Superpotentials and Geometric Invariants of Parallel/Complete Coincident/Part Coincident D-brane System on Compact Calabi-Yau Manifold

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Abstract: For D-brane system with three D-branes on compact Calabi-Yau threefolds, the dual F-theory fourfolds for parallel/complete coincident/part coincident D-brane system is constructed by the type II/F-theory duality. Complete coincident means that the three D-branes coincide and part coincident represents the coincident of two of the three D-branes. The low energy effective superpotentials are calculated by mirror symmetry, GKZ-system method and the type II/F-theory duality on the B-model side, respectively. Using the mirror symmetry, A-model superpotentials and the Ooguri-Vafa invariants are obtained from the B-model side. These results indicate that the superpotential contributed by one of three parallel D-branes is identical with the D-brane system with only one D-brane, which is a signal of decoupling of the parallel topological D-branes. However, the superpotential and Oogrui-Vafa invariants are different among parallel, complete coincident and part coincident D-brane system, which show the evidence of the phase transition due to the enhanced gauge symmetry in the low energy theories and the geometrical singularity.

Key words: Superpotentials; Oogrui-Vafa invariants; D-brane; F-theory

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1 Introduction

In the \(N = 2\) supersymmetric theories, the closed-string mirror symmetry gives an equivalence between A-model parameterized by Kähler moduli in the terms of the quantum geometry and B-model parameterized by complex structure moduli in terms of the classical geometry. Mirror symmetry provides many techniques for the variation of physical structures over their moduli spaces\cite{1,2}, and in the closed-string sector\cite{3,4} has a relatively perfect solution in the early works. With the appearance of D-branes, we take more attention to the open-string sector in recent years\cite{5}. The supersymmetry breaks into \(N = 1\) when D-brane is included and that leads the application of open-closed mirror symmetry\cite{6,7}, e.g., the quantum corrected domain wall tensions on the Calabi-Yau threefolds can be calculated with open-closed mirror for compact Calabi-Yau manifolds.

The D-brane superpotential is a section of a special holomorphic line bundles of the moduli space from the mathematical perspective. It is defined as the F-term of low-energy effective theory and determines the string vacuum structure from the physical perspective. The superpotential is the generating function of all disk instantons from the worldsheet point of view, and it can be calculated by relative period. The D-branes on A-model, which are called A-branes, wrap on the special Lagrangian submanifolds of Calabi-Yau manifold \(W_3\). Correspondingly, B-branes wrap on holomorphic submanifolds of Calabi-Yau manifolds \(M_3\). The expression of instanton expansion of superpotential on the A-model side encodes the number of BPS states which corresponds to the Ooguri-Vafa invariants mathematically\cite{23}. These invariants give an important mathematical language which has not been studied systematically in theory so far. It is hard to calculate in the A-model because of the non-perturbed quantum corrections. Nevertheless, we can figure it out in B-model, and then mirror it to A-model. For non-compact Calabi-Yau manifolds, several methods that can be used to calculate the D-brane effective superpotential are localization\cite{23,25}, topological vertex and direct integration related to \(N = 1\) special geometry\cite{26,27}. Further, for compact Calabi-Yau manifolds, some techniques have been evolved, e.g., mixed Hodge structure variation, Gauss-Manin connection\cite{28,29}, the blow-up method\cite{30} and the GKZ-generalized hypergeometric system for open-closed sectors. This paper we focus on is to calculate D-brane superpotentials for compact Calabi-Yau manifolds with open-closed mirror symmetry and generalized GZK system\cite{9,10}.

The dual description of the superpotential in Type-II string theory can be found in F-theory. To be precise, the D-brane superpotentials in the string theory of Type-
II is dual to the background flow in F-theory\cite{31,31}. This duality provides us with a method to calculate the D-brane superpotentials in Type-II string theory. Some works which are related to the calculation of the superpotentials near the limit point and the invariants for the system with two D-branes have been done\cite{9}. And it is mentioned that the parallel and the coincident D-branes phases correspond to the Coulomb branch and the Higgs branch of the non-Abelian gauge theory on the world-volume of D-brane system respectively. Motivated and guided by the work, we calculate and compare the system with three D-branes. The system with three branes are more generalized than two branes, and it complicates the research. The coincident D-branes phases are divided into complete coincident phase and part coincident phase. Complete coincident means that the three D-branes coincide and part coincident represents the coincident of two of the three D-branes.

The organization is as follow. In section 2, we review the background related to physics and mathematics. This section gives an overview of toric brane geometry and GKZ system. In section 3, we concentrate on two models, D-brane system on the mirror quintic and on hypersurface $X_2(1,1,2,2,2)$. The superpotentials of parallel, complete coincident and part coincident D-brane phase are calculated and discussed from each model. The Ooguri-Vafa invariants are extracted in different D-branes models as well. First, we calculate superpotentials and invariants of the parallel D-brane phase (Coulomb phase) with three parallel branes. Second, we calculate the superpotentials and invariants of complete coincident D-brane phase (Higgs phase) in which three parallel D-branes coincide. Third, we calculate the superpotential of part coincident D-brane phase (Coulomb phase-Higgs phase) in which two of the three parallel D-branes coincide. The last section is a brief summary.

2 Toric brane geometry and GKZ system

2.1 Mirror symmetry

Mirror symmetry is the duality between Type II A and Type II B topological string theory. The corresponding D-branes are A-brane and B-brane, respectively. The supersymmetry breaks from $N = 2$ to $N = 1$ when D-branes are included. Sometimes we also call $N = 1$ mirror symmetry as open-closed mirror symmetry since this involves the open and closed part.

Closed mirror symmetry is more fundamental than open-closed mirror symmetry. Taking the brane to infinity and we just suppose there is an infinity, then the open-closed mirror symmetry degenerates to the closed mirror symmetry. It is necessary to note is that there are two ways to explain the open-closed. One way to understand is that an open string falls on the brane and another one on the anti-brane (the brane and anti-brane can be regarded as two branes that carry opposite charges), then two discs was glued to be a sphere, which is closed string. Another way to understand is that closed string does not need brane since brane in string theory is just a boundary condition. Closed string is a Riemann surface without boundary, which can be projected into Calabi-Yau manifold, while open string is Riemann surface with boundary. The A-brane is wrapped on the special Lagrangian submanifold $L$ of the Calabi-Yau manifold $W$, while the B-brane is wrapped on the holomorphic submanifold $D$ of the Calabi-Yau manifold $M$. Mirrored symmetry with D-brane has a more precise description in mathematics: The equivalence between the derived category of coherent sheaves on a Calabi-Yau manifold and the Fukaya category of its mirror. $N = 1$ Mirror symmetry contacted two completely different D-brane geometries $(W, L)$ and $(M, D)$.

2.2 The effective superpotentials in Type-II string theory and F-theory

On the B-model side, the space-filling D5-branes wrap on a reducible curve $C = \sum C_i$ and $C$ embedded in a divisor $D$ of Calabi-Yau 3-fold $M_3$. The effective superpotential is:

$$ W_{N=1}(z, \tilde{z}) = \Pi_n(z, \tilde{z}) = \int_\gamma \Omega^{(3,0)}(z, \tilde{z}), \quad \gamma \in H_3(M_3, D) $$

The superpotential can be written as a linear combination of relative period\cite{26,27}:

$$ W_{N=1}(z, \tilde{z}) = \sum N_n \Pi_n(z, \tilde{z}) = W_{\text{open}}(z, \tilde{z}) + W_{\text{closed}}(z) = W_{\text{brane}}(z, \tilde{z}) + W_{\text{instanton}}(z), $$

$$ \Pi_n(z, \tilde{z}) = \int_\alpha \Omega^{(3,0)}(z, \tilde{z}). $$

On the A-model side, a general form of D-brane superpotential is:

$$ W_{N=1} = W_{\text{classical}}(t, \tilde{t}) + W_{\text{instanton}}(q, \tilde{q}) = (\frac{1}{2} K_{jk} t^j \tilde{t}^k + b) + \sum_{k,m} G_{k,m} q^k \tilde{q}^m = (\frac{1}{2} K_{jk} t^j \tilde{t}^k + b) + \sum_{k,m} \sum_{n} N_{k,m} \frac{1}{n^2 q^k \tilde{q}^m}, $$

where $t^i$ and $\tilde{t}^k$ are the closed and open Kähler moduli respectively, $q = \exp(2\pi i t)$, $\tilde{q} = \exp(2\pi i \tilde{t})$. $\{G_{k,m}\}$ are the open Gromov-Witten invariants labeled by relative homology class. $\bar{m}$ represent the elements of $H_1(L)$, $k$ represent the elements of $H_2(W_1)$, and $K_{jk}$ is the combination coefficient. $\{N_{k,m}\}$ are the Ooguri-Vafa invariants.
There is a duality between the type II string theory with D-brane systems on complex Calabi-Yau threefold $M_3$ and the F-theory compactified on the Calabi-Yau fourfold $M_4$ without any branes but with fluxes. The superpotential of 4-form flux $G_4$ in F-theory compactified on the Calabi-Yau 4-fold $M_4$ is a section of the Hodge line bundle in the complex structure moduli space $\mathcal{M}_{CS}(M_4)$. This superpotential is called Gukov-Vafa-Witten superpotential\cite{GVW, Witten}:

$$W_{GVW}(M_4) = \int_{M_4} G_4 \wedge \Omega^{(4,0)}$$

$$= \sum_{\Sigma} N_{\Sigma}(G_4) \Pi_{\Sigma}(z, \bar{z}) + \mathcal{O}(g_s) + \mathcal{O}(s^{-1/g_s}), \quad (4)$$

The leading term of above equation on the right-hand side is the D-brane superpotential $W_{N=1}$ and $g_s$ is the string coupling strength.

There is a duality between the periods of holomorphic $(4,0)$ form on the non-compact 4-fold $M_4$ and the relative periods of the brane geometry $(M_4, \mathcal{D})$. From mirror symmetry, one obtains the relation between the different compactifications:

$$(W_3, L) \xrightarrow{\text{mirror symmetry}} (M_3, \mathcal{D}) \xleftarrow{\text{mirror symmetry}} (M_4, \mathcal{D}) \xrightarrow{\text{mirror symmetry}} W_4$$

$$4f \xrightarrow{\text{dual}} 4f \xleftarrow{\text{dual}}$$

$$A\text{-branes} \quad B\text{-branes}$$

It is important that the moduli space $\mathcal{M}(M_3, \mathcal{D})$ of open-closed system and the moduli space $\mathcal{M}(\hat{M}_3)$ of non-compact 4-fold $\hat{M}_4$ is isomorphic. Thus the Picard-Fuchs equation can be used to calculate the superpotential by open-closed duality. Further, the structure of the mirror pair $(\hat{W}_4, \hat{M}_4)$ is as follows:

$$(W_3) \xrightarrow{\text{mirror symmetry}} \hat{W}_4 \xleftarrow{\text{mirror symmetry}} \hat{M}_4 \xrightarrow{\text{mirror symmetry}} \hat{W}_4$$

Where $\hat{M}_4$ is an elliptic fibration on a 3-fold $M_3$, while its mirror partner $\hat{W}_4$ is a fibration with a Calabi-Yau 3-fold $\hat{W}_3$ as fiber and a disk $T$ as the base. The mirror pair of four-fold $(\hat{W}_4, \hat{M}_4)$ are non-compact. To get the honest 4-dimensional F-theory compactification, a $P^1$ compactification of the non-compact base $T$ of $\hat{W}_4$ as follows:

$$(W_3) \xrightarrow{\text{mirror symmetry}} W_4 \xleftarrow{\text{mirror symmetry}} M_4$$

In this way one obtains a mirror pair of compact Calabi-Yau 4-fold $(W_4, M_4)$, where $M_4$ is the 4-fold for F-theory compactification and dual to the B-brane geometry $(M_3, \mathcal{D})$. The large volume limit $\text{Vol}(P^1) \to \infty$ maps under mirror symmetry to a weak coupling limit $g_s \to 0$. It is following in the figure:

$$W_4 \xleftarrow{\text{mirror symmetry}} M_4 \xrightarrow{\text{mirror symmetry}} W_4$$

In this limit, the D-brane superpotential $W_{N=1}$ is obtained from the GVW superpotential $W_{GVW}$ of F-theory as follows:

$$W_{N=1}(M_3, \mathcal{D}) = \sum_{\Sigma} N_{\Sigma}(G_4) \Pi_{\Sigma}(z, \bar{z})$$

$$\xrightarrow{g_s \to 0} \lim_{g_s \to 0} W_{GVW}(M_4), \quad (9)$$

The zero-order term of the GVW superpotential and the D-brane superpotential $W_{N=1}$ is equal at the weak coupling limit. The moduli space $\mathcal{M}(M_4)$ which is isomorphic to $\mathcal{M}(M_3, \mathcal{D})$ is a subspace of $\mathcal{M}(M_3)$. The moduli space $\mathcal{M}(M_4)$ is restricted to its subspace $\mathcal{M}(\hat{M}_4)$ by the weak coupling limit. Similarly, on the side of A model, the corresponding points in the moduli space $\mathcal{M}(W_4)$ is the large base limit $\text{Vol}(P^1)$\cite{Katz, Gukov}.

### 2.3 Toric geometry of D-branes system

The Calabi-Yau manifolds in this paper is the hypersurfaces in ambient toric variety. We will use the method of toric polyhedral to define the GLSM for the mirror pairs of toric brane geometries\cite{GK, Canas}.

Model: A-model B-model

Polyhedra: $\nabla_4 \quad \Delta_4$

Toric Variety: $P_{\Sigma(\nabla_4)} \quad P_{\Sigma(\Delta_4)}$

Hypersurfaces(CY 3-fold): $W_3 \quad M_3$

For each pair of reflexive polyhedral $(\nabla_4, \Delta_4)$ with a pair of complete fans $(\Sigma(\nabla_4), \Sigma(\Delta_4))$, there is a pair of toric varieties $(P_{\Sigma(\nabla_4)}, P_{\Sigma(\Delta_4)})$. The defining polynomial for the hypersurface $M_3$ is:

$$P = \sum_{i=0}^{p-1} a_i \prod_{k=1}^{4} X_k^{v_{i,k}}, \quad (10)$$

Here $X_k$ are local coordinates on an open torus $(C^*)^2 \subset P_{\Sigma(\Delta_4)}$, and $v_{i,k}$ is the k-th coordinate of the integral.
The coefficients $a_i$ are complex parameters related to the complex structure of $M_3$. For the homogeneous coordinates $x_j$ on toric ambient space, the polynomial $P$ can be rewritten as

$$P = \sum_{i=0}^{p-1} a_i \prod_{v \in \Delta_4} x_j^{(v,v_i^{*})+1}. \quad (11)$$

The $n$ parallel D-branes are defined by reducible divisor:

$$Q(D) = \prod_{m=0}^{n} (\phi_m a_0 + a_i \sum_{k=1}^{4} X^k v_i^{*} + \cdots)$$

$$= \sum_{k=0}^{n} b_k \prod_{v \in \Delta_4} x_j^{k(v,v_i^{*})+\cdots}, \quad (12)$$

The parallel D-brane geometry corresponds to the Coulomb phase of the gauge theory and the corresponding group is $U(1) \times U(1) \ldots U(1)$. The $U(1)$ group describes the electromagnetism that contains the Coulomb field. The "enhanced polyhedron" expands the dimension of $v_i$ from four to five, and the relevant polyhedron consists of the extended vertices:

$$\tilde{v}_i = (v_j,0) \quad (13)$$

The vertices of the parallel D-branes phase shaping the $\tilde{\nabla}_5$ are [19]

$$\tilde{v}_j = \begin{cases} (v_j,0) & j = 0, \ldots, p-1, \\ (mv_j,1) & j = p + m, 0 \leq m \leq n. \end{cases} \quad (14)$$

They define the non-compact 4-fold $\tilde{W}_4$. The graph is as follows

The gauge group $U(1) \times U(1) \ldots U(1)$ is promoted to the $U(N)$ group when parallel D-branes coincide, while the phase is translated to the Higgs branch.

In toric geometry, the singular curve corresponds to the inner integral lattice point on one-dimensional boundary of the dual polyhedron. Each newly added point corresponds to the exceptional divisor in the blow-up of the Calabi-Yau manifold. Conversely, the removal of the inner point corresponds to the blow-down of these exceptional divisors [35]. The $A_n$ singularity of the complex four-dimensional Calabi-Yau manifold is obtained by removing these vertices, which corresponds to the enhancement of the gauge symmetry group.

The compactifying point is chosen according to: 1, the origin is to be included in the enhanced polyhedron 2, The polyhedron and dual polyhedron are convex polyhedron.

2.4 The generalized GKZ systems and local solutions

The periodic integral satisfies the generalized GKZ system, from which the mirror map and the superpotential can be obtained. Combined with toric geometry, we
Using the Frobenius method, the complete cycle vector also known as the flat coordinate.

From this we can obtain the superpotential on B-model mirror map:}

\[ W = \sum_n C_n \Pi_{2,n}(z). \]  

(21)

The combination coefficient \( C_n \) in the formula can be determined by matching the leading term of the superpotential in the A-model and B-model. In A-model, the leading term or the classical part can be obtained by calculating the Kähler volume of the four-dimensional closed-chain \( \pi_4 \in H_4(W_4, \mathbb{Z}) \):

\[ \frac{1}{2} \int_{\pi_4} J \wedge J, \quad J = \sum_n t_n J_n. \]  

(22)

The instanton corrections are encoded as a power series expansion of \( q_i = \exp(2\pi i t_i) \):

\[ F^{\text{inst}}(t, \hat{t}) = \sum_{\vec{r}, \vec{m}} G_{\vec{r}, \vec{m}} q^{\vec{r}} \hat{q}^{\vec{m}} = \sum_{n} \sum_{\vec{r}, \vec{m}} \frac{N_{\vec{r}, \vec{m}}}{n^2} q^{n \vec{r}} \hat{q}^{n \vec{m}}. \]  

(23)

\( \{G_{\vec{r}, \vec{m}}\} \) are open Gromov-Witten invariants and \( \{N_{\vec{r}, \vec{m}}\} \) are Ooguri-Vafa invariants. \( \vec{m} \) represent the elements of \( H_1(L) \) and \( \vec{r} \) represent the elements of \( H_2(W_3) \).

3 Model

3.1 D-brane system on the mirror quintic

The quintic is defined as a hypersurface with polynomial P:

\[ P = a_1 x_1^5 + a_2 x_2^5 + a_3 x_3^5 + a_4 x_4^5 + a_5 x_5 + a_6 x_1 x_2 x_3 x_4 x_5. \]  

(24)

The degree 5 hypersurface P is in the ambient toric variety \( P(\Delta_5) \). And the toric variety is determined by the vertices of the polyhedron \( \Delta_4 \):

\[ v_1 = (4, -1, -1, -1), v_2 = (-1, 4, -1, -1), v_3 = (-1, -1, 4, -1), v_4 = (-1, -1, -1, 4), v_5 = (-1, -1, -1, -1). \]  

(25)
The vertex of its dual polyhedron \( \nabla_4 \) are:

\[
\begin{align*}
v_0^* &= (0,0,0,0), \\
v_1^* &= (1,0,0,0), \\
v_2^* &= (0,1,0,0), \\
v_3^* &= (0,0,1,0), \\
v_4^* &= (0,0,0,1). 
\end{align*}
\]

\[\text{(26)}\]

3.1.1 Parallel phase of three D-branes

The polyhedron \( \nabla_5 \) corresponding to the compact 4-fold \( W_4 \) of F theory compactification consists of vertices \( \nabla_3 \) and a extra vertex \( \tilde{v}_5^* \). However, the detail of the \( \mathbb{P}^3 \) compactification only dominates the subleading term in \( g_s \) and would be irrelevant in the decoupling limit.

We consider the reducible divisor \( \mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 \) defines the parallel D-branes, it can be written as a degree 15 homogeneous equations:

\[
Q = b_0(x_1x_2x_3x_4x_5)^3 + b_1x_1^2x_2^2x_3^2x_4^2x_5^2 + b_2x_1x_2 + b_3x_5^{15}
\sim \prod_{i=1}^3 (\phi_i a_0x_1x_2x_3x_4x_5 + a_1 x_1^5). \quad \quad \quad \text{(27)}
\]

The vertices of the enhanced polyhedron \( \tilde{\nabla}_5 \) for this open and closed system are as follows:

\[
\begin{align*}
\tilde{v}_0^* &= (0,0,0,0,0), \\
\tilde{v}_1^* &= (1,0,0,0,0), \\
\tilde{v}_2^* &= (0,1,0,0,0), \\
\tilde{v}_3^* &= (0,0,1,0,0), \\
\tilde{v}_4^* &= (0,0,0,1,0), \\
\tilde{v}_5^* &= (1,0,0,0,1), \\
\tilde{v}_6^* &= (2,0,0,0,1), \tilde{v}_7^* &= (3,0,0,0,1). 
\end{align*}
\]

\[\text{(28)}\]

The generators of Mori cone determined by \( \nabla_5 \) are given:

\[
\begin{align*}
&0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad c \\
l_1 = ( -2 \quad -2 \quad 1 \quad 1 \quad 1 \quad 1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \quad ) \\
l_2 = ( 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -2 \quad 1 \quad 0 \quad 0 \quad ) \\
l_3 = ( 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -2 \quad 1 \quad 0 \quad 0 \quad ) \\
l_4 = ( -1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -1 \quad 0 \quad 0 \quad ) \\
l_5 = ( 0 \quad -2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad ). 
\end{align*}
\]

The mixed inverse mirror maps with \( q_i = \exp(2\pi ik_i) \) and \( \{i = 1, 2, 3, 4\} \) are:

\[
\begin{align*}
z_1 &= q_1 - q_1q_4 - q_1q_3q_4 - q_1q_2q_3q_4 + q_1q_2q_3q_4^2 + 24q_1^2q_2q_3q_4^2 + 24q_1q_2^2q_3q_4^2 + 24q_1q_2q_3^2q_4^2 + \ldots \\
z_2 &= q_2 - 2q_2^2 + q_2q_3 + 3q_3^2 - 2q_2^2q_3 + q_2q_3^2 + q_2q_3^2q_4 + \ldots \\
z_3 &= q_3 - q_3^2 + 2q_3^2q_4 + q_2q_3q_4 + q_2^2q_3^2 + q_3^3 + 7q_2q_3^3 - 2q_2^2q_3^2 + \ldots \\
z_4 &= q_4 + q_3q_4 + q_4^2 + q_2q_3q_4 + q_3^2q_4^2 + q_3^3 + q_2^2q_4^2 + \ldots.
\end{align*}
\]

\[\text{(35)}\]

A suitable set of bases is selected to visualize the closed and open moduli.

\[
t = k_1 + k_2 + 2k_3 + 3k_4, \quad \hat{t}_1 = k_2 + k_3 + k_4, \quad \hat{t}_2 = k_3 + k_4, \quad \hat{t}_3 = k_4,
\]

the leading terms of the periods are:

\[
\begin{align*}
\hat{\Pi}^*_{2,1} &= \frac{5}{2} \hat{t}^2, \quad \hat{\Pi}^*_{2,2} = 2(t-\hat{t}_1)^2, \quad \hat{\Pi}^*_{2,3} = 2(t-\hat{t}_2)^2, \quad \hat{\Pi}^*_{2,4} = 2(t-\hat{t}_3)^2.
\end{align*}
\]

\[\text{(31)}\]
\[ \mathcal{W}_2(t, \hat{t}_2) = \Pi_{2,3} = 2(t - \hat{t}_2)^2 + \frac{2}{4\pi^2}(800q + 340000q^2 - 160\eta q_1^{-1} + 6600q^2 q_2^{-2} + 10\eta_2 \]
\[ - 58280q^2 \hat{q}_2^{-1} + \frac{5\hat{q}_2^2}{2} + 1020q \hat{q}_2 + \frac{10\hat{q}_2^3}{9} + \ldots ), \]
\[ \mathcal{W}_3(t, \hat{t}_3) = \Pi_{2,4} = 2(t - \hat{t}_3)^2 + \frac{2}{4\pi^2}(800q + 340000q^2 - 160\eta q_3^{-1} + 6600q^2 q_3^{-2} + 10\eta_3 \]
\[ - 58280q^2 \hat{q}_3^{-1} + \frac{5\hat{q}_3^2}{2} + 1020q \hat{q}_3 + \frac{10\hat{q}_3^3}{9} + \ldots . \] (36)

The disk invariants are Table 1.

| \[ n_1 n_2 n_3 n_4 \] | 1 | 2 | 3 | 4 |
|----------------|---|---|---|---|
| 1              | 0 | -320 | 0 | 0 |
| 2              | 0 | 0 | 13280 | 0 |

Table 1: Ooguri-Vafa invariants \( N_{n_1 n_2 n_3 n_4} \) from the off-shell superpotential \( \mathcal{W}_1(t, \hat{t}) \) contributed by one of three parallel branes on the mirror quintic.

The D-brane superpotential with one open deformation modulus defined by the divisor \( Q = (\phi, a_0 x_1 x_2 x_3 x_4 x_5 + a_1 x_1^{l_1}) \) and the two parallel D-branes superpotentials defined by the divisor \( Q = b_0(x_1 x_2 x_3 x_4 x_5)^3 + b_1 x_1^{l_1} x_2 x_3 x_4 x_5 + b_2 x_1^{l_2} \sim \prod_{j=1}^{5} (\phi a_0 x_1 x_2 x_3 x_4 x_5 + a_1 x_1^{l_1}) \) are the same as reference [17]. And so do the Ooguri-Vafa invariants.

3.1.2 Complete coincident phase of three D-branes

The coincidence can be two or all of them. As we know, the parallel D-branes geometry corresponds to the Coulomb branch of the gauge theory on the worldvolume. When parallel D-brane coincide, the gauge group \( U(1) \times U(1) \times U(1) \) is enhanced to the gauge group \( U(3) \), and the Coulomb branch transform to the Higgs branch of the gauge theory. It’s easy to deduce that three parallel branes is coincident, but two of the three parallel branes is coincident, which is more complicated, it present part coincident phase \( U(2) \times U(1) \). First, the result of complete coincidence (Higgs branch) of three parallel branes is listed here.

We ignore the interior point \( \tilde{v}_c \) and \( \tilde{v}_c^* \), and select \( \tilde{v}_c = (-1, 0, 0, 0, -1) \) as the compactifying point.

The generators of Mori cone determined by \( \nabla_5 \) are given:

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad c \]
\[ l^1 = ( -2 \quad -2 \quad 1 \quad 1 \quad 1 \quad 1 \quad -1 \quad 1 \quad 0 ) \]
\[ l^2 = ( -4 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 ) \]
\[ l^3 = ( 0 \quad -2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 ) \] (37)

A suitable set of bases is selected to visualize the closed and open moduli.

\[ t = k_1 + k_2 + 3k_3, \quad \hat{t}_1 = k_2 + k_3, \quad \hat{t}_2 = k_3. \] (41)

D-brane superpotentials in the A-model as follows:

\[ F_i(t) \equiv \Pi_{2,1} = \frac{5}{2} t^2 + \frac{1}{4\pi^2}(2875q + \frac{4876875}{4}q^2 + \frac{856457500}{9}q^3 + \ldots ), \]
\[ \mathcal{W}_1(t, \hat{t}) = \Pi_{2,2} = 2(t - \hat{t})^2 + \frac{2}{4\pi^2}(2580q + 866769q^2 + \frac{1879614800}{3}q^3 - 24\eta q^{-1} + 54q^2 \hat{q}^{-2} + 4\hat{q} + 13944q^2 \hat{q}^{-1} + \hat{q}^2 - 6552q\hat{q} + 10012416q^2 \hat{q}^{-1} + \frac{4\hat{q}^3}{9} + 5940q\hat{q}^2 + \ldots ). \] (39)

The disk invariants are Table 2.

3.1.3 Part coincident D-branes phase

\( "\tilde{v}_c = (-1, 0, 0, 0, -1)" \) is select as the compactifying point.

First, we ignore the interior point \( \tilde{v}_c \) on the one-dimensional edge with \( \tilde{v}_c^* \), \( \tilde{v}_c^* \) and \( \tilde{v}_c^* \) to obtain the new charge vectors. The generators of Mori cone determined by \( \nabla_5 \) are given:

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad c \]
\[ l^1 = ( -2 \quad -2 \quad 1 \quad 1 \quad 1 \quad 1 \quad -1 \quad 1 \quad 0 ) \]
\[ l^2 = ( 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -3 \quad 2 \quad 0 ) \]
\[ l^3 = ( -1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -1 \quad 0 ) \]
\[ l^4 = ( 0 \quad -2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 ) \] (40)

A suitable set of bases is selected to visualize the closed and open moduli.

\[ t_1 = k_1 + k_2 + 3k_3, \quad \hat{t}_1 = k_2 + k_3, \quad \hat{t}_2 = k_3. \] (41)

D-brane superpotentials in the A-model as follows:

\[ F_i(t) \equiv \Pi_{2,1} = \frac{5}{2} t^2 + \frac{1}{4\pi^2}(2875q + \ldots ), \]
\[ \mathcal{W}_1(t, \hat{t}) = \Pi_{2,2} = 2(t - \hat{t})^2 + \frac{2}{4\pi^2}(704q + 24\eta q_1^{-1} - 4\eta_1 \]
\[ - 5\eta_1^2 + 648q\eta_1 + 92\eta_1^2 \ldots ), \]
\[ \mathcal{W}_2(t, \hat{t}) = \Pi_{2,3} = 2(t - \hat{t})^2 + \frac{2}{4\pi^2}(800q - 160q_2^{-1} + 6600q^2 q_2^{-2} + 10\eta_2 + \frac{5\eta_2^2}{2} + 1020q\eta_2 + \frac{10\eta_2^3}{9} + \ldots ). \] (42)

Second, we ignore the interior point \( \tilde{v}_c \) on the one-dimensional edge with \( \tilde{v}_c^* \), \( \tilde{v}_c^* \) and \( \tilde{v}_c^* \) to obtain the new
3.2 D-brane system on hypersurface

The hypersurface is defined as the polynomial P:

\[ P = a_2 x_1^8 + a_2 x_2^8 + a_3 x_1^4 + a_3 x_2^4 + a_5 x_3^4 + a_0 x_1 x_2 x_3 x_4 x_5 + a_4 x_1^4. \]

(46)

The degree 8 hypersurface P is in the ambient toric variety \( P_{\mathbb{C}^2(\Delta_4)} \). And the toric variety is determined by the vertices of the polyhedron \( \Delta_4 \):

\[
\begin{align*}
v_1 &= (-1, -1, -1, -1), \\
v_2 &= (7, -1, -1, -1), \\
v_3 &= (-1, 3, -1, -1), \\
v_4 &= (-1, -1, 3, -1), \\
v_5 &= (-1, -1, -1, 3).
\end{align*}
\]

(47)

The vertex of its dual polyhedra \( \nabla_4 \) are:

\[
\begin{align*}
v_0^* &= (0, 0, 0, 0), \\
v_1^* &= (-1, -2, -2, -2), \\
v_2^* &= (1, 0, 0, 0), \\
v_3^* &= (0, 1, 0, 0), \\
v_4^* &= (0, 0, 0, 1), \\
v_5^* &= (0, -1, -1, -1).
\end{align*}
\]

(48)

3.2.1 Parallel phase of three D-branes

"\( \hat{v}^* = (0, -1, 0, 0, -1) \)" is select as the compactifying point.

We consider the reducible divisor \( D = D_1 + D_2 + D_3 \) defines the parallel D-branes, it can be written as a degree 15 homogeneous equations:

\[ Q = b_0(x_1 x_2 x_3 x_4 x_5)^3 + b_1 x_1^6 x_2^2 x_3^2 x_4^2 + b_2 x_3 x_1 x_2 x_4 x_5 + b_3 x_1^{12} \approx \prod_{i=1}^{3}(\phi_i a_0 x_1 x_2 x_3 x_4 x_5 + a_1 x_1^4). \]

(49)

The vertices of the enhanced polyhedron \( \hat{\nabla}_5 \) for this open and closed system are as follows:

\[
\begin{align*}
\hat{v}_0^* &= (0, 0, 0, 0, 0), \\
\hat{v}_1^* &= (-1, -2, -2, -2), \\
\hat{v}_2^* &= (1, 0, 0, 0), \\
\hat{v}_3^* &= (0, 1, 0, 0), \\
\hat{v}_4^* &= (0, 0, 0, 1), \\
\hat{v}_5^* &= (0, 1, 0, 0, 1), \\
\hat{v}_{10}^* &= (0, 1, 0, 0, 1), \\
\hat{v}_{10}^* &= (0, 3, 0, 0, 1).
\end{align*}
\]

(50)

The generators of Mori cone determined by \( \nabla_5 \) are given:

\[
\begin{align*}
l_1 &= (-1000 -2111 -1100100) \\
l_2 &= (01100002000000) \\
l_3 &= (000000010120100) \\
l_4 &= (0000000000101 -110) \\
l_5 &= (0000000000011101)
\end{align*}
\]

(51)

A suitable set of bases is selected to visualize the closed and open moduli.

\[
\begin{align*}
t_1 &= k_1 + k_3 + 2k_4 + 3k_5, \\
t_2 &= k_2. \\
\hat{t}_1 &= k_3 + k_4 + k_5, \\
\hat{t}_2 &= k_4 + k_5, \\
\hat{t}_3 &= k_5.
\end{align*}
\]

(52)
the leading terms of the periods are:

\[ \tilde{\Pi}_{2,1} = 2t_1^2, \quad \tilde{\Pi}_{2,2} = \frac{3}{2}(t_1 - \hat{t}_1)^2, \quad \tilde{\Pi}_{2,3} = \frac{3}{2}(t_1 - \hat{t}_2)^2, \]
\[ \tilde{\Pi}_{2,4} = \frac{3}{2}(t_1 - \hat{t}_3)^2. \] (53)

The \( \tilde{\Pi}_{2,1} \), that depends on the close moduli \( t \) is supposed to be the leading term of the bulk potential function \( F(t) \), while the \( \tilde{\Pi}_{2,2} \), \( \tilde{\Pi}_{2,3} \) and \( \tilde{\Pi}_{2,4} \), which depend on both open \( t \) and closed \( t \) parameters are supposed to lead the D-brane superpotential \( \mathcal{W}_1(t, \hat{t}_1) \), \( \mathcal{W}_2(t, \hat{t}_2) \) and \( \mathcal{W}_3(t, \hat{t}_3) \).

Using algebraic coordinates \( (16) \)

\[
z_1 = -\frac{a_4a_2b_3}{a_6b_0}, \quad z_2 = \frac{a_1a_2}{a_6}, \quad z_3 = \frac{b_0b_2}{b_6},
\]
\[
z_4 = \frac{b_3}{b_6}, \quad z_5 = -\frac{a_3b_2}{a_6b_3}, \quad (54)
\]

the fundamental period and the logarithmic periods:

\[
\Pi_0(z) = w_0(z; \rho),
\]
\[
\Pi_{1,i}(z) = \partial_{\rho_i} w_0(z; \rho)|_{\rho_i=0},
\]
\[
\Pi_{2,n}(z) = \sum_{i,j} K_{i,j,n} \partial_{\rho_i} \partial_{\rho_j} w_0(z; \rho)|_{\rho=0}. \quad (55)
\]

The flat coordinates are given by

\[
k_i = \frac{\Pi_{1,i}(z)}{\Pi_0(z)} = \frac{1}{2\pi i} \log z_i + \ldots. \quad (56)
\]

The mixed inverse mirror maps with \( q_i = e^{\pi i k_i} \) and \( \{ i = 1, 2, 3, 4, 5 \} \) are:

\[
z_1 = q_1 + q_1 q_2 - q_1 q_5 - q_1 q_5 q_6 - q_1 q_5 q_6 q_7 - q_1 q_5 q_6 q_7 q_8 - q_1 q_5 q_6 q_7 q_8 q_9 - q_1 q_5 q_6 q_7 q_8 q_9 q_{10} - \]
\[
q_1 q_5 q_6 q_7 q_8 q_9 q_{10} + q_1 q_5 q_6 q_7 q_8 q_9 q_{10} + 6q_1 q_5 q_6 q_7 q_8 q_9 q_{10} + q_1 q_5 q_6 q_7 q_8 q_9 q_{10} + 12q_1 q_5 q_6 q_7 q_8 q_9 q_{10} + 6q_1 q_5 q_6 q_7 q_8 q_9 q_{10} + \ldots
\]
\[
z_2 = q_2 - 2q_2^3 - 3q_2^3 - 48q_1 q_5 q_6 q_7 q_8 q_9 q_{10} + 240q_1 q_5 q_6 q_7 q_8 q_9 q_{10} + 624q_1 q_5 q_6 q_7 q_8 q_9 q_{10} + \ldots
\]
\[
z_3 = \frac{3}{4} \frac{q_3}{q_3}, \quad z_4 = \frac{q_4}{q_3}, \quad z_5 = \frac{q_5}{q_3} + \ldots
\]
\[
z_6 = \frac{q_6}{q_3}, \quad z_7 = \frac{q_7}{q_3}, \quad z_8 = \frac{q_8}{q_3}, \quad z_9 = \frac{q_9}{q_3}, \quad z_{10} = \frac{q_{10}}{q_3}, \quad \ldots
\] (57)

According to the leading terms \( (53) \), the relative periods which corresponds to the closed-string period and D-brane superpotentials in the A-model as follows:

\[
F(t) \equiv \Pi_{2,1} = 2t_1^2 + \frac{1}{4\pi^2} (4q_2 + q_2^3 + \frac{4}{9} q_2^3 + 640q_1q_2 + q_2^3(2222q_2 + 20224q_2^3) + \ldots),
\]
\[
W_1(t, \hat{t}_1) \equiv \Pi_{2,2} = \frac{3}{2}(t_1 - \hat{t}_1)^2 + \frac{1}{4\pi^2} (3q_2 + \frac{3}{4} q_2^3 + 12q_2 + 3q_2^2 - 27q_1q_2 - 66q_1q_2 - 63q_1q_2 - 360q_1q_2 + 282q_1q_2q_2 - 405 \frac{q_2^3}{4} \hat{q}_2^3 - 324q_2^3 \hat{q}_2^3 - \ldots),
\]
\[
W_2(t, \hat{t}_2) \equiv \Pi_{2,3} = \frac{3}{2}(t_1 - \hat{t}_2)^2 + \frac{1}{4\pi^2} (3q_2 + \frac{3}{4} q_2^3 + 12q_2 + 3q_2^2 - 27q_1q_2 - 66q_1q_2 + 360q_1q_2 + 282q_1q_2q_2 - 405 \frac{q_2^3}{4} \hat{q}_2^3 - 324q_2^3 \hat{q}_2^3 - \ldots),
\]
\[
W_3(t, \hat{t}_3) \equiv \Pi_{2,4} = \frac{3}{2}(t_1 - \hat{t}_3)^2 + \frac{1}{4\pi^2} (3q_2 + \frac{3}{4} q_2^3 + 12q_2 + 3q_2^2 - 27q_1q_2 - 66q_1q_2 + 360q_1q_2 + 282q_1q_2q_2 - 405 \frac{q_2^3}{4} \hat{q}_2^3 - 324q_2^3 \hat{q}_2^3 - \ldots). \quad (58)
\]

The disk invariants are Table \( (53) \)

| \( n_1 = n_2 = n_3 = 0 \) | \( n_5 \backslash n_4 \) |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 12 | 0 |
| 2 | 0 | 0 | 0 |

Table 3: Ooguri-Vafa invariants \( N_{n_1, n_2, n_3, n_4, n_5} \) from the off-shell superpotential \( W_1(t, \hat{t}) \) contributed by one of three parallel branes on the hypersurface \( X_{(1,2,2,2,2)} \).

The D-brane superpotential with one open deformation modulus defined by the divisor \( Q = b_0(x_1 x_2 x_3 x_4 x_5) + b_1 x_3^4 \) and two parallel D-branes superpotentials defined by the divisor \( Q = b_0(x_1 x_2 x_3 x_4 x_5)^2 + b_1 x_3^4 x_1 x_2 x_3 x_5 + b_2 x_3^8 \sim \prod_{i=1}^2 (\phi_i a_0 x_1 x_2 x_3 x_4 x_5 + a_3 x_3^4) \) are the same as reference \( (59) \).

3.2.2 Complete coincident phase of three D-branes

We ignore the interior point \( \hat{v}_c \) and \( \hat{v}_c^* \), and select \( \hat{v}_c^* = (0, -1, 0, 0, -1) \) as the compactifying point.
The generators of Mori cone determined by $\nabla_5$ are given:

\[
\begin{align*}
0 & 1 2 3 4 5 6 7 8 9 \\
\ell^1 &= (-1 0 0 -2 1 1 1 -1 1 0) \\
\ell^2 &= (0 1 1 0 0 0 -2 0 0 0) \\
\ell^3 &= (0 0 0 3 0 0 0 0 1 -10) \\
\ell^4 &= (0 0 0 -2 0 0 0 0 0 1 1).
\end{align*}
\]

A suitable set of bases is selected to visualize the closed and open moduli.

\[ t_1 = k_1 + k_3, \quad t_2 = k_2, \quad \hat{t} = k_3. \] (60)

D-brane superpotentials in the A-model as follows:

\[
F_1(t) = \Pi_{2,1} = 2t_1^2 + \frac{1}{4\pi^2}(4q_2 + q_2^* + 2q_2^* - 16q_1q_2 + \ldots),
\]

\[
W(t, \hat{t}) = \Pi_{2,2} = \frac{3}{2}(t_1 - \hat{t})^2 + \frac{1}{2}(3q_2 + q_2^* + 18q_1q_2^* - 84q_1q_2^* - 224q_1q_2^* - \ldots).
\] (64)

Second, we ignore the interior point $\tilde{v}_a^*$ on the one-dimensional edge with $\tilde{v}_a^*$ and $\tilde{v}_a^{10}$ to obtain the new charge vectors. The generators of Mori cone determined by $\nabla_5$ are given:

\[
\begin{align*}
0 & 1 2 3 4 5 6 7 8 9 \\
\ell^1 &= (-1 0 0 -2 1 1 1 -1 0 1 0) \\
\ell^2 &= (0 1 1 0 0 0 -2 0 0 0) \\
\ell^3 &= (0 0 0 0 0 0 1 -3 2 0) \\
\ell^4 &= (-1 0 0 1 0 0 0 0 1 -10) \\
\ell^5 &= (0 0 0 -2 0 0 0 0 0 1 1).
\end{align*}
\]

A suitable set of bases is selected to visualize the closed and open moduli.

\[ t_1 = 2k_1 + 3k_2, \quad t_2 = 2k_2, \quad \hat{t}_1 = k_3 + k_4, \quad \hat{t}_2 = k_4. \] (66)

D-brane superpotentials in the A-model as follows:

\[
F_1(t) = \Pi_{2,1} = 2t_1^2 + \frac{1}{4\pi^2}(4q_2 + q_2^* + 2q_2^* - 16q_1q_2 + \ldots),
\]

\[
W_2(t, \hat{t}) = \Pi_{2,2} = \frac{3}{2}(t_1 - \hat{t})^2 + \frac{1}{2}(3q_2 + q_2^* + 18q_1q_2^* - 84q_1q_2^* - 224q_1q_2^* - \ldots).
\] (65)

\begin{table}[h]
\begin{tabular}{cccc}
$N_{0,1,0}$ & $N_{1,0,0}$ & $N_{2,1,0}$ & $N_{3,1,0}$ \\
3 & 3 & 3 & 3 \\
\end{tabular}
\end{table}

Table 4: Ooguri-Vafa invariants $N_{a_1,a_2,a_3}$ from the off-shell superpotential $W_1(t, \hat{t})$ contributed by the complete coincident phase of three D-branes on the hypersurface $X_8(1,1,2,2,2)$.

3.2.3 Part coincident D-branes phase

"$\tilde{v}_a^* = (-1,0,0,0,0,0,1)$" is select as the compactifying point.

First, we ignore the interior point $\tilde{v}_a^*$ on the one-dimensional edge with $\tilde{v}_a^*$ and $\tilde{v}_a^{10}$ to obtain the new charge vectors. The generators of Mori cone determined by $\nabla_5$ are given:

\[
\begin{align*}
0 & 1 2 3 4 5 6 7 8 9 \\
\ell^1 &= (-1 0 0 -2 1 1 1 -1 0 1 0) \\
\ell^2 &= (0 1 1 0 0 0 -2 0 0 0) \\
\ell^3 &= (0 0 0 0 0 0 1 -3 2 0) \\
\ell^4 &= (-1 0 0 1 0 0 0 0 1 -10) \\
\ell^5 &= (0 0 0 -2 0 0 0 0 0 1 1).
\end{align*}
\]

A suitable set of bases is selected to visualize the closed and open moduli.

\[ t_1 = k_3 + 3k_4, \quad t_2 = k_2, \quad \hat{t}_1 = k_3 + k_4, \quad \hat{t}_2 = k_4. \] (63)

D-brane superpotentials in the A-model as follows:

\[
F_1(t) = \Pi_{2,1} = 2t_1^2 + \frac{1}{4\pi^2}(4q_2 + q_2^* + 2q_2^* - 16q_1q_2 + \ldots),
\]

\[
W_2(t, \hat{t}) = \Pi_{2,2} = \frac{3}{2}(t_1 - \hat{t})^2 + \frac{1}{2}(3q_2 + q_2^* + 18q_1q_2^* - 84q_1q_2^* - 224q_1q_2^* - \ldots).
\] (64)

4 Summary

The effective D-brane superpotential is of great significance to both physics and mathematics. In type II/F-theory compactification, the vacuum structure is determined by the superpotentials, whose second derivative gives the chiral ring structure, and which is the generating function of the Ooguri-Vafa invariants. Those invariants are closely related to the number of the BPS states, which count the holomorphic disks on Calabi-Yau manifolds mathematically.

In this paper, for the system with three D-branes on...
the mirror quintic and hypersurface $X_3(1,1,2,2,2)$, the off-shell effective superpotentials and relevant geometric invariants are calculated by type II/F-theory duality, open-closed mirror symmetry and GKZ-system method. The results show that there are three phases: parallel phase, part coincident phase and complete coincident phase, i.e., the Coulomb branch, the mixed Coulomb-Higgs branch and the Higgs branch, respectively. The phase transitions: the parallel phase → the part coincident phase → the complete coincident phase, correspond to the enhancement of gauge group $U(1) \times U(1) \times U(1) \rightarrow U(1) \times U(2) \rightarrow U(3)$ in the low energy effective theory. In the parallel phase, the effective superpotential contributed by one of the three D-branes is the same as that of the D-brane system with single brane in the same Calabi-Yau manifold. In the part coincident phase, i.e., the two of the three parallel D-branes approach to each other and melt into one, the superpotential and the invariant are different from either those of the system with three D-branes or those of the system with two D-branes. It gives the sign of the phase transition from the $U(1) \times U(1) \times U(1)$ to $U(1) \times U(2)$ in terms of gauge theory. Similarly, in the complete coincident D-brane phase, i.e., all parallel D-branes approach to each other and melt into one, our calculations indicate that the effective superpotentials and the Ooguri-Vafa invariants are completely different from those of the system with one D-brane, although there is no difference between the coincident D-branes and the single D-brane from the set theory view point. It shows the feature of the phase transition from $U(1) \times U(1) \times U(1)$ to $U(3)$ in terms of gauge theory. Thus, there are various phase structures in the low energy effective theory of the system with three D-branes on the compact Calabi-Yau manifold.

Furthermore, we are going to study the more general physical and mathematical properties of D-brane system on compact Calabi-Yau manifolds, especially in phenomenological applications of superpotential and the relevant geometric invariants.

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