\textbf{$\mathcal{P}\mathcal{T}$-symmetric non-commutative spaces with minimal volume uncertainty relations}

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Received 22 May 2012, in final form 23 July 2012
Published 31 August 2012
Online at stacks.iop.org/JPhysA/45/385302

\textbf{Abstract}
We provide a systematic procedure to relate a three-dimensional $q$-deformed oscillator algebra to the corresponding algebra satisfied by canonical variables describing non-commutative spaces. The large number of possible free parameters in these calculations is reduced to a manageable amount by imposing various different versions of $\mathcal{P}\mathcal{T}$-symmetry on the underlying spaces, which are dictated by the specific physical problem under consideration. The representations for the corresponding operators are in general non-Hermitian with regard to standard inner products and obey algebras whose uncertainty relations lead to minimal length, areas or volumes in phase space. We analyze in particular one three-dimensional solution which may be decomposed into a two-dimensional non-commutative space plus one commuting space component, and also into a one-dimensional non-commutative space plus two commuting space components. We study some explicit models on these types of non-commutative spaces.

PACS numbers: 11.10.Nx, 03.65.Fd

\section{1. Introduction}

The simplest and most commonly studied version of non-commutative spaces replaces the standard set of commuting coordinates by new ones obeying $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, with $\theta^{\mu\nu}$ being a constant antisymmetric tensor. However, even in the very first proposals on non-commutative spaces [1] the tensor $\theta^{\mu\nu}$ was taken to be a function of the position coordinates, i.e. $\theta^{\mu\nu}(x)$. Further possibilities arise when one breaks the Lorentz invariance of the tensor and allows for a general dependence of position and momenta [2–5]. It is known for some time that such a scenario leads to the interesting versions of a generalized version of Heisenberg’s uncertainty relations [2, 3]. In particular, when the commutation relations are modified in such a way that
their structure constants involve higher powers of the momenta or coordinates, one encounters minimal lengths or momenta, respectively. As a consequence, these types of relations lead to more radical changes in the interpretation of possible measurements than the conclusions usually drawn from the standard relations, whereas the conventional relations, which simply have Planck’s constant $\hbar$ as a structure constant, only prevent that two quantities commuting in this manner, e.g. $x$ and $p$, can be known simultaneously; the modified versions prohibit that the observables can be known below a certain scale, the minimal length or minimal momentum. This scale is usually identified to be of the order of the Planck scale. Combining some of these minimal lengths in two dimension leads to minimal areas and in three dimensions to minimal volumes. The need for such type of non-commutative space structures has arisen in many contexts, such as for instance in certain string theories [6] and models investigating gravitational stability [7]. For a general review on non-commutative quantum mechanics see for instance [8] and for a review on non-commutative quantum field theories see [9, 10].

By now many studies on the structure of such type of generalized canonical relations have been carried out [2, 3, 11–25], albeit mostly in dimensions less than 3. Besides leading to different physical results, a further crucial difference between the Snyder-type non-commutative spaces and those with broken Lorentz invariance is the way they are constructed. The former spaces can be thought of as arising naturally from deformations based on general twists [26, 27], whereas the construction of the latter is less systematic and is usually based on the deformation of oscillator algebras [28, 11, 3]. In [16, 18, 19], it was shown in one and two dimensions, respectively, how to map $q$-deformed oscillator algebras onto canonical variables. The main purpose of this paper is to extend these considerations to the full three-dimensional space. This approach has the advantage that it allows for the explicit construction of the entire Fock space [12–14].

Our paper is organized as follows. As an introduction we explain in section 2 how one can systematically construct canonical variables on a flat non-commutative space starting from standard generators in Fock space by exploiting $\mathcal{PT}$-symmetry to reduce the number of free parameters. In section 3, we extend these considerations to $q$-deformed oscillator algebras and present in particular one solution in more detail for which we construct in a non-trivial way $\mathcal{PT}$-symmetric versions of the harmonic oscillator on these spaces and in section 6 we state our conclusions.

## 2. Deformed oscillator algebras and non-commutative spaces

Our starting point is a $q$-deformed oscillator algebra for the creation and annihilation operators $A_i^+, A_i$, as studied for instance in [12–14, 16, 19]:

$$A_i A_j^\dagger = q^{\theta_{ij}/2} A_j^\dagger A_i = \delta_{ij}, \quad [A_i^\dagger, A_j] = 0, \quad [A_i, A_j] = 0, \quad \text{for } i, j = 1, 2, 3; \quad q \in \mathbb{R}. \quad (2.1)$$

In the limit $q \to 1$, we denote $A_i \to a_i$ and recover the standard Fock space commutation relations:

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = 0, \quad [a_i^\dagger, a_j^\dagger] = 0, \quad \text{for } i, j = 1, 2, 3. \quad (2.2)$$

We assume further that relations (2.2) are linearly related to the standard three-dimensional flat non-commutative space characterized by the relations

$$[x_0, y_0] = i\theta_1, \quad [x_0, z_0] = i\theta_2, \quad [y_0, z_0] = i\theta_3, \quad [x_0, p_{0\theta}] = i\hbar, \quad \text{for } \theta_1, \theta_2, \theta_3 \in \mathbb{R}. \quad (2.3)$$
with all remaining commutators to be zero and the $\theta_1, \theta_2, \theta_3$ denoting the non-commutative constants. The most general linear ansatz to relate the generators of relations (2.3) and (2.2) reads
\[
\varphi_i = \sum_{j=1}^{3} \kappa_{ij} a_j + \lambda_{ij} a_j^\dagger, \quad \text{for } \bar{\varphi} = \{x_0, y_0, z_0, p_{x_0}, p_{y_0}, p_{z_0}\},
\]
where the $\kappa_{ij}, \lambda_{ij}$ have dimensions of length or momentum for $i = 1, 2, 3$ or $i = 4, 5, 6$, respectively. The commutation relations obeyed by the canonical variables $X, Y, Z, P_x, P_y, P_z$ associated with the deformed algebra (2.1) are yet unknown and are subject to construction. The algebra they satisfy may be related to (2.1) by similar relations as (2.4), but since the constants $\kappa_{ij}$ and $\lambda_{ij}$ are in general complex, this amounts to finding 72 real parameters. To reduce this number to a manageable quantity one can utilize $\mathcal{PT}$-symmetry.

2.1. The role of $\mathcal{PT}$-symmetry

Whereas the momenta and coordinates in (2.3) are Hermitian operators acting on a Hilbert space with standard inner product, this is no longer true for the variables associated with the deformed algebra (2.1) as they become in general non-Hermitian with regard to these inner products. Thus, a quantum mechanical model or a quantum field theoretical model on these spaces will, in general, not be Hermitian in that space. However, it is by now well accepted that one may consider complex $\mathcal{PT}$-symmetric non-Hermitian systems as self-consistent descriptions of physical systems [29, 30]. Guided by these results one may try to identify this symmetry for the non-commutative space relations (2.3). In [31], the authors argue that this would not be possible and one is therefore forced to take the non-commutative constants to be complex. We reason here that this is incorrect and even the standard non-commutative $\mathcal{PT}$-symmetry is a known feature previously observed for many examples [33–35] in dimensions larger than 1. More restrictions and the explicit choice of symmetry result from the specific physical situation one wishes to describe. For instance $\mathcal{PT}_{\theta_1}$ might be appropriate when one deals with a problem in which one direction is singled out, $\mathcal{PT}_{\theta_2}$ requires the non-commutative constant $\theta_2$ to appear as a
parameter in the model and $\mathcal{PT}\sigma$ suggests a symmetry along the line $x_0 = z_0$. For the creation and annihilation operators, this symmetry could manifest itself in different ways, for instance as $a_i \rightarrow \pm a_i$, $a_i^\dagger \rightarrow \pm a_i^\dagger$ or by the permutation of indices $a_i \rightarrow a_j$, $a_i^\dagger \rightarrow a_j$, when they label for instance particles in different potentials, see e.g. [36]. Once again the underlying physics will dictate which version one should select. The general reasons for the occurrence of these different possibilities are just manifestations of the ambiguities in defining a metric to which the $\mathcal{PT}$-operator is directly related. What needs to be kept in mind is that we only require the symmetry of some antilinear involution [32] in order to obtain a meaningful quantum mechanical description.

2.2. Oscillator algebras of flat non-commutative spaces

Let us first see how to represent a three-dimensional oscillator algebra in terms of the canonical variables in three-dimensional flat non-commutative space. For definiteness we seek at first a description which is invariant under $\mathcal{PT}\pm$. The most generic linear ansatz for the creation and annihilation operators to achieve this is

$$a_1 = \alpha_1 x_0 + i \alpha_2 y_0 + \alpha_3 z_0 + i \alpha_4 p_x + \alpha_5 p_y + i \alpha_6 p_z,$$

$$a_2 = \alpha_7 x_0 + i \alpha_8 y_0 + \alpha_9 z_0 + i \alpha_{10} p_x + \alpha_{11} p_y + i \alpha_{12} p_z,$$

$$a_3 = \alpha_{13} x_0 + i \alpha_{14} y_0 + \alpha_{15} z_0 + i \alpha_{16} p_x + \alpha_{17} p_y + i \alpha_{18} p_z,$$

with dimensional real constants $\alpha_i$. We note that we have $\mathcal{PT}_\pm : a_i \rightarrow \pm a_i$, $a_i^\dagger \rightarrow \pm a_i^\dagger$ for $i = 1, 2, 3$. The non-sequential ordering of the constants in (2.8)–(2.10) is chosen to perform the limit to the two-dimensional case in a convenient way. For $\alpha_9, \ldots, \alpha_{18} \rightarrow 0$ we recover equation (2.4) in [19]. It is useful to invoke this limit at various stages of the calculation as a consistency check. We then compute that the operators (2.8)–(2.10), expressed on the three-dimensional flat non-commutative space (2.3), satisfy the standard Fock space commutation relations (2.2) provided that the following nine constraints hold:

$$1 = 2 \sum_{j=1}^{3} [(2 - j)\alpha_{2+j4}\alpha_{j+k} - (-1)^j/\hbar\alpha_{j+k}\alpha_{j+k+3}] \quad \text{for } k = 0, 6, 12, \quad (2.11)$$

$$0 = i(\alpha_p\alpha_q + \alpha_{p+2}\alpha_{q-2})\theta_2 + \sum_{j=1}^{2} (\alpha_{j+p}\alpha_{j+q-p+2} - \alpha_{j+p-1}\alpha_{j+q-2})\theta_2j-1$$

$$+ \sum_{j=1}^{3} (\alpha_{j+p+2}\alpha_{j+q-p-2} - \alpha_{j+p-1}\alpha_{j+q})\hbar \quad \text{for } \{p, q\} = \{1, 9\}, \{1, 15\}, \{7, 15\}, \quad (2.12)$$

$$0 = i(\alpha_p\alpha_q + \alpha_{p+2}\alpha_{q-2})\theta_2 - \sum_{j=1}^{2} (-1)^j(\alpha_{j+p}\alpha_{j+q-p+2} + \alpha_{j+p-1}\alpha_{j+q-2})\theta_2j-1$$

$$- \sum_{j=1}^{3} (-1)^j(\alpha_{j+p+2}\alpha_{j+q-p-2} + \alpha_{j+p-1}\alpha_{j+q})\hbar \quad \text{for } \{p, q\} = \{1, 9\}, \{1, 15\}, \{7, 15\}. \quad (2.13)$$

It turns out that when keeping $\theta_2 \neq 0$, these equations do not admit a non-trivial solution. However, setting $\theta_2$ to zero we can solve (2.11)–(2.13) for instance by

$$\alpha_2 = -\frac{\alpha_{14}(\alpha_2(2\alpha_{14}\alpha_{17} - 2\alpha_{13}\Delta') + 1) - 2\alpha_9\alpha_{13}\Delta''}{2\Delta\Delta''}. \quad (2.14)$$
where we abbreviated $\Delta := \alpha_3 \gamma - \alpha_1 \alpha_9$, $\Delta' := \hbar \alpha_{16} + \alpha_{14} \theta_1$, $\Delta'' := \hbar \alpha_{18} - \alpha_{14} \theta_3$. Thus, we still have nine parameters left at our disposal. In other words, the ansatz (2.8)–(2.10) together with (2.2) enforces the $\mathcal{PT}$-symmetry of the type (2.5).

Inverting relations (2.8)–(2.10) we may express the dynamical variables in terms of the creation and annihilation operators:

$$x_0 = \frac{\alpha_9 \alpha_1 - \alpha_3 \alpha_5}{2 \det M_1} (a_1 + a_1^\dagger) + \frac{\alpha_3 \alpha_5 - \alpha_5 \alpha_7}{2 \det M_1} (a_2 + a_2^\dagger) + \frac{\alpha_3 \alpha_1 - \alpha_5 \alpha_9}{2 \det M_1} (a_3 + a_3^\dagger),$$

$$y_0 = \frac{\alpha_9 \alpha_1 - \alpha_3 \alpha_5}{2i \det M_2} (a_1 - a_1^\dagger) + \frac{\alpha_3 \alpha_5 - \alpha_5 \alpha_7}{2i \det M_2} (a_2 - a_2^\dagger) + \frac{\alpha_3 \alpha_1 - \alpha_5 \alpha_9}{2i \det M_1} (a_3 - a_3^\dagger),$$

$$z_0 = \frac{\alpha_1 \alpha_3 - \alpha_3 \gamma}{2 \det M_1} (a_1 + a_1^\dagger) + \frac{\alpha_1 \alpha_5 - \alpha_3 \alpha_3}{2 \det M_1} (a_2 + a_2^\dagger) + \frac{\alpha_5 \gamma - \alpha_5 \alpha_1}{2 \det M_1} (a_3 + a_3^\dagger),$$

$$p_{x_0} = \frac{\alpha_9 \alpha_1 - \alpha_3 \alpha_5}{2i \det M_2} (a_1 - a_1^\dagger) + \frac{\alpha_3 \alpha_5 - \alpha_5 \alpha_7}{2i \det M_2} (a_2 - a_2^\dagger) + \frac{\alpha_3 \alpha_1 - \alpha_5 \alpha_9}{2i \det M_1} (a_3 - a_3^\dagger),$$

$$p_{y_0} = \frac{\alpha_9 \alpha_1 - \alpha_3 \alpha_5}{2 \det M_1} (a_1 + a_1^\dagger) + \frac{\alpha_3 \alpha_5 - \alpha_5 \alpha_7}{2 \det M_1} (a_2 + a_2^\dagger) + \frac{\alpha_3 \alpha_1 - \alpha_5 \alpha_9}{2 \det M_1} (a_3 + a_3^\dagger),$$

$$p_{z_0} = \frac{\alpha_1 \alpha_3 - \alpha_3 \gamma}{2 \det M_1} (a_1 + a_1^\dagger) + \frac{\alpha_1 \alpha_5 - \alpha_3 \alpha_3}{2 \det M_1} (a_2 + a_2^\dagger) + \frac{\alpha_5 \gamma - \alpha_5 \alpha_1}{2 \det M_1} (a_3 + a_3^\dagger),$$

$$p_{\theta_0} = \frac{\alpha_9 \alpha_1 - \alpha_3 \alpha_5}{2 \det M_1} (a_1 + a_1^\dagger) + \frac{\alpha_3 \alpha_5 - \alpha_5 \alpha_7}{2 \det M_1} (a_2 + a_2^\dagger) + \frac{\alpha_3 \alpha_1 - \alpha_5 \alpha_9}{2 \det M_1} (a_3 + a_3^\dagger),$$
where the matrices $M_{1/2}$ have entries
\[
(M_l)_{jk} = 6j + 2k + l - 8 \quad \text{for} \quad l = 1, 2.
\] (2.29)
These expressions satisfy the commutation relations (2.3) when we invoke the constraints (2.11)–(2.13) and the standard Fock space relation (2.2). In that case we also have the simple relation $\det M_1 \det M_2 = -1/8\hbar^3$. By changing the ansatz (2.8)–(2.10) appropriately one may also obtain $PT_{\theta M}$ or $PT_{\alpha M}$-invariant solutions.

3. Non-commutative spacetime from $q$-deformed creation and annihilation operators

Next, we construct the commutation relations for the deformed non-commutative space satisfied by the canonical variables $X$, $Y$, $Z$, $P_x$, $P_y$, $P_z$, which we express linearly in terms of the creation and annihilation operators obeying the deformed algebra (2.1). Guided by the fact that in the limit $q \to 1$ we should recover relations (2.23)–(2.28) of the previous subsection. We therefore make the ansatz
\[
X = \hat{k}_1(A^+_1 + A_1) + \hat{k}_2(A^+_2 + A_2) + \hat{k}_3(A^+_3 + A_3),
\] (3.1)
\[
Y = i\hat{\kappa}_4(A^+_1 - A_1) + i\hat{\kappa}_5(A^+_2 - A_2) + i\hat{\kappa}_6(A^+_3 - A_3),
\] (3.2)
\[
Z = \hat{k}_7(A^+_1 + A_1) + \hat{k}_8(A^+_2 + A_2) + \hat{k}_9(A^+_3 + A_3),
\] (3.3)
\[
P_x = i\hat{\kappa}_{10}(A^+_1 - A_1) + i\hat{\kappa}_{11}(A^+_2 - A_2) + i\hat{\kappa}_{12}(A^+_3 - A_3),
\] (3.4)
\[
P_y = \hat{k}_{13}(A^+_1 + A_1) + \hat{k}_{14}(A^+_2 + A_2) + \hat{k}_{15}(A^+_3 + A_3),
\] (3.5)
\[
P_z = i\hat{\kappa}_{16}(A^+_1 - A_1) + i\hat{\kappa}_{17}(A^+_2 - A_2) + i\hat{\kappa}_{18}(A^+_3 - A_3),
\] (3.6)
with $\hat{\kappa}_i = \kappa_i\sqrt{\hbar/(m\omega)}$ for $i = 1, \ldots, 9$ having the dimension of a length and $\hat{\kappa}_i = \kappa_i\sqrt{m\omega\hbar}$ for $i = 10, \ldots, 18$ possessing the dimension of a momentum. The constants $\kappa_i$ for $i = 1, \ldots, 18$ are therefore dimensionless. We deliberately keep here all dimensional constants different from 1. With the help of the $q$-deformed oscillator algebra (2.1), we compute
\[
[X, Y] = 2i \sum_{j=1}^{3} \hat{\kappa}_j \hat{\kappa}_{3+j}[1 + (q^2 - 1)]A^+_j A_j,
\] (3.7)
\[
[Y, Z] = -2i \sum_{j=1}^{3} \hat{\kappa}_{3+j} \hat{\kappa}_{6+j}[1 + (q^2 - 1)]A^+_j A_j,
\] (3.8)
\[
[X, P_x] = 2i \sum_{j=1}^{3} \hat{\kappa}_j \hat{\kappa}_{9+j}[1 + (q^2 - 1)]A^+_j A_j,
\] (3.9)
\[
[Y, P_y] = -2i \sum_{j=1}^{3} \hat{\kappa}_{3+j} \hat{\kappa}_{12+j}[1 + (q^2 - 1)]A^+_j A_j,
\] (3.10)
\[
[Z, P_x] = 2i \sum_{j=1}^{3} \hat{\kappa}_{6+j} \hat{\kappa}_{15+j}[1 + (q^2 - 1)]A^+_j A_j,
\] (3.11)
\[
[P_x, P_y] = -2i \sum_{j=1}^{3} \hat{\kappa}_{9+j} \hat{\kappa}_{12+j}[1 + (q^2 - 1)]A^+_j A_j,
\] (3.12)
We now make the assumption that the variables

\[ [P_y, P_z] = 2i \sum_{j=1}^{3} \hat{\kappa}_{1+q} \hat{\kappa}_{1+q} [1 + (q^2 - 1)] A_j^+ A_j, \quad (3.13) \]

\[ [X, P_z] = 2i \sum_{j=1}^{3} \hat{\kappa}_{2+q} \hat{\kappa}_{2+q} [1 + (q^2 - 1)] A_j^+ A_j, \quad (3.14) \]

\[ [Z, P_z] = 2i \sum_{j=1}^{3} \hat{\kappa}_{3+q} \hat{\kappa}_{3+q} [1 + (q^2 - 1)] A_j^+ A_j, \quad (3.15) \]

\[ [X, Z] = [P_x, P_z] = [X, P_z] = [Y, P_z] = [Y, P_z] = [Z, P_z] = 0. \quad (3.16) \]

Inverting now relations (3.1)–(3.6) we find that it is indeed possible to eliminate entirely the creations and annihilation operators from these relations. However, this leads to very lengthy expressions, which we will not present here. Instead, we report some special, albeit still quite general, solutions obtained by setting some of the constants to zero and imposing further constraints.

3.1. A particular \( \mathcal{PT}_\pm \)-symmetric solution

We now make the assumption that \( \kappa_1 = \kappa_4 = \kappa_5 = \kappa_8 = \kappa_{10} = \kappa_{12} = \kappa_{13} = \kappa_{14} = \kappa_{17} = \kappa_{18} = 0 \). This choice still guarantees that none of the canonical variables become mutually identical. The consistency with the direct limit \( q \to 1 \) in which we want to recover (2.3) enforces the constraints

\[ \hat{\kappa}_2 = \frac{\hbar}{2\kappa_1}, \hat{\kappa}_3 = \frac{\theta_1}{2\kappa_6}, \hat{\kappa}_9 = -\frac{\theta_1}{2\kappa_6}, \hat{\kappa}_{15} = -\frac{\hbar}{2\kappa_6}, \hat{\kappa}_{16} = \frac{\hbar}{2\kappa_7}. \quad (3.17) \]

The only non-vanishing commutators we obtain in this case are

\[ [X, Y] = i\theta_1 + i q^2 - 1 \theta_1 \left( \frac{m_0 \omega}{2\kappa_6^2} Y^2 + \frac{2\kappa_6^2}{m_0 \omega} P_x^2 \right), \quad (3.18) \]

\[ [Y, Z] = i\theta_3 + i q^2 - 1 \theta_1 \left( \frac{m_0 \omega}{2\kappa_6^2} Y^2 + \frac{2\kappa_6^2}{m_0 \omega} P_y^2 \right), \quad (3.19) \]

\[ [X, P_x] = i\hbar + i q^2 - 1 \theta_1 \left( \frac{m_0 \omega}{2\kappa_6^2} Y^2 + \frac{2\kappa_6^2}{m_0 \omega} P_x^2 \right), \quad (3.20) \]

\[ [Y, P_x] = i\hbar + i q^2 - 1 \theta_3 \left( \frac{1}{4m^2 \omega^2 \kappa_1^2} X^2 + \frac{1}{4m^2 \omega^2 \kappa_1^2} P_x^2 + \frac{\theta_1 \kappa_1^2}{\hbar^2} P_x^2 + 2\frac{\theta_1 \kappa_1^2}{\hbar} XP_x \right), \quad (3.21) \]

\[ [Z, P_x] = i\hbar + i q^2 - 1 \theta_3 \left( \frac{1}{4m^2 \omega^2 \kappa_1^2} X^2 + \frac{1}{4m^2 \omega^2 \kappa_1^2} P_x^2 + \frac{\theta_3 \kappa_1^2}{\hbar^2} P_x^2 + \frac{\theta_3 \kappa_1^2}{\hbar^2} ZP_x \right). \quad (3.22) \]

Note that we still have the three free parameters \( \kappa_6, \kappa_7 \) and \( \kappa_{11} \) at our disposal. It is easily verified that relations (3.18)–(3.22) are left invariant under a \( \mathcal{PT}_\pm \)-symmetry (2.3) in the variables \( X, Y, Z, P_x, P_y, P_z \).
3.1.1. Reduced three-dimensional solution for \( q \to 1 \). The solution (3.18)–(3.22) possesses a non-trivial limit leading to an even simpler set of commutation relations. For this purpose, we impose some additional constraints by setting first \( \kappa_{11} = m\omega_0 \kappa, \kappa_7 = 1/2\kappa_6, q = \exp(2\tau\kappa^2) \) and subsequently we take the limit \( \kappa_6 \to 0 \). Relations (3.18)–(3.22) then reduce to

\[
[X, Y] = i\theta_1 (1 + \hat{\tau} Y^2), \quad [Y, Z] = i\theta_2 (1 + \hat{\tau} Y^2), \quad [Z, X] = i\theta_3 (1 + \hat{\tau} Y^2),
\]

(3.23)

\[
[X, P_x] = i\hbar (1 + \hat{\tau} P_x^2), \quad [Y, P_y] = i\hbar (1 + \hat{\tau} P_y^2), \quad [Z, P_z] = i\hbar (1 + \hat{\tau} P_z^2),
\]

(3.24)

where \( \hat{\tau} = \tau m\omega_0 / \hbar \) has the dimension of an inverse squared length, \( \hat{\tau} = \tau / (m\omega_0) \) has the dimension of an inverse squared momentum and \( \tau \) is dimensionless. Note that once this limit has been carried out we loose the invertibility from the variables \( X, Y, Z, P_x, P_y, P_z \) to the \( A_i, \overline{A}_i \).

We find a concrete representation for this algebra in terms of the generators of the standard three-dimensional flat non-commutative space (2.3):

\[
X = (1 + \hat{\tau} P_x^2)x_0 + \frac{\theta_1}{\hbar} (\hat{\tau} p_{x_0}^2 - \hat{\tau} y_0^2)p_{x_0}, \quad P_x = p_{x_0},
\]

\[
Z = (1 + \hat{\tau} P_z^2)z_0 + \frac{\theta_2}{\hbar} (\hat{\tau} p_{z_0}^2 - \hat{\tau} y_0^2)p_{z_0}, \quad P_z = p_{z_0},
\]

\[
P_y = (1 + \hat{\tau} y_0^2)p_{y_0}, \quad Y = y_0.
\]

(3.25)

Evidently, the quantities \( X, Z \) and \( P_y \) are non-Hermitian in the space in which the \( x_0, y_0, z_0, p_{x_0}, p_{z_0}, P_{y_0} \) are Hermitian. In order to study concrete models, it is very convenient to carry out a subsequent Bopp-shift of the form \( x_0 \to x + \frac{\theta}{\hat{\tau}} p_{x_0}, y_0 \to y, z_0 \to z + \frac{\theta}{\hat{\tau}} p_{y_0}, p_{x_0} \to p_x, p_{y_0} \to p_y, p_{z_0} \to p_z \) and express the generators in (3.25) in terms of the standard canonical variables. Since there is no explicit occurrence of \( \theta_1 \), the representation (3.25) is trivially invariant under \( PT_{\pm} \) as well as \( PT_{\theta} \). Taking, however, the representation (3.25) and in addition \( \theta_2 \neq 0 \) this evidently changes, as by direct computation one of the commutation relations is altered to \( [X, Z] = i\theta_2 (1 + \hat{\tau} P_x^2)(1 + \hat{\tau} P_y^2) \). Setting furthermore \( \theta_1 = \theta_3 \) the representation in (3.25) is also invariant under the \( PT_{\pm} \)-symmetry stated in (2.7).

3.1.2. Reduction into a decoupled two-dimensional plus a one-dimensional space. The algebra (3.18)–(3.22) provides a larger three-dimensional setting for a non-commutative two-dimensional space decoupled from a standard one-dimensional space. This is achieved by parameterizing \( q = \exp(2\tau\kappa^2) \), setting \( \kappa_{11} = m\omega_0 \kappa, \theta := \theta_1 \) and subsequently taking the limit \( (\theta_2, \kappa_6) \to 0 \) reduces the algebra to a two non-commutative-dimensional spaces in the \( X-, Y- \)-directions

\[
[X, Y] = i\theta (1 + \hat{\tau} Y^2), \quad [Y, P_y] = i\hbar (1 + \hat{\tau} P_y^2), \quad [X, P_x] = i\hbar (1 + \hat{\tau} P_x^2)
\]

(3.26)

decoupled from a standard one-dimensional space in the \( Z \)-direction

\[
[Z, P_z] = i\hbar, \quad [Y, Z] = 0.
\]

(3.27)

As a representation for the algebra (3.26) in flat non-commutative space we may simply use (3.25) with the appropriate limit \( \theta_1 \to 0 \). Carrying out the corresponding Bopp-shift \( x_0 \to x + \frac{\theta}{\hat{\tau}} p_{x_0}, y_0 \to y, p_{x_0} \to p_x \) and \( p_{y_0} \to p_y \) yields the operators

\[
X = x + \theta \frac{p_x^2}{\hbar} + \hat{\tau} p_x^2 x - \theta \frac{\tau}{\hbar^2} p_{y_0} p_y, \quad Y = y, \quad P_x = p_x \quad \text{and} \quad P_y = p_y + \hat{\tau} y_0^2 p_{y_0},
\]

(3.28)

which are of course still non-Hermitian with regard to the standard inner product.
3.1.3. Reduction into three decoupled one-dimensional spaces. We conclude this section by noting that all three directions in the algebra (3.18)–(3.22) can be decoupled, of which one becomes a one-dimensional non-commutative space previously investigated by many authors, e.g. [2, 16]. It is easy to verify that this scenario is obtained from (3.18)–(3.22) when parameterizing \( q = \exp(2\pi\kappa_1^2\theta_1) \) and subsequently taking the limit \( (\theta_1, \theta_3, \kappa_{11}) \to 0 \). The remaining non-vanishing commutators are then

\[
[X, P_z] = i\hbar (1 + \tau P_x^2), \quad [Y, P_z] = i\hbar, \quad \text{and} \quad [Z, P_z] = i\hbar.
\]  

(3.29)

Thus all three space directions are decoupled from each other. It is known that the choices \( X = (1 + \tau p_x^2)\chi_x, P_x = p_x \) or \( X' = X' = \chi_x(1 + \tau p_x^2), P_x = p_x \) constitute representations for the commutation relations (3.29) in the \( X \)-direction.

3.2. A particular \( \mathcal{PT}_\theta \)-symmetric solution

Instead of solving the complicated relations (3.7)–(3.16) one may also start by making directly an ansatz of a similar form as in (3.25) without elaborating on the relation to the \( \varphi \)-deformed oscillator algebra. Proceeding in this manner with an ansatz with respect to the \( \mathcal{PT}_\theta \)-symmetry we find for instance the representation

\[
X = x_0 - \frac{\theta_1}{\hbar} \gamma_0 p_0 - \frac{\theta_2}{\hbar} \gamma_0 p_0, \quad P_x = p_0,
\]

\[
Z = z_0 + \frac{\theta_1}{\hbar} \gamma_0 p_0 + \frac{\theta_2}{\hbar} \gamma_0 p_0, \quad P_z = p_0,
\]

\[
P_y = p_y + \frac{\theta_3}{\hbar} \gamma_0 p_0, \quad Y = y_0,
\]

(3.30)

yielding the closed algebra

\[
[X, Y] = i\theta_1 (1 + \tau Y^2), \quad [X, Z] = i\theta_2 (1 + \tau Y^2), \quad [Y, Z] = i\theta_3 (1 + \tau Y^2),
\]

\[
[X, P_z] = i\hbar, \quad [Y, P_z] = i\hbar(1 + \tau Y^2), \quad [Z, P_z] = i\hbar.
\]

(3.31)

with all remaining commutators vanishing. Note that if we set \( \theta_1 = -\theta_3 \) the generators in (3.30) are also invariant under the \( \mathcal{PT}_{\varphi} \)-symmetry.

4. Minimal length, minimal areas and minimal volumes

Let us now investigate the generalized uncertainty relations associated with the algebras constructed above. In general, the uncertainties \( \Delta A \) and \( \Delta B \) resulting from a simultaneous measurement of two observables \( A \) and \( B \) have to obey the inequality

\[
\Delta A \Delta B \geq \frac{1}{\rho} |\langle [A, B] \rangle|.
\]

(4.1)

Here \( \langle . \rangle_\rho \) denotes the inner product on a Hilbert space with metric \( \rho \) in which the operators \( A \) and \( B \) are Hermitian, as discussed in more detail in [2, 16, 18, 19]. The minimal length \( \Delta A_{\text{min}} \) that is the precision up to which the observable \( A \) can be known by giving up all the information on \( B \) is then computed by minimizing \( \Delta A \Delta B - \frac{1}{\rho} |\langle [A, B] \rangle_\rho| \) as a function of \( \Delta B \).

In the standard scenario, i.e. when \( A \) and \( B \) commute up to a constant, the result is therefore usually zero. This outcome changes when the commutator \( [A, B] \) involves higher powers of \( \Delta B \), in which case we encounter the interesting scenario of non-vanishing \( \Delta A_{\text{min}} \). We now investigate some of the solutions presented above. Depending now on the question we ask, i.e. which quantities we attempt to measure, the minimal uncertainties for some specific operators turn out to be different.
4.1. A three-dimensional non-commutative space giving rise to minimal areas

We start with our simplest three-dimensional solution, that is, the algebra \((3.23)\)–\((3.24)\). If we just want to measure the position of the particle on such a space independent of its momentum we only have to investigate relations \((3.23)\). Taking \(\tau > 0\) and following the logic of \([16, 18, 19]\), we obtain from \((3.23)\) for a simultaneous measurement of all space coordinates the non-vanishing minimal length in two directions:

\[
\Delta X_{\min} = |\theta_1| \sqrt{\tau} \sqrt{1 + \frac{\tau}{2}(Y)^2_{\rho}}, \quad \Delta Y_{\min} = 0, \quad \text{and} \quad \Delta Z_{\min} = |\theta_1| \sqrt{\tau} \sqrt{1 + \frac{\tau}{2}(Y)^2_{\rho}}.
\]  

(4.2)

Thus any measurement of space will involve an unavoidable uncertainty of an area \(A\) of size \(\Delta A_0 = 4\tau |\theta_1|\) in the \(XZ\)-plane and no uncertainty in the \(Y\)-direction. Changing our question and attempt to measure instead all coordinates and all components of the momenta, we need to analyze the entire set of relations \((3.23)\)–\((3.24)\). The analysis of equations \((3.24)\) alone yields

\[
\Delta (P_{x})_{\min} = h \sqrt{\tau} \sqrt{1 + \frac{\tau}{2}(Y)^2_{\rho}}, \quad \Delta (P_{y})_{\min} = 0, \quad \text{and} \quad \Delta (P_{z})_{\min} = h \sqrt{\tau} \sqrt{1 + \frac{\tau}{2}(Y)^2_{\rho}}.
\]  

(4.3)

\[
\Delta (P_{x})_{\min} = 0, \quad \Delta (P_{y})_{\min} = h \sqrt{\tau} \sqrt{1 + \frac{\tau}{2}(Y)^2_{\rho}}, \quad \text{and} \quad \Delta (P_{z})_{\min} = 0.
\]  

(4.4)

Thus, depending now on whether \(|\theta_1|, |\theta_3| < 1\) or \(|\theta_1|, |\theta_3| > 1\) the uncertainties in \((4.2)\) or \((4.3)\) will be smaller, respectively. For any type of measurement the region of uncertainty will be an area.

4.2. A three-dimensional non-commutative space giving rise to minimal volumes

Let us now analyze our solution \((3.23)\)–\((3.24)\) before taking the limit \(q \to 1\). We compute the uncertainties with regard to a measurement of all components of the coordinates and all components of the momenta. Since now the quantities are all coupled, in the sense that we do not have any non-trivial subalgebra, we will encounter uncertainties for all of them and observe a different type of behavior as indicated in the previous subsection. Starting with a simultaneous \(Y, P_{y}\)-measurement we compute from \((4.1)\) with \((3.21)\) the uncertainties

\[
\Delta Y_{\min} = |\hat{k}_6| \sqrt{\frac{1}{2} (q^2 - q^{-2}) + (q - q^{-1})^2 \left( \frac{1}{4k_6^2} (Y)^2_{\rho} + \frac{k_6^2}{h^2} (P_{y})_{\rho} \right)}.
\]  

(4.5)

\[
\Delta (P_{y})_{\min} = \frac{h}{2 |\hat{k}_6|} \sqrt{\frac{1}{2} (q^2 - q^{-2}) + (q - q^{-1})^2 \left( \frac{1}{4k_6^2} (Y)^2_{\rho} + \frac{k_6^2}{h^2} (P_{y})_{\rho} \right)},
\]  

(4.6)

under the assumption that \(q > 1\). The absolute minimal uncertainties resulting from these expressions are therefore

\[
\Delta Y_0 = \frac{|\hat{k}_6|}{\sqrt{2}} \sqrt{q^2 - q^{-2}} \quad \text{and} \quad \Delta (P_{y})_0 = \frac{h}{2 \sqrt{2} |\hat{k}_6|} \sqrt{q^2 - q^{-2}}.
\]  

(4.7)

Next, we carry out a simultaneous \(X, Y\)-measurement and a \(Y, Z\)-measurement by employing \((3.18)\) and \((3.19)\), respectively. We find the minimal lengths

\[
\Delta X_{\min} = \left| \frac{\theta_1}{\hat{k}_6} \right| \sqrt{\frac{1}{2} \left( \frac{q - q^{-1}}{q + q^{-1}} \right)^2 + \left( \frac{q - q^{-1}}{q + q^{-1}} \right)^2 \left( \frac{1}{4k_6^2} (Y)^2_{\rho} + \frac{k_6^2}{h^2} (P_{y})_{\rho} + \Delta (P_{y})_0^2 \right)}.
\]  

(4.8)

\[
\Delta Z_{\min} = \left| \frac{\theta_3}{\hat{k}_6} \right| \sqrt{\frac{1}{2} \left( \frac{q - q^{-1}}{q + q^{-1}} \right)^2 + \left( \frac{q - q^{-1}}{q + q^{-1}} \right)^2 \left( \frac{1}{4k_6^2} (Y)^2_{\rho} + \frac{k_6^2}{h^2} (P_{y})_{\rho} + \Delta (P_{y})_0^2 \right)}.
\]  

(4.9)
There is no minimal length in the $Y$-direction resulting from these relations. Using the expression for $\Delta(P)_0$ from (4.7), the absolute minimal values for these uncertainties are

$$\Delta X_0 = \frac{1}{2\sqrt{2}} \frac{\theta_1}{\kappa_6} \sqrt{q^2 - q^{-2}}$$

and

$$\Delta Z_0 = \frac{1}{2\sqrt{2}} \frac{\theta_1}{\kappa_6} \sqrt{q^2 - q^{-2}}.$$  

(4.10)

Thus a measurement of the position in space will be accompanied by an uncertainty volume $V$ of the size

$$\Delta V_0 = \frac{1}{\sqrt{2}} \frac{\theta_1 \theta_2}{\kappa_6} (q^2 - q^{-2})^{3/2}.$$  

(4.11)

The evaluation for the simultaneous $X$, $P_x$, $P_z$-measurements is slightly more complicated due to the occurrence of the $XP_x$ and $ZP_z$ terms in (3.20) and (3.22), respectively. We proceed similarly as before and also make use of the well-known inequalities $|A + B| \geq |A| - |B|$ and $|\langle AB \rangle| \leq \Delta A \Delta B + |\langle A \rangle| |\langle B \rangle|$. We report here only the final result of the absolute minimal values:

$$\Delta (P)_0 = \frac{\gamma_i \Delta (P)_0 - \sqrt{\beta_i \left( \alpha_i \gamma_i^2 \Delta (P)_0^2 + \lambda_i (1 - 4\alpha_i \beta_i) \right)}}{4\alpha_i \beta_i - 1}$$

for $i = x, z$,  

(4.12)

with

$$\alpha_s = \alpha_2, \quad \beta_s = \alpha_{11}, \quad \gamma_s = \frac{2|\theta_1|}{h} \alpha_{11}, \quad \lambda_s = \frac{\hbar^2}{2} \alpha_{11} \frac{\theta_1^2}{\kappa_6} \Delta (P)_0^2,$$

$$\alpha_s = \alpha_7, \quad \beta_s = \alpha_{16}, \quad \gamma_s = \frac{|\theta_1|}{h} \alpha_{16}, \quad \lambda_s = \frac{\hbar^2}{2} \alpha_{16} \frac{\theta_1^2}{\kappa_6} \Delta (P)_0^2,$$  

(4.13)

where $\alpha_i = \kappa_i^2 (q - q^{-1})/(q + q^{-1}) \hbar$ for $i = 2, 7$ and $\alpha_i = \kappa_i^2 (q - q^{-1})/(q + q^{-1}) \hbar$ for $i = 11, 16$. Further restrictions do not emerge.

By similar reasoning one finds non-vanishing $\Delta X_{\text{min}}$, $\Delta Z_{\text{min}}$ and $\Delta P_{\text{min}}$ for the $\mathcal{PT}$-invariant algebra (3.31).

Note that in this section we have treated the variables $X, Y, Z, P_x, P_z, P$ as observables since they will give rise to the non-trivial uncertainty relations. In general, the set of observables is a matter of choice, especially in a non-Hermitian setting, see e.g. [30]. In the next section, we will also make use of the variables $x_0, y_0, z_0, p_{x_0}, p_{y_0}, p_{z_0}$ and $x_t, y_t, z_t, p_{x_t}, p_{y_t}, p_{z_t}$, but simply for technical reasons we do not treat them as observables.

5. Models on $\mathcal{PT}$-symmetric non-commutative spaces

5.1. The one-dimensional harmonic oscillator on a non-commutative space

We commence with the one-dimensional harmonic oscillator on the $\mathcal{PT}$-symmetric non-commutative space described by (3.29). The corresponding Hamiltonian

$$H_{\text{nch}} = \frac{p^2}{2m} + \frac{m\omega^2}{2} X^2 = H_{\text{ho}} + \frac{m\omega^2}{2} \left( \tilde{\tau} p^2 x_0^2 + \tilde{\tau} x_0 p^2 x_0 + \tilde{\tau}^2 p^2 x_0^2 p^2 x_0 \right) = H_{\text{ho}} + H_{\text{nch}}^{1D}$$  

(5.1)

is evidently non-Hermitian with regard to the standard inner product. However, it is $\mathcal{PT}$-symmetric, such that it might constitute a well-defined self-consistent description of a physical system. The associated Schrödinger equation $H_{\text{nch}}^{1D} \psi = E \psi$ is most conveniently solved in $p$-space, i.e. with $x_0 = i \hbar \theta_0$, it reads

$$\frac{m\omega^2}{2} \left( 1 + \tilde{\tau} p^2 \right) \psi'' + \tau \omega h p_x \left( 1 + \tilde{\tau} p_x \right) \psi' + \left( E - \frac{p_x^2}{2m} \right) \psi = 0.$$  

(5.2)
Using the transformation
\[
\mu = \frac{\sqrt{1 + 2E\hbar \tau}}{\tau}, \quad \nu = \frac{\sqrt{4 + \tau^2}}{2\tau} - \frac{1}{2} \quad \text{and} \quad z = ip_x \sqrt{\tau},
\]
we convert (5.2) into
\[
(1 - z^2)\psi'' - 2z\psi' + \left[\nu(v + 1) - \frac{\nu^2}{1 - z^2}\right]\psi = 0,
\]
which is the standard differential equation for the associated Legendre polynomials \(P_\nu^m(z)\) and \(Q_\nu^m(z)\) admitting the general solution
\[
\psi(z) = c_1 P_\nu^m(z) + c_2 Q_\nu^m(z). \tag{5.5}
\]
Seeking asymptotically vanishing solutions gives rise to the quantization condition \(\mu + \nu = -n - 1\) with \(n \in \mathbb{N}\). With (5.3), it follows therefore that the eigenenergies become
\[
E_n = \omega \hbar \left(\frac{1}{2} + n\right) \sqrt{1 + \frac{\tau^2}{4} + \frac{\tau \omega \hbar}{4}(1 + 2n + 2n^2)} \quad \text{for } n \in \mathbb{N}_0. \tag{5.6}
\]
The expression agrees with the one found in [2]. The polynomial \(Q_\nu^m(z)\) is not defined for these values, such that \(c_2 = 0\) and \(P_\nu^m(z)\) reduces to
\[
\psi_{2n-1}(z) = c_1 \sum_{k=0}^{2n-1} \frac{1}{k!} \prod_{l=0}^{\lfloor k/2 \rfloor} 2(n-1)(2n+2l+1) \left[\frac{z^2(1-z^2)^{k-\lfloor k/2 \rfloor}}{(1)^{k+1}(1-z^2)^k}\right], \tag{5.7}
\]
with \(i = 0, 1\). Clearly, the \(\psi_{2n-1}(z)\) vanishes for \(|z| \to \infty\) if \(\nu > -1\), which is always guaranteed for \(tm\omega > 0\). The Dyson map \(\eta\) which adjointly maps \(H_{\text{Dh}}\) to a Hermitian operator was easily found in [2, 16] to be \(\eta = (1 + \tau P_2^2)^{-1/2}\). In addition, we note that the solutions are square integrable \(\psi_{2n-1}(z) \in L^2(\mathbb{R})\) on \(|\nu|^{1/2}\) and form an orthonormal basis.

An exact treatment for models in the higher dimensions is more difficult, but we may resort to perturbation theory to obtain some useful insight into the solutions. As a quality gauge we compare here the exact solution against perturbation theory around the standard Fock space harmonic oscillator solution with normalized eigenstates:
\[
|n\rangle = \left(a_n^\dagger\right)^n |0\rangle, \quad a_n |0\rangle = 0, \quad a_n |n\rangle = \sqrt{n+1} |n+1\rangle, \quad a_n^\dagger |n\rangle = \sqrt{n} |n-1\rangle. \tag{5.8}
\]
A straightforward, albeit lengthy, computation yields the following corrections to the harmonic oscillator energy \(E_n^{(0)} = \omega \hbar \left(n + \frac{1}{2}\right)\) for the eigenenergies of \(H_{\text{Dh}}\):
\[
E_n^{(p)} = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + O(\tau^3) \tag{5.9}
\]
with
\[
E_n^{(1)} = \langle n|H_{\text{Dh}}^{(1)}|n\rangle = \frac{\tau \omega \hbar}{4}(1 + 2n + 2n^2) + \frac{\tau^2 \omega \hbar}{16}(3 + 8n + 6n^2 + 4n^3), \tag{5.10}
\]
\[
E_n^{(2)} = \sum_{p \neq n} \frac{\langle n|H_{\text{Dh}}^{(1)}|p\rangle \langle p|H_{\text{Dh}}^{(1)}|n\rangle}{E_n^{(0)} - E_p^{(0)}} = -\frac{1}{8} \tau^2 \omega \hbar (1 + 3n + 3n^2 + 2n^3) + O(\tau^3). \tag{5.11}
\]
As it should be, the expression for \(E_n\) in (5.6) when expanded up to order \(\tau^3\) coincides precisely with \(E_n^{(p)}\). We further note that also in a perturbative treatment the eigenenergies are strictly positive.

The validity of these expansions is governed by the well-known sufficient conditions for the applicability of the Rayleigh–Schrödinger perturbation theory to a Hamiltonian of the form \(H = H_0 + H_1\) around the solutions of \(H_0|n\rangle = E_n^{(0)}|n\rangle\):
\[
\left| \frac{\langle p|H_1|n\rangle}{E_n^{(0)} - E_p^{(0)}} \right| \ll 1 \quad \text{for all } p \neq n. \tag{5.12}
\]
This is guaranteed for (5.1) when \( \tau^2 \ll \frac{32/(2n+13)}{(4+n)(3+n)(2+n)(1+n)} \), such that perturbation theory will break down for large values of \( n \).

### 5.2. The two-dimensional harmonic oscillator on a non-commutative space

Next, we consider the two-dimensional harmonic oscillator on the \( \mathcal{PT}_\pm \)-symmetric non-commutative space described by the algebra (3.26). Using the representation (3.28) for this algebra, the corresponding Hamiltonian reads

\[
H_{\text{ncho}}^{2\text{D}} = \frac{1}{2m} (p_x^2 + p_y^2) + \frac{m \omega^2}{2} (x^2 + y^2)
\]

\[
= H_{\text{ncho}}^{2\text{D}} + \frac{\tau \omega}{2\hbar} \left[ \{p_x^2, x_0\} + \{\gamma_0^2 p_{x0}, p_{y0}\} + \frac{\theta}{\hbar} \{p^2_{x0}, p_{y0}, x_0\} - \frac{m^2 \omega^2 \theta}{\hbar} \{\gamma_0^2 p_{x0}, x_0\} \right]
\]

\[
+ \frac{\tau^2 \omega^2 m}{2\hbar^2} \left[ \{\gamma_0^2 p_{x0}, (1 + \Omega)\gamma_0 p_{x0} - \frac{\theta \gamma_2^2 p_{x0}}{\hbar} - \frac{\theta \gamma_2^2 p_{x0}}{\hbar^2} \right] + \left( \frac{p^2_{x0} p_{y0} + \theta p^2_{x0} p_{y0}}{m \omega} + \frac{\theta p^2_{x0} p_{y0}}{m \omega \hbar} \right)^2
\]

(5.13)

where we used the standard notation for the anticommutator \([A, B] = AB + BA\). Once again this Hamiltonian is non-Hermitian with regard to the inner product on the flat non-commutative space, but it respects a \( \mathcal{PT}_\pm \)-symmetry. In order to be able to perturb around the standard harmonic oscillator solution we still need to convert flat non-commutative space into the canonical variables \( x, y, p_x \), and \( p_y \). Thus, when using the representation (3.28) this Hamiltonian is converted into

\[
H_{\text{ncho}}^{2\text{D}} = H^{2\text{D}} + \frac{\tau \omega}{2\hbar} \left[ \{p_x^2, x_0\} + \{\gamma_0^2 p_{x0}, p_{y0}\} \right]
\]

\[
- \frac{m \theta \omega^2}{\hbar} \left[ \{\gamma_0^2 p_{x0}, x_0\} + \{\gamma_0^2 p_{x0}, p_{y0}\} \right]
\]

\[
+ \frac{\tau^2}{2} \left[ \left( \frac{1}{m} + \frac{m \theta \omega^2}{\hbar^2} \right) (\gamma_0^2 p_{y0})^2 - \frac{m \theta \omega^2}{\hbar} \{\gamma_0^2 p_{x0}, p_{y0}, x_0\} \right].
\]

In this formulation, we may now proceed as in the previous subsection and expand perturbatively around the standard two-dimensional Fock space harmonic oscillator solution with normalized eigenstates:

\[
|n_1 n_2\rangle = \frac{(a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}}{\sqrt{n_1! n_2!}} |00\rangle, \quad a_i |n_1 n_2\rangle = \sqrt{n_i + 1} |(n_i + 1) (n_1 + n_2)\rangle,
\]

(14)

\[
a_i |00\rangle = 0, \quad a_i |n_1 n_2\rangle = \sqrt{n_i} |(n_i - 1) (n_1 + n_2)\rangle,
\]

(15)

for \( i = 1, 2 \). The energy eigenvalues for the Hamiltonian \( H_{\text{ncho}}^{2\text{D}} \) then result to

\[
E_{nl} = E^{(0)}_{nl} + E^{(1)}_{nl} + E^{(2)}_{nl} + O(\tau^2)
\]

\[
= \hbar \omega (n + l + 1) + |n| H_{\text{ncho}}^{2\text{D}} |n| + \sum_{p,q \neq n+l} \langle n| H_{\text{ncho}}^{2\text{D}} |pq\rangle \langle pq| H_{\text{ncho}}^{2\text{D}} |n| \rangle + O(\tau^2)
\]

\[
= E^{(0)}_{nl} + \frac{\Omega \omega \hbar}{8} \left[ (3 + n + 5l) - \Omega \left( l + \frac{1}{2} \right) \right]
\]

\[
+ \frac{\tau}{2} \omega \hbar \left[ 1 + n + n^2 + l + l^2 \frac{\Omega}{4} (4 + 3n + n^2 + 7l + 4nl + 5l^2) \right] + O(\tau^2),
\]

(5.16)
where we introduced the dimensionless quantity $\Omega = m^2 \eta \omega^2 / \hbar^2$. Note that unlike in the one-dimensional case the perturbation beyond $H_{\text{ho}}^{(3D)}$ also involves terms of order $O(\tau^0)$, such that we need to compute also $E_{nlr}^{(2)}$ to achieve a precision of first order in $\tau$. We also note that the energy $E_{nlr}^{(0)}$ is only bounded from below for $\Omega < 5$. The minus sign is an indication that we will encounter exceptional points [37] and broken $\mathcal{PT}$-symmetry in some parameter range.

### 5.3. The three-dimensional harmonic oscillator on a non-commutative space

Let us finally consider the three-dimensional harmonic oscillator on the non-commutative space described by the algebra (3.18) and (3.19). Using the representation (3.25) together with a subsequent Bopp-shift, the corresponding Hamiltonian can be expressed in terms of the standard canonical coordinates:

\[
H_{\text{ncho}}^{(3D)} = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \frac{m \omega^2}{2}(x^2 + y^2 + z^2) = H_{\text{ho}}^{(3D)} + H_{\text{ncho}}^{(3D)}
\]

\[
= H_{\text{ho}}^{(3D)} + \frac{m \omega^2}{2\hbar} \left[ \theta_3(p_{y\gamma}, z_i) - \theta_1(x_i, p_{y\gamma}) + \frac{\theta_1^2 + \theta_2^2}{\hbar^2} p_{y\gamma}^2 \right] + \tau \frac{\omega^2}{2\hbar} \left[ p_{z\gamma}^2 x_i + \{ p_{z\gamma}^2 z_i, z_i \} + (1 + \Omega_1 + \Omega_3) \{ p_{z\gamma}^2 p_{y\gamma}, p_{y\gamma} \} \right] - \frac{\theta_1}{\hbar} (m^2 \omega^2 \{ x_i, p_{y\gamma} \} + \{ p_{z\gamma}^2 x_i, p_{y\gamma} \}) + \frac{\theta_1^2}{\hbar^2} (m^2 \omega^2 \{ z_i, p_{y\gamma} \} + \{ p_{z\gamma}^2 z_i, p_{y\gamma} \}) + \frac{\theta_1}{\hbar} (m^2 \omega^2 \{ z_i, p_{y\gamma} \} + \{ p_{z\gamma}^2 z_i, p_{y\gamma} \})
\]

\[
+ \frac{\theta_1}{\hbar} (m^2 \omega^2 \{ z_i, p_{y\gamma} \} + \{ p_{z\gamma}^2 z_i, p_{y\gamma} \})
\]

(5.17)

We expand now around the standard three-dimensional Fock space harmonic oscillator solution with normalized eigenstates

\[
|n_1 n_2 n_3\rangle = \prod_{i=1}^{3} \left( \frac{a_i^\dagger}{\sqrt{n_i!}} \right)^n |000\rangle, \quad a_i |n_1 n_2 n_3\rangle = \sqrt{n_i + 1} \prod_{j=1}^{3} (n_j + \delta_{ij})
\]

(5.18)

\[
a_i |000\rangle = 0, \quad a_i |n_1 n_2 n_3\rangle = \sqrt{n_i} \prod_{j=1}^{3} (n_j - \delta_{ij})
\]

(5.19)

for $i = 1, 2, 3$ and compute the energy eigenvalues for $H_{\text{ncho}}^{(3D)}$ to

\[
E_{nlr}^{(p)} = E_{nlr}^{(0)} + E_{nlr}^{(1)} + E_{nlr}^{(2)} + O(\tau^2)
\]

\[
= E_{nlr}^{(0)} + |nlr| H_{\text{ncho}}^{(3D)} |nlr\rangle + \sum_{s,p,q\neq n+l+r=p+q+r} \frac{|nlr| H_{\text{ncho}}^{(3D)} |pqrs\rangle \langle pqrs| H_{\text{ncho}}^{(3D)} |nlr\rangle}{E_{nlr}^{(0)} - E_{pqrs}^{(0)}} + O(\tau^2)
\]

\[
= \omega \hbar \left[ \frac{3}{2} + n + l + r + \frac{1}{8}(\Omega_1 + \Omega_3)(3 + 5l) - \frac{1}{16}(2l + 1)(\Omega_1 + \Omega_3)^2 + \frac{1}{8}(n\Omega_1 + r\Omega_3) + \frac{\tau}{2} \left( n^2 + n + l^2 + r^2 + 3l + r + \frac{1}{4}(n^2 + 4ln + 3n + 5l^2 + 7l + 4)\Omega_1 + \frac{1}{4}(5l^2 + 4rl + 7l + r^2 + 3r + 4)\Omega_3 + \frac{3}{2} \right) \right] \quad (5.20)
\]

As in the two-dimensional case we encounter negative terms in this expression, thus indicating that broken $\mathcal{PT}$-symmetry will be broken in some parameter range.
6. Conclusions

Contrary to some claims in the literature [31], we have demonstrated that it is indeed possible to implement $\mathcal{PT}$-symmetry on non-commutative spaces while keeping the non-commutative constants real. Starting from a generic ansatz for the canonical variables obeying a $q$-deformed oscillator algebra, we employed $\mathcal{PT}$-symmetry to limit the amount of free parameters. Relations (3.7)–(3.16) resulting from this ansatz turned out to be solvable. A specific $\mathcal{PT}_{\pm}$-symmetric solution was presented in (3.18)–(3.22). Clearly there exist more solutions with different kinds of properties. We constructed an explicit representation for the algebra obtained in the non-trivial limit $q \to 1$ in terms of the generators of a flat non-commutative space. With regard to the standard inner product for this space, the operators are non-Hermitian. We computed the minimal length and momenta resulting from the generalized uncertainty relations, which overall give rise to minimal areas or minimal volumes in phase space.

Despite being non-Hermitian, due to the built-in $\mathcal{PT}$-symmetry any model formulated in terms of these variables is a candidate for a self-consistent theory with real eigenvalue spectrum. We have studied the harmonic oscillator on these spaces in one, two and three dimensions. The perturbative computation of the energy eigenvalues indicates that there exists a parameter regime for which the $\mathcal{PT}$-symmetry is broken. It would be interesting to investigate this further and determine when this transition precisely occurs. The eigenvalues will also be useful in further investigations [38] allowing for the construction of coherent states related to the algebras presented in section 4.

Obviously, there are many more solutions to (3.7)–(3.16), which might be studied in their own right together with models formulated on them. Minor modifications would also allow us to investigate the occurrence of upper bounds, i.e. maximal length and momenta [24], giving rise to a second scale in special relativity, the so-called double special relativity [39] constituting a possibility to explain the observation of cosmic rays with energies above the GZK-threshold [40]. However, such an upper limit might not be required as recent experiments [41–43] seem to indicate at a consistency with the GZK-cutoff prediction, even though some inconsistencies still remain.

Acknowledgments

SD is supported by a City University Research Fellowship. LG is supported by the high energy section of the ICTP.

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