Topological Anderson Transitions in Chiral Symmetry Classes

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We study quantum phase transitions of three-dimensional disordered systems in the chiral classes (AIII and BDI) with and without weak topological indices. From our extensive numerical study, we show that the systems with a non-trivial weak topological index universally exhibit an emergent thermodynamic phase where wave functions are delocalized along one spatial direction but exponentially localized in the other two spatial directions, which we call the quasi-localized phase. This quasi-localized phase is distinct from the conventional metallic and Anderson-localized phases, where wave functions are delocalized and localized along all directions, respectively. We obtain the critical exponents of two different disorder-induced phase transitions, one between the metallic and quasi-localized phases and the other between the quasi-localized and Anderson-localized phases. These critical exponents are different from the critical exponent with no weak topological indices in the same symmetry classes, which signals the new universality classes induced by topology.

Introduction—The interplay of disorder and topology has been intensively studied for the quantum Hall plateau transitions [1-9] and disorder-driven phase transitions in topological insulators and semimetals. Universality classes of the disorder-driven metal-insulator transitions, known as the Anderson transitions, are characterized by the critical exponents and scaling functions, which are commonly believed to be determined solely by symmetry and spatial dimensions [10]. Many theories investigated whether topology can change the universality classes of the Anderson transitions [11-23]. Several previous works calculated the critical exponents of the Anderson transitions between metal and topological-insulator phases [17-21, 24-28] and between metal and trivial-insulator phases [12, 13, 25-27, 29-30] (see also Ref. [31]).

However, only a single critical exponent was found for each symmetry class and spatial dimension, regardless of topological indices. Thus, the role of topology in the Anderson transitions has been still elusive.

In this Letter, we elucidate that weak topological indices induce different universality classes of the Anderson transitions in three-dimensional (3D) chiral symmetry classes (AIII and BDI). Such weak topological indices are relevant to topological nodal-line semimetals. From our extensive numerical calculations, we evaluate the correlation-length critical exponents (see Fig. 1 and Table I) and find that they are distinct from the critical exponents in topologically trivial systems, signaling new universality classes induced by the topological indices. We clarify the origin of these new universality classes, showing that the non-trivial topological indices bring about an intermediate phase where wave functions are delocalized along one spatial direction but exponentially localized in the other two directions—quasi-localized phase (Fig. 1(a)). Introducing lower-dimensional inverse participation ratios of 3D wave functions, we demonstrate that the emergence of the quasi-localized phase in 3D systems is a consequence of finite inverse participation ratios at the topological phase transition of 1D chiral-symmetric systems. Similar discussions also suggest the absence of the intermediate phases or the different universality classes in layered quantum Hall systems.

Lyapunov exponents and topological indices—We study disorder-induced quantum phase transitions of 3D chiral-symmetric Hamiltonians \( \mathcal{H} \). The localization properties along the \( \mu \) direction (\( \mu = x, y, z \)) are efficiently captured by the Lyapunov exponents (LEs) along the \( \mu \) direction in the limit \( L \to \infty \), which are eigenvalues of [32, 33]

\[
\lim_{L \to \infty} \log \left( \frac{1}{\mu} \right)^{\frac{1}{L^2}}. \tag{1}
\]

Here, \( M \equiv M_{\mu, L} M_{\mu, L-1} \cdots M_1 \) is the product of transfer matrices along the \( \mu \) direction. The smallest positive LE gives the inverse of the localization length along the \( \mu \) direction [34]. In the limit \( L \to \infty \), the LEs of \( \mathcal{H} \) form

\[\begin{array}{c|c}
\nu = 0.82 \pm 0.02 & \nu = 1 \\
\text{metal} & \text{quasi-localized} \\
\text{phases} & \text{Anderson} \\
\text{insulator} & \\
\end{array}\]

\[\begin{array}{c|c}
\nu = 1.09 \pm 0.03 & \\
\text{metal} & \text{Anderson} \\
\text{insulator} & \\
\end{array}\]

![FIG. 1. Phase diagrams of 3D disordered Hamiltonians in the chiral symmetry classes (a) with and (b) without the weak topological index \( \nu_\perp \). The critical exponents \( \nu \) and localization lengths \( \xi_\perp, \xi_\parallel (\perp = x, y) \) along different directions are shown for different phases. The nontrivial critical exponents \( \nu = 0.82 \pm 0.02 \) and \( \nu = 1.09 \pm 0.03 \) are obtained for class BDI.](image-url)

\[\text{⊥}\]

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several continuous spectra. If the spectra do not include zero, the wave function is localized along the \( \mu \) direction. By contrast, if the spectra include zero, the localization length diverges, which means the delocalization of the wave function.

Symmetries of Hamiltonians give constraints on the spectrum of the LEs. For example, because of Hermiticity of \( \mathcal{H} \), the LEs come in opposite-sign pairs. Moreover, in the presence of chiral symmetry, \( \mathcal{H} \) can be brought into the block off-diagonal structure,

\[
\mathcal{H} = \begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix}, \tag{2}
\]

where the off-diagonal part \( h \) is assumed to be a square matrix. Because of chiral symmetry, the LEs of \( \mathcal{H} \) reduce to the LEs of \( h \) and \( h^\dagger \), which come in opposite-sign pairs. Consequently, we only need to calculate the product of the transfer matrices of \( h \).

In the following, we demonstrate that a weak topological index \( \nu_\mu \) imposes another constraint on the spectrum of the LEs and crucially changes the universality classes of the Anderson transitions. To introduce \( \nu_\mu \) along the \( \mu \) direction in the presence of disorder, let us insert a magnetic flux \( \phi_\mu \) through a closed loop along the \( \mu \) direction. Then, the weak topological index \( \nu_\mu \) is given by the winding of \( \det h(\phi_\mu) \) in Eq. (2) under an adiabatic insertion of a unit flux:

\[
\nu_\mu = \frac{i}{L^2} \int_0^{2\pi} d\phi_\mu \frac{\partial}{\partial \phi_\mu} \text{Tr} \left[ \log \left[ h(\phi_\mu) \right] \right], \tag{3}
\]

where \( L^2 \) is the system size within the two directions perpendicular to the \( \mu \) direction. Here, \( \nu_\mu \) is not necessarily quantized and takes an arbitrary real number. Notably, the weak topological index \( \nu_\mu \) and LEs of \( h \) are related to each other by:

\[
\nu_\mu = \frac{1}{2L^2} (N_{+,\mu} - N_{-,\mu}), \tag{4}
\]

where \( N_{+,\mu} \) and \( N_{-,\mu} \) are the numbers of positive and negative LEs of \( h \) along the \( \mu \) direction, respectively.

Suppose \( \mathcal{H} \) has a mobility gap around \( E = 0 \) and its zero-energy state is characterized by the weak topological indices \( \nu_x = 1, \nu_y = 0 \). From Eq. (4), a finite gap exists between the smallest positive LE and the largest negative LE such that \( N_{+,z} - N_{-,z} = 2L^2 \). By contrast, when disorder is strong enough, the zero-energy state is in a topologically-trivial localized phase with \( N_{+,z} = N_{-,z} \). Between the two localized phases, \( L^2 \) positive LEs of \( h \) cross zero, and \( \nu_z \) continuously changes from 1 to 0 with respect to the disorder strength, where the localization length \( \xi_z \) along the \( z \) direction always diverges. Within this finite range with divergent \( \xi_z \), the zero-energy state undergoes the Anderson transitions along the \( x \) and \( y \) directions.

Below, we obtain the critical exponents of these Anderson transitions induced by the weak topological indices \( \nu_\mu \) and demonstrate their new universality classes.

**Model**—As a prototypical example, we study a two-orbital tight-binding model on a 3D cubic lattice:

\[
\mathcal{H} = \sum_{r=(r_x,r_y,r_z)} \left\{ \epsilon_r \sigma_r^\dagger \sigma_r^c + \left[ \sum_{\mu=x,y} (t_{\mu} \sigma_{r+e_{\mu}}^\dagger \sigma_{r}^c + \text{H.c.}) \right] \right\}. \tag{5}
\]

Here, \( \sigma_r \) is a two-component annihilation operator at the cubic lattice site \( r \), \( \sigma_r^\dagger \) (\( \mu = x, y, z \)) are Pauli matrices, \( t_{\mu}, t_{\mu}^0 \) are real-valued parameters, and \( \epsilon_r \) is a random potential that distributes uniformly in \([-W/2, W/2] \). We assume \( t_{\mu}, t_{\mu}^0 > 0 \) for simplicity. This Hamiltonian respects time-reversal-invariant symmetry \( \mathcal{H} = \mathcal{H}^* \) and chiral symmetry \( \mathcal{H} = -\sigma_z \mathcal{H} \sigma_z \), and hence belongs to class BDI [10, 42]. In addition, the ensemble of Hamiltonians is statistically invariant under the combination of time reversal and reflection with respect to the \( xy \) plane, which requires \( N_{+,x} = N_{-,x}, N_{+,y} = N_{-,y} \) and \( \nu_z = \nu_y = 0 \), while \( \nu_x \) can be non-zero. In the clean limit, the Hamiltonian has an energy gap around \( E = 0 \) with \( \nu_z = 1 \) for \( 4t_{\mu} < 2|t_{\mu}^0| \). For \( 2|t_{\mu}^0| < 4t_{\mu} \), by contrast, the zero-energy state forms a nodal line in momentum space, resulting in \( 0 < \nu_z < 1 \). In the following, we focus on the nodal-line-semimetal phase for \( t_{\mu} = t_{\mu}^0 = 1/2 \) and \( t_{\mu} = 1 \) and study the Anderson transitions of the zero modes along all the directions. This nodal-line semimetal in class BDI can be realized by Bogoliubov quasi-particles in time-reversal-invariant superconductors [43] as well as in synthetic systems of electrical circuits [44] and cold atoms [34, 46].

**Localization length \( \xi_z \)**—Figure 2 shows a distribution of LEs \( \gamma \) of \( h \) in Eq. (2) for the nodal-line-semimetal model \( \mathcal{H} \) in Eq. (5) along the \( z \) direction in the quasi-1D geometry \( L \times L \times L_z \). The distribution consists of two separate spectra, each of which contains \( L^2 \) LEs. The upper spectrum is always \( \gamma = +\infty \) and irrelevant to the Anderson transitions. For \( W \leq W_c^{(z)} \) \( \approx 29 \), the lower spectrum includes zero \( \gamma = 0 \). Every positive LE in the lower spectrum for \( W < W_c^{(z)} \) crosses zero when we increase \( W \). At each crossing point, \( N_{-,z} \) changes by one. For \( L \rightarrow \infty \), the crossing points become dense and \( \nu_z = 1 - N_{-,z}/L^2 \) changes continuously with \( W \). For \( W > W_c^{(z)} \), all the LEs in the lower spectrum are negative (i.e., \( N_{-,z} = L^2 \)), and the system is in a localized phase with no weak topological index \( \nu_z = 0 \). At \( W = W_c^{(z)} \), the maximal LE in the lower spectrum crosses zero. Notably, \( W_c^{(z)} \) for \( L \rightarrow \infty \) cannot be determined by fitting \( \xi_z/L \) with a standard scaling function [e.g., see Eq. (S.12)] because \( \xi_z \) with finite \( L \) diverges at some \( W < W_c^{(z)} \). Instead, we map the non-Hermitian matrix \( h \) into a well-localized Hermitian matrix by a similarity transformation [31, 47], where the localization length obeys a scaling form in the strong disorder limit [33]. Then, we obtain the scaling form of the largest LE \( \gamma_{\text{max}}(W,L) \),

\[
\gamma_{\text{max}}(W,L) = a/L + \gamma_{\text{max}}(W,L = \infty). \tag{6}
\]
We numerically verify this scaling and determine the critical disorder strength \( W_c(x) = 29.45 \pm 0.05 \) (inset of Fig. 2).

**Localization length \( \xi_x, \xi_y \)—**The statistical symmetries mentioned above require LEs of \( h \) along the \( x \) and \( y \) directions to come in opposite-sign pairs. Thus, the localization length \( \xi_x \) along the \( x \) direction is always finite in the quasi-1D geometry with finite \( L \). As shown in Fig. 3, the normalized localization length \( \Lambda_x(W, L) \equiv \xi_x(L, W)/L \) shows scale-invariant behavior at a certain disorder strength \( W_c(x) \) far below \( W = W_c(x) \), indicating a quantum phase transition at \( W = W_c(x) \). To determine \( W_c(x) \) and the critical exponent \( \nu \), we use a finite-size scaling function and its polynomial expansion \([31, 49]\). The scaling function for \( \Lambda_x(W, L) \) is Taylor-expanded with respect to the relevant scaling variable \( \phi(w) \) and the least irrelevant scaling variable \( c \) up to the \( n \)th order and first order, respectively,

\[
\Lambda_x(W, L) = \sum_{i=0}^{n} \sum_{j=0}^{1} a_{i,j} \phi(w)^i L^{i/n} c^{j} L^{-y} W_c^{(x)/L},
\]

with \( w \equiv (W - W_c^{(x)})/W_c^{(x)} \) and the scaling dimension \(-y\) (\( < 0 \)) of the least irrelevant scaling variable around a saddle-point fixed point. The fitting is carried out by the \( \chi^2 \) fitting method, and the confidence error bars for the optimal parameters are determined by the Monte Carlo method \([31]\).

The first row in Table I shows the fitting results, where \( W_c^{(x)} = 27.24 \pm 0.05 \) is significantly smaller than \( W_c^{(z)} = 29.45 \pm 0.05 \). These two different critical disorder strengths illustrate the emergence of the three distinct phases as a function of the disorder strength \( W \) [Fig. 1(a)]. For \( W < W_c^{(x)} \), the localization lengths diverge along all directions (metallic phase). For \( W > W_c^{(z)} \), the localization lengths are finite along all directions (Anderson insulator phase). For \( W_c^{(x)} < W < W_c^{(z)} \), the localization lengths are finite along the \( x \) and \( y \) directions but diverge along the \( z \) direction, and \( \nu_z \) continues changing as \( W \) changes. From our extensive numerical calculations, we find that this intermediate phase with divergent \( \xi_z \) but finite \( \xi_x, \xi_y \) universally appears in different models with non-zero \( \nu_z \) \([31]\).

**Quasi-localized phase—**Now, we clarify the nature of the quasi-localized phase induced by the weak topological index \( \nu_\mu \). Let \( \Phi(r) = \langle r | \Phi \rangle \) be a normalized wave function. The wave function feels an effective disorder potential \( V_{\text{eff}} = \langle \Phi | V | \Phi \rangle = \sum_r V(r)|\Phi(r)|^2 \), whose strength is given by \( \langle V_{\text{eff}}^2 \rangle = W^2 P_2 \) with the inverse participation ratio \( P_2 \equiv \sum_r |\Phi(r)|^4 \). Here, \( \langle \ldots \rangle \) denotes the disorder average, and \( V(r) \) is assumed to be Gaussian: \( \langle V(r)V(r') \rangle = W^2 \delta_{r,r'} \). Let us introduce the integrated weight of the wave function in the \( n \)th layer by
TABLE I. Critical disorder strength $W_{c}^{(3)}$ and critical exponent $\nu$ for the 3D chiral classes, obtained by the polynomial fitting of the normalized localization length $\Lambda_{n} \equiv \xi_{n}/L$ along the $\mu$ direction ($\mu = x, y, z$) around critical points of different models with the quasi-one-dimensional geometry $L \times L \times L_{\mu}$. In the column “Topo”, “√” shows the nonzero weak topological index $\nu_{c}$ around the critical point, and “×” shows zero topological indices in all the directions. The square brackets denote the 95% confidence interval.

| Class       | Topo | $W_{c}^{(3)}$ | $\nu$  |
|-------------|------|---------------|--------|
| BDI         | √    | $27.241[27.194, 27.303]$ | 0.820 [0.783, 0.846] |
| AIII        | ×    | $9.143[9.125, 9.168]$ | 0.824 [0.776, 0.862] |
| BDI         | ×    | $23.220[23.167, 23.293]$ | 1.089 [1.005, 1.128] |
| BDI         | ×    | $23.170[23.098, 23.279]$ | 1.042 [0.943, 1.099] |
| AIII        | ×    | 8.091 [8.074, 8.096] | 1.024 [0.973, 1.070] |

$|\phi(z)|^{2} = \sum_{x,y} |\Phi(r)|^{2}$ and also the one-dimensional inverse participation ratio $P_{d}^{(1)} \equiv \sum_{r} |\phi(r)|^{4}$. $P_{z}^{(1)}$, $P_{y}^{(1)}$ can be defined in the same manner. $P_{d}^{(2)}$ measures the localization property of $\Phi(r)$ along the $\mu$ direction, giving an upper bound of $P_{d}$. $P_{d} \leq P_{d}^{(1)} (\mu = x, y, z)$ [31]. If the wave function is extended along the $z$ direction (i.e., $P_{z} \sim L_{z}^{-1}$ [11], $P_{z}$ and $\langle V_{z}^{2} \rangle$ should vanish for $L_{z} \rightarrow \infty$, and $\Phi(r)$ must be extended along all the directions. If $P_{z}$ is finite even for $L_{z} \rightarrow \infty$, by contrast, $P_{x}$ and $P_{y}$ should also be finite for $L_{x}, L_{y} \rightarrow \infty$. Otherwise, $\Phi(r)$ is extended within all the directions, which contradicts finite $P_{d}^{(2)}$. In the intermediate phase discussed above, we find that $\xi_{x}$ is finite but $\xi_{y}$ diverges. While finite $\xi_{x}$ means finite $P_{x}$ and $P_{z}^{(2)}$, divergent $\xi_{y}$ with finite $P_{y}^{(2)}$ means that the wave function $\Phi(r)$ must be quasi-localized along the $z$ direction. Thus, the wave function in the intermediate phase is localized within the $xy$ plane and delocalized only along the $z$ direction—quasi-localized phase. Here, $\Phi(r)$ along the $z$ direction shares the same localization properties as wave functions of 1D chiral-symmetric systems at a topological phase transition, where the 1D topological index changes [11] [36] [51] [51]. The 3D system in the intermediate phase is effectively decoupled into 1D wires because of finite $\xi_{x,y}$.

The emergence of the quasi-localized phase in 3D systems is a consequence of finite $P_{d}^{(2)}$ at the topological phase transition of 1D chiral-symmetric systems. Generally, when a $d'$-dimensional wave function $\Phi(R)$ in $R \equiv (r, s)$ with $r = (r_{1}, \ldots, r_{d})$ and $s = (s_{1}, \ldots, s_{d'}-d)$ ($d < d'$) is made out of coupled $d'$-dimensional wave functions $\psi(r)$ at the critical point, $\Phi(R)$ is more extended than $\psi(r)$ along the $r$ direction because of the interlayer coupling [51]. Thus, the effective disorder strength felt by the $d'$-dimensional wave function $\Phi(R)$ is bounded by the $d$-dimensional inverse participation ratio $P_{d}^{(1)}(r)$ of $\psi(r)$. When the wave function $\psi(r)$ has finite $P_{d}^{(1)}(r)$ at the critical point, the effective disorder strength can be finite, and $\Phi(R)$ can be either extended or localized within the $s$ direction. On the other hand, when $P_{d}^{(1)}(r)$ is zero at the transition point, e.g., 2D critical wave functions at the quantum Hall plateau transition, the effective disorder strength is zero, and the $d'$-dimensional wave function at the transition must be always extended in both $r$ and $s$ directions. Notably, the 1D topological phase transitions in all the three chiral classes are characterized by finite $P_{d}$ [10]. In the following, we demonstrate the quasi-localized phases also in the 3D chiral unitary class, which is consistent with the above argument.

Model without time-reversal symmetry—We add a time-reversal-breaking but chiral-symmetric disorder $\Delta H$ to the model $\mathcal{H}$ in Eq. (5):

$$\mathcal{H}_1 = \mathcal{H} + \Delta \mathcal{H}, \quad \Delta \mathcal{H} = \sum_{r} \epsilon_{r}^{x}c_{r}^{\dagger}c_{r}^{x} + \epsilon_{r}^{y}c_{r}^{\dagger}c_{r}^{y} + \epsilon_{r}^{z}c_{r}^{\dagger}c_{r}^{z} + \Delta \mathcal{H}_{s},$$

where the random potentials $\epsilon_{r}^{x}, \epsilon_{r}^{y}, \epsilon_{r}^{z}$ distribute uniformly for $\epsilon_{r}^{x} \leq \epsilon_{r}^{y} \leq \epsilon_{r}^{z} \leq W^2$. This model only respects chiral symmetry and belongs to class AIII, in which the weak topological indices are defined in the same manner. It shows a similar phase diagram as in the previous model in class BDI with $W_{c}^{(2)} = 9.8 \pm 0.1$ and $W_{c}^{(3)} = 9.14 \pm 0.01$ (see Fig. 1 and Table I). Despite the different critical disorder strength $W_{c}^{(2)}$ and $W_{c}^{(3)}$, the critical exponents are the same as those in the models in class BDI, which suggests possible super-universality in 3D systems in the chiral classes with the topological indices.

Models with trivial topological indices—To further clarify the role of the topological indices, we also study a topologically trivial model in class BDI with statistical symmetries. The statistical symmetry of time reversal combined with reflection with respect to the $xz$ or $yz$ plane forces all three topological indices vanish. In addition, LEs of $h$ along any direction come in opposite-sign pairs, and the localization lengths along the $x$ and $y$ directions are the same. On increasing the disorder strength, the model undergoes the Anderson transition, where the normalized localization lengths $\Lambda_{x}$ and $\Lambda_{y}$, along the $x$ and $z$ directions both show scale-invariant behaviors. The critical disorder strengths and critical exponents determined from $\Lambda_{x}$ and $\Lambda_{y}$ are consistent with each other (see Table I), which suggests that the scale-invariant behavior of $\Lambda_{x}$ and $\Lambda_{y}$ comes from the same quantum phase transition. The evaluated critical exponent $\nu = 1.089[1.005, 1.128]$ is different from $\nu$ of the topological model and consistent with Ref. [52]. We also evaluate the critical exponent in the chiral unitary class without weak topological indices as $\nu = 1.024[0.973, 1.070]$ (Table I), which is different from $\nu$ of the topological models in the same symmetry class and is consistent with Refs. [30] [52].

Summary and discussion—In this Letter, we show that in 3D systems in the chiral classes, the weak topological indices induce a disorder-driven quasi-localized phase where wave functions are delocalized only along one direction and localized along the other two directions (Fig. 1). Models with weak topological indices generally have three phases: (i) metal phase delocalized in all directions, (ii) quasi-localized phase found in this Letter,
and (iii) insulator phase localized in all directions. On the other hand, models without topological indices only exhibit the two conventional phases (i) and (iii). We demonstrate that the Anderson transition between the metal and quasi-localized phases is characterized by the critical exponent $\nu = 0.82 \pm 0.02$ while the transition between the quasi-localized and localized phases is characterized by $\nu' = 1$. Both of them are different from the critical exponent $\nu'' = 1.09 \pm 0.03$ of the Anderson transition with no topological indices, even in the same symmetry class and spatial dimensions. We believe that these conclusions hold in 3D systems in the chiral symplectic class (class CII) and leave the verification for future study.

While we focus on the weak topological indices in this Letter, 3D chiral-symmetric systems also host a strong topological index. However, we find that the strong index does not lead to the quasi-localized phases, not influencing the universality classes of the Anderson transitions [53]. It also remains to be explored whether the quasi-localized phase appears and whether the topological indices change the universality classes of the Anderson transitions in 2D systems.

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SUPPLEMENTAL MATERIAL FOR “TOPOLOGICAL ANDERSON TRANSITIONS IN CHIRAL SYMMETRY CLASSES”

This Supplemental Material is organized as follows. In Sec. [S.1], we summarize known critical exponents between metal and topological-insulator phases and between metal and ordinary-insulator phases. In Sec. [S.2], we introduce the inverse participation ratio along different directions and prove that coupling among low-dimensional systems makes wave functions more extended even in the small coupling limit. In Sec. [S.3], we review the polynomial fitting of the finite-size scaling function and details of Table I in the main text. In Sec. [S.4], we introduce the transfer matrix method and explain properties of transfer matrices for chiral-symmetric Hamiltonians. In Sec. [S.5], we summarize a relation between weak topological indices and distributions of Lyapunov exponents (LEs). In Sec. [S.6], we show how statistical symmetries require the weak topological indices to be zero and LEs of the non-Hermitian matrix (right upper part of a chiral-symmetric Hamiltonian) to come in opposite-sign pairs. In Sec. [S.7], we summarize a scaling form for the maximal and minimal LEs within a continuum spectrum and show numerical fittings based on this scaling form. In Secs. [S.8] and [S.9], we provide detailed numerical studies of the criticality in chiral-symmetric models with and without non-trivial topological indices, respectively.

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### S.1. Summary of known critical exponents

Table II summarizes known results of critical exponents of the Anderson transitions between metal and topological-insulator phases and between metal and trivial-insulator phases in the same spatial dimensions and symmetry classes. For each symmetry class and spatial dimensions, the evaluated critical exponents for the two types of the Anderson transitions are consistent with each other. The critical exponents between topological-insulator and trivial-insulator phases, as well as those between topological-semimetal and diffusive-metal phases, can be different from the ones in Table II. In our work, we focus on the Anderson transitions in the 3D chiral classes and demonstrate the different critical exponents due to the weak topological indices.

#### TABLE II. Correlation-length critical exponents of the Anderson transitions between metal and topological-insulator phases and those between metal and trivial-insulator phases in the same symmetry classes and spatial dimensions.

| Class | Metal-topological-insulator | Metal-trivial-insulator |
|-------|-----------------------------|-------------------------|
| 2D AII | 2.74 ± 0.12 | 2.73 ± 0.02 |
| D | 1.371 ± 1.11 | 1.3481 ± 1.302 |
| DIII | 1.5 ± 0.1 | 1.5 ± 0.1 |
| 3D A | 1.34 ± 1.33 | 1.443 ± 1.449 |
| AII | 1.311 ± 0.03 | 1.375 ± 0.014 |
| DIII | 0.85 ± 0.05 | 0.9030896, 0.9081 |

a 95% confidence interval.  
b Layered Chern insulator.  
c Axion insulator.  
d Reference 21 obtained the critical exponent between the metal and topological-insulator phases in a three-dimensional network model belonging to symmetry class AII. The critical exponent is different from the one between the metal and trivial-insulator phases in the same symmetry class and spatial dimensions obtained by the SU(2) model 13. However, a more careful error analysis is needed, because system sizes in Ref. 21 may not be large enough (≤ 10), and the difference between the two exponents is small.

### S.2. Inverse participation ratio

#### 1. Inverse participation ratio along different directions

The inverse participation ratio $P_2$ measures localization properties of a wave function $\Phi(r)$ in $d$ dimension, defined by

$$ P_2 \equiv \sum_r |\Phi(r)|^4 $$

(S.1)

with the normalization condition $\sum_r |\Phi(r)|^2 = 1$. We have $P_2 \leq 1$, where the equality holds only when $\Phi(r)$ is fully localized at one lattice site. $P_2$ in extended and localized phases shows the different scaling relations with the system size $L$,

$$ P_2 \sim \begin{cases} L^{-d} & [\Phi(r) \text{ is extended}], \\ \text{constant} < 1 & [\Phi(r) \text{ is localized}] \end{cases} $$

(S.2)

for $L \to \infty$. The one-dimensional inverse participation ratio $P_2^\mu (\mu = x, y, z)$ measures localization properties of $\Phi(x, y, z)$ along the $\mu$ direction. The integrated weight of the wave function at $z$, $|\phi(z)|^2 = \sum_{x,y} |\Phi(x, y, z)|^2$, is regarded as the squared one-dimensional normalized wave function along the $z$ direction and describes how the three-dimensional wave function $\Phi(x, y, z)$ is localized along the $z$ direction. Thus, the inverse participation ratio $P_2^z$ along the $z$ direction is introduced as

$$ P_2^z = \sum_z |\phi(z)|^4, \quad |\phi(z)|^2 = \sum_{x,y} |\Phi(x, y, z)|^2. $$

(S.3)

Notably, $P_2^z$ provides an upper bound of $P_2$,

$$ P_2 = \sum_z |\phi(z)|^4 \left[ \sum_{x,y} |\Phi(r)|^4 \right] \leq P_2^z, $$

(S.4)
where the equality holds only when we have \( \sum_{x,y} |\Phi(r)/\phi(z)|^4 = 1 \) for all \( z \). The one-dimensional inverse participation ratio \( P_2^{(x,y)} \) along the other two directions (\( \mu = x, y \)) is defined in the same manner. In a similar manner, the \( d \)-dimensional inverse participation ratio is defined for a normalized wave function in \( d' \) dimension (\( d < d' \)). For example, the following two-dimensional inverse participation ratio \( P_2^{(x,y)} \) measures the localization properties of \( \Phi(x, y, z) \) within the \( xy \) plane,

\[
P_2^{(x,y)} \equiv \sum_{x,y} |\phi(x, y)|^4, \quad |\phi(x, y)|^2 = \sum_z |\Phi(x, y, z)|^2,
\]

which satisfies

\[
P_2 \leq P_2^{(x,y)}.
\]

### 2. Wave-function hybridization and inverse participation ratio

Suppose that a \( d' \)-dimensional disordered non-interacting Hamiltonian \( \mathcal{H}^R \) in \( R \equiv (r, s) \) with \( r = (r_1, \ldots, r_d) \) and \( s = (s_1, \ldots, s_{d-d}) \) (\( d < d' \)) consists of \( d \)-dimensional Hamiltonians \( \mathcal{H}^R_s \) at different \( s \) and coupling \( \mathcal{H}' \) among the \( d \)-dimensional systems. On-site disorder potential \( V(R) \) is chosen to distribute uniformly in the range \([-W/2, W/2]\) for all the lattice sites \( R \). Then, \( s \) can be regarded as different disorder realizations from the same ensemble for the \( d \)-dimensional system with the disorder strength \( W \). In this section, we show that even in the small coupling limit, an eigenstate \( \Phi(R) \) of \( \mathcal{H}^R \) with eigenenergy \( E \) is more extended along the \( r \) direction than an eigenstate \( \psi(r) \) of \( \mathcal{H}^R_s \) with the same eigenenergy. Here, the small coupling limit means that the maximal eigenvalue of \( \mathcal{H}' \) is much smaller than the mean level spacing of \( \mathcal{H}^R \) around \( E \).

For \( \mathcal{H}' = 0 \), eigenstates of \( \mathcal{H}^R \) are given by eigenstates of \( \mathcal{H}^R_s \). In the small coupling limit, we can treat \( \mathcal{H}' \) perturbatively. We introduce an energy window \( [E - \Delta E, E + \Delta E] \) and choose \( \Delta E \) to be small enough that each \( \mathcal{H}^R_s \) has at most one eigenstate \( \psi_s(r) \) with eigenenergy \( E_s \) in the energy window and that we have \( |E_s - E| \ll \Delta E \). In the small coupling limit, the maximal eigenvalue of \( \mathcal{H}' \) can be much smaller than \( \Delta E \). Thus, in the lowest order of degenerate perturbation theory, \( \mathcal{H}' \) does not mix unperturbed eigenstates inside the energy window with those outside the energy window, and an eigenstate \( \Phi(R) \) of \( \mathcal{H}^R \) is given by a linear superposition of \( \psi_s(r) \) over different \( s \),

\[
\Phi(r, s) = \sum_{s'} a_{s'} \psi_{s'}(r) \delta_{s, s'} = a_s \psi_s(r),
\]

where \( \delta_{s, s'} \) is the Kronecker delta. Here, we impose the normalization conditions \( \sum_s |a_s|^2 = 1 \) and \( \sum_r |\psi_s(r)|^2 = 1 \), where we sum only over such \( s \) that \( \mathcal{H}^R_s \) has an eigenenergy inside the window \( [E - \Delta E, E + \Delta E] \). \( a_s \) is the \( s \)-component of an eigenstate of an effective Hamiltonian \( \mathcal{H}^R \) given as

\[
(H^R)_{s, s'} = \sum_r \sum_{r'} \psi_s^*(r) (H'_r)_{r, r'} \psi_{s'}(r') + E_s \delta_{s, s'}.
\]

In the following, we show that the inverse participation ratio \( P_2^x \) of \( \Phi(R) \) along the \( r \) direction is always smaller than the \( d \)-dimensional inverse participation ratio of \( \psi_s(r) \). The weight \( |\phi(r)|^2 \) of the wave function \( \Phi(R) \) on a hyperplane \( r = r_0 \) is given as

\[
|\phi(r)|^2 = \sum_s |\Phi(r, s)|^2.
\]

The inverse participation ratio \( P_2^x \) of \( \Phi(R) \) along the \( r \) direction measures the localization properties of \( \Phi(R) \) within the \( r \) direction and is given by the sum of the square of the weight over \( r \),

\[
P_2^x = \sum_r |\phi(r)|^4 = \sum_r \sum_{s_1} \sum_{s_2} |\Phi(r, s_1)|^4 |\Phi(r, s_2)|^2 \sum_{s_1} |a_{s_1}|^2 |a_{s_2}|^2 \left[ \sum_r |\psi_{s_1}(r)|^2 |\psi_{s_2}(r)|^2 \right]
\]

\[
\leq \frac{1}{2} \sum_{s_1} \sum_{s_2} |a_{s_1}|^2 |a_{s_2}|^2 \left[ |\psi_{s_1}(r)|^4 + |\psi_{s_2}(r)|^4 \right] = \sum_s |a_s|^2 \sum_r |\psi_s(r)|^4.
\]

Here, the equality holds true only when we have \( \psi_{s_1}(r) = \psi_{s_2}(r) \) for all \( r, s_1, \) and \( s_2 \). Notably, \( \sum_r |\psi_s(r)|^4 \) is the \( d \)-dimensional inverse participation ratio \( P_2^{(x,y)} \) of \( \psi_s(r) \), and \( \mathcal{H}^R_s \) at different \( s \) belongs to the same ensemble with
the same disorder strength $W$. In the thermodynamic limit ($N_r \equiv \sum_r \to \infty$), $\Delta E$ goes to zero as the mean level spacing goes to zero, and $P_2^{\psi(r)}$ at different $s$ takes the same value $P_2^{\psi(r)}$. Then, we have

$$P_2^r \leq P_2^{\psi(r)} \sum_s |a_s|^2 = P_2^{\psi(r)},$$

which proves that within the lowest order in $\mathcal{H}'$, the small coupling $\mathcal{H}'$ among the $d$-dimensional systems always makes $d$-dimensional wave functions spatially more extended.

### S.3. Polynomial fitting

In this section, we present more details about the polynomial fitting [Eq. (7) in the main text] of the normalized localization length $\Lambda_x(W, L) = \xi_x(W, L) / L$ and show details of Table I in the main text (see Table III). The scaling function for $\Lambda_x(W, L)$ is Taylor-expanded with respect to the relevant scaling variable $\phi(w)$ and the least irrelevant scaling variable $\psi(w)$ up to the $n$th order and first order, respectively,

$$\Lambda_x(W, L) = \sum_{i=0}^{n} \sum_{j=0}^{1} a_{i,j} (\phi(w) L^{1/\nu})^i (\psi(w) L^{-y})^j,$$

(S.12)

with $w \equiv (W - W_c^{(x)}) / W_c^{(x)}$, and the scaling dimension $-y$ ($< 0$) of the least irrelevant scaling variable around a saddle-point fixed point. The relevant scaling variable is further expanded around $w = 0$ up to the $m$th order, while only the zeroth-order in $w$ is kept for the irrelevant scaling variable $\psi(w)$,

$$\phi(w) = \sum_{k=1}^{m} b_k w^k, \quad \psi(w) = c.$$

(S.13)

Here, $\{W_c^{(x)}, \nu, y, a_{i,j}, b_k, c\}$ are the fitting parameters. To avoid the ambiguity in the Taylor expansion of the scaling function, we should set $a_{0,1} = a_{1,0} = 1$. Thus, the number $N_f$ of the free parameters in the fitting is $N_f = 2(n + 1) + m + 2$. We minimize $\chi^2$ statistics

$$\chi^2 = \sum_{j=1}^{N_D} \left( \frac{F_j - \Lambda_j}{\sigma_j} \right)^2,$$

(S.14)

where $\Lambda_j$ and $\sigma_j$ are the normalized localization length and its standard deviation for $(W, L)$ evaluated by the transfer matrix method, respectively, $F_j$ is the value of the polynomial fitting function for $(W, L)$, and $N_D$ is the number of data points. The confidence error bars for the optimal parameters are determined by the fittings for 1000 sets of synthetic data for $\Lambda_x(W, L)$. The synthetic data are generated according to a standard deviation from the transfer matrix calculation.

### S.4. Transfer matrix, Lyapunov exponents, and localization length

The transfer matrix method solves an eigenvalue problem of a non-interacting disordered Hamiltonian $\mathcal{H}$ recursively. This method is efficient for obtaining the localization length along one spatial direction, which we call the $\mu$ direction in the following. In this formulation, the Hamiltonian is decomposed into a layer structure along the $\mu$ direction,

$$\mathcal{H}_{i,j} = H_i \delta_{i,j} + V_{i,i+1} \delta_{i,j-1} + V_{i,i-1} \delta_{i,j+1},$$

(S.15)

where $i, j = 1, 2, \cdots, L_{\mu}$ are indices of the layers, $H_i$ is a block of matrix elements within the $i$th layer, and $V_{i,i\pm1}$ is a block of matrix elements between the $i$th layer and the $(i \pm 1)$th layer. The decomposition assumes that matrix elements appear only between the nearest neighboring layers or within each layer. In the presence of next-nearest hopping, one can redefine two neighboring layers as one layer. Let $H_i$, $V_{i,i\pm1}$ be $m$ by $m$ matrices and $(\cdots, A_{i-1}, A_i, A_{i+1}, \cdots)^T$ be an eigenvector of $\mathcal{H}$ for an eigenenergy $E$:

$$H_i A_i + V_{i,i-1} A_{i-1} + V_{i,i+1} A_{i+1} = E A_i.$$  

(S.16)
TABLE III. Polynomial fitting results of the normalized localization length Λ_µ ≡ ξ_µ/L along the µ direction (µ = x, y, z) around critical points of different models with the quasi-one-dimensional geometry L × L × L. "√" in the column “topology” shows that the weak topological index ν_i is non-zero around the critical point, and "×" shows that all the weak topological indices always vanish around the critical point. We show the critical disorder strength W_µ(ν), critical exponent ν, scaling dimension −y of the least irrelevant scaling variable, critical localization length Λ_c, the goodness of fitting (GOF), and Taylor-expansion order of (m, n) in Eqs. (S.12) and (S.13). The square brackets denote the 95% confidence interval. Note that Λ_c’s here are critical values in the presence of anisotropic spatial geometry and take non-universal values.

| symmetry class | topology | direction | m | n | GOF | W_µ(ν) | ν | y | Λ_c |
|---------------|----------|-----------|---|---|-----|--------|---|---|-----|
| BDI           | √        | µ = x     | 2 | 3 | 0.15 | 27.241[27.194,27.303] | 0.820[0.783,0.846] | 2.584[2.175,2.955] | 0.134[0.130,0.137] |
| BDI           | √        | µ = x     | 3 | 3 | 0.14 | 27.243[27.192,27.301] | 0.820[0.787,0.848] | 2.574[2.212,2.947] | 0.134[0.130,0.138] |
| BDI           | ×        | µ = z     | 2 | 3 | 0.47 | 9.143[9.125,9.168]   | 0.824[0.776,0.862] | 2.157[1.727,2.519] | 0.225[0.213,0.232] |
| BDI           | ×        | µ = z     | 3 | 3 | 0.19 | 23.220[23.167,23.293] | 1.089[1.005,1.128] | 1.926[1.074,2.034] | 0.374[0.352,0.385] |
| BDI           | ×        | µ = x     | 3 | 3 | 0.18 | 23.223[23.138,23.409] | 1.088[0.991,1.141] | 1.906[0.604,3.677] | 0.373[0.302,0.389] |
| BDI           | ×        | µ = x     | 2 | 3 | 0.23 | 23.170[23.098,23.279] | 1.042[0.943,1.099] | 1.591[0.889,2.543] | 0.281[0.254,0.293] |
| BDI           | ×        | µ = z     | 2 | 3 | 0.31 | 23.167[23.101,23.310] | 1.039[0.937,1.100] | 1.607[0.753,2.425] | 0.281[0.239,0.292] |
| AIII          | √        | µ = x     | 2 | 3 | 0.20 | 8.091[8.074,8.096]   | 1.024[0.973,1.070] | 0.470[0.450,1.481] | 0.650[0.639,0.706] |

For simplicity, suppose that the disorder terms are present only in the diagonal matrix elements and that V_i,i−1 = V_+ and V_i,i+1 = V_- are free from disorder. The eigenvectors are solved layer by layer recursively by a transfer matrix M_i,

\[
\begin{pmatrix}
A_{i+1} \\
A_i
\end{pmatrix} = M_i \begin{pmatrix}
A_i \\
A_{i-1}
\end{pmatrix}, \quad M_i = \begin{pmatrix}
-V^{-1}_- (H_i - E) & -V^{-1}_- V_+ \\
1 & 0
\end{pmatrix}.
\]

The product of the transfer matrices, M = M_{L_µ}, M_{L_µ-1} \cdots M_1, relates the components of the eigenvector at the (L_µ + 1)th and L_µth layers with the components at the first and zeroth layer. According to Oseledec’s theorem [54], the matrix

\[
P(E) = \lim_{L_µ \to \infty} \ln (M^L) \to \nu = - \lim_{L_µ \to \infty} \ln (M^{-1}M^{-1})\to \nu
\]

well converges in the limit L_µ → ∞. Eigenvalues of P(E) are known as Lyapunov exponents (LEs). If H is Hermitian, LEs come in opposite-sign pairs [54]. The inverse of the smallest positive or the largest negative LE corresponds to the localization length ξ_µ along the µ direction.

### 1. Transfer matrix of a chiral-symmetric Hamiltonian

Suppose that a 2n × 2n Hermitian Hamiltonian H satisfies chiral symmetry CHC⁻¹ = −H with a chiral operator C satisfying C² = 1. Eigenvalues of C are ±1, the numbers of which are assumed to be the same. Then, the unitary matrix C is diagonalized as C = \sum_{i=1}^{n} |v_i⟩⟨v_i| − \sum_{i=1}^{n} |u_i⟩⟨u_i|. Here, |v_1⟩, |v_2⟩, ..., |v_n⟩ and |u_1⟩, |u_2⟩, ..., |u_n⟩ are eigenvectors of C with eigenvalues +1 and −1, respectively. Because of chiral symmetry, we have ⟨u_i|H|v_j⟩ = ⟨u_i|H|v_j⟩ = 0. Thus, the 2n × 2n matrix H is decomposed into two n × n matrices h and h’ in the off-diagonal parts,

\[
H = \begin{pmatrix}
0 & h \\
h' & 0
\end{pmatrix},
\]

with

\[
(h)_{i,j} = ⟨v_i|H|v_j⟩, \quad (h')_{i,j} = ⟨u_i|H|v_j⟩,
\]

satisfying h' = h⁻¹.

Equation (S.20) does not determine h uniquely up to n × n unitary transformations, h → U⁻¹hU, where the unitary transformations U and U change bases among the n-fold degenerate eigenstates of C. Nonetheless, with a certain choice of the bases for |v_1⟩, ..., |v_n⟩ and |u_1⟩, ..., |u_n⟩, any Hermitian Hamiltonian with chiral symmetry can be decomposed into the off-diagonal form as Eq. (S.19). The off-diagonal parts thus introduced are non-Hermitian matrices, in general.

If the chiral operator C is diagonal with respect to the layer index and its matrix elements do not depend on the layer index, the Hamiltonian H_i within the i-th layer and the hopping matrix V± between the i-th layer and the (i ± 1)th.
layers also take the block off-diagonal structure,
\[ H_i = \begin{pmatrix} 0 & \tilde{h}_i \\ \tilde{h}_i^\dagger & 0 \end{pmatrix}, \quad V_+ = \begin{pmatrix} 0 & v_+ \\ v_+^\dagger & 0 \end{pmatrix}, \quad V_- = \begin{pmatrix} 0 & v_- \\ v_-^\dagger & 0 \end{pmatrix}, \]
for zero energy \( E = 0 \) reads
\[ M_i^{(H)} = \begin{pmatrix} -v_+^{-1} h_i & 0 \\ 0 & -v_-^{-1} h_i \end{pmatrix} = \begin{pmatrix} -v_+^{-1} h_i & 0 \\ 0 & -v_-^{-1} h_i \end{pmatrix}, \] (S.22)
where \( v_\pm \) are free from the disorder and independent of the layer index. The transfer matrix \( M_i^{(H)} \) of \( H \) and chiral symmetry comes in opposite-sign pairs. The LEs of \( h \) are the sum of the LEs of \( h \) and \( h' = h^\dagger \) are decomposed into the layer structure along the \( \mu \) direction,
\[ h_{i,j} = \tilde{h}_i \delta_{i,j} + v_+ \delta_{i,j-1} + v_- \delta_{i,j+1}, \]
\[ h'_{i,j} = \tilde{h}_i^\dagger \delta_{i,j} + v_+^\dagger \delta_{i,j-1} + v_-^\dagger \delta_{i,j+1}, \]
where \( i,j = 1, \ldots, L_\mu \) are the indices of layers. From Eqs. (S.16) and (S.17), we obtain \( M_i \) of \( h \) and \( M'_i \) of \( h' \) as in Eq. (S.23). Note that \( M'_i \) is equivalent to \((M_i^{(H)})^{-1}\) under a certain transformation,
\[ S = \begin{pmatrix} 0 & -v_-^\dagger \\ v_+^\dagger & 0 \end{pmatrix}, \]
\[ SM_i^{(H)}S^{-1} = (M_i^{(H)})^{-1} = \begin{pmatrix} 0 & -v_-^\dagger v_+^{-1} \\ v_+^\dagger & 0 \end{pmatrix}. \] (S.24)
Thus, the LEs obtained by the product of \( SM_i^{(H)}S^{-1} \) have signs opposite to the LEs obtained by the product of \( M_i^{(H)} \). The non-singular similarity transformation \( S \) does not change LEs. Thereby, the LEs of \( h \) and the LEs of \( h' = h^\dagger \) come in opposite-sign pairs. The LEs of \( H \) are the sum of the LEs of \( h \) and the LEs of \( h' \).

2. Transfer matrix of the nodal-line semimetal model

The Hamiltonian of the nodal-line semimetal model reads
\[ \mathcal{H} = \sum_{\mathbf{r}=(r_x,r_y,r_z)} \left\{ (\Delta + \epsilon_r)c_\mathbf{r}^\dagger \sigma_x c_\mathbf{r} + \sum_{\mu=x,y, z} \left( t_{l,\mu} c_\mathbf{r}^\dagger e_\mu \sigma_\mu c_\mathbf{r} + t_{l,\mu}^* c_\mathbf{r} e_\mu^\dagger \sigma_\mu c_\mathbf{r} + H.c. \right) \right\}, \] (S.25)
where \( c_\mathbf{r} \) is a two-component annihilation operator on the cubic lattice site \( \mathbf{r} \), \( \sigma_\mu(\mu = x, y, z) \) are the Pauli matrices, \( \Delta, t_{l,\mu}, t_{l,\mu}^* \) are real-valued parameters, and \( \epsilon_r \) is a random potential that distributes uniformly in \([-W/2,W/2]\), and \( e_x = (1,0,0), e_y = (0,1,0), \) and \( e_z = (0,0,1) \) are the unit vectors. Note that the Hamiltonian in Eq. (S.25) reduces to Eq. (5) in the main text for \( \Delta = 0 \). Depending on \( \Delta \) and the other parameters, Eq. (S.25) describes an ordinary insulator, topological insulator, and nodal-line semimetal [see Eq. (S.50)]. \( \mathcal{H} \) satisfies time-reversal symmetry \( \mathcal{H} = \mathcal{H}^* \) and chiral symmetry \( \mathcal{H} = -C^\dagger \mathcal{H}^\dagger C \) with a unitary operator \( C_{\mathbf{r},\mathbf{r}'} = \delta_{\mathbf{r},\mathbf{r}'} \sigma_z \) with \( C^* = C \), and thus belongs to the chiral orthogonal class (class BDI).

The chiral operator \( C \) has eigenvalues +1 and −1. Since \( C \) is diagonal with respect to the lattice site, eigenvectors of \( C \) can be labelled by the cubic-lattice site \( s \equiv (s_x,s_y,s_z) \):
\[ \langle \mathbf{r}|v_s \rangle = \delta_{\mathbf{r},\mathbf{s}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \langle \mathbf{r}|u_s \rangle = \delta_{\mathbf{r},\mathbf{s}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \] (S.26)
satisfying $\mathcal{C}|v_s\rangle = |v_s\rangle$ and $\mathcal{C}|u_s\rangle = -|u_s\rangle$. Following Eq. (S.20), we construct the right-upper part $h$ of $\mathcal{H}$ on the same cubic lattice as
\begin{equation}
(h)_{s+s} = \langle v_s | \mathcal{H} | u_s \rangle = \Delta + \varepsilon_s, \quad (S.27)
\end{equation}
\begin{equation}
(h)_{s+e_s} = \langle v_s | \mathcal{H} | u_s \rangle = \langle v_s + e_s | \mathcal{H} | u_s \rangle = \langle v_s | \mathcal{H} | u_s \rangle = t_\perp, \quad (S.28)
\end{equation}
\begin{equation}
(h)_{s+e_s} = \langle v_s + e_s | \mathcal{H} | u_s \rangle = t_{\parallel} + t'_\parallel, \quad (S.29)
\end{equation}
\begin{equation}
(h)_{s+s} = \langle v_s | \mathcal{H} | u_s \rangle = -t_{\parallel} + t'_\parallel, \quad (S.30)
\end{equation}
for $\mu = x, y$. All the other matrix elements of $h$ are zero. Notably, $h$ can be regarded as a single-orbital tight-binding model,
\begin{equation}
\begin{array}{c}
h = \sum_{r = (x, y, z)} \left[ (\Delta + \varepsilon_r) f_r^\dagger f_r + \sum_{\mu = x, y} \left( t_{\parallel} f_{r+e_{\mu}} f_r + \text{H.c.} \right) + (t'_{\parallel} + t_{\parallel}) f_{r+e_{\mu}} f_{r} + (t'_{\parallel} - t_{\parallel}) f_{r} f_{r+e_{\mu}} \right] \end{array}, \quad (S.31)
\end{equation}
where $f_r$ and $f_r^\dagger$ are annihilation and creation operators at site $r$. While $h$ respects $h = h^*$, we have $h \neq h^\dagger$ for $t_{\parallel} \neq 0$. Hence, $h$ generally belongs to the non-Hermitian symmetry class AI [30, 55].

The transfer matrix of $h$ along the $z$ direction is given by
\begin{equation}
M_i = \begin{pmatrix}
0_{m \times m} & 1_{m \times m} \\
-\frac{1}{t_{\parallel}} \tilde{h_i} & -\frac{t'_{\parallel} + t_{\parallel}}{t_{\parallel} + t'_L} 1_{m \times m} \\
0_{m \times m} & 0_{m \times m} \\
\end{pmatrix}, \quad (S.32)
\end{equation}
where $m = L^2$ is the degrees of freedom in each layer and the quasi-1D geometry ($L \times L \times L_z$, $L_z \gg L$) is considered. $\tilde{h_i}$ is the Hamiltonian within the $i$th layer, which has $\Delta + \varepsilon_s$ in its diagonal elements and $t_{\parallel}$ in its nearest-neighbor hopping. For $t'_{\parallel} - t_{\parallel} = 0$, $M_i$ is singular, and $m$ eigenvalues of $\frac{1}{L_z} \ln M \equiv \frac{1}{L_z} \ln(M_L, M_{L_z-1} \cdots M_1)$ diverge to $\infty$. In fact, $M_i^{-1}$ has zero eigenvalues with multiplicity $m$ for $t'_{\parallel} = t_{\parallel}$,
\begin{equation}
M_i^{-1} = \begin{pmatrix}
0_{m \times m} & 1_{m \times m} \\
\frac{1}{t'_{\parallel} + t_{\parallel}} \tilde{h_i} & -\frac{t'_{\parallel} + t_{\parallel}}{t_{\parallel} + t'_L} \tilde{h_i} \\
0_{m \times m} & 0_{m \times m} \\
\end{pmatrix} \rightarrow \begin{pmatrix}
0_{m \times m} & 1_{m \times m} \\
-\frac{1}{t_{\parallel} + t'} \tilde{h_i} & -\frac{1}{t_{\parallel} + t'_L} \tilde{h_i} \\
0_{m \times m} & 0_{m \times m} \\
\end{pmatrix}, \quad (S.33)
\end{equation}
where $M^{-1} = M_1^{-1} \cdots M_{L_z-1}^{-1} M_{L_z}^{-1}$ has zero eigenvalue with multiplicity at least $m$. Therefore, $m$ eigenvalues of $\frac{1}{L_z} \ln(M^{-1})^\dagger M^{-1}$ always diverge to $-\infty$, while $m$ eigenvalues of $\frac{1}{L_z} \ln M^\dagger M$ always diverge to $+\infty$. The other $m$ finite-valued LEs of $h$ are determined from the following product:
\begin{equation}
p = -\lim_{L_z \to \infty} \frac{1}{2L_z} \ln \left( \frac{1}{t'_{\parallel} + t_{\parallel}} \tilde{h_i} \cdots \frac{1}{t'_{\parallel} + t_{\parallel}} \tilde{h_i} L_z \tilde{h_i} \cdots \frac{1}{t'_{\parallel} + t_{\parallel}} \tilde{h_i} L_z \tilde{h_i} \cdots \frac{1}{t'_{\parallel} + t_{\parallel}} \tilde{h_i} \right), \quad (S.34)
\end{equation}

\begin{equation}
S.5. \text{ Weak topological indices and Lyapunov exponents}
\end{equation}

We summarize a relationship between the weak topological indices of chiral-symmetric Hamiltonians and the numbers of positive and negative LEs of its right-upper part $h$ in the canonical basis in Eq. (S.19) [30]. Consider a chiral-symmetric Hamiltonian $\mathcal{H}(\phi)$, in which a magnetic flux $\phi$ is inserted through a closed loop along the $\mu$ direction. Similarly to $\mathcal{H}$ in Eq. (S.19), $\mathcal{H}(\phi)$ takes a block off-diagonal structure in a basis where the chiral operator is diagonal,
\begin{equation}
\mathcal{H}(\phi) = \begin{pmatrix}
0 & h(\phi) \\
h(\phi)^\dagger & 0 \\
\end{pmatrix}. \quad (S.35)
\end{equation}

The right-upper block $h(\phi)$ is decomposed into a layer structure along the $\mu$ direction,
\begin{equation}
h(\phi) = \begin{pmatrix}
\tilde{h}_1 & v_- & 0 & \cdots & 0 & \frac{1}{t_{\parallel}} v_+ \\
v_+ & \tilde{h}_2 & v_- & \cdots & 0 \\
0 & v_+ & \tilde{h}_3 & v_- & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
v_- & 0 & \cdots & 0 & v_+ & \tilde{h}_{L_u} \\
\end{pmatrix}, \quad (S.36)
\end{equation}
with $z = e^{i\phi}$. We assume that the hopping appears only between the nearest neighboring layers or within each layer. In the presence of next-nearest neighbor hopping, we can redefine two neighboring layers as one layer. $v_-, v_+$, and $h_i(i = 1, 2, \cdots, L_\mu)$ are $m \times m$ matrices, where $m$ is the degrees of freedom of $h$ in each layer. The 1D winding number $w_\mu$ along the $\mu$ direction is defined in terms of $h(\phi)$,

$$w_\mu \equiv i \int_0^{2\pi} \frac{d\phi}{2\pi} \partial_\phi \text{Tr} [\log [h(\phi)]] .$$  \hfill (S.37)

The winding number $w_\mu$ is given by the contour integral

$$w_\mu = i \oint_{|z|=1} \frac{dz}{2\pi} \partial_z \text{Tr} [\log [h(z)]]$$

$$= m + i \oint_{|z|=1} \frac{dz}{2\pi} \partial_z \log [z^m \det [h(z)]] .$$  \hfill (S.38)

Here, $\det [h(z)]$ is a polynomial function in terms of $z$ with the lowest order $z^{-m}$ and the highest order $z^m$ if we have $\det v_- \neq 0$ and $\det v_+ \neq 0$. For simplicity, we assume $\det v_- \neq 0$ and $\det v_+ \neq 0$ while the following argument can be generalized to other cases. Then, $z^m \det [h(z)]$ is an analytic function of $z$, and $w_\mu$ is related to the number of zeros of $z^m \det [h(z)]$ within the circle $|z| = 1$,

$$w_\mu = i \oint_{|z|=1} \frac{dz}{2\pi} \partial_z \log [z^{-m} \det [h(z)]]$$

$$= m + i \oint_{|z|=1} \frac{dz}{2\pi} \partial_z \log [z^m \det [h(z)]]$$

$$= m - Z,$$  \hfill (S.39)

where $Z$ is the weighted number of the zeros, and the residue theorem is used in the last equality. $z = 0$ should not be a zero of $z^m \det [h(z)]$, since the lowest order of $\det [h(z)]$ is $z^{-m}$. Thus, $Z$ is equal to the weighted number of the zeros of $\det [h(z)]$ in the disk $|z| < 1$.

The number of the zeros of $\det [h(z)]$ is determined by the LEs of $h$ along the $\mu$ direction. For $\det [h(z)] = 0$, $h(z)$ has a zero mode. The presence of the zero modes is given by the transfer matrices of $h(z)$ along the $\mu$ direction. The transfer matrices for each layer are given by,

$$M_i(z) = \begin{cases} 
-\begin{pmatrix} v_1^{-1} & -\frac{1}{z} v_1^{-1} v_+ \\ 1 & 0 \end{pmatrix} & (i = 1), \\
-\begin{pmatrix} v_1^{-1} & -v_1^{-1} v_+ \\ 1 & 0 \end{pmatrix} = M_i & (i = 2, 3, \cdots, L_\mu - 1), \\
-\begin{pmatrix} \frac{1}{z} v_1^{-1} h_{L_\mu} & -\frac{1}{z} v_1^{-1} v_+ \\ 1 & 0 \end{pmatrix} & (i = L_\mu).
\end{cases}$$  \hfill (S.40)

Here, $M_1(z)$ and $M_{L_\mu}(z)$ satisfy

$$M_1(z)M_{L_\mu}(z) = \begin{pmatrix} \frac{1}{z} v_1^{-1} h_1 v_1^{-1} h_{L_\mu} - \frac{1}{z} v_1^{-1} v_+ & \frac{1}{z} v_1^{-1} H_1 v_1^{-1} v_+ \\
-\frac{1}{z} v_1^{-1} h_{L_\mu} & -\frac{1}{z} v_1^{-1} v_+ \end{pmatrix} = \frac{1}{z} M_1(z = 1)M_{L_\mu}(z = 1) .$$  \hfill (S.41)

Suppose that $(A_1, A_2, \cdots, A_{L_\mu})^T$ is a zero mode of $h(z)$ under the periodic boundary conditions. Then, we have

$$\begin{pmatrix} A_2 \\ A_1 \end{pmatrix} = M_1(z) \begin{pmatrix} A_1 \\ A_{L_\mu} \end{pmatrix} , \quad \begin{pmatrix} A_3 \\ A_2 \end{pmatrix} = M_2 \begin{pmatrix} A_2 \\ A_{L_\mu} \end{pmatrix} , \quad \cdots , \quad \begin{pmatrix} A_1 \\ A_{L_\mu} \end{pmatrix} = M_{L_\mu}(z) \begin{pmatrix} A_{L_\mu} \\ A_{L_\mu-1} \end{pmatrix} ,$$  \hfill (S.42)

and hence

$$\begin{pmatrix} A_{L_\mu} \\ A_{L_\mu-1} \end{pmatrix} = M_{L_\mu-1} \cdots M_2 M_1(z) M_{L_\mu}(z) \begin{pmatrix} A_{L_\mu} \\ A_{L_\mu-1} \end{pmatrix} \equiv \frac{1}{z} M \begin{pmatrix} A_{L_\mu} \\ A_{L_\mu-1} \end{pmatrix} ,$$  \hfill (S.43)
where $M = M_{L_0,1} \cdots M_2 M_1 (z = 1) M_{L_0} (z = 1)$ is the product of the transfer matrices without the magnetic flux ($\phi = 0, z = 1$). Since $M_1 (z = 1)$ and $M_{L_0} (z = 1)$ are statistically equivalent to $M_{L_0,1}, M_{L_0,2}, \cdots, M_3, \text{ and } M_2$, the eigenvalues of $\frac{1}{L_\mu} \ln M$ in the limit $L_\mu \to \infty$ are characterized by the LEs of $h$.

If $h(z)$ has a zero mode for a complex value $z$, $M$ has an eigenvalue of $z$ from Eq. (S.43), and vice versa. Thus, the number $Z$ of the zeros of $\det [h(z)]$ within the disk $|z| < 1$ is equivalent to the number of eigenvalues of $M$ whose absolute values are smaller than 1. The product $M$ of $L_\mu$ random matrices has eigenvalues $e^{\alpha_j} + i \beta_j$ ($j = 1, 2, \cdots, 2m; \alpha_j, \beta_j \in \mathbb{R}$), and generally, $\alpha_j$ grows linearly with $L_\mu$, satisfying

$$\gamma_j = \lim_{L_\mu \to \infty} \frac{\alpha_j}{L_\mu},$$  

(S.44)

with the LE $\gamma_j$ of $M \equiv [40, 54, 56]$. Thus, $Z$ is also the same as the number $N_-$ of the negative LEs of $h$:

$$Z = N_-.$$  

(S.45)

In terms of Eq. (S.39), the 1D winding number is given by

$$w_\mu = m - N_- = \frac{1}{2} (N_+ - N_-),$$  

(S.46)

where $N_+$ and $N_-$ are the numbers of positive and negative LEs, satisfying $N_+ + N_- = 2m$. The weak topological index along the $\mu$ direction is the 1D winding number normalized by the degrees of freedom of $h$ in each layer,

$$\nu_\mu = \frac{1}{m} w_\mu = \frac{1}{2m} (N_+ - N_-).$$  

(S.47)

Notably, if a LE is exactly zero, the localization length $\xi_\mu$ along the $\mu$ direction diverges and the winding number $w_\mu$ is ill defined. In the quasi-1D geometry of a 3D disordered Hamiltonian ($L \times L \times L_\mu, L_\mu \gg L$), $L^2$ LEs are distributed within a finite range and form a continuous spectrum in the thermodynamic limit $L \to \infty$. When the spectrum crosses zero with changing $W$, $w_\mu$ also changes from an integer to another integer, $\nu_\mu$ continuously changes with $W$, and the localization length always diverges.

1. **Winding number in the clean limit**

When a $d$-dimensional system has translation invariance in all the $d$-dimensional coordinates, Eq. (S.37) reduces to

$$w_\mu = i \int_0^{2\pi} \frac{dk_\mu}{2\pi} \partial_{k_\mu} \text{Tr} [\log [h(k)]] ,$$  

(S.48)

with the momentum $k \equiv (k_1, k_2, \cdots, k_{d-1}, k_d)$ and the Bloch Hamiltonian $h(k)$. The trace includes the sum over momenta along the directions complementary to the $\mu$ direction. For example, the Bloch Hamiltonian for the 3D nodal-line semimetal model is given by the two-by-two matrix

$$\mathcal{H}(k) = [\Delta + 2t_{\perp} (\cos k_x + \cos k_y) + 2t_{\parallel} \cos k_z] \sigma_z - 2t_{\parallel} \sin k_z \sigma_y.$$  

(S.49)

In the canonical basis where the chiral operator is diagonal, the matrix takes the block off-diagonal structure with

$$h(k) = 2it_{\parallel} \sin k_z - [\Delta + 2t_{\perp} (\cos k_x + \cos k_y) + 2t_{\parallel} \cos k_z].$$  

(S.50)

The complex number $h(k)$ winds around zero when $k_z$ changes from 0 to $2\pi$. For $|\Delta + 2t_{\perp} (\cos k_x + \cos k_y)| > 2|t_{\parallel}|$, $\mathcal{H}$ has an energy gap around $E = 0$ and $h(k)$ winds around zero clockwise for all $k_x$ and $k_y$ in the first Brillouin zone, leading to $w_z = L^2$ and $\nu_z = 1$. Here, $L^2$ is the system size within the $xy$ plane. For $|\Delta + 2t_{\perp} (\cos k_x + \cos k_y)| < 2|t_{\parallel}|$, $\mathcal{H}$ has a gap at $E = 0$, but $h(k)$ does not wind around zero for any $k_x$ and $k_y$, leading to $w_z = \nu_z = 0$. For $2|t_{\parallel}| - 4t_{\perp} < \Delta < 2|t_{\parallel}| + 4t_{\perp}$, zero modes of $\mathcal{H}$ form a nodal ring in momentum space, and the winding number is $+1$ (0) for the wave numbers $k_x, k_y$ inside (outside) the nodal ring, leading to $0 < \nu_z < 1$. 
S.6. Statistical symmetry

An ensemble of disordered Hamiltonians, as a whole, can be invariant under a symmetry operation even if each disorder realization in the ensemble breaks the symmetry. Such symmetry of the ensemble is dubbed statistical symmetry [57]. Statistical symmetry does not influence the symmetry class of the Hamiltonians since it is not a symmetry of each disordered Hamiltonian. An example of statistical symmetry is translation symmetry in 3D weak topological insulators with disorder [58].

Statistical symmetry can make the ensemble averages of physical observables or topological indices be zero. A prime example is zero Hall conductance due to statistical time-reversal symmetry. Suppose that \( H_\alpha \) is a time-reversal-breaking Hamiltonian in an ensemble with statistical time-reversal symmetry and has a finite Hall conductance \( \sigma_{xy} \). A time-reversed counterpart \( H_\alpha \) of \( H_\alpha \) exists in the same ensemble and has the opposite value \( -\sigma_{xy} \) of the Hall conductance. Such an ensemble has zero Hall conductance on average,

\[
\langle \sigma_{xy} \rangle = \frac{1}{2N_{\text{sample}}} \sum_\alpha (\sigma_{xy}^\alpha - \sigma_{xy}^\alpha) = 0.
\]  

(51)

1. Statistical symmetry, Lyapunov exponents, and one-dimensional winding number

In Sec. S.5, the 1D winding numbers and weak topological indices are defined in terms of the right-upper part \( h \) of the chiral-symmetric Hamiltonian \( \hat{H} \) in the canonical basis [see Eq. (S.19)]. Now, we introduce statistical symmetry of \( h \) and show that it requires LEs of \( h \) to come in opposite-sign pairs. Statistical symmetry also makes the 1D winding numbers and weak topological indices be zero as a whole. Suppose that an ensemble of \( h \) with different disorder realizations, \( \{ h | \epsilon_v \in [-W/2, W/2] \} \), is symmetric under transposition of \( h \) together with a certain unitary transformation \( \mathcal{U} \):

\[
\{ h | \epsilon_v \in [-W/2, W/2] \} = \{ h' | \epsilon_v \in [-W/2, W/2] \}, \quad \text{with} \quad h' \equiv \mathcal{U}h^T\mathcal{U}^T. 
\]  

(52)

Here, we assume that the unitary transformation \( \mathcal{U} \) is diagonal in a spatial coordinate \( r_\mu \) and is independent of \( r_\mu \) while it can be non-diagonal in the other coordinates \( s \),

\[
(\mathcal{U})_{r,r'} = \delta_{r_\mu,r'_\mu} (u)_{s,s'},
\]  

(53)

with \( r \equiv (s, r_\mu) \) and \( r' \equiv (s', r'_\mu) \). Then, LEs of \( h \) along the \( \mu \) direction come in opposite-sign pairs. The 1D winding number and weak topological index of \( h \) along the \( \mu \) direction vanish from Eqs. (S.46) and (S.47).

To see this, we decompose \( h \) into a layer structure along the \( \mu \) direction,

\[
(h)_{i,j} = \tilde{h}_i \delta_{i,j} + v_+ \delta_{i,j-1} + v_- \delta_{i,j+1},
\]  

(54)

with \( i,j = 1, \cdots, L_\mu \). \( \tilde{h}_i \) is a block of matrix elements of \( h \) within the \( i \)th layer, and \( v_\pm \) is a block of matrix elements of \( h \) between the \( i \)th layer and \((i \mp 1)\)th layer. \( v_\pm \) are free from disorder and independent of the layer index. If the degree of freedom in each layer of \( h \) is \( m \), \( \tilde{h}_i \) and \( v_\pm \) are \( m \times m \) matrices. Similarly, \( h' \equiv \mathcal{U}h^T\mathcal{U}^T \) is also decomposed into a layer structure along the \( \mu \) direction,

\[
(h')_{i,j} = u_+ \tilde{h}_i^T u_+^\dagger \delta_{i,j} + v_+^T u_+^\dagger \delta_{i,j-1} + v_-^T u_+^\dagger \delta_{i,j+1}.
\]  

(55)

From Eqs. (S.16) and (S.17), the transfer matrices of \( h \) and \( h' \) are obtained as

\[
M_i = \begin{pmatrix}
-v_+^{-1} \tilde{h}_i & -v_-^{-1} \tilde{h}_i \\
1 & 0
\end{pmatrix}, \quad M_i' = \begin{pmatrix}
-u(v_+^T)^{-1} \tilde{h}_i u_+ & -u(v_+^T)^{-1} v_+^T u_+ \\
1 & 0
\end{pmatrix}.
\]  

(56)

The two matrices are related to each other by the following symmetry,

\[
S M_i'^T S^{-1} = M_i^{-1},
\]  

(57)

with

\[
S \equiv \begin{pmatrix}
0 & -(v_-)^{-1} u^T \\
(v_+)^{-1} u^T & 0
\end{pmatrix}.
\]  

(58)
Since \( a \) and \( v_\pm \) in \( S \) are independent of the layer index, the same symmetry holds between the products of the transfer matrices, \( M = M_{L_\mu} \cdots M_1 \) and \( M' = M'_{L_\mu} \cdots M'_1 \),

\[
SM'T^{-1} = M^{-1}.
\]  

(S.59)

Since a non-singular similarity transformation \( S \) does not change LEs, eigenvalues of \( P'(0) \equiv \lim_{L_\mu \to \infty} \frac{1}{2L_\mu} \ln(M'M) \) are opposite to eigenvalues of \( P(0) \equiv \lim_{L_\mu \to \infty} \frac{1}{2L_\mu} \ln(M'M) \). Since \( h \) and \( h' \) are in the same ensemble, \( M_i \) and \( M'_j (i, j = 1, \cdots, L_\mu) \) are random matrices with the same probability distribution. Thus, according to Oseledec’s theorem [54], the eigenvalues of \( P(0) \) and \( P'(0) \) converge to the same values in the limit \( L_\mu \to \infty \). Then, the eigenvalues of \( P'(0) \), as well as the LEs of \( h \), must come in opposite-sign pairs in the limit \( L_\mu \to \infty \). Because of \( N_+ = N_- \) for the LEs along the \( \mu \) direction, \( \nu_\mu \) and \( w_\mu \) of \( h \) vanish from Eqs. (S.46) and (S.47).

As discussed in Sec. S.4, the LEs of \( h \) and the LEs of \( h^\dagger \) generally come in opposite-sign pairs. Thus, statistical Hermitian-conjugation symmetry of \( h \) also requires the LEs of \( h \) to come in the opposite-sign pairs. Suppose that an ensemble of \( h \) with different disorder realizations, \( \{ h | \epsilon_\mu \in [-W/2, W/2] \} \), is symmetric under Hermitian conjugation together with a certain unitary transformation \( U \) defined in Eq. (S.53):

\[
\begin{bmatrix} h \mid \epsilon_r \in [-W/2, W/2] \end{bmatrix} = \begin{bmatrix} h' \mid \epsilon_r \in [-W/2, W/2] \end{bmatrix}, \quad \text{with} \quad h' \equiv UhU^\dagger.
\]  

(S.60)

Then, the LEs of \( h \) along the \( \mu \) direction should come in opposite-sign pairs, leading to \( \nu_\mu = w_\mu = 0 \).

It is also notable that statistical symmetry of \( h \) leads to statistical symmetry of \( \mathcal{H} \). If an ensemble of \( h \) is invariant under the combination of transposition and a unitary operation in Eq. (S.52), the corresponding ensemble of \( \mathcal{H} \) is invariant under the combination of time reversal and a unitary operation:

\[
\begin{align*}
\{ \mathcal{H} = \begin{pmatrix} 0 & h^\dagger \\ h & 0 \end{pmatrix} \mid \epsilon_r \in [-W/2, W/2] \} = \{ \mathcal{H}' = \begin{pmatrix} 0 & h' \dagger \\ h' & 0 \end{pmatrix} \mid \epsilon_r \in [-W/2, W/2] \}, \\
\mathcal{H}' \equiv \begin{pmatrix} 0 & U \mid 0 \\ U^\dagger & 0 \end{pmatrix} \mathcal{H} \begin{pmatrix} 0 & U \mid 0 \\ U^\dagger & 0 \end{pmatrix}.
\end{align*}
\]  

(S.61)

If an ensemble of \( h \) is invariant under the combination of Hermitian conjugation and a unitary transformation in Eq. (S.60), the corresponding ensemble of \( \mathcal{H} \) is invariant under the following unitary operation:

\[
\begin{align*}
\{ \mathcal{H} = \begin{pmatrix} 0 & h^\dagger \\ h & 0 \end{pmatrix} \mid \epsilon_r \in [-W/2, W/2] \} = \{ \mathcal{H}' = \begin{pmatrix} 0 & h' \dagger \\ h' & 0 \end{pmatrix} \mid \epsilon_r \in [-W/2, W/2] \}, \\
\mathcal{H}' \equiv \begin{pmatrix} 0 & U \mid 0 \\ U^\dagger & 0 \end{pmatrix} \mathcal{H} \begin{pmatrix} 0 & U \mid 0 \\ U^\dagger & 0 \end{pmatrix}.
\end{align*}
\]  

(S.62)

2. Statistical symmetry in the nodal-line semimetal model

We show that the 3D nodal-line semimetal model has statistical transposition symmetries for the \( \mu = x, y \) directions. The nodal-line semimetal model takes a block off-diagonal structure in the canonical basis, where the upper-right part \( h \) is given by Eq. (S.31). Transposition of \( h \) exchanges \( t'_\parallel + t_\parallel \) and \( t'_\parallel - t_\parallel \) in Eq. (S.31). Since the disorder potential \( \epsilon_r \) is statistically equivalent for different lattice points \( r \), we can introduce a mirror operation with respect to the \( xy \) plane as a unitary transformation of Eqs. (S.52) and (S.53),

\[
U(t_{x, y, r_x, r_y} r'_x, r'_y, r'_z) = \delta_{r_x, r'_x} \delta_{r_y, r'_y} \delta_{r_z, r'_z}.
\]  

(S.63)

This mirror operation exchanges \( t'_\parallel + t_\parallel \) and \( t'_\parallel - t_\parallel \) as well as \( \epsilon_{r_x, r_y, r_z} \) and \( \epsilon_{r_x, r_y, -r_z} \) while \( \epsilon_r \) is statistically equivalent for different \( r \). Thus, an ensemble for \( h \) defined by Eq. (S.31) is statistically invariant under transposition with the unitary transformation. The symmetry of Eqs. (S.52) and (S.63) requires the LEs of \( h \) along the \( x \) (\( y \)) direction to come in opposite-sign pairs, leading to \( w_{x(y)} = \nu_{x(y)} = 0 \).

S.7. Finite-size scaling of Lyapunov exponents

The LEs of a chiral-symmetric Hamiltonian \( \mathcal{H} \) are the sum of LEs of the right-upper part \( h \) of \( \mathcal{H} \) in the canonical basis and their opposite-sign exponents [see Eq. (S.19)]. In the quasi-one-dimensional (quasi-1D) geometry, the LEs of \( \mathcal{H} \), as well as the LEs of \( h \), comprise continuum spectra for the limit \( L \to \infty \) [35]. In the nodal-line semimetal model \( \mathcal{H} \) with \( t_\parallel = t'_\parallel = 0 \), \( m = L^2 \) LEs of diverge to \( +\infty \), and the other \( m \) LEs of \( h \) form a finite spectrum around \( \gamma = 0 \).
Thus, the LEs of $h$ (see also Fig. 2 in the main text). For $t_{ij} \neq t'_{ij}$, on the other hand, all the $2m = 2L^2$ LEs of $h$ comprise either one or two continuum spectra around $\gamma = 0$, depending on the disorder strength (see Figs. 6(c) and 6(d)).

In the following discussion, we focus on the transfer matrix study of the non-Hermitian Hamiltonian $h$ in the quasi-1D geometry gives a discrete set of $2m$ LEs,

$$\{\gamma^{(1)}_{\text{min}}(W, L), \ldots, \gamma^{(1)}_{\text{max}}(W, L), \gamma^{(2)}_{\text{min}}(W, L), \ldots, \gamma^{(2)}_{\text{max}}(W, L)\},$$  \hspace{1cm} (S.64)

with $\gamma^{(1)}_{\text{min}}(W, L) < \ldots < \gamma^{(1)}_{\text{max}}(W, L) < \gamma^{(2)}_{\text{min}}(W, L) < \ldots < \gamma^{(2)}_{\text{max}}(W, L)$. In the limit $L \to \infty$, all the $m$ LEs from $\gamma^{(j)}_{\text{min}}(W, L)$ to $\gamma^{(j)}_{\text{max}}(W, L)$ ($j = 1, 2$) form a continuum spectrum that ranges from $\gamma^{(j)}_{\text{min}}(W)$ to $\gamma^{(j)}_{\text{max}}(W)$, satisfying

$$\lim_{L \to \infty} \gamma^{(j)}_{\text{min}}(W, L) \equiv \gamma^{(j)}_{\text{min}}(W), \quad \lim_{L \to \infty} \gamma^{(j)}_{\text{max}}(W, L) \equiv \gamma^{(j)}_{\text{max}}(W),$$  \hspace{1cm} (S.65)

with $j = 1, 2$. For some disorder strength, a finite gap $2\Delta \equiv \gamma^{(2)}_{\text{min}}(W) - \gamma^{(1)}_{\text{max}}(W)$ exists between the two continuum LEs spectra (see Figs. 6(c) and 6(d)).

When the gap $2\Delta$ is much larger than $L^{-1}$, $\gamma^{(1)}_{\text{max}}(W, L)$ and $\gamma^{(2)}_{\text{min}}(W, L)$ can be fitted well by the following scaling functions:

$$\gamma^{(1)}_{\text{max}}(W, L) = -\frac{a}{L} + \gamma^{(1)}_{\text{max}}(W),$$  \hspace{1cm} (S.66)

and

$$\gamma^{(2)}_{\text{min}}(W, L) = \frac{a}{L} + \gamma^{(2)}_{\text{min}}(W),$$  \hspace{1cm} (S.67)

Notably, this scaling holds irrespective of whether their limits $\gamma^{(2)}_{\text{min}}(W)$ and $\gamma^{(1)}_{\text{max}}(W)$ are close to zero. To see this, we first note that LEs of $h$ with different $t_{ij}$ and $t'_{ij}$ are related by an imaginary gauge transformation along the $z$ direction [47]. Let the imaginary gauge transformation with an imaginary gauge $ig$ act on $h$ by

$$h \to h_g \equiv V_g h V_g^{-1},$$  \hspace{1cm} (S.68)

where $V_g$ is a diagonal matrix whose diagonal element takes $e^{ig}$ in the $j$th layer along the $z$ direction:

$$V_g = \begin{pmatrix} e^{ig}1_{m \times m} & 0 & 0 & \cdots & 0 \\ 0 & e^{2g}1_{m \times m} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & e^{Lzg}1_{m \times m} \end{pmatrix}.$$  \hspace{1cm} (S.69)

The transfer matrix $M_i(g)$ of $h_g \equiv V_g h V_g^{-1}$ along the $z$ direction is obtained from Eq. (S.32) as

$$M_i(g) = -e^{g} \begin{pmatrix} -\frac{1}{t'_{ij} - t_{ij}} \tilde{h}_i & e^{2g} \frac{t'_{ij} + t_{ij}}{t'_{ij} - t_{ij}} 1_{m \times m} \\ 1_{m \times m} & 0_{m \times m} \end{pmatrix},$$  \hspace{1cm} (S.70)

and satisfies

$$M_i(g) = e^{gS} \begin{pmatrix} -\frac{1}{t'_{ij} - t_{ij}} \tilde{h}_i & e^{2g} \frac{t'_{ij} + t_{ij}}{t'_{ij} - t_{ij}} 1_{m \times m} \\ 1_{m \times m} & 0_{m \times m} \end{pmatrix} S^{-1} = e^{gS} M_S S^{-1}, \quad S = \begin{pmatrix} 1_{m \times m} & 0_{m \times m} \\ 0_{m \times m} & e^{-g} 1_{m \times m} \end{pmatrix}.$$  \hspace{1cm} (S.71)

Thus, the LEs of $h_g$ are obtained from the LEs of $h$ in Eq. (S.64),

$$\{\gamma^{(1)}_{\text{min}}(W, L) + g, \ldots, \gamma^{(1)}_{\text{max}}(W, L) + g, \gamma^{(2)}_{\text{min}}(W, L) + g, \ldots, \gamma^{(2)}_{\text{max}}(W, L) + g\}.$$  \hspace{1cm} (S.72)

Similarly, the LEs of $h_{g}^\dagger$ differ from the LEs of $h^\dagger$ by $-g$. Suppose that the gap between $\gamma^{(1)}_{\text{max}}(W)$ and $\gamma^{(2)}_{\text{min}}(W)$ is much larger than $L^{-1}$, and choose the imaginary gauge $g$ in such a way that a midpoint of the gap comes around zero,

$$\gamma^{(1)}_{\text{max}}(W) + g < 0 < \gamma^{(2)}_{\text{min}}(W) + g.$$  \hspace{1cm} (S.73)
Then, a zero mode of a chiral-symmetric Hamiltonian $H_g$ that has $h_g$ and $h_g^\dagger$ in the off-diagonal blocks is in the Anderson insulator phase, and its localization length $\xi_z$ is much shorter than $L$. Depending on $g$, the localization length is given by either

$$\frac{1}{\xi_z(W,L)} = -(\gamma^{(1)}_{\max}(W,L) + g), \quad \text{(S.74)}$$

or

$$\frac{1}{\xi_z(W,L)} = \gamma^{(2)}_{\min}(W,L) + g. \quad \text{(S.75)}$$

In the Anderson insulator phase, a finite-size scaling of the normalized localization length $\Lambda(L) \equiv \xi(L)/L$ is described by a function of the single parameter $\Lambda$ (i.e., single-parameter scaling) [48]:

$$\frac{d \ln \Lambda}{d \ln L} = \beta(\Lambda). \quad \text{(S.76)}$$

For small $\Lambda$, we have $\beta(\Lambda) \to -1$ and $\ln \Lambda \approx \ln \xi(L = \infty) = -\ln L$. When the localization length $\xi$ is much shorter than $L$, one may expand the $\beta$ function in small $\Lambda$, $\beta(\Lambda) = -1 + a\Lambda + O(\Lambda^2)$, and retain the zeroth and first order in $\Lambda$,

$$\frac{d \ln \Lambda}{d \ln L} = -1 + a\Lambda. \quad \text{(S.77)}$$

This differential equation may be solved by an integration in a domain of $[L, L_0]$ with $L \ll L_0$,

$$\frac{L}{\xi} \equiv \frac{1}{\Lambda} = a + \frac{L}{L_0} \left( \frac{1}{\xi_0} - a \right) \equiv a + \frac{L}{L_0} \left( \frac{L_0}{\xi_0} - a \right), \quad \text{(S.78)}$$

with $\xi_0 \equiv \xi(L = L_0)$. When $L_0$ goes to infinity, $\xi_0$ converges to finite $\xi(L = \infty) \equiv \xi(\infty)$. Thus, we obtain the lowest-order finite-size scaling of the quasi-1D localization length,

$$\frac{1}{\xi(L)} = \frac{a}{L} + \frac{1}{\xi(\infty)}. \quad \text{(S.79)}$$

Now that the localization length along the $z$ direction is much shorter than $L$ in Eqs. (S.74) and (S.75), we may use the same scaling function not only for $\xi_z(W,L)$ but also for $\gamma^{(2)}_{\min}(W,L)$ and $\gamma^{(1)}_{\max}(W,L)$. In fact, the scaling forms of Eqs. (S.66) and (S.67) work well for the numerical fittings. Note that the coefficient $a$ in Eq. (S.79) takes a non-universal value in general (see the fitting values in Table VIII). To obtain the scaling form for $\gamma^{(1)}_{\min}(W,L)$, let us choose large positive $g$ and make all the LEs of $h_g$ be positive,

$$0 < \gamma^{(1)}_{\min}(W,L) + g. \quad \text{(S.80)}$$

In the chiral-symmetric Hamiltonian $H_g$ that has such $h_g$ and its Hermitian conjugate $h_g^\dagger$, $E = 0$ is in the weak topological insulator phase ($\nu_z = 1$) and the localization length $\xi_z$ is given by $\gamma^{(1)}_{\min}(W,L) + g$. If we assume that the finite-size scaling of $\xi_z$ in the weak topological insulator phase is also described by the single parameter scaling function of $\Lambda_z \equiv \xi_z/L$, we also obtain the scaling function for $\gamma^{(1)}_{\min}(W,L)$ as

$$\gamma^{(1)}_{\min}(W,L) = \frac{a}{L} + \gamma^{(1)}_{\min}(W). \quad \text{(S.81)}$$

This scaling form also works well for the numerical data of $\gamma^{(1)}_{\min}(W,L)$.

1. Numerical fitting

To show the validity of the scaling forms in Eqs. (S.66) and (S.67), we use the standard $\chi^2$ fitting method to fit the data of $\gamma^{(i)}_{\min/\max}(W,L)$ ($i = 1, 2$) with larger $L$. For fixed $W$, we minimize the following $\chi^2$ function with respect to $a$ and $\gamma^{(i)}_{\min/\max}(W)$ in Eqs. (S.66) and (S.67),

$$\chi^2 = \sum_{j=1}^{D} \left( \frac{F_j - \gamma^{(i)}_{\min/\max}(W,L)}{\sigma_j} \right)^2, \quad \text{(S.82)}$$

FIG. S.1. Schematic pictures of Lyapunov exponents (LEs) of the nodal-line semimetal model \( \mathcal{H} \) and LEs of \( \mathcal{H}_g \) generated by an imaginary gauge transformation. The LEs of \( \mathcal{H} \) and \( \mathcal{H}_g \) are the sum of LEs of its right-upper non-Hermitian Hamiltonian \( h \) and their Hermitian conjugate \( h^\dagger \) in Eq. (S.19). The LEs calculated in the quasi-1D geometry \( (L \times L \times L_z, L_z \gg L) \) comprise two continuum LEs spectra in the limit \( L \to \infty \). The grey-shaded regions in the left (right) sides of the vertical axis denote the continuous spectra formed by LEs of \( h \) \( (h^\dagger) \). The dotted horizontal line denotes zero \( \gamma = 0 \). The LEs of \( \mathcal{H} \) and \( \mathcal{H}_g \), as a whole, are symmetric around zero. The smallest positive or the largest negative LE (marked in bold) corresponds to the inverse of the quasi-1D localization length. In (a), \( g \) is chosen such that finite \( \gamma_{\text{min}}^{(2)}(W) + g \) corresponds to the inverse of the localization length of \( \mathcal{H}_g \). In (b), \( g \) is chosen such that \( \gamma_{\text{min}}^{(1)}(W) + g \) corresponds to the inverse of the localization length.

The number \( D \) of the data points is typically 7 \((14 \leq L \leq 28)\) and 9 \((18 \leq L \leq 34)\). \( F_j \) is the fitted value from Eqs. (S.66) and (S.67) for different \( L \) specified by \( j \). \( \sigma_j \) is the standard deviation of \( \gamma_{\text{min}}^{(1)}(W,L) \) estimated from the transfer matrix calculation [34]. The finite-size scaling fit works well in the nodal-line semimetal models with or without time-reversal symmetry that are studied in this work. Some fitting results for the nodal-line semimetal models \((\Delta = 0, t_|| = t_\perp = 1/2, t_\perp = 1)\) that are studied in the main text are shown in Fig. S.2 (a,c) and Table IV. In Table IV, the Monte Carlo method is used to generate pseudo-data sets and evaluate the 95% confidence interval of the fitted values of \( \gamma_{\text{min}} \) and \( \gamma_{\text{max}}^{(1)} \).

Using \( \gamma_{\text{max}}^{(1)}(W) \) with the confidence interval, we determine the critical disorder strength \( W_c^{(z)} \) between the non-localized region and the localized phase. For example, the fitting results in Table IV show \( \gamma_{\text{max}}^{(1)} = 0.0020 \) \([0.0017, 0.0022]\) for \( W = 29.5 \) and \( \gamma_{\text{max}}^{(1)} = 0.0042 \) \([-0.0044, 0.0040]\) for \( W = 29.4 \) in the nodal-line semimetal model in symmetry class BDI. The results suggest that \( W_c^{(z)} \) must be between 29.4 and 29.5 with the 95% confidence: \( 29.45 \leq W_c^{(z)} \leq 29.5 \). \( W_c^{(z)} \) for the nodal-line semimetal models in symmetry classes BDI and AIII are summarized in Table IV. Table IV also shows \( W_c^{(z)} \) in the same nodal-line semimetal models for the same sets of parameters. Comparisons between \( W_c^{(z)} \) and \( W_c^{(x)} \) illustrate that \( W_c^{(x)} < W_c^{(z)} \) and \( |W_c^{(z)} - W_c^{(x)}| \) is around 10% of \( W_c^{(x)} \). This concludes the presence of the quasi-localized phase in these nodal-line semimetal models. Similarly, comparisons of \( W_c^{(z)} \) and \( W_c^{(x)} \) in other types of nodal-line semimetal models (Tables VII and IX) also suggest the presence of the quasi-localized phases (see below).
the basis that diagonalizes the chiral operator symmetry equivalent for different lattice points phases inside the non-localized region for all the models. The square brackets are the 95% confidence intervals determined by the Monte Carlo analyses.

\[
\begin{array}{cccccc}
\text{symmetry class} & W & L & \gamma_{\text{max}}(W, L = \infty) & a & \text{GOF} \\
\hline
\text{BDI} & 29.4 & 18 - 34 & 0.0020 [0.0017, 0.0022] & -3.890 [-3.893, -3.888] & 0.58 \\
\text{BDI} & 29.5 & 18 - 34 & -0.0042 [-0.0044, -0.0040] & -3.689 [3.691, -3.687] & 0.14 \\
\text{AIII} & 9.70 & 14 - 28 & 0.0143 [0.0128, 0.0156] & -1.921 [-1.948, -1.892] & 0.02 \\
\text{AIII} & 9.80 & 14 - 28 & 0.0015 [0.0000, 0.0029] & -1.929 [-1.957, -1.899] & 0.50 \\
\text{AIII} & 9.90 & 14 - 28 & -0.0114 [-0.0129, -0.0099] & -1.931 [-1.961, -1.904] & 0.22 \\
\end{array}
\]

TABLE V. Comparison between the critical disorder strength \(\nu_{\epsilon}(s)\) in the \(z\) direction (weak topological index \(\nu_s \neq 0\)) and the critical disorder strength \(W_{\epsilon}^{(s)}\) in the \(x\) direction (weak topological index \(\nu_x = 0\)) for the nodal-line semimetal models in symmetry classes BDI and AIII (\(\Delta = 0\), \(t|| = t'\parallel = 1/2\), \(t_{\perp} = 1\)). The square brackets denote the 95% intervals. The confidence intervals of \(W_{\epsilon}^{(s)}\) are determined by the 95% confidence intervals of \(\gamma_{\text{max}}(W)\).

\[
\begin{array}{cccc}
\text{symmetry class} & W_{\epsilon}^{(s)} & W_{\epsilon}^{(s)} \\
\text{BDI} & 27.24 [27.19, 27.30] & 29.45 [29.4, 29.5] \\
\text{AIII} & 9.14 [9.12, 9.17] & 9.8 [9.7, 9.9] \\
\end{array}
\]

S.8. Quasi-localized phases in chiral-symmetric models with weak topological indices

In Sec. S.7, we describe the finite-size scaling analysis of the LEs along the \(z\) direction in the nodal-line models with the weak topological index \(\nu_s \neq 0\). The analysis enables determinations of the phase boundary \(W_{\epsilon}^{(s)}\) of the non-localized region. The non-localized region comprises the metal and quasi-localized phases. In the nodal-line models with \(\nu_s = \nu_y = 0\), the phase transition between the metal and quasi-localized phases is characterized by the localization properties along the \(x\) or \(y\) direction. In this section, we discuss the localization properties along the \(x\) direction in the chiral-symmetric models with \(\nu_s \neq 0\) and \(\nu_x = 0\). We demonstrate the presence of the quasi-localized phases inside the non-localized region for all the models.

1. Nodal-line semimetal in class AIII

We discuss a nodal-line semimetal in Eq. (S.25) with the time-reversal-breaking disorder. The Hamiltonian \(H_1\) has the two types of random potentials,

\[
H_1 = \sum_{r = (r_x, r_y, r_z)} \left\{ (\Delta + \epsilon_r) c^\dagger_r \sigma_y c_r + \epsilon'_r c^\dagger_r \sigma_z c_r + \sum_{\mu = x, y} \left( t_\parallel \sigma^\dagger_r \epsilon_{r+e\mu} \sigma_y c_r + t'_\parallel \sigma^\dagger_r \epsilon_{r+e\mu} \sigma_z c_r + \text{H.c.} \right) \right\},
\]

where the random potential \(\epsilon_r (\epsilon'_r)\) respects (breaks) time-reversal symmetry and distributes uniformly in \(\epsilon^2_r + \epsilon^2'_r \leq W^2\). The parameters are chosen to be \(\Delta = 0\), \(t_\parallel = t'_\parallel = 1/2\), and \(t_{\perp} = 1\). The Hamiltonian \(H_1\) only satisfies chiral symmetry \(H_1 = -\sigma_x H_1 \sigma_x\), and hence belongs to class AIII.

According to Eq. (S.20), the chiral-symmetric Hamiltonian is decomposed into the block-off diagonal structure in the basis that diagonalizes the chiral operator \(C \equiv \sigma_z\). The right-upper part of \(H_1\) in this basis is given by

\[
h_1 = \sum_{r = (r_x, r_y, r_z)} \left[ (\Delta + i \gamma_r) f^\dagger_r f_r + \sum_{\mu = x, y} \left( t_\parallel f^\dagger_{r+e\mu} f_r + \text{H.c.} \right) + (t'_\parallel + t_\parallel) f^\dagger_{r+e\mu} f_r + (t'_\parallel - t_\parallel) f^\dagger_{r+e\mu} f_r \right].
\]

Transposition exchanges \(t'_\parallel + t_\parallel\) and \(t'_\parallel - t_\parallel\). Thus, as a unitary transformation in Eqs. (S.52) and (S.53), we can consider the mirror operation with respect to the \(xy\) plane as in Eq. (S.63). Since both \(\epsilon_r\) and \(\epsilon'_r\) are statistically equivalent for different lattice points \(r\), an ensemble of \(h_1\) defined in Eq. (S.84) is statistically invariant under the
FIG. S.2. (a,c) $\gamma_{\text{max}}(W, L)$ as a function of the system size $L$ for the different disorder strength $W$ in the nodal-line semimetal models ($\Delta = 0$, $t_{\parallel} = t_{\parallel} = 1/2$, $t_{\perp} = 1$) in (a) symmetry class BDI and (c) symmetry class AIII. The solid lines are the fitting curves from Eq. (S.67). A cross-section of the fitting curve at $1/L = 0$ determines $\gamma_{\text{max}}^{(1)}(W) \equiv \lim_{L \to \infty} \gamma_{\text{max}}^{(1)}(W, L)$.

(b,d) $\gamma_{\text{max}}^{(1)}(W)$ as a function of $W$ around $W = W_c(z)$ in the nodal-line semimetal models in (b) symmetry class BDI and (d) symmetry class AIII. Insets of (b,d): distributions of the Lyapunov exponents (LEs) of the right-upper part $h_{\text{r}}$ of the nodal-line semimetal model $H$ as a function of $W$ in the larger range of $\gamma$ and $W$. The LEs of the nodal-line semimetal models are the sum of the LEs of $h$ and their opposite-sign exponents.

combination of transposition and the mirror operation. The symmetry in Eqs. (S.52) and (S.63) requires the LEs of $h_1$ along the $x$ and $y$ directions to come in opposite-sign pairs, leading to $\nu_x = \nu_y = 0$.

We calculate the localization length $\xi_x$ of $h_1$ along the $x$ direction in the quasi-1D geometry ($L_x \times L \times L_x \gg L$). The normalized localization length $\Lambda_x \equiv \xi_x/L$ shows scale-invariant behavior around $W_c^{(x)} = 9.14 \pm 0.01$ (Fig. S.3). Fitting by the polynomial expansion of the finite-size scaling functions [see Eqs. (S.12) and (S.13)], we determine the critical disorder strength $W_c^{(x)}$ and the critical exponent (see Table III). Figure S.3 shows the normalized localization length for different $W$ and $L$ together with the fitting curves.

In Sec. S.7, we use the finite-size scaling of LEs to obtain the critical disorder strength $W_c^{(z)} = 9.8$ [9.7, 9.9]. For $W < W_c^{(z)}$, the localization length along the $z$ direction diverges. $W_c^{(x)}$ is well within the non-localized region, $W_c^{(x)} < W_c^{(z)} = 9.8$ [9.7, 9.9], demonstrating the presence of the quasi-localized phase in the nodal-line semimetal model without time-reversal symmetry.
FIG. S.3. Normalized localization length $\Lambda_x = \xi_x/L$ along the $x$ direction as a function of the disorder strength $W$ in the nodal-line semimetal model in class AII [Eq. (S.25) with $\Delta = 0$, $t_{\parallel} = t_{\parallel}' = 1/2$, $t_{\perp} = 1$]. $\xi_x$ is calculated in the quasi-1D geometry $(L \times L \times L_x)$. The black points are the raw data with the error bars. The solid lines for different $L$ and the dashed vertical line $W_c(\varepsilon) \approx 9.14$ with the error bars are the results of the fitting according to Eqs. (S.12) and (S.13) with $(m, n) = (2, 3)$. The dashed line $W_c(\varepsilon) \approx 9.8$ is evaluated by the fitting of the Lyapunov exponents along the $z$ direction by Eq. (S.66).

2. Nodal-line semimetals in class BDI

We also study the localization properties along the $x$ direction in other nodal-line semimetal models in class BDI: i) nodal-line semimetal model in Eq. (S.25) with the different parameters $\Delta = 0$, $t_{\parallel} = \sinh g$, $t_{\parallel}' = \cosh g$ ($g = 0.22, 1$), $t_{\perp} = 1$ and ii) another nodal-line semimetal model $H_2$ with an extended Fermi line running across the Brillouin zone,

$$H_2 = \sum_{r = (r_x, r_y, r_z)} \left\{ (\Delta + \epsilon_r c^\dagger_r \sigma_x c_r + \left[ t_{\parallel} c^\dagger_r c_{r+e_x} \sigma_x c_r + \sum_{\mu = y, z} \left( -it_{\parallel} c^\dagger_r c_{r+e_\mu} \sigma_y c_r + t_{\parallel}' c^\dagger_r c_{r+e_\mu} \sigma_z c_r \right) + \text{H.c.} \right) \right\}, \quad (S.85)$$

where $\epsilon_r$ takes real values and distributes uniformly in $[-W/2, W/2]$, and the parameters are chosen to be $\Delta = 0$, $t_{\parallel} = \sinh g$, $t_{\parallel}' = \cosh g$ ($g = 0.2$), $t_{\perp} = 1$. The Hamiltonian $H_2$ satisfies time-reversal symmetry $H_2 = H_2^*$ and chiral symmetry $H_2 = -\sigma_z H_2 \sigma_z$, and hence belongs to class BDI. In terms of Eq. (S.20), the chiral-symmetric Hamiltonian $H_2$ is decomposed into the block-off diagonal structure in the canonical basis where $\sigma_x$ is diagonalized. The right-upper part $h_2$ of $H_2$ in this basis is given by

$$h_2 = \sum_{r = (r_x, r_y, r_z)} \left\{ (\Delta + \epsilon_r) f^\dagger_r f_r + \left[ t_{\parallel} f^\dagger_r f_{r+e_x} + \sum_{\mu = y, z} \left( (t_{\parallel}' + t_{\parallel}) f^\dagger_r f_{r+e_\mu} + \bar{t}_{\parallel}' f^\dagger_r f_{r+e_\mu} + \text{H.c.} \right) \right] \right\}, \quad (S.86)$$

Transposition exchanges $t_{\parallel}' + t_{\parallel}$ and $t_{\parallel}' - t_{\parallel}$. Thus, as a unitary transformation in Eqs. (S.52) and (S.53), we can apply a $\pi$-rotation around the $x$ axis,

$$U_{(r_x, r_y, r_z | r_x', r_y', r_z')} = \delta_{r_x, r_{x'}} \delta_{r_y, r_y'} \delta_{r_z, r_{z'}.} \quad (S.87)$$
FIG. S.4. 2L² Lyapunov exponents (LEs) of the right-upper part h of the nodal-line semimetal model H in the canonical basis \( (t_\perp = 3/10, t'_\parallel = 1, t''_\parallel = 1/4, \Delta = 2) \) calculated along the \( z \) direction with the quasi-1D geometry \( L \times L \times L_z \). Distributions of the 2L² LEs are plotted as a function of the disorder strength \( W \). The LEs of the nodal-line semimetal model are the sum of the 2L² LEs of \( h \) and their opposite-sign exponents.

Since \( \epsilon_r \) is statistically equivalent for different lattice points \( r \), an ensemble of \( h_2 \) defined in Eq. (S.86) is statistically invariant under the combination of transposition and the \( \pi \)-rotation. The symmetry in Eqs. (S.52) and (S.87) requires the LEs of \( h_2 \) along the \( x \) direction to come in opposite-sign pairs, giving rise to \( \nu_x = 0 \). The hopping along the \( y \) direction and the hopping along the \( z \) direction are symmetric in Eq. (S.86). Thus, after transposition, we can also apply a mirror operation with respect to the plane with fixed \( y + z \):

\[
\mathcal{U}(r_x,r_y,r_z|r'_x,r'_y,r'_z) = \delta_{r_x,r'_x}\delta_{r_y,-r'_y}\delta_{r_z,-r'_z}.
\]  

(S.88)

Transposition exchanges \( t'_\parallel + t''_\parallel \) and \( t'_\parallel - t''_\parallel \) and the mirror operation puts them back. Thus, an ensemble of \( h_2 \) is statistically invariant under the combination of transposition and the mirror operation. The symmetry in Eqs. (S.52) and (S.88) requires the LEs of \( h_2 \) along the \( r_{(0,1,-1)} \equiv r_y - r_z \) direction to come in opposite-sign pairs, leading to \( \nu_y = \nu_z \).

For the directions with non-zero weak topological indices [i.e., \( z \) direction in Eq. (S.31) and \( y, z \) directions in Eq. (S.86)], we calculate the LEs of the right-upper part [i.e., \( h \) in Eq. (S.31) and \( h_2 \) in Eq. (S.86)] as a function of the disorder strength \( W \). In all these models, the distributions of the LEs for large \( L \) show the \( W \)-dependence described in Fig. S.4. For \( W = 0 \), the 2L² LEs form a continuum spectrum in the large \( L \) limit, including zero \( \gamma = 0 \). When \( W \) increases, the spectrum splits into the two continuous spectra. For \( W > W_c^{(z)} \), all the 2L² LEs in the lower spectrum become negative. The non-localized region extends from \( W = 0 \) to \( W = W_c^{(z)} \), while the Anderson insulator phase appears in \( W > W_c^{(z)} \). \( W_c^{(z)} \) is determined by the finite-size scaling of \( \gamma^{(1)}_{\text{max}} \) as in Sec. S.7, which is summarized in Table VII.
For the directions with zero weak topological indices [i.e., $x, y$ directions in Eq. (S.31) and $x$ direction in Eq. (S.86)], we calculate the localization length $\xi_x$ in the quasi-1D geometry $(L_x \times L \times L, L_x \gg L)$. Figure S.5 shows the normalized localization length $\Lambda_x \equiv \xi_x/L$ as a function of the disorder strength $W$. In all these models, $\Lambda_x$ shows scale-invariant behavior inside the non-localized region. Fitting $\Lambda_x$ around the scale-invariant points by the scaling functions of Eqs. (S.12) and (S.13), we evaluate the critical exponent $\nu$ and the critical disorder strength $W_{c}(x)$, as summarized in Table VI. The critical disorder strength $W_{c}(x)$ is far below $W_{c}(z)$, demonstrating the presence of the quasi-localized phases for $W_{c}(x) < W < W_{c}(z)$ in these three models (Table VII). The evaluated critical exponents are consistent with the critical exponent $\nu = 0.820/0.783, 0.846$ shown in the main text for the nodal-line semimetal in class BDI. This consistency suggests that all the phase transitions between the metal and quasi-localized phases in the 3D models in symmetry class BDI are of the same nature.

Note also that $\nu$ in the nodal-line semimetal model of Eq. (S.31) with $t_{\parallel} = \cosh g, t_{\perp}' = \sinh g$ ($g = 0.22$) shows the
TABLE VII. Comparison of the two critical disorder strengths, $W_{c}^{(x)}$ and $W_{c}^{(z)}$, in the nodal-line semimetal models $H$ [Eq. (S.25)] and $H_{2}$ [Eq. (S.83)]. The parameters of these models are the same as in Table VI. Both models are characterized by the parameters $\Delta = 0$, $t_{\perp} = \sinh g$, $t_{\parallel} = \cosh g$, $t_{\uparrow} = 1$. The column “parameter” specifies the value of $g$. In the two models, the non-localized regions extend from $W = 0$ to $W = W_{c}^{(x)}$, and the quasi-localized phases extend from $W_{c}^{(x)}$ to $W_{c}^{(z)}$.

| symmetry class | model | parameter | $W_{c}^{(x)}$ | $W_{c}^{(z)}$ |
|----------------|-------|-----------|---------------|---------------|
| BDI            | $H$   | $g = 0.22$| $21.73$       | $23.44$       |
| BDI            | $H_{2}$ | $g = 1$   | $41.36$       | $45.35$       |
| BDI            | $H_{2}$ | $g = 0.2$ | $23.29$       | $24.74$       |

larger error bars in their fitting results (see Table VI). These larger error bars may stem from a severe crossover effect. For $t_{\parallel} = \sinh g = 0$, the nodal-line semimetal model has an extra unitary symmetry $H = \sigma_{z}H\sigma_{z}$. The Hamiltonian can be block-diagonalized into two parts, and each block belongs to orthogonal class. For nonzero but small $g$, this unitary symmetry is only weakly broken. Thus, the finite-size systems with smaller $g$ must suffer from a stronger crossover effect.

3. Weak topological insulators and ordinary insulators in class BDI

Zero-energy states of the nodal-line semimetal model in Eq. (S.25) can be either in a topological insulator state with $(\nu_{x}, \nu_{y}, \nu_{z}) = (0, 0, 1)$ or in an ordinary insulator state with $(\nu_{x}, \nu_{y}, \nu_{z}) = (0, 0, 0)$, depending on its tight-binding parameters. For simplicity, we assume $\Delta, t_{\perp}, t_{\parallel} > 0$ in Eq. (S.25). For $\Delta + 4t_{\perp} < 2|t_{\parallel}| (\Delta - 4t_{\perp} > 2|t_{\parallel}|)$, the zero-energy states of $H$ in Eq. (S.25) are in the topological (ordinary) insulator state in the clean limit ($W = 0$).

In Ref. [41], the localization lengths of Eq. (S.25) along the $x$ direction were calculated with the quasi-1D geometry ($L_{x} \times L_{y} \times L_{z}$, $L_{x} \gg L$), and the two consecutive disorder-driven phase transitions were identified for the following set of parameters:

$$t_{\perp} = 3/10, \quad t_{\parallel} = 1, \quad t_{\parallel} = 1/4, \quad \Delta = 1/2 \quad \text{(topological insulator in the clean limit).} \quad (S.89)$$

The two phase transitions are i) a transition from the topological insulator phase to the diffusive metal phase at $W_{c,1}^{(x)} = 3.135 [3.132, 3.138]$ and ii) a transition from the diffusive metal phase to the Anderson insulator phase at $W_{c,2}^{(x)} = 11.96 [11.92, 12.02]$, respectively. In addition, Ref. [41] studied another parameter set,

$$t_{\perp} = 3/10, \quad t_{\parallel} = 1, \quad t_{\parallel} = 1/4, \quad \Delta = 4 \quad \text{(ordinary insulator in the clean limit),} \quad (S.90)$$

where a disorder-driven phase transition from the 3D ordinary band insulator phase to the diffusive metal phase [Fig. 6(b)] was found at $W_{c,1}^{(x)} = 4.76 [4.75, 4.77]$. The normalized localization length $\Lambda_{x} \equiv \xi_{x}/L$ shows scale-invariant behavior at these critical disorder strengths. From the finite-size scaling analyses, it was clarified that these phase transitions are universally characterized by the same critical exponent $\nu = 0.82 \pm 0.04$ [41], which is consistent with the evaluations of the disordered nodal-line semimetal models studied in the main text.

For these two sets of parameters, we study the localization length and the winding number along the $z$ direction. We calculate the LEs of the right-upper part $h$ of $H$ in the canonical basis [i.e., Eq. (S.31)] with the quasi-1D geometry ($L \times L \times L_{z}$, $L_{z} \gg L$). For the parameters in Eq. (S.89), the LEs show a $W$-dependence described as Fig. 6(c). The topological insulator is stable under weak disorder. For $W < W_{c,1}^{(x)}$, all the $2L^{2}$ LEs are positive, and the localization length is finite. The winding number $\nu_{z}$ along the $z$ direction is $L^{2}$, giving rise to $\nu_{z} = 1$. The non-localized region appears from $W = W_{c,1}^{(x)}$ to $W = W_{c,2}^{(x)} (> W_{c,1}^{(x)})$, where a continuous spectrum of LEs includes zero $\gamma = 0$. When $W$ increases from $W_{c,1}^{(x)}$ to $W_{c,2}^{(x)}$, $L^{2}$ positive LEs cross zero and become negative; $\nu_{z}$ changes from 1 to 0. For $W > W_{c,2}^{(x)}$, the $L^{2}$ LEs are positive and the other $L^{2}$ LEs are negative, leading to $\nu_{z} = 0$. For the parameters in Eq. (S.90), the LEs show a $W$-dependence described as Fig. 6(d), where the non-localized region appears from $W = W_{c,1}^{(x)}$ to $W = W_{c,2}^{(x)} (> W_{c,3}^{(x)})$. In the non-localized region, the number of positive LEs is greater than the number of negative ones, leading to $\nu_{z} > 0$.

From the finite-size scaling analyses of the minimal or maximal LEs by Eqs. (S.66), (S.67), or (S.81), we determine the phase boundaries of the non-localized regions as $W_{c,1}^{(x)} = 3.08 [3.07, 3.09]$, $W_{c,2}^{(x)} = 13.3 [13.2, 13.4]$, and $W_{c,3}^{(x)} = 4.56 [4.55, 4.57]$ (see Table VIII). From a comparison of these numbers with $W_{c}^{(x)}$ obtained in Ref. [41] (see Table IX), we conclude that the quasi-localized phases appear inside the non-localized regions.
\[ W = 0 \]

\begin{align*}
W_{c,1}^{(x)} &\approx 3.13 \\
W_{c,2}^{(x)} &\approx 12.0 \\
W_{c,1}^{(z)} &\approx 3.08 \\
W_{c,2}^{(z)} &\approx 13.3
\end{align*}

\textit{topological insulator} \quad \nu_z = 1

\textit{diffusive metal} \quad 1 > \nu_z > 0

\textit{Anderson Insulator} \quad \nu_z = 0

\textit{ordinary insulator} \quad \nu_z = 1

\begin{align*}
W_{c,3}^{(x)} &\approx 4.62 \\
W_{c,3}^{(z)} &\approx 4.56
\end{align*}

\textit{quasi-localized} \quad \textit{diffusive metal} \quad \textit{non-localized region}

\begin{align*}
&\nu_z = \frac{3}{10}, t'_{\parallel} = 1, t_{\perp} = \frac{1}{4}, \Delta = \frac{1}{2} \\
&\nu_z = \frac{3}{10}, t'_{\parallel} = 1, t_{\perp} = \frac{1}{4}, \Delta = 4
\end{align*}

**FIG. S.6.** (a,b) Schematic phase diagrams of \( \mathcal{H} \) in different parameter sets, (a) weak topological and (b) ordinary insulator sides. \( W_{c,i}^{(x)} (i = 1, 2, 3) \) stand for critical points of the Anderson transitions in the \( x \) or \( y \) direction. The weak topological index \( \nu_z \) is finite for \( W_{c,1}^{(x)} < W < W_{c,2}^{(x)} \) of (a) and for \( W_{c,3}^{(x)} < W \) of (b). The 2L^2 Lyapunov exponents of \( h \) in the same parameter sets as (a) and (b) are shown in (c) and (d), respectively.

For the parameters in Eq. (S.89) and

\[
\begin{align*}
W &< W_{c,1}^{(z)} \\
W_{c,1}^{(z)} &< W < W_{c,1}^{(x)} \\
W_{c,1}^{(x)} &< W < W_{c,2}^{(x)} \\
W_{c,2}^{(x)} &< W < W_{c,2}^{(z)} \\
W_{c,2}^{(z)} &< W < W_{c,4}^{(z)}
\end{align*}
\]

(S.91)

for the parameters in Eq. (S.90) and

\[
\begin{align*}
W &< W_{c,3}^{(z)} \\
W_{c,3}^{(z)} &< W < W_{c,4}^{(z)} \\
W_{c,4}^{(z)} &< W < \ldots
\end{align*}
\]

(S.92)

for Eq. (S.90). Here, “…” means that when \( W \) is further increased, the system undergoes a transition from the diffusive metal phase to the quasi-localized phase at \( W_{c,4}^{(x)} \), and a transition from the quasi-localized phase to the
TABLE VIII. Finite-size scaling analyses of $\gamma^{(1)}_{\text{max}}(W, L)$ or $\gamma^{(1)}_{\text{min}}(W, L)$ for the several disorder strength $W$ around $W = W_c(z)$ for the disordered topological insulator model with the parameters in Eq. (S.89) (shown as “P1” in the “parameter set”), and the disordered ordinary insulator model with the parameters in Eq. (S.90) (shown as “P3” in the “parameter set”). The square brackets are the 95% confidence error bars determined by the Monte Carlo analyses.

| parameter set | $W$ | $L$ | $\gamma^{(1)}_{\text{max}}(W, L)$ | $a$ | GOF  |
|---------------|-----|-----|----------------------------------|-----|------|
| P1            | 3.07| 24 - 60 | 0.0013 [0.0012, 0.0014] | 0.050 [0.046, 0.053] | 0.64 |
| P1            | 3.08| 24 - 60 | -0.0002 [-0.0003, -0.0000] | 0.051 [0.047, 0.054] | 0.90 |
| P1            | 3.09| 24 - 60 | -0.0015 [-0.0016, -0.0014] | 0.048 [0.045, 0.051] | 0.46 |

TABLE IX. Comparison of the critical disorder strengths, $W^{(z)}_c$ and $W^{(x)}_c$, in the disordered topological insulator model with the parameters in Eq. (S.89) (shown as “T1” and “T2” in the “transition”), and the disordered ordinary insulator model with the parameters in Eq. (S.90) (shown as “T3” in the “transition”). The square brackets are the 95% confidence error bars determined by the synthetic data.

| symmetry class | transition | $W^{(x)}_c$ | $W^{(z)}_c$ |
|----------------|------------|-------------|-------------|
| BDI            | T1         | 3.135 [3.132, 3.138] | 3.08 [3.07, 3.09] |
| BDI            | T2         | 11.96 [11.92, 12.02] | 13.3 [13.2, 13.4] |
| BDI            | T3         | 4.62 [4.60, 4.63] | 4.56 [4.55, 4.57] |

*From Ref. 41*

Anderson insulator phase at $W_c^{(z)}(z)$ but the respective critical disorder strengths $W^{(z)}_c$ and $W^{(x)}_c$ are not determined. The phase diagrams of the disordered topological insulator and ordinary insulator are shown in Figs. 6(a) and 6(b).

S.9. Anderson transitions in chiral-symmetric models with no weak topological indices

For comparison, we study three-dimensional (3D) chiral-symmetric models in symmetry classes BDI and AIII, where statistical symmetries enforce all the three topological indices to be zero. We refer to these models as non-topological models. Notably, the disordered ordinary insulator model in Eq. (S.25) with the parameters in Eq. (S.90) is a topological model because non-zero $\nu_z$ is induced by the disorder [see also Figs. 6(d) and 6(b)]. Non-topological models have the following three features that are distinct from the topological models in the same symmetry classes:

1. In the quasi-1D geometry ($L \times L \times L_{\mu}$, $L_{\mu} \gg L$), the localization length along any spatial direction is always finite with finite $L$. In the topological models with $\nu_z \neq 0$, the localization length along the $z$ direction can diverge for finite $L$ when the 1D winding $w_z$ changes.

2. In the thermodynamic limit $L \to \infty$, the localization lengths along all the spatial directions diverge at the same critical point, which implies no quasi-localized phase. In the topological models, the localization length along the $z$ direction and those along the other two directions diverge at different critical points in the thermodynamic limit, which gives rise to the quasi-localized phase.

3. The divergence of the localization length along all the directions is characterized by the same critical exponent in the non-topological models. In the topological models, the divergence of the localization length along the $z$ direction and those along the other directions are characterized by the different critical exponents. The two exponents in the topological models are also different from the exponents in the non-topological models in the same symmetry class.
Let us introduce the following non-topological chiral-symmetric model that belongs to symmetry class BDI or AIII,

\[ H_0 = \sum_{r=(x,y,z)} \left\{ (\Delta + \epsilon_r) c_{r}^\dagger \sigma_z c_r + \epsilon'_r c_{r}^\dagger \sigma_y c_r + \left( \sum_{\mu=x,y} t_{\mu} c_{r+e_{\mu}}^\dagger \sigma_0 c_{r+e_{\mu}} + \epsilon'_{r-x=0} c_{r+\epsilon_x}^\dagger \sigma_y c_r + \text{H.c.} \right) \right\}, \]

(S.93)

where \( c_r \) is a two-component annihilation operator at the cubic-lattice site \( r \equiv (r_x, r_y, r_z) \), \( e_{\mu} \)'s (\( \mu = x, y, z \)) are the unit vectors connecting the nearest neighbor cubic-lattice sites, \( \sigma_{\mu} \)'s (\( \mu = x, y, z \)) are the two-by-two unit matrix and Pauli matrices for the two orbitals, \( \Delta, t_\perp, t_\parallel, t'_\parallel \) are the real parameters, and \( \epsilon_r \) and \( \epsilon'_r \) are the real-valued on-site random potential. We choose the parameters to be \( \Delta = 0, t_\parallel = 1, t'_\parallel = 13/12, t'_\perp = 5/12 \). The model in the clean limit \( \epsilon_r \equiv \epsilon'_r \equiv 0 \) has a finite density of states at \( E = 0 \). \( H_0 \) respects chiral symmetry, \( H_0 = -C^\dagger H_0^0 C \) with the chiral operator

\[ C_{r,r'} \equiv (-1)^{x+y} \delta_{r,r'} \sigma_x, \]

(S.94)

satisfying \( C = C^T \). For \( \epsilon'_r = 0 \), \( H_0 \) respects time-reversal symmetry \( H_0 = H_0^0 \) and hence belongs to the chiral orthogonal class (class BDI). For \( \epsilon'_r \neq 0 \), time-reversal symmetry is broken, and \( H_0 \) belongs to the chiral unitary class (class AIII). For \( H_0 \) in class BDI, we choose \( \epsilon_r \) to be uniformly distributed in \([-W/2, W/2]\). For \( H_0 \) in class AIII, on the other hand, we choose \( \epsilon_r \) and \( \epsilon'_r \) to be uniformly distributed for \( \epsilon^2 + \epsilon'^2 \leq W^2 \).

Following Eq. (S.20), we decompose \( H_0 \) into the block-off-diagonal structure in a basis that diagonalizes the chiral operator \( C \). The right-upper part \( h_0 \) of \( H_0 \) is regarded as a single-orbital tight-binding model on the cubic lattice,

\[ h_0 = \sum_{r} \left\{ (\Delta + \epsilon_r + i\epsilon'_r) f_{r}^\dagger f_r + \sum_{\mu=x,y} (t_{\mu} f_{r+e_{\mu}}^\dagger f_{r} + \text{H.c.}) + \left( t_{\parallel} - (1)^{r_x+r_y} t'_{\parallel} \right) f_{r+e_z}^\dagger f_{r} + \left( t_{\parallel} - (1)^{r_x+r_y} t'_{\parallel} \right) f_{r}^\dagger f_{r+e_z} \right\}. \]

(S.95)

Transposition exchanges \( t_{\parallel} - (1)^{r_x+r_y} t'_{\parallel} \) and \( t_{\parallel} - (1)^{r_x+r_y} t'_{\parallel} \) in \( h_0 \). Thus, as a unitary transformation in Eqs. (S.52) and (S.53), we apply a spatial translation along the \( x \) or \( y \) direction by \( e_x \) or \( e_y \),

\[ U_{(r_x,r_y,r_z|\delta_{r_x},r'_z)} = \left\{ \begin{array}{ll} \delta_{r_x+r'_x+1} \delta_{r_y} \delta_{r_z}, \\ \delta_{r_x} \delta_{r_y+r'_y+1} \delta_{r_z}. \end{array} \right. \]

(S.96)

Instead of Eq. (S.96), we can also use a mirror operation with respect to the \( r_x = 1/2 \) plane or the \( r_y = 1/2 \) plane,

\[ U_{(r_x,r_y,r_z|\delta_{r_x},r'_z)} = \left\{ \begin{array}{ll} \delta_{r_x+r'_x+1} \delta_{r_y} \delta_{r_z}, \\ \delta_{r_x} \delta_{r_y+r'_y+1} \delta_{r_z}. \end{array} \right. \]

(S.97)

Since \( \epsilon_r \) (\( \epsilon'_r \)) at different lattice points \( r \) is statistically equivalent, an ensemble of \( h_0 \) defined in Eq. (S.95) is statistically invariant under the combination of transposition and any of these unitary transformations. These statistical symmetries require the LEs of \( h_0 \) along all the directions to come in opposite-sign pairs, leading to \( \nu_x = \nu_y = \nu_z = 0 \).

For \( H_0 \) in symmetry class BDI, we calculate the localization lengths \( \xi_z, \xi_x \) along the \( x, z \) directions with the quasi-1D geometry \( \mathbb{I}^2 \times L_\mu, \mu = x, z, L_\mu \gg L \). Because of chiral symmetry, it is sufficient to calculate the product of the transfer matrices of \( h_0 \) [see Eq. (S.23)]. Because of the statistical symmetries, the LEs of \( h_0 \) come in opposite-sign pairs. Both normalized localization length \( \Lambda_z = \xi_z/L \) and \( \Lambda_x \equiv \xi_x/L \) show scale-invariant behavior around the same critical disorder strength \( W \approx 23 \) (see Fig. (S.7)). From the fitting by the polynomial expansion of the finite-size scaling function [Eqs. (S.12) and (S.13)], we determine the critical disorder strength and the critical exponent (see the fourth to seventh rows of Table (III)).

The critical disorder strength and exponent determined from \( \Lambda_z \) and those determined from \( \Lambda_x \) are consistent with each other. The critical exponent is \( \nu = 1.089 [1.005, 1.128] \), and different from the two exponent \( \nu' = 0.820 [0.787, 0.848] \) and \( \nu' = 1 \) of the topological model in the same symmetry class (i.e., class BDI). Reference (62) studied the localization length of \( H_0 \) along the \( z \) direction with the different parameters \( \Delta = t_{\parallel} = t'_{\parallel} = 1, t_{\parallel} = 1/2 \) and evaluated the critical exponent to be \( \nu = 1.119 [0.973, 1.241] \), which is consistent with our evaluation.

For \( H_0 \) in symmetry class AIII, we calculate the normalized localization length \( \Lambda_z = \xi_z/L \) with the quasi-1D geometry \( \mathbb{I}^2 \times L_\mu \) and \( L_\mu \gg L \). \( \Lambda_z \) shows scale-invariant behavior around the critical disorder strength \( W \approx 8 \) (see Fig. (S.8)). From the fitting by the polynomial expansion of the finite-size scaling function [Eqs. (S.12) and (S.13)], we determine the critical disorder strength and critical exponents. The critical exponent is \( \nu = 1.024 [0.973, 1.070] \) (see the last row of Table (III)) different from the two critical exponents \( \nu = 0.824 [0.776, 0.862] \) and \( \nu' = 1 \) of the topological model in the same symmetry class (i.e., class AIII).
2. Another three-dimensional non-topological model in class AIII

Reference [52] introduced another 3D non-topological model in symmetry class AIII,  

\[ H'_0 = \sum_{r=(r_x,r_y,r_z)} \left\{ (\Delta + \epsilon_r) \sigma_z c_r \sigma_r + \left[ c^\dagger_{r+e_x} (t_1 \sigma_z + it_\perp \sigma_x) c_r + c^\dagger_{r+e_y} (t_2 \sigma_0 + it_\perp \sigma_y) c_r + t_\parallel c^\dagger_{r+e_z} \sigma_0 c_r + \text{H.c.} \right] \right\}, \]

where the disorder potential \( \epsilon_r \) distributes uniformly in \([-W/2, W/2]\), and the parameters are chosen to be \( \Delta = 0, t_\perp = 3/5, t_\parallel = 2/5, t_1 = t_2 = 1/2 \). \( H'_0 \) respects chiral symmetry \( H'_0 = -C^\dagger H'_0^\dagger C \) with a chiral operator \( C \)

\[ C_{r,r'} \equiv (-1)^{y+z} \delta_{r,r'} \sigma_y. \]

In terms of Eq. (S.20), \( H'_0 \) is decomposed into the block off-diagonal structure in a basis that diagonalizes the chiral operator. The right-upper part \( h'_0 \) of \( H'_0 \) in this basis is given by a single-orbital tight-binding model on the cubic lattice site,

\[ h'_0 = \sum_{r=(r_x,r_y,r_z)} \left[ (\Delta + \epsilon_r) f^\dagger_r f_r + (t_1 + (-1)^r \sigma_z t_\perp) f^\dagger_{r+e_x} f_r + (t_1 - (-1)^r \sigma_z t_\perp) f^\dagger_r f_{r+e_x} \right. \]

\[ + \left. (t_2 - (-1)^r \sigma_y t_\perp) f^\dagger_{r+e_y} f_r + (t_2 - (-1)^r \sigma_y t_\perp) f^\dagger_r f_{r+e_y} + \left( t_\parallel f^\dagger_r f_{r+e_z} + \text{H.c.} \right) \right]. \]

Hermitian conjugation exchanges \( t_1 + (-1)^r \sigma_z t_\perp \) and \( t_1 - (-1)^r \sigma_z t_\perp \), but transforms \( t_2 - (-1)^r \sigma_y t_\perp \) into \( t_2 + (-1)^r \sigma_y t_\perp \). Thus, as a unitary transformation in Eqs. (S.53) and (S.60), we apply a spatial translation along the \( y \)
FIG. S.8. Inverse $\Gamma_z \equiv 1/\Lambda_z \equiv \xi_z / L$ of the normalized localization length along the $z$ direction as a function of the disorder strength $W$ in the non-topological model $\mathcal{H}' (\Delta = 0, t_\perp = 1, t_\parallel = 13/12, t'_\parallel = 5/12)$ in class AIII [Eq. (S.93)] with the quasi-1D geometry ($L \times L \times L_z$). The black points are the raw data with the error bars. The solid lines for different $L$ are the results of the fitting according to Eqs. (S.12 and (S.13) with $(m,n) = (2,3)$.

or $z$ direction by $e_y$ or $e_z$,

$$U_{(r_x,r_y,r_z|r'_x,r'_y,r'_z)} = \begin{cases} 
\delta_{r_x,r'_x} \delta_{r_y,r'_y} \delta_{r_z,r'_z} + 1 & \delta_{r_x,r'_x} \delta_{r_y,r'_y} \delta_{r_z,r'_z} + 1, \\
\delta_{r_x,r'_x} \delta_{r_y,r'_y} \delta_{r_z,r'_z} & \delta_{r_x,r'_x} \delta_{r_y,r'_y} \delta_{r_z,r'_z} + 1.
\end{cases} \quad (S.101)$$

Instead of Eq. (S.101), we can also use a mirror operation with respect to the $r_y = 1/2$ plane and the $r_z = 1/2$ plane,

$$U_{(r_x,r_y,r_z|r'_x,r'_y,r'_z)} = \begin{cases} 
\delta_{r_x,r'_x} \delta_{r_y,r'_y} + r'_z, & \delta_{r_x,r'_x} \delta_{r_y,r'_y} + r'_z, \\
\delta_{r_x,r'_x} \delta_{r_y,r'_y} r'_z & \delta_{r_x,r'_x} \delta_{r_y,r'_y} r'_z + 1.
\end{cases} \quad (S.102)$$

Since $\epsilon_r$ is statistically equivalent for different lattice points $r$, an ensemble of $h'_0$ defined in Eq. (S.100) is statistically invariant under the combination of Hermitian conjugation and any of these unitary transformations. These symmetries require the LEs of $h'_0$ along all the directions to come in opposite-sign pairs, leading to $\nu_x = \nu_y = \nu_z = 0$. Reference [52] evaluated the critical exponent to be $1.059 [1.022, 1.100]$, which is consistent with our evaluation of the exponent in the non-topological model in symmetry class AIII.