Basic Principles of Thin-Walled Open Bars Taking into Account Where Influence Shifts of Cross Sections are Concerned

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Abstract. The finite element method is considered to be the most effective in relation to the calculation of strength and stability of buildings and engineering constructions. As a rule, for the modelling of supporting 3-D frameworks, finite elements with six degrees of freedom are used in each of the nodes. In practice, such supporting frameworks represent the thin-walled welded bars and hot-rolled bars of open and closed profiles in which cross-sectional deplanation must be taken into account. This idea was first introduced by L N Vorobjev and brought to one of the easiest variants of the thin-walled bar theory. The development of this approach is based on taking into account the middle surface shear deformation and adding the deformations of a thin-walled open bar to the formulas for potential and kinetic energy; these deformations depend on shearing stress and result in decreasing the frequency of the first tone of fluctuations to 13%. The authors of the article recommend taking into account this fact when calculating fail-proof dynamic systems.

1. Introduction

Progress in computer technologies stipulates widespread deployment of numerical methods, the finite-element method (FEM) being the most effective in structural analysis and car manufacturing analysis. By means of discretization procedures FEM represents construction elements as frameworks, lamellar or bulk systems, among which thin-walled welded bars and hot-rolled bars of open and closed profile are popular. (figure 1).

Recent research of deflected mode of thin-walled open bars is being taken in two concepts. The first concept is related to the improved theories where the difference is made between coordinates of initial and finite state of a thin-walled bar and where the influence of shifts and movements of torsion points and angles of cross sections on the values and distribution of internal forces is concerned.

The second concept presents the application of theory of thin-walled open bars to design analysis of space structures made of thin-walled bars in the state of both static and dynamic loading in terms of FEM [1-7].
Figure 1. Thin-walled open bars for machine engineering and building design: a, b – hot-rolled H-bars and U-bars; c, d – welded H-bars: c – with two symmetry axes; d – with one symmetry axis; e – with stability ribs providing general and local stability of the wall

2. The mathematical model of a thin-walled open bar
Following [1], we consider shear deformation influence and introduce into expression of deformation energy of a thin-walled open bar the part of deformation caused by work of tangential stresses. Let us determine its influence onto the “dynamic portrait” of the bar. Normal and tangential stresses in point A of a thin-walled bar are determined by the following formulas:

\[
\sigma = \frac{N}{A} + \frac{M_y}{I_y} y - \frac{M_x}{I_x} x - \frac{B_{\omega}}{I_\omega} \omega \tag{1}
\]

\[
\tau = \frac{2H}{I_d} h - \frac{Q_y S_y}{I_y \delta} - \frac{Q_x S_x}{I_x \delta} - \frac{M_{\omega} S_{\omega}}{I_\omega \delta} \tag{2}
\]

where \( h \) is distance from a point A to a middle surface of the bar with thickness \( \delta \) (figure 2).

Figure 2. Thin-walled open bar: a – initial state; b – deformed state in LCS \( oxyz \); c - coordinates of a point M \((z,s)\) on the middle surface of the bar; d – doubled area of elementary sector \( r\delta s \), \( \Theta \) - angle of torsion.
Torsion moment $H$ caused by ununiformed distribution of tangent stresses on wall thickness $\delta$ is expressed through angle of torsion $\Theta$, according to the formula:

$$H = GI_d \frac{d\Theta}{dz}$$  \hspace{1cm} (3)

$B$ (1)-(3) $G$ is elastic modulus under shearing; $d\Theta/dz$ - deplanation defined as a derivative from a torsion angle; $I_d = \alpha \sum_{i=1}^{n} \frac{h_i \delta_i^3}{3}$ - geometric characteristic of a cross section playing the same role as a polar inertia moment for bars with circular cross sections; the last summands in (1) and (2) $\sigma_\omega$ and $\tau_\omega$ - normal and tangential stresses in an open bar stipulated by its shear deformation and torsion where $B_\omega$ - bimoment which is similar to the expression of inner bending moment, with the difference that in equation (4) the arm, that elementary inner force $\sigma_\omega dA$ is multiplied by, is replaced by sectoral coordinate $\omega$ having dimension $\text{cm}^2 (\text{m}^2)$.

$$B_\omega = \int_{A} \sigma_\omega \omega dA = -E \Theta^\omega \int_{A} \omega^2 dA$$  \hspace{1cm} (4)

For ease of computations it is suitable to build an epure with 2-dimentional coordinates $\omega$, according to their sign convention. On this epure, their values are set on the normal, and their sign depends on location of the main sectorial zero point $M_0$ (figure 2):

$$\omega = \int_{s} d\omega = \int_{s} rds$$  \hspace{1cm} (5)

and product $rds$ in equation (5) equals to doubled area of elementary sector $AMm$ shown on figure 2d. Sectorial area $\omega$ is considered positive, if, when the point is moving along section profile from the reference point $M_0$, related position vector rotates counterclockwise relative to section in positive direction of axis $z$.

In both (1) and (2) $I_\omega$ is sectorial moment of inertia (dimension $\text{cm}^6$, $\text{m}^6$).

$$I_\omega = \int_{A} \omega^2 dA$$  \hspace{1cm} (6)

which is known as geometric characteristic of the section similar to axial moments of inertia $I_x$ and $I_y$; $M_0$ – flexure-torsion moment proper to stresses $\tau_\omega$ from formula (2). Amounting to pure torsion moment $M_0$ they create in the bar section internal torsion moment $M_\text{kp}$ which equalizes external force moment $M_z$, where $M_0 = H$ from (3),

$$M_z = M_\text{kp} = M_\omega + M_0$$  \hspace{1cm} (7)

where

$$M_\omega = \int_{A} \tau_\omega \delta d\omega = -E \Theta^\omega \int_{A} \omega^2 dA$$  \hspace{1cm} (8)

Besides mentioned above magnitudes characteristic to a thin-walled bar, in (2) $S_\omega$ is sectorial static moment of a cross section (dimension $\text{cm}^4$, $\text{m}^4$):

$$S_\omega = \int_{A} \omega dA$$  \hspace{1cm} (9)

From equations (4) and (8) it follows:

$$\frac{dB_\omega}{dz} = -EI_\omega \Theta^\omega = M_\omega$$
so, $B_\omega$ and $M_\omega$ and, consequently, $\sigma_\omega$ and $\tau_\omega$ can’t be determined without calculating torsion angle $\Theta=f(z)$ which is intrinsic to analysis of a thin-walled open bar affected by bending torsion. In order to determine $\Theta=f(z)$ let’s put into (7) values (3) and (8):

$$-EI_\omega \Theta'' + GI_\omega \Theta' = M_z \quad (10)$$

and further, subject to the first derivative from (10), we’ll get equation of a torsion angle of a bar:

$$\Theta'' - k^2 \Theta' = \frac{m}{EI_\omega} \quad (11)$$

where $k$ – flexure-torsion characteristic of a bar (dimension cm$^{-1}$, m$^{-1}$),

$$k = \sqrt{\frac{GI_\omega}{EI_\omega}} \quad (12)$$

$m$ – distributed external torsion moment.

Potential energy of internal forces in a bar is expressed through stresses $\sigma$ and $\tau$ from (1) and (2) in equation:

$$V = \int_0^l \left( \frac{\sigma^2}{2E} + \frac{\tau^2}{2G} \right) dAdz + \int_0^l \frac{\tau^2}{2G} dAdz$$

which, after transformation, becomes as:

$$V = \int_0^l \left( \frac{N^2}{2EA} + \frac{M_y^2}{2EI_y} + \frac{M_z^2}{2EI_z} + \frac{B^2}{2EI_a} + \frac{H^2}{2GI_d} \right) dZ +$$

$$+ \frac{1}{GA} \int_0^l \left( \frac{K_{xy}Q_x^2}{2} + \frac{K_{xz}Q_z^2}{2} + \frac{K_{y}M_z^2}{2} + K_{xy}Q_xQ_y + K_{xz}Q_zQ_x + K_{y}Q_yQ_x \right) dZ, \quad (13)$$

where $K_{ij}$ introduced into (13) are:

$$K_{ij} = \frac{A}{I} \int_0^l \frac{S_i S_j}{\delta^2} dA; \quad i, j = x, y, \omega \quad (14)$$

For symmetric profiles, if section of a thin-walled bar is symmetric relative to axis $x$, in (13) $K_{xy} = K_{yx} = 0$. If a symmetry axis is $y$, then $K_{xy} = K_{yx} = 0$. If section is symmetric relative to both main axes, then $K_{xy} = K_{yx} = K_{yx} = 0$ (figure 1).

Let’s consume $\zeta, \eta, \xi$ to be movements of points on the central line of bar bending along LCS axes $x, y$ and $z$.

Taking into account certain dependences

$$EA\xi' = N; \quad EI_x \xi'' = -M_x; \quad EI_y \xi'' = M_y; \quad -EI_y \eta'' = Q_y; \quad (15)$$

$$EI_y \xi'' = Q_y; \quad -EI_y \eta'' = Q_y; \quad -EI_\omega \Theta'' = B_\omega.$$
If a bar is affected by a distributed load \( q_x(z), q_y(z), q_z(z) \), which doesn’t change its direction in the process of a bar deformation, then, for values of potential energy of interior forces (16) linked differential equations of bending and torsion of a thin-walled open bar subject to middle surface shift will be (2):

\[
\begin{align*}
\left( K_{yy} + K_{yy} \right) q_x + \left( K_{yy} + K_{yy} \right) q_y &= m_x + \varphi''(z), \\
\left( K_{yy} + K_{yy} \right) q_y + \left( K_{yy} + K_{yy} \right) q_y &= m_y + \varphi''(z), \\
\left( K_{yy} + K_{yy} \right) \varphi + \left( K_{yy} + K_{yy} \right) \varphi &= m_{xy} + \varphi''(z),
\end{align*}
\]

where, according to (3) and (8), \( L = H + M_{m_{xy}} \).

Homogeneous equation (18) under homogeneous boundary conditions that determines eigen numbers and eigen functions of a boundary problem about free bending oscillations of a steel thin-walled open bar \( (\rho = 7.85 \text{ ton/m}^3) \) can be given as following:

\[
\begin{align*}
\eta(z, t) &= \eta(z) \sin(\omega t + \varphi) \\
\eta''(z) + K_{xy} \eta(z) - K_{yy} \eta(z) &= 0
\end{align*}
\]
\[
K_1 = \omega^2 \frac{\rho A}{EI_s} \left( \frac{EI_s}{GL_z} K_{xx} + \frac{I_z}{A} \right), \\
K = \omega^2 \frac{\rho A}{EI_s} \left( 1 - \omega^2 \frac{\rho I_s K_{xx}}{GL_z} \right).
\]

Answer (20) is the following:

\[
\eta(z) = c \cdot e^{\nu z}.
\]

After substituting (22) into (20) subject to (21) we receive eigen value equation

\[
s^4 + s^2 K_s - K = 0
\]

which corresponds to differential equation (20) and has four roots \( s_{1,2} = \pm a; \ s_{3,4} = \pm ib; \ i = \sqrt{-1} \), in which

\[
a = \sqrt{-\frac{K_s}{2} + \left( \frac{K_s}{2} \right)^2 + K}, \quad b = \sqrt{-\frac{K_s}{2} + \left( \frac{K_s}{2} \right)^2 + K}
\]

Therefore, total answer (20) subject to (22) and (24) will be [5]

\[
y(z) = \eta(z) = C_1 \cos (az) + C_2 \sin (bz) + C_3 \cos (bz) + C_4 \sin (bz)
\]

Taking into account the answer for expressions (19)–(24), let us consider console steel H-bar for which the use of boundary conditions

\[
\eta(0) = 0; \ \Theta_z(0) = 0; \ \eta''(l) = 0; \ \eta'''(l) = 0
\]

reduces equation (23) to transcendental equation subject to its eigen angular frequency \( \omega_1, \ c^{-1} \),

\[
a^4 + b^4 K + (a^2 b^2 + a^2 b^2 - a^2 b^2) \cos (al) + (a^2 - a^2 b^2 \cos (al) + (a^2 - a^2 b^2 \sin (bl)) = 0
\]

where subject to (14),

\[
a + a^3 \cdot K_{xx} \left( \frac{EI_s}{GL_d} \right) \\
- \frac{a^3 \cdot K_{xx}}{GR_d}
\]

middle surface of a section is not allowed, as well as inertia of its rotation \( (K_1=0, K_{xx}=0) \), and besides, in (24) \( a = b = \sqrt{K_s} \) where K is taken in terms of (21), equation (25) is assumed as following:

\[
\cos (al) = -1, \ \text{the least root of it is } al=1.8751. \ \text{Hence, we’ll find eigen oscillation } \omega_1 \text{of the first tone:}
\]

\[
\omega_1 = \left( \frac{1.8751}{l} \right)^2 \sqrt{\frac{EJ}{\rho A}}
\]

In particular, for H-bar having length \( l=0.6m, b=h=12cm, \delta_{shelf} = \delta_{wall} = 1cm \), with \( A=34cm^2, \ E=2.1 \cdot 10^5\text{MPa}, \ GL_d=10.9kN \cdot m^2, \ \rho=7.85\text{ton/m}^3 \), the answer for expression (25) subject to middle surface shift using Mathcad [6] gave eigen frequency of the first tone \( \omega_1=2142e^{-1} \), disregarding shift, according to formula (26) \( \omega_1=2464e^{-1} \).
3. Conclusion
So, accounting shifts of middle surface of cross sections in thin-walled open bars results in a decrease in first tone oscillation frequency to 13%. The authors of this article recommend taking into account this fact as an important factor for fail-proof dynamic systems. Besides, equations of bending and torsion (17) make it possible to create stiffness matrix $[K]$ and mass matrix $[M]$ of thin-walled open bars which can be built easily into any computing system for analysis of space metal frames of thin-walled bars using FEM.

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