REDUCING QUADRATIC FORMS BY KNEADING SEQUENCES

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Abstract. We introduce an invertible operation on finite sequences of positive integers and call it “kneading”. Kneading preserves three invariants of sequences – the parity of the length, the sum of the entries, and one we call the “alternant”. We provide a bijection between the set of sequences with alternant \( a \) and parity \( s \) and the set of Zagier-reduced indefinite binary quadratic forms with discriminant \( a^2 + (-1)^s \cdot 4 \), and show that kneading corresponds to Zagier reduction of the corresponding forms. We then make some observations and conjectures concerning the cycles of sequences whose entries have given sum.

1. Kneading sequences

We pinch an end of a finite sequence of positive integers by splitting 1 away from the first entry (or last, if it is the right end) as so

\[(4, 7, 3, 1, \ldots) \mapsto (1, 3, 7, 3, 1, \ldots)\]
or, when the entry is 1 to begin with, by pushing the 1 back onto its neighboring entry

\[(1, 3, 7, 3, 1, \ldots) \mapsto (4, 7, 3, 1, \ldots)\]

We will knead a finite sequence of positive integers by

- Removing the first entry, then
- Pinching both ends of what remains, then
- Placing the removed entry at the end of the result.

We note that in the second step, the ends can be pinched in either order and yield the same result.

Let us clarify what happens with sequences of length 1 or 2. We make the convention that the sequence (1) and the empty sequence are pinched by doing nothing. Then kneading fixes the sequences of length 1, and kneading the sequence \((a, b)\) will give the sequence \((1, b - 2, 1, a)\) when \(b \geq 3\) and \((b, a)\) when \(b = 1\) or 2.

Kneading is an invertible process – simply remove the entry from the back, pinch both ends of what remains, then place the removed entry at the front. It should also be apparent that kneading preserves the sum of the entries of a sequence. Thus, kneading permutes the finitely many sequences with a given sum, and each sequence lies in a finite kneading cycle.
**Kneading Example.** Repeated kneading of the sequence \((2,2,3,6)\) gives:

\[
(2,2,3,6) \mapsto (1,1,3,5,1,2) \mapsto (4,5,1,1,1,1) \mapsto (1,4,1,1,2,4) \mapsto (1,3,1,1,2,3,1,1) \\
\mapsto (1,2,1,1,2,3,2,1) \mapsto (1,1,1,1,2,3,3,1) \mapsto (2,1,2,3,4,1) \mapsto (3,3,5,2) \\
\mapsto (1,2,5,1,1,3) \mapsto (1,1,5,1,1,2,1,1) \mapsto (6,1,1,2,2,1) \mapsto (2,2,3,6)
\]

Aside from the sum of the entries, kneading preserves two other sequence invariants – the parity of the length, and a more subtle invariant we call the “alternant”, which is a positive integer built from continued fractions.

In our main result, Theorem 1 we provide a one-to-one correspondence between the set of sequences with given alternant \(a\) and length parity \(s = 0\) or \(1\) and the set of Zagier-reduced indefinite binary quadratic forms of discriminant \(a^2 \pm (-1)^s \cdot 4\), and show that kneading corresponds to Zagier reduction of the corresponding forms. It follows that the number of sequences with alternant \(a\) and parity \(s\) is finite and that the number of cycles of such sequences is an (imprimitive) class number, the number of classes of quadratic forms of discriminant \(a^2 \pm (-1)^s \cdot 4\), both primitive and imprimitive.

In the final section, we return to study of classifying kneading cycles by the sum of the entries. This invariant is not obvious when viewed from the perspective of quadratic forms. We do some computations and make some observations and conjectures, but prove little about this invariant.

For the curious, kneading came about while the author was developing a generalization to indefinite forms of the Hardy-Muskat-Williams algorithm for representing an integer by a positive-definite form. The correspondence between sequences and Zagier-reduced forms was discovered first. Once the correspondence was discovered, it became a natural question to see what operation on sequences corresponds to reduction of forms, and the kneading pattern was then spotted.

## 2. The Correspondence

A binary quadratic form is, for us, a polynomial \(Ax^2 + Bxy + Cy^2\) in indeterminates \(x\) and \(y\) with integer coefficients. We will now refer to them simply as “forms”. The question of which integers are obtained by inputting integers into a given form has motivated a tremendous amount of mathematics. Famous results include Fermat’s Two Squares Theorem that the prime numbers represented by the form \(x^2 + y^2\) are those congruent to \(1\) modulo \(4\) and the fact that for each nonsquare number \(D > 0\), the Pell Equation \(x^2 - Dy^2 = 4\) has a solution with \(y \neq 0\).

A general study of forms begins with the notion of Lagrange and Gauss of *equivalent forms*. Two forms are equivalent if one is transformed into the other by acting upon it with a \(2 \times 2\) matrix with integer coefficients and determinant \(1\), that is, a matrix in the group \(\text{SL}_2(\mathbb{Z})\). Specifically, a matrix

\[
M = \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\]

in \(\text{SL}_2(\mathbb{Z})\) transforms the form \(f(x, y)\) into the form

\[
f(\alpha x + \beta y, \gamma x + \delta y).
\]
This gives a right action of $\text{SL}_2(\mathbb{Z})$ on the set of all forms. The action preserves discriminants, so the forms with fixed discriminant $D$ split into classes of equivalent forms. The first major theorem in the theory of binary quadratic forms is that the number of equivalence classes with given discriminant is finite. For $D > 0$, the number of classes of forms all of which have relatively prime coefficients (primitive forms) is a class number, a central notion of algebraic number theory.

A reduction algorithm is a standard tool for determining when two forms are equivalent. Reduction is more complicated in the case of interest to us, when $D > 0$. In fact, there are competing notions of reduction in this case. We shall use Zagier reduction \[3\] rather than the more common reduction of Lagrange and Gauss.

Zagier declares a form $f = Ax^2 + Bxy + Cy^2$ to be reduced if

$$A > 0, \quad C > 0, \quad B > A + C.$$  

To perform a reduction step on $f$, we

1. Compute the “reducing number”, determined as the unique integer $n$ satisfying

$$n - 1 < \frac{B + \sqrt{D}}{2A} < n,$$

in which $D$ is the discriminant of $f$.

2. Act on the form $f$ with the matrix

$$\begin{bmatrix} n & 1 \\ -1 & 0 \end{bmatrix}$$

Zagier reduction is iteration of reduction steps.

Because for a reduced form $D \geq D - (B - 2A)^2 = 4A(B - A - C) > 0$, we see that the reduced forms with given discriminant $D$ have bounded $A$. The same inequalities then imply that $B$ must be bounded, hence $C$ must be as well. There are thus finitely many Zagier-reduced forms with given positive discriminant. Zagier shows that every form will reach a reduced form after finitely many reduction steps, after which it will continue through a cycle of reduced forms. He also shows that two reduced forms are equivalent if and only if each can be obtained from the other by reduction, that is, both must lie in the same cycle of forms. Thus, every equivalence class contains reduced forms, and the number of cycles of reduced primitive forms is a class number.

To define alternants, we turn to continued fractions. Every rational number $\frac{\alpha}{\beta} > 1$ can be expanded in two ways as a finite simple continued fraction:

$$\frac{\alpha}{\beta} = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{\ddots + \cfrac{1}{q_l}}}}$$

with positive integer quotients $q_1, \ldots, q_l$. (Switching between the two expansions is accomplished by pinching the right end of this sequence.)
We will denote the numerator of the continued fraction with sequence of quotients $q_1, \ldots, q_l$ by $[q_1, \ldots, q_l]$. Such expressions are called continuants.

**Definition.** The alternant of a finite sequence of positive integers $\overrightarrow{q} = (q_1, \ldots, q_l)$ with $l \geq 3$ is the difference

$$[\overrightarrow{q}]^* := [q_1, \ldots, q_l] - [q_2, \ldots, q_{l-1}]$$

We define directly the alternant of $(q_1)$ to be $q_1$ and of $(q_1, q_2)$ to be $q_1 q_2$.

We define the length parity of a finite sequence to be 0 if the number of terms in the sequence is even and 1 if the number is odd. All sequences in a kneading cycle have the same length parity.

For integers $a > 0$ and $s = 0$ or 1, excepting the cases $(a, s) = (1, 1)$ and $(2, 1)$, we define sets

$$S_{(a, s)} = \{ \text{Sequences of positive integers with alternant } a \text{ and length parity } s \}$$

$$Z_{(a, s)} = \{ \text{Zagier-reduced forms of discriminant } a^2 + (-1)^s \cdot 4 \}$$

We define a map $\psi_{(a, s)} : Z_{(a, s)} \to S_{(a, s)}$ as follows. If $f = Ax^2 + Bxy + Cy^2$ is in $Z_{(a, s)}$, we first compute $z = (a + B)/2$ (an integer) and expand the rational number $\frac{z}{A}$ into the unique continued fraction with sequence of quotients of length parity $s$. We set $\psi_{(a, s)}(f)$ to be this sequence of quotients. It will be shown to have alternant $a$ in Section 3.

We also define a map $\phi_{(a, s)} : S_{(a, s)} \to Z_{(a, s)}$. We define $\phi_{(a, s)}((q_1, \ldots, q_l))$ to be the form

$$(1) \quad [q_2, \ldots, q_l] x^2 + ([q_1, \ldots, q_l] + [q_2, \ldots, q_{l-1}]) xy + [q_1, \ldots, q_{l-1}] y^2$$

(When $l = 1$, we should interpret this as $\phi_{(a, s)}((q_1)) = x^2 + q_1 xy + y^2$.) In Section 3, the discriminant of the form (1) will be computed as $a^2 + (-1)^s \cdot 4$, where $a$ is the alternant of $(q_1, \ldots, q_l)$.

Our main theorem states:

**Theorem 1.** The maps $\psi_{(a, s)}$ and $\phi_{(a, s)}$ are inverses, and through them Zagier reduction of forms corresponds to kneading.

**Example.** Consider the form $f = 44x^2 + 114xy + 17y^2$, which has discriminant $100^2 + 4$. To compute the corresponding sequence (with $a = 100$ and $s = 0$), we compute $z = (114 + 100)/2 = 107$, then expand $\frac{107}{44}$ as a continued fraction with even length

$$\frac{107}{44} = 2 + \frac{1}{2 + \frac{1}{3 + \frac{1}{6}}}$$

Thus, $\psi_{(100,0)}(f) = (2, 2, 3, 6)$, the sequence from the kneading example in Section 1. We reduce $f$ by computing the integer $n$ for which $n - 1 < \frac{114 + \sqrt{10000}}{88} < n$, that is, $n = 3$, and then act on $f$ by the matrix $\begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$ to obtain the new form $f' = 71x^2 + 150xy + 44y^2$. To find $\psi_{(100,0)}(f')$, we
calculate \( z = (150 + 100)/2 = 125 \), then expand \( \frac{125}{11} \) as a continued fraction to obtain the sequence \( \psi(f') = (1, 1, 3, 5, 1, 2) \), the result of kneading \((2, 2, 3, 6)\).

Theorem (1) provides an efficient method for producing all sequences with given alternant and length parity from a known list of Zagier-reduced forms of a certain discriminant. Alternatively, from a known list of sequences with given alternant, we can compute the entire list of corresponding Zagier-reduced forms. For instance, it can be shown that the set of all sequences with given alternant and length parity from a known list of Zagier-reduced forms is fixed by kneading, and a single other kneading cycle:

\[
(1, 11) \mapsto (1, 9, 1, 1) \mapsto (1, 8, 2, 1) \mapsto (1, 7, 3, 1) \mapsto (1, 6, 4, 1) \mapsto (1, 5, 5, 1) \\
\mapsto (1, 4, 6, 1) \mapsto (1, 3, 7, 1) \mapsto (1, 2, 8, 1) \mapsto (1, 1, 9, 1) \mapsto (11, 1) \mapsto (1, 11)
\]

From these and (1), we obtain the entire list of Zagier reduced forms of discriminant 125. Representing the form \( Ax^2 + Bxy + Cy^2 \) by \((A, B, C)\), they are the imprimitive form \((5, 15, 5)\) and those in the reduction cycle

\[
(11, 13, 1) \mapsto (19, 31, 11) \mapsto (25, 45, 19) \mapsto (29, 55, 25) \mapsto (31, 61, 29) \mapsto (31, 63, 31) \\
\mapsto (29, 61, 31) \mapsto (25, 55, 29) \mapsto (19, 45, 25) \mapsto (11, 31, 19) \mapsto (1, 13, 11) \mapsto (11, 13, 1)
\]

Lemmermeyer (1) notes that often the middle terms of the triples in a cycle of Zagier-reduced forms steadily increase until they reach a maximum and then steadily decrease until they return to the minimum. In the above example, this phenomenon is illuminated by the clear pattern in the corresponding kneading cycle.

We note that kneading can be used to perform reduction on Zagier-reduced forms with arbitrary non-square discriminant \( D > 0 \). To accomplish this, begin by solving the Pell equation \( x^2 - Dy^2 = 4 \) for integers \( x \) and \( y \) with \( y \neq 0 \). If \( f \) is a Zagier-reduced form of discriminant \( D \), then consider the form \( yf \) obtained by multiplying all coefficients of \( f \) by \( y \). A glance at the Zagier reduction algorithm indicates that \( yf \) and \( f \) both have the same reducing number, hence if \( f' \) is the form obtained by reducing \( f \), then \( yf' \) is the form obtained by reducing \( yf \). Theorem (1) then implies that multiplication by \( y \) gives a bijection from the reduction cycle of \( f \) to that of \( yf \). On the other hand, the discriminant of \( yf \) is \( y^2D = x^2 - 4 \). Thus, we may reduce \( f \) by determining \( yf \) and kneading the corresponding sequence.

3. Proofs

To begin, we develop some properties of continued fractions and continuants.

Beginning with \([\cdot] = 1\) and \([q_1] = q_1\), continuants satisfy the recurrences

\[
[q_1, \ldots, q_i] = q_1 [q_2, \ldots, q_i] + [q_3, \ldots, q_i] \quad \text{or} \quad [q_1, \ldots, q_i] = q_i [q_1, \ldots, q_{i-1}] + [q_1, \ldots, q_{i-2}]
\]

(2)

We adopt, for now, the first as our definition and later show that it gives the numerator of an appropriate continued fraction. The equivalence with the
second recurrence and all other properties we will need follow elegantly from
the matrix identity:

\[
\begin{bmatrix}
q_1 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
q_2 & 1 \\
1 & 0
\end{bmatrix}
\cdots
\begin{bmatrix}
q_l & 1 \\
1 & 0
\end{bmatrix}
= \begin{bmatrix}
[q_1, \ldots, q_l] & [q_1, \ldots, q_{l-1}] \\
[q_{l+1}, \ldots, q_l] & [q_{l+1}, \ldots, q_{l-1}]
\end{bmatrix},
\]

which can be verified by induction using the first recursion (2).

Transposing both sides of (3) reveals the surprising symmetry \([q_1, \ldots, q_l] = [q_l, \ldots, q_1]\), from which follows the second recursion (2). Taking determinants in (3) yields another useful identity

\[
[q_1, \ldots, q_l] [q_2, \ldots, q_{l-1}] - [q_1, \ldots, q_{l-1}] [q_2, \ldots, q_l] = (-1)^l
\]

We also note the simplifications

\[
[q_1, q_2, \ldots, 0, q_{i+1}, \ldots, q_l] = [q_1, q_2, \ldots, q_{i-1}, q_i + q_{i+1}, q_{i+2}, q_l]
\]

\[
[q_1, q_2, \ldots, q_l] = [q_2, \ldots, q_1]
\]

\[
[q_1, q_2, \ldots, q_l] = [q_1 + 1, \ldots, q_l]
\]

The first follows from (3) and the computation

\[
\begin{bmatrix}
q_i & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
q_{i+1} & 1 \\
1 & 0
\end{bmatrix}
= \begin{bmatrix}
q_i + q_{i+1} & 1 \\
1 & 0
\end{bmatrix};
\]

and the others follow readily from the recursion (2).

Now let us return to continued fractions. We can prove inductively that

\[
q_1 + \frac{1}{q_2 + \frac{1}{1 + \cdots + \frac{1}{q_l}}} = \frac{[q_1, \ldots, q_l]}{[q_2, \ldots, q_l]}
\]

We see from (4) that this fraction is in lowest terms, so the continuant
\([q_1, \ldots, q_l]\) is the numerator when the continued fraction with partial quo-
tients \(q_1, \ldots, q_l\) is fully simplified.

Now we prove that for given integers \(a > 0\) and \(s = 0\) or \(1\) with \((a, s) \neq (1,1)\) or \((2,1)\), in order:

(i) If \((q_1, \ldots, q_l)\) has alternant \(a\), then \(\phi_{(a,s)}((q_1, \ldots, q_l))\) is Zagier-reduced
with discriminant \(a^2 + (-1)^l \cdot 4\)

(ii) \(\psi_{(a,s)} \circ \phi_{(a,s)}\) is the identity map on \(S_{(a,s)}\),

(iii) If \(f\) is a form of discriminant \(a^2 + (-1)^s \cdot 4\), then \(\psi_{(a,s)}(f)\) has alternant
\(a\) and length parity \(s\),

(iv) \(\phi_{(a,s)} \circ \psi_{(a,s)}\) is the identity map on \(Z_{(a,s)}\),

(v) Kneading corresponds to Zagier reduction of forms

(i): This is easily verified when \(l = 1\) or \(2\), so let \(l \geq 3\). Let \((q_1, \ldots, q_l)\) be
a sequence of positive integers with alternant \(a\), and let \(\phi_{(a,s)}((q_1, \ldots, q_l)) = Ax^2 + Bxy + Cy^2\) be the form (1). To see that it is reduced, note that
coefficients $A$ and $C$ are clearly positive, so we need only check that $B > A + C$. Using (2), we compute

$$B - C = (q_l - 1) [q_1, \ldots, q_{l-1}] + [q_1, \ldots, q_{l-2}] + [q_2, \ldots, q_{l-1}]$$

$$> (q_l - 1) [q_2, \ldots, q_{l-1}] + [q_2, \ldots, q_{l-2}] + [q_2, \ldots, q_{l-1}]$$

$$= q_l [q_2, \ldots, q_{l-1}] + [q_2, \ldots, q_{l-2}] = [q_2, \ldots, q_l] = A$$

For the discriminant, we compute, using (1) and (2),

$$([q_1, \ldots, q_l] + [q_2, \ldots, z_{l-1}])^2 - 4 [q_2, \ldots, z] [q_1, \ldots, q_{l-1}]$$

$$= [q_1, \ldots, q_l]^2 + [q_2, \ldots, q_{l-1}]^2 - 2 [q_2, \ldots, q_l] [q_1, \ldots, q_{l-1}] + (-1)^l \cdot 2$$

$$= (q_1 [q_2, \ldots, q_l])^2 + 2q_1 [q_2, \ldots, q_l] [q_3, \ldots, q_l] + [q_3, \ldots, q_l]^2 + [q_2, \ldots, q_{l-1}]^2$$

$$- 2q_1 [q_2, \ldots, q_l] [q_2, \ldots, q_{l-1}] - 2 [q_2, \ldots, q_l] [q_3, \ldots, q_{l-1}] + (-1)^l \cdot 2$$

$$= (q_1 [q_2, \ldots, q_l] - [q_2, \ldots, q_{l-1}] + [q_3, \ldots, q_l])^2 + (-1)^l \cdot 2$$

$$- (2 [q_2, \ldots, q_l] [q_3, \ldots, q_{l-1}] - 2 [q_2, \ldots, q_{l-1}] [q_3, \ldots, q_l])$$

$$= a^2 + (-1)^l \cdot 4$$

(ii): The definition of alternants and (1) show that the sequence

$$\psi(a, s)(\phi(a, s)((q_1, \ldots, q_l)))$$

is obtained by expanding in a continued fraction the rational number with denominator $[q_2, \ldots, q_l]$ and numerator $[q_1, \ldots, q_l]$. From (3) and the well-known uniqueness of continued fraction expansions, this sequence is $(q_1, \ldots, q_l)$ (which has the right length parity).

(iii): If $a = 1$ or $2$, so by our assumption $s = 0$, then it is not hard to show that the only Zagier-reduced forms of discriminant $a^2 + (-1)^s \cdot 4$ are $x^2 + 3xy + y^2$, $x^2 + 4xy + 2y^2$, and $2x^2 + 4xy + y^2$. Applying $\psi(1,0)$ to the first and $\psi(2,0)$ to the other two gives the sequences (1,1), (2,1), and (1,2) of alternants 1, 2, and 2 respectively. Thus (iii) holds in these cases.

Now choose a Zagier-reduced form $f = Ax^2 + Bxy + Cy^2$ of discriminant $D = a^2 + (-1)^s \cdot 4$ with $a > 2$, $s = 0$ or 1, and $D > 0$. By design, the length parity of $\psi(a,s)(f)$ is $s$, so we need only worry about the alternant.

First, $B$ and $D$ have the same parity, hence $a$ and $B$ do. The positive integer $z = (a + B)/2$ is thus a divisor of

$$\frac{B^2 - a^2}{4} = AC + \frac{D}{4} = \frac{a^2}{4} = AC + (-1)^s$$

Thus, $A$ is relatively prime to $z$ and $AC \equiv (-1)^{s+1} \pmod{z}$.

Note as well that $a^2 + (-1)^s \cdot 4 = B^2 - 4AC > (A-C)^2$ since $f$ is reduced. Then $a > |A - C|$ since $a > 2$, so $a + A > C$. Hence, using again that $f$ is reduced, we have $z > (a + A + C)/2 > C$. Since also $a + C > A$, we also have $z > A$.

Expand $z/A$ as a simple continued fraction with sequence of quotients $(q_1, \ldots, q_l)$, and choose the length so that $l$ and $s$ have the same parity. From (3), $z = [q_1, \ldots, q_l]$ and $A = [q_2, \ldots, q_l]$. Since $AC \equiv (-1)^{s+1} \pmod{z}$,
we also have from (4) the congruence
\[ C \equiv \left[ q_1, \ldots, q_{l-1} \right] \pmod{z}. \]
Since \( 0 < C < z \), it follows that \( C = \left[ q_1, \ldots, q_{l-1} \right] \).

From (4), we have
\[
[q_2, \ldots, q_{l-1}] = 2AC + (-1)^l \frac{a + B}{a + B} = \frac{B^2 - (a^2 + (-1)^l \cdot 4 + (-1)^l \cdot 4}{2(a + B)} = \frac{B - a}{2}
\]

Thus, the alternant of \( \psi(a,s)(f) \) is
\[
[q_1, \ldots, q_l] - [q_2, \ldots, q_{l-1}] = \frac{B + a}{2} - \frac{B - a}{2} = a
\]

(iv): Let \( f = Ax^2 + Bxy + Cy^2 \) be as in (iii). The verification of (iii) shows at least that the form \( \phi(a,s) \circ \psi(a,s)(f) \) is \( Ax^2 + B'xy + Cy^2 \) for some positive integer \( B' \). Also, (i) and (iii) show that \( B' \) satisfies \( B'^2 - 4AC = a^2 + (-1)^s \cdot 4 \).

But \( B \) is the unique such positive integer, thus \( \phi(a,s) \circ \psi(a,s)(f) = f \).

(v): Suppose that \( (q_1, \ldots, q_l) \) is a sequence with alternant \( a \) and length parity \( s \). The reducing number for Zagier reduction of \( \phi(a,s)((q_1, \ldots, q_l)) \) is
\[
\left\lceil \frac{[q_1, \ldots, q_l] + [q_2, \ldots, q_{l-1}] + \sqrt{D}}{2[q_2, \ldots, q_l]} \right\rceil
\]
where \( D = a^2 + (-1)^s \cdot 4 \) is the discriminant. When \( l = 1 \), so \( q_1 > 2 \), the number inside the ceiling is \( \frac{q_1 + \sqrt{q_1^2 - 4}}{2} \), making the value of the ceiling \( q_1 \).

When \( l = 2 \), the reducing number is
\[
\left\lceil \frac{q_1q_2 + 2 + \sqrt{(q_1q_2)^2 + 4}}{2q_2} \right\rceil
\]
A little algebra shows that this ceiling is \( q_1 + 1 \) when \( q_2 \geq 2 \) and \( q_1 + 2 \) when \( q_2 = 1 \). A direct check shows that in these cases, reducing the form using the appropriate matrix corresponds to kneading the corresponding sequence.

Otherwise, for \( l \geq 3 \) the term \( \sqrt{D} \) in the numerator of (9) is approximately \( a \), so the whole numerator is approximately
\[
2 [q_1, \ldots, q_l] = 2q_1 [q_2, \ldots, q_l] + 22 [q_3, \ldots, q_l],
\]
making, at least approximately, the expression in the ceiling in (9) between \( q_1 \) and \( q_1 + 1 \). Some algebra shows in fact that the exact quotient is between \( q_1 \) and \( q_1 + 1 \), so the reducing number is always \( q_1 + 1 \) in this case. Acting on \( \phi(a,s)((q_1, \ldots, q_l)) \) by the reduction matrix \( \begin{pmatrix} q_1 + 1 & 1 \\ -1 & 0 \end{pmatrix} \), the theorem follows by checking the formulas (5) shows these are appropriate even when \( q_2 \) or \( q_l \)
is 1):

\[ [q_2 - 1, q_3, \ldots, q_{l-1}, q_l, 1] = (q_1 + 1)^2, q_2, \ldots, q_l] - (q_1 + 1) ([q_1, \ldots, q_l] + [q_2, \ldots, q_{l-1}]) + [q_1, \ldots, q_{l-1}] \]

(10)

\[ [1, q_2 - 1, q_3, \ldots, q_{l-1}, q_l, 1] + [q_2 - 1, q_3, \ldots, q_{l-1}, q_l - 1, 1] = (2q_1 + 2) [q_2, \ldots, q_l] - ([q_1, \ldots, q_l] + [q_2, \ldots, q_{l-1}]) \]

(11)

\[ [1, q_2 - 1, q_3, \ldots, q_{l-1}, q_l, 1] = [q_2, \ldots, q_l] \]

(12)

First separating off a \( q_1 \) from the second and fourth continuants on the right side of (10), then repeatedly applying (2) simplifies it to

\[ (q_1 + 1) ([q_2, \ldots, q_l] - [q_3, \ldots, q_l]) - [q_2, \ldots, q_{l-1}] + [q_3, \ldots, q_{l-1}] \]

\[ = (q_2 - 1)(q_1 + 1) [q_3, \ldots, q_l] + (q_1 + 1) [q_4, \ldots, q_l] \]

\[ - (q_2 - 1) [q_3, \ldots, q_{l-1}] - [q_4, \ldots, q_{l-1}] \]

\[ = (q_1 + 1) [q_2 - 1, q_3, \ldots, q_l] - [q_2 - 1, q_3, \ldots, q_{l-1}] \]

\[ = (q_1 + 1) [q_2 - 1, q_3, \ldots, q_{l-1}, q_l - 1] + q_1 [q_2 - 1, q_3, \ldots, q_{l-1}, q_l - 1] \]

\[ = q_1 [q_2 - 1, q_3, \ldots, q_{l-1}, q_l - 1, 1] + [q_2 - 1, q_3, \ldots, q_{l-1}, q_l - 1] \]

\[ = [q_2 - 1, q_3, \ldots, q_{l-1}, q_l - 1, 1, q_1] \]

With this, (10) is verified. The verification of (11) is similar, but shorter, after first simplifying the left side to

\[ q_1 [q_2, \ldots, q_l] + [q_2, \ldots, q_{l-1}, q_l - 1] + [q_2 - 1, q_3, \ldots, q_l] \]

Equation (12) follows immediately from (7).

### 4. The sum invariant

We have identified three invariants of sequences under kneading: the length parity, the alternant, and the entry sum. On the other hand, there are two classical invariants of indefinite forms under reduction: the discriminant and the greatest common divisor of the coefficients. Theorem 1 shows that the length parity and alternant of a sequence determine the discriminant of the corresponding form. In this section, we make some observations concerning the entry sums of kneading sequences and greatest common divisors of the coefficients of the corresponding forms.

The number of sequences with sum \( n \) is the number of compositions (i.e. ordered partitions) of \( n \). It is well known that the number of such compositions with \( k \) terms is the binomial coefficient \( \binom{n-1}{k-1} \). It follows readily that there are \( 2^{n-2} \) sequences with sum \( n \) and even length and an equal number of sequences with odd length and sum \( n \).
Some data quickly reveals striking regularity in the kneading cycles of sequences with sum \( n \). The following observations hold for all such cycles of sequences with \( \text{even} \) length parity and \( 1 \leq n \leq 30 \):

(i) The vast majority of cycles have length \( n - 1 \)
(ii) All cycles have length a divisor of \( n - 1 \)
(iii) Sequences in cycles with length less than \( n - 1 \) correspond to imprimitive quadratic forms (i.e. forms that do not have relatively prime coefficients)

Let us call a kneading cycle with length less than \( n - 1 \), where \( n \) is the entry sum of the sequences in the cycle, a short cycle. The following formula for the number of short cycles with even length parity and entry sum \( n \) was produced with the On-Line Encyclopedia of Integer Sequences \[2\]:

\[
\frac{1}{2n} \sum_{d \mid n, d \text{ odd}} (\phi(d) - \mu(d)) 2^{n/d}
\]

in which \( \phi \) and \( \mu \) are the functions of Euler and Möbius. This formula holds for \( n \leq 30 \).

The kneading cycles of sequences with \( \text{odd} \) length parity exhibit a variety of lengths. However, it appears again that the vast majority of them fit one pattern. Namely, most such cycles pair with another cycle with the same alternant, and the sums of the lengths of the cycles in such pairs is \( n - 1 \) (where again \( n \) denotes the entry sum). There are sometimes a few cycles that do not pair with another. There can also be a few that pair with a partner for which the sums of the cycle lengths is less than \( n - 1 \). In this case, the sum seems always to divide \( n - 1 \), and those that do not pair off have cycle length a proper divisor of \( n - 1 \). We can consider both of these types as “short cycles”.

As an example, the cycles with odd parity and sum 6 are

\[
(1, 4, 1) \mapsto (1, 4, 1) \\
(6) \mapsto (6) \\
(2, 3, 1) \mapsto (1, 3, 2) \mapsto (1, 2, 1, 1) \mapsto (1, 1, 1, 2, 1) \mapsto (2, 3, 1) \\
(4, 1, 1) \mapsto (1, 1, 4) \mapsto (4, 1, 1) \\
(3, 2, 1) \mapsto (1, 2, 3) \mapsto (1, 1, 2, 1, 1) \mapsto (3, 2, 1) \\
(3, 1, 2) \mapsto (2, 1, 3) \mapsto (3, 1, 2) \\
(2, 2, 2) \mapsto (1, 1, 1, 1, 2) \mapsto (2, 1, 1, 1, 1) \mapsto (2, 2, 2)
\]

of lengths 1, 1, 4, 2, 3, 2, and 3 and alternants 2, 6, 6, 8, 8, 10, and 10 respectively. We see that all of these except the first pair off with cycles of the same alternant, and that the sum of the lengths of each pair is 5.

The question naturally arises: to what extent is a cycle determined by its length parity, alternant, entry sum, and the greatest common divisor of the coefficients of the corresponding form? Some cycles are not uniquely determined by these values. This happens with associate pairs of inequivalent
sequences like \((4, 3)\) and \((3, 4)\). Similarly, the pairs of sequences with odd length parity discussed above share all invariant values. Outside of these cases, it is hard to find counterexamples with small entry sum. But, for instance, the sequences \((5, 1, 1, 2)\) and \((4, 3, 1, 1)\) are not in associated cycles, each has even length, alternant 26, and entry sum 9, and the corresponding forms are primitive. Yet the sequences are not equivalent.

There are some readily observable patterns in the list of sequences with short cycles:

1. Every sequence of the form \((2, 1, \ldots, 1, 2)\) is fixed by kneading. The corresponding form is readily checked to be \(F_{n+3}x^2 + 3F_{n+3}xy + F_{n+3}y^2\), in which \(F_{n+3}\) is a Fibonacci number.

2. Every sequence of the form \((a, 1, a-1, 1, \ldots, 1, a-1, 1, a)\) with \(a \geq 2\) is fixed by kneading. To describe the corresponding forms, we define recursively a sequence of polynomials \(p_n\) for \(n \geq 1\) by

\[
p_1 = 1, \quad p_2 = a + 1, \quad \text{and} \quad p_k = (a + 1)p_{k-1} - p_{k-2}
\]

Then the form corresponding to \((a, 1, a-1, 1, \ldots, 1, a-1, 1, a)\) with \(k\) repeated \(a-1\)'s is

\[
p_{k+2} \cdot \left(x^2 + (a + 1)xy + y^2\right),
\]

as can be proved by induction.

3. Every sequence of the form \((a + 1, a, a, \ldots, a, a - 1, 1)\) with \(a \geq 2\) is in a short cycle of length \(a\). The cycle exhibits the following pattern:

\[
(a + 1, a, \ldots, a, a - 1, 1) \mapsto (1, a - 1, a, \ldots, a, a + 1) \mapsto (1, a - 2, a, \ldots, a, a, 1, 1) \mapsto \cdots
\]

\[
\mapsto (1, a - r, a, \ldots, a, r - 1, 1) \mapsto \cdots \mapsto (1, 1, \ldots, a, a - 2, 1) \mapsto (a + 1, a, \ldots, a, a - 1, 1)
\]

(Each sequence has the same number of repeated \(a\)'s.) We define the Fibonacci polynomials \(q_n\) by the recurrence

\[
q_1 = 1, \quad q_2 = a, \quad \text{and} \quad q_k = aq_{k-1} + q_{k-2}
\]

The form corresponding to the sequence \((a + 1, a, a, \ldots, a, a - 1, 1)\) with \(k\) repeated \(a\)'s in the middle is

\[
q_{k+2} \cdot \left(x^2 + (a + 2)xy + ay^2\right)
\]

Again, this follows by induction.

There are other more complex patterns than those given above. Lacking a general rule to produce all such patterns, we are content to stop with these three. Please note that each case validates the observations (ii) and (iii) made above.

Finally, we provide a table specifying the different short cycles whose sequences have even length parity and entry sum between 1 and 29. Each row corresponds to a cycle, listed in increasing order of entry sum. The second column gives the cycle length. The third and fourth columns together specify a form \(f\) in the corresponding cycle of forms. The third column contains \(d\), the gcd of the coefficients of \(f\), while the fourth column contains the triple \((A, B, C)\) where \(Ax^2 + Bxy + Cy^2\) is the primitive form obtained by dividing
each coefficient of $f$ by $d$. We have chosen the representative form in each case to be that with the smallest middle coefficient $B$.

5. ACKNOWLEDGEMENT

The author would like to thank Benjamin Dickman, Franz Lemmermeyer, and his father, Weldon Smith, for helpful comments during the preparation of this manuscript.
| Sum − 1 | Cycle Length | d | Form     | Sum − 1 | Cycle Length | d | Form     |
|---------|--------------|---|----------|---------|--------------|---|----------|
| 3       | 1            | 2 | (1,3,1)  | 24      | 8            | 901 | (7,34,9) |
| 5       | 1            | 5 | (1,3,1)  | 24      | 8            | 901 | (9,34,7) |
| 6       | 2            | 5 | (1,4,2)  | 24      | 8            | 677 | (7,34,17) |
| 7       | 1            | 13| (1,3,1)  | 24      | 8            | 1025| (8,38,13) |
| 9       | 1            | 34| (1,3,1)  | 24      | 8            | 1025| (13,38,8) |
| 9       | 3            | 10| (1,5,3)  | 24      | 8            | 1157| (10,40,11) |
| 10      | 2            | 29| (1,4,2)  | 24      | 8            | 1297| (12,46,17) |
| 11      | 1            | 89| (1,3,1)  | 24      | 8            | 1765| (13,52,18) |
| 12      | 4            | 17| (1,6,4)  | 25      | 1            | 75025| (1,3,1) |
| 12      | 4            | 37| (2,8,3)  | 25      | 5            | 8578| (1,5,2) |
| 13      | 1            | 233| (1,3,1) | 25      | 5            | 701 | (1,7,5) |
| 14      | 2            | 169| (1,4,2) | 25      | 5            | 6805| (3,11,3) |
| 15      | 1            | 610| (1,3,1) | 26      | 2            | 33461| (1,4,2) |
| 15      | 3            | 109| (1,5,3) | 27      | 1            | 196418| (1,3,1) |
| 15      | 5            | 130| (1,5,2) | 27      | 3            | 12970| (1,5,3) |
| 15      | 5            | 26 | (1,7,5) | 27      | 9            | 514 | (1,9,4) |
| 15      | 5            | 82 | (3,11,3)| 27      | 9            | 82  | (1,11,9) |
| 17      | 1            | 1597| (1,3,1)| 27      | 9            | 1154| (2,13,3) |
| 18      | 2            | 985| (1,4,2) | 27      | 9            | 1154| (3,13,2) |
| 18      | 6            | 17 | (1,8,6) | 27      | 9            | 442 | (3,23,7) |
| 18      | 6            | 101| (2,12,5)| 27      | 9            | 442 | (7,23,3) |
| 18      | 6            | 145| (3,14,4)| 27      | 9            | 3202| (4,23,8) |
| 18      | 6            | 145| (4,14,3)| 27      | 9            | 3202| (8,23,4) |
| 18      | 6            | 257| (5,20,7)| 27      | 9            | 3874| (5,25,7) |
| 19      | 1            | 4181| (1,3,1)| 27      | 9            | 626 | (5,27,5) |
| 20      | 4            | 305| (1,6,4) | 27      | 9            | 4610| (6,29,11)|
| 20      | 4            | 1405| (2,8,3)| 27      | 9            | 4610| (11,29,6)|
| 21      | 1            | 10946| (1,3,1)| 27      | 9            | 962 | (5,35,13)|
| 21      | 3            | 1189| (1,5,3) | 27      | 9            | 842 | (5,35,19)|
| 21      | 7            | 290| (1,7,3) | 27      | 9            | 7202| (9,35,9)|
| 21      | 7            | 514| (2,9,2) | 27      | 9            | 1370| (7,41,11)|
| 21      | 7            | 50 | (1,9,7) | 27      | 9            | 1370| (11,41,7)|
| 21      | 7            | 1154| (4,15,5)| 27      | 9            | 1090| (7,41,21)|
| 21      | 7            | 226| (3,17,5) | 27    | 9            | 1090| (21,41,7)|
| 21      | 7            | 226| (5,17,3) | 27    | 9            | 1522| (9,43,9)|
| 21      | 7            | 442| (5,25,9) | 27    | 9            | 2026| (11,51,13)|
| 21      | 7            | 362| (5,25,13)| 27    | 9            | 2026| (13,51,11)|
| 21      | 7            | 530| (7,27,7) | 27    | 9            | 1850| (11,51,17)|
| 22      | 2            | 5741| (1,4,2) | 27    | 9            | 2810| (17,63,17)|
| 23      | 1            | 28657| (1,3,1) | 27    | 9            | 2402| (17,63,23)|
| 24      | 8            | 65 | (1,10,8) | 27    | 9            | 3026| (13,65,23)|
| 24      | 8            | 197| (2,16,7) | 27    | 9            | 3722| (19,75,25)|
| 24      | 8            | 325| (3,20,6) | 27    | 9            | 3970| (21,79,27)|
| 24      | 8            | 325| (6,20,3) | 28    | 4            | 5473| (1,6,4)|
| 24      | 8            | 401| (4,22,5) | 28    | 4            | 53353| (2,8,3)|
| 24      | 8            | 401| (5,22,4) | 29    | 1            | 514229| (1,3,1)|
| 24      | 8            | 577| (5,30,16) |    |              |      |            |
| 24      | 8            | 677| (5,30,11) |    |              |      |            |
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