We propose a Lax equation for the non-linear sigma model which leads directly to the conserved local charges of the system. We show that the system has two infinite sets of such conserved charges following from the Lax equation, much like dispersionless systems. We show that the system has two Hamiltonian structures which are compatible so that it is truly a bi-Hamiltonian system. However, the two Hamiltonian structures act on the two distinct sets of charges to give the dynamical equations, which is quite distinct from the behavior in conventional integrable systems. We construct two recursion operators which connect the conserved charges within a given set as well as between the two sets. We show explicitly that the conserved charges are in involution with respect to either of the Hamiltonian structures thereby proving complete integrability of the system. Various other interesting features are also discussed.
1 Introduction:

The non-linear sigma model has been studied from various points of view \cite{1}. In the context of 1+1 dimensional integrable systems, it is one of the most enigmatic models. The equations for the principal chiral model, for example, take the forms

\[ \partial_\mu J_{\mu}^{\alpha} = 0, \]
\[ \partial_\mu J_{\nu}^{\alpha} - \partial_\nu J_{\mu}^{\alpha} - f_{\mu\nu}^{\alpha\beta\gamma} J_{\beta}^{\mu} J_{\gamma}^{\nu} = 0, \]

(1)

where \( \mu = 0, 1 \) and \( J_\mu^{\alpha} \) belongs to a simple Lie algebra, with \( \alpha \) taking values in the adjoint representation. Explicitly, the two equations can be written as

\[ \partial_t J_0^a = \partial_x J_1^a \]
\[ \partial_t J_1^a = \partial_x J_0^a + f_{abc} J_0^b J_1^c \]

(2)

It is well known that the equations (1) can be combined into a single zero curvature condition by introducing a one parameter family of connections that depend on a constant spectral parameter \cite{2}. The zero curvature representation is quite useful and leads directly to the infinite number of conserved non-local charges of the system \cite{3}. While an infinity of conserved charges is a hallmark of integrable systems, these non-local charges are not in involution with respect to the Hamiltonian structure of the system \cite{4}. Rather, they satisfy a Yangian algebra \cite{5}. While classical integrability of the non-linear sigma model is known from alternate arguments \cite{6, 7} (see \cite{8} for quantum integrability), a conventional proof of integrability through the existence of an infinite number of conserved charges in involution is lacking so far. Furthermore, it is not known whether the system is bi-Hamiltonian, as is the case with most integrable systems \cite{9, 10}.

There have been various attempts \cite{11} to construct conserved local charges for the system (1), but a systematic study through a Lax representation is lacking so far. In this letter, we will provide a Lax equation for the non-linear sigma model equations (1). Although the equations (1) cannot be characterized as dispersionless, they share many features common to a dispersionless model \cite{12}. In section 2, we discuss briefly the two component Riemann equation. This is a dispersionless system sharing some properties with the non-linear sigma model. In section 3, drawing from our experience in the Toda lattice \cite{10, 13}, we present a Lax representation for (1). Even though the Lax equation for the system is not that of a dispersionless system, it immediately leads to two infinite sets of local conserved charges, a common feature of dispersionless systems \cite{14, 15}. We point out various properties of these conserved charges that are quite useful in proving the integrability of the system. We also present the two Hamiltonian structures which are compatible and make the system bi-Hamiltonian. In section 4, we construct two recursion operators for the system which connect charges within any given set as well as between the two sets. We also show explicitly that the two sets of charges are in involution with respect to either of the Hamiltonian structures thereby proving the integrability of the system. In section 5, we summarize our results and discuss some other interesting aspects of this system.
2 Matrix Riemann equation:

Before discussing the Lax equation for the non-linear sigma model, we will briefly discuss, in this section, the two component Riemann equation. We recall that the Riemann equation

$$u_t = uu_x,$$

is an integrable dispersionless system \([14, 16]\) which belongs to the class of equations of hydrodynamic type. Even though the non-linear sigma model is not a dispersionless system, it shares some nice properties with the two component Riemann equation. Let us define a \(2 \times 2\) matrix

$$U = \begin{pmatrix} u & v \\ v & u \end{pmatrix} = uI + v\sigma_1,$$

where \(u, v\) are the two dynamical variables of the system and \(\sigma_1\) is the familiar Pauli matrix \((I\) denotes the identity matrix\). Then, the generalized two component Riemann equation takes the form

$$U_t = UU_x.$$

Let us next consider a matrix Lax function of the type

$$L = p^2 I + U,$$

where \(p\) denotes the momentum in the phase space. Then, from the special structure of \(U\) in (4), it is clear that

$$[\sigma_1, U] = 0, \quad [\sigma_1, L] = 0.$$

As a result, there is an ambiguity in taking the square root of the Lax function, namely,

$$L_1^{\frac{1}{2}} = p I + \frac{1}{2} U p^{-1} - \frac{1}{8} U^2 p^{-3} + \cdots.$$

Furthermore, a second square root has the form

$$L_2^{\frac{1}{2}} = \sigma_1 L_1^{\frac{1}{2}}.$$

We note that the square root (either one) of the Lax function commutes with \(\sigma_1\), as does \(L\). Both the square roots lead to consistent Lax equations \([13]\) as

$$\frac{\partial L}{\partial t_n} = \frac{(2n)!!}{(2n + 1)!!} \left\{ (L_1)^{2n+1} \right\},$$

$$\frac{\partial L}{\partial t_n} = \frac{(2n)!!}{(2n + 1)!!} \left\{ (L_2)^{2n+1} \right\},$$

(10)
where \( n = 0, 1, 2, \ldots \) and which lead to the hierarchy of Riemann equations of the form

\[
\begin{align*}
U_t &= U^n U_x, \\
U_t &= \sigma_1 U^n U_x,
\end{align*}
\]

respectively. In the second equation the effect of \( \sigma_1 \), on the right hand side, is to interchange \( u \leftrightarrow v \), and eq. \((12)\) can be thought of as the “elastic medium” equations \([14]\) associated with the hierarchy of Riemann equations \((11)\). It is worth noting here that, although we have written the equations as non-standard equations, the same equations also arise from the standard Lax equations \([17]\).

The conserved charges of the system can now be easily obtained from the Lax function. In fact, there are two infinite sets of conserved charges which arise as \([15]\)

\[
\begin{align*}
H_n &= \int dx \, h_n = \frac{(2n)!!}{(2n-1)!!} \text{Tr} \, L^{2n-1} = \frac{(2n)!!}{(2n-1)!!} \int dx \, \text{Res} \, L^{2n-1} = \int dx \, \text{tr} \, U^n, \\
\tilde{H}_n &= \int dx \, \tilde{h}_n = \frac{(2n)!!}{(2n-1)!!} \text{Tr} \sigma_1 L^{2n-1} = \frac{(2n)!!}{(2n-1)!!} \int dx \, \text{Res} \sigma_1 L^{2n-1} = \int dx \, \text{tr} \sigma_1 U^n,
\end{align*}
\]

where “\( \text{tr} \)” denotes the trace over matrix indices, “\( \text{Res} \)” (Residue) stands for the coefficient of the \( p^{-1} \) term in the expansion and \( n = 1, 2, 3, \ldots \). (Alternatively, we can think of the two sets of charges as the traces of powers of \( L_1 \) and \( L_2 \) respectively.) Explicitly, the two sets of conserved densities have the closed forms

\[
\begin{align*}
h_n &= 2 \sum_{k=0}^{\left[ \frac{n}{2} \right]} \left( \begin{array}{c} n \\ 2k \end{array} \right) u^{n-2k} v^{2k}, \\
\tilde{h}_n &= 2 \sum_{k=1}^{\left[ \frac{n+1}{2} \right]} \left( \begin{array}{c} n \\ 2k-1 \end{array} \right) u^{n-2k+1} v^{2k-1}.
\end{align*}
\]

The conserved quantities are, of course, defined up to overall normalization constants. Rescaling

\[
(H_n, \tilde{H}_n) \rightarrow \frac{1}{n(n-1)} (H_n, \tilde{H}_n), \quad n > 1,
\]

we see that the equations \((11)\) and \((12)\) can be written in the Hamiltonian form respectively as

\[
\begin{align*}
U_t &= \partial \frac{\delta H_{n+2}}{\delta U}, \\
U_t &= \partial \frac{\delta \tilde{H}_{n+2}}{\delta U}.
\end{align*}
\]

The conserved densities of the system are easily seen to satisfy the usual relations for the two
component hydrodynamic systems [14], namely,
\[ \frac{\partial^2 h_n}{\partial u \partial u} = \frac{\partial^2 h_n}{\partial v \partial v}, \quad \frac{\partial^2 \tilde{h}_n}{\partial u \partial u} = \frac{\partial^2 \tilde{h}_n}{\partial v \partial v}. \] \tag{17}
This system of equations is integrable. However, we will not get into the details of this. Our interest has been only to discuss it briefly so that we can bring out the similarities of this system with the non-linear sigma model.

3 Lax equation for the non-linear sigma model:

The non-linear sigma model is not exactly a dispersionless system. Consequently, we do not expect a Lax function in the phase space to describe such a system. We find that, like the Toda lattice, the non-linear sigma model does not have a scalar Lax description, but can be given a Lax description much like the Toda lattice [10, 13] in the following manner.

Let us consider the symmetric $2 \times 2$ (multiplicative) Lax operator of the form
\[ S_{ab} = \begin{pmatrix} J_0^a J_0^b + J_1^a J_1^b & J_0^a J_1^b + J_1^a J_0^b \\ J_0^a J_1^b + J_1^a J_0^b & J_0^a J_0^b + J_1^a J_1^b \end{pmatrix}. \]
\[ = (J_0^a J_0^b + J_1^a J_1^b) I + (J_0^a J_1^b + J_1^a J_0^b) \sigma_1 \]
\[ = L^a L^b, \] \tag{18}
where
\[ L^a = J_0^a I + J_1^a \sigma_1 = \begin{pmatrix} J_0^a & J_1^a \\ J_1^a & J_0^a \end{pmatrix}. \] \tag{19}
Let us also introduce an anti-symmetric $2 \times 2$ matrix operator
\[ U_{ab} = (\delta_{ab} \partial + f^{abc} J_c^f) \sigma_1 = \begin{pmatrix} 0 & \delta_{ab} \partial + f^{abc} J_c^f \\ \delta_{ab} \partial + f^{abc} J_c^f & 0 \end{pmatrix}. \] \tag{20}
Then, it is easy to check that the Lax equation of the form of the Toda lattice, namely,
\[ \frac{\partial S_{ab}}{\partial t} = - [S, U]_{ab} \equiv - (S_{ac} U_{cb} - U_{ac} S_{cb}), \] \tag{21}
leads to the equations for the non-linear sigma model [1]. ($U_{ab}$, here, is an operator and should not be confused with the matrix $U$ in [1] of the last section, which describes the dynamical variables of the system much like $S_{ab}$.) Parenthetically, we would like to remark here that the correspondence with the Toda lattice, however, is not quite complete as the operator $U_{ab}$ cannot be obtained as the Fréchet derivative of the equations for the sigma model [10, 13]. Just like the two component
Riemann equation of the last section, it is clear that
\[ [\sigma_1, L^a] = 0, \quad [\sigma_1, S^{ab}] = 0, \quad [\sigma_1, U^{ab}] = 0, \quad (22) \]
which defines an analogous symmetry in the non-linear sigma model.

Equation (21) defines the Lax equation for the non-linear sigma model. As in the Toda lattice, it follows that the conserved densities of the system can be obtained as traces of powers of the Lax operator \( S^{ab} \) [10, 13]. In fact, because of the symmetry (22) in the system, it is trivial to check that there are two infinite sets of local conserved densities in the system, namely,
\[ h_n = \frac{1}{4n} \text{Tr} S^n, \quad \tilde{h}_n = \frac{1}{4n} \text{Tr} \sigma_1 S^n. \quad (23) \]

Here, “Tr” denotes trace over the matrix indices as well as over the internal indices. Using the definition (18) as well as the cyclicity property of the trace, it is easy to see that we can write the conserved densities in the simpler form
\[ h_n = \frac{1}{4n} \text{tr} (L^2)^n, \quad \tilde{h}_n = \frac{1}{4n} \text{tr} \sigma_1 (L^2)^n, \quad (24) \]
where
\[ L^2 = L^a L^a = (J_0^2 + J_1^2)I + 2(J_0 J_1) \sigma_1, \quad (25) \]
and “tr” denotes, as before, trace over matrix indices. In (25) the internal indices of the terms inside the parenthesis are summed. The similarity of these with the conserved densities of the two component Riemann equation in (13) is remarkable (we have chosen a particular normalization for simplicity). The first few conserved densities for the non-linear sigma model have the explicit forms,
\[ h_1 = \frac{1}{2} (J_0^2 + J_1^2), \quad \tilde{h}_1 = (J_0 J_1), \]
\[ h_2 = \frac{1}{4} (J_0^2 + J_1^2)^2 + (J_0 J_1)^2, \quad \tilde{h}_2 = (J_0^2 + J_1^2)(J_0 J_1), \]
\[ h_3 = \frac{1}{6} (J_0^2 + J_1^2)^3 + 2(J_0^2 + J_1^2)(J_0 J_1)^2, \quad \tilde{h}_3 = (J_0^2 + J_1^2)^2 (J_0 J_1) + \frac{4}{3}(J_0 J_1)^3. \quad (26) \]

We see that the local conserved densities in (24) do not involve spatial derivatives, much like the conserved densities of dispersionless systems. It is also clear now that the two sets of conserved charges can be mapped into those of the two component Riemann equation (13) or (14) (up to normalization constants) with the identifications
\[ u = J_0^2 + J_1^2, \quad v = 2J_0 J_1. \quad (27) \]

With this mapping, even the equations for the sigma model (1) go over to those of the two component Riemann equation (12) for \( n = 0 \). However, such a map is not one to one and, therefore, is
not very useful. Nonetheless, it brings out an interesting relation between a dispersionless system and the non-linear sigma model.

As in hydrodynamic systems, in the case of the non-linear sigma model, it can be shown that any local conserved density, $Q$ (without spatial derivatives), must satisfy

$$\frac{\partial^2 Q}{\partial J_a^0 \partial J_b^0} = \frac{\partial^2 Q}{\partial J_a^1 \partial J_b^1},$$

(28)

which is the analogue of (17). The conserved densities in (24) can be seen to satisfy this. However, from the form of the conserved densities in (24), it can be checked that they satisfy additional relations which can be written collectively as

$$\frac{\partial^2 h_n}{\partial J_0^a \partial J_1^a} = \frac{\partial^2 \tilde{h}_n}{\partial J_0^a \partial J_1^a},$$

$$\frac{\partial^2 h_n}{\partial J_0^a \partial J_1^b} = \frac{\partial^2 \tilde{h}_n}{\partial J_0^b \partial J_1^b},$$

$$J_0^a \partial h_n / \partial J_0^a + J_1^a \partial h_n / \partial J_1^a = 2nh_n,$$

$$J_0^a \partial \tilde{h}_n / \partial J_0^a + J_1^a \partial \tilde{h}_n / \partial J_1^a = 2n\tilde{h}_n,$$

$$J_0^b \partial h_n / \partial J_0^b + J_1^b \partial h_n / \partial J_1^b = 2nh_n,$$

$$J_0^b \partial \tilde{h}_n / \partial J_0^b + J_1^b \partial \tilde{h}_n / \partial J_1^b = 2n\tilde{h}_n,$$

$$f^{abc} \left( J_0^b \partial h_n / \partial J_0^a + J_1^b \partial h_n / \partial J_1^a \right) = 0,$$

$$f^{abc} \left( J_0^b \partial \tilde{h}_n / \partial J_0^a + J_1^b \partial \tilde{h}_n / \partial J_1^a \right) = 0,$$

$$f^{abc} \left( J_0^b \partial h_n / \partial J_1^a + J_1^b \partial h_n / \partial J_1^a \right) = 0,$$

$$f^{abc} \left( J_0^b \partial \tilde{h}_n / \partial J_1^a + J_1^b \partial \tilde{h}_n / \partial J_1^a \right) = 0,$$

$$\left( \begin{array}{c} \partial h_n / \partial J_0^a \\ \partial h_n / \partial J_1^a \\ \partial \tilde{h}_n / \partial J_0^a \\ \partial \tilde{h}_n / \partial J_1^a \end{array} \right) = \sigma_1 \left( \begin{array}{c} \partial h_n / \partial J_0^b \\ \partial h_n / \partial J_1^b \\ \partial \tilde{h}_n / \partial J_0^b \\ \partial \tilde{h}_n / \partial J_1^b \end{array} \right).$$

(29)

These relations are quite useful in proving the complete integrability of the system as well as various other interesting properties, as we will see.

Once the two sets of local conserved charges have been obtained, it is straightforward to show that the non-linear sigma model has two Hamiltonian structures. Defining

$$D_1^{ab} = \left( \begin{array}{cc} 0 & \delta^{ab} \partial \\ \delta^{ab} \partial & -f^{abc} J_0^c \end{array} \right), \quad D_2^{ab} = \left( \begin{array}{cc} \delta^{ab} \partial & 0 \\ 0 & \delta^{ab} \partial + f^{abc} J_1^c \end{array} \right),$$

(30)

it is easy to see that the non-linear sigma model equations, (4), can be written in the Hamiltonian
form as
\[
\begin{pmatrix}
J_{0,t}^a \\
J_{1,t}^a
\end{pmatrix} = D_1^{ab} \begin{pmatrix}
\frac{\delta H_1}{\delta J_0^b} \\
\frac{\delta H_1}{\delta J_1^b}
\end{pmatrix} = D_2^{ab} \begin{pmatrix}
\frac{\delta \tilde{H}_1}{\delta J_0^b} \\
\frac{\delta \tilde{H}_1}{\delta J_1^b}
\end{pmatrix}.
\]
(31)

In fact, using the properties in (29), it is easy to show that all the higher order equations in the hierarchy can also be written as
\[
\begin{pmatrix}
J_{0,t}^a \\
J_{1,t}^a
\end{pmatrix} = D_1^{ab} \begin{pmatrix}
\frac{\delta H_n}{\delta J_0^b} \\
\frac{\delta H_n}{\delta J_1^b}
\end{pmatrix} = D_2^{ab} \begin{pmatrix}
\frac{\delta \tilde{H}_n}{\delta J_0^b} \\
\frac{\delta \tilde{H}_n}{\delta J_1^b}
\end{pmatrix}.
\]
(32)

The anti-symmetry of the Hamiltonian structures in (30) is manifest and it can be seen, using the method of prolongation [18], that they satisfy Jacobi identity. Therefore, these are genuine Hamiltonian structures of the system. Furthermore, it is also easy to check, using the method of prolongation, that
\[
D_1 + \xi D_2
\]
(33)
satisfies Jacobi identity, where \(\xi\) is an arbitrary constant parameter. As a result, the system has two compatible Hamiltonian structures and is truly a bi-Hamiltonian system. However, it is worth noting that, unlike in conventional integrable systems, here the two Hamiltonian structures act on different families of charges to yield the same dynamical equations. Since the criterion of Magri [9] applies to systems where the bi-Hamiltonian structures act on the same family of conserved charges, it is not a priori clear that the integrability of the system follows from the existence of a bi-Hamiltonian structure. This is a question that we will take up in the next section. We would like to point out here that under the action of \(D_1\), the conserved charge \(\tilde{H}_1\) generates spatial translations, as does \(H_1\) under the action of \(D_2\). This is another interesting feature of this system, namely, the “Hamiltonian” and the “momentum” (with respect to a given Hamiltonian structure) seem to be distributed into distinct families of conserved quantities in the non-linear sigma model.

4 Complete integrability of the system:

Before proving complete integrability of the non-linear sigma model, let us derive the recursion operators that relate the conserved charges of the two infinite sets. Using the relations in (29), it is straightforward to show that the Lax operator, \(S^{ab}\), defines a recursion operator for the system.
It can be explicitly checked that

\[
\begin{pmatrix}
  \frac{\delta H_{n+1}}{\delta J_0^a} \\
  \frac{\delta H_{n+1}}{\delta J_1^a}
\end{pmatrix}
= \mathcal{S}^{ab}
\begin{pmatrix}
  \frac{\delta H_n}{\delta J_0^a} \\
  \frac{\delta H_n}{\delta J_1^a}
\end{pmatrix},
\begin{pmatrix}
  \frac{\delta \tilde{H}_{n+1}}{\delta J_0^a} \\
  \frac{\delta \tilde{H}_{n+1}}{\delta J_1^a}
\end{pmatrix}
= \mathcal{S}^{ab}
\begin{pmatrix}
  \frac{\delta \tilde{H}_n}{\delta J_0^a} \\
  \frac{\delta \tilde{H}_n}{\delta J_1^a}
\end{pmatrix}.
\] (34)

Namely, the charges within each of the two sets are related recursively by the Lax operator, \( \mathcal{S}^{ab} \), itself.

Furthermore, let us define

\[
\tilde{\mathcal{S}}^{ab} = (D_1^{-1})^{ac} D_2^{cb} = \sigma_1 \partial^{-1} \begin{pmatrix}
  f^{abc} J_0^c & f^{abc} J_1^c \\
  0 & 0
\end{pmatrix}.
\] (35)

Using (29), it is easy to check that the second term on the right hand side of (35) gives zero acting on

\[
\begin{pmatrix}
  \frac{\delta H_n}{\delta J_0^b} \\
  \frac{\delta H_n}{\delta J_1^b}
\end{pmatrix}, \quad \text{or} \quad \begin{pmatrix}
  \frac{\delta \tilde{H}_n}{\delta J_0^b} \\
  \frac{\delta \tilde{H}_n}{\delta J_1^b}
\end{pmatrix}.
\] (36)

It follows then that

\[
\begin{pmatrix}
  \frac{\delta \tilde{H}_n}{\delta J_0^b} \\
  \frac{\delta \tilde{H}_n}{\delta J_1^b}
\end{pmatrix}
= \tilde{\mathcal{S}}^{ab}
\begin{pmatrix}
  \frac{\delta H_n}{\delta J_0^b} \\
  \frac{\delta H_n}{\delta J_1^b}
\end{pmatrix}
= \sigma_1
\begin{pmatrix}
  \frac{\delta H_n}{\delta J_0^a} \\
  \frac{\delta H_n}{\delta J_1^a}
\end{pmatrix}.
\] (37)

This is, in fact, already noted (the last relation) in (29). It is worth noting here that the recursion operator, \( \tilde{\mathcal{S}}^{ab} \), conventionally constructed from the two Hamiltonian structures in an integrable system, in this case relates charges of the two distinct sets. In fact, on the space of the gradients of the Hamiltonian, \( \tilde{\mathcal{S}}^{ab} \) gives zero acting on \( \begin{pmatrix}
  \frac{\delta H_n}{\delta J_0^b} \\
  \frac{\delta H_n}{\delta J_1^b}
\end{pmatrix} \).

\[
(\tilde{\mathcal{S}})^2 = I.
\] (38)

Namely, although the square of this recursion operator is not identity, its effect on the space is that of the identity operator. This is quite important in that it implies that there are only two infinite sets of local conserved charges (without derivatives) associated with the system.

Thus, we see that, in the case of the non-linear sigma model, there exist two recursion operators — \( \mathcal{S}^{ab} \) relates charges within a family whereas \( \tilde{\mathcal{S}}^{ab} \) relates charges between the two families of conserved charges. By construction \( \tilde{\mathcal{S}}^{ab} \) is related to the two Hamiltonian structures of the system. One can similarly ask if there is yet another Hamiltonian structure within a given family of charges
constructed in the conventional manner as
\[ \mathbf{D}^{ab} = D_1^{ac} S^{cb}. \]  

We have checked that this structure does not satisfy Jacobi identity. (In fact, this structure does not even have the required anti-symmetry property. If it is anti-symmetrized by hand, the new structure does not lead to the correct dynamical equations, in addition to the fact that the Jacobi identity does not hold.) This, therefore, remains an open question.

To prove complete integrability of the system, we note that
\[ \{H_n, H_m\}_1 = \int dx \left[ \frac{\partial h_n}{\partial J_0^a} \left( \frac{\partial h_m}{\partial J_1^a} \right) + \frac{\partial h_n}{\partial J_1^a} \left( \frac{\partial h_m}{\partial J_0^a} - f^{abc} J_0^c \frac{\partial h_m}{\partial J_1^c} \right) \right]. \]  

From the forms of the conserved charges in (24), it is easy to see that the term on the right hand side involving the structure constant does not contribute. As a result, we have
\[ \{H_n, H_m\}_1 = \int dx \left[ F^a J_0^a,x + G^a J_1^a,x \right], \]

where
\[ F^a = \frac{\partial h_n}{\partial J_0^a} \frac{\partial^2 h_m}{\partial J_0^b \partial J_1^b} + \frac{\partial h_n}{\partial J_1^a} \frac{\partial^2 h_m}{\partial J_0^b \partial J_1^b}, \]
\[ G^a = \frac{\partial h_n}{\partial J_0^b} \frac{\partial^2 h_m}{\partial J_0^a \partial J_1^b} + \frac{\partial h_n}{\partial J_1^b} \frac{\partial^2 h_m}{\partial J_0^a \partial J_1^b}. \]

It is straightforward now to check, using (29), that
\[ \frac{\partial F^a}{\partial J_1^b} - \frac{\partial G^a}{\partial J_0^b} = 0. \]

This implies that there exists a function \( X \) such that
\[ F^a = \frac{\partial X}{\partial J_0^a}, \quad G^a = \frac{\partial X}{\partial J_1^a}, \quad \frac{\partial^2 X}{\partial J_0^a \partial J_1^b} = \frac{\partial^2 X}{\partial J_0^b \partial J_1^a}. \]

Substituting this into (40), we obtain
\[ \{H_n, H_m\}_1 = \int dx \frac{dX}{dx} = 0, \]

with the usual assumptions on the asymptotic fall off of the fields. In a similar manner, it can be shown that
\[ \{H_n, \tilde{H}_m\}_1 = 0 = \{\tilde{H}_n, \tilde{H}_m\}_1. \]
Namely, all the charges are in involution with respect to the first Hamiltonian structure. The involution with respect to the second Hamiltonian structure is also easy to show along similar lines,

\[
\{H_n, H_m\}_2 = 0 = \{H_n, \tilde{H}_m\}_2 = \{\tilde{H}_n, \tilde{H}_m\}_2.
\] (47)

This proves that the non-linear sigma model is completely integrable.

5 Discussions:

In this letter, we have proposed a Lax equation for the non-linear sigma model which directly generates all the conserved local quantities (without any derivative) of the system. There are, in fact, two infinite sets of conserved charges and the similarities of this system with the two component Riemann equation are discussed. We have obtained the two Hamiltonian structures of the system and have shown that they are compatible thereby proving that the system is bi-Hamiltonian. However, unlike conventional integrable systems, here the two Hamiltonian structures act on charges in the different sets to give the same dynamical equations. As a result, integrability of the system does not follow in spite of the fact that it is a bi-Hamiltonian system. We have constructed two recursion operators for the system — one that relates conserved quantities within any given family and the other that relates the charges between the two families. We have explicitly shown that the two sets of charges are in involution with respect to either of the Hamiltonian structures, thereby proving the integrability of the system. We have also brought out several other interesting features associated with the system.

In the conventional discussion of the non-linear sigma model \([2, 3]\), there is a standard Hamiltonian structure given by

\[
\mathcal{D}_{std}^{ab} = \begin{pmatrix}
 f^{abc} J_0^c & \delta^{ab} \partial + f^{abc} J_1^c \\
 \delta^{ab} \partial + f^{abc} J_1^c & 0
\end{pmatrix}.
\] (48)

This is manifestly anti-symmetric and can be easily checked to satisfy Jacobi identity. On the other hand, this seems to be quite different from the two Hamiltonian structures that we have obtained in \((30)\). We note that the Hamiltonian structure in \((48)\) acting on \(H_1\) leads to the non-linear sigma model equations, \((1)\). On the other hand, we note that

\[
\mathcal{D}_{std}^{ab} - \mathcal{D}_1^{ab} = \begin{pmatrix}
 f^{abc} J_0^c & f^{abc} J_1^c \\
 f^{abc} J_1^c & f^{abc} J_0^c
\end{pmatrix}.
\] (49)

Using \((49)\), it is easy to see that this difference gives vanishing contribution on the space \((36)\). As a result, on the space \((36)\), the two structures \(\mathcal{D}_{std}^{ab}\) and \(\mathcal{D}_1^{ab}\) have identical effect. (It is interesting to note that, in spite of this, the two structures, \(\mathcal{D}_1^{ab}\) and \(\mathcal{D}_{std}^{ab}\) are not compatible.) This feature is completely new, namely, we have two Hamiltonian structures, satisfying Jacobi identity, which give the same equations acting on the same Hamiltonian (normally multi-Hamiltonian structures act on different Hamiltonians). It is not a priori clear, therefore, how one can select between these two structures. The difference in the two Hamiltonian structures lies in their action on the space of charges that are not gauge invariant (which have a free internal index). Namely, we know that
the non-linear sigma model has an infinite set of conserved non-local charges, the first of which is, in fact, local and is of the form

\[ Q^a = \int dx \, J^a_0. \] (50)

This charge is, in fact, supposed to generate the global rotations of the system through the action of the Hamiltonian structure of the system. It can be checked that the action of \( D^{ab}_{\text{std}} \) acting on this leads to the correct rotation while \( D^{ab}_1 \) does not. This may be a reason to choose the conventional Hamiltonian structure over \( D^{ab}_1 \). These are, however, open questions that need further study.

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