General formulation for electrostatic solitons in multicomponent nonthermal plasmas

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Received 4 February 2015, revised 28 June 2015
Accepted for publication 20 July 2015
Published 12 August 2015

Abstract
A general formulation is proposed for electrostatic solitons in multicomponent plasmas consisting of one cold species described by the fluid model and multiple hot components described by the kinetic equation. For hot species two types of generalized velocity distributions are adopted: the kappa function and the highly nonthermal distribution with two free parameters which incorporate the widely used nonthermal distribution proposed by Cairns \textit{et al} (1995 \textit{Geophys. Res. Lett.} \textbf{22} 2709) as a special limit. The general expressions for the \textit{KdV} equations and the Sagdeev potential, as well as the conditions for electrostatic solitons with positive or negative polarity, are derived, which may give rise to various solutions for acoustic solitons in multicomponent plasmas with generalized nonthermal velocity distributions.

Keywords: electrostatic soliton, multicomponent plasmas, nonthermal plasmas

(Some figures may appear in colour only in the online journal)
proposed by Cairns et al [26] (referred to as the Cairns et al distribution), where \(a\) is a free parameter shaping the distribution and \(v_{\text{in}} = \sqrt{\kappa_0 T/m}\). The latter distribution function may lead to the ESW with depleted density (known as non-thermal solitons) observed by the spacecraft [17, 19] while the Maxwellian distribution may give rise to only positive electrostatic solitons. Both types of nonthermal distributions have also been adopted by various authors to study the ESW in various plasma systems, such as the two-electron-temperature plasma [27-29], electron–positron–ion plasmas [30-36], and dusty plasmas [37-39].

Hau and Fu [40] found that for the condition \(v^2/(\kappa_0 v_{\text{in}}^2) \ll 1\) the kappa function may be expanded as \(f^\kappa(v) \propto [1 - v^2/(\kappa_0 v_{\text{in}}^2) + v^4/(2\kappa_0 v_{\text{in}}^4) + \ldots \exp(-v^2/\kappa_0 v_{\text{in}}^2)]\), referred to as the highly nonthermal distribution, which bears a similar form to the Cairns et al distribution and may recover the Maxwellian distribution in the limit \(\kappa \to \infty\). Note that both the Cairns et al distribution and the highly nonthermal distribution may describe the features of nonmonotonic shoulder-like or bump-on-tail velocity distribution [41] while the kappa function is simply monotonic with a high tail feature. All three types of nonthermal distributions may suitably describe the various velocity distributions of nonthermal electrons observed in space plasma environments [10-16, 41].

Chuang and Hau [42] have developed a general formulation for acoustic solitons in three-component plasmas consisting of one cold fluid and two hot components described by the combination of the kappa function and the highly nonthermal distribution function. In their study the charge of each component is a free parameter while the charge neutrality for background plasmas is imposed. In light of the observational evidence for the existence of multiple components in space plasma environments [5-9], a number of theoretical models for electrostatic solitons in four-component plasmas have been developed [43-46]. In the present study we extend the formulations of Chuang and Hau [25, 42] and Chuang [47] to more general cases with multiple (unlimited) hot components described by the combination of the kappa function and the highly nonthermal distributions with suitable choices of \(a_1\) and \(a_2\). The general expressions for the KdV equations and Sagdeev potentials are derived, which may give rise to various solutions for electrostatic acoustic solitons in multicomponent plasmas with different velocity distributions, which is a distinct feature of collisionless plasmas. Note that the term ‘multicomponent’ is more general than the ‘multispecies’ based on the consideration that the same species, such as electrons, may possess more than one distribution function [6, 11, 12, 48]. The development of such a unified formulation is important not only for nonlinear plasma physics but also for space and astrophysical applications. This paper is organized as follows. In section 2 the model equations, along with the nonthermal distribution functions adopted in the present study, are presented and the corresponding linear dispersion relations for acoustic waves are derived. Section 3 shows the derivation of the KdV equation and section 4 presents the Sagdeev potential analysis. The summary is given in section 5.

### 2. Model equations and distribution functions

We consider the multicomponent plasmas comprising one cold component and multiple hot components, described by the fluid and kinetic models, respectively [49-52]. The model equations include the continuity, momentum and Poisson’s equations.

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{u}) = 0, \quad \nabla \cdot \mathbf{u} = 0.
\]

\[
m_n \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -n_q \mathbf{E} + \nabla \phi,
\]

\[
\nabla^2 \phi = -\frac{1}{\epsilon_0} \left( q_n n + \sum_{\alpha} q_{\alpha n} n_{\alpha} + \sum_{\beta} q_{\beta n} n_{\beta} \right),
\]

for which \(\mathbf{E} = -\nabla \phi(x)\) has been used and the subscript \(c\) denotes the cold species. The variables \(n, m, u\), and \(q\) represent the number density, mass, flow velocity and electric charge of charged particles, respectively. The subscripts \(a\) and \(b\) denote the hot species with kappa and general nonthermal distributions, respectively. In particular, for the problem under consideration, the kappa function has the following form [40].

\[
f^\kappa(v, x) = \frac{n_{0a}}{(\pi n_{\text{in,a}})^{3/2}} \frac{\Gamma(\kappa_a + 1)}{\Gamma(\kappa_a - 1/2)} \left( 1 + \frac{v^2}{n_{\text{in,a}}} + \frac{q_{a} \phi(x)}{k_B T_a} \right)^{-(\kappa_a + 1)},
\]

in which \(n_{0a}\) is the unperturbed number density of \(a\) species; \(\Gamma\) is the gamma function; \(\kappa\) is the kappa parameter which for \(\kappa \to \infty\) corresponds to the Maxwellian case. Note that a slightly different kappa function from the above expression has been adopted and applied to study the acoustic solitons in two-component plasmas [53, 54]. Comparisons between two different kappa functions and the obtained results have also been made [25, 55]. As for the \(\beta\) species we adopt the more general nonthermal distribution with the following form.

\[
f^\beta(v, x) = \frac{n_{0\beta}}{(\pi n_{\text{in,\beta}})^{3/2}} \frac{1}{1 + (3a_1 \beta/2 + 15a_2 \beta/4) \exp \left( -v^2/v_{\text{in,\beta}}^2 + q_{\beta} \phi(x)/k_B T_\beta \right)} \left( 1 + a_1 \beta \left( \frac{v^2}{v_{\text{in,\beta}}} + \frac{q_{\beta} \phi(x)}{k_B T_\beta} \right) + a_2 \beta \left( \frac{v^2}{v_{\text{in,\beta}}} + \frac{q_{\beta} \phi(x)}{k_B T_\beta} \right)^2 \right).
\]

For \(a_1 = 0, a_2 = a\) and \(a_1 = -1/\kappa\), \(a_2 = 1/(2\kappa^2)\) the above expression may, respectively, recover the 3D Cairns et al nonthermal distribution and the slightly different nonthermal distribution proposed by Hau and Fu [40], while for \(a_1 = a_2 = 0\) it describes the Maxwellian distribution. Note that for uniform plasmas, i.e. \(\phi(x) = 0\), the distribution functions are only functions of velocity, in particular,
Figure 1. Profiles of two types of nonthermal velocity distribution functions. Panel (a) shows the 3D Cairns et al distribution in equation (8) for $a = 0.5$ (blue line), $a = 1$ (red line) and $a = 2$ (green line). Panel (b) shows the highly nonthermal distribution in equation (9) for $k' = 1$ (green solid line), $k' = 1.7$ (red solid line) and $k' = 5$ (blue solid line). For comparison, the kappa functions with $k = 1.7$ and $k = 5$ are shown by a red dashed line and a blue dashed line, respectively. The dotted curve in both panels is the Maxwellian distribution, which is the special case of $a = 0$ in the Cairns et al distribution and $k' \to \infty$ in the highly nonthermal distribution.

$$f^H(v) = \frac{n_0}{\left(\pi v_{th}^2\right)^{3/2}} \frac{1}{1 + (3a/2) + (15a/4)} \times \left[1 + a \frac{v^2}{v_{th}^2} + a_2 \frac{v^2}{v_{th}^2} \right] \exp\left(-\frac{v^2}{v_{th}^2}\right).$$

(6)

The corresponding thermal pressure for kappa distribution may be calculated as $p^H = n_0 k_B T [\kappa/\kappa - 3/2]$, implying that $\kappa > 3/2$. While for the more general nonthermal distribution the thermal pressure in uniform plasmas is

$$p^H = n_0 k_B T \left[1 + \frac{a_1 + 5a_2}{1 + (3a_1/2) + (15a_2/4)}\right].$$

(7)

Figures 1(a) and (b) show the profiles of the 3D Cairns et al distribution

$$f^C(v) = \frac{n_0}{\left(\pi v_{th}^2\right)^{3/2}} \frac{1}{1 + (15a/4)} \left[1 + a \frac{v^2}{v_{th}^2} \right] \exp\left(-\frac{v^2}{v_{th}^2}\right).$$

(8)

for various $a$ values, and the highly nonthermal function

$$f^\kappa(v) = \frac{n_0}{\left(\pi v_{th}^2\right)^{3/2}} \frac{1}{1 + 3/(8\kappa')} \left[1 - \frac{1}{\kappa'} \frac{v^2}{v_{th}^2} + \frac{1}{2\kappa'} \frac{v^2}{v_{th}^2} \right] \times \exp\left(-\frac{v^2}{v_{th}^2}\right).$$

(9)

for various $\kappa'$ values. As indicated, for small $\kappa'$ values the profile describes the nonmonotonic bump-on-tail distribution while for large $\kappa'$ values, say, $\kappa' > 10$, the kappa function is recovered. Note that both Cairns et al and the highly nonthermal distributions may describe nonmonotonic and bump-on-tail velocity distributions in contrast to the monotonic kappa distribution, and may suitably be termed as the highly nonthermal distribution in the present study. As shown in figure 1, the features for both distributions are not quantitatively the same; in particular, the high tail peak may possibly exceed the central distribution in the Cairns et al case, which is not considered in our previous study for three-component plasmas [42]. The combination of both distributions may give rise to more generalized functions which may flexibly be fitted with the observations or simulations by suitably adjusting the free parameters in equation (6).

Based on the distributions in equations (4) and (5), the number density, $n = \iiint f(v, x) dv$, for $\alpha$ species is

$$n_\alpha = n_0,\frac{A_{\alpha} k_B T}{1 + \frac{q_{\alpha} f}{k_B T}}^{1/2},$$

(10)

and for $\beta$ species is

$$n_\beta = n_{0, \beta} \left[1 + (A_{1, \beta} + 3A_{2, \beta}) \frac{q_{s, \beta} f}{k_B T} + A_{2, \beta} \frac{q_{s, \beta} f}{k_B T} \right] \exp\left(-\frac{q_{s, \beta} f}{k_B T}\right),$$

(11)

for which $A_{1, \beta} = a_{1, \beta} [1 + (3a_{1, \beta}/2) + (15a_{2, \beta}/4)]$ and $A_{2, \beta} = a_{2, \beta} [1 + (3a_{1, \beta}/2) + (15a_{2, \beta}/4)]$. Equations (1)–(3), (10) and (11) are the model equations used in this study. In addition, the quasi-neutrality condition for the unperturbed density is imposed, i.e. $q_{n, \alpha} n + \sum_{\beta} q_{n, \beta} n_{\alpha} + \sum_{\beta} q_{n, \beta} n_{\beta} = 0$.

It is important to obtain the characteristic speed of the model equations in uniform plasmas before analyzing the non-linear model. We assume that all perturbed quantities have the form $\exp[i(kx - \omega t)]$ and there is no background flow velocity. Linearization of the model equations then yields:

$$-\omega \delta n_c + k n_0 \delta c = 0,$$

(12)

$$m_c n_0, \omega \delta c = q_{n, 0, c} k \delta f,$$

(13)

$$k^2 \delta f = \delta \alpha \delta n_c,$$

(14)

$$\delta n_\alpha = n_0,\frac{1}{2k_x} - 1 \frac{q_{\alpha, s, \beta}}{k_B T} \delta f,$$

(15)

$$\delta n_\beta = n_{0, \beta} A_{1, \beta} + 3A_{2, \beta} - 1 \frac{q_{\beta, s, \beta}}{k_B T} \delta f.$$

(16)
Equations (12) and (13) yield \( \delta n_c = (k^2/\omega^2)(q_0 n_{0c} l_m c) \delta \phi \). The dispersion relation may be obtained from equations (14)–(16) as follows.

\[
\omega^2 = \left( \frac{\varepsilon_0 c_0^2 \varepsilon^2}{n_{0c}^2 Z_e^2 c^2 + m_c k_0 \varepsilon T_{eff}} \right)^{-1} = \frac{c_s^2}{\lambda_{D,eff} k^2 + 1},
\]

for which \( q = Zc \) has been used and \( c_{S,eff} = (k_0 T_{eff}/m_n c)^{1/2} \) and \( \lambda_{D,eff} = [\varepsilon_0 k_0^2 T_{eff}/(n_{0c}^2 c^2 \varepsilon^2)]^{1/2} \) are the effective sound speed and effective Debye length, respectively, with the effective temperature \( T_{eff} [56] \) being defined as

\[
T_{eff} = \left[ \sum_\alpha \left( 1 - \frac{1}{2k_{\alpha}} \right) Z^2_{\alpha} n_{0,\alpha} \frac{1}{n_{0c}} T_{\alpha} \right]^{-1} + \sum_\beta (1 - A_{1,\beta} - 3A_{2,\beta}) \left( \frac{Z^2_{\beta} n_{0,\beta} 1}{n_{0c} T_{\beta}} \right)^{-1}.
\]

The plasma frequency for cold species is simply \( \omega_{pe,c} = C_{S,eff}/\lambda_{D,eff} = [n_{0c} Z_e^2 \varepsilon^2/(\varepsilon_0 m_c)]^{1/2} \). Note that the effective temperature for multiple species defined in equation (18) differs from the earlier studies and has the advantage of yielding physically meaningful \( C_{S,eff} \) and \( \lambda_{D,eff} \) which will be used as the normalization constants in the following calculations. The correct definition for the effective temperature in equation (18) and the corresponding sound speed is important in that the model calculations would not then yield subsonic solitons as obtained in earlier studies [25, 30, 42, 57] which are not physically possible, since steady solitons require the balance between nonlinear steepening and dispersive effects in collisionless plasmas.

3. KdV equation

We now proceed to adopt the reductive perturbation technique [23] to derive the KdV equation from the model equations for describing the weakly nonlinear ESW. First, we introduce two stretched coordinates, \( \xi = e^{1/2}(x - u_0 t) \) and \( \tau = e^{3/2} t \) (where \( e \) is a smallness parameter and \( u_0 \) is the speed of ESW normalized by \( C_{S,eff} \)) with dimensionless variables, \( x' \equiv x/\lambda_{D,eff}, \quad t' \equiv t/\omega_{pe,c}, \quad \phi' \equiv \phi/(k_0 T_{eff} / l_c), \quad u'_c \equiv u_c / C_{S,eff}, \) and \( n'_c \equiv n_c / n_{0c} \) \((s = c, \alpha, \beta)\). The model equations (1)–(3), (10) and (11) then become

\[
e^\xi \frac{\partial \phi'}{\partial \xi} = u_0 \frac{\partial \phi'}{\partial \xi} + \frac{\partial}{\partial \xi} (n'_c u'_c) = 0,
\]

\[
e^\xi \frac{\partial \phi'}{\partial \tau} = u_0 \frac{\partial \phi'}{\partial \xi} + u'_c \frac{\partial \phi'}{\partial \xi} + Z_{e} \frac{\partial \phi'}{\partial \tau} = 0,
\]

\[
e^\xi \frac{\partial^2 \phi'}{\partial \xi^2} + \frac{1}{Z_e} \left( n'_c + \sum_\alpha R_{\alpha} a_{\alpha} + \sum_\beta R_{\beta} \phi \delta \right) = 0,
\]

\[
n'_c = R_{n,c} \left( 1 + \frac{1}{\kappa_a} Z_{e} \sigma \phi \right)^{-1/2},
\]

\[
n'_c = R_{n,c} \left[ 1 + (A_{1,\beta} + 3A_{2,\beta}) Z_{e} \sigma \phi \phi' + A_{2,\beta} Z_{e}^2 \sigma^2 \phi^2 \right] \exp(-Z_{e} \sigma \phi),
\]

for which \( R_{\alpha,b} = Z_c / C_{\alpha} \), \( R_{\alpha,b} = n_{\alpha,b} / n_{e,c} \), and \( \sigma_b = T_{\alpha,b} / T_0 \) \((s = \alpha, \beta)\). By expanding the variables \( n'_c, u'_c, \) and \( \phi' \) in terms of a smallness quantity \( e \), i.e. \( n'_c = 1 + e^{1/2} u_0 \phi + e^{1/2} u_0 \phi', \quad u'_c = e^{1/2} u_0 \phi + e^{1/2} u_0 \phi', \) and \( \phi' = e^{3/2} \phi + e^{3/2} \phi' \), and substituting the expanded variables into equations (19)–(23), we obtain the equations of the first order in \( e \) and the second order in \( \phi \), as follows.

\[
-e^\xi \frac{\partial n'_c}{\partial \xi} = 0,
\]

\[
-e^\xi \frac{\partial n'_c}{\partial \tau} - u_0 \frac{\partial n'_c}{\partial \xi} = 0,
\]

\[
-n'_c Z_e + b_1 \phi' = 0,
\]

\[
-e^\xi \frac{\partial \phi'}{\partial \xi} = u_0 \frac{\partial \phi'}{\partial \xi} + u'_c \frac{\partial \phi'}{\partial \xi} + Z_e \frac{\partial \phi'}{\partial \xi} = 0,
\]

\[
-e^\xi \frac{\partial \phi'}{\partial \tau} = u_0 \frac{\partial \phi'}{\partial \xi} + u'_c \frac{\partial \phi'}{\partial \xi} + Z_e \frac{\partial \phi'}{\partial \tau} = 0,
\]

\[
-e^\xi \frac{\partial^2 \phi'}{\partial \xi^2} + \frac{1}{Z_e} n'_c = 0,
\]

\[
b_1 = \left[ \sum_\alpha \left( 1 - \frac{1}{2k_{\alpha}} \right) R^2_{\alpha,\phi} R_{n,a,\sigma_a} \right] + \sum_\beta (1 - A_{2,\beta} - 3A_{2,\beta}) R^2_{\beta,\phi} R_{n,\beta,\sigma_b} = -1,
\]

\[
b_2 = \sum_\alpha \left( \frac{1}{2} - \frac{1}{8k_{\alpha}} \right) Z_e R^2_{\alpha,\phi} R_{n,\alpha,\sigma_a}^2 + \sum_\beta \left( \frac{1}{2} - A_{1,\beta} - 2A_{2,\beta} \right) Z_e R^2_{\beta,\phi} R_{n,\beta,\sigma_b}^2.
\]

Subject to the boundary conditions, \( n'_c = 1 \) and \( u'_c = \phi' = 0 \) as \( \xi \to \infty \), equations (24) and (25) may be solved for \( u'_c = Z_e \phi' / u_0 \) and \( n'_c = Z_e \phi' / u_0 \delta_S c_0 \) which result in \( u_0 = 1 \) in equation (26), i.e. the ESW propagates at the effective sound speed \( C_{S,eff} \). By using these identities and eliminating the second order variables in equations (27)–(29), the KdV equation may be obtained as

\[
e^\xi \frac{\partial \phi'}{\partial \xi} + c_1 \phi' \frac{\partial \phi'}{\partial \xi} + c_2 \frac{\partial^2 \phi'}{\partial \xi^2} = 0,
\]

for which \( c_1 = 3Z_e / 2 + b_2 \) and \( c_2 = 1/2 \). The solution for a stationary small-amplitude soliton can then be solved from

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equation (32) in a frame of reference moving with the speed 
\(v_0\) (normalized by \(C_{S,\text{eff}}\)) as

\[
\phi^{(1)} = \phi_n \sech \left( \frac{\zeta}{\Delta} \right),
\]

(33)

which satisfies the boundary conditions of 
\(\phi^{(1)} = d\phi^{(1)}/d\zeta = d^2\phi^{(1)}/d\zeta^2 = 0\) as \(|\zeta| \to \infty\), where \(\zeta\) is a transformed coordinate defined as \(\zeta = \xi - v_0 t\). In equation (33) the amplitude is \(\phi_n = 3v_0/c_1\) and the width is \(\Delta = (2/v_0)^{1/2}\) which requires \(v_0 > 0\). The polarity of \(\phi^{(1)}\) is thus determined by the sign of the coefficient

\[
c_1 = \frac{3}{2}Z_c + \sum_a \left( \frac{1}{2} - \frac{1}{8k_a^2} \right) Z_a R_{Z,a} R_{n,a} \sigma_a^2
+ \sum_{\beta} \left( \frac{1}{2} - A_{1,\beta} - 2A_{2,\beta} \right) Z_\beta R_{Z,\beta} R_{n,\beta} \sigma_\beta^2.
\]

(34)

in particular, \(c_1 > 0\) and \(c_1 < 0\) correspond to the solitons with positive \(\phi^{(1)} > 0\) and negative \(\phi^{(1)} < 0\) potentials, respectively. For comparison with the result from fully nonlinear calculations, we may write \(P_{Kaw} = c_1\).

### 4. Sagdeev potential

The fully nonlinear solutions for electrostatic acoustic solitons can be obtained by solving the dimensionless model equations in the stationary frame of reference, \(\zeta = x - M_0 t\), the result being

\[
-M_0 \frac{d n_c'}{d \zeta} + \frac{d}{d \zeta} \left( n_c' u_c' \right) = 0,
\]

(35)

\[
-M_0 \frac{d n_c'}{d \zeta} + u_c' \frac{d n_c'}{d \zeta} + Z_c \frac{d \phi'}{d \zeta} = 0,
\]

(36)

\[
\frac{d^2 \phi'}{d \zeta^2} = -\frac{1}{Z_c} \left( n_c' + \sum_{a} R_{Z,a} n_{a}' + \sum_{\beta} R_{Z,\beta} n_{\beta}' \right) = -\frac{dV(\phi')}{d\phi'}.
\]

(37)

for which the definitions for dimensionless variables are the same as in section 3. The Mach number \(M_0\) is defined as the moving speed of the ESW with respect to \(C_{S,\text{eff}}\), and \(V(\phi')\) is the Sagdeev potential. The expressions for \(n_a'\) and \(n_{\beta}'\) are given in equations (22) and (23). By imposing the boundary condition of \(n_c' = 1\) and \(u_c' = \phi' = 0\) as \(|\zeta| \to \infty\), the integration of equations (35) and (36) yields

\[
u_c' = M_0 \left( 1 - \frac{1}{n_c} \right)
\]

(38)

\[
n_c' = \left( 1 - \frac{2Z_c}{M_0^2} \phi' \right)^{-1/2}
\]

(39)

Equation (37) may be integrated as

\[
\frac{1}{2} \left( \frac{d\phi'}{d\zeta} \right)^2 + V(\phi') = 0,
\]

(40)

where

\[
V(\phi') = \frac{M_0^2}{Z_c} \left[ 1 - \left( 1 - 2Z_c f' M_0^2 \right)^{1/2} \right] + \sum_a \frac{1}{Z_c} R_{Z,a} \alpha_a \left( \frac{\kappa_a}{\kappa_a - 3/2} \right) \left[ 1 - \left( 1 + \frac{1}{\kappa_a} Z_c \sigma_c d\phi' \right)^{-\kappa_a + 3/2} \right] + \sum_{\beta} \frac{1}{Z_c} R_{Z,\beta} \alpha_\beta \left( 1 + A_{1,\beta} + 5A_{2,\beta} \right) \times \left[ 1 - \left( 1 + A_{1,\beta} + 5A_{2,\beta} Z_c \sigma_c d\phi' \right) \right] \left( 1 + A_{1,\beta} + 5A_{2,\beta} Z_c \sigma_c \phi' \right) \exp(-Z_c \sigma_c d\phi'),
\]

(41)

for which the condition of \(dV(\phi')/d\phi' = V(\phi') = 0\) at \(\phi' = 0\) has been used.

To form a soliton, the following conditions shall be satisfied: (i) \(d^2 V(\phi')/d\phi'^2 < 0\) at \(\phi' = 0\); (ii) a nonzero \(\phi_{\text{max}}\) (or \(\phi_{\text{min}}\)) exists for which \(V(\phi_{\text{max}}) = 0\) (or \(V(\phi_{\text{min}}) = 0\)), and \(V(\phi') < 0\) when \(0 < \phi' < \phi_{\text{max}}\) (or \(\phi_{\text{min}} < \phi' < 0\)); and (iii) the number density of cold species in equation (39) has to be real. The second derivative of \(V(\phi')\) with respect to \(\phi'\) is

\[
\frac{d^2 V(\phi')}{d\phi'^2} = \frac{1}{M_0^2} \left[ 1 - 2Z_c f' M_0^2 \right]^{-3/2} \left[ -\sum_a R_{Z,a} R_{n,a} \alpha_a \left( 1 - \frac{1}{2k_a} \right) \left( 1 + \frac{1}{\kappa_a} Z_c \sigma_c d\phi' \right)^{-\kappa_a - 1/2} \right.
- \sum_{\beta} R_{Z,\beta} R_{n,\beta} \alpha_\beta \left( 1 - A_{1,\beta} - 3A_{2,\beta} \right) \exp(-Z_c \sigma_c d\phi')
\times \left( 1 + A_{1,\beta} + 5A_{2,\beta} Z_c \sigma_c d\phi' \right) \left( 1 - A_{1,\beta} - 3A_{2,\beta} Z_c \sigma_c \phi' \right)^2 \left. \right] \left. \left( 1 + A_{1,\beta} + 5A_{2,\beta} Z_c \sigma_c \phi' \right) \right)^2,
\]

(42)

The first condition \(d^2 V/d\phi'^2 = 0\) then results in the constraint of \(M_0\) and its lower limit \(M_{0,\text{L}}\) as

\[
M_0 > \left[ \sum_a R_{Z,a} R_{n,a} \alpha_a \left( 1 - \frac{1}{2k_a} \right) + \sum_{\beta} R_{Z,\beta} R_{n,\beta} \alpha_\beta \left( 1 - A_{1,\beta} - 3A_{2,\beta} \right) \right]^{1/2} = M_{0,\text{L}} = 1.
\]

(43)

implying that the moving speed of nonlinear solitons has to be larger than the effective sound speed \(C_{S,\text{eff}}\). By imposing the third condition, i.e. \(\phi_{\text{crit}}\) being real, the threshold condition for the normalized electric potential \(\phi' \leq M_0^2/(2Z_c) = \phi_{\text{crit}}\), may be obtained from equation (39) and by replacing \(\phi'\) with \(\phi_{\text{crit}}\) in equation (41) we may obtain the upper limit value of \(M_0\).

The polarity condition for nonlinear solitons can be derived from the third derivative of \(V(\phi')\) at \(\phi' = 0\) which has the following form
\[ \frac{d^2V}{d\phi'^3} = z_c \left( 1 - \frac{z_c}{M_S} \phi' \right)^{3/2} \]

\[ + \sum_{\alpha} Z_{\alpha} R_{\alpha,0}^2 \sigma_{\alpha}^2 \left( 1 - \frac{1}{4\kappa_\alpha} \right) \left( 1 + \frac{z_c}{\kappa_\alpha} \phi' \right)^{-\kappa_\alpha / 3} \]

\[ + \sum_{\beta} Z_{\beta} R_{\beta,0}^2 \sigma_{\beta}^2 (1 - 2A_{1,\beta} - 4A_{2,\beta}) \exp(-Z_{\beta} \sigma_{\beta} \phi') \times \left( 1 + \frac{A_{1,\beta}}{2A_{1,\beta} - 4A_{2,\beta}} Z_{\beta} \sigma_{\beta} \phi' \right) \]

\[ + \frac{A_{2,\beta}}{1 - 2A_{1,\beta} - 4A_{2,\beta}} Z_{\beta}^2 \sigma_{\beta}^2 \phi'^2. \] (44)

By setting \( \phi' = 0 \) and \( M_S = M_{LL} = 1 \) in equation (44) and writing \( P_{SP} = (d^2V/d\phi'^3)_{\phi' = 0, M_{LL} = 1} \), the condition for the polarity of electrostatic solitons can be obtained as

\[ P_{SP} = 3Z_c + \sum_{\alpha} \left( 1 - \frac{1}{4\kappa_\alpha} \right) Z_{\alpha} R_{\alpha,0}^2 \sigma_{\alpha}^2 \]

\[ + \sum_{\beta} \left( 1 - 2A_{1,\beta} - 4A_{2,\beta} \right) Z_{\beta} R_{\beta,0}^2 \sigma_{\beta}^2. \] (45)

In particular, \( P_{SP} > 0 \) and \( P_{SP} < 0 \) correspond to the solitons with positive (\( \phi' > 0 \)) and negative (\( \phi' < 0 \)) potentials, respectively. A comparison between equation (34) and equation (45) shows that \( P_{SP} = 2A_{KdV} \), i.e. the criterion for the electric polarity of acoustic solitons derived from both the KdV and Sagdeev potential approaches is consistent. If we apply the condition to the electron–ion plasma with electrons being described by the kappa distribution, the condition for anomalous negative ion acoustic solitons is simply \( 2 + 1/(4\kappa^2) < 0 \), implying no real \( \kappa \) value to satisfy the condition [25]. It can further be shown that the more general nonthermal distribution including the Cairns et al and the highly nonthermal distributions proposed by Hau and Fu [40] may possibly lead to the formation of negative acoustic solitons [42].

5. Summary

In this paper we have obtained the general formulations for both weakly nonlinear and fully nonlinear electrostatic acoustic solitons in multicomponent nonthermal plasmas. In the model the cold species with finite inertia is described by the fluid model, and the hot components follow the kappa distribution and/or the newly proposed generalized nonthermal distributions which may recover various distribution functions such as the Cairns et al nonthermal distribution adopted in many earlier studies. In the formulation the charges for both cold and hot species are unspecified and the model can suitably be applied to various plasma systems such as the electron–positron–ion and dusty plasmas etc. The present study is an extension of our previous formulation for three-component plasmas [25, 42] and may also serve as a unified model for the existing studies of electrostatic solitary waves in two, three and four component plasmas with specified species. The present model may also recover the three models for three-component plasmas discussed in Chuang and Hau [42], respectively, for the parameter values of \( \alpha = \alpha_1, \alpha_2, \beta = 0 \) (model A), \( \alpha = 0, \beta = \beta_1, \beta_2 \) (model B), and \( \alpha = \alpha_1, \beta = \beta_1 \) (model C). Note that the correct definition for the effective temperature in multiple components of charged particles with different velocity distributions in equation (18) has yielded a physically meaningful effective sound speed and Debye length used as the normalization quantities in the calculations which are different from the previous definitions [25, 42]. As a result, in the KdV and Sagdeev potential analyses the propagation speeds of weakly and fully nonlinear solitons are, respectively, to be the effective sound speed and supersonic (greater than the effective sound speed). One important application of incorporating nonthermal effects in the formulation is due to its capability in producing soliton solutions with anomalous electric polarity. In this study the polarity conditions for both weakly and fully nonlinear solitons are derived, which are essentially the same and can be used to analyze the possible existence of anomalous solitons in general nonthermal multicomponent plasmas. In particular, equation (45) can easily be applied to ordinary electron–ion plasmas which shows that the model with nonthermal electrons being described by the kappa velocity distribution cannot yield anomalous solitons which, however, can be achieved by more general nonthermal distributions [25].

Acknowledgments

This research is supported by the Ministry of Science and Technology of the Republic of China under grant MOST-103-2111-M-008-015-MY3 to National Central University.

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