A note on the Manin-Mumford conjecture

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Abstract. In [PR1], R. Pink and the author gave a short proof of the Manin-Mumford conjecture, which was inspired by an earlier model-theoretic proof by Hrushovski. The proof given in [PR1] uses a difficult unpublished ramification-theoretic result of Serre. It is the purpose of this note to show how the proof given in [PR1] can be modified so as to circumvent the reference to Serre’s result. J. Oesterlé and R. Pink contributed several simplifications and shortcuts to this note.

0. Introduction.

Let $A$ be an abelian variety defined over an algebraically closed field $L$ of characteristic 0 and let $X$ be a closed subvariety. If $G$ is an abelian group, write Tor($G$) for the group of elements of $G$ which are of finite order. A closed subvariety of $A$ whose irreducible components are translates of abelian subvarieties of $A$ by torsion points will be called a torsion subvariety. The Manin-Mumford conjecture is the following statement:

The Zariski closure of Tor($A(L)$) ∩ $X$ is a torsion subvariety.

This was first proved by Raynaud in [R]. In [PR1], R. Pink and the author gave a new proof of this statement, which was inspired by an earlier model-theoretic proof given by Hrushovski in [H]. The interest of this proof is the fact that it relies almost entirely on classical algebraic geometry and is quite short. Its only non elementary input is a ramification-theoretic result of Serre. The proof of this result is not published and relies (see [Se] (pp. 33–34, 56–59)) on deep theorems of Faltings, Nori and Raynaud. In this note, we show how the reference to Serre’s result in [PR1] can be replaced by a reference to a classical result in the theory of formal groups (see Th. 4 (a)).

The structure of the paper is as follows. For the convenience of the reader, the text has been written so as to be logically independent of [PR1]. In particular, no knowledge of

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[PR1] is necessary to read it. Section 1 recalls various classical results on abelian varieties and also contains two less well-known, but elementary propositions (Prop. 1 and Prop. 3) whose proofs can be found elsewhere but for which we have included short proofs to make the text more self-contained. The reader is encouraged to proceed directly to section 2, which contains a complete proof of the Manin-Mumford conjecture and to refer to the results listed in section 1 as needed.

Notations. w.r.o.g. is a shortening of without restriction of generality; if $X$ is closed subvariety of an abelian variety $A$ defined over an algebraically closed field $L$ of characteristic 0, then we write $\text{Stab}(X)$ for the stabiliser of $X$; this is a closed subgroup of $A$ such that $\text{Stab}(X)(L) := \{a \in A(L) | a + X = X\}$; it has the same field of definition as $X$ and $A$; if $p$ is a prime number and $G$ is an abelian group, we write $\text{Tor}_p(G)$ for the set of elements of $\text{Tor}(G)$ whose order is prime to $p$ and $\text{Tor}_p(G)$ for the set of elements of $\text{Tor}(G)$ whose order is a power of $p$.

Acknowledgments. We want to thank J. Oesterlé for his interest and for suggesting some simplifications in the proofs of [PR1] (see [Oes]) which have inspired some of the proofs given here. Also, the proof of Prop. 3 in its present form is due to him (see the explanations before the proof). I am also very grateful to R. Pink, who carefully read several versions of the text and suggested many improvements and simplifications. In particular, Prop. 6 was suggested by him.

1. Preliminaries.

Lemma 0. Let $L \subseteq L'$ be algebraically closed fields of characteristic 0. Let $A$ be an abelian variety defined over $L$ and let $X$ be a closed $L$-subvariety of $A$. Then:
(a) $X$ is a torsion subvariety of $A$ iff $X_{L'}$ is a torsion subvariety of $A_{L'}$;
(b) the Manin-Mumford conjecture holds for $X$ in $A$ iff it holds for $X_{L'}$ in $A_{L'}$.

Proof: we first prove (a). To prove the equivalence of the two conditions, we only need to prove the sufficiency of the second one. The latter is a consequence of the fact that the morphism $\pi : A_{L'} \to A$ is faithfully flat and that any torsion point and any abelian
subvariety of $A_{L'}$ has a model in $A$ (see [Mi] (Cor. 20.4, p. 146)). To prove (b), let $Z := \text{Zar}(\text{Tor}(A(L)) \cap X)$ (resp. $Z' := \text{Zar}(\text{Tor}(A(L')) \cap X_{L'})$). Using again the fact that any torsion point in $A_{L'}$ has a model in $A$ and that $\pi$ is faithfully flat, we see that $\pi^{-1}(\text{Tor}(A(L)) \cap X) = \text{Tor}(A(L')) \cap X_{L'}$. From this and the fact that the morphism $\pi$ is open ([EGA] (IV, 2.4.10)), we get a set-theoretic equality $\pi^{-1}(\text{Tor}(A(L)) \cap X) = \text{Tor}(A(L')) \cap X_{L'}$. From this and the fact that the morphism $\pi$ is radicial, the underlying set of $\pi^*(Z) := Z_{L'}$ is $\pi^{-1}(Z)$ ([EGA] (I, 3.5.10)). Since $Z_{L'}$ is reduced ([EGA] (IV, 4.6.1)), we thus have an equality of closed subschemes $Z_{L'} = Z'$. Now, by (a), the closed subscheme $Z_{L'}$ is a torsion subvariety of $A_{L'}$ iff $Z$ is a torsion subvariety of $A$.

**Proposition 1 (Pink-Roessler).** Let $A$ be an abelian variety over $\mathbb{C}$ and let $F : A \to A$ be an isogeny. Suppose that the absolute value of all the eigenvalues of the pull-back map $F^*$ on the first singular cohomology group $H^1(A(\mathbb{C}), \mathbb{C})$ is larger than 1. Then any closed subvariety $Z$ of $A$ such that $F(Z) = Z$ is a torsion subvariety.

The following proof can be found in [PR1] (Remark after Lemma 2.6).

**Proof:** w.r.o.g., we may replace $F$ by one of its powers and thus suppose that each irreducible component of $Z$ is stable under $F$. We may thus suppose that $Z$ is irreducible. Notice that $F(\text{Stab}(Z)) \subseteq \text{Stab}(Z)$. Let us first suppose that $\text{Stab}(Z) = 0$.

Write $\text{cl}(Z)$ for the cycle class of $Z$ in $H^*(A(\mathbb{C}), \mathbb{C})$. We list the following facts:

1. the degree of $F$ is the determinant of the restriction of $F^*$ to $H^1(A(\mathbb{C}), \mathbb{C})$;
2. each eigenvalue of $F^*$ on $H^i(A(\mathbb{C}), \mathbb{C})$ is the product of $i$ distinct zeroes (with multiplicities) of the characteristic polynomial of $F^*$ on $H^1(A(\mathbb{C}), \mathbb{C})$; Facts (1) and (2) follow from the fact that for all $i \geq 0$ there is a natural isomorphism $\Lambda^i(H^1(A(\mathbb{C}), \mathbb{C})) \simeq H^i(A(\mathbb{C}), \mathbb{C})$ (see [Mu] (p.3, Eq. (4))).

Now notice that since $\text{Stab}(Z) = 0$, the varieties $Z + a$, where $a \in \text{Ker}(F)(\mathbb{C})$, are pairwise distinct. These varieties are thus the irreducible components of $F^{-1}(Z)$. Now we compute

$$\text{cl}(F^*(Z)) = \sum_{a \in \text{Ker}(F)} \text{cl}(Z + a) = \#\text{Ker}(F)(\mathbb{C}) \cdot \text{cl}(Z) = \text{deg}(F) \text{cl}(Z)$$
and thus \( \text{cl}(Z) \) belongs to the eigenspace of the eigenvalue \( \deg(F) \) in \( H^*(A(\mathbb{C}), \mathbb{C}) \). Facts (1), (2) and the hypothesis on the eigenvalues imply that \( \text{cl}(Z) \in H^{2 \dim(A)}(A(\mathbb{C}), \mathbb{C}) \), which in turn implies that \( Z \) is a point. This point is a torsion point since it lies in the kernel of \( F - \text{Id} \), which is an isogeny by construction.

If \( \text{Stab}(Z) \neq 0 \), then replace \( A \) by \( A/\text{Stab}(Z) \) and \( Z \) by \( Z/\text{Stab}(Z) \). The isogeny \( F \) then induces an isogeny on \( A/\text{Stab}(Z) \), which stabilises \( Z/\text{Stab}(Z) \). We deduce that \( Z/\text{Stab}(Z) \) is a torsion point. This implies that \( Z \) is a translate of \( \text{Stab}(Z) \) by a torsion point and concludes the proof.

**Corollary 2.** Let \( A \) be an abelian variety over an algebraically closed field \( K \) of characteristic 0. Let \( n \geq 1 \) and let \( M \) be an \( n \times n \)-matrix with integer coefficients. Suppose that the absolute value of all the eigenvalues of \( M \) is larger than 1. Then any closed subvariety \( Z \) of \( A^n \) such that \( M(Z) = Z \) is a torsion subvariety.

**Proof:** Because of Lemma 0 (a), we may assume w.r.o.g. that \( K \) is the algebraic closure of a field which is finitely generated as a field over \( \mathbb{Q} \). We may thus also assume that \( K \subseteq \mathbb{C} \). Prop. 1 then implies the result for \( Z_{\mathbb{C}} \) in \( A^n_{\mathbb{C}} \) and using Lemma 0 (a) again we can conclude.

**Proposition 3 (Boxall).** Let \( A \) be an abelian variety over a field \( K \) of characteristic 0. Let \( p > 2 \) be a prime number and let \( L := K(A[p]) \) be the extension of \( K \) generated by the \( p \)-torsion points of \( A \). Let \( P \in \text{Tor}_p(A(\overline{K})) \) and suppose that \( P \notin A(L) \). Then there exists \( \sigma \in \text{Gal}(\overline{L}/L) \) such that \( \sigma(P) - P \in A[p] \setminus \{0\} \).

A proof of a variant of Prop. 3 can be found in [B]. For the convenience of the reader, we reproduce a proof, which is a simplification by Oesterlé (private communication) of a proof due to Coleman and Voloch (see [Vo]).

**Proof:** let \( n \geq 1 \) be the smallest natural number so that \( p^n P \in A(L) \). For all \( i \in \{1, \ldots, n\} \), let \( P_i = p^{n-i} P \). Let also \( \sigma_1 \) be an element of \( \text{Gal}(\overline{L}/L) \) such that \( \sigma_1(p^{n-1} P) \neq p^{n-1} P \). Furthermore, let \( \sigma_i := \sigma_1^{p^{i-1}} \) and \( Q_i := \sigma_i(P_i) - P_i \).

First, notice that we have \( pQ_1 = \sigma_1(p^n P) - p^n P = 0 \) and \( Q_1 = \sigma_1(p^{n-1} P) - p^{n-1} P \neq 0 \).
0, hence \( Q_1 \in A[p]\setminus\{0\} \). We shall prove by induction on \( i \geq 1 \) that \( Q_i = Q_1 \) if \( i \leq n \). This will prove the proposition, since \( Q_n = \sigma_n(P) - P \).

So assume that \( Q_i = Q_1 \) for some \( i < n \). We have \( p^2(\sigma_i - 1)(P_{i+1}) = p(\sigma_i - 1)(P_i) = pQ_i = 0 \). Since any \( p \)-torsion point of \( A \) is fixed by \( \sigma \), and hence by \( \sigma_i \), we also have \( p(\sigma_i - 1)^2(P_{i+1}) = 0 \) and \( (\sigma_i - 1)^3(P_{i+1}) = 0 \). The binomial formula shows that, in the ring of polynomials \( \mathbb{Z}[T] \), \( T^p \) is congruent to \( 1 + p(T - 1) \) modulo the ideal generated by \( p(T - 1)^2 \) and \( (T - 1)^3 \) (notice that \( p \neq 2 \)). We thus have \( (\sigma_i^p - 1)(P_{i+1}) = p(\sigma_i - 1)(P_{i+1}) = (\sigma_i - 1)(P_i) \), id est \( Q_{i+1} = Q_i \). This completes the induction on \( i \).

Suppose now that \( K \) is a finite extension of \( \mathbb{Q}_p \), for some prime number \( p \) and let \( K^{unr} \) be its maximal unramified extension. Let \( k \) be the residue field of \( K \). Suppose that \( A \) is an abelian variety over \( K \) which has good reduction at the unique non-archimedean place of \( K \). Denote by \( A_0 \) the corresponding special fiber, which is an abelian variety over \( k \).

**Theorem 4.**

(a) The kernel of the homomorphism

\[
\text{Tor}(A(K^{unr})) \rightarrow A_0(k)
\]

induced by the reduction map is a finite \( p \)-group.

(b) The equality \( \text{Tor}^p(A(K^{unr})) = \text{Tor}^p(A(K)) \) holds.

**Proof:** for statement (b), see [Mi] (Cor. 20.8, p. 147). Statement (a), which is more difficult to prove, follows from general properties of formal groups over \( K \). See [Oes2] (Prop. 2.3 (a)) for the proof.

Let now \( \phi \in \text{Gal}(\overline{k}|k) \) be the arithmetic Frobenius map.

**Theorem 5 (Weil).** There is a monic polynomial \( Q(T) \in \mathbb{Z}[T] \) with the following properties:

(a) \( Q(\phi)(P) = 0 \) for all \( P \in A_0(\overline{k}) \);

(b) the complex roots of \( Q \) have absolute value \( \sqrt{\#k} \).

**Proof:** see [We].
2. Proof of the Manin-Mumford conjecture.

**Proposition 6.** Let $A$ be an abelian variety over a field $K_0$ that is finitely generated as a field over $\mathbb{Q}$. Then for almost all prime numbers $p$, there exists an embedding of $K_0$ into a finite extension $K$ of $\mathbb{Q}_p$, such that $A_K$ has good reduction at the unique non-archimedean place of $K$.

**Proof:** Since by assumption $K_0$ has finite transcendence degree over $\mathbb{Q}$, there is a finite map

$$\text{Spec } K_0 \to \text{Spec } \mathbb{Q}(X_1, \ldots, X_d),$$

for some $d \geq 0$ (notice that $d = 0$ is allowed). Let $V \to \mathbb{A}^d_Z$ be the normalisation of the affine space $\mathbb{A}^d_Z$ in $K_0$. The scheme $V$ is integral, normal and has $K_0$ as a field of rational functions. Furthermore, $V$ is finite and surjective onto $\mathbb{A}^d_Z$. There is an open subset $B \subseteq V$ and an abelian scheme $A \to B$, whose generic fiber is $A$. Choose $B$ sufficiently small so that its image $f(B)$ is open and so that $f^{-1}(f(B)) = B$ (this can be achieved by replacing $B$ by $f^{-1}(\mathbb{A}^d_Z \setminus f(V \setminus B))$). Let $U := f(B)$. This accounts for the square on the left of the diagram (*) below.

Now notice that $U(\mathbb{Q}) \neq \emptyset$, since $\mathbb{A}^d(\mathbb{Q})$ is dense in $\mathbb{A}^d_{\mathbb{Q}}$ and $U \cap \mathbb{A}^d_{\mathbb{Q}}$ is open and not empty. Thus, for almost all prime numbers $p$, we have $U(F_p) \neq \emptyset$. Let $p$ be a prime number with this property. Let $P \in U(F_p)$ and let $a_1, \ldots, a_d \in F_p$ be its coordinates. Choose as well elements $x_1, \ldots, x_d \in \mathbb{Q}_p$ which are algebraically independent over $\mathbb{Q}$. The elements $x_1, \ldots, x_d$ remain algebraically independent if we replace some $x_i$ by $\frac{1}{x_i}$ so we may suppose that $\{x_1, \ldots, x_d\} \subseteq \mathcal{O}_{\mathbb{Q}_p}$. Notice also that any element of the residue field $F_p$ of $\mathcal{O}_{\mathbb{Q}_p}$ is the reduction mod $p$ of an element of $\mathbb{Z} \subseteq \mathcal{O}_{\mathbb{Q}_p}$. Furthermore, the elements $x_1, \ldots, x_d$ remain algebraically independent if some $x_i$ is replaced by $x_i + m$, where $m$ is an integer. Hence, we may also suppose that $x_i \mod p = a_i$ for all $i \in \{1, \ldots, d\}$. The choice of the $x_i$ induces a morphism $e : \text{Spec } \mathcal{O}_{\mathbb{Q}_p} \to \mathbb{A}^d_Z$, which by construction sends the generic point of $\text{Spec } \mathcal{O}_{\mathbb{Q}_p}$ on the generic point of $\mathbb{A}^d_Z$ and hence of $U$ and sends the special point of $\text{Spec } \mathcal{O}_{\mathbb{Q}_p}$ on $P \in U(F_p)$. Hence $e^{-1}(U) = \text{Spec } \mathcal{O}_{\mathbb{Q}_p}$. This accounts for
the lowest square in (*).

The middle square in (*) is obtained by taking the fibre product of $U \rightarrow B$ and $\text{Spec } \mathcal{O}_{\mathbb{Q}_p} \rightarrow B$. The morphism $B_1 \rightarrow \text{Spec } \mathcal{O}_{\mathbb{Q}_p}$ is then also finite and surjective.

To define the arrows in the triangle next to it, consider a reduced irreducible component $B'_1$ of $B_1$ which dominates $\text{Spec } \mathcal{O}_{\mathbb{Q}_p}$. This exists, because the morphism $B_1 \rightarrow \text{Spec } \mathcal{O}_{\mathbb{Q}_p}$ is dominant. The morphism $B'_1 \rightarrow \text{Spec } \mathcal{O}_{\mathbb{Q}_p}$ will then also be finite and will thus correspond to a finite (and hence integral) extension of integral rings. Let $K$ be the function field of $B'_1$, which is a finite extension of $\mathbb{Q}_p$; the ring associated to $B'_1$ is by construction included in the integral closure $\mathcal{O}_K$ of $\mathcal{O}_{\mathbb{Q}_p}$ in $K$ and the arrow $\text{Spec } \mathcal{O}_K \rightarrow B_1$ is defined by composing the morphism induced by this inclusion with the closed immersion $B'_1 \rightarrow B_1$.

The morphism $\text{Spec } K \rightarrow \text{Spec } \mathbb{Q}_p$ has been implicitly defined in the last paragraph and the morphisms $\text{Spec } \mathbb{Q}_p \rightarrow \text{Spec } \mathcal{O}_{\mathbb{Q}_p}$ and $\text{Spec } K \rightarrow \text{Spec } \mathcal{O}_K$ are the obvious ones.

We have a commutative diagram (*):

$$
\begin{array}{cccccc}
\text{Spec } K_0 & 
\text{Spec } \mathbb{Q}_p & 
\text{Spec } \mathbb{Q}(X_1, \ldots, X_d) & 
\text{Spec } \mathcal{O}_{\mathbb{Q}_p} & 
\text{Spec } \mathcal{O}_K & 
\text{Spec } K \\
\downarrow \text{Cart.} & 
\downarrow \text{Cart.} & 
\downarrow \text{Cart.} & 
\downarrow \text{Cart.} & 
\downarrow \text{Cart.} & 
\downarrow \\
B & 
U & 
\mathbb{A}^d & 
\text{Spec } \mathcal{O}_{\mathbb{Q}_p} & 
B_1 & 
\text{Spec } \mathcal{O}_K
\end{array}
$$

The single-barreled continuous arrows ($\rightarrow$) represent dominant maps; the double-barreled continuous ones ($\Rightarrow$) represent finite and dominant maps; all the schemes in the diagram apart from $B_1$ are integral; the cartesian squares carry the label "Cart."

Now notice that the map $\text{Spec } K \rightarrow B$ obtained by composing the connecting morphisms sends $\text{Spec } K$ on the generic point of $B$; to see this notice that the maps $\text{Spec } K \rightarrow \text{Spec } \mathcal{O}_K$, $\text{Spec } \mathcal{O}_K \Rightarrow \text{Spec } \mathcal{O}_{\mathbb{Q}_p}$ and $\text{Spec } \mathcal{O}_{\mathbb{Q}_p} \rightarrow U$ are all dominant; hence $\text{Spec } K$ is sent on the generic point of $U$; since $B \rightarrow U$ is a finite map, this implies that $\text{Spec } K$ is sent on the generic point of $B$.

Thus the map $\text{Spec } K \rightarrow B$ induces a field extension $K|K_0$. Furthermore, as we
have seen, $K$ is a finite extension of $\mathbb{Q}_p$ and by construction, the abelian variety $A_K$ is the generic fiber of the abelian scheme $A \times_B \text{Spec } \mathcal{O}_K$. In other words $A_K$ is an abelian variety defined over $K$ which has good reduction at the unique non-archimedean place of $K$. 

Next, we shall consider the following situation. Let $p > 2$ be a prime number and let $K$ be a finite extension of $\mathbb{Q}_p$. Let $k$ be its residue field. Let $A$ be an abelian variety over $K$. Suppose that $A$ has good reduction at the unique non-archimedean place of $K$. Let $A_0$ be the corresponding special fiber, which is an abelian variety over $k$.

Recall that $K^{\text{unr}}$ refers to the maximal unramified extension of $K$. Let $\phi \in \text{Gal}(k/k)$ be the arithmetic Frobenius map and let $\tau \in \text{Gal}(K^{\text{unr}}/K)$ be its canonical lift.

**Proposition 7.** Let $X$ be a closed $K$-subvariety of $A$. Then the Zariski closure of $X_K \cap \text{Tor}(A(K^{\text{unr}}))$ is a torsion subvariety.

**Proof:** w.r.o.g. we may suppose that $\text{Tor}(A(K^{\text{unr}}))$ is dense in $X_K$ (otherwise, replace $X$ by the natural model of $\text{Zar}(X_K \cap \text{Tor}(A(K^{\text{unr}})))$ over $K$). By Th. 4 (a), the kernel of the reduction homomorphism $\text{Tor}(A(K^{\text{unr}})) \to A_0(k)$ is a finite $p$-group. Let $p^r$ be its cardinality and let $Y := p^r \cdot X$. Let $Q(T) := T^n - (a_nT^{n-1} + \ldots + a_0) \in \mathbb{Z}[T]$ be the polynomial provided by Th. 5 (i.e. the characteristic polynomial of $\phi$ on $A_0(k)$). Let $F$ be the matrix

$$
\begin{pmatrix}
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 1 \\
a_0 & a_1 & \ldots & a_{n-2} & a_{n-1}
\end{pmatrix}
$$

For any $a \in A(K^{\text{unr}})$, write $u(x) := (x, \tau(x), \tau^2(x), \ldots, \tau^{n-1}(x)) \in A^n(K^{\text{unr}})$. Let $\bar{Y} := \text{Zar}(\{u(a)|a \in (p^r \cdot \text{Tor}(A(K^{\text{unr}}))) \cap Y_K\})$. Th. 5 (a) and Th. 4 (a) imply that

$$F(u(a)) = u(\tau(a))$$

for all $a \in p^r \cdot \text{Tor}(A(K^{\text{unr}}))$. Furthermore, by construction,

$$\tau(p^r \cdot \text{Tor}(A(K^{\text{unr}}))) \subseteq p^r \cdot \text{Tor}(A(K^{\text{unr}})).$$
Hence $F(\tilde{Y}) = \tilde{Y}$. Now Th. 5 (b) implies that the absolute value of the eigenvalues of the matrix $F$ are larger than 1 and Cor. 2 then implies that $\tilde{Y}$ is a torsion subvariety of $A_K$. The variety $Y_K$ is the projection of $\tilde{Y}$ on the first factor and is thus also a torsion subvariety. Finally, this implies that $X_K$ is a torsion subvariety.

**Proposition 8.** Let $X$ be a closed $K$-subvariety of $A$. Then the Zariski closure of $X_K \cap \text{Tor}(A(K))$ is a torsion subvariety.

**Proof:** we may suppose w.r.o.g. that $K = K(A[p])$, that $X$ is geometrically irreducible and that $X_K \cap \text{Tor}(A(K))$ is dense in $X_K$. We shall first suppose that $\text{Stab}(X) = 0$. Let $x \in X_K \cap \text{Tor}(A(K))$ and suppose that $x \not\in A(K_{\text{unr}})$. Write $x = x^p + x_p$, where $x^p \in \text{Tor}^p(A(K))$ and $x_p \in \text{Tor}_p(A(K))$. By Th. 4 (b) $x^p \in A(K_{\text{unr}})$ and thus $x_p \not\in A(K_{\text{unr}})$. By Prop. 3, there exists $\sigma \in \text{Gal}(\overline{K}|K_{\text{unr}})$ such that

$$\sigma(x_p) - x_p = \sigma(x) - x \in A[p] \setminus \{0\}.$$ 

Now notice that for all $y \in X(\overline{K})$ and all $\tau \in \text{Gal}(\overline{K}|K_{\text{unr}})$, we have $\tau(y) \in X(\overline{K})$. Hence if the set $\{x \in X_K \cap \text{Tor}(A(K)) | x \not\in A(K_{\text{unr}})\}$ is dense in $X_K$ then $\text{Stab}(X)(\overline{K})$ contains an element of $A[p] \setminus \{0\}$. Since $\text{Stab}(X) = 0$, we deduce that the set $\{x \in X_K \cap \text{Tor}(A(K)) | x \not\in A(K_{\text{unr}})\}$ is not dense in $X_K$ and thus the set $X_K \cap \text{Tor}(A(K_{\text{unr}}))$ is dense in $X_K$. Prop. 7 then implies that $X_K$ is a torsion point. If $\text{Stab}(X) \neq 0$, then we may apply the same reasoning to $X/\text{Stab}(X)$ and $A/\text{Stab}(A)$ to conclude that $X_K$ is a translate of $\text{Stab}(X)(\overline{K})$ by a torsion point.

We shall now prove the Manin-Mumford conjecture. Let the terminology of the introduction hold. By Lemma 0 (b), we may assume w.r.o.g. that $L$ is the algebraic closure of a field $K_0$ that is finitely generated as a field over $Q$ and that $A$ (resp. $X$) has a model $A$ (resp. $X$) over $K_0$. By Prop. 6, there is an embedding of $K_0$ into a field $K$, with the following properties: $K$ is a finite extension of $Q_p$, where $p$ is a prime number larger than 2 and $A_K$ has good reduction at the unique non-archimedean place of $K$. Prop. 8 now implies that the Manin-Mumford conjecture holds for $X_K$ in $A_K$ and using Lemma 0 (b) we deduce that it holds for $X$ in $A$. 

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Remark. Let the notation of the introduction hold. Prop. 3. alone implies the statement of the Manin-Mumford conjecture, with Tor(A(L)) replaced by Tor_p(A(L)), for any prime number p > 2. To see this, we may w.r.o.g. assume that X is irreducible and that Tor_p(A(L)) ∩ X is dense in X. By an easy variant of Lemma 0 (b), we may w.r.o.g. assume that L is the algebraic closure of a field K that is finitely generated as a field over Q and that A (resp. X) has a model A (resp. X) over K. Finally, we may assume w.r.o.g. that K = K(A[p]). Suppose first that Stab(X) = 0. By the same argument as above, the set \{a ∈ Tor_p(A(L))|a /∈ A(K), a ∈ X\} is not dense in X. Hence the set \{a ∈ Tor_p(A(L))|a ∈ A(K), a ∈ X\} must be dense in X; the theorem of Mordell-Weil (for instance) implies that this set is finite and thus X consists of a single torsion point. If Stab(X) ≠ 0, then we deduce by the same reasoning that X/Stab(X) is a torsion point in A/Stab(X) and hence X is a translate of Stab(X) by a torsion point. This proof of a special case of the Manin-Mumford conjecture is outlined in [B] (Remarque 3, p. 75).

References.

[B] Boxall, J. Sous-variétés algébriques de variétés semi-abéliennes sur un corps fini in Number Theory, Paris 1992-3, S. David, ed., London Math. Soc. lecture note series 215, 69–89, Cambridge Univ. Press, 1995.

[EGA] Grothendieck, A. Éléments de géométrie algébrique. Inst. Hautes Études Sci. Publ. Math. 4, 8, 11, 17, 20, 24, 28, 32 (1960-1967).

[H] Hrushovski, E. The Manin-Mumford conjecture and the model theory of difference fields. Ann. Pure Appl. Logic 112 (2001), no. 1, 43–115.

[Mi] Milne, J. S. Abelian varieties. Arithmetic geometry (Storrs, Conn., 1984), 103–150, Springer, New York, 1986.

[Mu] Mumford, D. Abelian varieties. Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Oxford University Press, London, 1970.

[Oes] Oesterlé, J. Lettre à l’auteur (20/12/2002).
[Oes2] Oesterlé, J. Courbes sur une variété abélienne (d’après M. Raynaud). Séminaire Bourbaki, Vol. 1983/84. Astérisque No. 121-122 (1985), 213–224.

[PR1] Pink, R., Roessler, D. On Hrushovski’s proof of the Manin-Mumford conjecture. Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), 539–546, Higher Ed. Press, Beijing, 2002.

[R] Raynaud, M. Sous-variétés d’une variété abélienne et points de torsion. Arithmetic and geometry, Vol. I, 327–352, Progr. Math. 35, Birkhäuser Boston, Boston, MA, 1983.

[Se] Serre, J.-P. Oeuvres, vol. IV (1985-1998). Springer 2000.

[Vo] Voloch, J.-F. Integrality of torsion points on abelian varieties over $p$-adic fields. Math. Res. Lett. 3 (1996), no. 6, 787–791.

[We] Weil, A. Variétés abéliennes et courbes algébriques. Hermann 1948.