DIAGONAL BASES IN
ORLIK-SOLOMON TYPE ALGEBRAS

RAUL CORDOVIL AND DAVID FORGE

ABSTRACT. To encode an important property of the “no broken circuit bases” of the Orlik-Solomon-Terao algebras, András Szénés has introduced a particular type of bases, the so called “diagonal basis”. We prove that this definition extends naturally to a large class of algebras, the so called $\chi$-algebras. Our definitions make also use of an “iterative residue formula” based on the matroidal operation of contraction. This formula can be seen as the combinatorial analogue of an iterative residue formula introduced by Szénés. As an application we deduce nice formulas to express a pure element in a diagonal basis.

1. INTRODUCTION

We denote by $\mathcal{M} = \mathcal{M}([n])$ a matroid of rank $r$ on the ground set $[n] := \{1, 2, \ldots, n\}$. Let $V$ be a vector space of dimension $d$ over some field $\mathbb{K}$. A (central) arrangement (of hyperplanes) in $V$, $\mathcal{A}_k = \{H_1, \ldots, H_n\}$, is a finite listed set of codimension one vector subspaces. Given an arrangement $\mathcal{A}_k$ we suppose always fixed a family of linear forms $\{\theta H_i \mid \theta \in \mathcal{A}_k, \ker(\theta H_i) = H_i\}$, where $V^*$ denotes the dual space of $V$. We denote by $L(\mathcal{A}_k)$ the intersection lattice of $\mathcal{A}_k$; i.e., the set of intersections of hyperplanes in $\mathcal{A}_k$, partially ordered by reverse inclusion. There is a matroid $\mathcal{M}(\mathcal{A}_k)$ on the ground set $[n]$ determined by $\mathcal{A}_k$: a subset $D \subset [n]$ is a dependent set of $\mathcal{M}(\mathcal{A}_k)$ iff there are scalars $\zeta_i \in \mathbb{K}$, $i \in D$, not all nulls, such that $\sum_{i \in D} \zeta_i \theta H_i = 0$. A circuit is a minimal dependent set with respect to inclusion.

If $\mathbb{K}$ is an ordered field an additional structure is obtained: to every circuit $C$, $\sum_{i \in C} \zeta_i \theta H_i = 0$, we associate a partition (determined up to a factor $\pm 1$) $C^+ = \{i \in C : \zeta_i > 0\}, C^- = \{i \in C : \zeta_i < 0\}$. With this new structure $\mathcal{M}(\mathcal{A}_k)$ is said a (realizable) oriented matroid and denoted by $\mathcal{M}(\mathcal{A}_k)$. Set $\mathcal{M}(\mathcal{A}_k) = \mathcal{M}(\mathcal{A}_k)$. Oriented matroids on a ground set $[n]$, denoted $\mathcal{M}([n])$, are a very natural mathematical concept and can be seen as the theory of generalized hyperplane arrangements, see [2].

Set $\mathfrak{M}(\mathcal{A}_k) = V \setminus \bigcup_{H \in \mathcal{A}_k} H$. The manifold $\mathfrak{M}(\mathcal{A}_C)$ plays an important role in the Aomoto-Gelfand theory of multidimensional hypergeometric functions (see [12] for a recent introduction from the point of view of arrangement theory). Let $K$ be a commutative ring. In [10][11][12] the determination of the cohomology $K$-algebra $H^*(\mathfrak{M}(\mathcal{A}_C); K)$ from the matroid $\mathcal{M}(\mathcal{A}_C)$ is accomplished by first defining the Orlik-Solomon $K$-algebra $\text{OS}(\mathcal{A}_C)$ in terms of generators and relations which depends only on the matroid $\mathcal{M}(\mathcal{A}_C)$, and then by showing that this algebra is isomorphic to $H^*(\mathfrak{M}(\mathcal{A}_C); K)$. The Orlik-Solomon algebras have been then intensively studied.

2000 Mathematics Subject Classification: Primary: 52C35; Secondary: 05B35, 14F40. Keywords and phrases: arrangement of hyperplanes, broken circuit, cohomology algebra, matroid, Orlik-Solomon algebra.

The first author’s research was supported in part by FCT (Portugal) through program POCTI and the project SAPIENS/36563/99. The second author’s research was supported by FCT through the project SAPIENS/36563/99.
Descriptions of developments from the early 1980’s to the end of 1999, together with the contributions of many authors, can be found in [3, 19].

Aomoto suggested the study of the (graded) $\mathbb{K}$-vector space $AO(A_k)$, generated by the basis $\{Q(B_i)^{-1}\}$, where $I$ is an independent set of $M(A_k)$, $B_i := \{H \in A_k : i \in I\}$, and $Q(B_i) = \prod_{i \in I} \theta_H$ denotes the corresponding defining polynomial. To answer to a conjecture of Aomoto, Orlik and Terao have introduced in [13] a commutative $\mathbb{K}$-algebra, $OT(A_k)$, isomorphic to $AO(A_k)$ as a graded $\mathbb{K}$-vector space in terms of the equations $\{\theta_H : H \in A_k\}$.

A “combinatorial analogue” of the algebra of Orlik-Terao was introduced in [7]: to every oriented matroid $\mathcal{M}$ was associated a commutative $\mathbb{Z}$-algebra, denoted by $\mathcal{K}(\mathcal{M})$.

Here we consider a large class of algebras, the so called $\chi$-algebras, that contain the three just mentioned algebras: Orlik-Solomon, Orlik-Terao and the algebras $A_\chi(\mathcal{M})$, see [2] or Definition 2.11 below. Following Szanes [15], we define a particular type of bases of $A_\chi$, the so called “diagonal basis”, see Definition 2.10. There is a natural example of these bases, the “no circuit basis”. We construct the dual bases of these bases, see Theorem 2.11. Our definitions make also use of an “iterative residue formula” based on the matroidal operation of contraction, see Equation (2.6). This formula can be seen as the combinatorial analogue of an iterative residue formula” based on the matroidal operation of contraction, see Equation (2.10) of the standard one. In the standard definition $C$ of the standard one. In the standard definition $C \setminus \alpha$ can be empty.) A no broken circuit set of a matroid $\mathcal{M}$ is an independent subset of $[n]$ which does not contain any broken circuit. Let $NBC(\mathcal{M}) \subset \binom{[n]}{\ell}$ be the set of the no broken circuit sets of cardinal $\ell$ of $\mathcal{M}$. Set $NBC(\mathcal{M}) = \bigcup_{\ell \in \mathbb{N}} NBC(\mathcal{M})$. We denote by $L(\mathcal{M})$ the lattice of flats of $\mathcal{M}$. (We remark that the lattice map $\phi : L(A_k) \to L(M(A_k))$, determined by the one-to-one correspondence $\phi' : H_i \leftrightarrow \{i\}, i = 1, \ldots, n$, is a lattice isomorphism.) For an independent set $I$, let $\mathcal{C}(I)$ be the closure of $I$ in $\mathcal{M}$.

Fix a set $E = \{e_1, \ldots, e_n\}$. Let $\mathcal{E} = \mathbb{K} \oplus \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n$ be the graded algebra over the field $\mathbb{K}$ generated by the elements $1, e_1, \ldots, e_n$ and satisfying the relations $e_i^2 = 0$ for all $e_i \in E$ and $e_i \cdot e_j = \beta_{i,j} e_i \cdot e_j$ with $\beta_{i,j} \in \mathbb{K} \setminus \{0\}$ for all $i < j$. Both the exterior algebra (take $\beta_{i,j} = -1$) and the commutative algebra with squares zero (take $\beta_{i,j} = 1$) are such algebras and will be the only ones to be used in the examples. Let $X^\sigma = (i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(m)})$, $\sigma \in \mathfrak{S}_m$, denote the ordered set $i_{\sigma(1)} < \cdots < i_{\sigma(m)}$. When necessary we see the set $X = \{i_1, \ldots, i_m\}$, as the ordered set $X^\text{id}$. Set $X^\sigma \setminus x := (i_{\sigma(1)}, \ldots, \widehat{x}, \ldots, i_{\sigma(m)})$. If $Y^\beta = (J_{\beta(1)}, \ldots, J_{\beta(m')})$ and $X \setminus Y = \emptyset$, set $X^\sigma \ast Y^\beta$ the concatenation $(i_{\sigma(1)}, \ldots, i_{\sigma(m)}; J_{\beta(1)}, \ldots, J_{\beta(m')})$. In the sequel we will denote by $e_X$ the (pure) element $e_{i_1} e_{i_2} \cdots e_{i_m}$ of $\mathcal{E}$. Fix a mapping $\chi : \mathbb{N}^n \to \mathbb{K}$. Let us also define $\chi$ for ordered sets by $\chi(X^\sigma) = \text{sgn}(\sigma) \chi(X)$, where $\text{sgn}(\sigma)$ denotes the sign of the permutation $\sigma$.2
The $\chi$-boundary of an element $e_X \in \mathcal{E}$ is given by the equation
\[ \partial e_X = \sum_{p=m}^{p=m} (-1)^p \chi(X \setminus i_p) e_{X \setminus i_p}. \]

We extend $\partial$ to $\mathcal{E}$ by linearity. It is easy to see that for $\sigma \in \mathcal{S}_I$, we have
\[ \partial e_X = \text{sgn}(\sigma) \sum_{p=1}^{p=m} (-1)^p \chi(X^\sigma \setminus i_{\sigma(p)}) e_{X^\sigma \setminus i_{\sigma(p)}} \]
and also for any $x \notin X$,
\[ \pm \partial e_{X \cup x} = (-1)^{m+1} \chi(X) e_X + \sum_{p=1}^{p=m} (-1)^p \chi(X \setminus i_p \ast x) e_{X \setminus i_p \cup x}. \]

Given an independent set $I$, an element $a \in \text{cl}(I) \setminus I$ is said active in $I$ if $a$ is the minimal element of the unique circuit contained in $I \cup a$. We say that a subset $U \subset [n]$ is a unidependent of $\mathcal{M}$, if it contains a unique circuit, denoted $C(U)$.

Note that $U$ is unidependent iff $\text{rk}(U) = |U| - 1$. We say that an unidependent set $U$ is an inactive unidependent if $\text{min}(C(U))$ is the minimal active element of $U \setminus \text{min}(C(U))$. Let us remark that $U$ is a unidependent of $\mathcal{M}$ iff for some (or every) $x \in U$, $\text{rk}(x) \neq 0$, $U \cup x$ is a unidependent of $\mathcal{M}/x$.

**Definition 2.1** ([9]). Let $\mathcal{Z}_\chi(\mathcal{M})$ be the (right) ideal of $\mathcal{E}$ generated by the $\chi$-boundaries $\{\partial e_C : C \in \mathcal{C}(\mathcal{M}), |C| > 1\}$ and the set $\{e_i : \{i\} \in \mathcal{C}(\mathcal{M})\}$. We say that $\mathcal{A}_\chi(\mathcal{M}) := \mathcal{E}/\mathcal{Z}_\chi(\mathcal{M})$ is a $\chi$-algebra if $\chi$ satisfies the following two properties:

(UC1) $\chi(I) \neq 0$ if and only if $I$ is independent.
(UC2) For any two unidependents $U$ and $U'$ of $\mathcal{M}$ with $U' \subset U$ there is a scalar $\varepsilon_{U,U'} \in \mathbb{K} \setminus 0$, such that $\partial e_U = \varepsilon_{U,U'}(\partial e_{U'}) e_{U \setminus U'}$.

**Remark 2.2.** From (UC2) we conclude that $\mathcal{Z}_\chi(\mathcal{M})$ has the basis
\[ \{e_D : D \text{ dependent of } \mathcal{M}\} \cup \{\partial e_U : U \text{ inactive unidependent of } \mathcal{M}\}, \]
and that $\text{nbc} := \{[I]_A : I \in \text{NBC}(\mathcal{M})\}$ is a basis of the vector space $A = \mathcal{A}_\chi(\mathcal{M})$.

This fundamental property was first discovered for the Orlik-Solomon algebras [12], and then also for other classes of $\chi$-algebras, see [7, 13] and the following example for more details. Note also that this implies that $[X]_A \neq 0$ iff $X$ is an independent set of $\mathcal{M}$.

**Example 2.3** ([9]). Recall the three usual $\chi$-algebras. Let $\mathcal{E}$ be the graded algebra over the field $\mathbb{K}$ generated by the elements $1, e_1, \ldots, e_n$ and satisfying the relations $e_i^2 = 0$ for all $e_i$ in $E$ and $e_i \cdot e_j = \beta_{i,j} e_i \cdot e_j$ where $\beta_{i,j}$ denotes a non null scalar fixed for every pair $i < j$.

- Let $\mathcal{E}$ be the exterior algebra (taking $\beta_{i,j} = -1$). Setting $\chi(I^\sigma) = \text{sgn}(\sigma)$ for every independent set $I$ of a matroid $\mathcal{M}$ and every permutation $\sigma \in \mathcal{S}_I$, we obtain the Orlik-Solomon algebra, OS($\mathcal{M}$).
- Let $\mathcal{A}_\mathbb{K} = \{H_i : H_i = \text{Ker}(\theta_i), i = 1, 2, \ldots, n\}$ be an hyperplane arrangement and $\mathcal{M}(\mathcal{A}_\mathbb{K})$ its associated matroid. For every flat $F := \{f_1, \ldots, f_k\} \subset [n]$ of $\mathcal{M}(\mathcal{A}_\mathbb{K})$ we choose a basis $B_F$ of the vector subspace of $(\mathbb{K}^d)^*$ generated by $\{\theta_{f_1}, \ldots, \theta_{f_k}\}$. By taking for $\mathcal{E}$ the free commutative algebra with squares null (taking $\beta_{i,j} = 1$) and taking for any $\{i_1, \ldots, i_k\} = I \in \text{IND}_\mathbb{K}$, $\chi(I) = \det(\theta_{i_1}, \ldots, \theta_{i_k})$, where the vectors are expressed in the basis $B_{\text{cl}(I)}$, we obtain the algebra OT($\mathcal{A}_\mathbb{K}$), defined in [13].
- Let $\mathcal{M}([n])$ be an oriented matroid. For every flat $F$ of $\mathcal{M}([n])$, we choose (determined up to a factor $\pm 1$) a basis signature in the restriction of $\mathcal{M}([n])$ to $F$. We define a signature of the independents of an oriented matroid $\mathcal{M}([n])$ as
a mapping, \( \text{sgn} : \text{IND}(\mathcal{M}) \to \{\pm1\} \), where \( \text{sgn}(I) \) is equal to the basis signature of \( I \) in the restriction of \( \mathcal{M}([n]) \) to \( \ell(I) \). By taking \( \mathcal{E} \) the free commutative algebra over the rational field \( \mathbb{Q} \) with squares zero (take \( \beta_{i,j} = 1 \) and taking \( \chi(I) = \text{sgn}(I) \) (resp. \( \chi(X) = 0 \)) for every independent (resp. dependent) set of the matroid, we obtain the algebra \( \Lambda(\mathcal{M}) \oplus_{\mathbb{Q}} \mathbb{Q} \), where \( \Lambda(\mathcal{M}) \) denotes the \( \mathbb{Z} \)-algebra defined in [7].

For every \( X \subseteq [n] \), we denote by \([X]_{\Lambda} \) or shortly by \( e_X \) when no confusion will result, the residue class in \( \Lambda_{\chi}(\mathcal{M}) \) determined by the element \( e_X \). Since \( 3_{\chi}(\mathcal{M}) \) is a homogeneous ideal, \( \Lambda_{\chi}(\mathcal{M}) \) inherits a grading from \( \mathcal{E} \). More precisely we have \( \Lambda_{\chi}(\mathcal{M}) = \mathbb{K} \oplus A_1 \oplus \cdots \oplus A_r \), where \( A_\ell = \mathcal{E}_\ell / \mathcal{E}_{\ell-1} \cap 3_{\chi}(\mathcal{M}) \) denotes the subspace of \( \Lambda_{\chi}(\mathcal{M}) \) generated by the elements \( \{[I]_{\Lambda} : I \in \text{IND}_\ell(\mathcal{M}) \} \). Set \( \mathbf{nbc}_\ell := \{[I]_{\Lambda} : I \in \mathbf{NBC}_\ell(\mathcal{M}) \} \) and \( \mathbf{nbc} := \bigcup_{\ell=0}^{\infty} \mathbf{nbc}_\ell \). From Remark 2.2 we conclude that \( \mathbf{nbc} \) is a basis of the vector space \( \Lambda_{\ell} \).

**Proposition 2.4.** Let \( \Lambda_{\chi}(\mathcal{M}) \) be a \( \chi \)-algebra. For any non loop element \( x \) of \( \mathcal{M}([n]) \), we define the two maps:

\[
\chi_{\mathcal{M}\setminus x} : 2^{[n]} \setminus x \to \mathbb{K} \quad \text{by} \quad \chi_{\mathcal{M}\setminus x}(I) = \chi(I) \quad \text{and}
\]

\[
\chi_{\mathcal{M}/x} : 2^{[n]} / x \to \mathbb{K} \quad \text{by} \quad \chi_{\mathcal{M}/x}(I) = \chi(I \cdot x).
\]

Then \( \Lambda_{\chi_{\mathcal{M}\setminus x}}(\mathcal{M}/x) \) and \( \Lambda_{\chi_{\mathcal{M}/x}}(\mathcal{M}\setminus x) \) are \( \chi \)-algebras.

**Proof.** The deletion case being trivial, we will just prove the contraction case. We have to show that \( \chi_{\mathcal{M}/x} \) verifies properties (UC1) and (UC2). The first property is verified since \( I \) is independent in \( \mathcal{M}/x \) iff \( I \cup x \) is independent in \( \mathcal{M} \). To see that the second property is also verified, let \( U \) and \( U' \) be two unidependents of \( \mathcal{M}/x \) (iff \( U \cup x \) and \( U' \cup x \) are two unidependents of \( \mathcal{M} \)). We know that \( \partial e_{U \cup x} = \varepsilon_{U \cup x, U' \cup x} (\partial e_{U' \cup x}) e_{U \cup x, U'} \). We note \( \partial' \) the boundary defined by \( \chi_{\mathcal{M}/x} \) and so we will show that there is a scalar \( \varepsilon_{U,U'} \) such that \( \partial' e_U = \varepsilon_{U,U'} (\partial e_{U'} e_{U \cup x}) e_{U \cup x, U'} \) Let \( X, X' \subseteq [n] \) two disjoint subsets then \( \varepsilon_{X,X'} = \beta_{X,X'} e_{X \cup X'} \), where \( \beta_{X,X'} = \prod_{i \in X, j \in X', i > j} \beta_{i,j} \). We have with \( U = (i_1, \ldots, i_m) \) and \( U' = (j_1, \ldots, j_k) \):

\[
\pm \partial e_{U \cup x} = \sum_{p=1}^{m} (-1)^p \chi(U \setminus i_p \cdot x) e_{U \cup x \setminus i_p} + (-1)^{m+1} \chi(U) e_U,
\]

\[
\partial' e_U = \sum_{p=1}^{m} (-1)^p \chi(U \setminus i_p \cdot x) e_{U \setminus i_p},
\]

\[
\pm (\partial e_{U' \cup x}) e_{U \cup U'} = \sum_{p=1}^{k} (-1)^p \chi(U' \setminus j_p \cdot x) \beta_{U' \cup x \cup j_p, U \cup x \cup j_p} e_{U \cup U' \cup j_p} + (-1)^{k+1} \chi(U') \beta_{U' \cup U, U'} e_{U'},
\]

\[
(\partial' e_{U'}) e_{U \cup U'} = \sum_{p=1}^{k} (-1)^p \chi(U' \setminus j_p \cdot x) \beta_{U' \cup j_p, U \cup U'} e_{U \cup U'}.
\]

After remarking that \( \beta_{U' \cup x \cup j_p, U \cup U'} \beta_{U' \cup x, U \cup U'}^{-1} = \beta_{x, U \cup U'} \) does not depend on \( j_p \), we can deduce that \( \partial' e_U = \varepsilon_{U, U'} (\partial e_{U'}) e_{U \cup U'} \) with \( \varepsilon_{U \cup x, U' \cup x} = \pm e_{U, U'} \beta_{x, U \cup U'} \). \( \square \)

**Proposition 2.5.** For every non loop element \( x \) of \( \mathcal{M}([n]) \), there is a unique monomorphism of vector spaces, \( i_x : \Lambda(\mathcal{M} \setminus x) \to \Lambda(\mathcal{M}) \), such that such that, for every \( I \in \text{IND}((\mathcal{M} \setminus x) \setminus x) \), we have \( i_x(I_x) = e_I \).
For every non loop element $x$ of $\mathcal{M}([n])$, there is a unique epimorphism of vector spaces, $p_x : A(\mathcal{M}) \to A(\mathcal{M}/x)$, such that, for every $e_I$, $I \in \text{IND}(\mathcal{M})$, we have

\begin{equation}
\p_x(e_I) := \begin{cases}
e_{I \setminus x} & \text{if } x \in I, \\
x(I \cup y, x) x(I \cup y) & \chi e_{I \setminus y} \text{ if there is } y \in I \text{ parallel to } x, \\
0 & \text{otherwise.}
\end{cases}
\end{equation}

Proof. From Remark 2.2 it is enough to prove that $p_x(\partial e_U) = 0$, for all unidependent $U = \{i_1, \ldots, \bar{i}_m\}$. We recall that if $x \in U$ then $U \setminus \{x\}$ is a unidependent set of $\mathcal{M}/x$. There are only the following four cases:

\begin{itemize}
  \item If $U$ contains $x$ but no $y$ parallel to $x$ then:
    \begin{align*}
    \pm p_x(\partial e_U) &= p_x((-1)^m x(U \setminus x)e_{U \setminus x} + \sum_{i_p \in U \setminus x} (-1)^p x(U \setminus \{i_p, x\} \ast x)e_{U \setminus \{i_p\}}) \\
    &= \sum_{i_p \in U \setminus x} (-1)^p x(U \setminus \{i_p, x\} \ast x)e_{U \setminus \{i_p, x\}} = 0
    \end{align*}
    from Proposition 2.4.

  \item If $U$ does not contain $x$ but a $y$ parallel to $x$ then:
    \begin{align*}
    \pm p_x(\partial e_U) &= p_x((-1)^m x(U \setminus y)e_{U \setminus y} + \sum_{i_p \in U \setminus y} (-1)^p x(U \setminus \{i_p, y\} \ast y)e_{U \setminus \{i_p\}}) \\
    &= \sum_{i_p \in U \setminus y} (-1)^p x(U \setminus \{i_p, y\} \ast y) x(U \setminus \{i_p, y\} \ast y)e_{U \setminus \{i_p, y\}} = 0
    \end{align*}
    like previously since $U \setminus y$ is again a unidependent of $\mathcal{M}/x$.

  \item If $U$ contains $x$ and a $y$ parallel to $x$ then:
    \begin{align*}
    \pm p_x(\partial e_U) &= p_x(x(U \setminus \{x, y\} \ast y)e_{U \setminus x} - x(U \setminus \{x, y\} \ast x)e_{U \setminus y}) \\
    &= x(U \setminus \{x, y\} \ast y) x(U \setminus \{x, y\} \ast y)e_{U \setminus \{x, y\}} - x(U \setminus \{x, y\} \ast x)e_{U \setminus \{x, y\}} = 0.
    \end{align*}

  \item If $U$ does not contain $x$ nor a $y$ parallel to $x$ then:
    \begin{align*}
    p_x(\partial e_U) &= p_x(\sum_{i_p \in U} (-1)^p x(U \setminus i_p)e_{U \setminus i_p}) = 0.
    \end{align*}
\end{itemize}

Theorem 2.7. For every element $x$ of a simple $\mathcal{M}([n])$, there is a splitting short exact sequence of vector spaces

\begin{equation}
0 \to A(\mathcal{M} \setminus x) \overset{i_x}{\to} A(\mathcal{M}) \overset{p_x}{\to} A(\mathcal{M}/x) \to 0.
\end{equation}

Proof. From the definitions we know that $p_x \circ i_x$ is the null map so $\text{Im}(i_x) \subset \text{Ker}(p_x)$. We will prove the equality $\dim(\text{Ker}(p_x)) = \dim(\text{Im}(i_x))$. By a reordering of the elements of $[n]$ we can suppose that $x = n$. The minimal broken circuits of $\mathcal{M}/n$ are the minimal sets $X$ such that either $X$ or $X \cup \{n\}$ is a broken circuit of $\mathcal{M}$ (see the Proposition 3.2.e of [3]). Then

$$\text{NBC}(\mathcal{M}/n) = \{X : X \subset [n-1] \text{ and } X \cup \{n\} \in \text{NBC}(\mathcal{M})\}$$

and
A basis of the exact sequence (2.4) splits.

Similarly to [15] (see also [4]), we now construct, making use of iterated contractions,

\[ p_n^{-1} : \mathbb{A}(M/n) \to \mathbb{A}, \quad \text{where} \quad p_n^{-1}([I]_{\mathbb{A}(M/n)}) := [I \cup n]_{\mathbb{A}}, \forall I \in NBC(M/n). \]

It is clear that \( p_n \circ p_n^{-1} \) is the identity map. From Equation (2.5) we conclude that the exact sequence (2.4) splits. \( \square \)

Remark 2.9. I. There are two permutations \( \sigma \) and \( \tau \) such that \( \chi_{\sigma}(1) = 0 \) and \( \chi_{\tau}(1) \neq 0 \). We call the iterated residue with respect to the ordered independent set \( I' \). (It is clear that the map \( p_{I'} \) depends on the order chosen on \( I' \) and not only on the underlying set \( I \).) We associate to \( I' \) the flag of flats of \( M \),

\[ \text{Flag}(I') := \text{cl}(\{i_{\sigma(p)}\} \subseteq \text{cl}(\{i_{\sigma(p), i_{\sigma(p-1)}}\}) \subseteq \cdots \subseteq \text{cl}(I). \]

Proposition 2.8. Let \( J \in \text{IND}_\ell(M) \) then we have \( p_{I'}(e_J) \neq 0 \) iff there is a unique permutation \( \tau \in \mathfrak{S}_\ell \) such that \( \text{Flag}(J') = \text{Flag}(I') \). And in this case we have \( p_{I'}(e_J) = \chi(I')/\chi(J') \). In particular we have \( p_{I'}(e_I) = 1 \) for any independent set \( I \) and any permutation \( \sigma \).

Proof. The first equivalence is very easy to prove in both direction. To obtain the expression of \( p_{I'}(e_J) \) we just need to iterate \( \ell \) times the residue. This gives:

\[ p_{I'}(e_J) = \frac{\chi(J \setminus J_{\tau(1)} \ast i_{\sigma(1)}) \times \chi(J \setminus J_{\tau(1)} \ast j_{\tau(1)} \ast i_{\sigma(1)}) \times \cdots \times \chi(J \setminus J_{\tau(1)} \ast I' \setminus i_{\sigma(1)})}{\chi(I')} \]

After simplification we obtain the announced formula. And finally the last result comes from the fact that if \( I = J \) then clearly \( \tau = \sigma \). \( \square \)

Remark 2.9. The fact that \( p_{I'}(e_J) \) is null depends on the permutation \( \sigma \). For example, for any simple matroid of rank 2 we have \( p_{I_{13}}(e_{12}) = 0 \) and \( p_{I_{31}}(e_{12}) \neq 0 \). But if \( p_{I'}(e_J) \neq 0 \) then its value does not depend on \( \sigma \). We mean by this that if there are two permutations \( \sigma \) and \( \sigma' \) such that \( p_{I'}(e_J) \neq 0 \) and \( p_{I'}(e_J) \neq 0 \) then \( p_{I'}(e_J) = p_{I'}(e_J) \).

Definition 2.10 ([15]). We say that the subset \( \mathbb{I}_\ell \subset \{[I]_{\mathbb{A}} : I \in \text{IND}_\ell(M)\} \) is a diagonal basis of \( \mathbb{A}_\ell \) if and only if the following three conditions hold:

1. For every \( [I]_{\mathbb{A}} \in \mathbb{I}_\ell \) there is a fixed permutation of the set \( I \) denoted \( \sigma_I \in \mathfrak{S}_\ell \);
2. \( \|I\| \geq \dim(\mathbb{A}_\ell) \);
3. For every \( [I]_{\mathbb{A}}, [J]_{\mathbb{A}} \in \mathbb{I}_\ell \) and every permutation \( \tau \in \mathfrak{S}_\ell \), the equality \( \text{Flag}(J) = \text{Flag}(I') \) implies \( J = I \).

Theorem 2.11. Suppose that \( \mathbb{I}_\ell \) is a diagonal basis of \( \mathbb{A}_\ell \). Then \( \mathbb{I}_\ell \) is a basis of \( \mathbb{A}_\ell \) and \( \mathbb{I}^*_\ell := \{p_{I'} : [I]_{\mathbb{A}} \in \mathbb{I}_\ell\} \) is the dual basis of \( \mathbb{I}_\ell \).
Proof. Pick two elements \([I]_\Lambda, [J]_\Lambda \in \mathbb{I}_\ell\). Note that \(p_{f_{\tau^*}}(e_J) = \delta_{f_J}\) (the Kronecker delta), from Condition \((2.11)\) and Proposition \(2.8\). The elements of \(\mathbb{I}_\ell\) are linearly independent: suppose that \([J] = \sum \zeta_{ij}[I]_j\), \(\zeta_j \in \mathbb{K} \setminus \{0\}\); then \(1 = p_{f_{\tau^*}}([J]) = p_{f_{\tau^*}}\left(\sum \zeta_{ij}[I]_j\right) = 0\), a contradiction. It is clear also that \(\mathbb{I}_\ell^\ast\) is the dual basis of \(\mathbb{I}_\ell\).

The following result gives an interesting explanation of results of \([6]\) and \([7]\).

**Corollary 2.12.** \(\text{nbc}_\ell(M)\) is a diagonal basis of \(\mathbb{A}_\ell\) where \(\sigma_I\) is the identity for every \([I]_\Lambda \in \text{nbc}_\ell(M)\). For a given \([J]_\Lambda \in \mathbb{A}_\ell\), suppose that \((2.12)\) \([J]_\Lambda = \sum \xi(I, J)[I]_\Lambda\), where \([I]_\Lambda \in \text{nbc}_\ell(M)\) and \(\xi(I, J) \in \mathbb{K}\).

Then are equivalent:
- \(\xi(I, J) \neq 0\),
- \(\text{Flag}(I) = \text{Flag}(J^\ast)\) for some permutation \(\tau\).

If \(\xi(I, J) \neq 0\) we have \(\xi(I, J) = \frac{\chi(I)}{\chi(J)}\). In particular if \(\mathbb{A}\) is the Orlik-Solomon algebra then \(\xi(I, J) = \text{sgn}(\tau)\).

**Proof.** By hypothesis \((2.10)\) and \((2.12)\) are true. We claim that \(\text{nbc}_\ell(M)\) verifies \((2.10)\). Suppose for a contradiction that \(J \neq I, [J]_\Lambda, [I]_\Lambda \in \text{nbc}_\ell(M)\) and there is \(\tau \in \mathbb{S}_\ell\), such that \(\text{Flag}(J^\ast) = \text{Flag}(I)\). Set \(I = (i_1, \ldots, i_k)\) and \(J = (j_\tau(1), \ldots, j_\tau(k))\), and suppose that \(j_{\tau(m+1)} = i_{m+1}, \ldots, j_{\tau(\ell)} = i_\ell\) and \(i_m \neq j_{\tau(m)}\). Then there is a circuit \(C\) of \(\mathbb{A}\) such that

\[
i_m, j_{\tau(m)} \in C \subset \{i_m, j_{\tau(m)}, i_{m+1}, i_{m+2}, \ldots, i_k\}.
\]

If \(j_{\tau(m)} < i_m\) [resp. \(i_m < j_{\tau(m)}\)] we conclude that \(I \notin \text{nbc}_\ell(M)\) [resp. \(J \notin \text{nbc}_\ell(M)\)] a contradiction. So \(\text{nbc}_\ell(M)\) is a diagonal basis of \(\mathbb{A}_\ell\).

From Theorem 2.11 we conclude that \(\text{nbc}_\ell^\ast := \{p_I : [I]_\Lambda \in \text{nbc}\}\) is the dual basis of \(\text{nbc}\). Suppose now that \([J]_\Lambda = \sum \xi_I[I]_\Lambda\), where \([I]_\Lambda \in \text{nbc}_\ell(M)\) and \(\xi_I \in \mathbb{K}\). Then \(\xi_I = p_I(e_J)\) and the remaining follows from Proposition 2.8.

Making full use of the matroidal notion of iterated residue, see Equation \((2.10)\), we are able to prove the following result very close to Proposition 2.1 of \([10]\).

**Proposition 2.13.** Consider the set of vectors \(V := \{v_1, \ldots, v_k\}\) in the plane \(x_d = 1\) of \(\mathbb{K}^d\). Set \(A_k := \{H_i : H_i = \text{Ker}(v_i) \subset (\mathbb{K}^d)^*\}, i = 1, \ldots, k\) and let OT\((A_k)\) be its Orlik-Terao corresponding algebra. Fix a diagonal basis \(\mathbb{I}_\ell \subset \{[I]_\Lambda : I \in \text{IND}_\ell(M)\}\) of \(\mathbb{A}_\ell\) and let \(\mathbb{I}_\ell^\ast = \{p_{f_{\tau^*}} : [I]_\Lambda \in \mathbb{I}_\ell\}\) be the corresponding dual basis. Then, for any \(e_J \in \mathbb{A}_\ell \setminus 0\), we have

\[
\sum_{I \in \mathbb{I}_\ell} p_{f_{\tau^*}}(e_J) = \sum_{I \in \mathbb{I}_\ell} \langle p_{f_{\tau^*}}, e_J \rangle = 1.
\]

**Proof.** We have for any \(\ell + 1\)-subset of \(V, \sum_{p=1}^{\ell+1} (-1)^p \chi(U \setminus i_p) = 0\). (This is the development of a determinant with two lines of 1.) For any rank \(\ell\) unidependent \(U = \{i_1, \ldots, i_{\ell+1}\}\) of the matroid \(M(A_k)\), we have

\[
\partial e_U = \sum_{p=1}^{\ell+1} (-1)^p \chi(U \setminus i_p) e_{U \setminus i_p}.
\]

Since the sum of the coefficients in these relations is 0 and that these relations are generating, see Remark 2.2, we can deduce that the sum of the coefficients in any relation in OT\((A_k)\) is also equal to 0 which concludes the proof. □
Example 2.14. Consider the 6 points $p_1, \ldots, p_6$ in the affine plane $z = 1$ of $\mathbb{R}^3$, whose coordinates are indicated in Figure 1. Set $v_i := (0, p_i)$, $i = 1, \ldots, 6$. And let $\mathcal{A}$ be the corresponding arrangement of $(\mathbb{R}^3)^*$, $\mathcal{A} := \{ H_i = \text{Ker}(v_i), i = 1, \ldots, 6 \}$. Let $\mathcal{M}(\mathcal{A})$ [resp. $\mathcal{M}(\mathcal{A})$] be the corresponding rank three [resp. oriented] matroid.

Figure 1

Let $\mathcal{A}_\chi$ be a $\chi$-algebra on $\mathcal{M}(\mathcal{A})$. We know that

$$\text{nbc}_{3} = \{ e_{124}, e_{125}, e_{126}, e_{134}, e_{135}, e_{136} \}$$

together with $\sigma_{124} = \sigma_{125} = \sigma_{134} = \sigma_{135} = \sigma_{136} = \sigma_{156} = \text{id}$ is a diagonal basis of $\mathcal{A}_3$, from Corollary 2.12. Directly from the Definition 2.10 we see that $\mathcal{B}_3 = \{ e_{124}, e_{125}, e_{134}, e_{135}, e_{136}, e_{156} \}$ with $\sigma_{124} = \sigma_{134} = \sigma_{135} = \sigma_{136} = \sigma_{156} = \text{id}$ and $\sigma_{125} = (132)$ is also a diagonal basis of $\mathcal{A}_3$. We will look at expression on the basis $\text{nbc}_{3}$ (resp. $\mathcal{B}_3$) of the vector space $\mathcal{A}_3$, of some elements of the type $e_B$, $B$ basis of $\mathcal{M}(\mathcal{A})$, for the three $\chi$-algebras of Example 2.3. Especially, one can verify as stated in Remark 2.4 that $p_{125}^\ast(e_{235}) = p_{125}^\ast(132)(e_{235})$. Let also point out that for the Orlik-Terao algebra, we have $\sum_{i \in \mathbb{B}} p_{i}^\ast(e_{i}) = 1$ as proved in Proposition 2.13.

- Consider the basis $\text{nbc}_{3}$ of the $\mathbb{K}$-vector space $\mathcal{A}_3$. So we have:

  $$e_{235} = \text{sgn}(325)e_{125} + \text{sgn}(235)e_{135} = -e_{125} + e_{135} \quad \text{in OS}(\mathcal{M}(\mathcal{A})),$$

  $$e_{235} = \frac{\text{det}(125)}{\text{det}(325)}e_{125} + \frac{\text{det}(135)}{\text{det}(325)}e_{135} = -e_{125} + 2e_{135} \quad \text{in OT}(\mathcal{A}),$$

  $$e_{235} = \chi(125)\chi(325)e_{125} + \chi(135)\chi(235)e_{135} = -e_{125} + e_{135} \quad \text{in } \mathcal{A}(\mathcal{M}(\mathcal{A})).$$

  $$e_{156} = \text{sgn}(165)e_{125} + \text{sgn}(156)e_{126} = -e_{125} + e_{126} \quad \text{in OS}(\mathcal{M}(\mathcal{A})),$$

  $$e_{156} = \frac{\text{det}(125)}{\text{det}(165)}e_{125} + \frac{\text{det}(126)}{\text{det}(165)}e_{126} = \frac{3}{2}e_{125} - \frac{1}{2}e_{126} \quad \text{in OT}(\mathcal{A}),$$

  $$e_{156} = \chi(125)\chi(165)e_{125} + \chi(126)\chi(156)e_{126} = e_{125} - e_{126} \quad \text{in } \mathcal{A}(\mathcal{M}(\mathcal{A})).$$

- Consider now the basis $\mathcal{B}_3$ of the $\mathbb{K}$-vector space $\mathcal{A}_3$. So we have:

  $$e_{235} = \text{sgn}(152)\text{sgn}(352)e_{125} + \text{sgn}(235)e_{135} = -e_{125} + e_{135} \quad \text{in OS}(\mathcal{M}(\mathcal{A})),$$

  $$e_{235} = \frac{\text{det}(152)}{\text{det}(352)}e_{125} + \frac{\text{det}(135)}{\text{det}(352)}e_{135} = -e_{125} + 2e_{135} \quad \text{in OT}(\mathcal{A}),$$

  $$e_{235} = \chi(152)\chi(352)e_{125} + \chi(135)\chi(235)e_{135} = -e_{125} + e_{135} \quad \text{in } \mathcal{A}(\mathcal{M}(\mathcal{A})).$$

  $$e_{126} = \text{sgn}(162)\text{sgn}(152)e_{125} + \text{sgn}(126)e_{156} = e_{125} + e_{156} \quad \text{in OS}(\mathcal{M}(\mathcal{A})),$$

  $$e_{126} = \frac{\text{det}(152)}{\text{det}(162)}e_{125} + \frac{\text{det}(156)}{\text{det}(162)}e_{156} = 3e_{125} - 2e_{156} \quad \text{in OT}(\mathcal{A}),$$

  $$e_{126} = \chi(152)\chi(162)e_{125} + \chi(156)\chi(126)e_{156} = e_{125} - e_{156} \quad \text{in } \mathcal{A}(\mathcal{M}(\mathcal{A})).$$
DIAGONAL BASES

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Departamento de Matemática,
INSTITUTO SUPERIOR TÉCNICO
AV. ROVISCO PAIS - 1049-001 LISBOA - PORTUGAL
E-mail address: cordovil@math.ist.utl.pt

LABORATOIRE DE RECHERCHE EN INFORMATIQUE
Bâtiment 490 Université Paris Sud
91405 ORSAY CEDEX - FRANCE
E-mail address: forge@lri.fr