Complete and Deterministic Bell State Measurement Using Nonlocal Spin Products

Keiichi Edamatsu

1Research Institute of Electrical Communication, Tohoku University, Sendai 980-8577, Japan

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A simple protocol for complete and deterministic Bell state measurement is proposed. It consists of measurements of nonlocal spin product operators with the help of shared entanglement as an ancillary resource. The protocol realizes not only nonlocal Bell state measurement between a pair of distant qubits but also a complete Bell filter that transmits either one of the Bell states indicated by the measurement outcome. These schemes will be useful in quantum technologies where nonlocal Bell state measurement is indispensable.

INTRODUCTION

Bell state measurement, or Bell measurement, is an essential concept in quantum technologies [1]. The most simple Bell measurement scheme is illustrated in Fig. 1. Although this scheme is simple and instructive, it requires a nonlocal controlled-not (CNOT) operation between the two qubits. A deterministic CNOT operation and thus a complete Bell measurement would be possible in various qubit systems where two qubits are situated close to each other. However, Bell measurement between distant qubits is not possible using only local operation and classical communication (LOCC). More realistic implementations of Bell measurement using linear optics \(^2\) are sometimes applied to various proof-of-principle demonstrations of quantum information protocols such as quantum teleportation and entanglement swapping etc. \(^3\) \(^4\) \(^5\), but it is known that they cannot be complete and deterministic, i.e., the linear-optical implementations can only partly distinguish among the four possible Bell states \(^3\) \(^4\) \(^5\).

In this letter, a simple scheme for complete and deterministic Bell measurement is proposed. It consists of the measurements of nonlocal spin product operators, the members of nonlocal product observables \(^6\), with the help of shared entanglement as a resource. The protocol enables nonlocal Bell measurement between a pair of distant qubits. Furthermore, not only Bell measurement but also a complete Bell filter is realizable, where the output state turns out to be either one of the Bell states indicated by the measurement outcome.

![Fig. 1: A simple circuit model for the Bell measurement composed of a CNOT gate followed by a Hadamard gate.](image)

NONLOCAL SPIN PRODUCTS AND BELL BASES

We consider a bipartite qubit system where a pair of qubits is distributed between Alice and Bob. In general, the state vector \(\ket{\psi} = \sum_{\mu,\nu} c_{\mu\nu} \ket{\mu\nu}\), \(\ket{\mu\nu} = \ket{\mu}_A \otimes \ket{\nu}_B\) is the eigenstate of the Pauli operator \(\sigma_z\) on Alice’s (Bob’s) site having eigenvalues \(\nu\), \(\nu = \pm 1\). Note that \(\sum_{\mu,\nu} |c_{\mu\nu}|^2 = 1\). Hereafter we sometimes write just + and – for the eigenvalues +1 and –1, respectively. For instance, \(|++\rangle = \ket{+}_A \otimes \ket{+}_B\) and \(|++\rangle = \ket{+}_A \otimes \ket{-}_B\). Also note that we regard the states |+⟩ and |−⟩ as |0⟩ and |1⟩ in the standard qubit representation, respectively. The Bell bases are defined as

\[
\ket{\Phi^\pm} = \frac{1}{\sqrt{2}} (|++\rangle \pm |−−\rangle),
\]

\[
\ket{\Psi^\mp} = \frac{1}{\sqrt{2}} (|+−\rangle \pm |−+\rangle).
\]

Using the Bell bases, \(\ket{\psi}\) in (1) is rewritten as

\[
\ket{\psi} = c_1 \ket{\Phi^+} + c_2 \ket{\Phi^-} + c_3 \ket{\Psi^+} + c_4 \ket{\Psi^-},
\]

where \(c_1 = (c_{++} + c_{--})/\sqrt{2}\), etc.

The simple circuit for the Bell measurement given in Fig. 1 transforms the input states \(\ket{\Phi^+}, \ket{\Phi^-}, \ket{\Psi^+}\) and \(\ket{\Psi^-}\) into the output states \(\ket{++}, \ket{+−}, \ket{−+}\) and \(\ket{−−}\), respectively. Thus we can distinguish all the Bell states by observing the local measurement outcomes on Alice’s and Bob’s qubits. However, as noted earlier, the simple scheme requires nonlocal CNOT operation between distant qubits.

Nonlocal spin product operators, or nonlocal spin products, \(S_{ij}\) are expressed as

\[
S_{ij} \equiv \sigma_i \otimes \sigma_j, \quad (i, j = x, y, z)
\]

where the first and second Pauli operators act on Alice’s and Bob’s qubits, respectively. The eigenvalues of \(S_{ij}\) are
The measurement operators $M$ of compatible, we can make simultaneous local measurements on the complete Bell measurement. 

\[ S_{ij}, S_{jj} = 0. \]  

(8)

Thus, for instance, $S_{zz}$ commutes with $S_{zz}$. As a result, $S_{zz}$ and $S_{xx}$ are compatible having a complete orthonormal set of common eigenbases:

\[ |m = +1, n = \pm 1\rangle = |\Phi^\pm\rangle, \]

(9)

\[ |m = -1, n = \pm 1\rangle = |\Psi^\pm\rangle, \]

(10)

where $n$ refers to the eigenvalue of $S_{xx}$. These are nothing other than the Bell bases $\{2\}$ and $\{3\}$. Thus, by observing $S_{zz}$ and $S_{xx}$ for a given input state, we carry out a complete Bell measurement.

### MEASUREMENT OF THE SPIN PRODUCTS

Suppose Alice and Bob want to measure any nonlocal spin product $S_{ij}$ for an arbitrary system state $|\psi\rangle_S$ expressed in $\{1\}$ or $\{4\}$. Here, the suffix S after the state vector refers to the system to be measured.

Measuring a component of $S_{ij}$ is simple. Since the measurements of $\sigma_i$ by Alice and $\sigma_j$ by Bob are compatible, we can make simultaneous local measurements of $\sigma_i$ and $\sigma_j$, and then compute the product of Alice’s and Bob’s outcomes to obtain the measurement result of $S_{ij}$. Consider, for example, the measurement of $S_{zz}$. The measurement operators $M(\mu\nu)$ for the four possible combinations of Alice’s and Bob’s outcomes, $(\mu, \nu)$, are the projective operators:

\[ M(\mu\nu) = \Pi(\mu\nu) = |\mu\nu\rangle\langle\mu\nu|, \]

(11)

where $\Pi(...)$ is the projector to the state $|...\rangle$. The corresponding POVM (positive operator valued measure) is

\[ E(\mu\nu) = M^\dagger(\mu\nu)M(\mu\nu) = \Pi(\mu\nu). \]

Taking a product of $\mu$ and $\nu$, we obtain the outcome $m$ for the measurement of $S_{zz}$. The POVMs $E_{\pm}$ for $m = \pm 1$ are obtained as

\[ E_{+} = \Pi(++) + \Pi(--) = \Pi(\Phi^+) + \Pi(\Phi^-), \]

(12)

\[ E_{-} = \Pi(+-) + \Pi(-+) = \Pi(\Psi^+) + \Pi(\Psi^-). \]

(13)

We see that $E_{\pm}$ correspond to the projection to the eigenspaces of $m = \pm 1$ presented in $\{6\}$ and $\{7\}$, respectively.

However, this local measurement strategy is inappropriate to make simultaneous measurements of two or more components of $S_{ij}$, for instance, $S_{zz}$ and $S_{xx}$. Since the above-mentioned measurement of $S_{zz}$ projects the system state to either of the local product states $|++\rangle$, $|\pm\rangle$, $|\mp\rangle$ or $|\pm\rangle$, the succeeding measurement of $S_{xx}$ is no longer identical to that of the original input state. In other words, the local projective measurement of $\sigma_z$ on either Alice’s or Bob’s qubit is not compatible with $S_{xx}$.

Another strategy for the measurement of spin products is the nonlocal measurement making use of additional entanglement shared by Alice and Bob. Suppose Alice and Bob share a maximally entangled Bell state (ebit):

\[ |\xi\rangle_M = |\Phi^+\rangle_M. \]

(14)

The suffix M indicates that it is used as a meter to measure the system state. Alice and Bob use each qubit in $|\xi\rangle_M$ as a meter (probe) to measure their system qubit. To do so, each of them makes a CNOT gate between her/his qubits, as shown in Fig. 2 and then makes a projective $\sigma_z$ measurement on her/his meter qubit. Note that when her/his initial meter qubit was fixed as $|+\rangle$, the measurement would be the projective local measurement of $\sigma_z$. After the CNOT gates, the initial state $|\psi\rangle_S \otimes |\Phi^+\rangle_M$ is converted to

\[ |\psi\rangle_S \otimes |\Phi^+\rangle_M \rightarrow (c_1|\Phi^+\rangle + c_2|\Phi^-\rangle)_{S} \otimes |\Phi^+\rangle_M \]

\[ + (c_3|\Psi^+\rangle + c_4|\Psi^-\rangle)_{S} \otimes |\Psi^+\rangle_M. \]

(15)

Let Alice’s (Bob’s) outcome be $z_A (z_B)$. The first term of the right hand side of (15) corresponds to the case where the measurement outcome is $(z_A z_B) = (++)$ or $(--)$, while the second term to $(+-)$ or $(-+)$). By simply taking a product of the local meter outcomes of Alice and Bob, $z_A z_B = m = \pm 1$ is obtained and thus the measurement of $S_{zz}$ is complete. The measurement operators $M_{\pm}$ for $m = \pm 1$ are

\[ M_{+} = \Pi(\Phi^+) + \Pi(\Phi^-), \]

(16)

\[ M_{-} = \Pi(\Psi^+) + \Pi(\Psi^-). \]

(17)

The corresponding POVMs are identical to that presented in $\{12\}$ and $\{14\}$. $M_{\pm}$ project the system state to the eigenspaces of $m = \pm 1$ presented in $\{6\}$ and $\{7\}$.
FIG. 3: Two schemes for the complete Bell measurement making use of sequential spin product measurements, $S_{zz}$ and $S_{xx}$. (a) With nonlocal $S_{zz}$ and local $S_{xx}$ measurements. (b) With nonlocal measurements of $S_{zz}$ and $S_{xx}$. Note that the scheme (b) acts as a complete Bell filter.

respectively. Note that the system state is still a superposition of $|\Phi^+\rangle$ and $|\Phi^-\rangle$ (or $|\Psi^+\rangle$ and $|\Psi^-\rangle$), preserving sufficient information for the succeeding $S_{xx}$ measurement.

**COMPLETE BELL MEASUREMENT AND A BELL FILTER**

Subsequent to the nonlocal measurement of $S_{zz}$ described above, Alice and Bob make a $S_{xx}$ measurement on the system state. As shown in (14) and (15), the measurement of $S_{xx}$ combined with $S_{zz}$ discriminates $|\Phi^+\rangle$, $|\Phi^-\rangle$, $|\Psi^+\rangle$, or $|\Psi^-\rangle$ from $|\psi\rangle$. In this way, a complete Bell measurement can be carried out.

For the $S_{xx}$ measurement, either local or nonlocal strategy can be used. If only the measurement outcome matters, the simple local strategy shown in Fig. 3(a) is appropriate. In this case, the POVMs $E_{mn}$, where suffixes $m$ and $n$ indicate the measurement outcomes of the preceding $S_{zz}$ and the following $S_{xx}$, respectively, are written as

$$E_{++} = \Pi(\Phi^+), \quad E_{+-} = \Pi(\Phi^-),$$

$$E_{-+} = \Pi(\Psi^+), \quad E_{--} = \Pi(\Psi^-).$$

On the other hand, if the system state at the output should be preserved in one of the resultant eigenstates given in (14) and (15), i.e., one of the Bell bases, Alice and Bob can use nonlocal strategy at a cost of an additional cnot, as shown in Fig. 3(b). In this case, the measurement operators $M_{mn}$ are found to be

$$M_{++} = \Pi(\Phi^+), \quad M_{+-} = \Pi(\Phi^-),$$

$$M_{-+} = \Pi(\Psi^+), \quad M_{--} = \Pi(\Psi^-).$$

Again, the corresponding POVMs are identical to that presented in (18) and (19). Thus, the system state is projected into one of the Bell basis depending on the measurement outcomes. This procedure functions as a complete Bell filter, where the output state will be either one of the Bell bases indicated by the measurement outcome.

It is noteworthy that, in both strategies, all of the outcomes are deterministically obtained and thus the Bell measurement proposed here is complete and deterministic, at the cost of requiring one (for the Bell measurement) or two (for the Bell filter) cnot(s) as a resource.

**PROPOSED EXPERIMENTS**

The measurement schemes described above are applicable to any physical qubits between which we can prepare entanglement and a cnot operation. However, in cases where we can directly make a nonlocal cnot operation between qubits held in Alice and Bob, we could employ a simpler scheme as shown, for instance, in Fig. 4 to implement the Bell measurement. Nevertheless, our scheme is still useful when we are not able to use nonlocal cnot, or when we need the function of the Bell filter.
as well as the Bell measurement.

Another situation where our schemes may be useful is the case of photonic qubits. It is known that with linear optics we cannot implement deterministic CNOT gates between individual photonic qubits [8]. As a result, to date, we could not implement deterministic Bell measurement with linear optics. Nevertheless, employing the scheme described in this paper we will be able to implement the deterministic and complete Bell measurement between photonic qubits.

Suppose we provide a pair of photons to Alice and Bob, as shown in Fig. 4. The photons’ polarizations constitute the system state |ψ⟩S of interest. In order to measure the nonlocal spin products on their polarizations, we prepare their path degrees of freedom, i.e., path qubits, in the maximally entangled Bell state |Φ+⟩M. Entanglement in the path degrees of freedom could be directly generated by spatial entanglement between photons generated by parametric down-conversion [9], or could be converted from time-bin entanglement [10]. When the polarization qubits are also entangled, it is called a hyperentangled state [11]. Between the polarization qubit and the path qubit, Alice and Bob employ CNOT gates using polarizing beamsplitters (PBS). Thus, the nonlocal measurement of Szz on the photons’ polarization qubits is implemented and the measurement outcomes are encoded in photons’ output paths. Then Alice and Bob carry out the local σz measurement for their polarization qubits using, for instance, two additional PBSs. This part implements the local measurement strategy of Szz. At the last stage, Alice detects her photon at one of her four output paths, as does Bob at one of his four output paths. From the path information, they know the result of Szz and Sxx, and thus the complete Bell measurement is carried out in a deterministic way. One drawback of this linear optics implementation is that we use an ebit implemented in the path degree of freedom of the photon pair. As a result, it is difficult to apply this method to Bell measurement between independent photons as in a case of quantum teleportation. Nonetheless, this method will be useful in many situations of quantum technologies where nonlocal Bell measurement is indispensable.

Furthermore, the measurement protocol of nonlocal spin products can be extended to measurements at weak and any intermediate measurement strength [8]. Thus, it would be possible to realize generalized measurements of nonlocal spin products and Bell measurement at any measurement strength. In this context, strength-variable measurements of photon polarization and the measurement uncertainty relations have been demonstrated [12–17]. By extending the protocols described here, it would be possible to explore measurement uncertainty relations in the nonlocal product observables.

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* Electronic address: eda@riec.tohoku.ac.jp

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