Abstract—The information capacity of a distributed storage system is the amount of source data that can be reliably stored for long durations. Storage nodes fail over time and are replaced, and thus data is erased at an erasure rate. To maintain recoverability of source data, a repairer generates redundant data from data read from nodes, and writes redundant data to nodes, where the repair rate is the rate at which the repairer reads and writes data. We prove the information capacity approaches

\[
\left(1 - \frac{1}{2\sigma}\right) \cdot N \cdot s
\]

as \(N\) and \(\sigma\) grow, where \(N\) is the number of nodes, \(s\) is the amount of data each node can store, and \(\sigma\) is the repair rate to erasure rate ratio.

Index Terms—distributed information systems, data storage systems, data warehouses, information science, information theory, information entropy, error compensation, mutual information, channel capacity, channel coding, time-varying channels, error correction codes, Reed-Solomon codes, network coding, signal to noise ratio, throughput, distributed algorithms, algorithm design and analysis, reliability, reliability engineering, reliability theory, fault tolerance, redundancy, robustness, failure analysis, equipment failure.

I. OVERVIEW OF PRACTICAL SYSTEMS

We start by providing a brief overview of practical distributed storage systems and some of their properties. These examples motivate the mathematical model of distributed storage subsequently introduced and analyzed.

A distributed storage system generically consists of interconnected storage nodes, where each node can store a large quantity of data. A primary goal of a distributed storage system is to reliably store as much source data as possible for a long time.

Commonly, distributed storage systems are built using relatively inexpensive and generally not completely reliable hardware. For example, nodes can go offline for periods of time (transient failure), in which case the data they store is temporarily unavailable, or permanently fail, in which case the data they store is permanently erased. Permanent node failures are not uncommon, and transient node failures are frequent.

Although it is often hard to accurately model node failures, a random node failure model can provide insight into the strengths and weaknesses of a practical system, and can provide a first order approximation to how a practical system operates.

Distributed storage systems generally allocate a fraction of their raw capacity to storage overhead: they use erasure codes to generate redundant data from the source data, and take advantage of the storage overhead to store the redundant data in addition to source data to ensure the source data is recoverable even when nodes (or other components) fail.

Source data is maintained at the granularity of objects. For a \((n, k, r)\) erasure code, each object is segmented into \(k\) source fragments, an encoder generates \(r = n - k\) repair fragments from the \(k\) source fragments, and each of these \(n = k + r\) fragments is stored at a different node. An erasure code is MDS if the object can be recovered from any \(k\) of the \(n\) fragments.

Replication is an example of a trivial MDS erasure code, i.e., each fragment is a copy of the original object. For example, triplication can be thought of as using the simple \((3, 1, 2)\) erasure code, wherein the object can be recovered from any one of the three copies. Many distributed storage systems use replication.

Reed-Solomon codes \[2, 3, 5\] are MDS codes that are used in a variety of applications and are a popular choice for storage systems. For example, \[22\] and \[19\] use a \((9, 6, 3)\) Reed-Solomon code, and \[24\] uses a \((14, 10, 4)\) Reed-Solomon code. These are examples of small code systems, i.e., systems that use small values of \(n, k\) and \(r\).

A repairer maintains recoverability of source data as node failures occur, by reading, regenerating and writing fragments lost due to node failures. Since a small number \(r + 1\) of node failures can cause object data loss for small code systems, reactive repair is used, i.e., the repairer operates as quickly as practical to regenerate fragments lost from a node that permanently fails before another node fails, and typically reads \(k\) fragments to regenerate each lost fragment. Thus, the peak read repair rate is higher than the average read repair rate, and the average read repair rate is \(k\) times the node failure erasure rate.

The read repair rate needed to maintain source data recoverability for small code systems can be substantial. Modifications of standard erasure codes have been designed for storage systems to reduce this rate, e.g., local reconstruction codes \[20, 24\], and regenerating codes \[14, 17\]. Some versions of local reconstruction codes have been used in deployments, e.g., by Microsoft Azure.

Placement groups, each mapping \(n\) fragments to \(n\) of the \(N\) nodes, determine where fragments for objects are stored. An equal amount of object data should be assigned to each placement group, and an equal number of placement groups should map a fragment to each node. For small code systems, Ceph \[29\] recommends \(100 : N\) placement groups, i.e., 100 placement groups map a fragment to each node. A placement group should avoid mapping fragments to nodes with correlated failures, e.g., to the same rack. Pairs of placement groups should avoid mapping fragments to the same pair of nodes. Placement groups are continually remapped as nodes fail and are added. These and other issues make it challenging.

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to design placement groups for small code systems.

The paper [30] introduces liquid systems. Liquid systems use erasure codes with large values of \( n, k \) and \( r \). For example, \( n = N \) and a fragment is assigned to each node for each object, i.e., only one placement group is used for all objects. The RaptorQ code [4], [6] is an example of an erasure code that is suitable for a liquid system, since objects with large numbers of fragments can be encoded and decoded efficiently in linear time.

For a liquid system, \( r \) is typically large, and thus object data is not reliably recoverable only when a large number of nodes fail. The repairer is lazy, i.e., repair operates in the background to slowly over time regenerate fragments erased from nodes that have permanently failed. Generally the repairer reads \( k \) fragments for each object to regenerate around \( r \) fragments erased over time due to permanent node failures, and the peak read repair rate is close to the average read repair rate.

There are any number of possible strategies, such as the strategies outlined above, and others which have not yet been invented, that could be used to implement a distributed storage system. Our primary goal is to capture the properties and metrics essential to all strategies, and provide fundamental insights into what is possible, and impossible, to achieve.

II. SHANNON THEORY OF COMMUNICATION

Our distributed storage model and results are inspired by Shannon’s communication model [11] and results. Figure 1 shows an architectural overview of Shannon’s model. A transmitter produces a signal generated from a source data \( \mathcal{X} \) received from a source, and sends the signal over a channel. Noise perturbs the signal within the channel. A receiver receives signal \( \mathcal{Y} \), generates \( \mathcal{Z} \) from \( \mathcal{Y} \), and provides \( \mathcal{Z} \) to a destination, where the source data is reliably recovered if \( \mathcal{Z} = \mathcal{X} \).

![Fig. 1: Shannon communication architecture](image)

The theory proves every channel has an information capacity, which is the maximum rate at which source data can be transmitted over the channel and still be reliably recovered at the receiver.

The binary erasure channel (BEC) is the closest analog to our work, which can be characterized as follows. The signal rate \( S \) is the raw rate at which data can be transmitted over the channel, independent of reliability. The erasure rate \( \mathcal{E} \) is the rate at which data transmitted over the channel is erased. The signal to erasure rate ratio (SER) \( \sigma \) is defined as

\[
\sigma = \frac{S}{\mathcal{E}}.
\]

The BEC information capacity is

\[
S - \mathcal{E} = \left(1 - \frac{1}{\sigma}\right) \cdot S.
\]

This information capacity is asymptotically achievable as the source data size grows.

III. DISTRIBUTED STORAGE MODEL

We introduce a model of distributed storage which is inspired by properties inherent and common to systems described in Section I.

Figure 2 shows an architectural overview of the distributed storage model. A storer generates data from source data \( \mathcal{X} \) received from a source, and stores the generated data at nodes. The source data size \( m \) is the number of bits in \( \mathcal{X} \). Alternatively, \( m = H (\mathcal{X}) \), where \( H (\mathcal{X}) \) is the entropy of \( \mathcal{X} \).

![Fig. 2: Distributed storage architecture](image)

The storage overhead \( \beta \) is the fraction of raw capacity available beyond the source data size, i.e.,

\[
\beta = 1 - \frac{m}{c},
\]

and thus source data size \( m = (1 - \beta) \cdot c \).

![Fig. 3: Storage nodes and repairer model.](image)

As time passes, nodes fail and data stored at the failed nodes is erased. A failure process determines when and what nodes fail as time passes. All bits at a node are immediately erased when the node fails, and a node with all bits initialized to zeroes is immediately added to replace the failed node. Thus, at each point in time there are \( N \) nodes. The erasure rate is

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\[\mathcal{E} = \frac{s}{\Delta},\]
where $\Delta$ is the average time between node failures, since $s$ bits are erased each time a node fails.

As nodes fail and are replaced, a repairer continually reads data from the nodes, computes a function of the read data, and writes the computed data back to the nodes. The repairer tries to ensure that the source data can be recovered at any point in time from the data stored at the nodes. The read repair rate $R$ is the rate at which the repairer reads data, the write repair rate $W$ is the rate at which the repairer writes data, and the repair rate $S = R + W$.

The read repair rate to erasure rate ratio (rRER) is
\[ \sigma_r = \frac{R}{E}, \]
the write repair rate to erasure rate ratio (wRER) is
\[ \sigma_w = \frac{W}{E}, \]
and the repair rate to erasure rate ratio (RER) is
\[ \sigma = \sigma_r + \sigma_w = \frac{S}{E}. \]

After some amount of time $T$ passes, an accessor reads data $Y$ from the nodes, generates $Z$ to a destination, where $X$ is reliably recovered if $Z = X$. MTTDL (mean time to data loss) is the mean amount of time $T$ the accessor can reliably recover $X$.

The DS information capacity is the largest value of $m$ for which a large MTTDL is possible, for a given RER $\sigma$ and raw capacity $c$. We show the DS information capacity approaches
\[ \left(1 - \frac{1}{2\sigma}\right) \cdot c, \]
as $\sigma$ and the number of nodes $N$ grow. This is a consequence of the main results described in Section VI and the results in Section XI.

A. Connecting DS and BEC information capacity

We highlight connections between BEC information capacity (Equation 2) and DS information capacity (Equation 9).

The BEC signal rate $S$ plays two different roles in defining the BEC information capacity:

1) **Signal overcomes noise**: $S$ combats the BEC erasure rate $E$ to enable reliable delivery of source data (Equation (1)).

2) **Noise-free capacity**: $S$ is the BEC information capacity when there are no erasures (Equation (2)).

For the DS model, these roles are split between the DS repair rate and the DS raw capacity. The DS repair rate plays role (1), i.e., the DS repair rate combats the DS erasure rate to enable reliable storage of source data (Equation (8)). The DS raw capacity plays role (2), i.e., the DS raw capacity is the DS information capacity when there are no erasures (Equation (9)).

Given these connections, it is notable how similar in form the BEC information capacity Equation (2) is to the DS information capacity Equation (9).
loss of generality, \( Q \) is chosen and fixed before running the repairer.

The repairer has \( v \) bits of local memory that it can use to temporarily store and perform computations on data. Generally, \( v \ll c \), e.g., \( N = 10^4 \), \( v = 10^6 \), \( s = 10^{16} \) and \( c = N \cdot s = 10^{20} \), and thus \( \frac{v}{c} = 10^{-14} \).

Repairers are computationally unlimited for repair rate lower bounds. The node failure sequence up to index \( i \) is provided to the repairer operating at time \( t \), where \( i \) is the largest index such that \( t \geq t(i) \), and this determines the repairer actions up till \( t \).

We prove that if the read repair rate is too small then the source data is not reliably recoverable. The lower bound is on an average read repair rate \( R_{avg} \), which is also a lower bound on the peak read repair rate \( R_{peak} \), since \( R_{avg} \geq R_{peak} \).

Repair rate upper bounds describe efficient repairers that maintain source data recoverability for long durations. At all times the read repair rate is at most a peak rate \( R_{peak} \), which is also an upper bound on the average read repair rate \( R_{avg} \), since \( R_{avg} \leq R_{peak} \).

VI. STATEMENT OF MAIN RESULTS

Theorem 6.1 provides a lower bound on average read repair rate \( R_{avg} \) for fixed erasure rate \( E \) and storage overhead \( \beta \).

The upper bound on DS information capacity implied by Equation (40), and Table I in Section VII-E provides values on peak read repair rate \( R \) since \( R_{peak} \geq R_{avg} \).

Theorem 6.2 provides an essentially matching upper bound on peak read repair rate \( R_{peak} \) for fixed erasure rate \( E \) and storage overhead \( \beta \). The lower bound on DS information capacity implied by Equation (40) follows from Theorem 6.2 and Theorem 11.3.

The function \( b_{max}(2\beta) \) mentioned in Theorem 6.1 is defined by Equation (40), and Table I in Section VII-E provides values of \( b_{max}(2\beta) \) as a function of example values of \( \beta \).

**Theorem 6.1:** For every fixed \( \beta \leq \frac{1}{2} \), asymptotically as \( N \) grows, for source data size \( m = (1 - \beta) \cdot c \), for any repairer with average read repair rate \( R_{avg} \) with respect to any random node failure distribution with erasure rate \( E \), if the repairer is able to achieve a MTTL polynomial in \( N \) then

\[
\frac{R_{avg}}{E} \geq \frac{b_{max}(2\beta)}{2\beta}.
\]  

**Proof:** The proof of Theorem 6.1 is provided in Section VIII.

For example, from Table I, for \( \beta = 0.1 \), Inequality (10) is

\[
\frac{R_{avg}}{E} \geq \frac{0.834}{2\beta}.
\]

The advanced liquid repairer mentioned in Theorem 6.2 is described in Section IX-D and is based on the paper [30] that introduces liquid systems with features such as the repairer peak read rate is equal to the average read rate, and traffic generated by the repairer is uniformly distributed across the network.

**Theorem 6.2:** For every fixed \( \beta \), asymptotically as \( N \) grows, for source data size \( m = (1 - \beta) \cdot c \), the advanced liquid repairer with peak read repair rate \( R_{peak} \) with respect to any random node failure distribution with erasure rate \( E \) can achieve a MTTL exponential in \( N \) and satisfy

\[
\frac{R_{peak}}{E} \leq \frac{1 + 3\beta}{2\beta} - \frac{(1 + 3\beta \cdot (1 - \beta))}{2\beta}.
\]

**Proof:** The proof of Theorem 6.2 is provided in Section IX-D.

For example, for \( \beta = 0.1 \), Inequality (11) is

\[
\frac{R_{peak}}{E} \leq \frac{1.17}{2\beta}.
\]

The bounds on \( rRER \) \( \frac{R_{avg}}{E} \) in Theorem 6.1 and on \( rRER \) \( \frac{R_{peak}}{E} \) in Theorem 6.2 converge to \( \frac{1}{2\beta} \) as \( \beta \to 0 \).

A. Visualization of main results

Figures 4 and 5 provide visualizations of the main results. The horizontal axis in Figure 4 is the storage overhead, and the vertical axis is the corresponding bound on the rRER: “rRER lower bound” shows the lower bound on \( \frac{R_{avg}}{E} \) from Inequality (10), “rRER upper bound” shows the upper bound on \( \frac{R_{peak}}{E} \) from Inequality (11), and “1/2β curve” shows the \( \frac{1}{2\beta} \) curve for comparison, which is between the lower and upper bounds.

The horizontal axis in Figure 5 is the storage overhead, and the vertical axis is the rRER lower and upper bounds from Figure 4 normalized by multiplying by \( 2\beta \), which shows how these bounds converge to the “1/2β curve” as \( \beta \) approaches zero.

![Fig. 4: Visualization of rRER lower and upper bounds.](image-url)
VII. CORE READ REPAIR RATE LOWER BOUND

This section introduces the key ideas used to prove Theorem 6.1. The core read repair rate lower bounds use the function

$$b(\beta, \epsilon, \psi) = \frac{\epsilon \cdot b_m(\psi \cdot \beta)}{\beta},$$

(12)

where $\beta$, $\epsilon$, and $\psi$ satisfy the conditions in Subsection VII-A and $b_m(\zeta)$ is defined in Equation (35). Fixing $\epsilon = \frac{1}{2}$ and $\psi \leq \frac{1}{2}$, and Equation (12) simplifies to

$$b(\beta) = \frac{b_m(2\beta)}{2\beta}.$$  

(13)

Table II in Section VII-E provides example values of $b_m(2\beta)$.

**Theorem 7.1:** For every fixed $\beta \leq \frac{1}{2}$, for source data size $m = (1 - \beta) \cdot c$, for every repairer there is a node failure sequence with a fixed timing sequence and with erasure rate $E$ such that if the repairer has average read repair rate $R_{\text{avg}}$ and is always able to maintain recoverability of the source data then

$$\frac{R_{\text{avg}}}{E} \geq b(\beta).$$

(14)

**Proof:** The proof of Theorem 7.1 follows immediately from Lemma 7.2 in Section VII-C.

The following sections introduce the concepts and notation needed to prove Lemma 7.2 and thus Theorem 7.1.

A. Concepts and notation

We introduce the concepts and notation needed to prove lower bounds on read repair rate. Let

$$(t(0), \text{id}(0)), (t(1), \text{id}(1)), \ldots, (t(i), \text{id}(i)), \ldots$$

be a node failure sequence.

Let $J = \{0, 1, \ldots, c - 1\}$ be the $c$ node bit-locations, $s$ bit-locations for each of the $N$ nodes, and let $V = \{0, 1, \ldots, v - 1\}$ be the $v$ repairer local memory bit-locations. Let $i' \geq i \geq 0$.

Let $E(i, i')$ be the node bit-locations whose bits are erased due to the node failures with indices $(i, \ldots, i')$. Thus,

$$|E(i, i')| = |\cup_{j=i}^{i'} \text{id}(j)| \cdot s.$$

Let $C(i, i')$ be the node bit-locations whose bits are read by the repairer between the node failures with indices $(i, \ldots, i')$. Generally, the repairer can read at any point in time. We assume without loss in generality the repairer does not read at the instant of time of a node failure, and thus $C(i, i) = \emptyset$.

Let $R(i, i')$ be the node bit-locations whose bits are read by the repairer between the node failures with indices $(i, \ldots, i')$, and the read bits are the same as just before the node failure with index $i$. More formally, $R(i, i) = \emptyset$, and for $i' > i$,

$$R(i, i') = \cup_{j=i+1}^{i'} C(i, j) - E(i, j - 1).$$

Let $L(i, i')$ be the node bit-locations whose bits are erased before they are read by the repairer due to node failures with indices $(i, \ldots, i')$, i.e., these bit-locations are assigned to nodes with identifiers in $\cup_{j=i}^{i'} \text{id}(j)$, and these bit-locations have not been read by the repairer between $t(i)$ and the time the node to which they are assigned fails. More formally,

$$L(i, i') = E(i, i') - R(i, i').$$

Let $Y^+(i')$ be the bit at bit-location $j \in J \cup V$ at the instant just after the node failure with index $i'$, and let

$$Y^+(i') = (Y^+_j(i') : j \in J \cup V).$$

Let $G(i, i') = J - E(i, i')$ be the bit-locations that have not been erased due to the node failures with indices $(i, \ldots, i')$, and let $Y^+_j(i)$ be the bit at bit-location $j \in J \cup V$ at the instant just before the node failure with index $i$, and let

$$Y^-(i) = (Y^-_j(i) : j \in R(i, i') \cup G(i, i') \cup V).$$

Let $B$ be the random bits used to choose the node failure sequence for indices $(i, \ldots, i')$. Then, $Y^+(i')$ is a deterministic function of $(Y^-(i), B)$, and thus,

$$H(Y^+(i'), B) \leq H(Y^-(i), B)$$

$$\leq |R(i, i') \cup G(i, i') \cup V| + |B|$$

$$= c - |L(i, i')| + v + |B|,$$

where the last equality is because

$$R(i, i') \cup G(i, i') = J - L(i, i').$$

Source data $X$ is not reliably recoverable after the node failure with index $i'$ if $H(X|Y^+(i'), B) > 0$. Since

$$H(X|Y^+(i'), B) \geq H(X, B) - H(Y^+(i'), B)$$

and

$$H(X, B) = m + H(B),$$

this implies that source data $X$ is not reliably recoverable just after the node failure with index $i'$ if

$$H(Y^+(i'), B) < m + H(B).$$

(16)

From Equation (15), $X$ is not reliably recoverable after the node failure with index $i'$ if

$$c - |L(i, i')| + v < m.$$  

(17)

An equivalent formulation of Inequality (17) is that some source data is not reliably recoverable after the node failure

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with index \( i' \), independent of repairer computation, if
\[
|\mathcal{L}(i, i')| > \beta \cdot c + v. \tag{18}
\]
Since \( v << \beta \cdot c \), we assume \( v = 0 \) in the lower bounds.

### B. Intuition for lower bound
Let
\[
(0, \text{id}(0)), (\Delta, \text{id}(1)), \ldots, (\cdot \Delta, \text{id}(i)), \ldots
\]
be a node failure sequence with a fixed timing sequence, and consider the situation starting at the node failure with index 0, and let \( M = \beta \cdot N - 1 \). Assume for now:

1. Identifiers \( \text{id}(0), \ldots, \text{id}(M) \) are all distinct.
2. For each \( j \in \{0, \ldots, M\} \), the repairer has not read any bits from \( \text{id}(j) \) between the node failures with indices \( (0, \ldots, j) \).

Then \( \mathcal{L}(0, M) = \mathcal{E}(0, M) \) and \( |\mathcal{E}(0, M)| = \beta \cdot c \). Inequality (18) implies \( |\mathcal{R}(0, M)| \geq m = (1 - \beta) \cdot c \) or the source data is not reliably recoverable just after the node failure with index \( M \).

Under these assumptions, the rRER is
\[
\frac{|\mathcal{R}(0, M)|}{|\mathcal{E}(0, M)|} \geq 1 - \frac{\beta}{\beta}. \tag{19}
\]

Assumption (1) is close to true for Theorem 6.1. Assumption (2) is roughly “half-true”: When a random node fails at time \( j \cdot \Delta \), the average fraction of unread data on that node since time 0 is equal to the fraction of unread data from all nodes that have not failed since time 0.

The proofs are based on a bound \( b(\beta) \) (see Equation (13)) with \( M' = 2\beta \cdot N - 1 \), i.e., \( M' \approx 2M \). If for some \( j \in \{1, \ldots, M'\} \) the read repair rate is averaging above \( b(\beta) \cdot s \) per node failure at time \( j \cdot \Delta \) then the average over this interval is above the desired read repair rate bound \( b(\beta) \cdot s \).

If for all \( j \in \{1, \ldots, M'\} \) the read repair rate is averaging below \( b(\beta) \cdot s \) per node failure at time \( j \cdot \Delta \) then, for any fraction \( f \), the repairer has read at most
\[
f \cdot M' \cdot b(\beta) \cdot s \approx f \cdot c
\]
bits up till time \( f \cdot M' \cdot \Delta \). From the “half-true” observation and these linear in \( f \) relationships, at least half the erased bits on average on the \( M' + 1 \) nodes that fail are lost, i.e., the number of bits lost is at least
\[
\frac{(M' + 1) \cdot s}{2} = \frac{2\beta \cdot N \cdot s}{2} = \beta \cdot c.
\]
The source data is not reliably recoverable from Inequality (18). The factor of two difference in the denominator of Equation (19) compared to Equation (10) is explained by \( M' \approx 2 \cdot M \).

### C. Core lower bound lemma

**Lemma 7.2:** For every \( \beta \), for \( \epsilon \) and \( \psi \) that satisfy the conditions in Subsection VII-D for source data size \( m = (1 - \beta) \cdot c \), for every repairer, there is a node failure sequence
\[
(0, \text{id}(0)), (\Delta, \text{id}(1)), \ldots, (\cdot \Delta, \text{id}(i)), \ldots
\]
and a subsequence of the indices
\[
i_1 = 0 < i_2 < \cdots < i_\ell - 1 < i_\ell < \cdots,
\]
such that, for all \( \ell \geq 1 \):

- \( i_{\ell+1} - i_\ell \leq \psi \cdot \beta \cdot N - 1 \).
- If the source data is recoverable just before the node failure with index \( i_\ell \) then either
\[
\frac{|\mathcal{R}(i_\ell, i_{\ell+1})|}{|\mathcal{E}(i_\ell, i_{\ell+1} - 1)|} \geq b(\beta, \epsilon, \psi), \tag{20}
\]
or the source data is not reliably recoverable just after the node failure with index \( i_{\ell+1} \).

**Proof:** Based on the repairer, a node failure sequence is generated in a series of phases. In general, for all \( i \geq 0 \), how many and which bits the repairer reads between the node failures with indices \( i \) and \( i + 1 \) is repairer specific and can depend on the node failures with indices \( (0, \ldots, i) \), and as described below this is used to determine how to set the identifiers in a phase. In phase 0, \( \text{id}'(0) \) is set to 0 and \( \text{id}(0) \) is set to any identifier, e.g., \( \text{id}(0) = 0 \).

Phase \( \ell \) operates as follows for \( \ell \geq 1 \). When phase \( \ell \) begins, \( i_\ell \) and \( \text{id}(i_\ell) \) are set to \( \text{id}'(\ell - 1) \) and \( \text{id}(\ell - 1) \), respectively, as determined in phase \( \ell - 1 \). Phase \( \ell \) determines \( \text{id}'(\ell) \) and identifiers \( \text{id}(i_\ell + 1), \ldots, \text{id}(i_\ell') \).

Let
\[
R = \beta \cdot N, \tag{21}
\]
\[
M = \psi \cdot R - 1. \tag{22}
\]
For \( j \geq 1 \), what the repairer reads up to time \( (i_\ell + j) \cdot \Delta \) is determined by \( \text{id}(0), \ldots, \text{id}(i_\ell + j - 1) \), where \( \text{id}(0), \ldots, \text{id}(i_\ell) \) have been fixed in previous phases and are thus implicitly fixed in the remainder of the proof.

Let \( (n_1, \ldots, n_M) \) be \( M \) possible identifiers that are distinct from one another and from \( \text{id}(i_\ell) \). There are
\[
(N - 1) \cdot (N - 2) \cdots (N - M)
\]
sets of such \( M \) possible identifiers. For each \( j \in \{1, \ldots, M\} \) and for each set of \( M \) possible identifiers \( n_1, \ldots, n_j \), consider the inequality
\[
|\mathcal{R}(i_\ell, i_\ell + j)| \geq j \cdot b(\beta, \epsilon, \psi) \cdot s \tag{23}
\]
when \( (\text{id}(i_\ell + 1), \ldots, \text{id}(i_\ell + j)) \) is set to \( (n_1, \ldots, n_j) \).

If there is a \( j \) and \( n_1, \ldots, n_j \) that satisfies Inequality (23) then phase \( \ell \) sets \( i_\ell' = i_\ell + j \) and sets \( \text{id}(i_\ell + 1), \ldots, \text{id}(i_\ell') \) to \( (n_1, \ldots, n_j) \). With these settings, Inequality (20) is satisfied.

If there is no \( j \) and \( n_1, \ldots, n_j \) that satisfies Inequality (23) then for all \( j \) and for sets of \( j \) possible identifiers \( (n_1, \ldots, n_j) \),
\[
|\mathcal{R}(i_\ell, i_\ell + j)| < j \cdot b(\beta, \epsilon, \psi) \cdot s \tag{24}
\]
when \( (\text{id}(i_\ell + 1), \ldots, \text{id}(i_\ell + j)) \) is set to \( (n_1, \ldots, n_j) \).

We prove that if Inequality (24) is satisfied for all \( j \) and all sets of \( M \) possible identifiers \( (n_1, \ldots, n_M) \) then there is at least one set of identifiers \( (n_1, \ldots, n_M) \) for which the source data is not reliably recoverable just after the node failure with index \( i_\ell + M \) when \( (\text{id}(i_\ell + 1), \ldots, \text{id}(i_\ell + M)) \) is set to \( (n_1, \ldots, n_M) \). Phase \( \ell \) sets \( i_\ell' = i_\ell + M \) and sets \( \text{id}(i_\ell + 1), \ldots, \text{id}(i_\ell') \) to \( (n_1, \ldots, n_M) \).
To simplify notation, the proof below is for phase $\ell = 1$, where $n_0 = \text{id}(0)$ is an arbitrary identifier for index $i_1 = 0$. Let $(n_1, \ldots, n_M)$ be a set of $M$ possible identifiers for the subsequent $M$ node failures, where all $\psi \cdot R$ identifiers are distinct.

For each $j = \{1, \ldots, M\}$, define
\[
\delta R_j = R(0, j) - R(0, j - 1)
\]
for each fixed $(n_0, n_1, \ldots, n_{j-1})$. We have the disjoint union
\[
R(0, j) = \bigcup_{j'=1}^j \delta R_{j'}
\]
for each fixed $(n_0, n_1, \ldots, n_{j-1})$.

For $1 \leq j \leq M$,
\[
E(0, M) \cap \delta R_j = E(j, M) \cap \delta R_j
\]
for each fixed $(n_0, n_1, \ldots, n_M)$. Thus,
\[
L(0, M) = E(0, M) - \bigcup_{j=1}^M E(j, M) \cap \delta R_j
\]
for each fixed $(n_0, n_1, \ldots, n_M)$, where the sets in the union on the right are disjoint and form subsets of $E(0, M)$.

For all $j \in \{1, \ldots, M + 1\}$, define
\[
c_j = \frac{M+1-j}{N-j}.
\]
Consider fixed $(n_0, n_1, n_2, \ldots, n_{j-1})$ and the uniform distribution over distinct subsequent choices $(n_j, n_{j+1}, \ldots, n_M)$. Taking expectation over this distribution, for all $j \in \{1, \ldots, M\}$
\[
E[|E(j, M) \cap \delta R_j|] = c_j \cdot |\delta R_j|.
\]
Taking expectation over $(n_1, n_2, \ldots, n_M)$, we obtain
\[
E[|L(0, M)|] = \psi \cdot R \cdot s - \sum_{j=1}^M c_j \cdot E[|\delta R_j|].
\]
Since
\[
c_1 \geq c_2 \geq c_3 \geq \cdots \geq c_M \geq c_{M+1} = 0,
\]
this can be rewritten as
\[
E[|L(0, M)|] = \psi \cdot R \cdot s
\]
\[- \sum_{j=1}^M (c_j - c_{j+1}) \cdot \sum_{j'=1}^{j-1} E[|\delta R_{j'}|].
\]
From Inequality (24),
\[
\sum_{j=1}^j E[|\delta R_{j'}|] = E[|R(0, j)|] < j \cdot b(\beta, \epsilon, \psi) \cdot s
\]
for all $j \in \{1, \ldots, M\}$. Hence,
\[
E[|L(0, M)|] > \psi \cdot R \cdot s - \sum_{j=1}^M (c_j - c_{j+1}) \cdot j \cdot b(\beta, \epsilon, \psi) \cdot s
\]
\[= \left(\psi \cdot R - b(\beta, \epsilon, \psi) \cdot \sum_{j=1}^M c_j\right) \cdot s.
\]
With the definition of $b_{\text{core}}(\zeta)$ provided in Equation (35),
\[
b(\beta, \epsilon, \psi) \cdot \sum_{j=1}^M c_j \leq \frac{\epsilon \cdot \psi^2 \cdot R}{2},
\]
and thus
\[
E[|L(0, M)|] > \psi \cdot \left(1 - \frac{\epsilon \cdot \psi}{2}\right) \cdot R \cdot s.
\]

Using values for $\epsilon$ and $\psi$ that satisfy the constraints described in Section VII-D, Equation (26) simplifies to
\[
E[|L(0, M)|] > R \cdot s.
\]

Thus, from Equations (21), (3), and Inequality (18) the source data is not reliably recoverable in expectation after the node failure with index $M$ when averaging over all sets of $M$ possible identifiers. Thus, there is at least one set of $M$ identifiers $(n_1, \ldots, n_M)$ for which the source data is not be reliably recoverable.  

\section{D. Choosing the parameters}

In the proof of Lemma 7.2, there is some choice in the values of $\epsilon$ and $\psi$ for a given $\beta$ that provide the best read repair rate lower bound on the tradeoffs, subject to the following constraints.

Since $\psi \cdot R \leq N$ must be satisfied, the constraint
\[
\beta \cdot \psi \leq 1
\]
follows from Equation (21).

Another constraint, from Equations (26) and (27), is that $\epsilon$ and $\psi$ should satisfy
\[
\psi \cdot \left(1 - \frac{\epsilon \cdot \psi}{2}\right) \geq 1.
\]

For fixed $\beta$, one would like to set $\epsilon$ and $\psi$ to maximize $b(\beta, \epsilon, \psi)$ defined in Equation (12) subject to these constraints. For $\beta \leq \frac{1}{3}$, $\epsilon = \frac{1}{2}$ and $\psi = 2$ satisfy these constraints, and Equation (13) is based on this (these values do not maximize $b(\beta, \epsilon, \psi)$).

For $\zeta < 1$, define
\[
\text{lni}(\zeta) = \ln \left(\frac{1}{1 - \zeta}\right).
\]
Note that for $\zeta < 1$,
\[
\sum_{j=0}^{N-1} \frac{1}{N-j} < \text{lni}(\nu) < \sum_{j=1}^{N} \frac{1}{N-j}.
\]

With $\zeta = \psi \cdot \beta$, the lefthand side of Equation (25), excluding the factor of $b(\beta, \epsilon, \psi)$, is at most
\[
\sum_{j=1}^{\zeta \cdot N - 1} \frac{\zeta \cdot N - j}{N-j} \leq \sum_{j=1}^{\zeta \cdot N - 1} \frac{\zeta \cdot N - j}{\zeta \cdot N - j + (\zeta \cdot N - N)} \leq \frac{\zeta \cdot N - N \cdot (1 - \zeta) \cdot \text{lni}(\zeta)}{N \cdot e(\zeta)},
\]
where
\[
e(\zeta) = \zeta - (1 - \zeta) \cdot \text{lni}(\zeta),
\]
and where the second inequality in Equation (32) follows from Equation (31). Thus, the lefthand side of Equation (25) is at most
\[
b(\beta, \epsilon, \psi, N) \cdot N \cdot e(\psi \cdot \beta).
\] (34)

Equation (34) simplifies to the righthand side of Equation (25) when \(b_{\text{core}}(\zeta)\) is defined as
\[
b_{\text{core}}(\zeta) = \frac{\zeta^2 e(\zeta)}{2e(\zeta)}.
\] (35)

Note that for all \(\zeta \in [0, 1]\),
\[
\frac{1}{2} = b_{\text{core}}(1) \leq b_{\text{core}}(\zeta) \leq b_{\text{core}}(0) = 1,
\]
since, as \(\zeta \to 0\),
\[
e(\zeta) \to \frac{e^2}{2}.
\]

**E. Table of values for \(b_{\text{core}}\) and \(b_{\text{num}}\)**

Table I shows example values of \(b_{\text{core}}(2\beta)\) (see Equation (33)) and \(b_{\text{num}}(2\beta)\) (see Equation (40)) as a function of \(\beta \leq \frac{1}{2}\).

| \(\beta\) | \(b_{\text{core}}(2\beta)\) | \(b_{\text{num}}(2\beta)\) | \(\beta\) | \(b_{\text{core}}(2\beta)\) | \(b_{\text{num}}(2\beta)\) |
|---|---|---|---|---|---|
| 0.05 | 0.966 | 0.917 | 0.30 | 0.771 | 0.505 |
| 0.10 | 0.931 | 0.834 | 0.35 | 0.723 | 0.420 |
| 0.15 | 0.894 | 0.752 | 0.40 | 0.669 | 0.333 |
| 0.20 | 0.856 | 0.670 | 0.45 | 0.605 | 0.236 |
| 0.25 | 0.815 | 0.588 | 0.50 | 0.500 | \(\infty\) |

**VIII. MAIN REPAIR RATE LOWER BOUND PROOF**

The proof of Theorem 6.1 follows immediately from Lemma 8.1. The function \(b_{\text{num}}(2\beta)\) mentioned in Lemma 8.1 is defined by Equation (40).

**Lemma 8.1:** For every fixed \(\beta < \frac{1}{2}\), there are functions \(\rho\) and \(\gamma\) such that, asymptotically as \(N\) grows, \(\rho(N)\) approaches zero, and \(\gamma(N)\) is exponentially small in \(N\), with the following properties. For source data size \(m = (1 - \beta) \cdot c\), for any random node failure distribution interacting with any repairer there is a subsequence
\[
i_1 = 0 < i_2 < \cdots < i_{\ell - 1} < i_\ell < \cdots
\]
of the indices of the node failure sequence produced by the random node failure distribution, such that, for all \(\ell \geq 1\):

- \(i_{\ell + 1} - i_\ell \leq \Gamma \cdot \beta \cdot N\), where \(\Gamma > 1\) is a constant.
- If the source data is recoverable just before the node failure with index \(i_\ell\) then with probability at least \(1 - \gamma(N)\), either
\[
\frac{|R(i_\ell, i_{\ell + 1})|}{|E(i_\ell, i_{\ell + 1} - 1)|} \geq \frac{1 - \rho(N)}{2\beta},
\] (36)

or the source data is not reliably recoverable just after the node failure with index \(i_{\ell + 1}\).

**Proof:** Most of the main proof ideas are similar to the proof of Lemma 7.2. We now sketch the remaining details to prove Lemma 8.1. In this case, a Poisson timing distribution is used to choose the time increments between node failures, where \(\Delta = E[T]\) is the average increment with respect to random variable \(T\). In addition, uniform identifier distribution is used to choose the identifiers of which nodes fail.

One difference from Lemma 7.2 is that the number of node failures within a period of time is not deterministic, but instead depends on \(T\). Using standard concentration of distribution bounds, it can be seen that over a duration of time \(R \cdot \Delta\), the probability the number of node failures is not within a small relative interval around \(R\) is exponentially small in \(N\). Thus, the average time between failures over a period of duration \(R \cdot \Delta\) is approximately \(E[T] = \Delta\) with high probability.

Another difference is that not each node failure is a distinct failure, since the uniform identifier distribution is used to choose identifiers of nodes that fail. In Subsection VIII-A we show that the number of distinct failures within phases is tightly related to the overall number of node failures within phases.

Another difference is that the lower bound is not with respect to a carefully chosen identifying sequence that is dependent on the repairer, but instead with respect to the uniform identifier distribution independent of the repairer. In Subsection VIII-B we improve the result in Subsection VII-C proving that the source data is not reliably recoverable in expectation, to a much stronger result proving that the source data is not reliably recoverable with high probability.  

**A. Distinct failures count versus node failures**

The uniform identifier distribution implies that the expected number of node failures from a phase start until the distinct failure count reaches \(\ell\) is
\[
\sum_{i=0}^{\ell-1} \frac{N}{N-i} < N \cdot \text{In}
\left(\frac{\ell}{N}\right),
\] (37)

where \(\text{In}\) is defined in Equation (30), and where the inequality follows from Equation (31).

Thus, the expected number of node failures from a phase start until the distinct failure count reaches \(\ell = 2R = 2\beta \cdot N\) is
\[
\sum_{i=0}^{2\beta N-1} \frac{N}{N-i} < N \cdot \text{In}(2\beta).
\] (38)

For a fixed value of \(\beta\), asymptotically as \(N\) grows, it can be shown that the number of node failures from a phase start until there are \(2R\) distinct failures is with high probability close to the expected value shown in Equation (38).

However, the distinct failure count \(\ell\) can be much smaller than \(2R\) for many phases, and thus it is not the case that number of node failures in a phase is with high probability close to the expected value shown in Equation (37). Furthermore, the distinct failure count of phases depends on the repairer.

Instead, for the analysis we group together consecutive phases, where the aggregate number of node failures in the group is proportional to \(2R\), and relate the aggregate number of distinct failures summed over the group of phases to the number of distinct failures of a single phase that spans the group of phases.
Let $s_1, s_2, \ldots, s_i$ be a group of $i$ consecutive phases with respective distinct failure counts $\ell_1, \ell_2, \ldots, \ell_i$. Let $s$ be an phase that begins at the beginning of phase $s_1$ and ends at the end of phase $s_i$, and let $\ell$ be the distinct failure count of $s$. Then, because any node failure that is a distinct failure for the phase $s$ is also a distinct failure for an phase $s_j$ that contains the node failure,

\[
\sum_{j=1}^{i} \ell_j \geq \ell.
\]

(39)

If for example $\ell = 2R = 2\beta \cdot N$, then the total number of node failures within the phase $s$ concentrates to at most $N \cdot \ln(2\beta)$. Alternatively, if $s$ spans $N \cdot \ln(2\beta)$ node failures then $\ell \geq 2R$, and thus $\sum_{j=1}^{i} \ell_j \geq 2R$ from Equation (39).

One can formalize the above (many details are omitted from this outline) and argue that within consecutive sequences of $N \cdot \ln(2\beta)$ node failures there are approximately $2\beta \cdot N$ distinct failures among the phases spanning the node failures. One can argue that this occurs with high enough probability that discarding the portions of the timeline where this is not true does not affect the overall read repair rate lower bound.

Define

\[
b_{\text{num}}(\zeta) = b_{\text{num}}(\zeta) \cdot \frac{\zeta}{\ln(\zeta)},
\]

(40)

where $b_{\text{num}}(\zeta)$ is defined in Equation (35) and $\ln(\zeta)$ is defined in Equation (30). Then, combining the above with Equation (13) yields Equation (36).

B. Source data very likely unrecoverable

We consider a phase as described in the proof of Lemma 7.2 in Section VII-C. Let $M = \psi \cdot R - 1$ and let $n_0$ be the identifier of the node that fails at the beginning of the phase, and let $(n_1, \ldots, n_M)$ be a set of $M$ possible identifiers.

Suppose Inequality (24) is satisfied for all sets of $M$ possible identifiers. We prove that the source data is recoverable for at most a fraction $\gamma$ of these sets, where $\gamma > 0$ is an exponentially small in $N$ value. The proof is based on the observation that in the analysis described in Section VII the value of $|L(0, i)|$ as a function of $i \in \{1, \ldots, M\}$ defines a Sub-martingale process with respect to the random choices of $(n_1, \ldots, n_M)$. By setting $\epsilon = \frac{1}{2} (1 - \alpha)$ and $\psi = 2$, where $\alpha > 0$ is a small constant, one can see that

\[
E[|L(0, M)|] \geq (1 + \alpha) \cdot R \cdot s.
\]

Furthermore, for each increase in $i$ by one, the change in the value of $|L(0, i)|$ changes by at most $s$. Thus, Azuma’s inequality can be used to show that $|L(0, M)| \leq R \cdot s$ with exponentially small probability $\gamma$, i.e., the source data is reliably recoverable with probability at most $\gamma$.

Let $p$ be the probability that Inequality (36) is not satisfied with respect random choices of $(n_1, \ldots, n_M)$, and thus $1 - p$ is the probability that Inequality (36) is satisfied. By a small modification of the argument outlined in the previous paragraph, with probability at least $\max\{p - \gamma, 0\}$, the source data is not reliably recoverable after the node failure with identifier $n_M$. Thus, with probability at least $1 - \gamma$, either Inequality (36) is satisfied or the source data is not reliably recoverable after the node failure with identifier $n_M$.

IX. READ REPAIR RATE UPPER BOUND

A. Intuition for upper bound

We provide some intuition for Theorem 6.2. We first describe a simplified version of the system in [10] with a tradeoff between $\beta$ and $R_{\text{peak}}$ that is around a factor of two worse than optimal.

The liquid repairer operates as follows, where $R = \beta \cdot N$ and $K = (1 - \beta) \cdot N$. Source data $X$ of $m$ bits is partitioned into $R$ equal sized objects $X(0), \ldots, X(R - 1)$. An erasure code is applied to each object independently, where each object is partitioned into $K$ source fragments from which at least $R$ repair fragments can be generated.

An Encoding Fragment ID, or EFI, is used to uniquely identify each fragment of an object. In this case, EFIs 0, 1, \ldots, $K - 1$ can be used to identify the source fragments of a object, and EFIs $K$ or larger can be used to identify the repair fragments generated from source fragments.

Examples of erasure codes can be found in [5] and [6]. We assume that any $K$ fragments of an object, identified by their EFIs, can be used to decode the object. This is true for the Reed-Solomon code [5], and essentially true for the RaptorQ code [6].

For each $i = 0, \ldots, N - 1$, let $P(i) = i$ be the EFI assigned to node $i$. Using an erasure code, for each $j = 0, \ldots, R - 1$, for each $i = 0, \ldots, K + j$, the storer generates and writes fragment $P(i)$ for $X(j)$ to node $i$.

The storage overhead is $R = \beta \cdot N$, since each node has raw capacity to store a fragment for each object, and the storage overhead for all fragments of an object is $\frac{R}{N}$. Also, the size of each fragment is $\frac{R}{N}$, since each node has raw capacity $s$ to store up to $R$ fragments.

The following invariant is established by the storer and maintained by the repairer:

- Just before the next node failure, for each $j = 0, \ldots, R - 1$, $X(j)$ has at least $K + j + 1$ fragments stored at the nodes.

Since each node has at most one fragment for each object, a node failure can decrease the number of available fragments for an object by at most one. Since the invariant ensures that each object has at least $K + 1$ fragments available just before the next node failure, each object has at least $K$ fragments available just after the next node failure, and is thus recoverable, and thus the source data is recoverable at each point in time.

The repairer cycles through the objects, repairing after each node failure the object $X(0)$ with the least available fragments. Suppose a node fails and is replaced with a node with all $s$ bits initialized to zeroes. The repairer operates as follows to reestablish the invariant after the node failure. The repairer reads $K$ of the fragments available for $X(0)$, uses the erasure encoder to generate the missing up to $R$ fragments for $X(0)$, and writes these up to $R$ generated fragments to the nodes with their assigned EFIs.

After this repair, $X(0)$ has all $N$ fragments available at the nodes. The objects are logically reordered at this point, i.e., for $j = 1, \ldots, R - 1$, the old $X(j)$ becomes the new $X(j - 1)$, and
the old $\mathcal{X}(0)$ becomes the new $\mathcal{X}(R - 1)$. This reestablishes the invariant.

If the repairer reads at a steady rate of

$$R_{\text{peak}} = \frac{K}{R} \cdot \frac{s}{\Delta} = \frac{1 - \beta}{\beta} \cdot \mathcal{E} \quad (41)$$

then the repairer can read $K$ fragments between node failures and reestablish the invariant.

For small $\beta$, Equation (41) for the liquid repairer is around a factor of two worse than Inequality (11) of Theorem 6.2 for the advanced liquid repairer described in Section X-B. The intuition for this gap in optimality for the liquid repairer is that the maximum redundancy per object is $\beta$, the average redundancy needed per object is only $\frac{\beta}{2}$, i.e., object $\mathcal{X}(i)$ needs only $K + i + 1$ fragments for the repairer to succeed in maintaining recoverability of all objects. Thus $\frac{\beta}{2}$ of the allocated $\beta$ storage overhead is not in use at each point in time, as some nodes store less fragments than other nodes. On the other hand, the value of $R_{\text{peak}}$ in Equation (41) is inversely proportional to the maximum redundancy per object $\beta$.

The redundancy per object for the advanced liquid repairer varies similar to that for the liquid repairer. The intuition behind the better bound for the advanced liquid repairer is that fragments are generated and stored using a strategy that allows the unequal object redundancy to be distributed equally among the nodes at each point in time, while maintaining the property that the value of $R_{\text{peak}}$ is inversely proportional to the maximum redundancy per object. The advanced liquid repairer uses a maximum of around $2\beta$ redundancy per object, but the average redundancy per object is $\beta$, and this redundancy is equally distributed among the nodes at each point in time. Thus, $R_{\text{peak}}$ for the advanced liquid repairer is inversely proportional to $2\beta$ in Inequality (11) of Theorem 6.2 when the storage overhead is $\beta$.

B. Core upper bound results

As the core for proving the main read repair rate upper bound results, we prove read repair rate upper bounds for repairers with respect to any node failure sequence with a fixed timing sequence.

The advanced liquid repairer introduced below is based on the paper [30].

Theorem 9.1: For every fixed $\beta$, for source data size $m = (1 - \beta) \cdot c$, the advanced liquid repairer with peak read repair rate $R_{\text{peak}}$ with respect to any node failure sequence with a fixed timing sequence and with erasure rate $\mathcal{E}$ guarantees that the source data is always recoverable if

$$\frac{R_{\text{peak}}}{\mathcal{E}} \leq \frac{(1 + 3\beta) \cdot (1 - \beta)}{2\beta}.$$  \hspace{1cm} (42)

Proof: The advanced liquid repairer operates as follows. A value $R'$ is determined so that the following equation holds:

$$\beta = \frac{R' + 3}{2N + R' + 1}.$$  \hspace{1cm} (43)

Note that for large $N$,

$$R' \approx \frac{2\beta \cdot N}{1 - \beta}.$$  

Source data is initially stored as follows by the storer. Source data $\mathcal{X}$ of $m$ bits is partitioned into $N \cdot R'$ equal sized objects $\mathcal{X}(0, 0), \ldots, \mathcal{X}(0, R' - 1), \mathcal{X}(1, 0), \ldots, \mathcal{X}(N - 1, R' - 1)$.

Let $K' = N - 1$ be the number of source fragments into which each object is partitioned for erasure coding. An erasure code is applied to each object independently, where each object is partitioned into $K'$ source fragments from which at least $R'$ repair fragments can be generated.

An EFI is used to uniquely identify each fragment. In this case, EFIs $0, 1, \ldots, K' - 1$ can be used to identify the source fragments of an object, and EFIs $K'$ or larger can be used to identify the repair fragments generated from source fragments. We assume that any $K'$ fragments of an object, identified by their EFIs, can be used to erase decode the object.

For each $i = 0, 1, \ldots, N - 1$, let $P(i) = i$ be the EFI initially assigned to node $i$. There is also an EFIs queue of $R'$ unassigned EFIs, where initially these EFIs are $0, \ldots, N + R' - 1$. The values of the EFIs that are either assigned EFIs or unassigned EFIs change over time, but at any point in time they are all distinct.

Using an erasure code, for each $i = 0, \ldots, N - 1$, for each $j = 0, \ldots, R' - 1$, for each $\ell = 0, \ldots, N - 1$, the storer generates and writes fragment $P(\ell)$ for $\mathcal{X}(i, j)$ to node $\ell$.

For each $i = 0, \ldots, N - 1$ and for each $j = 0, \ldots, R' - 1$, the storer generates fragments for the first $j + 1$ unassigned EFIs for $\mathcal{X}(i, j)$ and writes these $j + 1$ fragments at node $i$.

The following invariants are established by the storer and maintained by the repairer:

- Just before the next node failure, for each $i = 0, \ldots, N - 1$, for each $j = 0, \ldots, R' - 1$, for each $\ell = 0, \ldots, N - 1$, fragment $P(\ell)$ for $\mathcal{X}(i, j)$ is stored at node $\ell$.

- Just before the next node failure, for each $i = 0, \ldots, N - 1$, for each $j = 0, \ldots, R' - 1$, fragments for the first $j + 1$ unassigned EFIs for $\mathcal{X}(i, j)$ are stored at node $i$.

From the first invariant, all $N \cdot R'$ objects can be recovered from the fragments associated with assigned EFIs stored at the nodes just after a node failure, and thus the source data is recoverable at each point in time.

The repairer operates as follows to reestablish these invariants after a node failure. Suppose for some $i = 0, \ldots, N - 1$, node $i$ fails and is replaced by a node $i$ with all bits set to zeroes. Let $F' = P(i)$.

The repairer dequeues the first EFI $F$ from the front of the unassigned EFIs queue, and assigns $F$ as the EFI $P(i)$ for node $i$. The repairer enqueues EFI $F''$ to the end of the unassigned EFIs queue, which reestablishes $R'$ as the number of unassigned EFIs.

The first repairer step operates as follows. For each $i' \neq i$ and for each $j = 0, \ldots, R' - 1$, the repairer moves fragment $F = P(i)$ for object $\mathcal{X}(i', j)$ from node $i'$ to node $i$. The number of fragments read by the repairer in the first step is $(N - 1) \cdot R'$, and the number of fragments written by the repairer in the first step is also $(N - 1) \cdot R'$.

After this first step, for each $i' \neq i$, $\mathcal{X}(i', 0)$ no longer has a fragment for any unassigned EFI, and for $j = 1, \ldots, R' - 1$, $\mathcal{X}(i', j)$ has fragments for the first $j$ unassigned EFIs. There
are no fragments for the last unassigned EFI $F'$ at the end of unassigned EFIs queue. Thus, the objects are logically reordered at this point, i.e., for $j = 1, \ldots, R'$, the old $X(i', j)$ becomes the new $X'(i', j-1)$, and the old $X(i', 0)$ becomes the new $X(i', R' - 1)$ (which currently has no fragments for unassigned EFIs).

The second repairer step operates as follows. For each $i' \neq i$, the repairer reconstructs $X'(i', R' - 1)$ from the fragments for the assigned EFIs at the $N$ nodes other than node $i$, generates fragments for the $R'$ unassigned EFIs and writes these fragments to node $i$. The number of fragments read by the repairer in the second step is $(N-1)^2$, and the number of fragments stored by the repairer in the second step is $(N-1) \cdot R'$.

The third repairer step operates as follows. For each $j = 0, \ldots, R' - 1$, the repairer reconstructs $X(i, j)$ from the fragments for the assigned EFIs at the $N$ nodes other than node $i$, generates fragments for $F = P(i)$ and the first $j+1$ unassigned EFIs for $X(i, j)$, and writes these fragments to node $i$. The number of fragments read by the repairer in the third step is $(N-1) \cdot R'$ and the number of fragments stored by the repairer in the third step is $R' \cdot (R' + 2)$.

The three repairer steps reestablish the invariants after a node failure.

The fragments read by the repairer in the first and second steps are somewhat overlapping, and thus the total number of fragments read from those two steps is $(N-1) \cdot (N+R' - 2)$, and the total number of fragments read from all three steps for each node failure is

$$A' = (N-1) \cdot (N + 2R' - 2),$$

and the total number of fragments stored from all three steps for each node failure is

$$S' = R' \cdot \left(2N + \frac{R' - 1}{2} \right).$$

The storage overhead of this system can be derived as follows. The aggregate number of source fragments is $(N-1) \cdot N \cdot R'$. At each point in time, there are $N \cdot R'$ fragments for assigned EFIs and $\frac{R' \cdot (R' + 1)}{2}$ fragments for unassigned EFIs at each node, and thus the aggregate number of fragments at each node is

$$s' = N \cdot R' + \frac{R' \cdot (R' + 1)}{2}.$$ 

Since there are $N$ nodes, the value of $\beta$ defined by Equation (43) is the storage overhead.

For large $N$, $A' \approx \frac{N^2 \cdot (1 + 3\beta)}{1 - \beta}$

and $S' \approx \frac{2\beta \cdot N^2 \cdot (2 - \beta)}{(1 - \beta)^2}$

and

$$s' \approx \frac{2\beta \cdot N^2}{(1 - \beta)^2}.$$ 

Thus, $\frac{A'}{s'} \approx \frac{(1 + 3\beta) \cdot (1 - \beta)}{2\beta}$.

If the read repair rate is at a steady rate of $R_{peak}^\text{read} = \frac{A' \cdot s}{s' \cdot \Delta}$ then the repairer can read $A'$ fragments between node failures and complete the three steps between node failures to reestablish the invariants. Putting this together, yields Inequality (42), and completes the proof of Theorem 9.1.

Similarly,$$\frac{S'}{s'} \approx 2 - \beta.$$ 

If the write repair rate is at a steady rate of $W_{peak}^\text{write} = \frac{S' \cdot s}{s' \cdot \Delta}$ then the repairer can store $S'$ fragments between node failures and complete the three steps between node failures to reestablish the invariants. Thus,

$$W_{peak}^\text{write} \leq (2 - \beta) \cdot \frac{s}{\Delta}.$$  

C. Comparison of core results

We compare Theorem 7.1 with Theorem 9.1. The right-hand side of Inequalities (14) and (42) converge to $\frac{1}{2\beta}$ as $\beta \to 0$. Even for larger $\beta$ the bounds are relatively tight, e.g., when $\beta = 0.25$, Inequality (14) is $\frac{R_{\text{avg}}}{\mathbb{E}} \geq 0.815$ and whereas Inequality (42) is $\frac{R_{\text{peak}}}{\mathbb{E}} \leq 1.3125$.

D. Main upper bound proof

In this section we sketch the proof of Theorem 6.2.

Proof: The advanced liquid repairer used for this proof is a small modification (described below) of the advanced liquid repairer introduced in Section IX-B. The proof is similar in outline to the proof of Theorem 9.1 provided in Section IX-B.

One thing that needs to be shown in the proof of Theorem 6.2 is that over a long enough period of time the expected number of node failures over that period and the actual number of node failures over that period differ by a relatively small amount with high probability.

The increments of time between node failures is chosen by independently sampling a random variable $T$ with $\mathbb{E}[T] = \Delta$. For a fixed $\zeta > 0$, and for a given $T$, asymptotically as $N$ grows, it can be shown that over periods of duration $\zeta \cdot N \cdot \Delta$ the probability that the number of node failures is not within a small relative interval around $\zeta \cdot N$ is exponentially small in $N$. Thus, the average time between node failures over a period of duration $\zeta \cdot N \cdot \Delta$ is approximately $\mathbb{E}[T] = \Delta$ with high probability.
Another issue is that the advanced liquid repairer described in Section [X-B] does not have any buffer of protection against node failures that occur at random times instead of at fixed known times. Thus, we use $K' = (1 - \alpha) \cdot N$ for a constant $\alpha > 0$, and then the advanced liquid repairer operates at a smooth read rate, but always makes sure that the three repairer steps described in Section [X-B] with reference to node $i$ remaining are completed before there are at most $\alpha \cdot N$ additional node failures after node $i$ failed. It can be guaranteed that this condition is not met with only an exponentially small in $N$ probability, where the MTTDL is inversely proportional to this probability. The overall storage overhead is then approximately $\beta + \alpha$.

X. FUNCTIONAL REPAIRERS

A functional repairer generalizes a repairer that reads bits from a node to a repairer that reads the output of a locally computed function of bits at a node. Functional repairers are conceptually more powerful than repairers. Some forms of functional repairers were introduced in [14], [17], which are discussed in Section [XIII].

The functional read repair rate is the rate at which output bits of locally computed functions are read by the functional repairer, and the read repair rate is the rate at which input bits to locally computed functions are read from nodes. We let $\mathcal{R}_{\text{peak}}$ and $\mathcal{R}_{\text{avg}}$ denote the peak and average functional read repair rate, and as before $\mathcal{R}_{\text{peak}}$ and $\mathcal{R}_{\text{avg}}$ denote the peak and average read repair rate.

Figure 6 shows an example of a distributed storage system using a functional repairer, where $F_n$ is the local function computed at node $n$. The input bits to $F_n$ read from node $n$ are included in the read repair rate, whereas the output bits of $F_n$ read by the functional repairer are included in the functional read repair rate.

![Figure 6: Storage nodes and example of functional repairer model.](image)

When the local functions are the identity functions at each node then a functional repairer is simply a repairer and the read repair rate and the functional read repair rate are the same. In general, a local function can compress input bits into a smaller number of output bits, and thus the read repair rate can be larger than the functional read repair rate for a functional repairer.

Algorithm 1 emulates a phase as described in Section VII-C with respect to a functional repairer, and describes the information available to reconstruct the source data. Steps 1 through 8 are initialization steps for the phase. $\mathcal{B}$ is the set of random bits used to choose the portion of the node failure sequence. Note that

$$\mathbf{H}(\mathcal{X}, \mathcal{B}) = m + \mathbf{H}(\mathcal{B}).$$

$\mathcal{R}_{\text{func}}$ is the set of output bits from local functions that have been read since the phase start and, for each $n \in \mathcal{N}'$, $\mathcal{R}_{\text{func}}(n)$ is the set of output bits read from the local function at node $n$ since the phase start. $\mathcal{V}$ is fixed to the set of bits stored in the repairer local memory at the phase start and, for each $n \in \mathcal{N}'$, $\mathcal{V}(n)$ is fixed to the set of $s$ bits stored at node $n$ at the phase start. $\mathcal{I}$ is the list of node failure indices chosen since the phase start. $\mathcal{N}'$ is the set of nodes that have not yet failed since the phase start.

The value of $\ell_{\text{inc}}$ is a lower bound on the amount of entropy lost due to node failures.

**Algorithm 1** Phase with respect to functional repairer

1: $\mathcal{B} = \{0,1\}^b =$ bits used to choose node identifiers
2: $\mathcal{R}_{\text{func}} \leftarrow \emptyset$
3: for all $n \in \mathcal{N}$ do $\mathcal{R}_{\text{func}}(n) \leftarrow \emptyset$
4: $\mathcal{V} = \{0,1\}^v =$ bits in repairer local memory
5: for all $n \in \mathcal{N}'$ do $\mathcal{V}(n) \in \{0,1\}^s =$ bits at node $n$
6: $\mathcal{I} \leftarrow \emptyset$
7: $\mathcal{N}' \leftarrow \mathcal{N}$
8: $\ell_{\text{inc}} = 0$ // initialize lost entropy
9: while $\mathcal{N}' \neq \emptyset$ do
10: if $S(\mathcal{V}, \mathcal{R}_{\text{func}}, \mathcal{I}) = \text{true}$ then
11: $n \leftarrow \text{Select}(\mathcal{N}', \mathcal{B})$
12: $\mathcal{N}' \leftarrow \mathcal{N}' - \{n\}$
13: $\mathcal{I} \leftarrow (\mathcal{I}, n)$
14: $\mathcal{V}(n) \leftarrow 0^v$
15: $\ell_{\text{inc}} \leftarrow \ell_{\text{inc}} + s - |\mathcal{R}_{\text{func}}(n)|$ // update lost entropy
16: $n \leftarrow G(\mathcal{V}; \mathcal{R}_{\text{func}}, \mathcal{I})$
17: if $n \in \mathcal{N}'$ then
18: $A \leftarrow F(\mathcal{V}, \mathcal{R}_{\text{func}}, \mathcal{I}, \mathcal{V}(n))$
19: $\mathcal{R}_{\text{func}} \leftarrow \langle \mathcal{R}_{\text{func}}, A \rangle$
20: $\mathcal{R}_{\text{func}}(n) \leftarrow \langle \mathcal{R}_{\text{func}}(n), A \rangle$

The functions $F$, $G$, and $S$ define the functional repairer actions as the phase proceeds. The inputs to $F$, $G$ and $S$ include the values of $(\mathcal{V}, \mathcal{R}_{\text{func}}, \mathcal{I})$, since these bits determine the action of the repairer at each point in time.

The function $S$ determines how much data is read between distinct failures. When all reads since the previous distinct failure are completed and it is time for the next distinct failure to occur, $S$ evaluates to true at step 10 triggering the next distinct failure.

When a distinct failure is triggered, $\text{Select}(\mathcal{N}', \mathcal{B})$ is called in step 11 to select node $n$ from the remaining set $\mathcal{N}'$ of node that have not yet failed since the phase start. In step 14, $\mathcal{V}(n)$ is zeroed, indicating that node $n$ has failed and been replaced with a node initially storing all zeroes.

After step 15 the value of $\ell_{\text{inc}}$ after the node failure with index $M$ is the generalization for functional repairers of $|\mathcal{L}(0,M)|$.

The function $G$ determines the node $n$ from which to read the next bits in step 16. There can be many bit reads from
many different nodes between each node failure, and thus \( G \) may be called multiple times (until \( S \) evaluates to true) between node failures. If node \( n \) has failed since the phase start, as determined in step [17] then reading bits from \( n \) does not provide additional information, since bits written to node \( n \) since it failed are determined by \((V, R_{\text{fs}}), I)\).

The function \( F \) determines which bits are read by the repairer. \( F \) is called each time just after \( G \) has determined the node \( n \) from which to read the next bits, and thus one of the inputs to \( F \) is \( Y(n) \). Each call to \( F \) at step [18] determines which function to apply to \( Y(n) \) at that call and how many bits are in the output of the call (which is the number of bits counted as read by the functional repairer due to the call). These parameters vary at each call to \( F \) depending on the values of the inputs \((V, R_{\text{fs}}, I)\). The output of each call to \( F \) is appended to both \( R_{\text{fs}} \) and \( R_{\text{fs}}(n) \). Since the inputs to \( F \) includes all the inputs to \( G \) and \( S \), \( F \) implicitly can calculate \( G \) and \( S \) at any point in time, and thus \( G \) and \( S \) are introduced for notational convenience.

Inequality [46] of Theorem [10.1] is the functional repairer equivalent of Inequality [18], and thus Theorem [10.1] formally justifies the functional analogs of all the results, i.e., the functional repairer analogs of Theorem [7.1], Lemma [7.2], Theorem [8.1] and Lemma [8.1] are exactly the same except that repairer is replaced with functional repairer and read repair rate \( R_{\text{fs}} \) is replaced with functional read repair rate \( R_{\text{fs}} \).

**Theorem 10.1:** Source data \( X \) is not reliably recoverable if
\[
\ell_{\text{inc}} > \beta \cdot c + v, \tag{46}
\]
where \( \ell_{\text{inc}} \) is defined by Algorithm [1].

**Proof:**
Consider the conditions just after \( \ell_{\text{inc}} \) is updated in step [15]. Let \( Y' \) be the concatenation for all \( n \in N' \) of \( Y(n) \), i.e., \( Y' \) is the concatenation of bits stored at not yet failed nodes. Let \( R_{\text{fs}}(n) \) be the concatenation of the portions of \( R_{\text{fs}}(n) \) that are the output of \( F \) when the input \( Y(n) \) to \( F \) is for some \( n \in N' \).

Let \( N'' = N - N' \) be the set of failed nodes. Let \( Y'' \) be the concatenation for all \( n \in N'' \) of \( Y(n) \), i.e., \( Y'' \) is the concatenation of the bits stored at failed nodes, and thus all bits of \( Y'' \) are zeroes. Let \( R_{\text{fs}}(n) \) be the concatenation of the portions of \( R_{\text{fs}}(n) \) that are the output of \( F \) when the input \( Y(n) \) to \( F \) is for some \( n \in N'' \). (At the time \( Y(n) \) is an input to \( F \), node \( n \) has not yet failed.) \( R_{\text{fs}}(n) \) can be rearranged as
\[ R_{\text{fs}} = (R_{\text{fs}}, R_{\text{fs}}(n)) \]
Note that \( N = |N'| + |N''| \) and \( |R_{\text{fs}}| = |R_{\text{fs}}| + |R_{\text{fs}}(n)| \).

The bits \( Z \) stored at the nodes and repairer local memory after step [15] is a deterministic function of \( Z' = (V, R_{\text{fs}}, R_{\text{fs}}(n), Y', Y'', B) \),
\[ \mathbf{H}(Z, B) \leq \mathbf{H}(Z'). \tag{47} \]
and thus
\[ \mathbf{H}(Z, B) \leq \mathbf{H}(Z'). \tag{48} \]
Since \( Y'' \) is all zeroes,
\[ Z' = (V, R_{\text{fs}}, R_{\text{fs}}(n), Y', B). \]

Note that
\[ \mathbf{H}(Z') = \mathbf{H}(V, R_{\text{fs}}, R_{\text{fs}}(n), Y', B) \]
\[ = \mathbf{H}(R_{\text{fs}}(n) | V, R_{\text{fs}}(n), Y', B) + \mathbf{H}(V, R_{\text{fs}}, Y', B). \]
However, \( R_{\text{fs}} \) is \( F \)-computable from \((V, R_{\text{fs}}, Y, B)\), and thus
\[ \mathbf{H}(R_{\text{fs}}(n) | V, R_{\text{fs}}(n), Y', B) = 0. \]
Because \( N = |N'| + |N''| \) and because
\[ \ell_{\text{inc}} = \sum_{n \in N''} (s - |R(n)|) = |N''| \cdot s - |R_{\text{fs}}(n)|, \]
\[ |R_{\text{fs}}(n), Y'| = |R_{\text{fs}}| + |N''| \cdot s = N \cdot s - \ell_{\text{inc}} = c - \ell_{\text{inc}}. \]
Thus,
\[ \mathbf{H}(Z') = \mathbf{H}(V, R_{\text{fs}}, Y', B) \leq c - \ell_{\text{inc}} + v + \mathbf{H}(B). \tag{49} \]

Source data \( X \) is not reliably recoverable if \( H(X|Z, B) > 0 \). From Inequalities [45], [48], and [49], if \( \ell_{\text{inc}} > \beta \cdot c + v \) just after step [15] then \( X \) is not reliably recoverable.

Note that the upper bounds on read repair rate \( R_{\text{fs}}(n) \) for the advanced liquid repairers of Theorem [7.1] and Theorem [6.2] essentially match the lower bounds on functional read repair rate \( R_{\text{fs}} \) provided by the functional repairer analogs of Theorem [7.1] and Theorem [6.1].

**XI. WRITE REPAIR RATE BOUNDS**

The read repair rate generally dominates the write repair rate, i.e., the write repair rate for a repairer is substantially less than the read repair rate when \( \beta \) is small. Thus, bounds with respect to write repair rates are generally less important than bounds with respect to read repair rates.

Let \( W(i, j) \) be the bit-locations to which the repairer writes in a phase that starts at a node failure with index \( i \) and ends with index \( j \). Then, by reasoning similar to that used to justify Inequality [17], the source data is not reliably recoverable after the node failure with index \( i ' \) if
\[ c - |E(i, i')| + |W(i, i')| < m. \tag{50} \]
From this it can be seen that if there are \( \zeta \cdot N \) distinct failures staring at index \( i \) and ending at \( i' \) and the source data is recoverable after the node failure with index \( i' \) then
\[ |W(i, i')| \geq (\zeta - \beta) \cdot c. \tag{51} \]

**Theorem 11.1:** For every fixed \( \beta < 1 \), for source data size \( m = (1 - \beta) \cdot c \), for every repairer there is a node failure sequence with a fixed timing sequence and with erasure rate \( E \) such that if the repairer has average write repair rate \( W_{\text{avg}} \) and is always able to maintain recoverability of the source data then
\[ \frac{W_{\text{avg}}}{E} \geq 1 - \beta. \tag{52} \]

**Proof:** For each phase, a permutation of the \( N \) possible identifiers, e.g., \((0, 1, 2, \ldots, N - 1)\), can be chosen as the identifiers of the nodes that fail, and thus each node failure is a distinct failure. Thus, from Inequality [51], after all \( N \) node
failures of the phase (which is of duration \((N - 1) \cdot \Delta\), and corresponds to \(\zeta = 1\)), the repairer must have written at least \(m = (1 - \beta) \cdot c\) bits if the source data is still recoverable. Thus, the average rate at which bits are written by the repairer must satisfy Inequality (52).

Theorem 11.2: For every fixed \(\beta \leq 1\), for source data size \(m = (1 - \beta) \cdot c\), for any repairer with average write repair rate \(W_{\text{avg}}\) with respect to any random node failure distribution with erasure rate \(\bar{E} = \frac{\zeta}{N}\), if the repairer is always able to maintain recoverability of the source data then

\[
\frac{W_{\text{avg}}}{\bar{E}} \geq \frac{1 - \beta}{2 \cdot \ln(1 + \beta)/2}. \tag{53}
\]

Proof: On average, after there have been \(\zeta \cdot N\) distinct failures since the phase start, the number of node failures overall is \(\ln\left(\frac{\zeta}{N}\right) \cdot N\). From Inequality (51), after \(\zeta \cdot N\) distinct failures since the phase start (which takes average time \(\ln\left(\frac{\zeta}{N}\right) \cdot N \cdot \Delta\)), the repairer must have written at least \(m = (\zeta - \beta) \cdot c\) bits if the source data is still recoverable. Thus,

\[
\frac{W_{\text{avg}}}{\bar{E}} \geq \frac{\zeta - \beta}{\ln(\zeta)}. \tag{54}
\]

Inequality (53) follows using the setting \(\zeta = \frac{1 + \beta}{2}\).

Note that Inequality (53) becomes

\[
\frac{W_{\text{avg}}}{\bar{E}} \geq 0.36 \cdot (1 - \beta)
\]

when \(\beta = \frac{1}{2}\),

\[
\frac{W_{\text{avg}}}{\bar{E}} \geq 0.72 
\]

when \(\beta = 0\), and

\[
\frac{1}{2 \cdot \ln(1 + \beta)/2}
\]

transitions from 0.36 to 0.72 as \(\beta\) transitions from \(\frac{1}{2}\) to 0.

Based on Equation (44), it can be shown that the advanced liquid repairer satisfies the following write repair rate upper bound.

Theorem 11.3: For every \(\beta < 1\), asymptotically as \(N\) grows, for source data size \(m = (1 - \beta) \cdot c\), the advanced liquid repairer with peak write repair rate \(W_{\text{peak}}\) with respect to any random node failure distribution with erasure rate \(\bar{E}\), can achieve MTTDL exponential in \(N\) and satisfy

\[
\frac{W_{\text{peak}}}{\bar{E}} \leq 2 - \beta. \tag{54}
\]

XII. ADDITIONAL DISTRIBUTED STORAGE MODELS

In this section we briefly discuss some additional DS models with different types of fail processes.

A. Variable numbers of node failures

One can extend the read repair rate lower bound results to an augmented random node failure distribution, where a variable number of nodes concurrently fail. Thus, the augmented random node failure distribution decides the timing of failures, the number of nodes to fail, and the pattern of node failures. For each next failure event, the augmented random node failure distribution first chooses a number \(N'\) of nodes to next fail from a probability distribution \(\mathcal{D}\) (where \(\mathcal{D}\) can be known to the repairer). Then, the augmented random node failure distribution chooses the timing of the next failure event, according to a Poisson distribution with rate \(\lambda(N', N)\), where \(\lambda(N', N)\) can depend on \(N'\) and \(N\) (where the Poisson distribution rate \(\lambda(N', N)\) associated with \((N', N)\) can be known to the repairer). Finally, the augmented random node failure distribution chooses \(N'\) nodes to fail uniformly and randomly from among the \(N\) nodes at the chosen timing of the next failure event.

Lemmas 7.2 and 8.1 extend to this model, where the erasure rate \(\bar{E} = \frac{1}{2}\) in the statement of the theorems is replaced with

\[
\bar{E} = \sum_{N'} \Pr[N' \in \mathcal{D}] \cdot \frac{s \cdot N'}{\lambda(N', N)},
\]

i.e., with the average rate at which bits are erased by the augmented random node failure distribution.

Let \(N'\) be an upper bound on the number \(N'\) of the nodes that can fail at one time. Then, the analog of Lemmas 7.2 and 8.1 holds for this model as long as \(\frac{N'}{N} \rightarrow 0\).

B. Failures at finer granularity

Small units of data may be corrupted on nodes, i.e., corruption may be at the granularity at which data units are read from nodes. Corruption of a data unit is treated as loss (erasure) when strong check sums are used to protect data stored at the nodes, i.e., a corrupted data unit is discarded when it fails the check sum condition. There are two possible scenarios:

Notified loss: corruption of a data unit is detected when it occurs.

Silent loss: corruption of a data unit is undetected until it is read.

The DS information capacity with respect to any node failure sequence with a fixed timing sequence and with notified loss is asymptotically at most

\[
\left(1 - \frac{1}{\sigma}\right) \cdot c,
\]

as \(\text{RER } \sigma\) grows, which can be proved similar to the arguments in Section VII-B. The DS information capacity with respect to a random node failure distribution with silent loss is asymptotically at least

\[
\left(1 - \frac{1}{\sigma}\right) \cdot c,
\]

as \(\text{RER } \sigma\) grows, which can be proved similar to the arguments in Section IX-A.

C. BSC

The Shannon information capacity for the binary symmetric channel (BSC) can be characterized as follows. The signal rate \(S\) is the raw rate at which data can be transmitted over the channel, independent of reliability. The error rate \(\bar{E}\) is the rate at which data transmitted over the channel is randomly
corrupted (each bit selected for corruption is equally likely to be zero or one if corrupted). The BSC SER is defined as
\[ \sigma = \frac{S}{E}. \] (55)

The BSC information capacity has been shown to be
\[ \left(1 - H\left(\frac{1}{\sigma}\right)\right) \cdot S, \] (56)
where \( H\left(\frac{1}{\sigma}\right) \) is the entropy of \( \frac{1}{\sigma} \). This information capacity is asymptotically achievable as the source data size grows.

One can consider a BSC random node failure distribution with error rate \( E \) similar to that for the BSC, and the RER \( \sigma \) with respect to the BSC random node failure distribution is
\[ \sigma = \frac{R}{E}. \] (57)
where \( R \) is the repair rate. It can be shown that the DS information capacity with respect to a BSC random node failure distribution is asymptotically at most
\[ \left(1 - H\left(\frac{1}{2\sigma}\right)\right) \cdot c, \]
as RER \( \sigma \) grows, which can be proved similar to the arguments in Sections VII and VIII. The DS information capacity with respect to a BSC random node failure distribution is asymptotically at least
\[ \left(1 - H\left(\frac{1}{\sigma}\right)\right) \cdot c, \]
as RER \( \sigma \) grows, which can be proved similar to the arguments in Section IX-D.

XIII. RELATED WORK

The groundbreaking research of Dimakis et al., described in [14] and [17], is closest to our work: An object-based distributed storage framework is introduced, and optimal tradeoffs between storage overhead and functional read repair rate are proved. We refer to the framework, the lower bounds, and the repairer described in [14] and [17] as the Regenerating framework, the Regenerating lower bounds, and the Regenerating repairer, respectively.

The Regenerating framework models repair of a single lost fragment, and is applicable to reactive repair of a single object. The Regenerating framework is based on \((n, k, d, \alpha, \gamma)\): \( n \) is the number of fragments for the object (each stored at a different node); \( k \) is the number of fragments from which the object must be recoverable; \( d \) is the number of fragments used to generate a lost fragment at a new node when a node fails; \( \alpha \) is the fragment size; and \( \gamma/d \) is the amount of data generated from each of \( d \) fragments needed to generate a fragment at a new node.

A. Regenerating repairers

We consider Minimum Storage Regenerating (MSR) settings (the object size is \( \alpha \cdot k \)), and fix \( d = n - 1 \), as these are typically thought of as the most practical settings that minimize storage overhead and minimize repairer traffic.

The Regenerating repairer [14], [17] is a functional repairer (Section X). When a node fails, at each of the \( n - 1 \) remaining nodes a network coding function is applied to the fragment read from that node to generate \( \frac{n-\gamma}{n-1} \) bits that are sent to the repairer, which uses all received bits from the nodes to generate and store a fragment at the new node. The \( \frac{n-\gamma}{n-1} \) bits sent to the repairer from each of the \( n - 1 \) nodes is counted in \( R_{\text{peak}} \), whereas reading the fragment of size \( \alpha \) to generate the \( \frac{n-\gamma}{n-1} \) bits at each node is counted in \( R_{\text{peak}} \) but not counted in \( R_{\text{avg}} \). At the optimal setting that minimizes \( R_{\text{peak}} \),
\[ \alpha \approx \frac{n - k}{n - 1}, \]
and thus \( R_{\text{peak}} = (n - k) \cdot R_{\text{peak}} \).

The Regenerating repairer [33] is much more advanced. When a node fails, at each of the \( n - 1 \) remaining nodes a selected subset of \( \frac{n-\gamma}{n-1} \) bits of the fragment is read from that node and sent to the repairer, which uses all received bits from the nodes to generate and store a fragment at the new node. This construction can use Reed-Solomon codes, and is efficient for small values of \( k \) and \( n \).

Each fragment is partitioned into sub-fragments for the Regenerating repairer [33], and the number of sub-fragments provably grows quickly as \( n - k \) grows or as the storage overhead \( \beta = \frac{n-k}{n} \) approaches zero. For example there are 1024 sub-fragments per fragment for \( k = 16 \) and \( n = 20 \), and the Regenerating repairer generates a fragment at a new node from receiving 256 non-consecutive sub-fragments from each of 19 nodes. Reading many non-consecutive sub-fragments directly from a node is sometimes efficient, in which case \( R_{\text{peak}} = R_{\text{avg}} \), but typically it is more efficient to read an entire fragment from a node and select the appropriate sub-fragments to send to the repairer, in which case \( R_{\text{peak}} = (n - k) \cdot R_{\text{peak}} \).

In contrast to the Regenerating repairers [14], [17], [33], the advanced liquid repairer of Theorem 6.2 directly reads unmodified data from nodes, for example using HTTP, and thus the upper bound on \( R_{\text{peak}} \) accounts for all data read from nodes.

When Regenerating repairers [14], [17], [33] are used to repair all objects in a system, \( \frac{R_{\text{peak}}}{E} \approx \frac{1}{\beta} \) with respect to any node failure sequence with a fixed timing sequence, which is around a factor of two above the lower bound on \( \frac{R_{\text{peak}}}{E} \) from the functional repairer analog of Theorem 7.1.

When Regenerating repairers [14], [17], [33] with fixed \( n \) and \( k \) are used to repair all objects in a system, \( \frac{R_{\text{peak}}}{E} \) necessarily grows as \( N \) grows in order to achieve a MTTDL exponential in \( N \) with respect to any random node failure distribution. (Compare to the bound on \( \frac{R_{\text{peak}}}{E} \) from Theorem 6.2.)

This is because there is a good chance that multiple node failures occur over a small interval of time with a Poisson timing distribution, implying that repair for a node failure must occur in a very short interval of time.

Thus, although the Regenerating repairers [14], [17], [33] optimally minimize \( R_{\text{peak}} \) with respect to the Regenerating framework for repairing individual objects using reactive repair, they do not provide optimal \( R_{\text{peak}} \) at the system level when used to store and repair objects, for fixed \( n \) and \( k \) as \( N \) grows.
B. Regenerating lower bounds

Regenerating lower bounds on the functional read repair rate prove necessary conditions on the Regenerating framework parameters to ensure than an individual object remains recoverable when using reactive repair. The bounds are based on a specially constructed acyclic graph, which corresponds to a specially constructed node failure sequence, and does not show for example a lower bound for uniformly chosen node failures. Also, the lower bounds are not extendable to non-trivial timing sequences, e.g., a Poisson timing distribution.

One could consider applying the Regenerating framework at the system level across all objects, e.g., \( n = N \), and, for MSR settings, \( k = K \) and \( \alpha = \frac{\alpha}{k} \) for source data of size \( m \). The following two examples illustrate how the Regenerating framework so applied isn’t suitably expressive, and thus the Regenerating lower bounds do not provide system level lower bounds.

The Regenerating framework requirement would be that all source data is recoverable from any \( K \) of the \( N \) nodes. However, as described in Section I, small code systems partition source data into objects, and fragments for objects are distributed equally to all \( N \) nodes, and thus small code systems read data from almost all \( N \) nodes to recover all source data, which violates this requirement.

The Regenerating framework requirement would be that data is only written to a node when it is added: Writing data incrementally to a node over time as nodes fail is not expressible. Liquid systems write data incrementally to a node over a large number of node failures after the node is added, which violates this requirement.

XIV. Future work

There are many ways to extend this research, accounting for practical issues in storage system deployments.

Failures in deployed systems can happen at a variable rate that is not known a priori. For example, a new batch of nodes introduced into a deployment may have failure rates that are dramatically different than previous batches.

Both time and spatial failure correlation is common in deployed systems. Failures in different parts of the system are not completely independent, e.g., racks of nodes fail concurrently, entire data centers go offline, power and cooling units fail, node outages occur due to rolling system maintenance and software updates, etc. All of these events introduce complicated correlations between failures of the different components of the system.

Intermittent node failures are common in deployed systems, accounting for a vast majority (e.g., 90%) of node failures. In the case of an intermittent node failure, the data stored at the node is lost for the duration of the failure, but after some period of time the data stored on the node is available again once the node recovers (the period of time can be variable, e.g., ranging from a few seconds to days). Intermittent failures can also affect entire data centers, a rack of nodes, etc.

Repairing fragments temporarily unavailable due to transient node failures wastes network resources. Thus, a timer is typically set to trigger a fixed amount of time after a node fails (e.g., 15 minutes), and the node is declared permanently failed and scheduled for repair if it has not recovered within the trigger time. Setting the trigger time can be tricky for a small code system; a short trigger time can lead to unnecessary repair, whereas a long trigger time can reduce reliability.

Data can silently be corrupted or lost without any notification to the repairer; the only mechanism by which a repairer may become aware of such corruption or loss of data is by attempting to read the data, i.e., data scrubbing. (The data is typically stored with strong checksums, so that the corruption or loss of data becomes evident to the repairer when an attempt to read the data is made.) For example, the talk [32] reports that read traffic due to scrubbing can be greater than all other read data traffic combined.

There can be a delay between when a node permanently fails and when a replacement node is added. For example, in many cases adding nodes is performed by robots, or by manual intervention, and nodes are added in batches instead of individually.

It is important in many systems to distribute the repair evenly throughout the nodes and the network, instead of having a centralized repairer. This is important to avoid CPU and network hotspots. Distributed versions of the algorithms described in Sections X-A, X-B and X-D distribute the repair traffic smoothly among all nodes of the system. Based on this, it can be seen that distributed versions of the lower bounds and upper bounds asymptotically converge as the storage overhead approaches zero.

Network topology is an important consideration in deployments, for example when objects are geo-distributed to multiple data centers. In these deployments, the available network bandwidth between different nodes may vary dramatically, e.g., there may be abundant bandwidth available between nodes within the same data center, but limited bandwidth available between nodes in different data centers. The paper [31] addresses these issues, and the papers [20], [22] introduce some erasure codes that may be used in solutions to these issues. An example of such a deployment is described in [26].

Enhancing the distributed storage model by incorporating the elements described above into the model and providing an analysis can be of value in understanding fundamental tradeoffs for practical systems.

XV. Conclusions

We introduce a mathematical model of distributed storage that captures some of the relevant features of practical distributed storage systems. Shannon in [11] introduced a model of communication and provided asymptotically matching upper and lower bounds on the rate at which information can be communicated. Shannon’s mathematical theory of communication has been of great importance in the understanding and design of practical communication systems. Our hope is that the model of distributed storage and the asymptotically matching upper and lower bounds on the tradeoffs between storage overhead and read repair rate described herein will be found to have importance in the understanding and design of practical distributed storage systems.
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REFERENCES

[1] C. Shannon. A Mathematical Theory of Communication. The Bell System Technical Journal, Vol. 27, pp. 279-423, pp. 623-656, July, October 1948.

[2] J. Bloemer, M. Kalfane, M. Kapiniski, R. Karp, M. Luby, D. Zuckerman. An XOR-Based Error-Resilient Coding Scheme. ICSI Technical Report, No. TR-95-048, August 1995.

[3] L. Rizzo. Effective erasure codes for reliable computer communication protocols. ACM SIGCOMM computer communication review, Vol. 27, No. 2, pp. 24-36, April 1997.

[4] A. Shokrollahi, M. Luby. Raptor Codes. Foundations and Trends in Communications and Information Theory, 2011, Vol. 6 No. 3-4, pp 213-322.

[5] J. Lakan, V. Roca, J. P incidental erasure repairs and small sub-packetization. IETF RFC5510. Reed-Solomon Forward Error Correction (FEC) Schemes. Internet Engineering Task Force, April 2009.

[6] M. Luby, A. Shokrollahi, M. Watson, T. Stockhammer, L. Minder. IETF RFC6330. RaptorQ Forward Error Correction Scheme for Object Delivery. Internet Engineering Task Force, August 2011.

[7] Y. Chen, J. Edler, A. Goldberg, A. Gottlieb, S. Sobi, P. Yianilos. A prototype implementation of archival Intermemory. Proceedings of the fourth ACM conference on Digital libraries, pp. 28-37, 1999.

[8] J. Kubiatowicz, D. Bindel, Y. Chen, S. Czervinski, P. Eaton, D. Geels, R. Gummadi, S. Rhea, H. Weatherspoon, W. Weimer, C. Wells, B. Zhao. OceanStore: an architecture for global-scale persistent storage. ACM SIGPLAN Notices, Vol. 35, No. 11, pp. 190-201, Nov. 2000.

[9] H. Weatherspoon, H. Kubiatowicz. Erasure coding vs. replication: A quantitative comparison. Peer-to-Peer Systems, pp. 328-337, Springer Berlin Heidelberg, 2002.

[10] A. Haebeler, A. Mislove, P. Druschel. Glacier: highly durable, decentralized storage despite massive correlated failures. NSDI ’05 Proceedings of the 2nd conference on Symposium on Networked Systems Design and Implementation, Vol. 2, pp. 143-158, 2005.

[11] A. Dimakis, P. Prabhakaran, K. Ramchandran. Distributed Fountain Codes for Networked Storage. Proceedings of 2006 IEEE International Conference on Acoustics, Speech and Signal Processing, ICASSP, Vol. 5, pp. 143-158, 2005.

[12] E. Sit, A. Haebeler, F. Babek, B. Chun, H. Weatherspoon, R. Morris, M. Kaashoek, J. Kubiatowicz. Proactive replication for data durability. Proceedings of the 5th Intl Workshop on Peer-to-Peer Systems, 2006.

[13] B. Chun, F. Babek, A. Haebeler, E. Sit, H. Weatherspoon, M. Kaashoek, J. Kubiatowicz, R. Morris. Efficient replica maintenance for distributed storage systems. Proceedings of the 3rd conference on Networked Systems Design and Implementation, USENIX Association, Berkeley, CA, Vol. 3, 2006.

[14] A. Dimakis, P. Godfrey, Y. Wu, M. Wainwright, K. Ramchandran. Network coding for distributed storage systems. IEEE Infocom, May 2007.

[15] S. Aly, Z. Kong, E. Soljanin. Raptor Codes Based Distributed Storage Algorithms for Wireless Sensor Networks. IEEE ISIT 2008, March 2009.

[16] A. Wildani, T. Schwarz, E. L. Miller, D. Long. Protecting Against Rare Event Failures in Archival Systems. Proceedings of the 17th IEEE International Symposium on Modeling, Analysis, and Simulation of Computer and Telecommunication Systems (MASCOTS 2009), September 2009.

[17] A. Dimakis, P. Godfrey, Y. Wu, M. Wainwright, K. Ramchandran. Network coding for distributed storage systems. IEEE Transactions on Information Theory, Vol. 56, No. 9, pp. 4539-4551, September 2010.

[18] M. Asteris, A. Dimakis. Repairable Fountain Codes. 2012 IEEE International Symposium on Information Theory Proceedings (ISIT), pp. 1752-1756, July 2012.

[19] D. Ford, F. Labelle, F. Popovic, M. Stokely, V. Truong, L. Barroso, C. Grimes, and S. Quinan. Availability in globally distributed storage systems. USENIX Symposium on Operating Systems Designs and Implementation, Oct. 2010.

[20] P. Gopalan, C. Huang, H. Simiciti, S. Yekhanin. On the Locality of Codeword Symbols. IEEE Trans. Inf. Theory, vol. 58, No. 11, pp. 6925-6934. Nov. 2012.

[21] K. Rashmi, N. Shah, D. Gu, H. Kuang, D. Borthakur, K. Ramchandran. A Solution to the Network Challenges of Data Recovery in Erasure-coded Distributed Storage Systems: A Study on the Facebook Warehouse Cluster. 5th USENIX Workshop on Hot Topics in Storage and File Systems, June 2013.

[22] C. Huang, H. Simiciti, Y. Xu, A. Ogus, B. Calder, P. Gopalan, J. Li, and S. Yekhanin. Erasure Coding in Windows Azure System. USENIX Annual Tech. Conference, Boston, MA, 2012.

[23] O. Khan, R. Burns, J. Plank, W. Pierce, and C. Huang. Rethinking erasure codes for cloud file systems: minimizing I/O for recovery and degraded reads. In Proceedings of the 10th USENIX conference on File and Storage Technologies (FAST’12), Berkeley, CA, 2012.

[24] M. Sathiamoorty, M. Asteris, D. Papailiopoulos, A. Dimakis, R. Vadali, S. Chen, and D. Borthakur. XORing Elephants: Novel Erasure Codes for Big Data. Proceedings of the VLDB Endowment, Vol. 6, No. 5, 2013.

[25] M. Silverstein, L. Ganesh, Y. Wang, L. Alvisi, M. Dahlin. Lazy Means Smart: Reducing Repair Bandwidth Costs in Erasure-coded Distributed Storage. Proceedings of International Conference on Systems and Storage, pp. 1-7, 2014.

[26] S. Muralidhar, W. Lloyd, S. Roy, C. Hill, E. Lin, W. Liu, S. Pan, S. Shankar, V. Sivakumar, L. Tang, S. Kumar. 14: Facebook’s warm BLOB storage system. 17th USENIX conference on Operating Systems Design and Implementation, pp. 383-398, 2014.

[27] T. Okpote, S. Yousefi. Locality-aware fountain codes for massive distributed storage systems. 2015 IEEE 14th Canadian Workshop on Information Theory (CWIT), July 2015.

[28] A. Dimakis. Online wiki bibliography for distributed storage papers. http://storagewiki.ece.utexas.edu.

[29] A. Koomey. August/2012/docs.ceph.com/docs/master/radost/configuration/pool-pg-config-ref/

[30] M. Luby, R. Padovani, T. Richardson, L. Minder, P. Aggarwal. Liquid cloud storage: large and lazy works best. unpublished, 2016.

[31] P. Gopalan, G. Hu, S. Saraf, C. Wang, S. Yekhanin. Maximally Recoverable Codes for Grid-like Topologies arXiv:1605.05412v1, cs.IT, May 18, 2016.

[32] J. Cowling. Dropbox’s Exabyte Storage System. https://code.facebook.com/posts/253562281667886/data-scale-june-2016-recap/

[33] B. Sasidharan, M. Vajha, V. Kumar. An explicit, coupled-layer construction of a high-rate MSR code with low sub-packetization level, small field size and all-node repair. arXiv:1607.07335v1 [cs.IT], July 25, 2016.

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