CHEBYSHEV NETS IN $\mathbb{R}^3$ AND MINIMAL TIMELIKE SURFACES IN $\mathbb{R}^4$

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Abstract. In this work we provide necessary and sufficient conditions for the existence of a minimal timelike strip in Lorentz-Minkowski space $\mathbb{R}^4_1$ containing a given lightlike curve and prescribed normal bundle. We also discuss uniqueness of solutions.

1. Introduction

The classical Björling problem can be formulated as follows: given a real analytic curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ and a unit normal vector field $V : I \to \mathbb{R}^3$, along $\alpha$, determine a minimal surface containing $\alpha(I)$ such that its normal vector along the curve is $V$. The problem was firstly proposed and solved by Björling himself in [3] (1844). It was mentioned by Schwarz in [10] (1875) who also solved it, using a representation based on holomorphic data, in [11] (1890).

Since then, many generalizations of this problem appeared in several Riemannian and pseudo-Riemannian ambient manifolds. In $\mathbb{R}^3_1$ Alas, Chaves and Mira studied maximal spacelike surfaces in [1] and timelike minimal surfaces were studied by Chaves, Dussan and Magid in [4], where both existence and uniqueness of solutions are established. Analogous results are proved in $\mathbb{R}^4_1$, for spacelike surfaces in [2] by Asperti and Vilhena and, for timelike surfaces, in [7] by Dussan, Padua and Magid. The same holds for timelike surfaces in $\mathbb{R}^4_2$ (see [8]). On Riemannian or Lorentzian Lie Groups, Mercuri and Onnis, in [9], and Cintra, Mercuri and Onnis, in [6], also obtained results on existence and uniqueness of solutions. In all those papers the authors make use of some kind of Weierstrass representation formula, over complex or split-complex domains.

The study of timelike minimal surfaces is important not only from the mathematical point of view but also in physics, since they are

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solutions for the wave equation and therefore can be regarded (classical) relativistic strings.

In this work, without use of those complex or split-complex representations, we provide necessary and sufficient conditions for the existence of a solution for the Björling problem for a timelike surface in \( \mathbb{R}^4_1 \), when the prescribed curve is lightlike. In this case we cannot expect uniqueness of solutions, which will be shown to be a certain lift of a Chebyshev net in euclidean space \( \mathbb{R}^3 \).

2. Algebraic preliminaries and the two kinds of Chebyshev nets

The space \( \mathbb{R}^4_1 \) is the vector space \( \mathbb{R}^4 \) equipped with the following semi-Riemannian metric tensor:

\[
ds^2_1 = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3.
\]

We write this tensor in the inner product notation \( \langle v, w \rangle = ds^2_1(v, w) \). The standard basis of \( \mathbb{R}^4_1 \) will be denoted by \( \{ \partial_0, \partial_1, \partial_2, \partial_3 \} \) and we set \( \epsilon_i = \langle \partial_i, \partial_i \rangle \). If \( v = \sum_{i=1}^4 v_i \partial_i \), we have \( v_i = \epsilon_i \langle \cdot, \partial_i \rangle \). A vector \( v \in \mathbb{R}^4_1 \) is spacelike if \( v = 0 \) or \( \langle v, v \rangle > 0 \), timelike if \( \langle v, v \rangle < 0 \) and lightlike if \( v \neq 0 \) and \( \langle v, v \rangle = 0 \). In the same way, a spacelike plane \( V \) of the space \( \mathbb{R}^4_1 \) is a 2-dimensional subspace for which the induced bilinear form, \( ds^2_1|_V \), is positive definite; we say that \( V \) is timelike plane if \( ds^2_1|_V \) is non-degenerate and indefinite and it is lightlike if \( ds^2_1|_V \) is degenerate.

Let \( \{ a, b \} \) be an orthonormal basis of a spacelike plane \( V \subset \mathbb{R}^4_1 \) and consider the unit timelike vector

\[
\tau = \frac{1}{\sqrt{1+a_0^2+b_0^2}} \left( \partial_0 + a_0 a^0 a + b_0 b \right)
\]

The standard wedge product of \( u, w, w \in \mathbb{R}^4_1 \) is \( u \wedge v \wedge w \in \mathbb{R}^4_1 \), the unique solution for \( \langle u \wedge v \wedge w, x \rangle = \det(x, u, v, w) \). In matrix notation we have the formal determinant

\[
u \wedge v \wedge w = \det \begin{pmatrix}
\partial_0 & \partial_1 & \partial_2 & \partial_3 \\
 u_0 & u_1 & u_2 & u_3 \\
 v_0 & v_1 & v_2 & v_3 \\
 w_0 & w_1 & w_2 & w_3
\end{pmatrix}.
\]

Setting \( \Delta_{ij} = a_i b_j - a_j b_i \) for \( 0 \leq i, j \leq 3 \), we have the unit spacelike vector

\[
\nu = -\tau \wedge a \wedge b = \Delta_{23} \partial_1 - \Delta_{13} \partial_2 + \Delta_{12} \partial_3.
\]
The 2-dimensional vector subspace $T = \text{span}\{\tau, \nu\}$ is a timelike plane which is the orthogonal complement of $V$. The 4-uple $(\tau, a, b, \nu)$ is a positive and future-directed frame, name Minkowski frame adapted to $\{a, b\}$.

Indeed, we see that $\langle \tau, \tau \rangle = -1$ and $\tau_0 = \sqrt{1 + a_0^2 + b_0^2} \geq 1$, with $\langle \tau, a \rangle = 0 = \langle \tau, b \rangle$. We also have that $\nu_0 = 0$, and $\langle \nu, \nu \rangle = 1$, because the set $\{\tau, a, b\}$ is an orthonormal subset of $\mathbb{R}^4$. For each lightlike vector $L = (L_0, L_1, L_2, L_3)$ we define its projection onto the unit sphere $S^2 \subset \{0\} \times \mathbb{R}^3$ by the formula

$$\pi(L) = (0, L_1/L_0, L_2/L_0, L_3/L_0).$$

The vectors $\tau \parallel \nu$ are lightlike. Hence we set

$$n_0 = \pi(\tau - \nu) = (1/\tau_0)(\tau - \nu) - \partial_0 \text{ and } n_3 = \pi(\tau + \nu) = (1/\tau_0)(\tau + \nu) - \partial_0$$

to define a trigonometric angle $\theta \in ]0, \pi]$ in $V$ by

$$\cos \theta = \langle n_0, n_3 \rangle = 1 - \frac{2}{\tau_0} = \frac{a_0^2 + b_0^2 - 1}{a_0^2 + b_0^2 + 1}.$$  \hspace{1cm} (5)

**Proposition 2.1.** For the angle $\theta$ above we have

$$\sin \theta = \frac{2\sqrt{a_0^2 + b_0^2}}{\tau_0}, \quad \sin(\theta/2) = \frac{1}{\tau_0}, \quad \text{and} \quad \cos(\theta/2) = \frac{\sqrt{a_0^2 + b_0^2}}{\tau_0}.$$  \hspace{1cm} (5)

The timelike plane $T = \text{span}\{\partial_0 + n_0, \partial_0 + n_3\}$ has induced metric tensor represented, in this isotropic basis, by

$$g_{ij} = \begin{bmatrix} 0 & -1 + \cos \theta \\ -1 + \cos \theta & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2/\tau_0^2 \\ -2/\tau_0^2 & 0 \end{bmatrix}.$$  \hspace{1cm} (5)

In the spacelike plane $E = \text{span}\{n_0, n_3\} \subset \{0\} \times \mathbb{R}^3$, with respect to the given basis, it has the form

$$\hat{g}_{ij} = \begin{bmatrix} 1 & \cos \theta \\ \cos \theta & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 - 2/\tau_0^2 \\ 1 - 2/\tau_0^2 & 1 \end{bmatrix}.$$  \hspace{1cm} (5)

Now, when $\tau_0 > 1$ (that is, $|a_0| + |b_0| \neq 0$) we define an orthonormal basis $\{\vec{e}_1, \vec{e}_2\}$ for the plane $V$ by

$$\vec{e}_1 = \frac{1}{\sqrt{a_0^2 + b_0^2}}(a_0 a + b_0 b) \quad \text{and} \quad \vec{e}_2 = \frac{1}{\sqrt{a_0^2 + b_0^2}}(-b_0 a + a_0 b).$$  \hspace{1cm} (6)

We note that $\text{span}\{\vec{e}_2\} = V \cap \{0\} \times \mathbb{R}^3$. Setting

$$e = \frac{1}{2\cos(\theta/2)}(n_0 + n_3) \in S^2$$  \hspace{1cm} (7)
we have the following result.

**Proposition 2.2.** On the above conditions, the following relations on the vectors of the (non-orthogonal) Minkowski frame \( \{ \tau, \tilde{e}_1, e, \nu \} \) hold:

\[
\tau = \frac{1}{\tau_0} \left( \partial_0 + \sqrt{a_0^2 + b_0^2} \tilde{e}_1 \right) = \tau_0 \partial_0 + \tau_0 \cos(\theta/2) e \quad \text{and} \\
\tilde{e}_1 = \cot(\theta/2) \partial_0 + \cosec(\theta/2) e.
\]

**Proof.** The first identity comes from equations (4) and (7), where we see that

\[
\cos(\theta/2) e = \frac{n_0 + n_3}{2} = \tau/\tau_0 - \partial_0.
\]

For the second one, observe that \( \tilde{e}_1 \) is orthogonal to \( \tau \) and \( \nu \). This means that \( \tilde{e}_1 = \alpha \partial_0 + \beta e \), for some \( \alpha, \beta \in \mathbb{R} \). From Proposition 2.1, since \( \partial_0 \) and \( e \) are mutually orthonormal, we have

\[
\alpha = -\langle \tilde{e}_1, \partial_0 \rangle = \sqrt{a_0^2 + b_0^2} = \cot(\theta/2) \\
\beta = \langle \tilde{e}_1, e \rangle = \tau_0 = \cosec(\theta/2),
\]

as stated. \( \square \)

Now, we will define Chebyshev nets as immersions in the Euclidean vector space \( \mathbb{E} = \{ 0 \} \times \mathbb{R}^3 \subset \mathbb{R}^4_1 \).

**Definition 2.3.** We say that an immersion \((M, X)\) from a connected open subset \( M \subset \mathbb{R}^2 \) into the Euclidean space \( \mathbb{E} \) is a Chebyshev net if and only if the coefficients of its first quadratic form, written as \( ds^2 = E(u, v) du^2 + 2F(u, v) du dv + G(u, v) dv^2 \), verifies, for all \((u, v) \in M,\)

\[
E(u, v) = G(u, v) = 1 \quad \text{and} \quad F(u, v) = \cos(\theta(u, v)) \in ]-1,1[.
\]

Associated to each Chebyshev net \((M, X)\) there is a timelike isotropic immersion \((M, f)\), the lift of \( X \), from \( M \) into \( \mathbb{R}^4_1 \) defined by the formula

\[
f(u, v) = (u + v) \partial_0 + X(u, v),
\]

whose induced metric tensor is

\[
g_{ij}(f) = \begin{bmatrix} 0 & -1 + F \\ -1 + F & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \sin^2(\theta/2) \\ -2 \sin^2(\theta/2) & 0 \end{bmatrix}
\]

If \((M, X)\) is a Chebyshev net, we consider the equivalent immersion \((M, X, T)\) obtained applying the linear change of coordinates \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) given by:

\[
t = u + v \quad \text{and} \quad s = -u + v, \quad \text{such that} \quad dt \wedge ds = 2 du \wedge dv.
\]
That is, $\overline{M} = T(M)$ and
\begin{equation}
X(t, s) = X\left(\frac{t-s}{2}, \frac{t+s}{2}\right) = X(u, v).
\end{equation}

Now the metric tensor is given by
\begin{equation}
\mathrm{d}s^2_X = \mathbb{T}\,\mathrm{d}t^2 + \mathbb{G}\,\mathrm{d}s^2 = \cos^2(\theta/2)\,\mathrm{d}t^2 + \sin^2(\theta/2)\,\mathrm{d}s^2.
\end{equation}

The correspondent lift immersion
\begin{equation}
\overline{f}(t, s) = t\partial_0 + X(t, s)
\end{equation}
has isothermal parameters and its induced metric is
\begin{equation}
\mathrm{d}s^2_f = \sin^2(\theta/2)(-\mathrm{d}t^2 + \mathrm{d}s^2).
\end{equation}

**Theorem 2.4.** Let $f(u, v) = (u + v)\partial_0 + X(u, v) \in \mathbb{R}_4^1$ be a lift of a Chebyshev net. The vector fields
\begin{equation}
\tilde{e}(u, v) = \frac{1}{\sin\theta(u, v)}((1 + \cos\theta(u, v))\partial_0 + X_u(u, v) + X_v(u, v))
\end{equation}
and
\begin{equation}
e_2(u, v) = \frac{1}{\sin\theta(u, v)}X_u(u, v) \times_{\mathbb{R}^3} X_v(u, v)
\end{equation}
form a spacelike orthonormal normal frame along $S = f(M)$. Moreover, the mean curvature vector $H_f(u, v)$ of the surface $S$ is pointwise parallel to the normal Gauss map $e_2(u, v)$ of the surface $X(M) \subset \mathbb{E}$.

**Proof.** Straightforward computations, using Chebyshev net properties, show the algebraic aspects of the statement.

The coefficients of induced metric tensor on $f(M)$ give the mean curvature vector
\begin{equation}
H_f = \frac{-1}{2\sin^2(\theta/2)}f_{uv} = \frac{-1}{2\sin^2(\theta/2)}X_{uv},
\end{equation}
which is orthogonal to $\tilde{e}$, hence parallel to $e_2$. \hfill \Box

**Proposition 2.5.** The Gaussian curvature of a lift such as in Theorem 2.4 is
\begin{equation}
K = \frac{\theta_u \theta_v - \theta_{uv} \sin\theta}{(1 - \cos\theta)^2}.
\end{equation}

**Proof.** From [12, p. 443], the Gaussian curvature of a parametric surface whose coordinates curves are lightlike is given by
\begin{equation}
K = -\frac{1}{g_{12}} \left( \frac{g_{12}}{g_{12}} \right)_u^v.
\end{equation}
In this case $g_{12} = -1 + \cos\theta$. \hfill \Box
Recall that Gaussian curvature of any Chebyshev net satisfies the equation $\theta_{uv} + K_T \sin \theta = 0$. Hence we may rewrite (12) as

$$K = \frac{\theta_u \theta_v + K_T \sin^2 \theta}{(1 - \cos \theta)^2}$$

(13)

Now we will give two examples of Chebyshev nets, the first has a lift with $Hf \equiv 0$ and the second is not a critical surface of $\mathbb{R}^3$.

Example 2.6 (Critical lift). Set $U = [-\pi/2, \pi/2]$ and consider the immersion $X : U \to \mathbb{E}$, given by

$$X(u, v) = \int_0^u (0, \cos \xi, \sin \xi, 0) \, d\xi + \int_0^v (0, 0, \sin \xi, \cos \xi) \, d\xi.$$

Direct calculations show that:

1. The first quadratic form or metric tensor is
   $$ds^2 = du^2 + 2 \sin u \sin v \, du \, dv + dv^2;$$

2. The normal Gauss map is
   $$e_2(u, v) = \frac{1}{\sqrt{1 - \sin^2 u \sin^2 v}} (0, \sin u \cos v, -\cos u \cos v, \cos u \sin v);$$

3. The second quadratic form is
   $$B = \frac{-1}{\sqrt{1 - \sin^2 u \sin^2 v}} (\cos v \, du^2 + \cos u \, dv^2);$$

4. The Gaussian curvature is
   $$K(u, v) = \frac{\cos u \cos v}{(1 - \sin^2 u \sin^2 v)^2} > 0.$$

The lift surface, $f(u, v) = (u + v) \partial_0 + X(u, v)$, has vanishing mean curvature: one can see this from $X_{uv} = 0$ in (11) or noting that $f$ is a sum of two lightlike curves (see [5, p. 68]).

Lemma 2.7. Let $(W, Y)$ be an immersion from a connected open subset $W \subset \mathbb{R}^2$ into $\mathbb{E}$ with induced metric given by

$$ds_Y^2 = E(t, s) \, dt^2 + G(t, s) \, ds^2.$$

The equivalent immersion $(M, X)$ defined by $X(u, v) = Y(u + v, -u + v)$ is a Chebyshev net if and only if

$$E(t, s) + G(t, s) = 1.$$

Proof. We only need to observe that:

$$X_u(u, v) = Y_t(u + v, -u + v) - Y_s(u + v, -u + v),$$

$$X_v(u, v) = Y_t(u + v, -u + v) + Y_s(u + v, -u + v).$$


Hence

\[ E(u, v) = G(u, v) = E(t, s) + G(t, s) \text{ and } \]
\[ F(u, v) = E(u + v, -u + v) - G(u + v, -u + v). \]

If \( E(t, s) + G(t, s) = 1 \) then \( E(u, v) = G(u, v) = 1 \) and, since \( |F(u, v)| \leq 1 \), we have a smooth real valued function \( \theta(u, v) \) from \( M \) such that \( F(u, v) = \cos \theta(u, v) \). The converse is trivial. \( \square \)

**Example 2.8 (Non-critical lift).** Let \( Y : [-\pi, \pi] \times I \to \mathbb{E} \) be the parametric surface given by

\[ Y(t, s) = (0, x(s) \cos t, x(s) \sin t, y(s)). \]

We have that the metric coefficient \( F \) verifies \( F(t, s) = 0 \). Suppose that the other coefficients satisfy \( E(t, s) + G(t, s) = 1 \) and . In this case, the lift surface \( f(t, s) = t\partial_0 + Y(t, s) \) is isothermal and timelike. In terms of equation (11), to obtain a non critical surface we must have the equivalent immersion \( X(u, v) \) satisfying \( X_{uv} \neq 0 \), that is, \( f_{tt} - f_{ss} = Y_{tt} - Y_{ss} \neq 0 \). The ordinary differential equation imposed by the condition \( E(t, s) + G(t, s) = 1 \) is

\[ x^2(s) + (x'(s))^2 + (y'(s))^2 = 1. \]

The functions

\[ x(s) = \frac{1}{2} \tanh s \quad \text{and} \quad y(s) = \frac{1}{2} \int_0^s \sqrt{4 - \tanh^2 \xi - \text{sech}^4 \xi} \, d\xi, \]

are a particular solution to this equation. Since, \( y'' \neq 0 \), we have \( f_{tt} - f_{ss} \neq 0 \) and \( H_f \neq 0 \).

**Definition 2.9.** We say that a Chebyshev net \((M, X)\) is a Chebyshev net of first kind if and only if

\[ X(u, v) = p_0 + \int_0^u T_1(\xi) \, d\xi + \int_0^v T_2(\xi) \, d\xi, \]

for any disjoint curves \( T_1 : I \to S^2 \subset \mathbb{E} \) and \( T_2 : J \to S^2 \subset \mathbb{E} \) such that

\[ \{(u, v) \in I \times J : T_1(u) = T_2(v)\} \cup \{(u, v) \in I \times J : T_1(u) = -T_2(v)\} = \emptyset. \]

**Remark:** Example 2.6 above uses a Chebyshev net of first kind.
3. The Cauchy problem for Chebyshev nets and timelike minimal surfaces in \( \mathbb{R}^4_1 \)

**Problem 3.1.** Given a real analytic lightlike curve \( c: ] - r, r[ \subset \mathbb{R} \rightarrow \mathbb{R}^4 \) and a spacelike distribution \( D(t) = \text{span} \{ m(t), n(t) \} \) normal along this curve, establish necessary and sufficient conditions for the existence of a timelike minimal immersion \( (M, f) \) from an open and connected subset \( I \times 0 \subset M \subset \mathbb{R}^2 \), such that

1. the curve \( c \) is the coordinate curve \( f(t, 0) = c(t) \),
2. the normal bundle of \( f(M) \) is the given distribution: \( N_{c(t)}f(M) = D(t) \).

What can we say about uniqueness?

We start obtaining an integral representation for an isotropic time-like minimal parametric surface \( S \subset \mathbb{R}^4_1 \). In other words, every timelike minimal surface in \( \mathbb{R}^4_1 \) is the lift of a Chebyshev net of first kind:

**Theorem 3.2.** For each timelike minimal surface \( S \subset \mathbb{R}^4_1 \) and each point \( P_0 \in S \) there exists an open connected subset \( I \times 0 \subset M\subset \mathbb{R}^2 \) and a function \( f: I \times J \rightarrow \mathbb{R}^4_1 \) such that \( f(I \times J) \) is an open subset of the surface \( S \), where

\[
(14) \quad f(u, v) = P_0 + (u + v)\partial_0 + \int_0^u n_0(\xi) \, d\xi + \int_0^v n_3(\xi) \, d\xi,
\]

and \( n_0: I \rightarrow S^2 \) and \( n_3: J \rightarrow S^2 \) are smooth curves on the unit sphere of the Euclidean space \( \mathbb{E} \) such that \( \{(u, v) \in I \times J: |\langle n_0(u), n_3(v)\rangle| = 1\} = \emptyset \).

**Proof.** It is well known (see [5, p. 68]) that any open neighborhood of a timelike surface of \( \mathbb{R}^4_1 \) admits a parametrization given by a sum of two lightlike curves

\[
p(t, s) = P_0 + X(t) + Y(s),
\]

where \( X(t) = X_0(t)\partial_0 + \hat{X}(t) \) and \( Y(s) = Y_0(s)\partial_0 + \hat{Y}(s) \), for curves \( \hat{X}(t), \hat{Y}(s) \in \mathbb{E} \), and

\[
\frac{d}{dt}X_0(t) > 0 \quad \text{and} \quad \frac{d}{ds}Y_0(s) > 0,
\]

for each \( (t, s) \in I' \times J' \). We define the functions \( t = t(u) \) and \( s = s(v) \) for \( (u, v) \in I \times J \) such that

\[
f(u, v) = P_0 + (u + v)\partial_0 + \hat{X}(t(u)) + \hat{Y}(s(v)),
\]

\[
n_0(u) = \frac{d}{dt}(\hat{X}(t(u))) \quad \text{and} \quad n_3(v) = \frac{d}{ds}(\hat{Y}(s(v))).
\]
Corollary 3.3. If \((I \times J, f)\) is given by formula (14) and \(w = (u, v) \in I \times J\) then,
\[
\frac{\partial f}{\partial u}(u, v) = \partial_0 + n_0(u) = l_0(u) \quad \text{and} \quad \frac{\partial f}{\partial v}(u, v) = \partial_0 + n_3(v) = l_3(v)
\]
are lightlike vectors, the induced metric is \(ds^2 = (-1 + \cos \theta(w)) du dv\), and the normal bundle has a basis given by Theorem 2.4 and formulas (6):
\[
\tilde{e}_1(w) = \cot(\theta(w)/2) \partial_0 + \cosec(\theta(w)/2) e(w) \quad \text{and}
\]
\[
e_2(w) = \frac{1}{\sin \theta(w)} n_0(u) \times_{\mathbb{R}^3} n_3(v),
\]
where \(e(w) = \frac{1}{2 \cos(\theta(w)/2)} (n_0(u) + n_3(v)) \in S^2\). The immersion \((I \times J, X)\) defined by
\[
X(w) = \int_0^u n_0(\xi) \, d\xi + \int_0^v n_3(\xi) \, d\xi,
\]
is then a Chebyshev net of first kind.

Now we can establish our main result:

Theorem 3.4. Let \(c : I \subset \mathbb{R} \rightarrow \mathbb{R}^4\), \(c(t) = (c_0(t), c_1(t), c_2(t), c_3(t))\) be a given real analytic lightlike curve, and \(\mathcal{D}(t) = \text{span}\{a(t), b(t)\}\) a normal and orthonormal spacelike distribution along this curve. A necessary and sufficient condition for the existence of a timelike minimal immersion \((I \times J, f)\) such that \(f(t, 0) = c(t)\) and the normal space along \(c(t)\) is \(N_{c(t)}f(M) = \mathcal{D}(t)\) is
\[
c'(t) = c'_0(t) \left(\partial_0 + n_0(t)\right) (16)
\]
where \(n_0(t) = \pi(\tau(t) - v(t))\), \(\pi\) is the projection defined by (3), and the vectors \(\tau\) and \(v\) are given by (1) and (2), respectively.

Proof. The condition is necessary: if we have such an immersion, it can be written as \(f(t, s) = P_0 + X(t) + Y(s)\) and, from \(f(t, 0) = c(t)\) it follows that \(c'(t) = f_1(t, s) = X_1(t)\) for each \(s \in J\), with \(\left<X_1(t), X_1(t)\right> = 0\). The normal bundle of \(f(I \times J), \mathcal{D}(t, s)\), restricted to the curve, that is \(s = 0\), implies that \(c'(t)\) defines a lightlike direction orthogonal to \(\mathcal{D}(t, 0)\). Let \(l_0(t)\) be this direction. Then \(c'(t)\) and \(l_0(t) = \partial_0 + n_0(t)\) must be parallel to each other. The scalar in (16) is \(c'_0(t)\), since the first coordinate of \(l_0(t)\) is 1.

The condition is also sufficient. Up to a changing of variables \(t \leftrightarrow u\), if needed, we can suppose that \(c'(u) = l_0(u)\). This defines a lightlike vector field \(l_3\) along the curve, whose first coordinate is 1 and such that \(\left<l_0(u), l_3(u)\right> < 0\) and the vector field \(n_3(u) = l_3(u) - \partial_0 = \pi(\tau + v) \in S^2\).
Now we need to extend the distribution $\mathcal{D}$, defined on $I$ to $\mathcal{D}(u,v)$, defined on $I \times J$.

To do so, consider the curve

$$\alpha(u) = c(u) - u\alpha_0 \in \{k\} \times \mathbb{R}^3 \equiv \mathbb{E}, \text{ for some } k \in \mathbb{R},$$

and let $\mathcal{F} = \{T(u) = n_0(u), N(u), B(u)\}$ be its Frenet frame. Since $\mathcal{F}$ is a basis of $\mathbb{E}$, there are functions $p, q : I \to \mathbb{R}$ such that, along $\alpha$, we have

$$n_3(u) = \cos \theta(u) T(u) + p(u) N(u) + q(u) B(u). \tag{18}$$

In particular, $p^2(u) + q^2(u) = \sin^2 \theta(u)$.

Our aim is to provide extensions of the vector fields $n_0$ and $n_3$ to $I \times J$ such that $n_0(u,v) = n_0(u)$ and $n_3(u,v) = n_3(v)$. For this, if such extension exists for $n_3$, we can extend, using the same notation, all of the functions in the coefficients of (18) to $I \times J$. The Frenet formulae for $\alpha$ lead to

$$0 = -(\theta_u \sin \theta) T + (\kappa \cos \theta) N + p_u N + p(-\kappa T + \tau B) + q_u B - q \tau N,$$

where $\kappa(u)$ and $\tau(u)$ are, respectively, the curvature and the torsion of $\alpha$. Hence the desired extensions must satisfy the following PDE system:

$$\begin{cases}
\theta_u(u,v) \sin \theta(u,v) + \kappa(u) p(u,v) &= 0 \\
p_u(u,v) + \kappa(u) \cos \theta(u,v) - \tau(u) q(u,v) &= 0 \\
q_u(u,v) + \tau(u) p(u,v) &= 0,
\end{cases} \tag{19}$$

with initial conditions $p(u,0) = p(u), q(u,0) = q(u)$ and $\theta(u,0) = \theta(u)$ along the interval $I$. Since $p^2(u,v) + q^2(u,v) = \sin^2 \theta(u,v)$ the above system is equivalent to

$$\begin{cases}
\kappa(u) p(u,v) &= -\theta_u(u,v) \sin \theta(u,v) \\
\tau(u) q(u,v) &= p_u(u,v) + \kappa(u) \cos \theta(u,v) \\
p^2(u,v) + q^2(u,v) &= \sin^2 \theta(u,v),
\end{cases} \tag{20}$$

with the same initial conditions. Hence, for each extension of the function $\theta$ to $I \times J$ we have functions $p, q$ determined.

We set

$$n_3(v) = \cos \theta(u,v) T(u) + p(u,v) N(u) + q(u,v) B(u),$$

which depends, by construction, only on $v$ allowing us to build the tangent lightlike vector, $l_3(v)$. In this way the immersion $f : I \times J \to \mathbb{R}^4_1$ given by (14) is a local solution to Question 3.1. \hfill \square
In system (20) if \( \theta_u(u,v) \neq 0 \) we see that \( \theta_u(u,v) = -\kappa(u) \) or \( p(u,v) \equiv 0 \), and \( q(u,v) \equiv 0 \). Since \( p \) and \( q \) cannot both vanish simultaneously, we have from last equation in (19) that \( \tau(u) \equiv 0 \), that is \( \alpha \) is a planar curve.

On the other side, if \( \theta_u(u,v) \equiv 0 \) then either \( \kappa(u) \equiv 0 \) or \( p(u,v) \equiv 0 \). The former case says the \( \alpha \) is a straight line in \( E \), implying that \( c(u) \) is a lightlike straight line in \( \mathbb{R}^4_1 \). Here the immersion has the form

\[
f(u,v) = u\vec{l}_0 + v\partial_0 + \int_0^v n_3(\xi) \, d\xi.
\]

for some constant lightlike vector \( \vec{l}_0 \). In the latter case, \( q(u,v) = \sin \theta(u,v) \) and, noting that \( \theta(u,v) = \theta(v) \), we have \( \tan(\theta(v)) = \kappa(u)/\tau(u) \). That is, both \( \theta(u,v) \) and \( \kappa(u)/\tau(u) \) are constants. In particular \( \alpha \) is an helix. From equation (12) in Proposition 2.5 we have that such surfaces are planar. From (13) we conclude that this timelike surface is the lift of a planar Chebyshev net in \( E \).

We finally observe that we obtain existence and non-uniqueness of solutions for the Bjrling problem in \( L^3 = \mathbb{R}^3_1 \) with initial data given by the lightlike curve \( \gamma: I \to L^3 \) and normal vector field \( n: I \to S^2 \), using Theorem 3.4 with \( c(t) = (\gamma(t),0) \), \( a(t) = (n(t),0) \) and \( b(t) = e_4 \). An explicit example of non-uniqueness is Example 3.2 in [4].

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