DONALDSON-THOMAS THEORY AND RESOLUTIONS OF TORIC TRANSVERSE $A$-SINGULARITIES

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Abstract. We prove the crepant resolution conjecture for Donaldson-Thomas invariants of toric Calabi-Yau 3-orbifolds with transverse $A$-singularities.

1. Introduction

1.1. Summary of Results. Motivated by ideas in mirror symmetry, Ruan’s crepant resolution conjecture (CRC) roughly states that the Gromov-Witten (GW) invariants of a Calabi-Yau (CY) orbifold $Z$ should be related to those of a crepant resolution $\pi : W \to Z$. Due to the conjectural relationship between GW theory and Donaldson-Thomas (DT) theory in dimension 3, first formulated by Maulik–Nekrasov–Okounkov–Pandharipande [MNOP06], one might expect that a similar relationship holds for DT invariants of 3-folds. In particular, the GW CRC has an especially nice formulation when $Z$ satisfies the hard-Lefschetz condition, due to Bryan–Graber [BG09], and this led Bryan–Cadman–Young to conjecture a similar formulation of the DT CRC for hard-Lefschetz 3-orbifolds [BCY12]. Explicitly, they made the following conjecture.

Conjecture 1.1 (BCY12 Conjecture 1). If $\pi : W \to Z$ is a crepant resolution of a hard-Lefschetz 3-orbifold, then there is an explicit change of variables such that

$$DT(Z) = \frac{DT(W)}{DT_{\text{exc}}(W)}$$

where $DT(-)$ denotes the reduced, multi-regular DT potential and $DT_{\text{exc}}(W)$ is obtained by restricting the DT potential to curves supported on the exceptional locus of $\pi$ (cf. Section 2 for precise definitions).

The simplest type of hard-Lefschetz orbifold in dimension 3 occurs when the orbifold structure is cyclic and supported on isolated lines in $Z$, we call these transverse $A$-singularities. The main result of this paper is the following.

Main Theorem (Theorem 2.2). Conjecture 1.1 is true for toric CY 3-orbifolds with transverse $A$-singularities.

We refer the reader to Section 2.2 for an explicit description of the change of variables. We mention that Conjecture 1.1 has also been proved in the
projective case by Calabrese [Cal12]. However, since toric CYs are never projective, their proof does not apply here.

The objects of interest in this paper are inherently algebro-geometric. However, the methods we employ are combinatorial in nature. More specifically, Bryan–Cadman–Young [BCY12], generalizing Okounkov–Reshetikhin–Vafa [ORV06] and Maulik–Nekrasov–Okounkov–Pandharipande [MNOP06], showed that the entire DT theory of toric CY 3-orbifolds can be recovered from a basic building block, the orbifold vertex. The orbifold vertex is a generating function of colored 3-d partitions associated to each torus fixed point in \( \mathbb{Z} \) and the DT potential can be recovered from the orbifold vertex via an explicit gluing algorithm. In the transverse A-singularity case, the orbifold vertex has an explicit expression in terms of loop Schur functions, reviewed in Section 2.2.

In order to prove the DT CRC of Theorem 2.2, we proceed in two steps. We first formulate and prove a local correspondence on the level of the orbifold vertex (Theorem 3.1). Using the vertex operator expressions for the orbifold vertex given by Bryan–Cadman–Young [BCY12], the local correspondence is proved through a manipulation of the vertex operators and careful book-keeping of the commutations that take place. The second step, carried out in Section 4, is to prove that the vertex CRC is compatible with the edge terms in the gluing algorithm.

1.2. Plan of the Paper. In Section 2, we review the basic definitions of DT theory in order to make precise the objects which play a role in Conjecture 1.1. We pay particularly close attention to the case of toric targets with transverse A-singularities in Section 2.2. In Section 3, after reviewing the vertex operator expression for the orbifold vertex, we prove the local correspondence by carefully commuting the operators in a prescribed way. In Section 4, we show that the local correspondence is compatible with the gluing algorithm and deduce the DT CRC for all toric CY 3-orbifolds with transverse A-singularities.

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2. DT Theory

In this section we review the basic definitions of DT theory for CY 3-orbifolds. We mostly follow Bryan–Cadman–Young [BCY12] in Section 2.1 where we define all of the key players in Conjecture 1.1. In Section 2.2, we turn to the case of toric targets with transverse A-singularities where we
set up notation for a precise statement of the main theorem. Section 2.3 reviews the basic definitions pertaining to the orbifold vertex.

2.1. DT Theory. Let $\mathcal{Z}$ be a CY 3-orbifold, i.e., a smooth, quasi-projective Deligne-Mumford stack of dimension three over $\mathbb{C}$ with generically trivial isotropy and trivial canonical bundle. Let $Z$ denote the coarse moduli space of $\mathcal{Z}$.

Let $F_1K(\mathcal{Z})$ denote the compactly supported elements of $K$-theory, up to numerical equivalence, supported in dimension at most one. Then for any $\gamma \in F_1K(\mathcal{Z})$, the corresponding DT invariant is defined as a weighted Euler characteristic

$$ DT_\gamma(\mathcal{Z}) := \sum_{k \in \mathbb{Z}} k e(\nu^{-1}(k)) $$

where $\nu : \text{Hilb}_\gamma(\mathcal{Z}) \to \mathbb{Z}$ is Behrend’s constructible function [Beh09] associated to the Hilbert scheme of substacks $V \subset \mathcal{Z}$ with $[\mathcal{O}_V] = \gamma$, and $e(-)$ is the topological Euler characteristic. Define the DT potential as the formal series

$$ \widehat{DT}(\mathcal{Z}) := \sum_{\gamma \in F_1K(\mathcal{Z})} DT_\gamma(\mathcal{Z}) q^\gamma $$

with formal parameter $q$ defined so that $q^{\gamma_1}q^{\gamma_2} := q^{\gamma_1 + \gamma_2}$.

It is usually more useful to work with an appropriately specialized and normalized series. In order to define it, we need to consider two important subgroups of $F_1K(\mathcal{Z})$. Define the multi-regular $K$-group $F_{mr}K(\mathcal{Z}) \subseteq F_1K(\mathcal{Z})$ to be classes represented by sheaves such that at a generic point of each curve in the support, the associated representation of the stabilizer is a multiple of the regular representation. Define $F_0K(\mathcal{Z}) \subseteq F_1K(\mathcal{Z})$ to be classes represented by sheaves with zero-dimensional supports. Then we have specialized series

$$ DT_{mr}(\mathcal{Z}) := \sum_{\gamma \in F_{mr}K(\mathcal{Z})} DT_\gamma(\mathcal{Z}) q^\gamma \quad \text{and} \quad DT_0(\mathcal{Z}) := \sum_{\gamma \in F_0K(\mathcal{Z})} DT_\gamma(\mathcal{Z}) q^\gamma $$

In this paper, we will only be concerned with the reduced, multi-regular DT potential which is defined as

$$ DT(\mathcal{Z}) := \frac{DT_{mr}(\mathcal{Z})}{DT_0(\mathcal{Z})} $$

Let $\pi : W \to Z$ be a crepant resolution by a smooth variety $W$. We assume that $\mathcal{Z}$ satisfies the hard-Lefschetz condition (cf. [BG09]), which, for 3-folds, is equivalent to the fibers of $\pi$ being at most one-dimensional.

Define $F_{exc}K(W) \subseteq F_1K(W)$ to be classes represented by sheaves supported on the exceptional curves of $\pi$. 

3
2.2. Toric targets with transverse $A$-singularities. In this section we describe some of the basic geometry of toric CY 3-orbifolds with transverse $A$-singularities. This description allows us to choose a basis for the relevant $K$-groups so we can describe the change of variables in the CRC explicitly.

2.2.1. Global Geometry. Let $Z$ be a toric CY 3-orbifold with transverse $A$-singularities (ie. cyclic isotropy supported on disjoint lines), and let $W$ be its toric resolution. Then to $Z$ (and $W$) we can associate a web diagram, a trivalent planar graph

$$\Gamma_Z = \{\text{Edges: } E_Z, \text{Vertices: } V_Z\}$$

where vertices correspond to torus fixed points in $Z$, edges correspond to torus invariant lines, and regions delineated by edges correspond to torus invariant divisors. The web diagram is essentially dual to the toric fan of $Z$. Additionally, we choose an orientation for each edge of $\Gamma_Z$. Let $n_e$ denote the order of the isotropy on the line $L_e$ corresponding to an edge $e$. We label the edges adjacent to each vertex $(e_1(v), e_2(v), e_3(v))$ requiring that

- if $v$ is adjacent to an edge $e$ with $n_e > 1$, then $e_3(v) = e$, and
- the labels $(e_1(v), e_2(v), e_3(v))$ are ordered counterclockwise.

In order to formulate the change of variables in the CRC, we must define a few additional factors at each edge. The normal bundle splits $\mathcal{N}_{L_e/Z} \cong \mathcal{N}_r \oplus \mathcal{N}_l$ where $\mathcal{N}_r$ $(\mathcal{N}_l)$ corresponds to the normal bundle summand in the direction of the torus invariant divisor to the right (left) of $e$. Let $p$ be a generic point on $L_e$ and $p_0, p_\infty$ the torus fixed points with corresponding vertices $v_0, v_\infty$. By the transverse $A_{n-1}$ condition, we can write

$$\mathcal{N}_l = \mathcal{O}(m[p] - \delta_0[p_0] - \delta_\infty[p_\infty])$$
$$\mathcal{N}_r = \mathcal{O}(m'[p] - \delta_0'[p_0] - \delta_\infty'[p_\infty])$$

where we define $\delta_0 = 1$ if the edge corresponding to the fiber of $\mathcal{N}_r$ over $p_0$ is labelled $e_3(v_0)$, and $\delta_0 = 0$ otherwise. Similarly, $\delta_0' = 1$ if the edge corresponding to the fiber of $\mathcal{N}_l$ over $p_0$ is labelled $e_3(v_0)$, and $\delta_0 = 0$ otherwise. Similar for $\delta_\infty$ and $\delta_\infty'$. We have $\delta_0 + \delta_0' \in \{0,1\}$ and the CY condition is equivalent to

$$m + m' - (\delta_0 + \delta_0' + \delta_\infty + \delta_\infty') = -2$$

Remark 2.1. The $\delta$ factors defined here are different than those in [BCY12] when $n = 1$. We define them as such to eliminate the need for the factors in [BCY12] involving $A_\lambda$.

For each edge, we define a set of formal variables

$$q_e := (q_{e,0}, q_{e,1}, \ldots, q_{e,e(n)}{n})$$

The variables $q_{e,i}$ satisfy the natural relation $q := q_{e,0}q_{e,1}\cdots q_{e,n_e-1}$ for any $e$. Geometrically, the $q_{e,i}$ index skyscraper sheaves supported on $e$ with $\mathbb{Z}_{n_e}$ acting by the $i$th irreducible representation, and $q$ corresponds to the skyscraper sheaf with regular representation (ie. the sheaves indexed by $q$ correspond to points which can be deformed away from $e$). We also have
Figure 1. The labeling of the web diagrams for orbifold (left) and resolution (right) near an edge $e$ with $n_e = 4$ and all horizontal edges oriented rightward. The toric surfaces corresponding to the parallelograms are Hirzebruch surfaces $H_{4m_e+6}$, $H_{4m_e+4}$, and $H_{4m_e+2}$ (from bottom to top) where $H_k := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k))$. With this labeling, the bottom edge of each parallelogram corresponds to the zero section of the corresponding Hirzebruch surface.

“Novikov” variables $v_e$ associated to each edge. The $v_e$ satisfy natural curve class relations coming from the geometry of $\mathcal{Z}$. $DT(\mathcal{Z})$ is a formal series in the variables $\{v_e, q_{e,k}\}$ where $e \in E_{\mathcal{Z}}$ runs over the set of edges in the web diagram and $0 \leq k \leq n_e - 1$.

Now let $\pi : W \to \mathcal{Z}$ be the toric resolution of $\mathcal{Z}$. Then $\pi$ is an isomorphism away from the lines $L_e$. In terms of web diagrams, for each edge $e \in E_{\mathcal{Z}}$, there are $n_e$ corresponding edges in the web diagram for $W$. If we orient $e$, then we can compute $m_e$ as above. The orientation also induces an orientation on the $n_e$ corresponding edges in $E_W$ and we label them $f_{e,0}, \ldots, f_{e,n_e-1}$ from right to left. We label the edge connecting the initial point of $f_{e,k}$ to the initial point of $f_{e,k+1}$ by $g_{e,k+1}$ and we label the edge connecting the terminal point of $f_{e,k}$ to the terminal point of $f_{e,k+1}$ by $h_{e,k+1}$ (see Figure 1). Let $u_f$, $u_g$, $u_h$ be formal variables corresponding to these edges so that $DT(W)$ is a formal series in $\{u_f, u_g, u_h, q\}$.

There are relations amongst the variables $u_f$, $u_g$, $u_h$ coming from the geometry of $W$. In particular, the edges $f_{e,k}, f_{e,k-1}, g_{e,k},$ and $h_{e,k}$ correspond to the toric boundary of the Hirzebruch surface

$H_{2m_e+2(n_e-k)} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n_em_e + 2(n_e-k)))$

Therefore, we have the relations

$u_{g_{e,k}} = u_{h_{e,k}}$

and

$u_{f_{e,k}} = u_{f_{e,k-1}} u_{g_{e,k}}^{2k-2n_e-m_en_e} = u_{f_{e,0}} \prod_{l=1}^{k} u_{g_{e,l}}^{2l-2n_e-m_en_e}$

We henceforth interpret $u_{f_{e,k}}$ as a function of $u_{f_{e,0}}, u_{g_{e,1}}, \ldots, u_{g_{e,k}}.$
With this notation, $DT(W)_{exc}$ is the formal function obtained from $DT(W)$ by setting $u_{f,e} = 0$ for all $e \in E_Z$:

$$DT(W)_{exc} := DT(W)|_{u_f=0}$$

2.2.2. The CRC. Our main theorem is the following correspondence.

**Theorem 2.2.** With notation as above,

$$DT(Z) = \frac{DT(W)}{DT(W)_{exc}}$$

after the change of variables $u_{g_e,i}, u_{h_e,i} \rightarrow q_e,i$, $q \rightarrow q$, and

$$u_{f,e,0} \rightarrow v_e \prod_{l=1}^{n_e-1} q_{e,l}^{(m_e+2)(n_e-l)}$$

**Remark 2.3.** The change of variables given in Theorem 2.2 depends apriori on the choice of orientation of each $e$. However, it is easy to check that it is independent of this choice.

2.3. The Orbifold Vertex. In the case of toric targets with transverse $A$-singularities, the DT generating series has a particularly nice combinatorial description which we recall in this section. This combinatorial description, proved by Bryan–Cadman–Young [BCY12], follows from applying an analog of the Atiyah-Bott localization theorem to the Hilbert scheme in order to reinterpret the weighted Euler characteristics (1) in terms of 3-d partitions (monomial ideals) at each torus fixed point. We begin by setting up notation as in Ross–Zong [RZ14], following Bryan–Cadman–Young [BCY12].

2.3.1. Partitions. Let $\tau$ denote a partition, i.e. a weakly decreasing sequence of nonnegative integers. We think of $\tau$ as a Young diagram (in English notation) where we index the rows and columns beginning with 0. As an example of our conventions, the partition $\tau = (4,3,1,1,0,0,\ldots)$ corresponds to the Young diagram

```
  1 1 1 1
  2 2
  3
  4
```

where the index of the shaded box is $(i,j) = (1,2)$. We define the size of $\tau$, denoted $|\tau|$, to be the number of boxes in the corresponding Young diagram. We also define the $l$-th diagonal of $\tau$ to be all boxes satisfying $j - i = l$. The conjugate partition $\tau'$ is obtained by reflecting the Young diagram along the 0-th diagonal. An $n$-strip of $\tau$ is a connected collection of $n$ boxes in the Young diagram which contains no $2 \times 2$ square, and a border strip is such a collection which lies entirely along the southeast border of the Young diagram. We define the height, $ht(\nu)$, of a border strip $\nu$ to be the number of rows occupied by the strip, minus one.
Let $\lambda = (\lambda_0, \ldots, \lambda_{n-1})$ denote an $n$-tuple of partitions and $|\lambda| := \sum |\lambda_i|$. To $\lambda$ we can canonically associate a partition $\bar{\lambda}$ of size $n|\lambda|$ via $n$-quotients, defined in Section 3.2. We think of $\bar{\lambda}$ as an $n$-colored Young diagram where the boxes in the $l$-th diagonal are colored by $(l \mod n) \in \{0, \ldots, n-1\}$. In fact, $n$-quotients give a perfect dictionary between $n$-tuples of partitions $\lambda$ and $n$-colored Young diagrams which are balanced in the sense that they contain exactly $|\lambda|$ boxes of each color. In particular, each such $\bar{\lambda}$ can be decomposed by successively pulling off $n$-border strips $(\nu_1, \ldots, \nu_{|\lambda|})$. We define the quantity

$$\frac{\chi_{\lambda}(n^d)}{\dim(\lambda)} := (-1)^{\sum h_t(\nu_i)} = \pm 1$$

where the notation originates from an interpretation in terms of the representation theory of the generalized symmetric group (see, for example, [RZ13] Section 6). It is easily checked that this quantity is independent of border strip decomposition.

Let $q = (q_0, \ldots, q_{n-1})$ be formal variables with indices computed modulo $n$ and define $q := q_0 \cdots q_{n-1}$. Define the variables $q_t$ recursively by the rules $q_0 = 1$ and

$$q_t = qt_{t-1}$$

so that

$$(\ldots, q_{-2}, q_{-1}, q_0, q_1, q_2, \ldots) = (\ldots, q_0^{-1} q_{-1}^{-1}, q_0^{-1}, 1, q_1 q_2, \ldots)$$

For the colored Young diagram $\bar{\lambda} = ((\bar{\lambda})_0 \geq (\bar{\lambda})_1 \geq (\bar{\lambda})_2 \geq \ldots)$, we define the sequence of variables $q_{\bar{\lambda}-\lambda}$ by

$$q_{\bar{\lambda}-\lambda} := (q_{-(\bar{\lambda})_0}, q_{1-(\bar{\lambda})_1}, q_{2-(\bar{\lambda})_2}, \ldots).$$

We denote $q_{\bar{\lambda}} := q_{\bar{\lambda}-\lambda}$ and we use $q_{\bar{\lambda}}$ to denote the variables obtained in this way from the conjugate colored partition $\bar{\lambda}'$ (one checks from the definitions in Section 3.2 that $\bar{\lambda}'$ has $n$-quotient $(\lambda'_{n-1}, \ldots, \lambda'_0)$). We use an overline on an expression in the $q$ variables to denote the exchange $q_i \leftrightarrow q_{-i}$.

Let $X$ denote the $A_{n-1}$ vertex $[\mathbb{C}^3/\mathbb{Z}_n]$ where the action has weights $(1, -1, 0)$.

**Definition 2.4 ([RZ14], following [BCY12]).** The (reduced, multi-regular) Donaldson-Thomas vertex for $X$ is

$$P^n_{\rho^+, \rho^-, \lambda}(q) := \frac{\chi_{\lambda}(n^d)}{\dim(\lambda)} s_{\chi}(q) \sum_{\omega} q_0^{-|\omega|} s_{(\rho^+, \rho^- \omega)}(q_{\bar{\lambda}-\lambda}) s_{(\rho^- \omega)}(q_{\bar{\lambda}-\lambda})$$

where $s_{\chi}(q)$ denotes the loop Schur function of $\bar{\lambda}$ in the variables $(q_0, \ldots, q_{n-1})$ and $s_{\rho \omega}$ denotes a usual skew Schur function.

When convenient, we use $X'$ in the superscript instead of $n$. For future reference, we also define a slight modification

$$\tilde{P}^n_{\rho^+, \rho^-, \lambda}(q) := \left( \frac{\chi_{\lambda}(n^d)}{\dim(\lambda)} \prod_{(i,j) \in \bar{\lambda}} q_{j-i}^{-1} \right) P^n_{\rho^+, \rho^-, \lambda}(q)$$
which is more closely related to the definition given in Bryan–Cadman–Young [BCY12].

**Remark 2.5.** Skew Schur functions are classical and the standard reference is Macdonald [Mac95]. Loop Schur functions were introduced by Lam–Pylyavskyy in [LP12] in the context of total positivity of matrix loop groups. References for loop Schur functions are a paper of Lam [Lam12], and more closely related to the topic at hand, a paper of the author [Ros12]. The $s_\lambda(q)$ in definition 2.4 are obtained from those defined in [Ros12] by specializing $x_{i,j} \to q^j_i$.

2.3.2. **Gluing Formula.** Since our conventions differ slightly from those of Bryan–Cadman–Young [BCY12], we briefly recall their gluing algorithm in our notation.

Let $\Lambda(\mathbb{Z})$ denote the set of edge assignments $\Lambda = \{\lambda(e)\}_{e \in E_{\mathbb{Z}}}$ where each $\lambda(e)$ is a $n_e$-tuple of partitions $\lambda(e) = (\lambda(e)_0, \ldots, \lambda(e)_{n_e-1})$. For an edge assignment $\Lambda$ and a vertex $v$, set $\lambda(v) = (\lambda_1(v), \lambda_2(v), \lambda_3(v))$ where

$$
\lambda_i(v) =
\begin{cases}
\lambda(e_i(v)) & \text{if } e_i(v) \text{ is outgoing} \\
\lambda(e_i(v))' & \text{if } e_i(v) \text{ is incoming}
\end{cases}
$$

and for each vertex $v$, set

$$
q_v =
\begin{cases}
q_{e_3(v)} & \text{if } e_v \text{ is outgoing} \\
q_{e_3(v)}' & \text{if } e_v \text{ is incoming}
\end{cases}
$$

Then Theorem 10 of Bryan–Cadman–Young [BCY12] can be restated as follows.

**Theorem 2.6** ([BCY12], Theorem 10). Define

$$
DT(\mathbb{Z}) := \sum_\Lambda \prod_e \hat{P}_{\lambda(e)}(q_v)
$$

where

$$
E_{\lambda,e} := v_e |\lambda| (-1)^{(n_e+\delta_e,0+\delta_e,\infty)|\lambda|} \prod_{(i,j) \in \lambda} q_{e,j-i}^{-m_e-j+m'_{i,j}+1}
$$

Then the (reduced, multiregular) DT partition function $DT(\mathbb{Z})$ is obtained from $DT(\mathbb{Z})$ by adding a minus sign to the variables $q_{e,0}$ (and hence also to $q$).

2.3.3. **Resolutions.** Let $Y$ be the toric resolution of $X$. Then $Y$ is a chain of $n-1$ $\mathbb{P}^1$’s which all have normal bundle $O \oplus O(-2)$. On each compact edge in the web diagram corresponding to one of the $\mathbb{P}^1$’s, choose the orientation for which $N_r = O$. Orient each noncompact edge outward. At each of the $n$ vertices, choose $e_3(v)$ to be edge corresponding to the fiber in the trivial direction. Then it is easy to check that the edge term in Theorem 2.6 becomes

$$
E_{\lambda,e} = (qv_e)^{|\lambda_e|}.
$$
Therefore, we can write

\[
\text{DT}(Y) = \sum_{\tau_1, \ldots, \tau_{n-1}} P^1_{\tau_1, \emptyset, 0}(q) (q v_1)^{|\tau_1|} P^1_{\tau_2, \tau_1, 0}(q) (q v_2)^{|\tau_2|} \]

(2)

\[
\cdots (q v_{n-2})^{|\tau_{n-2}|} P^1_{\tau_{n-1}, \tau_{n-2}, 0}(q) (q v_{n-1})^{|\tau_{n-1}|} P^1_{\emptyset, \tau_{n-1}, 0}(q)
\]

To obtain the analog of \( P^0_{\rho^+, \rho^-, \lambda}(q) \) on the resolution, we generalize (2) by modifying the vertex terms:

\[
P^Y_{\rho^+, \rho^-, \lambda}(q, v) := \sum_{\tau_1, \ldots, \tau_{n-1}} P^1_{\tau_1, \rho^-, \lambda_0}(q) (q v_1)^{|\tau_1|} P^1_{\tau_2, \tau_1, \lambda_1}(q) (q v_2)^{|\tau_2|} \]

(3)

\[
\cdots (q v_{n-2})^{|\tau_{n-2}|} P^1_{\tau_{n-1}, \tau_{n-2}, \lambda_{n-2}}(q) (q v_{n-1})^{|\tau_{n-1}|} P^1_{\rho^+, \tau_{n-1}, \lambda_{n-1}}(q)
\]

Again, for later convenience we define a slight modification

\[
\hat{P}^Y_{\rho^+, \rho^-, \lambda}(q, v) := \left( \prod_k \prod_{(i, j) \in \lambda_k} q^{-i} \right) P^Y_{\rho^+, \rho^-, \lambda}(q, v)
\]

which can also be defined by hatting all of the \( P \)s in (3).

3. The Vertex CRC

The crepant resolution conjecture for the DT vertex is the following.

**Theorem 3.1.** After the change of variables \( v_i \leftrightarrow q_i \),

\[
P^Y_{\rho^+, \rho^-, \lambda}(q) = \frac{P^Y_{\rho^+, \rho^-, \lambda}(q, v)}{P^0_{\emptyset, \emptyset, 0}(q, v)} \left( \prod_{(i, j) \in \lambda_k} (-1)^{n-k-1} q^{(n-k-1)(i-j)} \prod_{l>k} q^{n-l} \right)
\]

The rest of Section 3 is devoted to the proof of Theorem 3.1.

3.1. Vertex Operators. One of the key combinatorial tools in Bryan–Cadman–Young [BCY12] is a description of the DT vertex in terms of *vertex operators*. Following Section 7 of [BCY12], we give a concise review of their setup.

Let \( \mathcal{P} \) be the set of all partitions, \( \mathcal{R} \) the space of formal Laurent series in \( q_0, \ldots, q_{n-1} \), and \( \mathcal{R} \mathcal{P} \) the free \( \mathcal{R} \)-module generated over \( \mathcal{P} \). Vertex operators are defined to act on the space \( \mathcal{R} \mathcal{P} \). For two partitions \( \tau \) and \( \sigma \), we write \( \tau \succ \sigma \) if

\[
\tau_0 \geq \sigma_0 \geq \tau_1 \geq \sigma_1 \geq \ldots
\]

For \( x \) a monomial in \( q_i \), define the vertex operators \( \Gamma_\pm \) and \( Q_k \) by their actions:

\[
\Gamma_+(x) \tau := \sum_{\sigma < \tau} x^{||\tau|-|\sigma||} \sigma
\]

\[
\Gamma_+(x) \tau := \sum_{\sigma > \tau} x^{||\sigma|-|\tau||} \sigma
\]

\[
Q_k \tau := q^{|\tau|}_k \tau
\]

9
For $O$ an operator on $\mathcal{RP}$, we define the expectation $\langle \sigma | O | \tau \rangle$ to be the coefficient of $\sigma$ after applying the operator $O$ to $\tau$.

Let $\tilde{\lambda}(t)$ denote the slope sequence of $\bar{\lambda}$. It is defined by setting

$$S(\bar{\lambda}) =: \{ \bar{\lambda}_0 - 1, \bar{\lambda}_1 - 2, \bar{\lambda}_2 - 3, \ldots \}$$

and

$$\tilde{\lambda}(t) := \begin{cases} + & t \in S(\bar{\lambda}) \\ - & t \notin S(\bar{\lambda}) \end{cases}$$

The relevance of the slope sequence is that it tells us how to trace the boundary of the Young diagram $\bar{\lambda}$ (after rotating clockwise by $\pi/2$), see Figure 2.

**Remark 3.2.** We warn the reader that our conventions are slightly different than [BCY12] because our Young diagrams are positioned in English notation whereas theirs are in French.

Bryan–Cadman–Young prove the following identity.

**Lemma 3.3** ([BCY12], Proposition 8).

$$\hat{P}_{\rho^+, \rho^-, \lambda}(q) = \frac{V_{\rho^+, \rho^-, \lambda}(q)}{V_{0,0,0}(q)}$$

where $V$ is the vertex operator expectation

$$V_{\rho^+, \rho^-, \lambda}(q) := q_0^{-|\rho^+|} \left\langle \gamma \prod_{t \in \mathbb{Z}} \Gamma_{\lambda(t)} \left( q_{t^{\lambda(t)}} \right) \right\rangle$$

The crucial bit of information that we need in order to manipulate Lemma 3.3 is that the ordering of the operators matters. The arrow above the product indicates that the operators are listed from left to right as the index $t$ increases. In order to reorder the operators, we need to use the following important commutation relation for $i, j = \pm$:

$$\Gamma_i(a) \Gamma_j(b) = (1 - ab) \frac{\tilde{\gamma}(a,b)}{\tilde{\gamma}(b,a)} \Gamma_j(b) \Gamma_i(a)$$
For future reference, the other commutation relation we’ll need is:

\[ \Gamma_j(a) Q_k = Q_k \Gamma_j(a q^k_j) \]

3.2. Quotients. As it is essential in the proof of Theorem 3.1, we briefly recall the basic definitions from the theory of \( n \)-quotients. From \( \lambda = (\lambda_0, \ldots, \lambda_{n-1}) \), we obtain \( n \) slope sequences \((\lambda_0(t), \ldots, \lambda_{n-1}(t))\) as in the previous section. From these \( n \) slope sequences, we define a new slope sequence

\[
\tilde{\lambda}(t) := \lambda_t \left( \frac{t - \lfloor \frac{t}{n} \rfloor}{n} \right)
\]

where \( t := t \mod n \). Concretely, we are simply interlacing the slope sequences \((\lambda_0(t), \ldots, \lambda_{n-1}(t))\).

**Definition 3.4.** If the slope sequences for \( \lambda \) and \( \tilde{\lambda} \) are related as in (6), we say that \( \lambda \) is the \( n \)-quotient of \( \tilde{\lambda} \).

**Remark 3.5.** Usually when one speaks of \( n \)-quotients, they also mention \( n \)-cores. However, \( n \)-cores do not play a role in this paper because we only consider balanced partitions \( \tilde{\lambda} \) which have empty \( n \)-core.

It will be helpful later to have a better understanding of the \( n \)-quotient correspondence in terms of Young diagrams, rather than slope sequences. In particular, we make the following observation which can easily be checked.

**Observation.** Adding a single box to the \((i, j)\) position of \( \lambda_t \) corresponds to adding a length \( n \) border strip to the colored partition \( \tilde{\lambda} \). This border strip has exactly one box of each color, the northeasternmost box has color \( l \) and the unique color 0 box lies in the \( n(j - i) \) diagonal of \( \tilde{\lambda} \).

**Example 3.6.** Suppose \( n = 5 \), \( \lambda_0 = (1) \), \( \lambda_1 = \emptyset \), \( \lambda_2 = (2, 1) \), \( \lambda_3 = (2) \), and \( \lambda_4 = \emptyset \). Then one checks that \( \tilde{\lambda} \) is the Young diagram:

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
3 & 4 & 0 \\
2 & 3 & 4 \\
1 & 2 \\
0 & 1 \\
4 \\
3
\end{array}
\]
where the numbers denote the colors. Now if we add a box in the (1, 1)
position of $\lambda_2$, the colored Young diagram becomes

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
0 & 1 & 2 & 3 & 4 & 0 \\
\hline
4 & 0 & 1 & 2 & 3 & 4 \\
\hline
3 & 4 & 0 & 1 & 2 \\
\hline
2 & 3 & 4 & 0 \\
\hline
1 & 2 & 3 & 4 \\
\hline
0 & 1 \\
\hline
4 \\
\hline
3 \\
\hline
\end{array}
\]

We have suggestively decorated the second Young diagram. The highlighted
boxes are the new border strip, but we don’t want to interpret the modi-
fication in terms of simply adding these new boxes. Rather, a more useful
interpretation of the modification is that we add the boxes containing cir-
cles which lie along the base in the desired diagonals, then we push out the
pre-existing boxes along their respective diagonals.

3.3. Proof of Vertex CRC. Consider the formal function $\hat{P}_Y^{\rho^+,\rho^-}(q, v)$
defined in (1). Rewriting in terms of the operator expression, we obtain

\[
\frac{\hat{P}_Y^{\rho^+,\rho^-}(q, v)}{\hat{P}_0^{\rho^+,\rho^-}(q, v)} = \frac{V_Y^{\rho^+,\rho^-}(q, v)}{V_Y^{\rho^+,\rho^-}(q, v)}
\]

where

\[
V^{\rho^+,\rho^-}(q, v) = q^{-|\rho^+|} \sum_{\tau_1, \ldots, \tau_{n-1}} \left( \prod_{t \in \mathbb{Z}} \Gamma_{\lambda_0(t)} \left( q^{-t\lambda_0(t)} \right) \right) \left( \prod_{t \in \mathbb{Z}} \Gamma_{\lambda_1(t)} \left( q^{-t\lambda_1(t)} \right) \right) \cdots \left( \prod_{t \in \mathbb{Z}} \Gamma_{\lambda_{n-1}(t)} \left( q^{-t\lambda_{n-1}(t)} \right) \right)
\]

We now perform the change of variables $v_i \rightarrow q_i$, approximate the infinite
operator expressions with finite ones, and concatenate the expectations in
the natural way. We obtain

\[
V^{\rho^+,\rho^-}(q) = q^{-|\rho^+|} \lim_{N \rightarrow \infty} \left( \prod_{t} \Gamma_{\lambda_0(t)} \left( q^{-t\lambda_0(t)} \right) \right) \left( \prod_{t} \Gamma_{\lambda_1(t)} \left( q^{-t\lambda_1(t)} \right) \right) \cdots \left( \prod_{t} \Gamma_{\lambda_{n-1}(t)} \left( q^{-t\lambda_{n-1}(t)} \right) \right)
\]

where the index $t$ satisfies $-N \leq t \leq N - 1$. 

If we commute all of the $Q_k$ operators to the right, then we arrive at the following expression

$$V_{\rho^+,-\rho,-\lambda}(q) = q_0^{-|\rho^+|} \lim_{N \to \infty} \rho^+ \left| \prod_t \Gamma_{t}(\lambda_0(t)) \right| \prod_t \Gamma_{t}(\lambda_1(t)) $$

(7)

$$\cdots \prod_t \Gamma_{t}(\lambda_{n-1}(t)) \left| q_{nt+n-1}^{-\lambda_n(t)} \right| \rho^+')$$

The final step is to interlace the operators appearing in the expression so that the indices on the $q$ variables are increasing. By definition, this interlacing of slope sequences produces the slope sequence for $\lambda$. Therefore, we have

$$V_{\rho^+,-\rho,-\lambda}(q) = q_0^{-|\rho^+|} \lim_{N \to \infty} F_{\lambda}(N) \left| \prod_{-nN \leq t \leq nN-1} \Gamma_{t}(\lambda) \left| q_{t}^{-\lambda(t)} \right| \rho^+)' \right)$$

where the factor $F_{\lambda}(N)$ arises from commuting the $\Gamma$ operators – notice that this factor does not depend on $\rho^\pm$. The expectation on the right is simply the numerator in the vertex operator expression for $\hat{P}_{\rho^+,-\lambda}(q)$ in the limit $N \to \infty$. Therefore, in order to prove Theorem 8, it is left to analyze the limit

$$\lim_{N \to \infty} \frac{F_{\lambda}(N)}{F_{0}(N)}.$$ 

In particular, it suffices to prove the following

(8)

$$\prod_{(i,j) \in \lambda_k} \left( (-1)^{n-k-1} q^{(n-k)(i-j)+j} \prod_{l>k} q_{l}^{n-l} \right) \lim_{N \to \infty} \frac{F_{\lambda}(N)}{F_{0}(N)} = \frac{\chi_{\lambda}(n^d)}{\dim(\lambda)} \prod_{(i,j) \in \lambda} q_{j-i}^{i}$$

We prove (8) inductively by systematically removing $n$-border strips from $\lambda$. More specifically, let $\nu = (\nu_0, \ldots, \nu_{n-1})$ be an $n$-tuple of partitions such that $\nu_i = \lambda_i$ for $i \neq k$ and $\nu_k \setminus \nu_k$ is a single box in the $(i, j)$ position. Theorem 3.1 follows from the following proposition.

**Proposition 3.7.**

$$(-1)^{n-k-1} q^{(n-k)(i-j)+j} \prod_{l>k} q_{l}^{n-l} \lim_{N \to \infty} \frac{F_{\lambda}(N)}{F_{\nu}(N)} = (-1)^{ht(\lambda|\rho)} \prod_{(r,s) \in \lambda|\rho} q_{s-r}^{i}$$

**Proof.** We start by closely analyzing $\lim_{N \to \infty} \frac{F_{\lambda}(N)}{F_{\nu}(N)}$. Since we removed a box in the $(i, j)$ position of $\lambda$, the discrepancy in the operator expressions (7) for $V_{\rho^+,-\lambda}(q)$ can be expressed as the quotient

$$\frac{\Gamma_-(q_1 \cdots q_k q^{j-i-1}) \Gamma_+(q_1^{-1} \cdots q_k^{-1} q^{i-j-1})}{\Gamma_+(q_1^{-1} \cdots q_k^{-1} q^{j-i+1}) \Gamma_-(q_1 \cdots q_k q^{j-i})}$$

(9)

In other words, the factor $\lim_{N \to \infty} \frac{F_{\lambda}(N)}{F_{\nu}(N)}$ comes entirely from commuting $\Gamma_\pm$ operators through both the numerator and denominator of (9).
For \( l < k \), we must commute the operators \( \Gamma_{\lambda_i(t)} \left( q_{mt+i}^{-\lambda(t)} \right) \) through (9) from left to right for all \( t > j - i \) (and halfway for \( t = j - i \)). For \( t > j - i \), we compute from (5) that this commutation produces the factor

\[
(10) \quad \left\{ \begin{array}{ll}
1-q_{i+1}^{-1} \cdots q_k^{-1} q^{i-t} & \text{if } \lambda_i(t) = + \\
1-q_{i+1}^{-1} \cdots q_k^{-1} q^{j+i-t} & \text{if } \lambda_i(t) = -
\end{array} \right.
\]

and for \( t = j - i \) it produces the factor

\[
(11) \quad \left\{ \begin{array}{ll}
(1 - q_{l+1} \cdots q_k q^{-i-1})^{-1} & \text{if } \lambda_i(t) = + \\
(1 - q_{l+1}^{-1} \cdots q_k^{-1} q^{-1})^{-1} & \text{if } \lambda_i(t) = -
\end{array} \right.
\]

For \( t >> 0 \), we are always in the second case of (10) by the definition of slope functions. In particular, for \( N \) sufficiently large a quick analysis shows that for \( t = N - 1, N - 2, N - 3, \ldots \) the successive commutation factors in (10) cancel except for an initial term of \((1 - q_{l+1}^{-1} \cdots q_k^{-1} q^{N-j+i})^{-1}\) which tends to 1 as \( N \to \infty \). As we decrease \( t \), the cancellation continues to happen until we encounter a place where \( \lambda_i(t+1) = - \) and \( \lambda_i(t) = + \) with \( t \geq j - i \). At this point, the successive terms cancel modulo a multiplicative factor of

\[
(12) \quad -q_{l+1}^{-1} \cdots q_k^{-1} q^{-j+i-t+1}
\]

Similarly, whenever we encounter a place where \( \lambda_i(t+1) = + \) and \( \lambda_i(t) = - \) with \( t \geq j - i \), we obtain a factor of

\[
(13) \quad -q_{l+1} \cdots q_k q^{j-i-t-1}.
\]

Since the factors (12) and (13) alternate, they merely contribute factors of \( q \) except for possibly the last occurrence. From this, it is not hard to see that the overall factor appearing is

\[
(14) \quad \left\{ \begin{array}{ll}
-q_{l+1}^{-1} \cdots q_k^{-1} q^a & \text{if } \lambda_i(j-i) = + \\
q^a & \text{if } \lambda_i(j-i) = -
\end{array} \right.
\]

where \( a := \#\{t : t > j - i \text{ and } \lambda_i(t) = +\} \). In terms of the Young diagram \( \lambda_i \), we can describe the occurrence of each of the two cases combinatorially. In what follows, we use \( d_l(\tau) \) to denote the number of boxes in the \( l \)-th diagonal of \( \tau \). Then from the above analysis (and similar analysis for \( l > k \)) we can conclude that

\[
\lim_{N \to \infty} \frac{F_\lambda(N)}{F_\nu(N)} = \prod_{l \neq k} F_{\lambda \setminus \nu}(l)
\]

where \( F_{\lambda \setminus \nu}(l) \) is defined by the following combinatorial rules.

(A) If \( l < k \), and

(I) \( j - i < 0 \), then \( F_{\lambda \setminus \nu}(l) \) is equal to

(a) \( q^a \) if \( d_{j-i}(\lambda_i) = a + j - i \) and \( d_{j-i+1}(\lambda_i) = a + j - i + 1 \), and

(b) \( -q_{l+1}^{-1} \cdots q_k^{-1} q^a \) if \( d_{j-i}(\lambda_i) = d_{j-i+1}(\lambda_i) = a + j - i \).
(II) $j - i \geq 0$, then $F_{\lambda \mu}(l)$ is equal to
(a) $q^a$ if the $d_{j-i}(\lambda_l) = d_{j-i+1}(\lambda_l) = a$, and
(b) $-q_{l+1}^{-1} \cdots q_{k}^{-1} q^{a+1}$ if $d_{j-i}(\lambda_l) = a + 1$ and $d_{j-i+1}(\lambda_l) = a$.

(B) If $l > k$ and:

(I) $j - i \leq 0$, then $F_{\lambda \mu}(l)$ is equal to
(a) $q^a$ if $d_{j-i+1}(\lambda_l) = d_{j-i}(\lambda_l) = a$, and
(b) $-q_{k+1}^{-1} \cdots q_l^{-1} q^{a+1}$ if $d_{j-i-1}(\lambda_l) = a$ and $d_{j-i}(\lambda_l) = a + 1$.

(II) $j - i > 0$, then $F_{\lambda \mu}(l)$ is equal to
(a) $q^a$ if the $d_{j-i-1}(\lambda_l) = a - j + i + 1$ and $d_{j-i}(\lambda_l) = a - j + i$, and
(b) $-q_{k+1}^{-1} \cdots q_l^{-1} q^a$ if $d_{j-i-1}(\lambda_l) = d_{j-i}(\lambda_l) = a - j + i$.

Notice that the factor only ever depends on the $(j - i)$ and $(j - i + 1)$ diagonal lengths if $l < k$ and the $(j - i)$ and $(j - i - 1)$ diagonal lengths if $l > k$. It is left to compare these factors with the other terms appearing in Proposition 3.7.

Let $\lambda[k]$ be the $n$-tuple of partitions obtained from $\lambda$ by setting $\lambda_l = \emptyset$ for $l \neq k$. Then $\lambda[k]$ is a colored Young diagram which can be constructed entirely out of $n$-strips where the northeasternmost box of each strip has color $k$.

**Example 3.8.** Suppose $n = 4$ and $\lambda_1 = (3,3)$, then $\lambda_1$ and $\lambda[1]$ correspond as follows:

\[
\begin{array}{ccc}
\lambda_1 &=& \begin{array}{ccc}
\begin{array}{ccc}
a & b & c \\
d & e & f
\end{array}
\end{array} \\
\lambda[1] &=& \begin{array}{cccccccc}
0_a & 1_a & 2_b & 3_b & 0_b & 1_b & 2_c & 3_c & 0_c & 1_c \\
3_a & 0_e & 1_e & 2_f & 3_f & 0_f & 1_f \\
2_a & 3_e \\
1_d & 2_f \\
0_d \\
3_d \\
2_d
\end{array}
\end{array}
\]

where the lettering indicates which 4-strips correspond to each box in $\lambda_1$.

We now show that Proposition 3.7 holds for $\lambda[k]$. Generalizing Example 3.8, we see that the right side of Lemma 3.7 in the case of removing the $(i,j)$ box from $\lambda[k]$ is equal to

\[
\begin{cases}
(-1)^{n-i} q^{n(i-j)-k+j} q_{k-1} q_{k-2} \cdots q_{k+1}^{n-1} & \text{if } j - i < 0 \\
(-1)^{n-k-1} q^j q_{n-1} q_{n-2} \cdots q_{k+1}^{n-k-1} & \text{if } j - i = 0 \\
q^j & \text{if } j - i > 0
\end{cases}
\]

It is also straightforward to compute the left side of Proposition 3.7 in the case of removing the $(i,j)$ box from $\lambda[k]$. For example, if we take $j - i < 0$, then from the combinatorial rules (A) and (B) for $\lim_{N \to \infty} F_{\lambda}(N) / F_{\mu}(N)$ we have
Multiplying these factors together and combining with the remaining terms in the left side of Proposition 3.7, we obtain exactly the first case of (15). The other two cases are similar. This proves that Proposition 3.7 holds for $\lambda[k]$.

To finish the proof of Proposition 3.7, we analyze what happens when we build $\bar{\lambda}$ from $\lambda[k]$. By induction, suppose Proposition 3.7 holds for $\bar{\mu}$ where $\bar{\mu}$ is a partial reconstruction of $\lambda$, i.e., $\mu_l \subset \lambda_l$ for $l \neq k$ and $\mu_k = \lambda_k$. We must show that the proposition continues to hold if we add one more box to $\lambda_l \setminus \mu_l$ for some $l$.

For simplicity, assume henceforth that $j - i < 0$ and $l < k$, all other cases are similar. Then by the interpretation given in Example 3.6 the only way that adding a box to $\mu_l$ affects the border strip corresponding to the $(i, j)$ box of $\lambda_k$ is if the additional box is in the $(j - i)$th or $(j - i + 1)$th diagonal of $\mu_l$. If the new box is in the $(j - i)$th diagonal of $\mu_l$, then the boxes of color $k + 1, \ldots, n - 1, 0, \ldots, l$ in the strip corresponding to the $(i, j)$ box of $\lambda_k$ get shifted out. If the box is in the $(j - i + 1)$th diagonal $\mu_l$, then it shifts out the boxes of color $l + 1, \ldots, k$.

**Example 3.9.** Suppose $n = 5$, $\mu_0 = (1)$, $\mu_1 = (1)$, $\mu_2 = (2, 1)$, $\mu_3 = \emptyset$, $\mu_4 = \emptyset$, $k = 2$, and $(i, j) = (1, 0)$. Then $\bar{\mu}$ is the colored Young diagram

```
0 1 2 3 4 0 1 2
4 0 1
3 4 0
2 3 4
1 2 3
0 1 2
4
3
```

where we have shaded the special strip corresponding to the $(i, j)$ box of $\lambda_k$. Now if we add the $(1, 0)$ box to the $-1$ diagonal of $\mu_1$ it has the following
where the boxes with \( \bullet \) are the strip corresponding to the new box in \( \mu_1 \) while the shaded boxes are the strip corresponding to the \((i, j)\) box of \( \lambda_k \). We see that adding the box had the effect of shifting the color 3, 4, 0 boxes along their diagonals.

So what effect does this shift have on Proposition 3.7? On the right side of Proposition 3.7 it is not hard to see that the shift results in a factor

\[
\begin{cases} 
-q_{k+1} \cdots q_{n-1} q_0 \cdots q_l & \text{if } d_{j-i}(\mu_l) \text{ increases} \\
-q_{l+1} \cdots q_k & \text{if } d_{j-i+1}(\mu_l) \text{ increases}
\end{cases}
\]

The sign comes from the fact that the number of rows occupied by the strip is changing by exactly one. Notice that the only thing that changes on the left side is the factor \( \lim_{N \to \infty} \frac{F_{\lambda(N)}}{F_{\nu(N)}} \). By the combinatorial description of this factor which we derived above, if \( d_{j-i}(\mu_l) \) increases then we must be passing from (A.I.b) to (A.I.a) (with an increase by one of the value \( a \)). But the discrepancy in these factors is \( -q_{k+1} \cdots q_{n-1} q_0 \cdots q_l \), agreeing with (16). Similarly, if \( d_{j-i+1}(\mu_l) \) increases then we must be passing from (A.I.a) to (A.I.b) (with the same value \( a \)). This results in a factor of \( -q_{l+1} \cdots q_k \), agreeing again with (16).

A similar check of the other cases finishes the proof of Proposition 3.7 and, hence, Theorem 3.1.

\[ \square \]

4. Compatibility with Gluing

Having proved the vertex correspondence, we now turn to the task of proving that it is compatible with the edge terms in the gluing formula. We use the notation from Section 2.

**Proof of Theorem 2.2.** Edge terms naturally fall into two cases depending on whether an edge \( e \) in the web diagram of \( Z \) has \( n_e = 1 \) or \( n_e > 1 \).

The case \( n_e = 1 \) is relatively easy. Given such an oriented edge \( e \) and inducing the same orientation and labeling of vertices for \( f_e \), it is not hard
to see that \( m_{f_e} = m_e \), \( m'_{f_e} = m'_e \), and all \( \delta_e \) agree. Therefore, for any partition \( \rho \) the edge terms in Theorem 2.6 coincide and the contributions from such edges fit in the framework of Theorem 2.2.

The case \( n_e > 1 \) is a bit more subtle. Fix an orientation on \( e \) which induces an orientation on \( f_k := f_{e,k} \) (we drop the \( e \) subscripts from this point on). Then it is not hard to compute that

\[
m_{f_k} = nm + 2(n - k - 1)
\]

and

\[
m'_{f_k} = -nm - 2(n - k)
\]

We have implicitly chosen a labeling such that the \( f_k \) are the third edge at each of their vertices, hence all of the \( \delta \) are zero. Assume we have an edge assignment where the partition associated to each \( f_k \) is \( \lambda_k \). Then locally at \( e \) the contribution to the right side of Theorem 2.2 is given by

\[
E^W_e = \prod_{(i,j)\in\lambda_k} u_{f_k}(-1)^{nm} q^{(nm + 2(n - k))(i - j) + 2j + 1}
\]

and where \( u_g = (u_{g_1}, \ldots, u_{g_{n-1}}) \) while \( \overline{u}_k = (u_{h_{n-1}}, \ldots, u_{h_1}) \). After the change of variables, the edge factor becomes

\[
E^W_e = \prod_{(i,j)\in\lambda_k} v \left( \prod_{l=1}^{k} q_l^{l-(m+1)l} \prod_{l=1}^{n-1} q_l^{(m+2)(n-l)} \right) (-1)^{nm} q^{(nm + 2(n - k))(i - j) + 2j + 1}
\]

Applying Theorem 3.1, we have (after the change of variables)

\[
\frac{\hat{P}^Y_{\rho_0, \rho_0, (\lambda_0, \ldots, \lambda_{n-1})}(q, u_g)}{\hat{P}^Y_{0,0,0}(q, u_g)} = P^X_{\rho_0, \rho_0, (\lambda_0, \ldots, \lambda_{n-1})}(q)
\]

\[
\cdot \prod_{(i,j)\in\lambda_k} (-1)^{n-k-1} q^{(n-k)(i-j)} \prod_{l>k} q_l^{l-n}
\]

and

\[
\frac{\hat{P}^Y_{\rho_\infty, \rho_\infty, (\lambda_{n-1}, \ldots, \lambda_0)}(q, \overline{u}_k)}{\hat{P}^Y_{0,0,0}(q, \overline{u}_k)} = P^X_{\rho_\infty, \rho_\infty, (\lambda_{n-1}, \ldots, \lambda_0)}(\overline{q})
\]

\[
\cdot \prod_{(i,j)\in\lambda_k} (-1)^{k} q^{k(i-j)} \prod_{l\leq k} q_l^{l-1}
\]

Combining terms in (18), (19), and (20), (17) becomes

\[
P^X_{\rho_0, \rho_0, (\lambda_0, \ldots, \lambda_{n-1})}(q) \hat{E}_\lambda P^X_{\rho_\infty, \rho_\infty, (\lambda_{n-1}, \ldots, \lambda_0)}(\overline{q})
\]
where
\[ \tilde{E}_\lambda = \prod_{(i,j) \in \lambda} v \left( \prod_{l=1}^{k_0} q_{l}^{(m+1)i} \prod_{l=k+1}^{n-1} q_{l}^{(m+1)(n-l)} \right) (-1)^{n(m+1)+1} q^{n(m+1)(i-j)+1} \]

From the observations made in Section 3.2, the following identity is not hard to prove.
\[ \prod_{(i,j) \in \lambda} \left( \prod_{l=1}^{k_0} q_{l}^{(m+1)i} \prod_{l=k+1}^{n-1} q_{l}^{(m+1)(n-l)} \right) q^{n(m+1)(i-j)} = \prod_{(i,j) \in \tilde{\lambda}} q^{(m+1)(i-j)} \]

Therefore, (21) becomes
\[ \hat{P}^{X \rho \rho_0} \lambda (q) E^Z \rho_\rho_0 \lambda (q) \]

where
\[ E^Z \rho_\rho_0 \lambda (q) = v |\lambda| (-1)^{m|\lambda|} \prod_{(i,j) \in \tilde{\lambda}} q^{m_j+(m+2)i+1} \]

is the edge term from the gluing algorithm in Theorem 2.6. This concludes the proof of Theorem 2.2. 

References

[BCR13] A. Brini, R. Cavalieri, and D. Ross. Crepant resolutions and open strings. arXiv:1309.4438, 2013.

[BCY12] J. Bryan, C. Cadman, and B. Young. The orbifold topological vertex. Adv. Math., 229(1):531–595, 2012.

[Beh09] Kai Behrend. Donaldson-Thomas type invariants via microlocal geometry. Ann. of Math. (2), 170(3):1307–1338, 2009.

[BG09] J. Bryan and T. Graber. The crepant resolution conjecture. In Algebraic geometry—Seattle 2005. Part 1, volume 80 of Proc. Sympos. Pure Math., pages 23–42. Amer. Math. Soc., Providence, RI, 2009.

[Cal12] J. Calabrese. On the crepant resolution conjecture for Donaldson-Thomas invariants. arXiv:1206.6524, 2012.

[Lam12] T. Lam. Loop symmetric functions and factorizing matrix polynomials. In Fifth International Congress of Chinese Mathematicians. Part 1, 2, volume 2 of AMS/IP Stud. Adv. Math., 51, pt. 1, pages 609–627. Amer. Math. Soc., Providence, RI, 2012.

[LP12] T. Lam and P. Pylyavskyy. Total positivity in loop groups, I: Whirls and curls. Adv. Math., 230(3):1222–1271, 2012.

[Mac95] I. G. Macdonald. Symmetric functions and Hall polynomials. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[MNOP06] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. Gromov-Witten theory and Donaldson-Thomas theory. I. Compos. Math., 142(5):1263–1285, 2006.

[MOOP11] D. Maulik, A. Oblomkov, A. Okounkov, and R. Pandharipande. Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds. Invent. Math., 186(2):435–479, 2011.
[ORV06] A. Okounkov, N. Reshetikhin, and C. Vafa. Quantum Calabi-Yau and classical crystals. In The unity of mathematics, volume 244 of Progr. Math., pages 597–618. Birkhäuser Boston, Boston, MA, 2006.

[Ros12] D. Ross. The loop Murnaghan-Nakayama rule. J. Algebraic Combin. in press, 2012. Preprint: math:1208.4369.

[RZ13] D. Ross and Z. Zong. The gerby Gopakumar–Mariño-Vafa formula. Geom. Topol., 17(5):2935–2976, 2013.

[RZ14] D. Ross and Z. Zong. Cyclic Hodge integrals and loop Schur functions. arXiv:1401.2217, 2014.

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