Cosmological perturbations in the bulk and on the brane

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We study cosmological perturbations in a brane-world scenario where the matter fields live on a four-dimensional brane and gravity propagates in the five-dimensional bulk. We present the equations of motion in an arbitrary gauge for metric perturbations in the bulk and matter perturbations on the brane. Gauge-invariant perturbations are then constructed corresponding to perturbations in longitudinal and Gaussian normal gauges. Longitudinal gauge metric perturbations may be directly derived from three master variables (separately describing scalar, vector and tensor metric perturbations) which obey five-dimensional wave-equations. Gaussian normal gauge perturbations are directly related to the induced metric perturbations on the brane with the additional bulk degrees of freedom interpreted as an effective Weyl energy-momentum tensor on the brane. We construct gauge-invariant perturbations describing the effective density, momentum and pressures of this Weyl fluid at the brane and throughout the bulk. We show that there exist gauge-invariant curvature perturbations on the brane and in the bulk that are conserved on large-scales when three-dimensional spatial gradients are negligible.

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I. INTRODUCTION

The brane-world scenario offers a different approach to dimensional reduction from the traditional Kaluza-Klein model. Ordinary matter fields are assumed to be confined to a lower-dimensional brane in a higher-dimensional spacetime rather than being associated with bulk degrees of freedom. The gravity that the brane-bound observer feels is only the projected gravity of a higher-dimensional gravitational field. In this paper we will consider models where the bulk is described by Einstein gravity in which case we require a positive brane tension for the induced gravity to have a positive gravitational constant at low energies, and a negative cosmological constant in the bulk to have a vanishing induced cosmological constant on the brane. Randall and Sundrum demonstrated that in such a model it is possible to have a four-dimensional Minkowski brane-world in five-dimensional anti-de Sitter spacetime (AdS$_5$) where gravity is effectively four-dimensional at low-energies even if the bulk is non-compact.

Our aim in this paper is to describe arbitrary linear perturbations about 4D Friedmann-Robertson-Walker (FRW) cosmologies embedded in 5D Schwarzschild-Anti-de Sitter bulk and understand the effect of bulk metric perturbations on the brane. Modelling the generation and evolution of cosmological perturbations offers a means by which current brane-world scenarios can be empirically tested. Although considerable effort has already been devoted to this subject, the results known thus far are limited to special cases. The only cosmological background in which the five-dimensional perturbation equations are separable corresponds to a de Sitter (or Minkowski) brane In more general (and more realistic) cosmological settings current results are restricted to either small (3D) scales where the cosmological expansion may be neglected, or large scales where gradient terms are neglected, and even then our understanding is incomplete.

We will assume that our 4-dimensional world is described by a domain wall (3-brane) $(M, g_{\mu\nu})$ in the 5-dimensional spacetime $(\mathcal{M}, g_{AB})$. We will denote the vector unit normal to $M$ by $n^A$ and the induced metric on $M$ by $g_{AB}$.

$$g_{AB} = (5)g_{AB} - n_A n_B$$

The effective action in the 5-dimensional spacetime is

$$S = \int_M d^5x \sqrt{-(5)g} \left[ \frac{1}{2\kappa_5^2} ((5)R - 2\Lambda_5) \right] + \int_M d^4x \sqrt{-g} \left[ L_{\text{matter}} - \lambda \right],$$

where $x^\mu$ are the 4-dimensional coordinates induced on the brane world. The 5-dimensional Einstein equations, obtained by minimising the action with respect to variations in the bulk metric, are

$$(5)G_{AB} + (5)g_{AB} \Lambda_5 = 0.$$
The requirement of spatial homogeneity and isotropy on the brane is sufficiently restrictive to allow one to solve for the bulk geometry for an arbitrary brane cosmology \[7\]. For a non-static Friedmann-Robertson-Walker (FRW) cosmology on the brane the bulk geometry must be either (anti-)de Sitter or Schwarzschild-(anti-)de Sitter \[8,9,23\]. Thus the expansion of a FRW universe may be re-interpreted in the brane-world scenario as due to motion in a static bulk. The four-dimensional matter fields determine the brane trajectory in the bulk spacetime via the junction condition by producing the jump in the extrinsic curvature at the brane. Without loss of generality, the surface energy-momentum on the brane can be split into two parts, \( T_{\mu\nu} - \lambda g_{\mu\nu} \), where \( T_{\mu\nu} \) is taken to be the matter energy-momentum tensor and \( \lambda \) a constant brane tension. The junction condition is then \[24,1,2\]

\[
K_{\mu\nu}^+ - K_{\mu\nu}^- = -\kappa_5^2 \left( T_{\mu}^\nu - \frac{1}{3} g_{\mu}^\nu (T - \lambda) \right),
\]

where the extrinsic curvature of the brane is denoted by

\[
K_{\mu\nu} = g_{\mu}^C g_{\nu}^D (\nabla_C \nabla_D),
\]

where \( \nabla_C \) is the 5D covariant derivative. We will consider the case where the brane is located at a \( Z_2 \)-symmetric orbifold fixed point. Imposing the \( Z_2 \) symmetry in Eq. \[1.4\], we obtain

\[
K_{\mu\nu}^+ = -K_{\mu\nu}^- = -\kappa_5^2 \left( T_{\mu}^\nu - \frac{1}{3} g_{\mu}^\nu (T - \lambda) \right).
\]

Henceforth we will only refer to quantities on the ‘+’ side of the brane and will drop the + superscript. The Codazzi equation for a vacuum bulk,

\[
\nabla_\mu K_{\mu\nu} - \nabla_\nu K = 0,
\]

together with Eq. \[1.6\] enforces local energy-momentum conservation on the brane,

\[
\nabla_\mu T_{\mu\nu} = 0,
\]

where \( \nabla_\mu \) is the 4D covariant derivative.

Shiromizu, Maeda and Sasaki \[2\] showed that the effective four-dimensional Einstein equations in a brane world can be obtained by projecting the five-dimensional equations onto the 4-dimensional brane. Using the Gauss and Codacci equations, we can obtain the Einstein tensor for the four-dimensional metric induced on the brane

\[
(4) \, G_{\mu\nu} = \Lambda_4 + K K_{\mu\nu} - K_{\mu}^\alpha K_{\nu}\alpha - \frac{1}{2} g_{\mu\nu} (K^2 - K_{\alpha\beta} K^{\alpha\beta}) - E_{\mu\nu},
\]

where

\[
E_{\mu\nu} = (5) \, C^{E}_{AFB} n_A n_F g_{\mu}^A g_{\nu}^B.
\]

is the projected five-dimensional Weyl tensor.

The effective four-dimensional Einstein equations can then be written using Eqs. \[1.1,2\] and \[1.9\]

\[
(4) \, G_{\mu\nu} + \Lambda_4 g_{\mu\nu} = \kappa_4^2 T_{\mu\nu} + \kappa_4^4 \Pi_{\mu\nu} - E_{\mu\nu},
\]

where we define

\[
\Lambda_4 = \frac{1}{2} \Lambda_5 + \frac{\kappa_4^4}{12} \lambda^2, \tag{1.12}
\]

\[
\kappa_4^2 = \frac{\kappa_4^4}{6} \lambda, \tag{1.13}
\]

\[
\Pi_{\mu\nu} = -\frac{1}{4} T_{\mu\alpha} T_{\nu}^{\alpha} + \frac{1}{12} T T_{\mu\nu} + \frac{1}{8} g_{\mu\nu} T_{\alpha\beta} T^{\alpha\beta} - \frac{1}{24} g_{\mu\nu} T^2. \tag{1.14}
\]

It is clear that we require \( \lambda > 0 \) to have a positive induced gravitational coupling constant (\( \kappa_4^2 > 0 \)), and then \( \Lambda_5 < 0 \) to have \( \Lambda_4 = 0 \).
The power of this approach is that the above form of the 4D effective equations of motion is independent of the evolution of the bulk spacetime, being given entirely in terms of quantities defined on, or near, the brane. Thus these equations apply to brane-world scenarios with infinite or finite bulk, static or evolving. A limitation is that this may leave terms which are not completely determined by the local dynamics on the brane \[E_{\mu \nu}\). In particular the tensor $E_{\mu \nu}$ is due to the 5-dimensional Weyl tensor and so can be affected by gravitational waves propagating in the bulk. In the case where the induced 4D metric is required to be homogeneous and isotropic this strong symmetry constrains the traceless tensor, $E_{\mu \nu}$, to be covariantly conserved on the brane, so that it acts like non-interacting radiation. However we will be interested in arbitrary linear perturbations about spacetimes including an FRW brane and then we will need information from the full five-dimensional field equations to describe the evolution of inhomogeneous perturbations on the brane. Nonetheless the dynamics and effective gravity on the brane can be interpreted, and often most easily understood, in terms of the effective four-dimensional Einstein equations (1.11).

In section II we briefly summarise essential results for the evolution of FRW brane-world cosmologies in a Schwarzschild-anti-de Sitter bulk which we shall use as our unperturbed background solution. In section II we introduce linear perturbations of the bulk metric split into scalar, vector and tensor perturbations (defined with respect to 3D spatial hypersurfaces) in an arbitrary gauge. It is trivial to construct quantities independent of the 3D-spatial gauge, and this is sufficient to give gauge-invariant vector and tensor perturbations, but there are alternative choices of temporal and bulk gauges and hence different choices of gauge-invariant scalar perturbations. In section IV we define gauge-invariant quantities coinciding with metric perturbations in the 5D longitudinal gauge, whereas in section V we construct quantities in a Gaussian normal gauge where the 5D bulk metric perturbations coincide with the induced 4D metric perturbations on the brane. This is particularly suited for imposing the perturbed junction conditions, given in section VI, which allows us to interpret the effect of bulk metric perturbations as seen by the brane-bound observer in section VII. We show in section VIII that one can construct gauge-invariant perturbations in order to describe the evolution of the Weyl tensor perturbations throughout the bulk. Finally we summarise our results in section IX. To assist the reader we have given the full unpexpurgated tensor components in appendices and included in the main text only simplified expressions.

II. COSMOLOGICAL BACKGROUND SOLUTION

In order to study inhomogeneous bulk metric perturbations we will pick a specific form for the unperturbed 5D spacetime that accommodates any spatially flat FRW cosmological solution on the brane at $y = 0$,

$$\begin{align*}
ds_5^2 &= -n^2(\eta, y) d\eta^2 + a^2(\eta, y) \delta_{ij} dx^i dx^j + dy^2.
\end{align*}$$

This form, using a Gaussian normal bulk coordinate, $y$, which measures the proper distance from the brane, has been extensively studied in the literature and includes anti-de Sitter spacetime as a special case. In particular it was shown in Ref. [7] that it is possible to use the 5D Einstein equations to solve exactly for the dependence of the metric upon the bulk coordinate $y$ in order to write (for $\Lambda^5 < 0$) \([7]\)

$$\begin{align*}
a^2(\eta, y) &= \alpha(\eta) \cosh(\mu |y|) + \beta(\eta) \sinh(\mu |y|) - \gamma^2(\eta),
\end{align*}$$

where $\mu^2 = -2\Lambda_5/3$ describes the curvature of the 5D space, and $\alpha$, $\beta$ and $\gamma$ are functions only of time that are determined by the matching conditions at the brane. Equation (A3) requires $(a/n)' = 0$, and hence we can write

$$n(\eta, y) = \frac{a}{a_b H},$$

where $H(\eta) = \dot{a}_b/n_b a_b$ is the Hubble expansion on the brane and we use a dot to denote derivatives with respect to coordinate time $\eta$, prime to denote derivatives with respect to the bulk coordinate $y$, and the subscript $b$ to denote bulk quantities evaluated at the brane.

However the evolution of the FRW cosmology on the brane can be determined solely from the local 5D Einstein equations at the brane. The 4D components of the 5D Einstein tensor [Eqs. (A1) and (A5)] can be written in terms of the 4D Einstein tensor [Eqs. (A8) and (A9)] as

$$\begin{align*}
(5)G_0^0 &= (4)G_0^0 + E_0^0 + 3 \frac{a'^2}{a_5^2} = -\Lambda_5, \\
(5)G_j^j &= (4)G_j^j + E_j^j + 2 \frac{a'^2}{a_b^2} \frac{a_i b_i}{a I a B} \delta_j^j = -\Lambda_5 \delta_j^j,
\end{align*}$$

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where the projected 5D Weyl tensor is given from Eqs \((A6)\) and \((A7)\) by

\[
E^0_0 = 3 \frac{a''}{a} + \frac{1}{2} \Lambda_5 ,
\]

\[
E^i_j = -\frac{1}{3} E^0_0 \delta^i_j ,
\]

and we have simplified these expressions using the 5D Einstein equations \((A4), (A1),\) and \((A5)\) which give the constraint equation

\[
\frac{n''}{n} + 3 \frac{a''}{a} = -\frac{2}{3} \Lambda_5 .
\]

The extrinsic curvature, defined in Eq. \((1.5)\) above, of the FRW brane is

\[
K^0_0 = \frac{n'_b}{n_b} ,
\]

\[
K^i_j = \frac{a'_b}{a_b} .
\]

The energy-momentum tensor in a background with homogeneous density \(\rho\) and pressure \(P\) is given by

\[
T^0_0 = -\rho ,
\]

\[
T^i_j = P \delta^i_j .
\]

Hence the junction conditions given in Eq. \((1.6)\), using Eqs. \((2.9)\)\(2.12)\) for a FRW brane, require

\[
\frac{n'_b}{n_b} = \frac{\kappa^2}{2} \left( \frac{2}{3} \rho + P - \frac{1}{3} \lambda \right) ,
\]

\[
\frac{a'_b}{a_b} = -\frac{\kappa^2}{2} \left( \frac{1}{3} \rho + \frac{1}{3} \lambda \right) .
\]

Thus the 5D Einstein equations \((2.4)\) and \((2.5)\) at the brane are of the general form given in Eq. \((1.11)\) for the 4D effective Einstein equations where \(\Lambda_4\) is given by Eq. \((1.12)\) and the quadratic correction to the effective energy-momentum tensor of ordinary matter, \(\Pi^\mu_\nu\) defined in Eq. \((1.14)\), is given by

\[
\Pi^0_0 = -\frac{1}{12} \rho^2 ,
\]

\[
\Pi^i_j = \frac{1}{12} \rho (\rho + 2P) \delta^i_j .
\]

Note that local energy conservation follows from the 5D Einstein equation \((A3)\), using the junction conditions \((2.13)\)

\[
\dot{\rho} = -3 \frac{\dot{a}}{a} (\rho + P) .
\]

One can verify that the quadratic energy-momentum tensor \(\Pi^\mu_\nu\) is also conserved for a spatially homogeneous fluid \([2]\). The 4D Bianchi identities, \(\nabla_\mu (4) G^\mu_\nu = 0\) then ensure that the contribution of the 5D Weyl tensor to the effective 4D Einstein equations \((1.11)\), \(E^\mu_\nu\), is also conserved, and thus the FRW brane dynamics are completely determined once one has set the initial conditions for all these terms on the brane, without requiring any further knowledge of the bulk behaviour away from the brane.

III. LINEAR PERTURBATIONS

We now proceed to consider arbitrary linear perturbations about the background metric defined in Eq. \((2.1)\). In keeping with the standard approach in cosmological perturbation theory \([23]\) we will introduce scalar, vector and tensor perturbations defined in terms of their properties on the 3-spaces at fixed \(t\) and \(y\) coordinates.

We can write the most general perturbed metric to first-order as
where \( A, B, R, E, A_y \) and \( A_{yy} \) are scalars, \( S_i, F_i, \) and \( S_{yi} \) are (divergence-free) 3-vectors, and \( h_{ij} \) is a (transverse and traceless) 3-tensor. The reason for splitting the metric perturbation into these three types is that they are decoupled in the linear perturbation equations.

In the perturbed spacetime there is a gauge-dependence in the definition of the scalar and vector perturbations under a first-order coordinate transformation, \( x^A \rightarrow x^A + \xi^A \), which we will write as

\[
\begin{align*}
\eta &\rightarrow \eta + \delta \eta, \\
x^i &\rightarrow x^i + \delta x^i, \\
y &\rightarrow y + \delta y,
\end{align*}
\]

where \( \delta \eta, \delta x \) and \( \delta y \) are scalars and \( \delta x^i \) is a (divergence-free) 3-vector.

### 1. Scalars

The scalar perturbations transform as

\[
\begin{align*}
A &\rightarrow A - \dot{\delta} \eta - \frac{\dot{n} \delta \eta}{n} - \frac{n' \delta y}{n}, \\
R &\rightarrow R - \frac{\dot{a} \delta \eta}{a} - \frac{a' \delta y}{a}, \\
B &\rightarrow B + \frac{n^2}{a^2} \delta \eta - \dot{\delta} x, \\
B_y &\rightarrow B_y - \delta x^i - \frac{1}{a^2} \delta y, \\
E &\rightarrow E - \delta x, \\
A_y &\rightarrow A_y + n \delta \eta' - \frac{1}{n} \dot{\delta} y, \\
A_{yy} &\rightarrow A_{yy} - \delta y'.
\end{align*}
\]

To eliminate the spatial gauge dependence we introduce the spatially gauge-invariant combinations

\[
\begin{align*}
\sigma &\equiv -B + \dot{E}, \\
\sigma_y &\equiv -B_y + \dot{E}',
\end{align*}
\]

which are subject only to temporal and bulk gauge transformations

\[
\begin{align*}
\sigma &\rightarrow \sigma - \frac{n^2}{a^2} \delta \eta, \\
\sigma_y &\rightarrow \sigma_y + \frac{1}{a^2} \delta y.
\end{align*}
\]

The variable \( \sigma \) represents the shear of a unit timelike vector field projected onto the brane, whereas \( \sigma_y \) represents the shear with respect to a unit spacelike vector field orthogonal to the brane. We sometimes also use the spatially-gauge invariant combination

\[
B \equiv B' - \dot{B}_y = \dot{\sigma} y - \sigma'.
\]
\[ S_i \rightarrow S_i + \dot{x}_i, \]
\[ S_{yi} \rightarrow S_{yi} + \dot{x}_i', \]
\[ F_i \rightarrow F_i - \delta x_i. \]

It will be convenient to separate out time and bulk variation of vector perturbations from their spatial dependence:

\[ S_i = S(t, y) \hat{e}_i(x), \]
\[ S_{yi} = S(t, y) \hat{e}_i(x), \]
\[ F = F(t, y) \hat{e}_i(x), \]

where the vector \( \hat{e}_i(x) \) is divergence-free (\( \hat{e}_i \cdot \nabla \hat{e}_i = 0 \)). We can then introduce the gauge-invariant combinations

\[
\tau = S + \dot{F},
\]
\[
\tau_y = S_y + F',
\]
\[
S = S' - \dot{S}_y = \tau' - \tau_y,
\]

which eliminates any gauge-ambiguity for the vector perturbations.

3. Tensors

The tensor perturbations \( h_{ij} \) are automatically gauge-invariant.

\[ h_{ij} \rightarrow h_{ij}. \] (3.10)

IV. BULK PERTURBATIONS IN THE 5D-LONGITUDINAL GAUGE

The bulk and temporal gauges are fully determined by setting

\[ \tilde{\sigma} = 0, \]
\[ \tilde{\sigma}_y = 0, \] (4.1)

which has been termed the longitudinal gauge [16]. We will refer to this as the 5D-longitudinal gauge, to avoid later confusion with quantities in the 4D-longitudinal gauge on the brane.

From an arbitrary gauge, we can obtain quantities in the 5D-longitudinal gauge, using Eqs. (3.3) and (4.1), to define the specific gauge transformation

\[ \delta \tilde{\eta} = \frac{a^2}{n^2} \sigma, \]
\[ \delta \tilde{y} = -a^2 \sigma_y. \] (4.2)

Hence, from Eqs. (3.3) and (4.2), we can define the remaining metric perturbations in the 5D-longitudinal gauge as

\[ \tilde{A} = A - \frac{1}{n} \left( \frac{a^2}{n^2} \right)' + \frac{n'}{n} \left( a^2 \sigma_y \right), \]
\[ \tilde{R} = R - \frac{\dot{a}}{a} \frac{a^2}{n^2} \sigma + \frac{a'}{a} a^2 \sigma_y, \]
\[ \tilde{A}_y = A_y + n \left( \frac{a^2}{n^2} \sigma \right)' + \frac{1}{n} \left( a^2 \sigma_y \right)', \]
\[ \tilde{A}_{yy} = A_{yy} + (a^2 \sigma_y)' . \] (4.3)

These are equivalent to the gauge-invariant bulk perturbations originally introduced in covariant form by Mukohyama [11,26] and in a coordinate-based approach by van den Bruck et al [14].
1. Scalar perturbations

First note that the spatial trace part of the 5D Einstein equations, given in Eq. (B17), simplifies in the 5D-longitudinal gauge to

\[ \tilde{A} + \tilde{R} + \tilde{A}_{yy} = 0. \]  

(4.4)

We can therefore eliminate any one of these metric perturbations from the equations. This simple constraint has also been found to hold for the case of static bulk solutions with a scalar field in the bulk \[27\] and continues to hold in a cosmological setting in the absence of anisotropic stress in the bulk \[16\].

Mukohyama \[11\] (see also Ref. \[26\]) was the first to show that the perturbed 5D Einstein equations (B10), (B12) and (B13) in the absence of bulk matter perturbations \((\delta G_{AB} = 0)\) are solved in an anti-de Sitter background if the metric perturbations are derived from a “master variable”, \(\Omega\):

\[ \tilde{A} = -\frac{1}{6a} \left\{ 2\Omega'' - \frac{n'}{n} \Omega' + \frac{\Lambda}{6} + \frac{1}{n^2} \left( \tilde{\Omega} - \frac{n}{n} \tilde{\Omega} \right) \right\}, \]  

(4.5)

\[ \tilde{A}_y = \frac{1}{na} \left( \tilde{\Omega}' - \frac{n'}{n} \tilde{\Omega} \right), \]  

(4.6)

\[ \tilde{A}_{yy} = \frac{1}{6a} \left\{ \Omega'' - 2\frac{n'}{n} \Omega' + \frac{2}{n^2} \left( \tilde{\Omega} - \frac{n}{n} \tilde{\Omega} \right) - \frac{\Lambda}{6} \Omega \right\}, \]  

(4.7)

\[ \tilde{R} = \frac{1}{6a} \left\{ \Omega'' + \frac{n'}{n} \Omega' - \frac{1}{n^2} \left( \tilde{\Omega} - \frac{n}{n} \tilde{\Omega} \right) + \frac{\Lambda}{3} \Omega \right\}. \]  

(4.8)

The remaining perturbed 5D Einstein Eqs. (B9), (B11) and (B14) then yield a single wave equation governing the evolution of the master variable \(\Omega\) in the bulk:

\[ \left( \frac{1}{na^3} \tilde{\Omega} \right)' + \left( \frac{\Lambda_5}{6} - \nabla^2 \right) \frac{n}{a^3} \Omega = \left( \frac{n}{a^3} \Omega' \right)'. \]  

(4.9)

This 5D wave equation is separable only for a separable background solution which corresponds to the case of a de Sitter brane-world \[10,18\].

2. Vector perturbations

As for scalar perturbations, the evolution equation for vector metric perturbations, Eq. (B20), is automatically satisfied when the gauge invariant vector perturbations \(\tau\) and \(\tau_y\) are derived from a single “master variable” \[11,26,18\]

\[ \tau = \frac{n}{a^3} \Omega^{(vector)}, \]  

(4.10)

\[ \tau_y = \frac{1}{na^3} \dot{\Omega}^{(vector)}. \]  

(4.11)

The remaining perturbed 5D Einstein equations (B18) and (B21) then yield a single governing equation for \(\Omega^{(vector)}\),

\[ \frac{1}{n^2} \left[ \ddot{\Omega}^{(vector)} - \left( \frac{3}{a} \frac{\dot{\Omega}^{(vector)}}{a} + \frac{n}{n} \dot{\Omega}^{(vector)} \right) - \left[ \dot{\Omega}^{(vector)} - \left( 3 \frac{a'}{a} - \frac{n'}{n} \right) \Omega^{(vector)} \right] + \frac{k^2}{a^2} \Omega^{(vector)} = 0, \]  

(4.12)

for each Fourier mode, \(\nabla^2 \hat{e}_i = -k^2 \hat{e}_i\).

3. Tensor perturbations

The only Einstein equation for the gauge-invariant tensor perturbations, Eq. (B22), already has the form of the wave equation for a master variable. This can be made explicit if we separate out the 3-space dependence, writing

\[ h_{ij}(t, x^i, y) = \Omega^{(tensor)}(t, y) \hat{e}_i(x^i), \]  

(4.13)
where \( e_{ij}(x') \) is a transverse, tracefree harmonic on the spatially flat 3-space, \( \nabla^2 e_{ij} = -k^2 e_{ij} \). Equation (3.24) then yields

\[
\frac{1}{n^2} \left[ \Omega^{(5)}(\text{tensor}) + \left( \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) \Omega^{(3)}(\text{tensor}) \right] - \left[ \Omega^{(3)}(\text{tensor}) + \left( \frac{\dot{a'}}{a'} + \frac{n'}{n} \right) \Omega^{(1)}(\text{tensor}) \right] + \frac{k^2}{a^2} \Omega^{(5)}(\text{tensor}) = 0. \tag{4.14}
\]

This equation and its solutions for the special case of a de Sitter brane were studied in Ref. [15].

V. THE GAUSSIAN NORMAL GAUGE

An alternative choice of gauge often used [10, 13, 20] is a Gaussian normal (GN) gauge, where the bulk \( y \) coordinate measures the proper distance from the brane, up to and including first-order perturbations. This requires the metric perturbations \( B_y, A_y \) and \( A_{yy} \) to vanish. Starting from an arbitrary 5D gauge, this can be enforced throughout the bulk via a coordinate shift \( x^A \to \bar{x}^A = x^A + \delta x^A \), [see Eqs. (3.2) and (3.3)] such that,

\[
\begin{align*}
\delta \eta' &= \frac{1}{n^2} \tilde{\delta} \eta - \frac{1}{n} A_y, \\
\delta \tilde{y}' &= A_{yy}, \\
\delta \bar{x}' &= -\frac{1}{a^2} \delta \tilde{y} + B_y, \\
\delta \bar{x}'' &= -S_{yi}.
\end{align*}
\]

This does not completely fix the gauge, but only the \( y \)-derivative of the gauge-transformation, analogous to the synchronous gauge in conventional 4D perturbation theory.

The projected metric on any constant-\( y \) hypersurface is then

\[
g_{AB} = (5)g_{AB} - n_An_B = \begin{pmatrix}
-n^2(1 + 2A) & a^2(\tilde{B}_{ij} - \tilde{S}_{ij})/a^2 \\
a^2(\tilde{B}_{ij} - \tilde{S}_{ij})/a^2 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}, \tag{5.2}
\]

where we denote by a bar the metric perturbations in a Gaussian normal gauge.

A. Transverse-tracefree GN gauge

A technique commonly used to study the propagation of gravitational waves in a vacuum spacetime is to work in a gauge in which the perturbations are transverse and tracefree in the background spacetime. We shall now show that while the transverse and tracefree condition is certainly sufficient to fix the residual gauge-freedom in a Gaussian normal gauge, it actually over constrains the problem for a general cosmology and its usefulness is restricted to the cases of a maximally symmetric 4D (anti-)de Sitter brane.

The linearly perturbed five-dimensional Ricci tensor can be written as

\[
(5)\delta R_{BD} = \frac{1}{2} \left\{ (5)\nabla^A(5)\nabla_D(5)\delta g_{BA} + (5)\nabla_B(5)\delta g_{DA} \right\} - (5)\nabla^A(5)\nabla_A(5)\delta g_{BD} - (5)\nabla_D(5)\nabla_B(5)\delta g^A_{\,A} \right\}, \tag{5.3}
\]

where \( (5)\nabla_A \) is the five-dimensional covariant derivative. Swapping the order of differentiation of the first two terms, this can be rewritten as

\[
(5)\delta R_{AB} = \frac{1}{2} \left\{ (5)\nabla_A(5)\nabla_C(5)\delta g_{BC} + (5)\nabla_C(5)\delta g_{AC} \right\} - (5)\nabla_A(5)\nabla_B(5)\delta g_{CD} - (5)\nabla_C(5)\nabla_C(5)\delta g_{AB} \right\} + (5)R_{CABD}(5)\delta g^{CD}. \tag{5.4}
\]

\(^1\)In principle the choice of coordinates, and hence gauge transformations, on the brane may be chosen quite independently of the bulk [14, 19]. However in order to relate the bulk perturbations as directly as possible to the brane-world observer we will use the same coordinates \((t, x')\) on the brane as are used to follow perturbations in the bulk.
In the absence of any bulk matter perturbations, the perturbed 5D Einstein equations require $(\Box)^5 \delta R_{AB} = 0$. Thus when the 5D perturbations are transverse $\nabla_C (\Box)^5 \delta g_{AC} = 0$ and tracefree $(\Box)^5 \delta g_{CC} = 0$, the perturbed Einstein equations can be written as

$$ (\Box)^5 \delta g_{AB} = 2 (\Box)^5 R_{CADB} (\Box)^5 \delta g^{CD}, $$

where $(\Box)^5 = \nabla_C (\Box)^5 \nabla^C$. The background Riemann tensor in AdS$_5$ is given by

$$ (\Box)^5 R_{ABCD} = \frac{\Lambda}{5} (g_{AC} g_{BD} - g_{AD} g_{BC}). $$

Therefore to linear order in the metric perturbations and enforcing the transverse and traceless conditions the field equations in the absence of matter are given by

$$ (\Box)^5 \delta g_{AB} = -\frac{1}{3} \Lambda (\Box)^5 \delta g_{AB}. $$

### 1. Scalar perturbations

The tracefree condition, in a GN gauge, requires

$$ A + 3 \mathcal{R} + \nabla^2 E = 0. $$

The transverse condition in general gives rise to four constraint equations, which can be written, using Eq. (5.8) above, as

$$ \nabla^2 B + 2 \left( \dot{A} + 4 \frac{\dot{a}}{a} A \right) = 0, $$

$$ \left\{ \frac{a^2}{n^2} \left[ \left( \frac{\dot{n}}{n} - 5 \frac{\dot{a}}{a} \right) B - \dot{B} \right] - 2 \dot{A} - 4 \dot{\mathcal{R}} \right\}_i = 0, $$

$$ 2 \left( \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) A = 0. $$

Unless $(a/n)' = 0$, i.e., $a$ and $n$ have the same $y$-dependence, the five constraint equations require that the four Gaussian normal scalar metric perturbations are all identically zero. Using the background field equation $(\Box)^5 G_0^4 = 0$ [see Eq. (A3)], this implies that it is only possible to use the transverse and tracefree Gaussian normal gauge for a separable bulk metric, which corresponds to an (anti-)de Sitter or Minkowski brane.

Quite generally, one can see that the gauge shift required to impose the traceless $(\Box)^5 \delta g_{AB} = 0$ and transverse $(\Box)^5 \delta g_{AC} = 0$ conditions throughout the bulk, will only be compatible with the Gaussian normal gauge shift $(\Box)^5 \delta g_{CD} = 0$ if the metric perturbations are composed of separable functions of brane coordinates $\eta$ and $x^i$ and bulk coordinate $y$. And this requires that the background bulk metric is separable.

Thus, only in the special case of a (anti-)de Sitter or Minkowski brane, the wave equation (5.7) gives an evolution equation for the scalar metric perturbation:

$$ \frac{1}{n^2} \left\{ \ddot{A} - \left( \frac{\dot{n}}{n} - 7 \frac{\dot{a}}{a} \right) \dot{A} \right\} + \left[ 8 \left( \frac{\dot{a}}{an} \right)^2 + 2 \left( \frac{\dot{a}'}{a} - \frac{1}{6} \Lambda \right) A - \frac{1}{a^2} \nabla^2 A = A'' + 4 \frac{a'}{a} A, $$

and the remaining scalar metric perturbations can be deduced from the constraint Eqs. (5.8) and (5.9).

### 2. Vector perturbations

The transverse condition for the vector perturbations gives a single constraint equation,

$$ \nabla^2 F_i + \frac{a^2}{n^2} \left[ \dot{S}_i - \left( \frac{\dot{n}}{n} - 5 \frac{\dot{a}}{a} \right) S_i \right] = 0. $$

9
For this to hold throughout the bulk in a GN gauge, Eq. (5.1), restricts us to the special case of a separable bulk/de Sitter brane. The wave equation (5.7) then yields a decoupled equation of motion for the gauge-invariant vector perturbation \[ 1 \quad n^2 \left[ \ddot{\eta} + 3 \left( \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) \dot{\eta} \right] + \frac{k^2}{a^2} \tau = \tau'' + 4 \frac{a'}{a} \tau', \] (5.14) which is consistent with the more generally applicable wave equation for Mukohyama’s master variable, \( \Omega^\text{vector} \), given in Eq. (4.12).

3. Tensor perturbations

Tensor perturbations, \( h_{ij} \), that are transverse and tracefree on 3D spatial hypersurfaces, remain transverse and tracefree with respect to the 4D metric. These perturbations are gauge-invariant and their evolution equation is given in Eq. (4.14).

B. Brane-GN gauge

As well as allowing arbitrary first-order metric perturbations in the bulk, we should in general allow the coordinate location of the brane to acquire a first-order perturbation, so that the brane is located at \( y_b = \xi(x^\mu) \). The coordinate value of the brane location is a coordinate-dependent quantity, transforming as

\[ \xi \rightarrow \xi + \delta y_b, \] (5.15)

under the gauge-transformation given in Eq. (3.2). It can obviously be set to zero by a suitable choice of gauge. Nonetheless it is important to realise that such a gauge restriction is being imposed when the coordinate location of the brane is fixed. It is not possible to set the brane location to zero if the gauge has already been completely fixed, e.g., in the 5D-longitudinal gauge or the transverse-tracefree GN gauge, without placing physical restrictions on the perturbations that can be accommodated [16,28]. \( \xi \) represents a 4D degree of freedom that although not part of the intrinsic 5D metric perturbations, does appear in the 4D junction conditions.

The unit vector field orthogonal to the brane then becomes

\[ n_A = (-\xi_{,\mu}, 1 + A_{yy}). \] (5.16)

Note that we will identify the brane location with the \( Z_2 \)-symmetric orbifold fixed point, and hence the covariant form of the junction condition given in Eq. (1.6) to be imposed at the brane is unchanged. In practice, the junction conditions and \( Z_2 \) symmetry at the brane are most easily presented in terms of the metric perturbations at the brane in the Gaussian normal gauge, where the bulk coordinate \( y \) still measures the proper distance from the brane including first-order perturbations.

To fix the coordinate position of the brane as \( \bar{y}_b = \bar{\xi} = 0 \) requires in addition
\[ \delta y_b = -\xi. \] (5.17)

This then fixes completely \( \delta y \), i.e., the bulk-slicing, but leaves a residual temporal gauge-freedom, i.e., time-slicing, which can be eliminated by fixing the time-slicing on any constant-\( y \) hypersurface, e.g., \( \delta \eta_b \). This corresponds to the standard 4D temporal gauge-dependence of the metric perturbations on the brane.

The induced scalar metric perturbations on the brane are thus obtained from arbitrary gauge perturbations via the transformations in Eqs. (3.2) using the specific coordinate shifts in Eqs. (5.1) and (5.17), to give

\[
\bar{A}_b = A_b + n'_b \frac{n_b}{a_b} \xi, \\
\bar{\mathcal{R}}_b = \mathcal{R}_b + a'_b \frac{n_b}{a_b} \xi, \\
\bar{\sigma}_b = \sigma_b.
\] (5.18)

These are the standard gauge-dependent metric perturbations corresponding to lapse, curvature and shear perturbations in conventional 4D cosmological perturbation theory [29]. They transform in the standard way under temporal gauge-transformations on the brane, \( \eta_b \rightarrow \eta_b + \delta \eta_b \), to give

\[
\bar{A}_b \rightarrow \bar{A}_b - \frac{1}{n_b} (n_b \delta \eta_b), \\
\bar{\mathcal{R}}_b \rightarrow \bar{\mathcal{R}}_b - \frac{a_b}{a_b} \delta \eta_b, \\
\bar{\sigma}_b \rightarrow \bar{\sigma}_b - \frac{n^2}{a_b^2} \delta \eta_b.
\] (5.19)

One can recover the 4D-longitudinal gauge perturbations on the brane by choosing \( \delta \eta_b = (a_b/n_b)^2 \bar{\sigma}_b \) in the above equations, yielding \( \bar{\sigma}_b \rightarrow 0 \) and

\[
\Phi = \bar{A}_b - \frac{1}{n_b} (a^2 \bar{\sigma}_b), \\
\Psi = -\bar{\mathcal{R}}_b + \frac{a_b}{a_b} \bar{\sigma}_b,
\] (5.20)

where \( \Phi \) and \( \Psi \) correspond to the 4D metric potentials defined in Ref. [29] and are completely gauge-invariant. These 4D-longitudinal gauge perturbations on the brane do not in general coincide with the 5D-longitudinal gauge perturbations. Using the 5D-longitudinal gauge perturbations \( \bar{A} \) and \( \bar{\mathcal{R}} \) defined in Eq. (4.3) and evaluated at the brane we obtain from Eqs. (5.18) and (5.20)

\[
\Phi = \bar{A}_b + n'_b \bar{\xi}, \\
\Psi = -\bar{\mathcal{R}}_b - a'_b \bar{\xi},
\] (5.21)

which will only coincide with the 5D-longitudinal gauge metric perturbations if the brane location in the 5D-longitudinal gauge remains unperturbed, i.e., \( \bar{\xi} = 0 \).

The remaining scalar bulk metric perturbation at the brane in a Gaussian normal gauge is described by

\[
\bar{\sigma}_{yb} = \sigma_{yb} - \frac{1}{a^2} \bar{\xi},
\] (5.22)

which is completely gauge-invariant, but is not part of the induced metric perturbations on the brane. Instead we shall show in Section VII A that it appears as a source term in the effective Einstein equations (1.11). This can immediately be identified with the brane location in the 5D-longitudinal gauge, \( \bar{\xi} = -a^2 \bar{\sigma}_{yb} \). Note that in terms of the Gaussian normal metric perturbations \( \bar{E} \) and \( \bar{B} \) in Eq. (5.2) we can write \( \bar{\sigma}_y = \bar{E}' \), while the alternative gauge-invariant combination, defined in Eq. (3.6), is given in Gaussian normal gauge by \( \bar{B} = \bar{B}' = \bar{\sigma}_y - \bar{\sigma} = \bar{B}' \).

The vector perturbations, \( \tau \) and \( \tau_y \), and the tensor perturbations, \( h_{ij} \), are invariant under the transformations given in Eqs. (5.1) and (5.17). \( \tau \) and \( h_{ij} \) are the induced 4D vector and tensor metric perturbations, but \( \tau_y \), like \( \bar{\sigma}_y \), acts like a source term in the effective Einstein equations (1.11).

In the following sections we shall show the central role played by brane-GN gauge perturbations in interpreting the effect of bulk perturbations on the brane. To minimise the notation we henceforth drop the ‘\( b \)’ subscripts for metric perturbations evaluated at the brane.
VI. MATTER PERTURBATIONS

The energy-momentum tensor for arbitrary matter perturbations on the brane is given by

\[
\delta T^0_0 = - \delta \rho, \tag{6.1}
\]

\[
\delta T^0_i = \delta p_i, \tag{6.2}
\]

\[
\delta T^i_j = \delta P \delta^i_j + \delta \pi^i_j, \tag{6.3}
\]

\[
\delta T = 3 \delta P - \delta \rho, \tag{6.4}
\]

where \(\delta p_i\) is the perturbed three-momentum, \(\delta \pi^i_j\) is the traceless anisotropic stress, and \(\delta T \equiv \delta T^\mu_\mu\) is the perturbed four-trace of the energy-momentum tensor. The perturbed three-momentum can be decomposed into scalar and vector components as

\[
\delta p_i \equiv \delta p_s + \delta p^{(\text{vector})} \hat{e}_i, \tag{6.5}
\]

and the anisotropic stress can be decomposed as

\[
\delta \pi^i_j \equiv \left(\nabla^i \nabla_j - \frac{1}{3} \delta^i_j \nabla^2\right) \delta \pi + \delta \pi^{(\text{vector})} \left(\hat{e}_i \hat{e}_j + \hat{e}_j \hat{e}_i\right) + \delta \pi^{(\text{tensor})}_j, \tag{6.6}
\]

where \(\delta \pi, \delta \pi^{(\text{vector})}\) and \(\delta \pi^{(\text{tensor})}_j\) are the scalar, vector and tensor parts of the anisotropic stress tensor, respectively.

The matter quantities transform under the temporal gauge transformation, \(\eta \rightarrow \eta + \delta \eta\) on the brane, as \[31\]

\[
\delta \rho \rightarrow \delta \rho - \dot{\rho} \delta \eta, \tag{6.7}
\]

\[
\delta P \rightarrow \delta P - \dot{P} \delta \eta,
\]

\[
\delta \rho \rightarrow \delta \rho + (\rho + P) \delta \eta,
\]

respectively, and the anisotropic stress tensor, \(\delta \pi^i_j\), and the vector momentum, \(\delta p^{(\text{vector})} \hat{e}_i\), are gauge-invariant.

Equation (1.6) yields the orbifold boundary condition for metric perturbations at the brane:

\[
\delta K^\mu_\nu = - \kappa^2 \delta T^\mu_\nu - \frac{1}{3} \delta T \delta^\mu_\nu, \tag{6.8}
\]

which we again decompose into scalar, vector and tensor parts.

1. Scalar perturbations

The contribution of the scalar metric perturbations in the Gaussian normal gauge to the extrinsic curvature of the brane is simply given in terms of the normal derivatives:

\[
\delta K^0_0 = \dot{A}', \tag{6.9}
\]

\[
\delta K^0_i = - \frac{a_b^2}{2 n_b^2} \dot{B}_i, \tag{6.10}
\]

\[
\delta K^i_j = \sigma_{y, i j} + \delta^i_j \ddot{R}'. \tag{6.11}
\]

Substituting the gauge transformations (3.3) and (3.3) of the metric perturbations, using Eqs. (5.1) and (5.17), into the above expressions (6.9-6.11) yields an expression for the extrinsic curvature perturbations in an arbitrary gauge

\[
\delta K^0_0 = A' + \frac{1}{n_b^2} \left[ \ddot{\xi} - \frac{\dot{A}_y}{A_b} \xi + n_b A_y \right] - \frac{\dot{A}_y}{A_b} A_{yy} + \left( \frac{n'_b}{n_b} \right)' \xi, \tag{6.12}
\]

\[
\delta K^0_i = - \frac{1}{2 n_b^2} \left( a_b^2 B - n_b A_y - 2 \left( \ddot{\xi} - \frac{\dot{A}_y}{A_b} \xi \right) \right)_{,i}, \tag{6.13}
\]

\[
\delta K^i_j = \left\{ \ddot{R}' + \frac{1}{n_b^2} \frac{\dot{A}_y}{a_b} \left( \ddot{\xi} + n A_y \right) - \frac{a_b^4}{a_b} A_{yy} + \left( \frac{n'_b}{a_b} \right)' \xi \right\} \delta^i_j + \left\{ \sigma_y - \frac{1}{a_b^2} \xi \right\} \delta_{i j}, \tag{6.14}
\]
which introduces additional terms involving time derivatives of the metric perturbations and $\xi$.

The perturbed junction conditions in Eq. (6.8) relate the extrinsic curvature perturbations (6.9–6.11) to the matter perturbations on the brane given in Eqs. (6.1–6.4). In the brane-GN gauge we find

$$\bar{A}' = \frac{1}{6} \kappa_5^2 (2 \delta \rho + 3 \delta P),$$

$$\bar{B} = \kappa_5^2 \frac{n_5^2}{a_5^2} \delta p,$$

$$\nabla^2 \bar{\sigma}_y + 3 \bar{R}' = \frac{1}{2} \kappa_5^2 \delta \rho,$$

$$\bar{\sigma}_y = -\frac{1}{2} \kappa_5^2 \delta \pi,$$

(6.15) (6.16) (6.17) (6.18)

giving a direct relation between the metric perturbations and the matter perturbations. But in any other gauge, these relations involve time-derivatives of the perturbations and so are non-local in time [26,11]. Only the anisotropic stress $\delta \pi$ [Eq. (6.18)] yields a local boundary condition for $\xi$ and $\sigma_y$ in an arbitrary gauge.

Two of the junction conditions can be interpreted as definitions of brane density and momentum perturbations in terms of bulk metric perturbations at the brane, whose evolution can be determined through the 5D Einstein equations. However any non-adiabatic or anisotropic pressure perturbation is not determined from the Einstein equations and hence represent additional constraints on the evolution of the bulk metric perturbations [26].

2. Vector perturbations

The non-zero contribution of the vector perturbations to the perturbed extrinsic curvature is given by

$$\delta K^0_i = \frac{a_5^2}{2n_5^2} S \hat{e}_i,$$

$$\delta K^i_j = \frac{1}{2} \tau_y \left( \hat{e}_i^{j'} + \hat{e}_j^i \right).$$

(6.19) (6.20)

Equating these with the vector perturbations in the matter on the brane, Eqs. (6.2–6.3), we find

$$S = -\kappa_5^2 \frac{n_5^2}{a_5^2} \delta \rho^{(\text{vector})},$$

$$\tau_y = -\kappa_5^2 \delta \pi^{(\text{vector})}.$$ 

(6.21) (6.22)

As the vector perturbations are gauge-invariant, the local form of the boundary conditions remains unchanged in arbitrary bulk or temporal gauges.

3. Tensor perturbations

The contribution of the tensor perturbations to the extrinsic curvature tensor is

$$\delta K^i_j = \frac{1}{2} h_{ij}^{i'} = -\frac{\kappa_5^2}{2} \delta \pi_j^{(\text{tensor})}.$$ 

(6.23)

If, for example, matter perturbations on the brane are described by linear perturbations of a scalar field then $\delta K^\mu_{\nu}$ for the vector and tensor perturbations on the brane must vanish.

VII. THE VIEW FROM THE BRANE

A. Einstein equations on the brane

In order to make predictions for the behaviour of perturbations seen by the brane-world observer it is necessary to relate the five-dimensional metric perturbations to perturbations in the effective four-dimensional Einstein equations
on the brane. From Eq. (1.11) we see that perturbations of the bulk gravitational field only enter via the projected Weyl tensor, $E^\mu_\nu$, which contributes like an effective energy-momentum tensor in the effective Einstein equations on the brane. We will define an effective Weyl-fluid energy density in the background by

$$ \kappa^2_4 \tilde{\rho} = E^0_0 = 3 \frac{a'_b}{a_b} + \frac{1}{2} \Lambda_5, $$

(7.1)

with isotropic pressure $E^j_j = -(1/3) E^0_0 \delta^j_j$.

The perturbed Weyl-fluid yields effective density, pressure and momentum perturbations, which by analogy with Eqs. (B1), (B3), and the Weyl-fluid perturbations, above, as

$$ \delta E^0_0 = \kappa^2_4 \tilde{\rho}, $$

$$ \delta E^i_0 = - \kappa^2_4 \tilde{p}_i, $$

$$ \delta E^i_j = - \kappa^2_4 \left( \tilde{\rho} \delta^i_j + \tilde{\pi}^i_j \right), $$

(7.2)

where the isotropic pressure perturbation is $\tilde{\rho} = \tilde{\rho}/3$ and we can decompose the Weyl momentum and the Weyl anisotropic stress into scalar, vector and tensor parts according to

$$ \tilde{\rho}_i \equiv \tilde{\rho}_i + \tilde{\pi} \ (\text{vector}), $$

$$ \tilde{\pi}_j \equiv \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \nabla^2 \right) \tilde{\pi} + \left( \tilde{\epsilon}_i \partial_j + \tilde{\epsilon}_j \partial_i \right) \tilde{\pi} \ (\text{vector}) + \tilde{\pi} \ (\text{tensor}). $$

(7.3)

(7.4)

1. Scalar perturbations

The scalar contribution to the projected Weyl fluid perturbations defined in Eqs. (7.2) can be written using the 5D Einstein equations, (B9)-(B17), to simplify the perturbed projected Weyl tensor given in equations, (B23)-(B27). In terms of normal derivatives of the Gaussian normal metric perturbations the Weyl fluid perturbations are [13].

$$ \kappa^2_4 \tilde{\rho} = 3 \kappa^2_4 \tilde{\rho} = - \left( \tilde{A}'' + 2 \frac{n'_i}{n_b} \tilde{A}' \right), $$

(7.5)

$$ \kappa^2_4 \tilde{\pi} = - \frac{a^2}{n_b} \left( \tilde{B} + \left( 3 \frac{a'_i}{a_b} - \frac{n'_i}{n_b} \right) \tilde{B} \right), $$

(7.6)

$$ \kappa^2_4 \tilde{\pi} = - \frac{a^2}{n_b} \left( \tilde{\sigma}' + 2 \frac{a'_i}{a_b} \tilde{\sigma} \right). $$

(7.7)

where we have employed a useful constraint equation that may be constructed from the spatial trace of the 5D Einstein tensor, Eq. (B14), using Eqs. (B9) and (B14),

$$ \tilde{A}'' + 2 \frac{n'_i}{n_b} \tilde{A}' + 3 \left( \tilde{R}'' + 2 \frac{a'_i}{a_b} \tilde{R}' \right) + \nabla^2 \tilde{\sigma}' + 2 \frac{a'_i}{a_b} \nabla^2 \tilde{\sigma} = 0. $$

(7.8)

The perturbed 5D Einstein equations, (5) $\delta G^\mu_\nu = 0$, in the brane Gaussian normal gauge [see Eqs. (B3), (B10), (B16) and (B17)] can thus be written in terms of the perturbed 4D Einstein tensor, given in Eqs. (B1)-(B3) and the Weyl-fluid perturbations, above, as

$$ (5) \delta G^0_0 = (4) \delta G^0_0 + \kappa^2_4 \tilde{\rho} + 2 \frac{a'_i}{a_b} \left( \nabla^2 \tilde{\sigma}' + 3 \tilde{R}' \right) = 0, $$

(7.9)

$$ (5) \delta G^i_0 = (4) \delta G^i_0 - \kappa^2_4 \tilde{\rho}_i + 2 \frac{a'_i}{a_b}, \frac{a'_j}{a_b} \tilde{B}' = 0, $$

(7.10)

$$ (5) \delta G_T = (4) \delta G_T - \kappa^2_4 \tilde{P} + \frac{2}{3} \left( \frac{n'_i}{n_b} + \frac{a'_i}{a_b} \right) \left( \nabla^2 \tilde{\sigma}' + 3 \tilde{R}' \right) + 2 \frac{a'_i}{a_b} \tilde{A}' = 0, $$

(7.11)

$$ (5) \delta G_{TF} = (4) \delta G_{TF} - \frac{a'_i}{a_b}, \frac{n'_i}{n_b} \tilde{\sigma}' = 0. $$

(7.12)
These equations, \((\ref{eq:5})\)-\((\ref{eq:12})\), can be finally rewritten, using the junction conditions \((\ref{eq:2.13})\) and \((\ref{eq:6.15})\)-\((\ref{eq:6.18})\), to give the perturbed field equations on the brane, as the linear perturbation of Eq. \((\ref{eq:1.11})\),

\[
(5) \delta G^\mu_\nu = (4) \delta G^\mu_\nu + \delta E^\mu_\nu - \kappa_4^2 \delta T^\mu_\nu - \kappa_4^3 \delta \Pi^\mu_\nu = 0 , \tag{7.13}
\]

where \(\delta T^\mu_\nu\) is the perturbation of the conventional matter energy-momentum tensor given in Eqs. \((\ref{eq:B28})\)-\((\ref{eq:B29})\) and the perturbed quadratic energy momentum tensor is given by

\[
\begin{align*}
\delta \Pi^0_0 &= -\frac{1}{6} \rho \delta \rho , \\
\delta \Pi^0_i &= \frac{1}{6} \rho \delta p, \\
\delta \Pi^i_j &= \delta^i_j \delta \Pi_T + \left( \nabla^i \nabla_j - \frac{1}{3} \nabla^2 \right) \delta \Pi_{TF} ,
\end{align*}
\]

where \(\delta \Pi_T\) and \(\delta \Pi_{TF}\) are the trace and the traceless part of the anisotropic stress tensor,

\[
\begin{align*}
\delta \Pi_T &= \frac{1}{6} \left[ (\rho + P) \delta \rho + \rho \delta P \right] , \\
\delta \Pi_{TF} &= -\frac{1}{12} (\rho + 3P) \delta \pi .
\end{align*}
\]

There are three additional field equations for scalar perturbations which come from the additional \((5) \delta G^0_i\), \((5) \delta G^i_j\) and \((5) \delta G^i_i\) Einstein equations in the 5D setting. In the next section we will show how the first two of these yield energy and momentum conservation equations for ordinary matter on the brane. The third has no counterpart in the 4D theory, and is most neatly expressed in the brane-GN gauge in the form given in Eq. \((\ref{eq:7.8})\).

2. Vector perturbations

Using the 5D vacuum Einstein equations, \((\ref{eq:B18})\)-\((\ref{eq:B21})\), we can write the vector contribution to the projected Weyl tensor given in equations \((\ref{eq:B28})\)-\((\ref{eq:B29})\) as

\[
\kappa_4^2 \delta \rho \text{(vector)} \hat{e}_i = -\delta E^0_i = \frac{a_i^b}{2n_b} \left\{ S' + \left( 3 \frac{a_i^b}{ab} - \frac{n_i^b}{mb} \right) S \right\} \hat{e}^i ,
\]

\[
\kappa_4^2 \delta \pi \text{(vector)} (\hat{e}^i,j + \hat{e}^j,i) = -\delta E^i_j = \frac{1}{2} \left\{ \tau_g + 2 \frac{a_i^b}{ab} \tau_g \right\} (\hat{e}^i,j + \hat{e}^j,i) .
\]

The 5D Einstein tensor for the vector perturbations on the brane can be rewritten to give

\[
\begin{align*}
(5) \delta G^0_i &= (4) \delta G^0_i + \delta E^0_i - \frac{a_i^b a_i^b}{2n_b} S \hat{e}_i , \\
(5) \delta G^i_j &= (4) \delta G^i_j + \delta E^i_j - \frac{1}{2} \left( \frac{n_i^b}{mb} + \frac{a_i^b}{ab} \right) \tau_g (\hat{e}^i,j + \hat{e}^j,i) .
\end{align*}
\]

Using the junction conditions Eqs. \((\ref{eq:2.13})\), \((\ref{eq:2.21})\) and \((\ref{eq:6.22})\) the 5D Einstein equations can thus be rewritten as

\[
\begin{align*}
(5) \delta G^0_i &= (4) \delta G^0_i + \delta E^0_i - \kappa_4^2 \delta \rho \text{(vector)} \hat{e}_i - \frac{1}{6} \kappa_4^2 \rho \delta \rho \text{(vector)} \hat{e}_i , \\
(5) \delta G^i_j &= (4) \delta G^i_j + \delta E^i_j - \kappa_4^2 \delta \pi \text{(vector)} (\hat{e}^i,j + \hat{e}^j,i) + \frac{1}{12} \kappa_4^3 (\rho + 3P) \delta \pi \text{(vector)} (\hat{e}^i,j + \hat{e}^j,i) ,
\end{align*}
\]

where the final terms in the expression for \((5) \delta G^\mu_\nu\) can be identified with the vector perturbations in the quadratic energy-momentum tensor, \(\delta \Pi^\mu_\nu\), in Eq. \((\ref{eq:7.13})\).
3. Tensor perturbations

Using the 5D Einstein equation, (B22), we can write the tensor contribution to the perturbed Weyl-tensor given in equation, (B30), as

\[ \tilde{\delta \pi}_{ij}^{(\text{tensor})} = -\delta E_{ij} = \frac{1}{2} h''_{ij} + \frac{a^i}{a_b} h''_{ij}. \]  

(7.25)

The 5D Einstein tensor for the tensor perturbations on the brane can be rewritten as

\[ (5) \delta G_{ij} = (4) \delta G_{ij} + \delta E_{ij} - \frac{1}{2} \kappa_5^2 (\rho + 3P) \delta \pi_{ij}^{(\text{tensor})} = 0. \]  

(7.26)

which can be re-expressed as the Einstein equation

\[ (5) \delta G_{ij} = (4) \delta G_{ij} + \delta E_{ij} - \frac{1}{12} \kappa_5^4 (\rho + 3P) \delta \pi_{ij}^{(\text{tensor})} = 0. \]  

(7.27)

where again the final term in the expression for (5) \( \delta G_{ij} \) can be identified with the tensor perturbations in the quadratic energy-momentum tensor, \( \delta \Pi_{\mu \nu} \), in Eq. (7.13).

B. Energy-momentum conservation on the brane

The perturbed 5D Einstein equations \( (5) \delta G^4_{ab} = 0 \) and \( (5) \delta G^4_i = 0 \) in the brane-GN gauge [see Eqs. (B11) and (B13)] can be rewritten in terms of the matter perturbations using the junction conditions Eqs. (6.15)-(6.18) at the brane. We then get the standard 4D local energy and momentum conservation equations for scalar perturbations [21]

\[ \dot{\delta \rho} + 3 \frac{\dot{a}_b}{a_b} (\delta \rho + \delta P) + (\rho + P) \nabla^2 \dot{\bar{\sigma}} + \frac{\dot{a}_b}{a_b} \nabla^2 \delta \rho + 3(\rho + P) \dot{\bar{\mathcal{R}}} = \frac{12}{\kappa_5^3} \left( \frac{\dot{a}_b}{a_b} - \frac{\dot{a}_b}{a_b} \right) \bar{\mathcal{A}}, \]  

(7.28)

\[ \dot{\delta p} + \left( \frac{\dot{a}_b}{a_b} + 3 \frac{\dot{a}_b}{a_b} \right) \delta \rho + \delta P + (\rho + P) \bar{A} + \frac{2}{3} \nabla^2 \delta \pi = 0, \]  

(7.29)

where the term on the right-hand-side of Eq. (7.28) vanishes using the background energy-conservation Eq. (A3).

For the vector perturbations we get a momentum conservation equation from the vector part of \( (5) \delta G^4_i = 0 \),

\[ \dot{\delta p}^{(\text{vector})} + \left( \frac{\dot{a}_b}{a_b} + 3 \frac{\dot{a}_b}{a_b} \right) \delta p^{(\text{vector})} + \nabla^2 \delta \pi^{(\text{vector})} = 0, \]  

(7.30)

where we have used the junction conditions for vector perturbations Eq. (1.21) and (1.22). We see that the 5D Einstein equations ensure that ordinary energy-momentum conservation always holds on the brane in a vacuum bulk, consistent with the Codazzi equation (1.7).

From Eq. (1.11), using the contracted Bianchi identities \( (\nabla^\mu G^\mu_{\nu} = 0) \) and energy momentum conservation \( (\nabla_\mu T^\mu_{\nu} = 0) \) on the brane, we find

\[ \nabla_\mu E^\mu_{\nu} = \kappa_5^3 \nabla_\mu \Pi^\mu_{\nu}. \]  

(7.31)

This may be interpreted as the production of bulk gravitons due to high-energy effects on the brane [32]. Energy-momentum is transfered from the quadratic energy-momentum tensor, \( \Pi^\mu_{\nu} \), to the Weyl-fluid, \( E^\mu_{\nu} \), but ordinary energy-momentum, i.e., the tensor \( T^\mu_{\nu} \), is always conserved, even when bulk gravitons are excited.

Using the projected field equations we find conservation equations governing the evolution of the Weyl fluid. In the background we have from Eq. (7.31)

\[ \dot{\tilde{\rho}} + \frac{\dot{a}_b}{a_b} \tilde{\rho} = 0, \]  

(7.32)

where we used Eq. (2.10). Thus the ‘dark radiation’ term in a homogeneous FRW cosmology is conserved [2] and its energy-density redshifts away like ordinary radiation in an expanding universe [23,34].

For scalar perturbations we get an energy and an energy-momentum conservation equation from Eq. (7.31) [21]
\[
\dot{\delta \rho} + \frac{\dot{a}_b}{a_b} \delta \rho + \frac{n_b^2}{a_b^2} \nabla^2 \delta \rho + 4 \dot{\rho} \delta R + \frac{4}{3} \delta \nabla^2 \sigma = 0 ,
\]
\[
\dot{\tilde{\delta} p} + \left( \frac{\dot{a}_b}{n_b} + \frac{\dot{a}_b}{a_b} \right) \tilde{\delta} p + \frac{1}{3} \tilde{\delta} \rho + \frac{4}{3} \tilde{\delta} A + \frac{2}{3} \nabla^2 \tilde{\delta} \pi = \frac{\rho + P}{\lambda} \left\{ \delta \rho - 3 \frac{\dot{a}_b}{a_b} \delta \rho - \nabla^2 \delta \pi \right\} ,
\]
where we have used Eqs. (7.28) and (7.23). For the vector quantities we get a momentum conservation equation [18]
\[
\dot{\delta p}^{(\text{vector})} + \left( \frac{\dot{b}_b}{n_b} + \frac{\dot{a}_b}{a_b} \right) \delta p^{(\text{vector})} + \nabla^2 \delta \pi^{(\text{vector})} = \frac{6(\rho + P)}{\lambda} \left( \frac{\dot{a}_b}{a_b} \delta p^{(\text{vector})} + \frac{1}{2} \nabla^2 \delta \pi^{(\text{vector})} \right) .
\]

Thus the Weyl-fluid energy density is always conserved to linear order (cf. Eq. (7.28)), whereas the Weyl-fluid momentum is coupled to the comoving energy-density and anisotropic pressure of ordinary matter. At low energies \((\rho + P \ll \lambda)\) the momentum transfer too becomes negligible, but the presence of the anisotropic stress \(\tilde{\delta} \pi\), which appears as a free function in the 4D equations, means that the Weyl-fluid cannot be completely described as a conventional 4D fluid even at low energies.

### C. Curvature perturbations on the brane

Local energy conservation can be used to demonstrate that the intrinsic curvature perturbation on uniform density hypersurfaces is conserved for adiabatic perturbations on large scales independently of the gravitational field equations [35]. We define the gauge-invariant quantity on the brane [36, 21, 37]
\[
\zeta \equiv \dot{R} + \frac{\delta \rho}{3(\rho + P)} .
\]
From the perturbed energy conservation equation (7.28) we get an evolution equation for \(\zeta\) [32],
\[
\dot{\zeta} = -\frac{\dot{a}_b}{a_b} \left( \frac{\delta P - c_s^2 \delta \rho}{\rho + P} \right) - \frac{1}{3} \nabla^2 \left[ \tilde{\sigma} + \frac{n_b^2}{a_b^2} \frac{\delta \rho}{\rho + P} \right] ,
\]
where \(c_s^2 = \dot{P}/\dot{\rho}\) is the adiabatic sound speed in the background solution. Hence, the curvature perturbation on uniform density hypersurfaces remains constant on large scales, where the divergence of the comoving shear, \(\nabla^2 [\tilde{\sigma} + \delta \rho/(\rho + P)]\), can be neglected, if the non-adiabatic pressure perturbation vanishes, i.e.,
\[
\delta P_{\text{nad}} \equiv \delta P - c_s^2 \delta \rho = 0 .
\]

Using the junction conditions given above, Eqs. (2.13) and (6.17), the curvature perturbation can be re-expressed solely in terms of metric perturbations in the brane Gaussian normal gauge
\[
\zeta = \dot{R} + \left( \frac{a_b}{a_b} - \frac{n_b}{n_b} \right)^{-1} \left( \dot{\bar{R}} + \frac{1}{3} \nabla^2 \bar{\sigma}_y \right) .
\]
The conservation of \(\zeta\) for adiabatic perturbations on large scales then follows from the \((5)\delta G_{\bar{R}} = 0\) Einstein equation [31]. The adiabatic condition (7.37) for matter perturbations then corresponds to a condition on the bulk metric perturbations at the brane
\[
\frac{\dot{a}_b}{a_b} \left( \frac{a_b}{a_b} - \frac{n_b}{n_b} \right) \dot{\bar{A}} = \left( \frac{\dot{a}_b}{n_b} \right) \left( \dot{R} + \frac{1}{3} \nabla^2 \bar{\sigma}_y \right) .
\]
Arbitrary bulk metric perturbations would not respect this condition, but physically it will be enforced by the junction conditions (1.4) at the brane if the matter perturbations on the brane are adiabatic, e.g., for a perfect fluid.

By transforming back from the brane Gaussian normal metric perturbations in Eq. (7.38) using Eqs. (3.3) and (3.5), we can write \(\zeta\) in terms of the bulk metric perturbations written in an arbitrary gauge. We have
\[
\zeta = \bar{R} + \frac{\dot{a}_b}{a_b} \xi + \left( \frac{a_b}{a_b} - \frac{n_b}{n_b} \right)^{-1} \left[ \bar{R}' - \frac{a_b}{a_b} A_{yy} + \frac{1}{3} \frac{\dot{a}_b}{n_b} \left( \xi + n A_y \right) + \left( \frac{a_b}{a_b} \right) \xi + \frac{1}{3} \nabla^2 \left( \bar{\sigma}_y - \frac{1}{a_b} \xi \right) \right] .
\]
Even if the total matter perturbation is not adiabatic, or indeed if total energy is not conserved on the brane \[16\], it is still possible to define a curvature perturbation on hypersurfaces where a given component has uniform-density, and this curvature perturbation remains constant on large scales if this component is adiabatic and its energy is conserved \[35,21\]. For instance, in a Schwarzschild-Anti-de Sitter bulk (\(\bar{\rho} \neq 0\)) the curvature perturbation on hypersurfaces of uniform Weyl fluid effective density on the brane is \[21\]:

\[
\tilde{\zeta} \equiv \bar{R} + \frac{\delta \rho}{4\bar{\rho}},
\]  

(7.41)

which we can rewrite in terms of GN metric quantities, using the definition of the Weyl-fluid density \[7.1\] and its perturbation \[7.3\], as

\[
\tilde{\zeta} = \bar{R} - \frac{1}{2n_b^2} \left( 6\frac{a''_b}{a_b} + \Lambda_b \right)^{-1} \left( n_b^2 \bar{A}' \right)' .
\]  

(7.42)

Using the conservation equation for the perturbed Weyl fluid \[7.33\] we get an evolution equation for \(\tilde{\zeta}\) \[21\],

\[
\dot{\tilde{\zeta}} = -\frac{1}{3} \nabla^2 \left[ \bar{\sigma} + n_b^2 \frac{3\delta \rho}{4\bar{\rho}} \right].
\]  

(7.43)

Hence, on large scales when the divergence of the shear can be neglected, \(\tilde{\zeta}\) is constant, regardless of whether the matter perturbations are adiabatic.

In a conformally flat (Anti-de Sitter) bulk where \(\bar{\rho} = 0\) the Weyl-density perturbation, \(\delta \rho\), is automatically gauge-invariant and Eq. \[7.33\] reduces to

\[
\dot{\delta \rho} + 4 \frac{a''_b}{a_b} \delta \rho = -\frac{n_b^2}{a_b} \nabla^2 \delta \rho ,
\]  

(7.44)

so that, when the divergence of the Weyl-momentum, \(\nabla^2 \delta \rho\), can be neglected on large-scales we have

\[
\tilde{S} \equiv a^4_b \delta \rho = \text{constant}.
\]  

(7.45)

In a Schwarzschild-Anti-de Sitter bulk (\(\bar{\rho} \neq 0\)) it is the difference between the matter and Weyl-fluid curvature perturbations that represents the relative Weyl entropy perturbation \[21\]

\[
\tilde{S} \equiv \tilde{\zeta} - \zeta .
\]  

(7.46)

Any Weyl-fluid density perturbation in a Anti-de-Sitter background with \(\bar{\rho} = 0\) represents an effective entropy perturbation. In conventional 4D Einstein gravity, it is sufficient to know the curvature perturbation for ordinary matter, \(\zeta\), to describe the total gauge-invariant curvature perturbation, but in the brane-world context it is not only ordinary matter (including its quadratic corrections at high energy) but also the Weyl-fluid perturbation, \(\tilde{S}\), that shapes the 4D geometry.

For adiabatic matter perturbations, \(\delta P_{\text{nad}} = 0\), we expect both \(\zeta\) and \(\tilde{S}\) to remain constant on large scales, but a relative entropy perturbation can cause a change in the total curvature perturbation on the brane

\[
\zeta_{\text{tot}} = \zeta + \bar{w} \tilde{S} ,
\]  

(7.47)

where the weight given to the Weyl entropy perturbation is time-dependent and given by

\[
\bar{w} = \left\{ \begin{array}{ll}
\left[ 3n_b^4 (1 + \rho/\lambda)(\rho + P) \right]^{-1} & \text{for } \bar{\rho} = 0 \\
\frac{3}{4\bar{\rho}} \left[ 3n_b^4 (1 + \rho/\lambda)(\rho + P) + 4\bar{\rho} \right]^{-1} & \text{for } \bar{\rho} \neq 0
\end{array} \right. .
\]  

(7.48)

It is this total curvature perturbation \(\zeta_{\text{tot}}\) which is related to other definitions of the intrinsic metric perturbation on the brane such as the conformal Newtonian potential, \(\Phi\) defined in Eq. (5.20), and hence to observational data \[21\].
VIII. WEYL TENSOR IN THE BULK

In the preceding sections we have seen that the bulk metric perturbations are felt on the brane only through the projected Weyl tensor, $E_{\mu\nu}$, which can be thought of as an effective energy-momentum tensor appearing on the right-hand-side of the effective Einstein equations on the brane, Eq. (7.1). Although the brane-observer is only influenced by the Weyl tensor at the brane, it can be defined throughout the bulk. Any background metric of the form introduced in Eq. (2.1) necessarily defines a projected Weyl tensor, $E_{\mu\nu}$, on any constant-$y$ hypersurface in the 5D bulk, and these 4D spacetimes are then sliced into maximally symmetric 3-space with homogeneous Weyl-fluid density, $\tilde{\rho}$. Although this was defined in Eq. (7.1) in terms of the Weyl-fluid density on the brane, this definition extends into the bulk and with the brane-world density simply corresponding to $\tilde{\rho}(\eta, y)$. The 5D Einstein equations yield the evolution equations

$$\dot{\tilde{\rho}} + 4\frac{\dot{a}}{a} \tilde{\rho} = 0,$$
$$\tilde{\rho}' + 4\frac{a'}{a} \tilde{\rho} = 0,$$  \hspace{1cm} (8.1)

which can be integrated to give $a^4\tilde{\rho} =$constant throughout the bulk.

Similarly the perturbed tensor $\delta E_{\mu\nu}$ is defined throughout the bulk, and can be decomposed, as in Eqs. (7.5–7.7), into the perturbed Weyl-fluid density, momentum, and anisotropic pressure

$$\kappa^2_4 \delta \rho = \left\{ A'' + \frac{n'}{n} \left( 2A' - A_{yy} \right) - 2\frac{\dot{n}'}{n} A_{yy} \right. - \frac{1}{n^2} \left( A_{yy} - \frac{n'}{n} A_{y} \right) + \frac{1}{n} \left( \dot{A}_{y} + \frac{n'}{n} A_{y} + \left( \frac{\dot{n}'}{n} - \frac{\ddot{n}}{n^2} \right) A_{y} \right) \right\},$$
$$\kappa^2_4 \delta p = -\frac{1}{n^3} \left\{ B' + \left( \frac{3\dot{a}}{a} - \frac{n'}{n} \right) B + \frac{2}{a^2} \left( \dot{A}_{yy} - \frac{\dot{a}}{a} A_{yy} \right) + \frac{n}{a^2} \left[ \left( \frac{\dot{a}}{a} - 2\frac{n'}{n} \right) A_{y} - A_{y}' \right] \right\},$$
$$\kappa^2_4 \delta \pi = \frac{1}{a^2} \left( a^2 \sigma \right)' + \frac{1}{a^2} A_{yy},$$  \hspace{1cm} (8.2)

where the isotropic pressure perturbation is $\delta P = \delta \rho/3$. In any Gaussian normal gauge these reduce to the simple definitions given in Eqs. (7.2) at the brane.

The energy and momentum conservation equations (7.33) in an arbitrary gauge in the bulk can be written as

$$\dot{\delta \rho} + 4\frac{\dot{a}}{a} \delta \rho + \frac{n'^2}{a^2} \Delta^2 \delta \rho + 4\tilde{\rho} \dot{\cal R} + \frac{4}{3} \tilde{\rho} \Delta^2 \sigma = 0,$$
$$\dot{\delta p} + \left( \frac{n'}{n} + 3\frac{\dot{a}}{a} \right) \delta p + \frac{1}{3} \tilde{\rho}' + \frac{4}{3} \delta \rho + \frac{2}{3} \dot{\cal R} = \frac{1}{3} \left( \frac{\dot{a}'}{a} - \frac{n'}{n} \right) \left( \frac{\dot{\rho} - 3\tilde{\rho} + \frac{2}{\kappa_4^2 a^2} \Delta \sigma}{\kappa_4^2 a^2} \right),$$  \hspace{1cm} (8.3)

where $\cal R$ is the curvature perturbation in the 5D longitudinal gauge, defined in Eq. (5.3). For a separable background bulk the energy and momentum conservation equations (8.3) reduce to the standard 4D energy and momentum conservation equations (7.28) and (7.29) with the 5D effect being due to the anisotropic stress $\delta \pi$.

In an Anti-de Sitter bulk ($\tilde{\rho} = 0$) the perturbed Weyl-fluid density, momentum, and anisotropic pressure are all naturally gauge-invariant. However in a Schwarzschild-Anti-de Sitter bulk ($\tilde{\rho} \neq 0$) the perturbed density and momentum transform under general gauge transformations (8.4) as

$$\delta \rho \rightarrow \delta \rho + 4\tilde{\rho} \left( \frac{\dot{a}}{a} \delta \eta + \frac{a'}{a} \delta y \right),$$
$$\tilde{\rho} \rightarrow \tilde{\rho} + \frac{4}{3} \delta \rho \eta.$$  \hspace{1cm} (8.4)

The anisotropic stress, $\delta \pi$, remains gauge-invariant.

Because the density perturbation has the same gauge-transformation properties as the intrinsic curvature perturbation, $\cal R$ in Eq. (3.3), it is natural to construct the gauge-invariant Weyl-fluid density perturbation as

$$\tilde{\rho} + 4\tilde{\rho} \tilde{\rho} = 4\tilde{\rho} \tilde{\rho} \tilde{\rho},$$  \hspace{1cm} (8.5)
where $\tilde{\zeta}$ is the gauge-invariant curvature perturbation on uniform Weyl-fluid density hypersurfaces defined in Eq. (7.42). We find that $\tilde{\zeta}$ is a 5D gauge-invariant quantity constructed in terms of the metric perturbations on any 3D spatial hypersurface at fixed $y$ and $\eta$. In particular, it is independent of the brane location, $\xi$, unlike $\zeta$ in Eq. (7.40) which is defined with respect to ordinary matter on the brane.

Although our original motivation was to construct a conserved quantity on the brane, we see that $\tilde{\zeta}$ can be evaluated on any fixed $y$ and $\eta$ hypersurface, i.e., in any bulk gauge, and off the brane. Just as one can solve for the behaviour of $\tilde{\rho}(\eta, y)$ in the whole bulk due to the assumption of 3D-spatial homogeneity, so we see from Eq. (7.43) that $\tilde{\zeta}$ should be constant throughout the bulk when the spatial gradients which appear on the right-hand-side of Eq. (7.43) can be neglected. In this case it may be possible to extend the picture developed in conventional 4D cosmology to follow the evolution of large-scale perturbations as essentially separate FRW universe, to model the evolution in the bulk as being due to separate FRW slices of a 5D bulk. The virtue of having a gauge-invariant definition of the Weyl-density perturbation on spatially flat hypersurfaces, is that it can be calculated from bulk metric perturbations in other gauges, such as the 5D-longitudinal gauge master variable, $\Omega$ in Eq. (5.14), or the transverse-tracefree GN gauge $\tilde{A}$ in Eq. (5.12).

The Weyl-fluid momentum is independent of the bulk gauge, $\delta y$, but transforms in an Anti-de Sitter bulk under shifts in the temporal gauge, like the 3D shear perturbation, $\sigma$ in Eq. (3.3). Thus a simple gauge-invariant expression for the Weyl-fluid momentum perturbation in the bulk is $\delta \rho + 4a^2 \rho \sigma/(3n^2)$, which is the Weyl fluid momentum perturbation in the 5D longitudinal gauge. In conventional 4D Einstein gravity it is common to choose a temporal gauge with vanishing 3-momentum, $\delta \rho$, on which one can can construct gauge-invariant definitions of the comoving density or curvature perturbation [31]. But in the 5D bulk the density or curvature perturbations comoving with the Weyl-fluid momentum retain a residual gauge-dependence due to the different possible bulk slicings. In this sense the curvature perturbation on uniform Weyl-fluid density hypersurfaces (or, equivalently, the Weyl fluid density perturbation on uniform curvature hypersurfaces) is more fundamental in the 5D case.

IX. SUMMARY AND CONCLUSION

In this paper we have presented the governing equations for cosmological perturbations starting in Gaussian normal background coordinates, in which the brane is at fixed coordinate $y = 0$, but allowing arbitrary linear perturbations both of the bulk metric and of the matter perturbations on the brane. In order to make clear the comparison with the conventional coordinate-based approach in four-dimensional gravity [25], we have decomposed the perturbations into scalar, vector and tensor perturbations on the flat three-dimensional spatial hypersurfaces. Scalars, vectors and tensors then obey decoupled evolution and constraint equations that we have given in (3D) spatially gauge-invariant variables, but for arbitrary temporal and bulk gauges. We have then defined gauge-invariant perturbations that correspond to different possible gauge choices which have been employed to study bulk and brane perturbations.

One approach is to use a transverse and tracefree gauge in a Gaussian normal coordinate system [6,10,14,20]. This is particularly well-suited to the study of the free gravitational field on a de Sitter slicing where the bulk wave equation is separable and solutions can be written down in terms of harmonic functions on the maximally-symmetric 4D spacetime. However we have shown that for any other background cosmology the transverse-tracefree conditions in a GN gauge over-constrain the perturbations leaving no non-vanishing scalar perturbations.

An alternative approach to study the free gravitational field in the bulk has been developed by Mukohyama [11] and others [20] working in a 5D generalisation of the longitudinal gauge [16,27]. Gauge-invariant scalar, vector and tensor perturbations can be written in terms of three master variables which obey simple 5D wave equations in the bulk. Separability of the 5D equations has restricted analytic solutions to the case of a de Sitter brane-cosmology [13,18]. In the 5D longitudinal gauge, once the gauge-freedom in the bulk is eliminated, the perturbed brane location, $\xi(x^\mu)$, becomes a gauge-invariant 4D perturbation and this is not determined by the bulk field equations, but rather by the anisotropic stress of the matter on the brane.

Matter perturbations on the brane are coupled to the bulk metric perturbations through the Israel junction conditions at the brane. These junction conditions only have a simple form in a Gaussian normal coordinate system where the brane location is held fixed at $y = 0$, which we refer to as a brane GN gauge. Metric perturbations in the brane-GN gauge coincide with the induced metric perturbations seen by the brane bound observer. We have shown how the brane-world observer sees the effect of the bulk metric perturbations only through the contribution of the projected Weyl tensor to the induced Einstein equations. This ‘Weyl-fluid’ has an effective energy density, momentum and pressure given in terms of second derivatives with respect to the normal coordinate of the GN metric perturbations [13]. It is the Weyl-fluid perturbations that offer distinctive signatures of the brane-world scenario.

The four-dimensional brane observer’s perspective enables us to establish two important results for the evolution of perturbations on large scales. Firstly energy-conservation on the brane ensures that the curvature perturbation on
uniform-density hypersurfaces remains constant for adiabatic matter perturbations when gradient terms are negligible. Secondly, an analogous curvature perturbation on hypersurfaces of uniform Weyl-fluid density on the brane also remains constant on large scales. We have given gauge-invariant definitions of these quantities in terms of bulk metric perturbations. While the matter perturbations are necessarily defined only on specific bulk slicings (i.e., the brane or some arbitrary extension into the bulk) we can define Weyl-fluid perturbations that are gauge-invariant throughout the bulk allowing the Weyl tensor evolution to be described in terms of a local density, momentum and pressures.

Our partial understanding of brane-world metric perturbations thus far allows us to predict the initial amplitude of perturbations in idealised (de Sitter) models of inflation. For perturbations that remain constant on super-horizon scales we can estimate their initial amplitude before horizon re-entry. This amounts to calculating brane-world corrections to conventional 4D results. But we have yet to calculate the amplitude of any intrinsically 5D effects, such as the interplay between bulk and brane metric perturbations at horizon entry in the radiation dominated era. This is where novel effects might appear, even at low energies, which could be testable in astronomical surveys of cosmological backgrounds. The outstanding challenge is to make detailed predictions, probably requiring numerical solutions, of the evolution of the coupled matter and metric perturbations on cosmological time- and length-scales.

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[1] P. Binetruy, C. Deffayet and D. Langlois, Nucl. Phys. B 565 (2000) 269 [hep-th/9905012].
[2] T. Shiromizu, K. Maeda and M. Sasaki, Phys. Rev. D 62 (2000) 024012 [gr-qc/9910076].
[3] K. Maeda and D. Wands, Phys. Rev. D 62 (2000) 124009 [hep-th/0008188].
[4] A. Mennim and R. A. Battye, Class. Quant. Grav. 18 (2001) 2171 [hep-th/0008192].
[5] R. A. Battye, B. Carter, A. Mennim and J. Uzan, [hep-th/0105091].
[6] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690 [hep-th/9906064].
[7] P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B 477 (2000) 285 [hep-th/9910219].
[8] S. Mukohyama, T. Shiromizu and K. Maeda, Phys. Rev. D 62, 024028 (2000) [Erratum-ibid. D 63, 029901 (2000)] [hep-th/9912287].
[9] P. Bowcock, C. Charmousis and R. Gregory, Class. Quant. Grav. 17, 4745 (2000) [hep-th/0007177].
[10] J. Carriga and M. Sasaki, Phys. Rev. D 62, 043523 (2000) [hep-th/9912113].
[11] S. Mukohyama, [hep-th/0104185]; Class. Quant. Grav. 17 (2000) 4777 [hep-th/0006146]; Phys. Rev. D 62 (2000) 084015 [hep-th/0004067].
[12] R. Maartens, Phys. Rev. D 62, 084023 (2000) [hep-th/0004166]; C. Gordon and R. Maartens, Phys. Rev. D 63, 044022 (2001) [hep-th/0009010].
[13] D. Langlois, Phys. Rev. D 62, 126012 (2000) [hep-th/0005029]; Phys. Rev. Lett. 86, 2212 (2001) [hep-th/0010063].
[14] S. W. Hawking, T. Hertog and H. S. Reall, Phys. Rev. D 62, 043501 (2000) [hep-th/0003052].
[15] D. Langlois, R. Maartens and D. Wands, Phys. Lett. B 489, 259 (2000) [hep-th/0006007].
[16] C. van de Bruck, M. Dorca, R. H. Brandenberger and A. Lukas, Phys. Rev. D 62 (2000) 123515 [hep-th/0005032].
C. van de Bruck, M. Dorca, C. J. Martins and M. Parry, Phys. Lett. B 495 (2000) 183 [hep-th/0009050].
C. van de Bruck and M. Dorca, [hep-th/0012073]; M. Dorca and C. van de Bruck, [hep-th/0012114].
T. Boehm, R. Durrer and C. van de Bruck, [hep-th/0102144].
[17] K. Koyama and J. Soda, Phys. Rev. D 62, 123502 (2000) [hep-th/0005231].
[18] H. A. Bridgman, K. A. Malik and D. Wands, Phys. Rev. D 63 (2001) 084012 [hep-th/0010133].
[19] N. Deruelle, T. Dolezal and J. Katz, Phys. Rev. D 63, 083513 (2001) [hep-th/0010213]; N. Deruelle and J. Katz, [gr-qc/0104097].
[20] N. Deruelle and T. Dolezal, [gr-qc/010511].
[21] U. Gen and M. Sasaki, [gr-qc/0101178]; N. Sago, Y. Himemoto and M. Sasaki, [gr-qc/010403].
[22] D. Langlois, R. Maartens, M. Sasaki and D. Wands, Phys. Rev. D 63 (2001) 084009 [hep-th/0012041].
[23] N. Kaloper, Phys. Rev. D 60 (1999) 123506 [hep-th/9905210].
[24] L. A. Gergely and R. Maartens, [gr-qc/0105058].
APPENDIX A: THE BACKGROUND EINSTEIN AND WEYL TENSORS

The 5D Einstein tensor in the background given by the line element Eq. (2.1) is

\[
(5) G^0_0 = 3 \left\{ \frac{a''}{a} + \left( \frac{a'}{a} \right)^2 - \left( \frac{\dot{a}}{an} \right)^2 \right\},
\]

\[
(5) G^0_i = 0,
\]

\[
(5) G^4_0 = 3 \left( \frac{\dot{a}n'}{an} - \frac{\dot{a'}}{a} \right),
\]

\[
(5) G^4_i = 3 \left\{ \frac{a'n'}{an} + \left( \frac{a'}{a} \right)^2 - \left( \frac{\dot{a}}{an} \right)^2 - \frac{\ddot{a}}{an^2} + \frac{\dot{a}n}{an^3} \right\},
\]

\[
(5) G^i_j = \delta^i_j \left\{ \frac{n''}{n} + \frac{2a''}{a} + 2 \frac{a'n'}{an} + \left( \frac{a'}{a} \right)^2 - \frac{1}{n^2} \left[ 2 \frac{\ddot{a}}{a} - 2 \frac{\dot{a}n}{an} + \left( \frac{\dot{a}}{a} \right)^2 \right] \right\},
\]

and the background Weyl tensor is given by

\[
E^0_0 = \frac{1}{2} \left\{ \frac{a''}{a} - \frac{n''}{n} - \left( \frac{a'}{a} \right)^2 + \frac{a'n'}{an} + \frac{1}{n^2} \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{\dot{a}n}{an} - \frac{\ddot{a}}{a} \right] \right\},
\]

\[
E^i_j = -\frac{1}{3} E^0_0 \delta^i_j.
\]

The 4D Einstein tensor in the background is

\[
(4) G^0_0 = -3 \left( \frac{\dot{a}}{an} \right)^2,
\]

\[
(4) G^i_j = -\frac{1}{n^2} \left[ 2 \frac{\ddot{a}}{a} - 2 \frac{\dot{a}n}{an} + \left( \frac{\dot{a}}{a} \right)^2 \right] \delta^i_j.
\]

APPENDIX B: THE PERTURBED EINSTEIN TENSOR

1. Perturbed 4D-Einstein tensor

We reproduce here the form of the perturbed 4-D Einstein tensor on the brane.
a. Scalar perturbations

\begin{equation}
(4) \delta G_0^i = \frac{2}{n_b^2} \left[ 3 \left( \frac{\dot{a}_b}{a_b} \right) \left( -\ddot{R} + \left( \frac{\dot{a}_b}{a_b} \right) A \right) - \left( \frac{\dot{a}_b}{a_b} \right) \nabla^2 \sigma + \left( \frac{n_b}{a_b} \right)^2 \nabla^2 R \right], \quad (B1)
\end{equation}

\begin{equation}
(4) \delta G_i^0 = -\frac{2}{n_b^2} \left[ \left( \frac{\dot{a}_b}{a_b} \right) A - \ddot{R} \right], \quad (B2)
\end{equation}

\begin{equation}
(4) \delta G_{ij} = \frac{2}{n_b^2} \left[ \left\{ \frac{\dot{a}_b}{a_b} \right\}^2 - 2 \frac{\dot{a}_b \ddot{a}_b}{a_b n_b} \right] A + \left( \frac{\dot{a}_b}{a_b} \right) \dot{A} - \ddot{R} - \left( 3 \frac{\dot{a}_b}{a_b} - \frac{n_b}{a_b} \right) \dddot{R} \delta_{ij} + \frac{1}{n_b^2} \nabla^2 \left( \frac{n_b}{a_b} \right)^2 \left( \dddot{R} + A \right) + \left( \frac{n_b}{a_b} - 3 \frac{\dot{a}_b}{a_b} \right) \sigma - \dot{\sigma} \right] \delta_{ij} - \frac{1}{3} \left( \frac{n_b}{a_b} \right)^2 \left( \dddot{R} + A \right) + \frac{1}{n_b^2} \left( \frac{n_b}{a_b} - 3 \frac{\dot{a}_b}{a_b} \right) \sigma - \dot{\sigma} \right] \delta_{ij}, \quad (B3)
\end{equation}

It is useful to split the spatial part of the perturbed 4D-Einstein tensor \((4) \delta G_{ij}\) into a trace and a traceless part,

\begin{equation}
(4) \delta G^i_j \equiv \delta_{ij} \delta G_T + \left( \nabla^i \nabla_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \delta G_{TF}, \quad (B4)
\end{equation}

where \((4) \delta G_T\) is the spatial trace and the traceless part is \((4) \delta G_{TF},\)

\begin{equation}
(4) \delta G_T = \frac{2}{n_b^2} \left[ \left( \frac{\dot{a}_b}{a_b} \right)^2 - 2 \frac{\dot{a}_b \ddot{a}_b}{a_b n_b} \right] A + \left( \frac{\dot{a}_b}{a_b} \right) \dot{A} - \ddot{R} - \left( 3 \frac{\dot{a}_b}{a_b} - \frac{n_b}{a_b} \right) \dddot{R} \\
+ \frac{2}{3} \left\{ \frac{1}{a_b^2} \nabla^2 \left( \dddot{R} + A \right) + \frac{1}{n_b^2} \left[ \left( \frac{n_b}{a_b} - 3 \frac{\dot{a}_b}{a_b} \right) \nabla^2 \sigma - \nabla^2 \dot{\sigma} \right] \right\}, \quad (B5)
\end{equation}

\begin{equation}
(4) \delta G_{TF} = -\frac{1}{a_b^2} \left( \dddot{R} + A \right) + \frac{1}{n_b^2} \left[ \dot{\sigma} - \left( \frac{n_b}{a_b} - 3 \frac{\dot{a}_b}{a_b} \right) \sigma \right]. \quad (B6)
\end{equation}

b. Vector perturbations

\begin{equation}
(4) \delta G_i^0 = -\frac{\nabla^2 \epsilon_i}{2n_b^2}, \quad (B7)
\end{equation}

\begin{equation}
(4) \delta G_j^i = \left( \frac{1}{2n_b^2} \right) \left[ \ddot{\epsilon}^i_j + \dot{\epsilon}_j^i \right]. \quad (B7)
\end{equation}

c. Tensor perturbations

\begin{equation}
(4) \delta G_{ij} = \frac{1}{2n_b^2} \left\{ \dddot{h}_{ij} - \left( \frac{n_b}{a_b} - 3 \frac{\dot{a}_b}{a_b} \right) \dddot{h}_{ij} - \left( \frac{n_b}{a_b} \right)^2 \nabla^2 \dddot{h}_{ij} \right\}. \quad (B8)
\end{equation}

2. Perturbed 5D-Einstein tensor

We present here the first-order perturbations of the five-dimensional Einstein tensor obtained for the perturbed metric given in Eq. (3.1).
The perturbed Einstein tensor for the scalar perturbations in terms of spatially gauge invariant variables, but for arbitrary bulk and temporal gauges, is given by

\[ (5) \delta G^0_a = \frac{6}{n^2} \left[ \left( \frac{\dot{a}}{a} \right)^2 A - \frac{\dot{a}}{a} \dot{R} \right] + 3 \mathcal{R}'' + 12 \frac{a'}{a} \mathcal{R}' + \frac{2}{a^2} \nabla^2 \mathcal{R} - \frac{2 \dot{a}}{a^2} \nabla^2 \sigma + \nabla^2 \sigma_y + 4 \frac{a'}{a^2} \nabla^2 \sigma_y + \frac{1}{a^2} \nabla^2 A_{yy} \] (B9)

\[ -3 \left\{ \frac{a'}{a} A_{yy} + \frac{\dot{a}}{a} \frac{\dot{a}}{n} A_{yy} + 2 \left[ \left( \frac{a'}{a} \right)^2 + \frac{a''}{a} \right] A_{yy} - \frac{1}{n} \left[ \frac{\dot{a}}{a} A_y + \left( 2 \frac{a'}{a^2} + \frac{\dot{a}}{a} \right) A_y \right] \right\} , \]

\[ (5) \delta G^0_i = \left\{ \frac{2}{n^2} \left( -\frac{\dot{a}}{a} A + \dot{\mathcal{R}} \right) + \frac{a^2}{2n^2} \left[ B' + \left( 5 \frac{a'}{a} - \frac{n'}{n} \right) B \right] \right\}, \] (B10)

\[ -\frac{1}{2n} \left[ A_y' + \left( \frac{2n'}{n} + \frac{a'}{a} \right) A_y + \frac{1}{n^2} \left[ A_{yy} - \frac{\dot{a}}{a} A_{yy} \right] \right] , \]

\[ (5) \delta G^0_4 = \left\{ \frac{1}{a^2} \left( A + \mathcal{R} \right) + \frac{1}{n^2} \left[ \sigma' + 3 \left( \frac{\dot{a}}{a} - \frac{n}{n} \right) \sigma \right] - \sigma_y' - \left( 3 \frac{a'}{a} + \frac{n'}{n} \right) \sigma_y - \frac{1}{a^2} A_{yy} \right\} \]

\[ \left[ \frac{1}{2} \frac{\dot{a}}{a} A' + \frac{\dot{a}}{a} + \frac{n'}{n} A_y + \frac{1}{n^2} \left( \frac{\dot{a}}{a} A + \left( 2 \frac{\dot{a}}{a} + \frac{\dot{a}}{a} - \frac{\dot{a}}{a} \right) A \right) \right. \]

\[ + \mathcal{R}'' + \left( \frac{3 \frac{a'}{a} + \frac{n'}{n} A_y + 2 \frac{a'}{a} A_y + \frac{n'}{n} - \frac{n'}{n} \right) \mathcal{R} + \frac{1}{a^2} \nabla^2 (R + A) \]

\[ - \frac{1}{n^2} \left[ \nabla^2 \sigma + \left( \frac{\dot{a}}{a} - \frac{n}{n} \right) \nabla^2 \sigma_y + \frac{3 \frac{a'}{a} + \frac{n'}{n}}{n^2} \nabla^2 \sigma_y \right] \]

\[ + \frac{1}{n^2} \left[ A_{yy} + \left( \frac{n'}{n} + 2 \frac{a'}{a} \right) A_y + 2 \frac{\dot{a}}{a} A_y + \left( \frac{n'}{n} - \frac{n'}{n} + 2 \frac{\dot{a}}{a} + \frac{\dot{a}}{a} \right) A_y \right] \]

\[ - \frac{1}{n^2} \left[ A_{yy} - \left( \frac{n'}{n} - \frac{2 \dot{a}}{a} \right) A_y + \frac{1}{a^2} \nabla^2 A_{yy} - \left( \frac{n'}{n} + 2 \frac{a'}{a} \right) A_{yy} \right] \]

\[ - 2 \left( \frac{2 a''}{a} + \frac{a^2}{a^2} + \frac{2 a a'}{a n} - \frac{n''}{n} \right) A_{yy} \] (B12)

\[ (5) \delta G^4_i = \left\{ -A' + \left( \frac{a'}{a} - \frac{n'}{n} \right) A - 2 \mathcal{R}' - \frac{a^2}{2n^2} \left[ B + \left( \frac{\dot{a}}{a} - \frac{n}{n} \right) \mathcal{B} \right] \right\} \]

\[ - \frac{1}{2n} \left[ A_y + \frac{\dot{a}}{a} A_y + \left( \frac{n'}{n} + 2 \frac{a'}{a} \right) A_{yy} \right] , \] (B13)

\[ (5) \delta G^4_4 = \frac{1}{a^2} \nabla^2 A + 3 \frac{a'}{a} \mathcal{A}' + \frac{3}{n^2} \left[ \frac{\dot{a}}{a} A + \frac{\dot{a}}{a} + \left( \frac{n}{a} - \frac{a}{a} \right) \mathcal{R} - \frac{a}{a} \mathcal{R} \right] \]

\[ + 3 \left( \frac{n'}{n} + 2 \frac{a'}{a} \right) \mathcal{R}' + \frac{1}{a^2} \nabla^2 \mathcal{R} - \frac{1}{n^2} \left[ \nabla^2 \sigma + \left( \frac{\dot{a}}{a} - \frac{n}{n} \right) \nabla^2 \sigma \right] \]

\[ + \left( \frac{a'}{a} + \frac{n'}{n} \right) \nabla^2 \sigma_y + \frac{3}{n} \left[ \frac{a'}{a} A_y + \left( \frac{a'}{a} + 2 \frac{\dot{a}}{a} \frac{\dot{a}}{a} \right) A_y \right] - 6 \left[ \frac{a' a'}{a^2} + \frac{\dot{a} \dot{a}}{a \dot{a}} \right] A_{yy} \] (B14)

As in the 4D case, it is useful to split the spatial part of the perturbed Einstein tensor \( (5) \delta G^i_j \) into a trace and a traceless part,

\[ (5) \delta G^i_j \equiv \delta^i_j \left( (5) \delta G_T + \left( \nabla \nabla_j - \frac{1}{3} \delta^i_j \nabla^2 \right) (5) \delta G_{TF} \right) , \] (B15)
where \( (5) \delta G_T \equiv \frac{1}{3} (5) \delta G^k_k \) is the 3-spatial trace and the traceless part is \( (5) \delta G_{TF} \). We get

\[
(5) \delta G_T = \frac{1}{3} \left[ \frac{2}{n^2} \left( \frac{n - 3 \dot{a}}{a} \right) \nabla^2 \sigma - \nabla^2 \sigma \right] + \frac{2}{a^2} \nabla^2 (A + \mathcal{R}) + 2 \nabla^2 \sigma_y + 2 \left( \frac{n'}{n} + \frac{3 \dot{a}'}{a} \right) \nabla^2 \sigma_y \]

\[
+ \frac{2}{n^2} \left[ \frac{n - 3 \dot{a}}{a} \mathcal{R} - \ddot{\mathcal{R}} + \frac{\dot{a}}{a} A + \left( \frac{2 \ddot{a} + \dot{a}^2}{a^2} - \frac{2 \dot{a} \ddot{a}}{a} \right) A \right] + 2 \mathcal{R}' + \frac{1}{2} A'' + \left( \frac{n'}{n} + \frac{3 \dot{a}'}{a} \right) \mathcal{R}' + \left( \frac{a'}{a} + \frac{n'}{n} \right) A' \]

\[
+ \frac{1}{n} \left[ \dot{A}_y + \left( \frac{n'}{n} + \frac{2 \dot{a}'}{a} \right) \dot{A}_y + 2 \frac{\dot{a}}{a} A_y + \left( \frac{\dot{n}'}{n} - \frac{n'}{n^2} + 2 \frac{\dot{a} a'}{a^2} + 4 \frac{\dot{a}'}{a} \right) A_y \right] - \frac{1}{n^2} \left[ \dot{A}_{yy} - \left( \frac{n}{n} - \frac{2 \dot{a}}{a} \right) \dot{A}_{yy} \right] + \frac{2}{3a^2} \nabla^2 A_{yy} - \left( \frac{n'}{n} + \frac{2 \dot{a}'}{a} \right) A_{yy}
\]

\[
+ \frac{1}{n^2} \left[ \sigma + \left( \frac{3 \dot{a}}{a} + \frac{\dot{n}}{n} \right) \sigma \right] - \frac{1}{a^2} (A + \mathcal{R} + A_{yy}) - \sigma_y' - \left( \frac{n'}{n} + \frac{3 \dot{a}'}{a} \right) \sigma_y.
\]

\[
(5) \delta G_{TF} = \frac{1}{n^2} \left[ \sigma + \left( \frac{3 \dot{a}}{a} + \frac{\dot{n}}{n} \right) \sigma \right] - \frac{1}{a^2} (A + \mathcal{R} + A_{yy}) - \sigma_y' - \left( \frac{n'}{n} + \frac{3 \dot{a}'}{a} \right) \sigma_y.
\]

b. Vector perturbations

Using the spatially gauge-invariant vector perturbations we obtain the perturbed Einstein tensor components,

\[
(5) \delta G^0_0 = 0, \quad (5) \delta G^4_4 = 0, \quad (5) \delta G^i_i = 0 \quad (5) \delta G^4_0 = 0
\]

\[
(5) \delta G^0_i = \frac{1}{2n^2} \left\{ -\nabla^2 \tau + \frac{a^2}{a} \left[ -\left( 5 \frac{\dot{a}}{a} - \frac{n'}{n} \right) S - S' \right] \right\} \hat{e}_i,
\]

\[
(5) \delta G^i_j = -\frac{1}{2} \left\{ \left( \frac{n'}{n} + \frac{3 \dot{a}'}{a} \right) \tau_y + \tau_y' + \frac{1}{n^2} \left[ \left( \frac{\dot{n}}{n} - \frac{3 \dot{a}}{a} \right) \tau - \tau' \right] \right\} (\hat{e}_i' j + \hat{e}_j' i),
\]

\[
(5) \delta G^4_i = \frac{1}{2} \left\{ \nabla^2 \tau_y + \frac{a^2}{n^2} \left[ -\left( \frac{n}{n} - \frac{5 \dot{a}}{a} \right) S + S' \right] \right\} \hat{e}_i.
\]

c. Tensor perturbations

The only non-zero component of the 5D Einstein tensor for the tensor perturbations is

\[
(5) \delta G^i_j = -\frac{1}{2} \left\{ \hat{h}^{i j} \left( \frac{n'}{n} + \frac{3 \dot{a}'}{a} \right) + \frac{1}{n^2} \left( \frac{n}{n} - \frac{3 \dot{a}}{a} \right) \hat{h}^{i j} + \frac{1}{a^2} \nabla^2 \hat{h}^{i j} + \hat{h}''^{i j} - \frac{1}{n^2} \hat{h}^{i j} \right\}.
\]

3. Projected Weyl tensor

The contribution of metric perturbations in the bulk to the modified Einstein equations on the brane in the Gaussian normal gauge is given by the projected Weyl tensor \( E_{\mu \nu} \) in Eq. (11).

a. Scalar perturbations

For the scalar perturbations we have

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As for the Einstein tensor, it is useful to split the spatial part of the perturbed Weyl tensor into a trace and a traceless part,

\[ \delta E_{ij} = \delta^i_j \delta E_T + \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \nabla^2 \right) \delta E_{TF}, \]

where \( \delta E_T \) is the spatial trace and the traceless part is \( \delta E_{TF} \). We find

\[
\delta E_T = \frac{1}{18} \left\{ 3 \tilde{A}' - 3 \tilde{R}' - 3 \frac{n^i}{n_b} \frac{\partial}{\partial n_b} \tilde{R}' + 3 \left( \frac{n^i}{n_b} - \frac{\partial^i}{\partial n_b} \right) \tilde{A}' - \nabla^2 \tilde{\sigma}' - n_b \nabla^2 \tilde{\sigma}_y + \frac{1}{a_b} \nabla^2 (2 \tilde{R} - \tilde{A}) \right\},
\]

\[
\delta E_{TF} = \frac{1}{3} \left\{ \frac{1}{a_b^2} (\tilde{A} + \tilde{R}) - 2 \tilde{\sigma}' + \left( \frac{n^i}{n_b} - 3 \frac{\partial^i}{\partial n_b} \right) \tilde{\sigma}_y + \frac{1}{n_b^2} \left[ \frac{n_b}{n_b} - 3 \frac{\partial}{\partial n_b} \right] \tilde{\sigma}_y - \tilde{\sigma}_y \right\},
\]

for the trace and the traceless part of the projected Einstein tensor, respectively.

b. Vector perturbations

For the vector perturbations we obtain

\[
\delta E^{\alpha}_i = -\frac{a_b^2}{3n_b^2} \left\{ \mathcal{S}' + \left( 2 \frac{\partial^i}{\partial n_b} - \frac{n^i}{n_b} \right) S - \frac{1}{2a_b^2} \nabla^2 \tau \right\} \dot{e}_i,
\]

\[
\delta E^{\alpha}_j = -\frac{1}{6} \left\{ \frac{1}{n_b} \left[ \dot{\tau} + 3 \frac{\partial}{\partial n_b} \frac{n_b}{n_b} \right] + 2 \tau' + \frac{3}{a_b^2} \left( \frac{n^i}{n_b} - n_b \right) \tau_y \right\} \left( \dot{e}^i_j + \dot{e}_j^i \right).
\]

c. Tensor perturbations

The only non-zero component is gauge-invariant and yields

\[
\delta E^{ij} = \frac{1}{6} \left\{ h^{ij} \left( 2 \frac{n_b}{n_b} - 3 \frac{a_b}{a_b} \right) - \frac{1}{n_b} h^{ij} - 2h^{ni}_j + h^{ni}_j \left( \frac{n^i}{n_b} - 3 \frac{a^i}{a_b} \right) + \frac{1}{a_b^2} \nabla^2 h^{ij} \right\}.
\]