Timing Channel: Achievable Rate in the Finite Block-Length Regime

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I. INTRODUCTION

II. BASIC DEFINITIONS AND CONVENTIONS

- Symbols denoting random variables are capitalized but symbols denoting their realization are not.
- Bold face is used for vectors.
- Blackboard bold or non-italic face are used for sets, calligraphic face for system of sets.

We use \( \bar{\cdot} \) as a shorthand for the difference \( 1 - \cdot \) for \( x \in [0, 1] \).

\( Z \) is the set of all integers and \( Z_+ = \{ z \in Z : z \geq 0 \} \).

\( 2 \) is the set \( \{0, 1\} \).

\([n]\) denotes the set \( \{z \in Z : 1 \leq z \leq n\} \).

Given the input and output sets \( X^n \) and \( Y^n \) a channel is a function \( P_{Y^n|X^n}(\cdot) : \mathcal{A} \times X^n \rightarrow [0, 1] \) such that for each \( x^n \in X^n \) \( P_{Y^n|X^n}(|x^n|) \) is a probability measure on \( \mathcal{A} \). \( \mathcal{A} \) is some \( \sigma \)-algebra over \( Y^n \).

We define the finite sets \( X^n \) and \( Y^n \) to be \( Z_+ \times 2^{n-1} \) and \( 2^n \), respectively, and associate the \( \sigma \)-algebras \( \mathcal{P}(X^n) \) and \( \mathcal{P}(Y^n) \) with them. \( \mathcal{P} \) indicates the power set.

Given a distribution \( P_{X^n} \) on \( X^n \) we define the distribution \( P_{Y^n} = \int_{X^n} P_{Y^n|X^n}(y^n|x^n) P_{X^n}(dx^n) \) and the function \( i(x^n, y^n) = \log \frac{P_{Y^n|X^n}(y^n|x^n)}{P_{Y^n}(y^n)} \) for \( x^n \in X^n \) and \( y^n \in Y^n \). This function is called the information density.

The expected value of a random variable \( X \) is denoted by \( \mathbb{E}[X] \).

The mutual information between two random vectors \( X^n \) and \( Y^n \) is \( \mathbb{E}[i(x^n, y^n)] \).

The capacity \( C(\lambda) \) of the channel is \( \sup_{P_{X^n}} \mathbb{E}[i(x^n, y^n)] \) where \( P_{X^n} \) is such that the rate \( \mathbb{E}[\sum_{i=1}^{n-1} X_i]/(n-1) \) does not exceed \( \lambda \).

The binary entropy function \( H(p) = -p \log p - \bar{p} \log \bar{p} \).

We drop subscripts whenever they are clear from the context. For example \( P_{Y^n|X^n}(y^n|x^n) = P(y^n|x^n) \).

III. SYSTEM DESCRIPTION AND PRELIMINARIES

The communication channel we consider is an interesting example of a channel with memory. It is essentially a probabilistic single server queueing system with the length of the queue being the memory of channel. At each discrete time instance \( i \) the random variable \( X_i, i \in [n-1] \) indicates if there was an arrival at the back of the queue at time \( i \).

The initial length of the queue \( Q_0 \) is a non-negative integer-valued random variable with distribution \( P_{Q_0} \). The variable \( Q_0 \) together with the vector \( \bar{X}_i, i \in [n-1] \) is considered the channel input vector \( X^n \in X^n \). We define the random variables \( X_i, \bar{Y}_i, Y_i \) and \( Q_i \) for \( i \in [n-1] \cup \{0\} \) such that

\[
\begin{align*}
X_i &= \sum_{\mathclap{l=1}}^{i} \bar{X}_l \\
\bar{Y}_i &= \sum_{\mathclap{l=0}}^{i} \bar{Y}_l \\
Q_i &= Q_0 + X_i - Y_{i-1} = Q_{i-1} + \bar{X}_i - \bar{Y}_{i-1}
\end{align*}
\]

where the binary random variables \( \bar{Y}_i \) are conditionally independent given \( Q_i \) and are distributed according to the transition probability function \( P_{Y_i|Q_i} = \begin{cases} 1; & \bar{Y}_i = 0, Q_i = 0 \\
\mu; & \bar{Y}_i = 0, Q_i > 0 \\
\bar{\mu}; & \bar{Y}_i = 1, Q_i > 0 \end{cases} \).

The vector \( \bar{Y}_i, i \in [n-1] \cup \{0\} \) is considered the channel output vector \( Y^n \in Y^n \).

Clearly, \( X_i, Y_i \) counts the total number of arrivals (departures), \( Q_i \) denotes the length of the queue at time \( i \) and \( \bar{Y}_i \) indicates if there was a departure from the front of the queue at time \( i \). The described relationships are illustrated in Figure 1.

In principle the distribution on \( X^n \) could be chosen arbitrary but we will assume \( \bar{X}_i, i \in [n-1] \) to be i.i.d. Bernoulli(\( \lambda \)) random variables for the following reason:

**Theorem 1.** The distribution on \( X^n \) that achieves capacity makes \( X_i, i \in [n-1] \) i.i.d. Bernoulli(\( \lambda \)).
Proof: The proof can be found in [1].

Most of the analysis in this paper is built on the theory of
Markov chains and so clearly we have to start with a
proper definition of this kind of stochastic process. Given
a probability space $(\Omega, \mathcal{F}, P)$ a discrete time stochastic
process $\Phi$ on a state space $\mathcal{X}$ is, for our purposes, a collection
of random variables $\Phi_i$, $i \in \mathbb{N}_0$ where each $\Phi_i$ takes values
in the set $\mathcal{X}$. The random variables are measurable with
respect to some given $\sigma$-field $\mathcal{A}$ over $\mathcal{X}$. We only consider
countable state spaces here and can hence take $\Omega$ to be the
product space $\prod_{i=0}^{\infty} \mathcal{X}$ and take $\mathcal{F}$ to be the product $\sigma$-algebra
$\bigcap_{i=0}^{\infty} \sigma(\mathcal{X})$ defined in the usual way. If $P(\Phi_{i+1} = \phi_{i+1}|\Phi_i = \phi_i) = P(\Phi_{i+1} = \phi_{i+1}|\Phi_i = \phi_i)$ holds, we
call the process Markov. Given an initial measure $P_{\phi_0}$ and transition
probabilities $P_{\phi_{i+1}|\phi_i}(\phi_{i+1}|\phi_i)$ there exists a
probability law $P$ satisfying $P(\Phi_{i+1} = \phi_{i+1}, \Phi_i = \phi_i, \ldots, \Phi_0 = \phi_0)$ $= P_{\phi_0}(\phi_0) \prod_{i=0}^{\infty} P_{\phi_{i+1}|\phi_i}(\phi_{i+1}|\phi_i)$ by the Kolmogorov
extension theorem [2, 3].

The length of the queue $Q_i$ forms an irreducible Markov
chain $Q = \{\Omega, \mathcal{F}, Q_i, P_Q\}$ with transition probabilities

$$P_{Q_{i+1}|Q_i}(q_{i+1}|q_i) = \begin{cases} 
\lambda; & q_{i+1} = q_i + 1, q_i = 0 \\
\lambda; & q_{i+1} = q_i, q_i = 0 \\
\bar{\lambda} \mu; & q_{i+1} = q_i - 1, q_i > 0 \\
\lambda \mu; & q_{i+1} = q_i + 1, q_i > 0 \\
1 - \lambda \mu - \bar{\lambda} \mu; & q_{i+1} = q_i, q_i > 0 
\end{cases}$$

(5)

If and only if $\lambda < \mu$, there exists a probability measure $\pi_Q$
on $\mathbb{N}_0$ that solves the system of equations

$$\sum_{q_i \in \mathbb{N}_0} \pi_Q(q_i) P_{Q_{i+1}|Q_i}(q_{i+1}|q_i) = \pi_Q(q_{i+1})$$

(6)

for all $q_{i+1} \in \mathbb{N}_0$ and this measure is called the invariant
measure. Note that for irreducible Markov chains the existence
of such a probability measure is equivalent to positive recurrence.
For the transition probabilities given it can be checked that

$$\pi_Q(q_i) = \begin{cases} 
\frac{\lambda \bar{\mu} - \lambda \mu}{\lambda \mu - \bar{\lambda} \mu} \rho^q; & q_i = 0 \\
\frac{\lambda \bar{\mu} - \bar{\lambda} \mu}{\lambda \mu - \bar{\lambda} \mu} \rho^q; & q_i > 0 
\end{cases}$$

(7)

where we defined

$$\rho = \frac{\lambda \mu}{\bar{\lambda} \mu}.$$

Note that $\rho < 1$ if and only if $\lambda < \mu$. In the remainder of
the paper we will always assume that the arrival rate $\lambda$ is smaller
than the serving rate $\mu$.

**Theorem 2.** The distribution on $X^n$ that achieves capacity
makes $Q_0$ distributed according to $\pi_Q$.

**Proof:** The proof can be found in [1].

And we will hence always assume that $Q_0$ is distributed
according to $\pi_Q$.

Definition 1. Let $0 < \epsilon < 1$. An $\epsilon$-Code of size $N$ is defined as
a sequence $\{(x^{(i)}, D^{(i)}), i = 1, \ldots, N\}$ such that $x^{(i)} \in X^n$,
$D^{(i)} \subset Y^n$, $D^{(i)}$ are mutually disjoint and $P(D^{(i)}|x^{(i)}) > 1 - \epsilon \forall i$. Let $N(\epsilon, n)$ be the supremum of the set of integers
$N$ such that an $\epsilon$-Code of size $N$ exists.

Note that $\frac{\log N(\epsilon, n)}{n}$ equals the rate of the code.

**Theorem 3.** The expression for the capacity $C$ of the channel
simplifies to

$$C = H(\lambda) - \frac{\lambda}{\mu} H(\mu)$$

(9)

and

$$\log N(\epsilon, n) = n C + \epsilon O(n)$$

(10)

**Proof:** A proof of a similar result for the continuous time
case appeared in the landmark paper “Bits through Queues” [4]
and the stated discrete time result was proved in [1].

**Theorem 4.** (Burke’s Theorem) Given the queue $Q$ is in
equilibrium and the random variables $X_i, i \in [n-1]$ are i.i.d.
Bernoulli($\lambda$) then the random variables $Y_i, i \in [n-1] \cup \{0\}$
are also i.i.d. Bernoulli($\lambda$).

**Proof:** The proof is similar to the one for continuous time
queues and can be found in [5]. A concise proof is also given in
the appendix.

By the above theorem $P(y^n) = \prod_{i=0}^{n-1} P(\tilde{y}_i)$ and

$$P(\tilde{y}_i) = \lambda \cdot \tilde{y}_i + \bar{\lambda} \cdot \tilde{y}_i.$$ (11)

The following lemma uses arguments introduced by Feinstein
[6] to give a lower bound on $N(\epsilon, n)$.

**Lemma 1.** (Feinstein) $N(\epsilon, n) \geq e^\theta (e - P(i(x^n, y^n) \leq \theta))$ for all $\theta \in \mathbb{R}$.

**Proof:** The proof is short and elegant and reproduced in
the appendix.

IV. finite-length scaling

The distributions $P(y^n|x^n)$ and $P(y^n)$ factor and hence

$$i(x, y) = \sum_{i=0}^{n-1} \log \frac{P(y_i|q_i)}{P(y_i)} = \sum_{i=0}^{n-1} f(\tilde{y}_i, q_i)$$

(12)

where we defined

$$f(\tilde{y}_i, q_i) = \log \frac{P(y_i|q_i)}{P(y_i)}.$$ (13)

The composed state $\psi_i = (\tilde{y}_i, q_i)$ again forms a positive
recurrent Markov chain $\Psi = \{\Omega, \mathcal{F}, \psi_i, P_{\Psi}\}$ whose transition
probabilities are illustrated in Figure 2. The invariant measure
$\pi_{\Psi}$ for this chain is only a slight extension to $\pi_Q$:

$$\pi_{\Psi}(\tilde{y}, q) = P_{Y_i|Q_i}(\tilde{y}|q) \pi_Q(q)$$

(14)

The proof of the following theorem is one of the main
contributions of this paper because it can be used to proof
bounds and an asymptotic on the quantity $N(\epsilon, n)$.

**Theorem 5.** The asymptotic variance

$$\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \text{Var} \left( \sum_{i=0}^{n-1} f(\tilde{y}_i, q_i) \right)$$

(15)
is well defined, positive and finite, and
\[ \sigma^2 = \text{Var}(f(\Psi_0)) + 2 \sum_{i=1}^{\infty} \text{Cov}(f(\Psi_0), f(\Psi_i)). \] (16)

Further the following Berry-Esseen type bound holds:
\[ \sup_{\xi \in \mathbb{R}} \left| P \left( \frac{n^{-1/2} \sum_{i=0}^{n-1} f(\bar{y}_i, q_i) - n \pi(f)}{\sigma} \leq \xi \right) - \Phi(\xi) \right| \leq O(n^{-1/2}) \] (17)

**Proof:** A detailed proof can be found in appendix. We only give a sketch here. The Markov Chain $\Psi$ is aperiodic and irreducible. The state space of $\Psi_i$ can be chosen to be $X = 2 \times \mathbb{N} \cup \{(0, 0)\}$. First we verify that there exists a Lyapunov function $V : X \to (0, \infty]$, finite at some $\psi_0 \in X$, a finite set $S \subset X$, and $b < \infty$ such that
\[ E[V(\Psi_{i+1}) - V(\Psi_i)|\Psi_i = \psi] \leq -1 + b1_\infty(\psi), \quad \psi \in X \] (18)

The chain is skip-free and the found Lyapunov function is linear and hence also Lipschitz. These properties imply that the chain is geometric ergodic \[2, 7, 8\] and the bound in Equation (17) hence holds by arguments made in \[9\]. \[\blacktriangleleft\]

An explicit solution to the asymptotic variance of a general irreducible positive recurrent Markov chain is not available. Significant research in the area of steady-state stochastic simulation has focused on obtaining an expression for this quantity \[10-12\] and has yielded a closed form solution for the class of homogeneous birth-death processes when $f(\psi_i)$ simply returns the integer valued state itself.

We build up on an idea introduced in \[13\] to give an explicit closed form solution to the asymptotic variance in Equation (18).

**Theorem 6.** The closed form expressions in Equation (24) equals the asymptotic variance defined in Equation (18).

**Proof:** Again we only sketch the proof here and refer to the appendix for a detailed version. For the computation of the sum $\sum_{i=0}^{\infty} \text{Cov}(f(\Psi_0), f(\Psi_i))$ we will setup and solve a recursion. Grassmann proposed this approach in \[13\] to obtain the asymptotic variance of a continuous time finite state birth death process.

We define
\[ r(\psi, i) = \sum_{\psi' \in X} (f(\psi') - C)\pi(\psi')p_{\psi',\psi_0}(\psi|\psi') \] (19)

Clearly
\[ r(\psi, 0) = (f(\psi) - C)\pi(\psi) \] (20)

and
\[ \text{Cov}(f(\Psi_0), f(\Psi_i)) = \sum_{\psi \in X} (f(\psi) - C)r(\psi, i) = \sum_{\psi \in X} f(\psi)r(\psi, i) \] (21)

Note however that for the computation of the asymptotic variance we actually do not even need to know this covariance for each $i$. It is sufficient to know its sum. So we define
\[ R(\psi) = \sum_{i=0}^{\infty} r(\psi, i) \] (22)

write
\[ \sum_{i=0}^{\infty} \text{Cov}(f(\Psi_0), f(\Psi_i)) = \sum_{\psi \in X} f(\psi)R(\psi) \] (23)

and derive an expression for $R(\psi)$.

Using the result stated in Theorem 5 we can finally prove the core contribution of this paper:

**Theorem 7.**
\[ \log N(n, \epsilon) \geq nC - \sqrt{n}\sigma Q^{-1}(\epsilon) - \frac{1}{2} \log n + O(1) \] (29)

where $C = \pi(\psi)$ and $\sigma$ is defined as in Theorem 5.

**Proof:** By Theorem 5, for $A > 0$:
\[ |P((i(x, y) - nC)/\sqrt{n}\sigma^2 \leq \xi_1) - \Phi(\xi_1)| \leq A/\sqrt{n} \forall \xi_1 \in \mathbb{R} \] (30)
\[
\sigma^2 = -\text{Var}(f(\Psi_0)) + 2 \sum_{i=0}^{\infty} \text{Cov}(f(\Psi_0), f(\Psi_i))
\]
(24)

\[
\text{Var}(f(\Psi_0)) = \log^2(\frac{1}{\lambda})(\pi Q(0) + \log^{2}(\frac{\mu}{X})\mu \pi Q(0) + \log^{2}(\frac{\bar{\mu}}{X})\bar{\mu} \pi Q(0) - C^2
\]
(25)

\[
c_M = \frac{\lambda}{\mu} (\mu \log(\frac{\mu}{X}) + \bar{\mu} \log(\frac{\bar{\mu}}{X}) - C)
\]
(26)

\[
c_{\bar{M}} = \left\{ \frac{c_M}{\mu} + (C - \log(\frac{\mu}{X})\pi Q(0)) \right\}
\]
(27)

\[
\sum_{i=0}^{\infty} \text{Cov}(f(\Psi_0), f(\Psi_i)) = \log(\frac{\rho}{X} - c_{M0}) + \log(\frac{\bar{\rho}}{X} - c_{\bar{M}0}) + \log(\frac{\rho}{X} - \rho (c_{M0} - \rho c_{\bar{M}0})
\]
(28)

Let \( B > A \) and \( \xi_1 = \Phi^{-1}(\epsilon - \frac{B}{\sqrt{n}}) < \Phi^{-1}(\epsilon) = \xi_0 \). Set \( \theta = \sqrt{n}\sigma\xi_1 + nC \) and the application of Lemma 1 yields

\[
\log N(n, \epsilon) - nC - \sqrt{n}\sigma\xi_0 \geq \log \left( \epsilon - P\left( \frac{i(x, y) - nC}{\sqrt{n}\sigma} \geq \xi_1 \right) \right) + \sqrt{n}\sigma(\xi_1 - \xi_0)
\]
(31)

\[
\geq \log \left( \epsilon - \Phi(\xi_1) - \frac{A}{\sqrt{n}} \right) + \sqrt{n}\sigma(\xi_1 - \xi_0) + \sqrt{n}\sigma O(\frac{1}{\sqrt{n}})
\]
(32)

This confirms that \( C \) is the operational capacity of the channel and any rate \( R < C \) is achievable. The real beauty of Theorem 7 is, however, that we can use the asymptotic

\[
\frac{\log N(n, \epsilon)}{n} \sim C - n^{-1/2}\sigma Q^{-1}(\epsilon)
\]
(33)

as an approximation to the channel coding rate \( \frac{\log N(n, \epsilon)}{n} \) and then anticipate the achievable rate on this channel in the finite block length regime. For illustration we plotted this asymptotic for blocklengths ranging between 50 and 3000 and the example values \( \lambda = 0.2 \), \( \mu = 0.8 \) and \( \epsilon = 10^{-5} \) in Figure 3.

![Figure 3. Channel Coding Rate in the Finite Block-Length Regime](image)

Or equivalently

\[
\epsilon < P(D \cup F^c(x)|x) \leq P(D|x) + P(F^c(x)|x)
\]
(38)

Multiplying this inequality with \( P(dy) \) and integrating it over \( x \) then yields

\[
\epsilon \leq P(D) + P(i(x, y) \leq \theta)
\]
(39)

Putting everything together we obtain the result

\[
\epsilon - P(i(x, y) \leq \theta) \leq P(D) \leq Ne^{-\theta}
\]
(40)

V. PROOF OF THEOREM

VI. PROOF OF LEMMA

Proof: Assume \( N \) is the maximal size of an \( \epsilon \)-Code such that \( D^{(i)} \subset F(x^{(i)}) \), where

\[
F(x) = \{ y : i(x, y) > \theta \}.
\]
(34)

Then we have

\[
P(D^{(i)}) = \int_{D^{(i)}} P(dy) < \int_{D^{(i)}} e^{-\theta} P(dy|x) \leq e^{-\theta}
\]
(35)

and

\[
P(\cup_i D^{(i)}) \leq \sum_i P(D^{(i)}) \leq Ne^{-\theta}
\]
(36)

Let \( D = \cup_i D^{(i)} \). By the maximality of \( N \) it follows that

\[
P(D^c \cap F(x)|x) < 1 - \epsilon.
\]
(37)

VII. PROOF OF THEOREM

The Markov Chain \( \Psi_t \) is aperiodic and irreducible. The state space of \( \Psi_t \) can be chosen to be \( X = 2 \times N \cup \{(0, 0)\} \). First we verify that Foster’s Criterion holds

Lemma 2. There exists a Lyapunov function \( V : X \rightarrow (0, \infty) \), finite at some \( \psi_0 \in X \), a finite set \( S \subset X \), and \( b < \infty \) such that

\[
E[V(\Psi_{t+1}) - V(\Psi_t)|\Psi_t = \psi] \leq -1 + b1_S(\psi), \ \psi \in X
\]
(41)
Further this function V is Lipschitz, i.e., for some $\alpha > 0$

$$|V(y) - V(x)| \leq \alpha ||y - x|| \quad \forall y, x \in \mathbb{X} \quad (42)$$

and for some $\beta > 0$

$$\sup_{x \in \mathbb{X}} E[e^{\beta ||\psi_{i+1} - \psi_i||}] |\psi_i = x| < \infty \quad (43)$$

**Proof:** We need to find a function $V$ such that $E[V(\psi_{i+1}) - V(\psi_i)|\psi = \psi] \leq -1$ for all but a finite number of $\psi \in \mathbb{X}$. If we simply choose $V(\hat{y}, q) = cq$ for some sufficiently large constant $c > 0$ then the requirement is clearly satisfied for all $\psi \in \mathbb{X}$ such that $\hat{y} = 1$ but it fails to hold otherwise. To fix this shortcoming we reward the transitions to a state with $\hat{y} = 1$ by a decreasing difference $V(\Psi_{i+1}) - V(\Psi_i)$. In particular we choose $V(\hat{y}, q) = (q - \hat{y})/(\mu - \lambda)$. Standard calculations reveal that for that choice $E[V(\Psi_{i+1}) - V(\Psi_i)|\psi = \psi] = -1$ for all $\psi \in \mathbb{X}$ with $q > 1$. Linear functions are always Lipschitz and $||\psi_{i+1} - \psi_i||$ is bounded almost surely.

By the results in [1] or Proposition A.5.7. in [8] the chain $\Psi$ is then geometrically ergodic.

By Theorem A.5.8. in [8] the asymptotic variance $\sigma^2$ is well defined, non-negative and finite, and

$$\sigma^2 = \text{Var}(f(\psi_0)) + 2 \sum_{i=1}^{\infty} \text{Cov}(f(\psi_0), f(\psi_i)) \quad (44)$$

Finally, $f(\hat{y}, q)$ is a bounded, nonlattice (I still have to check that!), real-valued functional on the state space $\mathbb{X}$ and hence

$$P_{\psi_{i}} \left( \sum_{i=0}^{n-1} f(\hat{y}_{i}, q_{i}) - n \pi_\psi(f) \leq \xi \right) = \Phi(\xi) \quad (45)$$

and

$$= \frac{\rho(\xi)}{\sqrt{n}}\left[ \frac{n}{\sigma^2} (1 - \xi^2) - f(\psi_0) \right] + o(n^{-1/2}) \quad (46)$$

where $\rho(\xi)$ denotes the density of the standard Normal distribution $\Phi$. $\hat{f}$ is the solution to Poissons equation and $\eta$ is a constant [3]. The solution $\hat{f}$ can be chosen such that $\pi_\psi(\hat{f}) = 0$ and the claim follows by averaging out $\psi_0$.

**VIII. PROOF OF THEOREM 6**

Using the representation of $\sigma^2$ in Equation (44) it remains to find explicit expressions for $\text{Var}(f(\psi_0))$ and the sum $\sum_{i=0}^{\infty} \text{Cov}(f(\psi_0), f(\psi_i))$.

The term $\text{Var}(f(\psi_0))$ is easy to compute

$$\text{Var}(f(\psi_0)) = \log^2\left(\frac{1}{\lambda}\right) \pi Q(0) + \log^2\left(\frac{\mu}{\lambda}\right) \mu \pi Q(0)$$

$$+ \log^2\left(\frac{\bar{\mu}}{\lambda}\right) \bar{\mu} \pi Q(0) - C^2 \quad (47)$$

Equation (28) holds true.

**Proof:** For the computation of the sum $\sum_{i=0}^{\infty} \text{Cov}(f(\psi_0), f(\psi_i))$ we will setup and solve a recursion. Grassmann proposed this approach in [13] to obtain the asymptotic variance of a continuous time finite state birth death process.

We define

$$r(\psi, i) = \sum_{\psi' \in \mathbb{X}} (f(\psi') - C) \pi_\psi(\psi') p_{\psi,\psi_0}(\psi|\psi') \quad (48)$$

Clearly

$$r(\psi, 0) = (f(\psi) - C) \pi_\psi(\psi) \quad (49)$$

and

$$\text{Cov}(f(\psi_0), f(\psi_i)) = \sum_{\psi \in \mathbb{X}} (f(\psi) - C) r(\psi, i) = \sum_{\psi \in \mathbb{X}} f(\psi) r(\psi, i) \quad (50)$$

Note however that for the computation of the asymptotic variance we actually do not even need to know this covariance for each $i$. It is sufficient to know its sum. So we define

$$R(\psi) = \sum_{i=0}^{\infty} r(\psi, i) \quad (51)$$

write

$$\sum_{i=0}^{\infty} \text{Cov}(f(\psi_0), f(\psi_i)) = \sum_{\psi \in \mathbb{X}} f(\psi) R(\psi) \quad (52)$$

and derive a recursion for $R(\psi)$.

For mean ergodic Markov processes $p_{\psi,\psi_0}(\psi|\psi') \rightarrow \pi_\psi(\psi)$ as $i \rightarrow \infty$ and hence

$$\lim_{i \rightarrow \infty} r(\psi, i) = 0 \quad (53)$$

Summing $r(\psi, i + 1) - r(\psi, i)$ in $i$ from zero to infinity then clearly yields $(C - f(\psi)) \pi_\psi(\psi)$. By the Chapman-Kolmogorov equations

$$p_{\psi_{i+1}|\psi_0}(\psi|\psi') - p_{\psi_{i}|\psi_0}(\psi|\psi')$$

$$= \sum_{\psi' \in \mathbb{X}} \pi_{\psi'} \{p_{\psi_{i+1}|\psi_0}(\psi|\psi'') - \delta_{\psi, \psi''}\} \quad (54)$$

and thus

$$r(\psi, i + 1) - r(\psi, i)$$

$$= \sum_{\psi' \in \mathbb{X}} (f(\psi') - C) \pi_\psi(\psi') \{p_{\psi_{i+1}|\psi_0}(\psi|\psi') - p_{\psi_{i}|\psi_0}(\psi|\psi')\}$$

$$= \sum_{\psi'' \in \mathbb{X}} \pi_{\psi''} \{p_{\psi_{i+1}|\psi_0}(\psi|\psi'') - \delta_{\psi, \psi''}\} r(\psi'', i) \quad (55)$$

If this expression for $r(\psi, i + 1) - r(\psi, i)$ is also summed in $i$ from zero to infinity and then compared to the above result of the same sum, we obtain

$$(C - f(\psi)) \pi_\psi(\psi) = \sum_{\psi'' \in \mathbb{X}} \{p_{\psi_{i+1}|\psi_0}(\psi|\psi'') - \delta_{\psi, \psi''}\} R(\psi'') \quad (56)$$

For notational convenience we abbreviate the right hand-side of Equation (56) by $D(\psi)$.

For $\psi$ with $q > 0$ and $\hat{y} = 0$

$$D(\psi) = (\tilde{\lambda}_\mu - 1) R(q, 0) + \tilde{\lambda}_\mu R(q + 1, 1) + \lambda \mu R(q, 1)$$

$$+ \lambda \mu R(q - 1, 0) \quad (57)$$
For $\psi$ with $q > 0$ and $\tilde{y} = 1$

$$D(\psi) = (\lambda \mu - 1) R(q, 1) + \tilde{\lambda} \mu R(q + 1, 1) + \tilde{\lambda} \mu R(q, 0) + \lambda \mu R(q - 1, 0)$$

(58)

And for $\psi$ with $q = 0$ and $\tilde{y} = 0$

$$D(\psi) = -\lambda R(0, 0) + \tilde{\lambda} R(1, 1)$$

(59)

Adding Equations 57 and 58 yields

$$\tilde{\lambda} R(q + 1, 1) - \lambda R(q, 0) - \tilde{\lambda} R(1, 1) + \lambda R(q - 1, 0)$$

(60)

We now sum Equation 58 in two ways:

$$\sum_{\psi \in X: q' \leq q} D(\psi) = \tilde{\lambda} R(q + 1, 1) - \lambda R(q, 0) = M(0, q)$$

(61)

for $q \geq 0$ where we defined

$$M(q, 0) = \sum_{\psi \in X: q' \leq q} (C - f(\psi)) \pi_{\psi}(\psi)$$

(62)

and

$$\sum_{\psi' \in X: q' < q \mid q' = q, \tilde{y} = 1} D(\psi) = -\lambda \tilde{\mu} R(q - 1, 0) - \lambda \mu R(q, 1) + \tilde{\lambda} \mu R(q, 0) + \tilde{\lambda} \mu R(q + 1, 1) = M(q, 1)$$

(63)

for $q \geq 1$ where we defined

$$M(q, 1) = \sum_{\psi \in X: q' < q \mid q' = q, \tilde{y} = 1} (C - f(\psi)) \pi_{\psi}(\psi)$$

(64)

We can combine Equation 61 and 63 to obtain the first order recurrence

$$R(q + 1, 0) = \frac{\tilde{\lambda} \mu}{\lambda \mu} R(q, 0) + \tilde{M}(q)$$

(65)

for $q \geq 0$

$$\tilde{M}(q) = \frac{1}{\mu} M(q + 1, 1) - M(q + 1, 0) + \frac{\tilde{\lambda} \mu}{\lambda \mu} M(q, 0)$$

(66)

Note that

$$M(q, 0) = C \left( 1 - \frac{\pi Q(0)}{\mu} \left( \frac{\tilde{\lambda} \mu}{\lambda \mu} \right)^{q+1} - \sum_{\psi \in X: q' \leq q} f(\psi) \pi_{\psi}(\psi) \right)$$

$$\frac{\pi Q(0)}{\mu} \left( \frac{\tilde{\lambda} \mu}{\lambda \mu} \right)^{q+1} \left( \mu \log \left( \frac{\mu}{\lambda} \right) + \tilde{\mu} \log \left( \frac{\tilde{\mu}}{\lambda} \right) - C \right)$$

$$= \frac{\tilde{\lambda}}{\mu} \left( \frac{\tilde{\lambda} \mu}{\lambda \mu} \right)^{q+1} \left( \mu \log \left( \frac{\mu}{\lambda} \right) + \tilde{\mu} \log \left( \frac{\tilde{\mu}}{\lambda} \right) - C \right)$$

$$= c_{M0} \left( \frac{\tilde{\lambda} \mu}{\lambda \mu} \right)^{q+1}$$

(67)

and with this Equation 66 becomes

$$\tilde{M}(q) = \frac{1}{\mu} M(q + 1, 1)$$

(68)

But

$$M(q + 1, 1) = M(q, 0) + (C - \log \frac{\mu}{\lambda}) \pi Q(0) \left( \frac{\lambda \mu}{\tilde{\mu} \mu} \right)^{q+1}$$

$$= \left( \frac{\tilde{\lambda} \mu}{\lambda \mu} \right)^{q+1} \left\{ c_{M0} + (C - \log \frac{\mu}{\lambda}) \pi Q(0) \right\}$$

(69)

So we obtain

$$\tilde{M}(q) = \left( \frac{\tilde{\lambda} \mu}{\lambda \mu} \right)^{q+1} \left\{ c_{M0} + (C - \log \frac{\mu}{\lambda}) \pi Q(0) \right\}$$

(70)

We now define two new sequences $a_q$ and $b_q$ such that

$$R(q, 0) = a_q + b_q R(0, 0)$$

(71)

Clearly, $a_0 = 0$ and $b_0 = 1$. By substituting Equation 71 into Equation 65 we find that

$$a_{q+1} = \frac{\tilde{\lambda} \mu}{\lambda \mu} a_q + \tilde{M}(q)$$

(72)

and

$$b_{q+1} = \frac{\lambda \mu}{\lambda \mu} b_q$$

(73)

The solution to the recurrence $b_q$ is obvious

$$b_q = \left( \frac{\lambda \mu}{\lambda \mu} \right)^q$$

(74)

In order to obtain the solution to the recurrence $b_q$ we employ the generating function method [2]

$$\sum_{q \geq 0} a_{q+1} z^q = \frac{\tilde{\lambda} \mu}{\lambda \mu} \sum_{q \geq 0} a_q z^q + \sum_{q \geq 0} \tilde{M}(q) z^q$$

(75)

and therefore

$$z^{-1} A(z) = \frac{\tilde{\lambda} \mu}{\lambda \mu} z A(z) + \tilde{M}(z)$$

(76)

We can now solve this equation for $A(z)$ to obtain

$$A(z) = \frac{\tilde{M}(z)}{z^{-1} - \frac{\tilde{\lambda} \mu}{\lambda \mu} z} = c_{M0} \left( \frac{\tilde{\lambda} \mu}{\lambda \mu} z \right)^q$$

(77)

and the corresponding sequence

$$a_q = c_{M0} q \left( \frac{\tilde{\lambda} \mu}{\lambda \mu} \right)^q$$

(78)

Finally

$$R(q, 0) = a_q + b_q R(0, 0) = c_{M0} q \left( \frac{\tilde{\lambda} \mu}{\lambda \mu} \right)^q + \left( \frac{\tilde{\lambda} \mu}{\lambda \mu} \right)^q R(0, 0)$$

(79)
Further
\[
\sum_{\psi \in X} r(\psi, i) = \sum_{\psi' \in X} (f(\psi') - C) \pi_{\psi'}(\psi') \sum_{\psi \in X} p_{\psi'\psi_0}(\psi|\psi') = 0
\]  
(80)

and summing this equation in \( i \) yields
\[
\sum_{\psi \in X} R(\psi) = 0
\]  
(81)

This result now allows us to compute \( R(0, 0) \): Using Equation 61 we can write
\[
\sum_{\psi \in X} R(\psi) = \sum_{q \geq 0} R(q, 0) + \sum_{q \geq 0} \left( \frac{1}{\lambda} M(q, 0) + \frac{1}{\lambda} \bar{M}(q, 0) \right)
\]
\[
= \frac{1}{\lambda} \sum_{q \geq 0} (R(q, 0) + M(q, 0))
\]  
(82)

Combining Equations 71, 81 and 82 then yields
\[
R(0, 0) = -\frac{\sum_{q \geq 0} a_q + \sum_{q \geq 0} M(q, 0)}{\sum_{q \geq 0} b_q}
\]
\[
= -c_M \frac{\lambda \bar{\mu}}{1 - \lambda \mu} - c_{M0} \frac{\lambda \bar{\mu}}{\lambda \mu}
\]  
(83)

Using the expressions for \( R(j + 1, 1), R(j, 0) \) and \( M(j, 0) \) from Equations 61, 79 and 67 respectively, we are eventually in a position to simplify the expression in Equation 84
\[
\sum_{i=0}^{\infty} \text{Cov}(f(\Psi_0), f(\Psi_i)) = \log \frac{1}{\lambda} R(0, 0)
\]
\[
+ \log \frac{\mu}{\lambda} \left( \frac{1}{\lambda} \sum_{q \geq 0} R(q, 0) + \frac{1}{\lambda} \sum_{q \geq 0} M(q, 0) \right) + \log \frac{\bar{\mu}}{\lambda} \sum_{q > 0} R(q, 0)
\]  
(84)

where
\[
\sum_{q \geq 0} R(q, 0) = c_M \frac{\lambda \bar{\mu}}{1 - \lambda \mu} + \frac{R(0, 0)}{1 - \lambda \mu}
\]
\[
= -c_{M0} \frac{\lambda \bar{\mu}}{1 - \lambda \mu}
\]  
(85)

and
\[
\sum_{q \geq 0} M(q, 0) = c_{M0} \frac{\lambda \bar{\mu}}{1 - \lambda \mu}
\]  
(86)

The claim then follows.

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