A \textit{q}-\textsc{analogue of lehmer's congruence}

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\textsc{abstract.} We establish the \textit{q}-analogue of a classical congruence of Lehmer. Also, the \textit{q}-analogues of two congruences of Morley and Granville are given.

1. Introduction

In 1938, Lehmer [Leh] established an interesting congruence as follows:

\[
\sum_{j=1}^{(p-1)/2} \frac{1}{j} \equiv -2Q_p(2) + Q_p(2)^2 p \pmod{p^2}, \tag{1.1}
\]

where \( p \geq 3 \) is prime and \( Q_p(2) = (2^{p-1} - 1)/p \). Lehmer’s congruence can be considered as an extension of the Wolstenholme’s harmonic series congruence [W]

\[
\sum_{j=1}^{p-1} \frac{1}{j} \equiv 0 \pmod{p^2}. \tag{1.2}
\]

On the other hand, the \textit{q}-analogues of some arithmetic congruences have been investigated by several authors (e.g., see [A], [F], [C], [GZ] and [PS]). Recently, Shi and Pan [SP] proved the following \textit{q}-analogue of (1.2):

\[
\sum_{j=1}^{p-1} \frac{1}{[j]_q} \equiv \frac{p-1}{2} (1 - q) + \frac{p^2-1}{24} (1 - q)^2 [p]_q \pmod{[p]_q^2}, \tag{1.3}
\]

where \([n] = (1 - q^n)/(1 - q) = 1 + q + \cdots + q^{n-1} \). Obviously (1.2) is deduced from (1.3) when \( q \to 1 \).

The main purpose of the present paper is to establish the \textit{q}-analogue of Lehmer’s congruence. Set

\[
(a; q)_n = \begin{cases} 
(1 - a)(1 - aq) \cdots (1 - aq^{n-1}) & \text{if } n \geq 1, \\
1 & \text{if } n = 0.
\end{cases}
\]

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It is easy to see that for any \( m \geq 0 \) with \( p \nmid m \) we have a \( q \)-analogue of Fermat’s little theorem
\[
\frac{(q^m; q^m)_{p-1}}{(q; q)_{p-1}} \equiv 1 \pmod{[p]_q}. \tag{1.4}
\]
Indeed, since
\[
[m]_q = \frac{1 - q^m}{1 - q} = \frac{1 - q^n}{1 - q} = [n]_q \pmod{[p]_q}
\]
whenever \( m \equiv n \pmod{p} \),
\[
\frac{(q^m; q^m)_{p-1}}{(q; q)_{p-1}} = \prod_{j=1}^{p-1} \frac{1 - q^{jm}}{1 - q^j} = \prod_{j=1}^{p-1} \frac{[jm]_q}{[j]_q} \equiv 1 \pmod{[p]_q}.
\]
So we can define the \( q \)-Fermat quotient by
\[
Q_p(m, q) = \frac{(q^m; q^m)_{p-1}/(q; q)_{p-1} - 1}{[p]_q}.
\]

**Theorem 1.1.** Let \( p \) be an odd prime. We have
\[
2 \sum_{j=1}^{(p-1)/2} \frac{1}{[2j]_q} + 2Q_p(2, q) - Q_p(2, q)^2[p]_q 
\equiv \left( Q_p(2, q)(1 - q) + \frac{p^2 - 1}{8}(1 - q)^2 \right)[p]_q \pmod{[p]_q^2}. \tag{1.5}
\]

In 1895, with the help of De Moivre’s Theorem, Morley [M] proved that
\[
(-1)^{\frac{p-1}{2}} \left( \frac{p - 1}{(p - 1)/2} \right) \equiv 4^{p-1} \pmod{p^3}. \tag{1.6}
\]
for any prime \( p \geq 5 \). In [G1], Granville generalized the congruence of Morley and showed that
\[
(-1)^{(p-1)(m-1)/2} \prod_{k=1}^{m-1} \left( \frac{p - 1}{[kp/m]} \right) \equiv m^p - m + 1 \pmod{p^2}, \tag{1.7}
\]
for any \( m \geq 2 \), where \([x]\) denotes the greatest integer not exceeding \( x \). Now we can give the \( q \)-analogues of (1.6) and (1.7). For any \( m, n \in \mathbb{N} \), define the \( q \)-binomial coefficients by
\[
\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_m(q; q)_{n-m}}
\]
if \( n \geq m \), and if \( n < m \), then we let \( \begin{bmatrix} n \\ m \end{bmatrix}_q = 0 \). It is easy to see that \( \begin{bmatrix} n \\ m \end{bmatrix}_q \) is a polynomial in \( q \) with integral coefficients, since the \( q \)-binomial coefficients satisfy the recurrence relation
\[
\begin{bmatrix} n + 1 \\ m \end{bmatrix}_q = q^n \begin{bmatrix} n \\ m \end{bmatrix}_q + \begin{bmatrix} n \\ m-1 \end{bmatrix}_q.
\]
Theorem 1.2.
\[-1 \frac{p-1}{2} q^{\frac{p^2-1}{4}} \left[ \frac{p-1}{(p-1)/2} \right] \equiv (-q; q)_{p-1}^2 - \frac{p^2-1}{24}(1-q)^2 [p]_q^2 \pmod{[p]_q^3} \] (1.8)
for any prime \( p \geq 5 \).

Theorem 1.3. Let \( p \geq 5 \) be a prime and \( m \geq 2 \) be an integer with \( p \nmid m \). Then
\[-1 \frac{p-1}{2} q^M \prod_{k=1}^{m-1} \left[ \frac{p-1}{[kp/m]} \right] \equiv \frac{m(q^m; q^m)_{p-1}}{(q; q)_{p-1}} - m + 1 \pmod{[p]_q^2}, \] (1.9)
where
\[ M = m \sum_{k=1}^{m-1} \left( \left\lfloor \frac{kp}{m} \right\rfloor + 1 \right). \]

The proofs of Theorems 1.1, 1.2 and 1.3 will be given in the next sections.

2. Some Lemmas

In this section we assume that \( p \) is a prime greater than 3. And the following lemmas will be used in the proofs of Theorems 1.1 and 1.2.

Lemma 2.1.
\[ \sum_{j=1}^{p-1} \frac{1}{[j]_q} \equiv \frac{p-1}{2}(1-q) \pmod{[p]_q}, \] (2.1)
\[ \sum_{j=1}^{p-1} \frac{q^j}{[j]_q^2} \equiv -\frac{p^2-1}{12}(1-q)^2 \pmod{[p]_q} \] (2.2)
and
\[ \sum_{j=1}^{p-1} \frac{1}{[j]_q^2} \equiv -(p-1)(p-5) \pmod{[p]_q}. \] (2.3)

Proof. See Theorem 4 in [A] and Lemma 2 in [SP].

Lemma 2.2.
\[ q^{kp} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (1-q)^j [p]_q^j. \]

Proof.
\[ \sum_{j=0}^{k} (-1)^j \binom{k}{j} (1-q)^j [p]_q^j = (1-(1-q)[p]_q)^k = (1-(1-q^p))^k = q^{kp}. \]
\[ \square \]

From Lemma 2.2, we deduce that
\[ q^{kp} \equiv 1-k(1-q)[p]_q + \frac{k(k-1)}{2} (1-q)^2 [p]_q^2 \pmod{[p]_q^3}. \] (2.4)
Lemma 2.3.
\[
4 \sum_{1 \leq j < k \leq p-1} \frac{(-1)^k}{[j]_q[k]_q} \equiv \left( \sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q} \right)^2 + (p - 3)(1 - q) \sum_{k=1}^{p-1} \frac{(-1)^k}{[k]_q} + \frac{(p - 1)(p + 7)}{12} (1 - q)^2 \quad (\text{mod } [p]_q). \tag{2.5}
\]

Proof. Since \( p \) is odd,
\[
\left( \sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q} \right)^2 = \left( \sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q} \right) \left( \sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q} - (1 - q) \sum_{j=1}^{p-1} (-1)^j \right)
= \left( \sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q} \right) \left( \sum_{j=1}^{p-1} \frac{(-q)^j}{[j]_q} \right)
= \sum_{k=2}^{2p-2} (-1)^k \min\{k-1,p-1\} \frac{q^j}{[j]_q[k-j]_q}.
\]

Then we have
\[
\left( \sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q} \right) \left( \sum_{j=1}^{p-1} \frac{(-q)^j}{[j]_q} \right) - (-1)^p \sum_{j=1}^{p-1} \frac{q^j}{[j]_q[p-j]_q}
= \sum_{k=2}^{p-1} (-1)^k \sum_{j=1}^{k-1} \frac{q^j}{[j]_q[k-j]_q} + \sum_{k=p+1}^{2p-2} (-1)^k \sum_{j=k-p+1}^{p-1} \frac{q^j}{[j]_q[k-j]_q}
= \sum_{k=2}^{p-1} (-1)^k \sum_{j=1}^{k-1} \frac{q^j}{[j]_q[k-j]_q} + \sum_{l=2}^{p-1} (-1)^{2p-l} \sum_{j=p-l+1}^{p-1} \frac{q^j}{[j]_q[2p-l-j]_q}
= \sum_{k=2}^{p-1} (-1)^k \sum_{j=1}^{k-1} \frac{q^j}{[j]_q[k-j]_q} + \sum_{l=2}^{p-1} (-1)^l \sum_{i=1}^{l-1} \frac{q^{p+i-l}}{[p+i-l]_q[p-i]_q}.
\]
(here \( l = 2p - k \))
(\text{ here } i = l + j - p)

Note that
\[
\frac{q^{p+i-l}}{[p+i-l]_q[p-i]_q} \equiv \frac{q^{p+i-l}(1-q)^2}{(1-q^{p+i-l})(1-q^{p-i})} \equiv \frac{q^i(1-q)^2}{(1-q^i)(1-q^i)} \quad (\text{mod } [p]_q).
\]
It follows that

\[
\left( \sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q} \right) \left( \sum_{j=1}^{p-1} \frac{(-q)^j}{[j]_q} \right) + \sum_{j=1}^{p-1} \frac{q^j}{[j]_q[p-j]_q}
\]

\[
\equiv \sum_{k=2}^{p-1} (-1)^k \sum_{j=1}^{k-1} \frac{q^j}{[j]_q[k-j]_q} + \sum_{l=2}^{p-1} (-1)^l \sum_{i=1}^{l-1} \frac{q^i}{[i]_q[l-i]_q}
\]

\[
= 2 \sum_{k=2}^{p-1} (-1)^k \sum_{j=1}^{k-1} \frac{q^j(1-q)^2}{1-q^k} \sum_{j=1}^{k-1} \left( \frac{1}{q^{k-j}-q^k} + \frac{1}{1-q^{k-j}} \right)
\]

\[
= 2 \sum_{1 \leq j < k \leq p-1} \frac{(-1)^k(q^k + q^j)}{[j]_q[k]_q} \pmod{[p]_q}.
\]

We can write

\[
\sum_{1 \leq j < k \leq p-1} \frac{(-1)^k(q^k + q^j)}{[j]_q[k]_q}
\]

\[
= \sum_{1 \leq j < k \leq p-1} \frac{(-1)^k(2 - (1 - q^k) - (1 - q^j))}{[j]_q[k]_q}
\]

\[
= 2 \sum_{1 \leq j < k \leq p-1} \frac{(-1)^k}{[j]_q[k]_q} - (1-q) \left( \sum_{k=2}^{p-1} \frac{(-1)^k(k-1)}{[k]_q} + \sum_{j=1}^{p-2} \frac{1}{[j]_q} \sum_{k=j+1}^{p-1} (-1)^k \right)
\]

Now

\[
\sum_{k=2}^{p-1} \frac{(-1)^k(k-1)}{[k]_q} = \frac{1}{2} \sum_{k=1}^{p-1} \left( \frac{(-1)^k(k-1)}{[k]_q} + \frac{(-1)^{p-k}(p-k-1)}{[p-k]_q} \right)
\]

\[
= \frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^k(k-1)}{[k]_q} \left( \frac{1}{[k]_q} + \frac{1}{[p-k]_q} \right) + \frac{p-2}{2} \sum_{k=1}^{p-1} \frac{(-1)^k}{[k]_q}
\]

\[
= \frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^k(k-1)}{[k]_q} \left( \frac{[p]_q}{[k][p-k]_q} + (1-q) \right) + \frac{p-2}{2} \sum_{k=1}^{p-1} \frac{(-1)^k}{[k]_q}
\]

\[
\equiv \frac{p-1}{4} (1-q) + \frac{p-2}{2} \sum_{k=1}^{p-1} \frac{(-1)^k}{[k]_q} \pmod{[p]_q}.
\]
And from (2.1) we have
\[ \sum_{j=1}^{p-2} \frac{1}{[j]_q} \sum_{k=j+1}^{p-1} (-1)^k = \frac{1}{2} \sum_{j=1}^{p-1} \frac{1}{[j]_q} \left[ 1 - (-1)^j \right] \]
\[ \equiv \frac{(p-1)}{4} (1-q) - \frac{1}{2} \sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q} \pmod{[p]_q}. \]

Finally by (2.3),
\[ \sum_{j=1}^{p-1} \frac{q^j}{[j]_q[p-j]_q} = \frac{1}{2} \sum_{j=1}^{p-1} \frac{q^{p-j}}{[j]_q[p-j]_q} = \frac{1}{2} \sum_{j=1}^{p-1} \frac{q^p}{[j]_q([p]-[j]_q)} \]
\[ \equiv \frac{(p-1)(p-5)}{12} (1-q)^2 \pmod{[p]_q}. \]

Thus combining the equations and congruences above, we obtain that
\[ 4 \sum_{1 \leq j < k \leq p-1} (-1)^k \frac{1}{[j]_q[k]_q} - \left( \sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q} \right) \left( \sum_{j=1}^{p-1} \frac{(-q)^j}{[j]_q} \right) \]
\[ \equiv (p-3)(1-q) \sum_{k=1}^{p-1} \frac{(-1)^k}{[k]_q} + \frac{(p-1)(p+7)}{12} (1-q)^2 \pmod{[p]_q}. \]

\[ \square \]

**Lemma 2.4.**
\[ \sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q} \equiv 2 \sum_{j=1}^{(p-1)/2} \frac{1}{[2j]_q} - \frac{p-1}{2} (1-q) - \frac{p^2-1}{24} (1-q)^2 [p]_q \pmod{[p]_q^2} \] (2.6)

**Proof.** Clearly
\[ \sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q} = \sum_{j=1}^{(p-1)/2} \frac{1}{[2j]_q} - \sum_{j=1}^{(p-1)/2} \frac{1}{[2j-1]_q} = \sum_{j=1}^{(p-1)/2} \frac{1}{[2j]_q} - \sum_{j=1}^{(p-1)/2} \frac{1}{[p-2j]_q}. \]

Observe that
\[ \frac{1}{[p-2j]_q} = \frac{q^{2j} ([p]_q + [2j]_q)}{[p]_q - [2j]_q} = q^{2j} ([p]_q + [2j]_q) \pmod{[p]_q^2}. \]
And by (2.2), we have
\[- \frac{p^2 - 1}{12} (1 - q)^2 \equiv \sum_{j=1}^{p-1} \frac{q^j}{[j]_q^2} = \frac{1}{\sum_{j=1}^{(p-1)/2} \frac{q^{2j}}{[2j]_q^2} + \sum_{j=1}^{(p-1)/2} \frac{q^{p-2j}}{[p-2j]_q^2}}
= \sum_{j=1}^{(p-1)/2} \frac{q^{2j}}{[2j]_q^2} + \sum_{j=1}^{(p-1)/2} \frac{q^{p+2j}}{(\lfloor p \rfloor_q - [2j]_q)^2}
\equiv 2 \sum_{j=1}^{(p-1)/2} \frac{q^{2j}}{[2j]_q^2} \pmod{\lfloor p \rfloor_q^2}.
\]

Hence
\[
\sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q} \equiv \sum_{j=1}^{(p-1)/2} \frac{1}{[2j]_q} + \sum_{j=1}^{(p-1)/2} \frac{q^{2j}([\lfloor p \rfloor_q + [2j]_q)}{[2j]_q^2}
= \sum_{j=1}^{(p-1)/2} \frac{1 + q^{2j}}{[2j]_q} - \frac{p^2 - 1}{24} (1 - q)^2 \lfloor p \rfloor_q
\equiv 2 \sum_{j=1}^{(p-1)/2} \frac{1}{[2j]_q} - \frac{p - 1}{2} (1 - q) - \frac{p^2 - 1}{24} (1 - q)^2 \lfloor p \rfloor_q \pmod{\lfloor p \rfloor_q^2}.
\]

3. PROOFS OF THEOREMS 1.1 AND 1.2

Proof of Theorem 1.1. One can directly verify (1.5) when \( p = 3 \). So below we assume that \( p \geq 5 \). It is well-known (cf. Corollary 10.2.2 of [AAR]) that
\[(x; q)_n = \sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right]_q q^{(j)}(-x)^j.\]
Then we have
\[
\frac{(-1; q)_p - q^{(\frac{p}{2})}_q - 1}{[p]_q} = \frac{1}{[p]_q} \sum_{k=1}^{p-1} \left[ \begin{array}{c} p \\ k \end{array} \right]_q q^{(\frac{k}{2})}
= \sum_{k=1}^{p-1} \frac{1}{[k]_q} \prod_{j=1}^{k-1} \frac{q^j(1 - q^{p-j})}{1 - q^j}
= \sum_{k=1}^{p-1} \frac{1}{[k]_q} \prod_{j=1}^{k-1} \left( \frac{[p]_q}{[j]_q} - 1 \right)
\equiv [p]_q \sum_{1 \leq j < k \leq p-1} \frac{(-1)^k}{[j]_q[k]_q} - \sum_{k=1}^{p-1} \frac{(-1)^k}{[k]_q} \pmod{[p]_q^2}.
\]
Consequently
\[
\sum_{k=1}^{p-1} \frac{(-1)^k}{[k]_q} \equiv -\frac{(1; q)_{q} - q(\frac{p}{2})_q - 1}{[p]_q}\equiv\frac{2(-q; q)_{q-1} - 2}{[p]_q} - \frac{p - 1}{2} (1 - q) \pmod{[p]_q}.
\]
Thus applying Lemma 2.3, we have
\[
\sum_{1 \leq j < k \leq p-1} \frac{(-1)^k}{[j]_q[k]_q} \equiv \frac{(p - 1)(p + 7)}{48} (1 - q)^2
\]
\[
\equiv \frac{1}{4} \left( \sum_{k=1}^{p-1} \frac{(-1)^k}{[k]_q} \right) \left( \sum_{k=1}^{p-1} \frac{(-1)^k}{[k]_q} + (p - 3)(1 - q) \right)
\]
\[
\equiv \frac{1}{4} \left( -2Q_p(2, q) - \frac{p - 1}{2} (1 - q) \right) \left( -2Q_p(2, q) + \frac{p - 5}{2} (1 - q) \right)
\]
\[
= Q_p(2, q)^2 + Q_p(2, q)(1 - q) - \frac{(p - 1)(p - 5)}{16} (1 - q)^2 \pmod{[p]_q}.
\]
On the other hand, it follows from (2.4) that
\[
\frac{(1; q)_{q} - q(\frac{p}{2})_q - 1}{[p]_q} \equiv \frac{2(-q; q)_{q-1} - 2}{[p]_q} + \frac{p - 1}{2} (1 - q) - \frac{(p - 1)(p - 3)}{8} (1 - q)^2 [p]_q \pmod{[p]_q^2}.
\]
Then by Lemma 2.4,
\[
\sum_{k=1}^{p-1} \frac{(-1)^k}{[k]_q} + \frac{(1; q)_{q} - q(\frac{p}{2})_q - 1}{[p]_q}
\]
\[
\equiv 2 \sum_{j=1}^{(p-1)/2} \frac{1}{[2j]_q} + 2Q_p(2, q) - \frac{(p - 1)(p - 2)}{6} (1 - q)^2 [p]_q \pmod{[p]_q^2}.
\]
Combining (3.1), (3.2) and (3.3), the desired (1.5) is obtained. □

**Proof of Theorem 1.2.** Since
\[
\left[ \frac{p - 1}{(p - 1)/2} \right]_{q^2} = \prod_{j=1}^{(p-1)/2} \left[ \frac{p - j}{[j]_q^2} \right]_{q^2} = \prod_{j=1}^{(p-1)/2} \frac{[p]_q^2 - [j]_q^2}{q^2 [j]_q^2},
\]
we have
\[
(-1)^{\frac{p - 1}{2} q^{\frac{p - 1}{2}}} \left[ \frac{p - 1}{(p - 1)/2} \right]_{q^2}
\]
\[
= \prod_{j=1}^{(p-1)/2} \left( 1 - \frac{[p]_q^2}{[j]_q^2} \right)
\]
\[
\equiv 1 - \frac{1 + q^p}{1 + q} \sum_{j=1}^{(p-1)/2} \frac{[p]_q}{[j]_q^2} \left( 1 + q^p \right)^2 \sum_{1 \leq j < k \leq (p-1)/2} \frac{[p]_q^2}{[j]_q^2 [k]_q^2} \pmod{[p]_q^3}.
\]
From Theorem 1.1, we deduce that

\[
\frac{1 + q^p}{1 + q} \sum_{j=1}^{(p-1)/2} \frac{1}{[j]_q^2} = (1 + q^p) \sum_{j=1}^{(p-1)/2} \frac{1}{[2j]_q} \\
\equiv -(1 + q^p) Q_p(2, q) + \frac{1 + q^p}{2} \left( Q_p(2, q)^2 + Q_p(2, q)(1 - q) + \frac{p^2 - 1}{8} (1 - q)^2 \right) [p]_q
\]

\[
\equiv -(1 + q^p) Q_p(2, q) + Q_p(2, q)[p]_q + Q_p(2, q)(1 - q)[p]_q + \frac{p^2 - 1}{8} (1 - q)^2 [p]_q
\]

(mod \([p]_q^2\)).

Notice that

\[
\sum_{1 \leq j < k \leq (p-1)/2} \frac{1}{[j]_q^2[k]_q^2} = \frac{1}{2} \left( \left( \sum_{j=1}^{(p-1)/2} \frac{1}{[j]_q^2} \right)^2 - \sum_{j=1}^{(p-1)/2} \frac{1}{[j]_q^2} \right)
\]

\[
= \frac{(1 + q^p)^2}{2} \left( \left( \sum_{j=1}^{(p-1)/2} \frac{1}{[2j]_q} \right)^2 - \sum_{j=1}^{(p-1)/2} \frac{1}{[2j]_q} \right).
\]

And Theorem 1.1 implies that

\[
\sum_{j=1}^{(p-1)/2} \frac{1}{[2j]_q} \equiv -Q_p(2, q) \text{ (mod \([p]_q\))}.
\]

Then using (2.7),

\[
\sum_{j=1}^{(p-1)/2} \frac{1}{[2j]_q} = \sum_{j=1}^{(p-1)/2} \frac{q^{2j}}{[2j]_q^2} + (1 - q) \sum_{j=1}^{(p-1)/2} \frac{1}{[2j]_q}
\]

\[
\equiv - \frac{p^2 - 1}{24} (1 - q)^2 - Q_p(2, q)(1 - q) \text{ (mod \([p]_q\))}.
\]

Consequently

\[
\frac{2}{(1 + q)^2} \sum_{1 \leq j < k \leq (p-1)/2} \frac{1}{[j]_q^2[k]_q^2} \equiv Q_p(2, q)^2 + Q_p(2, q)(1 - q) + \frac{p^2 - 1}{24} (1 - q)^2 \text{ (mod \([p]_q\))}.
\]
Thus it follows from (3.4), (3.5) and (3.6) that
\[ (-1)^{\frac{p-1}{2}} q^{p^2-1} \left\lfloor \frac{p-1}{(p-1)/2} \right\rfloor_q^2 - 1 \]
\[ \equiv [p]_q^2 \cdot \frac{4}{(1+q)^2} \sum_{1 \leq j < k \leq (p-1)/2} \frac{1}{[j]_q [k]_q^2} - [p]_q \cdot \frac{1+q^p}{1+q} \sum_{j=1}^{(p-1)/2} \frac{1}{[j]_q^2} \]
\[ \equiv Q_p(2, q)^2 [p]_q^2 + (1+q^p)Q_p(2, q)[p]_q + Q_p(2, q)(1-q)[p]_q^2 - \frac{p^2-1}{24} (1-q)^2 [p]_q^2 \]
\[ \equiv ((-q; q)_{p-1} - 1)((-q; q)_{p-1} + 1) - \frac{p^2-1}{24} (1-q)^2 [p]_q^2 \pmod{[p]_q^3}. \]

□

4. Fermat Quotient

Lemma 4.1. Let \( p \) be an odd prime. Suppose that \( m \) is a positive integer with \( (m, p) = 1 \). Then
\[ Q_p(m, q) \equiv \sum_{j=1}^{p-1} \left\lfloor \frac{jm}{p} \right\rfloor - \frac{(p-1)(m-1)}{2}(1-q) \pmod{[p]_q}. \quad (4.1) \]

Proof. For each \( j \in \{1, 2, \ldots, p-1\} \), let
\[ r_j = jm - \left\lfloor \frac{jm}{p} \right\rfloor p. \]

Then
\[ \frac{(q^m; q^m)_{p-1}}{(q; q)_{p-1}} = \prod_{j=1}^{p-1} \frac{1-q^{jm}}{1-q^2} = \prod_{j=1}^{p-1} \left( \frac{1-q^{r_j}}{1-q^2} + \frac{q^{r_j}(1-q^{jm/p})}{1-q^2} \right) \]
\[ = \prod_{j=1}^{p-1} \frac{1-q^{r_j}}{1-q^2} \left( 1 + \frac{q^{r_j}(1-q^{jm/p})}{1-q^{r_j}} \right) \]
Since \( r_j \) runs through \( 1, 2, \ldots, p-1 \) as \( j \) does so, we have
\[ \frac{(q^m; q^m)_{p-1}}{(q; q)_{p-1}} = \prod_{j=1}^{p-1} \left( 1 + \frac{q^{r_j}(1-q^{jm/p})}{1-q^{r_j}} \right) \]
\[ \equiv 1 + (1-q^p) \sum_{j=1}^{p-1} \frac{q^{r_j}}{1-q^{r_j}} \cdot \frac{1-q^{jm/p}}{1-q^p} \]
\[ \equiv 1 + (1-q^p) \sum_{j=1}^{p-1} \left\lfloor \frac{jm}{p} \right\rfloor \frac{q^{r_j}}{1-q^{r_j}} \]
\[ \equiv 1 + [p]_q \sum_{j=1}^{p-1} \left\lfloor \frac{jm}{p} \right\rfloor \frac{q^{jm}}{[jm]_q} \pmod{[p]_q^2}. \]
Finally,
\[
\sum_{j=1}^{p-1} \left\lfloor \frac{jm}{p} \right\rfloor \frac{q^{jm}}{[jm]_q} = \sum_{j=1}^{p-1} \left\lfloor \frac{jm}{p} \right\rfloor - (1 - q) \sum_{j=1}^{p-1} \left\lfloor \frac{jm}{p} \right\rfloor
\]
\[
= \sum_{j=1}^{p-1} \left\lfloor \frac{jm}{p} \right\rfloor - \frac{(p-1)(m-1)}{2} (1 - q).
\]

We are done. □

Remark. Letting \( q \to 1 \) in (4.1), we obtain that
\[
\frac{mp - m}{p} \equiv \sum_{j=1}^{p-1} \frac{\left\lfloor \frac{jm}{p} \right\rfloor}{j} \pmod{p},
\]
which was firstly discovered by Lerch [Ler].

Proof of Theorem 1.3. We write
\[
\left[ \frac{p-1}{[kp/m]} \right] q^m = \prod_{j=1}^{\lfloor kp/m \rfloor} \frac{[p]_{q^m} - [j]_{q^m}}{q^m [j]_{q^m}} = q^{-m(\frac{\lfloor kp/m \rfloor}{2} + 1)} \prod_{j=1}^{\lfloor kp/m \rfloor} \left( \frac{[p]_{q^m}}{[j]_{q^m}} - 1 \right).
\]
As \( p \nmid m, [p]_q \) divides \([p]_{q^m} = (1 - q^{mp})/(1 - q^m)\). Thus
\[
(-1)^{(p-1)(m-1)/2} q^{\sum_{k=1}^{m-1} m(\frac{\lfloor kp/m \rfloor}{2} + 1)} \prod_{k=1}^{m-1} \left[ \frac{p-1}{[kp/m]} \right] q^m
\]
\[
= \prod_{k=1}^{m-1} \prod_{j=1}^{\lfloor kp/m \rfloor} \left( 1 - \frac{[p]_{q^m}}{[j]_{q^m}} \right)
\]
\[
\equiv 1 - [p]_{q^m} \sum_{k=1}^{m-1} \sum_{1 \leq j < kp/m} \frac{1}{[j]_{q^m}}
\]
(Here \( kp/m \not\in \mathbb{Z} \), so \( j \leq \lfloor kp/m \rfloor < kp/m \))
\[
= 1 - [p]_{q^m} \sum_{j=1}^{p-1} \frac{m - 1 - \left\lfloor \frac{jm}{p} \right\rfloor}{[j]_{q^m}} \pmod{[p]_{q}^2}.
\]

In view of (2.1) and Lemma 4.1, we have
\[
[p]_{q^m} \sum_{j=1}^{p-1} \frac{m - 1 - \left\lfloor \frac{jm}{p} \right\rfloor}{[j]_{q^m}}
\]
\[
= (m - 1)[mp]_q \sum_{j=1}^{p-1} \frac{1}{[jm]_q} - [mp]_q \sum_{j=1}^{p-1} \frac{\left\lfloor \frac{jm}{p} \right\rfloor}{[jm]_q}
\]
\[
\equiv (m - 1)[mp]_q \cdot \frac{(p-1)}{2} (1 - q) - [mp]_q Q_p(m, q) - \frac{(p-1)(m-1)}{2} (1 - q)[mp]_q
\]
\[
\equiv - m[p]_q Q_p(m, q) \pmod{[p]_{q}^2}.
\]
This concludes our proof. □

**Remark.** For further developments of Granville’s congruence (1.7), the reader is referred to [S].

5. **A conjecture of Skula**

Recently with help of polynomials over finite fields, Granville [G2] confirmed a conjecture of Skula:

\[
\left( \frac{2^{p-1} - 1}{p} \right)^2 \equiv - \sum_{j=1}^{p-1} \frac{2^{j}}{j^2} \ (\text{mod } [p]_q) \tag{5.1}
\]

for any prime \( p \geq 5 \). Using our \( q \)-analogue of Lehmer’s congruence, we have the following \( q \)-analogue of (5.1):

**Theorem 5.1.** Let \( p \geq 5 \) be a prime. Then

\[
\sum_{j=1}^{p-1} \frac{q^j(-q; q)_j}{[j^2]_q} + Q_p(2, q)^2 \\
\equiv - (p - 1)Q_p(2, q)(1 - q) - \frac{(7p - 5)(p - 1)}{24}(1 - q)^2 \ (\text{mod } [p]_q). \tag{5.2}
\]

**Lemma 5.2.**

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{\frac{n-k}{2}}(-q; q)_k = (-1)^n q^{\frac{n+1}{2}}. \tag{5.3}
\]

**Proof.** From the well-known \( q \)-binomial theorem (cf. Theorem 10.2.1 of [AAR]), we have

\[
\sum_{k=0}^{\infty} (-1)^k q^{\frac{k}{2}}(q; q)_k x^k = (x; q)_\infty
\]

and

\[
\sum_{k=0}^{\infty} \frac{(-q; q)_k}{(q; q)_k} x^k = \frac{(-qx; q)_\infty}{(x; q)_\infty}.
\]

Then by comparing the coefficient of \( x^n \) in the both sides of

\[
(x; q)_\infty \cdot \frac{(-qx; q)_\infty}{(x; q)_\infty} = (-qx; q)_\infty,
\]

we obtain that

\[
\sum_{k=0}^{n} \frac{(-1)^{n-k} q^{\frac{n-k}{2}}(-q; q)_k}{(q; q)_{n-k}(q; q)_k} = \frac{q^{\binom{n}{2} + n}}{(q; q)_n},
\]

which is an equivalent form of (5.3). □
Corollary 5.3. For any odd prime $p$, we have
\[ \sum_{j=1}^{p-1} \frac{q^j(-q;q)_j}{[j]_q} \equiv -2Q_p(2,q) - (p-1)(1-q) \pmod{[p]_q}. \] (5.4)

Proof. From Lemma 5.2, we deduce that
\[
\sum_{j=1}^{p-1} \frac{q^j(-q;q)_j}{[j]_q} \equiv \sum_{j=1}^{p-1} \frac{q^{p(p-1)/2-jp+j}(-q;q)_j}{[j]_q} \\
= -\frac{1}{[p]_q} \sum_{j=1}^{p-1} (-1)^j q^{(\frac{p}{2})j+j} \left[ \frac{p}{j} \right]_q (-q;q)_j \\
= -\frac{1}{[p]_q} \sum_{j=1}^{p-1} (-1)^j q^{(\frac{p}{2}-j)} \left[ \frac{p}{j} \right]_q (-q;q)_j \\
= -\frac{1}{[p]_q} ((-1)^p q^{(\frac{p+1}{2})} - q^{(\frac{p}{2})} - (-1)^p (-q;q)_p) \pmod{[p]_q}.
\]

Notice that
\[
\frac{2 - q^{(\frac{p+1}{2})} - q^{(\frac{p}{2})}}{[p]_q} \equiv \frac{p+1}{2}(1-q) + \frac{p-1}{2}(1-q) = p(1-q) \pmod{[p]_q},
\]

and that
\[
\frac{(-q;q)_p - 2}{[p]_q} = \frac{(-q;q)_{p-1}(1+q^p) - 2}{[p]_q} = \frac{2(-q;q)_{p-1} - 2}{[p]_q} - (1-q)(-q;q)_{p-1} \\
\equiv 2Q_p(2,q) - (1-q) \pmod{[p]_q}.
\]

Hence
\[
\sum_{j=1}^{p-1} \frac{q^j(-q;q)_j}{[j]_q} \equiv -\frac{(-q;q)_p - q^{(\frac{p+1}{2})} - q^{(\frac{p}{2})}}{[p]_q} \\
\equiv -2Q_p(2,q) - (p-1)(1-q) \pmod{[p]_q}.
\]

We are done. \(\blacksquare\)

Remark. Corollary 5.3 is the $q$-analogue of an observation of Glashier:
\[
\frac{2^{p-1} - 1}{p} \equiv -\sum_{j=1}^{p-1} \frac{2^{j-1}}{j} \pmod{p}. \] (5.5)
Lemma 5.4.

\[ \sum_{k=1}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} \binom{-q}{k}_q = q^{\binom{n}{2}} \sum_{k=1}^{n} \frac{(-q)^k - 1}{[k]_q}. \] (5.6)

**Proof.** We make an induction on \( n \). The case \( n = 1 \) is trivial. Assume that \( n > 1 \) and that (5.6) holds for the smaller values of \( n \). Then we conclude that

\[
\sum_{k=1}^{n} (-1)^k \begin{bmatrix} n \\ n-k \end{bmatrix}_q q^{\binom{n-k}{2}} \binom{-q}{k}_q = \sum_{k=1}^{n} (-1)^k \left( q^{\binom{n-1}{2}} \frac{(-q)_k}{[k]_q} + \frac{1}{[n]_q} \sum_{k=1}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k-1}{2}} \binom{-q}{k}_q \right) = q^{\binom{n}{2}} \sum_{k=1}^{n-1} (-1)^k \left( q^{\binom{n-1-k}{2}} \binom{-q}{k}_q \right) + \frac{1}{[n]_q} (-1)^n q^{\binom{n+1}{2}} - q^{\binom{n}{2}},
\]

where in the last step we apply the induction hypothesis and Lemma 5.2. \(\square\)

**Proof of Theorem 5.1.** Using Lemma 5.4, we have

\[
\sum_{j=1}^{p-1} \frac{q^j (-q)}{[j]_q^{2}} = \sum_{j=1}^{p-1} \frac{q^{p(p-1)/2-jp+j} (-q)}{[j]_q^{2}}
\]

\[
= -\frac{1}{[p]_q} \sum_{j=1}^{p-1} (-1)^j q^{\binom{p-j}{2}} \begin{bmatrix} p \\ j \end{bmatrix}_q \frac{(-q)_j}{[j]_q} = \frac{q^{\binom{p}{2}}}{[p]_q} \sum_{j=1}^{p} \frac{(-q)^j - 1}{[j]_q} + (-1)^p \frac{(-q)_p}{[p]_q^{2}}
\]

\[
= q^{\binom{p}{2}} \sum_{j=1}^{p-1} \frac{(-q)^j - 1}{[j]_q} -\frac{(-q)_p - q^{\binom{p+1}{2}} - q^{\binom{p}{2}}}{[p]_q^{2}} \pmod{[p]_q}.
\]
With help of (1.3) and Theorem 1.1,

\[
\sum_{j=1}^{p-1} \frac{(-q)^j - 1}{[j]_q} = -\sum_{j=1}^{(p-1)/2} \frac{2}{[2j-1]_q} \equiv -2Q_p(2, q) + Q_p(2, q)^2[p]_q + Q_p(2, q)(1 - q)[p]_q
\]

\[
+ \frac{p^2 - 1}{8} (1 - q)^2[p]_q - ((p - 1)(1 - q) + \frac{p^2 - 1}{12} (1 - q)^2[p]_q)
\]

\[
\equiv -\frac{(p - 1)(1 - q) - 1}{[p]_q} \sum_{j=1}^{p-1} \frac{(-q)^j - 1}{[j]_q} \equiv -\frac{(7p - 5)(p - 1)}{24} (1 - q)^2 \pmod{[p]_q}.
\]

And by (2.4) we have

\[
\frac{(-q; q)_p - q^{(p+1)/2} - q^{(p)}_q}{[p]_q^2} = \frac{2(-q; q)_{p-1} - q^{(p+1)/2} - q^{(p)}_q}{[p]_q^2} - \frac{(-q; q)_{p-1}}{[p]_q} (1 - q)
\]

\[
\equiv \frac{2(-q; q)_{p-1}}{[p]_q^2} - \frac{2}{[p]_q^2} + \frac{p}{[p]_q} (1 - q) - \frac{(p - 1)^2}{4} (1 - q)^2 - \frac{(-q; q)_{p-1}}{[p]_q} (1 - q)
\]

\[
\equiv 2Q_p(2, q)[p]_q + \frac{(p - 1)(1 - q)}{[p]_q} - \frac{(p - 1)^2}{4} (1 - q)^2 - Q_p(2, q)(1 - q) \pmod{[p]_q}.
\]

Therefore

\[
\sum_{j=1}^{p-1} \frac{q^j (-q; q)_j}{[j]_q^2} \equiv \left( \frac{(p - 1)(1 - q)}{2} - \frac{1}{[p]_q} \right) \sum_{j=1}^{p-1} \frac{(-q)^j - 1}{[j]_q} \frac{(-q; q)_p - q^{(p+1)/2} - q^{(p)}_q}{[p]_q^2}
\]

\[
\equiv -\frac{(p - 1)(1 - q) - Q_p(2, q)(1 - q) - Q_p(2, q)^2}{[p]_q^2} \equiv -\frac{(7p - 5)(p - 1)}{24} (1 - q)^2 \pmod{[p]_q}.
\]

\[\square\]

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