On the Existence of Pure, Broadband Toroidal Sources in Electrodynamics

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Multipoles are paramount for describing electromagnetic fields in many areas of nanoscale optics, playing an essential role in the design of devices in plasmonics and all-dielectric nanophotonics. Challenging the traditional division into electric and magnetic moments, toroidal moments are proposed as a physically distinct family of multipoles with significant contributions to the properties of matter. However, the apparent impossibility of separately measuring their response sheds doubt on their true physical significance. Here, the possibility of selectively exciting toroidal moments is confirmed without any other multipole. A set of general conditions is developed that any current distribution must fulfill to be entirely described by toroidal moments and prove the results in an analytically solvable case. The new theory allows to design and verify experimentally an artificial structure supporting a pure broadband toroidal dipole response in the complete absence of the electric dipole and other “ordinary” multipole contributions. In addition, a structure capable of supporting a novel type of non-radiating source is proposed—a “toroidal anapole,” originating from the destructive interference of the toroidal dipole with the unconventional electromagnetic sources known as mean square radii. The results in this work provide conclusive evidence on the independent excitation of toroidal moments in electrodynamics.

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1. Introduction
Multipole expansions constitute a central framework in many areas of modern physics and underpin our understanding of several phenomena in electrodynamics, nuclear physics, and condensed matter.[1–3] In optics, multipole expansions are widely exploited to characterize the radiation properties of a finite source. Any multipolar formalism typically involves electric and magnetic dipoles and increasingly more complex combinations, such as quadrupoles, octupoles, and so forth.[3,4] It is nowadays well-known that the so-called “Cartesian” multipole expansions, based on a Taylor series of the vector potential,[5,6] or equivalently the internal current distribution,[7] require the introduction of dynamic toroidal moments for a complete parametrization of any charge–current distribution, both classical and quantum.[1] The distinct constituents of this extended class of moments are...
generated by transverse current distributions associated with the curl of a magnetization vector, topologically similar to the poloidal currents flowing through toroidal surfaces.\textsuperscript{[8,9]}

Toroidal multipole moments have been shown to be important in several branches of physics where toroidal topology plays a role. The most recognizable member of the toroidal family is the toroidal dipole (TD), first identified\textsuperscript{[10]} and observed in a static current distribution. In condensed matter systems, the static TD was shown to naturally give rise to a new type of long-range ferroic order violating both space and time inversion symmetries.\textsuperscript{[1,10]} Similarly, in molecular physics, static toroidal moments were predicted to appear in the electronic structure of several molecules.\textsuperscript{[2]}

In modern electrodynamics, dynamic toroidal moments describe currents that vary in time. They are currently raising a renewed scientific and technological interest. In particular, they are important for describing light scattering from nanostructured matter in nanophotonics and meta-optics. By making use of artificially tailored media (metamaterials and their flat counterparts, metasurfaces), the ordinary electric dipole (ED) can be suitably designed to interfere destructively with the dynamic TD in the far field, leading to the formation of nontrivial, non-radiating sources, (dynamic anapoles\textsuperscript{[6,11]}), with enhanced near fields. Anapoles have also been proposed as candidates for observing the dynamic Aharonov–Bohm effect.\textsuperscript{[11]} Beyond the dynamic TD, higher order toroidal moments, also known as toroidal mean-square radii (MSR), have been recently proposed to realize new types of non-radiating anapole states.\textsuperscript{[6,12]}

The first observation of a dynamic TD in a plasmonic metamaterial,\textsuperscript{[13]} triggered many investigations aiming to isolate it and understand its fundamental properties. Nowadays, structures featuring a certain degree of dynamic TD contribution have been designed for virtually all frequency bands of interest. The required current distribution of the source can therefore have many physical origins, e.g., localized plasmons,\textsuperscript{[14]} surface plasmon polaritons,\textsuperscript{[15]} photonic (Mie) resonances,\textsuperscript{[16,17]} or spoof surface plasmons.\textsuperscript{[18]} In extended photonic metasurfaces, some eigenmodes with dynamic TD character correspond to symmetry-protected quasi-bound states in the continuum that couple weakly to the incident field, manifesting as a sharp Fano resonance in the transmission and reflection spectra.\textsuperscript{[17]} The latter holds promising prospects for enhancing light–matter interactions and sensing applications in both plasmonic,\textsuperscript{[19]} all-dielectric,\textsuperscript{[20–23]} and hybrid plasmonic-dielectric platforms. Dynamic TD meta-devices have been shown to outperform their elementary counterparts for a variety of tasks, such as polarization conversion,\textsuperscript{[24]} strong plasmon-exciton coupling,\textsuperscript{[25]} metasurface-enhanced waveguiding,\textsuperscript{[26]} or active optical switching.\textsuperscript{[27]}

However, the physical significance of the dynamic TD has been the subject of an active scientific debate.\textsuperscript{[8,28,29]} The main source of controversy is centered around the fact that no measurement of the fields radiated or scattered by a dynamic TD can distinguish them from the ordinary ED (and can be related to the observability condition of dynamical systems). The statement applies to all measurements performed outside the smallest spherical volume enclosing the source. This poses the question of whether there is any real need to define two separate quantities. More formally, as will be detailed later, the TD arises as the second-order term in a Taylor series of the exact electric coefficients.\textsuperscript{[28]} The split of such a coefficient is questionable since it does not lead to additional observables outside the source region. The debate has also motivated interesting theoretical proposals on how the ordinary ED and the dynamic TD “response” could be distinguished in an actual physical system in the far field.\textsuperscript{[30]} Unfortunately, these ideas remain untested so far.

Currently, three significant drawbacks prevent a more thorough investigation of dynamic toroidal moments. First, due to the absence of clear, rigorous rules justifying the separation of the ordinary ED and the dynamic TD, it is challenging to confidently claim the observation of a “pure” or “ideal” dynamic TD. Second, in most structures investigated until now, the dynamic TD response is often masked by the contributions of other electric and magnetic multipoles.\textsuperscript{[14,29,31–36]} Third, while some works have achieved a significant background-to-noise ratio of the dynamic TD with respect to the other multipoles,\textsuperscript{[13,16,18,17,37]} they rely on isolated resonances, limiting their operation to a reduced frequency range. Moreover, even in the most well-known examples, the dynamic TD is accompanied by an ordinary ED with a similar line shape as the TD but suppressed amplitude,\textsuperscript{[13,16,18,17,37]} hinting at the fact that the resonances are not “pure,” and shedding doubt whether the dynamic TD can be truly separated from the ordinary ED.

Here, we aim to demonstrate that the excitation of an ideal dynamic TD is indeed possible and relies on physically distinct constraints with respect to the ordinary ED. To do so, we first determine the exact conditions that a current distribution of arbitrary spatial extension must satisfy to excite a pure dynamic TD in the complete absence of the two most common caveats, i.e., ordinary ED and parasitic magnetic moments. Besides being divergenceless, we show that the curl of such currents must not have a radial component and importantly cannot induce a surface charge at the boundary of the source. If a current validates these general conditions, all the exact magnetic moments and the ordinary ED become zero, and the source is completely characterized by time-odd, spatial-odd moments, i.e., toroidal moments. The fields radiated by a pure TD in the far zone do not differ from a conventional point ED. However, within the smallest spherical shell enclosing the source, the topology of the electromagnetic fields is drastically altered. Although we focus on the case of the dynamic TD, the same rules apply to dynamic toroidal quadrupoles, octupoles, and so forth.

Second, based on our enhanced physical insight into the problem, we design and verify a pure, broadband, dynamic TD source with all the other multipoles, including the ordinary ED, suppressed by more than three orders of magnitude. Unlike all previous works, our antenna displays a dynamic TD spanning hundreds of MHz in the microwave frequency range since we do not rely on any resonant behavior. These results are validated through a direct probing of the internal fields inside the source region as well as far-field measurements for a broad range of frequencies.

In the last part of our study, following a similar design strategy as for the pure dynamic TD source, we demonstrate analytically and prove numerically the excitation of a novel type of non-radiating source, a “toroidal anapole”, originating from the destructive interference of the dynamic TD with the first order MSR of the structure.
By rigorously justifying the split between ordinary electric and dynamic toroidal moments, we conclude that the latter can be independently controlled from the former and thus deserves to be considered as separate, meaningful physical entities. Furthermore, in the near future, toroidal anapoles could play a unique role in the rapidly growing field of anapole electrodynamics.  

2. Spherical Multipole Moments  

First, we formulate the mathematical setting in which we base our analysis. In what follows, we will always consider a piecewise continuous current distribution \( \mathbf{J}(\mathbf{r}, t) \) bounded in a finite space region, such as \( \mathbf{J}(\mathbf{r}, t) = 0 \) everywhere outside \( \Omega \). The source is embedded in a homogeneous medium. We keep in mind that the Helmholtz theorem allows separating any field into its transverse (divergence-free, denoted by \( L_\perp \)) and longitudinal (curl-free \( L_\parallel \)) components \( \mathbf{J}(\mathbf{r}, t) = \mathbf{J}_L(\mathbf{r}, t) + \mathbf{J}_T(\mathbf{r}, t) \). Without loss of generality, we consider the Fourier transform of \( \mathbf{J}(\mathbf{r}, t) \) to yield monochromatic components, denoted as \( \mathbf{J}(\mathbf{r}, \omega) \). Only \( \mathbf{J}_L(\mathbf{r}, \omega) \) contributes to the electromagnetic fields radiated outside \( \Omega \) at frequency \( \omega \). \( \mathbf{J}_L(\mathbf{r}, \omega) \) can be further expanded into plane wave components:  

\[
\mathbf{J}_L(\mathbf{r}, \omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \mathbf{J}_L(\mathbf{k}, \omega) \exp(\mathbf{i}\mathbf{k} \cdot \mathbf{r}) d^3k 
\]  

(1)  

The contribution of \( \mathbf{J}_L(\mathbf{r}, \omega) \) to radiation is given only by those components of its plane wave expansion with wave vector \( \mathbf{k} \). Therefore, all radiating \( \mathbf{k} \) vectors define a spherical surface. It then becomes possible to expand the radiating \( \mathbf{J}_L \) in terms of vector spherical harmonics that form an orthonormal basis for transverse vector functions in the unit sphere, as:  

\[
\mathbf{J}_L(\mathbf{r}, \omega) = \sum_{lm} a_{lm} \mathbf{Z}_{lm}(\mathbf{r}, \omega) + b_{lm} \mathbf{X}_{lm}(\mathbf{r}, \omega) 
\]  

(2)  

where \( l \) is the total angular momentum, referred hereon as the multipole order, and \( m \) its projection to the z-axis. \( \mathbf{X}_{lm} \) and \( \mathbf{Z}_{lm} \) are provided in Equation 6 of Ref. [4]. The \( a_{lm}, b_{lm} \) are the spherical multipole moments of electric and magnetic type, respectively. Each spherical multipole is a spherical tensor of order \( l \), where \( l = 1 \) corresponds to the dipole, \( l = 2 \) to the quadrupole, and so forth. Straightforward formulas for \( a_{lm}, b_{lm} \) in terms of the original \( \mathbf{J}(\mathbf{r}, \omega) \) can be obtained by exploiting the orthogonality of the vector spherical harmonics. They can be converted into Cartesian tensors through a simple basis transformation to yield more familiar expressions. However, we emphasize that in this work, the term spherical multipole refers not to the basis of the tensor but to the fact that the coeﬃcients describe the weights of each vector spherical harmonic in the expansion of the current, as given by Equation 2. Here, we are interested in the spherical electric dipole \( \mathbf{d} \) (spherical ED), defined (in Cartesian coordinates) as:  

\[
\mathbf{d} = -\frac{1}{i\omega} \int \mathbf{J}_L d^3r 
\]  

(3)  

\[ j_1(\mathbf{k} \cdot \mathbf{r}) \] is the ith spherical Bessel function of the first kind, and \( \mathbf{J}(\mathbf{r}, \omega) \) has been rewritten as \( \mathbf{J}_L \) for compactness. Similarly, the spherical magnetic dipole moment \( \mathbf{m} \) (spherical MD) is given by:  

\[
\mathbf{m} = \frac{3}{2} \int d^3r \mathbf{r} \times \mathbf{J}_L \frac{j_1(\mathbf{k} \cdot \mathbf{r})}{\mathbf{k} \cdot \mathbf{r}} 
\]  

(4)  

The expressions are valid for a source of arbitrary extension and suffice to describe electromagnetic radiation everywhere outside \( \Omega \).  

3. The Physical Significance of Toroidal Moments  

Maxwell’s equations are invariant under both the space inversion \( \mathcal{P} \) and time reversal \( \mathcal{T} \). The first operation changes the sign of all polar vectors (e.g., position or electric field) while leaving pseudovectors such as the magnetic field invariant. Time operator \( \mathcal{T} \) reverses the flow of time. Thus, any moment describing a current distribution in Maxwell–Lorentz electrodynamics can be classified according to the operations in the space-time group. In principle, up to four possible combinations exist: the moments can be either even or odd under \( \mathcal{P} \) or \( \mathcal{T} \). However, in the absence of magnetic charges, obtaining a parity-even, time-even tensor in a localized source is impossible. We are left with three choices.  

At this point we regard \( \mathbf{m} \) in Equation 4. It is clear that \( \mathcal{P}[\mathbf{m}] = \mathbf{m} \), since the spherical MD is defined as a pseudovector. Conversely, the \( \mathcal{T} \) operation reverses the current flow so that \( \mathcal{T}[\mathbf{m}] = -\mathbf{m} \). The spherical MD is, therefore, a parity-even and time-odd tensor. On the other hand, the spherical ED in Equation 3 is odd under parity inversion \( \mathcal{P}[\mathbf{d}] = -\mathbf{d} \). Similarly, \( \mathcal{T}[\mathbf{d}] = \mathbf{d} \). We are missing an additional tensor that should be odd under parity and time reversal. To recover this tensor, we resort to the long wavelength limit and expand \( \mathbf{J}_L \) in a Taylor series with respect to the origin, yielding:  

\[
\mathbf{J}_L = -i\omega \delta(\mathbf{r}) + \mathbf{V} \times [\mathbf{m}_0 \delta(\mathbf{r}) - T\mathbf{V} \delta(\mathbf{r}) + \cdots] 
\]  

(5)  

where \( \delta(r) \) is a delta distribution centered at \( r = 0 \). The first term in Equation 5 is proportional to the ordinary ED moment \( \mathbf{p} \), while the second term is the ordinary MD moment \( \mathbf{m}_0 \). Mathematically, they correspond to the first order terms in a Taylor series of \( \mathbf{d} \) and \( \mathbf{m} \) in Equations 3 and 4, respectively. The ordinary ED moment has the textbook expression:  

\[
\mathbf{p} = -\frac{1}{i\omega} \int \mathbf{J}_L d^3r 
\]  

(6)  

The third term in Equation 5 is the dynamic TD moment, given by:  

\[
\mathbf{T} = \frac{1}{10} \int (\mathbf{r} \times \mathbf{J}_L) \mathbf{r} - 2\mathbf{r}\mathbf{r} \mathbf{J}_L d^3r 
\]  

(7)  

In what remains of this work, we will omit the adjective “dynamic” and refer to \( \mathbf{T} \) simply as “TD”. Crucially, while \( \mathbf{p} \) and \( \mathbf{m}_0 \) retain the same behavior under \( \mathcal{P} \) and \( \mathcal{T} \) as \( \mathbf{d} \) and \( \mathbf{m} \), this is no longer the case for \( \mathbf{T} \), which changes its sign under both space and time inversions. In the long wavelength approximation, the TD is thus the necessary moment that completes the set of allowed sign permutations under space and time inversion. Since the TD is
odd under parity inversion, it appears after $p$ in the Taylor series of $d$. However, it is evident from our discussion that the TD describes an elementary current with different symmetry properties in comparison with the ordinary ED moment. Such a distinction cannot be made directly with the spherical ED moment, which groups together the contributions of moments with the same parity, regardless of their behavior under time inversion. The physical meaning of the TD moment is clear: it describes a current distribution that changes its sign under both space and time inversions. The same argument can be used to justify the physical distinction between “ordinary” and “toroidal” terms of electric multipole coefficients of a higher-order. [14]

4. Conditions for an Ideal TD Source

We derive here the conditions for a current distribution to have only toroidal character. We show that, in this special case, the ordinary contributions to the electric spherical multipoles vanish exactly, and the source can only support toroidal moments and their MSR. To do so, we impose the conditions separately for the moments to be simultaneously odd under space and time inversions.

We start with time inversion. Equivalently, this constraint can be understood as the exact vanishing of all ordinary electric moments. In the case of the TD, it is necessary to make $p = 0$, which has never been achieved in practice. To determine the conditions for this to happen, we rewrite the ordinary ED in Equation 6 with the help of the continuity equation (refer to Section S1, Supporting Information):

$$p = \frac{-1}{i\omega} \int_{\Omega} J_\omega d^3r + \int_{\Omega} \sigma_n d^3r$$

Equation 8 shows that $p$ can be expressed as a combination of a volume and a surface integral involving solely moments of the charge density. $\rho_\omega$, $\sigma_n$ are the Fourier transforms of the volume and surface charge densities, respectively. $\sigma_n$ arises at the boundary of the source, $\partial \Omega$. It is produced by any current having a discontinuity in its normal component at the boundary, $J_n^\omega$.

From the interface conditions for the current density (Section S2, Supporting Information), it follows that $J_n^\omega = -i\omega \sigma_n$.

Equation 8 explicitly shows the link between the ordinary multipoles associated with the bulk poloidal currents and those associated with the charge densities. This connection arises naturally from the condition of the source being finite and is not the consequence of any simplification. The latter expression is a manifestation of the well-known Siegert theorem in photonuclear physics, i.e., in the long wavelength limit, nuclear transitions are determined by the time rate of change of the electric dipole operator. [40]

In the context of electrodynamics, it essentially tells us that the ordinary multipoles are proportional to the moments of the charge density.

We now arrive to our first important conclusion: a straightforward strategy to make $p$ vanish consists of imposing $\rho_\omega = 0$, or equivalently $V \times J_\omega = 0$. $J_n^\omega = 0$, everywhere in $\Omega$ and $\partial \Omega$ respectively. With this constraint, the moments describing our current are forced to be odd under time-reversal (see Figure 1a).

Next, we impose the moments to be parity-odd. As follows from the discussion in the previous section, this translates into making all magnetic moments vanish, as depicted in Figure 1b. By inspecting the general form of a spherical magnetic moment of arbitrary order (Section S5, Supporting Information), we show that $J_n^\omega$ must satisfy:

$$\mathbb{L} \cdot J_n^\omega = 0$$

In Equation 9, $\mathbb{L} = -i r \times V$ is the orbital angular momentum operator. To interpret Equation 9 geometrically, it is instructive to recast it as

$$r \cdot (V \times J_n^\omega) = 0$$

It turns out that the curl of the desired current must be tangential to an arbitrary sphere centered at the origin.

In summary, we have shown the most general conditions that a bounded current distribution must fulfill to become an ideal toroidal source. In order to be odd under time reversal, it cannot support neither the volume $\rho_\omega = 0$ nor surface charges $\sigma_n = 0$ at...
the boundary. This is equivalent to imposing that the current is divergence-less and $J_\rho = 0$ everywhere in $\partial \Omega$. To be odd under space (parity) inversion, the current must fulfill Equation 10, i.e., its curl cannot have a radial component. We emphasize that these conditions are not subject to any simplifications. In the next section, we will use them to derive a class of currents behaving as ideal toroidal sources.

5. Ideal Sources of Toroidal Moments

We consider an arbitrary, bounded current expressed in a “toroidal” system of coordinates, schematically shown in Figure 2a. A poloidal current takes the general form $J_\rho = A_\rho e_\rho$, where $A_\rho$ can be an arbitrary function of the coordinates (vanishing at some finite radius $\rho_0$) and $e_\rho$ is a poloidal unit vector, depicted in Figure 2a and defined in Section S3 (Supporting Information). Conveniently, this current distribution already fulfills $J_\rho = 0$, and is, therefore, a good starting point. However, it can have volume charges. It can be shown (see Section S4, Supporting Information), that imposing $\rho_m = 0$ and Equation 9, leads to:

$$J_m(\rho, \theta) = \frac{K(\rho)}{R_0 + \rho \cos \theta} e_\theta$$

Equation 11 describes a general family of poloidal currents in a torus with an ideal toroidal character. In the limit when the current is infinitesimally small, i.e. $K(\rho) \propto \delta(\rho - \rho_0)$ and $k R_0 << 1$, Equation 11 resembles the one originally proposed in the early works of Afanasiev\cite{41} as a prototypical TD. Interestingly, the current density in Equation 11 is not constant along the torus; it becomes more intense in the inner radius and decreases in the outer one, as depicted in the inset of Figure 2b. This condition is crucial to keep the current free of ordinary electric moments since otherwise, the current along every loop would be uncompensated, leading to charge accumulation. Critically, the family of currents proposed here can be of arbitrary spatial extension. Regardless of the size of the torus supporting it, a current density of the form given in Equation 11 is always completely characterized only by toroidal moments and their MSR.

To demonstrate this, we investigate an analytically solvable case. We consider that the current is localized inside a torus with minor radius $\rho_0$ and major radius $R_0$, distributed homogeneously along the $\rho$ direction (inset of Figure 2b). Then, $K(\rho) = I S(\rho, \rho_0)$, where $I$ is a constant (in Amperes) and $S(\rho, \rho_0)$ is a step function that vanishes for $\rho > \rho_0$. Remarkably, the ordinary ED moment in Equation 6 yields exactly $p = 0$. Similarly, all the magnetic moments are precisely zero, since the current fulfills Equation 10. The toroidal moments and their MSR remain nonzero. For example, the elementary TD (Equation 7) takes the form:

$$T = \frac{I \pi^2}{3} \rho_0^3 R_0 e_z$$

The 1st MSR of the TD (or second order toroidal dipole) was derived in\cite{6} and evaluates to:

$$T_2 = \frac{I \pi^2}{150} \rho_0^5 R_0 (3 \rho_0^2 + 5 R_0^2) e_z$$

Equations 12 and 13 are plotted in Figure 2b as a function of $k R_0$. Interestingly, the ratio between $T_2$ and $T$ scales with $3 \rho_0^2 + 5 R_0^2$. Obviously, shorter wavelengths and/or much larger sizes are required in order to obtain significant higher order toroidal MSR response. However, what is important is that, regardless of the extension of the source, the current is always completely characterized by toroidal moments.

Finally, it is also worth noting that Equation 11 does not correspond to the current flowing along a toroidal solenoid. The latter has for a long time been regarded as promising candidates for the observation of an ideal TD.\cite{8,12,42,43} However, as we show in...
Figure 3. a) Illustration of one copper loop and multipole decomposition of radiated power with dominating magnetic dipole. b) System of 8 copper loops arranged to form a torus. Each loop is supplied with the same current, in the same phase, forming a poloidal current distribution. The radiated power is almost completely described by a toroidal dipole with a small contribution from the poloidal MSR (the difference between orange and dashed red lines), while the ordinary ED is suppressed by more than three orders of magnitude in the whole range of $kr$ studied. c) The same system with one loop removed, resulting in broken symmetry and the appearance of a spherical MD comparable to the TD. d) The loop used in simulations and the electric field (in logarithmic scale) it produces. Arrows show the current flowing on the surface of the loop. (e) and (f) show the electric field formed by loops with current (in normal scale) respectively for eight loops and seven loops with broken symmetry in the $z = 0$ plane. The arrows show the magnetic field inside the “discrete” torus formed by the loops. More details on the simulations can be found in Section S11 (Supporting Information).

Section S10 (Supporting Information), a current flowing along a toroidal solenoid is not a pure TD because there is a non-zero net azimuthal current that produces a MD, that can even be comparable in magnitude to the TD. This MD vanishes only in the limit of tight winding (infinite number of turns of the wire), or by superposing two toroidal solenoids with opposite winding. Due to the complexity of the designs, such structures have never been realized. Hence, toroidal solenoids appear unsuitable for the task and should not be used as examples of ideal TDs.

We emphasize that the conclusions in this section are fundamentally new since, until now, the dynamic TD and its successive MSR were only analytically shown to correspond to current distributions confined to a point in space. Conversely, now we have generalized these results to currents of arbitrary spatial extension, which are no longer “idealized” toy models.

6. Design of an Intrinsic TD Source

We are now ready to envision a realistic structure presenting an ideal toroidal character. The most straightforward strategy is to find a toroidal-like geometry supporting a current as in Equation 11. Intuitively, a poloidal current can be viewed as the one produced by $N$ magnetic dipoles of equal magnitude arranged head-to-tail. To implement this idea, we model $N$ subwavelength metallic loops arranged in a torus (inset of Figure 3b), with a radius $r$. Since $\frac{r}{\lambda} \ll 1$, where $\lambda$ is the input wavelength, every isolated loop supports a circulating current distribution radiating as an ordinary MD given by $m = IA$, where $I$ is the current in the loop and $A$ is a vector normal to it, with magnitude equal to its cross-sectional area. Critically, every antenna is fed by a current through an input port located at their base, as depicted in the inset of Figure 3a (more details on the port settings can be found in Section S11, Supporting Information). This strategy allows us to independently excite each loop as to ensure a head-to-tail distribution of the MDs along the torus. Earlier works lacked this additional degree of freedom, resulting in the appearance of parasitic ordinary EDs and high order quadrupole moments.

First, we start with the simulation of one loop and calculate its cartesian multipole decomposition (Figure 3a). The origin of the decomposition is taken at the center of the loop. Multipole expansions are origin-dependent, but the number of terms is minimal at the center of symmetry of the charge–current distribution, hence the choice of origin. The result confirms the expected dominant MD response but also reveals a weak ordinary ED stemming from the port contribution, approximately an order of magnitude smaller. In the torus, however, the weak ED contribution of each loop is canceled out by the one in the opposite side. We further validate these results by inspecting the current within the loop and the near-field distribution (Figure 3a). It can be clearly seen how the field around the antenna has its highest intensity close to the metallic surface and distributes evenly around the wire following the circular current, except at the top and bottom. This small inhomogeneity is attributed to the port, acting as a capacitance.

We then construct a torus arrangement with $N = 8$ loops, which suffice to emulate a smooth poloidal current (Figure 3b).
We calculate the total power radiated by the structure and its multipolar decomposition in both the spherical and cartesian representations. The origin of the two decompositions is taken at the center of the toroidal source, once again due to symmetry.

Due to the symmetry of the current distribution, we note that the structure does not support a magnetic response, as predicted in the previous section. The contribution of the spherical ED moment is sufficient to reconstruct the radiated power visually. However, it gives no insight into the topology of the current distribution supported by the source besides its evident electric nature. Strikingly, the cartesian representation reveals that the TD is entirely responsible for the radiated power in the whole spectral range studied, with the ordinary ED suppressed everywhere by more than three orders of magnitude. To the best of our knowledge, this behavior has never been observed before. Our design constitutes the first broadband TD source. As a result, a TD moment will always be present.

It is also interesting to visualize the effect of symmetry breaking in the current. For that purpose, we “turn off” the port in one of the loops. Since the current no longer validates Equation 10, it can manifest a magnetic response. This is confirmed in the multipole decomposition in Figure 3c: the source displays a strong MD response comparable to the TD, which contributes significantly to the total radiated power.

To validate our results experimentally, we fabricated our structure and measured the near and far fields produced by it in the microwave frequency range (Figure 4). The loops are made of 1 mm thick copper wire, which behaves as a PEC in the frequency range under consideration (inset of Figure 4c). In practice, the discrete ports feeding the toroidal source are implemented with a waveguide connected to a power divider, as shown in the photograph in the inset of Figure 4c. This strategy ensures each loop is supplied with the same current, in the same phase. More details on the antenna and the experimental setup are given in Section S8 (Supplementary Information).

In Figure 4a,c, we provide a comparison of the simulated and measured magnetic fields for the frequency of 1000 MHz, where...
only the dominant azimuthal component is shown. The numerical and experimental results are in good agreement with each other. In both cases, we observe a circulating magnetic field inside the discrete torus formed by the loops, a clear signature of the TD. We emphasize that identical near-field patterns can be observed in a frequency range spanning hundreds of MHz, not only at 1000 MHz. In the experiment, this has been tested from 900 up to 1100 MHz (refer to Figure S2 in Section S9, Supplementary Information).

Figure 4b depicts the setup for the far-field measurements. The antenna was installed on a numerically controlled positioning device, which allows the rotating of a sample mounted on its table with a precision of 0.1°. The whole setup was placed in an anechoic chamber. A horn antenna with a lower frequency bound of 0.7 GHz was used as the far-field detector. The antenna was then placed 2.5 m away from the experimental model, which is enough to ensure the far-field region for the selected frequency range.

Figure 4d,e displays the measured radiation patterns in and out of the torus plane, respectively. In all the frequencies studied, the source indeed displays a characteristic ED pattern associated with the TD excitation. This is further confirmed by an inspection of the near fields, being identical to the one in Figure 4c (not shown). The combination of near and far field measurements completely characterizes the source and confirms the first realization of a pure, broadband dynamic toroidal source.

Summarizing the results of this section, we have designed and implemented a realistic antenna behaving as an ideal toroidal source in the experiment. In stark contrast with previous works, which focused on resonant structures, we have been able to achieve a broadband toroidal response spanning hundreds of MHz in the microwave frequency range in the complete absence (three orders of magnitude suppression) of the ordinary ED moment, as well as all magnetic moments. This result has been confirmed semi-analytically with the help of both spherical and cartesian multipole representations, reaching a perfect agreement with the numerical calculations. Furthermore, we have directly measured both the near and far field signature of our pure toroidal source in good qualitative agreement with the simulations.

7. Toroidal Anapole

The study of non-radiating charge–current configurations featuring nontrivial fields within a certain volume but zero at any other point in space is a fascinating research topic with a venerable history.

Nowadays, anapoles, the destructive interference of toroidal moments with their ordinary counterparts, have been extensively studied within the framework of nanophotonics.

The anapole takes place when the ordinary ED is cancelled by the TD according to \( p = -ikR \), assuming that the MSR is negligible. Recently, however, the authors of Ref. [12] tentatively introduced the concept of a toroidal nonradiating source, i.e., where the destructive interference would not involve any ordinary multipole and would take place mainly between a toroidal moment and its MSR. Its realization could pave the way toward interesting developments in metamaterials and photonics exploiting the non-trivial effects of non-radiating sources, such as the Aharonov–Bohm effect.

Within this work, for brevity, we shall call the latter a toroidal anapole (TA), in contrast with the conventional ED anapole (EDA). To the best of our knowledge, the TA remains purely a proposal and has not been studied theoretically beyond a toy model.

For the emergence of a TA, the contributions to radiation from the TD and the 1st MSR of the whole structure must vanish, so that:

\[
T = -k^2 T_2
\]  

We consider a current allowing the 1st MSR to be comparable to the TD. From Figure 2b, one can appreciate that the TD and 1st MSR are quite different in magnitude for the current in Equation 11. To enhance the MSR, we consider two concentric poloidal currents in two torii \( a \) and \( b \), with parameters \( \rho_r, R, I \) where each of the currents satisfies Equation 11. Equation 14 can be solved analytically for our system, and leads to the relation (refer to the Section S6, Supporting Information):

\[
I_k R_k \rho_k^b \left( 1 - \frac{k^2 P_k^b}{10} \right) = I_k R_k \rho_k^a \left( 1 - \frac{k^2 P_k^a}{10} \right)
\]  

In Equation 15, \( P_k^j \) are the intensities of the currents circulating along the surface of each torus. To envision a system where the TA can be observed, we consider a circular arrangement of N point MDs (PMDs), depicted in Figure 5b.

Assuming an equal number of PMDs in both loops, in an analogous fashion to Equation 15, the condition for the TA reduces to (Section S7, Supporting Information):

\[
r m \left( 1 - \frac{1}{10} k^2 r^2 \right) - r m u \left( 1 - \frac{1}{10} k^2 r'^2 \right) = 0
\]  

where \( r, r' \) are the radii of the largest and the smallest loop and \( m, m' \) are the magnitudes of PMDs on the largest and the smallest loop. To validate Equation 16, we numerically simulate this model for power versus \( m/m' \) for \( N = 16 \) PMDs, \( r' = 2r \). The results of the numerical experiments are shown in Figure 5d. One can see that the lines representing the power of TD (blue) and 1st MSR (red) are crossing at the theoretically predicted point, the power of the sum of TD and 1st MSR (orange line) is zero, and the total power (dashed line) has a minimum at that point, giving rise to a TA. In addition, when \( rm = r'm' \), (in this particular case, when \( m' = 2m \)), the TD vanishes, giving rise to a well-defined MSR response. This second result can be found by adding up the contributions to the TD of both rings, as discussed in Section S7 (Supporting Information).

In this section, we theoretically described and confirmed through numerical simulations a new type of non-radiating source, the TA. The proposed structure could be implemented with an array of resonant metamolecules with artificial magnetic response, either dielectric or plasmonic in nature. For instance, core-shell spheres or disks made of silicon and gold, such as the ones studied in,[30,31] allows to obtain a pure MD response. By arranging them in loops as in Figure 5, a TA could be experimentally realized in the visible. Alternatively, since our model of the TA is implemented with PMDs, potential candidates for its realization at the nanoscale are rare-earth ions, which can be used as magnetic dipole emitters. Well-known examples of such emitters are erbium ions (Er\(^{3+}\)) and europium ions (Eu\(^{3+}\)), whose
Figure 5. a) Schematic illustration of 8 PMDs. The PMDs are located equidistantly on the circle with a radius \( r \). The magnitude of each dipole is \( m \) while arrows show the direction of the magnetic moments. b) Scheme of 16 PMDs arranged in two circular loops (8 PMDs on each circumference). Each circular array represents a “discrete” version of the current in Equation 11. The magnetic moments of the red PMDs are reversed in direction with respect to the blue ones, have magnitude \( m' \), and are placed along a circumference with radius \( r' \). c) Results of numerical simulation of dependence of power on \( kr \) for 8 PMDs. Orange and purple lines are the power of TD and the sum of TD and 1st MSR, respectively. Blue circles correspond to the total power radiated by the two circular arrays. d) Radiated power versus the ratio \( m/m' \) for 16 PMDs. All legends are the same as in c), and the red line is 1st MSR. Parameters: \( kr = 0.24\pi, kr = 1.5\pi, \frac{r}{r'} = 2 \).

MD transitions can be enhanced with the help of nanophotonic platforms.\[53\]

8. Conclusion

Toroidal moments have been a subject of controversy ever since their introduction to electrodynamics. Until now, the separation of the spherical electric multipoles into ordinary and toroidal terms seemed somehow artificial and unnecessary for characterizing a current distribution. Here, we have shown those dynamic toroidal moments describe currents with a well-defined behavior under both space and time inversions. We have determined the exact conditions that any bounded current distribution must fulfill in order to be entirely characterized by dynamic toroidal moments and their MSR. Besides being divergence-less, we show that the curl of such currents must not have a radial component and, importantly cannot induce a surface charge at the boundary of the source. If a current validates these general conditions, all the exact spherical magnetic moments and the ordinary ED become zero, and the source is completely characterized by time-odd, spatial-odd moments, i.e., toroidal moments and their MSR. The fields radiated by a pure TD in the far zone do not differ from a conventional point ED. However, within the smallest spherical shell enclosing the source, the topology of the electromagnetic fields is drastically altered. The distinction cannot, however, be made unless one has access to the electromagnetic field inside the source volume.

Second, based on our physical insight into the problem, we have designed and verified a pure, broadband dynamic TD source in the microwave frequency range experimentally. The toroidal response spans hundreds of MHz, starkly contrasting with earlier attempts, where a pure TD could only be achieved at a single frequency.\[13,16\] Furthermore, all the other multipoles, including the conventional ED, are suppressed by more than three orders of magnitude.

In the last part of our study, following a similar design strategy as for the pure TD source, we have demonstrated analytically and proved numerically the excitation of a novel type of nonradiating source, a “toroidal anapole”, originating from the destructive interference of the TD with the first order MSR of the structure. Our results unambiguously prove the possibility of realizing an ideal toroidal source in electrodynamics. We expect our results to find application in several fields, from the design of novel nanophotonic devices to the characterization of biomolecules.

Supporting Information

Supporting Information is available from the Wiley Online Library or from the author.
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Conflict of Interest
The authors declare no conflict of interest.

Data Availability Statement
The data that support the findings of this study are available from the corresponding author upon reasonable request. The authors declare no conflict of interest.

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