CERTAIN CURVATURE CONDITIONS ON KENMOTSU MANIFOLDS ADMITTING A QUARTER-SYMMETRIC METRIC CONNECTION

Peibiao Zhao, Uday Chand De, Krishanu Mandal, and Yanling Han

Abstract. We study certain curvature properties of Kenmotsu manifolds with respect to the quarter-symmetric metric connection. First we consider Ricci semisymmetric Kenmotsu manifolds with respect to a quarter-symmetric metric connection. Next, we study $\xi$-conformally flat and $\xi$-concircularly flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection. Moreover, we study Kenmotsu manifolds satisfying the condition $\hat{Z}(\xi, Y) \cdot \hat{S} = 0$, where $\hat{Z}$ and $\hat{S}$ are the concircular curvature tensor and Ricci tensor respectively with respect to the quarter-symmetric metric connection. Then, we prove the non-existence of $\xi$-projectively flat and pseudo-Ricci symmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection. Finally, we construct an example of a 5-dimensional Kenmotsu manifold admitting a quarter-symmetric metric connection for illustration.

1. Introduction

A linear connection $\hat{\nabla}$ in a Riemannian manifold $M$ is said to be a quarter symmetric connection \cite{7} if the torsion tensor $T$ of the connection $\hat{\nabla}$

\begin{equation}
T(X, Y) = \hat{\nabla}_X Y - \hat{\nabla}_Y X - [X, Y]
\end{equation}

satisfies

\begin{equation}
T(X, Y) = \eta(Y)\phi_X - \eta(X)\phi_Y,
\end{equation}

where $\eta$ is a 1-form and $\phi$ is a $(1, 1)$ tensor field. A linear connection $\hat{\nabla}$ is called a metric connection of $M$ if

\begin{equation}
(\hat{\nabla}_X g)(Y, Z) = 0,
\end{equation}

2010 Mathematics Subject Classification: Primary 53C05; Secondary 53D15.

Key words and phrases: quarter-symmetric metric connection, Kenmotsu manifold, Ricci semisymmetric manifold, $\xi$-concircularly flat manifold, $\xi$-conformally flat manifold, $\xi$-projectively flat manifold, Pseudo Ricci-symmetric manifold.

Communicated by Andrey Mironov.
where $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on $M$. A linear connection $\tilde{\nabla}$ satisfying (1.2) and (1.3) is called a quarter-symmetric metric connection. If we change $\phi X$ by $X$, then the connection is called a semi-symmetric metric connection. Thus the notion of quarter-symmetric connection generalizes the notion of the semi-symmetric connection. Semi-symmetric metric connections have been studied by several authors such as Özgür and Sular [18, 19], Ozen et al [20, 21], Prvanović [23], Smaranda and Andonie [26], Singh and Pandey [27] and many others.

A transformation in an $n$-dimensional Riemannian manifold $M$, which transforms every geodesic circle of $M$ into a geodesic circle of $M$, is said to be a concircular transformation. A concircular transformation is always a conformal transformation. Here, we mean a geodesic circle by a curve in $M$ whose first curvature is constant and second curvature is identically zero. Thus, the geometry of concircular transformations is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism. An important invariant of a concircular transformation is the concircular curvature tensor $Z$, defined by

$Z(X, Y)W = R(X, Y)W - \frac{r}{n(n-1)}[g(Y, W)X - g(X, W)Y]$, (1.4)

for all $X, Y, W \in \chi(M)$, where $R$ is the Riemannian curvature tensor and $r$ is the scalar curvature with respect to the Levi-Civita connection. The importance of concircular transformation and concircular curvature tensor is very well known in the differential geometry of certain $F$-structure such as complex, almost complex, Kähler, almost Kähler, contact and almost contact structure etc., [3, 34, 33].

Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus, the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

The Weyl conformal curvature tensor is defined by

$C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y]$, (1.5)

where $S$ is the Ricci tensor of type $(0, 2)$ and $Q$ is the Ricci operator defined by $S(X, Y) = g(QX, Y)$.

Let $M$ be an $n$-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighborhood of $M$ and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then $M$ is said to be locally projectively flat. For $n \geq 3$, $M$ is locally projectively flat if and only if the well-known projective curvature tensor $P$ vanishes. Here $P$ is defined by

$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y]$, (1.6)
for all $X, Y, Z \in \chi(M)$. In fact, $M$ is projectively flat if and only if it is of constant curvature. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

A non-flat $n$-dimensional Riemannian manifold $(M, g)$, $n > 3$ is called pseudo Ricci-symmetric $[5]$ with respect to the quarter-symmetric metric connection if there exists a non-zero 1-form $\alpha$ on $M$ such that

$$\langle \tilde{\nabla}_X \tilde{S}(Y, U) \rangle = 2\alpha(X) \tilde{S}(Y, U) + \alpha(Y) \tilde{S}(X, U) + \alpha(U) \tilde{S}(Y, X),$$

where $X, Y, U \in \chi(M)$.

Quarter-symmetric metric connection in a Riemannian manifold have been studied by several authors such as Mandal and De [15], Rastogi [24, 25], Yano and Imai [32], Mukhopadhyay, Roy and Barua [16], Han et al [8, 9], Biswas and De [4] and many others. Recently, Sular, Özgür and De [29] studied quarter-symmetric metric connection in a Kenmotsu manifold.

Motivated by the above studies in the present paper, we study quarter-symmetric metric connection in a Kenmotsu manifold. The paper is organized as follows: In section 2, we give a brief account of Kenmotsu manifolds. In section 3, we give the curvature tensor and the Ricci tensor of a Kenmotsu manifold with respect to the quarter-symmetric metric connection. Next, in section 4, we consider Ricci semisymmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection and prove that a Ricci semisymmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection is an Einstein manifold with respect to the Levi-Civita connection. Section 5 is devoted to study $\xi$-conformally flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection and prove that a $\xi$-conformally flat Kenmotsu manifold with respect to the quarter-symmetric metric connection is an $\eta$-Einstein manifold with respect to the Levi-Civita connection. Section 6 deals with $\xi$-concircularly flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection and prove that if a Kenmotsu manifold is $\xi$-concircularly flat then the scalar curvature $r = -n(n - 1)$. Next, we study Kenmotsu manifolds satisfying the condition $Z(\xi, Y) \cdot \tilde{S} = 0$, where $Z$ and $\tilde{S}$ are the concircular curvature tensor and Ricci tensor respectively with respect to the quarter-symmetric metric connection and prove that in this case the manifold is $\eta$-Einstein with respect to the Levi-Civita connection. In section 7, we consider $\xi$-projectively flat and pseudo Ricci-symmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection. We obtain the non-existence of these type manifolds. Finally, we construct an example of a 5-dimensional Kenmotsu manifold admitting a quarter-symmetric metric connection to verify some theorems.

2. Kenmotsu Manifolds

Let $M$ be an $n=(2m+1)$-dimensional almost contact metric manifold with an almost contact metric structure $(\phi, \xi, \eta, g)$, where $\phi$ is a tensor field, $\xi$ is the associated vector field, $\eta$ is a 1-form and $g$ is the Riemannian metric satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0,$$
with respect to the quarter-symmetric metric connection and Levi-Civita connection respectively of a Kenmotsu manifold. Then the relation between $\tilde{\nabla}$ valued functions on a Kenmotsu manifold. $M$ (or, a nonzero constant; (2) that a Kenmotsu manifold of constant $\phi$ is a space of constant curvature $g$ for all vector fields $X, Y$. In addition to the above results in a Kenmotsu manifold the following conditions hold 

\begin{align}
(2.1) & \quad \nabla_X \xi = X - \eta(X)\xi, \\
(2.2) & \quad \nabla_X \eta = g(X, Y) - \eta(X)\eta(Y), \\
(2.3) & \quad \nabla_X \phi Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \quad \nabla_X \phi Y = -g(X, \phi Y), \\
& \quad g(X, \xi) = \eta(X), \\
(2.4) & \quad (\nabla_X \eta) Y = g(X, Y) - \eta(X)\eta(Y).
\end{align}

In addition to the above results in a Kenmotsu manifold the following conditions hold [12, 29, 11]

\begin{align}
(2.5) & \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \\
& \quad R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \\
& \quad S(X, \xi) = -(n-1)\eta(X), \\
& \quad Q\eta = -(n-1)\xi, \\
& \quad R(\xi, X)\xi = X - \eta(X)\xi, \\
(2.6) & \quad S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \\
(2.7) & \quad S(\phi X, Y) = -S(X, \phi Y),
\end{align}

where $R$ is the curvature tensor, $S$ is the Ricci tensor and $Q$ is the Ricci operator. A Kenmotsu manifold is normal, that is, the Nijenhuis tensor of $\phi$ equals $-2d\eta \otimes \xi$ but not Sasakian. Moreover, Kenmotsu manifold is not compact since from the equation (2.3) we have $\text{div} \xi = n-1$. In [12], Kenmotsu showed (1) that locally a Kenmotsu manifold is a warped product $I \times_f N$, where $I$ is an interval, $N$ is a Kähler manifold and $f$ is a warping function defined by $f(t) = se^t$, $s$ is a nonzero constant; (2) that a Kenmotsu manifold of constant $\phi$-sectional curvature is a space of constant curvature $-1$, hence it is hyperbolic space. Let $M$ be a Kenmotsu manifold. $M$ is said to be an $\eta$-Einstein manifold if there exists the real valued functions $a$ and $b$ such that $S(X, Y) = a g(X, Y) + b \eta(X)\eta(Y)$. For $b = 0$, the manifold $M$ is an Einstein manifold. Now we state the following:

**Lemma 2.1.** [12] Let $M$ be an $\eta$-Einstein Kenmotsu manifold of the form $S(X, Y) = a g(X, Y) + b \eta(X)\eta(Y)$. Then $a + b = -(n-1)$. If $b = \text{constant}$ (or, $a = \text{constant}$), then $M$ is an Einstein one.

Kenmotsu manifolds have been studied by several authors such as Pitis [22], De and Pathak [6], Jun, De and Pathak [11], Özgür and De [17], Kirichenko [18], Hong et al [10] and many others.

3. Curvature Tensor

The quarter-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection $\nabla$ on a Kenmotsu manifold are related by [29]

\begin{align}
(3.1) & \quad \tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y,
\end{align}

for all vector fields $X, Y$ on $M$. Let $\tilde{R}$ and $R$ be the Riemannian curvature tensor with respect to the quarter-symmetric metric connection and Levi-Civita connection respectively of a Kenmotsu manifold. Then the relation between $\tilde{R}$ and $R$ is
given by \cite{29}
\[ \tilde{R}(X, Y)Z = R(X, Y)Z + \eta(X)g(\phi Y, Z)\xi - \eta(Y)g(\phi X, Z)\xi - \eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X. \]

Also in a Kenmotsu manifold with respect to the quarter-symmetric metric connection the following relations hold \cite{29}
\begin{align*}
(3.2) & \quad \tilde{R}(X, Y)\xi = \eta(X)Y - \eta(Y)X - \eta(\phi X, Y)\xi - \eta(Y)\phi X, \\
(3.3) & \quad \tilde{R}(X, \xi)Y = g(X, Y)\xi - \eta(Y)X - g(\phi X, Y)\xi - \eta(Y)\phi X, \\
(3.4) & \quad \tilde{S}(X, Y) = S(X, Y) + g(\phi X, Y), \\
(3.5) & \quad \tilde{S}(X, \xi) = S(X, \xi) = -(n-1)\eta(X), \\
(3.6) & \quad \tilde{r} = r,
\end{align*}
where \(\tilde{S}\) and \(\tilde{r}\) are the Ricci tensor and the scalar curvature respectively with respect to the quarter-symmetric metric connection. Moreover, it is noted that the Ricci tensor \(\tilde{S}\) with respect to the quarter-symmetric metric connection is not symmetric. Using expressions (3.2) and (3.3), the following are easily obtained from (1.4)
\begin{align*}
(3.8) & \quad \tilde{Z}(\xi, Y)U = \left[1 + \frac{\tilde{r}}{n(n-1)}\right] \left[\eta(U)Y - g(Y, U)\xi\right] + g(\phi Y, U)\xi - \eta(U)\phi Y, \\
(3.9) & \quad \tilde{Z}(\xi, Y)\xi = \left[1 + \frac{\tilde{r}}{n(n-1)}\right] \left[Y - \eta(Y)\xi\right] - \phi Y.
\end{align*}

4. Ricci Semisymmetric Kenmotsu Manifolds with Respect to the Quarter-symmetric Metric Connection

In this section we consider Ricci semisymmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection \(\tilde{\nabla}\).

**Definition 4.1.** A manifold \(M\) is said to be Ricci semisymmetric manifold if it satisfies \(R(X, Y) \cdot S = 0\), where \(R\) is (1, 3)-type curvature tensor and \(S\) is the (0, 2)-type Ricci tensor.

Let us consider an \(n\)-dimensional Kenmotsu manifold which is Ricci semisymmetric with respect to the quarter-symmetric metric connection, that is,
\[ \tilde{R}(X, Y) \cdot \tilde{S} = 0, \]
where \(\tilde{R}(X, Y)\) denotes the derivation of the tensor algebra at each point of the manifold. This implies
\begin{align*}
(4.1) & \quad \tilde{S}(\tilde{R}(X, Y)U, V) + \tilde{S}(U, \tilde{R}(X, Y)V) = 0.
\end{align*}
Substituting \(X = \xi\) in (4.1) and using (3.5), we get
\[ S(\tilde{R}(\xi, Y)U, V) + S(U, \tilde{R}(\xi, Y)V) = g(\tilde{R}(\xi, Y)U, \phi V) + g(\phi U, \tilde{R}(\xi, Y)V) = 0. \]
Putting \(U = \xi\) in the above equation and in view of (3.3) and (3.4) yields
\begin{align*}
(4.2) & \quad S(Y, V) - S(\phi Y, V) + (n-2)g(\phi Y, V) + ng(Y, V) - \eta(Y)\eta(V) = 0.
\end{align*}
Interchanging $Y$ and $V$ in the above equation we have
\begin{equation}
S(V, Y) - S(\phi V, Y) + (n - 2)g(\phi Y, V) + ng(V, Y) - \eta(V)\eta(Y) = 0.
\end{equation}
Adding (4.2) and (4.3) and using the facts (2.3) and (2.7) we obtain
\begin{equation}
S(Y, V) = -ng(Y, V) + \eta(Y)\eta(V).
\end{equation}
Hence by Lemma 2.1, (4.4) implies that the manifold is an $\eta$-Einstein manifold.

Now, in this position we can state the following:

**Theorem 4.1.** Let $M$ be a Ricci semisymmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$. Then the manifold is an $\eta$-Einstein manifold with respect to the Levi-Civita connection.

Now, using Lemma 2.1 and Theorem 4.1 we can state the following:

**Theorem 4.2.** Let $M$ be a Ricci semisymmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$. Then the manifold is an Einstein manifold with respect to the Levi-Civita connection.

Since Ricci symmetric manifold ($\tilde{\nabla}\tilde{S} = 0$) with respect to the quarter-symmetric metric connection implies $\tilde{R} \cdot \tilde{S} = 0$, therefore we obtain the following:

**Corollary 4.1.** If a Kenmotsu manifold is Ricci symmetric with respect to the quarter-symmetric metric connection, then the manifold is an Einstein manifold with respect to the Levi-Civita connection.

5. $\xi$-conformally Flat Kenmotsu Manifolds with Respect to the Quarter-symmetric Metric Connection

$\xi$-conformally flat $K$-contact manifolds have been studied by Zhen et al [35]. Since at each point $p \in M^n$ the tangent space $T_p(M^n)$ can be decomposed into the direct sum $T_p(M^n) = \phi(T_p(M^n)) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the one-dimensional linear subspace of $T_p(M^n)$ generated by $\xi_p$, the conformal curvature tensor $C$ is a map
\begin{equation}
C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \phi(T_p(M^n)) \oplus \{\xi_p\}.
\end{equation}

**Definition 5.1.** [35] An almost contact metric manifold $M^n$ is called $\xi$-conformally flat if the projection of the image of $C$ onto $\{\xi_p\}$ is zero, that is, $C(X, Y)\xi = 0$, where $C$ is the conformal curvature tensor defined in (1.5).

This section deals with $\xi$-conformally flat Kenmotsu manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$, that is, $\tilde{C}(X, Y)\xi = 0$. From this and (1.5) we have
\begin{equation}
\tilde{R}(X, Y)\xi - \frac{1}{n-2}[\tilde{S}(Y, \xi)X - \tilde{S}(X, \xi)Y + \eta(Y)\tilde{Q}X - \eta(X)\tilde{Q}Y]
+ \frac{\tilde{r}}{(n-1)(n-2)}[\eta(Y)X - \eta(X)Y] = 0.
\end{equation}

With the help of (3.2), (3.5) and (5.0) we have from (5.1)
\begin{equation}
\eta(Y)QX - \eta(X)QY = \left[1 + \frac{\tilde{r}}{n-1}\right][\eta(Y)X - \eta(X)Y]
\end{equation}
Substituting $Y = \xi$ in the above equation and using (5.7), we have

$$(5.2) \quad QX = \left(1 + \frac{r}{n-1}\right)X - \left(n + \frac{r}{n-1}\right)\eta(Y)\xi + (n-3)\phi X.$$ 

Taking the inner product of (5.2) with $Y$, we have

$$S(X,Y) = \left(1 + \frac{r}{n-1}\right)g(X,Y) - \left(n + \frac{r}{n-1}\right)\eta(X)\eta(Y) + (n-3)g(\phi X,Y).$$

Adding (5.3) and (5.4) and using fact (2.3) yields

$$S(X,Y) = \left(1 + \frac{r}{n-1}\right)g(X,Y) - \left(n + \frac{r}{n-1}\right)\eta(X)\eta(Y).$$

Therefore by Lemma 2.1, (5.5) shows that the manifold under consideration is an $\eta$-Einstein manifold. This leads to the following:

**Theorem 5.1.** A $\xi$-conformally flat Kenmotsu manifold with respect to the quarter-symmetric metric connection is an $\eta$-Einstein manifold.

Since the conformally flatness implies $\xi$-conformally flat, therefore from the above theorem we state the following:

**Corollary 5.1.** A conformally flat Kenmotsu manifold with respect to the quarter-symmetric metric connection is an $\eta$-Einstein manifold.

### 6. $\xi$-concircularly flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Analogously to the definition of $\xi$-conformally flat almost contact metric manifold, we define $\xi$-concircularly flat Kenmotsu manifolds.

**Definition 6.1.** A Kenmotsu manifold $M$ is said to be $\xi$-concircularly flat with respect to the quarter-symmetric metric connection if it satisfies

$$\tilde{Z}(X,Y)\xi = 0,$$

for any vector fields $X, Y \in \chi(M)$ and $\tilde{Z}$ is the concircular curvature tensor defined by (1.4) with respect to the quarter-symmetric metric connection.

In this section we study $\xi$-concircularly flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection $\tilde{\nabla}$. Then from (1.4) and (6.1), we have

$$\tilde{R}(X,Y)\xi - \frac{\tilde{r}}{n(n-1)}[\eta(Y) X - \eta(X) Y] = 0.$$ 

Using (3.2) and (3.7) we obtain from (6.3)

$$\left[1 + \frac{r}{n(n-1)}\right][\eta(Y) Y - \eta(Y) X] - \eta(X)\phi Y + \eta(Y)\phi X = 0.$$
Taking the inner product of (6.3) with $U$ we obtain
\[
\begin{aligned}
\left[1 + \frac{r}{n(n - 1)}\right] & \left[\eta(X)g(Y, U) - \eta(Y)g(X, U)\right] \\
& - \eta(Y)g(\phi Y, U) + \eta(\phi Y, U) = 0.
\end{aligned}
\]
Now putting $Y = U = e_i$ in (6.4), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over $i = 1, 2, \ldots, n$ we get $\left[1 + \frac{r}{n(n - 1)}\right](n - 1)\eta(X) = 0$. Hence for $n \geq 3$, the scalar curvature $r = -n(n - 1)$. Therefore we conclude the following:

**Theorem 6.1.** If an $n$-dimensional ($n \geq 3$) Kenmotsu manifold is $\xi$-concircularly flat with respect to the quarter-symmetric metric connection, then the scalar curvature $r = -n(n - 1)$.

### 7. Kenmotsu Manifolds Satisfying the Condition $\tilde{Z}(\xi, Y) \cdot \tilde{S} = 0$

This section is devoted to study Kenmotsu manifold satisfying the condition
\[
\tilde{Z}(\xi, Y) \cdot \tilde{S} = 0,
\]
where $\tilde{Z}$ and $\tilde{S}$ are the concircular curvature tensor and Ricci tensor respectively with respect to the quarter-symmetric metric connection. Equation (7.1) implies
\[
\tilde{S}(\tilde{Z}(\xi, Y)U, V) + \tilde{S}(U, \tilde{Z}(\xi, Y)V) = 0.
\]
In view of (3.5) and (7.2) we have
\[
\begin{aligned}
S(\tilde{Z}(\xi, Y)U, V) - g(\tilde{Z}(\xi, Y)U, \phi V) \\
+ S(U, \tilde{Z}(\xi, Y)V) + g(\phi U, \tilde{Z}(\xi, Y)V) &= 0.
\end{aligned}
\]
Putting $U = \xi$ in (7.3), we get
\[
\begin{aligned}
S(\tilde{Z}(\xi, Y)\xi, V) - g(\tilde{Z}(\xi, Y)\xi, \phi V) + S(\xi, \tilde{Z}(\xi, Y)V) &= 0.
\end{aligned}
\]
With the help of (3.7), (3.8) and (3.9) we have from (7.4)
\[
\begin{aligned}
\left[1 + \frac{r}{n(n - 1)}\right][S(Y, V) + (n - 1)\eta(Y)\eta(V)] \\
- S(\phi Y, V) - \left\{1 + \frac{r}{n(n - 1)}\right\}g(Y, \phi V) - g(\phi Y, \phi V)
\end{aligned}
\]
\[
\begin{aligned}
- (n - 1)\left\{1 + \frac{r}{n(n - 1)}\right\}[\eta(Y)\eta(Y) - g(Y, Y)] + g(\phi Y, V) = 0.
\end{aligned}
\]
Interchanging $Y$ and $V$ in (7.5) yields
\[
\begin{aligned}
\left[1 + \frac{r}{n(n - 1)}\right][S(Y, V) + (n - 1)\eta(V)\eta(Y)] \\
- S(\phi Y, V) - \left\{1 + \frac{r}{n(n - 1)}\right\}g(V, \phi Y) - g(\phi V, \phi Y)
\end{aligned}
\]
\[
\begin{aligned}
- (n - 1)\left\{1 + \frac{r}{n(n - 1)}\right\}[\eta(Y)\eta(V) - g(Y, Y)] + g(\phi V, Y) = 0.
\end{aligned}
\]
Subtracting (7.5) from (7.6), we get
\[
S(\phi Y, V) - S(\phi V, Y) - \left[ 1 + \frac{r}{n(n-1)} \right] g(V, \phi Y) - g(Y, \phi V)] - (n-1)[g(\phi V, Y) - g(\phi Y, V)] = 0.
\]
In view of (2.7), (2.3) and (7.7) we have
\[
S(\phi V, Y) = \left[ 2 - n + \frac{r}{n(n-1)} \right] g(\phi V, Y).
\]
Substituting \( V = \phi V \) in (7.8) and using (2.6) and (2.2) we obtain
\[
S(Y, V) = \left[ 2 - n + \frac{r}{n(n-1)} \right] g(Y, V) - \left[ 1 + \frac{r}{n(n-1)} \right] \eta(Y)\eta(V).
\]
Hence by Lemma 2.1, (7.9) shows the manifold under consideration is an \( \eta \)-Einstein manifold.

From the above discussions we have the following:

**Theorem 7.1.** If a Kenmotsu manifold satisfies the condition \( \tilde{Z}(\xi, Y) \cdot \tilde{S} = 0 \) with respect to the quarter-symmetric metric connection, then the manifold is an \( \eta \)-Einstein manifold with respect to the Levi-Civita connection.

**Theorem 8.1.** If a Kenmotsu manifold satisfies the condition \( \tilde{Z}(\xi, Y) \cdot \tilde{S} = 0 \) with respect to the quarter-symmetric metric connection, then the manifold is an \( \eta \)-Einstein manifold with respect to the Levi-Civita connection.

**Theorem 8.2.** If a Kenmotsu manifold satisfies the condition \( \tilde{Z}(\xi, Y) \cdot \tilde{S} = 0 \) with respect to the quarter-symmetric metric connection, then the manifold is an \( \eta \)-Einstein manifold with respect to the Levi-Civita connection.

**Proof.** Let us suppose that there exists a \( \xi \)-projectively flat Kenmotsu manifold with respect to the quarter-symmetric metric connection \( \tilde{\nabla} \). Then we have \( \tilde{P}(X, Y)\xi = 0 \), for any vector fields \( X \) and \( Y \) on \( M \). From this and (1.6) we have
\[
\tilde{R}(X, Y)\xi = -\frac{1}{n-1} [\tilde{S}(Y, \xi)X - \tilde{S}(X, \xi)Y] = 0.
\]
With the help of (3.2) and (3.6) we obtain from (8.1)
\[
\eta(Y)\phi X - \eta(X)\phi Y = 0.
\]
Taking \( Y = \xi \) in (8.2) and using the facts \( \phi \xi = 0 \) and \( \eta(\xi) = 1 \), we get \( \phi X = 0 \), which is a contradiction. Therefore the statement of this theorem follows.

**Theorem 8.3.** If a Kenmotsu manifold satisfies the condition \( \tilde{Z}(\xi, Y) \cdot \tilde{S} = 0 \) with respect to the quarter-symmetric metric connection \( \tilde{\nabla} \). Then from (1.7) we have
\[
(\tilde{\nabla}_X \tilde{S})(Y, U) = 2\alpha(X)\tilde{S}(Y, U) + \alpha(Y)\tilde{S}(X, U) + \alpha(U)\tilde{S}(Y, X).
\]
Taking \( U = \xi \) in (8.3) and using (3.6) we obtain
\[
(\tilde{\nabla}_X \tilde{S})(Y, \xi) = -2(n-1)\alpha(X)\eta(Y) - (n-1)\alpha(Y)\eta(X) + \alpha(\xi)\tilde{S}(Y, X).
\]
On the other hand, by the covariant differentiation of the Ricci tensor ̂S with respect to the quarter-symmetric metric connection ̃∇, we have

\[(\nabla_X \tilde{S})(Y,U) = \tilde{\nabla}_X \tilde{S}(Y,U) - \tilde{S}(\nabla_X Y, U) - \tilde{S}(Y, \nabla_X U).\]

So putting \(U = \xi\) in (8.5) and using (8.3), (3.1) and (2.4) we get

\[(\tilde{\nabla}_X \tilde{S})(Y,\xi) = -(n-1)g(X,Y) - S(X,Y) - g(X,\phi Y).\]

Then comparing the right hand sides of equations (8.4) and (8.6), we obtain

\[-2(n-1)\alpha(X)\eta(Y) - (n-1)\alpha(Y)\eta(X) + \alpha(\xi)\tilde{S}(Y,X) - \alpha(\xi)\tilde{S}(X,Y) - \alpha(X,\phi Y).\]

Substituting \(X\) and \(Y\) with \(\xi\) in the above equation we find (since \(n > 3\))

\[\alpha(\xi) = 0.\]

Now we show that \(\alpha = 0\) holds for any vector field on \(M\). Taking \(Y = \xi\) in (8.3) and using (8.7) we have \((\tilde{\nabla}_X \tilde{S})(\xi,\xi) = -2(n-1)\alpha(X)\). By the use of (8.6) we find \(\alpha(X) = 0\) for every vector field \(X\) on \(M\), which implies that \(\alpha = 0\) on \(M\). This contradicts to the definition of pseudo Ricci-symmetry. \(\square\)

9. Example of a 5-dimensional Kenmotsu Manifold Admitting a Quarter-symmetric Metric Connection

We consider the 5-dimensional manifold \(M = \{(x, y, z, u, v) \in \mathbb{R}^5\}\), where \((x, y, z, u, v)\) are the standard coordinates in \(\mathbb{R}^5\). We choose the vector fields

\[e_1 = e^{-v} \frac{\partial}{\partial x}, \quad e_2 = e^{-v} \frac{\partial}{\partial y}, \quad e_3 = e^{-v} \frac{\partial}{\partial z}, \quad e_4 = e^{-v} \frac{\partial}{\partial u}, \quad e_5 = \frac{\partial}{\partial v},\]

which are linearly independent at each point of \(M\).

Let \(g\) be the Riemannian metric defined by

\[g(e_i, e_j) = 0, \quad i \neq j, \quad i, j = 1, 2, 3, 4, 5\]
\[g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1.\]

Let \(\eta\) be the 1-form defined by \(\eta(Z) = g(Z, e_5)\), for any \(Z \in \chi(M)\).

Let \(\phi\) be the \((1,1)\)-tensor field defined by

\[\phi e_1 = e_3, \quad \phi e_2 = e_4, \quad \phi e_3 = -e_1, \quad \phi e_4 = -e_2, \quad \phi e_5 = 0.\]

Using the linearity of \(\phi\) and \(g\), we have

\[\eta(e_5) = 1, \quad \phi^2(Z) = -Z + \eta(Z)e_5, \quad g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),\]

for any \(U, Z \in \chi(M)\). Thus, for \(e_5 = \xi\), \(M(\phi, \xi, \eta, g)\) defines an almost contact metric manifold. The 1-form \(\eta\) is closed. We have

\[\Omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = g\left(\frac{\partial}{\partial x}, \phi \frac{\partial}{\partial z}\right) = g\left(\frac{\partial}{\partial x}, -\frac{\partial}{\partial x}\right) = -e^{2v}.\]
Hence, we obtain \( \Omega = -e^{2\nu} dx \wedge dz \). Thus, \( d\Omega = -2e^{2\nu} dv \wedge dx \wedge dz = 2\eta \wedge \Omega \). Therefore, \( M(\phi, \xi, \eta, g) \) is an almost Kenmotsu manifold. It can be seen that \( M(\phi, \xi, \eta, g) \) is normal. So, it is a Kenmotsu manifold. Then we have

\[
[e_1, e_2] = [e_1, e_3] = [e_1, e_4] = [e_2, e_3] = 0, [e_1, e_5] = e_1,
[e_4, e_5] = e_4, [e_2, e_4] = [e_3, e_4] = 0, [e_2, e_5] = e_2, [e_3, e_5] = e_3.
\]

The Levi-Civita connection \( \nabla \) of the metric tensor \( g \) is given by Koszul’s formula which is given by

\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\
- g([X, Y], Z) - g(Y, [X, Z]) + g(Z, [X, Y]).
\]

Taking \( e_5 = \xi \) and using Koszul’s formula we get the following

\[
\nabla_{e_1} e_1 = -e_5, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \quad \nabla_{e_1} e_4 = 0, \quad \nabla_{e_1} e_5 = e_1,
\]
\[
\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_5, \quad \nabla_{e_2} e_3 = 0, \quad \nabla_{e_2} e_4 = 0, \quad \nabla_{e_2} e_5 = e_2,
\]
\[
\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = -e_5, \quad \nabla_{e_3} e_4 = 0, \quad \nabla_{e_3} e_5 = e_3,
\]
\[
\nabla_{e_4} e_1 = 0, \quad \nabla_{e_4} e_2 = 0, \quad \nabla_{e_4} e_3 = 0, \quad \nabla_{e_4} e_4 = -e_5, \quad \nabla_{e_4} e_5 = e_4,
\]
\[
\nabla_{e_5} e_1 = 0, \quad \nabla_{e_5} e_2 = 0, \quad \nabla_{e_5} e_3 = 0, \quad \nabla_{e_5} e_4 = 0, \quad \nabla_{e_5} e_5 = 0.
\]

Using the above relations in (3.1), we obtain

\[
\nabla_{e_1} e_1 = -e_5, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \quad \nabla_{e_1} e_4 = 0, \quad \nabla_{e_1} e_5 = e_1,
\]
\[
\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_5, \quad \nabla_{e_2} e_3 = 0, \quad \nabla_{e_2} e_4 = 0, \quad \nabla_{e_2} e_5 = e_2,
\]
\[
\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = -e_5, \quad \nabla_{e_3} e_4 = 0, \quad \nabla_{e_3} e_5 = e_3,
\]
\[
\nabla_{e_4} e_1 = 0, \quad \nabla_{e_4} e_2 = 0, \quad \nabla_{e_4} e_3 = 0, \quad \nabla_{e_4} e_4 = -e_5, \quad \nabla_{e_4} e_5 = e_4,
\]
\[
\nabla_{e_5} e_1 = -e_5, \quad \nabla_{e_5} e_2 = -e_4, \quad \nabla_{e_5} e_3 = e_1, \quad \nabla_{e_5} e_4 = e_2, \quad \nabla_{e_5} e_5 = 0.
\]

By the above results, we can easily obtain the non-vanishing components of the curvature tensors as follows:

\[
R(e_1, e_2)e_2 = R(e_1, e_3)e_3 = R(e_1, e_4)e_4 = R(e_1, e_5)e_5 = -e_1,
\]
\[
R(e_1, e_2)e_1 = e_2, \quad R(e_1, e_3)e_1 = R(e_5, e_3)e_5 = R(e_2, e_3)e_3 = e_3,
\]
\[
R(e_2, e_3)e_3 = R(e_2, e_4)e_4 = R(e_2, e_5)e_5 = -e_2, R(e_3, e_4)e_4 = -e_3,
\]
\[
R(e_2, e_5)e_2 = R(e_1, e_5)e_1 = R(e_4, e_5)e_4 = R(e_3, e_5)e_3 = e_5,
\]
\[
R(e_1, e_4)e_1 = R(e_2, e_4)e_2 = R(e_3, e_4)e_3 = R(e_5, e_4)e_5 = e_4,
\]
\[
\tilde{R}(e_1, e_2)e_2 = \tilde{R}(e_1, e_3)e_3 = \tilde{R}(e_1, e_4)e_4 = -e_1,
\]
\[
\tilde{R}(e_1, e_2)e_1 = e_2, \quad \tilde{R}(e_1, e_3)e_1 = \tilde{R}(e_2, e_3)e_3 = e_3,
\]
\[
\tilde{R}(e_2, e_3)e_3 = \tilde{R}(e_2, e_4)e_4 = -e_2, \tilde{R}(e_2, e_5)e_5 = e_4 - e_2,
\]
\[
\tilde{R}(e_3, e_4)e_4 = -e_3, \tilde{R}(e_2, e_5)e_2 = \tilde{R}(e_1, e_5)e_1 = \tilde{R}(e_4, e_5)e_4 = e_5,
\]
\[
\tilde{R}(e_3, e_5)e_5 = e_5, \tilde{R}(e_1, e_5)e_1 = \tilde{R}(e_2, e_4)e_2 = \tilde{R}(e_3, e_4)e_3 = e_4,
\]
\[
\tilde{R}(e_1, e_5)e_5 = e_3 - e_1, \tilde{R}(e_3, e_5)e_5 = -e_1 - e_3, \tilde{R}(e_4, e_5)e_5 = -e_2 - e_4.
\]
With the help of the above results we get the Ricci tensors as follows:

\begin{align}
S(e_1, e_1) &= S(e_2, e_2) = S(e_3, e_3) = S(e_4, e_4) = S(e_5, e_5) = -4 \\
\tilde{S}(e_1, e_1) &= \tilde{S}(e_2, e_2) = \tilde{S}(e_3, e_3) = \tilde{S}(e_4, e_4) = \tilde{S}(e_5, e_5) = -4.
\end{align}

Therefore \( r = \sum_{i=1}^{5} S(e_i, e_i) = -20 \) and \( \tilde{r} = \sum_{i=1}^{5} \tilde{S}(e_i, e_i) = -20. \)

From (9.2) it can be easily verified that the manifold is Ricci semisymmetric with respect to the quarter-symmetric metric connection. Also from (9.1) it follows that the manifold is Einstein with respect to the Levi-Civita connection. Therefore Theorem 4.2 is verified.

Also \( \tilde{r} = r = -20. \) Again from the expressions of the curvature tensor we can easily verify that the manifold is \( \xi \)-concircularly flat with respect to the quarter-symmetric metric connection. Hence Theorem 6.1 is verified.

Acknowledgments. Supported by National Natural Science Foundation of China No. 11871275 and No. 11371194 and by NUST Research Funding No. 30920140132035.

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Department of Applied Mathematics
Nanjing University of Science and Technology
Nanjing, P. R. China
pbzhao@njust.edu.cn

Department of Pure Mathematics,
University of Calcutta
Kol, West Bengal, India
uc_de@yahoo.com
krishanu.mandal013@gmail.com

School of Science,
Qilu University of Technology
Jinan, P. R. China
hanyanling1979@163.com