UNIVERSAL UPPER BOUND ON THE BLOWUP RATE OF NONLINEAR SCHRÖDINGER EQUATION WITH ROTATION

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Abstract. In this paper, we prove a universal upper bound on the blowup rate of a focusing nonlinear Schrödinger equation with an angular momentum under a trapping harmonic potential, assuming that the initial data is radially symmetric in the weighted Sobolev space. The nonlinearity is in the mass supercritical and energy subcritical regime. Numerical simulations are also presented.

1. Introduction

Consider the focusing nonlinear Schrödinger equation (NLS) with an angular momentum term in $\mathbb{R}^{1+n}$:

$$
\begin{cases}
  iu_t = -\Delta u + Vu - \lambda |u|^{p-1}u + L_A u \\
u(0, x) = u_0 \in H^1.
\end{cases}
$$

Here $u = u(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ denotes the wave function, $V(x) := \gamma^2 |x|^2$ ($\gamma > 0$) is a trapping harmonic potential that confines the movement of particles, and $\lambda$ is a positive constant indicating the self-interaction between particles is attractive. The nonlinearity has the exponent $1 \leq p < 2^* - 1$, where by convention $2^* := \frac{2n}{n-2}$ if $n \geq 3$; $\infty$ if $n = 1, 2$. The operator $L_A u := iA \cdot \nabla u$ is the angular momentum term, where $A = Mx$ with $M = (M_{j,k})_{1 \leq j,k \leq n}$ being an $n \times n$ real-valued skew-symmetric matrix, i.e., $M = -M^T$. It generates a rotation in $\mathbb{R}^n$ in the sense that $e^{-itL_A} f(x) = f(e^{iMx})$ for $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. The space $H^1 = H^{1,2}$ denotes the weighted Sobolev space

$$H^{1, r}(\mathbb{R}^n) := \{ f \in L^r(\mathbb{R}^n) : \nabla f, x f \in L^r(\mathbb{R}^n) \}$$

for $r \in (1, \infty)$, and the endowed norm is given by

$$\|f\|_{H^1, r} = \|\nabla f\|_r + \|xf\|_r + \|f\|_r,$$

where $\|\cdot\|_r := \|\cdot\|_{L^r}$ is the usual $L^r$-norm. The linear Hamiltonian $H_{A,V} := -\Delta + V + iA \cdot \nabla$ is essentially self-adjoint in $L^2$, whose eigenvalues are associated to the Landau levels as quantum numbers.

When $n = 3$, equation (1) is also known as Gross-Pitaevskii equation, which models rotating Bose-Einstein condensation (BEC) with attractive particle interactions in a dilute gaseous ultra-cold superfluid. The operator $L_A$ is usually denoted by $-\Omega \cdot L$, where $\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3$ is a given angular velocity vector and $L = -ix \wedge \nabla$. In this case the skew-symmetric matrix $M$ is equal to

$$
\begin{pmatrix}
  0 & -\Omega_3 & \Omega_2 \\
  \Omega_3 & 0 & -\Omega_1 \\
  -\Omega_2 & \Omega_1 & 0
\end{pmatrix}.
$$
Such system as given in (1), the rotational nonlinear Schrödinger equation (RNLS) describing rotating particles in a harmonic trap has acquired significance in connection with optics, plasma, quantized vortices, superfluids, spinor BEC in theoretical and experimental physics [1, 4, 5, 13, 18, 22]. Meanwhile, mathematical study of the solutions to equation (1) have been conducted in order to provide insight and rigorous understanding for the dynamical behaviors of such wave-matter. For $\lambda \in \mathbb{R}$ and $1 \leq p < 1 + \frac{4}{n - 2}$, the local well-posedness results of equation (1) were obtained in e.g. [3, 6, 17], see also [10, 11, 14, 27] for the treatment in a general magnetic setting. In the focusing case $\lambda > 0$ and $p \geq 1 + \frac{4}{n}$, there exist solutions that blowup in finite time [7, 8, 23, 25, 26].

Let $Q \in H^1$ be the unique positive, non-increasing and radial ground state solution of the elliptic equation
\[-\Delta Q + Q - Q^p = 0,
\]
where $H^1$ denotes the usual Sobolev space. In the mass-critical case $p = 1 + \frac{4}{n}$, the paper [6] showed that $\|Q\|_2$ serves as the sharp threshold for blowup and global existence for equation (1). Moreover, if $p = 1 + \frac{4}{n}$ and $\|u_0\|_2$ is slightly greater than $\|Q\|_2$, the paper [7] obtained the exact blowup rate $\|\nabla u(t)\|_2 = (2\pi)^{-1/2}\|\nabla Q\|_2 \sqrt{\log(\log(T - t))}$ as $t \to T = T_{\text{max}}$. The analogous results for the standard NLS were initially proven in [23] and [19], where $A = V = 0$. In [7] we apply the so-called $R$-transform method, which is a composite of the lens transform and a time-dependent rotation that allows to convert (1) into the standard NLS. We would like to mention that the case $A = 0, V = \gamma^2|x|^2$ was considered in [25, 29]. Also if the harmonic potential is repulsive, i.e., $V = -\gamma^2|x|^2$, there are similar blowup results for equation (1) without angular momentum, see e.g. [28].

The purpose of this article is to give a space-time universal upper bound on the blowup rate for the blowup solution to equation (1) with radial data in the mass-supercritical regime $p \in (1 + \frac{4}{n}, 1 + \frac{4}{n - 2})$. Our main result is stated as follows.

**Theorem 1.1.** Let $n \geq 3$ and $1 + \frac{4}{n} < p < 1 + \frac{4}{n - 2}$, or $n = 2$ and $3 < p < 5$. Let $u_0 \in \mathcal{X}^1$ be radially symmetric, and assume that the corresponding solution $u \in C((0, T), \mathcal{X}^1)$ blows up in finite time $T$. Then
\[
\int_t^T (T - s)\|\nabla u(s)\|_2^2 ds \leq C(T - t)^{-\frac{2(5 - p)}{p + 3(n - 1)p - 17}}.
\]

Theorem 1.1 is motivated by a similar result by Merle, Raphaël and Szefert [21], where they proved such an upper bound on the blowup rate for the standard NLS without potential or angular momentum. The proof of Theorem 1.1 mainly follows the idea in [21] but relies on a refined version of the localized virial identity (Lemma 3.1, Section 3) in the magnetic setting. Note that the $R$-transform introduced in [21] does not apply here for $p > 1 + \frac{4}{n}$. In Section 3 we shall give the proof of the main theorem. In Section 5 we include numerical figures to show the threshold of blowup for various cases of interest.

## 2. Preliminaries

In this section we recall the local well-posedness theory for equation (1) and a radial version of Gagliardo-Nirenberg inequality that we shall apply in the proof of our main theorem.
2.1. Local well-posedness of RNLS for $p \in [1, 1 + 4/(n - 2)]$. For $u_0 \in \mathcal{H}^1$, the local well-posedness of equation (1) was obtained as a special case in e.g., [11, 27] and [7]. The papers [11, 27] dealt with a general class of magnetic potentials and electric potentials where $A$ is sublinear and $V$ is subquadratic and essentially of positive sign. The case where $V$ is subquadratic of both signs, e.g., $V = \pm \sum_{j=1}^{n} \gamma_j^2 x_j^2$, $\gamma_j > 0$ were considered in [3] when $n = 2, 3$, and [7, 9] in higher dimensions.

Let $H_{A,V} = -\Delta + V + iA \cdot \nabla = -(\nabla - \frac{i}{2} A)^2 + V_c$, where $V_c(x) = V(x) - \frac{|A|^2}{4}$ and $\text{div} A = 0$. The proof for the local result relies on local in time dispersive estimates for the propagator $U(t) = e^{-itH_{A,V}}$ constructed in [23]. Alternatively, for $V = \pm \gamma^2 |x|^2$, this can also be done by means of

$$e^{-itH_{A,V}}(x, y) = \left(\frac{-\gamma}{2\pi i \sin(2\gamma t)}\right)^{\frac{1}{2}} e^{i \gamma^2 |x|^2 / (4t)} e^{-\gamma (\sin(2\gamma x) - 2\gamma x / (2\gamma t))},$$

the fundamental solution to $iu_t = H_{A,V} u$ if $V(x) = \gamma^2 |x|^2$; and replacing $\gamma \to i\gamma$ if $V(x) = -\gamma^2 |x|^2$. The above formula (4) can be obtained via the $\mathcal{R}$-transform, a type of pseudo-conformal transform in the rotational setting, see [7].

**Proposition 2.1.** For equation (1), we have the following known results on well-posedness and conservation laws. Let $r := p + 1$ and $q := \frac{4(p+1)}{n(p-1)}$.

(a) Well-posedness and blowup alternative:

(i) If $1 \leq p < 1 + \frac{4}{n}$, then equation (1) has an $\mathcal{H}^1$-bounded global solution $u \in C([0, T]; \mathcal{H}^1) \cap L^q_{\text{loc}}([0, T]; \mathcal{H}^{1,r})$.

(ii) If $1 + \frac{4}{n} \leq p < 1 + \frac{4}{n-2}$, then there exists $T = T_{\text{max}} > 0$ such that equation (1) has a unique maximal solution $u \in C([0, T), \mathcal{H}^1) \cap L^q_{\text{loc}}((0, T); \mathcal{H}^{1,r})$. If $T < \infty$, then $u$ blows up at $T$ with a lower bound

$$\|\nabla u(t)\|_2 \geq C(T - t)^{-\frac{n+2}{2p+2}}.$$  

(b) The followings are conserved on the maximal lifespan $[0, T)$:

(i) Mass: $M[u] = \int |u|^2$

(ii) Energy: $E[u] = \int \left( |\nabla u|^2 + V|u|^2 - \frac{2\lambda}{p+1} |u|^{p+1} + \pi L_A u \right)$

(iii) Angular momentum: $\ell_A[u] = \int \pi L_A u$.

**Proof.** Here we briefly outline the proof. In virtue of [24], the kernel representation for $U(t)$ is given by

$$U(t)f(x) = (2\pi i t)^{-n/2} \int e^{iS(t,x,y)} a(t, x, y)f(y)dy,$$

where $S(t, x, y)$ is real-valued in $C^\infty(I^*_\delta \times \mathbb{R}^{2n})$, $I^*_\delta := (-\delta, \delta) \setminus \{0\}$ for some positive constant $\delta$, and $a(t, x, y)$ is in $L^\infty \cap C^\infty(\mathbb{R} \times \mathbb{R}^{2n})$. Then, from (6) it follows the dispersive estimate for $0 \neq |t| < \delta$,

$$\|U(t)f\|_{\dot{H}^\infty} \lesssim \frac{1}{|t|^{n/2}} \|f\|_{L^1}.$$  

2.2. Radial Gagliardo-Nirenberg inequality. Let Lemma 2.2. of Gagliardo-Nirenberg inequality due to W. A. Strauss. According to [7, Proposition 4.5], if \( p > \frac{4}{n+2} \) the estimate (3) is sharp for equation (1), since the harmonic potential. For the standard NLS, such upper bound is sharp, which is assumed in Proposition 2.1. Special cases of (9) as \( (q, r, n) \neq (2, \infty, 2) \) and
\[
\frac{2}{q} + \frac{n}{r} = \frac{n}{2},
\]
with \( q' \) denoting the Hölder conjugate of \( q \). Hence the local in time existence of (1) holds. If \( p \in [1 + 4/n, 1 + 4/(n - 2)] \), the blowup alternative and the lower bound for blowup rate of (1) follow from standard argument as in [7]. □

Remark 1. The Strichartz estimates (8)-(9) generalize those obtained in [11] where \( V_\epsilon(x) \approx |x|^2, \beta > 0 \) as \( |x| \to \infty \). Here we allow \( V_\epsilon \) to be any quadratic function asymptotically \( V_\epsilon(x) \approx \sum \beta_j x_j x_j \) with \( \beta_j \in \mathbb{R} \). In the proof of [3]-[4], we directly study the action of \( U(t - s) \) in the weighted space \( H^{1, r} \) based on Lemma 3.1, an oscillatory integral operator (OIO) formula of Yajima in the magnetic setting. The OIO method was initially applied by Fujiiwa in treating electric potentials. Our approach allows to technically deal with the commuting issue between \( x, \nabla \) and \( U(t - s) \), and provides a treatment for general sublinear \( A \) and subquadratic \( V \) assumed in Proposition 2.1. Special cases of \( A \) and \( V \) for the RNLS were studied in the literature, see e.g., [3, 9, 10, 14, 17].

Let \( Q = Q_0 \) be the ground state solution of (2). In the \( L^2 \)-critical case \( p = 1 + 4/n \), from [6, 7] we know that \( \|Q_0\|_2 \) is the sharp threshold in the sense that:

(a) If \( \|u_0\|_2 < \|Q_0\|_2 \), then equation (1) has a unique global in time solution.
(b) For all \( c \geq \|Q_0\|_2 \), there exists \( u_0 \) with \( \|u_0\|_2 = c \) so that \( u \) is a finite time blowup solution of equation (1).

According to [7] Proposition 4.5], if \( p = 1+4/n \) and \( \|u_0\|_2 = \|Q_0\|_2 \), then all blowup solutions of (1) have the pseudo-conformal blowup rate
\[
\|\nabla u\|_2 = O((T - t)^{-1}), \quad \text{as } t \to T.
\]
The case \( p > 1 + 4/n \) are technically more challenging. As far as we know, there have not been results on the characterization for the blowup profile or blowup rate. Theorem 1.1 provides an upper bound for the rotational NLS (1) under a harmonic potential. For the standard NLS, such upper bound is sharp, which is shown by constructing a ring-blowup solution in [21]. However, we do not know if the estimate (3) is sharp for equation (1), since the \( R \) transform does not apply for the \( L^2 \)-supercritical case.

2.2. Radial Gagliardo-Nirenberg inequality. The following is a radial version of Gagliardo-Nirenberg inequality due to W. A. Strauss.

Lemma 2.2. Let \( u \in H^1 \) be a radial function. Then for \( R > 0 \), there holds true
\[
\|u\|_{L^\infty(\{|x| \geq R\})} \leq C \frac{\|\nabla u\|_2^{1/2} \|u\|_2^{1/2}}{R^{\frac{n}{2}}}.
\]
To prove Lemma 2.2, first note that when \( n = 1 \), the classical Gagliardo-Nirenberg inequality reads \( \|u\|_\infty \leq C \|u'\|_{L^2}^{1/2} \|u\|_2^{1/2} \). For general dimensions, since \( u \) is radial, we denote \( u(x) = v(|x|) = v(r) \) and note that

\[
\|u\|_2 \geq \|u\|_{L^2(|x| \geq R)} = C \left( \int_R^\infty |v(r)|^2 r^{n-1} dr \right)^{1/2} \geq CR^{n-1} \left( \int_R^\infty |v(r)|^2 dr \right)^{1/2} = CR^{\frac{n-1}{2}} \|v_R\|_2,
\]

where \( v_R = v\big|_{r \geq R} \). Similarly, we have \( \|\nabla u\|_2 \geq CR^{\frac{n-2}{2}} \|v_R'\|_2 \). Combining these with the above one-dimensional inequality we obtain

\[
\|u\|_{L^n(|x| \geq R)} = \|v_R\|_n \leq C \|v_R'\|_2^{1/2} \|v_R''\|_2^{1/2} \leq C \frac{\|\nabla u\|_2^{1/2} \|u\|_2^{1/2}}{R^{\frac{n-2}{2}}}.
\]

### 3. Localized virial identity

To prove Theorem 1.1, we derive certain localized virial identity associated to equation (1). This type of identities were shown in [20] in the case \( A = V = 0 \) and in [13, 15] for some general electromagnetic potentials. Here we present a direct proof for \( A = Mx \) (\( M \) skew-symmetric) and general \( V \), which is different than that in [13, 15]. Let \( C_0^\infty = C_0^\infty(\mathbb{R}^n) \) denote the space of \( C^\infty \) functions with compact support.

**Lemma 3.1** (Localized virial identity). Assume that \( u \in C([0, T), \mathcal{H}^1) \) is a solution to equation (1). Define \( J(t) := \int \varphi |u|^2 \) for any real-valued radial function \( \varphi \in C_0^\infty \).

Then

\[
J'(t) = 23 \int \pi \nabla \varphi \cdot \nabla u,
\]

and

\[
J''(t) = -\Delta^2 \varphi |u|^2 - \frac{2\lambda(p-1)}{p+1} \int \Delta \varphi |u|^{p+1} + 4 \int \left( \varphi^{''} - \varphi' \right) |x \cdot \nabla u|^2
\]

\[+ 4 \int \frac{x^2 - r^2}{r} |\nabla u|^2 - 2 \int \nabla \varphi \cdot \nabla V |u|^2.\]

**Proof.** Note that

\[
J'(t) = \int \varphi u_t \pi + \int \varphi u \pi_t = 2\Re \int \varphi \pi u_t
\]

\[
= 2\Re \int \varphi \pi (i \Delta u - i V u + i \lambda |u|^{p-1} u + A \cdot \nabla u) := 2(I_1 + I_2 + I_3 + I_4).
\]

The term \( I_1 \) is estimated as

\[
I_1 = \Re \left( i \int \varphi \pi \Delta u \right) = \Re \left( -i \int \varphi |\nabla u|^2 - i \int \pi \nabla \varphi \cdot \nabla u \right) = 3 \int \pi \nabla \varphi \cdot \nabla u.
\]

Obviously \( I_2 = \Re \left( -i \int \varphi V |u|^2 \right) = 0 \) and \( I_3 = \Re \left( i \lambda \int \varphi |u|^{p+1} \right) = 0 \). For \( I_4 \), we have

\[
I_4 = \Re \int \varphi \pi A \cdot \nabla u = \Re \left( - \int \nabla \varphi \cdot A |u|^2 - \int \varphi \nabla \pi \cdot A u - \int \varphi (\nabla \cdot A) |u|^2 \right)
\]

\[= - \int \nabla \varphi \cdot A |u|^2 - I_4 - \int \varphi \nabla \cdot A |u|^2.
\]
Since \( \varphi \) is radial and \( M \) is skew-symmetric, we know that (with \( r = |x| \))

\[
\nabla \varphi \cdot A = \varphi'(r) \frac{x}{r} \cdot A = \frac{\varphi'(r)}{r} x \cdot (Mx) = 0 \quad \text{and} \quad \nabla \cdot A = 0.
\]

So \( I_4 = -I_4 \) and this implies \( I_4 = 0 \). Hence (11) follows.

Differentiating (11) again, we have

\[
J''(t) = 2 \left( 3 \int \pi_t \nabla \varphi \cdot \nabla u + 3 \int \pi \nabla \varphi \cdot \nabla u_t \right) = 2 \left( 3 \int \pi_t \nabla \varphi \cdot \nabla u - 3 \int \nabla \cdot (\pi \nabla \varphi) u_t \right)
\]

\[
= 2 \left( -3 \int \Delta \varphi \pi u_t - 2 \int \nabla \varphi \cdot \nabla \pi u_t \right) = 2(-S - 2T).
\]

To estimate \( S \), first we write

\[
S = 3 \int \Delta \varphi \pi u_t = 3 \int \Delta \varphi \pi (i \Delta u - iV u + i\lambda |u|^{p-1} u + A \cdot \nabla u) = S_1 + S_2 + S_3 + S_4.
\]

Since

\[
S_1 = 3 \left( i \int \Delta \varphi \pi \Delta u \right)
\]

\[
= \int \Delta^2 \varphi |u|^2 + 3 \left( -i \int \Delta \varphi \Delta \pi u \right) - 2 \int \Delta \varphi \nabla u |u|^2 = \int \Delta^2 \varphi |u|^2 - S_1 - 2 \int \Delta \varphi \nabla u |u|^2,
\]

we have \( S_1 = \frac{1}{2} \int \Delta^2 \varphi |u|^2 - \int \Delta \varphi \nabla u |u|^2 \). Obviously, \( S_2 = - \int \Delta \varphi V |u|^2 \) and

\[
S_3 = \lambda \int \Delta \varphi |u|^{p+1}. \quad \text{In } S_4, \text{ since } \Delta \varphi \text{ is also radial, by (13) we note that}
\]

\[
\int \Delta \varphi \pi A \cdot \nabla u = - \int \nabla (\Delta \varphi) \cdot A |u|^2 - \int \Delta \varphi \nabla \pi \cdot A u - \int \Delta \varphi (\nabla \cdot A) |u|^2
\]

\[
\quad = - \int \Delta \varphi \pi A \cdot \nabla u,
\]

indicating \( \int \Delta \varphi \pi A \cdot \nabla u \) is imaginary. So \( S_4 = 3 \int \Delta \varphi \pi A \cdot \nabla u = -i \int \Delta \varphi \pi A \cdot \nabla u. \)

To estimate \( T \), first we write

\[
T = 3 \int \nabla \varphi \cdot \nabla (i \Delta u - iV u + i\lambda |u|^{p-1} u + A \cdot \nabla u) = T_1 + T_2 + T_3 + T_4.
\]

For \( T_1 \), one has

\[
T_1 = 3 \left( i \int \sum_{j,k} \varphi_{x_j x_k} \bar{u}_{x_j x_k} \right) = 3 \left( -i \int \sum_{j,k} \varphi_{x_j x_k} \bar{u}_{x_j x_k} - i \int \sum_{j,k} \varphi_{x_j x_k} \bar{u}_{x_j x_k} \right) = T_{1,1} + T_{1,2}.
\]

Since \( \varphi \) is radial, we have \( \varphi_{x_j} = \varphi'(r) \frac{x_j}{r} \) and \( \varphi_{x_j x_k} = \varphi''(r) \frac{x_k x_j}{r^2} + \varphi'(r) \frac{\delta_{j,k}}{r} - \varphi'(r) \frac{x_j x_k}{r^3} \), so

\[
T_{1,1} = - \int \frac{\varphi''}{r^2} |x \cdot \nabla u|^2 - \int \frac{\varphi'}{r} |\nabla u|^2 + \int \frac{\varphi'}{r^3} |x \cdot \nabla u|^2.
\]
Also,

\[ T_{1,2} = 3 \left( i \int \Delta \varphi |\nabla u|^2 + i \int \sum_{j,k} \varphi_{x_j} \delta_{x_k} u_{x_k x_j} \right) = \int \Delta \varphi |\nabla u|^2 - T_{1,2}, \]

which reveals \( T_{1,2} = \frac{1}{2} \int \Delta \varphi |\nabla u|^2. \) For \( T_2, \) one has

\[ T_2 = 3 \left( -i \int \nabla \varphi \cdot \nabla \mathbb{V} u \right) = \int \Delta \varphi V|u|^2 + \int \nabla \varphi \cdot \nabla V|u|^2 - T_2, \]

so \( T_2 = \frac{1}{2} \int \Delta \varphi V|u|^2 + \frac{1}{2} \int \nabla \varphi \cdot \nabla V|u|^2. \) For \( T_3, \) there is

\[ T_3 = 3 \left( i \lambda \int \nabla \varphi \cdot \nabla \mathbb{V} |u|^{p-1} u \right) = -\lambda \int \Delta \varphi |u|^{p+1} - p T_3, \]

and so \( T_3 = -\frac{\lambda}{p+1} \int \Delta \varphi |u|^{p+1}. \) For \( T_4, \) we have

\[ T_4 = 3 \int (\nabla \varphi \cdot \nabla \mathbb{V}) (A \cdot \nabla u) = 3 \left( -\int \mathbb{V} \Delta \varphi A \cdot \nabla u - \int \mathbb{V} \nabla \varphi \cdot (A \cdot \nabla u) \right) := T_{4,1} + T_{4,2}. \]

By (14) we obtain \( T_{4,1} = i \int \mathbb{V} \Delta \varphi A \cdot \nabla u. \) Also,

\[
T_{4,2} = 3 \left( \int \mathbb{V} \sum_j \varphi_{x_j} \left( \sum_{k,l} M_{k,l} \delta_{x_k} M_{x_k x_k} \right) \right)
= 3 \left( \int \mathbb{V} \sum_{j,k,l} \varphi_{x_j} M_{k,l} \delta_{x_k} u_{x_k} - \int \mathbb{V} \sum_{j,k,l} \varphi_{x_j} M_{k,l} \delta_{x_k} u_{x_k x_j} \right)
= 3 \left( \int \mathbb{V} \sum_{j,k} \varphi_{x_j} M_{k,j} u_{x_k} + \int \mathbb{V} \sum_{j,k} \varphi_{x_j} M_{k,j} u_{x_k x_j} \right.
\left. + \int \mathbb{V} \sum_{j,k,l} \varphi_{x_j} M_{k,l} \delta_{x_k} u_{x_j} \right)
:= T_{4,2,1} + T_{4,2,2} + T_{4,2,3} + T_{4,2,4}.
\]

Obviously \( T_{4,2,2} = -T_4, \) and the skew-symmetry of \( M \) implies \( T_{4,2,4} = 0. \) Note that

\[
T_{4,2,3} = 3 \left( \int \mathbb{V} \sum_{j,k,l} \varphi_{x_j} M_{k,l} \delta_{x_k} u_{x_k x_j} - \int \mathbb{V} \sum_{j,k,l} \varphi_{x_j} M_{k,l} \delta_{x_k} u_{x_j} \right)
= -3 \int A \cdot \nabla \varphi |\nabla u|^2 - T_{4,2,1} - 3 \int \mathbb{V} A \cdot \nabla \varphi \Delta u = -T_{4,2,1}.
\]

Hence \( T_{4,2} = -T_4 \) and so \( T_4 = \frac{i}{2} \int \mathbb{V} \Delta \varphi A \cdot \nabla u. \) Finally we obtain (12) by collecting all estimates on \( S's \) and \( T's. \) \( \square \)
4. Proof of the main theorem

Now we are ready to prove Theorem 1.1.

Proof. For a radial data $u_0$, let $u$ be a corresponding radial solution that blows up in finite time $T < \infty$. Then $x \cdot \nabla u = ru'$ and $|\nabla u| = |u'|$, and the localized virial identity \cite{12} can be written as, with $V = \gamma^2|x|^2$

\begin{equation}
J''(t) = -\int \Delta^2 \varphi |u|^2 - \frac{2\lambda(p-1)}{p+1} \int \Delta \varphi |u|^{p+1} + 4 \int \varphi'' |\nabla u|^2 - 4\gamma^2 \int x \cdot \nabla \varphi |u|^2.
\end{equation}

Choose a smooth radial function $\psi$ such that $\psi(x) = \frac{|x|^2}{2}$ if $|x| \leq 2$ and $\psi(x) = 0$ if $|x| \geq 3$. Pick a time $0 < \tau < T$ and a radius $0 < R = R(\tau) \ll 1$ (to be determined later). Let $\varphi(x) = R^2 \psi(\frac{x}{R})$. Then, with $r = |x|$,

$$
\nabla \varphi(x) = R\psi'(\frac{r}{R}) \frac{x}{r}, \quad \varphi''(r) = R^2 \psi''(\frac{r}{R}), \quad \Delta \varphi(x) = \Delta \psi(\frac{x}{R}), \quad \Delta^2 \varphi(x) = \frac{1}{R^2} \Delta^2 \psi(\frac{x}{R}),
$$

so

\begin{equation}
J''(t) = -\frac{1}{R^2} \int \Delta^2 \psi(\frac{x}{R})|u|^2 - \frac{2(p-1)}{p+1} \int \Delta \psi(\frac{x}{R}) |u|^{p+1} + 4 \int \psi''(\frac{r}{R}) |\nabla u|^2 - 4\gamma^2 R \int \psi'(\frac{r}{R}) r |u|^2
\end{equation}

\begin{equation}
= J_1 + J_2 + J_3 + J_4.
\end{equation}

Since $\Delta^2 \psi$ is bounded, and $\Delta^2 \psi(\frac{x}{R}) = 0$ if $|x| \leq 2R$ or $|x| \geq 3R$, we have $J_1 \leq \frac{C}{R^2} \int_{|x| \leq 3R} |u|^2$. Also, since $\Delta \psi$ is bounded, and $\Delta \psi(\frac{x}{R}) = n$ when $|x| \leq 2R$, there is

\begin{equation}
J_2 = -\int_{|x| \leq 2R} |u|^{p+1} - \frac{2\lambda(p-1)}{p+1} \int_{|x| > 2R} \Delta \psi(\frac{x}{R}) |u|^{p+1} = -\int_{|x| \leq 2R} |u|^{p+1} + \frac{2\lambda(p-1)}{p+1} \int_{|x| \geq 2R} |u|^{p+1} - \int_{|x| \geq 2R} \Delta \psi(\frac{x}{R}) |u|^{p+1}
\end{equation}

\begin{equation}
\leq -\int_{|x| \leq 2R} |u|^{p+1} + C \int_{|x| \geq 2R} |u|^{p+1}.
\end{equation}

By choosing $\psi$ such that $\psi'' \leq 1$, we have $J_3 \leq 4 \int |\nabla u|^2$. And last, since $\psi'(\frac{x}{R}) = \frac{x}{R}$ when $|x| \leq 2R$ and $\psi'(\frac{x}{R}) = 0$ when $|x| \geq 3R$, there is

\begin{equation}
J_4 = -\gamma^2 R \int_{|x| \leq 2R} \frac{r}{R} |u|^2 - 4\gamma^2 R \int_{2R < |x| \leq 3R} \psi'(\frac{r}{R}) r |u|^2
\end{equation}

\begin{equation}
\leq -4\gamma^2 R \int_{|x| \leq 2R} |x|^2 |u|^2 + CR \int_{2R < |x| \leq 3R} |x||u|^2 \leq CR \int_{2R < |x| \leq 3R} |x||u|^2.
\end{equation}

Collecting all these terms, we have

\begin{equation}
J''(t) \leq 4 \int |\nabla u|^2 - \frac{2n\lambda(p-1)}{p+1} \int |u|^{p+1}
\end{equation}

\begin{equation}
+ C \left( \frac{1}{R^2} \int_{|x| \leq 3R} |u|^2 + \int_{|x| \geq 2R} |u|^{p+1} + R \int_{2R < |x| \leq 3R} |x||u|^2 \right).
\end{equation}
Recall that the energy $E[u]$ is conserved, and $\ell_A[u] = 0$ since $u$ is radial and $L_A u = 0$, we obtain
\[
\int |u|^{p+1} = \frac{p+1}{2\lambda} \int |\nabla u|^2 + \frac{p+1}{2\lambda} \int V|u|^2 - \frac{p+1}{2\lambda} E[u_0],
\]
so
\[
4 \int |\nabla u|^2 - \frac{2n\lambda(p-1)}{p+1} \int |u|^{p+1} = n(p-1)E[u_0] - (n(p-1) - 4) \int |\nabla u|^2 - n(p-1) \int V|u|^2.
\]
This yields
\[
J''(t) \leq n(p-1)E[u_0] - (n(p-1) - 4) \int |\nabla u|^2 - n(p-1) \int V|u|^2 + C \left( \frac{1}{R^2} \int_{2R \leq |x| \leq 3R} |u|^2 + \int_{|x| \geq 2R} |u|^{p+1} \right).
\]
Since $R \ll 1$, we know $\frac{1}{R^2} \gg 3R^2$, so
\[
(n(p-1) - 4) \int |\nabla u|^2 + J''(t) \leq n(p-1)E[u_0] + C \left( \frac{1}{R^2} \int_{2R \leq |x| \leq 3R} |u|^2 + \int_{|x| \geq 2R} |u|^{p+1} \right).
\]
Also recall conservation of mass and $\frac{1}{R^2} \gg 1$, we have
\[
(n(p-1) - 4) \int |\nabla u|^2 + J''(t) \leq n(p-1)E[u_0] + C \left( \frac{1}{R^2} + \int_{|x| \geq 2R} |u|^{p+1} \right) \leq C \left( \frac{1}{R^2} + \int_{|x| \geq 2R} |u|^{p+1} \right).
\]
To control the last term in the above inequality, we apply Lemma 2.2 and again the conservation of mass to obtain
\[
\int_{|x| \geq 2R} |u|^{p+1} \leq \|u\|_{L^{p-1}_{\infty}(2R \leq |x| \leq 2R)}^{-1} \int_{|x| \geq 2R} |u|^2 \leq C \frac{\|\nabla u\|_{p-1}}{R^{\frac{n(p-1)}{p(-1)}}} \frac{C}{R^{\frac{2(n-1)(p-1)}{p}}}.
\]
where the last inequality is an application of Young’s inequality with $\frac{1}{p_1} + \frac{1}{p_2} = 1$.

By choosing $\delta > 0$ small enough, and noting that $\frac{2(n-1)(p-1)}{b-p} > 2$, we obtain
\[
\frac{n(p-1) - 4}{2} \|\nabla u\|_2^2 + J'(t) \leq C \frac{C(t-\tau)}{R^{\frac{2(n-1)(p-1)}{b-p}}}.
\]
Integrate the above inequality over $[\tau, t]$ to obtain
\[
\frac{n(p-1) - 4}{2} \int_\tau^t \|\nabla u(s)\|_2^2 + J'(t) \leq C\frac{(t-\tau)}{R^{\frac{2(n-1)(p-1)}{b-p}}} + J'(\tau).
\]
Recalling (11) and integrating the inequality again with respect to \(t\) over \([\tau, t_0]\), we have

\[
\frac{n(p-1)-4}{2} \int_{\tau}^{t_0} (t_0-s) \|\nabla u(s)\|^2_2 + \int \varphi|u(t_0)|^2 \\
\leq C \frac{(t_0-\tau)^2}{R^{2(n-1)(p-1)}} + 2(t_0-\tau) \int \nabla(\tau) \cdot \nabla u(\tau) + \int \varphi|u(t_0)|^2.
\]

Recall that \(\varphi(x) = R^2\psi(\frac{x}{R})\), so

\[
\int_{\tau}^{t_0} (t_0-s) \|\nabla u(s)\|^2_2 + \int \varphi|u(t_0)|^2 \\
\leq C \left( \frac{(t_0-\tau)^2}{R^{2(n-1)(p-1)}} + R(t_0-\tau) \left| \int \nabla(\tau) \cdot \nabla u(\tau) \right| + R^2 \int |u(t_0)|^2 \right)
\]

\[
\leq C \left( \frac{(t_0-\tau)^2}{R^{2(n-1)(p-1)}} + R(t_0-\tau) \left\| \nabla(\tau) \cdot \nabla u(\tau) \right\|_2 + R^2 \|u(t_0)\|_2 \right)
\]

\[
\leq C \left( \frac{(t_0-\tau)^2}{R^{2(n-1)(p-1)}} + R(t_0-\tau) \|\nabla u(\tau)\|_2 + R^2 \right).
\]

Letting \(t_0 \to T\) and applying Young’s inequality yield

\[
\int_{\tau}^{T} (T-s) \|\nabla u(s)\|^2_2 \\
\leq C \left( \frac{(T-\tau)^2}{R^{2(n-1)(p-1)}} + R(T-\tau) \|\nabla u(\tau)\|_2 + R^2 \right)
\]

\[
\leq C \left( \frac{(T-\tau)^2}{R^{2(n-1)(p-1)}} + R^2 \right) + (T-\tau)^2 \|\nabla u(\tau)\|_2^2.
\]

By setting \(\frac{(T-\tau)^2}{R^{2(n-1)(p-1)}} = R^2\), i.e., choosing \(R = (T-\tau)\frac{5-p}{p(n-1)(p-1)}\), we have

\[
\int_{\tau}^{T} (T-s) \|\nabla u(s)\|^2_2 \\
\leq C(T-\tau)^{\frac{2(5-p)}{p(n-1)(p-1)}} + (T-\tau)^2 \|\nabla u(\tau)\|_2^2.
\]

To solve this inequality, let \(g(\tau) = \int_{\tau}^{T} (T-s) \|\nabla u(s)\|^2_2\). Then the above inequality becomes

\[
g(\tau) \leq C(T-\tau)^{\frac{2(5-p)}{p(n-1)(p-1)}} - (T-\tau)g'(\tau),
\]

which is equivalent to

\[
\frac{d}{d\tau} \left( \frac{g(\tau)}{T-\tau} \right) \leq C(T-\tau)^{-\frac{2(n-1)(p-1)}{5-p(n-1)(p-1)}}.
\]

Integrating this with respect to \(\tau\) over \([0, t]\) yields \(g(t) \leq C(T-t)^{-\frac{2(n-1)(p-1)}{5-p(n-1)(p-1)}}\), the desired result in (3).

**Remark 2.** Theorem 1.1 is still valid if we replace the positive constant \(\lambda\) in equation (1) by a \(C^1\)-function \(\lambda(x)\) that satisfies the following three conditions:

(a) There exist \(\lambda_1, \lambda_2 > 0\) such that \(\lambda_1 \leq \lambda(x) \leq \lambda_2\);
(b) \(x \cdot \nabla \lambda \leq 0\);
(c) \(\nabla \lambda\) is bounded.
nonlinear Schrödinger equation \( \psi \) with given initial data increasing in both \( p \) where we note that the function \( \delta = \delta(p,n) = \frac{(n-1)(p-1)}{s-p+(n-1)(p-1)} \in \left( \frac{1}{2}, \frac{n-1}{2n-4} \right) \) is increasing in both \( p \) and \( n \), given \( p \in [1+4/n, 1+4/(n-2)] \). From [7] we know that for any initial data in \( H^1 \) one can derive a general lower bound for the collapse rate, namely, there exists \( C_{p,n} > 0 \) such that

\[
\| \nabla u(t) \|_2 \geq C(T-t)^{-\left(\frac{1}{p} - \frac{n-1}{2n-4}\right)}.
\]

In particular, if \( p = 1 + 4/n \), the estimate [3] is only valid for the lower bound \( (T-t)^{-1/2} \). Thus, comparing the mass-critical case, where the log-log law and pseudo-conformal blowup rate [10] can occur, the mass-supercritical case for larger data can be more subtle, see [7] Theorem 1.1] and [21].

5. Numerical results for mass-critical and mass-supercritical RNLS in 2D

In this section we show numerical simulations for the blowup of (1) with \( n = 2 \) and with initial data \( \psi_0 \) being a multiple of the ground state for the following nonlinear Schrödinger equation

\[
(16) \quad i\psi_t = -\frac{1}{2}\Delta \psi + \frac{1}{2}(\gamma_1^2 x^2 + \gamma_2^2 y^2)\psi - \lambda |\psi|^{p-1}\psi - i\Omega(y\partial_x - x\partial_y)\psi.
\]

Let \( Q = Q_{\Omega,V} \) be the ground state for (16) satisfying the associated Euler-Lagrange equation

\[
(17) \quad \omega Q = -\frac{1}{2}\Delta Q + \frac{1}{2}(\gamma_1^2 x^2 + \gamma_2^2 y^2)Q - \lambda |Q|^{p-1}Q - i\Omega(y\partial_x - x\partial_y)Q,
\]

where \( \omega \) is the chemical potential. The construction of the ground states can be found e.g., in [6] [12] if \( p \leq 1 + 4/n \). Here we use GPELab as introduced in [2] to do the computations and observe the blowup phenomenon for \( \psi_0 = C Q_{\Omega,V} \) with appropriate constant \( C \) for \( p = 3 \) and \( p = 6 \). Note that the case \( p = 6 \) is beyond the limit of exponents covered in Theorem [1.1]. For certain convenience from the software, we compute the solution \( \psi \) of equation (16) rather than (1) on the \( (x, y) \)-domain \([-3, 3] \times [-3, 3]\) in the plane. There is an obvious scaling relation between \( \psi \) and \( u \) of these two equations. From Subsection 2.1 we know that when \( p = 3 \), the mass of \( Q_{0,0} \) is the dichotomy that distinguishes the blow-up vs. global existence solutions. The main reason we use \( Q_{\Omega,V} \) in place of \( Q_{0,0} \) is that numerically the actual ground state \( Q_{\Omega,V} \) is easier to compute and save as a stable profile under a trapping potential.

(1) Isotropic case: \( \gamma_1 = \gamma_2 = 1, \lambda = 1, \Omega = 0.5 \). Let \( p = 3 \). We see in Figure 1 that the solution has energy concentration in short time and blows up with \( \psi_0 = 2.5 Q_{\Omega,V} \), but it shows stable smooth solution at the level \( \psi_0 = 2 Q_{\Omega,V} \). For \( p = 4 \), we observe in Figure 2 that using \( \psi_0 = 2 Q_{\Omega,V} \) yields a blowup solution; but there shows no blowup at \( 1.6 Q_{\Omega,V} \).
(a) $\psi_0 = 2.5 \ast Q_{\Omega,V}$ (max $|\psi|^2 \approx 1812$)  
(b) $\psi_0 = 2 \ast Q_{\Omega,V}$ (max $|\psi|^2 \approx 3.9$)

Figure 1. $|\psi|^2$ when $p = 3$, $(\gamma_1, \gamma_2) = (1, 1)$, $\Omega = 0.5$

(2) Anisotropic case: $\gamma_1 = 1$, $\gamma_2 = 2$, $\lambda = 1$, $\Omega = 0.5$. Let $p = 4$. We observe that the anisotropic harmonic potential may yield blowup at a lower level ground state. Figure 3 shows blowup when $\psi_0 = 1.8Q_{\Omega,V}$; while stable smooth solution at $\psi_0 = 1.5Q_{\Omega,V}$.

(2A) $\psi_0 = 2 \ast Q_{\Omega,V}$ (max $|\psi|^2 \approx 102$)  
(2B) $\psi_0 = 1.6 \ast Q_{\Omega,V}$ (max $|\psi|^2 \approx 1.63$).

Figure 2. $|\psi|^2$ when $p = 4$, $(\gamma_1, \gamma_2) = (1, 1)$, $\Omega = 0.5$

(2A) $\psi_0 = 1.8 \ast Q_{\Omega,V}$ (max $|\psi|^2 \approx 93$)  
(2B) $\psi_0 = 1.5 \ast Q_{\Omega,V}$ (max $|\psi|^2 \approx 1.85$)

Figure 3. $|\psi|^2$ when $p = 4$, $(\gamma_1, \gamma_2) = (1, 2)$, $\Omega = 0.5$
(3) If turning off the rotation, i.e., $\Omega = 0$, then Figure 4 shows that in the isotropic case $\gamma_1 = \gamma_2 = 1$, $p = 6$, $\lambda = 1$, then blowup threshold $\psi_0 = 1.565Q_{\Omega,V}$; and there exists a bounded solution in $H^1$ if $\psi_0 = 1.56Q_{\Omega,V}$. However, in the anisotropic case for $V$, $\gamma_1 = 1$, $\gamma_2 = 2$, the blowup threshold is at level $\psi_0 = 1.395Q_{\Omega,V}$; and there exists a bounded solution in $H^1$ if $\psi_0 = 1.39Q_{\Omega,V}$. The above results reveal that higher order exponent $p$ and anisotropic property for the potential contribute more to the wave collapse, which may make the system unstable at a lower level of mass. It is of interest to observe that in the presence of rotation (Figure 5), the threshold constants $C$ remain the same in both isotropic and anisotropic cases, although $|\psi|^2$, the energies and chemical potentials grow at larger magnitude.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{\hspace{0.5cm} (a) $(\gamma_1, \gamma_2) = (1, 1)$, $\psi_0 = 1.565 \ast Q_{\Omega,V}$ (b) $(\gamma_1, \gamma_2) = (1, 2)$, $\psi_0 = 1.395 \ast Q_{\Omega,V}$}
\end{figure}

Notice that if $p = 6$, then the behavior of wave-collapse is quite different than the case $p < 5$. The modulus square of $\psi(t,x)$ first forms growing singularity. Then it quickly reduces to normal level but with large energy and $\|\nabla \psi\|_2$ after collapsing time although it does not seem to admit proper self-similar profile of energy concentration.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{\hspace{0.5cm} (a) $(\gamma_1, \gamma_2) = (1, 1)$, $\psi_0 = 1.565 \ast Q_{\Omega,V}$ (b) $(\gamma_1, \gamma_2) = (1, 2)$, $\psi_0 = 1.395 \ast Q_{\Omega,V}$}
\end{figure}
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