Complexity-like properties and parameter asymptotics of $L_q$-norms of Laguerre and Gegenbauer polynomials

Jesús S Dehesa$^{1,2,*}$ and Nahual Sobrino$^{3,4}$

1 Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, Granada 18071, Spain
2 Departamento de Física Atómica, Molecular y Nuclear, Universidad de Granada, Granada 18071, Spain
3 Donostia International Physics Center, Paseo Manuel de Lardizabal 4, E-20018 San Sebastián, Spain
4 Nano-Bio Spectroscopy Group and European Theoretical Spectroscopy Facility (ETSF), Departamento de Polímeros y Materiales Avanzados: Física, Química y Tecnología, Universidad del País Vasco UPV/EHU, Avenida de Tolosa 72, E-20018 San Sebastián, Spain

E-mail: dehesa@ugr.es and nahualcsc@dipc.org

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Abstract
The main monotonic statistical complexity-like measures of the Rakhmanov’s probability density associated to the hypergeometric orthogonal polynomials (HOPs) in a real continuous variable, each of them quantifying two configurational facets of spreading, are examined in this work beyond the Cramér–Rao one. The Fisher–Shannon and LMC López-Ruiz–Mancini–Calvet (LMC) complexity measures, which have two entropic components, are analytically expressed in terms of the degree and the orthogonality weight-function’s parameter(s) of the polynomials. The degree and parameter asymptotics of these two-fold spreading measures are shown for the parameter-dependent families of HOPs of Laguerre and Gegenbauer types. This is done by using the asymptotics of the Rényi and Shannon entropies, which are closely connected to the $L_q$-norms of these polynomials, when the weight-function’s parameter tends toward infinity. The degree and parameter asymptotics of these Laguerre and Gegenbauer algebraic norms control the radial and angular charge and momentum distributions of numerous relevant multidimensional physical systems with a spherically-symmetric quantum-mechanical potential in the high-energy (Rydberg) and high-dimensional (quasi-classical) states, respectively. This is because the corresponding states’ wavefunctions are expressed by means

* Author to whom any correspondence should be addressed.
of the Laguerre and Gegenbauer polynomials in both position and momentum spaces.

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1. Introduction

The quantification of the spreading of the hypergeometric orthogonal polynomials (HOPs) along the support interval \( \Lambda \subseteq \mathbb{R} \) is interesting per se in the theory of special functions and approximation theory, and because of their numerous applications in quantum mechanics and mathematical physics [1–6]. A relevant reason for the latter is that the HOPs control the physical solutions of the non-relativistic and relativistic wave equations of a great deal of relevant quantum systems (oscillator-like systems, hydrogenic atoms, . . .) [1, 7, 8]. Here we study the spreading measures of the real HOPs, \( \{p_n(x)\} \), orthogonal with respect to the weight function \( h^\alpha(x) \) on the support interval \( \Lambda \). These quantities are defined by the corresponding measures of the normalized-to-unity Rakhmanov’s probability density \( \rho_n(x) = p_n^2(x) h^\alpha(x) \). This density function governs the \((n \to \infty)\)-asymptotics of the ratio of two polynomials with consecutive orders [9], and describes the quantum-mechanical probability density of the bound stationary states of a great deal of quantum systems in one and many dimensions [1, 10–13]. Indeed, it happens e.g. that the wavefunctions for the bound states of a large family of non-relativistic quantum-mechanical potentials are controlled by the three canonical HOPs families of Hermite \( H_n(x) \), Laguerre \( L^{\alpha\beta}(x) \) and Jacobi \( P^{\alpha\beta}_n(x) \) types [1, 5]. So, the associated Rakhmanov density \( \rho_n(x) \) may be often interpreted as the charge and/or the matter density of single-particle quantum systems. Consequently, the spreading measures of the HOPs characterize different fundamental and/or experimentally measurable properties of physical and chemical systems.

Beyond the dispersion measures (the standard deviation and its extensions, the ordinary and central moments), the spreading measures of a given probability density have an entropy-like origin. Contrary to the dispersion ones, the entropic measures do not depend on any specific point of the density’s support, so that they quantify spreading facets qualitatively different from the ones given by the dispersion measures. Then, the entropic measures quantify the different facets of the extent of the density along its support in a much more appropriate manner.

The entropy-like measures, each quantifying a single spreading facet, are of local (Fisher information) or global (Rényi and Shannon entropies) character depending on whether they are very sensitive to the fluctuations of the density or not, respectively. The Fisher information \( F(\rho) \), which is the most familiar and relevant local entropic measure [14, 15], is a functional of the derivative of the density \( \rho(x) \). Then, it controls the localization of the density around its nodes, appropriately grasping the oscillatory nature of the density. This allows it to characterize a great diversity of scientific phenomena which are closely connected to the kinetic and Weizsäcker energies [16–18] of the quantum systems. The Rényi entropies \( R_q[\rho], q \neq 1, 2 \) [19, 20], which depend on a real parameter \( q \), and its limiting case \( q \to 1 \), the Shannon entropy \( S[\rho] \) [21, 22], are the most important global spreading measures. They are \( q \)-power functionals of the density, closely related to the algebraic \( L_q \)-norms of the involved HOPs. Then, they can describe many quantities of great scientific and technological interest, such as e.g. the thermodynamical entropy in the case of a thermal ensemble and the disequilibrium when \( q \to 1 \) and
2, respectively; moreover, they are the basic variables of the classical and quantum information theories [22–24].

The knowledge of the entropic measures of the HOPs has been recently reviewed [25, 26]. Therein, the analytical expressions for the Fisher information and the Rényi and Shannon entropies are given for the three canonical families of the real HOPs at all \( n \). The ones for the global entropies are not handy in the sense that they only provide algorithmic expressions to compute them in a symbolic way because they require the calculation of the combinatorial Bessel polynomials evaluated at the HOP expansion coefficients or some multivariate hypergeometric functions evaluated at unity for the Rényi cases, and the calculation of the logarithmic potential of the HOPs evaluated at their zeros for the Shannon case. For the most complicated situations (i.e. when the polynomial degree \( n \) is high), however, the degree asymptotics \( (n \to \infty) \) of the Rényi and Shannon entropies and the related weighted \( L_q \)-norms of the HOPs allows one to obtain simple, transparent and compact expressions.

In this work we consider the monotonic complexity-like measures of the HOPs [27], which are quantities composed by two or more entropic factors, so that they can simultaneously quantify two or more different configurational facets of the spread of the HOPs along the orthogonality interval. The idea in mind is to quantify the simplicity/complexity of the HOPs as simpler and as possible by means of a single measure. Three measures have been recently proposed [28], the Cramer–Rao, Fisher–Shannon and LMC (López-Ruiz–Mancini–Calvet) complexities, which were originally introduced in a quantum-physical context (see e.g. the reviews [29–35]). Up until now, however, it is only known [28] the explicit expression of the Cramér–Rao complexity for the three canonical families of HOPs at all \( n \) and the asymptotical values for the three HOPs families when \( n \to \infty \), assuming the weight function’s parameters to be fixed.

The main goal of this paper is to find the Fisher–Shannon and LMC complexities for the Laguerre and Gegenbauer orthonormal polynomials with an arbitrary fixed degree when the weight function’s parameter tends to infinity. This issue is theoretically relevant in the theory of special functions, and because of its direct applications to compute the physical entropies of quasi-classical or high dimensional states of multidimensional systems in quantum physics and quantum technologies [36–39]; the latter is because the radial and spherical components of the state’s wavefunctions are controlled by Laguerre and Gegenbauer polynomials with a parameter which depends on the space dimensionality of the systems, respectively. The Gegenbauer polynomials are well known [1, 5] to be a particular family of the Jacobi polynomials. We will use the recent methodology of Temme et al [36, 40] which is based on the weight-function’s parameter asymptotics of the (unweighted) \( L_q \)-norms of the HOPs with a fixed degree. It is worth remarking that we do not consider here the complexity-like measures of polynomials with varying weights (i.e. polynomials whose weight-function’s parameter does depend on the polynomial degree), which are also of great mathematical and physical interest [41–43].

This paper has the following structure. In section 2 we describe the basic monotonic complexity measures with two entropic components of the Rakhmanov probability density for the HOPs. In sections 3 and 4 we obtain the asymptotic behavior for the Fisher–Shannon and LMC complexities of the Laguerre polynomials when \( (n \to \infty; \text{ fixed } \alpha) \) and when \( (\alpha \to \infty; \text{ fixed } n) \) in a simple, compact form. In sections 5 and 6 we find the corresponding issue for the Gegenbauer polynomials. Finally, some concluding remarks are pointed out and a number of open related issues.
In this section we briefly describe the three basic complexity measures of the HOPs.

### 2. Complexity measures of Rakhmanov’s density of HOPs

In this section we briefly describe the three basic complexity measures of the HOPs \( \{ p_n(x) \} \) orthogonal with respect to the weight function \( h^0(x) \) on the interval \( \Lambda \subseteq \mathbb{R} \); namely, the Cramér–Rao, Fisher–Shannon and López–Ruiz–Mancini–Calbet (LMC) measures. They are defined \([28]\) by the corresponding complexity measures of the associated Rakhmanov’s probability density

\[
\rho_n(x) = p_n^2(x) h^0(x),
\]

where the polynomials \( \{ p_n(x) \} \) fulfill the orthogonality condition \([1, 5]\)

\[
\int_\Lambda p_m(x) p_n(x) h^0(x) \, dx = \kappa_n^0 \delta_{m,n}, \quad \deg p_n = n
\]

and the weight function \( h^0(x) \) on the support \((a, b)\) and the normalization constant \( \kappa_n^0 \) of the HOPs \( \{ p_n(x) \} \) considered in this work are given in table 1. Note that \( \kappa_n^0 = 1 \) for the orthonormal polynomials \( p_n(x) \) of Hermite \( H_n(x) \), Laguerre \( L_n^{(\alpha)}(x) \), Jacobi \( P_n^{(\alpha,\beta)}(x) \) and Gegenbauer \( C_n^{(\lambda)}(x) \) types; so that the relation between the orthogonal and orthonormal HOP’s is \( p_n(x) = \tilde{p}_n(x)(\kappa_n^0)^{1/2} \).

The Cramér–Rao complexity of the polynomial \( p_n(x) \) is given \([44–46]\) by

\[
C_{CR}[p_n] = F[p_n] \times V[p_n],
\]

where \( F[p_n] \) and \( V[p_n] \) are the Fisher information \([14, 15]\) and the variance of the Rakhmanov density \( \rho_n(x) \) associated to \( p_n(x) \), which are defined as

\[
F[p_n] = \int_\Lambda \frac{|p_n'(x)|^2}{\rho_n(x)} \, dx, \quad \text{and} \quad V[p_n] = \langle \Delta x \rangle^2 = \langle x^2 \rangle - \langle x \rangle^2,
\]

respectively, with the expectation value \( \langle x^k \rangle = \int_\Lambda x^k \rho_n(x) \, dx \) for \( k = 1, 2 \). Then, the Cramér–Rao complexity quantifies the gradient content (so, the pointwise concentration of the Rakhmanov probability over its support interval) of \( \rho_n(x) \) jointly with the spreading of the probability around the centroid.

The Fisher–Shannon complexity of the polynomial \( p_n(x) \) is given \([47, 48]\) by

\[
C_{FS}[p_n] = F[p_n] \times \frac{1}{2 \pi e} e^{2S[p_n]} = \frac{1}{2 \pi e} F[p_n] \times (C_S[p_n])^2,
\]

where the symbol \( S[p_n] \) denotes the Shannon–like entropic functional of the polynomial \( p_n(x) \).
\[
S[p_n] = \lim_{q \to 1} R_q[p_n] = -\int_{\Lambda} \rho_n(x) \log \rho_n(x) \, dx,
\]
(5)

which is the limiting case \( q \to 1 \) of the Rényi entropy of \( p_n(x) \) defined as

\[
R_q[p_n] = \frac{1}{1-q} \log W_q[p_n], \quad \text{being} \quad W_q[p_n] = \int_{\Lambda} [\rho_n(x)]^q \, dx
\]
(6)

the \( q \)-th-order entropic moment or weighted \( L_q \)-norm of the associated Rakhmanov density (1). The symbol \( \mathcal{L}_S[p_n] = e^{S[p_n]} \) denotes the Shannon entropic power or Shannon spreading length of the polynomial \( p_n(x) \). Note that the Fisher–Shannon complexity \( C_{FS}[p_n] \) estimates the gradient content of the Rakhmanov probability density \( \rho_n(x) \) associated to the polynomial \( p_n(x) \), together with its total extent along the support interval \( \Lambda \) of the orthogonality weight function \( h^p(x) \). In addition, we remark that, from (1) and (5), one has that the Shannon-like entropic functional can be expressed as

\[
S[p_n] = E[p_n] + I[p_n],
\]
(7)

where the symbols \( I[p_n] \) and \( E[p_n] \) denote the integral functional

\[
I[p_n] = -\int_{\Lambda} [p_n(x)]^2 h^p(x) \log h^p(x) \, dx
\]
(8)

and the Shannon entropy of the polynomial \( p_n(x) \),

\[
E[p_n] = -\int_{\Lambda} [p_n(x)]^2 h^p(x) \log [p_n(x)]^2 \, dx,
\]
(9)

respectively. Note that the Shannon entropy of the orthogonal and orthonormal polynomials are related by

\[
E[\hat{p}_n] = \frac{1}{\kappa_n^2} E[p_n] + \log \kappa_n^p,
\]
(10)

and the corresponding relation for the Shannon-like entropic functionals is

\[
S[\hat{p}_n] = E[\hat{p}_n] + I[\hat{p}_n] = \frac{1}{\kappa_n^2} S[p_n] + \log \kappa_n^p,
\]
(11)

because \( I[\hat{p}_n] = \frac{1}{\kappa_n^2} I[p_n] \).

The LMC complexity of the polynomial \( p_n(x) \) is defined [49] as

\[
C_{LMC}[p_n] = W_2[p_n] \times e^{S[p_n]} = W_2[p_n] \times \mathcal{L}_S[p_n],
\]
(12)

which quantifies the combined balance of the disequilibrium of the associated Rakhmanov density or deviation from uniformity (as given by the averaging density \( \langle \rho \rangle \) or second-order entropic moment \( W_2[p_n] \), which is a measure of disorder), and its total extent (as given by the Shannon entropic power \( \mathcal{L}_S[p_n] \), which is a measure of disorder). Note for mathematical convenience that the disequilibrium of the orthogonal and orthonormal polynomials are mutually related by

\[
W_2[\hat{p}_n] = \frac{1}{(\kappa_n^2)^2} W_2[p_n].
\]
(13)
Table 2. First order asymptotics for the entropy-like ($F$, $L_S$, $W_2$) and complexity-like ($C_{FS}$, $C_{LMC}$) measures of the orthonormal Laguerre polynomials $\hat{L}_n^{(\alpha)}(x)$, $\alpha > -1$, when $n \to \infty$ and $\alpha \to \infty$.

| Measure of $\hat{L}_n^{(\alpha)}(x)$ | $n \to \infty$ | $\alpha \to \infty$ |
|-------------------------------------|-----------------|-------------------|
| $F[\hat{L}_n^{(\alpha)}]$          | $2\alpha$      | $2\alpha + 1$    |
| $L_S[\hat{L}_n^{(\alpha)}]$        | $2\alpha$      | $\sqrt{\frac{2\alpha}{\pi}} e^{\alpha + \frac{1}{2}}$ |
| $W_2[\hat{L}_n^{(\alpha)}]$        | $\log e^{n^{1/2}}$ | $\alpha^{2n} - \frac{1}{2(n^{1/2} + 1)}$ |
| $C_{FS}[\hat{L}_n^{(\alpha)}]$     | $\frac{1}{\sqrt{e^2}} \log n$ | $\alpha^{2n} \left( \frac{e^{\alpha + 1/2}}{2\pi(e n + 1/2)} \right)$ |
| $C_{LMC}[\hat{L}_n^{(\alpha)}]$    | $\frac{1}{e^2} \log n$ | $\alpha^{2n} \left( \frac{e^{\alpha + 1/2}}{2\pi(e n + 1/2)} \right)$ |

These three (dimensionless) complexity measures of the HOPs polynomial $p_n(x)$ turn out (a) to grasp the combined balance of two different configurational facets of the associated Rakhmanov density, (b) to be bounded from below by unity (when $\rho_n(x)$ is a continuous density in $\mathbb{R}$ in the Cramér–Rao and Fisher–Shannon cases, and for any $\rho_n(x)$ in the LMC case), (c) to be minimum for the two extreme (or least complex) distributions which correspond to perfect order (i.e. the extremely localized Dirac delta distribution) and maximum disorder (associated to a uniform or highly flat distribution), and (d) to fulfill invariance properties under replication, translation and scaling transformation [50, 51].

Finally, the Cramér–Rao complexity $C_{CR}[p_n]$ has been explicitly found at all $n$ [28] for the three canonical HOPs families $\{p_n(x)\}$ in an analytical compact form, basically because the variance and Fisher information of their associated Rakhmanov densities are expressed in an analytically handy way. Such is not the case for the weighted $L_2$ norm nor for the Shannon-like entropic functional $S[p_n]$, so that the Fisher–Shannon and LMC complexity-like measures have not yet been analytically determined for all $n$, but only for very high $n$ in the Fisher–Shannon case; the latter is basically because of the strong degree asymptotics of Aptekarev et al [52–54] for the Shannon entropy of HOPs polynomials of Hermite [55, 56], Laguerre [57] and Jacobi [59] polynomials.

In this work, we extend the asymptotical knowledge of the Fisher–Shannon and LMC complexity-like measures of Laguerre and Gegenbauer polynomials for very high $n$ (LMC) and very high weight-function parameter (Fisher–Shannon, LMC). This is done in the following sections by use of both the degree asymptotics mentioned above and the parameter asymptotics of Temme et al [36, 40] for the weighted $L_2$-norm and the Shannon entropy of the Laguerre and Jacobi polynomials. Let us advance that the main results obtained in the next four sections are collected in tables 2 and 3 for the Laguerre and Gegenbauer polynomials, respectively.
In this section we obtain simple analytical expressions for the Fisher–Shannon complexity of the orthonormal Laguerre polynomials $C_n^{(\lambda)}(x)$, \( \lambda > -\frac{1}{2} \), \( \lambda \neq 0 \), when \( n \to \infty \) and \( \lambda \to \infty \).

### Table 3. First order asymptotics for the entropy-like (\( F, \mathcal{L}_S, W_2 \)) and complexity-like (\( C_{\text{FS}}, C_{\text{LMC}} \)) measures of the orthonormal Gegenbauer polynomials $C_n^{(\lambda)}(x), \lambda > \frac{1}{2}$.

| Measure of $C_n^{(\lambda)}(x)$ | \( n \to \infty \) | \( \lambda \to \infty \) |
|-------------------------------|-----------------|-----------------|
| \( F[\hat{C}_n^{(\lambda)}] \) | $\frac{4n^3}{\lambda^2 - \lambda - 4} \pi^3$ | $\lambda = \frac{3}{2}$ \( (4n + 2)\lambda \) |
| \( \mathcal{L}_S[\hat{C}_n^{(\lambda)}] \) | $\frac{2^2\pi}{\Gamma(\lambda - 1/2)^2}$ | otherwise |
| \( W_2[\hat{C}_n^{(\lambda)}] \) | $\log n$ | $\lambda = \frac{1}{2}$ |
| \( C_{\text{FS}}[\hat{C}_n^{(\lambda)}] \) | $\frac{2\pi n^3}{\lambda^2 - \lambda - 3/4} \pi^3$ | $\lambda = \frac{3}{2}$ |
| \( C_{\text{LMC}}[\hat{C}_n^{(\lambda)}] \) | $\frac{\pi^{2-3\lambda}}{\log n}$ | $\lambda = \frac{1}{2}$ |

### 3. Fisher–Shannon complexity of the Laguerre polynomials

In this section we obtain simple analytical expressions for the Fisher–Shannon complexity of the orthonormal Laguerre polynomials $\hat{L}_n^{(\alpha)}(x)$ in the two following extreme situations: \( (\alpha \to \infty; \text{fixed } n) \) and \( (n \to \infty; \text{fixed } \alpha) \). This quantity is defined (4) as

\[
C_{\text{FS}}[\hat{L}_n^{(\alpha)}] = F[\hat{L}_n^{(\alpha)}] \times \frac{1}{2\pi e} e^{2S[\hat{L}_n^{(\alpha)}]} = \frac{1}{2\pi e} F[\hat{L}_n^{(\alpha)}] \times (\mathcal{L}_S[\hat{L}_n^{(\alpha)}])^2, \tag{14}
\]

where the Fisher information has been shown to have the values [58]

\[
F[\hat{L}_n^{(\alpha)}] = \begin{cases} 
4n + 1, & \alpha = 0, \\
\frac{(2n + 1)\alpha + 1}{\alpha^2 - 1}, & \alpha > 1, \\
\infty, & \alpha \in [-1, 1], \alpha \neq 0,
\end{cases}
\tag{15}
\]

and the Shannon entropy power or Shannon spreading length $\mathcal{L}_S[\hat{L}_n^{(\alpha)}] = e^{2S[\hat{L}_n^{(\alpha)}]}$ whose explicit expression is unknown despite multiple efforts (see e.g. the reviews [11, 26]). According to (7),
the Shannon-like entropic functional $S[L_n^{(\alpha)}]$ is given by
\[
S[L_n^{(\alpha)}] = -\int_0^\infty \left[ L_n^{(\alpha)}(x) \right]^2 h_n^\alpha(x) \log \left[ \left[ L_n^{(\alpha)}(x) \right]^2 h_n^\alpha(x) \right] \, dx
\]
\[
= E[L_n^{(\alpha)}] + I[L_n^{(\alpha)}],
\]
with the integral functional $[57, 60]$
\[
I[L_n^{(\alpha)}] = -\int_0^\infty \left[ L_n^{(\alpha)}(x) \right]^2 h_n^\alpha(x) \log h_n^\alpha(x) \, dx = 2n + \alpha + 1 - \alpha \psi(\alpha + n + 1),
\]
(17)

(\text{where } \psi(x) = \Gamma'(x)/\Gamma(x) \text{ is the digamma function}) and the Shannon entropy of $L_n^{(\alpha)}(x)$ defined by
\[
E[L_n^{(\alpha)}] = -\int_0^\infty \left[ L_n^{(\alpha)}(x) \right]^2 h_n^\alpha(x) \log \left[ L_n^{(\alpha)}(x) \right]^2 \, dx.
\]
(18)

The only existing approach to calculate this quantity requires the logarithmic potential of the Laguerre polynomials evaluated at their zeros, which is not analytically handy $[11].$

Thus, the explicit expression of the Fisher–Shannon complexity of the Laguerre polynomials for generic values $(n, \alpha)$ is yet to be known. However, as shown below in this section and tabulated in table 2, there are two extremal situations where the value of this quantity can be expressed in a simple and transparent way: $(\alpha \to \infty; \text{fixed } n)$ and $(n \to \infty; \text{fixed } \alpha).$

3.1. Asymptotics $\alpha \to \infty$

To obtain the asymptotics $(\alpha \to \infty; \text{fixed } n)$ of the Fisher–Shannon complexity $C_{FS}[L_n^{(\alpha)}]$, given by (14), we first take into account from (15) that $F[L_n^{(\alpha)}] \sim \frac{2\alpha+1}{n}$ and then, we determine the asymptotics the Shannon-like integral functional (16) of the orthonormal Laguerre polynomials $L_n^{(\alpha)}(x).$ To find the asymptotical $(\alpha \to \infty; \text{fixed } n)$ value of $E[L_n^{(\alpha)}]$ we express, following (10), this quantity in terms of the corresponding one $E[L_n^{(\alpha)}]$ for the orthogonal polynomials
\[
E[L_n^{(\alpha)}] = \frac{1}{\kappa_{n,\alpha}} E[L_n^{(\alpha)}] + \log \kappa_{n,\alpha},
\]
and then we use the following asymptotical value for the Shannon entropy of orthogonal Laguerre polynomials $[40, 61]$
\[
E[L_n^{(\alpha)}] \sim -\sqrt{2\pi} \frac{(\alpha)}{(n-1)!} \alpha^{n+1/2} \log \alpha, \quad \alpha \to \infty,
\]
(20)

and for the normalization constant $\kappa_{n,\alpha}$ given in the table 1, which fulfills (keep in mind that $\Gamma(z) \sim e^{-z} z^{(2z-1)/2}$, see equation 5.11.3 of [5])
\[
\kappa_{n,\alpha} \sim \kappa_{n,\infty} \equiv \sqrt{2\pi} \frac{(\alpha)}{n!} \alpha^{n+1/2}, \quad \alpha \to \infty,
\]
(21)

to finally obtain that
\[
E[L_n^{(\alpha)}] = \log \left( \frac{\kappa_{n,\infty}}{\alpha^n} \right) = \log \left( \frac{\sqrt{2\pi(\alpha)}}{n!} \left( \frac{\alpha}{e} \right)^n \right) + O(\alpha^{-1}), \quad \alpha \to \infty.
\]
(22)

\[
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\]
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The corresponding asymptotics for the functional $I[\hat{L}_n^{(\alpha)}]$ is given, according to (17) and taking into account that $\psi(z) \sim \log z - \frac{1}{2}z$ for $z \to \infty$ (see equation 5.11.2 of [5]) and $\alpha \log(\alpha + n + 1) = \alpha \log(\alpha) + n + 1 + \mathcal{O}(\alpha^{-1})$, as

$$I[\hat{L}_n^{(\alpha)}] = \log \left( \frac{e^{\alpha}}{\alpha} \right)^n + n + \frac{1}{2} + \mathcal{O}(\alpha^{-1}), \quad \alpha \to \infty. \quad (23)$$

Then, from equations (16), (22) and (23), we have that the asymptotics ($\alpha \to \infty$; fixed $n$) of the Shannon entropy power of the Laguerre polynomials $\mathcal{L}_S[\hat{L}_n^{(\alpha)}]$ is given by

$$\mathcal{L}_S[\hat{L}_n^{(\alpha)}] = e^{S[\hat{L}_n^{(\alpha)}]} = \left( \frac{e^{\alpha}}{\alpha} \right)^n \frac{K_{n,\infty}}{\alpha^n} e^{\frac{\pi}{2}+\frac{1}{2}} + \mathcal{O}(\alpha^{-1/2}), \quad \alpha \to \infty. \quad (24)$$

Finally, according to equations (14), (15) and (24), we have that the Fisher–Shannon complexity of the Laguerre polynomials behaves as

$$C_{FS}[\hat{L}_n^{(\alpha)}] = \frac{2n+1}{(n!)^2} e^{2n} + \mathcal{O}(\alpha^{-1}), \quad \alpha \to \infty. \quad (25)$$

We observe that the first dominant term does not depend on the Laguerre parameter, indicating uniformity (perfect disorder) for the Rakhmanov probability. This happens because the two entropic components of structure-order (Fisher information) and disorder (Shannon entropic power) qualitatively cancel when $\alpha \to \infty$. Eventually, we can go further away by obtaining the second asymptotical term. This requires to improve the asymptotical behavior of the Shannon entropy $S[\hat{L}_n^{(\alpha)}]$ of the Laguerre polynomials, what it is a feasible task following the lines of [40, 61]. The latter may be an interesting task for the future, not only in the theory of orthogonal polynomials but also for its physical consequences. Indeed, the determination of the complexity of the charge distribution of the quasi-classical (i.e. high-dimensional) states of quantum systems with a spherically-symmetric potential boils down to the mathematical computation of the complexity of the Laguerre polynomials. This is because the radial eigenfunction of the quasi-classical states are controlled by Laguerre polynomials with a parameter $\alpha$ which linearly depends on the space dimensionality of the system [36–39].

### 3.2. Asymptotics $n \to \infty$

In addition, for completeness, let us briefly show that the asymptotics ($n \to \infty$; fixed $\alpha$) of the Laguerre polynomials is known [57] to behave as

$$C_{FS}[\hat{L}_n^{(\alpha)}] \sim \begin{cases} 
\left( \frac{8\pi}{e^4} \right) n^3, & \alpha = 0, \\
\frac{\alpha}{\alpha^2 - 1} \left( \frac{4\pi}{e^4} \right) n^3, & \alpha > 1.
\end{cases} \quad (26)$$

Basically, this is because the Fisher information is given by (15) and the Shannon-like entropic functional $S[\hat{L}_n^{(\alpha)}]$ has the non-trivial value [11]

$$S[\hat{L}_n^{(\alpha)}] = (\alpha + 1) \log n - \alpha \psi(\alpha + n + 1) - 1 + \log(2\pi) + \mathcal{O}(1), \quad n \to \infty,$$
so that the Shannon entropic power $\mathcal{L}_S[L_n^{(\alpha)}]$ fulfills that

$$\mathcal{L}_S[L_n^{(\alpha)}] \sim \frac{2\pi}{e} n, \quad n \rightarrow \infty. \quad (27)$$

Note from (26) that the Fisher–Shannon complexity of the Laguerre polynomials follows a growth scaling law $n^2$ when $n \rightarrow \infty$, because the Fisher and Shannon components combine constructively since they behave as $n$ and $n^2$ for $(n \rightarrow \infty; \text{fixed } \alpha)$, respectively. Interestingly, this is specially useful to explain the charge complexity of highly-excited (Rydberg) states of the multidimensional Coulomb and oscillator-type systems [26, 38, 62–64]. Basically, this is because the radial eigenfunctions of these multidimensional quantum systems are controlled by Laguerre polynomials [10, 11].

### 4. LMC complexity of the Laguerre polynomials

In this section we obtain simple and compact expressions in two extreme situations, $(\alpha \rightarrow \infty; \text{fixed } n)$ and $(n \rightarrow \infty; \text{fixed } \alpha)$, for the LMC complexity of the orthonormal Laguerre polynomials $L_n^{(\alpha)}(x)$. This quantity, according to (12), is given by

$$C_{\text{LMC}}[L_n^{(\alpha)}] = W_2[L_n^{(\alpha)}] \times \mathcal{L}_S[L_n^{(\alpha)}]. \quad (28)$$

The explicit expressions of this quantity at generic values of $n$ and $\alpha$ is not yet known, although there are non-handy analytical expressions for the second-order entropic moment $W_2[L_n^{(\alpha)}]$ and the Shannon entropic power $\mathcal{L}_S[L_n^{(\alpha)}]$ which allow one to calculate them in an algorithmically symbolic manner. In fact, the computation of $W_2[L_n^{(\alpha)}]$ requires [28, 57] the evaluation of the four-variate Lauricella function $F_{\lambda}^{(4)}(\frac{1}{\lambda}, \frac{1}{\lambda}, \frac{1}{\lambda}, \frac{1}{\lambda})$ or the computation of the multivariate Bessel polynomials of combinatorics evaluated at the expansion coefficients of the Laguerre polynomials; and the computation of the Shannon entropic power $\mathcal{L}_S[L_n^{(\alpha)}]$ requires [11] the evaluation of the logarithmic potential of the Laguerre polynomials at their zeros.

#### 4.1. Asymptotics $\alpha \rightarrow \infty$

To obtain the asymptotics $(\alpha \rightarrow \infty; \text{fixed } n)$ of the LMC complexity $C_{\text{LMC}}[L_n^{(\alpha)}]$ we begin with the asymptotical expression (24) of $\mathcal{L}_S[L_n^{(\alpha)}]$ already shown in the previous section. Let us now tackle the asymptotics for the second-order entropic moment $W_2[L_n^{(\alpha)}]$ given by

$$W_2[L_n^{(\alpha)}] = \int_0^{\infty} \left( [L_n^{(\alpha)}(x)]^2 h_n^{(\alpha)}(x) \right)^2 \, dx = \int_0^{\infty} x^{2\alpha} e^{-2\alpha [L_n^{(\alpha)}(x)]^4} \, dx. \quad (29)$$

Now, we use the recent methodology of Temme et al [40]. Let $\alpha, \lambda, \kappa,$ and $\mu$ be positive real numbers; then, the following Rényi-like functional for orthogonal Laguerre polynomials fulfills the asymptotics

$$\int_0^{\infty} x^\mu e^{-\lambda x} |L_n^{(\alpha)}(x)|^\kappa \, dx \sim \frac{\alpha^{\kappa} \Gamma(\mu)}{\lambda^{\kappa}(\kappa)!}, \quad \alpha \rightarrow \infty. \quad (30)$$

Then, with the values $\mu = 2\alpha + 1, \lambda = 2$ and $\kappa = 4$, this general asymptotical formula provides the required asymptotics for $W_2[L_n^{(\alpha)}]$:

$$W_2[L_n^{(\alpha)}] \sim \frac{1}{(k_n^{(\alpha)})^2} \frac{\alpha^{4\kappa} \Gamma(2\alpha + 1)}{2^{2\alpha+1}(n!)^2}, \quad \alpha \rightarrow \infty. \quad (31)$$
Using now the (previously given) asymptotical estimate for the gamma function together with equation (21), one finds

$$W_2[\hat{L}_n^{(\alpha)}] = \alpha^{2n} \left( \frac{1}{2(n!)^2 \sqrt{\pi \alpha}} + O(\alpha^{-3/2}) \right), \quad \alpha \to \infty. \quad (32)$$

Finally, the combination of equations (24), (28) and (32) lead us to the following asymptotical values of the LMC complexity of the Laguerre polynomials:

$$C_{\text{LMC}}[\hat{L}_n^{(\alpha)}] = \alpha^{2n} \left( \frac{\alpha^{\frac{n}{2} + 1/2}}{21/2^{2}\left(n!\right)^3} + O \left( \alpha^{-\frac{1}{2}} \right) \right), \quad \alpha \to \infty. \quad (33)$$

Note that this quantity behaves as $\alpha^{2n}$ when $\alpha \to \infty$ because its two order (entropic moment $W_2[\hat{L}_n^{(\alpha)}]$) and disorder (Shannon spreading length $L_S[L_n^{(\alpha)}]$) components contribute constructively as $(\alpha^{2n-1/2}, \alpha^{1/2})$ at first asymptotical order. In fact, this expression can be improved by using higher terms in the asymptotical expression (30) following the method of Temme et al [40]. This is relevant per se and because this quantity allows us to determine the corresponding statistical complexity of the high-dimensional states of both multidimensional hydrogenic and oscillator systems. The latter is because the Laguerre polynomials control the radial eigenfunctions of the high-dimensional states of these quantum systems [36–39] as previously mentioned.

4.2. Asymptotics $n \to \infty$

Let us now tackle the asymptotics ($n \to \infty$; fixed $\alpha$) of the LMC complexity of the Laguerre polynomials. Then, we take into account the asymptotical value (27) for the Shannon entropic power $L_S[\hat{L}_n^{(\alpha)}]$, and to find the corresponding asymptotics of $W_2[\hat{L}_n^{(\alpha)}]$ we use the recent asymptotics for the generalized weighted $L_q$-norms of Aptekarev et al [62] which, in particular, gives

$$W_2[\hat{L}_n^{(\alpha)}] \sim \frac{\log n + O(1)}{\pi^2 n}, \; n \to \infty. \quad (34)$$

Finally, according to (27), (28) and (34), we obtain the following asymptotics for the LMC complexity of orthonormal Laguerre polynomials

$$C_{\text{LMC}}[L_n^{(\alpha)}] = \frac{2}{\pi e} \log n + O(n^{-1}), \; n \to \infty. \quad (35)$$

Thus, the LMC complexity of the Laguerre polynomials follows a logarithmic growth scaling law at large degree $n$; basically, because its two entropic components ($W_2$, $L_S$) behave as $(\frac{\log n}{\pi^2 n}, n)$, respectively. This mathematical result allows us to compute the corresponding radial charge complexity for the Rydberg quantum states of the hydrogenic and harmonic systems, because the radial eigenfunctions of such states are controlled by the Laguerre polynomials [10, 11].

5. Fisher–Shannon complexity of the Gegenbauer polynomials

In this section we obtain the Fisher–Shannon complexity (4) of the orthonormal Gegenbauer polynomials $\hat{C}_n^{(\lambda)}(x)$, $\lambda > -\frac{1}{2}$ when ($\alpha \to \infty$; fixed $n$) and for ($n \to \infty$; fixed $\alpha$). This quantity
Fisher information of the Gegenbauer polynomials are found to be functional of the orthonormal Gegenbauer polynomials corresponding quantity \( F \). The explicit expression of the Fisher–Shannon complexity of the Gegenbauer polynomials for generic values \((n, \lambda)\) is unknown up until now, basically because the Shannon entropy is also not known despite many efforts [65, 66]. However, there are two extremal situations where the value of this quantity can be analytically expressed when \((\lambda \to \infty; \text{fixed } n)\) and when \((n \to \infty; \text{fixed } \lambda)\). The goal of this section is to obtain both the parameter and degree asymptotics in a compact way.

The Fisher information of the Gegenbauer polynomials \( F[\hat{C}_n^{(\lambda)}] \) can be obtained from the corresponding quantity \( F[P_n^{(\alpha, \beta)}] \) of the Jacobi polynomials \( P_n^{(\alpha, \beta)}(x) \), given \([58, 59]\) by

\[
F[P_n^{(\alpha, \beta)}] = \begin{cases} 
2n(n+1)(2n+1), & \alpha, \beta = 0, \\
\frac{2n + \alpha + \beta + 1}{4(n + \alpha + \beta - 1)} \left[ \frac{n^2}{\beta + 1} + n + (4n + 1)(n + \beta + 1) + \frac{(n+1)^2}{\beta - 1} \right], & \alpha = 0, \beta > 1, \\
\frac{2n + \alpha + \beta + 1}{4(n + \alpha + \beta - 1)} \left[ n(n + \alpha + \beta - 1) \left( \frac{n + \alpha}{\beta + 1} + 2 + \frac{n + \beta}{\alpha + 1} \right) \right] + (n + 1)(n + \alpha + \beta) \left( \frac{n + \alpha}{\beta - 1} + 2 + \frac{n + \beta}{\alpha - 1} \right), & \alpha, \beta > 1, \\
\infty, & \text{otherwise}.
\end{cases}
\]

(37)

From this expression and taking into account the relation of the orthogonal/orthonormal Gegenbauer polynomials and the Jacobi polynomials given as

\[
\hat{C}_n^{(\lambda)}(x) = (\kappa_n^{(G)})^{1/2} C_n^{(\lambda)}(x),
\]

(38)

\[
C_n^{(\lambda)}(x) = c_{n, \lambda} P_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(x) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(2\lambda)} \frac{\Gamma(n + 2\lambda)}{\Gamma(n + \lambda + \frac{1}{2})} P_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(x),
\]

(39)

together with the exact identity \( c_{n, \lambda} \left( \kappa_n^{(G)} / \kappa_n^{(G, \lambda - 1/2, \lambda - 1/2)} \right)^{1/2} = 1 \), we have that the values of the Fisher information of the Gegenbauer polynomials are found to be

\[
F[\hat{C}_n^{(\lambda)}] = F[P_n^{(\lambda - 1/2, \lambda - 1/2)}] = \begin{cases} 
2n(n+1)(2n+1), & \lambda = \frac{1}{2}, \\
\frac{2(n + \lambda)(2\lambda - 1)(1 + 2\lambda + 2n(n + 2\lambda))}{(2\lambda - 3)(1 + 2\lambda)} & \lambda > \frac{3}{2}, \\
\infty, & \text{otherwise}.
\end{cases}
\]

(40)

In addition, the Shannon entropic power \( L_s[\hat{C}_n^{(\lambda)}] = e^{s \hat{C}_n^{(\lambda)}} \) where the Shannon-like integral functional of the orthonormal Gegenbauer polynomials \( C_n^{(\lambda)}(x) \), according to (7), is given by
From expression (40) we can obtain the following asymptotics:

$$S \left[ \hat{C}_n^{(\lambda)} \right] = - \int_{-1}^{+1} \left[ \hat{C}_n^{(\lambda)}(x) \right]^2 h_G^x(x) \log \left\{ \left[ \hat{C}_n^{(\lambda)}(x) \right]^2 h_G^x(x) \right\} \, dx$$

$$= E \left[ \hat{C}_n^{(\lambda)} \right] + I \left[ \hat{C}_n^{(\lambda)} \right], \quad (41)$$

with the integral functional [60]

$$I \left[ \hat{C}_n^{(\lambda)} \right] = - \int_{-1}^{+1} \left[ \hat{C}_n^{(\lambda)}(x) \right]^2 h_G^x(x) \log h_G^x(x)$$

$$= \left( \lambda - \frac{1}{2} \right) \left( -2 \psi(n + 2 \lambda) + \frac{1}{n + \lambda} + 2 \log(2) + 2 \psi(n + \lambda) \right), \quad (42)$$

and the Shannon entropy of $\hat{C}_n^{(\lambda)}(x)$ is defined by

$$E \left[ \hat{C}_n^{(\lambda)} \right] = - \int_{-1}^{+1} \left[ \hat{C}_n^{(\lambda)}(x) \right]^2 h_G^x(x) \log \left[ \hat{C}_n^{(\lambda)}(x) \right]^2 \, dx. \quad (43)$$

The analytical determination of the latter quantity is a formidable task [65, 66]. Indeed, it has been calculated for integer values of the polynomial’s parameter only and in a form that is not easily manipulated analytically. However, we find below that they can be expressed in a simple and compact way in the two following extremal situations; the main results have been collected in table 3.

5.1. Asymptotics $n \to \infty$

From expression (40) we can obtain the following asymptotics ($n \to \infty$; fixed $\lambda$) behavior for the Fisher information of the Gegenbauer polynomials $F[\hat{C}_n^{(\lambda)}]$:

$$F \left[ \hat{C}_n^{(\lambda)} \right] = \begin{cases} 4n^3 + O(n^2), & \lambda = \frac{1}{2}, \\ \frac{2 \lambda}{\lambda^2 - \lambda - \frac{\pi}{4}} n^3 + O(n^2), & \lambda > \frac{3}{2}, \\ \infty, & \text{otherwise}. \end{cases} \quad (44)$$

The asymptotics of $\mathcal{L}_S[\hat{C}_n^{(\lambda)}]$ requires to find the asymptotics of the Shannon entropy-like functional $S \left[ \hat{C}_n^{(\lambda)} \right]$ which, according to (41), involves the asymptotics ($n \to \infty$; fixed $\lambda$) of the Shannon entropy $E \left[ \hat{C}_n^{(\lambda)} \right]$ and the integral functional $I \left[ \hat{C}_n^{(\lambda)} \right]$ given by (42). The Shannon entropy of $\hat{C}_n^{(\lambda)}(x)$ has the following degree asymptotical behavior [11, 52, 54]:

$$E(\hat{C}_n^{(\lambda)}) \equiv - \int_{-1}^{+1} h_\lambda(x) \left[ \hat{C}_n^{(\lambda)}(x) \right]^2 \log \left[ \hat{C}_n^{(\lambda)}(x) \right]^2 \, dx$$

$$= \log \pi + (1 - 2 \lambda) \log 2 - 1 + O(n^{-1}), \quad n \to \infty \quad (45)$$

for fixed $\lambda$ [11, 52]. Moreover, from (42) and the previously given asymptotical expressions for the gamma and digamma functions, we find that the functional $I \left[ \hat{C}_n^{(\lambda)} \right]$ behaves for fixed
\[ I \left[ \hat{C}_n^{(\lambda)} \right] = (2\lambda - 1) \log(2) + O(n^{-1}), \quad n \to \infty. \quad (46) \]

Therefore, from (41), (45) and (46) we find that the asymptotics for the Shannon-like functional of the Gegenbauer polynomials is

\[ S \left[ \hat{C}_n^{(\lambda)} \right] \sim E \left[ \hat{C}_n^{(\lambda)} \right] = \log(\pi) - 1, \quad n \to \infty \quad (47) \]

so that the Shannon entropy power has the behavior

\[ \mathcal{L}_S[\hat{C}_n^{(\lambda)}] \sim \frac{n^3}{e^2}, \quad n \to \infty. \quad (48) \]

Finally, taking into account (36), (44) and (48) we have that the Fisher–Shannon complexity for the orthonormal Gegenbauer polynomials has the expression

\[ C_{FS} \left[ \hat{C}_n^{(\lambda)} \right] = \begin{cases} 
\frac{2\pi}{e^2} n^3 + O(n^2), & \lambda = \frac{1}{2}, \\
\frac{\pi}{2e^2} \left( \frac{2\lambda - 1}{(\lambda^2 - \lambda - \frac{3}{4})^2} \right) n^3 + O(n^2), & \lambda > \frac{3}{2}, 
\end{cases} \quad (49) \]

in the limit \( n \to \infty \). Further terms can be obtained by improving the asymptotics (46) of the Shannon entropy \( E \left[ \hat{C}_n^{(\lambda)} \right] \) as previously indicated [65, 66]. Note that the Fisher–Shannon complexity of the Gegenbauer polynomials behaves dominantly according to the scaling law \( n^3 \) for large degrees \( n \); so, similarly to the Laguerre case (see (26)) but for different reasons. Indeed, the entropic Fisher and Shannon components behave according to laws \((n^3, \text{constant})\) and \((n, n^2)\) for the Gegenbauer and Laguerre cases, respectively. This indicates that when \( n \to \infty \), the gradient content is much higher for the Gegenbauer polynomials than for the Laguerre polynomials, while the disequilibrium (i.e. deviation from the uniform distribution) in the Gegenbauer case is much lower than in the Laguerre case for any fixed degree.

Finally, let us mention that expression (49) allows one to compute the corresponding radial momentum complexity for the Rydberg quantum states of the hydrogenic and harmonic systems, because the radial eigenfunctions of such states are controlled by the Gegenbauer polynomials [10, 11].

5.2. Asymptotics \( \lambda \to \infty \)

Let us now determine the LMC complexity \( C_{FS}[\hat{C}_n^{(\lambda)}] \) in the limit \( \lambda \to \infty \) with fixed degree \( n \). For this purpose we first make use of Temme et al.’s ideas [40] to derive the Shannon entropy of \( \hat{C}_n^{(\lambda)}(x) \) from the corresponding asymptotics of the \( \mathcal{N}_p \)-norm of the orthogonal Gegenbauer polynomials, defined as

\[ \mathcal{N}_p \left[ C_n^{(\lambda)} \right] = \int_{-1}^{1} (1 - x^2)^{\lambda - \frac{1}{2}} |C_n^{(\lambda)}|^p \, dx. \quad (50) \]

This quantity can be analytically estimated for \( \lambda \to \infty \) by taking into account the known relation [5]

\[ \lim_{\lambda \to \infty} \frac{C_n^{(\lambda)}(x)}{C_n^{(\lambda)}(1)} = x^n, \quad (51) \]
with

\[ C_n^{(1)}(1) = \frac{(n + 2\lambda - 1)!}{n!(2\lambda - 1)!}. \]  

Then, from (50) and (51) we have

\[ \mathcal{N}_p[C_n^{(\lambda)}] \sim \frac{n! \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} + n \right)}{\Gamma \left( 1 + \lambda + \frac{n}{2} \right)}. \]  

Now, according to equations (9) and (53), one has that the Shannon entropy of the orthogonal \( C_n^{(\lambda)}(x) \) in the current limit is given as

\[ E[C_n^{(\lambda)}] = 2 \frac{d}{dp} \mathcal{N}_p[C_n^{(\lambda)}]_{p=2} \sim 2n^2_{n,\lambda} \left( \log \frac{(n + 2\lambda - 1)!}{n!(2\lambda - 1)!} \right) + \frac{n}{2} \psi \left( \frac{2n + 1}{2} \right) - \frac{n}{2} \psi(n + 2\lambda + 1), \]

so that we can express the Shannon entropy of the orthonormal Gegenbauer polynomials as

\[ E[C_n^{(\lambda)}] \sim 2 \left( \log \frac{(n + 2\lambda - 1)!}{n!(2\lambda - 1)!} + \frac{n}{2} \psi \left( \frac{2n + 1}{2} \right) - \frac{n}{2} \psi(n + 2\lambda + 1) \right) \]

\[ \sim 2 \log \left( \frac{\lambda n^2}{n!} \right). \]

In addition, the integral functional \( I[C_n^{(\lambda)}] \) given by (42) behaves as

\[ I[C_n^{(\lambda)}] = \frac{1}{2} + n + O(\lambda^{-1}), \quad \lambda \to \infty. \]

Then, according to equations (41), (55) and (56), we find the following asymptotics for the Shannon-like functional of the Gegenbauer polynomials

\[ S[C_n^{(\lambda)}] \sim E[C_n^{(\lambda)}] \sim 2 \log \left( \frac{\lambda n^2}{n!} \right), \quad \lambda \to \infty, \]

so that the Shannon entropy power of Gegenbauer polynomials behaves as

\[ \mathcal{L}_S[C_n^{(\lambda)}] \sim \frac{2\lambda^2 n^2}{(n!)^2}, \quad \lambda \to \infty. \]

In addition we determine the asymptotics (\( \lambda \to \infty \), fixed \( n \)) for the Fisher information of the Gegenbauer polynomials \( F[C_n^{(\lambda)}] \) from (40), obtaining:

\[ F[C_n^{(\lambda)}] = (2 + 4n)\lambda + 2 + 4n + 6n^2 + O(\lambda^{-1}), \quad \lambda \to \infty. \]

Finally, the substitution of the last two quantities into equation (36) gives rise to the following asymptotics for the Fisher–Shannon complexity of the orthonormal Gegenbauer polynomials:

\[ C_{FS}[C_n^{(\lambda)}] \sim \frac{2^{4n}(2n + 1)}{(n!)^2 \pi e} \lambda^{4n+1}, \quad \lambda \to \infty. \]
Note that the Fisher–Shannon complexity of the Gegenbauer polynomials \( \hat{C}_n^{(\lambda)}(x) \) behaves dominantly according to the scaling law \( \lambda^{\alpha+1} \) for large values of the parameter \( \lambda \); so, very different to the Laguerre case (where this quantity is constant according to (25)). This is because the entropic Fisher and Shannon components behave according to laws \( (\lambda, \lambda^{\alpha}) \) and \( (\alpha^{-1}, \alpha) \) for the Gegenbauer and Laguerre cases with a given parameter \( \lambda(\alpha) \), respectively. This indicates that when the orthogonality weight-function’s parameter goes to infinity, the pointwise concentration around the polynomial nodes (as given by the Fisher information) linearly/inversely depends on the parameter in the Gegenbauer and Laguerre cases, respectively. And the disequilibrium/order of the Rakhmanov probability follows a growth scaling law of \( \lambda^n \) and \( \alpha \) types for the Gegenbauer and Laguerre polynomials, respectively.

Finally, let us mention that expression (60) allows one to compute (a) the corresponding radial momentum complexity for the high-dimensional quantum states of the hydrogenic systems, because the radial eigenfunctions of such states are controlled by the Gegenbauer polynomials [10, 11, 64] in momentum space, and (b) the corresponding angular momentum complexity for the high-dimensional quantum states of the hydrogenic and harmonic systems, because the angular eigenfunctions of such states are also controlled by the Gegenbauer polynomials.

6. LMC complexity of the Gegenbauer polynomials

From expression (12), the LMC complexity of the orthonormal Gegenbauer polynomials \( \hat{C}_n^{(\lambda)}(x), \lambda > -\frac{1}{2} \), is given by

\[
C_{\text{LMC}}[\hat{C}_n^{(\lambda)}] = W_2[\hat{C}_n^{(\lambda)}] \times L_S[\hat{C}_n^{(\lambda)}],
\]

where the second-order entropic moment \( W_2[\hat{C}_n^{(\lambda)}] \) is, according to (6), given by

\[
W_2[\hat{C}_n^{(\lambda)}] = \int_{-1}^{1+1} \left( \left[ \hat{C}_n^{(\lambda)}(x) \right]^2 \right)^2 \, dx
= \int_{-1}^{1+1} (1 - x^2)^{2\lambda-1} \left[ \hat{C}_n^{(\lambda)}(x) \right]^4 \, dx.
\]

The explicit expression of this quantity at generic values of \( n \) and \( \lambda \) has not yet been determined in an analytically handy way, because neither \( W_2[\hat{C}_n^{(\lambda)}] \) nor the spreading length \( L_S[\hat{C}_n^{(\lambda)}] \) are analytically known. In this section we obtain simple and compact analytical expressions for \( C_{\text{LMC}}[\hat{C}_n^{(\lambda)}] \) in the two following extremal situations: when \( (\lambda \to \infty; \text{fixed } n) \) and when \( (n \to \infty; \text{fixed } \lambda) \). They are briefly summarized in table 3.

6.1. Asymptotics \( n \to \infty \)

To obtain the LMC complexity \( C_{\text{LMC}}[\hat{C}_n^{(\lambda)}] \) in the limit \( (n \to \infty; \text{fixed } \lambda) \) we first realize that the asymptotical expression of \( L_S[\hat{C}_n^{(\lambda)}] \) has been already found in the previous section. To determine the remaining component, \( W_2[\hat{C}_n^{(\lambda)}] \), when \( n \to \infty \) we use theorem 3 of
Aptekarev et al [64], obtaining

\[ W_2[\hat{C}_n^{(\lambda)}] \sim \begin{cases} 
  n^{1-2\lambda}, & -\frac{1}{2} < \lambda < \frac{1}{2}, \\
  \log n, & \lambda = \frac{1}{2}, \\
  \frac{3}{2\pi^{3/2}} \frac{\Gamma\left(\lambda - \frac{1}{2}\right)}{\Gamma(\lambda)}, & \lambda > \frac{1}{2}
\end{cases} \tag{63} \]

in the limit \( n \to \infty \).

This expression jointly with (48) and (61) gives rise to the following asymptotical behavior \( (n \to \infty) \) of the LMC complexity of the orthonormal Gegenbauer polynomials \( C_n^{(\lambda)}(x) \):

\[ \mathcal{C}_{\text{LMC}}[\hat{C}_n^{(\lambda)}] \sim \begin{cases} 
  \frac{\pi^{21-2\lambda}}{e} n^{1-2\lambda}, & -\frac{1}{2} < \lambda < \frac{1}{2}, \\
  \frac{\pi}{e} \log n, & \lambda = \frac{1}{2}, \\
  \frac{2^{-2\lambda}}{e} \frac{3}{\sqrt{\pi}} \frac{\Gamma\left(\lambda - \frac{1}{2}\right)}{\Gamma(\lambda)}, & \lambda > \frac{1}{2}
\end{cases} \tag{64} \]

Interestingly, the LMC complexity of the Gegenbauer polynomials follows a logarithmic growth scaling law (so, similarly to the Laguerre case (34)) at large degree \( n \) only for \( \lambda = \frac{1}{2} \). Nevertheless, this behavior has a qualitatively different origin. Indeed, the two entropic components \( (W_2, \mathcal{L}_S) \) behave according to laws \((\log n, \text{constant})\) and \((\frac{\log n}{n^2}, n)\) for the Gegenbauer and Laguerre cases, respectively. This indicates that when \( n \to \infty \), the gradient content is much higher for the Gegenbauer polynomials than for the Laguerre polynomials, while the disequilibrium (i.e. deviation from the uniform distribution) has the opposite behavior: in the Gegenbauer case it is much lower than in the Laguerre case for any fixed large degree. Moreover, note that the LMC complexity exponentially grows as \( n^{1-2\lambda} \) for \(-\frac{1}{2} < \lambda < \frac{1}{2}\) and has an uniform behavior (perfect disorder: non-dependence on \( n \)) for \( \lambda > \frac{1}{2} \) when \( n \to \infty \).

Here again we remark that this mathematical result has relevant applications when we determine the spatial charge LMC complexity measures for the high-energy (Rydberg) states of hydrogenic and harmonic systems, and the total momentum LMC complexity measures for the high-energy (Rydberg) hydrogenic states.

### 6.2. Asymptotics \( \lambda \to \infty \)

To determine the LMC complexity \( \mathcal{C}_{\text{LMC}}[\hat{C}_n^{(\lambda)}] \) in the limit \( \lambda \to \infty \); fixed \( n \) we first note that its first component, the spreading length \( \mathcal{L}_S[\hat{C}_n^{(\lambda)}] \), has been already obtained in equation (58) above in subsection 5.2.

Let us now tackle the second component, namely the second-order entropic moment \( W_2[\hat{C}_n^{(\lambda)}] \) given by equation (62). We use the limiting relation (51) into (62), obtaining for the orthogonal Gegenbauer polynomials the value

\[ W_2[\hat{C}_n^{(\lambda)}] \sim \left[ C_n^{(\lambda)}(1) \right]^2 \int_{-1}^{+1} (1 - x^2)^{2\lambda - 2} x^{2n} \, dx \]

\[ = \frac{(n + \lambda - 1)!^2 (1 + (-1)^{2n}) \Gamma\left(\frac{1}{2} + 2n\right) \Gamma(2\lambda)}{n!^2 (2\lambda - 1)!^2 \frac{2\pi}{\sqrt{2}} 2\Gamma\left(\frac{1}{2} + 2n + 2\lambda\right)}. \tag{65} \]
Then, according to \( (13) \) one has the following asymptotics for the second-order entropic power of the orthonormal Gegenbauer polynomials:

\[
W_2[C_n^{(\lambda)}] = \frac{1}{(K_n^{(\lambda)})^2} W_2[C_n^{(\lambda)}] = \frac{\Gamma \left( \frac{\lambda}{2} + 2n \right)}{\sqrt{2\pi n^2}} \lambda^{\frac{\lambda}{2}} + O(\lambda^{-1/2}), \quad \lambda \to \infty.
\]

Finally, the combination of expressions \((58), (61)\) and \((66)\) lead to the asymptotical behavior

\[
\mathcal{C}_{\text{LMC}}[C_n^{(\lambda)}] = \frac{2^{\frac{\lambda}{2}+1} \Gamma \left( \frac{\lambda}{2} + 2n \right)}{\pi n^{1/2}} \lambda^{\frac{\lambda}{2} + 1} + O \left( \lambda^{\frac{\lambda}{2}} \right), \quad \lambda \to \infty
\]

for the LMC complexity of the (orthonormal) Gegenbauer polynomials. Note that the LMC complexity of the Gegenbauer polynomials \( C_n^{(\lambda)}(x) \) behaves dominantly according to the scaling law \( \lambda^{(n+1)/2} \) for large values of the parameter \( \lambda \); so, different to the Laguerre polynomials \( L_n^{(\alpha)}(x) \) (where this quantity behaves as \( \alpha^{2n} \); see \((33)\)). This is because the two entropic components \( W_2, L_2 \) behave according to laws \( (\lambda^{1/2}, \lambda^{n/2}) \) and \( (\alpha^{2n-1/2}, \alpha^{1/2}) \) for the Gegenbauer and Laguerre cases with a given polynomial degree, respectively. This indicates that when the orthogonality weight-function’s parameter goes to infinity, the disequilibrium/order (as given by the second-order entropic moment) depends on the parameter as \( \lambda^{1/2} \) and \( \alpha^{2n-1/2} \) in the Gegenbauer and Laguerre cases, respectively. And the disorder of the Rakhmanov probability (as given by the Shannon entropy power) follows a growth scaling law of \( \lambda^{n/2} \) and uniform types for Gegenbauer and Laguerre polynomials, respectively. Finally, let us remark that this mathematical result has relevant applications when we determine the spatial charge LMC complexity measures for the high-dimensional (quasi-classical) states of hydrogenic and harmonic systems, and the total momentum LMC complexity measures for the high-dimensional hydrogenic states.

### 7. Conclusions

In this work we investigate the notions of simplicity/complexity and order/disorder for the parameter-dependent HOPs of Laguerre and Gegenbauer types. This is done by means of the Fisher–Shannon and LMC complexity measures of the associated Rakhmanov probability density of such polynomials. Each of these quantities capture two configurational facets of the HOPs: the Shannon spreading length or entropy power of the polynomials (which quantifies the equilibrium/disorder of the Rakhmanov probability) and the deviation from equilibrium or disequilibrium/order (which is measured by the Fisher information and the second-order entropic moment in the Fisher–Shannon and LMC complexity measures, respectively).

We have determined the Fisher–Shannon and LMC complexities of the Laguerre and Gegenbauer polynomials in the two following asymptotics at first order: when \( (n \to \infty; \text{fixed polynomial’s parameter}) \) and when \( (\text{parameter} \to \infty; \text{fixed } n) \). We have found the following results. First, in the asymptotics \( (n \to \infty; \text{fixed parameter}) \) the Fisher–Shannon measure of both Laguerre and Gegenbauer polynomials follow a simple exponential power \( (n^3) \)-law. However, the LMC complexities of these two sets of polynomials with high degree have a similar logarithmic behavior only for the Gegenbauer parameter \( \lambda = 1/2 \), while the LMC measure of the Gegenbauer polynomials follows an exponential and constancy (i.e. it does not depend on \( n \)) behavior for \( \lambda < 1/2 \), and \( >1/2 \), respectively.

Second, in the asymptotics \( (n \to \infty; \text{fixed } n) \) the Fisher–Shannon measure of Laguerre polynomials \( L_n^{(\alpha)}(x) \) attains constancy (i.e. it does not depend on \( \alpha \)), while the LMC measure of such polynomials follow the power law \( \alpha^n \). Moreover, something similar happens for the
Gegenbauer polynomials $\hat{C}_n^{(\lambda)}(x)$ when ($\lambda \to \infty$; fixed $n$); namely, the Fisher–Shannon and LMC measures behave according to the power laws $\lambda^{n+1}$ and $\lambda^{(n+1)/2}$, respectively.

Finally, these different scaling laws can be understood by observing the contributions of the two entropic components of the complexity measures in each case. Particularly interesting is the constancy of the Fisher–Shannon complexity of the Laguerre polynomials $L_n^{(\alpha)}(x)$ when ($\alpha \to \infty$; fixed $n$); this is because the $\alpha$-dependence of Shannon and Fisher components of this measure mutually cancel, indicating uniformity (so, perfect disorder) since the Fisher–Shannon complexity does not depend on $\alpha$.

These mathematical results are interesting per se and because of their applications to compute the physical entropy and complexity measures of the charge and momentum distributions of the high-dimensional (quasi-classical) and high-energy (Rydberg) quantum states of the multidimensional atomic systems, such as e.g. the Coulomb and harmonic systems as previously pointed out. This should not be surprising because the charge (momentum) probability density of (e.g.) multidimensional hydrogenic and harmonic oscillator systems can be represented by the Rakhmanov density of the Laguerre and Gegenbauer polynomials in position (momentum) space, respectively.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

ORCID iDs

Jesús S Dehesa https://orcid.org/0000-0003-4397-9426

References

[1] Nikiforov A F and Uvarov V B 1988 Special Functions of Mathematical Physics (Basel: Birkhäuser)
[2] Temme N M 1996 Special Functions: An Introduction to the Classical Functions of Mathematical Physics (New York: Wiley)
[3] Andrews G E, Askey R and Roy R 1999 Special Functions (Encyclopedia for Mathematics and its Applications) vol 16 (Cambridge: Cambridge University Press)
[4] Ismail M E H 2005 Classical and Quantum Orthogonal Polynomials in One Variable (Encyclopedia for Mathematics and its Applications) (Cambridge: Cambridge University Press)
[5] Olver F W J, Lozier D W, Boisvert R F and Clark C W 2010 NIST Handbook of Mathematical Functions (Cambridge: Cambridge University Press)
[6] Koekoek R, Lesky P A and Swarttouw R F 2010 Hypergeometric Orthogonal Polynomials and Their q-Analogues (Berlin: Springer)
[7] Bagrov V G and Gitman D M 1990 Exact Solutions of Relativistic Wavefunctions (Dordrecht: Kluwer)
[8] Cooper F, Khare A and Sukhature U 2001 Supersymmetry in Quantum Mechanics (Singapore: World Scientific)
[55] Sánchez-Moreno P, Dehesa J S, Manzano D and Yáñez R J 2010 J. Comput. Appl. Math. 233 2136
[56] Aptekarev A, Dehesa J, Sánchez-Moreno P and Tulyakov D 2012 Contemp. Math. 578 19
[57] Sánchez-Moreno P, Manzano D and Dehesa J S 2011 J. Comput. Appl. Math. 235 1129
[58] Sánchez-Ruiz J and Dehesa J S 2005 J. Comput. Appl. Math. 182 150
[59] Guerrero A, Sánchez-Moreno P and Dehesa J S 2010 J. Phys. A: Math. Theor. 43 305203
[60] Sánchez-Ruiz J and Dehesa J S 2000 J. Comput. Appl. Math. 118 311
[61] Belega E D and Tulyakov D N 2017 Russ. Math. Surv. 72 965
[62] Aptekarev A I, Tulyakov D N, Toranzo I V and Dehesa J S 2016 Eur. Phys. J. B 89 85
[63] Dehesa J S, Toranzo I V and Puertas-Centeno D 2017 Int. J. Quantum Chem. 117 48
[64] Aptekarev A I, Belega E D and Dehesa J S 2021 J. Phys. A: Math. Theor. 54 035305
[65] Buyarov V S, López-Artés P, Martínez-Finkelshtein A and Assche W V 2000 J. Phys. A: Math. Gen. 33 6549
[66] de Vicent J I, Gandy S and Sánchez-Ruiz J 2007 J. Phys. A: Math. Theor. 40 8345