Algorithms for $D$-modules — restriction, tensor product, localization, and local cohomology groups

Toshinori Oaku and Nobuki Takayama

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1 Introduction

The purpose of this paper is to describe algorithms for computing various functors for algebraic $D$-modules, i.e. systems of linear partial differential equations with polynomial coefficients. The algorithms enable us to perform actual computations (with limitation caused by the complexity) by using e.g. a program kan developed by the second author, as well as to establish theoretical computability of some fundamental functors in the $D$-module theory.

Let $K$ be an algebraically closed field of characteristic zero and let $X$ be the affine space $K^n$ with a positive integer $n$. We denote by $\mathcal{O}_X$ and $\mathcal{D}_X$ the sheaves on $X$ of rings of regular functions and of algebraic linear differential operators respectively (cf. Bernstein, Björk, Borel et al.). Let $\mathcal{M}$ and $\mathcal{N}$ be coherent left $\mathcal{D}_X$-modules.

Various functors are defined for (especially for holonomic) $D$-modules and play the fundamental role (see [3], [4], [5] and also e.g., Kashiwara, Mebkhout for their analytic counterparts). Among such functors, we are concerned with the following:

1. The cohomology groups of the restriction $\mathcal{M}_Y := \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{M}$ of $\mathcal{M}$ to $Y$ as left $\mathcal{D}_Y$-modules, where $Y$ is a non-singular subvariety of $X$ and $\otimes^L$ denotes the left derived functor (cf. [12]) of the tensor product.

2. The cohomology groups $\text{Ext}^i_{\mathcal{D}_X} (\mathcal{M}, K[[x_1, \ldots, x_n]])$ with coefficients in the formal power series solutions of $\mathcal{M}$, which equal to those with coefficients in the convergent power series solutions if $\mathcal{M}$ is regular holonomic (cf. Kashiwara-Kawai [13]).

3. The tensor product $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ and, more generally, the torsion groups $\text{Tor}^i_{\mathcal{O}_X} (\mathcal{M}, \mathcal{N})$, as left $\mathcal{D}_X$-modules.

4. The localization $\mathcal{M}[f^{-1}] := \mathcal{O}_X[f^{-1}] \otimes_{\mathcal{O}_X} \mathcal{M}$ of $\mathcal{M}$ as a left $\mathcal{D}_X$-module, where $f \in K[x_1, \ldots, x_n]$ is an arbitrary non-constant polynomial.

5. The (algebraic) local cohomology groups $\mathcal{H}^i_Y(\mathcal{M})$ with support $Y$ as left $\mathcal{D}_X$-modules, where $Y$ is an arbitrary algebraic set of $X$. 
It was proved by Kashiwara [14] that these are all holonomic systems (the second one is a finite dimensional vector space) if so are $\mathcal{M}$ and $\mathcal{N}$.

Let us remark that if $K = \mathbb{C}$ and $\mathcal{M}$ is Fuchsian along $Y$ in the sense of Laurent and Moteiro-Fernandes [22], which is the case if $\mathcal{M}$ is regular holonomic in the sense of [19], then there exists an isomorphism

$$R\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^\text{an}_X)|_Y \simeq R\text{Hom}_{\mathcal{D}_Y}(\mathcal{M}_Y^\bullet, \mathcal{O}^\text{an}_Y)$$

in the derived category of sheaves of $\mathbb{C}$-vector spaces; here $\mathcal{O}^\text{an}_X$ and $\mathcal{O}^\text{an}_Y$ denotes the sheaves of holomorphic functions on $X$ and on $Y$ respectively, and $R\text{Hom}$ the right derived functor of $\text{Hom}$. Thus roughly speaking, $\mathcal{M}_Y^\bullet$ corresponds to the system of partial differential equations which the solutions of $\mathcal{M}$ restricted to $Y$ satisfy. Similarly, $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ corresponds to the system which the product of solutions of $\mathcal{M}$ and of $\mathcal{N}$ satisfies.

As was observed by Galligo [11] and was developed by several authors (e.g. [7], [36], [37], [27], [28], [29], [1], [35]) the notion of Gröbner basis and the Buchberger algorithm [6] are essential in the algorithmic study of $\mathcal{D}$-modules as well as in computational algebraic geometry (cf. [8], [4]). By using Gröbner bases for the Weyl algebra, we give algorithms for computing the objects listed above under some conditions on $\mathcal{M}$ and $\mathcal{N}$, which are certainly satisfied if $\mathcal{M}$ and $\mathcal{N}$ are holonomic. These algorithms also apply to the analytic counterparts of these functors as long as the input $\mathcal{D}$-module is defined algebraically.

We first give an algorithm for the restriction (Algorithm 5.4) when $Y$ is a linear subvariety of arbitrary codimension under the condition that $\mathcal{M}$ is specializable along $Y$, which is the case with an arbitrary holonomic $\mathcal{D}_X$-module $\mathcal{M}$. Here $\mathcal{M}$ is specializable along $Y$ by definition if and only if there exists a nonzero $b$-function, or the indicial polynomial of $\mathcal{M}$ along $Y$. We also give an algorithm to compute the $b$-function (Algorithm 4.6).

Our method consists in computing a free resolution of $\mathcal{M}$ that is adapted to the so-called $V$-filtration associated with $Y$. Such a free resolution tensored with $\mathcal{D}_{Y\rightarrow X} := (\mathcal{D}_X)^\bullet_Y$ gives $\mathcal{M}_Y^\bullet$, but it is not a complex of coherent $\mathcal{D}_Y$-modules in general. Then we use information on the integral roots of the $b$-function to truncate the complex and obtain a complex of finitely generated free $\mathcal{D}_Y$-modules. The first author gave in [31] an algorithm for the case where $Y$ is of codimension one without using free resolution.

This algorithm for the restriction also solves the other problems by virtue of some isomorphisms provided by the $\mathcal{D}$-module theory, especially those described in [14]. See Algorithm 5.2 for the tensor product, Algorithm 5.4 for the localization, and Algorithm 7.3 for the algebraic local cohomology groups. Finally the computation of the restriction for the general case where $Y$ is not necessarily linear reduces to that of local cohomology through the so-called Kashiwara equivalence [14], which claims the equivalence of the category of coherent $\mathcal{D}_Y$-modules and that of coherent $\mathcal{D}_X$-modules supported by $Y$.

Algorithms for the local cohomology groups have been given in [31] when $Y$ is of codimension one, and by Walther [39] under the assumption that $\mathcal{M}$ is saturated with respect to $Y$. An algorithm for the localization has been given in [31] under the condition that $\mathcal{M}$ is $f$-saturated. These assumptions are removed in the present paper. As another application of the restriction algorithm, we can also obtain an algorithm for integration of
a module over the Weyl algebra, which enables us to compute the de Rham cohomology groups of some algebraic varieties. See [32] for details.

Finally we discuss how to get the free resolution mentioned above. For that purpose, we apply Schreyer’s method for free resolution in the polynomial ring (see e.g. [1]) to the ring of differential operators. In doing so, we need some modification because of the non-commutativity and the fact that the term order we use is not a well-order. We have two methods to cope with this difficulty: one is the homogenization with respect to the V- filtration by the first author ([28],[29]); the other is what we call the homogenized Weyl algebra which was introduced and implemented by the second author in the 2nd version of kan/sm1 [38] that was released in 1994, but has not been published in the literature. A similar method was employed by Assi et al. [1] independently and applied to the computation of the slopes of a D-module.

We have implemented the algorithms by using kan/sm1 [38] for computations of Gröbner bases and free resolutions in the Weyl algebra, and Risa/Asir [25] for factorization and primary decomposition in the polynomial ring.

2 V-filtration and free resolution

Let $K$ be an algebraically closed field of characteristic zero. We fix positive integers $d$ and $n$. Let $X$ be the affine space $K^{d+n}$ with the coordinate system $(t, x) = (t_1, \ldots, t_d, x_1, \ldots, x_n)$. We denote by $\partial_t = (\partial_{t_1}, \ldots, \partial_{t_d})$ and $\partial_x = (\partial_{x_1}, \ldots, \partial_{x_n})$ the corresponding derivations with $\partial_{x_i} = \partial/\partial x_i$, $\partial_{t_j} = \partial/\partial t_j$. We use the notation $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial_x^\beta := \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$, $t^\mu := t_1^{\mu_1} \cdots t_d^{\mu_d}$, $\partial_t^\nu := \partial_{t_1}^{\nu_1} \cdots \partial_{t_d}^{\nu_d}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ and $\mu = (\mu_1, \ldots, \mu_d)$, $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}^d$, where we put $\mathbb{N} := \{0, 1, 2, \ldots\}$. We also use the notation $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

Let $Y$ be the $d$-codimensional linear subvariety of $X$ given by $Y := \{(t, x) \in X \mid t = 0\}$. Let $\mathcal{O}_X$ and $\mathcal{O}_Y$ be the sheaves of regular functions on $X$ and on $Y$ respectively. We denote by $\mathcal{D}_X$ and $\mathcal{D}_Y$ the sheaves of rings of algebraic linear differential operators on $X$ and on $Y$ respectively.

Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$-module on $X$. Then the set of the global sections $M := \Gamma(X, \mathcal{M})$ is a finitely generated left module over the Weyl algebra $A_{d+n} := \Gamma(X, \mathcal{D}_X)$. Conversely, for a finitely generated left $A_{d+n}$-module $M$, its sheafification $\mathcal{M} := \mathcal{D}_X \otimes_{A_{d+n}} M$ is a coherent $\mathcal{D}_X$-module. More precisely, this correspondence gives an equivalence between the category of finitely generated $A_{d+n}$-modules and that of coherent $\mathcal{D}_X$-modules (cf. [3],[3]). Hence we could work only in the first category. However, as to e.g., the restriction functor, it would be preferable to work in the latter category since a coherent $\mathcal{D}_X$-module can be specializable along some Zariski open subset of $Y$ but not along whole $Y$ (cf. Section [1]). In any case, actual computations are done for modules over the Weyl algebra.

In the sequel, we define the notion of free resolution adapted to the V-filtration. Let $\mathcal{D}_X|_Y$ be the sheaf theoretic restriction of $\mathcal{D}_X$ to $Y$. Let $J_Y := \mathcal{O}_X t_1 + \cdots + \mathcal{O}_X t_d$ be the defining ideal of $Y$. Then for each integer $k$ we put

$$F^j_Y(\mathcal{D}_X) := \{ P \in \mathcal{D}_X|_Y \mid P(J_Y)^j \in (J_Y)^{j-k} \text{ for any } j \geq k\}$$
where $e_r$,..., $e_1$, $e_2, e_r$). More generally, given an $r$-vector $m := (m_1, ..., m_r)$ of integers, we put

$$F_Y^k[m](\mathcal{D}_X^r) := \bigoplus_{i=1}^r F_Y^{k-m_i}(\mathcal{D}_X)e_i,$$  

where $e_1, ..., e_r$ are the canonical generators of $\mathcal{D}_X^r$. We may assume that $\mathcal{M}$ has a presentation $\mathcal{M} = \mathcal{D}_X^r/\mathcal{N}$ on $X$, where $\mathcal{N}$ is a coherent left $\mathcal{D}_X$ submodule of $\mathcal{D}_X^r$. In fact, $\mathcal{M} := \Gamma(X, \mathcal{M})$ can be written in the form $\mathcal{M} = \mathcal{A}_{d+n}^r/N$ with an integer $r$ and an $\mathcal{A}_{d+n}$ submodule $\mathcal{N} := \mathcal{D}_X \otimes \mathcal{A}_{d+n}$ $\mathcal{N}$ satisfies the above property. Let $u_i$ be the residue class of $e_i \in \mathcal{D}_X^r$ in $\mathcal{M}$. Then for $m \in \mathbb{Z}^r$, we put

$$F_Y^k[m](\mathcal{N}) := \mathcal{N} \cap F_Y^k[m](\mathcal{D}_X^r),$$  

$$F_Y^k[m](\mathcal{M}) := F_Y^{k-m_1}(\mathcal{D}_X)u_1 + \cdots + F_Y^{k-m_r}(\mathcal{D}_X)u_r$$

for each integer $k \in \mathbb{Z}$. The graded ring and modules associated with these filtrations are defined by

$$\text{gr}_Y(\mathcal{D}_X) := \bigoplus_{k \in \mathbb{Z}} F_Y^k(\mathcal{D}_X)/F_Y^{k-1}(\mathcal{D}_X),$$  

$$\text{gr}_Y[m](\mathcal{D}_X^r) := \bigoplus_{k \in \mathbb{Z}} F_Y^k[m](\mathcal{D}_X^r)/F_Y^{k-1}[m](\mathcal{D}_X^r),$$  

$$\text{gr}_Y[m](\mathcal{N}) := \bigoplus_{k \in \mathbb{Z}} F_Y^k[m](\mathcal{N})/F_Y^{k-1}[m](\mathcal{N}).$$

Then $\text{gr}_Y[m](\mathcal{N})$ and $\text{gr}_Y[m](\mathcal{M})$ are coherent left $\text{gr}(\mathcal{D}_X)$-modules. If $m$ is the zero vector, we shall omit the notation $[m]$.

For a nonzero section $P$ of $\mathcal{D}_X^r|_Y$, let $k = \text{ord}_Y[m](P)$ be the minimum $k \in \mathbb{Z}$ such that $P \in F_Y^k[m](\mathcal{D}_X^r)$. (We put $\text{ord}_Y[m](0) := -\infty$.) Then let $\sigma_Y[m](P)$ be the residue class of $P$ in

$$\text{gr}_Y^k[m](\mathcal{D}_X^r) := F_Y^k[m](\mathcal{D}_X^r)/F_Y^{k-1}[m](\mathcal{D}_X^r).$$

**Definition 2.1** Let $\mathcal{M} = \mathcal{D}_X^r/\mathcal{N}$ be as above. Let us consider a free resolution

$$\mathcal{D}_X^{r_0} \xrightarrow{\psi_1} \mathcal{D}_X^{r_1} \xrightarrow{\psi_1^{-1}} \cdots \xrightarrow{\psi_2} \mathcal{D}_X^{r_2} \xrightarrow{\psi_1} \mathcal{D}_X^{r_1} \xrightarrow{\psi_1} \mathcal{D}_X^0 \xrightarrow{\varphi} \mathcal{M} \rightarrow 0$$

(2.1)

of $\mathcal{M}$, where $\psi_i$ are homomorphisms of left $\mathcal{D}_X$-modules, and $\varphi$ is defined by $\varphi(e_i) = u_i$ for $i = 1, ..., r_0$ with $r_0 = r$. This free resolution is said to be adapted (or strict with respect) to the V-filtration $F_Y[m]$ if and only if there exist vectors $m_1 \in \mathbb{Z}^{r_1}, ..., m_l \in \mathbb{Z}^{r_l}$ such that

$$\psi_{j+1}(F_Y^k[m_{j+1}](\mathcal{D}_X^{r_{j+1}})) \subset F_Y^k[m_j](\mathcal{D}_X^r)$$
holds for \( j = 0, 1, \ldots, l - 1 \) with \( m_0 = m \) and that

\[
F^k_Y[m_i](D^j_X) \xrightarrow{\psi_i} \cdots \xrightarrow{\psi_2} F^k_Y[m_1](D^i_X) \xrightarrow{\psi_1} F^k_Y[m_0](D^0_X) \xrightarrow{\varphi} F^k_Y[m](M) \to 0
\]

is an exact sequence for any \( k \in \mathbb{Z} \). We call \( m_1, \ldots, m_l \) the shift vectors associated with the free resolution \((2.1)\).

The definition above is general in the sense that it is local and also applies to the analytic case (cf. Section 3). However, from the computational viewpoint, working in the Weyl algebra would be more convenient: Let \( A_n \) and \( A_{d+n} \) be the Weyl algebras on the \( n \) variables \( x \) and on the \( d + n \) variables \( (t, x) \) respectively with coefficients in \( K \) (cf. [4]). Put \( L := \mathbb{N}^{2(d+n)} = \mathbb{N}^d \times \mathbb{N}^d \times \mathbb{N}^n \). An element \( P \) of \( A_{d+n}^r \) is written in a finite sum

\[
P = \sum_{i=1}^r \sum_{(\mu, \nu, \alpha, \beta) \in L} a_{\mu \nu \alpha \beta} t^\mu x^\alpha \partial^\beta \psi_i
\]

with \( a_{\mu \nu \alpha \beta} \in K \), \( e_1 := (1, 0, \ldots, 0), \ldots, e_r := (0, \ldots, 0, 1) \). Put

\[
F^k_Y(A_{d+n}) := \{ P = \sum_{|\mu| - |\nu| \leq k} a_{\mu \nu \alpha \beta}(x) t^\mu \partial^\nu \partial^\beta \in A_{d+n} \mid a_{\mu \nu \alpha \beta}(x) \in K[x]\},
\]

\[
F^k_Y[m](A_{d+n}^r) := \bigoplus_{i=1}^r F^k_Y[m](A_{d+n}) e_i,
\]

\[
F^k_Y[m](M) := F^k_Y[m](A_{d+n}) u_1 + \cdots + F^k_Y[m](A_{d+n}) u_r,
\]

where \( u_i \) is the residue class of \( e_i \) in \( M \).

The following lemma follows immediately from the definition:

**Lemma 2.2** Let \( p \) be a point of \( Y \) and \( P \) a germ of \( D_X \) at \( p \). Then \( P \) belongs to \( F^k_Y(D_X)_p \) if and only if there exists \( a(t, x) \in K[t, x] \) such that \( a(p) \neq 0 \) and \( a(t, x)P \in F^k_Y(A_{d+n}) \).

By using this lemma and the flatness of \( D_X \) over \( A_{d+n} \), we can easily get the following:

**Proposition 2.3** Let

\[
A_{d+n}^{r_1} \xrightarrow{\psi_1} A_{d+n}^{r_1-1} \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{i-1}} A_{d+n}^{r_i} \xrightarrow{\psi_i} A_{d+n}^{r_0} \xrightarrow{\varphi} M \to 0
\]

be a free resolution of \( M \) adapted to the \( F_Y[m] \)-filtration, i.e., \( \psi_i \) are homomorphisms of left \( A_{d+n} \)-modules, \( \varphi \) is defined by \( \varphi(e_i) = u_i \) for \( i = 1, \ldots, r_0 \) with \( r_0 = r \), and there exist \( m_1 \in \mathbb{Z}^{r_1}, \ldots, m_l \in \mathbb{Z}^{r_i} \) such that

\[
\psi_{j+1}(F^k_Y[m_{j+1}](A_{d+n}^{r_{j+1}})) \subset F^k_Y[m_j](A_{d+n}^{r_j})
\]

holds for \( j = 0, 1, \ldots, l - 1 \) with \( m_0 = m \) and that

\[
F^k_Y[m](A_{d+n}^{r_1}) \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_1} F^k_Y[m](A_{d+n}^{r_0}) \xrightarrow{\varphi} F^k_Y[m](M) \to 0
\]

is an exact sequence for any \( k \in \mathbb{Z} \). Under this assumption, the exact sequence \((2.3)\) tensored by \( D_X \) from the left gives a free resolution of \( M \) adapted to the \( F_Y[m] \)-filtration.
3 Gröbner bases and free resolution

The purpose of this section is to show that Gröbner bases homogenized with respect to the V-filtration provide a free resolution adapted to the V-filtration. An alternative and more efficient method will be described in Section 3.

We fix a natural number $r$ and a vector $m \in \mathbb{Z}^r$. Let $\prec$ be a well-order (i.e. a linear order) on $L \times \{1, \ldots, r\}$ which satisfies

$$(\bar{\alpha}, i) \prec (\bar{\beta}, j) \text{ implies } (\bar{\alpha} + \bar{\gamma}, i) \prec (\bar{\beta} + \bar{\gamma}, j)$$

for any $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in L$ and $i, j \in \{1, \ldots, r\}$. (3.1)

Then we define a total order $\prec_F$ on $L \times \{1, \ldots, r\}$ by

$$(\mu, \nu, \alpha, \beta, i) \prec_F (\mu', \nu', \alpha', \beta', j) \text{ if and only if } |\nu - \mu| + m_i < |\nu' - \mu'| + m_j \text{ or else }$$

$$|\nu - \mu| + m_i = |\nu' - \mu'| + m_j, \quad (\mu, \nu, \alpha, \beta, i) \prec (\mu', \nu', \alpha', \beta', j). \quad (3.2)$$

Let $P$ be a nonzero element of $A_{d+n}^r$ which is written in the form (2.2). Then the leading exponent $\text{lexp}_F(P) \in L \times \{1, \ldots, r\}$ of $P$ with respect to $\prec_F$ is defined as the maximum element of $\{(\mu, \nu, \alpha, \beta, i) \mid a_{\mu \nu \alpha \beta i} \neq 0\}$ in the order $\prec_F$. Moreover, for $(\mu, \nu, \alpha, \beta, i) = \text{lexp}_F(P)$, the leading coefficient of $P$ is defined by $\text{lcoef}_F(P) := a_{\mu \nu \alpha \beta i}$. The set of leading exponents $E_F(N)$ of a subset $N$ of $A_{n+1}^r$ is defined by

$$E_F(N) := \{\text{lexp}_F(P) \mid P \in N \setminus \{0\}\}.$$

**Definition 3.1** A finite set $G$ of generators of a left $A_{d+n}$-submodule $N$ of $A_{d+n}$ is called a Gröbner basis of $N$ with respect to $\prec_F$ (or an $F[m]$-Gröbner basis), if we have

$$E_F(N) = \bigcup_{P \in G} (\text{lexp}(P) + L),$$

where we write

$$(\bar{\alpha}, i) + L = \{(\bar{\alpha} + \bar{\beta}, i) \mid \bar{\beta} \in L\}$$

for $\bar{\alpha} \in L$ and $i \in \{1, \ldots, r\}$.

We define an order $\prec_H$ on $N \times L \times \{1, \ldots, r\}$ by

$$(\lambda, \bar{\alpha}, i) \prec_H (\lambda', \bar{\alpha}', i) \text{ if and only if } \lambda < \lambda' \text{ or else }$$

$$\lambda = \lambda', \quad (\lambda, \bar{\alpha}, i) \prec (\lambda', \bar{\alpha}', j), \quad (3.3)$$

where $\lambda, \lambda' \in N, \bar{\alpha}, \bar{\alpha}' \in L$ and $i, j \in \{1, \ldots, r\}$. It is easy to see that $\prec_H$ is a well-order and satisfies

**Lemma 3.2** If $|\nu - \mu| + m_i - \lambda = |\nu' - \mu'| + m_j - \lambda'$, then we have

$$(\lambda, \mu, \nu, \alpha, \beta, i) \prec_H (\lambda', \mu', \nu', \alpha', \beta', j)$$

if and only if

$$(\mu, \nu, \alpha, \beta, i) \prec_F (\mu', \nu', \alpha', \beta', j).$$
We introduce an indeterminate $t_0$ which commutes with any element of $A_{d+n}$ in order to define the homogenization.

**Definition 3.3** An element $P$ of $A_{d+n}[t_0]^r$ of the form

$$P = \sum_{i=1}^{r} \sum_{\lambda,\mu,\nu,\alpha,\beta} a_{\lambda\mu\nu\alpha\beta} t_0^{\lambda\mu\nu\alpha\beta} x^\alpha \partial_x^\beta e_i,$$

is said to be $F[m]$-homogeneous of order $k$ if $a_{\lambda\mu\nu\alpha\beta} = 0$ whenever $|\nu - \mu| - \lambda + m_i \neq k$.

**Definition 3.4** For an element $P$ of $A_{d+n}$ of the form (2.2), put

$$k := \min\{|\nu - \mu| + m_i | a_{\mu\nu\alpha\beta} \neq 0 \text{ for some } \alpha, \beta\}.$$

Then the $F[m]$-homogenization $h(P) \in A_{d+n}[t_0]^r$ of $P$ is defined by

$$h(P) := \sum_{i=1}^{r} \sum_{\mu,\nu,\alpha,\beta} a_{\mu\nu\alpha\beta} t_0^{\nu-\mu+k} t_0^{\mu} x^\alpha \partial_x^\beta e_i.$$

Then $h(P)$ is $F[m]$-homogeneous of order $k$.

When $m$ is the zero vector, we simply say $F$-homogeneous instead of $F[m]$-homogeneous.

**Lemma 3.5** If $P \in A_{d+n}[t_0]$ is $F$-homogeneous and $Q \in A_{d+n}[t_0]^r$ is $F[m]$-homogeneous, then $PQ$ is $F[m]$-homogeneous.

**Lemma 3.6** For $P_1, \ldots, P_k \in A_{d+n}$, put $P = P_1 + \cdots + P_k$. Then there exist $l, l_1, \ldots, l_k \in N$ so that

$$t_0^{l_0} h(P) = t_0^{l_1} h(P_1) + \cdots + t_0^{l_k} h(P_k).$$

Let us define $\varpi: N \times L \times \{1, \ldots, r\} \rightarrow L \times \{1, \ldots, r\}$ by $\varpi(\lambda, \mu, \nu, \alpha, \beta, i) = (\mu, \nu, \alpha, \beta, i)$. For a nonzero element $P = P(t_0) \in A_{d+n}[t_0]^r$, let us denote by $\text{lexp}_H(P) \in N \times L \times \{1, \ldots, r\}$ and $\text{lcoef}_H(P) \in K$ the leading exponent and the leading coefficient of $P$ with respect to $\prec_H$.

**Lemma 3.7**
1. If $P(t_0) \in A_{d+n}[t_0]^r$ is $F[m]$-homogeneous, then we have $\text{lexp}_F(P(1)) = \varpi(\text{lexp}_H(P(t_0)))$.
2. For any $P \in A_{d+n}^r$, we have $\text{lexp}_F(P) = \varpi(\text{lexp}_H(h(P)))$.

Since the Buchberger algorithm preserves the $F[m]$-homogeneity, we have

**Proposition 3.8** Let $N$ be a left $A_{d+n}[t_0]$-submodule of $A_{d+n}[t_0]^r$ generated by $F[m]$-homogeneous operators. Then there exists a Gröbner basis with respect to $\prec_H$ of $N$ consisting of $F[m]$-homogeneous operators. Moreover, such a Gröbner basis can be computed by the Buchberger algorithm.
The following proposition can be easily proved in the same way as [29, Theorem 3.12]

**Proposition 3.9** Let $N$ be a left $A_{d+n}$-submodule of $A'_{d+n}$ generated by $P_1,\ldots,P_l \in A'_{d+n}$. Let us denote by $h(N)$ the left $A_{d+n}[t_0]$-submodule of $A_{d+n}[t_0]$ generated by $h(P_1),\ldots,h(P_l)$. Let $G = \{Q_1(t_0),\ldots,Q_k(t_0)\}$ be a Gröbner basis of $h(N)$ with respect to $<_H$ consisting of $F[m]$-homogeneous operators. Then $G(1) := \{Q_1(1),\ldots,Q_k(1)\}$ is an $F[m]$-Gröbner basis of $N$.

Thus we have an algorithm of computing an $F[m]$-Gröbner basis for an arbitrary shift vector $m \in \mathbb{Z}^r$. We can prove the following in the same way as [29, Proposition 3.11]

**Proposition 3.10** Let $N$ and $Q_j(t_0)$ be as in Proposition 3.9 and put $N := D_X \otimes_{A_{d+n}} N \subset D'_X$. Then for any germ $P$ of $N$ at $p \in Y$, there exist germs $U_j$ of $D_X$ at $p$ such that $P = U_1Q_1(1) + \cdots + U_kQ_k(1)$ and $\ord_Y(m)(U_jQ_j(1)) \leq \ord_Y(m)(P)$ for $j = 1,\ldots,k$.

If the leading exponent of $P \in A_{d+n}[t_0]$ is $\text{lexp}(P) = (\lambda,\alpha,i) \in N \times L \times \{1,\ldots,r\}$, we define the leading position $l_P(H)$ of $P$ by $i$. For $\alpha,\beta \in L$ and $i \in \{1,\ldots,r\}$, we put

$$
(\alpha,i) \lor (\beta,i) := (\max\{\alpha_1,\beta_1\},\ldots,\max\{\alpha_{d+2n},\beta_{d+2n}\},i),
$$

$$
(\alpha,i) + (\beta,i) := (\alpha + \beta,i).
$$

Let $N$ and $Q_j(t_0)$ be as in Proposition 3.9 and put $\Lambda := \{(i,j) \mid 1 \leq i < j \leq k, \ l_P(H(Q_i(t_0))) = l_P(H(Q_j(t_0)))\}$. For $(i,j) \in \Lambda$, let $S_{ij}(t_0), S_{ji}(t_0) \in A_{d+n}[t_0]$ be monomials such that

$$
\text{lexp}(S_{ji}(t_0)Q_i(t_0)) \leq \text{lexp}(S_{ij}(t_0)Q_j(t_0)) \leq \text{lexp}(Q_i(t_0)) \lor \text{lexp}(Q_j(t_0)),
$$

$$
\text{lcoef}(S_{ji}(t_0)Q_i(t_0)) = \text{lcoef}(S_{ij}(t_0)Q_j(t_0)).
$$

Then by the Buchberger algorithm, there exist $F$-homogeneous $U_{ij}(t_0) \in A_{d+n}[t_0]$ so that we have

$$
S_{ji}(t_0)Q_i(t_0) - S_{ij}(t_0)Q_j(t_0) = \sum_{l=1}^k U_{ijl}(t_0)Q_l(t_0)
$$

and either $U_{ij}(t_0) \neq 0$ or else

$$
\text{lexp}(U_{ij}(t_0)Q_l(t_0)) <_H \text{lexp}(Q_i(t_0)) \lor \text{lexp}(Q_j(t_0))
$$

for each $l = 1,\ldots,k$.

The proof of the following proposition is similar to that of [29, Theorem 3.13]:

**Proposition 3.11** In the same notation as in Proposition 3.10, the left $A'_{d+n}$-module

$$
\text{Syz}(Q_1(1),\ldots,Q_k(1)) := \{(U_1,\ldots,U_k) \in A'_{d+n} \mid U_1Q_1(1) + \cdots + U_kQ_k(1) = 0\}
$$

is generated by $\{V_{ij}(1) \mid (i,j) \in \Lambda\}$ with

$$
V_{ij}(t_0) := (0,\ldots,S_{ji}(t_0),\ldots,0) - (U_{ij1}(t_0),\ldots,U_{ijk}(t_0)).
$$
Now let us describe an algorithm for computing a free resolution of $M$ which is adapted to the filtration $F_Y[m]$ (cf. Proposition 3.3). Let $N$ be a left $A_{d+n}$-submodule of $A_{d+n}$ such that $M = A_{d+n}/N$.

First, starting with a given $m \in \mathbb{Z}^r$, let $\{P_1, ..., P_r\}$ be an $F[m]$-Gröbner basis of $N$ constructed as in Proposition 3.9. Put
\[ m_1 := (\text{ord}_F[m](P_1), \ldots, \text{ord}_F[m](P_1)). \]
and define $\psi_1 : A_{d+n}^{r_1} \to A_{d+n}^{r_1}$ by
\[ \psi_1(Q_1, ..., Q_{r_1}) := Q_1P_1 + \cdots + Q_{r_1}P_{r_1}. \]
Then we get a set of generators of the kernel $\ker N := \ker \psi_1$. Proceeding in the same way, we can obtain a free resolution (2.3) which is adapted to the $F[m]$-filtration for any given $l \in \mathbb{N}$.

### 4 The $b$-function of a $D$-module

Let $M$ be a left coherent $D_X$-module on $X$. We assume that a left $A_{d+n}$-submodule $N$ of $A_{d+n}$ is given explicitly so that $M = D_X \otimes_{A_{d+n}} M$ holds with $M := A_{d+n}/N$. Set $N := D_X \otimes_{A_{d+n}} N \subset D_X^r$. We fix an arbitrary shift vector $m = (m_1, \ldots, m_r) \in \mathbb{Z}^r$ and put
\[
\text{gr}_Y^k[m](N) := F_Y^k[m](N)/F_Y^{k-1}[m](N),
\]
\[
\text{gr}_Y^k[m](M) := F_Y^k[m](M)/F_Y^{k-1}[m](M).
\]
They are left $\text{gr}_Y^0(D_X)$-modules. The $F[m]$-filtration and the associated graded module are defined also for $A_{d+n}$-modules. Moreover we have
\[
\text{gr}_Y^k[m](M) = \text{gr}_Y^k(D_X)\text{gr}_Y^0[m](M).
\]
We put $\vartheta := t_1\partial_{t_1} + \cdots + t_d\partial_{t_d}$. This is the unique vector field modulo $F_Y^{-1}(D_X)$ that operates on $J_Y/J_Y^2$ as identity. Let $\theta$ be a commutative variable corresponding to $\vartheta$. 

...
**Definition 4.1** The \( b \)-function (or the indicial polynomial) \( b(\theta, p) \in K[\theta] \) of \( M \) along \( Y \) with respect to the filtration \( F_Y[M] \) at \( p \in Y \) is the monic polynomial \( b(\theta, p) \in K[\theta] \) in \( s \) of the least degree, if any, that satisfies

\[
b(\theta, p)\text{gr}_Y^0[M](M)_p = 0.
\]

If such \( b(\theta, p) \) exists, \( M \) is called specializable along \( X \) at \( p \). If \( M \) is not specializable at \( p \), we put \( b(\theta, p) = 0 \). The global \( b \)-function \( b(\theta) \) of \( M \) along \( Y \) is defined to be the least common multiple of \( b(\theta, p) \) with \( p \) running through \( Y \).

It is known that the specializability does not depend on the shift vector \( m \) while the \( b \)-function can depend on it (cf. \[23\]). It is also known that if \( M \) is holonomic, then \( M \) is specializable at any \( p \in X \) (\[18\], \[19\], \[21\]).

First, we reduce to the case \( r = 1 \). For each \( i = 1, \ldots, r \), let \( \pi_i \) be the projection of \( \text{gr}_Y^0[M](D_X) \) to the \( i \)-th component and put

\[
\text{gr}_Y^0[M](N)^{(i)} := \{ P \in \text{gr}_Y^0[M](N) \mid \pi_j(P) = 0 \text{ for } j = i + 1, \ldots, r \}.
\]

Note that \( \text{gr}_Y^0[M](N)^{(i)} / \text{gr}_Y^0[M](N)^{(i-1)} \) can be regarded as a left ideal of \( \text{gr}_Y^0(D_X) \) by the projection to the \( i \)-th component. Then we get the following lemma:

**Lemma 4.2** Under the above notation, \( b(\theta, p) \) is a generator of the ideal

\[
\bigcap_{i=1}^{r} \left( K[\theta] \cap (\text{gr}_Y^0[M](N)^{(i)} / \text{gr}_Y^0[M](N)^{(i-1)})_p \right).
\]

Let us now assume that the order \( \prec \) satisfies

\[
(\tilde{\alpha}, i) \prec (\tilde{\alpha}', j) \text{ if } i < j \text{ for } \tilde{\alpha}, \tilde{\alpha}' \in L \text{ and } i, j \in \{1, \ldots, r\}.
\]

Let \( G \) be an \( F[m] \)-Gröbner basis of \( N \) with respect to \( \prec_F \) defined by \( \prec \) as in Section \[3\]. Then \( \hat{I} := \text{gr}_Y[m](N)^{(i)} / \text{gr}_Y[m](N)^{(i-1)} \) is generated by

\[
\{ \pi_i(\sigma_Y[m](P)) \mid P \in G, \sigma_Y[m](P) \in \text{gr}_Y[m](N)^{(i)} \}.
\]

Our next task is to compute the intersection \( \hat{I} \cap D_Y[t_1 \partial_1, \ldots, t_d \partial_d] \). For this purpose, we introduce commutative indeterminates \( v = (v_1, \ldots, v_d) \) and \( w = (w_1, \ldots, w_d) \), and work with the ring \( A_{d+n}[v, w] \). For an element \( P \) of \( A_{d+n} \) of the form

\[
P = \sum_{(\mu, \nu,\alpha,\beta) \in L} a_{\mu,\nu,\alpha,\beta} t_\mu \partial_\nu^\alpha x_\beta \partial_\xi^\beta,
\]

its multi-homogenization \( \text{mh}(P) \in A_{d+n}[v] \) is defined by

\[
\text{mh}(P) := \sum_{(\mu, \nu,\alpha,\beta) \in L} a_{\mu,\nu,\alpha,\beta} v_1^{\mu_1-\mu} \cdots v_d^{\mu_d-\mu_d} t_\mu \partial_\nu^\alpha x_\beta \partial_\xi^\beta.
\]
with \( \kappa_j := \min \{ \nu_j - \mu_j \mid a_{\mu \nu \alpha} \neq 0 \} \). Let \( \prec_{mh} \) be an order on \( \mathbb{N}^d \times \mathbb{N}^d \times L \ni (\rho, \sigma, \tilde{\alpha}) \) defined by

\[
(\rho, \sigma, \tilde{\alpha}) \prec_{mh} (\rho', \sigma', \tilde{\alpha}') \text{ if and only if } |\rho + \sigma| < |\rho' + \sigma'| \text{ or else } |\rho + \sigma| = |\rho' + \sigma'|, \tilde{\alpha} < \tilde{\alpha}'
\]

with an arbitrary well-order \( \prec \) on \( L \) satisfying (3.1). Fixing an \( i \in \{1, \ldots, d\} \), we assign weight 1 to \( w_i, \partial_{t_i} \), weight \(-1\) to \( v_i, t_i \), and weight 0 to all the other variables. An element of \( A_{d+n} [v, w] \) is said to be multi-homogeneous if it is homogeneous with respect to the weight above for each \( i = 1, \ldots, d \). Thus \( \text{mh}(P) \) is multi-homogeneous for any \( P \in A_{d+n} \). Put \( S_\kappa := S_1 \kappa_1 \cdots S_d \kappa_d \) for \( \kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{Z}^d \) with \( S_{ij} = \partial_{t_i} \) if \( j \geq 0 \) and \( S_{ij} := t_i^{-j} \) otherwise. Let \( s = (s_1, \ldots, s_d) \) be commutative indeterminates. Assume that \( P \in A_{d+n} \) is multi-homogeneous. Then we have

\[
S_\kappa P = Q(t_1 \partial_{t_1}, \ldots, t_d \partial_{t_d}, x, \partial_x)
\]

with some \( Q(s_1, \ldots, s_d, x, \partial_x) \in A_n [s_1, \ldots, s_d] \) and \( \kappa \in \mathbb{Z}^d \). We put

\[
\psi(P)(s_1, \ldots, s_d) := Q(s_1, \ldots, s_d).
\]

**Proposition 4.3** Let \( \hat{I} \) be a left ideal of \( \text{gr}_Y(\mathcal{D}_X) \). Let \( G_0 \) be a finite subset of \( \text{gr}_Y(\mathcal{D}_X) \) which generates \( \hat{I} \). Let \( G_1 \) be a Gröbner basis with respect to \( \prec_{mh} \) of the ideal of \( A_{d+n} [v, w] \) generated by

\[
\{ \text{mh}(P) \mid P \in G_0 \} \cup \{ 1 - v_i w_i \mid i = 1, \ldots, d \}.
\]

We may assume that \( G_1 \) consists of multi-homogeneous elements since so does the input. Then the left ideal \( \hat{I} \cap \mathcal{D}_Y[t_1 \partial_{t_1}, \ldots, t_d \partial_{t_d}] \) of \( \mathcal{D}_Y[t_1 \partial_{t_1}, \ldots, t_d \partial_{t_d}] \) is generated by

\[
G_2 := \{ \psi(P)(t_1 \partial_{t_1}, \ldots, t_d \partial_{t_d}) \mid P \in G_1 \cap A_{d+n} \}.
\]

**Proof:** Let \( P \) be an element of \( G_1 \cap A_{d+n} \). Since \( P \) is multi-homogeneous and free of \( v, w \), there exists \( \kappa \in \mathbb{Z}^d \) so that \( Q := \psi(P)(t_1 \partial_{t_1}, \ldots, t_d \partial_{t_d}) = S_\kappa P \). By definition, \( P \) belongs to the ideal generated by \( \text{mh}(G_0) \) and \( 1 - v_i w_i \) (\( i = 1, \ldots, d \)). Setting \( v_i = w_i = 1 \), we know that \( Q \) belongs to \( \hat{I} \).

Conversely, let \( P \) be an arbitrary germ of \( \hat{I} \cap \mathcal{D}_Y[t_1 \partial_{t_1}, \ldots, t_d \partial_{t_d}] \) at \( p \in Y \). Multiplying \( P \) by a polynomial in \( x \) which does not vanish at \( p \), we may assume \( P \in A_n [t_1 \partial_{t_1}, \ldots, t_d \partial_{t_d}] \). In view of the definition of the multi-homogenization and the fact that \( \text{mh}(P) = P \), there exists \( \rho \in \mathbb{N}^d \) so that \( v^\rho P \) belongs to the ideal generated by \( \text{mh}(G_0) \). This implies that \( P \) belongs to the ideal generated by \( \text{mh}(G_0) \) and \( \{ 1 - v_i w_i \mid i = 1, \ldots, d \} \) since

\[
P = (1 - v^\rho w^\rho) P + v^\rho w^\rho P
\]

and \( (1 - v^\rho w^\rho) \) belongs to the ideal generated by \( \{ 1 - v_i w_i \mid i = 1, \ldots, d \} \). Set \( G_1 \cap A_{d+n} = \{ P_1, \ldots, P_k \} \). Then by the definition of \( \prec_{mh} \) and \( G_1 \), there exist \( Q_1, \ldots, Q_k \in A_{d+n} \) so that

\[
P = Q_1 P_1 + \cdots + Q_k P_k.
\]
Lemma 4.4 Let each ideal on the right hand side can be computed easily by the following:

Then by the flatness of \( (O) \) generality: So let

This completes the proof. []

Let \( \hat{J} \) be as in Proposition \[13\]. Now we have obtained a set of generators \( G_2 \) of \( \hat{J} \cap \mathcal{D}_Y[t_1 \partial_{t_1}, \ldots, t_d \partial_{t_d}] \). We identify each \( t_i \partial_{t_i} \) with \( s_i \). Then from \( G_2 \), we can compute a set of generators \( G_3 \) of the ideal \( \hat{J} \cap \mathcal{O}_Y[s] \) of \( \mathcal{O}_Y[s] \) by eliminating \( \partial_x \) by means of Gröbner basis in the Weyl algebra (see e.g. [27] for details). Then it is easy to obtain a subset \( G_4 \) of \( K[x, \theta] \) which generates the sheaf of ideals

with \( \theta = s_1 + \cdots + s_d \) again by Gröbner basis in the polynomial ring. Let us denote by \( J \) the ideal of \( K[x, \theta] \) generated by \( G_4 \). Then \( b(\theta, p) \) is a generator of

Our final task is to compute the \( b \)-function at each point of \( Y \) by using the input \( G_4 \). This is achieved by primary decomposition. Let us state the method in a more general setting, where we replace the variable \( \theta \) by the variables \( s = (s_1, \ldots, s_d) \) for the sake of generality: So let \( J \) be an arbitrary ideal of \( K[x, s] \) whose generators are given. For each point \( p \) of \( Y \), put

which is an ideal of \( K[s] \). Let \( J = Q_1 \cap \cdots \cap Q_t \) be a primary decomposition in \( K[x, s] \). Then by the flatness of \( \mathcal{O}_Y[s] \) over \( K[x, s] \) we have

Each ideal on the right hand side can be computed easily by the following:

Lemma 4.4 Let \( Q \) be a primary ideal of \( K[x, s] \) and put

Then we have

Proof: First assume \( p \notin V_Y(Q) \). Then there exists \( a(x) \in K[x] \) such that \( a(p) \neq 0 \). This implies that \( B(Q, p) = K[s] \). Next assume \( p \in V_Y(Q) \) and \( b(s) \in B(Q, p) \). Then there exists \( a(x) \in K[x] \) so that \( a(x)b(s) \in Q \) and \( a(p) \neq 0 \). Suppose \( b(s) \) does not belong to \( Q \). Then we have \( a(x)b(s) \in Q \) with some \( j \in \mathbb{N} \) since \( Q \) is primary. This implies \( a(p) = 0 \), which is a contradiction. Thus we have \( B(Q, p) \subseteq Q \cap K[s] \). The converse inclusion is obvious. []
Lemma 4.5 Let \( J \) be an ideal of \( K[x, s] \). Then we have
\[
\bigcap_{p \in Y} B(J, p) = J \cap K[s].
\]

Proof: Let \( J = Q_1 \cap \cdots \cap Q_l \) be a primary decomposition. Then by the preceding lemma, we have
\[
\bigcap_{p \in Y} B(J, p) = \bigcap_{p \in Y} \bigcap_{j=1}^l B(Q_j, p)
\]
\[
= \bigcap_{j=1}^l (Q_j \cap K[s] \mid Q_j \cap K[x] \neq K[x])
\]
\[
= J \cap K[s]
\]
since \( Q_j \cap K[s] = K[s] \) if and only if \( Q_j \cap K[x] = K[x] \).

Returning back to the ideal \( J \) of \( K[s, \theta] \) generated by \( G_k \), we have only to apply Lemma 4.4 with \( s \) replaced by \( \theta \). This gives us an algebraic stratification of \( Y \) so that \( b(s, p) \) is constant on each stratum as a function of \( p \). Moreover, Lemma 4.5 tells us that the global \( b \)-function of \( \mathcal{M} \) along \( Y \) is simply a generator of \( J \cap K[\theta] \). Thus the algorithm is summarized as follows:

Algorithm 4.6 (The \( b \)-function of \( \mathcal{M} := \mathcal{D}_X \otimes_{A_{d+n}} M \))

Input: \( M = A_{r+d+n}/N \) with an \( A_{d+n} \)-submodule \( N \) of \( A_{r+d+n} \), and \( m \in \mathbb{Z}^r \).

1. Compute a Gröbner basis \( G \) of \( N \) with respect to the order \( \prec_F \) that is defined through (3.2) by using \( m \) and an order \( \prec \) satisfying (3.1) and (4.1).

2. For \( i = 1 \) to \( r \) do
   (a) \( G_i := \{ \pi_i(\sigma_Y[m](P)) \mid P \in G, \; \pi_j(\sigma_Y[m](P)) = 0 \text{ for any } j > i \} \).
   (b) Let \( G_{i1} \) be a Gröbner basis of the left ideal of \( A_{d+n}[v, w] \) generated by
   \[
   \{ mh(P) \mid P \in G_i \} \cup \{ 1 - v_i w_i \mid i = 1, \ldots, d \}
   \]
   with respect to an order \( \prec_{mh} \) satisfying (4.2).
   (c) \( G_{i2} := \{ \psi(P) \in A_n[s] \mid P \in G_{i1} \cap A_{d+n} \} \).
   (d) Compute \( J_i := \langle G_{i2} \rangle \cap K[x, \theta] \) first by eliminating \( \partial_x \), then eliminating \( s_2', \ldots, s_d' \) after substitution \( \theta = s_1 + \cdots + s_d \) and \( s_j' = s_j \) for \( j = 2, \ldots, d \); here \( \langle G_{i2} \rangle \) denotes the left ideal of \( A_n[s] \) generated by \( G_{i2} \).

3. The global \( b \)-function \( b(\theta) \) of \( \mathcal{M} \) is the generator of \( \bigcap_{i=1}^r (J_i \cap K[\theta]) \).

4. For \( i = 1 \) to \( r \) do
   (a) Compute a primary decomposition \( J_i = \bigcap_{j=1}^{l_i} Q_{ij} \) in \( K[x, \theta] \).
   (b) Compute (generators of) \( Q_{ij} \cap K[s] \) and \( Q_{ij} \cap K[x] \) for \( j = 1, \ldots, l_i \) by elimination.
5. For each $p \in Y$, the local $b$-function $b(\theta, p)$ is the generator of the ideal

$$\bigcap_{i=1}^{r} \bigcap_{j=1}^{l_i} \{ Q_{ij} \cap K[\theta] \mid g(p) = 0 \text{ for any } g(x) \in Q_{ij} \cap K[x] \}.$$  

Let us remark on the coefficient field: Suppose that the input is defined over a subfield $K_0$ of $K$. Then the steps 1–3 can be done over $K_0$ instead of $K$ and $b(\theta, p)$ divides $b(\theta)$ for any $p \in Y$. However, the primary decomposition in the step 4 must be one in $K[x, \theta]$ not in $K_0[x, \theta]$. In fact, we need a primary decomposition over an intermediate field $K_1$ with $K_0 \subset K_1 \subset K$ so that $b(\theta)$ factors into linear polynomials in $K_1[\theta]$. If, e.g., $K_0$ is the rationals $\mathbb{Q}$, such $K_1$ is computable. Hence the primary decomposition in the step 4 is certainly computable if the input is defined over $\mathbb{Q}$ in view of e.g., [2], [10], [34] and gives the local $b$-function at any $p \in Y = K^n$.

As a special case where all the computation can be done over $K_0$, suppose that the ideal $J \cap K_0[\theta]$ is generated by a polynomial which is a multiple of linear factors over $K_0$. Then the step 4 of Algorithm 4.6 can be computed over $K_0$ and the step 5 is true for any $p \in K^n$; one can easily verify this by considering a projection of $K$ to $K_0$. Note that this is exactly the case with the classical Bernstein-Sato polynomial (cf. [29], [30], [31] for algorithms) and $K_0 = \mathbb{Q}$ by virtue of Kashiwara's theorem on the rationality [13].

At this occasion, let us make a correction to [31]: Lemma 4.4 of [31] does not hold in general; we need field extension as explained above. This correction does not affect the rest of [31].

Example 4.7 Put $X := K^5 = \langle t_1, t_2, x, y, z \rangle$ and $Y := \{(t_1, t_2, x, y, z) \in X \mid t_1 = t_2 = 0\}$.

Put $M := A_5/I$ with the left ideal $I$ generated by

$$t_1 - x^3 + y^2, \quad t_2 - y^3 + z^2, \quad \partial_x + 3x^2\partial_t_1, \quad \partial_y - 2y\partial_t_1 + 3y^2\partial_t_2, \quad \partial_z - 2z\partial_t_2.$$  

Then the $b$-function $b(s, p)$ of $M := \mathcal{D}_X \otimes_{A_5} M$ along $Y$ at $p \in Y = K^3$ is given as follows:

$$b(s, p) = s \left( s - \frac{5}{18} \right) \left( s - \frac{1}{6} \right) \left( s - \frac{11}{18} \right) \left( s + \frac{1}{18} \right) \left( s + \frac{1}{6} \right) \times \left( s + \frac{5}{18} \right) \left( s + \frac{7}{18} \right) \left( s + \frac{11}{18} \right) \left( s + \frac{2}{3} \right)$$

if $p = (0, 0, 0) \in Y$; $b(s, p) = s$ if $p \in \{(x, y, z) \mid x^3 - y^2 = 0, \quad y^3 - z^2 = 0\}$; and $b(s, p) = 1$ otherwise.

5 Restriction of a $D$-module

We retain the notation of the preceding sections. Put

$$\mathcal{D}_{Y \to X} := \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{D}_X.$$  

Then $\mathcal{D}_{Y \to X}$ has a natural structure of $(\mathcal{D}_Y, \mathcal{D}_X)$-bimodule. Let $\mathcal{M} = \mathcal{D}_X \otimes_{A_{d+n}} M$ be a coherent $\mathcal{D}_X$-module with a finitely generated $A_{d+n}$-module $M = A_{d+n}^r/N$. Then the (D-module theoretic) restriction of $\mathcal{M}$ to $Y$ is defined by

$$\mathcal{M}^*_Y := \mathcal{D}_{Y \to X} \otimes^{\mathcal{L}}_{\mathcal{D}_X} \mathcal{M}.$$
in the derived category of left \( D_X \)-modules (see [12] for the derived category and derived functors).

In general, let \( \mathcal{L} \) be a \( K[t] \)-module and \( \mathcal{L}_j \) \( (j \in \mathbb{Z}) \) be additive subgroups of \( \mathcal{L} \) such that \( t_i \mathcal{L}_j \subset \mathcal{L}_{j-1} \) holds for \( i = 1, \ldots, d \) and \( j \in \mathbb{Z} \). Then for any integer \( k \), we define the Koszul complex associated with \( \mathcal{L}_* = \{ \mathcal{L}_j \}_{j \in \mathbb{Z}} \) and \( t_1, \ldots, t_d \) by

\[
0 \longrightarrow \mathcal{L}_{k+d} \otimes_{\mathbb{Z}} \mathbb{Z}^d \xrightarrow{\delta} \mathcal{L}_{k+d-1} \otimes_{\mathbb{Z}} \mathbb{Z}^d \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{L}_{k} \otimes_{\mathbb{Z}} \mathbb{Z}^d \longrightarrow 0,
\]

where \( \delta \) is defined by

\[
\delta(u \otimes e_{i_1} \wedge \cdots \wedge e_{i_j}) = \sum_{l=1}^{d} t_l u \otimes e_l \wedge e_{i_1} \wedge \cdots \wedge e_{i_j}
\]

for a subset \( \{i_1, \ldots, i_j\} \) of \( \{1, \ldots, d\} \) with the unit vectors \( e_1, \ldots, e_d \) of \( \mathbb{Z}^d \). We denote this complex by \( K^\bullet(\mathcal{L}_*[k], t_1, \ldots, t_d) \). When \( \mathcal{L}_j = \mathcal{L} \) for each \( j \), we also denote it simply by \( K^\bullet(\mathcal{L}, t_1, \ldots, t_d) \). Here we regard \( \mathcal{L}_{k+d-j} \otimes \mathbb{Z}^d \) as being placed at the degree \( -j \) to be compatible with the cohomology theory.

In particular, \( K^\bullet(D_X, t_1, \ldots, t_d) \) is quasi-isomorphic to \( D_{Y \to X} \) in the derived category of right \( D_X \)-modules. Hence we can identify \( M^*_Y \) with the complex

\[
K^\bullet(D_X, t_1, \ldots, t_d) \otimes_{D_X} \mathcal{M} = K^\bullet(M, t_1, \ldots, t_d).
\]

Our purpose below is to describe an algorithm to compute each cohomology group \( H^i(M^*_Y) \) (for \( i = 0, -1, \ldots, -d \) since it is zero for other \( i \)) under the assumption that \( M \) is specializable along \( Y \). Let \( b(\theta) \) be the global \( b \)-function of \( M \) with respect to the filtration \( F_Y[m] \) with a given \( m \in \mathbb{Z}^r \). In what follows, we can replace \( b(\theta) \) by the local \( b \)-function \( b(\theta, p) \) in order to compute \( M^*_Y \) locally, i.e., on a Zariski neighborhood of \( p \).

**Proposition 5.1** Let \( k \) be an integer such that \( b(k) \neq 0 \). Then the Koszul complex \( K^\bullet(\text{gr}^*_Y[m](M)[k], t_1, \ldots, t_d) \) associated with \( \{ \text{gr}^*_Y[m](M) \}_{j \in \mathbb{Z}} \) is exact.

We shall prove this proposition in a slightly more general situation. Let \( A_d := K[t] \langle \partial_t \rangle \) be the Weyl algebra on the variables \( t = (t_1, \ldots, t_d) \) and define a filtration on it and the associated graded module by

\[
F_k(A_d) := \{ \sum_{\mu, \nu \in \mathbb{N}^d} a_{\mu, \nu} t^\mu \partial_t^\nu | |\nu - \mu| \leq k \}, \quad \text{gr}_k(A_d) := F_k(A_d)/F_{k-1}(A_d).
\]

Note that \( \text{gr}(A_d) := \bigoplus_{k \in \mathbb{Z}} \text{gr}_k(A_d) \) is isomorphic to \( A_d \). In particular, we can regard \( \text{gr}_0(A_d) \) as a subring of \( A_d \).

**Proposition 5.2** Let \( \mathcal{L} = \bigoplus_{j \in \mathbb{Z}} \mathcal{L}_j \) be a graded \( \text{gr}(A_d) \)-module; i.e., assume \( \text{gr}_j(A_d) \mathcal{L}_i \subset \mathcal{L}_{i+j} \) for \( i, j \in \mathbb{Z} \). Assume moreover that there exists a nonzero polynomial \( b(\theta) \in K[\theta] \) which satisfies \( b(\theta + j) \mathcal{L}_j = 0 \) for any \( j \in \mathbb{Z} \) with \( \theta = t_1 \partial_{t_1} + \cdots + t_d \partial_{t_d} \). Let \( k \) be an integer such that \( b(k) \neq 0 \). Then \( K(\mathcal{L}_*[k], t_1, \ldots, t_d) \) is exact.
Proof: We argue by induction on $d$. First suppose $d = 1$. Then $\mathcal{K}^\bullet(\mathcal{L}_\bullet[k], t_1)$ is the complex

$$0 \longrightarrow \mathcal{L}_{k+1} \overset{t_1}{\longrightarrow} \mathcal{L}_k \longrightarrow 0.$$ 

Assume $u \in \mathcal{L}_{k+1}$ satisfies $t_1u = 0$. Then we have $u = 0$ since

$$0 = b(t_1 \partial_{t_1} + k + 1)u = b(\partial_{t_1} t_1 + k)u = b(k)u.$$ 

On the other hand, there exists $P \in \mathcal{A}_d$ so that $b(t_1 \partial_{t_1} + k) = t_1 P + b(k)$. Hence for an arbitrary $v \in \mathcal{L}_k$, we conclude $v \in t_1 \mathcal{L}_{k+1}$ from $b(t_1 \partial_{t_1} + k) v = 0$.

Now assume the proposition is true with $d$ replaced by $d - 1$. It is easy to see, as in the case of the usual Koszul complex (see e.g., [33, p.188]), that $\mathcal{K}^\bullet(\mathcal{L}_\bullet[k], t_1, \ldots, t_d)$ is quasi-isomorphic to the complex associated with the double complex

$$\begin{array}{c}
\mathcal{K}^\bullet(\mathcal{L}_\bullet[k + 1], t_1, \ldots, t_d) \\
\downarrow \quad t_d \\
\mathcal{K}^\bullet(\mathcal{L}_\bullet[k], t_1, \ldots, t_{d-1}).
\end{array}$$ (5.1)

Let us denote by $\mathcal{L}'_j$ and $\mathcal{L}''_j$ the kernel and the cokernel of $t_d : \mathcal{L}_{j+1} \longrightarrow \mathcal{L}_j$. Then $\mathcal{L}' := \bigoplus_{j \in \mathbb{Z}} \mathcal{L}'_j$ and $\mathcal{L}'' := \bigoplus_{j \in \mathbb{Z}} \mathcal{L}''_j$ are graded $\text{gr}(\mathcal{A}_{d-1})$-modules. For any $u \in \mathcal{L}'_j$, we have

$$0 = b(t_1 \partial_{t_1} + \cdots + t_d \partial_{t_d} + j + 1)u$$

$$= b(t_1 \partial_{t_1} + \cdots + t_{d-1} \partial_{t_{d-1}} + \partial_{t_d} t_d + j)u$$

$$= b(t_1 \partial_{t_1} + \cdots + t_{d-1} \partial_{t_{d-1}} + j)u.$$ 

On the other hand, for $v \in \mathcal{L}_j$, let $\overline{v}$ be its residue class in $\mathcal{L}''_j$. Then we have

$$0 = b(t_1 \partial_{t_1} + \cdots + t_d \partial_{t_d} + j)\overline{v} = b(t_1 \partial_{t_1} + \cdots + t_{d-1} \partial_{t_{d-1}} + j)\overline{v}.$$

Thus both $\mathcal{L}'$ and $\mathcal{L}''$ satisfy the conditions of the proposition with $d$ replaced by $d - 1$. By the induction hypothesis, the complexes $\mathcal{K}^\bullet(\mathcal{L}'_\bullet[k]; t_1, \ldots, t_d)$ and $\mathcal{K}^\bullet(\mathcal{L}''_\bullet[k]; t_1, \ldots, t_d)$ are exact. Hence the vertical chain map of (5.1) is a quasi-isomorphism, which implies that $\mathcal{K}^\bullet(\mathcal{L}_\bullet[k], t_1, \ldots, t_d)$ is exact. []

Under the assumption of Proposition 5.1, we have $b(\vartheta + j)\text{gr}_Y^j[\mathfrak{m}](\mathcal{M}) = 0$ for any $j \in \mathbb{Z}$. In fact, for $P \in \text{gr}_Y^j(\mathcal{D}_X)$, we easily get $b(\vartheta + j)P = Pb(\vartheta)$. This yields

$$b(\vartheta + j)\text{gr}_Y^j[\mathfrak{m}](\mathcal{M}) = b(\vartheta + j)\text{gr}_Y^j(\mathcal{D}_X)\text{gr}_Y^0[\mathfrak{m}](\mathcal{M})$$

$$= \text{gr}_Y^j(\mathcal{D}_X)b(\vartheta)\text{gr}_Y^0[\mathfrak{m}](\mathcal{M}) = 0.$$ 

Hence Proposition 5.1 is an immediate consequence of Proposition 5.2.

Now for $\mathfrak{m} \in \mathbb{Z}^r$, we define the $F_Y[\mathfrak{m}]$-filtration on $\mathcal{D}_{Y \rightarrow X}$ by

$$F_Y^k[\mathfrak{m}](\mathcal{D}_{Y \rightarrow X}) \ := \ F_Y^k[\mathfrak{m}](\mathcal{D}_X)/(t_1 F_Y^{k+1}[\mathfrak{m}](\mathcal{D}_X) + \cdots + t_d F_Y^{k+1}[\mathfrak{m}](\mathcal{D}_X))$$

$$\simeq \ {P = \sum_{i=1}^r \sum_{\nu, \beta} a_{\nu \beta}(x) \partial_\nu^i \partial_x^\beta e_i \mid a_{\nu \beta}(x) = 0 \text{ if } |\nu| > k - m_i}$$

$$= \bigoplus_{i=1}^r \bigoplus_{|\nu| \leq k - m_i} \mathcal{D}_Y.$$
Theorem 5.3 For an arbitrary $m \in \mathbb{Z}^r$, let us take a free resolution (2.1) of Definition 2.1 of length $l = d + 1$, which is adapted to the $F_Y[m]$-filtration. Let $m_1, \ldots, m_{d+1}$ be the associated shift vectors. Take two integers $k_0 \leq k_1$ so that the global $b$-function $b(\theta)$ of $M$ satisfies $b(k) \neq 0$ for any integer $k$ with $k < k_0$ or else $k > k_1$. Then $M_Y^\bullet$ is quasi-isomorphic to the complex

$$
\cdots \rightarrow F^k_Y[m_{d+1}](D^{d+1}_{r \rightarrow X}) \xrightarrow{\psi_{d+1}} \cdots \xrightarrow{\psi_1} F^k_Y[m_0](D^0_Y \rightarrow X) \rightarrow 0 \quad (5.2)
$$

with $r_0 = r$ and $m_0 = m$, where $\psi_j$ is a homomorphism induced by $\psi_j$. In particular, we have $M_Y^\bullet = 0$ if $b(k) \neq 0$ for any $k \in \mathbb{Z}$.

Proof: For any $k \in \mathbb{Z}$, the complex

$$
\cdots \rightarrow F^k_Y[m_{d+1}](D^{d+1}_X) \xrightarrow{\psi_{d+1}} \cdots \xrightarrow{\psi_1} F^k_Y[m_0](D^0_X) \rightarrow 0
$$

is quasi-isomorphic to $F^k_Y[m](M)$ in view of the exact sequence (2.1) of Definition 2.1. Hence we know that $K^\bullet(\text{gr}^Y_Y[m](M)[k], t_1, \ldots, t_d)$ is quasi-isomorphic to the complex associated with the double complex

$$
0 \\
\uparrow \\
K^\bullet(\text{gr}^Y_Y[m_0](D^0_X)[k], t_1, \ldots, t_d) \\
\uparrow \psi_1 \\
\vdots \\
\uparrow \psi_{d+1} \\
K^\bullet(\text{gr}^Y_Y[m_{d+1}](D^{d+1}_X)[k], t_1, \ldots, t_d) \\
\uparrow \\
\vdots
$$

On the other hand, we have a quasi-isomorphism

$$
K^\bullet(\text{gr}^Y_Y[m_i](D^i_X)[k], t_1, \ldots, t_d) \simeq \text{gr}^k_Y[m_i](D^i_Y \rightarrow X).
$$

Hence $K^\bullet(\text{gr}^Y_Y[m](M)[k], t_1, \ldots, t_d)$ is quasi-isomorphic to the complex

$$
\cdots \rightarrow \text{gr}^k_Y[m_{d+1}](D^{d+1}_Y \rightarrow X) \xrightarrow{\psi_{d+1}} \cdots \xrightarrow{\psi_1} \text{gr}^k_Y[m_0](D^0_Y \rightarrow X) \rightarrow 0. \quad (5.3)
$$

By virtue of Proposition 5.1, the complex (5.3) is exact if $b(k) \neq 0$. This implies the theorem since we have

$$
F^k_Y[m_i](D^i_Y \rightarrow X) = 0
$$
for sufficiently small \( k \in \mathbb{Z} \). \[ \]

Note that \( F_{Y_{\mathcal{Y}}}^{k_1}[m_i](\mathcal{D}_{Y_{\mathcal{Y}}})/F_{Y_{\mathcal{Y}}}^{k_0-1}[m_i](\mathcal{D}_{Y_{\mathcal{Y}}}) \) is a free \( \mathcal{D}_Y \)-module of rank

\[
\sum_{j=1}^{r_i} z^d \{ \nu \in \mathbb{Z}^d \mid k_0 - m_{ij} \leq |\nu| \leq k_1 - m_{ij} \}.
\]

Hence Theorem 5.3 gives us an algorithm to compute each cohomology group \( \mathcal{H}^i(\mathcal{M}_{\mathcal{Y}}^*) \). In fact, we have only to compute the cohomology groups of the complex (5.2) as left \( \mathcal{D}_Y \)-modules with \( \mathcal{D}_{Y_{\mathcal{Y}}} \) replaced by \( A_n[\partial_i] \). The flatness of \( \mathcal{D}_Y \) over \( A_n \) assures us that the generators of the cohomology group over \( A_n \) also give the ones over \( \mathcal{D}_Y \). The algorithm is summarized as follows:

**Algorithm 5.4** (The cohomology groups of the restriction of \( \mathcal{M} \) to \( Y \))

Input: \( M = A_{d+n}^r/N \) with an \( A_{d+n} \)-submodule \( N \) of \( A_{d+n}^r \).

Output: \( \mathcal{H}^{-i}(\mathcal{M}_{\mathcal{Y}}^*) = \mathcal{D}_Y \otimes_{\mathcal{A}_n} (A_{n}^i/I_i) \) for \( 0 \leq i \leq d \).

1. Choose an arbitrary \( m \in \mathbb{Z}^r \); e.g., take \( m = (0, \ldots, 0) \) by default.

2. Compute the global \( b \)-function \( b(\theta) \) of \( \mathcal{M} \) along \( Y \) by the steps 1–3 of Algorithm 4.6 with \( M \) and \( m \) as input.

3. If \( b(\theta) = 0 \), then \( \mathcal{M} \) is not globally specializable along \( Y \); quit.

4. Let \( k_0 \) and \( k_1 \) be the minimum and the maximum integral root of \( b(\theta) = 0 \). If there is no integral root, then we have \( \mathcal{H}^i(\mathcal{M}_{\mathcal{Y}}^*) = 0 \) for all \( i \); quit.

5. Compute a free resolution

\[
A_{d+n}^{r_{d+1}} \xrightarrow{\psi_{d+1}} A_{d+n}^{r_d} \xrightarrow{\psi_d} \cdots \xrightarrow{\psi_2} A_{d+n}^{r_1} \xrightarrow{\psi_1} A_{d+n}^{r_0} \xrightarrow{\varphi} M \rightarrow 0
\]

of \( M \) adapted to the \( F_Y[m] \)-filtration (cf. Proposition 2.3) and the shift vectors \( m_1, \ldots, m_{d+1} \) by using Proposition 3.1 successively, or by using Theorem 9.10.

6. Compute the induced complex

\[
\frac{F_{Y_{\mathcal{Y}}}^{k_1}[m_{d+1}](A_n[\partial_i]^{r_{d+1}})}{F_{Y_{\mathcal{Y}}}^{k_0-1}[m_{d+1}](A_n[\partial_i]^{r_{d+1}})} \xrightarrow{\psi_{d+2}} \cdots \xrightarrow{\psi_2} \frac{F_{Y_{\mathcal{Y}}}^{k_1}[m_0](A_n[\partial_i]^{r_0})}{F_{Y_{\mathcal{Y}}}^{k_0-1}[m_0](A_n[\partial_i]^{r_0})} \rightarrow 0
\]

as a complex of finitely generated free left \( A_n \)-modules, where \( A_n[\partial_i] \) is identified with \( A_{d+n}/(t_1 A_{d+n} + \cdots + t_d A_{d+n}) \). Put \( \psi_0 := 0 \).

7. Via Gröbner bases of modules over \( A_n \), compute the \(-i\)-th cohomology group \( \ker \psi_i/\text{Im} \psi_{i+1} \) of the above complex in the form \( A_n^i/I_i \) with a left \( A_n \)-module \( I_i \) for \( i = 0, \ldots, d \).
Note that in the step 5 of the above algorithm, only \( \psi_1, \ldots, \psi_{i_0+1} \) are needed if one wants to compute only the \(-i\)-th cohomology groups for \( i = 0, \ldots, i_0 \). In particular, one does not need the free resolution to compute only the 0-th cohomology as will be the case with Algorithm 6.4.

As a direct application of the algorithm above, we obtain an algorithm to compute the cohomology groups with coefficients in the formal power series solutions of \( \mathcal{M} \):

\[
\text{Ext}^i_{\mathcal{D}_X}(\mathcal{M}, K[[x]]) \quad (i = 0, \ldots, n)
\]

under the assumption that \( \mathcal{M} \) is specializable along \( Y := \{0\} \). In fact, we can easily verify that there exists an isomorphism (see e.g. [22, p.428] for the case \( \mathcal{M} = \mathcal{C} \))

\[
\text{Ext}^i_{\mathcal{D}_X}(\mathcal{M}, K[[x]]) \simeq \text{Ext}^i_{\mathcal{K}}(\mathcal{M}_Y^\bullet, K) \simeq H^i(\mathcal{M}_Y^\bullet).
\]

If \( \mathcal{M} = \mathcal{C} \) and \( \mathcal{M} \) is Fuchsian along \( Y \) in the sense of [22] (this condition holds if \( \mathcal{M} \) is regular holonomic in the sense of [19]), then by virtue of the comparison theorem, we have also an isomorphism

\[
\text{Ext}^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}\{x\}) \simeq H^i(\mathcal{M}_Y^\bullet),
\]

where \( \mathcal{C}\{x\} \) denotes the ring of convergent power series in \( x \). Note that Kashiwara’s index theorem ([19, p.127]) gives the local index

\[
\sum_{i \geq 0} (-1)^i \dim_{\mathcal{C}} \text{Ext}^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}\{x\})
\]

at \( 0 \in X \) in terms of some topological quantity associated with the characteristic cycle of \( \mathcal{M} \).

**Example 5.5** Let us consider \( \mathcal{M} := A_4/I \), where \( I \) is the left ideal of \( A_4 \) (with \( \mathcal{K} = \mathcal{C} \)) generated by

\[
x_3 \partial_3 + x_4 \partial_4 - a_1, \quad x_1 \partial_1 + x_3 \partial_3 - a_2, \quad x_2 \partial_2 + x_4 \partial_4 - a_3, \quad \partial_1 \partial_4 - \partial_2 \partial_3,
\]

where \( a_1, a_2, a_3 \in \mathcal{C} \) are parameters. Put \( \mathcal{M} := \mathcal{D}_X \otimes A_4 \mathcal{M} \) with \( X = \mathcal{C}^4 \) and \( \mathcal{Y}_1 := \{(x_1, x_2, x_3, x_4) \in X \mid x_1 = x_2 = x_3 = 0 \} \). The global b-function of \( \mathcal{M} \) along \( \mathcal{Y}_1 \) is \((s - a_2)(s + a_1 - a_2 - a_3)\). Hence the cohomology groups of the restriction of \( \mathcal{M} \) to \( \mathcal{Y}_1 \) all vanish unless \( a_2 \) or \( a_1 - a_2 - a_3 \) is an integer. If \( a_1 = a_2 = a_3 = 0 \), we have by Algorithm 5.4

\[
\mathcal{H}^i(\mathcal{M}_{Y_1}^\bullet) = \begin{cases} 
\mathcal{D}_{Y_1}/\mathcal{D}_{Y_1} x_4 \partial_4 & (i = 0), \\
(\mathcal{D}_{Y_1}/\mathcal{D}_{Y_1} x_4 \partial_4)^2 & (i = -1), \\
\mathcal{D}_{Y_1}/\mathcal{D}_{Y_1} x_4 \partial_4 & (i = -2), \\
0 & (i \leq -3).
\end{cases}
\]

The b-function of \( \mathcal{M} \) along the point \( Y_0 := \{(0, 0, 0, 0)\} \) is \( s - a_2 - a_3 \). Suppose \( a_1 = a_2 = a_3 = 0 \). Then the cohomology groups \( \mathcal{H}^i(\mathcal{M}_{Y_0}^\bullet) \) of the restriction of \( \mathcal{M} \) to \( Y_0 \) are \( \mathcal{C}, \mathcal{C}^3, \mathcal{C}^3, \mathcal{C}, 0 \) for \( i = 0, -1, -2, -3, -4 \) respectively. Since \( \mathcal{M} \) is regular holonomic, this implies that \( \text{Ext}^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}\{x\}) \) is \( \mathcal{C}, \mathcal{C}^3, \mathcal{C}^3, \mathcal{C}, 0 \) for \( i = 0, 1, 2, 3, 4 \) respectively.
Example 5.6 Put $X := \mathbb{C}^2 \ni (x, y)$ and $\mathcal{M} := \mathcal{D}_X \otimes_{A_2} (A_2/I)$ with $I$ being the left ideal generated by

$$x \partial_x - x(x \partial_x + y \partial_y + a)(x \partial_x + b_1), \quad y \partial_y - y(x \partial_x + y \partial_y + a)(y \partial_y + b_2).$$

Then by the computation of the restriction of $\mathcal{M}$ to $(0, 0)$, we get

$$\text{Ext}^i_{\mathcal{D}_X} (\mathcal{M}, C[[x, y]]) = \begin{cases} C & (i = 0), \\ C^2 & (i = 1), \\ C & (i = 2). \end{cases}$$

for generic parameters $a, b_1, b_2$ (this means that we perform the computation over the coefficient field $K := \mathbb{Q}(a, b_1, b_2)$). In particular, we have $\sum_{i=0}^2 (-1)^i \dim_C \text{Ext}^i_{\mathcal{D}_X} (\mathcal{M}, C[[x, y]]) = 0$. On the other hand, the characteristic cycle of $\mathcal{M}$ is

$$3\{\xi = \eta = 0\} + 4\{x = \eta = 0\} + 4\{y = \xi = 0\} + \{x - y = \xi + \eta = 0\} + 7\{x = y = 0\}$$

as a cycle in the cotangent bundle $T^*X = \{(x, y, \xi, \eta)\}$. Thus, by Kashiwara’s index theorem we have

$$\sum_{i=0}^2 (-1)^i \dim_C \text{Ext}^i_{\mathcal{D}_X} (\mathcal{M}, C[x, y]) = 3 - (4 + 4 + 1) + 7 = 1.$$

Hence $\mathcal{M}$ is not regular at $(0, 0)$.

$\dim_C \text{Ext}^0_{\mathcal{D}_X} (\mathcal{M}, C[[x, y]]) = 1$ implies that the system $\mathcal{M}$ admits one dimensional space of formal power series solutions at the origin. In fact, the (divergent) formal series

$$\sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(1)_m(1)_n} x^m y^n, \quad (c)_m := c(c+1) \cdots (c+m-1)$$

spans the solution space.

6 Tensor product and localization

In this and subsequent sections, we denote by $X$ the affine space $K^n$. First let us describe an algorithm to compute the tensor product and the torsion groups of two holonomic $\mathcal{D}_X$-modules $\mathcal{M}_1$ and $\mathcal{M}_2$. We suppose that left $A_n$-modules $N_1$ and $N_2$ are given so that

$$M_i := \Gamma(X, \mathcal{M}_i) = A_{n_i}^r/N_i \quad (i = 1, 2).$$

Let $\pi_1, \pi_2 : X \times X \to X$ be the projections to the first and the second component respectively and put

$$\mathcal{D}'_{X \times X} := \pi_1^{-1}\mathcal{D}_X \otimes_K \pi_2^{-1}\mathcal{D}_X.$$

Then the exterior tensor product is defined by

$$\mathcal{M}_1 \otimes \mathcal{M}_2 := \mathcal{D}_{X \times X} \otimes_{\mathcal{D}'_{X \times X}} (\pi_1^{-1}\mathcal{M}_1 \otimes_K \pi_2^{-1}\mathcal{M}_2).$$
First, let us describe this exterior tensor product more concretely. Let $u_1, \ldots, u_{r_1}$ be the residue classes of the unit vectors $e_1, \ldots, e_{r_1}$ of $A_{n_1}^*$, and $v_1, \ldots, v_{r_2}$ the residue classes of the unit vectors $e'_1, \ldots, e'_{r_2}$ of $A_{n_2}^*$. Then as a $D_{X \times X}$-module, $M_1 \hat{\otimes} M_2$ is generated by $u_i \otimes v_j$ $(1 \leq i \leq r_1, 1 \leq j \leq r_2)$. Let us denote by $(x, y)$ the coordinate system of $X \times X$. For $P = (P_1, \ldots, P_{r_1}) \in D^*_X$ and $Q = (Q_1, \ldots, Q_{r_2}) \in D^*_X$ we write

$$P \otimes Q := (P_i(x, \partial_x)Q_j(y, \partial_y))_{ij} \in D_{X \times X}^{r_1r_2}.$$ 

Let $\mathcal{I}'$ be the left $D'_{X \times X}$-submodule of $(D'_{X \times X})^{r_1r_2}$ generated by the set

$$\{P \otimes e'_j, e_i \otimes Q \mid P \in N_1, Q \in N_2, 1 \leq i \leq r_1, 1 \leq j \leq r_2\}$$

and put $\mathcal{I} := D_{X \times X} \otimes_{D'_{X \times X}} \mathcal{I}'$.

**Lemma 6.1** Under the above notation, there is an isomorphism

$$M_1 \hat{\otimes} M_2 \simeq D_{X \times X}^{r_1r_2}/\mathcal{I}.$$ 

Proof: Put

$$K' := \{(P_{ij})_{ij} \in (D'_{X \times X})^{r_1r_2} \mid \sum_{i,j} P_{ij}(u_i \otimes v_j) = 0 \text{ in } \pi_1^{-1}M_1 \otimes_K \pi_2^{-1}M_2\}.$$ 

Then it is easy to see that $\mathcal{I}' \subset K'$. Hence we have a commutative diagram

$$\begin{array}{ccc}
\pi_1^{-1}M_1 \times \pi_2^{-1}M_2 & \xrightarrow{\Phi} & (D'_{X \times X})^{r_1r_2}/\mathcal{I}' \\
\downarrow & & \downarrow \\
\pi_1^{-1}M_1 \otimes_K \pi_2^{-1}M_2 & \simeq (D'_{X \times X})^{r_1r_2}/K',
\end{array}$$

where $\Phi$ is a $K$-bilinear map defined by

$$\Phi(\sum_i P_i u_i, \sum_j Q_j v_j) = P \otimes Q \mod \mathcal{I}',$$ 

which is well-defined by the definition of $\mathcal{I}'$. In view of the universal property of the tensor product, we know that the vertical map above is an isomorphism. This completes the proof. []

Hence $M_1 \hat{\otimes} M_2$ is computable with $N_1$ and $N_2$ being given. Put $\Delta := \{(x, y) \in X \times X \mid x = y\}$ and identify $\Delta$ and $X$ by the map $\pi_1$. Then by [14, Proposition 4.7], which obviously applies to algebraic $D$-modules as well, we have

$$M_1 \otimes_{D_X} M_2 \simeq D_{\Delta \to X \times X} \otimes^{L}_{D_{X \times X}} (M_1 \hat{\otimes} M_2) = (M_1 \hat{\otimes} M_2)^\Delta.$$ 

Suppose that $M_1$ and $M_2$ are holonomic. Then it is easy to see that $M_1 \hat{\otimes} M_2$ is a holonomic $D_{X \times X}$-module since its characteristic variety is contained in the Cartesian product of those of $M_1$ and $M_2$. Hence $M_1 \hat{\otimes} M_2$ is specializable along $\Delta$ and the following algorithm is correct:
Algorithm 6.2 (The tensor product and torsion groups of two $\mathcal{D}_X$-modules)

Input: Holonomic systems $\mathcal{M}_i = \mathcal{D}_X \otimes A_n M_i$, where $M_i = A_r^i / N_i$ with an $A_n$-submodule $N_i$ of $A_r^i$ for $i = 1, 2$.

Output: $\text{Tors}^O_{dX}(\mathcal{M}_1, \mathcal{M}_2) = \mathcal{L}_k$ for $k = 0, \ldots, n$.

1. From sets $G_i$ of generators of $N_i$, compute

   $$ G_3 := \{ P \otimes e'_j, e_i \otimes Q \mid P \in G_1, Q \in G_2, 1 \leq i \leq r_1, 1 \leq j \leq r_2 \}. $$

2. Let $G_4$ be the result of the substitution $y_i = x_i + t_i$ ($i = 1, \ldots, n$) for each element of $G_3$; let $I_4$ be the left ideal of $A_2^n = K[t, x] \langle \partial_t, \partial_x \rangle$ generated by $G_4$ with $t = (t_1, \ldots, t_n)$

3. Apply Algorithm 5.4 with $A_2^n / I_4$ as input and $d = n$ to obtain

   $$ \mathcal{L}_k = \mathcal{H}^{-k}((\mathcal{D}_X \otimes A_2^n (A_r^1 \otimes A_r^2 / I_4))_{\{0\} \otimes X}). $$

   for $k = 0, \ldots, n$.

Example 6.3 Put $X := K$ and

$$ \mathcal{M} := \mathcal{D}_X / \mathcal{D}_X x \partial_x, \quad \mathcal{N} := \mathcal{D}_X / \mathcal{D}_X x. $$

First, the exterior tensor product is given by

$$ \mathcal{M} \otimes \mathcal{N} = \mathcal{D}_X / (\mathcal{D}_X x \partial_x + \mathcal{D}_X x y) $$

with $(x, y) \in X \times X$. Its global b-function along the diagonal is $s$, and by restricting $\mathcal{M} \otimes \mathcal{N}$ to the diagonal we get

$$ \text{Tors}^O_{dX}(\mathcal{M}, \mathcal{N}) = \mathcal{D}_X / \mathcal{D}_X x \quad (i = 0, 1). $$

In the same way, we get

$$ \text{Tors}^O_{dX}(\mathcal{D}_X / \mathcal{D}_X x, \mathcal{D}_X / \mathcal{D}_X x) = \begin{cases} 0 & (i = 0), \\ \mathcal{D}_X / \mathcal{D}_X x & (i = 1). \end{cases} $$

$$ \text{Tors}^O_{dX}(\mathcal{D}_X / \mathcal{D}_X (x \partial_x + 1), \mathcal{D}_X / \mathcal{D}_X x) = 0 \quad (i = 0, 1). $$

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module and let $f \in K[x]$ be an arbitrary non-constant polynomial. Then we immediately obtain an algorithm to compute the localization $\mathcal{M}[f^{-1}] := \mathcal{O}_X[f^{-1}] \otimes_{\mathcal{O}_X} \mathcal{M}$ by combining this algorithm with that of computing $\mathcal{O}[1/f]$ given in [30] since $\mathcal{O}_X[f^{-1}]$ is holonomic ([4, Theorem 1.3]). Since $\mathcal{O}_X[f^{-1}]$ is flat over $\mathcal{O}_X$, the higher torsion groups vanish.

Algorithm 6.4 (The localization $\mathcal{M}[f^{-1}]$)

Input: A holonomic system $\mathcal{M} = \mathcal{D}_X \otimes A_n (A_r^i / N)$ and a non-constant polynomial $f \in K[x]$.

Output: $\mathcal{M}[f^{-1}]$. 

1. Compute the global Bernstein-Sato polynomial $b_f(s)$ of $f$ as follows (cf. [29]), where $s$ is a single indeterminate:

   (a) Letting $t$ be a single variable and let $I$ be the left ideal of $A_{n+1}$ generated by $t - f(x)$ and $\partial x_i + (\partial f/\partial x_i)\partial t$ for $i = 1, \ldots, n$.

   (b) Let $b(\theta)$ be that in the step 3 of Algorithm 4.6 with $A_{n+1}/I$ as input and $d = 1$. Put $b_f(s) := b(-s - 1)$.

2. Compute a set of generators of $J_f := \{ P(s, x, \partial x) \mid A_n[s] \mid Pf = 0 \}$ by [30, Theorem 19].

3. Let $\nu$ be the minimum integer root of $b_f(s) = 0$ and put $J_f(\nu) := \{ P(\nu, x, \partial x) \mid P(s, x, \partial x) \in J_f \}$. Then $O_X[f^{-1}] \simeq D_X \otimes_{A_n} (A_n/J_f(\nu))$.

4. Compute $O_X[f^{-1}] \otimes_{O_X} M$, which is obtained as the output of Algorithm 6.2 with $A_n/J_f(\nu)$ and $A'_n/N$ as input and $i = 0$.

7 Algebraic local cohomology groups

Let $f_1, \ldots, f_d \in K[x]$ be arbitrary polynomials and put $Y := \{ x \in X \mid f_1(x) = \ldots = f_d(x) = 0 \}$ with $X := K^n$. Let $\mathcal{M}$ be a coherent $D_X$-module. Our purpose is to compute the algebraic local cohomology groups $\mathcal{H}'_{Y\gamma}(\mathcal{M})$ with support $Y$ defined by Grothendieck as $D_X$-modules. Recall that $\mathcal{H}'_{Y\gamma}(\mathcal{M})$ is defined as the $k$-th derived functor of the functor

$$\Gamma_{Y\gamma}(\mathcal{M}) := \lim_{\rightarrow} \mathcal{H}om_{O_X}(O_X/J^m_Y; \mathcal{M}),$$

where $J_Y$ is the defining ideal of $Y$ and the inductive limit is taken as $m$ tends to infinity. Note that if $\mathcal{M}$ is a holonomic $D_X$-module, then so is $\mathcal{H}'_{Y\gamma}(\mathcal{M})$ (cf. [14, Theorem 1.4]).

Put $\widetilde{X} := K^d \times X$ and identify $X$ with the linear subvariety $\{ 0 \} \times X$ of $\widetilde{X}$. We set

$$Z := \{ (t, x) \in \widetilde{X} \mid t_i = f_i(x) \ (i = 1, \ldots, d) \}.$$ 

Then $B[Z] := H'^d_{[Z]}(O_{\widetilde{X}})$ is isomorphic to $D_{\widetilde{X}}/I$, where $I$ is the left ideal generated by

$$t_j - f_j(x) \quad (j = 1, \ldots, d), \quad \partial x_i + \sum_{j=1}^d \frac{\partial f_j}{\partial x_i} \partial t_j \quad (i = 1, \ldots, n).$$

Let $\pi : \widetilde{X} \longrightarrow X$ be the projection and put

$$\Delta' := \{ (t, x, y) \in \widetilde{X} \times X \mid x = y \}.$$ 

Then we can identify $\widetilde{X}$ with $\Delta'$ by $\pi$. In the same way as [31, Lemma 6.3] we get
Lemma 7.1

\[ \text{Tor}_i^{\pi^{-1}\mathcal{O}_X}(\mathcal{B}_{[Z]}, \pi^{-1}\mathcal{M}) \simeq \begin{cases} \mathcal{H}^0((\mathcal{B}_{[Z]\hat{\otimes}}\pi^{-1}\mathcal{M})^\bullet) & (i = 0) \\ 0 & (i \neq 0) \end{cases} \]

with

\[ \mathcal{B}_{[Z]\hat{\otimes}}\mathcal{M} := \mathcal{D}_{X \times X} \otimes p_1^{-1}\mathcal{D}_{X} \otimes p_2^{-1}\mathcal{D}_{X} (p_1^{-1}\mathcal{B}_{[Z]} \otimes K p_2^{-1}\mathcal{M}), \]

where \( p_1 \) and \( p_2 \) are the projections of \( \tilde{X} \times X \) to \( \tilde{X} \) and to \( X \) respectively.

In fact, this lemma follows from the fact that \( \Delta' \) is non-characteristic with respect to \( \mathcal{B}_{[Z]\hat{\otimes}}\mathcal{M} \). The proof of [31, Theorem 6.1] yields the following:

Proposition 7.2 For any coherent \( \mathcal{D}_X \)-module \( \mathcal{M} \), we have an isomorphism

\[ \mathcal{H}^i_{[Y]}(\mathcal{M}) \simeq \mathcal{H}^{i-d}((\mathcal{B}_{[Z]} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M})^\bullet)_{\{0\} \times X} \]

as left \( \mathcal{D}_X \)-module for any \( i \geq 0 \).

Algorithm 7.3 (Algebraic local cohomology groups \( \mathcal{H}^i_{[Y]}(\mathcal{M}) \))

Input: Polynomials \( f_1, \ldots, f_d \in K[x] \) and a holonomic \( \mathcal{D}_X \)-module \( \mathcal{M} = \mathcal{D}_X \otimes_{A_n} (A^r_n/N) \) with an \( A_n \)-submodule \( N \) of \( A^r_n \) generated by \( G \).

Output: \( \mathcal{H}^i_{[Y]}(\mathcal{M}) = \mathcal{L}_{i-d} \) for \( i = 0, \ldots, d \) with \( Y := \{ x \in K^n \mid f_1(x) = \ldots = f_d(x) = 0 \} \).

1. Let \( I \) be the left \( A_{d+2n} \)-submodule of \( A^r_{d+2n} \) generated by

\[ G_1 := \{(t_j - f_j(x))e_k \mid 1 \leq j \leq d, \ 1 \leq k \leq r \} \]

\[ \cup \ (\partial_{x_i} + \sum_{j=1}^d \frac{\partial f_j}{\partial x_i} \partial_{t_j})e_k \mid 1 \leq i \leq n, \ 1 \leq k \leq r \} \]

\[ \cup \ \{P(y, \partial_y) \mid P \in G\}, \]

where \( e_1, \ldots, e_r \) are the unit vectors of \( A^r_n \).

2. Apply substitution \( y_i = x_i + z_i \) for \( i = 1, \ldots, n \) to \( G_1 \) and let the result be \( G_2 \). Let \( J_2 \) be the submodule of \( A^r_{d+2n} \) generated by \( G_2 \).

3. Compute the 0-th cohomology of the restriction of \( A^r_{d+2n}/J_2 \) to \( \{(t, x, z) \mid z = 0\} \) in the form \( \mathcal{D}_{X} \otimes_{A_{d+n}} (A^r_{d+n}/J_3) \) by Algorithm 5.4. Here we can assume \( k_0 = k_1 = 0 \) skipping the steps 1–4 of Algorithm 5.4.

4. Compute \( \mathcal{L}_{i} := \mathcal{H}^i((\mathcal{D}_{X} \otimes_{A_{d+n}} (A^r_{d+n}/J_3))^\bullet)_{\{0\} \times X} \) for \( -d \leq i \leq 0 \).
Finally, assume that $Y$ is non-singular and let $\iota: Y \to X$ be the embedding. Then for a coherent $\mathcal{D}_Y$-module $\mathcal{N}$,

$$\iota^* \mathcal{N} := \mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{N}$$

is a coherent $\mathcal{D}_X$-module with support in $Y$; here we put $\mathcal{D}_{X\leftarrow Y} := \mathcal{D}_X/(\mathcal{D}_X f_1 + \cdots + \mathcal{D}_X f_d)$, which has a structure of $(\mathcal{D}_X, \mathcal{D}_Y)$-bimodule. Moreover, the functor $\iota^*$ gives an equivalence between the category of coherent $\mathcal{D}_Y$-modules and that of coherent $\mathcal{D}_X$-modules supported by $Y$ ([14, Proposition 4.2]). In terms of this equivalence, we can compute the cohomology groups of the restriction of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ to $Y$ by using Algorithm 7.3 and the following isomorphism:

**Proposition 7.4** ([14, Proposition 4.3])

$$\iota^* \mathcal{H}^i(\mathcal{M}^\bullet_Y) = \mathcal{H}^{i+d}_{[Y]}(\mathcal{M}).$$

**Example 7.5** Put $K := \mathbb{C}^3 \ni (x, y, z)$ and

$$\mathcal{M} := \mathcal{D}_X/(\mathcal{D}_X \partial_x + \mathcal{D}_X \partial_y + \mathcal{D}_X(z^2 \partial_z + z)).$$

Then the local cohomology groups of $\mathcal{M}$ with support $Y := \{(x, y, z) \in X \mid xz = yz = 0\}$ are given by

$$\mathcal{H}^i_{[Y]}(\mathcal{M}) = \begin{cases} 
\mathcal{D}_X/(\mathcal{D}_X x + \mathcal{D}_X y + \mathcal{D}_X(z^2 \partial_z + 2z + 1)) & (i = 2), \\
0 & (i = 1), \\
\mathcal{D}_X/(\mathcal{D}_X \partial_x^2 + \mathcal{D}_X(x \partial_x - 1) + \mathcal{D}_X \partial_y + \mathcal{D}_X z) & (i = 0). 
\end{cases}$$

Moreover, $\mathcal{H}^0_{[Y]}(\mathcal{M}) = \mathcal{D}_X u$ is also isomorphic to

$$\mathcal{D}_X/(\mathcal{D}_X \partial_x + \mathcal{D}_X \partial_y + \mathcal{D}_X z) = \mathcal{D}_X v$$

by the correspondence $u \mapsto \partial_x v$ and $v \mapsto xu$, where $u$ and $v$ denote the residue classes of $1 \in \mathcal{D}_X$ in respective modules.

The localization of $\mathcal{M}$ by $z$ is given by

$$\mathcal{M}[1/z] = \mathcal{D}_X/(\mathcal{D}_X \partial_x + \mathcal{D}_X \partial_y + \mathcal{D}_X(z^2 \partial_z + z + 1)).$$

Note that the methods of [31] or [39] cannot be applied since $z : \mathcal{M} \to \mathcal{M}$ is not injective.

### 8 Functors in the analytic category

The functors studied in the preceding sections have analytic counterparts (cf. [14], [24]). Throughout this section, we assume $K = \mathbb{C}$ and denote by $\mathcal{O}^\text{an}_X$ and $\mathcal{D}^\text{an}_X$ the sheaves on $X$ of holomorphic functions and of rings of differential operators with holomorphic coefficients respectively. For a left $\mathcal{D}_X$-module $\mathcal{M} = \mathcal{D}_X/N$, we put $\mathcal{M}^\text{an} := \mathcal{D}^\text{an}_X \otimes_{\mathcal{D}_X} \mathcal{M}$. For $k \in \mathbb{Z}$, put

$$F^k_Y(\mathcal{D}^\text{an}_X) := \{ P \in \mathcal{D}^\text{an}_X \mid P(\mathcal{J}^\text{an}_Y)^j \in (\mathcal{J}^\text{an}_Y)^{j-k} \text{ for any } j \geq k \},$$

$$M^\text{an} := \mathcal{D}^\text{an}_X \otimes_{\mathcal{D}_X} \mathcal{M}.$$
where $\mathcal{J}_Y^\text{an}$ is the defining ideal of $Y$ in $\mathcal{O}_X^\text{an}$. Then for $m = (m_1, \ldots, m_r) \in \mathbb{Z}^r$, the $F_Y[m]$-filtrations are defined by

$$F_Y^k[m](\mathcal{D}_X^\text{an})^r := \bigoplus_{i=1}^r F_Y^{k-m_i}(\mathcal{D}_X^\text{an}),$$

$$F_Y^k[m](\mathcal{M}^\text{an}) := F_Y^{k-m_1}(\mathcal{D}_X^\text{an})u_1 + \cdots + F_Y^{k-m_r}(\mathcal{D}_X^\text{an})u_r,$$

where $u_1, \ldots, u_r$ are the residue classes of the unit vectors of $(\mathcal{D}_X^\text{an})^r$. The graded modules are defined in the same way as in Section 2. Put $\mathcal{D}_Y^\text{an} \to \mathcal{D}_X^\text{an} :\mathcal{O}_Y^\text{an} \otimes_{\mathcal{O}_X^\text{an}} \mathcal{D}_X^\text{an}$, which is a $(\mathcal{D}_Y^\text{an}, \mathcal{D}_X^\text{an})$-bimodule. Then the restriction of $\mathcal{M}^\text{an}$ to $Y$ is defined by

$$(\mathcal{M}^\text{an})_Y^\bullet := \mathcal{D}_Y^\text{an} \otimes_{\mathcal{D}_Y^\text{an}} \mathcal{M}^\text{an}.$$

The $b$-function of $\mathcal{M}^\text{an}$ along $Y$ at $p \in Y$ is defined to be the generator of the ideal

$$\{b(\theta) \in \mathbb{C}[\theta] \mid b(\theta)g_Y^0[m](\mathcal{M}^\text{an}) = 0\},$$

where $\theta$ is defined as in Section 4.

**Lemma 8.1** Let $\mathcal{N}$ be a coherent left $\mathcal{D}_X$-submodule of $\mathcal{D}_X^r$. Define the $F_Y[m]$-filtrations on $\mathcal{N}$ and on $\mathcal{N}^\text{an}$ as in Section 2 with a shift vector $m$. Then we have

$$g_Y^0[m](\mathcal{N}^\text{an}) = g_Y^0(\mathcal{D}_X^\text{an}) \otimes g_Y^0(\mathcal{D}_X) g_Y^0[m](\mathcal{N}) = \mathcal{D}_Y^\text{an} \otimes_{\mathcal{D}_Y} g_Y^0[m](\mathcal{N}).$$

Proof: We can prove the first equality by the same method (considering syzygies in the graded module) as [29, Theorem 3.16] (cf. also [1, Lemma 1.1.2]), where the case of $r = d = 1$ is treated; the argument applies to this case with trivial modifications. The second equality follows from

$$g_Y^0(\mathcal{D}_X) = \mathcal{D}_Y[t_1 \partial t_1, \ldots, t_d \partial t_d], \quad g_Y^0(\mathcal{D}_X^\text{an}) = \mathcal{D}_Y^\text{an}[t_1 \partial t_1, \ldots, t_d \partial t_d].$$

**Proposition 8.2** The $b$-function $b^\text{an}(\theta, p)$ of $\mathcal{M}^\text{an}$ and the $b$-function $b(\theta, p)$ of $\mathcal{M}$ with the same shift vector $m$ coincide for any $p \in Y$.

Proof: This follows from Lemma 8.1 and the faithful flatness of $\mathcal{O}_Y^\text{an}$ over $\mathcal{O}_Y$ (cf. [29, Lemma 4.4]).

**Proposition 8.3** We have for any $i \in \mathbb{Z}$,

$$\mathcal{H}^i((\mathcal{M}^\text{an})_Y^\bullet) = \mathcal{D}_Y^\text{an} \otimes_{\mathcal{D}_Y} \mathcal{H}^i(\mathcal{M}_Y^\bullet).$$
Proof: Since we can regard

\[ D_{Y \to X}^{an} = \{ \sum_{\nu, \beta} a_{\nu \beta}(x) \partial^\nu \partial^\beta \in D_X^{an} \mid a_{\nu \beta}(x) \in \mathcal{O}^an_Y \} , \]

we have an isomorphism \( D_{Y \to X}^{an} \cong D_Y^{an} \otimes_{D_Y} D_{Y \to X} \) as \( (D_X^{an}, D_Y^{an}) \)-bimodules. Combining this with the flatness of \( D_X^{an} \) over \( D_Y^{an} \), and that of \( D_Y^{an} \) over \( D_Y \), we get

\[
(M^{an})_{Y}^{\bullet} = D_{Y \to X}^{an} \otimes^L_{D_X^{an}} (D_X^{an} \otimes_{D_X} \mathcal{M}) \\
= D_{Y \to X}^{an} \otimes^L_{D_X} \mathcal{M} \\
= (D_Y^{an} \otimes_{D_Y} D_{Y \to X}) \otimes^L_{D_X} \mathcal{M} \\
= D_Y^{an} \otimes_{D_Y} (D_{Y \to X} \otimes^L_{D_X} \mathcal{M}) \\
= D_Y^{an} \otimes_{D_Y} \mathcal{M}_{Y}^{\bullet}.
\]

This implies the assertion since \( D_Y^{an} \) is flat over \( D_Y \). 

**Proposition 8.4** For \( D_X \)-modules \( \mathcal{M} \) and \( \mathcal{N} \), we have

\[
\text{Tor}_i^{O_X^{an}}(\mathcal{M}^{an}, \mathcal{N}^{an}) = D_X^{an} \otimes_{D_X} \text{Tor}_i^{O_X}(\mathcal{M}, \mathcal{N}).
\]

Proof: The assertion follows from Proposition 8.3 and [14, Proposition 4.7].

For an algebraic set \( Y \) of \( X \), the algebraic local cohomology groups of \( \mathcal{M}^{an} \) are defined to be the derived functors of the functor

\[
\Gamma[Y](\mathcal{M}^{an}) := \lim_{m \to \infty} \text{Hom}_{O_X^{an}}(O_X^{an}/(J_Y^{an})^m, \mathcal{M}^{an}).
\]

**Proposition 8.5** For a left \( D_X \)-module \( \mathcal{M} \), a polynomial \( f \in C[x] \), and an algebraic set \( Y \), we have

\[
\mathcal{M}^{an}[f^{-1}] = D_X^{an} \otimes_{D_X} \mathcal{M}[f^{-1}], \\
\mathcal{H}_Y^{\bullet}(\mathcal{M}^{an}) = D_X^{an} \otimes_{D_X} \mathcal{H}_Y^{\bullet}(\mathcal{M}).
\]

Proof: The first equality is an immediate consequence of Proposition 8.4. The second equality follows from Proposition 8.3 and Proposition 7.2 together with its analytic counterpart.

### 9 Homogenized Weyl algebra and Schreyer’s method for free resolution

In this section, we work in a framework more general than is needed in the preceding sections. Let \( A_n = K[x]/(\partial) \) be the Weyl algebra over a field \( K \) of characteristic zero with \( x = (x_1, \ldots, x_n) \), \( \partial = (\partial_1, \ldots, \partial_n) \), \( \partial_i = \partial/\partial x_i \). We introduce a weight vector
For each integer \( \nu \in \mathbb{Z} \), we put
\[
F^\nu_w(A_n) := \{ \mathcal{P} = \sum_{\alpha, \beta \in \mathbb{N}^n} a_{\alpha \beta} x^\alpha \partial^\beta \in A_n \mid a_{\alpha \beta} = 0 \text{ if } \sum_{i=1}^n w_i \alpha_i + \sum_{i=1}^n w_{n+i} \beta_i > \nu \},
\]
where \( a_{\alpha \beta} \in K \) and the sum with respect to \( \alpha, \beta \in \mathbb{N}^n \) is finite. For a nonzero \( \mathcal{P} \in A_n \), let \( \text{ord}_w(\mathcal{P}) \) be the minimum integer \( k \) such that \( \mathcal{P} \in F^k_w(A_n) \). It is easy to see that \( \text{ord}_w(\mathcal{P}Q) = \text{ord}_w(\mathcal{P}) + \text{ord}_w(Q) \) holds for nonzero \( \mathcal{P}, Q \in A_n \). More generally, for a shift vector \( \mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r \), we define a filtration \( F^k_w[\mathbf{m}] \) of \( A_n^r \) by
\[
F^k_w[\mathbf{m}](A_n^r) := \bigoplus_{i=1}^r F^{k-m_i}(A_n)e_i,
\]
where \( e_1, \ldots, e_r \) are the canonical generators of \( A_n^r \). For a nonzero \( \mathcal{P} \in A_n^r \), we put \( \text{ord}_w[\mathbf{m}](\mathcal{P}) := \min\{ k \in \mathbb{Z} \mid \mathcal{P} \in F^k_w[\mathbf{m}](A_n^r) \} \).

Now we introduce the homogenized Weyl algebra, which was introduced from the second version (1994) of kan/sm1 [38].

**Definition 9.1** Let \( A_n^{(h)} \) be the algebra over \( K \) generated by \( h, x = (x_1, \ldots, x_n) \), and \( \partial = (\partial_1, \ldots, \partial_n) \) which satisfy the relations
\[
\begin{align*}
x_i x_j - x_j x_i &= 0, & \partial_i \partial_j - \partial_j \partial_i &= 0, & x_i \partial_j - \partial_j x_i &= -\delta_{ij} h^2, \\
h x_i - x_i h &= 0, & h \partial_i - \partial_i h &= 0 & (i, j = 1, \ldots, n),
\end{align*}
\]
where \( \delta_{ij} = 1 \) and \( \delta_{ij} = 0 \) if \( i \neq j \). We call \( A_n^{(h)} \) the homogenized Weyl algebra. The substitution \( h = 1 \) defines a \( K \)-algebra homomorphism
\[
\rho : A_n^{(h)} \ni P \mapsto P|_{h=1} \in A_n.
\]

An element \( \mathcal{P} \) of \( A_n^{(h)} \) is uniquely expressed as a finite sum
\[
\mathcal{P} = \sum_{\lambda \in \mathbb{N}, \alpha, \beta \in \mathbb{N}^n} a_{\lambda \alpha \beta} h^\lambda x^\alpha \partial^\beta
\]
with \( a_{\lambda \alpha \beta} \in K \). The total degree of \( \mathcal{P} \in A_n^{(h)} \) is defined by
\[
\text{deg}(\mathcal{P}) := \max\{ \lambda + |\alpha| + |\beta| \mid a_{\lambda \alpha \beta} \neq 0 \}
\]
if \( \mathcal{P} \neq 0 \) and \( \text{deg}(\mathcal{P}) = -\infty \) if \( \mathcal{P} = 0 \).

Now let us take another vector \( \mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r \), which describes the shift with respect to the total degree. For \( \mathcal{P} = (P_1, \ldots, P_r) \in (A_n^{(h)})^r \), we put
\[
\text{deg}[\mathbf{n}](\mathcal{P}) := \max_{1 \leq i \leq r} (\text{deg}(P_i) + n_i).
\]

**Definition 9.2** 1. An element \( \mathcal{P} = \sum_{i=1}^r \sum_{(\lambda, \alpha, \beta) \in \mathbb{N}} a_{\lambda \alpha \beta i} h^\lambda x^\alpha \partial^\beta e_i \) of \( (A_n^{(h)})^r \) is said to be \( h[\mathbf{n}] \)-homogeneous if there exists \( k \in \mathbb{Z} \) so that \( a_{\lambda \alpha \beta i} = 0 \) unless \( \lambda + |\alpha| + |\beta| + n_i = k \).
2. For \( P = \sum_{i=1}^r \sum_{(\alpha, \beta) \in L} a_{\alpha \beta} x^{\alpha} \partial^\beta e_i \in A_n^r \), we define the \( h[\mathbf{n}] \)-homogenization \( h[\mathbf{n}](P) \in (A_n^{(h)})^r \) by

\[
h[\mathbf{n}](P) := \sum_{i=1}^r \sum_{(\alpha, \beta) \in L} a_{\alpha \beta} h^{k - |\alpha| - |\beta|} x^{\alpha} \partial^\beta e_i
\]

with \( k := \max\{ |\alpha| + |\beta| + n_i \mid a_{\alpha \beta} \neq 0 \} \). This is \( h[\mathbf{n}] \)-homogeneous.

If \( \mathbf{n} \) is the zero vector, we denote \( h[\mathbf{n}](P) \) simply by \( h(P) \).

**Lemma 9.3** For \( P \in A_n^r \) and \( Q \in A_n \), we have \( \rho(h[\mathbf{n}](P)) = P \) and \( h[\mathbf{n}](QP) = h(Q)h[\mathbf{n}](P) \).

**Definition 9.4** Let \( \prec \) be a monomial order (i.e. an order satisfying (3.1)) on \( L \times \{1, \ldots, r\} \) with \( L := \mathbb{N}^{2n} \). We denote by \( \text{lexp}_\prec(P) \) the leading exponent of \( P \in A_n^r \) with respect to \( \prec \). Then \( \prec \) is said to be adapted to the filtration \( F_w[\mathbf{m}] \) if \( \langle w, \alpha \rangle + m_i < \langle w, \beta \rangle + m_j \) implies \( (\alpha, i) \prec (\beta, j) \) for \( \alpha, \beta \in L \) and \( i, j \in \{1, \ldots, r\} \), and if \( \text{lexp}_\prec(e_i) \prec \text{lexp}_\prec(x_j \partial e_i) \) for any \( 1 \leq i \leq r \) and \( 1 \leq j \leq n \); here we write \( \langle w, \alpha \rangle = \sum_{i=1}^{2n} w_i \alpha_i \) for \( \alpha = (\alpha_1, \ldots, \alpha_{2n}) \).

We fix a monomial order \( \prec \) on \( L \times \{1, \ldots, r\} \) that is adapted to the \( F_w[\mathbf{m}] \)-filtration. Then we define an order \( \prec_{h} \) on \( \mathbb{N} \times L \times \{1, \ldots, r\} \) by

\[
(\lambda, \alpha, i) \prec_{h[\mathbf{n}]} (\mu, \beta, j)
\]

if and only if

\[
\lambda + |\alpha| + n_i < \mu + |\beta| + n_j \quad \text{or else}
\]

\[
\lambda + |\alpha| + n_i = \mu + |\beta| + n_j, \quad (\lambda, \alpha, i) \prec (\mu, \beta, j),
\]

for \( \lambda, \lambda' \in \mathbb{N}, \alpha, \beta \in L \) and \( i, j \in \{1, \ldots, r\} \). Then it is easy to see that \( \prec_{h[\mathbf{n}]} \) is a well-order. For a nonzero element

\[
P = \sum_{i=1}^r \sum_{(\alpha\beta)} a_{\lambda\alpha\beta} h^\lambda x^{\alpha} \partial^\beta e_i \tag{9.1}
\]

of \( (A_n^{(h)})^r \), its leading exponent \( (\lambda_0, \alpha_0, \beta_0, i_0) = \text{lexp}_{h[\mathbf{n}]}(P) \in \mathbb{N} \times L \times \{1, \ldots, r\} \) is defined as the maximum element of \( \{ (\lambda, \alpha, \beta, i) \mid a_{\lambda\alpha\beta i} \neq 0 \} \) with respect to \( \prec_{h[\mathbf{n}]} \). Then the leading position \( \text{lp}_{h[\mathbf{n}]}(P) \) and the leading coefficient \( \text{lcoef}_{h[\mathbf{n}]}(P) \) are defined to be \( i_0 \) and \( a_{\lambda_0, \alpha_0, \beta_0, i_0} \) respectively. We denote them simply by \( \text{lexp}(P), \text{lp}(P), \) and \( \text{lcoef}(P) \) if there is no fear of confusion. The following lemmas follow easily from Definition 9.4 and the definitions of \( A_n^{(h)} \) and \( \prec_{h[\mathbf{n}]} \):

**Lemma 9.5** For \( P \in (A_n^{(h)})^r \) and \( Q \in A_n^{(h)} \), we have \( \text{lexp}(QP) = \text{lexp}(Qe_k) + \text{lexp}(P) \) with \( k = \text{lp}(P) \).

**Lemma 9.6** If \( P \in (A_n^{(h)})^r \) is \( h[\mathbf{n}] \)-homogeneous and \( Q \in A_n^{(h)} \) is \( h[0] \)-homogeneous, then \( QP \) is \( h[\mathbf{n}] \)-homogeneous.
Applying the ring homomorphism by the Buchberger algorithm, there exist $h$-homogenized Weyl algebra and can employ the Buchberger algorithm, which preserves the $h[n]$-homogeneity:

**Definition 9.8** Let $N$ be a left $A_n^{(h)}$-submodule of $(A_n^{(h)})^r$. Then a finite subset $G$ of $N$ is called a Gröbner basis of $N$ with respect to $\prec_{h[n]}$ if

$$E(N) := \{\text{lexp}(P) \mid P \in N \setminus \{0\}\} = \bigcup_{P \in G} (\text{lexp}(P) + L).$$

Note that $G$ generates $N$ if $G$ is a Gröbner basis of $N$ since $\prec_{h[n]}$ is a well-order.

**Proposition 9.9** Let $N$ be a left $A_n$-submodule of $A_n^r$ generated by $P_1, \ldots, P_k$. Let $h[n](N)$ be the left $A_n^{(h)}$-submodule of $(A_n^{(h)})^r$ generated by $h[n](P_i)$ for $i = 1, \ldots, k$. Let $G = \{Q_1, \ldots, Q_s\}$ be a Gröbner basis of $h[n](N)$ with respect to $\prec_{h[n]}$. Then for any $P \in N$, there exist $U_j \in A_n$ such that $P = U_1 \rho(Q_1) + \cdots + U_s \rho(Q_s)$ and $\text{ord}_w[m](U_j \rho(Q_j)) \leq \text{ord}_w[m](P)$ for $j = 1, \ldots, s$.

**Proof:** There exists $\nu \in N$ such that $h^\nu h[n](P)$ belongs to $h[n](N)$. By the division algorithm in $(A_n^{(h)})^r$, we can find $h[n]$-homogeneous $U_1, \ldots, U_s \in (A_n^{(h)})^r$ such that $h^\nu h[n](P) = U_1 Q_1 + \cdots + U_s Q_s$ and $\text{lexp}(U_k Q_k) \prec_{h[n]} \text{lexp}(h^\nu h[n](P))$ for $k = 1, \ldots, s$. Applying the ring homomorphism $\rho$, we get $P = \rho(U_1 \rho(Q_1) + \cdots + U_s \rho(Q_s))$ and $\text{lexp}_{\prec}(\rho(U_k) \rho(Q_k)) = \text{lexp}(\rho(U_k Q_k)) \leq \text{lexp}(h^\nu h[n](P)) = \text{lexp}_{\prec}(P)$. Since $U_i$ and $Q_i$ are $h[n]$-homogeneous, this implies $\text{ord}_w[m](\rho(U_i Q_i)) \leq \text{ord}_w[m](P)$. $\square$

We use the same notation as in the preceding proposition. Put $\Lambda := \{(i, j) \mid 1 \leq i < j \leq s, \text{lp}(P_i) = \text{lp}(P_j)\}$. Then for $(i, j) \in \Lambda$, let $S_{ij}, S_{ji} \in A_n^{(h)}$ be monomials such that

$$\text{lexp}(S_{ji} P_i) = \text{lexp}(S_{ij} P_j) = \text{lexp}(P_i) \lor \text{lexp}(P_j), \quad \text{lcoef}(S_{ji} P_i) = \text{lcoef}(S_{ij} P_j).$$

By the Buchberger algorithm, there exist $h[n]$-homogeneous $U_{ijk} \in A_n^{(h)}$ so that we have

$$S_{ji} P_i - S_{ij} P_j = \sum_{k=1}^s U_{ijk} P_k$$

and either $U_{ijk} \neq 0$ or

$$\text{lexp}(U_{ijk} P_k) \prec_{h[n]} \text{lexp}(P_i) \lor \text{lexp}(P_j)$$

for each $k = 1, \ldots, s$. The following is an analogue of F.O. Schreyer’s theorem for the syzygies in the polynomial ring:
Theorem 9.10 In the notation above, let $\prec'$ be the order on $\mathbb{N} \times L \times \{1, \ldots, s\}$ defined by

$$(\alpha, \mu) \prec' (\beta, \nu) \text{ if and only if } \exp(P_\mu) + \alpha \prec h[n] \exp(P_\nu) + \beta$$
or else $\exp(P_\mu) + \alpha = \exp(P_\nu) + \beta$ and $\mu > \nu$}

for $\alpha, \beta \in \mathbb{N} \times L$ and $\mu, \nu \in \{1, \ldots, s\}$. Put

$$m' := (\text{ord}_w[m](P_1), \ldots, \text{ord}_w[m](P_s)), \quad n' = (\deg[n](P_1), \ldots, \deg[n](P_s)).$$

Then $\prec'$ is a well-order adapted to the $F_w[m']$-filtration and called the Schreyer order induced by $\prec_h[n]$.

1. For $(i, j) \in \Lambda$,

$$V_{ij} := (0, \ldots, S_{ji}, \ldots, -S_{ij}, \ldots, 0) - (U_{ij1}, \ldots, U_{ij3})$$
is $h[n']$-homogeneous and $\{V_{ij} | (i, j) \in \Lambda\}$ is a Gröbner basis with respect to $\prec'$ of the module

$$\text{Syz}(P_1, \ldots, P_s) := \{([U_1, \ldots, U_s] \in (A_n^{(h)})^s | U_1P_1 + \cdots + U_sP_s = 0\}.$$

2. Put

$$\text{Syz}(\rho(P_1), \ldots, \rho(P_s)) := \{([U_1, \ldots, U_s] \in A_n^s | U_1\rho(P_1) + \cdots + U_s\rho(P_s) = 0\}.$$

Then for any $P \in \text{Syz}(\rho(P_1), \ldots, \rho(P_s)) \cap F_w[m'](A_n^s)$ with $\nu \in \mathbb{Z}$, there exist $Q_{ij} \in A_n$ such that $P = \sum_{(i, j) \in \Lambda} Q_{ij} \rho(V_{ij})$ and that $Q_{ij} \rho(V_{ij}) \in F_w[m'](A_n^s)$.

Proof: For a nonzero $P \in (A_n^{(h)})^r$ of the form $[\bar{P}]$, we define its initial term by

$$\text{in}(P) := a_{\lambda_0\beta i} h^\lambda x^\alpha \xi^\beta e_i \in K[h, x, \xi]^r$$

with $(\lambda, \alpha, \beta, i) = \exp(P)$, where $\xi = (\xi_1, \ldots, \xi_n)$ are commutative indeterminates. Let $s_{ij}$ be the monomial in $K[h, x, \xi]$ obtained by substituting $\xi$ for $\partial$ in $S_{ij}$. Then we have $s_{ij}\text{in}(P_i) - s_{ij}\text{in}(P_j) = 0$ for $(i, j) \in \Lambda$. Now suppose $U = (U_1, \ldots, U_s) \in \text{Syz}(P_1, \ldots, P_s)$ and put $(\lambda_0, \alpha_0, \beta_0, i_0) := \max_{1 \leq j \leq s} \exp(U_jP_j)$. Define $u_j \in K[h, x, \xi]$ by

$$u_j e_{i_0} = \text{in}(U_j e_{i_0}) \quad \text{if } \exp(U_jP_j) = (\lambda_0, \alpha_0, \beta_0, i_0),$$

and put $u_j = 0$ otherwise. Then $\sum_{j=1}^s u_j \text{in}(P_j) = 0$ holds. By the definition of $\prec'$ we have

$$\exp_\prec(V_{ij}) = (\exp(s_{ij}), i), \quad \exp_\prec'(U) = \exp_\prec'((u_1, \ldots, u_s)).$$

(9.2)

On the other hand, since $(0, \ldots, s_{ij}, 0, \ldots, -s_{ij}, 0, \ldots, 0)$ for $(i, j) \in \Lambda$ constitute a Gröbner basis with respect to $\prec'$ of the syzygies on $\text{in}(P_i), \ldots, \text{in}(P_s)$ by virtue of Schreyer’s theorem for the polynomial ring (cf. [3, Theorem 15.10]), we know that

$$\exp_\prec'((u_1, \ldots, u_s)) \in \bigcup_{(i, j) \in \Lambda} ((\exp(s_{ij}), i) + (\mathbb{N} \times L)).$$
Theorem 9.11

Moreover, in view of the second part of Theorem 9.10, the sequence

\[ \ldots \xrightarrow{\psi_3} (A_n^{(h)})^{r_2} \xrightarrow{\psi_2} (A_n^{(h)})^{r_1} \xrightarrow{\psi_1} (A_n^{(h)})^{r_0} \xrightarrow{\varphi} (A_n^{(h)})^r / h(N) \longrightarrow 0 \]  

(9.3)

with \( r_0 := r \). Put \( m_0 = n_0 = 0 \) and

\[
\begin{align*}
  m_i &= (\text{ord}_u[m_{i-1}])(\rho(\psi_i(1,0,\ldots,0))), \ldots, \text{ord}_u[m_{i-1}])(\rho(\psi_i(0,\ldots,0,1))), \\
  n_i &= (\text{deg}[n_{i-1}])(\psi_i(1,0,\ldots,0)), \ldots, \text{deg}[n_{i-1}])(\psi_i(0,\ldots,0,1)).
\end{align*}
\]

Applying the homomorphism \( \rho \) to (9.3), we get an exact sequence

\[ \ldots \xrightarrow{\rho(\psi_3)} A_n^{r_2} \xrightarrow{\rho(\psi_2)} A_n^{r_1} \xrightarrow{\rho(\psi_1)} A_n^{r_0} \xrightarrow{\rho(\varphi)} A_n^r / N \longrightarrow 0. \]  

(9.4)

Moreover, in view of the second part of Theorem 9.10, the sequence

\[ \ldots \xrightarrow{\rho(\psi_3)} F^k_w[m_1](A_n^{r_1}) \xrightarrow{\rho(\psi_2)} F^k_w[m_0](A_n^{r_0}) \xrightarrow{\rho(\varphi)} F^k_w[0](A_n^r) / (N \cap F^k_w[0](A_n^r)) \longrightarrow 0 \]

is exact for any \( k \in \mathbb{Z} \). That is, the resolution (9.4) is adapted to the \( F_w \)-filtration. Furthermore, we can prove the following in the same way as its counterpart in the polynomial ring (Proposition 15.11)

**Theorem 9.11** By arranging the Gröbner bases appropriately, we can construct the free resolution (9.3) so that \( \psi_{2n+2} = 0 \).

It is known that there exists a shorter projective resolution. It is an open question to obtain shorter resolutions.

**Remark 9.12**

1. If each element of the weight \( w \) is non-negative and the order \( \prec \) adapted to \( F_w[m] \) is a well-order, then the above construction can be done directly in \( A_n^r \) without the homogenized Weyl algebra.

2. If the weight \( w \) satisfies \( w_i + w_{n+i} = 0 \) for \( i = 1, \ldots, n \), then we can work in \( A_n[x_0]^r \) as in Section 3 instead of the homogenized Weyl algebra. Then Theorem 9.10 holds with the \( h[n] \)-homogenization replaced by the \( F_w[m] \)-homogenization defined by

\[
h[n](P) := \sum_{i=1}^r \sum_{\alpha,\beta \in \mathbb{N}^n} a_{\alpha\beta} x_0^{(w, (\alpha, \beta)) + m_i - k} x^{\alpha} \partial^\beta e_i
\]

with \( k := \min \{ (w, (\alpha, \beta)) + m_i \mid a_{\alpha\beta} \neq 0 \} \) for \( P = \sum_{i=1}^r \sum_{\alpha,\beta \in \mathbb{N}^n} a_{\alpha\beta} x^{\alpha} \partial^\beta e_i \).

Theorem 9.11 also holds in this case.
In the computation of the free resolution described in this section, we can employ the method of La Scala and Stillman \cite{27}, which computes the ‘Schreyer frame’ (initial terms of the resolution) first, then computes the resolution by a selection strategy or in parallel. We have implemented this algorithm in kan/sm1. Most of the examples of this paper have been computed by using this implementation.

**Example 9.13** We give an example of the Shreyer resolution and explain the limits of our method (cf. Example 7.1 in \cite{39}) caused by the complexity. Let $I$ be the left ideal in

$$A_9 := K\langle x_1, x_2, x_3, x_4, x_5, x_6, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4}, \partial_{x_5}, \partial_{x_6}, t_1, t_2, t_3, \partial_{t_1}, \partial_{t_2}, \partial_{t_3} \rangle$$

generated by

$$x_4x_2 - x_5x_1 + t_1, x_4x_3 - x_6x_1 + t_2, x_5x_3 - x_6x_2 + t_3,$$

$$x_3\partial_{t_1} + x_6\partial_{t_2} - \partial_{x_1} - x_4\partial_{t_1} + x_6\partial_{t_3} + \partial_{x_2} - x_4\partial_{t_2} - x_5\partial_{t_3} + \partial_{x_3},$$

$$-x_2\partial_{t_1} - x_3\partial_{t_2} + \partial_{x_4} - x_1\partial_{t_1} - x_3\partial_{t_3} + \partial_{x_5} - x_1\partial_{t_2} + x_2\partial_{t_3} + \partial_{x_6}.$$ 

We want to obtain the restriction of $A_9/I$ along $t_1 = t_2 = t_3 = 0$, but we have not yet succeeded in getting the restrictions of all degrees except zero by using our algorithm and our implementation; we could get only a huge resolution. The hardest part in the computation of the restriction is to compute the quotient of the kernel over the image of the truncated complex (the step 7 of Algorithm \cite{7}). We cannot compute the higher order quotients by our algorithm and implementations.

We homogenize the ideal with the variable $s$;

$$x_4s + x_2s - x_5sx_1 + t_1, x_4s - x_6x_1s + t_2, x_5sx_3 - x_6x_2s + t_3,$$

$$x_3s\partial_{t_1} + x_6s\partial_{t_2} + \partial_{x_1} - x_4s\partial_{t_1} + x_6s\partial_{t_3} + \partial_{x_2} - x_4s\partial_{t_2} - x_5s\partial_{t_3} + \partial_{x_3},$$

$$-x_2s\partial_{t_1} - x_3s\partial_{t_2} + \partial_{x_4} - x_1s\partial_{t_1} - x_3s\partial_{t_3} + \partial_{x_5} - x_1s\partial_{t_2} + x_2s\partial_{t_3} + \partial_{x_6},$$

which we denote by $I'$. We compute the Schreyer resolution with the weight matrix

$$\begin{pmatrix}
  s & x_i & \partial_{x_i} & t_i \\
  1 & 0 & 0 & 0 \\
  0 & 1 & 1 & 1
\end{pmatrix}.$$ 

The reduced (non-$h$-homogenized) Gröbner basis of $I'$ consists of 55 elements. The Betti numbers (i.e., $r_1, r_2, \ldots$) of the Schreyer resolution are $630, 3329, \ldots$ where the reduced Gröbner basis of syzygies by the Schreyer order among the 55-elements of the reduced Gröbner basis of $I'$ consists of 630 elements. The syzygy among 3329 elements by the Schreyer order could not be obtained, because of the memory exhaustion, on three MMX Pentium PCs (166 MHz) with 64 Mega bytes of memory under the Linux operating system (version 2.0.30) with kan/sm1 (version 2.980129) and the open sm1 module for communication between distributed processors \cite{27}. The first Gröbner basis of $I'$ can be obtained in two seconds.
The number of Gröbner basis is smaller in this case if we compute in the homogenized Weyl algebra. We homogenize the ideal \( I \) in the homogenized Weyl algebra with the variable \( h \):

\[
\begin{align*}
 x_4x_2 - x_5x_1 + t_1h, & \quad x_4x_3 - x_6x_1 + t_2h, & \quad x_5x_3 - x_6x_2 + t_3h, \\
 x_5\partial_{t_1} + x_6\partial_{t_2} + h\partial_{x_1}, & \quad -x_4\partial_{t_1} + x_6\partial_{t_3} + h\partial_{x_2}, & \quad -x_4\partial_{t_2} - x_5\partial_{t_3} + h\partial_{x_3}, \\
 -x_2\partial_{t_1} - x_3\partial_{t_2} + h\partial_{x_4}, & \quad x_1\partial_{t_1} - x_3\partial_{t_3} + h\partial_{x_5}, & \quad x_1\partial_{t_2} + x_2\partial_{t_3} + h\partial_{x_6},
\end{align*}
\]

which we denote by \( I^h \). We compute the Schreyer resolution with the weight matrix

\[
\begin{pmatrix}
 0 & 0 & 1 & -1 \\
 1 & 1 & 1 & 0
\end{pmatrix}.
\]

The reduced Gröbner basis of \( I^h \) consists of 44 elements which are less than those of V-homogenized Gröbner basis. The basis can be obtained in 1.5 seconds by the same system. The Betti numbers of the Schreyer resolution are 506, 2422, ... .

We conjecture that in the homogenized Weyl algebra the minimal resolution exists and can be constructed by the algorithm in [20] since it is a graded algebra, but the boundary maps are not adapted to the filtration in general. For example, let us consider the following free resolution in the homogenized Weyl algebra

\[
0 \longrightarrow A_2^{(h)} \xrightarrow{\psi_2} (A_2^{(h)})^2 \xrightarrow{\psi_1} A_2^{(h)} \xrightarrow{\varphi} A_2^{(h)}/(p_1, p_2) \longrightarrow 0
\]

where

\[
A_2^{(h)} = \mathbb{C}\langle h, x, y, \partial_x, \partial_y \rangle,
\]

\[
\psi_1((1, 0)) = p_1, \quad \psi_2((0, 1)) = p_2, \quad \psi_2(1) = (-p_2, p_1)
\]

and

\[
p_1 = 2yh\partial_x + 3x^2\partial_y, p_2 = 2x\partial_x + 3y\partial_y + 6h^2.
\]

So, we cannot replace the Schreyer resolution by the minimal resolution.

Finally, the authors note the iteration approach to get local cohomology groups as in [39] may also improve the performance of our algorithms, but it is a future problem.

During preparing this paper, we knew the paper Castro-Jiménez et al., Homogenizing differential operators, 1997, preprint. The contents of this section may have an overlap with their paper.

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Toshinori Oaku
oaku@math.yokohama-cu.ac.jp
Department of Mathematical Sciences, Yokohama City University
Seto 22-2, Kanazawa-ku, Yokohama, 236-0027 Japan

Nobuki Takayama
takayama@math.kobe-u.ac.jp
Department of Mathematics, Kobe University
Rokko, Kobe, 657-8501 Japan