Abstract. We prove that the class of convolution-type kernels satisfying suitable decay conditions of the Fourier transform, appearing in the works of Christ [4], Christ-Rubio de Francia [6] and Duoandikoetxea-Rubio de Francia [13] gives rise to maximally truncated singular integrals satisfying a sparse bound by \((1 + \varepsilon, 1 + \varepsilon)\)-averages for all \(\varepsilon > 0\), with linear growth in \(\varepsilon^{-1}\). This is an extension of the sparse domination principle by Conde-Alonso, Culiuc, Ou and the first author [7] to maximally truncated singular integrals. Our results cover the rough homogeneous singular integrals on \(\mathbb{R}^d\)

\[ T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^d} f(x - t) \frac{\Omega(t/|t|)}{|t|^d} \, dt \]

with angular part \(\Omega \in L^\infty(S^{d-1})\) and having vanishing integral on the sphere. Consequences of our sparse bound include novel quantitative weighted norm estimates as well as Fefferman-Stein type inequalities. In particular, we obtain that the \(L^2(w)\) norm of the maximal truncation of \(T_\Omega\) depends quadratically on the Muckenhoupt constant \([w]_{A_2}\), extending a result originally by Hytönen, Roncal and Tapiola [16]. A suitable convex-body valued version of the sparse bound is also deduced and employed towards novel matrix weighted norm inequalities for the maximal truncated rough homogeneous singular integrals. Our result is quantitative, but even the qualitative statement is new, and the present approach via sparse domination is the only one currently known for the matrix weighted bounds of this class of operators.

1. Introduction and main results

Let \(\eta \in (0, 1)\). A countable collection \(S\) of cubes of \(\mathbb{R}^d\) is said to be \(\eta\)-sparse if there exist measurable sets \(\{E_I : I \in S\}\) such that

\[ E_I \subset I, \ |E_I| \geq \eta |I|, \quad I, J \in S, I \neq J \implies E_I \cap E_J = \emptyset. \]

Let \(T\) be a sublinear operator mapping the space \(L^\infty_0(\mathbb{R}^d)\) of complex-valued, bounded and compactly supported functions on \(\mathbb{R}^d\) into locally integrable functions. We say that \(T\) has the sparse \((p_1, p_2)\) bound [10] if there exists a constant \(C > 0\) such that for all \(f_1, f_2 \in L^\infty_0(\mathbb{R}^d)\) we may find...
a $\frac{1}{2}$-sparse collection $\mathcal{S} = S(f_1, f_2)$ such that
\[
|\langle T f_1, f_2 \rangle| \leq C \sum_{Q \in \mathcal{S}} |Q| \prod_{j=1}^{2} \langle f_j \rangle_{p_j, Q}
\]
in which case we denote by $\|T\|_{(p_1,p_2),\text{sparse}}$ the least such constant $C$. As customary,
\[
\langle f \rangle_{p, Q} = \frac{\| f 1_Q \|_p}{|Q|^\frac{1}{p}}, \quad p \in (0, \infty].
\]

Estimating the sparse norm(s) of a sublinear or multisublinear operator entails a sharp control over the behavior of such operator in weighted $L^p$-spaces; this theme has been recently pursued by several authors, see for instance [1, 8, 18, 20, 21, 33]. This sharp control is exemplified in the following proposition, which is a collection of known facts from the indicated references.

**Proposition 1.1.** Let $T$ be a sublinear operator on $\mathbb{R}^d$ mapping $L^p_0(\mathbb{R}^d)$ to $L^1_\text{loc}(\mathbb{R}^d)$. Then the following hold.

1. [7, Appendix B] Let $1 \leq p_1, p_2 < \infty$. There is an absolute constant $C_{p_2} > 0$ such that
\[
\|T : L^{p_1}(\mathbb{R}^d) \rightarrow L^{p_1,\infty}(\mathbb{R}^d)\| \leq C_{p_2}\|T\|_{(p_1,p_2),\text{sparse}}
\]

2. [12, Proposition 4.1] If
\[
(1.1) \quad \Psi(t) := \|T\|_{(1+\frac{1}{t},1+\frac{4}{t}),\text{sparse}} < \infty \quad \forall t > 1,
\]
then there is an absolute constant $C > 0$ such that
\[
\|T\|_{L^2(\mathbb{R}^d)} \leq C[w]_{A_2} \Psi(C[w]_{A_2}).
\]

In particular,
\[
\sup_{t \rightarrow 1} \Psi(t) < \infty \quad \implies \quad \|T\|_{L^2(\mathbb{R}^d)} \leq C[w]_{A_2}.
\]

In this article, we are concerned with the sparse norms (1.1) of a class of convolution-type singular integrals whose systematic study dates back to the celebrated works by Christ [4], Christ-Rubio de Francia [6], and Duoandikoetxea-Rubio de Francia [13], admitting a decomposition with good decay properties of the Fourier transform. To wit, let $\{K_s : \mathbb{R}^d \rightarrow \mathbb{C}, s \in \mathbb{Z}\}$ be a sequence of (smooth) functions with the properties that
\[
\sup \text{supp } K_s \subset A_s := \{x \in \mathbb{R}^d : 2^{s-d} < |x|_\infty < 2^{s-2}\},
\]
\[
\sup_{s \in \mathbb{Z}} 2^d \|K_s\|_\infty \leq 1,
\]
\[
\sup \sup \max_{s \in \mathbb{Z}, \xi \in \mathbb{R}^d} \{ |2^s \xi|^\alpha, |2^s \xi|^{-\alpha} \} |\hat{K}_s(\xi)| \leq 1,
\]
for some $\alpha > 0$. We consider truncated singular integrals of the type
\[
T f(x, t_1, t_2) = \sum_{t_1 < s \leq t_2} K_s * f(x), \quad t_1, t_2 \in \mathbb{Z},
\]
and their maximal version
\[
(1.3) \quad T_* f(x) := \sup_{t_1 \leq t_2} |T f(x, t_1, t_2)|.
\]
Theorem A. For all $0 < \varepsilon < 1$
\[ \|T_\star\|_{(1+\varepsilon, 1+\varepsilon), \text{sparse}} \leq \frac{1}{\varepsilon}, \]
with absolute dimensional implicit constant, in particular uniform over families $\{K_s\}$ satisfying (1.2).

Theorem A entails immediately a variety of novel corollaries involving weighted norm inequalities for the maximally truncated operators $T_\star$. In addition to, for instance, those obtained by suitably applying the points of Proposition 1.1, we also detail the quantitative estimates below, whose proof will be given in Section 7.

Theorem B. Let $T$ be a sublinear operator satisfying the sparse bound (1.1) with $\Psi(t) \leq Ct$.

1. For any $1 < p < \infty$,
\[ \|T\|_{L^p(w)} \leq [w]_{A_p}^\frac{1}{p} ([w]_{A_p}^\frac{1}{p} + [\sigma]_{A_\infty}^\frac{1}{p}) \max([\sigma]_{A_\infty}, [w]_{A_\infty}) \]
with implicit constant possibly depending on $p$ and dimension $d$; in particular,
\[ \|T\|_{L^p(w)} \leq [w]_{A_p}^{\max\{1, \frac{1}{p}\}} . \]

2. The Fefferman-Stein type inequality
\[ \|Tf\|_{L^p(w)} \leq p^2 (p')^\frac{1}{p} (r')^{1 + \frac{1}{p'}} \|f\|_{L^p(M^p, w)} , \quad r < p \]
holds with implicit constant possibly depending on $d$ only.

3. The $A_q$-$A_\infty$ estimate
\[ \|Tf\|_{L^p(w)} \leq [w]_{A_q}^{\frac{1}{q}} [w]_{A_\infty}^{1 + \frac{1}{p'}} \|f\|_{L^p(w)}, \]
holds for $q < p$ and $w \in A_q$, with implicit constant possibly depending on $p, q$ and $d$ only.

4. The following Coifman-Fefferman type inequality
\[ \|Tf\|_{L^p(w)} \leq \frac{[w]_{A_\infty}^2}{\varepsilon} \|M_{1+\varepsilon}f\|_{L^p(w)} \]
holds for all $\varepsilon > 0$ with implicit constant possibly depending on $p$ and $d$ only.

Remark 1.2. Take $\Omega : S^{d-1} \to \mathbb{C}$ with $\|\Omega\|_\infty \leq 1$ and having vanishing integral on $S^{d-1}$, and consider the associated truncated integrals and their maximal function
\[ T_{\Omega, \delta} f(x) := \int_{\delta < |t| < \frac{1}{\delta}} f(x - t) \frac{\Omega(t/|t|)}{|t|^d} \, dt, \quad T_{\Omega, \star} f(x) := \sup_{\delta > 0} \left| T_{\Omega, \delta} f(x) \right|, \quad x \in \mathbb{R}^d. \]
It is well known– for instance, see the recent contribution [16, Section 3]– that
\[ T_{\Omega, \star} f(x) \leq M f(x) + T_{\star} f(x), \quad x \in \mathbb{R}^d \]
with $T_{\star}$ being defined as in (1.3) for a suitable choice of $\{K_s : s \in \mathbb{Z}\}$ satisfying (1.2) with $\alpha = \frac{1}{d}$.

As $\|M\|_{(1,1), \text{sparse}} \leq 1$, a corollary of Theorem A is that
\[ \|T_{\Omega, \star}\|_{(1+\varepsilon, 1+\varepsilon), \text{sparse}} \leq \frac{1}{\varepsilon} . \]
as well. The main result of [7] is the stronger control

\[
\sup_{\delta > 0} \| T_{\Omega,\delta} \|_{(1,1+\varepsilon) \text{-sparse}} \lesssim \frac{1}{\varepsilon}.
\]

The above estimate, in particular, is stronger than the uniform weak type \((1,1)\) for the operators \(T_{\Omega,\delta}\), a result originally due to Seeger [31]. As the weak type \((1,1)\) of \(T_{\Omega,*}\) under no additional smoothness assumption on \(\Omega\) is a difficult open question, estimating the \((1,1+\varepsilon)\) sparse norm \(T_{\Omega,*}\) as in (1.8) seems out of reach.

The study of sharp weighted norm inequalities for \(T_{\Omega,\delta}\) (the uniformity in \(\delta\) is of course relevant here) was initiated in the recent article [16] by Hytönen, Roncal and Tapiola. Improved quantifications have been obtained in [7] as a consequence of the domination result (1.8), and further weighted estimates— including a Coifman-Fefferman type inequality, that is a norm control of \(T_{\Omega,\delta}\) by \(M\) on all \(L^p(w)\), \(0 < p < \infty\) when \(w \in A_{\infty}\)— have been later derived from (1.8) in the recent preprint by the third named author, Pérez, Roncal and Rivera-Ríos [28].

Although (1.7) is a bit weaker than (1.8), we see from comparison of (1.4) from Theorem B with the results of [7, 28] that the quantification of the \(L^2(w)\)-norm dependence on \([w]_{A_{\infty}}\) entailed by the two estimates is the same— quadratic; on the contrary, for \(p \neq 2\), (1.8) yields the better estimate \(\| T_{\Omega,\delta} \|_{L^p(w)} \lesssim [w]^{1/2}_{A_{\infty}}\). We also observe that the proof of the mixed estimate (1.5) actually yields the following estimate for the non-maximally truncated operators, improving the previous estimate given in [28]

\[
\| T_{\Omega,\delta} f \|_{L^p(w)} \lesssim [w]^{1/2}_{A_{\infty}} [w]^{1/p}_{A_p} \| f \|_{L^p(w)}.
\]

Finally, we emphasize that (1.7) also yields a precise dependence on \(p\) of the unweighted \(L^p\) operator norms. Namely, from the sparse domination, we get

\[
\| T_{\Omega,*} \|_{L^p(\mathbb{R}^d) \to L^{p',\infty}(\mathbb{R}^d)} \lesssim \max\{p, p'\}, \quad \| T_{\Omega,*} \|_{L^p(\mathbb{R}^d)} \lesssim pp' \max\{p, p'\}
\]

with absolute dimensional implicit constant, which improves on the implicit constants in [13]. Moreover, we note that the main result of [29] implies that if (1.9) is sharp, then our quantitative weighted estimate (1.4) is also sharp.

**Remark 1.3.** Comparing Theorem A with the sparse domination formula for commutators of Calderón-Zygmund operators with \(BMO\) symbols [23], all our weighted corollaries hold for commutators as well, with the help of John-Nirenberg inequality.

### 1.4. Matrix weighted estimates for vector valued rough singular integrals.

Let \((e_j)_{j=1}^d\), \(\langle \cdot, \cdot \rangle_{\mathbb{F}^n}\) and \(\| \cdot \|_{\mathbb{F}^n}\) be the canonical basis, scalar product and norm on \(\mathbb{F}^n\) over \(\mathbb{F}\), where \(\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}\).

A recent trend in Harmonic Analysis— see, among others, [2, 3, 9, 14, 30]— is the study of quantitative matrix weighted norm inequalities for the canonical extension of the (integral) linear operator \(T\)

\[
\langle Tf(x), e_j \rangle_{\mathbb{F}^n} := \langle T \otimes \text{Id}_{\mathbb{F}^n} f(x), e_j \rangle_{\mathbb{F}^n} = T(\langle f, e_j \rangle_{\mathbb{F}^n})(x), \quad x \in \mathbb{R}^d
\]

to \(\mathbb{F}^n\)-valued functions \(f\). In Section 6 of this paper, we introduce an \(L^p\), \(p > 1\), version of the convex body averages first brought into the sparse domination context by Nazarov, Petrichkl, Treil and Volberg [30], and use them to produce a vector valued version of Theorem A. As a corollary, we obtain quantitative matrix weighted estimates for the maximal truncated vector valued extension of the rough singular integrals \(T_{\Omega,\delta}\) from (1.6). In fact, the next corollary is a special case of the more precise Theorem E from Section 6.
Corollary E.1. Let $W$ be a positive semidefinite and locally integrable $L(\mathbb{F}^n)$-valued function on $\mathbb{R}^d$ and $T_{\Omega,\delta}$ be as in (1.6). Then

$$
(1.10) \quad \left\| \sup_{\delta>0} \left| W^\Delta T_{\Omega,\delta} f \right| \right\|_{L^2(\mathbb{F}^n)} \lesssim [W]_{A_2} \left\| W^\Delta f \right\|_{L^2(\mathbb{R}^d)}
$$

with implicit constant depending on $d$, $n$ only, where the matrix $A_2$ constant is given by

$$
[W]_{A_2} := \sup_{\text{cube of } \mathbb{R}^d} \left( \frac{1}{|Q|} \int_Q W(x) \, dx \right)^\frac{1}{2} \left( \frac{1}{|Q|} \int_Q W^{-1}(x) \, dx \right)^\frac{1}{2} \left\| Q \right\|_{L(\mathbb{F}^n)}^2.
$$

As the left hand side of (1.10) dominates the matrix weighted norm of the vector valued maximal operator first studied by Christ and Goldberg in [5], the finiteness of $[W]_{A_2}$ is actually necessary for the estimate to hold. To the best of the authors’ knowledge, Theorem E has no precedors, in the sense that no matrix weighted norm inequalities for vector rough singular integrals were known before, even in qualitative form. At this time we are unable to assess whether the power $\frac{3}{2}$ appearing in (1.10) is optimal. For comparison, if the angular part $\Omega$ is Hölder continuous, the currently best known result [30] is that (1.10) holds with power $\frac{3}{2}$; see also [9].

1.5. Strategy of proof of the main results. We will obtain Theorem A by an application of an abstract sparse domination principle, Theorem C from Section 3, which is a modification of [7, Theorem C]. At the core of our approach lies a special configuration of stopping cubes, the so-called stopping collections $Q$, and their related atomic spaces. The necessary definitions, together with a useful interpolation principle for the atomic spaces, appear in Section 2. In essence, Theorem C can be summarized by the inequality

$$
\|T_\star\|_{(p_1,p_2),\text{sparse}} \lesssim \|T_\star\|_{L(L^2(\mathbb{R}^d)))} + \sup_{Q_{t_1,t_2}} \left( \|Q_{t_1}\|_{X_{p_1} \times Y_{p_2}} + \|Q_{t_2}\|_{Y_{\sigma} \times X_{p_2}} \right)
$$

where the supremum is taken over all stopping collections $Q$ and all measurable linearizations of the truncation parameters $t_1, t_2$, and $Q_{t_i}^2$ are suitably adapted localizations of the adjoint form to the linearized versions of $T_\star$. In Section 4, we prove the required uniform estimates for the localizations $Q_{t_i}^2$ coming from Dini-smooth kernels. The proof of Theorem A is given in Section 5, relying upon the estimates of Section 4 and the Littlewood-Paley decomposition of the convolution kernels (1.2) whose first appearance dates back to [13].

Remark 1.6. We remark that while this article was being finalized, an alternative proof of (1.8) was given by Lerner [22]. It is of interest whether the strategy of [22], relying on bumped bilinear grand local maximal functions, can be applied towards estimate (1.7) as well.

Notation. With $q' = \frac{q}{q-1}$ we indicate the Lebesgue dual exponent to $q \in (1, \infty)$, with the usual extension $1' = \infty$, $\infty' = 1$. The center and the (dyadic) scale of a cube $Q \in \mathbb{R}^d$ will be denoted by $c_Q$ and $s_Q$ respectively, so that $|Q| = 2^{sd}$. We use the notation

$$
M_p(f)(x) = \sup_{Q \subset \mathbb{R}^d} \langle f \rangle_{p,Q} 1_Q(x)
$$

for the $p$-Hardy Littlewood maximal function and write $M$ in place of $M_1$. Unless otherwise specified, the almost inequality signs $\lesssim$ imply absolute dimensional constants which may be different at each occurrence.
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2. Stopping collections and interpolation in localized spaces

The notion of stopping collection $Q$ with top the (dyadic) cube $Q$ has been introduced in [7, Section 2], to which we send for details. Here, we recall that such a $Q$ is a collection of pairwise disjoint dyadic cubes contained in $3Q$ and satisfying suitable Whitney type properties. More precisely,

\[ (2.1) \quad \bigcup_{L \in Q} 9L \subset \text{sh}Q := \bigcup_{L \in Q} L \subset 3Q, \quad \text{cQ} := \{L \in Q : 3L \cap 2Q \neq \emptyset\}; \]

\[ (2.2) \quad L, L' \in Q, L \cap L' \neq \emptyset \implies L = L'; \]

\[ (2.3) \quad L \in Q, L' \in N(L) \implies |s_L - s_{L'}| \leq 8, \quad N(L) := \{L' \in Q : 3L \cap 3L' \neq \emptyset\}. \]

A consequence of (2.3) is that the cardinality of $N(L)$ is bounded by an absolute constant.

The spaces $Y_p(Q), X_p(Q), \hat{X}_p(Q)$ have also been defined in [7, Section 2]: here we recall that $Y_p(Q)$ is the subspace of $L^p(\mathbb{R}^d)$ of functions satisfying

\[ \text{supp} \ f \subset 3Q, \]

\[ (2.4) \quad \infty > \|f\|_{Y_p(Q)} := \begin{cases} \max \left\{ \|f\|_{\mathbb{R}^d \setminus \text{sh}Q}_\infty, \sup_{L \in Q} \inf_x M_{p,f}(x) \right\} & p < \infty \\ \|f\|_\infty & p = \infty \end{cases} \]

where $\hat{L}$ stands for the (non-dyadic) $2^5$-fold dilate of $L$, and that $X_p(Q)$ is the subspace of $Y_p(Q)$ of functions satisfying

\[ b = \sum_{L \in Q} b_L, \quad \text{supp} \ b_L \subset L. \]

Finally, we write $b \in \hat{X}_p(Q)$ if $b \in X_p(Q)$ and each $b_L$ has mean zero. We will omit $(Q)$ from the subscript of the norms whenever the stopping collection $Q$ is clear from context.

There is a natural interpolation procedure involving the $Y_p$-spaces. We do not strive for the most general result but restrict ourselves to proving a significant example, which is also of use to us in the proof of Theorem A.

Proposition 2.1. Let $B$ be a bisublinear form and $A_1, A_2$ be positive constants such that the estimates

\[ |B(b, f)| \leq A_1\|b\|_{X_p(Q)}\|f\|_{Y_p(Q)}, \quad |B(g_1, g_2)| \leq A_2\|g_1\|_{Y_p(Q)}\|g_2\|_{Y_p(Q)} \]

Then for all $0 < \varepsilon < 1$

\[ |B(f_1, f_2)| \leq (A_1)^{1-\varepsilon}(A_2)^{\varepsilon}\|f_1\|_{X_p(Q)}\|f_2\|_{Y_p(Q)}^\varepsilon, \quad p = \frac{2}{2 - \varepsilon}. \]
Proof. We may assume $A_2 < A_1$, otherwise there is nothing to prove. We are allowed to normalize $A_1 = 1$. Fixing now $0 < \varepsilon < 1$, so that $1 < p < 2$, it will suffice to prove the estimate

$$|B(f_1, f_2)| \lesssim (A_2)^p$$

for each pair $f_1 \in \dot{X}_p(Q), f_2 \in \mathcal{Y}_p(Q)$ with $\|f_1\|_{X_p} = \|f_2\|_{Y_p} = 1$ with implied constant depending on dimension only. Let $\lambda \geq 1$ to be chosen later. Using the notation $f_{>\lambda} := f 1_{|f| > \lambda}$, we introduce the decompositions

$$f_1 = g_1 + b_1, \quad b_1 := \sum_{Q \in \mathcal{Q}} \left( (f_1)_{>\lambda} - \frac{1}{|Q|} \int_Q (f_1)_{>\lambda} \right) 1_Q, \quad f_2 = g_2 + b_2, \quad b_2 := (f_2)_{>\lambda}$$

which verify the properties

$$g_1 \in \dot{X}_2(Q), \quad \|g_1\|_{X_p} \leq 1, \quad \|g_1\|_{X_1} \leq \lambda^{1-p}, \quad b_1 \in \dot{X}_1(Q), \quad \|b_1\|_{X_1} \leq \lambda^{1-p}$$

$$\|g_2\|_{X_1} \leq \lambda^{1-p}, \quad \|b_2\|_{X_1} \leq \lambda^{1-p}.$$

We have used that $b_1$ is supported on the union of the cubes $Q \in \mathcal{Q}$ and has mean zero on each $Q$, and therefore $g_1$ has the same property, given that $f_1 \in \dot{X}_p(Q)$. Therefore

$$|B(f_1, f_2)| \leq |B(b_1, b_2)| + |B(b_1, g_2)| + |B(g_1, b_2)| + |B(g_1, g_2)|$$

$$\leq \|b_1\|_{X_1} \|b_2\|_{Y_1} + \|b_1\|_{X_1} \|g_2\|_{Y_1} + \|g_1\|_{X_1} \|b_2\|_{Y_1} + A_2 \|g_1\|_{Y_1} \|g_2\|_{Y_1}
\leq \lambda^{2-2p} + 2\lambda^{1-p} + A_2 \lambda^{2-2p} \leq \lambda^{2-2p}(1 + A_2 \lambda^{p})$$

which yields (2.6) with the choice $\lambda = A_2^{-\frac{1}{2}}$. \qed

3. A sparse domination principle for maximal truncations

We consider families of functions $[K] = \{K_s : s \in \mathbb{Z}\}$ satisfying

$$\text{supp } K_s \subset \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| < 2^s\},$$

$$\|\{K_s\}\| := \sup_{s \in \mathbb{Z}} 2^{sd} \sup_{x \in \mathbb{R}^d} (\|K_s(x, \cdot)\|_\infty + \|K_s(\cdot, x)\|_\infty) < \infty$$

and associate to them the linear operators

$$T[K]f(x, t_1, t_2) := \sum_{t_1 < s \leq t_2} \int_{\mathbb{R}^d} K_s(x, y)f(y) \, dy, \quad x \in \mathbb{R}^d, \ t_1, t_2 \in \mathbb{Z}$$

and their sublinear maximal versions

$$T_{**}[K]f(x) := \sup_{t_1 \leq t_1 \leq t_2 \leq t_2} |T[K]f(x, \tau_1, \tau_2)|, \quad T_{*}[K]f(x) = \sup_{t_1 \leq t_2} |T[K]f(x, t_1, t_2)|.$$

We assume that there exists $1 < r < \infty$ such that

$$\|\{K_s\}\|_{L^r(\mathbb{R}^d)} < \infty.$$ 

For pairs of bounded measurable functions $t_1, t_2 : \mathbb{R}^d \to \mathbb{Z}$, we also consider the linear operators

$$T[K]_{t_1, t_2}f(x) := T[K]f(x, t_1(x), t_2(x)), \quad x \in \mathbb{R}^d.$$
**Remark 3.1.** From the definition (3.2), it follows that
\[ t_1, t_2 \in \mathbb{Z}, t_1 \geq t_2 \implies T[K]f(x, t_1, t_2) = 0. \]

In consequence, for the linearized versions defined in (3.4) we have
\[ \text{supp} \, T[K]^{\pm}_{t_1} f \subset \{ x \in \mathbb{R}^d : t_2(x) - t_1(x) > 0 \}. \]

A related word on notation: we will be using linearizations of the type \( T[K]^{\pm}_{t_1} \) and similar, where \( s_Q \) is the (dyadic) scale of a (dyadic) cube \( Q \). With this we mean we are using the constant function equal to \( s_Q \) as our upper truncation function. Finally, we will be using the notations \( t_2 \land s_Q \) for the linearizing function \( x \mapsto \min\{t_2(x), s_Q\} \) and \( t_1 \lor s_L \) for the linearizing function \( x \mapsto \max\{t_1(x), s_L\} \).

Given two bounded measurable functions \( t_1, t_2 \) and a stopping collection \( Q \) with top \( Q \), we define the localized truncated bilinear forms
\[
(3.5) \quad Q[K]^{\pm}_{t_i}(f_1, f_2) := \frac{1}{|Q|} \left[ \left\langle T[K]^{t_2 \lor s_Q}_{t_1}(f_1 1_Q), f_2 \right\rangle - \sum_{L \leq Q} \left\langle T[K]^{t_2 \lor s_Q}_{t_1}(f_1 1_L), f_2 \right\rangle \right].
\]

**Remark 3.2.** Note that we have normalized by the measure of \( Q \), unlike the definitions in [7, Section 2]. Observe that as a consequence of the support assumptions in (3.1) and of the largest allowed scale being \( s_Q \), we have
\[
Q[K]^{\pm}_{t_1}(f_1, f_2) = Q[K]^{\pm}_{t_1}(f_1 1_Q, f_2 1_Q).
\]

Similarly we remark that \( T[K]^{t_2 \lor s_Q}_{t_1}(f 1_L) \) is supported on the set \( 3L \cap \{ x \in \mathbb{R}^d : s_L - t_1(x) > 0 \} \); see Remark 3.1.

Within the above framework, we have the following abstract theorem.

**Theorem C.** Let \( [K] = \{ K_s : s \in \mathbb{Z} \} \) be a family of functions satisfying (3.1) and (3.3) above. Assume that there exist \( 1 \leq p_1, p_2 < \infty \) such that
\[
(3.6) \quad \sup_{\|b\|_{Y_{p_1}(Q)} = 1} \left| Q[K]^{\pm}_{t_1}(b, f) \right| + \sup_{\|f\|_{Y_{p_2}(Q)} = 1} \left| Q[K]^{\pm}_{t_1}(f, b) \right| =: C_L([K])(p_1, p_2) < \infty.
\]

hold uniformly over all bounded measurable functions \( t_1, t_2 \), and all stopping collections \( Q \). Then
\[
(3.7) \quad \| T_\star [K] \|_{(p_1, p_2), \text{sparse}} \lesssim \|[K]\|_{r_*} + C_L([K])(p_1, p_2).
\]

**Proof.** The proof follows essentially the same scheme of [7, Theorem C]; for this reason, we limit ourselves to providing an outline of the main steps.

**Step 1. Auxiliary estimate.** First of all, an immediate consequence of the assumptions of the Theorem is that the estimate
\[
(3.8) \quad \left| Q[K]^{\pm}_{t_1}(f_1, f_2) \right| \leq C_0([K], p_1, p_2) \|f_1\|_{Y_{p_1}(Q)} \|f_2\|_{Y_{p_2}(Q)}
\]
where \( \Theta_{(k_1, p_1, p_2)} := \|\Theta\|_{r, \ast} + C L\|K\|_{p_1, p_2} \), holds with \( C > 0 \) uniform over bounded measurable functions \( t_1, t_2 \). See [7, Lemma 2.7]. Therefore,

\[
\left| \left< T[K]_{t_1}^{\mathfrak{L}_{Q}}(f_1 1_{\Omega}), f_2 \right> \right| \leq C \Theta_{(k_1, p_1, p_2)}(Q) \left( \|f_1\|_{L^p(Q)} \|f_2\|_{L^p(Q)} + \sum_{L \in Q} \left| \left< T[K]_{t_1}^{\mathfrak{L}_{Q}}(f_1 1_{L}), f_2 \right> \right| \right)
\]

Step 2. Initialization. The argument begins as follows. Fixing \( f_j \in L^p(\mathbb{R}^d) \), \( j = 1, 2 \) with compact support, we may find measurable functions \( t_1, t_2 \) which are bounded above and below and a large enough dyadic cube \( Q_0 \) from one of the canonical \( 3^d \) dyadic systems such that \( \text{supp} f_1 \subset Q_0 \), \( \text{supp} f_2 \subset 3Q_0 \) and

\[
\left< T_*[K]f_1, f_2 \right> \leq 2 \left| \left< T[K]_{t_1}^{\mathfrak{L}_{Q_0}}(f_1 1_{Q_0}), f_2 \right> \right|
\]

and we clearly can replace \( f_2 \) by \( |f_2| \) in what follows.

Step 3. Iterative process. Then, the argument proceeds via iteration over \( k \) of the following construction, which follows from (3.9) and the Calderón-Zygmund decomposition and is initialized by taking \( S_k = \{Q_0\} \) for \( k = 0 \). Given a disjoint collection of dyadic cubes \( Q \in S_k \) with the further Whitney property that (2.3) holds for \( S_k \) in place of \( Q \), there exists a further collection of disjoint dyadic cubes \( L \in S_{k+1} \) such that

* (2.3) for \( S_k \) in place of \( Q \) continues to hold,
* each subcollection \( S_{k+1}(Q) = \{L \in S_{k+1} : L \subset 3Q\} \) is a stopping collection with top \( Q \),

and for which for all \( Q \in S_k \) there holds

\[
\left| \left< T[K]_{t_1}^{\mathfrak{L}_{Q}}(f_1 1_{Q}), f_2 \right> \right| \leq C |Q| \Theta_{(k_1, p_1, p_2)}(f_1) \left( \|f_1\|_{L^p(Q)} \|f_2\|_{L^p(Q)} + \sum_{L \in S_{k+1}(Q)} \left| \left< T[K]_{t_1}^{\mathfrak{L}_{Q}}(f_1 1_{L}), f_2 \right> \right| \right).
\]

More precisely, \( S_{k+1} \) is composed by the maximal dyadic cubes \( L \) such that

\[
9L \subset \bigcup_{Q \in S_k} E_Q, \quad E_Q := \left\{ x \in 3Q : \max_{j=1,2} \frac{M_{p_j}(f_{j3Q})(x)}{M_{p_j}(f_{j3Q})} > C \right\}
\]

for a suitably chosen absolute large dimensional constant \( C \). This construction, as well as the Whitney property (2.3) results into

\[
\left| Q \cap \bigcup_{L \in S_{k+1}} L \right| = \left| Q \cap \bigcup_{Q \in S_k : Q \in N(Q)} E_Q \right| \leq \frac{1}{2} |Q| \quad \forall Q \in S_k, k = 0, 1, ...
\]

guaranteeing that \( T_k := \bigcup_{k=0}^k S_k \) is a sparse collection for all \( k \). When \( k = \tilde{k} \) is such that \( \inf \{s_Q : Q \in S_k \} < \inf t_1 \), the iteration stops and the estimate

\[
\left< T_*[K]f_1, f_2 \right> \leq \Theta_{(k_1, p_1, p_2)} \sum_{Q \in T_k} \left| Q \right| \left< f_1, f_2 \right>_{p_1, 3Q}
\]

is reached. This completes the proof of Theorem C. \( \square \)
4. Preliminary localized estimates for the truncated forms (3.5)

We begin by introducing our notation for the Dini constant of a family of kernels \([K]\) as in (3.1). We write

\[
\|\|K\|\|_{\text{Dini}} := \|\|K\|\| + \sum_{j=0}^{\infty} \sigma_j([K])
\]

where

\[
\sigma_j([K]) := \sup_{s \in \mathbb{Z}} 2^{sd} \sup_{x \in \mathbb{R}^d} \sup_{h \in \mathbb{R}^d \atop |h| < 2^{j-3}} \left( \|K_s(x, x + \cdot) - K_s(x + h, x + \cdot)\|_\infty + \|K_s(x + \cdot, x) - K_s(x + \cdot, x + h)\|_\infty \right)
\]

The estimates contained within the lemmata that follow are meant to be uniform over all measurable functions \(t_1, t_2\) and all stopping collections \(Q\). The first one is an immediate consequence of the definitions: for a full proof, see [7, Lemma 3.2].

**Lemma 4.1.** Let \(1 < r < \infty\). Then

\[
|Q[K]^{1/2}_{t_1}(f_1, f_2)| \leq \|\|K\|\|_r \|f_1\|_{Y_r(Q)} \|f_2\|_{Y_r(Q)}
\]

The second one is a variant of [7, Lemma 3.2]; we provide a full proof.

**Lemma 4.2.** There holds

\[
|Q[K]^{1/2}_{t_1}(b, f)| \leq \|\|K\|\|_{\text{Dini}} \|b\|_{X_{t_1}(Q)} \|f\|_{Y_{t_1}(Q)}
\]

**Proof.** We consider the family \(K\) fixed and use the simplified notation \(Q_{t_1}^2\) in place of \(Q[K]_{t_1}^2\), and similarly for the truncated operators \(T[K]\). By horizontal rescaling we can assume \(|Q| = 1\). Let \(b \in X_{t_1}\). Recalling the definition (2.5) and using bilinearity of \(Q_{t_1}^2\) it suffices for each stopping cube \(R \in Q\) to prove that

\[
|Q_{t_1}^2(b, f)| \leq \|\|K\|\|_{\text{Dini}} \|b\|_{1} \|f\|_{Y_{t_1}}
\]

as \(\|b\|_{\infty} \leq |R| \|b\|_{X_{t_1}}\), and conclude by summing up over the disjoint \(R \in Q\), whose union is contained in \(3Q\). We may further assume \(R \subset Q\); otherwise \(Q_{t_1}^2(b, f) = 0\). In addition we can assume \(f\) is positive, by repeating the same argument below with the real and imaginary, and positive and negative parts of \(f\). Using the definition of the truncated forms (3.5) and the disjointness of \(L \in Q\),

\[
|Q_{t_1}^2(b, f)| = \left| \left\langle T_{t_1}^{1, \lambda Q}(b, f) - T_{t_1}^{1, \lambda Q}(b, f), f \right\rangle \right| = \left| \left\langle T_{t_1 \vee \delta Q}^{1, \lambda Q}(b, f), f \right\rangle \right| \leq \left\langle T_{t_1 \vee \delta Q}^{1, \lambda Q} b, f \right\rangle.
\]

Thus, if \(R_s\) denotes the cube concentric to \(R\) and whose sidelength is \(2^{10+s}\), using the support conditions and abbreviating a standard calculation

\[
|Q_{t_1}^2(b, f)| \leq \left\langle T_{t_1 \vee \delta Q}^{1, \lambda Q} b, f \right\rangle \leq \sum_{s = \delta_1 + 1}^{\delta Q} \int_{R_s} \int_{R_s} K_s(x, y) b_R(y) dy \right| f(x) dx
\]

which is bounded by the right hand side of (4.2).  \(\square\)
The third localized estimate is new. However, its roots lie in the well-known principle that the maximal truncations of a Dini-continuous kernel to scales larger than $s$ do not oscillate too much on a ball of radius $2^s$, see (4.7). This was recently employed, for instance, in [11, 16, 19].

**Lemma 4.3.** There holds

$$\left| Q[K]^2_{1i}(f, b) \right| \leq (\| [K] \|_{\text{Dini}} \lor \| K \|_{r,*}) \| f \|_{Y_\infty(Q)} \| b \|_{X_1(Q)}.$$  

**Proof.** We use similar notation as in the previous proof and again we rescale to $|Q| = 1$, and work with positive $b \in X_1$. We can of course assume that $\text{supp} f \subset Q$. We begin by removing an error term; namely, referring to notation (2.1), if

$$b_o = \sum_{R \in cQ} b_R$$

then

$$\left| Q[K]^2_{1i}(f, b_o) \right| \leq \left\langle T f(\cdot, s_Q - 1, s_Q), b_o \right\rangle \leq \| [K] \| \| b \|_{X_1} \| f \|_{Y_\infty}.$$  

The first inequality holds because $\text{dist}(\text{supp} f, \text{supp} b_o) > 2^{s_Q - 1}$, so at most the $s_Q$ scale may contribute, and in particular no contribution comes from cubes $L \subseteq Q$. The second inequality is a trivial estimate, see [7, Appendix A] for more details. Thus we may assume $b_R = 0$ whenever $R \notin cQ$. Then by support considerations

$$\left\langle T_{t_1^{s_Q \wedge s_L}}(f 1_L), b_R \right\rangle \neq 0 \implies L \in N(R).$$

Similarly,

$$\left\langle T_{t_1^{s_Q \wedge s_L}} f, b_R \right\rangle = \left\langle T_{t_1^{s_Q \wedge s_L}}(f 1_{Q}), b_R \right\rangle = \sum_{L \in N(R)} \left\langle T_{t_1^{s_Q \wedge s_L}}(f 1_L), b_R \right\rangle.$$  

In fact, using (2.1) we learn that $\text{dist}(\text{sh} Q, R) > 2^{s_R}$, whence the first equality. Therefore, subtracting and adding the last display to obtain the second equality,

$$Q_{s_1}^2(f, b_R) = \left\langle T_{t_1^{s_Q \wedge s_L}} f, b_R \right\rangle - \sum_{L \in N(R)} \left\langle T_{t_1^{s_Q \wedge s_L}}(f 1_L), b_R \right\rangle$$

$$= \left\langle T_{t_1^{s_Q \wedge s_L}} f, b_R \right\rangle - \sum_{L \in N(R)} \text{sign}(s_L - s_R) \left\langle T_{t_1^{s_Q \wedge s_L}}(f 1_L), b_R \right\rangle.$$  

Now, the summation in the above display is then bounded in absolute value by

$$\sum_{L \in N(R)} \left\langle T_{s_L \vee s_R} (f 1_L), b_R \right\rangle \lesssim \| [K] \| \sum_{L \in N(R)} \| b_R \|_1 (f)_1, L \lesssim \| [K] \|_{\text{Dini}} \| b_R \|_1 \| f \|_{Y_\infty},$$

using that $|s_L - s_R| \leq 8$ whenever $L \in N(R)$. Therefore when $R \in cQ$

$$|Q_{s_1}^2(f, b_R)| \leq \left\langle T_{t_1^{s_Q \wedge s_L}} f, b_R \right\rangle + C \| [K] \|_{\text{Dini}} \| b_R \|_1 \| f \|_{Y_\infty},$$  

with absolute constant $C$. Now, define the function

$$F(x) = \begin{cases} \sup_{S \leq T_1 \leq T_2 \leq s_Q} |T f(x, T_1, T_2)| & x \in R \in cQ, \\ 0 & x \notin \bigcup_{R \in cQ} R \end{cases}$$

for
and notice that $|T_{t_1 \vee s_k}^{{\xi, \alpha, \zeta}} f| \leq F$ on $R \in cQ$. Since $b$ is positive, using (4.3), summing (4.4) over $R \in cQ$ and using that this is a pairwise disjoint collection, we obtain that

$$
|Q^i_1(f, b)| \leq |Q^i_1(f, b_0)| + \sum_{R \in cQ} |Q^i_1(f, b_R)| 
$$

(4.5)

\[ \leq C[[K]]\|Dini\|b\|y_1\|f\|y_\infty \sum_{R \in cQ} \left|\left(T_{t_1 \vee s_k}^{{\xi, \alpha, \zeta}} f, b_R\right)\right| \]

\[ \leq C[[K]]\|Dini\|b\|x_1\|f\|y_\infty + \langle F, b_R \rangle = C[[K]]\|Dini\|b\|x_1\|f\|y_\infty + \langle F, b \rangle. \]

Therefore, we are left with bounding $\langle F, b \rangle$. This is actually done using both the $L'$ estimate and the Dini cancellation condition. In fact, decompose

$$
b = g + z, \quad g = \sum_{R \in cQ} g_R := \sum_{R \in cQ} \langle b \rangle_{1,R} 1_R, \quad z = \sum_{R \in cQ} z_R := \sum_{R \in cQ} (b - \langle b \rangle_{1,R}) 1_R
$$

so that

$$
\|g\|_{y_\infty} \leq \|b\|_{x_1}, \quad \|z\|_{x_1} \leq 2\|b\|_{x_1}.
$$

Then

(4.6) \[ \langle F, g \rangle \leq \langle T_{w} f, g \rangle \leq ||[K]||_r \|f\|_\infty \|g\|_\infty \leq ||[K]||_r \|g\|_{y_\infty} \|f\|_{y_\infty} \leq ||[K]||_r \|f\|_{y_\infty} \|b\|_{x_1}, \]

and we are left to control $\langle F, z \rangle$. We recall from [16, Lemma 2.1] the inequality

(4.7) \[ |T f(x, \tau_1, \tau_2) - T f(\xi, \tau_1, \tau_2)| \leq ||[K]||_{Dini} \sup_{s \geq \tau_R} \langle f \rangle_{1,R}, \quad x, \xi \in R, \tau_2 \geq \tau_1 \geq \tau_R, \]

where $R_s$ is the cube concentric with $R$ and side length $2^s$, whence for suitable absolute constant $C$

$$
F(x) \leq F(\xi) + C||[K]||_{Dini} f_{y_1}, \quad x, \xi \in R
$$

and taking averages there holds

$$
\sup_{x \in R} |F(x) - \langle F \rangle_{1,R}| \leq ||[K]||_{Dini} ||f||_{y_1}.
$$

Finally, using the above display and the fact that each $z_R$ has zero average and is supported on $R$,

(4.8) \[ |\langle F, z \rangle| \leq \sum_{R \in cQ} |\langle F, z_R \rangle| = \sum_{R \in cQ} |\langle F - \langle F \rangle_{1,R} 1_R, z_R \rangle| \leq ||[K]||_{Dini} ||f||_{y_1} \sum_{R \in cQ} \|z_R\|_{l_1} \]

and collecting (4.5), (4.6) and (4.8) completes the proof of the Lemma. \hfill \Box

By Proposition 2.1 applied to the forms $Q^i_1[K]$, we may interpolate the bound of Lemma 4.2 with the one of Lemma 4.1 with $r = 2$. A similar but easier procedure allows to interpolate Lemma 4.3 with Lemma 4.1 with $r = 2$. We summarize the result of such interpolations in the following lemma.

**Lemma 4.4.** For $0 \leq \varepsilon \leq 1$ and $p = \frac{2}{2-\varepsilon}$ there holds

$$
C_L[K](p, p) \leq (\|K\|_{Dini} \vee \|K\|_{l_2, *})^{1-\varepsilon} (\|K\|_{l_2, *})^\varepsilon
$$

where $C_L[K](p_1, p_2)$ is defined in (3.6).
Remark 4.5 (Calderón-Zygmund theory). Let $T$ be an $L^2(\mathbb{R}^d)$-bounded singular integral operator with Dini-continuous kernel $K$. Then its maximal truncations obey the estimate

$$T_* f(x) := \sup_{\delta > 0} \left| \int_{|\delta| < \frac{1}{2}} K(x, x + h) f(x + h) \, dh \right| \lesssim M f(x) + T_*[K] f(x),$$

with the family $[K] := \{K_s : s \in \mathbb{Z}\}$ defined by

$$K_s(x, x + h) := K(x, x + h)\psi(2^{-s}h), \quad x, h \in \mathbb{R}^d$$

where the smooth radial function $\psi$ satisfies

$$\text{supp } \psi \subset \{ h \in \mathbb{R}^d : 2^{-2} < |h| < 1 \}, \quad \sum_{s \in \mathbb{Z}} \psi(2^{-s}h) = 1, \quad h \neq 0.$$

We know from classical theory [32, Ch. I.7] that $\|[K]\|_{2, \ast} \lesssim \|T\|_{L^2(\mathbb{R}^d)} + \|K\|_{\text{Dini}}$. Therefore, in consequence of (4.9) and of the bound $\|M\|_{(1, 1), \text{sparse}} \lesssim 1$, an application of Theorem C in conjunction with Lemmata 4.2 and 4.3 yields that

$$\|T_*[K]\|_{(1, 1), \text{sparse}} \lesssim \|T\|_{L^2(\mathbb{R}^d)} + \|K\|_{\text{Dini}}.$$  

This is a well-known result. The dual pointwise version was first obtained in this form in [16] quantifying the initial result of Lacey [19]; see also [24]. An extension to multilinear operators with less regular kernels was recently obtained in [26].

5. Proof of Theorem A

In this section, we will prove Theorem A by appealing to Theorem C for the family $[K] = \{(x, y) \mapsto K_s(x - y) : s \in \mathbb{Z}\}$ of (1.2). First of all, we notice that the assumption (3.1) is a direct consequence of (1.2). It is known from e.g. [13] (and our work below actually reproves this) that, with reference to (1.3)

$$\|T_*[K]\|_{L^2(\mathbb{R}^d)} \lesssim 1,$$

which is assumption (3.3) with $r = 2$. Therefore, for an application of Theorem C with

$$p_1 = p_2 = p = \frac{2}{2 - \varepsilon}, \quad 0 < \varepsilon < 1$$

we are left with verifying the corresponding stopping estimates (3.6) hold with $C_L[K](p, p) \lesssim \varepsilon^{-1}$. We do so by means of a Littlewood-Paley decomposition, as follows. Let $\varphi$ be a smooth radial function on $\mathbb{R}^d$ with support in a sufficiently small ball containing the origin, having mean zero and such that

$$\sum_{k = -\infty}^{\infty} \hat{\varphi}_k(\xi) = 1, \quad \forall \xi \neq 0, \quad \varphi(\cdot) := 2^{-kd}\varphi(2^{-k}\cdot).$$

Also define

$$(5.1) \quad \phi_k(\cdot) := \sum_{\ell \geq k} \varphi_\ell(\cdot), \quad K_s,0 = K_s \ast \phi_s, \quad K_s,j = \sum_{\ell = \Delta(j-1)+1}^{\Delta j} K_s \ast \varphi_{s-\ell}, \quad j \geq 1.$$  

for some large integer $\Delta$ which will be specified during the proof. Unless otherwise specified, the implied constants appearing below are independent of $\Delta$ but may depend on $\alpha > 0$ from (1.2) and on the dimension. Note that $K_s,j$ are supported in $\{ |x| < 2^\ell \}$. Define now for all $j \geq 0$

$$[K^j] = \{(x, y) \mapsto K_{s,j}(x - y) : s \in \mathbb{Z}\}$$
and note that, with unconditional convergence

\[(5.2) \quad K_s(y) = \sum_{j=0}^{\infty} K_{s,j}(y), \quad y \in \mathbb{R}^d.\]

The following computation is carried out in [16, Section 3].

**Lemma 5.1.** There holds

\[(5.3) \quad \omega([K^j]) \lesssim \min\{1, 2^{\Delta_j-j}\}\]

and as a consequence \(\|[K^j]\|_{\text{Dini}} \lesssim 1 + \Delta j\) for all \(j \geq 0\).

It is also well-known that

\[(5.4) \quad \sup_{t_1, t_2 \in \mathbb{Z}} \left\| f \mapsto T[K^j]f(\cdot, t_1, t_2) = \sum_{t_1 \leq s \leq t_2} K_{s,j} * f \right\|_{L^2(\mathbb{R}^d)} \lesssim 2^{\alpha \Delta (j-1)},\]

however, we need a stronger estimate on the pointwise maximal truncations, which is implicit in [13].

**Lemma 5.2.** There holds

\[\|[K^0]\|_{2, \star} \lesssim 1, \quad \|[K^j]\|_{2, \star} \lesssim 2^{-\frac{\alpha}{2}(j-1)}, \quad j \geq 1.\]

**Proof.** Let \(\beta\) be a smooth compactly supported function on \(\mathbb{R}^d\) normalized to have \(\langle \beta, 1 \rangle = 1\), and write \(\beta_s(\cdot) = 2^{-sd} \beta(2^{-s} \cdot)\). By usual arguments it suffices to estimate the \(L^2(\mathbb{R}^d)\) operator norm of

\[f \mapsto \sup_{t_1, t_2 \in \mathbb{Z}} \left\| T[K^j]f(\cdot, s, t_2) \right\|_{L^2(\mathbb{R}^d)}\]

uniformly over \(t_1, t_2 \in \mathbb{Z}\). We then have

\[(5.5) \quad T[K^j]f(\cdot, s, t_2) = \beta_s * \left( \sum_{t_1 < k \leq t_2} K_k * (\phi_{k-\Delta_j} - \phi_{k-\Delta (j-1)}) \right) * f - \beta_s * \left( \sum_{t_1 < k \leq s} K_k * (\phi_{k-\Delta_j} - \phi_{k-\Delta (j-1)}) \right) * f + (\delta - \beta_s) * \left( \sum_{s < k \leq t_2} K_k * (\phi_{k-\Delta_j} - \phi_{k-\Delta (j-1)}) \right) * f =: I_{1,s} + I_{2,s} + I_{3,s},\]

For \(I_{1,s}\), by (5.4) we have

\[(5.6) \quad \left\| \sup_{t_1, t_2 \in \mathbb{Z}} |I_{1,s}| \right\|_2 \lesssim \|M(T[K^j]f(\cdot, t_1, t_2))\|_2 \lesssim 2^{-\alpha \Delta (j-1)} \|f\|_2.\]
Next we estimate the second and third contribution in (5.5). We have, using the third assertion in (1.2), that
\[
\left| \sum_{k \leq s} (\hat{\beta}(2^s \xi) \hat{K}(\xi)(\hat{f}(2^{k-\Delta_j} \xi) - \hat{f}(2^{k-\Delta(j-1)} \xi))) \right|
\]
\[
\lesssim \left| \sum_{n \leq s} \min\{1, |2^s \xi|^{-1}\} \min\{|2^k \xi|^\alpha, |2^k \xi|^{-\alpha}\} \cdot \sum_{l=\Delta(j-1)}^{\Delta_j} \min\{|2^l-\xi|, |2^l-\xi|^{-1}\} \right|
\]
\[
\lesssim \begin{cases} 
2^{-\alpha \Delta(j-1)}|2^s \xi|^{-1}, & |2^s \xi| > 1, \\
2^{-\Delta(j-1)}|2^s \xi|, & |2^s \xi| \leq 1
\end{cases}
\]
A similar computation reveals
\[
\left| \sum_{k \geq s} (1 - \hat{\beta}(2^s \xi) \hat{K}(\xi)(\hat{f}(2^{k-\Delta_j} \xi) - \hat{f}(2^{k-\Delta(j-1)} \xi))) \right| \leq 2^{-\alpha \Delta(j-1)/2} \min\{|2^s \xi|, |2^s \xi|^{-\alpha/2}\}.
\]
Thus by Plancherel, for $m = 2, 3$ we have
\[
(5.7) \quad \left\| \sup_{t_1 \leq s \leq t_2} |I_{m, t}| \right\|_2 \leq \left( \sum_{t=t_1}^{t_2} |I_{m, t}|^2 \right)^{1/2} \leq 2^{-\alpha \Delta(j-1)/2} \|f\|_2
\]
and the proof of the Lemma is completed by putting together (5.5)–(5.7).

We are now ready to verify the assumptions (3.6) for the truncated forms $Q[K]^j$ associated to a family $[K]$ satisfying the assumptions (1.2). By virtue of Lemma 5.1 and 5.2, Lemma 4.4 applied to the families $[K^j]$ for the value $\Delta = 2 \varepsilon^{-1} \alpha^{-1}$ yields that
\[
C_{\Lambda}[K^j](\rho, \rho) \leq (\|K^j\|_{\mathcal{D}_{\mathcal{M}}} \lor \|K^j\|_{\mathcal{L}_{\mathcal{M}}})^{1-\varepsilon} (\|K^j\|_{\mathcal{L}_{\mathcal{M}}})^{\varepsilon} \leq (1 + \Delta j)^{1-\varepsilon} 2^{-\frac{\Delta(j-1)}{\varepsilon}} \leq \varepsilon^{-1}(1 + j)2^{-j}.
\]
Therefore using linearity in the kernel family $[K]$ of the truncated forms $Q_n^j(K)$ and the decomposition (5.1)-(5.2)
\[
(5.8) \quad C_{\Lambda}[K](\rho, \rho) \leq \sum_{j=0}^{\infty} C_{\Lambda}[K^j](\rho, \rho) \lesssim \varepsilon^{-1}
\]
which, together with the previous observations, completes the proof of Theorem A.

6. Extension to vector-valued functions

In this section, we suitably extend the abstract domination principle Theorem C to (a suitably defined) $\mathbb{F}^n$-valued extension, with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, of the singular integrals of Section 3. In fact, the $\mathbb{C}^n$-valued case can be recovered by suitable interpretation of the $\mathbb{R}^{2n}$-valued one; thus, it suffices to consider $\mathbb{F} = \mathbb{R}$.  

6.1. Convex body domination. Let $1 \leq p < \infty$. To each $f \in L^p_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^n)$ and each cube $Q$ in $\mathbb{R}^d$, we associate the closed convex symmetric subset of $\mathbb{R}^n$
\[
(6.1) \quad \langle f \rangle_{p, Q} := \left\{ \frac{1}{|Q|} \int_Q f \varphi \, dx : \varphi \in \Phi_{p^*}(Q) \right\} \subset \mathbb{R}^n.
\]
where we used the notation $\Phi_{q}(Q) := \{ \varphi : Q \rightarrow \mathbb{R}, \langle \varphi \rangle_{p,q} \leq 1 \}$. It is easy to see that

$$\sup_{\xi \in \langle f \rangle_{p,Q}} |\xi| \leq \langle |f| \rangle_{p,Q}$$

where $\langle \cdot \rangle_{p,Q}$ on the right hand side is being interpreted in the usual fashion. A slightly less obvious fact that we will use below is recorded in the following simple lemma, which involves the notion of John ellipsoid of a closed convex symmetric set $K$. This set, which we denote by $E_K$, stands for the solid ellipsoid of largest volume contained in $K$; in particular, the John ellipsoid of $K$ has the property that

$$E_K \subset K \subset \sqrt{n}E_K \tag{6.2}$$

where, if $A \subset \mathbb{R}^n$ and $c \geq 0$, by $cA$ we mean the set $\{ca : a \in A\}$. We also apply this notion in the degenerate case as follows: if the linear span of $K$ is a $k$-dimensional subspace $V$ of $\mathbb{R}^n$, we denote by $E_K$ the solid ellipsoid of largest $k$-dimensional volume contained in $K$. In this case, (6.2) holds with $\sqrt{k}$ in place of $\sqrt{n}$, but then it also holds as stated, since $k \leq n$ and $E_K$ is also convex and symmetric.

**Lemma 6.2.** Let $f = (f_1, \ldots, f_n) \in L^p_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^n)$ and suppose that $E_{\langle f \rangle_{p,Q}} = B_1$, where $B_\rho = \{ a \in \mathbb{R}^n : |a|_{\mathbb{R}^n} \leq \rho \}$. Then

$$\sup_{j=1,\ldots,N} \langle f_j \rangle_{p,Q} \leq \sqrt{n}.$$  

**Proof.** By definition of $\langle \cdot \rangle_{p,Q}$ for scalar functions,

$$\langle f_j \rangle_{p,Q} = \sup_{\varphi \in \Phi_{p,Q}} \frac{1}{|Q|} \int_Q f_j \varphi \, dx = \frac{1}{|Q|} \int_Q f_j \varphi_* \, dx$$

for a suitable $\varphi_* \in \Phi_{p,Q}$. Thus $\langle f_j \rangle_{p,Q}$ is the $j$-th component of the vector $f_{\varphi_*} = \frac{1}{|Q|} \int_Q f_j \varphi_* \, dx \in \langle f \rangle_{p,Q}$. By (6.2), and in consequence of the assumption, $f_{\varphi_*} \in B_{\sqrt{n}}$, which proves the assertion. \hfill \Box

We now define the sparse $(p_1, p_2)$ norm of a linear operator $T$ mapping the space $L^p_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^n)$ into locally integrable, $\mathbb{R}^n$-valued functions, as the least constant $C > 0$ such that for each pair $f_1, f_2 \in L^p_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^n)$ we may find a $\frac{1}{2}$-sparse collection $S$ such that

$$\langle T f_1, f_2 \rangle \leq C \sum_{Q \in S} |Q| \langle f_1 \rangle_{p_1,Q} \langle f_2 \rangle_{p_2,Q}. \tag{6.3}$$

We interpret the rightmost product in the above display as the right endpoint of the Minkowski product $AB = \{ (a, b)_{\mathbb{R}^n} : a \in A, b \in B \}$ of the closed convex symmetric sets $A, B \subset \mathbb{R}^n$, which is a closed symmetric interval. We use the same familiar notation $||T||_{(p_1, p_2)\text{-sparse}}$ for such norm.

Within such framework, we have the following extension of Theorem C.

**Theorem D.** Let $[K] = \{ K_s : s \in \mathbb{Z} \}$ be a family of real-valued functions satisfying (3.1), (3.3), and (3.6) for some $1 < r < \infty, 1 \leq p_1, p_2 < \infty$. Then the $\mathbb{R}^n$-valued extension of the linearized truncations $T[K]_{t_1}$ defined in (3.4) admits a $(p_1, p_2)$ sparse bound, namely

$$||T[K]_{t_1} \otimes \text{Id}_{\mathbb{R}^n}||_{(p_1, p_2)\text{-sparse}} \leq |||K|||_r \ast + C_L[K](p_1, p_2). \tag{6.4}$$
with implicit constant possibly depending on $r$, $p_1$, $p_2$ and the dimensions $d$, $n$ only, and in particular uniform over bounded measurable truncation functions $t_1, t_2$.

**Remark 6.3.** The objects (6.1) for $p = 1$ have been introduced in this context by Nazarov, Petermichl, Treil and Volberg [30], where sparse domination of vector valued singular integrals by the Minkowski sum of convex bodies (6.1) is employed towards matrix-weighted norm inequalities. In [9], a similar result, but in the dual form (6.3) with $p_1 = p_2 = 1$ is proved for dyadic shifts via a different iterative technique which is a basic version of the proof of Theorem C. Subsequent developments in vector valued sparse domination include the sharp estimate for the dyadic square function [14]. The usage of exponents $p > 1$ in (6.1), necessary to effectively tackle rough singular integral operators, is a novelty of this paper.

### 6.4. Matrix-weighted norm inequalities

We now detail an application of Theorem 6.4 to matrix-weighted norm inequalities for maximally truncated, rough singular integrals. In particular, Corollary E from the introduction is a particular case of Theorem E below.

The classes of weights we are concerned with are the following. A pair of matrix-valued weights $W, V \in L^1_{\text{loc}}(\mathbb{R}^d; L(\mathbb{R}^n))$ is said to satisfy the (joint) matrix $A_2$ condition if

$$[W, V]_{A_2} := \sup_Q \|\sqrt{W_Q} \sqrt{V_Q}\|^2_{L(\mathbb{R}^n)} < \infty$$

supremum being taken over all cubes $Q \subset \mathbb{R}^d$ and

$$W_Q := \frac{1}{|Q|} \int_Q W(x) \, dx \in L(\mathbb{R}^n).$$

We simply write $[W]_{A_2} := [W, W^{-1}]_{A_2}$. We further introduce a directional matrix $A_{\infty}$ condition, namely

$$[W]_{A_{\infty}} := \sup_{\xi \in S^{n-1}} \langle |W\xi, \xi| \rangle_{\mathbb{R}^n} \leq \sup_{\xi \in S^{n-1}} \langle |W\xi, \xi| \rangle_{\mathbb{R}^n} \leq [W]_{A_2}.$$

where the second inequality is the content of [30, Lemma 4.3].

**Theorem E.** Let $W, V \in L^1_{\text{loc}}(\mathbb{R}^d; L(\mathbb{R}^n))$ be a pair of matrix weights, and $T_{\Omega, \delta}$ be defined by (1.6), with in particular $||\Omega||_{\infty} \leq 1$. Then

$$\sup_{\delta > 0} \|W^{1/2} T_{\Omega, \delta} (V^{1/2} f)\|_{L^2(\mathbb{R}^d)} \leq \max([W]_{A_{\infty}}, [V]_{A_{\infty}}) \sqrt{[W, V]_{A_2} [W]_{A_2} [V]_{A_2}} \|f\|_{L^2(\mathbb{R}^d; \mathbb{R}^n)}. \tag{6.5}$$

We now explain how an application of Theorem D reduces Theorem E to a weighted square function-type estimate for convex-body valued sparse operators. First of all, fix $f$ of unit norm in $L^2(\mathbb{R}^d; \mathbb{R}^n)$. We may then find $g$ of unit norm in $L^2(\mathbb{R}^d; \mathbb{R}^n)$ and bounded measurable functions $t_1, t_2$ such that the left hand side of (6.5) is bounded by twice the sum of

$$\left|\langle T[K]_{t_1} \otimes \text{Id}_{\mathbb{R}^n} (V^{1/2} f), W^{1/2} g\rangle\right| \tag{6.6}$$

and

$$\sup_{Q \ni x} \left|\langle W_Q^{1/2}V^{1/2} f, Q\rangle\right|_{L^2(\mathbb{R}^d)} \leq \sup_{Q \ni x} \left|\langle W_Q^{1/2}V^{1/2} f, Q\rangle\right|_{L^2(\mathbb{R}^d)},$$

where $[K]$ is a suitable decomposition satisfying (1.2). The latter expression is (the norm of) a two-weight version of the matrix weighted maximal function of Christ and Goldberg [5]. In the
one weight case when \( V = W^{-1} \in A_2 \), its boundedness has been proved in [5] and quantified in [17], which contains the explicit bound

\[ c_{d,n}[W]_{A_2} \| f \|_{L^2(\mathbb{R}^d; \mathbb{R}^n)} \]

and the implicit improvement

\[ c_{d,n}[W]^{1/2}_{A_2} [W^{-1}]^{1/2}_{A_\infty} \| f \|_{L^2(\mathbb{R}^d; \mathbb{R}^n)}, \]

where \( c_{d,n} \) is a dimensional constant. A straightforward modification of the same argument, using the splitting on the right of the previous display, gives the bound

\[ [W, V]^{1/2}_{A_2} [V]^{1/2}_{A_\infty} \| f \|_{L^2(\mathbb{R}^d; \mathbb{R}^n)} \]

in the two weight case. Roughly speaking, the first factor is controlled by the two weight \( A_2 \) condition and the second one by the \( A_\infty \) property of \( V \).

By virtue of the localized estimate (5.8) for \([K]\), an application of Theorem D tells us that (6.6) is bounded by \( C/\varepsilon \) times a sparse sublinear form as in (6.3) with \( p_1 = p_2 = 1 + \varepsilon, f_1 = V^{1/2} f \) and \( f_2 = W^{1/2} g \) for all \( \varepsilon > 0 \). Finally, we gather that

\[
\left\| \sup_{\delta > 0} \| W^{1/2} T_{\Omega, \delta}(V^{1/2} f) \|_{L^2_\varepsilon} \right\|_2 \leq \sqrt{[V, W]_{A_2} \max\{[W]_{A_\infty}, [V]_{A_\infty}\}} + \inf_{\varepsilon > 0} \frac{1}{\varepsilon} \sum_{Q \in S} |Q| (\langle V^{1/2} f \rangle_{1+\varepsilon, Q} \langle W^{1/2} g \rangle_{1+\varepsilon, Q})
\]

where the supremum is being taken over \( 1/2 \)-sparse collections \( S \), and the proof of Theorem E is completed by the following proposition.

**Proposition 6.5.** The estimate

\[
\inf_{\varepsilon > 0} \frac{1}{\varepsilon} \sum_{Q \in S} |Q| (\langle V^{1/2} f \rangle_{1+\varepsilon, Q} \langle W^{1/2} g \rangle_{1+\varepsilon, Q}) \leq \max\{[W]_{A_\infty}, [V]_{A_\infty}\} \sqrt{[V, W]_{A_2}[W]_{A_\infty} [V]_{A_\infty}}
\]

holds uniformly over all \( f, g \) of unit norm in \( L^2(\mathbb{R}^d; \mathbb{R}^n) \) and all \( 1/2 \)-sparse collections \( S \).

**Proof.** There is no loss in generality with assuming that the sparse collection \( S \) is a subset of a standard dyadic grid in \( \mathbb{R}^d \), and we do so. Fix \( \varepsilon > 0 \). By standard reductions, we have that

\[
|Q| (\langle V^{1/2} f_1 \rangle_{1+\varepsilon, Q} \langle W^{1/2} f_2 \rangle_{1+\varepsilon, Q}) \leq \sqrt{[V, W]_{A_2}} \sup_{\| f \|_{L^2(\mathbb{R}^d, \mathbb{R}^n)} = 1} \| S_{\varepsilon, \varepsilon} f_1 \|_2 \| S_{\varepsilon, \varepsilon} f_2 \|_2
\]

having defined the square function

\[ S_{\varepsilon, \varepsilon} f^2 = \sum_{Q \in S} \left( (|W_Q|^{-1/2} W^{1/2} f)_{1+\varepsilon, Q}^2 \right) \]

Now, if

\[ \varepsilon < 2^{-10} t_W, \quad t_W := (2^{2d+N+1} + [W]_{A_\infty})^{-1}, \quad p := \frac{1 + 2t}{(1 + \varepsilon)(1 + t)} \in (1, 2) \]

as a result of the sharp reverse Hölder inequality and of the Carleson embedding theorem there holds

\[ \| S_{\varepsilon, \varepsilon} f \|_2^2 \leq \sum_{Q \in S} |Q| |\langle f \rangle_{\mathbb{R}^n}|^{2/p} \| Q \|_{1+\varepsilon, Q}^2 \leq (p')^p \| f \|_{\mathbb{R}^n}^2 \leq [W]_{A_\infty} \| f \|_{\mathbb{R}^n}^2 \]
3.11 in conjunction with the $D$ then yields the slightly improved weighted estimate $6.5$.

**Remark 6.6.** We may derive a slightly stronger weighted estimate than (6.7) for the non-maximally truncated rough integrals $T_{\Omega, \delta}$, by applying Theorem D in conjunction with the $(1, 1 + \varepsilon)$ localized estimates proved in [7]. Namely, the estimate

$$
\|W^\frac{1}{2} T_{\Omega, \delta}(V^\frac{1}{2} f)\|_{L^2(\mathbb{R}^d; \mathbb{R}^n)} \leq \sup_{\|g\|_{L^2(\mathbb{R}^d; \mathbb{R}^n)} = 1} \inf_{\varepsilon > 0} \sup_{\delta > 0} \frac{1}{\varepsilon} \sum_{Q \in S} |Q| (V^\frac{1}{2} f)_{1, \tilde{Q}} (W^\frac{1}{2} g)_{1 + \varepsilon, \tilde{Q}}.
$$

holds uniformly in $\delta > 0$ for all $f$ of unit norm in $L^2(\mathbb{R}^d; \mathbb{R}^n)$. Repeating the proof of Proposition 6.5 then yields the slightly improved weighted estimate

$$
\sup_{\delta > 0} \|W^\frac{1}{2} T_{\Omega, \delta}(V^\frac{1}{2} f)\|_{L^2(\mathbb{R}^d; \mathbb{R}^n)} \leq \min([W]_{A_0}, [V]_{A_0}) \sqrt{[W, V]_{A_2} [W]_{A_0} [V]_{A_0}} \|f\|_{L^2(\mathbb{R}^d; \mathbb{R}^n)}.
$$

**6.7. Proof of Theorem D.** The proof of Theorem D is formally identical to the argument for the scalar valued case, provided that estimate (3.10) and the definition of $E_Q$ given in (3.11) are replaced by suitable vector valued versions. We begin with the second tool. The proof, which is a minor variation on [9, Lemma 3.3], is given below

**Lemma 6.8.** Let $0 < \eta \leq 1$, $Q$ be a dyadic cube and $f_j \in L^{p_j}(\mathbb{R}^d; \mathbb{R}^n)$, $j = 1, 2$. Then the set

$$
E_Q := \bigcup_{j=1}^2 \left\{ x \in 3Q : \eta(f_j) 1_{3Q} p_j \not\subset \langle f_j \rangle_{p_j, 3Q} \text{ for some cube } L \subset \mathbb{R}^d \text{ with } x \in L \right\}
$$

satisfies $|E_Q| \leq C \eta^{\min(p_1, p_2)} |Q|$ for some absolute dimensional constant $C$.

**Proof.** We may assume that $\text{supp } f_j \subset 3Q$. It is certainly enough to estimate the measure of each $j \in \{1, 2\}$ component of $E_Q$ by $C \eta^{p_j} |Q|$, and we do so: we fix $j$ and are thus free to write $f_j = f, p_j = p$. Let $L_f = \{L \subset \mathbb{R}^d : \eta(f) p_L \not\subset \langle f \rangle_{p, 3Q}\}$. By usual covering arguments it suffices to show that if $L_1, ..., L_m \in L_f$ are disjoint then

$$
\sum_{\mu=1}^m |L_\mu| \leq C \eta^p |Q|.
$$

Fix such a disjoint collection $L_1, ..., L_m$. Notice that if $A \in \text{GL}((\mathbb{R}^n)$ then $L_{A f} = L_f$. By action of $\text{GL}(\mathbb{R}^n)$ we may thus reduce to the case where $E_{\langle f \rangle_{p, 3Q}} = B_1$, and in particular

$$
B_1 \subset \langle f \rangle_{p, 3Q} \subset B_{\sqrt{n}}.
$$

By membership of each $L_\mu \in L_f$, we know that $\eta(f) p_{\mu} \not\subset B_1$. A fortiori, there exists $\varphi_\mu \in \Phi_{p'} (L_\mu)$ and a coordinate index $\ell_\mu \in \{1, ..., n\}$ such that

$$
\eta(F_\mu)_{\ell} > \frac{1}{\sqrt{n}}, \quad F_\mu := \int_{L_\mu} f \varphi_\mu \frac{dx}{|L_\mu|}.
$$
Let $M_\ell = \{\mu \in \{1, \ldots, m\} : \ell_\mu = \ell\}$. As $\{1, \ldots, m\} = \bigcup_{\ell = 1, \ldots, n} M_\ell$ it suffices to show that

\[ (6.12) \quad \frac{1}{|3Q|} \sum_{\mu \in M_\ell} |L_\mu| =: \delta < C \eta^p. \]

Using the membership $\varphi_\mu \in \Phi_p(L_\mu)$ for the first inequality and the disjointness of the supports for the second equality

\[ 1 \geq \frac{1}{\delta} \sum_{\mu \in M_\ell} \int_{L_\mu \cap 3Q} |\varphi_\mu| p' \, \frac{dx}{|3Q|} = \int_{3Q} |\varphi| p' \, \frac{dx}{|3Q|}, \quad \varphi := \delta^{-\frac{1}{p'}} \sum_{\mu \in M_\ell} \varphi_\mu 1_{3Q} \]

so that $\varphi \in \Phi_p(3Q)$. In particular, beginning with the right inclusion in (6.10) and using (6.11) in the last inequality

\[ \sqrt{n} \geq \int_{3Q} (f\varphi)_t \, \frac{dx}{|3Q|} = \delta^{-\frac{1}{p'}} \sum_{\mu \in M_\ell} |L_\mu| \frac{|F_\mu|}{|3Q|} > \frac{1}{\eta} \sqrt{n} \]

which rearranging yields (6.12) with $C = n^p$, thus completing the proof.

At this point, let $S_k$ be a collection of pairwise disjoint cubes as in Step 3 of the proof of Theorem C. The elements of the collection $S_{k+1}$ are defined to be the maximal dyadic cubes $L$ such that the same condition as in (3.11) holds, provided the definition of $E_Q$ therein is replaced with the one in (6.9). By virtue of Lemma 6.8, (3.12) still holds provided $\eta$ is chosen small enough. And, we still obtain that $S_{k+1}(Q) = \{L \in S_{k+1} : L \subset 3Q\}$ is a stopping collection. By definition of $S_{k+1}$, it must be that

\[ (6.13) \quad \langle f_1 1_{3Q} \rangle_{p, K} \subset C \langle f_j \rangle_{p, 3Q} \]

whenever the (not necessarily dyadic) cube $K$ is such that a moderate dilate $CK$ of $K$ contains $2^5 L$ for some $L \in S_{k+1}(Q)$. Fix $Q$ for a moment and let $A_j = (A_{j, \mu}^m : 1 \leq m, \mu \leq n) \in \text{GL}(\mathbb{R}^n)$, $j = 1, 2$ be chosen such that the John ellipsoid of $\langle \tilde{f}_j \rangle_{3Q, p}$ is $B_1$, or its intersection with a lower dimensional subspace in a degenerate case, and $A_j \tilde{f}_j := f_j$. It follows from (6.13) that if $2^5 L \subset CK$ then $\langle \tilde{f}_j 1_{3Q} \rangle_{p, K} \subset B_{C}$. This fact, together with Lemma 6.2 readily yields the estimates

\[ (6.14) \quad \| \tilde{f}_j \|_{\nu_{3Q}} \leq 1, \quad j = 1, 2. \]

We are ready to obtain a substitute for (3.10). In fact

\[ \left| \left\langle T[K]_{n}^{t_{\omega} \boldsymbol{1}_{Q}} \otimes \text{Id}_{\mathbb{R}^n}(f_1 1_{Q}), f_2 \right\rangle \right| \leq |Q| \left| \sum_{m=1}^{n} Q[K]_{n}^{t_{\mu_1}, f_2} \right| + \sum_{L \in S_{k+1(Q)}} \left| \left\langle T[K]_{n}^{t_{\omega} \boldsymbol{1}_{Q}} \otimes \text{Id}_{\mathbb{R}^n}(f_1 1_{L}), f_2 \right\rangle \right| \]

and by actions of $\text{GL}(\mathbb{R}^n)$, see the proof of [9, Lemma 3.4],

\[ \left| \sum_{m=1}^{n} Q[K]_{n}^{t_{\mu_1}, f_2} \right| = \left| \sum_{m, \mu_1, \mu_2} A_{1}^{m_{\mu_1}} A_{2}^{m_{\mu_2}} Q[K]_{n}^{t_{\mu_1}, \tilde{f}_2} \right| \]

\[ \lesssim \langle f_1 \rangle_{p_1, 3Q} \langle f_2 \rangle_{p_2, 3Q} \sup_{\mu_1, \mu_2} \| Q[K]_{n}^{t_{\mu_1}, \tilde{f}_2} \| \lesssim \langle f_1 \rangle_{p_1, 3Q} \langle f_2 \rangle_{p_2, 3Q} \]

where we also employed (3.8) coupled with (6.14) in the last line. Assembling together the last two displays yields the claimed vector-valued version of (3.10), and finishes the proof of Theorem D.
7. Proof of Theorem B

We begin with the proof of the first point. As a direct application of the main result of [27],

\[ |(Tf, g)| \leq c_{d,p} \varepsilon^{-1} [v]_{A_{\sigma_0}}^{\frac{1}{q}} \|g\|_{L_p} \|f\|_{L_{q'}}(\sigma), \]

where

\[ r = \left( \frac{1+\varepsilon}{p} \right) \left( \frac{p_{q'-1}}{p_{q'+1}} - 1 \right) + 1 = p + \frac{p_{q'-1}}{p_{q'+1} - 1} \varepsilon, \quad v = \sigma^{\frac{1+\varepsilon}{q'}} = w^{1+\frac{q'}{p'} - (1+\varepsilon)}, \quad u = w^{\frac{1+\varepsilon}{q'}}. \]

By definition,

\[ [v]_{A_{\sigma_0}}^{\frac{1}{q}} = \sup_{Q} \left( \frac{1}{|Q|} \int_{Q} w^{1+\frac{q'}{p'+1}} \right)^{\frac{1}{q}} \left( \frac{1}{|Q|} \int_{Q} \sigma^{1+\frac{q}{p'+1}} \right)^{\frac{1}{p'}}, \]

By the sharp reverse Hölder inequality [15], taking \( \varepsilon = \frac{1}{r_d \max[p, p'] \max[v]_{A_{\sigma_0}}, \sigma_{1, \infty}} \), we can conclude

\[ ||T||_{L^p(w)} \leq c_{d,p} [w]_{A_p}^{\frac{1}{q}} ([v]_{A_{\sigma_0}}^{\frac{1}{q}} + \sigma_{1, \infty}) \max\{\sigma_{1, \infty}, [w]_{A_{\sigma_0}}\} \leq c_{d,p} [w]_{A_p}^{\frac{2}{q} \max(1, \frac{1}{p'})}. \]

Next let us prove the Fefferman-Stein type inequality of the second point. Indeed, let \( A(t) = t^{p'/p} \) and \( \tilde{B}(t) = t^{q'(q'+1)} \), where \( 1 < r < p \) and \( \tilde{r} = \frac{p_{q'-1}}{p_{q'+1} - 1} \). Then

\[ \sup_{Q} \langle w' \rangle_{Q}^{\frac{1}{q}} \| (M_w)^{-r/p} \|_{B,Q}^{\frac{1}{q}} \leq \sup_{Q} \inf_{Q \in \mathcal{S}} \langle w' \rangle_{Q}^{\frac{1}{q}} \| (M_w)^{-r/p} \|_{B,Q} \leq 1. \]

Let \( v = M_w \cdot w \). Now we have,

\[ \tilde{r}' \sum_{Q \in \mathcal{S}} |Q| \langle f \rangle_{Q}^{\frac{1}{q}} \langle g w^\tilde{r} \rangle_{Q} \geq \tilde{r}' \sum_{Q \in \mathcal{S}} |Q| \langle f^\tilde{r} v^\tilde{r} g w^\tilde{r} \cdot w^\tilde{r} \rangle_{Q} \langle w^\tilde{r} \rangle_{Q} \]

\[ \leq \tilde{r}' \sum_{Q \in \mathcal{S}} \| f^\tilde{r} v^\tilde{r} \|_{B,Q} \| g^{\tilde{r}} \|_{A_{\sigma_0}} \| w^{\tilde{r}} \|_{A_{\sigma_0}} \| E_Q \| \]

\[ \leq 2 \tilde{r}' \sum_{Q \in \mathcal{S}} \| f^\tilde{r} v^\tilde{r} \|_{B,Q} \| g^{\tilde{r}} \|_{A_{\sigma_0}} \| E_Q \| \]

\[ \leq 2 \tilde{r}' \int M_{B}^{\tilde{r}}(f^\tilde{r} v^\tilde{r}) \| M_{A_{\tilde{r}}}(g) \|_{L_{q'}} \]

\[ \leq 2 \tilde{r}' \int M_{B}^{\tilde{r}}(f^\tilde{r} v^\tilde{r}) \| g \|_{L_{q'}} \]

\[ \leq c_{d,p} (p')^{\tilde{r}} (r')^{1+\tilde{r}} \| f \|_{L_{q'}(M_w)}. \]

By sparse domination formula and duality,

\[ ||T(f)||_{L^p(w)} \leq c_{d,p} (p')^{\tilde{r}} (r')^{1+\tilde{r}} \| f \|_{L_{q'}(M_w)}. \]

Notice that the \( A_{1-\infty} \) estimate just follows from the sharp reverse Hölder inequality, so that we may restrict to \( q > 1 \). The idea is still viewing \( A_q \) condition as a bumped \( A_p \) condition (see
Let \( C(t) = t^{\frac{p}{p+1}} \). We have
\[
 r' \sum_{Q \in S} |Q|(f)_{r,Q}^w g_w^{r,Q} \leq r' \sum_{Q \in S} |Q|(f^w w)^{\frac{1}{C} Q} g^w_{r,Q} (w^r)^{\frac{1}{C} Q} \langle g^w_{r,Q} (w^r)^{-\frac{1}{s'}} \rangle^{\frac{1}{s'}}.
\]
Take
\[
r = 1 + \frac{1}{8p(\frac{p}{q})^r_d[w]_{A_w}}, \quad s = 1 + \frac{1}{4(\frac{p}{q})^r_d[w]_{A_w}}.
\]

Then \( rs < 1 + \frac{1}{2p} < p' \), \( r < 1 + \frac{1}{8(\frac{p}{q})^r_d[w]_{A_w}} \) and \( (r - \frac{1}{s'}) < 1 + \frac{1}{\tau_d[w]_{A_w}} \). Then applying the sparse domination, and the sharp reverse Hölder inequality we obtain
\[
\|T(f)\|_{L^p(w)} \leq \sup_{\|g\|_{L^p(w)} = 1} r' \sum_{Q \in S} |Q|(f)_{r,Q}^w g_w^{r,Q} \leq \sum_{Q \in S} \sup_{\|g\|_{L^p(w)} = 1} c_{d,p,q}[w]_{A_w} \frac{1}{C} Q \sum_{Q \in S} (f^w w)^{\frac{1}{C} Q} g^w_{r,Q} (w^r)^{\frac{1}{C} Q} \langle g^w_{r,Q} (w^r)^{-\frac{1}{s'}} \rangle^{\frac{1}{s'}}
\]
where in the last step we have used the Carleson embedding theorem; we omit the routine details.

Finally, we prove the Coifman-Fefferman type inequality. Fix \( \varepsilon > 0 \) and denote \( \eta = 1 + \varepsilon \). Also let
\[
r = 1 + \frac{1}{8p(\frac{p}{q})^r_d[w]_{A_w}}, \quad s = 1 + \frac{1}{4\eta p}.
\]

Then again \( rs < 1 + \frac{1}{2p} < p' \), \( r < \eta \) and \( (r - \frac{1}{s'}) < 1 + \frac{1}{\tau_d[w]_{A_w}} \). Applying the sparse domination formula again, we obtain
\[
\|T(f)\|_{L^p(w)} \leq \sup_{\|g\|_{L^p(w)} = 1} \eta' [w]_{A_w} \int_{\mathbb{R}^d} M_{q_f} f M_{d,q_r,r}(g) w dx \leq \eta' [w]_{A_w} \|M_{q,f}\|_{L^p(w)}.
\]

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