Jeans instability criterion from the viewpoint of Kaniadakis’ statistics

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Abstract – In this letter we have derived the Jeans length in the context of the Kaniadakis statistics. We have compared this result with the Jeans length already obtained in the nonextensive Tsallis statistics (Jiulin D., Phys. Lett. A, 320 (2004) 347) and we discussed the main differences between these two models. We have also obtained the $\kappa$-sound velocity. Finally, we have applied the results obtained here to analyze an astrophysical system.

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The dynamical stability of a self-gravitating system usually can be described by the Jeans criterion of gravitational instability. The so-called Jeans length [1,2] is given by

$$\lambda_J = \sqrt{\frac{\pi k_B T}{\mu m_H G \rho_0}},$$

where $k_B$ is the Boltzmann constant, $T$ is the temperature, $\mu$ is the mean molecular weight, $m_H$ is the atomic mass of hydrogen, $G$ is the gravitational constant and $\rho_0$ is the equilibrium mass density. The critical value $\lambda_J$, eq. (1), is derived by considering a small perturbation in a set of four equations that are the equation of continuity, Euler’s equation, Poisson’s equation and the equation of state of an ideal gas. The Jeans length establishes that if the wavelength $\lambda$ of the density fluctuation is greater than $\lambda_J$, then the density will grow with time in an exponential form and the system will become gravitationally unstable. For more details see Jiulin in ref. [3].

Tsallis [4] has proposed an important extension of the Boltzman-Gibbs (BG) statistical theory. In a brief and technical terminology, this model is also currently referred to as a nonextensive (NE) statistical mechanics.

Tsallis thermostatistics formalism defines a nonadditive entropy as

$$S_q = k_B \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1} \left( \sum_{i=1}^{W} p_i = 1 \right),$$

where $p_i$ is the probability of the system to be in a microstate, $W$ is the total number of configurations and $q$, known in the current literature as being the Tsallis parameter or NE parameter, is a real parameter which quantifies the degree of nonextensivity. The definition of entropy in Tsallis statistics carries the usual properties of positivity, equiprobability, concavity and irreversibility and it also has motivated the study of multifractals systems. It is important to stress that Tsallis thermostatistics formalism contains the BG statistics as a particular case in the limit $q \to 1$ where the usual additivity of entropy is recovered.

Plastino and Lima [5] have derived a NE equipartition law of energy. It has been shown that the kinetic foundations of Tsallis’ NE statistics leads to a velocity distribution for free particles given by [6]

$$f_q(v) = B_q \left[ 1 - (1 - q) \frac{m v^2}{2 k_B T} \right]^{1/1-q},$$

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where $B_q$ is a normalization constant. Then, the expectation value of $v^2$ for each degree of freedom, is given by [7]

$$\langle v^2 \rangle_q = \frac{\int_0^\infty \int_0^\infty f_q v^2 dv}{\int_0^\infty \int_0^\infty f_q dv},$$

$$= \frac{2}{5} \frac{k_B T}{m}.$$  \hspace{1cm} (4)

The equipartition theorem is then obtained by using

$$E_q = \frac{1}{2} N m \langle v^2 \rangle_q,$$  \hspace{1cm} (5)

and we will arrive at

$$E_q = \frac{1}{5} \frac{1}{3q} Nk_B T.$$  \hspace{1cm} (6)

The range of $q$ is $0 \leq q < 5/3$. For $q = 5/3$ (critical value) the expression of the equipartition law of energy, eq. (6), diverges. It is also easy to observe that for $q = 1$, the classical equipartition theorem for each microscopic degrees of freedom can be recovered.

On the other hand, concerning entropy, the recent formalism proposed by Verlinde [8] obtains the gravitational acceleration by using the holographic principle and the well-known equipartition law of energy. His ideas relied on the fact that gravitation can be considered universal and independent of the details of the spacetime microstructure. Besides, he brought new concepts concerning holography since the holographic principle must unify matter, gravity and quantum mechanics.

The model considers a spherical surface as being the holographic screen, with a particle of mass $M$ positioned in its center. The holographic screen can be imagined as a storage device for information. The number of bits, which is the smallest unit of information in the holographic screen, is assumed to be proportional to the holographic screen area $A$ and can be written as

$$N = \frac{A}{\ell_P^2},$$  \hspace{1cm} (7)

where $A = 4\pi r^2$ and $\ell_P = \sqrt{G\hbar/c^3}$ is the Planck length and $\ell_P^2$ is the Planck area. In Verlinde’s framework one can suppose that the bits total energy on the screen is given by the equipartition law of energy,

$$E = \frac{1}{2} Nk_BT.$$  \hspace{1cm} (8)

It is important to notice that the usual equipartition theorem in eq. (8), can be derived from the usual BG thermostatistics. Let us consider that the energy of the particle inside the holographic screen is equally divided by all bits in such a manner that we can have the expression

$$Mc^2 = \frac{1}{2} Nk_BT.$$  \hspace{1cm} (9)

To obtain the gravitational acceleration, we can use eq. (7) and the Unruh temperature equation [9] given by

$$k_B T = \frac{\hbar a}{2\pi c}.$$  \hspace{1cm} (10)

Hence, we are able to obtain the (absolute) gravitational acceleration formula

$$a = \frac{l_P^2 c^3 M}{h} \frac{1}{r^2} = G \frac{M}{r^2}.$$  \hspace{1cm} (11)

From eq. (11) we can see that the Newton constant $G$ can be written in terms of the fundamental constants, $G = \ell_P^2 c^3/h$.

As an application of NE equipartition theorem in Verlinde’s formalism we can use the NE equipartition formula, i.e., eq. (6). Hence, we can obtain a modified acceleration formula given by [10]

$$a = G_q \frac{M}{r^2},$$  \hspace{1cm} (12)

where $G_q$ is an effective gravitational constant which is written as

$$G_q = \frac{5}{2} \frac{3q}{G}.$$  \hspace{1cm} (13)

From the result (13) we can observe that the effective gravitational constant depends on the NE parameter $q$. For example, when $q = 1$ we have $G_q = G$ (BG scenario) and for $q = 5/3$ we have the curious and hypothetical result $G_q = 0$. This result shows us that $q = 5/3$ is an upper bound limit when we are dealing with the holographic screen, as we have said before. But now we have other reasons to justify this classification.

Substituting $G$ by $G_q$, eq. (13), in (1), we can derive the same expression obtained by Jiulin [3] for the NE critical wavelength,

$$\lambda^q_c = \sqrt{\frac{2}{5-3q}} \frac{\pi k_BT}{\mu m H G \rho_0} = \sqrt{\frac{2}{5-3q}} \lambda_J.$$  \hspace{1cm} (14)

This NE modification of the Jeans criterion leads to a new critical length $\lambda^q_c$ that depends on the nonextensive $q$-parameter as follows:

- If $q = 1 \Rightarrow \lambda > \lambda^q_c = \lambda_J$, the usual Jeans criterion is recovered.

- If $0 < q < 1$ the Jeans criterion is modified as $\lambda > \lambda^q_c < \lambda_J$.

- If $1 < q < 5/3$, the Jeans criterion is modified as $\lambda > \lambda^q_c > \lambda_J$.

- If $q \rightarrow 5/3, \lambda^q_c \rightarrow \infty$, the self-gravitating system is always stable.

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Therefore, we have used Verlinde’s formalism and Tsallis’ thermostatistics in order to derive the NE Jeans length.

On the other hand, considering the well-known Kaniadakis statistics [11], also called $\kappa$-statistics, similarly to Tsallis thermostatistics formalism generalizes the standard BG statistics initially by the introduction of the $\kappa$-exponential and $\kappa$-logarithm functions.

The $\kappa$-entropy associated with this $\kappa$-framework is given by

$$S_\kappa = -k_B \sum_i W f_i^{1+\kappa} - p_i^{1-\kappa},$$

which recovers the BG entropy in the limit $\kappa \to 0$. It is important to mention here that the $\kappa$-entropy has satisfied the properties of concavity, additivity and extensivity. Tsallis’ entropy satisfies the property of concavity and extensivity but not additivity. This property is not fundamental, in principle. The $\kappa$-statistics has been successfully applied in many experimental fronts. As an example we can mention cosmic rays [12] and cosmic effects [13], quark-gluon plasma [14], kinetic models describing a gas of interacting atoms and photons [15] and financial models [16].

The kinetic foundations for the $\kappa$-statistics lead to a velocity distribution for free particles given by [17]

$$f_\kappa(v) = \left[ \sqrt{1 + \kappa^2 \left( \frac{mv^2}{2k_B T} \right)^2} - \kappa mv^2 / 2k_B T \right]^{\frac{1}{2}}.$$

Then, the expectation value of $v^2$, for each degree of freedom, is given by

$$\langle v^2 \rangle_\kappa = \frac{\int_0^\infty f_\kappa v^2 dv}{\int_0^\infty f_\kappa dv}.$$

Using the integral relation [11]

$$\int_0^\infty x^{r-1} \exp_\kappa(-x) dx = \frac{\Gamma(r) \Gamma\left(\frac{1}{\kappa} - \frac{r}{2}\right)}{\Gamma\left(\frac{1}{\kappa} + \frac{r}{2}\right) \Gamma\left(\frac{1}{\kappa} - \frac{r}{2}\right)},$$

we have that

$$\langle v^2 \rangle_\kappa = \frac{(1 + \frac{\kappa}{2}) \Gamma\left(\frac{1}{2\kappa} - \frac{3}{4}\right) \Gamma\left(\frac{1}{2\kappa} + \frac{1}{4}\right)}{(1 + \frac{2\kappa}{3}) 2\kappa \Gamma\left(\frac{1}{2\kappa} + \frac{1}{4}\right) \Gamma\left(\frac{1}{2\kappa} - \frac{1}{4}\right)} k_B T.$$  

The $\kappa$-equipartition theorem can be obtained by using

$$E_\kappa = \frac{1}{2} N m \langle v^2 \rangle_\kappa,$$

where we arrive at

$$E_\kappa = \frac{1}{2} N m \left(1 + \frac{\kappa}{2}\right) \Gamma\left(\frac{1}{2\kappa} - \frac{3}{4}\right) \Gamma\left(\frac{1}{2\kappa} + \frac{1}{4}\right) k_B T.$$

The range of $\kappa$ is $0 \leq \kappa < 2/3$. For $\kappa = 2/3$ (critical value) the expression of the equipartition law of energy, eq. (24), diverges, which is equivalent to $q = 5/3$ in the Tsallis formalism. For $\kappa = 0$, the classical equipartition theorem for each microscopic degrees of freedom can be recovered.

On the other hand, if we use the Kaniadakis equipartition theorem, eq. (24), in the Verlinde formalism, the modified acceleration formula is given by

$$a = G_\kappa \frac{M}{r^2},$$

where $G_\kappa$ is an effective gravitational constant which is written as

$$G_\kappa = \frac{(1 + \frac{3}{2\kappa}) 2\kappa \Gamma\left(\frac{1}{2\kappa} - \frac{3}{4}\right) \Gamma\left(\frac{1}{2\kappa} + \frac{1}{4}\right)}{(1 + \frac{2\kappa}{3}) 2\kappa \Gamma\left(\frac{1}{2\kappa} + \frac{1}{4}\right) \Gamma\left(\frac{1}{2\kappa} - \frac{1}{4}\right)} G.$$  

From the limits $\kappa = 0, q = 1$, with which we can recover the BG statistics, and $\kappa = 2/3, q = 5/3$. It is important to notice that these values come from a well-known [17–19] linear relation between $\kappa$ and $q$ written as

$$\kappa = q - 1.$$

In particular, the limits $\kappa = 0$, when $q = 1$, and $\kappa = 2/3$, when $q = 5/3$, lead to $G_q = G_\kappa$. It is clear that the relation (27) is valid only for the range $1 \leq q < 5/3$, given by eq. (13).

Using eq. (26), in (1), we have derived the expression for the critical wavelength in the Kaniadakis statistics

$$\lambda_\kappa^c = \sqrt{\frac{(1 + \frac{3}{2\kappa}) 2\kappa \Gamma\left(\frac{1}{2\kappa} - \frac{3}{4}\right) \Gamma\left(\frac{1}{2\kappa} + \frac{1}{4}\right)}{(1 + \frac{2\kappa}{3}) 2\kappa \Gamma\left(\frac{1}{2\kappa} + \frac{1}{4}\right) \Gamma\left(\frac{1}{2\kappa} - \frac{1}{4}\right)} \lambda_J}.$$  

This $\kappa$ modification of the Jeans’ criterion leads to a new critical length $\lambda_\kappa^c$ that depends on the $\kappa$-parameter as follows:

- If $\kappa = 0 \Rightarrow \lambda > \lambda_\kappa^c = \lambda_J$, the usual Jeans criterion is recovered.
- If $0 < \kappa < 2/3$, the Jeans criterion is modified as $\lambda > \lambda_\kappa^c > \lambda_J$.
- If $\kappa \to 2/3-$, $\lambda_\kappa^c \to \infty$, the self-gravitating system is always stable.

In fig. 1 we have plotted $\lambda_q^c$, eq. (14), and $\lambda_\kappa^c$, eq. (28), both normalized by $\lambda_J$, eq. (1), as a function of $\kappa$. For this, we have used relation (27).
It is easy to see that have the NE sound velocity \( \frac{2T}{5 - 3q} \). Consequently we have the NE sound velocity \( v_{\text{sq}} = \sqrt{\frac{k_B T_q}{m}} \).

We can show that \( T_q = T \) if \( q \to 1 \). Consequently we have the \( \kappa \)-sound velocity \( v_{\text{sk}} = \sqrt{\frac{k_B T_k}{m}} \).

Let us now apply the results developed above and investigate how non-Gaussian statistics can affect an astrophysical system. We will use 16 mass measurements of high X-ray luminosity clusters in the redshift range from 0.17 to 0.55. The X-ray luminosity of galaxies provides us some information concerning their evolution, but here let us specifically analyze the masses of these 16 clusters obtained in [20]. If the galaxies are in equilibrium, the virial mass of a cluster can be calculated as

\[
M = \frac{3}{G} \sigma_1^2 r, \tag{33}
\]

where \( \sigma_1 \) and \( r \) are the velocity dispersion of the cluster and the three-dimensional virial radius, respectively. The virial mass can be rewritten as

\[
M_q = \frac{3}{G_q} \sigma_q^2 r, \tag{34}
\]

for the scenario modified via Tsallis statistics and

\[
M_\kappa = \frac{3}{G_\kappa} \sigma_\kappa^2 r, \tag{35}
\]

for the framework modified by the Kaniadakis statistics. In eqs. (34) and (35), \( G_q, \kappa \) is given by eq. (13) and (26), respectively.

In [21] the author has analyzed systems with self-gravitating long-range interaction and their distribution functions in order to obtain an expression for the nonextensive parameter, which is given by

\[
(1 - q) \nabla \varphi + 2 \sigma \nabla \sigma = 0, \tag{36}
\]

which relates the nonextensive parameter, the velocity dispersion and the gravitational acceleration. For spherically symmetric systems, we can write eq. (36) as

\[
1 - q = -2 \frac{\sigma}{\sigma} \frac{d\sigma}{dr} = -2 \frac{\sigma}{\sigma} \frac{d\sigma}{GM(r)/r^2}, \tag{37}
\]

where we can see the case in which we have a variable mass, \( M = M(r) \). From (36) and (37) we can see that the parameter \( q \) is equal to unity if and only if \( \nabla \sigma = 0 \).

To justify our approach for the virial mass, eq. (34), we can rewrite eq. (37) in the form

\[
M = \frac{2 \sigma^2 d\sigma/dr}{G_q (1 - q)} \tag{38}
\]

Then, eq. (34) can be reobtained if we impose the condition

\[
-2 \sigma^2 d\sigma/dr = \frac{3}{G_q} \sigma_q^2 r, \tag{39}
\]

which leads to a solution for \( \sigma^2(r) \) given by

\[
\sigma^2(r) = 3(q - 1) \sigma_1^2 \ln(r) + \sigma_1^2. \tag{40}
\]

Here we would like to mention that in the limit \( q \to 1 \), we recover \( \sigma^2(r) = \sigma_1^2 \). This result is compatible with eq. (37). Therefore, if the velocity dispersion for the galaxies cluster satisfies eq. (40), then the nonextensive virial mass can be written as eq. (34). A similar procedure can be done to justify the form of the virial mass, eq. (35), using Kaniadakis statistics.

We will estimate from now on how the theoretical predictions given by eqs. (34) and (35) can fit the data for

![Fig. 1: Dashed line: Tsallis critical wavelength, \( \lambda_q \). Solid line: Kaniadakis critical wavelength, \( \lambda_\kappa \).](image-url)
Finally, we can say that in this work we have described both non-Gaussian statistical formalisms, Tsallis and Kaniadakis, in the light of the star formation criterion formulated by Jeans at the beginning of the last century. We have seen that the limit for the Jeans instability condition is coherent with the standard values of $q$ in the literature. After some computations, different values for the wavelength are compared through the curves. Different values for the sound speed were obtained. We have investigated how non-Gaussian effects can fit the masses of 16 galaxy clusters. We have found that the best fit value is completely compatible with BG statistics, but the non-Gaussian effects cannot be discarded with the present set of data. More specifically we note $(0.943)\ 0.968 \leq q \leq 1.01 (1.035)$ and $0 \leq \kappa \leq 0.034 (0.054)$ at $1\sigma \ (2\sigma)$ CL.

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