Abstract

The propagation of strings in cosmological space-time backgrounds is reviewed. We show the relation of a special class of cosmological backgrounds to exact conformal field theory. Particular emphasis is put on the singularity structure of the cosmological space-time and on the discrete duality symmetries of the string background.
1. Introduction

String theory is at the moment the most attractive candidate for a unified description of the basic constituents in nature and their interactions. In the past years, the main emphasis within this subject was put on the discussion of critical strings: $D = 10$ for the most promising case of the heterotic string \cite{I}. In order to get contact with low energy phenomenology, one can make the consistent assumption that the string background space has the form of four-dimensional flat Minkowski space-time times a six-dimensional internal, compact space with characteristic size of order of the Planck scale. More generally, four-dimensional heterotic strings are based on the tensor product of a trivial (super) conformal field theory having $c_L = 4$ ($c_R = 6$), corresponding to flat four-dimensional space time, with an internal (super) conformal field theory possessing $c_L = 22$ ($c_R = 9$). The form of the internal compact space or internal conformal field theory determines all particle physics properties of the four-dimensional string, such as the gauge group, the number of supersymmetries (gravitinos), matter representations, Yukawa couplings etc.

In order to investigate quantum gravity effects like processes shortly after the ‘Big Bang’, a further step is necessary: the flat four-dimensional Minkowski space-time has to be replaced by a curved target space. Consequently, this part of the string theory also corresponds to a non-trivial conformal field theory. In general relativity,
singularities in curved space-times are often unavoidable. In fact, the well-known singu-
arity theorems [2] prove the existence of singularities under certain assumptions of
the matter energy momentum tensor. For example, the standard Big Bang scenario
exhibits an initial singularity at $t = 0$. In string theory, there are several reasons to
believe that singularities in the target space are not harmful to the theory. Heuris-
tically, this believe is based on the fact that the string theory possesses a ‘minimal
length scale’ set by the extension of the string itself. A well known example of a finite
string theory with a singular background space is the compactification of strings on
orbifolds.

One of the keys in understanding the meaning of singularities and of the minimal
length scale in string theory may be given by the so-called duality symmetries [3].
Duality is, roughly spoken, the invariance of the string theory under the inversion
of certain characteristic length parameters. More generally, duality transformations
act on the background parameters of the string, like the space-time metric or dilaton
field. The existence of the duality symmetry was first shown in the context of string
compactification on a constant background, like a circle or torus. Here the stringy
nature of this symmetry can be clearly identified since duality involves the exchange
of momentum with stringy winding modes. The emergence of the duality symmetries
within string compactifications already proved to be very useful [4] to get information
about the effective Lagrangian in ‘flat’ four-dimensional string theories. For example
one can show [3] the certain orbifold compactifications cannot lead to the spectrum
of the minimal supersymmetric standard model without breaking the stringy duality
symmetries.

The duality symmetries are not limited to flat backgrounds; their existence was
shown [3,7,8] for many curved, for example time-dependent backgrounds. Even
for non-compact spaces, duality survives as discussed recently in ref.[3]. This ob-
servation is of particular interest in the context of discussing singularities in curved
string backgrounds. For non-compact spaces, the duality transformations can change
the topology of the space. This topology change is most dramatic when the duality
transformation maps a singular on a non-singular space, as shown first for the Eu-
clidean two-dimensional black-hole in ref.[10]. Since both spaces are equivalent from
the string point of view, the meaning of the singularity in the target space is unclear.
In this review we will discuss the propagation of strings in cosmological, i.e. time-dependent backgrounds [14]. In the next section we will investigate the time-dependent solutions of the string equations of motion including a non-trivial metric and dilaton background. In section 3 we will relate particular cosmological string backgrounds to exact conformal field theories, namely the gauged Wess-Zumino-Witten (WZW) models. Finally, in section four, we will discuss the cosmological background in the so-called Einstein frame. Particular emphasis is put on the duality symmetries.

2. Cosmological String Backgrounds

Let us consider the propagation of a (bosonic) string in the presence of a D-dimensional metric and dilaton background $G_{MN}(x)$ $(M, N = 0, \ldots, D - 1)$ and $\Phi(x)$. The string tree level effective action for these background fields has the form

$$ S_{\text{eff}} = \int d^D x \sqrt{-G} e^\Phi (R - (D\Phi)^2 + \Lambda). \quad (1) $$

The cosmological constant is related to the dimension $D$ of space time as $\Lambda = \frac{2(26-D)}{3}$. The effective action (1) leads to the following equations of motion

$$ R_{MN} + D_M D_N \Phi = 0, $$

$$ R + (D_M \Phi)^2 + 2D_M D^M \Phi = \Lambda. \quad (2) $$

These equations can be also obtained [18] as the conditions of vanishing $\beta$-functions in the corresponding two-dimensional $\sigma$-model

$$ S_{2-\text{dim}} = \int d^2 \sigma \sqrt{g} [g^{mn} G_{MN}(x) \partial_m X^M \partial_n X^N - \frac{\eta(2)}{4} \Phi(x)]. \quad (3) $$

As an ansatz for the metric and dilaton field we consider a cosmological, i.e. time dependent background of the form

$$ G_{00} = -1, \quad G_{ij} = \delta_{ij} R_i(t)^2 \quad (i, j = 1, \ldots, D - 1), \quad \Phi = \Phi(t). \quad (4) $$

Then the string field equations (2) can be rewritten as [14]

$$ \sum_i \frac{\dddot{R}_i}{R_i} + \dddot{\Phi} = 0, $$

$$ \frac{\dddot{R}_i}{R_i} + \sum_{j \neq i} \frac{\dddot{R}_i \dddot{R}_j}{R_i R_j} + \Phi \frac{\dddot{R}_i}{R_i} = 0, \quad (5) $$

$$ \dddot{\Phi} + \frac{1}{2} \Lambda + \frac{1}{2} \dddot{\Phi}^2 - \sum_{i<j} \frac{\dddot{R}_i \dddot{R}_j}{R_i R_j} = 0. $$
Now, it is not difficult to realize that the field equations (5) are invariant under the duality transformation

\[ R_i(t) \rightarrow \frac{1}{R_i(t)}, \]
\[ \Phi(t) \rightarrow \Phi(t) + 2 \log R_i(t). \] (6)

The non-trivial duality transformation behavior of the dilaton field implies that the time-dependent string coupling constant is transformed like \( g^2(t) = e^{-\phi(t)} \rightarrow g^2(t)R_i(t)^{-2}. \) This change of the string coupling constant agrees with the transformation of \( g^2 \) in the static case when one considers the genus expansion of the string partition function \([19]\). The origin of the duality invariance comes from the fact that the metric (4) is independent from the spatial coordinates \( x_i \) which leads to \( D - 1 \) Abelian isometries of the model. In this case it follows that the non-linear \( \sigma \)-model based on the dual background (6) corresponds to a conformal field theory provided that also the original background is conformal (at lowest order in \( \alpha' \)) \([20]\).

The explicit form of the solutions \([14]\) of the field equations (5) depends on the value of the cosmological constant \( \Lambda \). For \( \Lambda = 0 \) (\( D = 26 \)) one obtains a family of solutions

\[ R_i(t) = \alpha_i(t - t_0)^{p_i}, \quad \sum_{i=1}^{D-1} p_i^2 = 1 \]
\[ e^\Phi = \beta^2(t - t_0)^p, \quad p = 1 - \sum_{i=1}^{D-1} p_i, \] (7)

where \( \alpha_i, \beta, t_0 \) are arbitrary real parameters.

For \( \Lambda \neq 0 \), the cosmological solutions are given by

\[ R_i(t) = \alpha_i[\tanh \frac{\sqrt{-\Lambda}}{2}(t - t_0)]^{p_i}, \quad \sum_{i=1}^{D-1} p_i^2 = 1 \]
\[ e^\Phi = \beta^2[\cosh \frac{\sqrt{-\Lambda}}{2}(t - t_0)]^{2-p}[\sinh \frac{\sqrt{-\Lambda}}{2}(t - t_0)]^p, \quad p = 1 - \sum_{i=1}^{D-1} p_i. \] (8)

Alternatively, for \( \Lambda \neq 0 \), there is another solution, the so-called linear dilaton \([12]\):

\[ R_i = \text{const}, \quad \Phi = \sqrt{-\Lambda} t. \] (9)
To understand the singularity structure and the duality properties of the above cosmological solutions let us compute the Ricci-tensor. For $\Lambda = 0$ it has the form ($\alpha_i = 1, t_0 = 0$):

$$R_{00} = \frac{p}{t^2}, \quad R_{ij} = \delta_{ij} \frac{p_i p_j}{t^2 - 2p_i}.$$  \hspace{1cm} (10)

The corresponding scalar curvature is then

$$R = -\frac{p^2}{t^2}.$$  \hspace{1cm} (11)

We see that for arbitrary $p$, space-time is singular at $t = 0$. However for $p = 0$, space-time is Ricci-flat. However the curvature tensor is still singular at $t = 0$ for this case ($R_{ab}^{cd} \sim t^{-2}$). Only for the “two-dimensional” case $p_1 = 1, p_i = 0 (i \neq 1)$ space-time is completely flat. This is most easily seen by introducing lightcone coordinates like $u = -te^{-x_1}, v = te^{x_1}$. Then the metric becomes that of flat Minkowski space time, $ds^2 = du dv$, where the whole $t - x_1$ plane is mapped onto the forward/backward lightcones $uv < 0$.

For $\Lambda \neq 0$, the scalar curvature has the form

$$R = \frac{(4 - 4p)(\sinh \frac{\sqrt{-\Lambda}}{2} t)^2 - p^2}{(\sinh \frac{\sqrt{-\Lambda}}{2} t)^2(\cosh \frac{\sqrt{-\Lambda}}{2})^2}. \hspace{1cm} (12)$$

Thus for generic values of $p$ there is a singularity at $t = 0$. However for $p = 0$, $\Lambda < 0$, the scalar curvature is finite for all $t$ \[13\] whereas the curvature tensor is still singular at $t = 0$. Only in the “two-dimensional” case with $p_1 = 1$ and $p_i = 0 (i \neq 1)$ space-time is completely free of singularities in the $t - x_1$ plane.

The duality transformation $p_i \rightarrow -p_i, p \rightarrow p + 2p_i$ relates two inequivalent cosmological backgrounds. For generic $p$, the original space-time and also the dual transformed space-time possess a singular scalar curvature. However for $p = 0$, a space-time with singular scalar curvature is mapped onto a background without singularity in $R$. In particular, for $p_1 = 1$ and $\Lambda = 0$, flat Minkowski space-time is mapped onto a singular space, and for $p_1 = 1, \Lambda < 0$, a singular-free expanding space is mapped onto a singular, but contracting space-time. (For more discussion on this case see next chapter.)

The equations (2) and (5) are the tree-level field equations determining the string propagation in the graviton, dilaton vacuum background. However to obtain a realistic
cosmological scenario one has to include also matter energy density and pressure from the stringy matter. As discussed in ref. [13] this is determined by the one-loop free energy of the string. In an isotropic universe with $R_i(t) = R(t)$ $(l = 1, \ldots, d$, all other $R_i = \text{constant}$), $\Lambda = 0$, assuming that the matter evolves adiabatically, one can show [13] that the massless momentum modes with energy $E \propto R(t)^{-1}$ lead to the radiation dominated era of the standard cosmology with $R(t) \propto e^{\alpha t}$, and with a dilaton approaching a constant. However at very early times the contribution of the winding modes to the energy also becomes important. These modes oppose the expansion of the universe, and, as it was argued in [13], the universe oscillates for some time around the Planck scale, until by coincidence it starts expanding in a smaller number of dimensions ending in the radiation dominated era. Based on the thermodynamics of the winding modes one can give arguments [13] that the preferred number of expanding directions is smaller than four.

3. Gauged WZW Models

Now we want to relate the cosmological solutions of the string field equations to exact conformal field theories, namely to the gauged WZW models. We will show [16] that the “two-dimensional” model with $p_1 = \pm 1$ can be derived from a gauged Wess-Zumino-Witten (WZW) models [21] based on the non-compact coset space $SL(2, \mathbb{R})/SO(1,1)$. In fact it was first shown by Witten [22] that this type of coset space CFT has a very interesting target space interpretation, namely that of a two-dimensional black-hole [23]. However a simple change in the CFT, namely the change of sign of the level $k$ of the $SL(2, \mathbb{R})$ Kac-Moody algebra, will lead to the cosmological interpretation.

The gauged WZW model based on the coset $G/H$ is described by the action

$$S = \frac{k}{4\pi} \int d^2z \text{tr} (g^{-1} \partial g^{-1} \overline{g}) - \frac{k}{12\pi} \int_B \text{tr} (g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg)$$

$$+ \frac{k}{2\pi} \int d^2 \text{tr} (A \overline{g} g^{-1} - A g^{-1} \partial g - g^{-1} A g \overline{A}),$$

(13)

where the boundary of $B$ is the 2D worldsheet, $g$ is a group element of the group $G$, and $A$ are the gauge fields of $H$ transforming as $A \rightarrow h_L^{-1} (A + \partial) h_L$; $k$ is the level of the Kac–Moody algebra for the group $G$. To be specific, we
want to discuss a four-dimensional target-space based on the non-compact coset $SL(2, \mathbb{R}) \times SO(1, 1)^2/\text{SO}(1, 1)$. The central charge of this coset CFT (non-compact coset CFT’s were discussed in [24]) is given by

$$c = 4 + \frac{6}{k-2},$$

where the $k$, being a real number, is the level of the non-compact $SL(2, \mathbb{R})$ Kac–Moody algebra. For the case that the string theory is entirely given by this coset CFT, the condition $c = 26$ implies $k = 25/11$. However, we leave $k$ as a free parameter. Therefore we couple the $SL(2, \mathbb{R}) \times SO(1, 1)^2/\text{SO}(1, 1)$ coset CFT to an internal CFT with central charge

$$c_{\text{int}} = D - 4 = 22 - \delta c, \quad \delta c = 26 - D = \frac{6}{k-2}.$$  

(15)

Now we parametrize the group element of $SL(2, \mathbb{R})$ as $g = \begin{pmatrix} u & a \\ -b & v \end{pmatrix}$ with $uv + ab = 1$. Finally we assume for simplicity that $H = \text{SO}(1, 1)$ is entirely inside $SL(2, \mathbb{R})$. Performing a vector-like gauging, $a = \pm b$, the action (13) can be identified with a $\sigma$-model action of the form

$$S = \int d^2z G_{MN} \partial X^M \overline{\partial X^N}.$$  

(16)

In the semiclassical approximation $k \to \pm \infty$, the corresponding four-dimensional $\sigma$-model metric is given as [22], [8]

$$ds^2 = -k \frac{dudv}{1 - uv} + dx_2^2 + dx_3^2.$$  

(17)

The associated dilaton can be derived from the change of the integration measure in the path integral. It has the form

$$\Phi = \log(1 - uv)$$  

(18)

We recognize that the signature of the two-dimensional part of this metric depends just on the sign of the level $k$ of the $SL(2, \mathbb{R})$ Kac–Moody algebra. In fact, for $k \to +\infty$ space-time possesses a singularity in futures times $\tau = u + v$. In this case, $u$ and $v$ are Kruskal-like coordinates of a black-hole metric; more precisely the
metric (17) has the causal structure of a four-dimensional black-brane (see figure 1). The singularity at $uvw = 1$ originates from the fixed points of the modded vector gauge symmetry $H$ at this curve. However for $k \to -\infty$ the causal structure is completely different \[16\]. Specifically the causal structure for negative $k$ is obtained from the black-hole case by a $90^\circ$ rotation of the two-dimensional $u, v$-plane (see figure 2). Specifically, we see that, introducing the proper time $\tau = u - v$, there is no singularity for future times $\tau$ inside the light-cone $uvw < 0$ (region I in figure 2). The singularity at $uvw = 1$ is hidden behind the horizon $uvw = 0$. However signals in regions II, III may hit the singularity, and the singularity may also send signals through regions II, III into region I. Now one has a forward light-cone with the singularity behind it. As we will discuss in the following, this class of string backgrounds with negative $k$ describes an expanding Universe with singularity outside the visible horizon.*

Using eq.(15) is easy to determine for which number of dimensions, i.e. for what values of central charges $\delta c$, one has a black-hole or cosmological metric respectively. Specifically for $D < 26$ we deal with a black-hole. (In addition, for $D < 29$ one obtains again a black-hole metric. Then, however, the central charge of $SL(2,\mathbb{R})/SO(1,1)$ becomes negative, and one may expect serious problems with unitarity.) On the other hand, for $26 < D < 29$ one deals with the cosmological scenario. Regarding this theory as a non-critical Liouville string in the asymptotic limit, it would imply that the Liouville field is a time-like coordinate \[25\].

To make contact with the discussion of section 2, it is easy to show that inside the singularity free region I we have in fact an expanding Universe. Specifically, we introduce coordinates which cover exactly the region I,

$$u = e^{x_1/\sqrt{-k}}\sinh\left(\frac{t}{\sqrt{-k}}\right), \quad v = -e^{-x_1/\sqrt{-k}}\sinh\left(\frac{t}{\sqrt{-k}}\right).$$  \hspace{1cm} (19)

(The coordinates $t$ and $x_1$ are not geodesically complete.) Then the metric (17) in region I becomes

$$ds^2 = -dt^2 + \left[\tanh\left(\frac{t}{\sqrt{-k}}\right)\right]^2dx_1^2 + dx_2^2 + dx_3^2.$$  \hspace{1cm} (20)

* In ref.[13] it was discussed that for $k > 0$, $\Lambda > 0$, the interior region II of the two-dimensional black-hole has an interesting cosmological interpretation, in particular after going to an infinite cover fold of $SL(2,\mathbb{R})$. 

---

-8-
This metric is exactly of the form eq.(8) with \( p = 0, p_1 = 1 \), all other \( p_i = 0 \) and \( \Lambda = \frac{2\delta c}{3} = \frac{4}{k-2} \) after the necessary shift \( k \rightarrow k - 2 \) which originates from renormalization effects.

In the gauged WZW model, the duality operation corresponds to the exchange of gauging the axial \( SO(1, 1) \) subgroup of \( SL(2, \mathbb{R}) \) instead of gauging the vector-like subgroup \( SO(1, 1) \) \([7], [26]\). This implies that in the \( u, v \) coordinate system, duality acts as

\[
uv \rightarrow uv - 1.
\]  

(21)

This is just the exchange of the regions I and IV in figures 1 and 2, whereas the regions II and III are mapped onto themselves \([10], [26]\). In the \( t, x_1 \) coordinate system, as defined in eq.(19), the duality transformation (6) leads to the dual metric of the form

\[
ds_D^2 = -dt^2 + [\coth\left( \frac{t}{\sqrt{-k}} \right)]^2 dx_1^2 + dx_2^2 + dx_3^2.
\]

(22)

and the dual dilaton

\[
\Phi(t)_D = 2 \log \sinh\left( \frac{t}{\sqrt{-k}} \right).
\]

(23)

Therefore the dual metric of region IV possesses a singularity at \( t = 0 \).

The coset CFT based on \( SL(2, \mathbb{R})/SO(1, 1) \) with negative level \( k = -N \) is closely related to the CFT of the coset space \( SU(2)/U(1) \) of level \( N \) with the same central charge \( c = \frac{3N}{N+2} - 1 \). The gauged WZW model of \( SU(2)/U(1) \) leads, for large \( N \), to a \( \sigma \)-model with the following two-dimensional target space metric:

\[
ds^2 = N \frac{dz d\overline{z}}{1 - \overline{z} z}.
\]

(24)

Here \( z \) is a complex parameter of an \( SU(2) \) group element, \( z = x + i\tau \). Performing a change of coordinates like

\[
z = e^{ix_1/\sqrt{N}} \sin\left( \frac{x_0}{\sqrt{N}} \right),
\]

(25)

the metric (24) becomes

\[
ds^2 = dx_0^2 + [\tan\left( \frac{x_0}{\sqrt{N}} \right)]^2 dx_1^2.
\]

(26)

This Euclidean metric is just the analytic continuation of the cosmological metric (20), i.e \( x_0 = -it \). It leads to a singularity at \( x_0 = \pi/2 \). In contrast to its Minkowski counter
part, the Euclidean space is self-dual, which means that the duality transformation \( x_0 \rightarrow x_0 + \pi/2 \). It is interesting to note that conformal field theory \( SL(2,\mathbb{R})/U(1) \) with positive \( k \), which leads to the two-dimensional black-hole, is governed by an infinite dimensional, non-linear symmetry \( \widehat{W}_\infty(k) \). This algebra truncates for the cosmological case \( k = -N < 0 \) to the finite algebra \( W_N \). Both algebras linearize in the limit \( k, N \rightarrow \infty \), where the ordinary \( W_\infty \) algebra is recovered.

So far we have considered the semiclassical approximation of the gauged WZW model, i.e. the target space metric, as given in eq.(17), was derived for \( k \rightarrow \pm \infty \). This metric, together with the dilaton eq.(18) provides a solution of the string field equations eq.(2), which are valid at lowest order in \( \alpha' \) or \( 1/k \). Thus in the bosonic \( \sigma \)-model there are higher order corrections to the metric and dilaton background. However for models with extended \((n=4)\) worldsheet supersymmetries, there are no renormalization effects, and the lowest order background is supposed to be exact (see the discussion at the end of this chapter). For the bosonic case, the exact, i.e. conformal, background can be obtained by various methods [26,28,29]. For the coset \( SL(2,\mathbb{R})/SO(1,1) \times \mathbb{R}^2 \) the exact, all order metric has the following form [28]:

\[
\frac{d}{d\sigma^2} = \frac{k-2}{1-(1-2k)uv}[dudv + \frac{1}{2k(1-uv)}(vdu + udv)^2] + dx_2^2 + dx_3^2. \tag{27}
\]

Here \( k - 2 \) is the renormalized coupling constant. Obviously, this metric approaches the leading expression eq.(17) for \( k \rightarrow \pm \infty \). The exact dilaton has the form

\[
e^\Phi = (1-uv)\sqrt{1 + \frac{2uv}{k(1-uv)}}. \tag{28}
\]

To discuss the cosmological (or black-hole) interpretation of the exact metric, it is useful to again change the coordinates. For example, for the region IV, \( uv > 1 \), the new coordinates are given as

\[
u = e^{x_1/\sqrt{-k+2}}\cosh(\frac{t}{\sqrt{-k+2}}), \quad v = e^{-x_1/\sqrt{-k+2}}\cosh(\frac{t}{\sqrt{-k+2}}). \tag{29}\]

Then eq.(27) becomes [26],[28]

\[
d\sigma^2 = -dt^2 + \frac{1}{[\tanh(\frac{t}{\sqrt{-k+2}})]^2 - \frac{k}{2}}dx_1^2 + dx_2^2 + dx_3^2. \tag{30}\]

\[
-10-
\]
Examining this expression one observes the interesting fact that for the cosmological case, $k < 0$, there is no singularity in space-time anymore, since $1/(1 + \frac{2}{N}) < R_1(t)^2_{\text{Exact}} = \left(\left[\tanh\left(\frac{t}{\sqrt{-k+2}}\right)\right]^2 - \frac{2}{k}\right)^{-1} < N/2$ for all $t$ ($N = -k > 2$). (However in the $n = 4$ supersymmetric case the singularity is still present.) This is quite different from the lowest order metric eq.(22). Thus in the exact CFT, the physical relevance of the singularity of the lowest order solution seems to be rather low. This observation is also supported by the expectation that all physical amplitudes in the coset CFT will be non-singular.

The duality transformation for the exact metric is again given by $uv \to uv - 1$ due to the exchange of vector versus axial gauging. This implies that the exact dual metric is obtained from eq.(30) by replacing $\tanh(\frac{t}{\sqrt{-k+2}})$ by $\coth(\frac{t}{\sqrt{-k+2}})$. However note that this transformation does not anymore correspond to $R_1(t)^2_{\text{Exact}} \to 1/R_1(t)^2_{\text{Exact}}$.

At the end of this section let us discuss briefly some extended four-dimensional models. For example consider the gauged WZW model based on the coset $SL(2, \mathbb{R}) \times SU(2) \times U(1)$ [31], [16], [17], [32]. The central charge of this CFT is given by

$$c = \frac{3k_1}{k_1 - 2} + \frac{3k_2}{k_2 + 2} - 2.$$  \hspace{1cm} (31)

The corresponding background metric has the following form:

$$ds^2 = -k_1 \frac{1}{1 - uv} du dv + k_2 \frac{1}{1 - z \bar{z}} dz d\bar{z}.$$  \hspace{1cm} (32)

For the black-hole case with positive $k_1$ one can set $k_1 = k_2 + 4$. Then the model possesses an extended $n = 4$ world-sheet supersymmetry [32], and one obtains that $c = 4$ regardless of the value of $k_1$. Thus $\delta c = 0$ for all $k_1$ and the metric (32) is valid for all $k_1$, since there are no renormalization effects which can lead to higher order contributions in $1/k$.

For the cosmological case with negative $k_1$ we cannot set $k_1 = k_2 + 4$ ($k_2$ is always positive). As an alternative one can couple $SL(2, \mathbb{R})/SO(1,1)$ with negative

---

* For the black-hole case with $k > 2$ it was shown [30] that the exact metric is also free of singularities in the sense that the singularity is separated from the horizon by a Euclidean region.
$k_1$ to an Euclidean two-dimensional ‘black-hole’ based on the coset $SL(2, \mathbb{R})/U(1)$ with positive Kac–Moody level $k_2$. The central charge of this model is

$$c = \frac{3k_1}{k_1 - 2} + \frac{3k_2}{k_2 - 2} - 2. \quad (33)$$

and the background metric looks like

$$ds^2 = -k_1 \frac{1}{1 - uv} du dv + k_2 \frac{1}{1 + z \bar{z}} dz d \bar{z}. \quad (34)$$

Again one obtains $\delta c = 0$ for $k_1 = -k_2 + 4$, where the model is $n = 4$ supersymmetric.

4. The Einstein Frame

To get contact with standard gravity theory, which possesses a canonical Einstein term, one has to perform a Weyl rescaling of the $\sigma$-model metric eq.(17) by the exponential of the dilaton field. This Weyl rescaling however does not change the causal structure of the theory. Specifically, the metric in the Einstein frame is given as

$$ds^2 = e^\Phi ds_\sigma^2 = du dv + (1 - uv)(dx_2^2 + dx_3^2). \quad (35)$$

Here we have focused on the cosmological case with $k \to -\infty$, where we have absorbed the level $k$ in the metric. The effective action now has the form

$$S_{\text{eff}} = \int d^4x \sqrt{G} \left( \frac{1}{2} R - \frac{1}{4} D_M \Phi D^M \Phi - V(\Phi) \right), \quad V(\Phi) = 2e^{-\Phi}. \quad (36)$$

To show the cosmological behavior of the metric inside region I we again introduce coordinates which cover exactly the region I, namely

$$u = e^{x_1} t, \quad v = -e^{-x_1} t. \quad (37)$$

It follows that curves of constant times $t$ correspond in figure 2 to hyperbolae $uv = -t^2$, whereas the curves of constant $x_1$ are given by the straight lines $u/v = -e^{2x_1}$ (see figure 2). In these coordinates the metric now looks like

$$ds^2 = -dt^2 + t^2 dx_1^2 + (1 + t^2)(dx_2^2 + dx_3^2), \quad (38)$$

* There is no Weyl rescaling of the metric in two space-time dimensions.
and the dilaton has the form

$$\Phi(t) = \log(1 + t^2).$$

(39)

Clearly, the metric (38) describes an expanding Universe in region I with two different scale factors $R_1(t) = t$, $R_{2,3}(t) = \sqrt{1 + t^2}$, where $t$ is the cosmological time coordinate. For small $t$, the Universe expands unisotropically. However, for large times, one approaches an isotropic, linear expansion of the Friedmann-Robertson-Walker type with $R_i(t) = t$ ($i = 1, 2, 3$). However our solution has no initial singularity at $t = 0$ in contrast to the standard isotropic Robertson-Walker Universe. (The Ricci tensor in the coordinates (37) takes the form $R_{tt} = \frac{2}{(1+t^2)^2}$, $R^i_j = -\frac{2}{1+t^2}\delta^i_j$.)

Let us derive the energy momentum tensor of the dilaton matter field. Specifically consider the classical Einstein equations

$$R_{MN} - \frac{1}{2}G_{MN}R = -T_{MN}.$$  

(40)

The corresponding energy-momentum tensor from the dilaton matter field has the form

$$T_{MN} = \frac{1}{2}D_M\Phi D_N\Phi - G_{MN}\left(\frac{1}{4}G^{PQ}D_P\Phi D_Q\Phi + V(\Phi)\right).$$

(41)

Then we obtain, with $\Phi = \log(1 + t^2)$ and $V(\Phi) = \frac{2}{1+t^2}$, that

$$T_{tt} = \frac{(\partial_t \Phi)^2}{4} + V = \rho = \frac{3t^2 + 2}{(1+t^2)^2},$$

$$T_{ij} = \left(\frac{(\partial_t \Phi)^2}{4} - V\right)G_{ij} = pG_{ij} = -\frac{t^2 + 2}{(1+t^2)^2}G_{ij}.$$  

(42)

Here $\rho$ is the energy density of the dilaton matter system and $p$ is its pressure. Now it is easy to see that the quantity

$$\rho + 3p = -\frac{4}{(1+t^2)^2}$$

(43)

is negative for all $t$. It is interesting to observe that the form of $\rho + 3p$, being always negative, violates an assumption by Hawking and Ellis [2] on the form of the matter energy-momentum tensor, which, being satisfied, would always lead to a singular space-time. Thus, the absence of an initial singularity in the cosmological region
I can be understood from the specific form of the energy-momentum tensor of the dilaton matter system.

The duality transformation in the Einstein frame is expressed as

\[ t^2 \rightarrow -1 - t^2. \]  \hspace{1cm} (44)

Thus we see again that the cosmological region I is mapped to the cosmological region IV, which requires an analytic continuation to imaginary \( t \) values. However it is more convenient to use real coordinates which cover the region IV of the following form:

\[ u = e^{x_1} t, \quad v = e^{-x_1} t, \quad t^2 > 1. \]  \hspace{1cm} (45)

Then the dual metric can be written as

\[ ds_D^2 = -dt^2 + t^2 dx_1^2 + (t^2 - 1)(dx_2^2 + dx_3^2), \]  \hspace{1cm} (46)

and the duality transformed dilaton is obtained as

\[ \Phi(t)_D = \log(t^2 - 1). \]  \hspace{1cm} (47)

It is easy to show that this metric leads to a scalar curvature which is singular at \( t = 1 \). It is instructive to compute again the dual energy density and the dual pressure of the dual dilaton matter system:

\[ \rho_D = \frac{3t^2 - 2}{(t^2 - 1)^2}, \quad p_D = \frac{2 - t^2}{(t^2 - 1)^2}. \]  \hspace{1cm} (48)

Thus the quantity \( \rho_D + 3p_D = \frac{4}{(t^2 - 1)^2} \) is now positive which implies the existence of an initial singularity according to the theorem of $\mathbb{R}$.

5. Summary

We have reviewed the cosmological solutions of the string background equations and the action of the duality transformation on the cosmological backgrounds. In two dimensions (and possibly also in higher dimensions), the cosmological coordinates are not geodesically complete and there exist a Kruskal-like completion of space-time. The cosmological metric in the Kruskal coordinates can be obtained from an exact
CFT, the gauged WZW-models based on the coset \( SL(2, \mathbb{R})/SO(1,1) \). The same CFT leads to the two-dimensional black-hole when one changes the sign of the level of the underlying non-compact Kac–Moody algebra. For both signs, black-holes as well as cosmological backgrounds, the metric of the target space, at lowest order in \( \alpha' \), leads to space-time singularities. In the cosmological case the singularity is hidden behind the light-cone. However the singularity could send signals into the light-cone and therefore influence the expansion of the Universe. This is very similar to the initial singularity (Big Bang) in the standard Friedmann-Robertson–Walker Universe. It is important to stress that, seen from the CFT point of view, the singularities in the black-hole as well as in the cosmological frameworks have exactly the same origin. Therefore one could expect that the same type of quantum gravity effects are relevant near the black-hole as well as in the early Universe at times shortly after the initial singularity. These quantum gravity effects should in principle be determined from the underlying coset CFT. (For considerations in this direction for the black-hole case, see ref.[26].) Let us mention that the elimination of the negative-norm states for the gauged WZW model with negative level \( k \) has still to be demonstrated. However we believe that the unitarity of the spectrum is possible since it was already shown [12] that for the asymptotic region \( t \to \infty \), where one approaches the linear dilaton model, the elimination of the negative norm states is possible. In addition, the construction of a modular invariant partition function is still an interesting problem.

I would like to thank C. Kounnas for a very pleasant collaboration on many of the subjects presented here. I am also grateful to E. Kiritsis for useful discussions.
References

[1] D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, Nucl. Phys. B256 (1985) 253.

[2] S.W. Hawking and G.F.R. Ellis, “The Large Scale Structure of Space-Time”, Cambridge University Press, Cambridge, 1973.

[3] K. Kikkawa and M. Yamasaki, Phys. Lett. B149 (1984) 357; N. Sakai and I. Senda, Progr. Theor. Phys. 75 (1986) 692.

[4] S. Ferrara, D. Lüst, A. Shapere and S. Theisen, Phys. Lett. B225 (1989) 363.

[5] L. Ibáñez and D. Lüst, Nucl. Phys. B382 (1992) 305.

[6] T. Buscher, Phys. Lett. B194 (1987) 59, Phys. Lett. B201 (1988) 466; L.E. Ibáñez, D. Lüst, F. Quevedo and S. Theisen, unpublished notes (1990); E. Smith and J. Polchinski, Phys. Lett. B263 (1991) 59; G. Veneziano, Phys. Lett. B265 (1991) 287; A.A. Tseytlin, Mod. Phys. Lett. A6 (1991) 1721; X. de la Ossa and F. Quevedo, preprint NEIP-92-004.

[7] E.B. Kiritsis, Mod. Phys. Lett. A6 (1991) 2871.

[8] P. Ginsparg and F. Quevedo, Nucl. Phys. B385 (1992) 527.

[9] E. Kiritsis, preprint CERN-TH.6797/93; A. Giveon and E. Kiritsis, preprint CERN-TH.6816/93.

[10] A. Giveon, Mod. Phys. Lett. A6 (1991) 2843.

[11] R.C. Myers, Phys. Lett. B199 (1987) 371; K.A. Meissner and G. Veneziano, Phys. Lett. B267 (1991) 33; A. Sen, Phys. Lett. B271 (1991) 295; M. Gasperini, J. Maharana and G. Veneziano, Phys. Lett. B272 (1991) 277; M. Gasperini and G. Veneziano, Phys. Lett. B277 (1992) 256; A.A. Tseytlin, Class. Quant. Grav. 9 (1992) 979; A.A. Tseytlin, preprint DAMPT-92-15; A.A. Tseytlin, preprint DAMPT-92-36; I. Bars and K. Sfetsos, preprint USC-92/HEP-B1; K. Behrndt, preprints DESY 92-055; DESY 92-179; M. Gasperini and G. Veneziano, preprint CERN-TH.6572/92; H.J. de Vega and N. Sanchez, preprint LPTHE 92-31; H.J. de Vega, A.V. Mikhailov and N. Sanchez, preprint LPTHE 92-32.

[12] I. Antoniadis, C. Bachas, J. Ellis and D.V. Nanopoulos, Phys. Lett. B211 (1988) 393, Nucl. Phys. B328 (1989) 117.

[13] R. Brandenberger and C. Vafa, Nucl. Phys. B316 (1988) 391; A.A. Tseytlin and C. Vafa, Nucl. Phys. B372 (1992) 443.
[14] M. Muller, Nucl. Phys. B337 (1990) 37.
[15] K.A. Meissner and G. Veneziano, Mod. Phys. Lett. A6 (1991) 3397.
[16] C. Kounnas and D. Lüst, Phys. Lett. B289 (1992) 56.
[17] C. Nappi and E. Witten, Phys. Lett. B293 (1992) 309.
[18] C. Callan, D. Friedan, E. Martinec and M. Perry, Nucl. Phys. B262 (1985) 593.
[19] E. Alvarez and M. Osorio, Phys. Rev. 40 (1989) 1150.
[20] M. Rocek and E. Verlinde, Nucl. Phys. B373 (1992) 630; A. Giveon and M. Rocek, Nucl. Phys. B380 (1992) 128.
[21] E. Witten, Comm. Math. Phys. 92 (1984) 455; E. Witten, Nucl. Phys. B223 (1983) 422; K. Bardacki, E. Rabinovici and B. Saering, Nucl. Phys. B301 (1988) 151; D. Karabali and H.J. Schnitzer, Nucl. Phys. B329 (1990) 649.
[22] E. Witten, Phys. Rev. 44 (1991) 314.
[23] S. Elitzur, A. Forge and E. Rabinovici, Nucl. Phys. B359 (1991) 581; G. Mandal, A.M. Sengupta and S.R. Wadia, Mod. Phys. Lett. A6 (1991) 1685.
[24] L. Dixon, J. Lykken and M. Peskin, Nucl. Phys. B325 (1989) 325; I. Bars, Nucl. Phys. B334 (1990) 125; I. Bars and D. Nemeschansky, Nucl. Phys. B348 (1991) 89.
[25] J. Polchinski, Nucl. Phys. B324 (1989) 123.
[26] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B371 (1992) 269.
[27] I. Bakas and E. Kiritsis, Int. J. Mod. Phys. A7 (1992) 55; J. Ellis, N. Mavromatos and D.V. Nanopoulos, Phys. Lett. B272 (1991) 261, Phys. Lett. B284 (1992) 43.
[28] I. Bars and K. Sfetsos, preprints USC-92/HEP-B2, USC-92/HEP-B3.
[29] A.A. Tseytlin, preprints Imperial/TP/92-93/7, CERN-TH.6804/93.
[30] M.J. Perry and E. Teo, preprint DAMTP R93/1; P. Yi, preprint CALT-68-1852.
[31] P. Horava, Phys. Lett. B278 (1992) 101.
[32] C. Kounnas, preprints CERN-TH.6790/93, CERN-TH.6799/93.
Figure Captions

**Figure 1:** The causal structure of the two-dimensional slice of the black-hole metric equation (17) with positive Kac–Moody level $k$.

**Figure 2:** The causal structure of the two-dimensional slice of the cosmological metric equation (17) with negative Kac–Moody level $k$. 