Remarks on the embedding of spaces of distributions into spaces of Colombeau generalized functions

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August 3, 2018

Abstract

We present some remarks about the embedding of spaces of Schwartz distributions into spaces of Colombeau generalized functions. Following ideas of M. Nedeljkov et alii, we recall how a good choice of compactly supported mollifiers allows to perform globally the embedding of \( \mathcal{D}'(\Omega) \) into \( \mathcal{G}(\Omega) \). We show that this embedding is equal to the one obtained with local and sheaf arguments by M. Grosser et alii, this giving various equivalent technics to embed \( \mathcal{D}'(\Omega) \) into \( \mathcal{G}(\Omega) \).

Mathematics Subject Classification (2000): 46E10, 46E25, 46F05, 46F30

Keywords: Schwartz distributions, generalized functions, embedding.

1 Introduction

The question of embedding classical spaces such as \( C^0(\Omega), C^\infty(\Omega), \mathcal{D}'(\Omega) \) (where \( \Omega \) is an open subset of \( \mathbb{R}^d, d \in \mathbb{N} \)) in spaces of generalized functions arises naturally.

The embedding of \( C^\infty(\Omega) \) into \( \mathcal{G}(\Omega) \) is classically done by the canonical map

\[
\sigma : C^\infty(\Omega) \rightarrow \mathcal{G}(\Omega) \quad f \rightarrow [(f_\varepsilon)_\varepsilon] \text{ with } f_\varepsilon = f \text{ for } \varepsilon \in (0, 1]
\]

which is an injective homomorphism of algebras. \([(f_\varepsilon)_\varepsilon]\) denotes the class of \((f_\varepsilon)_\varepsilon\) in the factor algebra \( \mathcal{G}(\Omega) \): See section 2 for a short presentation of \( \mathcal{G}(\Omega) \) or [2], [3] for a complete construction.

For the embedding of \( \mathcal{D}'(\Omega) \) into \( \mathcal{G}(\Omega) \) the following additional assumption is required: If \( \iota_A \) is the expected embedding, one wants the following diagram to be commutative

\[
\begin{array}{ccc}
C^\infty(\Omega) & \rightarrow & \mathcal{D}'(\Omega) \\
\sigma \downarrow & & \downarrow \iota_A \\
\mathcal{G}(\Omega) & & \\
\end{array}
\]

that is \( \iota_A|_{C^\infty(\Omega)} = \sigma \).

In [3], this program is fulfilled by using the sheaf properties of Colombeau simplified algebras. Let us quote the main step of the construction. First, an embedding \( \iota_0 \) of \( \mathcal{E}'(\mathbb{R}^d) \) into \( \mathcal{G}(\mathbb{R}^d) \) is realized by convolution of compactly supported distributions with...
suitable mollifiers \((\rho_\varepsilon)_\varepsilon\) belonging to \(\mathcal{S}(\mathbb{R}^d)\). In fact, this map \(\iota_0\) can be considered as an embedding of \(\mathcal{E}'(\mathbb{R}^d)\) into \(\mathcal{G}_C(\mathbb{R}^d)\), the subalgebra of \(\mathcal{G}(\mathbb{R}^d)\) of compactly supported generalized functions, since the support of \(T \in \mathcal{E}'(\mathbb{R}^d)\) is equal to the support of its image by \(\iota_0\) (Proposition 1.2.12 of [9]). The following step of the construction of \(\iota_A\) is to consider for every open set \(\Omega \subset \mathbb{R}^d\) an open covering \((\Omega_\lambda)_\lambda\) of \(\Omega\) with relatively compact open sets and to embed \(\mathcal{D}'(\Omega)\) into \(\mathcal{G}(\Omega_\lambda)\) with the help of cutoff functions and \(\iota_0\). Using a partition of unity subordinate to \((\Omega_\lambda)_\lambda\), \(\iota_A\) is constructed by “gluing the bits obtained before together”. Finally, it is shown that the embedding \(\iota_A\) does not depend on the choice of \((\Omega_\lambda)_\lambda\) and other material of the construction, excepted the net \((\rho_\varepsilon)_\varepsilon\).

On one hand, the choice of \textit{ad hoc} not compactly supported mollifiers renders trivially the diagram \(\mathbb{I}\) commutative. On the other hand, \(\rho_\varepsilon\) cannot be convoluted with elements of \(\mathcal{D}'(\Omega)\) unrestrictedly, obliging to consider first compactly supported distributions, and then sheaf arguments.

In [9], the authors give an other construction which avoid this drawback. The main idea is to use compactly supported mollifiers enough close from the \textit{ad hoc} mollifiers \((\rho_\varepsilon)_\varepsilon\) of [8]. This is done by regular cutoff of \((\rho_\varepsilon)_\varepsilon\), this cutoff being defined with an other rate of growth than the net \((\rho_\varepsilon)_\varepsilon\), let say in \(\ln \varepsilon\) whereas the scale of growth of \((\rho_\varepsilon)_\varepsilon\) is in \(1/\varepsilon\). This permits to keep the good properties of the embedding in particular the commutativity of the diagram \(\mathbb{I}\). We present in details this construction in section 3 for the case \(\Omega = \mathbb{R}^d\).

In section 4 we show that these embeddings are in fact equal, consequently only depending on the choice of the mollifiers \((\rho_\varepsilon)_\varepsilon\). (This dependance is well known for the simplified Colombeau algebra.) We finally turn to the case of the embedding of \(\mathcal{D}'(\Omega)\) into the simplified Colombeau Algebra \(\mathcal{G}(\Omega)\) where \(\Omega\) is an arbitrary open subset of \(\mathbb{R}^d\) (Section 5). We show that for the global construction of [6] an additional cutoff applied to the elements of \(\mathcal{D}'(\Omega)\) is needed. We also give a local version (with no cutoff on the distribution) of the construction of [6].

This note comes from a workshop in Paris 7 and seminars of the team AANL of the laboratory AOC held in June and September 2003. I deeply think D. Scarpalezos and J.-A. Marti for the discussions about these constructions.

## 2 Preliminaries

### 2.1 The sheaf of Colombeau simplified algebras

Let \(\mathcal{C}^\infty\) be the sheaf of complex valued smooth functions on \(\mathbb{R}^d\) \((d \in \mathbb{N})\) with the usual topology of uniform convergence. For every open set \(\Omega\) of \(\mathbb{R}^d\), this topology can be described by the family of semi norms

\[
p_{K,l}(f) = \sup_{|\alpha| \leq l, K \subseteq \Omega} |\partial^\alpha f(x)|,
\]

where the notation \(K \subseteq \Omega\) means that the set \(K\) is a compact set included in \(\Omega\).

Let us set

\[
\mathcal{F}(\mathcal{C}^\infty(\Omega)) = \left\{ (f_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega)^{(0,1)} \mid \forall l \in \mathbb{N}, \forall K \subseteq \Omega, \exists q \in \mathbb{N}, p_{K,l}(f_\varepsilon) = o(\varepsilon^{-q}) \text{ for } \varepsilon \to 0 \right\},
\]

\[
\mathcal{N}(\mathcal{C}^\infty(\Omega)) = \left\{ (f_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega)^{(0,1)} \mid \forall l \in \mathbb{N}, \forall K \subseteq \Omega, \forall p \in \mathbb{N}, p_{K,l}(f_\varepsilon) = o(\varepsilon^p) \text{ for } \varepsilon \to 0 \right\}.
\]
Lemma 1 \([4] \text{ and } [5]\)

i. The functor \(F : \Omega \to \mathcal{F}(C^\infty(\Omega))\) defines a sheaf of subalgebras of the sheaf \((C^\infty)^{(0,1)}\).

ii. The functor \(N : \Omega \to \mathcal{N}(C^\infty(\Omega))\) defines a sheaf of ideals of the sheaf \(\mathcal{N}\).

We shall note prove in detail this lemma but quote the two main arguments:

i. For each open subset \(\Omega\) of \(X\), the family of seminorms \((p_{K,l})\) related to \(\Omega\) is compatible with the algebraic structure of \(C^\infty(\Omega)\); in particular:

\[
\forall l \in \mathbb{N}, \forall K \subseteq \Omega, \exists C \in \mathbb{R}_+, \forall (f,g) \in (C^\infty(\Omega))^2, p_{K,l}(fg) \leq C p_{K,l}(f) p_{K,l}(g),
\]

ii. For two open subsets \(\Omega_1 \subset \Omega_2\) of \(\mathbb{R}^d\), the family of seminorms \((p_{K,l})\) related to \(\Omega_1\) is included in the family of seminorms related to \(\Omega_2\) and

\[
\forall l \in \mathbb{N}, \forall K \subseteq \Omega_1, \forall f \in C^\infty(\Omega_2), p_{K,l}(f_{\mid \Omega_1}) = p_{K,l}(f).
\]

Definition 2

The sheaf of factor algebras

\[\mathcal{G}(C^\infty(\cdot)) = \mathcal{F}(C^\infty(\cdot)) / \mathcal{N}(C^\infty(\cdot))\]

is called the sheaf of Colombeau simplified algebras.

The sheaf \(\mathcal{G}\) turns to be a sheaf of differential algebras and a sheaf of modulus on the factor ring \(\mathbb{T} = \mathcal{F}(\mathbb{C}) / \mathcal{N}(\mathbb{C})\) with

\[
\mathcal{F}(\mathbb{K}) = \left\{(r_\varepsilon)_\varepsilon \in \mathbb{K}^{0,1} \mid \exists q \in \mathbb{N}, |r_\varepsilon| = o(\varepsilon^{-q}) \text{ for } \varepsilon \to 0\right\},
\]

\[
\mathcal{N}(\mathbb{K}) = \left\{(r_\varepsilon)_\varepsilon \in \mathbb{K}^{0,1} \mid \forall p \in \mathbb{N}, |r_\varepsilon| = o(\varepsilon^p) \text{ for } \varepsilon \to 0\right\},
\]

with \(\mathbb{K} = \mathbb{C}\) or \(\mathbb{K} = \mathbb{R}, \mathbb{R}_+\).

Notation 3

In the sequel we shall note, as usual, \(\mathcal{G}(\Omega)\) instead of \(\mathcal{G}(C^\infty(\Omega))\) the algebra of generalized functions on \(\Omega\).

2.2 Local structure of distributions

To fix notations, we recall here two classical results on the local structure of distributions which are going to be used in the sequel. We refer the reader to [7] chapter 3, specially theorem XXI & XXVI, for proofs and details. Let \(\Omega\) be an open subset of \(\mathbb{R}^d\) \((d \in \mathbb{N})\).

Theorem 4

For all \(T \in \mathcal{D}'(\Omega)\) and all \(\Omega'\) open subset of \(\mathbb{R}^d\) with \(\overline{\Omega'} \in \Omega\), there exists \(f \in C^0(\mathbb{R}^d)\) whose support is contained in an arbitrary neighborhood of \(\overline{\Omega'}\), \(\alpha \in \mathbb{N}^d\) such that \(T_{\mid \Omega'} = \partial^{\alpha} f\).

Theorem 5

For all \(T \in \mathcal{E}'(\Omega)\), there exists an integer \(r \geq 0\), a finite family \((f_\alpha)_{0 \leq |\alpha| \leq r}\) \((\alpha \in \mathbb{N}^d)\) with each \(f_\alpha \in C^0(\mathbb{R}^d)\) having its support contained in the same arbitrary neighborhood of the support of \(T\), such that \(T = \sum_{0 \leq |\alpha| \leq r} \partial^{\alpha} f_\alpha\).
3 Embedding of $\mathcal{D}'(\mathbb{R}^d)$ in $\mathcal{G}(\mathbb{R}^d)$

3.1 Construction of the mollifiers

Take $\rho \in \mathcal{S}(\mathbb{R}^d)$ even such that

$$
\int \rho(x) \, dx = 1, \quad \int x^m \rho(x) \, dx = 0 \text{ for all } m \in \mathbb{N} \setminus \{0\}
$$

and $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $\overline{B(0,1)}$ and $\chi \equiv 0$ on $\mathbb{R}^d \backslash B(0,2)$. Define

$$
\forall \varepsilon \in (0,1], \forall x \in \mathbb{R}^d, \quad \rho_\varepsilon(x) = \frac{1}{\varepsilon^d} \rho \left( \frac{x}{\varepsilon} \right)
$$

and

$$
\forall \varepsilon \in (0,1), \forall x \in \mathbb{R}^d, \quad \theta_\varepsilon(x) = \rho_\varepsilon(x) \chi (x | \ln \varepsilon|); \quad \theta_1(x) = 1.
$$

**Remark 6** The nets $(\rho_\varepsilon)_\varepsilon$ and $(\theta_\varepsilon)_\varepsilon$ defined above belong to $\mathcal{F}(\mathcal{C}^\infty(\mathbb{R}^d))$.

Let us verify this result for $(\theta_\varepsilon)_\varepsilon$ and $d = 1$. Fixing $\alpha \in \mathbb{N}$, we have

$$
\forall x \in \mathbb{R}, \quad \partial^\alpha \theta_\varepsilon(x) = \sum_{\beta=0}^\alpha C_\alpha^\beta \partial^\beta \rho_\varepsilon(x) \partial^{\alpha-\beta} \chi (x | \ln \varepsilon|),
$$

$$
= \sum_{\beta=0}^\alpha C_\alpha^\beta \varepsilon^{-1-\beta} \ln \varepsilon |^{\alpha-\beta} \rho^{(\beta)} \left( \frac{x}{\varepsilon} \right) \chi^{(\alpha-\beta)} (x | \ln \varepsilon|).
$$

For all $\beta \in \{0, \ldots, \alpha\}$, we have $\varepsilon^{-1-\beta} \ln \varepsilon |^{\alpha-\beta} = o(\varepsilon^{-2-\alpha})$ for $\varepsilon \to 0$. As $\rho^{(n)}$ and $\chi^{(n)}$ are bounded, there exists $C(\alpha)$ such that

$$
\forall x \in \mathbb{R}, \quad |\partial^\alpha (\theta_\varepsilon(x))| \leq C(\alpha) \varepsilon^{-2-\alpha}.
$$

Our claim follows from this last inequality.

**Lemma 7** With the previous notations, the following properties holds

$$
(\theta_\varepsilon)_\varepsilon - (\rho_\varepsilon)_\varepsilon \in \mathcal{N} \left( \mathcal{C}^\infty(\mathbb{R}^d) \right),
$$

$$
\forall k \in \mathbb{N}, \quad \int \theta_\varepsilon \, dx = 1 + o(\varepsilon^k) \text{ for } \varepsilon \to 0,
$$

$$
\forall k \in \mathbb{N}, \forall m \in \mathbb{N}^d \setminus \{0\}, \quad \int x^m \theta_\varepsilon \, dx = o(\varepsilon^k) \text{ for } \varepsilon \to 0.
$$

In other words, we have

$$
\left( \int \theta_\varepsilon \, dx - 1 \right) \varepsilon \in \mathcal{N}(\mathbb{R}) \quad \forall m \in \mathbb{N}^d \setminus \{0\}, \quad \left( \int x^m \theta_\varepsilon \, dx \right) \varepsilon \in \mathcal{N}(\mathbb{R}).
$$

**Proof.** We consider the case $d = 1$ in order to simplify notations.

**First assertion.-** We have, for all $x \in \mathbb{R}$ and $\varepsilon \in (0,1)$,

$$
|\rho_\varepsilon(x) - \theta_\varepsilon(x)| \leq \frac{1}{\varepsilon} \left| \rho \left( \frac{x}{\varepsilon} \right) \right| (1 - \chi (x | \ln \varepsilon|)) \leq \frac{1}{\varepsilon} \left| \rho \left( \frac{x}{\varepsilon} \right) \right|,
$$

(6)
Since $\rho$ belongs to $\mathcal{S}(\mathbb{R})$, for all integer $k > 0$ there exists a constant $C(k)$ such that

$$\forall x \in \mathbb{R}, \quad |\rho(x)| \leq \frac{C(k)}{1 + |x|^k}.$$ 

Then, for all $x \in \mathbb{R}$ with $|x| \geq 1/|\ln \varepsilon|$, \(\frac{1}{\varepsilon} \left| \frac{\rho(x)}{\varepsilon} \right| \leq \frac{C(k)}{\varepsilon^k + |x|^k} \leq C(k) |\ln \varepsilon|^k \varepsilon^{k-1} = o\left(\varepsilon^{k-2}\right) \quad (7)\)

According to remark (\ref{remark}) $(\rho_{\varepsilon} - \theta_{\varepsilon})_{\varepsilon} \in \mathcal{F}(\mathcal{C}^\infty(\mathbb{R}))$. Then we can conclude without estimating the derivatives that $(\rho_{\varepsilon} - \theta_{\varepsilon})_{\varepsilon} \in \mathcal{N}(\mathcal{C}^\infty(\mathbb{R}))$ by using theorem 1.2.3. of \[3\].

Second and third assertions.- According definition of $\rho$ we find that

$$\int \rho_{\varepsilon}(x) \, dx = 1; \quad \forall m \in \mathbb{N} \setminus \{0\}, \int x^m \rho_{\varepsilon}(x) \, dx = 0.$$ 

So, it suffices to show that, for all $m \in \mathbb{N}$ and $k \in \mathbb{N},$

$$\Delta_{m,\varepsilon}(x) = \int x^m (\rho_{\varepsilon}(x) - \theta_{\varepsilon}(x)) \, dx = o\left(\varepsilon^k\right) \text{ for } \varepsilon \to 0$$

to complete our proof. Let fix $m \in \mathbb{N}$. We have

$$\Delta_{m,\varepsilon}(x) = \int_{-\infty}^{1/|\ln \varepsilon|} x^m (\rho_{\varepsilon}(x) - \theta_{\varepsilon}(x)) \, dx + \int_{1/|\ln \varepsilon|}^{+\infty} x^m (\rho_{\varepsilon}(x) - \theta_{\varepsilon}(x)) \, dx.$$ 

Let us find an estimate of $(\ast\ast)$. We have, according to relations (6) and (7), \(\forall x > 0, \forall k \in \mathbb{N}, \quad x^m |\rho_{\varepsilon}(x) - \theta_{\varepsilon}(x)| \leq \frac{1}{\varepsilon} x^m \left| \frac{x}{\varepsilon} \right| \leq C(k) \varepsilon^{k-1} x^{m-k}.$$ 

Therefore, by choosing $k \geq m + 2$,

$$\left| \int_{1/|\ln \varepsilon|}^{+\infty} x^m (\rho_{\varepsilon}(x) - \theta_{\varepsilon}(x)) \, dx \right| \leq C(k) \varepsilon^{k-1} \int_{1/|\ln \varepsilon|}^{+\infty} x^{m-k} \, dx \leq \frac{C(k)}{k - 1 - m} \varepsilon^{k-1} |\ln \varepsilon|^{k-m-1} = o\left(\varepsilon^{k-2}\right), \text{ for } \varepsilon \to 0.$$ 

With a similar estimate for $(\ast)$ we obtain our claim. 

3.2 Construction of the embedding $\iota_S$

**Proposition 8** With notations of lemma 7, the map

$$\iota_S : \mathcal{D}'(\mathbb{R}^d) \to \mathcal{G}(\mathbb{R}^d) \quad T \mapsto (T * \theta_{\varepsilon})_{\varepsilon} + \mathcal{N}\left(\mathcal{C}^\infty(\mathbb{R}^d)\right)$$

is an injective homomorphism of vector spaces. Moreover $\iota_S|_{\mathcal{C}^\infty(\mathbb{R}^d)} = \sigma$. 

5
The integration is performed on the compact set supp $\theta$. Fix $\epsilon$ such that $\theta(\epsilon) = \epsilon$ uniformly on supp $\epsilon$. Indeed, taking $\epsilon(\theta)$ allows to define the map $\epsilon$. We get $T \ast \epsilon(\theta) = \langle T, \{ x \mapsto \epsilon(\theta) (y - x) \} \rangle$ for $\epsilon$ small enough.

Then, the function $x \mapsto \epsilon(\theta) (y - x)$ belongs to $C \ast \epsilon(\theta)$ and $\langle T, \epsilon(\theta) (y - \cdot) \rangle = \langle T_{\epsilon(\theta)}, \epsilon(\theta) (y - \cdot) \rangle$.

Using theorem $\epsilon$, we can write $T_{\epsilon(\theta)} = \partial^2 \epsilon f$ where $f$ is a continuous function compactly supported. Then $T \ast \epsilon(\theta) = f \ast \partial^2 \epsilon f$ and

$$\forall y \in K, \quad (T \ast \epsilon(\theta))(y) = \int_{\epsilon} f(y - x) \partial^2 \epsilon f(x) \, dx.$$ 

According to remark $\delta$ there exists $m(\alpha) \in \mathbb{N}$ such that

$$\forall x \in \mathbb{R}^d, \quad |\partial^2 \epsilon f(x)| \leq C \epsilon^{-m(\alpha)}.$$

We get

$$\forall y \in K, \quad |(T \ast \epsilon(\theta))(y)| \leq C \sup_{x \in \epsilon} |f(\epsilon)| \, \text{vol}(\epsilon) \epsilon^{-m(\alpha)},$$

and $\sup_{y \in K} |(T \ast \epsilon(\theta))(y)| = O(\epsilon^{-m(\alpha)})$ for $\epsilon \to 0$.

Since $\partial^3 (f \ast \partial^2 \epsilon) = f \ast \partial^3 \epsilon$ the same arguments applies to derivatives and the claim follows.

Let us now prove that $\epsilon$ is injective, id est

$$(T \ast \epsilon(\theta)) \in \mathcal{N} \left(C\ast \epsilon(\theta) \left(\mathbb{R}^d \right) \right) \Rightarrow T = 0.$$ 

Indeed, taking $\varphi \in C \ast \epsilon(\theta)$ we have $\langle T \ast \epsilon(\theta), \varphi \rangle \to \langle T, \varphi \rangle$ since $T \ast \epsilon(\theta) \to T$. But, $T \ast \epsilon(\theta) \to 0$ uniformly on supp $\varphi$ since $T \ast \epsilon(\theta) \in \mathcal{N} \left(C\ast \epsilon(\theta) \left(\mathbb{R}^d \right) \right)$.

Then $\langle T \ast \epsilon(\theta), \varphi \rangle \to 0$ and $\langle T, \varphi \rangle = 0$.

We shall prove the last assertion in the case $d = 1$, the general case only differs by more complicate algebraic expressions.

Let $f$ be in $C\ast \epsilon(\theta)$ and set $\Delta = \epsilon f (f) - \sigma(f)$. One representative of $\Delta$ is given by

$$\Delta : \mathbb{R} \to C \ast \epsilon(\theta) \left(\mathbb{R}^d \right) \quad y \to f(y \ast \epsilon) = \int_{\epsilon} f(y - x) \epsilon(x) \, dx - f(y).$$

Fix $K$ a compact of $\mathbb{R}$. Writing $\int_0^1 \epsilon(x) \, dx = 1 + \mathcal{N} \epsilon$ with $\mathcal{N} \epsilon \in \mathcal{N} \ast \epsilon(\theta)$ we get

$$\Delta(x) = \int_{\epsilon} (f(y - x) - f(y)) \epsilon(x) \, dx + \mathcal{N} \epsilon f(y).$$

The integration is performed on the compact set supp $\epsilon \subset [-2/|\epsilon|, 2/|\epsilon|]$.
Let \( k \) be an integer. Taylor’s formula gives

\[
f(y-x) - f(y) = \sum_{i=1}^{k} \frac{(-x)^i}{i!} f^{(i)}(y) + \frac{(-x)^k}{k!} \int_0^1 f^{(k+1)}(y - ux)(1-u)^k \, du
\]
and

\[
\Delta \varepsilon(y) = \sum_{i=1}^{k} \frac{(-1)^i}{i!} f^{(i)}(y) \int_{-2/|\ln \varepsilon|}^{2/|\ln \varepsilon|} x^i \theta \varepsilon(x) \, dx
\]

using theorem 1.2.3. of [3].

According to lemma 7, we have \( \left( \int x^i \theta \varepsilon(x) \, dx \right)_\varepsilon \in \mathcal{N}(\mathbb{R}) \) and consequently

\[
(P_\varepsilon(k,y))_\varepsilon \in \mathcal{N}(\mathbb{R}).
\]

Using the definition of \( \theta \varepsilon \), we have

\[
R_\varepsilon(k,y) = \frac{1}{\varepsilon} \int_{-2/|\ln \varepsilon|}^{2/|\ln \varepsilon|} \frac{(-x)^k}{k!} \int_0^1 f^{(k+1)}(y - ux)(1-u)^k \, du \rho \left( \frac{x}{\varepsilon} \right) \chi(x|\ln \varepsilon|) \, dx.
\]

Setting \( v = x/\varepsilon \) we get

\[
R_\varepsilon(k,y) = \varepsilon^{k+1} \int_{-2/(\varepsilon|\ln \varepsilon|)}^{2/(\varepsilon|\ln \varepsilon|)} \frac{(-v)^k}{k!} \int_0^1 f^{(k+1)}(y - \varepsilon uv)(1-u)^k \, du \rho(v) \chi(\varepsilon|\ln \varepsilon|v) \, dv.
\]

For \( (u,v) \in [0,1] \times [-2/(\varepsilon|\ln \varepsilon|), 2/(\varepsilon|\ln \varepsilon|)] \), we have \( y - \varepsilon uv \in [y-1, y+1] \) for \( \varepsilon \) small enough. Then, for \( y \in K \), \( y - \varepsilon uv \) lies in a compact \( K' \) for \( (u,v) \) in the domain of integration.

It follows

\[
|R_\varepsilon(k,y)| \leq \frac{\varepsilon^k}{k!} \sup_{\xi \in K'} f^{(k+1)}(\xi) \int_{-2/(\varepsilon|\ln \varepsilon|)}^{2/(\varepsilon|\ln \varepsilon|)} |v|^{k+1} |\rho(v)| \, dv,
\]

\[
\leq \frac{\varepsilon^k}{k!} \sup_{\xi \in K'} f^{(k+1)}(\xi) \int_{-\infty}^{\infty} |v|^{k+1} |\rho(v)| \, dv \leq C \varepsilon^k (C > 0).
\]

The constant \( C \) depends only on the integer \( k \), the compacts \( K, K', \rho \) and \( f \).

Finally, for all \( k > 0 \)

\[
\sup_{y \in K} \Delta \varepsilon(y) = o \left( \varepsilon^k \right) \text{ for } \varepsilon \to 0.
\]

As \( (\Delta \varepsilon)_\varepsilon \in \mathcal{F} \left( C^\infty(\mathbb{R}^d) \right) \) and \( \sup_{y \in K} \Delta \varepsilon(y) = o \left( \varepsilon^k \right) \) for all \( k > 0 \) and \( K \subset \mathbb{R} \), we can conclude directly (without estimating the derivatives) that \( (\Delta \varepsilon)_\varepsilon \in \mathcal{N} \left( C^\infty(\mathbb{R}^d) \right) \) by using theorem 1.2.3. of [3].
4 Comparison between \( i_A \) and \( i_S \)

As quoted in the introduction, the embedding \( i_A : \mathcal{D}'(\mathbb{R}^d) \to \mathcal{G}(\mathbb{R}^d) \) constructed in [3] depends on the choice of the chosen net \( \rho \in S(\mathbb{R}^d) \). This dependance is a well known fact for the simplified Colombeau algebra. Of course, \( i_S \) depends also on the choice of \( \rho \in S(\mathbb{R}^d) \), but not on the choice of \( \chi \). Moreover:

**Proposition 9** For the same choice of \( \rho \), we have: \( i_A = i_S \).

The proof is carried out in the two following subsections.

4.1 The case of the embedding of \( \mathcal{E}'(\mathbb{R}^d) \) in \( \mathcal{G}(\mathbb{R}^d) \)

In [3], the embedding of \( \mathcal{E}'(\mathbb{R}^d) \) in \( \mathcal{G}(\mathbb{R}^d) \) is realized with the map

\[
i_0 : \mathcal{E}'(\mathbb{R}^d) \to \mathcal{G}(\mathbb{R}^d), \quad T \mapsto (T \ast \rho_\varepsilon)_\varepsilon + N\left(C^\infty(\mathbb{R}^d)\right).
\]

We compare here with \( i_S|_{\mathcal{E}'(\mathbb{R}^d)} \). Let fix \( T \in \mathcal{E}'(\mathbb{R}^d) \). We have to estimate \((T \ast \rho_\varepsilon)_\varepsilon - (T \ast \theta_\varepsilon)_\varepsilon\).

Using theorem [5] we can write \( T = \sum f_{\text{finite}} \partial^\alpha f_\alpha \), each \( f_\alpha \) having a compact support. Using linearity, we only need to estimate for one summand, and we shall consider that \( T = \partial^\alpha f \). Setting \( \Delta_\varepsilon = T \ast \theta_\varepsilon - T \ast \rho_\varepsilon \) we have

\[
\forall y \in \mathbb{R}^d, \quad \Delta_\varepsilon(y) = \int f(y - x) (\partial^\alpha \theta_\varepsilon(x) - \partial^\alpha \rho_\varepsilon(x)) \, dx.
\]

Then

\[
|\Delta_\varepsilon(y)| \leq C \int |\partial^\alpha \theta_\varepsilon(x) - \partial^\alpha \rho_\varepsilon(x)| \, dx \quad \text{with } C = \sup_{\xi \in \mathbb{R}^d} |f(\xi)|,
\]

\[
\leq C \int_{\mathbb{R}^d \setminus B(0,1/|\ln \varepsilon|)} |\partial^\alpha \theta_\varepsilon(x) - \partial^\alpha \rho_\varepsilon(x)| \, dx.
\]

since \( \partial^\alpha \theta_\varepsilon = \partial^\alpha \rho_\varepsilon \) on \( B(0,1/|\ln \varepsilon|) \).

To simplify notations, we suppose \( d = 1 \) and for example \( \alpha = 1 \). We have

\[
\theta_\varepsilon(x) - \rho_\varepsilon(x) = \varepsilon^{-1} |\ln \varepsilon| \chi'(x|\ln \varepsilon|) \rho(\varepsilon^{-1}x) + \varepsilon^{-2} \rho'(\varepsilon^{-1}x) (\chi(x|\ln \varepsilon|) - 1).
\]

Since \( \rho \in S(\mathbb{R}) \), for all \( k \in \mathbb{N}, \) with \( k \geq 2 \), their exists \( C(k) \in \mathbb{R}_+ \) such that

\[
|\rho^{(i)}(x)| \leq \frac{C(k)}{1 + |x|^k} \quad (\text{for } i = 0 \text{ and } i = 1).
\]

Then, for all \( x \) with \( |x| \geq 1/|\ln \varepsilon| \) we get

\[
|\rho^{(i)}(\varepsilon^{-1}x)| \leq C(k) \varepsilon^k \frac{1}{\varepsilon^k + |x|^k} \leq C(k) \varepsilon^k |x|^{-k}.
\]

Since \( |\ln \varepsilon| \leq \varepsilon^{-1} \) for \( \varepsilon \in (0,1] \) and \( |\chi(x|\ln \varepsilon|) - 1| \leq 1 \) for all \( x \in \mathbb{R}, \) we get

\[
|\theta'(x) - \rho'_\varepsilon(x)| \leq |x|^{-k} \left(\varepsilon^{k-1} |\ln \varepsilon| \sup_{\xi \in \mathbb{R}} |\chi'(\xi)| |x|^{-k} C(k) |x|^{-k} + \varepsilon^{k-2} C(k) |x|^{-k}\right),
\]

\[
\leq \varepsilon^{k-2} C(k) \left( \sup_{\xi \in \mathbb{R}} |\chi'(\xi)| + 1 \right) |x|^{-k}.
\]
Then, we get a constant $C' = C'(k, \chi, f) > 0$ such that
\[
|\Delta_{\epsilon}(y)| \leq 2\epsilon^{k-2}C' \int_{1/|\ln \epsilon|}^{+\infty} |x|^{-k} \, dx = \frac{2C'}{k-1} \epsilon^{k-2} |\ln \epsilon|^{k-1}.
\]

Finally, we have $\sup_{y \in \mathbb{R}} |\Delta_{\epsilon}(y)| = o(\epsilon^{k})$ for all $k \in \mathbb{N}$.

As $(\Delta_{\epsilon})_{\epsilon} \in \mathcal{F}(C^\infty(\mathbb{R}^d))$, we finally conclude that $(\Delta_{\epsilon})_{\epsilon} \in \mathcal{N}(C^\infty(\mathbb{R}^d))$ by using theorem 1.2.3. of [3]. Then:

**Lemma 10** For the same choice of $\rho$, we have: $\iota_0 = \iota_S|\mathcal{E}'(\mathbb{R}^d)$.

### 4.2 The case of the embedding of $\mathcal{D}'(\mathbb{R}^d)$ in $\mathcal{G}(\mathbb{R}^d)$

**Notation 11** In this subsection we shall note $\mathbb{N}_m = \{1, \ldots, m\}$ for all $m \in \mathbb{N}\setminus\{0\}$.

Let us recall briefly the construction of [3]. Fix some locally finite open covering $(\Omega_{\lambda})_{\lambda \in \Lambda}$ with $\overline{\Omega_{\lambda}} \subseteq \mathbb{R}^d$ and a family $(\psi_{\lambda})_{\lambda \in \Lambda} \in \mathcal{D}(\mathbb{R}^d)^{\Lambda}$ with $0 \leq \psi_{\lambda} \leq 1$ and $\psi_{\lambda} \equiv 1$ on a neighborhood of $\Omega_{\lambda}$. For each $\lambda$ define
\[
\iota_{\lambda} : \mathcal{D}'(\mathbb{R}^d) \to \mathcal{G}(\Omega_{\lambda}) \quad T \to \iota_{\lambda}(T) = \iota_0(\psi_{\lambda} T|_{\Omega_{\lambda}}) = \left((\psi_{\lambda} T|_{\Omega_{\lambda}})\right) + \mathcal{N}(C^\infty(\Omega_{\lambda})).
\]

The family $(\iota_{\lambda})_{\lambda \in \Lambda}$ is coherent and by sheaf argument, there exists a unique $\iota_A : \mathcal{D}'(\mathbb{R}^d) \to \mathcal{G}(\mathbb{R}^d)$ such that
\[
\forall \lambda \in \Lambda, \quad \iota_A|_{\Omega_{\lambda}} = \iota_{\lambda}.
\]

Moreover, an explicit expression of $\iota_A$ can be given: Let $(\chi_j)_{j \in \mathbb{N}}$ be a smooth partition of unity subordinate to $(\Omega_{\lambda})_{\lambda \in \Lambda}$. We have
\[
\forall T \in \mathcal{D}'(\mathbb{R}^d), \quad \iota_A(T) = \left(\sum_{j=1}^{+\infty} \chi_j \left((\psi_{\lambda(j)} T|_{\Omega_{\lambda}})\right)\right) + \mathcal{N}(C^\infty(\mathbb{R}^d)).
\]

Let us compare $\iota_A$ and $\iota_S$. Using sheaf properties, we only need to verify that
\[
\forall \lambda \in \Lambda, \quad \iota_A|_{\Omega_{\lambda}} = \iota_S|_{\Omega_{\lambda}} \quad (= \iota_{\lambda})
\]

For a fixed $\lambda \in \Lambda$ and $T \in \mathcal{D}'(\mathbb{R}^d)$, we have $\iota_{\lambda}(T) = \iota_0(\psi_{\lambda} T|_{\Omega_{\lambda}})$ and
\[
\iota_A|_{\Omega_{\lambda}} - \iota_S|_{\Omega_{\lambda}}(T) = \iota_0(\psi_{\lambda} T) - \iota_0(\psi_{\lambda} T) + \iota_0(\psi_{\lambda} T) - \iota_0(T).
\]

(We omit the restriction symbol in the right hand side).

As $\psi_{\lambda} T \in \mathcal{E}'(\mathbb{R}^d)$, we have $\iota_0(\psi_{\lambda} T) = \iota_s(\psi_{\lambda} T)$ according to lemma [10]. It remains to show that $\iota_s(\psi_{\lambda} T) = \iota_s(T)$, that is to compare $((\psi_{\lambda} T))_{\epsilon}$ and $(T \ast \theta_{\epsilon})_{\epsilon}$. Let us recall that
\[
\forall y \in \Omega_{\lambda}, \quad ((\psi_{\lambda} T) \ast \theta_{\epsilon})(y) - (T \ast \theta_{\epsilon})(y) = (\psi_{\lambda} T - T, \{x \mapsto \theta_{\epsilon}(y - x)\}),
\]

for $\epsilon$ small enough.

Let consider $K$ a compact included in $\Omega_{\lambda}$. According to relation [8], we have $\text{supp} \theta_{\epsilon}(y - \cdot) \subseteq B(y, 2/|\ln \epsilon|)$. Using the fact that $\Omega_{\lambda}$ is open, we obtain that
\[
\forall y \in K, \quad \exists \epsilon_y \in (0, 1], \quad \forall \epsilon \in (0, \epsilon_y], \quad B(y, 2/|\ln \epsilon|) \subseteq \Omega_{\lambda}.
\]
The family \((B(y, 1/|\ln \varepsilon_y|))_{y \in K}\) is an open covering of \(K\) from which we can extract a finite one, \((B(y_l, 1/|\ln \varepsilon_l|))_{1 \leq l \leq n}\) (with \(\varepsilon_l = \varepsilon_{y_l}\)). Put

\[\varepsilon_K = \min_{1 \leq l \leq n} \varepsilon_l.\]

For \(y \in K\), there exists \(l \in \mathbb{N}_n\) such that \(y \in B(y_l, 1/|\ln \varepsilon_l|)\). Then, for \(\varepsilon \leq \varepsilon_K^2\), we have

\[\text{supp} \theta_\varepsilon (y - \cdot) \subset B(y, 2/|\ln \varepsilon|) \subset B(y_l, 1/|\ln \varepsilon_l|) \subset B(y_l, 2/|\ln \varepsilon_l|) \subset \Omega.\]

since \(d(y, y_l) < 1/|\ln \varepsilon_l|\).

For all \(y \in K\), \(\theta_\varepsilon (y - \cdot) \in \mathcal{D} (\Omega_\lambda)\) for \(\varepsilon \in (0, \varepsilon_K^2]\). Since \(T|_{\Omega_\lambda} = (\psi_\lambda T)|_{\Omega_\lambda}\) we finally obtain

\[\forall y \in K, \forall \varepsilon \in (0, \varepsilon_K^2], \quad \langle \psi_\lambda T - T, \{x \mapsto \theta_\varepsilon (y - x)\}\rangle = 0,\]

this showing that \(((\psi_\lambda T - T) \ast \theta_\varepsilon)\) lies in \(\mathcal{N} (C^\infty (\mathbb{R}^d))\).

## 5 Embedding of \(D' (\Omega)\) into \(G (\Omega)\)

All the embeddings of \(D' (\Omega)\) into \(G (\Omega)\) considered in the literature are based on convolution of distributions by \(C^\infty\) functions. This product is possible under additional assumptions in particular about supports. Let consider both constructions compared in this paper.

For the construction of \([3]\), the local construction with cutoff technics applied to the elements of \(D' (\Omega)\) is needed to obtain a well defined product of convolution between elements of \(\mathcal{E}' (\mathbb{R}^d)\) and \(\mathcal{S} (\mathbb{R}^d)\). Note that the cutoff is fixed once for all, and in particular does not depend on \(\varepsilon\).

The construction of \([3]\) allows a “global” embedding of \(D' (\mathbb{R}^d)\) into \(G (\mathbb{R}^d)\) since the convolution of elements of \(D' (\mathbb{R}^d)\) with \((\theta_\varepsilon)_\varepsilon \in (D (\mathbb{R}^d))^{(0,1]}\) is well defined. But, for the case of an open subset \(\Omega \subseteq \mathbb{R}^d\), previous arguments show that for \(y \in \Omega\), the functions \(\{x \mapsto \theta_\varepsilon (y - x)\}\) belong to \(D (\Omega)\) for \(\varepsilon\) smaller than some \(\varepsilon_y\) depending on \(y\). This does not allow the definition of the net \((T \ast \theta_\varepsilon)_\varepsilon\) for \(T \in D' (\Omega)\) not compactly supported. To overcome this difficulty, a net of cutoffs \((\kappa_\varepsilon)_\varepsilon \in (D (\mathbb{R}^d))^{(0,1]}\) such that \(\kappa_\varepsilon T \to T\) in \(D' (\Omega)\) is considered, giving a well defined convolution of elements of \(\mathcal{E}' (\mathbb{R}^d)\) with elements of \(D (\mathbb{R}^d)\). We present this construction below with small changes and another construction mixing local technics and compactly supported mollifiers of \([3]\).

### 5.1 Embedding using cutoff arguments

Let us fix \(\Omega \subset \mathbb{R}^d\) an open subset and set, for all \(\varepsilon \in (0, 1]\),

\[K_\varepsilon = \left\{ x \in \Omega \mid d(x, \mathbb{R}^d \setminus \Omega) \geq \varepsilon \text{ and } d(x, 0) \leq 1/\varepsilon \right\}.\]

Consider \((\kappa_\varepsilon)_\varepsilon \in (D (\mathbb{R}^d))^{(0,1]}\) such that

\[\forall \varepsilon \in (0, 1], \quad 0 \leq \kappa_\varepsilon \leq 1, \quad \kappa_\varepsilon \equiv 1 \text{ on } K_\varepsilon.\]

(Such a net \((\kappa_\varepsilon)_\varepsilon\) is obtained, for example, by convolution of the characteristic function of \(K_\varepsilon\) with a net of mollifiers \((\varphi_\varepsilon)_\varepsilon \in (D (\mathbb{R}^d))^{(0,1]}\) with support decreasing to \(\{0\}\).)
Proposition 12 With notations of lemma 7, the map

\[ \iota_S : \mathcal{D}'(\Omega) \to \mathcal{G}(\Omega) \quad T \mapsto ((\kappa_\varepsilon T) \ast \theta_\varepsilon) + \mathcal{N}(\mathcal{C}^\infty(\Omega)) \]  

is an injective homomorphism of vector spaces. Moreover \( \iota_S|_{\mathcal{C}^\infty(\mathbb{R}^d)} = \sigma \).

We shall not give a complete proof since it is a slight adaptation of the proof of proposition 8. We just quote here the main point. As seen above, many estimates have to be done on compact sets. Let \( K \) be a compact included in \( \Omega \) and \( \Omega' \) an open set such that \( K \subset \Omega' \subset \Omega \). There exists \( \varepsilon_0 \in (0,1] \) such that

\[ \forall \varepsilon \in (0,\varepsilon_0], \quad \Omega' \subset K_\varepsilon. \]

On one hand this implies that we have \( (\kappa_\varepsilon T)|_{\Omega'} = (\kappa_{\varepsilon_0} T)|_{\Omega'} \) for all \( T \in \mathcal{D}'(\Omega) \) and \( \varepsilon \in (0,\varepsilon_0] \). On the other hand, we already noticed that for \( y \in K \), the functions \( \{ x \to \theta_\varepsilon(y-x) \} \) belongs to \( \mathcal{D}'(\Omega') \) for all \( \varepsilon \in (0,\varepsilon_0], \varepsilon \) only depending on \( K \).

Thus a representative of \( \iota_S(T) \) is given, for all \( y \in K \), by the convolution of an element of \( \mathcal{E}'(\mathbb{R}^d) \) with an element of \( \mathcal{D}(\Omega) \) this being valid for \( \varepsilon \) smaller than \( \min(\varepsilon_0,\varepsilon_2,\varepsilon_K) \) only depending on \( K \).

Proof of propositions 8 and 9 can now be adapted using this remark.

Remark 13 For the presentation of the construction of \( \sigma \) we chose to consider first the case \( \Omega = \mathbb{R}^d \). In fact, we can unify the construction and consider for all \( \Omega \) (included in \( \mathbb{R}^d \)) the embedding defined by \( \sigma \). In the case \( \Omega = \mathbb{R}^d \), the cutoff functions \( \kappa_\varepsilon \) are equal to one on the closed ball \( \overline{B}(0,1/\varepsilon) \).

5.2 Embedding using local arguments

Let fix \( \Omega \) an open subset of \( \mathbb{R}^d \). Recall that relation 8 implies that

\[ \forall y \in \Omega, \quad \exists \varepsilon_y \in (0,1], \quad \forall \varepsilon \in (0,\varepsilon_y], \quad \text{supp} \theta_\varepsilon(y-\cdot) \subset B(y,2/|\ln \varepsilon|) \subset \Omega. \]

and consequently that \( \theta_\varepsilon(y-\cdot) \in \mathcal{D}(\Omega) \) for \( \varepsilon \in (0,\varepsilon_y] \). We consider here a local construction to overcome the fact that \( \varepsilon_y \) depends on \( y \).

Let \( \Omega' \) be an open relatively compact subset of \( \Omega \). As in subsection 5.2, we find \( \varepsilon_{\Omega'} \) such that, for all \( \varepsilon \leq \varepsilon_{\Omega'}^2 \) and \( y \in \Omega' \), we have supp \( \theta_\varepsilon(y-\cdot) \subset \Omega \) and \( \theta_\varepsilon(y-\cdot) \in \mathcal{D}(\Omega) \). For \( T \in \mathcal{D}'(\Omega) \), define, for all \( y \in \Omega' \)

\[ T_\varepsilon(y) = \langle T, \theta_\varepsilon(y-\cdot) \rangle \text{ for } \varepsilon \in (0,\varepsilon_{\Omega'}^2], \quad T_{\varepsilon}^2(y) = T_{\varepsilon^2}(y) \text{ for } \varepsilon \in (\varepsilon_{\Omega'}^2,1]. \]  

Lemma 14 The map

\[ \iota_{\Omega'} : \mathcal{D}'(\Omega) \to \mathcal{G}(\Omega') \quad T \mapsto T_\varepsilon(y) + \mathcal{N}(\mathcal{C}^\infty(\Omega')) \]

is an injective homomorphism of vector spaces.

The proof is very similar to proposition 8's one.

Consider now a locally finite open covering of \( (\Omega_\lambda)_{\lambda \in \Lambda} \) with \( \Omega_\lambda \subset \Omega \) and set \( \iota_\lambda = \iota_{\Omega_\lambda} \) for \( \lambda \in \Lambda \).

Lemma 15 The family \( (\iota_\lambda)_{\lambda \in \Lambda} \) is coherent.
Proof. Let us take \((\lambda, \mu) \in \Lambda^2\) with \(\Omega_\lambda \cap \Omega_\mu \neq \emptyset\). We have
\[
i_\lambda|_{\Omega_\lambda \cap \Omega_\mu} = i_\mu|_{\Omega_\lambda \cap \Omega_\mu}
\]
since, for all \(T\) in \(D'(\Omega')\) representatives of \(i_\lambda\) and \(i_\mu\), written in the form \((\mathbf{1})\), are equal for \(\varepsilon \leq \min\left(\varepsilon_{\Omega_\lambda}^2, \varepsilon_{\Omega_\mu}^2\right)\). ■

By sheaf property of \(G(\Omega)\) there exists a unique \(i'_S : D'(\Omega) \rightarrow G(\Omega)\) such that \(i'_S|_{\Omega_\lambda} = i_\lambda\) for all \(\lambda \in \Lambda\). Moreover, we can give an explicit formula: If \((\Psi_\lambda)_{\lambda \in \Lambda}\) is a partition of unity subordinate to \((\Omega_\lambda)_{\lambda \in \Lambda}\), we have
\[
\forall T \in D'(\Omega), \quad i'_S(T) = \sum_{\lambda \in \Lambda} \Psi_\lambda i_\lambda(T).
\]
This map \(i'_S\) realizes an embedding which does not depend on the particular choice of \((\Omega_\lambda)_{\lambda \in \Lambda}\) (proof left to the reader).

Remark 16 One may think that it is regrettable to come back here to local arguments, whereas they are avoided with cutoff technic. This is partially true but the advantage of compactly supported mollifiers remains: The convolution with any distribution is possible. This renders the local arguments very simple.

5.3 Final remark
Let \(\Omega\) be an open subset of \(\mathbb{R}^d\).

Proposition 17 For the same choice of \(\rho\), we have: \(i_A = i_S = i'_S\).

With notations of previous sections, we only have to prove the equality on each open set \(\Omega_\lambda\), where \((\Omega_\lambda)_{\lambda \in \Lambda}\) is a covering of \(\Omega\) with relatively compact open sets. As seen before, we shall have \((\kappa_\varepsilon T)|_{\Omega_\lambda} = T|_{\Omega_\lambda}\) and \(\theta_\varepsilon(y - \cdot) \in D(\Omega'_\lambda)\), for all \(y \in \Omega_\lambda\) and \(\varepsilon\) small enough. (\(\Omega'_\lambda\) is an open subset relatively compact such that \(\Omega_\lambda \subset \Omega'_\lambda \subset \Omega_\lambda\).) This remark lead to our result, since we obtain for \(T \in D(\Omega')\) representatives for \(i_S(T)\) and \(i'_S(T)\) equal for \(\varepsilon\) small enough.

Remark 18
i. Let \(B^\infty(\mathbb{R}^d)\) is the subset of elements of \(S^\infty(\mathbb{R}^d)\) satisfying \((\mathbf{2})\). We saw that there exists fundamentally one class of embeddings \((i_\rho)_{\rho \in B^\infty(\mathbb{R}^d)}\) of \(D'(\Omega)\) into \(G(\Omega)\) which renders the diagram \((\mathbf{7})\) commutative. For a fixed \(\rho \in B^\infty(\mathbb{R}^d)\), \(i_\rho\) can be described globally using technics of \((\mathbf{11})\) or locally using either technics of \((\mathbf{3})\) or of subsection 5.2 of this paper. This enlarges the possibilities when questions of embeddings arise in a mathematical problem.

ii. As mentioned in the introduction, \(i_0\) can be considered as an embedding of \(E'(\mathbb{R}^d)\) into \(G_C(\mathbb{R}^d)\). One has the following commutative diagram
\[
\begin{array}{cccc}
D & \overset{c}{\rightarrow} & E' & \overset{i_0 = i_S}{\rightarrow} & G_C \\
\downarrow_{c} & & \downarrow_{c} & & \downarrow_{i} \\
E = C^\infty(\Omega) & \overset{c}{\rightarrow} & D' & \overset{i_A = i_S = i'_S}{\rightarrow} & G
\end{array}
\]
where \(c\) denote the classical continuous embedding, and \(i\) the canonical embedding of \(G_C\) in \(G\).
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