Tug of war with noise: a game theoretic view of the $p$-Laplacian

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Abstract

Fix a bounded domain $\Omega \subset \mathbb{R}^d$, a continuous function $F : \partial \Omega \to \mathbb{R}$, and constants $\epsilon > 0$ and $1 < p, q < \infty$ with $p^{-1} + q^{-1} = 1$. For each $x \in \Omega$, let $u^\epsilon(x)$ be the value for player I of the following two-player, zero-sum game. The initial game position is $x$. At each stage, a fair coin is tossed and the player who wins the toss chooses a vector $v \in B(0, \epsilon)$ to add to the game position, after which a random “noise vector” with mean zero and variance $\frac{1}{p} |v|^2$ in each orthogonal direction is also added. The game ends when the game position reaches some $y \in \partial \Omega$, and player I's payoff is $F(y)$.

We show that (for sufficiently regular $\Omega$) as $\epsilon$ tends to zero the functions $u^\epsilon$ converge uniformly to the unique $p$-harmonic extension of $F$. Using a modified game (in which $\epsilon$ gets smaller as the game position approaches $\partial \Omega$), we prove similar statements for general bounded domains $\Omega$ and resolutive functions $F$.

These games and their variants interpolate between the tug of war games studied by Peres, Schramm, Sheffield, and Wilson ($p = \infty$) and the motion-by-curvature games introduced by Spencer and studied by Kohn and Serfaty ($p = 1$). They generalize the relationship between Brownian motion and the ordinary Laplacian and yield new results about $p$-capacity and $p$-harmonic measure.

1 Introduction

Given $p > 1$, a bounded domain $\Omega$ in $\mathbb{R}^d$ and a continuous function $F : \partial \Omega \to \mathbb{R}$, the $p$-Dirichlet problem consists of finding a continuous extension $u : \overline{\Omega} \to \mathbb{R}$ of $F$ which is $p$-harmonic, that is, $u$ minimizes $\int_{\Omega} |\nabla u(x)|^p \, dx$ subject to the given boundary conditions. (In general, such an extension exists only under a regularity condition on $\Omega$, and $\int |\nabla u(x)|^p \, dx$ should be minimized over compact subsets of $\Omega$, see Proposition 1.3) In the classical case $p = 2$, Kakutani and Doob discovered that the Dirichlet problem can be solved by starting a Brownian motion $B$ at $x$, running it until the hitting time $\tau$ of $\partial \Omega$, and taking $u(x) = \mathbb{E}_x[F(B(\tau))]$; see [5] for a comprehensive study.

In this paper, we develop an analogous interpretation of a $p$-harmonic extension $u(x)$ as the limit of the values of certain stochastic games. The extreme case $p = \infty$ was already considered in [14].

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1.1 Game definition

Fix a bounded domain \( \Omega \subset \mathbb{R}^d \) and a continuous function \( F : \partial \Omega \to \mathbb{R} \). Let \( \mu \) be a mean zero compactly supported probability measure on \( \mathbb{R}^d \) that is preserved by orthogonal transformations of \( \mathbb{R}^d \) that fix the first basis vector \( e_1 \). We call \( \mu \) the noise measure. (For example, \( \mu \) can be the uniform distribution on the sphere of radius \( r \) in the hyperplane orthogonal to \( e_1 \); for the relation to the \( p \)-Laplacian described below to hold, \( r \) should be \( \sqrt{(d-1)q/p} \).

For each \( v \in \mathbb{R}^d \) and Borel measurable \( S \subset \mathbb{R}^d \), define \( \mu_v(S) = \mu(\Psi^{-1}(S)) \) where \( \Psi \) is a constant \( c \) times some orthonormal transformation of \( \mathbb{R}^d \), chosen so that \( \Psi(e_1) = v \). The requirement that \( \Psi(e_1) = v \) clearly implies that \( c = |v| \). In particular, if \( v = ce_1 \) for some \( c > 0 \), then the law of a vector chosen from \( \mu_v \) will be simply the law of \( c \) times a vector chosen from \( \mu \). The fact that \( \mu \) is invariant under orthogonal transformations of \( \mathbb{R}^d \) that fix \( e_1 \) implies that our definition of \( \mu_v \) does not depend on the choice of \( \Psi \).

Let \( \alpha = 1 + \inf \{ R : \mu B(0, R) = 1 \} \), where \( B(z, R) \) denotes the ball of radius \( R \) centered at \( z \).

We now introduce a two-player zero-sum game, called tug of war (with noise), played as follows. Fix an initial game state \( x_0 = x \in \Omega \). At the \( k \)'th turn, a fair coin is tossed, and the player who wins the coin toss is allowed to make a move. If \( \text{dist}(x_{k-1}, \partial \Omega) > \alpha \epsilon \), then the moving player chooses \( v_k \in \mathbb{R}^d \) with \( |v_k| \leq \epsilon \) and sets \( x_k = x_{k-1} + v_k + z_k \) where \( z_k \) is a random “noise vector” sampled from \( \mu_{v_k} \). To understand the scaling, observe that the law \( \mu_{v_k} \) of the noise vector is supported on a set of radius \( (\alpha - 1)|v_k| \); thus, we expect \( z_k \) and \( v_k \) to be of the same order of magnitude.

If \( \text{dist}(x_{k-1}, \partial \Omega) \leq \alpha \epsilon \), then the moving player chooses an \( x_k \in \partial \Omega \) with \( |x_k - x_{k-1}| \leq \alpha \epsilon \) and the game ends, with player I receiving a payoff of \( F(x_k) \) from player II. The rules above ensure that if the game terminates at the \( k \)'th step, then \( x_j \in \Omega \) for all \( 0 \leq j < k \) and \( x_k \in \partial \Omega \). Both players receive a payoff of zero if the game never terminates. Tug of war in a domain of \( \mathbb{R}^d \) without noise (i.e., with \( \mu \) supported at the origin) was introduced in [14], where it was used to give uniqueness results for PDEs involving the infinity Laplacian and also extended to solve the optimal Lipschitz extension problem on general length spaces. Other games with random turn order were introduced earlier [15] [10].

1.2 Main results

Fix the game parameters \( F, \Omega, \) and \( \mu \) above, and denote by \( u_1^*(x) \) the supremum over all player I (measurable, pure) strategies (defined precisely in Section 2.1) of the infimum over all player II strategies of the expected payoff for player I when both players adopt these strategies. Denote by \( u_2^*(x) \) the infimum over all player II strategies of the supremum over all player I strategies of the expected payoff for player I when both players adopt these strategies. It is clear that \( u_1^* \leq u_2^* \). When \( u_1^* = u_2^* \) we say that the game has value \( u^* = u_1^* = u_2^* \).

A function \( u \) is \( p \)-harmonic in a domain \( \Omega \subset \mathbb{R}^d \) if \( u \) is continuous on \( \Omega \) and for every bounded subdomain \( \Omega_0 \subset \Omega \), the \( p \)-energy \( \int_{\Omega_0} |\nabla u(x)|^p dx \) of \( u \) in \( \Omega_0 \) is finite and as small as possible, given the values of \( u \) on \( \partial \Omega_0 \). (We give equivalent viscosity and weak \( p \)-Laplacian-based definitions in Section 1.4 where we will also define the terms “\( p-\)
superharmonic” and “p-subharmonic.”) Note that the continuity assumption can be replaced by a weaker regularity assumption, and that p-harmonic functions are the same as harmonic functions when p = 2.

Let $\Pi_i$ denote projection to the $i$th coordinate. The covariance matrix

$$C = \left(\int \Pi_i(x) \Pi_j(x) \, d\mu(x)\right)^d_{i,j=1}$$

of the noise measure $\mu$ is necessarily diagonal with $C_{i,i} = C_{j,j}$ for all $2 \leq i, j \leq d$. The main result of this paper is that as $\epsilon$ tends to zero, the functions $u_1^\epsilon$ and $u_2^\epsilon$ converge uniformly to the unique $p$-harmonic extension of $F$ (at least when $F$ is continuous and $\Omega$ is sufficiently regular), where $p$ is determined as follows: Given $\mu$, we define a constant $p = p(\mu) \in [1, \infty]$ by $p = \frac{C_{1,1} + C_{2,2} + 1}{C_{2,2}}$. Equivalently, $p$ is such that for some $\beta > 0$, we have $C_{1,1} + 1 = \beta q^{-1}$ and $C_{i,i} = \beta p^{-1}$ for $2 \leq i \leq d$, where $p^{-1} + q^{-1} = 1$. This $p$ is chosen so that if one player always chooses the vector $v_k$ to be some $v$ with $|v| = \epsilon$, and the other player always chooses $v_k$ to be $-v$, then for each $k$ the variance of $x_k - x_{k-1}$ is proportional to $q^{-1} \epsilon^2$ in the direction of $v$ and $p^{-1} \epsilon^2$ in each direction orthogonal to $v$. In other words, $q^{-1}$ and $p^{-1}$ are the relative sizes of the parallel and perpendicular variance when the players “tug” in opposite directions. The case $p = \infty$ corresponds to purely parallel variance and $p = 1$ to purely perpendicular variance.

We now describe the required regularity condition on $\Omega$. For a given noise measure $\mu$ and $p = p(\mu)$, a point $y \in \partial \Omega$ is called a game-regular boundary point of $\Omega$ if whenever the game starts near that $y$, player I has a strategy for making the game terminate near $y$ with high probability. More precisely, $y$ is game-regular if for every $\delta > 0$ and $\eta > 0$ there exists a $\delta_0$ and $\epsilon_0$ such that for every $x_0$ with $|x_0 - y| < \delta_0$ and $\epsilon < \epsilon_0$, player I has a strategy that
guarantees that an $\epsilon$-step game started at $x_0$ will terminate at a point on $\partial \Omega \cap B(y, \delta)$ with probability at least $1 - \eta$. We say $\Omega$ itself is game-regular if every $y \in \partial \Omega$ is game-regular. We say that a point $y \in \partial \Omega$ satisfies the cone property if there is a neighborhood $U$ of $y$ and a cone $C$ with tip $y$ such that $C \cap U$ is disjoint from $\Omega$. We say that $\Omega$ satisfies the cone property if every $y \in \partial \Omega$ satisfies the cone property. The following results will be proved in Section 2.

**Proposition 1.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ and fix a noise measure $\mu$ and $p = p(\mu) \in (1, \infty)$ as above. Then

(i) If $p > d$ then $\Omega$ is game-regular.

(ii) Every $y \in \partial \Omega$ that satisfies the cone property is game-regular.

(iii) If $d = 2$ and $\Omega$ is simply connected then $\Omega$ is game-regular.

**Theorem 1.2.** Let $\Omega$ be a bounded domain in $\mathbb{R}^d$. Fix a noise measure $\mu$ and $p = p(\mu) \in (1, \infty)$.

(i) Suppose $\Omega$ is game-regular and $F : \partial \Omega \to \mathbb{R}$ is continuous. Then as $\epsilon \to 0$, the game values $u^1_\epsilon$ and $u^2_\epsilon$ converge uniformly to the unique $p$-harmonic function $u$ that extends continuously to $F$ on $\partial \Omega$.

(ii) Conversely, if $\Omega$ is not game regular, then there exists a continuous function $F : \partial \Omega \to \mathbb{R}$ for which $u^1_\epsilon$ does not converge uniformly to a function that extends continuously to $\partial \Omega$.

Section 3 extends the above results to two natural variants of tug of war. The first is the same as the game described above except that the players alternate turns instead of deciding turn order with coin tosses. Alternating turn tug of war without noise is not interesting (since either player may choose a strategy of always undoing the other player’s moves, thereby preventing the game from terminating), but the values in alternating turn tug of war with noise converge to $p$-harmonic functions for an appropriate choice of $p$. The second variant interpolates between tug of war without noise ($p = \infty$) and certain optimal control processes introduced by Spencer nearly thirty years ago [16] and studied in detail by Kohn and Serfaty [8] ($p = 1$).

Analogous results for discontinuous boundary conditions and non-regular domains are presented in Section 4. Nonmeasurable strategies are briefly discussed in Section 5, and some unsolved problems are collected in the final section.

### 1.3 Motivation

In the case $p = \infty$, the tug of war games without noise were used to prove new uniqueness results for $\Delta_\infty u = g$, to solve the optimal Lipschitz extension problem on general length spaces, and to give various bounds on infinity harmonic measure [14]. We hope that the results of this paper will be similarly useful to the study of $p$-harmonic measure and $p$-capacity. In Section 4.3 we will see that some of the bounds in [14] translate to the $p$-harmonic case as well. In particular, the question of how the $p$-harmonic measure of a $\delta$
neighborhood of a Cantor set decays as $\delta$ tends to zero can be easily addressed using game theoretic arguments.

Furthermore, although the games in [15] were applied to solve problems in analysis, the authors of [15] (including the current authors) were originally motivated by the games themselves. We expect that tug of war games will have applications to political and economical modeling. They are natural models for situations in which the relevant state space is well summarized by finitely many parameters and in which opposing parties continually seek to improve their positions through incremental “tugs.” They are related to many of the differential games (in which players choose drift and diffusion terms at each point in a domain) used in economic modeling, but they are particularly simple in that the move sets are player-symmetric, and each player’s allowed set of incremental moves is independent of anything the other player does. Barron, Evans, and Jensen have recently developed some continuous time variants of tug of war without noise in which both players specify a control flow on the entire domain $\Omega$ in advance, and then the game position alternates randomly between evolving according to one player’s control and evolving according to the other player’s control until it reaches the boundary [3].

1.4 More definitions and background about $p$-harmonicity

On the space of real-valued functions on a subset $\Omega \subset \mathbb{R}^d$, we define the **infinity Laplacian** operator $\Delta_\infty$ by $\Delta_\infty u = |\nabla u|^{-2} \sum_{i,j} u_{ij} u_{ij}$. (Here $u_i = \frac{\partial u}{\partial x_i}$.)

When $u$ is twice differentiable and $\nabla u \neq 0$, the infinity Laplacian is the second derivative of $u$ in the gradient direction. For more on the infinity Laplacian, see the survey paper [1].

We similarly define the **1-Laplacian** $\Delta_1 = \Delta - \Delta_\infty$ where $\Delta u = \sum_{i=1}^d u_{ii}$ is the ordinary Laplacian. In other words, $\Delta_1 u$ is the sum of the second derivatives in each of the $d-1$ directions orthogonal to $\nabla u$.

For $1 < p < \infty$, a function that minimizes $\int |\nabla u(x)|^p dx$ (given its boundary values) solves the **Euler Lagrange equation**

$$\text{div}(|\nabla u|^{p-2} \nabla u) = 0. \quad (1)$$

When $u$ is smooth, we can write the left hand side as

$$\Delta_p^V u := \sum_{i=1}^d \frac{\partial}{\partial x_i} (|\nabla u|^{p-2} u_i)$$

$$= \sum_{i=1}^d |\nabla u|^{p-2} u_{ii} + \sum_{i=1}^d u_i \frac{\partial}{\partial x_i} (|\nabla u|^2)^{(p-2)/2})$$

$$= |\nabla u|^{p-2} \Delta u + \sum_{i=1}^d u_i \frac{p-2}{2} |\nabla u|^{p-4} \frac{\partial}{\partial x_i} |\nabla u|^2$$

$$= |\nabla u|^{p-2} \Delta u + (p-2)|\nabla u|^{p-4} \sum_{i=1}^d \sum_{j=1}^d u_i u_{ij} u_j$$

$$= p|\nabla u|^{p-2} (p^{-1} \Delta_1 + q^{-1} \Delta_\infty) u.$$
If \( \nabla u \neq 0 \), then the last expression vanishes precisely when \( p^{-1}\Delta_1 + q^{-1}\Delta_\infty = 0 \). Thus, for general \( 1 \leq p, q \leq \infty \) (with \( p^{-1} + q^{-1} = 1 \)), it is natural to define an operator \( \Delta^G \) to be the convex combination of \( \Delta_1 \) and \( \Delta_\infty \) given by \( \Delta^G := p^{-1}\Delta_1 + q^{-1}\Delta_\infty \). Since \( \Delta^G_p \) and \( \Delta^V_p \) differ only by a normalization factor of \( p|\nabla u|^{p-2} \), the equation \( \Delta^G_p u = 0 \) is equivalent in the classical sense to the Euler-Lagrange equation \( \Delta^V_p u = 0 \) (provided \( \nabla u \neq 0 \)). It is also equivalent in the weak sense as well as the viscosity sense, as discussed below. Note that if \( \nabla u(x) = 0 \), then \( \Delta^G_p u(x) \) is undefined in the classical sense (as is \( \Delta^V_p u(x) \) when \( p < 2 \)).

In much of the literature, the \( p \)-Laplacian is set to zero, so the distinction between \( \Delta^V_p \) and \( \Delta^G_p \) is irrelevant. The equations \( \Delta^V_p u = g \) for non-zero constant functions \( g \) have a natural variational interpretation in terms of minimizing the \( p \)-energy of \( u \) conditioned on the volume \( \int_{\Omega} u(x) dx \) bounded underneath \( u \). However, the solution \( u \) of \( \Delta^G_p u = g = -2f/\beta \) has a natural game-theoretic interpretation as the limiting value of a modified tug of war game in which in addition to the payoff player I receives when the game terminates, player I receives a “running payoff” of size \( f(x_k) \epsilon^2 \) at the \( k \)th step of the game. (See [14] for complete details in the case \( p = \infty \).) The normalization \( \Delta^G_p \) also has the aesthetic advantage of being a convex combination of \( \Delta_1 \) and \( \Delta_\infty \). Throughout the remainder of this paper, when we use the term \( p \)-Laplacian, or write \( \Delta_p \), we will always mean \( \Delta^G_p \).

We say that \( u \) is a viscosity subsolution to \( \Delta_p(\cdot) = g \) in an open set \( U \) if \( u \) is upper semi-continuous and for every \( x \in U \) and \( C^2 \) function \( \phi \) on a neighborhood of \( x \) such that

1. \( \phi(x) = u(x) \),
2. \( u \leq \phi \) in a neighborhood of \( x \),
3. \( \nabla \phi(x) \neq 0 \),

we have \( \Delta_p \phi(x) \geq g(x) \). We say \( u \) is a viscosity supersolution to \( \Delta_p(\cdot) = g \) if \( -u \) is a viscosity subsolution to \( \Delta_p(\cdot) = -g \). We make similar definitions for \( \Delta^V_p \) and observe that \( u \) is a viscosity subsolution to \( \Delta_p u = 0 \) if and only if it is a viscosity subsolution to \( \Delta^V_p u = 0 \).

(If the latter case, when \( p \geq 2 \), the relevant comparison functions \( \phi \) are those that have \( \Delta^V_p \phi(x) > 0 \) at \( x \), which in turn implies \( \nabla \phi \neq 0 \) and \( \Delta_p \phi(x) > 0 \).)

Both [7] and [6] define \( p \)-harmonic functions to be weak solutions to \( \Delta^V_p u = 0 \): precisely, a function \( u \) on a domain \( \Omega \) is called \( p \)-harmonic in \( \Omega \) (for \( 1 < p < \infty \)) if it is continuous and belongs to the Sobolev space \( W^{1,p}_{loc}(\Omega) \) (i.e., \( \nabla u \in L^p_{loc}(\Omega) \)) and

\[
\int_{\Omega} |\nabla u|^{p-2}(\nabla u, \nabla \phi) dx = 0
\]

for every \( \phi \in C^\infty_0(\Omega) \). (This is the distributional form of [11].) Say that \( u : \Omega \to (-\infty, \infty] \) is \( p \)-superharmonic if \( u \) is lower semicontinuous, \( u \neq \infty \), and \( u \) satisfies the following comparison principle on each bounded subdomain \( \Omega_0 \) with \( \overline{\Omega_0} \subset \Omega \): if \( h \in C(\overline{\Omega_0}) \) is \( p \)-harmonic in \( \Omega_0 \) and \( u \geq h \) on \( \partial \Omega_0 \), then \( u \geq h \) in \( \Omega_0 \). We say \( u \) is \( p \)-subharmonic if \( -u \) is \( p \)-superharmonic.

We now cite the following:

**Proposition 1.3.** When \( 1 < p < \infty \), a function is \( p \)-superharmonic in \( \Omega \) if and only if it is a viscosity \( p \)-super-solution. Moreover, \( u \) is \( p \)-harmonic (equivalently, a viscosity \( p \)-solution) if and only if \( u \) is a continuous solution to the variational problem, i.e., for every subdomain \( \Omega_0 \) with compact closure \( \overline{\Omega_0} \subset \Omega \), the integral \( \int_{\Omega_0} |\nabla u|^p \) is finite and as small as possible, given the values of \( u \) on \( \partial \Omega_0 \).
The viscosity and weak equivalence was proved in [7]. The equivalence of (2) with the
variational problem is classical Euler-Lagrange theory; see, e.g., the reference text [6] on non-
linear potential theory. Recall that \( W^{1,p}_0(\Omega) \) is the closure in \( W^{1,p}(\Omega) \) of the smooth functions
supported on compact subsets of \( \Omega \). From Chapter 3 and Chapter 5 of [6] we cite the
following:

**Proposition 1.4.** If \( \Omega \subset \mathbb{R}^d \) is bounded and \( \theta \in W^{1,p}(\Omega) \), then there exists a unique \( p \)-harmonic function \( u \) in \( \Omega \) such that \( u - \theta \in W^{1,p}_0(\Omega) \); moreover, this \( u \) minimizes the integral
\( \int_\Omega |\nabla u|^p \) among all functions \( v \in W^{1,p}(\Omega) \) for which \( v - \theta \in W^{1,p}_0(\Omega) \).

**Proposition 1.5.** If \( \Omega \) is bounded and \( \phi \in C(\overline{\Omega}) \) is \( p \)-harmonic in \( \Omega \), then any \( p \)-harmonic function which extends continuously to the boundary of \( \Omega \) and agrees with \( \phi \) on the boundary of \( \Omega \) is equal to \( \phi \).

We also quote the following smoothness result, which was proved by DiBendetto [4] and
independently by Tolksdorf [17], who extended earlier work by Morrey, Uhlenbeck, Evans
and others.

**Proposition 1.6.** If \( u \) is \( p \)-harmonic in \( \Omega \), then it is everywhere differentiable in \( \Omega \) and real
analytic wherever \( \nabla u \neq 0 \). Moreover, \( u \) has a Hölder continuous gradient—i.e., \( u \in C^{1,\gamma}(\Omega) \)
for some \( \gamma > 0 \).

## 2 Proofs

### 2.1 Strategy definition

When two players play tug of war, we define the **history up to step** \( k \) of the game to be
the sequence \( h_k = \{x_0, v_1, x_1, v_2, x_2, \ldots, v_k, x_k\} \). (If the game terminated at time \( j < k \), we
set \( v_m = 0 \) and \( x_m = x_j \) for \( m \geq j \).) This \( h_k \) belongs to the space \( H_k = \Omega \times (B(0,\epsilon) \times \overline{\Omega})^k \). The complete **history space** \( H_\infty \) is the set of all infinite game position sequences \( h = \{x_0, v_1, x_1, v_2, x_2, \ldots\} \).

We endow \( H_\infty \) with the product topology. It is easy to see that if \( F : \partial \Omega \to \mathbb{R} \) is Borel measurable, then the payoff for player I is a Borel measurable function on \( H_\infty \). A **measurable pure** strategy is a sequence of Borel measurable maps from \( H_k \) to \( B(0,\epsilon) \), giving
the move a player would make at the \( k \)th step of the game as a function of the game history.
A pair of strategies \( \sigma = (S_I, S_{II}) \) (where \( S_I \) is a strategy for player I and \( S_{II} \) is a strategy
for player II) and a starting point \( x \) determine a unique probability measure \( P_x \) in \( H1 \) (constructed using Kolmogorov’s extension theorem). Denoting the corresponding expectation by
\( E_x \), the expected payoff for this strategy pair is \( V(S_I, S_{II}) = E_x \left[ F(x_\tau) 1_{\tau < \infty} \right] \) where \( \tau \) is the
exit time of \( \Omega \). We then have \( u_1^\varepsilon = \sup_{S_I} \inf_{S_{II}} V(S_I, S_{II}) \) and \( u_2^\varepsilon = \inf_{S_{II}} \sup_{S_I} V(S_I, S_{II}) \).

The restriction to Borel measurable strategies is necessary since \( \mu \) is a Borel measure.
It is also natural, since measurable strategies are in some sense the only ones that can be
implemented in a constructive way. (We mention some relaxations of the measurability
condition in Section [5].)
2.2 Smooth case, non-vanishing gradient

If $x$ is a matrix or column vector, denote by $x^T$ the transpose of $x$. Suppose that $z$ has law $\mu$ and $z_1 = \Psi(z)$ has law $\mu_v$ (where $\Psi$ is a rotation multiplied by $|v|$ and $\Psi(e_1) = v$). As noted above, $z$ has mean zero and covariance matrix

$$C^z = (\beta q^{-1} - 1)e_1e_1^T + \beta p^{-1}(I - e_1e_1^T).$$

Therefore, $z_1$ has mean zero and covariance matrix

$$C^{z_1} = (\beta q^{-1} - 1)vv^T + \beta p^{-1}(|v|^2I - vv^T).$$

Next, we need some basic lemmas about quadratic functions. Let $\phi(x) = x^TAx + (\xi, x)$ (for $\xi \in \mathbb{R}^d \setminus \{0\}$) be a quadratic function. Without loss of generality, $A$ is a symmetric matrix, since only the quadratic form induced by $A$ is used. Suppose that $x_0 = 0$ and let $\psi(v)$ be the expected value of $\phi(x_1) = \phi(v + z_1)$ if player I chooses $v_1 = v$. Let $z_1^i$ denote the $i$th component of $z_1$. Since $\mathbb{E}z_1 = 0$, we have

$$\psi(v) := \mathbb{E}\phi(v + z_1) = \phi(v) + \sum_{i,j} A_{i,j} \mathbb{E}z_1^i z_1^j = \phi(v) + \sum_{i,j} A_{i,j} C^{z_1}_{i,j}.$$ 

Note that $\sum_{i,j}(vv^T)_{i,j}A_{i,j} = v^TAv$ and $\sum_{i,j} I_{i,j}A_{i,j} = \text{Tr}A$. Therefore

$$\psi(v) = \phi(v) + (\beta q^{-1} - 1)v^TAv + \beta p^{-1}(|v|^2\text{Tr}A - v^TAv)$$

$$= \phi(v) + (\beta q^{-1} - \beta p^{-1} - 1)v^TAv + \beta p^{-1} |v|^2\text{Tr}A$$

$$= (\beta q^{-1} - \beta p^{-1})v^TAv + \beta p^{-1} |v|^2\text{Tr}A + (\xi, v),$$

whence $\psi(v) = (\xi, v) + v^TBv$ where $B = (\beta q^{-1} - \beta p^{-1})A + \beta p^{-1}(\text{Tr}A)I$. In the following, we use the $L^2$ operator norm on matrices given by $\|B\| = \sup_{|v|=1} |Bv|$. 

**Lemma 2.1.** Fix $\xi \in \mathbb{R}^d \setminus \{0\}$, and let $\zeta = \frac{4|B|}{|\xi|^2}$. If $\epsilon < \zeta^{-1}$, then any $v \in \overline{B}(0, \epsilon)$ for which $\psi(v) = (\xi, v) + v^TBv$ is maximal within $\overline{B}(0, \epsilon)$, satisfies $|v| = \epsilon$ and $|v - (\epsilon|\xi|^{-1}\xi)| \leq \zeta \epsilon^2$.

**Proof.** We first observe that the maximum of $\psi$ within $\overline{B}(0, \epsilon)$ is obtained on the boundary $\partial B(0, \epsilon)$: Otherwise $\nabla\psi(v) = \xi + 2Bv$ must vanish at the maximum, but

$$|2Bv| \leq 2\|B\|\epsilon < 2\|B\|\zeta^{-1} < |\xi|.$$ 

Let $w = \epsilon|\xi|^{-1}\xi$. Then $\psi(v) \geq \psi(w) = \epsilon|\xi| + w^TBw$. This means

$$(\xi, v) \geq \epsilon|\xi| + w^TBw - v^TBv \geq \epsilon|\xi| - 2\|B\|\epsilon|v - w|.$$ 

Multiplying by $\epsilon|\xi|^{-1}$ gives

$$(w, v) \geq \epsilon^2 - \epsilon^2 \frac{|\xi|}{2}|v - w|,$$ 

whence

$$|v - w|^2 \leq 2\epsilon^2 - 2(w, v) \leq \zeta|v - w|\epsilon^2.$$
Lemma 2.2. Given $v, \psi, B$, and $\xi$ as in the statement Lemma 2.1, we have the following approximation of $\psi(v)$:

$$|\psi(v) - \psi(w) - \Delta_\infty \psi(0)\epsilon^2/2| \leq 16\|B\|_2^2 \epsilon \|\xi\|,$$

for all $\epsilon < 1$.

Proof. Since $\Delta_\infty \psi(0)$ is the second derivative of $\psi$ in the $\xi$ direction, the vector $w = \epsilon|\xi|^{-1}\xi$ satisfies $\psi(w) = \epsilon|\xi| + \Delta_\infty \psi(0)\epsilon^2/2$. Let $\zeta$ be as in Lemma 2.1. Recall that $|v| = |w| = \epsilon$ and $|w - v| < \zeta \epsilon^2$. By (3),

$$(w, w - v) \leq \frac{\epsilon^2 \zeta^2}{2} |v - w| \leq \frac{\epsilon^4 \zeta^2}{2}.$$

Therefore,

$$|\psi(v) - \psi(w)| \leq \frac{|\xi|}{\epsilon} |(w, w - v)| + 2\|B\|\epsilon|v - w| \leq \frac{|\xi| \epsilon^3 \zeta^2}{2} + 2\|B\|\zeta \epsilon^3 = \frac{16\|B\|_2^2 \epsilon^3}{|\xi|}.$$

If $\|B\|$ is bounded above and $|\xi|$ is bounded below, then Lemma 2.2 implies that the expected change in $\phi$ when player I makes an optimal move is the same as the expected change when player I moves in the gradient direction, up to an error of $O(\epsilon^3)$. These estimates deteriorate badly when $|\xi|$ tends to zero and $\|B\|$ gets large. However, they are enough to prove the following:

Lemma 2.3. Let $\phi(x) = x^T Ax + (\xi, x)$. Fix $k \geq 0$ and suppose that player I’s strategy in move $k + 1$ is to tug distance $\epsilon$ in the gradient direction of $\phi$ if player I wins the coin toss. Then regardless of player II’s strategy, we have

$$\mathbb{E}[\phi(x_{k+1})|h_k] \geq \phi(x_k) + \frac{\epsilon |\xi|}{2} \Delta_\infty \phi(x_k)\epsilon^2 - M\epsilon^3,$$

where

$$h_k = \{x_0, v_1, x_1, v_2, x_2, \ldots, v_k, x_k\}$$

is the game history up to step $k$ and $M = \frac{16\beta(d+1)^2\|A\|_2^2}{|\xi|}$.

Proof. Recall that $\psi(v) = (\xi, v) + v^T Bv$ where $B = (\beta q^{-1} - \beta p^{-1})A + \beta p^{-1}(\text{Tr}A)I$. Thus

$$\Delta_\infty \psi(0) = (\beta q^{-1} - \beta p^{-1})\Delta_\infty \phi + \beta p^{-1}\Delta_\infty \phi = \beta \Delta_\phi (0).$$

Lemma 2.2 implies that if player I adopts the strategy in the statement, then

$$\mathbb{E}[\phi(x_{k+1})|h_k^I] \geq \phi(x_k) + \epsilon|\xi| + \frac{\beta}{2} \Delta_\phi(x_k)\epsilon^2 - M\epsilon^3,$$

where $h_k^I$ indicates that in move $k + 1$, player I won the coin toss and chooses $v_{k+1}$, whence the conditional expectation is simply integrating with respect to $\mu_{v_{k+1}}$. Minimizing $\psi$ over
\( \mathcal{B}(0, \epsilon) \) is the same as maximizing \(-\psi\) there. Now \(-\psi\) is obtained from \(\psi\) by replacing \(\xi\) and \(B\) by \(-\xi\) and \(-B\), respectively. Therefore, Lemma 2.2 also implies that regardless of player II’s strategy, we have

\[
\mathbb{E}[\phi(x_{k+1})|h_k'] \geq \phi(x_k) - \epsilon|\xi| + \frac{\beta}{2} \Delta_p \phi(x_k) \epsilon^2 - M\epsilon^3,
\]

where the conditional expectation is integrating with respect to \(\mu_{x_k+1}\), as before, but now \(v_{k+1}\) was chosen by player II. Averaging the last two displayed equations proves the lemma.

Given a differentiable function \(u\) with non-vanishing gradient, the gradient strategy for player I is to always take

\[
v_k = \epsilon|\nabla u(x_{k-1})|^{-1} \nabla u(x_{k-1})
\]

at every step of the game regardless of what player II does. The gradient strategy for player II is to always take \(v_k = -\epsilon|\nabla u(x_{k-1})|^{-1} \nabla u(x_{k-1})\). We can interpret Lemma 2.3 as a bound on the expected change in \(u\) when the gradient strategy is employed.

**Theorem 2.4.** Suppose that \(u\) is \(p\)-harmonic on a domain \(\Omega'\) and \(\nabla u \neq 0\) throughout \(\Omega'\). Let \(\Omega\) be a domain whose closure is a compact subset of \(\Omega'\). Fix a noise measure \(\mu\). For \(x \in \Omega\), let \(u_1'(x)\) be the value for player I of the game in \(\Omega\), started at \(x_0 = x\), with boundary conditions \(F = u\) on \(\partial\Omega\). Then the functions \(u_1'\) converge uniformly to \(u\). In fact, \(|u_1' - u|_\infty = O(\epsilon)\). (The implied constant may depend on \(\mu\).)

**Proof.** We may assume (adding a constant if necessary) that \(u > 0\) on \(\overline{\Omega}\). By Proposition 1.6, \(u\) is real analytic in \(\Omega'\), and in particular its derivatives restricted to \(\Omega\) are all bounded. We thus have the Taylor expansion at any \(y \in \Omega\) given by \(u(x) = u(y) + (\nabla u, (x - y)) + \frac{1}{2}(x - y)^T (D^2 u)(x - y) + O((x - y)^3)\). Thus, at every \(x\), the function \(\phi(x) = (\nabla u, y) + (x - y)^T D^2 u(x - y)\) approximates \(u\) up to an error of \(O(\epsilon^3)\) within the ball of radius \(2(\alpha + 1)\epsilon\) of \(y\). The bounds in Lemma 2.3 then imply that player I, by adopting the gradient strategy, can ensure that for some constant \(c\) the sequence \(M_k = u(x_k) + c\epsilon k^3\) is a submartingale.

The non-vanishing gradient and compactness imply that for some \(c_1 > 0\), we have \(u(x_k) - u(x_{k-1}) \geq c_1 \epsilon\) provided player I wins the coin toss and \(\epsilon\) is small enough. Therefore \(\mathbb{E}[|M_k - M_{k-1}|^2|h_{k-1}] \geq \frac{1}{2} c_1^2 \epsilon^2\). Consequently, the difference \(M_k^2 - M_0^2 - c_2 \epsilon^2 k\), where \(c_2 = c_1^2/2\), is a submartingale, because

\[
\mathbb{E}[M_k^2 - M_{k-1}^2|h_{k-1}] = \mathbb{E}[(M_k - M_{k-1})^2|h_{k-1}] + \mathbb{E}[2(M_k - M_{k-1})M_{k-1}|h_{k-1}],
\]

and the second term on the right hand side is non-negative.

Suppose that player I adopts the gradient strategy and player II adopts some arbitrary strategy \(S_{II}\). Use \(\mathbb{P}_x\) to denote probability when the initial game position is \(x_0 = x\). Then have

\[
c_2 k \mathbb{P}_x[\tau \geq k] \leq c_2 \mathbb{E}_x[\tau \land k] \\
\leq \epsilon^{-2} \mathbb{E}_x[M_{\tau \land k}^2 - M_0^2] \\
\leq \epsilon^{-2} (8\|u\|_\infty^2 + 2c_2^2 k^2 \epsilon^6),
\]
so \( P_\epsilon(\tau > a\epsilon^{-2}) \leq 1/2 \) for some \( a \) (independent of \( \epsilon \)).

Since this holds uniformly in \( \epsilon \) and \( x \), iteration shows that \( \frac{\tau}{a\epsilon} \) can be bounded by a geometric(1/2) random variable. Thus, \( E_\epsilon[\tau] = O(\epsilon^{-2}) \). Applying the optional stopping time theorem for submartingales,

\[
\tau\epsilon^{-2} \leq 1/2 \text{ for some } a \text{ (independent of } \epsilon). 
\]

Thus, \( u_1(x) \geq u(x) - O(\epsilon) \). A symmetric argument gives \( u_2(x) \leq u(x) + O(\epsilon) \), so \( |u_1 - u|_\infty = O(\epsilon) \), since \( u_1' \leq u_2' \).

We now show that the solution \( u \) of \( \Delta_p u = g = -2f/\beta \) has a natural game-theoretic interpretation as the limiting value of a modified tug of war game in which in addition to payoff player I receives when the game terminates, player I receive s a “running payoff” of size \( f(x_k)\epsilon^2 \) at the \( k \)th step of the game.

**Theorem 2.5.** Suppose that \( u \) is smooth on a bounded domain \( \Omega' \supset \overline{\Omega} \) with non-vanishing gradient and \( \Delta_p u = g \), where \( f = -\frac{\beta}{2}g \) is bounded below on \( \Omega \) by a positive constant. Then \( u \) is the limit as \( \epsilon \to 0 \) of the functions \( u_1' \) defined for tug of war with noise on \( \Omega \), with running payoff \( f \) and boundary payoff given by \( F = u \) on \( \partial \Omega \).

**Proof.** The proof is exactly the same as in Theorem 2.4 (the estimates in Lemma 2.3 include the running payoff case), except that we need a different argument to show that the expected number of turns in the game is \( O(\epsilon^{-2}) \). But since \( f \) is bounded below, if player I adopts the strategy of always pulling distance \( \epsilon \) in the gradient direction, either the expected number of turns in the game is less than

\[
\sup_{\Omega} u(x) - \inf_{\partial \Omega} F(x) \epsilon^{-2},
\]

in which case the arguments of Theorem 2.4 apply, or it is greater than this, in which case player I’s expected payoff is even greater than \( u(x) \).

On the other hand, player II can adopt a strategy which makes the process \( u(x_k) + \sum_{i=0}^{k-1} f(x_i)\epsilon^2 - c\epsilon^3 \) a supermartingale (for some constant \( c \) independent of \( \epsilon \) and \( x_0 \)). Since this supermartingale is bounded below, by optional stopping

\[
\mathbb{E}\left[u(x_\tau) + \sum_{i=0}^{\tau-1} f(x_i)\epsilon^2 - c\tau\epsilon^3\right] \leq u(x_0).
\]

Hence

\[
\mathbb{E}\left[u(x_\tau) + \tau\left(\inf_{x\in\Omega} f(x)\epsilon^2 - ce^3\right)\right] \leq u(x_0).
\]

Since \( u(x_\tau) \) is bounded on \( \Omega \), this implies \( \mathbb{E}\tau = O(\epsilon^{-2}) \).

The remainder of this paper will focus on zero running payoff case \( f = 0 \).
2.3 Continuous boundary conditions, regular boundary

Even if the boundary function $F$ is continuous and $u'$ converges pointwise to a function $u$, it may not be the case that $u$ extends continuously to $F$ on $\partial \Omega$. For example, in dimension two, if $p \leq 2$ and $\Omega$ is the unit ball minus the origin and $f$ is the function which is 1 at the origin and 0 at the boundary of the unit ball, then it is not hard to see that as $\lim_{\epsilon \to 0} u_1^\epsilon = 0$ throughout $\Omega$, which is discontinuous at the origin. However, we will see that this cannot happen if $\Omega$ is sufficiently regular. This section extends the results of the previous section to sufficiently regular domains by proving Theorem 1.2.

Proof of Theorem 1.2. (i) Suppose that $\Omega$ is game-regular. Our strategy will be to use this fact to establish some a priori equicontinuity on the $u_1^\epsilon$ that will ensure (via a compactness argument) subsequential uniform convergence of the $u_1^\epsilon$ to a limit that extends continuously to $F$ on $\partial \Omega$. Then we will use comparison arguments and Theorem 2.4 to show that any such limit is a viscosity solution to $\Delta_\gamma u = 0$, which is unique by Proposition 1.5.

Fix a continuous $F$ and write $b_\epsilon = \min_{y \in \partial \Omega} F(y)$ and $b_U = \max_{y \in \partial \Omega} F(y)$. Given a constant $\gamma$, we can find a $\delta$ such that $|F(x) - F(y)| < \gamma/2$ whenever $|x - y| < \delta$. We then define $\eta = \frac{\gamma}{2(b_U - b_\epsilon)}$.

By game-regularity, for each $y \in \partial \Omega$ we can find $\delta_0 = \delta_0(y)$ and $\epsilon_0 = \epsilon_0(y)$ such that for every $x_0$ with $|x_0 - y| < \delta_0$ and $\epsilon < \epsilon_0$, player I has a strategy that guarantees that an $\epsilon$-step game started at $x_0$ will terminate at a point on $\partial \Omega \cap B(y, \delta)$ with probability at least $1 - \eta$. The probability that $F$ at the terminal point differs from the $F(y)$ by more than $\gamma/2$ is at most $\frac{\gamma}{2(b_U - b_\epsilon)}$, so we have that $|u_1^\epsilon(x) - F(y)| \leq \gamma$ for each $x \in B(y, \delta_0)$.

By compactness of $\partial \Omega$ (a closed bounded set), we can find a finite collection $y_1, \ldots, y_N$ such that

$$\partial \Omega \subset S = \bigcup_{i=1}^N B(y_i, \delta_0(y_i)/2).$$

Now, define

$$\overline{\delta}_0 = \text{dist}(\Omega \setminus S, \partial \Omega) \land \min_{1 \leq i \leq N} \delta_0(y_i)/2,$$

and $\overline{\tau}_0 = \min_{1 \leq i \leq N} \epsilon_0(y_i) \land \overline{\tau}_0/2(\alpha)$. Every point $x$ within distance $\overline{\delta}_0$ of $\partial \Omega$ has the property that $|u_1^\epsilon(x) - u_1^\epsilon(x')| \leq 2\gamma$ for every $x'$ with $|x - x'| \leq \overline{\delta}_0$ and $\epsilon < \overline{\tau}_0$, because $x \in B(y_i, \delta_0(y_i)/2)$ for some $i$, and hence $x' \in B(y_i, \delta_0(y_i))$. This in turn implies that

$$\text{for every } \epsilon < \overline{\tau}_0 \text{ and any } x, x' \in \Omega \text{ with } |x - x'| < \overline{\delta}_0, \text{ we have } |u_1^\epsilon(x) - u_1^\epsilon(x')| < 2\gamma. \quad (4)$$

The reason is that a player starting at $x'$ can always adopt the same strategy that would have been chosen starting at $x$ (i.e., define $x_j = x_j + x' - x$ and when the game position is some $x_{k-1}$ a player can choose the step $v_k$ that would be chosen if the game position were $x_{k-1}$ up until the first time that $\text{dist}(x_k, \partial \Omega) \land \text{dist}(x_k', \partial \Omega) < \overline{\delta}_0$. At that point we have $|u_1^\epsilon(x_k) - u_1^\epsilon(x_k')| \leq 2\gamma$, and the result follows. By symmetry, the analog of (4) with $u_1^\epsilon$ replaced by $u_2^\epsilon$ also holds. Consider $\underline{u}_1(x) := \liminf_{\epsilon \to 0} u_1^\epsilon(x)$ and $\overline{\underline{u}}_2(x) := \limsup_{\epsilon \to 0} u_2^\epsilon(x)$. The asymptotic equicontinuity (1) implies that $\underline{u}_1$ and $\overline{\underline{u}}_2$ are (uniformly) continuous functions on $\overline{\Omega}$ which agree on $\partial \Omega$. If we can show that these functions are both $p$-harmonic, then uniqueness of extensions (Proposition 1.5) and the general inequality $u_1^\epsilon \leq u_2^\epsilon$ will imply that the pointwise limits $\lim_{\epsilon \to 0} u_i^\epsilon(x)$ for $i = 1, 2$ exist and coincide everywhere in $\Omega$. (Another application of (4) will then yield that the convergence is uniform.)
Using symmetry, it only remains to show that \( u^i \) is \( p \)-harmonic. By Proposition 1.3 it is enough to show that \( u^i \) is a viscosity solution of \( \Delta_p u = 0 \) in \( \Omega \). We will verify that \( u^i \) is a viscosity subsolution; the proof that it is a supersolution is similar.

Let \( \phi \) be a \( C^2 \) function in a neighborhood \( V_0 \) of \( x \in \Omega \) such that \( \phi(x) = u^i(x) \), the inequality \( u^i \geq \phi \) holds in \( V_0 \) and \( \nabla \phi(x) \neq 0 \) in \( V_0 \). We must show that \( \Delta_p \phi(x) \geq 0 \). Otherwise, there exist \( r > 0 \) and \( \theta > 0 \) such that \( \Delta_p \phi \leq -\theta \) on \( B(x, r) \subset V_0 \). Let

\[
\tilde{\phi}(y) = \phi(y) + \frac{\theta}{4d}(y - x)^T(y - x) - \frac{\theta r^2}{8d},
\]

denote \( g = \Delta_p \tilde{\phi} \leq \Delta_p \phi + \theta/2 \leq -\theta/2 \), and consider noisy tug of war played in \( B(x, r) \) with stepsize \( \epsilon \), running payoff \( f = -\frac{\beta}{2} g \geq \frac{\beta \theta}{4} \) and boundary values \( \tilde{\phi} \) on \( \partial B(x, r) \). By Theorem 2.5, the value functions for player I in this game (which must be greater than \( u^i \) for small \( \epsilon \) due to the positive running payoff and the larger boundary values), converge as \( \epsilon \to 0 \) to \( \tilde{\phi} \). This is a contradiction, since \( \tilde{\phi}(x) < u^i(x) \). Thus we have shown that \( \Delta_p \phi(x) \geq 0 \), and the proof is complete.

**Proof of Theorem 1.2(ii).** Suppose that for every continuous \( F \), the \( u^i \) converge uniformly. Then we claim that \( y \) is game regular, i.e., i.e., that for every \( \delta > 0 \) and \( \eta > 0 \) there exists a \( \delta_0 \) and \( \epsilon_0 \) such that for every \( x_0 \) with \( |x_0 - y| < \delta_0 \) and \( \epsilon < \epsilon_0 \), player I has a strategy that guarantees that an \( \epsilon \)-step game started at \( x_0 \) will terminate at a point on \( \partial \Omega \cap B(y, \delta) \) with probability at least \( 1 - \eta \). To see this, define \( F(y') = -|y - y'| \) on \( \partial \Omega \) and let \( u \) be its extension. Given \( \delta \) and \( \eta \) we can choose \( \delta_0 \) such that \( |x_0 - y| < \delta_0 \) implies \( |u(x_0) - u(y)| < \eta \delta/2 \). By uniform convergence, we can find \( \epsilon_0 \) such that for \( \epsilon < \epsilon_0 \) we have \( ||u^i - u|\infty < \eta \delta/2 \) which implies that \( |u^i(x_0)| < \eta \delta \), and hence player one has a strategy that guarantees that the probability of ending \( \delta \) away from \( y \) is at most \( \eta \).

\[\square\]

### 2.4 Sufficient conditions for regularity

We first present a condition equivalent to game-regularity which is easier to verify.

**Lemma 2.6.** Fix \( p > 1 \), a measure \( \mu \) as in the introduction and a domain \( \Omega \subset \mathbb{R}^d \). A point \( y \in \partial \Omega \) is a game-regular boundary point of \( \Omega \) if and only if there is some \( \theta > 0 \) with the following property: for every \( \delta > 0 \) there exists a \( \delta_0 \) and \( \epsilon_0 \), such that for every \( x_0 \) with \( |x_0 - y| < \delta_0 \) and \( \epsilon < \epsilon_0 \), player I has a strategy that guarantees that an \( \epsilon \)-step game started at \( x_0 \) will terminate at a point on \( \partial \Omega \) before exiting \( B(y, \delta) \) with probability at least \( \theta \).

**Proof.** Given \( \theta \) as in the statement of the lemma and \( \eta \) as in the definition, find \( k \) such that \((1 - \theta)^k \eta < \eta \). For \( \epsilon_0 \) and \( \delta_0 \) as in the statement, there exist \( \delta_1 \) and \( \epsilon_1 \) such that for every \( x_0 \) with \( |x_0 - y| < \delta_1 \) and \( \epsilon < \epsilon_1 \), player I has a strategy that guarantees that an \( \epsilon \)-step game started at \( x_0 \) will terminate at a point on \( \partial \Omega \) before exiting \( B(y, \delta_0 - \alpha \epsilon) \), with probability at least \( \theta \). Iterating this argument \( k \) times proves the lemma.

\[\square\]

The following is needed for the proof of Proposition 1.1.

**Lemma 2.7.** Fix \( p \) with \( 1 < p \leq \infty \). Given positive constants \( r, s, t, \epsilon \) with \( 0 < s < 1 < t \), consider an \( \epsilon \)-game of tug of war with noise on an annulus \( B(0, tr) \setminus B(0, sr) \) with \( 1 < p \leq \infty \),
and $0 < s < 1 < t$. If the initial game position is some $x_0$ with $|x_0| - \alpha \epsilon \leq r \leq |x_0| + \alpha \epsilon$, and the ratio $t/s$ is held fixed, then as $\epsilon \rightarrow 0$, player I has a strategy that guarantees that the game position will terminate on $\partial B(0, sr)$ (instead of $\partial B(0, tr)$) with probability at least $b - O(\epsilon)$ where

$$b = b(s, t, p, d) := \frac{t^c - 1}{t^c - s^c}$$

where $c = c(p, d) := \frac{p-d}{p-1}$. (As in Theorem 2.4, the constant implied by $O(\cdot)$ may depend on $\mu$.)

**Proof.** Given $d \geq 1$ and $p > 1$ and $x \in \mathbb{R}^d \backslash \{0\}$, we write

$$\rho_{d,p}(x) := \begin{cases} |x|^{c(p,d)} & p \neq d \\ \log |x| & p = d \end{cases},$$

where $c = c(p, d) = \frac{p-d}{p-1}$. The reader may check that for each $1 < p < \infty$ the function $\rho_{d,p}(x)$ is a radially symmetric $p$-harmonic function on $\mathbb{R}^d \backslash \{0\}$. Taking $F$ to be the function that has these values on the boundary of the annulus and applying Theorem 2.4, we see that if $r$, $s$, and $t$ are fixed, player I can achieve a probability of reaching the inside first that is, up to an $O(\epsilon)$ error, given by

$$\frac{\rho_{d,p}(tr) - \rho_{d,p}(r)}{\rho_{d,p}(tr) - \rho_{d,p}(sr)} = \frac{(tr)^c - r^c}{(tr)^c - (sr)^c} = \frac{t^c - 1}{t^c - s^c}.$$

(A similar argument applies when $p = d$.) If only the ratio $t/s$ is fixed, then the fact that the error is $O(\epsilon)$ follows similarly from Theorem 2.4 and a rescaling that replaces $\epsilon$ with $\epsilon' = \epsilon/(rs)$ and replaces $r$ with $r' = r/(rs)$ (so that $r's = 1$ and $r't$ is fixed).

\[\square\]

**Proof of Proposition 1.1.** (i) We first prove that all domains are game-regular when $p > d$. Fix $s = 1/2$ and $t = 2$. Then since $c(p, d) > 0$, we have $b = b(s, t, p, d) > 1/2$. Consider the domain $\Omega = \mathbb{R}^d \backslash \{0\}$. For each integer $m \geq 0$, write $r_m = 2^m \alpha \epsilon$ and let $A_m$ denote the closed annulus centered at 0 with radii $r_m + \alpha \epsilon$ and $r_m - \alpha \epsilon$.

By Lemma 2.7, if the game position begins at $x_0 \in A_m$, and $m \geq 1$, then Player I can arrange, with probability at least $b - O(\epsilon/r_m) = b - O(2^{-m})$ to have the game position enter $A_{m-1}$ before entering $A_{m+1}$. Next, let $P_m$ denote the infimum over all starting points $x_0 \in A_m$ of the maximum probability with which player I can guarantee that the game position hits $\overline{B}(0, \alpha \epsilon)$ before hitting $A_{m+1}$. It is not hard to see that $P_m > 0$ for every $m \geq 1$. (If player I wins a long sequence of coin tosses—and chooses a length $\epsilon$ vector uniformly at random each time—then the game position will undergo a symmetric random walk that moves by at most $\alpha \epsilon$ at each step. It is enough to observe that any such random walk hits $\overline{B}(0, \alpha \epsilon)$ before $A_{m+1}$ in some fixed number of steps with a probability that is bounded below independently of the choice of starting position in $A_m$.) Since player I can arrange to have at least a $b_m = b - O(2^{-m})$ chance of reaching $A_{m-1}$ before $A_{m+1}$ when the game positions starts in $A_m$, we have

$$P_m \geq b_m \left( P_{m-1} + (1 - P_{m-1})P_m \right).$$
Rearranging terms, we get
\[
(1 - b_m + b_m P_{m-1}) P_m \geq b_m P_{m-1}
\]
and hence
\[
P_m \geq \frac{b_m P_{m-1}}{1 - b_m + b_m P_{m-1}} = \frac{P_{m-1}}{b_m - 1 + P_{m-1}}.
\]
The right hand side is less than or greater than \(P_{m-1}\) as \(P_{m-1}\) is, respectively, greater than or less than \(2 - b_m^{-1}\). From this it is not hard to see that \(P_m\) is bounded below independently of \(m\). In fact, \(\liminf P_m \geq 2 - b^{-1}\).

Now, to prove game-regularity, we must show that there exists \(\theta > 0\) such that for every \(\delta > 0\), there exists a \(\delta_0\) and \(\epsilon_0\) with the property that for every \(x_0\) with \(|x_0 - y| < \delta_0\) and any \(\epsilon < \epsilon_0\), player I has a strategy that guarantees that an \(\epsilon\)-step game started at \(x_0\) will terminate at a point on \(\partial \Omega \cap B(y, \delta)\) with probability at least \(\theta\). We may take \(\delta_0 = \delta/4\), so that for every sufficiently small \(\epsilon\) we have \(\delta_0 \leq r_m < r_{m+1} \leq \delta\) for some \(m\). Then if \(|x_0 - y| < \delta_0\), player I can arrange to reach \(\overline{B}(y, \epsilon)\) (or terminate the game sooner) before exiting \(B(y, \delta)\) with probability at least \(P_m\), which is bounded below by some constant \(\theta\) independent of \(m\).

\textbf{(ii)} The second part of the proposition states that all domains having the cone property are game-regular. Assume without loss of generality that \(y = 0 \in \partial \Omega\) is the tip of the cone, and we will argue that \(y\) is game-regular. There exists some constant \(\chi \in (0, 1)\) and \(R\) such that for every \(r < R\), there exists a ball of radius \(\chi r\) contained in \(\mathcal{C}\) whose center is of distance \(r\) from the origin. If the game position begins at a point \(x_0\) whose distance is \(r\) from the tip of the cone, then player I can adopt the strategy of pulling towards the center of a ball of radius \(\chi r\) centered at a point \(z\) in \(\mathcal{C}\) (which is distance at most \(2r\) away from \(x_0\)). Then by Lemma 27, for all sufficiently small \(\epsilon\) player one has a probability \(\theta\) (bounded below independently of \(\epsilon\) and \(r\)) of reaching this ball and terminating the game (or terminating the game even sooner) before exiting \(B(0, 4r)\). The result now follows from Lemma 26.

\textbf{(iii)} Finally, we prove that all simply connected domains are game-regular when \(d = 2\). By translation, we can suppose that \(0 \in \partial \Omega\). Our goal is to find \(\theta > 0\), so that for any initial game position \(x_0\), for sufficiently small \(\epsilon\) player I can ensure that the game will end with probability at least \(\theta\) before the game position reaches a point outside of \(B(0, 2|x_0|)\).

By rotating and scaling, we may and shall assume that \(x_0 = (-2, 3)\). Let \(L_1\) be the ordered line segment in \(\mathbb{R}^2\) from \((-2, 3)\) to \((1, -1)\). Let \(L_2\) be the ordered line segment from \((1, -1)\) to \((-1, -1)\) and let \(L_3\) be the ordered line segment from \((-1, -1)\) to \((2, 3)\). Let \(L\) be the concatenation of \(L_1\), \(L_2\), and \(L_3\) (a continuous path). Finally, let \(z_0 = (-1, 2), z_1, z_2, \ldots, z_{1200} = (1, 2)\) be an evenly spaced sequence of points along the path \(L\), and let \(r = \frac{|L_1| + |L_2| + |L_3|}{1200} = \frac{12}{1200} = 1/100\) be the distance between adjacent \(z_i\) along the path \(L\). If the game position begins at a point in \(B(z_i, r) \subset B(z_{i+1}, 2r)\) for \(0 \leq i < 1199\), then for any \(b'\) with \(0 < b' < (1/2, 2, p, d) + O(r^{-1})\) player I can arrange (for all \(\epsilon\) sufficiently small) to reach a point in \(B(z_{i+1}, r)\) before exiting \(B(0, 4r)\) with probability at least \(b'\). Thus, if the game position begins at \(z_0\), player I can arrange, with probability \(\theta = (b')^{1200}\), to hit each of the \(B(z_i, r)\) without ever reaching a point of distance more than \(4r\) from \(L\). Since \(4r < \text{dist}(L, 0)\), a simple topological argument (this is where \(d = 2\) is used) shows that if \(0\) lies on the boundary of \(\Omega\) and \(\Omega\) is simply connected, then the path obtained by joining
Figure 2: Path used to prove simply connected domains are game-regular

subsequent game positions $x_j$ in such a game sequence with line segments would have to surround 0 and intersect itself—and would thus have to cross $\partial \Omega$ since $\Omega$ is simply connected. However, this also implies that at some point the game position is within $\alpha \epsilon$ of $\partial \Omega$, and thus the game must have terminated. We conclude that for all sufficiently small $\epsilon$, player I can arrange for the game to end with probability at least $\theta = (b')^{1200}$ before the game position reaches a point outside of $B(0, 2|x_0|)$. The result follows from Lemma 2.6.

3 Variants

3.1 Alternating turns

We now introduce a two-player zero-sum game, called alternating turn tug of war (with noise). The rules are exactly the same as those of ordinary tug of war except that players alternate turns (with player I moving first) instead of determining turn order with coin tosses.

In the alternating turn game, we define $p = p(\mu) \in [1, \infty]$ by $p = \frac{C_{1,1} + C_{2,2}}{C_{2,2}}$ instead of $p = \frac{C_{1,1} + C_{2,2}}{C_{2,2}^2 + 1}$.

Equivalently, $p$ is such that for some $\beta > 0$, we have $C_{1,1} = \beta q^{-1}$ and $C_{i,i} = \beta p^{-1}$ for $i \geq 2$ for some $\beta > 0$, where $p^{-1} + q^{-1} = 1$. As in the random turn game, $p$ is chosen so that if one player always chooses the vector $v_k$ to be some $v$ with $|v| = \epsilon$, and the other player always chooses $v_k$ to be $-v$, then for each $k$ the variance of $x_k - x_{k-1}$ is proportional to $p\epsilon^2$ in the direction of $v$ and $q\epsilon^2$ in each direction orthogonal to $v$.

As in the random turn case, we let $\phi(x) = x^T Ax + (\xi, x)$ (for $\xi \in \mathbb{R}^d$) be a quadratic function. Suppose that $x_0 = 0$ and let $\psi(v)$ be the expected value of $\phi(x_1) = \phi(v_1 + z_1)$ if
player I chooses $v_1 = v$. Then $z_1$ has expectation zero and covariance matrix $C$ given by
\[ \beta q^{-1}vv^T + \beta p^{-1}(|v|^2I - vv^T). \]

Since $z_1$ has expectation zero, we have
\[ \psi(v) := \mathbb{E}[\phi(v + z_1)] = \phi(v) + \sum_{i,j} A_{i,j} \mathbb{E}[z_1^i z_1^j], \]

where $z_1^i$ is the $i$th component of $z_1$.

We can also write this as $\sum_{i,j} A_{i,j} C_{i,j}$. Note that $\sum(vv^T)_{i,j} A_{i,j} = v^T Av$ and $\sum I_{i,j} A_{i,j} = \text{Tr} A$. We can now write
\[
\psi(v) = \phi(v) + (\beta q^{-1})v^T Av + \beta p^{-1}(|v|^2\text{Tr} A - v^T Av)
= \phi(v) + (\beta q^{-1} - \beta p^{-1})v^T Av + \beta p^{-1}|v|^2\text{Tr} A
= (\beta q^{-1} - \beta p^{-1} + 1)v^T Av + \beta p^{-1}|v|^2\text{Tr} A + (\xi, v)
\]

and we can write $\psi(v) = (\xi, v) + v^T Bv$ where $B = (\beta q^{-1} - \beta p^{-1} + 1)A + \beta p^{-1}(\text{Tr} A)I$.

We give the following alternating turn analog of Lemma 2.3.

**Lemma 3.1.** If $\phi$ is quadratic, then for any even $k$ (in the alternating turn game) if player I makes the $k+1$th move in the gradient direction, then regardless of what player II’s strategy is, we have
\[ \mathbb{E}[\phi(x_{k+2})|h_k] \geq \phi(x_k) + \beta \Delta_p \phi(x_k)\epsilon^2 + O(\epsilon^3). \]

The constant in the $O(\epsilon^3)$ depends only $\|B\|$ and $|\xi| = |\nabla \phi(x_k)|$, as in Lemma 2.2.

**Proof.** It is enough to prove this for $k = 0$. We may assume $x_0 = 0$ and then with the gradient strategy $x_1 = \epsilon|\xi|^{-1}\xi$ and $\nabla \phi(x_1) = \xi + 2Ax_1$. Therefore
\[ |\nabla \phi(x_1)|^2 - |\nabla \phi(x_0)|^2 = 4\eta^T Ax_1 + O(\epsilon^2). \]

Dividing by $|\nabla \phi(x_0)| + |\nabla \phi(x_1)| = 2|\xi| + O(\epsilon)$ gives
\[ |\nabla \phi(x_1)| - |\nabla \phi(x_0)| \geq \frac{2\epsilon^T Ax_1}{|\xi|} + O(\epsilon^2) = \epsilon \Delta_{\infty} \phi(x_0) + O(\epsilon^2). \]

If player I is moving at the first turn, player I can arrange—by pulling directly in the gradient direction—to have
\[ \mathbb{E}[\phi(x_1)] - \phi(x_0) \geq |\nabla \phi(x_0)|\epsilon + \frac{\epsilon^2}{2} (\beta \Delta_p + \Delta_{\infty}) \phi(x_0) + O(\epsilon^3). \quad (5) \]

When player II makes the subsequent move, the expectation is smallest (up to $O(\epsilon^3)$ error) if player II always pulls in the minus gradient direction. In that case we have
\[
\mathbb{E}[\phi(x_2)|h_1] - \phi(x_1) \geq -|\nabla \phi(x_1)|\epsilon + \frac{\epsilon^2}{2} (\beta \Delta_p + \Delta_{\infty}) \phi(x_1) + O(\epsilon^3)
= -\left(|\nabla \phi(x_0)| + \epsilon \Delta_{\infty} \phi(x_0)\right)\epsilon + \frac{\epsilon^2}{2} (\beta \Delta_p + \Delta_{\infty}) \phi(x_0) + O(\epsilon^3).
\]
Taking expectations and adding (5) gives

\[ \mathbb{E}\phi(x_2) - \phi(x_0) \geq \beta \Delta \phi(x_0) \epsilon^2 + O(\epsilon^3), \]

as claimed.

The proofs of analogs of Theorem 2.4, Theorem 2.5, and Theorem 1.2 are exactly the same as in the random turn case.

3.2 Direction selection and Spencer’s game

The following game (along with other variants) was introduced by Spencer [16] and studied in detail by Kohn and Serfaty [8]. We focus on the case \( d = 2 \), although many natural variants exist in higher dimensions.

Fix \( \Omega, x, F \) as in tug of war and set \( x_0 = x \). At each turn \( k \), player II chooses a vector \( v_k \) of length exactly \( \epsilon \). Player I then chooses a sign \( \sigma_k \in \{-1, 1\} \) and sets \( x_k = x_{k-1} + \sigma_k v_k \). Kohn and Serfaty were primarily concerned with convex domains, constant running payoff \( f \) and zero boundary conditions \( F = 0 \) (i.e., one player seeks to maximize the number of steps before the boundary of the domain is reached and the other player seeks to minimize that number). In this case, the relevant operator is \( \Delta_1 \), i.e., we generally expect the limiting payoff function \( u \) to be a solution to \( \Delta_1 u = -2f \); however, many complicating issues arise in this setting that do not arise in tug of war, so the results here are somewhat more restrictive. Even on simple domains (such as the unit disc) with \( C^\infty \) boundary values \( F \), the equation \( \Delta_1 u = 0 \) may not have a unique solution. (For a concrete example, note that if the unit circle is parameterized by angle \( \theta \in [0, 2\pi] \), then the function \( \cos(2\theta) \) can be extended to the interior of the unit disc in such a way that it is constant on vertical lines; it can also be extended to the unit disc in such a way that it is constant on horizontal lines. Both extensions are solutions to \( \Delta_1 u = 0 \) — at least in the sense that they are smooth and satisfy \( \Delta_1 u = 0 \) wherever \( \nabla u \neq 0 \).) We refer the reader to [8] for more discussion along these lines.

Consider the following interpolation between tug of war and Spencer’s game: as before, we set an initial game position \( x_0 \in \Omega \). At the \( k \)th turn, we toss a \( p^{-1} \)-coin to determine how the game position is updated: with probability \( p^{-1} \), \( x_k \) is updated using the rule of Spencer’s game described above (with player II choosing a vector and player I choosing the sign); with probability \( q^{-1} \), it is updated using the rule of ordinary tug of war (i.e., a fair coin is tossed, and the player who wins the toss chooses a \( v_k \in B(0, \epsilon) \) and we set \( x_k = x_{k-1} + v_k \)).

In this setting, the analog of Theorem 2.5 is even simpler to prove than in the other two cases, since we may deal with the two cases separately. First, if it is decided that the game position will be updated at the \( k + 1 \)th step using the tug of war rule, then by pulling in the gradient direction, player I can arrange for \( \mathbb{E}[\phi(x_{k+1}) - \phi(x_k) \mid h_k] \) to be at least \( \Delta_\infty \phi/2 + O(\epsilon^3) \) when \( \phi \) is smooth (and player II can arrange for \( \mathbb{E}[\phi(x_{k+1}) - \phi(x_k) \mid h_k] \) to be at most \( \Delta_\infty \phi/2 + O(\epsilon^3) \)).

This follows as a special (\( \mu \) supported at the origin) case of the bounds given for tug of war with noise. Second, we claim if it is decided that the game position will be updated at the \( k + 1 \)th step using Spencer’s rule, it is similarly clear (using the Taylor expansion for \( \phi \)) that player II can arrange (by choosing the direction orthogonal to the gradient) for \( \mathbb{E}[\phi(x_{k+1}) - \phi(x_k) \mid h_k] \) to be at most \( \Delta_1 \phi/2 + O(\epsilon^3) \). If player I adopts the strategy of always
choosing a sign so that the inner product \((\sigma_k v_k, \nabla \phi(x_k))\) is non-negative, then a similar Taylor expansion shows that \(\mathbb{E}[\phi(x_{k-1} - \phi(x_k) \mid h_k] \) will be at least \(\Delta_1 \phi/2 + O(\epsilon^3)\). Thus, before it is decided which update rule will be used, either player can arrange for the expected change, given the history, to be \(\Delta_p \phi/2 + O(\epsilon^3)\). The remainder of the argument is the same as the one given for ordinary tug of war.

A complete proof of an analog of Theorem 1.2 in this context is also possible, but the definition of game regularity must be symmetrized to require the same for player II as is required for player I.

4 Irregular domains and discontinuous payoff functions

4.1 Resolutive functions

If either \(\Omega\) is game irregular or \(F\) is discontinuous (e.g., if \(\Omega\) is the unit disc and \(F\) is zero on points with rational angles and one on all other points), then we cannot expect there to exist a \(p\)-harmonic function on \(\Omega\) that extends continuously to \(F\) on \(\partial \Omega\). There is still a natural notion of \(p\)-harmonic extension of \(F\) provided \(F\) is resolutive (as defined below; when \(p = 2\), resolutivity follows from boundedness and Borel measurability). However, in this setting, the \(u^\epsilon\) defined above need not converge to that extension (as is clear in the example where \(\Omega\) is the unit disc and \(F\) is the characteristic function of the set of points of irrational angles).

To generalize \(p\)-harmonic extensions to general domains and boundary conditions, it is conventional to define the upper class \(\mathcal{U}_F\) of \(F\) as the set of all functions \(u\) such that \(u\) is \(p\)-superharmonic in \(\Omega\), \(u\) is bounded below, and \(\liminf_{x \to y} u(x) \geq F(y)\) for all \(y \in \partial \Omega\). The lower class \(\mathcal{L}_F\) can be analogously defined by writing \(v \in \mathcal{L}_F\) if \(-v \in \mathcal{U}_{-F}\). The function \(\overline{H}_F = \inf\{u : u \in \mathcal{U}_F\}\) is the upper Perron solution of \(F\) in \(\Omega\) and \(\underline{H}_F = \sup\{u : u \in \mathcal{L}_F\}\) is the lower Perron solution of \(F\) in \(\Omega\). The upper and lower Perron solutions are either identically \(\pm \infty\) or everywhere finite and \(p\)-harmonic. In particular, when \(F\) is bounded, both \(\overline{H}_F\) and \(\underline{H}_F\) are bounded and \(p\)-harmonic (Chapter 9 of [6]).

We say that \(F\) is resolutive if \(\overline{H}_F\) and \(\underline{H}_F\) agree and are \(p\)-harmonic in \(\Omega\). If \(\Omega\) is regular then every continuous function on \(\Omega\) is resolutive; in fact, if \(\Omega\) is regular, then every bounded and lower semi-continuous \(F\) on \(\partial \Omega\) is resolutive (Chapter 9 of [6]). If \(F_i\) are resolutive and \(F_i \to F\) uniformly then \(F\) is resolutive. It is known that when \(p = 2\) any bounded Borel measurable \(f : \partial \Omega \to \mathbb{R}\) is resolutive. For general \(p\), no such comparable result is known, and it is an open problem whether every Borel function is resolutive [6]. It is not hard to see that some simple discontinuous functions (e.g., the function that is 1 on points of the boundary with rational coordinates and 0 on all other points, etc.) are resolutive for all \(p\).

4.2 Shrinking step sizes

Consider the variant of tug of war called small-step-size tug of war played as follows: fix parameters \(\Omega, F, p\), and \(x\) as in ordinary (either random turn or alternating turn) tug of war. One player chooses any real \(\epsilon_0 > 0\) and the other player chooses any real \(\epsilon\) with \(0 < \epsilon \leq \epsilon_0\). The players then play tug of war using the parameter \(\epsilon\). Clearly, the payoff of this game is \(\liminf_{\epsilon \to 0} u^\epsilon\) if player I chooses \(\epsilon_0\) and player II chooses \(\epsilon\) and \(\limsup_{\epsilon \to 0} u^\epsilon\)

if player II chooses \( \epsilon_0 \) and player I chooses \( \epsilon \). The convergence in Theorem 1.2 implies the following:

**Corollary 4.1.** If \( \Omega \) is game-regular, then for every continuous function \( F : \partial \Omega \to \mathbb{R} \), the value of small-step-size tug of war as a function of the initial game position is given by the unique \( p \)-harmonic function \( u \) that extends continuously to \( F \) on \( \partial \Omega \).

Let \( v \) be the value for player I of the following game called shrinking-step-size tug of war. Players begin by choosing an \( \epsilon_0 \) and \( \epsilon \) as in small-\( \epsilon \) tug of war, with the requirement that \( d(x_0, \partial \Omega) > \alpha \epsilon \) (so that the game does not end instantly). Subsequently, at each time \( j \) such that either

1. \( d(x_j, \partial \Omega) \leq \alpha \epsilon \), or
2. for some \( m \geq 1 \), \( j \) is the smallest integer for which \( d(x_j, \partial \Omega) \leq 2^{-m} \),

player I chooses a new \( \epsilon_0 \) and player II chooses a new \( \epsilon < \epsilon_0 \) (again, with the requirement that \( d(x_j, \partial \Omega) > \alpha \epsilon \)), and the game continues. Clearly this game cannot terminate in finitely many steps, because the \( \epsilon \) used during a game step is always such that the distance from the boundary is at least \( \alpha \epsilon \) (and hence it is not possible to reach the boundary and end the game during that step). We define the payoff to be the infimum of \( F(y) \) over all limit points \( y \in \partial \Omega \) of the \( x_j \), and 0 if no limit point on \( \partial \Omega \) exists. Let \( \overline{v} \) be the value for player II of the game defined analogously except that player I chooses the \( \epsilon_0 \) and player II chooses the \( \epsilon \) (at each stage) and the payoff is the supremum of \( F(y) \) over all limit points of \( y \in \partial \Omega \), instead of the infimum. Clearly \( v \leq \overline{v} \). Now we have the following:

**Theorem 4.2.** In shrinking-step-size tug-of-war on a domain \( \Omega \) with bounded boundary function \( F \), as defined above, the following holds throughout \( \Omega \):

\[
H_F \leq v \leq \overline{v} \leq \overline{H}_F.
\]

In particular, if \( F \) is bounded and resolutive, then its \( p \)-harmonic extension \( H_F = \overline{H}_F \) is the value of the game independently of which player chooses the \( \epsilon_0 \) and whether the payoff is taken to be the supremum or the infimum of \( F \) on the set of limit points the \( x_k \).

We will make some more observations before proving this result. When \( F \) is the characteristic function \( 1_A \) for some \( A \subset \partial \Omega \), the value \( \overline{H}_F(x) \) is called the \( p \)-harmonic measure of \( A \) at \( x \) and written \( \omega_p(A, x, \Omega) \). It is well known (see, e.g., [6]) that if \( \omega_p(A, x_0, \Omega) = 0 \) for some \( x_0 \) in a connected domain \( \Omega \), then \( \omega_p(A, x, \Omega) = 0 \) for all points \( x \in \Omega \).

Theorem 4.2 is interesting in light of the fact that \( p \)-harmonic measure is non-additive even on null sets [11]. In fact, [11] exhibits a disjoint finite collection \( \{ A_i \} \) of resolutive sets with \( p \)-harmonic measure zero whose union is all of \( \partial \Omega \).

Interpreted game theoretically, this means that there can be a partition \( \{ A_i \} \) of \( \partial \Omega \) such that for each \( i \), there is a strategy that causes all of the limit points of the game play to lie outside of \( A_i \) with probability one. In other words, player I has a strategy for avoiding any one of the \( A_i \) with probability one (regardless of player II’s actions), even though player I clearly has no strategy for avoiding all of the \( A_i \) with even positive probability.
Following Chapter 2 of [6], given a compact subset $K$ of $\Omega$, let $W(K, \Omega) := \{u \in C_b(\Omega) : u \geq 1 \text{ on } K\}$ and define its $p$-capacity to be $\text{cap}_p(K, \Omega) = \inf_{u \in W(K, \Omega)} \int_{\Omega} |\nabla u|^p$. If $U$ is open, define $\text{cap}_p(U, \Omega)$ to be supremum of $\text{cap}_p(K, \Omega)$ over compact $K \subset U$; if $E$ is arbitrary, then $\text{cap}_p(E, \Omega)$ is the infimum of $\text{cap}_p(U, \Omega)$ over open sets $U \supset E$. From Chapter 9 of [6] we cite the following:

**Proposition 4.3.** The set of $p$-irregular boundary points has $p$-capacity zero.

Note also that a subset $A$ of $\Omega$ has $p$-capacity equal to zero if and only if $A \subset \partial(\Omega \setminus A)$ (i.e., $A$ has empty interior) and $\omega_p(A, x, \Omega) = 0$ for each $x \in \Omega \setminus A$ [6]. By Theorem 4.2, $\omega_p(A, x, \Omega)$ is an upper bound on the value function for a game with payoff 1 on $A$ and $\partial \Omega \setminus A$—with equality in the case that $A$ is resolutive (which holds, in particular, if $A$ is open or closed; see Chapter 9 of [6]).

Thus, Theorem 4.2 gives a new interpretation of what positive $p$-capacity means: a resolutive set has positive $p$-capacity if and only if for some $x \in \Omega \setminus A$, this value is non-zero. In other words, the resolutive subsets of $\Omega$ with positive $p$-capacity are precisely those sets $A$ that a player can reach—with positive probability—from some $x \in \Omega \setminus A$ in shrinking-step-size tug of war.

One definition of a $p$-regular point is a point such that $\lim_{x \to x_0} \overline{H}_F(x) = F(x_0)$ for each continuous $F : \partial \Omega \to \mathbb{R}$. A point is irregular if it is not regular. It easy to see that game-regularity implies $p$-regularity (since Theorem 4.2 defines a $p$-harmonic extension explicitly in this case), but the converse is not known. It may be that equivalence of game-regularity and $p$-regularity depends very sensitively on the precise termination rule used near the boundary. For this reason, we consider the shrinking-step game to be more natural than the ordinary game when the domain is wildly irregular.

**Proof of Theorem 4.2.** Fix a bounded Borel measurable function $F$ on the boundary of an arbitrary bounded domain $\Omega \subset \mathbb{R}^d$. Suppose that $u$ belongs to $U_F$, i.e., the set of all functions $u$ such that $u$ is $p$-superharmonic in $\Omega$, $u$ is bounded below, and $\lim \inf_{x \to y} u(x) \geq F(y)$ for all $y \in \partial \Omega$.

It is enough to show that $u$ is an upper bound bound on the value of shrinking-step-size tug-of-war for player one. Let $\Omega_m$ be the component of \{ $x \in \Omega : d(x, \partial \Omega) > 2^{-m}$ \} containing $x_0$ (which is non-empty for all sufficiently large $m$) and let $\hat{\Omega}_m$ be any smooth domain with $\Omega_m \subset \hat{\Omega}_m \subset \Omega_{m+1}$. Let $u_m$ be the $p$-harmonic extension to $\hat{\Omega}_m$ of the values of $u$ on $\partial \Omega_m$. Then we know (recall Propositions 1.3 and 1.4) that this extension is unique and that $u_m \leq u$ on $\Omega_m$. By Theorem 2.4 Player II can choose $\epsilon$ small enough to ensure that the expected value of $u(x_{k_m})$, where $k_m$ is the first $k$ for which $x_k \notin \hat{\Omega}_m$, is at most $u(x) \big( \text{up to an error that can be made arbitrarily small, say smaller than } \delta 2^{-m} \text{ for some fixed } \delta > 0 \big)$. This implies that $u(x_{k_m}) - \delta \sum_{i=1}^{m} 2^{-i}$ is a supermartingale in $m$; hence $u(x_{k_m})$ almost surely converges to a limit $V$ with $E(V) \leq u(x_0) + \delta$. Since the payoff for player I is at most $V$, this implies that the expected payoff for player I is at most $u(x_0) + \delta$.

**Corollary 4.4.** If $F$ is resolutive then the value of $F$ on any subset of $\partial \Omega$ of $p$-capacity zero of $\partial \Omega$ makes no difference to the value of shrinking-step-size tug of war. Zero capacity sets cannot be reached with positive probability—i.e., if $F$ is 1 on a set of $p$-capacity zero and 0 on the rest of $\partial \Omega$, then $u = 0$ identically.
Remark. As noted by the referee, similar results were proved by analytical means in the papers by Kurki [9] and Avilés-Manfredi [2].

4.3 Harmonic measure for $\Delta_p$ and porous sets

Here, we present an estimate for the $p$-harmonic measure of a porous set for $1 < p < \infty$, analogous to the estimate given for $\omega_\infty$ in [14]. Estimates of this type were derived analytically by O. Martio, see Theorem 11.27 in [6]. Nevertheless, we feel that the argument below shows how useful game theoretic intuition is for these problems.

Recall that a set $S$ in a metric space $Z$ is $\lambda$-porous if for every $r \in (0, \text{diam} Z)$ every ball of radius $r$ contains a ball of radius $\lambda r$ that is disjoint from $S$. An example of a porous set is the ternary Cantor set in $[0,1]$.

Theorem 4.5. Let $\Omega \subset \mathbb{R}^d$, $d > 1$ be the closed unit ball. Let $\lambda \in (0, 1/2)$ and let $\delta > 0$. Let $S$ be a $\lambda$-porous subset of $\partial \Omega$, and let $S_\delta$ be the closure of the $\delta$-neighborhood of $S$. Then for some constant $c = c(\lambda, p) > 0$ we have

$$\omega_p(S_\delta, 0, \Omega) \leq \delta^c.$$ 

The key to the proof of the analogous result in [14] is that in tug of war without noise, if player I chooses a strategy of always pulling towards a point $z$ at distance $r$ from the initial game position $x_0$, and $z \in B(z, R) \subset \Omega$, then the limit as $\epsilon \to 0$ of the probability that the game sequence $\{x_j\}$ will reach $z$ before exiting $B(z, R)$ is $(R - r)/R$; in particular, this limiting probability depends only on the ratio $r/R$.

In our setting, this does not hold when $p < d$. However, if instead of targeting $z$ one merely aims to reach the ball $B(z, r_0)$ for some $r_0 < r < R$, then the (small $\epsilon$) limit of the probability of reaching this ball is a positive constant depending only on $r/r_0$ and $R/r_0$.

The idea of the proof in [14] was to construct a finite sequence $x_0, \ldots, x_k$ of points—with $x_k = z$ and $x_k$ a boundary point in the complement of $S_\delta$—such that the distance between successive $x_i$ most some constant $r$ and the distance from any $x_i$ to $S$ is at most some larger constant $R$. Then player I can, with positive probability (call it $C$), reach $x_1$ before exiting $B_R(x_1)$. Subsequently, player I can, with probability $C$, reach $x_2$ before exiting $B_R(x_2)$. Repeating, we see that with probability at least $1 - (1 - C)^k$, player I can win the game before exiting $\cup B_R(x_i)$. The proof in [14] then shows that if player one fails at this, then the same strategy may be repeated on a smaller scale. The proof of Theorem 4.5 is identical to the one given in [14] except that instead of targeting a sequence of individual points (terminating at a point in one of the intervals of the complement of $S_\delta$) one targets a sequence of small-radius balls (terminating at a ball that lies just outside $\Omega$ and is incident to the center of one of the intervals of the complement of $S_\delta$).

5 Non-measurable strategies and finite additivity

Some of the measurability restrictions we have imposed on strategies and on $F$ can be relaxed by invoking some less standard mathematics (such as finitely additive probability and/or the Axiom of Determinacy). There is a sizable stochastic game theory literature on
these subjects (see, e.g., [13, 12] and the references therein) and we will only briefly mention what the main results imply in our setting.

First, the requirement that the players adopt measurable strategies is unnecessary if we require that all subsets of \( \mathbb{R}^d \) be \( \mu \) measurable (since the latter implies that all strategies are measurable). This can be achieved within Zermelo Fraenkel set theory with the Axiom of Choice (ZFC) by relaxing the requirement that \( \mu \) be \( \sigma \)-additive. In this context, [13, 12] define the expected payoff for every pair of strategies (mixed or pure), and the following is immediate from their more general result, which holds in ZFC:

**Proposition 5.1.** Let \( \Omega \) be an arbitrary domain, \( x \in \Omega \), and \( F \) a Borel measurable payoff function. Suppose we require that the \( \mu \) in the definition of tug-of-war is finitely additive and defined on all subsets of \( \mathbb{R}^d \), and that the players are allowed arbitrary finitely additive mixed strategies. Then all of the variants of tug of war we have introduced (alternating turn and random turn; fixed \( \epsilon \) and small-step-size and shrinking-step-size) have values \( u^\epsilon \). Moreover, \( u_1^\epsilon \leq u^\epsilon \leq u_2^\epsilon \), where \( u_1^\epsilon \) and \( u_2^\epsilon \) are defined (as in the introduction) using only pure measurable strategies.

Recall that even for continuous boundary data, we did not resolve the issue of whether \( u_1^\epsilon - u_2^\epsilon \) is constant under the restriction to measurable strategies (although we showed that \( |u_1^\epsilon - u_2^\epsilon|_\infty \to 0 \) as \( \epsilon \to 0 \)). Proposition 5.1 gives a value for every finite \( \epsilon \) and every Borel boundary function \( F \), which is at least compatible with our measurable-setting definition in that it is bounded between \( u_1^\epsilon \) and \( u_2^\epsilon \).

The results of [13, 12] also imply the existence of a value function for shrinking step size tug of war, and it is not hard to see (using comparison with smooth functions) that any such value function must be \( p \)-harmonic. We may interpret this function as a canonical \( p \)-harmonic extension of \( F \)—defined for every Borel measurable function \( F \)—(although we have not shown that it does not depend on \( \mu \)).

An even more exotic assumption—which allows us to use arbitrary payoff functions \( F \)—is the Axiom of Determinacy, which contradicts the Axiom of Choice but which implies that all stochastic games (in particular, tug of war games) have values for all bounded payoff functions on \( H_\infty \) [13]. (It also implies that all subsets of \( \mathbb{R} \) are Lebesgue measurable [13].) Under the Axiom of Determinacy, the value of shrinking-step-size tug of war exists (and is \( p \)-harmonic) for every bounded boundary function \( F : \partial \Omega \to \mathbb{R} \).

### 6 Open questions

We conclude with some simple open questions. We learned of the following question from [6]:

**Question 6.1.** Are all Borel measurable functions \( F \) of \( \partial \Omega \) resolutive when \( \Omega \) is a bounded domain in \( \mathbb{R}^d \)?

If the answer is no, then one may still attempt to construct a canonical \( p \)-harmonic extension of \( F \) by affirmatively answering the following:

**Question 6.2.** Does shrinking step size tug of war have a value (within standard ZFC, using measurable strategies) for all Borel measurable \( F \) that depends only on \( p \) (and not on any other properties of \( \mu \))?
We showed that game-regularity implied $p$-regularity but could not prove the converse.

**Question 6.3.** Is game-regularity equivalent to $p$-regularity?

In [14], a game theoretic argument based on tug of war was used to prove uniqueness of solutions to $\Delta_\infty u = g$ (given zero boundary conditions) for all sufficiently regular and strictly positive $g$—and to show that uniqueness failed in general if $g$ was allowed to assume values of both signs. It is natural to ask whether these arguments can be adapted to the $p$-Laplacian.

**Question 6.4.** Let $\Omega$ be a bounded, game-regular domain and $g$ a Lipschitz function on $\Omega$ with $\inf_{x \in \Omega} g(x) > 0$. Does there necessarily exist a unique solution to $\Delta_p u = g$? Does this uniqueness fail when $g$ is allowed to assume values of both signs?

Theorem 1.2 shows that $u^1$ and $u^2$ converge to the same limit, but we have not addressed the following:

**Question 6.5.** In the setting of Theorem 1.2, is it always the case that $u^1 = u^2$?

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