Overview. Enriques surfaces are a classical object of algebraic geometry. In this talk we will mostly concentrate on varieties over \(\mathbb{C}\). Then an Enriques surface \(Y\) is defined as quotient of a K3 surface \(X\) by a fixed point free involution \(\tau\):

\[
Y = X/\tau.
\]

Conversely the K3 surface \(X\) can be recovered as the universal covering of \(Y\):

\[
\pi : X \to Y.
\]

This covering opens the way for lattice theory as we can pull back the Néron-Severi group on \(Y\) to obtain a primitive embedding

\[
U(2) + E_8(-2) \hookrightarrow \text{NS}(X).
\]

(1)

Conversely, such an embedding implies that \(X\) has an Enriques involution unless there are \((-2)\)-vectors in the orthogonal complement in \(\text{NS}(X)\). Thus we obtain a precise abstract description of Enriques surfaces through K3 surfaces with the lattice polarisation (1), giving a ten-dimensional moduli space with a hypersurface removed. However, this method does not give us a good control of explicit geometric constructions as soon as the Picard number of \(X\) is increased. On the other hand, there are many classical geometric constructions, the first dating back to Enriques, where we cannot control the moduli very well. Here we propose a novel construction that aims to balance the geometric and moduli theoretic aspects, see [3].

Set-up. We start with a rational elliptic surface \(S \to \mathbb{P}^1\) with section. Consider a quadratic map \(f : \mathbb{P}^1 \to \mathbb{P}^1\). For shortness we only study the generic situation in order to avoid special cases. Then the pull-back of \(S\) by \(f\) leads to a K3 surface \(X\). On \(X\) we have the following involutions: the deck transformation \(\iota\) and the hyperelliptic involution \(-id\) defined fiberwise. Their composition \(j = -id \circ \iota\) is a Nikulin involution, and the quotient \(X/j\) has a K3 surface \(X'\) as minimal desingularisation. \(X'\) is actually the quadratic twist of \(S\) with respect to the base change \(f\), having \(I_0\) fibers at the two ramification points. We thus have a diagram

\[
\begin{array}{ccc}
X & \xleftarrow{\iota} & \mathbb{P}^1 \\
S & \xrightarrow{f} & X' \\
\mathbb{P}^1 & \xrightarrow{f} & \mathbb{P}^1 \\
\end{array}
\]

Generically \(X\) does not admit an Enriques involution as \(\text{NS}(X) = U + E_8(-2)\). To overcome this, we impose a section \(P'\) on \(X'\). Pulling back to \(X\), we obtain a \(\iota^\ast\)-anti-invariant section \(P\) on \(X\). In consequence, translation by \(P\) composed with \(\iota\) defines an involution \(\tau\) on \(X\). This involution is fixed point free if \(P\) does not
specialise to the zero section $O$ on the two fixed fibers of $\iota$. For instance, this holds if $P$ is two-torsion. Lattice theoretically we can encode this information on the quotient $X'$: the section $P'$ has to meet both $I_0^*$ fibers at other components than $O'$. That is, the decomposition $\text{NS}(X') = U + L$ induced by the elliptic fibration does not admit a direct summand $D_4$ inside the negative-definite lattice $L$. In summary, we can encode the Enriques involution $\tau$ on $X$ in a lattice polarisation on $X'$ while retaining a good control about the geometry. Note that our geometric construction is neither limited to K3 surfaces nor to base fields of characteristic zero.

**Applications.** We illustrate the above construction by discussing some applications.

*Nodal Enriques surfaces.* An Enriques surface $Y$ is called nodal if it contains a $(-2)$-curve. This curve features as a bisection for an elliptic fibration on $Y$. On the K3 cover $X$, this bisection splits into two disjoint sections $O, P$ as above (cf. [5]). Through the above construction, we can work out the induced elliptic fibration on $X$ explicitly [3, Lemma 3.2]. Immediately, we deduce the unirationality of the moduli space of nodal Enriques surfaces.

*Shioda–Inose structures.* The construction applies to the framework of Shioda–Inose structures. Here $X'$ is a Kummer surface of product type, and isogenies of the elliptic curves give rise to sections $P'$ as above. In particular, we obtain Enriques involutions on singular K3 surfaces and study the arithmetic of the quotient surfaces such as fields of definition and Galois actions on divisors [4].

*Brauer groups.* An Enriques surface has $\text{Br}(Y) = \mathbb{Z}/2\mathbb{Z}$. Beauville showed that generically $\pi^* \text{Br}(Y) = \mathbb{Z}/2\mathbb{Z}$ and gave lattice theoretic conditions on $\text{NS}(X)$ for the pull-back of $\text{Br}(Y)$ to vanish. The base change construction enables us to produce explicit examples over $\mathbb{Q}$ or number fields [3] or in interesting families [2].

*Enriques Calabi-Yau threefolds.* Starting from a K3 surface $X$ with an Enriques involution $\tau$, one can construct Calabi-Yau threefolds by pairing $X$ with an elliptic curve $E$ and dividing the product $X \times E$ by the fixed point free involution $\tau \times (-id)$. With our methods we can derive interesting Calabi-Yau threefolds by endowing $X$ with specific structures such as big Picard number, aiming for instance at arithmetic properties (e.g. modularity) or Picard-Fuchs equations.

**References**

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