Random groups do not have Property (T) at densities below 1/4

Calum J. Ashcroft

Abstract
We prove that random groups in the Gromov density model at density $d < 1/4$ do not have Property (T), answering a conjecture of Przytycki. We also prove similar results in the $k$-angular model of random groups.

1 Introduction

The density model of random groups was introduced by Gromov in [Gro93] to study a ‘typical’ group. Fix $n \geq 2, l \geq 1$, and $0 < d < 1$. Let $F_n$ be the free group of rank $n$. A random group on $n$ generators at density $d$ and length $l$ is defined by killing a uniformly random chosen set of $(2n - 1)ld$ cyclically reduced words of length $l$ in $F_n$. We write $G \sim G(n, l, d)$ if a random group has the distribution of this procedure, and with overwhelming probability (w.o.p.) if a property holds with probability tending to 1 as $l$ tends to infinity.

The study of Property (T) in random quotients was proposed in Gromov’s manuscript on hyperbolic groups [Gro87, §4.5C]. A famed theorem of Żuk states that w.o.p a random group at density $d > 1/3$ has Property (T) [Ż03] (see also [KK13, Ash21]). Żuk’s theorem was recently extended to random quotients of hyperbolic groups in [Ash22], answering a question of Gromov–Ollivier.

Random groups at density less than $1/6$ w.o.p. act freely and cocompactly on a CAT(0) cube complex [OW11] (they are hyperbolic w.o.p. [Gro93, Oll04] and hence linear w.o.p. [HW08, Ago13]). Montee proved that random groups w.o.p. act non-trivially and cocompactly on a CAT(0) cube complex at density less than $3/14$ [Mon21], extending the previous density bounds for such an action of Ollivier–Wise ($d < 1/5$, [OW11]) and Mackay–Przytycki ($d < 5/24$, [MPL15]).

A gap exists between the current known density bound for Property (T) and for an unbounded action on a CAT(0) cube complex. It has been conjectured by Przytycki that random groups at densities below 1/4 do not have Property (T). We prove this conjecture.

Theorem A. Let $G$ be a random group at density less than 1/4. Then, with overwhelming probability, $G$ acts with unbounded orbits on a finite dimensional CAT(0) cube complex, and hence does not have Property (T).

This conjecture was personally communicated to the author by Przytycki and has been strongly alluded to, for example, [MPL15] p. 398.
However, the question of what happens between densities $1/4$ and $1/3$ remains. Indeed, random groups at density $d < 1/6$ w.o.p. have the Haagerup property [OW11, Corollary 9.2], a strong negation of Property (T). There are therefore two questions to answer. Firstly, what is the optimum density bound for random groups to have the Haagerup property? And secondly, what is the optimum density bound for random groups to have Property (T)? More concretely, suppose that $0 < d_H \leq d_{(T)} < 1$ are optimal constants such that a random group at density $d < d_H$ w.o.p. has the Haagerup property and a random group at density $d > d_{(T)}$ w.o.p. has Property (T). Does $d_H = d_{(T)}$? This would mean that the Haagerup property and Property (T) would be ‘generically opposite’, as discussed in [Oll05, §IV.c]. Beyond Property (T), there are results on further fixed point properties in the density model, such as $F(L^p)$ [DM19], or actions on uniformly negatively curved Banach spaces [Opp21] (see also [dLdS21] for similar results in the triangular model). Questions then remain for the optimum density bounds for these fixed point properties.

Our approach to the problem is different to the previous methods used to obstruct Property (T) in random groups. The results of Ollivier–Wise, Mackay–Przytycki, and Montee were proved by building separating subspaces of the Cayley complex of a random group, and then employing the cubulation techniques of Sageev [Sag95]. These subspaces were constructed inductively by joining midpoints of edges to form ‘hypergraphs’, using the isoperimetric inequality for random groups (see [Oll07, Odr18]) to prove that these hypergraphs are 2-sided trees. As $d$ tends to $1/4$ these methods present increasing combinatorial difficulties. We instead approach the problem similarly to Osajda’s work on group cubization [Osa18], using covers of Cayley graphs to obstruct Property (T) in certain extensions of groups.

In Definition 2.4, we introduce the notion of a group being aspherically relator separated. It is well known that a group acting with unbounded orbits on a finite dimensional CAT(0) cube complex does not have Property (T) [NR97, Theorem B]. The main technical result of this paper is Theorem B, the proof of which occupies Section 2.

**Theorem B.** Let $G$ be an aspherically relator separated group. Then $G$ acts with unbounded orbits on a finite dimensional CAT(0) cube complex, and hence does not have Property (T).

We may therefore deduce Theorem A from Theorem B and the following lemma, the proof of which forms Section 3.

**Lemma 3.8.** Let $G$ be a random group at density less than $1/4$. Then, with overwhelming probability, $G$ is aspherically relator separated.

We study a further model of random groups that was introduced in [ARD20]. Fix $k \geq 2$ and $0 < d < 1$. A random group in the $k$-angular model at density $d$ is defined by choosing $G \sim G(n,k,d)$ and letting $n$ tend to infinity. This was preceded by the triangular model of Żuk [Z03] and the square model of Odrzygóźdź [Odr18]. The positive $k$-angular model, $G^+(n,k,d)$, is obtained
similarly, by choosing $n^{kd}$ positive words (words containing no inverse letters) of length $k$ in $F_n$. For any fixed $m$ and $k$, a random group $G \sim G^+(n, k, d)$ admits a homomorphism onto a finite index subgroup of a random group $H \sim G(m, nk, d)$ [KK13 §3.3]. Since Property (T) is preserved by passing to quotients and finite index extensions, we obtain the following corollary (writing w.o.p.(n) to mean that a property holds with probability tending to 1 as $n$ tends to infinity).

**Corollary C.** Let $G$ be a random group in the positive $k$-angular model at density $d < 1/4$. Then, w.o.p.(n), $G$ does not have Property (T).

It was recently shown that random groups in the $k$-angular model at density $d > (k + (-k \mod 3))/3k$ have Property (T) w.o.p.(n) [Ash21], extending the known results of: $d > 1/3$ for the triangular model [Z03]; $d > 1/3$ for the hexagonal model [Odr19]; and in fact $d > 1/3$ for any $k$-angular model where $k$ is divisible by 3 [Mon22]. Random groups do not have Property (T) w.o.p.(n) at densities: $d < 1/3$ in the triangular model [Z03]; $d < 3/8$ in the square model [Odr19]; and $d < 1/3$ in the hexagonal model [Odr19]. Random groups in the square model are in fact w.o.p.(n) virtually special (in the sense of Haglund–Wise) at density $d < 1/3$ [Duo17]. We prove the following result for the $k$-angular model.

**Theorem D.** Let $k \geq 2$ and $d < \lfloor k/2 \rfloor/2k$. Let $G$ be a random group in the $k$-angular model at density $d$. Then, w.o.p.(n), $G$ acts with unbounded orbits on a finite dimensional CAT(0) cube complex, and hence does not have Property (T).

We are dealing with asymptotics, hence frequently arrive at situations where $m$ is some parameter tending to infinity that is required to be an integer. If $m$ is not integer, then we will implicitly replace it by $\lfloor m \rfloor$. This will not affect the validity of any of our arguments.

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2 Using covers of graphs to obstruct (T)

We first discuss the construction of a specific covering of a given Cayley graph. Our construction is similar to Osajda’s group cubization [Osa18], which employs Wise’s double cover [Wis21 §9.d]. Throughout, given a set $S$, $F(S)$ will be the free group on the set $S$. Consider subsets $T, R \subseteq F(S)$ with $T \subseteq \langle R \rangle$. Let $G_T = \langle S\mid T \rangle$ and $K_R = \langle S\mid R \rangle$, so that there is an epimorphism $\xi : G_T \to K_R$
obtained by mapping each \( s \in S \) in \( G_T \) to the generator \( s \) in \( K_R \). Let \( \Gamma_T \) be the Cayley graph of \( G_T \) and \( \Sigma_R \) the Cayley graph of \( K_R \), so that \( \pi_1(\Gamma_T) = \langle\langle T \rangle\rangle \) and \( \pi_1(\Sigma_R) = \langle\langle R \rangle\rangle \). We have the inclusion \( \langle\langle T \rangle\rangle \hookrightarrow \langle\langle R \rangle\rangle \), and hence an induced map \( \iota : H_1(\langle\langle T \rangle\rangle) \to H_1(\langle\langle R \rangle\rangle) \). All homology groups in this text are taken with coefficients in \( \mathbb{Z}/2\mathbb{Z} \), and so homology groups are \( \mathbb{F}_2 \) vector spaces.

**Definition 2.1.** The double \( T \)-cover of \( \Sigma_R \) is the covering graph \( \hat{\Delta}_{T,R} \overset{\pi}{\to} \Sigma_R \) corresponding to the kernel of the map \( P_{T,R} : \pi_1(\Sigma_R) = \langle\langle R \rangle\rangle \to Q(T,R) = H_1(\langle\langle R \rangle\rangle)/\iota(H_1(\langle\langle T \rangle\rangle)). \)

Since \( Q(T,R) = \bigoplus_I \mathbb{Z}/2\mathbb{Z} \) for some index set \( I \), \( \hat{\Delta}_{T,R} \) is a normal cover of the Cayley graph \( \Sigma_R \). It follows immediately by the construction and Sabidussi’s theorem [Sab58] that \( \hat{\Delta}_{T,R} \) is the Cayley graph of the group \( \hat{H}_{T,R} = F(S)/\ker(P_{T,R}) \). We also call \( \hat{H}_{T,R} \) the double \( T \)-cover of \( K_R \) (note that \( G_T \to \hat{H}_{T,R} \to K_R \)).

**Remark 2.2.** The double cover is obtained by taking \( T = \emptyset \). In particular, given a group \( K_R = \langle S \mid R \rangle \), Osajda’s group cubization \( \hat{H} \) is the double \( T \)-cover of \( K_R \) for \( T = \emptyset \).

**Remark 2.3.** There are several different groups occurring in the above constructions. Particular care should be taken that we have surjections

\[
G_T = \langle S \mid T \rangle \to \hat{H}_{T,R} = \langle S \mid \ker(P_{T,R}) \rangle \to K_R = \langle S \mid R \rangle,
\]

whereas the relators give us inclusions of subgroups of the free group \( F(S) \),

\[
\langle\langle T \rangle\rangle \leq \ker(P_{T,R}) \leq \langle\langle R \rangle\rangle.
\]

In an attempt to reduce confusion where possible, we make the following conventions.

- Elements of \( G_T \) will always be denoted by \( g \), elements of \( \hat{H}_{T,R} \) will always be denoted by \( h \), and elements of \( K_R \) will always be denoted by \( k \) or \( \kappa \).
- Elements of \( T \) will always be written \( t \) or \( \tau \), whereas elements of \( R \) will always be written \( r \) or \( \rho \).
- Finally, paths in \( \Gamma_T \) will be denoted by \( \gamma \), paths in \( \hat{\Delta}_{T,R} \) will be denoted by \( \delta \), and paths in \( \Sigma_R \) will be denoted by \( \sigma \).

We now move to the central definition of this text, designed to give us control over the quotient group \( Q(T,R) \), and hence allows us to also obtain control over the covering graph \( \hat{\Delta}_{T,R} \). A group presentation \( L = \langle X \mid Y \rangle \) is aspherical if the corresponding presentation complex \( \mathcal{P}(X,Y) \) is topologically aspherical, or equivalently, if the second homology with integer coefficients of the universal cover of \( \mathcal{P}(X,Y) \), \( \hat{\mathcal{P}}(X,Y) \), is trivial.
**Definition 2.4.** Let \( L = \langle S \mid t_1, \ldots, t_N \rangle \) be a finite presentation such that each \( t_i \) is reduced without cancellation, and we have fixed a partition \( t_i = r_ir_i' \) for each word \( t_i \). Let \( H(L) = \langle S \mid r_1, r_1', \ldots, r_N, r_N' \rangle \). We say that \( L \) is aspherically relator separated if the presentations \( L \) and \( H(L) \) are aspherical, and \( H(L) \) is infinite.

Since \( L \) surjects \( H(L) \), the above also implies that \( L \) is infinite.

**Remark 2.5.** In the sequel, unless otherwise specified, we will always fix the partition \( t_i = r_ir_i' \) with \( |r_i| = |r_i'| \) if \( |t_i| \) is even, and \( |r_i| = |r_i| - 1 \) otherwise.

We begin by noting the following consequences of a group presentation being aspherical.

**Lemma 2.6.** [CCHS1] Let \( L = \langle X \mid Y \rangle \) be an aspherical group presentation.

i) No \( y \in Y \) is a proper power, and if \( y \in Y \) then no other member of \( Y \) is a cyclic conjugate of \( y \) or \( y^{-1} \) (including the trivial conjugates \( y, y^{-1} \)).

ii) If \( y, y', \gamma, \gamma' \in Y \) with \( y' \neq y^{-1} \) and \( yy' = \gamma \gamma' \), then \( y = \gamma \) and \( y' = \gamma' \).

**Proof.** Statement i) follows by [CCHS1 Proposition 1.3].

For Statement ii), we use the statement from [CCHS1 proof of Lemma 1.8], that Condition (I.1) of [LS15, §III.10.2] is valid for \( L \). Put simply, this implies that, since \( yy' \gamma^{-1} \gamma^{-1} = 1 \) with \( y 
eq y^{-1} \), either: \( y = \gamma \) and \( y' = \gamma' \); or \( y = \gamma' \) and \( y' = \gamma \).

Suppose that \( y = \gamma' \), \( y' = \gamma \). Then in the free group \( F(X) \) we have \( yy' = y' y \). In particular, \( y \) and \( y' \) commute, and so are powers of a common element \( v \). Since \( y, y' \) cannot be proper powers by statement i), we have \( y = y' = v \), so that \( y = y' = \gamma = \gamma' \). In particular, in either case we have \( y = \gamma \) and \( y' = \gamma' \). \( \square \)

**Theorem 2.7.** [CCHS1 Proposition 1.2] Let \( L = \langle X \mid Y \rangle \) be an aspherical group presentation and let \( \overline{Y} = \langle \langle Y \rangle \rangle^{ab} \) be the relation module. Then \( \overline{Y} \) decomposes as a free \( L \)-module into a direct sum of cyclic submodules \( M_y, y \in Y \), where each \( M_y \) is generated by \( \overline{y} = y(\langle Y \rangle, \langle Y \rangle) \).

In particular, \( H_1(\langle \langle Y \rangle \rangle) \) is easily described.

**Remark 2.8.** Let \( L = \langle X \mid Y \rangle \) be an aspherical group presentation. Then

\[
H_1(\langle \langle Y \rangle \rangle) = \bigoplus_{\ell \in L} \bigoplus_{y \in Y} \mathbb{Z}/2\mathbb{Z}(\overline{y}).
\]

**Remark 2.9.** For the remainder of this section, we will assume that we have fixed an aspherically relator separated group \( G = \langle S \mid T \rangle \). We further assume that \( T = \{t_1, \ldots, t_N\} \), where we have fixed for each \( i \) a partition \( t_i = r_i r_i' \), so that \( H(G) = \langle S \mid R \rangle \) for \( R = \{r_1, r_1', \ldots, r_N, r_N' \} \). Using our earlier notation, we write \( G = G_T \) and \( H(G) = K_R \). We construct the double \( T \)-covers: \( \hat{\Delta}_{T,R} \) for \( \Sigma_R \); and \( \hat{H}_{T,R} \) for \( K_R \).
We begin by analysing the group $Q(T, R) = H_1(\langle R \rangle)/\iota(H_1(\langle T \rangle))$. Recall that we have the surjection $\xi : G_T \rightarrow K_R$, and for $\rho \in R$ we defined $\overline{\rho} := \rho[\langle R \rangle, \langle R \rangle]$. Let $\Psi : H_1(\langle R \rangle) \rightarrow Q(T, R)$ be the obvious surjection.

**Lemma 2.10.** For each basis element $k\tau$ of

$$H_1(\langle R \rangle) = \bigoplus_{k \in K_R} \bigoplus_{\rho \in R} \mathbb{Z}/2\mathbb{Z}(k\overline{\rho})$$

as an $\mathbb{F}_2$ vector space, there exists exactly one other basis element $k'\overline{\tau}$ of $H_1(\langle R \rangle)$ with $\Psi(k\tau) = Q(T, R) \Psi(k'\overline{\tau})$. This basis element takes the form $k\overline{\tau}$, where $r' \neq r$ is the unique element of $R$ with $rr' \in T$ or $r'r$ lying in $T$.

Since $K_R$ is infinite, $\langle R \rangle$ is an infinite index normal subgroup of $F(S)$. Therefore, $\langle R \rangle$ is an infinite rank free group, and so $H_1(\Sigma_R) = \langle R \rangle$ is infinite rank. Lemma 2.10 implies that $Q(T, R)$ is of infinite rank, i.e. that it is infinite dimensional as an $\mathbb{F}_2$ vector space.

**Proof.** Using Remark 2.8, we see that $\iota(H_1(\langle T \rangle))$ is the subspace of

$$H_1(\langle R \rangle) = \bigoplus_{k \in K_R} \bigoplus_{\rho \in R} \mathbb{Z}/2\mathbb{Z}(k\overline{\rho})$$

spanned by the set $\mathcal{V} = \{ \xi(g)\overline{\rho} + \xi(g)\overline{\rho'} : g \in G_T, \rho, \rho' \in R, \rho \rho' \in T \}$. We have that $\Psi(k\tau) = Q(T, R) \Psi(k'\overline{\tau})$ if and only if $(k\tau + k'\overline{\tau})$ lies in the span of $\mathcal{V}$. By Lemma 2.6 and Remark 2.8, the vectors in $\mathcal{V}$ are pairwise orthogonal, that is when adding two distinct elements of $\mathcal{V}$, no cancellation occurs. Therefore, $(k\tau + k'\overline{\tau}) \in \text{Span}(\mathcal{V})$ if and only if $k = k'$ and either: $rr' \in T$ or $r'r \in T$; or $r = r'$ (in this latter case $k\tau + k'\overline{\tau} = Q(T, R)$). By Lemma 2.6, there is exactly one $r' \in R$ such that $rr'$ or $r'r$ is an element of $T$. Furthermore, by Lemma 2.6 no $t \in T$ is a proper power, so $r \neq r'$.

We now find cutsets in $\hat{\Delta}_{T, R}$. We will make great use of the following loops.

**Definition 2.11.** Let $r$ be an element of $R$. We define $\sigma_r$ as the loop based at the identity in $\Sigma_R$ and read by $r$.

**Lemma 2.12.** Let $\pi : \hat{\Delta}_{T, R} \rightarrow \Sigma_R$ be the double $T$-cover, and for each $r \in R$ let $\sigma_r$ as the loop based at the identity in $\Sigma_R$ and read by $r$. For each word $t = rr' \in T$, and each $k \in K_R$, the set $\pi^{-1}(k\sigma_r \cup k\sigma_{r'})$ is a separating subset of $\hat{\Delta}_{T, R}$.

Note that, for $t = rr' \in T$, the loop $\sigma_r \sigma_{r'}$ in $\Sigma_R$ lifts to a loop $\delta_{r, r'} = \sigma_r \sigma_{r'}$ in $\hat{\Delta}_{T, R}$; the content of the above lemma is that the orbit of this loop under the deck group forms a cutset. This is illustrated in Figure I
Definition 2.13. Let $t = r r' \in T$ and let $\sigma_r$ (respectively $\sigma_{r'}$) be the loop based at the identity in $\Sigma_R$ and read by $r$ (respectively $r'$). Suppose $\Sigma_R - (\sigma_r \cup \sigma_{r'})$
contains exactly one infinite component. Let $A_{r,r'}$ be the union of $\sigma_r \cup \sigma_{r'}$ together with the finite connected components of $\Sigma_R - (\sigma_r \cup \sigma_{r'})$.

By Lemma 2.12 $\Delta_{T,R} - \pi^{-1}(A_{r,r'})$ contains at least two components. It is easy to see that there exists a finite uniform bound $D$ to the diameter of $A_{r,r'}$ as $rr'$ ranges over the elements of $T$.

**Remark 2.14.** Suppose that $\Sigma_R - (\sigma_r \cup \sigma_{r'})$ contains exactly one infinite component. Then $\Sigma_R - A_{r,r'}$ consists of exactly one component, which is infinite. Therefore, all components of $\Delta_{T,R} - \pi^{-1}(A_{r,r'})$ are infinite.

**Definition 2.15.** Suppose that for $rr' \in T$, $\Sigma_R - (\sigma_r \cup \sigma_{r'})$ contains exactly one infinite component, and let $A_{r,r'}$ be as in Definition 2.12. Let $\mathcal{L}_{r,r'}$ be the finite set of loops in $\Sigma_R$ of the form $k\sigma_r$, where $k \in K_R$, $\rho \in R$, and $k\sigma_r \cap A_{r,r'}$ is non-empty.

There is a uniform upper bound $L$ to the size of $\mathcal{L}_{r,r'}$ as $rr'$ ranges across $T$.

**Remark 2.16.** Suppose that $\Sigma_R - A_{r,r'}$ consists of exactly one component. Then given any $x \in \Sigma_R - A_{r,r'}$, and any component $C$ of $\Delta_{T,R} - \pi^{-1}(A_{r,r'})$, there exists $\hat{x} \in C$ with $\pi(\hat{x}) = x$.

We now define a particular element of $Q(T,R)$.

**Definition 2.17.** Let $rr' \in T$ and suppose that $\Sigma_R - (\sigma_r \cup \sigma_{r'})$ contains exactly one infinite component. Let $A_{r,r'}$ be given as in Definition 2.12. Given $\hat{x}, \hat{y}$ in $\Delta_{T,R} - \pi^{-1}(A_{r,r'})$ with $\pi(\hat{x}) = \pi(\hat{y})$, let $\mathcal{q}(\hat{x}, \hat{y})$ be the element of $Q(T,R)$ defined as follows. Let $\delta$ be a path in $\Delta_{T,R}$ between $\hat{x}$ and $\hat{y}$, so that $\pi(\delta)$ is a loop in $\Sigma_R - A_{r,r'}$, and therefore gives and element $\phi_\delta$ in $(\langle R \rangle)$. The element $\mathcal{q}(x,y)$ is defined as $P_{T,R}(\phi_\delta)$.

Note that the above definition does not depend on the choice of the path $\delta$. This element encodes a lot of information about the components of $\Delta_{T,R} - \pi^{-1}(A_{r,r'})$, as we now show.

**Lemma 2.18.** Let $rr' \in T$ be such that $\Sigma_R - (\sigma_r \cup \sigma_{r'})$ contains exactly one infinite component. Let $A_{r,r'}$ be given as in Definition 2.12, and $\mathcal{q}(x,y)$ be given as in Definition 2.17. Suppose that $\hat{x}, \hat{x}', \hat{y}, \hat{y}'$ are such that:

- $\hat{x}, \hat{x}'$ lie in the same component of $\Delta_{T,R} - \pi^{-1}(A_{r,r'})$, and
- we have $\mathcal{q}(\hat{x}, \hat{y}) = \mathcal{q}(\hat{x}', \hat{y}')$.

Then the points $\hat{y}, \hat{y}'$ lie in the same component of $\Delta_{T,R} - \pi^{-1}(A_{r,r'})$.

**Proof.** Since $\hat{x}$ and $\hat{x}'$ lie in the same component of $\Delta_{T,R} - \pi^{-1}(A_{r,r'})$, there exists a path $\delta(\hat{x}', \hat{x})$ in $\Delta_{T,R}$ from $\hat{x}'$ to $\hat{x}$ not intersecting $\pi^{-1}(A_{r,r'})$. Write $\delta(\hat{x}', \hat{x})$ for the inverse of $\delta(\hat{x}', \hat{x})$.

Let $\delta(\hat{x}, \hat{y})$ be a path in $\Delta_{T,R}$ from $\hat{x}$ to $\hat{y}$ with inverse $\delta(\hat{y}, \hat{x})$, and define similarly $\delta(\hat{x}', \hat{y}')$, $\delta(\hat{y}', \hat{x}')$. 



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Let \( \delta' \) be the lift of \( \pi(\delta(\hat{x}', \hat{x})) \) to \( \hat{\Delta}_{T,R} \) starting at \( \hat{y}' \). This gives a path in \( \hat{\Delta}_{T,R} \) starting at \( \hat{y} \), defined as

\[
\zeta = \delta(\hat{y}, \hat{x})\delta(\hat{x}, \hat{x}')\delta(\hat{x}' \hat{y}')\delta'.
\]

In particular, we go from \( \hat{y} \) to \( \hat{x} \), then backwards along \( \delta(\hat{x}', \hat{x}) \) to \( \hat{x}' \) then to \( \hat{y}' \) and then along \( \delta \). Note that \( \pi(\zeta) \) is a loop in \( \Sigma_R \) and gives an element \( \phi_\zeta \in \langle (R) \rangle \). We similarly obtain elements \( \phi_{\delta(\hat{x}, \hat{x}')} \) etc.

By the definition of \( q(\hat{x}, \hat{y}) \) and by the assumptions in the statement of the lemma,

\[
P_{T,R}(\phi_{\delta(\hat{x}, \hat{y})}) = q(\hat{x}, \hat{y}) = q(\hat{x}' \hat{y}') = P_{T,R}(\phi_{\delta(\hat{x}', \hat{y}')}).
\]

Furthermore, \( \pi(\delta') = \pi(\delta(\hat{x}', \hat{x})) \).

Since we are taking homology with \( \mathbb{Z}/2\mathbb{Z} \) coefficients, we immediately deduce that

\[
P_{T,R}(\phi_\zeta) = Q(T,R) \mathbf{0}
\]

Hence \( \pi(\zeta) \) lies in \( \ker(P_{T,R}) \), and so \( \zeta \) is a loop in \( \hat{\Delta}_{T,R} \). Therefore \( \delta' \) connects \( \hat{y} \) and \( \hat{y}' \) in \( \hat{\Delta}_{T,R} \). Since \( \delta(\hat{x}, \hat{x}') \) does not intersect \( \pi^{-1}(A_{r,r'}) \), neither does \( \delta' \) and so \( \delta' \) is a path between \( \hat{y}' \) and \( \hat{y} \) not intersecting \( \pi^{-1}(A_{r,r'}) \). In particular, \( \hat{y} \) and \( \hat{y}' \) lie in the same component of \( \hat{\Delta}_{T,R} - \pi^{-1}(A_{r,r'}) \). \( \square \)

**Remark 2.19.** Suppose that \( \Sigma_R - (\sigma_r \cup \sigma_r') \) contains exactly one infinite component, so that \( \Sigma_R - A_{r,r'} \) is connected. Consider the components of \( \hat{\Delta}_{T,R} - \pi^{-1}(A_{r,r'}) \), of which there are at least two by Lemma 2.12. Let \( \hat{x} \) be a vertex of \( \hat{\Delta}_{T,R} - \pi^{-1}(A_{r,r'}) \). If the loop \( (k \gamma_p) \) is not an element of \( L_{r,r'} \), let \( \sigma \) be a simple path in \( \Sigma_R - A_{r,r'} \) from \( \pi(\hat{x}) \) to the start point of \( (k \gamma_p) \), which exists as \( \Sigma_R - A_{r,r'} \) is connected. Then the lift of the path \( \sigma(k \gamma_p)\pi^{-1} \) starting at \( \hat{x} \) ends at an element \( \hat{y} \) with \( q(\hat{x}, \hat{y}) = \Psi(k \gamma_p) \).

In particular, by repeatedly applying the above construction, we may deduce the following lemma.

**Lemma 2.20.** Let \( t = tr' \in T \), and let \( \hat{x} \) be a point in \( \hat{\Delta}_{T,R} - \pi^{-1}(A_{r,r'}) \). Let \( q \in Q(T,R) \) with

\[
(q, \Psi(k \sigma_p)) = 0
\]

for all \( k \sigma_p \leq L_{r,r'} \). Then there exists a point \( \hat{y} \) in the same connected component of \( \hat{\Delta}_{T,R} - \pi^{-1}(A_{r,r'}) \) as \( \hat{x} \), and with

\[
q(\hat{x}, \hat{y}) = Q(T,R) q.
\]

Therefore, given a point \( \hat{x} \) in \( \hat{\Delta}_{T,R} - \pi^{-1}(A_{r,r'}) \), we see that the connected component of \( \hat{\Delta}_{T,R} - \pi^{-1}(A_{r,r'}) \) in which \( \hat{x} \) lies is uniquely determined by \( (q(\hat{x}, \hat{y}), \Psi(k \gamma_p)) \), where \( \hat{y} \) ranges over the connected components of \( \hat{\Delta}_{T,R} - \pi^{-1}(A_{r,r'}) \) and \( k \gamma_p \) ranges over \( L_{r,r'} \). In particular, by applying Lemma 2.18, we may deduce the following.
Lemma 2.21. Suppose that $\Gamma_R - (\sigma_r \cup \sigma_{r'})$ contains exactly one infinite component. Let $A_{r,r'}$ be defined as in Definition 2.13 and $L_{r,r'}$ be defined as in Definition 2.15. There are at most $2|L_{r,r'}|$ connected components of $\hat{\Delta}_{T,R} - \pi^{-1}(A_{r,r'})$. 

Now, we begin to discuss the CAT(0) cube complex on which $G_T$ acts. The main element of interest is the following: given a space $X$, a wall is a pair $\{U, V\}$ where $U \cup V = X$.

Remark 2.22. Suppose that $\Sigma_R - (\sigma_r \cup \sigma_{r'})$ contains exactly one infinite component for each $rr' \in T$. Each translate $(k\sigma_r \cup k\sigma_{r'})$ for $k \in K_R$ and $rr' \in T$ determines some finite number, $N_{r,r'}$, of walls

$$A^{i}_{k,r,r'} = \left\{U^{i}_{k,r,r'} \cup \pi^{-1}(kA_{r,r'}), V^{i}_{k,r,r'} \cup \pi^{-1}(kA_{r,r'})\right\},$$

where $\{U^{i}_{k,r,r'}, V^{i}_{k,r,r'}\}$ ranges over the finite number of non-trivial partitions of connected components of $\hat{\Delta}_{T,R} - \pi^{-1}(kA_{r,r'})$. There are finitely many such choices by Lemma 2.21. There is clearly a uniform upper bound $N$ to the number $N_{r,r'}$ as $rr'$ ranges over $T$.

We will say that two walls $\{U, V\}, \{U', V'\}$ are transverse if each of $U \cap U'$, $U \cap V'$, $V \cap U$, $V \cap V'$ are non-empty. A wall $\{U, V\}$ separates two points $x, y$ if $x \in U - V$ and $y \in V - U$ (or vice versa).

Remark 2.23. Note that two walls $A^i_{k,r,r'}$ and $A^j_{k',r',r''}$ are transverse only if $kA_{r,r'} \cap k' \sigma_{r,r'}$ is non-empty. Since $A_{r,r'}$ has diameter bounded above by some uniform constant $D$, there is a finite upper bound to the size of any collection of pairwise intersecting $kA_{r,r'}$. Since each $kA_{r,r'}$ gives a bounded number of walls of the form $A^i_{k,r,r'}$, there is a finite upper bound $M$ to the number of pairwise transverse walls.

Definition 2.24. Suppose that $\Sigma_R - (\sigma_r \cup \sigma_{r'})$ contains exactly one infinite component for each $rr' \in T$. Let $A^i_{k,r,r'}$ be defined as in Remark 2.22. We define the collection of walls of $\hat{\Delta}_{T,R}$,

$$\mathcal{W} = \{A^i_{k,r,r'} : k \in K_R, r, r' \in R, r,r' \in T\}.$$ 

Note that $\hat{H}_{T,R}$ acts on a wall $A^i_{g,r,r'}$, in a clear manner, and $\hat{H}_{T,R} \mathcal{W} = \mathcal{W}$.

Remark 2.25. The graph $\hat{\Delta}_{T,R}$ is locally finite, and so Remark 2.23 on the size of any pairwise transverse collection of $A^i_{k,r,r'}$ implies that the pair $(\hat{\Delta}_{T,R}, \mathcal{W})$ is a wallspace in the language of Hruska–Wise [HW14 §2.2]. There exists a canonical CAT(0) cube complex, $\mathcal{C} = \mathcal{C}(\hat{\Delta}_{T,R}, \mathcal{W})$, on which $\hat{H}_{T,R}$ acts by isometries, built from $(\hat{\Delta}_{T,R}, \mathcal{W})$: see [HW14 §3] for an overview of its construction. Since any collection of pairwise transverse $\Lambda$ in $\mathcal{W}$ has cardinality at most $M$ by Remark 2.23, the cube complex $\mathcal{C}$ is finite dimensional [HW14 Corollary 3.13].

Definition 2.26. For two points $\hat{x}, \hat{y}$ in $\hat{\Delta}_{T,R}$, let $\#(\hat{x}, \hat{y})$ be the (finite) number of $\Lambda \in \mathcal{W}$ such that $\hat{x}$ and $\hat{y}$ are separated by $\Lambda$. 

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Remark 2.27. Given \( \hat{x}, \hat{y} \in \hat{\Delta}_{T,R} \) and \( h \in \hat{H}_{T,R} \) with \( h\hat{x} = \hat{y} \), there exist points \( c\hat{x}, c\hat{y} \in C \) with \( d_C(c\hat{x}, c\hat{y}) = \#(\hat{x}, \hat{y}) \) and \( hc\hat{x} = c\hat{y} \) [HW14, p. 480].

We may now prove Theorem 2B.

**Theorem 2B.** Let \( G \) be an aspherically relator separated group. Then \( G \) acts with unbounded orbits on a finite dimensional CAT(0) cube complex, and hence does not have Property (T).

**Proof of Theorem 2B.** If \( G = \langle S | T \rangle \) is aspherically relator separated, we may assume that \( |S \cup S^{-1}| \geq 4 \), as otherwise \( G \) is cyclic. Let \( G_T, \Gamma_T, K_r, \Sigma_R, \hat{H}_{T,R}, \Delta_{T,R}, \Sigma_R \) be defined by our choice of aspherically relator separated group \( G = \langle S | T \rangle \) and our fixed partition of each element \( t_i = r_i r'_i \in T \).

For \( r \in R \), let \( \sigma_i \) be defined as in Definition 2.11 i.e. the loop in \( \Sigma_R \) based at the identity and read by \( r_i \). First suppose that \( \Sigma_R - (\sigma_i \cup \sigma_i') \) contains exactly one infinite component for each \( r r' \in T \). Let \( A_{r,r'}, A_{r,r',r_i} ', \Lambda \) be as described in Lemma 2.12, Lemma 2.21 and Remark 2.22. Consider the CAT(0) cube complex \( C(\Sigma, W) \).

We know that \( \Sigma_R - A_{r,r'} \) is connected for each \( r r' \in T \). As discussed previously in Remark 2.22 for each \( k \in K_R \) and \( r r' \in T \), the set \( k A_{r,r'} \) gives us the finite collection of walls \( \{ A_{r,r'} \} \).

Let \( n \geq 1 \). Since \( K_R \) (and therefore \( \Sigma_R \)) is infinite, and \( K_R \) has at least two generators, we may choose a simple path \( \sigma \) with endpoints \( x, y \in V(\Sigma_R) \) such that: after removing loops from \( \sigma \) the resulting path is simple; there exist translates \( k_i A_{r_i r_i'} \), \( k_i A_{r_i r_i'} \), \( \ldots \), \( k_n A_{r_n r_n'} \), \( \ldots \) pairwise disjoint such that \( \sigma \) contains exactly one loop from each of \( \{ k_i \sigma_{r_i}, k_i \sigma_{r_i'} \}, \ldots , \{ k_n \sigma_{r_n}, k_n \sigma_{r_n'} \}; \)

and no other loops appear in \( \sigma \). Then \( \sigma \) lifts to a path \( \hat{\sigma} \) in \( \hat{\Delta}_{T,R} \) with endpoints \( \hat{x}, \hat{y} \) in \( V(\hat{\Delta}_{T,R}) \). By Lemma 2.12 \( \hat{x} \) and \( \hat{y} \) are separated by the walls \( \Lambda_{k_i r_i r_i'} \), \( \Lambda_{k_i r_i r_i'} \), \( \ldots \), \( \Lambda_{k_n r_n r_n'} \), \( \ldots \) for appropriate choices of \( j_1, \ldots , j_n \), so that \( \#(\hat{x}, \hat{y}) \geq n \). As \( \hat{\Delta}_{T,R} \) is the Cayley graph of \( \hat{H}_{T,R} \), there exists \( h \in \hat{H}_{T,R} \) with \( h\hat{x} = \hat{y} \). Therefore, by Remark 2.27, for all \( n \geq 1 \) there exist points \( c\hat{x}, c\hat{y} \in C \) and \( h c\hat{x} = c\hat{y} \), and so \( \hat{H}_{T,R} \) acts on \( C \) with unbounded orbits.

If the unbounded action of \( \hat{H}_{T,R} \) on \( C \) is given by the homomorphism \( \alpha : \hat{H}_{T,R} \to Isom(C) \), then \( G = G_T \) also acts with unbounded orbits on \( C \), via the action \( \alpha : G \to Isom(C) \).

If \( \Sigma_R - (\sigma_r \cup \sigma_r') \) contains at least two infinite components for some \( r r' \in T \), then we may repeat the above argument using \( \Sigma_R - (\sigma_r \cup \sigma_r') \) in place of \( \Delta_{T,R} - \pi^{-1}(\sigma_r \cup \sigma_r') \) to build a wallspace \( (\Sigma_R, W') \). In particular, for each \( k \in K_R \), we take the partitions of the (necessarily finite number of) components of \( \Sigma_R - k(\sigma_r \cup \sigma_r') \) as our walls, which gives us the wallspace \( (\Sigma_R, W') \). We then similarly deduce that the group \( K_R \) acts with unbounded orbits on the CAT(0) cube complex \( C(\Sigma, W') \). \( \square \)
3 Random groups are aspherically relator separated

We now turn to proving Lemma 3.8. We first introduce some auxiliary models of random groups to model the behaviour of \( \mathcal{H}(G) \) for \( G \sim G(n, l, d) \). For \( t, t'^{-1} \in S \cup S^{-1} \), let \( W(t, t') \) be the set of words in \( F(S) \) starting with \( t \) and ending with \( t' \).

Essentially we have the following issue. Given \( G = \langle S|T \rangle \sim G(n, 2l, d) \), we would like to say that \( \mathcal{H}(G) = \langle S|R \rangle \sim G(n, l, 2d) \). Recall that given \( T = \{ t_1, \ldots, t_N \} \), we will fix for each \( i \) the partition \( t_i = r_i r_i' \) with \( |r_i| = |r_i'| \) if \( |t_i| \) is even, and \( |r_i| = |r_i| - 1 \) otherwise. In particular, \( \mathcal{H}(G) = \langle S|R \rangle \) for \( R = \{ r_1, r_1', \ldots, r_N, r_N' \} \) for \( r_i, r_i' \) as described.

However, suppose that there exists some word \( r \in R \cap W(t_1, t_1') \). Then there exists a reduced word \( r' \in R \) such that the word \( t = rt \) is cyclically reduced without cancellation and lies in \( T \) (or possibly \( t = r'r \)). Therefore \( r' \in W(t_2, t_2') \) for some \( t_2^{-1} \neq t_1', t_1^{-1} \neq t_2' \). In particular, the relator set for \( \mathcal{H}(G) \) is not chosen uniformly at random. We are therefore required to take a slight diversion for technical purposes. In essence, we show that we can extend the relator set \( R \) slightly to get a ‘random enough’ group presentation. We define:

- \( \Theta(n, l, d) \) by choosing uniformly at random a set \( R_l \) of \( 2(2n-1)^ld \) reduced words of length \( l \), and considering \( H = F_n/\langle\langle R_l \rangle\rangle \),

- \( \Omega(n, l, d) \) by choosing uniformly at random a set \( R_l \) of \( (2n-1)^ld \) reduced words of length \( l+1 \), and considering \( H = F_n/\langle\langle R_l \cup R_{l+1} \rangle\rangle \),

- \( \overline{\Omega}(n, l, d) \) by choosing \( \langle S|R_l \rangle \sim \Theta(n, l, d) \) conditioned on, for all \( t, t^{-1} \), \( |R_l \cap W(t, t')| = 2(2n-1)^ld(2n)^{-2} \),

- \( \overline{\Omega}(n, l, d) \) by choosing \( \langle S|R_l \cup R_{l+1} \rangle \sim \Omega(n, l, d) \) conditioned on, for all \( t, t^{-1} \), \( |R_l \cap W(t, t')| = |R_{l+1} \cap W(t, t')| = (2n-1)^ld(2n)^{-2} \).

Let \( f, g : \mathbb{N} \to \mathbb{R}_+ \): we write \( f = o_m(g) \) if \( f(m)/g(m) \to 0 \) as \( m \to \infty \), and \( f = O_m(g) \) if there exist constants \( N \geq 0 \) and \( M \geq 1 \) such that \( f(m) \leq Ng(m) \) for all \( m \geq M \). We now discuss spherical diagrams.

**Definition 3.1.** Let \( G = \langle S|R \rangle \) be a group. A spherical diagram for \( G \) is a finite cellular decomposition of the sphere, \( D \), such that:

i) Each (oriented) 1-cell of \( D \) is labelled by some \( s \in S \) such that if \( e \) is labelled by \( s \) then \( e^{-1} \) is labelled by \( s^{-1} \).

ii) Each 2-cell \( C \) has a marked start point and orientation. Reading the boundary of \( C \) from the marked point and concatenating edge labels, the word obtained is reduced without cancellation and is equal to some \( r \in R \).

A spherical diagram \( D \) is unreduced if \( D \) contains 2-cells \( C_1, C_2 \), such that: \( C_1 \) and \( C_2 \) bear the same relator with opposite orientations; and \( C_1 \) and \( C_2 \) both
contain an edge $e \in D$ that represents the same letter in the relator (with respect to marked start points and orientations).

**Definition 3.2.** Following Gersten [Ger87], a presentation $G = \langle S | R \rangle$ is diagramatically reducible if every spherical diagram for $G$ is either unreduced or consists of a single point.

Note that being diagramatically reducible is preserved by removing relators from the set $R$. Importantly, if $G$ is diagramatically reducible, then it is aspherical [Ger87, Remark 3.2]. We have the following theorem due to Ollivier [Oll04]. This was originally proved only for the $G(n, l, d)$ model, but the result follows similarly for $\Theta(n, l, d)$ and $\Omega(n, l, d)$ (see [Oll05, §I.2.c] or [OW11, Remark 1.4]).

**Theorem 3.3.** [Oll04, pp. 614–615] Fix $d < 1/2$ and let $G \sim G(n, l, d)$, $G \sim \Theta(n, l, d)$, or $G \sim \Omega(n, l, d)$. Then w.o.p. $G$ is infinite and diagramatically reducible.

Using this, we may prove the following two lemmas. We will use the Chebyshev inequality [Tch67]: if $X$ is a random variable with finite expected value $\mu$ and finite variance $\sigma^2$, then for any $\kappa > 0$,

$$P(|X - \mu| \geq \kappa \sigma) \leq \kappa^{-2}.$$ 

**Lemma 3.4.** Fix $d < 1/2$ and let $G = \langle S | R \rangle \sim \Theta(n, l, d)$, or $G = \langle S | R \rangle \sim \Omega(n, l, d)$, conditioned on $\frac{1}{3}$

for all $t, t'$. Then w.o.p. $G$ is infinite and diagramatically reducible.

**Proof.** We prove this for $\Theta(n, l, d)$ only, since the proof is similar for $\Omega(n, l, d)$. First, let $R_t$ be a uniformly random choice of set of $2(2n - 1)^ld(2n)^{-2}$ reduced words of length $l$ in $F_n$, so that $\langle S | R_t \rangle \sim G(n, l, d)$. Fix $t, t^{-1}$. Define the random variable $X_{t, t'} = |R_t \cap W(t, t')|$. Then $X_{t, t'}$ has expectation $\mu = 2(2n - 1)^ld(2n)^{-2}$ and variance $\sigma^2 = O_l(\mu)$. Therefore, taking $\kappa = \kappa(l) = \mu^{1/100}$, by applying the Chebyshev inequality, we see that $P(|X_{t, t'} - \mu| \geq \kappa \sigma) \leq \kappa^{-2}$. In particular

$$|X_{t, t'} - \mu| \geq \kappa \sigma = O_l(\mu^{51/100})$$

with probability less than or equal to $\mu^{-2/100}$. Therefore

$$|X_{t, t'} - 2(2n - 1)^ld(2n)^{-2}| \leq (2n - 1)^{3ld/4}$$

with probability tending to 1 as $l$ tends to infinity. Taking the finite intersection of events across all pairs $t, t'$, it follows that for all $t, t'$ w.o.p.

$$|R_t \cap W(t, t')| - 2(2n - 1)^ld(2n)^{-2} \leq (2n - 1)^{3ld/4}.$$
In particular, we have shown that the set of group presentations obtainable by sampling from $\Theta(n, l, d)$ conditioned on $|R \cap W(t, t')| - 2(2n - 1)^{ld}(2n)^{-2} \leq (2n - 1)^{3d/4}$ for all $t, t'$ is of density $1 - o_l(1)$ in the set of group presentations obtainable by sampling from $\Theta(n, l, d)$.

By Theorem 3.3 we have that the set of group presentations obtainable by sampling from $\Theta(n, l, d)$ conditioned on the group being infinite and diagramatically reducible is also of density $1 - o_l(1)$ in the set of group presentations obtainable by sampling from $\Theta(n, l, d)$.

Taking the intersection of two subsets of density $1 - o_l(1)$ also gives a subset of density $1 - o_l(1)$, so the conclusion of the Lemma holds. In particular, given $G = \langle S | R \rangle \sim \Theta(n, l, d)$, conditioned on $|R \cap W(t, t')| - 2(2n - 1)^{ld}(2n)^{-2} \leq (2n - 1)^{3d/4}$ for all $t, t'$, we have that $G$ is w.o.p. infinite diagramically reducible.

Lemma 3.5. Fix $d < 1/2$ and let $H = \langle S | R \rangle \sim \ovl{\Theta}(n, l, d)$ or $H \sim \ovl{\Omega}(n, l, d)$. Then w.o.p. $H$ is infinite and diagramatically reducible.

Proof. We prove this for $\ovl{\Theta}(n, l, d)$ only; the proof is similar for $\ovl{\Omega}(n, l, d)$.

Let $R_l$ be a randomly chosen set of reduced words of length $l$ in $F_n$, conditioned on $|R_l \cap W(t, t')| = (2n - 1)^{ld}(2n)^{-2}$ for all $t, t'$. Let $d < d' < 1/2$, and let $R_l \subseteq R_l'$ be a randomly chosen set of reduced words of length $l$ in $F_n$, conditioned on $|R_l' \cap W(t, t')| - 2(2n - 1)^{ld'}(2n)^{-2} \leq (2n - 1)^{3d'/4}$ for all $t, t'$. Note that the distribution of $R_l'$ is uniform, so that $\langle S | R_l' \rangle \sim \Theta(n, l, d')$, conditioned on $|R_l' \cap W(t, t')| - 2(2n - 1)^{ld'}(2n)^{-2} \leq (2n - 1)^{3d'/4}$ for all $t, t'$. By Lemma 3.4 the group $\langle S | R_l' \rangle$ is w.o.p. infinite and diagramatically reducible. Being infinite and diagramatically reducible is preserved by removing relators, and so $H = \langle S | R_l \rangle$ is w.o.p. infinite and diagramatically reducible.

We now note a result on small cancellation in random groups: we refer the reader to [LS15, p. 240] for the definition of the $C'(\lambda)$ condition.

Lemma 3.6. [Oll07, Corollary 3] Let $d < 1/2$ and let $G \sim G(n, l, d)$. Then w.o.p. $G$ is $C'(2d)$. 

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We now analyse the group \( \mathcal{H}(G) \).

**Lemma 3.7.** Let \( d < 1/4 \) and \( \delta < 1/4 - d \). Let \( G \sim G(n, l, d) \), and let \( \mathcal{H}(G) = \langle S|R \rangle \). Then w.o.p. there exists a set \( R \subseteq R' \) such that \( \langle S|R' \rangle \sim \mathcal{O}(n, l/2, 2d + \delta) \) if \( l \) is even, and \( \langle S|R' \rangle \sim \mathcal{O}(n, (l - 1)/2, 2d + \delta) \) if \( l \) is odd.

**Proof.** We prove the first case only. Let \( G = \langle S|T \rangle \) and \( H(G) = \langle S|R \rangle \). By Lemma 3.6 w.o.p. \( R \) consists of exactly \( 2(2n - 1)^l d \) reduced words of length \( l/2 \). By the Chebyshev inequality, w.o.p. for each \( t, t'-1 \),

\[
|R \cap W(t, t')| - 2(2n - 1)^l d (2n)^{-2} \leq (2n - 1)^{3l/4} \]

so that w.o.p. for each \( t, t'-1 \),

\[
|R \cap W(t, t')| \leq (2n - 1)^{(2d + \delta)/2 (2n)^{-2}}.
\]

We may therefore choose uniformly at random a set \( R \subseteq R' \) with \( |R' \cap W(t, t')| = 2(2n - 1)^{(2d + \delta)/2 (2n)^{-2}} \) for all \( t, t' \), so that \( \langle S|R' \rangle \sim \Theta(n, l/2, 2d + \delta) \).

Using these results, we may immediately deduce Lemma 3.8.

**Lemma 3.8.** Let \( G \) be a random group at density less than \( 1/4 \). Then, with overwhelming probability, \( G \) is aspherically relator separated.

**Proof of Lemma 3.8.** Let \( d < 1/4 \), and \( G \sim G(n, l, d) \). Then \( G \) is w.o.p. infinite and diagramatically reducible by Theorem 3.3 and therefore infinite and aspherical w.o.p. by \cite[Remark 3.2]{Ger87}. If \( \mathcal{H}(G) = \langle S|R \rangle \) then w.o.p. there exists a set \( R' \) such that \( R \subseteq R' \) and \( \langle S|R' \rangle \) is infinite and diagramatically reducible by Lemma 3.7 and Lemma 3.7. Since both of these properties are preserved by removing relators, it follows that \( \mathcal{H}(G) \) is w.o.p. infinite and diagramatically reducible, hence infinite and aspherical w.o.p. by \cite[Remark 3.2]{Ger87}.

### 3.1 Extension to the \( k \)-angular model

The proof of Theorem D follows an identical method to the above proof of Theorem A, obtained by simply swapping results in the density model for the analogous results in the \( k \)-angular model. We list the replacements below.

The following theorem, which we use in place of Theorem 3.3, was originally only proved for \( G(n, l, d) \). However, the proof strategy of \cite[Theorem 3.11]{ARD20} can be extended to the models \( \Theta(n, l, d) \) and \( \Omega(n, l, d) \) with little alteration. We therefore omit the details of the proof of the following, and simply cite the result.

**Theorem 3.9.** \cite[Theorem 3.11]{ARD20} Let \( k \geq 2 \), \( d < 1/2 \), and let \( G \sim G(n, k, d) \), \( G \sim \Theta(n, k, d) \), or \( G \sim \Omega(n, k, d) \). Then w.o.p. \( G \) is infinite and diagramatically reducible.
Next, we use the following in place of Lemma 3.6, which follows by an application of the isoperimetric inequality of random $k$-angular groups, due to Odrzygóźdź.

**Lemma 3.10.** [Odr19, Theorem 2.6] Let $d < 1/2$ and let $G \sim G(n, k, d)$. Then w.o.p. $(n)$ $G$ is $C'(2d)$.

Finally, in place of Lemma 3.7, we make the following observation.

**Remark 3.11.** The main difference between $H(G) = \langle S | R \rangle$ for $G$ a random group in the density model, as opposed to the $k$-angular model, is the following. Let $G \sim (n, k, d)$ for $k$ even. Suppose that $r = r_1 \ldots r_{k/2} \in R$. Then we must have a word $r' = r'_1 \ldots r'_{k/2} \in R$ where $r'_1 \neq r^{-1}_{k/2}$, $r'_k/2 \neq r^{-1}_1$. In the case of a fixed number of generators, i.e. where we view $G$ as a random group in the density model (so that $k \to \infty$) there are approximately $(2n - 1)^{k/2}$ choices of such a word $r'$. However, there are $2n(2n - 1)^{kd - 1}$ reduced words of length $k$ that could be chosen from to obtain a random group in $\Theta(n, k/2, 2d)$. In particular, the relator set for $H(G)$ is not chosen independently.

Now suppose that we are in the case that $n \to \infty$, so that we view $G$ as a random group in the $k$-angular model. There are again approximately $(2n - 1)^{k/2}$ choice of $r' \in R$ with $rr' \in T$. Since $(2n - 1)^{k/2}/2n(2n - 1)^{k/2 - 1} \to 1$, the relator set of $H(G)$ is chosen asymptotically independently, so that we can view $H(G)$ as having the (asymptotic) distribution of $\Theta(n, k/2, 2d)$.

If $k$ is odd, then we may similarly deduce that $H(G)$ has the asymptotic distribution of $\Omega(n, (k - 1)/2, 2kd/(k - 1))$.

The proof of Theorem D now follows similarly to our proof of Theorem A.

**References**

[Ago13] Ian Agol. The virtual Haken conjecture. *Doc. Math.*, 18:1045–1087, 2013. With an appendix by Agol, Daniel Groves, and Jason Manning.

[ARD20] Calum J. Ashcroft and Colva M. Roney-Dougal. On random presentations with fixed relator length. *Comm. Algebra*, 48(5):1904–1918, 2020.

[Ash21] Calum J Ashcroft. Property (T) in density-type models of random groups. *arXiv preprint arXiv:2104.14986*, 2021.

[Ash22] Calum J Ashcroft. Property (T) in random quotients of hyperbolic groups at densities above $1/3$. *arXiv preprint arXiv:2202.12318*, 2022.

[CCH81] Ian M. Chiswell, Donald J. Collins, and Johannes Huebschmann. Aspherical group presentations. *Mathematische Zeitschrift*, 178(1):1–36, 1981.

[dLdlS21] Tim de Laat and Mikael de la Salle. Banach space actions and $L^2$-spectral gap. *Analysis & PDE*, 14(1):45–76, 2021.
(DM19) Cornelia Druțu and John M. Mackay. Random groups, random graphs and eigenvalues of p-laplacians. *Advances in Mathematics*, 341:188–254, 2019.

(Duo17) Yen Duong. *On Random Groups: The Square Model at Density d < 1/3 and as Quotients of Free Nilpotent Groups*. ProQuest LLC, Ann Arbor, MI, 2017. Thesis (Ph.D.)—University of Illinois at Chicago.

(Ger87) Steve M Gersten. Reducible diagrams and equations over groups. In *S. M. Gersten, editor, Essays in Group Theory*, volume 8 of *Mathematical Sciences Research Institute Publications*, pages 15–73. Springer, NY, 1987.

(Gro87) Mikhael Gromov. Word hyperbolic groups. In *S. M. Gersten, editor, Essays in Group Theory*, volume 8 of *Mathematical Sciences Research Institute Publications*, pages 75–264. Springer, NY, 1987.

(Gro93) M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

(HW08) F. Haglund and D.T. Wise. Special cube complexes. *Geometric and Functional Analysis*, 17(5):1551–1620, 2008.

(HW14) G.C. Hruska and D.T. Wise. Finiteness properties of cubulated groups. *Compositio Mathematica*, 50(3):453–506, 2014.

(KK13) Marcin Kotowski and Michał Kotowski. Random groups and property (T): Żuk’s theorem revisited. *Journal of the London Mathematical Society*, 88(2):396–416, 2013.

(LS15) Roger C Lyndon and Paul E Schupp. *Combinatorial Group Theory*, volume 89 of *Classics in Mathematics*. Springer Berlin / Heidelberg, Berlin, Heidelberg, 2015.

(Mon21) Murphy Kate Montee. Random groups at density $d < 3/14$ act non-trivially on a CAT(0) cube complex. *arXiv preprint arXiv:2106.14931*, 2021.

(Mon22) Murphy Kate Montee. Property (t) in k-gonal random groups. *Glasgow Mathematical Journal*, page 1–5, 2022.

(MP15) John M Mackay and Piotr Przytycki. Balanced walls for random groups. *The Michigan Mathematical Journal*, 64:397–419, 2015.

(NR97) G. Niblo and L. Reeves. Groups acting on CAT(0) cube complexes. *Geometry & Topology*, 1(1):1–7, 1997.

(Odr18) Tomasz Odrzygódz. Cubulating random groups in the square model. *Israel Journal of Mathematics*, 227(2):623–661, 2018.
[Odr19] Tomasz Odrzygózdź. Bent walls for random groups in the square and hexagonal model. arXiv preprint arXiv:1906.05417, 2019.

[Oll04] Y. Ollivier. Sharp phase transition theorems for hyperbolicity of random groups. Geometric and Functional Analysis, 14(3):595–679, 2004.

[Oll05] Yann Ollivier. A January 2005 invitation to random groups, volume 10 of Ensaios Matemáticos [Mathematical Surveys]. Sociedade Brasileira de Matemática, Rio de Janeiro, 2005.

[Oll07] Yann Ollivier. Some small cancellation properties of random groups. International Journal of Algebra and Computation, 17(01):37–51, 2007.

[Opp21] Izhar Oppenheim. Banach Żuk’s criterion for partite complexes with application to random groups. arXiv preprint arXiv:2112.02929, 2021.

[Osa18] Damian Osajda. Group cubization. Duke Mathematical Journal, 167(6):1049–1055, 2018.

[OW11] Y. Ollivier and D.T. Wise. Cubulating random groups at density less than 1/6. Transactions of the American Mathematical Society, 363(9):4701–4733, 2011.

[Sab58] Gert Sabidussi. On a class of fixed-point-free graphs. Proceedings of the American Mathematical Society, 9(5):800–804, 1958.

[Sag95] M. Sageev. Ends of group pairs and non-positively curved cube complexes. Proceedings of the London Mathematical Society, 363(9):585–617, 1995.

[Tch67] P. Tchébychef. Des valeurs moyennes (Traduction du russe, N. de Khanikof. Journal de Mathématiques Pures et Appliquées, pages 177–184, 1867.

[Wis21] Daniel T Wise. The structure of groups with a quasiconvex hierarchy. Princeton University Press, 2021.

[Ż03] A. Żuk. Property (T) and Kazhdan constants for discrete groups. Geometric and Functional Analysis, 13(3):643–670, 2003.

DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WB, UK. E-mail address: cja59@dpmms.cam.ac.uk