Which algebraic groups are Picard varieties?

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Abstract

We show that every connected commutative algebraic group over an algebraically closed field of characteristic 0 is the Picard variety of some projective variety having only finitely many non-normal points. In contrast, no Witt group of dimension at least 3 over a perfect field of prime characteristic is isogenous to a Picard variety obtained by this construction.

1 Introduction and statement of the main results

With any proper scheme $X$ over a field $k$, one associates the Picard scheme $\text{Pic}_{X/k}$ and its neutral component $\text{Pic}^0_{X/k}$, a connected group scheme of finite type which parameterizes the algebraically trivial invertible sheaves on $X$. When $k$ is perfect, the reduced neutral component of $\text{Pic}_{X/k}$ is an algebraic group, classically known as the Picard variety $\text{Pic}^0(X)$. One may ask whether any connected commutative algebraic group can be obtained in this way. In this article, we obtain a positive answer to that question when $k$ is algebraically closed of characteristic 0, and a negative partial answer in prime characteristics. The analogous question for the reduced neutral component of the automorphism group scheme is answered in the positive by [Br13, Thm. 1].

By general structure results, every connected commutative algebraic group $G$ over a perfect field sits in a unique exact sequence $0 \to U \times T \to G \to A \to 0$, where $U$ is a connected unipotent algebraic group, $T$ a torus, and $A$ an abelian variety. Conversely, given such an exact sequence, we shall construct a projective variety $X$ such that $\text{Pic}^0(X) \cong G$, under additional assumptions on the affine part $U \times T$. Our result holds more generally in the setting of the relative Picard functor (see [BLR90, Kl05]):

Theorem 1.1. Let $S$ be a locally noetherian scheme, and

\begin{equation}
0 \to V \times T \to G \to A \to 0
\end{equation}

an exact sequence of commutative $S$-group schemes, where $V$ is a vector group, $T$ a quasi-split torus, and $A$ an abelian scheme. Then there exists a proper flat $S$-scheme $X$ with integral geometric fibers, such that $G \cong \text{Pic}^0_{X/S}$. Moreover, $X$ may be taken locally projective over $S$, if $A$ is locally projective.

Here a vector group is the additive group scheme of a locally free sheaf of finite rank; a quasi-split torus is a group scheme $T$ such that the pull-back $T_{S'}$ under some finite étale...
Galois cover $S' \to S$ with group $\Gamma$ is isomorphic to a direct product of finitely many copies of $\mathbb{G}_{m,S}'$, which are permuted by $\Gamma$.

Under the assumptions of that theorem, we now sketch how to construct the desired scheme $X$ from the exact sequence (1). We use the process of pinching studied in [Fe03]; more specifically, we obtain $X$ by pinching an appropriate smooth $S$-scheme $X'$ along a finite subscheme $Y'$ via a morphism $\psi : Y' \to Y$. We then have an exact sequence

$$0 \to \mathbb{G}_{m,S} \to V^*_Y \to (\text{Pic}_{X'/S}, Y') \to \text{Pic}_{X'/S} \to 0,$$

where $V^*_Y$ is a smooth affine group scheme with connected fibers, defined by $V^*_Y(S') = \mathcal{O}(Y_{S'})^*$ for any scheme $S'$ over $S$; $\text{Pic}_{X'/S}$ stands for the relative Picard functor, and $(\text{Pic}_{X'/S}, Y')$ parameterizes the invertible sheaves on $X'$, rigidified along $Y'$ (see [BLR90 8.1]). There is of course an analogous sequence for $(X, Y)$; in addition, one easily obtains an isomorphism of rigidified Picard functors $(\text{Pic}_{X/S}, Y) \cong (\text{Pic}_{X'/S}, Y')$. All of this yields an exact sequence

$$0 \to V^*_Y / \psi^*(V^*_Y) \to \text{Pic}_{X/S} \to \text{Pic}_{X'/S} \to 0. \quad (2)$$

It remains to find $X'$, $Y'$ and $\psi$ so that (2) gives back the exact sequence (1). For this, we use a result of Önsiper: every extension of an abelian scheme by the direct product of a vector group and a split torus can be constructed as a rigidified Picard functor (see [Ön87]). A slight modification of that construction yields the desired objects; note that [Ön87] uses the notion of rigidifier as in [Ra70], which is weaker than that of [BLR90].

Over an algebraically closed field of characteristic 0, every connected commutative unipotent group is a vector group, and every torus is (quasi-)split; hence any connected commutative algebraic group is the Picard variety of some projective variety with finite singular locus. But this does not extend to prime characteristics:

**Theorem 1.2.** Let $W_n$ denote the Witt group of dimension $n$ over a perfect field $k$ of characteristic $p > 0$. Then $W_n$ is not isogenous to the Picard variety of any projective variety with finite non-normal locus, if $p \geq 5$ and $n \geq 2$ (resp. $p \leq 3$ and $n \geq 3$).

It should be noted that the affine part of the Picard variety of any proper reduced scheme $X$ over a perfect field $k$ has been described by Geisser in [Ge09]. In particular, the maximal torus of $\text{Pic}^0(X)$ has cocharacter module isomorphic to $H^1_{\acute{e}t}(X_{\overline{k}}, \mathbb{Z})$ as a Galois module (see [Ge09 Thm. 1], and [Al02 Thm. 4.1.7] for a closely related result). We do not know whether all tori (or equivalently, all Galois modules) can be obtained in this way. When the non-normal locus of $X$ is finite, the maximal torus of $\text{Pic}^0(X)$ must be stably rational, see Remark 4.8.

This article is organized as follows. In Section 2, we begin by gathering results taken from [Fe03] about pinching and Picard groups; then we obtain the exact sequence (2) together with representability of the associated Picard functors under suitable assumptions. Section 3 constructs some extensions of abelian schemes by adapting the results of [Ön87]; it concludes with the proof of Theorem 1.1. In Section 4, we study the quotients $\mu^B / \mu^A$, where $A \subset B$ are artinian algebras over a field and $\mu^A \subset \mu^B$ denote the associated unit group schemes. These quotients are exactly the affine parts of Picard varieties of projective varieties with finite non-normal locus, see Proposition 4.1. We conclude with the proof of Theorem 1.2.
2 Pinching and Picard functor

2.1 Pinched schemes

Throughout this section, we fix a locally noetherian base scheme $S$. Schemes are assumed to be separated and of finite type over $S$ unless otherwise mentioned.

Let $X'$ be a scheme, $\iota' : Y' \to X'$ the inclusion of a closed subscheme, and $\psi : Y' \to Y$ a finite morphism. We assume that the natural map $O_Y \to \psi_* (O_{Y'})$ is injective; in particular, $\psi$ is surjective. We also assume that $X'$ and $Y$ satisfy the following condition:

(AF) Every finite set of points is contained in an open affine subscheme.

Under these assumptions, there exists a cocartesian diagram of schemes

$$
\begin{array}{ccc}
Y' & \xrightarrow{\iota'} & X' \\
\psi \downarrow & & \psi \downarrow \\
Y & \xrightarrow{\iota} & X,
\end{array}
$$

where $\iota$ is a closed immersion, $\varphi$ is finite, and $X$ satisfies (AF). Moreover, $\varphi$ induces an isomorphism $X' \setminus Y' \to X \setminus Y$; in particular, $\varphi$ is surjective. We say that $X$ is obtained by pinching $X'$ along $Y'$ via $\psi$.

These results follow from [Fe03, Thm. 5.4, Prop. 5.6], except for the assertion that $X$ is of finite type over $S$, which is a consequence of [Bo64, Chap. V, §1, no. 9, Lem. 5]. If in addition $X'$ is proper over $S$, then so is $X$ (since $\varphi : X' \to X$ is finite and surjective). But projectivity is not preserved under pinching, as shown by the examples in [Fe03, Sec. 6].

Since the formation of $X$ is Zariski local on $S$, we may replace (AF) with a slightly weaker condition:

(IAF) $S$ is covered by open subschemes $S_i$ such that every finite set of points over $S_i$ is contained in an open affine subscheme.

This condition holds in particular for locally projective $S$-schemes.

2.2 Their invertible sheaves

With the notation and assumptions of Subsection 2.1, the data of an invertible sheaf $\mathcal{L}$ on $X$ is equivalent to that of a triple $(\mathcal{L}', s', \mathcal{M})$, where $\mathcal{L}'$ (resp. $\mathcal{M}$) is an invertible sheaf on $X'$ (resp. $Y'$), and $s' : \psi^*(\mathcal{M}) \to \iota'^*(\mathcal{L}')$ is an isomorphism. Namely, one associates with $\mathcal{L}$ the sheaves $\mathcal{L}' := \varphi^*(\mathcal{L})$, $\mathcal{M} := \iota^*(\mathcal{L})$ and the isomorphism

$$
\psi^*(\mathcal{M}) = \psi^* \iota^*(\mathcal{L}) \to \iota'^* \varphi^*(\mathcal{L}) = \iota'^*(\mathcal{L}')
$$

arising from the commutative diagram (3).

Moreover, the isomorphisms $\mathcal{L}_1 \to \mathcal{L}_2$ are equivalent to the pairs $(u, v)$, where $u : \mathcal{L}_1 \to \mathcal{L}_2$, $v : \mathcal{M}_1 \to \mathcal{M}_2$ are isomorphisms such that the diagram

$$
\begin{array}{ccc}
\psi^*(\mathcal{M}_1) & \xrightarrow{s'_1} & \iota'^*(\mathcal{L}_1') \\
\psi^*(\mathcal{M}_2) & \xrightarrow{s'_2} & \iota'^*(\mathcal{L}_2') \\
\end{array}
$$

is commutative.
commutes, with an obvious notation.

These results are consequences of [Fe03, Thm. 2.2] when $X'$ is affine; the general case follows by using the fact that $\varphi$ is affine, as alluded to in [loc. cit., 7.4] and explained in detail in [Ho12, Thm. 3.13].

In particular, for any $s' \in \mathcal{O}(Y')^*$ (the unit group of the ring of global sections $\mathcal{O}(Y')$), the triple $(\mathcal{O}_{X'}, s', \mathcal{O}_Y)$ corresponds to an invertible sheaf on $X$, which is trivial if and only if $s' = \psi^*(u)\psi^*(v)$ for some $u \in \mathcal{O}(X')^*$ and $v \in \mathcal{O}(Y)^*$.

Also, an invertible sheaf $\mathcal{L}'$ over $X'$ is the pull-back of some invertible sheaf on $X$ if and only if $\psi^*(\mathcal{L}') \cong \psi^*(\mathcal{M})$ for some invertible sheaf $\mathcal{M}$ on $Y$.

### 2.3 Their Picard functor

We keep the notation and assumptions of Subsection 2.1 and assume in addition the following two conditions:

- (PF) The structure map $f': X' \to S$ is proper and flat with integral geometric fibers.
- (FF) The structure maps $g: Y \to S$ and $g': Y' \to S$ are finite and faithfully flat.

The latter condition implies that $Y$ satisfies (LAF). Also, by [EGAIII, Prop. 7.8.6], the condition (PF) yields that $f'_*(\mathcal{O}_{X'}) = \mathcal{O}_S$ universally.

We now recall some notions and results from [BLR90, §8.1]. We denote by $\text{Pic}_{X'/S}$ the relative Picard functor, i.e., the fppf sheaf associated with the functor $S' \mapsto \text{Pic}(X'_S)$, where $X'_S := X' \times_S S'$. Since $f'^*: \mathcal{O}(S') \to \mathcal{O}(X'_S)$ is an isomorphism for any $S'$-scheme $S'$, the natural map $\mathcal{O}(X'_S) \to \mathcal{O}(Y'_S)$ is injective, and hence $Y'$ is a rigidifier of $\text{Pic}_{X'/S}$.

Also, the functor $S' \mapsto \mathcal{O}(Y'_S)$ is represented by a locally free ring scheme $V_{Y'}$, and the subfunctor of units, $S' \mapsto \mathcal{O}(Y'_S)^*$, by a group scheme, open in $V_{Y'}$. Clearly, $V_{X'} \cong \mathbb{G}_{m,S}$ and $V_{X'}^* \cong \mathbb{G}_{m,S}$. Also, note that

$$V_{Y'}^* = R_{Y'/S}(\mathbb{G}_{m,Y'}),$$

where $R$ denotes the Weil restriction. We have an exact sequence of sheaves for the étale topology

$$0 \longrightarrow V_{X'}^* \longrightarrow V_{Y'}^* \longrightarrow (\text{Pic}_{X'/S}, Y') \longrightarrow \text{Pic}_{X'/S} \longrightarrow 0,$$

where $(\text{Pic}_{X'/S}, Y')$ denotes the sheaf of isomorphism classes of invertible sheaves on $X'$, rigidified along $Y'$.

We record some easy additional properties of the unit group scheme $V_{Y'}$:

**Lemma 2.1.** With the above notation, we have:

- (i) $V_{Y'}^*$ is a smooth affine group scheme with connected fibers.
- (ii) If $Y'$ is the disjoint union of two closed subschemes $Y'_1, Y'_2$, then $V_{Y'} \cong V_{Y'_1} \times_S V_{Y'_2}$.
- (iii) $V_{Y'}^*$ is a torus if and only if $Y'$ is étale over $S$.

**Proof.** (i) Since $V_{Y'}$ is smooth, so is its open subscheme $V_{Y'}^*$. Also, Weil restriction preserves affineness in view of (FF) and [DG70, Chap. I, §1, Prop. 6.6]; in particular, $V_{Y'}^*$ is affine. Its fibers are connected by [Ra70, Prop. 2.4.3].

(ii) is readily checked.
(iii) If $V'_Y$ is a torus, then so are its fibers $(V'_x)_s = \mathcal{O}(Y'_s)^*$. It follows readily that the $k(s)$-algebra $\mathcal{O}(Y'_s)$ is separable (see Proposition 4.10 below for a more general result). Hence $Y'$ is étale over $S$. To show the converse implication, we may replace $S$ (resp. $Y'$) with $Y'$ (resp. $Y'$ with $Y' \times_S Y'$). Then $g' : Y' \to S$ has a section, so that $Y' = S \sqcup Y''$ for some scheme $Y''$, finite and étale over $S$. Thus, $V'_Y \cong \mathbb{G}_{m,S} \times V''_Y$ and we conclude by induction.

Next, observe that the structure map $f : X \to S$ also satisfies (PF): the properness has already been observed, while the flatness and the assertion on geometric fibers follow from [Ho12, Thm. 3.11]. Thus, $Y$ is a rigidifier of $\text{Pic}_{X/S}$ and the latter sits in an exact sequence of sheaves for the étale topology, analogous to (4). Morever, both sequences sit in a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{G}_{m,S} & \longrightarrow & V'_Y & \longrightarrow & (\text{Pic}_{X/S}, Y) & \longrightarrow & \text{Pic}_{X/S} & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{G}_{m,S} & \longrightarrow & V'_Y & \longrightarrow & (\text{Pic}_{X'/S}, Y') & \longrightarrow & \text{Pic}_{X'/S} & \longrightarrow & 0.
\end{array}
\]

We may now state a key observation:

**Lemma 2.2.** The map $\varphi^* : (\text{Pic}_{X/S}, Y) \to (\text{Pic}_{X'/S}, Y')$ is an isomorphism.

*Proof.* Consider an arbitrary $S$-scheme $S'$. Then the square obtained from (3) by base change to $S'$ is still cocartesian in view of [Ho12, Thm. 3.11]. Thus, the invertible sheaves on $X_{S'}$ can be described as in Subsection 2.2 in view of [Ho12, Thm. 3.13]. So it suffices to show that

\[ \varphi^* : (\text{Pic}(X), Y) \to (\text{Pic}(X'), Y') \]

is an isomorphism. Here $(\text{Pic}(X), Y)$ denotes the group of isomorphism classes of pairs $(L, \alpha)$, where $L$ is an invertible sheaf on $X$, and $\alpha : \mathcal{O}_Y \to \iota^*(L)$ is an isomorphism.

Let $(L', \alpha')$ be an invertible sheaf on $X'$, rigidified along $Y'$. Then the triple $(L', \alpha', \mathcal{O}_Y)$ corresponds by Subsection 2.2 to an invertible sheaf $L$ on $X$ such that $\varphi^*(L) = L'$ and $\iota^*(L) = \mathcal{O}_Y$. Moreover, $\varphi^*(L, 1) \cong (L', \alpha')$. Thus, $\varphi^*$ is surjective.

Next, let $(L, \alpha)$ be an invertible sheaf on $X$ rigidified along $Y$, such that $\varphi^*(L, \alpha)$ is trivial in $(\text{Pic}(X'), Y')$. In particular, $\varphi^*(L) \cong \mathcal{O}_X$, and $\iota^*(L) \cong \mathcal{O}_Y$. Thus, $L$ is isomorphic to the invertible sheaf associated with a triple $(\mathcal{O}_{X'}, s', \mathcal{O}_Y)$, where $s' \in \mathcal{O}(Y')^*$. Then $\alpha \in \mathcal{O}(Y)^*$; moreover, replacing $(\mathcal{O}_{X'}, s', \mathcal{O}_Y)$ with the isomorphic triple $(\mathcal{O}_{X'}, s'\psi^*(v), \mathcal{O}_Y)$ for $v \in \mathcal{O}(Y)^*$ replaces $\alpha$ with $\alpha v$. Thus, $(L, \alpha)$ is isomorphic to $(\mathcal{O}_X, 1)$, and $\varphi^*$ is injective.

Lemma 2.2 and the commutative diagram (5) yield readily the following:

**Corollary 2.3.** We have an exact sequence of sheaves for the étale topology

\[ 0 \longrightarrow V'_Y \xrightarrow{\psi^*} V'_{Y'} \longrightarrow \text{Pic}_{X/S} \xrightarrow{\varphi^*} \text{Pic}_{X'/S} \longrightarrow 0. \]
2.4 Their Picard scheme

We keep the assumptions (PF) and (FF) of Subsection 2.3, and assume in addition that $X'$ is locally projective over $S$.

**Proposition 2.4.** (i) $X$ is locally projective over $S$ as well.

(ii) The Picard functors $\text{Pic}_{X'/S}$, $\text{Pic}_{X/S}$ are represented by group schemes $\text{Pic}_{X'/S}$, $\text{Pic}_{X/S}$ which are locally of finite type.

(iii) Assume in addition the following condition:

(R) The homomorphism of group schemes $\psi^*: V_Y^* \to V_Y'^*$ is a closed immersion and its cokernel is represented by a group scheme.

Then the latter group scheme sits in an exact sequence

\[
0 \to V_Y^*/\psi^*(V_Y'^*) \to \text{Pic}_{X/S} \xrightarrow{\varphi^*} \text{Pic}_{X'/S} \to 0.
\]

**Proof.** (i) We may assume that $X'$ has an $S$-ample invertible sheaf $\mathcal{L}'$. In view of (FF), $\iota^*(\mathcal{L}')$ is trivial on the pull-back of some open affine covering of $S$. Thus, we may further assume that $\iota^*(\mathcal{L}') \cong \mathcal{O}_{Y'}$; then by Subsection 2.2, $\mathcal{L}' \cong \varphi^*(\mathcal{L})$ for some invertible sheaf $\mathcal{L}$ on $X$. Since $\varphi$ is finite, $\mathcal{L}$ is $S$-ample.

(ii) The assertion on $\text{Pic}_{X'/S}$ is a consequence of (PF) and the local projectivity assumption in view of [BLR90, 8.2 Thm. 1] (see also [Kl05, Thm. 9.4.8]). The assertion on $\text{Pic}_{X/S}$ follows similarly in view of (i).

(iii) is a direct consequence of Corollary 2.3.

**Remark 2.5.** The assumption (R) is satisfied when $S = \text{Spec}(k)$ for a field $k$, see the next remark. This assumption also holds when $\psi$ admits a section $\sigma$ (in view of the exact sequence $V_Y^* \xrightarrow{\psi^*} V_Y'^* \xrightarrow{\sigma^*} V_Y^*$) or when $Y$ is étale over $S$ (then $V_Y^*$ is a torus and the assertion follows from [SGA3, Exp. IX, Cor. 2.5]).

**Remark 2.6.** Consider the case where $S = \text{Spec}(k)$, where $k$ is a field. Then the assumptions (PF), (FF) and of local projectivity just mean that $X'$ is a projective $k$-variety equipped with a finite subscheme $Y'$ and with a morphism $\psi: Y' \to Y$ such that $\mathcal{O}_Y \leftarrow \psi_*(\mathcal{O}_{Y'})$. (By a variety, we mean a geometrically integral scheme.) Moreover, the group scheme $V_Y^*$ represents the functor $R \mapsto (R \otimes_k A)^*$ from $k$-algebras to groups, where $A := \mathcal{O}(Y)$ is an artinian $k$-algebra. We shall rather denote $V_Y^*$ by $\mu^A$, as in [DG70, Chap. II, §1, 2.3]; then $\mu^A$ is a connected affine algebraic group with Lie algebra the vector space $A$ equipped with the trivial bracket. This group is also considered in [Ru13], where it is denoted by $L_A$.

Let $A' := \mathcal{O}(Y')$; then the injective homomorphism of algebras $A \to A'$ induces a homomorphism of algebraic groups $\psi^*: \mu^A \to \mu^{A'}$ which is a closed immersion (since $\psi^*$ is injective on points over the algebraic closure of $k$, and on Lie algebras). Thus, the cokernel of $\psi^*$ is represented by a connected affine algebraic group, that we denote by $\mu^{A'/A}$. So the condition (R) is satisfied, and (6) yields an exact sequence

\[
0 \to \mu^{A'/A} \to \text{Pic}_{X/k} \xrightarrow{\varphi^*} \text{Pic}_{X'/k} \to 0.
\]

The analogous sequence for Picard groups is well-known (see e.g. [EGAIV] Prop. 21.8.5)].
If $X'$ is geometrically normal, then $\text{Pic}^0_{X'/k}$ is projective by [Kl05, Thm. 9.5.4]. Thus, $\mu^{A'/A}$ is the affine part of the Picard variety of $X$, if in addition $k$ is perfect.

Finally, all the above results extend without change to the case where $X'$ is a proper variety satisfying (AF). Indeed, the Picard functor $\text{Pic}_{X'/k}$ is still represented by a scheme locally of finite type, in view of [Kl05, Cor. 9.4.18.3].

Returning to the notation and assumptions of Proposition 2.4, assume that $Y$ is the disjoint union of two closed subschemes $Y_1, Y_2$. Then we also have $Y' = Y'_1 \sqcup Y'_2$, where $Y'_i := \psi^{-1}(Y'_i)$ for $i = 1, 2$. We may pinch $X'$ along the restriction $\psi_1 : Y'_1 \to Y_1$ to obtain a scheme $X_1$ satisfying all the assumptions of Subsection 2.3. Moreover, the induced morphism $Y'_1 \to X_1$ is a closed immersion (since $Y'_2 \subset X' \setminus Y'_1$), and $X$ is obtained by pinching $X_1$ along the restriction $\psi_2 : Y'_2 \to Y_2$. Likewise, $X$ is obtained by pinching $X_2$ along $\psi_1 : Y'_1 \to Y_1$; this yields a commutative diagram

\[ X' \xrightarrow{\varphi'_1} X_1 \]
\[ \varphi'_1 \downarrow \quad \varphi_1 \downarrow \]
\[ X_2 \xrightarrow{\varphi_2} X. \]

**Lemma 2.7.** With the above notation and assumptions, the map

\[ \varphi_1 \times \varphi_2 : \text{Pic}_{X/S} \to \text{Pic}_{X_1/S} \times_{\text{Pic}_{X'/S}} \text{Pic}_{X_2/S} \]

is an isomorphism.

This result follows easily from the exact sequence (6) and from the analogous exact sequences for $\varphi_1$ and $\varphi_2$. It will be used in the proof of Theorem 1.1 to reduce to the case where $U$ or $T$ is trivial.

## 3 Some extensions of abelian schemes

### 3.1 Extensions by vector groups

Throughout this section, we keep the standing assumptions of Section 2 on schemes. All group schemes are assumed to be commutative.

Let $A$ be an abelian scheme over $S$. By [FC90, Thm. 1.9], $A$ has a dual abelian scheme $\hat{A}$, and both satisfy (LAF). Also, recall that $\hat{A}$ is locally projective if so is $A$ (see e.g. [Kl05, Rem. 9.5.24]).

Consider a locally free sheaf $\mathcal{Q}$ of finite rank over $S$ and denote by $V = V(\mathcal{Q})$ its total space, i.e., the affine $S$-scheme associated with the sheaf of $\mathcal{O}_S$-algebras $\text{Sym}_{\mathcal{O}_S}(\mathcal{Q})$. Then $V$ is a vector group over $S$, i.e., a group scheme locally isomorphic to a direct product of copies of $\mathbb{G}_{a,S}$ and equipped with an action of $\mathbb{G}_{m,S}$ which restricts to the multiplication on each $\mathbb{G}_{a,S}$. For example, if $Y$ is a finite faithfully flat $S$-scheme, then $V_Y = V(g_* (\mathcal{O}_Y))$ with the notation of Subsection 2.3.

By [MM74, Chap. I, (1.9)], any extension of $S$-group schemes

\[ 0 \to V \to G \to A \to 0 \]
is classified by a morphism of $S$-group schemes
\[ \gamma : V(\omega^\vee_A) \rightarrow V, \]
where $\omega^\vee_A$ denotes the sheaf of (relative) differential 1-forms on $\hat{A}$, and $\omega^\vee$ its dual. (Note that the convention of [MM74] for vector groups is dual to ours.) When we take into account the structure of vector group of $V$ (or equivalently, the $\mathbb{G}_{m,S}$-action on that group scheme), the morphism $\gamma$ is in addition $\mathbb{G}_{m,S}$-equivariant, i.e., it comes from a morphism of locally free sheaves $Q \rightarrow \omega^\vee$. For simplicity, we still denote the dual morphism by
\[ \gamma : \omega^\vee_A \rightarrow Q^\vee. \]

Let $I_S(Q^\vee)$ denote the affine $S$-scheme associated with the sheaf of $\mathcal{O}_S$-algebras $\mathcal{O}_S \oplus \varepsilon Q^\vee$, where $\varepsilon^2 = 0$, and define similarly $I_S(\omega^\vee_A)$. Then the above morphism $\gamma$ yields a morphism of schemes $I_S(\gamma) : I_S(Q^\vee) \rightarrow I_S(\omega^\vee_A)$. Also, $I_S(\omega^\vee_A)$ may be viewed as a closed subscheme of $\hat{A}$, namely, the first infinitesimal neighborhood of the zero section. Thus, $I_S(\omega^\vee_A)$ makes sense.

Proposition 3.1. With the above notation and assumptions, the connected component of the zero section, $\text{Pic}^0_{X/S}$, exists and is isomorphic to $G$. If $A$ is locally projective, then so is $X$.

Proof. Since $Y = S$, the natural map $(\text{Pic}_{X/S}, Y) \rightarrow \text{Pic}_{X/S}$ is an isomorphism (as follows e.g. from the exact sequence (1)). Thus, $\text{Pic}_{X/S} \cong (\text{Pic}_{X/S}, Y')$ by Lemma 2.2. Also, we have an exact sequence of group schemes
\[ 0 \rightarrow \mathbb{G}_{m,S} \rightarrow V^*_Y \rightarrow V \rightarrow 0 \]
with the notation of Subsection 2.3 and hence an exact sequence of étale sheaves
\[ 0 \rightarrow V \rightarrow (\text{Pic}_{X/S}, Y') \rightarrow \text{Pic}_{X/S} \rightarrow 0 \]

by Corollary 2.3 (this also follows directly from the exact sequence (1)).

For each $s \in S$, we have $X'_s = \hat{A}_s \times_{k(s)} \mathbb{P}(Q^\vee_s \oplus k(s))$ and hence $\text{Pic}^0_{X'_s/k(s)} \cong A_s$. In particular, $\text{Pic}^0_{X'/k(s)}$ is smooth of dimension independent of $s$. By [Kol05, Prop. 9.5.20], it
follows that \( \text{Pic}^0_{X/S} \) exists and its fiber at any \( s \in S \) is \( \text{Pic}^0_{X'/k(s)} \). Thus, the projection \( \pi : X' \to \hat{A} \) yields an isomorphism

\[
\pi^* : A = \text{Pic}^0_{\hat{A}/S} \xrightarrow{\cong} \text{Pic}^0_{X'/S}.
\]

Moreover, \( \pi \) sits in a commutative diagram of rigidifiers in the (generalized) sense of [Ra70, Def. 2.1.1]

\[
\begin{array}{c}
Y' \xrightarrow{id} Y' \\
\downarrow \iota' \downarrow I_s(\gamma) \\
X' \xrightarrow{\pi} \hat{A}
\end{array}
\]

which yields a commutative diagram of exact sequences

\[
0 \longrightarrow V \longrightarrow (\text{Pic}^0_{\hat{A}/S}, Y') \longrightarrow \text{Pic}_{\hat{A}/S} \longrightarrow 0
\]

\[
0 \longrightarrow V \longrightarrow (\text{Pic}^0_{X'/S}, Y') \longrightarrow \text{Pic}_{X'/S} \longrightarrow 0
\]

in view of [On87, §1]. It follows that \( (\text{Pic}^0_{\hat{A}/S}, Y') \) and \( (\text{Pic}^0_{X'/S}, Y') \) exist and are isomorphic via the commutative diagram of exact sequences

\[
0 \longrightarrow V \longrightarrow (\text{Pic}^0_{\hat{A}/S}, Y') \longrightarrow \text{Pic}_{\hat{A}/S} \longrightarrow 0
\]

\[
0 \longrightarrow V \longrightarrow (\text{Pic}^0_{X'/S}, Y') \longrightarrow \text{Pic}_{X'/S} \longrightarrow 0.
\]

On the other hand, the commutative diagram of rigidifiers

\[
\begin{array}{c}
Y' \xrightarrow{I_s(\gamma)} I_s(\omega_{\hat{A}}) \\
\downarrow \downarrow \\
\hat{A} \xrightarrow{id} \hat{A}
\end{array}
\]

yields a commutative diagram of exact sequences

\[
0 \longrightarrow V(\omega_{\hat{A}}) \longrightarrow (\text{Pic}^0_{\hat{A}/S}, I_s(\omega_{\hat{A}})) \longrightarrow \text{Pic}^0_{\hat{A}/S} \longrightarrow 0
\]

\[
0 \longrightarrow V \longrightarrow (\text{Pic}^0_{\hat{A}/S}, Y') \longrightarrow \text{Pic}_{\hat{A}/S} \longrightarrow 0.
\]

Moreover, the top line in the above diagram is the universal vector extension of \( A \), in view of [MM74] Chap. I, (2.6)]. It follows that the bottom line is the extension (7). Finally, the local projectivity assertion follows from the construction and Proposition 2.4. □
3.2 Extensions by quasi-split tori

Consider a torus $T$ over $S$. We say that $T$ is quasi-split if there exists a finite étale Galois cover $f : S' \to S$ with group $\Gamma$, and a permutation $\mathbb{Z}[\Gamma]$-module $P$ satisfying

$$T_{S'} \cong \mathbb{G}_{m,S'} \otimes_{\mathbb{Z}} P$$

as group schemes over $S'$ equipped with an action of $\Gamma$, compatible with its action on $S'$. (Recall that a $\mathbb{Z}[\Gamma]$-module is said to be a permutation module if it admits a $\Gamma$-stable $\mathbb{Z}$-basis).

When $S = \text{Spec}(k)$ for a field $k$, the quasi-split tori are exactly the unit group schemes of finite étale $k$-algebras (see e.g. [Vo98, Chap. 2, §6.1, Prop. 1]). We shall extend this to an arbitrary base scheme $S$. Let $T$ be a quasi-split torus as above. We may decompose the permutation module $P$ as

$$P = \bigoplus_{i=1}^{m} \mathbb{Z}[\Gamma/\Gamma_i],$$

where $\Gamma_1, \ldots, \Gamma_m$ are subgroups of $\Gamma$. Consider the scheme $Z$ over $S = S'/\Gamma$ defined by

$$Z = \bigsqcup_{i=1}^{m} S'/\Gamma_i.$$

Alternatively, we have $Z = (\bigsqcup_{i=1}^{n} S')/\Gamma$, where $n$ denotes the rank of the free $\mathbb{Z}$-module $P$, and $\Gamma$ acts on $\bigsqcup_{i=1}^{n} S'$ by permuting the $n$ copies of $S'$ with orbits $\Gamma/\Gamma_1, \ldots, \Gamma/\Gamma_m$.

Lemma 3.2. With the above notation, the natural map $q : \bigsqcup_{i=1}^{n} S' \to Z$ is a finite étale Galois cover with group $\Gamma$. Also, $Z$ is finite étale over $S$, and $T \cong V_{Z}^*$ as $S$-group schemes. Conversely, if $Z'$ is a finite étale scheme and $S$ is connected, then $V_{Z'}^*$ is a quasi-split torus.

Proof. Since $S'/\Gamma_i \cong (S' \times \Gamma)/\Gamma_i$, where $\Gamma$ acts diagonally on $S' \times \Gamma/\Gamma_i$, we have

$$Z \cong (S' \times \bigsqcup_{i=1}^{m} \Gamma/\Gamma_i)/\Gamma,$$

where $\Gamma$ acts diagonally on the right-hand side. In view of [SGA7, Exp. V, Prop. 1.9], it follows that $q$ is a $\Gamma$-torsor.

To complete the proof of the first assertion, it suffices by descent to check that the base change $(V_Z)_{S'}$ is finite étale over $S'$, and $T_{S'} \cong (V_Z^*)_{S'}$ as $S'$-group schemes equipped with a compatible action of $\Gamma$. Since $V_Z^* = R_{Z/S}(\mathbb{G}_{m,Z})$ and Weil restriction commutes with base change, we have $(V_Z^*)_{S'} \cong R_{Z_{S'}/S'}(\mathbb{G}_{m,Z_{S'}})$. Moreover,

$$Z_{S'} = \bigsqcup_{i=1}^{m} S' \times_S (S'/\Gamma_i) \cong \bigsqcup_{i=1}^{m} (S' \times_S S')/\Gamma_i \cong \bigsqcup_{i=1}^{m} (S' \times \Gamma)/\Gamma_i \cong S' \times \bigsqcup_{i=1}^{m} \Gamma/\Gamma_i = \bigsqcup_{i=1}^{n} S',$$

where the first isomorphism follows from [SGA7, Exp. V, Prop. 1.9] again, and the second one comes from the isomorphism

$$S' \times \Gamma \xrightarrow{\cong} S' \times_S S', \quad (x, g) \mapsto (gx, x);$$

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the composed isomorphism is equivariant for the natural action of $\Gamma$ on $Z_{S'}$ and its action on $\bigsqcup_{i=1}^m S'$ by permuting the copies of $S'$. This yields the desired assertions in view of Lemma 2.1 (iii).

For the second assertion, we may assume that $Z'$ is connected, since the product of any two quasi-split tori is easily seen to be quasi-split. Then, by the classification of finite étale covers in terms of the étale fundamental group, there exist a finite étale Galois cover $Z'' \to S$ with group $\Gamma$, and a subgroup $\Gamma_1 \subset \Gamma$ such that $Z' \cong Z''/\Gamma_1$ and the structure map $Z' \to S$ is identified with the natural morphism $Z''/\Gamma_1 \to Z''/\Gamma = S$. Thus, $Z' \cong (Z'' \times \Gamma/\Gamma_1)/\Gamma$, where $\Gamma$ acts diagonally on $Z'' \times \Gamma/\Gamma_1$. Let $S' := Z'' \times \Gamma/\Gamma_1$; then the structure map $S' \to S$ is a finite étale Galois cover with group $\Gamma$. Moreover, by arguing as in the first part of the proof, we obtain $\Gamma$-equivariant isomorphisms

$$Z_{S'}' \cong S' \times_S S'/\Gamma_1 \cong S' \times \Gamma/\Gamma_1.$$  

It follows that $V_{Z''/S'}^* \cong \mathbb{G}_m.S' \otimes_Z \mathbb{Z}[\Gamma/\Gamma_1]$ as an $S'$-torus equipped with a compatible action of $\Gamma$. Since $V_{Z''/S'}^{**} \cong (V_{Z''/S'}^*)'$, this completes the proof. \hfill \Box

**Remark 3.3.** In the definition of a quasi-split torus $T$, we may replace $S'$ with any larger Galois cover. Keeping this in mind, the permutation module $P$ is uniquely determined by $T$; the split tori correspond of course to the trivial permutation modules. Thus, the direct image of $\mathcal{O}_Z$ under the structure map $Z \to S$ is uniquely determined by $T$ as well (this is in fact the Lie algebra of $T$). But the $\mathcal{O}_S$-algebra structure of $\mathcal{O}_Z$ is not uniquely determined by $T$; in fact, the orbits $\Gamma/\Gamma_1, \ldots, \Gamma/\Gamma_m$ are not unique, since the $\Gamma$-module $\mathbb{Z}[\Gamma/\Gamma_1]$ does not determine the subgroup $\Gamma_1 \subset \Gamma$ up to conjugacy (see [Sc93]).

Next, let $A$ be an abelian scheme and consider the group $\text{Ext}^1(A, T)$ classifying the extensions of $S$-group schemes

$$0 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0.$$  

(8)

**Lemma 3.4.** With the above notation, there is a canonical isomorphism

$$\text{Ext}^1(A, T) \xrightarrow{\cong} \mathfrak{A}(Z).$$  

(9)

**Proof.** By [SGA7] Exp. VIII, Prop. 3.7, we have a canonical isomorphism (given by push-out)

$$\text{Ext}^1(A, T) \xrightarrow{\cong} \text{Hom}(\widehat{T}, \widehat{A}),$$

where $\widehat{T}$ denotes the Cartier dual of $T$. Moreover, the pull-back map

$$\text{Hom}(\widehat{T}, \widehat{A}) \longrightarrow \text{Hom}^\Gamma(\widehat{T}_S', \widehat{A}_S')$$

is an isomorphism by descent theory (see [SGA1] Exp. VIII, Cor. 7.6, which applies since every $\Gamma$-orbit in $\widehat{T}_S'$ and in $\widehat{A}_S'$ is contained in an open affine subscheme). Also, $\widehat{T}_S'$ is isomorphic to the constant group scheme $\text{Hom}(P, \mathbb{Z})_{S'}$, equivariantly for the action of $\Gamma$, and hence

$$\text{Hom}^\Gamma(\widehat{T}_S', \widehat{A}_S') \cong (P \otimes_{\mathbb{Z}} \widehat{A}(S'))^\Gamma \cong \bigoplus_{i=1}^m \widehat{A}(S_i) = \widehat{A}(\bigsqcup_{i=1}^m S'/\Gamma_i) \cong \mathfrak{A}(Z).$$  

\hfill \Box
Remark 3.5. In view of the isomorphism $T \cong R_{Z/S}(G_{m,Z})$ and the Weil-Barsotti formula (see [Oo66, Thm. 18.1], the isomorphism $T \cong R_{Z/S}(G_{m,Z})$ may be rewritten as

$$\text{Ext}^1(A, R_{Z/S}(G_{m,Z})) \cong \text{Ext}^1(A_Z, G_{m,Z}).$$

Such an isomorphism has also been obtained by Russell (via a very different argument) when $S = \text{Spec}(k)$ for a field $k$, and $Z$ is finite but not necessarily étale; see [Ru13, Prop. 1.19]. In fact, Russell’s argument extends to our relative setting, and yields an isomorphism $\text{Ext}^1(A, V^*_Z) \cong \hat{A}(Z)$ for any finite flat $S$-scheme $Z$.

We now define $Y' := Z \sqcup S$.

Then $Y'$ is finite and étale over $S$. Moreover, any extension (8) yields a morphism $Z \to \hat{A}$ and hence a map $Y' \to \hat{A}$, where $S$ is sent to $\hat{A}$ via the zero section $s_0$. We also have a closed immersion $Z \to \text{Spec Sym}_{O_S}(A)$, and hence a closed immersion $Y' \to \mathbb{P}(A \oplus O_S)$, where $S$ is sent to the section at infinity. This yields a closed immersion $\iota' : Y' \to \hat{A} \times_S \mathbb{P}(A \oplus O_S)$.

Denoting by $\psi : Y' \to S := Y$ the structure map, we may again form the pinching diagram (9), where $\text{Pic}_{X/S}$, $\text{Pic}_{X'/S}$ are represented by group schemes $\text{Pic}_{X/S}$, $\text{Pic}_{X'/S}$. We now obtain the same statement as Proposition 3.1:

**Proposition 3.6.** With the above notation and assumptions, the connected component of the zero section, $\text{Pic}^0_{X/S}$, exists and is isomorphic to $G$. If $A$ is locally projective, then so is $X$.

**Proof.** As in the proof of Proposition 3.1 the natural map $(\text{Pic}_{X/S}, Y) \to \text{Pic}_{X/S}$ is an isomorphism, and $\text{Pic}_{X/S} \cong (\text{Pic}_{X'/S}, Y')$.

Consider first the case where $T \cong \mathbb{G}_{m,S}$ is split. Then with the notation of Subsection 2.3 the map $V^*_Y \to V^*_Y$, may be identified with the diagonal, $\delta : \mathbb{G}_{m,S} \to \mathbb{G}_{m,S}^{n+1}$. The latter sits in an exact sequence of group schemes

$$0 \longrightarrow \mathbb{G}_{m,S} \longrightarrow \mathbb{G}_{m,S}^{n+1} \longrightarrow \mathbb{G}_{m,S}^{n} = T \longrightarrow 0,$$

where $\gamma(x_1, \ldots, x_n, x_0) := (x_1x_0^{-1}, \ldots, x_nx_0^{-1})$. In view of Corollary 2.3 this yields an exact sequence

$$0 \longrightarrow T \longrightarrow (\text{Pic}_{X'/S}, Y') \longrightarrow \text{Pic}_{X'/S} \longrightarrow 0.$$

Next, arguing again as in the proof of Proposition 3.1 we obtain that the projection $\pi : X' \to A$ yields an isomorphism $\pi^* : A = \text{Pic}^0_{A/S} \cong \text{Pic}^0_{X'/S}$ which extends to an isomorphism of exact sequences

$$0 \longrightarrow T \longrightarrow \text{Pic}^0_{A/S}, Y') \longrightarrow \text{Pic}^0_{A/S} \longrightarrow 0,$$

where $\pi^* \downarrow \pi^* \downarrow \pi^* \downarrow \pi^* \downarrow \pi^* \downarrow \pi^*$
Moreover, the top extension $0 \to T \to (\text{Pic}_A^0/S, Y') \to A \to 0$ is sent to $(s_1, \ldots, s_n)$ by the isomorphism (9), as follows from [On87, Prop. 1] in the case where $n = 1$, and from (the proof of) [On87, Cor. 1.1] in the general case. This yields isomorphisms $G \cong (\text{Pic}_A^0/S, Y') \cong \text{Pic}^0_{X/S}$.

For an arbitrary quasi-split torus $T$, we reduce similarly to showing that the above extension corresponds to the map $Z \to \hat{A}$ under the isomorphism (9). But this holds after the Galois base change $f : S' \to S$ by the preceding step. Moreover, the pull-back map $\text{Ext}^1(A, T) \to \text{Ext}^1(\hat{A}_S, T_S)$ is injective, since it is identified under the isomorphism (9) to the map $\hat{A}(Z) \to \hat{A}(\prod_{i=1}^n S')$ induced by the natural morphism $q : \prod_{i=1}^n S' \to Z$; moreover, $q$ is finite and étale by Lemma 3.2, and hence is faithfully flat. \hfill \square

**Remark 3.7.** In Proposition 3.1 (resp. Proposition 3.6), we may replace $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_S)$ (resp. $\mathbb{P}(A \oplus \mathcal{O}_S)$) with any projective space bundle over $S$ that contains $Y'$. Here, by a projective space bundle, we mean the projectivization of a locally free sheaf of finite rank over $S$.

**Proof of Theorem 1.1.** Note that the quotients $G/T$, $G/V$ exist and sit in extensions

$$0 \to V \to G/T \to A \to 0, \quad 0 \to T \to G/V \to A \to 0.$$ 

The sum of these extensions is the extension (11), since the natural map $G \to G/T \times_A G/V$ is easily seen to be an isomorphism. Moreover, these extensions yield morphisms of schemes $Y'_1 := I_S(Q^\vee) \to \hat{A}$, where $V = V(Q^\vee)$, and $Y'_2 := Z \sqcup S \to \hat{A}$; in turn, this yields closed immersions $Y'_1 \hookrightarrow \hat{A} \times_S \mathbb{P}(Q^\vee \oplus \mathcal{O}_S)$ and $Y'_2 \hookrightarrow \hat{A} \times_S \mathbb{P}(A \oplus \mathcal{O}_S)$. Now consider the composition of the closed immersions

$$Y' := Y'_1 \sqcup Y'_2 \to \hat{A} \times_S (\mathbb{P}(Q^\vee \oplus \mathcal{O}_S) \sqcup \mathbb{P}(A \oplus \mathcal{O}_S)) \to \hat{A} \times_S \mathbb{P}(Q^\vee \oplus \mathcal{O}_S \oplus A \oplus \mathcal{O}_S) =: X',$n

and the natural map $Y' = Y'_1 \sqcup Y'_2 \to S \sqcup S =: Y$. Then the statement follows by combining Lemma 2.7 Propositions 3.1 and 3.6 and Remark 3.7.

### 4 Relative unit groups

#### 4.1 Definition and first properties

Throughout this section, we fix a base field $k$ and choose an algebraic closure $\bar{k}$. We denote by $k^{sep}$ the separable closure of $k$ in $\bar{k}$, and by $\Gamma$ the Galois group of $k^{sep}/k$.

We shall consider (commutative) artinian $k$-algebras. Given such an algebra $A$, we denote by $\mu^A$ its group scheme of units, introduced in Remark 2.6. Then $\mu^A = R_{A/k}(\mathbb{G}_{m, A})$, where $R_{A/k}$ denotes the Weil restriction (see e.g. [CGP10 App. A.5]). Thus, $\mu^A$ is a connected affine algebraic group with Lie algebra $A$. Also, we may uniquely decompose $A$ as a direct product $A_1 \times \cdots \times A_n$ of local $k$-algebras; then $\mu^A \cong \mu^{A_1} \times \cdots \times \mu^{A_n}$.

When $A$ is a subalgebra of an algebra $B$, we have $\mu^A \subset \mu^B$ (by Remark 2.6 again) and we set

$$\mu^{B/A} := \mu^B / \mu^A.$$
Then $\mu^{B/A}$ is a connected affine algebraic group, that we shall call the relative unit group; its Lie algebra is $B/A$. Any chain of algebras $A \subset B \subset C$ yields an exact sequence of algebraic groups
\begin{equation}
0 \longrightarrow \mu^{B/A} \longrightarrow \mu^{C/A} \longrightarrow \mu^{C/B} \longrightarrow 0.
\end{equation}
(10)

Also, note that $\mu^{(A \times A)/A} \cong \mu^A$ in view of the exact sequence
\begin{equation}
0 \longrightarrow \mu^A \longrightarrow \mu^{A \times A} = \mu^A \times \mu^A \overset{f}{\longrightarrow} \mu^A \longrightarrow 0,
\end{equation}
where $f(x, y) = xy^{-1}$.

Our main motivation for studying relative unit groups comes from the following:

**Proposition 4.1.** When $k$ is perfect, the algebraic groups of the form $\mu^{B/A}$ are exactly the affine parts of Picard varieties of projective varieties with finite non-normal locus.

**Proof.** Let $X$ be such a variety, and denote by $\varphi : X' \rightarrow X$ the normalization. Then $X'$ is projective, and we have an exact sequence $0 \rightarrow \mu^{B/A} \rightarrow \text{Pic}^0(X) \rightarrow \text{Pic}^0(X') \rightarrow 0$ for appropriate algebras $A \subset B$ (see Remark 2.6). Moreover, $\text{Pic}^0(X')$ is an abelian variety by [Kl05, Thm. 9.5.4, Rem. 9.5.6]. Thus, $\mu^{B/A}$ is the affine part of $\text{Pic}^0(X)$.

Conversely, given algebras $A \subset B$, we may embed $\text{Spec}(B)$ in some projective space $\mathbb{P}$, and form the pinching diagram

\[
\begin{array}{ccc}
\text{Spec}(B) & \longrightarrow & \mathbb{P} \\
\downarrow & & \downarrow \\
\text{Spec}(A) & \longrightarrow & X.
\end{array}
\]

Then $\mu^{B/A} = \text{Pic}^0(X)$ in view of Remark 2.6 again. \qed

Since relative unit groups are interesting in their own right, we shall consider them in more detail than is needed for applications to Picard varieties. We begin with the following:

**Examples 4.2.** (i) Let $K/k$ be a finite separable field extension. We may assume that $K \subset k^{\text{sep}}$; we then denote by $\Gamma_K \subset \Gamma$ the Galois group of $k^{\text{sep}}/K$. Then $\mu^K$ is a torus with character module $\mathbb{Z}[\Gamma/\Gamma_K]$. It follows that $\mu^{K/k}$ is a torus as well, with character module the kernel of the augmentation map $\mathbb{Z}[\Gamma/\Gamma_K] \rightarrow \mathbb{Z}$.

(ii) More generally, consider an algebra $A$ which is separable (or equivalently, étale). Then $\mu^A$ is a quasi-split torus; moreover, all quasi-split tori are obtained in this way, as recalled in Subsection 3.2.

(iii) Let $A := k \oplus I$, where $I$ is an ideal of square 0. Then $\mu^{A/k}$ is the vector group associated with $I$.

(iv) Assume that char$(k) = p > 0$ and $[k^{1/p} : k] = p$. Let $K := k^{1/p}$ and choose $t \in k \setminus k^p$. Then $\mu^{K/k}$ is isomorphic to the closed subgroup scheme of $\mathbb{G}_a^p$ defined by $x_0^p + tx_1^p + \cdots + t^{p-1}x_{p-1}^p = x_{p-1}$ (see [Oe84, Prop. VI.5.3]). In particular, $\mu^{K/k}$ is unipotent, and contains no copy of $\mathbb{G}_a$ in view of [Oe84, Lem. VI.5.1]. In other words, $\mu^{K/k}$ is $k$-wound in the sense of Tits (see [Oe84, V.3] and also [CGP10, Def. B.2.1, Cor. B.2.6]).
Next, we collect basic properties of relative unit groups, in a series of lemmas.

**Lemma 4.3.** (i) Let $I$ be an ideal of an algebra $A$. Then the quotient map $A \to A/I$ yields an epimorphism $\gamma : \mu^A \to \mu^{A/I}$. If $I$ is nilpotent, then $\ker(\gamma) = 1 + I$ with an obvious notation.

(ii) Let $I \subset A \subset B$, where $I$ is a nilpotent ideal of $B$. Then the natural map $\mu^B/A \to \mu^{B/I}/A/I$ is an isomorphism.

(iii) Let $A, A'$ be subalgebras of an algebra $B$. Then the natural map $\iota : \mu^{A'/A \cap A'} \to \mu^{B/A}$ is a closed immersion.

(iv) Let $K/k$ be a finite extension of fields. Then the base change $\mu^K_{B/A}$ is isomorphic to $\mu^{B\otimes K/A \otimes K}$ as a $K$-group scheme.

**Proof.** (i) To show that $\gamma$ is an epimorphism, it suffices to check that the induced map $\mu^A(\bar{k}) \to \mu^{A/I}(\bar{k})$ is surjective, since $\mu^A$ and $\mu^{A/I}$ are algebraic groups. Thus, we may assume that $k$ is algebraically closed; also, we may reduce to the case that $A$ is local. Then its maximal ideal $m$ is nilpotent, and $A^* = k^* \times (1 + m)$ while $(A/I)^* \simeq k^* \times (1 + m/I)$. So the map $A^* \to (A/I)^*$ is surjective as desired. The assertion on $\ker(\gamma)$ is obvious.

(ii) By (i), we have a commutative diagram of exact sequences

\[
\begin{array}{cccccc}
0 & \longrightarrow & 1 + I & \longrightarrow & \mu^A & \longrightarrow & \mu^{A/I} & \longrightarrow & 0 \\
\text{id} & \downarrow & \downarrow & & \gamma_A & \downarrow & \downarrow & \gamma_{A/I} & \downarrow & 0 \\
0 & \longrightarrow & 1 + I & \longrightarrow & \mu^B & \longrightarrow & \mu^{B/I} & \longrightarrow & 0
\end{array}
\]

which yields the assertion.

(iii) Clearly, $\iota$ induces an injective morphism on Lie algebras. Arguing as in the proof of (i), it suffices to show that $\iota$ is also injective on $\bar{k}$-points. But this follows from the equality $(A \cap A')^* = A^* \cap A'^*$.

(iv) Since exact sequences of group schemes are preserved by field extensions, it suffices to show that $\mu^K_A = \mu^{A \otimes_k K}$, where the right-hand side is understood as a $K$-group scheme. Let $R$ be a $K$-algebra; then $\mu^K_A(R) = \mu^A(R) = (A \otimes_k R)^* = (A \otimes_k K \otimes_K R)^* = \mu^{A \otimes_k K}(R)$. 

\[\square\]

**Lemma 4.4.** Let $A \subset B$ be algebras, $I$ (resp. $J$) the nilradical of $A$ (resp. $B$), and set $A_{\text{red}} := A/I$, $B_{\text{red}} := B/J$.

(i) $A_{\text{red}} \subset B_{\text{red}}$ and we have an exact sequence of algebraic groups

\[
0 \longrightarrow (1 + J)/(1 + I) \longrightarrow \mu^{B/A} \longrightarrow \mu^{B_{\text{red}}/A_{\text{red}}} \longrightarrow 0. \tag{11}
\]

(ii) Let $A_{\text{sep}} \subset A_{\text{red}}$ be the largest separable subalgebra, and define likewise $B_{\text{sep}}$. Then $A_{\text{sep}} = A_{\text{red}} \cap B_{\text{sep}}$ and the homomorphism $\iota : \mu^{B_{\text{sep}}/A_{\text{sep}}} \to \mu^{B_{\text{red}}/A_{\text{red}}}$ is a closed immersion. Moreover, the exact sequence (11) splits canonically over $\mu^{B_{\text{sep}}/A_{\text{sep}}}$.

(iii) $(1 + J)/(1 + I)$ has a composition series with subquotients $\mathbb{G}_a$. 

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Proof. (i) Since \( A \cap J = I \), the map \( A_{\text{red}} \to B_{\text{red}} \) is injective. Moreover, by Lemma 4.3 (i), we have a commutative diagram of exact sequences

\[
\begin{array}{ccccccc}
0 & \longrightarrow & 1 + I & \longrightarrow & \mu^A & \longrightarrow & \mu^{A_{\text{red}}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & 1 + J & \longrightarrow & \mu^B & \longrightarrow & \mu^{B_{\text{red}}} & \longrightarrow & 0.
\end{array}
\]

This yields the exact sequence (11).

(ii) Clearly, \( A_{\text{sep}} \subset A_{\text{red}} \cap B_{\text{sep}} \); also, the opposite inclusion holds since every subalgebra of a separable algebra is separable. This yields the desired equality, and in turn the assertion on \( \iota \) in view of Lemma 4.3 (iii).

Denote by \( B' \subset B \) the preimage of \( B_{\text{sep}} \) and define \( A' \subset A \) similarly; then \( A' = A_{\text{red}} \cap B' \).

By a special case of the Wedderburn-Malcev theorem (see e.g. [CR62, Thm. (72.19)]), the exact sequence of algebras \( 0 \to J \to B' \to B_{\text{sep}} \to 0 \) has a unique splitting. Thus, \( B' = B_{\text{sep}} \oplus J \supset A_{\text{sep}} \oplus I = A' \). This yields compatible splittings in the exact sequences

\[
\begin{array}{ccccccc}
0 & \longrightarrow & 1 + I & \longrightarrow & \mu^{A'} & \longrightarrow & \mu^{A_{\text{sep}}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & 1 + J & \longrightarrow & \mu^{B'} & \longrightarrow & \mu^{B_{\text{sep}}} & \longrightarrow & 0,
\end{array}
\]

and hence the desired splitting.

(iii) We may replace \( A \) (resp. \( B \)) with its subalgebra \( k \oplus I \) (resp. \( k \oplus J \)), and hence assume that \( A, B \) are local with residue field \( k \). Then the subspaces \( B_m := k \oplus (I + J^m) \), where \( m \geq 1 \), form a decreasing sequence of subalgebras of \( B \), with \( B_1 = B \) and \( B_m = A \) for \( m \gg 0 \). Using the exact sequence (10) and the inclusion \((I + J^m)^2 \subset I + J^{m+1}\), we may thus assume that \( J^2 \subset I \). Then \( I \) is an ideal of \( J \), and hence we may further assume that \( I = 0 \) by using Lemma 4.3 (ii). In that case, \((1 + J)/(1 + I) = 1 + J\) is a vector group, since \( J^2 = 0\).

Lemma 4.5. Let \( A \subset B \) be reduced algebras and write \( A = \prod_{i=1}^m K_i \), \( B = \prod_{j=1}^n L_j \), where \( K_i, L_j \) are fields. Then \( \mu^{B/A} \) has a composition series with subquotients \( \mu^{L_j/K_i} \) (where \( K_i \hookrightarrow L_j \)) and possibly \( \mu^{K_i} \). Moreover, all the \( \mu^{L_j/K_i} \) occur with multiplicity 1.

Proof. Let \( e_1, \ldots, e_m \) be the primitive idempotents of \( A \). Then

\[
A = \prod_{i=1}^m K_i = \prod_{i=1}^m A e_i \subset \prod_{i=1}^m B e_i = B,
\]

and each \( B e_i \) is a subalgebra of \( B \). Thus, \( \mu^{B/A} = \prod_{i=1}^m \mu^{B e_i/A e_i} \), and hence we may assume that \( A \) is a field, say \( K \). Then \( K \subset K^n \subset \prod_{j=1}^n L_j = B \), so that (10) yields an exact sequence

\[
0 \longrightarrow \mu^{K^n/K} \longrightarrow \mu^{B/A} \longrightarrow \prod_{j=1}^n \mu^{L_j/K} \longrightarrow 0.
\]
We may factor the diagonal inclusion $K \subset K^n$ as $K \subset K^2 \subset \cdots \subset K^n$, where each $K^i$ is embedded in $K^{i+1}$ via $(x_1, \ldots, x_i) \mapsto (x_1, \ldots, x_i, x_i)$. Thus, $\mu^{K^n/K}$ has a composition series with subquotients $\mu^{K^{i+1}/K^i}$. Moreover, the map

$$\mu^{K^{i+1}} = (\mu^K)^{i+1} \to \mu^K, \quad (x_1, \ldots, x_{i+1}) \mapsto x_1 x_{i+1}$$

is an epimorphism with kernel $\mu^{K^i}$, and hence yields an isomorphism $\mu^{K^{i+1}/K^i} \cong \mu^K$. □

4.2 Tori

We keep the notation of Subsection 4.1. We first record the following observation, probably well-known but that we could not locate in the literature:

**Lemma 4.6.** Let $K/k$ be a finite extension of fields and denote by $K_{\text{sep}}$ the separable closure of $k$ in $K$. Then $K_{\text{sep}} \otimes_k \bar{k}$ is the largest reduced subalgebra of $K \otimes_k \bar{k}$.

In particular, the nilradical of $K \otimes_k \bar{k}$ has dimension $[K:k] - [K_{\text{sep}}:k]$ as a $\bar{k}$-vector space; moreover, $\mu^{K_{\text{sep}}}$ is the maximal torus of $\mu^K$.

**Proof.** We have an isomorphism of $\bar{k}$-algebras $K_{\text{sep}} \otimes_k \bar{k} \cong \prod_{i=1}^m \bar{k}$, where $m := [K_{\text{sep}}:k]$. Also, we may assume that $k$ has characteristic $p > 0$ (since there is nothing to prove in characteristic 0). Then $x^n \in K_{\text{sep}}$ for $n \gg 0$ and $x \in K$. Thus, $x^n \in K_{\text{sep}} \otimes_k \bar{k}$ for $n \gg 0$ and all $x \in K \otimes_k \bar{k}$. It follows that $K \otimes_k \bar{k} = (K_{\text{sep}} \otimes_k \bar{k}) \oplus I$, where $x^n = 0$ for $n \gg 0$ and all $x \in I$. This yields the assertions on $K_{\text{sep}} \otimes_k \bar{k}$ and on the nilradical of $K \otimes_k \bar{k}$. As a consequence, $\mu^{K_{\text{sep}} \otimes_k \bar{k}}$ is the maximal torus of $\mu^{K \otimes_k \bar{k}}$; the assertion on $\mu^{K_{\text{sep}}}$ follows in view of Lemma 4.3 (iv). □

We may now describe the maximal tori of relative unit groups:

**Proposition 4.7.** Let $A \subset B$ be algebras, $I \subset J$ their nilradicals, $A_{\text{red}} := A/I \subset B/J =: B_{\text{red}}$ the associated quotients, and $A_{\text{sep}} \subset B_{\text{sep}}$ the largest separable subalgebras of these quotients.

(i) $\mu^{B_{\text{red}}/A_{\text{red}}}$ is the maximal torus of $\mu^{B/A}$.

(ii) If $B_{\text{red}} = B_{\text{sep}}$ (and hence $A_{\text{red}} = A_{\text{sep}}$; this holds e.g. if $k$ is perfect), then

$$\mu^{B/A} \cong (1 + J)/(1 + I) \times \mu^{B_{\text{red}}/A_{\text{red}}},$$

where $(1 + J)/(1 + I)$ is unipotent and $\mu^{B_{\text{red}}/A_{\text{red}}}$ is a torus.

**Proof.** (i) Given an exact sequence of connected algebraic groups $0 \to G_1 \to G \to G_2 \to 0$, the sequence of maximal tori $0 \to T(G_1) \to T(G) \to T(G_2) \to 0$ is exact as well. Thus, it suffices to show that $\mu^{B_{\text{red}}}$ is the maximal torus of $\mu^B$. For this, we may assume that $B$ is reduced, in view of Lemma 4.3 (i). Then $B$ is a direct product of fields, and we conclude by Lemma 4.6 (ii) follows from (i) in view of Lemma 4.3. □
Remark 4.8. With the notation of the above proposition, the maximal torus \( T \) of \( \mu_{B/A}^{1} \) sits in an exact sequence \( 0 \to \mu_{A_{\sep}}^{1} \to \mu_{B_{\sep}}^{1} \to T \to 0 \). Also, \( \mu_{A_{\sep}}^{1}, \mu_{B_{\sep}}^{1} \) are quasi-split tori, as seen in Example 4.2 (i). By [Vo98, Chap. 2, §4.7, Thm. 2], it follows that \( T \) is stably rational (this is a restrictive condition on tori, e.g., if the group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) is a quotient of \( \Gamma \), then some tori of dimension 3 are not stably rational; see [Vo98, Chap. 2, §4.10]). We do not know whether all stably rational tori can be obtained as relative unit groups.

Remark 4.9. Every connected affine algebraic group \( G \) over the field \( \mathbb{R} \) of real numbers is the Picard variety of some projective variety. Indeed, \( G \cong V \times T \), where \( V \cong \mathbb{G}_{m, \mathbb{R}}^{n} \) is a vector group, and \( T \) a torus; moreover, by [Vo98, Chap. 4, §10.1], \( T \) is isomorphic to a direct product of copies of \( \mu_{\mathbb{R}}, \mu_{\mathbb{C}}, \mu_{\mathbb{C}/\mathbb{R}} \). Using Theorem 1.1 and Lemma 2.7, we reduce to the cases where \( G = \mu_{\mathbb{C}} \) or \( G = \mu_{\mathbb{C}/\mathbb{R}} \). In the latter case, we may choose a smooth projective rational curve \( X' \) containing a closed point \( Y' \) with residue field \( \mathbb{C} \); pinching via the structure map \( Y' \to Y := \text{Spec}(\mathbb{R}) \) yields the desired variety, as can be checked by arguing as in the proof of Proposition 4.1. In the former case, we replace \( Y' \) with \( Z' \), where \( Z' \) is the disjoint union of \( Y' \) and a closed point with residue field \( \mathbb{R} \), and pinch via the structure map again.

The same result holds for any real closed field \( k \), with the same proof. Yet we do not know whether it extends to all connected (not necessarily affine) algebraic groups over \( k \). The example in [On87, p. 505] suggests a negative answer to that question.

Next, we characterize those relative unit groups that are tori:

**Proposition 4.10.** With the notation of Proposition 4.9, the following are equivalent:

(i) \( \mu_{B/A}^{1} \) is a torus.

(ii) \( I = J \) and \( B_{\red} \) is separable over \( k \) (hence so is \( A_{\red} \)).

**Proof.** (i)⇒(ii) We must have \( I = J \) by Lemma 4.3. In view of Lemma 4.3 (ii), we may thus assume that \( B \) (and hence \( A \)) is reduced. Write \( A = \prod K_{i} \) and \( B = \prod L_{j} \) as in Lemma 4.5. By that lemma, \( \mu_{L/K}^{1} \) must be a torus whenever \( K = K_{i} \hookrightarrow L_{j} = L \). Thus, the base change \( \mu_{L/K}^{1} \) is a torus over \( \bar{k} \). This is equivalent to \( \mu_{L \otimes_{k} \bar{k} / K \otimes_{k} \bar{k}}^{1} \) being a torus, in view of Lemma 4.3 (iv). Using the exact sequence

\[
0 \to \mu_{L \otimes_{k} \bar{k}}^{1} \to \mu_{L \otimes_{k} \bar{k}}^{1} \to \mu_{L \otimes_{k} \bar{k} / K \otimes_{k} \bar{k}}^{1} \to 0
\]

and Lemma 4.4, it follows that \( K \otimes_{k} \bar{k} \) and \( L \otimes_{k} \bar{k} \) have the same nilradical. By Lemma 4.6, this yields

\[
[K : k] - [K_{\sep} : k] = [L : k] - [L_{\sep} : k].
\]

Since \( K_{\sep} = K \cap L_{\sep} \), we have

\[
\dim_{k}(K + L_{\sep}) = [K : k] + [L_{\sep} : k] - [K_{\sep} : k] = [L : k],
\]

and hence \( K + L_{\sep} = L \); in particular, \( L = KL_{\sep} \). Since the extension \( L_{\sep}/K_{\sep} \) is separable and \( K/K_{\sep} \) is purely inseparable, \( L_{\sep} \) and \( K \) are linearly disjoint over \( K_{\sep} \) (as follows e.g. from Mac Lane’s criterion). As a consequence,

\[
[L : K_{\sep}] = [L_{\sep} : K_{\sep}][K : K_{\sep}].
\]
On the other hand, \([L : K_{\text{sep}}] = \dim_{K_{\text{sep}}} (K + L_{\text{sep}}) = [K : K_{\text{sep}}] + [L_{\text{sep}} : K_{\text{sep}}] - 1\). Thus, we obtain

\[ ([L_{\text{sep}} : K_{\text{sep}}] - 1)([K : K_{\text{sep}}] - 1) = 0, \]

and hence \(L_{\text{sep}} = K_{\text{sep}}\) or \(K = K_{\text{sep}}\). In the former case, we have \(L = K + L_{\text{sep}} = K\). In the latter case, \(L = L_{\text{sep}}\), i.e., \(L\) is separable over \(k\).

(ii)\(\Rightarrow\)(i) By Lemma 4.4 we have \(\mu^{B/A} \cong \mu^{B_{\text{red}}/A_{\text{red}}}\). Moreover, \(\mu^{B_{\text{red}}/A_{\text{red}}}\) is a torus in view of Proposition 4.7.

\[\square\]

### 4.3 Unipotent groups

Throughout this subsection, we consider algebras \(A \subset B\) with nilradicals \(I \subset J\) and associated quotients \(A_{\text{red}} = A/I \subset B/J = B_{\text{red}}\). We first obtain an (easy) characterization of those relative unit groups that are unipotent:

**Proposition 4.11.** (i) When \(\text{char}(k) = 0\), \(\mu^{B/A}\) is unipotent if and only if \(A_{\text{red}} = B_{\text{red}}\).

(ii) When \(\text{char}(k) = p > 0\), \(\mu^{B/A}\) is unipotent if and only if \(b^p \in A\) for \(n \gg 0\) and all \(b \in B\). Equivalently, the extension \(L/K\) is purely inseparable for any inclusion \(K \subset L\), where \(K\) (resp. \(L\)) is a residue field of \(A\) (resp. \(B\)).

**Proof.** (i) follows from Lemma 4.4 (ii), since \(\mu^{B_{\text{red}}/A_{\text{red}}}\) is a torus by Proposition 4.7.

(ii) Recall that \(\mu^{B/A}\) is unipotent if and only if its group of \(\overline{k}\)-points is \(p^n\)-torsion for \(n \gg 0\). Since \(\mu^{B/A}(\overline{k}) = (B \otimes_k \overline{k})^*/(A \otimes_k \overline{k})^*\), this is in turn equivalent to the condition that \(b^n \in (A \otimes_k \overline{k})^*\) for \(n \gg 0\) and all \(b \in (B \otimes_k \overline{k})^*\). As the \(\overline{k}\)-vector space \(B \otimes_k \overline{k}\) is spanned by \((B \otimes_k \overline{k})^*\), this is also equivalent to \(b^n \in A \otimes_k \overline{k}\) for \(n \gg 0\) and all \(b \in B \otimes_k \overline{k}\), and hence to \(b^n \in A\) for \(n \gg 0\) and all \(b \in B\).

The equivalence with the condition on residue fields follows readily from the structure of \(A\) and \(B\).

\[\square\]

**Remark 4.12.** The above results may be reformulated in terms of the morphism

\[\psi : Z := \text{Spec}(B) \longrightarrow \text{Spec}(A) =: Y\]

associated with the inclusion of algebras \(A \subset B\) (so that \(Y, Z\) are finite, and \(\psi\) is surjective). For example, Proposition 4.11 means that \(\mu^{B/A}\) is unipotent if and only if \(\psi\) is a universal homeomorphism.

Likewise, when \(A\) contains no ideal of \(B\), Proposition 4.10 means that \(\mu^{B/A}\) is a torus if and only if \(Y\) and \(Z\) are étale.

Also, Lemma 4.5 may be reformulated and slightly sharpened as follows: if \(Z\) (and hence \(Y\)) is reduced, then \(\mu^{B/A}\) has a composition series with subquotients \(\mu^{k(z)/k(y)}\), where \(y \in Y\) and \(z \in \psi^{-1}(y)\), and possibly \(\mu^{k(y)}\). Moreover, all the \(\mu^{k(z)/k(y)}\) occur with multiplicity \(1\), and \(\mu^{k(y)}\) with multiplicity \(|\psi^{-1}(y)| - 1\).

Next, we show that certain unipotent relative unit groups are \(k\)-wound, generalizing Example 4.12 (iv). For this, we shall need:

**Lemma 4.13.** Let \(k \subset K \subset L\) be a tower of finite extensions of fields, where \(K/k\) is separable. Then every homomorphism of algebraic groups \(h : \mathbb{G}_a \rightarrow \mu^{L/K}\) is constant.
Proof. Since $\mu^K$ is a torus, every extension $0 \to \mu^K \to G \to \mathbb{G}_a \to 0$ splits by \textbf{[SGA3, Exp. XVII, Thm. 6.1.1]}. In view of the exact sequence $0 \to \mu^K \to \mu^L \to \mu^{L/K} \to 0$, it follows that any homomorphism $h : \mathbb{G}_a \to \mu^L$ lifts to a homomorphism $\tilde{h} : \mathbb{G}_a \to \mu^{L/K}$. We may view $\tilde{h}$ as a $k[t]$-point of $\mu^L$, i.e., $\tilde{h} \in L[t]^* = L^*$. Since $\tilde{h}(0) = 1$, it follows that $\tilde{h}$ is constant. 

With the assumptions of the above lemma, if in addition $L/K$ is purely inseparable, then it follows that the unipotent group $\mu^{L/K}$ is $k$-wound (this also results from \textbf{[Oe84, Prop. V.7, Lem. VI.5.1]}). We do not know whether $\mu^{L/K}$ is $k$-wound when $K/k$ is no longer assumed to be separable.

Returning to the setting of algebras $A \subseteq B$ with nilradicals $I \subseteq J$, we now obtain a succession of elementary results which will readily imply Theorem 1.2.

\textbf{Lemma 4.14.} (i) The maximal ideals of $J$ are exactly the hyperplanes containing $J^2$.
(ii) There exists a flag of subspaces $I = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = J$ such that $I_i$ is a maximal ideal of $I_{i+1}$ for all $i$. In particular, $n = \dim(J) - \dim(I)$.
(iii) $J^{2^m} \subseteq I$.

Proof. (i) Let $K$ be a maximal ideal of $J$. Then $J/K$ is a nilpotent algebra having no proper ideal. Hence $\dim(J/K) = 1$ and $(J/K)^2 = 0$. In other words, $K$ is a hyperplane of $J$ containing $J^2$. Conversely, any such hyperplane is clearly a maximal ideal.

(ii) Let $m$ be the largest integer such that $J^m = 0$. Then we have a flag of subspaces $I \subseteq I + J^m \subseteq I + J^{m-1} \subseteq \cdots \subseteq I + J^2 \subseteq J$. Choose a complete flag of subspaces $I_i$ refining this partial flag. Then each $I_i$ can be written as $I + V$ for some subspace $V$ such that $J^{j+1} \subseteq V \subseteq J^j$ for some $j$. Since $(I + V)(I + J^j) = I^2 + IV + IJ^j + VJ^j \subseteq I + J^{j+1}$, we see that each $I + V$ is an ideal of $I + J^j$. This implies the assertion.

(iii) By (i), we have $I_i^2 \subseteq I_{i+1}$ for all $i$. This yields the statement by induction. 

Next, assume that $k$ has characteristic $p > 0$. Let $U := (1 + J)/(1 + I)$ and $n := \dim(U) = \dim(J) - \dim(I)$. Then $U$ is an iterated extension of $n$ copies of $\mathbb{G}_a$ by Lemma 4.13 (iii); hence the commutative group $U(\bar{k})$ is $p^n$-torsion. Let $m$ be the smallest positive integer such that $U(\bar{k})$ is $p^m$-torsion; then $m \leq n$. We say that $U$ has period $p^m$.

We shall use repeatedly the following observation:

\textbf{Lemma 4.15.} With the above notation, assume that $U$ has maximal period $p^n$. Let $I'$ be a subalgebra of $J$ containing $I$. Then the connected unipotent groups $(1 + I')/(1 + I)$ and $(1 + J)/(1 + I')$ have maximal period as well.

Proof. This follows readily from the exact sequence (a special case of (110))

$$0 \longrightarrow (1 + I')/(1 + I) \longrightarrow U \longrightarrow (1 + J)/(1 + I') \longrightarrow 0.$$ 

We now consider successively the cases where $p \geq 5$, $p = 3$ and $p = 2$ (the latter turns out to be much less straightforward):

\textbf{Lemma 4.16.} With the above notation, we have $m < n$ when $p \geq 5$ and $n \geq 2$. 

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Proof. We argue by contradiction, and assume that $U$ has maximal period. Since $n \geq 2$, there exists a subalgebra $I_2 \subset J$ such that $I \subset I_2$ and $\dim(I_2) = \dim(I) + 2$ (by Lemma 4.14 (ii)). By Lemma 4.15, the 2-dimensional subgroup $(1 + I_2)/(1 + I)$ is not $p$-torsion. On the other hand, $I_2^2 \subset I$ by Lemma 4.14 (iii). If $p \geq 5$, then $(1 + x)^p = 1 + x^p \in I$ for all $x \in I_2$, a contradiction.

Lemma 4.17. With the above notation, we have $m < n$ when $p = 3$ and $n \geq 3$.

Proof. We adapt the argument of Lemma 4.16. By Lemma 4.14, we may choose a subalgebra $I_3 \subset J$ such that $I \subset I_3$ and $\dim(I_3) = \dim(I) + 3$. By Lemma 4.15 again, the 3-dimensional subgroup $(1 + I_3)/(1 + I)$ is not 9-torsion, if $U$ has maximal period. But $I_3^2 \subset I$ by Lemma 4.14 again; this yields a contradiction.

Lemma 4.18. With the above notation, we have $m < n$ when $p = 2$ and $n \geq 3$.

Proof. We argue again by contradiction, and assume that $U$ has maximal period. We may reduce to the case where $n = 3$ as in the proof of Lemma 4.17. To analyze $(1 + J)/(1 + I)$, we begin with some further reductions.

If $I$ contains an ideal $J'$ of $J$, then the natural homomorphism

$$(1 + J)/(1 + I) \longrightarrow (1 + J/J')/(1 + I/J')$$

is an isomorphism by Lemma 4.3 (ii). Thus, we may assume that $I$ contains no nonzero ideal of $J$.

Also, if there exists a subalgebra $I'$ of $J$ such that $I + I' = J$, then the natural homomorphism

$$(1 + I')/(1 + I) \longrightarrow (1 + J)/(1 + I)$$

is an isomorphism, as follows from Lemma 4.3 (iii) in view of the equality $\dim(I'/I \cap I') = \dim(J/I)$. Thus, we may assume that there exists no proper subalgebra $I'$ of $J$ such that $I + I' = J$. By Lemma 4.14 (ii), this is equivalent to the assumption that $I \subset I'$. In view of Lemma 4.14 (i), we may thus assume that $I \subset J^2$.

By Lemma 4.15, it follows that the group $(1 + J)/(1 + J^2)$ has maximal period. But $(1 + J)/(1 + J^2) \cong 1 + J/J^2$ is a vector group, and hence has period 2. Hence $\dim(J/J^2) = 1$. By Nakayama’s lemma, we then have

$$J = tk[t]/(t^{m+1}) = \langle x, x^2, \ldots, x^m \rangle$$

for some $x \in J$ and a unique integer $m \geq 1$. Then

$$J^2 = \langle x^2 \rangle = \langle x^2, x^3, \ldots, x^m \rangle$$

is the unique maximal ideal of $J$. Moreover, our reductions mean that $I \subset \langle x^2, x^3, \ldots, x^m \rangle$ and $x^m \notin I$.

Consider $I' := \langle I, x^m \rangle \subset J$; this is a subalgebra of codimension 2 of $J$, which contains $I$ as a maximal ideal. By Lemma 4.14 (ii), $I'$ is a maximal ideal of $J^2$, hence $I' \supset J^4$ by that lemma, (iii). Since $J^4 = \langle x^4, x^5, \ldots, x^m \rangle$, there exist $a, b \in k$ such that $(a, b) \neq (0, 0)$ and

$$I' = \langle ax^2 + bx^3, x^4, x^5, \ldots, x^m \rangle.$$
Moreover, \(a \neq 0\): otherwise, \(I' = \langle x^3, x^4, x^5, \ldots, x^m \rangle\) is an ideal of \(J\), so that \((1+J)/(1+I')\) has dimension 2 and period 2; this yields a contradiction in view of Lemma 4.15.

By Lemma 4.14 (iii) again, we have \(I^2 \subset I\). Thus, \(I\) contains \(x^8, x^9, \ldots\) and also \((ax^2 + bx^3)x^5\); in particular, \(x^7 \in I\). Likewise, \((ax^2 + bx^3)x^4 \in I\) so that \(x^6 \in I\). By our reductions, it follows that \(x^6 = 0\). Also, \(a^2x^4 + b^2x^6 = (ax^2 + bx^3)^2 \in I\); thus, \(x^4 \in I\). Since \(x\) generates the nilpotent algebra \(J\), this yields \(y^4 \in I\) for all \(y \in J\). As a consequence, \((1 + J)/(1 + I)\) has period at most 4, a contradiction.

**Proof of Theorem 1.2.** We argue again by contradiction, and assume that \(W_n\) is isogenous to \(\text{Pic}_{X/k}\) for some projective variety \(X\) with finite non-normal locus. In particular, \(U := \text{Pic}^0(X)\) is unipotent. By \([Se59, \text{Chap. VII, no. 10, Prop. 9}]\), \(U\) has maximal period \(p^n\), where \(n := \dim(U)\). On the other hand, there exist algebras \(A \subset B\) such that \(U \cong \mu_{B/A}\), by Proposition 4.1. Since \(k\) is perfect and \(U\) is unipotent, we must have \(U \cong (1 + J)/(1 + I)\) by Lemma 4.4. But then Lemmas 4.16, 4.17 and 4.18 yield a contradiction.

**Remark 4.19.** Consider the algebra \(B := k[x]/(x^4)\), and its subalgebra \(A\) generated by \(x^2 + x^3\) (of square 0). Then \(U := \mu_{B/A}\) is a connected unipotent group of dimension 2. If \(p = 3\) (resp. \(p = 2\)), then \(U\) has period 9 (resp. 4) since \(x^3, x^2 \notin A\). By \([Se59, \text{Chap. VII, no. 10, Prop. 9}]\) again, it follows that \(U\) is isogenous to \(W_2\). In particular, the statement of Theorem 1.2 is optimal.

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