Differential quadrature method of nonlinear bending of functionally graded beam

Xu Gangnian¹, Ma Liansheng², Wang Youzhi¹, Yuan Quan¹ and You Weijie¹

1 School of Civil Engineering, Shandong University, Ji’nan 250061, China
2 School of Science, Lanzhou University of Technology, Lanzhou 730050, China
E-mail: wangyouzhi@sdu.edu.cn

Abstract. Using the third-order shear deflection beam theory (TBT), nonlinear bending of functionally graded (FG) beams composed with various amounts of ceramic and metal is analyzed utilizing the differential quadrature method (DQM). The properties of beam material are supposed to accord with the power law index along to thickness. First, according to the principle of stationary potential energy, the partial differential control formulae of the FG beams subjected to a distributed lateral force are derived. To obtain numerical results of the nonlinear bending, non-dimensional boundary conditions and control formulae are dispersed by applying the DQM. To verify the present solution, several examples are analyzed for nonlinear bending of homogeneous beams with various edges. A minute parametric research is in progress about the effect of the law index, transverse shear deformation, distributed lateral force and boundary conditions.

1. Introduction

It is very clear that analysis of the nonlinear bending for FGM beams under various shear deflection theories consists of two accented phases, for example, building the mathematical pattern. These problems are represented by a crowd of the partial differential equations, either nonlinear or linear, it is difficult to establish the analytical solutions. Therefore, the approximate numerical solution is adopted to meet the requirements of the project. At present, the focus of a large amount of literature is on FGM beams under the mechanical load.

For exact solutions, Sankar [1] obtained an exact elasticity solution for FG beams that considers the static transverse loading; this was later adopted for getting analytical solutions regarding stresses about a sandwich beam. Yu and Zhong [2] obtained a general solution for a beam with the uniform loading on the upper surface by using a semi-inverse method. Huang [3] et al. studied the exact solutions for anisotropic FG material beams with the different boundary conditions. Based on the shear deformation theory, Ma and Lee [4] gave a analytical solution about nonlinear responses of a FG beam. Lü et al. [5] obtained an exact solution for the bending of FGM beams to adopt the differential quadrature method.

For numerical solutions, using TBT, the shear, axial stresses and the transverse deflection in the thick FGM bar under an uniform loading for the various boundary conditions are studied in detailed [6]. By using the mathematical similarities among the governing equations, Li et al. [7] derived the analytical relationship of bending solutions. Li [8] obtained a new way to study the static responses of classic and first-order beams. Application of TBT and physical neutral surface, Zhang and Zhou [9] built a mathematical model of beams to obtain the numerical results of bending by the Multi-term Ritz method. Under mechanical loading, Niknam et al. [10] studied nonlinear static responses of the tapered beam under the various conditions. For the static and dynamic analysis of beams subjected to thermal load, Ma and Lee [11] carried out the related research by a shooting method. In this paper,
based on TBT, a numerical result by DQ method is obtained about nonlinear bending of FG beams subjected to mechanical loading under C-C and S-S boundary conditions.

2. Mathematical problem

2.1. Model material properties

Take into account a FGM beam (width \(b\), length \(l\), and height \(h\)) with a rectangular section (Figure 1), which is mainly made up of ceramics and metals. The material properties vary continuously along the height direction from the pure ceramic surfaces to the pure metal surfaces. Young’s modulus \(E\) and the passion ratio \(\nu\) are expressed as following:

\[
P(z) = P_c + (P_M - P_c)V_M
\]

where \(M\) and \(C\) denote the components of metal and ceramic; and \(V_M\) is the percentage of the volume of metal components. This formula changes according to below function:

\[
V_M = \left(\frac{1 - z}{2h}\right)^n
\]

\[\text{(2)}\]

**Figure 1.** Geometry model of the studied FG beam.

2.2. The equilibrium equations

Displacement field as following:

\[
\begin{align*}
    U_x(x, z) &= \phi(x) + u(x) - \beta z^3 \phi(x) - \beta z^4 w_z(x) \\
    U_z(x, z) &= w(x)
\end{align*}
\]

\[\text{(3)}\]

where, \(x\) is the horizontal ordinate along the longitudinal direction of model beam; \(U_x(x, z)\) and \(U_z(x, z)\) are the deformation of \(x\), \(z\) axial at arbitrary points of the studied beam; \(w(x)\) denotes the vertical deflection; \(\phi(x)\) expresses as the rotational angle.

Formula expression of displacement-strain are quoted by Heyliger and Reddy[12]:

\[
\begin{align*}
    \varepsilon_x &= z\phi_x + \frac{w_z}{2} + u_x - \beta z^3 (\phi_x + w_{z, x}) \\
    \gamma_{xz} &= w_x + \phi - 3\beta z^2 (\phi + w_x)
\end{align*}
\]

\[\text{(4)}\]

where \(\beta = \frac{4}{3h^2}\)

As a result of the materials of the FG beam obey Hooke’s law, we find the following stress-displacement relations:

\[
\sigma_x = E(z)\varepsilon_x
\]

\[\text{(5)}\]

\[
\tau_{xz} = \frac{E}{2(1+\nu)}\gamma_{xz}
\]

\[\text{(6)}\]
2.3. Third-order shear deformation theory (TBT)

Based on TBT, the equilibrium equations can be expressed by:

\[
\begin{align*}
&\frac{d^4u}{dx^4} + \frac{dW}{dx} \left( \frac{d^2w}{dx^2} - \frac{\beta E_1\Omega_1}{\Omega_4} \frac{d^3w}{dx^3} + \frac{\beta E_1\Omega_3}{\Omega_4} \right) \varphi = 0 \\
&\frac{d^2\varphi}{dx^2} - \frac{A_1\Omega_2}{\Omega_4} \frac{d^3w}{dx^3} \varphi = 0 \\
&\left( \frac{\beta E_1\Omega_2^2}{\Omega_4} - \beta F_1 + \beta F_1 \Omega_2 \right) \frac{d^4w}{dx^4} + \Omega_2 \Omega_3 \tilde{D}_{11} \frac{d\varphi}{dx} + \left[ A_1 \left( \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right) - \beta E_1 \frac{d\varphi}{dx} \right] \frac{d^3w}{dx^3} + q = 0
\end{align*}
\]

(7)

Where

\[
\Omega_1 = A_1 - \frac{\beta E_1^2}{\tilde{D}_{11}}, \quad \Omega_2 = 1 + \frac{\beta E_1}{\tilde{D}_{11}}, \quad \Omega_3 = \frac{A_1 \tilde{A}_{14}}{\tilde{D}_{11}}, \quad \varphi = \phi + \frac{dw}{dx}, \quad \tilde{A}_{14} = \tilde{A}_{14} - 3 \beta \tilde{D}_{14}, \quad \tilde{A}_{14} = A_{14} - 3 \beta D_{14},
\]

\[
\tilde{D}_{11} = D_{11} - \beta F_{11}, \quad \tilde{D}_{11} = D_{11} - \beta F_{11}.
\]

According to the above equations, the non-dimensional quantities are defined by:

\[
\begin{align*}
\xi &= \frac{x}{l}, \quad \omega = \frac{w}{h}, \quad \bar{\varphi} = \frac{\omega}{h}, \quad U = \frac{ul}{h}, \quad \beta_0 = \frac{l}{h}, \quad p = \frac{P_l^2}{D_l}, \quad S_{r_0} = \frac{\beta E_1^2 \Omega_2^2}{\Omega_4} - \beta F_{11} + \beta F_{11} \Omega_2, \quad S_{r_1} = \frac{\beta E_1 \Omega_2}{\Omega_4}, \\
S_{r_2} &= \frac{\beta E_1 \Omega_2^2}{\Omega_4}, \quad S_{r_3} = A_1 \Omega_2, \quad S_{r_4} = \Omega_2, \quad S_{r_5} = \Omega_3 \Omega_2, \quad S_{r_6} = \frac{A_1 h^2}{S_{r_0}}, \quad q_{r_0} = \frac{q l^4}{h S_{r_0}}.
\end{align*}
\]

From Eq. (7), we obtain the non-dimensional equilibrium equations as following:

\[
\begin{align*}
&\frac{d^4U}{d\xi^4} + \frac{dW}{d\xi} \left( \frac{d^2W}{d\xi^2} - S_{r_1} \frac{d^3W}{d\xi^3} - S_{r_2} \bar{\varphi} \right) = 0 \\
&\frac{d^2\bar{\varphi}}{d\xi^2} - S_{r_3} \frac{d^3W}{d\xi^3} - S_{r_4} \bar{\varphi} = 0 \\
&\frac{d^4W}{d\xi^4} + S_{r_5} \frac{d\bar{\varphi}}{d\xi} + S_{r_6} \left[ \left( \frac{dU}{d\xi} + \frac{1}{2} \left( \frac{dW}{d\xi} \right)^2 \right) - S_{r_2} \frac{d\bar{\varphi}}{d\xi} \right] \frac{d^3W}{d\xi^3} + q_{r_0} = 0
\end{align*}
\]

(8)

3. Mathematical formulation

3.1. The weighting coefficients

The DQM must discretize the definition domain into \( m \) points. Along the definition domain, a weighted nonlinear summation of all of the functional values in \( m \) points approximately expresses the derivatives of any point [13-14].

\[
\frac{d^k f(x_i)}{dx^k} = f_i^{(k)} = \sum_{j=1}^{m} A_j^{(k)} f_j, \quad i = 1, 2, \cdots, m
\]

(9)

where \( m \) denotes the number of grid points, \( A_j^{(k)} \) is the weighting coefficient for the \( k \)th derivative.

3.2. Selection of the sampling nodes

The sampling points of differential quadrature representations are selected as shifted Chebyshev-Gauss-Lobatto points, which are given as follows:

\[
x_i = \frac{1}{2} \left[ 1 - \cos \left( \frac{\pi (i-1)}{m-1} \right) \right], \quad i = 1, 2, \cdots, m
\]

(10)

4. Numerical results and discussions
For numerical analysis, we calculate an Si3N4-SUS304 FG beam under a uniformly distributed loading $q$(KN cm$^{-1}$), with a length of $l=100$ cm, a height of $h=1$ cm, a width of $b=1$ cm, and the number of nodes is $m=11$. The material properties: $E_{\text{Si3N4}} = 348.43$ GPa, $E_{\text{SUS304}} = 201.04$ GPa, $\nu_{\text{SUS304}} = 0.24$, $\nu_{\text{SUS304}} = 0.33$.

![Figure 2](image1.png)  
**Figure 2.** Central deflection-loading curves of a C-C FG beam.

![Figure 3](image2.png)  
**Figure 3.** Central deflection-loading curves of a S-S FG beam.

From figures 2 to 3, it can be observed that the shear deformation effects is the critical factor to cause an increase of the central deflection. For an Si3N4-SUS304 FG beam with the C-C and S-S boundary conditions, using a shear deformation beam theory, dropping the power law index will cut down the stiffness of FG beam, in succession, results in a decrease of the deflections.

5. Conclusions
In this article, based on TBT, the nonlinear bending of FG beams exerted under two different edges is analytically investigated. Nonlinear curves are gradually obvious with the increasing of load. Due to consider shear deformation, the central deflection of an FG beam using TBT is greater compared to homogeneous beam. The central deflection-load curves obtained by exerting an simply supported edge is faster, and gradually becomes smoothly with load increase. Using TBT, the maximum central deflections of Si$_3$N$_4$-SUS304 beams is not significantly different from the center deflections found using FBT. In addition, dropping the power law index will cut down the stiffness of FG beam, in succession, results in a decrease of the deflections.

References
[1] Sankar B V 2001 Compos. Sci. Technol. An elasticity solution for functionally graded beams 61 689-96
[2] Yu T and Zhong Z 2006 Acta Mech. Solida Sin. A general solution of a clamped functionally graded cantilever-beam under uniform loading 27 15-20
[3] Huang D J, Ding H J and Chen W Q 2007 Int. J. Solids Struct. Analytical solution for functionally graded magneto-electro-elastic plane beams 45 467-85
[4] Ma L S and Lee D W 2012 Eur. J. Mech. A/Solids Exact solutions for nonlinear static responses of a shear deformable FGM beam under an in-plane thermal loading 31 13-20
[5] Lü C F, Chen W Q, Xu R Q and Lim C W 2008 Int. J. Solids Struct. Semi-analytical elasticity solutions for bi-directional functionally graded beams using higher order shear deformation theory 45 258-75
[6] Kadoli R, Akhtar K and Ganesan N 2003 Appl. Math. Model. Static analysis of functionally graded beams from those of the homogenous Euler-Bernoulli beam 37 7077-85
[7] Li S R, Cao D F and Wan Z Q 2013 Appl. Math. Model. Bending solutions of FGM Timoshenko beams from those of the homogenous Euler-Bernoulli beam 37 7077-85
[8] Li X F 2008 J. Sound Vib. A unified approach for analyzing static and dynamic behaviors of functionally graded Timoshenko and Euler-Bernoulli beams 318 1210-29
[9] Zhang D G 2013 Compos. Struct. Nonlinear bending analysis of FGM beams based on physical neutral surface and higher-order shear deformation theory 100 121-6
[10] Niknam H, Fallah A and Aghdam M M 2014 Int. J. Nonlin. Mech. Nonlinear bending of functionally graded tapered beams subjected to thermal and mechanical loading 65 141-7
[11] Ma L S and Lee D W 2011 Compos. Struct. A further discussion of nonlinear mechanical behavior for FGM beams under plane thermal loading 93 831-42
[12] Heyliger P R and Reddy J N 1998 J. Sound Vib. A higher order beam finite element for bending and vibration problem 126 309-26
[13] Pu J P 2005 J. Zhejiang Univ. Technol. The differential quadrature method and its application 33 429-33
[14] Wang L 1983 J. Liaoning Arch. Eng. Large deflection calculation of beams 1 17-36