Optimal query complexity for private sequential learning

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Abstract
Motivated by privacy concerns in many practical applications such as Federated Learning, we study a stylized private sequential learning problem: a learner tries to estimate an unknown scalar value, by sequentially querying an external database and receiving binary responses; meanwhile, a third-party adversary observes the learner’s queries but not the responses. The learner’s goal is to design a querying strategy with the minimum number of queries (optimal query complexity) so that she can accurately estimate the true value, while the adversary even with the complete knowledge of her querying strategy cannot. Prior work has obtained both upper and lower bounds on the optimal query complexity, however, these upper and lower bounds have a large gap in general. In this paper, we construct new querying strategies and prove almost matching upper and lower bounds, providing a complete characterization of the optimal query complexity as a function of the estimation accuracy and the desired levels of privacy.

1 Introduction
Enabled by the rapid developments in machine learning and data science, organizations and individuals rely more and more on data to solve inference and decision problems. However, with the increasing awareness of data privacy, collecting such data from its owners becomes more costly and sometimes even infeasible. To ease the data owners’ privacy concerns, researchers and practitioners have proposed a new learning framework, known as learning with external workers, wherein the data is kept confidential by its owners from the learner, and the learner interacts with these data owners by submitting sequential queries and receiving responses [10, 12, 11, 3, 16]. This new framework has been implemented in practical systems such as Google’s Federated Learning, wherein Google tries to learn a model via interacting with mobile users while keeping the data on users’ mobile devices [13].

While this new framework mitigates data owner’s privacy risk [6, 14, 2], it exposes the learner to substantial risk of privacy breach, as a third-party adversary may easily steal the learned model by pretending as an external worker and eavesdropping queries. For example, to learn a model, a common approach is to optimize some loss function by running the gradient descent algorithm: at each iteration, the learner first broadcasts the model learned at the current iteration to external workers; and then each external worker computes the gradient at the current model based on the local data and transits it back to the learner; and finally the learner aggregates the received gradients and updates the learned model by running a gradient descent step. By pretending as an

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external worker, the third-party adversary completely observes the model learned (queries) even though not gradients (responses) at every iteration. With the advancements in the surveillance technologies, similar privacy breach scenarios are likely to arise more frequently and broadly whenever the learner has to interact with an external entity in an open environment and such interaction can be monitored by a third-party adversary. This privacy consideration immediately leads to two natural but fundamental questions:

1. How to learn, while ensuring that an adversary who observes queries but not responses does not learn?

2. What is optimal query complexity, that is the minimal number of queries needed, for an accurate and private learning?

Note that in the learning with external workers framework, communication bandwidth is a scarce resource, as the data transmission between the external workers and the learner typically suffers from high latency and low throughput. Thus, determining the optimal query complexity is of fundamental importance in both theory and practice.

In this work, we address these questions by studying a stylized private sequential learning model proposed by [17]. Suppose that a learner is trying to estimate an unknown value $X^* \in [0, 1]$, by submitting $n$ queries sequentially, $(q_1, \ldots, q_n) \in [0, 1]^n$, for some $n \in \mathbb{N}$. For each query $q_i$, the learner receives a binary response $r_i = \mathbb{1}\{X^* \geq q_i\}$, indicating the value $X^*$ relative to the query, where $\mathbb{1}\{\cdot\}$ denotes the indicator function. Meanwhile, there is an adversary who observes all of the learner’s queries $(q_1, \ldots, q_n)$, but not the responses $(r_1, \ldots, r_n)$, and also tries to estimate $X^*$. The learner’s goal is to design a querying strategy with a minimal $n$ (optimal query complexity) so that she can estimate $X^*$ up to an additive error of $\epsilon/2$ with probability 1, while the adversary even with the complete knowledge of the learner’s querying strategy cannot estimate $X^*$ up to an additive error of $\delta/2$ with probability larger than $1/L$ for some integer privacy level $L \geq 1$.

In the special case $L = 1$, this model (with its noisy response variant) reduces to the classical problem of sequential search with binary feedback, with numerous applications such as data transmission with feedback [7], finding the roots of a continuous function [18], and the game of “twenty questions” [9]. This model can be also viewed as a simple abstraction of the learning with external workers framework, where $X^*$ represents the model to learn; the quadratic function $(X^* - X)^2$ is the loss function to optimize; the query $q_i$ is the estimated model at the $i$-th iteration; and the response $r_i$ is the sign of the aggregated gradient. Here we assume the external workers have a large volume of data in total so that the aggregated gradient is effectively noiseless and equal to the sign of the true gradient: $\text{sign}(X^* - q_i)$.

There are many possible extensions to enrich this model, such as generalizing $X^*$ to be a multi-dimensional vector and the responses to be noisy. However, it turns out that this model, albeit simple, already captures the essential tension between learning and privacy-preserving. To see this, first consider the simple bisection strategy, which recursively queries the mid-point of the interval that the learner knows to contain $X^*$. It is query-efficient with only $\log(1/\epsilon)$ queries needed, but is almost non-private: if the adversary sets her estimator to be the learner’s last query, then $X^*$ is within a distance of at most $\epsilon$. On the contrary, consider the completely non-adaptive grid search strategy, where the learner partitions the whole interval $[0, 1]$ into sub-intervals of length at most $\epsilon$ and queries the mid-points of all sub-intervals. It is almost completely private, but has a very high query complexity $1/\epsilon$. This observation suggests that the more adaptive the querying strategy is, the faster to learn, but at the same time, the harder to protect privacy; the optimal strategy

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1Here and subsequently log refers to logarithm with base 2.
needs to submit queries adaptively at a very fine grain to achieve the best trade-off between query complexity and privacy.

Building upon this observation, previous work \cite{17} and \cite{19} proves both upper and lower bounds for the optimal query complexity under the deterministic setting where $X^*$ is deterministic but arbitrary and under the Bayesian setting where $X^*$ is uniformly distributed over $[0, 1]$. In particular, under the deterministic setting where $\delta = 1/L$, the upper and lower bounds almost coincide, showing that the optimal query complexity is about $\log(1/\epsilon) + 2L$ \cite{17}. In contrast, under the Bayesian setting where $\delta$ and $L$ stay fixed while $\epsilon$ tends to 0, the optimal query complexity is asymptotically $L \log(1/\epsilon)$ \cite{19}. Unfortunately, the previous upper and lower bounds can have a large gap in general, leaving the optimal query complexity elusive.

In this paper, we prove almost matching upper and lower bounds in both Bayesian and deterministic settings, providing a complete understanding of how the optimal query complexity $N(\epsilon, \delta, L)$ behaves as a function of $(\epsilon, \delta, L)$:

- **Deterministic setting:**
  
  $$N(\epsilon, \delta, L) \approx 2L + \log \frac{\max\{2^{-L}, \delta\}}{\epsilon};$$

- **Bayesian setting:**
  
  $$N(\epsilon, \delta, L) \approx L \log \frac{\delta}{\epsilon} + \log \frac{1}{L\delta}.$$

To prove our upper bounds, we construct new querying strategies that ensure estimation accuracy for the learner while maintaining a desired level of privacy against the adversary. To show our lower bounds, we prove that no strategy that submits fewer queries can achieve the same level of accuracy and privacy.

In passing, we remark that there is a large body of literature on private iterative learning, such as private stochastic gradient descent \cite{15, 1, 2}, private online learning \cite{8}, and private federated learning \cite{6, 14}, wherein the focus is to protect data owners’ privacy by preventing the adversary inferring about a data owner from the outputs of learning algorithms, under the notion of differential privacy \cite{5}. In this setting, a common privacy-preserving mechanism is to inject calibrated noise at each iteration of learning algorithms. In contrast, our work aims to protect the learner’s privacy by preventing the adversary inferring the learned model from the learner’s queries. As a result, our problem setup, privacy-preserving mechanisms, and main results are significantly different.

## 2 Problem formulation

Consider the problem of learning some unknown true value $X^* \in [0, 1]$. Let $\hat{X}$ be the learner’s estimator of $X^*$ and $\tilde{X}$ be the adversary’s. The learner submits queries $q_1, q_2, \ldots \in [0, 1]$ sequentially. Each time a query $q_i$ is submitted, the learner receives a response $r_i = 1\{X^* \geq q_i\}$.

The learner’s query $q_i$ can depend on all past queries and responses, and it is allowed to incorporate outside randomness. Since all random variables and all random vectors with finite alphabets can be simulated from a random variable uniformly distributed on $[0, 1]$, without loss of generality, let $Y \sim \text{Unif}[0, 1]$ be the random seed that the learner may use to generate queries. Then $q_i$ can be written as $f_{i-1}(q_1, \ldots, q_{i-1}, r_1, \ldots, r_{i-1}, Y)$ for some function $f_{i-1}$. Note that the first query $q_1$ is submitted without any information and is only a function of $Y$. Thus we have $q_2 = f_1(q_1, r_1, Y) = f_1(f_0(Y), r_1, Y) := \phi_1(r_1, Y)$. It is easy to see that all $q_i$ can be written iteratively as a function of only the past responses and $Y$, i.e., $q_i = \phi_{i-1}(r_1, \ldots, r_{i-1}, Y)$.
Then a querying strategy $\phi$ is defined by an initial $f_0 : [0, 1] \to [0, 1]$ used to generate $q_1$ from $Y$, a sequence of mappings $(\phi_i)_i$ with $\phi_i : \{0, 1\}^i \times [0, 1] \to [0, 1]$ used to generate the rest of the query sequence, and a final estimator $\hat{X}$, which can depend on $Y$ and all the queries and responses. The adversary $\tilde{X}$, on the contrary, only has access to the queries and the querying strategy $\phi$ but not the random seed $Y$.

The goal of the learner is to design a querying strategy to ensure that she can accurately estimate $X^*$, but the adversary cannot. Following [17], we consider both the Bayesian setting where $X^* \in [0, 1]$ is uniformly distributed on $[0, 1]$ and the setting where $X^*$ is deterministic. The two settings call for different definitions for accuracy and privacy, which we shall discuss separately.

**Bayesian setting** We assume $X^*$ is uniformly distributed on $[0, 1]$, which is independent from the random seed $Y$, as the learner does not know the true value $X^*$ a priori. We say a strategy $\phi$ is

- $\epsilon$-accurate for $\epsilon > 0$, if
  $$\Pr \left\{ \left| \hat{X} - X^* \right| \leq \epsilon/2 \right\} = 1;$$

- $(\delta, L)$-private for $\delta > 0$ and an integer $L \geq 2$, if there is no adversary $\tilde{X}$ such that
  $$\Pr \left\{ \left| \tilde{X} - X^* \right| \leq \delta/2 \right\} > \frac{1}{L}.$$

**Deterministic setting** Suppose $X^*$ is a deterministic but arbitrary number on $[0, 1]$. Then the only source of randomness in the querying strategy is from $Y$. We say a strategy $\phi$ is

- $\epsilon$-accurate for $\epsilon > 0$, if
  $$\Pr \left\{ \left| \hat{X} - X^* \right| \leq \epsilon/2 \right\} = 1, \quad \forall X^* \in [0, 1];$$

- $(\delta, L)$-private for $\delta > 0$ and an integer $L \geq 2$, if for each query sequence $\bar{q}$, the $\delta$-covering number $^2$ of the information set $\mathcal{I}(\bar{q})$ is at least $L$. The information set is defined as the set of all true values that could lead to the query sequence $\bar{q}$ under strategy $\phi$ with non-negligible probability. Note that the query sequence $q$ is a random vector that depends on $X^*$ and $Y$, i.e. $q = q(X^*, Y)$. Formally we define
  $$\mathcal{I}(\bar{q}) = \{ X^* \in [0, 1] : \Pr \{ q(X^*, Y) = \bar{q} \} > 0 \}.$$ 

Unlike the Bayesian setting, the definition for privacy no longer involves an adversary’s estimator $\tilde{X}$. However, one can easily argue (see [17, Appendix A] for a proof) that this definition of privacy is equivalent to the following: there is no adversary $\tilde{X}$ such that for each query sequence $\bar{q}$ and each $X^* \in \mathcal{I}(\bar{q})$,

$$\Pr \left\{ \left| \tilde{X} - X^* \right| \leq \delta/2 \right\} > \frac{1}{L}.$$

Compare this to the definition of $(\delta, L)$-privacy in the Bayesian setting, the difference is that when $X^*$ is deterministic, the adversary can no longer average over some prior distribution of $X^*$; instead, she needs to learn well for all possible query sequences and all admissible values of $X^*$. In this sense the learning task is harder for the adversary, which means that it is easier for the

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$^2$The $\delta$-covering number of a set $A \subseteq \mathbb{R}$ is defined as the size of the smallest set $\mathcal{N}$, such that $\bigcup_{r \in \mathcal{N}} [r-\delta, r+\delta] \supseteq A$. 

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learner to achieve $(\delta, L)$-privacy. This distinction from the Bayesian setting is reflected by a smaller optimal query complexity, as we will see.

For both Bayesian and deterministic setting, we define the optimal query complexity as

$$ N(\epsilon, \delta, L) = \min \{ n : \exists \phi \text{ that is both } \epsilon\text{-accurate and } (\delta, L)\text{-private and submits at most } n \text{ queries} \}.$$  

Note that for a larger $\delta$ or a larger $L$, the $(\delta, L)$-private constraint is a stronger requirement. Therefore $N(\epsilon, \delta, L)$ is monotone nondecreasing in $\delta$ and $L$.

Same as [17], we focus on the regime of parameters

$$ 2\epsilon \leq \delta \leq \frac{1}{L},$$

which is natural and without loss of generality. To see this, on the one end of the spectrum, if $\delta > 1/L$, then the adversary can make an arbitrary guess to break the privacy constraint: simply choosing $\tilde{X} = 1/2$ yields $\mathbb{P}\{|\tilde{X} - X^*| \leq \delta/2\} = \delta > 1/L$. In this regime the $(\delta, L)$-privacy constraint is too strong to be satisfied by any querying strategy. On the other end of the spectrum, if $\delta \leq 2\epsilon$, then we can argue that the privacy constraint is so weak that it becomes trivial. Indeed, suppose the learner adopts the aforementioned bisection querying strategy. Then from the sequence of queries, the adversary can deduce the binary responses except for the response to the last query. However, since she is missing the last bit of information, she can achieve $2\epsilon$-accuracy at best. Therefore, if $\delta \leq 2\epsilon$, the learner can simply run the bisection method while satisfying the privacy constraint, achieving the optimal querying complexity $\log(1/\epsilon)$.

3 Main results

In this section, we present our almost matching upper and lower bounds to the optimal query complexity in the Bayesian and deterministic settings separately.

**Bayesian setting** We first focus on the Bayesian setting. It was shown in [17, Proposition B.2] that if $2\epsilon < \delta \leq 1/L$, then

$$ N(\epsilon, \delta, L) \leq L \log \frac{1}{\delta} + L - 1. \quad (1)$$

Notice that this upper bound does not change with $\delta$. Since the private learning task becomes easier for smaller $\delta$, this upper bound is not tight.

Subsequently, it was shown in [19, Theorem 2.1] that if $4\epsilon < \delta < 1/L$, then

$$ N(\epsilon, \delta, L) \geq L \log \frac{1}{\epsilon} - L \log \frac{2}{\delta} - 3L \log \log \frac{\delta}{\epsilon}. $$

As pointed out by [19, Corollary 2.2], if $\delta, L$ stay as fixed constants while $\epsilon \to 0$, then the above upper and lower bounds imply that the optimal query complexity scales as $L \log(1/\epsilon)$. However if $\delta \to 0$ as well and is comparable to $\epsilon$, then the upper and lower bounds above can be quite far apart.

The following is our first main result. We relax the $\delta > 4\epsilon$ constraint in the lower bound to $\delta \geq 2\epsilon$ and obtain sharper upper and lower bounds that almost match in the entire parameter regime.

**Theorem 1** (Bayesian setting). If $2\epsilon \leq \delta \leq 1/L$, then

$$ \left\lfloor \log \frac{1}{\delta} \right\rfloor + L \left( \log \frac{\delta}{\epsilon} - 2 \right) - 1 \leq N(\epsilon, \delta, L) \leq \left\lfloor \log \frac{1}{\delta} \right\rfloor + L \left( \left\lfloor \log \frac{\delta}{\epsilon} \right\rfloor + 2 \right) - 1.$$
If $\delta$ is a constant multiple of $\epsilon$, Theorem 1 implies that the optimal query complexity scales as $\log(1/\epsilon)$ plus a constant multiple of $L$. Since the bisection method takes $\log(1/\epsilon)$ queries, in this regime the query complexity price to pay for the increased privacy, in terms of $L$, is additive in $L$, in contrast to multiplicative as suggested by (1).

**Deterministic setting**  
Next we shift to deterministic setting. It was shown in [17, Theorem 4.1] that
\[
\max \left\{ \log \frac{1}{\epsilon}, \log \frac{\delta}{\epsilon} + 2L - 4 \right\} \leq N(\epsilon, \delta, L) \leq \log \frac{1}{L\epsilon} + 2L.
\]
Again the upper bound cannot be tight because it does not vary with $\delta$. In the following theorem we sharpen both the upper and lower bounds, shrinking the gap between them to only 8 queries.

**Theorem 2** (Deterministic setting). If $2\epsilon \leq \delta \leq 1/L$, then
\[
2L + \left\lceil \log \max \left\{ \frac{2^{-L}}{\epsilon} \right\} \right\rceil - 8 \leq N(\epsilon, \delta, L) \leq 2L + \left\lceil \log \max \left\{ \frac{2^{-L}}{\epsilon} \right\} \right\rceil.
\]

We can see from Theorem 2 that the lower bound in (2) is tight when $\delta \geq 2^{-L}$. However when $\delta < 2^{-L}$, the optimal query complexity is roughly $2L + \log(2^{-L}/\epsilon) = L + \log(1/\epsilon)$; and the query complexity price to pay for the increased privacy, in terms of $L$, is an additive factor of $L$.

4 The querying strategies

In this section we first review the querying strategies deployed in [17]. We then give the construction of our querying strategies and argue heuristically why our constructions lead to the optimal query complexity. Section 6 contains the full proofs of our results, where these heuristic arguments are made precise. As before we discuss the Bayesian and deterministic settings separately.

4.1 The Bayesian setting

When constructing a querying strategy we want it to possess the merits of accuracy, meaning that the learner can learn $X^*$ well with probability one; and privacy, meaning that the adversary cannot learn $X^*$ well with probability greater than $1/L$; and efficiency, meaning that the strategy submits as few queries as possible. As mentioned in the introduction, the bisection method is the most efficient, but not private. The grid search, on the contrary, is almost completely private, but very inefficient. It was previously shown in [17, Proposition B.2] that a querying strategy named the *replicated bisection* method achieves the query complexity of $L\log(1/(Le)) + L - 1$ as given in (1). In a way, replicated bisection arises from combining the bisection and a grid search. The idea behind its construction is simple: first split $[0, 1]$ into $L$ equal-length subintervals, then run bisection search on each subinterval.

More specifically, the replicated bisection method contains two phases. In phase 1, the learner submits $L - 1$ queries $\{1/L, 2/L, \ldots, (L - 1)/L\}$, which is essentially a $(1/L)$-accurate grid search. In this phase the learner determines which of the $I_j := [(j-1)/L, j/L]$ subintervals $X^*$ is in. Note that the adversary gains no information from these $L - 1$ queries. In phase 2, the learner searches for the true value $X^*$ via the bisection method in the true subinterval, while conducting copied bisection searches in the other subintervals. For example say $X^* \in I_1$. The first batch queries the midpoints of all the subintervals. If upon receiving the responses to the first batch of queries, the learner learns that $X^*$ is greater than the midpoint of $I_1$. Then the second batch of queries contain
the third quartiles of all the subintervals, and so on. After \( \lceil \log(1/(L\epsilon)) \rceil \) batches of queries the learner achieves \( \epsilon \)-accuracy.

The replicated bisection method is private because the adversary cannot deduce from the queries which one of the \( L \) subintervals \( X^* \) is in. Therefore she cannot estimate \( X^* \) up to an additive error of \( 1/L \) with probability higher than \( 1/L \). The query complexity of the replicated bisection method is \( L - 1 + L\lceil \log(1/(L\epsilon)) \rceil \).

However note that the subintervals \( I_1, ..., I_L \) are all of length \( 1/L \), while the learner’s strategy only needs to safeguard \( X^* \) against the adversary making a \( \delta \)-accurate guess. As a result, it is expected to be rather wasteful to run the replicated bisection on these subintervals of length \( 1/L \), when \( \delta \ll 1/L \). This key observation motivates us to construct a more query-efficient learner’s strategy: first run the bisection search to locate \( X^* \) within an interval \( I \) of length roughly \( L\delta \); and then conduct the replicated bisection search on \( I \), dividing \( I \) into \( L \) subintervals of equal length. One small caveat is that it may be impossible to get \( |I| = L\delta \) exactly from a bisection search. To ensure \((\delta,L)\)-privacy we need the lengths of the subintervals to be at least \( \delta \), i.e. \( |I| \geq L\delta \). The precise construction of our querying strategy is in algorithm 1. The values of \( K_1, K_2 \) are chosen so that \( |I_j| \geq \delta \) and the learner achieves \( \epsilon \)-accuracy. An example of algorithm 1 can be found in Figure 1.

**Algorithm 1:** Our querying strategy under the Bayesian setting

\[
\begin{align*}
K_1 & := \lceil \log(1/(L\delta)) \rceil; \ I := [0,1]; \\
\text{for } i = 1 \text{ to } K_1 \text{ do} & \quad \begin{array}{l}
\quad q_i := \text{midpoint of } I = [a, b]; \\
\quad \text{if } r_i = 1 \text{ then } I := [q_i, b] \text{ else } I := [a, q_i]; \\
\end{array} \\
\text{end} \\
\text{for } i \text{ in } 1 \text{ to } L \text{ do} & \quad \begin{array}{l}
\quad I_i := [a + (i - 1)(b - a)/L, a + i(b - a)/L], \ \text{where } [a, b] = I; \\
\quad J_i := I_i; \\
\end{array} \\
\text{end} \\
\text{for } i \text{ in } 1 \text{ to } L-1 \text{ do} & \quad \text{// query the endpoints of } I_1, ..., I_L \\
\quad q_{K_1+i} := \text{right endpoint of } I_i; \\
\text{end} \\
\text{Inspect the responses to find } i^* \in \{1, ..., L\} \text{ such that } X^* \in I_{i^*}; \\
K_2 & := \lceil \log(\delta/\epsilon) \rceil + 1; \\
\text{for } i \text{ in } 1 \text{ to } K_2 \text{ do} & \quad \begin{array}{l}
\quad \text{// replicated bisection on } I_1, ..., I_L \\
\quad \text{for } j \text{ in } 1 \text{ to } L \text{ do} \quad \begin{array}{l}
\quad q_{K_1+L-1+(i-1)L+j} := \text{midpoint of } J_j; \\
\quad \text{end} \\
\quad \text{if } q_{K_1+L-1+(i-1)L+i^*} = 1 \text{ then} \\
\quad \quad \text{for } j \text{ in } 1 \text{ to } L \text{ do left endpoint of } J_j := q_{K_1+L-1+(i-1)L+j}; \\
\quad \quad \text{else} \\
\quad \quad \text{for } j \text{ in } 1 \text{ to } L \text{ do right endpoint of } J_j := q_{K_1+L-1+(i-1)L+j}; \\
\quad \text{end} \\
\end{array} \\
\text{end}
\end{align*}
\]

The total number of queries submitted under algorithm 1 is

\[
K_1 + L - 1 + LK_2 = \left\lfloor \log \frac{1}{L\delta} \right\rfloor + L \left( \left\lfloor \log \frac{\delta}{\epsilon} \right\rfloor + 2 \right) - 1. \tag{3}
\]
Figure 1: An example of algorithm 1 with $L = 5$, $K_1 = 3$, $K_2 = 3$. The learner first runs $K_1$ steps of bisection to locate $X^*$ within $I$. Divides $I$ into $L$ equal length subintervals $I_1, ..., I_L$. By querying the endpoints of the subintervals $q_4, ..., q_7$, the learner locates the subinterval $X^*$ is in, in this case $I_4$. She then proceeds to submit $K_2$ batches of queries. The first, second and third batches of queries submitted are labeled $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$ respectively. On $I_4$, the queries are submitted via bisection while copies are submitted on the other subintervals in parallel.

### 4.2 The deterministic setting

Under the deterministic setting, the previous upper bound in [17, Theorem 4.1] is obtained via a querying strategy named opportunistic bisection. First we sketch the opportunistic bisection method, which later motivates our construction of the querying strategy. The querying strategy we use is more query-efficient, although slightly more sophisticated. Other than describing the querying strategy, we will walk the reader through our thought process that goes behind its construction.

Like replicated bisection, the opportunistic bisection method also consists of two phases. In phase 1, the learner queries $L$ pairs of queries:

$$q_i = (i-1)/L; \quad q_i+L = q_i + \epsilon, \quad i = 1, ..., L.$$  

The authors of [17] described phase 1 as submitting $L$ guesses. For each guess submitted, the learner is effectively testing a hypothesis $X^* \in I_i := [q_i, q_{i+L})$. If none of the guesses are correct, i.e. $X^* \notin I_i$ for all $i \leq L$, then from the responses of the $L$ pairs of queries the learner finds an interval $J_i = [q_{i+L}, q_{i+1})$ that $X^*$ is in. In phase 2 the learner conducts a bisection search in $J_i$ to approximate $X^*$ up to $\epsilon$-accuracy. On the contrary if the learner makes a lucky guess in phase 1, say $X^* \in I_i$, then the accuracy requirement is fulfilled by taking $\hat{X}$ to be the midpoint of $I_i$. However to disguise this finding from the adversary, the learner cannot stop submitting queries. In phase 2 the learner randomly picks an interval $J_i$ from $i = 1, ..., L$, and conducts a “fake” bisection search on $J_i$. By “fake” bisection we mean a simulated bisection search where the binary responses $R_i$, instead of being $R_i = 1\{X^* \geq q_i\}$, are generated i.i.d. from the Bernoulli(1/2) distribution.

The opportunistic bisection method is private because regardless of whether any of the guesses in phase 1 is correct, from only the queries the adversary cannot rule out the possibilities that $X^* \in I_i$ for some $i = 1, ..., L$. Recall that for a query sequence $\bar{q}$, the information set is defined as $\mathcal{I}(\bar{q}) = \{X^* \in [0, 1] : \mathbb{P}(q(X^*, Y) = \bar{q}) > 0\}$. Thus for each $\bar{q}$ we have $\mathcal{I}(\bar{q}) \supseteq \cup_{i \leq L} I_i$. It is easy to see that the $\delta$-covering number for $\cup_{i \leq L} I_i = \cup_{i \leq L} ((i-1)/L, (i-1)/L + \epsilon)$ is at least $L$. That is, the opportunistic bisection is $(\delta, L)$-private by definition.

The query complexity of opportunistic bisection is $[\log(1/(L\epsilon))] + 2L$ as given by the upper bound in (2). In comparison, the upper bound $2L + [\log(\max\{2^{-L}, \delta/\epsilon\})]$ in Theorem 2 is always smaller. The querying strategy we use to achieve this tighter bound is obtained through further improving over the opportunistic bisection. We observe that phase 1 of opportunistic bisection can be especially wasteful, since it is effectively submitting $L$ guesses via a grid search. This observation
motivates us to consider a more economical opportunistic bisection, where the guesses in phase 1 are submitted via the bisection method. That is, the first \( L \) queries \( q_1, \ldots, q_L \) are submitted via the bisection method and \( q_{i+L} = q_i + \epsilon \) for \( i = 1, \ldots, L \).

However, this strategy has a caveat that it no longer ensures privacy. The opportunistic bisection is private because the adversary cannot tell whether \( X^* \) is in any of the guesses. That is no longer the case if phase 1 is simply changed to bisection. For example, note that \( X^* < \frac{1}{2} \) is equivalent to \( r_1 = 0 \), which is equivalent to \( q_2 = \phi_1(r_1) = 1/4 \). Therefore once the adversary observes \( q_2 = 1/4 \), she learns that \( X^* \) cannot be in the first guess \( I_1 = [1/2, 1/2 + \epsilon] \). To resolve this issue we improve on phase 1 of the opportunistic bisection in the following two aspects:

1. Reorder the queries so the learner can test the \( L \) guesses sequentially. Let \( q_{2i} = q_{2i-1} + \epsilon \) for \( i = 1, \ldots, L \). The intervals corresponding to the guesses are redefined as \( I_i = [q_{2i-1}, q_{2i}] \).

2. The odd queries \( q_1, q_3, \ldots, q_{2L-1} \) are now obtained as follows: start with the bisection method, at the same time testing each guess \( X^* \in I_i \). If the learner finds out that one of the guesses is correct, she stops running bisection with the responses \( r_{2i-1} = 1 \{ X^* \geq q_{2i-1} \} \), and transitions into a “fake” bisection using random responses distributed i.i.d. Bernoulli(1/2) to generate the rest of the odd queries.

The final improved opportunistic bisection is displayed in algorithm 2. An example can be found in Figure 2. Under this querying strategy, the adversary cannot rule out \( X^* \in I_i \) for any \( i = 1, \ldots, L \).

The query complexity of algorithm 2 is \( 2L + \lceil \log(2^{-L/\epsilon}) \rceil \). Unfortunately, it only applies to when \( \delta \leq 2^{-L} \). The reason is that since the guesses are submitted via bisection, when \( \delta > 2^{-L} \), the guesses become cluttered and the intervals \( I_1, \ldots, I_L \) may even overlap. It is possible that the \( \delta \)-covering number of \( \bigcup_{i \leq L} I_i \) is smaller than \( L \). In this case we need to change the way of submitting the \( L \) guesses, so that

(i) The intervals \( I_i = [q_{2i-1}, q_{2i}] \), \( i = 1, \ldots, L \) do not overlap, and their left endpoints are at least \( \delta \) from each other;

(ii) After phase 1, the learner can always narrow down the possibilities for \( X^* \) to an interval of length at most \( 2\delta \).

Of these two requirements, (i) is to guarantee \((\delta, L)\)-privacy and (ii) is so that phase 2 only needs \( \lceil \log(2\delta/\epsilon) \rceil \) queries to achieve \( \epsilon \)-accuracy. The way we fulfill (i) and (ii) is to submit the first \( K \) guesses through bisection for some suitably chosen \( K \), and the rest through a grid search. See algorithm 3 for the exact construction and Figure 3 for an example. The reader might notice that algorithm 3 starts with an initial guess \( X^* \in [0, \epsilon] \). This creates an \( \epsilon \)-length interval belonging to the information set, which is needed to ensure that the \( \delta \)-covering number of the information set is...
Algorithm 2: Our querying strategy in deterministic case when $\delta \leq 2^{-L}$

$\text{FoundIt} := \text{False}$;
$I := [0, 1]$;
for $i = 1$ to $L$ do  // submit $L$ guesses via (possibly partially fake) bisection
    $q_{2i-1} := \text{midpoint of } I = [a, b]$;
    $q_{2i} := q_{2i-1} + \epsilon$;
    if not FoundIt then
        Inspect the responses $r_{2i-1}$ and $r_{2i}$;
        if $r_{2i-1} = 1$ then $I := [q_{2i-1}, b]$ else $I := [a, q_{2i-1}]$;
        if $r_{2i-1} = 1$ and $r_{2i} = 0$ then $\text{FoundIt} := \text{True}$;
    else  // once guessed correctly, proceed with a fake bisection
        Sample $R \sim \text{Bernoulli}(1/2)$;
        if $R = 1$ then $I := [q_{2i-1}, b]$ else $I := [a, q_{2i-1}]$;
    end
end
$i := 2L + 1$;
while $|I| > \epsilon$ do  // run (potentially fake) bisection on $I$
    $q_i := \text{midpoint of } I = [a, b]$;
    if not FoundIt then
        if $r_i = 1$ then $I := [q_i, b]$ else $I := [a, q_i]$;
    else
        Sample $R \sim \text{Bernoulli}(1/2)$;
        if $R = 1$ then $I := [q_i, b]$ else $I := [a, q_i]$;
    end
    $i := i + 1$;
end
at least $L$ (See the derivations around (11) for details). Algorithm 3 achieves the query complexity of $2L + \lceil \log(\delta/\epsilon) \rceil$. See Section 6.2 for a rigorous proof.

**Algorithm 3**: Our querying strategy in deterministic case when $\delta > 2^{-L}$

```plaintext
q_1 := 0; q_2 := \epsilon; // submit initial guess $X^* \in [0, \epsilon)$
K := an integer solution in \{0, 1, ..., L - 1\} to $\ell_K = 2^{-K}/(L - K) \in [\delta, 2\delta]$;
if $r_1 = 1$ and $r_2 = 0$ then FoundIt := True else FoundIt := False;
I := [0, 1];
for $i=2$ to $K+1$ do // submit the next $K$ guesses via bisection
    $q_{2i-1}$ := midpoint of $I = [a, b]$;
    $q_{2i}$ := $q_{2i-1} + \epsilon$;
    if not FoundIt then
        if $r_{2i-1} = 1$ then $I := [q_{2i-1}, b]$ else $I := [a, q_{2i-1}]$;
        if $r_{2i-1} = 1$ and $r_{2i} = 0$ then FoundIt := True;
    else
        Sample $R \sim$ Bernoulli(1/2);
        if $R = 1$ then $I := [q_{2i-1}, b]$ else $I := [a, q_{2i-1}]$;
    end
end
for $i = (K+2)$ to $L$ do // submit the next $L - K - 1$ guesses via grid search
    $q_{2i-1}$ := $a + \ell_K(i - K - 1)$ where $a$ is the left endpoint of $I$;
    $q_{2i}$ := $q_{2i-1} + \epsilon$;
    if $r_{2i-1} = 1$ and $r_{2i} = 0$ then FoundIt := True;
end
i := 2$L$ + 1;
while $|I| > \epsilon$ do // run (potentially fake) bisection on $I$
    $q_i$ := midpoint of $I = [a, b]$;
    if not FoundIt then
        if $r_i = 1$ then $I := [q_i, b]$ else $I := [a, q_i]$;
    else
        Sample $R \sim$ Bernoulli(1/2);
        if $R = 1$ then $I := [q_i, b]$ else $I := [a, q_i]$;
    end
    i := i + 1;
end
```

### 5 Lower bound proof strategies

In this section, we introduce our lower bound proof strategies; the rigorous proofs are deferred to Section 6.

#### 5.1 Bayesian setting

The lower bound is shown by constructing an intelligent adversary so that the learner cannot disguise the location of $X^*$ without a certain number of queries. The adversary strategy considered...
in [19] is called proportional-sampling, which samples from all the queries proportionally. In particular, given an observed query sequence $q_1, \ldots, q_n$, the proportional-sampling estimator is defined as $\tilde{X} = q_J$, where $J \sim \text{Unif}\{1, \ldots, n\}$.

We argue that it is more advantageous for the adversary to apply the following truncated proportional-sampling: disregard the first $K = \lfloor \log(1/(L\delta)) \rfloor$ queries and proportionally sample from $q_{K+1}, \ldots, q_n$. Even though the adversary seems to throw away useful information, since the first $K$ queries are very unlikely to be close to $X^*$, it follows that $n > K$ and the adversary in fact achieves more accurate estimation of $X^*$.

For any querying strategy that is $(\delta, L)$-private, it must satisfy $\mathbb{P}\{|\tilde{X} - X^*| \leq \delta/2\} \leq 1/L$ for all adversary strategies. Suppose the adversary’s estimator $\tilde{X}$ is obtained through the truncated proportional-sampling, then we have

$$\mathbb{P}\{|\tilde{X} - X^*| \leq \delta/2\} = \frac{\sum_{i=K+1}^{n} \mathbb{P}\{|q_i - X^*| \leq \delta/2\}}{n-K}. \quad (4)$$

The numerator can be interpreted as the expected number of queries among $q_{K+1}, \ldots, q_n$ that are in $I = [X^* - \delta/2, X^* + \delta/2]$. Loosely speaking, since $I$ is a length-$\delta$ interval, the learner needs to submit at least $\log(\delta/\epsilon)$ queries in $I$ to estimate $X^*$ within $\epsilon$-accuracy. Out of these $\log(\delta/\epsilon)$ queries, we show that they are almost all taken from $q_{K+1}, \ldots, q_n$ because with high probability, the first $K$ queries cannot approximate $X^*$ well. As a result, $\log(\delta/\epsilon)$ roughly serves as a lower bound for the numerator in (4). Deduce from $\mathbb{P}\{|\tilde{X} - X^*| \leq \delta/2\} \leq 1/L$ that a lower bound for $n$ is $K + L \log(\delta/\epsilon)$, which only differs from the precise lower bound in Theorem 1 by an additive factor of $2L+1$. See Section 6.1 for a rigorous lower bound proof.

### 5.2 Deterministic setting

Under the deterministic setting, it was shown in [17, Theorem 4.1] that the optimal query complexity has the lower bound

$$N(\epsilon, \delta, L) \geq 2L + \log \frac{\delta}{\epsilon} - 4$$

Comparing to the upper bound in Theorem 2, this lower bound is almost tight when $\delta \geq 2^{-L}$. For completeness we briefly sketch their proof here. Fix an $\epsilon$-accurate and $(\delta, L)$-private querying strategy $\phi$ and let $I = [0, \delta]$. On the one hand, for some $X^* \in I$, there are at least $\log(\delta/\epsilon)$ queries
in \( I \) by the optimality of the bisection search method. On the other hand, note that for a point \( x \) to belong to the information set \( \mathcal{I}(\tilde{q}) \), there must be two queries that are at most \( \epsilon \) apart on opposite sides of \( x \); otherwise, the learner cannot be \( \epsilon \)-accurate. Since the \( \delta \)-covering number of the information set is at least \( L \), there are at least \( L \) pairs of queries that are at most \( \epsilon \) apart. The interval \( I \) is of length only \( \delta \), so almost all these \( L \) pairs of queries are outside of \( I \), yielding a total of roughly \( 2L + \log(\delta/\epsilon) \) queries.

For the lower bound proof of Theorem 2 we only need to consider the \( \delta \leq 2^{-L} \) case. Fix any querying strategy \( \phi \) that is both \( \epsilon \)-accurate and \((\delta,L)\)-private. Write \( \mathcal{Q}(X^*) \) for the set of queries when \( X^* \) is the truth. We want to show there is at least one \( X^* \) for which \( |\mathcal{Q}(X^*)| \geq L + \log(1/\epsilon) - 8 \).

On a high level, we prove this lower bound by finding an interval \( I \) of length \( 2\delta \) and \( X^* \in I \) such that if \( X^* \) is the true value, there are at least \( \log(\delta/\epsilon) \) queries in \( I \) and \( L + \log(\delta/\epsilon) - 8 \) queries outside of \( I \). Within \( I \), it takes at least \( \log(\delta/\epsilon) \) queries to approximate \( X^* \) up to \( \epsilon \) accuracy. Outside of \( I \), we show that there are roughly \( \log(1/\delta) \) queries that are at least \( \delta \) away from each other. At the same time, there are at least roughly \( L \) pairs of queries that are at most \( \delta \) apart to ensure that the \( \delta \)-covering number of the information set is at least \( L \), which contribute around \( L \) extra queries outside of \( I \). Hence there are at least around \( \log(\delta/\epsilon) + \log(1/\delta) + L = \log(1/\epsilon) + L \) queries needed by \( \phi \).

# 6 Proofs

## 6.1 Analysis under the Bayesian setting

**Proof of Theorem 1. Upper bound:** From (3), the multistage querying strategy described in Section 4.1 achieves the upper bound in the statement of Theorem 1. It suffices to show that it is both \( \epsilon \)-accurate and \((\delta,L)\)-private. First we establish accuracy. From the responses to all the queries, the learner can narrow down the possible values of \( X^* \) to an interval \( I^{(\text{final})} \) of length

\[
|I^{(\text{final})}| = \frac{1}{L} 2^{-(K_1 + K_2)} = \frac{1}{L} 2^{-([\log(1/L\delta)] + \log(\delta/\epsilon)) + 1} \leq \frac{1}{L} 2^{- \log(1/L\epsilon)} = \epsilon. \tag{5}
\]

The learner can then take \( \tilde{X} \) to be the midpoint of this interval so that \(|\tilde{X} - X^*| \leq \epsilon/2\).

Next we show privacy. Recall that the learner performs parallel bisections on the \( L \) intervals \( I_1, \ldots, I_L \). Since the adversary only observes the queries and the query strategy \( \phi \), it learns that \( X^* \) is contained in one of \( L \) intervals \( J_1, \ldots, J_L \) where \( J_j = [a_j, b_j] \subseteq I_j \). But it cannot tell which of them \( X^* \) is in. Therefore it cannot guess the location of \( X^* \) with probability higher than \( 1/L \). More precisely, the posterior distribution of \( X^* \) given all the queries and the query strategy is uniform over the union of \( J_1, \ldots, J_L \). Use \(| \cdot |\) to denote the Lebesgue measure of subsets of \([0,1]\). We have

\[
\mathbb{P}\{|\tilde{X} - X^*| \leq \delta/2 \mid \text{the queries}\} = \frac{(\bigcup_{j \leq L} J_j) \cap \left[ \tilde{X} - \delta/2, \tilde{X} + \delta/2 \right]}{|\bigcup_{j \leq L} J_j|} \tag{6}
\]

Since the queries on \( I_1, \ldots, I_L \) are exact copies of each other, \( J_1, \ldots, J_L \) are also equidistant translations on the real line. The left endpoints \( a_1, \ldots, a_L \) of \( J_1, \ldots, J_L \) satisfy \( a_{i+1} = a_i + |I_i| \) for all \( i \) where \(|I_1| = 2^{-K_1}/L \geq \delta \). Moreover, note that the lengths of all \( J_i \) are equal, and because the adversary does not observe the response to the last batch of queries, \(|J_i| = 2|I^{(\text{final})}|\). From (5) we have \(|J_i| \leq 2\epsilon \). Therefore under the assumption that \( \delta \geq 2\epsilon \), any interval of length \( \delta \) can only intersect with \( \bigcup_i J_i \) on a set of Lebesgue measure at most \(|J_i| \). Deduce that the right hand side of (6) is bounded by \(|J_i|/|\bigcup_i J_i| = 1/L \). Therefore

\[
\mathbb{P}\{|\tilde{X} - X^*| \leq \delta/2\} = \mathbb{E}\left( \mathbb{P}\{|\tilde{X} - X^*| \leq \delta/2 \mid \text{the queries}\} \right) \leq 1/L.
\]
**Lower bound:** Suppose $\phi$ is a $\epsilon$-accurate and $(\delta, L)$-private strategy that submits at most $n$ queries. Denote $n(X^*, Y)$ as the number of queries submitted when $X^*$ is the truth and the random seed is $Y$, so $n = \max_{X,Y} n(X^*, Y)$. The goal is to bound $n$ from below. Consider the querying strategy $\hat{\phi}$ that concatenates trivial queries at 0 to the query sequence so that the length of query sequence is always $n$, i.e., $\hat{q}_i = q_i$ for $i \leq n(X^*, Y)$ and $\hat{q}_i = 0$ for $n(X^*, Y) < i \leq n$. Clearly $\hat{\phi}$ is also $\epsilon$-accurate and $(\delta, L)$-private, because the trivial queries at 0 does not provide the adversary with any extra information. Moreover the maximum number of queries submitted by $\hat{\phi}$ equals that submitted by $\phi$. Hence for the rest of this proof, without loss of generality, we can assume that the learner always submits exactly $n$ queries under $\phi$.

Since $\phi$ is $(\delta, L)$-private, we have $\mathbb{P}\{|\hat{X} - X^*| \leq \delta/2\} \leq 1/L$ for each adversary $\hat{X}$. Consider the adversary that adopts the **truncated proportional-sampling** strategy described in section 5.1: let $\hat{X} = q_J$ where $J \sim \text{Unif}\{K + 1, \ldots, n\}$. Choose $K = \lceil \log(1/(L\delta)) \rceil $. Let us point out that $n$ must be larger than $K$ so truncated proportional-sampling can be run. We will show later in the proof that $n > K$ always holds for any strategy $\phi$ that is $\epsilon$-accurate. By construction,

$$\mathbb{P}\{|\hat{X} - X^*| \leq \delta/2\} = \mathbb{E} \frac{\sum_{i=K+1}^{n} \mathbb{I}\{|q_i - X^*| \leq \delta/2\}}{n-K} \leq \frac{1}{L}.$$  

Deduce that

$$n \geq K + L \left( \sum_{i=1}^{n} \mathbb{P}\{|q_i - X^*| \leq \delta/2\} - \sum_{i=1}^{K} \mathbb{P}\{|q_i - X^*| \leq \delta/2\} \right).$$

We claim that

1. $\sum_{i \leq n} \mathbb{P}\{|q_i - X^*| \leq \delta/2\} \geq \log(\delta/4\epsilon)$.  
2. $\sum_{i \leq K} \mathbb{P}\{|q_i - X^*| \leq \delta/2\} \leq 1/L$.  

(iii) $n > K$, so that the truncated proportional-sampling strategy is valid.

The desired lower bound immediately follows.

**Proof of (i) and (iii):** The statement (i) claims that on average, there are at least $\log(\delta/4\epsilon)$ queries in the interval $[X^* - \delta/2, X^* + \delta/2]$. One would expect this to be true because $[X^* - \delta/2, X^* + \delta/2]$ is an interval of length $\delta$. In order for the learner to achieve $\epsilon$-accuracy, it needs to submit at least $\log(\delta/\epsilon)$ queries by optimality of the bisection method. Next we make this argument rigorous. The randomness of the interval $[X^* - \delta/2, X^* + \delta/2]$ complicates the proof. We will instead show something stronger than (i). We claim that for each fixed interval $I \subseteq [0, 1]$, we have

$$\sum_{i \leq n} \mathbb{P}\{q_i \in I \mid X^* \in I\} \geq \log(|I|/2\epsilon). \quad (7)$$

To see why (i) follows from (7), note that for each interval $I$ of length $\delta/2$,

$$\sum_{i \leq n} \mathbb{P}\{|q_i - X^*| \leq \delta/2 \mid X^* \in I\} \geq \sum_{i \leq n} \mathbb{P}\{q_i \in I \mid X^* \in I\} \geq \log(|I|/2\epsilon) = \log(\delta/4\epsilon).$$

Moreover, claim (iii) also follows from (7) by taking $I = [0, 1]$:

$$n = \sum_{i \leq n} \mathbb{P}\{q_i \in [0, 1]\} \geq \log(1/2\epsilon) > \lceil \log(1/(L\delta)) \rceil = K,$$

where the strict inequality holds because by assumption $2\epsilon \leq \delta$ and $L \geq 2$. 

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It remains to show (7). Since $\phi$ is $\epsilon$-accurate, we have
\[
\mathbb{P} \left\{ |\hat{X} - X^*| > \epsilon/2 \mid X^* \in I, Y = y \right\} = 0
\]
for all but a negligible (zero-measure) set of the random seed $Y$, denoted as $\mathcal{N}^y$. For $y \notin \mathcal{N}^y$, conditioning on $Y = y$, the estimator $\hat{X}$ is only a function of the responses $r_1, ..., r_n$. Further conditioning on $X^* \in I$, since $X^*$ is independent from the random seed $Y$, $X^*$ is distributed uniform in $I$. By the continuous version of Fano inequality [4, Proposition 2],
\[
\mathbb{P} \left\{ |\hat{X} - X^*| > \epsilon/2 \mid X^* \in I, Y = y \right\} \geq 1 - \frac{I(X^*; r_1, ..., r_n \mid X^* \in I, Y = y) + \log 2}{\log(|I|/\epsilon)}.
\]
Hence
\[
H(r_1, ..., r_n \mid X^* \in I, Y = y) \geq I(X^*; r_1, ..., r_n \mid X^* \in I, Y = y) \geq \log(|I|/\epsilon) - \log 2 = \log(|I|/2\epsilon).
\]
Using the entropy chain rule, the left hand side can also be written as
\[
H(r_1, ..., r_n \mid X^* \in I, Y = y) = H(r_1 \mid X^* \in I, Y = y) + \sum_{i=1}^{n-1} H(r_{i+1} \mid X^* \in I, Y = y, r_1 = \rho_1, ..., r_i = \rho_i). \tag{9}
\]
Expand each summand:
\[
H(r_{i+1} \mid X^* \in I, Y = y) = \sum_{\rho_1, ..., \rho_i} \mathbb{P} \{ r_1 = \rho_1, ..., r_i = \rho_i \mid X^* \in I, Y = y \} \log \left( \frac{\mathbb{P} \{ r_1 = \rho_1, ..., r_i = \rho_i, X^* \in I, Y = y \}}{\mathbb{P} \{ r_1 = \rho_1, ..., r_i = \rho_i \}} \right). \tag{10}
\]
Write $I = [a, b]$. On the event $X^* \in I$, if $q_{i+1} = \phi(r_1, ..., r_i, y)$ is smaller than $a$, then $r_{i+1} = 1$. Similarly if $q_{i+1} > b$, then $r_{i+1} = 0$. In other words, the value of $r_{i+1}$ is completely determined by $\rho_1, ..., \rho_i$ if $q_{i+1} \notin I$ and $X^* \in I$. Hence (10) equals
\[
\sum_{\rho_1, ..., \rho_i : q_{i+1} \notin I} \mathbb{P} \{ r_1 = \rho_1, ..., r_i = \rho_i \mid X^* \in I, Y = y \} \log \left( \frac{\mathbb{P} \{ q_{i+1} \in I \mid X^* \in I, Y = y \}}{\mathbb{P} \{ q_{i+1} \in I \}} \right).
\]
With $Y = y$ fixed, we have $q_1 = f_0(y)$ is deterministic. Similarly argue that $H(r_1 \mid X^*, Y = y) \leq 1 \{ q_1 \in I \}$. Combine with (8) and (9) to deduce that
\[
\sum_{i \leq n} \mathbb{P} \{ q_i \in I \mid X^* \in I, Y = y \} \geq H(r_1, ..., r_n \mid X^* \in I, Y = y) \geq \log(|I|/2\epsilon).
\]
The above holds for all $y \notin \mathcal{N}^y$. Since $\mathcal{N}^y$ is a negligible set, we have
\[
\sum_{i \leq n} \mathbb{P} \{ q_i \in I \mid X^* \in I \} = \int_{[0,1] \setminus \mathcal{N}^y} \sum_{i \leq n} \mathbb{P} \{ q_i \in I \mid X^* \in I, Y = y \} dy \geq \log(|I|/2\epsilon).
\]
The proof of (7) is complete.

Proof of (ii): To show (ii) we introduce the notion of learner intervals, which stands for the sequence of intervals that the learner knows $X^*$ is in, as the learner submits queries sequentially.
Start from $I_0 = [0, 1]$. If $r_1 = 1$, then the learner learns that $X^* \in [q_1, 1]$ and $I_1$ is defined as $[q_1, 1]$. Otherwise $I_1 = [0, q_1]$. For all $i$,

$$\mathbb{P}\left\{|q_i - X^*| \leq \frac{\delta}{2}\right\} = \mathbb{E}\left[\mathbb{P}\left\{|q_i - X^*| \leq \frac{\delta}{2} \mid r_1, \ldots, r_{i-1}\right\}\right]$$

$$= \mathbb{E}\left[\frac{|I_{i-1} \cap [q_i - \delta/2, q_i + \delta/2]|}{|I_{i-1}|}\right] \leq \delta \mathbb{E}(1/|I_{i-1}|).$$

Next we show that $\mathbb{E}(1/|I_i|) \leq 2^i$ for all $i$ by induction. Suppose it is true for $i = 0, \ldots, k$. For $i = k + 1$,

$$\mathbb{E}(1/|I_{k+1}|) = \mathbb{E}\left[\mathbb{E}\left(1/|I_{k+1}| \mid r_1, \ldots, r_k\right)\right].$$

Conditioning on $r_1, \ldots, r_k$, the learner interval $I_k$ is deterministic and so is $q_{k+1}$. Let $I_k = [a_k, b_k]$. There are three possibilities for $I_{k+1}$:

1. $q_{k+1} \notin I_k$. In this case the $r_{k+1}$ provides no additional information on the location of $X^*$. Therefore $I_{k+1} = I_k$.

2. $q_{k+1} \in I_k$ and $R_{k+1} = 1$. The learner learns that $X^* \geq q_{k+1}$ and $I_{k+1} = [q_{k+1}, b_k]$.

3. $q_{k+1} \in I_k$ and $R_{k+1} = 0$. In this case $I_{k+1} = [a_k, q_{k+1}]$.

Therefore

$$\mathbb{E}\left(1/|I_{k+1}| \mid r_1, \ldots, r_k\right)$$

$$= \mathbb{I}\{q_{k+1} \notin I_k\} \frac{1}{|I_k|} + \mathbb{I}\{q_{k+1} \in I_k\} \mathbb{P}\{X^* \geq q_{k+1} \mid r_1, \ldots, r_k\} \frac{1}{b_k - q_{k+1}}$$

$$+ \mathbb{I}\{q_{k+1} \in I_k\} \mathbb{P}\{X^* < q_{k+1} \mid r_1, \ldots, r_k\} \frac{1}{q_{k+1} - a_k}$$

$$= \mathbb{I}\{q_{k+1} \notin I_k\} \frac{1}{|I_k|} + \mathbb{I}\{q_{k+1} \in I_k\} \frac{1}{|I_k|} + \mathbb{I}\{q_{k+1} \in I_k\} \frac{1}{|I_k|} \leq \frac{2}{|I_k|}.$$ 

Hence $\mathbb{E}(1/|I_{k+1}|) \leq \mathbb{E}(2/|I_k|) \leq 2 \cdot 2^k = 2^{k+1}$. Deduce that $\mathbb{P}\{|q_i - X^*| \leq \frac{\delta}{2}\} \leq \delta \mathbb{E}(1/|I_{i-1}|) \leq \delta 2^{i-1}$ for all $i$. Therefore

$$\sum_{i=1}^{K} \mathbb{P}\left\{|q_i - X^*| \leq \frac{\delta}{2}\right\} \leq \delta \sum_{i=1}^{K} 2^{i-1} \leq \delta 2^K \leq 1/L,$$

where the last inequality is from $K = \lceil \log(1/(L\delta)) \rceil \leq \log(1/(L\delta))$. 

\[\Box\]

6.2 Analysis under the deterministic setting

**Proof of Theorem 2. Upper bound:** First consider the case $\delta \leq 2^{-L}$. Under algorithm 2, phase 1 contains $2L$ queries. In phase 2 the learner conducts a bisection search within an interval of length $2^{-L}$. It takes $\lceil 2^{-L}/\epsilon \rceil$ queries to achieve $\epsilon$-accuracy. The total number of queries submitted is $2L + \lceil \log(1/\epsilon) \rceil$.

Because the guesses $I_1, \ldots, I_L$ are all of length $\epsilon$, algorithm 2 is $\epsilon$-accurate. It suffices to show it is also $(\delta, L)$-private. As argued in Section 4.2, the adversary cannot rule out any of the $L$ possibilities that $X^* \in I_i$ for $i = 1, \ldots, L$, the information set contains the union of $I_1, \ldots, I_L$. That is, for each query sequence $q$,

$$\mathcal{I}(q) \supseteq \cup_{i \leq L} [q_{2i-1}, q_{2i}).$$
When $\delta \leq 2^{-L}$, these $L$ intervals do not overlap. Since their left endpoints are submitted in via a bisection search, legitimate or fake, they are at least $2^{-L} \geq \delta$ apart from each other. Therefore the $\delta$-covering number for $\mathcal{I}(q)$ is at least $L$. We have shown that algorithm 2 is $(\delta, L)$-private.

Next consider the case $\delta > 2^{-L}$. Again algorithm 3 is clearly $\epsilon$-accurate. To show it is also $(\delta, L)$-accurate, recall that algorithm 3 is designed so that the $L$ guesses in phase 1 satisfy

(i) The intervals $I_i = [q_{2i-1}, q_{2i})$, $i = 1, \ldots, L$ do not overlap, and their left endpoints are at least $\delta$ from each other;

(ii) After phase 1, the learner can always narrow down the possibilities for $X^*$ to an interval of length at most $2\delta$.

We claim that (i) ensures $(\delta, L)$-privacy. As in the $\delta \leq 2^{-L}$ case, we have for each $q$, $\mathcal{I}(q) \supseteq \bigcup_{i \leq L} [q_{2i-1}, q_{2i})$. Assuming (i), the $\delta$-covering number of $\bigcup_{i \leq L} [q_{2i-1}, q_{2i})$ is at least $L$.

Assuming (ii), the learner only needs to submit at most $\lceil \log(2\delta/\epsilon) \rceil$ queries to achieve $\epsilon$-accuracy in phase 2. The total number of queries submitted under this strategy is at most $2L + \lceil \log(\delta/\epsilon) \rceil + 1$. Moreover, as we can see from Algorithm 3 the learner always submits $q_1 = 0$. Omit this first trivial query to obtain the desired query complexity upper bound $2L + \lceil \log(\delta/\epsilon) \rceil$.

We still to show (i) and (ii) are satisfied by algorithm 3. The first $K$ guesses locate $X^*$ within an interval $I$ of length $2^{-K}$. The remaining $L - K - 1$ odd queries then divide $I$ into $L - K$ subintervals of equal length. Therefore the closest pair of odd queries among $q_1, q_3, \ldots, q_{2L-1}$ are at distance $2^{-K}/(L - K)$. In phase 2, the learner conducts bisection in one of the $L - K$ subintervals, which is also of length $2^{-K}/(L - K)$. Therefore (i) and (ii) translate to $\delta \leq 2^{-K}/(L - K) \leq 2\delta$. It remains to show that we can find at least one $K \in \{0, 1, \ldots, L - 1\}$ for which

$$\ell_K := \frac{2^{-K}}{L - K} \in [\delta, 2\delta].$$

(11)

Observe that

1. $\ell_0 = 1/L \geq \delta$;
2. $\ell_{L-1} = 2^{-(L-1)} \leq 2\delta$;
3. for all $K < L - 1$,

$$\frac{\ell_K}{\ell_{K+1}} = \frac{2^{-K}}{2^{-(K+1)}} \cdot \frac{L - K}{L - K - 1} \leq 2.$$

These facts above ensure that there is at least one solution to (11) in $\{0, 1, \ldots, L - 1\}$.

**Lower bound:** When $\delta > 2^{-L}$, the lower bound has already been proved in [17, Theorem 4.1]. Thus we only need to prove the lower bound for the $\delta \leq 2^{-L}$ case. It suffices to show the lower bound holds for all realizations of the random seed $Y$ so the dependences on $Y$ are suppressed for the rest of the proof. Fix any querying strategy $\phi$ that is both $\epsilon$-accurate and $(\delta, L)$-private. Recall that $Q(X^*)$ stands for the set of queries when the true value is $X^*$. We want to show there is at least one $X^*$ for which $|Q(X^*)| \geq L + \log(1/\epsilon) - 8$. To this end, we will prove the following claims:

(i) There exists an interval $I$ of length $2\delta$ and $\tilde{Q} = \{\tilde{q}_1, \ldots, \tilde{q}_K\}$ where $K \geq \log(1/\delta) - 3$ and $|\tilde{q}_i - \tilde{q}_j| > \delta$ for all $i \neq j$, such that for each $X^* \in I$, $Q(X^*) \cap I \supseteq \tilde{Q}$.

(ii) For each interval $I$ of length $2\delta$ and each $X^* \in I$, there exist at least $L - 5$ distinct pairs of queries $\{s_1, t_1\}, \ldots, \{s_{L-5}, t_{L-5}\} \subseteq Q(X^*) \cap I$ for which $|s_i - t_i| \leq \epsilon$ for all $i$. 

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(iii) For each interval $I$ of length $2\delta$, there exists $X^* \in I$ such that $Q(X^*)$ contains at least $\log(\delta/\epsilon)$ queries in $I$.

Claims (i) and (ii) together imply that there exists an interval $I$ of length $2\delta$ such that for all $X^* \in I$,

$$Q(X^*) \setminus I \supseteq \tilde{Q} \cup \left( \bigcup_{i \leq L-5} \{ s_i, t_i \} \right).$$

Since all members of $\tilde{Q}$ are at least $\delta$-apart and $|s_i - t_i| \leq \epsilon$, at least one of $s_i$ and $t_i$ is outside of $\tilde{Q}$. To show that on top of $\tilde{Q}$, each pair $\{ s_i, t_i \}$ contributes at least one extra member to $Q(X^*) \setminus I$, we only need to rule out the case where two pairs $\{ s_i, t_i \}$ and $\{ s_j, t_j \}$ are such that $s_i, s_j \in \tilde{Q}$ and $t_i = t_j$. This can not happen because

$$\delta < |s_i - s_j| \leq |s_i - t_i| + |s_j - t_i| = |s_i - t_i| + |s_j - t_j| \leq \epsilon + \epsilon,$$

which contradicts with the assumption $\delta \geq 2\epsilon$. Thus $Q(X^*) \setminus I$ contains at least $K + L - 5$ distinct members. From claim (iii) there exists $X^* \in I$ for which $|Q(X^*) \cap I| \geq \log(\delta/\epsilon)$. We have

$$|Q(X^*)| = |Q(X^*) \cap I| + |Q(X^*) \setminus I| \geq \log(\delta/\epsilon) + K + L - 5 \geq L + \log \frac{1}{\epsilon} - 8,$$

which equals $2L + \log(\max\{2^{-L}, \delta\}/\epsilon) - 8$ when $\delta \leq 2^{-L}$. It remains to prove the three claims.

Proof of (i): To prove this claim, we first construct a subsequence $\tilde{q}$ of $q$ where all the queries in $\tilde{q}$ are at least $\delta$ apart from each other, for any given $X^* \in [0, 1]$.

Let $\tilde{q}_1 = q_1$. If $X^* \in [\tilde{q}_1 - \delta, \tilde{q}_1 + \delta]$, then declare the construction finished, i.e. the subsequence $\tilde{q} = (\tilde{q}_1)$ is of length one. Otherwise look at $q_2 = \phi_1(r_1)$. If $q_2 \in [\tilde{q}_1 - \delta, \tilde{q}_1 + \delta]$, then $\tilde{q}_1$ and $q_2$ must be on the same side of $X^*$ and $r_2 = 1\{X^* \geq q_2\}$ must be equal to $r_1$. Proceed to look at $q_3 = \phi_2(r_1, r_2) = \phi_2(r_1, r_1), q_4 = \phi_3(r_1, r_1, r_1)$ and so on, until $q_i \notin [\tilde{q}_1 - \delta, \tilde{q}_1 + \delta]$. Let $\tilde{q}_2 = q_i$. Similarly define the rest of $\tilde{q}$ as follows: For $k \geq 2$ if $\tilde{q}_k$ is chosen to be $q_i$, then let $\tilde{q}_{k+1} = q_{i_{k+1}}$,

$$i_{k+1} = \min\{ j > i_k : q_j \notin \cup_{k' \leq k} [\tilde{q}_{k'} - \delta, \tilde{q}_{k'} + \delta] \}.$$

Repeat this process until $[\tilde{q}_k - \delta, \tilde{q}_k + \delta]$ contains $X^*$. Note that such a $k$ always exists, as $\phi$ is $\epsilon$-accurate and hence there exists at least one query that is within $\epsilon$ distance to $X^*$.

Let $\tilde{r}_i = 1\{X^* \geq \tilde{q}_i\}$. Next we argue that $\tilde{q}$ is completely determined by $\tilde{r}$. Indeed, given $\tilde{r} = (\tilde{r}_1, ..., \tilde{r}_k)$, we have $\tilde{q}_j = q_{i_j}$ for all $j \leq k$, where $i_1 = 1$ and

$$i_2 = \min\{ j > i_1 : \phi_{j-1}(\tilde{r}_1, ..., \tilde{r}_1) \notin [\tilde{q}_1 - \delta, \tilde{q}_1 + \delta] \}.$$

Thus $\tilde{q}_2 = q_{i_2} = \phi_{i_2-1}(\tilde{r}_1, ..., \tilde{r}_1)$. To determine $i_3$, inspect $q_{i_2+1} = \phi_{i_2}(r_1, ..., r_{i_2}) = \phi_{i_2}(\tilde{r}_1, ..., \tilde{r}_1, \tilde{r}_2)$. If $q_{i_2+1} \notin \cup_{j=1,2}[\tilde{q}_j - \delta, \tilde{q}_j + \delta]$, the we have $i_3 = i_2 + 1$. Otherwise if $q_{i_2+1} \in [\tilde{q}_1 - \delta, \tilde{q}_1 + \delta]$, then we have $r_{i_2+1} = \tilde{r}_1$ and $q_{i_2+2} = \phi_{i_2+1}(\tilde{r}_1, ..., \tilde{r}_1, \tilde{r}_2, \tilde{r}_1)$; similarly if $q_{i_2+1} \in [\tilde{q}_2 - \delta, \tilde{q}_2 + \delta]$, then $q_{i_2+2} = \phi_{i_2+1}(\tilde{r}_1, ..., \tilde{r}_1, \tilde{r}_2, \tilde{r}_2)$. As such we can reconstruct the queries $q_{i_2+3}, q_{i_2+4}$ and so on until we find $j > i_2$ where $\phi_j \notin \cup_{j=1,2}[\tilde{q}_j - \delta, \tilde{q}_j + \delta]$. Then we have determined $i_3 = j$ and $\tilde{q}_3 = q_j$, which is completely determined by $(\tilde{r}_1, \tilde{r}_2)$. Following the same argument, the entire $\tilde{q}$ sequence can be reconstructed from $\tilde{r}$. Consequently,

$$||\{\tilde{q} : X^* \in [0, 1]\}|| \leq ||\{\tilde{r} : X^* \in [0, 1]\}||.$$

Suppose $K + 1$ is the maximum length of $\tilde{q} = \tilde{q}(X^*)$ among all $X^* \in [0, 1]$. Then the total number of distinct binary $\tilde{r}$ sequences is at most $\sum_{k \leq K+1} 2^k < 2^{K+2}$. In addition if $\tilde{q}$ is of length $k$, then $X^* \in [\tilde{q}_k - \delta, \tilde{q}_k + \delta]$ by construction. Hence

$$1 = ||[0, 1]|| \leq \bigcup_{X^* \in [0, 1]} [\tilde{q}_k - \delta, \tilde{q}_k + \delta] \leq 2\delta \cdot ||\{\tilde{q} : X^* \in [0, 1]\}|| \leq 2\delta \cdot 2^{K+2}.$$
Deduce that \( K \geq \log(1/\delta) - 3 \). In other words, there exists \( X^* \in [0, 1] \) for which \( \tilde{q} \) is of length \( k \) where \( k \geq K + 1 \geq \log(1/\delta) - 2 \). We choose \( I = [\tilde{q}_k - \delta, \tilde{q}_k + \delta] \) for such \( \tilde{q} \) and show that it satisfies the statement in (i). By construction all the queries in \( \tilde{q} \) are more than \( \delta \) apart; therefore, all the queries in \( \tilde{q} \) except \( \tilde{q}_k \) are all outside of \( I \). As a result for all \( X \in I \) and \( i \leq k - 1 \), \( \mathbb{I}\{X \geq \tilde{q}_i\} \) yields the same response as \( \mathbb{I}\{X^* \geq \tilde{q}_i\} \). Deduce that \( \tilde{q}(X) = \tilde{q}(X^*) \) for all \( X \in I \). To complete the proof of (i), take \( \tilde{Q} = \{\tilde{q}_1, ..., \tilde{q}_{k-1}\} \) to obtain a subset of \( \mathcal{Q}(X^*) \setminus I \) of size at least \( K \geq \log(1/\delta) - 3 \).

Proof of (ii): This part of the proof borrows the idea from the lower bound proof in [17, Theorem 4.1]. For \( q = (q_1, ..., q_n) \), let \( \mathcal{Q}(X^*) = \{q_1, ..., q_n, 0, 1\} \). The key observation is that for each \( x \) in the information set \( \mathcal{I}(q) \), there must be two queries \( s, t \in \mathcal{Q}(X^*) \) with \( s \leq x \), \( t > x \) and \( t - s \leq \epsilon \). Otherwise when \( x = \epsilon \) the learner could not have achieved \( \epsilon \)-accuracy through the query sequence \( q \). The inclusion of 0,1 in \( \mathcal{Q}(X^*) \) is because even if they are never queried, they could still serve in these \((s, t)\) pairs.

Let

\[
\mathcal{P} = \{(s, t) : s, t \in \mathcal{Q}(X^*), 0 < t - s \leq \epsilon\}
\]

denote the set of all pairs of queries that are no more than \( \epsilon \)-apart. We have

\[
\mathcal{I}(q) \subseteq \cup_{(s, t) \in \mathcal{P}} [s, t].
\]

From the definition of \((\delta, L)\)-privacy, the \( \delta \)-covering number of \( \cup_{(s, t) \in \mathcal{P}} [s, t] \) is at least \( L \), which immediately gives \( |\mathcal{P}| \geq L \). However since we want to lower bound the number of pairs \((s, t)\) where both \( s \) and \( t \) are outside of \( I \), the proof is slightly more complicated. Write \( I = [a, b] \). If one of \( s, t \) is in \( I \), then \( [s, t] \subseteq [a - \epsilon, b + \epsilon] \subseteq [a - \delta/2, b + \delta/2] \). This is an interval of length \( 3\delta \). We also need to discount the pairs that use 0 or 1 as one of the endpoints. Let

\[
\tilde{\mathcal{P}} = \{(s, t) : s, t \in \mathcal{Q}(X^*) \setminus (I \cup \{0, 1\}), 0 < t - s \leq \epsilon\}
\]

\[
\supseteq \{(s, t) : [s, t] \subseteq [0, 1] \setminus ([a - \delta/2, b + \delta/2] \cup [0, \delta] \cup [1 - \delta, 1])\}.
\]

The \( \delta \)-covering number for \([a - \delta/2, b + \delta/2] \cup [0, \delta] \cup [1 - \delta, 1]) \) is at most 5. Deduce that the \( \delta \)-covering number for \( \cup_{(s, t) \in \tilde{\mathcal{P}}} [s, t] \) is at least \( L - 5 \). Thus \( |\tilde{\mathcal{P}}| \geq L - 5 \).

Proof of (iii): The part of the proof is similar to the proof of (i). We take \( \tilde{q}(X^*) \) to be the subsequence of \( q(X^*) \) that contains all the queries in \( \mathcal{Q}(X^*) \) that are in \( I \). Let \( J(X^*) \) be the interval formed by the two queries in \( q(X^*) \) to the left and right of \( X^* \) that are the closest to \( X^* \). For all \( X^* \in I \), \( X^* \in J(X^*) \) and thus \( I \subseteq \cup_{X^* \in I} J(X^*) \). Since \( |I| = 2\delta \) and the querying strategy \( \phi \) is \( \epsilon \)-accurate so that \( |J(X^*)| \leq \epsilon \), we have that \( \{J(X^*) : X^* \in I\} \) contains at least \( 2\delta/\epsilon \) distinct members.

Let \( \tilde{r}_I(X^*) = \mathbb{I}\{X^* \geq \tilde{q}_I(X^*)\} \). Next we show that for each \( X^* \in I \), \( J(X^*) \) is completely determined by \( \tilde{r}(X^*) \). Indeed given any \( X^* \in I \), the responses to the queries outside of \( I \) can be deduced from their position relative to \( I \). Therefore from only \( \tilde{r}(X^*) \), which only contains responses to the queries in \( I \), one can reconstruct the entire query sequence \( q(X^*) \), from which one can infer \( J(X^*) \). Thus

\[
|\tilde{r}(X^*) : X^* \in I| \geq |J(X^*) : X^* \in I| \geq 2\delta/\epsilon.
\]

Suppose \( T \) is the maximal length of \( \tilde{q}(X^*) \) among all \( X^* \in I \). Then \( \tilde{r}(X^*) \) can take no more than \( \sum_{t \leq T} 2^t < 2^{T+1} \) distinct values. Deduce that \( T \geq \log(\delta/\epsilon) \). Recall that \( \tilde{q}(X^*) \) is a subsequence of \( q(X^*) \) and only contains queries in \( I \). Conclude that there exists \( X^* \in I \) for which the querying strategy \( \phi \) submits at least \( \log(\delta/\epsilon) \) queries that are in \( I \).
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