SYMBOLIC DYNAMICS, PARTIAL DYNAMICAL SYSTEMS, BOOLEAN ALGEBRAS AND $C^*$-ALGEBRAS GENERATED BY PARTIAL ISOMETRIES

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Abstract. We associate to each discrete partial dynamical system a universal $C^*$-algebra generated by partial isometries satisfying relations given by a Boolean algebra connected to the discrete partial dynamical system in question. We show that for symbolic dynamical systems like one-sided and two-sided shift spaces and topological Markov chains with an arbitrary state space the $C^*$-algebras usually associated to them can be obtained in this way.

As a consequence of this, we will be able to show that for two-sided shift spaces having a certain property, the crossed product of the two-sided shift space is a quotient of the $C^*$-algebra associated to the corresponding one-sided shift space.

1. Introduction

The history of associating $C^*$-algebras to symbolic dynamical systems is long and successful.

A good example of this is the crossed product of infinite minimal two-sided shift spaces which in [13] was used to classify infinite minimal shift spaces up till strong orbit equivalence and flip conjugacy (it is actually done for a bigger class of dynamical systems, namely Cantor systems, but we will in this paper only concern ourselves with symbolic dynamical systems).

Another very important example of a class of $C^*$-algebras associated to symbolic dynamical systems is the Cuntz-Krieger algebras [9], which in a natural way can be viewed as $C^*$-algebras associated to topological Markov chains with finite state space. The Cuntz-Krieger algebras have proved to be very important examples in the theory of $C^*$-algebras and have also let to invariants of shift of finite type such as the dimension group (cf. [18] and [9]).

The Cuntz-Krieger algebras have been generalized in many different ways. We will in this paper focus on two of those. The first one is due to Exel and Lace, who in [11] have generalized the Cuntz-Krieger algebras to topological Markov chains with arbitrary state space. The other generalization is due to Matsumoto, who in [20] associated a $C^*$-algebra to every shift space (called

Date: March 29, 2022.

2000 Mathematics Subject Classification. Primary 46L55; Secondary 46L05, 37B10, 06E99.

Key words and phrases. $C^*$-algebras, partial dynamical systems, symbolic dynamical systems, Boolean algebras.

This research has been supported by the EU-Network Quantum Spaces - Noncommutative Geometry (HPRN-CT-2002-00280).
Matsumoto's original construction associated a $C^*$-algebra to every \textit{two-sided} shift space, but it is more natural to view it as a way to associated a $C^*$-algebra to every \textit{one-sided} shift space (cf. \cite{matsumoto} and \cite{exel_laca}).

Topological Markov chains with finite state space are examples of one-sided shift spaces, and it turns out that the $C^*$-algebras associated to these kind of shift spaces are Cuntz-Krieger algebras, so in this way the class of $C^*$-algebras associated with shift spaces is a generalization of the class of Cuntz-Krieger algebras (cf. \cite[Section 8]{exel_laca}).

Thus we have three different classes of $C^*$-algebras associated to symbolic dynamical system, namely crossed products of two-sided shift spaces, Exel and Laca’s generalization of Cuntz-Krieger algebras and $C^*$-algebras associated to one-sided shift spaces. The main purpose of this paper is to unify these three constructions to one.

My original motivation for written this paper was to prove Theorem 8.18, which for shift spaces having a certain property relates the crossed product of the two-sided shift space and the $C^*$-algebra associated to the corresponding one-sided shift space. Doing this I found that the crossed product of two-sided shift spaces and the $C^*$-algebra associated to one-sided shift spaces have a common structure, which I also found in Exel and Laca’s generalization of Cuntz-Krieger algebras (which we for simplicity from now on just will call Cuntz-Krieger algebras). This structure can be described by partial representations of groups and Boolean algebras. More formal, what we will do is to associate to every so called discrete partial dynamical system a $C^*$-algebra and then to every symbolic dynamical system (both one-sided and two-sided) associate a discrete partial dynamical system in such a way that the $C^*$-algebras we get in this way for one-sided shift spaces, two-sided shift spaces and topological Markov chains are canonical isomorphic to the $C^*$-algebra associated to the one-sided shift space, the crossed product of the two-sided shift space and the unitization of the Cuntz-Krieger of the transition matrix of the topological Markov chain, respectively.

This construction is very natural, and I hope that beside the benefits from having a unified construction of these different classes of $C^*$-algebras associated to different kinds of symbolic dynamical systems, the construction will also clarify in which way the structure of the symbolic dynamical system is reflected in the structure of the associated $C^*$-algebra.

The paper is organized as follows: In Section 2, we will shortly introduce some notation which will be used throughout the paper, in Section 3 discrete partial dynamical systems will be defined, and we will see how we from one- and two-sided symbolic dynamical systems can construct discrete partial dynamical systems. We will then in Section 4 define the $C^*$-algebra of a discrete partial dynamical system and show some basic properties of it. In Section 5 we will show the $C^*$-algebra of a discrete partial dynamical system can be constructed as a crossed product of a $C^*$-partial dynamical systems, and we will in Section 6 construct a representation of the $C^*$-algebra of a discrete partial dynamical system as operators on a Hilbert space. Section 7 is the main section of this paper; here we will show that the crossed product of a two-sided shift space, the $C^*$-algebra of a one-sided shift space, and
Cuntz-Krieger algebras can be obtained as C*-algebras of discrete partial dynamical systems. We will then in Section 8 describe the ideal structure of the C*-algebra of a discrete partial dynamical system and then use this description to prove the above mentioned Theorem 8.18. The paper finish with three appendices in which we will give a short introduction to partial representations of groups, Boolean algebras and crossed products of C*-partial dynamical systems.

A previous version of this paper appeared in my Ph.D thesis [2]. Unfortunately that version contained a lot of mistakes, which hopefully have been fixed in this version. The most notable of these mistakes was that I claimed that the C*-algebra of a higher rank graph could be constructed as the C*-algebra of a discrete partial dynamical system. The proof of this is however false, but it is possible to construct the C*-algebras of a higher rank graph in a very similar way by using an action of an (discrete) inverse semigroup instead of a partial action of a (discrete) group. This will be proved in a forthcoming paper by Gwion Evans and the author.

Acknowledgment. The process of writing this paper has been very long. I started on it when I was a Ph.D-student at the University of Copenhagen, I continued working on it while I was a post Doc at the Institut Mittag-Leffler and at the Norwegian University of Science and Technology and I finally finished it as a post Doc at Universität Münster. I wish to thank all members of the operator algebra groups at these places for their kind hospitality and especially Søren Eilers, Christian Skau and Joachim Cuntz. I also wish to thank Aidan Sims for pointing out the above mentioned mistake.

2. Notation and preliminaries

Throughout this paper, e will denote the neutral element of a given group. We will by \( \mathbb{Z} \) denote the set of integers, \( \mathbb{N}_0 \) will denote the set of non-negative integers and \( \mathbb{N} \) will denote the set of positive integers.

If \( X \) is a set, then we will by \( \text{Id}_X \) denote the identity map on \( X \). For a map \( \sigma : X \to X \), we will for every \( k \in \mathbb{N} \) by \( \sigma^k \) denote the \( k \)-times composition of \( \sigma \) with itself, and we will set \( \sigma^0 = \text{Id}_X \). If \( \sigma \) is invertible, then we will for every \( k \in \mathbb{N} \) by \( \sigma^{-k} \) denote the map \( (\sigma^{-1})^k \). If \( A \) is a subset of \( X \), then \( 1_A \) denotes the characteristic function:

\[
1_A(x) = \begin{cases} 
1 & \text{if } x \in A, \\
0 & \text{if } x \notin A.
\end{cases}
\]

If \( \theta \) is a map defined on \( A \), then we will for another subset \( B \) of \( X \) by \( \theta(B) \) mean \( \theta(A \cap B) \), and we will by \( \theta|_B \) denote the restriction of \( \theta \) to \( A \cap B \).

If \( C \) is a subset of a vector space, then we will use \( \text{span}(C) \) to denote the linear span of \( C \), and if the vector space also comes with a topology, then \( \overline{\text{span}}(C) \) will denote the closure of the linear span of \( C \). For a subset \( D \) of a C*-algebra we denote the C*-subalgebra generated by \( D \), by \( C^*(D) \).

When \( a \) is a set, then we will by \( F_a \) denote the free group generated by \( a \). We will regard \( a \) as a subset of \( F_a \) and denote the subset \( \{a^{-1} \mid a \in a \} \) of \( F_a \) by \( a^{-1} \). We say that an element \( g \in F_a \) is written in reduced form \( b_1 b_2 \cdots b_k \) if \( b_1, b_2, \ldots, b_k \in a \cup a^{-1} \) and \( b_j = b \Rightarrow b_{j+1} \neq b^{-1} \) for every \( j \in \{1, 2, \ldots, k-1\} \).
We will call the family \((\partial \theta)_{\theta \in G}\) of \(N \times N\)
eventually periodic if there exist \(m, N \in \mathbb{N}\) such that \(x_{n+m} = x_n\) for \(n > N\).

3. Discrete partial dynamical systems

We are now going to define partial actions and discrete partial dynamical systems and look at some ways to construct discrete partial dynamical systems.

Partial actions have been defined and studied by Ruy Exel in \cite{10}, where he for any given group constructed an inverse semigroup such that there is a one-to-one correspondence between the actions of the inverse semigroup and the partial actions of the group.

**Definition 3.1.** Given a group \(G\) and a set \(X\), a partial action \(\theta\) of \(G\) on \(X\) is a pair

\[ ((\partial \theta)_{\theta \in G}, (\theta_g)_{G}) \]

where for each \(g \in G\), \(D_g\) is a subset of \(X\) and \(\theta_g\) is a bijective map from \(D_{g^{-1}}\) to \(D_g\), satisfying for all \(h\) and \(i\) in \(G\):

\[
\begin{align}
(3.1a) & \quad D_e = X \text{ and } \theta_e \text{ is the identity map on } X, \\
(3.1b) & \quad \theta_h(D_i) = D_h \cap D_{hi}, \\
(3.1c) & \quad \theta_h(\theta_i(x)) = \theta_{hi}(x) \text{ for } x \in D_{i^{-1}} \cap D_{h^{-1}i^{-1}i}.
\end{align}
\]

We will call the family \((\partial \theta)_{\theta \in G}\) the domains of \(\theta\) and the family \((\theta_g)_{G}\) the partial one-to-one maps of \(\theta\).
The triple \((X, \theta, G)\) is called a *discrete partial dynamical system*.

The reason that we have chosen to call such a triple a *discrete* partial dynamical system is that we do not require any structure of the system \((X, \theta, G)\) other than the above mentioned. It is in many cases (see for example [12]) natural to ask for the set \(X\) to be a topological space, the domains \((D_g)_{g \in G}\) open subsets, and the maps \((\theta_g)_{g \in G}\) homeomorphisms, but we will in this paper only consider the discrete case, where we do not require such a structure.

A very simple example of a discrete partial dynamical systems is if we for a given group \(G\) and a given set \(X\) for every \(g \in G\) let \(D_g = X\) and let \(\theta_g\) be the identity map on \(X\). We get slightly more interesting examples if we consider group actions:

**Example 3.2.** Let \(G\) be a group, \(X\) a set and \(\theta\) an action of \(G\) on \(X\), i.e., \(\theta_g\) is for every \(g \in G\) a map from \(X\) to \(X\) such that
\[
\begin{align*}
(3.2a) & \quad \theta_e \text{ is the identity map on } X, \\
(3.2b) & \quad \theta_h \circ \theta_i = \theta_{hi} \text{ for every } h, i \in G.
\end{align*}
\]

If we for every \(g \in G\) let \(D_g = X\), then \(\theta = (D_g)_{g \in G}, (\theta_g)_{g \in G}\) is a partial action of \(G\) on \(X\), and \((X, \theta, G)\) a discrete partial dynamical system.

As mentioned in the Introduction, we will in this paper mainly concern ourselves with partial dynamical systems which come from symbolic dynamical systems. We will now show how to get partial dynamical systems from symbolic dynamical systems; we will first see how to define a partial dynamical system from a *one-sided* symbolic dynamical system, and then how to define a partial dynamical system from a *two-sided* symbolic dynamical system.

3.1. **One-sided symbolic dynamical systems.** Let \((X^+, \sigma)\) be a one-sided symbolic dynamical system over the alphabet \(a\). That is: \(a\) is a set (finite or infinite), \(\sigma : a^{\mathbb{N}_0} \to a^{\mathbb{N}_0}\) is the map

\[
x_0x_1x_2\cdots \mapsto x_1x_2\cdots
\]

and \(X^+\) is a subset of \(a^{\mathbb{N}_0}\) such that \(\sigma(X^+) \subseteq X^+\). To turn \((X^+, \sigma)\) into a partial dynamical system we restrict \(\sigma\) to subsets of \(X^+\) such that \(\sigma\) is injective on these subsets. This is done in this way:

Let for every \(a \in a\), \(D_a\) be the subset \(\{(x_n)_{n \in \mathbb{N}_0} \in X^+ \mid x_0 = a\}\), \(D_{a^{-1}}\) be the subset \(\sigma(D_a)\), \(\theta_a : D_{a^{-1}} \to D_a\) be the map

\[
x \mapsto ax,
\]

and \(\theta_a : D_a \to D_{a^{-1}}\) be the map

\[
x \mapsto \sigma(x).
\]

Let \(\mathbb{F}_a\) be the free group generated by \(a\) and let for every \(g \in \mathbb{F}_a\) written in the reduced form \(b_1b_2\cdots b_k\), where \(b_1, b_2, \ldots, b_k \in a \cup a^{-1}\), \(D_g\) be the subset of \(X^+\) defined by

\[
D_g = \theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_k}(X^+),
\]

and \(\theta_g : D_{g^{-1}} \to D_g\) be the map defined by

\[
\theta_g = \theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_k}.
\]
Then \( \theta_{X^+} = \{(D_g)_{g \in \mathbb{F}_a^+}, (\theta_g)_{g \in \mathbb{F}_a} \} \) is a partial action of \( \mathbb{F}_a \) on \( X^+ \), and \((X^+, \theta_{X^+}, \mathbb{F}_a)\) is a discrete partial dynamical system.

**Definition 3.3.** Let \((X^+, \sigma)\) be a one-sided symbolic dynamical system over the alphabet \( a \). Then we call the discrete partial dynamical system \((X^+, \theta_{X^+}, \mathbb{F}_a)\) constructed above the **discrete partial dynamical system associated to** \((X^+, \sigma)\).

**Lemma 3.4.** Let \((X^+, \sigma)\) be a one-sided symbolic dynamical system over the alphabet \( a \) and let \((X^+, \theta_{X^+}, \mathbb{F}_a)\) be the discrete partial dynamical system associated to \((X^+, \sigma)\) as in Definition 3.3. Then the following holds for the partial action \( \theta_{X^+} = \{(D_g)_{g \in G}, (\theta_g)_{g \in G} \} \):

1. If \( g \in G \) and \( D_g \neq \emptyset \), then there exist \( u, v \in a^* \) such that \( g = uv^{-1} \).
2. If \( u, v \in a^* \) and the last (rightmost) letters of \( u \) and \( v \) are not equal (or either \( u \) or \( v \) is equal to the empty word), then we have that
   \[
   D_{uv^{-1}} = \left\{ x \in X^+ \mid x_{|0,v|} = v, \; ux_{|v|, \infty} \in X^+ \right\},
   \]
   and \( \theta_{uv^{-1}} \) is the map
   \[
   ux \mapsto vx
   \]
   from \( D_{uv^{-1}} \) to \( D_{vu^{-1}} \).
3. If \( u, v \in a^* \), \( |u| = |v| \) and \( u \neq v \), then \( D_u \cap D_v = \emptyset \).

**Proof.** (2) can easily be proved by induction over the length of \( u \) and \( v \). (3) then follows from (2), and (1) follows from the definition of \( D_g \) and (3).

### 3.2. Two-sided symbolic dynamical systems.

Let \((X, \tau)\) be a two-sided symbolic dynamical system over the alphabet \( a \). That is: \( a \) is a set (finite or infinite), \( \tau : a^Z \to a^Z \) is the map defined by

\[
(\tau((z_n)_{n \in Z}))_m = z_{m+1}
\]

for every \((z_n)_{n \in Z} \in a^Z \) and every \( m \in \mathbb{Z} \), and \( X \) is a subset of \( a^Z \) such that \( \tau(X) = X \).

Since \( \tau \) is bijective, \((\tau^k)_{k \in \mathbb{Z}}\) is an action of \( \mathbb{Z} \) on \( X \), so we could turn \((X, \tau)\) into a discrete partial dynamical system by the method of Example 3.2, but we can also do it by imitating the method we used to define the discrete partial dynamical system associated to a one-sided symbolic dynamical system, and that is what we will do here.

Let for every \( a \in a, D_a \) be the subset \( \{(z_n)_{n \in \mathbb{Z}} \in X \mid z_0 = a\} \), \( D_{a^{-1}} \) be the subset \( \{(z_n)_{n \in \mathbb{Z}} \in X \mid z_0 = a\} \), \( \theta_a : D_{a^{-1}} \to D_a \) be the restriction of \( \tau^{-1} \) to \( D_{a^{-1}} \), and \( \theta_{a^{-1}} : D_a \to D_{a^{-1}} \) be the restriction of \( \tau \) to \( D_a \).

Let \( \mathbb{F}_a \) be the free group generated by \( a \), and let for every \( g \in \mathbb{F}_a \) written in the reduced form \( b_1 b_2 \cdots b_k \), where \( b_1, b_2, \ldots, b_k \in \{a \cup a^{-1}\} \), \( D_g \) be the subset of \( X \) defined by

\[
D_g = \theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_k}(X),
\]

and let \( \theta_g : D_g \to D_g \) be the map defined by

\[
\theta_g = \theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_k}.
\]

Then \( \theta_X = \{(D_g)_{g \in \mathbb{F}_a^+}, (\theta_g)_{g \in \mathbb{F}_a^+} \} \) is a partial action of \( \mathbb{F}_a \) on \( X \), and \((X, \theta_X, \mathbb{F}_a)\) is a discrete partial dynamical system.
Definition 3.5. Let \((X, \tau)\) be a two-sided symbolic dynamical system over the alphabet \(a\). Then we call the discrete partial dynamical system \((X, \theta_X, \mathbb{F}_a)\) constructed above the discrete partial dynamical system associated to \((X, \tau)\).

Lemma 3.6. Let \((X, \tau)\) be a two-sided symbolic dynamical system over the alphabet \(a\) and let \((X, \theta_X, \mathbb{F}_a)\) be the discrete partial dynamical system associated to \((X, \tau)\) as in Definition 3.5. Then the following holds for the partial action \(\theta_X = ((D_g)_{g \in G}, (\theta_g)_{g \in G})\):

1. if \(g \in G\) and \(D_g \neq \emptyset\), then either \(g \in a^*\) or \(g^{-1} \in a^*\),
2. if \(u \in a^*\), then we have that \(D_u = \{ z \in X \mid z_{|0,|u|} = u \}\), \(D_u^{-1} = \{ z \in X \mid z_{|1-|u|,0} = u \}\), \(\theta_u\) is the restriction of \(\tau^{-|u|}\) to \(D_u^{-1}\) and \(\theta_u^{-1}\) is the restriction of \(\tau^{|u|}\) to \(D_u\),
3. if \(u, v \in a^*, |u| = |v|\) and \(u \neq v\), then \(D_u \cap D_v = \emptyset\) and \(D_u^{-1} \cap D_v^{-1} = \emptyset\).

Proof. (2) can easily be proved by induction over the length of \(u\). (3) then follows from (2), and (1) follows from the definition of \(D_g\) and (2). \(\square\)

4. THE \(C^*\)-ALGEBRA ASSOCIATED TO A DISCRETE PARTIAL DYNAMICAL SYSTEM

The main object of this paper is to associate to every discrete partial dynamical system \((X, \theta, G)\) a \(C^*\)-algebra \(C^*(X, \theta, G)\) in such a way that the class of \(C^*\)-algebras obtained in this way in a natural way generalizes Cuntz-Krieger algebras (both for finite and infinite matrices), \(C^*\)-algebras associated to one-sided shift spaces and crossed products of two-sided shift spaces.

We want the \(C^*\)-algebra \(C^*(X, \theta, G)\) associated to a discrete partial dynamical system \((X, \theta, G)\) to have the property that there is a bijective correspondence between the representations of \(C^*(X, \theta, G)\) and certain representations of \((X, \theta, G)\).

Let us as a motivating example look at the Cuntz-Krieger algebra \(\mathcal{O}_A\) of an \(n \times n\)-matrix \(A = (A(i, j))_{i,j=1}^n\) with entries in \(\{0, 1\}\) and without any zero rows. The matrix \(A\) gives raise to a one-sided symbolic dynamical system, namely the topological Markov chain with transition matrix \(A\):

\[X_A^+ = \left\{(x_n)_{n \in \mathbb{N}_0} \in \{1, 2, \ldots, n\}^{\mathbb{N}_0} \mid \forall i \in \mathbb{N}_0: A(x_i, x_{i+1}) = 1\right\}.
\]

Thus we have by Definition 3.3 a partial action \(\theta_{X_A^+}\) of the free group \(\mathbb{F}_n\) generated by \(n\) generators on \(X_A^+\). It turns out the structure of the discrete partial dynamical system \((X_A^+, \theta_{X_A^+}, \mathbb{F}_n)\) is reflected in the structure of \(\mathcal{O}_A\) in the following way:

Let \((S_i)_{i=1}^n\) denote the generators of \(\mathcal{O}_A\). Then the map

\[i \mapsto S_i\]

extends to a partial representation \(u\) of \(\mathbb{F}_n\) on \(\mathcal{O}_A\) (see Appendix A for a short introduction to partial representations). Since \(u\) is a partial representation, \((u(g))_{g \in \mathbb{F}_n}\) is a family of partial isometries with commuting range projections, so \(C^*(\{u(g)u(g)^* \mid g \in \mathbb{F}_n\})\) is a unital abelian \(C^*\)-algebra, and
so the set of projections in this $C^*$-algebra is in a natural way a Boolean algebra (see Appendix B for a definition of this Boolean algebra).

The discrete partial dynamical system $(X_A^+, \theta_{X_A^+}, F_n)$ also gives rise to a Boolean algebra: namely the Boolean algebra generated by all the domains $\{D_g \mid g \in F_n\}$ of the partial action $\theta_{X_A^+}$ (cf. Appendix B). It turns out that the map

$$D_g \mapsto u(g)u(g)^*$$

extends to a Boolean homomorphism of the Boolean algebra generated by the domains $\{D_g \mid g \in F_n\}$ of the partial action $\theta_{X_A^+}$ to the Boolean algebra of projections in the $C^*$-subalgebra of $O_A$ generated by $\{u(g)u(g)^* \mid g \in F_n\}$. It is natural to view this Boolean homomorphism together with the partial representation $u$ of $F_n$ as a representation of the partial dynamical system $(X_A^+, \theta_{X_A^+}, F_n)$. We will in Section 7 see that this representation of the partial dynamical system $(X_A^+, \theta_{X_A^+}, F_n)$ on $O_A$ is universal and thus completely characterizes $O_A$.

Similar characterizations hold for Cuntz-Krieger algebras of infinite matrices, the $C^*$-algebras associated to one-sided shift spaces and the crossed products of two-sided shift spaces.

Thus we are lead to the following definition of $C^*(X, \theta, G)$:

**Definition 4.1.** Let $(X, \theta, G)$ be a discrete partial dynamical system. Then $C^*(X, \theta, G)$ is the universal $C^*$-algebra generated by a family $(s_g)_{g \in G}$ of elements satisfying:

(4.1a) $(s_g)_{g \in G}$ is a family of partial isometries with commuting range projections,

(4.1b) $s_e = 1$,

(4.1c) $s_g^{-1} = s_g^*$ for every $g \in G$,

(4.1d) $s_h s_i = s_h s_i^* s_h i$ for every $h, i \in G$,

(4.1e) the map $D_g \mapsto s_g s_g^*$ extends to a Boolean homomorphism from the Boolean algebra $B(X, \theta, G)$ generated by the domains $\{D_g \mid g \in G\}$ of the partial action $\theta$ to the Boolean algebra of projections in the unital abelian $C^*$-algebra $C^*(\{s_g s_g^* \mid g \in G\})$.

We will call the family $(s_g)_{g \in G}$ the *generators* of $C^*(X, \theta, G)$.

**Remark 4.2.** It is not immediately clear that a $C^*$-algebra with the properties mentioned in Definition 4.1 exists, but we will in Section 5 for every discrete partial dynamical system constructed such a $C^*$-algebra as a crossed product of a $C^*$-partial dynamical system.

**Remark 4.3.** It follows from Appendix A that condition (4.1a)-(4.1d) are equivalent to the following 3 conditions:

(4.2a) $s_e s_e^* = 1$,

(4.2b) $s_g s_g^* s_g s_g^* = s_g s_g^* s_g s_g^*$ for every $g \in G$,

(4.2c) $s_h s_i s_i^* s_h s_h^* = s_h s_i s_i^* s_h $ for every $h, i \in G$
Proof. If \( s_e = 1 \) for every \( e \in G \), then the following series of equalities hold:

\[(4.3a) \quad (s_g)_{g \in G} \text{ is a family of partial isometries with commuting range projections,} \]

\[(4.3b) \quad s_{g^{-1}} = s_g^* \text{ for every } g \in G, \]

\[(4.3c) \quad s_h s_i s_{i^{-1}} = s_{h i} s_{i^{-1}} \text{ for every } h, i \in G, \]

(\cite{[12]}).

**Definition 4.4.** Let \((X, \theta, G)\) be a discrete partial dynamical system, \((D_g)_{g \in G}\) the domains of \( \theta \) and \((s_g)_{g \in G}\) the generators of \( C^*(X, \theta, G) \). It is clear that the Boolean homomorphism from \( B(X, \theta, G) \) to the Boolean algebra of projections in \( C^* \{(s_g s_g^* \in C^*(X, \theta, G) \mid g \in G)\} \) which extends the map \( D_g \mapsto s_g s_g^* \) is unique. We will denote it by \( \phi_{(X, \theta, G)} \).

We will in Corollary \cite{[2]} see that \( \phi_{(X, \theta, G)} \) is always injective, and thus that \( C^*(X, \theta, G) \neq 0 \) (unless \( X = \emptyset \)).

The following lemma describe how the structure of the discrete partial dynamical system \((X, \theta, G)\) is reflected in the structure of the \( C^*-\)algebra \( C^*(X, \theta, G) \).

**Lemma 4.5.** If \((X, \theta, G)\) is a discrete partial dynamical system, then we have that

\[
\phi_{(X, \theta, G)}(\theta_g(A)) = s_g \phi_{(X, \theta, G)}(A) s_g^*
\]

for every \( A \in B(X, \theta, G) \) and every \( g \in G \).

**Proof.** If \( \phi_{(X, \theta, G)}(\theta_g(A)) = s_g \phi_{(X, \theta, G)}(A) s_g^* \) and \( \phi_{(X, \theta, G)}(\theta_g(B)) = s_g \phi_{(X, \theta, G)}(B) s_g^* \), then the following series of equalities hold:

\[
\phi_{(X, \theta, G)}(\theta_g(A \cap B)) = \phi_{(X, \theta, G)}(\theta_g(A) \cap \theta_g(B))
\]

\[
= \phi_{(X, \theta, G)}(\theta_g(A)) \phi_{(X, \theta, G)}(\theta_g(B))
\]

\[
= s_g \phi_{(X, \theta, G)}(A) s_g^* s_g \phi_{(X, \theta, G)}(B) s_g^*
\]

\[
= s_g s_g^* s_g \phi_{(X, \theta, G)}(A) \phi_{(X, \theta, G)}(B) s_g^*
\]

\[
= s_g \phi_{(X, \theta, G)}(A \cap B) s_g^*,
\]

and so does the next:

\[
\phi_{(X, \theta, G)}(\theta_g(X \setminus A)) = \phi_{(X, \theta, G)}(D_g \setminus \theta_g(A))
\]

\[
= s_g s_g^* - s_g \phi_{(X, \theta, G)}(A) s_g^*
\]

\[
= s_g (1 - \phi_{(X, \theta, G)}(A)) s_g^*
\]

\[
= s_g \phi_{(X, \theta, G)}(X \setminus A) s_g^*.
\]

So it is enough to show that \( \phi_{(X, \theta, G)}(\theta_g(D_h)) = s_g s_h s_h^* s_g^* \) for every \( g, h \in G \).

So let \( g, h \in G \). Then we have that

\[
\phi_{(X, \theta, G)}(\theta_g(D_h)) = \phi_{(X, \theta, G)}(D_g \cap D_{gh})
\]

\[
= s_g s_g^* s_{gh} s_{gh}^*
\]

\[
= s_g s_g^* s_{gh} s_{gh}^* s_g s_g^*
\]

\[
= s_g s_h s_h^* s_g^*,
\]

which proves the lemma. \( \square \)
The next lemma, which we state for further reference, is an easy but useful consequent of the definition of the $C^*$-algebra of a discrete partial dynamical system.

**Lemma 4.6.** Let $(X, \theta, G)$ be a discrete partial dynamical system and let $g, h, i \in G$. If $D_g = \emptyset$, then $s_g = 0$, and if $D_h \cap D_i = \emptyset$, then $s_h^* s_i = 0$.

**Proof.** If $D_g = \emptyset$, then $s_g = s_g s_g^* s_g = \phi_{(X, \theta, G)}(D_g) s_g = 0$, and if $D_h \cap D_i = \emptyset$, then we have that

$$s_h^* s_i = s_h s_h^* s_i,$$

$$= s_h^* \phi_{(X, \theta, G)}(D_h) \phi_{(X, \theta, G)}(D_i) s_i,$$

$$= s_h^* \phi_{(X, \theta, G)}(D_h \cap D_i) s_i,$$

$$= 0.$$

\[\square\]

**Example 4.7.** If $\theta$ is an action of a discrete group $G$ on a set $X \neq \emptyset$ and $(X, \theta, G)$ is the discrete partial dynamical system defined in Example 3.2 then all the domains $(D_g)_{g \in G}$ of the partial action $\theta$ are equal to $X$. Thus the Boolean algebra generated by these domains only consists of $X$ and $\emptyset$, and so $C^*(X, \theta, G)$ is just the group $C^*$-algebra of $G$.

5. A Construction of $C^*(X, \theta, G)$

We will in this section for every discrete partial dynamical system $(X, \theta, G)$ construct the $C^*$-algebra $C^*(X, \theta, G)$ (cf. Definition 4.1) as a crossed product of a $C^*$-partial dynamical systems (see Appendix C for a short introduction to crossed product of $C^*$-partial dynamical systems, cf. also [12, 22, 25]).

We will first from the discrete partial dynamical system $(X, \theta, G)$ construct the $C^*$-partial dynamical systems $(\hat{X}, G, \hat{\theta})$ which $C^*(X, \theta, G)$ is a crossed product of.

Let $(D_g)_{g \in G}$ denote the domains and $(\theta_g)_{g \in G}$ the partial one-to-one maps of $\theta$. Recall that $B(\mathfrak{B}(X, \theta, G))$ is the Boolean algebra generated by the domains $\{D_g \mid g \in G\}$ of the partial action $\theta$. Notice that $\theta_h(D_i) = D_h \cap D_{hi} \in B(\mathfrak{B}(X, \theta, G))$ for every $h, i \in G$, so $\theta_g(A) \in B(\mathfrak{B}(X, \theta, G))$ for every $A \in B(\mathfrak{B}(X, \theta, G))$ and every $g \in G$.

Let $\hat{X}$ be the dual of $B(\mathfrak{B}(X, \theta, G))$: i.e., $\hat{X}$ is the closed subset

$$\left\{ \phi \in \{0, 1\}^{B(\mathfrak{B}(X, \theta, G))} \mid \phi \text{ is a Boolean homomorphism} \right\}$$

of the Cantor space $\{0, 1\}^{B(\mathfrak{B}(X, \theta, G))}$ endowed with the product topology of the discrete topology on $\{0, 1\}$. Then $\hat{X}$ is a totally disconnected compact Hausdorff space (a Boolean space, cf. [13, §18]).

For each $A \in B(\mathfrak{B}(X, \theta, G))$, let $\hat{A} = \{\phi \in \hat{X} \mid \phi(A) = 1\}$, and notice that $\hat{A}$ is a clopen subset of $\hat{X}$. We then have that the map $A \mapsto \hat{A}$ is a Boolean isomorphism between $\mathfrak{B}(X, \theta, G)$ and the Boolean algebra of clopen subsets of $\hat{X}$ (cf. [13, §18]).

**Lemma 5.1.** The system $\{\hat{D}_t \mid t \in G\}$ separates points in $\hat{X}$. 
Proof. Let \( \phi_1, \phi_2 \in \hat{X} \) and let \( \mathcal{A} \) be the subset of \( \mathcal{B}(X, \theta, G) \) defined by
\[
\mathcal{A} = \{ A \in \mathcal{B}(X, \theta, G) \mid \phi_1(A) = \phi_2(A) \}.
\]
Then \( \mathcal{A} \) is a Boolean subalgebra of \( \mathcal{B}(X, \theta, G) \). Assume that the following equivalence
\[
\phi_1 \in \bar{D}_g \iff \phi_2 \in \bar{D}_g
\]
holds for every \( g \in G \). That means that \( \bar{D}_g \in \mathcal{A} \) for every \( g \in G \) and thus that \( \mathcal{A} = \mathcal{B}(X, \theta, G) \). But then \( \phi_1 \) and \( \phi_2 \) must be equal. \( \square \)

For each \( g \in G \), let \( \hat{\theta}_g \) be the map given by
\[
\hat{\theta}_g(\phi)(A) = \phi(\theta_g^{-1}(A))
\]
for \( A \in \mathcal{B}(X, \theta, G) \) and \( \phi \in \bar{D}_{g^{-1}} \). It is easy to check that \( \hat{\theta}_g \) is a homeomorphism from \( \bar{D}_{g^{-1}} \) to \( \bar{D}_g \) with \( \hat{\theta}_{g^{-1}} \) as its inverse (cf. Appendix C).

Let \( \bar{X} = C(\hat{X}) \) and let for each \( g \in G \), \( \bar{D}_g \) be the subset of \( \bar{X} \) defined by
\[
\bar{D}_g = \left\{ f \in \bar{X} \mid f|_{\bar{X} \setminus \bar{D}_g} = 0 \right\},
\]
and let \( \hat{\theta}_g : \bar{D}_{g^{-1}} \to \bar{D}_g \) be defined by
\[
\hat{\theta}_g(f)(\phi) = \begin{cases} f(\hat{\theta}_{g^{-1}}(\phi)) & \text{if } \phi \in \bar{D}_g, \\ 0 & \text{if } \phi \in \bar{X} \setminus \bar{D}_g, \end{cases}
\]
for \( f \in \bar{D}_{g^{-1}} \) and \( \phi \in \hat{X} \). Then \( \hat{\theta} = (\bar{D}_g)_{g \in G}, (\hat{\theta}_g)_{g \in G} \) is a partial action of \( G \) on the \( C^* \)-algebra \( \hat{X} \) (cf. Appendix C). Thus \( (\hat{X}, G, \hat{\theta}) \) is a \( C^* \)-partial dynamical system.

**Theorem 5.2.** Let \( (X, \theta, G) \) be a discrete partial dynamical system and let \( \hat{X} \) and \( \hat{\theta} \) be as defined above. Then the partial crossed product \( \hat{X} \rtimes \hat{\theta} \) of the \( C^* \)-partial dynamical system \( (X, G, \theta) \) is generated by a family of elements \( (s_g)_{g \in G} \) satisfying condition \( 4.1a-4.1c \) of Definition 4.1 and if \( \mathcal{A} \) is another \( C^* \)-algebra with a family of elements \( (S_g)_{g \in G} \) satisfying condition \( 4.1a-4.1c \), then there exists a \( * \)-homomorphism from \( \hat{X} \rtimes \hat{\theta} \) to \( \mathcal{A} \) which maps \( s_g \) to \( S_g \) for every \( g \in G \).

**Proof.** Let for every \( g \in G \), \( s_g = \delta_g \), where \( \{ \delta_g \}_{g \in G} \) is a described in Appendix C. Since \( (\pi, u) \mapsto \pi \times u \) is a bijective correspondence between covariant representations of \( (X, G, \hat{\theta}) \) and non-degenerated representations of \( \hat{X} \rtimes \hat{\theta} \), there exists a covariant representation \( (\pi, u) \) of \( (X, G, \theta) \) on a Hilbert space \( H \) such that \( \pi \times u \) is a faithful non-degenerate representation of \( \hat{X} \rtimes \hat{\theta} \). We then have that \( u \) is a partial representation of \( G \) and so the family \( (u(g))_{g \in G} \) satisfies condition \( 4.1a-4.1c \) of Definition 4.1.

Denote for every subset \( B \) of \( \hat{X} \), the subspace
\[
\text{span}\{ \pi(T)\xi \mid T \in B, \ \xi \in H \}
\]
of \( H \) by \( [B] \) and the projection of \( H \) onto \( [B] \) by \( \text{proj}([B]) \).

Since the map sending an element \( A \) of \( \mathcal{B}(X, \theta, G) \) to \( \hat{A} \), the map sending a clopen subset \( V \) of \( \hat{X} \) to \( \{ f \in \hat{X} \mid f|_{\hat{X} \setminus V} = 0 \} \), and the map sending an ideal \( I \) of \( \hat{X} \) to \( \text{proj}([I]) \) all are Boolean homomorphism, so is the map...
sending an element \( A \) of \( \mathcal{B}(X, \theta, G) \) to \( \text{proj}([\{ f \in \hat{X} \mid f|_{\hat{X}\setminus\hat{A}} = 0 \}]) \), and since we have that
\[
\text{proj}([\{ f \in \hat{X} \mid f|_{\hat{X}\setminus\hat{D}_g} = 0 \}]) = \text{proj}(\hat{D}_g) = u(g)u(g)^*
\]
for every \( g \in G \), \( (u(g))_{g \in G} \) satisfies (4.1e) of Definition 4.1. Since \( \pi \times u \) is faithful and \( \pi \times u(s_g) = u(g) \) for every \( g \in G \), \( (s_g)_{g \in G} \) satisfies (4.1a)–(4.1d) of Definition 4.1.

Remember that \( \hat{X} \rtimes_{\hat{\theta}} G \) is generated by \( \hat{X} \) and \( (s_g)_{g \in G} \). It follows from Lemma 5.1 and the Stone-Weierstrass Theorem that the equality
\[
\text{span}\{1_{\hat{D}_g} \mid g \in G\} = \hat{X}
\]
holds, and since \( s_gs_g^* = 1_{\hat{D}_g} \) for every \( g \in G \), this shows that \( \hat{X} \rtimes_{\hat{\theta}} G \) is generated by \( (s_g)_{g \in G} \).

Now let \( \mathcal{A} \) be another \( C^* \)-algebra with a family \( (S_g)_{g \in G} \) of elements which satisfies condition (4.1a)–(4.1d). Let \( \psi \) be a non-degenerate faithful representation of \( \mathcal{A} \) on a Hilbert space \( H \). Since \( S_e \) is the unit of \( \mathcal{A} \), \( \psi(S_e)\xi = \xi \) for every \( \xi \in H \).

Let for every \( g \in G \), \( U(g) = \psi(S_g) \). Then \( U \) is a partial representation of \( G \) on \( H \). Since \( \text{span}\{1_{\hat{D}_g} \mid g \in G\} = \hat{X} \), since the map \( A \mapsto \hat{A} \) is a Boolean isomorphism between \( \mathcal{B}(X, \theta, G) \) and the Boolean algebra of clopen subsets of \( \hat{X} \), and since the map \( D_g \mapsto \tau_gS_g^* \) extends to a Boolean homomorphism from \( \mathcal{B}(X, \theta, G) \) to the set of projections in \( C^*(\{S_gS_g^* \mid g \in G\}) \), there exists by Lemma 3.1 a *-homomorphism from \( \hat{X} \) to \( \mathcal{A} \) which maps \( 1_{\hat{D}_g} \) to \( S_gS_g^* \) for every \( g \in G \). Let us denote the composition of this *-homomorphism with \( \psi \) by \( \eta \). Then \( \eta \) is a representation of \( \hat{X} \) on \( H \), and since \( \eta(1)\xi = \psi(S_e)\xi = \xi \) for every \( \xi \in H \), \( \eta \) is non-degenerate.

If \( g \in G \), then we have that
\[
U(g)U(g)^* = \psi(S_gS_g^*) = \eta(1_{\hat{D}_g}) = \text{proj}\left(\text{span}\{\eta(1_{\hat{D}_g})\xi \mid \xi \in H\}\right) = \text{proj}(\hat{D}_g).
\]

Let \( i, h \in G \). It then follows from (3.1b) that \( \hat{\theta}_h(1_{\hat{D}_{h^{-1}\cap\hat{D}_i}}) = 1_{\hat{D}_h\cap\hat{D}_i} \). Thus we have that
\[
\eta\left(\hat{\theta}_h(1_{\hat{D}_{h^{-1}\cap\hat{D}_i}})\right) = \eta(1_{\hat{D}_h\cap\hat{D}_i}) = U(h)U(h)^*U(hi)U(hi)^* = U(h)U(i)U(hi)^*
\]
\[
= U(h)U(i)U(i^{-1})U(i^{-1})^*U(i^{-1}h^{-1}) = U(h)U(h)^*U(h)U(i)U(i^{-1})U(h^{-1}) = U(h)\eta(1_{\hat{D}_{h^{-1}\cap\hat{D}_i}})U(h^{-1}),
\]
where the third equality follows from (4.1d), the forth from (4.1a) and (4.1c), and the fifth from (4.1a) and (4.1c). It follows from Lemma 5.1 and the
Stone-Weierstrass Theorem that \( \overline{\text{span}}\{1_{D_{h^{-1}} \cap D_i} \mid i \in G\} = \tilde{D}_{h^{-1}} \), so the above computation shows that the equality

\[
\eta(\tilde{\theta}_h(f)) = U(h)(\eta(f))U(h^{-1})
\]

holds for every \( f \in \tilde{D}_{h^{-1}} \).

Thus \( (\eta, U) \) is a covariant representation of \( (\tilde{X}, G, \tilde{\theta}) \), and so there exists a non-degenerate representation of \( \tilde{X} \times_G G \) on \( H \) which maps \( s_g \) to \( \psi(S_g) \) for every \( g \in G \), and thus a \( * \)-homomorphism from \( \tilde{X} \times_G G \) to \( \hat{A} \) which maps \( s_g \) to \( S_g \) for every \( g \in G \).

**Remark 5.3.** Let \( I \) be an arbitrary index set and let \( A = A(i, j)_{i, j \in I} \) be a matrix with entries in \( \{0, 1\} \) and having no zero rows. In [11] Exel and Lace associated to \( A \) a \( C^* \)-algebra \( \overline{O}_A \) (cf. Section 7.3). We will in Theorem 7.5 show that the unitization \( \overline{O}_A \) of \( O_A \) is isomorphic to \( C^*(\hat{X}^+_A, \theta_{\hat{X}^+_A}; \mathbb{F}_I) \) for a certain discrete partial dynamical system \( (\hat{X}^+_A, \theta_{\hat{X}^+_A}; \mathbb{F}_I) \).

Exel and Lace constructed in [11] a partial action \( (\Delta^A_g)_{g \in \mathbb{F}_I}, (h^A_g)_{g \in \mathbb{F}_I} \) of \( \mathbb{F}_I \) on a compact topological space \( \Omega_A \) such that \( \overline{O}_A \) is isomorphic to the crossed product of this partial action.

If we let \( (D_g)_{g \in G} \) and \( (\theta_g)_{g \in G} \) denote the domains and partial one-to-one maps of \( \theta_{\hat{X}^+_A} \), \( \hat{D}_g \) and \( \theta_g \) be as above, then one can show that the map

\[
\phi \mapsto \{ g \in \mathbb{F}_I \mid \phi(D_g) = 1 \}
\]

is a homeomorphism from \( \hat{X}^+_A \) to \( \Omega_A \) which for every \( g \in \mathbb{F}_I \) maps \( \hat{D}_g \) to \( \Delta^A_g \) and intertwines \( \theta_g \) and \( h^A_g \).

This fact lays the foundation for an alternative proof of Theorem 7.5.

6. A representation of \( C^*(X, \theta, G) \)

Let \( (X, \theta, G) \) be a discrete partial dynamical system and let \( (D_g)_{g \in G} \) denote the domains of \( \theta \) and \( (s_g)_{g \in G} \) the generators of \( C^*(X, \theta, G) \). If \( \pi \) is a representation of \( C^*(X, \theta, G) \), then \( \pi \) will induce a Boolean homomorphism from the Boolean algebra of projections in the unital abelian \( C^* \)-algebra \( C^*(\{s_g s^*_g \in C^*(X, \theta, G) \mid g \in G\}) \) to the Boolean algebra of projections in the unit abelian \( C^* \)-algebra \( C^*(\{\pi(s_g s^*_g) \mid g \in G\}) \) which maps \( s_g s^*_g \) to \( \pi(s_g s^*_g) \) for every \( g \in G \), and by composing this Boolean homomorphism with \( \phi_{(X, \theta, G)} \), we get a Boolean homomorphism from \( \mathcal{B}(X, \theta, G) \) to the Boolean algebra of projections in the unital abelian \( C^* \)-algebra \( C^*(\{\pi(s_g s^*_g) \mid g \in G\}) \) which maps \( D_g \) to \( \pi(s_g s^*_g) \) for every \( g \in G \).

We will in this section for every discrete partial dynamical system \( (X, \theta, G) \) construct a representation \( \pi \) of \( C^*(X, \theta, G) \) such that the Boolean homomorphism mentioned above is injective. As a corollary of this, we see that the Boolean homomorphism \( \phi_{(X, \theta, G)} \) is injective and thus that \( C^*(X, \theta, G) \neq 0 \) (unless \( X = \emptyset \)).

Let \( (X, \theta, G) \) be a discrete partial dynamical system and let \( (D_g)_{g \in G} \) and \( (\theta_g)_{g \in G} \) denote the domains and partial one-to-one maps of \( \theta \). Let \( (e_x)_{x \in X} \) be an orthonormal basis for the Hilbert space \( l_2(X) \), and define for each
$g \in G$ an operator $S_g$ by letting
\[ S_g \left( \sum_{x \in X} \lambda_x e_x \right) = \sum_{x \in D_g} \lambda_{\theta_g^{-1}(x)} e_x \]
for every $\sum_{x \in X} \lambda_x e_x \in l_2(X)$.

It is straightforward to check that these operators are partial isometries, that $S_{e} = 1$ and that for every $h,i \in G$, $S_h^* S_i = S_h S_h^* S_i$ and $S_h S_h^* = \text{proj}(\overline{\text{span}}\{e_x \mid x \in D_h\})$, where $\text{proj}(\overline{\text{span}}\{e_x \mid x \in D_h\})$ is the orthogonal projection of $l_2(X)$ onto $\overline{\text{span}}\{e_x \mid x \in D_h\}$. Since the map
\[ A \mapsto \text{proj}(\overline{\text{span}}\{e_x \mid x \in A\}) \]
is a Boolean homomorphism from $B(X, \theta, G)$ to the Boolean algebra of projections in the unital abelian $C^*$-algebra $C^*(\{S_g S_g^* \mid g \in G\})$, the family $(S_g)_{g \in G}$ of operators satisfies condition (4.1a)–(4.1e) of Definition 4.1.

Thus we have:

**Proposition 6.1.** Let $(X, \theta, G)$ be a discrete partial dynamical system, let $(s_g)_{g \in G}$ denote the generators of $C^*(X, \theta, G)$ and $(S_g)_{g \in G}$ the operators defined by (6.1). Then there is a $*$-homomorphism from $C^*(X, \theta, G)$ to the $C^*$-algebra of bounded operators on the Hilbert space $l_2(X)$, sending $s_g$ to $S_g$ for every $g \in G$.

**Corollary 6.2.** The Boolean homomorphism $\phi_{(X, \theta, G)}$ (cf. Definition 4.4) is for every discrete partial dynamical system $(X, \theta, G)$ injective.

**Proof.** Since the Boolean homomorphism
\[ A \mapsto \text{proj}(\overline{\text{span}}\{e_x \mid x \in A\}) \]
from $B(X, \theta, G)$ to the set of projections in $C^*(\{S_g S_g^* \mid g \in G\})$ is injective, so is $\phi_{(X, \theta, G)}$.

\[ \square \]

### 7. $C^*$-Algebras Associated to Symbolic Dynamical Systems

We will now show that the class of $C^*$-algebras associated to discrete partial dynamical systems generalizes Cuntz-Krieger algebras (both for finite and infinite matrices), $C^*$-algebras associated to one-sided shift spaces and crossed products of two-sided shift spaces.

We will do that by regarding one-sided and two-sided shift spaces and topological Markov chains as symbolic dynamical systems and thus associated to them the discrete partial dynamical system of Definition 3.3 and 3.5 and then show that the $C^*$-algebras of these discrete partial dynamical systems are isomorphic to the $C^*$-algebra associated to the one-sided shift space, the crossed product of the two-sided shift space, and the unitization of the Cuntz-Krieger algebra of the transition matrix of the Markov chain, respectively.

Before we do that let us briefly look at the general structure of $C^*(X, \theta, G)$ when $(X, \theta, G)$ is the discrete partial dynamical system associated to a one-or two-sided symbolic dynamical system as in Definition 3.3 and 3.5.

**Lemma 7.1.** Let $(X^+, \sigma)$ be a one-sided symbolic dynamical system over the alphabet $a$ and let $(X^+, \theta_{X^+}, \mathcal{F}_a)$ be the discrete partial dynamical system
associated to \((X^+, \sigma)\) as in Definition 3.4. Then the following holds for the generators \((s_g)_{g \in F_a}\) of \(C^*(X^+, \theta_{X^+}, F_a)\):
(1) \(s_{b_1} s_{b_2} \cdots s_{b_k} = s_g\) for \(g \in F_a\) written in reduced form, \(b_1 b_2 \cdots b_k\),
(2) if \(s_g \neq 0\), then there exist \(u, v \in \alpha^*\) such that \(g = uv^{-1}\),
(3) \(s^*_k s_v = 0\) if \(u, v \in \alpha^*, |u| = |v|\) and \(u \neq v\).

Proof. If \(b \in a \cup a^{-1}\) and \(g \in F_a\) written in reduced form does not begin with \(b\), then \(D_{bg} \subseteq D_b\), so we have that
\[
s_{b_1} s_{b_2} \cdots s_{b_k} = s_g\quad \text{for } g \in F_a \text{ written in reduced form } b_1 b_2 \cdots b_k.
\]
(2) and (3) easily follow from Lemma 3.6 and 4.6 \(\square\)

Lemma 7.2. Let \((X, \tau)\) be a two-sided symbolic dynamical system over the alphabet \(a\) and let \((X, \theta_X, F_a)\) be the discrete partial dynamical system associated to \((X, \tau)\) as in Definition 3.5. Then the following holds for the generators \((s_g)_{g \in F_a}\) of \(C^*(X, \theta_X, F_a)\):
(1) \(s_{b_1} s_{b_2} \cdots s_{b_k} = s_g\) for \(g \in F_a\) written in the reduced form \(b_1 b_2 \cdots b_k\),
(2) if \(s_g \neq 0\), then either \(g \in a^*\) or \(g^{-1} \in a^*\),
(3) \(s^*_k s_i = s_h s^*_i = 0\) if \(h, i \in F_a\), \([h] = [i]\) and \(h \neq i\).

Proof. (1) can be proved in exactly the same way (1) was proved in Lemma 7.1 (2) easily follows from Lemma 3.6 and 4.6 and (3) follows from (2) and Lemma 3.6 and 4.6 \(\square\)

7.1. Crossed products of two-sided shift spaces. Let \((X, \tau)\) be a two-sided shift space over the finite alphabet \(a\) (cf. [17] and [19]). That is:
\[
\tau : a^Z \to a^Z
\]
is the map defined by
\[
(\tau((z_n)_{n \in Z}))_m = z_{m+1}
\]
for every \((z_n)_{n \in Z} \in a^Z\) and every \(m \in Z\), and \(X\) is a closed (in the product topology of the discrete topology of \(a\)) subset of \(a^Z\) such that \(\tau(X) = X\).

Let \(\tau^*\) be the automorphism on \(C(X)\) defined by \(f \mapsto f \circ \tau\) and let \(C(X) \rtimes_{\tau^*} Z\) be the full crossed product of the \(C^*\)-dynamical system \(C(X, \tau^*, Z)\) (cf. [23] 7.6.5]). Thus \(C(X) \rtimes_{\tau^*} Z\) is the universal \(C^*\)-algebra generated by a copy of \(C(X)\) and an unitary operator \(U\) which satisfies that \(UfU^* = f \circ \tau\) for every \(f \in C(X)\).

Since \((X, \tau)\) is a two-sided symbolic dynamical system, we can associate to it the discrete partial dynamical system \((X, \theta_X, F_a)\) of Definition 3.5. We then have the following theorem:

Theorem 7.3. Let \((X, \tau)\) be a two-sided shift space and let \((X, \theta_X, F_a)\) be the discrete partial dynamical system associated to \((X, \tau)\) as done in Definition 3.5. Then \(C^*(X, \theta_X, F_a)\) is isomorphic to the crossed product \(C(X) \rtimes_{\tau^*} Z\).

More precisely: if \((D_g)_{g \in F_a}\) denotes the domains of \(\theta_X\), \((s_g)_{g \in F_a}\) denotes the generators of \(C^*(X, \theta_X, F_a)\) and \(U\) is as above, then we have that
\[
C(X) = \overline{\text{span}}\{1_{D_g} \mid g \in F_a\},
\]
and there exists a \(*\)-isomorphism from \(C(X) \times_{\tau^*} \mathbb{Z}\) to \(C^*(X, \theta_X, F_a)\) which maps \(1_{D_g}\) to \(s_g s_g^*\) for every \(g \in F_a\) and \(U\) to \(\sum_{a \in a} s_a\).

Proof. We will first show that the Boolean algebra \(\mathcal{B}(X, \theta_X, \mathbb{F}_a)\) generated by \(\{D_g \mid g \in F_a\}\) is the Boolean algebra of clopen subsets of \(X\). It easily follows from Lemma 3.6 that every set in \(\mathcal{B}(X, \theta_X, \mathbb{F}_a)\) is clopen. In the other direction, we have by Lemma 3.6 that
\[
\{ z \in X \mid z[\|u\|, \|v\|] = uv \} = D_u^{-1} \cap D_v \in \mathcal{B}(X, \theta_X, \mathbb{F}_a)
\]
for \(u, v \in \mathfrak{a}^*\), and since the system consisting of sets of this form is a basis for the topology of \(X\), every clopen set is a finite union of sets of this form and thus belongs to \(\mathcal{B}(X, \theta_X, \mathbb{F}_a)\).

So it follows from the Stone-Weierstrass theorem that
\[
C(X) = \overline{\text{span}} \{ 1_{D_g} \mid g \in F_a \}.
\]
Since \(\phi_{(X, \theta_X, \mathbb{F}_a)}\) (cf. Definition 4.4) is a Boolean homomorphism from \(\mathcal{B}(X, \theta, \mathbb{F}_a)\) to the Boolean algebra of projections in the unital abelian \(C^*\)-algebra
\[
C^* \left( \{ s_g s_g^* \in C^*(X, \theta_X, \mathbb{F}_a) \mid g \in F_a \} \right)
\]
which maps \(D_g\) to \(s_g s_g^*\), it follows from Lemma 5.1 that there exists a \(*\)-homomorphism \(\eta\) from \(C(X)\) to \(C^*(X, \theta_X, \mathbb{F}_a)\) which maps \(1_{D_g}\) to \(s_g s_g^*\) for every \(g \in F_a\). Let \(u\) be the element \(\sum_{a \in a} s_a\) in \(C^*(X, \theta_X, \mathbb{F}_a)\). Then we have that
\[
u \eta_X(1_{D_g}) u^* = \left( \sum_{a \in a} s_a \right) \left( s_g s_g^* \right) \left( \sum_{a' \in a} s_{a'}^* \right)
\]
\[
= \sum_{a \in a, a' \in a} s_a s_g s_g^* s_{a'} s_{a'}^* s_{a'}^* s_{a'}
\]
\[
= \sum_{a \in a} s_a s_g s_g^* s_a
\]
\[
= \sum_{a \in a} s_a s_g s_g^* s_{a g} s_{a g}^* s_a
\]
\[
= \sum_{a \in a} \eta_X(1_{D_a \cap D_{a g}})
\]
\[
= \sum_{a \in a} \eta_X(1_{\theta_a(D_g)})
\]
\[
= \eta_X(1_{\bigcup_{a \in a} \theta_a(D_g)})
\]
\[
= \eta_X(1_{\tau^{-1}(D_g) \circ \tau})
\]
for every \(g \in F_a\), where the second equality follows from (4.1a) and (4.1e), the third from (4.1a) and Lemma 7.2(3), the forth from (4.1a) and the sixth from (3.1b). Since \(C(X) = \overline{\text{span}} \{ 1_{D_g} \mid g \in F_a \} \), this shows that
\[
u \eta_X(f) u^* = \eta_X(f \circ \tau)
\]
for every \(f \in C(X)\).
Thus it follows from the universal property of \( C(X) \times_{\tau^*} Z \) that there exists a \( * \)-homomorphism \( \tilde{\eta}_x \) from \( C(X) \times_{\tau^*} Z \) to \( C^*(X, \theta_X,F_a) \) which is equal to \( \eta_x \) on \( C(X) \) and sends \( U \) to \( u = \sum_{a \in a} S_a \).

Let us now look at \( C(X) \times_{\tau^*} Z \). Let for every \( a \in a \), \( S_a \) be the element \( 1_{D_a}U \) of \( C(X) \times_{\tau^*} Z \), \( S_{a^{-1}} = S_a^* \), and let \( S_{c} = 1 \) and \( S_g = S_{b_1}S_{b_2} \cdots S_{b_k} \), where \( g \in F_a \) is written in the reduced form \( b_1b_2 \cdots b_k \). Then \((S_g)_{g\in F_a}\) is a family of elements from \( C(X) \times_{\tau^*} Z \) which clearly satisfies (1.1b) and (1.1c) of Definition 1.1. We will now show that \((S_g)_{g\in F_a}\) also satisfies (1.1a), (1.1d) and (1.1e) of Definition 1.1.

Remember (cf. Section 2) that \([ \cdot ]\) is the unique homomorphism from \( F_a \) to \( Z \) such that \([a] = 1\) for every \( a \in a \). According to Lemma 3.6, the map \( \theta_g \) is for every \( g \in F_a \) equal to the restriction of \( \tau^{-[g]} \) to \( D_{g^{-1}} \), so we have that

\[
D_h \cap \tau^{-[h]}(D_i) = \theta_h(D_i) = D_{hi} \cap D_h
\]

for all \( h, i \in F_a \).

If \( a \in a \) and \( g \in F_a \) written in reduced form does not begin with \( a^{-1} \), then \( D_{ag} \subseteq D_a \) and \( D_a \cap \tau^{-1}(D_g) = D_{ag} \cap D_a = D_{ag} \) and so we have that

\[
S_a1_{D_g}U^g = 1_{D_a}U1_{D_g}U^g
\]

\[= 1_{D_a}U1_{D_g}U^g + 1 \]

\[= 1_{D_a}U^{g+1} \]

\[= 1_{D_{ag}}U^{g+1} \]

and if \( g \) written in reduced form does not begin with an \( a \), then \( D_{a^{-1}g} \subseteq D_{a^{-1}} \) and \( \tau(D_a \cap D_g) = \theta_a^{-1}(D_g) = D_{a^{-1}} \cap D_{a^{-1}g} = D_{a^{-1}g} \), so we have that

\[
S_a^{-1}1_{D_g}U^g = U^*1_{D_a}1_{D_g}U^g
\]

\[= 1_{\tau(D_a \cap D_g)}U^{g-1} \]

\[= 1_{D_{a^{-1}}U^{g-1}} \]

This shows that \( S_g = 1_{D_g}U^g \) and thus that \( S_gS_g^* = 1_{D_g} \) for every \( g \in F_a \). Hence \((S_g)_{g\in F_a}\) satisfies (1.1a) and (1.1d) of Definition 1.1. If \( h, i \in F_a \), then we have that

\[
S_hS_i = 1_{D_h}U^{h[i]}1_{D_i}U^{i[i]}
\]

\[= 1_{D_h}U^{h[i]}1_{D_i}U^{-h[i]}U^{h[i]} \]

\[= 1_{D_h \cap \tau^{-h[i]}(D_i)}U^{h[i]} \]

\[= 1_{D_{hi} \cap \tau_{D_h}U^{h[i]}}, \]

\[= 1_{D_{hi}}U^{h[i]} \]

\[= S_hS_iS_{hi}, \]

which shows that \((S_g)_{g\in F_a}\) also satisfies (1.1e) of Definition 1.1.

Thus it follows from the universal property of \( C^*(X, \theta_X, F_a) \) that there is a \( * \)-homomorphism \( \psi \) from \( C^*(X, \theta_X, F_a) \) to \( C(X) \times_{\tau^*} Z \) such that \( \psi(s_g) = S_g = 1_{D_g}U^g \) for every \( g \in F_a \).
We have that
\[ \psi(\tilde{\eta}_X(U)) = \psi \left( \sum_{a \in \mathcal{A}} s_a \right) = \sum_{a \in \mathcal{A}} 1_{D_a} U = U, \]
and that
\[ \psi(\tilde{\eta}_X(1_{D_a})) = \psi(s_g s_g^* s_h) = 1_{D_a} U^{[g]} U^{-[g]} 1_{D_g} = 1_{D_g} \]
for every \( g \in \mathcal{F}_a \), and since \( C(X) \rtimes \mathbb{Z} \) is generated by \( U \) and \( \{ 1_{D_g} \mid g \in \mathcal{F}_a \} \), this shows that \( \psi \circ \tilde{\eta}_X = \text{Id}_{C(X) \rtimes \mathbb{Z}} \).

We also have that
\[ \tilde{\eta}_X(\psi(s_g)) = \tilde{\eta}_X(1_{D_g} U^{[g]}) = s_g s_g^* \left( \sum_{a \in \mathcal{A}} s_a \right)^{[g]} \]
for every \( g \in \mathcal{F}_a \). It follows from Lemma 7.2 that we have that
\[ \left( \sum_{a \in \mathcal{A}} s_a \right)^{[g]} = \sum_{h \in \mathcal{F}_a} s_h, \]
and that
\[ s_g s_g^* s_h = \begin{cases} s_g & \text{if } g = h, \\ 0 & \text{if } g \neq h, \end{cases} \]
for every \( h \in \mathcal{F}_a \) with \([h] = [g]\). Thus it follows that
\[ \tilde{\eta}_X(\psi(s_g)) = s_g s_g^* \left( \sum_{a \in \mathcal{A}} s_a \right)^{[g]} = s_g \]
for every \( g \in \mathcal{F}_a \), which shows that \( \tilde{\eta}_X \circ \psi = \text{Id}_{C^*(X, \theta_X, \mathcal{F}_a)} \).

Hence \( \tilde{\eta}_X \) is an isomorphism from \( C(X) \rtimes \mathbb{Z} \) to \( C^*(X, \theta_X, \mathcal{F}_a) \) which maps \( 1_{D_g} \) to \( s_g s_g^* \) for every \( g \in \mathcal{F}_a \) and \( U \) to \( \sum_{a \in \mathcal{A}} s_a \). \( \square \)

**7.2.** \textbf{\( C^* \)-algebras associated to one-sided shift spaces.} \( (X^+, \sigma) \) be a one-sided shift space over the finite alphabet \( \mathcal{A} \) (cf. [17] and [19] §13.8). That is: \( \sigma : \mathcal{A}^{\mathbb{N}_0} \to \mathcal{A}^{\mathbb{N}_0} \) is the map
\[ (7.2) \quad x_0 x_1 x_2 \cdots \mapsto x_1 x_2 \cdots, \]
and \( X^+ \) is a closed (in the product topology of the discrete topology of \( \mathcal{A} \)) subset of \( \mathcal{A}^{\mathbb{N}_0} \) such that \( \tau(X^+) = X^+ \).

As far as the author know, Kengo Matsumoto was in [20] the first to consider \( C^* \)-algebras associated to shift spaces. Matsumoto’s construction is however in the opinion of the author not the optimal one (see [6] for a discussing of this matter). In [3] the author considered a different construction of \( C^* \)-algebras associated to shift spaces which for some shift spaces gives a slightly different \( C^* \)-algebra than Matsumoto’s (and the \( C^* \)-algebra considered in [6], cf. [7] Section 7). We will in this paper work with the \( C^* \)-algebra \( C_{X^+} \) of [3] (it is isomorphic to the \( C^* \)-algebra \( D_{X^+} \mathbb{C} \) considered in [7], cf. [4] Remark 9)). It can be characterized in the following way (cf. [3] Remark 7.3):

Let \( \mathcal{A}^* \) denote the set of finite words with letters from \( \mathcal{A} \). For \( u, v \in \mathcal{A}^* \), let \( C(u, v) \) be the subset of \( X^+ \) defined by
\[ C(u, v) = \{ x \in X^+ \mid x_{[0, |v|]} = v, \; ux_{[|v|, \infty]} \in X^+ \}. \]
We let $\mathcal{B}(X^+)$ be the abelian $C^*$-algebra of all bounded functions on $X^+$, and $D_{X^+}$ the $C^*$-subalgebra of $\mathcal{B}(X^+)$ generated by $\{1_{C(u,v)} \mid u, v \in a^*\}$.

Then the $C^*$-algebra $O_{X^+}$ associated to the one-sided shift space $(X^+, \sigma)$ is the universal $C^*$-algebra generated by a family of partial isometries $(S_a)_{a \in a}$ which satisfies that the map

$$1_{C(u,v)} \mapsto S_u^*S_u S_v$$

extends to a $*$-homomorphism from $D_{X^+}$ to $O_{X^+}$, where $S_a = S_{u_1} S_{u_2} \cdots S_{u_k}$ for $u = u_1 u_2 \cdots u_k \in a^*$ with $u_1, u_2, \ldots, u_k \in a$, and $S_v = S_{v_1} S_{v_2} \cdots S_{v_l}$ for $v = v_1 v_2 \cdots v_l \in a^*$ with $v_1, v_2, \ldots, v_l \in a$. We will denote this $*$-isomorphism (which in fact is injective) by $\eta_O$.

One should notice (cf. [7, Theorem 12]) that when $X^+$ is a topological Markov chain with transition matrix $A$ (cf. Section 7.8), then $O_{X^+}$ is equal to the Cuntz-Krieger algebra $O_A$ (or to be more precise: to the universal Cuntz-Krieger algebra considered by an Hufer and Raeburn in [15], cf. also [9]).

We will for each $a \in a$, by $\lambda_a$ denote the map on $D_{X^+}$ given by

$$\lambda_a(f)(x) = \begin{cases} f(ax) & \text{if } ax \in X^+, \\ 0 & \text{if } ax \notin X^+, \end{cases}$$

and by $\phi_a$ the map on $D_{X^+}$ given by

$$\phi_a(f)(x) = \begin{cases} f(\sigma(x)) & \text{if } x \in D_a, \\ 0 & \text{if } x \notin D_a, \end{cases}$$

for $f \in D_{X^+}$ and $x \in X^+$ (cf. [3, Proposition 4.3 and Lemma 8.2]).

Since $(X^+, \sigma)$ is a one-sided symbolic dynamical system, we can associate to it the discrete partial dynamical system $(X^+, \theta_{X^+}, F_a)$ of Definition 3.3 We then have the following theorem:

**Theorem 7.4.** Let $(X^+, \sigma)$ be a one-sided shift space and let $(X^+, \theta_{X^+}, F_a)$ be the discrete partial dynamical system associated to $(X^+, \sigma)$ as done in Definition 3.3. Then $C^*(X^+, \theta_{X^+}, F_a)$ is isomorphic to the $C^*$-algebra $O_{X^+}$ associated to $(X^+, \sigma)$.

More precisely: if $(D_g)_{g \in F_a}$ denotes the domains of $\theta_{X^+}$, $(s_g)_{g \in F_a}$ denotes the generators of $C^*(X^+, \theta_{X^+}, F_a)$, and $D_{X^+}$, $\eta_O$ and $(S_a)_{a \in a}$ are as above, then we have that

$$\text{span}\{1_{D_g} \mid g \in F_a\} = D_{X^+},$$

and that there exists a $*$-isomorphism from $O_{X^+}$ to $C^*(X^+, \theta_{X^+}, F_a)$ which maps $\eta_O(1_{D_g})$ to $s_g s_{a}$ for every $g \in F_a$, and $S_a$ to $s_a$ for every $a \in a$.

**Proof.** To see that $\text{span}\{1_{D_g} \mid g \in F_a\} = D_{X^+}$ notice first that it follows from Lemma 3.4 that if $u, v \in a^*$, then we have that

$$C(u, v) = \theta_v(D_{u^{-1}}) = D_v \cap D_{v^{-1}}.$$ 

So $D_{X^+} \subseteq \text{span}\{1_{D_g} \mid g \in F_a\}$.

If $A$ is a subset of $X^+$ such that $1_A \in D_{X^+}$, then $1_{\theta_A} = \lambda_a(1_A) \in D_{X^+}$ and $1_{\theta_a^{-1}} = \phi_a(1_A) \in D_{X^+}$ for $a \in a$. So since we have that

$$D_g = \theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_k}(X^+)$$
for every $g \in \mathbb{F}_a$ written in the reduced form $b_1b_2\cdots b_k$, we have that $\text{span}\{1_{D_g} | g \in \mathbb{F}_a\} \subseteq D_{\mathcal{X}^+}$. Thus $D_{\mathcal{X}^+} = \text{span}\{1_{D_g} | g \in \mathbb{F}_a\}$.

Since $\phi(\mathcal{X}^+, \theta_{\mathcal{X}^+}, \mathbb{F}_a)$ is a Boolean homomorphism from $\mathcal{B}(\mathcal{X}^+, \theta_{\mathcal{X}^+}, \mathbb{F}_a)$ to the set of projections in $C^*(\mathcal{X}^+, \theta_{\mathcal{X}^+}, \mathbb{F}_a)$ which maps $D_g$ to $s_g s_g^*$, it follows from Lemma 3.1 that there exists a $*$-homomorphism $\gamma$ from $D_{\mathcal{X}^+}$ to $C^*(\mathcal{X}^+, \theta_{\mathcal{X}^+}, \mathbb{F}_a)$ which maps $1_{D_g}$ to $s_g s_g^*$ for every $g \in \mathbb{F}_a$.

For $g \in \mathbb{F}_a$ and $a \in \mathcal{A}$, we have that
\[
\gamma(1_{\theta_{\mathcal{X}^+}(D_g)}) = \gamma(1_{D_a \cap D_{\mathcal{X}^+}}) = s_a s_a^* s_a s_a^* s_a^* = s_a s_a^* s_a s_a^* s_a^*,
\]
and that
\[
\gamma(1_{\theta_{\mathcal{X}^+}(D_{\mathcal{X}^+})}) = \gamma(1_{D_a \cap D_{\mathcal{X}^+}}) = s_a^* s_a^* s_a^* s_a^* s_a^* = s_a^* s_a^* s_a^* s_a^* s_a^* s_a^* s_a^*.
\]

Since $\text{span}\{1_{D_g} | g \in \mathbb{F}_a\} = D_{\mathcal{X}^+}$, this shows that $\gamma(\phi_a(f)) = s_a \gamma(f) s_a^*$ and $\gamma(\lambda_a(f)) = s_a^* \gamma(f) s_a$ for $f \in D_{\mathcal{X}^+}$ and $a \in \mathcal{A}$. Thus we have that
\[
\gamma(1_{C(u,v)}) = \gamma(1_{\theta_{\mathcal{X}^+}(D_{\mathcal{X}^+})}) = \gamma(1_{\theta_{\mathcal{X}^+}(D_{\mathcal{X}^+})}) = s_{u_1 u_2 \cdots u_k} \circ \phi_{v_1} \circ \phi_{v_2} \circ \cdots \circ \phi_{v_k} \circ \lambda_{u_1} \circ \cdots \circ \lambda_{u_k} (1)
\]
for $u = u_1 u_2 \cdots u_k$ and $v = v_1 v_2 \cdots v_k$ in $\mathcal{A}^*$. Hence it follows from the universal property of $\mathcal{O}_{\mathcal{X}^+}$ that there is a $*$-homomorphism $\tilde{\gamma}$ from $\mathcal{O}_{\mathcal{X}^+}$ to $C^*(\mathcal{X}^+, \theta_{\mathcal{X}^+}, \mathbb{F}_a)$ such that $\tilde{\gamma} \circ \eta_\mathcal{O} = \gamma$ (and hence $\tilde{\gamma}(\eta_\mathcal{O}(1_{D_g})) = s_g s_g^*$ for every $g \in \mathbb{F}_a$) and $\tilde{\gamma}(S_a) = s_a$ for every $a \in \mathcal{A}$.

Let us now look at $\mathcal{O}_{\mathcal{X}^+}$. Let $S_a^{-1} = S_a^*$ for every $a \in \mathcal{A}$, let $S_e = 1$ and let for $g \in \mathbb{F}_a$ written in the reduced form $b_1b_2\cdots b_k$, $S_g = S_{b_1} S_{b_2} \cdots S_{b_k}$. Then $(S_g)_{g \in \mathbb{F}_a}$ is a family of elements from $O_{\mathcal{X}^+}$ which clearly satisfies (4.1a) and (4.1e) of Definition 4.1. We will show that $(S_g)_{g \in \mathbb{F}_a}$ also satisfies (4.1b), (4.1d) and (4.1e) of Definition 4.1.

It is easy to check that $\eta_\mathcal{O}(\lambda_a(f)) = S_a^* \eta_\mathcal{O}(f) S_a$ and $\eta_\mathcal{O}(\phi_a(f)) = S_a \eta_\mathcal{O}(f) S_a^{-1}$ for $a \in \mathcal{A}$ and $f \in D_{\mathcal{X}^+}$. Thus if we let $\omega_b$ and $\omega_{b^{-1}}$ be defined for $b \in \mathcal{A}$ by
\[
\omega_b = \begin{cases} 
\phi_b & \text{if } b \in \mathcal{A}, \\
\lambda_{b^{-1}} & \text{if } b^{-1} \in \mathcal{A},
\end{cases}
\]
then we have that
\[
\eta_\mathcal{O}(1_{D_g}) = \eta_\mathcal{O}(1_{\theta_{\mathcal{X}^+}(D_g)}) = \eta_\mathcal{O}(\omega_{b_1} \circ \omega_{b_2} \circ \cdots \circ \omega_{b_k} (1)) = S_{b_1} S_{b_2} \cdots S_{b_k} S_{b_k}^* \cdots S_{b_1}^* = S_g S_g^*.
\]
for every $g \in \mathbb{F}_a$ written in the reduced form $b_1b_2 \cdots b_k$. This shows that $(S_g)_{g \in \mathbb{F}_a}$ satisfies (4.1a) and (4.1b) of Definition 4.1.

Let $h, i \in \mathbb{F}_a$, and let $b_1b_2 \cdots b_k$ be the reduced form of $h$ and $b'_1b'_2 \cdots b'_k$, be the reduced form of $i$. Consider those $l \in \{1, 2, \ldots, k\}$ for which $k+1-l \leq k'$ and $b'_1 = b_k^{-1}$, $b'_2 = b_k^{-1}$, \ldots, $b'_{k+1-l} = b_l^{-1}$, and for which $b'_{k+1-l} \neq b'_l$ if $l \neq 1$ and $k+1-l \neq k'$. Notice that if such an $l$ exists, then it is necessarily unique. In this case, we let $j = b_l b_{l+1} \cdots b_k$. If no such $l$ exists, then we let $j$ be equal to the neutral element.

Let $h' = hj^{-1}$ and $i' = ji$. We then have that the reduced form of $h'$ is $b_1b_2 \cdots b_{l-1}$, the reduced form of $i'$ is $b'_{k+2-l}b'_{k+3-l} \cdots b'_k$, and the reduced form of $hi = h'i'$ is $b_1b_2 \cdots b_{l-1}b_{l+1}b_{l+2} \cdots b'_k$ (if $j = e$, then $h' = h$, $i' = i$ and the reduced form of $hi = h'i'$ is $b_1b_2 \cdots b_kb'_1b'_2 \cdots b'_k$). Thus $S_{h'}S_{i'} = S_{h'i'} = S_{hi}$, and we have that

\[
S_{h}S_{i} = S_{h}S_{j}S_{j}^{*}S_{i'} \\
= S_{h}S_{h'}S_{j}S_{j}^{*}S_{i'} \\
= S_{h}S_{j}S_{j}^{*}S_{i'}S_{h'} \\
= S_{h}S_{j}^{*}S_{h'} \\
= S_{h}S_{hi}.
\]

This shows that $(S_g)_{g \in \mathbb{F}_a}$ satisfies (4.1d) of Definition 4.1.

Hence it follows from the universal property of $C^*(X^+, \theta_X, \mathbb{F}_a)$ that there is a $*$-homomorphism $\psi$ from $C^*(X^+, \theta_X, \mathbb{F}_a)$ to $O_{X^+}$ such that $\psi(s_g) = S_{b_1s_{b_2} \cdots s_{b_k}}$ for every $g \in \mathbb{F}_a$, where $g \in \mathbb{F}_a$ is written in the reduced form $b_1b_2 \cdots b_k$.

We have that $\psi(\tilde{\gamma}(S_a)) = \psi(s_a) = S_a$ for every $a \in a$, and since $O_{X^+}$ is generated by $(S_a)_{a \in a}$, this shows that $\psi \circ \tilde{\gamma} = \text{Id}_{O_{X^+}}$.

We also have by Lemma 7.1 that

\[
\tilde{\gamma} (\psi (s_g)) = \tilde{\gamma}(S_{b_1s_{b_2} \cdots s_{b_k}}) = s_{b_1}s_{b_2} \cdots s_{b_k} = s_g
\]

for every $g \in \mathbb{F}_a$, where $g \in \mathbb{F}_a$ is written in the reduced form $b_1b_2 \cdots b_k$, and since $C^*(X^+, \theta_X, \mathbb{F}_a)$ is generated by $(s_g)_{g \in \mathbb{F}_a}$, this shows that $\tilde{\gamma} \circ \psi = \text{Id}_{C^*(X^+, \theta_X, \mathbb{F}_a)}$.

Thus $\tilde{\gamma}$ is a $*$-isomorphism from $O_{X^+}$ to $C^*(X^+, \theta_X, \mathbb{F}_a)$ which maps $\eta_{O}(1_{D_g})$ to $s_g s_{g}^{*}$ for every $g \in \mathbb{F}_a$ and $S_a$ to $s_a$ for every $a \in a$. \hfill \Box

7.3. Cuntz-Krieger algebras. Let $I$ be an arbitrary index set and let $A = (A(i,j))_{i,j \in I}$ be a matrix with entries in $\{0,1\}$ and having no zero rows. Exel and Laca have in [14] defined a $C^*$-algebra $O_A$ associated with $A$. It is the universal $C^*$-algebra generated by a family $(S_i)_{i \in I}$ of partial
isometries satisfying the following 4 conditions:

\( (7.3) \)

for each pair of finite subsets \( X \) and \( Y \) of \( \mathcal{I} \) such that the number \( A(X, Y, j) \) defined by

\[
A(X, Y, j) = \prod_{x \in X} A(x, j) \prod_{y \in Y} (1 - A(y, j))
\]

is zero for all but a finite number of \( j \)'s in \( \mathcal{I} \), the following equation holds:

\[
\prod_{x \in X} S_x^* S_x \prod_{y \in Y} (1 - S_y^* S_y) = \sum_{j \in \mathcal{I}} A(X, Y, j) S_j S_j^*,
\]

\( (7.4) \)

\( S_i^* S_i \) and \( S_j^* S_j \) commute for all \( i, j \in \mathcal{I} \),

\( (7.5) \)

\( S_i^* S_j = 0 \), if \( i \neq j \in \mathcal{I} \),

\( (7.6) \)

\( S_i^* S_j = A(i, j) S_j \), for all \( i, j \in \mathcal{I} \).

We will call the family \( (S_i)_{i \in \mathcal{I}} \) the generators of \( \mathcal{O}_A \).

If \( A \) is a finite matrix, then the conditions \( (7.3) - (7.6) \) reduce to the ordinary Cuntz-Krieger relations, and so \( \mathcal{O}_A \) is isomorphic to the usual Cuntz-Krieger algebra of \( A \) \([9]\), or the universal Cuntz-Krieger algebra considered in \([11]\) (cf. \([11, \text{Examples 8.9}]\)).

Let \( (X_A^+, \sigma) \) be the topological Markov chain associated to \( A \). That is: \( X_A^+ \) is the set defined by

\[
X_A^+ = \{ (x_n)_{n \in \mathbb{N}_0} \in \mathcal{I}^\mathbb{N}_0 \mid \forall k \in \mathbb{N}_0 : A(x_k, x_{k+1}) = 1 \},
\]

and \( \sigma : X_A^+ \rightarrow X_A^+ \) is the map

\[
x_0 x_1 x_2 \cdots \mapsto x_1 x_2 \cdots.
\]

Since \( (X_A^+, \sigma) \) is a one-sided symbolic dynamical system, we can associate to it the discrete partial dynamical system \( (X_A^+, \theta_{X_A^+}, F_F) \) of Definition \([3.3]\).

We then have the following theorem:

**Theorem 7.5.** Let \( \mathcal{I} \) be an arbitrary index set, let \( A = (A(i,j))_{i,j \in \mathcal{I}} \) be a matrix with entries in \( \{0,1\} \) and having no zero rows, and let \( (X_A^+, \theta_{X_A^+}, F_F) \) be the discrete partial dynamical system defined above. Then \( C^*(X_A^+, \theta_{X_A^+}, F_F) \) is isomorphic to the unitization \( \tilde{\mathcal{O}}_A \) of \( \mathcal{O}_A \).

More precisely: if \( (s_g)_{g \in \mathbb{F}_2} \) denotes the generators of \( C^*(X_A^+, \theta_{X_A^+}, F_F) \) and \( (S_i)_{i \in \mathcal{I}} \) denotes the generators of \( \mathcal{O}_A \), then \( C^*(X_A^+, \theta_{X_A^+}, F_F) \) is generated by its unit and \( \{ s_i \mid i \in \mathcal{I} \} \), and there exists a unital \(*\)-isomorphism from \( C^*(X_A^+, \theta_{X_A^+}, F_F) \) to \( \tilde{\mathcal{O}}_A \) which maps \( s_i \) to \( S_i \) for every \( i \in \mathcal{I} \).

**Proof.** Let \( (D_g)_{g \in \mathbb{F}_2} \) and \( (\theta_g)_{g \in \mathbb{F}_2} \) denote the domains and partial one-to-one maps of \( \theta_{X_A^+} \).

Using the facts that \( s_i s_i^* s_i = s_i \) for every \( i \in \mathcal{I} \) and that the map \( D_g \mapsto s_g s_g^* \) extends to a Boolean homomorphism from the Boolean algebra \( \mathcal{B}(X_A^+, \theta_{X_A^+}, F_F) \) to the Boolean algebra of projections in the unital abelian \( C^*\)-algebra

\[
C^*\left( \{ s_g s_g^* \in C^*(X_A^+, \theta_{X_A^+}, F_F) \mid g \in \mathbb{F}_2 \} \right).
\]
it is easy to check that the family \((s_i)_{i \in I}\) of elements of \(C^*(X^+_A, \theta_{X^+_A}, F_I)\) satisfies condition (1.3), (1.4), (7.5) and (7.6) above. Thus it follows from the universal property of \(O_A\) that there is a \(*\)-homomorphism \(\phi\) from \(O_A\) to \(C^*(X^+_A, \theta_{X^+_A}, F_I)\) such that \(\phi(s_i) = s_i\) for all \(i \in I\).

If \(O_A\) is not unital, then \(\phi\) extends to a unital \(*\)-homomorphism \(\tilde{\phi}\) from \(\tilde{O}_A\) to \(C^*(X^+_A, \theta_{X^+_A}, F_I)\). If \(O_A\) is unital (in which case \(\tilde{O}_A = O_A\)), then it follows from [11, Proposition 8.5] that there exist finite subsets \(X\) and \(Y\) of \(I\) such that the equation

\[
X_A^+ = \bigcup_{x \in X} D_x \cup \bigcup_{y \in Y} D_{y^{-1}}
\]

holds. That means that the unit of \(C^*(X^+_A, \theta_{X^+_A}, F_I)\) is contained in the \(C^*\)-subalgebra generated by \(\{s_i \mid i \in I\}\) and thus is in the image of \(\phi\), and so \(\phi\) is unital.

So we have in both cases that there exists a unital \(*\)-homomorphism \(\tilde{\phi}\) from \(\tilde{O}_A\) to \(C^*(X^+_A, \theta_{X^+_A}, F_I)\) which maps \(S_i\) to \(s_i\) for all \(i \in I\).

Let us now turn towards \(\tilde{O}_A\). We let for every \(i \in I\), \(S_{i-1} = S_i^*\) and we let \(S_e = 1\) and \(S_g = S_{b_1}S_{b_2} \cdots S_{b_k}\), where \(g = b_1b_2 \cdots b_k \in F_I\) is written in reduced form. It then follows from condition (1.3), (7.5) and (7.6) that the map

\[
g \mapsto S_g, \quad g \in F_I
\]

is a partial representation of \(F_I\) (see [11 Proposition 3.2] for a proof of this) and thus that \((S_g)_{g \in F_I}\) is a family of partial isometries which satisfies condition (4.1a)–(4.1d) of Definition 4.1. We will now show that \((S_g)_{g \in F_I}\) also satisfies (4.1e) of Definition 4.1.

It follows from Lemma (1.4) that \(s_g = s_{b_1}s_{b_2} \cdots s_{b_k}\) for every \(g \in F_I\) written in the reduced form \(b_1b_2 \cdots b_k\). Thus the \(*\)-homomorphism \(\tilde{\phi} : \tilde{O}_A \to C^*(X^+_A, \theta_{X^+_A}, F_I)\) mentioned above maps \(S_g\) to \(s_g\) for every \(g \in F_I\), and thus induces a Boolean homomorphism from the Boolean algebra of projections in the unital abelian \(C^*\)-algebra

\[
C^*\left(\{S_gS_g^* \in \tilde{O}_A \mid g \in F_I\}\right)
\]

to the Boolean algebra of projections in the unital abelian \(C^*\)-algebra

\[
C^*\left(\{s_gS_g^* \in C^*(X^+_A, \theta_{X^+_A}, F_I) \mid g \in F_I\}\right)
\]

which maps \(S_gS_g^*\) to \(s_gS_g^*\). Let us denote the Boolean subalgebra of the Boolean algebra of projections in the unital abelian \(C^*\)-algebra

\[
C^*\left(\{S_gS_g^* \in \tilde{O}_A \mid g \in F_I\}\right)
\]

generated by \(\{S_gS_g^* \mid g \in F_I\}\) by \(B(O_A)\). Remember (cf. Definition 1.4 and Corollary 6.2) that \(\phi(X^+_A, \theta_{X^+_A}, F_I)\) is an injective Boolean homomorphism from \(B(X^+_A, \theta_{X^+_A}, F_I)\) to the Boolean algebra of projections in the unital abelian \(C^*\)-algebra

\[
C^*\left(\{s_gS_g^* \in C^*(X^+_A, \theta_{X^+_A}, F_I) \mid g \in F_I\}\right)
\]
where $\eta$ is a Boolean homomorphism from $\mathcal{B}(\mathcal{O}_A)$ to $\mathcal{B}(\mathcal{O}_A, \theta_{X^*_A})$ which maps $S_g$ to $D_g$. We claim that $\eta$ is injective.

To prove this we will use that the family $(S_g)_{g \in \mathcal{F}}$ has the following properties (remember that we by $\mathcal{I}^*$ denote the set of finite words with letters from $\mathcal{I}$, and that we identify $\mathcal{I}^*$ with the unital sub-semigroup of $\mathcal{F}$ generated by $\mathcal{I}$, cf. Section 2):

(7.7) $S^*_h S^*_i S^*_l S^*_k = S^*_l S^*_i S^*_h S^*_j$ for all $h, i, j, k \in \mathcal{F}$,
(7.8) $S^*_u S^*_v = S^*_w$ for all $u, v, w \in \mathcal{I}^*$,
(7.9) $S^*_u S^*_v = 0$ if $u, v \in \mathcal{I}^*$, $|u| = |v|$ and $u \neq v$,
(7.10) $S^*_u S^*_v = A(u_1, u_2) \cdots A(u_{|u| - 1}, u_{|u|}) S^*_u S^*_v$ for all $u, v \in \mathcal{I}^*$,
(7.11) if $g \neq 0$ then $g = uv^{-1}$ and $S_g = S_u S_v$ for some $u, v \in \mathcal{I}^*$.

Property (7.11) follows from the fact that $g \mapsto S_g$ is a partial representation of $\mathcal{I}^*$, (7.8) follows from the definition of $S_g$, (7.7) is Claim 2 and (7.10) is Claim 1 in the proof of Proposition 3.2. To see (7.11) notice that it follows from Proposition 3.1 that if $g \neq 0$, then $g = uv^{-1}$ for some $u, v \in \mathcal{I}^*$, and if we choose $u$ and $v$ such that the last symbol of $u$ is different from the first letter of $v$ (or $u$ or $v$ is the empty word), then $S_g = S_u S_v$ by definition.

Let for every $u \in \mathcal{I}^*$ and every pair $(I, J)$ of (possible empty) finite subsets of $\mathcal{I}$, $C(u, I, J)$ be the subset of $X^*_A$ defined by

$$C(u, I, J) = \theta(u) \left( \bigcap_{i \in I} D_i \setminus 1 \right) \cap \left( \bigcap_{j \in J} X^*_A \setminus 1 \right)$$

$$= \{ ux \in X^*_A \mid \forall i \in I : ix \in X^*_A, \forall j \in J : jx \notin X^*_A \}.$$

Notice that we have that

$$\eta \left( S_u \prod_{i \in I} S^*_i S^*_j \prod_{j \in J} (1 - S^*_j S^*_j) S^*_u \right) = C(u, I, J)$$

for finite subsets $I, J$ of $\mathcal{I}$ and $u \in \mathcal{I}^*$.

We will now prove that every element of $\mathcal{B}(\mathcal{O}_A, \theta_{X^*_A})$ can be written as a finite union of elements of the form

$$S_u \prod_{i \in I} S^*_i S^*_j \prod_{j \in J} (1 - S^*_j S^*_j) S^*_u \left( \prod_{k=1}^{n} \left( 1 - S^*_u \prod_{i \in I} S^*_i S^*_j \prod_{j \in J} (1 - S^*_j S^*_j) S^*_u \right) \right)$$

where $u, u^1, u^2, \ldots, u^n \in \mathcal{I}^*$ and $I, J, I_1, J_1, I_2, J_2, \ldots, I_n, J_n$ all are finite (possibly empty) subsets of $\mathcal{I}$.

Notice first that if $u \in \mathcal{I}^*$ and $i \in \mathcal{I}$, then we have that

$$S_u S_i S_i S_u = \begin{cases} S^*_i S^*_i S^*_u & \text{if } u = \varepsilon, \\ S^*_u S^*_i S^*_u & \text{if } u \neq \varepsilon, \end{cases}$$

and thus that $S_u S^*_i S_i S^*_u$ belongs to $\mathcal{B}(\mathcal{O}_A)$. Since $\mathcal{B}(\mathcal{O}_A)$ is closed under intersection and complement, and the complement of an element $p$ of $\mathcal{B}(\mathcal{O}_A)$
is defined to be \(1 - p\) and the intersection of two elements \(p, q \in \mathcal{B}(O_A)\) is \(pq\), we also have that
\[
S_u(1 - S_j^* S_j) S_u^* = S_u S_u^*(1 - S_u S_j^* S_j S_u^*) \in \mathcal{B}(O_A)
\]
for every \(u \in I^*\) and every \(j \in I\).

Let \(u, \tilde{u} \in I^*\) and let \(I, J, \tilde{I}, \tilde{J}\) be subsets of \(I\). It then follows from (7.6) and (7.14) that the element
\[
S_u \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_u^* S_u \prod_{i \in \tilde{I}} S_i^* S_i \prod_{j \in \tilde{J}} (1 - S_j^* S_j) S_u^*
\]
is equal to
\[
S_u \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_u^*
\]
if \(u = \tilde{u} v\) for some \(v = v_1 v_2 \cdots v_{|v|} \in I^*\) which satisfies that \(A(\tilde{I}, \tilde{J}, v_1) = 1\), to
\[
S_u \prod_{i \in \tilde{I}} S_i^* S_i \prod_{j \in \tilde{J}} (1 - S_j^* S_j) S_u^*
\]
if \(\tilde{u} = uv\) for some \(v = v_1 v_2 \cdots v_{|v|} \in I^*\) which satisfies that \(A(I, J, v_1) = 1\), to
\[
S_u \prod_{i \in I \cup \tilde{I}} S_i^* S_i \prod_{j \in J \cup \tilde{J}} (1 - S_j^* S_j) S_u^*
\]
if \(u = \tilde{u}\), and to 0 otherwise.

Thus if we let \(Z\) be the set
\[
\left\{ S_u \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_u^* \left| u \in I^*, I, J \text{ are finite subset of } I^* \right. \right\},
\]
then \(Z\) is a subset of \(\mathcal{B}(O_A)\), and it is closed under intersection. So if we let
\[
\mathcal{Y} = \left\{ Z_0 \cap (1 - Z) \left| Z_0 \in Z, \mathcal{F} \text{ is a finite subset of } \mathcal{Z} \right. \right\},
\]
then \(\mathcal{Y}\) is also closed under intersection. Let \(\mathcal{X}\) be the set of elements in \(\mathcal{B}(O_A)\) which can be written as a finite union of elements from \(\mathcal{Y}\). Then \(\mathcal{X}\) is closed under union and intersection, and since we also have that \(1 - Y\) belongs to \(\mathcal{X}\) if \(Y\) belongs to \(\mathcal{Y}\), we also have that \(\mathcal{X}\) is closed under complement and thus is a Boolean subalgebra of \(\mathcal{B}(O_A)\). It follows from (7.8), (7.10) and (7.11) that if \(g \in \mathbb{F}_I\) and \(S_g \neq 0\), then \(g = uv^{-1}\) for suitable \(u, v \in I^*\) and that
\[
S_g^* S_g = S_v S_v^* S_i S_i^* = S_v S_i^* S_i S_v^* \in \mathcal{X}
\]
where \(i\) is the last letter of \(u\) (if \(u\) is the empty word, then \(S_g^* S_g = S_v S_v^* \in \mathcal{X}\)). Thus every element in \(\mathcal{B}(O_A)\) belongs to \(\mathcal{X}\), and is thus a finite union of elements of the form
\[
S_u \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_u^* \left( \prod_{k=1}^n \left(1 - S_u S_i^* S_i S_u^* \prod_{j \in J_k} (1 - S_j^* S_j) S_u^* \right) \right)
\]
where \(u, u^1, u^2, \ldots, u^n \in I^*\) and \(I, J, J_1, J_2, \ldots, I_n, J_n\) all are finite (possible empty) subsets of \(I\).
In order to prove that $\eta$ is injective, it is therefore enough to show that if $u, u^1, u^2, \ldots, u^n \in I^*$ and $I, J, I_1, J_1, I_2, J_2, \ldots, I_n, J_n$ all are finite (possible empty) subsets of $I$ and

$$C(u, I, J) \cap \left( \bigcap_{k=1}^{n} X^*_A \setminus C(u^k, I_k, J_k) \right) = \emptyset,$$

then we have that

$$S_u \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_u \left( \prod_{k=1}^{n} \left( 1 - S_u^k \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_v^* \right) \right) = 0.$$

Let $u, v \in I^*$ with $|u| > |v|$ (remember that $|u|$ and $|v|$ denote the length of $u$ and $v$, respectively, cf. Section 2) and let $I, J, I', J'$ be finite subsets of $I$. Then either the equality

$$C(u, I, J) \cap X^*_A \setminus C(v, I', J') = C(u, I, J)$$

holds, or $u = v\alpha$ for some $\alpha = \alpha_1 \alpha_2 \cdots \alpha_{|u|} \alpha_{|v|} \in I^*$ which satisfies that $A(I', J', \alpha_1) = 1$. In the latter case, we have that

$$S_u S_u^* S_v \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_v^* = S_u S_u^*,$$

and thus that

$$S_u \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_u \left( 1 - S_v \prod_{i \in I'} S_i^* S_i \prod_{j \in J'} (1 - S_j^* S_j) S_v^* \right) = 0.$$

So in order to prove that the condition

$$C(u, I, J) \cap \left( \bigcap_{k=1}^{n} X^*_A \setminus C(u^k, I_k, J_k) \right) = \emptyset,$$

implies that

$$S_u \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_u \left( \prod_{k=1}^{n} \left( 1 - S_u^k \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_u^* \right) \right) = 0,$$

we may assume that $|u^k| \geq |u|$ for every $k \in \{1, 2, \ldots, n\}$.

So let $u, u^1, u^2, \ldots, u^n \in I^*$ with $|u^k| \geq |u|$ for every $k \in \{1, 2, \ldots, n\}$ and let $I, J, I_1, J_1, I_2, J_2, \ldots, I_n, J_n$ be finite subsets of $I$ such that

$$C(u, I, J) \cap \left( \bigcap_{k=1}^{n} X^*_A \setminus C(u^k, I_k, J_k) \right) = \emptyset.$$

**Claim.** We claim that there for every $m > |u|$ exists a finite subset $W_m$ of the set

$$\left\{ v \in I^* \mid |v| = m, \forall x \in X^*_A : A(v_m, x_0) = 1 \implies v x \in C(u, I, J) \cap \left( \bigcap_{k=1}^{n} X^*_A \setminus C(u^k, I_k, J_k) \right) \right\}.$$
such that the following equality holds:

\[
S_u \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_u^* \left( \prod_{k=1}^n \left( 1 - S_u^* \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_u^k \right) \right)
= \sum_{v \in W_m} S_u^* S_v^* \prod_{k=1}^n \left( 1 - S_u^k \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_u^k \right).
\]

It follows from this claim that the equality

\[
S_u \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_u^* \left( \prod_{k=1}^n \left( 1 - S_u^k \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_u^k \right) \right) = 0,
\]

holds, because since \(A\) has no identically zero row, there is for every \(v = v_1 v_2 \cdots v_m \in T^*\), an \(x \in X_A^*\) such that \(A(v_m, x_0) = 1\), and it follows from this that the set

\[
\left\{ v \in T^* \mid |v| = m, \forall x \in X_A^* : A(v_m, x_0) = 1 \implies vx \in C(u, I, J) \cap \left( \bigcap_{k=1}^n X_A^* \setminus C(u^k, I_k, J_k) \right) \right\}
\]

is empty for \(m > \max\{|u^k| \mid k \in \{1, 2, \ldots, n\}\}\).

To prove the claim, we need a little lemma:

**Lemma 7.6.** Let \(R\) be a unital ring, let \(k \in \mathbb{N}\) and let \(x_1, x_2, \ldots, x_n, \in R\). Then we have that

\[
1 - \prod_{i=1}^n x_i = \sum_{E \in \mathcal{F}_n \setminus \{E \mid i \in \{1, 2, \ldots, n\} \setminus E \} \} \prod_{j \in E} (1 - x_j)
\]

where \(\mathcal{F}_n\) is the set of non-empty subsets of \(\{1, 2, \ldots, n\}\).

**Proof.** We will prove the lemma by induction. The lemma obviously holds for \(n = 1\). Assume now that the lemma holds for \(n = m\). Then we have that

\[
1 - \prod_{i=1}^{m+1} x_i = \left( 1 - \prod_{i=1}^m x_i \right) x_{m+1} + \prod_{i=1}^m x_i (1 - x_{m+1})\left( 1 - \prod_{i=1}^m x_i \right) (1 - x_{m+1})
= \sum_{E \in \mathcal{F}_m \setminus \{E \mid i \in \{1, 2, \ldots, m\} \setminus E \} \} \prod_{i \in E} x_i \prod_{j \in E} (1 - x_j) x_{m+1} + \prod_{i=1}^m x_i (1 - x_{m+1})
+ \sum_{E \in \mathcal{F}_m \setminus \{E \mid i \in \{1, 2, \ldots, m\} \setminus E \} \} \prod_{i \in E} x_i \prod_{j \in E} (1 - x_j) (1 - x_{m+1})
= \sum_{E \in \mathcal{F}_{m+1} \setminus \{E \mid i \in \{1, 2, \ldots, m+1\} \setminus E \} \} \prod_{i \in E} x_i \prod_{j \in E} (1 - x_j).
\]

\[\square\]
Proof of the claim. We will now prove the claim by induction. First let \( m = |u| + 1 \). Let \( K_u \) be the subset of \( \{1, 2, \ldots, n\} \) defined by
\[
K_u = \{ k \in \{1, 2, \ldots, n\} \mid u^k = u \},
\]
and let for each \( k \in K_u \), \( \mathcal{F}_k \) be the set of non-empty subsets of \( I_k \cup J_k \). We let \( \mathcal{E} \) be the set of families \((E_k)_{k \in K_u}\) where for each \( k \in K_u \), \( E_k \in \mathcal{F}_k \), and we let for \( E = (E_k)_{k \in K_u} \in \mathcal{E} \), \( I_E \) be the finite subset of \( \mathcal{I} \) defined by
\[
I_E = I \cup \{u_{|u|}\} \cup \bigcup_{k \in K_u} (J_k \cap E_k) \cup (I_k \setminus E_k),
\]
and we let \( J_E \) be the finite subset of \( \mathcal{I} \) defined by
\[
J_E = J \cup \bigcup_{k \in K_u} (I_k \cap E_k) \cup (J_k \setminus E_k).
\]
We then have that if \( y \in X^+_A \) and \( A(I_E, J_E, y_0) = 1 \), then
\[
uy \in C(u, I, J) \cap \left( \bigcap_{|u^k| = |u|} X^+_A \right) \subseteq \bigcup_{|u^k| > |u|} C(u^k, I_k, J_k),
\]
so if we let \( W_m = \{u_j \mid j \in \mathcal{I}, \exists E \in \prod_{k \in B_u} \mathcal{F}_k : A(I_E, J_E, j) = 1\} \), then \( W_m \) is a subset of the set
\[
\left\{ v \in \mathcal{I}^* \mid |v| = m, \forall x \in X^+_A : A(v_m, x_0) = 1 \implies vx \in C(u, I, J) \cap \left( \bigcap_{|u^k| < m} X^+_A \right) \right\}.
\]
Since \( A \) has no zero row, there is for every \( j \in \mathcal{I} \) an \( y \in X^+_A \) such that \( y_0 = j \). Now if \( u_j \in W_m \), then we have, as mentioned above, that
\[
uy \in \bigcup_{|u^k| > |u|} C(u^k, I_k, J_k),
\]
which means that
\[
j = y_0 \in \{(u^k)_m \mid k \in \{1, 2, \ldots, n\}, |u^k| > |u|\}.
\]
So \( W_m \) is finite.

If \( |u^k| = |u| \), but \( u^k \notin K_u \), then \( S^*_u S^*_{u^k} = 0 \) by (7.9), and so we have that
\[
S_u \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S^*_u \left( 1 - S_{u^k} \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S^*_u \right) = S_u \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S^*_u,
\]
Thus we have that

\[ S_u \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_u^* \]

\[ \left( \prod_{k=1}^{n} \left( 1 - S_u^k \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_u^k \right) \right) = \]

\[ S_u \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_u^* S_u \]

\[ \left( \prod_{k \in K_u} \left( 1 - \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) \right) \right) S_u^* = \]

\[ S_u \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_u^* S_u \]

\[ \left( \prod_{k \in K_u \setminus E_k \neq F_k} \left( \sum_{i \in (J_k \cap E_k) \setminus (I_k \setminus E_k)} \prod_{j \in (I_k \cap E_k) \setminus (J_k \setminus E_k)} \right) \right) S_u^* = \]

\[ S_u \sum_{E \in \mathcal{E}} \left( \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_u^* S_u \right) \]

\[ \left( \prod_{k \in K_u \setminus E_k \neq F_k} \left( \sum_{i \in (J_k \cap E_k) \setminus (I_k \setminus E_k)} \prod_{j \in (I_k \cap E_k) \setminus (J_k \setminus E_k)} \right) \right) S_u^* = \]

\[ S_u \sum_{E \in \mathcal{E}} \left( \prod_{i \in I_E} S_i^* S_i \prod_{j \in J_E} (1 - S_j^* S_j) S_u^* S_u = \sum_{v \in W_n} S_v S_v^*, \right. \]

where the second equality follows from (7.7) and the facts that \( u^k = u \) for \( k \in K_u \) and that \( S_u \) is a partial isometry, the third follows from Lemma 7.6, the fourth from the (7.10) (if \( A(u_i, u_{i+1}) = 0 \) for some \( i = \{1, 2, \ldots, |u| - 1\} \),
then $S_u = 0$ according to (7.6) and (7.8), and the equality still holds), the fifth from the distribute law, the sixth from the definition of $I_E$ and $J_E$, the seventh from (7.4) (that $A(I_E, J_E, j)$ vanishes for all but a finite number of $j$’s in $I$ follows from the fact that $W_m$ is finite), and the eight from (7.8) and the definition of $W_m$. Thus we have proved the claim in the case where $m = |u| + 1$.

Assume now that $m > |u|$ and that

$$S_u \prod_{i \in I} S_i^* \prod_{j \in J} (1 - S_j^* S_j) S_u^* \left( \prod_{k=1}^n (1 - S_u^k \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_u^k) \right)$$

$$= \sum_{v \in W_m} S_v S_v^* \prod_{k=1}^n (1 - S_u^k \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_u^k)$$

for some finite subset $W_m$ of the set

$$\left\{ v \in I^* \mid |v| = m, \forall x \in X_A^+ : A(v_m, x_0) = 1 \implies vx \in C(u, I, J) \bigcap \left( \prod_{k=1}^n X_A^+ \setminus C(u^k, I_k, J_k) \right) \right\}.$$ 

Let for every $\gamma \in W_m$, $K_\gamma$ be the subset of $\{1, 2, \ldots, n\}$ defined by

$$K_\gamma = \{ k \in \{1, 2, \ldots, n\} \mid u^k = \gamma \},$$

and let for each $k \in K_\gamma$, $F_k$ be the set of non-empty subset of $I_k \cup J_k$. We let $E$ be the set of families $(E_k)_{k \in K_\gamma}$ where $E_k \in F_k$ for each $k \in K_\gamma$, and we let for $E = (E_k)_{k \in K_\gamma} \in E$, $I_E$ be the finite subset of $I$ defined by

$$I_E = \{ \gamma_m \} \cup \bigcup_{k \in K_\gamma} (J_k \cap E_k) \cup (I_k \setminus E_k)$$

and $J_E$ be the finite subset of $J$ defined by

$$J_E = \bigcup_{k \in K_\gamma} (I_k \cap E_k) \cup (J_k \setminus E_k).$$

We then have that if $y \in X_A^+$ and $A(I_E, J_E, y_0) = 1$, then

$$\gamma y \in \bigcap_{k=1}^n X_A^+ \setminus C(u^k, I_k, J_k),$$

and since we have that

$$\gamma \in W_m \subseteq \left\{ v \in I^* \mid |v| = m, \forall x \in X_A^+ : A(v_m, x_0) = 1 \implies vx \in C(u, I, J) \bigcap \left( \prod_{k=1}^n X_A^+ \setminus C(u^k, I_k, J_k) \right) \right\},$$
it follows that
\[ \gamma y \in C(u, I, J) \cap \left( \bigcap_{k=1}^{n} X^+_A \setminus C(u^k, I_k, J_k) \right) \subseteq \bigcup_{k=1}^{n} C(u^k, I_k, J_k). \]

Let \( \mathcal{W}_\gamma \) be the subset of \( \mathcal{I} \) defined by
\[
\mathcal{W}_\gamma = \left\{ \gamma j \mid j \in \mathcal{I}, \ \exists E \in \prod_{k \in K_\gamma} \mathcal{F}_k : A(X_E, Y_E, j) = 1 \right\}.
\]

Since \( A \) has no zero row, there is for every \( j \in \mathcal{I} \) an \( y \in X^+_A \) such that \( y_0 = j \).
Now if \( \gamma j \in \mathcal{W}_\gamma \), then we have, as mentioned above, that
\[
\gamma y \in \bigcup_{k=1}^{n} C(u^k, I_k, J_k), \quad |u^k| > m
\]
which means that
\[
j = y_0 \in \{(u^k)_m \mid k \in \{1, 2, \ldots, n\}, \ |u^k| > m\}.
\]
Thus \( \mathcal{W}_\gamma \) is a finite set. Hence if we let \( \mathcal{W}_{m+1} = \bigcup_{\gamma \in \mathcal{W}_m} \mathcal{W}_\gamma \), then \( \mathcal{W}_{m+1} \) is a finite subset of the set
\[
\left\{ v \in \mathcal{I}^* \mid |v| = m + 1, \ \forall x \in X^+_A : A(v_{m+1}, x_0) = 1 \implies vx \in C(u, I, J) \cap \left( \bigcap_{k=1}^{n} X^+_A \setminus C(u^k, I_k, J_k) \right) \right\}.
\]

Let \( \gamma \in \mathcal{W}_m \). If \( |u^k| = m \), but \( u^k \notin K_\gamma \), then \( S^{\gamma^*}S_{u^k} = 0 \) by (7.9), so the equality
\[
S_{\gamma^*}S_{\gamma} \left( 1 - S_{u^k} \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j)S^*_u \right) = S_{\gamma^*}S_{\gamma}
\]
holds.
Thus we have that
\[
S_\gamma S_\gamma^* \left( \prod_{k=1}^{n} \left( 1 - S_{u_k} \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_{u_k}^* \right) \right) =
\]
\[
S_\gamma S_\gamma^* \left( \prod_{k \in K_\gamma} \left( 1 - S_{u_k} \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_{u_k}^* \right) \right) =
\]
\[
S_\gamma S_\gamma^* \left( \prod_{k \in K_\gamma} \left( 1 - \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) \right) \right) S_\gamma^* =
\]
\[
S_\gamma S_\gamma^* \left( \prod_{k \in K_\gamma} \left( \sum \left( \prod S_i^* S_i \prod (1 - S_j^* S_j) \right) \right) \right) S_\gamma^* =
\]
\[
S_\gamma S_\gamma^* \left( \prod_{k \in K_\gamma} \left( \sum \left( \sum S_i^* S_i \prod (1 - S_j^* S_j) \right) \right) \right) S_\gamma^* =
\]
\[
S_\gamma \sum_{E \in \mathcal{E}} \left( \prod_{i \in I_E} S_i^* S_i \prod_{j \in J_E} (1 - S_j^* S_j) \right) S_\gamma^* =
\]
\[
S_\gamma \sum_{E \in \mathcal{E}} \sum_{J \in \mathcal{I}} \sum_{A(I_E,J_E,j)=1} S_j S_j^* S_\gamma^* = \sum_{\eta \in \mathcal{W}_\gamma} S_\eta S_\eta^*,
\]
where the second equality follows from (7.7), the fact that \( u^k = \gamma \) for \( k \in K_\gamma \) and that \( S_\gamma \) is a partial isometry, the third follows from Lemma (7.8), the fourth from (7.10) (if \( A(\gamma_i, \gamma_{i+1}) = 0 \) for some \( i = \{1, 2, \ldots, |\gamma| - 1\} \), then \( S_\gamma = 0 \) according to (7.10) and (7.8), and the equality still holds), the fifth from the distribution law, the sixth from the definition of \( I_E \) and \( J_E \), the seventh from (7.11) (that \( A(I_E,J_E,j) \) vanishes for all but a finite number of \( j \)'s in \( I \) follows from the fact that \( \mathcal{W}_\gamma \) is finite), and the eighth from (7.8) and the definition of \( \mathcal{W}_\gamma \).

Thus we have that
\[
S_u \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_u^* \left( \prod_{k=1}^{n} \left( 1 - S_{u_k} \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_{u_k}^* \right) \right) =
\]
\[
S_u \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_u^* \left( \prod_{k=1}^{n} \left( 1 - S_{u_k} \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_{u_k}^* \right) \right) =
\]
\[
S_u \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_u^* \left( \prod_{k=1}^{n} \left( 1 - S_{u_k} \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_{u_k}^* \right) \right) =
\]
which finalizes the induction proof of the claim. 

Hence the Boolean homomorphism \( \eta \) from \( B(\mathcal{O}_A) \) to \( B(\mathcal{X}^+_{A^+}, \mathcal{F}_I) \) is injective, and since it obviously also is surjective, it is invertible. Thus there is a Boolean homomorphism from \( B(\mathcal{X}^+_{A^+}, \mathcal{F}_I) \) to the Boolean algebra of projections in the unital abelian \( C^* \)-algebra

\[
C^*\left(\{S_gS_g^* \in \tilde{\mathcal{O}}_A \mid g \in \mathbb{F}_I\}\right),
\]

sending \( D_g \) to \( S_gS_g^* \) for every \( g \in \mathbb{F}_I \). Hence \( (S_g)_{g \in \mathbb{F}_I} \) also satisfies Definition 4.1 of \( \tilde{\mathcal{O}}_A \). So it follows from the universal property of \( C^*(\mathcal{X}^+_{A^+}, \mathcal{F}_I) \) that there is a unital \(*\)-homomorphism \( \psi \) from \( C^*(\mathcal{X}^+_{A^+}, \mathcal{F}_I) \) to \( \tilde{\mathcal{O}}_A \), such that \( \psi(s_g) = S_{b_1}S_{b_2} \cdots S_{b_k} \) for \( g \in \mathbb{F}_I \) written in the reduced form \( b_1b_2 \cdots b_k \).

We have that \( \psi(\tilde{\phi}(S_i)) = S_i \) for every \( i \in \mathcal{I} \) and that \( \psi(\tilde{\phi}(1)) = 1 \), and since \( \tilde{\mathcal{O}}_A \) is generated by \( \{S_i \mid i \in \mathcal{I}\} \cup \{1\} \), this shows that \( \psi \circ \tilde{\phi} = \text{Id}_{\tilde{\mathcal{O}}_A} \).

According to Lemma 5.1 \( s_g = s_{b_1}s_{b_2} \cdots s_{b_k} \) for every \( g \in \mathbb{F}_I \) written in the reduced form \( b_1b_2 \cdots b_k \). Thus we have that

\[
\tilde{\phi}(\psi(s_g)) = \tilde{\phi}(S_{b_1}S_{b_2} \cdots S_{b_k}) = s_{b_1}s_{b_2} \cdots s_{b_k} = s_g
\]

for every \( g \in \mathbb{F}_I \) written in the reduced form \( b_1b_2 \cdots b_k \), and since \( C^*(\mathcal{X}^+_{A^+}, \mathcal{F}_I) \) is generated by \( (s_g)_{g \in \mathbb{F}_I} \), this shows that \( \tilde{\phi} \circ \psi = \text{Id}_{C^*(\mathcal{X}^+_{A^+}, \mathcal{F}_I)} \).

Thus \( \psi \) is a unital \(*\)-isomorphism from \( C^*(\mathcal{X}^+_{A^+}, \mathcal{F}_I) \) to \( \tilde{\mathcal{O}}_A \) which maps \( s_i \) to \( S_i \) for every \( i \in \mathcal{I} \), and since \( \tilde{\mathcal{O}}_A \) is generated by its unit and \( \{S_i \mid i \in \mathcal{I}\} \), \( C^*(\mathcal{X}^+_{A^+}, \mathcal{F}_I) \) is generated by its unit and \( \{s_i \mid i \in \mathcal{I}\} \).

**8. The ideal structure of \( C^*(X, \theta, G) \)**

One of the advantages of having a unified construction of the \( C^* \)-algebras associated to one-sided shift spaces, crossed product of two-sided shift spaces and Cuntz-Krieger algebras is that it is easy to obtain results which holds for all of these \( C^* \)-algebras (and of course other \( C^* \)-algebras which can be constructed as \( C^* \)-algebras of discrete partial dynamical systems) and results which relate these different kind of \( C^* \)-algebras to each other.

We will in this section for every discrete partial dynamical system \((X, \theta, G)\) show how a \( \theta \)-invariant (see the definition of \( \theta \)-invariant below) subset of \( X \) gives rise to an ideal in \( C^*(X, \theta, G) \), and from this construct an injective order preserving map between certain \( \theta \)-invariant subsets of \( X \) and ideals of \( C^*(X, \theta, G) \). This will enable us to recover some well-known result about the ideal structure of crossed product of two-sided shift spaces and Cuntz-Krieger algebras, and will shade new light on the ideal structure of \( C^* \)-algebra associated to one-sided shift spaces.

We will also obtain a result which relates the \( C^* \)-algebras of two different partial dynamical systems and use this to show that for two-sided shift spaces having a certain property, the crossed product of the two-sided shift space is a quotient of the \( C^* \)-algebra associated to the corresponding one-sided shift space. This lays the ground for a description of the \( K \)-theory of the \( C^* \)-algebra associated to the one-sided shift space which is explained in [4] and [5].
Definition 8.1. Let \((X, \theta, G)\) be a discrete partial dynamical system and \((\theta_g)_{g \in G}\) the partial one-to-one maps of \(\theta\). Then we say that a subset \(Y\) of \(X\) is \(\theta\)-invariant if \(\theta_g(Y) \subseteq Y\) for all \(g \in G\).

Let \((X, \theta, G)\) be a discrete partial dynamical system, \((D_g)_{g \in G}\) the domains and \((\theta_g)_{g \in G}\) the partial one-to-one maps of \(\theta\), and let \(Y\) be an \(\theta\)-invariant subset of \(X\). Then we also have that \(\theta_g(X \setminus Y) \subseteq X \setminus Y\) for all \(g \in G\), so if we for every \(g \in G\) by \(\theta_g|_{X \setminus Y}\) denote the restriction of \(\theta_g\) to \(D_g^{-1} \cap X \setminus Y\), then the triple
\[
((D_g \cap X \setminus Y)_{g \in G}, (\theta_g|_{X \setminus Y})_{g \in G})
\]
is a partial action of \(G\) on \(X \setminus Y\) which we will denote by \(\theta_{|X \setminus Y}\). Thus \((X \setminus Y, G, \theta_{|X \setminus Y})\) is a discrete partial dynamical system.

Proposition 8.2. Let \((X, \theta, G)\) be a discrete partial dynamical system and let \(Y\) be a \(\theta\)-invariant subset of \(X\).

Let \(\mathcal{I}(Y)\) be the ideal of \(C^*(X, \theta, G)\) generated by the set
\[
\{\phi_{(X, \theta, G)}(A) \mid A \in \mathcal{B}(X, \theta, G), \ A \subseteq Y\}.
\]
Then the quotient \(C^*(X, \theta, G)/\mathcal{I}(Y)\) is isomorphic to \(C^*(X \setminus Y, \theta_{|X \setminus Y}, G)\).

More precisely: if \((s^X_g)_{g \in G}\) denotes the generators of \(C^*(X, \theta, G)\), and \((s^{X \setminus Y}_g)_{g \in G}\) denotes the generators of \(C^*(X \setminus Y, \theta_{|X \setminus Y}, G)\), then the map
\[
s^X_g + \mathcal{I}(Y) \mapsto s^{X \setminus Y}_g
\]
extends to a \(*\)-isomorphism from \(C^*(X, \theta, G)/\mathcal{I}(Y)\) to \(C^*(X \setminus Y, \theta_{|X \setminus Y}, G)\) which maps \(\phi_{(X, \theta, G)}(A) + \mathcal{I}(Y)\) to \(\phi_{(X \setminus Y, \theta_{|X \setminus Y}, G)}(A \cap X \setminus Y)\) for every \(A \in \mathcal{B}(X, \theta, G)\).

Proof. Let \((D_g)_{g \in G}\) denote the domains of \(\theta\) and let \(\mathcal{B}(X \setminus Y, \theta_{|X \setminus Y}, G)\) be the Boolean algebra on \(X \setminus Y\) generated by \(\{D_g \cap X \setminus Y \mid g \in G\}\). We claim that the following identity holds:
\[
(8.1) \quad \mathcal{B}(X \setminus Y, \theta_{|X \setminus Y}, G) = \{A \cap X \setminus Y \mid A \in \mathcal{B}(X, \theta, G)\}.
\]
To see this, notice first that \(\{A \cap X \setminus Y \mid A \in \mathcal{B}(X, \theta, G)\}\) is a Boolean algebra on \(X \setminus Y\). Then let \(g \in G\). Since \(D_g \in \mathcal{B}(X, \theta, G)\), we have that
\[
D_g \cap X \setminus Y \in \{A \cap X \setminus Y \mid A \in \mathcal{B}(X, \theta, G)\},
\]
and therefore that \(\mathcal{B}(X \setminus Y, \theta_{|X \setminus Y}, G) \subseteq \{A \cap X \setminus Y \mid A \in \mathcal{B}(X, \theta, G)\}\).

It is easy to check that \(\{A \subseteq X \mid A \cap X \setminus Y \in \mathcal{B}(X \setminus Y, \theta_{|X \setminus Y}, G)\}\) is a Boolean algebra on \(X\), and since we have that
\[
D_g \in \{A \subseteq X \mid A \cap X \setminus Y \in \mathcal{B}(X \setminus Y, \theta_{|X \setminus Y}, G)\}
\]
for every \(g \in G\), it follows that \(\mathcal{B}(X, \theta, G) \subseteq \{A \subseteq X \mid A \cap X \setminus Y \in \mathcal{B}(X \setminus Y, \theta_{|X \setminus Y}, G)\}\), which shows that \(8.1\) holds.

It follows from \(8.1\) that the map
\[
\pi : A \mapsto A \cap X \setminus Y
\]
is a Boolean homomorphism from \(\mathcal{B}(X, \theta, G)\) to \(\mathcal{B}(X \setminus Y, \theta_{|X \setminus Y}, G)\) which maps \(D_g\) to \(D_g \cap X \setminus Y\) for every \(g \in G\). Thus \(\phi_{(X \setminus Y, \theta_{|X \setminus Y}, G)} \circ \pi\) is a Boolean homomorphism from \(\mathcal{B}(X, \theta, G)\) to the Boolean algebra of projections in the unital abelian \(C^*\)-algebra \(C^*\left(\{s^X_g s^{X \setminus Y}_g \mid g \in G\}\right)\) which maps \(D_g\)
to $s^X \setminus Y s^X \setminus Y^*$ for every $g \in G$. So it follows from the universal property of $C^*(X, \theta, G)$ that there exists a $*$-homomorphism $\psi$ from $C^*(X, \theta, G)$ to $C^*(X \setminus Y, \theta|_{X \setminus Y}, G)$ which maps $s^X_g$ to $s^X_{g|Y}$ for every $g \in G$, and $\phi_{(X, \theta, G)}(A)$ to $\phi_{(X \setminus Y, \theta|_{X \setminus Y}, G)} \circ \pi(A)$ for every $A \in B(X, \theta, G)$. We are done with the proof when we have shown that $\ker \psi = \mathcal{I}(Y)$.

If $A \subseteq Y$, then $\pi(A) = \emptyset$ and therefore $\psi(\phi_{(X, \theta, G)}(A)) = 0$. Thus $\mathcal{I}(Y) \subseteq \ker \psi$.

The quotient map from $C^*(X, \theta, G)$ to $C^*(X, \theta, G)/\mathcal{I}(Y)$ induces a Boolean map $\pi$ from the Boolean algebra of projections in the unital abelian $C^*$-algebra

$$C^* \left( \{ s^X_g s^X_{g|Y} \mid g \in G \} \right)$$

to the Boolean algebra of projections in the unital abelian $C^*$-algebra

$$C^* \left( \{ s^X_g s^X_{g|Y} + \mathcal{I}(Y) \in C^*(X, \theta, G)/\mathcal{I}(Y) \mid g \in G \} \right)$$

which maps $p$ to $p + \mathcal{I}(Y)$ for every projection $p$ in $C^* \left( \{ s^X_g s^X_{g|Y} \mid g \in G \} \right)$. The map $\pi \circ \phi_{(X, \theta, G)}$ is then a Boolean homomorphism from $B(X, \theta, G)$ to the Boolean algebra of projections in the unital abelian $C^*$-algebra

$$C^* \left( \{ s^X_g s^X_{g|Y} + \mathcal{I}(Y) \in C^*(X, \theta, G)/\mathcal{I}(Y) \mid g \in G \} \right)$$

which maps $D_g$ to $s^X_g s^X_{g|Y} + \mathcal{I}(Y)$ for every $g \in G$. Since $\pi \circ \phi_{(X, \theta, G)}(A) = 0$ if $A \subseteq Y$, we have that $\pi \circ \phi_{(X, \theta, G)}$ induces a Boolean homomorphism from $B(X \setminus Y, \theta|_{X \setminus Y}, G)$ to the Boolean algebra of projections in the unital abelian $C^*$-algebra

$$C^* \left( \{ s^X_g s^X_{g|Y} + \mathcal{I}(Y) \in C^*(X, \theta, G)/\mathcal{I}(Y) \mid g \in G \} \right)$$

which maps $D_g \cap X \setminus Y$ to $s^X_g s^X_{g|Y} + \mathcal{I}(Y)$ for every $g \in G$. Thus it follows from the universal property of $C^*(X \setminus Y, \theta|_{X \setminus Y}, G)$ that there exists a $*$-homomorphism $\tau$ from $C^*(X \setminus Y, \theta|_{X \setminus Y}, G)$ to $C^*(X, \theta, G)/\mathcal{I}(Y)$ which maps $s^X_g$ to $s^X_{g|Y} + \mathcal{I}(Y)$ for every $g \in G$.

The $*$-homomorphism $\psi$ from $C^*(X, \theta, G)$ to $C^*(X \setminus Y, \theta|_{X \setminus Y}, G)$ induces a $*$-homomorphism $\tilde{\psi}$ from $C^*(X, \theta, G)/\ker(\psi)$ to $C^*(X \setminus Y, \theta|_{X \setminus Y}, G)$ which maps $s^X_g + \ker(\psi)$ to $s^X_{g|Y} + \mathcal{I}(Y)$. Thus $\tau \circ \tilde{\psi}$ is a $*$-homomorphism from $C^*(X, \theta, G)/\ker(\psi)$ to $C^*(X, \theta, G)/\mathcal{I}(Y)$ which maps $s^X_g + \ker(\psi)$ to $s^X_{g|Y} + \mathcal{I}(Y)$. This shows that $\ker(\psi) \subseteq \mathcal{I}(Y)$. □

Let $(X, \theta, G)$ be a discrete partial dynamical system and $Y$ a $\theta$-invariant subset of $X$. Clearly, the ideal $\mathcal{I}(Y)$ from Proposition 5.2 only depends of the set $\{ A \in B(X, \theta, G) \mid A \subseteq Y \}$. We will for a subset $Y$ of $X$ call the set

$$\bigcup \{ A \in B(X, \theta, G) \mid A \subseteq Y \}$$

for the $\theta$-admissible core of $Y$, and we will call $Y$ $\theta$-admissible if it is equal to its $\theta$-admissible core. We then have for every $\theta$-invariant subset $Y$ of $X$ that the ideal $\mathcal{I}(Y)$ from Proposition 5.2 is identical to the ideal $\mathcal{I}(Y^\circ)$ where $Y^\circ$ denotes the $\theta$-admissible core of $Y$.

Notice that a subset $Y$ of $X$ is $\theta$-admissible if and only if there for every $x \in Y$ exists a $A \in B(X, \theta, G)$ such that $x \in A \subseteq Y$. 

We have just seen how an \( \theta \)-invariant \( \theta \)-admissible subset of \( X \) gives raise to an ideal of \( C^*(X, \theta, G) \). We will now go the other way and from an ideal of \( C^*(X, \theta, G) \) construct a \( \theta \)-invariant \( \theta \)-admissible subset of \( X \).

**Proposition 8.3.** Let \( (X, \theta, G) \) be a discrete partial dynamical system and let \( I \) be an ideal in \( C^*(X, \theta, G) \). Then the set
\[
U(I) = \bigcup \{ A \in B(X, \theta, G) \mid \phi_{(X, \theta, G)}(A) \in I \}
\]
is a \( \theta \)-invariant \( \theta \)-admissible subset of \( X \).

*Proof.* The set \( U(I) \) is clearly a \( \theta \)-admissible subset of \( X \), and it follows from Lemma 8.3 that it is \( \theta \)-invariant. \( \square \)

**Proposition 8.4.** Let \( (X, \theta, G) \) be a discrete partial dynamical system and let \( Y \) be a \( \theta \)-invariant \( \theta \)-admissible subset of \( X \). Then we have that
\[
Y = U(I(Y)),
\]
where \( I(Y) \) is as in Proposition 8.2 and \( U(I(Y)) \) is as in Proposition 8.3.

*Proof.* Since \( Y \) and \( U(I(Y)) \) both are \( \theta \)-admissible subsets of \( X \), it is enough to prove that the equivalence
\[
A \subseteq Y \iff \phi_{(X, \theta, G)}(A) \in I(Y)
\]
holds for all \( A \in B(X, \theta, G) \). It is clear that \( \phi_{(X, \theta, G)}(A) \in I(Y) \) if \( A \subseteq Y \).

Assume that \( A \) is not a subset of \( Y \). Then \( A \cap Y \) is non-empty, and since \( \phi_{(X, X \setminus Y, G)} \) is injective, \( \phi_{(X \setminus Y, \theta_{X \setminus Y}, G)}(A \cap X \setminus Y) \) is a non-zero element of \( C^*(X \setminus Y, \theta_{X \setminus Y}, G) \). By Proposition 8.2, this implies that \( \phi_{(X, \theta, G)}(A) + I(Y) \) is non-zero, and thus that \( \phi_{(X, \theta, G)}(A) \) is not in \( I(Y) \). \( \square \)

From Proposition 8.4 now directly follows the promised theorem about the existence of an injective order preserving map between certain \( \theta \)-invariant subsets of \( X \) and ideals of \( C^*(X, \theta, G) \):

**Theorem 8.5.** Let \( (X, \theta, G) \) be a discrete partial dynamical system and let for every \( \theta \)-invariant \( \theta \)-admissible subset \( Y \) of \( X \), \( I(Y) \) be the ideal of \( C^*(X, \theta, G) \) generated by the set
\[
\{ \phi_{(X, \theta, G)}(A) \mid A \in B(X, \theta, G), \ A \subseteq Y \}.
\]

Then the map
\[
Y \mapsto I(Y)
\]
is an injective order preserving (i.e., \( Y \subseteq Z \Rightarrow I(Y) \subseteq I(Z) \)) map from the set of \( \theta \)-invariant \( \theta \)-admissible subsets of \( X \) to the set of ideals of

We will now present a result which relate the \( C^* \)-algebras of two partial dynamical systems to each other:

**Theorem 8.6.** Let \( G \) be a group and let for \( i \in \{1, 2\} \), \( X_i \) be a set, \( \theta_i \) a partial action of \( G \) on \( X_i \) and \( (\theta^N_g)_{g \in G} \) the partial one-to-one maps of \( \theta_i \). If there exists a Boolean homomorphism \( \eta \) from \( B(X_1, \theta_1, G) \) to \( B(X_2, \theta_2, G) \) such that the identity
\[
\eta(\theta^N_{X_1}(A)) = \theta^N_{X_2}(\eta(A))
\]
holds for all $A \in \mathcal{B}(X_1, \theta_1, G)$ and $g \in G$, then $C^*(X_2, \theta_2, G)$ is a quotient of $C^*(X_1, \theta_1, G)$.

More precisely: if for $i \in \{0, 1\}$, $(s^X_g)_{g \in G}$ denotes the generators of $C^*(X_i, \theta_i, G)$, then there exists a surjective $*$-homomorphism from $C^*(X_1, \theta_1, G)$ to $C^*(X_2, \theta_2, G)$ which maps $s^X_{g_1}$ to $s^X_{g_2}$ for every $g \in G$, and $\phi(x, \theta_i, G)(A)$ to $\phi(x, \theta_2, G)(\eta(A))$ for every $A \in \mathcal{B}(X_1, \theta_1, G)$.

The kernel of this $*$-homomorphism is the ideal generated by the set
\[ \{\phi(x, \theta_1, G)(A) \mid A \in \mathcal{B}(X_1, \theta_1, G), \, \eta(A) = \emptyset\}. \]

Proof. Notice that $Y = \bigcup\{A \in \mathcal{B}(X_1, \theta_1, G) \mid \eta(A) = \emptyset\}$ is a $\theta_1$-invariant $\theta_1$-admissible subset of $X_1$ and that if we let $\mathcal{I}(Y)$ be as in Proposition 8.2 then $\mathcal{I}(Y)$ is the ideal generated by the set
\[ \{\phi(x, \theta_1, G)(A) \mid A \in \mathcal{B}(X_1, \theta_1, G), \, \eta(A) = \emptyset\}. \]

It directly follows from the universal property of $C^*(X_1, \theta_1, G)$ that there exists a $*$-homomorphism $\psi$ from $C^*(X_1, \theta_1, G)$ to $C^*(X_2, \theta_2, G)$ which maps $s^X_{g_1}$ to $s^X_{g_2}$ for every $g \in G$ and $\phi(x, \theta_i, G)(A)$ to $\phi(x, \theta_2, G)(\eta(A))$ for every $A \in \mathcal{B}(X_1, \theta_1, G)$, and it is clear that the set
\[ \{\phi(x, \theta_1, G)(A) \mid A \in \mathcal{B}(X_1, \theta_1, G), \, \eta(A) = \emptyset\} \]
is contained in the kernel of $\psi$. Since $\psi$ maps $s^X_{g_1}$ to $s^X_{g_2}$, $\psi$ is surjective, so we only have to show that the kernel of $\psi$ is contained in $\mathcal{I}(Y)$. We will do that by proving that there exists a $*$-homomorphism from $C^*(X_1, \theta_1, G)/\ker \psi$ to $C^*(X_1, \theta_1, G)/\mathcal{I}(Y)$ which maps $s^{X_1}_{g_1} + \ker \psi$ to $s^{X_2}_{g_2} + \mathcal{I}(Y)$ for every $g \in G$.

Let us denote the generators of $C^*(X_1 \setminus Y, \theta_{1|X_1 \setminus Y}, G)$ by $(s^{X_1}_{g|Y})_{g \in G}$. Since $\psi$ is surjective, it induces a $*$-isomorphism from $C^*(X_1, \theta_1, G)/\ker \psi$ to $C^*(X_2, \theta_2, G)$ which maps $s^{X_1}_{g}$ to $s^{X_2}_{g}$ and it follows from Proposition 8.2 that there exists a $*$-isomorphism from $C^*(X_1, \theta_1, G)/\mathcal{I}(Y)$ to $C^*(X_1 \setminus Y, \theta_{1|X_1 \setminus Y}, G)$ which maps $s^{X_1}_{g} + \mathcal{I}(Y)$ to $s^{X_1}_{g|Y}$ for every $g \in G$. So all we have to do is to show that there exists a $*$-homomorphism from $C^*(X_2, \theta_2, G)$ to $C^*(X_1 \setminus Y, \theta_{1|X_1 \setminus Y}, G)$ which maps $s^{X_2}_{g}$ to $s^{X_1}_{g|Y}$ for every $g \in G$.

Let $(D^X_g)_{g \in G}$ denote the domains of $\theta_1$ and $(D^{X_2}_g)_{g \in G}$ denote the domains of $\theta_2$. We then have that
\[ \eta(D^X_g) = \eta(\theta^X_g(X_1)) = \theta^X_g(\eta(X_1)) = \theta^{X_2}_g(X_2) = D^{X_2}_g \]
for every $g \in G$, $\eta$ is surjective, and if $A, B \in \mathcal{B}(X_1, \theta_1, G)$ and $\eta(A) = \eta(B)$, then $A \cap X_1 \setminus Y = B \cap X_1 \setminus Y$. So the map
\[ \eta(A) \mapsto A \cap X_1 \setminus Y \]
is a well-defined map from $\mathcal{B}(X_2, \theta_2, G)$ to $\mathcal{B}(X_1 \setminus Y, \theta_{1|X_1 \setminus Y}, G)$ which maps $A$ to $A \cap X_1 \setminus Y$ for every $g \in G$. It is easy to check that it is a Boolean homomorphism, so it follows from the universal property of $C^*(X_2, \theta_2, G)$ that there exists a $*$-homomorphism from $C^*(X_2, \theta_2, G)$ to $C^*(X_1 \setminus Y, \theta_{1|X_1 \setminus Y}, G)$ which maps $s^{X_2}_g$ to $s^{X_1}_g$ for every $g \in G$. \qed
8.1. The ideal structure of $C(X) \rtimes_{\tau^*} \mathbb{Z}$. Let $(X, \tau)$ be a two-sided shift space over the finite alphabet $a$. As it is proved in the proof of Theorem 7.3, the Boolean algebra $B(X, \theta_X, F_a)$ is equal to the Boolean algebra of clopen subsets of $X$, and since the clopen subsets generate the topology of $X$, a subset of $X$ is $\theta_X$-admissible if and only if it is open. It is easy to check that a subset $Y$ of $X$ is $\theta_X$-invariant if and only if $\tau(Y) = Y$.

Thus we recover from Theorem 7.3 and 8.5 that there exists an injective order preserving map from the set of open $\tau$-invariant subsets of $X$ to the set of ideals of $C(X) \rtimes_{\tau^*} \mathbb{Z}$. If $(X, \tau)$ is free (meaning that $\tau^n(x) \neq x$ for every $x \in X$ and every $n \in \mathbb{Z} \setminus \{0\}$) then this map is bijective, cf. [26] Proposition 5.10 and Théorème 5.15.

8.2. The ideal structure of $\tilde{\Omega}_A$. Let $I$ be an arbitrary index set and let $A = (A(i,j))_{i,j \in I}$ be a matrix with entries in $\{0,1\}$ and having no zero rows. There is, as mentioned in Remark 5.1, a homeomorphism from $\tilde{X}_A$ to $\tilde{\Omega}_A$. Thus there is a bijective correspondence between elements of $B(X_A^+, \theta_{X_A^+}, F_I)$ and clopen subsets of $\tilde{\Omega}_A$. This correspondence extend to a correspondence between $\theta_{X_A^+}$-admissible subsets of $X_A^+$ and open subsets of $\tilde{\Omega}_A$, and this correspondence takes $\theta_{X_A^+}$-invariant subsets to invariant subsets of $\tilde{\Omega}_A$.

Thus we get from Theorem 7.3 and 8.5 an injective order preserving map from the set of open invariant subsets of $\tilde{\Omega}_A$ to the set of ideals of $\tilde{\Omega}_A$. Exel and Laca have in [11, Theorem 15.1] proved that this map is bijective if the directed graph $Gr(A)$ of $A$ has no transitory circuits (cf. [11] Section 12).

8.3. The ideal structure of $O_{X^+}$. Let $(X^+, \sigma)$ be a one-sided shift space over the finite alphabet $a$, and let $\mathcal{L}(X^+)$ be the language of $X^+$ (cf. [19] §1.3), that is

$$\mathcal{L}(X^+) = \{ u \in a^* \mid \exists x \in X^+, 0 \leq k \leq l : x_k x_{k+1} \cdots x_l = u \}.$$

We let $(X^+, \theta_{X^+}, F_a)$ be the discrete partial dynamical system of Theorem 8.2 and $(D_g)_{g \in F_a}$ be the domains and $(\theta_g)_{g \in F_a}$ the partial one-to-one maps of $\theta_{X^+}$.

Following [21], we let for every $x \in X^+$ and every $k \in \mathbb{N}_0$, $\mathcal{P}_k(x)$ be the subset of $\mathcal{L}(X^+)$ defined by

$$\mathcal{P}_k(x) = \{ u \in \mathcal{L}(X^+) \mid ux \in X^+, |u| = k \},$$

and define for every $l \in \mathbb{N}_0$ an equivalence relation $\sim_l$ on $X^+$ by

$$x \sim_l x' \iff \mathcal{P}_l(x) = \mathcal{P}_l(x').$$

We then define for every $(k, l) \in \mathbb{N}_0^2$ an equivalence relation $k \sim_l$ on $X^+$ by

$$x \sim_l y \iff x|_{[0,k]} = y|_{[0,k]} \land \mathcal{P}_l(x|_{[k,\infty]}) = \mathcal{P}_l(y|_{[k,\infty]}),$$

and let $[x]|_l$ denote the equivalence class of $x$ under this equivalence relation. We then define an order $\leq$ on $\mathbb{N}_0^2$ by

$$(k_1, l_1) \leq (k_2, l_2) \iff k_1 \leq k_2 \land l_1 \leq l_2 - k_2.$$
Lemma 8.7. holds for all $x$ and $x$

Proof. Let $\neg P$ where $P$.
We must then show that $\forall x \in X^+$ and notice that if $(k_1, l_1) \leq (k_2, l_2)$, then $k_2[x]_{l_2} \subseteq k_1[x]_{l_1}$ for all $x \in X^+$.

Notice also that $k[x]_l \in B(X^+, \theta_{X^+}, F_a)$ for all $x \in X^+$ and all $(k, l) \in \mathbb{N}_0^2$ because we have that

$$k[x]_l = \theta_{x[k,l]} \left( \left( \bigcap_{u \in \mathcal{P}(x[k,\infty])} (D_{u-1}) \right) \cap \left( \bigcap_{u \in \mathcal{P}(x[k,\infty])} (X^+ \setminus D_{u-1}) \right) \right),$$

where $\mathcal{P}(x[k,\infty]) = \{ u \in \mathcal{L}(X^+) \mid |u| = l, u \notin \mathcal{P}(x[k,\infty]) \}$.

Lemma 8.8. There exists for every $A \in B(X^+, \theta_{X^+}, F_a)$ a $(k, l) \in \mathbb{N}_0^2$ such that the implication

$$x \in A \Rightarrow k[x]_l \subseteq A$$

holds for all $x \in X^+$.

Proof. Let $A$ be the subset of $B$ defined by

$$A = \{ A \in B(X^+, \theta_{X^+}, F_a) \mid \exists (k, l) \in \mathbb{N}_0^2 \forall x \in A : k[x]_l \subseteq A \}.$$

We must then show that $A = B(X^+, \theta_{X^+}, F_a)$. Clearly $X^+ \in A$. Assume that $A, B \in A$ and choose $(k_a, l_a), (k_b, l_b) \in \mathbb{N}_0^2$ such that the two implications

$$x \in A \Rightarrow k_a[x]_{l_a} \subseteq A$$

and

$$x \in B \Rightarrow k_b[x]_{l_b} \subseteq B$$

hold for all $x \in X^+$.

Let $k = \max\{k_a, k_b\}$ and $l = \max\{l_a - k_a, l_b - k_b\} + k$. Then we have that $(k_a, l_a), (k_b, l_b) \preceq (k, l)$, from which it follows that the implication

$$x \in A \cap B \Rightarrow k[x]_l \subseteq k_a[x]_{l_a} \cap k_b[x]_{l_b} \subseteq A \cap B,$$

holds for all $x \in X^+$. This shows that $A \cap B \in A$. We also have that the three implications

$$x \in X^+ \setminus A \Rightarrow k_a[x]_{l_a} \subseteq X^+ \setminus A,$$

$$x \in \theta_a(A) \Rightarrow k_a[x]_{l_a} \subseteq \theta_a(A)$$

and

$$x \in \theta_{a^{-1}}(A) \Rightarrow k_a[x]_{l_a+1} \subseteq \theta_{a^{-1}}(A)$$

hold for all $x \in X^+$. Thus the sets $X^+ \setminus A, \theta_a(A)$, and $\theta_{a^{-1}}(A)$ all belong to $A$. Hence $A$ is a Boolean algebra containing $D_g$ for every $g \in F_a$, which means that $A = B(X^+, \theta_{X^+}, F_a)$.

Lemma 8.8. A subset $Y$ of $X^+$ is $\theta_{X^+}$-admissible if and only if there for every $x \in Y$ exists a $(k, l) \in \mathbb{N}_0^2$ such that $k[x]_l \subseteq Y$.

Proof. Let $A$ be the subset of $X^+$ defined by

$$A = \{ Y \subseteq X^+ \mid \forall x \in Y \exists (k, l) \in \mathbb{N}_0^2 : k[x]_l \subseteq Y \}. $$

We will show that a subset $Y$ of $X^+$ belongs to $A$ if and only if it is $\theta_{X^+}$-admissible.

It is clear that if $(Y_i)_{i \in I}$ is a family of elements of $A$, then $\bigcup_{i \in I} Y_i \in A$, and since it follows from Lemma 8.7 that $B(X^+, \theta_{X^+}, F_a) \subseteq A$, we have that every $\theta_{X^+}$-admissible subset of $X^+$ belongs to $A$. $\blacksquare$
Now let $Y \in A$. It is in order to show that $Y$ is $\theta_{X^+}$-admissible enough to show that there for every $x \in Y$ exists a $A \in \mathcal{B}(X^+, \theta_{X^+}, \mathbb{F}_a)$ such that $x \in A \subseteq Y$, but we have that $x \in k[x]_l \subseteq Y$, and $k[x]_l \in \mathcal{B}(X^+, \theta_{X^+}, \mathbb{F}_a)$ for some $(k, l) \in \mathbb{N}_0^2$, so we are done. \hfill \Box

It is not difficult to see that a subset $Y$ of $X^+$ is $\theta_{X^+}$-invariant if and only if $\sigma(Y) \subseteq Y$ and $\sigma^{-1}(Y) \subseteq Y$. Combining this with Theorem 8.4, Theorem 8.5 and Lemma 8.8 we get:

**Theorem 8.9.** Let $(X^+, \sigma)$ be a one-sided shift space over the finite alphabet $a$, let $\mathcal{H}(X^+)$ denote the set of subsets $Y$ of $X^+$ which satisfies the following 3 conditions:

1. $\sigma(Y) \subseteq Y$,
2. $\sigma^{-1}(Y) \subseteq Y$,
3. $\forall x \in Y \exists (k, l) \in \mathbb{N}_0^2 : k[x]_l \subseteq Y$,

and let for every subset $Y$ in $\mathcal{H}(X^+)$, $\mathcal{I}(Y)$ be the ideal of $C^*(X^+, \theta_{X^+}, \mathbb{F}_a)$ generated by the set

$$\{ \phi_{(X^+, \theta_{X^+}, \mathbb{F}_a)}(A) \mid A \in \mathcal{B}(X^+, \theta_{X^+}, \mathbb{F}_a), \ A \subseteq Y \}.$$ 

Then the map

$$Y \mapsto \mathcal{I}(Y)$$

is an injective order preserving (i.e., $Y \subseteq Z \Rightarrow \mathcal{I}(Y) \subseteq \mathcal{I}(Z)$) map from $\mathcal{H}(X^+)$ to the set of ideals of $\mathcal{O}_{X^+}$.

If $Y$ is an open subset of $X^+$ then there exists for every $x \in Y$ a $(k, l) \in \mathbb{N}_0^2$ such that $k[x]_l \subseteq Y$ (in fact, one can choose $l$ to be 0). There may on the other hand be subsets $Y$ of $X^+$ which are not open, but with the property that $\forall x \in Y \exists (k, l) \in \mathbb{N}_0^2 : k[x]_l \subseteq Y$. If however $X^+$ is a shift of finite type (cf. [19, §2.1]), then this can not happen (in fact according to [19, Theorem 1], $X^+$ is of finite type if and only if $\sigma$ is an open map, and it is not difficult to show that this is equivalent to $k[x]_l$ being clopen for all $x \in X^+$ and all $(k, l) \in \mathbb{N}_0^2$). Thus if $X^+$ is a shift of finite type, then the set $\mathcal{H}(X^+)$ from Theorem 8.9 is the set of all open subsets $Y$ of $X^+$ with the property that $\sigma(Y) \subseteq Y$ and $\sigma^{-1}(Y) \subseteq Y$. Hence we get the following corollary:

**Corollary 8.10.** Let $(X^+, \sigma)$ be a one-sided shift of finite type and let for every subset $Y$ of $X^+$, $\mathcal{I}(Y)$ be the ideal of $C^*(X^+, \theta_{X^+}, \mathbb{F}_a)$ generated by the set

$$\{ \phi_{(X^+, \theta_{X^+}, \mathbb{F}_a)}(A) \mid A \in \mathcal{B}(X^+, \theta_{X^+}, \mathbb{F}_a), \ A \subseteq Y \}.$$ 

Then the map

$$Y \mapsto \mathcal{I}(Y)$$

is an injective order preserving (i.e., $Y \subseteq Z \Rightarrow \mathcal{I}(Y) \subseteq \mathcal{I}(Z)$) map from the set of open subsets $Y$ of $X^+$ with the property that $\sigma(Y) \subseteq Y$ and $\sigma^{-1}(Y) \subseteq Y$ to the set of ideals of $\mathcal{O}_{X^+}$. 


8.4. Connections between the \( C^* \)-algebras of one- and two-sided shift spaces. We will in this section let \( X \) be a two-sided shift space, and
\[
X^+ = \{(z_n)_{n \in \mathbb{N}_0} \mid (z_n)_{n \in \mathbb{Z}} \in X\}
\]
be the corresponding one-sided shift space. We then have that map
\[
\pi : (z_n)_{n \in \mathbb{Z}} \mapsto (z_n)_{n \in \mathbb{N}_0}
\]
is a surjective continuous map from \( X \) to \( X^+ \) and \( \pi \circ \tau = \sigma \circ \pi \), where \( \tau \) and \( \sigma \) are the maps defined by \eqref{section 7.1} and \eqref{section 7.2}, respectively.

We will throughout this section let \((X, \theta_X, F_a)\) be the discrete partial dynamical system associated to \((X, \tau)\) as done in Theorem \[7.3\] and \((X^+, \theta_{X^+}, F_a)\) be the discrete partial dynamical system associated to \((X^+, \sigma)\) as done in Theorem \[7.4\]. We let \((D^X_g)_{g \in F_a}\) and \((\theta^X_g)_{g \in F_a}\) denote the domains and partial one-to-one maps of \( \theta_X \), and \((D^{X^+}_g)_{g \in F_a}\) and \((\theta^{X^+}_g)_{g \in F_a}\) denote the domains and partial one-to-one maps of \( \theta_{X^+} \).

Remark 8.11. If \( X \) only contains finitely many elements, then every element of \( X \) is periodic (meaning that there exists an \( n \in \mathbb{N} \) such that \( \tau^n(z) = z \)), so \( \pi \) is bijective, and since it also satisfies that \( \theta_g \circ \pi = \pi \circ \theta_g \) for every \( g \in F_a \), it easily follows from Theorem \[7.3\] and \[7.4\] that there exists a \( \ast \)-isomorphism from \( O_{X^+} \) to \( C(X) \rtimes_{\tau^+} Z \) which maps \( \sum_{a \in a} S_a \) to \( U \) and \( \eta_O(1_{D^X_g}) \) to \( 1_{D^X_g} \) for every \( g \in F_a \), where \( U \) is as in Section \[7.1\] and \( (S_a)_{a \in a} \) and \( \eta_O \) are as in Section \[7.2\].

If \( X \) contains infinitely many elements, then \( C(X) \rtimes_{\tau^+} Z \) and \( O_{X^+} \) are, as we will see below, in general quit different. We will however show later that if \( X \) has a certain property, then \( C(X) \rtimes_{\tau^+} Z \) is a quotient of \( O_{X^+} \).

Remark 8.12. If \( Y \) is an open subset of \( X^+ \) with the property that \( \sigma(Y) \subseteq Y \) and \( \sigma^{-1}(Y) \subseteq Y \), then \( \pi^{-1}(Y) \) is an open subset of \( X \) and \( \pi(\pi^{-1}(Y)) = \pi^{-1}(Y) \). This explains why \( C(X) \rtimes_{\tau^+} Z \) in general have more ideals than \( O_{X^+} \) if \( X \) (and thus \( X^+ \)) is of finite type.

It is in fact easy to construct an example of a shift of finite type \( X \) such that \( O_{X^+} \) is simple, but \( X \) contains infinitely many open \( \tau \)-invariant subset and thus \( C(X) \rtimes_{\tau^+} Z \) infinitely many ideals according to Section \[5.2\]. The full two-shift \( \{0,1\}^\mathbb{Z} \) will for example do the trick (in this case \( O_{X^+} \) will be the Cuntz-algebra \( O_\mathbb{Z} \), cf. \[8\]).

If \( X \) and \( X^+ \) are not of finite type, then it might happen that even though \( X^+ \) is minimal (meaning that the only closed subsets \( Y \) of \( X^+ \) such that \( \sigma(Y) \subseteq Y \) are \( X^+ \) and \( \emptyset \), cf. \[19\], \[13.7\]), there exists a subset \( Y \) of \( X^+ \) which is neither equal to \( X^+ \) nor \( \emptyset \) and with the property that \( \sigma(Y) \subseteq Y \), \( \sigma^{-1}(Y) \subseteq Y \) and \( \forall x \in Y \exists (k,l) \in I : k[x]_l \subseteq Y \), and thus that \( O_{X^+} \) is not simple. We will in Example \[8.19\] see an example of this phenomenon.

We will now describe a class of shift spaces for which \( C(X) \rtimes_{\tau^+} Z \) is a quotient of \( O_{X^+} \), and we will then describe a subclass of this class for which we can show that the ideal of this quotient is a direct sum of a finite number of the compact operators \( K \).

This result has been used in \[4\] and \[5\] to compute the \( K \)-theory of \( O_{X^+} \) and relate it to the \( K \)-theory of \( C(X) \rtimes_{\tau^+} Z \) for these classes of shift spaces.
Definition 8.13. We say that a shift space \( X \) has property (\( \ast \)) if for every \( u \in \mathcal{L}(X^+) \) there exists an \( x \in X^+ \) such that \( \mathcal{P}_i(x) = \{u\} \).

An element \( z \in X \) is called left special if there exists \( z' \in X \) such that \( \pi(z) = \pi(z') \), but \( z_{-1} \neq z'_{-1} \).

Notice that \( z \in X \) is left special if and only if \( \mathcal{P}_1(\pi(z)) \) consists of a least two elements.

Definition 8.14. We say that a shift space \( X \) has property (\( \ast \ast \)) if it has property (\( \ast \)) and the number of left special elements of \( X \) is finite, and no such left special word is periodic.

It has been proved that every finite shift space and every minimal shift space with a finite number of left special elements (for example every shift space of a primitive substitution and every shift space of a Sturmian sequence) have property (\( \ast \ast \)), and that the shift space of a non-regular Toeplitz sequence has property (\( \ast \)), but not necessarily property (\( \ast \ast \)).

We say that \( z, z' \in X \) are right shift tail equivalent if there exist \( m, N \in \mathbb{Z} \) such that \( z_n = z'_n + m \) for all \( n > N \).

Lemma 8.15. Let \( X \) be a shift space which has property (\( \ast \ast \)) and let \( (k, l) \in \mathbb{N}_0^2 \). Then the set

\[
\{ x \in X^+ \mid \mathcal{P}_l(x[k,\infty)) \text{ contains more than one element} \}
\]

is finite, and every element of it is of the form \( \pi(z) \), where \( z \) is an element of \( X \) which is right shift tail equivalent to a left special element of \( X \).

Proof. Assume that \( u, v, w \in a^* \) are such that the last letter of \( u \) is different from the last letter of \( v \) and \( uw, vw \in \mathcal{P}_l(x[k,\infty]) \) (\( w \) might be the empty word). Then \( \mathcal{P}_l(wx[k,\infty]) \) consists of a least two elements so \( wx[k,\infty] \) is equal to \( \pi(z') \) for some left special element \( z' \).

Thus the set

\[
\{ x \in X^+ \mid \mathcal{P}_l(x[k,\infty)) \text{ contains more than one element} \}
\]

is finite, and every element of it is of the form \( \pi(z) \), where \( z \) is an element of \( X \) which is right shift tail equivalent to a left special element of \( X \). \( \square \)

Lemma 8.16. Let \( X \) be a shift space which has property (\( \ast \ast \)), and let \( z \in X \) be right shift tail equivalent to a left special element. Then \( \pi(z) \) is not eventually periodic, meaning that there exist no \( k, N \in \mathbb{N} \) such that \( z_n = z_{n+k} \) for all \( n > N \).

Proof. Assume that \( m, N_1 \in \mathbb{Z} \) and \( z_n = z'_{n+m} \) for all \( n > N_1 \) with \( z' \) being a left special element, and that \( k, N_2 \in \mathbb{N} \) are such that \( z_n = z_{n+k} \) for all \( n > N_2 \). We then have that \( z_{n+m} = z'_{n+m+k} \) for \( n > \max\{N_1, N_2\} \), so the set \( \{ n \in \mathbb{Z} \mid z'_n \neq z'_{n+k} \} \) is bounded above, and since \( z' \) is left special and thus not periodic, this set is not empty. So we can define \( n_0 \) by the equation

\[
n_0 = \max\{ n \in \mathbb{Z} \mid z'_n \neq z'_{n+k} \} + 1.
\]

We then have that \( z'_{n_0-1} \neq z'_{n_0-1+k} \) and that \( z'_n = z'_{n+k} \) for \( n \geq n_0 \).

We define a sequence \( z'' \) by letting \( z''_{n_0+i+k} = z'_{n_0+i} \) for \( i \in \{0, 1, \ldots, k-1\} \) and \( l \in \mathbb{Z} \). We then have for all \( n \in \mathbb{Z} \) that \( z''_{[l,\infty]} = z'_{[l+k,\infty]} \), where \( l \in \mathbb{Z} \) is chosen such that \( n + lk \geq n_0 \), and thus that \( z'' \in X \). We also have that \( z'' \)
is periodic, that \( z''_{n_0-1} = z'_{n_0-1+k} \neq z'_{n_0-1} \) and that \( z''_{[n_0,\infty]} = z'_{[n_0,\infty]} \). Thus \( \sigma^{n_0}(z'') \) is an element of \( X \) which is both periodic and left special, but that contradicts the assumption that \( X \) has property (**).

**Lemma 8.17.** Let \( X \) be a shift space which has property (**). Then we have that
\[
\{ \pi(z) \} \in \mathcal{B}(X^+, \theta_{X^+}, \mathbb{F}_a)
\]
for every \( z \in X \) which is right shift equivalent to a left special element.

**Proof.** Let \( z \) be an element of \( X \) which is right shift equivalent to a left special element. Then there exist a left special element \( z' \) and \( \alpha, \beta \in \mathbb{Z} \) such that \( z_n = z'_{n+\alpha} \) for all \( n > N \). Since \( z' \) is left special, there exist \( a, b \in a \) such that \( a \neq b \) and \( a, b \in P_1(\pi(z')) \). That means that
\[
\pi(z') \in D_{a-1}^{X^+} \cap D_{b-1}^{X^+},
\]
and since \( X \) only contains a finite number of special elements, \( D_{a-1}^{X^+} \cap D_{b-1}^{X^+} \) is finite. Thus there exists a \( k > 0 \) such that
\[
\{ z'_{[k,\infty]} \} = D_{(a+1)-1}^{X^+} \cap D_{(b+1)-1}^{X^+},
\]
where \( u = z'_{[0,k]} \).

Let us denote \( \max\{m + N, 0, m - k\} \) by \( n_0 \) and let \( v = z'_{[k,k+n_0]} \) and \( w = z_{[0,n_0]} \). We then have that \( \pi(z) = wy \) where \( y \) is the unique element in \( X^+ \) such that \( vy = z'_{[k,\infty]} \). Thus it follows from Lemma 3.3 that
\[
\{ \pi(z) \} = \theta_{w^{-1}}^{X^+} \left( D_{(a+1)-1}^{X^+} \cap D_{(b+1)-1}^{X^+} \right) \in \mathcal{B}(X^+, \theta_{X^+}, \mathbb{F}_a).
\]

Remember that there exist an inclusion \( \eta_\mathcal{O} \) of \( \mathcal{D}_{X^+} \) into \( \mathcal{O}_{X^+} \) (cf. Section 7.2) and a that \( C(X) \) sits inside \( C(X) \rtimes_{\tau^*} \mathbb{Z} \) (cf. Section 7.4). We will denote the inclusion of \( C(X) \) into \( C(X) \rtimes_{\tau^*} \mathbb{Z} \) by \( \eta_X \). We then have the follow result:

**Theorem 8.18.** Let \( X \) be a two-sided shift space which has property (*) and let \( X^+ \) be the corresponding one-sided shift space defined by \( \mathcal{D} \). Then there are surjective \( * \)-homomorphisms \( \kappa : \mathcal{D}_{X^+} \rightarrow C(X) \) and \( \rho : \mathcal{O}_{X^+} \rightarrow C(X) \rtimes_{\tau^*} \mathbb{Z} \) making the diagram
\[
\begin{array}{ccc}
\mathcal{D}_{X^+} & \xrightarrow{\kappa} & C(X) \\
\eta_\mathcal{O} \downarrow & & \downarrow \eta_X \\
\mathcal{O}_{X^+} & \xrightarrow{\rho} & C(X) \rtimes_{\tau^*} \mathbb{Z}
\end{array}
\]
commute. We furthermore have that
\[
\kappa(1_{C(u,v)}) = \begin{cases} 
1_{\{x \in X| x_{[0,|u|]} = v\}} & \text{if } \exists w \in a^* : v = uw, \\
1_{\{x \in X| x_{[|v|-|u|,|v|]} = u\}} & \text{if } \exists w \in a^* : u = vw, \\
0 & \text{else,}
\end{cases}
\]
for every \( u, v \in a^* \), and that \( \rho(S_a) = \eta_a(1_{\mathcal{D}^X})U \) for every \( a \in a \) where \( U \) is as in Section 7.2 and \( (S_a)_{a \in a} \) and \( C(u, v) \) are as in Section 7.2.
If $X$ also has property (**) then the kernel of $\rho$ is isomorphic to $K^{P}$, where $K$ is the $C^*$-algebra of compact operators on an infinite dimensional separable Hilbert space and $n_x$ is the number of right shift tail equivalence classes of $X$ containing a left special element.

Proof. Let $(X, \theta_X, F_a)$ be the discrete partial dynamical system associated to $(X, \tau)$ as done in Theorem 7.3 and $(X^+, \theta_{X^+}, F_a)$ be the discrete partial dynamical system associated to $(X^+, \sigma)$ as done in Theorem 7.4. We let $\theta^X_g, g \in F_a$ be the domains and partial one-to-one maps of $\theta_X$, and $(\theta^X_g^+, g \in F_a)$ the domains and partial one-to-one maps of $\theta_{X^+}$.

Let for every $A \in B(X^+, \theta_{X^+}, F_a)$, $\psi(A)$ be the subset of $X$ defined by

$$\psi(A) = \left\{ z \in X \mid \forall (k, l) \in I \exists x \in A : x_{[0,k]} = z_{[0,k]} \right\}.$$  

We claim that the map $\psi$ is a Boolean homomorphism from $B(X^+, \theta_{X^+}, F_a)$ to $B(X, \theta_X, F_a)$ which satisfies that

$$\psi(\theta^X_g^+(A)) = \theta^X_g(\psi(A))$$

for all $A \in B(X^+, \theta_{X^+}, F_a)$ and $g \in F_a$.

We will prove that by establishing a sequence of claims.

Claim 1. The map $\psi$ is a Boolean homomorphism.

Proof. We will prove that $\psi$ is a Boolean homomorphism by showing that

$$\psi(A \cap B) = \psi(A) \cap \psi(B)$$

and

$$\psi(X^+ \setminus A) = X \setminus \psi(A)$$

for all $A, B \in B(X^+, \theta_{X^+}, F_a)$.

Let $A, B \in B(X^+, \theta_{X^+}, F_a)$. It is obvious that $\psi(A \cap B) \subseteq \psi(A) \cap \psi(B)$. Assume that $z \in \psi(A) \cap \psi(B)$. There exist by Lemma 8.7 $(k_a, l_a), (k_b, l_b) \in N_0^2$ such that the two implications

$$x \in A \Rightarrow k_a[x]_{l_a} \subseteq A$$

and

$$x \in B \Rightarrow k_b[x]_{l_b} \subseteq B,$$

hold for all $x \in X^+$.

Let $(k_0, l_0) \in N_0^2$ and choose $(k, l) \in N_0^2$ such that we have that

$$(k, l) \geq (k_a, l_a), (k_b, l_b), (k_0, l_0),$$

and choose $x^A \in A$ and $x^B \in B$ such that we have that

$$z_{[0,k]} = x^A_{[0,k]} = x^B_{[0,k]}$$

and

$$\text{P}_l(x^A_{[k,\infty]}) = \text{P}_l(x^B_{[k,\infty]}) = \{z_{[k-l,k]}\}.$$

Then $x^A_{k-l} \sim x^B_{k-l}$, so $x^A \in A \cap B$, and since $x^A_{[0,k_0]} = z_{[0,k_0]}$ and $\text{P}_{l_0}(x^A_{[k_0,\infty]}) = \{z_{[k_0-l_0,k_0]}\}$, this shows that $z \in \psi(A \cap B)$. Thus $\psi(A \cap B) = \psi(A) \cap \psi(B)$.

Assume now that $z \in \psi(X^+ \setminus A)$. We then have that if $x \in X^+$, $z_{[0,k_0]} = x_{[0,k_0]}$ and $\text{P}_{l_0}(x_{[k_0,\infty]}) = \{z_{[k_0-l_0,k_0]}\}$, then $x \in X^+ \setminus A$. Thus $z \in X \setminus \psi(A)$.

If $z \in X \setminus \psi(A)$, then there is a $(k_z, l_z) \in N_0^2$ such that we have for every $x \in A$ have that

$$z_{[0,k_z]} \neq x_{[0,k_z]} \text{ or } \text{P}_{l_z}(x_{[k_z,\infty]}) \neq \{z_{[k_z-l_z,k_z]}\}.$$

Let $(k_l, l_0) \in N_0^2$ and choose $(k, l) \geq (k_z, l_z), (k_0, l_0)$. Since $X$ has property (**), there exists an $x \in X^+$ such that $z_{[0,k]} = x_{[0,k]}$ and $\text{P}_{l}(x_{[k,\infty]}) = \{z_{[k-l,k]}\}$.
This $x$ must belong to $X^+ \setminus A$, and since $z_{[0,k_0]} = x_{[0,k_0]}$ and $P_{\ell_0}(x_{[k_0,\infty]}) = \{z_{[k_0-\ell_0, k_0]}\}$, this shows that $z \in \psi(X^+ \setminus A)$. Thus $\psi(X^+ \setminus A) = X \setminus \psi(A)$. □

Claim 2. We have that $\psi(\theta_a^{X^+}(A)) = \theta_a^X(\psi(A))$ for every $A \in \mathcal{B}(X^+, \theta_X, F_a)$ and every $a \in \mathfrak{a}$.

Proof. Let $z \in \psi(\theta_a^{X^+}(A))$ and let $(k, l) \in \mathbb{N}_0^2$. Then there is an $x \in \theta_a^{X^+}(A)$ such that $x_{[0,k+1]} = z_{[0,k+1]}$ and $P_l(x_{[k+1,\infty]}) = \{z_{[k+1-,l,k+1]}\}$. That means that there is a $y \in A \cap D_{\alpha-1}^{X^+}$ such that $x = \theta_a^{X^+}(y)$. We then have that $z_0 = x_0 = a$, that $y_{[0,k]} = x_{[1,k+1]} = z_{[1,k+1]}$ and that

$$P_l(y_{[k,\infty]}) = P_l(x_{[k+1,\infty]}) = \{z_{[k+1-,l,k+1]}\}.$$

Thus $\tau(z) \in \psi(A)$, and $z_0 = a$, which shows that $z \in \theta_a^{X^+}(\psi(A))$.

Now let $z \in \psi(\theta_a^{X^+}(\psi(A)))$. Then there is a $z' \in \psi(A)$ such that $z = \theta_a^X(z')$. That means that there is a $y \in A \cap D_{\alpha-1}$ such that $x = \theta_a^{X^+}(y)$, and we then have that

$$z_{[0,k+1]} = ax_{[0,k]} = ax_{[0,k+1]} = (\theta_a^{X^+}(x))_{[0,k+1]}$$

and that

$$P_l((\theta_a^{X^+}(x))_{[k+1,\infty]}) = P_l(x_{[k,\infty]}) = \{z_{[k+1-,l,k+1]}\}.$$

which shows that $z \in \psi(\theta_a^{X^+}(A))$. □

Claim 3. We have that $\psi(\theta_a^{X^+}(A)) = \theta_a^{X^+}(\psi(A))$ for every $A \in \mathcal{B}(X^+, \theta_X, F_a)$.

Proof. Let $z \in \psi(\theta_a^{X^+}(A))$ and let $(k, l) \in \mathbb{N}_0^2$. Then there is an $x \in \theta_a^{X^+}(A)$ such that $x_{[0,k+1]} = z_{[0,k+1]}$ and $P_l(x_{[k+1,\infty]}) = \{z_{[k+1-,l,k+1]}\}$. That means that there is a $y \in A \cap D_{\alpha-1}^{X^+}$ such that $x = \theta_a^{X^+}(y)$, and we then have that

$$a = y_0 \in P_1(x) = \{z_{-1}\},$$

and thus that

$$y_{[0,k]} = ax_{[0,k-1]} = z_{[-1,k-1]}$$

and $P_l(y_{[k,\infty]}) = P_l(x_{[k-1,\infty]}) = \{z_{[k-1-,l,k-1]}\}$ if $k > 0$, and

$$P_l(y_{[k,\infty]}) = P_l(ax) = \{z_{[k-1-,l,k-1]}\}$$

if $k = 0$. Thus $\tau^{-1}(z) \in \psi(A)$ and $z_{-1} = a$, which shows that

$$z = \theta_a^{X^+}(\tau(z)) \in \theta_a^{X^+}(\psi(A)).$$

Now let $z \in \psi(\theta_a^{X^+}(\psi(A)))$. We then have that there is a $z' \in \psi(A) \cap D_{\alpha}^{X^+}$ such that $z = \theta_a^{X^+}(z')$. That means that there is an $x \in \theta_a^{X^+}(A)$ such that $z_{[0,k+1]} = x_{[0,k+1]}$ and $P_l(x_{[k+1,\infty]}) = \{z_{[k+1-,l,k+1]}\}$. We especially have that $x_0 = z_0 = a$, so $x \in A \cap D_{\alpha}$, and we also have that

$$z_{[0,k]} = x_{[1,k+1]} = x_{[1,k+1]} = (\theta_a^{X^+}(x))_{[0,k]}.$$
and that
\[ \mathcal{P}_l\left( \left( \theta_{a^{-1}}^+(x) \right)_{[k,\infty]} \right) = \mathcal{P}_l(x_{[k+1,\infty]}) = \{ z'_{[k+1-l, k+1]} \} = \{ z_{[k-l, l]} \}. \]
This shows that \( z \in \psi\left( \theta_{a^{-1}}^+(A) \right) \).

It follows from Claim 2 and Claim 3 and the definition of \( \theta_g^X \) and \( \theta_g^X \) that
\[ \psi\left( \theta_g^X(A) \right) = \theta_g^X\left( \psi(A) \right) \]
for every \( A \in \mathcal{B}(X^+, \theta_{X^+}, F_a) \) and every \( g \in F_a \).

Thus if we let \( (s_g^X)_{g \in F_a} \) denote the generators of \( C^*(X^+, \theta^X, F_a) \) and \( (s_g^X)_{g \in F_a} \) the generators of \( C^*(X, \theta_X, F_a) \), then it follows from Theorem 8.6 that there exists a surjective \(*\)-homomorphism \( \phi \) from \( C^*(X^+, \theta_{X^+}, F_a) \) to \( C^*(X, \theta_X, F_a) \) which maps \( s_g^X \) to \( s_g^X \) for every \( g \in F_a \), and thus from Theorem 3.3 and 3.4 that there is a surjective \(*\)-homomorphism \( \rho : \mathcal{O}_{X^+} \rightarrow C(X) \rtimes_r \mathbb{Z} \) which maps \( \sum_{a \in a} S_a \) to \( U \) and \( \eta_\mathcal{O}(1_{D_a}) \) to \( \eta_X(1_{D_a}) \) for every \( g \in F_a \). We then have that
\[ \rho(S_a) = \rho \left( \eta_\mathcal{O} \left( 1_{D_a} \sum_{a' \in a} S_{a'} \right) \right) = \eta_X(1_{D_a}) \]
for every \( a \in a \), and that
\[ \rho(\eta_\mathcal{O}(1_{C(u,v)})) = \rho(\eta_\mathcal{O}(1_{D_v})) \rho(\eta_\mathcal{O}(1_{D_{vu}})) \]
\[ = \eta_X(1_{D_u}) \eta_X(1_{D_{vu-1}}) \]
\[ = \begin{cases} \eta_X \left( 1 \{ x \in X^* | x_{[0,|v|]} = v \} \right) & \text{if } \exists w \in a^* : v = wu, \\ \eta_X \left( 1 \{ x \in X^* | x_{[-|u|,|v|]} = u \} \right) & \text{if } \exists w \in a^* : u = wv, \\ 0 & \text{else,} \end{cases} \]
for \( u, v \in a^* \), according to Lemma 3.1 and 3.6. Since \( \mathcal{D}_{X^+} \) is generated by \( \{ 1_{C(u,v)} \mid u, v \in a^* \} \) and \( C(X) \) is generated by the set
\[ \left\{ 1 \{ x \in X^* | x_{[k-|u|, k]} = u \} \mid u \in a^*, k \in \mathbb{Z} \right\}, \]
it follows that there exists a surjective \(*\)-homomorphism \( \kappa : \mathcal{D}_{X^+} \rightarrow C(X) \) with the desired properties.

Assume now that \( X \) has property (\(*\)). Let \( \mathcal{J}_X \) be the set of right shift tail equivalence classes of \( X \) which contains a left special element. We will show that the kernel of \( \phi \), and thus the kernel of \( \rho \), is isomorphic to \( n_X \) copies of \( K \) by constructing a family
\[ \left( e_{x,y} \right)_{j \in \mathcal{J}_X, x,y \in \pi(j)} \]
of non-zero elements of \( C^*(X^+, \theta_{X^+}, F_a) \) such that the equations
\[ (e_{x,y})^* = e_{y,x} \quad \text{and} \quad e_{x,y} e_{x',y'} = \begin{cases} e_{x,y} & \text{if } j = j' \text{ and } y = x', \\ 0 & \text{else,} \end{cases} \]
hold for all \( j, j' \in \mathcal{J}_X \) and all \( x, y \in \pi(j) \) and \( x', y' \in \pi(j') \), and such that we have that
\[ \text{span} \left\{ e_{x,y} \mid j \in \mathcal{J}_X, x,y \in \pi(j) \right\} = \ker \phi. \]
So let \( j \in J_X \) and \( x, y \in \pi(j) \). Then there exist \( n, m \in \mathbb{N} \) such that \( x_{[n, \infty]} = y_{[m, \infty]} \). It follows from Lemma 8.17 that \( \{x\} \) and \( \{y\} \) belong to \( B(X^+, \theta_X^+, F_a) \) so we can define an element \( e_{x, y}^j \) in \( C^*(X^+, \theta_X^+, F_a) \) by the equation

\[
e_{x, y}^j = \phi(X^+, \theta_X^+, F_a)(\{x\}) \cdot s_{X([0, n][0, m])} X^+ \cdot \phi(X^+, \theta_X^+, F_a)(\{y\}).
\]

Notice that \( e_{x, y}^j \) does not depend on the choice of \( n \) and \( m \), because if \( k, l \in \mathbb{N} \) also satisfy that \( x_{[k, \infty]} = y_{[l, \infty]} \), then we have that

\[
x_{n+l+i} = y_{m+l+i} = x_{m+k+i}
\]

for all \( i \in \mathbb{N} \), and since \( x \) is not eventually periodic according to Lemma 8.16, we have that \( n + l = m + k \) and thus that

\[
x_{[0, n][0, m]} = x_{[0, n+l][0, m+l]} = x_{[0, m+k][0, l]} = x_{[0, k][0, l]}.
\]

It is clear that \( e_{x, y}^j = e_{y, x}^j \) and that \( e_{x, y}^j \cdot e_{x', y'}^j = 0 \) if \( y \neq x' \). Assume now that \( j \in J_X \), that \( x, y, z \in \pi(j) \) and that \( k, l, m, n \in \mathbb{N} \) are such that \( x_{[n, \infty]} = y_{[m, \infty]} \) and \( y_{[k, \infty]} = z_{[l, \infty]} \). Then we have that

\[
y \in D_X^+ x_{[0, n][0, m]}^{-1} \quad \text{and} \quad \theta_X^+ x_{[0, n][0, m]}^{-1} (y) = x.
\]

Thus it follows from Lemma 8.15 that we have that

\[
s_{y_{[0, n][0, m]}} X^+ \cdot \phi(X^+, \theta_X^+, F_a)(\{y\}) = s_{X_{[0, n][0, m]}} X^+ \cdot \phi(X^+, \theta_X^+, F_a)(\{y\}) \cdot s_{y_{[0, n][0, m]}} X^+
\]

and

\[
\phi(X^+, \theta_X^+, F_a)(\{x\}) \cdot s_{x_{[0, n][0, m]}} X^+ = \phi(X^+, \theta_X^+, F_a)(\{x\}) \cdot s_{x_{[0, n][0, m]}} X^+.
\]

We also have that \( x \in D_X^+ x_{[0, n][0, m]}^{-1} \), and thus that

\[
\phi(X^+, \theta_X^+, F_a)(\{x\}) \cdot s_{x_{[0, n][0, m]}} X^+ = \phi(X^+, \theta_X^+, F_a)(\{x\}) \cdot s_{x_{[0, n][0, m]}} X^+.
\]

according to (4.14) and (4.15). Thus we have that

\[
e_{x, y}^j \cdot e_{y, z}^j = \phi(X^+, \theta_X^+, F_a)(\{x\}) \cdot s_{x_{[0, n][0, m]}} X^+ \cdot \phi(X^+, \theta_X^+, F_a)(\{y\}) \cdot s_{y_{[0, k][0, l]}} X^+ \cdot \phi(X^+, \theta_X^+, F_a)(\{z\})
\]

\[
= \phi(X^+, \theta_X^+, F_a)(\{x\}) \cdot s_{x_{[0, n][0, m]}} X^+ \cdot \phi(X^+, \theta_X^+, F_a)(\{y\}) \cdot s_{y_{[0, k][0, l]}} X^+ \cdot \phi(X^+, \theta_X^+, F_a)(\{z\})
\]

\[
= \phi(X^+, \theta_X^+, F_a)(\{x\}) \cdot s_{x_{[0, n][0, m]}} X^+ \cdot \phi(X^+, \theta_X^+, F_a)(\{y\}) \cdot s_{y_{[0, k][0, l]}} X^+ \cdot \phi(X^+, \theta_X^+, F_a)(\{z\})
\]

\[
= e_{x, z}^j.
\]
Finally we notice that it follows from Corollary 5.2 that \( c_{x,y}^j \) is non-zero, because we have that
\[
\pi^* c_{x,y}^j = c_{x,x}^j = \phi(x^+, \theta_{x^+, F_a})(\{x\}).
\]

It follows from Theorem 8.6 that the kernel of \( \phi \) is generated by the set
\[
\{ \phi(x^+, \theta_{x^+, F_a}(A) \mid \psi(A) = \emptyset \},
\]
and since \( \psi(\phi(x^+, \theta_{x^+, F_a})(\{x\})) = \emptyset \) if \( x \in \pi(j) \) for some \( j \in J_X \), we have that the set
\[
\text{span}\{ c_{x,y}^j \mid j \in J_X, \ x, y \in \pi(j) \}
\]
is contained in the kernel of \( \phi \).

Assume that \( A \in B(X^+, \theta_{X^+, F_a}) \) and that \( \psi(A) = \emptyset \). Choose by Lemma 8.7 \((k, l) \in \mathbb{N}_0^2 \) such that the implication
\[
x \in A \Rightarrow k[x]_l \subseteq A
\]
holds for all \( x \in X^+ \). Then we have that
\[
A = \bigcup_{x \in A} k[x]_l.
\]

Assume that \( x \in A \) and that the number of elements of \( \mathcal{P}_l(x[k, \infty]) \) is one. Then it follows from Lemma 3.6 that
\[
\psi(k[x]_l) = \psi\left( \theta_{x_0, k}^X \left( D_{v, l}^{X^+} \cap \left( \bigcap_{u \in -\mathcal{P}_l(\sigma^k(x))} X^+ \setminus D_{u, l}^{X^+} \right) \right) \right) = \theta_{x_0, k}^X \left( D_{v, l}^{X^+} \cap \left( \bigcap_{u \in -\mathcal{P}_l(\sigma^k(x))} X^+ \setminus D_{u, l}^{X^+} \right) \right) \neq 0,
\]
where \( v \) is the unique element of \( \mathcal{P}_l(x[k, \infty]) \) and \( -\mathcal{P}_l(x[k, \infty]) = \{ u \in \mathcal{L}(X^+) \mid |u| = l, \ u \notin \mathcal{P}_l(x[k, \infty]) \} \), but this contradict our assumption that \( \psi(A) = \emptyset \).

Thus \( \mathcal{P}_l(x[k, \infty]) \) consists of at least two elements for every \( x \in A \), so it follows from Lemma 8.17 that \( A \) is finite and that every element of \( A \) is of the form \( \pi(z) \) for some \( z \in X \) which is right shift tail equivalent to a left special element. Hence we get by Lemma 8.17 that
\[
\phi(x^+, \theta_{x^+, F_a})(A) = \sum_{x \in A} \phi(x^+, \theta_{x^+, F_a})(\{x\}) = 
\sum_{x \in A} e_{x,x}^x \in \text{span}\{ e_{x,y}^j \mid j \in J_X, \ x, y \in \pi(j) \}.
\]

where for every \( x \in A \), \( x \) denotes the right shift tail equivalence class which contains an element \( z \) such that \( \pi(z) = x \).

So to show that the equality
\[
\text{span}\{ e_{x,y}^j \mid j \in J_X, \ x, y \in \pi(j) \} = \ker \phi,
\]
holds, we just have to show that the subset
\[
\text{span}\{ e_{x,y}^j \mid j \in J_X, \ x, y \in \pi(j) \}
\]
is an ideal of $C^*_a(X^+, \theta_{X^+}, \mathbb{F}_a)$, and since $C^*_a(X^+, \theta_{X^+}, \mathbb{F}_a)$ is generated by the set $\{s_g^1 \mid g \in \mathbb{F}_a\}$, it is enough to prove that $s_g^1 e_{x,y}^1$ and $c_{x,y}^1 s_g^1$ belong to $	ext{span}\{c_{x',y'}^1 \mid j \in J_X, x', y' \in \pi(j)\}$ for every $g \in \mathbb{F}_a$, $j \in J_X$ and $x, y \in \pi(j)$.

So let $g \in \mathbb{F}_a$, $j \in J_X$ and $x, y \in \pi(j)$, and let $n, m \in \mathbb{N}$ such that $x[n, \infty] = y[m, \infty]$. It follows from Lemma 4.3 that if $s_g^1 \neq 0$, then there exist $u, v \in a^*$ such that $g = uv^{-1}$. If $y$ does not belong to $D_{uv^{-1}}^+$, then we have that

$$\phi(X^+, \theta_{X^+}, F_a)\{\{y\}\} s_{uv^{-1}}^+ = \phi(X^+, \theta_{X^+}, F_a)\{\{y\}\} s_{uv^{-1}}^+ = \emptyset,$$

from which it follows that

$$e_{x,y}^1 s_g^1 = \phi(X^+, \theta_{X^+}, F_a)\{\{x\}\} s_{x[y, y]}^+ x_{[0,n]\{y\}[0,m]}^+ \phi(X^+, \theta_{X^+}, F_a)\{\{y\}\} s_{uv^{-1}}^+ = 0.$$

If $y \in D_{uv^{-1}}^+$, then we have according to Lemma 3.4 and 4.5 that $\theta_{uv^{-1}}^+ (y) \in \pi(j)$, $\theta_{uv^{-1}}^+ (y)[m + |v|, \infty] = x[n + |u|, \infty]$ and that

$$\phi(X^+, \theta_{X^+}, F_a)\{\{y\}\} s_{uv^{-1}}^+ = \phi(X^+, \theta_{X^+}, F_a)\{\{y\}\} s_{uv^{-1}}^+ \phi(X^+, \theta_{X^+}, F_a)\{\{y\}\} s_{uv^{-1}}^+ = s_{uv^{-1}}^+ \phi(X^+, \theta_{X^+}, F_a)\{\{y\}\} s_{uv^{-1}}^+ = s_{uv^{-1}}^+ \phi(X^+, \theta_{X^+}, F_a)\{\{\theta_{uv^{-1}}^+ (y)\}\}.$$

We further more have that

$$y_{[0,m + |u|]}^{-1} u v^{-1} = \left(u y_{[|u|, m + |u|]}^{-1}\right) u v^{-1} = y_{[|u|, m + |u|]}^{-1} u v^{-1} = \left(\theta_{uv^{-1}}^+ (y)_{[0, m + |v|]}^{-1}\right),$$

and hence that

$$s_{x[y, y]}^+ x_{[0,n + |u|][y, y]}^{-1} s_{uv^{-1}}^+ = \phi(X^+, \theta_{X^+}, F_a) D_{x[y, y]}^{-1} x_{[0,n + |u|][y, y]}^{-1} \phi(X^+, \theta_{X^+}, F_a)\{\{\theta_{uv^{-1}}^+ (y)_{[0, m + |v|]}^{-1}\}\}.$$
Thus we have that
\[
e^j_{x,y} s^X^+ = \phi(x^+, \theta^+_X, F_a) (\{x\}) \ s^X^+_{[0,n][y_0,m]} \ \phi(x^+, \theta^+_X, F_a) (\{y\}) \ s^X^+_{[u,v]-1} \\
= \phi(x^+, \theta^+_X, F_a) (\{x\}) \ s^X^+_{[0,n][y_0,m]} \ \phi(x^+, \theta^+_X, F_a) (\{\theta^+_{vu-1}(y)\}) \\
= \phi(x^+, \theta^+_X, F_a) (\{x\}) \ s^X^+_{[0,n+|u|][y_0,m+|u|]} \ \phi(x^+, \theta^+_X, F_a) (\{\theta^+_{\phi_{vu-1}(y)}\}) \\
= \phi(x^+, \theta^+_X, F_a) (\{x\}) \ s^X^+_{[0,n+|u|][y_0,m+|u|]} \ \phi(x^+, \theta^+_X, F_a) (\{\theta^+_{vu-1}(y)\}) \\
= e^j_{x,\phi_x^+} (\{x\}).
\]

So we have in all cases that \(e^j_{x,y} s^X^+ \in \text{span}\{e^j_{x',y'} \mid j \in J_X, x', y' \in \pi(j)\}\).

One can in a similar way prove that we have that
\[
s^X^+ e^j_{x,y} \in \text{span}\{e^j_{x',y'} \mid j \in J_X, x', y' \in \pi(j)\}\.
\]

Hence \(\text{span}\{e^j_{x,y} \mid j \in J_X, x, y \in \pi(j)\}\) is an ideal of \(C^*(X^+, \theta^+_X, F_a)\) and thus equal to the kernel of \(\phi\).

**Example 8.19.** Let \(X_\eta\) be the shift space of an aperiodic proper substitution \(\eta\). It then follows from [23] page 90 and 107 and [1] Theorem 3.9 that \(X_\eta\) is minimal (and thus also \(X^+_\eta\)) and contains a finite, but nonzero, number of left special elements. Thus according to [3] Example 3.6, \(X_\eta\) has property (**) and since \(n_{X_\eta}\) is nonzero, it follows from Theorem 5.18 that \(\mathcal{O}_{X^+_\eta}\) is not simple. On the other hand \(C(X_\eta) \cong \mathbb{Z}\) is simple because \(X_\eta\) is minimal.

**Appendix A. Partial representations of groups**

A *partial representation* of a group \(G\) was defined in [10] to be a map \(u\) from \(G\) to a \(C^*\)-algebra \(A\) such that the following conditions hold:

(A.1a) \(u(e) = 1\),
(A.1b) \(u(g^{-1}) = u(g)^*\) for every \(g \in G\),
(A.1c) \(u(h)u(i)u(i^{-1}) = u(hi)u(i^{-1})\) for every \(h, i \in G\).

In [25] another definition of a partial representation of a group was given, namely as a map \(u\) from \(G\) to a \(C^*\)-algebra \(A\) such that the following conditions hold:

(A.2a) \((u(g))_{g \in G}\) is a family of partial isometries with commuting range projections,
(A.2b) \(u(e)u(e)^* = 1\),
(A.2c) \(u(g)^*u(g) = u(g^{-1})u(g)^{-1}\) for every \(g \in G\),
(A.2d) \(u(h)u(i)u(i)^*u(h)^* = u(h)u(i)u(hi)^*\) for every \(h, i \in G\).
It was in [25, Lemma 1.8 and Remark 1.9] also noticed that the conditions (A.2a)–(A.2d) are equivalent to the conditions:

(A.3a) \((u(g))_{g \in G}\) is a family of partial isometries with commuting range projections,

(A.3b) \(u(e) = 1\),

(A.3c) \(u(g) = u(g^{-1})\) for every \(g \in G\),

(A.3d) \(u(h)u(i) = u(h)u(h)^*u(hi)\) for every \(h, i \in G\).

As mentioned in [12], the conditions (A.1a)–(A.1c) are equivalent to the conditions (A.2a)–(A.2d) and thus to the conditions (A.3a)–(A.3d), but since we have not been able to find a complete proof of this, we will now give one:

**Proposition A.1.** Let \(u\) be a map from a group \(G\) to a \(C^*\)-algebra \(A\). Then the following are equivalent:

1. \(u\) satisfies the conditions (A.1a)–(A.1c),
2. \(u\) satisfies the conditions (A.2a)–(A.2d),
3. \(u\) satisfies the conditions (A.3a)–(A.3d).

**Proof.** As mentioned above, the equivalence of (2) and (3) follows from [25, Lemma 1.8 and Remark 1.9].

Assume that \(u\) satisfies the conditions (A.1a)–(A.1c). Then we have that

\[
 u(g)u(g)^*u(g) = u(g)u(g^{-1})u(g) = u(e)u(g) = u(g),
\]

so \(u(g)\) is a partial isometry for all \(g \in G\).

If \(h, i \in G\), then we have that

\[
 u(h)u(h)^*u(i)u(i)^* = u(h)u(h^{-1}i)u(i)^* = u(i)u(i^{-1}h)u(h^{-1}i)u(i)^* = (u(h^{-1}i)u(i^{-1}))^*(u(h^{-1}i)u(i^{-1})) = u(i)u(i)^*u(h)u(h)^*u(i)u(i)^*,
\]

from which it follows that

\[
 u(i)u(i)^*u(h)u(h)^* = (u(h)u(h)^*u(i)u(i)^*)^* = (u(i)u(i)^*u(h)u(h)^*u(i)u(i)^*)^* = u(i)u(i)^*u(h)u(h)^*u(i)u(i)^* = u(i)u(i)^*u(h)u(h)^*u(i)u(i)^*,
\]

This shows that \(u\) satisfies condition (A.2a).

It is clear that \(u\) satisfies condition (A.2b) and (A.2c). If \(h, i \in G\), then we have that

\[
 u(h)u(i)u(i)^*u(h)^* = u(h)(u(h)u(i)u(i)^*)^* = u(h)(u(hi)u(i)^*)^* = u(h)u(i)u(hi)^*,
\]

so \(u\) also satisfies condition (A.2d). Thus (1) implies (2).
Assume now that $u$ satisfies the conditions (A.3a)–(A.3d), and that $h, i \in G$. Then we have that
\[
u(h)u(i)u(i)^* = (u(i)u(i)^*u(h^{-1}))^* = (u(i)u(i^{-1}h^{-1}))^* = u(h)i^{-1},
\]
which shows that (3) implies (1).

\[\square\]

**Appendix B. Boolean Algebras**

We recommend [14] for a very nice introduction to Boolean algebras. A **Boolean algebra** is a set $\mathcal{B}$ with two distinct elements $0, 1 \in \mathcal{B}$ which act like the empty set and the whole set, respectively, and three operations $\lor : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, $\land : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $\neg : \mathcal{B} \to \mathcal{B}$ which act like union, intersection and complement, respectively. To be precise, they satisfy the following axioms:

\[
\begin{align*}
(B.1) & \quad -0 = 1 & -1 &= 0 \\
(B.2) & \quad A \land 0 = 0 & A \lor 1 &= 1 \\
(B.3) & \quad A \land 1 = A & A \lor 0 &= 0 \\
(B.4) & \quad A \land -A = 0 & A \lor -A &= 1 \\
(B.5) & \quad -\neg A = A \\
(B.6) & \quad A \land A = A & A \lor A &= A \\
(B.7) & \quad -(A \land B) = -A \lor -B & -(A \lor B) = -A \land -B \\
(B.8) & \quad A \land B = B \land A & A \lor B &= B \lor A \\
(B.9) & \quad A \land (B \land C) = (A \land B) \land C & A \lor (B \lor C) &= (A \lor B) \lor C \\
(B.10) & \quad A \land (B \lor C) = (A \land B) \lor (A \land C) \\
(B.11) & \quad A \lor (B \land C) = (A \lor B) \land (A \lor C)
\end{align*}
\]

This set of axioms is not the shortest one possible. In fact one could do with for example just axiom (B.3), (B.4), (B.8), (B.10) and (B.11). We will call $A \lor B$ for the intersection of $A$ and $B$, $A \land B$ for the union of $A$ and $B$, and $\neg A$ the complement of $A$.

The generic example of a Boolean algebra is of course the power set of a set $X$, where $0 = \emptyset$, $1 = X$, $\lor = \cup$, $\land = \cap$ and $\neg A = X \setminus A$.

If $A$ is a subset of a Boolean algebra $\mathcal{B}$, then we call it a **Boolean subalgebra** of $\mathcal{B}$ if $0, 1 \in A$ and $A \lor B, A \land B, \neg A \in A$ for every $A, B \in A$. In this case $A$ is of course itself a Boolean algebra with operations inherited from $\mathcal{B}$. When $X$ is a set, then we will by a **Boolean algebra on $X$** mean a Boolean subalgebra of the power set of $X$.

An example of this which we will use in this paper, is if $X$ is a topological space. Then the set of clopen subsets of $X$ is a Boolean subalgebra of the power set of $X$ and thus a Boolean algebra on $X$.

If $A$ is some subset of a Boolean algebra $\mathcal{B}$, then we will by the **Boolean algebra generated by $A$** mean the smallest Boolean subalgebra of $\mathcal{B}$ containing $A$. Notice that if a subset $A$ of a Boolean algebra $\mathcal{B}$ is closed under intersection (respectively union) and complement, then it also closed under union (respectively intersection) and thus is a Boolean subalgebra.
Other examples of Boolean algebras which we will meet in this paper is the set \( \{0, 1\} \) where

\[
0 \lor 0 = 0 \land 0 = 0 \land 1 = 1 \land 0 = -1 = 0,
1 \land 1 = 1 \lor 1 = 1 \lor 0 = 0 \lor 1 = -0 = 1,
\]

and the set of projections in a unital abelian \( C^* \)-algebra, where

\[
p \lor q = pq \quad p \land q = p + q - pq \quad \neg p = 1 - p.
\]

A map \( \phi \) between two Boolean algebras \( B \) and \( B' \) is called a Boolean homomorphism if \( \phi(A \lor B) = \phi(A) \lor \phi(B) \), \( \phi(A \land B) = \phi(A) \land \phi(B) \) and \( \phi(-A) = -\phi(A) \) for every \( A, B \in B \). In fact, the first (respectively the second) together with the last equality imply the second (respectively the first), so in order to verify that \( \phi \) is a Boolean homomorphism, it is enough to check these two equalities. Notice that when \( \phi \) is a Boolean homomorphism, then \( \phi(1) = 1 \) and \( \phi(0) = 0 \).

If \( A \) is a unital abelian \( C^* \)-algebra and \( B \) is the Boolean algebra of projections of \( A \), then it is easy to check that \( \text{span}(B) \) is a \( * \)-subalgebra of \( A \) and thus that \( \text{span}(B) \) is a \( C^* \)-subalgebra of \( A \). Hence \( \text{span}(B) = C^*(B) \).

**Lemma B.1.** If for \( i \in \{1, 2\}, A_i \) is a unital abelian \( C^* \)-algebra and \( B_i \) is the Boolean algebra of projections of \( A_i \), and \( \phi : B_1 \to B_2 \) is a Boolean homomorphism, then there is a uniquely determined \( * \)-homomorphism from \( \text{span}(B_1) \) to \( \text{span}(B_2) \) which maps \( A \) to \( \phi(A) \) for \( A \in B_1 \).

**Proof.** Since \( B_1 \) generates \( \text{span}(B_1) \), there can at most be one \( * \)-homomorphism from \( \text{span}(B_1) \) to \( \text{span}(B_2) \) which maps \( A \) to \( \phi(A) \) for \( A \in B_1 \).

Let \( F \) denote the set of finite subset of \( B_1 \). We then have that \( \text{span}(B_1) \) is the closure of the set

\[
\bigcup_{C \in F} \text{span}(C),
\]

and since \( \text{span}(C) \) is the \( C^* \)-subalgebra of \( A_1 \) generated by \( C \), it is enough to show that there for every \( C \in F \) exists a \( * \)-homomorphism from \( \text{span}(C) \) to \( \text{span}(B_2) \) which maps \( A \) to \( \phi(A) \) for \( A \in C \).

So let \( C \in F \). There then exists a finite family \( p_1, p_2, \ldots, p_n \) of mutually orthogonal projections in \( C \) such that every element of \( \text{span}(C) \) uniquely can be written as

\[
\sum_{i=1}^{n} \lambda_i p_i
\]

with \( \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C} \). The map

\[
\sum_{i=1}^{n} \lambda_i p_i \mapsto \sum_{i=1}^{n} \lambda_i \phi(p_i)
\]

is therefore a well-defined \( * \)-homomorphism from \( \text{span}(C) \) to \( \text{span}(B_2) \) which maps \( A \) to \( \phi(A) \) for \( A \in C \). \( \square \)
Appendix C. Crossed products of $C^*$-partial dynamical systems

A $C^*$-partial dynamical system has been defined in [25] to be a triple $(A,G,\alpha)$ where $A$ is a $C^*$-algebra, $G$ is a discrete group and $\alpha$ is a partial action of $G$ on $A$. That means that $\alpha$ consists of a family $(D_g)_{g \in G}$ of closed ideals of $A$ and a family $(\alpha_g)_{g \in G}$ of isomorphisms $\alpha_g : D^{-1}_g \rightarrow D_g$ such that

\[(C.1) \quad D_e = A,\]
\[(C.2) \quad \alpha_{hi} \text{ extends } \alpha_{h} \alpha_{i} \text{ for all } h, i \in G \text{ (where the domain of } \alpha_{h} \alpha_{i} \text{ is } \alpha_{i}^{-1}(D_{h^{-1}})).\]

$C^*$-partial dynamical systems have been studied in [12,22,25] (the definition of $C^*$-partial dynamical systems in [22] is a bit different from the abovementioned, but it is shown in [25, Remark 1.9] that the two definitions are equivalent).

A covariant representation of a $C^*$-partial dynamical system $(A,G,\alpha)$ on a Hilbert space $H$ is a pair $(\pi,u)$ where $\pi$ is a non-degenerate representation of $A$ on $H$, and $u$ is a partial representation (cf. Appendix A) of $G$ on $H$ such that for each $g \in G$, $u(g)u(g)^*$ is the projection of $H$ onto the subspace $\text{span}(\pi(D_g)H)$, and $\pi(\alpha_g(a)) = u(g)\pi(a)u(g^{-1})$ for $a \in D_g$. As it is the case with $C^*$-partial dynamical systems, the definition of a covariant representation in [22] is a bit different from the above mentioned, but it is shown in [25, Remark 1.12] that the two definitions are equivalent.

The crossed product $A \rtimes_{\alpha} G$ of a $C^*$-partial dynamical system $(A,G,\alpha)$ is a $C^*$-algebra which is generated by a copy of $A$ and a family $(\delta_g)_{g \in G}$ of elements such that there exists a bijective map $(\pi,u) \mapsto \pi \times u$ between covariant representations of $(A,G,\alpha)$ on $H$ and non-degenerated representations of $A \rtimes_{\alpha} G$ on $H$ such that $(\pi \times u)(a\delta_g) = \pi(a)u(g)$ for $g \in G$ and $a \in D_g$.

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