The Kraichnan-Kazantsev dynamo

D. Vincenzi
CNRS, Observatoire de la Côte d’Azur, B.P. 4229, 06304 Nice Cedex 4, France
Dipartimento di Fisica, Università di Genova, Via Dodecaneso, 33, I-16142, Genova, Italy
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The problem of the dynamo effect for a Kraichnan incompressible helicity-free velocity field is considered. Exploiting a quantum formalism first introduced by Kazantsev (A.P. Kazantsev, Sov. Phys. JETP 26, 1031-1034 (1968)), we show that a critical magnetic Reynolds number exists for the presence of dynamo. The value of the Prandtl number influences the spatial distribution of the magnetic field and its growth in time. The magnetic field correlation length is always the largest between the diffusive scale and the viscous scale of the flow. In the same way the field growth is characterized by a time scale that corresponds to the largest between the diffusive and the viscous characteristic time.

**Keywords**: Turbulent transport, Magnetohydrodynamics, Dynamo effect, Kraichnan statistical ensemble

### I. INTRODUCTION

The study of magnetic fields generated by the turbulent motion of a charged conducting fluid is relevant for several astrophysical applications [1]. An inhomogeneous flow of a charged fluid is able to locally produce a magnetic field. The advected field in turn generates electric currents in the fluid and these dissipate the magnetic energy because of the finite resistivity. Depending on the properties of the flow, magnetic field creation or dissipation can prevail. To determine in general terms the presence or not of dynamo is a daunting task. However, as we will see, there are particular models which allow for a detailed treatment.

The evolution of an initially given magnetic field \( B(r, 0) \) in an incompressible flow of a conducting fluid is determined by the following equations [1]

\[
\begin{align*}
\frac{\partial B}{\partial t} + (\mathbf{v} \cdot \nabla) B &= (B \cdot \nabla) \mathbf{v} + \kappa \nabla^2 B \\
\nabla \cdot B &= 0
\end{align*}
\]

where \( \mathbf{v}(r, t) \) is the velocity field. The magnetic diffusivity \( \kappa \), which is assumed to be uniform and constant, is proportional to the inverse of the electric conductivity of the fluid.

In Eqs. (1) the term \( (\mathbf{v} \cdot \nabla) B \) is a purely advective contribution that preserves the magnetic energy. The stretching term \( (B \cdot \nabla) \mathbf{v} \) acts either as an energy source or as a sink depending on the local properties of the flow. Finally, the diffusive term \( \kappa \nabla^2 B \) is responsible for the ohmic dissipation at small scales and balances with the inertial terms at the diffusive scale \( r_d \).

The relative importance of the two contributions on the right-hand side of (1) is given by the magnetic Reynolds number \( R_e = U L / \kappa \), where \( L \) denotes the integral scale of the flow and \( U \) is the characteristic velocity at such scale. \( R_e \) can be regarded as a dimensionless measure of the fluid conductivity. For \( R_e \to 0 \) the diffusion dominates and the magnetic energy density (proportional to \( B^2 \)) always decays to zero in time. In the opposite limit, \( R_e \to \infty \), the diffusion term is relevant only at very small scales and the magnetic field is almost frozen in the fluid. We can expect that at high magnetic Reynolds numbers the flow is able to enhance the magnetic field, producing a consequent growth in time of \( B^2 \). The last process is called dynamo effect, referring to the energy transfer from the velocity field to the magnetic one.

The field \( \mathbf{B} \) acts on the velocity by means of the Lorentz force, which yields a term proportional to \( (\mathbf{B} \cdot \nabla) \mathbf{v} \) in Navier-Stokes equations. In general, it would be necessary to take into account such feedback action on \( \mathbf{v} \). However, since we are interested in understanding if the initial generation of the magnetic field is a persistent situation or not, we can assume for the initial conditions \( B^2 \ll v^2 \) and neglect the Lorentz force contribution. Under this hypothesis the evolution equations (1) are totally uncoupled from Navier-Stokes equations. Following this kinematic approach, we thus proceed as if \( \mathbf{v} \) was an assigned random field: given the initial condition \( \mathbf{B}(r, 0) \) and appropriate boundary conditions, Eqs. (1) completely determine the magnetic field evolution.

For the prescribed velocity we refer to the Kraichnan statistical ensemble [2], in which \( \mathbf{v} \) is assumed Gaussian, homogeneous, isotropic and \( \delta \)-correlated in time. The reason for this choice is that analytical results can be obtained [3]. A real turbulent flow is characterized by two scales: the integral scale \( L \) and the viscous scale \( \eta \), at which the dissipation term and the transport one balance in Navier-Stokes equations. The velocity structure function is supposed to be smooth for \( r \ll \eta \), to scale as \( r^\xi \) (0 \( \leq \xi \leq 2 \)) in the inertial range \( \eta \ll r \ll L \) and to approach a constant value at scales much larger than \( L \). The parameter \( \xi \) represents the Hölder exponent of the structure function and can be thought of as a measure of the field roughness: for \( \xi = 2 \) the velocity is smooth in space, while with \( \xi = 0 \) we describe a diffusive field. It is well known that the magnetic dynamo can emerge for an helical flow due to the \( \alpha \)-effect [4]. Here we will restrict to a parity invariant statistical ensemble, which does not give rise to
The analysis of dynamo effect is made easier by a simple quantum mechanics formulation, first introduced by Kazantsev \cite{3}. Indeed, on account of the $\delta$-correlation in time of the Kraichnan velocity field, the single time correlation function for the magnetic field $\langle B_i(x,t)B_j(x+r,t) \rangle$ can be expressed in terms of a function that satisfies a one-dimensional Schrödinger-like equation. The problem of the dynamo effect can thus be mapped into that of studying the bound states of a quantum particle in a given potential that only depends on the velocity correlation function. In particular, the ground state energy $E_0$ of such potential will turn out to be the asymptotic magnetic field growth rate. The technique used to compute $E_0$ for different quantum potentials is the variation-iteration method described in appendix A.

The aim of this paper is to single out the role that the velocity scales $L$ and $\eta$ play in dynamo theory. To this purpose it is interesting to study the magnetic field generation as the magnetic Reynolds number $Re_m$ and the Prandtl number $Pr = \nu/\kappa$ are varied ($\nu$ denotes the viscosity of the fluid). Indeed, they are related to the relative importance of the characteristic scales in the physical problem by the expressions $Re \simeq L/r_d$ and $Pr \simeq (\eta/r_d)^\xi$.

In Ref. \cite{3} Kazantsev finally restricts himself to the limiting case of $Re_m \to \infty$ and $Pr \to 0$ and proves that dynamo can take place only for a velocity scaling exponent in the range $1 \leq \xi \leq 2$. Here we show that in this range of $\xi$ the characteristic time of the dynamo effect is of order of the diffusive time $t_d = r_d^2/\kappa = O(|B|/|\eta \nabla^2 B|)$ and the magnetic field correlation length is of order $r_d$. We also provide a numerical computation of the growth rate vs $\xi$.

Then, we analyze the case of finite $Re_m$ and prove that a critical magnetic Reynolds number exists: if $Re_m$ is sufficiently small, the dynamo does not ever take place, even for $1 \leq \xi \leq 2$.

Finally, we show that the Prandtl number does not affect the presence of dynamo, but only determines the magnetic field correlation length and the characteristic growth time. If $Pr < 1$, the field $B$ has a correlation length of order $r_d$ and it grows with a characteristic time scale of order $t_d$. On the contrary, if $Pr > 1$, the correlation length is of order $\eta$ and the characteristic time of order $t_v$, where $t_v$ represents the viscous time for the velocity field: $t_v = \eta^2/\nu = O(|\mathbf{v}|/|\nu \nabla^2 \mathbf{v}|)$.

This paper is organized as follows. After this general introduction to the problem, in section II we define more precisely the Kraichnan model and, following Kazantsev \cite{3}, we describe the quantum formalism mentioned above. In particular we give the Schrödinger equation that is the central point of this quantum approach. In section III we revisit the case of infinite magnetic Reynolds number and zero Prandtl number. Then, starting from these results, we analyse the effect of finite $Re_m$ and we study how a nonzero $Pr$ influences the magnetic field evolution in time.

**II. THE KRAICHNAN-KAZANTSEV MODEL**

In this section we recall in detail the quantum formalism introduced by Kazantsev in Ref. \cite{3}. The random velocity field is assumed to be incompressible, Gaussian, homogeneous, isotropic, parity invariant, and $\delta$-correlated in time. Under these hypotheses it is completely defined by its correlation matrix

$$
\langle v_i(x,t)v_j(x',t) \rangle = \delta(t-t')D_{ij}(r)
= \delta(t-t')\left[ D_{ij}(0) - S_{ij}(r) \right] \quad (r = x - x'),
$$

where $S_{ij}(r)$ denotes the structure function of the field $\mathbf{v}$.

The $\delta$-correlation in time of $\mathbf{v}$ is an essential property in order to write a closed equation for the magnetic field correlation function that, under a suitable transformation, reduces to a Schrödinger-like equation.

We impose homogeneous and isotropic initial conditions for $B$. Therefore, on account of the translational and rotational invariance of Eqs. \cite{1}, the magnetic field maintains homogeneous and isotropic statistics at every time $t$. Its correlation tensor has thus the form (see, e. g., Ref. \cite{3})

$$
\langle B_i(x,t)B_j(x',t) \rangle = G_1(r,t)\delta_{ij} + G_2(r,t)\frac{r_ir_j}{r^2}.
$$

Because of the solenoidality condition $\nabla \cdot \mathbf{B} = 0$, the functions $G_1$ and $G_2$ are related by the following differential equation

$$
\frac{\partial G_1}{\partial r} = -\frac{1}{r^2} \frac{\partial}{\partial r} (G_2 r^2). \quad (4)
$$
The covariance of $B$ is then completely described by a single scalar function, e. g., its trace $H(r, t) = 3G_1(r, t) + G_2(r, t)$. The dynamo effect will correspond to an unbounded growth in time of $H(r, t)$.

The correlation function $H(r, t)$ can be transformed into another function $\Psi(r, t)$ that solves the imaginary time Schrödinger equation

$$-\frac{\partial \Psi}{\partial t} + \left[ \frac{1}{m(r)} \frac{\partial^2}{\partial r^2} - U(r) \right] \Psi = 0 \tag{5}$$

in which the mass and the potential depend on $r$ only through $S_\mu(r)$. (For the details see the appendix \cite{8} and Ref. \cite{3}).

To study the dynamo effect it is useful to put in evidence the time dependence of $\Psi$. As usual in quantum mechanics, we thus expand the ‘wave function’ $\Psi$ in terms of the ‘energy’ eigenfunctions $\Psi(r, t) = \sum_E \psi_E(r)e^{-Et}dE$ (or $\Psi(r, t) = \sum_E \psi_E(r)e^{-Et}$ for discrete energy levels) and get the ‘stationary’ equation

$$\frac{1}{m(r)} \frac{d^2\psi_E}{dr^2} + [E - U(r)]\psi_E = 0. \tag{6}$$

Referring to the meaning of $\Psi$, it is clear that an unbounded growth of the magnetic field corresponds to the existence of negative energies in Eq. (6). In particular, it is the sign of the ground state energy $E_0$ that determines the presence of dynamo and its value eventually represents the asymptotic growth rate of the magnetic field. Indeed, in this case it is the ground state $\psi_E, e^{-E_0t}$ that dominates the growth in time. (Recall that the negative energy levels of a Schrödinger equation are always discrete).

By looking at the variational expression for the eigenvalues in (6)

$$E = \frac{\int mU\psi^2_Edr + \int (\psi'_E)^2dr}{\int m\psi^2_Edr}, \tag{7}$$

one can easily conclude that the presence of dynamo effect is equivalent to the existence of bound states for a quantum particle of unit ($r$-independent) mass in the potential $V(r) = m(r)U(r)$. Therefore, in order to state if dynamo can take place for a given velocity field, it is sufficient to study the properties of $V$.

Having summarized the quantum mechanics formalism for a magnetic field transported by a Kraichnan turbulent flow, in the next section we study the dynamo effect for a velocity correlation function that mimics the real physical situation. In particular, we numerically compute $E_0$ and describe the properties of the ground state eigenfunction as $Re_m$ and $Pr$ are varied. From this analysis we are able to obtain information about the critical magnetic Reynolds number, the correlation length of the magnetic field, the asymptotic behaviours of its correlation function, and the characteristic time-scale of the magnetic field growth.

### III. TURBULENT DYNAMO

We consider the realistic situation of a structure function $S_\mu(r)$ that scales as $r^2$ for $r \ll \eta$, as expected in the viscous range, as $r^\xi$ ($0 \leq \xi \leq 2$) in the inertial range $\eta \ll r \ll L$, and tends to a constant value $D_\mu(0)$ for $r \gg L$.

The case $\xi = 0$ corresponds to the diffusive behaviour, while the other limit $\xi = 2$ describes a velocity field that is smooth at all scales below the correlation length $L$. For the other values of $\xi$, in the inertial range $S_\mu(r)$ is only an Hölder continuous function of $r$ with exponent $\xi/2$. The parameter $\xi$ thus represents a measure of the field roughness.

An explicit expression for the velocity correlation tensor, which has the desired scaling properties, is, for example,

$$D_{ij}(r) = \int e^{ik \cdot r}D_{ij}(k)d^3k \tag{8}$$

with

$$D_{ij}(k) = D_0 \frac{e^{-\eta k}}{(k^2 + L^{-2}\eta \xi + 3\eta /2)}P_{ij}(k). \tag{9}$$

The solenoidal projector $P_{ij}(k) = (\delta_{ij} - k_ik_j/k^2)$ ensures the incompressibility of the velocity field.

In what follows we refer to Eq. (9) whenever we show numerical computations that exemplify our conclusions. However, it should be noted that our results are general: they depend only on the qualitative properties of $S_\mu(r)$ and not on its explicit form.
A. Fully developed turbulent dynamo

We first consider the limiting case of $Re_m \to \infty$ and $Pr \to 0$. Under these conditions the diffusive scale $r_d$ is in the inertial range and the presence of the cutoffs $L$ and $\eta$ is neglected: only the scaling behaviour $r^\xi$ ($0 \leq \xi \leq 2$) is considered for the velocity structure function.

The general expression of $S_{ij}(r)$ for an homogeneous, isotropic, parity invariant, incompressible field that scales as $r^\xi$ is

$$
\lim_{\eta \to 0} S_{ij}(r) = D_1 r^\xi \left( (2 + \xi) \delta_{ij} - \xi \frac{r_i r_j}{r^2} \right)
$$

(10)

where the coefficient $D_1$ has the dimensions of length$^{(2-\xi)}/$time.

In this limit the total energy $D_{\alpha}(0)$ diverges with the infrared cutoff as $L^\xi$.

In order to analyze the existence of the dynamo, let us turn to the quantum formulation described above. The potential $V$ has the following asymptotic behaviours (see the appendix A and Ref. [3] for the complete expression)

$$
V(r) \sim \begin{cases} 
2/r^2 & r \ll r_d \\
(2 - \frac{1}{2} \xi - \frac{3}{4} \xi^2)/r^2 & r_d \ll r.
\end{cases}
$$

(11)

For sufficiently small $\xi$ the potential is positive for all $r$, it does not generate bound states and therefore the dynamo cannot take place. For larger $\xi$, $V$ is repulsive up to $r \simeq r_d$ and becomes attractive at infinity (Fig. 3). A quantum mechanical analysis based on asymptotic behaviours [4] allows to establish that $\xi = 1$ is the exact threshold for the dynamo effect [4].

If $0 \leq \xi \leq 1$ the turbulent flow alone is unable to increase the magnetic field and $B^2$ decays in time. For those values of $\xi$, the presence of a forcing term in Eq. (1) is necessary to obtain a statistically stationary state. See Vergassola [5] for the detailed analysis.

From now on we restrict to the values $1 \leq \xi \leq 2$, for which the dynamo is present.

If Eq. (6) is rewritten in a rescaled form by means of the transformation

$$
\psi(r) = \eta(r, t) \sqrt{D_0(0)} / \kappa
$$

(12)

where $\psi(r)$ depends only on the scaling exponent $\xi$ and $t_d$ is the characteristic time of magnetic diffusion.

We have already noted that the ground state eigenfunction dominates the evolution in time and that $E_0$ is the asymptotic magnetic growth rate. We numerically compute $\epsilon_0(\xi)$ as a function of $\xi$ by the variation-iteration method described in the appendix A. The quantity $\epsilon_0$ grows with $\xi$ as shown in Fig. 3. When $\xi$ tends to one, $\epsilon_0$ approaches zero and the bound states disappear. In the other limit, $\epsilon_0$ reaches the value $15/2$ according to the results of Kazantsev [3]. An estimation for $\epsilon_0$ vs $\xi$ already appears in Ref. [3], but there the results are limited to the values $1.25 \leq \xi \leq 2$. Moreover, the numerical computations in that paper are performed by a variational method based on the particular guess $r^2 e^{-\beta r}$ for the eigenfunction $\psi_{E_0}$. This ansatz is correct for $r \ll r_d$, but it fails to capture the right behaviour for $r \gg r_d$. Indeed, if we insert the asymptotic behaviours (13) in Eq. (4), we find that, for $1 < \xi < 2$, $\psi_{E_0}(r)$ shows for $r > r_d$ a stretched exponential decay with characteristic scale $r_d$ and stretching exponent $(2 - \xi)/2$ (Fig. 3).

The variation-iteration method we used (see the appendix A) presents the big advantage of not requiring an explicit form for $\psi_{E_0}$. The algorithm provides as results both the eigenvalue and the corresponding eigenfunction.

From the expressions of $\psi_{E_0}(r)$ we can recover the behaviour of $H(r, t)$ (see the definition (B2) in appendix B). We have that, for $r \ll r_d$, $H(r, t)$ is approximately constant, while, if $1 < \xi < 2$, the magnetic field correlation function decays for $r \gg r_d$ as a stretched exponential with scale $r_d$

$$
H(r, t) \propto e^{-\beta (r/r_d)^{(2-\xi)/2}} \quad (r_d \ll r \ll L).
$$

(13)

The prefactor

$$
\beta = \frac{\sqrt{2|\epsilon_0(\xi)|}}{2 - \xi}
$$

(14)

depends on the growth rate $\epsilon_0(\xi)$. We can thus conclude that, for $1 < \xi < 2$, the magnetic field has a spatial distribution characterized by structures whose scales are of order $r_d$. The cases $\xi = 2$ and $\xi = 1$ have to be treated separately. Indeed, the asymptotic properties cannot be deduced
directly from Eq. (11).

The smooth case is solved by Chertkov et al. in Ref. [11] by a Lagrangian approach that relates the growth rate to the Lyapunov exponents. There is a big difference between the situation of a smooth velocity field and one that is just Hölder continuous. In the former case the correlation function is found to depend on the spatial coordinate as $H(r,t) \propto r^{-5/2}$ (equivalent to $\psi_{E_0}(r) \propto r^{1/2}$), which implies the presence of structures with at least one dimension of inertial range size. Actually the magnetic field in the smooth case has been shown to be characterized by strip-like objects.

Finally, the case $\xi = 1$ can be solved exactly. Indeed, the appropriate ground state eigenfunction of Eq. (11) is (recall that $\xi = 1$ is the threshold for dynamo and hence $E_0 = 0$)

$$\psi_0(x) = C \frac{\sqrt{1+x} \left(-2x + (2+x) \log(1+x)\right)}{x}, \quad (x = r/r_d),$$

(15)

where the constant $C$ is related to the value of $H(0,\cdot)$ by the relation $C = 3\sqrt{\kappa} r_d^2 H(0,\cdot)$.

If we neglect logarithmic corrections, the asymptotic behaviour of $\psi_0$ for $r \gg r_d$ is $\psi_0(r) \propto r^{3/2}$, which yields again $H(r,\cdot) \propto r^{-5/2}$ for $r \gg r_d$.

The results we have outlined in this section will be useful in the following to describe the general case where the velocity energy spectrum has an infrared and an ultraviolet cutoff. Indeed, we will study a structure function that for $r \ll \eta$ scales as $r^2$ and so takes the $\xi = 2$ behaviour, while for $r \gg L$ tends to a constant value like in the diffusive case $\xi = 0$.

**B. Finite Reynolds effect**

Let us analyze the situation of finite $Re_m$ (and zero Prandtl number). The principal fact is that a large scale cutoff $L$ appears for velocity field correlations. The diffusive scale $r_d$ is again within the inertial range of the velocity fluctuations, and the presence of the viscous cutoff can be neglected. The velocity structure function therefore scales as $r^2$ for $r \ll L$ and tends to $D_{ii}(0)$ for $r \gg L$.

Therefore, the potential $V$ behaves as in the previous case for $r \ll L$, while it takes the $\xi = 0$ behaviour for $r \gg L$

$$V(r) \sim \begin{cases} 
2/r^2 & r \ll r_d \\
\left(2 - \frac{3}{2} \xi - \frac{3}{4} \xi^2\right)/r^2 & r_d \ll r \ll L \\
2/r^2 & L \ll r.
\end{cases}$$

(16)

The main consequence of a finite $Re_m$ is that $V$ is repulsive also at large scales. It is thus clear that, for sufficiently high $Re_m$, a potential well is present at scales of order $r_d$. On the contrary, if $Re_m$ is too small, the well can be absent or anyway not deep enough to generate bound states (see Fig. 4). Therefore, we can conclude that for sufficiently small $Re_m$, the dynamo does not take place, even for $1 < \xi < 2$.

The effect of a large scale cutoff on the velocity energy spectrum is thus the presence of a critical Reynolds number $Re_m^{(crit)}$. For $Re_m$ smaller than that value the potential $V$ has not bound states or equivalently, on account of our quantum mechanic interpretation, the velocity field is unable to favour the magnetic field growth and the ohmic dissipation eventually prevails on stretching.

The dependence of the dimensionless rate-of-growth $E_0 t_d^{-1}$ on $Re_m$ is shown in Fig. 3 for $Re_m > Re_m^{(crit)}$ in the case of the scaling $\xi = 4/3$. Notice that, for $Re_m \gg Re_m^{(crit)}$, $E_0$ takes the inertial range behaviour $E_0 \simeq \epsilon_0/\eta t_d^{-1}$.

We can again deduce from Eq. (11) some properties of the function $H(r,t)$. The correlation length of the magnetic field is again of order $r_d$ and, at $r \gg L$, $H(r,t)$ shows an exponential decay

$$H(r,t) \propto e^{-\gamma (r/L)} \quad (L \ll r),$$

(17)

with $\gamma = E_0 \left[L^2/(2\bar{\kappa})\right]^{-1}$, $\bar{\kappa} = \kappa + D_{ii}(0)/6$.

**C. Nonzero Prandtl effect**

Finally, we consider the situation of nonzero Prandtl number (at infinite Reynolds number). This is equivalent to look at the effect of the viscous scale on the dynamo effect.

If $Pr < 1$, the diffusive scale $r_d$ is in the inertial range, while, if $Pr > 1$, it lies within the viscous range. The structure
function $S_{ii}(r)$ scales as $r^2$ for $r \ll \eta$ and as $r^4$ for $r \gg \eta$.

From the previous considerations we can expect for the potential $V$ the same asymptotic behaviours for $r \to \infty$ as in the case of $Pr = 0$. Therefore, the Prandtl number does not influence the presence of dynamos. What is sensitive to $Pr$ is the correlation length of the magnetic field, that approximately corresponds to the scale at which the function $\psi_E_0$ begins its exponential-like decay. When $Pr < 1$, the potential has nearly the same shape as in the case $Pr = 0$

$$V(r) \sim \begin{cases} 2/r^2 & r \ll r_d \\ (2 - \frac{3}{2} \xi - \frac{3}{4} \xi^2)/r^2 & r_d \ll r \end{cases}$$

and the correlation length is of order $r_d$.

On the contrary, when $Pr > 1$, the potential well is modified by an attractive $\xi = 2$ contribution

$$V(r) \sim \begin{cases} 2/r^2 & r \ll r_d \\ -4/r^2 & r_d \ll r \ll \eta \\ (2 - \frac{3}{2} \xi - \frac{3}{4} \xi^2)/r^2 & \eta \ll r. \end{cases}$$

For these $Pr$ the function $\psi_E_0(r)$ grows as $r^2$ for $r \ll r_d$, as $r^{1/2}$ in the range $r_d \ll r \ll \eta$ and has a stretched exponential decay for $\eta \ll r$. We can thus conclude that, when $Pr > 1$, the magnetic field correlation length is of order $\eta$.

On account of what we have just seen, we expect that for $Pr \ll 1$ the ground state energy will be proportional to the diffusive time: $E_0 \simeq \epsilon_0 \xi t_d^{-1}$. In the other limit, $Pr \gg 1$, we can predict an approximate expression for $E_0$ by a simple scaling argument. Indeed, for large $Pr$ the potential $V$ behaves like in the case $\xi = 2$ and we can expect $E_0 \propto D_1$ (see Ref. [11] for the discussion of the smooth case). Knowing that $S_{ii}(r) \propto r^2$ for $r \ll \eta$ and $S_{ii}(r) \propto D_1 r^4$ in the inertial range, we can match the previous behaviours to obtain $D_1 \propto \eta^{3-2}$. Finally, we recall that from dimensional arguments we have $\eta \simeq (\nu/D_1)^{1/2}$. Summarizing the previous considerations, it is easily seen that, for $Pr \gg 1$, the relation $E_0 \propto \nu^{-1}$ holds (the time $t_v$ is the characteristic one for the velocity diffusion). The Prandtl number $Pr \simeq (t_v/t_d)^{1/2}$ thus influences also the magnetic field rate-of-growth: in presence of dynamo, $B^2$ increases with a characteristic time-scale determined by the largest between the viscous and the diffusive time (see Fig. [3]).

To conclude this section, we discuss a result that emerges from numerical computations: for $Pr \simeq 1$ the magnetic growth rate reaches a maximum (Fig. [3]). We can easily guess this behaviour, if we refer once more to the Kazantsev quantum formalism. For $Pr < 1$ the $\xi = 2$ behaviour is practically absent in the potential $V$, while, when $Pr$ approaches the value 1, the scale $\eta$ begins to come into play yielding a strongly attractive $-4/r^2$ contribution at scales $r_d \ll r \ll \eta$. The $\xi = 2$ potential is more attractive than that of other $\xi$ and the ground state energy increases in absolute value. Then, as $Pr$ becomes larger, $|E_0|$ decreases as explained above. In other words as long as the viscous behaviour affects only the potential shape around $r_d$, its only effect is to make the well deeper and so to favour the dynamo. When viscosity becomes very large, the level of velocity fluctuations lowers significantly, inducing eventually the depletion of the rate-of-growth.

IV. CONCLUSIONS

The presence of dynamo effect for a magnetic field advected by a conducting fluid is strongly dependent on the properties of the turbulent flow. We have highlighted in the framework of the Kraichnan ensemble the consequences of considering a viscous and an integral cutoff for the velocity. Using the quantum formalism introduced by Kazantsev, we have found that a critical magnetic Reynolds number exists. By the same method, we have shown that the Prandtl number is the parameter that determines the correlation length of the magnetic field and the characteristic time of its growth. Finally, we have argued that in the presence of dynamo the magnetic growth rate is maximum for Prandtl number of order unity. As already noted, the previous analysis depends only on the qualitative properties of the velocity structure function, so we expect our conclusions to hold for a generic turbulent flow with same statistical symmetries and therefore to be relevant for real applications.

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APPENDIX A: VARIATION-ITERATION METHOD

For the numerical analysis of Schrödinger equation (1) we make the transformation $y = a^{-r}$ ($a > 1$) which maps $(0, \infty)$ on the finite interval $(0, 1)$. (The constant $a$ should be chosen to properly resolve this interval). Eq.(6) can thus be rewritten in the form

$$\mathcal{L} \psi = \lambda \mathcal{M} \psi$$

(A1)

where

$$\mathcal{L} = -(\ln a)^2 \left( \frac{d^2}{dy^2} + \frac{d}{dy} \right) + \frac{m(y)}{y} (U(y) - U_{\text{min}})$$

$$\mathcal{M} = \frac{m(y)}{y} \quad \lambda = E - U_{\text{min}}$$

(A2)

and $U_{\text{min}}$ denotes the minimum value of $U$. $\mathcal{L}$ and $\mathcal{M}$ are positive-definite self-adjoint operators defining a spectrum of eigenvalues $\lambda$ bounded from below and which extends to infinity. Moreover, $\mathcal{L}$ is invertible on all functions twice differentiable on $(0, 1)$ and vanishing at the boundaries of the interval. Under these hypotheses the variation-iteration method described in Ref. [12] provides a valuable tool to compute the lowest eigenvalue $\lambda_0$ of Eq. (A1) and the corresponding eigenfunction $\varphi_0$. Indeed, let $\varphi_0$ be an initial trial function such that $\int_0^1 \psi_0 \mathcal{M} \varphi_0 \, dy \neq 0$ and define the $n$th iterate $\varphi_n$ as

$$\varphi_n \equiv (\mathcal{L}^{-1} \mathcal{M})^{n-1} \varphi_0.$$  

(A3)

Then, as $n$ is increased, the sequence $\varphi_n$ converges to the eigenfunction $\varphi_0$. The $n$th approximation to $\lambda_0$ is given by the following variational expression employing $\varphi_n$ as trial function

$$\lambda_0^{(n)} = \frac{\int_0^1 \varphi_n \mathcal{L} \varphi_n \, dy}{\int_0^1 \varphi_n \mathcal{M} \varphi_n \, dy}.$$  

(A4)

The set $\lambda_0^{(n)}$ form a monotonic sequence of decreasing values, approaching $\lambda_0$ from the above. The advantage of the variation-iteration technique is that no expression is required a priori for the function $\varphi_0$. We only have to choose any guess for initial function $\varphi_0$ and then improve the result by iterating the method for sufficiently large $n$. The convergence is more rapid the smaller is the ratio between $\lambda_0$ and the following eigenvalue.

Finally, for the numerical implementation of the method, we exploited the first order discrete expression of $\mathcal{L}$ preserving the boundary conditions on $\psi$. If $(0, 1)$ is divided in intervals of length $\Delta$ and $y_i = i \Delta$, we have

$$\mathcal{L}_{ij} = \frac{m(y_i)}{y_i} (U(y_i) - U_{\text{min}}) + \frac{(\ln a)^2}{2\Delta^2} \left\{ \begin{array}{ll}
(-y_{i-1} - y_i) & \text{if } i = j + 1 \\
(y_{i-1} + 2y_i + y_{i+1}) & \text{if } i = j \\
(-y_i - y_{i+1}) & \text{if } i = j - 1.
\end{array} \right.$$  

(A5)

APPENDIX B: THE SCHRÖDINGER EQUATION IN THE DYNAMO THEORY

In the present appendix we refer to the notation adopted in the body of the paper. So, the trace of the correlation tensor $\langle B_i(x, t) B_j(x + r, t) \rangle$ will be denoted by $H(r, t)$. As a consequence of the velocity $\delta$-correlation in time, $H$ satisfies a closed equation that, under a suitable transformation, takes on the form of a one-dimensional Schrödinger-like equation. In order to exploit this fact, let us denote
\[ s(r) = S_{ii}(r) \] and define the following quantities
\[ \varpi(r) = \frac{1}{r^2} \int_0^r s(\rho) \rho^2 d\rho, \]
\[ \Lambda(r) = \kappa + \varpi(r), \quad \Lambda_1(r) = \Lambda(r) + 3\kappa + \frac{s(r)}{2}. \] (B1)

Then, the function
\[ \Psi(r, t) = \sqrt{\kappa} \exp \left( \int_0^r \Lambda_1(\rho) \rho d\rho \right) \frac{1}{r^3} \int_0^r H(\rho, t) \rho^2 d\rho \] (B2)
solves the imaginary time Schrödinger equation
\[ -\frac{\partial \Psi}{\partial t} + \left[ \frac{m}{m(r)} \frac{\partial^2}{\partial r^2} - U(r) \right] \Psi = 0 \] (B3)
where
\[ m = \frac{1}{2\Lambda}, \quad U = -\frac{1}{r} \frac{ds}{dr} + \frac{1}{2r^2} \frac{\Lambda_1^2}{\Lambda} + \Lambda \frac{d}{dr} \left( \frac{\Lambda_1}{r\Lambda} \right). \] (B4)

(See Ref. [3] for the detailed derivation). If we expand \( \Psi \) in terms of the energy eigenfunctions \( \Psi(r, t) = \int \psi_E(r) e^{-iE\eta} \rho(E) dE \), we get the stationary equation
\[ \frac{1}{m(r)} \frac{d^2 \psi_E}{dr^2} + [E - U(r)] \psi_E = 0. \] (B5)

The dynamo effect corresponds to the presence of negative eigenvalues in Eq. (B5).

The correlation function \( H(r, \cdot) \) must tend to a constant value as \( r \to 0 \) and decreases to zero as \( r \to \infty \). From the definition (B2) we have therefore that Eq. (B5) must be solved with the boundary conditions that \( \psi_E(r) \) vanishes as \( r \to 0 \) and increases as \( r \to \infty \) slowly enough to guarantee that \( H(r, \cdot) \) decreases to zero. In particular, if \( s(r) \) tends to a constant as \( r \to \infty \), \( \psi_E(r) \) cannot increase more rapidly than \( r \).

We consider now the explicit expression
\[ D_{ij}(r) = \int e^{ik \cdot r} \tilde{D}_{ij}(k) d^3k \] (B6)
with
\[ \tilde{D}_{ij}(k) = D_0 \frac{e^{-\eta k}}{(k^2 + L^2)(\xi + 3)/2} P_{ij}(k), \quad (\alpha > -1). \] (B7)

The transverse projector \( P_{ij}(k) = (\delta_{ij} - k_i k_j / k^2) \) ensures the incompressibility of the velocity field.

In the limits \( \eta \to 0 \) and \( L \to 0 \), \( S_{ij}(r) \) takes the form
\[ \lim_{\eta \to 0, L \to \infty} S_{ij}(r) = D_1 r^\xi \left[ (2 + \xi)\delta_{ij} - \xi \frac{r_j r_j}{r^2} \right] \] (B8)
with
\[ D_1 = \frac{4\pi \cos \left( \frac{\pi}{2} \right) \Gamma(-1 - \xi)}{\xi + 3} D_0. \] (B9)

(The function \( \Gamma \) is the Euler function).

If we insert \( s(r) = 2(\xi + 3)D_1 r^\xi \) in (B4), the transformation (B2) takes on the form
\[ \Psi(r, t) = \left( \kappa + D_1 r^\xi \right)^{1/2} \frac{1}{r^3} \int_0^r H(\rho, t) \rho^2 d\rho, \] (B10)

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while its inverse reads
\[
H(r, t) = \frac{(2\kappa - D_1 r^\xi (\xi - 2))}{2r^2 (\kappa + D_1 r^\xi)^{\frac{3}{2}}} \Psi(r, t) + 2r (\kappa + D_1 r^\xi) \Psi'(r, t). \tag{B11}
\]

For the mass and the potential we obtain the following expressions
\[
m(r) = \frac{1}{2(\kappa + D_1 r^\xi)}, \tag{B12}
\]
\[
U(r) = \frac{4\kappa^2 + A(\xi)\kappa D_1 r^\xi + B(\xi)D_1^2 r^{2\xi}}{r^2 (\kappa + D_1 r^\xi)}, \tag{B13}
\]
with \(A(\xi) = (8 - 3\xi - \xi^2)\) and \(B(\xi) = (4 - 3\xi - \frac{3}{2}\xi^2)\).

For the sake of completeness we write also the expressions of the trace \(s(r)\), which we used to compute \(E_0\) respectively in the case of finite \(R_m\) and in the case of nonzero \(Pr\)
\[
\lim_{\eta \to 0} s(r) = \frac{4\pi D_0 L^2}{\Gamma\left(\alpha + \frac{\xi + 1}{2}\right)} \left[ \Gamma\left(\frac{1 + \alpha}{2}\right) \Gamma\left(\frac{\xi}{2}\right) - \sqrt{\pi \frac{L}{r}} G_{1,3}^{2,1} \left(\frac{r^2}{4L^2} \left| \frac{1 - \alpha}{2}, \frac{1}{2}, 0 \right. \right) \right], \tag{B14}
\]
\[
\lim_{L \to \infty} s(r) = 8\pi D_0 \eta^\xi \left( \Gamma(-\xi) + \frac{\eta}{r} \left(1 + \frac{r^2}{\eta^2}\right)^{\frac{1+\xi}{2}} \right) \Gamma(-1 - \xi) \sin\left[ (1 + \xi) \arctan\left(\frac{r}{\eta}\right) \right]. \tag{B15}
\]
(The function \(G\) denotes the \(G\)-Meijer’s function of argument \(r^2/(4L^2)\). See Ref. [13] for the exact definition). The explicit expressions of the mass and the potential can be derived from [14].

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FIGURES CAPTIONS

Fig. 1: The dependence of the magnetic growth rate $\epsilon_0 = E_0 t_d^{-1}$ on the scaling exponent $\xi$ in the limit of infinite $Re_m$ and zero $Pr$, as computed by the variation-iteration method described in appendix A.

Fig. 2: The shape of the quantum potential $V$ in the limit $Re_m \to \infty$ and $Pr \to 0$ for a value of $\xi$ ($\xi = 2$) for which dynamo is present and for a value ($\xi = 0.91$) for which there is no dynamo effect.

Fig. 3: The asymptotic behaviours of the ‘stationary wave function’ $\psi_{E_0}$ in the limit of infinite $Re_m$ and zero $Pr$. The maximum at $r \approx r_d$ determines the magnetic field correlation length.

Fig. 4: A qualitative picture of the quantum potential shape for $Re_m$ respectively above and below the critical value $Re_m^{(crit)}$ ($1 < \xi < 2$).

Fig. 5: The dependence of the magnetic growth rate on the magnetic Reynolds number for $Pr \to 0$ and $\xi = 4/3$. The numerical computation is performed using expression (9) for the correlation tensor of the magnetic field.

Fig. 6: The dependence of the magnetic growth rate on the Prandtl number for $\xi = 4/3$ and in the limit $Re_m \to \infty$. The numerical computation is performed using expression (9) for the correlation tensor of the magnetic field.
FIG. 1. D. Vincenzi
FIG. 2. D. Vincenzi
\[ \psi(r) = \frac{r^2}{\sqrt{2\pi \sigma}} e^{-\beta \frac{(r-r_0)^2}{2\sigma^2}} \]

FIG. 3. D. Vincenzi
$V(r) \sim -\frac{2}{r^2} + \frac{(2-3\xi^2/2 + 3\xi^2/4)}{r^2}$

FIG. 4. D. Vincenzi
FIG. 5. D. Vincenzi
\[ E_0 \propto t_d^{-1} \]

\[ E_0 \propto t_v^{-1} \]

FIG. 6. D. Vincenzi