HYDRODYNAMICS OF SPINNING PARTICLES. †

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Abstract – In this note, we first obtain the decomposition of the non-relativistic field
velocity into the classical part (i.e., the velocity \( \vec{\omega} = \vec{p}/m \) OF the center-of-mass (CM),
and the so-called quantum part (i.e., the velocity \( \vec{V} \) of the motion IN the CM frame
(namely, the internal spin–motion or Zitterbewegung), these two parts being orthogonal.
Our starting point is the Pauli current. Then, by inserting such a composite expression
of the velocity into the kinetic energy term of the non-relativistic newtonian lagrangian,
we get the appearance of the so-called “quantum potential” (which makes the difference
between classical and quantum behaviour) as a pure consequence of the internal motion.
Such a result carries further evidence about the possibility that the quantum behaviour
of micro-systems be a direct consequence of the fundamental existence of spin.

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1. Madelung fluid: A variational approach

The lagrangian for a non-relativistic scalar particle may be assumed to be:

\[ L = \frac{i\hbar}{2}(\psi^* \partial_t \psi - (\partial_t \psi^*)\psi) - \frac{\hbar^2}{2m} \nabla \psi^* \cdot \nabla \psi - U \psi^* \psi \quad (1) \]

where \( U \) is the external potential energy and the other symbols have the usual meaning. It is known that, by taking the variations of \( L \) with respect to \( \psi, \psi^* \), one can get the Schrödinger equations for \( \psi^* \) and \( \psi \), respectively.

By contrast, since a generic scalar wavefunction \( \psi \in \mathbb{C} \) can be written as

\[ \psi = \sqrt{\rho} \exp[i\varphi/\hbar] \quad (2) \]

with \( \rho, \varphi \in \mathbb{R} \), we take the variations of

\[ L = - \left[ \partial_t \varphi + \frac{1}{2m} (\nabla \varphi)^2 + \frac{\hbar^2}{4m} \left( \frac{\nabla \rho}{\rho} \right)^2 + U \right] \rho \rho \quad (3) \]

with respect to (w.r.t.) \( \rho \) and \( \varphi \). We then obtain\([1-3]\) the two equations for the so-called Madelung fluid\([4]\) (which, taken together, are equivalent to the Schrödinger equation):

\[ \partial_t \varphi + \frac{1}{2m} (\nabla \varphi)^2 + \frac{\hbar^2}{8m} \left( \frac{\nabla \rho}{\rho} \right)^2 + U = 0 \quad (4) \]

and

\[ \partial_t \rho + \nabla \cdot \left( \rho \nabla \varphi/m \right) = 0 \quad (5) \]

which are the Hamilton–Jacobi and the continuity equation for the “quantum fluid”, respectively; where

\[ \frac{\hbar^2}{4m} \left[ \frac{1}{2} \left( \frac{\nabla \rho}{\rho} \right)^2 - \frac{\Delta \rho}{\rho} \right] \equiv - \frac{\hbar^2}{2m} \frac{\Delta |\psi|}{|\psi|} \quad (6) \]

is often called the “quantum potential”. Such a potential derives from the last-but-one term in the r.h.s. of eq.(3), that is to say, from the (unique) “non-classical term”

\[ \frac{\hbar^2}{8m} \left( \frac{\nabla \rho}{\rho} \right)^2 \quad (7) \]

entering our lagrangian \( L \).

Notice that we got the present hydrodynamical reformulation of the Schrödinger theory directly from a variational approach.\([3]\) This procedure, as we are going to see, offers us a physical interpretation of the non-classical terms appearing in eqs.(3) or (4). On the contrary, eqs.(4-5) are ordinarily obtained by inserting relation (2) into the Schrödinger
equation, and then separating the real and the imaginary part: a rather formal procedure, that does not shed light on the underlying physics.

Let us recall that an early physical interpretation of the so-called “quantum” potential, that is to say, of term (6) was forwarded by de Broglie’s pilot-wave theory[5]; in the fifties, Bohm[6] revisited and completed de Broglie’s approach in a systematic way [and, sometimes, Bohm’s theoretical formalism is referred to as the “Bohm formulation of quantum mechanics”, alternative and complementary to Heisenberg’s (matrices and Hilbert spaces), Schroedinger’s (wave-functions), and Feynman’s (path integrals) theory]. From Bohm’s up to our days, several conjectures about the origin of that mysterious potential have been put forth, by postulating “sub-quantal” forces, the presence of an ether, and so on. Well-known are also the derivations of the Madelung fluid within the stochastic mechanics framework:[7,2] in those theories, the origin of the non-classical term (6) appears as substantially kinematical. In the non-markovian approaches,[2] for instance, after having assumed the existence of the so-called zitterbewegung, a spinning particle appears as an extended-like object, while the “quantum” potential is tentatively related to an internal motion.

But we do not need following any stochastic approach, even if our philosophical starting point is the recognition of the existence[8-12] of a zitterbewegung (zbw) or diffusive or internal motion [i.e., of a motion observed in the center-of-mass (CM) frame, which is the one where \( p = 0 \) by definition], besides of the [external, or drift, or translational, or convective] motion of the CM. In fact, the existence of such an internal motion is denounced, besides by the mere presence of spin, by the remarkable fact that in the standard Dirac theory the particle impulse \( p \) is in general not parallel to the velocity: \( v \neq p/m \); moreover, while \([\hat{p}, \hat{H}] = 0\) so that \( p \) is a conserved quantity, quantity \( v \) is not a constant of the motion: \([\hat{v}, \hat{H}] \neq 0\) (\( v \equiv \alpha \equiv \gamma^0 \gamma \) being the usual vector matrix of Dirac theory).

For dealing with the zbw it is highly convenient[10,12] to split the motion variables as follows (the dot meaning derivation with respect to time):

\[
\dot{x} = \dot{\xi} + \dot{X} ; \quad \ddot{x} \equiv v = w + V ,
\]

where \( \xi \) and \( w \equiv \dot{\xi} \) describe the motion of the CM in the chosen reference frame, whilst \( X \) and \( V \equiv \dot{X} \) describe the internal motion referred to the CM frame (CMF). From a dynamical point of view, the conserved electric current is associated with the helical trajectories[8-10] of the electric charge (i.e., with \( x \) and \( v \equiv \dot{x} \)), whilst the center of the particle coulombian field is associated with the geometrical center of such trajectories (i.e., with \( \xi \) and \( w \equiv \dot{\xi} = p/m \)).

Going back to lagrangian (3), it is now possible to attempt an interpretation of the non-classical term \( \frac{\hbar^2}{8m} (\nabla \rho/\rho)^2 \) appearing therein. The first term in the r.h.s. of eq.(3) represents, apart from the sign, the total energy

\[
\partial_t \varphi = -E ;
\]
whereas the second term is recognized to be the kinetic energy $\frac{p^2}{2m}$ of the CM, if one assumes that

$$p = -\nabla \varphi.$$  \hfill (10)

The third term, that gives origin to the quantum potential, will be shown below to be interpretable as the kinetic energy in the CMF, that is, the internal energy due to the zbw motion. It will be soon realized, therefore, that in Lagrangian (3) the sum of the two kinetic energy terms, $\frac{p^2}{2m}$ and $\frac{1}{2m}V^2$, is nothing but a mere application of the König theorem. We are not going to exploit, as often done, the arrival point, i.e. the Schrödinger equation; by contrast, we are going to exploit a non-relativistic (NR) analogue of the Gordon decomposition[13] of the Dirac current: namely, a suitable decomposition of the Pauli current.[14] In so doing, we shall meet an interesting relation between zbw and spin.

2. The “quantum” potential as a consequence of spin and zbw

Let us start from the familiar expression of the Pauli current[14] (i.e., from the Gordon decomposition of the Dirac current in the NR limit):

$$j = \frac{i\hbar}{2m}[(\nabla \psi^\dagger)\psi - \psi^\dagger \nabla \psi] - \frac{eA}{m} \psi^\dagger \psi + \frac{1}{m} \nabla \wedge (\psi^\dagger \hat{s} \psi).$$ \hfill (11)

A spinning NR particle can be simply factorized into

$$\psi \equiv \sqrt{\rho} \Phi,$$ \hfill (12)

$\Phi$ being a Pauli 2-component spinor, which has to obey the normalization constraint

$$\Phi^\dagger \Phi = 1$$

if we want to have $|\psi|^2 = \rho$.

By definition $\rho \mathbf{s} \equiv \psi^\dagger \hat{s} \psi \equiv \rho \Phi^\dagger \hat{s} \Phi$; therefore, introducing the factorization $\psi \equiv \sqrt{\rho} \Phi$ into the above expression (14) for the Pauli current, one obtains:

$$j \equiv \rho \mathbf{v} = \rho \frac{p - eA}{m} + \frac{\nabla \wedge (\rho \mathbf{s})}{m \rho}$$ \hfill (13)

which is nothing but the decomposition of $\mathbf{v}$ got by Hestenes[15] by employing the Clifford algebras language:

$$\mathbf{v} = \frac{p - eA}{m} + \frac{\nabla \wedge (\rho \mathbf{s})}{m \rho}$$ \hfill (14)

where the light speed $c$ is assumed equal to 1, quantity $e$ is the electric charge, $\mathbf{A}$ is the external electromagnetic vector potential, $\mathbf{s}$ is the spin vector $\mathbf{s} \equiv \rho^{-1} \psi^\dagger \hat{s} \psi$, and $\hat{s}$ is the spin operator usually represented in terms of Pauli matrices as

$$\hat{s} \equiv \frac{\hbar}{2}(\sigma_x; \sigma_y; \sigma_z).$$ \hfill (15)
[Hereafter, every quantity is a local or field quantity: $\mathbf{v} \equiv \mathbf{v}(\mathbf{x};t)$; $\mathbf{p} \equiv \mathbf{p}(\mathbf{x};t)$; $\mathbf{s} \equiv \mathbf{s}(\mathbf{x};t)$; and so on]. As a consequence, the internal (zbw) velocity reads:

$$\mathbf{V} \equiv \frac{\nabla \wedge (\rho \mathbf{s})}{m \rho}.$$  \hfill (16)

The Schroedinger sub-case [i.e., the case in which the vector spin field $\mathbf{s} = \mathbf{s}(\mathbf{x};t)$ is constant in time and uniform in space] corresponds to spin eigenstates; so that one needs now a wave-function factorizable into the product of a “non-spin” part $\sqrt{\rho} e^{i \varphi}$ (scalar) and of a “spin” part $\chi$:

$$\psi \equiv \sqrt{\rho} e^{i \varphi} \chi,$$

$\chi$ being constant in time and space. Therefore, when $\mathbf{s}$ has no precession (and no external field is present: $\mathbf{A} = 0$), we have $\mathbf{s} \equiv \chi^\dagger \hat{\mathbf{s}} \chi = \text{constant}$, and

$$\mathbf{V} = \frac{\nabla \rho \wedge \mathbf{s}}{m \rho} \neq 0 .$$ \hfill (Schroedinger case) (18)

One can notice that, even in the Schroedinger theoretical framework, the zbw does not vanish, except for plane waves, i.e., for the non-physical case of $\mathbf{p}$-eigenfunctions, when not only $\mathbf{s}$, but also $\rho$ is constant and uniform, so that $\nabla \rho = 0$.

But let us go on. We may now write

$$\mathbf{V}^2 = \left( \frac{\nabla \rho \wedge \mathbf{s}}{m \rho} \right)^2 = \frac{(\nabla \rho)^2 \mathbf{s}^2 - (\nabla \rho \cdot \mathbf{s})^2}{(m \rho)^2}$$ \hfill (19)

since in general it holds

$$(a \wedge b)^2 = a^2 b^2 - (a \cdot b)^2 .$$ \hfill (20)

Let us now observe that, from the smallness of the negative-energy component (the so-called “small component”) of the Dirac bispinor it follows the smallness also of: $\nabla \cdot (\rho \mathbf{s}) \simeq 0$. This was already known from the Clifford algebra approach to Dirac theory, that yielded[15] ($\beta$ being the Takabayasi angle[16]): $\nabla \cdot (\rho \mathbf{s}) = -m \rho \sin \beta$, which in the NR limit corresponds to $\beta = 0$ (“pure electron”) or $\beta = \pi$ (“pure positron”), so that one gets $\nabla \cdot (\rho \mathbf{s}) = 0$ and in the Schroedinger case [$\mathbf{s} = \text{constant}$; $\nabla \cdot \mathbf{s} = 0$]:

$$\nabla \rho \cdot \mathbf{s} = 0.$$ \hfill (21)

By putting such a condition into eq.(19), it assumes the important form

$$\mathbf{V}^2 = \mathbf{s}^2 \left( \frac{\nabla \rho}{m \rho} \right)^2 ,$$ \hfill (22)

which does finally allow us to attribute to the so-called “non-classical” term, eq.(7), of our lagrangian (3) the simple meaning of kinetic energy of the internal (zbw) motion [i.e., of kinetic energy associated with the internal (zbw) velocity $\mathbf{V}$], provided that

$$\hbar = 2 \mathbf{s} .$$ \hfill (23)
In agreement with the already mentioned König theorem, such an internal kinetic energy does appear, in lagrangian (3), as correctly added to the (external) kinetic energy $p^2/2m$ of the CM besides to the total energy (9) and the external potential energy $U$.

Vice-versa, if we assume (within a zbw philosophy) that $V$, eq.(22), is the velocity attached to the kinetic energy term (7), then we can deduce eq.(23), i.e., we deduce that actually:

$$|s| = \frac{1}{2} \hbar .$$

Let us mention, by the way, that in the stochastic approaches the ("non-classical") stochastic, diffusion velocity is $V \equiv v_{\text{dif}} = \nu (\nabla \rho / \rho)$, quantity $\nu$ being the diffusion coefficient of the "quantum" medium. In those approaches, however, one has to postulate that $\nu \equiv \hbar/2m$. In our approach, on the contrary, if we just adopted for a moment the stochastic language, by comparison of our eqs.(7), (22) and (23) we would immediately deduce that $\nu = \hbar/2m$ and therefore the interesting relation

$$\nu = \frac{|s|}{m} .$$  \hspace{1cm} (24)

Let us explicitly remark that, because of eq.(22), in the Madelung fluid equation (and therefore in the Schrödinger equation) quantity $\hbar$ is naturally replaced by $2|s|$, the presence itself of the former quantity being no longer needed; we might say that it is more appropriate to write $\hbar = 2|s|$, rather than $|s| = \hbar/2 \ldots$!

We conclude by stressing the following. We first achieved a non-relativistic, Gordon-like decomposition of the field velocity within the ordinary tensorial language. Secondly, we derived the "quantum" potential (without the postulates and assumptions of stochastic quantum mechanics) by simply relating the "non-classical" energy term to zbw and spin. Such results carry further evidence that the quantum behaviour of micro-systems may be a direct consequence of the existence of spin. In fact, when $s = 0$, the quantum potential does vanish in the Hamilton–Jacobi equation, which then becomes a totally classical and newtonian equation. We have also seen that quantity $\hbar$ itself enters the Schrödinger equation owing to the presence of spin. We are easily induced to conjecture that no scalar quantum particles exist that are really elementary; but that scalar particles are always constituted by spinning objects endowed with zbw.

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