LOGARITHMIC CAPACITY OF RANDOM $G_\delta$ SETS

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Abstract. We study the logarithmic capacity of $G_\delta$ subsets of the interval $[0, 1]$. Let $S$ be of the form

$$S = \bigcap_m \bigcup_{k \geq m} I_k,$$

where each $I_k$ is an interval in $[0, 1]$ with length $l_k$ that decrease to 0. We provide sufficient conditions for $S$ to have full capacity, i.e. $\text{Cap}(S) = \text{Cap}([0, 1])$. We consider the case when the intervals decay exponentially and are placed in $[0, 1]$ randomly with respect to some given distribution. The random $G_\delta$ sets generated by such distribution satisfy our sufficient conditions almost surely and hence, have full capacity almost surely. This study is motivated by the $G_\delta$ set of exceptional energies in the parametric version of the Furstenberg theorem on random matrix products. We also study the family of $G_\delta$ sets $\{S(\alpha)\}_{\alpha > 0}$ that are generated by setting the decreasing speed of the intervals to $l_k = e^{-k\alpha}$. We observe a sharp transition from full capacity to zero capacity by varying $\alpha > 0$.

1. Introduction

1.1. The setting. Finite signed Borel measures on $\mathbb{C}$ form a vector space over $\mathbb{R}$. Given two finite signed measures Borel measures $\mu$ and $\nu$ on $\mathbb{C}$ we define their interaction by

$$I(\nu, \mu) := \iint (-\log |z - w|) d\nu(z) d\mu(w). \quad (1.1)$$

The interaction is a bilinear form on the vector space of finite signed Borel measures on $\mathbb{C}$ with the following properties:

1. $I(\nu, \mu) = I(\mu, \nu)$,
2. $I(\nu, \mu) > 0$, if $\nu$ and $\mu$ are probability measures and the union of their support has diameter of at most 1,
3. $I(\nu, \mu + \mu') = I(\nu, \mu) + I(\nu, \mu')$ and $I(\nu, c\mu) = cI(\nu, \mu)$.

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This bilinear form is a generalization of the energy of measure $\mu$, which is defined as $I(\mu) := I(\mu, \mu)$. We can think of the energy of a measure as self-interacting. In physics $\mu$ is considered as a charge distribution on $\mathbb{C}$ and $I(\mu)$ as the total energy of $\mu$ on $\mathbb{C}$ (see [10, pg. 56]).

The logarithmic capacity of a subset $E \subset \mathbb{C}$ is then defined by minimizing the energy:

$$\text{Cap}(E) = \exp(-\inf\{I(\mu)\}),$$

where the infimum is taken over the set of Borel probability measures whose support is a compact subset of $E$ (we interpret $e^{-\infty}$ as 0). Capacity gauges how far away a set is from being polar. A set $E$ is said to be polar if $I(\mu) = \infty$ for every non-trivial measure $\mu$ with compact support in $E$.

Most of the literature on capacity has been devoted to the study of compact sets in $\mathbb{C}$, but non-compact sets has also been studied (see [10], [11, Appendix A]). Moreover, $G_\delta$ sets have also been of interest (see [3], [7], [10]). In [2, Section 9], the authors discuss the applications of potential theory to spectral theory.

Our focus will be on the capacity of $G_\delta$'s of the form:

$$S = \bigcap_{m} \bigcup_{k \geq m} I_k,$$  \hspace{1cm} (1.2)

where each $I_k$ is an open interval of length $l_k$ with center at $c_k \in (0, 1)$. The sequence $\{l_k\}$ is taken to approach 0 as $k \to \infty$. It is immediate that $S$ is a $G_\delta$ subset of $[0, 1]$. Under certain assumptions, we will show that the set $S$ has full capacity on the unit interval:

**Definition 1.** Let $J \subset \mathbb{C}$. A set $E \subset J$ is said to have full capacity on $J$ if

$$\text{Cap}(E) = \text{Cap}(J).$$

The capacity of an interval $J$ is $\text{Cap}(J) = \frac{|J|}{4}$ (see, e.g. [10, pg. 135], [11, Example A.17]).

Our results and methods can be extended to higher dimensions, but we do not elaborate on that here. In this paper, we will be focused on $G_\delta$ subsets of an interval of the real line. We are mostly interested in one-dimension because our motivation came from the one-dimensional random $G_\delta$ sets of exceptional energies in the parametric version of the Furstenberg theorem (see Section 1.3).

1.2. Main results. Our first main result is devoted to the random setting, being a “toy model” for the exceptional energies in the parametric version of the Furstenberg theorem (see Section 1.3). Namely, the set of exceptional parameters in [6] is generated by exponentially small intervals, that are asymptotically distributed with respect to some (dynamically defined) measure. However, their positions are random.
A random $G_\delta$ set is obtained by viewing the centers $\{c_k\}$ as random variables. As a toy model, it is reasonable to consider first the set generated by random intervals, that are placed independently (with the same “reasonable” distribution of their centers) - instead of some complicated definition coming from the random dynamical systems.

**Theorem 1.** Let $\{c_k\}$ be i.i.d. with an absolutely continuous distribution on any interval $J$ with almost everywhere positive and uniformly bounded density function. Take $l_k = e^{-\lambda k}$ for some fixed $\lambda > 0$ and let $S$ be the corresponding $G_\delta$-set (1.2). Then, almost surely $S$ has full capacity on the unit interval:

$$\text{Cap}(S) = \text{Cap}([0,1]) > 0.$$  

**Remark 1.** Full capacity is a property that is inherited when restricted to subintervals (see [7, Proposition 1.6]): If $E$ is a subset of interval $J$ such that $\text{Cap}(E) = \text{Cap}(J)$, then given any subinterval $J' \subset J$, one has $\text{Cap}(E \cap J') = \text{Cap}(J')$.

**Remark 2.** In Theorem 1 (and Theorem 3 below), the interval $[0,1]$ may be replaced with any bounded interval $J$ due to the fact that $\text{Cap}(\beta \cdot J) = \beta \cdot \text{Cap}([0,1])$ for some $\beta > 0$. Without loss of generality, we will only be working on the interval $[0,1]$. So, we will take $dx$ to be the restriction to the unit interval: $dx|_{[0,1]}$.

Now, take the centers $\{c_k\}$ from Theorem 1 and let us vary the lengths of the intervals as a function of the parameter $\alpha \in (0,1]$:

$$l_k = e^{-\lambda k \alpha}.$$  

It turns out that at $\alpha = 1$ the capacity undergoes a (sharp) phase transition:

**Theorem 2** (Random phase transition). Let $\{c_k\}$ be i.i.d. with an absolutely continuous distribution with density function that is bounded and positive almost everywhere with respect to the Lebesgue measure. Let $S$ be generated by $l_k := e^{-\lambda k \alpha}$ for $\lambda > 0$ and $\alpha > 0$. Then

1. $\text{Cap}(S) = \text{Cap}([0,1]) > 0$ for $0 < \alpha \leq 1$ almost surely,
2. $\text{Cap}(S) = 0$ for $1 < \alpha$.

**Remark 3.** The phase transition in Theorem 2 (and in Theorem 4 below) is analogous to the one observed in [7] (see Equation (1.7) and Theorem 6 in Section 1.3). It is interesting to note that in the present paper we actually establish the full capacity at the critical point $\alpha = 1$, while for the setting in [7] full capacity at the critical value $\alpha = 2$ was only a conjecture.

Theorem 1 and Theorem 2 follow from our deterministic results. Our first main deterministic result provides sufficient conditions for a $G_\delta$ set $S$, defined by (1.2), to have full capacity on the unit interval:

**Theorem 3** (Sufficient conditions for full capacity). Assume that the intervals $\{I_k\}$ from (1.2) have exponentially decreasing lengths $l_k = e^{-\lambda k}$ for
some fixed $\lambda > 0$ and satisfy the assumptions $A.1$ - $A.3$ below. Then the $G_\delta$ set $S$ has full capacity on the unit interval:

$$\text{Cap}(S) = \text{Cap}([0, 1]) > 0.$$ 

The assumptions that are imposed in this theorem, roughly speaking, state that these intervals are sufficiently uniformly placed and sufficiently well-spaced (both in terms of “average” and minimal distances between their centers).

First, we will pack intervals $\{I_k\}$ into groups with the indices from

$$A_n := \{n, \ldots, 2n - 1\},$$

and then pack these into larger groups:

$$A_{n,q(n)} := A_n \cup A_{2n} \cup \cdots \cup A_{2^qn},$$

where $q \in \mathbb{N}$. We will assume the following:

$A.1$ (distribution) The centers are distributed with respect to some density function $\phi(x) \in L^1([0, 1], dx)$, where $\phi(x) > 0$ a.e.. Namely, for every $f \in C([0, 1])$, we have

$$\frac{1}{\#(A_n)} \sum_{k \in A_n} f(c_k) \to \int_0^1 f(x)\phi(x) \, dx \quad \text{as} \quad n \to \infty.$$

Also, there exists a sequence $q(n)$ of integer numbers, such that $q(n) \to \infty$ as $n \to \infty$ and that the following two conditions hold:

$A.2$ (log-average spacing) For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $n$ large enough for any $n', n'' \in \{n, 2n, \ldots, 2^qn\}$ we have

$$\frac{1}{\#(A_{n'} \times A_{n''})} \sum (- \log |c_i - c_j|) < \varepsilon,$$

where the sum is over $i \in A_{n'}, j \in A_{n''}$ such that $i \neq j$ and $|c_i - c_j| < \delta$.

$A.3$ (gap control) For every $\varepsilon > 0$, for all large enough $n$, we have that for every $i, j \in A_{n,q(n)}$ with $i \neq j$ the following holds:

$$\frac{l_i + l_j}{2|c_i - c_j|} < \varepsilon.$$

Applying Theorem 3, we obtain our next result, a deterministic phase transition for the capacity. Again, take the centers $\{c_k\}$ that assumptions $A.1$-$A.3$ are satisfied for the choice of lengths $l_k = e^{-\lambda k}$ for some fixed $\lambda > 0$ and varying the speed at which the lengths of intervals decrease, we observe a sharp phase transition in the deterministic setting:

**Theorem 4** (Deterministic phase transition). Let the centers $\{c_k\}$ be the same centers from Theorem 3. Let $S$ be generated by $l_k := e^{-\lambda k^\alpha}$ for $\alpha > 0$. Then
(1) \( \text{Cap}(S) = \text{Cap}([0,1]) > 0 \) for \( 0 < \alpha \leq 1 \),
(2) \( \text{Cap}(S) = 0 \) for \( 1 < \alpha \).

The next theorem states that both the random theorems (Theorem 1 and Theorem 2) follow from the deterministic theorems (Theorem 12 and Theorem 4):

**Theorem 5.** Let \( \{c_k\} \) be i.i.d. with an absolutely continuous distribution on any interval \( J \) with almost everywhere positive and uniformly bounded density function. Take \( l_k = e^{-\lambda k} \) for some fixed \( \lambda > 0 \). Then, almost surely assumptions A.1-A.3 are satisfied.

1.3. **Motivation and historical background.** In this section, we will discuss the motivation behind our project and the historical background.

Gorodetski and Kleptsyn in [6, Section 1.2] studied the set of exceptional energies in the parametric version of the Furstenberg theorem. Consider

\[
T_{n,\omega,a} := A_{\omega_n}(a) \ldots A_{\omega_1}(a)
\]

where matrices \( A_{\omega_k}(a) \in SL(2,\mathbb{R}) \) are i.i.d., depending on a parameter \( a \), taking values in some interval \( J \subset \mathbb{R} \). Furstenberg’s theorem implies that for every \( a \in J \), for almost every \( \omega \), we get

\[
\lim_{n \to \infty} \frac{1}{n} \log \| T_{n,\omega,a} \| = \lambda_F(a) > 0.
\] (1.5)

Questions on switching the quantifiers in the limit appear naturally in spectral theory, specifically, in Anderson localization proofs.

In [6, Theorem 1.5], the authors proved that almost surely switching the quantifiers leads to the occurrence of a different kind of behavior. Namely, under some technical assumptions, it was shown that for almost every \( \omega \), there exists some random exceptional energies subset of parameters \( S_e(\omega) \subset J \) such that (1.5) does not hold. Additionally, there also exists a smaller set of parameters \( G_\delta \)-set \( S_0(\omega) \) such that for all \( a \in S_0(\omega) \), we have

\[
\liminf_{n \to \infty} \frac{1}{n} \log \| T_{n,\omega,a} \| = 0.
\]

Both these sets are random \( G_\delta \)'s of the form (1.2).

Additionally, in [6] it was shown that the set \( S_e(\omega) \) (and thus \( S_0(\omega) \)) have zero Hausdorff dimension. Capacity is a finer measurement than the Hausdorff dimension in the sense that any set \( E \subset \mathbb{C} \) that has zero capacity must have zero Hausdorff dimension. The question as to what is the capacity of both \( S_e(\omega) \) and \( S_0(\omega) \) is still open. If one can show that those sets satisfy assumptions A.1-A.3 (and this is what we conjecture), our Theorem 3 will imply that these sets have full capacity, that is \( \text{Cap}(S_e(\omega)) = \text{Cap}(S_0(\omega)) = \text{Cap}(J) \), in the same way as we get full capacity in the “toy model” Theorem 2.
The capacity of such $G_\delta$’s is also interesting because it showcases a phase transition. That is, a drastic transition from zero capacity to full capacity precisely when the series 
\[ \sum_k \frac{1}{|\log l_k|} \quad (1.6) \]
transitions from convergent to divergent. As we mentioned above, capacity gauges how far away a set is from being polar. Hence, as we change the speed of intervals so that the series (1.6) transitions from convergent to divergent, $S$ goes from being polar to being as far away as possible from polar, there is no middle ground. This transition was first noticed by Kleptsyn and Quintino (see [7]) in the case when the centers $\{c_k\}$ are equidistributed in the following way: for every $n$ we consider $n$ equally spaced centers:
\[ c_{j,n} = \frac{2j + 1}{2n} \quad \text{for every} \quad j = 0, \ldots, n - 1, \]
and with the restriction that the corresponding interval $J_{j,n}$ have the same length $r_n$ for $j = 0, \ldots, n - 1$. The uniform $G_\delta$-set $\tilde{S}$, corresponding to the sequence $r_n$, is given by
\[ \tilde{S} := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{j=0}^{n-1} J_{j,n}. \quad (1.7) \]
Any uniform $G_\delta$ set $\tilde{S}$ may be written in the generic setting (1.2) by ordering $J_{j,n}$ and re-labeling. They noticed that there is a “phase transition” in which $\tilde{S}$ goes from having zero capacity to full capacity:

**Theorem 6** (Phase transition [7, Theorem 1.2]). For $r_n = e^{-n^\alpha}$,

(1) if $\alpha > 2$, then $\text{Cap}(\tilde{S}) = 0$,
(2) if $\alpha < 2$, then $\text{Cap}(\tilde{S}) = \text{Cap}([0,1])$.

We refer to $\alpha = 2$ as the critical case because it is precisely when the sum (1.6) transitions from convergent to divergent. Note that there are $n$ intervals of length $e^{-n^\alpha}$ in (1.7), and that is why the critical case is $\alpha = 2$ in Theorem 6 and not $\alpha = 1$. Also, note that full capacity of $\tilde{S}$ in the critical case in [7] was conjectured, but not proved; contrary to this, in the setting of the present paper the analogous statement for the critical $\alpha = 1$ is established (see Remark 3).

The zero capacity part in all the theorems above goes back to the works in the first half of twentieth century: a 1918 paper by Lindeberg [8] and 1937 by Erdős and Gillis [5]. They were working on connecting the notion of the $h$-volume of a set with the logarithmic capacity of a set. A function $h$ that is defined in some right neighborhood of 0 is called a measuring function provided that $h$ is continuous, positive, increasing, concave, and $h(0) = 0$. 
The \( h \)-volume of a set \( E \subset \mathbb{R} \) is defined as
\[
m_h(E) := \lim_{\varepsilon \to 0^+} \inf \left\{ \sum_j h(r_j) \middle| \mathcal{I}(E, \varepsilon) \right\},
\]
where the infimum is taken over the set \( \mathcal{I}(E, \varepsilon) \) of covers of \( E \) by balls of diameter less than \( \varepsilon \):
\[
\mathcal{I}(E, \varepsilon) = \left\{ (x_j, r_j)_{j \in \mathbb{N}} \mid \bigcup_j U_{r_j}(x_j) \supset E, \quad \forall j \quad r_j < \varepsilon \right\}.
\]
In particular, the function \( h_0(x) = \frac{1}{|\log x|} \) provided the following link:

**Theorem 7** (Erdős and Gillis [5, p. 187], generalizing Lindeberg [8, p. 27]).

*If for a set \( E \) one has \( m_{h_0}(E) < +\infty \), then \( \text{Cap}(E) = 0 \).*

This was later re-proved by Carleson [1], and noticed in [7, Thm. 1.3] to be a corollary of the Cauchy-Schwarz inequality. Theorem 7 immediately implies (see [7, Corollary 1.4]):

**Corollary 8.** Let \( S \) be defined by (1.2). If the series \( \sum_k \frac{1}{|\log k|} \) converges, then the set \( S \) is of zero capacity.

In the same 1937 paper, Erdős and Gillis [5, (C), p. 186] have mentioned a conjecture, going back to Nevanlinna’s paper [9], that aimed at generalizing Theorem 7 to other \( h \)-volume settings. This conjecture was disproved by Ursell [12]; the re-distribution construction that was used in [7] and that we are using in the present paper can be seen as an extension of his technique.

1.4. **Sketch of the proof and plan of the paper.** In this section, we will give a sketch of the proofs and end with the plan of the paper.

The statement in the phase transition theorems 2 and 4 for \( \alpha > 1 \) is a result from [7] and does not require assumptions A.1 - A.3 (see Theorem 7 above).

Due to monotonocity of capacity, the statement in the phase transition theorems 2 and 4 for \( 0 < \alpha < 1 \) follows by establishing full capacity for \( \alpha = 1 \). For the deterministic phase transition this is Theorem 3. For the random phase transition the result follows from Theorem 3 by showing that assumptions A.1-A.3 hold almost surely, that is Theorem 5.

In Section 2, we will show that the centers from the random phase transition satisfy assumptions A.1 - A.3 for \( \alpha = 1 \) (Theorem 5). Hence, the random phase transition holds.

Thus, the main task is to show that \( S \) has full capacity for \( \alpha = 1 \) (Theorem 3). The method that we will employ to show full capacity is the re-distribution technique under assumptions A.1-A.3 for \( \alpha = 1 \).

We introduce this technique in Section 3.1. Namely, we will begin with the equilibrium measure \( \nu_J \) on the interval \( J \), then we will construct a probability measure \( \nu_1 \) such that the energy \( I(\nu_1) \) approximates the energy \( I(\nu_J) \)
and whose support is a subset of $\text{supp} \, \mu_f$ and is a finite union of intervals \( \{I_k\} \). Then we will construct another probability measure $\mu_2$ such that the energy $I(\mu_2)$ approximates the energy $I(\mu_1)$ and whose support is a subset of $\text{supp} \, \mu_1$ and is a finite union of intervals \( \{I_k\} \). Inductively, repeating this procedure, we get a sequence of probability measures that have their energies that are arbitrarily close to $I(\nu_f)$ and such that their supports create a decreasing sequence of compact subsets. After passing to the weak-limit we obtain a measure supported on $S$ (Proposition 12), thus proving the desired full capacity for the set $S$ (see Section 3.2 for the proof). Proposition 13 states that the above technique is applicable when assumptions A.1-A.3 are satisfied. Hence, Theorem 3 follows from Proposition 12 and Proposition 13.

Finally, in Section 4 we develop the tools to prove Proposition 13. In Section 4.5 we conclude with the proof of Proposition 13.

2. In the random setting A.1-A.3 are a.s. satisfied

This section is devoted to the proof of Theorem 5: we assume that the centers \( \{c_k\} \) are i.i.d. random variables and show that if their distributions are nice, then assumptions A.1-A.3 are satisfied.

The distribution immediately follows from the law of large numbers:

**Lemma 9.** Under the assumptions of Theorem 5, assumption A.1 is almost surely satisfied.

Now, take $q(n) = \lfloor \log_2(\log n) \rfloor$. The uniform gap control can be obtained by a straightforward estimate of the probability of two random centers being close to each other:

**Lemma 10.** Under the assumptions of Theorem 5, for $q(n) = \lfloor \log_2(\log n) \rfloor$, assumption A.3 is almost surely satisfied.

**Proof.** Let $K$ be the upper bound for the density of the distribution, and let $\varepsilon > 0$ be fixed. For any $i \neq j$, $i, j \in \mathcal{A}_{n,q(n)}$, if

$$\frac{l_i + l_j}{2|c_i - c_j|} < \varepsilon$$

does not hold, it implies that

$$|c_i - c_j| \leq \frac{l_i + l_j}{2\varepsilon} < \frac{1}{\varepsilon} e^{-\lambda n},$$

and the probability of such an event (for any given $i$ and $j$) does not exceed $2K \varepsilon e^{-\lambda n}$. As there are less than $2^{q(n)+1}n < 2n^2$ possible indices $i$ and $j$, the total probability that the condition is violated for a given $n$ does not exceed $4n^4 \cdot \frac{2K}{\varepsilon} e^{-\lambda n}$. The series

$$\sum_n 4n^4 \cdot \frac{2K}{\varepsilon} e^{-\lambda n}$$
converges, and the application of the Borel-Cantelli Lemma concludes the proof.

Finally, the log-averages of spaces also can be controlled quite directly:

\textbf{Lemma 11.} Under the assumptions of Theorem 5, for \( q(n) = \lfloor \log_2(\log n) \rfloor \), assumption \textbf{A.2} is almost surely satisfied.

\textit{Proof.} Given \( \varepsilon > 0 \) be given and set

\[ G(X, Y) = (-\log |X - Y|) \mathbb{1}_{(0, \delta)}(|X - Y|), \]

where \( \delta > 0 \) and \( G(X, X) = 0 \). We have that

\[ \frac{1}{\#(A_{n'} \times A_{n''})} \sum_{0 < |c_i - c_j| < \delta} (-\log |C_i - C_j|) = \frac{1}{\#(A_{n'} \times A_{n''})} \sum_{i \in A_{n'} \times j \in A_{n''}} G(C_i, C_j). \]

Suppose the law of large numbers holds for \( G(X, Y) \): as \( n \to \infty \) we have

\[ \frac{1}{\#(A_{n'} \times A_{n''})} \sum_{i \in A_{n'}, j \in A_{n''}} G(C_i, C_j) \to \mathbb{E} G(C_1, C_2), \]

where for \( n', n'' \in \{n, 2n, \ldots, 2^{q(n)}n\} \). Then we may find a \( \delta \) such that \textbf{A.2} holds.

The law of large numbers holds by considering the difference:

\[ H(x, y) := G(x, y) - c - \mathbb{E}[G(x, y) - c|y] - \mathbb{E}[G(x, y) - c|x], \]

where \( c = \mathbb{E} G(x, y) \) and their average

\[ \frac{1}{n'n''} S_{n', n''} = \frac{1}{n'n''} \sum_{i \neq j} H(x, y). \]  \quad (2.1)

Now, consider the fourth power of \( S_{n', n''} \) and take its expectation:

\[ \mathbb{E}(S_{n', n''})^4 = \sum \mathbb{E}[H(c_{i_1}, c_{i_2})H(c_{j_1}, c_{j_2})H(c_{k_1}, c_{k_2})H(c_{l_1}, c_{l_2})]. \]  \quad (2.2)

The function \( H(x, y) \) has the property that \( \mathbb{E}[H(x, y)|y] = \mathbb{E}[H(x, y)|x] = 0 \).

Hence, if a term has an independent random variable, say \( c_{i_1} \), then

\[ \mathbb{E}[H(c_{i_1}, c_{i_2})H(c_{j_1}, c_{j_2})H(c_{k_1}, c_{k_2})H(c_{l_1}, c_{l_2})] = 0. \]

On the other hand, when every random variable is depended on another random variable, we can count the non-vanishing terms. There are \( (n'n'')^2 \) terms of the form \( \mathbb{E}[H(c_{i_1}, c_{i_2})^4] \). There are \( (n'n'')^2 \) terms of the form

\[ \mathbb{E}[H(c_{i_1}, c_{i_2})^2 H(c_{j_1}, c_{j_2})^2]. \]

There are at most \( (n'n'')(n' + n'')^2 \) terms of the form

\[ \mathbb{E}[H(c_{i_1}, c_{i_2})^2 H(c_{j_1}, x) H(c_{k_1}, y)], \]

where \( x, y \in \{c_{i_1}, c_{i_2}\} \). Lastly, there are at most \( n'n'/(n' + n'')^2 \) terms of the form

\[ \mathbb{E}[H(c_{i_1}, c_{i_2}) H(c_{i_1}, c_{j_2}) H(c_{k_1}, c_{i_2}) H(c_{k_1}, c_{j_2})]. \]
Since $E[H(x,y)^4] < \infty$, then

$$\mathbb{E} S_{n',n''}^4 \leq C' \max\{(n'n'')^2, n'^3n'', n'n'^3\},$$

where $C' > 0$ is some constant. An application of the Chebyshev inequality implies that

$$\mathbb{P}\left(|S_{n',n''}| > \varepsilon(n'n'')\right) \leq \frac{C'}{\varepsilon^4} \max\left\{\frac{1}{(n'n'')^2}, \frac{1}{n'^3n''}, \frac{1}{n'^3n''} \right\}.$$

Since $n \leq n', n''$, then

$$\mathbb{P}\left(|S_{n',n''}| > \varepsilon(n'n'')\right) \leq \frac{C'}{\varepsilon^4} \frac{1}{n^4}.$$We have that

$$\sum_{n=1}^{\infty} \sum_{n'=n}^{2^{2q(n)}n} \sum_{n''=n}^{n''=n} \mathbb{P}\left(|S_{n',n''}| > \varepsilon(n'n'')\right) \leq \frac{C'}{\varepsilon^4} \sum_{n=1}^{\infty} \frac{2^{2q(n)}n}{n^2}$$

$$\leq \frac{C'}{\varepsilon^4} \sum_{n=1}^{\infty} \frac{(\log n)^2}{n^2},$$

which is finite. By Borel-Cantelli lemma, $|S_{n',n''}| > \varepsilon(n'n'')$ does not occur infinitely often with probability 1. Let $\varepsilon_k$ be a sequence of positive numbers that decreases to 0 as $k \to \infty$. For each $\varepsilon_k$, $|S_{n',n''}| > \varepsilon_k(n'n'')$ does not occur infinitely often with probability 1. Since the countable intersection of sets of full measure has full measure, then for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, for every $n', n'' \in \{n, 2n, \ldots, 2^{2q(n)}n\}$, we have $|S_{n',n''}| < \varepsilon(n'n'')$ with probability 1. That is, the average (2.1) goes to 0 as $n \to \infty$.

Together, lemmas 9, 10, 11 imply Theorem 5.

3. The re-distribution technique

3.1. Introducing the technique. Our main tool for establishing full capacity for a set $S$ (what is needed for the proof of Theorem 3) will be the re-distribution technique that was introduced in [7]. We will recall the method in this section. The main property that allows it to work will be the following one:

**Definition 2.** We say that $S$ (in the generic setting (1.2)) is re-distributable if the following holds: for every probability measure $\nu$ with piecewise continuous density that is supported on a finite collection of intervals in $[0,1]$ and for every $\varepsilon > 0$ and every $m \in \mathbb{N}$, there exists another probability measure $\nu'$ with piecewise continuous density such that

1. $I(\nu') < I(\nu) + \varepsilon$, 

(2) \( \nu' \) is supported on \( \text{supp} \, \nu \cap V_n \) for some \( n \geq m \), where \( V_n \) is a finite union of \( I_k \)'s with \( k \geq n \).

The following proposition then allows us to establish full capacity:

**Proposition 12.** If \( S \) is re-distributable, then \( S \) has full capacity on the unit interval:

\[
\text{Cap}(S) = \text{Cap}([0,1]).
\]

**Proposition 13.** Assume A.1 - A.3 for interval lengths \( l_k = e^{-\lambda k} \) for some \( \lambda > 0 \). Then the set \( S \) is re-distributable.

Section 4 is devoted to the proof of Proposition 13.

3.2. **Proof of Proposition 12.** In this section, we will prove that \( S \) has full capacity when \( S \) is re-distributable.

As we have mentioned in Section 1.4, the proof of Proposition 3 is obtained by inductively constructing a sequence of measures with smaller and smaller support. Let us make these arguments formal:

**Proof of Proposition 12.** The density function for the equilibrium measure for the unit interval is

\[
f_{[0,1]}(x) = \frac{1}{\pi \sqrt{x(1-x)}},
\]

for \( x \in (0,1) \) and 0 otherwise (see e.g. [11, Eq. (A.53)]). Given \( \varepsilon > 0 \), there exists a continuous density function \( f \) such that

\[
I(f(x)dx) < I(f_{[0,1]}(x)dx) + \varepsilon.
\]

Let \( d\nu_0(x) := f(x) \, dx \) with support \([0,1]\). Applying Definition 2 to \( \nu_0 \), there exists \( \nu_1 \) with support \( V_{n_1} \) and

\[
I(\nu_1) < I(\nu_0) + \varepsilon/2^2.
\]

Apply Definition 2 to \( \nu_1 \), there exists \( \nu_2 \) with support \( V_{n_1} \cap V_{n_2} \) and

\[
I(\nu_2) < I(\nu_1) + \varepsilon/2^3.
\]

and \( n_1 < n_2 \). By induction and applying Definition 2, for each \( m \in \mathbb{N} \) there exists a Borel probability measure \( \nu_m \) that is supported on

\[
C_m := V_{n_1} \cap \cdots \cap V_{n_m}.
\]

We consider the telescoping sum:

\[
I(\nu_m) - I(\nu_0) = \sum_{i=1}^{m} (I(\nu_i) - I(\nu_{i-1})) < \sum_{i=1}^{m} \frac{\varepsilon}{2^{i+1}}.
\]

It follows that

\[
I(\nu_m) < I(\nu_0) + \varepsilon.
\]
As in [7], any weak* limit will work. Assume that \( \nu_\infty \) is a weak* limit of \( \{ \nu_m \} \). Passing to a weak* limit can only decrease the energy (see [10, Lemma 3.3.3]):
\[
I(\nu_\infty) \leq \liminf_{m \to \infty} I(\nu_m) < I(\nu_0) + \varepsilon < I(f_{[0,1]}(x) \, dx) + 2\varepsilon.
\]
Since \( \varepsilon > 0 \) is arbitrary, we have that
\[
I(\nu_\infty) \leq I(f_{[0,1]}(x) \, dx).
\]
If \( \nu_\infty \) has compact support contained in \( S \), then we are done. The weak* limit only allows us to conclude that \( \nu_\infty \) has compact support contained in
\[
\bar{C}_\infty := \bigcap_m \text{Cl}(C_m).
\]
However, \( \bar{C}_\infty \) differs from
\[
C_\infty := \bigcap_m C_m \subset S,
\]
by at most a countable set \( P \) (the collection of boundary points of each \( C_m \)). Since \( I(\nu_\infty) < \infty \), then \( \nu_\infty(P) = 0 \) (see [10, Theorem 3.2.3]). By regularity of Borel measures, we may find a Borel probability measure with compact support contained in \( C_\infty \) that differs from \( I(\nu_\infty) \) as small as we want. Hence,
\[
I(f_{[0,1]}(x) \, dx) = \inf\{I(\nu) : \nu \in \mathcal{P}(C_\infty)\}.
\]
Since \( C_\infty \subset S \), then \( S \) has full capacity:
\[
\text{Cap}(S) = \text{Cap}([0,1]).
\]

\[\square\]

4. Proving Proposition 13: \( \text{A.1-A.3} \) imply re-distribution

4.1. Properties of assumptions \( \text{A.1-A.3} \). In this section, we will discuss some of the properties of assumptions \( \text{A.1-A.3} \) that will be needed in the proofs.

The gap control property (assumption \( \text{A.3} \)) is aimed at controlling the gaps between two distinct intervals in a collection of intervals in a uniform way. Let \( I, I' \subset (0,1) \) be two disjoint intervals with centers \( c, c' \), then the gap between \( I \) and \( I' \) is
\[
\text{dist}(I, I') = |c - c'| - \frac{1}{2}(|I| + |I'|) > 0. \quad (4.1)
\]
We will control the gaps by controlling the ratio of the average of the lengths and the distance between their centers (see \( \text{A.3} \)).
Remark 4. Notice that by letting \( \varepsilon < 1 \) in A.3, we get (4.1). Hence, gap control implies that the intervals in

\[ \{ I_k : k \in 2n, \ldots, 2^{q(n)}n - 1 \}, \]

are pairwise disjoint. Each measure that we construct in Section 4.2 will be supported on pair-wise disjoint collection:

\[ \{ I_k : k \in \mathcal{A}_n \} = \{ I_n, I_{n+1}, \ldots, I_{2n-1} \}. \]

In Section 4.5, we will construct a measure that is an average of measures from Section 4.2. Hence, the average measure will be supported on:

\[ \{ I_n, I_{n+1}, \ldots, I_{2n-1} \} \]
\[ \{ I_{2n}, I_{2n+1}, \ldots, I_{2^2n-1} \} \]
\[ \vdots \]
\[ \{ I_{2^{q(n)-1}n}, I_{2^{q(n)-1}n+1}, \ldots, I_{2^{q(n)}n} \}. \]

Assumption A.3 allows the collection of intervals above to be disjoint.

The distribution property (assumption A.1) requires the centers to be \textit{distributed} with respect to some function \( \phi(x) \in L^1([0, 1], dx) \), where \( \phi(x) > 0 \) a.e.:

\[
\frac{1}{\#(\mathcal{A}_n)} \sum_{k \in \mathcal{A}_n} f(c_k) \to \int_0^1 f(x)\phi(x) \, dx \quad \text{as} \quad n \to \infty, \quad (4.2)
\]

for every continuous function \( f \in C([0,1]) \). This definition is a generalization of equidistributed sequences.

Remark 5. Note that (4.2) will hold for piecewise continuous functions \( f \) since we may approximate such functions from above and below by continuous functions in \( L_1 \). Equation (4.2) extends to 2-dimensions: Let \( f \) be any piecewise continuous function. Then

\[
\frac{1}{\#(\mathcal{A}_n \times \mathcal{A}_{n'})} \sum_{i \in \mathcal{A}_n, j \in \mathcal{A}_{n'}} f(c_i)f(c_j) \to \int_0^1 \int_0^1 f(x)f(y)\phi(x)\phi(y) \, dx \, dy,
\]

\( (4.3) \)

as \( m \to \infty \) and \( n, n' \geq m \).

To show that \( S \) is re-distributable (see Definition 2), we will show that the following statement holds:

P.1 For each positive continuous function \( f \) on the interval \([0,1]\), there exists a sequence of probability measures \( \{ \mu^n \} \) so that each \( \mu^n \) has a
piecewise continuous density with support contained in a finite union of disjoint $I_k$’s with $k \geq n$ and with asymptotic behavior:

$$I(\mu^n) = \frac{I(f(x) \, dx)}{(\int_0^1 f(x) \, dx)^2} + o(1).$$

With the distribution assumption A.1 and log-average spacing assumption A.2, we can see that the centers have the asymptotic behavior that is needed in P.1:

**Lemma 14.** Under assumptions A.1 and A.2, for every $f \in C([0, 1])$, as $n \to \infty$ and $n \leq n', n''$, we have that

$$\frac{1}{\#(\mathcal{A}_{n'} \times \mathcal{A}_{n''})} \sum_{i \neq j} (-\log |c_i - c_j|) f(c_i) f(c_j) \to I(f(x) \phi(x) \, dx), \quad (4.4)$$

where the sum is taken over $(i, j) \in \mathcal{A}_{n'} \times \mathcal{A}_{n''}$ and $i \neq j$. Moreover,

$$I(f(x) \phi(x) \, dx) < \infty.$$

**Remark 6.** The density function $\phi(x)$ in assumption A.1 is not to be confused with the continuous density function $f$ in Definition 2 and in P.1. Once the centers are distributed with respect to $\phi(x)$, the function $\phi(x)$ is fixed. The continuous density function $f$ in Definition 2 and in P.1 is arbitrary.

**Proof of Lemma 14.** Given $\varepsilon > 0$, let $\delta > 0$ satisfy assumption A.2. For $s > 0$, define $f_s(x) = -\log x$ for $x \geq s$ and 0 otherwise. Using Fatou’s lemma, we get

$$\int \int_{|x - y| < \delta} (-\log |x - y|) \phi(x) \phi(y) \, dx \, dy \leq \liminf_{s \to 0^+} \int \int_{|x - y| < \delta} f_s(|x - y|) \phi(x) \phi(y) \, dx \, dy.$$

Using assumption A.1 for $n', n'' \in \{n, 2n, \ldots , 2^{\varrho(n)} n\}$, we have

$$\liminf_{s \to 0^+} \int \int_{|x - y| < \delta} f_s(|x - y|) \phi(x) \phi(y) \, dx \, dy = \liminf_{s \to 0^+} \lim_{n \to \infty} \sum_{0<|c_i - c_j|<\delta} \frac{f_s(|c_i - c_j|)}{\#(\mathcal{A}_{n'} \times \mathcal{A}_{n''})} \leq \lim_{n \to \infty} \sum_{0<|c_i - c_j|<\delta} \frac{(-\log |c_i - c_j|)}{\#(\mathcal{A}_{n'} \times \mathcal{A}_{n''})} \leq \varepsilon,$$

where the last holds by assumption A.2 for some $\delta > 0$. Hence, for every continuous function $f$, we have finite energy:

$$I(f(x) \phi(x) \, dx) = \int \int (-\log |x - y|) f(x) \phi(x) f(y) \phi(y) \, dx \, dy < \infty.$$
Note that
\[
\sum_{i \neq j} \frac{(- \log |c_i - c_j|) f(c_i) f(c_j)}{\#(A_n' \times A_{n''})} = \sum_{i \neq j} \frac{(- \log |c_i - c_j|) f_\delta(c_i) f_\delta(c_j)}{\#(A_n' \times A_{n''})}.
\]

Let \( n \to \text{inf} \), then by assumptions \textbf{A.1} and \textbf{A.2}, we have
\[
\lim_{n \to \infty} \sum_{i \neq j} \frac{(- \log |c_i - c_j|) f(c_i) f(c_j)}{\#(A_n' \times A_{n''})} - \int \int f_\delta(|x - y|) f(x) f(y) \phi(x) \phi(y) \leq \varepsilon \cdot (\max |f|)^2.
\]

By letting \( \delta \to 0 \), we have that
\[
\lim_{n \to \infty} \sum_{i \neq j} \frac{(- \log |c_i - c_j|) f(c_i) f(c_j)}{\#(A_n' \times A_{n''})} - I(f(x) \phi(x) \, dx) \leq \varepsilon \cdot \max |f|.
\]

Since \( \varepsilon > 0 \) is arbitrary, we get (4.4).

\[
4.2. \textbf{Construction of a single-level re-distribution.} \quad \text{Our first step in constructing the probability measures in \textbf{P.1} is to construct a single-level re-distribution probability measure. This section is devoted to the construction of such probability measures.}
\]

We begin with a “re-distribution” type of measure \( f(x) \, dx|_{[0,1]} \) onto a single interval:
\[
\mu_k = \frac{f(x) \, dx|_{I_k}}{|I_k|}.
\]

We do not call this a re-distribution as in [7] because the measure is not necessarily a probability measure. We consider the average of \( \mu_k \)'s:
\[
\mu_{A_n} := \frac{1}{\#A_n} \sum_{k \in A_n} \mu_k,
\]

where \( A_n \) are defined in (1.3) and satisfy the gap control property \textbf{A.3}. Recall that gap control implies that for large enough \( n \), our collection of intervals are disjoint (see Remark 4). Notice that each measure \( \mu_{A_n} \) is not necessarily a probability measure. To correct that, let us define
\[
V_n = \bigcup_{k \in A_n} I_k.
\]
Then, we consider the single-level re-distribution probability measure:

\[ \hat{\mu}_n := \frac{\mu_{A_n}}{\mu_{A_n}(V_n)}, \]  

which is supported on \( V_n \) and it’s energy is

\[ I(\hat{\mu}_n) = \frac{1}{(\mu_{A_n}(V_n))^2} I(\mu_{A_n}). \]

Thus, we are interested in the asymptotic behavior of \( \mu_{A_n}(V_n) \) and the asymptotic behavior of:

\[ I(\mu_{A_n}) = \frac{1}{(#A_n)^2} \sum_{k \in A_n} I(\mu_k) + \frac{1}{(#A_n)^2} \sum_{i \neq j} I(\mu_i, \mu_j). \]  

(4.6)

The first sum is referred to as the self-interaction sum because in \( I(\mu_k) = I(\mu_k, \mu_k) \) the same measure is interacting with itself. The second sum is referred to as outer-interaction sum because we have two measures with disjoint supports interacting with each other in \( I(\mu_k, \mu_j) \).

In Section 4.3, we will discuss the asymptotic behavior of the outer-interaction and in Section 4.4 we will work on controlling the asymptotic behavior of the self-interaction sum. In Section 4.5, we will put the two together. We will finish the section with the asymptotic behavior of \( \mu_{A_n}(V_n) \):

**Lemma 15.** If A.1 and A.3 hold, then

\[ \mu_{A_n}(V_n) = \int f(x) \phi(x) \, dx + o(1). \]

**Proof.** Since for large \( n \) the intervals in

\[ \{I_k : k \in A_n\} \]

are disjoint (see Remark 4), then \( \mu_k(V_n) = \frac{1}{|I_k|} \int_{I_k} f(x) \, dx \). Hence,

\[ \mu_{A_n}(V_n) = \frac{1}{#A_n} \sum_{k \in A_n} \frac{1}{|I_k|} \int_{I_k} f(x) \, dx. \]

Due to the uniform continuity of \( f \) on the interval \([0, 1]\), for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[ |f(x) - f(y)| < \varepsilon \quad \text{if} \quad |x - y| < \delta. \]

Since the lengths of the intervals \( I_k \) approach 0, then there exists \( N \in \mathbb{N} \) such that for every \( k \geq N \) we have

\[ |f(x) - f(c_k)| < \varepsilon \quad \text{if} \quad x \in I_k. \]

Therefore, for every \( n \geq N \) and every \( k \in A_n \), we have

\[ |f(x) - f(c_k)| < \varepsilon \quad \text{if} \quad x \in I_k. \]
It follows that for large enough \( n \), we have
\[
\left| \mu_{A_n}(V_n) - \frac{1}{\# A_n} \sum_{i \in A_n} f(c_i) \right| < \varepsilon.
\]
As the centers are distributed with respect to \( \phi(x) \) \( A.1 \), it follows that
\[
\left| \mu_{A_n}(V_n) - \left( \int f(x)\phi(x) \, dx + o(1) \right) \right| < \varepsilon.
\]
Since \( \varepsilon > 0 \) was arbitrary, then the result holds. \( \square \)

4.3. Asymptotic behavior of outer-interaction. In this section, we are interested in the asymptotic behavior of the outer-interaction sum:
\[
\frac{1}{(\# A_n)^2} \sum_{i \neq j} I(\mu_i, \mu_j),
\]
where the sum is taken over \( i, j \in A_n \) and \( i \neq j \). It is the outer-interaction sum that gives the limit point in \( P.1 \):

**Lemma 16.** We have
\[
\frac{1}{(\# A_n)^2} \sum_{i \neq j} I(\mu_i, \mu_j) = I(f(x)\phi(x) \, dx) + o(1),
\]
where \( i, j \in A_n \).

To show Lemma 16, we want to estimate \((- \log |x - y|)\) by \((- \log |c - c'|)\) where \( x \) and \( y \) are in intervals with centers \( c \) and \( c' \), respectively. The next lemma allows us to do that.

**Lemma 17.** Let \( f \) be any continuous function on \([0, 1]\) and let \( J, J' \) be two disjoint intervals in \([0, 1]\) with lengths \( r, r' \) and centers \( c, c' \), respectively. Define
\[
\mu := \frac{1}{r} f(x) \, dx|_J \quad \text{and} \quad \mu' := \frac{1}{r'} f(x) \, dx|_{J'}.
\]
Let \( \varepsilon > 0 \). If
\[
\frac{r + r'}{2|c - c'|} \leq (1 - e^{-\varepsilon}),
\]
then
\[
\left| I(\mu, \mu') - (- \log |c - c'|)f(c)f(c') \right| \leq (2K(- \log |c - c'|) + K^2)\varepsilon,
\]
where \( K = \| f \|_\infty \).

**Proof.** Let us first prove that for \( a, b > 0 \), we have
\[
| \log a - \log b | \leq \varepsilon,
\]
if
\[
|a - b| \leq b(1 - e^{-\varepsilon}).
\]
We have that
\[ |\log a - \log b| \leq \varepsilon \]
holds if and only if
\[ -\varepsilon \leq \log(a/b) \leq \varepsilon, \]
if and only if
\[ be^{-\varepsilon} \leq a \leq be^\varepsilon, \]
if and only if
\[ -b(1 - e^{-\varepsilon}) \leq a - b \leq b(e^\varepsilon - 1). \]
Since \((1 - e^{-\varepsilon}) \leq (e^\varepsilon - 1)\), then
\[ |\log a - \log b| \leq \varepsilon \]
holds when
\[ |a - b| \leq b(1 - e^{-\varepsilon}). \]

For any two disjoint intervals \(J, J' \in [0, 1]\) with centers \(c, c'\) and with lengths \(r, r'\) respectively, we have
\[ ||x - y| - |c - c'|| \leq |(x - c) + (c' - y)| \leq \frac{r + r'}{2}. \]
Since assumption (4.8) is equivalent to
\[ \frac{r + r'}{2} \leq |c - c'|(1 - e^{-\varepsilon}), \]
then
\[ |(- \log |x - y|) - (- \log |c - c'|)| < \varepsilon. \]

Since \(f\) is uniformly continuous on \([0, 1]\), then there exists \(\delta > 0\) such that
\[ |f(a) - f(b)| < \varepsilon \text{ if } |a-b| < \delta. \] If \(0 < r, r' < \delta\), we have that \(|f(x) - f(c)| < \varepsilon\)
and \(|f(y) - f(c')| < \varepsilon\). We would like to combine the three inequalities.

Suppose \(A, a, B, b \in \mathbb{R}\) and \(\varepsilon_a, \varepsilon_b > 0\) such that
\[ |A - a| < \varepsilon_a \quad \text{and} \quad |B - b| < \varepsilon_b. \]
We have that
\[ |AB - ab| \leq |AB - A\varepsilon| + |A\varepsilon - ab| \leq (|A|\varepsilon_b + |b|\varepsilon_a). \]
One application of the above gives
\[ |f(x)f(y) - f(c)f(c')| \leq 2K\varepsilon, \]
where \(K = \|f\|_\infty\). A third application yields
\[ |(- \log |x - y|)f(x)f(y) - (- \log |c - c'|)f(c)f(c')| < (2K\varepsilon(- \log |c - c'|) + K^2\varepsilon). \]
Integrating by
\[
\frac{1}{r} \, dx|_J \quad \text{and} \quad \frac{1}{r'} \, dy|_{J'}
\]
finishes the proof. \(\square\)

Now that we can estimate \((- \log |x - y|)\) by \((- \log |c - c'|)\) where \(x\) and \(y\) are in intervals with centers \(c\) and \(c'\), respectively, we are ready to estimate
\[
\frac{1}{(\#A_n)^2} \sum_{i \neq j} I(\mu_i, \mu_j),
\]
by
\[
\frac{1}{(\#A_n)^2} \sum_{i \neq j} (- \log |c_i - c_j|) f(c_i) f(c_j).
\]

Let us go back to Lemma 16 and prove the asymptotic behavior of the outer-interaction:

**Proof of Lemma 16.** Let \(\epsilon > 0\) be given and let \(K = \|f\|_{\infty}\). The gap control \(A.3\) guarantees that there exists \(N \in \mathbb{N}\) such that for every \(n \geq N\) and every \(i \neq j\), where \(i, j \in A_n\), we have
\[
\frac{l_i + l_j}{2|c_i - c_j|} \leq (1 - e^{-\epsilon}),
\]
which is the condition (4.8) in Lemma 17. Since \(l_k\) decrease to 0 as \(k \to \infty\), then for all large enough \(n\) we may apply Lemma 17 to get
\[
|I(\mu_i, \mu_j) - (- \log |c_i - c_j|) f(c_i) f(c_j)| \leq (2K(- \log |c_i - c_j|) + K^2)\epsilon,
\]
for every \(n \geq N\) and every \(i \neq j\), where \(i, j \in A_n\). Adding this up for \(i \neq j\) where \(i, j \in A_n\) and then dividing by \((\#A_n)^2\), gives us:
\[
\left| \frac{1}{(\#A_n)^2} \sum_{i \neq j} I(\mu_i, \mu_j) - \frac{1}{(\#A_n)^2} \sum_{i \neq j} (- \log |c_i - c_j|) f(c_i) f(c_j) \right| 
\leq \frac{2K \epsilon}{(\#A_n)^2} \sum_{i \neq j} (- \log |c_i - c_j|) + K^2 \epsilon.
\]

We will apply Lemma 14 twice to the last two sums. We apply the lemma to the last sum by taking \(f = 1\) in Lemma 14, and then we apply the lemma again for arbitrary \(f\) in Lemma 14 to get:
\[
\left| \lim_{n \to \infty} \frac{1}{(\#A_n)^2} \sum_{i \neq j} I(\mu_i, \mu_j) - I(f(x)\phi(x)dx) \right| 
\leq (2KI(\phi(x)dx) + K^2)\epsilon.
\]
Lemma 14 also informs us that the energy of $\phi(x)dx$ is finite, hence $0 < (2KI(\phi(x)dx) + K^2)\varepsilon < \infty$. As $\varepsilon > 0$ is arbitrary, it follows that

$$
\frac{1}{(\#A_n)^2} \sum_{i \neq j} (-\log |c_i - c_j|) f(c_i)f(c_j) \to I(f(x)\phi(x)dx).
$$

Lemma 16 holds.

4.4. Asymptotic behavior of self-interaction. In this section, we will control the self-interaction:

Lemma 18. If $l_k = e^{-\lambda k}$ where $\lambda > 0$, then

$$
\frac{1}{(\#A_n)^2} \sum_{k \in A_n} I(\mu_k) = O\left(\frac{1}{(\#A_n)^2} \sum_{k \in A_n} k + o(1)\right).
$$

Proof. By shifting and a change of variables, we get

$$
I\left(\frac{1}{l_k} dx|_{l_i}\right) = -\log l_k + I(dx|_{[0,1]}) = -\log l_k \cdot (1 + o(1)).
$$

If $l_k = e^{-\lambda k}$, then adding the above over $k \in A_n$ gives us our result.

If the self-interaction sum vanishes in the limit, then we will be able to finish the proof with a single-level re-distribution. Let us see what the self-interaction tells us:

Lemma 19. If $A_n := \{n, \ldots, p(n) - 1\}$ where $p(n)$ is an integer-valued function such that $p(n) \geq 2n$, then

$$
\frac{1}{(\#A_n)^2} \sum_{k \in A_n} k = \frac{p(n) + n - 1}{2(p(n) - n)}.
$$

If $p(n) >> n$, then the right-hand side is close to $\frac{1}{2}$.

Proof. Let

$$
S = \sum_{k \in A_n} k.
$$

Then the arithmetic sum becomes

$$
S = \frac{(p(n) + n - 1)(\#A_n)}{2}.
$$

Since $\#A_n = p(n) - n$, then dividing by $(\#A_n)^2$ we get what we want.

Remark 7. Lemma 19 tells us that no matter how many intervals are included in our single-level of re-distribution, the self-interaction sum will never vanish.
But since we can bound the self-interaction sum uniformly for all $n$, we will be able to apply a multi-level re-distribution in Section 4.5. That is, we will take the average of measures $\hat{\mu}_n$ to handle the self-interaction sum.

4.5. The proof of Proposition 13. In this section, we will use a multi-level re-distribution to show $\textbf{P.1}$ holds and prove Proposition 13.

Let us first see where the asymptotic behavior of a single-level re-distribution leads:

**Proposition 20** (Single-level re-distribution). Let $l_k := e^{-\lambda k}$ and $\lambda > 0$. If assumptions $A.1$-$A.3$ are satisfied, then

$$I(\hat{\mu}_n) = \frac{1}{(\mu_{A_n}(V_n))^2} I(\mu_{A_n}) + O\left(\frac{1}{(\int f(x)\phi(x)\,dx)^2}\right) + o(1) = O(1),$$

where each $\hat{\mu}_n$ is the corresponding measure defined in (4.5).

**Proof.** We have that

$$I(\hat{\mu}_n) = \frac{1}{\#A_n} \sum_{k \in A_n} I(\mu_k) + \frac{1}{\#A_n^2} \sum_{i \neq j} I(\mu_i, \mu_j).$$

Breaking down the last, we get

$$I(\mu_{A_n}) = \frac{1}{\#A_n^2} \sum_{k \in A_n} I(\mu_k) + \frac{1}{\#A_n^2} \sum_{i \neq j} I(\mu_i, \mu_j).$$

Applying Lemma 16 to the outer-interaction sum and Lemma 19 to the self-interaction sum, and lastly, applying Lemma 15 to the normalization $\mu_{A_n}(V_n)$ completes the proof.

□

Remark 7 tells us that using a single level re-distribution will not render $S$ to be re-distributable no matter how many intervals are included in $\{I_k : k \in A_n\}$. We will need to take the average of $q(n)$ single-level re-distribution measures, where $q(n)$ is an integer-valued function such that $q(n) \to \infty$ as $n \to \infty$.

Let $\hat{\mu}_n$ be a single-level re-distribution as defined in (4.5). For each $n$, we consider a *multi-level re-distribution* probability measure:

$$\mu^m := \frac{1}{\#\mathcal{B}_m} \sum_{s \in \mathcal{B}_m} \hat{\mu}_{2^m s},$$

where

$$\mathcal{B}_m := \{0, \ldots, q(m) - 1\}.$$ 

Since each $\hat{\mu}_n$ is supported on

$$V_n := \bigcup_{k \in A_n} I_k = \bigcup_{k=n}^{2n-1} I_k,$$
then \( \mu^m \) is supported on \( V_n, V_{2n}, V_{2^2n}, \ldots, V_{2^{\varepsilon(n)}-1} \).

See Remark 4 for details.

Our convex measure can now be partitioned into a new self-interaction sum and a new outer-interaction sum:

\[
I(\mu^m) = \frac{1}{(\# B_m)^2} \sum_{s \in B_m} I(\hat{\mu}_{2^s m}) + \frac{1}{(\# B_m)^2} \sum_{s,t \in B_m \atop s \neq t} I(\hat{\mu}_{2^s m}, \hat{\mu}_{2^t m}).
\]

Proposition 20 tells us that

\[
I(\hat{\mu}_n) = O(1).
\]

Hence,

\[
\frac{1}{(\# B_m)^2} \sum_{s \in B_m} I(\hat{\mu}_{2^s m}) = \frac{1}{(\# B_m)^2} \sum_{s \in B_m} O(1)
\]

\[
\leq \frac{O(1)}{(\# B_m)} \to 0 \text{ as } m \to \infty.
\]

That is, the self-interaction sum vanishes. The outer-interaction sum gives what we aim:

**Lemma 21.** Assume A.1-A.3. As \( m \to \infty \) we have that

\[
I(\hat{\mu}_n) = O(1).
\]

We will leave the proof of Lemma 21 to the end of the section. Note that the vanishing of the self-interaction sum and Lemma 21 gives us:

**Proposition 22.** For every \( f \in C([0,1]) \), we have that

\[
I(\mu^m) = I_2(f(x)\phi(x) \, dx) \left( \int f(x)\phi(x) \, dx \right)^2 + o(1).
\]

Notice that with Proposition 22 we can show that \( S \) is re-distributable when \( \phi(x) \equiv 1 \). In order to remove \( \phi(x) \), we will need to apply Proposition 22 to a continuous approximation of \( 1/\phi(x) \) and take an appropriate subsequence:

**Proposition 23.** Suppose for each continuous function \( f \), there exists a sequence of probability measures \( \{\mu^n\} \) so that each \( \mu^n \) has a piecewise continuous density with support \( V_n \), where \( V_n \) is a finite union of disjoint \( I_k \)'s with \( k \geq n \) with asymptotic behavior:

\[
I(\mu^n) = I_2(f(x)\phi(x) \, dx) \left( \int f(x)\phi(x) \, dx \right)^2 + o(1).
\]

Then property P.1 holds.
Let us go back to show that $S$ is re-distributable using a multi-level re-distribution before we prove Proposition 23.

**Proof of Proposition 13.** Since $\phi(x) > 0$ almost everywhere, then combining Proposition 22 and Proposition 23 shows $P.1$ holds.

Given any probability measure $\nu$ with piecewise continuous density that is supported on a finite collection of intervals in $[0, 1]$ and given any $\varepsilon > 0$, we may apply $P.1$ to a continuous $L^1$ approximation of the density function of $\nu$ to show that there exists a probability measure $\nu'$ satisfying properties (1) and (2) in Definition 2. Thus, $S$ is re-distributable.

$\square$

Now, let us analyze the asymptotic behavior of the new outer-interaction sum:

**Proof of Lemma 21.** The goal is to show that

$$I(\hat{\mu}_n, \hat{\mu}_{n'}) = \frac{I(\mu_{A_n}, \mu_{A_{n'}})}{\mu_{A_n}(V_n) \cdot \mu_{A_{n'}}(V_{n'})} \to \frac{I(f(x)\phi(x) \, dx)}{(\int f(x)\phi(x) \, dx)^2},$$

as $m \to \infty$ and independently of our choice of $n, n' \in \{2^s m : s \in B_m\}$. Once we accomplish this, then

$$\frac{1}{(#B_m)^2} \sum_{s \neq t} I(\hat{\mu}_{2^s m}, \hat{\mu}_{2^t m}) \to \frac{I(f(x)\phi(x) \, dx)}{(\int f(x)\phi(x) \, dx)^2},$$

as $m \to \infty$.

Lemma 15 shows that

$$\frac{1}{\mu_{A_n}(V_n) \cdot \mu_{A_{n'}}(V_{n'})} \to \frac{1}{(\int f(x)\phi(x) \, dx)^2},$$

as $m \to \infty$ and independently of our choice of $n, n' \in \{2^s m : s \in B_m\}$.

Let us focus on $I(\mu_{A_n}, \mu_{A_{n'}})$. Given $n \neq n'$ where $n, n' \in \{2^s m : s \in B_m\}$, we have that

$$I(\mu_{A_n}, \mu_{A_{n'}}) = \frac{1}{(#A_n)(#A_{n'})} \sum_{i \neq j} I(\mu_i, \mu_j), \quad (4.10)$$

where $(i, j) \in A_n \times A_{n'}$. By the gap control assumption $A.3$, we know that for all large enough $m$ and every $i \neq j$, where $i, j \in \{m, \ldots, 2^{q(m)} m - 1\}$, we have

$$\frac{l_i + l_j}{2|c_i - c_j|} \leq (1 - e^{-\varepsilon}),$$

which is the needed condition (4.8) to apply Lemma 17 for all large enough $m$. Lemma 17 gives us

$$|I(\mu_i, \mu_j) - (-\log |c_i - c_j|)f(c_i)f(c_j)| \leq (2K(-\log |c_i - c_j|) + K^2)\varepsilon,$$
for every \( i \neq j \) where \( i, j \in \{m, \ldots, 2^q(m) - 1\} \) and for all large enough \( m \). Adding this up over \((i, j) \in \mathcal{A}_n \times \mathcal{A}_{n'}\) and then dividing by \((\#\mathcal{A}_n)(\#\mathcal{A}_{n'})\) gives us:

\[
\left| \frac{1}{(\#\mathcal{A}_n)(\#\mathcal{A}_{n'})} \sum_{i \neq j} I(\mu_i, \mu_j) \right| - \frac{1}{(\#\mathcal{A}_n)(\#\mathcal{A}_{n'})} \sum_{i \neq j} (-\log |c_i - c_j|) f(c_i)f(c_j) \leq \frac{2K\varepsilon}{(\#\mathcal{A}_n)(\#\mathcal{A}_{n'})} \sum_{i \neq j} (-\log |c_i - c_j|) + K^2\varepsilon.
\]

We remark that the inequality holds independently of our choice of \( n, n' \in \{2^s m : s \in \mathcal{B}_m\} \) for all large enough \( m \). By the distribution assumption \( A.1 \) and the log-average spacing assumption \( A.2 \), we can apply Lemma 14 twice to the last two sums above. One application yields:

\[
\frac{1}{(\#\mathcal{A}_n)(\#\mathcal{A}_{n'})} \sum_{i \neq j} (-\log |c_i - c_j|) f(c_i)f(c_j) \rightarrow I(f(x)\phi(x) \, dx),
\]

as \( m \to \infty \) with \( n, n' \geq m \). For the second application we take \( f = 1 \) in Lemma 14 to get

\[
\frac{1}{(\#\mathcal{A}_n)^2} \sum_{i \neq j} (-\log |c_i - c_j|) \rightarrow I(\phi(x) \, dx) < \infty,
\]

as \( m \to \infty \) with \( n, n' \geq m \). Therefore, for \( n, n' \in \{2^s m : s \in \mathcal{B}_m\} \), we have

\[
\left| \lim_{m \to \infty} I(\mu_{A_n}, \mu_{A_{n'}}) - I(f(x)\phi(x) \, dx) \right| \leq 2K\varepsilon I(\phi(x) \, dx) + K^2\varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary and \( I(\phi(x) \, dx) < \infty \), then for \( n, n' \in \{2^s m : s \in \mathcal{B}_m\} \), we have

\[
\left| \lim_{m \to \infty} I(\mu_{A_n}, \mu_{A_{n'}}) - I(f(x)\phi(x) \, dx) \right| = 0.
\]

Therefore, as \( m \to \infty \), then

\[
I(\hat{\mu}_n, \hat{\mu}_{n'}) \rightarrow \frac{I(f(x)\phi(x) \, dx)}{(\int f(x)\phi(x) \, dx)^2},
\]

where \( n, n' \in \{2^s m : s \in \mathcal{B}_m\} \), which completes the proof. \( \square \)

**Proof of Proposition 23.** For every continuous function \( h \), set

\[
f(x) := \frac{h(x)}{\phi(x)},
\]
when $\phi \neq 0$ and 0 otherwise. For each $\varepsilon > 0$, there exists a continuous function $f'$ such that

$$\left| \frac{I(f_0(x)\phi(x) \, dx)}{\int_0^1 f_0(x)\phi(x) \, dx} - \frac{I(f(x)\phi(x) \, dx)}{\int_0^1 f(x)\phi(x) \, dx} \right| < \varepsilon/2.$$  

Applying (4.9) to $f_0$, gives us that for each $\varepsilon > 0$, there exists $N$ such that

$$\left| \frac{I(f_0(x)\phi(x) \, dx)}{\int_0^1 f_0(x)\phi(x) \, dx} - I(\mu^n) \right| < \varepsilon/2,$$

where each $\mu^n$ is a probability measure with a piecewise continuous density with support in $V_n$, where $V_n$ is a finite unions of disjoint $I_k$’s with $k \geq n$.

Since $f(x)\phi(x) = h(x)$ a.e., then for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ as large as we need such that

$$\left| I(\mu^n) - \frac{I(h(x) \, dx)}{\int_0^1 h(x) \, dx} \right| \leq \left| I(\mu^n) - \frac{I(f_0(x)\phi(x) \, dx)}{\int_0^1 f_0(x)\phi(x) \, dx} \right| + \left| \frac{I(f_0(x)\phi(x))}{\int_0^1 f_0(x)\phi(x) \, dx} - \frac{I(f(x)\phi(x))}{\int_0^1 f(x)\phi(x) \, dx} \right| < \varepsilon.$$  

Hence, there exists a subsequence $n_k$ and probability measures $\mu^{n_k}$ with piecewise continuous density supported in $V_{n_k}$ such that

$$I(\mu^{n_k}) = \frac{I(h(x) \, dx)}{\int_0^1 h(x) \, dx} + o(1). \quad (4.11)$$

Each $\mu^{n_k}$ is a probability measure with a piecewise continuous density with support contained in $V_{n_k}$, that is a finite union of disjoint $I_j$’s with $j \geq k$. Hence, these $\{\mu^{n_k}\}$ satisfy $\textbf{P.1}$.

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