A non-equilibrium Ising model of turbulence

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ABSTRACT

We introduce a model of interacting lattices at different resolutions driven by the two-dimensional Ising dynamics with a nearest-neighbor interaction. We study this model both with tools borrowed from equilibrium statistical mechanics as well as non-equilibrium thermodynamics. Our findings show that this model keeps the signature of the equilibrium phase transition. The critical temperature of the equilibrium models corresponds to the state maximizing the entropy and delimits two out-of-equilibrium regimes, one satisfying the Onsager relations for systems close to equilibrium and one resembling convective turbulent states. Since the model preserves the entropy and energy fluxes in the scale space, it seems a good candidate for parametric studies of out-of-equilibrium turbulent systems.

1. Introduction

Equilibrium systems can often be described through well founded statistical mechanics, built up from the classical laws of thermodynamics. In this sense, they represent an exception with respect to most systems found in nature, which are subject to flux of matter and energy to and from other systems and/or to chemical reactions. Most systems are, therefore, intrinsically out of equilibrium and their description often remains beyond the scope of present statistical mechanics [1]. Among all out-of-equilibrium systems, we may single out those leading to (out-of-equilibrium) steady states, called non-equilibrium steady states (NESS), which exhibit a constant average energy. Intense research efforts have thus focused on understanding how much such states differ from the equilibrium states of corresponding ideal systems (without say, forcing and dissipation). For example, there is now strong evidence that the large scales of the steady states of two-dimensional (2D) or quasi-2D turbulent flows can be described as equilibrium states of the corresponding Euler equation [2,3], whereas such flows are subject to non-zero energy or enstrophy fluxes through scales, characteristic of out-of-equilibrium dynamics. On the other hand, there has been recent numerical and theoretical evidence of steady states in out-of-equilibrium systems that do not correspond to any equilibrium states, such as an Ising model driven by a temperature gradient [4], and a quantum open system in contact with two reservoirs [5]. One way to get some insights about why and how NESS approach their equilibrium counterpart relies on the study of analytically tractable toy models like the symmetric simple exclusion processes or the zero range processes, which have become a paradigm of non-equilibrium toy models on a lattice [6]. Such models are, however, not appropriate to understand the turbulent NESS, because they lack of an essential ingredient: the development of an energy cascade over a wide range of scales. This was recognized by [7], who devised a dissipative forced zero...
range process on a Cayley (self-similar) tree identified in the article to a scale space. This process exhibits a transition between a quasi-equilibrium regime and a far-from equilibrium regime where net fluxes through scales were observed. However, this example does not exactly describe turbulent NESS, for which both the non-zero energy flux and the equilibrium state are superposed.

In the present article, we propose a new toy model of ‘turbulent process’ based on the Ising model [8,9]. It approximates magnetic dipole moments of atomic spins through discrete variables with a Boolean-like distribution (+1 or −1), interacting with their nearest neighbors on a lattice. The properties of the model at thermal equilibrium are well established and allow for the identification of second-order phase transitions in 2D [10,11]. The idea of exploring the existence of NESS of the Ising model as a paradigm for other phenomena hails from the work of Glauber [12]. In the same spirit, other modifications of the Ising model have been suggested by [4,13–18] where the addition of a temperature gradient within the lattice sets the system out of equilibrium. However, none of these variants of the Ising model so far contain the dynamics of energy or enstrophy fluxes through scales. In order to take into account this feature, we propose a model of two interacting lattices. Lattice A is the reference: its dynamics is iterated at temperature $T_A$, then the configuration is copied to lattice B and an iteration is performed at $T_B$. Finally, the first lattice is updated with the changes applied to lattice B. The currents are computed as the number of spins changed within these dynamical steps.

In turbulent systems, ranging from laboratory experiments (von Kármán swirling flows, Rayleigh-Benard flows, plane Couette Flow) to geophysical systems, energy is injected at large scales and dissipated at small scales. In order to simulate these effects, the resolution of the lattice B can be downgraded in our model. In such case, the dynamics on the lattice B corresponds to the large-scale dynamics of a turbulent flow, whereas the dynamics on the lattice mimics the small-scale interactions. Our setup is useful to understand how the dynamics is set out of equilibrium through coarse-grain. This is particularly relevant in climate dynamics where high resolutions regional models are embedded in low resolution global models to resolve fine scale features of regional climates [19].

The paper is organized as follows: we introduce the two-lattice model in Section 2. We then perform two different analyses: (i) Section 3 is devoted to the study of possible traces of classical equilibrium phase transitions in the model dynamics; (ii) Section 4 contains the non-equilibrium thermodynamic analysis of the model, where we show that coarse-grain effects alone introduce non-equilibrium effects and that such dynamics depend both on the temperature gradient between the lattices A and B and on the temperature $T_A$ of the grid at higher resolution. Finally, in Section 5, we discuss the possible physical implications of our findings.

2. Model of two interacting lattices at different temperatures

The standard Hamiltonian for an Ising system includes only nearest-neighbor interactions. For each lattice site $k$, there is a discrete variable $\sigma_k \in \{+1, -1\}$ representing the site’s spin. The standard Hamiltonian is:

$$H = -J \sum_{\text{neighbors}} \sigma_j \sigma_i.$$

Here, we consider an Ising system on square grids, where each spin interacts directly with four neighbors (above, below, to the right and to the left). Our model consists of two square lattices A and B. The two lattices evolve at different temperatures $T_A$ and $T_B$ and may be of different sizes $L_A$ and $L_B$. A schematic representation of the dynamics of the model is reported in Figure 1 and detailed below:
2.1. Dynamics on the reference lattice $A$

First, the lattice $A$ is updated following the Metropolis algorithm with a full Monte Carlo step at temperature $T_A$. We place the $N$ particles in a random configuration. Then we move each of the particles in succession: we calculate the change in the energy of the system $\Delta E$, which is caused by the move. If $\Delta E < 0$, i.e. if the move would bring the system to a state of lower energy, we allow the move and put the particle in its new position. If $\Delta E > 0$, we allow the move only with probability $\exp (- \Delta E / (kT_A))$, being $k$ the Boltzmann constant. A full Monte Carlo step consists of performing a number of such operations of the same order of the size of the lattice.

2.2. Coarse-grain operation

Then, if $L_A \neq L_B$, a coarse-grain operation is performed: for each spin of the lattice at lower resolution, we count the number of corresponding spins at full resolution. And if the majority of the spin is positive (black in the figure), we assign a positive spin to the corresponding coarse-grain position. If half is positive an half is negative, we chose randomly with equal probability to assign a positive or negative value to the coarse-grain site. We then perform a Monte Carlo step with the Metropolis algorithm at the temperature $T_B$ on lattice $B$.

2.3. Final update of the fine grid

Eventually, we update grid $A$ by reversing all the spins which correspond to a change in lattice $B$. Non-equilibrium effects can be tracked measuring non-zero average fluxes $j(t)$ circulating in the two-lattice system. The most natural definition for $j$ computes the number of spins changed on grid

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**Figure 1.** Schematic representation of the two lattices model. In this representation, the size $L_B$ of grid $B$ is smaller than $L_A$. Vertical steps are dynamical and related to the iteration of the Ising dynamics at temperature $T_A$ or $T_B$. The horizontal steps describe the way the operations of coarse-grain and fine-grain are performed.
A after a complete iteration of the model. For clarity, a complete iteration means: (1) Monte Carlo full step at temperature $T_A$ on the lattice $A$, (2) coarse-grain operation from $A$ to $B$, (3) Monte Carlo full step at temperature $T_B$ on the lattice $B$ and (4) update of the lattice $A$ according to $B$. In the following, we will describe some of the system properties as the role of non-equilibrium effects and of the phase transitions. All the quantities will be computed for the lattice $A$ unless explicitly specified in the text. We will see that there is no trivial complete description of the non-equilibrium effects using solely statistical tools devised for equilibrium dynamics. However, relevant quantities of equilibrium dynamics still play an important role when the system is set out of equilibrium.

We have investigated the dynamics of the systems for different temperatures $T_A$ and $T_B$ and different sizes $L_A$ and $L_B$ from $L = 8$ to $L = 256$. On this range, we have found that results do not qualitatively depend on the resolution. Therefore, in the following, we will consider two typical experiments: one where the two grids are at the same resolution but at two different temperatures $L_A = L_B = 64$ and another one where the two grids are at different temperatures and resolutions $L_A = 64$ and $L_B = 8$. The temperature range analyzed is $(T_A, T_B) \in [T_c - 1, T_c + 1]^2$, $T_c = 2/\ln (1 + \sqrt{2})$ being the reduced critical temperature for 2D square lattices.

Far from the transition, the system settles in a steady state after few hundreds of full-dynamical-steps of our model. Close to the transition the steady state is reached after few thousands steps. Our definition for reaching the steady state is based on the analysis of time series of magnetization

$$M(t) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i.$$ 

As specified before, this quantity is computed on the reference lattice $A$ only. We consider that we have reached the steady state when $M(t)$ fluctuates around a well-defined average value. In order to be sure to take averages only when the system has reached the stationary state, we discard the first 10,000 full steps of the computation. The value of 10,000 has been chosen as it is twice as longer as the longest time needed to settle in stationary state near the transition (about 5000 steps).

### 3. Equilibrium analysis and phase transitions

It is interesting to see whether such steady states still undergo a phase transition at a certain temperature set $(T_A, T_B)$; and to assess how such a transition depends on the coarse-grain effects. In order to do so, for each experiment we have computed the time and space average magnetization $\langle \mathcal{M} \rangle$, and the first zero of the auto-correlation function $C(\tau)$ of the time series of magnetization. For a classical Onsager square lattice, $C(\tau)$ is known to decrease exponentially with a characteristic time which diverges following a power law near the critical temperature $T_c$, whereas $\langle \mathcal{M} \rangle$ satisfies a critical power-law scaling near the same critical temperature $T_c$. Both quantities are represented in Figure 2 for the non coarse-grain experiment (left plots) and the coarse-grain experiments (right).

We first observe that the absolute value of $\langle \mathcal{M} \rangle$ (upper panels of Figure 2) reproduces the equilibrium Ising critical transition but with a higher pseudo-critical temperature $T_c(T_A, T_B)$. When no coarse-grain is applied to the grid $L_B$ the picture looks symmetric with respect $T_A$ and $T_B$. In this sense, one could define $T_c(T_A, T_B)$ by following the curve on the $T_A, T_B$ plane where $\langle \mathcal{M} \rangle$ changes abruptly. On the other hand, when the resolution of $L_B$ is decreased (top right panel of the same figure), this curve does not look symmetric with respect to $T_A$ and $T_B$. The transition looks sharper when the temperature of the coarse-grain lattice $L_B$ is varied. Our model also respects the symmetry of the magnetization with respect to zero: positive and negative magnetization are obtained with equal probability.

Further signatures of phase transitions can be found computing the auto-correlation function $C(\tau)$ of the magnetization time series $\langle M(t) \rangle$. The first zero of $C(\tau)$ is reported in the lower panels of Figure 2. In both cases, we observe an increase of the resulting time scale in a region of the $(T_A, T_B)$
plane which can be readily identified as a critical temperature. This region matches the region of maximal variations of the average magnetization \( \langle M \rangle \).

We now turn to analyze the effect of the temperature as a function of the entropy \( S \) and the energy \( E \). In fact, as predicted by Onsager, if the system remains close to equilibrium, one must find:

\[
\frac{\partial E}{\partial S} \bigg|_M = T_A,
\]

where \( M \) is the magnetization at the end of the simulation. The entropy used here is the usual Shannon Entropy computed in information theory which measures the disorder of lattice \( A \) at the end of the simulation:

\[
S = - \sum_{i,j} p(i,j) \log_2 (p(i,j))
\]

with \( p(i,j) \) the probability of observing a spin up or down at the position \( (i,j) \). The linearity of such a relation can be verified in Figure 3 where the quantities \( E, S, T_A \) are plotted for four different values of \( M \).

In the next section, we will perform a non-equilibrium analysis considering thermodynamic quantities such as the entropy and the currents, and understand whether such a phase transition is still relevant when the system is set out of equilibrium.

### 4. Non-equilibrium analysis

We start the analysis by verifying that our model possesses a genuine non-equilibrium dynamics and then comment on some of its relevant thermodynamic properties. We use the definition of currents already described above by defining \( j \) as the number of spins changed on lattice \( A \) after the steps performed on lattice \( B \). We rescale \( j \) to include only the effects of the temperature change.

In Figure 4, we present the behavior of \( j \) for the two experiments detailed before: no coarse-grain on the left panels and coarse-grain on the right. Colors represent the temperature \( T_B \).
the non-coarse-grain dynamics, we immediately see that, for each temperature – each color – \( T_B \)
the current \( j \) increases proportionally to \( \Delta T = T_B - T_A \). This is the signature of Fourier’s relation
when the system is driven out of equilibrium and the heat transfer can be considered conductive. However, for higher temperature gradients and depending on \( T_B \), this regime is broken and currents saturate at a specific value. This phenomenon is facilitated for coarse-grained dynamics (right panel) where the saturation is indeed observed at all temperatures \( T_B \).

Another important proof of far-from-equilibrium properties relies on the existence of a maximum of entropy production and the relation linking this maximum to the temperatures \( T_A \) and \( T_B \) and to the currents and their conductive/convective nature. In fact, as pointed out in [20], for other out-of-equilibrium models such as the zero range process, a maximum of entropy production is

**Figure 3.** (Colour online) Numerical verification of the Onsager relations close to equilibrium. The temperature \( T \) is a good linear function of the Shannon Entropy \( S \) and the energy \( E \), when the magnetization \( M \) is kept constant. Different colors correspond to different \( M \) intervals as described in the legend.

**Figure 4.** (Colour online) Current \( j \) between grids \( B \) and \( A \), as a function of the temperature gradient \( \Delta T = T_B - T_A \). The color scale represents temperature \( T_B \). Left: lattices without coarse-grain \( L_A = L_B = 64 \). Right: coarse-grain with \( L_A = 64 \), \( L_B = 8 \).
clearly recognizable and falls between these two regimes. In our model, we will define the macroscopic entropy production $\sigma$ as the product of the fluxes by the corresponding thermodynamics forces responsible for such fluxes. It is important to point out that this entropy is different from the microscopic entropy defined above. In terms of currents and temperatures, the entropy production is written as:

$$\sigma = j \left( \frac{1}{T_B} - \frac{1}{T_A} \right),$$

where $j$ is the current. Figure 5 represents $\sigma$ as a function of the currents $j$ and it is colored according to $T_A$ (upper panels) and $T_B$ (lower panels). We analyze first the non coarse-grained dynamics (on the left panels) where maxima of $\sigma$ are observed for each curve of $T_B$. Moreover, each curve has the maximum located at almost the same temperature $T_A$ which roughly corresponds to the critical temperature of the equilibrium Ising model $T_c$. Moreover, the maxima always fall in between the conductive and convective regimes. This is even more evident when analyzing the plots for the coarse-grained dynamics (right panels) as the separation between these two regimes corresponds to a sharp change of $\sigma$ especially at lower temperatures $T_B$. In this case, the location of the maximum also corresponds to the critical temperature $T_c$.

This fact is indeed relevant for two reasons: first, it suggests that equilibrium properties, e.g. the critical temperature $T_c$, still play an important role when the system is set out of equilibrium. Second it correlates to the phenomenon of self-organized criticality widely invoked to explain the fact that many physical systems set into non-equilibrium states close to their critical temperature actually display exact critical properties [21,22]. Here, the states obtained following a maximum entropy production principle coincide with the ones obtained invoking the self organized criticality.

Finally, we can study the distribution of $\sigma$ in the plane $(T_A, T_B)$ as shown in Figure 6. Note that the scales of $\sigma$ for the non coarse-grained dynamics (left panel) is different from the one of the

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*Figure 5.* (Colour online) Entropy production $\sigma$ as a function of the currents $j$. Left panels: lattice without coarse-grain $L_A = L_B = 64$. Right panels: coarse-grain with $L_A = 64$, $L_B = 8$. The color scale represents $T_A$ (upper panels) and $T_B$ (lower panels).
coarse-grain simulations (right panels). The latter case shows a sharper $\sigma$ gradient than the former case. This is consistent with what is observed in Figure 5.

5. Discussion

In this paper, we have introduced an effective way to obtain a genuine non-equilibrium Ising dynamics with an interaction in the scale space reproducing some properties of turbulent systems.

Previously, other authors have investigated the possibility of building such a model by using only a one lattice dynamics and introduced the temperature gradient within the sites of the lattice. Although such systems possesses a rich dynamics with an interesting variety of metastable and stable states, they do not mimic interaction among different scales and present discontinuities in the temperature field which hinder simple tests of Onsager relations as well as out-of-equilibrium conjectures like the existence of maxima of entropy production.

Our model satisfies such constraints by composing the dynamics of two interacting lattices at different temperatures. This model not only respects Fourier's law $dE/dS|_N = T_A$, but also features the possibility of having the second lattice at a coarse-grained resolution, as it happens in several physical phenomena. Moreover, clear differences arise when comparing the coarse-grained dynamics with the one of two lattices having the same spatial resolutions. For example, the coarse-grain dynamics exhibit a breakdown of Fourier's law (and therefore of conductive heat transfer) introducing, at high temperature gradients, convective fluxes confirmed by the appearance of a negative temperature in the energy-entropy relation. The fact that the coarse-grain operations induces a strong non-equilibrium behavior justifies to test the stability of physical properties in turbulent and geophysical models against the resolution of the model [23–25]. Moreover, the states maximizing the entropy production are located between conductive and convective regimes and correspond to the critical temperature of the equilibrium Ising dynamics.

The convective states are new features for the Ising dynamics and cannot be observed e.g. by adding an external magnetic field to the system. In addition, the introduction of the temperature gradient between the two different lattices preserves the symmetry spin up/down which is broken with the addition of magnetic fields.

Despite its relative simplicity, our model displays coexistence of quasi-equilibrium states at the largest scale, with non-zero average fluxes at smaller scale. It therefore shows that turbulent flows may not be so exceptional, and rather be representative of a wide class of out-of-equilibrium systems with both spatial and scale dynamics.

An interesting outcome of our model could be the development of more realistic models of natural phenomena. For example, several models of atmospheric convection are based on the

![Figure 6. (Colour online) Entropy production $\sigma$ (in color scale) as a function of the temperatures $T_A$ and $T_B$. Left panel: lattice without coarse-grain $l_A = l_B = 64$. Right panel: coarse-grain with $l_A = 64, l_B = 8$.](image_url)
equilibrium Ising dynamics [26,27]. We believe that a better representation of atmospheric transport can be achieved introducing our non-equilibrium dynamics modeling both conductive and convective heat transfer. On the other hand, one can imagine that the second lattice is just some coarse-grained properties of the first one and that temperature $T_B$ maps some properties in the scale space.

Finally, the fact that the maxima of entropy production correspond to states at critical temperature for the lattice suggests that there is a link between the maximum entropy production principle and self-organized criticality. The latter is often invoked to explain why out-of-equilibrium systems settle into states close to their critical manifolds but without a solid physical underlying motivation. We believe that such a connection might represent a step in solving this issue, despite requiring further investigations.

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