NEGATIVE PHASE VELOCITY IN ISOTROPIC DIELECTRIC–MAGNETIC MEDIUMS VIA HOMOGENIZATION

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ABSTRACT: We report on a strategy for achieving negative phase velocity (NPV) in a homogenized composite medium (HCM) conceptualized using the Bruggeman formalism. The constituent material phases of the HCM do not support NPV propagation. The HCM and its constituent phases are isotropic dielectric–magnetic mediums; the real parts of their permittivities/permeabilities are negative–valued whereas the real parts of their permeabilities/permittivities are positive–valued.

Keywords: negative phase velocity, negative refraction, Bruggeman formalism

1. INTRODUCTION

A plane wave is said to propagate with negative phase velocity (NPV) if its phase velocity is projected opposite to the time–averaged Poynting vector. A host of exotic electromagnetic phenomenons follow as consequence of NPV, most notably negative refraction [1], as is well–documented elsewhere [2, 3].
In the absence of readily available, naturally occurring materials which support NPV propagation\(^1\), the realization of artificial *metamaterials* which are effectively homogeneous and which support NPV propagation has been the focus of considerable attention [3]. NPV metamaterials for performance in the microwave regime have been realized [6]–[9], and progress towards the same goal in the optical regime continues to be made.

The simplest medium which supports NPV propagation is the idealization represented by the nondissipative isotropic dielectric–magnetic medium with relative permittivity \( \epsilon < 0 \) and relative permeability \( \mu < 0 \). In reality, the effects of dissipation necessitate that \( \epsilon \) and \( \mu \) are complex–valued. In a dissipative isotropic dielectric–magnetic medium, NPV is indicated by the satisfaction of the inequality [2, 10]

\[
\frac{\text{Re} \{\epsilon\}}{\text{Im} \{\epsilon\}} + \frac{\text{Re} \{\mu\}}{\text{Im} \{\mu\}} < 0.
\]

(1)

In this communication we address the question: can the NPV condition (1) be satisfied by a homogenized composite medium (HCM) which arises from constituent material phases which do not themselves support NPV propagation?

**2. HOMOGENIZATION**

Let us consider the homogenization of two constituent material phases: phase \( a \) and phase \( b \). Both material phases are taken to be isotropic dielectric–magnetic mediums with relative permittivities \( \epsilon_{a,b} \) and relative permeabilities \( \mu_{a,b} \). The relative permittivity and relative permeability of the HCM are written as \( \epsilon_{HCM} \) and \( \mu_{HCM} \), respectively. In accordance with the principle of causality and because of the implicit time–dependence \( \exp(-i\omega t) \), we have

\[
\begin{align*}
\text{Im} \{\epsilon_\ell\} &> 0, \\
\text{Im} \{\mu_\ell\} &> 0,
\end{align*}
\]

(\( \ell = a, b, HCM \))

(2)

as the mediums under consideration are assumed to be passive. The volume fraction of phase \( \ell \) is denoted by \( f_\ell \in (0, 1) \) (\( \ell = a, b \)) with \( f_a + f_b = 1 \).

Conventional approaches to homogenization of particulate materials, such as provided by

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\(^1\)We note the possibility of NPV induced by the effects of (a) special relativity in simple dielectric–magnetic mediums [4] and (b) general relativity in vacuum [5].
the Bruggeman and the Maxwell Garnett formalisms [11], run into difficulties within the context of NPV–supporting HCMs. These formalisms have been shown to be inappropriate if \( \text{Re} \{ \epsilon_a \} \text{Re} \{ \epsilon_b \} < 0 \) or \( \text{Re} \{ \mu_a \} \text{Re} \{ \mu_b \} < 0 \), at least in the weakly dissipative regime [12]. There are no such difficulties provided that \( \text{Re} \{ \epsilon_a \} \text{Re} \{ \epsilon_b \} > 0 \) and \( \text{Re} \{ \mu_a \} \text{Re} \{ \mu_b \} > 0 \). However, in view of (2), it is clear that the NPV condition (1) cannot be satisfied if \( \text{Re} \{ \epsilon \} > 0 \) and \( \text{Re} \{ \mu \} > 0 \). We therefore explored the prospects of achieving NPV in a HCM arising from components with either

\[
\text{Case I: } \text{Re} \{ \epsilon_{a,b} \} < 0 \text{ and } \text{Re} \{ \mu_{a,b} \} > 0, \text{ or}
\]

\[
\text{Case II: } \text{Re} \{ \epsilon_{a,b} \} > 0 \text{ and } \text{Re} \{ \mu_{a,b} \} < 0.
\]

Cases I and II are duals of each other, and only one of the two needs further investigation.

The relative permittivity and relative permeability of the corresponding HCM are estimated by the Bruggeman formalism as [13]

\[
\epsilon_{\text{HCM}} = \frac{f_a \epsilon_a (\epsilon_b + 2\epsilon_{\text{HCM}}) + f_b \epsilon_b (\epsilon_a + 2\epsilon_{\text{HCM}})}{f_a (\epsilon_b + 2\epsilon_{\text{HCM}}) + f_b (\epsilon_a + 2\epsilon_{\text{HCM}})}
\]

\[
\mu_{\text{HCM}} = \frac{f_a \mu_a (\mu_b + 2\mu_{\text{HCM}}) + f_b \mu_b (\mu_a + 2\mu_{\text{HCM}})}{f_a (\mu_b + 2\mu_{\text{HCM}}) + f_b (\mu_a + 2\mu_{\text{HCM}})}.
\]

As a representative example, let

\[
\epsilon_a = -6 + 0.9i, \quad \mu_a = 1.5 + 0.2i
\]

\[
\epsilon_b = -1.5 + i, \quad \mu_b = 2 + 1.2i
\]

in accordance with Case I. The Bruggeman estimates \( \epsilon_{\text{HCM}} \) and \( \mu_{\text{HCM}} \) are plotted as functions of the volume fraction \( f_a \) in Figures 1 and 2, respectively. Whereas \( \text{Re} \{ \epsilon_{\text{HCM}} \} \) follows an almost linear progression between its constraining values of \( \text{Re} \{ \epsilon_b \} \) at \( f_a = 0 \) and \( \text{Re} \{ \epsilon_a \} \) at \( f_a = 1 \), and similar dependences are evinced by both \( \text{Re} \{ \mu_{\text{HCM}} \} \) and \( \text{Im} \{ \mu_{\text{HCM}} \} \), \( \text{Im} \{ \epsilon_{\text{HCM}} \} \) displays a markedly nonlinear relationship with respect to \( f_a \).

In Figure 3, the NPV parameter

\[
\rho_\ell = \frac{\text{Re} \{ \epsilon_\ell \}}{\text{Im} \{ \epsilon_\ell \}} + \frac{\text{Re} \{ \mu_\ell \}}{\text{Im} \{ \mu_\ell \}}, \quad (\ell = a, b, \text{HCM})
\]

(5)
is graphed against the volume fraction $f_a$. For the constituent material phases we have the constant values $\rho_a = 0.83$ and $\rho_b = 0.17$; i.e., neither constituent phase supports NPV propagation. In contrast, $\rho_{HCM}$ is negative–valued for $0.28 < f_a < 0.92$. Thus, we see that the HCM supports NPV propagation across a wide range of volume fractions.

3. CONCLUDING REMARKS

In answer to the question posed in Section 1: an HCM which supports NPV propagation, arising from constituent material phases which do not support NPV propagation, may be conceptualized through homogenizing components with $\text{Re} \{\epsilon_{a,b}\} < 0$. In view of (1), it may be inferred that the prospects for NPV propagation are increased through considering constituent phases with relatively small $\text{Im} \{\epsilon_{a,b}\}$ and relatively large $\text{Im} \{\mu_{a,b}\}$.

Since the relative permittivities and relative permeabilities are decoupled within the Bruggeman formalism, our demonstration with Case I also holds for Case II. However, we note that in practice suitable materials with $\text{Re} \{\epsilon_{a,b}\} < 0$ may be more readily available than those with $\text{Re} \{\mu_{a,b}\} < 0$.

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Figure 1: The real (top) and imaginary (bottom) parts of the relative permittivity $\epsilon_{HCM}$ of the HCM, as estimated using the Bruggeman formalism, plotted (solid curve) against volume fraction $f_a$. The dashed horizontal line represents $\epsilon_a$ and the broken dashed horizontal line represents $\epsilon_b$. 
Figure 2: The real (top) and imaginary (bottom) parts of the relative permeability \( \mu_{HCM} \) of the HCM, as estimated using the Bruggeman formalism, plotted (solid curve) against volume fraction \( f_a \). The dashed horizontal line represents \( \mu_a \) and the broken dashed horizontal line represents \( \mu_b \).
Figure 3: The NPV parameter $\rho_{HCM}$ of the HCM, as estimated using the Bruggeman formalism, plotted (solid curve) against volume fraction $f_a$. The dashed horizontal line represents $\rho_a$ for phase $a$ and the broken dashed horizontal line represents $\rho_b$ for phase $b$. 