On Vanishing Fermat Quotients and a Bound of the Ihara Sum

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Abstract

We improve an estimate of A. Granville (1987) on the number of vanishing Fermat quotients $q_p(\ell)$ modulo a prime $p$ when $\ell$ runs through primes $\ell \leq N$. We use this bound to obtain an unconditional improvement of the conditional (under the Generalised Riemann Hypothesis) estimate of Y. Ihara (2006) on a certain sum, related to vanishing Fermat quotients. In turn this sum appears in the study of the index of certain subfields of cyclotomic fields $\mathbb{Q}(\exp(2\pi i/p^2))$.

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1 Introduction

For a prime $p$ and an integer $u$ with $\gcd(u, p) = 1$ we define the Fermat quotient $q_p(u)$ as the unique integer with

$$q_p(u) \equiv \frac{u^{p-1} - 1}{p} \pmod{p}, \quad 0 \leq q_p(u) \leq p - 1.$$ 

We also define $q_p(u) = 0$ for $u \equiv 0 \pmod{p}$.

Fermat quotients appear and play a major role in various questions of computational and algebraic number theory and thus have been studied in
a number of works, see, for example, [1, 2, 3, 5, 6, 7, 9, 10] and references therein. Amongst other properties, the $p$-divisibility of Fermat quotients $q_p(a)$ by $p$ is important for many applications and in particular, the smallest value $\ell_p$ of $u \geq 1$ with $q_p(u) \neq 0$, has been studied in a number of works, see [1, 2, 3, 5, 9]. For example, in [1], improving the previous estimate $\ell_p = O((\log p)^2)$ of Lenstra [9] (see also [3, 6, 7]), the following bounds have been given:

$$\ell_p \leq \begin{cases} (\log p)^{463/252+o(1)} & \text{for all primes } p, \\ (\log p)^{5/3+o(1)} & \text{for almost all primes } p, \end{cases}$$

(where “almost all primes $p$” means for all primes $p$ but a set of relative density zero).

Here we use some results of [1], combined with the approach of Granville [4] to obtain new estimates on the cardinality of the sets

$$Q_p(N) = \{n \leq N : q_p(n) = 0\},$$

$$R_p(N) = \{\ell \leq N : \ell \text{ prime, } q_p(\ell) = 0\},$$

which for small $N$ improve that of [4]. We apply these improvements to study the sums

$$S_p = \sum_{n \in Q_p(p)} \frac{\Lambda(n)}{n}$$

introduced by Ihara [7], where, as usual,

$$\Lambda(n) = \begin{cases} \log \ell, & \text{if } n \text{ is a power of a prime } \ell, \\ 0, & \text{otherwise}, \end{cases}$$

be the von Mangoldt function.

We note that in [7, Corollary 7], under the Generalised Riemann Hypothesis, the bound

$$S_p \leq 2 \log \log p + 2 + o(1)$$

as $p \to \infty$, has been obtained. Here we give an unconditional proof of a stronger bound.

Throughout the paper, the implied constants in the symbols ‘$O$’, and ‘$\ll$’ may occasionally depend on the real positive parameter $\alpha$ and are absolute otherwise (we recall that the notation $U \ll V$ is equivalent to $U = O(V)$).


\section{Preparations}

We recall that for any integers \(m\) and \(n\) with \(\gcd(mn, p) = 1\) we have

\[q_p(mn) \equiv q_p(m) + q_p(n) \pmod{p},\]

(2)

see, for example, [2, Equation (2)].

Let \(\mathcal{G}_p\) be the group of the \(p\)th power residues in the unit group \(\mathbb{Z}_{p^2}^*\) of the residue ring \(\mathbb{Z}_{p^2}\) modulo \(p^2\).

\textbf{Lemma 1.} For any \(u \in \mathbb{Z}_{p^2}^*\) the conditions \(q_p(u) = 0\) and \(u \in \mathcal{G}_p\) are equivalent.

\begin{proof}
Clearly \(q_p(u) = 0\) for \(u \in \mathbb{Z}_{p^2}^*\) is equivalent to \(w^{p-1} \equiv 1 \pmod{p^2}\), which in turn is equivalent to \(u \in \mathcal{G}_p\).
\end{proof}

Let \(T_p(K)\) be the number of \(w \in [1, K]\) such that their residues modulo \(p^2\) belong to \(\mathcal{G}_p\). The following estimate follows immediately from [1, Equation (12)].

\textbf{Lemma 2.} For any fixed \(\alpha > \frac{463}{252}\) and \(K \geq p^\alpha\) we have

\[T_p(K) \ll K/p.\]

Let \(\tau_s(n)\) be the number of representations of \(n\) as a product of \(s\) positive integers:

\[\tau_s(n) = \# \left\{ (n_1, \ldots, n_s) \in \mathbb{N}^s \mid n = n_1 n_2 \ldots n_s \right\}.\]

We also need the following upper bound from [III]:

\textbf{Lemma 3.} Uniformly over \(n\) and \(s\) we have

\[\tau_s(n) \leq \exp \left( \frac{\log n (\log s)}{\log \log n} \left( 1 + O \left( \frac{\log \log \log n + \log s}{\log \log n} \right) \right) \right).\]

In particular, we have:

\textbf{Corollary 4.} If \(s = (\log n)^{o(1)}\) then

\[\tau_s(n) \leq n^{o(1)}.\]

as \(n \to \infty\).
3 Distribution of vanishing Fermat quotients

Here we estimate the cardinality of the sets $Q_p(N)$ and $R_p(N)$. For large values of $N$, namely for $N \geq p^\alpha$ with $\alpha > 463/252$ such a bound is given by Lemma 2. However here we are mostly interested in small values of $N$.

We note that Granville [4] has given a bound on the cardinality of the set $R_p(N)$. Namely, it is shown in [4] that for $u = 1, 2, \ldots$

$$\# R_p(p^{1/u}) \leq u p^{1/2u}.$$  \hfill (3)

We note that the argument used in the proof of (3) can be used to estimate $\# R_p(p^{1/u})$ for any $u \geq 1$.

We derive now upper bounds on $\# Q_p(N)$ and $\# R_p(N)$ that improve (3).

Theorem 5. For any fixed

$$\alpha > \frac{463}{252},$$

for $1 \leq u = (\log p)^{o(1)}$, where

$$u = \frac{\log p}{\log N},$$

we have

$$\# Q_p(N) \ll u N p^{-(1+o(1))}/[\alpha u],$$

as $p \to \infty$.

Proof. We put

$$s = [\alpha u].$$

We consider $(\# Q_p(N))^s$ products $n = n_1 \ldots n_s$ where $(n_1, \ldots, n_s) \in Q_p(N)^s$. By [2] we see that

$$q_p(n) = q(n_1) \ldots q_p(n_s) = 0.$$

Besides, using Corollary [4] we see that each $n \leq N^s \leq p^{o+1}$ has at most

$$\tau_s(n) = p^{o(1)}$$

such representations. We also note that $N^s \geq p^\alpha$. Therefore, combining Lemmas [1] and [2] we derive

$$(\# Q_p(N))^s \leq T_p(N^s)p^{o(1)} \leq N^sp^{-1+o(1)},$$

which implies the desired result. \hfill $\square$
Corollary 6. If
\[
\frac{\log p}{\log N} = (\log p)^{o(1)} \quad \text{and} \quad \frac{\log p}{\log N} \to \infty
\]
then
\[
\# Q_p(N) \leq N^{211/463 + o(1)}
\]
as \( p \to \infty \).

For the set \( R_p(N) \) we have a bound in a wider range of \( u \).

Theorem 7. For any fixed
\[
\alpha > \frac{463}{252},
\]
for \( u \geq 1 \), where
\[
u = \frac{\log p}{\log N},
\]
we have
\[
\# R_p(N) \ll uNp^{-1/\lceil \alpha u \rceil}
\]
as \( p \to \infty \).

Proof. The proof is the same as that of Theorem 5 except that instead of Corollary 4 we note that there are at most \( s! \) products of \( s \) primes \( \ell_1 \ldots \ell_s \) that take the same value. So, we derive
\[
(\# R_p(N))^s \ll s!T_p(N^s) \ll s!N^s p^{-1},
\]
and the result now follows. \( \square \)

Corollary 8. If \( N < p \) and
\[
\frac{\log p}{\log N} \to \infty
\]
then
\[
\# R_p(N) \leq N^{211/463 + o(1)} \log p
\]
as \( p \to \infty \).
4 Ihara sums

First we consider approximations of $S_p$ by partial sums

$$S_p(N) = \sum_{n \in Q_p(N)} \frac{\Lambda(n)}{n}.$$

**Theorem 9.** For $N = p^{o(1)}$ we have

$$S_p = S_p(N) + O(N^{-252/463+o(1)} \log p)$$

as $p \to \infty$.

**Proof.** Clearly, we have

$$S_p - S_p(N) = \sum_{\ell \not\in Q_p(N)} \frac{\log \ell}{\ell} + O(N^{-1} \log N). \quad (4)$$

We now see from Corollary 5 that for any

$$L < N^3$$

we have

$$\sum_{2L \leq \ell > L} \frac{\log \ell}{\ell} \leq \frac{\log L}{L} \sum_{\ell \in R_p(2L)} 1 \leq \frac{\log L}{L} L^{211/463+o(1)} \log p = L^{-252/463+o(1)} \log p. \quad (5)$$

For

$$p \geq L > N^3$$

we choose

$$\alpha = \frac{463}{251}$$

and note that for $u \geq 1$ we have

$$[\alpha u] \leq \frac{3}{2} \alpha u.$$
Thus Theorem 7 implies the bound
\[
\# \mathcal{R}_p(L) \ll L^{1-2/3\alpha} \log p \ll L^{2/3} \log p.
\]

Hence in the above range, we have
\[
\sum_{2L \leq \ell > L, \ell \in \mathcal{R}_p(p)} \frac{\log \ell}{\ell} \leq \frac{\log L}{L} \sum_{\ell \in \mathcal{R}_p(2L)} 1 \leq \frac{\log L}{L} L^{2/3} \log p = L^{-1/3+o(1)} \log p.
\]

Thus covering the range \([N, p]\) by dyadic intervals of the form \([L, 2L]\) and using the bounds (5), and (6) we derive
\[
\sum_{\ell > N, \ell \in \mathcal{R}_p(p)} \frac{\log \ell}{\ell} \leq N^{-252/463+o(1)} \log p,
\]
which after the substitution in (4) implies the desired estimate. \(\square\)

Since by the Mertens formula (see, for example, \(\text{[8, Equation (2.14)]}\))
\[
S_p(N) \leq \sum_{n \leq N} \frac{\Lambda(n)}{n} = \log N + O(1),
\]
we derive from Theorem 9:

**Corollary 10.** For \(N = p^{o(1)}\) we have
\[
S_p \leq \log N + O(N^{-252/463+o(1)} \log p + 1)
\]
as \(p \to \infty\).

We now obtain an unconditional improvement of the conditional estimate (1).

**Corollary 11.** We have
\[
S_p \leq (463/252 + o(1)) \log \log p
\]
as \(p \to \infty\).

**Proof.** Taking \(N = \lceil (\log p)^{\alpha} \rceil\) with \(\alpha > 463/252\) in the bound of Corollary 10 leads to the estimate
\[
S_p \leq \alpha \log \log p + O(1).
\]
Since \(\alpha\) is arbitrary, the result now follows. \(\square\)
5 Index of some subfields of cyclotomic fields

We recall that the index $I(K)$ of an algebraic number field $K$ is the greatest common divisor of indexes $[O_K : \mathbb{Z}[\xi]]$ taken over all $\xi \in O_K$, where $O_K$ is the ring of integers of $K$.

As in [7], we denote by $I_p$ the index of the field $K_p$, which is the unique cyclic extension of degree $p$ over $\mathbb{Q}$ that is contained in the cyclotomic field $\mathbb{Q}(\exp(2\pi i/p^2))$.

It has been shown in [7, Proposition 4 (i)] that under the Generalised Riemann Hypothesis the bound
\[
\log I_p \leq (1 + o(1))p^2 \log \log p
\]
holds as $p \to \infty$. Also [7, Proposition 5] gives an unconditional but weaker bound
\[
\log I_p \leq (1/4 + o(1))p^2 \log p.
\]

We use Corollary 11 to obtain an unconditional improvement of (7).

**Theorem 12.** We have
\[
\log I_p \leq \left(\frac{463}{504} + o(1)\right) p^2 \log \log p
\]
as $p \to \infty$.

**Proof.** By [7, Equation (2.4.1)] we have
\[
\log I_p = \sum_{n \in \mathbb{Q}_p(p)} \alpha_p(n) \Lambda(n),
\]
where
\[
\alpha_p(n) = \left\lfloor \frac{p}{n} \right\rfloor \left( p - \frac{1}{2} n - \frac{1}{2} \left\lfloor \frac{p}{n} \right\rfloor \right).
\]
Since
\[
\alpha_p(n) = \left\lfloor \frac{p}{n} \right\rfloor \left( p - \frac{1}{2} n \left( 1 + \left\lfloor \frac{p}{n} \right\rfloor \right) \right) \leq \left\lfloor \frac{p}{n} \right\rfloor \frac{p}{2} \leq \frac{p^2}{2n},
\]
we see from (8) that
\[
\log I_p \leq \frac{p^2}{2} S_p.
\]
Using Corollary 11 we conclude the proof. \qed

One certainly expects that $I_p$ is much smaller, than the bound given in Theorem 12 however no unconditional lower bound seems to be known (see [7, Proposition 4 (ii)] for a conditional estimate).

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