ON THE GENERAL ADDITIVE DIVISOR PROBLEM

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Abstract. We obtain a new upper bound for \( \sum_{h \leq H} \Delta_k(N, h) \) for \( 1 \leq H \leq N \), \( k \in \mathbb{N}, k \geq 3 \), where \( \Delta_k(N, h) \) is the (expected) error term in the asymptotic formula for \( \sum_{N<n \leq 2N} d_k(n)d_k(n+h) \), and \( d_k(n) \) is the divisor function generated by \( \zeta(s)^k \). When \( k = 3 \) the result improves, for \( H \geq N^{1/2} \), the bound given in the recent work [1] of Baier, Browning, Marasingha and Zhao, who dealt with the case \( k = 3 \).

In honor of Professor A.A. Karatsuba’s 75th birthday

1. Introduction

Let \( d_k(n) \) denote that (generalized) divisor function, which represents the number of ways \( n \) can be written as a product of \( k (\in \mathbb{N}) \) factors. Thus

\[
\sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} = \zeta(s)^k \quad (\Re s > 1),
\]

where \( \zeta(s) \) is the familiar zeta-function of Riemann. In particular \( d_1(n) \equiv 1 \) and \( d_2(n) = \sum_{\delta|n} 1 \) is the number of positive divisors of \( n \). The function \( d_k(n) \) is a multiplicative function of \( n \), and

\[
d_k(p^\nu) = (-1)^\nu \binom{-k}{\nu} = \frac{k(k+1) \cdots (k+\nu-1)}{\nu!}
\]

for primes \( p \) and \( \nu \in \mathbb{N} \). The general divisor problem deals with the estimation of \( \Delta_k(x) \), the error term in the asymptotic formula (see Chapter 13 of Ivić [3] and Chapter 12 of Titchmarsh [15] for an extensive discussion)

\[
D_k(x) := \sum_{n \leq x} d_k(n) = xp_{k-1}(\log x) + \Delta_k(x),
\]

where

\[
p_{k-1}(\log x) = \text{Res}_{s=1} \left( \zeta(s) k \frac{x^{s-1}}{s} \right).
\]

Since \( \zeta(s) \) is regular in \( \mathbb{C} \) except at \( s = 1 \) where it has a simple pole with residue 1, it transpires that \( p_{k-1}(y) \) is a polynomial of degree \( k - 1 \), whose coefficients may

\[
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\]

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be explicitly evaluated, and in particular $p_1(y) = y + 2\gamma - 1$, where $\gamma$ is Euler’s constant. The important constants $\alpha_k, \beta_k$ are defined as

$$\alpha_k := \inf \left\{ a_k : \Delta_k(x) \ll x^{a_k} \right\},$$

$$\beta_k := \inf \left\{ b_k : \int_1^X |\Delta_k(x)|^2 \, dx \ll X^{1+2b_k} \right\}.$$  

It is known that $\alpha_k \geq \beta_k \geq (k-1)/(2k)$ for all $k \in \mathbb{N}$, and the conjecture that $\alpha_k = \beta_k = (k-1)/(2k)$ for all $k \in \mathbb{N}$ is equivalent to the Lindelöf hypothesis that $\zeta(\frac{1}{2} + it) \ll \varepsilon (|t| + 1)\varepsilon$. Here and later $\varepsilon$ ($>0$) denotes arbitrarily small constants, not necessarily the same ones at each occurrence, while $\ll_{a,b,\ldots}$ means that the implied constant in the $\ll$-symbol depends on $a,b,\ldots$.

The general additive divisor problem is another important problem involving the divisor function $d_k(n)$. It consists of the estimation of the quantity $\Delta_k(x, h)$, given by the formula

$$\sum_{n \leq x} d_k(n)d_k(n+h) = x P_{2k-2}(\log x; h) + \Delta_k(x, h).$$

In (1.4) it is assumed that $k \geq 2$ is a fixed integer, and $P_{2k-2}(\log x; h)$ is a suitable polynomial of degree $2k-2$ in $\log x$, whose coefficients depend on $k$ and $h$, while $\Delta_k(x, h)$ is supposed to be the error term. This means that we should have

$$\Delta_k(x, h) = o(x) \quad \text{as} \quad x \to \infty,$$

but unfortunately (1.5) is not yet known to hold for any $k \geq 3$, even for fixed $h$. However, when we consider the sum

$$\sum_{h \leq H} \Delta_k(x, h),$$

we may reasonably hope that a certain cancelation will occur among the individual summands $\Delta_k(x, h)$, since there are no absolute value signs in (1.6). It turns out that it is precisely the estimation of the sum in (1.6) which is relevant for bounding the integral

$$\int_0^T |\zeta(\frac{1}{2} + it)|^{2k} \, dt,$$

which is of great importance in the theory of the Riemann zeta-function (see the monographs [3, 4, 15]).

For $k = 1$ the sum in (1.6) is trivial, while for $k = 2$ it was extensively studied by many authors, including Kuznetsov [10], Motohashi [13], Ivić & Motohashi [8] and Meurman [12]. The natural next step in (1.6) is to deal with the case $k = 3$, but the works of A.I. Vinogradov and Takhtadžian [19, 20] and A.I. Vinogradov [16, 17, 18] show that the analytic problems connected with the Dirichlet series generated by $d_3(n)d_3(n+h)$ are overwhelmingly hard. The ensuing problems are connected with the group $SL(3,\mathbb{Z})$, and they are much more difficult than the corresponding problems connected with the group $SL(2,\mathbb{Z})$ which appear in the case $k = 2$. The latter involve the spectral theory of the non-Euclidean Laplacian, which was extensively developed in recent times by Kuznetsov (see e.g., [11]), Iwaniec and others (see Motohashi’s monograph [14] for applications of spectral theory to the
theory of $\zeta(s)$). Thus at present in the case $k = 2$ we have sharp explicit formulas, while in the case $k > 2$ we have none.

A.I. Vinogradov [18] conjectured that $\Delta_k(x, h) \ll x^{1 - 1/k}$, without stating for which range of $h$ this sharp bound should hold. Very likely this bound is too strong, and (even for fixed $h$) it seems probable that a power of a logarithm should be included on the right-hand side. More importantly, one hopes that the bound

$$
\sum_{h \leq H} \Delta_k(x, h) \ll_{k, \varepsilon} H x^{1 - 1/k + \varepsilon}
$$

holds uniformly in $H$ for fixed $k \geq 3$ and some $\delta_k > 0$, which was stated in [5]. Note that Vinogradov’s conjecture in the form $\Delta_k(x, h) \ll_{k, \varepsilon} x^{1 - 1/k + \varepsilon}$ trivially implies (1.8), but the important point is that there are no absolute value signs in the sum in (1.8). One can also assume (1.8) to hold in the case $k = 2$ for $1 \leq H \leq x^{(k - 2)/k + \delta_k}$.

For $k = 2$ and fixed $h$ this conjecture was proved by Motohashi [13]. As usual, $f(x) = \Omega(g(x))$ means that $\lim_{x \to \infty} f(x)/g(x) \neq 0$.

The general additive divisor problems is connected to the power moments of $|\zeta(\frac{1}{2} + it)|$ (see e.g., [3] and [4] for an extensive account). In 1996 the first author [5] proved that

$$
\int_0^T |\zeta(\frac{1}{2} + it)|^6 \, dt \ll_{\varepsilon} T^{1 + \varepsilon} + T^{(\alpha + 3\beta - 1)/2 + \varepsilon}
$$

provided that

$$
\sum_{h \leq H} \Delta_3(x, h) \ll_{\varepsilon} H^\alpha x^{\beta + \varepsilon}
$$

holds for $1 \leq H \leq x^{1/3 + \delta_3}$ for some constant $\delta_3 > 0$, $0 \leq \alpha, \beta \leq 1$, $\alpha + \beta \geq 1$. The conjecture (1.8) with $k = 3$ means that we can take $\alpha = 1, \beta = 2/3$ in (1.11) so that the sixth moment in the form

$$
\int_0^T |\zeta(\frac{1}{2} + it)|^6 \, dt \ll_{\varepsilon} T^{1 + \varepsilon}
$$

follows. Note that the best known exponent of $T$ for the right-hand side of the above integral is $5/4$ (see [3, Chapter 8]).

In [6] the research begun in [5] was continued, and a plausible heuristic evaluation of the polynomial $P_{2k - 2}(x; h)$ in (1.4) was made. Yet another (heuristic) evaluation
of the sum in (1.5) was made later by Conrey and Gonek [2] in 2001. Moreover, it was shown in [6] that, for a fixed integer \( k \geq 3 \) and any fixed \( \varepsilon > 0 \), we have

\[
\int_0^T |\zeta(\frac{1}{2} + it)|^{2k} \, dt \ll_{k,\varepsilon} T^{1+\varepsilon} \left( 1 + \sup_{T^{1+\varepsilon} < M < T^{k/2}} \frac{G_k(M;T)}{M} \right),
\]

if, for \( T^{1+\varepsilon} \leq M \ll T^{k/2} \) and \( M < M' \leq 2M \),

\[
G_k(M;T) := \sup_{M \leq x \leq M', 1 \leq t \leq M^{1+\varepsilon}/T} \left| \sum_{h \leq t} \mathbb{D}_k(x, h) \right|.
\]

This result, which generalizes (1.11), provides a direct link between upper bounds for the \( 2^k \)-th moment of \(|\zeta(\frac{1}{2} + it)|\) and sums of \( \mathbb{D}_k(x, h) \) over the shift parameter \( h \). The result also gives an insight as to the limitations of the attack on the \( 2^k \)-th moment of \(|\zeta(\frac{1}{2} + it)|\) via the use of estimates for \( \Delta_k(x, h) \). Of course the problem greatly increases in complexity as \( k \) increases, and this is one of the reasons why in [5] only the case \( k = 3 \) was considered. The case \( k = 2 \) was not treated, since for the fourth moment of \(|\zeta(\frac{1}{2} + it)|\) we have an asymptotic formula with precise results for the corresponding error term (see e.g., [7] and [14]). Note that (1.13)–(1.14) again lead to the sixth moment bound (1.12) if the conjecture (1.8) holds with \( k = 3 \).

2. The general additive divisor problem

The main objective of this note is to study the averaged sum (1.6), when \( k \geq 3 \) is a fixed integer. To this end we introduce more notation, defining

\[
D_k(N, h) := \sum_{N < n \leq 2N} d_k(n) d_k(n + h),
\]

and letting henceforth

\[
\Delta_k(N; h) := D_k(N, h) - \int_N^{2N} \mathfrak{S}_k(x, h) \, dx,
\]

so that \( \Delta_k(N; h) \) in (2.2) differs slightly from (1.4); in fact it equals \( \Delta_k(2N, h) - \Delta_k(N, h) \) in the notation of (1.4). Here we follow the notation of [1], based on the approach of Conrey and Gonek [2], who made conjectures on the high moments of \(|\zeta(\frac{1}{2} + it)|\). Let us also define

\[
\mathfrak{S}_k(x, h) := \sum_{q=1}^{\infty} \frac{c_q(h)}{q^2} Q_k(x, q)^2,
\]

where \( \mu(n) \) is the Möbius function, \( c_q(h) := \sum_{d|\varphi(q)} d\mu(q/d) \) is the Ramanujan sum and \( Q_k(x, q) \) is defined as follows. If \( \varphi(n) \) is the Euler totient function, set

\[
\Psi_{d,e}(s, q, k) := \frac{d\mu(d)\varphi(e)}{\varphi(d)e} \prod_{p|\varphi(q)} \left\{ \left( 1 - \frac{1}{p^s} \right)^k \sum_{\nu=0}^{\infty} \frac{d_k\left( p^{\nu + \nu_{\varphi(q)/d}} \right)}{p^{\nu s}} \right\},
\]
where here and later \( \nu_p(m) \) is the \( p \)-adic valuation of \( m \). Then we define

\[
Q_k(x, q) := \frac{1}{2\pi i} \int_{|s-1|=1/8} \zeta(s)^k \sum_{d|q} \sum_{e|d} \Psi_{d,e}(s, q, k) \left( \frac{ex}{dq} \right)^{s-1} ds
\]

\[
= \text{Res}_{s=1} \left\{ \zeta(s)^k \sum_{d|q} \sum_{e|d} \Psi_{d,e}(s, q, k) \left( \frac{ex}{dq} \right)^{s-1} \right\},
\]

by the residue theorem. Thus \( Q_k(x, q) \) is a polynomial of degree \( 2k-2 \) whose coefficients depend on \( q \), and may be explicitly evaluated. The work of Conrey-Gonek (op. cit.) predicts, as stated in (2.2), that \( D_k(N, h) \) is well approximated by

\[
\int_{N-2N}^N S_k(x, h) \, dx = N \times \text{a polynomial in } \log N \text{ of degree } 2k-2, \quad \text{all of whose coefficients depend on } h \text{ and } k.
\]

This is in agreement with [5] (when \( k = 3 \) and [6] (in the general case), although the shape of the polynomial in question is somewhat different. Conrey and Gonek even predict that uniformly

\[
\Delta_k(N; h) \ll \varepsilon N^{1/2+\varepsilon} \quad \text{for } 1 \leq h \leq N^{1/2}.
\]

Remark 1. Note that (2.3), in the range \( N^{1/6+\varepsilon} \leq H \leq N^{1-\varepsilon} \), provides an asymptotic formula for the averaged sum \( \sum_{h \leq H} D_3(N, h) \) (see (2.1)). However, it should be noted that no asymptotic formula for the individual \( D_3(N, h) \) has been found yet, and in general for \( \Delta_k(N; h) \) when \( k \geq 3 \). In fact, it is worth pointing out that when \( 1 \leq H \leq N^{1/6} \), the bound in (2.3) is worse than the trivial bound \( HN^{1+\varepsilon} \). Namely we have

\[
\sum_{h \leq H} D_k(N, h) \ll \varepsilon \sum_{h \leq H} \sum_{N<n<2N} (n+h)^{\varepsilon/2} \ll \varepsilon (HN)^{1+\varepsilon/2} \ll \varepsilon HN^{1+\varepsilon}.
\]

On the other hand we have

\[
\sum_{h \leq H} \int_{N}^{2N} \mathcal{S}_k(x, h) \, dx \ll \varepsilon HN^{1+\varepsilon},
\]
which is obvious from (3.8). Hence by (2.2) it follows that
\begin{equation}
\sum_{h \leq H} \Delta_k(N; h) \ll_{\varepsilon} HN^{1+\varepsilon} \quad (1 \leq H \leq N),
\end{equation}
and clearly (2.5) for \( k = 3 \) improves (2.3) for \( 1 \leq H \leq N^{1/6} \). The aim of this note is to give a bound for the sum in (1.6), or equivalently for the average of (2.2), which for \( k = 3 \) improves (2.3) for a certain range of \( H \). The result is contained in the following

**Theorem 1.** For fixed \( k \geq 3 \) we have
\begin{equation}
\sum_{h \leq H} \Delta_k(N; h) \ll_{\varepsilon} N^\varepsilon (H^2 + N^{1+\beta_k}) \quad (1 \leq H \leq N),
\end{equation}
where \( \beta_k \) is defined by (1.3).

Note that we have \( \beta_3 = 1/3, \beta_4 = 3/8 \) (see Chapter 13 of [3]), \( \beta_5 \leq 9/20 \) (see Zhang [21]), \( \beta_6 \leq 1/2 \), etc. For a discussion of the values of \( \alpha_k \) and \( \beta_k \), see also the paper by Ouellet and Ivić [9].

**Corollary 1.** We have, for \( 1 \leq H \leq N \),
\begin{align*}
\sum_{h \leq H} \Delta_3(N; h) &\ll_{\varepsilon} N^\varepsilon (H^2 + N^{4/3}), \\
\sum_{h \leq H} \Delta_4(N; h) &\ll_{\varepsilon} N^\varepsilon (H^2 + N^{11/8}), \\
\sum_{h \leq H} \Delta_5(N; h) &\ll_{\varepsilon} N^\varepsilon (H^2 + N^{29/20}), \\
\sum_{h \leq H} \Delta_6(N; h) &\ll_{\varepsilon} N^\varepsilon (H^2 + N^{3/2}).
\end{align*}

**Remark 2.** Since it is known that \( \beta_k < 1 \) for any \( k \), this means that the bound in (2.6) improves on the trivial bound \( HN^{1+\varepsilon} \) in the range \( N^{\beta_k+\varepsilon} \leq H \leq N^{1-\varepsilon} \). Our result thus supports the assertion that \( \Delta_k(N; h) \) is really the error term in the asymptotic formula for \( D_k(N, h) \), as given by (3.1) and (3.2). In the case when \( k = 3 \), we have by (2.7) an improvement of (2.3) when \( H \geq N^{1/2} \).

3. **Proof of Theorem 1**

We begin by noting that obviously
\[
\sum_{h \leq H} d_k(n + h) = \sum_{m \leq n + H} d_k(m) - \sum_{m \leq n} d_k(m).
\]
Therefore by (1.1)–(1.2) and (2.1)–(2.2) we can write
\begin{align*}
\sum_{h \leq H} \Delta_k(N, h) &= \sum_{N < n \leq 2N} d_k(n) \sum_{h \leq H} d_k(n + h) - \sum_{h \leq H} \int_N^{2N} \mathcal{S}_k(x, h) \, dx \\
&= M_k(N, H) + R_k(N, H) - \sum_{h \leq H} \int_N^{2N} \mathcal{S}_k(x, h) \, dx,
\end{align*}
(3.1)
say, where

\[
M_k(N, H) := \sum_{N < n \leq 2N} d_k(n) \text{Res}_{s=1} \left( \zeta(s)^k \frac{(n + H)^s - n^s}{s} \right),
\]

\[
R(N, H) := \sum_{N < n \leq 2N} d_k(n) (\Delta_k(n + H) - \Delta_k(n)),
\]

where \(\Delta_k(x)\) is defined by (1.1). It is rather easy to estimate \(R_k(N, H)\). Namely since \(d_k(n) \ll n^\varepsilon\), we have trivially

\[
R_k(N, H) \ll \varepsilon N \sum_{n < 3N} |\Delta_k(n)|.
\]

For \(n < t < n + 1\), we obviously have

\[
\Delta_k(n) - \Delta_k(t) = t p_{k-1}(\log t) - n p_{k-1}(\log n) \ll (\log n)^{k-1}.
\]

Thus

\[
R_k(N, H) \ll \varepsilon N \sum_{n < 3N} \int_n^{n+1} |\Delta_k(n)| \, dt \ll \varepsilon N \int_n^{n+1} |\Delta_k(t)| \, dt + N^{1+\varepsilon}
\]

\[
\ll \varepsilon N \int_1^{4N} |\Delta_k(t)| \, dt + N^{1+\varepsilon} \ll \varepsilon N \left( \int_1^{4N} |\Delta_k(t)|^2 \, dt \right)^{1/2} + N^{1+\varepsilon} \ll \varepsilon N^{1+\beta_k + \varepsilon},
\]

where we used the Cauchy-Schwarz inequality for integrals and the mean square bound (1.3) in the last step.

To estimate \(M_k(N, H)\), set

\[
u_k(x) := \text{Res}_{s=1} \left( \zeta(s)^k \frac{(x + H)^s - x^s}{s} \right)\]

Then we can write

\[
M_k(N, H) = \int_N^{2N+0} u_k(x) \, dD_k(x).
\]

But we have, since

\[D_k(x) = \text{Res}_{s=1} \left( \zeta(s)^k \frac{x^s}{s} \right) + \Delta_k(x)\]

in view of (1.1) and (1.2),

\[
M_k(N, H) = \int_N^{2N} u_k(x) \text{Res}_{s=1} \left( \zeta(s)^k \frac{x^{s-1}}{s} \right) \, dx + \int_N^{2N} u_k(x) \, d\Delta_k(x).
\]
Further note that

\begin{equation}
\begin{aligned}
\left. u_k(x) = y p_{k-1}(\log y) \right|_{x}^{x+H} & \ll H (\log x)^{k-1}, \\
\left. u'_k(x) = \Res_{s=1} \zeta(s)^k \{ (x + H)^{s-1} - x^{s-1} \} \right|_{x} & \ll x^\varepsilon.
\end{aligned}
\end{equation}

On integrating by parts and using (1.3) and (3.4) we obtain, similarly to (3.2),

\begin{equation}
\begin{aligned}
\int_{N}^{2N} u_k(x) \, d\Delta_k(x) = u_k(x) \Delta_k(x) \bigg|_{N}^{2N} & - \int_{N}^{2N} \Delta_k(x) u'_k(x) \, dx \\
& \ll \varepsilon H N^{\alpha_k} + N^{1+\beta_k}.
\end{aligned}
\end{equation}

As for the other integral in (3.3), note that

\begin{equation}
\frac{(x + H)^s - x^s}{s} = x^s \left\{ 1 + \frac{s H}{x} + \left( \frac{s}{2} \right) \frac{H^2}{x^2} + \cdots - 1 \right\}.
\end{equation}

This gives

\begin{equation}
\begin{aligned}
\int_{N}^{2N} u_k(x) \Res_{s=1} (\zeta(s)^k x^{s-1}) \, dx = H \int_{N}^{2N} \left( \Res_{s=1} \zeta(s)^k x^{s-1} \right)^2 \, dx & + O_\varepsilon \left( H^2 N^\varepsilon \right).
\end{aligned}
\end{equation}

Therefore from (3.3), (3.5) and (3.6) we obtain

\begin{equation}
\begin{aligned}
M_k(N, H) = H \int_{N}^{2N} \left( \Res_{s=1} \zeta(s)^k x^{s-1} \right)^2 \, dx & + O_\varepsilon \left( H^2 N^\varepsilon + NH^{\alpha_k} + N^{1+\beta_k} \right).
\end{aligned}
\end{equation}

Next we shall prove that

\begin{equation}
\begin{aligned}
\sum_{h \leq H} \int_{N}^{2N} \mathcal{G}_k(x, h) \, dx = H \int_{N}^{2N} \left( \Res_{s=1} \zeta(s)^k x^{s-1} \right)^2 \, dx & + O_\varepsilon \left( N^{1+\varepsilon} \right).
\end{aligned}
\end{equation}

The case of $k = 3$ has been treated in [1]. Here we repeat the same argument with some simplification in the general case, obtaining (3.8).

First write

\begin{equation}
x^{s-1} = \sum_{n=0}^{\infty} \frac{(\log x)^n}{n!}(s - 1)^n.
\end{equation}

Since $\Psi_{d,e}(s, q)$ and $(s - 1)^n \zeta(s)^k$ with $n \geq k$ are holomorphic for $\Re s > 0$, Cauchy’s theorem allows us to deduce that

\begin{equation}
Q_k(x, q) = \frac{1}{2\pi i} \sum_{n=0}^{k-1} \sum_{|s-1|=1/8} \zeta(s)^k \sum_{d|q} \sum_{e|d} \Psi_{d,e}(s, q) \frac{(\log(dx/eq))^n}{n!}(s - 1)^n \, ds.
\end{equation}
Clearly for $\Re s > \frac{1}{2}$, we have

$$
\Psi_{d,e}(s, q) \leq \frac{d}{\varphi(d)e} \prod_{p|(eq/d)} \left\{ \left( 1 + \frac{1}{p^{1/2}} \right)^k \sum_{\nu=0}^{\infty} \frac{d_k(p^{\nu+\nu_p(eq/d)})}{p^{\nu s}} \right\} \\
\leq \varepsilon \frac{d}{\varphi(d)e} \prod_{p|(eq/d)} \left\{ \left( 1 + \frac{1}{p^{1/2}} \right)^k p^{\nu_p(eq/d)/4} \sum_{\nu \geq 0} \frac{p^{\nu e/4}}{p^{\nu/2}} \right\} \\
\leq \varepsilon q^{\varepsilon/2}.
$$

Thus

(3.9) \quad Q_k(x, q) \leq \varepsilon, k q^{\varepsilon} (\log x)^{k-1},

where the implied constant depends only on $\varepsilon$ and $k$.

In view of (3.9) and the bound $|c_q(h)| \leq (h, q)$, we have

(3.10) \quad \sum_{h \leq H} \sum_{q > H} \frac{c_q(h)}{q^2} Q_k(x, q)^2 \leq (\log x)^{k-1} \sum_{h \leq H} \sum_{q > H} \frac{(h, q)}{q^{2-\varepsilon}}

\leq \varepsilon H^{\varepsilon} (\log x)^{k-1}.

On the other hand, it is well known that $\sum_{h \leq q} c_q(h) = 0$ if $q > 1$. From this it is easy to deduce that

$$
\sum_{h \leq H} c_q(h) = \begin{cases} 
H + O(1) & \text{if } q = 1, \\
O_\varepsilon(q^{1+\varepsilon}) & \text{if } q > 1.
\end{cases}
$$

With the help of this relation and (3.9), we can write

(3.11) \quad \sum_{h \leq H} \sum_{q \leq H} \frac{c_q(h)}{q^2} Q_k(x, q)^2

\begin{align*}
&= \{ H + O(1) \} Q_k(x, 1)^2 + O((\log x)^{k-1} \sum_{1 < q \leq H} \frac{1}{q^{1-\varepsilon}}) \\
&= H \left( \text{Res}_{s=1} \zeta(s)^k x^{s-1} \right)^2 + O((\log x)^{k-1} H^\varepsilon),
\end{align*}

where we have used the fact that

$$
Q_k(x, 1) = \text{Res}_{s=1} \left( \zeta(s)^k x^{s-1} \right) \ll_k (\log x)^{k-1}.
$$

By combining (3.10) and (3.11), we obtain (3.8).

From (3.1), (3.2), (3.7) and (3.8) we obtain

(3.12) \quad \sum_{h \leq H} \Delta_k(N, H) \ll \varepsilon N^\varepsilon \left( H^2 + H N^{\alpha_k} + N^{1+\beta_k} \right) \quad (1 \leq H \leq N).

But we always have

(3.13) \quad \alpha_k \leq \frac{1}{2} + \frac{1}{2} \beta_k.
To see this note that, for $1 \leq H \leq x$, the defining relation (1.1) and $d_k(n) \ll \varepsilon n^\varepsilon$ give

$$
\Delta_k(x) - \frac{1}{H} \int_x^{x+H} \Delta_k(y) \, dy = \frac{1}{H} \int_x^{x+H} (\Delta_k(x) - \Delta_k(y)) \, dy
\ll \varepsilon \frac{1}{H} \int_x^{x+H} \{|D_k(x) - D_k(y)| + O(x^\varepsilon)\} \, dy
\ll \varepsilon H x^\varepsilon.
$$

This gives, by the Cauchy-Schwarz inequality for integrals and (1.3),

$$
\Delta_k(x) \ll \varepsilon \frac{1}{H} \int_x^{x+H} |\Delta_k(y)| \, dy + H x^\varepsilon
\ll \varepsilon x^{1+\beta_k+\varepsilon} H^{-1} + H x^\varepsilon
\ll \varepsilon x^{(1+\beta_k)/2+\varepsilon}
$$

with $H = x^{(1+\beta_k)/2}$. Hence

$$
\Delta_k(x) \ll \varepsilon x^{(1+\beta_k)/2+\varepsilon}
$$

and (3.13) follows. Now in (3.12) we have $HN^{\alpha_k} \leq H^2$ for $H \geq N^{\alpha_k}$. If $H \leq N^{\alpha_k}$, then $HN^{\alpha_k} \leq N^{2\alpha_k} \leq N^{1+\beta_k}$ by (3.13). Thus the term $HN^{\alpha_k}$ in (3.12) can be discarded, and (2.6) follows. This completes the proof of the Theorem.

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