HAUSDORFF DIMENSIONS FOR GRAPH-DIRECTED MEASURES
DRIVEN BY INFINITE ROOTED TREES

KAZUKI OKAMURA

Abstract. We give upper and lower bounds for the Hausdorff dimensions for a class of
graph-directed measures when its underlying directed graph is the infinite $N$-ary tree.
These measures are different from graph-directed self-similar measures driven by finite
directed graphs and are not necessarily Gibbs measures. However our class contains sev-
eral measures appearing in fractal geometry and functional equations, specifically, mea-
sures defined by restrictions of non-constant harmonic functions on the two-dimensional
Sierpiński gasket, the Kusuoka energy measures on it, and, measures defined by solutions
of de Rham’s functional equations driven by linear fractional transformations.

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1. INTRODUCTION AND MAIN RESULTS

Let $N \geq 2$. Let $\Sigma_N := \{0, 1, \ldots, N - 1\}$. Let $Y$ be a set and $Z$ be a measurable
space. For $i \in \Sigma_N$, let $G_i : Y \to [0, 1]$ and $H_i : Y \to Y$ be maps and $F_i : Z \to Z$ be
measurable maps. Then we consider the following equation for a family of probability
measures $\{\mu_y\}_{y \in Y}$ on $Z$:

$$\mu_y = \sum_{i \in \Sigma_N} G_i(y)\mu_{H_i(y)} \circ F_i^{-1},$$

under the assumption that

$$\sum_{i \in \Sigma_N} G_i(y) = 1.$$ 

This family of measures can be regarded as the Markov type measures in Edgar-Mauldin
[EM92], a self-similar family of measures in Strichartz [St93, Definition 2.2] and the graph
directed self-similar measures in Olsen [Ol94, Section 1.1]. Let $(V,E)$ be a directed graph
where multi-edges and self-loops are allowed. [EM92], [St93] and [Ol94] focus on the
case that $V$ is finite. Several arguments in these references such as the Perron-Frobenius
theorem and ergodic theorem for finite Markov chains depend on the fact that $V$ is finite.
Here we consider the case that $(V,E)$ is the infinite $N$-ary tree on which each edge is
equipped with a direction from a vertex closer to the root to its descendants.

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de Rham’s functional equations.
If $V = Y(y) = \{y\}$ (See (1.3) below for the definition of $Y(y)$.), then,
\[\mu_y = \sum_{i \in \Sigma_N} G_i(y) \mu_y \circ F_i^{-1}.\]

If $\{F_i\}_i$ is a family of contractions on a complete metric space $Z$, then, $\mu_y$ is the invariant measure of the iterated function system $(Z, \{F_i\}_i \in \Sigma_N)$ equipped with probability weights $(G_i(y))_i \in \Sigma_N$. However, this paper rather focuses on the case that $\mu_y$ is not an invariant measure of an iterated function system. Our class contains several measures which are not not necessarily invariant or Gibbs measures and appear in fractals and functional equations, specifically, measures defined by restrictions of non-constant harmonic functions on the two-dimensional standard Sierpinski gasket, the Kusuoka energy measures on it, and furthermore measures defined by solutions of de Rham's functional equations driven by linear fractional transformations.

We give upper and lower bounds for the Hausdorff dimensions for $\{\mu_y\}_{y \in Y}$ under certain regularity conditions for the functions $\{G_i\}_i \in \Sigma_N$ and $\{H_i\}_i \in \Sigma_N$ on $Y$ (Definition 1.1) and the assumption that $\{F_i\}_i \in \Sigma_N$ is an iterated function system on a complete metric space $Z$ satisfying certain conditions (Assumption 1.2). We now state some applications of our results. We extend the main result of [ADF14] considering restrictions of non-constant harmonic functions on the two-dimensional standard Sierpinski gasket. Our proof gives an alternative proof of singularity of the Kusuoka energy measures on the standard 2-dimensional Sierpinski gasket [Ku89]. Furthermore, de Rham's functional equations driven by linear fractional transformations considered in [Ok14] are also generalized. Specifically, we deal with the case that an equation is driven by $N$ linear fractional transformations which are weak contractions. [Ok14] deals with the case that an equation is driven by only two linear fractional transformations which are contractions. We also discuss singularity of $\mu_y$ with respect to self-similar measures of iterated function systems equipped with probability weights which are not a canonical measure on $Z$.

1.1. Main results. Now we start to describe assumptions and main results. We first give notation and assumptions for $Y$. For $y \in Y$, let
\[Y(y) := \{H_{i_1} \circ \cdots \circ H_{i_l}(y) \mid i_1, \ldots, i_l \in \Sigma_N, l \geq 1\}. \quad (1.3)\]
This corresponds to the infinite $N$-ary tree with root $y$, specifically, the set of vertices is $Y(y)$, and the set of directed edges is given by \[\{(H_{i_1} \circ \cdots \circ H_{i_l}(y), H_{i_{l+1}} \circ \cdots \circ H_{i_1}(y)) \mid i_1, \ldots, i_l, i_{l+1} \in \Sigma_N, l \geq 1\},\] where we let $H_i \circ \cdots \circ H_1(y) := y$ if $l = 0$.

Definition 1.1. For each $y \in Y$,
(i) We say that (A-y) holds if
\[0 < \inf_{i \in \Sigma_N, z \in Y(y)} G_i(z) \leq \sup_{i \in \Sigma_N, z \in Y(y)} G_i(z) < 1.\]
(ii) We say that (wA-y) holds if for some $i \in \Sigma_N$ and $c > 0$,
\[c \leq \inf_{z \in Y(y)} G_i(z) \leq \sup_{z \in Y(y)} G_i(z) \leq 1 - c.\]
(iii) (disjointness) We say that (B-y) holds if there exist $\epsilon_0 \in (0, 1/N)$, $l \geq 1$ and $i_1, \ldots, i_l \in \Sigma_N$ such that
\[Y(y) \cap \bigcap_{i \in \Sigma_N} G_i^{-1}\left(\left[0, \frac{1}{N} - \epsilon_0, \frac{1}{N} + \epsilon_0\right]\right) \cap \bigcap_{j \in \Sigma_N} (G_j \circ H_{i_1} \circ \cdots \circ H_{i_l})^{-1}\left(\left[0, \frac{1}{N} - \epsilon_0, \frac{1}{N} + \epsilon_0\right]\right) = \emptyset. \quad (1.4)\]
(iv) We say that (sB-y) holds if there exist $\epsilon_0 \in (0, 1/N)$, $l \geq 1$ such that (1.4) holds for every $i_1, \ldots, i_l \in \Sigma_N$.
In the above, we regard $Y$ simply as a set, but in order to check the conditions in Definition 1.1, we will often put a metric structure and a linear structure on $Y$. Our features are that $Y$ is only a set and irrelevant to $Z$, $Y$ contains countably many points, $G_i$ and $H_i$, $i \in \Sigma_N$, are not constant functions, and furthermore, $\mu_y$ is not an invariant measure of a certain iterated system equipped with probability weights.

Second, we give notation and assumptions for $Z$. Let $f_i, i \in \Sigma_N$, be contractive maps on a complete metric space $(M, d)$, and $K$ be the attractor of the iterated function system \{\(f_i\)\}_{i \in \Sigma_N}. We put the Borel $\sigma$-algebra on $K$ induced by the metric $d$ on $M$. Let $f_{i_1 \cdots i_m} = f_{i_1} \circ \cdots \circ f_{i_m}$ and $K_{i_1 \cdots i_m} = f_{i_1 \cdots i_m}(K)$.

**Assumption 1.2.** There exists constants $r \in (0, 1)$, $c_1, c_2 > 0$ and $D > 0$ such that

(i) For every $m \geq 1$, $\text{diam}(K_{i_1 \cdots i_m}) \leq c_1 r^m$.

(ii) For every $x \in K$ and $m \geq 1$, $|\{(i_1 \cdots i_m) \in (\Sigma_N)^m \mid B(x, r^m) \cap K_{i_1 \cdots i_m} \neq \emptyset\}| \leq D$.

Let $\dim_H A$ be the Hausdorff dimension of $A \subset K$ and the Hausdorff dimension of $\mu$ be $\dim_H \mu := \inf \{\dim_H K \mid \mu(K) = 1\}$. The following is our main result.

**Theorem 1.3.** Let $Z = K$ and $F_i = f_i, i \in \Sigma_N$. Suppose that Assumption 1.2 holds for $(Z, \{F_i\}_{i \in \Sigma_N})$. Then, a family of solutions $\{\mu_y\}_{y \in Y}$ of (1.1) exists and is unique. Furthermore we have the following:

(i) Assume both of (A-$y$) and (B-$y$) or, assume (sB-$y$) only. Then,

$$
\dim_H \mu_y < \frac{\log N}{\log(1/r)}.
$$

(ii) If (wA-$y$) holds, then, $\dim_H \mu_y > 0$.

In examples we deal with later, (1.5) implies that $\mu_y$ is singular with respect to a “canonical” probability measure on $Z$ whose Hausdorff dimension is $\log N/\log(1/r)$. As an outline of our proof, we follow [Ok14, Theorem 1.2]. However, our method is more transparent than [Ok14]. Specifically, we do not use four kinds of random subsets of natural numbers as in [Ok14, Lemma 3.3]. See Lemma 2.6 below for an alternative way. We emphasize that not only Theorem 1.3 is applicable to the models described above, but also there is potential for applications to different models. Indeed, we deal with some special examples other than the models described above.

1.2. **Comparison with related results.** Our purpose is to know whether $\mu_y$ is absolutely continuous or singular with respect to a natural canonical probability measure on $Z$. In several examples, we deal with the case that $Z = [0, 1]$. If $\mu_y$ is singular, then, the monotone increasing function which is a distribution function of $\mu_y$ is singular\(^1\). The singularity problem proposed by Kaimanovich [Kai03] is a little similar to our motivation. [Kai03] considers whether the harmonic measure on a boundary of Markov chain is singular with respect to a “canonical” measure on the boundary. It is also interesting to consider whether not only the harmonic measure but also other measures on the boundary defined in natural ways are singular or not with respect to the canonical measure. Indeed, in [Kai03, p.180], investigating dynamical properties for the Kusuoka energy measure is proposed, and recently [JOP17] considers them. The Kusuoka energy measure is not a Gibbs measure, and therefore, techniques of the thermodynamic formalism are not suitable to apply. See [JOP17, Section 3] for more details.

If the Hausdorff dimension of $\mu_y$ is strictly smaller than the Hausdorff dimension of a canonical measure, then, $\mu_y$ is singular with respect to the canonical measure. So, it is desirable to know an exact value or a good upper bound of the Hausdorff dimension of $\mu_y$. In general, for iterated function systems with place-dependent probability weights,\(^1\)

\(^1\) A singular function is a continuous, increasing function on $[0, 1]$ whose derivative is zero almost surely with respect to the Lebesgue measure.
deriving a dimension formula for invariant measures is valuable. For a class of iterated function systems which are driven by non-linear weak contractions and equipped with place-dependent probability weights, Jaroszewska-Rams [JR08] gave an upper bound for the Hausdorff dimension of an invariant measure in the form of the entropy divided by the Lyapunov exponent. For a large class of iterated function systems driven by similitudes with considerable overlaps, Hochman [Ho14] obtained the exact dimension formula. In order to give an upper bound for the Hausdorff dimension, we need to estimate the entropy and the Lyapunov exponent, both of which are the values of integrands with respect to μy, which might be singular with respect to the canonical measure on Z. However, intricated calculations are actually required for estimating the Hausdorff dimensions of invariant measures of an iterated function system, in particular when the probability weights of the iterated function system are place-dependent or the iterated function system is driven by non-similitudes. See Bárány-Pollicott-Simon [BPS12, Sections 7 and 8] and Bárány [Ba15, Section 5] for example.

In the case that Y consists of only one point and furthermore all Fi, i ∈ ΣN, are similitudes on Z, the open set condition holds for (Z, {Fi}) and μy is a self-similar measure where the corresponding probabilities {G1i} are not place-dependent, by Bandt and Graf [BG92] and Assumption 1.2. However, we mainly focus on the case that Y contains at least countably many points and the case that μy might not be a Gibbs measure or an invariant measure of an iterated function system.

In the case that Y contains countably many points and has a linear structure, the values of G1(y) and H1(y), i ∈ ΣN, can be non-linear functions on Y. So, even if we have a form of dimension formula for μy expressed by {G1i}, {H1i}, and μy, it is difficult for estimating dimH μy from possible dimension formulae. In this paper, we focus on the issue whether μy is singular or not with respect to the canonical measure on Z whose Hausdorff dimension is logN/log(1/r), rather than pursuing a form of dimension formula for μy. We give upper and lower bounds for the Hausdorff dimensions of μy without deriving any forms of dimension formulae.

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.3. Section 3 is devoted to deal with various examples including the restriction of harmonic functions, the energy measures on the Sierpinski gasket and de Rham’s functional equations. In Section 4, we state open problems.

2. Proof of Theorem 1.3

Let N := {1, 2, . . .}. Henceforth we put the product σ-algebra on a symbolic space (ΣN)N. Let π: (ΣN)N → K be a surjective map such that

\[ \pi(ix) = f_i(\pi(x)) \]  

for every i ∈ ΣN and x ∈ (ΣN)N. (2.1)

π is uniquely determined and is called the natural projection.

For n ∈ N, let Xn(x) be the projection of x ∈ (ΣN)N to n-th coordinate. Let \( F_n := \sigma(X_1, \ldots, X_n). \) This is a σ-algebra on (ΣN)N. For n ≥ 1, let a cylinder set

\[ I(i_1, \ldots, i_n) := \left\{ w \in (\Sigma_N)^N \mid X_k(w) = i_k, 1 \leq k \leq n \right\}, \]  
i_1, \ldots, i_n ∈ ΣN.

Consider the case that Z = (ΣN)N and F1(x) = ix.

By (1.2) and the Kolmogorov extension theorem, there exists a unique probability measure νy on (ΣN)N such that

\[ \nu_y(I(i_1, \ldots, i_n)) = \prod_{k=1}^{n} G_{i_k} \circ H_{i_{k-1}} \circ \cdots \circ H_{i_1}(y). \]
Then, \( \{ \nu_y \}_{y \in Y} \) is a family of solutions of (1.1). Then, by (2.1), a family of the push-forward measures \( \{ \nu_y \circ \pi^{-1} \}_{y \in Y} \) is a family of solutions of (1.1) for \( Z = K \) and \( F_i = f_i \).

Since each \( f_i \) is contractive, then, by following the proof of [F97, Theorem 2.8], we can show that a family of solutions of (1.1) for \( Z = K \) and \( F_i = f_i \) is unique. It follows that for every \( y \in Y \),

\[
\nu_y \circ \pi^{-1} = \mu_y. \tag{2.2}
\]

Let

\[
R_{y,n}(x) := \nu_y (I(X_1(x), \ldots, X_n(x))).
\]

It follows from induction in \( n \) that

**Lemma 2.1.**

\[
R_{y,n+1}(x) = R_{y,n}(x) G_{X_{n+1}}(x) \circ H_{X_n}(x) \circ \cdots \circ H_{X_1}(x)(y).
\]

Let

\[
P_N := \{(p_0, \ldots, p_{N-1}) \in [0, 1]^N \mid \sum_{i \in \Sigma_N} p_i = 1 \}.
\]

Define an entropy \( s_N : P_N \to \mathbb{R} \) by

\[
s_N(p_0, \ldots, p_{N-1}) := \sum_{i \in \Sigma_N} -p_i \log p_i.
\]

Here we put \( 0 \log 0 = 0 \).

It can occur that \( R_{y,n-1}(x) = 0 \), however, \( \nu_y(\{ x \in (\Sigma_N)^N \mid R_{y,n-1}(x) = 0 \}) = 0 \) holds for every \( n \geq 1 \). Hence if we say “\( \nu_y \)-a.s.\( x \)”, then, we can assume that \( R_{y,n-1}(x) > 0 \) for every \( n \). Then, by Lemma 2.1,

**Lemma 2.2.**

\[
E^{\nu_y} \left[ -\log \left( \frac{R_{y,n}}{R_{y,n-1}} \right) \bigg| \mathcal{F}_{n-1} \right] (x) = s_N \left( (G_j \circ H_{X_n-1}(x) \circ \cdots \circ H_{X_1}(y))_{j \in \Sigma_N} \right)
\]

holds for \( \nu_y \)-a.s.\( x \).

Let \( M_{y,0} = 0 \). For \( n \geq 1 \), let

\[
M_{y,n} - M_{y,n-1} := -\log \frac{R_{y,n}}{R_{y,n-1}} - E^{\nu_y} \left[ -\log \left( \frac{R_{y,n}}{R_{y,n-1}} \right) \bigg| \mathcal{F}_{n-1} \right].
\]

(If \( R_{y,n-1}(x) = 0 \), then, we let \( (M_{y,n} - M_{y,n-1})(x) := 0 \). but such \( x \) does not affect integrations with respect to \( \nu_y \).)

Then, \( \{M_{y,n}, \mathcal{F}_n\}_{n \geq 0} \) is a martingale under \( \nu_y \) and we have that

**Lemma 2.3.**

\[
\lim_{n \to \infty} \frac{M_{y,n}}{n} = 0, \text{ } \nu_y \text{-a.s.}
\]

**Proof.** This part will be shown in the same manner as [Ok14, Lemma 2.3 (2)] by using Jensen’s inequality and Doob’s submartingale inequality. Let

\[
C := \sup_{(p_i) \in P_N} \sum_{i \in \Sigma_N} p_i (-\log p_i)^2 < +\infty.
\]

Then, for every \( n \geq 1 \),

\[
E^{\nu_y} \left[ \left( \log \frac{R_{y,n+1}}{R_{y,n}} \right)^2 \right] < C.
\]

By this and Jensen’s inequality,

\[
\sup_{n \geq 0} E^{\nu_y} \left[ (M_{y,n+1} - M_{y,n})^2 \right] \leq 4C.
\]
By Doob’s submartingale inequality, we have that for every \( \epsilon > 0 \) and every \( n \geq 1 \),

\[
\nu_y \left( \max_{1 \leq k \leq 2^n} |M_{y,k}| \geq \epsilon 4^n \right) \leq \frac{\sum_{k \leq 2^n} \nu_y \left( (M_{y,k} - M_{y,k-1})^2 \right)}{\epsilon 4^n} \leq \frac{C}{\epsilon 4^{n-1}}.
\]

Therefore we have

\[
\limsup_{n \to \infty} \frac{|M_{y,n}|}{n} \leq \sqrt{\epsilon}, \ \nu_y\text{-a.s.}
\]

\( \square \)

For \( i \geq 1, y \in Y \) and \( x \in (\Sigma_N)^N \), let

\[
h_i(y; x) := H_{X_i(x)} \circ \cdots \circ H_{X_1(x)}(y),
\]

and

\[
p_i(y; x) := (G_j \circ h_i(y, x))_{j \in \Sigma_N} = (G_j \circ H_{X_i(x)} \circ \cdots \circ H_{X_1(x)}(y))_{j \in \Sigma_N}.
\]

By Lemmas 2.2 and 2.3,

**Proposition 2.4.**

\[
\limsup_{n \to \infty} \frac{-\log R_{y,n}(x)}{n} = \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} s_N \left( p_{i-1}(y; x) \right), \ \nu_y\text{-a.s.}
\]

and,

\[
\liminf_{n \to \infty} \frac{-\log R_{y,n}(x)}{n} = \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} s_N \left( p_{i-1}(y; x) \right), \ \nu_y\text{-a.s.}
\]

### 2.1. Proof of (i).

Now we assume both of (A-y) and (B-y) or, assume (sB-y) only.

**Lemma 2.5.** If

\[
\limsup_{n \to \infty} \frac{-\log R_{y,n}(x)}{n} \leq a, \ \nu_y\text{-a.s.},
\]

then, there exists a Borel subset \( B_0 \subset K \) such that \( \mu_y(B_0) = 1 \) and

\[
\dim_H(B_0) \leq \frac{a}{\log(1/r)}.
\]

**Proof.** Let \( A_{n,\epsilon} := \{ -\log R_n \leq n(a + \epsilon) \} \subset (\Sigma_N)^N \). Then, by the assumption and (2.2),

\[
\mu_y \left( \pi \left( \bigcap_{k \geq 1} \bigcup_{m \geq 1} \bigcap_{n \geq m} A_{n,1/k} \right) \right) = 1.
\]

Now it suffices to show that

\[
\dim_H \pi \left( \bigcap_{k \geq 1} \bigcup_{m \geq 1} \bigcap_{n \geq m} A_{n,1/k} \right) \leq \frac{a}{\log(1/r)}.
\]  \( \text{(2.3)} \)

Let

\[
A(n, s) := \{(i_1, \ldots, i_n) \in (\Sigma_N)^n \mid \nu_y(I(i_1, \ldots, i_n)) \geq \exp(-n(a + s)) \}.
\]

Since

\[
\sum_{(i_1, \ldots, i_n) \in (\Sigma_N)^n} \nu_y(I(i_1, \ldots, i_n)) = 1,
\]

\[|A(n, s)| \leq \exp(n(a + s)).\]
Fix $k \geq 1$ and $m \geq 1$. By Assumption 1.2 (i), $\text{diam} \left( \pi(I(i_1, \ldots, i_n)) \right) \leq c_1 r^n$. Then, for every $n \geq m$,
\[
\pi \left( \bigcap_{n \geq m} A_{n,1/k} \right) \subset \bigcup_{(i_1, \ldots, i_n) \in \mathcal{A}(n, 1/k)} \pi \left( I(i_1, \ldots, i_n) \right),
\]
and,
\[
\sum_{(i_1, \ldots, i_n) \in \mathcal{A}(n, 1/k)} \text{diam} \left( \pi \left( I(i_1, \ldots, i_n) \right) \right)^s \leq c_1^s |\mathcal{A}(n, 1/k)| r^{sn}
\]
\[
\leq c_1^s \exp \left( \left( \frac{a + 1}{k} + s \log r \right) n \right).
\]
Hence,
\[
\mathcal{H}_s \left( \pi \left( \bigcap_{n \geq m} A_{n,1/k} \right) \right) = 0, \text{ if } s > \frac{a + 1/k}{\log(1/r)},
\]
where we let $\mathcal{H}_s$ be the $s$-dimensional Hausdorff measure on $M$. Hence,
\[
\dim_H \pi \left( \bigcup_{m \geq 1} \bigcap_{n \geq m} A_{n,1/k} \right) \leq \frac{a + 1/k}{\log(1/r)}.
\]
Hence (2.3) follows. \hfill \Box

The following is different from a part of the proof of [Ok14, Theorem 1.2 (ii)], specifically, [Ok14, Lemma 3.3].

**Lemma 2.6.** Assume that (1.4) holds for $(i_k)_{1 \leq k \leq l}$. Then, for every $i \in \mathbb{N}$ and $x \in (\Sigma_N)^\mathbb{N}$ satisfying that $X_{i+k}(x) = i_k$, $1 \leq k \leq l$,
\[
\sum_{k=1}^l s_N (p_{i+k}(y; x)) \leq (l - 1) \log N + \sup \left\{ s_N ((p_j)_{j \in \Sigma_N}) \left| \sum_{j \in \Sigma_N} p_j - \frac{1}{N} \right| > \epsilon_0 \right\}.
\]
If there exist $\epsilon_0 \in (0, 1/N)$, $l \geq 1$ such that (1.4) holds for every $i_1, \ldots, i_l \in \Sigma_N$, then this inequality holds without the constraint that $X_{i+k}(x) = i_k, 1 \leq k \leq l$.

**Proof.** In this proof, $\| \cdot \|$ denotes the $\ell^1$-norm on $\mathbb{R}^l$.

Case 1. Since
\[
\left\| p_i(y; x) - \left( \frac{1}{N}, \ldots, \frac{1}{N} \right) \right\| > \epsilon_0,
\]
\[
s_N(p_i(y; x)) \leq \sup \left\{ s_N ((p_j)_{j \in \Sigma_N}) \left| \sum_{j \in \Sigma_N} p_j - \frac{1}{N} \right| > \epsilon_0 \right\}.
\]
Therefore,
\[
\sum_{k=1}^l s_N (p_{i+k}(y; x)) \leq (l - 1) \log N + \sup \left\{ s_N ((p_j)_{j \in \Sigma_N}) \left| \sum_{j \in \Sigma_N} p_j - \frac{1}{N} \right| > \epsilon_0 \right\}.
\]

Case 2. If
\[
\left\| p_i(y; x) - \left( \frac{1}{N}, \ldots, \frac{1}{N} \right) \right\| \leq \epsilon_0,
\]
and moreover $X_{i+k}(x) = i_k, 1 \leq k \leq l$, then, by (1.4),
\[
\left\| p_{i+l}(y; x) - \left( \frac{1}{N}, \ldots, \frac{1}{N} \right) \right\| \geq \left| G_{i+l}(h_{i+l}(y; x)) - \frac{1}{N} \right| > \epsilon_0,
\]
\[
\sum_{k=1}^l s_N (p_{i+k}(y; x)) \leq (l - 1) \log N + \sup \left\{ s_N ((p_j)_{j \in \Sigma_N}) \left| \sum_{j \in \Sigma_N} p_j - \frac{1}{N} \right| > \epsilon_0 \right\}.
\]
and hence,
\[ s_N(p_{i+l}(y; x)) \leq \sup \left\{ s_N \left( (p_j)_{j \in \Sigma_N} \right) \left| \sum_{j \in \Sigma_N} \left| p_j - \frac{1}{N} \right| \right| > \epsilon_0 \right\}. \]

Therefore,
\[ \sum_{k=1}^{l} s_N(p_{i+k}(y; x)) < (l-1) \log N + \sup \left\{ s_N \left( (p_j)_{j \in \Sigma_N} \right) \left| \sum_{j \in \Sigma_N} \left| p_j - \frac{1}{N} \right| \right| > \epsilon_0 \right\}. \]

Thus we have the assertion. \qed

**Lemma 2.7.** For every \( i_1, \ldots, i_l \in \Sigma_N \), there exists a non-random constant \( c_1 > 0 \) such that for \( \nu_y \)-a.s., there exists a random subset \( I(x) \subset \mathbb{N} \) such that
\[ \lim \inf_{n \to \infty} \frac{|I(x) \cap \{1, \ldots, n\}|}{n} \geq c_1, \]
and furthermore \( X_{i+k}(x) = i_k \) holds for every \( i \in I(x) \) and every \( 1 \leq k \leq l \).

**Proof.** Fix \( i_1, \ldots, i_l \in \Sigma_N \). Let
\[ \tilde{c} = \tilde{c}(y) := \inf_{i \in \Sigma_N, x \in Y(y)} G_i(z). \tag{2.4} \]

By (A-y), \( \tilde{c}(y) > 0 \). Let \( C(n) := \{ X_{(n-1)l+k} = i_k, 1 \leq k \leq l \} \). Let \( \tilde{M}_0 := 0 \) and for \( n \geq 1 \), \( \tilde{M}_n - \tilde{M}_{n-1} = 1_{C(n)} - \tilde{c} \). Then, \( |\tilde{M}_n - \tilde{M}_{n-1}| \leq 2 \), and, \( \{\tilde{M}_n, \mathcal{F}_n\}_n \) is a submartingale.

Then, by Azuma’s inequality \([A67]\),
\[ \nu_y \left( \sum_{k=1}^{n} 1_{C(k)} < -\frac{n \tilde{c}^2}{2} \right) = \nu_y \left( \tilde{M}_n < -\frac{n \tilde{c}^2}{2} \right) \leq \exp \left( -\frac{n \tilde{c}^2}{32} \right). \]

Hence by the Borel-Cantelli lemma,
\[ \lim \inf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} 1_{C(k)} \geq \frac{\tilde{c}^2}{2}, \nu_y \text{-a.s.} \]

**Proof of Theorem 1.3 (i).** By Lemmas 2.6 and 2.7, there exists \( \epsilon_1 > 0 \) such that
\[ \lim \sup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} s_N(p_k(y; x)) \leq \log N - \epsilon_1, \nu_y \text{-a.s.}, x \in (\Sigma_N)^N. \]

By this and Proposition 2.4,
\[ \lim \sup_{n \to \infty} -\frac{\log R_{y,n}}{n} \leq \log N - \epsilon_1, \nu_y \text{-a.s.} \]

By this and Lemma 2.5,
\[ \dim H \mu_y \leq \frac{\log N - \epsilon_1}{\log(1/r)}. \]

**2.2. Proof of (ii).** We now assume (wA-y).

**Lemma 2.8.** Assume
\[ \lim \inf_{n \to \infty} -\frac{\log R_{y,n}}{n} \geq a_2, \nu_y \text{-a.s.} \]

Then, \( \mu_y(K_1) = 0 \) holds for every Borel subset \( K_1 \) of \( K \) such that
\[ \dim H(K_1) < \frac{a_2}{\log(1/r)}. \tag{2.5} \]
\textbf{Proof.} Let $n \geq 1$ and $\delta > 0$. Assume (2.5) holds. Then, we can take open sets \{\(U_{n,l}\)\} in \(K\) such that for every \(l \geq 1\), \(\text{diam}(U_{n,l}) \leq c2^n\), \[K_1 \subset \bigcup_{l \geq 1} U_{n,l},\] (2.6) and \[
abla \text{diam}(U_{n,l})^{a_2/\log(1/r)} < \delta.\] (2.7)

Let \(k(n,l)\) be an integer such that \(c2^{k(n,l)} \leq \text{diam}(U_{n,l}) < c2^{k(n,l)-1}\). Then, by \(k(n,l) \geq n\), we have that
\[
\nu_y \left( I_{k(n,l)}(x) \right) \leq \exp(-k(n,l)a_2) \leq \text{diam}(U_{n,l})^{a_2/\log(1/r)}
\]
holds for every \(x \in \bigcap_{k \geq n} \{ -\log R_{y,k} \geq ka_2 \}\). By this and Assumption 1.2 (ii),
\[
\nu_y \left( \pi^{-1}(U_{n,l}) \cap \bigcap_{k \geq n} \{ -\log R_{y,k} \geq ka_2 \} \right) \leq D \text{ diam}(U_{n,l})^{a_2/\log(1/r)}.
\]
By this and (2.6) and (2.7),
\[
\nu_y \left( \pi^{-1}(K_1) \cap \bigcap_{k \geq n} \{ -\log R_{y,k} \geq ka_2 \} \right) \leq D\delta.
\]
By this, (2.2) and the assumption, \(\mu_y(K_1) = 0\). \qed

By Lemma 2.8, Proposition 2.4, and (wA-y),
\[
\dim_H \mu_y \geq \frac{-c \log c - (1-c)\log(1-c)}{\log(1/r)} > 0,
\]
where \(c\) is the constant in (wA-y).

3. Examples

This section is devoted to state various examples.

As we can see in [Kig95, Corollary 1.3], the Sierpinski gasket, the Sierpinski carpet, the Koch curve, and the Lévy curve satisfy Assumption 1.2. Let \(V_m := \bigcup_{i_1,\ldots,i_m \in \Sigma_N} f_{i_1,\ldots,i_m}(V_0)\). Let \(r_0 := \max_{i \in \Sigma_N} \text{Lip}(f_i)\).

\textbf{Lemma 3.1.} Assume that there exists \(D > 1\) such that for every large \(n\),
\[
\sup_{x \in V_n} \left| V_n \cap B(x, 2r_0^n \text{diam}(K)) \right| \leq D. \tag{3.1}
\]
Then, Assumption 1.2 (ii) holds for \(r = r_0\).

\textbf{Proof.} Assume \(\text{diam}(U) \leq \text{diam}(K)r_0^n\). Let \(x, y \in U\). Then there exist points \(x_n, y_n \in V_n\) such that \(\max\{d(x, x_n), d(y, y_n)\} \leq r_0^n \text{diam}(K)\). Then \(d(x_n, y_n) \leq 2r_0^n \text{diam}(K)\). By the assumption, there are at most \(D\) sets of forms \(f_{i_1,\ldots,i_n}(K)\) covering \(U\). \qed

\textbf{Example 3.2.} We consider the Euclid metric.

(i) If \(f_i(z) = (z + \iota)/N\), \(z \in \mathbb{R}\), then, \(K = [0, 1]\), and, Assumption 1.2 holds for \(r = 1/N\).

(ii) If \(N = 4\), \(f_i(z) = (z + q_i)/2\), \(z \in \mathbb{R}^2\), where \(\{q_i\}_i = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x, y \leq 1\}\), then, \(K = [0, 1]^2\), and, Assumption 1.2 holds for \(r = 1/2\).

(iii) If \(N = 3\), \(f_i(z) = (z + q_i)/2\), \(z \in \mathbb{R}^2\), where \(\{q_i\}_i\) forms an equilateral triangle, then, \(K\) is a 2-dimensional Sierpinski gasket, and, Assumption 1.2 holds for \(r = 1/2\).

(iv) If \(N = 8\), \(f_i(z) = (z + q_i)/3\), \(z \in \mathbb{R}^2\), where \(\{q_i\}_i = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x, y \leq 2\} \setminus \{(1, 1)\}\), then, \(K\) is a 2-dimensional Sierpinski carpet, and, Assumption 1.2 holds for \(r = 1/3\).
Proof. Let \( L := \{ \sum_i a_i q_i : a_i \in \mathbb{Z} \} \). This is a discrete subset of \([0,1]\) or \(\mathbb{R}^2\). By the definition of \( f_i \), it follows that for every \( n \), \( V_n \subset N^{-n} L \). Hence, \( \sup_{x,y \in V_n} d(x,y) \geq cN^{-n} \), and, (3.1) holds for some \( D \).

If \( Y \) is a one-point set, then, (1.1) does not depend on \( y \), so we drop the notation. If (A) holds and \( (K, \{ f_i \}_{i \in \Sigma_N}) \) is an iterated function system, then, the solution \( \mu \) of (1.1) is an invariant measure of the iterated function system.

Hereafter, if \( (G_0, \ldots, G_N) = (p_0, \ldots, p_N) \), then, we call \( \nu \) and \( \mu \) the \((p_0, \ldots, p_N)\)-Bernoulli measure and the \((p_0, \ldots, p_N)\)-self-similar measure, respectively. We denote them by \( \nu_{\{p_0, \ldots, p_N\}} \) and \( \mu_{\{p_0, \ldots, p_N\}} \), respectively.

3.1. Singularity with respect to self-similar measures.

Proposition 3.3 (Singularity with respect to self-similar measures). Let \( p_0, \ldots, p_{N-1} \) be positive numbers satisfying that \( \sum_{i \in \Sigma_N} p_i = 1 \). Assume that for some \( i_1, \ldots, i_\ell \),

\[
Y(y) \cap \bigcap_{i \in \Sigma_N} \left( G_i^{-1}([p_i - \epsilon_0, p_i + \epsilon_0]) \cap \bigcap_{j \in \Sigma_N} \left( G_j \circ H_{i_1} \circ \cdots \circ H_{i_\ell} \right)^{-1}([p_j - \epsilon_0, p_j + \epsilon_0]) = \emptyset. \tag{3.2}
\]

Then,

(i) \( \nu_y \) is singular with respect to \( \nu_{\{p_0, \ldots, p_{N-1}\}} \).

(ii) If moreover \( \pi^{-1}(\pi(A)) \setminus A \) is at most countable for every subset \( A \) of \( \{\Sigma_N\}^N \), \( \mu_y \) is singular with respect to \( \mu_{\{p_0, \ldots, p_{N-1}\}} \).

Proof. By Azuma’s inequality, \( \nu_{\{p_0, \ldots, p_{N-1}\}} \)-a.s., there are infinitely many \( i \) such that \( X_{i+k} = i_k \) for every \( 1 \leq k \leq l \). By (3.2), \( \nu_{\{p_0, \ldots, p_{N-1}\}} \)-a.s., there are infinitely many \( i \) such that

\[
\| p_i(y;x) - (p_0, \ldots, p_{N-1}) \| \geq \epsilon_0.
\]

Now assertion (i) follows from this and [Hi04, Theorem 4.1]. Assertion (ii) follows from (i), (2.2) and the assumption. \( \square \)

3.2. Energy measures on Sierpiński gaskets. [Kn89] shows that energy measures for canonical Dirichlet forms on Sierpiński gaskets are singular with respect to the Hausdorff measure on them. It is generalized by [BST99], [Hi04]. Recently, [JOP17] considers the Kusuoka measure from an ergodic theoretic viewpoint. Their framework covers a general class of measures that can be defined by products of matrices.

Let \( V_0 := \{q_0, q_1, q_2\} \) be the set of vertices of an equilateral triangle in \( \mathbb{R}^2 \). Let \( K \) be a 2-dimensional Sierpiński gasket, that is, the attractor of \( K = \cup_{i=0,1,2} f_i(K) \), where we let \( f_i(z) := (z + q_i) / 2, z \in \mathbb{R}^2 \).

Let \( a_i \in K, i = 0, 1, 2 \), be unique fixed points of \( F_i, i = 0, 1, 2 \), respectively. Let

\[
A_0 := \begin{pmatrix} 3/5 & 0 \\ 0 & 1/5 \end{pmatrix}, A_1 := \begin{pmatrix} 3/10 & \sqrt{3}/10 \\ \sqrt{3}/10 & 1/2 \end{pmatrix}, A_2 := \begin{pmatrix} 3/10 & -\sqrt{3}/10 \\ -\sqrt{3}/10 & 1/2 \end{pmatrix}.
\]

They are regular matrices and define linear transformation of \( Y \).

Let \( \| \cdot \| \) be the Euclid norm on \( \mathbb{R}^2 \). Let

\[
Y := S^1 = \{ x \in \mathbb{R}^2 : \|x\| = 1 \}.
\]

We regard \( Y \) as a topological space with respect to the Euclid distance on \( Y \subset \mathbb{R}^2 \). For \( y \in Y \) and \( i = 0, 1, 2 \), let

\[
G_i(y) := \frac{5}{3} \| A_i y \|^2, \text{ and, } H_i(y) := \frac{A_i y \| A_i y \|}.
\]

Lemma 3.4. (i)

\[
A_0^2 + A_1^2 + A_2^2 = \frac{3}{5} I_2,
\]
where $I_2$ denotes the identity matrix. In particular, (1.2) holds.

(ii) $(A-y)$ holds for every $y$.

Proof. (i) is immediately seen.

(ii) The set of eigenvalues of $A_0, A_1, A_2$ are $\{1/5, 3/5\}$. Hence, for every $i$ and $y$,

$$\frac{1}{15} \leq G_i(y) \leq \frac{3}{5}. \square$$

Lemma 3.5. (i) If $G_0(y) = G_0 \circ H_0(y)$, then, $y \in \{(\pm 1,0), (0,\pm 1)\}$ and furthermore

$$G_0(y) = G_0 \circ H_0(y) \in \left\{ \frac{1}{15}, \frac{3}{5} \right\}.$$

In particular, if $G_0(y) = 1/3$, then,

$$G_0 \circ H_0(y) \neq \frac{1}{3}.$$

(ii) $(B-y)$ holds for every $y \in Y$.

Proof. (i) is easy to see. For (ii), by using (i) and that fact that $G_i$ and $H_i$ are continuous on $Y$ and $Y$ is compact, $(B-y)$ follows. $\square$

Lemma 3.6. Assume $G_i(y) = p_i \in (0,1)$, $i = 0,1,2$. Then,

(i) If $y \notin \{(\pm 1,0), (0,\pm 1)\}$, then,

$$G_0(H_0(y)) \neq p_0.$$

(ii) If $y \in \{(\pm 1,0), (0,\pm 1)\}$, then,

$$G_0(H_1(y)) \neq p_0.$$

Now we can apply Theorem 1.3 and Proposition 3.3 to this case, ($N = 3, l = 2$)

Proposition 3.7. It holds that

$$0 < \dim_H \mu_y < \frac{\log 3}{\log 2}.$$

Furthermore, $\mu_y$ is singular with respect to every $(p_0,p_1,p_2)$-self-similar measure on $K$.

Let $f$ be a harmonic function on $K$. Let $h_1$ and $h_2$ be the harmonic functions on $K$ such that

$$(h_1(q_0), h_1(q_1), h_1(q_2)) = (0, \sqrt{2}, \sqrt{2}) \quad \text{and} \quad (h_2(q_0), h_2(q_1), h_2(q_2)) = (0, \sqrt{2/3}, -\sqrt{2/3}).$$

Let $v$ be the components of $f$ in $(h_1, h_2)$. Let $y = v/\|v\|$. Then, the energy measure associated with $f$ is $\mu_y$. (Cf. [BST99].)

Remark 3.8. In a formal level, the framework adopted in [Hi04] is interpreted as follows. See [Hi04, Section 2] for details of Dirichlet forms. Let $(K, \Sigma_N, \{\psi_i\}_{i \in \Sigma_N})$ be a self-similar structure and $\mu$ be the invariant measure. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(K, \mu)$. Assume (A1)-(A6) in [Hi04, Section 2].

$$Y := \mathcal{F},$$

$$G_i(f) := \frac{\mathcal{E}(f \circ \psi_i, f \circ \psi_i)}{\mathcal{E}(f,f)} \quad \text{if} \; \mathcal{E}(f,f) > 0,$$

$$G_i(f) := \frac{1}{N} \quad \text{if} \; \mathcal{E}(f,f) = 0,$$

$$H_i(f) := f \circ \psi_i.$$

Then, for $f \in Y$, $\mu_f$ is the normalized energy measure.
3.3. Restriction of harmonic function on Sierpiński gasket. [ADF14] considers the restriction on $[0, 1]$ of harmonic functions on the Sierpiński gasket, and shows that they are singular functions\(^2\) whenever they are monotone. The restrictions are among a wider class of functions containing several functions such as Lebesgue singular functions.

[ADF14, Notation 6] is interpreted as follows in our framework: Let

$$Y = [0, 1],$$

$$G_0(y) = \frac{2 + y}{5}, \quad G_1(y) = 1 - G_0(y) = \frac{3 - y}{5},$$

and

$$H_0(y) = \frac{1 + 2y}{2 + y}, \quad H_1(y) = \frac{y}{3 - y}.$$ \hspace{1cm} (3.5)

Then, \(f_y\) in [ADF14, Subsection 2.1] is the distribution function of \(\mu_y\). [ADF14, Theorem 3] is their main theorem, and, [ADF14, Lemma 24 and Theorem 25] restates it in a more general framework.

This is not of de Rham type in the subsection below, even if we exchange \(H_0\) with \(H_1\). Theorem 1.3 (i) gives the following improvements for [ADF14, Theorem 25]. (A-y) corresponds to [ADF14, assumption (a) in Lemma 24]. (sB-y) corresponds to [ADF14, assumption (b) in Lemma 24].

We loosen the assumptions in two ways and simultaneously obtain a stronger conclusion. The first way for weakening the assumption is adopting (B-y), which is strictly weaker than [ADF14, assumption (b) in Lemma 24]. The second way is the case that (A-y) fails but (sB-y) holds.

**Theorem 3.9** (Application to singularity for real functions). Let \(Y = Z = [0, 1]\) and \(f_i(z) = (z + i)/N, z \in [0, 1], i \in \Sigma_N\). Let \(y \in [0, 1]\). Assume \(\mu_y\) has no atoms. Let \(\varphi_y\) be the distribution function of \(\mu_y\). Then,

(i) If (A-y) and (B-y) hold, then, \(\dim_H \mu_y < 1\).

(ii) If (B-y) holds, then, \(\varphi_y\) does not have non-zero derivative at almost every point with respect to every \((p_0, \ldots, p_{N-1})\)-self-similar measure.

(iii) If (sB-y) holds, then, \(\varphi_y\) does not have non-zero derivative at every point, and, \(\dim_H \mu_y < 1\).

**Proof.** Assume that \(\varphi_y\) has non-zero derivative at \(\pi(x) \in (0, 1)\). Then,

$$\lim_{n \to \infty} G_{X_n(x)} \circ H_{X_{n-1}(x)} \circ \cdots \circ H_{X_1(x)}(y) = \frac{1}{2}. \hspace{1cm} (3.6)$$

Assume (B-y). Let \(\mu_{p_0,\ldots,p_{N-1}}\) be a Bernoulli measure on \(\{0,1\}^N\). Then by Azuma’s inequality,

$$\mu_{p_0,p_1} \left( \bigcap_{n \geq 1} \bigcup_{m \geq n} \bigcap_{k=1}^l \{ X_{m+k} = i_k \} \right) = 1.$$  

By this and (B-y), (3.6) fails for \(\mu_{p_0,\ldots,p_{N-1}}\)-a.e. \(x \in \{0,1\}^N\). Thus (ii) follows. (sB-y) implies that (3.6) fails for every \(x \in \{0,1\}^N\). \(\square\)

**Remark 3.10.** If we see the proof, assertion (ii) above holds even if we replace an arbitrarily Bernoulli measure with an arbitrarily measure satisfying that there exists \(c \in (0, 1)\) such that for every \(m, k \geq 1,\)

$$c \mu \left( \left[ \frac{k - 1}{2^m}, \frac{k}{2^m} \right] \right) \leq \mu \left( \left[ \frac{k - 1}{2^m}, \frac{2k - 1}{2^{m+1}} \right] \right) \leq (1 - c) \mu \left( \left[ \frac{k - 1}{2^m}, \frac{k}{2^m} \right] \right).$$

In the same manner as in the proof of assertion (ii) of Proposition 3.21 below,

\(^2\)A singular function is a continuous, increasing function on \([0, 1]\) whose derivative is zero almost surely with respect to the Lebesgue measure.
Proposition 3.11 (Singularity with respect to self-similar measures). If (3.3), (3.4) and (3.5) hold, then, $\mu_y$ is singular with respect to every $(p_0, p_1)$-self-similar measure.

Remark 3.12. Our main results might be applicable to different models of Sierpinski gaskets such as level 3 Sierpinski gaskets. [ADF14] deals with the two-dimensional standard Sierpinski gaskets.

Remark 3.13. The restrictions of harmonic functions on the attractors of different iterated function systems have also been considered. [ES16, Theorem 3.2] states that the restrictions of harmonic functions on the Hata tree is singular. Assume the framework of [ES16, Theorem 3.2]. Denote a restriction by $\phi_1$. Then, $\mu_1$ satisfies (1.1) for $x \\leq 1/N,

\phi(x) = \begin{cases} g_0(\phi(Nx)) & 0 \leq x \leq 1/N, \\ g_1(\phi(Nx - 1)) & 1/N \leq x \leq 2/N, \\ \ldots \\ g_{N-1}(\phi(Nx - (N - 1))) & (N - 1)/N \leq x \leq 1, \end{cases}

where each $g_i$ is a weak contraction (its definition is given below), $g_0(0) = 0$, $g_{N-1}(1) = 1$ and $g_i(0) = g_i(1)$ for each $i$. Solutions of de Rham’s functional equations give parameterizations of several self-similar sets such as the Koch curve and the Polya curve, etc. Some singular functions such as the Cantor, Lebesgue, etc. functions are solutions of such functional equations.

3.4. de Rham’s functional equations. Before we apply our results to a class of de Rham’s functional equations, we give a short review. De Rham [dR56, dR57] considered a certain class of functional equations. He considered the solution $\phi$ of the following functional equation which takes its values in a certain metric space:

$$\phi(x) = \begin{cases} 0 \leq x \leq 1/N, \\ 1/N \leq x \leq 2/N, \\ \ldots \\ (N - 1)/N \leq x \leq 1, \end{cases}$$

(3.8)

where each $g_i$ is a weak contraction (its definition is given below), $g_0(0) = 0$, $g_{N-1}(1) = 1$ and $g_i(0) = g_i(1)$ for each $i$. Solutions of de Rham’s functional equations give parameterizations of several self-similar sets such as the Koch curve and the Polya curve, etc. Some singular functions such as the Cantor, Lebesgue, etc. functions are solutions of such functional equations.

We give a short review of some known results. [BK00] considers self-similarity, inversion and composition of de Rham’s functions, and points out a connection with Collatz’s problem. [Kr09] shows connections between sums related to the binary sum-of-digits function and the Lebesgue’s singular function, and its partial derivatives with respect to the parameter. [Kaw11] investigates the set of points where Lebesgue’s singular function has the derivative zero. [P04] regards a de Rham curve as the limit of a polygonal arc by repeatedly cutting off the corners, obtain a formula for the local Hölder exponent of a
de Rham curve at each point, and describe the sets of points with given local regularity. \cite{Be08, Be12} consider multifractal and thermodynamic formalisms for the de Rham function. Recently, \cite{BKK} performs the multifractal analysis for the pointwise Hölder exponents of zipper fractal curves on $\mathbb{R}^d$ generated by affine mappings on $\mathbb{R}^d$, $d \geq 2$. \cite{BKK} and \cite{N04} consider the Hausdorff dimension of the image measure of the Lebesgue measure on an interval by the de Rham function. \cite{PV17, Section 9.3} also consider de Rham curves in terms of matrix products. \cite{DL91, DL92-1, DL92-2, P06} are related to wavelet theory. The length of de Rham curve is investigated in \cite{Me98} and \cite{DMS98}. \cite{TGD98} curves in terms of matrix products. \cite{DL91, DL92-1, DL92-2, P06} are related to wavelet

3.4.1. Dimension formula. Contrary to the examples in the above subsections, in this framework, $\mu_{g}$ is the invariant measure of a certain iterated function system with place-dependent probabilities, and furthermore, we have a form of dimension formula for $\mu_{g}$ for a large class of $(g_{i})_{i}$, thanks to Fan-Lau \cite{FL99}. By the Lebesgue-Stieltjes integral, for every bounded Borel measurable function $F : [0,1] \rightarrow \mathbb{R}$, we have

$$\int_{[0,1]} F(x) d\mu_{g}(x) = \sum_{i} \int_{[0,1]} g_{i}'(\varphi(x)) F \left( \frac{x + i}{N} \right) d\mu_{g}(x).$$

We assume that $g_{i} \in C^{2}([0,1])$ and $0 < g_{i}'(z) < 1$ hold for each $i \in \Sigma_{N}$ and every $z \in [0,1]$. Then, $\varphi$ is Hölder continuous and

$$\int_{0}^{1} \sup_{t} \left\{ \log g_{i}'(\varphi(s_{1})) - \log g_{i}'(\varphi(s_{2})) : |s_{1} - s_{2}| \leq t \right\} dt \leq \sup_{s \in [0,1]} \frac{|g_{i}''(s)|}{g_{i}'(s)} \int_{0}^{1} t c^{-1} dt < +\infty,$$

where we let $c = c_{\varphi}$ be a positive number.

Let $h$ be a positive function on $[0,1]$ such that

$$h(x) = \sum_{i \in \Sigma_{N}} g_{i}'(g_{i}(\varphi(x))) h \left( \frac{x + i}{N} \right),$$

$$h \circ \varphi^{-1}(x) = \sum_{i} g_{i}'(g_{i}(x)) h \circ \varphi^{-1}(g_{i}(x)).$$

This is unique under the constraint that

$$\int_{[0,1]} h(x) d\mu_{g}(dx) = 1.$$
See [FL99, Theorem 1.1].
Then, by [FL99, Corollary 3.5],
\[
\dim_H \mu_\varphi = \frac{\sum_{i \in \Sigma_N} \int_{[i/N,(i+1)/N]} h(x) \log \left( 1/g_i'(\varphi(x)) \right) \mu_\varphi(dx)}{\log N}.
\]
We have that
\[
\dim_H \mu_\varphi = \frac{\sum_{i \in \Sigma_N} \int_{[0,1]} H(g_i(y)) g_i'(y) \log \left( 1/g_i'(g_i(y)) \right) \ell(dy)}{\log N},
\]
where \( \ell \) is the Lebesgue measure on \([0,1]\) and \( H \) is a function on \([0,1]\) satisfying the following conditions:
\[
H(y) = \sum_{i \in \Sigma_N} g_i'(g_i(y)) H(g_i(y)), \quad \text{and} \quad \int_{[0,1]} H(y) \ell(dy) = 1. \tag{3.9}
\]
It is interesting to investigate properties for \( H \). If each \( g_i \) is linear, in other words, \( g_i' \) is a constant function, then \( \sum_{i \in \Sigma_N} g_i' = 1 \), and hence, \( H \equiv 1 \) satisfies (3.9).
We focus on the case that \( g_i \) is not affine. In that case, it is difficult for knowing whether
\[
\sum_{i \in \Sigma_N} \int_{[0,1]} H(g_i(y)) g_i'(y) \log \left( 1/g_i'(g_i(y)) \right) \ell(dy) < \log N.
\]
In the following subsection, we give a necessary and sufficient condition for \( \dim_H \mu_\varphi < 1 \) for a specific choice for \((g_i)_i\), by using Theorem 1.3. It is interesting to investigate properties for \( H \), but in this paper we do not analyze \( H \) directly.
Furthermore, by [FL99, Theorem 1.6],
\[
\lim_{n \to \infty} \frac{-\log \mu_\varphi(I_n(x))}{n} = \sum_{i \in \Sigma_N} \int_{[0,1]} h(z) \log \left( 1/g_i'(\varphi(z)) \right) d\mu_\varphi(z), \quad \mu_\varphi \text{-a.s.} x \in [0,1], \tag{3.10}
\]
where we let \( I_n(x) := [i/2^n, (i+1)/2^n] \) such that \( x \in [i/2^n, (i+1)/2^n] \). However, we do not see how to estimate the integrand in the right hand side of (3.10). Arguments in [FL99] depend on the fact \( \mu_\varphi \) is an invariant measure of an iterated function system, and we are not sure whether a convergence corresponding to (3.10) holds for the examples in the above subsections.

3.4.2. De Rham’s functional equations driven by \( N \) linear fractional transformations. [Ok14] considers de Rham’s functional equations driven by two linear fractional transformations, here we consider not only the case that \( N = 2 \) but also the case that \( N \geq 3 \). Our outline is similar to the one in [Ok14], however several additional considerations are needed.
In the following, we consider the equation (3.8) for the case that all \( g_i \) are linear fractional transformations. Let \( \Phi(A; z) := \frac{az+b}{cz+d} \) for a \( 2 \times 2 \) real matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( z \in \mathbb{R} \). Let
\[
g_i(x) := \Phi(A_i; x), \quad x \in [0,1], \quad i = 0, 1, \ldots, N-1,
\]
such that \( 2 \times 2 \) real matrices \( A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \quad i = 0, 1, \) satisfy the following conditions \((A1) - (A3)\).
\((A1)\) \( \Phi(A_0; 0) = 0, \Phi(A_{N-1}; 1) = 1, \) and
\[
\Phi(A_i; i) = \Phi(A_{i-1}; 1), \quad i = 1, \ldots, N-1.
\]
\((A2)\)
\[
\det A_i = a_id_i - b_ic_i > 0, \quad i = 0, 1, \ldots, N-1.
\]
Lemma 3.14. For each \( i \), \( \Phi (A_i; x) \) is a weak contraction on \([0, 1]\).

**Proof.** If \( c_i = 0 \), then, \( \Phi (A_i; x) = k ax + b_i \). By (A2) \( a_1 > 0 \). By (A1), \( a_i + b_i = b_{i+1} < 1 \) and \( b_i \leq 0 \) for \( i \leq N - 2 \), and, \( a_i + b_i \leq 1 \) and \( b_i > 0 \) for \( i \geq 1 \). Hence \( a_i < 1 \). Thus \( \Phi (A_i; x) \) is a contraction on \([0, 1]\).

If \( c_i \neq 0 \), then, by (A3), for every \( t > 0 \),

\[
\min_{i<|y-x|} \frac{a_i - b_i c_i}{(c_i x + 1)(c_i y + 1)} < 1.
\]

Hence, \( \Phi (A_i; x) \) is a weak contraction. 

Therefore, (A1) - (A3) guarantee that (3.8) has a unique continuous solution \( \varphi \). The above (sA3) is identical with (A3) in [Ok14].

We remark that our framework contains the cases that the technique of [La73] appearing in [Ha85, Theorem 7.3] and [SLK04, Proposition 3.1] is not applicable. By (A1) - (A3), \( b_i - c_i \leq 1 - a_i \leq 1 - b_i c_i \). Hence,

\[
(1 - a_i)^2 + 4b_i c_i \geq \min \{(b_i + c_i)^2, (1 + b_i c_i)^2\} \geq 0.
\]

Let

\[
\alpha := \min \left\{ 0, \frac{c_i}{1 - a_i}, \frac{a_i - 1 + \sqrt{(1 - a_i)^2 + 4b_i c_i}}{2b_i} \right\}, \quad 1 \leq i \leq N - 1.
\]

If \( a_0 = 1 \), then, we replace \( c_0/(1 - a_0) \) with \(-1\).

Let

\[
\beta := \max \left\{ 0, \frac{c_i}{1 - a_i}, \frac{a_i - 1 + \sqrt{(1 - a_i)^2 + 4b_i c_i}}{2b_i} \right\}, \quad 1 \leq i \leq N - 1.
\]

If \( a_0 = 1 \), then, we replace \( c_0/(1 - a_0) \) with \(+\infty\).

Let \( Y := [a, \beta] \). We consider the topology of \( Y \) defined by the Euclid metric. For \( k \in \Sigma_N \) and \( y \in Y \), let

\[
G_k(y) := \frac{(a_k - b_k c_k)(y + 1)}{(b_k y + 1)((a_k + b_k) y + c_k + 1)}, \quad H_k(y) := \frac{a_k y + c_k}{b_k y + 1}.
\]

If \( a_0 = 1 \), then, \( G_0(+\infty) := 1, G_k(+\infty) := 0, H_0(+\infty) := +\infty, H_k(+\infty) := \frac{a_k}{b_k}, 1 \leq k \leq N - 1.\)

**Lemma 3.15.** (i) If \( a_0 = 1 \), then, \( \alpha = -1 \) and \( \beta = +\infty \). If \( a_0 < 1 \) and \( b_{N-1} + c_{N-1} = 0 \), then, \(-1 \leq \alpha \leq \beta < +\infty \). If \( a_0 < 1 \) and \( b_{N-1} + c_{N-1} > 0 \), then, \(-1 < \alpha \leq \beta < +\infty \).

(ii) \( H_i(y) \in Y \) for \( i \in \Sigma_N, y \in Y \).

(iii) \( (3.12) \) holds.

**Proof.** (i) By (A1), it follows that if \( 1 \leq i \leq N - 2 \), then,

\[
\frac{a_i - 1 - \sqrt{(1 - a_i)^2 + 4b_i c_i}}{2b_i} < -1 < \frac{a_i - 1 + \sqrt{(1 - a_i)^2 + 4b_i c_i}}{2b_i}.
\]

(3.11)

By (A2) and (A3), \( b_{N-1} + c_{N-1} \geq 0 \). By this and (A1), it follows that if \( i = N - 1 \), then,

\[
\frac{a_{N-1} - 1 - \sqrt{(1 - a_{N-1})^2 + 4b_{N-1} c_{N-1}}}{2b_{N-1}} = -1,
\]

(3.12)
and,

\[-1 \leq \frac{c_{N-1}}{b_{N-1}} = \frac{a_{N-1} - 1 + \sqrt{(1 - a_{N-1})^2 + 4b_{N-1}c_{N-1}}}{2b_{N-1}}.\]  \hspace{1cm} (3.13)

Hence, if \(a_0 = 1\), then, \(\alpha = -1\). If \(a_0 < 1\), then, by (A1), \(a_0 < c_0 + 1\). Hence, if \(a_0 < 1\) and \(b_{N-1} + c_{N-1} > 0\), then, \(\alpha > -1\).

(ii) Let \(i = 0\). First we remark that \(H_0(z) = a_0z + c_0\). Assume \(a_0 < 1\). Then, by \(\alpha \leq c_0/(1 - a_0) \leq \beta\), we see that \(\alpha \leq H_0(\alpha) \leq H_0(\beta) \leq \beta\). If \(a_0 = 1\), then, \(\beta = +\infty\). It is easy to see that \(\alpha \leq H_0(\alpha)\).

Let \(1 \leq i \leq N - 1\). Then, \(H_i(z) \geq z\) if and only if

\[\frac{a_i - 1 - \sqrt{(1 - a_i)^2 + 4b_ic_i}}{2b_i} \leq z \leq \frac{a_i - 1 + \sqrt{(1 - a_i)^2 + 4b_ic_i}}{2b_i}.\]  \hspace{1cm} (3.14)

Hence, \(H_i(\beta) \leq \beta\).

By (A2), for every \(k\), \(H_k(z)\) is monotone increasing on \(z \geq -1\). By this, (3.11), (3.12), (3.13), and (3.14), \(H_i(\alpha) \geq \alpha\).

(iii) It is easy to see by calculations that for every \(i \geq 0\),

\[G_i(y) = \frac{(y + 1)(b_{i+1} - b_i)}{(b_{i+1}y + 1)(b_iy + 1)}.\]

The assertion follows from this and (A1).

**Lemma 3.16.** If (sA3) holds, then, (A-y) holds for \(y = 0\).

**Proof.** By (sA3), \(1 > a_0\) and \(b_{N-1} + c_{N-1} > 0\). By this and Lemma 3.15 (i), we have that \(-1 < \alpha \leq \beta < +\infty\). Therefore,

\[0 < G_0(\alpha) \leq G_0(\beta) < 1.\]

Let \(i \geq 1\). Since \(\alpha > -1\) and \(0 < b_i < 1\),

\[\inf_{y \geq \alpha} G_i(y) > 0.\]

By the definition of \(G_i\), we can show that if \(y \geq -1\), then, \(G_i(y) < 1\). Hence, by continuity of \(G_i\),

\[\sup_{y \in [\alpha, \beta]} G_i(y) < 1.\]

Hereafter we denote the set of fixed points of a map \(f\) by \(\text{Fix}(f)\).

**Remark 3.17.** (i) In the above proof, we have used \(b_{N-1} + c_{N-1} > 0\). However, if \(i < N - 1\), then, \(b_i + c_i > 0\) may fail.

(ii) If \(i \geq 1\), \(G_i\) may not be increasing.

(iii)

\[\bigcap_{i \in \Sigma_N} \text{Fix}(H_i) \leq |\text{Fix}(H_0)| = 1.\]

(iv) If \(a_0 < 1\), then, \(\text{Fix}(H_0) = \{c_0/(1 - a_0)\}\). If \(a_0 = 1\), then, by extending the domain of \(H_0\), \(\text{Fix}(H_0) = \{+\infty\}\).

**Lemma 3.18.** Assume \(\{\nu_y\}_{y \in Y}\) satisfies (1.1) for \(Z = (\Sigma_N)^N\) and \(F_i(x) = ix\). Then,

\[\pi(x) = \sum_{i \geq 1} \frac{X_i(x)}{N^i}, \quad \text{and, } \mu_\nu = \nu_0 \circ \pi^{-1}.\]

**Proof.** Let

\[
\begin{pmatrix}
p_{n}(x) & q_{n}(x) \\
r_{n}(x) & s_{n}(x)
\end{pmatrix} := A_{X_1(x)} \cdots A_{X_n(x)}, \quad n \geq 1.
\]
Theorem 3.19 (Upper bound for Hausdorff dimension). (i) If either
\[ \bigcap_{i \in \Sigma} G_i^{-1} \left( \frac{1}{N} \right) \neq \emptyset \]  
(3.15)
or
\[ \bigcap_{i \in \Sigma} G_i^{-1} \left( \frac{1}{N} \right) \subset \bigcap_{i \in \Sigma} \text{Fix}(H_i) \]  
(3.16)
fails, then,
\[ \dim_H \mu < 1. \]
(ii) If (3.15) and (3.16) hold, then, the distribution function of \( \mu \) is given by
\[ f(x) = \frac{x}{1 - C_{N,0}(x - 1)}, \]  
where we let \( C_{N,0} := \frac{c_0 N}{N - 1} \).  
(3.17)

In other words, if the solution \( \varphi \) of (3.8) is not of the form of (3.17), then, \( \dim_H \mu_{\varphi} < 1 \), and hence, \( \varphi \) is a singular function.

Proof. (i) If (3.15) fails and \( \beta < +\infty \), then, by the continuity of \( G_i \) and the compactness of \( Y \), \( \inf_{y \in Y} \sum_{i \in \Sigma} |G_i(y) - N^{-1}| > 0 \). This holds even when \( \beta = +\infty \), by recalling the definition of \( G_i(+\infty) \). Hence, 
\[ \sup_{x \in (\Sigma N)^{m^i}, y \in Y, i \in I} s_N(p_i(y; x)) < \log N. \]
Now the assertion follows from Proposition 2.4 and Lemma 2.5.

Assume that (3.15) holds and (3.16) fails. Using (3.15) and
\[ G_0^{-1} \left( \frac{1}{N} \right) = \left\{ \frac{1 + c_0 - a_0 N}{a_0(N - 1)} \right\}, \]
\[ \bigcap_{i \in \Sigma} G_i^{-1} \left( \frac{1}{N} \right) = \left\{ \frac{1 + c_0 - a_0 N}{a_0(N - 1)} \right\}. \]  
(3.18)
Since (3.16) fails, for some \(i\),
\[
H_i \left( \frac{1 + c_0 - a_0 N}{a_0(N - 1)} \right) \neq \frac{1 + c_0 - a_0 N}{a_0(N - 1)}.
\]
Since \(G_0\) is monotone increasing, (1.4) holds for \(i_1 = i\).

(ii) By the assumption, (3.18) holds, and hence it follows that
\[
G_i \left( \frac{1 + c_0 - a_0 N}{a_0(N - 1)} \right) = \frac{1}{N},
\]
and
\[
H_i \left( \frac{1 + c_0 - a_0 N}{a_0(N - 1)} \right) = \frac{1 + c_0 - a_0 N}{a_0(N - 1)}.
\]
By using them, we see that for every \(i \in \Sigma_N\),
\[
a_i = \frac{(N + 1)C_{N,0}b_i + 1}{N}, \quad \text{and} \quad c_i = \frac{C_{N,0}(N - 1 - C_{N,0}b_i)}{N}. \quad (3.19)
\]
We now show that
\[
b_i = \frac{i}{(N - i)C_{N,0} + N}. \quad (3.20)
\]
by induction in \(i\). We remark that by (A2) and (A3), \(c_0 + 1 > 0\), and hence \((N - i)C_{N,0} + N > 0\) for every \(i \leq N - 1\).

If \(i = 0\), then, by (A1), \(b_0 = 0\). Assume that (3.20) holds for \(i = k\). By (3.19), (3.20), and (A1),
\[
b_{k+1} = \frac{k + 1}{(N - k + 1)C_{N,0} + N}.
\]
Now it is easy to see that \(\varphi\) given by (3.17) satisfies (3.8).

Remark 3.20 (Lower bound for Hausdorff dimension). Assume \(a_0 < 1\) and \(b_{N-1} + c_{N-1} > 0\). Let \(\tilde{c}\) be the constant in (2.4). Then, by Proposition 2.4 and Lemma 2.8,
\[
\dim_H \mu_\varphi \geq \inf \left\{ \frac{\log s_N((p_j)_{j \in \Sigma_N})}{\log N} \mid (p_j)_{j \in \Sigma_N} \in P_N, \tilde{c} \leq p_j \leq 1 - \tilde{c}, \forall j \right\} > 0.
\]

Proposition 3.21 (Regularity and singularity with respect to self-similar measures). Let \(p_i \in (0, 1), 0 \leq i \leq N - 1\). Let \(e_0 := c_0/(1 - a_0)\) and assume \(a_0 < 1\). Then,
(i) If there exist \(p_i \in (0, 1), 0 \leq i \leq N - 1\) such that
\[
a_i = \frac{p_i + e_0 \sum_{j=0}^{i-1} p_j}{1 - \sum_{j=0}^{i-1} p_j} e_0 + 1, \quad (3.21)
\]
\[
b_i = \frac{\sum_{j=0}^{i-1} p_j}{1 - \sum_{j=0}^{i-1} p_j} e_0 + 1, \quad (3.22)
\]
\[
c_i = \frac{e_0 \left( \left(1 - \sum_{j=0}^{i-1} p_j \right) e_0 + 1 - p_i \right)}{1 - \sum_{j=0}^{i-1} p_j} e_0 + 1. \quad (3.23)
\]
hold for every \(i\), then, \(\mu_\varphi\) is absolutely continuous with \(p_0, \ldots, p_{N-1}\)-self-similar measure \(\mu_{(p_0, \ldots, p_{N-1})}\).

(ii) If the assumptions of (i) fails, then, \(\mu_\varphi\) is singular with respect to every \(p_0, \ldots, p_{N-1}\)-self-similar measure.

We do not know about an explicit expression for the Radon-Nikodym derivative \(d\mu_\varphi/d\mu_{(p_0, \ldots, p_{N-1})}\).

Proof. (i) We first remark that in this case, \(0 < a_0 = p_0 < 1\) and \(\beta < +\infty\). By computation,
\[
H_i(e_0) = e_0, \quad \text{and} \quad H_i'(e_0) = p_i,
\]
where $H'_i$ denotes the derivative of $H_i$.

Hence, each $H_i$ is contractive on a neighborhood $U$ of $e_0$. Since $H_0$ is contractive on $[\alpha, \beta]$, there exists $M$ such that $H_0^M(x) \in U$ holds for every $x \in [\alpha, \beta]$.

By Azuma’s inequality,

$$\nu_0 \left( \bigcup_{i \geq 1} \bigcap_{j=1}^M \{ X_{i+j} = 0 \} \right) = 1.$$ 

Hence,

$$\lim_{n \to \infty} H_{X_n(x)} \circ \cdots \circ H_{X_{i}(x)}(0) = e_0, \quad \nu_0\text{-a.s.}$$

and this convergence is exponentially fast\(^5\). Hence,

$$\sum_{n \geq 1} 1 - \sum_{i \in \Sigma_N} \sqrt{p_i G_i \left( H_{X_{n}(x)} \circ \cdots \circ H_{X_{i}(x)}(0) \right)} < +\infty, \quad \nu_0\text{-a.s.}$$

By this and [Sh80, Theorem VII.6.4], $\nu_0$ is absolutely continuous with respect to $\nu_{(p_0, \ldots, p_{N-1})}$.

Hence, $\mu_{\varphi}$ is absolutely continuous with respect to $\mu_{(p_0, \ldots, p_{N-1})}$.

(ii) Let $\mu_{(p_0, \ldots, p_{N-1})}$ be $(p_0, \ldots, p_{N-1})$-self-similar measure. By the definition of $\pi$, we have that $\pi^{-1}(\pi(A)) \setminus A$ is at most countable for every $A$. First we consider the case that $p_0 \neq a_0$. Then, $\text{Fix}(H_0) = e_0 \neq G_0^{-1}(p_0)$. Therefore, if $G_0(y) = p_0$, then, $G_0(H_0(y)) \neq p_0$. Thus (3.2) holds for $l = 1$ and $i_1 = 0$.

Assume that $p_0 = a_0$. Then, either (a) $G_i(e_0) = p_i, i \in \Sigma_N$, or (b) $H_i(e_0) = e_0, i \in \Sigma_N$, fails, because if both (a) and (b) hold, then, (3.21), (3.22) and (3.23) follow.

Assume (a) fails. Then, by using that $p_0 = a_0$ and $G_0$ is strictly increasing, $\cap_i G_i^{-1}(p_i) = \emptyset$. Since each $G_i$ is continuous and $Y$ is compact, (3.2) holds for every $l$ and $i_1, \ldots, i_l$.

Assume (b) fails. Then, $H_i(e_0) \neq e_0$ for some $i$, and (3.2) holds for $l = 1$ and $i_1 = i$. □

**Example 3.22** (Linear case). If all $c_i$ are zero, that is, all $g_i$ are affine maps, then, $\alpha = \beta = 0$, and hence $Y = \{0\}$ and $G_i(0) = a_i$. Then, $\mu_{\varphi}$ is $(a_0, \ldots, a_{N-1})$-self-similar measure. We have that

$$\dim_H \mu_{\varphi} = \frac{s_N((a_0, \ldots, a_{N-1}))}{\log N}.$$

**Example 3.23** ((B-y) fails but (sB-y) holds). Let $N = 2$. Consider (3.8) for

$$A_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix},$$

then, $g_0(x) = x/(x+1)$ and $g_1(x) = 1/(2-x)$. Then, (A1)-(A3) holds, and hence, a unique solution $\varphi$ of (3.8) exists. $\varphi$ is the inverse function of Minkowski’s question-mark function [Mi1904]. But (sA3) fails. Then, $Y = [-1, +\infty],

$$G_0(x) = \frac{x+1}{x+2}, \quad H_0(x) = x+1, \quad H_1(x) = -\frac{1}{x+2},$$

and, $\mu_{\varphi} = \mu_0$. In this case, (A-y) may fail, but (1.4) holds for every $i_1, i_2 \in \{0, 1\}$. By Lemma 2.6, there is a constant $\epsilon_1 > 0$ such that for every $i \in \mathbb{N}$ and every $x \in \{0, 1\}^N$, $\epsilon_1 + s_2(p_i(y; x)) + s_2(p_{i+1}(y; x)) < 2 \log 2$.

Hence, by Theorem 1.3 (i), we have that $\dim_H \mu_{\varphi} < 1$. In this case, (wA-y) fails for $y = 0$. We are not sure whether $\dim_H \mu_{\varphi} > 0$. By Proposition 3.3, $\mu_{\varphi}$ is singular with respect to every $(p_0, p_1)$-self-similar measure.

\(^5\)the speed of convergence depends on the choice of $x$. 

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Remark 3.24. If $\hat{\mu}$ is the measure on $[0, 1]$ such that its distribution function is Minkowski’s question-mark function, then, by [Kin60] it is known that

$$\dim_H \hat{\mu} = \frac{\log 2}{2 \int_{[0,1]} \log(1+x) \hat{\mu}(dx)} > \frac{1}{2}. \tag{3.24}$$

Remark 3.25. (i) In the case that each $g_i$ is a linear fractional transformation, we can take $Y$ as a subset of the set of real numbers. However, if some $g_i$ are not linear fractional transformations, then, we may not be able to take $Y$ as a subset of $\mathbb{R}$. Therefore, our approach may not work well, at least in a direct manner. One difficulty is that a set of functions can be very large, informally speaking.

(ii) The approaches in [Ha85, SLK04, Ok16] are different from the one used here. As an outline level, they are somewhat similar to each other.

3.5. Other examples.

Example 3.26. We give an example that (1.4) fails for every $i_1, \ldots, i_l = 1, 2$ both, but (1.4) holds for every $i_1, \ldots, i_l = 3$. Let $N = 2$. Let $Y = \mathbb{R}$. Let

$$G_0(x) := \frac{1}{6} 1_{\{x<0, x>1\}} + \left( x + \frac{1}{6} \right) 1_{\{0 \leq x \leq 1/2\}} + \left( \frac{7}{6} - x \right) 1_{\{1/2 \leq x \leq 1\}}.$$ 

Let

$$H_0(y) = H_1(y) = \frac{5 - 3y}{6}.$$ 

Then,

$$G_0 \left( \frac{1}{3} \right) = G_0 \left( H_0 \left( \frac{1}{3} \right) \right) = G_0 \left( \frac{2}{3} \right) = \frac{1}{2},$$

$$G_0 \left( H_1^2 \left( \frac{1}{3} \right) \right) = G_0 \left( \frac{1}{2} \right) = \frac{2}{3},$$

$$G_0 \left( H_0^3 \left( \frac{2}{3} \right) \right) = G_0 \left( H_0 \left( \frac{1}{2} \right) \right) = G_0 \left( \frac{7}{12} \right) = \frac{7}{12}.$$ 

Example 3.27. Let $N = 2$ and $Y = Z = [0, 1]$. Let

$$G_0(x) = H_0(x) = \frac{1}{4} 1_{\{x<1/4\}} + x 1_{\{1/4 \leq x \leq 3/4\}} + \frac{3}{4} 1_{\{x>3/4\}}.$$ 

Let $H_1(x) = 1 - G_0(x)$. Then,

$$G_0 \left( \frac{1}{2} \right) = G_1 \left( \frac{1}{2} \right) = H_0 \left( \frac{1}{2} \right) = H_1 \left( \frac{1}{2} \right) = \frac{1}{2},$$

and for every $y \neq 1/2$, $\dim_H \mu_y < 1$. Therefore, if we did not introduce $Y(y)$ in the condition (B-y) and simply assumed that the intersection of $\bigcap_{i \in \Sigma_N} G_i^{-1} \left( \left[ \frac{1}{N} - \epsilon_0, \frac{1}{N} + \epsilon_0 \right] \right)$ and $\bigcap_{j \in \Sigma_N} (G_j \circ H_{i_1} \circ \cdots \circ H_{i_n})^{-1} \left( \left[ \frac{1}{N} - \epsilon_0, \frac{1}{N} + \epsilon_0 \right] \right)$ is empty, then, the converse of Theorem 1.3 (i) would not hold.

Example 3.28. Fix $p \in (0, 1)$. Let $N = 2$ and

$$Y = [- \min\{p, 1-p\}^2, \min\{p, 1-p\}^2]$$

and $Z = [0, 1]$. Let

$$G_0(y) = p + \sqrt{|y|}, \text{ and } H_0(y) = H_1(y) = \frac{|y|}{|y| + 1}.$$ 

Then, for every $x \in (\Sigma_N)^N$ and $y \in Y$,

$$\lim_{n \to \infty} G_0 \circ H_{X_n(x)} \circ \cdots \circ H_{X_1(x)}(y) = p.$$
By this and Proposition 2.4,
\[ \lim_{n \to \infty} \frac{-\log R_{y,n}(x)}{n} = \lim_{n \to \infty} s_2 \left( (G_j \circ H_{X_0(x)} \circ \cdots \circ H_{X_1(x)}(y))_j \right) = s_2(p), \text{ } \nu_y\text{-a.s.}x. \]

By using Example 3.2 (i) and Lemmas 2.5 and 2.8, we can show that for every \( y \in Y \), \( \dim_{H} \mu_y = s_2(p)/\log 2 \).

Furthermore, \( \mu_y \) is the product measure on \( \{0,1\}^N \) such that
\[ \mu_y(X_n = 0) = p + \sqrt{H_0^{n-1}(y)} = p + \sqrt{\frac{|y|}{(n-1)|y|+1}}. \]

If \( y = 0 \), then, \( \mu_y \) is \((p,1-p)\)-self-similar measure. If \( y \neq 0 \), then, by Kakutani’s dichotomy [Kak48], \( \mu_y \) is singular with respect to \((p,1-p)\)-self-similar measure.

**Example 3.29.** Let \( Z = [0,1] \) and \( N = 2 \). Let \( Y = \mathbb{R} \). Let
\[ G_0(y) := \max \left\{ \frac{1}{2} - |y|, 0 \right\}, H_0(y) := (1 - \epsilon)y \text{, and, } H_1(y) := \frac{y}{\epsilon}. \]

Let \( y \neq 0 \). Then, \((A-y)\) fails. By using that \( H_0 \) is contractive and \( G_0(0) = G_1(0) = 1/2 \) and 0 is the fixed points of \( H_0 \) and \( H_1 \) both, we see that \((B-y)\) fails.

We will show that \( \dim_{H} \mu_y < 1 \). For simplicity we assume \( y = 1/4 \). By Azuma’s inequality, if we take sufficiently small \( \epsilon > 0 \), then, there exists \( D > 1 \) such that for every \( n \geq 1 \),
\[ \ell \left( \left\{ x \in \{0,1\}^N \bigg| \left| H_{X_n(x)} \circ \cdots \circ H_{X_1(x)} \left( \frac{1}{4} \right) \right| < \frac{1}{2} \right\} \right) \leq D^{-n}, \]

where \( \ell \) is the \((1/2,1/2)\)-Bernoulli measure on \( \{0,1\}^N \). Therefore, by the definition of \( \nu_y \), for each \( n \),
\[ \left| \{(i_1, \ldots, i_n) \in \{0,1\}^n \mid \nu_y(I(i_1, \ldots, i_n,0)) > 0 \} \right| \]
\[ = \left| \{i_1, \ldots, i_n) \in \{0,1\}^n \mid H_{i_n} \circ \cdots \circ H_{i_1} \left( \frac{1}{4} \right) \right| \leq \frac{1}{2} \right| \leq \left( \frac{2}{D} \right)^n. \]

Hence, by taking sum with respect to \( n \),
\[ \left| \{(i_1, \ldots, i_m) \in \{0,1\}^m \mid \nu_y(I(i_1, \ldots, i_m)) > 0 \} \right| \]
\[ = \sum_{n<m} \left| \{(i_1, \ldots, i_n) \in \{0,1\}^n \mid \nu_y(I(i_1, \ldots, i_n,0,1,1, \ldots)) > 0 \} \right| \]
\[ \leq \sum_{n<m} \left( \frac{2}{D} \right)^n \leq C \left( \frac{2}{D} \right)^m. \]

By using this and recalling that \( D > 1 \), \( \dim_{H} \mu_{1/4} < 1 \).

**Example 3.30.** Let \( Z = [0,1] \) and \( N = 2 \) and \( f_i(z) = (z+i)/2 \). It is easy to construct an example such that \( \mu_y \) is not absolutely continuous or singular with respect to \((p,1-p)\)-self-similar measure \( \mu_{(p,1-p)} \). Let \( Y = \{0,1,2\} \). Let
\[ H_0(0) = 1, H_1(0) = 2, \text{ and, } H_1(j) = j, j = 1, 2, i = 0, 1. \]

Let
\[ G_0(j) = \frac{1}{2}, j = 1, 2. \]

Let \( \mu_1 = \mu_{(p,1-p)} \) and \( \mu_2 \) be a probability measure which is singular with respect to \( \mu_{(p,1-p)} \). Then, by the uniqueness of the Lebesgue decomposition, \( \mu_0 \) is not absolutely continuous or singular with respect to \( \mu_{(p,1-p)} \).
4. Open problems

(1) Give estimates for the upper and lower local dimensions of $\mu_y$ and consider the Hausdorff dimensions for the level sets. Consider other notions of dimensions for $\mu_y$, such as $L^p$-dimensions. See [St93].

(2) If $Z = [0, 1]$, then, under what conditions is $\mu_y$ a Rajchman measure [R28], that is, a measure whose Fourier transform vanishes at infinity? See [Ly95] for more information of Rajchman measures. Recently, [JS16] shows that the Fourier coefficients of $\tilde{\mu}$ decay to zero, by considering conditions which assure a given measure invariant with respect to the Gauss map is a Rajchman measure, and using (3.24).

(3) Let $Z = [0, 1]$. It is natural to consider structures of $L^2([0, 1], \mu_y)$. For example, find a subset $P$ of $\mathbb{N}$ such that $\{\exp(2\pi i k) : k \in P\}$ forms an orthonormal basis of $L^2([0, 1], \mu_y)$. [JP98] considers this question for a class of self-similar measures.

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SCHOOL OF GENERAL EDUCATION, SHINSHU UNIVERSITY, 3-1-1, ASAHI, MATSUMOTO, NAGANO, 390-8621, JAPAN.

E-mail address: kazukio@shinshu-u.ac.jp