Cosmological solution moduli of bigravity

Nejat Tevfik Yılmaz

Department of Electrical and Electronics Engineering, Yaşar University, Selçuk Yaşar Kampüsü, Üniversite Caddesi, No. 35–37, AğaçlıYol, 35100, Bornova, İzmir, Turkey
E-mail: nejat.yilmaz@yasar.edu.tr

Received March 11, 2015
Revised July 24, 2015
Accepted August 27, 2015
Published September 29, 2015

Abstract. We construct the complete set of metric-configuration solutions of the ghost-free massive bigravity for the scenario in which the $g$–metric is the Friedmann-Lemaitre-Robertson-Walker (FLRW) one, and the interaction Lagrangian between the two metrics contributes an effective ideal fluid energy-momentum tensor to the g-metric equations. This set corresponds to the exact background cosmological solution space of the theory.

Keywords: modified gravity, dark energy theory, gravity, dark matter theory

ArXiv ePrint: 1503.02811
1 Introduction

The massive gravity theory which was constructed in [1, 2] as a two-parameter family of actions, is a nonlinear generalization of the Fierz-Pauli massive gravity model [3]. In [2] it was shown that one of the actions of this theory is Boulware-Deser (BD)-ghost-free [4, 5] up to the fourth order in metric perturbations around flat space. An extension of this massive gravity model which originally admits a flat reference metric was also constructed in [6–8] for a general background or reference metric. In [7], it was shown that the two-parameter family of actions given in [2] are BD-ghost-free at the complete nonlinear level for all perturbation orders. In [8], the analysis is extended to show that the general reference metric massive gravity theory in [6] is also BD-ghost-free at the complete nonlinear level for all orders. Later on, a ghost-free massive bimetric theory in which the interaction term between the foreground and the background metrics arises from the mass terms was proposed by introducing a copy of general relativity (GR) dynamics for the background metric [9]. Within this theory, a particular class of cosmological solutions of the coupled field equations of the two-metric-sectors [9–12] have been extensively studied in recent years [12–26]. The solution space corresponding to an effective decoupling of the two metric sectors of the theory is constructed in [27]. The elements of the decoupling solution moduli that is derived in [27] give rise to self-accelerating cosmologies in the $g$–sector via the contribution of an effective cosmological constant. On the other hand, in this work we will construct the general cosmological solution space of the massive bigravity theory. Throughout our derivation, we will follow a parallel route with those of [28, 29] in which cosmological solutions of massive gravity are derived in a formal fashion without explicit classification. On the contrary, in what follows likewise in [27], our main perspective will be to derive the entire set of metric couples $(f, g)$ explicitly, which will consistently solve all the field equations of the theory, and which will enable a semi-decoupling of the $f$–metric from the $g$–metric sector. Since we will consider a homogeneous, and an isotropic scenario in the $g$–sector apart from
assigning a Friedmann-Lemaître-Robertson-Walker (FLRW) form for $g$ which corresponds to the cosmological background metric of the universe, we will also take the overall contribution of the interaction Lagrangian to the $g$–metric equations as an effective ideal fluid energy-momentum tensor. In this respect, the Bianchi identity of the interaction terms in the $g$–metric equations will naturally transform to be the continuity equation of the effective fluid. We will show that when one proposes such a solution ansatz then the only remaining task to derive the set of metric couples which allow this picture is to find the general solutions of an inhomogeneous cubic matrix equation. We will derive the entire set of solutions of this matrix equation whose coefficients are functions of the elementary symmetric polynomials of the solutions themselves rather than being constants. By using the general solutions of this matrix equation one can construct the solution moduli of the metrics $(f, g)$ which admit a cosmological scenario in the $g$–sector.

In section one, starting from the bigravity dynamics by assuming a cosmological $g$–sector and thus, introducing the above-mentioned ansatz for the contribution of the interaction terms in the $g$–metric equations we will derive an algebraic matrix equation whose solutions will lead us to the metric couples which are compatible with the field equations of the cosmological picture. In section two, we will derive the Jordan normal form solutions of this equation. By using this complete set of Jordan normal form solutions, and other three sets constructed from them we will discuss in section three that when special form of similarity transformations are used one can obtain the general set of solutions of the above-mentioned ansatz-generated matrix equation. An outline of the proof of the completeness of this set of solutions will be given in the appendix. Then, again in section three as a consequence of the completely-derived general solution space of the ansatz equation we will be able to define the cosmological solution moduli space of bigravity. We reserve section four for the derivation of the equations of the cosmological dynamics in $g$–sector, as well as a discussion about the associated $f$–dynamics, and its solution methodology. We will also present the outline of the $f$–sector solution construction of an example.

2 The set-up

The ghost-free bigravity action can be given as [9–12]

$$S = -\frac{1}{16\pi G} \int dx^4 \sqrt{-g} \bigg[ R^g + \Lambda^g - 2m^2 \mathcal{L}_{\text{int}}(\sqrt{\Sigma}) \bigg] + S^g_M$$

$$- \frac{\kappa}{16\pi G} \int dx^4 \sqrt{-f} \bigg[ R^f + \Lambda^f \bigg] + eS^f_M,$$  

(2.1)

where $g$ is the foreground, and $f$ is the background metric which are coupled to two types of matter via the actions $S^g_M, S^f_M$, respectively. $\Lambda^g, \Lambda^f$ are the cosmological constants in each sector. $R^g, R^f$ are the corresponding Ricci scalars. The Lagrangian which describes the interaction between the two metrics above reads

$$\mathcal{L}_{\text{int}}(\sqrt{\Sigma}) = \beta_1 e_1(\sqrt{\Sigma}) + \beta_2 e_2(\sqrt{\Sigma}) + \beta_3 e_3(\sqrt{\Sigma}),$$  

(2.2)

where $\{e_n\}$ are the elementary symmetric polynomials

$$e_1 \equiv e_1(\sqrt{\Sigma}) = \text{tr} \sqrt{\Sigma},$$

$$e_2 \equiv e_2(\sqrt{\Sigma}) = \frac{1}{2} \left( (\text{tr} \sqrt{\Sigma})^2 - \text{tr}(\sqrt{\Sigma})^2 \right),$$

$$e_3 \equiv e_3(\sqrt{\Sigma}) = \frac{1}{6} \left( (tr\sqrt{\Sigma})^3 - 3 \text{tr} \sqrt{\Sigma} \text{tr}(\sqrt{\Sigma})^2 + 2 \text{tr}(\sqrt{\Sigma})^3 \right),$$  

(2.3)
corresponding to the square-root-matrix
\[ \sqrt{\Sigma} = \sqrt{g^{-1}f}. \] (2.4)

Originally, the interaction Lagrangian also contains the terms \( \beta_0 e_0 = \beta_0 \), and \( \beta_4 e_4 = \beta_4 \det \sqrt{\Sigma} \). However, we combine their contributions with the cosmological constants \( \Lambda_g \), and \( \Lambda_f \), respectively. Eq. (2.2) reduces to the Fierz-Pauli form in the weak-field limit when one chooses
\[ \beta_1 + 2\beta_2 + \beta_3 = -1. \] (2.5)

From eq. (2.1) one can obtain the field equations for the metric \( g \) as
\[ R^g_{\mu
u} - \frac{1}{2} R^g g_{\mu\nu} - \frac{1}{2} \Lambda^g g_{\mu\nu} + m^2 T^g_{\mu\nu} = 8\pi G T^g_{M\mu\nu}. \] (2.6)

Whereas, the field equations of the metric \( f \) become
\[ \kappa \left[ R^f_{\mu\nu} - \frac{1}{2} R^f f_{\mu\nu} - \frac{1}{2} \Lambda^f f_{\mu\nu} \right] + m^2 T^f_{\mu\nu} = \epsilon 8\pi G T^f_{M\mu\nu}. \] (2.7)

The corresponding energy-momentum tensors arising from the interaction term eq. (2.2) are
\[ T^g_{\mu\nu} = -g_{\mu\rho} \tau^\rho_{\nu} + \mathcal{L}_{\text{int}} g_{\mu\nu}, \] (2.8)

and
\[ T^f_{\mu\nu} = \frac{\sqrt{-g}}{\sqrt{-f}} f_{\mu\rho} \tau^\rho_{\nu}. \] (2.9)

The matrix \( \tau \) with the entries \( \{ \tau^\rho_{\nu} \} \) is defined to be
\[ \tau = \beta_3 (\sqrt{\Sigma})^3 - (\beta_2 + \beta_3 e_1)(\sqrt{\Sigma})^2 + (\beta_1 + \beta_2 e_1 + \beta_3 e_2) \sqrt{\Sigma}. \] (2.10)

The effective energy-momentum tensors in eqs. (2.6), and (2.7) are ought to satisfy the Bianchi identities
\[ \nabla^\mu (T^g)^\mu_{\nu} = 0, \quad \nabla^\mu (T^f)^\mu_{\nu} = 0. \] (2.11)

If a solution configuration satisfies one of these equations then the other one is automatically satisfied \([15, 16]\). Now, let us focus on the cosmological solutions in the \( g \)-sector of the action eq. (2.1). Thus, we will take \( g \) as the FLRW metric
\[ g = -dt^2 + \frac{a^2(t)}{1 - kr^2} dr^2 + a^2(t)r^2 d\theta^2 + a^2(t)r^2 \sin^2 \theta d\phi^2. \] (2.12)

Let us also consider the solutions for which the effective energy-momentum tensor entering into the \( g \)-metric equations in eq. (2.6) takes the form
\[ T^g_{\mu\nu} = (\tilde{\rho}(t) + \tilde{p}(t)) U_\mu U_\nu + \tilde{p}(t) g_{\mu\nu}, \] (2.13)

of an ideal fluid. For an ideal fluid the on-shell Lagrangian can be taken as \([29]\)
\[ \mathcal{L}_{IF} = \tilde{p}, \] (2.14)

so that
\[ T^g_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_{IF})}{\delta(g^{\mu\nu})}. \] (2.15)
Therefore, for the solutions which generate eq. (2.13) in the field equations, eq. (2.6) a close inspection of the interaction term in the action eq. (2.1), and eq. (2.15) shows us that we must have

$$\mathcal{L}_{\text{int}} = \bar{p}. \quad (2.16)$$

This is due to the fact that while eq. (2.8) is derived by varying the interaction term in the action eq. (2.1) with respect to the metric $g$, eq. (2.15) is obtained by varying the Lagrangian eq. (2.14) with respect to $g$, and by using the first law of thermodynamics [29]. Our effective fluid that is introduced in eq. (2.13) will certainly obey the first law of thermodynamics as, it must satisfy the conservation equation in eq. (2.11) which will result in an ordinary continuity or fluid equation when eq. (2.13) is substituted in it. If we take the effective ideal fluid four-velocity vector as $U_\mu = (1, 0, 0, 0)$ in the rest frame of the fluid, and use the FLRW metric eq. (2.12) we obtain

$$g^{-1}T^g = \begin{pmatrix} \tilde{\rho} & 0 & 0 & 0 \\ 0 & \tilde{p} & 0 & 0 \\ 0 & 0 & \tilde{p} & 0 \\ 0 & 0 & 0 & \tilde{p} \end{pmatrix}. \quad (2.17)$$

Index raising on both sides of eq. (2.8) by the metric $g$ gives

$$(T^g)_{\mu}^\nu = -\tau_{\mu}^\nu + \mathcal{L}_{\text{int}} \delta_{\mu}^\nu, \quad (2.18)$$

where $(T^g)_{\mu}^\nu = [g^{-1}T^g]_{\mu}^\nu$. By using eqs. (2.16), and (2.17) in this expression we obtain the matrix equation

$$A(\sqrt{\Sigma})^3 + B(\sqrt{\Sigma})^2 + C\sqrt{\Sigma} + D = 0, \quad (2.19)$$

with

$$D = \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.20)$$

where

$$A = \beta_3, \quad B = -\beta_2 - \beta_3 e_1, \quad C = \beta_1 + \beta_2 e_1 + \beta_3 e_2, \quad D = -\tilde{\rho} - \tilde{p}. \quad (2.21)$$

3 Classification of the solutions

Next, we will derive and classify the Jordan canonical form solutions of the cubic matrix equation (2.19). This is a highly non-trivial matrix equation for two reasons: first, it is not in a polynomial form, and second, its coefficients are functions of the elementary symmetric polynomials $e_1, e_2$ of its solutions $\sqrt{\Sigma}$ rather than being constants. For this reason, in this section we will derive the diagonal and the nondiagonal Jordan form solutions of it, and show in the next section that they can be used to generate the entire solution space. Firstly, let us define the polynomials

$$Ax^3 + Bx^2 + Cx + D = 0, \quad (3.1)$$
and
\[ x(Ax^2 + Bx + C) = 0, \quad (3.2) \]
whose roots we will generally call \( \alpha_i \) and \( \lambda_j \) respectively. Since, in general for any Jordan canonical form matrix \( J \) (diagonal or nondiagonal) when it is substituted into the eq. \((2.19)\) the eigenvalues namely the diagonal elements of \( J \) must satisfy one copy of eq. \((3.1)\), and three copies of the polynomial in eq. \((3.2)\) the multiplicity of \( \alpha_i \) in the diagonal of \( J \) must be one. This is obvious for the diagonal Jordan forms. Besides, if we have a nondiagonal Jordan canonical form as
\[
J = \begin{pmatrix} \alpha_i & 1 & 0 & 0 \\ 0 & \alpha_i & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ddots \\ \end{pmatrix},
\]
when this solution ansatz is used in eq. \((2.19)\) then the diagonal entries would lead to two inconsistent equations \( A(\alpha_i)^3 + B(\alpha_i)^2 + C(\alpha_i) + D = 0 \), and \( A(\alpha_i)^3 + B(\alpha_i)^2 + C(\alpha_i) = 0 \).\(^1\) Therefore, following this observation we can conclude that the Jordan form solutions of eq. \((2.19)\) are partitioned as
\[
J = \begin{pmatrix} \alpha_i & 0 \\ 0 & H_{3 \times 3} \end{pmatrix},
\]
where \( H_{3 \times 3} \) is a three by three Jordan normal form matrix which satisfies the matrix polynomial equation
\[
A(H_{3 \times 3})^3 + B(H_{3 \times 3})^2 + CH_{3 \times 3} = 0. \quad (3.5)
\]
The diagonal ones are naturally in the form of eq. \((3.4)\), and the nondiagonal ones must be in this form due to the partitioning no go fact discussed above. \( H_{3 \times 3} \) must be constructed by its eigenvalues \( \{\lambda_j\} \). As a result, we deduce that the classification of the Jordan form solutions of eq. \((2.19)\) must be based on the classification of the solutions of eq. \((3.5)\). We should remark one important point here that, in deriving these solutions we will exclude the cases which arise from the conditions \( B = 0 \), and/or \( C = 0 \). As it will be clear in section four, such conditions would lead to extra constraints on the equation of state \( \tilde{p} = \tilde{p}(\tilde{\rho}) \) of the effective fluid that would cause it to be nondynamical. In addition, we will also exclude the trivial case of \( H_{3 \times 3} = 0 \).

### 3.1 \( \Delta > 0 \) solutions

Let us first consider the solutions when \( \Delta = B^2 - 4AC > 0 \). In this case there are three distinct roots of the polynomial in eq. \((3.2)\), they are \( \{0, \lambda_1, \lambda_2\} \). Now, none of the Jordan normal forms which satisfy eq. \((3.5)\) may have a repeated root of its minimum polynomial which is formed by a subset of the factors in eq. \((3.5)\). That is to say, for these cases if we write eq. \((3.5)\) in the form
\[
H(H - \lambda_1 1_3)(H - \lambda_2 1_3) = 0, \quad (3.6)
\]
where \( 1_3 \) is the unit \( 3 \times 3 \) matrix then we see that the solutions of this equation must make the product of three factors, or any two factors, or just a single factor vanish. Thus, there are \( 1 + 3 + 3 \) distinct classes of solutions where one of them is trivial. For each class the vanishing combination of factors become the minimum polynomial of the corresponding \( 3 \times 3 \)

\(^1\)Obviously, these two equations can be satisfied simultaneously if \( D = 0 \) which would correspond to \( \tau = 0 \) cases. However, we take \( D \neq 0 \) so that the effective fluid is not simply an effective cosmological constant. Thus, we exclude the \( \tau = 0 \) solutions here as they are completely derived in [27].
matrix solution. Therefore, we also observe that for each of these classes the roots of the corresponding minimum polynomial are not repeated (as $\lambda_1, \lambda_2$ are distinct).\(^2\) Since a matrix is diagonalizable if and only if its minimum polynomial has no repeated roots we deduce that in this case any solution satisfying eq. (3.5) must be a diagonalizable one. On the other hand, if a Jordan canonical form satisfies eq. (3.5) its similarity equivalence class also does. Hence, combining these two facts we conclude that all the Jordan forms which satisfy eq. (3.5) must be diagonal. The reader may also verify this result by direct substitution. In other words, none of the non diagonal Jordan canonical forms whose diagonal elements are chosen from the set \{0, $\lambda_1, \lambda_2$\} satisfies eq. (3.5). Therefore, upon this identification of the $3 \times 3$ sectors, the corresponding $4 \times 4$ Jordan normal forms which satisfy eq. (2.19) can be listed as

$$N_1 = \begin{pmatrix} \alpha_i & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}, \quad N_2 = \begin{pmatrix} \alpha_i & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}, \quad N_3 = \begin{pmatrix} \alpha_i & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

\(N_4 = \begin{pmatrix} \alpha_i & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N_5 = \begin{pmatrix} \alpha_i & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N_6 = \begin{pmatrix} \alpha_i & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \) (3.7)

as well as the ones,

$$N_7 = \begin{pmatrix} \alpha_i & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}, \quad N_8 = \begin{pmatrix} \alpha_i & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}, \quad N_9 = \begin{pmatrix} \alpha_i & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \) (3.8)

in which both of the roots $\lambda_1, \lambda_2$ appear. By direct substitution, the reader may verify that these matrices do solve eq. (2.19), and the corresponding non diagonal Jordan forms that share the same eigenvalues can not satisfy eq. (2.19) in this case when $\Delta > 0$. To find the explicit form of these solutions we have to know $e_1, e_2$ which constitute both the coefficients given in eq. (2.21) (of the matrix equation that these solutions must satisfy), and the entries of these solution matrices listed above. In other words, we have to solve $e_1, e_2$ in terms of the $\{\beta_i\}$-parameters of the action eq. (2.4), and the constituents of the solution ansatz eq. (2.13) so that eq. (2.19) is satisfied. We will first consider the cases $N_{1,2,3,4,5,6}$. If we take the trace of these solutions we get

$$e_1 = tr(N_a) = \alpha_i + n\lambda_j,$$

(3.9)

where $n = 3, 3, 2, 1, 2, 1$ for $N_1, N_2, N_3, N_4, N_5, N_6$, respectively. We also have

$$tr(N_a)^2 = (e_1)^2 - 2e_2 = (\alpha_i)^2 + n(\lambda_j)^2.$$

(3.10)

By using eq. (3.9), and singling out $e_2$ from this expression we get

$$e_2 = -\frac{n + n^2}{2}(\lambda_j)^2 + ne_1\lambda_j.$$

(3.11)

\(^2\)We should note that, if $\lambda_1$, or $\lambda_2$ vanish then there will be a repeated root (the zero root). In this case a single nondiagonalizable Jordan form exits but this case requires $C = 0$ condition that we exclude as we pointed above.
If we substitute this result into eq. (3.2) we obtain the relation
\[ a(\lambda_j)^2 + b\lambda_j + c = 0, \]  
where
\[ a = \frac{(2 - n - n^2)}{2} \beta_3, \quad b = (n - 1)\beta_3 e_1 - \beta_2, \quad c = \beta_1 + \beta_2 e_1. \]  
(3.13)

Since from eq. (3.9) we have \( \alpha_i = e_1 - n\lambda_j \), substituting this into eq. (3.1), and successive usage of eq. (3.12) leads us to the relation
\[
\lambda_j = \left[ \frac{(n^2 - n)\beta_3}{2} \left( \frac{b}{a} e_1 - \frac{b^2}{a^2} + \frac{n}{a} c + \beta_2 n e_1 + \beta_2 n^2 \frac{b}{a} - \beta_1 n \right) \right]^{-1} \times \left[ \frac{(n^2 - n)\beta_3 e_1}{2a} + \frac{n(n^2 - n)\beta_3 bc}{2a^2} - \frac{n^2 c\beta_2}{a} - \beta_1 e_1 - D \right].
\]  
(3.14)

Finally, when we use eqs. (3.14), and (3.11) back in eq. (3.12) and we refer to the definitions in eq. (3.13) we obtain an equation for \( e_1 \) solely in terms of the \( \beta_i \)-coefficients. For the \( n = 3 \) cases this equation reads
\[ a_3(e_1)^4 + b_3(e_1)^3 + c_3(e_1)^2 + d_3 e_1 + f_3 = 0, \]  
(3.15)

where we define
\[
\begin{align*}
a_3 &= 3(\beta_3)^2(-3(\beta_2)^2 + 4\beta_1 \beta_3), \\
b_3 &= 6\beta_3(-15(\beta_2)^2 + 20\beta_1 \beta_2 \beta_3 + 2(\beta_3)^2 D), \\
c_3 &= -216(\beta_2)^4 + 159\beta_1(\beta_2)^2 \beta_3 + 172(\beta_1)^2(\beta_3)^2 + 102\beta_2(\beta_3)^2 D, \\
d_3 &= 204\beta_1 \beta_2(-3(\beta_2)^2 + 4\beta_1 \beta_3) + \beta_3(249(\beta_2)^2 + 20\beta_1 \beta_3) D, \\
f_3 &= -432(\beta_1)^2(\beta_2)^2 + 576(\beta_1)^3 \beta_3 + 36(\beta_2)^3 D + 240\beta_1 \beta_2 \beta_3 D - 125(\beta_3)^2 D^2.
\end{align*}
\]  
(3.16)

For the \( n = 2 \) cases we get
\[ a_2(e_1)^3 + b_2(e_1)^2 + c_2 e_1 + d_2 = 0, \]  
(3.17)

where
\[
\begin{align*}
a_2 &= -(\beta_2)^3 \beta_3 + \beta_1 \beta_2 (\beta_3)^2, \\
b_2 &= -6(\beta_2)^4 + 4\beta_1(\beta_2)^2 \beta_3 + 2(\beta_1)^2(\beta_3)^2 + 2\beta_2(\beta_3)^2 D, \\
c_2 &= -21\beta_1(\beta_2)^3 + 21(\beta_1)^2 \beta_2 \beta_3 + 8(\beta_2)^2 \beta_3 D - 2\beta_1(\beta_3)^2 D, \\
d_2 &= -18(\beta_1)^2 (\beta_2)^2 + 18(\beta_1)^3 \beta_3 + 3(\beta_2)^3 D + 6\beta_1 \beta_2 \beta_3 D - 4(\beta_3)^2 D^2.
\end{align*}
\]  
(3.18)

For each real root of eq. (3.15) we have the solutions \( N_1, N_2 \), and for each real root of eq. (3.17) we have the solutions \( N_3, N_5 \) of eq. (2.19) with the corresponding entries that can be read from
\[
\lambda_j = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad \alpha_i = e_1 - n\lambda_j.
\]  
(3.19)

The domain of validity of these solutions are determined by the conditions \( B^2 - 4AC > 0 \), and \( b^2 - 4ac \geq 0 \), together with the reality conditions of the corresponding roots of the eqs. (3.15), and (3.17) which can be obtained from the definitions of the coefficients in
eqs. (3.16), and (3.18). These conditions will define a validity domain for each particular solution in the union of the \(\{\beta_i\}\)-parameter space, and the state space of the effective ideal fluid. On the other hand, for the cases with \(n = 1\), namely for the solutions of the form \(N_4, N_6\) we have a simpler picture. In these cases, eq. (3.12) gives

\[
\lambda_j = e_1 + \frac{\beta_1}{\beta_2},
\]

and from eq. (3.11) we have

\[
e_2 = -\left( e_1 + \frac{\beta_1}{\beta_2} \right)^2 + e_1 \left( e_1 + \frac{\beta_1}{\beta_2} \right).
\]

Now, by using eq. (3.20) in eq. (3.9) we get

\[
\alpha_i = -\frac{\beta_1}{\beta_2}.
\]

Substituting this result, together with eq. (3.21) into eq. (3.1) gives us

\[
e_1 = -\frac{2\beta_1}{\beta_2} + \frac{D}{\beta_1}.
\]

The eqs. (3.20), (3.22), and (3.23) define the explicit form of the entries of \(N_4, N_6\) in terms of the parameters of the theory, and the state of the effective fluid. The validity domain of these solutions is defined from the condition \(B^2 - 4AC > 0\) without extra requirements.

Now, let us consider the solutions \(N_7, N_8\). Taking the trace of these solutions we get

\[
e_1 = trN_a = \alpha_i + 2\lambda_j + \lambda_k = \alpha_i + \lambda_j - \frac{B}{A},
\]

where we label the excess or the repeated root on the diagonal of the solution by \(\lambda_j\). By referreing to the definitions in eq. (2.21) we find that

\[
\alpha_i = -\lambda_j - \frac{\beta_2}{\beta_3}.
\]

We also have

\[
tr(N_a)^2 = (e_1)^2 - 2e_2 = (\alpha_i)^2 + 2(\lambda_j)^2 + (\lambda_k)^2.
\]

By using the identity

\[
(\lambda_j)^2 + (\lambda_k)^2 = \frac{B^2}{A^2} - \frac{2C}{A},
\]

in the above equation we see that for these solutions

\[
\alpha_i\lambda_j = \frac{(\beta_2)^2}{(\beta_3)^2} - \frac{\beta_1}{\beta_3}.
\]

By using this result in eq. (3.25) we obtain the equation

\[
- (\lambda_j)^2 - \frac{\beta_2}{\beta_3} \lambda_j + \frac{\beta_1}{\beta_3} - \frac{(\beta_2)^2}{(\beta_3)^2} = 0,
\]
for $\lambda_j$. Its solutions are

$$
\lambda_j^\pm = -\frac{1}{2} \left( \frac{\beta_2}{\beta_3} \pm \sqrt{-3 \frac{\beta_2^2}{(\beta_3)^2} + \frac{4 \beta_1}{\beta_3}} \right). \quad (3.30)
$$

Using eqs. (3.29), and (3.30) in eq. (3.2) will enable us to write $e_2$ in terms of $e_1$. After some algebra we get

$$
e_2 = \left( -\frac{3}{2} \frac{\beta_2}{\beta_3} \pm \frac{1}{2} \sqrt{-3 \frac{\beta_2^2}{(\beta_3)^2} + \frac{4 \beta_1}{\beta_3}} \right) e_1 \mp \frac{\beta_2}{\beta_3} \sqrt{-3 \frac{\beta_2^2}{(\beta_3)^2} + \frac{4 \beta_1}{\beta_3} - 2 \frac{\beta_1}{\beta_3}}. \quad (3.31)
$$

We note that, when the (+) solution is taken in eq. (3.30) the opposite sign must be chosen in eq. (3.31), and vice versa. Finally, substituting expressions (3.25), (3.30), and (3.31) into eq. (3.1) will lead us to the explicit value of $e_1$ for either of the solutions in eq. (3.30). For the solutions $\lambda_j^\pm$ this computation reads

$$
e_1^\pm = -\frac{2 \beta_2}{\beta_3} - \frac{2(\beta_3)^2 D}{C^\pm}, \quad (3.32)
$$

where we defined

$$C^\pm = 3(\beta_2)^2 \beta_3 - 4\beta_1(\beta_3)^2 \pm \beta_2(\beta_3)^2 \sqrt{-3 \frac{\beta_2^2}{(\beta_3)^2} + \frac{4 \beta_1}{\beta_3}}. \quad (3.33)
$$

In summary, for these latest cases we find that

$$N_7 = \begin{pmatrix} \alpha_i^+ & 0 & 0 & 0 \\ 0 & \lambda_j^+ & 0 & 0 \\ 0 & 0 & \lambda_j^+ & 0 \\ 0 & 0 & 0 & \lambda_k^+ \end{pmatrix}, \quad N_8 = \begin{pmatrix} \alpha_i^- & 0 & 0 & 0 \\ 0 & \lambda_j^- & 0 & 0 \\ 0 & 0 & \lambda_j^- & 0 \\ 0 & 0 & 0 & \lambda_k^- \end{pmatrix}, \quad (3.34)
$$

are the solutions of eq. (2.19). Explicitly, together with eqs. (3.30), and (3.32) we have

$$
\alpha_i^\pm = -\lambda_j^\pm - \frac{\beta_2}{\beta_3}, \quad \lambda_k^\pm = e_1^\pm + \frac{\beta_2}{\beta_3} - \lambda_j^\pm. \quad (3.35)
$$

The domain of validity of these solutions in the parameter space of the action eq. (2.1), and the state space of the ansatz eq. (2.13) is governed by the conditions $B^2 - 4AC > 0$, and $-3(\beta_2)^2/(\beta_3)^2 + 4\beta_1/\beta_3 \geq 0$ with the respective substitutions of $e_2$, and $e_1$ from eqs. (3.31), and (3.32). The final solution we have to derive explicitly in this class is $N_9$. If we take its trace we find that

$$e_1 = tr N_9 = \alpha_i + \lambda_1 + \lambda_2 = \alpha_i - \frac{B}{A}. \quad (3.36)
$$

From this relation by referring to eq. (2.21) we see that

$$
\alpha_i = -\frac{\beta_2}{\beta_3}. \quad (3.37)
$$

Also,

$$tr(N_9)^2 = (e_1)^2 - 2e_2 = (\alpha_i)^2 + (\lambda_1)^2 + (\lambda_2)^2. \quad (3.38)
$$
By using the identity (3.27) this relation reduces to the condition
\[(\beta_2)^2 - \beta_1\beta_3 = 0.\] (3.39)

Now, substitution of eq. (3.37) into the polynomial eq. (3.1) gives \(e_2\) in terms of \(e_1\) explicitly. It reads
\[e_2 = -2\frac{\beta_2}{\beta_3}e_1 - 2\frac{(\beta_2)^2}{(\beta_3)^2} - \frac{\beta_1}{\beta_3} + \frac{D}{\beta_2}.\] (3.40)

We see in this formulation that \(e_1\) remains completely an arbitrary spacetime field. For a particular choice of it one can read \(e_2\) from eq. (3.40), and construct \(A, B, C\) explicitly in terms of \(\{\beta_i\}\), and \(D\) via their definitions in eq. (2.21), then one can explicitly obtain the entries of \(N_9\) from
\[\lambda_{1,2} = -\frac{B \pm \sqrt{B^2 - 4AC}}{2A},\] (3.41)
and eq. (3.37). For this solution to exist the conditions eq. (3.39), and \(B^2 - 4AC > 0\) must be satisfied. Again, the second of these defines a domain in the union of the action-parameter space of the theory, and the state space of the effective fluid.

### 3.2 \(\Delta = 0\) solutions

We now turn our attention on the cases when \(\Delta = B^2 - 4AC = 0\). In these cases there is a repeated root \(\lambda' = -B/2A\) of the polynomial eq. (3.2). The roots of eq. (3.2) become \(\{0, \lambda', \lambda'\}\). Since, when it is factorized eq. (3.5) has a repeated factor some of the Jordan normal forms which satisfy eq. (3.5) may have a repeated root of their minimum polynomials. Thus, when \(\Delta = 0\) we have nondiagonal, as well as diagonal Jordan normal forms which satisfy eq. (3.5). We again, do not consider the solutions arising from \(B = 0\), and/or \(C = 0\) conditions. Bearing this fact in mind, therefore, in this case the list of all the possible Jordan normal forms which satisfy eq. (2.19) can be given as

\[
\begin{align*}
N_{10} &= \begin{pmatrix}
\alpha_i & 0 & 0 & 0 \\
0 & \lambda' & 0 & 0 \\
0 & 0 & \lambda' & 0 \\
0 & 0 & 0 & \lambda'
\end{pmatrix}, &
N_{11} &= \begin{pmatrix}
\alpha_i & 0 & 0 & 0 \\
0 & \lambda' & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, &
N_{12} &= \begin{pmatrix}
\alpha_i & 0 & 0 & 0 \\
0 & \lambda' & 0 & 0 \\
0 & 0 & \lambda' & 0 \\
0 & 0 & 0 & \lambda'
\end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
N_{13} &= \begin{pmatrix}
\alpha_i & 0 & 0 & 0 \\
0 & \lambda' & 1 & 0 \\
0 & 0 & \lambda' & 0 \\
0 & 0 & 0 & \lambda'
\end{pmatrix}, &
N_{14} &= \begin{pmatrix}
\alpha_i & 0 & 0 & 0 \\
0 & \lambda' & 1 & 0 \\
0 & 0 & \lambda' & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\end{align*}
\] (3.42)

The reader may again verify that these matrices satisfy eq. (2.19) by direct substitution. If we take the trace of the matrices in eq. (3.42) we find
\[e_1 = tr N_a = \alpha_i - \frac{nB}{2A},\] (3.43)
where \(n = 3, 1, 2, 3, 2\), for \(N_{10}, N_{11}, N_{12}, N_{13}, N_{14}\), respectively. When the definitions in eq. (2.21) are used this relation yields
\[\alpha_i = \frac{n\beta_2}{2\beta_3} + \frac{2 - n}{2} e_1.\] (3.44)
We also have
\[ tr(N_a)^2 = (e_1)^2 - 2e_2 = (\alpha_i)^2 + n \left( \frac{B}{2A} \right)^2, \]
which can be written in the form
\[ e_2 = -\frac{n^2 + n (\beta_2)^2}{8 (\beta_3)^2} - \frac{n^2 - n \beta_2}{4 \beta_3} e_1 - \frac{n^2 - 3n}{8} (e_1)^2. \]  
(3.46)
by using eqs. (2.21), and (3.44). Now, substituting eqs. (3.44), and (3.46) into eq. (3.1) yields
\[ a'(e_1)^3 + b'(e_1)^2 + c' e_1 + d' = 0, \]  
(3.47)
where
\[
\begin{align*}
    a' &= (-2n + 3n^2 - n^3)(\beta_3)^3, \\
    b' &= (4n + 3n^2 - 3n^3)\beta_2(\beta_3)^2, \\
    c' &= (16 - 8n)\beta_1(\beta_3)^2 + (6n - 3n^2 - 3n^3)(\beta_2)^2 \beta_3, \\
    d' &= -(3n^2 + n^3)(\beta_2)^3 - 8n\beta_1\beta_2\beta_3 + 16(\beta_3)^2 D.
\end{align*}
\]  
(3.48)
On the other hand, for these solutions to exist we have the condition \( \Delta = B^2 - 4AC = 0 \). Again, by using the definitions in eq. (2.21), and by expressing \( e_2 \) in terms of \( e_1 \) via \( \Delta = 0 \) condition, then by substituting the result in the equation (3.46) we obtain
\[ \frac{n^2 - 3n + 2}{8} (e_1)^2 + \frac{n^2 - n - 2 \beta_2}{4 \beta_3} e_1 + \frac{n^2 + n + 2 (\beta_2)^2}{8 (\beta_3)^2} - \frac{\beta_1}{\beta_3} = 0. \]  
(3.49)
For \( n = 1 \), this equation is reduced to a linear one and it has the solution
\[ e_1 = -\frac{2\beta_1}{\beta_2} + \frac{\beta_2}{\beta_3}, \]  
(3.50)
for \( n = 2 \), the \( e_1 \)-terms vanish and it boils down to the condition
\[ (\beta_2)^2 - \beta_1\beta_3 = 0, \]  
(3.51)
and for \( n = 3 \), it has the solutions
\[ e_1 = -\frac{2\beta_2}{\beta_3} \pm 2 \sqrt{\frac{\beta_1}{\beta_3} \frac{3 (\beta_2)^2}{4 (\beta_3)^2}}, \]  
(3.52)
provided that \( \beta_1/\beta_3 - 3(\beta_2)^2/4(\beta_3)^2 \geq 0 \). When eqs. (3.50), and (3.52) are substituted into the equation (3.47) one finds the equation of state for the effective fluid. However, at this stage we need not explicitly derive these expressions since such solutions cannot lead to evolving scale factors for these cases. We will explain the reason why this occurs for the \( n = 1 \), and \( n = 3 \) cases, and thus, why we can disregard them in the next section. On the other hand, for the \( n = 2 \) cases namely for the solutions \( N_{12}, N_{14} \) eq. (3.49) does not fix \( e_1 \) in terms of the \( \beta_i \)-coefficients but only results in a condition on them. For these cases \( e_1 \) must be solved from eq. (3.47) thus, the equation of state of the effective fluid is not fixed. Solving eq. (3.47) which reduces to be a quadratic equation when \( n = 2 \) for \( e_1 \) yields
\[ e_1^\pm = \left[ \frac{-c' \pm \sqrt{(c')^2 - 4b'd'}}{2b'} \right]_{n=2}. \]  
(3.53)
Having found $e_1$ now, we can explicitly express the entries of the solutions $N_{12}, N_{14}$ via

$$\lambda_{\pm} = -\frac{B}{2A} ± \frac{\beta_2 + e_1}{2}, \quad \alpha_i = -\frac{\beta_2}{\beta_3},$$

(3.54)

where $e_1$ must be substituted from eq. (3.53). We see that, there are two sets of solutions for each of $N_{12}$, and $N_{14}$. For the existence of these solutions, there are two conditions to be satisfied; one of them is eq. (3.51), and the other one is $(c')^2 - 4b'd' \geq 0$.

### 3.3 $\Delta < 0$ solutions

When $\Delta = B^2 - 4AC < 0$ the polynomial (3.2) has complex roots. The roots of eq. (3.2) are $\{0, \lambda, \lambda^*\}$. Upon factorization, eq. (3.5) has complex root factors too. The non-trivial Jordan normal forms which satisfy eq. (3.5) by causing a minimum polynomial that is a sub-factor in the factorization of eq. (3.5) to vanish must have complex eigenvalues. Thus, they are nondiagonal. In this case, if $\lambda$ is an eigenvalue (or a root of the corresponding minimum polynomial) of any Jordan normal form which satisfy eq. (3.5) $\lambda^*$ must also be an eigenvalue. Beside this fact, by assuming $B \neq 0$, and $C \neq 0$, also by considering the form in eq. (3.4) we conclude that, the only possible Jordan normal form in this class that would satisfy eq. (2.19) is

$$N_{15} = \begin{pmatrix} \alpha_i & 0 & 0 & 0 \\ 0 & R & I & 0 \\ 0 & -I & R & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(3.55)

where we define $\lambda = R + Ii$ with

$$R = -\frac{B}{2A}, \quad I = \frac{\sqrt{4AC - B^2}}{2A}.$$ 

(3.56)

Again, it can be verified that eq. (3.55) satisfies eq. (2.19) via direct substitution. Now, by taking the trace of $N_{15}$, and also by referring to eq. (2.21) we find that

$$\alpha_i = -\frac{\beta_2}{\beta_3}.$$ 

(3.57)

Furthermore,

$$tr(N_{15})^2 = (e_1)^2 - 2e_2 = (\alpha_i)^2 + 2(R^2 - I^2),$$

(3.58)

which reduces to the condition

$$(\beta_2)^2 - \beta_1\beta_3 = 0,$$

(3.59)

upon using the definitions in eq. (2.21). Substituting eq. (3.57) into eq. (3.1) leads us to

$$e_2 = -\frac{2\beta_2}{\beta_3}e_1 - \frac{2(\beta_2)^2}{(\beta_3)^2} - \frac{\beta_1}{\beta_3} + \frac{D}{\beta_2}.$$ 

(3.60)

We realize that, in this solution $e_1$ remains to be an arbitrary spacetime function. When one specifies $e_1$ one can express $e_2$ in terms of it from eq. (3.60), then one can explicitly obtain the matrix entries of $N_{15}$ by using eq. (2.21) in eq. (3.56), and from eq. (3.57). The conditions of existence of $N_{15}$ are eq. (3.59), and $\Delta = B^2 - 4AC < 0$ in which the particular choice of $e_1$, and the corresponding $e_2$ must be used.
4 The solution space

In the previous section, we have explicitly constructed the entire set of nontrivial Jordan canonical form solutions of eq. (2.19). We have disregarded the trivial case of $\sqrt{\Sigma} = \text{diag}(\alpha_i, 0, 0, 0)$ which results in nonphysical $f$–metric solutions. Before defining the solution space of eq. (2.19), let us discuss one last constraint on the solutions that we constructed in the previous section. In obtaining the Jordan normal form of the solutions, although we used the conditions on the elementary symmetric polynomials $e_1$, and $e_2$ we did not refer to the $e_3$–structure of the solutions. This is a necessary, and a crucial point, as our solution ansatz eq. (2.13) brings the constraint eq. (2.16) on $e_1, e_2, e_3$ values of the solutions which we have not yet considered. To impose this condition on the solutions we simply take the trace of eq. (2.19). Following the trace operation on eq. (2.19) if we make use of the eqs. (2.3), and (2.16) we get the relation

$$-2\beta_1 e_1 - \beta_2 e_2 - \tilde{\rho} + 2\tilde{p} = 0. \quad (4.1)$$

When, for each solution the appropriate value of $e_1 = e_1(\beta, \tilde{\rho}, \tilde{p})$, and $e_2 = e_2(\beta, \tilde{\rho}, \tilde{p})$ of that particular solution are used in eq. (4.1) the above expression fixes the equation of state of the effective fluid that is $\tilde{\rho} = \tilde{p}(\tilde{\rho})$ for the solution chosen. At this stage, we can explain why we have to exclude the solutions of eq. (2.19) for the $B = 0$, and/or $C = 0$ cases. These conditions via the definitions in eq. (2.21) will bring an extra constraint on $\tilde{\rho}, \tilde{p}$. Thus, when solved simultaneously with eq. (4.1) this constraint will cause $\tilde{\rho}, \tilde{p}$ to be constants. Besides, the $n = 1, 3$ cases of the $\Delta = 0$ solutions of the previous section also can be eliminated as cosmological solutions. Similarly, for those solutions we have seen that the $\Delta = 0$ condition already caused the determination of the equation of state of the effective fluid composing the cosmological solution ansatz eq. (2.13). For these cases, when one solves the resulting conditions on $\tilde{\rho}, \tilde{p}$ coming from eqs. (3.47), and (3.50), or (3.52) together with eq. (4.1) one sees that both $\tilde{\rho}$, and $\tilde{p}$ must be constants again. Therefore, as it will be clear in the next section for all of these cases the scale factor can not evolve hence, it results in static, and nonphysical cosmological solutions. Now, firstly let us define the set of Jordan normal form solutions of eq. (2.19)

$$\mathcal{J}_1 = \left\{ N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{12}, N_{14}, N_{15} \right\}. \quad (4.2)$$

Next, we introduce the matrix field

$$P_1 = \begin{pmatrix} m(x^\mu) & 0 \\ 0 & P_{3 \times 3}(x^\nu) \end{pmatrix}, \quad (4.3)$$

where $P(x^\mu)$ is an invertible $3 \times 3$ matrix field, and $m(x^\mu)$ is a scalar field which can simply be taken as $m(x^\mu) = 1$ without loss of generality. Since any element $J \in \mathcal{J}_1$ is a solution of eq. (2.19) if we perform a similarity transformation on both sides of eq. (2.19) we get

$$A(P_1^{-1}JP_1)^3 + B(P_1^{-1}JP_1)^2 + C P_1^{-1}JP_1 + \mathcal{D} = 0, \quad (4.4)$$

where we have used the fact that $P_1^{-1}\mathcal{D}P_1 = \mathcal{D}$. Therefore, we see that $P_1^{-1}JP_1$ is also a solution of eq. (2.19) for any $J \in \mathcal{J}_1$, and for any matrix field of the form eq. (4.3). In that regard, we can define a subset of the solution space of eq. (2.19) as

$$\mathcal{M}_1 = \left\{ \sqrt{\Sigma} \mid \sqrt{\Sigma} = P_1^{-1}JP_1 \mid \forall J \in \mathcal{J}_1, \text{ and } \det P_1 \neq 0 \right\}. \quad (4.5)$$
in which $P_1$ is any matrix field of the form eq. (4.3). We should remark that eq. (4.1) that is obtained for the element $J \in J_1$ remains the same for also the corresponding element $P_1^{-1}J P_1 \in M_1$ as the elementary symmetric polynomials do not vary under similarity transformations. Now, let us consider the matrix equations

$$A(J)^3 + B(J)^2 + CJ + D_i = 0,$$

(4.6)

where $i = 2, 3, 4$. Here, $D_2 = \text{diag}(0, D, 0, 0)$, $D_3 = \text{diag}(0, 0, D, 0)$, and $D_4 = \text{diag}(0, 0, 0, D)$. The Jordan canonical form solution spaces namely $J_i$ of these equations can be constructed from the elements of $J_1$. In particular, for example, the diagonal elements of $J_2$ are obtained by placing $\alpha_i$ to the second diagonal entry, and by shifting the rest of the diagonal entries diagonally in the elements of $J_1$. Also, the two nondiagonal elements of $J_2$ are obtained again, by placing $\alpha_i$ in the second diagonal entry and by shifting the primary blocks diagonally in $N_{14}, N_{15}$. The elements of $J_3$, and $J_4$ can be obtained in a similar fashion. We should state that all these diagonal shifting operations which are used to generate the elements of $J_{2,3,4}$ from the elements of $J_1$ do not change the elementary symmetric polynomials of the corresponding element as we keep the diagonal content in these operations. Thus, the parametrization derived in section three for the elements of $J_1$ are also valid for the elements of $J_{2,3,4}$. Next, let us define the invertible $4 \times 4$ transformation matrix functions

$$P_2 = \begin{pmatrix} 0 & t_2 & 0 & 0 \\ 0 & 0 & \bullet & \bullet \\ 0 & \bullet & 0 & \bullet \\ \bullet & \bullet & \bullet & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & t_3 & 0 \\ 0 & \bullet & 0 & \bullet \\ \bullet & 0 & \bullet & \bullet \\ \bullet & \bullet & \bullet & 0 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 0 & 0 & 0 & t_4 \\ 0 & \bullet & \bullet & 0 \\ \bullet & 0 & \bullet & \bullet \\ \bullet & \bullet & \bullet & 0 \end{pmatrix},$$

(4.7)

where $t_{2,3,4}$ are arbitrary functions of $x^\mu$ like $m(x^\mu)$, and the entries which are not specified in the above $4 \times 4$ matrices form up partitioned $3 \times 3$ invertible matrix functions. If now, we define the solution spaces

$$M_2 = \left\{ \sqrt{\Sigma} \mid \sqrt{\Sigma} = P_2 J P_2^{-1} \quad \forall J \in J_2, \text{ and } \det P_2 \neq 0 \right\},$$

$$M_3 = \left\{ \sqrt{\Sigma} \mid \sqrt{\Sigma} = P_3 J P_3^{-1} \quad \forall J \in J_3, \text{ and } \det P_3 \neq 0 \right\},$$

$$M_4 = \left\{ \sqrt{\Sigma} \mid \sqrt{\Sigma} = P_4 J P_4^{-1} \quad \forall J \in J_4, \text{ and } \det P_4 \neq 0 \right\},$$

(4.8)

then

$$M = M_1 \cup M_2 \cup M_3 \cup M_4,$$

(4.9)

becomes the general solution space of the eq. (2.19). We will give a sketch of the proof of this fact in the appendix. Now that we have found the complete solution space of eq. (2.19), we can turn our attention on the background metric solutions of the action eq. (2.1). By referring to eq. (2.4) we now have

$$f = g \Sigma,$$

(4.10)

where $\Sigma$ is the square of any element in $M$, and $g$ is the FLRW metric. However, not all elements of $M$ which solve eq. (2.19) will lead to symmetric results in eq. (4.10) thus, physically acceptable background metrics. We have to impose the condition

$$g \Sigma = \Sigma^T g,$$

(4.11)
which guarantees the symmetry of \( f \). Therefore, we define the cosmological solution moduli of the action eq. (2.1) as the set

\[
\Gamma_C = \{(g, f) \mid f = gX^2, X \in \mathcal{M}, \text{ and } gX^2 = (X^T)^2g\}.
\] (4.12)

In special, when one chooses the diagonal elements in \( J_1, J_2, J_3, J_4 \), then squares them, and substitutes the result in eq. (4.10) one directly obtains the exact background metric solutions in a concise way without being obliged to concern the symmetry condition. On the other hand, for the more general cases one has to choose a special form for the matrices \( P_1, P_2, P_3, P_4 \) to satisfy eq. (4.11). Since, the symmetry requirement in eq. (4.11) becomes

\[
gP_i^T J^2 P_i^{-1} = (P_i^T J^2 P_i^{-1})^T g,
\] (4.13)

a closer inspection denotes that for a particular choice of \( J \in J_i \) this equation brings three algebraic constraint conditions on the function-entries of the solution-generating \( P_i \)-matrices (in particular, their unspecified \( 3 \times 3 \) partitions) which enable us to determine three of the entries of these partitions in terms of the other six entries which remain arbitrary. Next, we will give a summary of the cosmological dynamics.

5 Cosmological dynamics

In the \( g \)-sector beside the effective ideal fluid energy-momentum tensor that is introduced in eq. (2.13), we will also take the physical matter as a perfect fluid with the energy-momentum tensor

\[
T^g_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu},
\] (5.1)

where \( p = p(t) \), and \( \rho = \rho(t) \) are the pressure, and the energy density of the \( g \)-matter fluid, respectively. Now, by using the physical \( g \)-matter, and the effective energy-momentum tensors together with the FLRW metric eq. (2.12) in the \( g \)-metric equations eq. (2.6) leads us to the \( t \)-component equation

\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho - \frac{m^2}{3} \tilde{p} - \frac{\Lambda^g}{6},
\] (5.2)

as well as the three identical spatial–component equations

\[
\frac{2\ddot{a}}{a} = -\left( \frac{\dot{a}}{a} \right)^2 - \frac{k}{a^2} - 8\pi Gp + m^2 \tilde{p} - \frac{\Lambda^g}{2},
\] (5.3)

which become the modified Friedmann equations. By using eq. (5.2) in eq. (5.3) we can obtain the modified cosmic acceleration equation as

\[
\frac{\dddot{a}}{a} = -\frac{4\pi G}{3}(3p + \rho) + \frac{m^2}{6}(3\tilde{p} + \tilde{\rho}) - \frac{\Lambda^g}{6}.
\] (5.4)

We observe that the Friedmann, and cosmic acceleration equations are in the canonical form, only getting additional contributions from the effective fluid pressure, and energy density which are the reflections of the interaction Lagrangian term in eq. (2.1) which is proportional to the squared graviton mass. The matter-fluid equation

\[
\dot{\rho} = -\frac{3\dot{a}}{a}(p + \rho),
\] (5.5)
is the consequence of the matter energy-momentum conservation law namely, $\nabla^\nu T^\mu_\nu = 0$ that is derived for the FLRW $g$–metric. Besides, a similar continuity equation 
\[ \dot{\tilde{\rho}} = - \frac{3\dot{a}}{a} (\tilde{\rho} + \tilde{P}), \]  
(5.6)
for the effective fluid follows from the substitution of the effective energy-momentum tensor eq. (2.13) into the corresponding Bianchi identity in eq. (2.11) upon using the FLRW $g$–metric. On the other hand, in this solution scheme the $f$–metric equation becomes 
\[ \kappa \left[ R^I_{\mu\nu} - \frac{1}{2} R^f f_{\mu\nu} - \frac{1}{2} \Lambda^f f_{\mu\nu} \right] + m^2 \frac{\sqrt{-g}}{\sqrt{-f}} f^\mu_{\rho\nu} (\tilde{\rho} + \tilde{P}) \delta^\rho_\beta \delta^0_\nu = \epsilon 8\pi G T^f_{\mu\nu}, \]  
(5.7)
where we have used eq. (2.19) in eq. (2.9), and substituted the result in eq. (2.7). When eq. (5.6) is satisfied one does not have to consider the second of the Bianchi identities in eq. (2.11) as it is also automatically satisfied [15, 16]. Since the two metric sectors are efficiently decoupled from each other one can solve the $g$–sector equations independently without making an assumption on the $f$–matter. The solution methodology should start by first choosing which similarity equivalence class representatives in $J_{1,2,3,4}$ links the two metrics. Fixing $J$ in this way determines the equation of state of the effective fluid via eq. (4.1) by substituting the appropriate elementary symmetric polynomials of the chosen $J$. Then, by using the equations of state of the effective, and the $g$–matter (corresponding to various eras) one can solve eqs. (5.2), (5.5), and (5.6) to find out the evolution of the scale factor, and the state of the effective fluid, and the matter ideal fluid. The reader should appreciate that, our solutions are justified only if one finds also solutions of the $f$–metric equations eq. (5.7). In general, one can now use $a, \rho, \tilde{P}$ (which are completely determined) in eq. (4.10) to read the associated $f$–metric which has an implicit dynamical link in eq. (4.10) to the $g$–sector via barely, the metric $g$, and the elements of $J_{1,2,3,4}$ which are not only functions of the $\beta$–parameters of the theory but also the effective pressure, and the energy density of the effective ideal fluid whose functional forms are solved from the cosmological equations of the $g$–metric sector. At this point we have to remark that, although we have previously mentioned a degree of arbitrariness in constructing $f$ in the set eq. (4.12) via six arbitrary entries of the matrices $P_i$ these arbitrary functions may also be used to fix the form of $f$ now, so that it will satisfy eq. (5.7) when one chooses the form of $f$–matter in it. However, this route is not the only one to be followed in general. On the contrary, to exemplify a solution outline in the $f$–sector let us consider the solutions $N_1, N_2$. If they are used in eq. (4.10) one obtains 
\[ f = -N(t)dt^2 + \frac{b^2(t)}{1 - kr^2} dr^2 + b^2(t)r^2 d\theta^2 + b^2(t)r^2 sin^2 \theta d\varphi^2, \]  
(5.8)
which is in the generalized FLRW form with a lapse function $N(t)$. Here, we see that $N(t) = (\alpha_t)^2$, and the $f$–scale factor can be read from $b^2(t) = (\lambda_{1,2})^2 a^2(t)$. We can also read $\alpha_t = \alpha_t[\beta_j, \tilde{\rho}, \tilde{P}]$, and $\lambda_{1,2} = \lambda_{1,2}[\beta_j, \tilde{\rho}, \tilde{P}]$ from eq. (3.19). Since, as we discussed above from the $g$–sector equations $a, \tilde{\rho}, \tilde{P}$ are completely solved $N(t)$, and $b(t)$ in eq. (5.8) are also determined. Thus, in this case the $f$–metric is fixed as a generalized FLRW one. Let us also take the $f$–matter in the perfect fluid form, and consider $\Lambda^f = \Lambda^f(t)$. With these choices, and the substitution of the $f$–metric from eq. (5.8) the $f$–metric equation (5.7) becomes 
\[ \kappa \left[ G^I_{\mu\nu} - \frac{1}{2} \Lambda^f(t) f_{\mu\nu} \right] - m^2 \tilde{\Lambda}^f(t) N(t) \delta_{\mu0} \delta_{0\nu} = \epsilon 8\pi G \left( (\rho_f + p_f) U_{\mu} U_{\nu} + p_f f_{\mu\nu} \right), \]  
(5.9)
where

\[ \tilde{\Lambda}^f(t) = \sqrt{\frac{1}{\det(N^2_{1,2})}(\tilde{\rho} + \tilde{p})}. \]  

(5.10)

We should remind the reader that, in eq. (5.9) the only unknown functions are \( \Lambda^f(t), \rho_f(t), p_f(t) \) as the scale factor \( b(t) \) is predetermined. The third term on the left hand side in eq. (5.9) will only contribute a time-dependent effective cosmological constant to the 00-component but not to the spatial component equations. However, both the 00-, and the spatial-component equations will get extra contributions from the lapse function with respect to the FLRW ones. Therefore, from eq. (5.9) we will get two modified Friedmann equations which are algebraic (since the scale factor is already determined) for the unknown functions \( \Lambda^f(t), \rho_f(t), p_f(t) \). There is also a first-order ordinary differential equation arising from the fluid equation of the \( f \)-matter perfect fluid, namely, from \( \nabla^f_{\mu}(T^f_{\mu\nu}) = 0 \). Finally, from these two algebraic, and one first-order ordinary differential equations we can solve the unknown functions \( \Lambda^f(t), \rho_f(t), p_f(t) \) to complete the \( f \)-sector solution.

6 Concluding remarks

For the massive bigravity theory \[9\] we constructed the complete solution moduli space of the \((f, g)\) couples of metrics which admit a FLRW cosmology in the \(g\)-sector via the presence of an effective ideal fluid contribution coming from the interaction Lagrangian of the mass terms in addition to the matter one. We employed the cosmological solution ansatz by choosing the energy-momentum tensor of the interaction terms in the \(g\)-metric equations in the form of an effective ideal fluid one. This choice resulted in a cubic matrix equation for the building block matrix of the interaction Lagrangian that is composed of the two metrics. By deriving the general solution space of this nontrivial matrix equation (whose coefficients are also functions of the elementary symmetric polynomials of its solutions) we were able to construct and define the complete solution space of the \((f, g)\) metric configurations which enable FLRW cosmologies in the \(g\)-sector that is modified by an effective ideal fluid whose contributions are proportional to the square of the graviton mass. Although, we obtained the general solutions of the ansatz matrix equation we also discussed that one still has to impose a symmetry condition on these solutions to construct a symmetric result for the \(f\)-metric. Therefore, in spite of the existence of a matrix field degree of freedom in constructing \(f\)-solutions out of the \(g\)-sector fields one has to render three out of nine function components of this arbitrary matrix field to satisfy the symmetry condition we mentioned. Furthermore, we also discussed that one might also have to fix the remaining degrees of freedom of the \(f\)-metric in satisfying the \(f\)-metric sector field equations in the presence of \(f\)-type matter. We have shown that, the cosmological solution moduli of bigravity that we constructed is composed of similarity equivalence classes which do not differ from each other only in their functional form but also in the equations of state that they impose on the associated effective ideal fluid they give rise to. Finally, in the last section, we presented the resulting cosmological equations of the \(g\), and the \(f\)-metrics for which we shortly discussed the solution flow chart dictated by the semi-decoupling of the two metric sectors.

The known exact solutions of bigravity can in general be divided into three groups \[20\]. There is a class of solutions in which both metrics are proportional to each other. There exists another class of spherically symmetric solutions which has a nondiagonal background metric. There are also solutions including both diagonal but not proportional \(f\), and \(g\) metrics. In this work, we present the complete cosmological background solution space of the
theory. Massive bigravity as a ghost-free massive gravity theory promises to possess cosmological solutions which can exhibit late time self-acceleration which could compensate the dark energy problem in standard cosmology. The background cosmological solutions [12–26], and their perturbations and stability issues [14, 22, 23, 30, 31] arising from the above-listed known solutions have gained a considerable interest and they are under extensive inspection recently. It has been shown that although there are stability problems and the perturbations of these solutions differ from the ones of GR these problems can still be overcome by turning on the $f$–type matter which is heuristically interpreted as dark matter [14, 22, 23, 30, 31]. We believe that, apart from its mathematical legitimacy of completeness which presents an extensive amount of new cosmological solutions of the theory our derivation of the cosmological background solution space can also serve for the phenomenology of the theory. We have found that, in the general similarity equivalence class structure of the solutions there is a rich variety of functional relations between the spatial parts of the two metrics unlike the case in the particular cosmological solution which is widely studied in the literature. The behavior of the ratio of the $g$, and $f$–scale factors of this particular solution (which we believe must be related to the $N_1$, or $N_2$ solutions we have discussed) causes early time instabilities of the perturbations which differ from the GR ones. Therefore, we hope that among the variety of complete background solutions we have derived there may exist ones which may admit acceptable perturbation behavior with respect to GR perturbations. To explicitly construct, and study the solutions in this direction one may follow two main routes, one may either inspect the solution behavior in the various similarity classes one by one or one may attempt to design particular form of cosmological solutions with or without $f$–matter which exhibit a stable nature of perturbations within the solution construction methodology we have discussed. However, we should also state that in our generally-constructed solution space, majority of the $f$–metric solutions may fail to exhibit homogeneity, and/or isotropy behavior. On the other hand, one may question the necessity of homogeneity, and isotropy in the $f$–sector since opposite cases may have acceptable results from the $g$–metric perturbation theory point of view, and in addition they may lead to interesting variety of dark matter scenarios. Finally, we point out a possible direction in which one can extend the results of the present work to study the cosmological solutions within the newly proposed formalism of ghost-free effective-metric-matter coupling [32–34].

A On the completeness of $\mathcal{M}$

In the appendix, we will give a sketch of the proof of the statement we made in section four that any solution of eq. (2.19) must belong to the solution set (4.9). First, let us assume that eq. (2.19) has a diagonalizable solution $X_D$ such that

$$\tau(X_D) + \mathcal{D} = 0. \quad (A.1)$$

If we do a similarity transformation which brings $X_D$ to a diagonal Jordan form $J_D$ on the above equation then we get

$$P^{-1} \tau(X_D) P + P^{-1} \mathcal{D} P = \tau(J_D) + P^{-1} \mathcal{D} P = 0, \quad (A.2)$$

where $J_D = P^{-1} X_D P$. We can directly observe that $P^{-1} \mathcal{D} P$ must be a diagonal matrix. Furthermore, since the eigenvalues of $\mathcal{D}$ must be invariant under a similarity transformation we can conclude that the diagonal matrix $P^{-1} \mathcal{D} P$ must be in one of the forms; diag($D, 0, 0, 0$),

– 18 –
diag(0, D, 0, 0), diag(0, 0, D, 0), or diag(0, 0, 0, D). From this observation we deduce that \( P \) must be in one of the forms in eqs. (4.3), or (4.7). This result proves that any diagonalizable solution of eq. (2.19) must be an element of \( \mathcal{M} \). On the other hand, let us consider nondiagonalizable solutions of eq. (2.19) with real eigenvalues. They also satisfy

\[
\tau(X_{ND}) + D = 0, \tag{A.3}
\]

which can be brought to a form

\[
P^{-1}\tau(X_{ND})P + P^{-1}DP = \tau(J_{ND}) + P^{-1}DP = 0, \tag{A.4}
\]

where \( J_{ND} = P^{-1}X_{ND}P \) is one of the nondiagonal Jordan canonical forms

\[
J_1 = \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix}, \quad J_2 = \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & a_4 \end{pmatrix}, \quad J_3 = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix},
\]

\[
J_4 = \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & a_4 \end{pmatrix}, \quad J_5 = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & a_4 \end{pmatrix}, \quad J_6 = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & e \end{pmatrix}, \tag{A.5}
\]

where in \( J_{2,3,4,5,6} \) the distinct diagonal elements \( a_{1,2,3,4} \) may be equal to \( e \) or a different value than \( e \). Here, we observe that since \( \tau(J_{ND}) \) is in uppertriangular form

\[
P^{-1}DP = \begin{pmatrix} u_1 & u_5 & u_9 & u_{13} \\ u_2 & u_6 & u_{10} & u_{14} \\ u_3 & u_7 & u_{11} & u_{15} \\ u_4 & u_8 & u_{12} & u_{16} \end{pmatrix} \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ t_5 & t_6 & t_7 & t_8 \\ t_9 & t_{10} & t_{11} & t_{12} \\ t_{13} & t_{14} & t_{15} & t_{16} \end{pmatrix},
\]

\[
= \begin{pmatrix} u_1 t_1 & u_1 t_2 & u_1 t_3 & u_1 t_4 \\ u_2 t_1 & u_2 t_2 & u_2 t_3 & u_2 t_4 \\ u_3 t_1 & u_3 t_2 & u_3 t_3 & u_3 t_4 \\ u_4 t_1 & u_4 t_2 & u_4 t_3 & u_4 t_4 \end{pmatrix} \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{pmatrix}, \tag{A.6}
\]

must be in uppertriangular form too. Thus, its diagonal elements which are its eigenvalues must be \( \{0, 0, 0, D\} \) as under similarity transformations eigenvalues and their algebraic multiplicities are preserved. Therefore, \( P^{-1}DP \) must be in one of the forms

\[
\begin{pmatrix} D & \cdots & \cdots \\ 0 & \cdots & \cdots \\ 0 & \cdots & \cdots \\ 0 & \cdots & \cdots \end{pmatrix}, \quad \begin{pmatrix} 0 & \cdots & \cdots \\ 0 & D & \cdots \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \cdots & \cdots \\ 0 & \cdots & \cdots \\ 0 & \cdots & \cdots \\ 0 & \cdots & \cdots \end{pmatrix}, \quad \begin{pmatrix} 0 & \cdots & \cdots \\ 0 & \cdots & \cdots \\ 0 & \cdots & \cdots \\ 0 & \cdots & \cdots \end{pmatrix}. \tag{A.7}
\]

Let us assume that the nondiagonalizable solution is such that \( P^{-1}DP \) becomes the second form above. In this case, since \( u_2 \) and \( t_2 \) can not be zero we see from eq. (A.6) that \( u_3, u_4, t_1 \) must vanish. The corresponding Jordan form must be \( J_6 \) because in all the other cases via eq. (A.4) the diagonal entries would lead to two inconsistent equations \( Ae^3 + Be^2 + Ce + D = 0 \), and \( Ae^3 + Be^2 + Ce = 0 \).\footnote{We refer the reader to the first footnote in section three.} In this restriction, again from eq. (A.6) we see that as \( t_2 \) is not
zero $u_1$ must be zero, also, as $u_2$ is not zero $t_3$ must be zero. Besides, eq. (A.4) denotes that when $J_6$ is used in eq. (A.4) since $u_2 \neq 0, t_4$ must be zero. Furthermore, $P^{-1}P = PP^{-1} = 1$ yields $t_6 = t_{10} = t_{14} = u_6 = u_{10} = u_{14} = 0$. Therefore, we conclude that in this case $P = P_2$, and $P^{-1}DP = \text{diag}(0, D, 0, 0)$. If we consider the nondiagonalizable solutions which generate the fourth matrix in eq. (A.7) for $P^{-1}DP$ we realize that since $u_4, t_4$ are nonzero $t_1, t_2, t_3$ must be zero. Also, similar to the previous case, since they would result in the inconsistent equations $Ae^3 + Be^2 + Ce + D = 0$, and $Ae^3 + Be^2 + Ce = 0$ the Jordan canonical forms $J_1, J_3, J_6$ must be excluded in this case. For this reason, since $t_4 \neq 0, u_3$ must be zero as a result of the substitution of the possible Jordan forms $J_2, J_4, J_5$ in eq. (A.4). In addition, since $\tau(J_2, J_4, J_5)$ can not have nonzero elements at the fourth column except the fourth row, since $t_4 \neq 0$ via eq. (A.6) we see that $u_1, u_2$ must be zero. Upon these substitutions, $P^{-1}P = PP^{-1} = 1$ gives $t_8 = t_{12} = t_{16} = u_8 = u_{12} = u_{16} = 0$. Hence, we observe that for this case $P = P_4$, and $P^{-1}DP = \text{diag}(0, 0, 0, D)$. Two straightforward, and similar analysis show also that for the first, and the third cases in eq. (A.4) the transformation matrices must be $P_1$, and $P_3$, also, $P^{-1}DP = \text{diag}(D, 0, 0, 0)$, and $P^{-1}DP = \text{diag}(0, 0, D, 0)$, respectively. Besides, the possible Jordan forms for these cases are $J_3, J_5, J_6$, and $J_4$, respectively. We observe also that, $J_1$ is not possible for any of the nondiagonalizable solutions with real eigenvalues. Therefore, this analysis proves that any nondiagonalizable solution of eq. (2.19) with real eigenvalues must be contained in $\mathcal{M}$. Now, let us consider the nondiagonalizable soltions of eq. (2.19) with complex eigenvalues. By applying an appropriate similarity transformation on eq. (A.3) we get

$$P^{-1}\tau(X_{ND})P + P^{-1}DP = \tau(K_{ND}) + P^{-1}DP = 0,$$

(A.8)

where $K_{ND} = P^{-1}X_{ND}P$ is one of the nondiagonal Jordan canonical forms

$$K_1 = \begin{pmatrix} R_1 & I_1 & 0 & 0 \\ -I_1 & R_1 & 0 & 0 \\ 0 & 0 & R_2 & I_2 \\ 0 & 0 & -I_2 & R_2 \end{pmatrix}, \quad K_2 = \begin{pmatrix} R & I & 0 & 0 \\ -I & R & 0 & 0 \\ 0 & 0 & e & 1 \\ 0 & 0 & 0 & e \end{pmatrix},$$

$$K_3 = \begin{pmatrix} R & I & 0 & 0 \\ -I & R & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}, \quad K_4 = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & R & I & 0 \\ 0 & 0 & -I & R \\ 0 & 0 & 0 & a_4 \end{pmatrix},$$

$$K_5 = \begin{pmatrix} e & 1 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & R & I \\ 0 & 0 & -I & R \end{pmatrix}, \quad K_6 = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & R & I \\ 0 & 0 & 0 & -I \end{pmatrix}.$$

(A.9)

We should state that, when substituted into $\tau$ all these matrices keep their forms with entries changed. For example,

$$\tau(K_1) = \begin{pmatrix} R_1 & I_1 & 0 & 0 \\ -I_1 & R_1 & 0 & 0 \\ 0 & 0 & R_2 & I_2 \\ 0 & 0 & -I_2 & R_2 \end{pmatrix}.$$

(A.10)

If we use this in eq. (A.8), and refer to eq. (A.6) we see that $u_1 t_1 = u_2 t_2$, and $u_2 t_1 = -u_1 t_2$ which give $t_1 = t_2 = 0$. Also, $u_3 t_3 = u_4 t_4$, and $u_4 t_3 = -u_3 t_4$ which give $t_3 = t_4 = 0$. However,
now $P$ becomes a zero-matrix, hence, it becomes singular, and can not perform any similarity transformation. Therefore, this case must be excluded (there exists no nonsingular matrix which can bring a nondiagonalizable solution of eq. (2.19) into $K_1$-form since, this results in an inconsistency). Let us consider the case $K_4$ which leads to the form

$$\tau(K_4) = \begin{pmatrix} a'_1 & 0 & 0 & 0 \\ 0 & R' & I' & 0 \\ 0 & -I' & R' & 0 \\ 0 & 0 & 0 & a'_4 \end{pmatrix}.$$ (A.11)

Similarly, now, from eq. (A.8) we have $u_2 t_2 = u_3 t_3$, and $u_2 t_3 = -u_3 t_2$ which give $t_3 = t_2 = 0$. From eq. (A.6) we deduce that, for a consistent nontrivial $P$ we can not have $t_1 = 0$, and $t_4 = 0$ at the same time. Also, $u_{1,2,3,4}$ can not vanish simultaneously. These facts leave us two cases. Either; $t_1 = 0, t_4 \neq 0$, but $u_{1,2,3} = 0, u_4 \neq 0$, or $t_1 \neq 0, t_4 = 0$, but $u_{2,3,4} = 0, u_1 \neq 0$ to have consistency when eq. (A.11) is substituted into eq. (A.8). The first case gives $P = P_4$, and $P^{-1}DP = \text{diag}(0,0,0,D)$ (via the preservation of the eigenvalues under similarity transformations). Whereas, the second case corresponds to $P = P_1$, and $P^{-1}DP = \text{diag}(D,0,0,0)$. Next, let us consider $K_2$. Similar to the previous cases above now, for the consistency of eq. (A.8) we must have $t_1 = t_2 = 0$. Hence, $P^{-1}DP$ must be in uppertriangular form with diagonal elements as its eigenvalues which must be the set \{D,0,0,0\} where $D$ must be either at the third, or the fourth diagonal entry. However, for either of these cases eq. (A.8) leads us to two inconsistent equations $Ae^3 + Be^2 + Ce + D = 0$, and $Ae^3 + Be^2 + Ce = 0$ as we assumed $D \neq 0$. Therefore, this case must be excluded. $K_3$ on the other hand, leads to the conditions $u_1 t_1 = u_2 t_2$, and $u_2 t_1 = -u_1 t_2$ that give $t_1 = t_2 = 0$. Again, eq. (A.6) shows that for a consistent nontrivial $P$ we can not have $t_3 = 0$, and $t_4 = 0$ at the same time, as well as $u_{1,2,3,4}$ can not vanish all. Thus, either; $t_3 = 0, t_4 \neq 0$, but $u_{1,2,3} = 0, u_4 \neq 0$, or $t_3 \neq 0, t_4 = 0$, but $u_{1,2,4} = 0, u_3 \neq 0$ to have consistency when $\tau(K_3)$ is used in eq. (A.8). The first case gives $P = P_4$, and $P^{-1}DP = \text{diag}(0,0,0,D)$, and the second case corresponds to $P = P_3$, and $P^{-1}DP = \text{diag}(0,0,D,0)$. A very similar line of reasoning denotes that $K_5$ is not possible, also, $K_6$ is possible with either; $P = P_1$, and $P^{-1}DP = \text{diag}(D,0,0,0)$, or $P = P_2$, and $P^{-1}DP = \text{diag}(0,D,0,0)$. Therefore, we conclude that any nondiagonalizable solution with complex eigenvalues of eq. (2.19) must also be contained in $\mathcal{M}$. As a final remark, in summary, in the appendix we showed that any diagonalizable or nondiagonalizable solution of eq. (2.19) must be an element of $\mathcal{M}$ via proving that their Jordan canonical forms must satisfy one of the four equations in eq. (2.19), and eq. (4.6).

**Acknowledgments**

We thank Merete Lillemark for useful communications.
References

[1] C. de Rham and G. Gabadadze, Generalization of the Fierz-Pauli Action, 
*Phys. Rev. D* 82 (2010) 044020 [arXiv:1007.0443] [inSPIRE].

[2] C. de Rham, G. Gabadadze and A.J. Tolley, Resummation of Massive Gravity, 
*Phys. Rev. Lett.* 106 (2011) 231101 [arXiv:1011.1232] [inSPIRE].

[3] M. Fierz and W. Pauli, On relativistic wave equations for particles of arbitrary spin in an 
electromagnetic field, *Proc. Roy. Soc. Lond. A* 173 (1939) 211 [inSPIRE].

[4] D.G. Boulware and S. Deser, Can gravitation have a finite range?, *Phys. Rev. D* 6 (1972) 3368 [inSPIRE].

[5] D.G. Boulware and S. Deser, Inconsistency of finite range gravitation, 
*Phys. Lett.* B 40 (1972) 227 [inSPIRE].

[6] S.F. Hassan and R.A. Rosen, On Non-Linear Actions for Massive Gravity, 
*JHEP* 07 (2011) 009 [arXiv:1103.6055] [inSPIRE].

[7] S.F. Hassan and R.A. Rosen, Resolving the Ghost Problem in non-Linear Massive Gravity, 
*Phys. Rev. Lett.* 108 (2012) 041101 [arXiv:1106.3344] [inSPIRE].

[8] S.F. Hassan, R.A. Rosen and A. Schmidt-May, Ghost-free Massive Gravity with a General 
Reference Metric, *JHEP* 02 (2012) 026 [arXiv:1109.3230] [inSPIRE].

[9] S.F. Hassan and R.A. Rosen, Bimetric Gravity from Ghost-free Massive Gravity, 
*JHEP* 02 (2012) 126 [arXiv:1109.3515] [inSPIRE].

[10] V. Baccetti, P. Martin-Moruno and M. Visser, Massive gravity from bimetric gravity, 
*Class. Quant. Grav.* 30 (2013) 015004 [arXiv:1205.2158] [inSPIRE].

[11] V. Baccetti, P. Martin-Moruno and M. Visser, Null Energy Condition violations in bimetric 
gravity, *JHEP* 08 (2012) 148 [arXiv:1206.3814] [inSPIRE].

[12] V. Baccetti, P. Martin-Moruno and M. Visser, Gordon and Kerr-Schild ansatze in massive and 
bimetric gravity, *JHEP* 08 (2012) 108 [arXiv:1206.4720] [inSPIRE].

[13] D. Comelli, M. Crisostomi, F. Nesti and L. Pilo, FRW Cosmology in Ghost Free Massive 
Gravity, *JHEP* 03 (2012) 067 [Erratum ibid. 1206 (2012) 020] [arXiv:1111.1983] [inSPIRE].

[14] D. Comelli, M. Crisostomi and L. Pilo, Perturbations in Massive Gravity Cosmology, 
*JHEP* 06 (2012) 085 [arXiv:1202.1986] [inSPIRE].

[15] M. von Strauss, A. Schmidt-May, J. Enander, E. Mortsell and S.F. Hassan, Cosmological 
Solutions in Bimetric Gravity and their Observational Tests, *JCAP* 03 (2012) 042 
[arXiv:1111.1655] [inSPIRE].

[16] M.S. Volkov, Cosmological solutions with massive gravitons in the bigravity theory, 
*JHEP* 01 (2012) 035 [arXiv:1110.6153] [inSPIRE].

[17] M.S. Volkov, Exact self-accelerating cosmologies in the ghost-free bigravity and massive gravity, 
*Phys. Rev. D* 86 (2012) 061502 [arXiv:1205.5713] [inSPIRE].

[18] Y. Akrami, T.S. Koivisto and M. Sandstad, Accelerated expansion from ghost-free bigravity: a 
statistical analysis with improved generality, *JHEP* 03 (2013) 099 [arXiv:1209.0457] 
[inSPIRE].

[19] M.S. Volkov, Hairy black holes in the ghost-free bigravity theory, 
*Phys. Rev. D* 85 (2012) 124043 [arXiv:1202.6682] [inSPIRE].

[20] M.S. Volkov, Self-accelerating cosmologies and hairy black holes in ghost-free bigravity and 
massive gravity, *Class. Quant. Grav.* 30 (2013) 184009 [arXiv:1304.0238] [inSPIRE].
[21] F. Koennig, A. Patil and L. Amendola, Viable cosmological solutions in massive bimetric gravity, *JCAP* **03** (2014) 029 [arXiv:1312.3208] [inSPIRE].

[22] A. De Felice, A.E. Gümrukçuoğlu, S. Mukohyama, N. Tanahashi and T. Tanaka, Viable cosmology in bimetric theory, *JCAP* **06** (2014) 037 [arXiv:1404.0008] [inSPIRE].

[23] F. Koennig, Y. Akrami, L. Amendola, M. Motta and A.R. Solomon, Stable and unstable cosmological models in bimetric massive gravity, *Phys. Rev. D* **90** (2014) 124014 [arXiv:1407.4331] [inSPIRE].

[24] S.F. Hassan, A. Schmidt-May and M. von Strauss, Particular Solutions in Bimetric Theory and Their Implications, *Int. J. Mod. Phys.* **D 23** (2014) 1443002 [arXiv:1407.2772] [inSPIRE].

[25] T. Katsuragawa, Properties of Bigravity Solutions in a Solvable Class, *Phys. Rev. D* **89** (2014) 124007 [arXiv:1312.1550] [inSPIRE].

[26] M. Fasiello and A.J. Tolley, Cosmological Stability Bound in Massive Gravity and Bigravity, *JCAP* **12** (2013) 002 [arXiv:1308.1647] [inSPIRE].

[27] N.T. Yilmaz, Decoupling Solution Moduli of Bigravity, arXiv:1502.00463 [inSPIRE].

[28] N.T. Yilmaz, Effective Matter Cosmologies of Massive Gravity I: Non-Physical Fluids, *JCAP* **08** (2014) 037 [arXiv:1405.6402] [inSPIRE].

[29] N.T. Yilmaz, Effective matter cosmologies of massive gravity: Physical fluids, *Phys. Rev. D* **90** (2014) 124034 [arXiv:1412.4919] [inSPIRE].

[30] D. Comelli, M. Crisostomi and L. Pilo, FRW Cosmological Perturbations in Massive Bigravity, *Phys. Rev. D* **90** (2014) 084003 [arXiv:1403.5679] [inSPIRE].

[31] A. De Felice, T. Nakamura and T. Tanaka, Possible existence of viable models of bi-gravity with detectable graviton oscillations by gravitational wave detectors, *PTEP* **2014** (2014) 043E01 [arXiv:1304.3920] [inSPIRE].

[32] C. de Rham, L. Heisenberg and R.H. Ribeiro, On couplings to matter in massive (bi-)gravity, *Class. Quant. Grav.* **32** (2015) 035022 [arXiv:1408.1678] [inSPIRE].

[33] A.E. Gümrukçuoğlu, L. Heisenberg and S. Mukohyama, Cosmological perturbations in massive gravity with doubly coupled matter, *JCAP* **02** (2015) 022 [arXiv:1409.7260] [inSPIRE].

[34] S.F. Hassan, M. Kocic and A. Schmidt-May, Absence of ghost in a new bimetric-matter coupling, arXiv:1409.1909 [inSPIRE].