\sigma - \eta_{\mu\sigma} s_{\nu\rho} - \eta_{\mu\rho} s_{\nu\sigma}) = 0 \\
(P_\mu, s_{\nu\rho}) = (x_\mu, s_{\nu\rho}) = 0
\]  
(18)
The localization observables \(X_\mu\) and \(S_\mu\) obey hermiticity conditions, even if they are not self-adjoint \([18,22]\). Relations \((13,17)\) thus show that canonical variables \(x_\mu\) and \(s_\mu\) are not hermitian. This is an important output of our quantum approach to the localization problem.

One may define either hermitian observables with non-canonical commutation relations or canonical variables which lead to simpler explicit calculations but are not hermitian.

From now on, we focus our attention on canonical variables. They can be seen as quantum algebraic versions of the position parameters and spin matrices of Dirac theory, as it will become clear in forthcoming computations. We emphasize that 4 positions in space and in time have been defined in contrast with previous studies of the localization problem where only positions in space were introduced \([24,25]\). This means that the requirement enounced by Schrödinger \([3]\) has been met: Lorentz transformations may now be properly described within quantum theory. Precisely, canonical positions are quantum observables which are transformed according to classical laws of special relativity
\[
(P_\mu, x_\nu) = -\eta_{\mu\nu}, \quad (D, x_\mu) = -x_\mu \\
(J_{\mu\nu}, x_\rho) = \eta_{\mu\rho} x_\nu - \eta_{\mu\nu} x_\rho
\]  
(19)

**IV. CLIFFORD ALGEBRA**

We narrow still more the scope by considering electrons which are fermions with a spin number \(s = \frac{1}{2}\).

We notice that the involution \(\gamma\) commutes with the squared mass \(M^2 = P^2\) while \(M\) commutes with \(\gamma^2 = 1\). These conditions are fulfilled as soon as \(\gamma\) and \(M\) commute or anticommute. We will assume in the following that \(\gamma\) and \(M\) are anticommuting variables. This entails that \(M\) may be written as follows
\[
M = \varepsilon |M|, \quad |M| = \sqrt{P^2} \\
(P_\mu, \varepsilon) = (J_{\mu\nu}, \varepsilon) = (D, \varepsilon) = 0 \\
\varepsilon^2 = 1, \quad \varepsilon \cdot \varepsilon = 0
\]  
(20)
The modulus \(|M|\) is equal to the norm of the energy-momentum vector while the sign \(\varepsilon\) is a further dimensionless Lorentz scalar with a square equal to unity. Clearly, the two eigenvalues \(\pm \varepsilon\) of \(\varepsilon\) are associated with the electron/positron components so that \(\varepsilon\) corresponds to charge in Dirac theory \([21]\). The fact that \(\gamma\) and \(\varepsilon\) anticommute or that \(\gamma\) and \(M\) anticommute is an important property. It means that the orientation \(\gamma\) is an operator changing the mass sign \(\varepsilon\) into its opposite or, equivalently, that \(\varepsilon\) interchanges the two spin orientations. We show in the following that this property is sufficient, when taken together with the general properties of conformal algebra, to build up a quantum algebraic theory of electrons.

Velocity observables \(V_\mu\) may be defined as ratios of momenta \(P_\mu\) to mass \(M\) or, equivalently, by applying the derivation operator \((\, , M)\) to hermitian positions \(X_\mu\) \([24]\)
\[
V_\mu = \frac{P_\mu}{M} = (X_\mu, M)
\]  
(21)
Further quantities \(\gamma_\mu\) may analogously be defined as derivatives of canonical positions \(x_\mu\)
\[
\gamma_\mu = (x_\mu, M) = V_\mu - 2\gamma \frac{S_\mu}{\hbar} = \frac{W_\mu}{M}
\]  
(22)
Notice that \(S_\mu\) and \(\gamma\) anticommute. Two velocities have been defined which have quite different properties. The velocities \([21]\) defined from hermitian positions have the standard form of mechanical velocities with the mass being however treated as a quantum observable. Their different components commute with each other. The velocities \(\gamma_\mu\) defined from canonical positions involve the mechanical velocities as well as spin terms. Their components do not commute but have commutators directly related to the self-dual spin tensor
\[
s_{\mu\nu} = -\frac{\hbar^2}{4} (\gamma_\mu, \gamma_\nu)
\]  
(23)
These velocities also allow to write the mass \(M\) as a linear expression of momenta as in standard Dirac theory
\[
M = P^\mu \gamma_\mu = \gamma_\mu P^\mu
\]  
(24)
The two foregoing relations suggest that the velocities \(\gamma_\mu\) are the extensions in our quantum algebraic framework of the Clifford matrices of Dirac theory.

This statement can effectively be put on firm grounds. To this aim, we evaluate the component of the spin vector \(S_\mu\) measured along an arbitrary unit vector \(n^\mu\) transverse to momentum
\[
S_\mu n^\mu = -\frac{\hbar}{2} \gamma_\mu n^\mu
\]  
(25)
For a spin \(\frac{1}{2}\), this component can only take the two values \(\pm \frac{\hbar}{2}\). This spectral condition may also be written as a characteristic polynomial
\[
S_\mu S_\nu = -\frac{\hbar^2}{4} (\eta_{\mu\nu} - \frac{P_\mu P_\nu}{M^2})
\]  
(26)
One deduces that the velocities \([22]\) generate a Clifford algebra with 4 generators.
\[ \gamma_{\mu} \cdot \gamma_{\nu} = \eta_{\mu\nu} \]  

(27)

One then derives that they commute with momenta and canonical positions

\[ (P_{\mu}, \gamma_{\nu}) = (x_{\mu}, \gamma_{\nu}) = 0 \]  

(28)

These results show how the velocities \( \gamma_{\mu} \) may be considered as quantum algebraic extensions of Clifford matrices. Notice that the mass \( M \) is now a quantum operator which anticommutes with \( \gamma \) and has a non vanishing commutator with \( D \), in consistency with the appropriate dimension of a mass. Hence this operator is distinct from the classical mass constant \( m \) of Dirac theory and, accordingly, the Clifford generators \( (24) \) cannot be confused with Clifford matrices of Dirac theory. Clifford matrices are fundamental entities in standard Dirac theory. Here, the expression \( (22) \) of Clifford generators has been derived from a few basic assumptions associated with symmetry principles and fermionic character of electrons.

A few remarks are worth of consideration at this point. First the mass \( M \) has different signs for electrons and positrons so that its modulus \( |M| \) rather than itself has to be interpreted as the quantum counterpart of the classical constant \( m \). Then spin terms cannot be considered as small corrections in \( (22) \) since Clifford velocities have a magnitude always equal to the velocity of light, due to \( (27) \), whereas mechanical velocities are usually much smaller than the velocity of light. Finally \( \gamma \) may be written as the product of Clifford generators and it anticommutes with each of them

\[ \gamma = i \gamma_0 \gamma_1 \gamma_2 \gamma_3 \quad \gamma \cdot \gamma_{\mu} = 0 \]  

(29)

The description of electrons presented here has been written down in a purely quantum algebraic manner. In this respect, it presents interesting analogies with the description of leptons in Connes’ non commutative geometry \( (13) \) where the “Dirac operator” \( P^\mu \gamma_{\mu} \) and the involution \( \gamma \) also play primary roles. Notice however that the present description is built on fundamental symmetry principles embodied in the conformal algebra. The expressions \( (22) \) of Clifford generators, the Dirac relation \( (24) \) and the Clifford properties \( (27) \) have not been assumed but rather derived from symmetries. Moreover, the effects of acceleration will be derived in the next section from the same principles and not postulated as a further separate assumption.

V. ACCELERATED FRAMES

We now aim to describe quantum electrons not only in inertial frames but also in uniformly accelerated frames.

As a first step, we complete the conformal algebra by considering the generators of transformations to accelerated frames and adding the following commutation relations to the already known ones \( (13) \)

\[ (P_{\mu}, C_{\nu}) = -2\eta_{\mu\nu} D - 2J_{\mu\nu} \]
\[ (J_{\mu\nu}, C_{\rho}) = \eta_{\rho\nu}C_{\mu} - \eta_{\rho\mu}C_{\nu} \]
\[ (D, C_{\mu}) = -C_{\mu} \quad (C_{\mu}, C_{\nu}) = 0 \]  

(30)

The quantities \( C_{\mu} \) generate relativistic shifts under infinitesimal transformations to uniformly accelerated frames. In particular, the first line of \( (30) \) gives the redshift laws for energy-momentum operators. As already discussed, the foregoing relations also represent the quantum commutation rules between \( C_{\mu} \) and other observables. Then, finite frame transformations are described by conjugations in the group built on the conformal algebra. Precisely, the shift of an observable from \( A \) in a frame to \( \overline{A} \) in another one is read, for transformations to accelerated frames, as

\[ \overline{A} = \exp \left( -\frac{\alpha^\rho C_{\rho}}{i\hbar} \right) A \exp \left( \frac{\alpha^\rho C_{\rho}}{i\hbar} \right) \]  

(31)

The parameters \( \alpha^\rho \) are classical accelerations along the 4 space-time directions.

Clearly, this conjugation preserves the structure of quantum algebraic relations known in inertial frames. For example, position and momentum observables are transformed under \( (31) \) but the canonical commutators between them are preserved since \( \eta_{\mu\nu} \) is a classical number invariant under conjugations. Canonical commutators have thus the same form in accelerated and inertial frames and can be written in terms of the same Minkowski metric. This result had to be expected in a quantum algebraic approach but it clearly stands in contradiction with covariant techniques of classical relativity. However the metric properties of classical relativity will be recovered or, more properly, generalized to a quantum algebraic framework in the forthcoming developments based on the evaluation of the redshifts of observables under the group conjugation \( (31) \).

Explicit expressions of the redshifts obviously depend on the relations existing between generators \( C_{\mu} \) on one hand, Poincaré and dilatation generators on the other hand. A general study of such relations, relying on general properties of the conformal algebra, is already available \( (20) \). We obtain now more specific results by assuming that electrons are fundamental particles with a spin number \( s = \frac{1}{2} \) preserved under frame transformations. Preservation of the spin number entails that generators \( C_{\mu} \) have closed expressions in terms of Poincaré and dilatation generators \( (20) \). These expressions take a simple form when written with canonical variables

\[ C_{\mu} = 2D \cdot x_{\mu} - P_{\mu} \cdot x^2 + 2x^{\rho} \cdot s_{\rho\mu} \]  

(32)

We may now derive redshifts \( (31) \) through straightforward algebraic computations, using the expression \( (32) \) of \( C_{\mu} \) and the simple commutation relations of canonical variables. As a first important output, the mass \( M \) is found to vary as a position-dependent conformal factor
\[ \overline{\mathbf{M}} = M \cdot \frac{1}{\lambda} \quad \frac{1}{\lambda} = 1 - 2\alpha^\mu x_\mu + \alpha^2 x^2 \]
\[ M = \overline{\mathbf{M}} \cdot \lambda \quad \lambda = 1 + 2\alpha^\mu \tau_\mu + \alpha^2 \tau^2 \] (33)

The conformal factor \( \lambda \) also appears in the transformation of canonical positions which has the form expected from classical relativity
\[ \overline{\mathbf{x}}^\nu = \lambda \left(x^\mu - x^2 \alpha^\mu\right) \quad x^\mu = \frac{1}{\lambda} \left(\overline{\mathbf{x}}^\mu + \overline{\mathbf{x}}^2 \alpha^\mu\right) \] (34)

Its interpretation as a conformal factor is confirmed by the relation between the metric tensors evaluated in the two frames
\[ g_{\mu\nu} = \partial_\mu \overline{\mathbf{x}} \eta_{\rho\sigma} \partial_\nu \overline{\mathbf{x}} = \lambda^2 \eta_{\mu\nu} \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} \]
(35)

To fix ideas, we have supposed the coordinate frame \( \overline{\mathbf{x}}^\mu \) to be inertial and the coordinate frame \( x^\mu \) to be accelerated. We however emphasize that the description of frame transformations by equations (33-35) is completely reciprocal in accordance with the group structure embodied in (31).

The mass \( \overline{\mathbf{M}} \) has the same expression as in conformally invariant generalizations of Dirac theory \[27,30\]. But such generalizations were based on an \textit{ad hoc} prescription where mass is a classical parameter varying according to the classical metric factor. This is why these generalizations were never able to reach the standards of full quantum consistency. The same remark holds for generalizations of Dirac theory to Riemannian space \[31,32\]. The results obtained here have a quite different status since they have been derived in a consistent quantum framework. In fact, the metric factor has been derived from the variation of mass and it has been obtained as a function of quantum canonical positions. Therefore this metric factor is itself a quantum observable.

Using the shift laws obtained for \( M \) and \( x \), we deduce the transformation of other quantities, in particular of the tetrad of Clifford generators
\[ \overline{\mathbf{e}}^\nu_\mu = \lambda e^\nu_\mu \gamma_\nu \]
\[ e^\nu_\mu \left(x\right) = \overline{\partial}_\mu x^\nu = \frac{1}{\lambda^2} \left(\partial^\nu \gamma_\mu\right) \] (36)

This transport law, written in terms of a \textit{vierbein} matrix \( e^\nu_\mu \), ensures that Clifford relations \[27\] are preserved under frame transformations as well as the orientation \( \gamma \) of the tetrad.

The redshift of energy-momentum is then seen to involve a spin-dependent part
\[ \overline{\mathbf{P}}_\mu = e^\nu_\mu \cdot P_\nu + \frac{1}{2} \left(\partial^\nu e^\nu_\mu\right) s_{\nu\rho} \] (37)

This means that translations are transformed as a covariant vector provided that a spin term is added which has the form of a connection \[\overline{\mathbf{e}}^\nu_\mu\gamma_\nu\]. However, the connection appearing in \[\overline{\mathbf{e}}^\nu_\mu\gamma_\nu\] is a quantum operator written as a function of canonical position and spin variables. Moreover, its expression is not a further assumption but an output of conformal algebra.

Expressions \[\overline{\mathbf{M}},\overline{\mathbf{P}}\] have a simple form when written in terms of canonical variables but, as already noticed, they involve non-hermitian operators. Alternatively, the shifts may be written in terms of hermitian observables \( X_\mu \) and \( S_{\mu\nu} \), and then involve momentum-dependent corrections
\[ \overline{\mathbf{M}} = M \cdot \left(1 - 2\alpha^\mu X_\mu + \alpha^2 \left(X^2 + \frac{3h^2}{4P^2}\right)\right) \]
\[ \overline{\mathbf{P}}_\mu = E^\nu_\mu \cdot P_\nu + \frac{1}{2} \partial^\rho E^\nu_\mu \cdot S_{\rho\nu} + \frac{3h^2}{32} \partial_\nu \partial^\rho E^\nu_\mu \cdot P_\rho P^2 \]
\[ E^\nu_\mu = e^\nu_\mu \left(X\right) \] (38)

The function \( e^\nu_\mu \) being defined from the tetrad transformation \( \overline{\mathbf{e}}^\nu_\mu \) has been used to build \( E^\nu_\mu \) through a substitution of hermitian positions \( X \) to canonical ones \( x \). Since this function is a second-order polynomial form, an ordering has to be chosen when writing it. However, this ordering does not matter in the expression of \( \overline{\mathbf{P}}_\mu \) due to the form \[\overline{\mathbf{e}}^\nu_\mu\gamma_\nu\] of the commutators between hermitian positions.

**VI. DISCUSSION**

A new quantum algebraic description of electrons has been presented in this paper. This description fulfills the relativistic and quantum requirements discussed in the introduction in a completely consistent theoretical framework. We have not specified the quantum field theory except for the basic properties that it is a conformally invariant description of spin \( \frac{1}{2} \) fermions. The definition of localization observables, the expression of Clifford generators, the evaluation of quantum commutators and relativistic shifts have all been derived within a single calculus built on conformal algebra.

Frame transformations have been described as group conjugations. This ensures that the quantum algebraic relations defined in inertial frames may be exported to uniformly accelerated frames. Although it is quite different from covariance techniques of classical relativity, this description has allowed to recover in a quantum framework a lot of geometric laws known from classical relativity. Mass has been found to experience a redshift which fits the expectation deduced from Einstein equivalence principle. This mass redshift may be considered as defining a quantum conformal factor which has also been shown to enter the expression of a quantum metric tensor. Expressions obtained for other observables not only translate the geometric laws of classical relativity into a quantum theoretical framework but they also change these laws through the addition of spin terms.

These results should open the way to a renewed treatment of effects of acceleration or gravitation on quantum systems which could in particular show useful for
analysing high precision tests of inertial or gravitational effect on atomic clocks [34].

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