Holographic quantum criticality from multi-trace deformations

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Abstract

We explore the consequences of multi-trace deformations in applications of gauge
gravity duality to condensed matter physics. We find that they introduce a powerful
new “knob” that can implement spontaneous symmetry breaking, and can be used to
construct a new type of holographic superconductor. This knob can be tuned to drive
the critical temperature to zero, leading to a new quantum critical point. We calculate
nontrivial critical exponents, and show that fluctuations of the order parameter are
‘locally’ quantum critical in the disordered phase. Most notably the dynamical critical
exponent is determined by the dimension of an operator at the critical point. We
argue that the results are robust against quantum corrections and discuss various
generalizations.
1 Introduction

Over the past couple of years, gauge/gravity duality has been applied to a number of problems in condensed matter physics (for reviews see [1, 2, 3]). An important feature of some condensed matter systems is the existence of quantum critical points, marking continuous phase transitions at zero temperature. One goal of the present work is to introduce and study a new mechanism for generating quantum critical points in the context of gauge/gravity duality. We will see that the behavior near the critical points is described by nontrivial critical exponents and goes beyond the usual Landau-Ginzburg-Wilson description of phase transitions at zero temperature.

A second goal is to introduce a new type of holographic superconductor. The key ingredient in constructing a gravitational dual of a superconductor is to find an instability which breaks a $U(1)$ symmetry at low temperature and causes a condensate to form. Previous constructions have started with a charged anti de Sitter (AdS) black hole which has such an instability when coupled, e.g., to a charged scalar field [4, 5, 6]. We will show that there is another source of instability which applies even for Schwarzschild AdS black holes. So these new superconductors can exist even at zero chemical potential and no net charge density.

Both of these goals are achieved by adding a multi-trace operator to the dual field theory action$^1$. For example, given a (single trace) scalar operator $\mathcal{O}$ of dimension $\Delta_- < 3/2$ in a $2 + 1$ dimensional field theory (which will be our main focus) one can modify the action

$$S \rightarrow S - \int d^3x \bar{\kappa} \mathcal{O}^\dagger \mathcal{O}$$

(1.1)

where for convenience later the coupling will be rescaled as $\bar{\kappa} = 2(3 - 2\Delta_-)\kappa$. Since this is a relevant deformation, it is unnatural to exclude such a term, and it has important consequences. If $\mathcal{O}$ is the operator dual to the bulk charged scalar field in conventional holographic superconductors, then adding this term (with $\kappa > 0$) makes it harder to form the condensate and lowers the critical temperature. We will see that in some cases $T_c$ vanishes at a finite value of $\kappa = \kappa_c$. This defines a new quantum critical point which we will study in

$^1$For another recent discussion of multi-trace operators in gauge/gravity duality, see [7].
detail. Since $T_c$ can be quite large at $\kappa = 0$, adding this double trace perturbation introduces a sensitive new knob for adjusting the critical temperature.

For $\kappa > \kappa_c$ (and nonzero chemical potential $\mu$), the ground state is described by the extremal Reissner-Nördstrom (RN) AdS black hole which has an emergent $AdS_2$ geometry in the IR. For $\kappa < \kappa_c$, there are various possible IR geometries depending on details of the bulk potential. However, as $\kappa$ approaches $\kappa_c$ from below, the bulk solution develops an intermediate $AdS_2$ geometry. It is this intermediate region which controls the behavior near the critical point. For example, we will show that the critical exponents do not take mean field values, but are determined by the scaling dimension of certain operators in the $0 + 1$ dimensional CFT dual to this region. This is closely analogous to the way properties of holographic non-Fermi liquids [8, 9, 10] were described in terms of a dual $0 + 1$ dimensional CFT [11, 12]. In addition, the instability for $\kappa < \kappa_c$ can be interpreted as turning on a double trace term with negative coefficient in the $0 + 1$ dimensional CFT dual to this region.

Let us contrast this with the usual argument for why the RN AdS black hole becomes unstable at low temperature in the presence of a scalar field [13]. In the $AdS_2$ near horizon geometry of the $T = 0$ solution, the scalar field has an effective mass $m_{\text{eff}}$ which depends on the original mass $m$ and charge $q$ of the scalar field. When this effective mass squared is below the Breitenlohner-Freedman (BF) bound [14] for $AdS_2$, this near horizon region becomes unstable. Since $m^2$ is above the BF bound of the asymptotic $AdS_4$ geometry, the asymptotic region is stable, and the solution settles down to a black hole with scalar hair. It is now clear that this argument is sufficient but not necessary. It overlooks the possibility of instabilities with $m_{\text{eff}}^2$ above the BF bound which are allowed due to modified boundary conditions for the scalar field. The boundary conditions may be modified due to the addition of a multi-trace deformation in the dual field theory [15, 16], or simply due to alternative quantization of the bulk theory [17].

It was widely believed that if one added a double trace term with $\kappa < 0$, then the theory would not have a stable ground state. However, we have recently shown that this is not necessarily the case [18]. For a large class of dual gravity theories, there is still a stable ground state with $\langle O \rangle \neq 0$ when the boundary conditions correspond to $\kappa < 0$. As one increases the temperature, there is a second order phase transition to a state with $\langle O \rangle = 0$. This provides a new mechanism for spontaneously breaking a $U(1)$ symmetry and constructing novel holographic superconductors. This mechanism does not require a charged black hole and works for Schwarzschild AdS as well. In other words, one can set $\mu = 0$ and still break the $U(1)$ symmetry at low temperature. The critical temperature is now set by $\kappa$. We will discuss some properties of these novel holographic superconductors in section 2.
It is worth pointing out that the coupling $\kappa$, as the coefficient of the square of the order parameter, is the usual tuning parameter in the context of Landau-Ginzburg theory. Also if $\mathcal{O}$ is a gauge invariant trace of a fermion bilinear then the double trace is a 4 fermion interaction, a natural interaction to consider.

Although our discussion so far has focussed on the case where the operator $\mathcal{O}$ is charged, our results apply equally well when $\mathcal{O}$ is neutral. In this case, the ordered phase breaks a $Z_2$ symmetry. More generally, one can imagine different symmetry breaking scenarios where for example $\mathcal{O}$ could be part of a triplet of operators forming a representation of $SU(2)$ which is spontaneously broken at low temperature. This is a particular attractive possibility as outlined in [19, 20], since in many condensed matter systems $SU(2)$ spin is a global symmetry (ignoring spin orbit effects.) Including an exact global $SU(2)$ symmetry then allows us to model magnetism in a holographic setup. The triplet in which one embeds $\mathcal{O}$ can be interpreted as the staggered order parameter associated with anti-ferromagnetic transitions. Since the boundary theory has a global $SU(2)$ symmetry the bulk will have an $SU(2)$ gauge symmetry distinct from the $U(1)$ electromagnetic charge. For the rest of this paper the $SU(2)$ gauge fields and triplet structure of the order parameter will not play a roll.

It is useful to bear this possibility in mind, in particular so we can compare our results to quantum phase transitions in metallic systems, where anti-ferromagnetic order plays an important role. The “standard” theory of which was given in [21, 22] is based on the renormalization group Landau-Ginzburg paradigm. However experimental measurements (see for example [23, 24], and references therein) of heavy fermion systems with quantum critical points show that the “standard” theory can break down, as a result new theoretical methods are required. Subsequently several different methods were developed (see for example [25, 26, 27, 28].) One such method [25, 26] which is formally justified in a large $d$ expansion [29] shows local quantum critical behavior similar to the new quantum critical point that we find. It will be useful to compare and contrast our results to those of the standard theory and the other theoretical methods used in the study of quantum criticality in heavy fermion systems.

Since the finite density normal phase we consider is governed by $AdS_2 \times R^2$ in the IR, at zero temperature the theory has a finite entropy density. This has lead many people to suggest that this state must not be the true ground state, since otherwise one finds unnatural violations of the third law of thermodynamics. Of course this may be natural in the context of applications to heavy fermion system where superconductivity instabilities can be observed close to criticality. So it may be that the state we work in is the correct one for a large range of temperatures, but ultimately at low temperatures something else takes over.
Indeed as is discussed and extended in this paper $AdS_2$ has many possible forms of instability. This however motivates us to attempt to extend our results in various directions to directly address this problem. One extension we consider is adding a magnetic field. We show that while a magnetic field suppresses the superconducting instability, it can enhance the neutral (anti-ferromagnetic) instability. Another extension we consider is replacing $AdS_2 \times R^2$ with other possible IR geometries, such as a Lifshitz geometry which does not have finite entropy density at zero temperature. We find our results are rather robust here.

The organization of the paper is as follows: In the next section we show how double trace deformations can induce spontaneous symmetry breaking and use this to construct a new type of holographic superconductor with zero net charge density. We then extend this to the finite density case and show that the coefficient of the double trace deformation provides a sensitive knob by which one can tune the critical temperature $T_c$ to zero. In section 3 we study the new quantum critical point that arises and analytically compute the nontrivial critical exponents. In section 4 we numerically construct the backreacted geometries that correspond to the ordered (condensed) phase away from the phase transition. We confirm the critical exponents near the critical point. In the discussion section, we summarize our results and discuss generalizations to magnetic fields and Lifshitz normal phases. The Appendices contain additional details.

2 Double trace deformations

In this section we begin by emphasizing the simple under appreciated fact that double trace deformations are useful for studying symmetry breaking in gauge/gravity duality. We then note that this system provides a simple holographic model for superconductivity with zero total charge density. With nonzero charge density, we show that double trace deformations introduce a new parameter by which one can tune the critical temperature of the superconductor.

2.1 Gravity setup and boundary conditions

The theory we study is gravity in $3 + 1$ dimensions with a negative cosmological constant, a $U(1)$ gauge field, and a scalar field $\Psi$ which may or may not be charged under the $U(1)$ symmetry. By general arguments of gauge/gravity duality this theory is dual to a $CFT_{2+1}$ with a conserved current operator $J^\mu$ and a scalar operator $\mathcal{O}$.

The action is that of the Einstein-Abelian Higgs model with a negative cosmological constant, where we parameterize the phase and modulus of the charged scalar as $\Psi = \psi e^{i\theta}$,
following e.g. \[30\]

\[ S = \int d^4x \sqrt{-g} \left( R - \frac{1}{4} G(\psi) F^2 - (\nabla \psi)^2 - J(\psi)(\nabla \theta - qA)^2 - V(\psi) \right) \]  

(2.1)

We require that the coupling functions \( G \) and \( J \) and the potential \( V \) be even functions of \( \psi \), since we need to preserve our \( U(1) \) symmetry. We will assume an expansion of the form

\[ V = -6 + m^2 \psi^2 + \mathcal{O}(\psi^4), \quad G = 1 + g \psi^2 + \mathcal{O}(\psi^4), \quad J = \psi^2 + \mathcal{O}(\psi^4) \]  

(2.2)

where we have set the \( AdS_4 \) radius to one. The coefficient of \( \psi^2 \) in \( J \) is fixed by regularity at \( \Psi = 0 \). This is all that we will need to determine the behavior near the critical point in section 3. We will specify the potential and coupling functions more fully later when they are needed to construct the ordered phase away from the critical point.

The mass \( m \) of the field around the symmetric point \( \psi = 0 \) determines the conformal dimension of the dual operator \( \mathcal{O} \) in the \( CFT_{2+1} \)

\[ \Delta_{\pm} = 3/2 \pm \sqrt{9/4 + m^2} \]  

(2.3)

This also controls the asymptotic behavior of the scalar field

\[ \psi(r) = \frac{\alpha}{r^{\Delta_-}} + \frac{\beta}{r^{\Delta_+}} + \ldots \quad \text{as} \quad r \to \infty \]  

(2.4)

with the metric asymptotically approaching

\[ ds^2 = r^2(-dt^2 + dx_i dx^i) + \frac{dr^2}{r^2} \]  

(2.5)

In order for us to be able to add a double trace operator as in (1.1) in a controlled fashion (without destroying the \( AdS_4 \) asymptotics) we require that the mass be in the range:

\[ -9/4 < m^2 < -5/4 \]  

(2.6)

In this range, both sets of modes are normalizable and one has a choice of boundary conditions for quantizing the bulk theory. \textit{Standard} quantization corresponds to setting \( \alpha = 0 \), and \( \mathcal{O} \) has dimension \( \Delta_+ \). \textit{Alternative} quantization corresponds to \( \beta = 0 \), and \( \mathcal{O} \) has dimension \( \Delta_- \). \cite{17}. We will refer to these two theories as \( AdS_4^{(\text{std.})} \) and \( AdS_4^{(\text{alt.})} \). We want to start with alternative quantization, so the dimension of \( \mathcal{O} \) is

\[ 1/2 < \Delta_- < 3/2 \]  

(2.7)
and add a double trace deformation. In this range, adding the double trace operator (1.1) amounts to studying the gravitational theory in asymptotically $AdS_4$ space with new boundary conditions for the scalar $\beta = \kappa \alpha$ (2.8)

Up to an overall normalization, $\alpha = \langle O \rangle$. Note that $\kappa$ has dimension $\Delta_+ - \Delta_- = 3 - 2\Delta_-$ and is thus a relevant coupling in the range of interest (2.7). Hence adding the perturbation will induce an RG flow from the original CFT to a new theory in the IR which can be understood by sending $\kappa \to \infty$. Dividing (2.8) by $\kappa$ as we take this limit we see that we arrive at $\alpha = 0$, the theory with standard boundary conditions. If we had started with $\kappa = 0$ we would stay at the unstable fixed point in alternative quantization. Many details of this flow have been studied (see, e.g., [31]).

### 2.2 Symmetry breaking from double trace deformations

We will first study a simple model of spontaneous symmetry breaking. We will work with zero chemical potential, and hence can set the bulk Maxwell field to zero. This is natural in a theory which preserves charge conjugation. Adding a double trace term (1.1) with negative coefficient is expected to destabilize the vacuum. This is easily seen as follows. Consider the two-point function for the operator $O$ in the vacuum (at zero chemical potential, temperature, and double-trace coefficient.) The retarded Green’s function is (up to overall normalization)

$$G^R_{\pm}(p) = p^{2\Delta_\pm - 3} \quad p^2 = -(\omega + i\epsilon)^2 + \vec{p}^2$$

where the sign $+(-)$ indicates the Green’s function for the standard (alternative) quantized theory. Also $\omega$ and $\vec{p}$ are the energy and momentum respectively. If we start with the alternative quantized theory and add a double-trace term of the form (1.1), one finds\(^2\)

$$G^R(p) = \frac{1}{G^R(p) + \kappa} + O(1/N^2).$$

(2.10)

For $\kappa > 0$, this just introduces a new massive pole with a width at $p^2_{\text{pole}} = (-\kappa)^{1/(3 - 2\Delta_-)}$. However, for $\kappa < 0$, we have a tachyonic instability, with a pole at real positive $p^2_{\text{pole}}$. All of this is directly analogous to a massless free scalar field getting a massive deformation of either sign. In [32] an exponentially growing tachyonic mode was explicitly found in the bulk

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\(^2\)This can be calculated either in the field theory at large $N$ by summing up a geometric series of diagrams, or on the gravity side with proper treatment of boundary conditions.
precisely when \( \kappa \) had the wrong sign. It was widely believed that theories with \( \kappa < 0 \) would not have a stable vacuum, but in [18] it was proven that for many scalar gravity theories, there was a stable ground state with nonzero \( \langle O \rangle \).

The stability of the dual gravitational system depends on the global existence of a superpotential \( P_c(\psi) \), and the zero temperature broken symmetry ground state is given entirely by \( P_c(\psi) \). The details can be found in [18]. The key result is the behavior of the off-shell potential \( V(\alpha) \). It turns out that

\[
V = 2(\Delta_+ - \Delta_-)(W + W_0),
\]

(2.11)

\( W(\alpha) \) is given by our boundary conditions \( \beta = W'(\alpha) \), and for our double trace deformation is simply \( W(\alpha) = \kappa \alpha^2/2 \). \( W_0(\alpha) \) is found from a scaling limit of smooth horizonless static solitons in global \( AdS \), again see [18] for details. At \( T = \mu = 0 \), scale invariance implies

\[
W_0(\alpha) = \frac{s_c \Delta_-}{3} |\alpha|^{3/\Delta_-}.
\]

(2.12)

The coefficient \( s_c \) depends on the full bulk potential \( V(\psi) \), and is generally positive. If \( s_c \) is negative the theory is somewhat sick since \( W_0 \) is unbounded, and the alternative quantized theory is unstable, as it has states with arbitrarily negative energy. We do not consider this case any further. The full off shell potential is thus

\[
V(\alpha) = (\Delta_+ - \Delta_-) \left( \kappa \alpha^2 + \frac{2s_c \Delta_-}{3} |\alpha|^{3/\Delta_-} \right).
\]

(2.13)

For our range of interest (2.7) the second term dominates at large \( \alpha \), and we have a classic example of spontaneous symmetry breaking with a saddle-shaped potential for negative \( \kappa \) (see Fig. 1). The ground state which minimizes (2.13) has

\[
\alpha = \langle O \rangle = \left( -\frac{\kappa}{s_c} \right)^{\Delta_-/(3-2\Delta_-)}, \quad V_{\text{min}} = -\frac{1}{3} \left( \frac{3 - 2\Delta_-}{s_c^{\Delta_-/(3-2\Delta_-)}} \right)^2 (-\kappa)^{3/(3-2\Delta_-)}.
\]

(2.14)

As mentioned above, the gravitational description of this ground state is uniquely determined by the superpotential \( P_c(\psi) \).

Putting the theory at finite temperature can lift this instability. The detailed calculation is in appendix [13], but the result is that as we heat the system up to

\[
T_c = k_1 (-\kappa)^{1/(3-2\Delta_-)}, \quad \text{with } k_1 = \frac{3}{4\pi} \left( \frac{\Gamma((\Delta_+ - \Delta_-)/3)\Gamma(\Delta_-/3)^2}{\Gamma((\Delta_- - \Delta_+)/3)\Gamma(\Delta_+/3)^2} \right)^{1/(3-2\Delta_-)}
\]

(2.15)

3Strictly speaking, [32] studied the theory on a sphere, where coupling to background curvature induces a positive double trace term for scalars. In that case, \( \kappa \) needed to be sufficiently negative to find the instability. Since we are studying the theory on Minkowski space, the critical point is simply when \( \kappa \) changes sign.

4The overall factor of 2 (which was not present in [13]) arises since we do not have a 1/2 in front of our action (2.1).
the system returns to the symmetry preserving state. Note that everything scales as a power of $\kappa$, since this is the only scale in the problem. Another way of studying this system at $T > 0$ is to construct the finite temperature generalization of $\mathcal{V}$. This can be obtained from the family of hairy black holes in the bulk, as described in [33].

2.3 A novel holographic superconductor

We saw above that adding a renormalizable double trace coupling can break a $U(1)$ symmetry at low temperature. This provides a new mechanism for constructing holographic superconductors. Unlike the previous approach, which required a nonzero charge density to generate a low temperature condensate, we can now work at zero net charge. In this case, the Maxwell field remains strictly zero, even when the charged scalar hair is present in the bulk.

We computed the critical temperature above. To study the system away from $T_c$, we need to specify the full nonlinear bulk potential. Working in units of $\kappa$ (which is analogous to working in units of $\mu$ or $\rho$ in cases of finite density), we find that the order parameter behaves just as it does in the case where we find an instability by lowering $T/\mu$ in the standard holographic superconductor setup. We can also calculate the difference between the free energy of the hairy black hole and the normal black hole, which is simply AdS-Schwarzschild, and find generically that below the critical temperature the hairy black hole is always preferred. An example which comes from a consistent string theory truncation is shown in Fig. 2.

As usual, to compute the conductivity, one starts by perturbing the Maxwell field and
Figure 2: The order parameter $\alpha = \langle O \rangle$ and free energy density $f$ across the second order phase transition down to zero temperature. In the figure on the right, the dashed red line is the free energy of the normal phase (Schwarzschild AdS solution) and the black line is the free energy of the condensed phase. We used a case with $\Delta_+ = 1$, and bulk potential $V(\psi) = \sinh^2(\psi/\sqrt{2})(\cosh(\sqrt{2}\psi) - 5) = -6 - 2\psi^2 + O(\psi^4)$ [34] [35]. In [18] it was found that $s_c = 0.56$ for this potential. From (2.15), we have $T_c/(-\kappa) \approx 0.62$.

As shown in Appendix A the conductivity can be simply related to the reflection coefficient in a one dimensional Schrodinger problem:

$$ -b''(z) + V_{Sch}(z)b(z) = \omega^2 b, $$

(2.16)

where $\delta A_x = a_x e^{-i\omega t}$ and $b = \sqrt{G(\psi)}a_x$. The Schrodinger potential is bounded (and given explicitly in (A.24)). $z$ is a new radial coordinate that vanishes at infinity and goes to minus infinity at the horizon. To obtain the required ingoing wave boundary condition at the horizon, we assume $b = e^{-i\omega z} + \mathcal{R}e^{i\omega z}$ near $z = 0$ so that $b = T e^{-i\omega z}$ near the horizon. The conductivity is simply given by

$$ \sigma = \frac{1 - \mathcal{R}}{1 + \mathcal{R}}. $$

(2.17)

Since the potential is bounded, this will produce the usual behavior of the optical conductivity. At low temperature there will typically be a gap at frequencies below the height of the potential, and at higher frequencies the conductivity will approach its normal state value. There will be a delta function at $\omega = 0$ in the condensed phase, which can be seen from a pole in the imaginary part of the conductivity.

The key difference from the holographic superconductors at nonzero charge density, is that there is no delta function in $\text{Re}[\sigma]$ at $\omega = 0$ in the normal phase. This awkward feature of the previous construction arose since a state with net charge can be boosted, yielding a

\footnote{Since the Schrodinger potential is no longer positive definite, one can sometimes get peaks at low frequency in the conductivity [36].}
nonzero current with zero applied electric field. This implies infinite DC conductivity. Since we can now start in a state with zero charge, we no longer have this problem. Mathematically, the delta function arose since the Schrodinger potential was nonzero in the normal phase due to a contribution from the background electric field. Here, the Schrodinger potential vanishes in the normal phase. The background is just the Schwarzschild AdS black hole, and $\sigma = 1$ with no delta-function contributions.

### 2.4 Non-zero density and stability conditions

In addition to providing another way to construct holographic superconductors, the addition of a double trace perturbation provides a new knob for adjusting the critical temperature of traditional holographic superconductors. As discussed in the introduction, adding a term with $\kappa > 0$ makes it harder to condense the operator $\mathcal{O}$ and lowers the critical temperature. We will see below that in some cases, $T_c \to 0$ as $\kappa$ approaches a finite value, $\kappa_c$. This is a new quantum critical point which will be studied in detail in the next section.

With a nonzero charge density, the normal phase is described by the Reissner-Nördstrom-AdS (RN-AdS) black hole. The critical temperature is determined by looking for a static normalizable mode of the scalar field in this background [13]. This marks the onset of the instability to form scalar hair. This problem only requires the leading terms in the functions $V(\psi)$, $G(\psi)$, $J(\psi)$ given in (2.2). The addition of the double trace term changes the critical temperature since it changes the boundary condition on the normalizable mode.

For reference, the RN AdS black hole is described by the metric and gauge potential,

$$ds^2 = -f dt^2 + r^2 d\vec{x}^2 + \frac{dr^2}{f}, \quad \phi = \mu - \frac{\rho}{r}, \quad f = r^2 - \frac{m_0}{2r} + \frac{\rho^2}{4r^2} \quad (2.18)$$

$$m_0 = \frac{2\rho^3}{\mu^3} + \frac{\rho \mu}{2}, \quad T = \frac{\mu}{4\pi} \left( \frac{3\rho}{\mu^2} - \frac{\mu^2}{4\rho} \right) \quad (2.19)$$

The horizon is located at $r_0 = \rho/\mu$, where $\rho$ and $\mu$ are the charge density and chemical potential respectively.

On this background the linear fluctuations are given by,

$$(r^2 f \psi')' = \left( m^2 r^2 + \vec{p}^2 - \frac{q \rho^2}{2r^2} - \frac{r^2 (\omega + q \phi)^2}{f} \right) \psi \quad (2.20)$$

where we have included momentum dependence $\vec{p}$ and frequency dependence $\omega$ and $\Psi = \psi e^{-i\omega t + i\vec{x} \cdot \vec{p}}$. We will need these in Section 3 but for now they can be set to zero. Before going onto the case of double trace boundary conditions, we first recall some known results on stability conditions. One natural question to ask is what is the condition for the absence of
an instability at any $T$. In other words, when is the normal state stable at zero temperature? A necessary condition was identified in [13]: the extremal RN black hole in the near horizon limit becomes $AdS_2 \times R^2$, so if the effective mass of $\psi$ derived from (2.20) is below the $AdS_2$ BF bound, the RN BH will be unstable below some critical temperature. So one condition for stability is demanding $m_{\text{eff}}^2 > -\frac{1}{4}$ where

$$m_{\text{eff}}^2 = \frac{m^2}{6} - \frac{q^2}{3} - g$$

This is equivalent to demanding the conformal dimension of the operator dual to $\psi$ in the $AdS_2$ region, $\delta_{\pm}$, are real, where

$$\delta_{\pm} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + m_{\text{eff}}^2}. \quad (2.22)$$

This condition is too weak however, and we would like to refine it. As we will see below, the stability condition $m_{\text{eff}}^2 > -\frac{1}{4}$ is sufficient for standard boundary conditions for the scalar $\alpha = 0$ ($\kappa = \infty$). However since alternative boundary conditions $\beta = 0$ ($\kappa = 0$) are weaker, the scalar field can still be unstable to forming hair despite the BF bound in $AdS_2$ being satisfied. This is because for any effective mass, there are always unstable modes in $AdS_2$. It is just that they are usually thrown out by the boundary conditions. With alternative boundary conditions in the asymptotic $AdS_4$ region, some of these unstable modes are allowed.

As further indication of the fact that alternative boundary conditions are more unstable, it was noticed in [13] that $T_c$ diverges as one approaches the unitarity bound $\Delta_- = 1/2$. As shown in Fig. 3, this divergence actually takes the form

$$T_c \sim \frac{\mu q}{(\Delta_- - 1/2)^{1/2}} \quad (2.23)$$

Interestingly, for neutral scalar fields there is no divergence and $T_c$ approaches a finite limit as $\Delta_- \to 1/2$.

We now want to include the effect of nonzero $\kappa$. Working at fixed $\mu$ the relevant scale invariant quantity that we will vary is $\kappa/\mu^{\Delta_+ - \Delta_-}$ (as well as $T/\mu$). It turns out to be easy to study $T_c$ as a function of $\kappa$ simply by changing the definition of “normalizable” \footnote{Amusingly this is a much simpler problem than the usual shooting problem. Rather than adjusting $T$ to find a static normalizable mode we can simply fix $T = T_c$, shoot to the boundary and read off $\kappa(T_c)$.} Increasing $\kappa$ always decreases $T_c$. If the mass and charge of the scalar field is such that extreme RN AdS is unstable with standard boundary conditions, then $T_c$ remains nonzero for all $\kappa$. However, if extreme RN AdS is stable with standard boundary conditions, then $T_c$ must vanish at a
finite value \( \kappa = \kappa_c \). Both cases are illustrated in Fig. 4. Since \( T_c \) can be arbitrarily large at \( \kappa = 0 \) and vanish at \( \kappa_c \), we see that \( \kappa \) is a very sensitive knob to adjust the critical temperature of the superconductor. The point \( \kappa = \kappa_c \) is the quantum critical point that we will study in the next section. Notice that as \( \kappa \) becomes large and negative in Fig. 4, \( \mu \) becomes less important, and both curves approach the scaling (2.15).

3 Quantum critical point

We now turn to a more precise discussion of the point \( \kappa = \kappa_c \) where the critical temperature \( T_c/\mu \to 0 \). Below we will outline the various requirements on bulk parameters to achieve this critical point. Note in particular we need not fine tune these bulk parameters. The coupling that we do tune \( \kappa/\mu^{\Delta_+ - \Delta_-} \) is a well defined boundary theory coupling.

3.1 The flow of double trace couplings and the 2 point function

It is useful to think of the extremal RN black hole as representing a flow from a \( CFT_{2+1} \) in the UV to a \( CFT_{0+1} \) in the IR. This flow is induced in the UV by turning on a source \( (\mu) \) for the charge density operator \( J' \). The IR CFT can be seen by taking a scaling limit of (2.18)
Figure 4: The critical temperature, in units of chemical potential, as a function of the UV double trace coupling $\kappa$ for fixed $\Delta_\perp = 1$ and $q = 1/2$. The top curve has $g = 0.2$ and has nonzero critical temperature for all $\kappa$. The lower curve has $g = -0.2$ and ends at a quantum critical point.

towards $r \to r_0$ at extremality.\footnote{This limit can be taken more carefully, keeping a finite but small $T$. See Appendix C. The scaling limit was discussed in \cite{11} and the discussion here follows that paper closely.} Rescale coordinates as:

$$
\hat{t} = \epsilon(r - r_*) \quad \hat{\tau} = \epsilon t \quad \hat{x} = r_* x
$$

where $r_*$ is the location of the horizon at extremality $r_* \equiv r_0|_{T=0} = \rho_* / \mu$ with $\rho_* \equiv \rho|_{T=0} = \mu^2 / \sqrt{12}$. Formally we can scale towards $r \to r_*$ by expanding in $\epsilon$ then setting $\epsilon = 1$. This yields

$$
ds_0^2 = \left( -6\hat{r}^2 d\hat{\tau}^2 + \frac{d\hat{x}^2}{6\hat{r}^2} \right) + (d\hat{x}^2 + d\hat{y}^2), \quad A_0 = \sqrt{12}\hat{r} d\hat{\tau}
$$

This is the classic $AdS_2 \times R^2$ geometry found in the IR of many extremal black hole solutions. Notice in particular that the scale $\mu$ has dropped out. Since this geometry is supposed to be dual to a scale invariant theory, this had to be the case. The only knowledge that this theory has of the scale $\mu$ is encoded in higher order irrelevant terms which we have dropped. For example keeping the next order terms in the expansion in $\epsilon$ one finds:

$$
ds^2 = ds_0^2 + \delta_h ds_1^2 + \ldots, \quad A = A_0 + \delta_h A_1 + \ldots
$$

where $ds_1^2$ and $A_1$ have energy scaling dimension 1 under the $AdS_2$ scaling. The chemical potential now appears through $\delta_h = 1/\mu$. Note that since $\delta_h$ has dimensions of $-1$ this
represents an irrelevant coupling, which when turned on induces a flow in the UV to AdS$_4$. It is useful to think of $\delta_h$ as opening up the $R^2$ directions of the metric.

We now return to linearized fluctuations of $\psi$ in the extremal RN background. Our goal will be to compute the two point function of the order parameter at small frequencies and momenta $\omega, p \ll \mu$. We proceed heuristically, leaving details to Appendix C. Following [11] we do a matched asymptotic expansion where we split the geometry into two regions. In both regions we do a perturbative expansion in $\epsilon$ where we redefine

$$\omega \to \epsilon \omega, \quad \vec{p} \to \epsilon \vec{p}$$

so as to access small frequencies and momenta. In the inner region we rescale coordinates as in (3.1). In the outer region we leave the coordinates unscaled. A systematic expansion in both regions is defined in this way, matching occurs in an intermediate region connecting the two.

At zeroth order the outer region simply follows from setting $\omega = 0, T = 0, \vec{p} = 0$ in the full RN background. A general solution to the resulting equation can be characterized by the behavior at the AdS$_4$ boundary and at the extremal horizon,

$$\psi(r \to \infty) \to \alpha_0 r^{-\Delta_-} + \beta_0 r^{-\Delta_+}, \quad \psi(r \to r_*) \to \hat{\alpha}_0 (r - r_*)^{-\delta_-} + \hat{\beta}_0 (r - r_*)^{-\delta_+}$$

(3.5)

$$\begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = L \begin{pmatrix} \hat{\alpha}_0 \\ \hat{\beta}_0 \end{pmatrix} \equiv \begin{pmatrix} a^+ & a^- \\ b^+ & b^- \end{pmatrix} \begin{pmatrix} \hat{\alpha}_0 \\ \hat{\beta}_0 \end{pmatrix}$$

(3.6)

where $a^\pm$ and $b^\pm$ are constants (in units of $\mu$) and can only be computed numerically.\(^8\) If we impose linear boundary conditions (2.8) on the allowed fluctuations, then this maps into the following condition near $r_*$,

$$\frac{\hat{\beta}_0}{\hat{\alpha}_0} = \frac{(a^+)^2 (\kappa - \kappa_c)}{\det L - a^- a^+ (\kappa - \kappa_c)} \quad \text{where} \quad \kappa_c \equiv b^+/a^+ \quad \text{and} \quad \det L = \mu^2 \frac{\delta_+ - \delta_-}{\Delta_+ - \Delta_-}$$

(3.7)

We would like to make the identification of $\hat{\beta}_0/\hat{\alpha}_0$ above with the value of a double trace coupling in the IR AdS$_2$ CFT, $\kappa_{\text{IR}}$. We will only be interested in $\kappa$ close to $\kappa_c$ such that,

$$\kappa_{\text{IR}} = \frac{(a^+)^2}{\det L (\kappa - \kappa_c)}$$

(3.8)

It is then natural to identify $\kappa = \kappa_c$ as the critical point that we observed in the previous section. One main reason for this identification is the fact that for $\kappa < \kappa_c$ the double trace coupling in the IR is negative, and thus analogous to the discussion in Section 2, there will be a new state with lower free energy and scalar hair.

---

\(^8\) In [11] these same constants were called $a^{(0)}_{\pm}, b^{(0)}_{\pm}$. 

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We are now in a position to complete the computation of the two point function of the order parameter. The retarded Green’s function follows from imposing incoming boundary conditions at the extremal horizon in the inner region. Then to zeroth order in the $\epsilon$ expansion one finds that $\tilde{\beta}_0/\tilde{\alpha}_0 = \Sigma_R(\omega)$ where $\Sigma_R$ is the retarded $AdS_2$ Green’s function for fluctuations on the background (3.2). The Green’s function in the full CFT can be computed using any of the usual prescription [37, 38] generalized to include nonstandard boundary conditions. The result is up to overall normalization,

$$\chi_R(\omega, \vec{p}) = \frac{\alpha}{-\beta + \kappa \alpha} = \frac{Z + \ldots}{\kappa_{IR} - \Sigma_R(\omega) + \mathcal{X}(\omega, \vec{p}) + \ldots} Z = (a^+)^2 / \det L \quad (3.9)$$

where we have included higher order terms that can be important in $\mathcal{X}$. These higher order terms always come from perturbative corrections in the outer region and are thus real. In contrast, $\Sigma_R$ is in general complex. The ellipses above represent even higher order terms that we have dropped. We compute $\mathcal{X}$ in Appendix C. The result can be written as,

$$\mathcal{X}(\omega, \vec{p}) = c_p \vec{p}^2 - c_\omega \omega^2 - c_T \kappa cT + c_q q \left( -\omega + \frac{2\pi}{\sqrt{3}} T q \right) \quad (3.10)$$

where $c_i$ are constants in units of $\mu$. They have the following positivity constraints depending on the value of $\delta_-$. $c_p > 0, c_T > 0$ always, $c_\omega > 0$ for $\delta_- < -1/2$ and $c_q > 0$ for $\delta_- < 0$.

The $AdS_2$ Green’s function plays the role of the self energy in (3.9) and is given by

$$\Sigma_R(\omega, T = 0) = h e^{i\phi}(-i\omega)^{1-2\delta_-} \quad (3.11)$$

where $h$ is a real positive number, and $e^{i\phi}$ is a phase, the precise form of which does not matter. We can generalize the above discussion to finite but small $T$. At finite temperature $\Sigma_R$ takes the form of a nontrivial scaling function,

$$\Sigma_R(\omega, T) = \left( \frac{2\pi T}{3} \right)^{\delta_+ - \delta_-} \frac{\Gamma(\delta_+ - \delta_-) \Gamma(\delta_+ - i q / \sqrt{3}) \Gamma(\delta_+ + i q / \sqrt{3} - i \omega / (2\pi T))}{\Gamma(\delta_- - \delta_+) \Gamma(\delta_- - i q / \sqrt{3}) \Gamma(\delta_- + i q / \sqrt{3} - i \omega / (2\pi T))} \quad (3.12)$$

Importantly the $AdS_2$ Green’s function always satisfies the constraint,

$$\omega \text{Im}\Sigma_R(\omega, T) > 0 \quad (3.13)$$

which is necessarily true for any bosonic spectral density.

Generally speaking since the quantities computed in the IR $AdS_2$ geometry depend simply on two numbers $q$ and $\delta_-$ we will call these quantities “universal”. Since they are associated with a CFT this language seems appropriate. Other quantities that come from the outer region such as $a^\pm, b^\pm$ and the $c_i$ are “nonuniversal”, they can be computed only numerically.
3.2 Properties of the critical point

Given the two point function (3.9) we can now understand the physics close to the critical point. First we study the phase boundary in the \((\kappa, T)\) plane where the order parameter condenses, or equivalently, where the correlation length diverges.

We again look for a static normalizable mode at \(T = T_c\) which manifests itself as a zero frequency pole in (3.9). Since the instability kicks in first for the homogenous mode we can take \(\vec{p} = 0\). There are two cases depending on the IR conformal dimension \(\delta_-\). For \(0 < \delta_- < 1/2\) we can ignore \(X\) altogether, however for \(\delta_- < 0\) the analytic correction \(\propto T\) is larger than \(\Sigma_R \propto T^{1-2\delta_-}\). Thus we find for small \(\kappa_{IR}\),

\[
\begin{align*}
(0 < \delta_- < 1/2) & \quad T_c = k_2 (-\kappa_{IR})^{1/(1-2\delta_-)} \\
(\delta_- < 0) & \quad T_c = k_3 (-\kappa_{IR})
\end{align*}
\]

where,

\[
k_2 = \frac{3}{2\pi} \left( -\frac{\Gamma(\delta_- - \delta_+)}{\Gamma(\delta_+ - \delta_-)} \frac{\Gamma(\delta_- - iq/\sqrt{3})}{\Gamma(\delta_+ - iq/\sqrt{3})} \right)^{1/(1-2\delta_-)}
\]

\[
k_3 = \frac{1}{-c_T \kappa_c + c_q q^2 \pi/\sqrt{3}}
\]

Note that while \(k_2\) is a universal number (one to be compared with (2.15)) \(k_3\) is nonuniversal, depending on quantities \(c_T\) and \(c_q\) defined in the outer region. Indeed it is not clear the sign of \(k_3\) is fixed. Although \(c_q > 0\) and \(c_T > 0\) for the range of dimensions of interest, \(\kappa_c\) does not have a fixed sign. Generically it seems that for a critical point occurring with \(\delta_- < 0\) and \(q = 0\) then \(\kappa_c < 0\) so that \(k_3\) is fixed to be positive in such a case. However we do not know a proof of the positivity of \(k_3\).

To check the scaling relations (3.14) and (3.15), we have numerically computed the critical temperature. In Fig. 3 we plot \(T_c\) as a function of \(\kappa\) for various \(\delta_-\). The results are perfectly consistent with our scaling relations.

Now we move away from the phase boundary and study the disordered phase at zero temperature. Here we can examine the structure of the retarded Green’s function in the complex \(\omega\) plane. We will be particularly interested in the dispersion of the mode that becomes tachyonic on the ordered side. Again depending on the value of \(\delta_-\) the dispersion will differ, also now there will be a difference if the order parameter is charged or not. Examining (3.9) at \(T = 0\) for the charged case the dispersion of the pole in the complex
Figure 5: The critical temperature close to $\kappa_c$ for different values of $\delta_-$. These are for theories with $q = 0$, $\Delta_- = 1$, and from left to right: $\delta_- = 0.45, 0.30, 0.26, 0.15, 0, -0.15$. Note that when $\delta_- > 0$ the critical temperature vanishes with a power law, but for $\delta_- \leq 0$ it vanishes linearly.

plane is,

$$
(0 < \delta_- < 1/2) \quad \omega_* = ie^{-i\phi/(1-2\delta_-)} \left( \frac{c_p p^2 + \kappa_{\text{IR}}}{\hbar} \right)^{1/(1-2\delta_-)} \tag{3.17}
$$

$$
(\delta_- < 0; q \neq 0) \quad \omega_* = \omega_R(p) - \frac{\Sigma_R(\omega_R(p), T = 0)}{qc_q}, \quad \omega_R(p) = \frac{c_p p^2 + \kappa_{\text{IR}}}{qc_q} \tag{3.18}
$$

where in the last case the width of the quasiparticle scales like $\text{Im}\Sigma_R(\omega_R) \propto |\omega_R|^{1-2\delta_-}$, which is smaller than the energy $|\omega_R|$. In this case we have a genuine quasiparticle with a mass $\propto \kappa_{\text{IR}}$. In the first case (3.17) the width scales like the energy and thus the pole does not represent a genuine quasiparticle.

For the neutral case we again get the same behavior as in (3.17) however now for the range of conformal dimensions $-1/2 < \delta_- < 1/2$. For the remaining neutral case,

$$
(-1/2 < \delta_- < 0) \quad \omega_*^2 = \omega_R(p)^2 - \frac{\Sigma_R(\omega_R(p), T = 0)}{c_\omega}, \quad \omega_R^2(p) = \frac{c_p p^2 + \kappa_{\text{IR}}}{c_\omega} \tag{3.19}
$$

where there are now two quasiparticles at positive and negative energies $\pm \omega_R$. The width of these quasiparticles scales as $|\omega_R|^{-2\delta_-}$.

As expected, $\kappa_{\text{IR}}$ is playing the role of a mass (or an energy gap.) For all cases one can show via application of the constraint (3.13) that for $\kappa_{\text{IR}} > 0$ the quasi particle pole always
lies in the lower half plane and there is no instability. There is a “mass gap” $E_g$, for all cases which is roughly given by the closest approach of the pole to $\omega = 0$. For the less universal cases (3.18) and (3.19) one finds the expected results, $E_g \propto \kappa_{\text{IR}}$ and $\kappa_{\text{IR}}^{1/2}$ for $q \neq 0$ and $q = 0$ respectively. However for the more interesting “critical” case the gap scales as,

$$E_g \sim (\kappa_{\text{IR}})^{\frac{1}{2(2\epsilon_+ - 1)}}$$

(3.20)

The correlation length for all cases scales as $\xi \propto \kappa_{\text{IR}}^{-1/2}$.

When $\kappa_{\text{IR}} < 0$, application of the constraint (3.13) shows that the pole always lies in the upper half plane for momenta $p < \sqrt{-\kappa_{\text{IR}}/c_p}$, representing an instability.

Exactly at the critical point $\kappa_{\text{IR}} = 0$ we find a free gapless mode which disperses in the complex plane as

$$\omega \sim |\vec{p}|^z \text{ with } z = \frac{2}{1 - 2\delta_-}$$

(3.21)

for the “critical” case, and $z = 2$ for $q \neq 0$ and $\delta_- < 0$ and $z = 1$ for $q = 0$ and $\delta_- < -1/2$. We thus conclude that the physics of the critical point has a nontrivial dynamical critical exponent determined by the dimension of an operator $\delta_-$ in the IR $AdS_2$ CFT. This identification is consistent with the relationship $E_g \propto \xi^{-z}$ in all cases. Interestingly $z$ has a lower bound in this model, with $z > 2$ for the charged case and $z > 1$ for the neutral case.

To summarize, in the most interesting “critical case” where $0 < \delta_- < 1/2$ for $q \neq 0$ and $-1/2 < \delta_- < 1/2$ for $q = 0$ the two point function close to the critical point has the universal scaling form,

$$\chi_R = \frac{Z}{\kappa_{\text{IR}} + c_\delta \vec{p}^2 + T^{2/z} g(\omega/T)}$$

(3.22)

where $z = \frac{2}{1 - 2\delta_-}$ and $g(\omega/T)$ is a universal scaling function that follows from (3.12). Note that since the correlation function is analytic in $\vec{p}$ (the self energy is momentum independent) the critical point is ‘locally’ quantum critical [25]. This type of criticality goes beyond the usual Landau Ginzburg paradigm often applied to quantum critical points, due to the existence of the locally critical modes associated with $AdS_2$. These results are compatible with experimental measurements of the spin susceptibility in a heavy fermion compound $CeCu_{6-\delta}Au_{\delta}$ at criticality [39]. In order to compare the spin susceptibility to the two point function of the triplet staggered order parameter (which is effectively $\chi_R$), one must shift the momentum in (3.22) by the ordering vector associated to the anti-ferromagnetic order $\vec{p} \rightarrow \vec{p} - \vec{K}$. Very similar results were also found theoretically for the spin susceptibility in

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9 This is not a genuine gap, since there will always be gapless incoherent junk coming from the $AdS_2$ Green’s function. However it does represent a gap to the coherent part of the 2 point function, which is represented by the dispersing pole.
The most important feature for comparing to experiments was $\omega/T$ scaling of the susceptibility at the ordering vector $\vec{p} = \vec{K}$ and a nontrivial exponent $z \approx 2.7$ or $\delta_- \approx .13$. Indeed we capture both features here, although since our $\delta_-$ does not take a universal value, it is hard to make a prediction for this exponent without a real string embedding where $\delta_-$ will be fixed.

### 3.3 Renormalization group interpretation

We would like to now give an RG interpretation of the above results and in so doing try to understand what to expect of the zero temperature ordered phase when $\kappa_{\text{IR}} < 0$.

We have already discussed a major aspect of the RG flow: the extreme RN black hole represents a flow from one $2 + 1$ CFT in the UV to a $0 + 1$ CFT in the IR. We would like to now understand how this picture changes in the presence of the scalar $\psi$. For this purpose there is clearly a set of a distinct cases depending on the value of $\delta_-$ the conformal dimension of the operator dual to $\psi$ in the IR CFT. Firstly for $\delta_-$ complex the IR CFT will never be realized and there will be no critical point. We do not consider this case further here. There are two remaining cases $0 < \delta_- < 1/2$ and $\delta_- < 0$ which we turn to now.

#### $0 < \delta_- < 1/2$

This range of dimensions is in the “critical” range identified above where the 2 point function takes the more universal form $\Sigma_{\text{IR}}$ for both neutral and charged cases. Here we argue that this result is universally controlled by the $AdS_2$ theory supplemented by double trace deformations. The bulk field $\psi$ in the $AdS_2 \times R^2$ geometry is dual to a set of operators $\Psi_{\vec{p}}$ where $\vec{p}$ labels the momentum in the $R^2$ direction, which one should think of as a charge under the KK reduction down to $AdS_2$. $\Psi_{\vec{p}}$ can be viewed as the image of $O(t, \vec{p})$ under RG flow.

The operator dimension of $\Psi_{\vec{p}}$ in the $0 + 1$ dimensional CFT is given by $\delta_\pm + O(p^2)$, where in this range of dimensions we can take either value. The two point function of $\Psi_{\vec{p}}$ is then,

$$\Sigma_{\text{IR}}^\pm \propto \omega^{2\delta_\pm - 1} \quad (3.23)$$

where the previously defined $AdS_2$ Green’s function $\Sigma_{\text{IR}}^+$ is $\Sigma_{\text{IR}}^+ = \Sigma_{\text{IR}}$. If we take the dimension of $\Psi_{\vec{p}}$ to be $\delta_-$ such that we are working in alternative quantization then we can reproduce the result of $\Sigma_{\text{IR}}$ at $T = 0$ simply by including the following deformation of the $0 + 1$ CFT $\Sigma_{\text{IR}}$

$$\mathcal{S}_{0+1} \to S_{0+1} - \int dt \frac{d^2\vec{p}}{(2\pi)^2} (\kappa_{\text{IR}} + c_p \vec{p}^2) \Psi_{\vec{p}}^\dagger \Psi_{\vec{p}} \quad (3.24)$$
Note for example when the deformation is zero, $\kappa_{\text{IR}} + c_\bar{p} \bar{p}^2 = 0$, the two point function (3.22) scales as $\omega^{2\delta_+ - 1}$ consistent with $\Psi_{\bar{p}}$ having dimension $\delta_-$. At zero momentum there is a single relevant double trace coupling $\kappa_{\text{IR}}$ at the critical point, that does not explicitly break the symmetry. Just as in the $AdS_4$ case discussed in Section 2, a positive $\kappa_{\text{IR}}$ will induce a flow to standard quantization where the operators $\Psi_{\bar{p}}$ now have dimension $1 - \delta_+ = \delta_-$. On the other hand, a negative $\kappa_{\text{IR}}$ will induce an instability which will lead to a symmetry broken state. Since $c_\bar{p} > 0$ the zero momentum mode will go unstable first, so in the ordered phase the homogenous mode $\Psi_{\bar{p}=0}$ develops a vev.

The ordered state can then be studied using gravity with the bulk field $\psi$ turned on. For $\delta_- > 0$, $m_{\text{eff}}^2$ defined in (2.21) is negative which means that for the disordered phase the scalar $\psi$ is sitting at a maximum of an effective potential; in the ordered phase it will roll away from this maximum. The resulting geometry will tell us where the theory ends up in the deep IR. In fact there are many possibilities which will depend on the bulk functions $V(\psi), G(\psi)$ and $J(\psi)$. One possibility which we highlight in the next section is the theory flows to a different $AdS_2 \times R^2$ geometry, where the field $\psi$ sits at the minimum of a particular effective potential to be discussed later. In Section 4 we will outline some other possible examples of deep IR geometries. At this stage to be appropriately noncommittal we call this geometry $X$.

The above discussion can be understood within the context of the UV completion of $AdS_2$ (a fancy name for the RN black hole) where we identify the coefficient of the double trace operator $\kappa$ as the microscopic control parameter which allows us to probe the critical point. The critical value $\kappa_c$ is thus not special from the perspective of the UV theory. The blue region of Fig. 6 is a pictorial description of the RG flows represented by the (extremal) RN black hole phase of the theory. In the large-$N$ limit, this flow is trivial, only effecting the fluctuations of fields in the bulk through boundary conditions. When $\kappa < \kappa_c$, the double trace coupling runs negative in the IR theory, and the instability ensues. The full $AdS_4$ theory then allows us to view the end point of this instability - the geometry in the extreme IR flows to a new attractive fixed point $X$. These flows are represented by the white region in Fig. 6.

Finally we make an argument as to how the order parameter behaves in the condensed phase. The argument is based on an RG analysis close to the critical fixed point and relies on the picture given in Fig. 6. We will confirm the result with a numerical calculation in the next section. Consider shooting from the theory $X$ in the extreme IR up to $AdS_4$. As suggested by Fig. 6 there will be two tuning parameters, of which only an appropriate scale

\footnote{A similar argument appeared in \cite{10} in the context of a BKT type transition, and related arguments appear in \cite{11}.}
Figure 6: Renormalization group flow interpretation of the quantum critical point (denoted
by the red dot) for $0 < \delta_- < 1/2$. This picture is heuristic, since the couplings are only
well defined close to each fixed point. The blue region is the disordered phase which we
have studied in this section. A distinction is made between the theory in alternative (alt)
and standard (std) quantizations. The white region denotes the ordered phase which we
will study more carefully in the next section. The theories denoted by $X$ and $Y$ depend on bulk
couplings. $Y$ is the end point of the flow induced by turning on a negative double trace
coupling in the $AdS_4$ theory and setting $\mu = 0$. These flows were considered in Section 2. $X$
is discussed in the text above.

Invariant ratio of the two will matter. As we tune this ratio we can shoot closer and closer
to the $AdS_2$ critical theory. The dashed line in Fig. 6 is an example of such a flow. In the
$AdS_2$ region, the metric and Maxwell field are given by (3.2) and the scalar field takes the
form

$$\psi(\hat{r}) \sim v \hat{r}^{-\delta_-} + w \hat{r}^{-\delta_+}$$

(3.25)

Here $v$ and $w$ are constrained by scale invariance:

$$w = -s_c^{IR} v^{\delta_+ / \delta_-}$$

(3.26)

where $s_c^{IR}$ is a number which can be computed numerically. It is analogous to the $s_c$ in (2.12).

We can now match (3.25) onto linear fluctuations in the extreme RN background which
we have considered above (what we called the outer region above, see for example (3.5)).

$$\hat{r} = \Lambda (r - r_*) , \quad \hat{t} = t / \Lambda , \quad \hat{\alpha}_0 = v \Lambda^{-\delta_-} , \quad \hat{\beta}_0 = w \Lambda^{-\delta_+}$$

(3.27)

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where we have allowed for an arbitrary rescaling \( \Lambda \) which will drop out in the end. Additionally we have for \( \kappa \) close to \( \kappa_c \),

\[
\alpha_0 = a^+ \hat{\alpha}_0, \quad \hat{\beta}_0 = \kappa_{\text{IR}} \hat{\alpha}_0
\]

(3.28)

Putting these results together we find,

\[
\langle O \rangle = \alpha \approx a^+ \left( -\frac{\kappa_{\text{IR}}}{s_{\text{IR}}^c} \right)^{\frac{s}{1-2\delta_-}},
\]

(3.29)

This is precisely an IR analog of the scaling we derived in section 2 for a negative double trace perturbation (2.14). Note we have made an assumption that \( s_{\text{IR}}^c \) is positive. This is a nontrivial assumption and will depend on the bulk couplings, as for example was the case for the analogous \( \text{AdS}_4 \) parameter \( s_c \) [13]. However whereas in the \( \text{AdS}_4 \) case, a negative \( s_c \) meant a somewhat sick theory, we suspect a negative \( s_{\text{IR}}^c \) will simply result in a first order transition. Since we are working with the assumption of a continuous transition we cannot say much about the case with \( s_{\text{IR}}^c \) negative.

More generally we can consider a black hole solution with a non-zero temperature which has \( \psi \) nonzero and comes close to \( \text{AdS}_2 \times \mathbb{R}^2 \). Again this is a nontrivial shooting problem. Now there is a one parameter family of solutions labeled by the temperature \( \hat{T} \) and asymptotic to (3.25). Scale invariance imposes the following relationship between \( v \) and \( w \),

\[
w = -S \left( \frac{\hat{T}}{v^{1/\delta_-}} \right) v^{\delta_+ / \delta_-}
\]

(3.30)

where \( S \) is a scaling function with \( S(0) = s_{\text{IR}}^c \). It can only be computed numerically. Going through the same matching procedure and rescaling \( \hat{T} = \Lambda T \) one finds the following scaling relation,

\[
-\kappa_{\text{IR}} = \left( \langle O \rangle / a^+ \right)^{\frac{1-2\delta_-}{s}} S \left( \frac{T}{\langle O \rangle / a^+} \right)^{1/\delta_-}
\]

(3.31)

This is of course consistent with the dimension of the IR operator which gets a vev

\[
\langle \Psi_{\vec{p}} \rangle = \hat{\alpha}_0 \hat{\alpha}^2 \langle \vec{p} \rangle
\]

(3.32)

being \( \delta_- \) in alternative quantization and \( \kappa_{\text{IR}} \) having dimensions \( 1 - 2\delta_- \). In the next section we will construct these flows on the condensed side. We will confirm amongst other things the result (3.29).

Actually this story is incomplete, for \( 0 < \delta_- < 1/4 \) the \( \text{AdS}_2 \) CFT has higher order multi-trace operators that are relevant (first \( \hat{\alpha}^4 \) then \( \hat{\alpha}^6 \) etc.). In this situation the scaling
arguments above fail, since the critical point is now multi-critical. Away from the multi-critical point, on the continuous side we now expect a mean field relationship \( \langle O \rangle \sim (-\kappa_{\text{IR}})^{1/2} \). The argument for this goes as follows. Nonlinear corrections \( \mathcal{O}(\psi^3) \) to the linear equation for \( \psi \) in the outer region produce additional terms in the matching \(^{11}(3.27)\). Most importantly

\[
\hat{\beta} = \hat{\beta}_0 + \hat{\beta}_{\text{NL}} + \ldots \approx \kappa_{\text{IR}} \hat{\alpha}_0 + u_{\text{IR}} \hat{\alpha}_0^3 = w \Lambda^{-\delta_-}, \quad \hat{\alpha} = \hat{\alpha}_0 + \hat{\alpha}_{\text{NL}} + \ldots \approx \hat{\alpha}_0 = v \Lambda^{-\delta_-}
\]  

(3.33)

where \( u_{\text{IR}} \) can be computed along the lines of Appendix C. It is obvious that we should interpret \( u_{\text{IR}} \) as the RG flow of the quadruple trace operator from \( AdS_4 \) to \( AdS_2 \) analogous to \( \kappa \to \kappa_{\text{IR}} \). Thus at zero temperature we find,

\[
\kappa_{\text{IR}} \hat{\alpha}_0 + u_{\text{IR}} \hat{\alpha}_0^3 = -s_{\text{IR}} c (\hat{\alpha}_0) (1 - \delta_-)/\delta_- \]  

(3.34)

For \( \delta_- > 1/4 \), the \( u_{\text{IR}} \) term is not important for small \( \hat{\alpha}_0 \) and we reproduce the result \(^{11}(3.29)\). However for \( \delta_- < 1/4 \) the non-analytic term in \( \hat{\alpha}_0 \) is less important and we get the mean field answer assuming that \( u_{\text{IR}} > 0 \),

\[
\langle O \rangle = a^+ (-\kappa_{\text{IR}}/u_{\text{IR}})^{1/2}
\]  

(3.35)

for \( u_{\text{IR}} < 0 \) we get a first order transition that we cannot say much about. These nonlinear corrections to the linear \( \psi \) equation come from potential terms, as well as from back-reaction on gravity. They are rather complicated, however they are most likely computable in the probe approximation introduced in \(^{19}\) for the neutral case. We leave their explicit computation to future work.

\( \delta_- < 0 \)

In this case, \( m_{\text{eff}}^2 \) defined in \(^{21}(2.21)\) is positive, so the IR \( AdS_2 \) with \( \psi = 0 \) is stable. However, one can still have phase transitions which turn on \( \psi \) at larger radius. In this case, we will see the critical exponents are not governed by the \( 0 + 1 \) CFT but take mean field values.

We can reproduce the more general 2 point function \(^{3}(3.9)\) for \( X \) given in \(^{3}(3.10)\) using the following semi-holographic action \(^{20}\),

\[
S = S_{0+1} + \int d^3x \left( |\partial_i \Phi|^2 + g_4 \left( i \Phi^\dagger \partial_i \Phi + h.c. \right) - c^2 |\Phi|^2 - \kappa_{\Phi} |\Phi|^2 - \frac{1}{2} u_{\Phi} |\Phi|^4 \right)
\]  

(3.36)

where

\[
S = S_{0+1} + \int dt \left( \frac{d^2 \vec{\rho}}{(2\pi)^2} \left( \Phi^\dagger \vec{\rho} \Phi + h.c. \right) \right)
\]  

(3.37)

\(^{11}\)We thank Kristan Jensen for drawing our attention to this possibility.
where $\Phi$ is a boundary field which we have coupled to the $AdS_2$ CFT operator $\Psi$ and $\Phi_\vec{p}$ is the spatial fourier transform of $\Phi$. To reproduce (3.9) the dimension of $\Psi$ must be $\delta_+ + O(\vec{p}^2)$. The two point function for $\Phi$ will then agree with $\chi_R$ with the following identifications,

$$|\eta|^2 = 1/c_\omega, \quad \kappa_\Phi = \kappa_{1R}/c_\omega, \quad q_\Phi = q c_q/c_\omega \quad c^2 = c_p/c_\omega, \quad \Phi \sim O(\sqrt{c_\omega/Z})$$

(3.38)

Note that $O$ will also have an overlap with $\Psi$ but this will lead to a subdominant correction to the two point function. We have also included a nonlinear interaction term $u_\Phi$ in (3.36) which should be generated in the flow from $AdS_4$ to $AdS_2$. We will assume that $u_\Phi > 0$, but this need not be the case.

In order to construct the ordered phase we first assume that we can treat $\eta$ perturbatively. We will show this is a consistent assumption. So for now we set $\eta = 0$ and work in the mean field approximation for $\Phi$. We do this because we are working in the classical gravity approximation. (The whole action above should be multiplied by $1/G_N$, and for small $G_N$ mean field applies.) Then for $\kappa_\Phi$ negative, $\Phi$ develops a vev:

$$\langle \Phi \rangle = \sqrt{-\kappa_\Phi/u_\Phi}$$

(3.39)

Turning on $\eta$, this will now act as a source for the homogenous mode of the operator $\Psi$, $\Psi_\vec{p}$,

$$S = S_{0+1} + \eta \sqrt{-\kappa_\Phi/u_\Phi} \int dt d^2\vec{p}/(2\pi)^2 \delta^2(\vec{p}) \left( \Psi_\vec{p}^\dagger + \text{h.c.} \right)$$

(3.40)

Since the dimension of $\Psi$ is irrelevant ($\delta_+ > 1$) this source will scale away in the IR. Thus we do not expect the vev of $\Phi$ to back-react on the $AdS_2$ CFT which is now a stable fixed point. This is consistent with the fact that for $\delta_- < 0$, the effective mass square $m_{\text{eff}}^2 > 0$, and the bulk field $\psi$ sits at a minimum in the disordered phase and has nowhere to go in the ordered phase. Thus the ordering is controlled by the boundary field $\Phi$.

It is now clear how to construct the ordered phase from gravity, we simply shoot from $AdS_2$ with a non-zero source term for the irrelevant operator $\Psi$. (One also needs to turn on the irrelevant coupling $\delta_\hbar$ discussed around (3.3).) This situation is depicted in Fig. 7. Again we can match this onto the perturbations of the extreme RN black hole with

$$\hat{\alpha} \approx \hat{\alpha}_0 = \eta \sqrt{-\kappa_\Phi/u_\Phi}, \quad \hat{\beta} \approx \hat{\beta}_0 + u_{\text{IR}} \hat{\alpha}_0^3 = 0$$

(3.41)

where we have included an important nonlinear correction as in (3.33). The condition $\hat{\beta} = 0$ (which is the requirement that $\psi$ not blow up in the IR) and $\hat{\beta}_0 = \kappa_{\text{IR}} \hat{\alpha}_0$ then allow us to match $u_\Phi$,

$$u_\Phi = u_{\text{IR}}/c_\omega^2$$

(3.42)
The vev then follows from the value of the source term $\hat{\alpha}_0$,

$$\langle \mathcal{O} \rangle = \alpha = (a^+/c_\omega)(-\kappa_{\mathrm{IR}}/u_\Phi)^{1/2} = a^+(-\kappa_{\mathrm{IR}}/u_{\mathrm{IR}})^{1/2}$$

(3.43)

which is same as (3.35).

Figure 7: RG interpretation for $\delta_- < 0$. In this case $m_{\mathrm{eff}}^2 > 0$, so from the gravity perspective $AdS_2$ is stable and the three fixed points on the right side are all the same. The condensation is controlled by an $AdS_2$ boundary field $\Phi$. The three cases are distinguished by whether $\Phi$ has a nonzero vev (bottom), is part of the IR dynamics but does not condense (middle), or is massive so it drops out of the IR theory (top).

3.4 Parametric dependence on bulk couplings

In this section we compile some numerical results relating to the critical point. The universal features discussed above depend on two parameters $q$ and $\delta_-$. However the location of the critical point itself $\kappa_c$ and other nonuniversal constants appearing in the dynamic susceptibility depend on three bulk parameters, $m^2, g, q$. In Fig. 8 and Fig. 9 we plot $\kappa_c$ through two different slices of this three parameter space, one with $q = 0$ and the other with $\Delta = 1$.

4 Constructing the ordered phase

We have left many of the details of constructing the ordered phase to this section. We will consider the full back-reaction of the condensing field $\psi$ on the metric so the problem is highly
Figure 8: Contour plots of $\kappa_c$ for $q = 0$. The lines $\delta_- = 0$ and $\kappa_c = 0$ are shown. The solid (blue) region represents the excluded BF bound in $AdS_2$ and $AdS_4$. Note that positive $\kappa_c$ tends to occur close to this region since then the theory is more unstable. For theories with $\kappa_c$ negative, introducing a chemical potential fails to destabilize the theory with alternative boundary conditions. So these theories are more stable.

Figure 9: Contour plots of $\kappa_c$ for $\Delta_- = 1$. The lines $\delta_- = 0$ and $\kappa_c = 0$ are shown. Clearly increasing $q$ tends to destabilize the theory since $\kappa_c$ increases. Decreasing the bulk coupling $g$ also tends to increase stability.

nonlinear and we must proceed numerically. We will focus on the zero temperature case where the basic problem will be to find the appropriate IR solution and to integrate outwards from there. Perturbing around the IR solution one finds various shooting parameters that take the form of irrelevant couplings. These couplings generate flows in the UV to an asymptotically $AdS_4$ solution, representing the UV fixed point of the theory. From here we can read off various data such as the vev of the order parameter, the double trace couplings $\kappa$ and the free energy.
As one tunes the irrelevant couplings in the IR we find that one can shoot closer and closer to the critical point that we identified in the previous section. As we will see this involves a flow whose geometry is described, for a large chunk of radial proper distance, by the $\text{AdS}_2 \times \mathbb{R}^2$ critical solution.

### 4.1 Ansatz for the background

We will use the following metric and field ansatz,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + h^2(r)d\vec{x}^2; \quad A = \phi(r)dt, \quad \psi = \psi(r)$$

(4.1)

This metric differs from the one used in [6] in which the radial coordinate was chosen to be $\sqrt{g_{xx}}$. The form (4.1) is more convenient since it allows $\text{AdS}_2 \times \mathbb{R}^2$ as a solution. Also numerically it is more convenient to work with this metric as we will demonstrate later.

The equations of motion which follow from the action (2.1) are given in Appendix A (A.5-A.8). For our ansatz (4.1) this reduces to the system of ODEs (A.10-A.13). There are three important reparameterizations that leave the metric form invariant, thus allowing us to generate new solutions from old solutions. They are:

\begin{align*}
\text{Shift} : & \quad r \rightarrow r + a. \quad (4.2) \\
\text{Conformal rescaling} : & \quad r \rightarrow \Lambda r, \quad (t, \vec{x}) \rightarrow (t, \vec{x})/\Lambda, \quad f \rightarrow \Lambda^2 f, \\
& \quad h \rightarrow \Lambda h, \quad \phi \rightarrow \Lambda \phi. \quad (4.3) \\
\text{Spatial rescaling} : & \quad \vec{x} \rightarrow \vec{x}/s, \quad h \rightarrow sh. \quad (4.4)
\end{align*}

We will fix these reparameterizations by demanding certain asymptotic boundary conditions, which we specify in the next subsection.

### 4.2 Asymptotic $\text{AdS}_4$ data and the free energy

The asymptotic UV fixed point will always be $\text{AdS}_4$. For $\Delta_+ < 3$ there are no irrelevant (single trace) operators within our truncation so $\text{AdS}_4$ is always an attractive fixed point in the UV. A complete expansion about $\text{AdS}_4$ can be systematically derived as discussed in Appendix D. The leading terms include

\begin{align*}
\psi &= \alpha r^{-\Delta_-} + \beta r^{-\Delta_+} + \ldots, \quad \phi = \mu - \rho r^{-1} + \ldots \\
h &= r \left(1 + 0 r^{-1} + \ldots\right), \quad fh^{-2} = \left(1 - (m_0/2) r^{-3} + \ldots\right)
\end{align*}

(4.5)

Additional terms in these expansions will be needed for some of the manipulations which follow, however due to their cumbersome form we leave these corrections to the appendix.
A general solution is parameterized by 5 constants $\alpha, \beta, \mu, \rho, m_0$. We have used the shift (4.2) to fix the sub-leading $r^{-1}$ terms in the metric to zero, and used the spatial rescaling in (4.4) to fix the normalization of $h$ and thus the spatial components of the boundary metric. The remaining conformal symmetry (4.3) can be used to fix one of these five constants. We will work with fixed chemical potential $\mu$, and often set $\mu = 1$. Numerical results of dimensionful quantities will be quoted in units of $\mu$.

We will need to compute the thermodynamic potential, which in our ensemble will be the grand potential $G$. We will go through this in some detail by computing the Euclidean on-shell action. While the results for general scalar boundary conditions are known using other methods the inclusion of a gauge field is new.

The grand potential is $G/T = S_E + S_{ct}$ where $S_E$ is the Euclidean action and $S_{ct}$ are boundary counter terms. These counter terms are required for a good variational problem, as well as to regulate divergences in $S_E$. We want to keep the metric fixed on the boundary (necessitating the Gibbons Hawking term), and the chemical potential fixed. For the scalar field $\psi$ we would like to require $\alpha, \beta$ to be constrained by $\beta = W'(\alpha)$. We will work initially with a fixed source $\beta$ in alternative quantization where $W(\alpha) = \beta \alpha$, and generalize later. The counter terms in the case of fixed $\beta$, $S^\beta_{ct}$, and $S_E$ are given in Appendix D, with the free energy density evaluating to:

$$g^\beta = -\frac{1}{2}m_0 + \frac{2}{3}\alpha \beta \Delta_+ (\Delta_+ - \Delta_-)$$

(4.6)

where we have defined $g = G/V$ and $V$ is the field theory volume.

We can also compute how the free energy varies as we move in solution space parameterized by changes in the five integration constants $\delta \alpha, \delta \beta, \delta \mu, \delta \rho, \delta m_0$. Varying the on shell action we derive an analog of the first law of thermodynamics,

$$\delta g^\beta = -s \delta T - \rho \delta \mu + 2(\Delta_+ - \Delta_-) \alpha \delta \beta$$

(4.7)

where $s$ is the entropy density. From (4.7) it is clear that $g^\beta$ is stationary at fixed $T, \mu, \beta$. To generalize the boundary conditions for $\psi$ we simply define

$$g_W = g^\beta - 2(\Delta_+ - \Delta_-)(\alpha \beta - W)$$

(4.8)

who’s variation is given by,

$$\delta g_W = -s \delta T - \rho \delta \mu - 2(\Delta_+ - \Delta_-) \delta \alpha (\beta - W'(\alpha))$$

(4.9)

Notice that $g_W$ is stationary if $\beta = W'(\alpha)$ (and temperature and chemical potential are fixed). The free energy $g_W$ is what we will be concerned with in this section, the preferred state of the system must have lowest $g_W$. 29
To summarize, in order to have a well defined variational problem and a finite on shell action, for boundary conditions determined by an arbitrary function $W(\alpha)$ with fixed chemical potential, we require the following boundary counter terms:

$$S^W_{ct} = S^\beta_{ct} - 2(\Delta_+ - \Delta_-) \int d^3x (\alpha\beta - W)$$

(4.10)

where $S^\beta_{ct}$ is given in Appendix D.

At zero temperature we will not be minimizing the energy $\epsilon_W$ but rather the appropriate free energy at fixed chemical potential $\epsilon_W - \rho\mu = g_W(T = 0)$. To do this we present one final manipulation which effectively fixes the constant $m_0$ in terms of the other integration constants. Consider the free energy at fixed $\alpha$ (and $T = 0$):

$$g_\alpha = g_\beta - 2(\Delta_+ - \Delta_-)\alpha\beta \rightarrow dg_\alpha = -\rho d\mu - 2(\Delta_+ - \Delta_-)\beta d\alpha$$

(4.11)

Using scale invariance we can write $g_\alpha = \mu^3w(\alpha/\mu^{\Delta_-})$. Plugging this into (4.11) we find,

$$w'(a) = -2(\Delta_+ - \Delta_-)b, \quad \rho/\mu^2 = 3w(a) - \Delta_-aw'(a)$$

(4.12)

where we have defined $a = \alpha/\mu^{\Delta_-}$ and $b = \beta/\mu^{\Delta_+}$. The first equation above tells us that $g_\alpha$ can be found as an integral,

$$g_\alpha - g_{RN} = -2(\Delta_+ - \Delta_-) \int_0^\alpha d\alpha' \beta(\alpha') \equiv 2(\Delta_+ - \Delta_-)W_0(\alpha)$$

(4.13)

where the integral should be understood as occurring at fixed $\mu$ (and $T = 0$). When we turn off the scalar field altogether $\alpha = 0, \beta = 0$ there is a unique solution: the extremal RN black hole. This fixes the integration constant in (4.13). Using (4.8) and (4.11), the full free energy density is

$$g_W - g_{RN} = 2(\Delta_+ - \Delta_-) [W_0(\alpha) + W(\alpha)]$$

(4.14)

Notice that the right hand side is identical to the off shell potential (2.11). The above discussion provides a thermodynamic derivation (generalized to nonzero chemical potential) of this result.

The second equation in (4.12) multiplied by $\mu^3$ becomes,

$$-\mu\rho = 3g_\alpha(T = 0) + 2\Delta_-(\Delta_+ - \Delta_-)\alpha\beta$$

(4.15)

Using (4.6) and (4.11) we then find that $\mu\rho = (3/2)m_0$. More generally with nonzero $T$, scale invariance imposes a thermodynamic relationship which can be derived analogous to the above manipulations. The result is,
\[ m_0 = \frac{2}{3} (\mu \rho + Ts) \rightarrow g_W = -\frac{1}{3} (Ts + \mu \rho) + 2(\Delta_+ - \Delta_-) \left( -\frac{1}{3} \Delta_- \alpha \beta + W(\alpha) \right) \] (4.16)

The first equation was derived in [42] using the Noether charge associated with the conformal rescaling. Practically speaking these last two equations are the most useful for reading off the free energy from numerical data. Since the \( m_0 \) coefficient is highly subleading it is more accurate to use \( \mu, T \) and \( s \) to find \( m_0 \) using the above equation.

### 4.3 IR fixed point and shooting

To begin with we would like to construct the zero temperature ground state. One reason to do this is so we can see how the order parameter \( \langle O \rangle \) behaves as we tune across the critical point.

To understand the ground state in the ordered phase we must find the final IR fixed point geometry. This should not be confused with the \( AdS_2 \) critical point governing the behavior seen in section 3, but instead is the extreme infrared limit of the ordered phase we called \( X \) in section 3. Depending on the specifics of one’s model, there are many possibilities for infrared fixed points, and we will not attempt to classify all possibilities here. The case of unbounded \( V(\psi) \) was studied in [44], which found various null singularities in the zero temperature limit. When the bulk potential \( V(\psi) \) is regulated, in the sense that it has a global minimum, the system is better behaved. The ground state depends on whether the field is charged or not. For the charged case, this problem was first considered in [42] where the possible IR solutions were \( AdS_4 \) solutions based about the new minimum or Lifshitz solutions.

In this section we will mostly focus on the neutral case \( q = 0 \). Since the field \( \psi \) does not carry any charge, there are no possible sources for the electric field lines. So there are one of two possible outcomes, either the horizon carries charge (the electric field is sourced on the horizon) or the field \( \psi \) runs to a point where the coupling function \( G(\psi) \) blows up.\(^{12}\)

Although this later case would be an interesting possibility we will leave this problem to future work. For now since we have a regulated potential and we are assuming that \( G(\psi) \) is smooth then this later case will not be realized. In the former case since the horizon is charged, at zero temperature, there must necessarily be an \( AdS_2 \times R^2 \) solution in the IR, with constant scalar field [19]. We will refer to this solution as \( \tilde{AdS}_2 \).

\(^{12}\)Another possibility, outlined recently in [45], is that a curvature singularity develops in the IR which sources the gauge field, one which is physically sensible [46]. Such behavior was seen for unbounded potentials at \( q = 0 \) in [44], though not thoroughly understood. This would be an interesting state to consider since then the ordered phase could be insulating (gapped to charged excitations.)
When \( q = 0 \) the coupling function \( J(\psi) \) does not play a role. The equations of motion simplify greatly, reducing to

\[
\begin{align*}
\phi' &= \frac{\rho}{h^2 G(\psi)}, \\
\psi'' + \left( \frac{f'}{f} + \frac{2h'}{h} \right) \psi' - \frac{V'(\psi)}{2f} + \frac{\rho^2 G'(\psi)}{4fh^4 G(\psi)^2} &= 0, \\
\frac{h''}{h} + h' \psi'^2/2 &= 0, \\
\frac{h'^2}{h} + \frac{h'f'}{f} - \frac{h\psi'^2}{2} + \frac{hV(\psi)}{2f} + \frac{\rho^2}{4fh^3 G(\psi)} &= 0, \\
f'' + \frac{2f'h'}{h} - \frac{\rho^2}{2h^4 G(\psi)} + V(\psi) &= 0. \tag{4.17}
\end{align*}
\]

where the last equation is a redundant dynamical equation. The first equation is easily integrated (with \( \phi = 0 \) at the horizon) after a solution to the other equations is obtained. A solution to (4.17) with constant scalar field and \( AdS_2 \times \mathbb{R}^2 \) metric is easily found with the ansatz

\[
f = f_0 r^2, \quad h = h_0, \quad \psi = \psi_0. \tag{4.18}
\]

The equations of motion reduce to

\[
f_0 = -V(\psi_0), \quad \frac{\partial}{\partial \psi_0} \left( G(\psi_0) V(\psi_0) \right) = 0, \quad \rho^2 = -2h_0^4 G(\psi_0) V(\psi_0). \tag{4.19}
\]

Note that the field \( \psi \) sits at a minimum of an effective potential

\[
V_{\text{eff}}(\psi) = V(\psi) G(\psi). \tag{4.20}
\]

This is our IR \( \widetilde{AdS}_2 \) solution. To flow in the UV towards \( AdS_4 \) we must add irrelevant perturbations. Since \( \widetilde{AdS}_2 \) is stable, there are no relevant perturbations other than the one which leads to finite temperature.

The method of studying deformations of fixed points is standard perturbation theory. Starting with a known fixed point \( \Phi_0 = (\psi_0, f_0, h_0) \), consider an expansion

\[
\Phi = \Phi_0 + \delta \Phi_1 + \delta^2 \Phi_2 + \ldots \tag{4.21}
\]

We call the perturbation relevant if the expansion is convergent at large radius and divergent at small radius, and irrelevant if it is divergent at large radius but convergent at small radius\footnote{The technical definition is that if the scaling dimension of the deformation is greater than the space-time dimension of the CFT, it is irrelevant and classically scales away in the infrared. If it is less than the field theory space-time dimension it is relevant, and becomes important in the infrared.}. Schematically, the equations of motion will be of the form

\[
D^2 \Phi_1 = 0, \quad D^2 \Phi_2 = \Phi_1^2, \ldots \tag{4.22}
\]
One can carry out this procedure to arbitrary order to increase the precision of one’s numerics. Carrying this out for $\tilde{A}dS_2$ we find

$$h = h_0 + \epsilon (h_1 r) + \mathcal{O}(\epsilon^2),$$

$$\psi = \psi_0 + \epsilon \left( \tilde{\alpha} r^{-\tilde{\Delta} -} + \tilde{\beta} r^{-\tilde{\Delta} +} + c_0 h_1 r \right) + \mathcal{O}(\epsilon^2),$$

$$f = f_0 r^2 + \epsilon \left( a + br + c_1 h_1 r^3 + c_2 \tilde{\alpha} r^{2-\tilde{\Delta} -} + c_3 \tilde{\beta} r^{2-\tilde{\Delta} +} \right) + \mathcal{O}(\epsilon^2),$$

(4.23)

where

$$\tilde{\Delta}_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{h_0^4 V'_eff(\psi_0)}{\rho^2}},$$

(4.24)

and the constants $c_0$, $c_1$, $c_2$ and $c_3$ are unenlightening constants fixed by the equations of motion. We follow the method of identifying relevant and irrelevant deformations in [42] in terms of the $\tilde{A}dS_2$ scaling. Since $V''_{eff}(\psi_0) > 0$, $\tilde{\Delta}_+ > 1$, we see perturbing the scalar field is irrelevant, and so we will want to turn on the mode vanishing in the infrared, namely the $\tilde{\alpha}$ mode. Similarly changing $h$ is relevant, and we can turn on $h_1$. The deformations $a, b$ in $f$ correspond to turning on a finite temperature, which we do not wish to do. With this it is clear that $\tilde{A}dS_2$ is a totally attractive fixed point, and we therefore expect it to be the true ground state. To flow to $AdS_4$ at zero temperature we then only want to turn on $\tilde{\alpha}$ and $h_1$, since they vanish in the infrared. Thanks to the conformal symmetry we in fact only have one shooting parameter, $\tilde{\alpha}/h_1^{\tilde{\Delta} -}$, and we can use our scaling symmetries to work at fixed $\mu$.

Integrating (4.17) to large radius with this small perturbation at small radius we will find an asymptotically $AdS_4$ solution. Note that we have only presented the shooting method (4.23) to leading order in perturbation theory. Often (when $\tilde{\Delta}_+ \gg 1$) it is necessary to work to higher order to have trustworthy numerical results.

As we tune $\tilde{\alpha}/h_1^{\tilde{\Delta} -}$ we fill out a curve in the $\beta,\alpha$ plane. We give some examples in Fig. 10. The basic strategy for reading off the possible states for a given $W(\alpha)$ is to find the intersection of the boundary condition curve $\beta = W'(\alpha)$ with the $\beta(\alpha)$ curves. One can then read off the vev of the operator $\langle O \rangle = \alpha$ and the free energy using (4.16). For example, for a double trace deformation $W(\alpha) = (1/2) \kappa \alpha^2$, we look along lines $\beta = \kappa \alpha$. An example of the resulting $\langle O \rangle$ as a function of $\kappa$ is shown in Fig. 11.

4.4 Confirming the scaling relations close to the critical point

We now examine the above results close to the critical point. This point can be seen in Fig. 11 as the value of $\kappa = \kappa_c$ where $\langle O \rangle$ vanishes. As expected, this value of $\kappa_c$ agrees well with the calculation of where $T_c = 0$ in section 2. For $\kappa > \kappa_c$ there are no solutions with scalar
Figure 10: The left figure shows $\beta(\alpha)$ curves computed numerically as outlined above. The model used has $V(\psi) = -6 + \Delta(\Delta - 3)\psi^2 + \lambda\psi^4$ with $\Delta = 1$, $\lambda = 1$, and $G(\psi) = \text{sech}([-(2g)^{1/2}\psi])$. The curves from top to bottom represent three different choices of $-g = .085, .125, .140$. These values were chosen to give representative critical point conformal dimensions. The gray line shows an example double trace coupling $W' = \kappa\alpha$ for $\kappa = -.015$. The right figure shows the effective potential $W_0 + W$ for the same models when $\kappa = -.015$. Note that the minima occur at the same points as the intersection points in the left figure. Also note that the difference in free energies in the broken phase is negative.

Figure 11: This figure represents a redrawing of the left panel of Fig 10 using the relation $\kappa = \beta/\alpha$ and $\langle O \rangle = \alpha$. The critical point $\kappa_c$ is where $\langle O \rangle \to 0$. Interestingly these curves roughly interpolate between mean field behavior and a transition of infinite order (BKT transitions.)

hair, so the ground state is simply the extremal RN black hole. For $\kappa < \kappa_c$ the free energy of the ordered phase is smaller, so this is preferred.

First we study the metric functions of the ordered phase close to this point. The goal here
is to give a picture where it is clear that the bulk geometry comes close to an intermediate critical point. Fig. 12 shows $f/r^2$ and $\psi$ for flows that have $\kappa$ close to $\kappa_c$. Three separate regions are clearly visible in the figure. For small $r$, $\psi$ is nonzero and the solution approaches $\text{AdS}_2 \times R^2$. At intermediate $r$, $\psi = 0$ and the solution is approximately $\text{AdS}_2 \times R^2$. This is the critical region. At large $r$, we approach $\text{AdS}_4$.

Figure 12: Metric function $f/r^2$ (top curves) and the scalar field $\psi$ (bottom curves) for different values of $\kappa$ close to the critical point. The model is the same as in Fig. 10 with a fixed $g = -0.125$. The two important scales are clear, the chemical potential $\mu$ which represents a cutoff on the critical $\text{AdS}_2$ theory and $\Lambda = (-\kappa_{IR})^{1/(1-2\delta_-)}$ where $\delta_-$ is the critical IR dimension of operator dual to $\psi$.

We then use precision numerics to plot the order parameter and the free energy difference close to the critical point. These are shown in Fig. 13 for different values of the scaling dimension $\delta_-$ given in (2.22). This scaling dimension is distinct from $\tilde{\Delta}_-$, and its importance was made clear in Section 3. As Fig. 13 shows, the data fits well to the following scaling relations to within 5% error (this error could be reduced by going closer to the critical point):

$$\langle O \rangle \propto (\kappa_c - \kappa)^{\delta_-} \quad \text{and} \quad g_W - g_{RN} \propto (\kappa_c - \kappa)^{\delta_-}$$

(4.25)
The former was predicted in (3.29), and the latter can be derived from the former using (4.13).

Finally one point we would like to emphasize is that at the end of the day the details of bulk coupling functions $V, G, J$ and the IR geometry $X$ will not be important for understanding physics near the critical point. Although, as already mentioned, there are a

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14 Similar “3 shelf” structure was seen in [17] for a probe brane model. It would be interesting to understand better the critical theory involved in that case.
Figure 13: The left figure shows the order parameter close to the critical point for the three examples considered previously in Fig. 10 (with the order of the curves flipped.) For large negative $\kappa$ the data all follow the same curve determined by the UV conformal dimension $\Delta_-$. However for $\kappa \to \kappa_c$ the curves diverge depending on their critical conformal dimension, $\delta_- = .26, .37, .45$. The right figure shows the free energy difference.

few consistency conditions which depend on the bulk coupling functions and are required in order to realize this continuous second order transition. These are the conditions $s_c > 0$ defined in (2.12) and $s_c^{\text{IR}} > 0$ defined in (3.26). Also when it is important we require $u_c^{\text{IR}} > 0$ defined in (3.33). Although we have not attempted to systematically compute these, as we have shown above it is not hard to find models in which these conditions are satisfied.

4.5 Critical solution

The size of the intermediate $AdS_2$ region grows without bound as $\kappa \to \kappa_c$ from below. At the critical point, the asymptotic $AdS_4$ part of the solution completely decouples from the $AdS_2$ IR region. There are two limiting solutions: One is the standard extreme RN AdS black hole which keeps the asymptotic $AdS_4$ region, and the other is a solution which asymptotically approaches $AdS_2 \times R^2$ in the UV.

Since we can compactify $R^2$ into $T^2$, this seems to contradict earlier arguments that there are no nontrivial solutions with asymptotically $AdS_2$ (times compact) boundary conditions [48]. One way to see the potential problem is to consider a congruence of null geodesics at constant $\vec{x}$ in the metric (4.1). Their convergence is $c \propto \pm h'/h$. As long as $h$ is not constant everywhere, either the left or right moving geodesics will have $c > 0$ somewhere. However, the Raychaudhuri equation and the null energy condition imply that if $c > 0$ at one point, it must diverge in finite affine parameter causing a spacetime singularity. The resolution is
that our limiting solution does have a singularity, but it is inside the Poincare horizon of the \( \text{AdS}_2 \) IR region. In fact, its Penrose diagram is similar to the extreme RN AdS solution. In short, the obstruction [48] applies only to solutions which are asymptotically \( \text{AdS}_2 \) in both the left and right asymptotic regions, and our critical solution has only one asymptotic region.

We have focussed on a neutral scalar field in this section for simplicity. If \( q \neq 0 \), then the natural IR geometry is a new \( \text{AdS}_4 \) geometry with \( \psi \) sitting at the global minimum of the potential. In this case, the critical solution flows from \( \text{AdS}_2 \) in the UV to \( \text{AdS}_4 \) in the IR, just the opposite of the standard RN AdS case.

## 5 Discussion

In the growing literature on applications of gauge/gravity duality to condensed matter, one class of relevant operators has been largely ignored: multi-trace deformations. We have started to remedy the situation by studying various effects of these deformations. This includes spontaneous symmetry breaking and new quantum critical points. Below we first summarize the behavior near the quantum critical points, and then discuss various applications and generalizations of our results.

### 5.1 Summary of critical exponents

We have derived various scaling relations which hold near the quantum critical point which we now summarize. Let the operator dual to our bulk scalar, \( O \), have dimension \( \Delta_- \), and \( \kappa \) be the coefficient of a double trace perturbation as in (1.1). Then for \( \mu = 0 \), the critical point is \( \kappa = 0 \) and near this point (from (2.14) and (2.15)):

\[
\langle O \rangle \propto (-\kappa)^{\Delta_-/(3-2\Delta_-)}, \quad T_c \propto (-\kappa)^{1/(3-2\Delta_-)} \tag{5.1}
\]

For \( \mu \neq 0 \), the critical point is typically at a nonzero \( \kappa_c \). For \( \kappa \) close to \( \kappa_c \) the bulk solution has a large intermediate \( AdS_2 \times R^2 \) region. Let \( \delta_- \) be the dimension of the operator dual to the bulk scalar in the corresponding 0 + 1 dimensional CFT. There are a few cases depending on \( \delta_- \). For \( 1/4 < \delta_- < 1/2 \), the critical point can be viewed as turning on a double trace operator with negative coefficient in the 0 + 1 CFT. The exponents are given by (3.29) and (3.14) which are just the IR analogs of the ones above. Setting \( \kappa_{IR} \propto \kappa - \kappa_c \):

\[
\langle O \rangle \propto (-\kappa_{IR})^{\delta_-/(1-2\delta_-)}, \quad T_c \propto (-\kappa_{IR})^{1/(1-2\delta_-)} \tag{5.2}
\]
For $0 < \delta_- < 1/4$, there are relevant higher multi-trace deformations. If the phase transition remains second order, then (3.35) gives

$$\langle O \rangle \propto (-\kappa_{\text{IR}})^{1/2}, \quad T_c \propto (-\kappa_{\text{IR}})^{(1-2\delta_-)}$$

(5.3)

For $\delta_- < 0$, we have mean field behavior (from (3.43) and (3.15))

$$\langle O \rangle \propto (-\kappa_{\text{IR}})^{1/2}, \quad T_c \propto (-\kappa_{\text{IR}})$$

(5.4)

Note that the exponent of $\langle O \rangle$ is continuous at $\delta = 1/4$, and the exponent of $T_c$ is continuous at $\delta_- = 0$.

We have also found that at the critical point, $\kappa_{\text{IR}} = 0$, there is a free gapless mode which satisfies $\omega \sim |\vec{p}|^z$ where the dynamical critical exponent $z$ is given by

$$z = 2 \quad q \neq 0 \text{ and } \delta_- < 0$$  

(5.5)

$$z = 1 \quad q = 0 \text{ and } \delta_- < -1/2$$  

(5.6)

$$z = \frac{2}{1-2\delta_-} \quad \text{otherwise}$$  

(5.7)

### 5.2 Applications of our results

As we discussed in section two, one main application of our results (with $q \neq 0$) is to holographic superconductors. Since double trace deformations can break a $U(1)$ symmetry even with zero net charge density, they provide a new mechanism for constructing gravitational duals of superconductors. One appealing aspect of the new construction is that, despite translation invariance, the DC conductivity in the normal phase is finite. We also showed that adding a double trace deformation to the traditional construction of holographic superconductors introduces a sensitive knob for adjusting the critical temperature.

Another possible application is to neutral order parameters (neutral under the $U(1)$ charge density). As mentioned in the introduction these can be used to model the onset of anti-ferromagnetic order [19]. Interestingly we found examples where we could drive the critical temperature to zero, revealing a quantum critical point characterized by the disappearance of anti-ferromagnetic order. The nature of the critical point is rather mysterious, however it’s locally quantum nature discussed around (3.22) has appeared in previous theoretical studies of heavy fermion criticality [25, 26]. Indeed measurements [39] of the dynamic susceptibility of a certain heavy fermion material $CeCu_{6-x}Au_x$ close to criticality seem consistent with the form of the two point function that we derive. Optimistically our results based on $AdS_2 \times R^2$ might shed some light on the robustness of the large dimension limit used...
to justify the theoretical results of [25, 26]. Or turning this the other way, we might learn something about the mysterious CFT dual to $AdS_2 \times R^2$. Such possibilities were discussed recently in a related context [49]. The relationship of $AdS_2$ to local quantum criticality was discussed in [50].

We now would like to discuss some immediate generalizations.

5.3 Magnetic fields

Studying the theory when a magnetic field is turned on is often difficult, due to the formation of superconducting droplets at nonzero $q$. However, we can study the theory near the critical point following [1, 51] by studying linearized analysis around a dyonic black hole. The dyonic RN AdS black hole is a simple generalization of (2.18), which now we are writing in terms of the horizon radius $r_0$, the chemical potential $\mu = \rho / r_0$, and the magnetization $M = B / r_0$ (not to confused with the mass of the scalar field $m$)

$$\text{ds}^2 = -f dt^2 + r^2(dx^2 + dy^2) + \frac{dr^2}{f}, \quad A = \mu \left(1 - \frac{r_0}{r}\right) dt + Mr_0 x dy,$$

$$f = r^2 - \frac{m_0}{2r} + \left(\frac{\mu^2 + M^2}{4r^2}\right) r_0^2, \quad m_0 = \frac{(\mu^2 + M^2) r_0}{2} + 2r_0^3, \quad T = \frac{3r_0}{4\pi} - \frac{\mu^2 + M^2}{16\pi r_0}.$$

We can generalize our calculations of the critical temperature by studying linearized fluctuations, a simple generalization of [1, 51] [19].

For now let us restrict to $q = 0$, and study the effect of a nontrivial $G$. The important equation is the generalization of (2.20), with $\Psi = R(r)e^{iky - i\omega t}$, which leads to the wave equation

$$(r^2 f R')' + \left[\frac{(r\omega)^2}{f} + \frac{g(\mu^2 - M^2)r_0^2}{2r^2} - m^2 r^2\right] R = 0.$$  

(5.10)

This can be solved either numerically or by a matched asymptotic expansion. Since we are working at $q = 0$, turning on a magnetic field effectively shifts $g$ for our linearized fluctuations. If we restrict to zero temperature and frequency, we find that the effective $AdS_2$ mass $m_{eff}^2$ changes as a function of $M$,

$$m_{eff}^2 = \frac{m^2}{6} + g \left(\frac{M^2 - \mu^2}{M^2 + \mu^2}\right),$$

(5.11)

where again $\delta_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} + m_{eff}^2}$ is the infrared dimension. It is clear that turning on a magnetic field changes the infrared dimension of the operator, and can possibly change it.

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15See [52] for a detailed study of the transport properties of the dyonic black hole.
This defines a critical magnetization where \( m_{\text{eff}}^2 = -1/4 \),

\[
M_c = \mu \sqrt{\frac{12g - 2m^2 - 3}{12g + 2m^2 + 3}},
\]

(5.12)

where there is an infinite order holographic BKT transition\(^{17}\), just as that found in \([40, 19]\) and studied further in \([53, 54, 55]\) (for a field theoretic discussion see \([56]\).) The critical point at \( \kappa = \kappa_c \) turns into a second order phase boundary, which terminates at \( M_c \). See figure 14 for example phase diagrams.

Figure 14: Phase diagrams when turning on a magnetic field. On the left, when \( g \) is negative and \( q \) is zero, turning on a magnetic field helps destabilize the disordered phase. On the right, when \( q \) is nonzero and \( g \) is zero, turning on a magnetic field helps stabilize the disordered phase. The red dashed line is the holographic BKT transition at \( M = M_c \), an infinite order transition. The solid blue line is the non-mean field transition controlled by \( \kappa_{IR} \) changing sign, where \( \delta_- \lesssim 1/2 \). At the holographic BKT line the critical exponents diverge.

Along our \( \kappa_c \) transition line, moving towards the BKT transition, \( \delta_- \to 1/2 \) which implies the critical exponents of the transition \((5.2, 5.7)\) diverge. This must match onto the known holographic BKT behavior \([19]\), where

\[
T_c, \langle O \rangle \sim \exp \left[ -\frac{\pi}{\sqrt{-3/2 - 6m_{\text{eff}}^2}} \right].
\]

(5.13)

It would be interesting to understand the behavior of the order parameter two point function as we approach the holographic BKT transition, and how it matches on to the known behavior along the second order transition line. At nonzero \( q \), \((5.11)\) generalizes to

\[
m_{\text{eff}}^2 = \frac{m^2}{6} - \frac{q^2 \mu^2}{3(M^2 + \mu^2)} + \frac{qM}{\sqrt{3(M^2 + \mu^2)}} + g \left( \frac{M^2 - \mu^2}{M^2 + \mu^2} \right).
\]

(5.14)

\(^{16}\)For \( q \neq 0 \) this was pointed out in \([19]\), and we see that a nontrivial \( G \) can cause this to occur for \( q = 0 \) as well. The magnetic field at which this occurs when \( G = 1 \), \( q \neq 0 \) is the \( B_c \) found in \([6]\).

\(^{17}\)For a different type of quantum phase transition that occurs as a function of a magnetic field, see \([65]\).
5.4 Quantum corrections

We will now discuss the effect of quantum corrections governed by an expansion in the gravitational coupling $G_N$. These correspond to $1/N$ corrections in the field theory. We will be particularly interested in how robust our results for the critical exponents are to bulk quantum corrections.

We start by considering the disordered phase. Firstly it would be interesting to understand one loop corrections to the thermodynamics and transport that come from fluctuations of $\psi$. Methods developed in [57, 58, 59, 60, 12, 61] would be useful for this purpose. In the large $N$ limit, quite often bulk geometric contributions to thermodynamics and transport dominate since they are of order $1/G_N$, so 1 loop contributions which are of order 1 in the $G_N$ expansion will be suppressed. However the mode that becomes gapless at the critical point will give large contributions that may be isolated from the geometric contributions due to strong IR behavior. This philosophy was used in [57, 58, 60] and in [12] to extract thermodynamics (most notably quantum oscillations) and charge transport of holographic non-Fermi liquids.

It is also important to understand how these modes renormalize bulk couplings when running in loops. Although this would require a fairly complicated bulk one loop calculation, we can argue what the outcome should be, based on the low energy semi-holographic model that we wrote down in (3.36). Note that in the disordered phase this model reproduces the 2 point function $\chi_R$ for all $\delta_-$ real. Consider here the neutral case $q = 0$, then ignoring the coupling $\eta$ between the boundary mode $\Phi$ and the CFT operator $\Psi$ what we wrote done in (3.36) was simply a $\Phi^4$ theory in $2+1$ dimensions. Thus we expect large quantum corrections to mean field at low energies, since $u_\Phi$ should run to strong coupling (at the Wilson Fisher fixed point). Low energies should be $\omega \sim G_N$ in this case.

We believe the real situation when $\eta$ is non-zero will be slightly different since the modes in the 2 point function for $\Phi$ now have a different critical exponent $z = \max(1, 2/(1-2\delta_-))$. This changes the RG power counting of the non-linear coupling $\Phi^4$ effectively softening the running. One can express this in terms of an effective dimensionality $d_{\text{eff}} = 2 + z$, where for $d_{\text{eff}} > 4$ one is above the upper critical dimension of $\Phi^4$ theory and non-linear effects should not be important (except in the case that they are dangerously irrelevant, which here they are not.) Thus the scaling exponents we derived should not receive corrections as long as $z > 2$. Interestingly when $z < 2$ we should be able to see a Wilson Fisher type fixed point as an expansion in $\delta_-$, analogous to the classic $d-4$ expansion since $d_{\text{eff}} \approx 4 + 4\delta_-$ for small $\delta_- < 0$. This will then change the mean field predictions we found above, for example in

\footnote{This discussion was inspired by similar considerations in [26].}
We leave details for future investigation. This softening of nonlinear interactions occurs in a similar context for holographic non-fermi liquids, leading to suppression of the BCS instability. 

On the ordered side, of course the second order mean field transition that occurs along the line \( T = T_c(\kappa) \) will not survive corrections beyond mean field theory. In 2+1 dimensions at finite temperature fluctuations of a continuous order parameter destroy long range order. This is a consequence of the Coleman-Mermin-Wagner theorem studied in a holographic context in [61]. Ultimately on very long distance scales (\( \sim e^{\#}/G_N \)) this mean field transition should be replaced by a (conventional) BKT transition. It would be very interesting to understand how these corrections scale close to the critical point. At zero temperature since fluctuations in the temporal direction become important it is now possible to have continuous order. (One should not be fooled by the fact that the critical theory seems to be controlled by a 0+1 dimensional theory. The IR of the zero temperature ordered phase has nothing to do with this 0+1 dimensional theory, and besides from the bulk perspective the \( R^2 \) directions of \( AdS_2 \times R^2 \) are infinite and still allow for fluctuations.)

### 5.5 Lifshitz normal phase

One possible generalization of the critical point story is to replace the normal phase, which for us was the RN black hole, with a more general charged black hole solution, which is however still asymptotically \( AdS_4 \). There are several motivations for doing this, one is that this may solve the finite entropy problem at extremality, that plagues the RN black hole. In this vein one expects quantum corrections (orthogonal to the ones discussed above) to lift the finite degeneracy associated with this finite entropy. In so doing, these corrections must replace the \( AdS_2 \times R^2 \) critical solution with something else. This is exactly what happens when one studies the back reaction of fermions in the bulk [62], where the IR \( AdS_2 \times R^2 \) solution is replaced by a Lifshitz geometry [63] with a dynamical critical exponent \( z_G \sim \mathcal{O}(1/G_N) \) (we use a subscript \( G \) to denote a geometric exponent, to be contrasted with our \( z \).) For \( z_G \to \infty \) one recovers the \( AdS_2 \times R^2 \) geometry. So finite \( z_G \) is playing the role of a regulator and we expect this to be a general phenomenon. Note for example now the entropy scales as \( S \sim T^{2/z_G} \) at low temperatures [64].

Another motivation is simply to have a more general understanding of what holographic models can do at finite density. One simple generalization is to add a dilaton field \( \sigma \) which couples to the gauge field, considered first in the AdS/CMT context in [64].[19] We will consider this model here as a concrete example. In our action (2.1) one replaces \( F^2 \to e^{2\alpha \sigma} F^2 \)

[19]See [43] for further generalizations.
and adds a kinetic term for $\sigma$ with normalization $-2(\nabla \sigma)^2$. We will not add a potential for $\sigma$. There are many choices for how to couple $\sigma$ to $\psi$, we begin for simplicity by assuming no extra couplings between $\sigma$ and $\psi$ other than through the replacement above. However we will argue that other choices of couplings should produce similar results.

The normal phase has $\psi = 0$. The appropriate solution which is charged and asymptotically $AdS_4$ in the UV was identified in \[64\]. It has a Lifshitz like geometry in the IR, replacing the $AdS_2 \times R^2$ solution of the RN black hole. Roughly speaking the dilaton runs $\sigma \to \infty$ such that the coupling and the charge density redshift away in the IR. So there is no need for a charge on the horizon to source the electric field. The critical exponent is determined by the dilatonic coupling $z_G = 1 + 2/\alpha^2$. The IR scaling geometry at finite temperature is,

$$ds^2 = -\hat{f} dt^2 + \frac{d\hat{r}^2}{\hat{f}} + \hat{r}^{2/z_G} dx^2, \quad \hat{f} = \mathcal{N}\hat{r}^2 \left(1 - (\hat{r}/\hat{r}_T)^{1+2/z_G}\right)$$

$$\sigma = -\frac{\sqrt{z_G - 1}}{z_G} \log(\hat{r}/\mu), \quad A = \mathcal{M}\mu^{-2/z_G} \left(\hat{r}^{1+2/z_G - \hat{r}_T^{1+2/z_G}}\right) dt$$

where $\mathcal{N}, \mathcal{M}$ are numbers determined by $z$ and $\hat{r}_T = (2/3)\pi(1 + 1/z_G)T$. This solution should be compared to (C.2). Note that the solution is not precisely scale invariant due to the form of both the gauge field and the dilaton, to emphasize this we have included explicit factors of $\mu$.

In this background we can now repeat our construction of the critical point. We will only consider the disordered phase, where the 2 point function determines most of the physics. The linear fluctuations of $\psi$ on (5.15) have a scale invariant form for $\hat{r} \ll \mu$, with dimensions $[\omega] = z_G[\hat{p}] = [T] = [\hat{r}]$. This is a nontrivial statement because the background is not scale invariant, so in the limit $\hat{r} \ll \mu$ various couplings disappear from the equation for $\psi$. For example since the gauge potential $q\phi$ redshifts away compared to the energy $\omega$, the coupling $q$ will not play a role in IR physics. Including more general couplings between $\sigma$ and $\psi$ will result in different IR scaling limits; the only assumption one needs in order to extend these results, is that this limit exists (and is nontrivial.)

Imposing incoming boundary conditions on the solution to this scale invariant differential equation, for large $\hat{r}$ one finds a generalization of (C.3).

$$\psi(\hat{r}) \sim \left(\hat{r}^{-\delta_-} + \Sigma_R(\omega, T, \hat{p})\right) = \left(\hat{r}^{-\delta_-} + \hat{T}^{\delta_+ - \delta_-} g \frac{(\omega/T, \hat{p}^2/T^{2/z_G})}{\hat{r}^{1+2/z_G}}\right)$$

where $\Sigma_R$ is the IR retarded Green’s function, which now has general nonanalytic dependence

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20 One may also need to generalize the scaling to $[\omega] = z_G[\hat{p}] = n[T] = n[\hat{r}]$ for some $n$. 

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on $\vec{p}^2$. The critical conformal dimensions are given by,

$$\delta_{\pm} = \frac{d_{\text{eff}}^{G}}{2z_{G}} \pm \sqrt{m_{\text{eff}}^2 + \left(\frac{d_{\text{eff}}^{G}}{2z_{G}}\right)^2}, \quad m_{\text{eff}}^2 = \frac{d_{\text{eff}}^{G}}{z_{G}} \left(\frac{\Delta(\Delta - 3)}{6} \left(1 + \frac{1}{z_{G}}\right) - g \left(1 - \frac{1}{z_{G}}\right)\right)$$

(5.18)

where $d_{\text{eff}}^{G} = 2 + z_{G}$. Note in particular any dependence on $q$ has vanished. We do not however expect the expression for $m_{\text{eff}}$ to be universal when considering other couplings of $\psi$ to the dilaton. Matching to the outer region we find for the IR limit of the two point function,

$$\chi_{R} = \frac{Z}{\kappa_{\text{IR}} + c_{p}\vec{p}^2 - T^{2/z_{G}} g(\omega/T, \vec{p}^2/T^{2/z_{G}})}$$

(5.19)

where $z = 2/(\delta_{+} - \delta_{-})$. Note that this result can also be obtained semi-holographically $[20]$. Again there should be some matching between $\kappa_{\text{IR}}$ and $(\kappa - \kappa_{c})$ in the outer region. Importantly we have included an analytic in $\vec{p}^2$ correction which will also come from the outer region. This correction will be important if $z < z_{G}$ as we discuss now.

The coherent part of the two point function comes from the dispersing mode with $\vec{p}^2 \sim T^{2/z_{G}} \sim \omega^{2/z_{G}}$. Along this mode the second argument in the scaling function $g$ vanishes, $\vec{p}^2/T^{2/z_{G}} \rightarrow 0$. That is, we can ignore the momentum dependence in the scaling function $g$. Thus we are forced back to a locally quantum critical theory, where the 2 point function is effectively analytic in $\vec{p}$. The critical exponent $z$ is still determined by the conformal dimension of the operator dual to $\psi$ in the Lifshitz IR theory.

Of course this analyticity in momentum and the critical exponent $z$ will not persist away from the coherent peak, and the 2 point function will have signatures of the $z_{G}$ Lifshitz scaling for $\vec{p}^2 \sim \omega^{2/z_{G}}$. These signatures will be incoherent and subdominant. They are analogous to the gapless modes that are present in the $AdS_{2}$ case, and also present in the holographic non-Fermi liquids $[11]$. This makes sense in the context of the limit $z_{G} \rightarrow \infty$ where one recovers $AdS_{2}$.

For $z > z_{G}$ then the analytic $\vec{p}^2$ term is subdominant and the two point function will have strong nonanalyticity in both frequency and momentum. The critical point is then truly governed by the Lifshitz geometry.

So to summarize, for the case $z < z_{G}$ we see that our results for the $AdS_{2}$ critical point are robust under generalization to a Lifshitz IR scaling geometry. Further it is easy to see that the phase boundary is still controlled by the conformal dimension $T_{c} \sim (-\kappa_{\text{IR}})^{1/(\delta_{+} - \delta_{-})}$ and the order parameter turns on as $\langle \mathcal{O} \rangle \sim (-\kappa_{\text{IR}})^{\delta_{-}/(\delta_{+} - \delta_{-})}$ which is similar to what we found earlier, although now we have $\delta_{+} + \delta_{-} = d_{\text{eff}}^{G}/z_{G}$. A more thorough discussion is in order, but we leave this to future work.
Acknowledgements

It is a pleasure to thank Hong Liu, Don Marolf, and Joe Polchinski for discussions. TF
would also like to thank Kristan Jensen and Andreas Karch for collaboration on related
matters. This work was supported in part by the US National Science Foundation under
Grant No. PHY08-55415, Grant No. PHY05-51164 and the UCSB Physics Department.

A Equations of motion

A.1 Field equations

We begin with the action
\[ S = \int d^4x \sqrt{-g} \left( R - \frac{1}{4} G(\psi) F^2 - (\nabla \psi)^2 - J(\psi) (\nabla \theta - q A)^2 - V(\psi) \right), \] (A.1)
where we have decomposed a complex scalar \( \Psi = \psi e^{i\theta} \). We have normalized \( A \) such that
\[ G = 1 + g \psi^2 + \mathcal{O}(\psi^4), \] (A.2)
and periodicity of \( \theta \sim \theta + 2\pi \) implies
\[ J = \psi^2 + \mathcal{O}(\psi^4). \] (A.3)
Lastly we assume
\[ V = -6 + m^2 \psi^2 + \mathcal{O}(\psi^4), \] (A.4)
where we have chosen units where the AdS radius is unity. The wave equations for the scalar
are
\[ \nabla^2 \psi - \frac{1}{2} V'(\psi) - \frac{1}{8} G'(\psi) F^2 - \frac{1}{2} J'(\psi) (\nabla \theta - q A)^2 = 0, \] (A.5)
\[ \nabla_\mu [J(\psi)(\nabla^\mu \theta - q A^\mu)] = 0. \] (A.6)
Maxwell’s equation is
\[ \nabla_\mu [G(\psi) F^{\mu \nu}] + 2 q J(\psi)(\nabla^\nu \theta - q A^\nu) = 0. \] (A.7)
Einstein’s equation is
\[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} \left( R - \frac{1}{4} G(\psi) F^2 - (\nabla \psi)^2 - J(\psi) (\nabla \theta - q A)^2 - V(\psi) \right) \]
\[ - \frac{1}{2} G(\psi) F_{\mu \rho} F^\rho_{\nu} - \nabla_\mu \psi \nabla_\nu \psi - J(\psi)(\nabla_\mu \theta - q A_\mu)(\nabla_\nu \theta - q A_\nu) = 0. \] (A.8)
A.2 Our ansatz

When studying static black hole solutions, we make the ansatz

\[ ds^2 = -f(r)dt^2 + dr^2/f(r) + h(r)^2(dx^2 + dy^2), \]
\[ A = \phi(r)dt, \quad \psi = \psi(r), \quad \theta = 0. \]  

(A.9)

This leads to the equations of motion

\[ \psi'' + \left( \frac{f'}{f} + \frac{2h'}{h} \right) \psi' - \frac{V'\psi}{2f} + \frac{G'(\psi)\phi'^2}{4f} + \frac{q^2 \phi^2 J'(\psi)}{2f^2} = 0, \]  

(A.10)

\[ \phi'' + \left( \frac{2h'}{h} + \frac{G'(\psi)\psi'}{G(\psi)} \right) \phi' - \frac{2q^2 J(\psi)}{G(\psi)f} \phi = 0, \]  

(A.11)

\[ h'' + \frac{\psi'^2 h}{2} - \frac{q^2 J(\psi)\phi^2}{2f^2} h = 0, \]  

(A.12)

\[ \frac{h'^2}{h} + \frac{f'h'}{f} - \frac{h\psi'^2}{2} + \frac{hV(\psi)}{2f} + \frac{G(\psi)h\phi'^2}{4f} - \frac{q^2 hJ(\psi)\phi^2}{2f^2} = 0. \]  

(A.13)

There is a third component of Einstein’s equations which is a dynamical equation for \( f \) and can be derived from the above equations,

\[ f'' + \frac{2f'h'}{h} + V(\psi) - \frac{G(\psi)\phi'^2}{2} - \frac{2q^2 J(\psi)\phi^2}{f} = 0. \]  

(A.14)

The equations of motion are invariant under the following symmetries, which leave invariant the scalar field, the line element, and the Maxwell one-form: First, a shift symmetry,

\[ r \rightarrow r + a. \]  

(A.15)

Second, a conformal rescaling,

\[ r \rightarrow \Lambda r, \quad (t, x_i) \rightarrow (t, x_i)/\Lambda, \quad f \rightarrow \Lambda^2 f, \quad h \rightarrow \Lambda h, \quad \phi \rightarrow \Lambda \phi. \]  

(A.16)

Lastly, a spacial rescaling

\[ x_i \rightarrow x_i/s, \quad h \rightarrow sh. \]  

(A.17)

A.3 Calculating conductivity

To calculate conductivity in this background, we study linearized modes

\[ \delta A_x = a_x(r)e^{-i\omega t}, \quad \delta g_{tx} = g_{tx}(r)e^{-i\omega t}. \]  

(A.18)
These have the following equations of motion
\[
a''_x + \left( \frac{f'}{f} + \frac{G'\psi'}{G(\psi)} \right) a'_x + \left( \frac{\omega^2}{f^2} - \frac{2q^2J'G'}{fG(\psi)} \right) a_x + \frac{\phi'}{f} g'_{tx} - \frac{2h'\phi'}{fh} g_{tx} = 0, \\
g'_{tx} - \frac{2h'}{h} g_{tx} + G(\psi)\phi' a_x = 0.
\] (A.19)

We can eliminate \(g_{tx}\) and find
\[
a''_x + \left( \frac{f'}{f} + \frac{G'\psi'}{G(\psi)} \right) a'_x + \left( \frac{\omega^2}{f^2} - \frac{G(\psi)\phi'^2}{f} - \frac{2q^2J'}{fG(\psi)} \right) a_x = 0. \tag{A.20}
\]

There is now a factor of \(G(\psi)\) multiplying the Maxwell term in the action, but since \(\psi = \alpha/r^\Delta + \ldots\) and \(\Delta > 1/2\), \(G\) goes to one rapidly enough at the boundary that the analysis of [6] follows and we can read off the conductivity from the asymptotics of \(a_x\),
\[
a_x = a_x^{(0)} + a_x^{(1)}/r + \ldots, \sigma = -ia_x^{(1)}/\omega a_x^{(0)}. \tag{A.21}
\]

As in [44] we can rewrite this as Schrödinger’s equation. Using a new radial variable and rescaling the gauge field
\[
dz = \frac{dr}{f}, \quad b = \sqrt{G(\psi)}a_x,
\] (A.22)
we find that (A.20) reduces to
\[
-b''(z) + V_{Sch}(z)b(z) = \omega^2 b, \tag{A.23}
\]
where
\[
V(z) = f \left[ G(\psi)\phi'^2 + \frac{2q^2J(\psi)}{G(\psi)} + \frac{1}{\sqrt{G(\psi)}} \left[ \frac{fG'(\psi)\psi_r}{2\sqrt{G(\psi)}} \right] \right]. \tag{A.24}
\]

We can relate conductivity to the reflection amplitude. In the new coordinates, near the boundary at \(z = 0\), where \(z = -1/r + \ldots\), we assume \(b = e^{-i\omega z} + \mathcal{R}e^{i\omega z}\) and near the horizon at \(z = -\infty\) ingoing boundary conditions require \(b = \mathcal{T}e^{-i\omega z}\). We find
\[
\sigma = \frac{1 - \mathcal{R}}{1 + \mathcal{R}}. \tag{A.25}
\]

We once again see that the necessary condition to find a hard gap in the real conductivity is that \(V(z)\) not vanish on the horizon. In [44] it was proven that for \(J = \psi^2, G = 1\) the Schrödinger potential must vanish on the horizon. It is an interesting question as to whether one can evade this theorem with more general \(J\) and \(G\).
B Critical temperature at zero chemical potential

We can find the critical temperature by looking for static linearized scalar hair on AdS satisfying our linear boundary conditions. In the background of a \( d + 1 \) dimensional AdS-Schwarzschild black hole, the wave equation is

\[
\psi'' + \left( \frac{f'}{f} + \frac{d-1}{r} \right) \psi' - \frac{\Delta(\Delta - d)}{f} \psi + \left( \frac{\omega^2}{f^2} - \frac{k^2}{r^2 f} \right) \psi = 0, \tag{B.1}
\]

where \( f = r^2[1 - (r_0/r)^d] \) and \( T = dr_0/4\pi \). We have dropped the subscript and are working with \( \Delta = \Delta_- \). The static case \( k = \omega = 0 \) can be solved via hypergeometric functions. Writing \( z = r_0/r \),

\[
\psi = c_1 z^\Delta \, _2F_1[\Delta/d, \Delta/d, 2\Delta/d; z^d] + c_2(\Delta \leftrightarrow d - \Delta) \tag{B.2}
\]

which at large \( r \) falls off as

\[
\psi = c_1(r_0/r)^\Delta + c_2(r_0/r)^{d-\Delta} + \ldots \tag{B.3}
\]

and therefore satisfies the linear boundary condition

\[
\frac{\beta}{\alpha} = \kappa = \frac{c_2}{c_1} (4\pi T/d)^{d-2\Delta} \tag{B.4}
\]

Now recalling that \(_2F_1[a, b, a + b; y] \propto \log(1 - y)\) we must choose the appropriate linear combination to cancel the logarithmic term for regularity on the horizon (this logarithmic branch is coming from the \( \omega = 0 \) limit of ingoing and outgoing waves on the horizon.) The appropriate combination is

\[
\frac{c_2}{c_1} = \frac{\Gamma(2\Delta/d)\Gamma(1 - \Delta/d)^2}{\Gamma(2 - 2\Delta/d)\Gamma(\Delta/d)^2} \tag{B.5}
\]

This tells us that there is a second order phase transition as we heat the system up, and above the critical temperature

\[
T_c = \frac{d}{4\pi} \left( -\frac{\Gamma(\Delta/d)^2\Gamma(1 - 2\Delta/d)}{\Gamma(1 - \Delta/d)^2\Gamma(-1 + 2\Delta/d)} \right)^{1/(d-2\Delta)} (-\kappa)^{1/(d-2\Delta)} \tag{B.6}
\]

the system returns to the symmetry preserving state.
C More on the 2 point function

Here we give details of the computation of the two point function in the ordered phase that we left out from Section 3. Most importantly we include a discussion of finite but small temperature, and compute the quantity $\mathcal{X}$ given in (3.10).

The small frequency limit can be supplemented with a small $T$ limit, where $\omega/T$ is held fixed. To do this, as in (3.4), we redefine,

$$\omega \to \epsilon \omega, \quad T \to \epsilon T, \quad \vec{p} \to \epsilon \vec{p}$$

(C.1)

and proceed to expand in $\epsilon$ (setting $\epsilon = 1$ at the end.) The inner region now has the metric of the $AdS_2$ black hole:

$$ds_0^2 = -\hat{f} dt^2 + \frac{d\hat{r}^2}{\hat{f}} + d\vec{x}^2$$

$$\hat{f} = 6\hat{r} (\hat{r} - (2/3)\pi T), \quad A_0 = \sqrt{12}(\hat{r} - (2/3)\pi T)d\hat{t}$$

(C.2)

In this background we can compute the $AdS_2$ Green’s function for $\psi$, imposing incoming boundary conditions at the horizon. Asymptotically at the $AdS_2$ boundary one has,

$$\psi_0(\hat{r}) = \mathcal{N}(\hat{r}^{-\delta} + \Sigma_R(\omega, T)\hat{r}^{-\delta})$$

(C.3)

The answer is found in Appendix D Eq. (27) of [11]. It is reproduced here in (3.12) using our current conventions. Higher order corrections to the inner region will never be important, so we do not consider them.

In the outer region we expand $\psi(r) = \psi_0(r) + \epsilon \psi_1(r) + \epsilon^2 \psi_2(r) + \ldots$ Where:

$$D \psi_0 = 0, \quad D \psi_1 = X_1(\psi_0), \quad D \psi_2 = X_2(\psi_0, \psi_1) \ldots$$

(C.4)

Where at zeroth order we have,

$$D = \partial_r (r^2 f_0 \partial_r) - m^2 r^2 + \frac{g\rho_0^2}{2r^2} + q^2 r^2 \phi_0^2 / f_0, \quad \phi_0 = \mu - \frac{\rho_0}{r}, \quad f_0 = r^2 - \frac{\rho_0 \mu}{3r} + \frac{\rho_0^2}{4r^2}$$

(C.5)

where we recall that $\rho_0 = \mu^2/\sqrt{12}$. And for the next two orders the source terms are:

$$X_1(\psi_0) = -\frac{T}{\mu} \frac{2\pi}{\sqrt{3}} D(r \psi_0') + 2q \left( -\omega + \frac{2\pi}{\sqrt{3}} T q \right) \frac{\phi_0 r^2}{f_0} \psi_0$$

(C.6)

$$X_2(\psi_0, \psi_1) = \left( \tilde{p}^2 - \frac{\omega^2 r^2}{f_0} \right) \psi_0 + \mathcal{O}(\omega, T) \psi_1$$

(C.7)

where in $X_2$ we have ignored nonleading corrections. These corrections will be of order $q \omega^2, q \omega T, T^2$, which are sub-leading compared to $X_1$. Note that we have kept an $\omega^2$ term in
\[ X_2 \text{ since for the neutral case this is the leading correction in } \omega. \text{ Given this we can now treat } \psi_1 + \epsilon \psi_2 \approx \psi_L \text{ together (} L \text{ stands for leading corrections). So we now write,} \\
\begin{equation}
D \left( \psi_L + \frac{2\pi T}{\sqrt{3}\mu} r \psi'_0 \right) = X_L \psi_0, \quad X_L = 2q \left( -\omega + \frac{2\pi}{\sqrt{3}T} q \right) \frac{\phi_0 r^2}{f_0} + \epsilon \left( \beta^2 - \omega^2 r^2 \right) \tag{C.8} \end{equation}
\]

The horizon and boundary behavior of \( \psi_0 \) was given in (3.5). Since we can always add a homogenous solution to \( \psi_L \) we demand that at the horizon \( \psi_L \) has the following behavior

\[ \psi_L = B_2 (r - r_*)^{-\delta_+ - 2} + B_1 (r - r_*)^{-\delta_+ - 1} + 0 \cdot (r - r_*)^{-\delta_+} \]

\[ + A_2 (r - r_*)^{-\delta_- - 2} + A_1 (r - r_*)^{-\delta_- - 1} + 0 \cdot (r - r_*)^{-\delta_-} + \ldots \tag{C.9} \]

\[ \text{ where } A_{1,2} \text{ and } B_{1,2} \text{ are completely determined by } \psi_0 \text{ through the source terms in (C.4). The nontrivial requirement relates to the terms we have set to zero above. Given this constraint we can then match to the inner region via (C.3) and (3.5), we find } \hat{\beta}_0/\hat{\alpha}_0 = \Sigma_R \text{ with no contribution from } \psi_L \text{ (and here we have set } \epsilon = 1.) \text{ Note the diverging terms in (C.9) can be matched to lower order } 1/\hat{r} \text{ corrections to (C.3).} \]

Asymptotically at the boundary of \( AdS_4 \) we have,

\[ \psi_L = \alpha_L r^{-\Delta_-} + \beta_L r^{-\Delta_+} \tag{C.11} \]

and from this and (3.5) we can read off the 2 point function:

\[ \chi_R = \frac{\alpha}{-\beta + \kappa \alpha} = \frac{\alpha_0 + \epsilon \alpha_L + \ldots}{-\beta_0 + \kappa \alpha_0 + \epsilon (-\beta_L + \kappa \alpha_L) + \ldots} \tag{C.12} \]

Close to the critical point where \( \kappa \approx \kappa_c \text{ the leading correction comes only from } -\beta_L + \kappa_c \alpha_L \text{ which we intend to compute now. Consider the normalizable mode } \psi_{\kappa_c} \text{ at the critical point defined as:} \]

\[ D\psi_{\kappa_c} = 0, \quad \psi_{\kappa_c} = 1(r - r_0)^{-\delta_-} + 0(r - r_0)^{-\delta_+}, \quad \psi_{\kappa_c} = a^+(r^{-\Delta_-} + \kappa_c r^{-\Delta_+}) \tag{C.13} \]

where the last two equations are the boundary conditions at the horizon and \( AdS_4 \) boundary respectively. Then if we multiply the first equation in (C.8) by \( \psi_{\kappa_c} \) and integrate by parts twice we find,

\[ W \left[ \psi_{\kappa_c}; \psi_L + \frac{2\pi T}{\sqrt{3}\mu} r\psi'_0 \right]_{r_*}^{\infty} = \int_{r_*}^{\infty} \psi_{\kappa_c} X_L \psi_0 \tag{C.14} \]

where \( W[\psi_A; \psi_B] = f_0 r^2 (\psi_A \psi'_B - \psi_B \psi'_A) \) is the Wronskian. Also close to the critical point \( \Sigma_R \) is small so for the purposes of computing this correction we can set \( \hat{\beta}_0 = 0 \) such that \( \psi_0 = \hat{\alpha}_0 \psi_{\kappa_c}. \) Evaluating the Wronskian we find the desired result,

\[ \frac{(-\beta_L + \kappa_c \alpha_L)}{\hat{\alpha}_0} = -\frac{2\pi T}{\sqrt{3}\mu} a^+(\Delta_+ - \Delta_-) \kappa_c + \frac{1}{a^+(\Delta_+ - \Delta_-)} \int_{r_*}^{\infty} dr \psi_{\kappa_c} X_L \psi_{\kappa_c} \tag{C.15} \]
where the integral is understood to be regulated. Possible divergences come only from the horizon and the regulators actually come from the Wronskian in (C.14) evaluated as \( r \to r_* \), when one includes the diverging terms given in (C.9).

Finally the leading contribution is,

\[
-\beta_0 + \kappa \alpha_0 \over \beta_0 = a_+ (\kappa - \kappa_c) - \frac{\det L}{a_+} \Sigma_R
\]

(C.16)

Compiling all of these results and using (C.12) we arrive at the result quoted in (3.9) where the constants in X are given by,

\[
c_T = \frac{(a^+)^2}{\det L} \frac{2\pi}{\sqrt{3\mu}} (\Delta_+ - \Delta_-)
\]

(C.17)

\[
c_p = \langle g^{xx} \rangle, \quad c_\omega = \langle |g^{tt}| \rangle, \quad c_q = \langle A_t |g^{tt}| \rangle
\]

(C.18)

where the angle brackets define the following:

\[
\langle Y \rangle \equiv \frac{1}{\det L (\Delta_+ - \Delta_-)} \int_{r_*(\text{reg})}^{\infty} \sqrt{-g} \psi_{\kappa c} Y \psi_{\kappa c}
\]

(C.19)

and where the metric components above are all for the extremal black hole metric. Note that these integrals are not always divergent, depending on the value of \( \delta_- \). When they are not divergent since all the integrands are positive it follows that the constants in (C.17) are positive. This is always the case for \( c_p \), it is only the case for \( c_\omega \) when \( \delta_- < -1/2 \) and it is only the case for \( c_q \) when \( \delta_- < 0 \).

D Complete AdS\(_4\) expansion and boundary terms

We give here the asymptotic solution to (A.10 - A.13) with \( V = -6 + m^2 \psi^2 + (\lambda/2) \psi^4 \). The expansion will be complete up to and including \( O(r^{-3}) \) assuming that \( 3/2 < \Delta_+ < 3 \). To this order, it suffices to take \( G = 1 \) and \( J = \psi^2 \). The result is only a slight generalization of expressions that can be found in [66]. The generalization consists of allowing for a nonzero chemical potential (we also work in a different coordinate system.) These are then used to compute the on shell action and boundary terms that are needed in Section 4.2.
\[
\psi = \alpha r^{-\Delta_-} \left(1 - \frac{q^2 \mu^2}{2(2\Delta_- - 1)} r^{-2}\right) + \beta r^{-\Delta_+} \left(1 - \frac{q^2 \mu^2}{2(2\Delta_+ - 1)} r^{-2}\right) + \frac{\alpha^3 c_1}{(3\Delta_- - 4)} r^{-3\Delta_-} + \ldots \\
\phi = \mu - \rho r^{-1} + \frac{q^2 \mu^2 \alpha^2}{\Delta_-(2\Delta_- - 1)} r^{-2\Delta_-} + \ldots \\
h r^{-1} = 1 - \frac{\alpha \beta \Delta_- \Delta_+}{6} r^{-3} - \frac{\alpha^2 \Delta_-}{4(2\Delta_- - 1)} r^{-2\Delta_-} + \frac{\alpha^4 c_2}{(3\Delta_- - 4)} r^{-4\Delta_-} + \ldots , \\
f h^{-2} = 1 - (m_0/2) r^{-3} + \ldots 
\] (D.1)

where the two constants are:
\[c_1 = \frac{\lambda}{2\Delta_-} + \frac{\Delta_- (5\Delta_- - 3)}{4(2\Delta_- - 1)}, \quad c_2 = -\frac{3\lambda}{8(4\Delta_- - 1)} - \frac{\Delta_-^2 (26\Delta_- - 15)}{32(2\Delta_- - 1)(4\Delta_- - 1)}\] (D.2)

Following [6] the on shell Euclidean action can be shown to be a total derivative. The boundary terms give:
\[S_E = \beta V \left(-2 h h' f \bigg|_{r=\infty}\right)\] (D.3)
where \(V\) is the volume of space and \(\beta\) is the inverse temperature. The horizon does not give any contribution since \(f\) vanishes on the horizon.

Following [6] the boundary terms needed in Section 4 are (they are all located at the boundary of \(AdS_4\)):
\[S^3_{ct} = \int d^3 x \sqrt{-g_{\infty}} \left(-2 K + A_{\mu n_\nu} F^{\mu\nu} - 2 \psi n^\mu \partial_\mu \psi + \Lambda_B + m_B \psi^2 + \lambda_B \psi^4\right) \bigg|_{r=\infty}\] (D.4)
\[= \beta V h^2 \left(-f' - 4 h h^{-1} f + \phi' \phi - 2 f \psi' \psi + \sqrt{f} (\Lambda_B + m_B \psi^2 + \lambda_B \psi^4)\right) \bigg|_{r=\infty}\] (D.5)
where \(g_{\infty}\) is the metric on a fixed \(r\) slice as \(r \to \infty\). \(K\) is the trace of the extrinsic curvature and \(n^\mu\) is the outward pointing unit normal to the boundary. By demanding that the \(S_E + S^3_{ct}\) is finite one can show that the constants in (D.4) are,
\[\Lambda_B = 4, \quad m_B = -\Delta_-, \quad \lambda_B = -\frac{3(3\Delta_-^2 + 4\lambda)}{8(4\Delta_- - 3)}\] (D.6)

Note that the \(\psi^4\) term in (D.4) is only required when \(\Delta_- < 3/4\) exactly when \(O^4\) is relevant in alternative quantization. Plugging (D.1) into (D.3) and (D.4) one finds that the free energy is given by (4.6).
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