We study the infinite time dynamics of a class of nonlinear Schrödinger/Gross–Pitaevskii equations. In a previous paper, we prove the asymptotic stability of the nonlinear ground state in a general situation which admits degenerate neutral modes of arbitrary finite multiplicity, a typical situation in systems with symmetry. Neutral modes correspond to purely imaginary (neutrally stable) point spectrum of the linearization of the Hamiltonian partial differential equation (PDE) about a critical point. In particular, a small perturbation of the nonlinear ground state, which typically excites such neutral modes and radiation, will evolve toward an asymptotic nonlinear ground state soliton plus decaying neutral modes plus decaying radiation. In the present article, we give a much more detailed, in fact quantitative, picture of the asymptotic evolution. Specifically, we prove an equipartition law.

The asymptotic soliton which emerges, $\phi^{λ_∞}$, has a mass which is equal to the initial soliton mass plus one half the mass, $|z_0|^2$, contained in initially perturbing neutral modes:

$$\|\phi^{λ_∞}\|_{L^2}^2 = \|\phi^{λ_0}\|_{L^2}^2 + \frac{1}{2} |z_0|^2 + o(|z_0|^2).$$

1 Introduction

In this paper, we study the nonlinear Schrödinger/Gross–Pitaevskii (NLS/GP) equations in $\mathbb{R}^3$.
\[ i\partial_t \psi = -\Delta \psi + V \psi - |\psi|^{2\sigma} \psi, \]

(1.1)

where \( \sigma \geq 1 \), \( V: \mathbb{R}^3 \to \mathbb{R} \) is a real, smooth function decaying rapidly at spatial infinity. We study the large time distribution of mass/energy of solutions with initial data

\[ \psi(x,0) = \psi_0, \]

(1.2)

which are sufficiently small in the \( H^2(\mathbb{R}^3) \) norm. (Since our results are in the low energy/small amplitude regime, our analysis goes through without change for the nonlinearities of the form \( +g|\psi|^{2\sigma} \psi \) for any fixed real \( g \). Here, we have taken \( g = 1 \).) NLS/GP arises in many physical contexts. In quantum physics, it describes a mean-field limit, \( N \to \infty \), of the linear quantum description of \( N \)-weakly interacting bosons. Here, \( \psi \) is a collective wave-function and \( V \), a trapping potential, and the nonlinear potential arises due to the collective effect of many quantum particles on a representative particle [4, 8]. In classical electromagnetics and nonlinear optics, NLS/GP arises via the *paraxial approximation* to Maxwell's equations, and governs the slowly varying envelope, \( \psi \), of a nearly monochromatic beam of light, propagating through a waveguide [1, 7]. The waveguide has a linear refractive index profile, determining the potential, \( V \), and cubic (\( \sigma = 1 \)) nonlinear refractive index, due to the optical Kerr effect.

NLS/GP is an infinite-dimensional Hamiltonian system and a unitary evolution in \( L^2(\mathbb{R}) \). In the \( N \)-body quantum setting the time-invariant \( L^2 \) norm corresponds to the conservation of mass. In the electromagnetic setting, it is the conservation of energy (optical power). In this paper, we prove an equipartition law (Theorem 3) for the \( L^2 \) mass/energy small (weakly nonlinear) solutions. Hence, we may refer to this result equipartition of energy or equipartition of mass.

The mathematical set-up is as follows. We choose a spatially decaying potential \( V \) for which the Schrödinger operator, \( -\Delta + V \), has only two negative eigenvalues

\[ e_0 < e_1 < 0. \]

\( e_1 \) is chosen to be closer to the continuous spectrum than to \( e_0 \) (permitting coupling via nonlinearity of discrete and continuum modes at quadratic order in the nonlinear coupling coefficient, \( g \)):

\[ 2e_1 - e_0 > 0. \]

The excited state eigenvalue \( e_1 \) may be degenerate with multiplicity \( N \). (In Section 5, we allow for nearly degenerate excited state eigenvalues.) Denote the corresponding
For $\phi_{\text{lin}}$, $\xi_{\text{lin}}$, $\xi_{\text{lin}}^1, \ldots, \xi_{\text{lin}}^N$. (1.3)

More specifically, there exists an open interval $I$, with $e_0$ as an endpoint, such that for any $\lambda \in I$, NLS/GP (1.1) has solutions of the form

$$\psi(x, t) = e^{i\lambda t}\phi^\lambda(x), \quad (1.4)$$

where $\phi^\lambda$ is asymptotically collinear to $\phi_{\text{lin}}$ for small $H^2$ norm and $\lambda \to -e_0, \lambda \in I$.

The excited state eigenvalues give rise, in the linear approximation, to neutral modes, $(\xi, \eta)^T$, and therefore linearized time-dependent solutions, which are undamped (neutral) oscillations about $\phi^\lambda$:

$$e^{i\lambda t}(\phi^\lambda + (i\Re z) \cdot \xi + i(\Im z) \cdot \eta), \quad (1.5)$$

where $z \in \mathbb{C}^N$.

In [6], also referred to in this paper as GW1, we proved the asymptotic stability of the ground states. Namely, if the initial condition is of the form

$$\psi_0 = e^{i\gamma_0}[\phi^\lambda_0 + R_0] \quad (1.6)$$

for some $\gamma_0 \in \mathbb{R}$ and $R_0 : \mathbb{R}^3 \to \mathbb{C}$ satisfying $\|\langle x \rangle^4 R_0\|_{H^2} \ll \|\phi^\lambda_0\|_2$, then generically there exists a $\lambda_\infty \in I$ such that

$$\min_{\gamma \in \mathbb{R}} \|\psi(t) - e^{i\gamma}\phi^\lambda_\infty\|_\infty \to 0 \quad \text{as } t \to \infty. \quad (1.7)$$

In particular, the neutral oscillatory modes eventually damp to zero as $t \to \infty$ via the coupling and transfer of their energy to the nonlinear ground state and to continuum radiation modes. When the neutral mode is simple, that is, $N = 1$ in (1.3), similar results have been obtained in [2, 3, 5, 9, 10, 12].

In the present paper, we seek a more detailed, quantitative description of the large time dynamics. We consider a special class of initial conditions to which the results of GW1, in particular, (1.7) apply:

$$\psi_0 = e^{i\gamma_0}\phi^\lambda_0 + \text{neutral modes} + R_0$$
with
\[ \| \phi^{\lambda_0} \|_2 \gg \| \text{neutral modes} \|_2 \gg \langle x \rangle^4 R_0 \|_{H^2}. \]

The main result of this paper, proved by a considerable refinement of the analysis in [6], is that the emerging asymptotic ground state has, up to high order corrections, a mass equal to its initial mass plus one-half of the mass of the initial excited state mass:
\[ \| \phi^{\lambda_\infty} \|_2^2 = \| \phi^{\lambda_0} \|_2^2 + \frac{1}{2} \| \text{neutral modes} \|_2^2 (1 + o(1)). \]

Thus, half of the excited state mass goes into forming a limiting, more massive, ground state, \( \phi^{\lambda_\infty} \) and the other half of the excited state mass is radiated away to infinity. We call this the \textit{mass- or energy- equipartition}. That this phenomenon is expected, was discussed in [9–11]. The main achievement of the present work is a \textit{rigorous quantification} of the asymptotic (\( t \to \infty \)) mass/energy distribution.

The paper is organized as follows: In Section 2, we review results on the existence and properties of the ground state manifold, and on the spectral properties of the linearized NLS/GP operator about the ground state. In Section 3, we state and discuss Theorem 3.1 on equipartition. In Section 4, we present the proofs, using technical estimates established in the appendices, for example, Sections G–I. In Section 5, we present a generalization of the Theorem 3.1 to the case of nearly degenerate case, and an outline of its proof. A more extensive list of references and a discussion of related work on NLS/GP appears in GW1.

1.1 Notation

1. \( \alpha_+ = \max\{\alpha, 0\}, [r] = \max_{\eta \in \mathbb{Z}} \{\eta \leq r\} \).
2. \( \Re z = \text{real part of } z, \Im z = \text{imaginary part of } z. \)
3. Multi-indices
\[ z = (z_1, \ldots, z_N) \in \mathbb{C}^N, \quad \bar{z} = (\bar{z}_1, \ldots, \bar{z}_N), \]
\[ a \in \mathbb{N}^N, \quad z^a = z_1^{a_1} \cdots z_N^{a_N}, \]
\[ |a| = |a_1| + \cdots + |a_N|. \] (1.8)

4. \( Q_{m,n} \) denotes an expression of the form
\[ Q_{m,n} = \sum_{|a|=m, |b|=n} q_{a,b} z^a \bar{z}^b = \sum_{|a|=m, |b|=n} q_{a,b} \prod_{k=1}^N z_k^{a_k} \bar{z}_k^{b_k}. \]
(5) \[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \quad L = JH = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}. \]

(6) \( \sigma_{\text{ess}}(L) = \sigma_c(L) \) is the essential (continuous) spectrum of \( L \), \( \sigma_{\text{disc}}(L) \) is the discrete spectrum of \( L \).

(7) Riesz projections: \( P_{\text{disc}}(L) \) and \( P_c(L) = I - P_{\text{disc}}(L) \), \( P_{\text{disc}}(L) \) projects onto the discrete spectral part of \( L \), \( P_c(L) \) projects onto the continuous spectral part of \( L \).

(8) \( \langle f, g \rangle = \int f(x)\overline{g(x)} \, dx \).

(9) \( \| f \|_p^p = \int_{\mathbb{R}^3} |f(x)|^p \, dx, \quad 1 \leq p \leq \infty. \)

(10) \( \| f \|_{L^2(\mathbb{R}^3)}^2 = \int (I - \Delta)^{s/2} f(x)^2 \, dx. \)

(11) \( \| f \|_{H^s(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} |(x)^\nu (I - \Delta)^{s/2} f(x)|^2 \, dx. \)

2 Review of the set-up

In this section, we review the setting presented in detail in [6].

2.1 Assumptions on the potential, \( V(x) \)

We assume that the Schrödinger operator \( -\Delta + V \) has the following properties:

(V1) \( V \) is real valued and decays sufficiently rapidly, for example, exponentially, as \( |x| \) tends to infinity.

(V2) \( -\Delta + V \) has two eigenvalues \( e_0 < e_1 < 0. \)

\( e_0 \) is the lowest eigenvalue with ground state \( \phi_{\text{lin}} > 0 \), the eigenvalue \( e_1 \) is degenerate with multiplicity \( N \) and eigenvectors \( \xi_{\text{lin}}^1, \xi_{\text{lin}}^2, \ldots, \xi_{\text{lin}}^N \).

2.2 Bifurcation of ground states from \( e_0 \)

**Proposition 2.1.** Suppose that the linear operator \( -\Delta + V \) satisfies the conditions above in subsection 2.1. Then there exists a constant \( \delta_0 > 0 \) and a nonempty interval \( I \subset [e_0 - \delta_0, e_0) \) such that for any \( \lambda \in I \), NLS/GP (1.1) has solutions of the form

\[ \psi(x, t) = e^{i\lambda t} \phi^\lambda \in L^2 \]

with

\[ \phi^\lambda = \delta(\phi_{\text{lin}} + O(\delta^{2\sigma})) \quad \text{and} \quad \delta = \delta(\lambda) = |e_0 + \lambda|^{1/2\sigma} \left( \int \phi_{\text{lin}}^{2\sigma+2} \right)^{-1/2\sigma}. \]
2.3 Linearization of NLS/GP about the ground state

If we write \( \psi(x,t) = e^{i\lambda t}(\phi^\lambda + u + iv) \), then we find the linearized perturbation equation to be:

\[
\frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} = L(\lambda) \begin{pmatrix} u \\ v \end{pmatrix} = JH(\lambda) \begin{pmatrix} u \\ v \end{pmatrix},
\]

where

\[
L(\lambda) := \begin{pmatrix} 0 & L_-(\lambda) \\ -L_+(\lambda) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} L_+(\lambda) & 0 \\ 0 & L_-(\lambda) \end{pmatrix} = JH(\lambda).
\]

Here, \( L_+ \) and \( L_- \) are given by

\[
L_-(\lambda) := -\Delta + \lambda + V - (\phi^\lambda)^{2\sigma},
\]

\[
L_+(\lambda) := -\Delta + \lambda + V - (2\sigma + 1)(\phi^\lambda)^{2\sigma}.
\]

The following results on the point spectrum of \( L(\lambda) \) appear in [6]; see Proposition 4.1, p. 275 and Propositions 5.1–5.2, p. 277.

**Lemma 2.1.** Let \( L(\lambda) \), or more explicitly, \( L(\lambda(\delta), \delta) \) denote the linearized operator about the bifurcating state \( \phi^\lambda, \lambda = \lambda(\delta) \). Note that \( \lambda(0) = -e_0 \). Corresponding to the degenerate e-value, \( e_1 \), of \( -\Delta + V \), the matrix operator \( L(\lambda = -e_0, \delta = 0) \) has degenerate eigenvalues \( \pm i E(-e_0) = \pm i(e_1 - e_0) \), each of multiplicity \( N \). For \( \delta > 0 \) and small these bifurcate to (possibly degenerate) eigenvalues \( \pm i E_1(\lambda), \ldots, \pm i E_N(\lambda) \) with neutral modes

\[
\begin{pmatrix} \xi_1 \\ \pm i\eta_1 \\
\xi_2 \\ \pm i\eta_2 \\
\vdots \\
\xi_N \\ \pm i\eta_N 
\end{pmatrix}
\]

satisfying the estimates

\[
\langle \xi_m, \eta_n \rangle = \delta_{m,n}, \quad \langle \xi_m, \phi^\lambda \rangle = \langle \eta_m, \partial_\lambda \phi^\lambda \rangle = 0
\]

and

\[
0 \neq \lim_{\lambda \to -e_0} \xi_n = \lim_{\lambda \to -e_0} \eta_n \in \text{span}\{\xi_n^{\text{lin}}, n = 1, 2, \ldots, N\} \text{ in } H^k \text{ spaces for any } k > 0.
\]

**Remark 2.1.** Since \( E(-e_0) = e_1 - e_0 \), it follows that if \( 2e_1 - e_0 > 0 \), then for sufficiently small \( \delta \), \( 2E_n(\lambda) > \lambda, n = 1, 2, \ldots, N \). This ensures nonlinear coupling of discrete to continuous spectrum at second order (in the nonlinearity coefficient, \( g \)). Thus, to ensure such coupling, we assume:
\[ 2e_1 - e_0 > 0. \] (2.7)

\[ \square \]

**Lemma 2.2.** Assume the potential \( V = V(|x|) \) and the functions \( \xi_{n}^{\text{lin}} \) admit the form \( \xi_{n}^{\text{lin}} = (x_n/|x|)\xi(|x|) \) for some function \( \xi^{\text{lin}} \), then \( \phi^{\lambda} \), hence \( \partial_{\lambda}\phi^{\lambda} \), is spherically symmetric, \( E_n = E_1 \) for any \( n = 1, 2, \ldots, N = d \) and we can choose \( \xi_n, \eta_n \) such that \( \xi_n = (x_n/|x|)\xi(|x|) \) and \( \eta_n = (x_n/|x|)\eta(|x|) \) for some real functions \( \xi \) and \( \eta \).

\[ \square \]

In this paper, we make the following assumptions on the spectrum of the operator \( L(\lambda) \):

**(SA)** The linearized operator \( L(\lambda) \) has a discrete spectrum given by:

(a) an eigenvalue 0 with generalized eigenspace spanned by 
\[
\left\{ \begin{pmatrix} 0 \\ \phi^{\lambda} \end{pmatrix}, \begin{pmatrix} \partial_{\lambda}\phi^{\lambda} \\ 0 \end{pmatrix} \right\},
\]
(b) neutral eigenvalues \( \pm i E(\lambda), E(\lambda) > 0 \), satisfying the condition \( 2E(\lambda) > \lambda \) and with corresponding eigenvectors \( \begin{pmatrix} \xi_n \pm i\eta_n \end{pmatrix}, n = 1, \ldots, N \).

For the nonself-adjoint operator \( L(\lambda) \) the (Riesz) projection onto the discrete spectrum subspace of \( L(\lambda) \), \( P_d = P_d(L(\lambda)) = P_d^{\lambda} \) is given explicitly in [6, Proposition 5.6, p. 280]:

\[
P_d = \frac{2}{\partial_{\lambda}\|\phi^{\lambda}\|^2} \left( \begin{pmatrix} 0 \\ \phi^{\lambda} \end{pmatrix} \begin{pmatrix} 0 \\ \partial_{\lambda}\phi^{\lambda} \end{pmatrix} + \begin{pmatrix} \partial_{\lambda}\phi^{\lambda} \\ 0 \end{pmatrix} \begin{pmatrix} \phi^{\lambda} \\ 0 \end{pmatrix} \right) \right.
\]

\[
+ \frac{1}{2} \sum_{n=1}^{N} \begin{pmatrix} \xi_n \\ i\eta_n \end{pmatrix} \begin{pmatrix} -i\eta_n \\ -i\xi_n \end{pmatrix} \begin{pmatrix} \xi_n \\ i\eta_n \end{pmatrix} \right)
\]

and the projection onto the essential spectrum by \( P_c \equiv 1 - P_d \).

The large time analysis of NLS/GP requires good decay estimates on the linearized evolution operator, \( e^{\lambda t}P^{\lambda}_c \). An obstruction to such estimates are, so-called, threshold resonances (see [6] and references therein), which we preclude with the following hypothesis.

**(Thresh)\( _{\lambda} \)** Assume \( L(\lambda) \) has no resonances at \( \pm i\lambda \).

For small solitons, \( \delta \) sufficiently small, \( (\text{Thresh} _{\lambda}) \) follows from the absence of a zero energy resonance for \( -\Delta + V \).
2.4 Second-order (nonlinear) Fermi Golden Rule

In this subsection, we review the definitions and constructions presented in detail in [6, pp. 281–282]. The amplitudes and phases of the neutral modes are governed by the complex-valued vector parameter \( z: \mathbb{R}^+ \to \mathbb{C}^N \), first arising in the linear approximation of solution \( \psi \), see, for example, (1.5). Its precise definition is seen in the decomposition of the solution \( \psi \) in (3.1), under the condition (3.6), below, from which it follows that

\[
\partial_t z = -i E(\lambda)z - \Gamma(z, \bar{z})z + \Lambda(z, \bar{z})z + O((1 + t)^{-3/2 - \delta}), \quad \delta > 0, \tag{2.9}
\]

where \( \pm i E(\lambda) \) are complex conjugate \( N \)-fold degenerate neutral eigenfrequencies of \( L(\lambda) \), \( \Gamma \) is nonnegative symmetric and \( \Lambda \) is skew symmetric.

In what follows, we define the nonnegative, Fermi Golden Rule matrix, \( \Gamma \). Define vector functions \( G_k \), \( k = 1, 2, \ldots, N \), as

\[
G_k(z, x) := \begin{pmatrix} B(k) \\ D(k) \end{pmatrix} \tag{2.10}
\]

with the functions \( B(k) \) and \( D(k) \) defined as

\[
B(k) := -i \sigma (\phi^\lambda)^{2\sigma - 1} [(z \cdot \xi_k) \eta + (z \cdot \eta) \xi_k],
\]

\[
D(k) := -\sigma (\phi^\lambda)^{2\sigma - 1} [3(z \cdot \xi) \xi_k - (z \cdot \eta) \eta_k] - 2\sigma (\sigma - 1)(\phi^\lambda)^{2\sigma - 1}(z \cdot \xi) \xi_k,
\]

where

\[
z \cdot \xi := \sum_{n=1}^{N} z_n \xi_n, \quad z \cdot \eta := \sum_{n=1}^{N} z_n \eta_n.
\]

In terms of the column 2-vector, \( G_k \), we define a \( N \times N \) matrix \( Z(z, \bar{z}) \) as

\[
Z(z, \bar{z}) = (Z^{(k,l)}(z, \bar{z})), \quad 1 \leq k, l \leq N \tag{2.11}
\]

and

\[
Z^{(k,l)} = -(L(\lambda) + 2i E(\lambda) - 0)^{-1} P_c G_l, i J P_c G_k). \tag{2.12}
\]

Finally, we define \( \Gamma(z, \bar{z}) \) as follows:

\[
\Gamma(z, \bar{z}) := \frac{1}{2} [Z(z, \bar{z}) + Z^*(z, \bar{z})]. \tag{2.13}
\]

Thus,

\[
[\Gamma(z, \bar{z})]_{kl} = -\Re((L(\lambda) + 2i E(\lambda) - 0)^{-1} P_c G_l, i J P_c G_k). \tag{2.14}
\]

By (2.9) and (2.14) we find

\[
\partial_t |z(t)|^2 = -2z^* \Gamma(z, \bar{z})z + \cdots. \tag{2.15}
\]
In GW1, $\Gamma$ was shown to be nonnegative and we require it to be positive definite. In particular, we shall require the following Fermi Golden Rule hypothesis.

Let $P_c^{\text{lin}}$ denote the spectral projection onto the essential spectrum of $-\Delta + V$. Then

(FGR) We assume that there exists a constant $C > 0$ such that

$$-\Re\langle i[-\Delta + V + \lambda - 2E(\lambda) - i0]^{-1}P_c^{\text{lin}}(\phi^{\text{lin}})^{2\sigma-1}(z \cdot \xi^{\text{lin}})^2, (\phi^{\text{lin}})^{2\sigma-1}(z \cdot \xi^{\text{lin}})^2 \rangle \geq C|z|^4 \text{ for any } z \in \mathbb{C}^N.$$  

The assumption FGR implies that there exists a constant $C_1 > 0$ such that for any $z \in \mathbb{C}^N$

$$z^*\Gamma(z, \bar{z})z \geq C_1\|\phi^\lambda\|_{L^\infty}^{4\sigma-2}|z|^4. \hspace{1cm} (2.16)$$

Note that for each fixed $z$, smallness of $|\lambda + e_0|$ together with (2.1) and (2.6) imply that the leading term in $z^*\Gamma(z, \bar{z})z$ is

$$z^*\Gamma_0(z, \bar{z})z = -2\sigma^2(\sigma + 1)^2\delta^{4\sigma-2}(\lambda)$$

$$\times \Re\langle i[-\Delta + V + \lambda - 2E(\lambda) - i0]^{-1}P_c^{\text{lin}}(\phi^{\text{lin}})^{2\sigma-1}(z \cdot \xi^{\text{lin}})^2, (\phi^{\text{lin}})^{2\sigma-1}(z \cdot \xi^{\text{lin}})^2 \rangle. \hspace{1cm} (2.17)$$

3 Main Theorem

In this section, we state our main results, Theorems 3.1 and 3.2.

**Theorem 3.1.** Assume a cubic nonlinearity, $\sigma = 1$, in (1.1). If the spectral conditions (SA) (Thres$_\lambda$) and (FGR) are satisfied, then there exists a constant $\delta$ such that if the initial condition $\psi_0$ satisfies the condition

$$\psi_0(x) = e^{i\gamma_0}[\phi^{\lambda_0} + \alpha_0 \cdot \xi + i\beta_0 \cdot \eta + R_0]$$

for some real constants $\gamma_0$ and $\lambda_0$, real $N$ vectors $\alpha_0$ and $\beta_0$, function $R_0 : \mathbb{R}^3 \to \mathbb{C}$, such that for $\epsilon \leq \delta$: $|\lambda_0 - |e_0|| \leq \epsilon, |\alpha_0| + |\beta_0| \lesssim \epsilon\|\phi^{\lambda_0}\|_2, \|\langle x \rangle^4 R_0\|_{L^2} \lesssim |\alpha_0|^2 + |\beta_0|^2 = O(\epsilon^2)$, for $\epsilon \leq \delta$, then there exist smooth functions

$$\lambda(t) : \mathbb{R}^+ \to I, \quad \gamma(t) : \mathbb{R}^+ \to \mathbb{R}, \quad \beta(t) : \mathbb{R}^+ \to \mathbb{C}^N,$$

$$R(x, t) : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{C}$$
such that the solution of NLS evolves in the form:

\[ \psi(x, t) = e^{i \int_0^t \lambda(s) \, ds} e^{i \gamma(t)} \times \left[ \phi^\lambda + a_1(z, \bar{z}) \partial_\tau \phi^\lambda + i a_2(z, \bar{z}) \phi^\lambda + (\text{Re} \, \bar{z}) \cdot \xi + i (\text{Im} \, \bar{z}) \cdot \eta + R \right], \tag{3.1} \]

where \( \lim_{t \to \infty} \lambda(t) = \lambda_\infty \), for some \( \lambda_\infty \in \mathcal{I} \).

Here, \( a_1(z, \bar{z}), a_2(z, \bar{z}) : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{R} \) and \( \bar{z} - z : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}^N \) are some polynomials of \( z \) and \( \bar{z} \), beginning with terms of order \( |z|^2 \).

(A) The dynamics of mass/energy transfer is captured by the following reduced dynamical system for the key modulating parameters, \( \lambda(t) \) and \( z(t) \):

\[
\frac{d}{dt} \| \phi^\lambda(t) \|^2_2 = z^* \Gamma_0(z, \bar{z}) z + S_\lambda(t),
\]

\[
\frac{d}{dt} |z(t)|^2 = -2 z^* \Gamma_0(z, \bar{z}) z + S_z(t),
\]

where \( z^* \Gamma_0(z, \bar{z}) z \) is given in (2.17), and

\[ S_\lambda(t) \lesssim (1 + t)^{-19/10} \quad \text{and} \quad S_z(t) \lesssim (1 + t)^{-12/5}. \tag{3.4} \]

Furthermore,

\[ \int_0^\infty S_\lambda(\tau) \, d\tau, \quad \int_0^\infty S_z(\tau) \, d\tau = o(|z_0|)^2. \tag{3.5} \]

(B) \( \vec{R}(t) = (\text{Re} \, R(t), \text{Im} \, R(t))^T \) lies in the essential spectral part of \( L(\lambda(t)) \). Equivalently, \( R(\cdot, t) \) satisfies the symplectic orthogonality conditions:

\[
\omega(R, i \phi^\lambda) = \omega(R, \partial_\tau \phi^\lambda) = 0,
\]

\[
\omega(R, i \eta_n) = \omega(R, \xi_n) = 0, \quad n = 1, 2, \ldots, N,
\]

where \( \omega(X, Y) := \text{Im} \int X \bar{Y} \).

(C) Decay estimates: For any time \( t \geq 0 \)

\[ \| (1 + x^2)^{-\nu} \vec{R}(t) \|_2 \leq C (\| \langle x \rangle^4 \psi_0 \|_{H^2}) (1 + t)^{-1}, \tag{3.7} \]

\[ \| \vec{R}(t) \|_{H^2} \leq \epsilon_\infty, \tag{3.8} \]

\[ |z(t)| \leq C (\| \langle x \rangle^4 \psi_0 \|_{H^2}) (1 + t)^{-1/2}. \tag{3.9} \]

(D) Mass/energy equipartition: Half of the mass of the neutral modes contributes to forming a more massive asymptotic ground state and half is
radiated away
\[ \|\phi^{\pm}\|_2^2 = \|\phi^{\pm0}\|_2^2 + \frac{1}{2}(|\alpha_0|^2 + |\beta_0|^2) + o(|\alpha_0|^2 + |\beta_0|^2). \] (3.10)

The following result applies to the case where \( \sigma > 1 \).

**Theorem 3.2.** Assume the general nonlinearity \( \sigma > 1 \). Then statements (A)–(D) of Theorem 3.1 hold provided, in addition to the assumptions of Theorem 3.1, we assume the following:

1. in the case where the neutral modes are degenerate \( (N > 1) \), the potential \( V \) is spherically symmetric and the eigenvectors \( \xi_n^{\text{lin}} \), \( n = 1, 2, \ldots, N = d \), admit the form \( \xi_n^{\text{lin}} = \frac{\xi_0}{|x|} x |\xi(x)| \) for some function \( \xi \);
2. \[ |\alpha_0|^2 + |\beta_0|^2 \leq \left\| \phi^{\pm0} \right\|_2^2 C(\sigma) \] for some sufficiently large constant \( C(\sigma) \).

The statements (B) and (C) are obtained in \([6, \text{Theorem 7.1, p. 284}]\). Equation (3.8) is from \([6, \text{Proof, Line 18, p. 306}]\) that \( R_4(T) := \max_{0 \leq t \leq T} \| \tilde{R}(t) \|_{H^2} \ll 1; R_4 \) is defined in (11.2).

The bounds on \( S_\lambda(t) \) and \( S_\xi(t) \), (3.4), of statement (A) were proved in \([6, \text{see equations (8–9) and (8–11) p. 286}]\). For the estimate \( |\text{Remainder}| \lesssim (1 + t)^{-19/10} \), see \([6, \text{line 9, p. 306}]\). The remaining assertions in (A) will be reformulated as Theorems 4.1 and 4.2, and proved in Section 4. Statement (D) is proved just below.

**Remark 3.1 (Mass equipartition).** It is straightforward to interpret (3.10) as implying equipartition of the neutral mode mass. Indeed, since \( \phi^{\pm} \) is orthogonal to \( \xi_m \) (see (2.5)) and since mass is conserved for NLS/GP, that is, \( \|\psi(t)\|_2 = \|\psi_0\|_2 \), we have
\[ \|\psi(\cdot, t)\|_2^2 = \|\psi_0\|_2^2 = \|\phi^{\pm0}\|_2^2 + |\alpha_0|^2 + |\beta_0|^2 + o(|\alpha_0|^2 + |\beta_0|^2) \quad \text{for all } t. \]

The theorem implies that \( \psi(t, \cdot) \) has a weak-\( L^2 \) limit, \( \phi^{\pm\infty} \), whose mass is given by (3.10). Thus, half of the mass of the neutral modes is transferred to the ground states while the other half is radiated to infinity.

We now use Statement (A) of Theorem 3.1 to prove Statement (D).

**Proof of Mass equipartition.** Twice the first plus the second equation in (3.4) yields:
\[ \frac{d}{dt}(2\|\phi^{\pm(t)}\|_2^2 + |z(t)|^2) = 2S_\lambda(t) + S_\xi(t). \] (3.11)
Integration of (3.11) with respect to \( t \) from zero to infinity and use of the decay of \( z(t) \), (3.9), imply

\[
2\|\phi^{\lambda}(\infty)\|_2^2 = 2\|\phi^{\lambda}(0)\|_2^2 + |z(0)|^2 + \int_0^\infty (2S_{\lambda}(t') + S_z(t')) \, dt'.
\]

Dividing by two and estimating the integral, using (3.5), completes the proof of Statement D.

\[\Box\]

**Remark 3.2 (Generic data in a neighborhood of the origin).** For the case of cubic nonlinearity, \( \sigma = 1 \), the condition \( |\alpha_0|^2 + |\beta_0|^2 \ll \|\phi^{\lambda_0}\|_2^2 \) can be improved to a state about generic (low energy) initial conditions satisfying

\[
|\alpha_0|^2 + |\beta_0|^2 \approx \|\phi^{\lambda_0}\|_2^2.
\]

We impose the stronger condition in the present paper to simplify the treatment and to apply directly the results in GW1 [6]. We refer to [10, 13]. See also our Remarks 4.1 and F.1.

*The generality of the nonlinearity in (1.1).* Our results hold not only for focusing nonlinearity, that is, \(-|\psi|^{2\sigma} \psi\) in (1.1). In fact all of the results in Theorems 3.1 and 3.2 can be transferred to the general cases \( g|\psi|^{2\sigma} \psi, \ g \in \mathbb{R}\setminus\{0\} \) without difficulty. We restrict ourselves to the present consideration in order not to clutter our arguments by discussing various constants.

\[\Box\]

### 3.1 Relation to previous work

Theorems 3.1 and 3.2 are derived from a refinement of the analysis of [6] and a generalization to arbitrary nonlinearity parameter \( \sigma \geq 1 \). In this subsection, we explain this.

The overall plan for proofs of asymptotic stability can be broken into two parts, motivated by a view of the soliton as an interaction between discrete and continuum modes:

**Part 1:**

(a) We seek a natural decomposition of the solution into a component evolving along the manifold of solitons and a component which is dispersive. However, since the linearization about the soliton may have neutral modes, nondecaying time periodic states, we incorporate these degrees of freedom among the discrete degrees of freedom in the Ansatz. The dispersive components of the evolution lie in the subspace bi-orthogonal, in fact symplectic-orthogonal, to the discrete modes. The result is a *strongly coupled system*
governing the discrete degrees of freedom and dispersive wave field, $R(t)$. Mathematically, we decomposed the solution $\psi$ as in (A.1), and by the orthogonal conditions (2.5) and (3.6), we derive equations for $\dot{\lambda}$, $\dot{\gamma}$, $z$, and $\vec{R}$. These are taken from [6] and displayed in Appendix A.

(b) We solve explicitly for the leading order components of $R(t)$, which arise due to resonant forcing by new, nonlinearity-generated, discrete mode frequencies. To achieve this, we find the leading order, that is second order in $z$ and $\bar{z}$ contributions to $R(x, t)$. This is presented in Appendix B.7.

(c) This leading order behavior is substituted into the equations governing the discrete modes, leading to a (to a leading order) closed equation for the discrete modes, implying estimates for $\dot{\lambda}$ and $\dot{\gamma}$. This is Proposition F.2.

(d) The latter is put into a normal form, via a finite sequence of near-identity changes of variables, in which the energy transfer mechanisms are made explicit. This is achieved via the introduction of $z \mapsto a_1(z, \bar{z}), a_2(z, \bar{z}), p(z, \bar{z})$, and $q(z, \bar{z})$ in Appendix B.

Part 2: The full coupled system is now in the form of a finite dimensional system of (normal form) one-dimensional equations (ODEs), with non-resonant terms removed by near identity changes of variables, with rapidly time-decaying corrections, determined by the dispersive part, weakly coupled to a dispersive PDE, with rapidly decaying and/or oscillating source terms, coming from the discrete components of the solution. The latter is essentially treatable by low-energy scattering methods.

In GW1 [6], we proved that the neutral mode mass and $\lambda(t)$, which through $\|\phi^{\lambda(t)}\|_2^2$ controls the ground state mass, are governed by

$$\frac{d}{dt} \lambda(t) = \text{Rem}_\lambda(t), \quad (3.12)$$

$$\frac{d}{dt} |z(t)|^2 = -2z^* \Gamma(z, \bar{z})z + \text{Rem}_z(t), \quad (3.13)$$

where $\text{Rem}_\lambda(t)$ and $\text{Rem}_z(t)$ satisfy an estimate of the form:

$$|\text{Rem}_\lambda(t)| \lesssim |z(t)|^4 + \|\langle x \rangle^{-4} R(t)\|_{H^2}^2 + \|R(t)\|_\infty^2 + |z(t)| \|\langle x \rangle^{-4} \bar{R}(t)\|_2,$$

$$|\text{Rem}_z(t)| \lesssim |z(t)|^5 + |z(t)| \|\langle x \rangle^{-4} R(t)\|_{H^2}^2 + |z(t)| \|R(t)\|_\infty^2 + |z(t)|^2 \|\langle x \rangle^{-4} \bar{R}(t)\|_2,$$  

(3.14)

where $\bar{R}$ is defined in (B.8), and for $t \gg 1$ we have

$$|z(t)| \sim t^{-1/2}, \quad \|\langle x \rangle^{-4} R(t)\|_{H^2} \sim t^{-1}, \quad \|R(t)\|_\infty \sim t^{-1}, \quad \|\langle x \rangle^{-4} \bar{R}(t)\|_2 \sim t^{-7/5}.$$
Since $\text{Rem}_z(t) = O(t^{-2-\tau})$, $\tau > 0$, $\text{Rem}_z(t)$ is dominated by the first term on the right-hand side of (3.13), which is $O(t^{-2})$ and strictly negative, by the Fermi Golden Rule resonance hypothesis (FGR). Furthermore, $\text{Rem}_\lambda(t)$ is integrable in $t$, $\lambda(t)$ has a limit, $\lambda_\infty$.

4 Refinements of the analysis and outline of the proof

In view of the results of GW1, we focus on the refinements required. These concern the terms $S_\lambda$ and $S_z$ in (3.2) and (3.3) and their estimation in (3.5), for the proofs of Theorems 3.1 and 3.2. In this section we derive $S_\lambda$ and $S_z$ and estimate them.

Technically the main effort in the present paper is to improve the estimates for the various terms on the right-hand side of (3.12) and (3.13). It is relatively easier to improve the estimate for $\partial_t |z|^2$, since the term $-2z^* \Gamma(z, \bar{z}) z$ already measures the decreasing of $|z|^2$. What is left is to prove the term $\text{Rem}_z$ is indeed a small correction in certain sense.

To improve the estimates of the terms on the right-hand side of (3.12) is more involved. From (3.12) we cannot tell the increasing or decreasing of the parameter $\lambda$. For that purpose, we expand the right-hand side of (3.12) to fourth order in $z$ and $\bar{z}$ to find some sign. This in turn is achieved by expansion of the function $R$ or $\tilde{R}$ further to third order in $z$ and $\bar{z}$. For that purpose, we define the third-order terms in (B.9) and introduce remainder by $R_{\geq 4}$ in (B.13).

We next present some precise estimates on $R_{\geq 4}$, $z$ and $\dot{\lambda}$, which are defined in Appendices A–B. To facilitate later discussions, we define the constant $\delta_\infty$ by

$$\delta_\infty := \|\phi^\lambda\|^2_{L^2} = O(|\lambda_\infty + e_0|^{1-\frac{1}{2\sigma}}) = O(\delta(\lambda(t))) \quad \text{for any time } t,$$

where the last estimate follows from the fact that the soliton manifold is stable (see [6]). Recall the constant $\delta(\lambda) \equiv \delta$ defined and estimated in (2.1), and recall $\lim_{t \to \infty} \lambda(t) = \lambda_\infty$ in Theorem 3.1. We have the following.

**Proposition 4.1.** Suppose that $|z_0|/\delta_\infty \ll 1$ for $\sigma = 1$ and $|z_0| \leq \delta^{\sigma}_{\infty}$ for $\sigma > 1$ and $C(\sigma)$ is a sufficiently large constant. Then the following results hold: there exists a constant $C > 0$ such that for any time $t \geq 0$

$$|z(t)| \leq C(|z_0|^{-2} + \delta^{4\sigma-2}_{\infty} t)^{-1/2},$$

(4.2)
if \( \sigma = 1 \), then

\[
\|x\|_{R_t^4}^2 \lesssim |z_0|^2(1 + t)^{-3/2} + \delta_\infty |z_0|^2 |z(t)|^2, \tag{4.3}
\]

\[
|\dot{\lambda}| \lesssim \delta_\infty |z(t)|^4 + 2\delta_\infty |z_0|^2 |z(t)|^3 + \delta_\infty |z_0|^4(1 + t)^{-3} + \delta_\infty |z_0|^2 |z(t)|(1 + t)^{-3/2}, \tag{4.4}
\]

and if \( \sigma > 1 \), then

\[
\|x\|_{R_t^4}^2 \lesssim |z_0|^2(1 + t)^{-3/2} + \|z_0|^2 + |z_0|^{2\sigma - 1}] |z(t)|^2, \tag{4.5}
\]

\[
|\dot{\lambda}| \lesssim |z|^{2\sigma + 1} + |z(t)|^4 + \delta_\infty |z_0|^2 |z(t)|^3 + |z_0|^4(1 + t)^{-3} + |z_0|^2 |z(t)|(1 + t)^{-3/2}. \tag{4.6}
\]

This proposition will be formulated as different parts of Propositions F.1 and F.2 in Appendix F.

In the next two subsections, we find and estimate the functions \( S_x \) and \( S_\sigma \) of (3.2) and (3.3).

4.1 Definition of \( S_x \) and its estimate

In this part, we define and estimate the function \( S_x \) in (3.3).

It was proved in [6, p. 293] (and can also be derived from (A.7) and (A.8)) that \( z \) satisfies the equation

\[
\partial_t z + i E(\lambda) z = - \Gamma(z, \bar{z}) z + \Lambda(z, \bar{z}) z + K, \tag{4.7}
\]

where \( \Gamma(z, \bar{z}) \) is positive definite and \( \Lambda(z, \bar{z}) \) is skew symmetric, \( K = (K_1, \ldots, K_M)^T \) is defined as

\[
K := - \left[ \partial_t p_n + i E(\lambda) \sum_{k + l = 2, 3} (k - l) P_{k,l}^{(n)} \right] - i \left[ \partial_t q_n + i E(\lambda) \sum_{k + l = 2, 3} (k - l) Q_{k,l}^{(n)} \right] + \left\langle JN(\vec{\mathbf{R}}, p, z) - \sum_{m + n = 2, 3} JN_{m,n} \left( \begin{array}{c} \eta_n \\ -i\xi_n \end{array} \right), Y_{1,1} \left[ (q \cdot \eta, \eta_n) - (i p \cdot \xi, \xi_n) \right] \rightangle \\
+ \left\langle JN(\vec{\mathbf{R}}, \bar{\mathbf{R}}, \bar{\mathbf{p}}, \eta, \eta_n) - i \left( (\alpha + \mathbf{p}) \cdot \xi, \xi_n \right) \right\rangle \\
- \lambda \left[ a_1 \langle \partial_\phi \phi^\lambda, \eta_n \rangle + (\alpha + \mathbf{p}) \cdot \partial_\xi \eta_n \right] + \left[ a_2 \langle \partial_\eta \phi^\lambda, \xi_n \rangle + i \langle (\beta + \mathbf{q}) \cdot \partial_\eta \xi_n, \eta_n \rangle \right] \\
+ \left\langle \vec{\mathbf{R}}, \lambda \left( \begin{array}{c} \partial_\eta \eta_n \\ -i\partial_\xi \xi_n \end{array} \right), -\dot{\gamma} \left( \begin{array}{c} i\xi_n \\ \eta_n \end{array} \right) \right\rangle.
\]

Recall that \( |z|^2 \) measures the neutral mode mass. By direct computation, we find

\[
\frac{d}{dt} |z|^2 = -2z^* \Gamma(z, \bar{z}) z + 2 \text{Re} z^* \cdot \bar{\mathcal{K}} = -2z^* \Gamma_0(z, \bar{z}) z + S_x
\]
with the function $S_z$ defined by

$$S_z := -2z^* \Gamma(z, \bar{z})z + 2z^* \Gamma_0(z, \bar{z})z + 2\text{Re} z^* \cdot K. \quad (4.8)$$

We now estimate different terms on the right-hand side of (4.8).

Lemma 4.1. For $\sigma \geq 1$

$$z^* \Gamma(z, \bar{z})z = z^* \Gamma_0(z, \bar{z})z + O(\delta^{|\sigma|} \sigma^{|\sigma|} |z|^4). \quad (4.9)$$

If $\sigma = 1$, then

$$|K| \lesssim \delta_\infty |z(t)|^2 + \delta_\infty \|x\|^4 R_{2+4} R_{2+4}^2; \quad (4.10)$$

if $\sigma > 1$, then

$$|K| \lesssim |z(t)|^{2\sigma + 1} + |z(t)|^4 + \|x\|^4 R_{2+4} R_{2+4}^2 + |z(t)||\langle x\rangle|^4 R_{2+4} R_{2+4}^2. \quad (4.11) \square$$

Equation (4.9) will be proved in Appendix G, (4.10) and (4.11) will be incorporated into Proposition F.2. By the above estimates, we have the following.

Theorem 4.1.

$$\int_0^\infty S_z(s) \, ds = o(|z_0|^2). \quad (4.12) \square$$

Proof. The following two estimates together with Lemma 4.1 are sufficient to prove the theorem:

$$\int_0^\infty |z(s)||K(s)|| \, ds = o(|z_0|^2) \quad (4.13)$$

and

$$\int_0^\infty \delta^{|\sigma|} |z|^4(s) \, ds \leq C |z_0|^2. \quad (4.14)$$

We next focus on proving the two inequalities (4.13) and (4.14). The proof of (4.14) is relatively easy; it follows applying the estimate of $z$ in (4.2) and direct computation.

We now turn to (4.13). For $\sigma = 1$ we use (4.10) and (4.3) to obtain

$$|K| \lesssim \delta_\infty |z|^4 + \delta_\infty |z_0|^2 |z|(1 + t)^{-3/2} + \delta_\infty^2 |z|^2 |z_0|^2 + |z_0|^4 \delta_\infty (1 + t)^{-3}. \quad (4.15)$$

Together with the assumption on the initial condition $|z| \ll \delta_\infty = O(\delta_0)$ (see (4.1)) and (4.2), we have

$$\int_0^\infty |z||K(s)|| \, ds = o(|z_0|^2). \quad (4.15)$$
For the case, $\sigma > 1$, the estimate is easier to obtain by applying the stronger condition $|z_0| \leq O(\delta_{0}^{C(\sigma)}) = O(\delta_{\infty}^{C(\sigma)})$ with $C(\sigma)$ being sufficiently large. This completes the proof. ■

4.2 Definition of $S_\lambda$ and its estimate

After expanding the dispersive part $\tilde{R}$ into the third order in $z$ and $\bar{z}$, we derive in Appendix C an equation for $(d/dt)\|\phi^\lambda\|_2^2$:

$$
\frac{d}{dt}\|\phi^\lambda\|_2^2 = -z^* \Gamma_0(z, \bar{z})z + S_\lambda
$$

with $S_\lambda$ defined as

$$
S_\lambda := 2\Psi + (2\Pi_{2,2} + z^* \Gamma_0(z, \bar{z})z + 2 \sum_{m+n=4,5; m\neq n} \Pi_{m,n}
$$

Here, $\sum_{m+n=4,5} \Pi_{m,n}$ is a collection of fourth- and fifth-order terms

$$
\sum_{m+n=4,5} \Pi_{m,n} := -\left( \sum_{m+n=4} JN_{m,n} \left( \begin{matrix} \phi^\lambda \\ 0 \end{matrix} \right) \right) + \gamma_{1,1} \left( \sum_{m+n=2,3} R_{m,n} \left( \begin{matrix} 0 \\ \phi^\lambda \end{matrix} \right) \right) + \gamma_{1,1} \langle q \cdot \eta, \phi^\lambda \rangle
$$

and $Z_{2,1} := -\Gamma(z, \bar{z})z + \Lambda(z, \bar{z})z$ with the latter defined in (2.9); and $\Psi$ is defined as

$$
\Psi := (\dot{\gamma} - \gamma_{1,1}) \left[ \left( \begin{matrix} \dot{R} \\ 0 \end{matrix} \right) \right] + (\beta + q) \cdot \eta, \phi^\lambda \right] + \gamma_{1,1} \left( R_{-4,4} \left( \begin{matrix} 0 \\ \phi^\lambda \end{matrix} \right) \right) + \dot{\lambda} \left( \begin{matrix} \partial_\lambda \phi^\lambda \\ 0 \end{matrix} \right) - \dot{\lambda} A_1 (\dot{\lambda}^2 \phi^\lambda, \phi^\lambda) - \dot{\lambda} (\alpha + p) (\partial_\lambda \xi, \phi^\lambda)
$$

and

$$
+ \langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle \left[ \partial_\lambda a_1 + iE(\lambda) \sum_{m+n=2,3} (m-n) A_{m,n}^{(1)} - \partial_\lambda a_1 \cdot Z_{2,1} - \partial_\lambda a_1 \cdot \bar{Z}_{2,1} \right]
$$

Here, we used the convention made in (B.1) and the definitions of Appendix B.

To control these terms in $S_\lambda$ we use the following results: (Recall $\delta_{\infty} = \|\phi^{\lambda,\infty}\|_2$, defined in (4.1).)
Lemma 4.2.

\[ |\Psi| \lesssim |z|^{2\sigma - 1} \|x\|^{-4} R_{\geq 4} + \|x\|^{-4} R_{\geq 4}^{2} + \delta^{2\sigma - 1} |z|^5, \]  
(4.17)

\[ 2 \Pi_{2,2} + z^a \Gamma_0(z, \bar{z}) z = O(\delta^{4\sigma - 1} |z|^4), \]  
(4.18)

\[ \sum_{m+n=4,5, m \neq n} \int_0^\infty \Pi_{m,n}(s) \, ds \lesssim \sum_{m+n=4,5, m \neq n} \int_0^\infty |\partial_{\lambda} \Pi_{m,n}| |\dot{\lambda}|(s) + |\partial_z \Pi_{m,n}| |\dot{z}| + i E(\lambda) |z|(s) \, ds \]
\[ + \, o(|z_0|^2). \]  
(4.19)

The bound (4.17) will be proved in Appendix F, (4.19) in Section H and (4.18) in Section I. We now briefly present the ideas in the proof.

1. \( \Psi \) is defined in term of functions \( \dot{\lambda}, \dot{\gamma}, z, \) and \( \bar{R} \). They satisfy a coupled system. This system must be put in matrix form and decoupled. In the end, we bound the functions \( \dot{\lambda} \) and \( \dot{\gamma} \) by the functions of \( \bar{R} \) (or \( R_{\geq 4} \)) and \( z \).

2. All the integrands in (4.18) are of order \( |z|^4 \) in \( z \) and \( \bar{z} \). What makes the terms different is the sizes of the coefficients. These depend smoothly on the functions \( \phi^\lambda, \partial_{\lambda} \phi^\lambda, \xi, \) and \( \eta \), which in turn depend smoothly on the small parameter \( \delta(\lambda) = O(\delta_{\infty}) \); see Proposition D.1. The estimate (4.18) follows from a perturbation expansion in the parameter \( \delta(\lambda) \).

3. For (4.19) the important observation is that, if \( m \neq n \), then function \( \Pi_{m,n} \) is a sum of the functions of the form \( C(\lambda) z^m \bar{z}^n = C(\lambda) \prod_k z_k^{m_k} \prod_l \bar{z}_l^{n_l} \) with \( m = \sum_k m_k, \, n = \sum_l n_l \). These are “almost periodic” with period \( 2\pi (E(\lambda)(m - n))^{-1} \neq 0 \) since \( z \) satisfies the equation \( \dot{z} = -i E(\lambda) z + \cdots \). This nontrivial oscillation enables us to integrate by parts in the variable \( s \) to derive smallness. The term \( o(|z_0|^2) \) in (4.19) is due to a boundary term obtained in this way.

Based on the estimates in Lemma 4.2 we will prove the following.

**Theorem 4.2.** \( S_\lambda \) satisfying the estimate in (3.2), that is,

\[ \int_0^\infty S_\lambda(s) \, ds = o(|z_0|^2). \]  
(4.20)

**Proof.** The result follows directly from Lemma 4.2 and the following two estimates:
We next prove estimates (4.20) and (4.21). In the proof we consider the case $\sigma = 1$. That of $\sigma > 1$ is different, but easier due to the stronger condition $|z_0| \leq \delta_\infty^{C(\sigma)}$ for some sufficiently large $C(\sigma)$, and hence omit the details.

We start with (4.20), by estimating three different terms in the estimate of $\Psi$ in (4.17) on the right-hand side. By applying the estimates for $z$ in (4.2)
\[ \int_0^\infty \delta_\infty |z|^5(s) \, ds \lesssim \int_0^\infty \delta_\infty (|z_0|^{-2} + \delta_\infty^2 s)^{-5/2} \, ds = \frac{2}{3} \delta_\infty^{-1}|z_0|^3 = o(|z_0|^2), \] (4.22)
where the assumption on the initial condition $|z_0| \ll \delta_0 = O(\delta_\infty)$ was used.

By the estimate of $R_{\geq 4}$ in (4.3) and $|z(t)|$ in (4.2)
\[ \int_0^\infty \delta_\infty |z(s)||\langle x \rangle^{-4} R_{\geq 4}(s)||^2_2 \, ds \lesssim \delta_\infty^2 |z_0|^2 \int_0^\infty (1 + s)^{-3/2}(|z_0|^2 + \delta_\infty^2 s)^{-1/2} \, ds + \delta_\infty^2 |z_0|^2 \int_0^\infty (|z_0|^2 + \delta_\infty^2 s)^{-3/2} \, ds \]
\[ = o(|z_0|^2). \]
The third term can be similarly estimated. Assembling the above estimates yields
\[ \int_0^\infty |\Psi(t)| \, dt = o(|z_0|^2). \]

To prove (4.21), we use the equations for $\dot{z}$ and $\dot{\lambda}$ in (4.7) and (4.4) to find that if $m + n = 4, 5$ and $m \neq n$, then
\[ |\partial_z \Pi_{m,n}||\dot{\lambda}(s)| + |\partial_{\lambda} \Pi_{m,n}||\dot{z} + iE(\lambda)z(s)| \lesssim |z||\dot{\lambda}| + |z|^6 + |z|^3 |K|. \]
Using the estimates in (4.10) and (4.11) for $K$, and the estimate (4.4) and similar techniques above we prove (4.21). This is straightforward, but tedious, hence we omit the details. ■

Remark 4.1. In the last step of (4.22) we used $|z_0| \ll \delta_\infty$ to control $\int_0^\infty \delta_\infty |z|^5(s) \, ds$. If $\sigma = 1$, then this can be relaxed to $|z_0| \leq \|\phi_k\|_\infty = O(\delta_\infty)$ by inspecting closely the terms forming $\delta_\infty|z|^5$. The term actually is a part of $\{JN_{\geq 5}, \phi_i\}$, and can be written as $\sum_{m+n=5} K_{m,n}$ for some properly defined $K_{m,n}$. To evaluate $\sum_{m+n=5} \int_0^\infty K_{m,n}(s) \, ds$, we
observe that $K_{m,n}, m+n=5$, are “almost periodic” as $\Lambda_{m,n}$ of (4.19). Hence, by integrating by parts as in the proof of (4.19) it is easy to obtain the desired estimate

$$\sum_{m+n=5} \int_0^\infty K_{m,n}(s) \, ds = o(|z_0|^2).$$

Note that the terms $K_{m,n}, m+n=5$, may not be well defined if $\sigma \notin \mathbb{N}$. □

5 Extension to the case of nearly degenerate neutral modes

In [6], and the main part of the present paper, we have proved that if the neutral modes are degenerate and their eigenvalues are sufficiently close to the essential spectrum, then the ground state is asymptotically stable and its mass will grow by half of that of the neutral modes.

In what follows, we extend the results to the cases where the neutral modes are nearly degenerate, that is, a cluster of approximately equal eigenfrequencies. For technical simplicity, we consider the case of cubic nonlinearity, $\sigma = 1$. The main result is Theorem 5.1. The key ideas of the proof will be presented after its statement.

5.1 New assumptions on the spectrum and definition of FGR

As in Section 2.1, we assume that the linear operator $-\Delta + V$ has the following properties:

(V1) $V$ is real valued and decays sufficiently rapidly, for example, exponentially, as $|x|$ tends to infinity.

(V2) The linear operator $-\Delta + V$ has $N + 1$ (counting multiplicity if degenerate) eigenvalues $e_0, e_k, k = 1, 2, \ldots, N$, with $e_0 < e_k$.

$e_0$ is the lowest eigenvalue with ground state $\phi_{\text{lin}} > 0$, the eigenvalues $\{e_k\}_{k=1}^N$ are possibly degenerate with eigenvectors $\xi_{1}^{\text{lin}}, \xi_{2}^{\text{lin}}, \ldots, \xi_{N}^{\text{lin}}$.

(V3) Moreover, for any $k = 1, 2, \ldots, N$ we assume

$$2e_k - e_0 > 0. \quad (5.1)$$

Then the nonlinear equation (1.1) admits a family of ground states solution $e^{it\lambda} \phi^\lambda$ with properties as described in Proposition 2.1. The linearized operators about the ground states, $L(\lambda)$, takes the same form as in (2.3). The excited states of $-\Delta + V$ bifurcate to the neutral modes $\left( \xi_k \pm i\eta_k \right)$ of $L(\lambda)$ with eigenvalues $\pm iE_k(\lambda), k = 1, \ldots, N$. The ground states $\phi^\lambda$ and neutral modes satisfy all the estimates in Lemma 2.1 and the estimates (D.1)–(D.4).
Assumption (SA) on the spectrum of $L(\lambda)$ is generalized, in the case where near-degeneracy is admitted, as

(SA) The discrete spectrum of the linearized operator $L(\lambda)$ consists of the eigenvalue 0 with generalized eigenvectors $\left( \begin{array}{c} 0 \\ \phi_\lambda \end{array} \right)$, $\left( \begin{array}{c} \partial_\lambda \phi_\lambda \\ 0 \end{array} \right)$, and eigenvalues $\pm i E_k(\lambda)$, $E_k(\lambda) > 0$, $k = 1, 2, \ldots, N$.

A consequence of nonzero neutral mode frequency differences is a slightly different system for the neutral mode amplitudes, $z(t)$. The solution $\psi(t)$ is decomposed as in (A.1). Following the same procedure as in [6], we derive

$$\partial_t z = -i E(\lambda) z - \Gamma(z, \bar{z}) z + \Lambda(z, \bar{z}) z + \cdots,$$

(5.2)

where $E(\lambda) = \text{Diag}[E_1(\lambda), \ldots, E_N(\lambda)]$ is a diagonal $N \times N$ matrix, $\Gamma$ is symmetric and $\Lambda$ is skew symmetric.

We now describe the matrix $\Gamma$, which takes a different form from the degenerate case: Define vector functions $G(k, m)$, $k, m = 1, 2, \ldots, N$, as

$$G(k, m) := \begin{pmatrix} B(k, m) \\ D(k, m) \end{pmatrix}$$

(5.3)

with the functions $B(k, m)$ and $D(k, m)$ defined as

$$B(k, m) := -i \phi^I_{\lambda} [z_m \xi_k + z_k \eta_m \xi],$$

$$D(k, m) := -\phi^I_{\lambda} [3z_m \xi_k - z_m \eta_m \eta].$$

In terms of the column 2-vector, $G(k, m)$, we define a $N \times N$ matrix $Z(z, \bar{z})$ as

$$Z(z, \bar{z}) = (Z^{(kl)}(z, \bar{z})), \quad 1 \leq k, l \leq N$$

(5.4)

and

$$Z^{(kl)} = -\left( \sum_{m=1}^N (L(\lambda) + iE_l(\lambda) + iE_m(\lambda) - 0)^{-1} P_c G(l, m), iJ P_c \sum_{m=1}^N G(k, m) \right).$$

(5.5)

Finally, we define $\Gamma(z, \bar{z})$ as follows:

$$\Gamma(z, \bar{z}) := \frac{1}{2} [Z(z, \bar{z}) + Z^*(z, \bar{z})].$$

(5.6)

We shall require the following Fermi Golden Rule hypothesis: Let $P_c^{\text{lin}}$ be the projection onto the essential spectrum of $-\Delta + V$ then
We assume that there exists a constant $C > 0$ such that
\[
-\text{Re} \left( i[-\Delta + V + \lambda - 2E_1(\lambda) - i0]^{-1} P_c^\text{lin} \phi_{\text{lin}}(z \cdot \xi_{\text{lin}})^2, \phi_{\text{lin}}(z \cdot \xi_{\text{lin}})^2 \right) \geq C |z|^4
\]
for any $z \in \mathbb{C}^N$.

The assumption FGR implies that there exist constants $C_1 > 0$ and $\delta_0 > 0$ such that if \( \sup_{k,l} |E_k(\lambda) - E_l(\lambda)| \leq \delta_0 \), then for any $z \in \mathbb{C}^N$
\[
z^* \Gamma(z, \bar{z}) z \geq C_1 \| \phi^\lambda \|_\infty^2 |z|^4.
\] (5.7)

We now introduce the leading order contribution to $\Gamma(z, \bar{z})$. For each fixed $z$, we use the fact of $|\lambda + e_0|$ being small and use (2.1) and (2.6) to find the leading term in $z^* \Gamma_0(z, \bar{z}) z$ defined as
\[
z^* \Gamma_0(z, \bar{z}) z = -8\delta^2(\lambda) N \left( \sum_{m,n \leq N} \left( i \sum_{m,n \leq N} \right. \right.
\left. \left. \left[ -\Delta + V + \lambda - E_m(\lambda) - E_n - i0 \right]^{-1} \times P_c^\text{lin} \phi_{\text{lin}}(z_m \xi_{\text{lin}})(z_n \xi_{\text{lin}}), \phi_{\text{lin}}(z \cdot \xi_{\text{lin}})^2 \right) \right).
\] (5.8)

5.2 Main Theorem in nearly degenerate case and strategy of proof

Recall that we only consider the case $\sigma = 1$, that is, the cubic nonlinearity.

**Theorem 5.1.** There exists a constant $\delta_0$ independent of the initial condition $\psi_0$ of (1.1) such that if $\max_{k,l} |E_k(\lambda) - E_l(\lambda)| \leq \delta_0$ then all the results in Theorem 3.1 hold with $z^* \Gamma_0(z, \bar{z}) z$ replaced by the expression in (5.8). Moreover, all the remainder estimates in (3.2)–(3.10) hold and are independent of the size of $\delta_0$.

In the next we show how to recover all the estimates. To simplify the treatment we only consider the case $N = 2$ with eigenfrequencies $E_1(\lambda)$ and $E_2(\lambda)$.

There are some differences between the degenerate and the nearly degenerate cases. Among them, the most outstanding one are terms, which previously vanished identically, which now need to be estimated. These terms include, for example, \( \langle \text{Im} N_{1,1}, \phi^\lambda \rangle \), which was proved to be zero in [6, Lemma 9.4, p. 291] which we see below is nonzero in the nearly degenerate case. To treat such terms, the key observation is that these terms have a factor $E_1(\lambda) - E_2(\lambda)$ in their coefficient enabling us to re-express $[E_1(\lambda) - E_2(\lambda)] z_1 \bar{z}_2$ as $-i \frac{d}{dt} z_1 \bar{z}_2 + o(|z|^4)$. Thus, these terms can be removed via integration by parts and a redefinition of the normal form transformation.
5.3 Normal form transformation and asymptotic stability of ground states

We decompose the initial condition in exactly the same way as in (A.1). All equations (A.1)–(A.6), (A.9) and (A.10) hold. The equations for $\dot{z}$ are slightly different since $z_j$ each have different associated frequencies. Consequently, instead of (A.7) and (A.8) we have

$$
\partial_t(\alpha_n + p_n) - E_n(\lambda) (\beta_n + q_n) + \ldots, \quad \partial_t(\beta_n + q_n) + E_n(\lambda) (\alpha_n + p_n) + \ldots,
$$

requiring a different near-identity/normal form transformation.

To illustrate the main difference in the calculation we study the equation for $\dot{\lambda}$. Recall that the function $\dot{\lambda}$ satisfies the equation

$$
\dot{\lambda} + \partial_t a_1 = -\frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} (\text{Im} N(\tilde{R}, z), \phi^\lambda) + \ldots
$$

and we want to remove the second and third order terms in $z$ and $\tilde{z}$ from the equation by defining some polynomial $a_1$ in $z$ and $\bar{z}$:

$$
a_1 := \sum_{m+n=2,3} A_{m,n}^{(1)}.
$$

In the degenerate case, we set $A_{1,1}^{(1)} = 0$ (see (B.2)) due to the fact $\langle \text{Im} N_{1,1}, \phi^\lambda \rangle = 0$. When the latter no longer holds $A_{1,1}^{(1)}$ has to be redefined. Following steps in [6, p. 291], we use the fact $\langle \xi_n, i\eta_m \rangle, n = 1, 2$, are eigenvectors of $L(\lambda)$ to obtain

$$
\langle \text{Im} N_{1,1}, \phi^\lambda \rangle := \frac{1}{2i} \sum_{n=1}^{2} \sum_{m=1}^{2} \tilde{z}_n z_m \int (\phi^\lambda)^2 (\xi_n \eta_m - \xi_m \eta_n) = \frac{1}{4i} \sum_{n=1}^{2} \sum_{m=1}^{2} \tilde{z}_n z_m \langle (L_- - L_+) \xi_n, \eta_m \rangle - \langle (L_- - L_+) \xi_m, \eta_n \rangle = \frac{1}{4i} [E_1(\lambda) - E_2(\lambda)] |z_1 \bar{z}_2 - z_2 \bar{z}_1| \langle \eta_1, \eta_2 \rangle + \langle \xi_1, \xi_2 \rangle].
$$

To remove (5.9) from the equation of $\dot{\lambda}$ we define

$$
A_{1,1}^{(1)} := -\frac{1}{4\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} [z_1 \bar{z}_2 + z_2 \bar{z}_1] \langle \eta_1, \eta_2 \rangle + \langle \xi_1, \xi_2 \rangle = O(\|\phi^\lambda\|^2 |z|^2), \quad (5.10)
$$

where in the last step the estimate (D.4) and the fact $\xi_1^\text{lin} \perp \xi_2^\text{lin}$ are used.

For the other terms in $a_1$, we only re-define $A_{2,0}^{(1)}$ to illustrate the differences: Decompose $(1/\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle) \langle \text{Im} N_{2,0}, \phi^\lambda \rangle$ as $K_1 z_1^2 + K_2 z_1 z_2 + K_3 z_2^2$, then instead of the
definition in (B.3) we define
\[ A^{(1)}_{2,0} = -\frac{i}{2E_1(\lambda)} K_1 z_1^2 - \frac{i}{E_1(\lambda) + E_2(\lambda)} K_2 z_1 z_2 - \frac{i}{2E_2(\lambda)} K_2 z_2^2. \]

The new normal forms enable the proof of asymptotic stability of the ground states to go through, as well as all results in Section D, that is, all the statements in Theorem 3.1 except (A) and (D), which we discuss in the next subsection.

5.4 Equipartition of Energy

In this subsection, we recover the Statements (A) and (D). Most of the arguments proved in the degenerate regime still hold. As presented above certain newly nonzero terms enter different places. In what follows, we present the strategy to handle such terms.

To illustrate the idea we only study one term whose counterpart is \( H_{2,2} \) in (G.7)
\[
D := \sum_{m+n=2} D(m, n, m', n')
\]
with \( D(m, n, m', n') \) being a real function:
\[
D(m, n, m', n') := \Re(i(-\Delta + V + \lambda + mE_1(\lambda) + nE_2(\lambda))^{-1} P_{\text{lin}} \phi^\lambda(z_1 \xi_1)^m(z_2 \xi_2)^n, \phi^\lambda(z_1 \xi_1)^{m'}(z_2 \xi_2)^{n'}). \]

If \( E_1(\lambda) = E_2(\lambda) \) then we use the observation in (G.7) to prove
\[
D(m, n, m', n') = \overline{D(m, n, m', n')} = -D(m', n', m, n) \text{ implies } D = 0.
\]

When \( E_1(\lambda) \neq E_2(\lambda) \), we use the following result to recover the desired estimate.

Lemma 5.1.
\[
\int_0^\infty D(s) \, ds = o(|z_0|^2). \tag{5.11}
\]

Proof. The facts that \( D(m', n', m, n) \) is real and \((-\Delta + V + \lambda + m'E_1(\lambda) + n'E_2(\lambda))^{-1}\) is self-adjoint imply
\[
D(m', n', m, n) = \overline{D(m', n', m, n)} = -\Re(i(-\Delta + V + \lambda + m'E_1(\lambda) + n'E_2(\lambda))^{-1} P_{\text{lin}} \phi^\lambda(z_1 \xi_1)^m(z_2 \xi_2)^n, \phi^\lambda(z_1 \xi_1)^{m'}(z_2 \xi_2)^{n'}). \]
The crucial step is to find the presence of $E_1(\lambda) - E_2(\lambda)$ in the coefficient:

$$D(m, n, m', n') + D(m', n', m, n) = -[(m - m')E_1(\lambda) + (n - n')E_2(\lambda)]\text{Re } H$$

$$= -[m - m'][E_1(\lambda) - E_2(\lambda)]\text{Re } H,$$

where $H$ is defined as

$$H := \langle i[-\Delta + V + \lambda + m'E_1(\lambda) + n'E_2(\lambda)]^{-1}[-\Delta + V + \lambda + mE_1(\lambda) + nE_2(\lambda)]^{-1}$$

$$\times P_c^{\text{lin}}\phi^+(z_1\xi_1)^m(z_2\xi_2)^n, \phi^+(z_1\xi_1)^{m'}(z_2\xi_2)^{n'} \rangle,$$

and in the last step the fact $m + n = m' + n' = 2$ is used.

Equation (5.12) enables us to use the same trick as in (4.19), namely integration by parts, to obtain the desired estimate

$$\int_0^\infty D(m, n, m', n') + D(m', n', m, n) \, ds$$

$$= \int_0^\infty \frac{d}{ds} \text{Re}(iH) \, ds + \int_0^\infty O(|\dot{\lambda}| |z|^4 + \|\phi^+\|^2|z|^6) \, ds + o(|z_0|^2).$$

The proof is complete.

In summary, as outlined above, all the estimates obtained in the degenerate can be proved in the nearly degenerate case.

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**A Decomposition of the Solution $\psi$**

This section is based upon [6, pp. 286–287]. As stated in (3.1), for any time $t$ the solution $\psi(x, t)$ can be decomposed as
\[
\psi(x, t) = e^{i\gamma(t)}e^{i\int_0^t \lambda(s) \, ds} \phi^{(t)}(x) + a_1(t) \partial_x \phi^{(t)}(x) + i a_2(t) \phi^{(t)}(x) + i \sum_{n=1}^N (\alpha_n(t) + p_n(t)) |\xi_n^{(t)}(x) + i \sum_{n=1}^N (\beta_n(t) + q_n(t)) |\eta_n^{(t)}(x) + R(x, t) \]

(A.1)

for some polynomials \(a_1, a_2, p_n,\) and \(q_n\) (will be defined explicitly in Appendix B) and the function \(R\) satisfies the symplectic orthogonality conditions (3.6). By this

\[
\mathbf{R} := \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{pmatrix} := \begin{pmatrix} \text{Re} \mathbf{R} \\ \text{Im} \mathbf{R} \end{pmatrix} \in \mathcal{P}_c(\lambda) L^2
\]

satisfies the equation

\[
\frac{d}{dt} \mathbf{R} = L(\lambda(t)) \mathbf{R} - P_c^{\lambda(t)} J \tilde{N}(\mathbf{R}, z) + L_{(\dot{\lambda}, \dot{\gamma})} \mathbf{R} + \mathcal{G}.
\]

(A.2)

Here,

\[
J \tilde{N}(\mathbf{R}, z) := \begin{pmatrix} \text{Im} N(\mathbf{R}, z) \\ -\text{Re} N(\mathbf{R}, z) \end{pmatrix},
\]

(A.3)

\[
\text{Im} N(\mathbf{R}, z) := |\phi^\lambda + I_1 + I_2|^{2\sigma} I_2 - (\phi^\lambda)^{2\sigma} I_2,
\]

\[
\text{Re} N(\mathbf{R}, z) := [(\phi^\lambda + I_1 + I_2)^{2\sigma} - (\phi^\lambda)^{2\sigma}] (\phi^\lambda + I_1) - 2\sigma (\phi^\lambda)^{2\sigma} I_1,
\]

\[
I_1 := \alpha \cdot \xi + a_1 \partial_x \phi^\lambda + p \cdot \xi + R_1,
\]

\[
I_2 := \beta \cdot \eta + a_2 \phi^\lambda + q \cdot \eta + R_2.
\]

The operator \(L_{(\dot{\lambda}, \dot{\gamma})}\) and the vector function \(\mathcal{G}\) are defined as

\[
L_{(\dot{\lambda}, \dot{\gamma})} := \dot{\lambda}(\partial_x P_c^{\lambda(t)}) + \dot{\gamma} P_c^{\lambda(t)} J,
\]

(A.4)

\[
\mathcal{G} := P_c^{\lambda(t)} \left[ \begin{array}{c}
\dot{\gamma} - \gamma_{1,1} (\beta + q) \cdot \eta - \dot{\lambda} a_1 \partial_x \phi^\lambda - \dot{\lambda} (\alpha + p) \cdot \partial_x \xi \\
-\dot{\lambda} a_2 \phi^\lambda - \dot{\lambda} (\beta + q) \cdot \partial_x \eta
\end{array} \right] + \gamma_{1,1} P_c^{\lambda(t)} \left( \begin{array}{c}
\beta + q \cdot \eta \\
-\alpha + p \cdot \xi
\end{array} \right),
\]

(A.5)

where \(\gamma_{1,1}\) is defined as

\[
\gamma_{1,1} := \frac{\langle (\phi^\lambda)^{2\sigma - 1} [(2\sigma^2 + \sigma)|z \cdot \xi|^2 + \sigma |z \cdot \eta|^2], \partial_x \phi^\lambda \rangle}{2 \langle \phi^\lambda, \partial_x \phi^\lambda \rangle}
\]

(A.6)

with

\[
z := (z_1, \ldots, z_N)^T, \quad \alpha_n := \alpha_n + i \beta_n, \quad n = 1, \ldots, N,
\]

and

\[
\xi := (\xi_1, \ldots, \xi_N)^T, \quad \eta := (\eta_1, \ldots, \eta_N)^T.
\]
By the orthogonality conditions (3.6) and (2.5), we derive equations for \( \dot{\lambda}, \dot{\eta}, \) and 
\[ z_n = \alpha_n + i \beta_n, \quad n = 1, \ldots, N, \] as

\[
\begin{align*}
\partial_t (\alpha_n + p_n) - E(\lambda) (\beta_n + q_n) + (\text{Im} N(\tilde{R}, z), \eta_n) &= F_{1n}, \\
\partial_t (\beta_n + q_n) + E(\lambda) (\alpha_n + p_n) - (\text{Re} N(\tilde{R}, z), \xi_n) &= F_{2n}, \\
\dot{\gamma} + \partial_t a_2 - a_1 - \frac{1}{\langle \phi^\lambda, \phi^\lambda \rangle} (\text{Re} N(\tilde{R}, z), \partial_\alpha \phi^\lambda) &= F_3, \\
\dot{\lambda} + \partial_t a_1 + \frac{1}{\langle \phi^\lambda, \phi^\lambda \rangle} (\text{Im} N(\tilde{R}, z), \phi^\lambda) &= F_4,
\end{align*}
\] (A.7) (A.8) (A.9) (A.10)

where the scalar functions \( F_{j,n}, j = 1, 2, n = 1, 2, \ldots, N, F_3, F_4, \) are defined as

\[
\begin{align*}
F_{1n} &= \dot{\gamma} \langle (\beta + q) \cdot \eta, \eta_n \rangle - \dot{\lambda} a_1 \langle \partial_\alpha^2 \phi^\lambda, \eta_n \rangle - \dot{\lambda} \langle (\alpha + p) \cdot \partial_\beta \xi, \eta_n \rangle - \dot{\gamma} \langle R_2, \eta_n \rangle + \dot{\lambda} \langle R_1, \partial_\gamma \eta_n \rangle, \\
F_{2n} &= -\dot{\gamma} \langle (\alpha + p) \cdot \xi, \xi_n \rangle - \dot{\lambda} a_2 \langle \phi^\lambda, \xi_n \rangle - \dot{\lambda} \langle (\beta + q) \cdot \partial_\beta \eta, \xi_n \rangle + \dot{\gamma} \langle R_1, \xi_n \rangle + \dot{\lambda} \langle R_2, \partial_\beta \xi_n \rangle, \\
F_3 &= \frac{1}{\langle \phi^\lambda, \phi^\lambda \rangle} [\dot{\lambda} \langle R_2, \phi^\lambda \rangle - \dot{\gamma} \langle R_1, \phi^\lambda \rangle - \langle \gamma (\alpha + p) \cdot \xi + \lambda a_2 \phi^\lambda, \partial_\beta (\beta + q) \cdot \partial_\beta \eta, \phi^\lambda \rangle], \\
F_4 &= \frac{1}{\langle \phi^\lambda, \phi^\lambda \rangle} [\dot{\lambda} \langle R_1, \phi^\lambda \rangle + \dot{\gamma} \langle R_2, \phi^\lambda \rangle + \langle \gamma (\beta + q) \cdot \eta - \dot{\lambda} a_1 \partial_\alpha \phi^\lambda - \dot{\lambda} (\alpha + p) \cdot \partial_\beta \xi, \phi^\lambda \rangle].
\end{align*}
\] (A.11)

B The Normal Form Expansion

All the results in this appendix, except the definitions of \( R_{m,n}, JN_{m,n}, m + n = 3, \) are taken from [6]. Specifically, the definitions of \( a_1, a_2, p_k, q_k, \; k = 1, 2, \ldots, N, \) are taken from [6, (9-12) and (9-13), p. 288]; the definitions of \( R_{m,n}, JN_{m,n}, m + n = 2, \) from [6, (9-18)–(9-21), p. 290].

Before defining various functions, we introduce the following convention on notation: we always use \( z \) to stand for a complex \( N \)-dimensional vector \( z = (z_1, z_2, \ldots, z_N) \) and an upper case letter or a Greek letter with two subindices, for example, \( Q_{m,n} \) to represent

\[
Q_{m,n}(\lambda) = \sum_{|a|=m, |b|=n} q_{a,b}(\lambda) \prod_{k=1}^{N} z_k^{a_k} \bar{z}_k^{b_k},
\] (B.1)

where \( a, b \in \mathbb{N}^N, |a| := \sum_{k=1}^{N} a_k. \) We refer to this kind of term as \( (m, n) \) term.

In what follows, we define \( R_{m,n}, m + n = 2, 3, JN_{m',n}, m' + n' = 2, 3, 4, \) and the polynomials \( a_1, a_2, p_k, q_k, \; k = 1, 2, \ldots, N, \) by induction.
Definitions of Polynomials $a_1, a_2, p_k, q_k, k = 1, 2, \ldots, N$.

We define the polynomials $a_1, a_2, p_k, q_k, k = 1, 2, \ldots, N$, in (A.1) as

$$a_k(z, \bar{z}) := \sum_{m+n=2,3, m \neq n} A_{m,n}^{(k)}(\lambda), \quad k = 1, 2,$$

$$p_k(z, \bar{z}) := \sum_{m+n=2,3} P_{m,n}^{(k)}(\lambda), \quad k = 1, 2, \ldots, N,$$

$$q_k(z, \bar{z}) := \sum_{m+n=2,3} Q_{m,n}^{(k)}(\lambda), \quad k = 1, 2, \ldots, N,$$

where the terms on the right-hand side take the form:

$$2iE(\lambda)A_{2,0}^{(1)} := \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \langle N_{2,0}^{\text{Im}}, \phi^\lambda \rangle,$$

$$3iE(\lambda)A_{3,0}^{(1)} := \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \langle N_{3,0}^{\text{Im}}, \phi^\lambda \rangle,$$

$$iE(\lambda)A_{2,1}^{(1)} := \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \left[ \langle N_{2,1}^{\text{Im}}, \phi^\lambda \rangle - \frac{i}{2} \gamma_{1,1} \langle \xi \cdot \xi, \phi^\lambda \rangle \right],$$

$$-2iE(\lambda)A_{2,0}^{(2)} - A_{2,0}^{(1)} := \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \langle N_{2,0}^{\text{Re}}, \partial_\lambda \phi^\lambda \rangle,$$

$$-3iE(\lambda)A_{3,0}^{(2)} - A_{3,0}^{(1)} := \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \langle N_{3,0}^{\text{Re}}, \partial_\lambda \phi^\lambda \rangle,$$

$$-iE(\lambda)A_{2,1}^{(2)} - A_{2,1}^{(1)} := \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \left[ \langle N_{2,1}^{\text{Re}}, \partial_\lambda \phi^\lambda \rangle - \frac{1}{2} \gamma_{1,1} \langle \xi \cdot \xi, \partial_\lambda \phi^\lambda \rangle \right],$$

$$A_{k,l}^{(n)} := \overline{A_{l,k}^{(n)}} \quad \text{for } n = 1, 2, \quad k + l = 2, 3, \quad k \neq l$$

and

$$-2iE(\lambda)P_{2,0}^{(n)} - E(\lambda)Q_{2,0}^{(n)} := -\langle N_{2,0}^{\text{Im}}, \eta_n \rangle,$$

$$-2iE(\lambda)Q_{2,0}^{(n)} + E(\lambda)P_{2,0}^{(n)} := \langle N_{2,0}^{\text{Re}}, \xi_n \rangle,$$

$$-3iE(\lambda)P_{3,0}^{(n)} - E(\lambda)Q_{3,0}^{(n)} := -\langle N_{3,0}^{\text{Im}}, \eta_n \rangle,$$

$$-3iE(\lambda)Q_{3,0}^{(n)} + E(\lambda)P_{3,0}^{(n)} := \langle N_{3,0}^{\text{Re}}, \xi_n \rangle,$$

$$2iE(\lambda)P_{1,2}^{(n)} - 2E(\lambda)Q_{1,2}^{(n)} := -\langle N_{1,2}^{\text{Im}}, \eta_n \rangle + i\langle N_{1,2}^{\text{Re}}, \xi_n \rangle,$$

$$+ i \gamma_{1,1} \sum_{k=1}^d \bar{z}_k (\langle \eta_k, \eta_n \rangle - \langle \xi_k, \xi_n \rangle),$$

$$-E(\lambda)Q_{1,1}^{(n)} := -\langle N_{1,1}^{\text{Im}}, \eta_n \rangle,$$

$$E(\lambda)P_{1,1}^{(n)} := \langle N_{1,1}^{\text{Re}}, \xi_n \rangle,$$

$$P_{k,l}^{(n)} := \overline{P_{l,k}^{(n)}}, \quad Q_{l,k}^{(n)} := \overline{Q_{k,l}^{(n)}}.$$

The functions $J N_{m,n} = \left( \begin{array}{cc} N_{m,n}^{\text{Im}} & N_{m,n}^{\text{Re}} \\ -N_{m,n}^{\text{Re}} & N_{m,n}^{\text{Im}} \end{array} \right)$ used above will be defined in the next subsection.
Expansion of $J \tilde{N}$ and $\tilde{R}$. For $m + n = 2$, we define

$$R_{m,n} := \begin{pmatrix} R^{(1)}_{m,n} \\ R^{(2)}_{m,n} \end{pmatrix} := [L(\lambda) + iE(\lambda)(m - n) - 0]^{-1} P_c J N_{m,n}. \quad (B.7)$$

We denote the remainder of the second-order expansion by $\tilde{R}$, that is,

$$\tilde{R} = R - \sum_{m+n=2} R_{m,n}. \quad (B.8)$$

For $m + n = 3$, we define

$$R_{m,n} := [L(\lambda) + iE(\lambda)(m - n) - 0]^{-1} P_c [J N_{m,n} + X_{m,n}], \quad (B.9)$$

where $\sum_{m+n=3} X_{m,n} := \gamma_{1,1} \left( -\frac{\beta \cdot \eta}{\alpha \cdot \xi} \right)$ and recall the definition of $\gamma_{1,1}$ in (A.6).

We define the quadratic terms $J N_{m,n}$, $m + n = 2$, as

$$\sum_{m+n=2} J N_{m,n} = \sigma (\phi^{\lambda})^{2\sigma - 1} \begin{pmatrix} 2 (\alpha \cdot \xi)(\beta \cdot \eta) \\ -[2\sigma + 1](\alpha \cdot \xi)^2 - (\beta \cdot \eta)^2 \end{pmatrix} \quad (B.10)$$

and define $J N_{m,n}$, $m + n = 3$, by

$$\sum_{m+n=3} J N_{m,n} := \sum_{m+n=3} \begin{pmatrix} N^{\text{Im}}_{m,n} \\ -N^{\text{Re}}_{m,n} \end{pmatrix}, \quad (B.11)$$

where $N^{\text{Im}}_{m,n}$ and $N^{\text{Re}}_{m,n}$ are defined as

$$\sum_{m+n=3} N^{\text{Im}}_{m,n} := 2\sigma (\phi^{\lambda})^{2\sigma - 1}(\alpha \cdot \xi) \sum_{m+n=2} [A^{(2)}_{m,n} \phi^{\lambda} + Q_{m,n} \cdot \eta + R^{(2)}_{m,n}]$$

$$+ 2\sigma (\phi^{\lambda})^{2\sigma - 1}(\beta \cdot \eta) \sum_{m+n=2} [A^{(1)}_{m,n} \phi^{\lambda} + P_{m,n} \cdot \xi + R^{(1)}_{m,n}]$$

$$+ \sigma (\phi^{\lambda})^{2\sigma - 2}[(\alpha \cdot \xi)^2 + (\beta \cdot \eta)^2][\beta \cdot \eta] + 2\sigma (\phi^{\lambda})^{2\sigma - 2}(\alpha \cdot \xi)^2(\beta \cdot \eta)$$

and

$$\sum_{m+n=3} N^{\text{Re}}_{m,n} := 2\sigma (2\sigma + 1)(\phi^{\lambda})^{2\sigma - 1}(\alpha \cdot \xi) \sum_{m+n=2} [A^{(1)}_{m,n} \phi^{\lambda} + P_{m,n} \cdot \xi + R^{(1)}_{m,n}]$$

$$+ 2\sigma (\phi^{\lambda})^{2\sigma - 1}(\beta \cdot \eta) \sum_{m+n=2} [A^{(2)}_{m,n} \phi^{\lambda} + Q_{m,n} \cdot \eta + R^{(2)}_{m,n}]$$

$$+ \sigma \left[ 2\sigma - 1 + \frac{4}{3}(\sigma - 1)(\sigma - 2) \right] (\phi^{\lambda})^{2\sigma - 2}(\alpha \cdot \xi)^3$$

$$+ \sigma (2\sigma - 1)(\phi^{\lambda})^{2\sigma - 2}(\alpha \cdot \xi)(\beta \cdot \eta)^2.$$
Now we expand $J \tilde{N}$ to fourth order:

$$\sum_{m+n=4} JN_{m,n} := \sum_{m+n=4} \left( \frac{N_{m,n}^{\text{Im}}}{-N_{m,n}^{\text{Re}}} \right)$$  \hspace{1cm} (B.12)$$

with $N_{m,n}^{\text{Im}}$ and $N_{m,n}^{\text{Re}}$ being defined as

$$\sum_{m+n=4} N_{m,n}^{\text{Im}} = 2\sigma (\phi^\lambda)^{2\sigma-1} (\alpha \cdot \xi) \sum_{m+n=3} [A_{m,n}^{(2)} \phi^\lambda + Q_{m,n} \cdot \eta + R_{m,n}^{(2)}]$$

$$+ 2\sigma (\phi^\lambda)^{2\sigma-1} (\beta \cdot \eta) \sum_{m+n=3} [A_{m,n}^{(1)} \partial_k \phi^\lambda + P_{m,n} \cdot \xi + R_{m,n}^{(1)}]$$

$$+ 2\sigma (\phi^\lambda)^{2\sigma-1} \sum_{m+n=2} [A_{m,n}^{(1)} \partial_k \phi^\lambda + P_{m,n} \cdot \xi + R_{m,n}^{(1)}] \sum_{m+n=2} [A_{m,n}^{(2)} \phi^\lambda + Q_{m,n} \cdot \eta + R_{m,n}^{(2)}]$$

$$+ \sigma (\phi^\lambda)^{2\sigma-2} (2\sigma - 1)(\alpha \cdot \xi)^2 + 3(\beta \cdot \eta)^2 \sum_{m+n=2} [A_{m,n}^{(2)} \phi^\lambda + Q_{m,n} \cdot \eta + R_{m,n}^{(2)}]$$

$$+ 2\sigma (\sigma - 1)(\phi^\lambda)^{2\sigma-3} \left[ \frac{2\sigma - 1}{3} (\alpha \cdot \xi)^3 (\beta \cdot \eta) + (\alpha \cdot \xi)(\beta \cdot \eta)^3 \right]$$

and

$$\sum_{m+n=4} N_{m,n}^{\text{Re}} := 2\sigma (2\sigma + 1)(\phi^\lambda)^{2\sigma-1} (\alpha \cdot \xi) \sum_{m+n=3} [A_{m,n}^{(1)} \partial_k \phi^\lambda + P_{m,n} \cdot \xi + R_{m,n}^{(1)}]$$

$$+ 2\sigma (\phi^\lambda)^{2\sigma-1} (\beta \cdot \eta) \sum_{m+n=3} [A_{m,n}^{(2)} \phi^\lambda + Q_{m,n} \cdot \eta + R_{m,n}^{(2)}]$$

$$+ \sigma (2\sigma + 1)(\phi^\lambda)^{2\sigma-1} \left[ \sum_{m+n=3} (A_{m,n}^{(1)} \partial_k \phi^\lambda + P_{m,n} \cdot \xi + R_{m,n}^{(1)}) \right]^2$$

$$+ \sigma (\phi^\lambda)^{2\sigma-1} \left[ \sum_{m+n=2} (A_{m,n}^{(2)} \phi^\lambda + Q_{m,n} \cdot \eta + R_{m,n}^{(2)}) \right]^2$$

$$+ \sigma (\phi^\lambda)^{2\sigma-2} [3C_1(\sigma)(\alpha \cdot \xi)^2 + C_2(\sigma)(\beta \cdot \eta)^2] \sum_{m+n=2} [A_{m,n}^{(1)} \partial_k \phi^\lambda + P_{m,n} \cdot \xi + R_{m,n}^{(1)}]$$

$$+ 2\sigma (\phi^\lambda)^{2\sigma-2} C_3(\sigma)(\beta \cdot \eta)(\alpha \cdot \xi) \sum_{m+n=2} [A_{m,n}^{(2)} \phi^\lambda + Q_{m,n} \cdot \eta + R_{m,n}^{(2)}]$$

$$+ (\phi^\lambda)^{2\sigma-3} [C_4(\sigma)(\alpha \cdot \xi)^4 + C_5(\sigma)(\beta \cdot \eta)^4 + C_6(\sigma)(\alpha \cdot \xi)^2 (\beta \cdot \eta)^2],$$

where $C_k(\sigma), \ k = 1, 2, \ldots, 6$ are real constants, $C_k(1) = 1$ if $k = 1, 2, 3,$ and $C_l(1) = 0$ if $l = 4, 5, 6.$
Remark B.1. Note that if in (1.1) $\sigma > 1$ the function $JN_{m,n}$, $m+n = 4$ might NOT be well defined: in the last lines of definitions of $N_{m,n}^\text{Im}$ and $N_{m,n}^\text{Re}$ we have terms of the form $(\phi^\lambda)^{2\sigma-3} (\alpha \cdot \xi)^4$, where the power $2\sigma - 3$ of $\phi^\lambda$ might be negative. Still the definitions are useful because later we will take inner production with $JN_{m,n}$, $m+n = 4$ and $(\phi^0_\lambda)$, see (E.5).

To facilitate later discussions, we define

$$R_{\geq 4} := \bar{R} - \sum_{m+n=2,3} R_{m,n} \tag{B.13}$$

and

$$JN_{\geq 5} := J\bar{N} (\bar{R}, z) - \sum_{m+n=2}^4 JN_{m,n}. \tag{B.14}$$

C Derivation of Equation (4.16)

By the equation for $\dot{\lambda}(t)$ in (A.10) we derive the following modulation equation:

$$\frac{1}{2} \frac{d}{dt} \|\phi^{\lambda(t)}\|_2^2 = \langle \phi^\lambda, \phi^{\lambda(t)} \rangle \dot{\lambda} = - \langle \text{Im} \, N(\bar{R}, z), \phi^\lambda \rangle + \langle \phi^\lambda, \phi^\lambda \rangle F_4 - \langle \phi^\lambda, \phi^\lambda \rangle \partial_1 a_1. \tag{C.1}$$

To see the increasing of the mass on the ground state, or $\|\phi^{\lambda(t)}\|_2^2$ we resort to expand the terms on the right-hand side to fourth order in $z$ and $\bar{z}$:

1. The definitions of $a_1$ in (B.2) and (B.3) imply

$$\partial_t a_1 = \partial_t a_1 + iE(\lambda) \sum_{m+n=2,3} (m-n) A_{m,n}^{(1)}$$

$$- \frac{1}{\langle \phi^\lambda, \partial_\xi \phi^\lambda \rangle} \left[ \sum_{m+n=2,3} JN_{m,n} \left( \begin{array}{c} \phi^\lambda \\ 0 \end{array} \right) - \gamma_{1,1} (\beta \cdot \eta, \phi^\lambda) \right], \tag{C.2}$$

where in the second step the fact $\langle N_{1,1}^{\text{Im}}, \phi^\lambda \rangle = 0$, proved in [6, Lemma 9.4 on p. 291], is used. Extracting the lower order terms, we find

$$\partial_t a_1 = \partial_\xi a_1 \cdot Z_{2,1} + \partial_\xi a_1 \cdot \bar{Z}_{2,1} - \frac{1}{\langle \phi^\lambda, \partial_\xi \phi^\lambda \rangle} \left[ \sum_{m+n=2,3} JN_{m,n} \left( \begin{array}{c} \phi^\lambda \\ 0 \end{array} \right) - \gamma_{1,1} (\beta \cdot \eta, \phi^\lambda) \right]$$

$$+ \partial_t a_1 + iE(\lambda) \sum_{m+n=2,3} (m-n) A_{m,n}^{(1)} - \partial_\xi a_1 \cdot Z_{2,1} - \partial_\xi a_1 \cdot \bar{Z}_{2,1} \tag{C.3}$$

with $Z_{2,1} = - \Gamma (z, \bar{z}) z + \Lambda (z, \bar{z}) z$ defined in (4.7).
(2) Separate the terms of order $|z|^3$, $|z|^4$, and $|z|^5$ from $F_4$ and obtain
\[
\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle_{F_4} = \Upsilon_{1,1} \sum_{m+n=2,3} \left\langle R_{m,n} \left( \begin{pmatrix} 0 \\ \phi^\lambda \end{pmatrix} \right) \right\rangle + \gamma_{1,1} \langle (\beta + q) \cdot \eta, \phi^\lambda \rangle + (\dot{\gamma} - \gamma_{1,1}) \left[ \left[ \tilde{R}, \left( \begin{pmatrix} 0 \\ \phi^\lambda \end{pmatrix} \right) \right] + \gamma_{1,1} \left( R_{\geq 4}, \left( \begin{pmatrix} 0 \\ \phi^\lambda \end{pmatrix} \right) \right) \right] 
\]
\[
+ \dot{\lambda} \left[ \tilde{R}, \left( \begin{pmatrix} \partial_\lambda \phi^\lambda \\ 0 \end{pmatrix} \right) \right] - \dot{\lambda} a_1 (\partial_\lambda^2 \phi^\lambda, \phi^\lambda) - \dot{\lambda} (\alpha + p) \langle \partial_\lambda \xi, \phi^\lambda \rangle.
\] (C.4)

(3) The definitions of $J_{N_{m,n}}$, $m + n = 2, 3, 4$, and $J_{N \geq 5}$ in (B.10)–(B.12) and (B.14) imply that
\[
\langle \text{Im} N(\tilde{R}, z), \phi^\lambda \rangle = \sum_{m+n=2}^4 J_{N_{m,n}} \left( \begin{pmatrix} \phi^\lambda \\ 0 \end{pmatrix} \right) + J_{N \geq 5} \left( \begin{pmatrix} \phi^\lambda \\ 0 \end{pmatrix} \right).
\] (C.5)

Equations (C.1)–(C.5) and cancellation of terms in sum leads to (4.16).

D Estimates on the Eigenvectors of $L(\lambda)$ and the Parameters of Normal Form Transformation

Precise estimations of $S_x$ and $S_\lambda$ defined in Theorems 4.1 and 4.2 require control on coefficients depending on norms of $\phi^\lambda$, its derivatives as well as such norms of the neutral modes. Recall the definition of $\delta_\infty$ and the fact $\delta_\infty = O(\delta(\lambda(t)))$ in (4.1).

The result is as follows.

Proposition D.1. There exist constants $C_0$, $C_1$, $C_2 \in \mathbb{R}$ such that in the space $\langle x \rangle^{-4} H^2$

\[
\phi^\lambda = C_0 |e_0 + \lambda|^{\frac{1}{2}} \phi_{\text{lin}} + O(|e_0 + \lambda|^{1 + \frac{1}{2\sigma}}) = O(\delta_\infty),
\]
\[
\partial_\lambda \phi^\lambda = C_1 |e_0 + \lambda|^{\frac{1}{2\sigma} - 1} \phi_{\text{lin}} + O(|e_0 + \lambda|^{\frac{1}{2\sigma}}) = O(\delta_\infty^{(2\sigma - 1)}),
\] (D.1)
\[
\partial_\lambda^2 \phi^\lambda = C_2 |e_0 + \lambda|^{\frac{1}{2\sigma} - 2} \phi_{\text{lin}} + O(|e_0 + \lambda|^{\frac{1}{2\sigma} - 1}) = O(\delta_\infty^{(4\sigma - 1)}).
\] (D.2)

For the neutral modes we have

\[
\| \langle x \rangle^4 \partial_\lambda \xi_n \|_{2}, \quad \| \langle x \rangle^4 \partial_\lambda \eta_n \|_{2} \lesssim 1,
\] (D.3)
\[
\| \langle x \rangle^4 (\eta_m - \xi_{\text{lin}}^m) \|_{H^2}, \quad \| \langle x \rangle^4 (\xi_m - \xi_{\text{lin}}^m) \|_{H^2}, \quad \| \langle x \rangle^4 (\xi_m - \eta_m) \|_{H^2} = O(\delta_\infty^{2\sigma}).
\] (D.4)
Recall $P_c^\text{lin}$ is the orthogonal project onto the essential spectrum of $-\Delta + V$

$$P_c = P_c^\text{lin} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}(\delta^{2\sigma}).$$ (D.5)

The function $\Upsilon_{1,1}$ in (A.6) satisfies the estimate

$$|\Upsilon_{1,1}| \lesssim \delta_\infty^{2\sigma-3} |z|^2.$$ (D.6)

In what follows, we estimate various functions defined in Appendix B.

For $m+n=2$,

$$\| \langle x \rangle^4 J_{m,n} \|_2, \quad \| \langle x \rangle^{-4} R_{m,n} \|_2 \lesssim \delta_\infty^{2\sigma-1} |z|^2,$$

$$|A_{m,n}^{(1)}| \lesssim \delta_\infty^{2(2\sigma-1)} |z|^2,$$

$$|A_{m,n}^{(2)}| \lesssim \delta_\infty^{2\sigma-2} |z|^2, \quad |P_{m,n}^{(k)}|, \quad |Q_{m,n}^{(k)}| \lesssim \delta_\infty^{2\sigma-1} |z|^2, \quad k = 1, 2, \ldots, N.$$

For $m+n=3$,

$$\| \langle x \rangle^4 J_{m,n} \|_2, \quad \| \langle x \rangle^{-4} R_{m,n} \|_2 \lesssim \delta_\infty^{2\sigma-2} |z|^3.$$

$$|A_{m,n}^{(1)}| \lesssim \delta_\infty^{2\sigma-3} |z|^3,$$

$$|A_{m,n}^{(2)}| \lesssim \delta_\infty^{2\sigma-3} |z|^3,$$

$$|P_{m,n}^{(k)}|, \quad |Q_{m,n}^{(k)}| \lesssim \delta_\infty^{2\sigma-2} |z|^3, \quad k = 1, 2, \ldots, N.$$

For $m+n=4$ and $\sigma = 1$

$$\| \langle x \rangle^4 J_{m,n} \|_2 \lesssim \delta_\infty |z|^4.$$

**Proof.** Since all the functions are defined in terms of $\phi^\lambda$, $\xi$, and $\eta$ and their derivatives, we start with deriving estimates for them, or proving (D.1)–(D.4).

The key observation is these functions can be constructed perturbatively, as can be found in the known results in the spaces $H^k$, $k=1,2$, (see, e.g. [12]). In what follows, we re-do the proof in the desired space.

We start with (D.1) by decomposing $\phi^\lambda$ as

$$\phi^\lambda = \delta \phi_{\text{lin}} + \phi_{\Re} \quad \text{with } \langle \phi_{\text{lin}}, \phi_{\Re} \rangle_{L^2} = 0.$$

On the subspace parallelling to $\phi^\lambda$ and its orthogonal, we derive two equations

$$\delta^{-1} \phi_{\Re} = \delta^{2\sigma} (-\Delta + V + \lambda)^{-1} P_c(\phi_{\text{lin}} + \delta^{-1} \phi_{\Re})^{2\sigma+1},$$

$$(\lambda + e_0) = \delta^{2\sigma} \langle \phi_{\text{lin}}, (\phi_{\text{lin}} + \delta^{-1} \phi_{\Re})^{2\sigma+1} \rangle_{L^2},$$ (D.7)
where $\lambda + e_0$ is a fixed small positive constant, and $1 - P_c$ is the projection onto the $L^2$ 1-dimensional subspace $\{\phi_{\text{lin}}\}$.

We prove the existence of the solutions in appropriate Sobolev spaces by applying the contraction mapping theorem. Its applicability is fairly routine except observing the map

$$(-\Delta + V + \lambda)^{-1} P_c = (-\Delta + \lambda)^{-1} P_c - (-\Delta + V + \lambda)^{-1} P_c : \langle x \rangle^{-4} H^2 \to \langle x \rangle^{-4} H^2$$

is bounded. By the contraction mapping theorem, it is easy to construct the small solutions $\delta^{-1} \phi_{\text{Re}}$ and $\delta^{2\sigma}$ and find they are functions of $\lambda + e_0$ with differentiability $C^3$. (Actually if $\sigma$ is an integer then the functions are analytic in $\lambda + e_0$.) The dependence on $\lambda + e_0$ can be displayed by rewriting (D.7)

$$
\delta^{-1} \phi_{\text{Re}} = \delta^{2\sigma} (-\Delta + V + \lambda)^{-1} P_c (\phi_{\text{lin}})^{2\sigma + 1} + \delta^{2\sigma} \kappa (1 + \mathcal{O}(\delta^{2\sigma})),
\delta^{2\sigma} = (\lambda + e_0)^{-1} \left( \int \phi_{\text{lin}}^2 \right)^{-1} + C (\lambda + e_0)^2 + O(\lambda + e_0)^3
$$

with $\kappa$ being some function in $\langle x \rangle^{-4} H^2$ and $C$, a constant. From these it is easy to derive (D.1). The estimates (D.3) and (D.4) can be proved similarly.

Equation (D.2) is implied by (D.1). The estimates (D.5) and (D.6) are implied by their definitions (2.8) and (A.6) and the estimates (D.1)–(D.4).

Recall the definitions of $J_{N_{m,n}}$, $m + n = 2, 3, 4$, in (B.10)–(B.12). The desired estimates are simple applications of (D.1)–(D.6). Since all the other functions are defined in terms of $J_{N_{m,n}}$ and the estimates are straightforward, we omitted the details here. This completes the proof.

E The Estimate on $J_{N \geq 5}$

In this section, we estimate the remainder of $J_N$ after expanding it to the fourth order. Recall the definitions of $J_{N_{m,n}}$, $m + n = 2, 3, 4$, in (B.10)–(B.12).

**Proposition E.1.** If $\sigma = 1$, then

$$J_{N \geq 5} = \text{Loc} + \text{NonLoc}$$

with $\text{NonLoc} := [R_1^2 + R_2^2]J \tilde{R}$ and

$$
\| \langle x \rangle^4 \text{Loc} \|_{L^1}, \quad \| \langle x \rangle^4 \text{Loc} \|_{L^2} \lesssim |z|^5 + |z| (\delta_{\infty} + |z|) \| \langle x \rangle^{-4} R_{\geq 4} \|_2 + (\delta_{\infty} + |z|) \| \langle x \rangle^{-4} \tilde{R} \|_2
$$

(E.2)
and

$$\left| \left\langle J_{N_{\geq 5}}, \left( \phi_0^\lambda \right) \right\rangle \right| \lesssim \delta_\infty |z|^5 + \delta_\infty |z||\langle x \rangle|^{-4} R_{\geq 4} \|_2 + \delta_\infty \|\langle x \rangle\|^{-4} R_{\geq 4} \|_2^2. \quad (E.3)$$

\[ \square \]

**Proof.** Recall the decomposition of $\psi$ in (A.1) and the fact that the nonlinearity of (1.1) is cubic if $\sigma = 1$. Hence each term in Loc must be product of three terms taken from $\phi_0^\lambda$, $a_0 \phi_0^\lambda$, $\sum_{n=1}^N (\alpha_n + p_n) \xi_n$, $i \sum_{n=1}^N (\beta_n + q_n) \eta_n$, $R_{m,n}$, and $R_{\geq 4}$. By considering all the possibilities and using Proposition D.1 we obtain (E.2). The procedure is tedious but not difficult, hence is omitted here. Equation (E.3) follows easily from (E.2) and the fact $\phi_0^\lambda = O(\delta_\infty)$. This completes the proof. \[ \square \]

Now, we study the case $\sigma > 1$.

**Proposition E.2.** If $\sigma > 1$, then

$$\left\| J_{\tilde{N}} - \sum_{m+n=2,3} J_{N_{m,n}} \right\|_{L^1} + \left\| J_{\tilde{N}} - \sum_{m+n=2,3} J_{N_{m,n}} \right\|_{L^2} \lesssim |z|^{2\sigma+1} + |z|^4 + |z||\langle x \rangle|^{-4} R_{\geq 4} \|_2 + |z|^{\sigma+1} (\|\langle x \rangle\|^{-4} \vec{R}_\|_{\sigma} + \|\langle x \rangle\|^{-4} \vec{R}_\|_{2\sigma}) + \| \vec{R}_\|_{2\sigma+1} + \| \vec{R}_\|_{2(2\sigma+1)} \quad (E.4)$$

and

$$\left| \left\langle J_{\tilde{N}}(\vec{R}, z) - \sum_{m+n=2}^4 J_{N_{m,n}, } \left( \phi_0^\lambda \right) \right\rangle \right| \lesssim |z|^{2\sigma+2} + |z|^5 + \delta_\infty^{2\sigma-1} |z||\langle x \rangle|^{-4} R_{\geq 4} \|_2 + \|\langle x \rangle\|^{-4} R_{\geq 4} \|_2^2. \quad (E.5)$$

\[ \square \]

**Proof.** As in the proof of the case $\sigma = 1$ the basic idea in proving (E.4) and (E.5) is to Taylor expand the function $J_{N}$ in $z$ and $\bar{z}$. What makes the present situation different is that if $\sigma \notin \mathbb{N}$, then the nonlinearity $|\psi|^{2\sigma} \psi$ is not smooth at $\psi = 0$. Technically, the decomposition of $\psi$ in (A.1) makes $|\psi|^{2\sigma}$ to be of the form

$$|\psi|^{2\sigma}(x, t) = |(\phi_0^\lambda)^2(x) + \epsilon(x, t)|^\sigma$$

for some $\epsilon \in H^2(\mathbb{R})$. Since the inequality $|\epsilon| < (\phi_0^\lambda)^2$ does not hold for all $x \in \mathbb{R}^3$, we find that after expanding in $\epsilon$ to certain orders some undesired negative powers of $\phi_0^\lambda$ will be encountered if $\sigma \notin \mathbb{N}$. To prevent the negative powers from appearing in the final form
we have to adopt some tricks, namely compare the sizes of $\phi^\lambda$ and $\epsilon$ and discuss several regimes.

Now we start proving the proposition. Recall that $|\psi|^{2\sigma} = [(\phi^\lambda)^2 + 2\phi^\lambda(\alpha \cdot \xi) + I]^\sigma$ with $I := 2\phi^\lambda(I_1 - \alpha \cdot \xi) + I_1^2 + I_2^2$, $I_1$ and $I_2$ defined in (A.3). To control the remainder of the expansion around $\phi^\lambda$, we consider separately two regimes

$$(\phi^\lambda)^2 \geq 4|\phi^\lambda(\alpha \cdot \xi)| + 2|I| \quad \text{and} \quad (\phi^\lambda)^2 < 4|\phi^\lambda(\alpha \cdot \xi)| + 2|I|.$$  

For the second regime, we have

$$\left| \sum_{m+n=2,3} JN_{m,n} \right| \leq \left| \sum_{m+n=2,3} JN_{m,n} \right| \leq |z|^{2\sigma+1} + |z|^4 + |\bar{R}|^{2\sigma+1}. \quad (E.6)$$

For the first regime, that is,

$$(\phi^\lambda)^2 \geq 4|\phi^\lambda(\alpha \cdot \xi)| + 2|I|.$$  

we only study one term $(\phi^\lambda)^{2\sigma+1} O(\epsilon^4)$ with

$$\epsilon := (\phi^\lambda)^{-2}[2\phi^\lambda(\alpha \cdot \xi) + I] \leq \frac{1}{2}.$$

It is easy to see that this term is the remainder after expanding $\epsilon$ to the third order.

We claim that

$$|(\phi^\lambda)^{2\sigma+1} O(\epsilon^4)| \lesssim |\alpha \cdot \xi|^{2\sigma+1} + |\alpha \cdot \xi|^4 + |I|^{\sigma+1/2}. \quad (E.8)$$

We compute directly to obtain

$$|(\phi^\lambda)^{2\sigma+1} O(|\epsilon|^4)| \lesssim (\phi^\lambda)^{\sigma-7} I^4 + (\phi^\lambda)^{2\sigma-3}(\alpha \cdot \xi)^4.$$  

Apply (E.7) to control the first term on the right-hand side

$$(\phi^\lambda)^{\sigma-7} I^4 \lesssim |I|^{\sigma+1/2}.$$  

To bound the second term, we have two possibilities: $2\sigma - 3 \geq 0$ and $2\sigma - 3 < 0$. For the first, we have

$$(\phi^\lambda)^{2\sigma-3}|(\alpha \cdot \xi)^4| \lesssim (\alpha \cdot \xi)^4,$$

hence (E.8) holds trivially; for the second apply (E.7) to obtain

$$(\phi^\lambda)^{2\sigma-3}(\alpha \cdot \xi)^4 = |\alpha \cdot \xi|^{2\sigma+1}(\phi^\lambda)^{2\sigma-3}|\alpha \cdot \xi|^{3-2\sigma} \lesssim |\alpha \cdot \xi|^{2\sigma+1}.$$  

By collecting the estimates above, we prove (E.8).
This completes the proof for the first regime, and moreover this together with (E.6) completes the proof for (E.4).

Now we turn to (E.5). Here the function \( JN \) has to be expanded to one more order to obtain the desired estimate. This is enabled by the fact that the function was taken as an inner product with function \( \phi^\lambda \). The technique is similar to the proof of (E.4), hence we omit the details here. The proof is complete. 

\[ \square \]

F Proof of Proposition 4.1, Equations (4.17), (4.10), and (4.11)

Proposition 4.1 and Equations (4.17), (4.10), and (4.11) are included in Propositions F.1 and F.2. Recall the definitions of \( \delta_\infty, R \geq 4 \) in (4.1) and (B.13), respectively.

**Proposition F.1.** If \( |z_0|/\delta_\infty \ll 1 \) for \( \sigma = 1 \) and for \( \sigma > 1 \), \( |z_0| \leq \delta_\infty^{C(\sigma)} \) with \( C(\sigma) \) sufficiently large, then the following results hold: \( \vec{R} \) and \( R_{\geq 4} \) satisfy the estimates

\[ \| \vec{R} \|_2^2 \lesssim |z_0|^2, \quad \| \vec{R} \|_\infty \lesssim |z_0|^2(1 + t)^{-3/2} + \delta_\infty^{2\sigma - 1}|z(t)|^2, \quad \| (x)^{-4} \vec{R} \|_{H^2} \lesssim |z_0|^2(1 + t)^{-3/2} + \delta_\infty^{2\sigma - 1}|z(t)|^2. \]  

If \( \sigma = 1 \), then

\[ \| (x)^{-4} R_{\geq 4} \|_2 \lesssim |z_0|^2(1 + t)^{-3/2} + \delta_\infty |z_0|^2|z(t)|^2; \]  

and if \( \sigma > 1 \), then

\[ \| (x)^{-4} R_{\geq 4} \|_2 \lesssim |z_0|^2(1 + t)^{-3/2} + [\|z_0|^2 + |z_0|^{2\sigma - 1}]|z(t)|^2. \]  

\[ \square \]

The proposition will be proved shortly.

We prepare for the proof by defining some functions. To bound various functions in the proposition, we define the following estimating functions:

\[ M_1(T) := \max_{0 \leq t \leq T} \| \vec{R}(t) \|_\infty [ |z_0|^2(1 + t)^{-3/2} + \delta_\infty^{2\sigma - 1}|z(t)|^2]^{-1}, \]

\[ M_2(T) := \max_{0 \leq t \leq T} \| (x)^{-4} \vec{R}(t) \|_{H^2} [ |z_0|^2(1 + t)^{-3/2} + \delta_\infty^{2\sigma - 1}|z(t)|^2]^{-1}, \]

\[ M_4(T) := \max_{0 \leq t \leq T} \| \vec{R} \|_2 |z_0|^{-1}. \]

For \( \sigma = 1 \), we define

\[ M_3(T) := \max_{0 \leq t \leq T} \| (x)^{-4} R_{\geq 4}(t) \|_2 [ |z_0|^2(1 + t)^{-3/2} + \delta_\infty |z_0|^2|z(t)|^2]^{-1}; \]
for $\sigma > 1$

$$M_3(T) := \max_{0 \leq t \leq T} \|\langle x \rangle^{-4} R_{\geq 4}(t)\|_2 \|z_0\|^2 (1 + t)^{-3/2} + (\|z_0\|^2 + \|z_0\|^{2\sigma - 1})|z(t)|^2 \right)^{-1}.$$ 

In the present paper, we use $|z|$ as a gauge to measure sizes of different functions. This makes it necessary to obtain lower and upper bounds for $|z|$. Recall the definition $\Gamma(z, \bar{z}) \equiv \Gamma^\lambda(z, \bar{z})$ in (2.15). By Equation (D.1) and the assumption (FGR) there exist constants $C_{\pm}$ such that

$$C_+ \delta^2 \|z\|^4 \leq 2z^* \Gamma^\lambda(z, \bar{z})z \leq C_- \delta^2 \|z\|^4.$$

We define the upper and lower bounds $z_{\pm}$ by

$$z_{\pm}(t) := (\|z_0\|^2 + C_{\pm} \delta^2 \|z\|^4 t)^{-1/2}.$$

Recall the equation for $\dot{\lambda}$, the definitions for $\Psi$ and $K$ in (4.16) and (4.7), and recall the equation for $\dot{\gamma}$ in (A.9), and the definition of $\Upsilon_{1,1}$ in (A.6). In the rest of the paper, we use Remainder to represent different terms satisfying the estimate

$$|\text{Remainder}| \lesssim |z(t)|^4 + \|\langle x \rangle^{-4} \bar{R}\|_2^2 + |z(t)||\langle x \rangle^{-4} R_{\geq 4}\|_2.$$

Define a constant $\epsilon_{\infty}$ by

$$\epsilon_{\infty} := \max_{0 \leq t \leq \infty} \|\bar{R}(t)\|_{H^2}.$$ 

By (3.8) it is a small constant. The result is as follows.

**Proposition F.2.** Suppose that $|z_0|/\delta_{\infty} \ll 1$ for $\sigma = 1$ and $|z_0| = \delta_{\infty}^{C(\sigma)}$ for $\sigma > 1$ with $C(\sigma)$ being sufficiently large. Then

$$\sum_{k=1}^4 \mathcal{M}_k \leq (\delta_{\infty} + \epsilon_{\infty})^{-1/2} \quad \text{and} \quad 10|z_+| \geq |z| \geq \frac{1}{10}|z_-| \quad \text{in the time interval} \quad [0, \delta],$$

then in the same interval the following results hold:

1. The function $|z|$ admits lower and upper bounds

$$\frac{1}{5}|z_-(t)| \leq |z(t)| \leq 5|z_+(t)| \quad \text{for any time} \ t.$$ (F.10)
The functions $K, \Psi, \dot{\lambda}$, and $\dot{\gamma} - \gamma_{1,1}$ satisfy the following estimates:

$$|\Psi| \lesssim |z|^{2\sigma-1} \| \langle x \rangle^{-4} R_{\infty} \|_2 + \| \langle x \rangle^{-4} R_{\infty} \|_2^2 + \delta_\infty^{2\sigma-1} |z|^5,$$  \hfill (F.11)

if $\sigma = 1$, then

$$|\dot{\lambda}|, \quad |K| \lesssim \delta_\infty \text{ remainder}, \quad |\gamma - \gamma_{1,1}| \lesssim \text{ remainder}. \quad \hfill (F.12)$$

If $\sigma > 1$, then

$$|\dot{\lambda}|, \quad |\dot{\gamma} - \gamma_{1,1}|, \quad |K| \lesssim |z|^{2\sigma+1} + \text{ remainder}. \quad \hfill (F.14)$$

The estimates (F.10)–(F.14), (F.17) and (F.18) of the proposition will be proved in later sections. The proofs of the other two estimates are almost the same to the corresponding estimates of [6], specifically [6, Propositions 11.3 and 11.5, pp. 299 and 300, respectively], hence are omitted here.

**Proof of Proposition F.1.** By the local wellposedness of (1.1) there exists some interval $[0, \delta]$ such that for any $T \in [0, \delta]$

$$\sum_{k=1}^{4} \mathcal{M}_k(T) \leq (\delta_\infty + \epsilon_\infty)^{-1/2}$$

and

$$10|z_+| \geq |z| \geq \frac{1}{10}|z_-|.$$  \hfill (F.19)

Then the estimates (F.10)–(F.18) hold in this time interval, which include

$$5|z_+| \geq |z| \geq \frac{1}{5}|z_-|.$$  \hfill (F.19)

Now we turn to the estimates on $\mathcal{M}_k, k = 1, 2, 3, 4$. By substituting estimates of $\mathcal{M}_1$ and $\mathcal{M}_4$ of (F.15) and (F.18) into (F.17) we obtain

$$\mathcal{M}_3(T) \leq 1 + (\epsilon_\infty + \delta_\infty + |z_0|) P(\mathcal{M}_1(T), \ldots, \mathcal{M}_4(T))$$

where $P$ is a polynomial in the variables $\mathcal{M}_1, \ldots, \mathcal{M}_4$.
with \( P(x_1, \cdots, x_4) \) being some polynomial of positive coefficient. This together with the other estimates implies
\[
\mathcal{M} \lesssim 1 + (\epsilon_\infty + \delta_\infty + |z_0|) P_1(M) \quad (F.20)
\]
with \( \mathcal{M} := \sum_{k=1}^{4} M_k \) and \( P_1 \) being some polynomial.

By the assumption on the initial condition we find \( \sum_{n=1}^{4} M_n(0) \lesssim 1 \), which together with (F.20) implies
\[
\mathcal{M} \lesssim 1 \text{ in } [0, \delta]. \quad (F.21)
\]

Equations (F.21) and (F.19) and the local wellposedness of (1.1) imply that (F.10)–(F.18) hold in a larger interval \([0, \delta_2]\), so do (F.21) and (F.10)–(F.14) hold in \([0, \infty)\).

To conclude the proof, the fact \( \mathcal{M} \lesssim 1 \) in \([0, \infty)\) implies Proposition F.1.

The proof is complete. \( \blacksquare \)

F.1 Proof of (F.11)–(F.14)

In this subsection, we study the derivatives of the normal form transformation, which appear in \( K \) of (4.7) and \( \Psi \) of (4.16).

Recall the equation for \( z \) in (2.9). Define
\[
Z_{2,1} := -\Gamma(z, \bar{z})z + \Lambda(z, \bar{z})z \quad (F.22)
\]
and define

\[
A_1 := \partial_t a_1 + i \epsilon(\lambda) \sum_{m+n=2,3} (m-n) A_{m,n}^{(1)} + \partial_z a_1 \cdot Z_{2,1} + \partial_{\bar{z}} a_1 \cdot \overline{Z_{2,1}},
\]

\[
A_2 := \partial_t a_2 + i \epsilon(\lambda) \sum_{m+n=2,3} (m-n) A_{m,n}^{(2)} + \partial_z a_2 \cdot Z_{2,1} + \partial_{\bar{z}} a_2 \cdot \overline{Z_{2,1}},
\]

\[
P_k := \partial_t p_k + i \epsilon(\lambda) \sum_{m+n=2,3} (m-n) P_{m,n}^{(k)} + \partial_z p_k \cdot Z_{2,1} + \partial_{\bar{z}} p_k \cdot \overline{Z_{2,1}},
\]

\[
Q_k := \partial_t q_k + i \epsilon(\lambda) \sum_{m+n=2,3} (m-n) Q_{m,n}^{(k)} + \partial_z q_k \cdot Z_{2,1} + \partial_{\bar{z}} q_k \cdot \overline{Z_{2,1}}.
\]

The result is as follows.

Lemma F.1.
\[
|K|, |A_1|, |A_2|, |P_k|, |Q_k| \lesssim |z|(|\dot{\lambda}| + |\dot{\gamma} - \gamma_{1,1}|) + \delta^{C(\sigma)}_\infty \text{ remainder} \quad (F.23)
\]
with \( C(1) = 1 \), \( C(\sigma) = 0 \) if \( \sigma > 1 \), and \( k = 1, 2, \ldots, N \). \( \square \)
Proof. The definitions of $p_k$ in (B.2), the equation for $z$ in (4.7) imply

$$|P_k| \leq |\dot{\lambda}| |\partial_\lambda p_k| + |\partial_z p_k| |K| \leq |\dot{\lambda}| |z| + |z||K|$$

and

$$|Q_k| \leq |\dot{\lambda}| |\partial_\lambda q_k| + |\partial_z q_k| |K| \leq |\dot{\lambda}| |z| + |z||K|.$$ 

Apply Propositions D.1, E.1, and E.2 to obtain

$$|K| \leq (|z| + \|\langle x\rangle^{-4} \vec{R}\|_2)(|\dot{\lambda}| + |\dot{\gamma} - \gamma_{1,1}|) + \delta_\infty^{\sigma} \text{remainder} \quad \text{(F.24)}$$

with $C(1) = 1$ and $C(\sigma) = 0$ if $\sigma > 1$, which together with the estimate above implies the estimates for $K$, $P_k$ and $Q_k$ in (F.23).

By almost identical arguments we produce the estimates for $A_1$ and $A_2$.

The proof is complete. ■

Proof of (F.11)–(F.14)

Proof. We start with estimating $\dot{\lambda}$, $\dot{\gamma} - \gamma_{1,1}$: as in [6, pp. 291–293], we put the equations into a matrix form to find

$$[\text{Id} + \Pi(z, \vec{R}, a, p, q)] \begin{pmatrix} \dot{\lambda} \\ \dot{\gamma} - \gamma_{1,1} \end{pmatrix} = \Omega + \text{remainder} \begin{pmatrix} \delta_\infty \\ 1 \end{pmatrix}, \quad \text{(F.25)}$$

where, recall the definition of remainder in (F.9), and the terms on the right-hand side were obtained by the following arguments:

1. The matrix $\Omega$ is defined as

$$\Omega := - \begin{pmatrix} \partial_t a_1 + iE(\lambda) \sum_{m+n=2,3} (m-n) A^{(1)}_{m,n} \\ \partial_t a_2 + iE(\lambda) \sum_{m+n=2,3} (m-n) A^{(2)}_{m,n} \end{pmatrix}$$

and is controlled by applying the results in Lemma F.1

$$\Omega = O(|z|(|\dot{\lambda}| + |\dot{\gamma} - \gamma_{1,1}|) + \delta_\infty \text{remainder}) + \begin{pmatrix} O(\delta_\infty |z|^4) \\ O(|z|^4) \end{pmatrix}; \quad \text{(F.26)}$$

2. Id is the $2 \times 2$ identity matrix, the matrix $\Pi$ is defined as

$$\Pi(z, \vec{R}, a, p, q) := \Pi_1 + \Pi_s,$$
where $\Pi_s$ is a matrix depending on $z, \vec{R}, a, p, q$, and satisfies the estimate

$$\|\Pi_s(z, \vec{R}, a, p, q)\| \lesssim |z_0| + \delta_{\infty}$$

by the conditions $\sum_{k=1}^{M} \mathcal{M}_k(t) \leq (\delta_{\infty} + \epsilon_{\infty})^{-1/2}$ of Proposition F.2; the matrix $\Pi_1$ is defined and estimated as

$$\Pi_1 := \begin{pmatrix} 0 & 0 \\ -\langle R_2, \partial_x^2 \phi^\lambda \rangle & 0 \end{pmatrix} = O(|z_0|^{1/2}).$$

To prove this, we used the observations that $R_2 = \text{Im} R \perp \partial_x \phi^\lambda$ in (3.6) and $\partial_x \phi^\lambda$ and $\partial_x^2 \phi^\lambda$ are almost collinear to each other, proved in (D.1). By these and (D.2) we obtain

$$(1/\langle \phi^\lambda, \partial_x \phi^\lambda \rangle) \langle R_2, \partial_x^2 \phi^\lambda \rangle \leq \delta_{\infty}^{-2\sigma} \|x\|^{-4} \vec{R} \|2. The assumption $\mathcal{M}_2 \leq \delta_{\infty}^{-1/2}$ in Proposition F.2 implies that $\|<x>^4 \vec{R}\|2 \leq |z_0|^{3/2}$. Consequently,

$$\frac{1}{\langle \phi^\lambda, \partial_x \phi^\lambda \rangle} \langle R_2, \partial_x^2 \phi^\lambda \rangle \lesssim \delta_{\infty}^{-1-2\sigma} |z_0|^{3/2} \leq |z_0|^{1/2}. \quad (F.27)$$

(3) The term remainder $\delta_{\infty}$ is produced by

$$\frac{1}{\langle \phi^\lambda, \partial_x \phi^\lambda \rangle} \begin{pmatrix} \mathcal{Y}_{1,1} \langle R_2, \phi^\lambda \rangle + \mathcal{Y}_{1,1} \langle q \cdot \eta, \phi^\lambda \rangle - \left< \text{Im } N - \sum_{m+n=2,3} \text{Im } N_{m,n} \phi^\lambda \right> \\ -\mathcal{Y}_{1,1} \langle R_1, \partial_x \phi^\lambda \rangle - \mathcal{Y}_{1,1} \langle p \cdot \xi, \partial_x \phi^\lambda \rangle + \left< \text{Re } N - \sum_{m+n=2,3} \text{Re } N_{m,n} \partial_x \phi^\lambda \right> \end{pmatrix}.$$ 

To prove this, we use the results in Propositions D.1 and E.1 and the fact $\langle \text{Im } N_{1,1}, \phi^\lambda \rangle = 0$. Here the “almost orthogonality” or “almost collinear condition” between functions $\xi_k$ and $\eta_k$, $\phi^\lambda$, and $\partial_x \phi^\lambda$, implied by Proposition D.1, were used to approximate the orthogonal conditions (2.5) and (3.6). An example is in proving (F.27). We omit the details here.

Now inverting the matrix

$$[\text{Id} + \Pi]^{-1} = \text{Id} - \Pi + O(\Pi^2)$$

in (F.25), we obtain the desired estimates on $\dot{\lambda}$ and $\dot{\gamma} - \mathcal{Y}_{1,1}$, which are (F.12)–(F.14) except these on $\mathcal{K}$.

The estimate on $\mathcal{K}$ are implied by (F.23) and (F.12)–(F.14).

By similar arguments we prove (F.11).

The proof is complete.
F.2 Proof of Equation (F.10)

**Proof.** As usual we only prove the case $\sigma = 1$, the cases $\sigma > 1$ is easier by using the stronger condition $|z_0| \leq \delta_\infty^{C(\sigma)}$ for some sufficiently large $C(\sigma)$.

By (4.7) and the estimate for $\mathcal{K}$ in (F.23)

$$\frac{\mathrm{d}}{\mathrm{d}t} |z|^2 = -2 \text{Re} \ z^* \Gamma(z, \bar{z}) z + 2 \text{Re} \ z^* \cdot \text{remainder}$$

(F.28)

with $\text{Re} \ z^* \cdot \text{remainder}$ satisfying the estimate

$$|\text{Re} \ z^* \cdot \text{remainder}| \lesssim \delta_\infty |z| \text{ remainder}$$

$$\lesssim \delta_\infty |z|^5 + \delta_\infty |z|^2 (|z_0|^2 (1 + t)^{-3/2} + \delta_\infty |z_0|^2 |z(t)|^2) M_3$$

$$+ \delta_\infty |z| (|z_0|^4 (1 + t)^{-3} + \delta_\infty^2 |z_0|^4 |z(t)|^4) M_3^2.$$

Observe that $|\text{Re} \ z^* \cdot \text{remainder}|$ is NOT a higher order correction of $\text{Re} \ z^* \Gamma(z, \bar{z}) z = \mathcal{O}(\delta_\infty^2 |z|^4)$ in a neighborhood of $t = 0$. This forces us to divide the region $t \in [0, \infty)$ into two parts $t \leq \delta_\infty^{-2} |z_0|^{-2}$ and $t > \delta_\infty^{-2} |z_0|^{-2}$.

In the finite time interval, we define

$$|\bar{z}|^2(t) := |z|^2(t) e^{-\int_0^t \frac{2 \text{Re} \ z^* \cdot \text{remainder}}{|z|^2} \mathrm{d}s} \approx |z|^2(t).$$

The assumptions $|z| \geq \frac{1}{10} |z_-(t)| \gtrsim (|z_0|^{-2} + \delta_\infty^{-1} t)^{-1/2}$ and $M_3 \leq \delta_\infty^{-1/2}$ imply that for any time $t \leq \delta_\infty^{-2} |z_0|^{-2}$,

$$0 \leq \int_0^t 2 |\text{Re} \ z^* \cdot \text{remainder}| \frac{\mathrm{d}s}{|z|^2} = \mathcal{O} \left( \frac{|z_0|}{\delta_\infty} \right) \ll 1. \quad (F.29)$$

Moreover by (F.28) and the estimate of $\text{Re} \ z^* \Gamma(z, \bar{z}) z$ in (F.7)

$$\frac{3}{4} C_- \delta_\infty^2 \leq \frac{\mathrm{d}}{\mathrm{d}t} |\bar{z}|^{-2} \leq \frac{5}{4} C_+ \delta_\infty^2. \quad (F.30)$$

Consequently, when $t \leq \delta_\infty^{-2} |z_0|^{-2}$ we have the desired estimate

$$2 |z_+(t)| \geq |\bar{z}(t)| \geq \frac{1}{2} |z_-(t)| \quad \text{hence} \quad 3 |z_+(t)| \geq |z(t)| \geq \frac{1}{3} |z_-(t)|, \quad (F.31)$$

where recall the definitions of $z_\pm(t) = (|z_0|^{-2} + C_\pm \delta_\infty^{-2} t)^{-1/2}$ in Equation (F.8).

When $t \geq \delta_\infty^{-2} |z_0|^{-2}$, we consider (F.28) in the regime $|\delta_\infty^{-2} |z_0|^{-2}, \infty)$ with the initial condition satisfying the estimate in (F.31). This is easier by the fact that the second term in (F.28) is a true correction to the first: Use $(1 + t)^{-1} \leq 2 (|z_0|^{-2} \delta_\infty^{-2} t) + 2 \delta_\infty^2 (|z_0|^{-2} + \delta_\infty^2 t)^{-1}$ to find $\delta_\infty^2 |z|^4 \gg |\text{Re} \ z^* \cdot \text{Remainder}|$. Thus, there exists an $0 \leq \epsilon \ll 1$ such that

$$-(2 + \epsilon) \text{Re} \ z^* \Gamma(z, \bar{z}) z \leq \frac{\mathrm{d}}{\mathrm{d}t} |z|^2 \leq -(2 - \epsilon) \text{Re} \ z^* \Gamma(z, \bar{z}) z$$

which together with the condition in (F.31) at $t = \delta_\infty^{-2} |z_0|^{-2}$ enables us to obtain (F.10).
The proof is complete.

**Remark F.1.** In the proof above, specifically (F.29), we used \(|z_0| \ll \delta_\infty = O(\|\phi^{\lambda_0}\|_2)\) to show \(\delta_\infty |z|^5 \ll 2 \text{Re} z' \Gamma(z, \bar{z}) z\). Actually this condition can be weakened to be \(|z_0| \leq \|\phi^{\lambda_0}\|_2\) by refining normal form transformation: Namely examine closely the equation of \(z\) to find that

\[
\partial_t z + iE(\lambda)z = -\Gamma(z, \bar{z}) z + \Lambda(z, \bar{z}) z + \sum_{m+n=4} Z_{m,n} + \text{remainder 2},
\]

where remainder2 satisfies the estimate

\[
|\text{remainder 2}| \lesssim \delta_\infty |z|^5 + \delta_\infty |z|\|\langle x \rangle^{-4} R\|_2 + \delta_\infty \|\langle x \rangle^{-4} R\|_2^2.
\]

The fourth-order term \(Z_{m,n}\) can be removed by choosing a new parameter \(\tilde{z}\) by

\[
\tilde{z} = z + \sum_{m+n=4} \frac{1}{iE(\lambda)(m-n-1)} Z_{m,n} \approx z.
\]

By studying the equation for \(\tilde{z}\) we obtain the desired estimate. \(\square\)

**F.3 The Estimate of \(\|\vec{R}\|^2_2\): Proof of (F.18)**

We only prove the case \(\sigma = 1\), the case \(\sigma > 1\) is easier by using the stronger condition \(|z_0| \leq \delta_\infty^{C(\sigma)}\) for some sufficiently large \(C(\sigma)\).

By taking time derivative on \(\|\vec{R}\|^2_2\) and using the equation for \(\vec{R}\) in (A.2) we find

\[
\frac{d}{dt} \langle \vec{R}, \vec{R} \rangle = K_1 + K_2
\]

with \(K_n, n = 1, 2\), defined as

\[
K_1 := \langle (L(\lambda) + \gamma J) \vec{R}, \vec{R} \rangle + \langle \vec{R}, (L(\lambda) + \gamma J) \vec{R} \rangle + \lambda \langle P_{c_1} \vec{R}, \vec{R} \rangle + \lambda \langle \vec{R}, P_{c_1} \vec{R} \rangle,
\]

\[
K_2 := -\langle P_{c_1}^J N(\vec{R}, z), \vec{R} \rangle - \langle \vec{R}, P_{c_1}^J N(\vec{R}, z) \rangle + \langle P_{c_1}^J \mathcal{G}, \vec{R} \rangle + \langle \vec{R}, P_{c_1}^J \mathcal{G} \rangle.
\]

By the observation \(J^* = -J\) and the fact that \(JL(\lambda)\) is self-adjoint, we cancel all the nonlocalized terms in \(K_1\) and obtain:

\[
|K_1| \lesssim \|\langle x \rangle^{-4} \vec{R}\|_2^2.
\]

Recall the definition of \(P_{c_1}^J \mathcal{G}\) in (A.2). By various estimates in Proposition D.1 and the estimates in previous subsections we obtain

\[
|K_2| \lesssim \delta_\infty |z|^2 \|\langle x \rangle^{-4} \vec{R}\|_2 + \|\langle x \rangle^{-4} \vec{R}\|_2^2 + \|\vec{R}\|_4^4.
\]
Consequently,
\[
\left| \frac{d}{dt} \langle \vec{R}, \vec{R} \rangle \right| \lesssim \delta_\infty |z|^2 \| \langle x \rangle^{-4} \vec{R} \|_2 + \| \langle x \rangle^{-4} \vec{R} \|_2^2 + \| \vec{R} \|_4^4 \\
\lesssim \delta_\infty^2 |z|^4 + \| \langle x \rangle^{-4} \vec{R} \|_2^2 + \| \vec{R} \|_4^4.
\]
Integrate the equation from 0 to \( T \) and use the fact \( \| \vec{R}(0) \|_2^2 \lesssim |z_0|^2 \) to obtain
\[
\| \vec{R}(T) \|_2^2 \lesssim |z_0|^2 + \int_0^T \delta_\infty^2 |z(s)|^4 + \| \langle x \rangle^{-4} \vec{R}(s) \|_2^2 + \| \vec{R}(s) \|_4^4 \, ds. \tag{F.32}
\]
Now we estimate the different terms inside the integral.

We start with the first term. By the assumption \( 10|z_+| \geq |z| \geq \frac{1}{10}|z_-| \), we obtain
\[
\int_0^T \delta_\infty^2 |z(s)|^4 \, ds \lesssim \int_0^T \delta_\infty^2 (|z_0|^{-2} + \delta_\infty^2 s)^{-2} \, ds \leq |z_0|^2.
\]

For the second term, we use the definition \( \vec{R} = \sum_{m+n=2,3} R_{m,n} + R_{\geq 4} \) to obtain
\[
\| \langle x \rangle^{-4} \vec{R} \|_2 \leq \sum_{m+n=2,3} \| \langle x \rangle^{-4} R_{m,n} \|_2 + \| \langle x \rangle^{-4} R_{\geq 4} \|_2.
\]
The two terms on the right-hand side admit the following estimate.

1. By \( \| \langle x \rangle^{-4} R_{m,n} \|_2 \lesssim \delta_\infty |z|^2 \) proved in Proposition D.1 and the assumption \( 10|z_+| \geq |z| \geq \frac{1}{10}|z_-| \)
   \[
   \int_0^T \| \langle x \rangle^{-4} R_{m,n}(s) \|_2^2 \, ds \lesssim \delta_\infty^2 \int_0^\infty (|z_0|^{-2} + \delta_\infty^2 s)^{-2} \, ds = |z_0|^2.
   \]

2. By the definition of \( \mathcal{M}_3 \)
   \[
   \int_0^T \| \langle x \rangle^{-4} R_{\geq 4}(s) \|_2^2 \, ds \leq \mathcal{M}_3^2(T)|z_0|^3.
   \]

For the third term \( \int_0^T \| \vec{R}(s) \|_4^4 \, ds \)
\[
\int_0^T \| \vec{R}(s) \|_4^4 \, ds \leq \int_0^T \| \vec{R}(s) \|_\infty^2 \| \vec{R}(s) \|_2^2 \, ds \leq \mathcal{M}_4^2(T)\mathcal{M}_3^2(T)|z_0|^3.
\]
Collecting the estimates above, we complete the proof.

F.4 The Estimate of \( \| \langle x \rangle^{-4} R_{\geq 4} \|_2 \): Proof of (F.17)

We only prove the case \( \sigma = 1 \), the cases \( \sigma > 1 \) is different, but easier by using the stronger condition \( |z_0| \leq \delta_\infty^{C(\sigma)} \) for some sufficiently large \( C(\sigma) \).
Use the definition of $R_{\geq 4}$ in (B.13) and equation (A.2) to derive an equation for $R_{\geq 4}$
\[
\partial_t R_{\geq 4} = L(\lambda) R_{\geq 4} + L(\tilde{\lambda}, \tilde{\gamma}) R_{\geq 4} - P_c^\lambda J N_{\geq 4} + G_1 \tag{F.33}
\]
with the terms $J N_{\geq 4}$ and $G_1$ defined as
\[
J N_{\geq 4} := J N(\bar{R}, \tilde{p}, \tilde{z}) - \sum_{m+n=2,3} J N_{m,n},
\]
\[
G_1 := G - \Upsilon_{1,1}(\beta \cdot \eta \leftarrow -\alpha \cdot \xi) + P_c S_{\geq 4},
\]
and
\[
P_c S_{\geq 4} := L(\tilde{\lambda}, \tilde{\gamma}) \sum_{m+n=2,3} R_{m,n} + \sum_{m+n=2,3} \left[ L(\lambda) R_{m,n} \right. \left. - \frac{d}{dt} R_{m,n} - P_c^\lambda J N_{m,n} \right].
\]
The functions in $G_1$ take certain forms.

**Lemma F.2.** There exist some functions $\phi(m, n, k)$ such that
\[
G_1 = \sum_{m+n=2,3, k=0,1,2} [L(\lambda) + iE(\lambda)(m-n) - 0]^{-k}\phi(m, n, k), \tag{F.34}
\]
where $\phi(m, n, k)$ are smooth functions admitting the estimate
\[
\|\langle x \rangle^4 \phi(m, n, k)\|_2 \lesssim |\lambda| |z| + |z| |\dot{\gamma}| - \Upsilon_{1,1} + \delta_\infty |z|^4. \tag{F.35}
\]

**Proof.** Equation (F.34) is taken from [6, Theorem 8.1, p. 285].

Equation (F.35) is an improvement from [6] by the fact one has to find that the lowest order term in $\phi(m, n, k)$ is of the order $\delta_\infty |z|^4$. By Proposition D.1 and direct computation, we find it is generated by $\Upsilon_{1,1}(\beta \cdot \eta \leftarrow -\alpha \cdot \xi)$, $\Upsilon_{1,1} \sum_{m+n=2} JR_{m,n}$, and part of $\sum_{m+n=2} [L(\lambda) R_{m,n} - \frac{d}{dt} R_{m,n} - P_c^\lambda J N_{m,n}]$. The procedure is tedious, but easy. We omit the detail here.

The proof is complete. \(\blacksquare\)

Rewrite the equation for $R_{\geq 4}$ in (F.33) as
\[
\partial_t R_{\geq 4} = L(\lambda) R_{\geq 4} + L(\tilde{\lambda}, \tilde{\gamma}) R_{\geq 4} - P_c^\lambda (\text{Loc} + \text{NonLoc}) + P_c^\lambda G_1, \tag{F.36}
\]
where recall the definitions of Loc and NonLoc in (E.1).
Following the steps in [6, p. 302], we now derive an integral equation for \( P_{c}^{λ_{1}} R_{≥4} \). Rewrite \( L(λ(t)) \) as \( L(λ(t)) = L(λ_{1}) + L(λ(t)) - L(λ_{1}) \) with \( λ_{1} := λ(T) \) for some fixed time \( T \), and rewrite (F.36) one more time to obtain

\[
\frac{d}{dt} P_{c}^{λ_{1}} R_{≥4} = L(λ_{1}) P_{c}^{λ_{1}} R_{≥4} + [\dot{γ} + λ - λ_{1}]i(P_{+} - P_{-}) R_{≥4} + P_{c}^{λ_{1}} O_{1} R_{≥4} + P_{c}^{λ_{1}} G_{1} - P_{c}^{λ_{1}} [\text{Loc} + \text{NonLoc}]. \tag{F.37}
\]

Here for the terms on the right-hand side we have the following:

1. \( O_{1} \) is the operator defined by

\[
O_{1} := \dot{λ} P_{c} + L(λ) - L(λ_{1}) + \dot{γ} P_{c}^{λ} J - [\dot{γ} + λ - λ_{1}]i(P_{+} - P_{-})
\]

and the function \( O_{1} R_{≥4} \) satisfies the estimate: when \( t ≤ T \) then apply Proposition F.2 to obtain

\[
\| ⟨x⟩^{4} O_{1} R_{≥4} \|_{2} \lesssim |\dot{λ}| + |\dot{γ}| + |λ - λ_{1}| \| ⟨x⟩^{-4} R_{≥4} \|_{2}
\]

\[
\lesssim |z_{0}|^{2}(1 + s)^{-3/2} + δ_{∞}|z_{0}|^{2}(|z_{0}|^{-2} + δ_{∞}^{2} s)^{-1}[1 + δ_{∞} M_{3} + δ_{∞}^{3} M_{3}^{2}]. \tag{F.38}
\]

2. Recall that \( L(λ) \) has two branches of essential spectrum \([iλ, i∞)\) and \((-i∞, -iλ]\), we use \( P_{+} \) and \( P_{-} \) to denote the projection operators onto these two branches of the essential spectrum of \( L(λ(T)) \).

Then we have the following lemma.

**Lemma F.3.** For any function \( h \) and any large constant \( ν > 0 \), we have

\[
\| ⟨x⟩^{ν} (P_{c}^{λ_{1}} J - i(P_{+} - P_{-}))h \|_{2} \leq c \| ⟨x⟩^{-4} h \|_{2}. \tag{F.39}
\]

\[\square\]

The following estimates are taken from [6, Theorem 5.7, p. 280].

**Lemma F.4.** There exists a constant \( c \) such that if the parameter \( λ_{1} \) satisfies the estimate \( |λ_{1} + ε_{0}| \ll 1 \), then for any function \( h \) and \( t \geq 0 \) we have

\[
\| ⟨x⟩^{-4} e^{t L(λ_{1})} (L(λ_{1}) ± ik E (λ_{1}) - 0)^{-n} P_{±} h \|_{2} \leq c(1 + t)^{-3/2} \| ⟨x⟩^{4} h \|_{2} \tag{F.40}
\]

with \( n = 0, 1, 2, k = 2, 3; \)

\[
\| ⟨x⟩^{-4} e^{t L(λ_{1})} P_{±} h \|_{2} \lesssim (1 + |t|)^{-3/2}(\| h \|_{1} + \| h \|_{2}). \tag{F.41}
\]

\[\square\]
Apply Duhamel’s principle on Equation (F.37) and use the observation that the operators \( P_+, P_- \) and \( L(\lambda_1) \) commute with each other to find

\[
\| \langle x \rangle^{-4} P^k_{\pm} R_{\geq 4} \|_2 \leq \sum_{i=1}^{4} A_i,
\]  

with

\[
A_1 := \| \langle x \rangle^{-4} e^{t \lambda \lambda_1 + i a(t,0)(P_+ + P_-)} P^k_{\pm} R_{\geq 4} \|_2,
\]

where \( a(t, s) := \int_s^t [\gamma(\tau) + \lambda(\tau) - \lambda_1] \, d\tau \in \mathbb{R} \),

\[
A_2 := \int_0^t \| \langle x \rangle^{-4} e^{(t-s)L(\lambda_1) + i a(t, s)(P_+-P_-)} P^k_{\pm} [O_1 R_{\geq 4} - P^k_{\pm} \text{Loc}] \|_2 \, ds,
\]

\[
A_3 := \int_0^t \| \langle x \rangle^{-4} e^{(t-s)L(\lambda_1) + i a(t, s)(P_+-P_-)} P^k_{\pm} G_1(s) \|_2 \, ds,
\]

and

\[
A_4 := \int_0^t \| \langle x \rangle^{-4} e^{(t-s)L(\lambda_1) + i a(t, s)(P_+-P_-)} P^k_{\pm} \text{NonLoc} \|_2 \, ds.
\]

Now we estimate \( A_k, \ k = 1, 2, 3, 4 \). In what follows, we use repeatedly the assumption \( 10|z_+| \geq |z| \geq \frac{1}{10}|z_-| \) and the fact

\[
e^{i a(t, s) (P_+ + P_-)} = e^{i a(t, s) P_+} + e^{-i a(t, s) P_-} : \langle x \rangle^k L^2 \to \langle x \rangle^k L^2
\]

is uniformly bounded for any \( k \in \mathbb{R} \).

1. By the propagator estimate (F.40), we have

\[
A_1 \leq \| \langle x \rangle^{-4} e^{t \lambda \lambda_1} P_{\pm} \tilde{R}(0) \|_2 + \| \langle x \rangle^{-4} e^{t \lambda \lambda_1} P_{\pm} \sum_{m+n=2,3} R_{m,n}(0) \|_2
\]

\[
\lesssim (1 + t)^{-3/2}[\| \langle x \rangle^4 \tilde{R}(0) \|_2 + |z_0|^2]
\]

\[
\lesssim (1 + t)^{-3/2} |z_0|^2.
\]

2. Equation (F.43) below and the estimates of \( O_1 R_{\geq 4} \) in (F.38), Loc in Proposition E.1 imply that for any time \( t \leq T \)

\[
A_2 \lesssim \int_0^t (1 + t - s)^{-3/2}[|z_0|^2(1 + s)^{-3/2} + \delta_{\infty} |z_0|^2(|z_0|^{-2}
\]

\[
+ \delta_{\infty}^2 s)^{-1}[1 + \delta_{\infty} \mathcal{M}_3 + \delta_{\infty}^3 \mathcal{M}_3^2] ds
\]

\[
\lesssim |z_0|^2(1 + t)^{-3/2} + \delta_{\infty} |z_0|^2(|z_0|^{-2} + \delta_{\infty}^2 t^{-1})[1 + \delta_{\infty} \mathcal{M}_3 + \delta_{\infty}^3 \mathcal{M}_3^2].
\]
Lemma F.5. Collecting the estimates above, we have
\[ A_3 \lesssim \int_0^t (1 + t - s)^{-3/2}[\delta_\infty |z(s)|^4 + \text{remainder}(s)|z(s)|] \, ds \]
\[ \lesssim |z_0|^2(1 + s)^{-3/2} + \delta_\infty |z_0|^2(|z_0|^{-2} + \delta_\infty^2 s)^{-1}][1 + \delta_\infty M_3 + \delta_\infty^3 M_2^2]. \]

(4) We control \( A_4 \) by the form of NonLoc in Proposition E.1 and the propagator estimate (F.41):
\[ A_4 \lesssim \int_0^t (1 + t - s)^{-3/2}[\| \vec{R}(s) \|_3^3 + \| \vec{R}(s) \|_6^3] \, ds \]
\[ \lesssim \int_0^t (1 + t - s)^{-3/2}[\| \vec{R}(s) \|_2^2 \| \vec{R} \|_\infty + \| \vec{R}(s) \|_2^2 \| \vec{R} \|_2] \, ds \]
\[ \lesssim |z_0|^2 \int_0^t (1 + t - s)^{-3/2}[(1 + s)^{-3/2} + \delta_\infty (|z_0|^{-2} + \delta_\infty^2 s)^{-1}] \, ds[M_2^2 M_1 + M_4 M_1^2]. \]

Apply (F.43) to obtain
\[ A_4 \lesssim |z_0|^2[(1 + t)^{-3/2} + \delta_\infty (|z_0|^{-2} + \delta_\infty^2 t)^{-1}][M_2^2 M_1 + M_4 M_1^2]. \]

Collecting the estimates above, we have
\[ \| (x)^{-4} R_{\geq 4} \|_2 \lesssim |z_0|^2(1 + t)^{-3/2} + \delta_\infty |z_0|^2(|z_0|^{-2} + \delta_\infty^2 t)^{-1}][1 + M_4 M_1^2 + M_4 M_1^2 + \delta_\infty (M_3 + M_3^2)]. \]

The proof is complete by the definition of \( M_3 \) in Equation (F.17).

In the proof we used the following lemma.

Lemma F.5.
\[ \int_0^t \frac{1}{(1 + t - s)^{3/2}}(|z_0|^{-2} + \delta_\infty^2 s)^{-1} \, ds \lesssim (|z_0|^{-2} + \delta_\infty^2 t)^{-1}. \] (F.43)

Proof. We divide the regime into two parts \( 0 \leq s \leq \frac{t}{2} \) and \( \frac{t}{2} \leq s \leq t \). For the latter
\[ \int_{t/2}^t \frac{1}{(1 + t - s)^{3/2}}(|z_0|^{-2} + \delta_\infty^2 s)^{-1} \, ds \lesssim 2(|z_0|^{-2} + \delta_\infty^2 t)^{-1} \]
\[ \int_{t/2}^t \frac{1}{(1 + t - s)^{3/2}} \, ds \lesssim (|z_0|^{-2} + \delta_\infty^2 t)^{-1}. \]
The estimate of the first is more involved. By direct computation we have

\[
I := \int_0^{t/2} \frac{1}{(1 + s - t)^{3/2}}(z_0^{-2} + \delta_\infty^2 s)^{-1} \, ds \\
\leq 2\sqrt{2}(1 + t)^{-3/2} \int_0^{t/2} (z_0^{-2} + \delta_\infty^2 s)^{-1} \, ds \\
= 2\sqrt{2}\delta_\infty^{-2}(1 + t)^{-3/2} \ln \left[ 1 + \frac{1}{2} |z_0|^2 \delta_\infty^2 t \right].
\]

Change variable \( u = \delta_\infty^2 |z_0|^2 t \) to obtain

\[
I \lesssim \delta_\infty^{-2}(1 + \delta_\infty^{-2} |z_0|^{-2} u)^{-3/2} \ln(1 + u) \\
\lesssim |z_0|^2 u^{-1}(1 + u)^{-1/2} \ln(1 + u) \\
\lesssim |z_0|^2 (1 + u)^{-1} = (|z_0|^{-2} + \delta_\infty^2 t)^{-1}.
\]

The proof is complete. ■

G Proof of (4.9)

Proof. In the following, we study \( z^* \Gamma(z, \bar{z}) z \) defined in (2.15). To prepare for the proof we list a few estimates.

1. If \( n = -2, -1, 0, 1, 2 \) then

\[
[L(\lambda) + i n E(\lambda) + 0]^{-1} P_c = [(-\Delta + V + \lambda) J + i n E(\lambda) + 0]^{-1} P_c \ln \left( \begin{array}{c}
1 \\
0
\end{array} \right) \\
+ \mathcal{O}(\|\langle x \rangle^4 |\varphi\lambda\|_{H^2})
\]

(4.1)

as operators mapping from space \( (x)^{-4} L^\infty \) to space \( (x)^{-4} L^\infty \). This is resulted from the estimate of \( P_c - P_c^{\text{lin}} \) in (D.5) and the fact \( L(\lambda) \) defined in (2.3) is of the form \( L(\lambda) = (-\Delta + V + \lambda) J + \mathcal{O}(\|\varphi_\lambda\|_{2^\gamma}) \).

2. To diagonalize the matrix \( J \) we define a unitary \( 2 \times 2 \) matrix \( U \) as

\[
U := \frac{1}{\sqrt{2}} \left( \begin{array}{cc}
1 & i \\
i & 1
\end{array} \right)
\]

(4.2)

which makes

\[
U^* J U = i\sigma_3
\]

with \( \sigma_3 \) being the Pauli matrix.
(3) Apply (D.4) to derive the leading orders from $\sum_{k=1}^{N} z_k G_k$ to obtain

$$\sum_{k=1}^{N} z_k G_k = JN_{2,0}$$

$$= \sigma (\phi^\lambda)^{2\sigma - 1} \begin{pmatrix} -2i(z \cdot \xi)(z \cdot \eta) \\ -(2\sigma + 1)(z \cdot \xi)^2 + (z \cdot \eta)^2 \end{pmatrix}$$

$$= -2\sigma \delta_\infty^{2\sigma - 1} (\phi_{\text{lin}})^{2\sigma - 1} (z \cdot \xi_{\text{lin}})^2 \begin{pmatrix} i \\ \sigma \end{pmatrix} + O(\delta_\infty^{4\sigma - 1}|z|^2). \quad (G.3)$$

Now, we begin perturbation—expanding the function $z^* \Gamma(z, \bar{z})z$ in the variable $\delta(\lambda)$. Use (G.1)–(G.3) to obtain

$$z^* \Gamma(z, \bar{z})z = -4\sigma^2 \delta_\infty^{4\sigma - 2}(\lambda) \text{Re} \left[ (-\Delta + V + \lambda) J + 2iE(\lambda) - 0 \right]^{-1} \sigma_{\text{lin}}^2 (z \cdot \xi_{\text{lin}})^2 \begin{pmatrix} i \\ \sigma \end{pmatrix}$$

$$\times i(\phi_{\text{lin}})^{2\sigma - 1} (z \cdot \xi_{\text{lin}})^2 J \begin{pmatrix} i \\ \sigma \end{pmatrix} + O(\delta_\infty^{4\sigma - 1}|z|^4)$$

$$= -4\sigma^2 \delta_\infty^{4\sigma - 2}(\lambda) \text{Re} \left[ U^* [(-\Delta + V + \lambda) J + 2iE(\lambda) - 0]^{-1} U \sigma_{\text{lin}}^2 (z \cdot \xi_{\text{lin}})^2 \begin{pmatrix} i \\ \sigma \end{pmatrix} \right]$$

$$\times (z \cdot \xi_{\text{lin}})^2 U^* \begin{pmatrix} i \\ \sigma \end{pmatrix}, \quad (\phi_{\text{lin}})^{2\sigma - 1} (z \cdot \xi_{\text{lin}})^2 U^* \begin{pmatrix} i \\ \sigma \end{pmatrix} + O(\delta_\infty^{4\sigma - 1}|z|^4).$$

In the new expression certain terms can be computed explicitly:

$$U^* [(-\Delta + V + \lambda) J + 2iE(\lambda) - 0]^{-1} U$$

$$= \begin{pmatrix} -i[(-\Delta + V + \lambda + 2E(\lambda)]^{-1} & 0 \\ 0 & i[-\Delta + V + \lambda - 2E(\lambda) - i0]^{-1} \end{pmatrix}, \quad (G.4)$$

and

$$iU^* \begin{pmatrix} i \\ \sigma \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma - 1 \\ \sigma + 1 \end{pmatrix}, \quad U^* \begin{pmatrix} i \\ \sigma \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i(1 - \sigma) \\ \sigma + 1 \end{pmatrix}. \quad (G.5)$$

Put this back into the expression to find

$$z^* \Gamma(z, \bar{z})z = z^* \Gamma_0(z, \bar{z})z + H_{2,2} + O(\delta_\infty^{4\sigma - 1}|z|^4) \quad (G.6)$$

with

$$H_{2,2} := -2\delta_\infty^{4\sigma - 2}(\lambda) \sigma^2 (\sigma - 1)^2$$

$$\times \text{Re} [i(-\Delta + V + \lambda + 2E(\lambda)]^{-1} \sigma_{\text{lin}}^2 (z \cdot \xi_{\text{lin}})^2, (\phi_{\text{lin}})^{2\sigma - 1} (z \cdot \xi_{\text{lin}})^2 \right).$$
We observe that

\[ H_{2,2} = 0 \]  

by the fact that the operator \((-\Delta + V + \lambda + 2E(\lambda))^{-1}P_{\text{lin}}^c\) is self-adjoint, hence that \(\langle (-\Delta + V + \lambda + 2E(\lambda))^{-1}P_{\text{lin}}^c\Omega, \Omega \rangle\) is real for any \(\Omega\). Hence (G.6) is the desired estimate and the proof is complete.

H Proof of Equation (4.19)

Recall the ideas present after Lemma 4.2, which basically is that the functions \(\Pi_{m,n}, m \neq n\) are almost periodic with period \(2\pi/E(m-n)\).

Compute directly to obtain

\[
\int_0^\infty \Pi_{m,n} \, ds = \int_0^\infty \Pi_{m,n} + \frac{1}{iE(\lambda)(m-n)} \Pi_{m,n} \, ds - \int_0^\infty \frac{d}{ds} \left( \frac{1}{iE(\lambda)(m-n)} \Pi_{m,n} \right) \, ds \\
= \int_0^\infty \left[ \frac{\partial}{\partial \lambda} \left( \frac{1}{iE(\lambda)(m-n)} \Pi_{m,n} \right) + \frac{1}{iE(\lambda)(m-n)} \partial_z \Pi_{m,n} \cdot \left( \hat{z} + iE(\lambda)z \right) \\
+ \frac{1}{iE(\lambda)(m-n)} \partial_{\bar{z}} \Pi_{m,n} \cdot \left( \hat{\bar{z}} - iE(\lambda)\bar{z} \right) \right] \, ds + \frac{1}{iE(\lambda)(m-n)} \Pi_{m,n}|_{t=0}. \tag{H.1}
\]

It is easy to see that the last term satisfies the estimate

\[
\frac{1}{iE(\lambda)(m-n)} \Pi_{m,n}|_{t=0} = o(|z_0|^2).
\]

Collecting the estimates above we prove (4.19).

The proof is complete.

I Proof of Equation (4.18)

To facilitate our discussions, we define

\[
\rho := z \cdot \xi, \quad \omega := z \cdot \eta.
\]

Recall the definition of \(\Pi_{2,2}\) in (4.16). By direct computation, we find

\[
\Pi_{2,2} = -\left( JN_{2,2}, \begin{pmatrix} \phi^i \\ 0 \end{pmatrix} \right) + \gamma_{1,1} \left( R_{1,1}, \begin{pmatrix} 0 \\ \phi^i \end{pmatrix} \right) + \gamma_{1,1} \sum_{n=1}^N Q_{1,1}^{(n)}(\phi^i, \eta^i).\tag{I.1}
\]

Further expand the first term to obtain

\[
\left( JN_{2,2}, \begin{pmatrix} \phi^i \\ 0 \end{pmatrix} \right) = \sum_{n=1}^7 (\Phi_n + \overline{\Phi_n}) + \Omega_{\sigma > 1}. \tag{I.2}
\]
Lemma I.1. The terms defined above satisfy the following estimate:

\[ \Omega_{\sigma > 1} \text{ only appears in } \sigma > 1 \]

\[ \Omega_{\sigma > 1} := \frac{1}{8i} \sigma (\sigma - 1)(2\sigma - 1)((\phi^2)^{2\sigma - 2}|\rho|^2, \omega\rho - \tilde{\omega}\rho) - \frac{3}{8i} \sigma (\sigma - 1)((\phi^2)^{2\sigma - 2}|\omega|^2, \tilde{\omega}\rho - \omega\rho). \]

The terms defined above satisfy the following estimate:

Lemma I.1.

\[ \sum_{n=2}^7 |\Phi_n + \overline{\Phi}_n| + |\gamma_{1,1} (R_{1,1}^{(2)}, \phi^\lambda)| + |\gamma_{1,1} \sum_{n=1}^d Q_{1,1}^{(n)} (\phi^\lambda, \eta^n)| + |\Omega_{\sigma > 1}| \lesssim \delta^{4\sigma - 1} |z|^4. \]  

(I.3)

The lemma will be proved in the later part of this section.

Now we prove (4.18).
Proof of Equation (4.18). Collecting the estimates in Equations (I.1), (I.2), and Lemma I.1 we have

\[ \Pi_{2,2} = -2 \text{Re}(R_{2,0}, K_{2,0}) + \mathcal{O}(\delta_{\infty}^{4\sigma-1}|z|^4) \quad (I.4) \]

with

\[ K_{2,0} := \sigma (\phi^*)^{2\sigma-1} \begin{pmatrix} -\frac{i}{2}(2\sigma - 1)\rho \omega \\ -\frac{3}{4}\rho^2 + \frac{1}{4}(2\sigma - 1)\rho^2 \end{pmatrix} \]

\[ = \frac{\sigma}{2} (\phi^*)^{2\sigma-1} \rho^2 \begin{pmatrix} -i(2\sigma - 1) \\ \sigma - 2 \end{pmatrix} + \mathcal{O}[\delta_{\infty}^{4\sigma-1}|z|^2] \]

\[ = \frac{\sigma}{2} \delta^{2\sigma-1}(\lambda) \phi^{\text{lin}}(z \cdot \xi^{\text{lin}})^2 \begin{pmatrix} -i(2\sigma - 1) \\ \sigma - 2 \end{pmatrix} + \mathcal{O}[\delta_{\infty}^{4\sigma-1}|z|^2] \]

and recall \( \delta(\lambda) \) defined in (2.1).

Use (G.1) to expand \( R_{2,0} \) in the space \( \langle x \rangle^4 L^\infty \)

\[ R_{2,0} = (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c JN_{2,0} \]

\[ = [(-\Delta + V + \lambda)J + 2iE(\lambda) - 0]^{-1} P_c^{\text{lin}} JN_{2,0} + \mathcal{O}(\delta_{\infty}^{4\sigma-1}|z|^2). \]

The estimate for \( JN_{2,0} \) in (G.3) implies

\[ -2 \text{Re}(R_{2,0}, K_{2,0}) = M_{2,2} + \mathcal{O}(\delta_{\infty}^{4\sigma-1}|z|^4) \]

with

\[ M_{2,2} := 2\sigma^2 \delta^{4\sigma-2}(\lambda) \left[ [(-\Delta + V + \lambda)J + 2iE(\lambda) - 0]^{-1} P_c^{\text{lin}} (\phi^{\text{lin}})^{2\sigma-1}(z \cdot \xi^{\text{lin}})^2 \begin{pmatrix} i \\ \sigma \end{pmatrix} , \right. \]

\[ \times \phi^{\text{lin}}(z \cdot \xi^{\text{lin}})^2 \begin{pmatrix} -i(2\sigma - 1) \\ \sigma - 2 \end{pmatrix} \right]. \]

To make the expression easier, we diagonalize the matrix operator \([(-\Delta + V + \lambda)J + 2iE(\lambda) - 0]^{-1}\) which is essential to diagonalize the matrix \( J \). Recall the definition of the unitary matrix \( U \) in (G.2). Put \( UU^* = \text{Id} \) into appropriate places to obtain

\[ M_{2,2} = 2\sigma^2 \delta^{4\sigma-2}(\lambda) \left[ U^* [(-\Delta + V + \lambda)J + 2iE(\lambda) - 0]^{-1} U P_c^{\text{lin}} (\phi^{\text{lin}})^{2\sigma-1}(z \cdot \xi^{\text{lin}})^2 U^* \begin{pmatrix} i \\ \sigma \end{pmatrix} , \right. \]

\[ \times \phi^{\text{lin}}(z \cdot \xi^{\text{lin}})^2 U^* \begin{pmatrix} -i(2\sigma - 1) \\ \sigma - 2 \end{pmatrix} \right]. \]
A few terms in the expression can be computed explicitly: \( U^* [-\Delta + V + \lambda] J + 2iE(\lambda) - 0)^{-1} U \) was computed in (G.4),

\[
U^* \left( \frac{i}{\sigma} \right) = \frac{1}{\sqrt{2}} \left( \frac{i(1 - \sigma)}{1 + \sigma} \right), \quad U^* \left( \frac{-i(2\sigma - 1)}{\sigma - 2\sigma} \right) = -\frac{1}{\sqrt{2}} \left( \frac{3i(\sigma - 1)}{\sigma + 1} \right).
\]

Put this back into the expression of \( M_{2,2} \) to obtain

\[
M_{2,2} = \frac{1}{2} z^* \Gamma_0(z, \bar{z}) z + \tilde{M}_{2,2}
\]  
(I.5)

with

\[
\tilde{M}_{2,2} := -3\delta^4\sigma^{-2}(\lambda)\sigma^2(\sigma - 1)^2
\]

\[
\times \text{Re} \{i(-\Delta + V + \lambda + 2E(\lambda))^{-1} P_c^{\text{lin}}(\phi_{\text{lin}})^{2\sigma - 1}(z \cdot \xi_{\text{lin}})^2, (\phi_{\text{lin}})^{2\sigma - 1}(z \cdot \xi_{\text{lin}})^2 \},
\]

where recall the definition of \( z^* \Gamma_0(z, \bar{z}) z \) in (2.17). Observe \( \tilde{M}_{2,2} = 0 \) by the same argument as in proving (G.7). Hence

\[
-2\text{Re} \{R_{2,0}, K_{2,0} \} = \frac{1}{2} z^* \Gamma_0(z, \bar{z}) z + \mathcal{O}(\delta^4\sigma^{-1}|z|^4).
\]

These together with (I.4) imply Equation (4.18).

The proof is complete.

In the rest of this section, we prove Lemma I.1 by considering the cases \( \sigma = 1 \) and \( \sigma > 1 \) separately.

**Proof of Lemma I.1 for \( \sigma = 1 \).** In what follows, we only study the term \( \Phi_5 \) and part of \( \Phi_4 \). The estimates on the other terms are similar and easier.

We start with analyzing \( JN_{2,1} \) since the terms are related to it. The definition in (B.11) and the fact \( \|\rho - \omega\|_{H^2} = \mathcal{O}(\delta^2_{\infty}|z|) \) imply that in the space \( \langle \chi \rangle^{-4} H^2 \)

\[
JN_{2,1} = -\frac{1}{2} \rho^2 \tilde{\rho} \left( \frac{i}{1} \right) + \mathcal{O}(\delta_{\infty}|z|^3).
\]  
(I.6)

Now we turn to \( \Phi_5 \). Recall the definition of \( R_{2,1} \) from (B.9). By the facts \( P_c \left( \begin{smallmatrix} \rho \\ 0 \end{smallmatrix} \right) = P_c \left( \begin{smallmatrix} 0 \\ \omega \end{smallmatrix} \right) = 0 \) and \( \rho - \omega = \mathcal{O}(\delta^2_{\infty}|z|) \) in (D.4) we obtain

\[
\gamma_{1,1} P_c \left( \begin{smallmatrix} i\omega \\ \rho \end{smallmatrix} \right) = \gamma_{1,1} P_c \left( \begin{smallmatrix} i(\omega - \rho) \\ \rho - \omega \end{smallmatrix} \right) = \mathcal{O}(\delta^2_{\infty}|z|^3).
\]
This together with (I.6) and (G.1) implies that

$$R_{2,1} = -\frac{1}{2} \left[ (-\Delta + V + \lambda) J + iE(\lambda) \right]^{-1} P_{c} \rho^{2} \tilde{\rho} \left( \begin{array}{c} i \\ 1 \end{array} \right) + O(\delta_{\infty} |z|^3).$$

Consequently,

$$\Phi_{5} = \Theta_{2,2} + O(\delta_{3} |z|^4) \tag{I.7}$$

with

$$\Theta_{2,2} := -\frac{1}{2} \left[ \left( \begin{array}{c} i \sigma_{3} \left[ \left( -\Delta + V + \lambda \right) J + iE(\lambda) \right]^{-1} P_{c} \rho^{2} \tilde{\rho} \left( \begin{array}{c} i \\ 1 \end{array} \right), (\phi^{\lambda})^{2} \rho \left( \begin{array}{c} -i \\ 1 \end{array} \right) \right) \right].$$

Now we claim

$$\Theta_{2,2} = 0,$$

which trivially implies the desired estimate on $\Phi_{5}$.

To prove the claim we diagonalize the matrix $J$ to obtain a convenient form. Recall the definition of the unitary matrix $U$ in (G.2). Insert $UU^{*} = \text{Id}$ into appropriate places and use the fact $U^{*} J U = i\sigma_{3}$ to obtain

$$\Theta_{2,2} = \frac{1}{2} \left\{ \left( i \sigma_{3} \left[ \left( -\Delta + V + \lambda \right) J + iE(\lambda) \right]^{-1} P_{c} \rho^{2} \tilde{\rho} \left( \begin{array}{c} i \\ 1 \end{array} \right), (\phi^{\lambda})^{2} \rho \left( \begin{array}{c} -i \\ 1 \end{array} \right) \right) \right\}$$

$$= \left\{ i \sigma_{3} \left[ \left( -\Delta + V + \lambda \right) J + iE(\lambda) \right]^{-1} P_{c} \rho^{2} \tilde{\rho} \left( \begin{array}{c} 0 \\ 1 \end{array} \right), (\phi^{\lambda})^{2} \rho \left( \begin{array}{c} -i \\ 0 \end{array} \right) \right\}$$

$$= 0. \tag{I.8}$$

This last line follows from the observations that the column vector functions $(\begin{array}{c} 0 \\ 1 \end{array})$ and $(\begin{array}{c} 1 \\ 0 \end{array})$ are “disjoint”, and the operator $\sigma_{3} \left[ \left( -\Delta + V + \lambda \right) J + iE(\lambda) \right]^{-1}$ is diagonal.

Now we choose a “difficult” term in $\Phi_{4}$ to study:

$$D_{2,2}^{(1)} := \left( \begin{array}{c} A_{2,1}^{(1)} \partial_{x} \phi^{\lambda} \\ A_{2,1}^{(2)} \phi^{\lambda} \end{array} \right) \times \left( \begin{array}{c} -i(\phi^{\lambda})^{2} \omega \\ (\phi^{\lambda})^{2} \rho \end{array} \right).$$

Put the definitions of $A_{2,1}^{(1)}$ and $A_{2,1}^{(2)}$ in (B.2) into the expression to find

$$D_{2,2}^{(1)} = \frac{1}{iE(\lambda)\langle \phi^{\lambda}, \partial_{x} \phi^{\lambda} \rangle} D_{2,2}^{(1)} - \frac{1}{2E(\lambda)\langle \phi^{\lambda}, \partial_{x} \phi^{\lambda} \rangle} D_{2,2}^{(2)} + \frac{A_{2,1}^{(1)}(\phi^{\lambda}, (\phi^{\lambda})^{2} \rho)}{iE(\lambda)}$$

with

$$D_{2,2}^{(1)} := \langle N_{2,1}^{\text{Im}}, \phi^{\lambda} \rangle \langle \partial_{x} \phi^{\lambda}, \phi^{\lambda} \rangle + \langle N_{2,1}^{\text{Re}}, \partial_{x} \phi^{\lambda} \rangle \langle \phi^{\lambda}, (\phi^{\lambda})^{2} \rho \rangle.$$
and

\[D^{(2)}_{2,2} := \gamma_{1,1}[(\omega, \phi^\lambda)\langle \partial_{\lambda} \phi^\lambda, -i(\phi^\lambda)^2 \omega \rangle + i(\rho, \partial_{\lambda} \phi^\lambda)\langle \phi^\lambda, (\phi^\lambda)^2 \rho \rangle]
\]

\[= \gamma_{1,1}[(\omega - \rho, \phi^\lambda)\langle \partial_{\lambda} \phi^\lambda, -i(\phi^\lambda)^2 \omega \rangle + i(\rho - \omega, \partial_{\lambda} \phi^\lambda)\langle \phi^\lambda, (\phi^\lambda)^2 \rho \rangle],\]

where in the last step the facts \(\phi^\lambda \perp \rho, \partial_{\lambda} \phi^\lambda \perp \omega\) in (2.5) are used.

Now, we estimate all the three terms in the definition of \(D^{(2)}_{2,2}\).

It is easy to see that the third term is of the order \(O(\delta^3_\infty |z|^4)\) by the estimate of \(A^{(1)}_{2,1}\) in Proposition D.1.

By the estimate \(\rho - \omega = O(\delta^2_\infty |z|)\) it is not hard to obtain

\[D^{(2)}_{2,2} - \frac{1}{2E(\lambda)\langle \phi^\lambda, \partial_{\lambda} \phi^\lambda \rangle}D^{(2)}_{2,2} = O(\delta^3_\infty |z|^4).\]

To estimate \(D^{(1)}_{2,1}\), we use the definitions of \(N^{Re}_{2,1}\) and \(N^{Im}_{2,1}\) in (B.11) and various estimates in Proposition D.1 to find

\[N^{Im}_{2,1} = -\frac{1}{4}\rho^2 \bar{\rho} + O(\delta_\infty |z|^3), \quad N^{Re}_{2,1} = -\frac{1}{4}\rho^2 \bar{\rho} + O(\delta_\infty |z|^3).\]

Put this into the expression of \(D^{(1)}_{2,1}\) and use (2.1) and (D.1) on \(\phi^\lambda\) and \(\partial_{\lambda} \phi^\lambda\), after the cancelation of the terms of order \(\delta^2_\infty |z|^4\) we obtain

\[D^{(1)}_{2,1}, \quad \frac{1}{iE(\lambda)\langle \phi^\lambda, \partial_{\lambda} \phi^\lambda \rangle}D^{(1)}_{2,1} = O(\delta^3_\infty |z|^4).\]

Collecting all the estimates, we obtain

\[D_{2,2} = O(\delta^3_\infty |z|^4).\]

The proof is complete. \(\blacksquare\)

**Proof of Lemma I.1 for \(\sigma > 1\).** In the case \(\sigma > 1\), the strategy has to be different since some observations for \(\sigma = 1\), for example (I.8), do not hold any more. Instead, we use an important observation resulted from Theorem 3.2 whose second statement requires that \(\xi^{lin}_n = (x_n/|x|)\xi(|x|), n = 1, 2, \ldots, N = d\), for some function \(\xi^{lin}\). By Lemma 2.2 this implies that

\[\left( \begin{array}{c} \xi_n \\ i\eta_n \end{array} \right) = \frac{x_n}{|x|} \left( \begin{array}{c} \xi(|x|) \\ i\eta(|x|) \end{array} \right)\]

for some real functions \(\xi\) and \(\eta\).

This makes \(\Omega_{\sigma > 1} = 0\) by observing \(\xi_m \eta_n - \xi_n \eta_m = 0\).

In estimating the other terms on the right-hand side of (I.3), we only study \(\Phi_5\), the estimation on the other terms are similar.
After some manipulation similar to that in (I.7) we find

\[ \Phi_5 = iC_3(\sigma)D_1 + iC_4(\sigma)D_2 + O(\delta^{-1}_\infty |x|^4) \]

for some \( C_3, C_4 \in \mathbb{R} \), \( \delta^{-1}_\infty \)

where \( D_1 \) and \( D_2 \) are defined as

\[
\begin{align*}
D_1 &= \langle [-\Delta + V + \lambda + E(\lambda)]^{-1} P_\text{lin} (\phi^\lambda)^{2\sigma - 2} \rho^2 \rho, (\phi^\lambda)^{2\sigma} \rho \rangle, \\
D_2 &= \langle [-\Delta + V + \lambda - E(\lambda)]^{-1} P_\text{lin} (\phi^\lambda)^{2\sigma - 2} \rho^2 \rho, (\phi^\lambda)^{2\sigma} \rho \rangle.
\end{align*}
\]

We claim that

\[ D_1, D_2 \in \mathbb{R}. \]

If the claim holds then it together with (I.9) yields the desired estimate

\[ \text{Re} \Phi_5 = O(\delta^{-1}_\infty |x|^4). \]

Now we prove the claim for \( D_1 \), the proof for \( D_2 \) is almost the same. The facts

\[ \xi_k = (x_k/|x|)\xi(|x|) \]

and the potential \( V \) and \( \phi^\lambda \) are spherically symmetric imply

\[
D_1 = \sum_{k,l \leq N = d, k \neq l} z_k^2 |z_l|^2 D(k,l) + \sum_{k,l \leq N = d, k \neq l} |z_k|^2 |z_l|^2 D(k,l) + \sum_{k \leq N} |z_k|^4 D(k,k)
\]

\[
= D(1,2) \sum_{k,l \leq N = d, k \neq l} z_k^2 |z_l|^2 + D(1,2) \sum_{k,l \leq N = d, k \neq l} |z_k|^2 |z_l|^2 + D(1,1) \sum_{k \leq N} |z_k|^4,
\]

where \( D(k,l) := \langle [-\Delta + V + \lambda + E(\lambda)]^{-1} P_\text{lin} (\phi^\lambda)^{2\sigma - 2} \xi_k^2 \xi_l^2 \rangle \). The last line follows from the observations \( D(k,l) = D(1,2) \in \mathbb{R} \) if \( k \neq l \) and \( D(k,k) = D(1,1) \in \mathbb{R} \) resulted from the permutation of coordinates. By observing \( \sum_{k,l \leq N = d, k \neq l} z_k^2 |z_l|^2 = | \sum_k z_k^2 | - \sum_l |z_l|^4 \in \mathbb{R} \) we have \( D_1 \in \mathbb{R} \).

The proof is complete. \[ \blacksquare \]

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