Four Kähler Moduli Stabilisation in type IIB Orientifolds with K3-fibred Calabi-Yau threefold compactification

Dieter Lüst\textsuperscript{a,b} and Xu Zhang\textsuperscript{a,c}

\textsuperscript{a}Max-Planck-Institute for Physics, Föhringer Ring 6, D-80805 Munich, Germany
\textsuperscript{b}Ludwig-Maximilians-Universität, Arnold-Sommerfeld-Center, Theresienstrasse 37, D-80333 Munich, Germany
\textsuperscript{c}State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China

E-mail: dieter.luest@lmu.de, luest@mppmu.mpg.de, xuzhang@mpp.mpg.de

Abstract: We present a concrete and consistent procedure to generate one kind of non-perturbative superpotential, including the gaugino condensation corrections and polyinstanton corrections, in type IIB orientifold compactification with four Kähler Moduli. Then we use this kind of superpotential as well as the $\alpha'$-corrections to Kähler potential to fix all of the four Kähler moduli on a general Calabi-Yau manifold with typical K3-fibred volume form. In our construction, the considered Calabi-Yau threefolds are K3-fibred and admit at least one del Pezzo surface and one W-surface. Searching through all existing four dimensional reflexive lattice polytopes, we find 23 of them fulfilling all the requirements.

Keywords: Moduli stabilisation, K3-fibred Calabi-Yau Manifolds
1 Introduction

The moduli parameters in string theory correspond to massless scalars in 4-dimensional effective supergravity and hence will lead to long range interactions. The couplings of these scalars to matter fields are in general not universal, which implies that different matter fields will obtain different accelerations from these long range forces. Obviously, this phenomenon violates the principle of equivalence, which has been tested by the ratio of inertial to gravitational mass up to $10^{-13}$ [1]. Therefore a “fifth force” must be very weak or sufficiently short ranged, and a very natural consequence is that all of the moduli should be massive. Furthermore, string theory loses any predictability, if the vacuum expectation values of the moduli fields, especially for the volume modulus, can take arbitrary values, since many physical parameters in the low energy theory depend on the specific value of moduli.

In the type IIB orientifold compactifications with $O7/O3$-planes, there are two proposed mechanisms to stabilise all of the moduli, at least in the case of a few Kähler moduli, i.e $h^{1,1}$ is small. One is called KKLT strategy [2], and the other one is the LARGE volume scenario (LVS) [3]. In both cases, one first stabilises the axion-dilaton and complex structure moduli by appropriate choice of background fluxes, more concretely by the Gukov-Vafa-Witten superpotential induced by these fluxes; then one fixes the values of the Kähler moduli by non-perturbative effects such as D-brane instantons and gaugino condensation. Many explicit constructions in type IIB orientifold compactification have been investigated for both cases, see for example [4–7]. The key difference between these two mechanisms is that the LVS admits a non-supersymmetric anti-de Sitter minimum instead of the supersymmetric one in KKLT, and the fixed value of the Calabi-Yau manifold volume is exponentially large with respect to the size of the “small” four-cycle, which is
usually a del-Pezzo surface and supports the $D$-brane instanton or the gaugino condensation. Furthermore the value of fixed “small” four-cycle volume is independent of the flux superpotential $W_0$ at fixed $g_s$, which implies that this non-perturbative stabilisation of the Kähler moduli will not disturb the complex structure stabilisation. This also avoids the fine tuning of $W_0$ and the necessity of a large gauge group as in the KKLT strategy. A more elaborated survey of moduli stabilisation mechanisms is presented in the appendix of [8].

The key requirement for a LVS model is to find a Calabi-Yau threefold with $h^{1,2} > h^{1,1} > 1$, where the volume can be expressed according to the strong ‘Swiss cheese’ type or K3-fibred type Calabi-Yau threefolds. In addition, there must be divisors which can support non-perturbative effects in the Calabi-Yau threefold. In this case, it is possible to make some of the four-cycles small while keeping the volume large. Even in the large volume limit, the induced non-perturbative effects can compete against the $\alpha'$-corrections to the Kähler potential. Actually quite some progress has been made in this context, especially for $h^{1,1} \leq 3$. For the case of strong ‘Swiss cheese’ type Calabi-Yau, one stabilises the volume plus one “small” four cycle by the $\alpha'$-corrections and the non-perturbative effects, and the third one is fixed by poly-instanton effects, e.g. [6]. For the case of a K3-fibred Calabi-Yau, the third one can be fixed either by the string-loop effects to Kähler potential [7], or by the poly-instanton effects [5].

Particularly in the case of a K3-fibred Calabi-Yau with poly-instanton corrections, we may obtain an anisotropic extra dimensions and a TeV string scale, which is advantageous for embedding extra-dimensional models of particle physics into type IIB string theory, as shown in [5]. And this case is also very useful in the cosmology model constructions, for example the single-field inflation models [7, 9], the double-field inflation models using the curvaton mechanism [10] or the modulation mechanism [11], and the quintessence models for dark energy [12]. However, there is no concretely analysis of the condition for generating the poly-instanton corrections in these papers. Actually, within the most simple constructions, the poly-instanton effects in these papers should be absent, as pointed out in [13].

On the other hand, most of particle physics models and multi-field inflationary models in type IIB theory require at least four Kähler moduli. For particle physics models, the moduli stabilisation seems to be more complicated, since one needs to consider the tension between non-perturbative effects and chirality[14] also together with the $D$-term problem [15]. Thus the moduli stabilisation in this case must be done for an explicit Calabi-Yau threefold, for example [15, 16]. For the double-field inflation model in [10, 11], we do not need to worry about the tension between non-perturbative effects and chirality and either the $D$-term problem, but one needs a K3-fibred Calabi-Yau threefold with two del Pezzo surfaces, which is not explicitly presented in these papers either.

In the present paper, we will address the issue of moduli stabilisation for a K3-fibred Calabi-Yau threefold with four Kähler moduli, which is a crucial ingredient for realistic particle physics models or for the multi-field inflation model construction in the type IIB orientifold framework. Within the stabilisation mechanism presented here, all of the Kähler moduli will be fixed by the non-perturbative effects, including poly-instanton corrections,
and $\alpha'$-corrections. Moreover this procedure will not be spoiled by the string-loop corrections to Kähler potential. We will concretely check all of the conditions for generating non-perturbative corrections, especially poly-instanton corrections, to the superpotential for an explicit toric K3-fibred Calabi-Yau threefold, making sure that this kind of superpotential can indeed be generated in the type IIB orientifolds, at least in the simple cases where the background fluxes are ignored.

The paper is organized as follows. In section 2, we briefly review some definitions relevant to $N = 1$ type IIB orientifold compactifications with fluxes as well as the general structure of LVS. In section 3, we will analyse the conditions for the presence of non-perturbative corrections, including poly-instanton corrections, to the superpotential in detail. First, we will reiterate the neutral zero-mode and charged zero-mode issues of instantons, then we will briefly mention the tools for calculating the cohomology group as well as its splitting properties. Then we will present an explicit Calabi-Yau manifold with an appropriate choice of orientifold action, which can satisfy all of the conditions for generating the expected corrections to the superpotential. In section 4, we will systematically present the procedure of moduli stabilisation in type IIB orientifolds, in which the general compact Calabi-Yau manifolds have the same structure as the one presented in section 3.1. Here we will find that we can not get a consistent result by using the minimal superpotential at the end of section 3.1. But using instead the racetrack superpotential, everything can be made consistent. We will also discuss that the string-loop effects will not spoil this procedure. Finally in section 5 we present our conclusions followed by an appendix providing the list of K3 fibrations with del Pezzo and W-surface.

2 Brief review of LARGE volume scenario

In the framework of type IIB string theory compactified on a Calabi-Yau threefold with RR and NS-NS 3-form fluxes(see [17] for reviews), we need orientifold planes to reduce the supersymmetry, such that the low-energy effective theory is a $N = 1$ supergravity theory. The presence of orientifold planes is also crucial to cancel the RR tadpoles [18]. Depending on the transformation properties of the holomorphic three form $\Omega$ on Calabi-Yau threefold, there are two different symmetry operations $O$ to generate the orientifold planes. Here we take the choice as follows, which can generate $O3/O7$-planes

$$O = (-1)^{F_L} \Omega_p \sigma^*, \quad \sigma^* \Omega = -\Omega, \quad \sigma^* J = J \quad (2.1)$$

where $F_L$ is the spacetime fermion number in the left-moving sector, $\Omega_p$ denotes the worldsheet parity, $\sigma^*$ is the pull-back of involution $\sigma : x_i \mapsto -x_i$, and the fixed point loci are defined as $O3/O7$-planes. Note that since $\sigma$ is a holomorphic involution, the cohomology groups $H^{(p,q)}$ split into two eigenspaces under the action $\sigma^*$, namely $H^{(p,q)} = H_+^{(p,q)} \oplus H_-^{(p,q)}$. The transformation properties for all of fields in type IIB supergravity can be found in [19].

The presence of $O3/O7$-planes wrapping a divisor gives rise to tadpoles for the RR form, which can be canceled by introducing suitable D3/D7-branes. Because we only discuss moduli stabilisation, we assume for simplicity that there are no gauge fluxes on the
D7-brane. In this case we can avoid the Freed-Witten anomaly \[20\] by choosing suitable background B-field, and the cancellation conditions read as

\[
\sum_i N_i([D7_i] + [D7_i])' = 8[O7],
\]
\[
N_{D3} + \frac{1}{2} N_{\text{flux}} - \frac{1}{4} N_{O3} = \frac{\chi(X)}{24}.
\] (2.2)

Here \([D7]\) and \([O7]\) denote the divisors wrapped by D7-branes and O7-planes respectively, \([D7]'\) denotes the orientifold image of \([D7]\), \(N_{D3}\) is the net number of D3-brane, namely the difference between the number of D3-branes and the number of \(\overline{D3}\)-branes and \(N_{\text{flux}} = \frac{1}{(2\pi)^4 \alpha'} \int H_3 \wedge F_3\) and \(\chi(X)\) is the Euler characteristic of the elliptically fibered Calabi-Yau fourfold \(X\). This framework can be viewed as a limit of F-theory compactified on \(X\), whose Euler characteristic is related to the D7-branes and the O7-planes in type IIB theory as follows \[21\]

\[
2\chi(X) = \chi_o([D7]) + 4\chi([O7]).
\] (2.3)

The modified Euler characteristic \(\chi_o([D7])\) is defined as follows

\[
\chi_o([D7]) = 24 \int \Gamma_{\text{pure}D7},
\] (2.4)

where \(\Gamma_{\text{pure}D7}\) is the charge of a pure D7-brane wrapping \([D7]\), and it is shown that \(\chi_o([D7] + [D7]') = 2\chi([D7])\), so the tadpole cancellation condition for \(C_4\)-form in Eq.(2.2) reduces to

\[
N_{D3} + \frac{1}{2} N_{\text{flux}} = \frac{1}{4} N_{O3} + \frac{1}{4} \chi([O7]).
\] (2.5)

Therefore, we can set eight D7-branes right on top of the O7-plane to cancel the tadpole for \(C_8\) form and the condition for \(C_4\) form can serve as a consistency check that the number is indeed a integer.

The 4-dimensional effective action in type IIB theory, which is compactified on a Calabi-Yau orientifold, can be expressed into the standard \(N = 1\) supergravity form, namely the action can be completely determined by a Kähler potential \(K\), a holomorphic superpotential \(W\) and a holomorphic gauge-kinetic coupling functions \(f\). Here we only talk about the terms which are concerned with moduli and we must stress that all of the variables involved in the following are in Einstein frame\(^1\). To leading order in \(g_s\) and \(\alpha'\), the Kähler potential in Einstein frame is given as \[19, 22\]

\[
K = -2 \log \left[ V + \frac{\hat{\xi}}{2} \right] - \log \left[ -i \int \Omega \wedge \bar{\Omega} \right] - \log \left[ -i (\tau - \bar{\tau}) \right],
\] (2.6)

where \(V\) is the volume of the Calabi-Yau manifold and the \(\hat{\xi}\)-term, which comes from \(\alpha'\)-corrections, is expressed as

\[
V = \frac{1}{6} \kappa_{\alpha\beta\gamma} t^\alpha t^\beta t^\gamma, \quad \hat{\xi} = -\frac{\zeta(3) \chi(M)}{2(2\pi)^3 g_s^{3/2}}
\] (2.7)

\(^1\)The relation between string frame and Einstein frame can see the appendix of [10]
respectively. In the previous equations, we have used the fact that the string coupling \( g_s = e^\phi \), and \( \zeta(3) \approx 1.202 \) is the approximate value of Riemann \( \zeta \)-function, \( \chi(M) \) is the Euler characteristic of the Calabi-Yau manifold. In order to perform the LARGE volume scenario, we must require that \( \hat{\xi} > 0 \), namely \( h^{2,1} > h^{1,1} \). \( \kappa_{\alpha\beta\gamma} \) denotes the intersection numbers of the Calabi-Yau manifold, and \( t^\alpha \) denotes the coefficients of Kähler form on the basis of \( H_+^{1,1} \). Furthermore, the holomorphic three-form \( \Omega \) in Eq. (2.6) only depends on the complex structure moduli, and the dilaton \( \tau \) and Kähler moduli \( T_\alpha \) take the following definitions:

\[
\tau = C_0 + i e^{-\phi}, \quad G^a = e^a - \tau b^a, \quad \zeta_\alpha = -\frac{i}{\tau} \kappa_{abc} C^b (G - \bar{G})^c, \quad a, b, c = 1, \ldots, h_+^{1,1}, \\
T_\alpha = \frac{1}{2} \kappa_{\alpha\beta\gamma} t^\beta t^\gamma + i \rho_\alpha - \zeta_\alpha, \quad \alpha, \beta, \gamma = 1, \ldots, h_+^{1,1}.
\]

(2.8)

c^a and \( b^a \) are the coefficients of \( C_2 \) and \( B_2 \) on the basis of \( H_+^{1,1} \) respectively, \( \rho_\alpha \) is the coefficient of \( C_4 \) on the basis of \( H_+^{2,2} \). Note that the basis of \( H_+^{1,1} \) and \( H_+^{2,2} \) are dual to each other and \( \pm \) denotes the two eigenspaces of splitting cohomology groups \( H^{(p,q)} \) under the involution \( \sigma^* \). Under our constructions, we can always set \( h_+^{1,1} = 0 \), so that the Kähler moduli can be simplified as

\[
T_\alpha = \frac{1}{2} \kappa_{\alpha\beta\gamma} t^\beta t^\gamma + i \rho_\alpha.
\]

(2.9)

Note that \( \tau_\alpha \equiv \frac{1}{2} \kappa_{\alpha\beta\gamma} t^\beta t^\gamma \) can be viewed as the volume of divisor \( D_\alpha \in H_4(M, \mathbb{Z}) \).

Ignoring gauge sectors, for orientifolds with \( h_+^{1,1} = 0 \), the superpotential \( W \) in the perturbative theory was shown to be the Gukov-Vafa-Witten superpotential: [23]

\[
W = \int \Omega \wedge G_3, \quad G_3 = F_3 - \tau H_3.
\]

(2.10)

Note that the superpotential is independent of the Kähler moduli, and the Kähler potential possesses the well-known no-scale structure. Thus the scalar potential of \( N = 1 \) supergravity

\[
V = e^K \left[ K^{IJ} D_I W \bar{D}_J \bar{W} - 3|W|^2 \right]
\]

(2.11)

is positive definite and only depends on the dilaton and the complex structure moduli. We can fix both of them by solving

\[
D_a W \equiv \partial_a W + K_a W = 0.
\]

(2.12)

Here \( a \) runs over the dilaton and complex structure moduli. Actually this moduli fixing can be done for appropriate choice of the fluxes [17], and from now on we denote the value of \( W \) following this step as \( W_0 \).

Therefore, in order to fix Kähler moduli, we must introduce some non-perturbative effect, such as instanton corrections or gaugino condensation effects to the superpotential as suggested by [3]. In the next section we will discuss the condition for generating such effects in details.
3 Superpotential with non-perturbative effects

The superpotential with non-perturbative corrections, including instanton effects, poly-instanton effects or gaugino condensation takes the following form

\[
W = W_0 + A_i \exp(-a_i T_i + A_j e^{-2\pi T_j})
\]

where the instantons or the gaugino condensation are supported by the divisors \(D_i\), the poly-instantons are supported by the divisors \(D_j\), the corresponding Kähler moduli are \(T_i, T_j\) respectively, and \(A_i, A_j\) are one-loop determinants, which depend on complex structure moduli. \(a_i = \frac{2\pi N}{N}, N \in \mathbb{Z}_+,\) where for D-brane instantons \(N = 1\), while for gaugino condensation the value of \(N\) depends on the rank of the gauge group. In this section we will systematically analyse the condition for generating such kind of superpotential, using the methods in [13, 24, 25].

Each BPS D-brane instanton is 1/2 BPS, and thus locally breaks 4 out of 8 the supersymmetries. These broken supersymmetries manifest themselves in the volume of the instanton as Goldstinos, namely as fermionic zero modes. They are conventionally denoted by \(\theta^a\) and \(\bar{\tau}_\dot{a}\). Depending on the divisor wrapped by the instanton, some other neutral zero modes may also be present. In addition to the geometric Calabi-Yau background, various other ingredients, such as branes, orientifolds and fluxes, maybe change the spectrum of zero modes (See [26] for a brief review on D-brane instanton). In this paper we only consider the geometric Calabi-Yau background as in [13, 24, 25]. The general structure for the neutral zero modes of an \(O(1)\) instanton is showed in the table 1, in which \(\gamma_\alpha\) and \(\bar{\gamma}_{\dot{\alpha}}\) denote the Wilson line Goldstinos; \(\chi_\alpha\) and \(\bar{\chi}_{\dot{\alpha}}\) denote the deformation Goldstinos. If the instanton contributes to the holomorphic superpotential \(W\), the anti-holomorphic zero modes have to be removed, namely \(h^{n,0} = 0, n = 0, 1, 2, \cdots\), and they should be no more other zero modes, i.e. \(h^{1,0}(D) = h^{2,0}(D) = 0\). For the contribution from gaugino condensation, the condition of the divisor is the same as before, actually in this case we have an ordinary gauge instanton for a \(\text{Sp}(2N)\) or \(\text{SO}(N)\) gauge group. Considering these constraints and the realization of the LARGE volume scenario, the divisor which supports an instanton or gaugino condensation has to be a del-Pezzo surface \(dP_n\), since they are arbitrarily contractible to a point without affecting the rest of geometry on a Calabi-Yau threefold [27], since a brane wrapping such a surface has no adjoint matter and no extra fermionic modes. For the contribution from poly-instanton, the former Wilson line Goldstinos of the E1 instanton, for an E3 instanton, can arise from either Wilson line or deformation Goldstinos, which are counted by \(h^{1,0}_+ (D) + h^{2,0}_+(D)\). So we can summarize the conditions for the zero mode structure of an instanton and a poly-instanton contribution for the superpotential.

| Number | \(X_{\mu}, \theta^\alpha\) | \(\bar{\tau}_\dot{a}\) | \(\gamma_\alpha\) | \(\bar{\gamma}_{\dot{\alpha}}\) | \(\chi_\alpha\) | \(\bar{\chi}_{\dot{\alpha}}\) |
|--------|----------------------------|----------------|--------------|---------------|------------|------------|
| \(h^{0,0}_+ (D)\) | \(h^{1,0}_- (D)\) | \(h^{1,0}_+(D)\) | \(h^{1,0}_-(D)\) | \(h^{2,0}_+(D)\) | \(h^{2,0}_-(D)\) |

Table 1. Neutral zero mode structure for an \(O(1)\)-instanton wrapping a divisor \(D\) [25]
in the following table 2. Actually as argued in [13], only the divisor that admits a single

| Instanton | $h^0_{+0}(D)$ | $h^1_{+0}(D)$ | $h^2_{+0}(D)$ | $h^3_{+0}(D)$ |
|-----------|---------------|---------------|---------------|---------------|
| 1         | 1             | 0             | 0             | 0             |
| Poly-Instanton | 1 or 0        | 0 or 1        | 0             | 0             |

Table 2. Neutral zero mode structure of an instanton and a poly-instanton wrapping on a divisor $D$ in order to contribute to the superpotential [13].

complex Wilson line Goldstino can really support a poly-instanton correction. They call it a W-surface, which is characterized by $(h^{0,0}, h^{1,0}, h^{2,0}) = (1, 1, 0)$.

Because of the presence of D7-branes, so in addition of open strings going from the instanton to itself, which gives rise to neutral zero modes, there are also open strings going from instanton to D7 branes, which generates charged zero modes transforming in the fundamental or anti-fundamental representation of D7-brane gauge group. These charged zero modes can couple to the matter fields on D7-brane, after integrating over these zero modes can induce some effective operators involving matter fields in the low energy effective theory. It implies that in order to know exactly the physical charged zero modes, we need to understand the structure of Yukawa couplings in our compactification. And that is beyond the scope of this paper, so we will simply require the absence of charged zero modes on the instanton.

Consider an instanton $A$ and a background D7-brane wrapping different divisors, that intersect over the curve $C = [A] \cdot [D7]$. The spectrum of charged zero modes from the open string going from the instanton $A$ and the D7-brane originates from the cohomology group

$$ (\alpha, \bar{\beta}) \in (H^0(C, K_C^{1/2}), H^1(C, K_C^{-1/2})) ,$$

(3.2)

where $\alpha$ and $\bar{\beta}$ denote the modes in the fundamental and anti-fundamental representation of the D7-brane gauge group, and $K_C$ stands for the anticanonical bundle of $C$. We can ensure there are no charged zero modes at least in the following two cases [25]:

- $C = 0$, namely there is no intersection between the instanton and the D7-brane,
- $C = \mathbb{P}^1$.

From the above analysis, almost all of the conditions have been translated into the language of cohomology. Hence in the following we will briefly discuss the tools to calculate the cohomology group and its splitting under the orientifold involution. First we have the usual holomorphic Euler characteristic of the divisor $D$

$$ \chi(D, O_D) = \sum_{i=0}^{2} (-1)^i h^{i,0}(D), $$

(3.3)

and it is easy to compute using the Riemann-Roch formula:

$$ \chi(D, O_D) = \int_D Td(TD). $$

(3.4)
$Td(TD)$ is the Todd classes of tangent bundle to $D$. On the other hand, recalling the splitting $H^i(D,\mathcal{O}_D) = H^i_+(D,\mathcal{O}_D) \oplus H^i_-(D,\mathcal{O}_D)$, we immediately have $h^{i,0}(D) = h^{i,0}_+(D) + h^{i,0}_-(D)$, and we also have the Lefschetz’s equivariant genus for the orientifold involution $\sigma$ as

$$
\chi^\sigma(D,\mathcal{O}_D) = \sum_{i=0}^{2} (-1)^i (h^{i,0}_+(D) - h^{i,0}_-(D)).
$$

(3.5)

On the other hand, we can easily compute the Lefschetz’s equivariant genus from the Lefschetz fixed point theorem, where one can see some details of the theorem in the appendix of [25],

$$
\chi^\sigma(D,\mathcal{O}_D) = \frac{1}{4} N_{O3} - \frac{1}{4} \int_{D} [D],
$$

(3.6)

where $N_{O3}$ is the number of isolated fixed points on $D$, $\mathcal{C}^\sigma = [D] \cap D$ are the fixed curves on $D$, and $[D] \in H^2(M)$ denotes the Poincare dual to the divisor $D$. For the equivariant Betti number, a similar theorem applies, which leads to

$$
L^\sigma(M) = \sum_{i=0}^{4} (-1)^i (b^i_+ - b^i_-) = N_{O3} + \chi(\mathcal{C}^\sigma).
$$

(3.7)

In most of the cases, using the above equations, we can determine all equivariant cohomology classes. For other cases, we can employ the tools presented in [29] for the computation of line bundle cohomology over toric varieties, where we obtain the polytope information, which is crucial to construct the Calabi-Yau threefold, by the help of PALP package [30].

Using the tools mentioned above, searching through the 158 examples of four dimensional reflexive lattice polytopes presented in the appendix, which admit a K3-fibred Calabi-Yau hypersurface with four Kähler moduli where at least one of them is a del Pezzo surface, we find that 23 of them can also admit one W-surface. We will present all of them in the appendix.

Next we will pick one of the reflexive lattice polytopes, namely No.3 in the appendix, to show explicitly that the del Pezzo surfaces and W-surfaces can indeed support an instanton and a poly-instanton, which contribute to superpotential.

### 3.1 An explicit example

The toric ambient space can be defined by homogeneous coordinates and their equivalence relations, which are all encoded in the following weight matrix:

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $D_H$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 2     | 1     | 6     | 1     | 2     | 0     | 0     | 0     | 12    |
| 2     | 1     | 6     | 0     | 1     | 0     | 2     | 0     | 12    |
| 2     | 0     | 6     | 1     | 1     | 0     | 0     | 2     | 12    |
| 1     | 0     | 3     | 0     | 1     | 1     | 0     | 0     | 6     |

We can show that the surface of this reflexive lattice polytope admit 4 maximal triangulations, considering that we may construct a fan from each triangulation, so we can obtain several toric varieties for this weight matrix. Actually the different triangulations
are not isolated from each other, they maybe related to each other via flop transitions. Here we stick to one of the triangulations, which is encoded in the following Stanley-Reisner (SR) ideal

\[
\text{SR} = \{x_2x_4, x_2x_5, x_4x_5, x_5x_6, x_2x_7, x_4x_8, x_1x_3x_6x_7, x_1x_3x_6x_8, x_1x_3x_7x_8\}. \tag{3.8}
\]

The Calabi-Yau hypersurface in this toric ambient space has Hodge number \((h^{1,1}, h^{1,2}) = (4, 70)\) and Euler characteristic \(\chi = -132\). From the SR-ideal and the weight matrix, we can calculate the triple intersection numbers for a basis of divisor classes of the toric variety on Calabi-Yau. For simplifying the expression of the volume, we choose the following basis \((\eta_1, \eta_2, \eta_3, \eta_4) = (D_2, D_4, D_5, D_5 + 6D_4 + 3D_7)\), where divisors \(D_i \equiv \{x_i = 0\}\). The triple intersection numbers can be expressed in the following polynomial:

\[
I_3 = \eta_3^3 + \eta_2^3 + 18\eta_3\eta_4^2. \tag{3.9}
\]

The generators \(C_i\) of the Mori cone of the toric variety are

\[
\int_{C_i} D_j = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 \\
1 & 0 & 0 & 3 \\
0 & -1 & 0 & 0 \\
0 & 1 & 0 & 3
\end{pmatrix}. \tag{3.10}
\]

We know that the Kähler cone and Mori cone are dual to each other, and from this we can get the generators of the Kähler cone as follows:

\[
\Gamma_1 = \eta_3, \\
\Gamma_2 = 2\eta_3 + \eta_4, \\
\Gamma_3 = -3\eta_2 + 2\eta_3 + \eta_4, \\
\Gamma_4 = -3\eta_1 - 3\eta_2 + 2\eta_3 + \eta_4, \\
\Gamma_5 = -3\eta_1 + 2\eta_3 + \eta_4. \tag{3.11}
\]

This explicitly shows that the polytope is non-simplicial since \(i\) runs from one to five instead of four. Next we write the Kähler form in the basis of \(\{\eta_i\}\) and \(\{\Gamma_i\}\) respectively

\[
J = \sum_{i=1}^{4} t_i \eta_i = \sum_{i=1}^{5} r_i \Gamma_i \quad \text{with} \quad r_i > 0, \tag{3.12}
\]

where \(r_i > 0\) ensures that the stabilisation is within the Kähler cone. Now we can obtain the volume form in terms of two-cycle volume \(t_i\),

\[
\mathcal{V} = \frac{1}{3!} \int J \wedge J \wedge J = \frac{1}{6} \kappa_{ijk} t^i t^j t^k = \frac{1}{6} t_1^3 + \frac{1}{6} t_2^3 + 9t_3t_4^2. \tag{3.13}
\]
and we can also express $t_i$ in terms of $r_i$, from which we can determine the sign of $t_i$ and the linear combination of them.

$$t_1 = -3r_4 - 3r_5 < 0,$$
$$t_2 = -3r_3 - 3r_4 < 0,$$
$$t_3 = r_1 + 2r_2 + 2r_3 + 2r_4 + 2r_5 > 0,$$
$$t_4 = r_2 + r_3 + r_4 + r_5 > 0. \quad (3.14)$$

Defining the volumes $\tau_i$ of the four-cycle $D_i$,

$$\tau_i = \frac{1}{2} \int_{D_i} J \wedge J = \frac{1}{2} \kappa_{ijk} t^j t^k, \quad (3.15)$$

we find that

$$\tau_1 = \frac{1}{2} t_1^2,$$
$$\tau_2 = \frac{1}{2} t_2^2,$$
$$\tau_3 = 9t_4^2,$$
$$\tau_4 = 18t_3t_4. \quad (3.16)$$

Taking into account the Kähler cone condition (3.14), we can rewrite the volume form in terms of four-cycle’s volumes\footnote{Note that if we choose a basis including the W-surface divisor, the ‘minus’-part in the volume form will be similar to the examples in [13]}

$$V = \frac{1}{6} \sqrt{\tau_3 \tau_4} - \frac{\sqrt{2}}{3} \tau_1^{3/2} - \frac{\sqrt{2}}{3} \tau_2^{3/2}. \quad (3.17)$$

Next we will analyze the properties of the divisors $D_i, i = 1, \cdots, 8$, and show that there are two del Pezzo surfaces, one W-surface, and also the Calabi-Yau hypersurface is indeed K3-fibred. First of all, after computing the Hodge diamonds, we find that both $D_2$ and $D_4$ have the topological data of $dP_8$, namely $(h^{0,0}, h^{1,0}, h^{2,0}, h^{1,1}) = (1, 0, 0, 9)$ and $\chi = 11$, and also the following triple intersection structures with the other divisors:

$$\begin{array}{cccccccc}
D_1 & D_2 & D_3 & D_4 & D_5 & D_6 & D_7 & D_8 \\
\hline
D_2^2 & -1 & 1 & -3 & 0 & 0 & -1 & -2 \\
D_4^2 & -1 & 0 & -3 & 1 & 0 & -1 & -2 & 0
\end{array}$$

The divisors $D_2$ and $D_4$ have triple self-intersections $D_2^3 = 1, D_4^3 = 1$, any other intersection numbers are either vanishing or negative. This reads

$$\int_{C=D_i \cap S} c_1(S) = \int_M D_i \wedge S \wedge (-c_1(N_{S|M})) = -S^2 \cdot D_i > 0 \quad \forall C \neq \emptyset, \quad (3.18)$$

where $D_i \neq S$. It is a necessary condition for the divisor $S$ to be a rigid and shrinkable divisor, i.e. del Pezzo surface. This confirms that these two divisors should be $dP_8$. 

- 10 -
Actually we can also algebraically show that these two divisors are indeed $dP_8$. Following the procedure showed in [32], we can get the representation of these two divisors

| $D_2$ | $x_1$ | $x_3$ | $x_6$ | $x_7$ | $x_8$ | $D_H|D_2$ |
|-------|-------|-------|-------|-------|-------|-----------|
|       | 2     | 6     | 0     | 2     | 2     | 12        |
|       | 1     | 3     | 1     | 0     | 2     | 6         |

| $D_4$ | $x_1$ | $x_3$ | $x_6$ | $x_7$ | $x_8$ | $D_H|D_4$ |
|-------|-------|-------|-------|-------|-------|-----------|
|       | 2     | 6     | 0     | 2     | 2     | 12        |
|       | 1     | 3     | 1     | 0     | 2     | 6         |

Hence $D_2$ and $D_4$ are both $dP_8$-surface.

As a next step, we find out the K3 divisor. We can easily compute that $\int_{D_5} c_1(D_5) \wedge i^* D_i = -D_5^2 D_i = 0$ and $\int_{D_5} i^* c_2(M) = \int_{D_5} (10\eta_1^2 - 28\eta_1\eta_2 + 18\eta_2^2 - 64\eta_1\eta_3 - 28\eta_2\eta_3 + \frac{3}{2}\eta_3^2 - 8\eta_1\eta_4 + 8\eta_2\eta_4 + 6\eta_3\eta_4 + \frac{1}{2}\eta_4^2) = 24 > 0$, the main theorem of [33] implies that this Calabi-Yau threefold is a K3 fibration over $\mathbb{P}^1$ with typical fibre $D_5$. The explicit computation of the Hodge diamond of $D_5$ leads to $(h^{0,0}, h^{1,0}, h^{2,0}, h^{1,1}) = (1, 0, 1, 20)$ and $\chi = 24$, which is exactly the topological data of K3 surface. And we can also compute the Hodge diamond of $D_6$, $(h^{0,0}, h^{1,0}, h^{2,0}, h^{1,1}) = (1, 1, 0, 4)$ and $\chi = 2$, which is exactly the topological data of W-surface.

Finally let us analyse the splitting properties of the cohomology group under the orientifold involution $\sigma$. We restrict that the orientifold involution $\sigma$ just flips the sign of one homogeneous coordinates, i.e. $\sigma : x_i \mapsto -x_i$, and we find that there are three inequivalent involutions $\sigma : \{x_2 \mapsto -x_2, x_4 \mapsto -x_4, x_5 \mapsto -x_5\}$, such that the W-surface has the appropriate splitting properties and $h^{1,1}(M) = 0$. In the following we take the involution $\sigma : x_5 \mapsto -x_5$ as an example. Following the algorithm presented in [32], we obtain the following fixed point set of the ambient space

$$\{\text{Fixed}\}_{\text{Ambient}} = \{D_5, D_2, D_4, D_1 \cdot D_3 \cdot D_6, D_6 \cdot D_7 \cdot D_8\}. \quad (3.20)$$

From the generic hypersurface equation, only the following subset of the fixed point intersects the invariant hypersurface

$$\{\text{Fixed}\}_{\text{CY}} = \{D_5, D_2, D_4, D_6 \cdot D_7 \cdot D_8\}. \quad (3.21)$$

Hence we have three O7-planes wrapping the divisors $D_2, D_4, D_5$ respectively and two O3-planes, since the intersection number $D_6 \cdot D_7 \cdot D_8 = 2$. And we need to put eight D7-branes right on top of each O7-plane to cancel the D7-brane tadpole. Since the D7-branes are wrapping on the del Pezzo surfaces $D_2$ and $D_4$, which are pointwise invariant under the involution $\sigma$, so the only possible non-perturbative corrections to superpotential are from gaugino condensations on the D7-branes instead of D-brane instantons. Meanwhile applying Eq.(2.5), the contribution to D3-brane tadpole is

$$N_{D3} + \frac{1}{2} N_{\text{flux}} = \frac{1}{4}(2 + 11 + 11 + 24) = 12, \quad (3.22)$$
which is indeed integer as required. The splitting Hodge number of the del-Pezzo surfaces and W-surface under the involution $\sigma$ is as follows:

$$D_2 : (h^{0,0}, h^{1,0}, h^{2,0}, h^{1,1}) = (1_+, 0, 0, 9_+) ,$$

$$D_4 : (h^{0,0}, h^{1,0}, h^{2,0}, h^{1,1}) = (1_+, 0, 0, 9_+) ,$$

$$D_6 : (h^{0,0}, h^{1,0}, h^{2,0}, h^{1,1}) = (1_+, 1_+, 0, 4_+).$$

(3.23)

So we have the correct topological data for the neutral zero modes. Furthermore, we can read from the SR-ideal that the divisors supporting D7-branes, which lie on top of O7-planes, do not intersect with each other, and also there is no intersection between $D_5$ and $D_6$. Furthermore, the intersection between $D_2$ and $D_6$ is a surface with genus 1, which can be determined to be $T^2$ according to the classification theorem of the closed surface. We can summarize all this topological and geometrical information in the table 3.

| Divisor   | $(h^{0,0}, h^{1,0}, h^{2,0}, h^{1,1})$ | Intersection Curves |
|-----------|----------------------------------------|---------------------|
| $D_2 = dP_8$ | $(1_+, 0, 0, 9_+)$                      | $D_6 : C_g=1$       |
| $D_4 = dP_8$ | $(1_+, 0, 0, 9_+)$                      | $D_6 : C_g=1$       |
| $D_5 = K3$   | $(1_+, 0, 1_+, 20_+)$                   | Null                |
| $D_6 = W$    | $(1_+, 1_+, 0, 4_+)$                    | $D_2 : C_g=1, D_4 : C_g=1$ |

Table 3. Divisors with topological and geometrical information.

Since $W$ intersects the D7-branes over a $T^2$ and $h^*(T^2, \mathcal{O}) = (1, 1)$, there will be extra vector-like zero modes. If there is a non-trivial Wilson line on $T^2$, these zero modes can pair up and become massive [34]. For this purpose, one must have the freedom to turn on an additional gauge bundle on the divisor $[D7]$, whose restriction on the intersection curve $T^2$ is a non-trivial Wilson line. As argued in [13], an additional gauge bundle which is supported only on 2-cycles $C_i \subset [D7]$, which are topological trivial in M but do intersect with the curve $T^2$, allows one to avoid these extra zero modes. Considering that both $D_2$ and $D_4$ have more 2-cycles than the Calabi-Yau M, since $h^{1,1}(D_2) = h^{1,1}(D_4) > h^{1,1}(M)$, they must therefore exist such trivial 2-cycles.

Finally after checking all of constraints, we can ensure that the following superpotential can indeed be generated by the gaugino condensations on D7-branes and the poly-instanton effects

$$W = W_0 + A_1 e^{-a_1 T_1} + A_1 A_6 e^{-a_1 T_1} e^{-2\pi(T_3-T_1-T_2)} + A_2 e^{-a_2 T_2} + A_2 A_6 e^{-a_2 T_2} e^{-2\pi(T_3-T_1-T_2)} .$$

(3.24)

Note that $D_6 = \eta_3 - \eta_1 - \eta_2$ and the Kähler moduli $T_i, i = 1, 2, 3$ in the previous equation is associate to the volume modulus of basis divisors $\eta_i$ respectively.

4 Moduli stabilisation

In this section, we will discuss the Kähler moduli stabilisation using the superpotential (3.24) and the Kähler potential (2.6) with the general volume form

$$V = \alpha \sqrt{3} \tau_4 - \beta_1 \tau_1^{3/2} - \beta_2 \tau_2^{3/2}$$

(4.1)
which has the same volume structure as the explicit example (3.17) and $\alpha, \beta_1, \beta_2$ are some real constants.

As suggested in [3], the scalar potential (2.11) in the LARGE volume scenario can be divided into three parts $V_{np1}, V_{np2}$ and $V_{\alpha'}$:

$$V = V_{np1} + V_{np2} + V_{\alpha'},$$

$$V_{np1} = e^K K^{ij} \partial_i W \partial_j W,$$

$$V_{np2} = e^K K^{ij} (\partial_i W K_{j\bar{j}} W + \partial_{\bar{j}} W K_{i\bar{j}}),$$

$$V_{\alpha'} = e^K \left( K^{ij} K_i \bar{K}_j - 3 - \frac{3}{4} \xi^2 \right) |W|^2 = e^K \left[ 3\xi \xi^2 + 7\xi^3 \chi + V^2 \right].$$

Note that in the large volume limit we can ignore the $\alpha'$-corrections to the Kähler potential in the expression of $V_{np1}$ and $V_{np2}$, and the $V_{\alpha'}$ term reduces to

$$V_{\alpha'} = e^K \frac{3\xi}{4V^3} |W|^2 = e^K \frac{3\xi}{4V^3} |W|^2. \quad (4.3)$$

We also expect the divisors $\eta_1$ and $\eta_2$ to be the small divisors in the large volume limit.

In order to perform the calculation of the scalar potential, first of all we need to get the Kähler metric and its inverse. The Kähler metric is given by the following symmetric matrix

$$K_{ij} = \left( \begin{array}{cccc}
\frac{3}{8V^{3/2}} & \frac{3\beta_1 \beta_2 \sqrt{\tau_1 \tau_2}}{8V^{3/2}} & -\frac{3\beta_1 \sqrt{\tau_1}}{8V^{3/2}} & -\frac{3\alpha \beta_1 \sqrt{\tau_1 \tau_3}}{4\sqrt{V^3}} \\
\frac{3\beta_1 \beta_2 \sqrt{\tau_1 \tau_2}}{8V^{3/2}} & \frac{3}{8V^{3/2}} & \frac{3\beta_2 \sqrt{\tau_2}}{8V^{3/2}} & -\frac{3\alpha \beta_2 \sqrt{\tau_2 \tau_3}}{4\sqrt{V^3}} \\
-\frac{3\beta_1 \sqrt{\tau_1}}{8V^{3/2}} & \frac{3\beta_2 \sqrt{\tau_2}}{8V^{3/2}} & \frac{3}{8V^{3/2}} & \frac{3\alpha \beta_1 \sqrt{\tau_1 \tau_3}}{4\sqrt{V^3}} \\
-\frac{3\alpha \beta_1 \sqrt{\tau_1 \tau_3}}{4\sqrt{V^3}} & -\frac{3\alpha \beta_2 \sqrt{\tau_2 \tau_3}}{4\sqrt{V^3}} & \frac{3\alpha \beta_1 \sqrt{\tau_1 \tau_3}}{4\sqrt{V^3}} & \frac{3}{4\sqrt{V^3}} + \frac{3\alpha \beta_1 \sqrt{\tau_1 \tau_3}}{4\sqrt{V^3}} \\
\end{array} \right), \quad (4.4)$$

where we have used the expression of the volume in the large volume limit, $V = \alpha \sqrt{\tau_3 \tau_4}$, and dropped the subleading terms in orders of $V$.

The inverse of Kähler metric in the large volume limits reads as

$$K^{ij} = \left( \begin{array}{cccc}
\frac{8V^{3/2}}{4\sqrt{\tau_1}} & \frac{4\tau_1 \tau_2}{4\sqrt{\tau_3}} & \frac{4\tau_1 \tau_3}{4\sqrt{\tau_2}} & \frac{4\sqrt{\tau_1 \tau_3 \tau_4}}{4\sqrt{\tau_2}} \\
\frac{4\tau_1 \tau_2}{4\sqrt{\tau_3}} & \frac{8V^{3/2}}{4\sqrt{\tau_2}} & \frac{4\tau_2 \tau_3}{4\sqrt{\tau_1}} & \frac{4\sqrt{\tau_1 \tau_3 \tau_4}}{4\sqrt{\tau_2}} \\
\frac{4\tau_1 \tau_3}{4\sqrt{\tau_2}} & \frac{4\tau_2 \tau_3}{4\sqrt{\tau_1}} & \frac{8V^{3/2}}{4\sqrt{\tau_1}} & \frac{4\sqrt{\tau_1 \tau_3 \tau_4}}{4\sqrt{\tau_2}} \\
\frac{4\sqrt{\tau_1 \tau_3 \tau_4}}{4\sqrt{\tau_2}} & \frac{4\sqrt{\tau_1 \tau_3 \tau_4}}{4\sqrt{\tau_2}} & \frac{4\sqrt{\tau_1 \tau_3 \tau_4}}{4\sqrt{\tau_2}} & \frac{3}{4\sqrt{\tau_3}} + \frac{3\beta_1 \sqrt{\tau_1 \tau_3 \tau_4}}{4\sqrt{\tau_2}} \\
\end{array} \right). \quad (4.5)$$

So we can now calculate the scalar potential. In the LARGE volume scenario, we can set $e^{-a_i \tau_i} \propto V^{-1}, i = 1, 2$ and $e^{-2\pi(\tau_1 - \tau_1 - \tau_2)} \propto V^{-p}$, namely $e^{-2\pi \tau_3} \propto V^{-m_3 - m_2 - m_1}$ with $a_i = \frac{2\pi}{m_i}, m_i \in \mathbb{Z}_+, i = 1, 2$. Then we stabilize the moduli order by order in $1/V$. All the possible orders in $V$ in the expression of scalar potential are

$$\{\text{Possible Orders}\} = \{-3, -3 - p, -3 - 2p, -4, -4 - p, -4 - 2p, -5, -5 - p, -5 - 2p\}. \quad (4.6)$$

Observing the structure of the inverse matrix of Kähler metric and the structure of the superpotential, we can easily obtain that $\tau_3$ is only involved in the term whose order
includes \( p \), and other terms are independent of \( \tau_3 \). This observation is very important to determine the value of \( p \) in the end.

At the leading order, i.e. \( O(\lambda^{-3}) \), the scalar potential explicitly reads as

\[
V_{O(\lambda^{-3})} = \sum_{i=1}^{2} \left( \frac{8 a_i^2 A_i^2 \sqrt{\tau_i} e^{-2a_i \tau_i}}{3 \beta_i V} + \frac{4 a_i A_i W_0 \tau_i e^{-a_i \tau_i}}{V^2} \cos (a_i \rho_i) \right) + \frac{3 W_0^2 \hat{\xi}}{4 V^3}.
\]

(4.7)

We start with minimising the scalar potential with respect to the axions \( \rho_1, \rho_2 \), assuming that all involved parameters are real and positive. It is obvious that the minimal values lie at \( a_i \rho_i = (2k + 1)\pi, k \in \mathbb{Z} \), namely

\[
\rho_i = \frac{1}{2} (2k + 1) m_i, \quad k \in \mathbb{Z}, i = 1, 2.
\]

(4.8)

Then we minimise scalar potential with respect to the Kähler metric \( \tau_1, \tau_2 \), and the relevant derivatives equal to 0 can be reduced as

\[
\frac{\partial V}{\partial \tau_i} = 0 : \quad a_i A_i V (1 - 4a_i \tau_i) = 3 \beta_i W_0 e^{a_i \tau_i} (1 - a_i \tau_i)^{1/2}, \quad i = 1, 2
\]

\[
\frac{\partial V}{\partial V} = 0 : \quad \sum_{i=1}^{2} \left( \frac{32 a_i^2 A_i^2}{3 \beta_i} \sqrt{\tau_i} e^{-2a_i \tau_i} V^2 + 32 a_i A_i W_0 \tau_i e^{-a_i \tau_i} V \right) = 9 W_0^2 \hat{\xi}.
\]

(4.9)

Considering that we require \( a_i \tau_i \gg 1 \) to reduce the higher instanton corrections, the first equation in (4.9) reduces to

\[
4 a_i A_i V = 3 \beta_i W_0 e^{a_i \tau_i} \tau_i^{1/2}.
\]

(4.10)

In order to obtain a consistent value of the volume, we can take a very natural assumption that \( a_1 \tau_1 = a_2 \tau_2 \). Taking this identity into the second equation of (4.9), we obtain

\[
2 (\beta_1 \tau_1^{3/2} + \beta_2 \tau_2^{3/2}) = \hat{\xi}.
\]

(4.11)

Taking into account the assumption \( a_1 \tau_1 = a_2 \tau_2 \), we get the stabilisation value of \( \tau_1, \tau_2 \) and \( V \):

\[
a_i \langle \tau_i \rangle = \left( \frac{\xi}{2J} \right)^{2/3}, \quad i = 1, 2 \quad \text{with} \quad J = \sum_{i=1}^{2} \beta_i a_i^{-3/2},
\]

\[
\langle V \rangle = \frac{3 \beta_i W_0}{4 a_i A_i} e^{a_i \langle \tau_i \rangle \langle \tau_i \rangle^{1/2}} \quad \forall i = 1, 2.
\]

(4.12)

Next we consider the stabilisation of the fibration \( \tau_3 \). The first order containing \( T_3 \) in the scalar potential is \( O(\lambda^{-3-p}) \), which reads as

\[
V_{O(\lambda^{-3-p})} = - \frac{4 A_i W_0 e^{-(a_1-2\pi) \tau_1 - (a_2-2\pi) \tau_2 - 2\pi \tau_3}}{V^2} \left[ \tau_1 ((2\pi - a_1) A_1 e^{a_1 \tau_2} + 2\pi A_2 e^{a_1 \tau_1}) \right. \\
+ \tau_2 (2\pi A_1 e^{a_1 \tau_2} + (2\pi - a_2) A_2 e^{a_1 \tau_1} - 2\pi \tau_3 (A_1 e^{a_2 \tau_2} + A_2 e^{a_1 \tau_1})] \cos ((1 - m_1 - m_2) \pi + 2\pi \rho_1) \\
- \frac{16 A_i e^{-2((a_1-\pi) \tau_1 + (a_2-\pi) \tau_2 + 2\pi \tau_3)}}{3 \beta_1 \beta_2 V} \left[ a_1 A_1 A_2 \sqrt{\tau_i} e^{a_2 \tau_2} ((2\pi - a_1) A_1 e^{a_2 \tau_2} + 2\pi A_2 e^{a_1 \tau_1}) \\
+ a_2 A_2 A_1 \sqrt{\tau_i} e^{a_1 \tau_1} (2\pi A_1 e^{a_2 \tau_2} + (2\pi - a_2) A_2 e^{a_1 \tau_1})] \cos ((m_1 + m_2) \pi - 2\pi \rho_1),
\]

(4.13)
where we have used the stabilised value of $\rho_i, i = 1, 2$ in this equation. Now the scalar potential in the order of $\mathcal{V}^{-3}$ can be rearranged in the following form

$$V_{O(\mathcal{V}^{-3})} = (C_1 + C_2 \tau_3) \cos(2\pi \rho_3) e^{-2\pi \tau_3},$$

where $C_1$ and $C_2$ can be identified from the original expression of $V_{O(\mathcal{V}^{-3})}$ easily. It is obvious that the minimal of $V_{O(\mathcal{V}^{-3})}$ lies at

$$\langle \tau_3 \rangle = \frac{1}{2\pi} - \frac{C_1}{C_2} \quad (4.15)$$

and $\rho_3 \in \mathbb{Z}$ or $\rho_3 \in \mathbb{Z}/2$ depends on the specific value of $C_1$ and $C_2$. Unfortunately, under the relation of $(4.10)$, we can show that $C_1 = 0$, namely $\langle \tau_3 \rangle = \frac{1}{2\pi}$, which is manifest that $\tau_3$ is out of the Kähler cone, since it will lead to the negative value of the volume of $W$-surface.

After checking the procedure of the stabilisation, we find, if we use the precise relation derived from $\frac{\partial V}{\partial \tau_i} = 0$ instead of the approximation relation $(4.10)$, the value of $C_1$ is indeed non-zero. However, it seems that $C_1$ and $C_2$ have mostly the same sign. Even when they are in different sign, with $C_1 \ll C_2$, it can not solve the problem.

In order to solve the problem, we need introduce more parameters by using the racetrack superpotential as suggested in [5]. In this case, the superpotential reads as

$$W = W_0 + A_1 e^{-a_1 \tau_1} + A_1 A_6 e^{-a_1 \tau_1} e^{-2\pi(T_3 - T_1 - T_2)} + A_2 e^{-a_2 \tau_2} + A_2 A_6 e^{-a_2 \tau_2} e^{-2\pi(T_3 - T_1 - T_2)} - B_1 e^{-b_1 \tau_1} - B_1 B_6 e^{-b_1 \tau_1} e^{-2\pi(T_3 - T_1 - T_2)} - B_2 e^{-b_2 \tau_2} - B_2 B_6 e^{-b_2 \tau_2} e^{-2\pi(T_3 - T_1 - T_2)}, \quad (4.16)$$

where $a_i = \frac{2\pi}{m_i}, b_i = \frac{2\pi}{b_i}, m_i, l_i \in \mathbb{Z}_+, i = 1, 2$. Taking this racetrack superpotential, we can repeat the previous procedures. First of all, the possible orders in $\mathcal{V}$ in the scalar potential Eq.(2.11) are the same as before, namely $(4.6)$, and also only the terms whose order involves $p$ are relevant to the modulus $\tau_3$.

The leading order in $\mathcal{V}$ of the scalar potential reads

$$V_{O(\mathcal{V}^{-3})} = \sum_{i=1}^{2} \frac{1}{\mathcal{V}} \left[ \frac{8}{3\beta_i} \sqrt{\tau_i} \left( a_i^2 A_i^2 e^{-2a_i \tau_i} - 2a_i A_i b_i B_i e^{-a_i \tau_i - b_i \tau_i} \cos(a_i \rho_i - b_i \rho_i) + b_i^2 B_i^2 e^{-2b_i \tau_i} \right) \right]$$

$$+ \sum_{i=1}^{2} \frac{1}{\mathcal{V}^2} \left[ 4W_0 \tau_i \left( b_i B_i e^{-b_i \tau_i} \cos(b_i \rho_i) - a_i A_i e^{-a_i \tau_i} \cos(a_i \rho_i) \right) \right] + \frac{3W_0^2}{4\mathcal{V}^3} \dot{\xi}. \quad (4.17)$$

As before, let us find out the stabilised value of the axions $\rho_i, i = 1, 2$, where the relevant derivatives are:

$$\frac{\partial V}{\partial \rho_i} = \frac{1}{\mathcal{V}} \left[ \frac{16}{3\beta_i} a_i A_i b_i B_i \sqrt{\tau_i} (a_i - b_i) e^{-a_i \tau_i - b_i \tau_i} \sin(a_i \rho_i - b_i \rho_i) \right]$$

$$+ \frac{1}{\mathcal{V}^2} \left[ 4W_0 \tau_i \left( b_i^2 B_i e^{-b_i \tau_i} \sin(b_i \rho_i) - a_i^2 A_i e^{-a_i \tau_i} \sin(a_i \rho_i) \right) \right],$$

$$\frac{\partial^2 V}{\partial \rho_i^2} = \frac{1}{\mathcal{V}} \left[ \frac{16}{3\beta_i} a_i A_i b_i B_i \sqrt{\tau_i} (a_i - b_i)^2 e^{-a_i \tau_i - b_i \tau_i} \cos(a_i \rho_i - b_i \rho_i) \right]$$

$$+ \frac{1}{\mathcal{V}^2} \left[ 4W_0 \tau_i \left( b_i^2 B_i e^{-b_i \tau_i} \cos(b_i \rho_i) - a_i^2 A_i e^{-a_i \tau_i} \cos(a_i \rho_i) \right) \right]. \quad (4.18)$$
Note that \( \frac{\partial V}{\partial \rho_i} \) vanishes at \( \rho_i = 0 \), which is a minimum, if

\[
\frac{\partial^2 V}{\partial \rho_i^2} \bigg|_{\rho_i=0} = \frac{1}{V} \left[ \frac{16}{3 \beta_i} a_i A_i b_i B_i \sqrt{\tau_i} (a_i - b_i)^2 e^{-a_i \tau_i - b_i \tau_i} \right] \\
+ \frac{1}{V^2} \left[ 4 W_0 \tau_i (b_i^2 B_i e^{-b_i \tau_i} - a_i^3 A_i e^{-a_i \tau_i}) \right] > 0. \tag{4.19}
\]

For simplicity, we assume this condition to be true. Now we start analysing the stabilisation of \( \tau_i, i = 1, 2 \). As before, we first take the limit of \( a_i \tau_i \gg 1 \), then the vanishing of \( \frac{\partial V}{\partial \tau_i} \) implies

\[
e^{-b_i \tau_i} = \frac{3 \beta_i W_0 \tau_i^{1/2}}{4Z_i V}, \quad \text{with} \quad Z_i = b_i B_i - A_i a_i e^{-n_i \tau_i}, \tag{4.20}
\]

where we have written \( a_i = b_i + n_i \). In addition \( Z_i > 0 \) can ensure that the condition (4.19) is satisfied. Inserting this identity into the equation \( \frac{\partial V}{\partial \tau_i} = 0 \), this equation reduces to

\[
2 \left( \beta_1 \tau_1^{3/2} + \beta_2 \tau_2^{3/2} \right) = \xi, \tag{4.21}
\]

which is the same as the case with minimal superpotential. Therefore we obtain the same value for \( \langle \tau_i \rangle \) as before, and the value of the volume \( V \) can be determined by the identity (4.20).

The next step is to fix the value of \( \tau_3 \) and \( \rho_3 \) by the scalar potential \( V_{O(3^3-3^3)} \), which read as

\[
V_{O(3^3-3^3)} = \left\{ \begin{array}{l}
\frac{16}{3 \beta V^{3/2}} \cos(2 \pi \rho_3) e^{-2 \pi \tau_3} \left[ a_i^2 A_i^2 B_i^2 e^{2(\pi-a_i) \tau_1 + 2 \pi \rho_2} - 2 \pi a_i A_i e^{2(\pi-a_i) \tau_1 + 2 \pi \rho_2} \\
- 2 \pi a_i A_i e^{2(\pi-a_i) \tau_1 + 2 \pi \rho_2} - a_i A_i b_i B_i e^{-2 \pi \tau_2 - \tau_1 (a_i + b_i - 2 \pi) e^{-a_i \tau_1 - b_i \tau_2}} + 2 \pi A_i A_i b_i B_i e^{-2 \pi \tau_2 - \tau_1 (a_i + b_i - 2 \pi) e^{-a_i \tau_1 - b_i \tau_2}} \\
- a_i b_i B_i e^{-2 \pi \tau_2 - \tau_1 (a_i + b_i - 2 \pi) e^{-a_i \tau_1 - b_i \tau_2}} + 2 \pi a_i A_i B_i B_i e^{-2 \pi \tau_2 - \tau_1 (a_i + b_i - 2 \pi) e^{-a_i \tau_1 - b_i \tau_2}} + 2 \pi a_i A_i B_i B_i e^{-2 \pi \tau_2 - \tau_1 (a_i + b_i - 2 \pi) e^{-a_i \tau_1 - b_i \tau_2}} \\
+ 2 \pi A_i A_i e^{-2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} + 2 \pi b_i B_i B_i e^{-2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} - 2 \pi b_i B_i B_i e^{-2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} \\
- 2 \pi b_i B_i B_i e^{-2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} \\
\right. \\
+ \frac{1}{V^2} 4 W_0 \tau_3 \left[ 2 A_i A_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} + 2 A_i A_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} + 2 A_i A_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} + 2 A_i A_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} + 2 A_i A_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} + 2 A_i A_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} \\
\left. + 2 A_i A_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} + 2 A_i A_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} + 2 A_i A_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} + 2 A_i A_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} + 2 A_i A_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} + 2 A_i A_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} \\
- B_i B_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} - B_i B_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} - B_i B_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} - B_i B_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} - B_i B_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} - B_i B_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} \\
- B_i B_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} - B_i B_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} - B_i B_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} - B_i B_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} - B_i B_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} - B_i B_i e^{2 \pi \tau_2 - (b_i - 2 \pi) \tau_2} \\
\right] \tag{4.22}
\right.
\]

where we have used the stabilised value of the axions \( \rho_i \) in this expression. The scalar potential in the order of \( V^{3-3} \) can be rearranged in the form,

\[
V_{O(3^3-3^3)} = (C'_1 + C'_2 \langle \tau_3 \rangle) \cos(2 \pi \rho_3) e^{-2 \pi \tau_3} \tag{4.23}
\]

as before and the constant \( C'_1 \), \( C'_2 \) can be easily identified from the original expression of \( V_{O(3^3-3^3)} \). Hence its minimum lies at \( \langle \tau_3 \rangle = \frac{1}{2 \pi} - \frac{C'_1}{C'_2} \), and \( \rho_3 \in \mathbb{Z} \) or \( \rho_3 \in \mathbb{Z}/2 \) depends on the specific value of \( C'_1 \) and \( C'_2 \). Unfortunately, when we take the approximate relation (4.20), \( C'_1 \) will vanish again. However in this racetrack superpotential case, we indeed can
get a suitable ratio of $C_1'$ and $C_2'$ by solving the equation \( \frac{\partial V_{\mathcal{O}(V^{-3-p})}}{\partial \tau_i} = 0 \) exactly instead of the approximate relation (4.20). In this case the relation will be modified as

\[
e^{-b_i \tau_i} = \frac{3\beta_i W_0 \tau_i^{1/2}}{4Z_i V} f_i^{\text{corr}}
\]

(4.24)

where

\[
f_i^{\text{corr}} = 1 - \frac{3\epsilon_i}{-\epsilon_i + 1 + n_i \left( \frac{1}{b_i} - \frac{B_i}{Z_i} \right)} \quad \text{with} \quad \epsilon_i := \frac{1}{4b_i \tau_i} \ll 1.
\]

(4.25)

One subtle thing is that we cannot solve the fixed values of $\tau_i, i = 1, 2$ analytically any more, so we have to solve them numerically. Then we obtain the stabilised value of the volume $V$ by using the identity (4.24). Note that although the relation has been modified, the stabilised value of the volume $V$ and $\tau_1, \tau_2$ are still independent of $\tau_3$. After that, we can get the stabilised value of $\tau_3$ numerically by minimising the scalar potential $V_{\mathcal{O}(V^{-3-p})}$.

We can also determine the value of $p$ as follows:

\[
p = \frac{l_i (\tau_3 - \tau_1 - \tau_2)}{\tau_i}, \quad \forall i = 1, 2.
\]

(4.26)

Here we must stress that the definition of the $p$ is different from the one in [5], as we have a much more complex structure of $V_{\mathcal{O}(V^{-3-p})}$. This definition originates from the direct comparison of these two relations, $e^{b_i \tau_i} \propto V, e^{2\pi (\tau_3 - \tau_1 - \tau_2)} \propto V^p$.

We will give some benchmark models in table 4 and the fixed values of divisor volume moduli corresponding to the respective benchmark models in table 5, to show that this procedure of moduli stabilisation does work consistently.

| Nos. | $g_s$ | $a_i$ | $b_i$ | $A_i$ | $B_i$ | $A_6$ | $B_6$ | $W_0$ | $f_i^{\text{corr}}$ | $Z_i$ | $p$ |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-----------------|------|-----|
| I    | 0.1   | $\sqrt{\frac{2\pi}{1}}$ | $\frac{2 \pi}{8}$ | 0.1   | 0.1   | 7     | 4.5   | 1     | 0.57            | 0.021 | 1.00 |
| II   | 0.1   | $\sqrt{\frac{2\pi}{1}}$ | $\frac{2 \pi}{8}$ | 0.1   | 0.2   | 6     | 4.37  | 1     | 0.79            | 0.065 | 0.51 |
| III  | 0.01  | $\sqrt{\frac{2\pi}{1}}$ | $\frac{2 \pi}{8}$ | 0.1   | 8     | 21    | 7.96  | 1     | 0.97            | 4.80  | 0.21 |
| IV   | 0.01  | $\sqrt{\frac{2\pi}{1}}$ | $\frac{2 \pi}{8}$ | 0.5   | 12    | 5.45  | 1     | 0.96            | 0.19  | 0.57 |

**Table 4.** Parameters for four benchmark models

| Nos. | $\tau_1$ | $\tau_3$ | $\rho_3$ | $\mathcal{V}$ | $M_s(\text{GeV})$ |
|------|----------|----------|----------|---------------|-----------------|
| I    | 4.07     | 8.65     | $\mathbb{Z}$ | 473.33        | $1.30 \times 10^{16}$ |
| II   | 3.44     | 7.13     | $\mathbb{Z}/2$ | 174.76        | $2.13 \times 10^{16}$ |
| III  | 31.02    | 62.96    | $\mathbb{Z}/2$ | $4.92 \times 10^{11}$ | $4.02 \times 10^{11}$ |
| IV   | 31.17    | 64.56    | $\mathbb{Z}$ | $4.27 \times 10^{11}$ | $4.32 \times 10^{11}$ |

**Table 5.** The fixed values of divisor volume moduli corresponding to the respective benchmark models

The string scale in the table is defined as usual $M_s = \frac{M_p}{\sqrt{4\pi V}}$. And we have set the constants $\alpha, \beta_1, \beta_2$ in the expression of volume as $\alpha = \frac{1}{b}, \beta_1 = \beta_2 = \frac{\sqrt{2}}{3}$, so that the volume is just the same as the explicit example presented in section 3.1.
Observing the table of benchmark models, one should note that we always choose the symmetric value of $\tau_1$ and $\tau_2$. If we only consider the stabilisation of the Kähler moduli by minimising the scalar potential, it seems that this kind of symmetry is not necessary. However when we consider the geometrical and topological properties of these two divisors in the Calabi-Yau threefold, it seems that this kind of symmetry indeed exists. Of course, one may search the possible parameters, which breaks this kind of symmetry.

**String loop corrections:** Except the $\alpha'$-corrections and the non-perturbative corrections mentioned above, there are also corrections to the scalar potential from the string loop effects. In this final part, let us analyse the affect of this string loop correction to our procedure of moduli stabilisation.

The string loop corrections to the scalar potential read as

$$\delta V(g_s) = \frac{g_s^2}{V^2} \left[ \frac{3\beta_1}{8V}\frac{(C^{KK})^2}{\sqrt{\tau_1}} + \frac{3\beta_2}{8V}\frac{(C^{KK})^2}{\sqrt{\tau_2}} + \frac{(C^{KK})^2}{4\tau_3^2} \right],$$

where $C^{KK}_i$ and $C^{WI}_i$ are constants which depend on the complex structure moduli, $K_{ii}$ is the Kähler metric for $\tau_i$ and $a_{il}t_l$ denotes a linear combination of the basis 2-cycle volumes $t_l$. The first part of the correction comes from the exchange of closed strings which carry Kaluza-Klein momentum between D7 and D3-branes and the second part comes from the exchange of winding strings between intersecting stacks of D7-branes. Applying this to the explicit example presented in section 3.1, since there are no intersection between every pair of the divisors $D_2, D_4, D_5$, which are wrapped by the D7-branes, it is obvious that $\delta K^{W}_{(g_s),\tau_i} = 0$. Therefore, in our constructions, the string loop corrections to the scalar potential are:

$$\delta V(g_s) = g_s^2 \frac{W_0^2}{V^2} \left[ \frac{3\beta_1}{8V}\frac{(C^{KK})^2}{\sqrt{\tau_1}} + \frac{3\beta_2}{8V}\frac{(C^{KK})^2}{\sqrt{\tau_2}} + \frac{(C^{KK})^2}{4\tau_3^2} \right].$$

Since $g_s < 1$, the LARGE volume scenario will be safe as long as $W_0 \sim O(1)$. One can also see the effects of the string loop corrections to various moduli stabilisation mechanism in the appendix of [8]. For the stabilisation of $\tau_3$, the leading order scalar potential relevant to the divisor $\tau_3$ seems from this string loop corrections, since superficially it scales as $V^{-2}$, but we can estimate the scale of the last term in Eq. (4.28), denoted as $\delta V^{(3)}_{(g_s)}$, by using the 1-loop Coleman-Weinberg potential as in [5], which is showed that $\delta V^{(3)}_{(g_s)} \sim \Lambda^2 STr(M^2) \sim (M^{6D}_{KK})^2 m_{3/2}^2 \sim \frac{\tau_4}{\tau_4^2}$, where the cut-off $\Lambda$ in the 1-loop Coleman-Weinberg potential given by the 6D Kaluza-Klein scale $M^{6D}_{KK} \sim M_P \frac{\sqrt{\tau_4}}{\sqrt{V}}$ and $m_{3/2} \sim \frac{M_P}{\sqrt{V}}$. Considering that the leading order term relevant to $\tau_3$ in the poly-instanton potential scales as $V^{-3-p}$, we can conclude that the string loop corrections of the scalar potential will not spoil the procedure of the moduli stabilisation.

### 5 Conclusions

In this paper, we have presented one consistent procedure to generate the superpotential in types of Eq. (3.24), including the gaugino condensation and poly-instanton effects, in type
IIB orientifold compactification. And then we use this kind of superpotential as well as the $\alpha'$-corrections to the Kähler potential to stabilise all four Kähler moduli, where the volume form of the compact Calabi-Yau is in the general form Eq.(4.1). For this purpose we first searched all the possible Calabi-Yau threefolds which have one W-surface to support the poly-instanton effects from the 158 examples of reflexive lattice polytopes, which admit a K3-fibred Calabi-Yau hypersurface in [31]. We find that only 23 of them admit a W-surface, where the result has been listed in the appendix. Then we analysed all the topological and geometrical conditions of the non-perturbative superpotential induced by the gaugino condensation and poly-instanton effect for one explicit Calabi-Yau threefold. Finally we systematically studied the stabilisation procedure and discussed that this procedure is safe from the string-loop corrections.

One advantage of K3-fibred Calabi-Yau threefolds in type IIB theory is that the extra dimensions can be anisotropic. Thus we can try to embed the supersymmetric extra dimensional models of particle physics into this frame, such as [5]. However, one of the constraints for this kind of embedding enforces the string scale to be around TeV-scale, namely the volume of the compactified manifold should be around $10^{28}$. In our procedure of moduli stabilisation, the volume is proportion to the exponential of $\tau_i$, whose value is mainly determined by the value of $\xi$. As the Calabi-Yau threefold for compactification has been uniquely chosen, the only possible change is in the string coupling $g_s$. If we demand the volume to be around $10^{28}$, the string coupling should be around $g_s \sim 0.004$, which is unnaturally small. Of course for the embedding of a concrete particle physics model, not all the Kähler moduli should be fixed by this procedure, because some of moduli or their linear combinations can be fixed by the $D$-terms, which are generated from the D7-brane gauge theory in the visible sectors.

In addition, we can also expect that this procedure can be used in the string cosmology, especially for various inflationary models, in which the Kähler moduli will serve as inflaton field in the single field inflation scenarios, or both inflaton and curvaton(or light modulating field) in the double-field inflation scenarios (for an overview of this point see [35]). To constrain the K3-fibred Calabi-Yau threefold in the single field inflation scenarios, either one of the del Pezzo surfaces or the K3-divisor can serve as the inflaton, as in [7, 9]. For the double-field inflation scenarios, more precisely for the curvaton mechanism, one of the del-Pezzo surfaces serves as the inflaton and the fibre K3 divisor servers as curvaton [10], and for the modulation mechanism, the fibre K3 divisor can serve as the inflaton, while the W-surface can serve as the light modulating field [11]. The explicit example present in this paper can either be used for the single-field inflation or for the double-field scenario, since at the leading order, the structure of the scalar potential is similar as the one in these papers. For the same reason, one can also expect that the explicit example present in this paper to be used in some quintessence models for dark energy, in which the quintessence field can be identified as the fibre K3 divisor [12].

Finally we need to point out that the racetrack superpotential used in the procedure of moduli stabilisation in this paper is just an assumption, instead of a concrete construction as for the superpotential Eq.(3.24). To get a racetrack superpotential, we need introduce the gauge flux on the D7-branes to split the original gauge group into two parts with
different ranks, and then have gaugino condensation on both of them. Note that the
different ranks is not a general requirement for generating racetrack superpotenial, but it’s
necessary to moduli stabilisation, since we need that $a_i \neq b_i$. However turning on the gauge
flux will destroy the favourable zero-modes of instantons, so we must carefully choose the
gauge flux to cancel the extra zero-modes. That is much more complicated, and beyond
the simple setups in this paper, so we leave it to the later works.

Acknowledgments

We thank P. Shukla and X. Gao for useful discussions on the non-perturbative effects to
the superpotential. And also we thank T. Weigand and M. Cicoli for very useful comments
on the manuscript. X. Zhang is supported by the MPG-CAS Joint Doctoral Promotion
Program.
A List of all the K3 fibrations with del Pezzo and W-surface

In this appendix we give all the 23 four dimensional reflexive lattice polytopes, which admit a K3-fibred Calabi-Yau hypersurface with four Kähler moduli in which at least one del Pezzo surface and one W-surface.

We must stress that the difference between the weight matrices of the polytopes listed here and those listing in [31], is due to the fact that the reflexive lattice polytopes which can be defined without having to specify all the weights. But in order to make comparisons between each other, we will preserve the mark number.

| Nos. | $\sum_{A}$ | $\omega_{A}^{i}$ | $\sum_{B}$ | $\omega_{B}^{i}$ | $\sum_{C}$ | $\omega_{C}^{i}$ | $\sum_{D}$ | $\omega_{D}^{i}$ |
|------|------------|----------------|------------|----------------|------------|----------------|------------|----------------|
| 1    | 8          | 212120000       | 8          | 21102002       | 8          | 20112020       | 4          | 10101100       |
| 3    | 12         | 216120000       | 12         | 21601020       | 12         | 20611020       | 6          | 10301100       |
| 6    | 8          | 122021000       | 5          | 11101010       | 2          | 10010000       | 4          | 01101001       |
| 10   | 12         | 421230000       | 6          | 21101100       | 6          | 20101020       | 3          | 10000001       |
| 12   | 12         | 620121000       | 8          | 41011010       | 6          | 31001001       | 4          | 20110000       |
| 17   | 14         | 720311000       | 16         | 80142100       | 8          | 40021001       | 4          | 20010010       |
| 21   | 24         | 811202100       | 18         | 60911100       | 12         | 40601010       | 6          | 20300001       |
| 32   | 6          | 310101000       | 10         | 50112010       | 8          | 40010120       | 4          | 20000011       |
| 33   | 8          | 10122020       | 5          | 10011110       | 3          | 01001010       | 4          | 00011011       |
| 34   | 6          | 210101110      | 6          | 21001200       | 3          | 10110000       | 3          | 10000101       |
| 35   | 6          | 21110100       | 6          | 21200010       | 6          | 11011200       | 3          | 00001101       |
| 41   | 8          | 41000120       | 4          | 20101000       | 10         | 50021110       | 4          | 20000011       |
| 56   | 3          | 10000110       | 8          | 01120220       | 7          | 01011220       | 4          | 00010111       |
| 60   | 8          | 41001110       | 10         | 50101120       | 8          | 40012100       | 4          | 20001001       |
| 62   | 10         | 51101020       | 12         | 62001120       | 6          | 31000011       | 8          | 40110100       |
| 63   | 10         | 51102010       | 8          | 41000120       | 4          | 20011000       | 4          | 20000011       |
| 69   | 8          | 41201000       | 6          | 31100100       | 8          | 40111010       | 4          | 20100001       |
| 71   | 6          | 310000110      | 8          | 40101110       | 6          | 20011110       | 7          | 30001111       |
| 77   | 6          | 30110010       | 4          | 20001100       | 3          | 10000101       | 2          | 01000100       |
| 79   | 12         | 61002120       | 4          | 20100010       | 6          | 30011100       | 6          | 30001011       |
| 95   | 12         | 61002210       | 8          | 40101110       | 4          | 20010100       | 6          | 30001101       |
| 99   | 12         | 61201020       | 10         | 51100120       | 6          | 30100011       | 4          | 20010010       |
| 106  | 8          | 11401010       | 4          | 10200100       | 8          | 01410020       | 4          | 00200011       |

References

[1] C. Will, “The Confrontation between General Relativity and Experiment,” Living Reviews in Relativity 4 (May, 2001) 4, arXiv:gr-qc/0103036.

[2] S. Kachru, R. Kallosh, A. D. Linde, and S. P. Trivedi, “De Sitter vacua in string theory,” Phys. Rev. D68 (2003) 046005, arXiv:hep-th/0301240 [hep-th].

[3] V. Balasubramanian, P. Berglund, J. P. Conlon, and F. Quevedo, “Systematics of moduli stabilisation in Calabi-Yau flux compactifications,” JHEP 0503 (2005) 007, arXiv:hep-th/0502058 [hep-th].
[4] D. Lust, S. Reffert, E. Scheidegger, W. Schulgin, and S. Stieber, “Moduli Stabilization in Type IIB Orientifolds (II),” *Nucl.Phys.* **B766** (2007) 178–231, arXiv:hep-th/0609013 [hep-th].

[5] M. Cicoli, C. Burgess, and F. Quevedo, “Anisotropic Modulus Stabilisation: Strings at LHC Scales with Micron-sized Extra Dimensions,” *JHEP* **1110** (2011) 119, arXiv:1105.2107 [hep-th].

[6] R. Blumenhagen, X. Gao, T. Rahn, and P. Shukla, “Moduli Stabilization and Inflationary Cosmology with Poly-Instantons in Type IIB Orientifolds,” *JHEP* **1211** (2012) 101, arXiv:1208.1160 [hep-th].

[7] M. Cicoli, C. Burgess, and F. Quevedo, “Fibre Inflation: Observable Gravity Waves from IIB String Compactifications,” *JCAP* **0903** (2009) 013, arXiv:0808.0691 [hep-th].

[8] M. Cicoli, J. P. Conlon, and F. Quevedo, “Systematics of String Loop Corrections in Type IIB Calabi-Yau Flux Compactifications,” *JHEP* **0801** (2008) 052, arXiv:0708.1873 [hep-th].

[9] M. Cicoli, F. G. Pedro, and G. Tasinato, “Poly-instanton Inflation,” *JCAP* **1112** (2011) 022, arXiv:1110.6182 [hep-th].

[10] C. Burgess, M. Cicoli, M. Gomez-Reino, F. Quevedo, G. Tasinato, et al., “Non-standard primordial fluctuations and nongaussianity in string inflation,” *JHEP* **1008** (2010) 045, arXiv:1005.4840 [hep-th].

[11] M. Cicoli, G. Tasinato, I. Zavala, C. Burgess, and F. Quevedo, “Modulated Reheating and Large Non-Gaussianity in String Cosmology,” *JCAP* **1205** (2012) 039, arXiv:1202.4580 [hep-th].

[12] M. Cicoli, F. G. Pedro, and G. Tasinato, “Natural Quintessence in String Theory,” *JCAP* **1207** (2012) 044, arXiv:1203.6655 [hep-th].

[13] R. Blumenhagen, X. Gao, T. Rahn, and P. Shukla, “A Note on Poly-Instanton Effects in Type IIB Orientifolds on Calabi-Yau Threefolds,” *JHEP* **1206** (2012) 162, arXiv:1205.2485 [hep-th].

[14] R. Blumenhagen, S. Moster, and E. Plauschinn, “Moduli Stabilisation versus Chirality for MSSM like Type IIB Orientifolds,” *JHEP* **0801** (2008) 058, arXiv:0711.3389 [hep-th].

[15] M. Cicoli, C. Mayrhofer, and R. Valandro, “Moduli Stabilisation for Chiral Global Models,” *JHEP* **1202** (2012) 062, arXiv:1110.3333 [hep-th].

[16] A. Collinucci, M. Kreuzer, C. Mayrhofer, and N.-O. Walliser, “Four-modulus ‘Swiss Cheese’ chiral models,” *JHEP* **0907** (2009) 074, arXiv:0811.4599 [hep-th].

[17] M. Grana, “Flux compactifications in string theory: A Comprehensive review,” *Phys.Rept.* **423** (2006) 91–158, arXiv:hep-th/0509003 [hep-th].

[18] S. B. Giddings, S. Kachru, and J. Polchinski, “Hierarchies from fluxes in string compactifications,” *Phys.Rev.* **D66** (2002) 106006, arXiv:hep-th/0105097 [hep-th].

[19] T. W. Grimm and J. Louis, “The Effective action of N = 1 Calabi-Yau orientifolds,” *Nucl.Phys.* **B699** (2004) 387–426, arXiv:hep-th/0403067 [hep-th].

[20] D. S. Freed and E. Witten, “Anomalies in string theory with D-branes,” *Asian J.Math.* **3** (1999) 819, arXiv:hep-th/9907189 [hep-th].
[21] A. Collinucci, F. Denef, and M. Esole, “D-brane Deconstructions in IIB Orientifolds,” JHEP 0902 (2009) 005, arXiv:0805.1573 [hep-th].
[22] K. Becker, M. Becker, M. Haack, and J. Louis, “Supersymmetry breaking and alpha-prime corrections to flux induced potentials,” JHEP 0206 (2002) 060, arXiv:hep-th/0204254 [hep-th].
[23] S. Gukov, C. Vafa, and E. Witten, “CFT’s from Calabi-Yau four folds,” Nucl.Phys. B584 (2000) 69–108, arXiv:hep-th/9906070 [hep-th].
[24] F. Denef, M. R. Douglas, and B. Florea, “Building a better racetrack,” JHEP 0406 (2004) 034, arXiv:hep-th/0404257 [hep-th].
[25] M. Cvetic, I. Garcia-Etxebarria, and J. Halverson, “On the computation of non-perturbative effective potentials in the string theory landscape: IIB/F-theory perspective,” Fortsch.Phys. 59 (2011) 243–283, arXiv:1009.5386 [hep-th].
[26] R. Blumenhagen, M. Cvetic, S. Kachru, and T. Weigand, “D-Brane Instantons in Type II Orientifolds,” Ann.Rev.Nucl.Part.Sci. 59 (2009) 269–296, arXiv:0902.3251 [hep-th].
[27] A. Iqbal, A. Neitzke, and C. Vafa, “A Mysterious duality,” Adv.Theor.Math.Phys. 5 (2002) 769–808, arXiv:hep-th/0110108 [hep-th].
[28] E. Sharpe, “Lectures on D-branes and sheaves,” arXiv:hep-th/0307245 [hep-th].
[29] R. Blumenhagen, B. Jurke, T. Rahn, and H. Roschy, “Cohomology of Line Bundles: Applications,” J.Math.Phys. 53 (2012) 012302, arXiv:1010.3717 [hep-th].
[30] M. Kreuzer and H. Skarke, “PALP: A Package for analyzing lattice polytopes with applications to toric geometry,” Comput.Phys.Commun. 157 (2004) 87–106, arXiv:math/0204356 [math-sc].
[31] M. Cicoli, M. Kreuzer, and C. Mayrhofer, “Toric K3-Fibred Calabi-Yau Manifolds with del Pezzo Divisors for String Compactifications,” JHEP 1202 (2012) 002, arXiv:1107.0383 [hep-th].
[32] C. Mayrhofer, Compactifications of Type IIB String Theory and F-Theory Models by Means of Toric Geometry. PhD thesis, Vienna University of Technology, 2010.
[33] K. Oguiso, “On Algebraic Fiber Space Structures on Calabi-Yau 3-fold,” Int. J. of Math. 4 (1993) 439–465.
[34] R. Blumenhagen, V. Braun, T. W. Grimm, and T. Weigand, “GUTs in Type IIB Orientifold Compactifications,” Nucl.Phys. B815 (2009) 1–94, arXiv:0811.2936 [hep-th].
[35] M. Cicoli and F. Quevedo, “String moduli inflation: An overview,” Class.Quant.Grav. 28 (2011) 204001, arXiv:1108.2659 [hep-th].