Erratic Extremism causes Dynamic Consensus
(a new model for one-dimensional opinion dynamics)

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Abstract

A society of agents, with ideological positions, or “opinions” measured by real values ranging from $-\infty$ (the “far left”) to $+\infty$ (the “far right”), is considered. At fixed (unit) time intervals agents repeatedly reconsider and change their opinions if and only if they find themselves at the extremes of the range of ideological positions held by members of the society. Extremist agents are erratic: they become either more radical, and move away from the positions of other agents, with probability $\varepsilon$, or more moderate, and move towards the positions held by peers, with probability $(1 - \varepsilon)$. The change in the opinion of the extremists is one unit on the real line. We prove that the agent positions cluster in time, with all non-extremist agents located within a unit interval. However, the consensus opinion is dynamic. Due to the extremists’ erratic behavior the clustered opinion set performs a “sluggish” random walk on the entire range of possible ideological positions (the real line). The inertia of the group, the reluctance of the society’s agents to change their consensus opinion, increases with the size of the group. The extremists perform biased random walk excursions to the right and left and, in time, their actions succeed to move the society of agents in random directions. The “far left” agent effectively pushes the group consensus toward the right, while the “far right” agent counter-balances the push and causes the consensus to move toward the left.

We believe that this model, and some of its variations, has the potential to explain the real world swings in societal ideologies that we see around us.
1 Introduction

Over the years, social psychologists proposed numerous explanations for the complex behavior emerging in large groups of supposedly intelligent agents, like tribes and nations. They proposed models and principles of individual behavior and some of these models were even amenable to mathematical analysis enabling predictions about long-term behavior and the inevitable emergence of surprising global economic or political phenomena.

The ideas of balance theory [Cartwright and Harary 1956] and social dissonance [Festinger 1962] led to the consideration of several basic mathematical models, attempting to incorporate the idea that individuals, or agents attempt to reach an equilibrium between their drives, opinions and “local comfort” and those in their neighborhood. They do so by adjusting their position (ideological, political, economic, or spatial) to be similar, or comfortably near the position of their neighbors.

Simplified mathematical models for multi-agent interaction consider a group, colony, society or swarm of agents, each agent associated with a quantity which can be a real number, or a vector, describing the “state”, opinion or position of the agent. The state of the whole group (at time $t$) is specified by the vector
\[ X(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T, \]
where $x_k(t)$ is the state of agent $k$ at time $t$, and the group comprises $N$ agents.

Then, models postulate that, from some initialization $X(0)$ at time $t = 0$, the state of the system evolves, and, if we consider that changes happen at equal intervals (arbitrarily set to one), we obtain general discrete time evolution models of the following form
\[
\begin{align*}
X(t+1) &= \Psi(X(t)) \\
X(0) &= \text{initial condition}
\end{align*}
\]

Here $\Psi$ describes the way each agent $k$, determines its state at time $(t+1)$ given the states of all agents at time $t$.

The inter-agent interaction function $\Psi$ is designed to reflect the assumed influence of agents on their peers. [DeGroot 1974] postulated that $\Psi$ should be a fixed matrix $A$ acting on $X$ with columns displaying the influence each agent has on every other agent. Rows of the matrix then display how the next state of agent $k$ at time $(t+1)$ will be computed as a weighted combination of the states of all agents at time $t$. If $A$ is constant (and independent of the state at all times) the vector $X$ has a linear evolution, with dynamics completely determined only by the eigen-structure of $A$ and the initial state.

When positive entries and convex combination of states are postulated, $A$ is a stochastic matrix, and then one readily has, under quite general conditions, that the system asymptotically achieves consensus, i.e. as $t \to \infty$ we have that all $x_k(t)$’s will evolve to have the same value.

This model is highly appealing, however it assumes that each agent always adjusts its state according to a fixed convex combination of its own state and all other states. Since real individuals in any group are well known to possess a certain reluctance in considering far-away positions of others, and tend to stick to their initial opinions, models that took such tendencies into consideration...
soon emerged. The very popular Hegselmann-Krause (HK) model [Hegselmann
and Krause 2002] postulates that

\[ x_k(t + 1) = \frac{1}{|N_k|} \sum_{l \in N_k} x_l(t), \]

where the index set \( N_k \) is defined as \( \{ l : \|x_k(t) - x_l(t)\| < \varepsilon_k \} \), i.e. \( N_k \) defines an \( \varepsilon_k \)-neighborhood of the \( k \)-th agent position, \( x_k(t) \), at time \( t \), and \( |N_k| \) is the size of
the set \( N_k \).

This model leads, in general, to clusters of agents in local consensus at
different state values/positions, a phenomenon often observed in society. Several
variations based on this model were put forth in the literature and a lot of
research is still devoted to study their convergence and properties.

Another interesting variation of the DeGroot model was proposed in [Fried-
kin and Johnsen 1990]. This model assumes that each agent \( k \) remains faithful
to its initial position to a certain degree \( g_k \), \( 0 \leq g_k \leq 1 \) and has a susceptibility
of \( 1 - g_k \) to be socially influenced by the other agents. The classical linear model
then becomes, in a matrix notation:

\[ X(t + 1) = G X(0) + (I - G) A X(t), \]

Here \( G \) is a diagonal matrix with \( g_k \)-s on the main diagonal, and \( I \) is the
identity matrix. This model leads to a spread of steady state positions that can
be predicted by a simple matrix inversion.

Following the footsteps of [DeGroot 1974], [Friedkin and Johnsen 1990] and
Hegselmann and Krause 2002, a considerable number of interesting “opinion
dynamics”, “multi-agent” and “consensus”/“gathering” models have been pro-
posed. Over the years the research in the field split into several branches. Today
researchers of Autonomous Swarms and Swarm Intelligence invent local inter-
action models to achieve “gathering”, “geometric consensus”, “collective area
sweeps” and “cooperative search and pursuit” with simple autonomous mobile
agents (see e.g. [Reynolds 1987], [Dudek et al. 1993], [Vicsek et al. 1995],
Ando et al. 1999, Camazine et al. 2001, Jadbabaie et al. 2003, Olfati-
Saber and Murray 2004, Sahin 2004). An overview of this field is provided in
Barel et al. 2016. Computer Scientists are interested in agreement and com-
mon knowledge in distributed computer networks ([Halpern and Moses 1990],
Shoham and Tennenholtz 1995, Flocchini et al. 2012), while communication
engineers consider distributed coordination and collaboration in large, ad-hoc
networks of “cellular-phone” agents ([Krishna et al. 1997], [Chen et al. 2002],
Chong and Kumar 2003). Biologists analyze and try to understand and model
colonies of ants, flocks of starlings, schools of fish and swarms of locusts ([Okubo
1986], Camazine et al. 2001, Couzin and Krause 2003, Sumpter 2006).

Social science researchers continue to be interested in simulating and an-
alyzing human agent interactions, voting patterns and social opinion dynam-
ics. A recent survey by [Lorenz 2017] nicely presents the advancements and
clearly describes some of the issues of interest in the field. Stochastic models,
explicitly dealing with random behavior of agents with parameters probabilis-
tically characterizing their open-mindedness (the agent’s probability of chang-
ing/reconsidering opinions), are currently being investigated. In his concluding
remarks, Lorentz states
"Agent-based models for the evolution of ideological landscapes are still in infancy and it remains to show if they can add interesting insight to political dynamics."

In this vein, we here propose a new, probabilistic, opinion dynamics model, in part based on some early ideas of [Festinger 1954]. He introduced a qualitative social psychology theory, supported by a vast corpus of data collected. The theory suggests that the majority of agents hold neutral opinion on subjects at hand. This majority is rather unmoved by extreme opinions, while the “extremists” are unstable and tend to fluctuate, moving most probably in the direction of a social norm.

We model opinions or ideological positions as real numbers and allow only extreme agents to change opinions at discrete times by a constant quantum value arbitrarily set to one in any direction. Changes in the positions of the “extremists” in the direction of the “social norm”, (represented by all agents except the two “extremists”), are assumed to be highly probable. In the opposite direction the erratic “extremists” may move, but with smaller probabilities. We show that for any initial spread of agent opinions, a consensus opinion arises. The “core” group in consensus spreads over an interval of size smaller than the quantum change in the opinion of the extreme agents. The “core” is not stationary and, over time, moves at random. In the society of agents “extremist” is not a sticky label. From time to time one of the “extremists” becomes a part of the “core” of normal agents; a previously “normative” moderate agent finds itself to be at one of the extremes. It is these role-changes between “extremists” and “moderates” that moves the “core” over time.

This paper is organized as follows. Section 2 presents the mathematical model of opinion dynamics and states our main results. Section 3 reviews and proves some basic facts about biased random walks. Section 4 analyzes the gathering process by first considering a unilateral case in which we assume that only one extremal agent is active, then a decoupling trick enables us to use the unilateral results for the analysis of the problem when both extremal agents are in action. Section 5 presents extensive simulation results confirming the theoretical predictions and showing that our bounds are quite loose due to the need to decouple the action of the extremal agents in order to enable the theoretical results. The final Section 6 discusses possible interesting extensions of the model presented along with some initial simulation results in two dimensions.
2 Model Description

Suppose a set of point agents, the individuals in the society, called $p_1, p_2, \ldots, p_N$ are, at the beginning of time, i.e. at $t = 0$, on the real line (the range of positions or opinions) at locations $x_1(0), x_2(0), \ldots, x_N(0) \in \mathbb{R}$. The agents are identical and indistinguishable points and perform the following algorithm:

| Agent decision rule ($\varepsilon \in [0, \frac{1}{2}]$): |
|---|
| 1 For $p_k$ located at $x_k(t)$ at discrete time $t$ define intervals $pR \triangleq (x_k(t), \infty)$ and $pL \triangleq (-\infty, x_k(t))$. |
| 2 if in both intervals $pL$ and $pR$ there are other agents then |
| $x_k(t+1) = x_k(t)$, i.e. $p_k$ stays put. |
| 4 else |
| 5 if $pR$ is empty then |
| $p_k$ makes a probabilistic jump, setting |
| $x_k(t+1) = \begin{cases} x_k(t) + 1, & \text{w.p. } \varepsilon \\ x_k(t) - 1, & \text{w.p. } (1 - \varepsilon) \end{cases}$ |
| 8 if $pL$ is empty then |
| $p_k$ makes a probabilistic jump, setting |
| $x_k(t+1) = \begin{cases} x_k(t) + 1, & \text{w.p. } (1 - \varepsilon) \\ x_k(t) - 1, & \text{w.p. } \varepsilon \end{cases}$ |
| end |

Under the rule defined above only the two agents with extremal positions $x_{\min}(t)$ and $x_{\max}(t)$ will move, and their tendency will be to approach the agents in between. After each jump, carried out at discrete integer times, we rename the identical agents to have them always indexed in the increasing order of their $x$-locations. Hence at all discrete time instances $t = 1, 2, \ldots$ we have the ordered agents $\{p_1, p_2, \ldots, p_N\}$ with $x_1(t) \leq x_2(t) \leq \ldots \leq x_N(t)$, where $p_1$ and $p_N$ are extremists and probabilistic jumps will be carried out by extremists only (see Figure 1).

\[
\begin{align*}
\varepsilon (1 - \varepsilon) & \quad (1 - \varepsilon) \varepsilon \\
-1 & +1 \\
\text{L = p}_1 p_2 \ldots p_i p_{i+1} \ldots p_{n-1} p_N = \text{R} \\
x_1(t) & \quad x_i(t) = [x_i] + \{x_i\} \\
& \quad x_N(t)
\end{align*}
\]

Figure 1: N agents on the line

The process defined above evolves the constellation of points in time and we clearly expect that a gathering of the agents will occur, since extremal agents are probabilistically “attracted” toward their peers.

Indeed, if $\varepsilon$ would be exactly zero, the deterministic jumps carried out by the extremal agents $p_1 \equiv p_L$ and $p_N \equiv p_R$ would always be toward the interior of the interval $(x_1, x_N)$, shortening it while $(x_N(t) - x_1(t)) > 1$. However, note
that when the \([x_1(t), x_N(t)]\) interval reaches a value of 1 or less, interesting things start to happen, since \(p_1\) and \(p_N\) while jumping cross each other, in such a way that the spread about the (time invariant, in this case) centroid of points may increase and decrease in a way that depends on the specific spread of the initial point locations’ fractional parts. We therefore expect similar things to happen when randomness is introduced as \(\varepsilon\) of the initial point locations’ points may increase and decrease in a way that depends on the specific spread about the \((0)\) centroid of \([x_1(t), x_N(t)]\) interval.

If \(\varepsilon = 0\) we have the constellation at time \(t\), \(\{p_1, p_2, \ldots, p_N\}\), described by the ordered set of point locations \(x_1(t) < x_2(t) < \ldots < x_N(t)\) and their centroid and variance behave as follows: for the centroid arbitrarily chosen to be \(0\) at time \(0\) we have

\[
C(t + 1) \triangleq \frac{1}{N} \sum_{i=1}^{N} x_i(t + 1) = \frac{1}{N} \sum_{i=2}^{N-1} x_i(t) + \frac{1}{N} (x_1(t) + 1 + x_N(t) - 1) = C(t),
\]

and \(C(t + 1) = C(t) = \ldots C(0) \triangleq 0\), hence the centroid is an evolution invariant.

The variance of the constellation about the centroid at \(0\) is \(\sigma^2(t) = \frac{1}{N} \sum_{i=1}^{N} x_i^2(t)\), therefore

\[
\sigma^2(t + 1) = \frac{1}{N} \sum_{i=2}^{N-1} x_i^2(t) + \frac{1}{N} \left( x_1^2(t) + 2x_1(t) + 1 + x_N^2(t) - 2x_N(t) + 1 \right)
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} x_i^2(t) - 2 \frac{1}{N} \left[ (x_N(t) - x_1(t)) - 1 \right]
\]

\[
= \sigma^2(t) - \frac{2}{N} \left[ (x_N(t) - x_1(t)) - 1 \right]
\]

While \((x_N - x_1) > 1\) the variance monotonically decreases, however when \((x_N - x_1) \leq 1\) we have \(\sigma^2(t + 1) > \sigma^2(t)\). Hence after gathering, or reaching consensus (i.e. when \(|x_N - x_1| \leq 1\), oscillations in \(\sigma^2(t)\) subsequently occur, but the constellation remains gathered around \(0\).

For the probabilistic case we expect a somewhat similar behavior. We shall see that a “dynamic” consensus is reached. Agents on a line behaving according to the probabilistic rule discussed above evolve to a dynamic constellation that is “gathered” and the group of agents move on the line as follows:

1) For a given \(\varepsilon\), \(0 < \varepsilon < 1/2\), we have

\[
C(t + 1) = C(t) + \begin{cases} 
\frac{2}{N}, & \text{with probability } \varepsilon(1 - \varepsilon) \\
0, & \text{with probability } 1 - 2\varepsilon(1 - \varepsilon) \\
-\frac{2}{N}, & \text{with probability } \varepsilon(1 - \varepsilon)
\end{cases}
\]

2) The “core” group of moderate agents, i.e. \(\{p_2, p_3, \ldots, p_{N-1}\}\) eventually gathers to reside within a “dynamic” interval of length less than one.

3) The extremal agents \(p_1\) and \(p_N\) perform random excursions to the left and right of the core group, with motion biased towards the core. Their bias ensures that they will be mostly near the core, the total distance between them being a sum of random variables, one always less than one and two others bounded by positive random variables with a geometric distribution.
3 Some Basic Facts About Random Walk

In order to analyze the gathering process due to the random behavior of the extremal points \( p_L \triangleq p_1 \) and \( p_R \triangleq p_N \) in case \( \varepsilon > 0 \) we need to first recall some basic facts about random walks on the line. Suppose an agent performs a (biased) random walk from an initial location (denoted by \( x(0) = 0 \)) on the real line, making, at discrete time instants \( t = 0, 1, 2, \ldots \) moves to the left with probability \((1 - \varepsilon)\) and to the right with probability \(\varepsilon\). If \( \varepsilon = 1/2 \) the walk is the unbiased, symmetric random walk, while \(\varepsilon < 1/2\) biases the motion of the agent towards the left. Let us define \( \alpha \) as the positive departure of \((\varepsilon)\) and \((1 - \varepsilon)\) from \(1/2\), i.e.

\[ \varepsilon = \frac{1}{2} - \alpha \Leftrightarrow 1 - \varepsilon = \frac{1}{2} + \alpha. \]

Clearly, \( \alpha \in (0, 1/2) \), since we assume \( 0 < \varepsilon < \frac{1}{2} \). In this notation \( \alpha \) quantifies the bias towards left of the agents’ motion and we have the following results.

3.1 The probability of reaching \((-1)\) from \(0\).

The probability that the agent hits \((-1)\) is given by the following expression:

\[
P(\text{walk hits } (-1)) = \sum_{k=0}^{\infty} P(\text{walk hits } (-1) \text{ at } (2k+1) \text{ for the first time})
\]

\[
= \sum_{k=0}^{\infty} \left( \text{step to the left after making } k \text{ steps to the right and } k \text{ steps to the left in any order, i.e. returning to 0, without having been at } (-1) \right)
\]

\[
= \sum_{k=0}^{\infty} \left( \frac{1}{2} + \alpha \right) \cdot C_k \left( \frac{1}{2} - \alpha \right)^k \left( \frac{1}{2} + \alpha \right)^k
\]

Here \( C_k \) counts the number of possible paths of length \( k \) from 0 to 0 never reaching \((-1)\), which is given by the \( k \)th Catalan number.

It is well known \cite{Stanley2015}, \cite{HiltonPedersen1991} that, the generating function of the series \( \{C_k\} \) is given by:

\[
\sum_{k=0}^{\infty} C_k x^k = \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} x^k = \frac{1 - \sqrt{1 - 4x}}{2x}
\]

Hence we have, for \( \alpha > 0 \),

\[
P(\text{walk hits } (-1)) = \left( \frac{1}{2} + \alpha \right) \frac{1 - \sqrt{1 - 4(\frac{1}{2} - \alpha)^2}}{2(\frac{1}{2} + \alpha)(\frac{1}{2} + \alpha)} = \left( \frac{1}{2} + \alpha \right) \left( \frac{1}{2} - 2\alpha \right) = 1
\]

This is totally expected : a left biased random walk will almost surely (i.e. with probability 1) reach \((-1)\), when started at 0.
3.2 The probability of reaching (+1) from 0.

We have, similarly:

\[ P(\text{walk hits } (+1)) = \sum_{k=0}^{\infty} P(\text{walk hits } (+1) \text{ at step } 2k+1 \text{ for the first time}) \]

\[ = \sum_{k=0}^{\infty} P(\text{last step to the right after making } k \text{ steps to the left and } k \text{ steps to the right (i.e. returning to 0) without having been at } (+1)) \]

\[ = \sum_{k=0}^{\infty} \left( \frac{1}{2} - \alpha \right) \cdot C_k \left( \frac{1}{2} + \alpha \right)^k \left( \frac{1}{2} - \alpha \right)^k \]

\[ = \left( \frac{1}{2} - \alpha \right) \frac{1-2\alpha}{(1-\alpha)(1+2\alpha)} = \frac{1-2\alpha}{1+2\alpha} < 1 \]

Hence, while the walk almost surely reaches (−1), there is a non-zero probability, given by \( 1 - \frac{1-2\alpha}{1+2\alpha} = \frac{1-2\epsilon}{1-\epsilon} \) of never reaching (+1).

3.3 The expected number of steps to first reach (-1).

Using the generating function for \( \{C_k\} \) we can readily calculate the expected number of steps to reach (−1) from 0. Hence we have the following, (quite well known) result:

\[ E(\text{steps to first hit } (-1)) = \sum_{k=0}^{\infty} (2k+1) \cdot P(\text{walk hits } (-1) \text{ at step } 2k+1) \]

\[ = \sum_{k=0}^{\infty} (2k+1) \left( \frac{1}{2} + \alpha \right) \cdot C_k \left( \frac{1}{2} - \alpha \right)^k \left( \frac{1}{2} + \alpha \right)^k \]

\[ = \left( \frac{1}{2} + \alpha \right) \sum_{k=0}^{\infty} (2k+1) \left( \frac{1}{2} - \alpha \right)^k C_k \]

To compute this value explicitly we use

\[ \sum_{k=0}^{\infty} k C_k x^k = \frac{d}{dx} \left( \sum_{k=0}^{\infty} C_k x^k \right) = \frac{d}{dx} \left( \frac{1 - \sqrt{1 - 4x}}{2x} \right) = \frac{1 - 2x - \sqrt{1 - 4x}}{2x^2 \sqrt{1 - 4x}} \]

hence we have,

\[ \sum_{k=0}^{\infty} k C_k x^k = \sum_{k=0}^{\infty} \frac{k}{k+1} \binom{2k}{k} x^k = \frac{1 - 2x - \sqrt{1 - 4x}}{2x\sqrt{1 - 4x}} \] \hspace{1cm} (2)

This yields, setting \( x \) to \( \left( \frac{1}{2} + \alpha \right) \left( \frac{1}{2} - \alpha \right) \), after some algebra,

\[ E(\text{steps to first hit } (-1)) = \frac{1}{2\alpha} = \frac{1}{1 - 2\epsilon} \] \hspace{1cm} (3)

Of course, the expected number of steps to reach (+1) is infinite, since there is a strictly positive probability given by \( \frac{1-2\epsilon}{1-\epsilon} \) of never getting there. But, we know for sure that the biased random walk (more likely moving to the left) will reach (−1) from 0 in the, above calculated, finite expected number of steps.
3.4 The expected farthest excursion to the right on the way from 0 to first reaching (-1).

Another result that we shall need in analyzing the evaluation of the agents’ behavior is the following result on the excursions that biased random walks make in the direction opposite to their preferred direction: the expected farthest excursion to the right on the way from 0 to first reaching \((-1)\) in the left biased random walk is bounded by

\[
E(\text{farthest right excursion}) \leq \sum_{k=0}^{\infty} k \cdot P\left(\text{walk makes } k \text{ right steps and } (k+1) \text{ left steps to first reach } -1\right)
\]

\[
= \sum_{k=0}^{\infty} kC_k \left(\frac{1}{2} + \alpha\right) \left(\frac{1}{2} - \alpha\right)^k \left(\frac{1}{2} + \alpha\right)^k
\]

The above inequality can be explained as follows: any excursion that starts at 0 and eventually ends in \(-1\) is necessarily of the odd length \(2k + 1\) for some \(k\). No matter what the actual order of steps is, the walk makes \(k\) steps to the right and \(k + 1\) steps to the left (with obvious limitations on the order of the steps). Therefore, the farthest to the right such an excursion could get is a distance \(k\) from 0. Hence, left-hand side of the inequality above is a clear upper bound.

Using the previously established relation \(\sum_{k=0}^{\infty} kC_k x^k = \frac{1-2x-\sqrt{1-4x}}{2x}\) we obtain

\[
E(\text{farthest right excursion}) \leq \sum_{k=0}^{\infty} kC_k \left(\frac{1}{2} + \alpha\right) \left(\frac{1}{2} - \alpha\right)^k \left(\frac{1}{2} + \alpha\right)^k
\]

\[
= \left(\frac{1}{2} + \alpha\right) \sum_{k=0}^{\infty} kC_k \left(\frac{1}{2} - \alpha\right)^k \left(\frac{1}{2} + \alpha\right)^k \quad (4)
\]

\[
= \left(\frac{1}{2} + \alpha\right) \frac{(\frac{1}{2} - \alpha)^2}{2\alpha(\frac{1}{4} - \alpha)(\frac{1}{2} + \alpha)} = \frac{\frac{1}{2} - \alpha}{2\alpha} = \frac{\epsilon}{1-2\epsilon}
\]
4 Analysis of the Dynamic Gathering Process

4.1 Unilateral Action Results

In order to analyze the gathering process, let us first consider a one sided version where only the rightmost agent moves at each moment and all other agents stay put. Furthermore assume that to the left of \( p_1(0) \) at time \( t = 0 \) we put a “beacon agent” \( p_0 \) at \( x_0(0) < x_1(0) \). The rightmost agent at times \( t = 1, 2, 3, \ldots \) makes a unit jump to the left with high probability \((1 - \varepsilon)\), or a jump to the right with probability \( \varepsilon \). Suppose the agents are initially located at: \( x_1(0), x_2(0), \ldots, x_{N-1}(0), x_N(0) \). Clearly the rightmost agent \( p_R \equiv p_N \) will first reach, with probability 1, \( x_{N-1}(0) - 1 \) in \( \frac{1}{1 - 2\varepsilon} \) expected number of steps, then from \( x_{N-1}(0) - 1 \) it will reach a.s. \( x_{N-1}(0) - 2 \) in further \( \frac{1}{1 - 2\varepsilon} \) expected steps etc. until, at some point it will jump over \( x_{N-1}(0) - 1 \) to land somewhere in the interval \( (x_{N-1}(0) - 1, x_{N-1}(0)) \), making the agent at \( x_{N-1}(0) - 1 \) the rightmost agent. This will happen with probability 1, after a number of steps, which we shall denote as \( T_{\text{jump}} \), having the expected value of \( \frac{1}{1 - 2\varepsilon} \left[ \left| x_N(0) - x_{N-1}(0) \right| + 1 \right] \) number of steps.

![Figure 2: The agent \( p_N \) will jump over \( p_{N-1} \) after an expected number of steps equal to \( \frac{1}{1 - 2\varepsilon} \left[ \left| x_N(0) - x_{N-1}(0) \right| + 1 \right] \).](image)

Now it will be the turn of the former \( p_{N-1} \) agent, which is now “renamed” \( p_N \equiv p_R \), to start its biased random walk and it will reach \( x_{N-1}(0) - 1 \) in \( \frac{1}{1 - 2\varepsilon} \) expected steps (clearly jumping over at least the “current” position of the “former” moving agent) to land in the interval \( (x_1(0) - 1, x_N(T_{\text{jump}})) \) defined by the “renamed” agents \( (p_1, p_2, \ldots, p_{N-1}) \). Clearly the new rightmost agent (which might be the former random walker or another agent located to the left of \( x_{N-1}(0) \) in the initial configuration) will do the same.

Recall that we assume, for simplicity, that agents’ initial locations have all distinct fractional parts, so that one agent will never land on top of another!

From the above description it is clear that the “erratic extremist” random walk of rightmost agents will eventually “sweep” all the agents towards the left, and in a finite expected number of steps equal to:

\[
E(T_{\{x_0(0), x_1(0), \ldots, x_N(0)\}}) = \frac{1}{1 - 2\varepsilon} \sum_{k=1}^{N} (|x_k(0) - x_0(0)| + 1),
\]

all agents will be to the right of the “beacon” \( p_0 \) after having jumped over \( x_0(0) \) exactly once, making the “beacon” \( p_0 \) the rightmost agent for the first time!

Indeed, note that while jumping one over the other (to the left) all the agents to the right of \( p_0 \) will have carried out (perhaps with interruptions due to reordering, following jumps over the agent called \( p_{N-1} \)) a biased random walk from their initial locations \( x_1(0), x_2(0), \ldots, x_N(0) \) until each one of them, for the first time, jumped over the fixed “beacon” point \( p_0 \) at \( x_0(0) \). Subsequently, the
agents will stop and wait for the “beacon” \( p_0 \) to become the rightmost agent. This will happen when the last of all the agents (that were \( p_0 \)’s initial right neighbors) completes its random walk by jumping over \( p_0 \).

An important byproduct of this analysis is the fact that, the moment after the last right neighbor jumps over \( p_0 \), all the other agents have made exactly one left jump over \( x_0(0) \). Hence all the agents will be located in the interval \((x_0(0) - 1, x_0(0))\). Therefore we proved:

**Theorem 1.** If \( p_0, p_1, \ldots, p_N \) are located at \( t = 0 \) at \( x_0(0), x_1(0), \ldots, x_N(0) \) with \( x_0(0) < x_1(0) < \ldots < x_N(0) \), and the rightmost agent performs random walk biased toward the left with probability of a left unit jump of \( 1 - \varepsilon \), the agents first gather to the interval \((x_0(0) - 1, x_0(0))\), with probability 1, in a finite expected number of steps given by

\[
\mathbb{E} \left( T_{\{x_0(0), x_1(0), \ldots, x_N(0)\}} \right) = \frac{1}{1 - 2\varepsilon} \sum_{k=1}^{N} \left( \left\lfloor x_k(0) - x_0(0) \right\rfloor + 1 \right)
\]

Note that we could have chosen in this description the “beacon” to be the leftmost agent \( p_1 \) located at \( x_1(0) \) and then in a finite expected time of

\[
\mathbb{E} \left( T_{\{x_1(0), x_2(0), \ldots, x_N(0)\}} \right) = \frac{1}{1 - 2\varepsilon} \sum_{k=2}^{N} \left( \left\lfloor x_k(0) - x_1(0) \right\rfloor + 1 \right)
\]

the agent \( p_1 \) becomes the rightmost agent. If, beyond the “first gathering” to the left of \( p_1 \), the process continues indefinitely, the group of agents will be pushed to the left due to the rightmost agent’s actions with an average speed of about \( 1 - 2\varepsilon/N \).

Note also that we have the corresponding symmetric result for agent groups where only the leftmost agent is moving and it sweeps all agents, by the action of its biased random walk, towards the right, after gathering the group to an interval of length bounded by 1.

### 4.2 Bilateral Action Results

So far we have seen that a unilateral random-walk, biased toward the group of agents, carried out either by the rightmost or by the leftmost agent results in gathering the agents into a cluster with a span upper bounded by 1 (i.e. the step size). Something slightly more complex happens when both extremal agents are jointly herding the group. Of course we expect gathering to happen, and even faster than in the case when only one extremal agent is at work. This is indeed the case, however the simultaneous work of the extremal agents leads to interactions that slightly complicate the proofs.

Suppose we have a constellation of agents \( p_1, p_2, \ldots, p_N \) located at time \( t = 0 \) at \( x_1(0) < x_2(0) < \ldots < x_N(0) \), as before. The “erratic extremists”, the leftmost and rightmost agents \( p_L \triangleq p_1 \) and \( p_R \triangleq p_N \) perform biased steps by simultaneously jumping, towards the agents \( \{p_2, p_3, \ldots, p_{N-1}\} \) with probability \( 1 - \varepsilon \) or away from them with probability \( \varepsilon \).

The results below represent the main contribution of this paper. **Theorem 2** states that if the internal agents are gathered in an interval smaller than the step size, they never spread beyond this size. **Theorem 3** bounds the expected time
to shrink the excess distance, beyond one, between \( p_2 \) and \( p_{N-1} \) (the internal agent span) by one half. Theorem 4 then uses the fact that, once less than 2, the distances \( |x_{N-1}(t) - x_2(t)| \) can only take a finite set of values, to show that the inner agents gather to an interval of length less than 1 in finite expected time. Theorem 5 uses the bounds on the expected excursions of biased random walks in the direction opposite to the bias to prove that, with high probability, the total span of all the agents will have a small value as the process continues to evolve after the “core” gathered.

**Theorem 2.** Suppose at \( t = T \) the internal agents \( \{p_2, p_3, \ldots, p_{N-1}\} \) are all close, so that \( x_{N-1}(T) - x_2(T) \leq 1 \), then \( x_{N-1}(T+1) - x_2(T+1) \leq 1 \). Hence for all \( t > T \) we will have \( x_{N-1}(t) - x_2(t) \leq 1 \).

**Proof.** Assume \( x_{N-1}(T) - x_2(T) \leq 1 \). Designate by \( A_L \) and \( A_R \) the agents \( x_2(T) \) and \( x_{N-1}(T) \), respectively. After jumps by extremal agents we can have at \( t = T+1 \) the following cases:

- \( A_L \) and \( A_R \) both remained internal. Then all the internal agents are still inside the interval \( [x_2(T), x_{N-1}(T)] \) with assumed length of at most one.

- \( A_L \) and \( A_R \) both became extremal. This case is even simpler: all the internal agents at time \( T+1 \) are now strictly inside the interval \( [x_2(T), x_{N-1}(T)] \) with assumed length of at most one.

- Either \( A_L \) or \( A_R \) only became an extremal agent. Assume w.l.o.g. that agent \( A_L \) at location \( x_2(T) \) became extremal, i.e. \( x_1(T+1) = x_2(T) \). In this case all the internal agents are contained in either \( [x_2(T), x_{N-1}(T)] \) (because the left extremal agents moved into it, see Figure 3a) or \( [x_2(T), x_1(T) + 1] \) (because the left extremal agent over-jumped all the previous internal agents, see Figure 3b). In both cases, the interval containing new internal agents is of length at most one.

Hence in all possible cases the span of the gathered agents at the next step never exceeds one.

The next theorem demonstrates that the size of internal agents’ interval, if bigger than one, will be reduced in finite expected time by one-half of the difference between the interval size and 1. We shall then exploit the fact that the number of agents is finite and that the shrinkage can not be infinitesimal, to show that the interval indeed will attain a size less than 1, in finite expected time.
Then, if for some $i$ for all $j$, the interval $\left(\frac{x_{i-1}(0) - x_{i}(0)}{2}, \frac{x_{i}(0) - x_{i+1}(0)}{2}\right)$ is swept a distance of at most $E$, one must move all the way to reach the fence. Hence by $\min_{x} \{T : x_{n-1}(t) - x_{n}(t) \leq 1 + \frac{S_n}{2}\}$ be the first time, when all the internal agents are inside an interval bounded by $1 + \frac{S_n}{2}$, then

$$E(T) < \frac{1}{1 - 2\varepsilon} \left(\left(N - 2\right) \left[\frac{S_0}{2}\right] + \left(x_N(0) - x_1(0) - 1\right)\right).$$

Proof. Locate two fictional “beacon agents” $p^L_F$ and $p^R_F$ at the locations defined as follows:

(a) $p^L_F$ at $x^L_F(0) = x_2(0) + \frac{S_0}{2}$

(b) $p^R_F$ at $x^R_F(0) = x_{n-1}(0) - \frac{S_0}{2}$

Obviously, $x^R_F(0) - x^L_F(0) = 1 + S_0 - 2\frac{S_0}{2} = 1$.

Now consider the agents to the right of $x^R_F(0)$ and the action of $p_R$ and the agents to the left of $x^L_F(0)$ and the action in time by $p_L$. Clearly there will be no interaction between the two dynamic processes to the left and to the right of the interval $[x^L_F(0), x^R_F(0)]$ until one of the agents $p_R$ or $p_L$ will fully sweep all agents located in either the interval $(-\infty, x^L_F(0))$ or in the interval $(x^R_F(0), \infty)$, into the unit interval $[x^L_F(0), x^R_F(0)]$. Indeed no agents from the left can cross into the right region until all of them “jumped the fence” at $x^L_F(0)$ and the same happens in the opposite direction.

Therefore we have that in a finite expected time upper bounded by

$$\frac{1}{1 - 2\varepsilon} \left(\left(N - 2\right) \left[\frac{S_0}{2}\right] + \left(x_N(0) - x_1(0) - 1\right)\right)$$

the span of the “internal”, non-mobile agents will shrink to be at most $1 + \frac{S_n}{2}$.

The bound is explained as follows: if we denote by $T_L$ - a random time it takes the agents left of $x^L_F(0)$ to “jump the fence” and by $T_R$ - the random time it takes the agents right of $x^R_F(0)$ to “jump the fence”, then clearly $T$, the first moment when one of the $\frac{S_n}{2}$ intervals will be cleared of agents is bounded above by $\min\{T_L, T_R\}$. We have then, that in the worst case, we will need at most all internal agents to be swept a distance of at most $\left[\frac{S_n}{2}\right]$, and also an extremal one must move all the way to reach the fence. Hence $E(T = \min\{T_L, T_R\}) < E(\text{worst extremal excursion time})$, which is the expression above.

We next prove the following simple fact

**Lemma 1.** Let $x_1, x_2, \ldots, x_n$ be a set of real numbers, such that $\{x_i\} \neq \{x_j\}$ for all $i \neq j$ (i.e. their fractional parts are all different). Define

$$d := \min_{i \neq j} \{|\{x_i\} - \{x_j\}|, 1 - |\{x_i\} - \{x_j\}|\}$$

Then, if for some $i, j$ $|x_i - x_j| > 1$, we must have that: $|x_i - x_j| \geq 1 + d$. 

Proof. Write $x_i = s_i + r_i$, where $r_i \in [0, 1)$ and $s_i \in \mathbb{Z}$. Then $|x_i - x_j| = |(s_i - s_j) + (r_i - r_j)|$ and $-1 < r_i - r_j < 1$.

If $x_i - x_j > 1$, then two cases are possible:

- $r_i > r_j$, then $x_i - x_j = (s_i - s_j) + (r_i - r_j)$, $s_i - s_j \geq 1$ and $r_i - r_j = |r_i - r_j| = |(x_i) - (x_j)| \geq d$. This yields $x_i - x_j \geq 1 + d$.

- $r_i < r_j$, then $x_i - x_j = (s_i - s_j - 1) + (1 - (r_j - r_i))$, $(s_i - s_j - 1) \geq 1$ and $1 - (r_j - r_i) = 1 - |r_j - r_i| = 1 - |(x_j) - (x_i)| \geq d$. This again yields $x_i - x_j \geq 1 + d$.

In case $x_i - x_j < -1$, it follows $x_j - x_i > 1$ and we apply previous argument by exchanging roles of indexes $i$ and $j$. Hence in both cases the claim follows.

Note, if $\{x_i(0)\}$ are the fractional parts of the initial locations of the agents on the line, then these fractional parts are invariant under the evolution process since agents jump unit steps.

Assuming, as we do, that all initial fractional parts are distinct we have the following result: Define $d$ as in Lemma 1 to be the smallest fractional difference of all the initial agent pair locations. If $x_{N-1}(t) - x_2(t) > 1$, then necessarily $x_{N-1}(t) - x_2(t) \geq 1 + d$.

Theorem 4. In the setting of Theorem 3, let $T = \inf\{ t : x_{N-1}(t) - x_2(t) \leq 1 \}$, i.e. the first time when all the internal agents are inside an interval bounded by 1, then

$$
E(T) < \frac{1}{1 - 2^\varepsilon} (N \cdot (S_0 + \lceil \log_2 \frac{S_0}{d} \rceil) + (x_N(0) - x_1(0) - S_0 - 1)).
$$

Proof. From the Theorem 3, given that at time 0, $(x_{N-1}(0) - x_2(0)) = 1 + S_0$, we have at a random time $T_1$ with finite expectation that $(x_{N-1}(T_1) - x_2(T_1)) \leq 1 + \frac{S_0}{d}$. We next consider the process with the constellation of agents at the moment where one of the active extremal agents cleared out an interval of length $\frac{S_0}{d}$ on one side of the span of “internal agents”. At this moment ($T_1$, the initial time for the next phase) all internal agents are spanning an interval of length at most $1 + \frac{S_0}{d}$. Therefore by Theorem 3, after a random time span of $T_2$, again having finite expectation, we find the internal points gathered within an interval of $1 + \frac{S_0}{d}$, etc.

After $k$ such steps, each with finite expected duration, we shall find the internal agents within an interval of length at most $1 + \frac{S_0}{d^k}$. The decrease of the upper bound value on the span of internal agents at step $k$ will be at least $\frac{S_0}{2d^k}$. Recall now that $d$ is the smallest fractional difference of all possible agent pair locations. Suppose at step $k_f$ (at time $T^* := T_1 + T_2 + \ldots + T_{k_f}$), we attain for the first time $\frac{S_0}{2d^k} < d$ but, still we have $x_{N-1}(T^*) - x_2(T^*) > 1$. By Lemma 1 we must have

$$
x_{N-1}(T^*) - x_2(T^*) \geq 1 + d
$$

However, since

$$
x_{N-1}(T^*) - x_2(T^*) \leq 1 + \frac{S_0}{2d^k} < 1 + d
$$

14
leads to a contradiction, we must have an interval $x_{N-1}(T^*) - x_2(T^*) \leq 1$ and $T \leq T^*$. This proves that, at some step before $k_f = \lceil \log_2 \frac{S_0}{d} \rceil$ all the internal points will be gathered in an interval of unit length.

Using the upper bound for every $T_1, T_2, \ldots$ we obtain

$$\mathbb{E}(T) \leq \mathbb{E}(T_1) + \mathbb{E}(T_2) + \ldots + \mathbb{E}(T_{k_f})$$

$$\leq \frac{N \lceil \frac{S_0}{1-2\varepsilon} \rceil}{1-2\varepsilon} + \frac{N \lceil \frac{S_0}{1-2\varepsilon} \rceil}{1-2\varepsilon} + \ldots + \frac{N \lceil \frac{S_0}{1-2\varepsilon} \rceil}{1-2\varepsilon} + \Delta$$

We still need to evaluate $\Delta$ here. Starting at each time $T_1, T_2, \ldots$ extremal agents need to sweep by their biased walk distances of $\lceil \frac{S_0}{1-2\varepsilon} \rceil$, $\lceil \frac{S_0}{1-2\varepsilon} \rceil$, $\ldots$, respectively, with the exception of the first interval $T_1$, when an additional initial “gap” had to be traversed by one of extremal agents. The possible initial “gaps” were, $x_2(0) - x_1(0)$ and $x_N(0) - x_{N-1}(0)$. For the upper bound we take the initial traversal length to be the sum of these quantities. After reordering we have for $\Delta = \frac{(x_N(0) - x_{N-1}(0)) - 1 - S_0}{1-2\varepsilon}$, hence we obtain

$$\mathbb{E}(T) < \frac{N \cdot (S_0 + \lceil \log_2 \frac{S_0}{d} \rceil) + (x_N(0) - x_1(0) - S_0 - 1)}{1-2\varepsilon}$$

To summarize, we have the following results so far:

- Consider the span of the non-extremal agents’ constellation at time $t = 0$ on $\mathbb{R}$ as $L(0) \triangleq x_{N-1}(0) - x_2(0) \triangleq 1 + S_0$

and with $S_0 > 0$. Due to the actions of the “erratic extremist” agents, while the span of the “core” agents is greater than 1 (i.e. it is $L(t) = 1 + S$, with $S > 0$), we have that $x_2(t)$, the location of the second agent in the reordered naming of agents, can only increase, and similarly $x_{N-1}(t)$ can only decrease. Hence, while $L(t)$ is bigger than one, it will be a non-increasing sequence in time. In finite expected time $L(t)$ becomes less than 1 and the subsequent actions of the extremists can never make it exceed 1.

- Following the gathering of the “core” agents to a consensus interval less than 1 after a finite expected time, the total distance between $p_1$ and $p_N$ will be a sum of three parts: the interval occupied by “core” agents of size at most 1 and two distances from the consensus “core” interval to the left and right “extremists”.


4.3 The total span of agents after gathering

In subsection 3.4 we provided a bound of $\varepsilon/(1-2\varepsilon)$ on the expected length of maximal excursions of an extremal agent from a fixed point. Since expectation is linear we can provide a rough bound on the total span of agent locations as the sum of 1, (which upper bounds the span of the gathered “core”, or consensus agents) and expected maximal excursions to the left and right made by the “extremist” agents. This argument yields, roughly

$$E(x_N(t) - x_1(t)) \leq 1 + \frac{2\varepsilon}{1-2\varepsilon}$$

The Markov’s inequality ($\forall a > 0 \quad P(X \geq a) \leq \frac{E(X)}{a}$) then provides

$$P(x_N(t) - x_1(t) \geq k) \leq \frac{1}{k} + \frac{2\varepsilon}{k \cdot 1 - 2\varepsilon} \approx \frac{1}{k}$$

Therefore, we have qualitatively that $P(x_N(t) - x_1(t) \in [k,k+1]) = \Theta(1/k^2)$. However, we can do even better.

**Figure 4: Left-biased bounded random walk used to bound extremal agent distance from the internal core agents**

Let us introduce a left-biased, partially reflective and bounded-from-the-left random walk on the state space $\{1, 2, \ldots\}$ (see Figure 4). Further, consider each state as representing the extremal agent’s current distance from the farthest internal agent rounded to the closest bigger integer. The probability to move right, i.e. away from the “core” (which is the gathered, internal agents span), at every state is $\varepsilon$, and the probability to move closer to the “core” is $(1-\varepsilon)$. The right extremal agent can, with probability $(1-\varepsilon)$, jump to the left, but at state 1 such a jump constitutes a move over all the internal agents. In this case, a new extremal agent “emerges”, maintaining the distance from the farthest left internal agent just below 1 (e.g. Theorem 2). This can happen in two ways. Either the other extremal agent jumps over the “core”, or the closest internal agent becomes “exposed” and turns into the right extremal one.

After the convergence of the internal agents, suppose we couple the (right) extremal agent’s moves to the above defined random walk, i.e. the random walk proceeds exactly following the decisions of extremal agent.

**Claim 4.1.** The random walk defined above provides an upper bound on the distance of the extremal agent (at $x_N(t)$) from the farthest internal agents (at $x_2(t)$).

**Proof.** Let $X(t)$ denote the state of random walk at time $t$. Suppose at time $t = T$, $X(T) \geq x_N(T) - x_2(T)$, i.e. random walk is at state ‘at least the distance of right extremal agent from the farthest internal agent’. At $t = T + 1$ one of the following things can happen.
bullet The right extremal agent decides to jump right. In such a case, the distance to the extremal agent increases by at most 1, which corresponds to an increase in the random walk position.

\[ X(T + 1) = X(T) + 1 \geq (x_N(T) + 1) - x_2(T) \]

where \( x_N(T + 1) - x_2(T) \geq x_N(T + 1) - x_2(T + 1) \)

The last inequality follows from the fact, that the left-most internal agent can only “move” to the right, due to the action of the left “extremist”.

bullet The right extremal agent decides to jump left, but remains the right extremal agent at \( t = T + 1 \), and \( x_N(T) - x_2(T) > 1 \). Therefore,

\[ X(T + 1) = X(T) - 1 \geq (x_N(T) - 1) - x_2(T) \]

The last inequality is explained as in the preceding case.

bullet The right extremal agent decides to jump left and stops being the right extremal agent at \( t = T + 1 \), and \( x_N(T) - x_2(T) \geq 1 \). We assumed, that \( X(T) \geq x_N(T) - x_2(T) \) which is equivalent to \( X(T) \geq 2 \), hence by definition of coupling \( X(T + 1) \geq 1 \). Two situations are possible: the internal agent at \( x_{N-1}(T) \) “emerged” to be the right extremal agent at time \( T + 1 \) or the left extremal agent at time \( T + 1 \) jumps over all the other agents to the right and becomes the right extremal one. We have \( x_2(T + 1) \geq x_2(T) \), since the right extremal agent becomes the internal agent. Also \( x_N(T + 1) = x_1(T) + 1 \leq x_2(T) + 1 \), since only the extremal agents actually move. In both cases it follows that

\[ x_N(T + 1) - x_2(T + 1) \leq (x_2(T + 1) - x_2(T) = 1 \leq X(T + 1) \]

bullet The right extremal agent decides to jump left and \( x_N(T) - x_2(T) \leq 1 \). In such a case \( X(T) \geq 1 \) and \( X(T + 1) \geq 1 \), because 1 is the lowest value the random walk could attain. The distinctive difference from the previous case is that the right extremal agent moves over all the internal agents. We have then three cases. The first is when the right extremal agent becomes the left-most internal agent, hence

\[ x_2(T + 1) = x_N(T) - 1 \geq x_{N-1}(T) - 1 = x_N(T + 1) - 1. \]

In the second and third cases, it becomes the left extremal agent. We differentiate between those two cases considering the new role of the previous left extremal agent. If it becomes a new right extremal agent, we have

\[ x_N(T + 1) = x_1(T) + 1 \leq x_2(T) + 1 = x_2(T + 1) + 1. \]

Otherwise,

\[ x_N(T + 1) = x_{N-1}(T) \leq (x_N(T) - 1) + 1 \]

\[ x_1(T + 1) + 1 \leq x_2(T + 1) + 1 \]

In all the above cases, we conclude \( x_N(T + 1) - x_2(T + 1) \leq 1 \leq X(T + 1) \), as claimed.
Returning to analyze the “upper bounding” random walk we have the following: if \( \varepsilon < \frac{1}{2} \) the above random walk is positive recurrent, and aperiodic, hence has a stationary distribution \( \pi \) that is determined by the balance equations

\[
\varepsilon \cdot \pi(k) = (1 - \varepsilon) \cdot \pi(k + 1)
\]

Along with the normalization condition \( \sum_{k=1}^{\infty} \pi(k) = 1 \), this provides the steady state distribution \( \pi = [\pi(1) \ \pi(2) \ \ldots] \) with

\[
\pi(k) = \left( \frac{\varepsilon}{1 - \varepsilon} \right)^{k-1} \frac{1 - 2\varepsilon}{1 - \varepsilon} \quad \forall k \in \{1, 2, \ldots \}.
\]

The above analysis is symmetrically applicable to the random walk of the left extremal agent. We then have two independent and identically distributed walks, upper bounding the distance of the right and left extremal agents from the “core’s” left and right boundaries. Denoting them by \( X(t) \) and \( Y(t) \), we have

\[
x_N(t) - x_1(t) \leq (x_N(t) - x_2(t)) + (x_{N-1}(t) - x_1(t)) = X(t) + Y(t)
\]

Here we are interested in assessing \( P(x_N(t) - x_1(t) \leq k) \), hence we can estimate a lower bound for \( P(x_N(t) - x_1(t) \leq k) \) by \( P(X(t) + Y(t) \leq k) \). Therefore, consider

\[
P(X + Y \geq k) = \sum_{i=1}^{k-2} P(X = i)P(Y \geq k - i) + P(X \geq k - 1)
\]

In the steady state we have that \( P(X = k) \) is just a \( \pi(k) \), and \( P(X \geq k) = \sum_{i=k}^{\infty} \pi(i) \). Therefore,

\[
\sum_{i=k}^{\infty} \pi(i) = \sum_{i=k}^{\infty} \left( \frac{\varepsilon}{1 - \varepsilon} \right)^{i-1} \frac{1 - 2\varepsilon}{1 - \varepsilon} = \left( \frac{\varepsilon}{1 - \varepsilon} \right)^{k-1}
\]

Thus, we obtained the following simple expression

\[
P(X \geq k) = \left( \frac{\varepsilon}{1 - \varepsilon} \right)^{k-1}
\]

Using this result in (7) produces for \( k \) greater than two

\[
P(X + Y \geq k) = \sum_{i=1}^{k-2} \left( \frac{\varepsilon}{1 - \varepsilon} \right)^{i-1} \frac{1 - 2\varepsilon}{1 - \varepsilon} \cdot \left( \frac{\varepsilon}{1 - \varepsilon} \right)^{k-i-1} + \left( \frac{\varepsilon}{1 - \varepsilon} \right)^{k-2},
\]

which after few algebraic manipulations provides

\[
P(X + Y \geq k) = \left( \frac{\varepsilon}{1 - \varepsilon} \right)^{k-2} \left( k - 2 \cdot \frac{1 - 2\varepsilon}{1 - \varepsilon} + 1 \right).
\]

We summarize these findings as follows:

**Theorem 5.** After the internal agents gather in an interval of length 1, the distribution of interval lengths containing all the agents is lower bounded by

\[
P(x_N(t) - x_1(t) < k) \leq P(X + Y < k) \approx 1 - k \left( \frac{\varepsilon}{1 - \varepsilon} \right)^{k-2}.
\]
4.4 On arbitrary initial position of agents

In proving Theorem 4 we have assumed all the agents locations’ fractional parts are different. We can slightly change the model to accommodate for cases in which some agents may share the same location. Of course, the problem arises when several agents find themselves sharing extremal locations. In such cases their motions must be specified and disambiguated. Suppose several agents share the same place and all other agents are located on exactly one side either to the left, or to the right. We assume that only one of these extremal agents will become “erratic”, and move at a given time. We can then readily prove a claim equivalent to Theorem 4 in this new model.

**Theorem 6.** Let agents \( p_1, p_2, \ldots, p_N \) be initially located at \( x_1(0), x_2(0), \ldots, x_N(0) \), and define

\[
T := \inf \{ t : x_{N-1}(t) - x_2(t) \leq 1 \}
\]

be the first time, when all the internal agents are inside the interval bounded by 1, then with the modified rule of behavior we have \( E(T) < \infty \).

**Proof.** Since we are not assuming that fractional parts are all different, it is possible that there will be more than one agent with the same fractional part of their initial (and subsequent) locations. Let \( \Delta \) be the minimal fractional non-zero distance between two agents.

\[
\Delta := \min_{\{x_j(0) \neq x_k(0)\}} \{\{x_j(0) - x_k(0)\}, 1 - \{x_j(0) - x_k(0)\}\}
\]

In case all the agents share the same fractional part, simply set \( \Delta := 1 \).

Step 1. Define a new process with the following initial coordinates:

\[
y_k(0) = x_k(0) + \frac{(k-1)\Delta}{N}
\]

for all \( k \in \{1, 2, \ldots, N\} \). It is not difficult to see that the “newly defined locations” \( y_1(0), y_2(0), \ldots, y_N(0) \) fulfill the requirements of Theorem 4.

Step 2. The Theorem 4 proves that all agents \( (y_k)_{k=1}^n \) gather in expected finite time to the interval of unit length. Denote this time by \( T_y \).

- By separately handling cases of same and different initial fractional part of the location one can show that for all \( k \geq 2 \),

\[
x_2(t) \leq x_k(t)
\]

In the same manner, for all \( k \leq N - 1 \)

\[
x_k(t) \leq x_{N-1}(t)
\]

We can then conclude, that all the correspondingly indexed \( x \) and their “shadow \( y \)-agents” will be called inner and extremal in both models at the same time.
Suppose $y_2(T_y)$ and $y_{N-1}(T_y)$ are agents which originally had the same fractional part of their respective location. Due to the way we mapped the coordinates, we know that $y_2(T_y) < x_2(T_y) + \Delta \leq x_2(T_y) + 1$ and that $y_{N-1}(T_y) \geq x_{N-1}(T_y)$, hence $|x_{N-1}(T_y) - x_2(T_y)| < 2$. But, since $x_2(T_y)$ and $x_{N-1}(T_y)$ have the same fractional part, we conclude that we have $x_{N-1}(T_y) - x_2(T_y) \leq 1$.

If $y_2(T_y)$ and $y_{N-1}(T_y)$ are not agents which originally had the same fractional part of their respective location, then one of two cases is possible. If $y_{N-1}(T_y) < x_2(T_y) + 1$ we have all agents in original model inside the interval $[x_2(T_y), x_2(T_y) + 1)$. Otherwise $x_2(T_y) + 1 \leq y_{N-1}(T_y) \leq y_2(T_y) + 1$. But, due to definition of $\Delta$, only points, which have the same fractional part, as $x_2(T_y)$ could fall between $x_2(T_y) + 1$ and $y_2(T_y) + 1$. Hence the latter case is impossible.

It follows, that in all cases $x_{N-1}(T_y) - x_2(T_y) \leq 1$, which implies $T \leq T_y$, and by Theorem 4 $T_y$ has a finite expectation.

5 Simulation results

We next present some simulation results to showcase the validity of the above-presented theoretical predictions. In Figure 5 we present the result of simulation runs with a different values for $\varepsilon$ and fixed number of agents $N$ located at random points uniformly distributed in an initial interval of size $1 + S_0$. The simulations measured the time to convergence of the inner agents to an interval of length one. The theory predicts that the expected time to gathering is bounded as follows

$$
E(T) \leq \frac{N \cdot (S_0 + \lceil \log_2 \frac{S_0}{\varepsilon} \rceil) + (x_N(0) - x_1(0) - S_0 - 1)}{1 - 2\varepsilon}
$$

As predicted, the average convergence times exhibit a hyperbolic dependence on $\varepsilon$. Figure 6 clearly showcases the linear functional dependence between the convergence time and the initial span of internal agents, and implicitly to the initial span of all agents. Varying the number of agents $N$ supplies another linear dependency as can be seen in Figure 7.
Figure 5: (a) Convergence times as a function of probability of motion in the wrong direction ($\varepsilon$), $N = 400$, $S_0 = 500$. (b) Theoretical upper bound to measured convergence time ratio vs $\varepsilon$. Each point on the actual results’ line is an average of 100 different simulations with the same parameters set.
Figure 6: (a) Convergence times as a function of initial span ($S_0$). (b) Ratio between the theoretical upper bound to measured convergence time vs $S_0$. Each point on the actual results’ line is an average of 100 different simulations with the same parameter set.
Figure 7: (a) Convergence times as a function of number of agents ($N$). (b) Theoretical upper bound to measured convergence time vs $N$. Each point on the actual results’ line is an average of 100 different simulations with the same parameters set.
In all above experiments we notice that our theoretical bounds are roughly eight times higher than the actual measurements. Recall, that we derived our results based on overly cautious assumptions, namely that one extremal agent is doing the constructive work, while the second extremal agent is randomly wandering outside the interval containing the internal agents. In reality, this is not the case: both agents independently and concurrently contribute to convergence. Hence we should focus on the stochastic process, which is in some sense “the distance between two independent random walks biased towards each other”.

Let $X, Y$ be two independent biased random walks with a probability $\varepsilon$ to jump right. For any one of mentioned random walks $\mathbb{E}$(step length) = $2\varepsilon - 1$. On the other hand, for the process $Z \triangleq X + Y$, $\mathbb{E}$(step length of $Z$) = $2(2\varepsilon - 1)$. Note, that the “contraction” process $Z$ describes the distance between two extremal agents, with each extremal agent sweeping the internal agents in the direction of its counterpart. Furthermore, on the average, the core convergence will happen approximately around the middle of the initial interval. And we should finally recall the assumed uniform initial spread of agents inside the initial interval at the beginning, implying that each extremal agent will need “to push” only half of the initial agents. Thus until convergence we have two stochastic processes of the kind we analyzed in this paper, and each starts with half the number of agents, and half the initial interval, and the process will proceed at least twice as fast. Hence, we have 3 factors that each improve by 2 the time to convergence (hence the 8!)

Another aim of our simulations was to assess on the bounds on the total span of agents after gathering. As can be seen in Figure 8 using a semi-logarithmic scaling, the probability to find an extremal agent at a specific distance is indeed decreasing exponentially fast, according to the bound of Theorem 5. The same is true about the results predicted in subsection 4.3. In Figure 8b we show that application of Theorem 5 gives a much better bound than the crude evaluation of (10).

Figure 9 presents a typical behavior of the Gathered Core’s center of mass and of the two extremal agents. The period before gathering is shown in Figure 9a followed by a display of “post-gathering” typical behavior in Figure 9b. Simulations for different number of agents are shown in Figure 10 and Figure 11. Unsurprisingly, Figures 10b and 11b prove a much higher inertia of the Core center of mass to the actions of extremal agents, when the number of agents is five times higher.
Figure 8: (a) Long-term (steady-state) distribution of agents’ total span. (b) Long-term cumulative distribution of total span with lower bounds from subsection 3.4 and subsection 4.3. The simulations were done for different values of N and $\varepsilon = 0.1$. 

(a) Long-term distribution of agent span

(b) Cumulative distribution of spans
Figure 9: Typical “core” center and extremal agent location vs. time. From the beginning (a) and after the gathering (b). Simulations with $N = 21$, $\varepsilon = 0.1$. 
Figure 10: “Core” center location and both extremal agents vs. time (after gathering, starting from $T = 10,000$). Simulations with $\varepsilon = 0.3$. Note that the “core” of gathered agents is much more easily moved by “extremists” when the population is small ($N = 21$)
Table 11: Typical evolution of “core”’s center of mass location and extremal agents’ locations in time (after gathering, starting from $T = 1,000,000$. Simulations done for $\epsilon = 0.1$). Note that in both cases the initial center of mass of all agents was at 0. Note that the “inertia” of society is much higher when $N = 1000$ than in case $N = 200$. 

(a) Evolution of “core” and extremal agents (zoomed out). $N = 200$

| Iteration | Core-Center | Left Extremal Point | Right Extremal Point |
|-----------|-------------|---------------------|----------------------|
| 0         | 0           | 0                   | 0                    |
| 1000000   | 0.0000      | -0.0000             | 0.0000               |
| 2000000   | 0.0000      | -0.0000             | 0.0000               |
| 3000000   | 0.0000      | -0.0000             | 0.0000               |
| 4000000   | 0.0000      | -0.0000             | 0.0000               |
| 5000000   | 0.0000      | -0.0000             | 0.0000               |

(b) Evolution of “core” and extremal agents (zoomed out). $N = 1000$

| Iteration | Core-Center | Left Extremal Point | Right Extremal Point |
|-----------|-------------|---------------------|----------------------|
| 0         | 0           | 0                   | 0                    |
| 1000000   | 0.0000      | -0.0000             | 0.0000               |
| 2000000   | 0.0000      | -0.0000             | 0.0000               |
| 3000000   | 0.0000      | -0.0000             | 0.0000               |
| 4000000   | 0.0000      | -0.0000             | 0.0000               |
| 5000000   | 0.0000      | -0.0000             | 0.0000               |
6 Concluding remarks

We here proposed a mathematical model of randomly interacting particles on the line, that could describe opinion dynamics in a society of presumably intelligent agents. Equipped with a simple decision rule, agents eventually get together to a drifting gathered constellation, in finite expected time. All the agents of the system, except two, constitute a core of moderate agents that remain closely clustered from that point on. The two “erratic extremist” agents perform random walks biased toward the “quasi-stationary” core; once in a while the roles of extremal agents change, when an “erratic extremist” walker joins the core. We have derived expression for the expected convergence time and the distribution of distances of extremal agents. Computer simulations support our findings.

We believe that the model presented will further help analyze two and higher dimensional models which have a practical importance in a number of areas in multi-agent studies.

An interesting two-dimensional model corresponding to the random evolution process analyzed in this paper could be the following. Assume that the agents’ locations are points in the plane $\mathbb{R}^2$. For a group of $N$ agents in the plane the “extremists” are the ones that define the convex hull of the points. Suppose at each time instant an agent that realizes it is an extreme vertex of the convex hull (by sensing the bearing only to all other agents!) decides to move a unit distance along the bisector of the corresponding convex hull angle either toward the other agents (i.e. into the convex hull), with probability $(1 - \varepsilon)$, or in the opposite direction, with probability $\varepsilon$. (see Figure 12)

![Figure 12: Group of Agents, Convex Hull and zoom on Extremal Agent movement options](image)
Preliminary simulations with this model show that indeed the population gathers to a small region in the plane (see Figure 13) and the gathered group performs a random walk in the plane (Figures 14 and 15). We plan to study this and several variations of such models in the near future.

Figure 13: Typical evolution of the system \( N = 400, \varepsilon = 0.1 \) from (a) beginning till (d) 400\(^{th}\) iteration. (Convex Hull is depicted for convenience)
Figure 14: Evolution of the Center of Mass of the system and the last Convex Hull after (a) 400 and (b) 1000 iterations
Figure 15: Evolution of the Center of Mass of the system split by (a) X direction and (b) Y direction.
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