BOUND STATES
OF
1+1 DIMENSIONAL FIELD THEORIES

T. Fujita, M. Hiramoto and H. Takahashi
Department of Physics, Faculty of Science and Technology
Nihon University, Tokyo, Japan

ABSTRACT

We discuss the bound states of the massive Thirring model. Here, the periodic boundary condition equations for the Bethe ansatz solutions are numerically solved. It is found that the massive Thirring model has only one bound state and the bound state spectrum as the function of the coupling constant agrees with that of infinite momentum frame prescription by Fujita and Ogura. Boson boson states (2p−2h states) appear only as the continuum spectrum without making any bound states.

Further, the finite size correction to the vacuum energy is estimated. The evaluated central charge is found to be close to unity.
1. Introduction

The sine-Gordon field theory or the massive Thirring model is believed to
be solved exactly. Dashen, Hasslacher and Neveu presented their solutions
to the quantum sine-Gordon model [1]. Although they use semiclassical
approximations, they consider their solutions to be exact. This spectrum
is translated into the massive Thirring model and the bound state mass $\mathcal{M}$
(vector boson) is written as

$$
\mathcal{M} = 2m \sin \frac{\pi}{2} \left( 1 + \frac{\frac{n}{2 \pi}}{1 + \frac{2g_0}{\pi}} \right)
$$

(1.1)

where $n$ is an integer and runs from 1 to $(1 + \frac{2g_0}{\pi})$. $m$ is the fermion mass of
the massive Thirring model. $g_0$ is the coupling constant with Schwinger’s
normalization.

Further, this spectrum is confirmed by the Bethe ansatz solution [2]. This
was very important since the Bethe ansatz wave function is indeed exact. In
their paper, Bergknoff and Thacker presented their solutions of the massive
Thirring model based on the string hypothesis when they solve equations of
the periodic boundary conditions (PBC) from Bethe ansatz wave functions.

However, Fujita and Ogura [3] have recently presented their solutions of the
massive Thirring model employing infinite momentum frame prescription.
Their spectrum is quite different from eq. (1.1). There is only one bound
state. However, the bound state energy is rather close to the lowest energy
of eq.(1.1). The deviation is about $10 \sim 20 \%$ from each other depending
on the coupling constant. The boson mass $\mathcal{M}$ is given as

$$
\tan \frac{\alpha}{2} - \alpha = \frac{g}{\pi} \left[ 1 + \frac{1}{\cos^2 \alpha} (1 - \frac{g}{4\pi}) \right]
$$

(1.2)

where the boson mass $\mathcal{M}$ is related to $\alpha$ as,

$$
\mathcal{M} = 2m \cos \alpha.
$$
$g$ is a coupling constant of the massive Thirring model with Johnson’s normalization.

It turns out that the solution eq.(1.2) has all the proper behaviors of the weak and strong coupling limits. Instead, if one checks eq.(1.1) carefully, then one sees that the semiclassical result of eq.(1.1) does not have a proper weak coupling limit. There, the important point is that one has to take into account current regularizations in a correct way [3].

On the other hand, the Bethe ansatz wave function is well known to be exact. This is a strong reason why people have believed for almost two decades that the bound state spectrum obtained from the semiclassical approximation is exact in spite of the fact that they took into account only the lowest quantum fluctuations in the path integral.

In this report, we reexamine the Bethe ansatz solutions for the massive Thirring model and discuss problems in the treatment by Bergknoff and Thacker [2]. In particular, we show that the string configurations taken by Bergknoff and Thacker do not satisfy the PBC equations. The reason why they have to introduce the string picture is because they solve the PBC equations for the density of states. Therefore, they could not determine proper rapidities for the positive energy particles.

Here, we have solved the PBC equations numerically. We consider a few hundred particles to a few thousand particles to make a vacuum. Then, we make one particle-one hole pairs, two particle two hole pairs and so on. It is found that there is only one bound state for one particle-one hole ($1p − 1h$) configuration. There is no bound state for two particle two hole cases. Further, the bound state energy calculated from the Bethe ansatz PBC equations turns out to be consistent with that of Fujita-Ogura’s solution [eq. (1.2)] though we can solve only a limited region of the coupling
constant.

Further, we find the boson boson scattering states in $2p-2h$ configurations. Here, it is important to note that the boson boson scattering states have rapidity variables which are all real. Therefore, there is no string-like solution which satisfies the PBC equations.

2. Solutions of PBC equations

The massive Thirring model is a 1+1 dimensional field theory with current current interactions. Its lagrangian density can be written as

$$\mathcal{L} = \bar{\psi}(i\gamma_\mu \partial^\mu - m_0)\psi - \frac{1}{2}g_0 j_\mu j_\mu$$

(2.1)

where the fermion current $j_\mu$ is written as

$$j_\mu = :\bar{\psi}\gamma_\mu \psi :.$$  

(2.2)

Choosing a basis where $\gamma_5$ is diagonal, the hamiltonian is written

$$H = \int dx \left[ -i(\psi_1^\dagger \frac{\partial}{\partial x} \psi_1 - \psi_2^\dagger \frac{\partial}{\partial x} \psi_2) + m_0(\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1) + 2g_0 \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1 \right].$$

(2.3)

The hamiltonian eq.(2.3) can be diagonalized by the Bethe ansatz wave functions. The Bethe ansatz wave functions satisfy the eigenvalue equation. However, they still do not have proper boundary conditions. The simplest way to define field theoretical models is to put the theory in a box of length $L$ and impose periodic boundary conditions (PBC) on the states.

Therefore, we demand that $\Psi(x_1, \ldots, x_N)$ be periodic in each argument $x_i$. This gives the boundary condition

$$\Psi(x_i = 0) = \Psi(x_i = L).$$  

(2.4)
This leads to the following PBC equations,

$$\exp(i m_0 L \sinh \beta_i) = \exp(-i \sum_j \phi(\beta_i - \beta_j)). \quad (2.5)$$

Taking the logarithm of eq.(2.5), we obtain

$$m_0 L \sinh \beta_i = 2\pi n_i - \sum_j \phi(\beta_i - \beta_j) \quad (2.6)$$

where $n_i$’s are integer. These are equations which we should now solve.

First, we want to make a vacuum. We write the PBC equations for the vacuum which is filled with negative energy particles ($\beta_i = i\pi - \alpha_i$),

$$\sinh \alpha_i = \frac{2\pi n_i}{L_0} - \frac{2}{L_0} \sum_{j \neq i} \tan^{-1} \left[ \frac{1}{2} g_0 \tanh \frac{1}{2} (\alpha_i - \alpha_j) \right], \quad (i = 1, \ldots, N). \quad (2.7)$$

Now, $n_i$ runs as

$$n_i = 0, \pm 1, \pm 2, \ldots, \pm N_0.$$  

We fix the values of $L_0$ and $N$, and then can solve eq.(2.7). This determines the vacuum. In this case, the vacuum energy $E_v$ can be written as

$$E_v = -\sum_{i=-N_0}^{N_0} m_0 \cosh \alpha_i. \quad (2.8)$$

Next, we want to make one particle-one hole (1p−1h) state. That is, we take out one negative energy particle ($i_0$-th particle) and put it into a positive energy state. In this case, the PBC equations become

$$i \neq i_0$$

$$\sinh \alpha_i = \frac{2\pi n_i}{L_0} - \frac{2}{L_0} \sum_{j \neq i} \tan^{-1} \left[ \frac{1}{2} g_0 \coth \frac{1}{2} (\alpha_i + \beta_{i_0}) \right]$$

$$- \frac{2}{L_0} \sum_{j \neq i, i_0} \tan^{-1} \left[ \frac{1}{2} g_0 \tanh \frac{1}{2} (\alpha_i - \alpha_j) \right] \quad (2.9a)$$

$$i = i_0$$

$$\sinh \beta_{i_0} = \frac{2\pi n_{i_0}}{L_0} + \frac{2}{L_0} \sum_{j \neq i_0} \tan^{-1} \left[ \frac{1}{2} g_0 \coth \frac{1}{2} (\beta_{i_0} + \alpha_j) \right]. \quad (2.9b)$$
where $\beta_{i_0}$ can be a complex variable as long as it can satisfy eqs. (2.9).

These PBC equations determine the energy of the one particle-one hole states which we denote by $E_{1p1h}^{(i_0)}$,

$$E_{1p1h}^{(i_0)} = m_0 \cosh \beta_{i_0} - \sum_{i=-N_0}^{N_0} m_0 \cosh \alpha_i.$$  \hspace{1cm} (2.10)

In the same way, we can set up the PBC equations for $2p-2h$ states.

3. Numerical results

These PBC equations are solved numerically and we obtain the energies for the vacuum as well as the $1p-1h$ and the $2p-2h$ states. The bare numbers of the calculated energies are shown in Table 1.

We now define the excitation energies with respect to the vacuum energy

$$\Delta E^{(0)} = E_{1p1h}^{(0)} - E_v$$

$$\Delta E^{(1)} = E_{1p1h}^{(1)} - E_v.$$  \hspace{1cm} (3.1a)

It turns out that these energies can be parametrized as

$$\Delta E_{1p1h}^{(0)} = m_0 \left( A_0 + B_0 \left( \frac{\rho}{m_0} \right)^\alpha \right) \hspace{1cm} (3.1a)$$

$$\Delta E_{1p1h}^{(1)} = m_0 \left( A_1 + B_1 \left( \frac{\rho}{m_0} \right)^\alpha \right). \hspace{1cm} (3.1b)$$

Now, we want to let $\rho \to \infty$, keeping $\Delta E_{1p1h}^{(0)}$ and $\Delta E_{1p1h}^{(1)}$ finite. Since $\alpha$ is smaller than unity, we can make a fine-tuning of $m_0$ such that

$$m_0^{1-\alpha} \rho^\alpha = \text{finite}.$$  \hspace{1cm} (3.2)

In this case, we can identify the mass of the bound state $M$ as

$$M = 2m \lim_{\rho \to \infty} \left( \frac{\Delta E_{1p1h}^{(0)}}{\Delta E_{1p1h}^{(1)}} \right) = 2m \frac{B_0}{B_1}.$$  \hspace{1cm} (3.2)
Here \( m \) is the physical fermion mass.

In Table 2, we show our calculated result of the boson state as the function of the coupling constant \( g \). As can be seen from the Table 2, the calculated boson masses agree with those of the infinite momentum frame calculation by Fujita and Ogura.

We have also calculated the 2p–2h configuration and found that there is no bound state in this configuration. Instead, the boson boson states appear as the continuum spectrum.

4. Finite size correction

Since we solve the field theory in the box with its length \( L \), we can calculate the finite size correction to the vacuum energy.

For the massless case, the finite size correction is written as

\[
\Delta E = -\frac{\pi}{6} \frac{c}{L} + ...
\]

where \( c \) denotes central charge and should be equal to unity for the Thirring model. In our calculations, the values of \( c \) are found to be unity for the negative values of the coupling constant \( g \). This is a good evidence that we solved the PBC equations properly.

5. Discussions and further studies

We have presented a new interpretation of the Bethe ansatz solutions of the massive Thirring model. Here, we solve the PBC equations directly but
numerically without referring to the density of states or *string* hypothesis. It is found that the Bethe ansatz solutions produce one bound state (a boson). This spectrum as the function of the coupling constant is consistent with Fujita-Ogura’s solution.

Also, it is shown that the *string* configurations taken by Bergknoff and Thacker do not satisfy the PBC equations and thus their *string* is not a solution of the PBC equations. In this way, the present result rules out a belief that the semiclassical result for the massive Thirring model is exact. Also, the strong coupling expansion is performed in ref.[5] and the analytic expressions are obtained for the vacuum state energy as well as the boson boson scattering states. There, it turns out that the boson boson scattering states which are made of continuum states coincide with the twice of the boson mass. Therefore, we also learn from the strong coupling expansion that the $2p - 2h$ states do not give any bound states.

Now, we want to discuss the S-matrix method by Zamolodchikov and Zamolodchikov [6]. This factorized S-matrix method is also known to give the same spectrum as the semiclassical result for the sine-Gordon field theory or the massive Thirring model. Concerning the factorization of the S-matrix for the particle-particle scattering in the massive Thirring model, one may convince oneself that the factorization is indeed satisfied.

However, there is a serious problem for the S-matrix factorization of the particle hole scattering. The problem is that the rapidity variables determined for $n$–particle $n$–hole states are different from each other as well as those determined for the vacuum. Since the Lagrangian of the massive Thirring model satisfies the charge conjugation, one tends to believe that the crossing symmetry should be automatically satisfied. Indeed, the crossing symmetry itself should hold. However, we should be careful whether the
crossing symmetry can commute with the factorization of the S-matrix or not. Recent calculations in ref.[7] show that the crossing symmetry and the factorization of the S-matrix do not commute with each other. Therefore, it turns out that the S-matrix factorization for the particle hole scattering does not hold. In a sense, it is reasonable that the S-matrix factorization is consistent with the semiclassical results since it is indeed due to the consequence of the neglect of the operator commutability.
We plot the calculated energies of $E_v$, $E_{1p1h}^{(n)}$ ($n = 0, 6$) and $E_{2p2h}^{(n,-n)}$ ($n = 1, 6$) for some values of the coupling constant $\frac{g}{\pi}$ with the fixed $L_0 = 100$. The number of particles here is $N = 1601$. Note that we put $m_0 = 1$ in our calculations.

|        | $\frac{g}{\pi} = 1$ | $\frac{g}{\pi} = 1.25$ | $\frac{g}{\pi} = 1.5$ | $\frac{g}{\pi} = 1.7$ |
|--------|----------------------|-------------------------|----------------------|----------------------|
| $E_v$  | -9095.31             | -6215.70                | -4205.83             | -2995.13             |
| $E_{1p1h}^{(0)}$ | -9089.43             | -6210.69                | -4201.76             | -2991.83             |
| $E_{1p1h}^{(1)}$ | -9080.78             | -6197.08                | -4182.54             | -2966.95             |
| $E_{1p1h}^{(2)}$ | -9080.72             | -6197.02                | -4182.48             | -2966.89             |
| $E_{1p1h}^{(3)}$ | -9080.66             | -6196.96                | -4182.42             | -2966.82             |
| $E_{1p1h}^{(4)}$ | -9080.59             | -6196.90                | -4182.35             | -2966.76             |
| $E_{1p1h}^{(5)}$ | -9080.53             | -6196.83                | -4182.29             | -2966.70             |
| $E_{1p1h}^{(6)}$ | -9080.47             | -6196.77                | -4182.23             | -2966.64             |
| $E_{2p2h}^{(1,-1)}$ | -9066.23             | -6178.55                | -4159.13             | -2938.79             |
| $E_{2p2h}^{(2,-2)}$ | -9066.10             | -6178.43                | -4158.99             | -2938.66             |
| $E_{2p2h}^{(3,-3)}$ | -9065.97             | -6178.30                | -4158.86             | -2938.52             |
| $E_{2p2h}^{(4,-4)}$ | -9065.85             | -6178.17                | -4158.74             | -2938.38             |
| $E_{2p2h}^{(5,-5)}$ | -9065.72             | -6178.04                | -4158.61             | -2938.25             |
| $E_{2p2h}^{(6,-6)}$ | -9065.59             | -6177.91                | -4158.49             | -2938.12             |
### Table 2

| $\mathcal{M}$ | Present Calculation | Fujita Ogura | Dashen et al. | Bergknoff Thacker |
|----------------|---------------------|--------------|---------------|-------------------|
| $\frac{g}{\pi} = 0.8$ ($g_0 = 4.19$) | $1.01m$ | $0.98m$ | $0.83m$ | $0.51m$ |
| $\frac{g}{\pi} = 1.0$ ($g_0 = 6.28$) | $0.75m$ | $0.77m$ | $0.62m$ | $0.34m$ |
| $\frac{g}{\pi} = 1.25$ ($g_0 = 10.5$) | $0.47m$ | $0.54m$ | $0.41m$ | $0.20m$ |

We plot the predicted values of the boson mass $\mathcal{M}$ by the present calculation, by the infinite momentum frame calculation (Fujita-Ogura), by the semiclassical method (Dashen et al.) and by the Bethe ansatz technique with *string* hypothesis (Bergknoff - Thacker).

**Reference**

1. R. F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. **D11** (1975), 3432
2. H. Bergknoff and H.B. Thacker, Phys. Rev. Lett. **42** (1979), 135
3. T. Fujita and A. Ogura, Prog. Theor. Phys. **89** (1993), 23
4. W. Thirring, Ann. Phys. (N.Y) **3** (1958), 91
5. T. Fujita, C. Itoi and H. Mukaida, to be published.
6. A.B.Zamolodchikov and A.B.Zamolodchikov, Ann. Phys. **120** (1979), 253
7. T. Fujita and M. Hiramoto, to be published.