Perturbations of self-gravitating, ellipsoidal superfluid-normal fluid mixtures

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We study the perturbation modes of rotating superfluid ellipsoidal figures of equilibrium in the framework of the two-fluid superfluid hydrodynamics and Newtonian gravity. Our calculations focus on linear perturbations of background equilibria in which the two fluids move together, the total density is uniform, and the densities of the two components are proportional to one another, with ratios that are independent of position. The motions of the two fluids are coupled by mutual friction, as formulated by Khalatnikov. We show that there are two general classes of modes for small perturbations: one class in which the two fluids move together and the other in which there is relative motion between them. The former are identical to the modes found for a single fluid, except that the rate of viscous dissipation, when computed in the secular (or “low Reynolds number”) approximation under the assumption of a constant kinematic viscosity, is diminished by a factor $f_N$, the fraction of the total mass in the normal fluid. The relative modes are completely new, and are studied in detail for a range of values for the phenomenological mutual friction coefficients, relative densities of the superfluid and normal components, and, for Roche ellipsoids, binary mass ratios. We find that there are no new secular instabilities connected with the relative motions of the two fluid components. Moreover, although the new modes are subject to viscous dissipation (a consequence of viscosity of the normal matter), they do not emit gravitational radiation at all.

I. INTRODUCTION

The problem of the equilibrium and stability of rotating neutron stars is encountered in various astrophysical contexts, ranging from the limiting frequencies of rapidly rotating isolated millisecond pulsars, emission of gravitational waves in neutron star-neutron star or neutron star-black hole binaries, to the generation of $\gamma$-rays in the neutron star mergers and X-rays in accreting systems \[1\]. Considerable current interest is attached to the problem of neutron star binary inspiral, which would be the primary source of gravitational wave radiation for detection by future laser interferometers. Such a detection, apart from testing the general theory of relativity, potentially could provide useful information on the equation of state of superdense matter. Also the stability criteria for rapidly rotating neutron stars are essential for placing firm upper limits on the frequencies to which millisecond pulsars can be spun up thereby constraining the range of admissible equations of states.

High precision, fully relativistic treatments of rapidly rotating isolated neutron stars and binaries comprising two neutron stars have become available in recent years \[1\]. Nevertheless, the development of simpler models that provide a fast and transparent insight into the underlying physics is needed when the basic set of equations is modified to include new effects.

A systematic framework for the treatment of the equilibrium and stability of rotating liquid masses bound by self-gravitation in the Newtonian theory is contained in Chandrasekhar’s Ellipsoidal Figures of Equilibrium \[2\] (hereafter EFE). The tensor virial method, developed most extensively by Chandrasekhar and co-workers, transforms the local hydrodynamical equations into global virial equations that contain the full information on the structure and stability of the Newtonian self-gravitating system as a whole. The method describes, in a coherent manner, the properties of solitary ellipsoids with and without intrinsic spin, and ellipsoids in binaries subject to Newtonian tidal fields. It is especially useful for studying divergence-free displacements of uniform ellipsoids from equilibrium, in which case the each perturbed virial equation yields (in the absence of viscous dissipation) a different set of normal modes.

Recent alternative formulations of the theory of ellipsoids are based mainly on either the energy variation method \[3,4\] or the affine star model \[5\] or the (Eulerian) two potential formalism \[6\]. A large class of incompressible and compressible ellipsoidal models has been studied using these methods \[3,4,5,6\]. The energy variation method employs the observation that an equilibrium configuration is possible if the energy of the ellipsoid is an extremum for variations

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of the ellipsoidal semiaxis at a constant volume; the ellipsoidal figure is stable only if the energy is a true local minimum. In the affine star method the figures are described by a time-dependent Lagrangian as a function of a deformation matrix and its derivative. The structure of the star at any particular time is related conformally to the initial unperturbed sphere via a quadratic form constructed from the deformation matrix.

The purpose of this paper is to extend previous studies to a treatment of the oscillation modes of ellipsoidal figures of equilibrium that contain a mixture of normal fluid and superfluid. Many-body studies of the pair correlations in neutron star matter show that the baryon fluids in their ground state form superfluid condensates in the bulk of the star. The superfluid phases, in the hydrodynamic limit, can be treated as a mixture of superfluid condensate and normal matter. The superfluid rotation is supported by the Feynman-Onsager vortex lattice state, and on the average leads to quasi-rigid body rotation of the superfluid component. The corresponding time-dependent two-fluid hydrodynamic equations are completely specified by two independent velocities for the superfluid and the normal component and corresponding densities of the constituents. The two fluids are coupled to one another by mutual friction forces, which we model phenomenologically according the prescription given by Khalatnikov [17]. Because of the additional degrees of freedom in this system, there are twice as many modes as for a single fluid. A natural question is whether the new modes affect the stability criteria previously deduced from studying perturbations of a single self-gravitating fluid.

In our treatment of perturbations of a mixture of normal fluid and superfluid, we shall follow most closely Chandrasekhar’s formulation. However, since the basic equations of motion for the two fluids will include mutual friction between them, the system we study is inherently dissipative. Nevertheless, since the frictional forces only depend linearly on the relative velocity between the two fluids and vanish in the background, where the two fluids move together, one can still derive relations that resemble Chandrasekhar’s tensor virial equations. Because these equations include dissipation, we shall prefer to regard them as moments of the fluid equations, rather than tensor virial equations. In fact, we shall relegate the derivation of perturbation equations from these moment equations to the Appendix of the paper, and instead derive the necessary equations for the fluid displacements directly by taking moments of linearized equations of motion for the two fluid components. In this paper, we concentrate on two-fluid variants of the classical Maclaurin, Jacobi and Roche ellipsoids.

Although our main aim is to access the oscillation modes and instabilities of neutron stars within the ellipsoidal approximation, the results obtained here may be of significance in other contexts (Ref. [1], the epilogue). One example is the understanding of rapidly rotating nuclei in the spirit of the Bohr-Wheeler model of a charged incompressible liquid droplet [3]. In this case, the stability is determined by the competition between the attractive surface tension, the repulsive Coulomb potential, and the centrifugal stretching due to the rotation [3,4]. Another example is the stability of rotating superfluid liquid drops of Bose condensed atomic gases, where the stability is determined through an interplay among the pressure of the condensate, the confining potential of the magnetic trap and the centrifugal potential [1].

Previous work on the oscillations of superfluid neutron stars concentrated mainly on perturbations of non-rotating or slowly rotating isolated neutron stars [4–7], and used methods entirely different from the one adopted here. The propagation of acoustic waves in neutron star interiors, including those related to the relative motion of neutron-proton superfluids, was studied by Epstein [4], who found the compressional and shear modes related to short-wavelength oscillations of neutron star matter. The small-amplitude pulsation modes of superfluid neutron stars were derived by Lindblom and Mendell [3], who found that the lowest frequency modes were almost indistinguishable from the normal modes of a single fluid star. Their analytical solutions also reveal the existence of a spectrum of modes which are absent in a single fluid star. Subsequent work concentrated on numerical solutions for the radial and non-radial pulsations of the two-fluid stars and identified distinct superfluid modes in the absence of rotation [14]. The r modes of slowly rotating two-fluid neutron stars have been derived by Lindblom and Mendell [3], who find that they are identical to their ordinary-fluid counterparts to the lowest order in their small-angular-velocity expansion. The linear oscillations of general relativistic stars composed of two non-interacting fluids in a non-rotating static background have been studied by Comer et al. [15].

Our calculations allow arbitrary fast rotation, in the context of (incompressible) Newtonian fluid models. One may anticipate that the effects of superfluidity on oscillation modes, if any, should be affected by the underlying vortex structure of the rotating superfluid. In our treatment, dissipation arises because of the drag forces experienced by the vortex lines as they move through the normal fluid (and there is no dissipation if the drag force is zero). We ignore the motions related to the isospin degrees of freedom in the core of a neutron star and, hence, the mutual entrainment of the neutron and proton condensates, as well as forces arising from deviations from β equilibrium. The two-fluid equations used in the remainder of this work can adequately describe the mutual friction of a two-condensate fluid in the core of a neutron star, since the entrainment effect renormalizes the effective superfluid densities and the frictional coefficients, i.e. the phenomenological input in the two-fluid equations. While the ellipsoidal approximation to a superfluid neutron star is restrictive, it allows us to study the effects of vorticity on the oscillation modes of a self-gravitating star in a transparent manner, avoiding complications due to the star’s inhomogeneity (multi-layer
The equations of motion for a mixture of two fluids may be summarized simply as
\[ \rho_{\alpha} D_{\alpha} u_{\alpha,i} = -\frac{\partial \rho_{\alpha}}{\partial x_i} - \rho_{\alpha} \frac{\partial \phi}{\partial x_i} + \frac{1}{2} \rho_{\alpha} \frac{\partial |\Omega \times x|^2}{\partial x_i} + 2 \rho_{\alpha} \epsilon_{i\alpha \alpha} u_{\alpha,i} \Omega_m + F_{\alpha \beta, i}, \]
where the subscript \( \alpha \in \{ S, N \} \) identifies the fluid component, and Latin subscripts denote coordinate directions; \( \rho_{\alpha}, \rho_{\alpha}, u_{\alpha} \) are the density, pressure, and velocity of fluid \( \alpha \); \( \phi \) is the gravitational potential, and \( F_{\alpha \beta} \) is the mutual friction force on fluid \( \alpha \) due to fluid \( \beta \). These equations have been written in a frame rotating with angular velocity \( \Omega \) relative to some inertial coordinate reference system. The total time derivative operator
\[ D_{\alpha} = \frac{\partial}{\partial t} + u_{\alpha,j} \frac{\partial}{\partial x_j}. \]

The gravitational potential \( \phi \) is derived from
\[ \nabla^2 \phi = \nabla^2 (\phi_S + \phi_N) = 4\pi G (\rho_S(x) + \rho_N(x)); \]
the individual fluid potentials \( \phi_\alpha \) obey \( \nabla^2 \phi_\alpha = 4\pi G \rho_\alpha \). The two fluids are coupled to one another via the frictional force \( F_{\alpha \beta} \) which is antisymmetric on interchange of \( \alpha \) and \( \beta \). For a normal-superfluid mixture
\[ F_{SN} = -F_{NS} \equiv \rho_{S\omega S} \{ \beta' \nu \times (u_S - u_N) + \beta \nu \times [\nu \times (u_S - u_N)] - \beta'' \nu \cdot (u_S - u_N) \}, \]
where \( \beta, \beta' \) and \( \beta'' \) are coupling coefficients, and \( \omega_S = \nu \omega_S \equiv \nabla \times u_S \); in components we have
\[ F_{SN, i} = -\rho_{S\omega S} \delta_{ij} (u_{S,j} - u_{N,j}), \]
where, from Eq. (4),
\[ \beta_{ij} = \beta \delta_{ij} + \beta' \epsilon_{ijm} u_m + (\beta'' - \beta) \nu_i \nu_j. \]

The net rate at which this force does work is
\[ u_S \cdot F_{SN} + u_N \cdot F_{NS} = -\rho_{S\omega S} \{ \beta \nu \times (u_S - u_N) \}^2 - \beta'' [\nu \cdot (u_S - u_N)]^2 \}
there is no dissipation associated with the term proportional to \( \beta' \) in \( F_{\alpha \beta} \). Throughout this paper, we assume that \( \beta, \beta' \) and \( \beta'' \) are independent of position in the fluid mixture. In Eq. (4), we have neglected the effects of the vortex tension, and expressed the mutual friction force in terms of the phenomenological coefficients \( \beta, \beta' \) and \( \beta'' \). While these parameters determine the macroscopic behavior of the fluid system, they are not the optimal ones for connecting microscopic parameters of the mixture to its macroscopic motion. Instead, the macroscopic results can be parametrized in terms of frictional coefficients \( \eta \) and \( \eta' \), which connect \( \beta \) and \( \beta' \) to the drag on individual superfluid vortices via the relations
\[ \beta = \frac{\eta \rho_{S\omega S}}{\eta^2 + (\rho_{S\omega S} - \eta')^2}, \quad \beta' = 1 - \frac{\rho_{S\omega S} (\rho_{S\omega S} - \eta')}{\eta^2 + (\rho_{S\omega S} - \eta')^2}. \]

The physical meaning of \( \eta \)'s is apparent from the equation of motion of a single vortex line
\[ \rho_{S\omega S} [(u_S - u_L) \times \nu] - \eta (u_L - u_N) + \eta' [(u_L - u_N) \times \nu] = 0, \]
where \( u_L \) is the vortex velocity. Equation (8) states that the Magnus force, which represents a lifting force due to the superflow imposed on the vortex circulation, is balanced by the viscous friction forces along the vortex motion (the term \( \sim \eta \)) and perpendicular to the vortex motion (the term \( \sim \eta' \)); these latter forces arise from the scattering of the normal quasiparticles off the vortex line. The characteristic dynamical relaxation time scale related to the vortex motion can be defined as
\[ \tau \approx \frac{\eta}{\eta'}. \]

1 The inertial mass of the vortex is neglected in the standard formulation of the two-fluid superfluid hydrodynamics.
\[
\tau_D = \frac{1}{\langle \omega_S \rangle} \left( \frac{\eta}{\rho_S \omega_S} + \frac{\rho_S \omega_S}{\eta} \right),
\]

where \( \langle \omega_S \rangle \) is the superfluid circulation averaged over macroscopic scales; e.g. for uniformly rotating superfluid \( \langle \omega_S \rangle = 2 \Omega_S \), where \( \Omega_S \) is the superfluid rotation frequency. For fixed density \( \rho_S \), \( \tau_D \to \infty \) asymptotically, when \( \eta \gg \rho_S \omega_S \) (strong coupling limit) and \( \rho_S \omega_S \ll \eta \) (weak coupling limit). Its minimal value is attained when \( \eta/\rho_S \omega_S = 1 \).

The relations (10) do not determine \( \beta'' \), which, if nonzero, implies friction along vortex lines, which would be possible if vortices oscillate or are deformed in the plane perpendicular to the rotation axis. Generally, we assume in this paper that \( \beta'' \ll \beta \) and \( \beta' \), but occasionally we retain nonzero (and not necessarily negligible) \( \beta'' \) to examine its effects on the modes. Moreover, if we choose to include the effects of viscous dissipation in the normal fluid, which we shall not do in this paper, we must resort to a low Reynolds number approximation.

Following the example set by Chandrasekhar, we could take moments of the fluid equations (1) to obtain tensor virial theorems of various orders, and then perturb them to find linear modes for uniform ellipsoids. We have derived the necessary moment equations in this way (see the Appendix), by analogy to Chandrasekhar’s treatment for a single fluid, but here we present a somewhat different (and possibly more transparent approach) to their derivation. We begin with the equation of motion for the displacement of fluid \( \alpha \) from equilibrium, a direct generalization of Eq. (107) in Chap. 2, Sec. 14 in EFE:

\[
\rho_{\alpha} \frac{d^2 \xi_{\alpha,i}}{dt^2} = -\frac{\partial \Delta_{\alpha} Q_{\alpha}}{\partial x_i} - \frac{\partial \Delta_{\alpha} p_{\alpha}}{\partial x_i} - \rho_{\alpha} \frac{\partial \Delta_{\alpha} \phi}{\partial x_i} + \rho_{\alpha} \frac{\partial}{\partial x_i} \left[ \frac{\xi_{\alpha,i}}{2} \left( \frac{\partial |\Omega \times x|^2}{\partial x_i} \right) \right] + 2 \rho_{\alpha} \epsilon_{ilm} \Omega_m \frac{d \xi_{\alpha,l}}{dt} + F_{\alpha \beta, i},
\]

where the Lagrangian variation \( \Delta_{\alpha} Q \) denotes the change in \( Q \) seen by a moving element of fluid \( \alpha \); in particular, it is easy to show that \( \Delta_{\alpha} \phi(x) \) separates into an Eulerian part, \( \delta \phi \), plus \( \xi_{\alpha,i} \partial \phi/\partial x_i \), where

\[
\delta \phi \equiv -G \sum_{\gamma=\alpha,\beta} \int_V d^3 x' \rho_{\gamma}(x') \xi_{\gamma, i}(x') \partial \left( \frac{1}{|x - x'|} \right)
\]

is the Eulerian potential perturbation. We shall assume that the mutual friction force is \( F_{\alpha \beta, i} = 0 \) in the background solution, i.e., either the two fluids are stationary in the rotating frame for the background or have identical fluid velocities in this frame. Our strategy will be to take moments of Eq. (11) by multiplying by appropriate factors of \( x_i \) and integrating over the (common) volume of the unperturbed background configuration. Note that this does not restrict the volumes of the perturbed fluids in any way.

Although the method we shall use to derive the perturbations resembles Chandrasekhar’s, there is an important distinction due to the mutual friction force. Chandrasekhar’s method of solution yields exact modes only in the dissipationless limit, where the modes of uniform ellipsoids are, respectively, linear, quadratic, cubic, etc. functions of the coordinates. Viscous terms would prevent exact solution in this manner, and one resorts to an approximation in which they are evaluated by substitution of the inviscid eigenfunctions (EFE, Chap. 5, §37(b)). However, it is possible to employ the moment method to find exact modes for the normal fluid-superfluid mixture coupled by mutual friction because \( F_{\alpha \beta} \) is a linear function of the velocity difference between the two fluids, not of their spatial derivatives (as is the case for viscous dissipation). By taking moments, we can derive the analogue of tensor virial theorems of various orders, but because the equations involve manifestly dissipative mutual friction forces, we prefer to think of these merely as moments of the original equations of motion. Of course, if we never choose to include the effects of viscous dissipation in the normal fluid, which we shall not do in this paper, we must resort to a low Reynolds number approximation, as was done in EFE. We discuss this briefly in Sec. [V].

Using the above result for the perturbed gravitational potential, we can rewrite Eq. (11) as
\[
\rho \frac{d^2 \xi_{\alpha,i}}{dt^2} = -\frac{\partial \xi_{\alpha,l}}{\partial x_i} \frac{\partial p_{\alpha}}{\partial x_l} - \xi_{\alpha,l} \frac{\partial^2 p_{\alpha}}{\partial x_i \partial x_l} - \frac{\partial \delta p_{\alpha}}{\partial x_i} - \rho \frac{\partial \delta \phi}{\partial x_i} - \rho \xi_{\alpha,l} \frac{\partial^2 \phi}{\partial x_i \partial x_l} + \rho \xi_{\alpha,l} \frac{\partial \delta \phi}{\partial x_i} + \rho \frac{\partial \delta \phi}{\partial x_i} + \frac{\partial \delta p_{\alpha}}{\partial x_i} - \rho \frac{\partial \delta \phi}{\partial x_i} - \rho \xi_{\alpha,l} \frac{\partial^2 \phi}{\partial x_i \partial x_l} + \rho \xi_{\alpha,l} \frac{\partial \delta \phi}{\partial x_i} + \frac{\partial \delta p_{\alpha}}{\partial x_i} + 2 \rho \epsilon_{ilm} \Omega_m \frac{d\xi_{\alpha,l}}{dt} + F_{\alpha\beta,i}. \tag{13}
\]

Then if we define
\[
\xi_+ = f_S \xi_S + f_N \xi_N \quad \xi_- = \xi_S - \xi_N, \tag{14}
\]
we find
\[
\rho \frac{d^2 \xi_{\alpha,i}}{dt^2} = -\frac{\partial \xi_{\alpha,l}}{\partial x_i} \frac{\partial p_{\alpha}}{\partial x_l} - \xi_{\alpha,l} \frac{\partial^2 p_{\alpha}}{\partial x_i \partial x_l} - \frac{\partial \delta p_{\alpha}}{\partial x_i} - \rho \frac{\partial \delta \phi}{\partial x_i} - \rho \xi_{\alpha,l} \frac{\partial^2 \phi}{\partial x_i \partial x_l} + \rho \xi_{\alpha,l} \frac{\partial \delta \phi}{\partial x_i} + \rho \frac{\partial \delta \phi}{\partial x_i} - \rho \omega_S \left(1 + \frac{f_S}{f_N}\right) \beta_{il} \frac{d\xi_{\alpha,l}}{dt}, \tag{15}
\]
where \( \delta p = \delta p_S + \delta p_N \), and
\[
\rho \frac{d^2 \xi_{\alpha,i}}{dt^2} = -\frac{\partial \xi_{\alpha,l}}{\partial x_i} \frac{\partial p_{\alpha}}{\partial x_l} - \xi_{\alpha,l} \frac{\partial^2 p_{\alpha}}{\partial x_i \partial x_l} - \frac{1}{f_S} \frac{\partial \delta p_{\alpha}}{\partial x_i} + \frac{1}{f_N} \frac{\partial \delta p_{\alpha}}{\partial x_i} + \rho \xi_{\alpha,l} \frac{\partial^2 \phi}{\partial x_i \partial x_l} + \rho \xi_{\alpha,l} \frac{\partial \delta \phi}{\partial x_i} + 2 \rho \epsilon_{ilm} \Omega_m \frac{d\xi_{\alpha,l}}{dt} - \rho \omega_S \left(1 + \frac{f_S}{f_N}\right) \beta_{il} \frac{d\xi_{\alpha,l}}{dt}. \tag{16}
\]

Equation (15) is identical to what is found for a single fluid, and therefore contains the well-known modes documented by Chandrasekhar: if we define
\[
V_{i;j} = \int_V d^3 x \rho \xi_{\alpha,i} x_j, \tag{17}
\]
then we find
\[
\frac{d^2 V_{i;j}}{dt^2} = 2 \epsilon_{ilm} \Omega_m \frac{dV_{i;j}}{dt} + \Omega^2 V_{i;j} - \Omega_i \Omega_k V_{kj} + \delta_{ij} \delta \Pi - \pi G \rho \left(2B_{ij} V_{ij} - a_i^2 \delta_{ij} \sum_{l=1}^3 A_{il} V_{il}\right), \tag{18}
\]
where \( \delta \Pi \equiv \delta \Pi_S + \delta \Pi_N \) and all other quantities are defined exactly as in EFE. All of the new modes of a mixture of normal fluid and superfluid are contained in Eq. (16) for their relative displacements. One noteworthy feature of Eq. (16) is that the Eulerian gravitational potential does not appear. Consequently, the new normal modes of the system only depend on the unperturbed gravitational potential; for perturbations of homogeneous ellipsoids, only the coefficients \( A_i \) defined by EFE, Chap. 3, Eqs. (18) and (40), will appear.

For the most part, we shall be interested in displacements that are linear functions of \( x_i \) in this paper. For the homogeneous ellipsoids, we can find the new modes that result from the differential displacements of normal fluid and superfluid by taking the first moment of Eq. (16), e.g. by multiplying by \( x_j \) and integrating over the unperturbed volume. If we define
\[
U_{i;j} = \int_V d^3 x \rho \xi_{\alpha,i} x_j, \tag{19}
\]
then we find

\[\text{In the nomenclature of EFE, this is the second order virial equation corresponding to Eq. (16).}\]
\[
\frac{d^2 U_{ij}}{dt^2} = 2\epsilon_{ilm} \Omega_m \frac{dU_{ij}}{dt} + \Omega^2 U_{ij} - \Omega_i \Omega_k U_{kj} + \delta_{ij} \left( \frac{\delta \Pi_S}{f_S} - \frac{\delta \Pi_N}{f_N} \right) \\
- 2\pi G \rho A U_{ij} - \omega_S \left( 1 + \frac{f_S}{f_N} \right) \beta_{ik} \frac{dU_{kj}}{dt},
\]

(20)

where

\[
\delta \Pi_{\alpha} \equiv \int_V d^3 x \delta p_{\alpha}.
\]

(21)

To obtain Eq. (20), various surface terms can be eliminated using the conditions that \( p_{\alpha} \) and \( \Delta_{\alpha} p_{\alpha} = \delta p_{\alpha} + \xi_{\alpha,\ell} \partial p_{\alpha} / \partial x_\ell \) vanish on the boundary; also, the equation of hydrostatic equilibrium for the unperturbed configuration,

\[
0 = \frac{\partial p}{\partial x_i} + \rho \frac{\partial \phi}{\partial x_i} - \rho \frac{\partial}{\partial x_i} (|\Omega \times x|^2 / 2),
\]

(22)

must be used. It is straightforward to compute higher moments of Eq. (16). For example, taking its second moment by multiplying by \( x_j x_k \) and integrating over the unperturbed volume gives

\[
\frac{d^2 U_{i;jk}}{dt^2} = \delta_{ij} \left( \frac{\delta \Pi_{S,k}}{f_S} - \frac{\delta \Pi_{N,k}}{f_N} \right) + \delta_{ik} \left( \frac{\delta \Pi_{S,j}}{f_S} - \frac{\delta \Pi_{N,j}}{f_N} \right) \\
- 2\pi G \rho A U_{ij} + (\Omega^2 \delta_{il} - \Omega_i \Omega_l) U_{lj} \\
+ \left[ 2\epsilon_{ilm} \Omega_m - \omega_S \left( 1 + \frac{f_S}{f_N} \right) \beta_{il} \right] \frac{dU_{lj}}{dt},
\]

(23)

where, by analogy to definitions in EFE for a single fluid,

\[
\begin{align*}
U_{i;jk} & = \int_V d^3 x \rho \xi_{-;j} x_k x_l, \\
U_{ij} & = U_{i;jk} + U_{j;ki} + U_{k;ij}, \\
\delta \Pi_{\alpha,k} & = \int_V d^3 x \delta p_{\alpha}.
\end{align*}
\]

(24)

Equations (20) and (23) only apply to unperturbed states that are static in the rotating frame.

### III. NEW MODES OF A TWO-FLUID MIXTURE

In this section, we derive the characteristic equations for the normal modes for relative fluid displacements implied by Eq. (20) assuming \( U_{i;j} \propto \exp(\lambda t) \). We list the results separately for perturbations of Maclaurin, Jacobi and Roche ellipsoids. Note that the modes described by Eq. (18) are identical to those treated in EFE.

The superfluid state of the matter does not affect the equilibrium figure. In equilibrium, Eq. (20) is satisfied trivially, for all virials \( U_{ij} = 0 \) in the absence of relative motion between the superfluid and normal components. The equilibrium figure follows from Eq. (18) by dropping the temporal variations and is identical to its non-superfluid counterpart.

For irrotational ellipsoids we can set \( \omega_S = 2\Omega \) in Eq. (21) and using the compact notations

\[
\delta \Pi = \left( \frac{\delta \Pi_S}{f_S} - \frac{\delta \Pi_N}{f_N} \right), \quad \tilde{\beta}_{ik} = \left( 1 + \frac{f_S}{f_N} \right) \beta_{ik},
\]

(25)

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3 This is the third order virial equation in the nomenclature of EFE.

4 Here, we have taken \( p_{\alpha} = f_{\alpha} p \) in the background state, which is a mathematically convenient idealization. More realistically, contributions from the pair condensation energy and the energy density of the superfluid vortex lattice, which distinguish the superfluid from the normal fluid, could play a role in both the equilibria and the perturbations. We need not restrict our attention to displacements for which the pressure perturbations of the two fluids are still proportional to one another, although, as argued in [4,5] for the solenoidal displacements considered here, this turns out to be the case.
we rewrite the Eq. (20) as
\[ \frac{d^2U_{ij}}{dt^2} = 2\epsilon_{ilm} \Omega_m \frac{dU_{ij}}{dt} + \Omega^2 U_{ij} - \Omega_i \Omega_k U_{kj} + \delta_{ij} \delta \Pi - 2\pi G \rho A_i U_{ij} - 2\Omega \tilde{\beta}_k \frac{dU_{kj}}{dt}. \] (26)

For time-dependent Lagrangian displacements of the form
\[ \xi_\alpha(x, t) = \xi_\alpha(x_1)e^{it}, \] (27)
Eq. (20) becomes
\[ \lambda^2 U_{ij} - 2\epsilon_{ilm} \Omega_m \lambda U_{ij} = +\Omega^2 U_{ij} - \Omega_i \Omega_k U_{kj} + (\pi \rho G)^{-1} \delta_{ij} \delta \Pi - 2A_i U_{ij} - 2\Omega \lambda \tilde{\beta}_k U_{kj}; \] (28)
here all frequencies are measured in the units \((\pi \rho G)^{1/2}\). Equation (28) contains all the second harmonic modes of the relative oscillation of Maclaurin and Jacobi ellipsoids, and only requires a slight modification for the application to Roche ellipsoids.

A. Superfluid Maclaurin spheroid

Next, we specialize Eq. (20) to Maclaurin spheroids, the equilibrium figures of a self-gravitating fluid with two equal semi-major axes, say \(a_1\) and \(a_2\), uniformly rotating about the third semi-major axis \(a_3\) (i.e. the \(x_3\) axis). The sequence of quasi-equilibrium figures can be parametrized by the eccentricity \(\epsilon^2 = 1 - a_3^2/a_1^2\), with (squared) angular velocity \(\Omega^2 = 2\epsilon^2 B_{13}\), in units of \((\pi \rho G)^{1/2}\).

Surface deformations related to various modes can be classified by corresponding terms of the expansion in surface harmonics labeled by the indexes \(l, m\). Second order harmonic deformations correspond to \(l = 2\) with five distinct values of \(m\), \(-2 \leq m \leq 2\). The 18 equations represented by Eq. (20) separate into two independent subsets which are odd and even with respect to index 3. The corresponding oscillation modes can be treated separately.

1. Relative transverse shear modes

These modes correspond to surface deformations with \(|m| = 1\) and represent relative shearing of the northern and southern hemispheres of the ellipsoid. The components of Eq. (28), which are odd in index 3, are
\[ \lambda^2 U_{3;1} = -2A_3 U_{31} - 2\Omega \tilde{\beta}' \lambda U_{3;1}, \] (29)
\[ \lambda^2 U_{3;2} = -2A_3 U_{32} - 2\Omega \tilde{\beta}' \lambda U_{3;2}, \] (30)
\[ \lambda^2 U_{1;3} = 2\Omega \lambda U_{2;3} = -2A_1 + \Omega^2 \) \(U_{13} - 2\Omega \lambda U_{1;3} - 2\Omega \tilde{\beta}' \lambda U_{2;3}, \] (31)
\[ \lambda^2 U_{2;3} = 2\Omega \lambda U_{1;3} = -2A_1 + \Omega^2 \) \(U_{23} - 2\Omega \lambda U_{2;3} + 2\Omega \tilde{\beta}' \lambda U_{1;3}. \] (32)

Note that because of the degeneracy in indexes 1 and 2 for the Maclaurin spheroid \(A_1 = A_2\). We sum Eqs. (29), (31) and (30), (32), respectively, and use the symmetry properties of \(U_{ij}\) combined with Eqs. (20), (22). We find
\[ \left[ \left(\lambda^2 + 2\Omega \tilde{\beta}' \lambda \right) \left(\lambda^2 + 2\Omega \tilde{\beta}' \lambda \right) + 2 \left(\lambda^2 + 2\Omega \tilde{\beta}' \lambda \right) \right] U_{13} + \left(\lambda^2 + 2\Omega \tilde{\beta}' \lambda \right) \left(2A_1 - \Omega^2 \right) U_{13} = 0, \] (33)
\[ \left[ \left(\lambda^2 + 2\Omega \tilde{\beta}' \lambda \right) \left(\lambda^2 + 2\Omega \tilde{\beta}' \lambda \right) + 2 \left(\lambda^2 + 2\Omega \tilde{\beta}' \lambda \right) \right] U_{23} + \left(\lambda^2 + 2\Omega \tilde{\beta}' \lambda \right) \left(2A_1 - \Omega^2 \right) U_{23} + 2\Omega \lambda \left(1 - \tilde{\beta}' \right) \left(\lambda^2 + 2\Omega \tilde{\beta}' \lambda + 2A_3 \right) U_{13} = 0. \] (34)

It is instructive to consider first the limit of zero mutual friction, in which case Eqs. (33)-(34) reduce to
\[ \lambda \left[ \lambda^2 + \left(2A_1 + 2A_3 - \Omega^2 \right) \right] U_{13} - 2\Omega \left(\lambda^2 + 2A_3 \right) U_{23} = 0, \]
\[ \lambda \left[ \lambda^2 + \left(2A_1 + 2A_3 - \Omega^2 \right) \right] U_{23} + 2\Omega \left(\lambda^2 + 2A_3 \right) U_{13} = 0, \] (35)
excluding the trivial mode \(\lambda = 0\). The characteristic equation can be factorized by substituting \(\lambda = i\sigma\).
decreases to zero for \( \eta \) is diminished as strong and weak coupling limits are discussed after Eq. (10). The damping of the modes is maximal for \( \lambda (\text{symmetry is imposed}) \) then the characteristic equation is third order:

\[
\sigma^3 \mp \mp - \Omega \sigma^2 + \left[-2(A_1 + A_3) + \Omega^2\right] \mp = 4A_3 \Omega = 0.
\]

Along the entire sequence parametrized in terms of the eccentricity the three modes are real \( ^5 \) and are given by

\[
\sigma_1 = \frac{2\Omega}{3} + (s_+ + s_-), \quad \sigma_{2,3} = \frac{2\Omega}{3} - \frac{1}{2}(s_+ - s_-),
\]

where

\[
s_\pm = \frac{8\Omega^3}{27} + \frac{2\Omega(A_1 - 2A_3) - \Omega^3}{3} \pm \left[\left(\frac{4\Omega^2}{9} - \frac{2(A_1 + A_3) - \Omega^2}{3}\right)^3 + \left(\frac{8\Omega^3}{27} + \frac{2\Omega(A_1 - 2A_3) - \Omega^3}{3}\right)^2\right]^{1/2}.
\]

Three complementary modes follow from Eqs. (38)-(39) via the replacement \( \Omega \to -\Omega \). The frictionless modes are real and are shown in Fig. 1. The two high frequency, frictionless modes are roughly twice as large as the transverse shear modes of ordinary Maclaurin spheroids. The third low frequency mode corresponds to the nearly rotational mode and indeed coincides with \( \Omega \) in the limits \( \epsilon \to 0 \) and \( A_1 \to A_3 \), but not generally. In the dissipative case the characteristic equation is of sixth order:

\[
\lambda^6 + 4\Omega(\overline{\beta} + \overline{\beta''})\lambda^5 + \left[4A_3 + 2\Omega(1 - \overline{\beta}r)^2 + 4\Omega^2(\overline{\beta} + \overline{\beta''})^2 - 2(2A_1 - \Omega^2)\right] \lambda^4
\]

\[
+ [16A_3\overline{\beta}\Omega + 8A_3\overline{\beta''}\Omega + 8\overline{\beta''}(1 - \overline{\beta})^2\Omega^2 + 16\overline{\beta}^2\overline{\beta''}\Omega^3
\]

\[
+ 16\overline{\beta}\overline{\beta''}\Omega - 4\overline{\beta}\Omega(2A_1 - \Omega^2) - 8\overline{\beta''}\Omega(2A_1 - \Omega^2) \lambda^3
\]

\[
+ \left[4A_3^2 + 8A_3\Omega(1 - \overline{\beta})^2 + 16A_3\overline{\beta}\Omega^2 + 32A_3\overline{\beta''}\Omega^2 + 8\overline{\beta''}(1 - \overline{\beta})^2\Omega^3 + 16\overline{\beta}^2\overline{\beta''}\Omega^4
\]

\[
- 4A_3(2A_1 - \Omega^2) - 16\overline{\beta}\overline{\beta''}\Omega^2(2A_1 - \Omega^2) - 8\overline{\beta''}\Omega^2(2A_1 - \Omega^2) + (2A_1 - \Omega^2)^2\right] \lambda^2
\]

\[
+ \left[16A_3^2\overline{\beta}\Omega + 16A_3\overline{\beta''}(1 - \overline{\beta})^2\Omega^2 + 32A_3\overline{\beta^2}\Omega^3 - 8A_3\overline{\beta}\Omega(2A_1 - \Omega^2)
\]

\[
- 8A_3\overline{\beta''}\Omega(2A_1 - \Omega^2) - 16\overline{\beta}\overline{\beta''}\Omega^2(2A_1 - \Omega^2) + 4\overline{\beta''}\Omega(2A_1 - \Omega^2)^2 \lambda
\]

\[
+ \left[8A_3^2(1 - \overline{\beta})^2 + 16A_3^2\overline{\beta}\Omega^2 - 16A_3\overline{\beta}\Omega^2(2A_1 - \Omega^2) + 4\overline{\beta''}\Omega^2(2A_1 - \Omega^2)^2\right] = 0.
\]

The real and imaginary parts of the relative transverse shear modes are shown in the Fig. 1 for several values of \( \eta \) and \( \eta' = 0 = \beta'' \) (here and below we scale \( \eta \) in units of \( \rho_{B_{20}} \)). The real part of the low frequency rotational modes is diminished as \( \eta \) is increased; the high frequency modes are unaffected except in the strong coupling limit \( \eta \geq 50 \) [the strong and weak coupling limits are discussed after Eq. (10)]. The damping of the modes is maximal for \( \eta = 1 \) and decreases to zero for \( \eta \to 0 \) and \( \eta \to \infty \). Note that in the limiting cases the vortices are locked either in the superfluid \( \eta \to 0 \) or the normal component \( \eta \to \infty \) and hence the damping is ineffective. The transverse shear modes are stable for arbitrary values of the eccentricity of the spheroid.

2. Relative toroidal modes

These modes correspond to \( |m| = 2 \) and the motions in this case are confined to the planes parallel to the equatorial plane. The components of Eq. (28), which are even in index 3, are:

\[^5\text{This is easy to prove directly from Eq. (33). Write the dispersion relation as } f(\lambda^2) = 0. \text{ Then show that (i) } f(\lambda^2) \to \pm \infty \text{ as } \lambda^2 \to \pm \infty, (ii) } f(0) > 0, \text{ and (iii) the two extrema of } f(\lambda^2) \text{ are both at } \lambda^2 < 0. \text{ Thus, the zeros of } f(\lambda^2) \text{ are all at } \lambda^2 < 0, \text{ so } \sigma = i\lambda \text{ must be real.}\]
\[ \lambda^2 U_{1:1} = (\pi G \rho)^{-1} \delta \Pi - 2A_1 U_{3:3} - 2\Omega \tilde{\beta}' \lambda U_{3:3}, \]  
(41) 
\[ \lambda^2 U_{1:2} = 2\Omega \lambda U_{2:1} = (\pi G \rho)^{-1} \delta \Pi + (\Omega^2 - 2A_1) U_{11} - 2\Omega \tilde{\beta} \lambda U_{1:1} - 2\Omega \tilde{\beta}' \lambda U_{2:1}, \]  
(42) 
\[ \lambda^2 U_{2:2} + 2\Omega U_{1:2} = (\pi G \rho)^{-1} \delta \Pi + (\Omega^2 - 2A_1) U_{22} - 2\Omega \tilde{\beta} \lambda U_{2:2} + 2\Omega \tilde{\beta}' \lambda U_{1:2}, \]  
(43) 
\[ \lambda^2 U_{1:2} = -2\Omega \lambda U_{2:2} = (-2A_1 + \Omega^2) U_{12} - 2\Omega \tilde{\beta} \lambda U_{1:2} - 2\Omega \tilde{\beta}' \lambda U_{2:2}, \]  
(44) 
\[ \lambda^2 U_{2:1} + 2\Omega U_{1:1} = (-2A_1 + \Omega^2) U_{21} - 2\Omega \tilde{\beta} \lambda U_{2:1} + 2\Omega \tilde{\beta}' \lambda U_{1:1}. \]  
(45) 

We add Eqs. (44) and (43) and subtract Eqs. (42) and (43) to find the following coupled equations for the toroidal modes (note that \( A_1 = A_2 \) for Maclaurin spheroids)

\[ \left( \lambda^2 + 2\Omega \tilde{\beta} \lambda + 4A_1 - 2\Omega^2 \right) (U_{11} - U_{22}) - 4\Omega \lambda (1 - \tilde{\beta}') U_{12} = 0, \]  
(46) 
\[ \left( \lambda^2 + 2\Omega \tilde{\beta} \lambda + 4A_1 - 2\Omega^2 \right) U_{12} + \Omega \lambda (1 - \tilde{\beta}') (U_{11} - U_{22}) = 0. \]  
(47) 

The characteristic equation for the toroidal modes is

\[ \lambda^4 + 4\tilde{\beta} \lambda^3 \Omega + \lambda^2 (8A_1 + 4\tilde{\beta}^2 \Omega^2 - 8\tilde{\beta}' \Omega^2 + 4\tilde{\beta}'' \Omega^2) \]  
\[ + \lambda (16A_1 \tilde{\beta} \Omega - 8\tilde{\beta}' \Omega^2) + 16A_1^2 - 16A_1 \Omega^2 + 4\Omega^4 = 0. \]  
(48) 

In the frictionless limit this can be written as

\[ (\lambda^2 + 4A_1 - 2\Omega^2)^2 + 4\Omega^2 \lambda^2 = 0, \]  
(49) 

which is factorized by writing \( \lambda = i\sigma \). The two solutions are then

\[ \sigma_{1,2} = \Omega \pm \sqrt{4A_1 - \Omega^2}, \]  
(50) 

and there are two complementary modes which are found by substituting \(-\Omega\) for \(\Omega\). The modes are always real because \(4A_1 > \Omega^2\) for incompressible Maclaurin spheroids. This result is in contrast to the modes of ordinary Maclaurin spheroids which become dynamically unstable at \(4B_{12} = \Omega^2\), \(\epsilon = 0.953\). Our model of the superfluid Maclaurin spheroid also becomes dynamically unstable at the same point, but only via the toroidal modes derived from the perturbation equations for \(V_{ij}\), just as in EFE.

The real and imaginary parts of the dissipative toroidal modes are shown in the Fig. 2, for the same values of \(\eta\) as in Fig. 1. The real parts of the modes tend towards each other and merge in the large friction limit. Note that there are no neutral points associated with these modes and the necessary condition for a point of bifurcation is not satisfied. The damping of the modes is maximal, as in the case of the transverse shear modes for \(\eta = 1\), and decreases in both limits of \(\eta \to 0\) and \(\eta \to \infty\). In contrast to ordinary Maclaurin spheroids, which become secularly unstable at the bifurcation point where \(2B_{12} = \Omega^2\) and \(\epsilon = 0.813\), the new toroidal modes are stable at all values of the eccentricity.

Our main conclusion is that the toroidal modes associated with the relative motions of the superfluid and the normal components always remain stable for incompressible Maclaurin spheroids. In the case of the compressible Maclaurin spheroids the point of the onset of secular instability may vary as a function of the adiabatic index (in the case of a polytropic type of equation of state) and hence the conclusions reached above should be verified for these models separately.

### 3. The relative pulsation mode

To find the pulsation modes, which correspond to \(m = 0\), we first add Eqs. (42)-(43) and subtract from the result the Eq. (41). In this manner we find that

\[ \left( \lambda^2/2 + \Omega \tilde{\beta} \lambda - \Omega^2 + 2A_1 \right) (U_{11} + U_{22}) \]  
\[ + 2\Omega \lambda (1 - \tilde{\beta}') (U_{1:2} - U_{2:1}) - (\lambda^2 + 4A_3 + 2\Omega \tilde{\beta}'' \lambda) U_{3:3} = 0. \]  
(51) 

Subtracting Eqs. (43) and (44) one finds

\[ \left( \lambda^2 + 2\Omega \tilde{\beta} \lambda \right) (U_{1:2} - U_{2:1}) - \Omega \lambda (1 - \tilde{\beta}') (U_{11} + U_{22}) = 0. \]  
(52)
Equations (51) and (52) can be further combined to a single equation:

\[
\left[ \left( \lambda^2 + 2\Omega \beta \lambda - 2\Omega^2 + 4A_1 \right) (\lambda^2 + 2\Omega \beta \lambda) + 4\Omega^2 \lambda^2 (1 - \beta')^2 \right] (U_{11} + U_{22}) \\
-2 \left[ \left( \lambda^2 + 2\Omega \beta \lambda \right) \left( \lambda^2 + 2\Omega \lambda \beta'' + 4A_3 \right) \right] U_{33} = 0.
\]  

(53)

The solution is found by supplementing these equations by the divergence free condition

\[
\frac{U_{11}}{a_1^2} + \frac{U_{22}}{a_2^2} + \frac{U_{33}}{a_3^2} = 0
\]

or, in terms of the eccentricity \( \epsilon = 1 - a_3^2/a_2^2 \),

\[
(U_{11} + U_{22})(1 - \epsilon^2) + U_{33} = 0.
\]

(54)

(55)

The third order characteristic equation is

\[
(3 - 2\epsilon^2)\lambda^2 + (8\hat{\beta} \Omega + 4\hat{\beta}'' \Omega - 4\hat{\beta} \epsilon^2 \Omega - 4\hat{\beta}'' \epsilon^2 \Omega)\lambda + (4A_1 + 8A_3 - 8A_3 \epsilon^2 - 2\Omega \lambda \beta'' - 8\hat{\beta} \beta'' \epsilon^2 \Omega) \lambda \\
+ 2\Omega^2 + 4\hat{\beta} \epsilon^2 \Omega + 4\hat{\beta}'' \epsilon^2 \Omega + 8\hat{\beta} \hat{\beta}'' \epsilon^2 \Omega - 8\hat{\beta} \hat{\beta}'' \epsilon^2 \Omega) \lambda \\
+ 8A_1 \hat{\beta} \hat{\beta}'' + 16A_3 \hat{\beta} \hat{\beta}'' - 16A_3 \hat{\beta} \hat{\beta}'' - 4\hat{\beta} \Omega^3 = 0,
\]

where the trivial mode \( \lambda = 0 \) is neglected. In the frictionless limit we find \( \lambda = i\sigma \) as before

\[
\sigma = \pm \left[ \frac{2\Omega^2 + 4A_1 + 8A_3 (1 - \epsilon^2)}{(3 - 2\epsilon^2)} \right]^{1/2}.
\]

(57)

The pulsation modes for a sphere follow in the limit \( \epsilon, \Omega \rightarrow 0 \): for a sphere \( A_i/(\pi \rho G) = 2/3 \), and Eq. (57) reduces to \( \sigma^2 = 8/3 \) [\( \sigma \) is given in units of \( (\pi \rho G)^{1/2} \)]. This result could be compared with the pulsation modes of an ordinary sphere: \( \sigma^2 = 16/15 \). Thus a superfluid sphere, apart from the ordinary pulsations, shows pulsations at frequencies roughly twice as large as the ordinary ones.

The real and imaginary parts of the dissipative pulsation modes of a superfluid Maclaurin spheroid are shown in the Fig. 3. The real parts of the modes are weakly affected by mutual friction and closely resemble those of an ordinary Maclaurin spheroid in the frictionless limit. These are located, however, at higher frequencies. The symmetry of the damping rate as a function of \( \eta \) observed for the transverse shear and toroidal modes is again observed in Fig. 3. Note that the results in Fig. 6 were obtained in the case \( \beta'' = 0 \). The pulsation modes of the superfluid Maclaurin spheroid are stable, as is the case for the ordinary Maclaurin spheroids.

B. Modes of superfluid Jacobi ellipsoid

The sequence of the Jacobi ellipsoids emerges from the Maclaurin sequence at the bifurcation point \( \epsilon = 0.813 \) via a spontaneous breaking of symmetry in the plane perpendicular to the rotation \( (a_1 \neq a_2) \). The superfluid equilibrium figures are again identical to their ordinary fluid counterparts and the defining relations \( a_1^2 a_2^2 A_{12} = a_3^4 A_3 \) and \( \Omega^2 = 2B_{12} \) are unchanged. Ordinary Jacobi ellipsoids are known to be stable against second order harmonic perturbations while they become dynamically unstable against transformation into Poincarè’s pear shaped figures through a mode belonging to third order harmonic perturbations. If the sequence of Jacobi ellipsoids is parametrized in terms of the variable \( \cos^{-1}(a_3/a_1) \), it is stable between the point of bifurcation from the Maclaurin sequence, \( \cos^{-1}(a_3/a_1) = 54.36 \), and the point where Poincarè’s figures bifurcate, \( \cos^{-1}(a_3/a_1) = 69.82 \). Here, by an explicit calculation, we verify that superfluid Jacobi ellipsoids do not develop new instabilities via second order harmonic modes of the relative displacements.

1. Relative odd modes

The treatment of the oscillations of the Maclaurin spheroid of the previous sections can be readily extended to the Jacobi ellipsoids by lifting the degeneracy in indexes 1 and 2 and imposing \( A_1 \neq A_2 \). The equations odd in index 3 are
\[ \lambda^2 U_{3;1} = -2 A^2 U_{3;1} - 2 \Omega \beta'' \lambda U_{3;1}, \]  
(58) 
\[ \lambda^2 U_{3;2} = -2 A^2 U_{3;2} - 2 \Omega \beta'' \lambda U_{3;2}, \]  
(59) 
\[ \lambda^2 U_{1;3} - 2 \Omega \lambda U_{2;3} = \left(-2 A + \Omega^2\right) U_{1;3} - 2 \Omega \beta \lambda U_{1;3}, \]  
(60) 
\[ \lambda^2 U_{2;3} + 2 \Omega \lambda U_{1;3} = \left(-2 A + \Omega^2\right) U_{2;3} - 2 \Omega \beta \lambda U_{1;3}. \]  
(61) 

Combining Eqs. (58) and (60) and, similarly, Eqs. (59) and (61), after some manipulation we find 
\[
\left[\left(\lambda^2 + 2 \Omega \beta'' \lambda\right) \left(\lambda^2 + 2 \Omega \beta \lambda\right) + 2 \left(\lambda^2 + 2 \Omega \beta \lambda\right) A_3 + \left(\lambda^2 + 2 \Omega \beta'' \lambda\right) (2 A_1 - \Omega^2)\right] U_{1;3} - 2 \Omega \lambda (1 - \beta') \left(\lambda^2 + 2 \Omega \beta'' \lambda + 2 A_3\right) U_{2;3} = 0, 
\]
(62) 
\[
\left[\left(\lambda^2 + 2 \Omega \beta'' \lambda\right) \left(\lambda^2 + 2 \Omega \beta \lambda\right) + 2 \left(\lambda^2 + 2 \Omega \beta \lambda\right) A_3 + \left(\lambda^2 + 2 \Omega \beta'' \lambda\right) (2 A_2 - \Omega^2)\right] U_{2;3} + 2 \Omega \lambda (1 - \beta') \left(\lambda^2 + 2 \Omega \beta'' \lambda + 2 A_3\right) U_{1;3} = 0. 
\]
(63) 

Equations (52) and (63) are sufficient to determine the symmetric parts of the virials, and any two of Eqs. (58)-(61) can be used to find the antisymmetric parts. The sixth order characteristic equation is 
\[
\lambda^6 + 4 \Omega (\beta + \beta'') \lambda^5 + \left[4 A_3 + 2 \Omega (1 - \beta')^2 + 4 \Omega^2 (\beta + \beta'')^2 - (2 A_1 - \Omega^2) - (2 A_2 - \Omega^2)\right] \lambda^4 
+ \left[16 A_3 \beta \Omega + 8 A_3 \beta'' \Omega + 8 \beta'' \Omega (1 - \beta')^2 + 16 \beta'' \Omega + 16 \beta'' \Omega^3 \right] \lambda^3 
- 2 \beta \Omega (2 A_1 - \Omega^2) - 2 \beta \Omega (2 A_2 - \Omega^2) - 4 \beta' \Omega (2 A_1 - \Omega^2) - 4 \beta'' \Omega (2 A_2 - \Omega^2) \right] \lambda^2 
+ \left[4 A_3^2 + 8 A_3 \Omega (1 - \beta')^2 + 16 A_3 \beta \Omega + 32 A_3 \beta'' \Omega + 8 \beta'' \Omega^3 (1 - \beta')^2 + 16 \beta'' \Omega^3 \right] \lambda 
+ \left[16 A_3^2 \beta \Omega + 16 A_3 \beta'' \Omega (1 - \beta')^2 + 32 A_3 \beta \Omega + 8 \beta'' \Omega (1 - \beta')^2 - 4 A_3 \beta' \Omega (2 A_1 - \Omega^2) - 4 A_3 \beta'' \Omega (2 A_2 - \Omega^2) \right] 
- 4 A_3 \beta'' \Omega (2 A_1 - \Omega^2) - 8 \beta'' \Omega (2 A_1 - \Omega^2) - 4 A_3 \beta' \Omega (2 A_2 - \Omega^2) - 8 \beta'' \Omega (2 A_2 - \Omega^2) \right] \lambda 
+ \left[8 A_3^2 \Omega (1 - \beta')^2 + 16 A_3 \beta \Omega + 8 A_3 \beta'' \Omega (2 A_1 - \Omega^2) \right] 
- 8 A_3 \beta'' \Omega (2 A_2 - \Omega^2) + 4 \beta'' \Omega (2 A_1 - \Omega^2) (2 A_2 - \Omega^2) = 0. \]
(64) 

The real and imaginary parts of the dissipative odd parity modes are shown in the Fig. 4. The Jacobi sequence is parametrized in terms of $\cos^{-1}(a_3/a_1)$ starting off from the point of bifurcation of the Jacobi ellipsoid from the Maclaurin sequence. The low frequency mode resembles the rotational mode of the ellipsoid; its frequency decreases with increasing friction. One of the remaining two distinct high frequency modes is almost unaffected by the dissipation, while the other is suppressed close to the bifurcation point in a monotonic manner. The damping rates of the odd modes are maximal at $\eta = 1$ and tend to zero for both large and small friction. The modes are damped along the entire sequence; hence, we conclude that superfluid Jacobi ellipsoids are stable against the odd second harmonic modes of oscillations.

2. Relative even modes

The explicit form of the even parity modes for the Jacobian sequence is 
\[ \lambda^2 U_{3;3} = (\pi \rho G)^{-1} \delta \Pi - 2 A^2 U_{3;3} - 2 \Omega \beta'' \lambda U_{3;3}, \]  
(65) 
\[ \lambda^2 U_{1;1} - 2 \Omega \lambda U_{2;1} = (\pi G \rho)^{-1} \delta \Pi - 2 A^2 U_{1;1} + \Omega^2 U_{11} - 2 \Omega \beta \lambda U_{1;1} - 2 \Omega \beta' \lambda U_{2;1}, \]  
(66) 
\[ \lambda^2 U_{2;2} + 2 \Omega \lambda U_{1;2} = (\pi G \rho)^{-1} \delta \Pi - 2 A^2 U_{2;2} + \Omega^2 U_{22} - 2 \Omega \beta \lambda U_{2;2} + 2 \Omega \beta' \lambda U_{1;2}, \]  
(67) 
\[ \lambda^2 U_{1;2} - 2 \Omega \lambda U_{2;2} = (\Omega^2 - 2 A^2) U_{12} - 2 \Omega \beta \lambda U_{1;2} - 2 \Omega \beta' \lambda U_{2;2}, \]  
(68) 
\[ \lambda^2 U_{2;1} + 2 \Omega \lambda U_{1;1} = (\Omega^2 - 2 A^2) U_{12} - 2 \Omega \beta \lambda U_{2;1} + 2 \Omega \beta' \lambda U_{1;1}. \]  
(69)
These equations can be reduced to a simpler set of equations through manipulations which eliminate the variations of the pressure tensor. Explicitly, in the first step we subtract the Eqs. (66) and (67); in the second we sum Eqs. (66) and (67) and subtract twice Eq. (65). The result is

\[
(\lambda^2/2 + \Omega \tilde{\beta} \lambda - \Omega^2 + 2A_1)U_{11} - (\lambda^2/2 + \Omega \tilde{\beta} \lambda - \Omega^2 + 2A_2)U_{22} - 2\Omega \lambda (1 - \tilde{\beta}') U_{12} = 0, \tag{70}
\]

\[
(\lambda^2/2 + \Omega \tilde{\beta} \lambda - \Omega^2 + 2A_1)U_{11} + (\lambda^2/2 + \Omega \tilde{\beta} \lambda - \Omega^2 + 2A_2)U_{22} - (\lambda^2 + 2\Omega \tilde{\beta}'' \lambda + 4A_3)U_{33} + 2\Omega \lambda (1 - \tilde{\beta}') (U_{12} - U_{2;1}) = 0. \tag{71}
\]

Further we add and subtract Eqs. (68) and (69) to find

\[
\left[ \lambda^2 + \Omega \tilde{\beta} \lambda - 4B_{12} + 2(A_1 + A_2) \right] U_{12} + \Omega (1 - \tilde{\beta}') \lambda (U_{11} - U_{22}) = 0, \tag{72}
\]

\[
(\lambda^2 + 2\Omega \tilde{\beta} \lambda) (U_{1;1} - U_{2;1}) - \Omega (1 - \tilde{\beta}') \lambda (U_{11} + U_{22}) + 2(A_1 - A_2)U_{12} = 0. \tag{73}
\]

Equations (70)-(73), supplemented by the divergence free condition, Eq. (54), are sufficient to determine the modes. The characteristic equation is of seventh order, excluding the trivial root \( \lambda = 0 \); in the frictionless limit the characteristic equation is of third order. The explicit form of these equations is cumbersome and will not be given here.

The real and imaginary parts of the dissipative even parity modes are shown in Fig. 5. For each member of the sequence, the eigenvalues of the two high frequency modes are suppressed and that of the low-frequency mode is amplified as the dissipation increases. As in the case of the odd modes the damping rates of the even parity modes are maximal at \( \eta = 1 \) and tend to zero for both large and small friction. The damping rates are again positive along the entire sequence and we conclude that superfluid Jacobi ellipsoids are stable against the even parity second harmonic modes of oscillations.

C. Modes of superfluid Roche ellipsoid

In this section we extend the previous discussion of isolated ellipsoids to binary star systems, and consider the simplest case – the Roche problem. The classical Roche binary consists of a finite size ellipsoid (primary of mass \( M \)) and a point mass (secondary of mass \( M' \)) rotating about their common center of mass with an angular velocity \( \Omega \). The new ingredient in the problem of the equilibrium and stability of the primary is the tidal Newtonian gravitational field of the secondary. Place the center of the coordinate system at the center of mass of the primary with the \( x_1 \)-axis pointing to the center of mass of the secondary and \( x_3 \)-axis along the vector \( \mathbf{M} \). The equation of motion for a fluid element of the primary in the frame rotating with angular velocity \( \Omega \) is, then (EFE, Chap. 8, Sec. 55)

\[
\rho_\alpha D_\alpha u_{\alpha,i} = -\frac{\partial \rho_\alpha}{\partial x_i} - \rho_\alpha \frac{\partial (\phi + \phi')}{\partial x_i} + \frac{1}{2} \rho_\alpha \frac{\partial |\mathbf{M} \times \mathbf{x}|^2}{\partial x_i} + 2\rho_\alpha \epsilon_{ilm} u_{\alpha,l} \Omega_m + F_{\alpha \beta, i}, \tag{74}
\]

where the tidal potential of the secondary, up to quadratic terms in \( x_i/R \), is

\[
\phi' = \frac{GM'}{R} \left( 1 + \frac{x_1}{R} + \frac{2x_1^2 - x_2^2 - x_3^2}{2R^2} \right). \tag{75}
\]

The modified Keplerian rotation frequency for circular orbits, which is consistent with the first order virial equations, is [6]

\[
\Omega^2 = (1 + P) \phi_0 (1 + \delta), \tag{76}
\]

where \( P = M/M' \) is the mass ratio, \( \phi_0 = GM'/R^3 \) is the tidal potential at the origin of the primary, and \( \delta \) is the quadrupole part of the tidal field. The latter correction to the Keplerian frequency is maximal at the Roche limit where \( \delta \sim 0.13 \) [5]. For the sake of simplicity this correction is dropped in the following, as it does not enter into the analysis of the stability of the Roche ellipsoid for displacements that are linear functions of the coordinates. However, the relation between the frequency \( \Omega \) and the orbital separation is now determined within an accuracy \( \delta \ll 1 \). As in the case of the solitary ellipsoids, we find that the equilibrium figure of the superfluid Roche ellipsoid is identical to its ordinary fluid counterpart.

To treat Roche ellipsoids we modify Eq. (28) to
The dissipative odd parity modes are shown in Fig. 6, for the case of an equal mass binary (limit of slow rotation; in the opposite limit the modes remain unaffected by the dissipation. There are three distinct frequency decreases with increasing friction. The high frequency modes tend towards each other and merge in the $-\delta$ parametrized in terms of $\cos \beta$. The Roche sequence for other values of the mass ratio display behavior similar to the $-\delta$ case. We have seen that Maclaurin spheroids do not develop any instabilities (i.e. neither dynamical odd modes of relative oscillation. We conclude that superfluid Roche ellipsoids do not develop instabilities via the second order harmonic friction, as was the case for the Maclaurin and Jacobi ellipsoids. The damping rates are positive along the entire theory above to superfluid Roche ellipsoids, as we show now, does not reveal any new instabilities in the presence of superfluid dissipation, again in contrast to the analysis based on the ordinary viscous dissipation.

1. Relative odd modes

The equations determining the modes even and odd in index 3 form separate sets. We start with the modes belonging to $l = 2$ and $m = -1, 1$ displacements, which are odd in index 3; for these,

$$\lambda^2 U_{i3;1} = -(2 A_3 + \phi_0) U_{i1} - 2 \Omega \beta \lambda^2 U_{i3;1},$$  
(78)

$$\lambda^2 U_{i3;2} = -2 (A_3 + \phi_0) U_{i2} - 2 \Omega \beta \lambda^2 U_{i3;2},$$  
(79)

$$\lambda^2 U_{13} - 2 \Omega \beta \lambda U_{23} = -(2 A_1 + \Omega^2 + 2 \phi_0) U_{13} - 2 \Omega \beta \lambda U_{13} - 2 \Omega \beta \lambda U_{23},$$  
(80)

$$\lambda^2 U_{23} - 2 \Omega \beta \lambda U_{13} = -(2 A_2 + \Omega^2 - \phi_0) U_{23} - 2 \Omega \beta \lambda U_{23} + 2 \Omega \beta \lambda U_{13},$$  
(81)

On combining Eqs. (78) and (80) and, similarly, Eqs. (79) and (81) we obtain

$$\left[ \left( \lambda^2 + 2 \Omega \beta \lambda + 2 A_3 + \phi_0 \right) \left( \lambda^2 + 2 \Omega \beta \lambda \right) + \left( \lambda^2 + 2 \Omega \beta \lambda \right) \right] U_{13} = 0,$$  
(82)

$$\left[ \left( \lambda^2 + 2 \Omega \beta \lambda + 2 A_3 + \phi_0 \right) \left( \lambda^2 + 2 \Omega \beta \lambda \right) + \left( \lambda^2 + 2 \Omega \beta \lambda \right) \right] U_{23} = 0.$$  
(83)

The $U_{ij}$ are symmetric under interchange of their indexes, and Eqs. (82) and (83) completely determine the modes [any two of Eqs. (78)-(81) may be used to find the antisymmetric parts of $U_{ij}$]. The real and imaginary parts of the dissipative odd parity modes are shown in Fig. 6, for the case of an equal mass binary ($P = 1$). The relative modes of Roche ellipsoids in terms of values of the mass ratio display behavior similar to the $P = 1$ case. The Roche sequence is parametrized in terms of $\cos^{-1}(a_3/a_1)$. The low frequency mode resembles the rotational mode of the ellipsoid; its frequency decreases with increasing friction. The high frequency modes tend towards each other and in the limit of slow rotation; in the opposite limit the modes remain unaffected by the dissipation. There are three distinct rates for the damping of oscillations. These are maximal at $\eta = 1$ and tend to zero in both limits of large and small friction, as was the case for the Maclaurin and Jacobi ellipsoids. The damping rates are positive along the entire sequence. We conclude that superfluid Roche ellipsoids do not develop instabilities via the second order harmonic odd modes of relative oscillation.

2. Relative Even Modes

As is well known, Roche ellipsoids develop a dynamical instability via the second order even parity modes beyond the Roche limit, which is the point of closest approach of the primary to the secondary. If viscous dissipation is allowed for, Roche ellipsoids become secularly unstable at the Roche limit via an even parity mode and before dynamical instability sets in. We have seen that Maclaurin spheroids do not develop any instabilities (i.e. neither dynamical nor secular) via the modes associated with the $U_{ij}$ in the presence of superfluid dissipation. The extension of the theory above to superfluid Roche ellipsoids, as we show now, does not reveal any new instabilities in the presence of superfluid dissipation, again in contrast to the analysis based on the ordinary viscous dissipation.

The explicit form of the even parity modes for the Roche sequence is

$$\lambda^2 U_{3;3} = (\pi \rho G)^{-1} \delta \Phi - (2 A_3 + \phi_0) U_{33} - 2 \Omega \beta \lambda U_{3;3},$$  
(84)

$$\lambda^2 U_{1;1} - 2 \Omega \lambda U_{2;1} = (\pi G \rho)^{-1} \delta \Phi - (2 A_1 - \Omega^2 - 2 \phi_0) U_{11} - 2 \Omega \beta \lambda U_{1;1} - 2 \Omega \beta \lambda U_{2;1},$$  
(85)

$$\lambda^2 U_{2;2} - 2 \Omega \lambda U_{1;2} = (\pi G \rho)^{-1} \delta \Phi - (2 A_2 - \Omega^2 + \phi_0) U_{22} + \Omega^2 U_{22} - 2 \Omega \beta \lambda U_{2;2} + 2 \Omega \beta \lambda U_{1;2},$$  
(86)

$$\lambda^2 U_{1;2} - 2 \Omega \lambda U_{2;2} = (\Omega^2 + 2 \mu - 2 A_1) U_{12} - 2 \Omega \beta \lambda U_{1;2} - 2 \Omega \beta \lambda U_{2;2},$$  
(87)

$$\lambda^2 U_{2;1} + 2 \Omega \lambda U_{1;1} = (\Omega^2 - \phi_0 - 2 A_2) U_{12} - 2 \Omega \beta \lambda U_{2;1} + 2 \Omega \beta \lambda U_{1;1},$$  
(88)
These equations can be reduced to a simpler set of equations through manipulations which eliminate variations of the pressure tensor. Using the symmetry properties of the virials we first subtract Eqs. (85) and (86), then sum Eqs. (87) and (88) and subtract twice Eq. (84) to obtain

\[
(\lambda^2/2 + \Omega \tilde{\beta} \lambda - \Omega^2 + 2 \phi_0 + 2 A_1)U_{11} - (\lambda^2/2 + \Omega \tilde{\beta} \lambda - \Omega^2 + \phi_0 + 2 A_2)U_{22} - 2 \Omega \lambda (1 - \tilde{\beta}') U_{12} = 0,
\]

(89)

\[
(\lambda^2/2 + \Omega \tilde{\beta} \lambda - \Omega^2 + 2 \phi_0 + 2 A_1)U_{11} + (\lambda^2/2 + \Omega \tilde{\beta} \lambda - \Omega^2 + \phi_0 + 2 A_2)U_{22} - (\lambda^2 + 2 \Omega \tilde{\beta}' \lambda + 2 \phi_0 + 4 A_3)U_{33} + 2 \Omega \lambda (1 - \tilde{\beta}') (U_{1:2} - U_{2:1}) = 0.
\]

(90)

Further we add and subtract Eqs. (87) and (88) to find

\[
\left[ \lambda^2 + 2 \lambda \tilde{\beta} \lambda + 2 (A_1 + A_2) - 2 \Omega^2 - \phi_0 \right] U_{12} + \Omega (1 - \tilde{\beta}') \lambda (U_{11} - U_{22}) = 0,
\]

(91)

\[
(\lambda^2 + 2 \Omega \tilde{\beta} \lambda)(U_{1:2} - U_{2:1}) - \Omega (1 - \tilde{\beta}') \lambda (U_{11} + U_{22}) - [3 \phi_0 - 2 (A_1 - A_2)] U_{12} = 0.
\]

(92)

Equations (91)-(92), supplemented by the divergence free condition, Eq. (54), are sufficient to determine the unknown virials.

The real and imaginary parts of the dissipative even parity modes are shown in Fig. 7. For each member of the sequence the eigenvalues of the two high frequency modes are suppressed and that of the low frequency one is amplified with increasing dissipation. In effect these modes merge in the slow rotation limit. The modes do not become neutral at any point along the frictionless sequence and, hence, the necessary condition for the onset of dynamical instability is not achieved. Note that our model for the Roche ellipsoids is dynamically unstable as is its classical counterpart, via the modes governed by the Eq. (18) modified appropriately to include the external tidal potential. We do not repeat the mode analysis for the virials \(V_{ij}\) as it is a complete analogue of the analysis in EFE. As in the case of odd modes the damping rates of even parity modes are maximal at \(\eta = 1\) and tend to zero in both limits of large and small friction. The damping rates are positive along the entire sequence and we conclude that superfluid Roche ellipsoids are secularly stable against the even parity second harmonic modes of oscillations associated with the relative motions between the superfluid and normal components.

### IV. VISCOSITY AND GRAVITATIONAL RADIATION

Above, we neglected viscous dissipation in computing normal modes of a normal fluid-superfluid mixture. However, for a single fluid, viscous dissipation is important for understanding stability, for it is responsible for secular instability. Although viscous terms spoil the calculation of the modes of uniform ellipsoids from moment equations formally, when the dissipative time-scale is long, one can include them perturbatively (e.g. EFE, Chap. 5, §37b).

The inclusion of viscosity is slightly more complicated for a mixture of superfluid and normal fluid because viscous dissipation only operates on the normal fluid. The separation of Eq. (11) for the fluid displacements into Eqs. (13) and (14) for the common and differential fluid displacements, \(\xi_\pm\), is possible because the only form of dissipation, the mutual friction force, included in Eq. (14) only depends on \(d\xi_\pm/dt\). Viscous dissipation depends on \(\xi_N\) only, and, in a formal sense, the dynamics no longer separate into the independent dynamics of \(\xi_\pm\).

If we assume that the time-scale associated with viscous dissipation is relatively long, then we can include it perturbatively. The calculation is a bit more subtle than for a single fluid, because we have to deduce \(\xi_N\) for the modes. This brings up an issue that we glossed over earlier, in setting up the calculation of modes in \(\xi_\pm\) even when computing the modes that arise from the equation for \(\xi_\pm\) alone, the equation for \(\xi_N\) must be satisfied, and vice versa. The simplest way for this to work is for \(\xi_+\) to vanish when \(\xi_-\) is nonzero and vice versa. In fact it is easy to show that this is a reasonable solution provided that the Eulerian pressure perturbations are \(\delta p_\alpha = -\xi_\alpha,\partial p_\alpha/\partial x_\alpha\), a situation that arises naturally for adiabatic perturbations, where the Lagrangian pressure perturbations are \(\Delta_\alpha p_\alpha = -\Gamma_\alpha \partial \xi_\alpha/\partial x_\alpha\), and the perturbations are solenoidal, so that \(\partial \xi_\alpha/\partial x_\alpha = 0\), as is true for all modes considered in this paper. To see how this works, consider a mode of Eq. (17), and examine under what conditions Eq. (17) will be satisfied. Then, using

### 6

More realistically, one would also need to consider non-adiabatic effects, such as might arise from perturbations from \(\beta\)-equilibrium; see e.g. Lindblom and Mendell [13]. These would tend to couple \(\xi_\pm\), but if small, could be computed perturbatively, as we do here for viscosity, which also couples \(\xi_\pm\).
\[
\delta p_\alpha = -\xi_{\alpha,l} \frac{\partial p_\alpha}{\partial x_l} = -f_\alpha \xi_{\alpha,l} \frac{\partial p}{\partial x_l}
\]  
(93)

and [substituting the definition of \(\xi_+\), and \(\rho_\alpha = f_\alpha \rho\) into Eq. (12)]

\[
\delta \phi \equiv -G \int_V d^3x' \rho(x') \xi_{+,l}(x') \frac{\partial}{\partial x'_l} \left( \frac{1}{|x - x'|} \right)
\]  
(94)

it is easy to see that Eq. (13) only depends on \(\xi_+\). But since the normal modes of Eq. (16) have different frequencies than normal modes of Eq. (13), we must have \(\xi_+ = 0\) when \(\xi_- \neq 0\). Since, in general,

\[
\xi_S = \xi_+ + f_N \xi_- \quad \xi_N = \xi_+ - f_S \xi_- ,
\]  
(95)

we conclude that, for modes of Eq. (14), \(\xi_S = f_N \xi_-\) and \(\xi_N = -f_S \xi_-\), when Eulerian pressure perturbations are given by Eq. (93). Similarly, since Eq. (16) only depends on \(\xi_-\), for modes with \(\xi_+ \neq 0\), we must have \(\xi_- = 0\) and, therefore, \(\xi_S = \xi_N = \xi_+\), assuming Eq. (93). In particular, if the kinematic viscosity \(\nu\) is held constant [see EFE, Chap. 5, Sec. 36, Eq. (111)], for modes with \(\xi_+ \neq 0\), the viscous dissipation rate is smaller by a factor of \(f_N\) than it is for a single fluid with the same background and displacement \(\xi_+\), as might have been expected qualitatively (i.e. for small normal fluid density, the viscous dissipation must be diminished). For perturbations with \(\xi_- \neq 0\) and displacements that are linear functions of the coordinates, we must add

\[
-5f_S \nu \left( \frac{1}{a_2^2} \frac{dU_{i,j}}{dt} + \frac{1}{a_i^2} \frac{dU_{i,j}}{dt} \right)
\]  
(96)

to the right hand side of Eq. (20) to include the effects of viscous dissipation.

Perturbations with \(\xi_- \neq 0\) emit no gravitational radiation because \(\xi_+ = 0\) for them, and therefore there are no perturbations of the quadrupole moment or any other net mass currents associated with them. Gravitational radiation is emitted by the modes in which the two fluids move together at the same rate as for a single fluid (e.g. ref. [18]). Thus, none of the new modes of a superfluid-normal fluid mixture found here is affected by gravitational radiation at all.

V. CONCLUSIONS

Despite a number of simplifying assumptions, the study of the oscillation modes of uniform ellipsoids is useful for understanding the equilibrium and stability of real neutron stars, at least qualitatively. Moreover, the theory is simple enough that it can be extended readily to include modifications to the underlying physics; here, we have considered new features that arise because a neutron star contains a mixture of normal fluid and superfluid coupled by mutual friction. The theory of uniform ellipsoids is interesting from the viewpoint of mathematical physics because it is solvable exactly, and it may also have applications to other physical systems, such as the physics of trapped, rotating Bose-Einstein condensates.

In this paper we have extended previous treatments of the oscillation modes of ellipsoidal figures of equilibrium to the case of a mixture of normal fluid and superfluid. The basic equations of motion for the two fluid hydrodynamics include mutual friction exactly, since the frictional forces depend linearly on the relative velocity between the two fluids, and vanish in the background where the two fluids move together. In addition the fluids are coupled via mutual gravitational attraction, which is also treated without further approximations. As a result our relations closely resemble Chandrasekhar’s tensor virial equations, even though they are intrinsically dissipative due to the mutual friction. While we have developed these moment equations for general underlying equilibria, they are most useful for perturbations around uniform backgrounds, in which case the moment equations of various orders decouple and yield exact solutions for the normal modes.

Quite generally, there are two classes of modes for small perturbations, one class in which the two fluids move together and the other in which there is relative motion between them. The former are identical to the modes found for a single fluid. As a result our models of superfluid Maclaurin and Roche ellipsoids undergo dynamical instabilities with respect to these modes which are indistinguishable from what is found for their classical counterparts. When ordinary viscous dissipation is included they are also subject to secular instabilities related to the modes in which two fluids move together. If the kinematic viscosity is held constant, then the rate of viscous dissipation, when computed in the “low Reynolds number” approximation, is diminished by a factor \(f_N\), the fraction of the total mass in the normal fluid (see however below).
The modes involving the relative motion between the fluids are completely new and are shown to be stable along the entire sequences of the incompressible Maclaurin, Jacobi and Roche ellipsoids independent of the magnitude of the phenomenological mutual friction. These modes also do not become neutral at selected points along any sequence and the necessary condition for the point of bifurcation is not achieved. Our main conclusion is that mutual friction does not drive secular instabilities in incompressible and irrotational ellipsoids. In addition we find that even though the new modes are subject to viscous dissipation (a consequence of viscosity of the normal matter), they do not emit gravitational radiation, and are therefore immune to any instabilities associated with gravitational radiation, irrespective of their modal frequencies.

The results summarized above hold within a combined framework of two-fluid superfluid hydrodynamics, Newtonian gravity, and the ellipsoidal approximation, as formulated in EFE. Each of these elements of our approach contains a number of simplifying approximations which need to be relaxed in realistic applications to neutron stars. For example, to treat the mixture of neutron and proton superfluids in the neutron star cores, the one-constituent two-fluid superfluid hydrodynamics must be replaced by the hydrodynamics of the multi-constituent superfluid mixtures, in which case the mutual entrainment of the superfluids and deviations from $\beta$-equilibrium must be accounted for (see Ref. [13] for a discussion of these effects and their impact on neutron star oscillations within the real energy functional method).

We anticipate that these effects can be incorporated in the tensor virial approach in a perturbative manner and the results of previous sections will hold in leading order of the perturbation expansion. On the other hand, relaxing the incompressible approximation and, hence, including the partial pressures of the superfluid and the normal fluid, will lead to non-perturbative effects as the pressure terms significantly alter the balance between gravitational attraction and centrifugal stretching. As is well known, the points of the onset of the dynamical and secular instabilities of compressible ellipsoids depend on the adiabatic index of the underlying polytropic equation of state. As noted in Sec. 3, the conclusions reached with respect to the stability of the incompressible ellipsoids should be verified for the compressible models anew. The differences in the pressures (or equations of states) of the normal and superfluid phases, however, are typically small in neutron stars, since the condensation energy is negligible compared to the degeneracy pressures of interacting Fermi-liquids. The coupling between the partial pressures of the normal fluid and superfluid, therefore, can be treated perturbatively.

Another effect that needs to be included is the density dependence of the mutual friction coefficients and the kinematic viscosity. For example, in neutron stars the kinematic viscosity is density dependent in general, explicitly as a result of to the density dependence of the phase space of normal quasiparticles undergoing collisions, implicitly because of the density dependence of the in-medium scattering amplitudes (see for further details Ref. [19]). The ramification for comparison of the secular instabilities of the normal fluid and superfluid ellipsoids is that the modifications of the rate of the viscous dissipation will depend in a non-trivial manner on the fraction of the normal fluid in the system.

The entrainment, $\beta$-nonequilibrium, compressibility, e. t. c. will couple the relative and center-of-mass modes in general. Such a “mixing”, as discussed in Sec. 5, implies $\xi_+ \neq 0$ for the relative modes and, similarly, $\xi_- \neq 0$ for the center-of-mass modes. Therefore, the mutual friction might tend to drive the center-of-mass modes secularly unstable; if they emit gravitational radiation, the mutual friction will suppress the gravitational radiation induced instabilities. One of the important issues to be addressed by the future work is the magnitude of the “mixing” of the modes for general equilibria and the corresponding times scales.

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APPENDIX A: “VIRIAL” EQUATIONS AND PERTURBATIONS

The two fluids need not occupy the same volume, and we shall suppose that fluid $\alpha$ occupies a volume $V_\alpha$. Taking the zeroth moment of Eq. (4) amounts to integrating over $V_\alpha$; doing so, we obtain the “first order ‘virial’ equation”.

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7 We assume that $p_\alpha = 0$ on the bounding surface of $V_\alpha$.
8 We put the word “virial” in quotes because the equations are dissipative.
\[
\frac{d}{dt} \left( \int_{V_\alpha} d^3 x \rho_\alpha u_{\alpha,i} \right) = 2\epsilon_{ilm} \Omega_m \int_{V_\alpha} d^3 x \rho_\alpha u_{\alpha,i} + (\Omega^2 \delta_{ij} - \Omega_i \Omega_j) \int_{V_\alpha} d^3 x \rho_\alpha x_j \\
- (1 - \delta_{\alpha\beta}) \int_{V_\alpha} d^3 x \rho_\alpha \frac{\partial \phi_\beta}{\partial x_i} + \int_{V_\alpha} d^3 x F_{\alpha\beta,i}. \quad (A1)
\]

Apart from inertial forces, which do not couple the two fluids, there are two forces that do couple them: gravity and friction. The net, mutual gravitational force between the fluids only vanishes if they (i) occupy the same volume and (ii) have densities that are proportional to one another (i.e. \( \rho_S \propto \rho_N \)). The mutual friction force is nonzero as long as the fluids move relative to one another. Thus we see that, for a two fluid mixture, the zeroth moment of Eq. (A1) is not trivial, as it would be for a single fluid (as in EFE). Note, though, that the center of mass motion of the combined system is trivial; i.e., if the center of mass of the combined system starts out at \( x = 0 \) with zero velocity, it does not move.

Taking the first moment of Eq. (A1) results in the second order “virial” equation

\[
\frac{d}{dt} \left( \int_{V_\alpha} d^3 x \rho_\alpha x_j u_{\alpha,i} \right) = 2\epsilon_{ilm} \Omega_m \left( \int_{V_\alpha} d^3 x \rho_\alpha x_j u_{\alpha,l} \right) + \Omega^2 I_{\alpha,ij} - \Omega_i \Omega_k I_{\alpha,kj} \\
+ 2T_{\alpha,ij} + \delta_{ij} \Pi_\alpha + M_{\alpha,ij} + (1 - \delta_{\alpha\beta})M_{\alpha\beta,ij} + F_{\alpha\beta,ij}, \quad (A2)
\]

where

\[
I_{\alpha,ij} \equiv \int_{V_\alpha} d^3 x \rho_\alpha x_i x_j \\
\Pi_\alpha \equiv \int_{V_\alpha} d^3 x \rho_\alpha \\
T_{\alpha,ij} \equiv \frac{1}{2} \int_{V_\alpha} d^3 x \rho_\alpha u_{\alpha,i} u_{\alpha,j} \\
M_{\alpha,ij} \equiv -\frac{G}{2} \int_{V_\alpha} d^3 x d^3 x' \rho_\alpha(x) \rho_\alpha(x') (x_i - x'_i) (x_j - x'_j) \left| x - x' \right|^3 \\
M_{\alpha\beta,ij} \equiv -G \int_{V_\alpha} d^3 x \int_{V_\beta} d^3 x' \rho_\alpha(x) \rho_\beta(x') x_j (x_i - x'_i) \left| x - x' \right|^3 \\
F_{\alpha\beta,ij} \equiv \int_{V_\alpha} d^3 x x_j F_{\alpha\beta,i}. \quad (A3)
\]

When there is only one fluid present, this equation reduces to the results found in Chap. 2 of EFE. There are two new terms here: there is a term that arises from the mutual gravitational forces of the two fluids \( (M_{\alpha\beta,ij}) \), and also a term from the mutual friction \( (F_{\alpha\beta,ij}) \).

Consider first the variation of the first order virial equation under the influence of perturbations. Most terms are simple to compute, but we must take special care in computing

\[
- \delta \int_{V_\alpha} d^3 x \rho_\alpha \frac{\partial \phi_\beta}{\partial x_i} = -\delta G \int_{V_\alpha} d^3 x \int_{V_\beta} d^3 x' \frac{\rho_\alpha(x') \rho_\beta(x)(x_i - x'_i)}{\left| x - x' \right|^3}. \quad (A4)
\]

In computing the necessary variation, think of \( x \) as having a label \( \alpha \) and \( x' \) as having a label \( \beta \). It is then easy to find that

\[
- \delta \int_{V_\alpha} d^3 x \rho_\alpha \frac{\partial \phi_\beta}{\partial x_i} = -G \int_{V_\alpha} d^3 x \rho_\alpha(x) \xi_{\alpha,i}(x) \frac{\partial}{\partial x_l} \int_{V_\beta} d^3 x' \frac{\rho_\beta(x')(x_i - x'_i)}{\left| x - x' \right|^3} \\
+ G \int_{V_\beta} d^3 x \rho_\beta(x) \xi_{\beta,i}(x) \frac{\partial}{\partial x_l} \int_{V_\alpha} d^3 x' \frac{\rho_\alpha(x')(x_i - x'_i)}{\left| x - x' \right|^3}. \quad (A5)
\]

\(^{9}\)There is a slight subtlety that has not been stated explicitly in deriving Eq. (A4). The mutual friction force is nonzero only in the overlap volume of the two fluids. This restriction is necessary to derive conservation of total momentum for the combined fluids. We might be interested in situations involving long range coupling, such as between the core superfluid and crustal normal fluid. For such a coupling to occur, we need to introduce long range fields (e.g. magnetic fields) capable of transmitting forces between fluid elements at different points in space.
which is manifestly antisymmetric on \( \alpha \leftrightarrow \beta \). Assuming \( V_\alpha = V_\beta = V \) and \( \rho_\alpha = f_\alpha \rho(x) \) in the background equilibrium, we can simplify this to

\[
- \delta \int_V d^3x \rho_\alpha \frac{\partial \phi_\beta}{\partial x_i} = G f_\alpha f_\beta \int_V d^3x \rho(x) \left[ \xi_{\beta,i}(x) - \xi_{\alpha,i}(x) \right] \frac{\partial}{\partial x_i} \int_V \frac{d^3x' \rho(x')(x_i - x'_i)}{|x - x'|^3}.
\]  

(A6)

Gathering terms, we find that the perturbed first order virial equation is

\[
\frac{d^2}{dt^2} \left( f_\alpha \int_V d^3x \rho(x) \right) = 2 \epsilon_{\alpha \mu \Omega \nu} \frac{d}{dt} \left( f_\alpha \int_V d^3x \rho(x) \right) + (\Omega^2 \delta_{ij} - \Omega_i \Omega_j) f_\alpha \int_V d^3x \rho(x) \partial \frac{\partial \phi(x)}{\partial x_i} \int_V \frac{d^3x' \rho(x')(x_i - x'_i)}{|x - x'|^3}
\]

\[
+ \delta \int_V d^3x F_{\alpha \beta, i}.
\]  

(A7)

Although we simplified the final answer by assuming that the fluids occupy identical volumes and have proportional densities in the background state, we could not have derived the correct perturbation of the first order virial theorem if we had not allowed the volumes to differ.

For uniform ellipsoids, we can simplify the mutual gravitational term further. First, we recognize that

\[
G \int_V \frac{d^3x' \rho(x')(x_i - x'_i)}{|x - x'|^3} = \frac{\partial \phi(x)}{\partial x_i},
\]  

so we can write the gravitational term generally as

\[
f_\alpha f_\beta \int_V d^3x \rho(x) \left[ \xi_{\beta,i}(x) - \xi_{\alpha,i}(x) \right] \frac{\partial^2 \phi(x)}{\partial x_i \partial x_i}.
\]  

(A9)

Second, recall that the potential at any interior point of a homogeneous ellipsoid is (Theorem 3 in Chap. 3 of EFE)

\[
\phi(x) = -\pi G \rho \left( I - \sum_{k=1}^3 A_k x_k^2 \right),
\]  

where \( I \) is a constant; consequently

\[
\frac{\partial^2 \phi}{\partial x_i \partial x_i} = 2 \pi G \rho A_i \delta_{ii}.
\]  

(A11)

Thus, the mutual gravitational contribution to the equation of motion for the perturbed center of mass is

\[
2 \pi G \rho^2 A_i f_\alpha f_\beta \int_V d^3x (\xi_{\beta,i} - \xi_{\alpha,i}).
\]  

(A12)

A sufficient condition for this to vanish is

\[
\int_V d^3x \xi_\alpha = \int_V d^3x \xi_\beta,
\]  

(A13)

which is just the statement that the perturbations have the same center of mass, specialized to the case of uniform density.

In Eq. (A7), we did not compute the variation in the last term. To do this, let us write

\[
F_{SN,i} = -\rho_S \omega_S \beta_{ij} (u_{S,j} - u_{N,j});
\]  

(A14)

in the background state, the two fluids move together (and may even be stationary in the rotating frame) so we have

\[
\delta \int_V d^3x F_{\alpha \beta, i} = -S_{\alpha \beta} \frac{d}{dt} \left( f_S \int_V d^3x \rho(x) \omega_S \beta_{ij} (\xi_{S,j} - \xi_{N,j}) \right),
\]  

(A15)

where \( S_{\alpha \beta} = 0 \) if \( \alpha = \beta, 1 \) if \( \alpha = S \) and \( \beta = N \), and \(-1 \) if \( \alpha = N \) and \( \beta = S \), and [see Eq. (B)] \( \beta_{ij} = \beta \delta_{ij} + \beta' \epsilon_{ijm} \nu_m + \beta'' \nu_i \nu_j \). It is clear that the centers of mass of the two fluids remain stationary if the fluid
For the uniform ellipsoids \[ \text{(EFE, Chap. 3, Eqs. (125) and (126))} \]
the integrated mutual friction force will be zero as long as displacements are identical. However, there may be other conditions under which they remain stationary. For example, with this substitution we get

\[
\delta M_{\alpha\beta,ij} = -G \frac{\partial}{\partial x_l} \int_V d^3 \rho(x) \xi_{\alpha}(x) \frac{d^3 \rho(x')}{|x-x'|^3} \left( x_i - x'_i \right) \left( x_j - x'_j \right)
\]

\[
+ \int_V d^3 \rho(x) \left( \xi_{\alpha,l}(x) - \xi_{\beta,l}(x) \right) \frac{d^3 \rho(x')}{|x-x'|^3} \left( x_i - x'_i \right) x'_j \right) \}
\]

where we have specialized to backgrounds with proportional densities and identical bounding volumes. The first term in the brackets can be combined with \( \delta M_{\alpha,ij} \) and we find

\[
\delta M_{\alpha,ij} + (1 - \delta_{\alpha\beta}) \delta M_{\alpha\beta,ij} = -G f_{\alpha} \int_V d^3 \rho(x) \xi_{\alpha}(x) \frac{d^3 \rho(x')}{|x-x'|^3} \left( x_i - x'_i \right) \left( x_j - x'_j \right)
\]

\[
- G f_{\alpha} f_{\beta} \int_V d^3 \rho(x) \left( \xi_{\alpha,l}(x) - \xi_{\beta,l}(x) \right) \frac{d^3 \rho(x')}{|x-x'|^3} \left( x_i - x'_i \right) x'_j \right) \}
\]

which is less restrictive than the requirement of identical displacements. We found the same condition for the vanishing of the integrated, mutual gravitational force for perturbations of uniformly dense ellipsoids. Equation (A17) may be true, in fact, for all of the perturbations considered in our paper, since both sides may vanish identically.

Next, consider variations of the second order virial equation. Most of the terms are varied exactly as for single fluids; one exception is

\[
\frac{\partial}{\partial x_i} \left( \frac{d^3 \rho(x')}{|x-x'|^3} \right) \left( x_i - x'_i \right) \left( x_j - x'_j \right)
\]

\[
\frac{\partial}{\partial x_i} \int_V d^3 \rho(x) \xi_{\alpha,l}(x) \frac{d^3 \rho(x')}{|x-x'|^3} \left( x_i - x'_i \right) x'_j \right) \}
\]

The last equation can be written more compactly in terms of the functions

\[
B_{ij} \equiv G \int_V d^3 \rho(x')\left( x_i - x'_i \right) \left( x_j - x'_j \right) \frac{d^3 \rho(x')}{|x-x'|^3}
\]

\[
D_{ij} \equiv -G \int_V d^3 \rho(x') \left( x_i - x'_i \right) \left( x_j - x'_j \right) \frac{d^3 \rho(x')}{|x-x'|^3}
\]

The last result can be rewritten using EFE, Chap. 2 Eq. (28) i.e.

\[
\frac{\partial D_{ij}}{\partial x_i} = B_{ij} - x_{ij} \frac{\partial \phi}{\partial x_i}
\]

with this substitution we get

\[
\delta M_{\alpha,ij} + (1 - \delta_{\alpha\beta}) \delta M_{\alpha\beta,ij} = -G f_{\alpha} \int_V d^3 \rho(x) \xi_{\alpha,l}(x) \frac{d^3 \rho(x')}{|x-x'|^3} \left( x_i - x'_i \right) \left( x_j - x'_j \right)
\]

\[
+ f_{\alpha} f_{\beta} \int_V d^3 \rho(x) \left( \xi_{\alpha,l}(x) - \xi_{\beta,l}(x) \right) \frac{d^3 \rho(x')}{|x-x'|^3} \left( x_i - x'_i \right) x'_j \right) \}
\]

For the uniform ellipsoids \[ \text{(EFE, Chap. 3, Eqs. (125) and (126))}, \]

\[
\frac{D_{ij}}{\pi G \rho} = a_i^2 x_j \left( A_j - \sum_{k=1}^{3} A_{jk} x_k^2 \right)
\]

\[
\frac{B_{ij}}{\pi G \rho} = 2 B_{ij} x_i x_j + a_i^2 \delta_{ij} \left( A_i - \sum_{k=1}^{3} A_{ik} x_k^2 \right)
\]
\[ (\pi G \rho)^{-1} \frac{\partial^2 D_i}{\partial x_j \partial x_i} = -2a_j^2(\delta_{ij}x_i A_{jl} + \delta_{lj}x_i A_{ji} + \delta_{i}x_j A_{jl}) \]
\[ = -2a_j^2[A_{ij}(\delta_{ij}x_i + \delta_{lj}x_i) + \delta_{i}x_j A_{jl}], \]
\[ (\pi G \rho)^{-1} \frac{\partial B_{ij}}{\partial x_l} = 2B_{lj}(\delta_{ij}x_i + \delta_{lj}x_l) - 2a_l^2\delta_{ij}A_{il}x_l. \quad (A25) \]

Using these results and the definitions [EFE, Chap. 2, Eqs. (122), (124), and (125)]

\[ V_{\alpha,ij} = \int_V d^3x \rho f_{\alpha,i}x_j \quad V_{\alpha,ij} = V_{\alpha,ij} + V_{\alpha,j;i}, \quad (A26) \]

we find

\[ \frac{\delta M_{\alpha,ij} + (1 - \delta_{\alpha\beta})\delta M_{\alpha\beta,ij}}{\pi G \rho} = -f_s \left( 2B_{ij}V_{\alpha,ij} - a_i^2\delta_{ij} \sum_{l=1}^3 A_l V_{\alpha,il} \right) \]
\[ -a_j^2f_s\rho [2A_{ij}(V_{\alpha,ij} - V_{\beta,ij}) + \delta_{ij} \sum_{l=1}^3 A_l(V_{\alpha,il} - V_{\beta,il})]. \quad (A27) \]

It is possible to write this more compactly, but the form of Eq. (A27) makes clear which terms depend on the differences between the displacements of the two fluids and which do not.

The other new moment we need is

\[ \delta F_{\alpha\beta,ij} = \delta \int_{V_\alpha} d^3x x_j F_{\alpha\beta,i}. \quad (A28) \]

Since the two fluids move together in the unperturbed state, this moment is first order in the perturbations at the largest. We then find

\[ \delta \int_{V_\alpha} d^3x x_j F_{\alpha\beta,i} = -S_{\alpha\beta,f} \rho \int_{V_\alpha} d^3x \omega_S x_j \beta_{ik} \left( \frac{d\xi_{S,k}}{dt} - \frac{d\xi_{N,k}}{dt} \right). \quad (A29) \]

For perturbations of uniform ellipsoids, \( \omega_S \) and \( \rho \) are independent of position in the unperturbed background, and we may also assume that \( \beta_{ij} \) is constant; then,

\[ \delta \int_{V_\alpha} d^3x x_j F_{\alpha\beta,i} = -S_{\alpha\beta,f} \rho \omega_S \beta_{ik} \left( \frac{dV_{\alpha,k;i}}{dt} - \frac{dV_{\beta,k;i}}{dt} \right), \quad (A30) \]

For backgrounds in which there are no fluid motions, the last term is absent and

\[ \delta \int_{V_\alpha} d^3x x_j F_{\alpha\beta,i} = -S_{\alpha\beta,f} \rho \omega_S \beta_{ik} \left( \frac{dV_{\alpha,k;i}}{dt} - \frac{dV_{\beta,k;i}}{dt} \right), \quad (A31) \]

using the definition in Eq. (A26).

When there are no fluid motions of the unperturbed star in the rotating frame, the second order virial equations are

\[ f_S \frac{d^2V_{S,ij}}{dt^2} = 2\epsilon_{ilm}\Omega_m f_S \frac{dV_{S,k;j}}{dt} + \Omega^2 f_S V_{S,ij} - \Omega_i \Omega_k f_S V_{S,k;j} + \delta_{ij}\delta \Pi_S \]
\[ - f_s \pi G \rho \left( 2B_{ij}V_{S,ij} - a_i^2\delta_{ij} \sum_{l=1}^3 A_l V_{S,il} \right) \]
\[ - a_j^2f_s\rho \left( 2A_{ij}(V_{S,ij} - V_{N,ij}) + \delta_{ij} \sum_{l=1}^3 A_l(V_{S,il} - V_{N,il}) \right) \]
\[ - f_s \rho \omega_S \beta_{ik} \left( \frac{dV_{S,k;i}}{dt} - \frac{dV_{N,k;i}}{dt} \right), \]
\[ f_N \frac{d^2V_{N,ij}}{dt^2} = 2\epsilon_{ilm}\Omega_m f_N \frac{dV_{N,k;j}}{dt} + \Omega^2 f_N V_{N,ij} - \Omega_i \Omega_k f_N V_{N,k;j} + \delta_{ij}\delta \Pi_N \]
\[ - f_N \pi G \rho \left( 2B_{ij}V_{N,ij} - a_i^2 \delta_{ij} \sum_{l=1}^{3} A_{il}V_{N,ii} \right) \]
\[ - a_j^2 f_S \pi G \rho \left[ 2A_{ij}(V_{N,ij} - V_{S,ij}) + \delta_{ij} \sum_{l=1}^{3} A_{il}(V_{N,ii} - V_{S,ii}) \right] \]
\[ - f_S \omega_S \beta_{ik} \left( \frac{dV_{N,kj}}{dt} - \frac{dV_{S,kj}}{dt} \right). \] (A32)

We can replace these equations with a different set by defining

\[ V_{ij} \equiv f_S V_{S,ij} + f_N V_{N,ij}, \quad U_{ij} \equiv V_{S,ij} - V_{N,ij}. \] (A33)

In terms of these new quantities we find Eqs. (18) and (20).

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[1] Relativistic Astrophysics and Cosmology, Proceedings of the 18th Texas Symposium on Relativistic Astrophysics, edited by A. Olinto, J. Friedman, and D. Schramm, (Singapore, World Scientific, 1998).
[2] S. Chandrasekhar, Ellipsoidal Figures of Equilibrium (Yale University Press, New Haven, 1969).
[3] Ya. B. Zel’dovich and I. D. Novikov, Relativistic Astrophysics (University of Chicago Press, Chicago, 1983).
[4] S. L. Shapiro and S. A. Teukolsky, Black Holes, White Dwarfs and Neutrons Stars (Wiley, New York, 1983).
[5] B. Carter, and J. P. Luminet, Mon. Not. R. Astron. Soc. 212, 23, (1985); J. P. Luminet and B. Carter, Astrophys. J. Suppl. Ser. 61, 219 (1986).
[6] J. R. Ipser and L. Lindblom, Astrophys. J., 355, 226 (1990); Astrophys. J., 379, 285 (1991); L. Lindblom and J. R. Ipser, Phys. Rev., D59, 044009 (1999).
[7] D. Lai, F. A. Rasio, and S. L. Shapiro, Astrophys. J. Suppl. Ser. 88, 205 (1993).
[8] N. Bohr and J. A. Wheeler, Phys. Rev. 56, 426 (1939).
[9] S. Cohen, F. Plasil, and W. J. Swiatecki, Ann. Phys. (N.Y.) 82, 557 (1974).
[10] G. Rosensteel, Ann. Phys. (N.Y.) 186, 230 (1988); A. L. Goodman, Phys. Rev. Lett. 73, 416 (1994).
[11] Rotating Bose-Einstein condensates are review by A. L. Fetter, in Bose-Einstein Condensation in Atomic Gases edited by M. Inguscio, S. Stzinger, and C. M. Wieman (IOS Press, 1999). cond-mat/9811366.
[12] R. Epstein, Astrophys. J. 333, 880 (1988).
[13] L. Lindblom and G. Mendell, Astrophys. J. 421, 689 (1994); \( r \)-modes are studied by L. Lindblom and G. Mendell, Phys. Rev. D 61, 104003, (2000).
[14] U. Lee, Astron. Astrophys. 303, 515 (1995).
[15] G. L. Comer, D. Langlois, and L. M. Lin, Phys. Rev., D 60, 104025 (1999).
[16] M. A. Ruderman, in Unsolved Problems in Astrophysics, edited by J. N. Bahcall and J. P. Ostriker (Princeton University Press, Princeton, 1996), p. 283; M. A. Ruderman, Nature (London) 222, 228 (1970).
[17] I. M. Khalatnikov, Introduction to the Theory of Superfluidity (Addison Wesley, New York, 1989).
[18] S. Chandrasekhar, Phys. Rev. Lett., 24, 611 (1970), Astrophys. J., 161, 561 (1970); 161, 571 (1970); S. Chandrasekhar and F. P. Esposito, ibid 160, 153 (1970).
[19] C. Cutler and L. Lindblom, Astrophys. J. 314, 234 (1987).
Fig 1: The real (upper panel) and imaginary (lower panel) parts of the relative transverse-shear modes of a superfluid Maclaurin spheroid as a function of eccentricity for three values of $\eta = 0.5, 1, 50$. The $\eta$ parameter is scaled in units of $\omega S\rho S$. The imaginary parts of the modes for $\eta = 50$ are magnified by a factor of 10. The grey lines show the frictionless solutions. To relate the $\beta$-coefficients to the rescaled $\tilde{\beta}$-coefficients we have set $f_S/f_N = 0.2$. The results are insensitive to the choice of this ratio.
Fig. 2: The relative toroidal modes of superfluid Maclaurin spheroids. Conventions are the same as in Fig. 1.
Fig. 3: The relative pulsation modes of superfluid Maclaurin spheroids. Conventions are the same as in Fig. 1.
Fig. 4: The relative odd-parity modes of superfluid Jacobi ellipsoids as a function of $\cos^{-1}(a_3/a_1)$. Conventions are the same as in Fig. 1.
Fig. 5: The relative even-parity modes of superfluid Jacobi ellipsoids as a function of $\cos^{-1}(a_3/a_1)$. Conventions are the same as in Fig. 1.
Fig 6: The relative odd-parity modes of superfluid Roche ellipsoids as a function of $\cos^{-1}(a_3/a_1)$ for $P = 1$. Conventions are the same as in Fig. 1.
Fig 7: The relative even-parity modes of superfluid Roche ellipsoids as a function of $\cos^{-1}(a_3/a_1)$ for $P = 1$. Conventions are the same as in Fig. 1.