¿How do 9 points looks like in $\mathbb{E}^3$?

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Abstract

The aim of this note is to give an elementary proof of the following fact: given 3 red convex sets and 3 blue convex sets in $\mathbb{E}^3$, such that every red intersects every blue, there is a line transversal to the reds or there is a line transversal to the blues. This is a special case of a theorem of Montajano and Karasev [3] and generalizes, in a sense, the colourful Helly theorem due to Lovász (cf. [2]).

1 Introduction.

Consider 9 points in the euclidian 3-dimensional space $\mathbb{E}^3$, and order them, by free will, in a $3 \times 3$ matrix:

$$
\begin{bmatrix}
\mathbb{E}^3 & \mathcal{U} & \mathcal{V} & \mathcal{W} \\
\mathcal{A} & \begin{pmatrix} a & \alpha & x \\ b & \beta & y \\ c & \gamma & z \end{pmatrix} \\
\mathcal{B} & \\
\mathcal{C} & 
\end{bmatrix}
$$

Now, consider the convex hull of the triples of its rows, say 3 blue triangles $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, and of its columns, say 3 red triangles $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$; observe that each red triangle intersect each blue triangle.

The aim of this note is to exhibit an elementary proof of the following fact:

Theorem. There is a line transversal to the red triangles, or there is a line transversal to the blue triangles.

This was first observed by Luis Montejano [4] and later generalised in collaboration with Roman Karasev [3]. Their proof uses multiplication formulas for Schubert cocycles, the Lusternik-Schnirelmann category of the Grassmannian, different versions of the colorful Helly theorem by Báránny and Lovász,
and a bit of Separoid Theory (see [1, 5]). In contrast, here I only use very elementary facts to prove this case: the non-planarity of the graph $K_{3,3}$, the fact that given $d + 2$ convex sets in $\mathbb{R}^n$ either they admit a $d$-transversal or every subset of them is separated from its complement, and the following result—which is interesting on its own— which can be proved using only basic properties of the interior product (Section 3).

**Basic Lemma.** Let $H_A^+, H_U^+, H_W^+$ and $H_C^+$ be four affine semispaces in $\mathbb{E}^n$, with unitary normal vectors $A, U, W$ and $C$, respectively. If

$$(H_A^+ \cap H_U^+) \setminus (H_W^+ \cup H_C^+) \neq \emptyset,$$

and

$$(H_W^+ \cap H_C^+) \setminus (H_A^+ \cup H_U^+) \neq \emptyset,$$

then

$$\text{pos}(A, U) \cap \text{pos}(W, C) = \emptyset,$$

Here, and in the sequel, $\text{pos}()$ denotes the positive span.

## 2 Proof of Theorem.

Consider 3 red triangles and 3 blue triangles in $\mathbb{E}^3$ as in the Introduction (each red intersects each blue), and suppose that there are no line transversals neither to the red, nor the blue. Then, each red triangle is separated by a plane from the other two red triangles, and each blue triangle is separated too — each triangle must be disjoint from the convex hull of the other two triangles of the same color. Let us denote by $H_A^+$ the semispace, with (blue) normal vector $A$, that is the witness of the separation $A \mid B, C$; that is, $A \subset H_A^+$ and $(B \cup C) \cap H_A^+ = \emptyset$. Analogously, we consider the witnesses of all separations, and take the 3 blue vectors $A, B$ and $C$, and the 3 red vectors $U, V$ and $W$ in the unitary sphere of $\mathbb{E}^3$.

Now, join each blue vector, with a spherical segment, to each red vector and, abusing the notation, name the segment with that point of the original configuration that corresponds to the intersection of the respective triangles; e.g., the unitary vectors $A$ and $U$ are joined with the spherical segment $a$ while the unitary vectors $W$ and $C$ are joined with the spherical segment $z$. We have just drawn $K_{3,3}$ in the 2-sphere, therefore there must be a crossing, which we may suppose is the crossing of the spherical segments $a$ and $z$.

However, we have that our original points $a$ and $z$ satisfy

$$a \in (H_A^+ \cap H_U^+) \setminus (H_W^+ \cup H_C^+),$$

$$z \in (H_W^+ \cap H_C^+) \setminus (H_A^+ \cup H_U^+),$$
and
\[ z \in (H_W^+ \cap H_C^+) \setminus (H_A^+ \cup H_U^+), \]
thus the crossing contradicts the Basic Lemma.

3 Proof of Lemma.

Let
\[ a \in (H_A^+ \cap H_U^+) \setminus (H_W^+ \cup H_C^+), \]
\[ z \in (H_W^+ \cap H_C^+) \setminus (H_A^+ \cup H_U^+), \]
and suppose
\[ \eta \in \text{pos}(A, U) \cap \text{pos}(W, C). \]

Then, there exist \( i, j, r, s > 0 \) such that \( \eta = iA + jU = rW + sC. \)

We may suppose that \( 0 \in H_A \cap H_U \) and \( p \in H_W \cap H_C \) —here \( H_A \) denotes
the hyperplane bounding \( H_A^+ \), and respectively with \( H_U \) and the others.
Then, the hypothesis of the lemma can be rewritten as the following eight
inequalities:
\[
\begin{align*}
    a \cdot A &> 0 & z \cdot A &< 0 \\
    a \cdot U &> 0 & z \cdot U &< 0 \\
    (a - p) \cdot W &< 0 & (z - p) \cdot W &> 0 \\
    (a - p) \cdot C &< 0 & (z - p) \cdot C &> 0
\end{align*}
\]

But, on the one hand we have that \( \eta \cdot a = (iA + jU) \cdot a > 0 \) and \( \eta \cdot (a - p) = (rW + sC) \cdot (a - p) < 0 \), therefore
\[ 0 < \eta \cdot a < \eta \cdot p; \]
on the other hand we have that \( \eta \cdot z = (iA + jU) \cdot p < 0 \) and \( \eta \cdot (z - p) = (rW + sC) \cdot (z - p) > 0 \), therefore
\[ 0 > \eta \cdot z > \eta \cdot p, \]
a clear contradiction which arises from the supposition of the existence of \( \eta. \)

4 Remarks and Open Problems.

It is easy to see that the basic lemma can be generalised to arbitrary large
hyperplane pencils — the proof of this is totally analogous to that in Section 3.
Lemma Consider two pencils of semispaces $\mathcal{R} = \{H_{v_i}^+ : i = 1, \ldots, r\}$ and $\mathcal{S} = \{H_{u_j}^+ : j = 1, \ldots, s\}$, with normal unitary vectors $\{v_1, \ldots, v_r\}$ and $\{u_1, \ldots, u_s\}$, respectively. If $\cap_i \mathcal{R} \setminus \cup_i \mathcal{S} \neq \emptyset$ and $\cap_i \mathcal{S} \setminus \cup_i \mathcal{R} \neq \emptyset$, then $\text{pos}(v_1, \ldots, v_r) \cap \text{pos}(u_1, \ldots, u_s) = \emptyset$.

But, how can we use this to prove the general case? What about the special case $4 \times 3$

$$\begin{align*}
\mathbb{R}^4 & \quad U \quad V \quad W \\
A & \begin{pmatrix} a & \alpha & x \\ b & \beta & y \\ c & \gamma & z \\ d & \delta & w \end{pmatrix} & \\
\end{align*}$$

Here the conclusion is that: there exists a 2-plane transversal to the four triangles, or a line transversal to the three tetrahedra — recall that, given 4 convex sets, either there is a 2-plane transversal to those 4 sets, or each of them is separated from the other three and each pair is separated from the other two (cf. the definition of Radon dimension in [5]).

For, consider the uniform rank 3 hypergraph $K_{4,3}^3 = (R \cup B, \Delta)$ consisting of seven vertices, four blue and three red, and all those triangles which can be formed by taking two blue vertices and one red. Observe that $K_{4,3}^3$ cannot be embedded into the 3-sphere without crossings; the link of each blue vertex is a drawing of a $K_{3,3}$ in the 2-sphere.

There are other two lines of research which deserves attention:
— Is the Theorem true in the context of oriented matroids? That is, can it be settled for all oriented matroids of order 9 and rank 4? (In this context, we may need to change the notion of line transversal to the coloured triangles by the convex hull of a coloured triangle is not separated from the convex hull of the others.)
— Is the basic lemma true in such a context? That is, can we change “semispaces” by “pseudosemispaces”?

The first question seems easy to check — there are 9,276,601 uniform oriented matroids to be checked — while the second may deserve a deeper understanding of the topological representation theorem.
References

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