TYPE II BLOW UP FOR THE ENERGY SUPERCRITICAL WAVE EQUATION

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Abstract. We consider the non-linear focusing wave equation
\[ \partial_t u - \Delta u - u|u|^{p-1} = 0 \]
in large dimensions \( d \geq 11 \) and for radially symmetric data. For \( p > p(d) \) large enough in the energy super critical zone
\[ s_c = \frac{d}{2} - \frac{2}{p-1} > 1, \]
we exhibit a family of \( C^\infty \) finite time blow up solutions which concentrate a universal bubble
\[ u(t,x) \sim \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{r}{\lambda(t)}\right) \]
where \( Q \) is the soliton profile, at the quantized blow up rates:
\[ \lambda(t) \sim c_\alpha(T-t)^{\frac{2}{\alpha}}, \ \ell \in \mathbb{N}^*, \ \ell > \alpha = \alpha(d,p). \]
The blow up is of type II i.e all norms below scaling remain bounded
\[ \limsup_{t \to T} \|\nabla^s u(t), \nabla^{s-1} \partial_t u(t)\|_{L^2} < +\infty \text{ for } 1 \leq s < s_c. \]

Our analysis adapts the robust energy method developed for the study of energy critical bubbles [23], [14], [24], [25] and the analogous result for the energy supercritical NLS problem [15].

1. Introduction

1.1. The NLW problem. We study in this paper the focusing nonlinear wave equation:
\[
\begin{align*}
(NLW) \quad \left\{ \begin{array}{ll}
\partial_t u - \Delta u - u|u|^{p-1} = 0, \\
u|_{t=0} = u_0, \ \partial_t u|_{t=0} = u_1 \\
(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \ u(t,x) \in \mathbb{R}.
\end{array} \right.
\end{align*}
\]

This canonical dissipative model conserves the total energy
\[ E(u(t)) = \frac{1}{2} \int |\nabla u|^2 + |\partial_t u|^2 - \frac{1}{p+1} \int |u|^{p+1} = E(u_0). \]

It admits a scaling symmetry: if \( u(t,x) \) is a solution then so is \( u_\lambda(\lambda t, \lambda x) = \lambda^\frac{2}{p-1} u(\lambda t, \lambda x) \) for \( \lambda > 0 \). This scaling is an isometry of the homogeneous Sobolev critical space
\[ \|u_\lambda(\lambda t, \cdot), \partial_t (u_\lambda(\lambda t, \cdot))\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}} = \|u(\lambda t, \cdot), (\partial_t u)(\lambda t, \cdot)\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}} \]
for \( s_c = \frac{d}{2} - \frac{2}{p-1} \). We shall in this paper focus onto the energy super critical models:
\[ s_c > 1 \ \text{ i.e } \ p \geq 2^* - 1 = \frac{d+2}{d-2}, \ d \geq 3 \]
which are well posed in \( H^{s_c} \). If the nonlinearity is analytic
\[ p = 2q + 1, \ q \in \mathbb{N}^*, \]
then the flow propagates Sobolev regularity and there holds the blow up alternative: 

\[ T < +\infty \implies \lim_{t \uparrow T} \|u(t)\|_{H^s} = +\infty \text{ for } s > s_c. \]

1.2. **Blow up for the wave equation.** The question of singularity formation for (NLW) has attracted a considerable attention for the past thirty years since the pioneering works by John [6]. Most known results deal with the subcritical case \( s_c < 1 \). The recent works by Merle and Zaag [17, 18] gives in particular a complete description of the local singularity in this subcritical regime corresponding to so called type I blow up bubbles, and we refer to this monumental series of works for complete references on the history of the energy subcritical problem. Recently also, general upper bounds on the blow up rates have been obtained [8, 2].

The situation is much more poorly understood in the energy critical and super critical regime \( s_c \geq 1 \). In this case, a new stationary solution arises: the soliton profile \( Q \) which is the unique up to scaling radially symmetric solution to

\[ \Delta Q + Q^p = 0, \quad d \geq 3, \quad s_c \geq 1. \]

The first construction of blow up solution in the energy critical setting goes back to [11] in dimension 3 where blow up bubbles of the type

\[ u(t, r) \sim \frac{1}{\lambda(t)^{\frac{d}{2}}} Q\left( r \frac{r}{\lambda(t)} \right), \quad \lambda(t) \sim (T - t)^{\nu} \]

for all \( \nu > 0 \) large enough are constructed (the result is improved in an optimal way in [9]). This result is a by product of the approach developed for the 2-dimensional energy critical wave map problem in the seminal work [10]. A different approach is implemented in [5] in the continuation of the energy method developed by Merle, Raphaël and Rodnianski for the study of the energy critical wave map problem [23] and the energy critical Schrödinger map problem [14]. In particular Hillairet and Raphaël obtained in the energy critical case in dimension 4 blow up bubbles of the form

\[ u(t, r) \sim \frac{1}{\lambda(t)^{\frac{d}{2}}} Q\left( r \frac{r}{\lambda(t)} \right), \quad \lambda(t) \sim (T - t)e^{-\sqrt{\log(T - t)}}. \]

An essential difference between these two constructions is the rigidity in the law (1.4) with respect to the continuum of blow up speeds (1.3) which reflects the fact that all solutions corresponding to (1.4) are arbitrarily smooth, while the continuum (1.3) generically corresponds to the propagation of a singularity on the light cone. The full family of discrete blow up rates for \( C^\infty \) data is constructed in [25] for the energy critical heat flow and could be propagated to the energy critical wave equation as well.

In the energy super critical setting \( s_c > 1 \), and as is now standard for the heat equation, type I or ODE blow up solutions can be constructed which correspond to the trivial constant blow up profile. The existence and stability of these solutions is addressed in [1]. These solutions correspond to a complete blow up

\[ \lim_{t \to T} \left\| u(t), \partial_t u(t) \right\|_{H^1 \times L^2} \to +\infty. \]

In the case of the heat equation, another type of blow up solution was predicted in large dimension \( d \geq 11 \) and large nonlinearities \( p \geq p(d) \) in the pioneering work by Herrero and Velasquez [3]. These so called type II blow up bubbles are rigorously constructed in [21, 22] using the breakthrough approach developed by Matano and Merle [12, 13]. The collection of these works yields a complete classification of
the type II blow up scenario for the radially symmetric energy supercritical heat equation. The main restriction of these techniques however is the systematic use of the maximum principle which cannot be extended to the dispersive setting.

1.3. Statement of the result. In the breakthrough work [15], the authors fully revisit the construction of type II blow up bubbles and show how the energy critical approach developed in [23, 14, 24, 25] can be extended to the energy critical setting to construct type II blow up solutions for the energy supercritical NLS problem. Our main claim in this paper is that this analysis can be propagated to the wave equation to construct the first family of type II blow up bubbles in the energy supercritical setting.

Before stating the result, we need to introduce some numbers attached to the super critical numerology. Let \( d \geq 11 \) and let the Joseph-Lundgren exponent be

\[
p_{JL} = 1 + \frac{4}{d - 4 - 2\sqrt{d - 1}} \quad \text{for} \quad d \geq 11.
\]

Then for \( p > p_{JL} \), the soliton profile admits an asymptotic expansion

\[
Q(r) = \frac{c_\infty}{r^{\frac{p}{p-1}}} + \frac{a_1}{r^\gamma} + o\left(\frac{1}{r^\gamma}\right), \quad a_1 \neq 0,
\]

with

\[
c_\infty = \left[\frac{2}{p-1} \left( d - 2 - \frac{2}{p-1}\right) \right]^{\frac{1}{p-1}}, \quad \gamma = \frac{1}{2} (d - 2 - \sqrt{\triangle}) > 0
\]

and where

\[
\triangle = (d - 2)^2 - 4pc_\infty^{p-1} > 0 \quad \text{for} \quad p > p_{JL}.
\]

These numbers are essential in the description of type II blow up bubbles and we claim:

**Theorem 1.1** (Type II blow up for the energy super critical wave equation). Let \( d \geq 11 \) and a nonlinearity

\[
p = 2q + 1, \quad q \in \mathbb{N}^*, \quad p > p_{JL}.
\]

Let \( \gamma \) be given by (1.7) and define:

\[
\alpha = \gamma - \frac{2}{p-1}.
\]

Assume moreover:

\[
\left( \frac{d}{2} - \gamma \right) \notin \mathbb{N}
\]

Pick an integer

\[
\ell \in \mathbb{N} \quad \text{with} \quad \ell > \alpha,
\]

and an arbitrarily large Sobolev exponent

\[
s^+ \in \mathbb{N}, \quad s_+ \geq s(\ell) \to +\infty \quad \text{as} \quad \ell \to +\infty.
\]

Then there exists a radially symmetric initial data \((u_0, u_1) \in H^{s^+} \times H^{s^++1}(\mathbb{R}^d)\) such that the corresponding solution to (1.1) blows up in finite time \(0 < T < +\infty\) by concentrating the soliton profile:

\[
u(t, r) = \frac{1}{\lambda(t)^{\frac{p}{p-1}}} \left( Q + \varepsilon \right) \left( \frac{r}{\lambda(t)} \right)
\]
with:

(i) Blow up speed:
\[ \lambda(t) = c(u_0)(1 + o(1))(T - t)^\frac{\ell}{2}, \quad c(u_0) > 0; \]  
\[ (1.13) \]

(iii) Asymptotic stability above scaling:
\[ \lim_{t \uparrow T} \| \varepsilon(t, \cdot), \lambda(\partial_t u)(t, \cdot) \|_{\dot{H}^s \times \dot{H}^{s-1}} = 0 \quad \text{for all} \quad s_c < s \leq s_+; \]  
\[ (1.14) \]

(iv) Boundedness below scaling:
\[ \limsup_{t \uparrow T} \| u(t), \partial_t u(t) \|_{\dot{H}^s \times \dot{H}^{s-1}} < +\infty \quad \text{for all} \quad 1 \leq s < s_c; \]  
\[ (1.15) \]

(v) Behavior of the critical norms:
\[ \| u(t) \|_{\dot{H}^{s_c}} = \left[ c(d, p)\sqrt{\ell} + o_1(T)(1) \right] \sqrt{\log(T - t)}; \]  
\[ \limsup_{t \uparrow T} \| \partial_t u(t) \|_{\dot{H}^{S-1}} < +\infty. \]  
\[ (1.16) \]

Comments on Theorem 1.1

1. On the assumptions on \( p \). As in [15], the assumption \( (1.10) \) is generic but technical and avoids the presence of logarithmic losses in the sequence of weighted Hardy inequalities which we will use to close our energy estimates. Unlike the situation in the critical case [23, 14], we claim that these logarithms are irrelevant in our setting and in this sense the assumption \( (1.10) \) could be removed. The assumption \( p = 2q + 1 \) makes the nonlinearity analytic and hence \( C^\infty \) regularity is propagated by the flow. For a nonlinearity with limited regularity, given a large integer \( \ell \), a blow up solution satisfying \( (1.13) \) can be constructed for any \( p \geq p(\ell) \) large enough using the same methodology.

2. The manifold construction. Our construction relies on a soft Schauder type compactness argument. We however have a complete understanding of the system of ODE’s underlying the type II blow regime which clearly exhibits some explicit finite codimensional instability. The rigorous construction of the associated center stable manifold should be amenable with the techniques developed in this paper and will be addressed in a forthcoming work.

3. On quantization of blow up rates. The quantization of blow up rates \( (1.13) \) is verbatim the same like the one obtained in the case of the heat equation through a complete classification theorem in [22], see also [24]. The quantization is a consequence of the regularity and decay associated to our initial data which in particular can be chosen in \( C^\infty_c(\mathbb{R}^d) \).

The strength and robustness of our approach is that it relies first on the derivation of the universal system of ODE’s driving the blow up bubble which avoids any sort of matching procedure, second that the control of the flow is performed using energy estimates only as opposed to more commonly used but more delicate spectral estimates on the propagator associated to the linearized flow near \( Q \). For both these reasons, we expect that our analysis can be propagated to the non radial problem as well, this will be addressed in a forthcoming work.

Acknowledgment. The author is supported by the ERC advances grant BLOWDISOL. This paper is part of the author PhD, and I would like to thank my advisor P.
Raphaël for his guidance and advice during the preparation of this work.

**Notations:** We collect here the main notations and facts which are used all along the paper.

**Super critical numerology:** Given $d \geq 11$, $p > p_{JL}$, we let $\alpha$ and $\alpha_2$ be the roots of the polynomial $X^2 - (d - 2 - \frac{2}{p-1})X + 2(d - 2 - \frac{2}{p-1})$ satisfying: $\alpha < \alpha_2$. One can check that the condition $p > p_{JL}$ ensures the reality of $\alpha$ and $\alpha_2$, and that they are not equal (see Lemma A.1). This definition is coherent with the formula (1.9). We recall the following relation:

$$\alpha = \gamma - \frac{2}{p-1} > 2,$$

where $\gamma$ was defined in (1.7). We define

$$\begin{cases} k_0 := E[\frac{d}{2} - \gamma] > 1, \\ \delta_0 := \frac{d}{2} - \gamma - k_0, 0 < \delta_0 < 1. \end{cases}$$

(1.18)

Because we are assuming $\left(\frac{d}{2} - \gamma\right) \notin \mathbb{N}$. We will use many times the following relation:

$$d = 2\gamma + 2k_0 + 2\delta_0.$$  

(1.19)

We let

$$g := \min(\alpha, \alpha_2 - \alpha_1) - \epsilon > 0$$

(1.20)

and

$$g' := \min(g, 2, 1 + \delta_0 - \epsilon) > 0$$

(1.21)

be the two real numbers that will quantify some gain in the asymptotics of our objects later on. $\epsilon$ stands for a very small constant $0 < \epsilon \ll 1$ that can be chosen independently of the sequel. The presence of $-\epsilon$ and $1 + \delta_0$ is just a way to simplify the writing of results later on.

**Notations for the analysis:** For the sake of simplicity, we will use the following equivalent formulation for the focusing nonlinear wave equation (NLW):

$$(NLW) \begin{cases} \partial_t u = F(u), \\ u_{|t=0} = u_0 \end{cases}, (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, u(t, x) : \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}.$$  

(1.22)

We refer to the coordinates of a function $u$ as $u^{(1)}$ and $u^{(2)}$:

$$u = \begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix}.$$  

(1.23)

We let the expression $F$ be:

$$F(u) := \begin{pmatrix} u^{(3)} \\ \Delta u^{(1)} + f(u^{(1)}) \end{pmatrix}.$$  

(1.24)

Here $f$ stands for the non-linearity:

$$f(u) := |u|^{p-1}u.$$  

We make an abuse of notation by still denoting the stationary state introduced earlier by $Q$:

$$Q := \begin{pmatrix} Q \\\ 0 \end{pmatrix},$$

where we recall the definition of the entire part $E[x] \leq x < E[x] + 1$, $E(x) \in \mathbb{Z}$.
Given a large integer \( L \gg 1 \), we define the Sobolev exponent:

\[
s_L := k_0 + 1 + L. \tag{1.25}
\]

We will use the standard scalar product on \( L^2(\mathbb{R}^d) \):

\[
\langle u, v \rangle := \int_{\mathbb{R}^d} uv,
\]

which induces a canonical scalar product on \( L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \) for which we keep the same notation:

\[
\langle u, v \rangle := \int_{\mathbb{R}^d} u^{(1)} v^{(1)} + \int_{\mathbb{R}^d} u^{(2)} v^{(2)}.
\]

Let \( 0 < \lambda \), we denote the renormalized function by:

\[
u_{\lambda}(x) := \left( \frac{\lambda^{2p-1} u^{(1)}(\lambda y)}{\lambda^{p-1} + 1 u^{(2)}(\lambda y)} \right).
\]

The rescaled coordinates are then:

\[
u_{\lambda} := \left( \begin{array}{c} u^{(1)}_{\lambda} \\ u^{(2)}_{\lambda} \end{array} \right). \tag{1.26}
\]

We let the generator of the scaling be:

\[
\Lambda u := \left( \begin{array}{c} \Lambda^{(1)} u^{(1)} \\ \Lambda^{(2)} u^{(2)} \end{array} \right) := \left( \begin{array}{c} \left( \frac{2}{p-1} + y \cdot \nabla \right) u^{(1)} \\ \left( \frac{2}{p-1} + 1 + y \cdot \nabla \right) u^{(2)} \end{array} \right).
\]

We introduce the renormalized space variable:

\[
y := \frac{r}{\lambda}
\]

Given \( b_1 > 0 \), we define:

\[
B_0 := \frac{1}{b_1}, B_1 := B_0^{1+\eta} \tag{1.28}
\]

where \( \eta \) is a small number \( 0 < \eta \ll 1 \) which will be choosen later. We denote by

\[
\mathcal{B}^n(R) := \{ x = (x_1, ..., x_n) \in \mathbb{R}^n, \sum_{i=1}^d x_i^2 \leq R^2 \},
\]

\[
\mathcal{S}^n(R) := \{ x = (x_1, ..., x_n) \in \mathbb{R}^n, \sum_{i=1}^d x_i^2 = R^2 \},
\]

\[
\mathcal{C}^n(r,R) := \{ x = (x_1, ..., x_n) \in \mathbb{R}^n, \ r^2 \leq \sum_{i=1}^d x_i^2 \leq R^2 \},
\]

the standard closed ball, sphere and ring of the standard euclidian \( n \)-dimension real space. For \( u \in \mathbb{R}^n \) we denote the standard euclidian norm by:

\[
|u| := \left( \sum_{i=1}^n u_i^2 \right)^{\frac{1}{2}}.
\]

We introduce a generic radial, \( C^\infty \) cut-off function:

\[
\chi \equiv 1 \text{ on } \mathcal{B}^d(1), \quad \chi \equiv 0 \text{ on } \mathbb{R}^d \setminus \mathcal{B}^d(2). \tag{1.29}
\]

And we adjust the zone of the cut by denoting, for \( B > 0 \):

\[
\chi_B : y \mapsto \chi \left( \frac{y}{B} \right).
\]

We use the Kronecker delta notation:

\[
\delta_{i,j} := \left\{ \begin{array}{ll} 1 & \text{if } i = j, \\ 0 & \text{otherwise}. \end{array} \right. \tag{1.31}
\]
Linearized operator: The linearized operator near $Q$ of equation (1.22) is given by:

$$H\varepsilon := \begin{pmatrix} -\varepsilon^{(2)} & -pQ^{p-1}\varepsilon^{(1)} \\ -\Delta^{(1)} - pQ^{p-1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -\Delta - pQ^{p-1} \end{pmatrix} \varepsilon,$$

so that:

$$F(Q + \varepsilon) = -H\varepsilon + NL.$$  

Here $NL$ stands for the purely nonlinear term:

$$NL := \begin{pmatrix} 0 & f(Q + \varepsilon^{(1)}) - f(Q) - pQ^{p-1} \varepsilon^{(1)} \\ 0 \end{pmatrix}.$$  

We define:

$$\mathcal{L} := -\Delta - pQ^{p-1},$$

so that:

$$H = \begin{pmatrix} 0 & -1 \\ \mathcal{L} & 0 \end{pmatrix}.$$  

Eventually, we note the potential:

$$V := pQ^{p-1}.$$  

1.4. Strategy of the proof. We start by a summary of the main ideas involved in the proof of Theorem 1.1. We use the same notations as in the critical settings, and in particular the formulation of (NLW) via (1.22). The reasoning is simplified, as giving more precision in the various approximations, quantifications and constructions would add unnecessary complications.

(i) Construction of an approximate blow-up profile: We want to study the dynamics close to the family of stationary profiles $(Q_{\frac{1}{\lambda}})_{\lambda > 0}$. Therefore we look for a perturbation $T_1$ such that at first order the dynamics moves along the branch:

$$-\lambda_1 \Delta Q = \lambda_1 \frac{\partial}{\partial \lambda} \left( Q_{\frac{1}{\lambda}} \right)_{|\lambda = 1} \sim \partial_t (Q + b_1 T_1) = F(Q + b_1 T_1) \sim -b_1 H(T_1).$$  

So $T_1$ is given by: $T_1 = -H^{-1} \Delta Q$. When applying a scale change we get:

$$F \left( Q_{\frac{1}{\lambda}} + b_1 T_{1, \frac{1}{\lambda}} \right) \sim -b_1 \frac{\partial}{\partial \lambda} \left( Q_{\frac{1}{\lambda}} \right)_{|\lambda = \lambda_1}.$$  

Consequently, for the approximate solution $Q_{\frac{1}{\lambda_1}} + b_1(t) T_{1, \frac{1}{\lambda_1}}$, the evolution of the scaling is given by $\lambda_t = -b_1$. $b_1$ is supposed to be a small parameter. In the previous equations, we omitted the time evolution of $b_1 T_{1, \frac{1}{\lambda}}$, and the non linear terms $NL$ because we expect them to be of higher order. We now include them in (1.38) to look for a time evolution of $b_1$ given by higher order terms:

$$b_1, t T_{1, \frac{1}{\lambda}} + \frac{b_1^2}{\lambda} \Delta T_{1, \frac{1}{\lambda}} \sim \partial_t \left( b_1 T_{1, \frac{1}{\lambda}} \right) \sim NL.$$  

Surprisingly, as will be explained just afterwards, one has that $\Delta T_{1, \frac{1}{\lambda}} \sim (1 - \alpha) T_{1, \frac{1}{\lambda}}$, and that $NL$ is negligible compared to $\frac{b_1^2}{\lambda} T_{1, \frac{1}{\lambda}}$. So we end up with: $b_{1,t} = -\frac{1 - \alpha}{\lambda} b_1^2$. In short: we have a perturbation that at first order makes the solution move along the branch, and at second order influences its own time evolution, the error in this approximation being of third order.

Still in the same spirit, to allow additional movement along $T_1$ we let $T_2 =$. 

$-\mathcal{H}^{-1}(T_1)$, and do the same matching technique for the profile $Q_{b,\lambda} + b_1 T_{1,b} + b_2 T_{2,b}$ with $b_2$ of order $b_1^2$ since we already know that $\lambda b_1$ should be of this order. This gives: $\lambda_t = - b_1$, $b_{1,t} = \frac{1}{\lambda}((1 - \alpha)b_1^2 + b_2)$, and $b_{2,t} = -\frac{2}{\lambda^2} b_1 b_2$.

Letting $T_i = (-1)^i \mathcal{H}^{-i} \Lambda Q$ and considering a general approximate profile of the form $Q_{b,\lambda} := Q_{b,\lambda} + \sum_{i=1}^{L} b_i T_{i,b}$ gives in turn at first order:

$$
\begin{cases}
\lambda_t = - b_1, \\
b_{i,t} = \frac{1}{\lambda}(- (i - \alpha)b_1 b_1 + b_{1,t})
\end{cases}
$$

We point out at this stage that what we are doing is to build an approximate center manifold $\mathcal{M}_{ap} = \{(Q_{b,\lambda})_{b,\lambda}\}$ close to $(Q_{b,\lambda})_{\lambda > 0}$, tangent to the vector space $\text{Span}(T_i)$ being the generalized kernel of the operator $\mathcal{H}$. This manifold is determined by $L + 1$ parameters. Thanks to a matching technique we have an insight for the parameters behavior under the dynamics of (NLW): their time evolution should be given by (1.39). We now explain what is the matching technique.

(ii) Tail dynamics: When constructing the profiles $T_i$ one has:

$$T_i(r) \sim r^{-\gamma + i - (i \mod 2)} \text{, as } r \to +\infty.$$  

Hence for $i$ big enough it has an irrelevant growth at infinity. For this reason, to obtain a reasonable approximate profile we cut the $T_i$’s in the zone $y \sim B_1$ because it is the zone where $b_i T_i$ has the same size than $\Lambda Q$. The true approximate profile is in fact of the form $(Q + \chi b_i \sum_{i=1}^{L} T_i)_{b}$. Therefore, via scale change, most of the analysis is done in the zone $y \sim B_1$ where $b_i T_i$ behaves like (1.40) (as $T_i(B_1)$ and $B_1 \gg 1$). As we will see later:

$$\Lambda T_i \sim (i - \alpha) T_i \text{, as } r \to +\infty.$$  

This explains why we say that $\Lambda T_i \sim (i - \alpha) T_i$: their difference is of lower order in the relevant zone $y \sim B_1$. To truly understand that point, one has to read the analysis to see how the size of a profile in the zone $y \sim B_1$ is directly related to a polynomial size in terms of the main parameter $b_1$ for some importants norms of this profile.

This way, the system of ODE’s (1.39) is just computed on the asymptotics of the profiles. This heuristic has been extensively used in blow-up problems.

(iii) Approximate blow-up profiles: The natural question is: what type of special solutions does the approximate dynamics possess? For $\ell > \alpha$, one notices that there exists a solution $(\lambda^\ell(t), b^\ell(t))$ of (1.39) such that $\lambda^\ell(t)$ goes to 0 in finite time with asymptotics $\lambda^\ell \sim (T - t)^{\ell}$.

This means that $Q_{b^\ell(t), \lambda^\ell(t)}$ blows up in finite time. It is the approximate blow-up profile we are going to work with. We note that for this special solution, the parameters have the following size:

$$b_i^\ell \sim (b_1^\ell)^i \text{, } b_i^\ell \sim (b_1^\ell)^{i+1}$$

\[\text{with the convention } b_{i+1} = 0.\]

\[\text{Where mod stands for the Euclidean division } a = a \mod 2 + b, 0 \leq b \leq 1.\]

\[\text{See Lemma 2.15.}\]
We write the approximate dynamics under the form:

\[
F(Q_{b, \lambda}) = -b_1 \frac{\partial}{\partial x}(Q_{b, \lambda})|_{\lambda = \lambda} + \frac{1}{\lambda} \sum_{i=1}^{L}(-(i-\alpha)b_i + b_{i+1}) \frac{\partial}{\partial x}(Q_{\lambda, \lambda'})|_{\lambda' = \lambda} + \tilde{\psi}
\]

where \( \tilde{\psi} \) denote the remainder which is of higher order.

(ii) Obtention a blow-up solution for the full dynamics: We now want to prove that this special solution persists in the full (NLW) dynamics. We look for a true solution under the form \( u(t) = Q_{b(t), \lambda(t)} + \varepsilon(t) \). \( \varepsilon \) is the error term "orthogonal" to the manifold \( \mathcal{M}_{ap} \). \( b(t) = b^f(t) + b'(t) \) and \( \lambda(t) = \lambda^s(t) + \lambda'(t) \) are perturbation of the special trajectory \( (b^f(t), \lambda^s(t)) \), they represent the projection of \( u \) on the manifold \( \mathcal{M}_{ap} \). We hope to find a solution for which \( \varepsilon, b' \) and \( \lambda' \) stay small, so that the blow-up still happens.

To do that we use a bootstrap technique. We look at all the solutions starting in a neighborhood \( \mathcal{O} \) of the curve \( \mathcal{(Q_{b^f(t), \lambda^s(t)}), 0 \leq t \leq T} \subset \mathcal{M}_{ap} \), and we prove that at least one has to stay in this neighborhood. We write:

\[
\mathcal{O} = \mathcal{(Q_{b^f(t), \lambda^s(t)}), 0 \leq t \leq T} + \mathcal{O}_1 \times \mathcal{O}_2,
\]

meaning that \( U \in \mathcal{O} \) if and only if \( \varepsilon \in \mathcal{O}_1 \) and \( (\lambda', b') \in \mathcal{O}_2 \). To measure the size of the objects, as \( \mathcal{(1.41)} \) holds, \( b_1 \) will be the quantity of reference. Our analysis has three main steps.

Modulation: We compute the time evolution of the parameters \( \lambda \) and \( b \). We show an inequality of the type:

\[
\left| (b^e_i + b'^e_i)_t + \frac{1}{\lambda} ((i-\alpha)(b^e_i + b'^e_i)(b^e_i + b'^e_i) - (b^e_{i+1} + b'^e_{i+1})) \right| \leq \frac{1}{\lambda} (\| \varepsilon \|_{loc} + b_{1}^{L+3}).
\]

\( \| \varepsilon \|_{loc} \) comes from a local interaction term. It means that as long as \( \varepsilon \) stays in \( \mathcal{O}_1 \), it does not influence too much the evolution of the parameters. That is to say, as long as \( u(t) \in \mathcal{O} \), the dynamics of \( \lambda^s + \lambda' \) and \( b_e + b' \) are given at first orders by \( \mathcal{(1.39)} \).

Energy method: We want to estimate the size of the error term \( \varepsilon \). Its time evolution is given by:

\[
\lambda \partial_t \varepsilon = -H_{+} \varepsilon + NL + \psi + \tilde{\psi},
\]

where \( \tilde{\psi} \) is a corrective term as \( \varepsilon \) is orthogonal to \( \mathcal{M}_{ap} \). Under the smallness assumption \( (b', \lambda') \in \mathcal{O}_2 \), \( \psi \) can be estimated, and under the smallness assumption \( \varepsilon \in \mathcal{O}_1 \) so can be \( \tilde{\psi} \).

To measure the size of \( \varepsilon \) we introduce two norms. The first one at high regularity:

\[
\mathcal{E}_{sL} = \int \varepsilon^{(1)} L^{s_L} \varepsilon^{(1)} + \int \varepsilon^{(2)} L^{s_L-1} \varepsilon^{(2)}.
\]

This quantity is coercive, and in particular it controls the usual Sobolev norm (see Corollary \( \mathcal{(E.4)} \)):

\[
\mathcal{E}_{sL} \geq \| \varepsilon \|_{H^s_L \times H^{s-1}_L}^2.
\]
The second norm we use is at a low regularity level:

$$\mathcal{E}_\sigma = \int |\nabla \sigma \varepsilon(1)|^2 + \int |\nabla \sigma^{-1} \varepsilon(2)|^2$$

for \(\sigma > s_c\) slightly supercritical. The first one is the most essential for the analysis, because it is involved in all the estimates for the modulation, and because the error term has better estimates at this regularity level. We exhibit a Lyapunov type monotonicity formula for this term:

$$\frac{d}{dt} \left\{ \mathcal{E}_{s_L} \frac{b^2 L^{1+\delta}}{\lambda^{2(s_L-s_c)+1}} \right\} \lesssim b^2 L^{1+\delta}$$

for \(\delta = \delta(d, p, L) > 0\). This can be integrated to obtain:

$$\mathcal{E}_{s_L} \lesssim b^2 L^{1+\delta}.$$

When deriving this estimate, we need to control derivatives at a lower level to deal with the non-linear term. This is why we also aim at controlling \(\mathcal{E}_\sigma\). For this norm we exhibit a similar estimate:

$$\frac{d}{dt} \left\{ \mathcal{E}_\sigma \frac{b^2 L^{1+\delta'}}{\lambda^{2(s-L-s_c)+1}} \right\} \lesssim b^2 L^{1+\delta'}.$$

When integrated in time it gives:

$$\mathcal{E}_\sigma \lesssim b^2 L^{1+\delta''}.$$

When establishing the monotonicity formula for \(\mathcal{E}_{s_L}\), we also need to control a local term that cannot be estimated directly with \(\mathcal{E}_{s_L}\) and \(\mathcal{E}_\sigma\). This is done through the use of a third tool: a Morawetz type quantity whose time evolution controls this local term.

All these estimates show the following fact: as long as \(u \in O\), \(\varepsilon\) enjoy in fact better estimates: \(\varepsilon \in \frac{1}{2}O_1\).

**Conclusion a la Brouwer:** We recapitulate what we have shown so far in the analysis: as long as \(u(t) \in O\), the parameters evolve according to (1.39) plus a small perturbation, and the error enjoys a better estimate \(\varepsilon \in \frac{1}{2}O_1\). So a solution escapes from \(O\) if and only if \((b', \lambda')\) escape from \(O_2\). We look at the dynamics given by (1.39) in the set \((\lambda^c, b^c) + O_2\). It admits \((\lambda^c, b^c)\) as an hyperbolic equilibrium. From standard argument a la Brouwer, even perturbed this equilibrium should persist in some sense: there must exist at least one orbit staying forever in \((\lambda^c, b^c) + O_2\). This ends the proof of the existence of a true blow-up profile.

The paper is organized as follows. In section 2 we present the main tools to understand the linear operator \(H\). After that we are able to construct or primary approximate profile in Proposition 2.11. We then localize this profile in the zone \(y \leq B_1\) and estimate the remainder in the approximate dynamics in Proposition 2.13. We end this section by studying the special solutions of the approximate dynamics: the existence of special solutions for (1.39) is done in Lemma 2.15; their linear stability is studied in Lemma 2.16. In section 3 we implement our bootstrap method and state our main result of existence in Proposition 3.2. First we explain how to "project" the full (NLW) on the manifold of approximate solutions in Lemma 3.1. Then we estimate the impact of \(\varepsilon\) on the dynamics of the parameters \(b\) and \(\lambda\) by computing the modulations equations in Lemmas 3.4 and 3.6. In the second part
we estimate the error term $\varepsilon$. We start by deriving the monotonicity formula for the low Sobolev norm in Proposition 3.7, then we do it for the high regularity norm in Proposition 3.8 which is the main result of the section. We end the section with deriving a Morawetz identity to control a local term that appeared earlier in the computations in Proposition 3.10. In section 4 we end the proof of Proposition 3.2. We show that in fact better bounds hold for the error term $\varepsilon$ in Lemma 4.2. We then examine the dynamics for the parameters in Lemmas 4.4 and 4.5, we show the existence of a true blow-up solution by topological arguments. For the completeness of the result we study the behavior of Sobolev norms in subsection 4.2.

2. The linearized dynamics and the construction of the approximate blow-up profile

To understand the dynamics close to the 1-parameter family of ground states $(Q_{\lambda})_{\lambda>0}$ we study first its linearization. In this section we start by the presentation of appropriate notions, and technical lemmas about the linearized operator $H$. Once we have these tools, we are able to create an approximate blow up profile in the second part of this section.

2.1. The stationary state. From standard argument, all smooth radially symmetric solutions to:

$$-\Delta \phi - \phi^p = 0,$$

are dilates of a given normalized ground state profile:

$$\phi = Q_{\lambda}, \; \lambda > 0, \left\{ \begin{array}{l}
-\Delta Q - Q^p = 0 \\
Q(0) = 1.
\end{array} \right.$$

We will now recall the asymptotic behavior of $Q$. Most of them are known properties, see [27], [7].

**Lemma 2.1** (Asymptotic expansion of the ground state). Let $p > p_{JL}$ (defined in (1.5)). We recall that $g > 0$, $c_{\infty}$ and $\gamma$ are defined in (1.7) and (1.20), one has:

(i) Asymptotics at infinity:

$$\forall k \geq 0, \; \partial^k_y Q = \partial^k_y \left[ \frac{c_{\infty}}{y^{p-2}} + \frac{a_1}{y^{\gamma}} \right] + O \left( \frac{1}{y^{\gamma+g+k}} \right), \text{ as } y \to +\infty, \quad (2.1)$$

for a non null constant $a_1 \neq 0$.

(ii) Degeneracy:

$$\forall k \geq 0, \; \partial^k_y \Lambda^{(1)} Q = \partial^k_y \left[ \frac{c}{y^{\gamma}} \right] + O \left( \frac{1}{y^{\gamma+g+k}} \right), \text{ as } y \to +\infty, \quad (2.2)$$

for a non null constant $c \neq 0$.

(iii) Positivity of $L$:

$$L > \frac{\delta(p)}{y^2} > 0 \text{ on } H^1(\mathbb{R}^d), \quad (2.3)$$

(iv) Positivity of $\Lambda^{(1)}$: $\Lambda^{(1)} Q > 0.$

**Proof of lemma 2.1** Only the fact that $a_1 \neq 0$ is not proven in the references we quoted. To prove it, we have to enter in details in their proof of the asymptotic expansion. This is done in Lemma B.1 of Appendix B. □
2.2. factorization of $\mathcal{L}$. The positivity of $\Lambda^{(1)} Q > 0$ implies from a direct calculation the factorization of this operator.

**Lemma 2.2 (Factorization of $\mathcal{L}$).** Let:

$$W := \partial_y (\log(\Lambda^{(1)} Q)), \quad (2.5)$$

and define the first order operators on radial functions:

$$A : u \mapsto -\partial_y u + W u, \quad A^* : u \mapsto \frac{1}{y^{d-1}} \partial_y (y^{d-1} u) + W u. \quad (2.6)$$

Then we have:

$$\mathcal{L} = A^* A. \quad (2.7)$$

**Remark 2.3.** The adjunction is taken with respect to the radially symmetric Lebesgue measure:

$$\int_{y > 0} (Au) vy^{d-1} dy = \int_{y > 0} u(A^* v) vy^{d-1} dy.$$  

**Proof of Lemma 2.2** This factorization relies on the fact that $\Lambda^{(1)} Q > 0$, and then it is a standard property of Schrodinger operators with a non-vanishing zero. One can compute:

$$A^* Au = -\Delta u + \left(\frac{d-1}{y} W + \partial_y W + W^2\right) u.$$  

Then the result follows from:

$$\frac{d-1}{y} W + \partial_y W + W^2 = \frac{\Delta \Lambda^{(1)} Q}{\Lambda^{(1)} Q} - \frac{\mathcal{L} \Lambda^{(1)} Q - V \Lambda^{(1)} Q}{\Lambda^{(1)} Q} = -V,$$

where we used the fact that $\mathcal{L} \Lambda^{(1)} Q = 0$.  

We collect here the informations about the asymptotic behavior of the potentials $V$ and $W$ which will be used many times in the sequel. These results are a direct implication of the previous Lemma 2.1.

**Lemma 2.4. (Asymptotic behavior of the potentials:)** We have the following expansions:

(i) Asymptotics:

$$\partial_y^k V = \begin{cases} O(1) \text{ as } y \to 0 \\
\frac{c_k}{y^{1+k}} + O\left(\frac{1}{y^{2+k\alpha}}\right) \text{ as } y \to +\infty \end{cases}, \quad (2.8)$$

$$\partial_y^k W = \begin{cases} O(1) \text{ as } y \to 0 \\
\frac{c'_k}{y^{1+k}} + O\left(\frac{1}{y^{2+k\alpha}}\right) \text{ as } y \to +\infty \end{cases}, \quad (2.9)$$

with $c_k \neq 0$, $c'_k \neq 0$ and $c'_1 = -\gamma$.

(ii) Degeneracy:

$$\partial_y\left(\frac{d}{d\lambda}[(Q\lambda)^{p-1}]_{\lambda=1}\right) \equiv O\left(\frac{1}{y^{2+k\alpha+k}}\right) \text{ as } y \to +\infty. \quad (2.10)$$
2.3. Inverting \( H \) on radially symmetric functions. We first start by inverting \( L \). We are only considering radially symmetric functions, so \( \Delta = \partial_{yy} + (d - 1)\frac{\partial}{y} \), and we can apply basic results from ODE theory. We will do this thanks to the explicit knowledge of the kernel of \( L \). Indeed from the rewriting:

\[
A : u \mapsto -\Lambda^{(1)}Q \partial_y \left( \frac{u}{\Lambda^{(1)}Q} \right), \quad A^* : u \mapsto \frac{1}{y^{d-1}\Lambda^{(1)}Q} \partial_y (y^{d-1}\Lambda^{(1)}Qu),
\]

we note that:

\[
Au = 0 \text{ iff } u \in \text{Span}(\Lambda^{(1)}Q), \quad A^*u = 0 \text{ iff } u \in \text{Span} \left( \frac{1}{y^{d-1}\Lambda Q} \right).
\]

It implies that for radially symmetric functions:

\[
Lu = 0 \text{ iff } u \in \text{Span}(\Lambda^{(1)}Q, \Gamma),
\]

with:

\[
\Gamma(y) := \Lambda^{(1)}Q(y) \int_y^\infty \frac{dx}{x^{d-1}(\Lambda^{(1)}Q(x))^2};
\]

We already knew \( \Lambda^{(1)}Q \) was in the kernel of \( L \) since it is the tangent vector to the branch of stationary solutions \((Q_{\lambda})_{\lambda > 0}\). We just found the second vector in the kernel: \( \Gamma \). From the asymptotic behavior \((2.12)\) of \( \Lambda^{(1)}Q \), we deduce the following asymptotic for \( \Gamma \):

\[
\Gamma \sim y \to 0 - \frac{c}{y^{d-2}} \text{ and } \Gamma \sim y \to +\infty \frac{c'}{y^{d-2}},
\]

\( c \) and \( c' \) being two positive constants. Both results are obtained from \((2.14)\), with the fact that \( \Lambda^{(1)}Q > 0 \) and the asymptotic \((2.2)\) that implies:

\[
0 < \int_1^{+\infty} \frac{dx}{x^{d-1}(\Lambda^{(1)}Q)^2} \leq C \int_1^{+\infty} \frac{dx}{x^{d-1-2\gamma}} < +\infty,
\]

where we used the relation from \((1.7)\): \( d - 1 - 2\gamma > 1 \).

Now that we know the Green’s functions of \( L \) we can introduce the formal inverse:

\[
L^{-1}f := -\Gamma(y) \int_0^y f\Lambda^{(1)}Q \int_0^x dx + \Lambda^{(1)}Q(y) \int_0^y f\Gamma x^{d-1}dx.
\]

One can check that for \( f \) smooth and radial we have indeed \( L(L^{-1}f) = f \). As we do not have uniqueness for the equation \( Lu = f \), one may wonder if this definition is the "right" one. The answer is yes because this inverse has the good asymptotic behavior at the origin and \(+\infty\), see Lemma \ref{lem:2.7}. To compute easily the asymptotic, we will use the following computational lemma.

Lemma 2.5. (Inversion of \( L \)) Let \( f \) be a \( C^{\infty} \) radially symmetric function, and denote by \( u \) its inverse by \( L \): \( u = L^{-1}f \) given by \((2.16)\), then:

\[
Au = \frac{1}{y^{d-1}\Lambda^{(1)}Q} \int_0^y f\Lambda^{(1)}Q x^{d-1}dx, \quad u = -\Lambda^{(1)}Q \int_0^y \frac{Au\Gamma x^{d-1}}{\Lambda^{(1)}Q}dx.
\]

This lemma says that to compute \( u = L^{-1}f \), we can do it in a rather easy way in two times: first we compute \( Au \), then compute \( u \) knowing \( Au \).

Proof of Lemma \ref{lem:2.5} We compute from the definition of \( \Gamma \) \((2.14)\):

\[
A\Gamma = -\partial_y \Gamma + \frac{\partial_y (\Lambda^{(1)}Q)}{\Lambda Q} \Gamma = -\frac{1}{y^{d-1}\Lambda^{(1)}Q}.
\]
We therefore apply $A$ to the definition of $u$ given by (2.16), and using the cancellation $A(\Lambda Q) = 0$, we find:

$$Au = \frac{1}{y^{d-1}\Lambda(1)Q} \int_0^y f \Lambda(1)Qx^{d-1}dx.$$ 

which, together with the definition of $A$ (2.11) gives:

$$u = -\Lambda(1)Q \int_0^y \frac{Au}{\Lambda(1)Q}dx + c_u\Lambda Q,$$

c$_u$ being an integration constant. But from (2.16) we see that: $u = O(y^2)$ and $Au = O(y)$ as $y \to 0$. From that we deduce the nullity of the constant: $c_u = 0$, which establishes the formula.

Knowing how to invert $L$, we define the inverse of $H$ by the following formula:

$$H^{-1} := \begin{pmatrix} 0 & L^{-1} \\ -1 & 0 \end{pmatrix}.$$ 

2.4. Adapted derivatives, admissible and homogeneous functions. The usual derivatives, that is to say the $\nabla^k$ ones, are not fit for the study of (NLW) close to the family of ground states $(Q_\lambda)_{\lambda > 0}$, because they do not commute with the linearized operator $L$. In this subsection we describe the adapted derivatives we will use. The asymptotic behavior of the adapted derivatives of the profiles, at the origin and at infinity, is going to play an important role. The second significant property is the vectorial position (when a function $f$ has only one of its coordinate being non null).

For the profiles we will use later, these informations are contained in the notion of admissible function. Given a radial function $f(x) = f(|x|)$, we define the sequence:

$$f_k = A^k f$$

of adapted derivatives of $f$ by induction:

$$f_0 := f$$

and

$$f_{k+1} := \begin{cases} A f_k & \text{for } k \text{ even,} \\ A^* f_k & \text{for } k \text{ odd.} \end{cases}$$

Definition 2.6. (Admissible functions:) Let $p_1$ be a positive integer, $p_2$ be a real number, and $\iota$ an indice $\iota \in \{0;1\}$.

We say that a vector of functions $f = \begin{pmatrix} f^{(1)}(\cdot) \\ f^{(2)}(\cdot) \end{pmatrix}$ of two $C^\infty$ radially symmetric functions is admissible of degree $(p_1, p_2, \iota)$ if:

(i) $\iota$ is the position:

$$f = \begin{pmatrix} f^{(1)}(\cdot) \\ 0 \end{pmatrix}$$

(ie $f^{(2)} = 0$) if $\iota = 0$, and

$$f = \begin{pmatrix} 0 \\ f^{(2)}(\cdot) \end{pmatrix}$$

(ie $f^{(1)} = 0$) if $\iota = 1$. 

(ii) $p_1$ describes the behavior near 0:

$$\forall 2p \geq p_1, \quad f(y) = \sum_{k=p_1-\iota, \text{ even}}^{2p} c_k y^k + O(y^{2p+2}), \text{ as } y \to 0.$$ 

(iii) $p_2$ describes the behavior at infinity:

$$\forall k \in \mathbb{N}, \quad |f_k(y)| = O(y^{p_2-\gamma-\iota-k}) \text{ as } y \to +\infty.$$ 

The actions of $H$ and $H^{-1}$ on admissible functions enjoy these properties:
Lemma 2.7. (Action of $H$ and $H^{-1}$ on admissible functions.) Let $f$ be an admissible function of degree $(p_1,p_2,ι)$, with $p_2 \geq -1$ then:

(i) $∀i \geq 0$, $H^{-i}f$ is admissible of degree $(\max(p_1-i,ι),p_2-i,ι+ιmod2)$.
(ii) $∀i \geq 0$, $H^{-i}f$ is admissible of degree $(p_1+i,p_2+i,ι+ιmod2)$.

Proof of Lemma 2.7. We compute:

$H^{2k} = (-1)^k \begin{pmatrix} L^k & 0 \\ 0 & L^k \end{pmatrix}$, and $H^{2k+1} = (-1)^k \begin{pmatrix} 0 & -L^k \\ L^{k+1} & 0 \end{pmatrix}$. \hspace{1cm} (2.23)

So that the property we claim holds by a direct check at the definitions of adapted derivatives and admissible functions.

We are going to prove the property by induction on $i$. We will prove it for $i = 0$, the proof being the same for $i = 1$. We can suppose without loss of generality that $p_1$ is even. The property is true, of course, for $i = 0$. Suppose now it is true for $i$. If $i$ is even, then:

$H^{-(i+1)}f = H^{-1}H^{-i}f = \begin{pmatrix} 0 & L^{-1} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} (H^{-i}f)^{(1)} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -(H^{-i}f)^{(1)} \end{pmatrix}$.

The induction hypothesis for $H^{-i}f$ implies that $H^{-(i+1)}f$ is of degree $(p_1+i+1,p_2+i+1,ι)$. Suppose now $i$ is odd. Then we have:

$H^{-(i+1)}f = \begin{pmatrix} 0 & L^{-1} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ (H^{-i}f)^{(2)} \end{pmatrix} = \begin{pmatrix} L^{-1}(H^{-i}f)^{(2)} \\ 0 \end{pmatrix}$.

We write $u = L^{-1}(H^{-i}f)^{(2)}$. We have from the induction hypothesis:

$(H^{-i}f)^{(2)} = \sum_{k=p_1+i-1, k \text{ even}}^{2p} c_k y^k + O(y^{2p+2})$, as $y \to 0$.

From (2.16) one can see the gain:

$u = \sum_{k=p_1+i+1, k \text{ even}}^{2p} c_k y^k + O(y^{2p+2})$, as $y \to 0$,

and since $ι(H^{-(i+1)}f) = 0$, we get $p_1(H^{-(i+1)}f) = p_1 + 1$.

From the induction hypothesis for $H^{-i}f$, and the relation $u_k = (H^{-i}f)^{(2)}_{k-2}$ for $k \geq 2$, the asymptotic (2.22) at $+∞$ for $u$ is true for $k \geq 2$. One only needs to check the asymptotic at $+∞$ for $k = 0$ and $k = 1$. We use the computational Lemma 2.5

$Au = \frac{1}{y^{2-ι}\Lambda^{(1)}Q} \int_0^y (H^{-i}f)^{(2)}\Lambda^{(1)}Qx^{d-1}dx = O \left(\frac{1}{y^{2-ι}} \int_0^y x^{p_2+i-1-2γ+d-1}dx\right) = O(y^{p_2+i-γ})$,

where we used the asymptotic (2.2) of $\Lambda^{(1)}Q$. Indeed the integral in the right hand side is divergent from:

$p_2 + i - 1 - 2γ + d = p_2 + i + \sqrt{Δ} + 1 > 0$.

We then do the same for $u$:

$u = -\Lambda^{(1)}Q \int_0^y \frac{Au}{\Lambda^{(1)}Q} dx = O \left(y^{-γ} \int_0^y x^{p_2+i-γ+γ}dx\right) = O(y^{p_2+i+1-γ})$,

and from $ι(H^{-i}f) = 0$ we deduce $p_2(H^{-i}f) = p_2 + i + 1$. □
Then:

Let profiles given by:

Let us now assume that definitions of the degree.

Proof of Lemma 2.8. From the degenerescence (2.2) and the fact that \( \Lambda Q = 0 \), \( \Lambda Q \) is admissible of degree \((0, 0, 0)\). Hence due to the properties of the action of \( \Lambda^{-1} \) on admissible functions, the previous Lemma 2.7 we get that \( T_i \) is admissible of degree \((i, i, \text{imod} 2)\).

To prove the lemma about the \( \Theta_i \)'s we will proceed by induction. The asymptotic behavior of the solitary wave (2.2) ensures that the property is true for \( \Theta_0 = \Lambda(\Lambda Q) + \alpha \Lambda Q \). For \( i \) odd we have:

\[
\Theta_i = \begin{pmatrix} 0 \\ \Lambda(\Lambda Q) - (i - \alpha)T_i^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ -((\Lambda(1) + 1)T_{i-1}^{(1)} - (i - 1 + 1 - \alpha)T_{i-1}^{(1)}) \end{pmatrix} = \begin{pmatrix} 0 \\ -\Theta_{i-1}^{(1)} \end{pmatrix}.
\]

So if the property is true for \( i \) even, it is true for \( i + 1 \) from a direct check at the definition of the degree.

Let us now assume that \( i \) is even, \( i \geq 2 \). We compute the following relation:

\[
\mathcal{L}(\Lambda(1)u) = 2\mathcal{L}u + \Lambda(1)\mathcal{L}u + (2V + y, \nabla V)u.
\]

The asymptotic behavior of the potential, Lemma 2.4 implies the improved decay:

\[
2V + y, \nabla V = O\left(\frac{1}{y^{2+\alpha}}\right).
\]

We then compute:

\[
\mathcal{L}(\Theta_i^{(1)}) = -\Theta_i^{(1)} + (2V + y, \nabla V)T_i^{(1)}.
\]

The induction hypothesis, together with the decay property of the potential and the degree of \( T_i \) give that \( H\Theta_i \) is of degree \((i - 1, i - 1 - g', 1)\). As \( 0 < g' \leq 2 \) we have that \( p_2(H\Theta_i) = i - 1 - g' \geq -1 \) and we can apply the inversion Lemma 2.7 about admissible functions: \( H^{-1}(H\Theta_i) \) is of degree \((i, i - g', 0)\). One has...
\( L^{-1}L(\Theta_i) = \Theta_i + a\Lambda^{(1)}Q + b\Gamma \), with \( a \) and \( b \) two integration constants. As \( \Theta_i(y) \to 0 \), \( L^{-1}L(\Theta_i) \to 0 \), \( \Lambda^{(1)}Q(y) \to c > 0 \) and \( \Gamma(y) \to +\infty \) one deduces \( a = b = 0 \). This means that \( \Theta_i = L^{-1}L(\Theta_i) \) is of degree \((i, i - g', 0)\). □

In the following, we will have to deal with polynomial functions of the coefficients \( b_i \). Knowing in advance that \( b_i \approx b_i' \) for the approximate blow-up profile\(^5\), we have that \( \prod b_i^i \approx b_i^\sum iJ_i \). Given a \( L \)-tuple \( J \) of integers, we define:

\[
|J|_1 = \sum_{i=1}^{L} J_i, \quad \text{and} \quad |J|_2 = \sum_{i=1}^{L} iJ_i. \tag{2.30}
\]

**Definition 2.9 (Homogeneous functions).** \( b \) denotes a \( L \)-tuple \((b_i)_{1\leq i \leq L}\). \( p_1 \) is an integer, \( p_2 \) is a real number, \( \iota \) is an indice \( \iota \in \{0; 1\} \), and \( p_3 \) is an integer.

We say that a function \( S(b, y) \) is homogeneous of degree \((p_1, p_2, \iota, p_3)\) if it can be written as a finite sum:

\[
S = \sum_{J \in \mathcal{J}, \ |J|_2 = p_3} \left( \prod_{i=1}^{L} b_i^i S_J(y) \right),
\]

\( \# \mathcal{J} < +\infty \), where for each \( J \), \( S_J \) is an admissible function of degree \((p_1, p_2, \iota)\).

Because of the asymptotics of the potential \( W \), see (2.24), asking that \( \mathcal{A}^k f \) behave like \( y^{-\gamma+k+p_2} \) at infinity is equivalent to say that \( \partial_y^k f \) behaves the same way. As a consequence, the asymptotics can be multiplied, derived etc... which is the object of the following computational lemma. It is a straightforward application of Lemma C.1 from the appendix.

**Lemma 2.10 (Calculus on homogeneous functions.)** Let \( f = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ 0 \end{pmatrix} \) be homogeneous of degree\(^6\) \((p_1, p_2, 0, p_3)\) and \((p_1', p_2', 0, p_3')\) \((p_1 \text{ and } p_1' \text{ even})\). Then:

(i) Multiplication: the product \( fg := \begin{pmatrix} fg_1 \\ 0 \end{pmatrix} \) is an homogeneous profile of degree \((p_1 + p_1', p_2 + p_2' - \gamma, 0, p_3 + p_3')\).

(ii) Multiplication by the potentials involved in the analysis: \( fQ^k := \begin{pmatrix} fQ^k_1 \\ 0 \end{pmatrix} \) is an homogeneous profile of degree \((p_1, p_2 - k\frac{\gamma}{p-1}, 0, p_3)\).

### 2.5. Slowly modulated blow profiles and growing tails.

We have displayed previously all the tools we needed to construct an approximate blow up profile. We do this in two steps. First, we construct an approximate blow-up profile that generates an approximate blow up locally around the origin, but far away nonetheless it is irrelevant because it has polynomial growth (Proposition 2.14). Secondarily we cut this profile in a relevant zone to avoid the preceding problems (Proposition 2.13). This cutting procedure creates additional error terms which will be estimated.

To manipulate the topological properties of the dynamics we will make use of the

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5see Lemma 2.15

6we just state the result for \( \iota = 0 \) as in (NLW) the nonlinearity only acts on the first coordinate.
following adapted norms for $k \in \mathbb{N}$:

$$
\| u \|_k^2 = \| u_{k_0+1+k} \|_{L^2}^2 + \| u_{k_0+k} \|_{L^2}^2 = \int u_{k_0+1+k}(t) \, \text{d}t + \int u_{k_0+k}(t) \, \text{d}t,
$$

involving the $k$-th adapted derivative of $u$ defined in (2.19). We will also use local norms:

$$
\| u \|_{k, y \leq M}^2 = \| u_{k_0+1+k} \|_{L^2(|y|\leq M)}^2 + \| u_{k_0+k} \|_{L^2(|y|\leq M)}^2.
$$

As the scale $\lambda$ of our solution is changing with time, we want to work with the appropriate space variable $y = r_\lambda$. The appropriate renormalized time is:

$$
s(t) = s_0 + \int_{t_0}^t \frac{1}{\lambda(\tau)} \, d\tau.
$$

Let $u$ be a solution of (NLW) on the time interval $[0, T]$, and $\lambda : [0, T] \to \mathbb{R}_+$ be a $C^1$ function. We define the associated renormalized solution by:

$$
v(y, s) = u_{\lambda(t)}(y, t).
$$

The time evolution of $v$ is then given by:

$$
\partial_s v = F(v) + \frac{\lambda_s}{\lambda} \Lambda v.
$$

It is often easier to work with this renormalized flow.

In the next proposition we state the existence of a primary blow up profile. This construction is related to the so-called center manifold in the finite dimensional case. The idea is to construct a manifold, tangent to the vector space of the generalized kernel of the linearized operator at the point $Q_\lambda$, displaying a special dynamics. At the linear level, this dynamics is driven by the linearized operator. At the quadratic level it is driven by the scaling. The non linear terms only affect the dynamics at higher order, thus being invisible as we work in a perturbative setting

Proposition 2.11. (Construction of the approximate profile) Let a very large odd integer

$$
L \gg 1
$$

and let $b = (b_1, \ldots, b_L)$ denote a $L$-tuple of real numbers, with $b_1 > 0$. There exists a $L$-dimensional manifold of $C^\infty$ radially symmetric functions $(Q_b)_{b \in \mathbb{R}_+^* \times \mathbb{R}^{L-1}}$ satisfying the following identity:

$$
F(Q_b) = b_1 \Lambda Q_b + \sum_{i=1}^L \left( - (i - \alpha) b_1 b_i + b_{i+1} \right) \frac{\partial Q_b}{b_i} - \psi_b,
$$

where we used the convention $b_{L+1} = 0$. $\psi_b$ stands for a higher order remainder term situated on the second coordinate:

$$
\psi_b = \begin{pmatrix} 0 \\ \psi_b \end{pmatrix}
$$

Let $B_1$ be defined by (1.28). In the regime in which $|b_i| \lesssim |b_1|^i$, $0 < b_1 \ll 1$, it enjoys the following estimates (the adapted norm is defined by (2.32)):

---

7 this point will be made clearer when studying the full non-linear dynamics.
8 we take $L$ to be odd just to know the coordinates of the objects we are manipulating, but it is not important.
Proof of Proposition 2.11.

Step 1: Computation of the error. We take a profile having the form (2.40) and compute the following identity:

$$-F(Q_b) + b_1 \Lambda Q_b = A_1 - A_2,$$

with:

$$A_1 := b_1 \Lambda Q + \sum_{i=1}^{L} [T_i + b_1 HT_i + b_1 b_1 \Lambda T_i] + \sum_{i=2}^{L+2} [HS_i + b_1 \Lambda S_i],$$

$$A_2 := \begin{pmatrix} 0 \\ f(Q + \alpha_b^{(1)}) - f(Q) + f'(Q)\alpha_b^{(1)} \end{pmatrix}.$$

Knowing in advance the fact that $S_i \sim \delta_i$ and $b_i \sim \delta_i$ we rearrange all the terms according to the power of $b_i$:

$$A_1 = b_1(\Lambda Q + HT_1) + \sum_{i=1}^{L-1} [b_i b_1 \Lambda T_i + b_1 \Lambda HT_{i+1} + HS_{i+1} + b_1 \Lambda S_i] + b_1 b_1 \Lambda T_i + b_1 \Lambda HT_{i+1} + HS_{i+1} + b_1 \Lambda S_i$$

$$+ b_1 b_1 \Lambda T_i + b_1 \Lambda HT_{i+1} + HS_{i+1} + b_1 \Lambda S_i] + b_1 b_1 \Lambda T_i + b_1 \Lambda HT_{i+1} + HS_{i+1} + b_1 \Lambda S_i] + b_1 b_1 \Lambda T_i + b_1 \Lambda HT_{i+1} + HS_{i+1} + b_1 \Lambda S_i] + b_1 b_1 \Lambda T_i + b_1 \Lambda HT_{i+1} + HS_{i+1} + b_1 \Lambda S_i] + b_1 b_1 \Lambda T_i + b_1 \Lambda HT_{i+1} + HS_{i+1} + b_1 \Lambda S_i] + b_1 b_1 \Lambda T_i + b_1 \Lambda HT_{i+1} + HS_{i+1} + b_1 \Lambda S_i] + b_1 b_1 \Lambda T_i + b_1 \Lambda HT_{i+1} + HS_{i+1} + b_1 \Lambda S_i].$$

\[9\] Here the zone $y \leq B_1$ is called global because we will cut the profile $Q_b$ in the next section at this precise location.
Because we have assumed \( p \) to be an integer, and from the localization of the \( T_i \)'s, we can expand \(^{10}\( A_2 \) as a sum of polynomials of order higher or equal to 2:

\[
A_2^{(2)} = \sum_{j=2}^p C_j Q^{p-j} (a^{(1)}_b)^j = \sum_{j=2}^p C_j Q^{p-j} \left( \sum_{i=2, i \text{ even}}^{L-1} b_i T_i + \sum_{i=2}^{L+2} S_i^{(1)} \right)^j .
\]

Again, we reorder these polynomials according to:

\[
A_2^{(2)} = \sum_{i=2}^{L+2} P_i + R.
\]

where:

\[
P_i = \sum_{j=2}^p C_j Q^{p-j} \left( \sum_{J,J_1=J,J_2=1,k}^{L-1} T_k J_k \prod_{k=2}^{L+2} (S_k^{(1)})_J \right),
\]

where here \( J = (J_2, ..., J_{L-1}, J_2, ..., J_{L+2}) \) and the way to count the powers of \( b_1 \) is: \( |J|_2 = \sum_{k=1}^{L+2} 2kJ_2k + \sum_{k=1}^{L+2} kJ_k \). The remainder is:

\[
R = \sum_{j=2}^p C_j Q^{p-j} \sum_{J,J_1=J,J_2=1,k}^{L+2} L+3 \left( \sum_{k=2}^{L-1} b_k^{j} T_k \prod_{k=2}^{L+2} (S_k^{(1)})_J \right).
\]

We make an abuse of notation by denoting \( P_i := \begin{pmatrix} 0 \\ P_i \end{pmatrix} \) and \( R := \begin{pmatrix} 0 \\ R \end{pmatrix} \). The error term \( \psi_b \) has then the following expression (anticipating then that \( \frac{\partial S_{j+1}}{\partial b_i} = 0 \) for \( j \leq i \)):

\[
\psi_b = \sum_{i=1}^L (-i-\alpha)b_1 b_i + (b_{i+1}) \frac{\partial Q_k}{\partial b_i} + A_1 - A_2
\]

\[
= \sum_{i=1}^L (-i-\alpha)b_1 b_i + (b_{i+1}) \left[ T_i + \sum_{i=i+1}^{L+2} \frac{\partial S_{i+1}}{\partial b_i} \right] + A_1 - A_2
\]

\[
= \sum_{i=1}^L \left[h(S_{i+1}) + b_1 b_i \Theta_i + b_1 \Lambda S_i + P_{i+1} + \sum_{j=2}^{i-1} (j-\alpha)b_1 b_j + b_{j+1} \frac{\partial S_{j+1}}{\partial b_j} \right]
\]

\[
+ h(S_{L+2}) + b_1 \Lambda S_{L+1} + P_{L+2} + \sum_{j=2}^L (j-\alpha)b_1 b_j + b_{j+1} \frac{\partial S_{j+2}}{\partial b_j} + R_i.
\]

\[
(2.45)
\]

Step 2: Expression of the \( S_i \)'s, simplification of \( \psi_b \). We define the \( S_i \)'s by induction, in order to cancel the terms with a power of \( b_1 \) less than \( L + 2 \) in \( \psi_b \):

\[
\begin{cases}
S_1 &= 0, \\
S_i &= -H^{-1}(\Phi_i) \text{ for } 2 \leq i \leq L + 2,
\end{cases}
\]

(2.46)

with the following expression for the profiles \( \Phi_i \):

\[
\begin{cases}
\Phi_{i+1} &= b_1 b_i \Theta_i + b_1 \Lambda S_i + P_{i+1} + \sum_{j=1}^{i-1} (j-\alpha)b_1 b_j + b_{j+1} \frac{\partial S_{j+1}}{\partial b_j} \text{ for } 1 \leq i \leq L, \\
\Phi_{L+2} &= b_1 \Lambda S_{L+1} + P_{L+2} + \sum_{j=1}^{L-1} (j-\alpha)b_1 b_j + b_{j+1} \frac{\partial S_{j+2}}{\partial b_j}.
\end{cases}
\]

(2.47)

The \( S_i \)'s being defined by \( \psi_b \), \( \psi_b \) has now the following expression:

\[
\psi_b = b_1 \Lambda S_{L+2} + \sum_{j=1}^L (j-\alpha)b_1 b_j + b_{j+1} \frac{\partial S_{L+2}}{\partial b_j} + R.
\]

(2.48)

Step 3: Properties of the \( S_i \)'s. We claim the following facts (the homogeneity is defined in Definition 2.9):

\( ^{10} \)For the moment we include all the \( S_i^{(1)} \) because we still have not proved their localization.
We now turn to the expression of the error term. We have
\[
\psi_b = \left( b_1 \Lambda^{(2)} S_{L+2}^{(2)} + \sum_{j=1}^{L} (- (j - \alpha) b_1 b_j + b_{j+1}) \frac{\partial S_j^{(2)}}{\partial b_{j+2}} + R \right).
\]

Step 4: Bounds for the error term. We now turn to the expression of the error term. We have
\[
\psi_b = \left( b_1 \Lambda^{(2)} S_{L+2}^{(2)} + \sum_{j=1}^{L} (- (j - \alpha) b_1 b_j + b_{j+1}) \frac{\partial S_j^{(2)}}{\partial b_{j+2}} + R \right).
\]

We start by estimating the first two terms. We already know that \( b_1 \Lambda S_{L+2} \) and \( \sum_{j=1}^{L} (- (j - \alpha) b_1 b_j + b_{j+1}) \frac{\partial S_j^{(2)}}{\partial b_{j+2}} \) are of degree \((L + 2, L + 2 + g', 1, L + 3)\). This
leads to the following estimates (the local adapted norm was defined in (2.32)):
\[ \| b_1 \mathbf{A} S_{L+2} + \sum_{j=1}^{L}(-(j - \alpha)b_1b_j + b_{j+1}) \frac{\partial S_{L+2}}{\partial b_i} \|_j (\leq B_1) \]
\[ \leq C(L) \int_{B_1} \left| \frac{b_1}{y^j} \right| \left( e^{-y\gamma f + L + 2 - 1 - g' - k} \right) dy \]
\[ = C(L) b_1^{L+6} \int_{B_1} y^{2\delta_0 + 2g' + 2L + 2 - 2j} \frac{dy}{y^{d-1}} \]
\[ = C(L) b_1^{2j + 2 + 2(1 - \delta_0) + 2g'} \]

The integral in the right hand side is always divergent as \( j \leq L \), and as \( 1 + \delta_0 - g' \geq 0 \) (see the definition of \( g' (1.21) \), the presence of \( 1 + \delta_0 \) was made to produce this result). We now prove the local estimates. We recall that we proved in step 3 that \( b_1 \mathbf{A} S_{L+2} + \sum_{j=1}^{L}(-(j - \alpha)b_1b_j + b_{j+1}) \frac{\partial S_{L+2}}{\partial b_i} \) is homogeneous of degree \( p_3 = L + 3 \). This means that:
\[ b_1 \mathbf{A} S_{L+2} + \sum_{j=1}^{L}(-(j - \alpha)b_1b_j + b_{j+1}) \frac{\partial S_{L+2}}{\partial b_i} = \sum_{|J|_2 = L+3} b^J f_J, \]
for a finite number of functions \( f_J \) such that \( |\partial y^k f_J| \leq y^{-\gamma + L + 2 - 1 - g' - k} \) at infinity, and with \( b^J = \prod b_i^{J_i} \). Hence the brute force upper bound:
\[ \left| \partial y^k \left( b_1 \mathbf{A} S_{L+2} + \sum_{j=1}^{L}(-(j - \alpha)b_1b_j + b_{j+1}) \frac{\partial S_{L+2}}{\partial b_i} \right) \right| \leq b_1^{L+3}(1 + y)^{-\gamma + L + 2 - 1 - g' - k} \]
which gives the local result. We now turn to the bounds for the \( R \) term. Again, thanks to the homogeneity property of the \( S_i \)’s, \( R \) is of the form:
\[ R = \sum_{|J|_2 \geq L+3} \prod_{i=1}^{L} b_i^{J_i} g_J, \]
for a finite number of functions \( g_J \) whose derivatives have polynomial growth at infinity. This directly implies the local bounds. For the global bounds, we rewrite \( R \) as a linear sum of terms of the form:
\[ Q^{p-j} \left( \prod_{i=2, i \text{ even}}^{L} b_i^{J_i} T_i^{J_i} \prod_{i=2, i \text{ even}}^{L} S_i^{J_i} \right), \]
for \( |J|_2 \geq L + 3 \) and \( 2 \leq j \leq p \). Using again the Calculus Lemma for admissible functions [2.10] each term has the asymptotic behavior:
\[ Q^{p-j} \left( \prod_{i=2, i \text{ even}}^{L} b_i^{J_i} T_i^{J_i} \prod_{i=2, i \text{ even}}^{L} S_i^{J_i} \right) = O \left( \frac{b_1^{\frac{|J_2|}{2}}}{1 + y^{2\gamma + (j - 1)\alpha + (\sum J_i)g' - |J_2| + k}} \right). \]

For all \( k \in \mathbb{N} \):
\[ \partial_y^k \left( Q^{p-j} \left( \prod_{i=2, i \text{ even}}^{L} b_i^{J_i} T_i^{J_i} \prod_{i=2, i \text{ even}}^{L} S_i^{J_i} \right) \right) = O \left( \frac{b_1^{\frac{|J_2|}{2}}}{1 + y^{2\gamma + (j - 1)\alpha + (\sum J_i)g' - |J_2| + k}} \right). \]

From the fact that \( (j - 1)\alpha > 2 \geq g' \) we conclude that the global estimates of the term \( R \) are in all cases better (ie with a higher power of \( b_1 \), \( b_1 \) being small \( 0 < b_1 < 1 \)) than the ones for \( b_1 \mathbf{A} S_{L+2} + \sum_{j=1}^{L}(-(j - \alpha)b_1b_j + b_{j+1}) \frac{\partial S_{L+2}}{\partial b_i} \), which concludes the proof. \( \square \)
As we have seen with the previous estimates of the error term $\psi_b$, we have a good approximate dynamics for $y \leq B_1$. However, as

$$T_i \sim y^{-\gamma + 1 - \delta_{\text{odd}}} \to +\infty \text{ as } y \to +\infty \text{ (as soon as } i \geq \gamma + 1),$$

the approximate dynamic is irrelevant far away of the origin. Consequently, we will now localise the profiles of Proposition 2.11 in the zone $y \leq B_1$, where $\frac{B_1}{\Lambda^{1/2}}$ is nearly of order 1. To do this, we will simply multiply by a cut-off function. This cut will create additional error terms that we will estimate in the next proposition. We recall that our cut-off function $\chi$ is defined by (1.29). We denote by $\chi_{B_1}\alpha_b$:

$$\chi_{B_1}\alpha_b := \begin{pmatrix} \chi_{B_1}\alpha_b^{(1)} \\ \chi_{B_1}\alpha_b^{(2)} \end{pmatrix}. \quad (2.49)$$

**Proposition 2.13** (Localization of the approximate profile). We use the assumptions and notations of Proposition 2.11. Let $I = ]s_0, s_1[$ denote a renormalized time interval, and

$$b : I \to \mathbb{R}^L, \quad s \mapsto (b_i(s))_{1 \leq i \leq L}$$

be a $C^1$ map such that: $|b_i| \lesssim b_1$ with $0 < b_1 < 1$. Assume the a priori bound:

$$|b_{1,s}| \lesssim b_1^2. \quad (2.50)$$

Let $\tilde{Q}_b$ denote the localized profile, given by:

$$\tilde{Q}_b = Q + \chi_{B_1}\alpha_b. \quad (2.51)$$

Then for $0 < \eta \ll 1$ small enough one has the following identity ($\text{Mod}(t)$ being defined by (2.43)):

$$\partial_s \tilde{Q}_b - F(\tilde{Q}_b) + b_1 \Lambda \tilde{Q}_b = \tilde{\psi}_b + \chi_{B_1} \text{Mod}(t). \quad (2.52)$$

$\tilde{\psi}_b$, the new error term, satisfies (the adapted norm being defined in (2.32)):

(i) Global weighted bounds:

$$\forall 0 \leq j \leq L - 1, \quad \|\tilde{\psi}_b\|_j^2 \lesssim C(L)b_1^{2j + 2 + 2(1 - \delta_b) - C_j \eta}, \quad (2.53)$$

for $j = L$, $\|\tilde{\psi}_b\|_L^2 \lesssim C(L)b_1^{2L + 2 + 2(1 - \delta_b)(1 + \eta)}$. \quad (2.54)

(ii) Local improved bounds: For $x \leq \frac{B_1}{2}$, $\tilde{\psi}_b(x) = \psi_B(x)$, where $\psi_B$ is the former error term of Proposition 2.11. Hence $\forall j \geq 0, \forall 1 \leq B \leq \frac{B_1}{2}$:

$$\int_{|y| \leq B} |\nabla^j \tilde{\psi}_b^{(1)}|^2 + |\nabla^j \tilde{\psi}_b^{(2)}|^2 \lesssim \int_{|y| \leq B} |\nabla^j \psi_b^{(2)}|^2 \lesssim C(L,j)B^{C(L,j)}b_1^{2L + 6}. \quad (2.55)$$

**Remark 2.14.** When comparing the estimates given by this proposition, and the ones given in the proposition 2.11 we note a loss. Indeed the first non cut profile creates an error seen on the corrective terms $S_{L+2}$ and $R$ which enjoy additional gains $y^{-\gamma'}$ or $y^{-\alpha}$ away from the origin compared to the $T_i$’s. When cuttting, we see in the additional error term the profiles $T_i$’s, giving a worst estimate as they do not have this additional gain.

However, the error created in the zone $B_1 \leq B_1$ is left unperturbed by the cut. The fact that the error enjoys two different estimate: a good one in the zone $y \leq B_1$ and a bad one in the zone $B_1 \leq y \leq 2B_1$ will be helpful in the analysis later.
Proof of Proposition 2.13. We compute the error in localizing:

\[ \partial_s \tilde{Q}_b - F(\tilde{Q}_b) + b_1 \Delta \tilde{Q}_b = \chi_{B_1} \psi_b + \chi_{B_1} M_{\text{Mod}}(t) + \partial_s (\chi_{B_1}) \alpha_b + b_1 (\Delta \tilde{Q}_b - \chi_{B_1} \Lambda Q_b) - (F(\tilde{Q}_b) - F(Q) - \chi_{B_1} (F(Q_b) - F(Q))). \]

So we have the following expression for the new error term:

\[ \tilde{\psi}_b = \chi_{B_1} \psi_b + \partial_s (\chi_{B_1}) \alpha_b + b_1 (\Lambda \tilde{Q}_b - \chi_{B_1} \Lambda Q_b) - (F(\tilde{Q}_b) - F(Q) - \chi_{B_1} (F(Q_b) - F(Q))), \]

(2.56)

and we aim at estimating all these terms in global and local norms.

Local bounds: From (2.56) we clearly see that \( \tilde{\psi}_b \equiv \psi_b \) for \( |y| \leq \frac{B_1}{2} \), because the new error terms appearing when cutting are created in the zone \( B_1 \leq |y| \leq 2B_1 \). Therefore the local bounds are a direct consequence of the local ones established in (2.39).

Global bounds: We recall that \( \| f \|_2^2 = \|

\sum_{j=-L}^{L} f_j(1) \|_{L^2}^2 + \|

\sum_{j=-L-1}^{L-1} f_j(2) \|_{L^2}^2 \) where the \( j \)-th adapted derivative of a function is defined by (2.19). We will now compute this norm for all the terms in the right hand side of (2.56).

• \( \chi_{B_1} \psi_b \) term: When applying the differential operators \( A \) or \( A^* \) to any product \( \chi_{B_1} f \), we have:

\[ A(\chi_{B_1} f) = \chi_{B_1} f_1 - b_1^{1+i} \partial_y \chi \left( \frac{y}{B_1} \right) f, \]

\[ A^* A(\chi_{B_1} f) = \chi_{B_1} f_2 + b_1^{1+i} \partial_y \chi \left( \frac{y}{B_1} \right) f_1 - \left[ b_1^{2+2i} \partial_y^2 \chi \left( \frac{y}{B_1} \right) + b_1^{1+i} \partial_y \chi \left( \frac{y}{B_1} \right) \left( 2W + \frac{d-1}{2} \right) \right] f. \]

(2.57)

And so on for higher powers of \( A \) and \( A^* \). Because of the asymptotic of \( W \), see Lemma 2.3, the general expression is of the form:

\[ (\chi_{B_1} f)_i = \chi_{B_1} f_1 + 1_{B_1 \leq y \leq 2B_1} \sum_{j=1}^i a_j f_j, \]

where \( a_i(y) = O(y^{-(i-j)}) \). It means that deriving \( \chi_{B_1} \) amounts to dividing by \( B_1 \) and localizing in the zone \( B_1 \leq y \leq 2B_1 \). Hence for \( 0 \leq j \leq L \):

\[ \| \chi_{B_1} \psi_b \|_2^2 = \int \left( (\chi_{B_1} \psi_b(2))_{k_0+j} \right)^2 \]

\[ \leq C(L) \sum_{i=1}^{k_0+j} \int_{B_1 \leq |y| \leq 2B_1} b_1^{2(1+\eta)i} \left| \psi_{b,k_0+j-i}^{(2)} \right|^2 + \int_{|y| \leq 2B_1} \left| \psi_{b,k_0+j}^{(2)} \right|^2 \]

\[ \leq C(L) b_1^{2+2i} + \left( \frac{y}{B_1} \right)^{-(L+j+1)} \sum_{|y| \leq 2B_1} \left| \psi_{b,k_0+j}^{(2)} \right|^2 \]

(2.58)

thanks to the Proposition 2.11.

• \( \partial_s (\chi_{B_1}) \alpha_b \) term: We have from the assumption \( |b_{1,s}| \leq b_1^2 \):

\[ \partial_s (\chi_{B_1}) = (1 + \eta) b_1 b_n y \partial_y \chi \left( \frac{y}{B_1} \right) \leq b_1 b_1^{1+i} \partial_y \chi \left( \frac{y}{B_1} \right). \]

Again, deriving \( y \partial_y \chi \left( \frac{y}{B_1} \right) \) amounts to dividing by \( B_1 \), we get:

\[ \| \partial_s (\chi_{B_1}) \alpha_b \|_2^2 = \int \| (\partial_s (\chi_{B_1}) \alpha_{b_1}^{(1)}_{k_0+j+1})^2 + \| (\partial_s (\chi_{B_1}) \alpha_{b_1}^{(2)}_{k_0+j+1})^2 \]

\[ \leq C(L) b_1^2 \int_{B_1 \leq |y| \leq 2B_1} \left| \alpha_{b,k_0+j+1}^{(1)} \right|^2 + \left| \alpha_{b,k_0+j+1}^{(2)} \right|^2. \]

(2.59)
We estimate the two terms using the asymptotic of the $T_i$’s from Lemma 2.8 and 2.11 for the $S_i$’s:

$$\int_{B(B_1)} |a_{b,k_0+j+1}|^2 \leq \int_{B(B_1)} (2.63) \leq C(L) \sum_{i=2}^{L-1} b_{2i}^1 \int_{B(B_1)} y^{\gamma-2i} dy + \frac{1}{y^{d-1}} dy + C(L) \sum_{i=2}^{L-1} b_{2i}^1 \int_{B(B_1)} y^{\gamma-2i} dy + \frac{1}{y^{d-1}} dy \leq (2.60)$$

Similarly:

$$\int_{B(B_1)} |a_{b,k_0+1}|^2 \leq C(L) \sum_{i=1}^{L} b_{2i}^1 \int_{B(B_1)} y^{\gamma-2i} dy + \frac{1}{y^{d-1}} dy + C(L) \sum_{i=2}^{L-1} b_{2i}^1 \int_{B(B_1)} y^{\gamma-2i} dy + \frac{1}{y^{d-1}} dy \leq (2.61)$$

The first upper bound (2.59), combined with the two we just proved, (2.60) and (2.61), lead to (because $0 < \delta_0 < 1$ avoids a possible log-term in the first sum):

$$\| \partial_y (\chi B_1) a_b \|^2 \leq C(L) \sum_{i=1}^{L} b_{2i}^1 \int_{B(B_1)} y^{\gamma-2i} dy + \frac{1}{y^{d-1}} dy + C(L) \sum_{i=2}^{L-1} b_{2i}^1 \int_{B(B_1)} y^{\gamma-2i} dy + \frac{1}{y^{d-1}} dy \leq (2.62)$$

for $\eta$ small enough.

- **$F(\tilde{Q}_b) - F(Q) - \chi B_1 (F(Q_b) - F(Q))$ term:** We compute:

$$F(\tilde{Q}_b) - F(Q) - \chi B_1 (F(Q_b) - F(Q)) = (\Delta(\chi B_1 a_{b}^{(1)}) - \chi B_1 \Delta(a_{b}^{(1)}) + f(\tilde{Q}_b) - f(Q) - \chi B_1 (f(Q_b) - f(Q))).$$

We estimate the two terms in the right hand side of (2.63):

$$\Delta(\chi B_1 a_{b}^{(1)}) - \chi B_1 \Delta(a_{b}^{(1)}) = \partial_y (\chi B_1) \partial_y (a_{b}^{(1)}) + \Delta(\chi B_1) a_{b}^{(1)} = b_{1+\eta} \partial_y (\chi B_1) \partial_y (a_{b}^{(1)}) + b_{2(1+\eta)} \Delta(\chi B_1) a_{b}^{(1)}.$$

Considering the asymptotics of $a_{b}^{(1)}$ we have:

$$\int (\Delta(\chi B_1 a_{b}^{(1)}) - \chi B_1 \Delta(a_{b}^{(1)})) \leq C(L) b_{2(1+\eta)} \int_{B(B_1)} b_{2i}^1 y^{\gamma-2i} - 2i - 2 \leq C(L) b_{2(1+\eta)} \int_{B(B_1)} b_{2i}^1 y^{\gamma-2i} - 2i - 2 \leq C(L) b_{2(1+\eta)} \int_{B(B_1)} b_{2i}^1 y^{\gamma-2i} - 2i - 2 \leq (2.64)$$

because $i < L - 1$ in the sum concerning the $T_i$’s and because of the gain $g' > 0$ in the one of the $S_i$’s. The second term is:

$$f(\tilde{Q}_b) - f(Q) - \chi B_1 (f(Q_b) - f(Q)) = \sum_{k=1}^{p} C_k Q^{p-k} \chi B_1 (a_{b}^{(1)})^k - \chi B_1 \sum_{k=1}^{p} C_k Q^{p-k} \alpha_{b}^{(1)} = \chi B_1 \sum_{k=1}^{p} C_k Q^{p-k} (\chi B_1 - 1) \alpha_{b}^{(1)}.$$
For each $2 \leq k \leq p$, we can expand the polynomial and we have a linear sum of terms of the form:

$$
\chi B_1 Q^{p-k}(\chi B_1^{-1} - 1) \prod_{i=2, \, i \text{ even}}^{L-1} (b_i T_i)^{J_i} \prod_{i=2, \, i \text{ even}}^{L+1} (S_i)^{\tilde{J}_i},
$$

for $|J_1| = k$. We have according to the calculus Lemma 2.10 for homogeneous functions:

$$
\partial_y^k \left( Q^{p-k} \prod_{i=2, \, i \text{ even}}^{L-1} (b_i T_i)^{J_i} \prod_{i=2, \, i \text{ even}}^{L+1} (S_i)^{\tilde{J}_i} \right) = O \left( \frac{b_i^{J_i}}{y^{\frac{p-k}{2} + k + \sum J_i \gamma - |J| + 1}} \right) \quad \text{as } y \to +\infty.
$$

As we have seen before, the presence of the term $\chi B_1$ does not affect the computation (deriving $\chi B_1$ amounts to divide by $y$):

$$
\int_{B_1} \left| (Q^{p-k} \prod_{i=2, \, i \text{ even}}^{L-1} (b_i T_i)^{J_i} \prod_{i=2, \, i \text{ even}}^{L+1} (S_i)^{\tilde{J}_i})_{j+k_0} \right|^2 \leq C(L) b^{2+2j+2(1-\delta_0)(1+\eta)} \quad \text{for } 0 \leq j \leq L.
$$

The primary decomposition (2.63), with the bounds (2.64) and (2.66) implies the bound we were looking for:

$$
\| F(\tilde{Q}_b) - F(Q) - \chi B_1 (F(Q_b) - F(Q)) \|_j^2 \leq \begin{cases} 
C(L) b^{2+2j+2(1-\delta_0)(1+\eta)} & \text{for } 0 \leq j \leq L, \\
C(L) b^{2+2L+2(1-\delta_0)(1+\eta)} & \text{for } j = L.
\end{cases}
$$

- $b_1 (\Lambda \tilde{Q}_b - \chi B_1 \Lambda Q_b)$ term: We compute:

$$
\Lambda \tilde{Q}_b - \chi B_1 \Lambda Q_b = (1 - \chi) \Lambda Q + y \partial_y (\chi B_1) \alpha_b.
$$

We have that:

$$
y \partial_y (\chi B_1) = b_1^{1+\eta} y \partial_y \chi \left( \frac{y}{B_1} \right).
$$

So the term $y \partial_y (\chi B_1) \alpha_b$ behaves the same way as the term $\partial_y (\chi B_1) \alpha_b$ previously treated and enjoys the same estimations. Finally we estimate the soliton contribution, because of which we had to derive $k_0$ times at least in order to have integrability. We again use the fact that deriving $k$ times $\chi B_1$ amounts to divide by $y^k$ and to localize in the zone $B_1 \leq y \leq 2B_1$:

$$
\int |b_1 (1 - \chi B_1) \Lambda^{(1)} Q_{k_0+j+1}|^2 \leq C(L) b_1^{2+2j+2(1-\delta_0)(1+\eta)} \int_{B_1} y^{-2\gamma - 2k_0 - 2j + d - 1} dy \leq C(L) b_1^{2+2j+2(1-\delta_0)(1+\eta) + (2j + 2(1-\delta_0)) \eta}.
$$

So that finally:

$$
\| b_1 (\Lambda \tilde{Q}_b - \chi B_1 \Lambda Q_b) \|_j^2 \leq \begin{cases} 
C(L) b^{2j+2(1+\eta)(1-\delta_0) - C_j \eta} & \text{for } j \leq L - 1, \\
C(L) b^{2L+2(1+\eta)(1-\delta_0)} & \text{for } j = L.
\end{cases}
$$

The decomposition (2.56), with the bounds for each term (2.58), (2.62), (2.67) and (2.68) give the global bounds (2.53) and (2.54) we had to prove. □
2.6. Study of the dynamical system governing the evolution of the parameters \((b_i)_{1 \leq i \leq L}\). We have constructed in the preceding propositions \([2.11, 2.13]\) a manifold of functions near the solitary wave such that:

\[
F(\tilde{Q}_b) \sim b_1 \Lambda \tilde{Q}_b + \sum_{i=1}^{L} (-\alpha b_1 b_i + b_{i+1}) \frac{\partial \tilde{Q}_b}{\partial b_i}.
\]

By applying scaling, and the identity \(\frac{\partial (f\lambda)}{\partial \lambda} = \frac{1}{\lambda} \lambda f\lambda\) we have that:

\[
F(\tilde{Q}_{b,\lambda}) \sim \frac{b_1}{\lambda} (\Lambda \tilde{Q}_b)_{,\lambda} + \sum_{i=1}^{L} \frac{1}{\lambda} (-\alpha b_1 b_i + b_{i+1}) \frac{\partial \tilde{Q}_b}{\partial b_i}.
\]

Hence approximately a solution of (NLW) on this manifold gives:

\[-\frac{\lambda}{\Lambda} (\Lambda \tilde{Q}_b)_{,\lambda} + \sum b_{i,t} \left(\frac{\partial \tilde{Q}_b}{\partial b_i}\right)_{,\lambda} = \partial_t (\tilde{Q}_{b,\lambda})_{,\lambda} \sim F(\tilde{Q}_{b,\lambda}) = \frac{b_1}{\lambda} (\Lambda \tilde{Q}_b)_{,\lambda} + \sum (-\alpha b_1 b_i + b_{i+1}) \left(\frac{\partial \tilde{Q}_b}{\partial b_i}\right)_{,\lambda}.
\]

By identifying the terms (the functions involved being linearly free) it gives:

\[
\left\{ \begin{array}{l}
\lambda_t = -b_1, \\
b_{i,t} = \frac{1}{\lambda} (-\alpha b_1 b_i + b_{i+1}) \quad \text{for} \quad 1 \leq i \leq L + 1, \\
b_{L,t} = -\frac{1}{\lambda} (\Lambda b_1) b_L.
\end{array} \right.
\]

We thus want to study the behavior of the solutions of this dynamical system in order to understand the behavior of a real solution close to the manifold of approximate solutions. Writing it in renormalized variables (the renormalized time being defined by \((2.33)\)), the evolution of the \(b_i\)'s is given by:

\[
\left\{ \begin{array}{l}
b_{i,s} = (\alpha b_i b_i + b_{i+1}) \quad \text{for} \quad 1 \leq i \leq L - 1, \\
b_{L,s} = -(\Lambda b_1) b_L.
\end{array} \right.
\]

We show in this section that this dynamical system admits exceptional solutions leading to an explosive scenario, and that the stability of such solutions can be explicitly computed.

Lemma 2.15. (Special solutions for the dynamical system:) Let \(\ell\) be an integer such that \(\alpha < \ell\). Then \([11] b^\ell : [0, +\infty] \to \mathbb{R}^L\) given by:

\[
\left\{ \begin{array}{l}
b_i^\ell(s) = \frac{c_i}{\sqrt{s}} \quad \text{for} \quad 1 \leq i \leq \ell, \\
b_i^\ell \equiv 0 \quad \text{for} \quad \ell < i,
\end{array} \right.
\]

with the constant \(c_i\) given by:

\[
c_i = \frac{\ell}{\ell - \alpha} \quad \text{and} \quad c_{i+1} = -\frac{\alpha (\ell - i)}{\ell - \alpha} c_i \quad \text{for} \quad 1 \leq i \leq \ell - 1,
\]

is a solution of \((2.70)\). Moreover, if the renormalised time \(s\) and the scaling satisfy:

\[
\frac{ds}{dt} = \frac{1}{\lambda}, \quad s(0) = s_0 > 0, \quad \frac{d\lambda}{dt} = -b_1, \quad \lambda(0) = 1,
\]

then there exists \(T > 0\) with \(s(t) \to +\infty\) as \(t \to T\), and there holds:

\[
\lambda(t) \sim (T - t)^{\frac{\ell}{\alpha}}
\]

\(^{11}\)We forget the dependence with \(\ell\) and write \(b^\ell\) to avoid additional notations, as \(\ell\) will be fixed throughout the paper.
We do not write here the proof as it is a direct computation. When dealing with the real equation (NLW), we want these special solutions to persist. A real solution will imply a corrective term "orthogonal" to the manifold \((\tilde{Q}_{b,\lambda})_{b,\lambda}\) and a corrective term for the parameters. Therefore, to understand the time evolution of the part of the error on the manifold \((\tilde{Q}_{b,\lambda})_{b,\lambda}\), we have to understand the dynamics of (2.70) close to the special solution \((b^e(s))_{s > 0}\).

**Lemma 2.16.** (Linearization around the special trajectories) Let us denote a perturbed solution around \(b^e\) by:

\[
b_k(s) = b^e_k(s) + \frac{U_k(s)}{s^k}, \text{ for } 1 \leq k \leq L,
\]

and note \(U = (U_1, ..., U_L)\) the perturbation. Suppose \(b^e\) is a solution of (2.70), then the evolution of \(U\) is given by:

\[
\partial_t U = \frac{1}{s} A_\ell U + O \left( \frac{|U|^2}{s} \right),
\]

with:

\[
A_\ell = \begin{pmatrix}
-(1 - \alpha)c_1 + \alpha \frac{\ell - 1}{\ell - \alpha} & 1 \\
-\ell (1 - \alpha)c_\ell & \alpha \frac{\ell - 1}{\ell - \alpha} & 1 \\
0 & 0 & 1 \\
\vdots & \vdots & \vdots \\
0 & 0 & 0 & \alpha \frac{\ell - 1}{\ell - \alpha} \\
\end{pmatrix}
\]

\[
A_\ell \text{ is diagonalizable into the matrix } \text{diag}(-1, \frac{2\alpha}{\ell - \alpha}, ..., \frac{\ell - \alpha}{\ell - \alpha}, 1, ..., \frac{\ell - \alpha}{\ell - \alpha}). \text{ We denote the eigenvector associated to the eigenvalue } -1 \text{ by } v_1 \text{ and the eigenvectors associated to the unstable modes } \frac{2\alpha}{\ell - \alpha}, ..., \frac{\ell - \alpha}{\ell - \alpha} \text{ by } v_2, ..., v_\ell. \text{ They are a linear combination of the } \ell \text{ first components only. That is to say there exists a } L \times L \text{ matrix coding a change of variables:}
\]

\[
P_\ell := \begin{pmatrix}
P'_\ell & 0 \\
0 & \text{Id}_{L-\ell}
\end{pmatrix},
\]

with \(P'_\ell\) an invertible \(\ell \times \ell\) matrix and \(\text{Id}_{L-\ell}\) the \(L - \ell \times L - \ell\) identity such that:

\[
P_\ell A_\ell P_\ell^{-1} = \begin{pmatrix}
-1 & 0 & q_1 \\
\frac{2\alpha}{\ell - \alpha} & \frac{\ell - \alpha}{\ell - \alpha} & q_2 \\
\vdots & \vdots & \vdots \\
\frac{\ell - \alpha}{\ell - \alpha} & \frac{\ell - \alpha}{\ell - \alpha} & q_\ell \\
0 & 0 & 1 \\
\end{pmatrix}.
\]
Proof of Lemma 2.16 step 1: Linearization. We compute:

\[
0 = b_{k,s} + (k - \alpha)b_{k} - b_{k+1}
\]

\[
= \frac{s}{s+1} [s(U_{k,s} - kU_{k}) + (k - \alpha)c_{k}U_{k} - U_{k+1} + O(U_{1}U_{k})]
\]

which gives the expression of \( A_{\ell} \).

step 2: Diagonalization. We will compute by induction the characteristic polynomial. The case \( \ell = 3 \) can be done by hand. We now assume \( \ell \geq 4 \) and let:

\[
\mathcal{P}_{\ell}(X) = \det(A_{\ell} - XId).
\]

We first notice that: \( \mathcal{P}_{\ell}(X) = \det(A'_{\ell} - XId)\det(A''_{\ell} - XId) \) where \( A'_{\ell} \) stands for the \( \ell \times \ell \) matrix on the top left corner, and \( A''_{\ell} \) for the \((L - \ell) \times (L - \ell)\) matrix on the bottom right corner:

\[
A'_{\ell} = \begin{pmatrix}
-(1 - \alpha)c_{1} + \frac{\ell - 1}{\ell - \alpha} & 1 & 0 \\
-(i - \alpha)c_{i} & \alpha \frac{\ell - i}{\ell - \alpha} & 1 \\
-(\ell - \alpha)c_{\ell} & 0 & 0
\end{pmatrix},
\tag{2.78}
\]

\[
A''_{\ell} = \begin{pmatrix}
-\frac{\alpha}{\ell - \alpha} & 1 & 0 \\
-\alpha \frac{\ell}{\ell - \alpha} & 1 \\
0 & -\alpha \frac{\ell - \alpha}{\ell - \alpha}
\end{pmatrix}.
\tag{2.79}
\]

We have:

\[
\det(A''_{\ell} - XId) = \prod_{i=\ell+1}^{L} (-1) \left( X + \frac{(i - \ell)\alpha}{\ell - \alpha} \right).
\tag{2.80}
\]

We write \( \mathcal{P}_{\ell}' = \det(A'_{\ell} - XId) \). We develop this determinant with respect to the last row and iterate this process. It gives for \( \mathcal{P}_{\ell}' \) an expression of the form:

\[
\mathcal{P}_{\ell}' = (-1)^{\ell+1}(-1)(\ell - \alpha)c_{\ell} + (-X)\left[ (-1)^{\ell}(-1)(\ell - 1 - \alpha) + \frac{\alpha}{\ell - \alpha} - X \right] \times \left[ (-1)^{\ell-1}(-1)(\ell - 2 - \alpha)c_{\ell-2} + \frac{2\alpha}{\ell - \alpha} - X ][... \right]
\]

We let for \( 1 \leq i \leq \ell \):

\[
A_{i} := (-1)^{\ell+2-i}(-1)(\ell + 1 - i - \alpha)c_{\ell+1-i},
\tag{2.81}
\]

\[
B_{i} := (i - 1)\frac{\alpha}{\ell - \alpha} - X.
\tag{2.82}
\]

We then rewrite:

\[
\mathcal{P}_{\ell}' = A_{1} + B_{1} (A_{2} + B_{2} [A_{3} + B_{3} [...]]).
\]

We now let for \( 1 \leq i \leq \ell - 1 \):

\[
C_{i} := (-1)^{\ell+1-i}(X(\ell - i - \alpha)c_{\ell-i} + \frac{\ell - \alpha}{i}c_{\ell-i+1}).
\tag{2.83}
\]

We have the following relation for \( 1 \leq i \leq \ell - 2 \):

\[
C_{i} + B_{i}B_{i+2}A_{i+2} = B_{i+2}C_{i+1}.
\tag{2.84}
\]
Indeed we compute:

\[ C_i + B_1 B_2 A_{i+2} = (-1)^{\ell+1-i} (X(\ell - i - \alpha)c_{\ell-i} + \frac{\ell}{\ell - \alpha} c_{\ell-i+1}) \]
\[ + (X)(\alpha - X)(-1)^{\ell-i}(-\ell + 1 - \alpha)c_{\ell-i+1} \]
\[ = (-1)^{\ell-i}(-X(\ell - i - \alpha)c_{\ell-i} - \frac{\ell}{\ell - \alpha} c_{\ell-i+1}) \]
\[ + X(i+1)(-i\frac{\alpha}{\ell \alpha} - \frac{\alpha}{\ell \alpha} - X)(\ell - i - \alpha)c_{\ell-i-1}) \]
\[ = B_{i+2}(-1)^{\ell-i}(\ell - i - 1 - \alpha)c_{\ell-i} \]
\[ + (-1)^{\ell-i}[X(\ell - i - \alpha)c_{\ell-i} - \alpha c_{\ell-i}] \]
\[ - i \frac{\alpha}{\ell \alpha} X(\ell - i - 1 - \alpha)(-\frac{\ell}{\ell + 1}(\ell - \alpha))c_{\ell-i}] \]
\[ = B_{i+2}(-1)^{\ell-i}(\ell - i - 1 - \alpha)c_{\ell-i} \]
\[ + (-1)^{\ell-i}c_{\ell-i}(-X(\ell - i - \alpha) + \alpha + \frac{i}{\ell \alpha}(\ell - i - 1 - \alpha)X) \]
\[ = B_{i+2}(\ell-i)^{\ell-i}(\ell - i - 1 - \alpha)c_{\ell-i} + (-1)^{\ell-i} c_{\ell-i} j_{\ell-i} B_{i+2} \]
\[ = B_{i+2}(\ell-i) + A_1 B_2 = C_1. \]

By iterations we get:

\[ P_\ell' = A_1 + B_1 A_2 + B_1 B_2 A_3 + B_1 B_2 B_3(A_4 + B_4(...)) \]
\[ C_1 + B_1 B_2 A_3 + B_1 B_2 B_3(A_4 + B_4(...)) \]
\[ C_2 B_3 + B_1 B_2 B_3(A_4 + B_4(...)) = B_3(C_2 + B_1 B_2(A_4 + B_4(...)) \]
\[ B_3 B_4 C_3 + B_1 B_2 B_4(A_5 + B_5(...)) = B_3 B_4(C_3 + B_1 B_2(A_5 + B_5(...)) \]
\[ ... \]
\[ B_3 ... B_{\ell}(C_{\ell-1} + B_1 B_2). \]

We compute the last polynomial:

\[ C_{\ell-1} + B_1 B_2 = X(1 - \alpha)c_1 + \frac{\ell - \alpha}{\ell - 1} c_2 + (-X) \left( \frac{\alpha}{\ell - \alpha} - X \right) = (X+1) \left( X - \frac{\alpha \ell}{\ell - \alpha} \right). \]

So:

\[ P_\ell' = (X + 1) \prod_{i=2}^{\ell} \left( \frac{i\alpha}{\ell - \alpha} - X \right). \]

This result, together with the result concerning \( P_\ell'' \), shows that \( A_\ell \) is diagonalizable and that its eigenvalues are: \((-1, \frac{2\alpha}{\ell - \alpha}, ..., \frac{\ell \alpha}{\ell - \alpha}, ..., \frac{(\ell - 1) \alpha}{\ell - \alpha}).\)

In addition, from the form of \( A_\ell \), one sees that the \( \ell \) first components do not affect the \( L - \ell \) last ones: \( P_{(\ell+1,L)} A P_{(1,\ell)} = 0 \) where \( P_{(\ell+1,L)} \) and \( P_{(1,\ell)} \) are the projectors:

\[ P_{(\ell+1,L)}(U_1, ..., U_L) = (0, ..., 0, U_{\ell+1}, ..., U_L) \]
\[ P_{(1,\ell)}(U_1, ..., U_L) = (U_1, ..., U_{\ell}, 0, ..., 0). \]

This gives the last result stated in the lemma. The \( v_i \)'s are a linear combination of the \( \ell \) first components only. \( \square \)

3. The trapped regime

In this section we are considering a real solution of (NLW). We fix 1 \( \ll L \) odd and \( \alpha < \ell \). Our aim is to show that the approximate solution \( \left( \hat{Q}_{\alpha} \right)_t \) constructed in the last section does persist. That is to say that there exists an orbit of the (NLW) equation that stays asymptotically (with respect to renormalized time \( s \)) close to the family of special approximate solutions \( \left( \hat{Q}_{\alpha} \right)_t \). Note that we do not prescribe in advance the behavior of the scaling \( \lambda \), but it will be shown to have the same
asymptotical behavior as $\lambda^\varepsilon$.

In order to do that, we need to understand how the full dynamics affects the approximate one we exhibited in the last section. We decompose a true solution under the form $u(t) = (\tilde{Q}_b + \varepsilon)\gamma$. We aim at estimating the contribution of the error $\varepsilon$ on the parameters dynamics, and at estimating the size of $\varepsilon$ in adapted norms.

The special approximate solutions $(\tilde{Q}_b)_{\varepsilon}^\lambda$ for $\lambda \sim \lambda^\varepsilon$, generate a reasonable error term, because as $|b_i^{e}| \lesssim s^{-i} \approx (b_i^{e})^i$ the estimates on the error term $\psi_b$ in Proposition 2.13 apply. But they are not stable along the unstable directions $(v_2, ..., v_\ell)$, and if the parameters $b_i$’s move too much, the error term in the approximate dynamics grows too big, consequently making a control over $\varepsilon$ impossible. Consequently we cannot work close to the full approximate manifold $(\tilde{Q}_b,\lambda)_{b,\lambda}$: we are restricted to work close to the subset of these approximate trajectories $(\tilde{Q}_b^{(s)},\lambda)_{s>0,\lambda>0}$. We work in a neighborhood of these approximate trajectories, study all the real trajectories starting from that neighborhood, and show that at least one must stay in that neighborhood for all time. We make a proof based on a bootstrap technique. We in particular argue "forward" in time what allows us to measure precisely the stabilities and instabilities.

The fact that staying in an appropriate neighborhood of a special approximate solution leads to a blow-up, whose blow-up rate and asymptotic behavior can be computed, will be shown in the next section.

3.1. Setting up the bootstrap. We are now going to define in which neighborhood of the family of approximate solutions $(\tilde{Q}_b^{(s)},\lambda)_{s,\lambda}$ we want to work. We start by defining how we decompose our solution into the sum $u = (\tilde{Q}_b + \varepsilon)\gamma$. After that we describe the neighborhood and state the main Proposition of the paper claiming the existence of an orbit staying in that neighborhood.

3.1.1. Projection onto the approximate solutions manifold. Close to $Q$, the manifold $(Q_{b,\lambda})_{b,\lambda}$ is tangent to the vector space $\text{Span}(T_i)$. It is consequently appealing to ask $\langle T_i, \varepsilon \rangle = 0$ for all $i$. However, the $T_i$’s are not in appropriate functional spaces, and in particular cannot be used to generate orthogonality conditions. Instead, we will create a sequence of profiles with compact support that approximate such orthogonality conditions. We let the adjoint of $H$ be the operator:

$$H^* = \begin{pmatrix} 0 & L \\ -1 & 0 \end{pmatrix}. \quad (3.1)$$

We have the following relations: $\langle Hu, v \rangle = \langle u, H^*v \rangle$, and

$$H^{*2i} = \begin{pmatrix} (-1)^i L^i & 0 \\ 0 & (-1)^i L^i \end{pmatrix}, \quad H^{*(2i+1)} = \begin{pmatrix} 0 & (-1)^i L^{i+1} \\ (-1)^{i+1} L^i & 0 \end{pmatrix}. \quad (3.2)$$

We recall that $L$ is an odd, large integer. We let $M$ be a large constant, and define:

$$\Phi_M = \sum_{p=0}^L c_p M H^{*p}(\chi_M \Lambda Q), \quad (3.3).$$
Proof of Lemma 3.1.

Proof of the orthogonality conditions: The profile $\Phi_M$ is located on the first coordinate:

$$\Phi_M = \begin{pmatrix} \Phi_M \\ 0 \end{pmatrix},$$

because for $1 \leq k = 2i + 1 \leq L$ an odd integer one has $c_{k,M} = 0$. Moreover the following bounds hold:

$$\begin{align*}
|\langle \Phi_M, \Lambda Q \rangle| &\sim cM^2\kappa_0 + 2\kappa_0, \\
c_{p,M} &\leq CM^p, \\
\int \Phi_M^2 &\leq CM^{2\kappa_0 + 2\kappa_0}.
\end{align*}$$

for two positive constants $c, C > 0$. In addition, the following orthogonality conditions are met for $1 \leq j \leq L$ and $i \in \mathbb{N}$:

$$\langle \Phi_M, H^j T_j \rangle = \langle \chi_M \Lambda Q, \Lambda Q \rangle \delta_{i,j}. \tag{3.7}$$

Proof of Lemma 3.1. Proof of the orthogonality conditions:

$$\langle \Phi_M, \Lambda Q \rangle = c_{0,M} \langle \chi_M \Lambda Q, \Lambda Q \rangle + \sum_{p=1}^L c_{p,M} \langle \chi_M \Lambda Q, H^p(\Lambda Q) \rangle$$

$$\sim cM^{d-2\gamma},$$

c > 0, from the asymptotic $\Lambda(1)Q \sim \frac{y'}{y}$, $c' \neq 0$. This proves the first property of (3.6). The orthogonality with respect to the $T_i$’s is created on purpose by the definition of the constants $c_{p,M}$:

$$\langle \Phi_M, T_k \rangle = \sum_{p=0}^{k-1} c_{p,M} \langle \chi_M \Lambda Q, T_k \rangle + c_{k,M} \langle \chi_M \Lambda Q, H^k T_k \rangle = 0.$$

Hence by duality:

$$\langle \Phi_M, H^i T_j \rangle = \langle \chi_M \Lambda Q, \Lambda Q \rangle \delta_{i=j}. \tag{3.7}$$

This proves (3.7).

bounds on the constants: We notice by induction that $c_{p,M} = 0$ for $p$ odd. This implies that $\Phi^{(2)}_M = 0$. We prove the estimate of the constants by induction. Since $c_{0,1}$, the estimation is true for $k = 0$. We assume now $k$ to be even. By definition we have:

$$|c_{k,M}| = \frac{|\sum_{p=0}^{k-1} \langle H^p(\chi_M \Lambda Q), T_k \rangle |}{|\langle \chi_M \Lambda Q, \Lambda Q \rangle|}$$

$$\leq CM^{-d+2\gamma} \sum_{p=0}^{k-1} |c_{p,M}| |\langle H^p(\chi_M \Lambda Q), T_k \rangle|$$

$$= CM^{-d+2\gamma} \sum_{p=0}^{k-1} |c_{p,M}| |\langle \chi_M \Lambda Q, T_{k-p} \rangle|.$$

In the sum, for $k-p$ odd this term equals 0. So we have $k-p \geq 2$. Using the asymptotics $\Lambda(1)Q \sim cy^{-\gamma}$ and $T_{k-p} \sim cy^{-\gamma+k-p}$ the integral in the scalar product is divergent and we estimate:

$$|\langle \chi_M \Lambda Q, T_{k-p} \rangle| \sim cM^{d-2\gamma+k-p}. \tag{3.6}$$

Using the induction hypothesis we get:

$$M^{-d+2\gamma} |c_{p,M}| |\langle H^p(\chi_M \Lambda Q), T_k \rangle| \leq CM^k,$$
and so the estimate is true for $c_{k,M}$. We have proven the second assertion of (3.6).

$L^{2}$ estimate: $\int |\Phi_{M}|^{2}$ is a finite sum of terms of the following form enjoying the bound:

$$|\langle c_{p1,M} H^{p1}(\chi_{M}A\Phi), c_{p2,M} H^{p2}(\chi_{M}A\Phi) \rangle| \leq CM^{p1+p2}|\langle L^{p1+p2}(\chi_{M}A\Phi), \chi_{M}A\Phi \rangle|\leq CM^{p1+p2} \int_{M}^{2M} \frac{1}{y^{p1+p2}}y^{d-1} \leq CM^{-2\gamma+d},$$

because we assumed $\gamma - \gamma$ not to be an integer. It implies the last bound in (3.6) \hfill $\square$

3.1.2. Modulation: We want to decompose a function $u$ close to $Q_{\lambda}$ as a unique sum $u = (Q_{b} + \varepsilon)$, with $\varepsilon$ ”orthogonal” to the manifold $(Q_{b,\lambda})_{b,\lambda}$. We make the following change of variable for the parameter $b$: $b_{1} := (b_{1},0,...,0)$ and $\tilde{b}_{1} = (b_{1},0,...,0,b_{1},0,...,0)$. We denote by $\phi$ the application $\phi : (\lambda,b) \mapsto ((\tilde{Q}_{b}, H^{*i}\Phi_{M}))_{0 \leq i \leq L}$. We denote by $D\phi$ the jacobian matrix of $\phi$ at the point $(1, (0,...,0))$ in the $(\lambda, \tilde{b})$ basis. From the properties (3.6) and (3.7) of the profile $\Phi_{M}$ that we previously established, one has:

$$D\phi = \langle \Lambda Q, \chi_{M}A\Phi \rangle \begin{pmatrix}
1 & 0 & \cdots & (0) \\
& 1 & 1 & \cdots \\
& & 1 & \cdots \\
& & & 1
\end{pmatrix}.$$ 

This proves that $\phi$ is a local diffeomorphism around $(1, (0,...,0))$. The implicit function theorem gives for $u$ close enough to $Q$ the existence of a unique decomposition:

$$u = (\tilde{Q}_{b}) + \varepsilon = (\tilde{Q}_{b}) + \varepsilon,$$

with $\varepsilon$ verifying the $L + 1$ orthogonality conditions:

$$\langle \varepsilon, H^{*i}\Phi_{M} \rangle = 0, \text{ for } 0 \leq i \leq M.$$ 

(3.9)

Hence for a real solution to (NLW) starting close enough to $Q$, and by scaling argument, we have as long as $u$ is close enough to $Q_{\lambda}$ a decomposition:

$$u = (\tilde{Q}_{b(t)} + \varepsilon)_{\chi_{M}},$$

(3.10)

with $b$ and $\lambda$ being $C^{1}$ in time, and $\varepsilon$ satisfying (3.9).

3.1.3. Adapted norms: We quantify the smallness of $\varepsilon$ through the following norms:

i. **High order Sobolev norm adapted to the linearized operator:** Remember that $s_{L} = L + k_{0} + 1$ and that the $k$-th adapted derivative of a function $f$, $f_{k}$, is defined in (2.19). We define:

$$E_{s_{L}} := \int |\varepsilon(1)|_{k_{0}+L+1}^{2} + \int |\varepsilon(2)|_{k_{0}+L}^{2} = \int \varepsilon(1)^{2}L^{k_{0}+L+1}(1) + \int \varepsilon(2)^{2}L^{k_{0}+L}(2),$$

(3.11)

which is coercive thanks to the result of Lemma 2.3. In particular:

$$E_{s_{L}} \geq \| \varepsilon \|_{H^{s_{L}} \times H^{s_{L}-1}}^{2}.$$ 

As we will see later on in this paper, a local part of this norm will have to be treated separately. Let $N > 0$, we define:

$$E_{s_{L},loc} := \int_{y \leq N} |\varepsilon(1)|_{k_{0}+L+1}^{2}.$$ 

(3.12)

12 The closeness assumption is described in the next subsection and is compatible with what we are saying here.

13 As the dynamic will be smooth.
(ii) **Low order slightly supercritical Sobolev norm:** We choose a real number $\sigma$ such that:

$$0 < \sigma - s_c \ll 1,$$

and we define:

$$E_\sigma := \int |\nabla^\sigma \varepsilon^{(1)}|^2 + \int |\nabla^{\sigma-1} \varepsilon^{(2)}|^2.$$

Estimates we want to bootstrap and main Proposition: Let $s_0$ denote a large enough real number $s_0 \gg 1$. We recall the definition of the renormalized variables:

$$y = \frac{r}{\lambda(t)}, \ s(t) = s_0 + \int_0^t \frac{d\tau}{\lambda(\tau)}.$$

We introduce notations for the decomposition of the solution in both real and renormalized time:

$$u = \tilde{Q}_{b(t), \frac{1}{\lambda(t)}} + w = \frac{\tilde{Q}_{b(s)} + \varepsilon(s)}{\lambda(t)}.$$

The parameters $b_i$ are choosen as a perturbation of the solution $b^c$:

$$b_i(s) = b_i^c(s) + \frac{U_i(s)}{s^i}.$$

To treat the stable and unstable modes separately, we employ the change of variables coded by the matrix $P_\ell$ defined by (2.76). Instead of $U_1, ..., U_\ell$ we consider:

$$V_i := (P_\ell U)_i \text{ for } 1 \leq i \leq \ell.$$

We assume initially:

(i) Smallness of the unstable modes: Let $0 < \tilde{\eta}$ be a constant to be defined later.

$$(V_2(s_0), ..., V_\ell(s_0)) \in B^{\ell-1} \left( \frac{1}{s_0} \right).$$

(ii) Smallness of the stable modes:

$$V_i(s_0) \leq \frac{1}{10s_0}, \text{ and } |b_i(s_0)| \leq \frac{\varepsilon_i}{10s_0^{1-\alpha_{c_i}}} \text{ for } \ell + 1 \leq i \leq L.$$

(iii) Smallness of the initial perturbation in high and low Sobolev norms:

$$E_{s_L}(s_0) + E_{\sigma}(s_0) < \frac{1}{s_0^{2L+2+2(1-\delta_0)(1+\alpha)}}.$$

(iv) Normalization: up to a fix rescaling, we may always assume:

$$\lambda(s_0) = 1.$$

(v) Regularity and compact support of the initial data:

$$\varepsilon(s_0) \equiv -Q \text{ away from the origin, and } \varepsilon(s_0) \in C^\infty.$$

**Proposition 3.2.** (Existence of an initial data for which the solution stays in a trapped regime:) There exists universal constants for the analysis:

$$0 < \eta = \eta(d, p, L) \ll 1, \ M = M(d, p, L) \gg 1, \ N = N(d, p, L, M) \gg 1, \ K_i = K_i(d, p, L, M) \gg 1, \text{ for } i = 1, 2, \ s_0 = s_0(l, d, p, L, M, K) \gg 1,$$

\footnote{The choice of the constants is done in the next proposition.}

\footnote{The $\frac{1}{10}$ is arbitrary: we just want the initial condition to be smaller than the information we want to bootstrap, see next proposition.}
that such the following fact holds. Given \( \varepsilon(s_0) \) satisfying (3.29), (3.21) and (3.23), and stable parameters \( V_1(s_0), (b_{\ell+1}(s_0), \ldots, b_L(s_0)) \) satisfying (3.20), there exists initial conditions for the unstable parameters \( (V_2(s_0), \ldots, V_\ell(s_0)) \) satisfying (3.19) for which the solution to (NLW) with initial data \( Q_{b(s_0)} + \varepsilon(s_0) \) with:

\[
b(s_0) = b^*(s_0) + (0, \ldots, 0, b_{\ell+1}(s_0), \ldots, b_L(s_0)) + \left( (P^{-1}(V_1(s_0), \ldots, V_{\ell}(s_0), 0, \ldots, 0))_{s_0}, \ldots, (P^{-1}(V_1(s_0), \ldots, V_{\ell}(s_0), 0, \ldots, 0))_{s_0} \right),
\]

admits the following bounds for all \( s \geq s_0 \):

- control of the part on the approximate profiles manifold: for the unstable modes:

\[
(V_2(s), \ldots, V_\ell(s)) \in \mathcal{B}^{\ell-1} \left( \frac{1}{s^\eta} \right).
\]

for the stable modes:

\[
|V_1(s)| \leq \frac{1}{s^\eta}, \quad |b_k(s)| \leq \frac{\varepsilon_k}{s^{k+\eta}}, \quad \text{for } \ell + 1 \leq k \leq L.
\]

- control of the error term:

\[
\mathcal{E}_{\delta}(s) \leq K_1 b_1^{2L + 2(1 - \delta_0)(1 + \eta)}, \\
\mathcal{E}_{\sigma}(s) \leq K_2 b_1^{2(\sigma - s_\ell)\delta \eta}. 
\]

To prove Proposition 3.2 we argue by contradiction and suppose that for all initial data of the unstable modes \( (V_2, \ldots, V_\ell) \in \mathcal{B}^{\ell-1}(s_0^{\eta}) \), the conditions are not met for all time:

\[
s^* = s^*(\varepsilon(s_0), s_0, V_1(s_0), \ldots, V_{\ell}(s_0), b_{\ell+1}(s_0), \ldots, b_L(s_0)) = \sup \{ s \geq s_0 \text{ such that (3.28), (3.26) and (3.27) hold on } [s_0, s] \} < +\infty.
\]

By continuity of the flow and the smallness of the initial perturbation, we know that \( s^* > 0 \). We perform a three steps reasoning to prove the contradiction:

(i) First we show that as long as \( \varepsilon \) is controlled by the estimates (3.28), it does not perturb too much the dynamical system (2.70). That is to say we have a sufficient control over the evolution of the \( b_i \)'s to show that the perturbation \( U \) of the trajectory \( b^* \) evolves according to the linearisation at the leading order.

(ii) (i) has given us control over the part of the solution on the approximate manifold, this allows us to compute the evolution of the scale \( \lambda \). Under the bootstrap conditions we know the size of the error term \( \tilde{\psi}_b \) generated by the approximate dynamics. Once we know the behavior of \( \tilde{\psi}_b \) and \( \lambda \), we can look for better informations about \( \varepsilon \). Indeed we apply an energy method and find out that we control the time evolution of \( \mathcal{E}_{\delta L} \) and \( \mathcal{E}_{\sigma} \). As \( \varepsilon \) is a stable perturbation, we find that we have in fact a better estimate for this term: \( \varepsilon \) is smaller than the estimate given by (3.28). Hence at time \( s^* \) we have:

\[
\mathcal{E}_{\delta L}(s^*) < K_1 b_1^{2L + 2(1 - \delta_0)(1 + \eta)}, \\
\mathcal{E}_{\sigma}(s^*) < K_2 b_1^{2(\sigma - s_\ell)\delta \eta}.
\]

This implies that the exit of the trapped regime is only when the parameters do not satisfy the estimates (3.27) and (3.27) anymore.
(iii) With the estimates we have found regarding the parameters dynamics in (i) we are able to say that this is impossible. Indeed, the stable parameters cannot go away because their dynamics is stable. It is possible for some unstable parameters to go away, but they cannot all leave the ball $B_{\epsilon_0}^{\ell - 1} \left( \frac{1}{\ell - 1} \right)$ in finite time. We have seen in Lemma 2.16 that the $V_i$'s for $2 \leq i \leq \ell$ evolve as a linearized system around a repulsive equilibrium. The true dynamics, adding a small error term to their time evolution, preserves this structure. The dynamics in our case cannot expulse all the orbits away from the equilibrium point: we will show how in that case it would be a contradiction to Brouwer’s fixed point theorem.

Remark 3.3. We are working with nice objects: $C^\infty$ and with a good decay at infinity. Indeed since $\alpha_{b(s_0)}$ is $C^\infty$ with compact support, and $\epsilon(s_0) \equiv -Q$ away from the origin, this means that our original data is $C^\infty$ with compact support. The finite speed of propagation of (NLW) implies that the solution stays with compact support. For (NLW) it is well known that as long as the $L^\infty$ norm of a solution stays bounded, it belongs to $C^\infty$. Because $\alpha_b$ and $Q$ are bounded in $L^\infty$ and so is $\epsilon(s)$ under the bootstrap estimates (see Lemma 2.17), it implies that $\epsilon(s) \in C^\infty$ for $s_0 \leq s < s^\prime$. These two informations imply that all the calculations that will be done throughout the proof are legitimate.

3.2. Equations of evolution for $\epsilon$ and $w$: We recall that we are studying a solution under the form:

$$u = \tilde{Q}_b(t, \frac{1}{\epsilon} \psi_b(t)) + w = (\tilde{Q}_b(s) + \epsilon(s)) \frac{1}{\epsilon} \psi_b(s) ,$$

where $\tilde{Q}_b$ is defined by (2.31) and $\epsilon$ satisfies the orthogonality conditions (3.19), this decomposition being explained in Subsubsection 3.1.2. The evolution of $\epsilon$ and $w$ is given by:

$$\partial_s \epsilon - \frac{1}{\epsilon} \Lambda \epsilon + H(\epsilon) = -Mod(t) + (\frac{\lambda}{\epsilon} + b_1) \Lambda \tilde{Q}_b - \tilde{\psi}_b + F(\tilde{Q}_b + \epsilon) - F(\tilde{Q}_b) - H_b(\epsilon) \quad \text{:= NL}(\epsilon) \quad (3.31)$$

$$+ H(\epsilon) - H_b(\epsilon) \quad \text{:= L}(\epsilon) ,$$

where $H_b$ denotes the linearization close to $\tilde{Q}_b$:

$$H_b := \begin{pmatrix} 0 & -1 \\ -\Delta - pQ_{b}^{p-1} & 0 \end{pmatrix} ; (3.32)$$

and:

$$\partial_t w + H_{\frac{1}{\epsilon}} w = \frac{1}{\epsilon} (-\frac{\lambda}{\epsilon} + b_1) \Lambda \tilde{Q}_b - \frac{\lambda}{\epsilon} \tilde{\psi}_b + F(\tilde{Q}_b, w) - F(\tilde{Q}_b) - H_b \frac{1}{\epsilon} w \quad \text{:= NL}(w) \quad (3.33)$$

$$+ H_{\frac{1}{\epsilon}} w - H_b \frac{1}{\epsilon} w \quad \text{:= L}(w) ,$$

where:

$$H_{\frac{1}{\epsilon}} := \begin{pmatrix} 0 & -1 \\ -\Delta - p(Q_{\frac{1}{\epsilon}})^{p-1} & 0 \end{pmatrix} , \quad \text{and} \quad H_{b,\frac{1}{\epsilon}} := \begin{pmatrix} 0 & -1 \\ -\Delta - p(Q_{b,\frac{1}{\epsilon}})^{p-1} & 0 \end{pmatrix} . (3.34)$$

We notice that the $NL$ and $L$ terms are situated on the second coordinate:

$$NL(\epsilon) = \begin{pmatrix} 0 \\ NL(\epsilon) \end{pmatrix} , \quad NL(w) = \begin{pmatrix} 0 \\ NL(w) \end{pmatrix} , \quad L(\epsilon) = \begin{pmatrix} 0 \\ L(\epsilon) \end{pmatrix} , \quad L(w) = \begin{pmatrix} 0 \\ L(w) \end{pmatrix} . (3.35)$$
We let the new modulation term that now includes the scale change be:

\[
\tilde{Mod}(t) := Mod(t) - \left( \frac{\lambda_s}{\lambda} + b_1 \right) \Lambda \dot{Q}_b.
\]  

### 3.3. Modulation equations

In this section we compute the influence of \( \varepsilon \) on the equations governing the evolution of the parameters \( \lambda \) and \( b \).

**Lemma 3.4** (Modulation estimates). Assume that all the constants involved in Proposition 3.2 are fixed in their range\(^{[9]}\) except \( s_0 \). Then for \( s_0 \) large enough there holds the bounds for \( s_0 \leq s < s^* \):

\[
\left\| \frac{\lambda_s}{\lambda} + b_1 \right\| + \sum_{i=1}^{L} |b_{i,s} + (i - \alpha) b_1 b_i + b_{i+1}| \leq C(M) b_1^{L+3} + C(L, M) b_1 \sqrt{E_{st,1}},
\]

\[
|b_{L,s} + (L - \alpha) b_1 b_L| \leq C(M) \sqrt{E_{st,1}} + C(M) b_1^{L+3}.
\]

**Remark 3.5.** Under the assumption on the smallness of \( \varepsilon \)\(^{[10]}\) This implies in particular that:

\[
\frac{\lambda_s}{\lambda} = -b_1 + O(b_1^2)
\]

and

\[
b_{i,s} = -(i - \alpha) b_1 b_i + b_{i+1} + O(b_1^{i+2})
\]

for \( 1 \leq i \leq L - 1 \). If we had also \( b_{L,s} = -(L - \alpha) b_1 b_L + O(b_1^{L+1+c}) \) for a small constant \( c > 0 \), this would be enough to conclude that the dynamics of the parameters is given at the first order by (2.70). Unfortunately this last condition is not met. We will see how to skirt this problem in the next Lemma 3.6.

**Proof of Lemma 3.4.** We let:

\[
D(t) = \left\| \frac{\lambda_s}{\lambda} + b_1 \right\| + \sum_{i=1}^{L} |b_{i,s} + (i - \alpha) b_1 b_i + b_{i+1}|.
\]

For \( 0 \leq i \leq L \) we take the scalar product of (3.31) with \( H^{*i} \Phi_M \):

\[
\langle \tilde{Mod}(t), H^{*i} \Phi_M \rangle = \langle -H(\varepsilon), H^{*i} \Phi_M \rangle + \langle \frac{\lambda_s}{\lambda} \Lambda \varepsilon, H^{*i} \Phi_M \rangle - \langle \tilde{\psi}_b, H^{*i} \Phi_M \rangle + \langle NL(\varepsilon), H^{*i} \Phi_M \rangle + \langle L(\varepsilon), H^{*i} \Phi_M \rangle.
\]

**Step 1: law for \( \lambda \):** We take \( i = 0 \) in the preceding equation (3.40) and compute all the terms. As \( \Phi_M \) is located on the first coordinate, see (3.4), it gives:

\[
\langle NL(\varepsilon), \Phi_M \rangle = \langle L(\varepsilon), \Phi_M \rangle = 0.
\]

\( \Phi_M \) is of compact support in \( \|y\| \leq M \) and situated on the first coordinate. For \( b_1 \) small enough one has \( \tilde{\psi}_b(y) = \psi_b(y) \), and \( \psi_b \) is situated on the second coordinate from (2.37). Hence:

\[
\langle \tilde{\psi}_b, \Phi_M \rangle = 0.
\]

The linear term is equal to 0 because of the orthogonality conditions (3.9):

\[
\langle -H(\varepsilon), \Phi_M \rangle = 0.
\]

The left hand side, the modulation term, is the one catching the evolution of \( \lambda_s \):

\[
\langle \tilde{Mod}(t), \Phi_M \rangle = \left\langle \frac{\lambda_s}{\lambda} + b_1, \Lambda \dot{Q}_b, \Phi_M \right\rangle + \sum_{i=1}^{L} (b_{i,s} + (i - \alpha) b_1 b_i + b_{i+1})(T_i + \sum_{j=i+1}^{L} \frac{\partial S_i}{\partial n_j}, \Phi_M) = \left\langle \frac{\lambda_s}{\lambda} + b_1, \Lambda Q, \Lambda Q \right\rangle + O(b_1 D(t)).
\]

\(^{[10]}\)It means that, for example, if we wrote \( 0 < C \ll 1 \) that \( C \) is fixed very small
We now estimate the scaling term:
\[ |\left(\frac{\lambda}{\lambda} \Lambda \varepsilon, \Phi_M\right)| \leq |\frac{\lambda}{\lambda} + b_1| |\Lambda^{(1)}\varepsilon^{(1)}, \Phi_M| + b_1 |\Lambda^{(1)}\varepsilon^{(1)}, \Phi_M| \]
\[ \leq (b_1 + D(t)) \|\Lambda \varepsilon^{(1)}\|_{L^2(\mathbb{R}^N)} \|\Phi_M\|_{L^2}. \]
We use the coercivity estimate from Corollary E.4 to relate the $L^2$ norm on the compact set $y \leq M$ to $\mathcal{E}_{sL}$:
\[ \int_{y \leq M} |\varepsilon^{(1)}|^2 = \int_{y \leq M} (1 + y)^{2k_0+2L+2} \frac{|\varepsilon^{(1)}|^2}{1 + y^{2k_0+2L+2}} \leq C(M)\mathcal{E}_{sL}, \]
\[ \int_{y \leq M} |\partial_y \varepsilon^{(1)}|^2 \leq \int_{y \leq M} (1 + y)^{2k_0+2L+2} \frac{|\partial_y \varepsilon^{(1)}|^2}{1 + y^{2k_0+2L+2}} \leq C(M)^2(1+L)^2\mathcal{E}_{sL}. \]
This gives:
\[ |\left(\frac{\lambda}{\lambda} \Lambda^{(1)}\varepsilon^{(1)}, \Phi_M\right)| \leq C(M)(b_1 + D(t))\sqrt{\mathcal{E}_{sL}}. \] (3.45)

Now that we have computed all the terms in (3.40) for $i = 0$, in (3.41), (3.42), (3.43), (3.44) and (3.45), we end up with:
\[ \left|\frac{\lambda}{\lambda} + b_1\right| = O(b_1D(t)) + O((b_1 + D(t))C(M)\sqrt{\mathcal{E}_{sL}}). \] (3.46)

**Step 2:** law of $b_i$ for $1 \leq i \leq L - 1$. We take again equation (3.40) and do the same computations. The $\tilde{\text{Mod}}$ term represents the approximate dynamics:
\[ \langle \tilde{\text{Mod}}(t), \allowbreak H^{s_i} \Phi_M\rangle = \langle \Lambda Q, \Phi_M\rangle (b_{i,s} + (i - \alpha)b_1b_1 - b_{i+1}) + O(b_1D(t)). \] (3.47)

The linear term still disappears because of the orthogonality conditions:
\[ \langle -H(\varepsilon), H^{s_i} \Phi_M\rangle = 0. \] (3.48)

For the scale changing term, as before, thanks to the coercivity of $\mathcal{E}_{sL}$ and to (3.46):
\[ |\left(\frac{\lambda}{\lambda} \Lambda \varepsilon, H^{s_i} \Phi_M\right)| \leq (b_1 + D(t))C(M)\sqrt{\mathcal{E}_{sL}}. \] (3.49)

The error contribution, as $\bar{\psi}_b = \psi_b$ for $y \leq 2M$ (for $s_0$ small enough) is estimated thanks to Proposition 2.11
\[ |\left(\bar{\psi}_b, H^{s_i} \Phi_M\right)| \leq C(M)b_1^{L+3}, \] (3.50)

We now want to estimate the nonlinear contribution. Since $\text{NL}$ is a linear sum of terms of the form $\tilde{Q}_b^{p-k}\varepsilon^{(1)}k$ for $k \geq 2$ we estimate using Cauchy-Schwarz, the $L^\infty$ estimate given in Lemma 2.1 and again the coercivity estimate:
\[ |\langle \tilde{Q}_b^{p-k}\varepsilon^{(1)}k, H^{s_i} \Phi_M\rangle| \leq C(M)\|\varepsilon^{(1)}\|_{L^\infty}^2 \mathcal{E}_{sL} \]
\[ = o(b_1\sqrt{\mathcal{E}_{sL}}), \] (3.51)

in the regime (3.28). Because $(\tilde{Q}_b^{(1)})^{p-1} - Q^{p-1} = O(b_1)$ there holds for the small linear term:
\[ |\langle L(\varepsilon), H^{s_i} \Phi_M\rangle| \leq b_1C(M)\sqrt{\mathcal{E}_{sL}}. \] (3.52)

We have estimated all the terms in (3.40) for $1 \leq i \leq L - 1$, in (3.47), (3.48), (3.49), (3.50), (3.51) and (3.52), it yields:
\[ |b_{i,s} - (i - \alpha)b_1b_1| \leq O(b_1D(t)) + C(M)b_1^{L+3} + C(M)b_1\sqrt{\mathcal{E}_{sL}}. \] (3.53)

**Step 3:** the law of $b_L$. We compute:
\[ \langle \tilde{\text{Mod}}(t), H^{s_L} \Phi_M\rangle = O(b_1D(t)) + (b_{L,s} + (L - \alpha)b_1b_L)\langle \Lambda Q, \Phi_M\rangle. \]
The terms that we previously estimated still admit the same bounds. But the linear term does not disappear in this case. We recall that we have chosen $L$ odd. From the identity (2.23) relating $H^k$ to $L$:

$$\langle H(\varepsilon), H^{*L} \Phi_M \rangle = |\langle H^{L+1} \varepsilon, \Phi_M \rangle| = \left| \int \mathcal{L}_{2+1}^{L+1} \varepsilon \Phi_M \right| \leq C(M) \sqrt{\mathcal{E}_{sL}}.$$  

This gives:

$$\left| \frac{\langle H(\varepsilon), H^{*L} \Phi_M \rangle}{\langle \Phi_M, \Lambda Q \rangle} \right| \lesssim M^{-\delta_0} \sqrt{\mathcal{E}_{sL}}. \quad (3.54)$$

We then conclude that:

$$\left| b_{L,s} - (L - \alpha) b_1 b_L \right| \leq C(M)(b_1 D(t) + b_1^{L+3}) + C(M) \sqrt{\mathcal{E}_{sL}}. \quad (3.55)$$

step 4: reinjection of the bounds. By summing (3.55), (3.53) and (3.46) we find that:

$$D(t) = O(\sqrt{\mathcal{E}_{sL}} + b_1^{L+3}). \quad (3.56)$$

This allows us to go back to the previous estimate of the law of $\lambda$, of the $b_i$’s (3.53), and of $b_L$ (3.56) to obtain the desired estimates (3.37) and (3.38). □

3.4. Improved modulation equation for $b_L$. We have seen in remark 3.5 that the control over the evolution of $b_L$ we found in the last Lemma 3.3 is not sufficient. In fact, this is because our orthogonality conditions approximate a true orthogonal decomposition (which would have been to ask $\langle \varepsilon, T_i \rangle = 0$) and would have implied the vanishing of the bad term $\langle H \varepsilon, T_L \rangle = \langle \varepsilon, -T_{L-1} \rangle = 0$. Nevertheless, we are able to determine which part of $\varepsilon$ contributes in the worst way to the evolution of $b_L$ and to control it. This is the subject of the following lemma:

**Lemma 3.6 (Improved modulation equation for $b_L$).** We recall that $B_0$ is given by (1.28). Assume all the constants involved in Proposition 3.2 are fixed in their range except $s_0$. Then for $s_0$ large enough there hold\(^{17}\) for $s_0 \leq s < s^*$:

$$b_{L,s} + (L - \alpha) b_1 b_L \leq \frac{d}{ds} \left[ \frac{\langle H^L \varepsilon, \chi_{B_0} \Lambda Q \rangle}{\langle \chi_{B_0} \Lambda^{(1)} Q, \Lambda^{(1)} Q \rangle - \frac{\delta_0}{3} \partial_{b_L} \downarrow \downarrow \left( \frac{s_{L+3}}{2} \right)_{L-1} \rangle} \right] \quad (3.57)$$

where $g'$ is the gain in the asymptotic of the profiles $S_i$ defined by (1.21).

**Proof of Lemma 3.6** Step 1: Expression of the time derivative of the numerator. We first compute the time evolution of the numerator of the new term we introduced in (3.6): $\langle H^L \varepsilon, \chi_{B_0} \Lambda Q \rangle$. From the evolution equation for $\varepsilon$ given by (3.31):

$$\frac{d}{ds} \left( \langle H^L \varepsilon, \chi_{B_0} \Lambda Q \rangle \right) = \langle H^L \varepsilon_s, \chi_{B_0} \Lambda Q \rangle + \langle H^L \varepsilon, b_{1,s} y \partial_y \chi \left( \frac{y}{B_0} \right) \Lambda Q \rangle \quad (3.58)$$

We will now compute each term in the right hand side. We first estimate the second term. From the modulation equation (3.37), and under the bootstrap assumptions (3.28) one has $|b_{1,s}| \leq C b_1^2$. We use the expression of $H^L$ given by (2.23), $L$ being

\(^{17}\)The denominator being non null from (3.70).
odd, and the coercivity of $E_{sL}$, see Corollary \ref{cor:coercivity}:

$$
\left| \langle H^L \varepsilon, b_{1,x} \partial_y \chi(\frac{y}{b_0}) \Lambda Q \rangle \right| = \left| \int (1)^{\frac{L+1}{p}} L^{L+1} \varepsilon(2) b_{1,y} \partial_y \chi(\frac{y}{b_0}) \Lambda(1) Q \right|
\leq C b_1^2 \int_{B_0} |\varepsilon(2) \frac{y}{|y|} |
\leq C(M) \int_{B_0} \sqrt{E_{sL}(2k_0 + \delta_0 + 2)}
\leq C(M) \sqrt{E_{sL}} b_1^{2k_0 + \delta_0}
$$

(3.59)

where we used the asymptotic \eqref{eq:2.2} of $\Lambda(1) Q$ (and we recall that $f_k$ stands for the $k$-th adapted derivative of $f$ given by \eqref{eq:2.19}). We now aim at estimating the other term in the right hand side of \eqref{eq:3.58}. We compute using again the expression of $H^L$ given by \eqref{eq:2.23} and the fact that $L$ is odd:

$$
\left| \int \chi_{B_0} \Lambda(1) Q \left( -L \varepsilon(1) + \frac{1}{\gamma} \Lambda(2) \varepsilon(2) - Mod(t)(2) - \psi(2) + NL(\varepsilon) + L(\varepsilon) \right) \right|_{L-1},
$$

(3.60)

and we now estimate all the terms in the right hand side.

- \textbf{$\mathcal{L} \varepsilon(1)$ term:} There holds using coercivity and the fact that $A(1) Q = 0$:

$$
\left| \int \chi_{B_0} \Lambda(1) Q(\mathcal{L} \varepsilon(1))_{L-1} \right| \leq C \int \frac{2B_0}{y^{1+\gamma}} |\varepsilon(1)| \leq C(M) \sqrt{E_{sL}} b_1^{2k_0 + \delta_0}
$$

(3.61)

- \textbf{$\Lambda(2) \varepsilon(2)$ term:} Again, using the same arguments, as $\frac{|\lambda|}{\gamma} \leq C b_1$ from \eqref{eq:3.37}:

$$
\left| \int \chi_{B_0} \Lambda(1) Q \frac{1}{\gamma} (\Lambda(2) \varepsilon(2))_{L-1} \right| \leq C b_1 \int \frac{2B_0}{y^{1+\gamma}} |\varepsilon(2)| \leq C(M) b_1 \sqrt{E_{sL}} b_1^{2k_0 + \delta_0 + 2}
$$

(3.62)

- \textbf{$\bar{\psi}(2)$ term:} Because we are in the zone $B_0$ we do not see the bad tail. We can then use the improved bound of Proposition \ref{prop:2.11}:

$$
\left| \int \chi_{B_0} \Lambda(1) Q(\bar{\psi}(2))_{L-1} \right| = \left| \int \chi_{B_0} \Lambda(1) Q(\bar{\psi}(2))_{L-1} \right| \leq \left| \Lambda(1) Q \right|_{L^2(\leq B_0)} \left| \psi(2) \right|_{L^2(\leq B_0)} \leq C b_1^{L + 2 - 2k_0 - 2\delta_0 + 2}
$$

(3.63)

- \textbf{$NL(\varepsilon)$ term:} By duality we put all the derivatives on $\Lambda(1) Q$:

$$
\left| \int \chi_{B_0} \Lambda(1) Q(NL(\varepsilon))_{L-1} \right| = \left| \int (\chi_{B_0} \Lambda(1) Q)_{L-1} NL(\varepsilon) \right| \leq C \int \frac{2B_0}{y^{1+\gamma}} |NL(\varepsilon)|
$$

We know that $NL(\varepsilon)$ is a sum of terms of the form: $Q^{p-k} \varepsilon(1)^k$ for $k > 2$. So from the asymptotic \eqref{eq:2.11} of $Q$ and using coercivity:

$$
\left| \int \frac{2B_0}{y^{1+\gamma}} Q^{p-k} \varepsilon(1)^k \right| \leq C \parallel \varepsilon(1) \parallel_{L^\infty} \int \frac{2B_0}{y^{1+\gamma}} b_1^{p-k} \parallel \varepsilon(1) \parallel_{L^2(\leq B_0)} \leq C(M) \parallel \varepsilon(1) \parallel_{L^\infty} b_1^{2k_0 + \delta_0 + 2} + C b_1^{p-k}.
$$
We now use the estimate provided by Lemma F.1:
\[
\| \varepsilon^{(1)} \|_{L^\infty} \leq C(M, K_1, K_2) \sqrt{E_{\sigma} b_1^T} \frac{d - \sigma + \frac{1}{2}}{L} + O(\frac{1}{L^{n/4}})
\]
\[
\leq C(M, K_1, K_2) \left( \frac{\varepsilon_0}{b_1^T} \right) b_1^T \frac{1}{L} + O(\frac{1}{L^{n/4}}).
\]

Therefore:
\[
\left| \int_{B_0}^{2B_0} \frac{1}{y^T + L - 1} Q^p - k \varepsilon^{(1)} \right| \leq C(M, K_1, K_2) \left( \frac{\varepsilon_0}{b_1^T} \right)^{k-1} \sqrt{E_{sL} b_1}^{-(2k_0 + \delta_0)} + O(\frac{1}{L^{n/4}}).
\]

Under the bootstrap estimate, for \( s_0 \) small enough this gives:
\[
\left| \int \chi_{B_0} \Lambda^{(1)}(1) QNL(\varepsilon^{(1)}) \leq 1 \right| \leq \sqrt{E_{sL} b_1}^{-(2k_0 + \delta_0)}. \tag{3.64}
\]

Indeed, the constant \( s_0 \) being chosen after all the other constants, we can increase \( s_0 \) to erase the dependence on the other constant in the preceding equation.

\[ \bullet \text{ } L(\varepsilon) \text{ term:} \]
\[
\left| \int \chi_{B_0} \Lambda^{(1)}(1) Q(L(\varepsilon)) \right| \leq C \int_{B_0}^{2B_0} \frac{1}{y^T + L - 1} |Q_b^{p-1} - Q^{p-1}| \varepsilon^{(1)}.
\]

We use the degeneracy of the potential: \( Q_b^{p-1} - Q^{p-1} \leq \frac{C}{1 + y^T + L} \) to estimate:
\[
\left| \int \chi_{B_0} \Lambda^{(1)}(1) Q(L(\varepsilon)) \right| \leq C \int_{B_0}^{2B_0} \frac{1}{y^T + L + \frac{1}{2}} |\varepsilon^{(1)}| \leq C(M) \sqrt{E_{sL} b_1}^{-(2k_0 + \delta_0)} b_1^Q. \tag{3.65}
\]

\[ \bullet \text{ } Mod(t)^{(2)} \text{ term:} \]

From the localization of the \( T_i \) and \( S_i \)'s (2.26) and (2.41):
\[
\int \frac{\text{Mod}(t)^{(2)}}{\Lambda^{(1)}(1) Q} L_1 \chi_{B_0} = \int \frac{\text{Mod}(t)^{(2)}}{\Lambda^{(1)}(1) Q} L_1 \chi_{B_0}.
\]

We compute from the fact that \( H(T_L) = (-1)^L \Lambda Q \):
\[
\int (T_L) L_1 \chi_{B_0} \Lambda^{(1)}(1) Q = (-1)^{L-1} \int |\Lambda^{(1)}(1) Q|^2 \chi_{B_0}.
\]

For \( i < L \), as \( (T_i)_{L-1} = 0 \) we have:
\[
\left| \int (T_i) \delta_{\text{mod}2,1} + \sum_{j \geq i+1, \text{ j odd}} \frac{\partial S_j^{(2)}}{\partial b_i} \right| L_1 \chi_{B_0} \Lambda^{(1)}(1) Q = \left| \int \sum_{j \geq i+1} \frac{\partial S_j^{(2)}}{\partial b_i} \right| L_1 \chi_{B_0} \Lambda^{(1)}(1) Q \leq CB_1^{(2k_0 + 2\delta_0)}.
\]

And for the last term there holds the bound:
\[
\left| \int (\Lambda^{(2)} \chi_{B_0} \Lambda^{(1)}(1) Q) \right| \leq CB_1^{(2k_0 + 2\delta_0)}
\]
We then conclude, using the majoration obtained in the previous Lemma \(3.4\) for the evolution of the \(b_i\)'s and \(\lambda\), that for the \(M^{\hat{d}}(t)\) term:

\[
\int M^{\hat{d}}(t)^{L-1} \chi B_0 \Lambda^{(1)} Q
\]

\[
= (b_{L,s} + (L - \alpha) b_1 b_L) \left[ \left(-\frac{L-1}{2}\right) \int (\Lambda^{(1)} Q)^2 \chi B_0 + \left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right) L-1 \chi B_0 \Lambda^{(1)} Q \right] + O(\sqrt{\mathcal{E}_s b_1^{(2k_0+\delta_0)}} + b_1^{L+3-(2k_0+\delta_0)}) \quad (3.66)
\]

(From now on we use the \(O()\) notation, the constants hidden depending only on \(M\)). We now collect all the estimates \((3.61), (3.62), (3.63), (3.64), (3.65)\) and \((3.66)\) and inject them in \((3.60)\) to find that the first term in the right hand side of \((3.60)\) is:

\[
\langle H^L (\varepsilon, \chi B_0 \Lambda Q) \rangle = (b_{L,s} + (L - \alpha) b_1 b_L) \left\langle \chi B_0 \Lambda^{(1)} Q, \Lambda^{(1)} Q + (-1) \left(-\frac{L-1}{2}\right) \left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right) L-1, \chi B_0 \Lambda^{(1)} Q \right\rangle + O(\sqrt{\mathcal{E}_s b_1^{(2k_0+\delta_0)}} + b_1^{L+1-2k_0-2\delta_0+g'}). \quad (3.67)
\]

With the two computations \((3.67)\) and \((3.59)\), the time evolution of the numerator given by \((3.68)\) is now:

\[
\frac{d}{dt}(H^L \varepsilon, \chi B_0 \Lambda Q) = (b_{L,s} + (L - \alpha) b_1 b_L) \left\langle \chi B_0 \Lambda^{(1)} Q, \Lambda^{(1)} Q + (-1) \left(-\frac{L-1}{2}\right) \left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right) L-1, \chi B_0 \Lambda^{(1)} Q \right\rangle + O(\sqrt{\mathcal{E}_s b_1^{(2k_0+\delta_0)}} + b_1^{L+1-2k_0-2\delta_0+g'}). \quad (3.68)
\]

Step 2: end of the computation. We have thanks to our previous estimate \((3.68)\):

\[
\frac{d}{ds} \left[ \langle H^L \varepsilon, \chi B_0 \Lambda Q \rangle \right] = \langle H^L \varepsilon, \chi B_0 \Lambda Q \rangle + \frac{O(\sqrt{\mathcal{E}_s b_1^{(2k_0+\delta_0)}} + b_1^{L+1-(2k_0+2\delta_0)+g'})}{\langle \chi B_0 \Lambda^{(1)} Q, \Lambda^{(1)} Q + (-1) \left(-\frac{L-1}{2}\right) \left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right) L-1, \chi B_0 \Lambda^{(1)} Q \rangle} - \langle H^L \varepsilon, \chi B_0 \Lambda Q \rangle \quad (3.69)
\]

From the asymptotic of \(\Lambda^{(1)} Q\) and \(S_{L+2}\), the denominator has the following size:

\[
\left\langle \chi B_0 \Lambda^{(1)} Q, \Lambda^{(1)} Q + (-1) \left(-\frac{L-1}{2}\right) \left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right) L-1, \chi B_0 \Lambda^{(1)} Q \right\rangle \sim C b_1^{-2k_0-2\delta_0}, \quad (3.70)
\]

for some constant \(C > 0\). So the second term in the right hand side of \((3.69)\) is:

\[
\left| \frac{O(\sqrt{\mathcal{E}_s b_1^{(2k_0+\delta_0)}} + b_1^{L+1-(2k_0+2\delta_0)+g'})}{\langle \chi B_0 \Lambda^{(1)} Q, \Lambda^{(1)} Q + (-1) \left(-\frac{L-1}{2}\right) \left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right) L-1, \chi B_0 \Lambda^{(1)} Q \rangle} \right| \leq C(M) \left( b_1^{-\delta_0} \sqrt{\mathcal{E}_s} + b_1^{L+1+g'} \right) \quad (3.71)
\]
We now estimate the third term in the right hand side of (3.69). We have by coercivity of the adapted norm:

$$
|\langle H^L \varepsilon, \chi B_0 \Lambda Q \rangle| \leq C \int_{B_0}^{2B_0} \frac{\varepsilon(2)}{y^7 + L-1} \leq C(M) \sqrt{E_{sL}} b_1^{s_0 - 2k_0 + (2k_0 + \delta_0) - 1}.
$$

(3.72)

As $\frac{\partial S^{(2)}_{L'))}{\partial b_L}$ does not depend on $b_L$, we obtain using the modulation bound (3.37) for $b_1, \ldots, b_{L-1}$:

$$
\left| \frac{d}{ds} \left[ \frac{\langle H^L \varepsilon, \chi B_0 \Lambda Q \rangle}{\langle \chi B_0 \Lambda Q, \Lambda Q \rangle} \right] \right| \leq C(M) b_1.
$$

The third term in the right hand side of (3.69) then admits the bound:

$$
\left| \frac{d}{ds} \left[ \frac{\langle H^L \varepsilon, \chi B_0 \Lambda Q \rangle}{\langle \chi B_0 \Lambda Q, \Lambda Q \rangle} \right] \right| \leq \frac{C(M) b_1^\delta}{\sqrt{E_{sL}}}.
$$

(3.73)

The identity (3.69), with the bounds on the terms (3.71) and (3.73), gives:

$$
\frac{d}{ds} \left[ \frac{\langle H^\nu \varepsilon, \chi B_0 \Lambda Q \rangle}{\langle \chi B_0 \Lambda Q, \Lambda Q \rangle} \right] = (b_L, \sigma + (L - \alpha) b_L)
$$

$$
+ O(\sqrt{E_{sL}} b_1^{s_0} + b_1^{L+1+\delta'}),
$$

the constant hidden in the $O()$ depending on $M$ (and $L$ of course but we do not track the dependence on this constant anymore).

3.5. Lyapunov monotonicity for the low Sobolev norm: As it appeared in the previous subsections, the key estimate in our analysis is the one concerning the high Sobolev norm. Nonetheless, to have an idea how the lower derivatives behave, and to close an estimate for the nonlinear term in the next section, we start by computing an energy estimate on the low Sobolev norm. We define:

$$
\nu := \frac{\alpha}{L - \alpha},
$$

(3.74)

so that $1 + \nu = \frac{L}{L - \alpha}$ and that the condition (3.28) for $E_\sigma$ can be rewritten as:

$$
E_\sigma \leq K_2 b_1^{2(s - s_c)(1 + \nu)}
$$

(3.75)

**Proposition 3.7.** (Lyapunov monotonicity for the low Sobolev norm) Assume all the constants involved in Proposition 3.2 are fixed in their range, except $s_0$ and $\eta$. Then for $s_0$ large enough and $\eta$ small enough there holds for $s_0 \leq s < s^*$:

$$
\frac{d}{dt} \left\{ \frac{E_\sigma}{\chi^{2(\sigma - s)}} \right\} \leq b_1 \sqrt{E_\sigma} b_1^{(\sigma - s_c)(1 + \nu)} \left[ \frac{\sigma - s_c}{\nu} + b_1^{s_0 + O\left(\frac{\sigma - s_c}{\nu}\right)} \right] b_1^{s_0 + O\left(\frac{\sigma - s_c}{\nu}\right)} + \sum_{k=2}^{p} \left[ \frac{\nu \sqrt{E_\sigma}}{b_1^{k-s_c}} \right]^{k-1}
$$

(3.76)

(the norm $E_\sigma$ was defined in (3.14)).
Proof of Proposition 3.7.\footnote{\(\text{\cite{2213}}\)} To prove this proposition we will compute the derivative with respect to time of \(\frac{E_\sigma}{\lambda^{\sigma-s_c}}\) and estimate it in the trapped regime using (3.28) and the size of the error given by Proposition 2.13. From the evolution of \(w\) given by (3.33) we first compute the following identity:

\[
\frac{d}{dt} \left\{ \frac{\xi}{\lambda^{\sigma-s_c}} \right\} = \frac{d}{dt} \left\{ \int |\nabla^\sigma w(1)|^2 + |\nabla^\sigma -1 w(2)|^2 \right\} = \int \nabla^\sigma w(1).\nabla^\sigma (w(2) + \frac{1}{\lambda}(-\text{Mod}(t))^{(1)} - \psi(b(1))) + \int \nabla^\sigma -1 w(2).\nabla^\sigma -1 (-\mathcal{L}u(1) + \frac{1}{\lambda}(-\text{Mod}(t))^{(2)} - \psi(b(2)) + NL(w) + L(w))\tag{3.77}
\]

Step 1: estimate on each term. We will now estimate everything in the right hand side of (3.77).

\begin{itemize}
\item Linear terms: Because the norm we are using is adapted to a wave equation we have:
\[
\int \nabla^\sigma w(1).\nabla^\sigma w(2) - \nabla^\sigma -1 w(2).\nabla^\sigma -1 \mathcal{L}u(1) = \int \nabla^\sigma -1 w(2).\nabla^\sigma -1 pQ^\sigma -1 w(1) \leq \| \nabla^\sigma w(2) \|_{L^2} \| \nabla^\sigma -2 (Q^\sigma -1 w(1)) \|_{L^2}.
\]

We now use the asymptotic behavior \(Q^\sigma - 1 \sim \frac{c}{x^2}\) \((c > 0)\) and the weighted Hardy estimate from Lemma 3.2.

\[
\| \nabla^\sigma -2 (Q^\sigma -1 w(1)) \|_{L^2} \leq C \| \nabla^\sigma w(1) \|_{L^2} = C \frac{\sqrt{E_\sigma}}{\lambda^{\sigma-s_c}}.
\]

The other term is estimated by interpolation. Indeed as \(\| \nabla^{sL-1} \|_{L^2} \leq C \| \nabla^\sigma w(2) \|_{L^2} \leq C \frac{\sqrt{E_\sigma}}{\lambda^{\sigma-s_c}}\) from Corollary 3.4

\[
\| \nabla^\sigma w(2) \|_{L^2} \leq \frac{C(M)}{\lambda^{\sigma-s_c+1}} \sqrt{E_\sigma}^{1-\frac{1}{sL-\sigma}} \sqrt{E_\sigma}^{1-\frac{1}{sL-\sigma}}.
\]

We have the following estimate under the bootstrap conditions (3.28):

\[
\sqrt{E_\sigma}^{1-\frac{1}{sL-\sigma}} \sqrt{E_\sigma}^{1-\frac{1}{sL-\sigma}} \leq C(K_1, K_2, M)b_1^{(\sigma-s_c)}(1+\nu) b_1^{\frac{1}{sL-\sigma}}(L+(1-\delta_0)(1+\eta)-(\sigma-s_c)(1+\nu))
\]

and from: \(\frac{L+(1-\delta_0)(1+\eta)-(\sigma-s_c)(1+\nu)}{sL-\sigma} = 1 + \frac{(1-\delta_0)\nu}{L} + O\left(\frac{1}{L}\right)\) we conclude that:

\[
\| \nabla^\sigma w(1).\nabla^\sigma w(2) - \nabla^\sigma -1 w(2).\nabla^\sigma -1 \mathcal{L}u(1) \|_{L^2} \leq C(K_1, K_2, M)b_1^{\frac{1}{sL-\sigma}}(1+\nu) \| \nabla^{(\sigma-s_c)}(1+\nu) \|_{L^2} \leq C(K_1, K_2, M)b_1^{\frac{1}{sL-\sigma}}(1+\nu) \| \nabla^{(\sigma-s_c)}(1+\nu) \|_{L^2} + O\left(\frac{1}{L}\right).	ag{3.78}
\]

\item Mod(t) terms: We only treat the \(\text{Mod}(t)^{(2)}\) terms, the computation being the same for the first coordinate.

\[
\left| \frac{1}{\lambda} \int \nabla^\sigma -1 w(2),\nabla^\sigma -1 \text{Mod}(t)^{(2)} \right| \leq \frac{1}{\lambda^{2(\sigma-s_c)}} \frac{1}{\lambda} \sqrt{E_\sigma} \| \nabla^\sigma -1 \text{Mod}(t) \|_{L^2}.
\]

We compute thanks to the previous estimate on the modulation, see Lemma 3.3.

\[
\| \nabla^\sigma -1 \text{Mod}(t)^{(2)} \|_{L^2} \leq (\sqrt{E_\sigma} + b^{L+3}) \left( \sum_{i<j \leq L+2} \| \nabla^\sigma -1 \chi B_1 \frac{\partial \mathcal{S}^{(2)}}{\partial s_0} \|_{L^2} + \sum_{l=0}^{L} \chi B_1 \nabla^\sigma -1 T_1^{(2)} \right) \|_{L^2} \leq C(M)\left( 1+\frac{1}{\lambda} \right) \| \nabla^\sigma -1 \text{Mod}(t) \|_{L^2}.
\]

\[
\leq \frac{C(M)}{\lambda^{2(\sigma-s_c)}} \| \nabla^\sigma -1 \text{Mod}(t) \|_{L^2}.
\]

\end{itemize}
Hence, treating similarly the other coordinate:

$$\left| \frac{1}{\lambda} \int \nabla^{\sigma-1} w^{(2)} \cdot \nabla^{\sigma-1} \tilde{M}(t)_{b,0}^{(2)} + \nabla^{\sigma} w^{(1)} \cdot \nabla^{\sigma} \tilde{M}(t)_{b,0}^{(1)} \right| \leq C(M) \frac{b_1 \sqrt{E_\sigma} b_1^{(\sigma-s_\varepsilon) + \alpha}}{\lambda^{2(\sigma-s_{\varepsilon})+1}}. \tag{3.79}$$

- $\tilde{\psi}_b$ term: Again we just compute for the first coordinate $\tilde{\psi}_b^{(1)}$, because we can treat the second one exactly the same way.

$$\left| \frac{1}{\lambda} \int \nabla^{\sigma} w^{(1)} \cdot \nabla^{\sigma} \tilde{\psi}_b^{(1)} \right| \leq \frac{1}{\lambda^{\alpha}} \frac{1}{\lambda \sqrt{E_\sigma}} \| \nabla^{\sigma} \tilde{\psi}_b^{(1)} \|_{L^2}. \tag{3.80}$$

We can estimate using proposition 2.13

$$\| \nabla^{\sigma} \tilde{\psi}_b^{(1)} \|_{L^2} \leq C b_1^{(1-\delta_b)+\sigma-k_0-C\eta} = C b_1^{(\sigma-s_{\varepsilon})+\alpha-C\eta+1}.$$ 

Hence for $\eta$ small enough:

$$\left| \frac{1}{\lambda} \int \nabla^{\sigma} w^{(1)} \cdot \nabla^{\sigma} \tilde{\psi}_b^{(1)} \right| \leq \frac{C}{\lambda^{\alpha}} \frac{b_1 \sqrt{E_\sigma}}{\lambda} \| \nabla^{\sigma} \tilde{\psi}_b^{(1)} \|_{L^2}.\tag{3.80}$$

The same computation for the second coordinate gives the same result, hence the error’s contribution is:

$$\left| \frac{1}{\lambda} \int \nabla^{\sigma} w^{(1)} \cdot \nabla^{\sigma} \tilde{\psi}_b^{(1)} + \frac{1}{\lambda} \int \nabla^{\sigma-1} w^{(2)} \cdot \nabla^{\sigma-1} \tilde{\psi}_b^{(2)} \right| \leq \frac{C}{\lambda^{\alpha}} \frac{b_1 \sqrt{E_\sigma} b_1^{(\sigma-s_{\varepsilon}) \frac{2\alpha}{\lambda}}}{\lambda}.$$ 

- $L(w)$ term: First using Cauchy-Schwarz:

$$\left| \int \nabla^{\sigma-1} w^{(2)} \cdot \nabla^{\sigma-1} (L(w)) \right| \leq \frac{\sqrt{E_\sigma}}{\lambda^{2(\sigma-s_{\varepsilon})+1}} \| \nabla^{\sigma-1} L(w) \|_{L^2}.\tag{3.80}$$

Now we have that $L(w) = (pQ^{p-1} - p\tilde{Q}_{b_0}^{p-1}) w^{(1)}$. From the asymptotics of the profiles $T_i$ and $S_i$, the potential here enjoys the following bounds:

$$\left| \partial_y^k (pQ^{p-1} - \tilde{Q}_{b_0}^{p-1}) \right| \leq C b_1 \frac{1}{y^{1+\alpha-C(L)\eta}}.$$ 

It allows us to use the fractional Hardy estimate from Lemma 1.2

$$\| \nabla^{\sigma-1} L(w) \|_{L^2} \leq C b_1 \| \nabla^{\sigma+\frac{1}{p-1}} w^{(1)} \|_{L^2},$$

because $\sigma + \frac{1}{p-1} < \frac{d}{2}$, and because for $\eta$ small enough one has: $\alpha \geq C(L)\eta \geq \frac{1}{p-1}$ (as $\alpha > 2$). In the trapped regime one has by interpolation:

$$\| \nabla^{\sigma+\frac{1}{p-1}} w^{(1)} \|_{L^2} \leq \frac{C(M) \sqrt{E_\sigma}}{\lambda^{\alpha}} \sqrt{E_{\sigma L}}^{p-1} \sqrt{E_{\sigma L}^{(\sigma-s_{\varepsilon})}} \frac{1}{b_1^{(\sigma-s_{\varepsilon})+1+O(\frac{1}{\lambda})}}.$$ 

Therefore we end up with the following bound on the small linear term:

$$\left| \int \nabla^{\sigma-1} w^{(2)} \cdot \nabla^{\sigma-1} (L(w)) \right| \leq C(K_1, K_2, M) \frac{b_1 \sqrt{E_\sigma}}{\lambda^{2(\sigma-s_{\varepsilon})+1}} b_1^{(\sigma-s_{\varepsilon})+1+O(\frac{1}{\lambda})}.\tag{3.81}$$

- $NL$ term: We start by integrating by parts and using Cauchy-Schwarz:

$$\leq \frac{1}{\lambda^{\alpha}} \| \nabla^{\sigma} \tilde{M}(t)_{b,0}^{(2)} \|_{L^2} \| \nabla^{\sigma-2+(k-1)(\sigma-s_{\varepsilon})} \|_{L^2}.$$

$$\leq \frac{1}{\lambda^{\alpha}} \| \nabla^{\sigma} \tilde{M}(t)_{b,0}^{(2)} \|_{L^2} \| \nabla^{\sigma-2+(k-1)(\sigma-s_{\varepsilon})} \|_{L^2}.$$

$$\leq \frac{1}{\lambda^{\alpha}} \| \nabla^{\sigma} \tilde{M}(t)_{b,0}^{(2)} \|_{L^2} \| \nabla^{\sigma-2+(k-1)(\sigma-s_{\varepsilon})} \|_{L^2}.$$

$$\leq \frac{1}{\lambda^{\alpha}} \| \nabla^{\sigma} \tilde{M}(t)_{b,0}^{(2)} \|_{L^2} \| \nabla^{\sigma-2+(k-1)(\sigma-s_{\varepsilon})} \|_{L^2}.$$
The first term is estimated via interpolation, and gives under the bootstrap conditions:
\[
\| \nabla^{-(k-1)(\sigma-s_c)} \varepsilon^{(2)} \|_{L^2} \leq C(M) \sqrt{\varepsilon}^{-(1-(k-1)(\sigma-s_c))} \varepsilon^{\frac{1-(k-1)(\sigma-s_c)}{2}} \sqrt{E^{(1-(k-1)(\sigma-s_c))}} L^{1-(k-1)(\sigma-s_c)}.
\]
\[
\leq C(M, K_1, K_2) b_1^{-(1+(k-1)(\sigma-s_c))} + O\left(\frac{\varepsilon}{M+1}\right).
\]
(3.83)

We now estimate the second one. We know that \(NL(\varepsilon)\) is a linear combination of terms of the form: \(\tilde{Q}_b^{(1)(p-k)} \varepsilon^{(1)} k\) for \(2 \leq k \leq p\). We know also that here we have:
\[
\partial^j \tilde{Q}_b^{(1)(p-k)} \leq \frac{\varepsilon}{\sqrt{y^p + (p-k)}}.
\]
So using the weighted and fractional hardy estimate of Lemma D.2,
\[
\| \nabla^{p-2+(k-1)(\sigma-s_c)} (Q^{p-k} \varepsilon^{(1)} k) \|_{L^2} \leq C \| \nabla^{2-\frac{2}{p}=(p-k)+(k-1)(\sigma-s_c)} (\varepsilon^{(1)} k) \|_{L^2}.
\]
We let \(\tilde{\sigma} = E[\sigma - 2 + \frac{2}{p-1} (p-k) + (k-1)(\sigma-s_c)]\) so that:
\[
\sigma - 2 + \frac{2}{p-1} (p-k) + (k-1)(\sigma-s_c) = \tilde{\sigma} + \delta_\sigma,
\]
with \(0 \leq \delta_\sigma < 1\). Developing the entire part of the derivative yields:
\[
\| \nabla^{2-\frac{2}{p}(p-k)+(k-1)(\sigma-s_c)} (\varepsilon^{(1)} k) \|_{L^2} = \| \nabla^{\delta_\sigma} (\nabla^{(1)} k) \|_{L^2}.
\]
We develop the \(\nabla^{\delta_\sigma} (v^{(1)} k)\) term: it is a linear combination of terms of the form:
\[
\prod_{j=1}^k \nabla^{l_j} \varepsilon^{(1)},
\]
for \(\sum_{j=1}^k l_j = \tilde{\sigma}\). We recall the standard commutator estimate:
\[
\| \nabla^{\delta_\sigma} (uv) \|_{L^2} \leq C \| \nabla^{\delta_\sigma} u \|_{L^{p_1}} \| v \|_{L^{p_2}} + C \| \nabla^{\delta_\sigma} v \|_{L^{p_1}} \| u \|_{L^{p_2}},
\]
for \(\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q}\) provided \(1 < q, p_1, p_2 < +\infty\) and \(1 \leq p_2, p_2 \leq +\infty\). So by iteration we have that:
\[
\| \nabla^{2-\frac{2}{p}(p-k)+(k-1)(\sigma-s_c)} (\varepsilon^{(1)} k) \|_{L^2} \leq C \sum_{j=1}^k \prod_{i=1}^k \| \nabla^{l_j} \varepsilon^{(1)} \|_{L^{p(j)_i}},
\]
with \(l(j)_i = l_i + \delta_\sigma \delta_{i=j}\) and with \(\sum \frac{1}{p(j)_i} = \frac{1}{2}\). We have for any \(i\) and \(j\): \(l(j)_i < \sigma\). Hence we can use Sobolev injection to find:
\[
\nabla^{l(j)} \varepsilon^{(1)} \in L^{p(j)_i},
\]
for \(p(j)_i^* = \frac{2d}{d-2\sigma+2l(j)_i}\). We compute (the strategy was designed to obtain this):
\[
\frac{1}{p(j)_i^*} = \sum_{i=1}^k \frac{1}{p(j)_i^*} = \frac{1}{2}.
\]
So we take \(p(j)_i = p(j)_i^*\). We then have:
\[
\| \nabla^{2-\frac{2}{p}(p-k)+(k-1)(\sigma-s_c)} (NL(\varepsilon)) \|_{L^2} \leq C \sqrt{\varepsilon}^{\frac{k}{2}}.
\]
(3.84)
The Cauchy-Schwarz inequality (3.82), with the estimates for the two terms (3.83) and (3.84) give eventually:
\[
\leq \frac{C(K_1, K_2, M b_1 \sqrt{\varepsilon})}{\lambda^{2(\sigma-s_c)+1}} \left( \frac{\varepsilon}{b_1} \right)^{\frac{1}{2}} \| \nabla^{\sigma-1} (NL(w)) \|_{L^2} \leq C \lambda^{2(\sigma-s_c)+1} \left( \frac{\varepsilon}{b_1} \right)^{\frac{1}{2}} \| \nabla^{\sigma-1} (NL(w)) \|_{L^2} + O\left(\frac{\varepsilon}{M+1}\right).
\]
(3.85)
Step 2: Gathering the bounds. We have made the decomposition (3.77) and have found an upper bound for all terms in the right hand side in (3.78), (3.79), (3.80), (3.81) and (3.85). So we get:

$$\frac{d}{dt} \left\{ \frac{E}{\lambda^{s-c}} \right\} \leq \frac{C(K_1,K_2,M)}{\lambda^{s+c}} \sqrt{\mathcal{E}_s} b_1 \lambda^{s-c} + b_1^{\frac{4}{p}+O\left( \frac{1}{L} \right)} (1+\nu) \left( b_1^{\frac{4}{p}+O\left( \frac{1}{L} \right)} + b_1^{\frac{1}{1+\nu}+O\left( \frac{1}{L} \right)} + b_1^{\frac{1}{1+\nu}+O\left( \frac{1}{L} \right)} \right) 18$$

We see that if one choose $\sigma - s_c$ small enough there holds:

$$\frac{\alpha}{2L} < \min \left( \frac{\alpha}{L} + O\left( \frac{\sigma - s_c}{L} \right), \alpha - \nu(\sigma - s_c), \frac{3}{4} \alpha - \nu(\sigma - s_c), \frac{1}{p-1} + O\left( \frac{1}{L} \right) \right).$$

In the trapped regime we recall that $b_1 \sim E$ is small, so that $b_1^{\alpha} \ll b_1^\alpha$ if $b \ll a$. Consequently by taking $s_0$ big enough to "erase" the constants, (3.86) combined with (3.87) give the result of the proposition.

3.6. Lyapunov monotonicity for the high Sobolev norm: We have seen that in order to control the evolution of the parameters, we need to control the high Sobolev norm $\mathcal{E}_s$. Indeed, the law of $b_s$ is computed when projecting the dynamics onto $H^s$. Which involves at least to control $L$ derivative. Why do we look at the $k_0 + 1 + L$-th derivative? Because it is only when deriving at least $k_0 + 1$ more times that we gain something on the error term $\psi_k$: the $\eta$ gain (see proposition 2.13). However, if we look at a higher order derivative (> $k_0 + L + 1$) we loose the control of the solution by lack of Hardy inequalities (Corollary 4.3 does not work at a higher level of regularity). For these reasons, the choice $L + k_0 + 1$ is sharp.

We state here a control on the evolution of $\mathcal{E}_s$, and prove it. We will not be able to estimate it directly, a local part will require the study of a Morawetz type quantity. This is the subject of the following subsection.

**Proposition 3.8.** (Lyapunov monotonicity for the high Sobolev norm:) Suppose all the constants of Proposition 3.2 are fixed, except $s_0$ and $\eta$. Then for $s_0$ large enough and $\eta$ small enough there holds for $s_0 \leq s < s^*$:

$$\frac{d}{dt} \left\{ \frac{\mathcal{E}_s}{\lambda^{s-c}} \right\} + O\left( \frac{\mathcal{E}_s b_1^{s-c}}{\lambda^{s-c}} \right) \leq C(M) \lambda^{2(s-c)} b_1 + O\left( \frac{\mathcal{E}_s b_1^{s-c}}{\lambda^{s-c}} \right) \left( \mathcal{E}_s b_1^{s-c} + \mathcal{E}_s b_1^{s-c} \right) 18 \sum_{k=2}^p \left( \frac{C}{b_1} \right)^{k-1}$$

$$+ C(N)\mathcal{E}_{s_{L,loc}} + \frac{\mathcal{E}_s}{N^{1/2}} + \frac{\mathcal{E}_s b_1^{s-c}}{N^{1/2}}$$

the constant hidden in the $O()$ in the left hand side depending on $M$ (the norms $\mathcal{E}_s$ and $\mathcal{E}_{s_{L,loc}}$ are defined by (3.11) and (3.12)).

18 this is the reason why we need or approximate profile to expand till the zone $y \sim B_1$. 
Proof of Proposition 3.8: First we compute the time evolution of $\mathcal{E}_{s_L}$:

$$
\frac{d}{dt} \left( \frac{\mathcal{E}_{s_L}}{\lambda^{2(s_L-n_c)}} \right) = \frac{d}{dt} \left( \int |w_{k_0+1}^{(1)}|^2 + |w_{k_0+1}^{(2)}|^2 \right)
= \frac{d}{dt} \left( \int w^{(1)}_{k_0+1} L_{k_0+L+1}^{L+1} w^{(1)} + w^{(2)}_{k_0+1} L_{k_0+L}^{L+1} w^{(2)} \right)
= 2 \int w^{(1)} L_{k_0+L+1}^{L+1} w^{(1)} + w^{(2)}_{k_0+L} L_{k_0+L}^{L+1} w^{(2)}
+ \sum_{i=1}^{k_0+L+1} \int w^{(1)} \lambda_{L_1}^{L+1} d \left( \frac{\lambda}{\lambda} \right) L_{k_0+L+1}^{L+1} w^{(1)}
+ \sum_{i=1}^{k_0+L} \int w^{(2)} \lambda_{L_1}^{L+1} d \left( \frac{\lambda}{\lambda} \right) L_{k_0+L}^{L+1} w^{(2)}
\quad (3.89)
= 2 \int w^{(1)} L_{k_0+L+1}^{L+1} (w^{(2)} - \frac{1}{2} \hat{\psi}_b^{(1)} - \frac{1}{X} \hat{\text{Mod}}(t)^{(1)})
+ 2 \int w^{(2)} L_{k_0+L}^{L+1} (L_{k_0+L}^{L+1} w^{(1)} - \frac{1}{2} \hat{\psi}_b^{(2)} - \frac{1}{X} \hat{\text{Mod}}(t)^{(2)} + L(w) + NL(w))
+ \sum_{i=1}^{k_0+L+1} \int w^{(1)} \lambda_{L_1}^{L+1} d \left( \frac{\lambda}{\lambda} \right) L_{k_0+L+1}^{L+1} w^{(1)}
+ \sum_{i=1}^{k_0+L} \int w^{(2)} \lambda_{L_1}^{L+1} d \left( \frac{\lambda}{\lambda} \right) L_{k_0+L}^{L+1} w^{(2)}
+ \frac{b_1}{\lambda^{(s_L-n_c)+1}} \left[ O\left( \sqrt{\mathcal{E}_{s_L}^{(1)}} \right) (1-\delta_0)^{(1+n)} + O\left( \mathcal{E}_{s_L}^{1/2} + O\left( \sum_{k=2}^{\infty} \frac{1}{b_1} \right) \right) \right].
\quad (3.90)
$$

where the constant hidden in the first $O()$ does not depend on $K_1$ and $K_2$. We now prove this intermediate estimate.

- **The linear term:** Because this norm is adapted to the flow of the wave equation we have the fundamental cancellation:

$$
\int w^{(1)} L_{k_0+L+1}^{L+1} w^{(2)} + w^{(2)}_{k_0+L} L_{k_0+L}^{L+1} w^{(1)} = 0.
\quad (3.91)
$$

- **the $\hat{\psi}_b$ term:** It is this term that gives the eventual estimate for $\mathcal{E}_{s_L}$ we want to prove. We recall that $f_j$, the $j$-th adapted derivative of a function $f$, is defined in (2.19). We just use Cauchy-Schwarz and the estimate provided in Proposition 2.13

$$
\left| \int \left( \frac{1}{X} \int w_{k_0+1}^{(1)} L_{k_0+L+1} \hat{\psi}_b^{(1)} + w_{k_0+1}^{(2)} L_{k_0+L} \hat{\psi}_b^{(2)} \right) \right|
\leq C \left( \frac{1}{X} \int \mathcal{E}_{s_L} b_1^{1/2} + (1-\delta_0)^{(1+n)} \right).
\quad (3.92)
$$

for a constant $C$ depending on $L$ only.
The non linear term: We begin by Cauchy-Schwarz inequality and by doing a change on the scaling:

\[
\left| \int w^{(2)} L^{k_0 + L} NL(w) \right| \leq \frac{1}{\lambda^{2(s_L - s_{\sigma}) + 1}} \sqrt{\mathcal{E}_{s_L}} \| NL(\varepsilon)_{k_0 + L} \|_{L^2}.
\]

We aim at estimating the last term in the right hand side. We know that \( NL(\varepsilon) \) is a sum of terms of the form \( \tilde{Q}_b^{(p-k)} \varepsilon^{(1)} k \) for \( 2 \leq k \leq p \). So by now we have to study quantities of the form: \( \tilde{Q}_b^{(p-k)} \varepsilon^{(1)} k \). For \( l = (l_0, \ldots, l_k) \) we recall the notation: \( |l|_1 = \sum_{i=0}^k l_i \). Close to the origin, we have from the equivalence between Sobolev norms and adapted norms (Lemma \ref{lemma:equivalence}), and because \( H^{s_L}(y \leq 1) \) is an algebra:

\[
\int_{y \leq 1} (NL(\varepsilon)_{k_0 + L})^2 \leq C \sum_{k=2}^p \| \varepsilon^{(1)} \|_{H^{s_L}(y \leq 1)}^2 \leq C(M) \mathcal{E}_{s_L} \leq C(M) \sqrt{\mathcal{E}_{s_L} b_L^2}.
\]

For \( y \geq 1 \) we notice that when applying \( A \) and \( A^* \):

\[
(\tilde{Q}_b^{(p-k)} \varepsilon^{(1)} k)_{k_0 + L} = \sum_{|l|_1 = k_0 + L} f_{l_0} \partial_y^l (\tilde{Q}_b^{(p-k)} \varepsilon^{(1)} k) \prod_{i=1}^k \partial_y^i \varepsilon^{(1)}.
\]

with \( f_{l_0} \sim \frac{1}{1 + y^{l_0}} \). We have the following asymptotic for the potential:

\[
\partial_y^l (\tilde{Q}_b^{(p-k)} \varepsilon^{(1)}) \leq \frac{C}{1 + y^{p^{-1}(p-k) + l_0}}.
\]

So, putting together the decay given by \( \partial_y^l (\tilde{Q}_b^{(p-k)} \varepsilon^{(1)}) \) and \( f_{l_0} \) and renaming \( l_0 := l_0 + l_0 \) we need to study integrals of the following form:

\[
\int_{y \geq 1} |NL(\varepsilon)_{s_L - 1}|^2 \leq \sum_{k=2}^p \sum_{|l|_1 = k_0 + L} \int_{y \geq 1} \frac{\prod_{i=1}^k |\partial_y^i \varepsilon^{(1)}|^2}{1 + y^{p^{-1}(p-k) + 2l_0}}, \quad (3.93)
\]

for \( \sum_{i=0}^k l_i = s_L - 1 \). We order the coefficient \( l_i \) for \( 1 \leq i \leq k \) by increasing order: \( l_1 \leq l_2 \leq \ldots \leq l_k \).

Case 1: we suppose that: \( \frac{2}{p-1}(p-k) + l_0 + l_k \leq s_L \). It implies the integrability \( \frac{\partial_y^l \varepsilon^{(1)}}{1 + y^{p^{-1}(p-k) + l_0}} \in L^2(y \geq 1) \) by the improved Hardy inequality from Lemma \ref{lemma:hardy}.

There also holds in that case for all \( 1 \leq i \leq k-1 \) that \( l_i < s_L - \frac{p}{2} \) which implies \( \partial_y^l \varepsilon^{(1)} \in L^\infty(y \geq 1) \). We then estimate:

\[
\left\| \prod_{i=1}^k |\partial_y^i \varepsilon^{(1)}|^2 \right\|_{L^2(y \geq 1)} \leq C \left\| \frac{\partial_y^l \varepsilon^{(1)}}{1 + y^{p^{-1}(p-k) + l_0}} \right\|_{L^2(y \geq 1)} \prod_{i=1}^{k-1} \left\| \partial_y^i \varepsilon^{(1)} \right\|_{L^\infty(y \geq 1)}.
\]

For \( 1 \leq i \leq k-1 \), from the equivalence between Laplace and \( \partial_y \) derivatives for \( y \geq 1 \):

\[
\partial_y^l \varepsilon^{(1)} = \sum_{j=0}^l f_j D^j \varepsilon^{(1)},
\]

with \( \partial_y^l f_j = O \left( \frac{1}{1 + x_{l_1}^{\sigma_1} \ldots x_{l_k}^{\sigma_k}} \right) \) for \( y \geq 1 \), we deduce:

\[
\| \partial_y^l \varepsilon^{(1)} \|_{L^\infty(y \geq 1)} \leq C \sum_{j=0}^l \| \frac{D^j \varepsilon^{(1)}}{1 + x_{l_1}^{\sigma_1} \ldots x_{l_k}^{\sigma_k}} \|_{L^\infty} \leq C \sqrt{\mathcal{E}_{s_L}} \leq \sqrt{\mathcal{E}_{s_L}} \frac{L^\infty}{L^\infty}.
\]
We used Sobolev injection, interpolation and coercivity. For \( i = k \) from Lemma F.1
\[
\left\| \frac{\partial^k_y \varepsilon^{(1)}}{1 + y^{\frac{1}{p-1}(p-k)+2l_0}} \right\|_{L^2(y \geq 1)} \leq C(M) \sqrt{\mathcal{E}_\sigma} \frac{s_L - l_k - l_0 - \frac{2}{p-1}(p-k)}{s_L - \sigma} \sqrt{\mathcal{E}_{s_L}} \frac{l_k + l_0 + \frac{2}{p-1}(p-k) - \sigma}{s_L - \sigma}.
\]
So that when combining the last two estimates we find:
\[
\left\| \prod_{i=1}^k \frac{|\partial^i_y \varepsilon^{(1)}|^2}{1 + y^{\frac{1}{p-1}(p-k)+2l_0}} \right\|_{L^2(y \geq 1)} \leq C(M) \sqrt{\mathcal{E}_\sigma} \frac{\sum_{i=1}^{k-1} \left( s_L - l_i - \frac{d}{2} \right)}{s_L - \sigma} + \frac{s_L - l_k - l_0 - \frac{2}{p-1}(p-k)}{s_L - \sigma} \frac{\sum_{i=1}^{k-1} \left( l_i + \frac{d}{2} - \sigma \right)}{s_L - \sigma} + \frac{l_k + l_0 + \frac{2}{p-1}(p-k) - \sigma}{s_L - \sigma}.
\]
We can calculate the coefficients:
\[
\sum_{i=2}^k \left( \frac{s_L - l_i - \frac{d}{2}}{s_L - \sigma} \right) + \frac{s_L - l_k - l_0 - \frac{2}{p-1}(p-k)}{s_L - \sigma} = \frac{(k-1)(s_L - \frac{d}{2} + 1 - \frac{2}{p-1}(p-k))}{s_L - \sigma},
\]
\[
\sum_{i=1}^{k-1} \left( \frac{l_i + \frac{d}{2} - \sigma}{s_L - \sigma} \right) + \frac{l_k + l_0 + \frac{2}{p-1}(p-k) - \sigma}{s_L - \sigma} = 1 + \frac{1 - (k-1)(\sigma - s_c)}{s_L - \sigma}.
\]
Under the bootstrap assumptions \(3.328\) it gives:
\[
\left\| \prod_{i=1}^k \frac{|\partial^i_y \varepsilon^{(1)}|^2}{1 + y^{\frac{1}{p-1}(p-k)+2l_0}} \right\|_{L^2} \leq C(K_1, K_2, M) b_1 \sqrt{\mathcal{E}_{s_L}} \left( \frac{\sqrt{\mathcal{E}_\sigma}}{b_1^{\frac{\sigma}{2} - s_c}} \right)^{k-1} b_1^{\alpha} + O\left( \frac{\sigma - s_c}{L} \right). \quad (3.94)
\]
\( \circ \) Case 2: if the last condition does not hold, it implies that \( l_k + l_0 = s_L - 1 \) with \( \frac{2}{p-1}(p-k) - 1 > 0 \), and that consequently for \( 1 \leq i \leq k-1, l_i = 1 \). It means that we have to estimate an integral of the following form:
\[
\int_{y \geq 1} |\varepsilon^{(1)}|^{2(k-1)} \frac{|\partial^k_y \varepsilon^{(1)}|^2}{1 + y^{\frac{1}{p-1}(p-k)+2l_0}}.
\]
We rewrite it as:
\[
\int_{y \geq 1} |\varepsilon^{(1)}|^{2(k-2)} \frac{\varepsilon^{(1)}}{1 + y^{\frac{1}{p-1}(p-k)-2} + y^2 + 2l_0}.
\]
The \( L^\infty \) norm of \( \varepsilon^{(1)} \) is estimated in Lemma F.1
\[
\| \varepsilon^{(1)} \|_{L^\infty} \leq C(M, K_1, K_2) \sqrt{\mathcal{E}_\sigma} b_1^{\left( \frac{d}{2} - \sigma \right) + \frac{2\alpha}{p-1} + 1} + O\left( \frac{\sigma - s_c}{L} \right).
\]
We use the improved Hardy estimate from Lemma F.1 to estimate:
\[
\left\| \frac{\partial^i_y \varepsilon^{(1)}}{1 + y^{l_0 + l_0}} \right\|_{L^2(y \geq 1)} \leq C(M) \sqrt{\mathcal{E}_{s_L}}.
\]
And finally we use the weighted \( L^\infty \) estimate (still from Lemma F.1):
\[
\left\| \frac{|\varepsilon^{(1)}|}{1 + y^{\frac{2}{p-1}(p-k)-1}} \right\|_{L^\infty} \leq C(M, K_1, K_2) \sqrt{\mathcal{E}_\sigma} b_1^{2(p-k) - 1 + \left( \frac{d}{2} - \sigma \right) + \frac{2\alpha}{p-1} + 1} + O\left( \frac{\sigma - s_c}{L} \right).
\]
With these last three estimates we have:

\[
\left\| \varepsilon^{(1)}(k-2) \right\|_{1+y^{p-1}(p-k)-1} \leq \sqrt{C_{eL}} \sqrt{\varepsilon_{eL}} b_{1} (k-2)^{2} \lambda_{1}^{2} \left( \frac{\varepsilon_{eL}}{b_{1}^{1}} \right)^{k-1} \frac{1}{1+y^{p-1}(p-k)-1} \left\| \frac{\partial L^\varepsilon}{\partial y} \right\|_{2} \leq C(M, K_{1}, K_{2}) \sqrt{C_{eL}} b_{1} \left( \frac{\varepsilon_{eL}}{b_{1}^{1}} \right)^{k-1} \frac{1}{1+y^{p-1}(p-k)-1} \left\| \frac{\partial L^\varepsilon}{\partial y} \right\|_{2} + O(\varepsilon_{eL}).
\]

We now come back to (3.93) and inject the bounds we have found. Putting together the result obtained in case 1, (3.94) and the result obtained in the second case, (3.95), gives for the non linear term:

\[
\left\| w^{(2)} L^{k_{0}+L} (NL(u)) \right\| \leq C(K_{1}, K_{2}, M) b_{1} \lambda_{1}^{2} \left( \frac{\varepsilon_{eL}}{b_{1}^{1}} \right)^{k-1} \frac{1}{1+y^{p-1}(p-k)-1} \left\| \frac{\partial L^\varepsilon}{\partial y} \right\|_{2} + O(\varepsilon_{eL}).
\]

We now recapitulate: we have found directs bounds for the quadratic term (3.91), for the error term (3.92), and for the non linear term (3.94). We inject them in (3.90) to obtain the intermediate identity (3.90), which we claimed in step 1.

**Step 2:** Terms for which only a local part is problematic. The small linear term and the scale changing term involve a potential that, in both cases, has a better decay than \( \frac{1}{y} \) far away of the origin. So away from the origin we can control them directly. Unfortunately, close to the origin we cannot. This is why we will have to use an additional tool, the study of a Morawetz type quantity, which will be done in the next subsection. We claim that (3.90) yields:

\[
\frac{d}{dt} \left( \frac{\varepsilon_{eL}}{\lambda^{2(s_{L}-s_{c})+1}} \right) = 2 \int w^{(1)} L^{k_{0}+L+1} (-\frac{1}{\lambda} \text{Mod}_{\lambda} (t)^{(1)}) + w^{(2)} L^{k_{0}+L} (-\frac{1}{\lambda} \text{Mod}_{\lambda} (t)^{(2)}) b_{1} \lambda_{1}^{2} \left( \frac{\varepsilon_{eL}}{b_{1}^{1}} \right)^{k-1} \frac{1}{1+y^{p-1}(p-k)-1} \left\| \frac{\partial L^\varepsilon}{\partial y} \right\|_{2} + O(\varepsilon_{eL}) \left( \frac{\varepsilon_{eL}}{\lambda^{2(s_{L}-s_{c})+1}} \right)^{k-1} + \frac{b_{1}}{\lambda^{2(s_{L}-s_{c})+1}} O \left( \frac{\varepsilon_{eL}}{\lambda^{2(s_{L}-s_{c})+1}} \right) + C(N) \varepsilon_{eL, \text{loc}}.
\]

We are now going to prove this identity (3.97) by establishing bounds on the small linear term and the scale changing term in (3.90).

- **The \( L(w) \) term:** We start by rescaling and using Cauchy-Schwarz:

\[
\left\| \int w^{(2)} L^{k_{0}+L} (L(w)) \right\| \leq \frac{1}{\lambda^{2(s_{L}-s_{c})+1}} \sqrt{\varepsilon_{eL}} \left\| (L(\varepsilon)) b_{0} + L \right\|_{L^{2}}.
\]

We have: \( L(\varepsilon) = p(Q^{p-1} - Q_{b}(1)) \varepsilon^{(1)} \). From the asymptotic of the the profiles \( T_{i} \) and \( S_{i} \) there holds the degeneracy:

\[
|\partial y^{(p)}(Q^{p-1} - Q_{b}^{(1)(p-1)})| \leq C(L) \frac{b_{1}}{1+y^{1+\alpha+j+C(L)\eta}}.
\]

Let\( \delta = \frac{b_{1}}{y} \). We first estimate the integral close to the origin. \( H^{s_{L}}(y \leq 1) \) is an algebra, from the equivalence between Laplace based derivatives and adapted ones

\[\text{We cannot expect to gain the weight } y^{-\alpha} \text{ because if } \alpha \text{ is too big the weighted coercivity does not apply. The limiting case is } \delta \text{ hence our choice for } \delta.\]
(see Lemma (\ref{lem:coercivity}), and from the weighted coercivity (Lemma (\ref{lem:coercivity})):  
\[
\int_{y \leq 1} (L(\varepsilon))_{s \ell -1}^2 \leq Cb_1^2 \int_{y \leq 1} \sum_{i=0}^{s \ell} |D^i \varepsilon(1)|^2 \leq C(M)b_1^2 \int \frac{|\varepsilon_{s \ell}^1|}{1 + y^2} .
\]

Away from the origin we estimate using the weighted coercivity and the equivalence between \(\partial_y\) derivatives and adapted derivatives (Lemma (\ref{lem:coercivity})).

\[
\| (L(\varepsilon))_{k_0 +1} \|_{L^2(y \geq 1)}^2 \leq C \sum_{i=0}^{s \ell -1} \| \frac{\partial_y \varepsilon_{s \ell}^i}{1 + y^{2+\alpha}} \|_{L^2(y \geq 1)}^2 \leq (M)b_1^2 \| \frac{\varepsilon_{s \ell}^i}{1 + y^2} \|_{L^2}^2 .
\]

With the two estimates, close and away from the origin, we have shown:

\[
\| (L(\varepsilon))_{s \ell -1} \|_{L^2}^2 \leq b_1^2 \| \frac{\varepsilon_{s \ell}^i}{1 + y^2} \|_{L^2}^2 .
\]

We now split the term of the right hand side in two parts, one before \(N\) and the other after, where \(N > 0\) is the large constant used in the definition of the local adapted norm (see (\ref{eq:local-adapted-norm}):

\[
\left\| \frac{\partial_y \varepsilon_{s \ell}^i}{1 + y^2} \right\|_{L^2} \leq b_1 \| \varepsilon_{s \ell}^i \|_{L^2(\leq N)} + b_1 \frac{1}{N^\delta} \| \varepsilon_{s \ell}^i \|_{L^2(\geq N)} .
\]

Finally:

\[
\left| \int w^{(2)} L_{s \ell}^{s \ell -1}(L(w)) \right| \leq \frac{C(M)}{\lambda^2(s \ell - s \ell_c)} \frac{\varepsilon_{s \ell}}{\lambda} \left( \frac{\varepsilon_{s \ell}}{N^\delta} + C(N) \varepsilon_{s \ell, loc} \right) .
\]

We now use Youngs inequality to reformulate it as:

\[
\left| \int w^{(2)} L_{s \ell}^{s \ell -1}(L(w)) \right| \leq \frac{C(M)}{\lambda^2(s \ell - s \ell_c)} \frac{\varepsilon_{s \ell}}{\lambda} \left( \frac{\varepsilon_{s \ell}}{N^\delta} + C(N) \varepsilon_{s \ell, loc} \right) .
\]

\bullet \hspace{1em} \textbf{The scale changing term:} The same reasoning applies to the scale changing term. Indeed one has:

\[
\frac{d}{dt} (L_{s \ell}^1) = -\frac{\lambda s}{\lambda^2} p Q^{p-1}(\Lambda^{(1)}) Q^1 .
\]

So that using the modulation bound (\ref{eq:modulation-bound}) stating that \(b_1 \approx -\frac{\lambda s}{\lambda^2}\) and the degeneracy (\ref{eq:degeneracy}) one has:

\[
\left| \partial_y \left( \frac{d}{dt} (L_{s \ell}^1) \right) \right| \leq \frac{C b_1}{\lambda y^2 + \alpha + j} .
\]

For a constant \(C\) independent of the other constants. Consequently we have the same gain of a weight \(y^{-\alpha}\) we had for the small linear term. Using verbatim the same techniques one obtain:

\[
\left| \int \sum_{i=1}^{s \ell} w^{(1)} L_{s \ell - i}^{s \ell - i}(L_{s \ell - i}^1) L_{s \ell - i}^1 \partial_y \right| \leq \frac{C(M)}{\lambda^2(s \ell - s \ell_c)} \frac{\varepsilon_{s \ell}}{\lambda} \left( \frac{\varepsilon_{s \ell}}{N^\delta} + C(N) \varepsilon_{s \ell, loc} \right) .
\]

We now come back to the identity (\ref{eq:identity}) established in step 1, and inject the bounds on the small linear term (\ref{eq:small-linear-term}) and on the scale changing term (\ref{eq:scale-changing-term}). This gives the identity (\ref{eq:claim}) we claimed in this step 2.
Step 3: Managing the modulation term. Eventually, we have to estimate the influence of the modulation term on \((3.97)\). We claim that:

\[
\int w^{(1)} L^s_t \frac{1}{\lambda} \hat{M}od(t)^{(1)} + \int w^{(2)} L^s_{t-1} \frac{1}{\lambda} \hat{M}od(t)^{(2)} = \frac{d}{dt} O \left[ \frac{\epsilon_s t}{\lambda^2(1-s) + \epsilon_s t + \epsilon_s t^2} \right] + O \left( \frac{b_1 \epsilon_s t}{\lambda^2(1-s) + \epsilon_s t + \epsilon_s t^2} \right).
\]

Once this bound is proven, we can finish the proof of the proposition by injecting it in \((3.97)\). So to finish to proof, we will now prove \((3.100)\). For \(1 \leq i \leq L - 1\), the bound \((3.37)\) we found for the modulation equations provides a sufficient estimate for the terms \((b_{1,s} + (i-\alpha)b_1 b_j - b_{i+1})(T_i + \sum \frac{\partial S_i}{\partial b_i})\). Indeed, pick an indice \(1 \leq i \leq L - 1\) and suppose it is even (the odd case being exactly the same). We calculate:

\[
\left| \frac{1}{\lambda} \int w^{(1)} L^s_t ((b_{1,s} + (i-\alpha)b_1 b_j - b_{i+1})\chi B_1(T_i + \sum_{j=i+1, j \text{ even}}^{L+2} \frac{\partial S_i}{\partial b_i})) \right| \leq \frac{C(M)\sqrt{\epsilon_s t}}{\lambda^2(1-s)} \left| \chi B_1(T_i + \sum_{j=i+1, j \text{ even}}^{L+2} \frac{\partial S_i}{\partial b_i}) \right| \left| \chi B_1(T_i + \sum_{j=i+1, j \text{ odd}}^{L+2} \frac{\partial S_i}{\partial b_i}) \right| \leq C b_1^{(L-i)}.
\]

and that we assumed \(i < L\), this bound implies the following identity for the modulation term:

\[
\int w^{(1)} L^s_t \frac{1}{\lambda} \hat{M}od(t)^{(1)} + \int w^{(2)} L^s_{t-1} \frac{1}{\lambda} \hat{M}od(t)^{(2)} = \frac{1}{\lambda} \int w^{(1)} L^s_t ((b_{1,s} + (L-\alpha)b_1 b_L)\chi B_1(\frac{\partial S_{L+1}}{\partial b_L})) + O \left( \frac{b_1 \epsilon_s t}{\lambda^2(1-s) + \epsilon_s t + \epsilon_s t^2} \right) + \frac{1}{\lambda} \int w^{(2)} L^s_{t-1} ((b_{1,s} + (L-\alpha)b_1 b_L)\chi B_1(T_L + \frac{\partial S_{L+2}}{\partial b_L}) \right) \]

The bad term is the last one for \(i = L\). But we know by the improved bound for the evolution of \(b_L\), see Lemma \(3.6\) that \(b_{1,s} + (L-\alpha)b_1 b_L\) is small enough up to the derivative in time of the projection of \(\epsilon\) onto \(H^4 L \chi B_1 \Lambda Q\). Let\(^{20}\)

\[
\xi := C(\xi) \left[ \chi B_1 \left( T_L + \frac{\partial S_{L+1}}{\partial b_L} + \frac{\partial S_{L+2}}{\partial b_L} \right) \right] \frac{1}{\lambda}.
\]

\(^{20}\)\(\xi\) can be seen as the coordinate of \(\epsilon\) along the vector \(\chi B_0 T_L\).
We claim that the bad part of the \( L \)-th modulation term can be integrated in time in the following way:

\[
\frac{d}{dt} \left( \int w^{(1)} \mathcal{L}^{s_L} \xi^{(1)} + \int w^{(2)} \mathcal{L}^{s_L-1} \xi^{(2)} \right) + \frac{1}{2} \int \xi^{(1)} \mathcal{L}^{s_L} \xi^{(1)} + \frac{1}{2} \int \xi^{(2)} \mathcal{L}^{s_L-1} \xi^{(2)}
\]

\[
= \frac{1}{2} \int w^{(1)} \mathcal{L}^{s_L}(b_{L,s} + (L - \alpha) b_L \chi_{B_1} (\frac{\partial S_{L+1}}{\partial b_L}))) \xi^{(1)}
\]

\[
+ \frac{1}{2} \int w^{(2)} \mathcal{L}^{s_L-1}(b_{L,s} + (L - \alpha) b_L \chi_{B_1} (T_L + \frac{\partial S_{L+2}}{\partial b_L}))) \xi^{(2)}
\]

\[
+ \frac{b_1}{\chi^{2(\gamma_L-y_L)}(1+2\eta)} O(\xi^{(1)} \mathcal{L}^{s_L} b_1^{\eta(1-\delta_0)}) + \frac{b_1}{\chi^{2(\gamma_L-y_L)}(1+2\eta)} O(\mathcal{L}^{s_L-1} b_1^{(1+2\eta)(1-\delta_0)})
\]

(3.103)

We will prove this identity at the end of this step 3. Once it is established, it allows us to prove the identity (3.100). Indeed, (3.101) can be rewritten as:

\[
\int w^{(1)} \mathcal{L}^{s_L} \xi^{(1)} + \int w^{(2)} \mathcal{L}^{s_L-1} \xi^{(2)}
\]

\[
= \frac{d}{dt} \left( \int w^{(1)} \mathcal{L}^{s_L} \xi^{(1)} + \int w^{(2)} \mathcal{L}^{s_L-1} \xi^{(2)} + \frac{1}{2} \int \xi^{(1)} \mathcal{L}^{s_L} \xi^{(1)} + \frac{1}{2} \int \xi^{(2)} \mathcal{L}^{s_L-1} \xi^{(2)}
\]

\[\quad \quad \quad \quad + \frac{b_1}{\chi^{2(\gamma_L-y_L)}(1+2\eta)} O(\xi^{(1)} \mathcal{L}^{s_L} b_1^{\eta(1-\delta_0)}) + \frac{b_1}{\chi^{2(\gamma_L-y_L)}(1+2\eta)} O(\mathcal{L}^{s_L-1} b_1^{(1+2\eta)(1-\delta_0)})
\]

(3.104)

We just have to check the gain obtained by the time integration. From the two estimates (3.70) and (3.72) we used in the proof of the improved modulation equation, one has the following size for the coefficient \( C(\xi) \):

\[
|C(\xi)| \leq \sqrt{\mathcal{E}_{s_L} b_1^{\delta_0-1}}.
\]

(3.105)

From the construction of the profiles \( S_i \) in Proposition 2.11 one has the following asymptotics:

\[
\left| \partial_y \left( \frac{\partial S_{L+1}}{\partial b_L} \right) \right| \leq C(L) \frac{b_1}{1 + y^{\gamma_L-\gamma_L-1+\gamma_L}} \quad \text{and} \quad \left| \partial^2_y \left( \frac{\partial S_{L+2}}{\partial b_L} \right) \right| \leq C(L) \frac{b_1^2}{1 + y^{\gamma_L-\gamma_L-1+\gamma_L}}.
\]

(3.106)

The cancellation \( \mathcal{L}^{s_L+1} T_L = 0 \) implies that the support of \( (\chi_{B_1} T_L)_{s_L-1} \) is in \( B_1 \leq y \leq 2B_1 \), hence \( \| (\chi_{B_1} T_L)_{s_L-1} \|_{L^2} \leq b_1^{(1-\delta_0)(1+\eta)} \). The two last estimates imply:

\[
\left| \int w^{(1)} \mathcal{L}^{s_L} \xi^{(1)} + \int w^{(2)} \mathcal{L}^{s_L-1} \xi^{(2)} \right| \leq C(M) \frac{\mathcal{E}_{s_L} b_1^{\gamma(1-\delta_0)}}{\chi^{2(\gamma_L-y_L)}}.
\]

(3.107)

For the same reasons:

\[
\left| \frac{1}{2} \int \xi^{(1)} \mathcal{L}^{s_L} \xi^{(1)} + \frac{1}{2} \int \xi^{(2)} \mathcal{L}^{s_L-1} \xi^{(2)} \right| \leq \frac{1}{\chi^{2(\gamma_L-y_L)}} \mathcal{E}_{s_L} b_1^{2\eta(1-\delta_0)},
\]

(3.108)

The injection of these last bounds (3.107) and (3.108) in the previous identity (3.104) yields the identity (3.100) we claimed in this step 3. To end the proof of the proposition, it just remains to prove (3.103), what we are now going to do. Using
the improved modulation bound \((3.37)\) for \(b_{L,s}\) one calculates:

\[
\frac{d}{dt} \left( \int w(t) \mathcal{L}_x^{s_L-1} \xi(t) + \int w(t) \mathcal{L}_x^{s_L} \xi(t) \right) =
\]

\[
\frac{1}{\alpha} \int w(t) \mathcal{L}_x^{s_L} \left( (b_{L,s} + (L - \alpha) b_1 b_L) \chi_{B_1} \frac{\partial S_{L+1}}{\partial t} \right) \nu_x
\]

\[
+ \frac{1}{\alpha} \int w(t) \mathcal{L}_x^{s_L-1} \left( (b_{L,s} + (L - \alpha) b_1 b_L) \chi_{B_1} (T_L + \frac{\partial S_{L+2}}{\partial t}) \right) \nu_x
\]

\[
+ O(\sqrt{\varepsilon_{s_L} + b_{L+1}^{1+\gamma}}) \left[ \int w(t) \mathcal{L}_x^{s_L} (\chi_{B_1} \frac{\partial S_{L+1}}{\partial t} \right) \nu_x
\]

\[
+ \int w(t) \mathcal{L}_x^{s_L+1} (\chi_{B_1} \frac{\partial S_{L+1}}{\partial t} \right) \nu_x
\]

\[
\leq C(L, M) \frac{b_1}{\sqrt{\varepsilon_{s_L}} b_1^{1-\delta_0} + b_1^{L+1-\delta_0} (1+\eta+y')}
\]

We show that all the other terms are small. From the modulation \((3.37)\) equations for \(b_i\) for \(i < L\) one has: \(|\lambda_n L^{-1}| \lesssim b_i, |b_{i,s}| \lesssim b_i^{1+1}\). As \(\xi\) does not depend on \(b_{L,s}\), this gives us the following bounds when the time derivative applies to \(\xi\) or \(\mathcal{L}\):

\[
\left| \int w(t) \mathcal{L}_x^{s_L-1} (\chi_{B_1} \frac{\partial S_{L+1}}{\partial t} \right) \nu_x
\]

\[
+ \int w(t) \mathcal{L}_x^{s_L+1} (\chi_{B_1} \frac{\partial S_{L+1}}{\partial t} \right) \nu_x
\]

\[
\leq C(L, M) \frac{b_1}{\sqrt{\varepsilon_{s_L}} b_1^{1-\delta_0} + b_1^{L+1-\delta_0} (1+\eta+y')}
\]

where we used coercivity, \((3.105)\) and \((3.106)\) and the fact that \(\partial_t (\mathcal{L}_x^{s_L-1} \chi_{B_1} T_L)\) has its support in \(B_1 \leq y \leq 2B_1\). We have now to estimate the terms involving \(\mathcal{L}\) in \((3.109)\). We do exactly the same things we did to the proof of Lemma \(3.3\). For the sake of simplicity we will only do it for the second coordinate, the first one being the same. We first compute the expression:

\[
\int w(t) \mathcal{L}_x^{s_L-1} \xi(t) = \int -\mathcal{L}_x^{1} w(t) \mathcal{L}_x^{s_L-1} \xi(t) + \int -\frac{1}{\alpha} \left( \psi_b^{0}(2) + \bar{M} \tilde{d}(t)^2 \right) \mathcal{L}_x^{s_L-1} \xi(t)
\]

\[
+ \int (L(w) + NL(w)) \mathcal{L}_x^{s_L-1} \xi(t).
\]

We use the bootstrap assumptions to put an upper bound on everything except the \(b_{L,s}\). For the linear term:

\[
\left| \int -\mathcal{L}_x^{1} w(t) \mathcal{L}_x^{s_L-1} \xi(t) \right| \leq \frac{1}{\sqrt{\varepsilon_{s_L}}} \sqrt{\varepsilon_{s_L} \| \xi(t) \|_{L^2}} \leq C(M) \frac{b_1}{\sqrt{\varepsilon_{s_L}}} b_1^{1-\delta_0}.
\]

Using the bounds on the error \(\tilde{\psi}_b\) from Proposition \(2.13\):

\[
\left| \int -\frac{1}{\alpha} \left( \psi_b^{0}(2) \right) \mathcal{L}_x^{s_L-1} \xi(t) \right| \leq \frac{C(M)b_1}{\sqrt{\varepsilon_{s_L}}} \sqrt{\varepsilon_{s_L} b_1^{1-\delta_0} (1+\eta+y')}.
\]

The small linear term gives the same estimate as the linear one:

\[
\left| \int L(w) \mathcal{L}_x^{s_L-1} \xi(t) \right| \leq \frac{C(M)b_1}{\sqrt{\varepsilon_{s_L}}} b_1^{1-\delta_0}.
\]
Finally, we start by decomposing the nonlinear term as a sum of term of the form: $G_{b_i}^{(1)}(p-k) w_i^{(1)k}$ for $2 \leq k \leq p$. we treat each term by letting all the derivatives on $\xi^{(2)}$:

$$
\left| \int NL(w) \mathcal{L}^{s_L-1} \xi^{(2)} \right| \lesssim \frac{1}{\lambda^{2s_L-s_c} + 1} \sqrt{\mathcal{E}_{s_L}} b_1^{\delta_0 - 1} \int \frac{\left| \varepsilon^{(1)k} \right|}{1 + y^{(p-k)}} (\chi B_1(T_L + \frac{\partial S_{L+2}}{\partial b_L}))_{2s_L-2}.
$$

We know by construction of the profiles that: $(T_L + \frac{\partial S_{L+2}}{\partial b_L})_{2s_L-2} = O(\frac{1}{1+y^{7+L+1+2k_0}})$, and by using the coercivity of the adapted norm and the $L^\infty$ estimate for $w_i^{(1)}$:

$$
\int \frac{\left| \varepsilon^{(1)k} \right|}{1 + y^{(p-k)}} (\chi B_1(T_L + \frac{\partial S_{L+2}}{\partial b_L}))_{2s_L-2} \leq C \int \frac{\left| \varepsilon^{(1)k} \right|}{1 + y^{(p-k)}} (\chi B_1(T_L + \frac{\partial S_{L+2}}{\partial b_L}))_{2s_L-2} \leq C \int (M) b_i^{-L+\gamma-1} \gamma \mathcal{E}_{s_L} \parallel \varepsilon^{(1)} \parallel_{L^\infty} \leq C(M, K_1, K_2) \mathcal{E}_{s_L} b_i^{-L+1+\alpha+O(1)} \left( \frac{\lambda^2}{b_i^{s_0}} \right)^{k-2}
$$

where the integral in $y$ we used with the Cauchy-Schwarz inequality was indeed divergent. Under the bootstrap assumptions it leads to:

$$
\int \frac{\sqrt{\mathcal{E}_{s_L}} b_1^{1-\delta_0}}{\lambda^{2(s_L-s_c)+1}} \int \frac{\left| \varepsilon^{(1)k} \right|}{1 + y^{(p-k)}} (\chi B_1(T_L + \frac{\partial S_{L+2}}{\partial b_L}))_{2s_L-2} \leq \frac{b_1 \varepsilon_{s_L}}{\lambda^{2(s_L-s_c)+1}} b_1^{\gamma(1-\delta_0) + \frac{\gamma}{2}}
$$

(as $C(M, K_1, K_2) b_i^{s_0} \leq b_i^{\frac{s_0}{2}}$ for $s_0$ large enough). Therefore for the nonlinear term we have:

$$
\left| \int NL(w) \mathcal{L}^{s_L-1} \xi^{(2)} \right| \leq \frac{b_1 \varepsilon_{s_L}}{\lambda^{2(s_L-s_c)+1}} b_1^{\gamma(1-\delta_0) + \frac{\gamma}{2}}.
$$

We now treat the modulation terms, preserving the $L$-th one. With the bound \ref{3.37} on the modulation for $1 \leq i \leq L - 1$, one has:

$$
\left| \int \frac{1}{\lambda} M \mathcal{L}^{s_L-1} \xi^{(2)} \right| = \int \frac{1}{\lambda} (b_{L,s} + (L-\alpha) b_1 b_L) (\chi B_1(T_L + \frac{\partial S_{L+2}}{\partial b_L}))_{2s_L-1} \xi^{(2)}
\leq C(M) b_i^{b_1 \frac{\sqrt{\mathcal{E}_{s_L}}}{\lambda^{2(s_L-s_c)+1}} (\sqrt{\mathcal{E}_{s_L}} b_1^{\gamma(1-\delta_0)} + b_1^{L+3})
$$

\ref{3.116}.

Coming back to the expression \ref{3.111} of the term involving $w_i^{(2)}$, and injecting the bounds we have found for each term, \ref{3.112}, \ref{3.113}, \ref{3.114} and \ref{3.115} yields the identity:

$$
\int w_i^{(2)} L^{s_L-1} \xi^{(2)} = \int \frac{1}{\lambda} (b_{L,s} + (L-\alpha) b_1 b_L) \left( \chi B_1(T_L + \frac{\partial S_{L+2}}{\partial b_L}) \right)_{L} \mathcal{L}^{s_L-1} \xi^{(2)}
+ \frac{b_1 \mathcal{E}_{s_L}}{\lambda^{2(s_L-s_c)+1}} O \left( \mathcal{E}_{s_L} b_1^{\gamma(1-\delta_0)} + \sqrt{\mathcal{E}_{s_L}} b_1^{L+1+\delta_0(1+2\eta)} \right)
$$

\ref{3.117}.

As we said, the same computation can be done using verbatim the same techniques for the first coordinate, yielding:

$$
\int w_i^{(1)} L^{s_L-1} \xi^{(2)} = \int \frac{1}{\lambda} (b_{L,s} + (L-\alpha) b_1 b_L) \left( \chi B_1(T_L + \frac{\partial S_{L+1}}{\partial b_L}) \right)_{\lambda} \mathcal{L}^{s_L} \xi^{(1)}
+ \frac{b_1 \mathcal{E}_{s_L}}{\lambda^{2(s_L-s_c)+1}} O \left( \mathcal{E}_{s_L} b_1^{\gamma(1-\delta_0)} + \sqrt{\mathcal{E}_{s_L}} b_1^{L+(1+\delta_0)(1+2\eta)} \right)
$$

\ref{3.118}.

Now we look back at the identity \ref{3.109}. We have estimated all terms in the right hand side in \ref{3.110}, \ref{3.117} and \ref{3.118}. Therefore it gives the intermediate
identity:

\[ \frac{d}{dt} \left( \frac{1}{2} \int \xi^{(1)} \Lambda^{sL}_{\alpha} \xi^{(1)} + \frac{1}{2} \int \xi^{(2)} \Lambda^{sL}_{\alpha} \xi^{(2)} \right) = \int \xi^{(1)} \partial_{t} (\xi^{(1)}) + \int \xi^{(2)} \partial_{t} (\xi^{(2)}) \]

\[ = \left( b_{L,s} + (L - \alpha)b_{1} b_{L} \right) \int \xi^{(1)} \Lambda^{sL}_{\alpha} (\chi_{B_{1}} \frac{\partial S_{L}^{+1}}{\partial b_{L}}) + \int \xi^{(2)} \Lambda^{sL}_{\alpha} (\chi_{B_{1}} (T_{L} + \frac{\partial S_{L}^{+2}}{\partial b_{L}})) + O(b_{1} \sqrt{\varepsilon_{L}} + b_{L}^{1+1+y'}) \left( \int \xi^{(1)} \Lambda^{sL}_{\alpha} (\chi_{B_{1}} \frac{\partial S_{L}^{+1}}{\partial b_{L}}) + \int \xi^{(2)} \Lambda^{sL}_{\alpha} (\chi_{B_{1}} (T_{L} + \frac{\partial S_{L}^{+2}}{\partial b_{L}})) \right) \]

\[ + O \left( \frac{C(\xi)}{2} \int \xi^{(1)} \partial_{t} \left( \Lambda^{sL}_{\alpha} (\chi_{B_{1}} \frac{\partial S_{L}^{+1}}{\partial b_{L}}) \right) + \frac{C(\xi)}{2} \int \xi^{(2)} \partial_{t} \left( \Lambda^{sL}_{\alpha} (\chi_{B_{1}} (T_{L} + \frac{\partial S_{L}^{+2}}{\partial b_{L}})) \right) \right) \]

We will now integrate in time the remaining term involving \( b_{L,s} + (L - \alpha)b_{1} b_{L} \). From the improved modulation equation (3.57) for \( b_{L} \), one computes using (3.119):

\[ \frac{d}{dt} \left( \frac{1}{2} \int \xi^{(1)} \Lambda^{sL}_{\alpha} \xi^{(1)} + \frac{1}{2} \int \xi^{(2)} \Lambda^{sL}_{\alpha} \xi^{(2)} \right) = \int \xi^{(1)} \partial_{t} (\xi^{(1)}) + \int \xi^{(2)} \partial_{t} (\xi^{(2)}) \]

\[ = \left( b_{L,s} + (L - \alpha)b_{1} b_{L} \right) \int \xi^{(1)} \Lambda^{sL}_{\alpha} (\chi_{B_{1}} \frac{\partial S_{L}^{+1}}{\partial b_{L}}) + \int \xi^{(2)} \Lambda^{sL}_{\alpha} (\chi_{B_{1}} (T_{L} + \frac{\partial S_{L}^{+2}}{\partial b_{L}})) + O(b_{1} \sqrt{\varepsilon_{L}} + b_{L}^{1+1+y'}) \]

\[ \left( \int \xi^{(1)} \Lambda^{sL}_{\alpha} (\chi_{B_{1}} \frac{\partial S_{L}^{+1}}{\partial b_{L}}) + \int \xi^{(2)} \Lambda^{sL}_{\alpha} (\chi_{B_{1}} (T_{L} + \frac{\partial S_{L}^{+2}}{\partial b_{L}})) \right) \]

Using verbatim the same techniques employed throughout this step 3 we estimate the remaining terms in this identity and end up with:

\[ \frac{d}{dt} \left( \frac{1}{2} \int \xi^{(1)} \Lambda^{sL}_{\alpha} \xi^{(1)} + \frac{1}{2} \int \xi^{(2)} \Lambda^{sL}_{\alpha} \xi^{(2)} \right) = \left( b_{L,s} + (L - \alpha)b_{1} b_{L} \right) \int \xi^{(1)} \Lambda^{sL}_{\alpha} (\chi_{B_{1}} \frac{\partial S_{L}^{+1}}{\partial b_{L}}) + \int \xi^{(2)} \Lambda^{sL}_{\alpha} (\chi_{B_{1}} (T_{L} + \frac{\partial S_{L}^{+2}}{\partial b_{L}})) + O \left( \frac{b_{1} \sqrt{\varepsilon_{L}}}{2(\varepsilon_{L}-\varepsilon_{\text{sc}})} + b_{L}^{L+(1-\delta')(1+2n)} y' \right) \]

\[ \left( \int \xi^{(1)} \Lambda^{sL}_{\alpha} (\chi_{B_{1}} \frac{\partial S_{L}^{+1}}{\partial b_{L}}) + \int \xi^{(2)} \Lambda^{sL}_{\alpha} (\chi_{B_{1}} (T_{L} + \frac{\partial S_{L}^{+2}}{\partial b_{L}})) \right) \]

We can now end the proof: combing the intermediate estimates (3.120) and (3.119) yields the identity (3.103)

3.7. **Control from a Morawetz type quantity:** As will be clear when we reintegrate the bootstrap equation in the next section, the term we still do no control in the monotonicity formula for the high regularity norm is the local one. We control it here via the study of a Morawetz type quantity. This term contributes to the time evolution of a bounded quantity (compared with \( \varepsilon_{sL} \)), so when we integrate it with respect to time it should remain small. For \( A > 0 \) and \( \delta > 0 \) let:

\[ \phi_{A}(x) := \int_{0}^{x} \chi_{A}(x') x'^{(1-\delta)} dx' \]

be the primitive of the function \( \chi_{A}(x)x^{1-\delta} \) and we still denote by \( \phi_{A} \) its radial extension. The quantity we will now study is (we recall that the adapted derivative \( f_{k} \) of a function is defined in (2.19)):

\[ M = - \int [\nabla \phi_{A} \nabla \varepsilon_{sL-1} + \frac{\Delta \phi_{A} \varepsilon_{sL-1}^{(1)} \varepsilon_{sL-1}^{(2)}}{2}] \]

\[ \frac{d}{dt} \left( \frac{1}{2} \int \xi^{(1)} \Lambda^{sL}_{\alpha} \xi^{(1)} + \frac{1}{2} \int \xi^{(2)} \Lambda^{sL}_{\alpha} \xi^{(2)} \right) \]
As $\nabla \phi_A$ and $\Delta \phi_A$ are bounded with compact support, it is clear that this quantity is controlled by the high Sobolev norm (from Corollary \[E.4\]):

$$|M| \leq C(A, M)\mathcal{E}_{sL}$$

(3.122)

We start by a lemma describing how this quantity controls the local norm $\mathcal{E}_{sL, loc}$ thanks to the fact that $\mathcal{L} > 0$ on $H^1$.

**Lemma 3.9.** (control from the Morawetz identity at the linear level) For $A$ big enough, $\delta$ small enough, there holds the following control:

$$-\int [\nabla \phi_A \cdot \nabla \varepsilon_{sL-1}^{(1)} + \frac{\Delta \phi_A}{2} \varepsilon_{sL-1}^{(1)}](-\mathcal{L} \varepsilon_{sL-1}^{(1)}) - \int [\nabla \phi_A \cdot \nabla \varepsilon_{sL-1}^{(2)} + \frac{\Delta \phi_A}{2} \varepsilon_{sL-1}^{(2)}] \varepsilon_{sL-1}^{(2)} \geq \frac{\mathcal{E}_{sL}}{2N^8 A^3} \mathcal{E}_{sL, loc} \int C(M) \mathcal{E}_{sL, loc}.$$  

(3.123)

**Proof of Lemma 3.9.** We calculate each term in the left hand side of (3.123). The second one is null:

$$\int [\nabla \phi_A \cdot \nabla \varepsilon_{sL-1}^{(2)} + \frac{\Delta \phi_A}{2} \varepsilon_{sL-1}^{(2)}] \varepsilon_{sL-1}^{(2)} = 0.$$  

(3.124)

For the first one we start by calculating:

$$-\int [\nabla \phi_A \cdot \nabla \varepsilon_{sL-1}^{(1)} + \frac{\Delta \phi_A}{2} \varepsilon_{sL-1}^{(1)}](-\mathcal{L} \varepsilon_{sL-1}^{(1)}) = \int \partial_y^2 \phi_A |\nabla \varepsilon_{sL-1}^{(1)}|^2 - \frac{1}{4} \int \Delta^2 \phi_A |\varepsilon_{sL-1}^{(1)}|^2 + \frac{1}{2} \int \nabla \cdot \nabla \phi_A |\varepsilon_{sL-1}^{(1)}|^2.$$  

(3.125)

We are now going to show that locally, the first term of the right hand side is bigger than the two others and control $\mathcal{E}_{sL, loc}$. We have $\partial_y^2 (\psi_A) = (\frac{1-\delta}{y^2}) A + \frac{1-\delta}{A} \partial_y \chi(y)$ which leads to:

$$\int \partial_y^2 \phi_A |\nabla \varepsilon_{sL-1}^{(1)}|^2 = (1-\delta) \int \chi_A |\nabla \varepsilon_{sL-1}^{(1)}|^2 + O \left( \frac{1}{A^3} \mathcal{E}_{sL} \right).$$  

(3.126)

We claim the following weighted Hardy inequality for radial functions:

$$\int \frac{\chi_A}{y^{d-\delta}} |\nabla u|^2 \geq \frac{(d-2-\delta)^2}{4} \int \chi_A \frac{u^2}{y^{2+\delta}} - C \int \frac{|y \partial_y \chi(y)| \partial_y u}{y^{2+\delta}} u^2.$$  

(3.127)

We prove this general inequality now. For smooth radial functions we compute, performing integration by parts:

$$\int \frac{\chi_A}{y^{1+\delta}} u \partial_y u = -\frac{d-2-\delta}{2} \int \frac{u^2}{y^{2+\delta}} \chi_A - \frac{1}{2} \int \frac{u^2 \partial_y \chi(y)}{A} \partial_y u.$$  

(3.128)

We can control the left hand side by using Cauchy-Schwarz and Young’s inequality:

$$\left| \int \frac{\chi_A}{y^{1+\delta}} u \partial_y u \right| \leq \frac{\epsilon}{2} \int \frac{\chi_A}{y^{1+\delta}} u^2 + \frac{1}{2 \epsilon} \int \frac{\chi_A}{y^\delta} |\nabla u|^2.$$  

(3.129)

Combining the two equations (3.128) and (3.129) with the choice $\epsilon = \frac{d-2-\delta}{2}$ gives the analysis bound (3.127) we claimed. We now come back to the identity (3.126), which gives the following control thanks to the Hardy inequality (3.127) we just proved:

$$\int \partial_y^2 (\psi_A) |\nabla \varepsilon_{sL-1}^{(1)}|^2 \geq (\delta-\delta^2) \int \chi_A \frac{|\nabla \varepsilon_{sL-1}^{(1)}|^2}{y^2} + (1-\delta)^2 \frac{(d-2-\delta)^2}{4} \int \chi_A \frac{|\varepsilon_{sL-1}^{(1)}|^2}{y^{2+\delta}} + O \left( \frac{1}{A^3} \mathcal{E}_{sL} \right).$$  

(3.130)
where we used the relation $1 - \delta = (1 - \delta)^2 + \delta - \delta^2$. With this control coming from the "gradient" part, the equation (3.125) can be rewritten as:

\[
- \int \left[ \nabla \phi_A \cdot \nabla \varepsilon_{sL-1}^{(1)} + \frac{\Delta \phi_A}{2} \varepsilon_{sL-1}^{(1)} \right] \left(- \mathcal{L} \varepsilon_{sL-1}^{(1)} \right)
\geq (\delta - \delta^2) \int \frac{\chi_A}{y^{d+\delta}} \frac{\nabla \varepsilon_{sL-1}^{(1)}}{y^{d+\delta}} + \frac{(1 - \delta)(1 - \delta^2)}{4} \int \frac{\chi_A}{y^{d+\delta}} \frac{\varepsilon_{sL-1}^{(1)}}{y^{d+\delta}} + O \left( \frac{1}{A^\delta} \varepsilon_{sL} \right) \tag{3.131}
\]

for some $\delta > 0$, because the potential is strictly smaller than the Hardy potential from Lemma 2.1. The expressions (3.132) and (3.133) imply that (3.131) can be rewritten as:

\[
- \Delta^2 (\phi_A) = \frac{\delta(d - 2)(d - 2 - \delta)}{2} \frac{\chi_A}{y^{d+\delta}} + O \left( \frac{1}{A^\delta} 1_{A \leq y \leq 2A} \right). \tag{3.132}
\]

We now prove that the last two terms are controlled by the two first ones. We calculate:

\[
\int \frac{\chi_A}{y^{d+\delta}} \frac{\nabla \varepsilon_{sL-1}^{(1)}}{y^{d+\delta}} + \frac{(1 - \delta)(1 - \delta^2)}{4} \int \frac{\chi_A}{y^{d+\delta}} \frac{\varepsilon_{sL-1}^{(1)}}{y^{d+\delta}} + O \left( \frac{1}{A^\delta} \varepsilon_{sL} \right).
\]

We now come back to the left hand side of (3.123). We have estimated the two terms in (3.121) and (3.134). For $N \ll A$ this gives the identity (3.123) we had to prove.

We can now state the control in the full nonlinear wave equation:

**Proposition 3.10.** (Control of the local term by the Morawetz identity) We suppose all the parameters of Proposition 3.2 are fixed in their range, except $s_0$. For $s_0$ and $A$ large enough, there holds for $s_0 \leq s < \ast$:

\[
\frac{d}{ds} M \geq \frac{\delta}{2N^\delta} \mathcal{E}_{sL, loc} - \frac{C(M)}{A^\delta} \mathcal{E}_{sL} - C(A) \sqrt{\mathcal{E}_{sL}} b_1^{L+3}, \tag{3.135}
\]

($\mathcal{E}_{sL}$ and $\mathcal{E}_{sL, loc}$ were defined in (3.11) and (3.12)).

**Remark 3.11.** As:

\[
\frac{d}{dt} \left( \frac{M}{\lambda^{2(sL-s_0)}} \right) = 2(s_L - s_0) \frac{b_1 M}{\lambda^{2(sL-s_0)+1}} + \frac{1}{\lambda^{2(sL-s_0)+1}} \frac{d}{ds} M,
\]

from the control (3.122) the result of the lemma implies (remember $b_1 \leq \frac{1}{A}$ in the bootstrap regime, and that $s_0$ is chosen in last so that $b_1$ can be arbitrarily small compared to the other constants):

\[
\frac{d}{dt} \left( \frac{M}{\lambda^{2(sL-s_0)}} \right) \geq \frac{1}{\lambda^{2(sL-s_0)+1}} \left( \frac{\delta}{2N^\delta} \mathcal{E}_{sL, loc} - \frac{C(M)}{A^\delta} \mathcal{E}_{sL} - C(A, M) \sqrt{\mathcal{E}_{sL}} b_1^{L+3} \right).
This is because the impact of the scale changing in the estimate we want to prove is of lower order, so we can work both at level $\varepsilon$ or $u$.

**Proof of Proposition 3.10.** The control comes from the previous lemma, and the new terms in the full (NLW) will be showed to be negligible. The time evolution of $M$ is ($f_k$ being the $k$-th adapted derivative of $f$ defined in (3.19)):

$$
\frac{d}{ds} M = - \int \nabla \phi_A \nabla \left[ \left( -\frac{\Lambda}{\varepsilon} \right) \varepsilon^{(1)} + \varepsilon^{(2)} - \psi_b^{(1)} - \text{Mod}(t)^{(1)} \right] \varepsilon^{(2)}_{s_{L-1}} - \int \frac{\Delta \phi_A}{2} \left( -\frac{\Lambda}{\varepsilon} \right) \varepsilon^{(1)} + \varepsilon^{(2)} - \psi_b^{(1)} - \text{Mod}(t)^{(1)} \varepsilon^{(2)}_{s_{L-1}} - \int \nabla \phi_A \nabla \varepsilon^{(1)}_{s_{L-1}} \left[ -L \varepsilon^{(1)} - \frac{\Lambda}{\varepsilon} \varepsilon^{(2)} - \psi_b^{(2)} - \text{Mod}(t)^{(2)} + L(\varepsilon) + NL(\varepsilon) \right]_{s_{L-1}} - \int \frac{\Delta \phi_A}{2} \varepsilon^{(1)}_{s_{L-1}} \left[ -L \varepsilon^{(1)} - \frac{\Lambda}{\varepsilon} \varepsilon^{(2)} - \psi_b^{(2)} - \text{Mod}(t)^{(2)} + L(\varepsilon) + NL(\varepsilon) \right]_{s_{L-1}}.
$$

And we aim at computing the effect of the right hand side. The linear part produces exactly the control we want thanks to the previous Lemma 3.9:

$$
- \int \left[ \nabla \phi_A \cdot \nabla \varepsilon^{(1)}_{s_{L-1}} + \frac{\Delta \phi_A}{2} \varepsilon^{(1)}_{s_{L-1}} \right] \left[ -L \varepsilon^{(1)} - \frac{\Lambda}{\varepsilon} \varepsilon^{(2)} - \psi_b^{(2)} - \text{Mod}(t)^{(2)} + L(\varepsilon) + NL(\varepsilon) \right]_{s_{L-1}} \geq \frac{\phi}{2N} E_{s_{L-1}}^\varepsilon - \frac{\phi}{M} E_{s_{L}},
$$

(3.137)

We are now going to show that all the other terms are of smaller order. As we work on a compact support, from the coercivity (3.21) and the fact that $\frac{\Lambda}{\varepsilon} \sim -b_1$ from (3.37):

$$
\left| \int \left[ \nabla \phi_A \cdot \nabla \left( \frac{\Lambda}{\varepsilon} \varepsilon^{(1)}_{s_{L-1}} \right) + \frac{\Delta \phi_A}{2} \varepsilon^{(1)}_{s_{L-1}} \right] \left[ -L \varepsilon^{(1)} - \frac{\Lambda}{\varepsilon} \varepsilon^{(2)} - \psi_b^{(2)} - \text{Mod}(t)^{(2)} + L(\varepsilon) + NL(\varepsilon) \right]_{s_{L-1}} \right| + \left| \int \left[ \nabla \phi_A \cdot \nabla \left( \frac{\Lambda}{\varepsilon} \varepsilon^{(1)}_{s_{L-1}} \right) + \frac{\Delta \phi_A}{2} \varepsilon^{(1)}_{s_{L-1}} \right] \left[ -L \varepsilon^{(1)} - \frac{\Lambda}{\varepsilon} \varepsilon^{(2)} - \psi_b^{(2)} - \text{Mod}(t)^{(2)} + L(\varepsilon) + NL(\varepsilon) \right]_{s_{L-1}} \right| \leq b_1 C(A) E_{s_{L}},
$$

(3.138)

so with $b_1$ small enough it is negligible. Still from the compactness of the support of $\phi_A$, for $b_1$ small enough we do not see the bad tail of $\psi_b$ (remember that for $y \leq B_1$, $\psi_b = \psi_b$). Hence:

$$
\left| \int \left[ \nabla \phi_A \cdot \nabla \left( \frac{\Lambda}{\varepsilon} \varepsilon^{(1)}_{s_{L-1}} \right) + \frac{\Delta \phi_A}{2} \varepsilon^{(1)}_{s_{L-1}} \right] \left[ -L \varepsilon^{(1)} - \frac{\Lambda}{\varepsilon} \varepsilon^{(2)} - \psi_b^{(2)} - \text{Mod}(t)^{(2)} + L(\varepsilon) + NL(\varepsilon) \right]_{s_{L-1}} \right| \leq C(A) \sqrt{E_{s_{L}}} \| \psi_{b,s_{L-1}}^{(1)} \|_{L^2(\leq A)} + \| \psi_{b,s_{L-1}}^{(2)} \|_{L^2(\leq A)} \leq C(A) \sqrt{E_{s_{L}}} b_1 L^{3+}.\n$$

The small linear term is also estimated easily. Indeed, we have that:

$$L(\varepsilon) = p(Q^{p-1} - \tilde{Q}^{p-1}) \varepsilon^{(1)} = b_1 \varepsilon^{(1)} O(1)$$

for $y \leq A$ for $b_1$ small enough. This gives using Cauchy-Schwarz:

$$
\left| \int \nabla \phi_A \cdot \nabla \varepsilon^{(1)}_{s_{L-1}} + \frac{\Delta \phi_A}{2} \varepsilon^{(1)}_{s_{L-1}} L(\varepsilon)_{s_{L-1}} \right| \leq C(A) b_1 E_{s_{L}}.
$$

(3.140)

For the nonlinear term we use what we already showed during the proof of the monotonicity formula for the high Sobolev norm, see (3.96):

$$
\left| \int \nabla \phi_A \cdot \nabla \varepsilon^{(1)}_{s_{L-1}} + \frac{\Delta \phi_A}{2} \varepsilon^{(1)}_{s_{L-1}} NL(\varepsilon)_{s_{L-1}} \right| \leq C(A) \sqrt{E_{s_{L}}} \| NL(\varepsilon)_{s_{L-1}} \|_{L^2} \leq C(A) b_1 E_{s_{L}},
$$

(3.141)

which is negligible for $b_1$ small enough as we said before. Finally it just remains to control the modulation terms. We just compute for the second coordinate, a similar estimate holding for the first one. Let $i$ be odd, $1 \leq i \leq L$. As $A \ll B_1$ for $s_0$ large enough, we do not see the the cut $\chi_{B_1}$ in the integral: $\chi_{B_1} \equiv 1$ for
\[ y \leq 2A. \text{ Because } H^i T_i = (T_i)_{i-1} (1 - \frac{1}{4}) = (1 - \frac{1}{4}) H, \text{ this term cancels in the integral because } (T_i)_{i-1} = (T_i)_{i-1} = 0 \text{ as } s_L - i = L + k_0 - i \geq 1. \]
\[
\int \left( \nabla \phi_A \nabla (\xi_{s_L-1}^{(1)}) + \frac{\Delta \phi_A}{2} \xi_{s_L-1}^{(1)} (b_{i,s} + (i - \alpha) b_1 b_i - b_{i+1} \chi B_1 T_i_{s_L-1}^{(2)}) = 0. \right.
\]

For the terms of the form \( \frac{\partial S}{\partial b_i} \) we always have at least one parameter \( b_i \) involved in this expression, which gives that for \( y \leq A \) there holds: \( \frac{\partial S}{\partial b_i}(y) \leq C(A)b_1 \). We then use the modulation equation proven in Lemma 3.11 to estimate:
\[
\left| \int \nabla \phi_A \nabla (\xi_{s_L-1}^{(1)}) + \frac{\Delta \phi_A}{2} \xi_{s_L-1}^{(1)} (b_{i,s} + (i - \alpha) b_1 b_i - b_{i+1} \chi B_1 \frac{\partial S}{\partial b_i}(y)) \right| \leq C(A,M)\xi_{s_L} b_L + C(A,M) \sqrt{\xi_{s_L} b_L^3}.\]

As we said, the same reasoning applies to treat the first coordinate. Consequently we have the following bound for the modulation terms:
\[
\left| \int \nabla \phi_A \nabla [\text{Mod}(t)]_{s_L-1}(2) \xi_{s_L-1}^{(2)}(2) + \int \frac{\Delta \phi_A}{2} [\text{Mod}(t)]_{s_L-1}(2) \xi_{s_L-1}^{(1)}(2) \right| \leq C(A,M)\xi_{s_L} b_L + C(A,M) \sqrt{\xi_{s_L} b_L^3}.\]

We now come back to our initial decomposition (3.139). We have the expected control from the linear term in (3.137), and have estimated all the other terms in (3.138), (3.139), (3.140), (3.141) and (3.142). It gives the desired result.

4. End of the proof:

4.1. End of the Proof of Proposition 3.2. We are now going to end the proof of the proposition 3.2. The hardest has already been done. Now we just have to reintegrate all the equations displayed so far: the ones about the evolution of the parameters and the ones about the evolution of the norms for the error term. The definition of the minimal time \( s^* \) for which the bootstrap assumptions are violated implies that at time \( s^* \) at least one of the three facts is true:

(i) \text{ The error term has grown too big: } \xi_{s_L}(s^*) = K_1 b_1(s^*)^{2L+2(1-\delta_b)(1+\eta)} \text{ or } \xi_{\sigma}(s^*) = K_2 b_1(s^*)^{2(1-\delta_b)(1+\eta)}.

(ii) \text{ Exit of the stable modes: } \xi_{s_L}(s^*) = \frac{1}{(s^*)^\eta} \text{ or } |b_k(s^*)| = \frac{\epsilon_b}{(s^*)^{k+\eta}}.

(iii) \text{ Exit of the unstable modes: } \xi_{s_L}(s^*) = (V_1(s^*)^2, \ldots, V_k(s^*)) \in \mathcal{S}^{-1} \left( \frac{1}{(s^*)^\eta} \right).\]

We will show in this section that the cases (i) and (ii) never happen for any initial solution. Indeed, the estimates of the error term can be improved using all the preceding monotonicity formulas, and are in fact smaller than what we asked for. The exit of the stable modes is impossible because their evolution is governed by a linear equation for which 0 is an attractor, plus a force term whose size is under control.

There are initial data leading to the exit of the unstable modes because they are driven by unstable dynamics. Indeed from the study of the linearized equation for the parameters we have seen that 0 is a repulsive equilibrium for these modes.
However this equilibrium must persist when we add the perturbative term to the equation, because the contrary would go against Brouwer fixed point theorem. This part will be made clearer in our precise case later on.

We begin with integrating the scaling equations.

**Lemma 4.1** (law for the scaling in the trapped regime). *Up to time* \( s^* \) *there holds the following estimations for the scaling:*

\[
\lambda(s) = \left( \frac{s_0}{s} \right)^{\frac{\ell}{1-\alpha}} \left[ 1 + O \left( \frac{1}{s_0^{\eta}} \right) \right].
\]  

(4.1)

**Proof of Lemma 4.1.** Untill \( s^* \), we have under the bootstrap assumptions (3.27) and (3.26) for the parameters that \( b_i(s) = b^e_i + U_i s^{\nu} \) with \( U_i \leq \frac{1}{s_0^{\eta}} \). So we use the modulation equation proved in Lemma 3.4:

\[
-\frac{\lambda_s}{\lambda} = b_1 + O \left( b_1 \varepsilon_{sL} + b_1^{L+3} \right) = \frac{\ell}{(\ell - \alpha)s} + O \left( \frac{1}{s^{1+\eta}} \right).
\]

We rewrite this equation as:

\[
\left| \frac{d}{ds}(\log(s^{\frac{\ell}{1-\alpha}} \lambda)) \right| \lesssim \frac{1}{s^{1+\eta}}.
\]

After integration gives:

\[
\lambda(s) = \left( \frac{s_0}{s} \right)^{\frac{\ell}{1-\alpha}} \left[ 1 + O \left( \frac{1}{s_0^{\eta}} \right) \right].
\]

\( \square \)

We now rule out the case (i). We recall that \( K_1 \) and \( K_2 \) are used to quantify the control of the error term \( \varepsilon \) in the trapped regime of proposition 3.2.

**Lemma 4.2** (Integrating the evolution equations for the norms). *Assume all the other constants of Proposition 3.2 are fixed in their range. There exist \( K_1, K_2 > 0, N > 0, \nu > 0 \) and \( \varepsilon \) such that for \( s_0 \) big enough, \( \eta \) small enough, under the bootstrap assumptions untill time \( s^* \) the norms enjoy a better estimation. There holds in fact:*

\[
\varepsilon_{sL} \leq \frac{K_1}{2} b_1^{2L+2(1-\delta_0)(1+\eta)},
\]

(4.2)

and:

\[
\varepsilon_\sigma \leq \frac{K_2}{2} b_1^{2(\sigma-s_c)} \frac{\ell}{1-\alpha}.
\]

(4.3)

**Remark 4.3.** The constant \( \frac{1}{2} \) is not really important, we could have stated it for any constant.

**Proof of Lemma 4.2.** The low Sobolev norm: We recall the result of Proposition 3.7

\[
\frac{d}{dt} \left\{ \frac{\varepsilon_\sigma}{\lambda^{2(\sigma-s_c)+1}} \right\} \leq b_1 \frac{\sqrt{\varepsilon_\sigma} b_1^{(\sigma-s_c)(1+\nu)}}{\lambda^{2(\sigma-s_c)+1}} + b_1^{\frac{\nu}{p}} + O \left( \frac{\sigma-s_c}{\lambda} \right) + b_1^{\frac{\nu}{p}} + O \left( \frac{\sigma-s_c}{\lambda} \right) \sum_{k=2}^{p} \left( \frac{\sqrt{\varepsilon_\sigma}}{b_1^{\sigma-s_c}} \right)^{k-1}
\]

21 this is a way of speaking, there is no fixed point but one trajectory staying bounded.
with $\nu = \frac{3}{2} - \alpha$. One has $\sum_{k=2}^{p} \left( \frac{\nu}{b_1^{1-k}} \right)^{k-1} \ll 1$ under the bootstrap conditions \((3.28)\). Therefore, we see that there exists a small constant $0 < \delta \ll 1$, such that if one chooses $s_0$ large enough, this equation can be rewritten as:

$$
\frac{d}{ds} \left\{ \frac{\mathcal{E}_\sigma}{\lambda^{2(s_1-s_0)}} \right\} \leq \frac{1}{\lambda^{2(s_1-s_0)}} b_1 \sqrt{\mathcal{E}_\sigma} b_1^{(s_1-s_0)} \frac{\ell}{1+\delta}.
$$

Still under the bootstrap assumption we can integrate this equation:

$$
\mathcal{E}_\sigma(s) \leq \mathcal{E}_\sigma(0) \lambda^{2(s_1-s_0)} + \lambda^{2(s_1-s_0)} \int_{s_0}^{s} \frac{b_1}{\lambda^{2(s_1-s_0)}} \sqrt{K_2 b_1^{2(\sigma-s_0)}} \frac{\ell}{1+\delta}.
$$

(4.4)

We recall that $\lambda(0) = 1$ and from \((4.1)\) and the bootstrap assumptions \((3.27)\) and \((3.26)\) on $b_1$:

$$
\left| \lambda(s) - \left( \frac{s}{s_0} \right)^{\frac{\ell}{1+\delta}} \right| \leq \frac{1}{s_{C_0}} \left( \frac{s}{s_0} \right)^{\frac{\ell}{1+\delta}} \quad \text{and} \quad \left| b_1 - \frac{C_1}{s} \right| \leq \frac{1}{s^{1+\delta}}.
$$

It implies: $\lambda(s) \leq \frac{C}{s^{L-\alpha}}$ and $b_1 \sim \frac{1}{s}$. Consequently:

$$
\mathcal{E}_\sigma(0) \lambda^{2(s_1-s_0)} \leq C \mathcal{E}_\sigma(0) b_1^{2(s_1-s_0)} \frac{\ell}{1+\delta}.
$$

Given the initial condition \((3.21)\) on $\mathcal{E}_\sigma(0)$ it yields:

$$
\mathcal{E}_\sigma(0) \lambda^{2(s_1-s_0)} \leq b_1^{2(s_1-s_0)} \frac{\ell}{1+\delta}.
$$

(4.5)

For the integral term one has:

$$
\lambda^{2(s_1-s_0)} \int_{s_0}^{s} \frac{b_1}{\lambda^{2(s_1-s_0)}} b_1^{2(\sigma-s_0)+\delta} \leq C \lambda^{2(s_1-s_0)} \leq Cb_1^{2(s_1-s_0)} \frac{\ell}{1+\delta}
$$

because the integral is convergent ($\frac{b_1}{\lambda^{2(s_1-s_0)}} b_1^{2(\sigma-s_0)+\delta} \leq s^{1-\delta}$). Therefore:

$$
\lambda^{2(s_1-s_0)} \int_{s_0}^{s} \frac{b_1}{\lambda^{2(s_1-s_0)}} b_1^{2(\sigma-s_0)+\delta} \sqrt{K_2} \leq C \sqrt{K_2} b_1^{2(s_1-s_0)} \frac{\ell}{1+\delta}.
$$

(4.6)

Injecting the two estimates \((4.5)\) and \((4.6)\) we found in \((4.4)\) gives:

$$
\mathcal{E}_\sigma(s) \leq b_1^{2(s_1-s_0)} \frac{\ell}{1+\delta} \left( 1 + C \sqrt{K_2} \right),
$$

and $(1 + C \sqrt{K_2}) \leq \frac{K_2}{2}$ for $K_2$ large enough. 

**the high Sobolev norm:** We recall the estimate of Proposition \((3.8)\) with $C$ independent of $N$. In the trapped regime \((3.28)\), by taking $s_0$ large enough one has:

$$
\mathcal{E}_{s_1} b_1^{\frac{s}{L}} + O\left( \frac{z_{s_0}}{L} \right) \sum_{k=2}^{p} \left( \sqrt{\mathcal{E}_\sigma} b_1^{(1-\delta_0)} \right)^{k-1} \leq C \mathcal{E}_{s_1} \frac{N_{s_0}}{b_1^{s_1-s_0}}.
$$

So the previous equation becomes:

$$
\frac{d}{ds} \left\{ \frac{\mathcal{E}_{s_1}}{\lambda^{2(s_1-s_0)}} + O \left( \frac{\mathcal{E}_{s_1} b_1^{(1-\delta_0)}}{\lambda^{2(s_1-s_0)}} \right) \right\} \leq \frac{C b_1}{\lambda^{2(s_1-s_0)}} \times \left\{ \frac{\mathcal{E}_{s_1}}{N_{s_0}} + \sqrt{\mathcal{E}_{s_1}} b_1^{L+(1-\delta_0)(1+\eta)} + C(N) \mathcal{E}_{s_1,loc} \right\}.
$$
(by multiplying the constant $C$ by 2). We also have by the Proposition 3.10

$$
\frac{d}{ds} \left( \frac{\mathcal{M}}{\lambda^{2(s_L-s_c)}} \right) \geq \frac{\delta}{2 N\lambda^{2(s_L-s_c)}} \mathcal{E}_{s_L,loc} - \frac{C}{A^d \lambda^{2(s_L-s_c)}} \mathcal{E}_{s_L} - \frac{C(A, N)\mathcal{E}_{s_L}}{\lambda^{2(s_L-s_c)}} b_1^{L+3}.
$$

Let $a > 0$. Once $N$, $K_1$ and $A$ are chosen, for $s_0$ small enough we have:

$$
\frac{CC(N)b_1}{\lambda^{2(s_L-s_c)}} \mathcal{E}_{s_L,loc} \leq \frac{CN}{a} \left( \frac{d}{ds} \left( \frac{\mathcal{M}}{\lambda^{2(s_L-s_c)}} \right) \right) + \frac{C(N, M)b_1}{\lambda^{2(s_L-s_c)}} \mathcal{E}_{s_L} + \frac{C(A, N)}{\lambda^{2(s_L-s_c)}} \mathcal{E}_{s_L},
$$

which gives for the evolution of the high Sobolev norm the following monotonicity formula:

$$
\frac{d}{ds} \left\{ \frac{\mathcal{E}_{s_L}}{\lambda^{2(s_L-s_c)}} + O \left( \frac{\mathcal{E}_{s_L}}{\lambda^{(1-\delta_0)}b_1} \right) \right\} \leq \frac{b_1}{\lambda^{2(s_L-s_c)}} + \frac{b_1}{\lambda^{2(s_L-s_c)}} \frac{CN}{a} \left[ \mathcal{E}_{s_L} b_1^{L+1-\delta_0} \right] + \frac{C(N, M)}{\lambda^{2(s_L-s_c)}} b_1^{L+1-\delta_0} + \mathcal{E}_{s_L} b_1^{L+1-\delta_0}.
$$

with $C$ independent of $a'$. We will now integrate it in time as we did for the low Sobolev norm, using the bootstrap assumption (3.28):

$$
\mathcal{E}_{s_L}(s) \leq C(s_0)(\mathcal{E}_{s_L}(s_0) + |\mathcal{M}(s_0)|\lambda^{2(s_L-s_c)} + \frac{b_1}{a}|\mathcal{M}(s)| + \lambda^{2(s_L-s_c)} \int_{s_0}^{s} \frac{b_1}{\lambda^{2(s_L-s_c)}} \left( \frac{K_1}{a} + C\sqrt{K_1} \right) b_1^{L+1-\delta_0} (1+\eta)^{\frac{1}{2}} \mathcal{E}_{s_L}).
$$

We recall that $|\mathcal{M}| \leq C(A)\mathcal{E}_{s_L}$, so:

$$
\frac{|\mathcal{M}|}{a'} \leq \frac{C(A)\mathcal{E}_{s_L}}{a'}.
$$

We then compare using the equivalents for $b_1$ and $\lambda$:

$$
b_1^{2L+2(1-\delta_0)(1+\eta)} \approx \frac{1}{s^{2L+2(1-\delta_0)(1+\eta)}}.
$$

$$
\lambda^{2(s_L-s_c)} \sim \frac{1}{s^{2L+2(1-\delta_0)(1+\eta)}} \leq \frac{1}{s^{2L+2(1-\delta_0)(1+\eta)}}.
$$

This implies $\lambda^{2(s_L-s_c)} = o(b_1^{2(L+1-\delta_0)(1+\eta)})$ (remember that $\ell \ll L$). Because of the initial bound (3.21) on $\mathcal{E}_{s_L}(0)$ there holds for all $s_0 \leq s \leq s^*$:

$$
C(s_0)(\mathcal{E}_{s_L}(0) + |\mathcal{M}(s_0)|)\lambda^{2(s_L-s_c)} \leq b_1^{2L+2(1-\delta_0)(1+\eta)}.
$$

We now treat the integral term using the equivalents for $\lambda(s)$ and $b_1(s)$:

$$
\lambda^{2(s_L-s_c)} \int_{s_0}^{s} \frac{b_1}{\lambda^{2(s_L-s_c)}} b_1^{2(L+1-\delta_0)(1+\eta)} \leq C s^{-2(L+1-\delta_0)(1+\eta)} \int_{s_0}^{s} \frac{b_1}{s^{2L+2(1-\delta_0)(1+\eta)} + \lambda^{2(s_L-s_c)}} + \frac{C s^{2L+2(1-\delta_0)(1+\eta)} / a'}{s^{2L+2(1-\delta_0)(1+\eta)}},
$$

with the constant $C$ just depending on $c_1$ and $s_0$. The integral is indeed divergent from $-2(L+1-\delta_0)(1+\eta)) + 2(s_L-s_c) / \ell > 0$ (as $\ell \ll L$). Eventually the three estimations we have shown allow us to conclude:

$$
\left( 1 - \frac{C(N)}{a'} \right) \mathcal{E}_{s_L}(s) \leq b_1^{2L+2(1-\delta_0)(1+\eta)} \left( \frac{C}{a'} K_1 + C\sqrt{K_1} + C \right).
$$
For $a'$ and $K_1$ big enough one has:

$$
\frac{C}{a'} K_1 + C \sqrt{K_1} + C \leq \frac{K_1}{2}
$$

(remember that here $\frac{C(N)}{a'} = \frac{C(N)}{a}$ and since we choose $a$ after $M$ this term can be arbitrarily small.

We now rule out case (ii). We recall that the small coefficients $(\epsilon_i)_{i+1\leq i\leq L}$ are used to quantify the control over the stable modes in the trapped regime of Proposition 3.2.

**Lemma 4.4** (control of the stable modes). *After having chosen the other constants correctly, there exists small enough constants $\bar{\eta}$, and $(\epsilon_i)_{i+1\leq i\leq L}$ such that for $s_0$ big enough, until time $s^*$ there holds:*

$$
|V_1| \leq \frac{1}{2s^{q_1}}, \text{ and } |b_k(s)| \leq \frac{\epsilon_i}{2s^{k+\bar{\eta}}} \text{ for } \ell + 1 \leq k \leq L.
$$

(4.7)

**Proof of Lemma 4.4.** The stable modes for $\ell + 1 \leq i \leq L - 1$: Let $i$ be an integer, $\ell + 1 \leq i \leq L - 1$. We recall that the evolution of $b_i$ is given by:

$$
b_{i,s} = -(i - \alpha) b_i b_i + b_{i+1} + O(b_1 \sqrt{E_{sl}} + b_{1+3}^{(i+3)})
$$

$$
= -\frac{c_1(i-\alpha)}{s} b_i - (i - \alpha) U b_i - b_{i+1} + O(s^{-L-1-(1-\delta_0)})
$$

$$
= -\frac{c_1(i-\alpha)}{s} b_i + b_{i+1} + O(s^{-1+i-2\bar{\eta}}),
$$

for $\bar{\eta}$ small enough, because $U_1 b_1 = O(s^{-2\bar{\eta}})$ under the bootstrap assumptions. Hence for $s_0$ large enough it gives:

$$
|b_{i,s} + (i - \alpha) c_1 \frac{b_i}{s}| \leq \frac{2\epsilon_{i+1}}{s^{i+1+\bar{\eta}}},
$$

which we rewrite as:

$$
\left| \frac{d}{ds}(s^{(i-\alpha)c_1} b_i) \right| \leq 2\epsilon_{i+1}s^{(i-\alpha)c_1 -(i+1+\bar{\eta})}.
$$

(4.8)

We notice that $(i - \alpha)c_1 = \frac{c_1(i-\alpha)}{i-\alpha} > i$. So for $\bar{\eta}$ small enough one has $(i - \alpha)c_1 \geq i + \bar{\eta}$. With these two facts in mind we integrate the last equation and estimate using the initial condition (3.20):

$$
|b_i(s)| \leq b_i(0) s^{(i-\alpha)c_1} + \frac{2\epsilon_{i+1}}{s^{i-1+c_1-i\alpha}} \int_{s_0}^{s} (i-\alpha)c_1 -(i+1+\bar{\eta}) d\tau
$$

$$
\leq \frac{\epsilon_i}{10s^{i+c_1}} + \frac{2\epsilon_{i+1}}{s^{i-1+c_1-i\alpha}},
$$

the integral that appeared being divergent. We therefore see here that we can choose the constants of initial smallness $(\epsilon_i)_{i+1\leq i\leq L}$ one after each other: once $\epsilon_i$ is chosen we can take $\epsilon_{i+1}$ small enough to produce $\frac{\epsilon_i}{10} + \frac{2\epsilon_{i+1}}{s^{i-1+c_1-i\alpha}} < \frac{\epsilon_i}{5}$. This, of course, makes only sense if one is able to bootstrap the estimate on the last parameter $b_L$.

The stable mode $i = L$: We recall the improved modulation equation for $b_L$:

$$
\left| b_{L,s} + (L - \alpha) b_L b_L - \frac{d}{d\alpha} \left[ \frac{\langle H^{\infty} e, X_{b_0} AQ \rangle}{\chi_{b_0}(\Lambda^{1})\Lambda^{1}(-1) L+(-1) \left( \frac{\partial_{\alpha} + 1}{\partial_{\alpha}} + \left( \frac{\partial_{\alpha}^2}{\partial_{\alpha}} + 1 \right) \Lambda_{L-1} \right)} \right] \right|
$$

$$
\leq \frac{1}{R_0^2} C(M) \left[ \sqrt{\mathcal{E}_{sl}} + b_{1}^{L+(1-\delta_0)(1+\bar{\eta})} \right].
$$

(4.9)
We have seen in (3.105) that:

\[
\frac{\langle H^L \varepsilon, \chi_{bL} \Lambda Q \rangle}{\langle \chi_{bL} \Lambda^{(1)} Q, \Lambda^{(1)} Q \rangle} \leq C \sqrt{E_{sL}} b_{0}^{L-1} \lesssim s^{-L-1-\eta(1-\delta_0)},
\]

We integrate the time evolution of \( b_L \) the same way we did for the other stable modes. This time, however, the force term comes from the error \( \varepsilon \). We reformulate (4.11)

\[
\frac{d}{ds} (s^{(L-a)c_1} b_L) = s^{(L-a)c_1} \frac{d}{ds} O \left( \frac{1}{s^{L+\eta(1-\delta_0)}} \right) + s^{(L-a)c_1} O \left( \frac{1}{s^{L+1+\eta(1-\delta_0)}} \right).
\]

Then as for the \( b_i \)'s for \( \ell + 1 \leq i \leq L - 1 \), we integrate and use integration by parts to find, under the initial smallness assumption on \( b_L \) and for \( \tilde{\eta} \) small enough:

\[
|b_L(s)| \leq \frac{\epsilon_L}{10s^{1+\tilde{\eta}}} + \frac{C}{s^{L+\eta(1-\delta_0)}},
\]

where \( C \) is just some integration constant. Hence by choosing \( s_0 \) large enough and \( \tilde{\eta} < \eta(1-\delta_0) \) we have: \( |b_L(s)| \leq \frac{\epsilon_1}{2s^{1+\tilde{\eta}}} \).

control of \( V_1 \). We recall that \( V_1 \) is the eigenvector associated to the eigenvalue \(-1\) of the linearized operator \( A_{\ell} \), defined by (3.18): \( V_1 = (P_{\ell} U)_1 = \sum_{i \leq \ell} p_{1,i} U_i \). We first calculate the time evolution of the \( U_i \)'s for \( 1 \leq i \leq \ell \) thanks to the modulation equation (3.3):

\[
U_{i,s} = \frac{(AU)_i}{s} + O \left( \frac{|U|^2|}{s} \right) + s^i O \left( b_1 C(M) \sqrt{E_{sL}} + C(M) b_1^{L+3} \right)
\]

where \( g_i(s) \) stands for the terms added in the full equation. It leads to the following expression for the time evolution of \( V_1 \):

\[
V_{1,s} = -\frac{1}{s} V_1 + O \left( \frac{|V|^2|}{s} \right) + \sum_{j=1}^{L} p_{1,j} s^j g_j(s) + q_1 s^j b_{\ell+1},
\]

where \( q_1 \) is a constant defined by (2.77). We reformulate it under the bootstrap assumptions as:

\[
\frac{d}{ds} (s V_1) = s O \left( \frac{1}{s^{1+2\tilde{\eta}}} + \frac{1}{s^{L-\ell}} \right) + s q_1 s^j b_{\ell+1}.
\]

As \( |b_{\ell+1}| \leq \epsilon_{\ell+1} s^{-\tilde{\eta}} \) under the bootstrap assumptions, for \( s_0 \) large enough the time integration gives:

\[
|V_1(s)| \leq \frac{s_0 |V_1(s_0)|}{s} + O \left( \frac{\epsilon_{\ell+1}}{s^{\tilde{\eta}}} \right).
\]

We recall the initial assumption \( V_1(s_0) \leq \frac{1}{10s_0} \). For \( \epsilon_{\ell+1} \) small enough the last equation becomes:

\[
|V_1(s)| \leq \frac{1}{2s^{\tilde{\eta}}}.
\]

We now fix all the constants of the analysis, and the constants of smallness, so that the last two lemmas hold. We just allow us to increase the initial time \( s_0 \) if necessary, as it still make these two lemmas hold. At this point we now know that \( s^* \) is characterized by:

\[
(V_2(s^*), ..., V_{\ell}(s^*)) \in \mathcal{S}^{\ell-1} \left( \frac{1}{s^{\tilde{\eta}}} \right).
\]
We fix $\varepsilon(s_0)$, $V_1(s_0)$ and $b_1(s_0)$ satisfying the smallness assumptions \(3.20\) and \(3.21\). We define the following application:

\[
\begin{align*}
  f : \mathcal{D}(f) \subset B^{\ell-1} \left( \frac{1}{s_0} \right) & \rightarrow S^{\ell-1} \left( \frac{1}{s_0} \right) \\
  (V_2(s_0), ...V_\ell(s_0)) & \mapsto (s^\mu)_{s_0} \left( V_2(s^*), ..., V_\ell(s^*) \right),
\end{align*}
\]

\[(4.12)\]

With domain:

\[
\mathcal{D}(f) = \left\{ (V_2(s_0), ..., V_\ell(s_0)) \in B^{\ell-1} \left( \frac{1}{s_0} \right), \text{ such that } s^* < +\infty \right\}.
\]

\[(4.13)\]

We prove in the following lemma that $\mathcal{D}$ is non empty, open in $B^{\ell-1} \left( \frac{1}{s_0} \right)$, that $f$ is continuous and equivalent to the identity on the sphere $S^{\ell-1} \left( \frac{1}{s_0} \right)$.

**Lemma 4.5.** (Topological properties of $f$) The following properties hold:

(i) $\mathcal{D}(f)$ is non empty and open, satisfying $S^{\ell-1} \left( \frac{1}{s_0} \right) \subset \mathcal{D}(f)$.

(ii) $f$ is continuous and is the identity on the sphere $S^{\ell-1} \left( \frac{1}{s_0} \right)$.

**Proof of Lemma 4.5.** We recall that $V_i$ is the projection of $U$ on the unstable direction $v_i$ associated to the eigenvalue $\lambda \alpha_i$ of the matrix $A_\ell$, see Lemma (2.16). To ease notation we will write $\mu_i := \frac{\lambda \alpha_i}{\lambda - \alpha}$ the eigenvalues. From the time evolution of $U_i$ for $1 \leq i \leq \ell$ computed in (4.11) we get that the time evolution of $V_i$ is:

\[
V_i,s = \frac{\mu_i}{s_0^\ell} V_i + O(s^{-1-2\eta}) + O(s^{L-\ell}) + O(\epsilon_{\ell+1}s^{-1-\eta})
\]

\[
\ell = \frac{\mu_i}{s_0} V_i + O(\epsilon_{\ell+1}s^{-1-\eta}).
\]

Let $(V_2(s_0), ..., V_\ell(s_0)) \in S^{\ell-1} \left( \frac{1}{s_0} \right)$ be an initial data on the sphere. We claim that $s^* = 0$ which implies of course:

\[
f((V_2(s_0), ...V_\ell(s_0))) = (V_2(s_0), ...V_\ell(s_0)).
\]

This will prove that $\mathcal{D}(f)$ is non empty and that $f$ is equivalent to the identity on $S^{\ell-1} \left( \frac{1}{s_0} \right)$. To prove that, we just compute the scalar product between the time derivative of $(V_2(s), ...V_\ell(s))$ and an outgoing normal vector to the sphere at the point $(V_2(s_0), ...V_\ell(s_0))$:

\[
(V_2(s_0), ...V_\ell(s_0)).(V_2,s(s_0), ...V_\ell,s(s_0)) = \sum_{i=2}^{\ell} \frac{\mu_i}{s_0} |V_i|^2 + O(\epsilon_{\ell+1}s_0^{-1-2\eta}) > 0
\]

for $\epsilon_{\ell+1}$ small enough. In addition, this inequality uniformly holds on the sphere. For any small time $s'$, we have that $(V_2(s_0 + s'), ...V_\ell(s_0 + s'))$ is outside the ball, which implies $s^* = s_0$.

At $s = s_0$, this says that close to the border of the ball $B^{\ell-1} \left( \frac{1}{s_0} \right)$ the force term whose size is $O(\epsilon_{\ell+1}s_0^{-1-\eta})$ is overtaken by the linear repulsive dynamics. We are going to show that this is also true for $s_0 \leq s \leq s^*$.

We now prove that $f$ is continuous. Let $s$ be such that $s_0 \leq s \leq s^*$ and let
be an initial data such that at time $s$, $\frac{1}{2s^\theta} \leq (V_2(s), ..., V_i(s))$. The same computation gives:

$$\frac{d}{ds}|V|^2 = (V_2(s), ..., V_i(s)), (V_2, s(s), ..., V_{\ell, s}(s)) \geq \min((\mu_i)_{2 \leq i \leq \ell}) \frac{1}{(\ell+1)^2} + O(\frac{\epsilon_{\ell+1}}{s^{\ell+1}}) > 0,$$

once again provided one has taken $\epsilon_{\ell+1}$ small enough. It implies that at time $s$, fixed, there exists a small enough time $s^+ > 0$ and a small enough distance $r > 0$ such that:

$$\frac{1}{s^\theta} - r \leq |V(s)| \leq \frac{1}{\eta} \text{ implies } s \leq s^* \leq s^+,$$

ie the orbit leaves the ball $B^{\ell-1}(\frac{1}{s^\theta})$ in finite time. Let now $(V_2(s_0), ..., V_i(s_0))$ be an initial data such that $s^* < +\infty$. Since the time evolution of $V$ is a lipischitz continuous function of our problem, there is local continuity of the trajectories. Take $s^- < s^*$ close enough to $s^*$ so that $1/s^\theta - \frac{\pi}{2} \leq |V(s^-)|$, there exists a small enough distance $r_0 > 0$ such that if $|V'(s_0) - V(s)| < r_0$ then $|V'(s) - V(s)| < \frac{\pi}{4}$ for $s_0 \leq s \leq s^-$. The exit result we just stated implies that $s^- < s^*(V')$ and that $1/s^\theta - \frac{3\pi}{4} \leq V'(s^-)$. So that $s^- \leq s^*(V') \leq s^- + s^+$. We have proven that $D(f)$ is open.

From direct inspection, with the use of continuity properties, it is easy to prove in the same spirit that the function $s^*$ is continuous on $D$, and that $f$ is continuous too on $D(f)$.

We have reached the end of the proof. Indeed, if for all choices of initial data $(V_2(s_0), ..., V_i(s_0))$ we had $s^* < +\infty$, ie that no solution stayed in the trapped regime for all time, then $f$ would be a continuous function from the ball $B^{\ell-1}(\frac{1}{s^\theta})$ onto the sphere $S^{\ell-1}(\frac{1}{s^\theta})$ being equal to the identity at the border. This would be a contradiction to Brouwer’s fixed point theorem. It implies the existence of at least one initial data $(V_2(s_0), ..., V_i(s_0)) \in B^{\ell-1}(\frac{1}{s^\theta})$ such that the solution of (NLW) stays in the trapped regime described by Proposition 3.2.

We now end the proof of the main theorem. We know from Proposition 3.2 that there exists an orbit satisfying the assumptions of the trapped regime. We have computed that in that case there exists a constant $c > 0$ such that:

$$\frac{1}{c} s^\frac{\pi}{s^\theta} \leq \lambda \leq cs^\frac{\pi}{s^\theta}.$$

Since $\frac{ds}{dt} = \frac{c}{s}$ it gives:

$$\frac{1}{c} s^\frac{\pi}{s^\theta} \leq \frac{ds}{dt} \leq c' s^\frac{\pi}{s^\theta}.$$

This is an explosive ODE, we have that there exists a maximal time $T$ with:

$$s \sim C(u(0))(T - t)^\frac{\pi}{s^\theta} \text{ as } t \to T.$$

This implies:

$$\frac{1}{c} (T - t)^\frac{\pi}{s^\theta} \leq \lambda(t) \leq c(T - t)^\frac{\pi}{s^\theta} \text{ as } t \to T.$$
4.2. Behavior of Sobolev norms near blow-up time. We now prove the convergence of the norms (1.17), (1.14) and (1.15). First note that our analysis relies only on the study of supercritical Sobolev norms \((\dot{H}^\sigma \cap \dot{H}^{s_L}) \times (\dot{H}^{\sigma -1} \cap \dot{H}^{s_L -1})\) for the perturbative term \(\dot{\alpha}_b + w\). For this reason, the finiteness of the \(H^1 \times L^2\) norm of the initial data is not a requirement. Still, it is worth studying the behavior of lower order Sobolev norms because it applies when taking "nice" initial data, say smooth and with compact support, and because their asymptotic really corresponds to the concentration of a critical object. We now use the following decompositions:

\[
\begin{align*}
\dot{u} &= Q_\frac{1}{\lambda} + \dot{w} = (Q + \dot{\varepsilon})_\frac{1}{\lambda}, \quad \dot{\varepsilon} = w + \dot{\alpha}_b, \quad \varepsilon = \varepsilon + \dot{\alpha}_b, \quad (4.14) \\
\dot{u} &= \chi Q_\frac{1}{\lambda} + w' = (\chi \frac{1}{\lambda} Q + \varepsilon')_\frac{1}{\lambda}, \quad w' = \dot{w} + ((1 - \chi \frac{1}{\lambda})Q)_\frac{1}{\lambda}, \quad \varepsilon' = \varepsilon + (1 - \chi \frac{1}{\lambda})Q. \quad (4.15)
\end{align*}
\]

We recall that the subscript \(\frac{1}{\lambda}\) has a different meaning when it applies to \(\chi\), see (1.30). First note that because of (3.28) and because \(E_{s_L}\) controls the usual Sobolev norms, see (E.25), one has by interpolation:

\[
\|\varepsilon\|_{\dot{H}^s \times \dot{H}^{s-1}} \rightarrow 0 \text{ for all } s \leq s_L. \quad (4.16)
\]

Moreover, this convergence is also true for the perturbation on the manifold of approximate blow-up solutions:

\[
\|\dot{\alpha}_b\|_{\dot{H}^s \times \dot{H}^{s-1}} \rightarrow 0 \text{ for all } s \leq s_L.
\]

so we get for the perturbation:

\[
\|\varepsilon\|_{\dot{H}^s \times \dot{H}^{s-1}} \rightarrow 0 \text{ for all } s \leq s_L. \quad (4.17)
\]

We suppose from now on that \(\|u(0)\|_{H^1 \times L^2}\) is finite. This implies: \(\|\varepsilon'(0)\|_{\dot{H}^s \times L^2} = \|w'(0)\|_{\dot{H}^1 \times L^2} \leq C(u(0))\). We show first that this last quantity stays bounded.

**Lemma 4.6** (Boundedness in \(\dot{H}^1 \times L^2\)). Suppose \(u\) is a solution described by Proposition 3.2 such that \(u(0) \in \dot{H}^1 \times L^2\). Then there exists a constant \(C(u(0))\) such that for all \(0 \leq t < T\):

\[
\|u\|_{\dot{H}^1 \times L^2} \leq C(u(0)) \quad (4.18)
\]

**Proof of Lemma 4.6.** We first compute that under the decomposition (4.15), the soliton’s contribution to the \(\dot{H}^1\) norm is finite:

\[
\|\chi Q_\frac{1}{\lambda}\|_{\dot{H}^1} = \frac{1}{\lambda^{1-s_c}} \|\chi Q\|_{\dot{H}^1} \leq \frac{1}{\lambda^{1-s_c}} C \left( \int \frac{1}{\lambda} y^{d-\frac{1}{\sigma}-2} \right)^{\frac{1}{\sigma}} \leq C. \quad (4.19)
\]

Therefore, the lemma is proven once we show that the \(\dot{H}^1 \times L^2\) norm of \(w'\) stays finite. We are going to prove this by computing its time evolution under the bootstrap regime. We claim that:

\[
\frac{d}{dt} \|w'\|_{\dot{H}^1 \times L^2} \leq C \|w\|_{\dot{H}^1 \times L^2} + C \sum_{k=1}^p \left( \|w\|_{\dot{H}^1 \times L^2}^{2-c_k} \|w\|_{\dot{H}^1 \times L^2}^{c_k} \|w\|_{\dot{H}^{\sigma} \times \dot{H}^{s-1}}^\sigma \right). \quad (4.20)
\]

where for each \(k\), \(0 < c_k \leq 2\). We start by proving this bound. The time evolution of \(w'\) is:

\[
\partial_t w' = L + \frac{1}{\lambda} \mathcal{F}_\frac{1}{\lambda} + \frac{1}{\lambda} I_\frac{1}{\lambda} \quad (4.21)
\]
where $L$ is the linear part, $L := \left( \frac{w^{(2)}}{\Delta w^{(1)}} \right)$, $\mathcal{F}$ is the force term:

$$
\mathcal{F} = \left( \chi^\frac{1}{x} Q^p(\chi^\frac{1}{x})^{p-1} - 1 + (\lambda^2(\partial_{rr} \chi))^{\frac{1}{x}} + \frac{d-1}{r} \lambda(\partial_r \chi) \right) Q + 2\lambda(\partial_r \chi) Q),
$$

and $I$ is the interaction term: $I = \left( \sum_{k=1}^{p} C_k(\chi^\frac{1}{x} Q)^{p-k}(\varepsilon^{(1)})^k \right)$. It leads to the following expression for the time derivative of the norm:

$$
\frac{d}{dt} \|w'\|_{H^1 \times L^2}^2 = 2 \int \nabla w'(1) \cdot \nabla (L^{(1)} + \frac{1}{\lambda} \mathcal{F}^{(1)}) + 2 \int w'(2)(L^{(2)} + \frac{1}{\lambda} \mathcal{F}^{(2)} + \frac{1}{\lambda} I^{(2)}).
$$

We now want to estimate everything in the right hand side of (4.22). The linear term’s contribution is null:

$$
\int \nabla w'(1) \cdot \nabla w'(2) + w'(2) \Delta w'(1) = 0. \tag{4.23}
$$

We then compute the size of the force term. For the first coordinate:

$$
\int \frac{1}{1^2} |\nabla \mathcal{F}^{(1)}|^2 = \frac{1}{1^2} \chi^{\frac{1}{x}} \int (\frac{1}{1^2} \chi^{\frac{1}{x}} Q^p(\chi^{\frac{1}{x}})^{p-1} - 1)
$$

because $\alpha > 2$ and $\lambda_t = b_1 \to 0$ as $t \to T$. For the second coordinate:

$$
\int \frac{1}{2^2} |\nabla \mathcal{F}^{(2)}|^2 \leq C \int \chi^{\frac{1}{x}} Q^p(\chi^{\frac{1}{x}})^{p-1} \lambda^2 (Q^p(\chi^{\frac{1}{x}})^{p-1}) + 2\lambda(\partial_r \chi) Q^2
\leq C \int \frac{1}{\chi^{(2-x)}} \int y^{d-4}\frac{d-1}{r} dy 
\leq C \frac{1}{\chi^{(2-x)}} \int y^{d-4} dy = C.
$$

The bounds (4.24) and (4.25) imply the bound for the force term’s contribution:

$$
\int \frac{1}{\lambda} \nabla w'(1) \cdot \nabla \mathcal{F}^{(1)} + \frac{1}{\lambda} w'(2) \mathcal{F}^{(2)} \leq C \|w'\|_{H^1 \times L^2}. \tag{4.26}
$$

We now turn to the $L^2$ norm of the interaction term. First we rescale:

$$
\int \frac{1}{\lambda} \int w'(2) I^{(2)} \leq \frac{C}{\lambda^{1+2(1-s_c)}} \int \sum_{k=1}^{p} \int |\varepsilon^{(2)}(\chi^\frac{1}{x} Q)^{p-k}|(\varepsilon^{(1)})^k. \tag{4.27}
$$

We first take $k = 1$. Because of the asymptotic $Q^{p-1} \sim \frac{1}{\lambda}$, we use Hardy inequality and interpolation:

$$
\int |\varepsilon^{(2)}(\chi^\frac{1}{x} Q)^{(p-1)}(\varepsilon^{(1)})| \leq C \|\varepsilon^{(2)}\|_{L^2} \|\nabla^2 \varepsilon^{(1)}\|_{L^2} \leq C \|\varepsilon^{(2)}\|_{L^2} \|\varepsilon^{(1)}\|_{H^{\frac{3}{2}}} \|\varepsilon^{(1)}\|_{H^{3/2}}. \tag{4.28}
$$

As $\frac{d-2}{2} \int (1-s_c) + \frac{d-2}{2} = 2-s_c$ this gives the the estimate when applying the scale change:

$$
\int \frac{1}{\lambda^{1+2(1-s_c)}} \int |\varepsilon^{(2)}(\chi^\frac{1}{x} Q)^{(p-1)}(\varepsilon^{(1)})| \leq C \|w'\|_{L^2} \|w'(1)\|_{H^1} \|w'(1)\|_{H^1} \|w'(1)\|_{H^{3/2}}. \tag{4.29}
$$

Now let $k$ be an integer, $2 \leq k \leq p$. We have the asymptotic: $Q^{p-k} \sim \frac{1}{\lambda^{p-k}}$. We put this weighted decay on $\varepsilon^{(2)}$, use Hardy inequality and interpolation:

$$
\|\chi^\frac{1}{x} Q^{p-k} \varepsilon^{(2)}\|_{L^2} \leq C \|\nabla^\frac{2(p-k)}{p-1} \varepsilon^{(2)}\|_{L^2} \leq C \|\varepsilon^{(2)}\|_{L^2} \|\varepsilon^{(2)}\|_{L^2} \|\nabla \varepsilon^{(2)}\|_{L^2} \tag{4.29}
$$
for $\theta = \frac{2(p-k)}{(p-1)(\sigma-1)}$. Now by Sobolev injection one has that $|\varepsilon'(1)|^k \in L^q$ for $q \in [\frac{2d}{k(d-2)} : \frac{2d}{k(d-2\sigma)}]$. Because we work in a high dimension $d \geq 11$ and $p$ is an integer $\geq 2$ one has:

$$\frac{2d}{k(d-2)} \leq 2 \leq \frac{2d}{k(d-2\sigma)} = \frac{(p-1)d}{2k} + O(\sigma - s_c).$$

This implies that $\varepsilon'(1)^k \in L^2$ with the estimate:

$$\|\varepsilon'(1)^k\|_{L^2} \leq \|\varepsilon'(1)\|_{L^{2\kappa'}} \leq C \|\varepsilon'(1)\|_{H^1}^{1-\theta} \|\varepsilon'(1)\|_{H^\sigma},$$

for $(1-\theta')(d-2) + \frac{\theta'(d-2\sigma)}{2d} = \frac{1}{2\kappa}$. The estimates (4.29) and (4.30) allow us to apply Cauchy Schwarz and find:

$$\int |\varepsilon'(2)(\chi \frac{1}{\sqrt{x}} Q)^{(p-k)}\|\varepsilon'(1)^k | \leq C \|\varepsilon'(2)\|_{L^2} \|\varepsilon'(2)\|_{H^{\sigma-1}} \|\varepsilon'(1)\|_{H^1}^{1-\theta} \|\varepsilon'(1)\|_{H^\sigma}.$$  

We now compute: $(1-\theta)(1-s_c) + \theta(\sigma - s_c) + k(1-\theta')(1-s_c) + k\theta'(\sigma - s_c) = 1 + 2(1-s_c)$. Hence when applying the scale change the last estimate gives:

$$\frac{1}{\lambda^{1+2(1-s_c)}} \int |\varepsilon'(2)(\chi \frac{1}{\sqrt{x}} Q)^{(p-k)}\|\varepsilon'(1)^k | \leq C \|w\|_{H^1 \times L^2} \|w\|_{H^{\sigma-1} \times H^\sigma}.$$  

we compute the power involved for the $\|w\|_{H^1 \times L^2}$ term:

$$1 - \theta + k(1-\theta') = 2 - \frac{1}{\sigma - 1} - (k-1)(\sigma - s_c) = 2 - c_k.$$  

We now go back to the expression (4.27). We have computed the right hand side for the linear case in (4.28), and in the non linear case in (4.31). We have computed the coefficient condition for the non linear case in the last equation (it is straightforward in the linear case). Therefore we have the following estimate for the interaction term:

$$\left| \frac{1}{\lambda} \int w^{(2)}(t^2) \right| \leq C \sum_{k=1}^{p} \|w\|_{H^1 \times L^2}^{2-c_k} \|w\|_{H^{\sigma-1} \times H^\sigma}.$$  

We now come back to the identity (4.22). We have estimated all terms in the right hand side in (4.23), (4.26) and (4.32). This proves the bound (4.20) we claimed. We now want to integrate this equation in time. We recall that $\dot{w} = w + \tilde{\alpha}_b \frac{1}{x} + (1 - \chi \frac{1}{\sqrt{x}} Q)^{\frac{1}{2}}$. We take $s$ slightly supercritical: $s_c < s \leq \sigma$. The profile $\tilde{\alpha}_b$ has finite supercritical norm:

$$\|\tilde{\alpha}_b\|_{H^s \times H^{s-1}} \to 0.$$  

The tail of the soliton has also a bounded size:

$$\|((1 - \chi \frac{1}{\sqrt{x}} Q)^{\frac{1}{2}}\|_{H^s \times H^{s-1}} \leq C.$$  

From the bound (3.28), the same property holds for $w$ for $s = \sigma$: $\|w\|_{H^s \times H^{s-1}} \leq C$. Consequently, we have the boundedness of the $\sigma$ Sobolev norm for $w'$:

$$\|w'\|_{H^s \times H^{s-1}} \leq C.$$  

Coming back to the identity (4.20) it gives:

$$\frac{d}{dt} \|w'\|_{H^1 \times L^2}^2 \leq C \|w'\|_{H^1 \times L^2}^2 + C \sum_{k=1}^{p} \|w\|_{H^1 \times L^2}^{2-c_k}.$$  

The growth of this quantity is sub linear: it stays bounded until time $T$. □
We now know from the previous Lemma 4.6 that our solution stays bounded in $L^2$ until blow-up time. Using (4.19) we have that:
\[ \| w' \|_{H^1 \times L^2} \leq C. \]
This implies for the renormalized error:
\[ \| \varepsilon' \|_{H^1 \times L^2} \leq \lambda^{1-s_c} C. \]
On the other hand, the bootstrap bound (3.28) gives:
\[ \| \varepsilon' \|_{H^s \times H^{s-1}} \leq \lambda^{s-s_c} C. \]
By interpolation, we get that for any $1 \leq s \leq \sigma$:
\[ \| \varepsilon' \|_{H^s \times H^{s-1}} \leq \lambda^{s-s_c} C. \]
We now come back to the decomposition: $\varepsilon' = \varepsilon + \tilde{\alpha} b + (1 - \chi_{\frac{1}{\lambda}} Q)$. From (4.33) and (4.34) the perturbation $\tilde{\alpha} b$ and the tail of the solitary waves enjoy the bound:
\[ \| \tilde{\alpha} b + (1 - \chi_{\frac{1}{\lambda}} Q) \|_{H^s \times H^{s-1}} \leq \lambda^{s-s_c} C. \]
Combined with the previous bound (4.35), it gives for the original error term:
\[ \| \varepsilon \|_{H^s \times H^{s-1}} \leq \lambda^{s-s_c} C \rightarrow 0 \text{ as } t \rightarrow T. \]
This proves the convergence to 0 of the renormalized perturbation in slightly supercritical norms:
\[ \| \tilde{\varepsilon} \|_{H^s \times H^{s-1}} \rightarrow 0 \text{ as } t \rightarrow T, \text{ for } s_c < s \leq \sigma. \]
We now put (4.17) and (4.36) together: for any $s_c < s \leq s_L$,
\[ \| \tilde{\varepsilon} \|_{H^s \times H^{s-1}} \rightarrow 0 \text{ as } t \rightarrow T. \]
Now we turn to subcritical Sobolev norms. Let $s$ be such that $1 \leq s < s_c$. From (4.35), the perturbation has finite subcritical norms:
\[ \| w' \|_{H^s \times H^{s-1}} \leq C. \]
As the localized soliton also has finite subcritical norms:
\[ \| (\chi_{\frac{1}{\lambda}} Q) \|_{H^s \times H^{s-1}} \leq C, \]
this means that the full solution stays bounded in subcritical norms:
\[ \| u \|_{H^s \times H^{s-1}} \leq C(u(0)). \]
We now turn the the critical norm. From (4.35), the perturbation has finite critical and slightly supercritical norms:
\[ \| w' \|_{H^s \times H^{s-1}} \leq C(u(0)) \text{ for } s_c \leq s \leq \sigma \]
As the soliton is located on the first coordinate, this implies the boundedness of the time derivative in the critical and slightly critical spaces:
\[ \| \partial_t u^{(1)} \|_{H^{s-1}} = \| u^{(2)} \|_{H^{s-1}} \leq C(u(0)) \text{ for } s_c \leq s \leq \sigma \]
The critical norm for the first coordinate comes then from the soliton cut in a fixed zone:
\[ \| u^{(1)} \|_{H^{s_c} \times L^2} \| \chi Q_{\frac{1}{\lambda}} \|_{H^{s_c}} = C(d, p) \sqrt{T} \sqrt{\log(T-t)} (1 + o(1)) \text{ as } t \rightarrow T. \]
Appendix A. Numerology

We recall here some numerical facts about the phenomenological numbers that describe the soliton profile $Q$. $\alpha$, $p_{JL}$ and $\gamma$ are defined in the introduction in (1.9), (1.5) and (1.7). We give a proof of these very basic facts (see [27] for example) for the reader’s convenience.

Lemma A.1 (Value of $\alpha$). Let $d \geq 11$. Then:

(i) the condition $p > p_{JL}$ is equivalent to:

$$2 + \sqrt{d - 1} < s_c < \frac{d}{2}.$$

(ii) $\alpha$ is real if and only if $p > p_{JL}$. Because:

$$\Delta(p_{JL}) = 0, \quad \Delta(p) > 0 \text{ for } p > p_{JL}.$$

(iii) there holds the bounds on the parameter $\alpha$:

$$2 < \alpha < \frac{d}{2} - 1.$$

Proof of Lemma A.1. The proof just involves computations on polynomials.

Proof of (i): $s_c(p) = \frac{d}{2} - \frac{2}{p-1}$ is a strictly increasing function of $p$. This is why $p > p_{JL}$ is equivalent to:

$$s_c(p) > s_c(p_{JL}) = 2 + \sqrt{d - 1}.$$

Proof of (ii): With the result of (i) we rewrite:

$$\Delta(p) = \Delta(s_c(p)) = (2s_c - 2)^2 - 8 \left(\frac{d}{2} + s_c - 2\right),$$

which is a polynomial of degree 2 in $s_c$. Its leading order coefficient is positive, and its roots are:

$$2 - \sqrt{d - 1} \text{ and } 2 + \sqrt{d - 1} = s_c(p_{JL}).$$

So it is zero for $p = p_{JL}$, and strictly positive for $s_c > s_c(p_{JL}) \Leftrightarrow p > p_{JL}$.

Proof of (iii) We recall that $\alpha$ is the smaller root of the second order polynomial (this fact can be easily checked by the reader):

$$X^2 - \left(d - 2 - \frac{4}{p-1}\right) X + 2 \left(d - 2 - \frac{2}{p-1}\right) = P.$$

The second root is given by $\alpha_2 = \frac{d}{2} - 1 + \frac{\sqrt{d-1}}{4}$, with $\alpha_2 > 2$ because $d \geq 11$. As the leading order coefficient of the polynomial is positive, we just have to check that $P(\alpha) > 0$ to prove $2 < \alpha$. And indeed:

$$P(2) = 4 + \frac{4}{p - 1} > 0.$$

Appendix B. Properties of the stationary state

We state here the fundamental decomposition for the asymptotic of the stationary state $Q$. These results are now standard, see [27] [28] for example, and see also [15] [15] for its role in type II blow-up involving $Q$ in other equations. An important fact, the non nullity of the second term in the expansion, is however not proven in these works. We therefore prove it hereafter.
Lemma B.1. (Asymptotic expansion for the stationary state:) We have the expansion:

\[ \partial_y^k Q(y) = \partial_y \left( \frac{c_\infty}{y^{p-1}} + \frac{a_1}{y^1} \right) + O \left( \frac{1}{y^{\gamma+q+k}} \right) \quad \text{as } y \text{ goes to } +\infty, \quad (B.1) \]

with \( a_1 \) being a strictly negative (in particular \( a_1 \neq 0 \)) coefficient:

\[ a_1 < 0 \quad (B.2) \]

In [28] and references therein, the authors show the expansion, but they do not show that \( a_1 \neq 0 \). This appendix is devoted to prove this fact. In the paper the authors show the following result:

Lemma B.2 (Gui Ni Wang, [28], Theorem 2.5). We recall that \( 0 < \alpha_1 < \alpha_2 \) are the roots of the polynomial:

\[ X^2 - \left( d - 2 - \frac{4}{p-1} \right) X + 2 \left( d - 2 - \frac{2}{p-1} \right). \quad (B.3) \]

Then the following expansion is true.

(i) If \( \alpha_1 \notin \mathbb{N} \), then for all \( k_1, k_2 \in \mathbb{N} \), as \( y \to +\infty \) one has:

\[ Q(y) = \frac{c_\infty}{y^{p-1}} + \sum_{i,j=1}^{k_1,k_2} \frac{a_{i,j}}{y^{(i+1)\alpha_1 + k_1 \alpha_2}} + O \left( \frac{1}{y^{\gamma+q+(k_1+1)\alpha_1}} \right). \quad (B.4) \]

(ii) If \( \frac{\alpha_1}{\alpha_2} = k_1 + 1 \in \mathbb{N} \): then as \( y \to +\infty \) one has:

\[ Q(y) = \frac{c_\infty}{y^{p-1}} + \sum_{i=1}^{k_1+1} \frac{a_{i+1}}{y^{p-1+\alpha_1}} + \frac{\alpha_1 \log(y) + a_1}{y^{p-1+\alpha_1}} + O \left( \frac{1}{y^{\gamma+q+(k+1)\alpha_1}} \right). \quad (B.5) \]

As in the previous case the expansion can be continued to higher terms, but it does not matter for the analysis of the present paper.

(iii) This expansion adapts for higher derivatives of \( Q \).

This proves the expansion of Lemma [B.1]. The rest of this section is devoted to the proof that \( a_1 \) is strictly negative.

Proof of the assertion [B.2]. As a consequence of the previous lemma we get that, noting \( k := E[\alpha_1] \) if \( \alpha_1 \notin \mathbb{N} \), and \( k := \frac{\alpha_1 - 1}{\alpha_1} \) if \( \alpha_1 \in \mathbb{N} \) we have in both cases:

\[ \Lambda^{(1)} Q = \sum_{i=1}^k -ia_1 \frac{a_1}{y^{p-1+\alpha_1}} + O \left( \frac{\log(y)}{y^{p-1+\alpha_2}} \right), \quad (B.6) \]

and:

\[ \partial_y \Lambda^{(1)} Q = \sum_{i=1}^k (ia_1) \left( \frac{2}{p-1} + ia_1 \right) \frac{a_1}{y^{p-1+\alpha_1+1}} + O \left( \frac{\log(y)}{y^{p-1+\alpha_2+1}} \right). \quad (B.7) \]

The key point is that the coefficient \( a_i \) are linked with a recurrence relation:

Lemma B.3. For \( 1 \leq i \leq k \), \( a_i \) is given by \( a_i = P_i(a_1) \) where \( P_i \) is a polynomial such that \( P_i(0) = 0 \) for all \( 1 \leq i \leq k \).

This lemma is proved later. Hence we have the following alternative:

\[ \text{either } a_1 \neq 0 \text{ or } \partial_y \Lambda^{(1)} Q = O \left( \frac{\log(y)}{y^{p-1+\alpha_2+1}} \right). \quad (B.8) \]
The remainder term of (B.7) is in $L^2$. Indeed, we compute:

$$d - 2 - \frac{2}{p - 1} - 2\alpha_2 - 2 = -\sqrt{\Delta} < 0.$$  

So if $a_1 = 0$ then $\Lambda^{(1)}Q \in \dot{H}^1$. The term associated to $a_1$ is not in $L^2$ because $d - 2 - \frac{2}{p - 1} - 2\alpha_2 - 2 = \sqrt{\Delta} > 0$, see (B.7).

But we know from [7] that $\mathcal{L}$ is positive definite on $\dot{H}^1$, and that $\mathcal{L}\Lambda^{(1)}Q = 0$. We then must have $\Lambda^{(1)}Q \notin \dot{H}^1$. Considering what was said previously, this implies $a_1 \neq 0$.

We also know from [7] that $\Lambda^{(1)}Q > 0$. From the expansion (B.6) This implies that $a_1$ is strictly negative. □

We now give the proof of the recurrence relation between the $a_i$'s stated in Lemma B.3.

**Proof of Lemma B.3.** We use here the ideas developped in [27]. In this paper or in references therin, the following facts are proven:

**Lemma B.4 ([27] Lemmas 4.3 and 4.4).** The following holds:

1. The solitary wave exists and has $C^\infty$ regularity.
2. $y^{\frac{d}{2}}Q(y)$ has a limit as $y \to +\infty$, denoted $c_\infty$.
3. If we renormalise the space variable by $y = e^t$ and define:

$$W(t) = y^{\frac{d}{2}}Q(y) - c_\infty.$$  

$W$ then satisfies the differential equation for $t$ large:

$$W_{tt} + \left(d - 2 - \frac{4}{p - 1}\right) + 2\left(\frac{d - 2}{p - 1}\right)W + P(W) = 0,$$  

where $P$ denotes the polynomial:

$$(X + c_\infty)^p - c_\infty^p - p c_\infty^{p-1}X.$$  

4. $W$ has the following begining of expansion at infinity:

$$W(t) = \begin{cases} 
    a_1 e^{-\alpha_1 t} + O(e^{-\alpha_2 t}) & \text{if } \alpha_2 < 2\alpha_1 \\
    a_1 e^{-\alpha_1 t} + O(te^{-\alpha_2 t}) & \text{if } \alpha_2 = 2\alpha_1 \\
    a_1 e^{-\alpha_1 t} + O(e^{-2\alpha_1 t}) & \text{if } \alpha_2 > 2\alpha_1.
\end{cases}$$  

We will now compute the other coefficients of the expansion. As $W$ is a solution of (B.10), basic ODE theory states that there exists two coefficients $a$ and $b$ such that:

$$W(t) = a e^{-\alpha_1 t} + b e^{-\alpha_2 t} + \frac{1}{\alpha_2 - \alpha_1} \int_{t_0}^t (e^{\alpha_2(s-t)} - e^{\alpha_1(s-t)}) P(W)ds.$$  

We now prove lemma [B.3] by iteration. Our iteration hypothesis is the following for $1 \leq j \leq k - 1$:

$$\mathcal{H}(j) : \quad W(t) = \sum_{i=1}^j a_i e^{-\alpha_1 t} + O(e^{(j+1)\alpha_1 t}), \text{ with } a_i = P_i(a_1),$$

$P_i$ being a polynomial such that $P_i(0) = 0$.

Initialization: For $i = 1$, $a_1 = P_1(a_1)$ with $P_1 = X$ and of course $P_1(0) = 0$. Because of the preliminary expansion (iv), the property is true for $j = 1$. 
Heredy: We now suppose it is true for $1 \leq j \leq k - 1$. We then plug the expansion (B.14) into (B.13). It gives the following expression for $W$:

$$
W(t) = ae^{-\alpha t} + \frac{1}{\alpha_2 - \alpha_1} + \int_{T_0}^t (e^{\alpha_2(s-t)} - e^{\alpha_1(s-t)})P(W)ds + O(e^{(j+1)\alpha_1 t}),
$$

(B.15)
since $(j+1)\alpha_1 < \alpha_2$ (because $1 \leq j \leq k - 1$). But with the definition (B.11) of $P$ and the hypothesis (B.14) on the $a_i$ for $i \leq j$ we have that:

$$
P(W(t)) = \sum_{i=2}^{j+1} \tilde{a}_i e^{-i\alpha_1 t} + O(e^{(j+2)\alpha_1 t}),
$$

where $\tilde{a}_i = \tilde{P}_1(a_i)$ with $\tilde{P}_1$ being a polynomial such that $\tilde{P}_1(0) = 0$. We now put this expression in (B.15) and compute the integral of the right hand side. For $2 \leq i \leq j + 1$:

$$
\int_{T_0}^t (e^{\alpha_2(s-t)} - e^{\alpha_1(s-t)})e^{-i\alpha_1 s}ds = \frac{1}{\alpha_2 - \alpha_1} e^{-\alpha_2 t} - \frac{1}{(i-1)\alpha_1} e^{-\alpha_1 t}
+ \left(\frac{1}{\alpha_2 - \alpha_1} + \frac{1}{(i-1)\alpha_1}\right) e^{-i\alpha_1 t},
$$

and:

$$
\int_{T_0}^t (e^{\alpha_2(s-t)} - e^{\alpha_1(s-t)})O(e^{-(j+2)\alpha_1 t})ds = e^{-\alpha_2 t} \int_{T_0}^t O(e^{(\alpha_2-(j+2)\alpha_1) s})ds
- e^{-\alpha_1 t} \int_{T_0}^t O(e^{-(j+1)\alpha_1 s})ds. 
$$

(B.16)

Since $\alpha_2 > (j+2)\alpha_1$ the first integral diverges, the second term is integrable. Hence:

$$
\int_{T_0}^t (e^{\alpha_2(s-t)} - e^{\alpha_1(s-t)})O(e^{-(j+2)\alpha_1 s})ds = e^{-\alpha_2 t} O(\int_{T_0}^t e^{(\alpha_2-(j+2)\alpha_1) s}ds)
- e^{-\alpha_1 t} (\int_{T_0}^\infty O(e^{-(j+1)\alpha_1 s})ds
- \int_{T_0}^\infty O(e^{-(j+1)\alpha_1 s})ds).
$$

So we finally get for a constant $C$:

$$
W(t) = Ce^{-\alpha_1 t} + \sum_{i=2}^{j+1} \tilde{a}_i \frac{1}{\alpha_2 - \alpha_1} \left(\frac{1}{\alpha_2 - \alpha_1} - \frac{1}{(i-1)\alpha_1}\right) e^{-i\alpha_1 t} + O(e^{-(i+2)\alpha_1 t}).
$$

(B.17)

By identifying this last identity with the expansion (B.14) given by the induction hypothesis, one finds that in fact $C = a_1$ and $a_i = \tilde{a}_i$ for $i \leq j$. Therefore the property $H(j)$ is true for $j + 1$.

By induction, we have proved that (B.14) is valid for $j = k - 1$. To finish the proof one needs to do the same computation that we did before for the case $j = k - 1$

(i) If $\frac{e}{\alpha_1} \not\in \mathbb{N}$. Then the only things that changes is that we do not have $e^{-\alpha_2 t} = O(e^{-(k+1)\alpha_1 t})$, so we cannot throw away the terms involving $e^{-\alpha_2 t}$ and we get:

$$
W(t) = C e^{-\alpha_1 t} + \sum_{i=2}^k \tilde{a}_i e^{-i\alpha_1 t} + O(e^{-(k+1)\alpha_1 t}).
$$

(ii) If $\frac{e}{\alpha_1}$ is an integer, and $k = \frac{e}{\alpha_1} - 1$ to go from $k - 1$ to $k$ we also do the same computations as before. Now what changes is that we have a $t$ corrective term in (B.16):

$$
\int_{T_0}^t e^{\alpha_2(s-t)}O(e^{-(k+1)\alpha_1 t}) = O(te^{-\alpha_2 t}).
$$

which is what produces the log term in the expansion of $Q$ in that case.
The two following propositions are equivalents:

- The functions $a$ we have just proven is the fact that for any integer $i$,
- For the equivalence of the weighted norms away from the origin, we note that what lemma by induction.

We recall that $f$ is a radial function. We divide or multiply by a potential similar to $A$. So we suppose:

**Proof of Lemma C.1.** We just show that (i) implies (ii), the other implication being similar. So we suppose:

$$f \in C_{rad}^\infty,$$ with $\forall k \geq 0$, $f_k = O \left( \frac{1}{y^{p_2+k}} \right)$ as $y \to +\infty$.

We are going to show to following property by induction: for $i$ an integer, for all $0 \leq j \leq i$ and $k \in \mathbb{N}$ there holds:

$$\mathcal{H}(i) \quad \partial_y^k f_j = O \left( \frac{1}{y^{p_2+j+k}} \right) \quad \text{for all } 0 \leq j \leq i \text{ and } k \in \mathbb{N}.$$  

The property $\mathcal{H}(0)$ is obviously true from the supposition on $f$. Suppose now $\mathcal{H}(i)$ is true for $i$, and let $k \in \mathbb{N}$, suppose in addition that $i$ is odd. Then:

$$\partial_y^k f_{i+1} = \partial_y^k (A^* f_i) = \partial_y^k \left[ \partial_y f_i + \left( \frac{d-1}{y} + W \right) f_i \right].$$

As $\partial_y^k \left( \frac{d-1}{y} + W \right) = O \left( \frac{1}{y^{p_2+j+k}} \right)$ the property $\mathcal{H}(i+1)$ is then true. If $i$ is even, then replacing $A^*$ by $A$ leads to the same result as they have the same structure (they divide or multiply by a potential similar to $y^{-1}$) at infinity. We have proven that if $\mathcal{H}(i)$ is true then so is $\mathcal{H}(i+1)$. Hence we have showed the first proposition of the lemma by induction.

For the equivalence of the weighted norms away from the origin, we note that what we have just proven is the fact that for any integer $i$:

$$\partial_y^k f = \sum_{j=0}^{i} a_{i,j} f_j \quad \text{and} \quad f_i = \sum_{j=0}^{i} \tilde{a}_{i,j} \partial_y^j f,$$

the functions $a_{i,j}$ and $\tilde{a}_{i,j}$ being radial and $C^\infty$ outside the origin, with $a_{i,j} = O(y^{-(i-j)})$ and $\tilde{a}_{i,j} = O(y^{-(i-j)})$ as $y \to +\infty$. This implies (C.1). 

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**Appendix C. Equivalence of norms**

In this subsection we show that the notion of degree for admissible functions (see Definition 2.3) is equivalent for usual derivatives and adapted ones. We also show that the weighted usual Sobolev norms are equivalent, to some extent, to the weighted adapted ones.

**Lemma C.1.** (equivalence of the degree) Let $p_2$ be a real number and $f$ a $C^\infty$ radial function. We recall that $f_k$ is the $k$-th adapted derivative defined in (2.4.11).

The two following propositions are equivalents:

1. $\forall k \geq 0$, $\partial_y^k f = O \left( \frac{1}{y^{p_2+k}} \right)$ as $y \to +\infty$.
2. $\forall k \geq 0$, $f_k = O \left( \frac{1}{y^{p_2+k}} \right)$ as $y \to +\infty$.

Let $a \in \mathbb{R}$. For any $u \in C_{rad}^\infty$ there holds

$$\sum_{i=0}^{k} \int_{y \geq 1} \frac{|\partial^i u|^2}{1 + y^{2k-2i+2a}} \sim \sum_{i=0}^{k} \int_{y \geq 1} \frac{|u|^2}{1 + y^{2k-2i+2a}}. \quad (C.1)$$

**Proof of Lemma C.1.** We just show that (i) implies (ii), the other implication being similar. So we suppose:

$$f \in C_{rad}^\infty,$$ with $\forall k \geq 0$, $f_k = O \left( \frac{1}{y^{p_2+k}} \right)$ as $y \to +\infty$.

We are going to show to following property by induction: for $i$ an integer, for all $0 \leq j \leq i$ and $k \in \mathbb{N}$ there holds:

$$\mathcal{H}(i) \quad \partial_y^k f_j = O \left( \frac{1}{y^{p_2+j+k}} \right) \quad \text{for all } 0 \leq j \leq i \text{ and } k \in \mathbb{N}.$$  

The property $\mathcal{H}(0)$ is obviously true from the supposition on $f$. Suppose now $\mathcal{H}(i)$ is true for $i$, and let $k \in \mathbb{N}$, suppose in addition that $i$ is odd. Then:

$$\partial_y^k f_{i+1} = \partial_y^k (A^* f_i) = \partial_y^k \left[ \partial_y f_i + \left( \frac{d-1}{y} + W \right) f_i \right].$$

As $\partial_y^k \left( \frac{d-1}{y} + W \right) = O \left( \frac{1}{y^{p_2+j+k}} \right)$ the property $\mathcal{H}(i+1)$ is then true. If $i$ is even, then replacing $A^*$ by $A$ leads to the same result as they have the same structure (they divide or multiply by a potential similar to $y^{-1}$) at infinity. We have proven that if $\mathcal{H}(i)$ is true then so is $\mathcal{H}(i+1)$. Hence we have showed the first proposition of the lemma by induction.

For the equivalence of the weighted norms away from the origin, we note that what we have just proven is the fact that for any integer $i$:

$$\partial_y^k f = \sum_{j=0}^{i} a_{i,j} f_j \quad \text{and} \quad f_i = \sum_{j=0}^{i} \tilde{a}_{i,j} \partial_y^j f,$$

the functions $a_{i,j}$ and $\tilde{a}_{i,j}$ being radial and $C^\infty$ outside the origin, with $a_{i,j} = O(y^{-(i-j)})$ and $\tilde{a}_{i,j} = O(y^{-(i-j)})$ as $y \to +\infty$. This implies (C.1). 

---

22 the quantity need not be finite. By $x \sim y$ we mean here $\frac{x}{y} \leq c \leq cx$ for $c > 0$.
We recall that the Laplace based derivatives of a $C^\infty$ functions are:

$$D^{2k} u := \Delta^k u, \text{ and } D^{2k+1} u := \partial_y \Delta^k u.$$ 

**Lemma C.2.** (Equivalence of weighted adapted norms) There holds for all $u \in C^\infty$ radial function and integer $k$:

$$\sum_{i=0}^k \int \frac{u_i^2}{1+y^{2k-2i}} \sim \sum_{i=0}^k \int \frac{|D^i u|^2}{1+y^{2k-2i}}. \quad (C.2)$$

**Proof of Lemma C.2** step 1: **Leibnitz rule.** Let $f$ and $u$ be $C^\infty$ radial, with:

$$\partial_y^k f = O\left(y^{a-k}\right) \text{ as } y \to +\infty,$$

for some real number $a$. We will show the following property by induction: for any integer $i$:

$$\mathcal{H}(i) : \quad (fu)_i = \sum_{j=0}^i V_{i,j}(f)u_j,$$

where $V_{i,j}(f) \in C^\infty$ depending just on $f$, with $\partial_y^k V_{i,j}(f) \sim y^{a-(j-i)-k}$, and with $\frac{V_{ij}(f)}{y} \in C^\infty$ for $i - j$ odd.

The property $\mathcal{H}(0)$ is obviously true. Suppose now it is true for $i$ odd:

$$(fu)_{i+1} = A^*((fu)_i) = \sum_{j=0}^i A^*(V_{i,j}u_j) + \sum_{j=0, j \text{ even}}^i A^*(V_{i,j}u_j)$$

$$= \sum_{j=0}^i \left( -A + 2W + \frac{d-1}{y} \right) (V_{i,j}u_j) + \sum_{j=0, j \text{ odd}}^i \partial_y V_{i,j}u_j + V_{i,j}u_{j+1}$$

$$= \sum_{j=0}^i \partial_y V_{i,j}u_j + \left( \partial_y V_{i,j} + 2WV_{i,j} + \frac{(d-1)V_{i,j}}{y} \right) u_j + \sum_{j=0, j \text{ odd}}^i \partial_y V_{i,j}u_j + V_{i,j}u_{j+1}$$

$$= \sum_{j=0, (i+1-j) \text{ even}} (\partial_y V_{i,j} + 2WV_{i,j} + \frac{d-1}{y} V_{i,j} + V_{i,j-1}) u_j + \sum_{j=0, (i+1-j) \text{ odd}} \partial_y V_{i,j}u_j + V_{i,j-1}u_j.$$

For the terms in the first sum we have: $\partial_y V_{i,j} + 2WV_{i,j} + \frac{d-1}{y} V_{i,j} + V_{i,j-1} \in C^\infty$ because of the property for $i$, and it satisfies the decay property:

$$\partial_y^k \left( \partial_y V_{i,j} + 2WV_{i,j} + \frac{d-1}{y} V_{i,j} + V_{i,j-1} \right) = O\left(y^{a-(i+1-j)-k}\right).$$

For the second one the asymptotic property is also true from the induction hypothesis $\mathcal{H}(i)$, and we have indeed: $\frac{1}{y}(\partial_y(V_{i,j}) + V_{i,j-1}) \in C^\infty$. We have showed that if $\mathcal{H}(i)$ is true for $i$ odd, then $\mathcal{H}(i+1)$ is true. For $i$ even a similar reasoning gives also that $\mathcal{H}(i)$ implies $\mathcal{H}(i+1)$. Consequently, the property $\mathcal{H}(i)$ holds for all $i \in \mathbb{N}$.

**Step 2:** passing from one derivation to the other: We now claim that for any integer $i$ another property holds:

$$\mathcal{H}'(i) : \quad D^i u = \sum_{j=0}^i V_{i,j}u_j,$$

with $V_{i,j} \in C^\infty$ satisfying $\partial_y^k V_{i,j} \sim y^{-(i-j)-k}$, and for $j - i$ odd $\frac{1}{y} V_{i,j} \in C^\infty$. We show this property also by induction. It is true for $i = 0, 1, 2$. Suppose now it is
true for $i \geq 2$. Suppose $i$ even, then:

$$D^{i+1}u = \partial_y(D^i u) = \sum_{j=0, j \text{ even}}^{i} (-A + W)(V_{i,j} u_j) + \sum_{j=0, j \text{ odd}}^{i} (A^* - W - \frac{d-1}{y})(V_{i,j} u_j)$$

$$= \sum_{j=0, j \text{ even}}^{i} V_{i,j} u_{j+1} + \partial_y V_{i,j} u_j + \sum_{j=0, j \text{ odd}}^{i} V_{i,j} u_{j+1} + (\partial_y V_{i,j} - W V_{i,j} - \frac{d-1}{y} V_{i,j}) u_j.$$

The asymptotic behavior of the potentials is easily checked from the induction hypothesis. For $i+1-j$ odd we have: $V_{i+1,j} = \partial_y V_{i,j} + V_{i,j-1}$, which verifies indeed $\frac{1}{y} V_{i+1,j} \in C^\infty$ from the induction hypothesis $\mathcal{H}(i)$. Hence $\mathcal{H}'(i+1)$ is true. We have shown $\mathcal{H}(i)$ implies $\mathcal{H}'(i+1)$ for $i$ even and claim that for $i$ odd a very similar proof shows the heredity. Therefore, the propriety $\mathcal{H}'(i)$ is true for any integer $i$

This implies:

$$\int |D^i u|^2 \leq C \sum_{j=0}^{i} \int \frac{u_j^2}{1 + y^{2(i-j)}},$$

which implies the control of the Laplace derivatives by adapted derivatives in the Lemma. The other inequality of the equivalence can be proved exactly the same way. The opposite formula holds indeed also:

$$u_i = \sum_{j=0}^{i} \tilde{V}_{i,j} D^j u,$$

with $\tilde{V}_{i,j} \in C^\infty$, $\partial_y \tilde{V}_{i,j} \sim y^{-(i-j)-k}$ and $\frac{1}{y} \tilde{V}_{i,j} \in C^\infty$ if $i-j$ odd. The proof is left to the reader.

\[\square\]

**Appendix D. Hardy inequalities**

In this subsection we recall the standard Hardy estimates we used in the paper, in order to make this paper self contained. We use them to derive Hardy type estimates for the adapted norms, see next subsection. These analysis results, used to relate a norm that is adapted to a linear flow to the standard $L^2$ norms for usual derivatives, is now used in a canonical way in some works about blow-up, see for example [20] in a more subtle critical setting, [13] [10] in supercritical settings.

**Lemma D.1.** (Hardy inequality with best constant)

(i) Hardy near the origin: Let $u \in \cap_{0<r<1} H^1(C(r,1))$, then \[23\]

$$\int_{y \leq 1} |\partial_y u|^2 y^{d-1} dy \geq \frac{(d-2)^2}{4} \int_{y \leq 1} \frac{u^2}{y^2} y^{d-1} dy - C(d) u^2(1). \quad (D.1)$$

(ii) Hardy away from the origin, non critical exponent: Let $p > 0$, $p \neq \frac{d-2}{2}$, and $u \in \cap_{1<R} H^1(C(1,R))$. If $p$ is supercritical, $p > \frac{d-2}{2}$ then \[24\]

$$\int_{y \geq 1} |\partial_y u|^2 y^{d-1} dy \geq \left(\frac{d-2p+2}{2}\right)^2 \int_{y \geq 1} \frac{u^2}{y^{2p+2}} y^{d-1} dy - C(d,p) u^2(1). \quad (D.2)$$

\[23\text{Note that the quantities can be infinite.}\]

\[24\text{Note that the quantities can be infinite.}\]
If $p$ is subcritical, $0 < p < \frac{d-2}{2}$, if:

$$
\int_{y \geq 1} \frac{|\partial_y u|^2}{y^{2p+2}} y^{d-1} dy < +\infty,
$$

then:

$$
\int_{y \geq 1} |\partial_y u|^2 y^{d-1} dy \geq \left( \frac{d - (2p + 2)}{2} \right)^2 \int_{y \geq 1} \frac{u^2}{y^{2p+2}} y^{d-1} dy.
$$

Proof of Lemma (D.7) Proof of (i): Let $r > 0$ be a small number, and suppose $u \in C^1(C(r, 1))$. We integrate by parts and use Cauchy-Schwarz inequality to compute that:

$$
\int_1^r \frac{u^2}{y^{2p}} y^{d-1} dy = \frac{1}{d-2} \int_1^r u^2 \partial_y (y^{d-2-2p}) dy
$$

$$
= \frac{1}{d-2} \frac{u^2}{y^{d-2-2p}} \bigg|_1^r - \frac{2}{d-2} \int_1^r \frac{\partial_y u}{y^{d-1}} y^{d-1} dy
$$

$$
\leq C(d)u^2(1) + \frac{2}{d-2} \left( \int_1^r \frac{u^2}{y^{d-1}} y^{d-1} dy \right)^{\frac{1}{2}} \left( \int_1^r |\partial_y u|^2 y^{d-1} dy \right)^{\frac{1}{2}}
$$

$$
\leq C(d)u^2(1) + \frac{2}{d-2} \epsilon \left( \int_1^r \frac{u^2}{y^{d-1}} y^{d-1} dy + \frac{1}{\epsilon} \int_1^r |\partial_y u|^2 y^{d-1} dy \right),
$$

where we used Young’s inequality for the last computation. By density this inequality is still true for $u \in H^1(C(r, 1))$. Taking $\epsilon = \frac{d-2}{2}$ we get:

$$
\int_1^r \frac{u^2}{y^{2p}} y^{d-1} dy \geq \frac{(d - 2p)^2}{4} \int_1^r \frac{u^2}{y^{2p+2}} y^{d-1} dy - C(d)u^2(1).
$$

We then let $r \to 0$ and use Lebesgue’s monotone convergence theorem to pass to the limit.

Proof of (ii):

- **We first suppose $p$ subcritical, i.e $0 < p < \frac{d-2}{2}$.** Let $u$ satisfy (D.3) and $R$ denote a large real number. We do the same kind of computation as previously:

$$
\int_1^R \frac{u^2}{y^{d-2p}} y^{d-1} dy = \frac{1}{d-2p+2} \int_1^R u^2 \partial_y (y^{d-2-2p}) dy
$$

$$
= \frac{1}{d-2p+2} \left[ u^2 y^{d-2-2p} \right]_1^R - \frac{2}{d-2p+2} \int_1^R u \partial_y u \frac{y^{d-1}}{y^{2p+2}} dy
$$

$$
\leq \frac{R^{d-2p-2}}{d-2p+2} u^2(R) - \frac{u^2(1)}{d-2p+2}
$$

$$
+ \frac{2}{d-2p+2} \left( \int_1^R \frac{u^2}{y^{d-2p}} y^{d-1} dy \right)^{\frac{1}{2}} \left( \int_1^R |\partial_y u|^2 y^{d-1} dy \right)^{\frac{1}{2}}.
$$

We get rid of the negative term $-\frac{u^2(1)}{d-2p+2}$ and apply Young inequality:

$$
2 \left( \int_1^R \frac{u^2}{y^{d-2p+2}} dy \right)^{\frac{1}{2}} \left( \int_1^R \frac{|\partial_y u|^2}{y^{d-2p+2}} dy \right)^{\frac{1}{2}} \leq \epsilon \int_1^R \frac{u^2}{y^{2p+2}} dy + \frac{1}{\epsilon} \int_1^R |\partial_y u|^2 y^{2p+2}.
$$

Taking with $\epsilon = \frac{d-2-2p}{2}$ and injecting it in the previous inequality yields:

$$
\frac{1}{2} \int_1^R \frac{u^2}{y^{2p+2}} \leq \frac{R^{d-2p-2}}{d - (2p + 2)} u^2(R) + \frac{2}{(d - 2 - 2p)^2} \int_1^R |\partial_y u|^2 y^{2p+2}.
$$

(D.5)

The integrability of $\frac{u^2}{y^{2p+2}}$ ensures the existence of a sequence $(R_n)_{n \in \mathbb{N}}$, $R_n \to +\infty$ such that $R_n^{d-2p-2} u^2(R_n) \to 0$. When going to the limit with this sequence we end up with:

$$
\int_1^{+\infty} \frac{u^2}{y^{2p+2}} y^{d-1} dy \leq \frac{2}{d - 2 - 2p} \left( \int_1^{+\infty} \frac{u^2}{y^{2p+2}} y^{d-1} dy \right)^{\frac{1}{2}} \left( \int_1^{+\infty} |\partial_y u|^2 y^{d-1} dy \right)^{\frac{1}{2}},
$$

we need integrability this time, a constant function violates this rule for example.
which gives the desired result.

- When $p$ is supercritical $p > \frac{d-2}{2}$ we can perform the same computations as did before. But in this case the quantity $\frac{1}{p+2} - \frac{1}{p}$ has changed sign. We then obtain:

$$
\int_1^R \frac{u^2}{y^{2p+2}} y^{d-1} \, dy \leq \frac{u^2(1)}{2p+2-d} \int_1^R \frac{u^2}{y^{2p+2}} y^{d-1} \, dy + \frac{2}{2p+2-d} \left( \int_1^R \frac{u^2}{y^{2p+2}} y^{d-1} \, dy \right)^{\frac{1}{2}} \left( \int_1^R \frac{|\partial_y u|^2}{y^{2p}} y^{d-1} \, dy \right)^{\frac{1}{2}}.
$$

This time, we get rid of the term $\frac{u^2(1)}{2p+2-d} u^2(R)$:

$$
\int_1^R \frac{u^2}{y^{2p+2}} y^{d-1} \, dy \leq \frac{2}{2p+2-d} \left( \int_1^R \frac{u^2}{y^{2p+2}} y^{d-1} \, dy \right)^{\frac{1}{2}} \left( \int_1^R \frac{|\partial_y u|^2}{y^{2p}} y^{d-1} \, dy \right)^{\frac{1}{2}}.
$$

We now do the same reasoning we did close to the origin. First we use Young’s inequality with $\epsilon = \frac{2p-2-d}{2}$:

$$
\frac{1}{2p+2-d} \left( \int_1^R \frac{u^2}{y^{2p+2}} y^{d-1} \, dy \right)^{\frac{1}{2}} \leq \frac{1}{4} \int_1^R \frac{u^2}{y^{2p+2}} y^{d-1} \, dy + \frac{2}{2p+2-d} \int_1^R \frac{|\partial_y u|^2}{y^{2p}} y^{d-1} \, dy.
$$

We then inject this last estimate in (D.6), yielding:

$$
\frac{(2p+2-d)^2}{4} \int_1^R \frac{u^2}{y^{2p+2}} y^{d-1} \, dy \leq \int_1^R \frac{|\partial_y u|^2}{y^{2p}} y^{d-1} \, dy + C(d,p) u^2(1). \quad (D.7)
$$

We eventually pass to the limit $R \to +\infty$ using Lebesgue’s monotone convergence theorem to obtain (ii) in the supercritical case.

We now state a useful refined version of Hardy inequality for arbitrary weight function and number of derivatives. We denote an element $x \in \mathbb{R}^d$ by $x := (x_1, \ldots, x_d)$. We introduce a notation for the partial derivatives of a function:

$$
\partial^\kappa f = \frac{\partial^\kappa f}{\partial x_1^{\kappa_1} \cdots \partial x_d^{\kappa_d}} \quad (D.8)
$$

for a $d$-tuple $\kappa := (\kappa_1, \ldots, \kappa_d)$ with $|\kappa|_1 = \sum_{i=1}^d \kappa_i$.

**Lemma D.2.** (Weighted Fractional Hardy :) Let:

$$
0 < \nu < 1, \quad k \in \mathbb{N} \text{ and } 0 < \alpha \text{ satisfying } \alpha + \nu + k < \frac{d}{2},
$$

and let $f$ be a smooth function with decay estimates:

$$
|\partial^\kappa f(x)| \leq \frac{C(f)}{1 + |x|^{\alpha + i}}, \text{ for } |\kappa|_1 = i, \quad i = 0, 1, \ldots, k + 1, \quad (D.9)
$$

then for $\varepsilon \in \dot{H}^{\alpha+k+\nu}$, there holds $\varepsilon f \in \dot{H}^{\nu+k}$ with:

$$
\| \nabla^{\alpha+k+\nu} \varepsilon f \|_{L^2} \leq C(C(f), \nu, k, \alpha, d) \| \nabla^{\alpha+k+\nu} \varepsilon \|_{L^2}. \quad (D.10)
$$

If $f$ is a smooth radial function satisfying:

$$
|\partial^i_x f(|x|)| \leq \frac{C(f)}{1 + |x|^{\alpha+i}}, \quad i = 0, 1, \ldots, k + 1, \quad (D.11)
$$

then (D.10) holds.
Proof of Lemma D.2. We first prove for $f$ satisfying the non radial condition (D.9), and show after that for a radial function, this condition is equivalent to (D.11) the radial condition mentioned in the Lemma. Step 1: case for $k = 0$. We let $\varepsilon \in C^\infty_c$. We will prove the result for $\varepsilon$ and then conclude by density. We recall the Aronszajn-Smith formula:

$$
\| \nabla^\nu (\varepsilon f) \|_{L^2}^2 = C(\nu, d) \int_{x,y} |f(x)\varepsilon(x) - f(y)\varepsilon(y)|^2 \frac{dx dy}{|x-y|^{d+2\nu}}
$$

We now split into two zones: near and away from the singularity:

$$
\| \nabla^\nu (\varepsilon f) \|_{L^2}^2 \leq \int_{|x-y| \leq \frac{|x|}{2}} [\ldots] + \int_{|x-y| \leq \frac{|y|}{2}} [\ldots] + \int_{|x-y| \geq \frac{|y|}{2}} [\ldots].
$$

We want to bound all terms in the right hand side by $\| \nabla^{\alpha+\varepsilon} \|_{L^2}^2$. We first study the zone away from the singularity. We recall the standard Hardy estimate for $0 \leq s < \frac{d}{2}$:

$$
\int \frac{u^2}{|x|^{2s}} \leq C(d,s) \| \nabla^s u \|_{L^2}^2.
$$

We apply it to find:

$$
\int \frac{|f(x)\varepsilon(x) - f(y)\varepsilon(y)|^2}{|x-y|^{d+2\nu}} dx dy 
\leq C \int_{|x-y| \geq \frac{|y|}{2}} \frac{|f(x)|^2|\varepsilon(x)|^2}{|x-y|^{d+2\nu}} dx dz + C \int_{y, |x-y| \geq \frac{|y|}{2}} \frac{|f(y)|^2|\varepsilon(y)|^2}{|x-y|^{d+2\nu}} dx dz
\leq C \int \frac{|f(x)|^2|\varepsilon(x)|^2}{|x|^{2\nu}} dx + C \int \frac{|f(y)|^2|\varepsilon(y)|^2}{|y|^{2\nu}} 
\leq C(\nu, d, \alpha) \| \nabla^{\nu+\alpha} \varepsilon \|_{L^2}^2,
$$

where we used the decay property (D.9) of $f$. Now we study the integrals close to the singularity in (D.12). By symmetry we just prove the estimate for one. We decompose:

$$
\int_{|x-y| \leq \frac{|y|}{2}} \frac{|f(x)\varepsilon(x) - f(y)\varepsilon(y)|^2}{|x-y|^{d+2\nu}} dx dy 
\leq \int_{|x-y| \leq \frac{|y|}{2}} \frac{|f(x)|^2|\varepsilon(x) - \varepsilon(y)|^2}{|x-y|^{d+2\nu}} dx dy + \int_{|y-x| \leq \frac{|x|}{2}} \frac{|f(x)|^2|\varepsilon(y)|^2}{|y-x|^{d+2\nu}} dx dy.
$$

We start with the second term. We estimate thanks to the decay (D.9) of $\nabla f$:

$$
|f(x) - f(y)| \leq |x-y| \sup_{z \in B(x, \frac{|x|}{2})} |\nabla f| \leq |y-x| \frac{1}{1 + |x|^{1+\alpha}}.
$$

Now, in the zone $|y-x| \leq \frac{|x|}{2}$ one has $|y| \sim |x|$, therefore:

$$
\int_{|y-x| \leq \frac{|x|}{2}} \frac{|f(x) - f(y)|^2|\varepsilon(y)|^2}{|y-x|^{d+2\nu}} dx dy 
\leq C \int_{|y-x| \leq \frac{|x|}{2}} \frac{|f(x)|^2|\varepsilon(y)|^2}{|y-x|^{d+2\nu}} dx dy
\leq C \int \frac{|f(x)|^2|\varepsilon(y)|^2}{|y|^{2\nu}} dx dy
\leq C(\nu, d, \alpha, \nu) \| \nabla^{\alpha+\nu} \varepsilon \|_{L^2}^2.
$$

The first term in (D.15) is a bit harder:

$$
\int_{|x-y| \leq \frac{|y|}{2}} \frac{|f(x)|^2|\varepsilon(x) - \varepsilon(y)|^2}{|x-y|^{d+2\nu}} dx dy 
\leq C \int \frac{|\varepsilon(x) - \varepsilon(x+z)|^2}{|z|^{d+2\nu}(1 + x^{2\alpha})} dx dz.
$$
We let $v_z = \varepsilon(x + z) - \varepsilon(x)$, so that:

$$
\int_{x, |z| \leq \frac{|(x) - v(x)|}{2}} |(x) - v(z)| = \int_{x, |z| \geq 2\varepsilon\frac{|v(z)|}{1/2}\varepsilon dx dz
\leq C \int_{x, |z| \leq 1/2\varepsilon} f \int_{|z| \geq 2\varepsilon\frac{|v(z)|}{1/2}\varepsilon} dx dz
= C \int_{x, |z| \leq 1/2\varepsilon} f \int_{|z| \geq 2\varepsilon\frac{|v(z)|}{1/2}\varepsilon} dx dz
= C \left\| \nabla^{\nu + \alpha \varepsilon} \right\|_{L^2}^2.
$$

\begin{equation}
\text{(D.17)}
\end{equation}

We turn back to the decomposition \text{(D.15)}, for which we have estimated the two terms in \text{(D.16)} and \text{(D.17)}. It yields:

$$
\int_{|x - y| \leq \frac{|x|}{2}} \frac{|f(x)\varepsilon(x) - f(y)\varepsilon(y)|^2}{|x - y|^{d+2\nu}} dxdy \leq C(C(f), d, \nu, \alpha) \left\| \nabla^{\nu + \alpha \varepsilon} \right\|_{L^2}^2.
$$

\begin{equation}
\text{(D.18)}
\end{equation}

By symmetry, this is the estimate for the first two term in \text{(D.12)} we were looking for. Combined with the estimate for the third term obtained in \text{(D.14)}, it gives the result of the lemma for $k = 0$.

Step 2: Proof for $k \geq 1$. Let $f$, $\varepsilon$, $\alpha$, $\nu$ and $k$ satisfying the conditions of the lemma, with $k \geq 1$. Using Liebnitz rule for the integer part of the derivation:

$$
\left\| \nabla^{\nu + k}(\varepsilon f) \right\|_{L^2} \leq C \sum_{(\kappa, \tilde{\kappa}), |\kappa|_1 + |\tilde{\kappa}|_1 = k} \left\| \nabla^{\nu}(\partial^{\kappa \tilde{\kappa} \varepsilon} \partial^{\tilde{\kappa} \varepsilon} f) \right\|_{L^2}.
$$

\begin{equation}
\text{(D.19)}
\end{equation}

We can now apply the result obtained in the case $k = 0$ to the norms $\left\| \nabla^{\nu}(\partial^{\kappa \tilde{\kappa} \varepsilon} \partial^{\tilde{\kappa} \varepsilon} f) \right\|_{L^2}$ in \text{(D.19)}. We have indeed that $\partial^{\kappa \tilde{\kappa} \varepsilon} \in H^{\alpha + k_2 + \nu}$, and that $\partial^{\tilde{\kappa} \varepsilon}$ satisfies the decay property from \text{(D.9)}. It implies that for all $\kappa, \tilde{\kappa}$:

$$
\left\| \nabla^{\nu}(\partial^{\kappa \tilde{\kappa} \varepsilon} \partial^{\tilde{\kappa} \varepsilon} f) \right\|_{L^2} \leq C \left\| \nabla^{\nu + \alpha k} \right\|_{L^2}^2.
$$

which implies the result: $\left\| \nabla^{\nu + k}(\varepsilon f) \right\|_{L^2} \leq C(C(f), d, k, \alpha) \left\| \nabla^{\nu + \alpha k} \right\|_{L^2}^2$.

Step 3: equivalence between the decay properties. We want to show that \text{(D.9)} and \text{(D.11)} are equivalents for radial smooth functions, therefore implying the last assertion of the lemma. Suppose that $f$ is smooth, radial, and satisfies \text{(D.9)}. Then one has:

$$
\partial_{y} f(y) = \partial_{f_1} (\|y| e_1)
$$

where $e_1$ stands for the unit vector $(1, \ldots, 0)$ of $\mathbb{R}^d$. From this formula, we see that the condition \text{(D.9)} on $\partial_{\tilde{y}, f_1} (\|y| e_1)$ implies the radial condition \text{(D.11)}. We now suppose that $f$ is a smooth radial function satisfying the radial condition \text{(D.11)}. Then there exists a smooth radial function $\phi$ such that:

$$
f(y) = \phi(y^2).
$$

With a proof by iteration left to the reader one has that the decay property \text{(D.11)} for $f$ implies the following decay property for $\phi$:

$$
|\partial_{y} \phi(y)| \leq \frac{C(f)}{1 + y^{2i+1}}, \quad i = 0, 1, \ldots, k + 1,
$$

Now the standard derivatives of $f$ are easier to compute with $\phi$. We claim that for all $d$-tuple $\kappa$ there exists a finite number of polynomials $P_i(x) := C_i x_1^{i_1} \ldots x_d^{i_d}$, for $1 \leq i \leq l(\kappa)$, such that:

$$
\partial^{\kappa} f(x) = \sum_{i=1}^{l(\kappa)} P_i(x) \partial^{\kappa(i)} |x|^2 \phi(|x|^2)
$$
with for all $i$, $2q(i) - \sum_{j=1}^{d} i_j = |\kappa|_1$. This fact is also left to the reader. The decay property for $\phi$ then implies:

$$|P_i(x) \partial_{|x|}^q(\phi(|x|^2))| \leq \frac{C}{1 + y^{\alpha+2q(i)-\sum_{j=1}^{d} i_j}} = \frac{C}{1 + y^{\alpha+|\kappa|_1}},$$

which implies the property (D.9). □

**Appendix E. Coercivity of the adapted norms**

Here we derive Hardy type inequalities for the operators $A$, $A^*$ and $L$. Such quantities are easier to manipulate for the linear flow of the operator $H$ (defined in (1.32)). As for the previous section of the appendix, this kind of bounds is now standard and we refer to the papers quoted therein for the use of similar techniques.

We start with $A^*$, then $A$, and after that we are able to deal with the coercivity of the adapted norms.

We recall that the profile $\Phi_M$ is defined by equation (3.3). Its main properties that we will use in this section are its localization on the first coordinate and its non-orthogonality with respect to $\Lambda_0Q$ (from (3.5) and (3.6)):

$$\Phi_M = (\Phi_M, \Lambda_0Q) = (\Phi_m, \Lambda(1)Q) \sim CM^{2k_0+2\delta_0} > 0 (C > 0).$$

(E.1)

We also recall the structure of the two first order differential operators on radial functions $A$ and $A^*$:

$$A^* = \partial_y + \left( \frac{d-1}{y} + W \right), \quad A = -\partial_y + W;$$

(E.2)

where $W$ is a smooth radial function with the asymptotic at infinity from (2.9):

$$W = -\frac{\gamma}{y} + O\left( \frac{1}{y^{1+g}} \right) \text{ as } y \to +\infty$$

(E.3)

**Lemma E.1.** (Weighted coercivity for $A^*$). Let $p$ be a non negative real number. Then there exists a constant $c_p > 0$ such that for all radial $u \in H^1_{loc}(\mathbb{R}^d)$ there hold\(^{26}\)

$$\int \frac{|A^* u|^2}{1 + y^{2p}} \geq c_p \left[ \int \frac{u^2}{y^2(1+y^{2p})} + \int \frac{\partial_y u|^2}{1 + y^{2p}} \right].$$

(E.4)

**Proof of Lemma E.1.** We take $u$ satisfying the conditions of the lemma.

**step 1:** Subcoercivity for $A^*$. We claim the subcoercivity lower bound:

$$\int \frac{|A^* u|^2}{1 + y^{2p}} \geq c \left[ \int \frac{u^2}{y^2(1+y^{2p})} + \int \frac{\partial_y u|^2}{1 + y^{2p}} \right]$$

(E.5)

$$-\frac{1}{c} \int u^2(1) + \int \frac{u^2}{1 + y^{2p+g}},$$

\(^{26}\)The quantities need not be finite.
for a universal constant $c = c(d, p) > 0$. We introduce the operator: $\bar{W} := W + \frac{d}{y}$. First we estimate close to the origin:

\[
\int_{y \leq 1} |A^* u|^2 = \int_{y \leq 1} (|\partial_y u|^2 + \bar{W}^2 u^2 + 2\bar{W} u \partial_y u) = \int_{y \leq 1} |\partial_y u|^2 + \int_{y \leq 1} u^2 \left( \bar{W}^2 - \frac{d-1}{y^2} (y^{d-1} \bar{W}) \right) + W(1)^2 u(1)^2 \geq \int_{y \leq 1} |\partial_y u|^2 + \int_{y \leq 1} u^2 \left( \frac{(d-1)(d-2)}{y^2} + O(1) \right) = \int_{y \leq 1} |\partial_y u|^2 + (d-1) \int_{y \leq 1} \frac{u^2}{y^2} + O(\int_{y \leq 1} u^2).
\]

(E.6)

Away from the origin, from the asymptotic (E.9):

\[
\int_{1}^{R} \frac{|A^* u|^2}{y^{2p}} = \int_{1}^{R} \frac{1}{y^{2p}} (|\partial_y u| + \frac{d-1}{y} u + O(\frac{1}{y^2}))^2 = \int_{1}^{R} \frac{1}{y^{2p}} |\partial_y u| + \frac{d-1}{y} u + O\left(\frac{1}{y^{2p+2}}\right). \]

We now use Cauchy-Schwarz and Young inequalities on better decaying term:

\[
\int_{1}^{R} \frac{1}{y^{2p+2(d-1)-\gamma}} |\partial_y(y^{d-1-\gamma} u)|^2 \geq \int_{1}^{R} \frac{|\partial_y u|^2}{y^{2p+2}} + C \left( \int_{1}^{R} \frac{|\partial_y u|^2}{y^{2p+2}} + C' \left( \int_{1}^{R} \frac{|\partial_y u|^2}{y^{2p+2}} \right)^{\frac{1}{2}} \right) \int_{1}^{R} \frac{u^2}{y^{p+2}}.
\]

Combining the last two estimates gives:

\[
\int_{1}^{R} \frac{1}{y^{2p+2(d-1)-\gamma}} |\partial_y(y^{d-1-\gamma} u)|^2 \geq c \left( \int_{1}^{R} \frac{u^2}{y^{2p+2}} + \int_{1}^{R} \frac{|\partial_y u|^2}{y^{2p}} \right) - C' u^2(1), \tag{E.8}
\]

for a constant $c > 0$. We come back to (E.7) and inject the bound (E.8), it yields:

\[
\int_{1}^{R} \frac{|A^* u|^2}{y^{2p}} \geq c \left( \int_{1}^{R} \frac{u^2}{y^{2p}} + \int_{1}^{R} \frac{|\partial_y u|^2}{y^{2p}} - \frac{1}{c} u^2(1) \right.
\]

\[
\left. + \int_{1}^{R} u O\left( \frac{|\partial_y u|}{y^{2p+2}} \right) \left( \partial_y u + u O\left( \frac{1}{y} \right) \right) \right).
\]

(E.9)

We now use Cauchy-Schwarz and Young inequalities on better decaying term:

\[
\left| \int_{1}^{R} u O\left( \frac{1}{y^{2p+1}+\gamma} \right) (\partial_y u + u O\left( \frac{1}{y} \right)) \right| \leq C \epsilon \int_{1}^{R} \frac{|\partial_y u|^2}{y^{2p}} + C \epsilon \int_{1}^{R} \frac{|u|^2}{y^{2p+2+g}} + C \int_{1}^{R} \frac{|u|^2}{y^{2p+2+g}}.
\]

Taking $\epsilon$ small enough and combining this bound with (E.9) gives for a constant $c > 0$:

\[
\int_{1}^{R} \frac{|A^* u|^2}{y^{2p}} \geq c \left( \int_{1}^{R} \frac{u^2}{y^{2p+2}} + \int_{1}^{R} \frac{|\partial_y u|^2}{y^{2p}} - \frac{1}{c} \left( u^2(1) + \int_{1}^{R} \frac{u^2}{y^{2p+2+g}} \right) \right).
\]

Because of the additional decay in the last term we have that if $\frac{u^2}{y^{2p}}$ or $\frac{|\partial_y u|^2}{y^{2p}}$ is non integrable at infinity, then going to the limit $R \to 0$ gives that $\frac{|A^* u|^2}{y^{2p}}$ is non integrable. Therefore in that case all quantities in (E.4) are infinite and the inequality is proven. Now, if they are integrable, then going to the limit $R \to +\infty$ in the last inequality and combining it with the estimate close to the origin (E.6) we proved
earlier gives the subcoercivity bound \((E.5)\).

\[\int \frac{|A^* u|^2}{1 + y^{2p}} \leq \frac{1}{n}, \text{ and } \int \frac{u^2}{y^2(1 + y^{2p})} + \int |\partial_y u|^2 \leq 1 \quad \text{ (E.10)}\]

From the subcoercivity estimate \((E.5)\) it implies that:

\[u_n^2(1) + \int \frac{u_n^2}{1 + y^{2p+2+g}} \gtrsim 1.\]

And by \((E.10)\) we have that \(u_n\) is uniformly bounded in \(H_1^{\text{rad,loc}}[r,R]\). Hence by compactness and by an extraction argument there exists a limit profile \(u_\infty \in H_1^{\text{loc}}\) such that up to a subsequence,

\[u_n \rightharpoonup u_\infty \text{ in } H_1^{\text{loc}}.\]

From continuity of functions in \(H^1\) in one dimension, and from compactness of the injection \(H^1 \hookrightarrow L^2\) on compact sets we have also:

\[u_n \to u_\infty \text{ in } L^2_{\text{loc}}, \quad u_n(1) \to u_\infty(1).\]

We now show that \(u_\infty \neq 0\). We have that \(u_n^2(1) \to u_\infty^2(1)\). Indeed the continuity of the \(H_1^{\text{loc}}\) functions in one dimension, the strong convergence \(L^2\) and of the equi-continuity of the family \(\{u_n\}\) implies the convergence \(L^\infty\). If \(u_\infty^2(1) = 0\), then \(u_\infty \neq 0\). If \(u_\infty(1) = 0\) then the subcoercivity bound implies that \(\int \frac{u_n^2}{1 + y^{2p+2+g}} \gtrsim 1\).

The local \(L^2\) convergence, and the fact that \(\int \frac{u_n^2}{y^{(1+y^{2p})}}\) is uniformly bounded implies that:

\[\int \frac{u_n^2}{1 + y^{2p+2+g}} \to \int \frac{u_\infty^2}{1 + y^{2p+2+g}}.\]

Hence \(\int \frac{u_\infty^2}{1 + y^{2p+2+g}} > 0\) so \(u_\infty \neq 0\). In any cases we have found: \(u_\infty \neq 0\). On the other hand from semi-continuity again we have that:

\[A^* u_\infty = 0.\]

This equation has for unique solution in \(H^1\) the function \(\Gamma\) up to multiplication by a scalar. Hence:

\[u_\infty = c\Gamma.\]

\(c\) is non zero because \(u_\infty\) is non zero. But:

\[\int_{y \leq 1} \frac{\Gamma^2}{y^2} \gtrsim \int_{y \leq 1} \frac{y^{d-1}}{y^{2(d-2)+2+d}} dy = +\infty,\]

which contradicts \((E.11)\).

We now focus on the coercivity of the operator \(A\).

**Lemma E.2. (Weighted coercivity for \(A\))** \(\text{Let } p \text{ be a non negative real number. Let } k_0 \text{ and } \delta_0 \text{ be defined by (1.18) } (\delta_0 > 0). \text{ Then:}\)

\((i)\) case \(p\) small: if \(0 \leq p < k_0 + \delta_0 - 1\), then there exists a constant \(c_p > 0\) such that for all \(u \in H_1^{\text{rad,loc}}(\mathbb{R}^d)\) satisfying:

\[\int_{y \geq 1} \frac{u^2}{y^{2p+2}} < +\infty,\]

\[\int \frac{\Gamma^2}{y^2} \gtrsim \int \frac{y^{d-1}}{y^{2(d-2)+2+d}} dy = +\infty,\]

which contradicts \((E.11)\). \(\square\)
there holds the coercivity\(^{27}\)
\[
\int \frac{|Au|^2}{1 + y^{2p}} \geq c_k \left[ \int \frac{|
abla_y u|^2}{1 + y^{2p}} + \frac{u^2}{y^2(1 + y^{2p})} \right]. \tag{E.12}
\]

(ii) case \(p\) large: let \(p > k_0 + \delta_0 - 1\), let \(M\) be large enough (depending on \(d\) and \(p\) only), then there exists \(c_{M,p} > 0\) such that if \(u \in H^1_{\text{rad,loc}}\) satisfies:
\[
\langle u, \Phi_M \rangle = 0. \tag{E.13}
\]

then\(^{28}\)
\[
\int \frac{|Au|^2}{1 + y^{2p}} \geq c_{M,p} \left[ \int \frac{|
abla_y u|^2}{1 + y^{2p}} + \frac{u^2}{y^2(1 + y^{2p})} \right]. \tag{E.14}
\]

Proof of Lemma \(E.2\). As for \(A^*\) we first show a subcoercivity bound and then show that if we want to violate the Hardy type inequality, one must get closer and closer to the zero of \(c > 0\).

We apply Cauchy-Schwarz and Young inequality to control the last term:
\[
\int_1^R 1 \frac{|Au|^2}{y^{2p}} y^{2p} \geq c M, p\left[ \int \frac{|
abla_y u|^2}{1 + y^{2p}} + \frac{u^2}{y^2(1 + y^{2p})} \right]. \tag{E.15}
\]

for a universal constant \(c > 0\). We start by computing close to the origin using (D.1), with the help of the Hardy inequality close to the origin (D.1):
\[
\int_{y \leq 1} |Au|^2 = \int_{y \leq 1} |
abla_y u|^2 + \int_{y \leq 1} O(u^2) + \int u \partial_y u O(1) \\
\geq c \left( \int_{y \leq 1} |
abla_y u|^2 + \frac{u^2}{y^2} \right) - \frac{c}{\epsilon} \left( u^2(1) + \int_{y \leq 1} u^2 \right) + \int u \partial_y u O(1).
\]

We apply Cauchy-Schwarz and Young inequality to control the last term:
\[
\int u \partial_y u O(1) \leq \epsilon C \int_{y \leq 1} |
abla_y u|^2 + \frac{C}{\epsilon} \int_{y \leq 1} u^2.
\]

Taking \(\epsilon\) small enough gives close to the origin:
\[
\int_{y \leq 1} |Au|^2 \geq c \left( \int_{y \leq 1} |
abla_y u|^2 + \frac{u^2}{y^2} \right) - \frac{c}{\epsilon} \left( u^2(1) + \int_{y \leq 1} u^2 \right) \tag{E.16}
\]

Away from the origin, we use the asymptotics (E.3) of the potential \(W\) to derive:
\[
\int_{y \geq 1} 1 \frac{|Au|^2}{y^{2p}} y^{2p} = \int_{y \geq 1} 1 \frac{1}{y^{2p}} \left( \partial_y u + \frac{u}{y} \right)^2 + O \left( \frac{u^2}{y^{2p+2}} \right) \left( \partial_y u + \frac{u}{y} \right), \tag{E.17}
\]

This time we let \(v = y^\gamma u\), and \(2p' = 2p + 2\gamma\). We observe: \(2p' - (d - 2) = 2p - 2k_0 + 2 - 2\delta_0 < 0\) in the case \(p\) small and \(> 0\) in the case \(p\) large. For \(p\) small we have from (D.5):
\[
\int_{y \geq 1} 1 \frac{1}{y^{2p}} \left( \partial_y u + \frac{u}{y} \right)^2 = \int_{y \geq 1} 1 \frac{1}{y^{2p}} \left( \partial_y u + \frac{u}{y} \right)^2 + \int_{y \geq 1} O \left( \frac{u^2}{y^{2p+2}} \right) \left( \partial_y u + \frac{u}{y} \right) \tag{E.18}
\]

As we did in the proof of the sub-coercivity estimate for \(A^*\), the identity (E.17) and the control (E.18) imply using Cauchy-Schwarz and Young inequality:
\[
\int_{y \geq 1} 1 \frac{|Au|^2}{y^{2p}} y^{2p} \geq c \left( \int_{y \geq 1} \frac{u^2}{y^{2p+2}} + \frac{|
abla_y u|^2}{y^{2p}} \right) - \frac{1}{c} \left( \frac{R^{d-2p'-2}}{d-2-2p} - \frac{R^{d-2K}}{d-2-2p} \right) \tag{E.19}
\]

\(^{27}\)the quantities in the coercivity estimate need not be finite.

\(^{28}\)idem.
The integrability condition \( (E.11) \) gives that along a sequence \( R_n \) the \( u(R_n) \) term goes to zero. This allow us to conclude that if \( \frac{|\partial_y u|^2}{y^{2p}} \) is not integrable, then \( \frac{|A^* u|^2}{y^{2p}} \) is not integrable neither. This gives the Hardy inequality in the case the quantities are infinite. We can now suppose that the involved quantities are finite. We go to the limit in the previous equation along \( R_n \) and combine it with \( (E.16) \) to obtain the subcoercivity estimate.

For \( p \) large we are in the supercritical case in the standard Hardy inequality for \( v \). We can do verbatim the same reasoning we did for the proof of the subcoercivity estimate for \( A^* \).

**step 2 Coercivity.** We argue by contradiction. If the hardy inequality we want to show was wrong, there would exist a sequence \( (u_n)_{n \in \mathbb{N}} \), such that:

\[
\int |\partial_y u_n|^2 + \frac{u_n^2}{1 + y^{2p}(1 + y^{2p})} = 1, \quad \int \frac{|Au|^2}{1 + y^{2p}} \to 0.
\]

From the subcoercivity estimate implies:

\[
u_n^2(1) + \int \frac{u_n^2}{1 + y^{2p+2+g}} \gtrsim 1,
\]

and \( u_n \rightharpoonup u_\infty \) in \( H^1_{loc}([0, +\infty[) \). The quantities go the same way to the limit and we find that \( u_\infty \) is not zero and must satisfy:

\[
Au = 0.
\]

This implies \( u_\infty = c\Lambda^{(1)}Q, \ c \neq 0 \).

If \( k \geq k_0 \) then the orthogonality condition goes to the limit with the weak topology and we find \( \langle u_\infty, \Phi_M \rangle = 0 \) which violates \( (E.1) \). If \( k \leq k_0 - 1 \), we have from lower semi continuity that:

\[
\int \frac{u_\infty^2}{1 + y^{2p+2+g}} < +\infty,
\]

but \( \Lambda^{(1)}Q \) does not satisfy this inequality because as \(-2\gamma - 2p - 2 + d = 2(k_0 - p) - 2(1 - \delta_0) > 0 \) we have:

\[
\int \frac{\Lambda^{(1)}Q^2}{1 + y^{2p+2}} = +\infty.
\]

In both cases there is a contradiction. Hence the lemma are proven.\( \square \)

Once the coercivity properties of \( A \) and \( A^* \) have been established, we can turn to the core of this part: the coercivity estimates for the adapted norms provided some orthogonality conditions are satisfied.

**Lemma E.3 (Coercivity of \( \mathcal{E}_k \)).** We still assume \( \delta_0 \neq 0 \). \( k \) denotes an integer. We recall that \( u_j \), the \( j \)-th adapted derivative of \( u \), is defined in \((2.19)\).

(i) case \( k \) small Let \( 0 \leq k \leq k_0 \) and \( 0 \leq \delta < \delta_0 \). Then there exists a constant \( c_{k, \delta} > 0 \) such that for all \( u \in H^k_{rad, loc}(\mathbb{R}^d) \) satisfying:

\[
\sum_{p=0}^{k} \int \frac{u_p^2}{1 + y^{2k-2p}} < +\infty,
\]

there holds:

\[
\int \frac{u_k^2}{1 + y^{2\delta}} \geq c_k \sum_{p=0}^{k-1} \int \frac{u_p^2}{1 + y^{2k-2p+2\delta}}.
\]
(ii) case $k$ large Let $k \geq k_0 + 1$ and $0 \leq \delta < \delta_0$, let $j = E\left(\frac{k-1}{2}\right)$. Then for $M = M(k)$ large enough, there exists $c_{M,k} > 0$ such that for all $H^{k}_{\text{loc},\text{rad}}(\mathbb{R}^d)$ satisfying:

$$\sum_{p=0}^{k} \frac{u_p^2}{1 + y^{2k-2p}} < +\infty \text{ and } \langle u, \mathcal{L}^p \Phi_M \rangle = 0, \text{ for } 0 \leq p \leq j - 1,$$

(E.21)

there holds:

$$\int \frac{u_p^2}{1 + y^{2k-2p}} \geq c_{M,k} \sum_{p=0}^{k-1} \int \frac{u_p^2}{1 + y^{2k-2p+2\delta}}.$$

(E.22)

**Corollary E.4** (Coercivity of $\mathcal{E}_{s_L}$). Let $L$ and $\sigma$ be defined by (2.23) and (3.13) ($L$ is odd) and $0 \leq \delta < \delta_0$. Then there exists a constant $c > 0$ such that for all radial $\varepsilon \in H^{s_L} \times H^{s_L-1} \cap H^\sigma \times H^{\sigma-1}$ satisfying:

$$\langle \varepsilon, H^s \Phi_M \rangle = 0 \text{ for } 0 \leq i \leq L,$$

(E.23)

there holds:

$$\sum_{p=0}^{s_L} \int \frac{|\varepsilon_p|^2}{1 + y^{2s_L-2p+2\delta}} + \sum_{p=0}^{s_L} \int \frac{|\varepsilon_p|^2}{1 + y^{2s_L-2-2p+2\delta}} \leq c \left( \int \frac{|\varepsilon_{s_L}|^2}{1 + y^{2\delta}} + \int \frac{|\varepsilon_{s_L-1}|^2}{1 + y^{2\delta}} \right),$$

(E.24)

$$\| \varepsilon \|^2_{H^{s_L} \times H^{s_L-1}} \leq c \mathcal{E}_{s_L} < +\infty,$$

(E.25)

the adapted derivatives $u_k$ being defined by (2.19) and $\mathcal{E}_{s_L}$ being defined by (3.11).

**Proof of Corollary E.4.** Step 1: Proof that $\mathcal{E}_{s_L} < +\infty$. From the equivalence between Laplace derivatives and adapted ones, (C.2), one has:

$$\int |\varepsilon_{s_L}^{(1)}|^2 \leq C \sum_{i=0}^{s_L} \int \frac{|D\varepsilon_{s_L}^{(1)}|^2}{1 + y^{2s_L-2i}}.$$

For $\sigma \leq i \leq s_L$ one has by interpolation $\int |D\varepsilon_{s_L}^{(1)}|^2 < +\infty$, hence $\int \frac{|D\varepsilon_{s_L}^{(1)}|^2}{1 + y^{2s_L-2i}} < +\infty$. For $0 \leq i \leq \sigma$ one has $\frac{|D\varepsilon_{s_L}^{(1)}|}{1 + y^{2s_L-2}} \in L^2$ from the Hardy inequality (D.10). Consequently in that case we also have $\frac{D\varepsilon_{s_L}^{(1)}}{1 + y^{2s_L-2}} \in L^2$. This proves:

$$\int |\varepsilon_{s_L}^{(1)}|^2 < +\infty.$$

Similarly one has $\int |\varepsilon_{s_L-1}^{(2)}|^2 < +\infty$, implying $\mathcal{E}_{s_L} < +\infty$. Step 2: Proof of the coercivity estimate. We want to apply the previous Lemma E.3 for $k = s_L$. We have seen in the previous step 1 that the integrability condition (E.21) is met. Now from the formula (2.23) giving the powers of $H^*$ we compute that the orthogonality condition (E.23) implies:

$$\langle \varepsilon^{(1)}, \mathcal{L}^i \Phi_M \rangle = \langle \varepsilon^{(2)}, \mathcal{L}'^i \Phi_M \rangle = 0 \text{ for } 0 \leq i \leq \frac{L - 1}{2}.$$

We compute: $E\left[\frac{k-k_0}{2}\right] = E\left[\frac{L+k_0+1-k_0}{2}\right] = \frac{L+1}{2}$. Therefore the Lemma E.3 applies and gives the bound (E.24). Now we use the equivalence between Laplace and adapted derivatives (C.2), with the bound we just proved for E.21 for $\delta = 0$ and it yields (E.25).
Proof of Lemma E.2, case k small: We suppose $1 \leq k \leq k_0$, and that $u$ is a function satisfying the conditions of the lemma. We have, depending on the parity of $k$:

$$u_k = Au_{k-1} \text{ or } u_k = A^*u_{k-1}.$$ 

In both cases, the conditions required to apply to $u_{k-1}$ Lemma 9.2 or Lemma 9.1 are fulfilled. Consequently:

$$\int \frac{u_k^2}{1 + y^{2k}} \geq \int \frac{u_{k-1}^2}{1 + y^{2k+2\delta}}.$$

If $k - 1 = 0$ we have finished. If not, then again, $u_{k-1} = Au_{k-2}$ or $u_{k-1} = A^*u_{k-2}$ and in both cases we can apply Lemma 9.2 or Lemma 9.1 which gives:

$$\int \frac{u_k^2}{1 + y^{2k}} \geq \int \frac{u_{k-1}^2}{1 + y^{2k+2\delta}} \geq \int \frac{u_{k-2}^2}{1 + y^{4k+2\delta}}.$$

We can iterate $k$ times what we did previously to obtain:

$$\int \frac{u_k^2}{1 + y^{2k}} \geq \int \frac{u_{k-1}^2}{1 + y^{2k+2\delta}} \geq \cdots \geq \int \frac{u_1^2}{1 + y^{2(k-2) + 2\delta}} \geq \int \frac{u_0^2}{1 + y^{2k+2\delta}},$$

which gives the result in that case.

Case $k$ large: Suppose first that $k \geq k_0 + 1$ and that $j = \frac{k-k_0}{2} \in \mathbb{N}^*$, so $k = k_0 + 2j$. We can apply the result for $k$ small we just showed to derive:

$$\int \frac{u_k^2}{1 + y^{2k}} \geq \int \frac{u_{k-k_0}^2}{1 + y^{2k_0+2\delta}} = \int \frac{u_{2j}^2}{1 + y^{2k_0+2\delta}}.$$

Since $2j$ is even we know that: $u_{2j} = A^*A...A^*Au = A^*u_{2j-1}$ and we can apply Lemma 9.1 to find:

$$\int \frac{u_{2j}^2}{1 + y^{2k_0+2\delta}} \geq \int \frac{u_{2j-1}^2}{1 + y^{2k_0+2\delta}} = \frac{Au_{2j-2}}{1 + y^{2k_0+2\delta}}.$$

We need an orthogonality condition for $u_{2j-2}$ in order to go on. This is given by the orthogonality condition on $u$. Indeed:

$$\langle u_{2j-2}, \Phi_M \rangle = \langle u, L_j^{-1}\Phi_M \rangle = 0.$$

Hence:

$$\int \frac{u_{2j-1}^2}{1 + y^{2(k_0+1) + \delta}} \geq \int \frac{u_{2j-2}^2}{1 + y^{2(k_0+2) + \delta}}.$$

We need exactly the $j$ orthogonality conditions to iterate like that till we reach 0.

Suppose now that $k = k_0 + 2j + 1$. Then it works the same, indeed without use of orthogonality conditions:

$$\int \frac{u_k^2}{1 + y^{2k}} \geq \int \frac{u_{k-1}^2}{1 + y^{2k+2\delta}} \geq \cdots \geq \int \frac{u_{k-k_0}^2}{1 + y^{2k_0+2\delta}} = \int \frac{|Au_{2j}|^2}{1 + y^{2k_0+2\delta}}.$$

We have exactly $j$ orthogonality conditions to go down to zero as we did before:

$$\int \frac{|Au_{2j}|^2}{1 + y^{2k_0+2\delta}} \geq \int \frac{u_{2j}^2}{1 + y^{2k_0+2\delta}} \geq \cdots \geq \int \frac{u^2}{1 + y^{2k+2\delta}}.$$

This ends the proof. \qed
Appendix F. Specific bounds for the analysis

We make use of the tools established in the last subsection to control $\varepsilon$. Again, the use of such estimates is standard in blow-up issues, and we refer to the papers quoted in Appendix D. Although their proofs are not very hard to write once one has the previous results, we put it here for the reader’s convenience. As the papers quoted in Appendix D. Although their proofs are not very hard to write once

$$
\text{We claim now that:}
$$

and we conclude by interpolation. We make use here of the tools established in the last subsection to control

$$
W e make use here of the tools established in the last subsection to control
$$

Proof of Lemma F.1.\text{Proof of (i):} \text{Let } j \in \mathbb{N} \text{ and } p > 0 \text{ satisfying } \sigma \leq j + p \leq s_L:\n\int_{y \geq 1} \frac{\left| \partial_y \varepsilon^{(1)} \right|^2}{1 + y^{2p}} \leq C(M) \mathcal{E}_\sigma s_L^{-j+\frac{d}{2}} \mathcal{E}_{s_L}^{\frac{j}{2} - \sigma}, \tag{F.1}
$$



(ii) $L^\infty$ control:

$$
\| \varepsilon^{(1)} \|_{L^\infty} \leq C(K_1, K_2, M) \sqrt{\mathcal{E}_\sigma b_1} \left( \frac{d}{2} - \sigma \right) + \frac{\mathcal{E}_\sigma}{y^\alpha} + O\left( \frac{\varepsilon}{y^a} \right), \tag{F.2}
$$

(iii) Weighted $L^\infty$ bound: for $0 < a < \frac{d}{2}$

$$
\left\| \varepsilon^{(1)} \right\|_{1 + x^a} \leq C(K_1, K_2, M) \sqrt{\mathcal{E}_\sigma b_1} \left( \frac{d}{2} - \sigma \right) + \frac{\mathcal{E}_\sigma}{y^\alpha} + O\left( \frac{\varepsilon}{y^a} \right). \tag{F.3}
$$

Proof of Lemma F.1. proof of (i): Let $j \in \mathbb{N}$ and $p$ satisfying $\sigma \leq j + p \leq s_L$. For a slow decaying potential, i.e., if $p$ satisfies in addition $p < \frac{d}{2}$ then the equivalence between Laplace derivatives and $\partial_y$ ones away from the origin, together with the weighted Hardy inequality (Lemma D.2) gives:

$$
\int \frac{\left| \partial_y \varepsilon^{(1)} \right|^2}{1 + y^{2p}} \leq C \int |\nabla^{j+p} \varepsilon^{(1)}|^2,
$$

and we conclude by interpolation. We claim now that:

$$
\sum_{i=0}^{s_L} \int_{y \geq 1} \frac{\left| \partial_y \varepsilon^{(1)} \right|^2}{1 + y^{2(s_L - i)}} \leq C(M) \mathcal{E}_{s_L}.
$$

Indeed, from the equivalence between $\partial_y$ and adapted derivatives (Lemma C.1), and from coercivity we have:

$$
\sum_{i=0}^{s_L} \int_{y \geq 1} \frac{\left| \partial_y \varepsilon^{(1)} \right|^2}{1 + y^{2(s_L - i)}} \sim \sum_{i=0}^{s_L} \int_{y \geq 1} \frac{\left| \varepsilon^{(1)} \right|^2}{1 + y^{2(s_L - i)}} \leq C(M) \mathcal{E}_{s_L}.
$$

This claim implies that for a fast decaying potential, i.e., $p = s_L - j$:

$$
\int \frac{\left| \partial_y \varepsilon^{(1)} \right|^2}{1 + y^{2p}} \leq \mathcal{E}_{s_L}.
$$

Now, for $\frac{d}{2} \leq p \leq s_l - j$ we interpolate the last two results, as for $a \leq b \leq c$:

$$
\frac{\left| \varepsilon^{(1)} \right|^2}{1 + y^{2b}} \sim \left( \frac{\left| \varepsilon \right|^2}{1 + y^{2a}} \right)^{\frac{b-a}{b}} \left( \frac{\left| \varepsilon \right|^2}{1 + y^{2b}} \right)^{\frac{a-b}{b}}
$$

and this gives (i).

proof of (ii). We prove it for $\varepsilon^{(1)}$, the proof for the second coordinate being similar. By the coercivity bound (F.25) we have that:

$$
\| \nabla^{s_L} \varepsilon^{(1)} \|^2 \leq C(M) \mathcal{E}_{s_L}.
$$
We have by interpolation that for all \( \sigma \leq k \leq s_L \), \( \nabla^k \varepsilon^{(1)} \in L^2 \) with the control
\[
\| \nabla^k \varepsilon^{(1)} \|_{L^2}^2 \leq C(M) \varepsilon_{\sigma, sL}^k \varepsilon_{sL-\sigma}^k.
\]
Denoting by \( \hat{\varepsilon}^{(1)} \) the Fourier transform of \( \varepsilon^{(1)} \) we have:
\[
| \varepsilon^{(1)}(y) | \leq \int_{|\xi| \leq 1} \frac{|\varepsilon^{(1)}(\xi)|^2}{|\xi|^s} + \int_{|\xi| \geq 1} \frac{|\varepsilon^{(1)}(\xi)|^2}{|\xi|^s} \lesssim \| \nabla^{k_1} \varepsilon^{(1)} \|_{L^2} + \| \nabla^{k_2} \varepsilon^{(1)} \|_{L^2}
\]
with \( \sigma < k_1 < \frac{d}{2} < k_2 < s_L \). Using the interpolation bound previously derived and taking \( k_1, k_2 \to \frac{d}{2} \) gives:
\[
| \varepsilon^{(1)}(y) | \leq C \varepsilon_{\sigma, sL}^{\frac{sL-d}{sL-\sigma}} \varepsilon_{sL-\sigma}^{\frac{sL-d}{sL-\sigma}} \leq C \varepsilon_{\sigma, b_1} \left( \frac{2}{sL} \right)^s \left( \frac{2}{sL} \right) O \left( \frac{2}{sL} \right) + O \left( \frac{2}{sL} \right)
\]
which gives the result.

Proof of (iii) Take \( a \geq 1, \alpha \leq a \ll s_L \). Then from (i):
\[
\| \nabla^{E \left[ \frac{d}{2} + 1 \right]} E^{(1)} \|_{L^2}^2 \sim \int \left| D^{E \left[ \frac{d}{2} + 1 \right]} \left( \frac{E^{(1)}(y)}{1+y^a} \right)^2 \right| \leq C(M) \varepsilon_{\sigma, sL-\sigma}^{E \left[ \frac{d}{2} + 1 \right] + a} \varepsilon_{sL-\sigma}^{E \left[ \frac{d}{2} + 1 \right] + a-\sigma}.
\]
And we estimate the same way \( \| \nabla^{E \left[ \frac{d}{2} + 1 \right]} E^{(1)} \|_{L^\infty}^2 \). We can interpolate this two estimations to have an estimate for \( \| E^{(1)}(y) \|_{L^\infty} \). By calculating the exponents the same way we did for the proof of (ii) we get the result of the lemma for \( a \). Now we can interpolate this result with (ii) to conclude for any exponent \( 0 \leq a \leq s_L \).

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