TOPOLOGICAL ORDERS AND EDGE EXCITATIONS IN FQH STATES

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Abstract: Fractional quantum Hall (FQH) liquids contain extremely rich internal structures which represent a whole new kind of ordering. We discuss the characterization and classification of the new orders (which is called topological orders). We also discuss the edge excitations in FQH liquids, which form the so-called chiral Luttinger liquids. The chiral Luttinger liquids at the edges also have very rich structures as a reflection of the rich topological orders in the bulk. Thus, edge excitations provide us a practical way to measure topological orders in experiments.

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1. Introduction

The fractional quantum Hall (FQH) effect appears in two-dimensional electron systems in a strong magnetic field. Since its discovery in 1982 [1, 2], experiments on FQH systems have continued to reveal many new phenomena and surprises. These, together with the observed rich hierarchical structures [3], indicate that electron systems that demonstrate a fractional quantum Hall effect (those systems are called FQH liquids) contain extremely rich internal structures. In fact FQH liquids represent a whole new state of matter. One needs to develop new concepts and new techniques to understand this new kind of states.

The first attempt to characterize the internal structures of FQH liquids was proposed in Ref. [4], where it was shown that a primary FQH liquid (with an inverse filling fraction $\nu^{-1} =$ integer) contains a off-diagonal-long-range-order in a non-local operator. Such an observation lead to a description of FQH liquids in terms of Ginzburg-Landau-Chern-Simons effective theory[5]. Those developments have lead to many interesting and important results and deeper understanding of FQH liquids. However, in this article I will try to describe the internal structures of FQH liquids from a more general point of view. It appears that some internal structures of FQH liquids (especially those in the so-call non-abelian FQH liquids) cannot be described by the Ginzburg-Landau-Chern-Simons effective theory and the associated off-diagonal-long-range-orders. Thus we need to develop more general concepts and formulations for the internal structures in FQH liquids. Let us start with an general discussion of correlated systems.

Gases are one of simplest systems. The motion of a molecule in a gas hardly depends on the positions and motion of other molecules. Thus gases are weakly correlated systems. As we lower the temperature, the motions of molecules become more and more correlated. Below a certain temperature the motions of molecules are completely correlated and the molecules form a crystal which is a strongly correlated system. In a crystal, an individual atom can hardly move by itself. Excitations in a crystal always correspond to collective motions of many atoms (which are called phonons).

FQH Hall liquids are another type of strongly correlated structure. Since electrons are much lighter than atoms, they have much stronger quantum motions. These quantum fluctuations prevent electrons in FQH systems from forming a crystal. Nevertheless, the motions of electrons in FQH liquids are highly correlated. For example let us look at Laughlin’s wave function for a filling fraction 1/3 FQH liquid [2]

$$\Psi_0 = \prod_{i < j} (z_i - z_j)^3 \exp\left(-\frac{1}{4} \sum_i |z_i|^2\right) \quad (1.1)$$

where $z_i = x_i + iy_i$ is the coordinate of the $i^{th}$ electron. We see from Laughlin’s wave function that

1) all electrons are in the first Landau level and doing their own cyclotron motion; and
2) the wave function has a third order zero as any pair of electrons approach each other, hence electrons try to stay away from each other as much as possible.

The Laughlin wave function represents a particular state in which all electrons do their own cyclotron motion in the first Landau level and at same time avoid collision with each other. This requires all the electrons in the Laughlin state to dance collectively following a strict global dancing pattern. This collective dancing minimizes the ground state energy and gives rise to a very stable correlated state. The strict collective dancing pattern makes FQH liquids highly organized systems which contain well defined internal orders. The highly correlated nature of FQH liquids also makes them incompressible.
liquids. A compression of FQH liquids create first order zeros in the Laughlin state and breaks the global dancing pattern of electrons, hence costing finite energies.

We would like to remark that above picture of collective dancing is quite vague and general. Collective motions of electrons exist in all correlated systems. To have a more concrete description of the internal structures in FQH liquids, one need to find measurable quantum numbers that reflect the internal correlations in FQH liquids. As we will see later that the correlations between electrons in FQH liquids are very special that we need a completely new set of quantum numbers to describe them.

Laughlin states represent only the simplest dancing pattern. Different FQH liquids contain different dancing patterns. Rich families of FQH liquids observed in experiments indicate that the possible dancing patterns are very rich. The internal orders (i.e., the dancing patterns) of FQH liquids are very different from the internal orders in other correlated systems, such as crystal, superfluids, etc. The internal orders in the latter systems can be described by order parameters associated with broken symmetries. As a result, the ordered states can be described by the Ginzburg-Landau effective theory. The internal order in FQH liquids is a new kind of ordering which cannot be described by order parameters associated with broken symmetries. Thus it would be very interesting to see what kind of effective theory describe the low energy physics of FQH liquids.

Despite the absence of broken symmetries and associated order parameters, the internal orders in FQH liquids are universal in the sense that they are robust against arbitrary perturbations. These (universal) internal orders characterize a phase in phase diagram. This makes FQH liquids a new state of matter. A concept of “topological order” was introduced to describe this new kind of ordering in FQH liquids [6,7]. Later we will give a more precise definition of the topological orders. We will introduce measurable quantities to characterize different topological orders. The universality of the topological orders means that these characterization quantities are independent of various perturbations.

It is also useful to introduce the notion of topological fluids to describe quantum liquids with non-trivial topological orders. FQH liquids are not the only kind of topological fluid. Other examples of topological fluids include anyon superfluids [8], chiral spin states [9,6], and short-ranged RVB states for spin systems [10]. A general but brief discussion of topological order can be found in Ref. [11].

It is instructive to compare FQH liquids with crystals. FQH liquids are similar to crystals in the sense that they both contain rich internal patterns (or internal orders). The main difference is that the patterns in the crystals are static related to the positions of atoms, while the patterns in FQH liquids are dynamic associated with the ways that electrons “dance” around each other. However, many of the same questions for crystal orders can also be asked and should be addressed for topological orders. One important question is: How do we characterize and classify the topological orders? Another is: How do we experimentally measure the topological orders?

1.1 Characterization of topological orders

In the above, the concept of topological order (the dancing pattern) is introduced through the ground state wave function. This is not quite correct because the ground state wave function is not universal. To establish a new concept, such as topological order, one needs to find physical characterizations or measurements of topological orders. Or, in other words, one needs to find universal quantum numbers which are independent of details of interactions, effective mass, etc., but can take different values for different classes of FQH liquids. The existence of such quantum numbers implies the existence of topological orders.
One way to show the existence of the topological orders in FQH liquids is to study their ground state degeneracies (in the thermodynamical limit). FQH liquids have a very special property. Their ground state degeneracy depends on the topology of space \[12,13,7\]. For example, the \( \nu = \frac{1}{q} \) Laughlin state has \( q^g \) degenerate ground states on a Riemann surface of genus \( g \) \[7\]. The ground state degeneracy in FQH liquids is not a consequence of symmetry of the Hamiltonian. The ground state degeneracy is robust against arbitrary perturbations (even impurities that break all the symmetries in the Hamiltonian)\[7\]. The robustness of ground state degeneracy indicates that the internal structures that give rise to ground state degeneracy is universal, hence demonstrating the existence of universal internal orders – topological orders.

The ground state degeneracy can be regarded as a new quantum number that (partially) characterize the topological orders in FQH states. In particular some hierarchical states with the same filling fraction can have different ground state degeneracies, indicating different topological orders in those hierarchical states. Thus the ground state degeneracies provide new information, in addition to the filling fraction, about the internal orders of FQH liquids. According to this picture we would like to suggest that the different hierarchical FQH states are characterized by the different topological orders instead of by their symmetry properties.

In compact space, the low-energy physics of FQH liquids is very unique. There is only a finite number of low-energy excitations (i.e., the degenerate ground states), yet the low-energy dynamics is non-trivial since the ground state degeneracy depends on the topology of the space. Such special low-energy dynamics, which depend only on the topology of the space, is described by the so called topological field theory which was under intensive study recently in the high-energy physics community\[14,15\]. Topological theories are effective theories for FQH liquids just as the Ginzburg-Landau theory for superfluids (or other symmetry broken phases).

The dependence of the ground-state degeneracy on the topology of the space indicates the existence of some kind of long range order (the global dancing pattern mentioned above) in FQH liquids, despite the absence of long-range correlations for all local physical operators. In some sense we may say that FQH liquids contain hidden long range orders.

1.2 Classification of topological orders

A long standing problem has been how to label and classify the rich topological orders in FQH liquids. We are able to classify all crystal orders because we know the crystal orders are described by symmetry group. But our understanding of topological orders in FQH liquids is very poor, and the mathematical structure behind the topological orders is unclear.

Nevertheless, we have been able to find a simple and unified treatment for a class of FQH liquids – abelian FQH liquids. It has been shown that the internal structures (the topological orders) in such fluids can be characterized by a symmetric integer matrix \( K \), a charge vector \( \mathbf{t} \) and a spin vector \( \mathbf{s} \) \[16,17,18,19,20,21\]. Various physical quantities can be determined from \((K, \mathbf{t}, \mathbf{s})\) that include ground state degeneracy, quasiparticle quantum numbers, structure of the edge excitations, exponents of electron and quasiparticle propagators along the edge, etc. All quasiparticle excitations in this class of FQH liquids have abelian statistics, which leads to the name abelian FQH liquids. The low energy effective theory of abelian FQH liquids is a topological Chern-Simons theory with several \( U(1) \) gauge fields \( a_I \mu \) \[16,7,18,19,20\]

\[
\mathcal{L} = \frac{1}{4\pi} K_{IJ} a_I \mu \partial_\nu a_J \lambda \varepsilon^{\mu \nu \lambda} - e A_\mu t_I \partial_\nu a_I \lambda \varepsilon^{\mu \nu \lambda} \tag{1.2}
\]
The above effective theory is the dual form of the Ginzburg-Landau-Chern-Simons effective theory discovered earlier [5]. Almost all FQH liquids observed in experiments were believed to be in this class.

We know that there are many different schemes to construct FQH states at filling fractions different from $1/m$ [3]. Different constructions in general lead to different wave functions, even when they have the same filling fraction. From the construction schemes themselves, it is usually difficult to see whether or not the FQH states described by those different wave functions belong to the same universality class. The above characterization (using $K$, $t$, and $s$) of the topological orders has an advantage in that it is independent of specific construction schemes and provides a universal description of the universality classes of abelian FQH liquids. One can usually derive an effective theory for the constructed FQH states from the construction schemes [18,17,19,20,21]. If two different constructions lead to the same $K$-matrix, charge vector $t$, and the spin vector $s$ (up to a field-redefinition of the gauge fields), then we can say that the two constructions generate the same FQH liquid. This approach allows us to determine [18,17] that the two 2/5 states obtained by hierarchical construction and Jain’s construction [3] belong to the same universality class.

The physical meaning of the $K$-matrix is quite transparent in a class of multi-layer FQH states obtained by generalizing Laughlin’s construction for the 1/m state [22]. The wave function of those multi-layer FQH states have the form

$$\prod_{a,b,i,j} (z_{ai} - z_{bj})^{\frac{1}{2}k_{ab}} \exp\left(-\sum |z_{ai}|^2\right)$$

where $z_{ai}$ is the coordinate of $i^{th}$ electron in the $a^{th}$ layer, and $k_{ab}$ are integers satisfying $k_{ab} = k_{ba}$ and $k_{aa} =$odd. The $K$-matrix that describes the multi-layer FQH state is nothing but the matrix formed by the integer exponents in (1.3): $K = (k_{ab})$. Thus, $K$ describes the pattern of zeros in the wave function which determines the way that electrons dance around each other. The zeros in the wave function can also be interpreted as a certain number of flux quanta being attached to each electron. For example the third order zero in the Laughlin wave function in (1.1) implies that three flux quanta are attached to the electrons, since the phase of the wave function changes by $3 \times 2\pi$ as one electron moves around another. In this sense $K$ also describes the way we attach flux quanta to electrons. It is obvious that the multi-layer FQH states naturally admit the $K$-matrix description. However, it is less obvious that the hierarchical states and the many states obtained in Jain construction also admit the $K$-matrix description.

The above classification of FQH liquids is not complete. Not every FQH state is described by $K$-matrices. In the last few years a new class of FQH states – non-abelian FQH states – was proposed [23,24,25]. A non-abelian FQH state contains quasiparticles with non-abelian statistics. The observed filling fraction 5/2 FQH state [26] is very likely to be one such state [27]. Recent studies in Refs. [23,28,25] reveal that the topological orders in some non-abelian FQH states can be described by conformal field theories. However, we are still quite far from a complete classification of all possible topological orders in non-abelian states.

1.3 Edge excitations – a practical way to measure topological orders in experiments

FQH liquids as incompressible liquids have a finite energy gap for all their bulk excitations. However, FQH liquids of finite size always contain one-dimensional gapless edge excitations, which is another unique property of FQH fluids. The structures of edge excitations are extremely rich which reflect the rich bulk topological orders. Different bulk topological orders lead to different structures of edge excitations. Thus we can study and measure the bulk topological orders by studying structures of edge excitations.
The edge excitations of integer quantum Hall (IQH) liquids was first studied by Halperin [29]. He found that edge excitations are described by 1D Fermi liquids. In the last few years we have begun to understand the edge excitations of FQH liquids. We found, due to the non-trivial bulk topological order, that electrons at the edges of (abelian) FQH liquids form a new kind of correlated state – chiral Luttinger liquids [30,31,32,18,33]. The electron propagator in chiral Luttinger liquids develops an anomalous exponent: \( \langle c^\dagger(t,x)c(0) \rangle \propto (x - vt)^{-g} \), \( g \neq 1 \). (For Fermi liquids \( g = 1 \)). The exponent \( g \), in many cases, is a topological quantum number which does not depend on detailed properties of edges. Thus, \( g \) is a new quantum number that can be used to characterize the topological orders in FQH liquids. Recently a Maryland-IBM group successfully measured the exponent \( g \) through the temperature dependence of tunneling conductance between two edges, [34] which was predicted to have the form \( \sigma \propto T^{2g-2} [35] \). This experiment demonstrates the existence of new chiral Luttinger liquids and opens the door to experimental study of the rich internal and edge structures of FQH liquids.

The edge states of non-abelian FQH liquids form more exotic 1D correlated systems which are not yet named. These edge states were found to be closely related to conformal field theories at 1+1 dimensions [25].

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* The discussion presented here applies only to the FQH states whose edge excitations all propagate in the same direction. This requires, for abelian states, all the eigenvalues of
have the form,

\[ S(f) \sim \sum_a C_a |f - f_a|^\gamma_a \]  

(1.4)

The frequencies and the exponents of the singularities \((f_a, \gamma_a)\) are determined by the topological orders. For the abelian state characterized by the matrix \(K\) and charge vector \(t\), the allowed values of the pair \((f_a, \gamma_a)\) are given by

\[ f_a = \frac{I^T}{e\nu} t^T K^{-1} l, \quad \gamma_a = 2l^T K^{-1} l - 1 \]  

(1.5)

where \(I^T = (l_1, l_2, \ldots)\) is an arbitrary integer vector, and \(\nu = t^T K^{-1} t\) is the filling fraction. The singularities in the noise spectrum are caused by quasiparticle tunneling between the two edges. The frequency of the singularity \(f_a\) is determined by the electric charge of the tunneling quasiparticle \(Q_q\): 

\[ f_a = \frac{I^T Q_q}{e\nu} \]  

The exponent \(\gamma_a\) is determined by the statistics of the tunneling quasiparticle \(\theta_q\): 

\[ \gamma = 2|\theta_q| - 1 \]  

Thus, the noise spectrum measures the charge and the statistics of the allowed quasiparticles.

![Diagram of a FQH fluid passing through a constriction](image)

Fig. 1.2: A FQH fluid passing through a constriction will generate narrow band noises due to the back scattering of the quasiparticles.

### 1.4 Organization

In Chapter 2 we will construct the effective theory of FQH liquids, through which we will demonstrate that the abelian FQH liquids are characterized by \(K\)-matrices and charge vectors. In Chapter 3 we will derive the low energy effective theory for the gapless edge excitations in FQH liquids from the bulk effective theory, and demonstrate the close connection between the bulk topological orders and the structures of edge states. We will also study some experimental consequence of the edge excitations. In particular we will discuss the effects of long range Coulomb interaction, smooth edge potential, and impurities on the structures of FQH edges. Chapters 4 and 5 contain some further study of topological orders, where we will introduce spin vectors and discuss classification of the abelian states. In Chapter 6, we make an attempt to describe non-abelian FQH states and their edge excitations using an operator algebraic approach.

I have tried to make each chapter more or less self-contained. Because of this, there are some overlaps between different chapters.
2. Effective theory of FQH liquids

One way to understand the topological orders is to construct a low-energy effective theory for FQH liquids. The effective theory should capture the universal properties of FQH liquids and provide hints on how to characterize and label different topological orders in FQH liquids. In the following we will review a way to construct effective theories that is closely related to the hierarchical construction proposed by Haldane and Halperin [3]. The first effective field theory for the Laughlin states was written in the form of a Ginzburg-Landau theory with a Chern-Simons term [5]. In this chapter we will adopt a different form of the effective theory, which contains only pure Chern-Simons terms. The pure Chern-Simons form is more compact and more convenient for studying hierarchical FQH states. The two forms of effective theory is related by duality transformations [36].

2.1 Effective theory of the hierarchical FQH states

In this section we will consider only single-layer spin-polarized quantum Hall (HQ) systems. To construct the effective theory for the hierarchical states, we will start with the effective theory (in the pure Chern-Simons form) of the Laughlin state. Then we will use the hierarchical construction to obtain the effective theory for the hierarchical states.

First let us review the effective theory for the Laughlin state [7,16,5]. Consider a charged boson or fermion system in a magnetic field,

\[ \mathcal{L} = -e A \cdot J + \text{Kinetic Energy} \]  

(2.1)

where

\[ J = \sum_i v_i \delta(x - x_i) \]

\[ J^0 = \sum_i \delta(x - x_i) \]  

(2.2)

are the current and the density of the particles and \((x_i, v_i)\) are the position and the velocity of the \(i^{th}\) particle. We have also assumed that the charge of each particle is \(-e\).

At a filling fraction \(\nu = 1/m\), where \(m\) is an even integer for boson and an odd integer for fermion, the ground state of the above system is given by the Laughlin wave function[2]

\[ \left[ \prod_i (z_i - z_j)^m \right] e^{-\frac{1}{2} \sum_i |z_i|^2} \]  

(2.3)

To construct the effective theory of such a state, we note that the state (2.3) is an incompressible fluid and that the particle number current \(J_\mu\) has the following response to a change of electromagnetic fields: \(\dagger\)

\[ -e \delta J_\nu = \sigma_{xy} \varepsilon^{\mu \nu \lambda} \partial_\nu \delta A_\lambda = \frac{\nu e^2}{2\pi} \varepsilon^{\mu \nu \lambda} \partial_\nu \delta A_\lambda \]  

(2.4)

as a result of finite Hall conductance \(\sigma_{xy} = \frac{\nu e^2}{h}\) (in this paper we always choose \(\hbar = 1\)).

In a hydrodynamic approach to the incompressible Hall liquid, we can write the effective

\(\dagger\) Here we use the convention that the Greek letters \(\mu, \nu\) represent space-time indices 0, 1, 2 and the Rome letter \(i, j\) represent spatial indices 1, 2.
theory in terms of the current. We choose the Lagrangian in such a way that it produces the equation of motion (2.4). It is convenient to introduce a \( U(1) \) gauge field \( a_\mu \) to describe the conserved particle number current:

\[
J_\mu = \frac{1}{2\pi} \partial_\nu a_\lambda \varepsilon^{\mu\nu\lambda} \tag{2.5}
\]

The current defined this way automatically satisfies the conservation law. Then the effective Lagrangian that produces (2.4) takes the following form

\[
\mathcal{L} = \left[ -m \frac{1}{4\pi} a_\mu \partial_\nu a_\lambda \varepsilon^{\mu\nu\lambda} - \frac{e}{2\pi} A_\mu \partial_\nu a_\lambda \varepsilon^{\mu\nu\lambda} \right] \tag{2.6}
\]

(2.6) describes only the linear response of the ground state to the external electromagnetic fields. To have a more complete description of the topological fluid such as QH liquid, we need to introduce the boson or fermion excitations in our effective theory.

In the effective theory (2.6), inserting the following source term which carries a \( a_\mu \)-charge \( q \),

\[
qa_0 \delta(x - x_0) \tag{2.7}
\]

will create an excitation of charge \( Q = -qe/m \). This can be seen from the equation of motion \( \frac{\delta\mathcal{L}}{\delta a_0} = 0 \),

\[
j_0 = \frac{1}{2\pi} \varepsilon_{ij} \partial_i a_j = -\frac{e}{2\pi m} B + \frac{q}{m} \delta(x - x_0) \tag{2.8}
\]

The first term indicates that the filling fraction \( \nu \equiv 2\pi \frac{j_0}{eB} \) is indeed \( \nu = 1/m \), and the second term corresponds to the increase of the particle density associated with the excitation.

We also see that the excitation created by the source term (2.7) is associated with \( q/m \) unit of the \( a_\mu \)-flux. Thus, if we have two excitations carrying \( a_\mu \)-charge \( q_1 \) and \( q_2 \), moving one excitation around the other will induce a phase \( 2\pi \times (\text{number of } a_\mu \text{-flux quanta}) \times a_\mu \text{-charge}, \)

\[
2\pi \times \frac{q_1}{m} \times q_2 \tag{2.9}
\]

If \( q_1 = q_2 \equiv q \), the two excitations will be identical. Interchanging them will induce half of the phase in (2.9),

\[
\theta = \pi \frac{q^2}{m} \tag{2.10}
\]

Here \( \theta \) is nothing but the statistical angle of the excitation that carries \( q \) unit of the \( a_\mu \)-charge.

Our bosons or fermions carry a charge of \(-e\). From the above discussion, we see that a charge \(-e\) excitation can be created by inserting a source term of \( m \) units of the \( a_\mu \)-charge. Such an excitation has a statistical angle \( \theta = \pi m \) (see (2.10)); thus it is a boson if \( m \) is even and a fermion if \( m \) is odd. Therefore, we can identify the excitations of \( m \) units of the \( a_\mu \)-charge with the particles (the bosons or the fermions) that form the QH liquid. We would like to stress that the identification of the fundamental particles (the bosons or the fermions) in the effective theory is very important. It is this identification, together with the effective Lagrangian, that provides a complete description of the topological properties of the QH liquid. We will see below that this identification allows us to determine the fractional charge and the fractional statistics of the quasiparticle excitations.
A quasihole excitation at a position described by a complex number \( \xi = x_1 + ix_2 \) is created by multiplying \( \prod_i (\xi - z_i) \) to the ground state wave function (2.3). Note that the phase of the wave function changes by \( 2\pi \) as a boson or a fermion going around the quasihole. Thus the quasihole also behaves like a vortex in the condensate of the bosons or the fermions.

Now let us try to create an excitation by inserting a source term of \( q \) units of the \( a_\mu \)-charge. Moving a boson or a fermion around such an excitation will induce a phase \( 2\pi q \) (see (2.9)). The single-value property of the boson or fermion wave function requires such a phase to be multiples of \( 2\pi \) and hence \( q \) must be quantized as an integer for allowed excitations. From the charge of the excitations, we find that \( q = -1 \) corresponds to the fundamental quasihole excitation described above, while \( q = 1 \) corresponds to the fundamental quasiparticle excitation. The quasiparticle excitation carries an electric charge \( -e/m \) and the quasihole \( e/m \). Both have statistics \( \theta = \pi/m \), as one can see from (2.10). We see that the effective theory reproduces the well known results for quasiparticles in Laughlin states [37]. The full effective theory with quasiparticle excitations is given by

\[
\mathcal{L} = \left[ -m \frac{1}{4\pi} a_\mu \partial_\nu a_\lambda \varepsilon^{\mu\nu\lambda} - \frac{e}{2\pi} A_\mu \partial_\nu a_\lambda \varepsilon^{\mu\nu\lambda} \right] + la_\mu j_\mu + \text{Kinetic Energy} \tag{2.11}
\]

where \( j_\mu \) is the current of the quasiparticles which has the form in (2.2). For fundamental quasiparticles the integer \( l \) in (2.11) has a value of \( l = 1 \), and for fundamental quasiholes \( l = -1 \). \( l \) takes other integer values for composite quasiparticles. (2.11), together with the quantization condition on \( l \), is a complete low energy effective theory which captures the topological properties of the \( 1/m \) Laughlin state. It can be shown that the effective theory (2.11) is simply the dual form [36,7] of the Ginzburg-Landau-Chern-Simons effective theory discovered earlier [5].

Now consider a \( 1/m \) FQH state formed by electrons, which corresponds to the fermion case discussed above. Let us increase the filling fraction by creating the fundamental quasiparticles, which are labeled by \( l = 1 \). (2.11) with \( l = 1 \) describes the \( 1/m \) state in the presence of these quasiparticles. Now, two equivalent pictures emerge:

a) In a mean-field-theory approach, we may view the gauge field \( a_\mu \) in (2.11) as a fixed background and do not allow \( a_\mu \) to respond to the inserted source term \( j_\mu \). In this case the quasiparticle gas behaves like bosons in the “magnetic” field \( b = \partial_i a_j \varepsilon_{ij} \), as one can see from the second term in (2.11). These bosons do not carry any electric charge since the quasiparticle number current \( j_\mu \) does not directly couple with the electromagnetic gauge potential \( A_\mu \). When the boson density satisfies

\[
j_0 = \frac{1}{p_2} \frac{b}{2\pi} \tag{2.12}
\]

where \( p_2 \) is even, the bosons have a filling fraction \( \frac{1}{p_2} \). The ground state of the bosons can again be described by a Laughlin state. The final electronic state that we obtained is just a second level hierarchical FQH state constructed by Haldane [3].

b) If we let \( a_\mu \) respond to the insertion \( j_\mu \), then quasiparticles will be dressed by the \( a_\mu \) flux. The dressed quasiparticles carry an electric charge of \( e/m \) and a statistics of \( \theta = \pi/m \). When the quasiparticles have the density

\[
j_0 = \frac{1}{(p_2 + \frac{\theta}{\pi})2\pi m} \frac{eB}{2\pi m} \tag{2.12}
\]
where $p_2$ is even, the quasiparticle will have a filling fraction $\frac{1}{(p_2 + \frac{\theta}{\pi})}$. In this case the quasiparticle system can form a Laughlin state described by the wave function

$$\prod_{i<j} (z_i - z_j)^{p_2 + \frac{\theta}{\pi}}$$

(2.13)

The final electronic state obtained this way is again a second level hierarchical FQH state. This construction was first proposed by Halperin [3]. The two constructions in a) and b) lead to the same hierarchical state and are equivalent.

In the following we will follow Haldane’s hierarchical construction to derive the effective theory of hierarchical FQH states. Notice that under the assumption a), the boson Lagrangian (the second term of (2.11) with $l = 1$) is just (2.1) with an external electromagnetic field $\mp A_\mu$ replaced by $a_\mu$. Thus, we can follow the same steps from (2.3) to (2.11) to construct the effective theory of the boson Laughlin state. Introducing a new $U(1)$ gauge field $\bar{a}_\mu$ to describe the boson current, we find that the boson effective theory takes the form

$$\mathcal{L} = -\frac{p_2}{4\pi} \bar{a}_\mu \partial_\nu \bar{a}_\lambda \varepsilon^{\mu\nu\lambda} + \frac{1}{2\pi} a_\mu \partial_\nu a_\lambda \varepsilon^{\mu\nu\lambda}$$

(2.13)

In (2.13) the new gauge field $\bar{a}_\mu$ describes the density $j^0$ and the current $j^i$ of the bosons and is given by

$$j^\mu = \frac{1}{2\pi} \partial_\nu \bar{a}_\lambda \varepsilon^{\mu\nu\lambda}$$

(2.14)

This reduces the coupling between $a_\mu$ and the boson current, $a_\mu j_\mu$, to a Chern-Simons term between $a_\mu$ and $\bar{a}_\mu$ (which becomes the second term in (2.13)). The total effective theory (including the original electron condensate) has the form

$$\mathcal{L} = \left[ -\frac{p_1}{4\pi} a_\mu \partial_\nu a_\lambda \varepsilon^{\mu\nu\lambda} - \frac{e}{2\pi} A_\mu \partial_\nu a_\lambda \varepsilon^{\mu\nu\lambda} \right] + \left[ -\frac{p_2}{4\pi} \bar{a}_\mu \partial_\nu \bar{a}_\lambda \varepsilon^{\mu\nu\lambda} + \frac{1}{2\pi} a_\mu \partial_\nu \bar{a}_\lambda \varepsilon^{\mu\nu\lambda} \right]$$

(2.15)

where $p_1 = m$ is an odd integer. (2.15) is the effective theory of a second level hierarchical FQH state.

The second level hierarchical FQH state contains two kinds of quasiparticles. One is quasihole (or vortex) in the original electron condensate, and the other is quasihole (or vortex) in the new boson condensate. The two kinds of the quasiholes are created by inserting the source terms $-j^\mu a_\mu$ and $-\bar{j}^\mu \bar{a}_\mu$, respectively, where $j^\mu$ and $\bar{j}^\mu$ have a similar form as in (2.2). The first kind of quasihole is created by multiplying $\prod_i (\xi - z_i)$ to the electron wave function, while the second kind is created by multiplying $\prod_i (\eta - \xi_i)$ to the boson Laughlin wave function (here $\xi_i$ are the complex coordinates of the boson and $\eta$ is the position of the quasihole). The effective theory for the quasiholes in the second level hierarchical states has the form

$$\bar{j}^\mu \bar{a}_\mu + j^\mu a_\mu + \text{Kinetic Energy}$$

(2.16)

We can use the effective theory in (2.15) and (2.16) to calculate the quantum numbers of the quasiholes.

The total filling fraction can be determined from the equation of motion $\frac{\delta \mathcal{L}}{\delta a_0} = \frac{\delta \mathcal{L}}{\delta \bar{a}_0} = 0$,

$$-eB = p_1 b - \bar{b}, \quad b = p_2 \bar{b}$$

(2.17)
We find
\[ \nu = \frac{b}{-eB} = \frac{1}{p_1 - \frac{1}{p_2}}. \] (2.18)

(2.15) can be written in a more compact form by introducing \((a_{1\mu}, a_{2\mu}) = (a_{\mu}, \bar{a}_{\mu})\),
\[
\mathcal{L} = -\sum_{I, I'} \frac{1}{4\pi} K_{II'} a_{I\mu} \partial_{\nu} a_{I'\lambda} \epsilon^{\mu \nu \lambda} - \frac{e}{2\pi} A_{\mu} \partial_{\nu} t_I a_{I\lambda} \epsilon^{\mu \nu \lambda}
\] (2.19)

where the matrix \( K \) has integer elements,
\[
K = \begin{pmatrix} p_1 & -1 \\ -1 & p_2 \end{pmatrix}
\] (2.20)

and \( t^T = (t_1, t_2) = (1, 0) \) will be called the charge vector. The filling fraction (2.18) can be rewritten as \( \nu = t^T K^{-1} t \).

A generic quasiparticle is labeled by two integers that consist of \( l_1 \) number of quasiparticles of the first kind and \( l_2 \) number of quasiparticles of the second kind. Such a quasiparticle carries \( l_1 \) units of the \( a_{1\mu} \) charge and \( l_2 \) units of the \( a_{2\mu} \) charge and is described by
\[
(l_1 a_{1\mu} + l_2 a_{2\mu}) j^\mu
\] (2.21)

After integrating out the gauge fields, we find that such a quasiparticle carries \( \sum_J K_{IJ} l_J \) units of the \( a_I^{\mu} \)-flux. Hence the statistics of such a quasiparticle are given by
\[
\theta = \pi t^T K^{-1} t = \frac{1}{p_2 p_1 - 1} (p_2 l_1^2 + p_1 l_2^2 + 2 l_1 l_2)
\] (2.22)

and the electric charge of the quasiparticle is
\[
Q_q = -e t^T K^{-1} t = -e \frac{p_2 l_1 + l_2}{p_2 p_1 - 1}
\] (2.23)

The above construction can be easily generalized to the level-3 hierarchical FQH states with the filling fraction
\[
\nu = \frac{1}{p_1 - \frac{1}{p_2} - \frac{1}{p_3}}
\] (2.24)

by allowing the boson (described by \( \bar{j} \) current) in (2.16) to condense into a \( 1/p_3 \) Laughlin state. The effective theory of this third level hierarchical state still has the form in (2.19), but now \( I, I' \) run from 1 to 3. The new \( K \)-matrix has the form
\[
K = \begin{pmatrix} p_1 & -1 & 0 \\ -1 & p_2 & -1 \\ 0 & -1 & p_3 \end{pmatrix}
\] (2.25)

and the charge vector \( t^T = (1, 0, 0) \). One can go further to obtain the effective theories of the \( n^{th} \) level hierarchical states. In the hierarchical construction one always assumes that the quasiparticles from the last condensate “condense” to get the next level hierarchical state. Therefore the \( K \)-matrix \( K \) for a hierarchical FQH state has a tri-diagonal form
\[
K_{II'} = p_I \delta_{I, I'} - \delta_{I, I'-1} - \delta_{I, I'+1}
\] (2.26)
with the charge vector given by

\[ t_I = \delta_{1I} \tag{2.27} \]

In (2.26) \( p_1 \) is odd and \( p_i |_{i>1} \) are even. The filling fraction of such a state is given by [18]

\[ \nu = t^T K^{-1} t = \frac{1}{p_1 - \frac{1}{p_2 - \frac{1}{\ddots - \frac{1}{p_n}}} \tag{2.28} \]

We can also construct FQH states which are more general than those obtained from the standard hierarchical construction. The effective theory for these more general FQH states still has the form in (2.19) but now \( I \) runs from 1 to an integer \( n \) (\( n \) will be called the level of the FQH state). To obtain the form of the matrix \( K \), let us assume that at level \( n-1 \) the effective theory is given by eq. (2.19) with \( a_{I\mu} \), \( I = 1, \ldots, n-1 \) and \( K = K^{(n-1)} \). The quasiparticles carry integer charges of the \( a_{I\mu} \) gauge fields. (This can always be achieved by properly normalizing the gauge fields.) Now consider an \( n \)th level hierarchical state which is obtained by “condensation” of quasiparticles with the \( a_{I\mu} \) charge \( l_{I|I=1,\ldots,n-1} \). The effective theory of this level \( n \) hierarchical state will be given by eq. (2.19) with \( n \) gauge fields. The \( n \)-th gauge field \( a_{n\mu} \) comes from the new condensate. The matrix \( K \) is given by

\[ K^{(n)} = \begin{pmatrix} K^{(n-1)} & -l \\ -l^T & p_n \end{pmatrix} \tag{2.29} \]

with \( p_n = \text{even} \). The charge vector \( t \) is still given by (2.27). By iteration, we see that the generalized hierarchical states are always described by integer symmetric matrices, with \( K_{II} = \text{even} \) except \( K_{11} = \text{odd} \). The new condensate gives rise to a new kind of quasiparticle which again carries integer charge of the new gauge field \( a_{n\mu} \). Hence, a generic quasiparticle always carries integral charges of the \( a_{I\mu} \) field. Assume those charges are \( l_I \). Then the electric charge and the statistics of the quasiparticle are given by the following general formulae

\[ \theta = \pi t^T K^{-1} l, \quad Q_q = -e t^T K^{-1} l \tag{2.30} \]

The filling fraction is given by

\[ \nu = t^T K^{-1} t \tag{2.31} \]

From the above discussion we see that more general hierarchical states, in principle, can be obtained by assuming condensation of different types of quasiparticles. In the standard Haldane-Halperin hierarchical scheme one assumes the quasiparticles from the last condensate condense to generate next level hierarchical states. This choice of condensing quasiparticles is valid if such quasiparticles have the smallest energy gap. However, in principle one can not exclude the possibility that a different type of quasiparticles (rather than the one from the last condensate) might have the smallest energy gap. In this case, we will obtain a new hierarchical state from the condensation of different quasiparticles. Certainly for the FQH systems used in real experiments, there are good reasons to believe that the quasiparticles from the last condensate do have the smallest energy gap and it is those quasiparticles that generate the next level hierarchical states. It would be interesting to find out under which conditions other types of quasiparticles may have smaller energy gaps.
From the generic effective theory in (2.19) we obtain one of the important results in this paper; that the generalized hierarchical states can be labeled by an integer valued $K$-matrix and a charge vector $t$. Now we would like to ask a question: Do different $(K,t)$s describe different FQH states? Notice that, through a redefinition of the gauge fields $a_{I\mu}$, one can always diagonalize the $K$-matrix into one with $\pm 1$ as the diagonal elements. Thus, it seems that all $K$-matrices with the same signature describe the same FQH states, since they lead to the same effective theory after a proper redefinition of the gauge fields. Certainly this conclusion is incorrect. We would like to stress that the effective Lagrangian (2.19) alone does not provide a proper description of the topological orders in the hierarchical states. It is the effective Lagrangian (2.19) together with the quantization condition of the $a_{I\mu}$ charges that characterize the topological order. A $U(1)$ gauge theory equipped with a quantization condition on the allowed $U(1)$ charges is called compact $U(1)$ theory. Our effective theory (2.19) is actually a compact $U(1)$ theory with all $U(1)$ charges quantized as integers. Thus the allowed $U(1)$ charges form an $n$-dimensional cubic lattice which will be called the charge lattice. Therefore, when one considers the equivalence of two different $K$-matrices, one can use only the field redefinitions that keep the charge quantization condition unchanged (i.e., keep the charge lattice unchanged). The transformations that map the charge lattice onto itself belong to the group $SL(n,Z)$ (a group of integer matrices with a unit determinate)

$$a_{I\mu} \rightarrow W_{IJ}a_{J\mu}, \quad W \in SL(n,Z)$$

(2.32)

From the above discussion we see that the two FQH states described by $(K_1,t_1)$ and $(K_2,t_2)$ are equivalent (i.e., they belong to the same universality class) if there exists a $W \in SL(n,Z)$ such that

$$t_2 = Wt_1, \quad K_2 = WK_1W^T$$

(2.33)

This is because, under (2.33), an effective theory described by $(K_1,t_1)$ simply changes into another effective theory described by $(K_2,t_2)$.

We would like to remark that in the above discussion we have ignored another topological quantum number – spin vector. Because of this, the equivalence condition in (2.33) does not apply to clean systems. However (2.33) does apply to disordered FQH systems because the angular momentum is not conserved in disordered systems and the spin vector is not well defined. A more detailed discussion of the spin vectors will be given in chapter 4.

In the following we list the $K$-matrix, the charge vector $t$, and the spin vector $s$ for some common single-layer spin-polarized FQH states:

$$\nu = 1/m, \quad t = (1), \quad K = (m), \quad s = (m/2)$$

$$\nu = 1-1/m, \quad t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 1 \\ 1 & -(m-1) \end{pmatrix}, \quad s = \begin{pmatrix} 1/2 \\ (1-m)/2 \end{pmatrix}$$

$$\nu = 2/5, \quad t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad K = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}, \quad s = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

$$\nu = 3/7, \quad t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad K = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad s = \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix}$$

(2.34)

2.2 Effective theory of simple multi-layer FQH states

The same approach used to construct the effective theory of the hierarchical states can also be used to construct the effective theory for the multi-layer FQH states. The
connection between the FQH wave function and the $K$-matrix becomes very transparent for the multi-layer FQH states. In this section we will concentrate on double-layer FQH states. However, the generalization to the $n$-layer FQH states is straightforward.

We would like to construct an effective theory for the following simple double-layer FQH state,

$$\prod_{i<j} (z_{1i} - z_{1j})^l \prod_{i<j} (z_{2i} - z_{2j})^m \prod_{i,j} (z_{1i} - z_{2j})^n e^{-\frac{1}{4}(\sum_i |z_{1i}|^2 + \sum_j |z_{2j}|^2)} \tag{2.35}$$

where $z_{Ii}$ is the complex coordinate of the $i^{th}$ electron in the $I^{th}$ layer. Here $l$ and $m$ are odd integers so that the wave function is consistent with the Fermi statistics of the electrons, while $n$ can be any non-negative integer. The above wave function was first suggested by Halperin as a generalization of the Laughlin wave function [22]. It appears that these wave functions can explain some of the dominant FQH filling fractions observed in double layer FQH systems.

We start with a single layer FQH state in the first layer,

$$\prod_{i<j} (z_{1i} - z_{1j})^l e^{-\frac{1}{4} \sum_i |z_{1i}|^2} \tag{2.36}$$

which is a $1/l$ Laughlin state and is described by the effective theory

$$\mathcal{L} = \left[-l \frac{1}{4\pi} a_{1\mu} \partial_\nu a_{1\lambda} \epsilon^{\mu\nu\lambda} - \frac{e}{2\pi} A_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} \right] \tag{2.37}$$

where $a_{1\mu}$ is the gauge field that describes the electron density and current in the first layer.

Examining the wave function in (2.35) we see that an electron in the second layer is bounded to a quasihole excitation in the first layer. Such a quasihole excitation is formed by $n$ fundamental quasihole excitations and carries an $a_{1\mu}$-charge of $-n$. A gas of the quasiholes is described by the following effective theory

$$\mathcal{L} = -na_{1\mu} j_\mu + \text{Kinetic Energy} \tag{2.38}$$

where $j_\mu$ has the form in (2.2). As we mentioned before, in the mean-field theory, if we ignore the response of the $a_{1\mu}$ field, (2.38) simply describe a gas of bosons in a magnetic field $-nb_1$ where $b_1 = \epsilon_{ij} \partial_i a_{1j}$. Now we would like to attach an electron (in second layer) to each quasiholes in (2.38). Such an operation has two effects: a) the bound state of the quasiholes and the electrons can directly couple with the electromagnetic field $A_\mu$ since the electron carries the charge $-e$; and b) the bound state behaves like a fermion. The effective theory for the bound states has the form

$$\mathcal{L} = (-eA_\mu - na_{1\mu}) j_\mu + \text{Kinetic Energy} \tag{2.39}$$

which now describes a gas of fermions (in the mean field theory). These fermions see an effective magnetic field $-eB - nb_1$.

When the electrons (i.e., the fermions in (2.39)) in the second layer have a density $\frac{1}{m} \frac{-eB - nb_1}{2\pi} \ 	ext{(i.e., have an effective filling fraction}\ 1/m)$, they can form a $1/m$ Laughlin state, which corresponds to the $\prod_{i<j} (z_{2i} - z_{2j})^m$ part of the wave function. (Note here $eB < 0$.) Introducing a new gauge field $j_\mu = \frac{1}{2\pi} \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda}$ to describe the fermion
current $j_\mu$ in (2.39), the effective theory of the $1/m$ state in the second layer has the form
\[ \mathcal{L} = -m \frac{1}{4\pi} a_{2\mu} \partial_\nu a_{2\lambda} \varepsilon^{\mu\nu\lambda} \] (2.40)

Putting (2.37), (2.39) and (2.40) together we obtain the total effective theory of the double layer state which has the form in (2.19) with the $K$-matrix and the charge vector $t$ given by
\[ K = \begin{pmatrix} l & n \\ n & m \end{pmatrix}, \quad t = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \] (2.41)

We see that the $K$-matrix is nothing but the exponents in the wave function. The filling fraction of the FQH state is still given by (2.31).

There are two kinds of (fundamental) quasihole excitations in the double layer state. The first kind is created by multiplying $\prod_i (\xi - z_{1i})$ to the ground state wave function and the second kind is created by $\prod_i (\xi - z_{2i})$. As the vortices in the two electron condensates in the first and the second layers, a first kind of quasihole is created by the source term $-a_{1\mu} j_\mu$, and the second kind of quasihole by $-a_{2\mu} j_\mu$. Thus, a generic quasiparticle in the double layer state is a bound state of several quasiholes of the first and the second kind and is described by (2.21). The quantum numbers of such quasiparticles are still given by (2.30).

In general, a multi-layer FQH state of type (2.35) is described by a $K$-matrix whose elements are integers and whose diagonal elements are odd integers. The charge vector has the form $t^T = (1, 1, ..., 1)$.

People usually label the double layer FQH state (2.35) by $(l, m, n)$. In the following we list the $K$-matrix, the charge vector $t$, and the spin vector $s$ for some simple double layer states:

\[
\begin{align*}
\nu &= 1/m, \quad & t &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad & K &= \begin{pmatrix} m & m \\ m & m \end{pmatrix}, \quad & s &= \begin{pmatrix} m/2 \\ m/2 \end{pmatrix} \\
\nu &= 2/5, \quad & t &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad & K &= \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \quad & s &= \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix} \\
\nu &= 1/2, \quad & t &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad & K &= \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad & s &= \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix} \\
\nu &= 2/3, \quad & t &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad & K &= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad & s &= \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix} \\
\nu &= 2/3, \quad & t &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad & K &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad & s &= \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}
\end{align*}
\] (2.42)

From the above we see that the (332) double layer state has the filling fraction 2/5 – a filling fraction that also appears in single layer hierarchical states. Now a question arises: Do the double layer 2/5 state and the single layer 2/5 belong to the same universality class? This question has experimental consequences. We can imagine the following experiment. We start with a (332) double layer state in a system with very weak interlayer tunneling. As we make the interlayer tunneling stronger and stronger, while keeping the filling fraction fixed, the double layer state will eventually become a single layer 2/5 state. The question is whether the transition between the double layer (332) state and the single layer 2/5 is a smooth crossover or a phase transition. If we ignore the spin vector we see that the $K$-matrices and the charge vectors of the two 2/5 states are equivalent since they are related by a $SL(2, Z)$ transformation. Therefore, in the presence of disorders (in which case the spin vector is not well defined) the two 2/5
states can change into each other smoothly. However, the cleaner samples may have a smaller activation gap for the DC conductance in the crossover region. When we include the spin vector, the two 2/5 states are not equivalent and, for pure systems, they are separated by a first order phase transition.

The transition between the above two 2/5 states should be similar to the transition in the following system. Let us consider an electron system with a strange kinetic energy such that the first Landau level has an energy $E_1$ and the second level has $E_2$. Originally $E_1 < E_2$. Assume that by changing a certain parameter we can make $E_1 > E_2$. This will cause a transition between the $\nu = 1$ state in the first Landau level and the the $\nu = 1$ state in the second Landau level. For pure systems, such a transition must be first order since the two $\nu = 1$ states have different spin vectors. For disordered systems the transition can be a smooth crossover. Certainly, it is always possible that some other states may appear between the two $\nu = 1$ (or $\nu = 2/5$) states. The bottom line is that the two $\nu = 1$ states cannot be smoothly connected in a clean sample.

From (2.42) we also see that there are two different 2/3 double layer states. When the intra-layer interaction is much stronger than the interlayer interaction (a situation in real samples), the ground state wave function prefers to have higher order zeros between electrons in the same layers. Thus, the (330) state should have lower energy than the (112) state. The $K$-matrix and the charge vector of the single layer 2/3 state is equivalent to those of the (112) state, and is not equivalent to those of the (330) state. Thus, to change a double layer 2/3 state (i.e., the (330) state) into the single layer 2/3 state, one must go through a phase transition for both random and pure systems.

After replacing the layer index by the $S_z$ spin index, all the results obtained in this section can be directly applied to describe spin unpolarized FQH states of spin-1/2 electron systems. In particular, one can show that the $\nu = 2/5$ and the second $\nu = 2/3$ states in (2.42) describe two spin singlet FQH states [38]. Due to the different spin vectors, both states are inequivalent to their spin polarized counterpart – the $\nu = 2/5$ and the $\nu = 2/3$ FQH states in (2.34). Thus for pure system, the above spin singlet FQH states and the spin polarized FQH states are separated by first order transitions. However for dirty system the first order transition may be smeared into a smooth crossover.

The double layer ($mmm$) state is an interesting state since det($K$) = 0. It turns out that the ($mmm$) state (in the absence of interlayer tunneling) spontaneously break a $U(1)$ symmetry and contain a superfluid mode.[39] More detailed discussions (including their experimental implications) can be found in Ref. [39,40].

3. Edge excitations in abelian FQH liquids

Due to the repulsive interaction and strong correlation between the electrons, a FQH liquid is an incompressible state despite the fact that the first Landau level is only partially filled. All the bulk excitations in FQH states have finite energy gaps. FQH states and insulators are very similar in the sense that both states have finite energy gaps and short ranged electron propagators. Because of this similarity people were puzzled by the fact that FQH systems apparently have very different transport properties than ordinary insulators. Halperin first pointed out that the integral quantum Hall (IQH) states contain gapless edge excitations [29]. Although the electronic states in the bulk are localized, the electronic states at the edge of the sample are extended (i.e., the electron propagator along the edge is long-ranged) [41]. Therefore, the nontrivial transport properties of the IQH states come from the gapless edge excitations [29,42] e.g., a two probe measurement of a QH sample can result in a finite resistance only when the source and the drain are connected by the edges. If the source and the drain are
not connected by any edge, the two probe measurement will yield an infinite resistance at zero temperature, a result very similar to the insulators. The edge transport picture has been supported by many experiments \[43\]. Halperin also studied the dynamical properties of the edge excitations of the IQH states and found that the edge excitations are described by a chiral 1D Fermi liquid theory.

Using the gauge argument in Ref. \[44,29,45\], one can easily show that FQH states also support gapless edge excitations. Thus it is natural to conjecture that the transport in FQH states is also governed by the edge excitations \[46,47\]. However, since FQH states are intrinsically many-body states, the edge excitations in the FQH states cannot be constructed from a single-body theory. Or in other words, the edge excitations of FQH states should not be described by a Fermi liquid. In this case we need completely new approaches to understand the dynamical properties of the edge states of FQH liquids. Recent advances in fabrication of small devices make it possible to study in detail the dynamical properties of the edge states in FQH liquids. Thus it is important to develop a quantitative theory for FQH edge states to explain new experimental data.

There is another motivation to study the edge states in FQHE. We know that different FQH states were generally labeled by their filling fractions. However, it becomes clear that FQH states contain extremely rich internal structures that the filling fraction alone is not enough to classify all the different universality classes of FQH states \[6,7\]. One can easily construct different FQH states with the same filling fraction \[3,18,17,19,20\]. From the last chapter we see that (generalized) hierarchical states and simple double layer states can be labeled by $K$-matrices and charge vectors $t$. But in the last chapter, $K$ and $t$ merely appear as parameters in some theoretical effective theory. An important question is that can one find experiments that measure $K$ and $t$. Certainly a combination of $K$ and $t$ can be measured through the filling fraction, and the determinant of $K$ can be measured through the ground state degeneracy of the FQH liquid on a torus \[13,7\]. Experimentally, however one can never put an HQ state on a torus. Thus, results in Ref. \[6,7,13\] can be checked only in numerical calculations. In the following we will see that edge excitations in FQH states provide an important (probably the only practical) probe to detect the topological orders in the bulk FQH states. Through tunneling experiments between FQH edges one can measure many different combinations of $K$ and $t$. Using the edge excitations we also can tell whether a FQH state is an abelian FQH state or a non-abelian FQH state. Thus, the edge states provide us with a practical window through which we can look into the internal structures in FQH states. The measurements of the edge states can provide us new quantum numbers, in addition to the filling fractions, to characterize different quantum Hall states.

\subsection{Hydrodynamical approach to the edge excitations – 1/m Laughlin state}

The simplest (but not complete) way to understand the dynamics of edge excitations is to use the hydrodynamical approach. In this approach, one uses the fact that QH (IQH or FQH) states are incompressible ir-rotational liquids that contain no low-energy bulk excitations. Therefore, the only low lying excitations (below the bulk energy gap) are surface waves on a HQ droplet. These surface waves are identified as edge excitations of the HQ state \[48,49,31\].

In the hydrodynamical approach we first study the classical theory of the surface wave on the HQ droplet. Then we quantize the classical theory to obtain the quantum description of the edge excitations. It is amazing that the simple quantum description obtained from the classical theory provides a complete description of the edge excitations at low energies and allows us to calculate the electron and the quasiparticle propagators along the edges.
Consider a QH droplet with a filling fraction \( \nu \) confined by a potential well. Due to the non-zero conductance, the electric field of the potential well generates a persistent current flowing along the edge,

\[
\mathbf{j} = \sigma_{xy} \hat{z} \times \mathbf{E}, \quad \sigma_{xy} = \nu \frac{e^2}{h}
\]

This implies that the electrons near the edge drift with a velocity

\[
v = \frac{E}{Bc}
\]

where \( c \) is the velocity of the light. Thus, the edge wave also propagates with the velocity \( v \). Let us use one dimensional density \( \rho(x) = nh(x) \) to describe the edge wave, where \( h(x) \) is the displacement of the edge, \( x \) is the coordinate along the edge, and \( n = \frac{\nu}{2\pi l_B^2} \) is the two dimensional electron density in the bulk. We see that the propagation of the edge waves are described by the following wave equation,

\[
\partial_t \rho - v \partial_x \rho = 0
\]

Notice that the edge waves always propagate in one direction, there are no waves that propagate in the opposite direction.

The Hamiltonian (i.e., the energy) of the edge waves is given by

\[
H = \int dx \frac{1}{2} e h \rho E = \int dx \frac{\pi v}{\nu} \rho^2
\]

In the momentum space (3.3) and (3.4) can be rewritten as

\[
\dot{\rho}_k = i \nu k \rho_k \\
H = 2\pi \frac{\nu}{\nu} \sum_{k>0} \rho_k \rho_{-k}
\]

where \( \rho_k = \int dx \frac{1}{\sqrt{L}} e^{ikx} \rho(x) \), and \( L \) is the length of the edge. Comparing (3.5) with the standard Hamiltonian equation,

\[
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}
\]

we find that if we identify \( \rho_k |_{k>0} \) as the “coordinates”, then the corresponding canonical “momenta” can be identified as \( p_k = i2\pi \rho_{-k}/\nu k \). We would like to stress that because the edge waves are chiral, the displacement \( h(x) \) contains both the “coordinates” and the “momenta”.

Knowing the canonical coordinates and momenta, it is easy to quantize the classical theory. We simply view \( \rho_k \) and \( p_k \) as operators that satisfy \( [p_k, \rho_{k'}] = i \delta_{kk'} \). Thus after quantization we have

\[
\left[ \rho_k, \rho_{k'} \right] = \frac{\nu}{2\pi} k \delta_{k+k'} \\
\left[ H, \rho_k \right] = \nu \rho_k
\]

\[
k, k' = \text{integer} \times \frac{2\pi}{L}
\]
The above algebra is called the \((U(1))\) Kac-Moody (K-M) algebra \([50]\). A similar algebra has also appeared in the Tomonaga model \([51]\). Notice that \((3.7)\) simply describes a collection of decoupled harmonic oscillators (generated by \((\rho_k, \rho_{-k})\)). Thus \((3.7)\) is an one dimensional free phonon theory (with only a single branch) and is exactly soluble. We will show later that \((3.7)\) provides a complete description of the low lying edge excitations of the HQ state.

To summarize, we find that the edge excitations in the QH states are described by a free (chiral) phonon theory at low energies. We not only show the existence of the gapless edge excitations, we also obtain the density of states of the edge excitations. The specific heat (per unit length) of the edge excitations is found to be \(T_s / 6\). The edge excitations considered here do not change the total charge of the system and hence are neutral. In the following, we will discuss the charged excitations and calculate the electron propagator from the K-M algebra \((3.7)\).

The low lying charge excitations obviously correspond to adding (removing) electrons to (from) the edge. Those charged excitations carry integer charges and are created by electron operators \(\Psi^\dagger\). The above theory of the edge excitations is formulated in terms of a 1D density operator \(\rho(x)\). So the central question is to write the electron operator in terms of the density operator. The electron operator on the edge creates a localized charge and should satisfy

\[ [\rho(x), \Psi^\dagger(x')] = \delta(x - x') \Psi^\dagger(x') \quad (3.8) \]

Since \(\rho\) satisfies the Kac-Moody algebra \((3.7)\), one can show that the operators that satisfy \((3.8)\) are given by \([30]\)

\[ \Psi \propto e^{i \frac{1}{\nu} \phi} \quad (3.9) \]

where \(\phi\) is given by \(\rho = \frac{1}{2 \pi} \partial_x \phi\).

\((3.8)\) implies only that the operator \(\Psi\) carries the charge \(e\). In order to identify \(\Psi\) as an electron operator we need to show that \(\Psi\) is a fermionic operator. Using the K-M algebra \((3.7)\) we find that

\[ \Psi(x) \Psi(x') = (-)^{1/\nu} \Psi(x') \Psi(x) \quad (3.10) \]

We see that the electron operator \(\Psi\) in \((3.9)\) is fermionic only when \(1/\nu = m\) is an odd integer, in which case the QH state is a Laughlin state \([30,52]\).

In the above discussion we have made an assumption that is not generally true. We have assumed that the incompressible QH liquid contains only one component of incompressible fluid which leads to one branch of edge excitations. The above result implies that, when \(\nu \neq 1/m\), the edge theory with only one branch does not contain the electron operators and is not self-consistent. Therefore we conclude that the FQH states with \(\nu \neq 1/m\) must contain more than one branch of edge excitations. (Here we have ignored the possibility of the pairing between the electrons.) Later (in section 6.1) we will see that the one-branch assumption is true only for a simple Laughlin state with filling fraction \(\nu = 1/m\). For hierarchical FQH states, there are several condensates corresponding to several components of incompressible fluid. Each component gives rise to a branch of the edge excitations. Thus a generic QH state may contain many branches of the edge excitations,\([47,32]\) even when electrons are all in the first Landau level (see section 3.2 and 3.3).

Now let us calculate electron propagator along the edge of the Laughlin states with \(\nu = 1/m\). In this case the above simple edge theory is valid. Because \(\phi\) is a free phonon field with a propagator

\[ \langle \phi(x, t) \phi(0) \rangle = -\nu \ln(x - vt) + \text{const.} \quad (3.11) \]
the electron propagator can be easily calculated as

$$G(x, t) = \langle T(\Psi^\dagger(x, t)\Psi(0)) \rangle = \exp\left[\frac{1}{\nu^2}(\phi(x, t)\phi(0))\right] \propto \frac{1}{(x - vt)^m} \tag{3.12}$$

The first thing we see is that the electron propagator on the edge of a FQH state acquires a non-trivial exponent $m = 1/\nu$ that is not equal to one. This implies that the electrons on the edge of the FQH state are strongly correlated and cannot be described by Fermi liquid theory. We will call this type of an electron state chiral Luttinger liquid.

The K-M algebra (3.7) and the electron operator (3.9) provided a complete description of both neutral and charged edge excitations at low energies. We would like to remark that the propagator (3.12) is correct only for large $x$ and $t$. At a short distance the form of the propagator depends on the details of the electron interactions and the edge potentials. We would also like to emphasize that the exponent $m$ of the edge propagator is determined by the bulk state. Such an exponent is a topological number that is independent of electron interactions, edge potential, etc. The quantization of the exponent is directly related to the fact that the exponent is linked to the statistics of the electrons (see (3.10)). Thus the exponent can be regarded as a quantum number that characterizes the topological orders in the bulk FQH states.

In the momentum space the electron propagator has the form

$$G(k, \omega) \propto \frac{(vk + \omega)^{m-1}}{vk - \omega} \tag{3.13}$$

The anomalous exponent $m$ can be measured in tunneling experiments. The tunneling density of states of electron is given by

$$N(\omega) \propto |\omega|^{m-1} \tag{3.14}$$

This implies that deferential conductance has the form $\frac{dI}{dV} \propto V^{m-1}$ for a metal-insulator-FQH junction.

### 3.2 Hydrodynamical approach to the edge excitations – 2/5 and 2/3 states

In this section we will use the hydrodynamical approach discussed above to study edge structures of second level hierarchical states. We will concentrate on the 2/5 and 2/3 states as examples. In particular we will study the structures of the electron and the quasiparticle operators on the edges of the hierarchical states. We will also see that the 2/3 state contains two edge modes that propagate in the opposite directions, which is quite counter intuitive.

First let us consider the $\nu = \frac{2}{5}$ FQH state. According to the hierarchical theory, the $\nu = \frac{2}{5}$ FQH state is generated by the condensation of quasiparticles on top of the $\nu = \frac{1}{3}$ FQH state. Thus the 2/5 state contains two components of incompressible fluids. To be definite let us consider a special edge potential such that the FQH state consists of two droplets, one is the electron condensate with a filling fraction $\frac{1}{3}$ and radius $r_1$, and the other is the quasiparticle condensate (on top of the 1/3 state) with a filling fraction $\frac{1}{15}$ (note $\frac{1}{3} + \frac{1}{15} = \frac{2}{5}$) and radius $r_2 < r_1$.

When $r_1 - r_2 \gg l_B$, the two edges are independent. Generalizing the hydrodynamical approach in section 3.1, we can show that there are two branches of the edge
excitations whose low energy dynamics are described by
\[ [\rho_{I k}, \rho_{J k'}] = \frac{v_I}{2\pi} k \delta_{IJ} \delta_{k+k'} \]
\[ H = 2\pi \sum_{I, k>0} \frac{v_I}{v_I} \rho_{I k} \rho_{I-k} \]  \hspace{1cm} (3.15)

where \( I = 1, 2 \) labels the two branches, \((\nu_1, \nu_2) = (\frac{1}{3}, \frac{1}{15})\) are filling fractions of the electron condensate and the quasiparticle condensate, and \( v_I \) are the velocity of the edge excitations. \( \rho_I \) in (3.15) are the 1D electron densities given by \( \rho_I = h_I \nu_I \frac{1}{2\pi} \) where \( h_I \) are the amplitude of the edge waves on the two droplets.

Because the electrons are interacting with each other, the edge velocities are determined by \( v_I = cE^*_I/B \) where \( E^*_I \) are the effective electric fields that include contributions from both the edge potential and the electrons. In order for the Hamiltonian to be bounded from below, we require \( v_I > 0 \). We find that the stability of the \( \nu = \frac{2}{5} \) FQH state requires both \( v_I \) to be positive.

Generalizing the discussion in section 3.1, the electron operators on the two edges are found to be
\[ \Psi_I = e^{i \frac{1}{\nu_I} \phi_I(x)} \hspace{1cm} I = 1, 2 \] \hspace{1cm} (3.16)
with \( \partial_x \phi_I = \frac{1}{2\pi} v_I \). The electron propagators have the form
\[ \langle T(\Psi_I(x, t)\Psi^*_I(0)) \rangle = e^{ik_I x} \frac{1}{(x-v_I t)^{-1/|\nu_I|}} \hspace{1cm} I = 1, 2 \] \hspace{1cm} (3.17)
where \( k_I = \frac{r_I}{2\pi B} \).

According to the hierarchical picture, the \( \nu = \frac{2}{3} \) FQH state is also formed by two condensates, an electron condensate with a filling fraction 1 and a hole condensate with a filling fraction \(-\frac{1}{3}\). Thus, the above discussion can also be applied to the \( \nu = \frac{2}{3} \) FQH state by choosing \((\nu_1, \nu_2) = (1, -\frac{1}{3})\). Again there are two branches of the edge excitations but now with opposite velocities if the Hamiltonian is positive definite. This result, although surprising, is not difficult to understand. The stability of both the electron droplet and the hole droplet requires \( E^*_1 \) and \( E^*_2 \) to have opposite signs.

As we bring the two edges together \((r_1 - r_2 \sim l_B)\) the interaction between the two branches of the edge excitations can no longer be ignored. In this case the Hamiltonian has the form
\[ H = 2\pi \sum_{I, J, k>0} V_{IJ} \rho_{I k} \rho_{J-k} \] \hspace{1cm} (3.18)
(The Hamiltonian may also contain terms that describe the electron hopping between edges. But those terms are irrelevant at low energies due to the chiral property of the edge excitations. For example one can show that those terms can never open an energy gap[45])

The Hamiltonian (3.18) can still be diagonalized. For \( \nu_1 \nu_2 > 0 \), we may choose
\[ \tilde{\rho}_{1 k} = \cos(\theta) \frac{1}{\sqrt{|\nu_1|}} \rho_{1 k} + \sin(\theta) \frac{1}{\sqrt{|\nu_2|}} \rho_{2 k} \]
\[ \tilde{\rho}_{2 k} = \cos(\theta) \frac{1}{\sqrt{|\nu_2|}} \rho_{2 k} - \sin(\theta) \frac{1}{\sqrt{|\nu_1|}} \rho_{1 k} \] \hspace{1cm} (3.19)
\[ \tan(2\theta) = 2 \frac{|\nu_1 \nu_2| V_{12}}{|\nu_1 V_{11} - |\nu_2 V_{22}|} \]
One can check that $\tilde{\rho}$s satisfy
\[
[H, [\tilde{\rho}_{Ik}, \tilde{\rho}_{Jk'}]] = \frac{\text{sgn}(\nu_I)}{2\pi}k\delta_{IJ}\delta_{k+k'}
\]
where the new velocities of the edge excitations $\tilde{v}_I$ are given by
\[
\text{sgn}(\nu_1)\tilde{v}_1 = \frac{\cos^2(\theta)}{\cos(2\theta)}|\nu_1|V_{11} - \frac{\sin^2(\theta)}{\cos(2\theta)}|\nu_2|V_{22}
\]
\[
\text{sgn}(\nu_2)\tilde{v}_2 = \frac{\cos^2(\theta)}{\cos(2\theta)}|\nu_2|V_{22} - \frac{\sin^2(\theta)}{\cos(2\theta)}|\nu_1|V_{11}
\]

We see that there are still two branches of the edge excitations. However in this case the edge excitations with a definite velocity are mixtures of those on the inner edge and the outer edge. One can also show that as long as the Hamiltonian (3.18) is bounded from below, the velocities of the two branches $\tilde{v}_I$ are always positive. This result has been confirmed by numerical calculations [53].

After rewriting the electron operator $\Psi_I$ in terms of $\tilde{\rho}_I$ by inverting (3.19), we can calculate their propagators using (3.21)
\[
\langle T(\Psi_I(x, t)\Psi_I^\dagger(0)) \rangle = e^{ik_Ix} \frac{1}{(x - \tilde{v}_1t)^{\alpha_I}(x - \tilde{v}_2t)^{\beta_I}}
\]

where
\[
(\alpha_1, \alpha_2) = (\frac{1}{|\nu_1|} \cos^2 \theta, \frac{1}{|\nu_2|} \sin^2 \theta), \quad (\beta_1, \beta_2) = (\frac{1}{|\nu_1|} \sin^2 \theta, \frac{1}{|\nu_2|} \cos^2 \theta)
\]

However, when the two edges are close to each other within the magnetic length, the $\Psi_I$ are no longer the most general electron operators on the edge. The generic electron operator may contain charge transfers between the two edges. For the $\nu = 2/5$ FQH state, the inner edge and the outer edge are separated by the $\nu = \frac{1}{3}$ Laughlin state. Thus, the elementary charge transfer operator is given by
\[
\eta(x) = e^{i(\phi_1 - \frac{\nu_2}{\nu_1}\phi_2)} = (\Psi_1 \Psi_2^\dagger)^{\nu_1}
\]

which transfers a $\nu_1 e = e/3$ charge from the outer edge to the inner edge. The generic electron operator then takes the form
\[
\Psi(x) = \sum_{n=-\infty}^{+\infty} c_n \psi_n(x)
\]
\[
\psi_n(x) = \Psi_1(x)\eta^n(x)
\]

To understand this result, we notice that each operator $\psi_n$ always creates a unit localized charge and is a fermionic operator regardless of the value of the integer $n$. Therefore, each $\psi_n$ is a candidate for the electron operator on the edge. For a generic interacting system the electron operator on the edge is expected to be a superposition of different
\(\psi\)'s as represented in (3.25). Note \(\Psi_2 = \psi - \frac{1}{v_I}\). The propagator of \(\psi_n\) can be calculated in a similar way as outlined above and is given by

\[
\langle T(\psi_n(x,t)\psi_m^\dagger(0)) \rangle \propto \delta_{n,m} e^{i[k_1+n\nu_1(k_2-k_1)]x} \prod_I (x - v_I t)^{-\gamma_{In}} \tag{3.26}
\]

where \(\gamma_{In}\) are

\[
\gamma_{1n} = \left( n + \frac{1}{|\nu_1|} \right) \sqrt{|\nu_1|} \cos \theta - \frac{n\nu_1}{\nu_2} \sqrt{|\nu_2|} \sin \theta \right)^2
\]

\[
\gamma_{2n} = \left( n + \frac{1}{|\nu_1|} \right) \sqrt{|\nu_1|} \sin \theta + \frac{n\nu_1}{\nu_2} \sqrt{|\nu_2|} \cos \theta \right)^2 \tag{3.27}
\]

From (3.25) and (3.26) we see that the electron propagator has singularities at discrete momenta \(k = k_1 + n\nu_1(k_2 - k_1)\). They are analogs to the \(k_F, 3k_F, ...\) singularities of the electron propagator in the interacting 1D electron systems.

For the \(\nu = \frac{2}{3}\) FQH state, \(\nu_1\nu_2 < 0\). In this case we need to choose

\[
\tilde{\rho}_{1k} = \text{ch}(\theta) \frac{1}{\sqrt{|\nu_1|}} \rho_{1k} + \text{sh}(\theta) \frac{1}{\sqrt{|\nu_2|}} \rho_{2k}
\]

\[
\tilde{\rho}_{2k} = \text{ch}(\theta) \frac{1}{\sqrt{|\nu_2|}} \rho_{2k} + \text{sh}(\theta) \frac{1}{\sqrt{|\nu_1|}} \rho_{1k} \tag{3.28}
\]

\[
\text{th}(2\theta) = 2 \sqrt{|\nu_1\nu_2| V_1 V_2} \left/ |\nu_1| V_1 + |\nu_2| V_2 \right.
\]

to diagonalize the Hamiltonian. One can check that \(\tilde{\rho}_I\) also satisfies the K-M algebra (3.20) but now

\[
\text{sgn}(\nu_1) \tilde{\nu}_1 = \frac{\text{ch}^2(\theta)}{\text{ch}(2\theta)} |\nu_1| V_1 - \frac{\text{sh}^2(\theta)}{\text{ch}(2\theta)} |\nu_2| V_2
\]

\[
\text{sgn}(\nu_2) \tilde{\nu}_2 = \frac{\text{ch}^2(\theta)}{\text{ch}(2\theta)} |\nu_2| V_2 - \frac{\text{sh}^2(\theta)}{\text{ch}(2\theta)} |\nu_1| V_1 \tag{3.29}
\]

Again as long as the Hamiltonian \(H\) is positive definite, the velocities of the edge excitations \(\tilde{\nu}_I\) always have opposite signs. The electron operator still has the form (3.25) with \(\eta = (\Psi_1\Psi_2^\dagger)^{\nu_1}\). The propagator of \(\psi_n\) is still given by (3.26) with

\[
\gamma_{1n} = \left( n + \frac{1}{|\nu_1|} \right) \sqrt{|\nu_1|} \text{ch} \theta + \frac{n\nu_1}{\nu_2} \sqrt{|\nu_2|} \text{sh} \theta \right)^2 \tag{3.30}
\]

\[
\gamma_{2n} = \left( n + \frac{1}{|\nu_1|} \right) \sqrt{|\nu_1|} \text{sh} \theta + \frac{n\nu_1}{\nu_2} \sqrt{|\nu_2|} \text{ch} \theta \right)^2
\]

From (3.27) and (3.30), we see that exponents in the electron propagator depend on the inter-edge interactions. However, we can show that the exponents satisfy a sum rule:

\[
\sum_I \text{sgn}(\nu_I) \gamma_{In} \equiv \lambda_n = \left( n + \frac{1}{|\nu_1|} \right)^2 \nu_1 + \frac{n^2\nu_1^2}{\nu_2} \tag{3.31}
\]
\( \lambda_n \) always take odd-integer values and are independent of the details of the electron system. The quantization of \( \lambda_n \) is again due to the fact that \( \lambda_n \) are directly linked to the statistics of the electrons:

\[
\psi_n(x)\psi_n(x') = (-)^{\lambda_n}\psi_n(x')\psi_n(x)
\]  

(3.32)

We would like to point out that the values of \( \lambda_n \) are determined by the internal correlations (topological orders) of the bulk FQH state. \( \lambda_n \) can be changed only by changing the bulk topological orders through a \textit{two dimensional} phase transition, despite the fact that they are properties of 1D edge excitations. Therefore \( \lambda_n \) are topological quantum numbers that can be used to characterize and to experimentally measure the 2D bulk topological orders.

The total exponents \( g_e^{(n)} \) of the electron operators

\[
g_e^{(n)} = \sum_I \gamma_{1n}
\]  

(3.33)

are every important quantities. For the 2/5 state, the minimum value of the exponents

\[
g_e \equiv \text{Min}(g_e^{(n)}) = g_e^{(0)} = g_e^{(-1)} = 3
\]  

(3.34)

controls the scaling properties of tunneling of electrons between two edges. For example the tunneling conductance scales at finite temperatures as [35]

\[
\sigma \propto T^{2g_e-2}
\]  

(3.35)

For the 2/5 state the sum rule (3.31) implies that \( g_e^{(n)} \) and \( g_e \) are topological quantum numbers that are independent of details of electron interactions and edge potential. However for 2/3 state the sum rule cannot fix the value of \( g_e \), and \( g_e \) is not universal. Later in section 3.5 we will see that in the presence of long range Coulomb interaction and/or in the presence of edge impurity, \( g_e \) for the 2/3 state will take a universal value \( g_e = 2 \) [54].

### 3.3 Bulk effective theory and the edge states

In this section we will directly derive the macroscopic theory of the edge excitations from the effective theory of the bulk FQH states [18,52,20]. In this approach we do not rely on a specify construction of the FQH states. The relation between the bulk topological orders and edge states becomes quite clear.

We know a hierarchical (or generalized hierarchical) FQH state contains many different condensates, electron condensate forms the Laughlin states, and additional quasiparticle condensates on top of that give rise to a hierarchical state. Each condensate corresponds to one component of the incompressible fluid. The idea is to generalize the hydrodynamical approach in section 3 to multi-component fluids and to obtain the low energy effective theory of the edge excitations. To accomplish this, we would first like to write down the low energy effective theory of the bulk FQH state. The effective theory contains information about the internal structures (the topological orders) in the bulk states, such as the number of the condensates and how different condensates interact with each other.

The different condensates in FQH states are not independent. The particles in one condensate behave like a flux tube to the particles in other condensates. To describe
such a coupling, it is convenient to use $U(1)$ gauge fields to describe the density and the current of each condensate. In this case the couplings between different condensates are described by the Chern-Simons term of the gauge fields (see chapter 2) [18,19,20]. The most general abelian FQH states of the electrons are classified by integer valued symmetry matrices $K$ and described by the following effective theory

$$\mathcal{L} = -\frac{1}{4\pi} K_{IJ} a_I a_J \delta^{\mu\nu} - \frac{e}{2\pi} I A_\mu \partial_\nu a_I \delta^{\mu\nu}$$

In the following we will choose the so called symmetric basis with the charge vector $t^T = (1, 1, ..., 1)$. In this case the diagonal elements of $K$ must be odd integers. We can always change the hierarchical basis with $t = (1, 0, ..., 0)$ to the symmetric basis with $t = (1, 1, ..., 1)$ by a field-redefinition of the $U(1)$ gauge fields. Let $\kappa = \text{dim}(K)$. Then the FQH state described by (3.36) contains $\kappa$ different condensates and there are $\kappa$ kinds of different quasiparticle excitations.

The quasiparticle excitations can be viewed as vortices in different condensates. A generic quasiparticle is labeled by $\kappa$ integers $l_I | I = 1, ..., \kappa$, and can be generated by a source term

$$\mathcal{L} = l_I a_I \mu j^\mu$$

Such a quasiparticle will be denoted as $\psi_1$. $j^\mu$ in (3.37) and has the form

$$j(x) = v \delta(x - x_0)$$
$$j^0(x) = \delta(x - x_0)$$

which create a quasiparticle at $x_0$ with velocity $v$. The density and the current of the $I^{th}$ component of incompressible fluid (i.e., the $I^{th}$ condensate) are given by

$$J_I^\mu = \frac{1}{2\pi} \varepsilon_{\mu\alpha\beta} \partial^\alpha a_{I\beta}$$

As we create a quasiparticle $\psi_1$, it will induce a change in the density of all the condensates, $\delta J_{I0}$. From the equation of motion of (3.36) and (3.37), we find that $\delta J_{I0}$ satisfies

$$\int d^2x \delta J_{I0} = l_I (K^{-1})_{IJ} \int d^2x j_0 = l_I (K^{-1})_{IJ}$$

The charge and the statistics of the quasiparticle $\psi_1$ are given by

$$\theta_1 = \pi l^T K^{-1} l, \quad Q_1 = -e t_I \int d^2x \delta J_{I0} = -e l^T K^{-1} t$$

A generic electron excitation can be viewed as a special kind of (generic) quasiparticles, $\psi_e = \psi_{le}$, where the components of $l_e$ are given by

$$l_e I = K_{IJ} L_J, \quad L_I = \text{integers}, \quad \sum_I L_I = 1$$

We can show that these electron excitations satisfy the following properties: a) they carry a unit charge (see (3.41)); b) they have the fermionic statistics; c) moving an electron excitation $\psi_e = \psi_I$ around any quasiparticle excitations $\psi_1$ always induces a phase of multiple of $2\pi$; and d) the excitations defined in (3.42) are all the excitations satisfying the above three conditions.
We would like to point out that the effective theory (3.36) not only applies to the standard QH system in which all electrons are spin polarized and are in the first Landau level, it also applies to the QH system in which electrons may occupy several Landau levels and/or occupy several layers and/or carry different spins. In this case the index $I$ may label the condensates in different Landau levels, in different layers and/or with different spins.

To understand the relation between the effective theory and the edge states, let us first consider the simplest FQH state of the filling fraction $\nu = 1/q$ and try to re-derive the results in section 3.1 from the bulk effective theory. Such a FQH state is described by the $U(1)$ Chern-Simons theory with the action:[7,52]

$$S = \frac{q}{4\pi} \int a_\mu \partial_\nu a_\lambda \varepsilon^{\mu\nu\lambda} d^3 x$$  \hspace{1cm} (3.43)

Suppose that our sample has a boundary. For the simplicity we shall assume that the boundary is the $x$-axis and the sample covers the lower half-plane.

There is one problem with the effective action (3.43) for FQH liquids with boundaries. It is not invariant under gauge transformations $a_\mu \rightarrow a_\mu + \partial_\mu f$ due to the presence of the boundary: $\Delta S = \frac{q}{4\pi} \int_{y=0} dx dt f (\partial_0 a_1 - \partial_1 a_0)$. To solve this problem we will restrict the gauge transformations to be zero on the boundary $f(x, y = 0, t) = 0$. Due to this restriction some degrees of freedom of $a_\mu$ on the boundary become dynamical.

We know that the effective theory (3.43) is derived only for a bulk FQH state without boundary. Here we will take (3.43) with the restricted gauge transformation as the definition of the effective theory for a FQH state with boundary. Such a definition is definitely self-consistent. In the following we will show that such a definition reproduces the results obtained in section 3.1.

One way to study the dynamics of gauge theory is to choose the gauge condition $a_0 = 0$ and regard the equation of motion for $a_0$ as a constraint. For the Chern-Simons theory such a constraint becomes $f_{ij} = 0$. Under this constraint, we may write $a_i$ as $a_i = \partial_i \phi$. Plugging this into (3.43), one obtains[15] an effective 1D theory on the edge with an action

$$S_{\text{edge}} = \frac{m}{4\pi} \int \partial_t \phi \partial_x \phi dx dt$$  \hspace{1cm} (3.44)

This approach, however, has a problem. It is easy to see that a Hamiltonian associated with the action (3.44) is zero and the boundary excitations described by (3.44) have zero velocity. Therefore, this action cannot be used to describe any physical edge excitations in real FQH samples. The edge excitations in FQH states always have finite velocities.

The appearance of finite velocities of edge excitations is a boundary effect. The bulk effective theory defined by Eq. (3.36) does not contain the information about the velocities of the edge excitations. To determine the dynamics of the edge excitations from the effective theory we must find a way to input the information about the edge velocity. The edge velocities must be treated as the external parameters that are not contained in the bulk effective theory. The problem is how to put these parameters in the theory.

Let us now note that the condition $a_0 = 0$ is not the only choice for the gauge fixing condition. A more general gauge fixing condition has the form

$$a_\tau = a_0 + v a_x = 0$$  \hspace{1cm} (3.45)

Here $a_x$ is the component of the vector potential parallel to the boundary of the sample and $v$ is a parameter that has a dimension of velocity.
It is convenient to choose new coordinates that satisfy
\[
\tilde{x} = x - vt \\
\tilde{t} = t, \quad \tilde{y} = y
\]
In these coordinates the components of the gauge field are given by
\[
\tilde{a}_t = a_t + v a_x \quad \tilde{a}_x = a_x \quad \tilde{a}_y = a_y
\]
(3.47)
The gauge fixing condition becomes the one discussed before in the new coordinates. It is easy to see that the form of the Chern-Simons action is preserved under the transformation (3.46) and (3.47):
\[
S = \frac{q}{4\pi} \int d^3 x a_\mu \partial_\nu a_\lambda \varepsilon^{\mu \nu \lambda} = \frac{q}{4\pi} \int d^3 x \tilde{a}_\mu \partial_\nu \tilde{a}_\lambda \varepsilon^{\tilde{\mu} \tilde{\nu} \tilde{\lambda}}
\]
(3.48)
Repeating the previous derivation, we find that the edge action is given by
\[
S = \frac{q}{4\pi} \int d\tilde{t} d\tilde{x} \partial_\tilde{t} \phi \partial_\tilde{x} \phi
\]
(3.49)
In terms of the original physical coordinates the above action acquires the form
\[
S = \frac{q}{4\pi} \int dt dx (\partial_t + v \partial_x) \phi \partial_x \phi
\]
(3.50)
which is a chiral boson theory [55]. It is easy to see that the edge excitations described by (3.50) have a non-zero velocity. The quantization of chiral boson theory has been discussed in detail in Ref. [55]. The canonical momentum \(\pi(x)\) is equal to \(\pi = \frac{\delta L}{\delta \phi_t} = \frac{q}{4\pi} \partial_x \phi\). The “coordinate” \(\phi\) and “momentum” \(\pi\) obey the commutation relations:
\[
[\pi(x), \phi(y)] = 2 \frac{\delta(x - y)}{\pi} \\
[\phi(x), \phi(y)] = -\frac{\pi}{q} \text{sgn}(x - y)
\]
(3.51)
The Hamiltonian of the theory (3.50) is given by
\[
H = -\frac{qv}{4\pi} \int dx \partial_x \phi \partial_x \phi
\]
(3.52)
The Hilbert space contains only left-moving degrees of freedom (or right moving degrees of freedom if \(v < 0\)). The theory (3.51) and (3.52) describes free left (or right) moving phonons (i.e., the edge density waves). One can easily show that (3.51) and (3.52) are equivalent to the K-M algebra (3.7) by identifying \(\rho = \frac{1}{2\pi} \partial_x \phi\).

In the following we would like to show that \(\rho = \frac{1}{2\pi} \partial_x \phi\) can actually be interpreted as the 1D electron density on the edge. First we notice that the coupling between the electrons and the external electromagnetic potential is given by \(\int A_\mu J_\mu d^3 x = \int \frac{1}{2\pi} A \partial a d^3 x\) (see (3.39)). But for a finite sample such a coupling should be written as
\[
\int A_\mu J_\mu d^3 x \sim \int \frac{1}{2\pi} a_\mu \partial_\nu A_\lambda \varepsilon^{\mu \nu \lambda} d^3 x
\]
(3.53)
which is invariant under the gauge transformation of the electromagnetic field \( A_\mu \) and under the restricted gauge transformation of the gauge field \( a_\mu \). Assuming that \( A_\mu \) is independent of \( y \) and \( \tilde{y} \) and \( A_y = A_{\tilde{y}} = 0 \), we see that, from \( \tilde{a}_i = \partial_i \phi \),

\[
\int \frac{1}{2\pi} a_\mu \partial_\nu A_\lambda \varepsilon^{\mu\nu\lambda} d^3 \tilde{x} = -\int d\tilde{x}d\tilde{y} d\tilde{t} \frac{1}{2\pi} \partial_\tilde{y} \phi (\partial_{\tilde{t}} A_\tilde{t} - \partial_\tilde{x} A_\tilde{x})
\]

\[
= -\int d\tilde{x}d\tilde{t} \frac{1}{2\pi} \phi (\partial_{\tilde{t}} A_\tilde{t} - \partial_\tilde{x} A_\tilde{x})|_{\tilde{y}=0}
\]

\[
= \int dx dt \frac{1}{2\pi} (A_t + vA_x) \partial_x \phi |_{\tilde{y}=0}
\]

where we have used the equation of motion \( (\partial_t + v\partial_x)\phi = \partial_t \phi = 0 \) and the transformations (3.46) and (3.47). (3.54) clearly indicates that the 1D edge electron density is given by \( \frac{1}{2\pi} \partial_x \phi = \rho \). This identification, together with the algebra (3.51) and (3.52), completes our proof that our treatment of the effective theory (3.43) and (3.45) reproduces the edge theory obtained in section 3.1.

The velocity of the edge excitations \( v \) enters our theory through the gauge fixing condition. Notice that under the restricted gauge transformations the gauge fixing conditions (3.45) with different \( v \) cannot be transformed into each other. They are physically inequivalent. This agrees with our assumption that that \( v \) in the gauge fixing condition is physical and actually determines the velocity of the edge excitations.

The Hamiltonian (3.52) is bounded from below only when \( vq < 0 \). The consistency of our theory requires \( v \) and \( q \) to have opposite signs. Therefore the sign of the velocity (the chirality) of the edge excitations is determined by the sign of the coefficient in front of the Chern-Simons terms.

The above results can be easily generalized to the generic FQH states described by (3.36) because the matrix \( K \) can be diagonalized. The resulting effective edge theory has the form

\[
S_{\text{edge}} = \frac{1}{4\pi} \int dt dx [K_{IJ} \partial_I \phi_I \partial_x \phi_J - V_{IJ} \partial_x \phi_I \partial_x \phi_J]
\] (3.55)

The Hamiltonian is given by

\[
H_{\text{edge}} = \frac{1}{4\pi} \int dt dx V_{IJ} \partial_x \phi_I \partial_x \phi_J
\] (3.56)

Therefore \( V \) must be a positive definite matrix. Using this result one can show that a positive eigenvalue of \( K \) corresponds to a left moving branch and a negative eigenvalue corresponds to a right moving one.

The effective theory of the \( \nu = 2/5 \) FQH state is given by

\[
K = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}
\] (3.57)

Since \( K \) has two positive eigenvalues, the edge excitations of the \( \nu = 2/5 \) FQH state have two branches moving in the same direction. The \( \nu = 1 - \frac{4}{n} \) FQH state is described by the effective theory with

\[
K = \begin{pmatrix} 1 & 0 \\ 0 & -n \end{pmatrix}
\] (3.58)

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The two eigenvalues of $K$ now have opposite signs, hence, the two branches of the edge excitations move in opposite directions.

### 3.4 Charged excitations and electron propagator on the edges of generic FQH states

In the last two sections we studied the dynamics of edge excitations in FQH liquids at low energies. We found that the low lying edge excitations are described by a free phonon theory that is exactly soluble. In this section we will concentrate on the generic charge excitations. In particular we will calculate the propagators of the electrons and the quasiparticles for the most general (abelian) FQH states [20]. (See chapter 5) The key point again is to write the electron or the quasiparticle operators in terms of the phonon operator $\rho_I$. Once we do so, the propagators can be easily calculated because the phonons are free (at low energies and long wave length).

We know that for the FQH state described by (3.36), the edge states are described by the action (3.55). The Hilbert space of the edge excitations forms a representation of K-M algebra

$$[\rho_{Ik}, \rho_{Jk'}] = (K^{-1})_{IJ} \frac{1}{2\pi} k \delta_{k+k'}$$  \hspace{1cm} (3.59)

where $\rho_I = \frac{1}{2\pi} \partial_x \phi_I$ is the edge density of the $I^{th}$ condensate in the FQH state, $I, J = 1, \ldots, \kappa$, and $\kappa$ is the dimension of $K$. The electron density on the edge is given by

$$\rho_e = -e \sum_I \rho_I$$  \hspace{1cm} (3.60)

The dynamics of the edge excitations are described by the Hamiltonian:

$$H = 2\pi \sum_{IJ} V_{IJ} \rho_{I,k} \rho_{J,-k}$$  \hspace{1cm} (3.61)

where $V_{IJ}$ is a positive definite matrix.

Let us first try to write down the quasiparticle operator $\Psi_I$ on the edge which creates a quasiparticle labeled by $l_I$. We know that inserting the quasiparticle on the edge will cause a change $\delta \rho_I$ in the edge density of the $I^{th}$ condensate (see (3.40)) that satisfies

$$\int dx \delta \rho_I = l_J (K^{-1})_{JI}.$$  \hspace{1cm} (3.62)

Because $\Psi_I$ is a local operator that only causes a local change of the density, we have

$$[\rho_I(x), \Psi_I(x')] = l_J (K^{-1})_{JI} \delta(x-x') \Psi_I(x')$$  \hspace{1cm} (3.63)

Using the Kac-Moody algebra (3.59) one can show that the quasiparticle operators that satisfy (3.63) are given by

$$\Psi_I \propto e^{i\phi_I l_I}$$  \hspace{1cm} (3.64)

The charge of the quasiparticle $\Psi_I$ is determined from the commutator $[\rho_e, \Psi_I]$ and is found to be

$$Q_I = -e \sum_{IJ} l_J (K^{-1})_{IJ}$$  \hspace{1cm} (3.65)
From (3.42) and (3.64), we see that the electron operator can be written as

$$\Psi_{e,L} \propto \prod_I \Psi_I^l \propto e^{i \sum_I l_I \phi_I}$$

$$l_I = \sum_J K_{IJ} L_J, \quad \sum_I L_I = 1 \quad (3.66)$$

The above operators carry a unit charge as one can see from (3.65). The commutation of $\Psi_{e,L}$ can be calculated as

$$\Psi_{e,L}(x)\Psi_{e,L}(x') = (-)^\lambda \Psi_{e,L}(x')\Psi_{e,L}(x)$$

$$\lambda = \sum_{IJ} L_I K_{IJ} L_J \quad (3.67)$$

Because the diagonal elements of $K$ are odd integers, we can show that $(-)^\lambda = -1$. The electron operators defined in (3.66) are indeed fermion operators.

Since all the operators $\Psi_{e,L}$ for different choices of $L_I$ carry a unit charge and are fermionic, each $\Psi_{e,L}$ can be a candidate for the electron operators. In general the true electron operator is a superposition of $\Psi_{e,L}$s

$$\Psi_e = \sum_L C_L \Psi_{e,L} \quad (3.68)$$

In this paper, when we say there are many different electron operators on the edge, we really mean that the true physical electron operator is a superposition of the those operators.

Using the K-M algebra (3.59) and the Hamiltonian (3.61), we can calculate the propagators of a generic quasiparticle operator

$$\Psi_1 \propto e^{i \sum_I l_I \phi_I} \quad (3.69)$$

(which includes the electron operators for suitable choices of $l$). First we notice that after a suitable redefinition of $\rho_I$:

$$\tilde{\rho}_I = \sum_J U_{IJ} \rho_J \quad (3.70)$$

$K$ and $V$ can be simultaneously diagonalized, i.e., in terms of $\tilde{\rho}_I$ (3.59) and (3.61) become

$$[\tilde{\rho}_{Ik}, \tilde{\rho}_{Jk'}] = \sigma_I \tilde{\delta}_{IJ} \frac{1}{2\pi} \delta_{k+k'}$$

$$H = 2\pi \sum_I |v_I| \tilde{\rho}_{I,k} \tilde{\rho}_{I,-k} \quad (3.71)$$

where $\sigma_I = \pm 1$ is the sign of the eigenvalues of $K$. The velocity of the edge excitations created by $\tilde{\rho}_I$ is given by $v_I = \sigma_I |v_I|$.

To prove the above result we first redefine $\rho_I$ to transform $V$ into the identity matrix: $V \rightarrow U_1 V U_1^T = 1$. This is possible because $V$ is a positive definite symmetric matrix. Now $K$ becomes a new symmetric matrix $K_1 = U_1 K U_1^T$ whose eigenvalues have the same sign as the eigenvalues of $K$ (although the absolute values may differ). Then we
make an orthogonal transformation to diagonalize \( K_1 \): \((K_1)_{IJ} \to \sigma_I v_I \delta_{IJ} \). After a trivial rescaling of the densities, we obtain \((3.71)\). In terms of \( \tilde{\rho}_I \) the operator \( \Psi_1 \) has the form

\[
\Psi_1 \propto e^{i \sum_I \tilde{l}_I \phi_I} \\
\tilde{l}_I = \sum_I l_I U_{IJ}^{-1}
\]

From \((3.71)\) and \((3.72)\) we see that the propagator of \( \Psi_1 \) has the following general form

\[
\langle \Psi_1^\dagger(x, t) \Psi_1(0) \rangle \propto e^{i l_I k_I x} \prod_I (x - v_I t + i \sigma_I \delta)^{-\gamma_I}, \quad \gamma_I = \tilde{l}_I^2
\]

where \( v_I = \sigma_I |v_I| \) is the velocities of the edge excitations. \( \gamma_I \) in \((3.73)\) satisfy the sum rule

\[
\sum_I \sigma_I \gamma_I \equiv \lambda_1 = \sum I K_{IJ}^{-1} l_J
\]

In order to prove the above sum rule, we have used the relation

\[
(U K U^T)_{IJ} = \sigma_I \delta_{IJ}
\]

From \((3.67)\) and \((3.74)\) we see that the sum rule is directly related to the statistics of \( \Psi_1 \) and \( \lambda_1 \) is a topological quantum number. If \( \Psi_1 \) represents the electron operator, \( \lambda_1 \) will be an odd integer. From \((3.73)\) we also see that the operator \( \Psi_1 \) creates an excitation with momentum near \( \sum I l_I k_I \).

As an application of the formalism developed above, let us discuss the hierarchical states with the filling fractions \( \nu = \frac{p}{pq+1} (q = \text{even}) \) in more detail. Those states include \( \nu = 2/5, 3/7, 2/9, \ldots \) FQH states. The hierarchical states with \( \nu = \frac{p}{pq+1} \) is described by the \( p \times p \) matrices \( K = 1 + q C \), where \( C \) is the pseudo identity matrix: \( C_{IJ} = 1 \), \( I, J = 1, \ldots, p \). \( K^{-1} = 1 - \frac{q}{pq+1} C \). Because all the edge excitations move in the same direction, we have

\[
\langle \Psi_1^\dagger(x = 0, t) \Psi_1(0) \rangle \propto t^{-\lambda_1}
\]

where \( \lambda_1 \) is given by \((3.74)\). The fundamental quasiparticle is given by \( I^T = (1, 0, \ldots, 0) \) and carries the charge \( \frac{1}{pq+1} \). The exponent in its propagator is \( \lambda_1 = 1 - \frac{q}{pq+1} \). The quasiparticle with the smallest exponent is given by \( I^T = (1, \ldots, 1) \) and carries the charge \( \frac{p}{pq+1} \). The exponent is \( \lambda_I = \frac{p}{pq+1} \) that is less than \( 1 - \frac{q}{pq+1} \) (note we have \( q \geq 2 \) and \( p \geq 1 \)). Such a quasiparticle (with the charge \( \frac{p}{pq+1} \)) dominates the tunneling between two edges of the same FQH fluid at low energies.

The electron operators are given by \( \Psi_{e,L} = \psi_1 \) with \( I \) satisfying \( \sum_I l_I = pq + 1 \). The exponent in the propagator is given by \( \lambda_1 = \sum I l_I^2 - q(pq+1) \). The electron operator with a minimum exponent in its propagator is given by \( I^T = (q, \ldots, q, q+1) \). The value of the minimum exponent is \( \lambda_1 = q + 1 \). Such an electron operator dominates the tunneling between edges of two different FQH fluid at low energies.

3.5 Some simple phenomenological consequences of chiral Luttinger liquids at FQH edges

In the last four sections we have shown that the electrons at the edges of a FQH liquid form a chiral Luttinger liquid. As one of the characteristic properties of chiral Luttinger liquids, the electron and the quasiparticle propagators obtain anomalous exponents:

\[
\langle \Psi_e^\dagger(t, x = 0) \Psi_e(0) \rangle \sim t^{-g_e}, \quad \langle \Psi_q^\dagger(t, x = 0) \Psi_q(0) \rangle \sim t^{-g_q}
\]

(3.77)
Those anomalous exponents can be directly measured by tunneling experiments between the FQH edges.

Two situations need to be considered separately: I) two edges of FQH liquids are separated by an insulator; and II) two edges are separated by a FQH liquid. In case I, only electron can tunnel between the edges, and in case II the quasiparticles supported by the FQH liquid that separates the two edges can tunnel. The tunneling operator $A$ is given by $A \propto \Psi_{e1} \Psi_{e2}^\dagger$ for case I, where $\Psi_{e1}$ and $\Psi_{e2}$ are the electron operators on the two edges. The tunneling operator $A$ has the form $A \propto \Psi_{q1} \Psi_{q2}^\dagger$ for case II, where $\Psi_{q1}$ and $\Psi_{q2}$ are the quasiparticle operators on the two edges. The physical properties of the tunneling can be calculated from the correlation of the tunneling operator, which in turn can be expressed as a product of the electron or quasiparticle propagators on the two edges.

Let $g = g_e$ for case I and $g = g_q$ for case II. Then at zero temperature one can show that the anomalous exponents of the electron and quasiparticle operators lead to non-linear tunneling $I$-$V$ curve,

$$I \propto V^{2g-1}$$

The noise spectrum of the tunneling current also contains a singularity at the frequency $f = QV/h$,

$$S(f) \sim |f - \frac{QV}{h}|^{2g-1}$$

where $V$ is the voltage difference between the two edges and $Q$ is the electric charge of the tunneling electron or the tunneling quasiparticle. At finite temperature $T$, the zero bias conductance also has a power law dependence:

$$\sigma \propto T^{2g-2}$$

We see that the anomalous exponents can be easily measured by the tunneling experiments. The noise spectrum further reveals the charges of the tunneling particles.

The exponents $g_e$ and $g_q$ are calculated in section 3.4. To summarize the results in a simple way, it is convenient to divide the FQH edges into three classes. A) all edge excitations move in one direction – the direction of the normal drift velocity along the edge. Such a direction will called the normal direction. B) One branch of edge excitations propagates in the normal direction, and all other branches propagate in the opposite direction. C) There are two or more branches propagating in the normal direction and one or more branches propagating in the opposite direction.

For the class A edges, the exponents $g_e$ and $g_q$ are directly related to the statistics of the electrons and quasiparticles. In this case $g_e$ and $g_q$ are universal and independent of details of electron interactions and edge potentials. The following table lists some FQH states which support the class A edge states, as well as the corresponding values of the exponents $g_{e,q}$ and the electric charge of the associated particles (note only the minimum values of $g_e$ and $g_q$ are listed):

| FQH states : $\nu$ | $g_e$ | $g_q$ | charge $Q_q/e$ |
|------------------|------|------|----------------|
| $\nu = 1/m$      | $m$  | $1/m$| $1/m$          |
| $\nu = 2/5$      | $3$  | $2/5$| $2/5$          |
| $\nu = \frac{n}{2n+1}$ | $3/3$ | $3/3$| $3/3$          |
| $\nu = \frac{n}{2n+1}$ | $3/8$ | $3/8$| $3/8$          |
| $\nu = \frac{n}{2n+1}$ | $2/5$ | $2/5$| $2/5$          |

We also see from section 3.4 that, due to the presence of the modes propagating in both directions, $g_e$ and $g_q$ are not universal for the class B and class C edges. Their values depend on the electron interactions and the edge potentials. However the electrons
in the real FQH sample interact through a quite special interaction – the long range Coulomb interaction. In this case the edge Hamiltonian has a special form,

\[ H = \sum [V_{ij} \rho_i \rho_j + V_c (\sum t_i \rho_i)^2] \] (3.81)

with \( V_c \gg V_{ij} \) (in fact \( V_c \sim -\ln k \) at long wave lengths). In (3.81) \( t_i \) is the charge vector, and \( \sum t_i \rho_i \) is the total charge density on the edge. In the limit \( V_c \gg V_{ij} \) the edge has a special structure. One edge mode propagates at a large velocity \( \sim V_c \) in the normal direction. All other modes propagate at smaller velocity \( v \sim V_{ij} \). What is more important is that only the fast mode carries charge and all the other slow modes are neutral. The separation between the charge and the neutral modes in the presence of the long range Coulomb interaction make it very difficult to detect more than one edge branch in the edge magnetoplasma experiments [56]. In general, the edge magnetoplasma experiments can see only a single charge mode. The neutral modes are hard to detect due to the weak coupling. Thus, the edge magnetoplasma experiment in Ref. [57] does not contradict our picture that the 2/3 FQH state contains edge modes that propagate in both directions.

In addition to separating the charge and the neutral modes, the long range Coulomb interaction also makes the exponents \( g_e \) and \( g_q \) universal for the class B edges. The exponent \( g_l \) in the propagator of a quasiparticle labeled by an integer vector \( l \) can be calculated as follows. The calculation applies only to the class B edges in the limit \( V_c \gg V_{ij} \). We first separate \( l \) into a charge part and a neutral part,

\[ l = l_c + l_n \] (3.82)

where the neutral part \( l_n \) satisfies

\[ t^TK^{-1}l_n = 0 \] (3.83)

Therefore, we have

\[ l_n = 1 - t^TK^{-1}l_c, \quad l_c = t^TK^{-1}l \] (3.84)

One can show that the operator \( \sum_i l_{n,i} \rho_i \) generates the slow neutral modes and \( \sum_i l_{c,i} \rho_i \) generates the fast charge mode. Following the calculation outlined in section 3.4, one can show that

\[ g_l = |l_c^TK^{-1}l_c| + |l_n^TK^{-1}l_n| \]

\[ = |l^T M l| \] (3.85)

where

\[ M = K^{-1} - \frac{2}{t^TK^{-1}t} K^{-1}tt^TK^{-1} \] (3.86)

The simple result (3.85) is due to two facts: a) the charge and the neutral modes separate, and b) all the neutral modes propagate in one direction. From (3.85) we can calculate the minimum values of \( g_e \) and \( g_q \). The following table lists some FQH states which support the class B edge states, and the corresponding values of the exponents \( g_{e,q} \) (again only the minimum values of \( g_e \) and \( g_q \) are listed).

| FQH states | \( \nu = 2/3 \) | \( \nu = 1 - 1/m \) | \( \nu = 3/5 \) | \( \nu = \frac{n+1}{2n+1} \) | \( \nu = 2/7 \) |
|------------|----------------|----------------|----------------|----------------|----------------|
| \( g_e \)  | 2              | \( \frac{m+1}{m-1} \) | 7/3            | 3 - \( \frac{2}{n+1} \) | 4              |
| \( g_q \)  | 2/3            | \( \frac{1}{m} \) | 3/5            | \( \frac{n+1}{2n+1} \) | 2/7            |
| charge \( Q_q/e \) | 1/3, 2/3 | 1/m               | 3/5            | \( \frac{n+1}{2n+1} \) | 1/7, 2/7 |
In section 3.3 we argued that the edge excitations of the a FQH state characterized by a matrix $K$ are described by $U(1)$ K-M algebras characterized by the same matrix (see (3.59)). This result is correct only for sharp edges where the electron density drops to zero in a range of order magnetic length near the edge. However, the edge potentials in real QH samples are very smooth, and the electron density gradually drops to zero in a range of a few thousands angstrom in order to minimize the electrostatic energy [58]. Thus the edge structures for smooth and sharp edge potentials are quite different. It was shown in Ref. [59] that as the edge potential changes from a sharp one to a smooth one, the QH edge will undergo a so called edge reconstruction. One or more pairs of edge branches are generated after the edge reconstruction. Each pair contains two branches that propagate in opposite directions, and behaves like the usual non-chiral Luttinger liquid in one dimension. After the edge reconstruction, even a $\nu = 1$ IQH state can have several edge branches. The edge excitations on a smooth edge are generally described by the following matrix:[59]

$$K_{\text{edge}} = K_{\text{bulk}} \oplus \begin{pmatrix} K' & 0 \\ 0 & -K' \end{pmatrix}$$

where $K_{\text{bulk}}$ is the matrix characterizing the bulk FQH state and $K'$ is another integer matrix that depends on the edge potential. For example, the edge excitations on a smooth 2/3 edge could be described by

$$K_{\text{edge}} = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}$$

In an ideal FQH sample where electrons and/or quasiparticles are not allowed to be scattered between different edge branches, a bar of the FQH liquid has a quantized two-terminal conductance only when the FQH liquid has a class A edge state. However, in the presence of the long range Coulomb interaction, any FQH liquids will have a (approximately) quantized two-terminal conductance $\sigma = \nu e^2/h$, due to the separation of the charge and the neutral modes in the presence of the long range interaction.

In the real sample, there are many mechanisms that allow electrons and quasiparticles to scatter between different edge branches. These mechanisms include inelastic scattering of phonons and elastic scattering of impurities. These scatterings equilibrate different edge branches. Using the sum rule obtained in Ref. [45,47], one can show that equilibrated edge always gives quantized two-terminal conductance. Thus in addition to the long range Coulomb interaction, there are many mechanisms to cause the two-terminal conductance to be quantized at $\sigma = \nu e^2/h$, even for the class B and C edges.

In the following we will briefly discuss some effects of impurities on the properties of FQH edge states. We know that the electrons in the non-chiral Luttinger liquids are localized in the presence of impurities. Because of this, the additional pairs of the edge branches generated by the smooth edge will be localized by the impurities. Thus it is possible that the dirty smooth edges may have very similar physical properties to the clean sharp edges at low enough energies. It is also obvious that the impurities cannot localize the electrons in the class A edge due to the lack of back scattering.

The situation becomes very interesting for the edge state of the $\nu = 2/3$ FQH liquid, which contains two branches propagating in opposite directions. In this case the impurities do cause back scattering. However the back scattering will not cause localization of the electrons on the edge. Mobile edge excitations are required to cancel the gauge anomaly from the bulk Chern-Simons term, so that the effective action for a finite QH
system will be invariant under the gauge transformations of the electromagnetic gauge potential [45]. A detailed dynamical theory for the disordered 2/3 edge has been proposed in Ref. [54], where it was shown that even at the strong-disordered fixed point the electrons on the edge remain delocalized. It was further argued that the strong-disordered fixed point has a very special dynamical property – the charge mode and the neutral mode separate even in the absence of long range interaction. The separation of the charge and the neutral modes leads to the universal $g_{e,q}$ whose values coincide with those calculated above for the long range interaction. Ref. [54] also studied in detail some experimental consequences of the disordered edges. However, a different point of view was raised in Ref. [60].

4. Shift and spin vectors – New topological quantum numbers for FQH liquids

In Chapter 2 we studied the effective theory of (abelian) FQH liquids. We found that the effective theory depends on the $K$-matrix and the charge vector $t$. This result seems to suggest that the (abelian) FQH liquids are completely characterized by $K$ and $t$. However, in this chapter we will see that there is a new kind of topological quantum numbers for FQH liquids. This new type of quantum numbers does not directly appear in the effective theory on the plane. However they can still be measured by experiments in planar systems, due to a new set of selection rules induced by these quantum numbers.

4.1 Quantum Hall states on a sphere and the shift

To understand the appearance of new quantum numbers – spin vectors, let us first consider the $\nu = 1$ IQH state on a sphere. Assume that the magnetic field is uniform on the sphere and let $N_{\phi}$ be the total number of the flux quanta passing through the sphere. The Hamiltonian has an $su(2)$ symmetry associated with the rotation of the sphere. Thus, the degenerate eigenstates form irreducible representations of the $su(2)$ group. The ground states form a representation with a $su(2)$-spin $S^{su(2)} = N_{\phi}/2$ and thus have $2S^{su(2)} + 1 = N_{\phi} + 1$ folds degeneracy [12]. These $N_{\phi} + 1$ states form the first Landau level on the sphere. The next energy level has $N_{\phi} + 3$ degeneracy and forms the $S^{su(2)} = \frac{N_{\phi}}{2} + 1$ representation of the $su(2)$, which corresponds to the second Landau level. In general the states in the $n^{th}$ Landau level form the $S^{su(2)} = \frac{N_{\phi}}{2} + n - 1$ representation of the $su(2)$ rotation of the sphere.

The electron wave functions on the sphere can be expressed in a simple form through the following spinor coordinates[12]

\[ (u, v) = (\cos(\theta)e^{i\phi/2}, \sin(\theta)e^{-i\phi/2}) \]  \hspace{1cm} (4.1)

The $N_{\phi} + 1$ states in the first Landau level are described by the wave functions $u^{m}v^{n}$, $m = N_{\phi} - n = 0, ..., N_{\phi}$, which form the $S^{su(2)} = m + n$ representation with $S^{su(2)} = m$.

We see that the $\nu = 1$ IQH state with a filled first Landau level contains $N_{e} = N_{\phi} + 1$ number of electrons. Thus $N_{e}/N_{\phi}$ is not exactly equal to one. We will call $S$ in the relation

\[ N_{\phi} = \nu^{-1}N_{e} - S \]

the shift. The shift of a QH state depends on the topology of the space. The $\nu = 1$ IQH state has a shift $S = 1$ on a sphere, and a shift $S = 0$ on a torus [13].
We can construct another \( \nu = 1 \) IQH state with filled second Landau level. Such a state contains \( N_e = N_\phi + 3 \) electrons and has a shift \( S = 3 \) on the sphere. On the torus the shift still vanishes.

Both the above two IQH states are described by the same effective theory with \( K = 1 \) and \( t = 1 \) on the plane. This indicates that the topological quantum number \( S \) cannot be determined from \( K \) and \( t \). Thus the \( K \)-matrix and the charge vector do not provide a complete description of the QH liquids. We need to find some additional quantum numbers which will at least enable us to determine the shift [21,61,62].

To gain some understanding of the origin of the new quantum numbers, let us consider in more detail the above two \( \nu = 1 \) IQH states. The two IQH states differ only by the different cyclotron motions of the electrons. We know that as the Landau level index increases, the orbital angular momentum carried by the cyclotron motion also increases. In the following we will call such orbital angular momentum – orbital spin. Note here that the orbital spin is associated with the \( U(1) \) rotation of the plane and should be distinguished from the ordinary spin quantum number of the electron and the \( su(2) \) quantum \( S^{su(2)} \) discussed above. In this chapter we will consider only spinless electrons (which correspond to the spin polarized electrons in experiments).

As an object with non-zero orbital spin moves along a loop \( C \) on a curved 2D space, a non-zero Berry’s phase will be induced. The Berry’s phase is given by

\[
\phi = S_{ob} \oint_C dx^i \omega_i = S_{ob} \int_{S_C} d^2 x R
\]

where \( S_C \) is the area enclosed by the loop \( C \). Here \( R \) is the curvature and \( \omega_i \) the connection whose curl gives rise to the curvature. The relation between \( \omega_i \) and \( R = \varepsilon_{ij} \partial_i \omega_j \) is the same as relation between the gauge potential \( a_i \) and the “magnetic field” \( b = \varepsilon_{ij} \partial_i a_j \). When a 2D space is embedded in a 3D space, the integral \( \oint_C dx^i \omega_i = \int_{S_C} d^2 x R \) has a geometric meaning. The norm vector of the 2D surface spans a solid angle \( \Omega \) as it moves along the loop \( C \). The above integral is simply determined by this solid angle:

\[
\oint_C dx^i \omega_i = \int_{S_C} d^2 x R = \Omega
\]

Thus the total curvature of a sphere is \( \int_{sphere} d^2 x R = 4\pi \).

From the above discussion we see that because electrons carry a non-zero orbital spin, the total flux seen by the electron is the sum of the magnetic flux and the Berry’s phase induced through the coupling of the orbital spin to the curvature of the space. It is this Berry’s phase that causes the shift. Thus, the electrons with different orbital spin will cause different shifts. According to this picture we also obtain the well known fact that the shift always vanishes on torus [13], since the curvature on torus is zero.

According to the semiclassical picture of the cyclotron motion, the electrons in the \( n^{th} \) Landau level have an orbital spin of \( S_{ob} = n + \text{const} \). Thus, the orbital spins of the electrons in the first and the second Landau levels differ by one, and the induced Berry’s phase differ by two flux quanta on the sphere. Therefore, the shifts of the \( \nu = 1 \) state in the first landau level and the second Landau level differ by two.

To determine the absolute value of the orbital spin for each Landau level, let us move an electron in the \( n^{th} \) Landau level around a loop \( C \) on the sphere. There are two ways to calculate the phase \( \Phi \) induced by such a motion. The phase \( \Phi \) can be obtained as the sum of the phase induced by the magnetic field and the Berry’s phase induced by the orbital spin

\[
\Phi = \oint_C (dx^i - eA_i + S_{ob} \omega_i) = 2\pi (N_\phi + 2S_{ob}) \frac{\Omega}{4\pi}
\]
where $\Omega$ is the solid angle spanned by the loop $C$ on the sphere. The phase $\Phi$ can also be calculated from the fact that the electrons in the $n^{th}$ Landau level form an $S^{su(2)} = \frac{N_\phi}{2} + n - 1$ representation of the $su(2)$. An electron in the $n^{th}$ Landau level and at position $(\theta, \phi)$ on the sphere is an eigenstate of $n \cdot S^{su(2)}$ with an eigenvalue $\frac{N_\phi}{2} + n - 1$, where $n$ is the unit vector in the direction $(\theta, \phi)$ and $S^{su(2)}$ are the three generators of the $su(2)$ rotation of the sphere. As $n$ traces out the loop $C$, a Berry’s phase will be induced due to the non-zero $su(2)$ spin $S^{su(2)}$. The induced phase is

$$\Phi = S^{su(2)} \Omega = \left[ N_\phi + 2(n - 1) \right] \frac{\Omega}{2}$$

(4.6)

Comparing (4.5) with (4.6) we find the orbital spin for the $n^{th}$ Landau level to be

$$S_{ob} = n - 1$$

(4.7)

The orbital spin for first Landau level vanishes. (4.5), (4.6) and (4.7) reveal a direct relation between the following quantities: a) the orbital spin of electrons; b) the number of effective flux quanta, $N^*_\phi = N_\phi + 2S_{ob}$, seen by the electrons; c) the $S^{su(2)}$ quantum number of the electrons, $S^{su(2)} = N^*_\phi / 2$, d) the number of states in the corresponding Landau level, $N = 2S^{su(2)} + 1 = N^*_\phi + 1$, and e) the value of the shift, $S = N - N_\phi = 2S_{ob} + 1$, for the $\nu = 1$ state.

4.2 Spin vectors in QH liquids

In the last section we saw that the shift is closely related to the coupling between the orbital spin and the curvature of the space. In this section we will attempt to include such coupling in the effective theory so that we will be able to calculate the shift from the effective theory. In the process of building such an effective theory we find that it is necessary to introduce a new topological quantum number – spin vector – in our description of QH liquids.

First let us consider a system of bosons or fermions on a sphere with a uniform magnetic field. We also assume that the bosons or fermions carry an orbital spin $S_{ob}$. Those particles are described by

$$\mathcal{L} = -et_1 A_\mu J^\mu + S_{ob} \omega_i J^i + \text{Kinetic Energy}$$

(4.8)

Here in addition to the orbital spin, we have also included two other complications. The first one is that we allow the magnetic field $B$ to have either positive or negative values. The second one is that we include $t_1$ in (4.8). $t_1$ is equal to 1 if the particles carry a charge of $-e$, and is equal to $-1$ if the particles carry a charge $e$.

Let $N_\phi = -eB/\Phi_0$ be the number of the flux quanta. If $N_\phi t_1 > 0$, then a filling-fraction-$1/m$ Laughlin state of the boson or the fermion is described by the wave function

$$\Phi = \prod_{i<j} (v_i u_i - v_j u_j)^m$$

(4.9)

where $(u, v)$ is the spinor coordinate (see (4.1)). If $t_1 N_\phi < 0$, the $1/m$ Laughlin state will have a form

$$\Phi = \prod_{i<j} (v_i^* u_i^* - v_j^* u_j^*)^m$$

(4.10)
Here we always assume $m > 0$. For a given $i$, $\Phi$ is a homogeneous polynomial of $(u_i, v_i)$ (or $(u^*_i, v^*_i)$) of degree $m(N_p - 1)$ where $N_p$ is the number of the bosons or fermions. Thus the single-particle states form an $S^{su(2)} = m(N_p - 1)/2$ representation of the $su(2)$ rotation of the sphere. Knowing that all particles are in the first Landau level we conclude that the total number of the effective flux quanta seen by the particles is given by $N^* = 2S^{su(2)} = m(N_p - 1) = |t_1N_\phi + 2S_{ob}|$. This implies that $|N_\phi| = mN_p - [m + 2S_{ob}sgn(t_1N_\phi)]$. The shift is defined through $|N_\phi| = mN_p - S$ when $N_\phi$ can be negative. So the shift of our $1/m$ state is given by

$$S = 2S_{ob}sgn(t_1N_\phi) + m \quad (4.11)$$

(4.11) is valid for both positive and negative $t_1$ and $N_\phi$. We also would like mention that the number of the particles $N_p$ is assumed to be positive in the above discussion.

Now the question is how to reproduce the shift (4.11) from the effective theory. Here we propose the Chern-Simons effective theory of the $1/m$ state on the sphere to have the form

$$\mathcal{L} = \left[-m\text{sgn}(t_1N_\phi)\frac{1}{4\pi}a_\mu \partial_\nu a_\lambda \varepsilon^{\mu\nu\lambda} - \frac{e t_1}{2\pi}A_\mu \partial_\nu a_\lambda \varepsilon^{\mu\nu\lambda} + s\omega_i \partial_\nu a_\lambda \varepsilon^{i\nu\lambda}\right] \quad (4.12)$$

where the third term describes the coupling of the curvature. The new quantum number $s$ will be called the spin vector (here it has only one component). From the equation of motion $\frac{\partial \mathcal{L}}{\partial a_0} = 0$, the total number of the particles can be shown to be

$$N_p = \int d^2x \frac{1}{2\pi} \varepsilon_{ij} \partial_i a_j = \int d^2x \frac{1}{2m\pi\text{sgn}(t_1N_\phi)} \varepsilon_{ij} \partial_i (t_1\partial_j A_j - 2\pi s\omega_j)$$

$$= \frac{1}{m\text{sgn}(t_1N_\phi)} (t_1N_\phi + 2s) \quad (4.13)$$

We see that the inclusion of $\text{sgn}(t_1N_\phi)$ in (4.12) ensure that $N_p > 0$. (4.13) implies that $N_p = \frac{1}{m}[|N_\phi| + 2\text{sgn}(t_1N_\phi)s]$. Thus the shift is simply given by $S = 2\text{sgn}(t_1N_\phi)s$ and we find the spin vector for the above $1/m$ state to be (by comparing with (4.11))

$$s = S_{ob} + \frac{m}{2}\text{sgn}(t_1N_\phi) \quad (4.14)$$

It is interesting to see that the spin vector receives two contributions. The first contribution, coming from the orbital spin $S_{ob}$, is easy to understand. One way to understand the second contribution is the following. We know that the $1/m$ state can be viewed as a boson condensation of composite bosons. A composite boson is a bound state of the boson or the fermion with $m$ units of flux quanta. The binding of the flux not only changes the statistics of the particle, it also changes the orbital spin of the boson or the fermion to a new value, and $s$ in (4.14) can be regarded as the orbital spin of the composite bosons.

We would like to mention that if the finite orbital spin $S_{ob}$ is due to the bosons or fermions occupying the $n^{th}$ Landau level, then

$$S_{ob} = (n - 1)\text{sgn}(t_1N_\phi) \quad (4.15)$$

which generalizes (4.7) to include the possibility that $t_1$ and $N_\phi$ may be negative. Thus (4.14) can be rewritten as

$$s = (n - 1 + \frac{m}{2})\text{sgn}(t_1N_\phi) \quad (4.16)$$
(4.16) only applies to the Laguian,

\[
L = \left[ -K_{11} \frac{1}{4\pi} a_\mu \partial_\nu a_\lambda \varepsilon^{\mu\nu\lambda} - \frac{e_{1}}{2\pi} A_\mu \partial_\nu a_\lambda \varepsilon^{\mu\nu\lambda} + s\omega_i \partial_\nu a_\lambda \varepsilon^{i\nu\lambda} \right]
\]  

(4.17)

with \( K_{11} \) and \( t_1 N_\phi \) to have the same sign. In this case, \( \epsilon_{ij} \partial_i a_j / 2\pi \) is positive and can be regarded as the density of the particles. When \( K_{11} \) and \( t_1 N_\phi \) have the opposite signs, we should view \(-\epsilon_{ij} \partial_i a_j / 2\pi \) as the density of the particles. Or we can make a transformation \( a_\mu \rightarrow -a_\mu \) and \( t_1 \rightarrow -t_1 \) in (4.17) to make \( K_{11} \) and \( t_1 N_\phi \) to have the same sign. Thus (4.16) can be generalized to

\[
s = (n - 1 + \frac{|K_{11}|}{2}) \text{sgn}(t_1 N_\phi)
\]

(4.18)

which applies to the 1/m state described by (4.17) where \( K_{11} \) can have any signs. Here we have assumed that the condensed particles occupy the \( n^{th} \) Landau level.

Applying the above results to the electron systems we see that the 1/m state in the first Landau level has a spin vector \( s = m/2 \) while a 1/m state in the \( n^{th} \) Landau level has a spin vector \( s = n - 1 + \frac{m}{2} \) (Here we have assumed \( t_1 > 0 \) and \( N_\phi > 0 \)).

Now, knowing the effective theory of the first level hierarchical states, we can include the quasiparticle excitations (or electron excitations in the next layer), and construct the effective theory of the second level hierarchical states (or double layer FQH states).

Consider a \( \kappa^{th} \) level hierarchical state (or a \( \kappa \)-layer FQH state) of an electron system on a sphere. Motivated by the effective theory of the Laughlin state (4.17), let us assume the effective theory for the FQH state to have a form

\[
L = -\frac{1}{4\pi} K_{II} a_{I\mu} \partial_\nu a_{I\lambda} \varepsilon^{\mu\nu\lambda} - \frac{e}{2\pi} t_I A_\mu \partial_\nu a_{I\lambda} \varepsilon^{\mu\nu\lambda} + s_I \omega_i \partial_\nu a_{I\lambda} \varepsilon^{i\nu\lambda}
\]

(4.19)

where the curvature couples with all the condensates through the spin vector \( s^T = (s_1, s_2, ...) \). From the equation of motion \( \frac{\partial L}{\partial a_0} = 0 \), we obtain the number of electrons

\[
N_e = t^T K^{-1} t N_\phi + 2t^T K^{-1} s
\]

(4.20)

For an electron system \( t^T K^{-1} t N_\phi = \nu N_\phi \) is always positive, and \( N_e \) is positive as expected. However for a hole system (where the particles carry a charge of \( e \)), \( t^T K^{-1} t N_\phi = \nu N_\phi \) is always negative. To include this possibility (4.20) should be generalized to

\[
N_e = |t^T K^{-1} t N_\phi + 2t^T K^{-1} s|
\]

(4.21)

Now the positive \( N_e \) can be regarded as the number of electrons or holes. We can see that the shift, for electron or hole systems, is given by

\[
S = \frac{2t^T K^{-1} s}{t^T K^{-1} t} \text{sgn}(N_\phi)
\]

(4.22)

on the sphere. We see that the spin vector determines the shift of the QH state.

Now the question is how to determine the value of the spin vector. The idea is that the FQH state described by (4.19) contain \( \kappa \) components, and each component is
described by a Laughlin state. Thus we may use the result (4.18) to calculate the spin vector \( s \).

Let us first concentrate on the first component described by \( a_{1\mu} \). (4.19) can be rewritten as

\[
\mathcal{L} = -\frac{1}{4\pi} K_{11} a_{1\mu} \partial_\nu a_{1\lambda} \varepsilon^{\mu\nu\lambda} + \left( -\sum_{I>1} \frac{1}{2\pi} K_{1I} a_{I\mu} \right) \partial_\nu a_{1\lambda} \varepsilon^{\mu\nu\lambda} + s_1 \omega_i \partial_\nu a_{1\lambda} \varepsilon^{\nu\lambda} + \ldots
\]

where “…” represents terms that do not contain \( a_{1\mu} \). (4.23) and (4.17) have the same form except \(-e t_1 A_\mu\) in (4.17) is replaced by \(-e t_1 A_\mu - \sum_{I>1} K_{1I} a_{I\mu}\). Thus from (4.18) we see that

\[
s_1 = \frac{|K_{11}|}{2} \text{sgn} (t_1 N_\phi - \sum_{I>1} K_{1I} \frac{b_I}{2\pi})
\]

if we assume the condensate occupies the first Landau level. Since \( \frac{b_I}{2\pi} = (K^{-1})_{IJ} t_{IJ} N_\phi \), (4.24) can be rewritten as

\[
s_1 = \frac{|K_{11}|}{2} \text{sgn} (t_1 - \sum_{I=2, J=1}^{\kappa} K_{IJ} (K^{-1})_{IJ} t_{IJ} N_\phi)
\]

\[
= \frac{|K_{11}|}{2} \sum_{J} (K^{-1})_{1J} t_{J} N_\phi
\]

Generalizing the above to a general condensate, we get

\[
s_I = \frac{K_{II}}{2} \text{sgn} (\sum_{J} (K^{-1})_{IJ} t_{IJ} N_\phi)
\]

Note \( t_I \sum_{J} (K^{-1})_{IJ} t_{IJ} N_\phi \) is the number of charges (in the unit of electron charge \(-e\)) in the \( I^{th} \) condensate.

To obtain (4.26) we have assumed that the each condensate forms a Laughlin state in the first Landau level, which is a part of assumptions in the hierarchical construction. To construct more general states we may assume (although it may not be energetically favorable) that the \( I^{th} \) condensate forms a Laughlin state in the \( n_I^{th} \) Landau level. In this case we should replace (4.26) by

\[
s_I = \frac{K_{II}}{2} \text{sgn} (\sum_{J} (K^{-1})_{IJ} t_{IJ} N_\phi) + (n_I - 1) \text{sgn} (K_{II} \sum_{J} (K^{-1})_{IJ} t_{IJ} N_\phi)
\]

where \( n_I \) is an integer.

An important consequence of (4.27) is the quantization of the spin vector. We see that two times the spin vector, \( 2s \), is always an integer vector.

We would like to point out that (4.26) and (4.27) apply to both the hierarchical FQH states and the multi-layer FQH state. In general we expect the stable states to have \( n_I = 1 \). For the multi-layer states with \( t_I = 1 \) and \( K_{IJ}\)s all positive, \( (K^{-1})_{IJ} t_{IJ} N_\phi \) is the number of electrons in the \( I^{th} \) layer which is positive. Therefore

\[
s_I = \frac{K_{II}}{2}
\]
In particular for the \((l m n)\) state

\[
s = \left( \begin{array}{c} l/2 \\ m/2 \end{array} \right)
\] (4.29)

For the hierarchical states with \(t^T = (1, 0, 0..., \) (4.26) reduces to

\[
s_I = \frac{K_{II}}{2} \text{sgn}((K^{-1})_{I1}N_\phi)
\] (4.30)

which allows us to calculate the spin vectors for the hierarchical states listed in (2.34).

Combining (4.28) and (4.30) with (4.22), we can calculate the shift for the hierarchical or the multi-layer states. Notice that the shift \(S\) is independent of \(\text{sgn}(N_\phi)\) due to a cancelation, which is expected.

In the following we will calculate the orbital spin quantum number of quasiparticle excitations\[21,61,62\] from the new effective theory (4.19) that contains the spin vector. Let us create a generic quasiparticle labeled by \(l_I\) in the FQH state (4.19) on the sphere. The charge and the statistics of the quasiparticle are given by (2.30). In the presence of the quasiparticle the relation between the number of electrons and the number of the flux quanta is no longer given by (4.21). There is an additional shift due to the quasiparticle. Again from the equation of motion \(\frac{\partial(L + L_{q})}{\partial a_0} = 0\), we find

\[N_e = t^T K^{-1} t N_\phi + 2t^T K^{-1} s + t^T K^{-1} l\]

(4.31)

To calculate the orbital spin of the quasiparticle, we move the quasiparticle around a loop \(C\) which spans a solid angle \(\Omega\). There are two ways to calculate the induced phase \(\Phi\). First the phase \(\Phi\) can be obtained as a sum of the phase induced by a magnetic field and the phase induced by the orbital spin,

\[\Phi = \frac{Q_q}{e} N_\phi \frac{\Omega}{2} + 2 S_{ob}^q \frac{\Omega}{2}\]

(4.32)

where \(Q_q = et^T K^{-1} l\) is the charge and \(S_{ob}^q\) is the orbital spin of the quasiparticle. In fact, (4.32) can be viewed as the definition of the orbital spin \(S_{ob}^q\) of the quasiparticle. Note \(N_\phi\) in (4.32) is the number of the flux quanta in the presence of the quasiparticle and is given by (4.31). The same phase can also be calculated from the coupling

\[l_I a_{I\mu} j_\mu\]

(4.33)

First let us assume that the densities of each condensate (i.e., the field strength of \(a_{I\mu}\)) is uniform on the sphere even in the presence of the quasiparticle. This can be achieved by including the term

\[\mathcal{L}_K = g(f^I_{\mu\nu})^2, \quad f^I_{\mu\nu} = \partial_\mu a_{I\nu} - \partial_\nu a_{I\mu}\]

(4.34)

in the effective theory (4.19) and assuming \(g\) is very very large. In this limit, the phase induced by the coupling (4.33) can be easily calculated. From the equation of motion we find,

\[\int d^2 x b_I = \int d^2 x \epsilon_{ij} \partial_i a_{Ij} = 2\pi K^{-1}_{IJ} (t_J N_\phi + 2s_J + l_J)\]

(4.35)

where we integrate over the whole sphere. Since \(b_I\) are constants on the sphere, we have

\[\Phi = l_I \frac{\Omega}{4\pi} \int d^2 x b_I = l_I K^{-1}_{IJ} (t_J N_\phi + 2s_J + l_J) \frac{\Omega}{2}\]

(4.36)
Although $\Phi$ in (4.36) is calculated in the large $g$ limit, in the following we would like to show that $\Phi$ is topological and is independent of $g$. Notice that the solid angle $\Omega$ spanned by the loop $C$ is defined only up to a multiple of $4\pi$. The consistency of the theory requires that the ambiguity of $\Omega$ should only cause an ambiguity of multiples of $2\pi$ in the phase $\Phi$. Thus the coefficient in front of $\Omega/2$ must be quantized as an integer for any value of $g$,

$$1^TK^{-1}tN_{\phi} + 21^TK^{-1}s + 1^TK^{-1}l = \text{integer} \quad (4.37)$$

This implies that the coefficient in front of $\Omega/2$ is independent of $g$ and that (4.36) is valid for all values of $g$. Comparing (4.32) and (4.36) we find

$$S_{ob} = \frac{1}{2}1^TK^{-1}l + 1^TK^{-1}s \quad (4.38)$$

It is interesting to see that the orbital spin of the quasiparticle receives two contributions. The first term $\frac{1}{2}1^TK^{-1}l = \frac{\theta}{2\pi}$ is directly related to the statistics of the quasiparticle $\theta$ and satisfies the spin-statistics theorem. The second term is due to the spin vector of the condensates, which violate the spin-statistics theorem [21,61,62].

We would like to remark that for a given $K$-matrix, (4.37) sometimes cannot be satisfied for any integer choice of $N_{\phi}$, if $l$ takes certain values. This implies that the quasiparticle labeled by these $l$ cannot be created individually on a sphere [63]. For example, one cannot create a single charge-1/4 quasiparticle on top of the $\nu = 1/2$ (331) FQH state on a sphere, although one can create two of them by inserting a unit flux. This selection rule, however, does not invalidate the above calculation of $S_{ob}$, because mathematically we can always choose a non-integer value for $N_{\phi}$.

From the construction discussed above we can easily obtain the spin vectors for the FQH states listed in (2.34) and (2.42). Using (4.38) we can calculate the orbital spin quantum numbers of various excitations. In particular, one can show that some neutral plasma modes carry non-trivial orbital spin quantum numbers. The conservation of the angular momentum implies that to excite these plasma modes through, e.g., Raman scattering, the photon must transfer a certain amount of angular momentum to the plasma mode. By performing angle-resolved circular-polarized Raman scattering we can measure the amount of transferred angular momentum, which in turn measures the orbital spin of the plasma mode. Such measurement allows us to obtain information about the spin vector of the FQH state [61].

4.3 Wave functions and $(K, t, s)$ of FQH liquids

In this section we will study the relation between $(K, t, s)$ (the $K$-matrix, charge vector $t$, and the spin vector $s$) and the wave functions of FQH liquids. The results obtained here allow us to easily calculate $(K, t, s)$ for various FQH liquids obtained in various construction schemes.

Let $\Psi_{K_1t_1s_1}(\{z_i\})$ and $\Psi_{K_2t_2s_2}(\{w_i\})$ be two wave functions of two FQH liquids labeled by $(K_1, t_1, s_1)$ and $(K_2, t_2, s_2)$. Putting them together we obtain a new FQH liquid labeled by $(K_3, t_3, s_3)$

$$\Psi_{K_1t_1s_1}(\{z_i\})\Psi_{K_2t_2s_2}(\{w_i\}) = \Psi_{K_3t_3s_3}(\{z_i\}, \{w_i\}) \quad (4.39)$$
Obviously

\[ K_3 = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} = K_1 \oplus K_2 \]

\[ t_3 = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = t_1 \oplus t_2 \quad \text{(4.40)} \]

\[ s_3 = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = s_1 \oplus s_2 \]

We also know that \( \Psi_{Kts}^\dagger (\{ z_i \}) \) is a wave function for the holes (with charge \( e \)) which will be called the charge conjugate state of the \( \Psi_{Kts} \) state. The Hall conductance of the charge conjugate state has an opposite sign to that of the original state. Notice that the following transformation in the effective theory (4.19):

\[ t \to -t, \quad s \to -s, \quad K \to -K, \quad a_{I\mu} \to a_{I\mu} \quad \text{(4.41)} \]

leaves the equation motion unchanged. However, the transformation changes the sign of the Hall conductance and the sign of electric current \( J_{\mu}^e = -\frac{e}{2\pi} t_I \partial_{\mu} a_{I\lambda} \epsilon^{\mu\nu\lambda} \). Thus,

\[ \Psi_{Kts}^\dagger = \Psi_{-K,-t,-s} \quad \text{(4.42)} \]

i.e., the charge conjugation changes the signs of \( t, s, \) and \( K \).

Given a spin-polarized FQH state in a single layer at a filling fraction \( \nu_{Kt} \) and labeled by \( (K, t, s) \), there is a particle-hole conjugate state at the filling fraction \( \nu = 1 - \nu_{Kt} \). This conjugate state is obtained simply by putting the \( \nu = 1 \) IQH state and the charge conjugate of the \( (K, t, s) \) state together. Thus the label of the \( 1 - \nu_{Kt} \) state is given by

\[ K_{1-\nu_{Kt}} = 1_{1 \times 1} \oplus (-K), \quad t_{1-\nu_{Kt}} = 1 \oplus (-t), \quad s_{1-\nu_{Kt}} = \frac{1}{2} \oplus (-s) \quad \text{(4.43)} \]

Similarly, given a spin-polarized FQH state in double layers at a filling fraction \( \nu_{Kt} \) and labeled by \( (K, t, s) \), there is a particle-hole conjugate state at the filling fraction \( \nu = 2 - \nu_{Kt} \). Such a conjugate state is labeled by

\[ K_{2-\nu_{Kt}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus (-K), \quad t_{2-\nu_{Kt}} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \oplus (-t), \quad s_{2-\nu_{Kt}} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \oplus (-s) \quad \text{(4.44)} \]

From the two FQH states \( \Psi_{K_1 t_1 s_1} (\{ z_i \}) \) and \( \Psi_{K_2 t_2 s_2} (\{ w_i \}) \), we can construct a new FQH state by multiplying the two wave functions together and setting \( z_i = w_i \):

\[ \Psi_{K_1 t_1 s_1} (\{ z_i \}) \Psi_{K_2 t_2 s_2} (\{ w_i \}) |_{z_i = w_i} = \Psi_{K_3 t_3 s_3} (\{ z_i \}) \quad \text{(4.45)} \]

To calculate \( (K_3, t_3, s_3) \) it is convenient to use the \( SL(n, Z) \) transformation

\[ K \to WKW^T, \quad t \to Wt, \quad s \toWs \quad \text{(4.46)} \]

to transform \( t_1 \) and \( t_2 \) into \( t_{1,2}^T = (1, 0, 0, \ldots) \). This is possible if the elements in \( t_{1,2} \) do not contain common integer factors (i.e., the largest common divisor is 1). In the new basis \( K_{1,2} \) have the form

\[ K_{1,2} = \begin{pmatrix} m_{1,2} & k_{1,2}^T \\ k_{1,2} & K_{1,2} \end{pmatrix} \quad \text{(4.47)} \]
where $m_{1,2}$ are odd integers, $k_{1,2}$ are integer vectors in dim$K_{1,2} - 1$ dimensions, and $\tilde{K}_{1,2}$ are (dim$K_{1,2} - 1$) $\times$ (dim$K_{1,2} - 1$) integer matrices with even diagonal elements (see chapter 5). Accordingly the spin vectors have the form,

$$ s_{1,2} = \left( \begin{array}{c} s_{1,2}^1 \\ \tilde{s}_{1,2} \end{array} \right) $$

where $s_{1,2}^1$ are half odd integers and $\tilde{s}_{1,2}$ are integer vectors in dim$K_{1,2} - 1$ dimensions.

Using the $(K, t, s)$ in the new basis, one can show that $(K_3, t_3, s_3)$ are given by

$$ K_3 = \begin{pmatrix} m_1 + m_2 & k_1^T \\ k_1 & K_1 \\ k_2 & 0 \end{pmatrix} $$

$$ t_3^T = (1, 0, 0, ... ) $$

$$ s_3 = \left( \begin{array}{c} s_1^1 + s_2^1 \\ s_1 \\ \tilde{s}_2 \end{array} \right) $$

The results in (4.49) are obtained from the following consideration. Before imposing the constraint $z_i = w_i$, the effective theory only describes the two independent FQH states and has the form

$$ L = -\frac{1}{4\pi} K_{11} a_{1\mu}^1 \partial_\nu a_{1\nu}^1 \varepsilon^{\mu\nu\lambda} + s_1 \omega_1 \partial_\nu a_{1\nu}^1 \varepsilon^{\nu\lambda} $$

$$ -\frac{1}{4\pi} K_{21} a_{1\mu}^2 \partial_\nu a_{1\nu}^2 \varepsilon^{\mu\nu\lambda} + s_2 \omega_1 \partial_\nu a_{1\nu}^2 \varepsilon^{\nu\lambda} $$

After imposing the constraint $z_i = w_i$, the charge currents in the two FQH states $j_{1,2}^\mu = \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu a_{1\nu}^{1,2}$ are always equal to each other $j_1^\mu = j_2^\mu$ and the relative fluctuations $j_1^\mu - j_2^\mu$ are not allowed. Thus the constraint $z_i = w_i$ can be realized in the effective theory by setting $a_{1\mu}^1 = a_{1\mu}^2$. This changes the effective theory (4.50) into one with $(K, s)$ given by (4.49). The charge vector can be obtained by realizing that $A_\mu$ only couples with the charge current $j_1^\mu = j_2^\mu$.

We would like to remark that (4.49) is not always valid. Let $\chi_n$ be the wave function of the $\nu = n$ integer QH state with first $n$ Landau levels filled. Then the FQH state described by $\chi_1(\{z_i\}) (\chi_2(\{z_i\}))^2$ is in fact a $\nu = 1/2$ non-abelian state rather than an abelian state as implied by (4.49). More detailed discussions can be found in Ref. [64].

However (4.49) does apply to the following wave function, $(\chi_1)^p \Psi_{Kts}$. Notice that $(\chi_1)^p$ is just the wave function of the $1/p$ Laughlin state. From (4.49) one can show that multiplying $(\chi_1)^p$ to a FQH wave function $\Psi_{Kts}$ changes $(K, t, s)$ to

$$ K \rightarrow K + ptt^T, \ t \rightarrow t, \ s \rightarrow s + \frac{p}{2}t $$

A sequence of FQH states, $\nu = 1/3, 2/5, 3/7, ..., \frac{n}{\lfloor n/2 \rfloor}, ..., \underbrace{\frac{n}{\lfloor n/2 \rfloor}, ..., \frac{n}{\lfloor n/2 \rfloor}}_{\lfloor n/2 \rfloor}$, under Jain’s construction, are described by the wave functions $\chi_1^2 \chi_n$. The $(K, t, s)$ for the IQH state $\chi_n$ are given by

$$ K = 1_{n \times n}, \ t^T = (1, ..., 1), \ s^T = (1/2, 3/2, 5/2, ..., (2n - 1)/2) $$
Thus the \((K, t, s)\) for the \(\nu = \frac{n}{2n+1}\) state are

\[
K = 1_{n \times n} + 2C, \quad t^T = (1, \ldots, 1), \quad s^T = (3/2, 5/2, 7/2, \ldots, (2n + 1)/2)
\] (4.53)

where \(C\) is a matrix with all its elements equal to 1. The hierarchical construction can also generate a sequence of FQH states with filling fraction \(\nu = \frac{n}{2n+1}\). The \(K\)-matrix obtained from the hierarchical construction is a \(n \times n\) tri-diagonal matrix with off diagonal elements equal to \(-1\) and diagonal elements equal to 2, except \(K_{11} = 3\). The charge and the spin vectors are given by \(t^T = (1, 0, \ldots)\) and \(s^T = (3/2, 1, 1, \ldots)\). Although the two sets of \((K, t, s)\) obtained from Jain’s construction and the hierarchical construction are very different, one can show that they are equivalent under a \(SL(n,\mathbb{Z})\) transformation. In fact the matrix \(W_{IJ} = \delta_{IJ} - \delta_{I-1,J}\) transforms the \((K, t, s)\) in (4.53) into the ones obtained from the hierarchical construction through (4.46). Therefore the \(\nu = \frac{n}{2n+1}\) states obtained from the Jain’s construction and the hierarchical construction belong to the same universality class.

Another interesting example is that the following three sets of \((K, t, s)\)s describe the same \(\nu = 2/3\) FQH state:

\[
K = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad t^T = (1, 1), \quad s^T = (1/2, -1/2)
\]

\[
K = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}, \quad t^T = (1, 0), \quad s^T = (1/2, -1)
\] (4.54)

\[
K = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}, \quad t^T = (1, -1), \quad s^T = (1/2, -3/2)
\]

In fact \(W = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\) and \(W = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\) map the first set of \((K, t, s)\) into the second and the third sets, respectively. The first set of \((K, t, s)\) is obtained from the Jain’s construction with a wave function \(\chi^*_2(\chi_1)^2\), the second set from the hierarchical construction, and the third set is obtained by simply putting a \(\nu = 1\) QH state of electrons and a \(\nu = -1/3\) QH state of holes together.

The above three sets of \((K, t, s)\)s describe the same spin-polarized single layer 2/3 state. However a double layer or a spin singlet 2/3 state is described by

\[
K = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad t^T = (1, 1), \quad s^T = (1/2, 1/2)
\] (4.55)

which is inequivalent to the \((K, t, s)\) of the spin-polarized single layer 2/3 state. The two kinds of the 2/3 states are separated by a first order transition (or other FQH states) in clean systems.

5. Classification of Abelian Hall States

In chapter 2 and 4, we have derived the effective theories and the form of \((K, t, s)\) for some FQH liquids from some specific construction schemes. In this chapter we will derive the form of the \((K, t, s)\) from some general principles that an electron system must satisfy. This allows us to obtain a classification of most general abelian FQH states.
For a class of generalized hierarchical states, the effective theory is given by (4.19) with \((K, t, s)\) satisfying

\[
\begin{align*}
K^h_{11} &= \text{odd integer}, \\
K^h_{II} \bigg|_{I > 1} &= \text{even integer}, \\
K^h_{IJ} &= \text{integer, for } I \neq J, \\
t^h_I &= \delta_{1I}, \\
s^h_I &= \pm \frac{K^h_{II}}{2}.
\end{align*}
\]  

(5.1)

We will call this basis the hierarchical basis, as indicated by the superscript \(h\). We also constructed the effective theory for the simple multi-layer FQH states with \((K, t, s)\) having the form

\[
\begin{align*}
K^s_{II} &= \text{odd integer}, \\
K^s_{IJ} &= \text{integer, for } I \neq J, \\
t^s_I &= 1, \\
s^s_I &= \pm \frac{K^s_{II}}{2}.
\end{align*}
\]  

(5.2)

Such a basis will be called the symmetric basis, which is indicated by the superscript \(s\).

First we would like to mention that using the transformation (4.46) and the matrix \(W_{IJ} = \delta_{IJ} - \delta_{I-1,J}\) in the group \(SL(\dim(K), Z)\), we can transform \((K^s, t^s)\) in the symmetric basis satisfying (5.2) into a one in the hierarchical basis satisfying (5.1). \(W^{-1}\) is also an element of \(SL(\dim(K), Z)\) and will transform in the opposite direction. Thus the matrices satisfying (5.1) and (5.2) have a one-to-one correspondence. However, in general the spin vectors in the symmetric basis do not transform into the corresponding spin vector in the hierarchical basis. To obtain the spin vectors in (5.1) and (5.2) we have assumed that the quasiparticles and the electrons always condense into Laughlin states in the “first” Landau level. If we allow them to form Laughlin states in higher Landau levels more general spin vectors will be generated:

\[
\begin{align*}
s^h_I &= \pm \frac{K^h_{II}}{2} + \text{integer} \\
s^s_I &= \pm \frac{K^s_{II}}{2} + \text{integer}
\end{align*}
\]  

(5.3)

In this case \((K, t, s)\) in the two basis will have a one-to-one correspondence under the transformation \(W\).

Even though the \((K, t, s)\) given above are quite general, it is not clear whether these \((K, t, s)\) cover all possible abelian FQH states or not. In the following we will derive the effective theory of the most general abelian FQH states from some general principles. This discussion will help us understand why the \((K, t, s)\) have the form described in (5.1) and (5.2) and what the physical reasons are behind it. We will show that the generalized hierarchical states and the multi-layer states represent the most general abelian FQH states (if we generalize the spin vectors to the ones in (5.3)).

Our first working assumption is that the effective theory of abelian FQH states is described by a Lagrangian of the form in (4.19). Certainly this assumption alone is not enough. For arbitrary choices of \((K, t, s)\) the Lagrangian in (4.19) may not describe an electron system. So the problem we are facing is not how to derive the effective theory
of an electron system, but the reverse, how to determine whether an effective theory is consistent with the underlying electron system or not. This leads us to our second working assumption: In order for the effective theory to describe an electron system, the effective theory must contain \( \kappa \) independent electron operators, where \( \kappa \) denotes the rank of \( K \). (Here we ignore the possibility of electron pairing.) We will discuss why we need \( \kappa \) electron operators later.

To implement our second assumption, it is convenient to work in the electron basis:

\[
\mathcal{L} = -\frac{1}{4\pi} \tilde{a}_{I\mu} \tilde{K}_{IJ} \varepsilon^{\mu\nu\lambda} \partial_\nu \tilde{a}_{J \lambda} - \frac{e}{2\pi} \tilde{t}_I A_\mu \varepsilon^{\mu\nu\lambda} \partial_\nu \tilde{a}_{I \lambda} + s_I \omega_i \varepsilon^{i\nu\lambda} \partial_\nu \tilde{a}_{I \lambda} + \tilde{a}_{I\mu} \tilde{\gamma}_I^\mu. \tag{5.4}
\]

We have put the twiddle sign (~) to remind ourselves that we are in the electron basis. The electron basis is defined by the requirement that a source term that carries a unit \( \tilde{a}_{I\mu} \) charge generates an electron. Thus \( \tilde{j}_I \) in (5.4) represents the current of the \( I \)th electron excitations above the ground state.

In order for the electrons to have Fermi statistics, we must have

\[
(K^{-1})_{IJ} = \begin{cases} 
\text{odd integer} & \text{for } I = J \\
\text{integer} & \text{for } I \neq J 
\end{cases} 
\tag{5.5}
\]

Furthermore, in order for the electrons to carry unit charge, \( \tilde{t} \) must satisfy

\[
\sum_J (K^{-1})_{IJ} \tilde{t}_J = 1 \quad \text{for all } I. \tag{5.6}
\]

This equation implies that

\[
\tilde{t}_J = \sum_L \tilde{K}_{JL}. \tag{5.7}
\]

To find the condition imposed on the spin vector \( \tilde{s}_I \), let us put the FQH state on a sphere. We know that the ground state on the sphere form the singlet representation of the \( su(2) \) rotation of the sphere. Any neutral excitations above the ground state must carry integer \( su(2) \) spins. As we add an electron to the system, the electron excitation must form an \( S^{su(2)} = N_\phi^2 + \text{integer} \) representation of the \( su(2) \). From the relation \( S^{su(2)} = \frac{N_\phi^2}{2} + S_{ob} \) we find that the orbital spin \( S_{ob} \) of an electron excitation must be an integer. Therefore (see (4.38))

\[
S_{ob}^I = \sum_J (K^{-1})_{IJ} \tilde{s}_J + \frac{1}{2}(K^{-1})_{II} = \text{integer} \tag{5.8}
\]

Given the effective theory in the electron basis, we now need to determine the allowed quasiparticle excitations. A generic quasiparticle is created by a source term

\[
\mathcal{L} = c_I a_{I\mu} j^\mu \tag{5.9}
\]

whose \( a_I \)-charge is equal to \( c_I \). Since the electron wave function must be single valued (even in the presence of the quasiparticle), the phase induced by moving an electron around the quasiparticle must be a multiple of \( 2\pi \). This requires \( c_I \) to satisfy

\[
\sum_{J=1}^\kappa (K^{-1})_{IJ} c_J = \text{integer for all } I. \tag{5.10}
\]
(Note moving a quasiparticle described by \( c_I \) around the one described by \( c'_I \) induces a phase \( 2\pi c_I (\tilde{K}^{-1})_{IJ} c'_J \).) The set of all \( c_J \)s satisfying (5.10) form a \( \kappa \)-dimensional lattice, with the basis vectors of the lattice \( c^{(L)}_J, L = 1, 2, \ldots, \kappa \). The components of \( c^{(L)}_J \) are given by

\[
c^{(L)}_J = \tilde{K}_{IJ} c_J.
\]  

(5.11)

The \( a_{I\mu} \)-charge \( c \) of any allowed excitations, including the electron excitations, will be linear combinations of \( c^{(L)}_J \) with integer coefficients:

\[
c = \sum_L l_L c^{(L)}
\]  

(5.12)

Such a quasiparticle (labeled by \( \psi \)) is created by the following source term,

\[
L = j^\mu a_{I\mu} (\tilde{K}^{-1})_{IJ} l_J
\]  

(5.13)

Now let us redefine the gauge fields and change to the so-called quasiparticle basis, in which the fundamental quasiparticles described by \( c^{(L)}_J \) will carry the unit charge of the new gauge fields. From (5.11) we see that the substitution \( a = \sum J \tilde{K}_{IJ} a_J \) in (5.4) and (5.13) will lead to

\[
L = -\frac{1}{4\pi} a_{I\mu} K_{IJ} \varepsilon^{\mu\nu\lambda} \partial_\nu a_{J\lambda} - \frac{e}{2\pi} l_I A_{\mu} \varepsilon^{\mu\nu\lambda} \partial_\nu a_{I\lambda} + s_i \omega_i \varepsilon^{i
u\lambda} \partial_\nu a_{I\lambda} + l_I a_{I\mu} j^\mu
\]  

(5.14)

The \((K, t, s)\) in (5.14) are related to \((\tilde{K}, \tilde{t}, \tilde{s})\) in (5.4) through

\[
K = \tilde{K}^{-1}, \quad t = K \tilde{t}, \quad s = K \tilde{s}.
\]  

(5.15)

(5.14) is exactly the same as the effective theory (4.19) that we wrote down in chapter 4, with \((K, t, s)\) satisfying (5.2) and (5.3) as one can see from from (5.6) and (5.8).

We have reached an important result: the most general abelian FQH states of unpaired electrons are described by the effective theory (5.14) with \((K, t, s)\) satisfying (5.2). In other words, the topological orders in the abelian FQH states are labeled by integer valued symmetric matrices with odd diagonal elements and a spin vector \( s \), up to an equivalency condition \( K \sim WKW^T \) and \( s \sim Ws \) with \( W \) an element of \( SL(\kappa, \mathbb{Z}) \) which leaves the vectors \( t = (1, 1, \ldots, 1) \) invariant. The determinant of \( W \) must be equal to 1 in order to preserve the charge quantization condition.

The above discussions also reveal that the condition \( K_{II}^\kappa = \text{odd} \) is a consequence of the Fermi statistics of the underlying electrons. FQH liquids formed by bosons will have \( K_{II}^\kappa = \text{even} \) in the symmetric basis.

We would next like to say a few words about why we need \( \kappa = \dim(K) \) independent electron operators. If we had less than \( \kappa \) electron operators, the condition (5.10) would become

\[
\sum_{J=1}^{\kappa'} (\tilde{K}^{-1})_{IJ} c_J = \text{integer only for } I = 1, \ldots, \kappa', \text{ where } \kappa' < \kappa.
\]

This is not enough to fix \( c_J \) on a lattice. In this case, excitations with arbitrarily small charge would be allowed. Such excitations may be continuously connected to the ground state and would be, we believe, gapless. This argument suggests that in order for the effective theory to have finite energy gap, we require the presence of \( \kappa \) different electron operators. (Of course, we also require \( \det K \neq 0 \) in order for the gauge fluctuations to have finite gaps.)
We would like to remark that here we have only shown that a generic abelian state must be described by \((K, t, s)\) that satisfies (5.1) or (5.2). However \((K, t, s)\) here merely appear as some parameters in the effective theory. Only certain combinations are physically measurable. Thus different \((K, t, s)\) may describe the same topological orders. At least the \((K, t, s)\)s related through the \(SL(n, Z)\) transformation (4.46) describe the same topological order. However, it may be possible that \((K, t, s)\)s with even different dimensions might describe the same topological order. More details can be found in Ref. [60]. An attempt to describe topological orders using only measurable quantities can be found in Ref. [6]. Clearly more work needs to be done to obtain a more complete and more satisfactory picture of abelian topological orders.

6. Algebraic approach and non-abelian FQH liquids

So far in this article we have discussed only a small class of FQH liquids – abelian FQH liquids. It was first pointed out by Moore and Read[23] that another type of FQH liquids with quasiparticles of non-abelian statistics is also possible. Non-abelian FQH liquids cannot be described by abelian Chern-Simons effective theory; thus the approaches previously used to study the edge states and quasiparticle quantum numbers can no longer be used for non-abelian states. We need to develop new methods to calculate the physical properties of non-abelian states.

At the moment, we know two ways to construct non-abelian FQH states. The first method uses the parton construction and the second method uses the conformal field theory (CFT). The parton construction allows us to construct FQH states whose effective theory is a non-abelian Chern-Simons theory [24,28,33]. For example \(\chi_1(\chi_2)^2\) is a \(\nu = 1/2\) non-abelian state described by a \(U(1) \times SU(2)\) Chern-Simons theory. The CFT approach allows us to construct more general non-abelian states [23,28,65,25]. The effective theories of many non-abelian states obtained from CFT are not known at this time.

In this chapter we will concentrate on the edge properties of non-abelian FQH states. We will develop an algebraic method which allows us to calculate the properties of the edge states for the FQH states constructed using CFT. Our approach is motivated by an observation by Moore and Read[23] that the Laughlin wave function can be written as a correlation function between the charge operators (i.e., the vertex operators) in the \(U(1)\) K-M algebra. The \(U(1)\) K-M algebra is a CFT which is also called the Gaussian model. (For readers not familiar with CFT, we suggest Ref. [66].) Moore and Read’s observation links the Laughlin wave function to a CFT. We will call this CFT the bulk CFT. From our discussion in chapter 3 we see that the edge excitations of the Laughlin state are also described by a CFT – the \(U(1)\) K-M algebra. Such a CFT will be called the edge CFT of the FQH liquid. It is very interesting to see that for a Laughlin state, the bulk CFT is identical to the edge CFT. It turns out that such a relation can be extended to more general abelian and non-abelian states. In this chapter we will explain how the bulk CFT and the edge CFT are connected and how to use such a relation to calculate physical properties of non-abelian edge states.

However, we would like to stress that to relate the bulk CFT that generates the bulk wave function to the edge CFT that describes the edge spectrum, we need to introduce a concept of minimal CFT for the bulk CFT. This is because once a FQH wave function can be written as a correlation function in a certain CFT, then the wave function can also be expressed as a correlation in many other CFTs which contain the original CFT. The minimal (bulk) CFT for a FQH state not only reproduces the wave function, it is also contained by any other CFT that reproduces the wave function. As we will see later, it is this minimal bulk CFT that is identical to the edge CFT that describes the
edge excitations of the corresponding FQH state. Thus it is very important to identify the minimal bulk CFT.

6.1 Ideal Hamiltonians, zero energy states, and edge excitations

In this section we will describe a microscopic theory for the edge excitations in the Laughlin states. Let us consider an electron gas in first Landau level. We choose the interaction between electrons to be

\[ V(\vec{r}) \propto -\partial^2 \delta(\vec{r}). \]

Because A) \( H_V = \sum_{ij} V(\vec{r}_i - \vec{r}_j) \) is positive definite (i.e., \( \langle \psi | H_V | \psi \rangle \geq 0 \) for any \( |\psi\rangle \)) and B) \( H_V \) have zero expectation value for the \( \nu = 1/3 \) Laughlin wave function

\[ \Phi_3(\{z_i\}) = \prod_{i<j} (z_i - z_j)^3 \prod_k e^{-\frac{1}{4}|z_k|^2} \quad (6.1) \]

Thus \( \Phi_3 \) is an exact ground state of our Hamiltonian with zero energy. Such Hamiltonian \( H_V \) will be call the ideal Hamiltonian of the \( 1/3 \) Laughlin state. However the Laughlin state (6.1) is not the only state with zero energy. One can easily check that the following type of states all have zero energy:

\[ \Phi(\{z_i\}) = P(\{z_i\})\Phi_3(\{z_i\}) \quad (6.2) \]

where \( P(z_i) \) is a symmetric polynomial of \( z_i \) so that \( \Phi(\{z_i\}) \) remains an anti-symmetric function. In fact, the reverse is also true: all the zero energy states are of the form in (6.2). This is because, in order for a fermion state to have zero energy, \( \Phi \) must vanish at least as fast as \( (z_i - z_j)^3 \) when any two electrons \( i \) and \( j \) are brought together (the possibility of \( (z_i - z_j)^2 \) is excluded by the fermion statistics). The Laughlin wave function is zero only when \( z_i = z_j \); therefore \( P = \Phi/\Phi_3 \) is a finite function. Since \( \Phi \) and \( \Phi_3 \) are both anti-symmetric functions in the first Landau level, \( P \) is a finite symmetric holomorphic function that can only be a symmetric polynomial.

Among all the states in (6.2), the Laughlin state describes a circular droplet with the smallest radius. All other states, having higher angular momenta, are deformation and/or inflation of the droplet of the Laughlin state. Thus the states generated by \( P \), being low energy excitations above the ground state \( \Phi_3 \), correspond to the edge excitations of the Laughlin state.

Now let us study the structures of the zero energy space (i.e., the space of symmetric polynomials). We know that the space of symmetric polynomials is generated by the following polynomials \( s_n = \sum_i z_i^n \) (through multiplication and addition). Let \( M_0 = \frac{3N(N-1)}{2} \) be the total angular momentum of the Laughlin state (6.1). Then the state \( \Phi \) will have an angular momentum \( M = \Delta M + M_0 \) where \( \Delta M \) is the order of the symmetric polynomial \( P \). Since we have only one order-zero and order-one symmetric polynomial \( s_0 = 1 \) and \( s_1 = \sum_i z_i \), the zero energy states for \( \Delta M = 0, 1 \) are non-degenerate. However, when \( \Delta M = 2 \) we have two zero energy states generated by \( P = s_2 \) and \( P = s_1^2 \). For general \( \Delta M \) the degeneracy of the zero energy states is given by

\[ \Delta M : \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \]
\[ \text{degeneracy} : \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 7 \quad 11 \quad (6.3) \]

Here we would like to point out that the degeneracy in (6.3) is exactly what we expected from the macroscopic theory. We know for a circular droplet, the angular momentum \( \Delta M \) can be regarded as the momentum along the edge \( k = 2\pi \Delta M/L \) where \( L \) is the perimeter of the QH droplet. According to the macroscopic theory the (neutral)
edge excitations are generated by the density operators $\rho_k$. One can easily check that the edge states generated by the density operators have same degeneracies as those in (6.3) for every $\Delta M$, e.g., the two states at $\Delta M = 2$ are generated by $\rho_{k_0}^2$ and $\rho_{2k_0}$ where $k_0 = 2\pi/L$. Therefore the space generated by the K-M algebra (3.7) and the space of the symmetric polynomials are identical.

There seems to be one problem in the above identification. The states generated by symmetric polynomials $P$ all have zero energy, while the edge excitations in the macroscopic theory have finite energies: $E \propto k \propto \Delta M$. This is because we did not include the confining potential in our ideal Hamiltonian. A confining potential for the FQH droplet will lift the degeneracy and give edge excitation an energy $E \propto \Delta M$.

Now let us ask a physical question. Do the symmetric polynomials generate all the low energy states? If this is true, the space of the low energy edge excitations will be identical to the space of symmetric polynomial (or the zero energy space) [48]. Unfortunately, up to now we do not have analytic proof of the above statement. This is because, although states orthogonal to the states generated by the symmetric polynomials have non-zero energies, it is not clear that those energies remain finite in the thermodynamical limit. It might be possible that the energy gap approaches zero in the thermodynamical limit. To resolve this problem, for now we have to rely on numerical calculations. In Fig. 6.1 we present the energy spectrum of a system of six electrons in the first 22 orbits for the Hamiltonian introduced at the beginning of this chapter. The degeneracies of the zero energy states at $M = 45, ..., 51$ (or $\Delta M = 0, ..., 6$) are found to be $1, 1, 2, 3, 5, 7, 11$, which agrees with (6.3). More importantly, we can clearly see a finite energy gap that separates all other states from the zero energy states. Thus, the numerical results imply that all the low lying edge excitations of the Laughlin state are generated by the symmetric polynomials or the K-M algebra (3.7). Using the above assumption of the energy gap, we see that the edge excitations in an FQH liquid can be studied through the zero energy states of the ideal Hamiltonian for that FQH liquid.

![Fig. 6.1: The energy spectrum of a system of six electrons in the first 22 orbits for the Hamiltonian $H_V$. The degeneracies of the zero energy states at $M = 45, ..., 51$ are found to be $1, 1, 2, 3, 5, 7, 11$.](image)

**6.2 Edge excitations and current algebra**

In this section we are going to study edge excitations of the Laughlin state using a current algebra approach. The algebraic approach developed here can be easily generalized.
to study edge excitations of non-abelian states.

We note that the $\nu = 1/m$ Laughlin state,

$$\prod_{i<j}(z_i - z_j)^m e^{-\frac{1}{4}\sum_i |z_i|^2} \equiv \Phi_m(\{z_i\}) e^{-\frac{1}{4}\sum_i |z_i|^2}$$  \hspace{1cm} (6.4)$$

is a zero-energy state of a Hamiltonian with the following two-body interaction$[67]$, $V(z_1 - z_2) = -\sum_{n=1}^{(m-1)/2} C_n \partial_{z_1}^{2n-1} \delta(z_1 - z_2) \partial_{z_2}^{2n-1}$  \hspace{1cm} (6.5)$$

where $C_n > 0$. (The Haldane’s pseudo-potentials$[12]$ for the above interaction are given by $V_{2l+1} > 0$ for $2l + 1 < m$ and $V_{2l+1} = 0$ for $2l + 1 \geq m$.) Thus (6.5) defines an ideal Hamiltonian for the $1/m$ Laughlin state. Other zero-energy states of the ideal Hamiltonian form the edge excitation of the $1/m$ state. From (6.5) we see that any anti-symmetric wave function of form

$$\tilde{\Phi}_m(\{z_i\}) e^{-\frac{1}{4}\sum_i |z_i|^2}$$  \hspace{1cm} (6.6)$$

has zero energy, if and only if $\tilde{\Phi}_m$ does not contain zeros of order less than $m$, i.e.,

$$\tilde{\Phi}_m(z_i) = O((z_1 - z_2)^m)$$  \hspace{1cm} (6.7)$$

as $z_1 \to z_2$.

Now the question is how to construct the Hilbert space of these zero-energy states. One approach is to use the symmetric polynomial discussed in the last section. Here we will use a more complicated approach – current algebra in CFT.

Let us concentrate on the holomorphic part of the wave function ($\Phi_m$ or $\tilde{\Phi}_m$ in (6.4) and (6.6)). To construct the edge states we need to construct holomorphic functions that satisfy (6.7). We notice that the Laughlin-Jastrow function $\Phi_m$ can be written as a correlation of the vertex operators (or primary fields), $\psi_e$, in the Gaussian model as follows:$[23,68]$

$$\Phi_m = \lim_{z_\infty \to \infty} (z_\infty)^{2h_N} \langle \Psi_N(z_\infty) \prod_{i=1}^N \psi_e(z_i) \rangle ,$$  \hspace{1cm} (6.8)$$

where

$$\psi_e(z) = e^{i\sqrt{m}\phi(z)}, \quad \Psi_N(z) = e^{-iN\sqrt{m}\phi(z)}.$$  \hspace{1cm} (6.9)$$

Here $N$ is the number of electrons, and $h_N = mN^2/2$ is the conformal dimension of $\Psi_N$. The scalar field, $\phi$, in the Gaussian model is normalized so that $\langle \phi(z)\phi(0) \rangle = \ln z$. (Note the $\phi$ field here is just the $\phi$ field we introduced in the $U(1)$ K-M algebra (3.9) in section 3.1) The factor $(z_\infty)^{2h_N}$ is included in (6.8) so that the limit $z_\infty \to \infty$ gives rise to a finite function $[69]$. Let $j(z) \equiv \partial \phi$ be the $U(1)$ current in the Gaussian model. Since $j$ is a local operator, we find that the correlation

$$\tilde{\Phi}_m \propto \langle \oint dz \alpha(z) j(z) \Psi_N(z_\infty) \prod_{i=1}^N \psi_e(z_i) \rangle ,$$  \hspace{1cm} (6.10)$$
with an appropriately chosen holomorphic function $\alpha(z)$, satisfies (6.7) because $\psi_e(z)$ has the following operator product expansion (OPE) as $z \to 0$:

$$
\psi_e(z)\psi_e(0) \propto z^m e^{i 2\sqrt{m}\phi(z)} + O(z^{m+1}) ,
$$

$$
j(z)\psi_e(0) \propto \frac{1}{z}\psi_e(0) + O(1) .
$$

(6.11)

The OPE between $\psi_e$ guarantees the correlation to have $n^{th}$ orders zero as $z_i \to z_j$. The integration contour of $z$ in (6.10) encloses all the $\psi_e(z_i)$ operators but not $\Psi_N$. If $\alpha(z)$ has no poles except at $z_\infty$, then (6.10) is finite for finite $z_i$, since $z_\infty \to \infty$. Therefore (6.10) is a wave function with zero energy, hence a wave function for the edge excitations. Introducing

$$
j(z) = \sum_{n=-\infty}^{\infty} \frac{j_n}{(z - z_\infty)^{n+1}}
$$

(6.12)

and choosing

$$
\alpha(z) = (z - z_\infty)^{-n} \quad (n \geq 0) ,
$$

(6.13)

we find that, by shrinking the contour around $z_\infty$ and letting $z_\infty \to \infty$, (6.10) becomes

$$
\Phi_m^{(n)} = \lim_{z_\infty \to \infty} (z_\infty)^{2h_N+2n}\langle \Psi_N^{(n)}(z_\infty) \prod_{i=1}^{N} \psi_e(z_i) \rangle ,
$$

(6.14)

where

$$
\Psi_N^{(n)}(z_\infty) = j_n \Psi_N(z_\infty) .
$$

(6.15)

Again the factor $z_\infty^{2h_N+2n}$ is included in (6.15) to ensure the existence of a finite limit [69].

The above discussion can be generalized to the case with several insertions of the current operator. The general statement is the following. The ground state wave function can be written as a correlation of $\psi_e(z_i)$ with a primary field $\Psi_N$ inserted at infinity. To generate edge excitations above the ground state, we simply replace the primary field $\Psi_N$ by its current descendants[70], which are generally of the form (with $n_1, n_2, \cdots, \geq 0$)

$$
\Psi_N^{(n_1,n_2,\cdots)} = (j_{-n_1}j_{-n_2}\cdots)\Psi_N .
$$

(6.16)

Let us call $l = \sum_i n_i$ the level of the descendant field $\Psi_N^{(n_1,n_2,\cdots)}$. The wave function associated with it is given by

$$
\Phi_m^{(n_1,n_2,\cdots)} = \lim_{z_\infty \to \infty} (z_\infty)^{2h_N+2l}\langle \Psi_N^{(n_1,n_2,\cdots)}(z_\infty) \prod_{i=1}^{N} \psi_e(z_i) \rangle .
$$

(6.17)

Such wave function has zero energy and can be identified as an edge excitation.

We would like to show that all the edge states generated by the level-$l$ descendant fields have a total angular momentum $L = M_0 + l$, where $M_0 = mN(N - 1)/2$ is the total angular momentum of the ground state. We first note that the angular momentum of the ground state can be expressed in terms of conformal dimensions, $h_e$ and $h_N$, of $\psi_e$ and $\Psi_N$. Under a conformal transformation $z \to w = f(z)$, the correlation of primary fields satisfies

$$
\langle \Psi_N(w_\infty) \prod_{i=1}^{N} \psi_e(w_i) \rangle = (f'(z_\infty))^{h_N} \prod_{i} (f'(z_i))^{h_e} \langle \Psi_N(w_\infty) \prod_{i=1}^{N} \psi_e(w_i) \rangle .
$$

(6.18)

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Choosing \( f(z) = \lambda z \), we have

\[
\langle \Psi_N(\lambda z_\infty) \prod_{i=1}^N \psi_e(\lambda z_i) \rangle = \lambda^{h_N-Nh_e} \langle \Psi_N(z_\infty) \prod_{i=1}^N \psi_e(z_i) \rangle,
\]

which implies that

\[
\Phi_m(\lambda z_i) = \lambda^{h_N-Nh_e} \Phi_m(z_i) .
\]

For \( \lambda = e^{i\theta} \), this transformation is nothing but a rotation by angle \( \theta \) in the complex plane. Thus the angular momentum of the ground state is

\[
M_0 = h_N - Nh_e .
\]

With \( h_N = mN^2/2 \) and \( h_e = m/2 \), we see that \( M_0 = mN(N-1)/2 \), as expected for the Laughlin \( 1/m \)-state. More generally, since the dimension of a current descendant \( \Psi_{N(n_1,n_2,\ldots)}(z) \) is the sum \( h_N + l \)[70] where \( l \) is its level, we find that under \( z \rightarrow w = \lambda z \),

\[
\Psi_{N(n_1,n_2,\ldots)}(z) = \lambda^{h_N+l} \Psi_{N(n_1,n_2,\ldots)}(w) ;
\]

Thus, from (6.17) and (6.22) we can see that the edge-excited state described by \( \Phi_{m(n_1,n_2,\ldots)} \) carries an angular momentum of \( L = h_N + l - Nh_e = M_0 + l \). As a general rule, valid for any FQH states that can be generated by CFT, the angular momentum of an edge excitation is equal to that of the ground state plus the level of the descendant level of the associated insertion at infinity.

Note that the descendant fields \( \Phi_{m(n_1,n_2,\ldots)} \) with different \( (n_1,n_2,\ldots) \) may not be linearly independent. Using the standard approach in CFT, one can show that, from the OPE \( j(z)j(0) = 1/z^2 \), \( j_n \) satisfies the \( U(1) \) K-M algebra

\[
[j_n, j_m] = 2\pi n \delta_{m+n} .
\]

From the \( U(1) \) K-M algebra (6.23), it is easy to show that the number, \( D_l \), of linearly independent descendant fields, (6.16), at level \( l \) is given by the partition number of \( l \). The values of \( D_l \) for small \( l \) are given in (6.3). The values of \( D_l \) for generic \( l \) can be expressed in terms of the character of the \( U(1) \) K-M algebra for the primary field \( \Psi_N \)[71]:

\[
\text{ch}_N(\xi) \equiv \xi^{h_N} \sum_l D_l \xi^l = \xi^{h_N} \frac{1}{\prod_{n>0}(1-\xi^n)} .
\]

If there is a one-to-one correspondence between the edge states and the descendant fields of \( \Psi_N \), then we can use the above formula to obtain the number, \( D_L \), of edge excitations at any given angular momentum \( L \):

\[
\text{Ch}_N(\xi) \equiv \sum_L D_L \xi^L = \text{ch}_N(\xi)^{-Nh_e}
\]

\[
= \frac{1}{\prod_{n>0}(1-\xi^n)} \xi^{M_0} .
\]

The one-to-one correspondence between the edge states and the descendant fields of \( \Psi_N \) means that linearly independent descendant fields always generate linearly independent wave functions of \( z_i \) through (6.17). However, we do not expect this correspondence
to be true for finite $N$ and arbitrarily large level $l = \sum_k n_k$. On the other hand, it is conceivable that this one-to-one correspondence holds for any finite level $l$ when $N$ is very large or in the limit $N \to \infty$. Though we do not know, at present, how to prove this statement within the CFT approach for a generic abelian FQH state, its validity for the $1/m$-states can be seen in the following way: It is known[48,49] that the edge states can be generated by multiplying the ground state wave function by symmetric polynomials of electron coordinates $z_i$. Mathematically, the number of linearly independent symmetric polynomials of degree $l$ is precisely given by the partition number of the integer $l$, which is just the number of linearly independent descendant fields.

In the above we have generated the edge excitations of the Laughlin state by inserting the current operator. One may try to generate more edge states by inserting some other operators, such as the energy-momentum tensor $T$, because the insertions of the energy-momentum tensor also maintain the structure of zeros (6.7) of the wave functions. But one can show that the edge states generated by the energy-momentum tensor is contained in those generated by the current since $T \propto j^2$. For the Laughlin state, $j$ is the “fundamental operator” that generates all the edge excitations.

From the discussions in this section, we see a close relationship between the (minimum) bulk CFT that generates the bulk wave function and the edge CFT that generates the spectrum of the edge excitations, at least for the $1/m$ Laughlin states. In next section, we will show that this relationship can be generalized to non-abelian FQH systems.

6.3 Non-abelian FQH liquids and their edge excitations

In this section we are going to study the physical properties of some simple non-abelian FQH liquids,[23,64,65,25] using the algebraic approach. In general, the holomorphic part of the many-body wave function for a non-abelian FQH state consists of two factors. One of them is of usual Laughlin-Jastrow form, $\prod_{i<j} (z_i - z_j)^r$ ($r$ is a fraction or integer), which we call the $U(1)$ part. The connection of this part to the $U(1)$ K-M algebra (or the Gaussian model) is well understood. Here we concentrate on the other part, which is not of the Laughlin-Jastrow form. We call it the non-abelian part, because it is this part that is presumably responsible to the appearance of non-abelian statistics for quasiparticles.

Let us first consider a simple non-abelian FQH state, namely the p-wave paired FQH state for spinless electrons discussed in Ref. [23]. The total wave function is given by the product of a Pfaffian wave function $\Phi_{Pf}$ and a Laughlin wave function $\Phi_m$

$$\Phi_{p} = \Phi_{Pf} \Phi_m$$

$$\Phi_{Pf} = A \left( \frac{1}{(z_1 - z_2)(z_3 - z_4)} \ldots \right)$$

$$\Phi_m = \left( \prod_{i<j} (z_i - z_j)^m \right) e^{-\frac{1}{4} \sum_i |z_i|^2}$$

(6.26)

where $A$ is the anti-symmetrization operator, and $m$ is an even integer. The filling fraction of the p-wave paired state is $\nu = 1/m$.

Let us first analyze the structure of zeros of this wave function assuming, for simplicity, $m = 2$. Let $z_1 = z_3 + \delta_1$ and $z_2 = z_3 + \delta_2$, we find $\Phi_p$ has the following expansion
(we expand $\delta_2$ first)

$$
\Phi_p = \sum_{k=\text{odd}} (\delta_2^k) A_k(z_1, z_3, \ldots) \\
= \sum_{k=\text{odd}} (\delta_2^k) \sum_l (\delta_1^l) A_{kl}(z_3, z_4, \ldots)
$$

(6.27)

One can directly check that the coefficients

$$
A_{12} = 0
$$

(6.28)

Therefore $\Phi$ is the exact ground state of Hamiltonian $H$ with the following three-body potential:

$$
V = -V_1 \partial_{z_2} \delta(z_2 - z_3) \partial_{z_2} \partial_{z_1}^2 \delta(z_1 - z_3) \partial_{z_1}^2
$$

(6.29)

This is because the Hamiltonian $H$ is positive definite if $V_1 > 0$, and $\Phi$ is a zero-energy state of $H$. It was checked numerically that $\Phi$ is the non-degenerate ground state of $H$ (on a sphere) with a finite energy gap. This implies that the constraint (6.28) uniquely fixes the ground state wave function [64,72].

On a plane, the ground state of $H$ is identified as the minimum angular momentum state with zero energy. Numerical results indicate that the ground state on the plane is non-degenerate [64,72]. Other zero-energy states all have higher angular momenta and are identified as the edge excitations above the ground state.

It was shown in Ref. [23] that the so-called non-abelian part, $\Phi_{Pf}$, can be written as a correlation of the dimension-$1/2$ primary fields $\psi$ in the Ising model. $\psi$ is just the fermion field in a free Majorana fermion theory. Therefore the total ground state wave function (the holomorphic part) can be written as

$$
\Phi_p = \lim_{z_\infty \to \infty} (z_\infty)^{2h_N} \langle \Psi_N(z_\infty) \prod_{i=1}^N \psi_e(z_i) \rangle,
$$

(6.30)

where

$$
\psi_e(z) = \psi(z) e^{i \sqrt{m} \phi(z)},
$$

(6.31)

$$
\Psi_N(z_\infty) = e^{-iN \sqrt{m} \phi(z_\infty)}.
$$

(6.32)

Here $\phi(z)$ is the chiral scalar field in the Gaussian model whose correlation produces the $U(1)$ part $\Phi_m$. In (6.30) $h_N = mN^2/2$ is the dimension of $\Psi_N$ and $N$ the total number of the electrons.

Let $z_1 = z_3 + \delta_1$ and $z_2 = z_3 + \delta_2$ and assume $m = 2$, we find the OPE of the operators $\psi_e$ has the following form

$$
\psi_e(z_1) \psi_e(z_2) \psi_e(z_3) = \psi_e(z_1) \delta_2 e^{2i \sqrt{m} \phi(z_3)} + \ldots
$$

(6.33)

$$
= \delta_2 (\delta_1^4) \psi_e(z_3) e^{3i \sqrt{m} \phi(z_3)} + \ldots
$$

We find that the lower-order zeros (such as the term containing $\delta_1^4 \delta_2^2$) are absent as described by (6.28). Thus the correlations between $\psi_e$ automatically generate wave functions that satisfy (6.28) and become the zero energy states of the three-body Hamiltonian $H$.  

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From the above we see that the wave function of a p-wave paired FQH state can be written as a correlation in the $U(1) \times $ Ising model. Since the bulk CFT of the p-wave paired state is not the Gaussian model, this motivated Moore and Read to suggest that the p-wave state is a non-abelian state. Indeed it was shown that the quasiparticles in the p-wave state have very different structures from those in abelian states [23,72].

To have some basic understanding of the new structures of the quasiparticles in the non-abelian state, let us start with the charge $e/2$ quasihole created by inserting a unit flux quantum:

$$\prod (\xi - z_i)\Phi_p(\{z_i\})$$  \hspace{1cm} (6.34)

Due to the presence of the pairing wave function $\Phi_{pf} = \mathcal{A} \prod_{i=\text{odd}} \frac{1}{z_i-z_{i+1}}$ we can split the quasihole in (6.34) into two excitations each carrying a charge of $e/4$:

$$\Phi(\{z_i\}; \xi_1, \xi_2) = \mathcal{A} \prod_{i=\text{odd}} \frac{(z_i - \xi_1)(z_{i+1} - \xi_2)}{z_i - z_{i+1}} \Phi_m(\{z_i\})$$  \hspace{1cm} (6.35)

(Note $\Phi(\{z_i\}; \xi, \xi)$ is just the wave function in (6.34).) An interesting phenomenon happens when we create four $e/4$ quasiholes. One can write down three different wave functions for the same four quasiholes at the same positions:

$$\Phi_1(\{z_i\}; \xi_1, \xi_2, \xi_3, \xi_4) = \mathcal{A} \prod_{i=\text{odd}} \frac{(z_i - \xi_1)(z_{i+1} - \xi_2)(z_i - \xi_3)(z_{i+1} - \xi_4)}{z_i - z_{i+1}} \Phi_m(\{z_i\})$$

$$\Phi_2(\{z_i\}; \xi_1, \xi_2, \xi_3, \xi_4) = \mathcal{A} \prod_{i=\text{odd}} \frac{(z_i - \xi_1)(z_{i+1} - \xi_3)(z_i - \xi_2)(z_{i+1} - \xi_4)}{z_i - z_{i+1}} \Phi_m(\{z_i\})$$

$$\Phi_3(\{z_i\}; \xi_1, \xi_2, \xi_3, \xi_4) = \mathcal{A} \prod_{i=\text{odd}} \frac{(z_i - \xi_1)(z_{i+1} - \xi_2)(z_i - \xi_4)(z_{i+1} - \xi_3)}{z_i - z_{i+1}} \Phi_m(\{z_i\})$$  \hspace{1cm} (6.36)

However only two of them are linearly independent [23]. It is a unique property of non-abelian states that the four-quasihole states have a two-fold degeneracy even when all the positions of the quasiholes are fixed. As we exchange different pairs of quasiholes, non-abelian Berry’s phases may be generated.

However, although we have found the existence of the degeneracy, at the moment, we are still unable to calculate the non-abelian Berry’s phase due to the lack of the plasma analog for the pairing wave function $\Phi_p$. But the numerical results in Ref. [72] do provide evidences for the existence of the non-abelian statistics in the p-wave state.

In the following we will study the edge excitations of the p-wave state. We find that the edge structure of the p-wave state is also very different from that of abelian states. This provides additional evidences for the existence of the new topological orders in the p-wave state.

From (6.30) we see that the holomorphic part of the ground state wave function can be written as a correlation of $\psi_e$ with the operator $\Phi_N$ inserted at infinity. Following the discussion in the last section, we can create edge excitations by inserting other operators. Here we can use both $\psi$- and $j$-insertions Repeating the derivation in the last section, we find that edge excitations [i.e., the zero energy states of the three body-Hamiltonian (6.29)] are generated by the descendant fields of $\Psi_N$:

$$\Phi^{\text{edge}}_p = \lim_{z_\infty \rightarrow \infty} \left( \psi_e \right)^{2h_N+2l} (\Psi_N^{(m_1,..,m_l,..)}(z_\infty) \prod_{i=1}^N \psi_e(z_i))$$  \hspace{1cm} (6.37)
where

\[ l = \sum_i n_i + \sum_k m_k \]  \hspace{1cm} (6.38)

is the level of the descendant field

\[ \Psi^{(n_1 \ldots m_1 \ldots)}_N(z_\infty) = (\psi_{-n_1 \ldots j\ldots}) \Psi_N(z_\infty) \]  \hspace{1cm} (6.39)

and the \( \psi_n \) are operators in the expansion of \( \psi(z) \)

\[ \psi(z) = \sum_n \frac{\psi_n}{z^{n+\frac{1}{2}}} \quad n = \text{integer} + \frac{1}{2} \]  \hspace{1cm} (6.40)

Note that in order for the correlation to be non-zero, \( \Phi^{(n_1 \ldots m_1 \ldots)}_N \) must contain even or odd numbers of \( \psi_n \) operators if there are even or odd number of electrons (i.e., \( N=\text{even or odd} \)).

Not all descendant fields \( \Phi^{(n_1 \ldots m_1 \ldots)}_N \) with different indices are linearly independent. The linearly independent descendant fields generated by \( j_n \) can be calculated from the K-M algebra (6.23) as we did in the last section. The linearly independent descendant fields generated by \( \psi_n \) can be calculated from the fermion algebra satisfied by \( \psi_n \) fields

\[ \{ \psi_n, \psi_m \} = 2\pi \delta_{n+m} \]  \hspace{1cm} (6.41)

The algebra (6.41) can be derived form the OPE ,

\[ \psi(z_1)\psi(z_2) = -\psi(z_2)\psi(z_1) = \frac{1}{z_1 - z_2} \]  \hspace{1cm} (6.42)

Thus, the space of the descendant fields is isomorphic to the direct product of the space of the K-M algebra generated by \( j \) and the space of the Majorana fermion theory generated by \( \psi \). This is because \( \psi \) and \( j \) commute with each other. Similar to the pure \( U(1) \) case in the last section, the edge excitations generated by level-\( l \) descendant fields can be shown to have angular momentum \( L = M_0 + l \), where \( M_0 = h_N - Nh_e = \frac{n}{2} N(N - 1) - \frac{N}{2} \) is the angular momentum of the ground state \( \Phi_p \). Here we have used (6.21), and \( h_e = \frac{m+1}{2} \), \( h_N = \frac{m}{2} N^2 \) (see Ref. [65]).

The number of the linear independent descendant fields at each level can be easily calculated by counting the number of states in a free boson theory (generated by \( j \)) and a free fermion theory (generated by \( \psi \)). The results can expressed through characters. The sectors with even and odd \( \psi_n \)s should be considered separately. Let us first concentrate on the space generated by \( \psi_n \). Notice that the character for the two states generated by \( \psi_n \) is \( 1 + \xi^n \). The total character for both the even and the odd sectors is given by

\[ \prod_{n=1/2}^{\infty} (1 + \xi^n) = \text{ch}_{\text{even}}(\xi) + \text{ch}_{\text{odd}}(\xi) \]  \hspace{1cm} (6.43)

To obtain \( \text{ch}_{\text{even,odd}} \) separately we need to keep track of the number of the fermion operators. This can be achieved by introducing a generalized character \( 1 + \xi^n \eta \) which leads to

\[ \text{ch}(\xi, \eta) = \prod_{n=1/2}^{\infty} (1 + \xi^n \eta) \]  \hspace{1cm} (6.44)
It is easy to see that the power of $\eta$ counts the numbers of fermions. Thus the characters for the even and the odd sectors are

$$
\text{ch}_{\text{even}} = \frac{1}{2} (\text{ch}(\xi, 1) + \text{ch}(\xi, -1)) = \frac{1}{2} \left( \prod_{n=1/2}^{\infty} (1 + \xi^n) + \prod_{n=1/2}^{\infty} (1 - \xi^n) \right) \xi^{M_0^{(e)}}
$$

$$
\text{ch}_{\text{odd}} = \frac{1}{2} (\text{ch}(\xi, 1) - \text{ch}(\xi, -1)) = \frac{1}{2} \left( \prod_{n=1/2}^{\infty} (1 + \xi^n) - \prod_{n=1/2}^{\infty} (1 - \xi^n) \right) \xi^{M_0^{(o)}}
$$

(6.45)

The characters for edge excitations of the p-wave paired state, after including the Gaussian (or abelian) part generated by $\tau$, can be written as products of the $U(1)$ character (6.24) and the non-abelian character (6.45), and are given by

$$
\text{Ch}_{\text{even}} = \frac{1}{2 \prod_{n>0}^{1}(1 - \xi^n)^2} \left( \prod_{n=1/2}^{\infty} (1 + \xi^n) + \prod_{n=1/2}^{\infty} (1 - \xi^n) \right) \xi^{M_0^{(e)}}
$$

$$
\text{Ch}_{\text{odd}} = \frac{1}{2 \xi^{1/2} \prod_{n>0}^{1}(1 - \xi^n)^2} \left( \prod_{n=1/2}^{\infty} (1 + \xi^n) - \prod_{n=1/2}^{\infty} (1 - \xi^n) \right) \xi^{M_0^{(o)}}
$$

(6.46)

where $M_0^{e,o}$ is the angular momentum for the ground state with an even or odd number of electrons.

Having edge states generated in this way, two questions immediately come to our mind. First, are the edge states generated with different descendants linearly independent to each other? Second, do the insertions with all descendant fields generated by $\tau$ and $\psi$ exhaust all possible edge states? These are very hard and important questions. (6.46) is actually a counting of linearly independent descendant fields. To apply (6.46) to count the physical excitations at the edge we need to address the above two questions. Let us examine them in turns.

For the first question, obviously the descendants of primary fields with different dimensions (which are related to the angular momentum quantum numbers) generate linearly independent edge states. The hard part of the question is whether different descendants at the same level will generate linearly independent states or not. Generally in CFT, linearly independent descendants, as operators, should have different (or linearly independent) sets of correlations which contain arbitrary numbers of electron operators $\psi_{\pm}$. But when the number $N$ of electrons is finite and fixed, one can not claim that different descendant fields generate linearly independent correlations, in particular for descendant fields at arbitrarily large levels. So we suggest that the insertions with different operators generate different edge excitations in the thermodynamic limit or in the large-$N$ limit; though at the moment we do not know how to prove it within CFT.

Now we turn to the question of whether the descendant fields discussed above can generate all possible edge excitations in the system. We would like to first point out that the edge wave functions constructed above not only preserve the structure of zeros in the ground state as two or three electrons approach each other, they may also preserve the structure of higher-order zeros for four or more electrons approaching each other. If the wave functions generated by the descendant fields do not exhaust the zero-energy sector of a three-body Hamiltonian, in principle it is possible to construct a more restrictive Hamiltonian that contains additional two-body, three-body, four-body, etc. interactions, so that the new Hamiltonian makes the wave functions constructed above be and exhaust its zero-energy states. Thus, the question of whether the space of edge states contains
more states depends on the dynamics of electron interactions, and cannot be addressed by merely studying the wave functions. In the following, we will assume that the Hamiltonian satisfies the above conditions.

Under the above two assumptions we see that there is a one-to-one correspondence between the descendant fields and the edge states (in the large $N$ limit). Thus one can use the known results (6.46) about the descendant fields derived from the CFT to obtain the spectrum of low-lying edge states. Numerical diagonalizations for small systems have been done to test the predictions [64]. It was found that indeed the numerical results shows the violation of the suggested correspondence for finite $N$ at large level $l$. On the other hand, they verify the validity of the correspondence at the levels less than a certain number of order $N$.

From (6.46) one finds the numbers of states (NOS) of the low lying edge excitations at total angular momenta $M_0 + \Delta M$ (where $M_0$ is the angular momentum of the ground state) [64]

$$
\begin{align*}
\Delta M : & \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \\
\text{sector} & \\
\text{NOS} : & \quad 1 \quad 1 \quad 3 \quad 5 \quad 10 \quad 16 \quad 28 \quad 43 \quad 70 \quad N = \text{even} \\
\text{NOS} : & \quad 1 \quad 2 \quad 4 \quad 7 \quad 13 \quad 21 \quad 35 \quad 55 \quad 86 \quad N = \text{odd}
\end{align*}
$$

(6.47)

The spectrum (6.47) is very different from the edge spectrums of the abelian states. The spectrum of one branch of edge excitations in abelian states is given by

$$
\begin{align*}
\Delta M : & \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \\
\text{NOS} : & \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 7 \quad 11 \quad 15 \quad 22
\end{align*}
$$

(6.48)

while the spectrum of two branches of abelian edge excitations is

$$
\begin{align*}
\Delta M : & \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \\
\text{NOS} : & \quad 1 \quad 2 \quad 5 \quad 10 \quad 20 \quad 36 \quad 65 \quad 110 \quad 180
\end{align*}
$$

(6.49)

It is clear that in addition to its special dependence on $N$, the number of the edge excitations in the p-wave state is more than one branch but less than two branches of edge excitations in abelian states. The specific heat (per unit length) of the p-wave edge state turns out to be $3/2$ times of $\frac{\pi T}{U_b}$ – the specific heat of one branch of edge excitations of the abelian states. In this sense, we may say that the p-wave state contains one and a half branches of edge excitations.

In the above we merely discussed the spectrum of the edge excitations. We can go one step further. Both the Gaussian model and the Majorana fermion theory are equipped with a natural definition of inner product in their Hilbert space. We may assume the natural inner product in the CFT happens to be the physical inner product between the edge wave functions generated by the corresponding operators (in the large $N$ limit). Under this assumption we may use the known correlation functions of the CFT to calculate the correlation functions of physical operators near the edge. For example, the electron operator is given by $\psi_e$ in (6.31). From the the CFT results $\langle \psi_e(z)\psi_e(0) \rangle = 1/z^{m+1}$ we find the electron propagator on the edge to have the form

$$
G_e(x, t) \sim \frac{1}{(x-vt)^{g_e}}, \quad g_e = m + 1
$$

(6.50)

This result can be confirmed through a calculation of the electron occupation number $n_l$ in the single particle state $| l \rangle$. Here $l$ is the angular momentum. For the p-wave state of 10 electrons with $m = 2$, it was found [64] that $n_{l_0} : n_{l_0-1} : \ldots = 1 : 3.06 : 5.86 : 8.63 : \ldots$. The theoretical predictions of $n_l$ calculated from the Green function (6.50) for $g_e = m + 1 = 3$ are given by $| 1 : 3 : 6 : 10 : \ldots \rangle$. Here $l_0$ is the angular momentum of
the last occupied orbit \((l_0 = 17\) for 10 electrons). The agreement suggests the validity of our assumption about the correspondence between the inner products [64].

Next let us consider the quasiparticle operator. The CFT that describes a p-wave edge state – the \(U(1) \times Ising\) model – contains the following local operators: \(e^{i\alpha \phi}, \psi\), and \(\sigma\). Here \(\sigma\) is the disorder operator in the \(Ising\) model which changes the boundary condition of the fermion \(\psi\). Not all the above operators are physical, \(i.e.,\) create an allowed excitation in the electron system. A physical quasiparticle operator must have a single-valued correlation function with the electron operator. This condition is closely related to the single-value property of the electron wave function in the presence of the quasiparticle. The condition can be expressed through the operator product expansion between the electron and quasiparticle operators:

\[
\psi_e(w_1)\psi_q(w_2) \propto (w_1 - w_2)^\gamma \hat{O}(w_2)
\]  

(6.51)

where we require \(\gamma\) to be an integer. From the OPE in the \(U(1) \times Ising\) model:

\[
e^{i\alpha \phi(w)}e^{i\beta \phi(0)} \propto w^{\frac{1}{8}[(\alpha + \beta)^2 - \alpha^2 - \beta^2]}e^{i(\alpha + \beta)\phi(0)}
\]

\[
\psi(w)\sigma(0) \propto w^{\frac{1}{2}\mu}(0)
\]  

(6.52)

we find that the following operators can be identified as quasiparticle operators: \(\psi_q(x) = e^{\frac{i}{2\sqrt{m}}\phi(x)}\sigma(x)\), and \(\psi'_q(x) = e^{\frac{i}{2\sqrt{m}}\phi(x)}\). \(\psi_q\) carries charge \(\frac{e}{2m}\) and corresponds to the non-abelian quasiparticle discussed above [23]. \(\psi'_q\) carries charge \(\frac{1}{m}\) and has an abelian statistics \(\theta = \frac{\pi}{m}\). \(\psi'_q\) is created by inserting a unit flux. From the CFT result \(\langle \sigma(w)\sigma(0) \rangle \propto w^{1/8}\), we find \(\langle \psi_q(x,t)\psi_q(0) \rangle \propto (x - vt)^{-g_q}\) with \(g_q = \frac{m + 2}{8m}\). The exponents \(g_e\) and \(g_q\) can be measured in tunneling experiments between the edges and provide an experimental test of the non-abelian states.

Another important non-abelian FQH state is the Haldane-Rezayi (HR) state for spin-1/2 electrons [27]. The HR state is important because it is a candidate for the \(\nu = 5/2\) FQH state observed in experiments[26]. The HR state is a d-wave-paired spin-singlet FQH state:

\[
\Phi_{HR}(z_1, w_1) = \Phi_m(z_1, w_1)\Phi_{ds}(z_1, w_1)
\]

\[
\Phi_{ds}(z_1, w_1) = \mathcal{A}_{z,w} \left( \frac{1}{(z_1 - w_1)^2} \right)\left( \frac{1}{(z_2 - w_2)^2} \right)...
\]

\[
\Phi_m(z_1, w_1) = \left( \prod_{i < j} (z_i - z_j)^m \prod_{i < j} (w_i - w_j)^m \prod_{i,j} (z_i - w_j)^m \right) e^{\frac{1}{4} \sum_i (|z_i|^2 + |w_i|^2)}
\]  

(6.53)

which has a filling fraction \(1/m\) with \(m\) an even integer. Here \(z_i\) (\(w_i\)) are the coordinates of the spin-up (down) electrons, and the operator \(\mathcal{A}_{z,w}\) performs separate anti-symmetrizations among \(z_i\)s and among \(w_i\)s.

The non-abelian part of the bulk wave function of the HR state was shown to be described by a CFT with central charge \(c = -2\) [65]. Using this \(c = -2\) CFT and the \(U(1)\) Gaussian model for the abelian part, one can calculate the spectrum of edge excitations. One finds that the edge excitations of an \(N\)-electron system with total spin \(s\) and a fixed \(S_z\) component \(\sigma\) are described by the following character

\[
\text{Ch}_{N,s}(\xi) = \frac{1 - \xi^2s+1}{\prod_n (1 - \xi^n)^2} \xi^{M_0(s)}
\]  

(6.54)
where
\[ M_0^{(s)} = h_{N,s} - Nh_e = \frac{m}{2}N(N-1) + \frac{1}{8}[(4s+1)^2 - 1] - N \]  
\[ (6.55) \]
is the minimum angular momentum in the spin-s sector. Here \( h_{N,s} = \frac{m}{2}N^2 + \frac{1}{8}[(4s+1)^2 - 1] \), and \( h_e = 1 + \frac{m}{2} \). According to (6.54), the number of edge excitations in the spin-s sector at angular momentum \( L = M_0^{(s)} + l \), is given by

\[
\begin{array}{cccccc}
  l : & 0 & 1 & 2 & 3 & 4 & 5 \\
  s = 0 & 1 & 1 & 3 & 5 & 10 & 16 \\
  s = 1/2 & 1 & 2 & 4 & 8 & 15 & 26 \\
  s = 1 & 1 & 2 & 5 & 9 & 18 & 31 \\
  s = 3/2 & 1 & 2 & 5 & 10 & 19 & 34 \\
\end{array}
\]

which again has very different structures from the abelian edge states.

In summary non-abelian FQH states are a fascinating class of topological fluids, which very likely exist in real experimental samples (such as the \( \nu = 5/2 \) state). Our understanding for the non-abelian is quite limited. In particular we do not understand the general mathematical structures behind the non-abelian topological orders, and this limits our ability to calculate the physics properties of non-abelian states.

### 7. Remarks and acknowledgments

Many important aspects of topological fluids were not covered. I recommend two review articles Ref. [73] (which is fun to read) and Ref. [74] that address some similar issues discussed in this paper with different emphasizes. They also cover some subjects which is not discussed here. Many early papers on topological properties of QH states can be found in the book in Ref. [75]. In this paper I include only what I believe to be the most fundamental properties of topological orders in FQH liquids, in particular those properties that might have experimental consequences. Other important and/or actively studied properties of FQH fluids include:

1) Phase transitions and phase structures of FQH systems [76];
2) Hierarchical structures in multi-layer FQH systems [77];
3) Superfluid mode and symmetry breaking in some double layer FQH states [40];
4) Structure of ground states of topological fluids on closed space [13,6,7]; For an abelian FQH state characterized by \( K \)-matrix \( K \), one can show it has a \((\det K)^g\)-fold degeneracy on the genus \( g \) Riemann surface;
5) \( W_\infty \) algebra in FQH states [78];
6) Spin dynamics and symmetries in FQH states [38];
7) ... ...

FQH liquids demonstrate extremely rich internal structures. The more we probe, the more we are fascinated by the endless richness that the nature reveals to us.

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References

1. D.C. Tsui, H.L. Stormer, and A.C. Gossard, *Phys. Rev. Lett.* **48**, (1982) 1559.
2. R.B. Laughlin, *Phys. Rev. Lett.* **50**, (1983) 1395.
3. F.D.M. Haldane, *Phys. Rev. Lett.* **51**, (1983) 605; B. I. Halperin, *Phys. Rev. Lett.* **52**, (1984) 1583; S. Girvin, *Phys. Rev. B** **29**, (1984) 6012; A.H. MacDonald and D.B. Murray, *Phys. Rev. B* **32**, (1985) 2707; M.P.A. Fisher and D.H. Lee, *Phys. Rev. Lett.* **63**, (1989) 903; J.K. Jain, *Phys. Rev. Lett.* **63**, (1989) 199; *Phys. Rev.* **B41**, (1991) 7653.
4. S. M. Girvin and A. H. MacDonald, *Phys. Rev. Lett.* **58**, (1987) 1252.
5. S. C. Zhang, T. H. Hansson and S. Kivelson, *Phys. Rev. Lett.* **62**, (1989) 82; N. Read, *Phys. Rev. Lett.* **62**, (1989) 86; Z.F. Ezawa and A. Iwazaki, *Phys. Rev.* **B43**, (1991) 2637.
6. X.G. Wen, *Int. J. Mod. Phys. B* **2**, (1990) 239; *Phys. Rev.* **B40**, (1989) 7387.
7. X.G. Wen and Q. Niu, *Phys. Rev.* **B41**, (1990) 9377.
8. A. Fetter, C. Hanna and R. Laughlin, *Phys. Rev.* **B39**, (1989) 9679; W. Chen, F. Wilczek, E. Witten and B. Halperin, *J. Mod. Phys.* **B3**, (1989) 1001; X.G. Wen and A. Zee, *Phys. Rev.* **B44**, (1991) 274.
9. V. Kalmyer and R.B. Laughlin, *Phys. Rev. Lett.* **59**, (1987) 2095; X.G. Wen, F. Wilczek and A. Zee, *Phys. Rev.* **B39**, (1989) 11413.
10. S.A. Kivelson, D.S. Rokhsar and J.P. Sethna, *Phys. Rev. B* **35**, (1987) 8865; D.S. Rokhsar and S.A. Kivelson, *Phys. Rev. Lett.* **61**, (1988) 2376; N. Read and B. Chakraborty, *Phys. Rev.* **B40**, (1989) 7133; X.G. Wen, *Phys. Rev.* **B**, (1991) B44, 2664.
11. X.G. Wen, *Int. J. Mod. Phys.* **B5**, (1991) 1641.
12. F.D.M. Haldane, *Phys. Rev. Lett.* **51**, (1983) 605.
13. F.D.M. Haldane and E.H. Rezayi, *Phys. Rev.* **B31**, (1985) 2529; T. Einarsson, *Mod. Phys. Lett. B* **5**, (1991) 675; D. Li, *Mod. Phys. Lett. B* **7**, (1993) 1103; E. Keski-Vakkuri, and X.G. Wen, *Int. J. Mod. Phys.* **B7**, (1993) 4227; G. Christofano, G. Maiella, R. Musto and F. Nicodemi, *Phys. Lett.* **B262**, (1991) 88; *Mod. Phy. Lett.* **A6**, (1991) 1779; *ibid A6*, 2985 (1991); *ibid A7*, 2583 (1992).
14. E. Witten, *Comm. Math. Phys.* **121**, (1989) 351; J. Fröhlich and C. King, *Comm. Math. Phys.* **126**, (1989) 167.
15. S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, *Nucl. Phys.* **B326**, (1989) 108.
16. X.G. Wen and A. Zee, *Nucl. Phys.* **B15**, (1990) 135.
17. N. Read, *Phys. Rev. Lett.* **65**, (1990) 1502.
18. B. Blok and X.G. Wen, *Phys. Rev.* **B42**, (1990) 8133; *Phys. Rev.* **B42**, (1990) 8145.
19. J. Fröhlich and A. Zee, *Nucl. Phys.* **B364**, (1991) 517.

20. X.G. Wen and A. Zee, *Phys. Rev.* **B46**, (1992) 2290.

21. X.G. Wen and A. Zee, *Phys. Rev. Lett.* **69**, (1992) 953; (E) *Phys. Rev. Lett.* **69**, (1992) 3000.

22. B. I. Halperin, *Helv. Phys. Acta* **56**, (1983) 75.

23. G. Moore and N. Read, *Nucl. Phys.* **B360**, (1991) 362.

24. X.G. Wen, *Phys. Rev. Lett.* **66**, (1991) 802.

25. X.G. Wen, Y.S. Wu, and Y. Hatsugai, *Nucl. Phys.* **B422**, (1994) 476.

26. R. Willett, J.P. Eisenstein, H.L. Strörmer, D.C. Tsui, A.C. Gossard, and J.H. English, *Phys. Rev. Lett.* **59**, (1987) 1776.

27. F.D.M. Haldane and E.H. Rezayi, *Phys. Rev. Lett.* **60**, (1988) 956; **60**, E1886 (1988).

28. B. Blok and X.G. Wen, *Nucl. Phys.* **B374**, (1992) 615.

29. B.I. Halperin, *Phys. Rev.* **B25**, (1982) 2185.

30. X.G. Wen, *Phys. Rev. B**41**, (1990) 12838.

31. D.H. Lee and X.G. Wen, *Phys. Rev. Lett.* **66**, (1991) 1765.

32. X.G. Wen, *Mod. Phys. Lett. B**5**, (1991) 39.

33. X.G. Wen, *Int. J. Mod. Phys.* **B6**, (1992) 1711.

34. F.P. Milliken, C.P. Umbach and R.A. Webb, unpublished.

35. X.G. Wen, *Phys. Rev.* **B44**, (1991) 5708; C. L. Kane and M. P. A. Fisher *Phys. Rev. Lett.* **68**, (1992) 1220; *Phys. Rev. B**46**, (1992) 15233.

36. M.P.A. Fisher and D.H. Lee, in Ref. [3]; X.G. Wen and A. Zee, *Int. J. Mod. Phys.* **B4**, (1990) 437.

37. D. Arovas, J. R. Schrieffer, and F. Wilczek, *Phys. Rev. Lett.* **53**, (1984) 722.

38. T. Chakraborty, and F.C. Zhang, *Phys. Rev. B**29**, (1984) 7032; *Phys. Rev. B**30**, (1984) 7320;
   R.C. Clark, *et al., Phys. Rev. Lett.* **60**, (1988) 1747; *Phys. Rev. Lett.* **62**, (1989) 1536;
   S. L. Sondhi and S. A. Kivelson, *Phys. Rev.* **B46**, (1992) 13319;
   A. Balatsky and M. Stone, *Phys. Rev.* **B43**, (1991) 8038.

39. S.Q. Murphy, J.P. Eisenstein, G.S. Boebinger, L.N. Pfeiffer and K.W. West, *Phys. Rev. Lett.* **72**, (1994) 728;
   X.G. Wen and A. Zee, *Phys. Rev. Lett.* **69**, (1992) 1811; *Phys. Rev. B**47**, (1993) 2265;
   K. Yang, K. Moon, L. Zheng, A.H. MacDonald, S.M. Girvin, D. Yoshioka and S.-C. Zhang, *Phys. Rev. Lett.* **72**, (1994) 732;
   N. Read, preprint [cond-mat/9501010].

40. H. Fertig, *Phys. Rev.* **B40**, (1989) 1087; A.H. MacDonald, P.M. Platzman, and G.S. Boebinger, *Phys. Rev. Lett.* **65**, (1990) 775;
L. Brey, *Phys. Rev. Lett.* **65**, (1990) 903;
S. Sondhi, A. Karlhede, S.A. Kivelson and E.H. Rezayi, *Phys. Rev. B47*, (1993) 16419.

41. Q. Niu and D.J. Thouless, *Phys. Rev. B35*, (1987) 2188.

42. S.A. Trugman, *Phys. Rev. B27*, (1983) 7539;
A.H. MacDonald, and P. Streda, *Phys. Rev. B29*, (1984) 1616;
P. Streda, J. Kucera and A.H. MacDonald, *Phys. Rev. Lett. 59*, (1987) 1973;
M. Buttiker, *Phys. Rev. B38*, (1988) 9375;
J.K. Jain and S.A. Kivelson, *Phys. Rev. B37*, (1988) 4276; *Phys. Rev. Lett. 60*, (1988) 1542.

43. P.L. McEuen, *et al.*, *Phys. Rev. Lett. 64*, (1990) 2062;
A.M. Chang and J.E. Cunningham, *Solid State Comm.* **72**, (1989) 651;
L.P. Kouwenhoven *et al.*, *Phys. Rev. Lett. 64*, (1990) 69;
J.K. Wang and V.J. Goldman, *Phys. Rev. Lett. 67*, (1991) 749.

44. R. Laughlin, *Phys. Rev. B23*, (1981) 5632.

45. X.G. Wen, *Phys. Rev. B43*, (1991) 11025.

46. C.W.J. Beenakker, *Phys. Rev. Lett. 64*, (1990) 216.

47. A.H. MacDonald, *Phys. Rev. Lett. 64*, (1990) 220.

48. F.D.M. Haldane, *Bulletin of APS 35*, (1990) 254.

49. M. Stone, *Phys. Rev. B 42*, (1990) 8399; *Annals of Physics 207*, (1991) 38; *Int. J. Mod. Phys. B 5*, (1991) 509.

50. See, for example,
P. Goddard and D. Olive, “Workshop on Unified String Theories, 1985”, eds. M. Green and D. Gross (World Scientific, Singapore), p. 214; *Inter. J. Mod. Phys. 1*, (1986) 303;
V.G. Kac, “Infinite dimensional Lie algebra”, (Birkhauser, Boston, 1983).

51. S. Tomonaga, *Prog. Theor. Phys. (Kyoto) 5*, (1950) 544.

52. J. Fröhlich and T. Kerler, *Nucl. Phys. B354*, (1991) 369.

53. M.D. Johnson and A.H. MacDonald, *Phys. Rev. Lett. 67*, (1991) 2060.

54. C. L. Kane, M.P.A. Fisher and J. Polchinski, *Phys. Rev. Lett. 72*, (1994) 4129.
C. L. Kane, and M.P.A. Fisher, preprint [cond-mat/9409028](cond-mat/9409028).

55. See for example, D. Gross *et al.*, *Nucl. Phys. B256*, (1985) 253;
R. Floreanini and R. Jackiw, *Phys. Rev. Lett. 59*, (1988) 1873.

56. V.K. Talyanskii *et al.*, *Surface Science 229*, (1990) 40;
M. Wassermeier *et al.*, *Phys. Rev. B41*, (1990) 10287.

57. R.C. Ashoori, H. Stormer, L. Pfeiffer, K. Baldwin and K. West, *Phys. Rev. B45*, (1992) 3894.

58. D. B. Chklovskii, B.I. Shklovskii, L. I. Glazman, *Phys. Rev. B46*, (1992) 4026;
J. Dempsey, B.Y. Gelfand, and B.I. Halperin, *Phys. Rev. Lett. 70*, (1993) 3639.

59. C. Chamon and X.G. Wen, *Phys. Rev. B49*, (1994) 8227.

60. F.D.M. Haldane, Princeton preprint [cond-mat/9501007](cond-mat/9501007).
61. D.H. Lee and X.G. Wen, Phys. Rev. B 49, (1994) 11066.

62. T. Einarsson, S. L. Sondhi, S. M. Girvin, and D. P. Arovas, preprint (cond-mat/9411078).

63. T. Einarsson, in Ref. [13]; D. Li, in Ref. [13].

64. X.G. Wen, Phys. Rev. Lett. 70, (1993) 355.

65. X.G. Wen and Y.S. Wu, Nucl. Phys. B419, (1994) 455.

66. For readers not familiar with CFT, we recommend the reprint book, “Conformal Invariance and Applications to Statistical Mechanics”, ed. C. Itzykson, H. Saleur and J.B. Zuber, (World Scientific, 1988); and the review article by P. Ginsparg, in Lectures at Les Houches Summer School (1988), Vol. XLIX, ed. E. Brezín and J. Zinn-Justin (North Holland, 1989).

67. S.A. Trugman and S. Kivelson, Phys. Rev. B26, (1985) 3682; V.L. Pokrovsky and A.L. Talapov, J. Phys. C18, (1985) L691.

68. S. Fubini, Int. J. Mod. Phys. A5, (1990) 3553; S. Fubini and C.A. Lütken, Mod. Phys. Lett. A6, (1991) 487.

69. Consider an operator Ψ of dimension ℏ. The correlation function ⟨Ψ(z)⟩ (here “...” representing other operators) as z → ∞ is proportional to ⟨Ψ(z)Ψ†(0)⟩(1 + o(z−1)), where Ψ† is the conjugate of Ψ. Thus ⟨Ψ(z)⟩ ∝ z−2ℏ as z → ∞.

70. V.G. Knizhnik and A.B. Zamolochikov, Nucl. Phys. B247, (1984) 83.

71. A. Rocha-Caridi, “Vacuum Vector Representations of the Virasoro Algebra”, in Vertex Operators in Mathematics and Physics”, MSRI Publications # 3 (Springer, Heidelberg, 1984), p. 451.

72. N. Read and E.H. Rezayi, Yale and CSU preprint, May 1993.

73. A. Zee, “Quantum Hall Fluids”, cond-mat/9501022.

74. J. Fröhlich and U.M. Studer, Rev. of Mod. Phys. 65, (1993) 733; J. Fröhlich, U.M. Studer and E. Thiran, lectures presented by J.F. at the 1994 Les Houches Summer School “Fluctuating Geometries in Statistical Mechanics and Field Theory”, cond-mat/9508062.

75. M. Stone, Ed. Quantum Hall effects (World Scientific, Singapore, 1991).

76. H.P. Wei, D.C. Tsui, M. Paalanen, and A.M.M. Pruisken, Phys. Rev. Lett. 61, (1988) 1294; S. Koch, R. Haug, K. v. Klitzing, and K. Ploog, Phys. Rev. Lett. 67, (1991) 883; H.P Wei, S.W. Hwang, D.C. Tsui, and A.M.M. Pruisken, Surf. Sci. 229, (1990) 34; H.W. Jiang, C.E. Johnson, K.L. Wang, and S.T. Hannahs, Phys. Rev. Lett. 71, (1993) 1439; A.M.M. Pruisken, Phys. Rev. Lett. 61, (1988) 1297; S.A. Kivelson, D-H Lee and S-C Zhang, Phys. Rev. B46, (1992) 2223; D-H Lee, Z. Wang and S.A. Kivelson, Phys. Rev. Lett. 70, (1993) 4130; Ziqiang Wang, Dung-Hai Lee, and Xiao-Gang Wen, Phys. Rev. Lett. 72, (1994) 2454.

77. Y.W. Suen et al., Phys. Rev. Lett. 68, (1992) 1379; J.P. Eisenstein et al., Phys. Rev. Lett. 68, (1992) 1383.
78. A. Cappelli, C. A. Trugenberger and G. R. Zemba, Nucl. Phys. B396, (1993) 465; Phys. Lett. B306, (1993) 100; A. Cappelli, V.G. Dunne, C.A. Trugenberger and G. Zemba, Nucl. Phys. 398B, (1993) 531; S. Iso, D. Karabali and B. Sakita, Phys. Lett. B296, (1992) 143; D. Karabali, Nucl. Phys. B419, (1994) 437; Nucl. Phys. B428, (1994) 531.
Figure captions

Fig 1.1 A 1D crystal passing an impurity will generate a narrow band noise in the voltage drop.

Fig 1.2 A FQH fluid passing through a constriction will generate narrow band noises due to the back scattering of the quasiparticles.

Fig 6.1 The energy spectrum of a system of six electrons in the first 22 orbits for the three-body Hamiltonian. The degeneracies of the zero energy states at $M = 45, ..., 51$ are found to be $1, 1, 2, 3, 5, 7, 11$. 