Abstract.

In this work we show that a Legendre transformation is nothing but a mere change of symplectic polarization from the point of view of contact geometry. Then, we construct a set of Riemannian and pseudo-Riemannian metrics on a contact manifold by introducing almost contact and para-contact structures and we analyze their isometries. We show that it is not possible to find a class of metric tensors which fulfills two properties: on the one hand, to be polarization independent i.e. the Legendre transformations are the corresponding isometries and, on the other, that it induces a Hessian metric into the corresponding Legendre submanifolds. This second property is motivated by the well known Riemannian structures of the geometric description of thermodynamics which are based on Hessian metrics on the space of equilibrium states and whose properties are related to the fluctuations of the system. We find that to define a Riemannian structure with such properties it is necessary to abandon the idea of an associated metric to an almost contact or para-contact structure. We find that even extending the contact metric structure of the thermodynamic phase space the thermodynamic desiderata cannot be fulfilled.
1. Introduction

Contact geometry is the natural setting to describe processes satisfying a constraint akin to the first law of thermodynamics. A thermodynamic process is realized as a smooth curve connecting two points on a contact manifold. Each point of the manifold is referred as a thermodynamic state. In this sense, we call a contact manifold a thermodynamic phase space (TPS) where the constraint given by the first law defines its contact structure [1, 2].

A contact structure on an odd-dimensional manifold is a maximally non-integrable distribution of co-dimension one hyperplanes. That is, for every pair of points on the manifold, there exists a curve connecting them such that its tangent vector at each point along the path lies in a hyperplane of the distribution. Furthermore, in every open set around each point on the manifold, there is a set of local coordinates such that the 1-form generating the contact distribution resembles the first law of thermodynamics [see (3), below]. This motivates us to call those curves whose tangent vector at each point is annihilated by this 1-form, thermodynamic processes [2, 3].

From a thermodynamic perspective, maximal dimensional embedded integral submanifolds such that its tangent bundle is entirely contained in the contact distribution correspond to the realization of specific thermodynamic systems. That is, those where the embedding is defined by a thermodynamic fundamental relation of a given set of independent thermodynamic variables, e.g. entropy, volume, temperature, pressure, volume, etc [2, 4]. In the context of geometric thermodynamics these are called equilibrium spaces [5–7]. Such submanifolds, when equipped with a metric whose components correspond to the Hessian of the fundamental relation, are the subject of geometric thermodynamics [8, 9]. In particular, it has been established that the Riemannian structure corresponding to a specific choice of thermodynamic potential encodes information stemming from thermodynamic fluctuation theory [10–14].

Each thermodynamic potential defines a different Legendre submanifold and, since its metric description arises from fluctuation theory, it yields a different Riemannian structure, signaling a non-equivalence of ensembles in the geometric characterization of fluctuations [15]. For instance, it may occur that different potentials may yield non Hessian metrics [16]. Thus, albeit a thermodynamic system in equilibrium does not depend on the choice of thermodynamic potential describing it, its geometric structure – related to its fluctuations – does [17,18]. This motivated the search for a potential independent form of the Riemannian structures for the equilibrium spaces, that is, metrics sharing the Legendre symmetry of the thermodynamic description [15,19–21].

In this manuscript we show that, when viewed from the thermodynamic phase space, each potential choice corresponds to a symplectic polarization for the contact distribution. Indeed, given a fundamental relation written in terms of a particular thermodynamic potential, the corresponding embedding provides a specific choice of ‘position’ and ‘momentum’ coordinates for the symplectic structure of the contact planes. Thus, a Legendre transformation, interchanging the role of a pair of conjugate thermodynamic variables, can be understood as a change of symplectic polarization [22]. As expected, this is a symmetry of the contact structure. We also obtain a class of metrics in the thermodynamic phase space invariant under a change of symplectic polarizations, generalizing the metric contact structure.

This work is structured as follows: In section 2 we explore the thermodynamic phase space in terms of a basis of vector fields satisfying the Heisenberg algebra commutation relations and recall the notion of symplectic polarization. In section 3 two different kind of horizontal contact Hamiltonians and the symmetry transformations generated by their corresponding contact Hamiltonian vector fields are explored, namely rotation and polarization scalings. In section
we introduce the canonical almost contact structure and three different almost para-contact structures. We study their symmetry properties under the transformations generated by the contact Hamiltonian vector fields of the horizontal contact Hamiltonians. In section 5 metrics for the contact manifold are constructed using the almost contact and almost para-contact structures and their properties are analyzed. In section 6 we present an automorphism on the tangent spaces of the contact manifold which allows us to construct a family of metrics for which the Legendre transformations are a set of isometries. Finally, in section 7 we provide some closing remarks and discuss the main conclusions of this work.

2. Thermodynamic Phase Space and the Heisenberg Group

Let us consider a \((2n+1)\) dimensional manifold \(\mathcal{T}\) together with a set of vector fields providing a basis for each tangent space such that they satisfy the commutation relations of the \(n\)th Heisenberg group algebra [23]. That is, for each point \(p \in \mathcal{T}\), the basis for the tangent space \(T_p \mathcal{T}\) is given by the linearly independent \(2n+1\) vector fields \(\{Q_a, P^a, \xi\}\) whose Lie bracket satisfies

\[
[P^a, Q_b] = \delta^a_b \xi, \quad [\xi, P^a] = 0 \quad \text{and} \quad [\xi, Q_a] = 0. \tag{1}
\]

This is the simplest example of a bracket generating distribution [24], namely, let \(\mathcal{D} \subset T\mathcal{T}\) be the \(2n\)-dimensional distribution generated by \(\{P^a, Q_a\}_{a=1}^n\), then

\[
T\mathcal{T} = \text{span} (P^a, Q_a, [P^a, Q_b]). \tag{2}
\]

Condition (2) implies that any two points \(p, q \in \mathcal{T}\) can be joined by a curve such that at each point along the path its tangent vector lies in \(\mathcal{D}\).

Historically, such condition was realized in connection with thermodynamic processes in the following sense: if each point in the manifold \(\mathcal{T}\) corresponds to a possible thermodynamic state characterized by \(2n+1\) quantities, then for any two states \(p, q \in \mathcal{T}\) there is a process joining them such that at each point along the path the first law of thermodynamics is satisfied. Indeed, \(\mathcal{D}\) is a contact distribution corresponding to the kernel of a 1-form \(\eta \in T^*\mathcal{T}\). Thus, around each point \(p \in \mathcal{T}\) there is a local set of coordinates \(\{w, q^a, p_a\}\) in which the the 1-form \(\eta\) is written as [25]

\[
\eta = dw - \sum_{a=1}^{n} p_a dq^a. \tag{3}
\]

This is known as Darboux theorem.

Any vector field \(X \in T\mathcal{T}\) such that

\[
\eta(X) = \eta \left[ X^w \frac{\partial}{\partial w} + \sum_{a=1}^{n} \left( X^a \frac{\partial}{\partial q^a} + X^p \frac{\partial}{\partial p_a} \right) \right] = X^w - \sum_{a=1}^{n} p_a X^a = 0, \tag{4}
\]
is a combination of $2n$ vector fields, that is

$$X = \left( \sum_{a=1}^{n} p_a X^a_q \right) \frac{\partial}{\partial w} + \sum_{a=1}^{n} \left( X^a_q \frac{\partial}{\partial q^a} + X^p_a \frac{\partial}{\partial p_a} \right)$$

$$= \sum_{a=1}^{n} \left[ X^a_q \left( \frac{\partial}{\partial q^a} + p_a \frac{\partial}{\partial w} \right) + X^p_a \frac{\partial}{\partial p_a} \right]$$

$$= \sum_{a=1}^{n} \left[ X^a_q Q_a + X^p_a P^a \right], \quad (5)$$

where

$$P^a = \frac{\partial}{\partial p_a}, \quad \text{and} \quad Q_a = \frac{\partial}{\partial q^a} + p_a \frac{\partial}{\partial w} \quad (6)$$

It is straightforward to verify that the Lie bracket of the vector fields $(6)$ satisfies the Heisenberg commutation relations $(1)$, that is

$$[P^a, Q_b](f) = \left[ \frac{\partial}{\partial p_a}, \frac{\partial}{\partial q^b} + p_b \frac{\partial}{\partial w} \right] (f)$$

$$= \left[ \frac{\partial}{\partial p_a}, \frac{\partial}{\partial q^b} \right] (f) + \left[ \frac{\partial}{\partial p_a}, p_b \frac{\partial}{\partial w} \right] (f)$$

$$= \frac{\partial}{\partial p_a} \left( p_b \frac{\partial f}{\partial w} \right) + p_b \frac{\partial}{\partial w} \left( \frac{\partial f}{\partial p_a} \right)$$

$$= p_b \frac{\partial^2 f}{\partial p_a \partial w} + \delta^a_b \frac{\partial f}{\partial w} - p_b \frac{\partial^2 f}{\partial w \partial p_a}$$

$$= \delta^a_b \xi(f), \quad (7)$$

while the vector field $\xi$ satisfies

$$d\eta(\xi, X) = 0 \quad \text{and} \quad \eta(\xi) = 1 \quad (8)$$

for any vector field $X \in T\mathcal{T}$. In the literature, the vector field satisfying $(8)$ is called the Reeb vector field $[25]$. 

The restriction of the exterior derivative of $\eta$ to $\mathcal{D}$ corresponds to a symplectic polarization $[22]$

$$d\eta|_{\mathcal{D}} (X, Y) = - \sum_{a=1}^{n} \mathrm{d} p^a \wedge \mathrm{d} q^a (X, Y) = \omega(X, Y) \quad (9)$$

for any pair $X, Y \in \mathcal{D}$, where $\omega$ is a symplectic 2-form for $\mathcal{D}$.

3. Horizontal Contact Hamiltonians and Symmetry Generators

As we have seen, the contact distribution $\mathcal{D}$ is generated by $\ker (\eta)$. In fact, this distribution is given by an equivalence class of 1-forms $[\eta]$ with respect to the module of a conformal factor, ie.
η \sim \eta' \text{ if } \eta' = \lambda \eta \text{ with } \lambda \text{ a differentiable non-vanishing function on } \mathcal{T}. \text{ Therefore, } \mathcal{D} \text{ is independent of the choice of } \eta \text{ in the same equivalence class. In fact, if a mapping } \Phi : \mathcal{T} \mapsto \mathcal{T} \text{ preserves } \mathcal{D}, \text{ i.e., }

\Phi^* \eta = f_\Phi \eta, \quad (10)

we say that } \Phi \text{ corresponds to a contact transformation [26]. Here, } f_\Phi \text{ is a non vanishing function and } \Phi^* \text{ is the pullback induced by } \Phi. \text{ In the case } f_\Phi = 1, \text{ we say } \Phi \text{ is a strict contact transformation. Moreover, if a vector field } X \text{ satisfies }

\mathcal{L}_X \eta = \tau_X \eta, \quad (11)

where } \tau_X : \mathcal{T} \mapsto \mathbb{R}, \text{ we say that } X \text{ corresponds to an infinitesimal contact transformation. If, in addition, }

\mathcal{L}_X \eta = 0, \quad (12)

we say that the infinitesimal contact transformation is strict. Since the notion of a strict contact transformation depends on the chosen 1-form } \eta \text{ in the class generating the contact distribution, a strict contact transformation constitutes a symmetry of the contact form, alone.}

Consider a real valued function } h \in C^\infty(\mathcal{T}). \text{ A contact Hamiltonian system } (\mathcal{T}, \mathcal{D}, \eta, h) \text{ is defined by the relation [1, 22, 26] }

\eta(X_h) = h, \quad (13)

where } X_h \text{ is a unique vector field called the contact Hamiltonian vector field. As } X_h \text{ generates a contact transformation it must satisfy equation (11), then it can be shown that }

i_\xi \mathcal{L}_{X_h} \eta = i_\xi (d \circ i_{X_h} \eta + i_{X_h} \circ d \eta),

= i_\xi (d \eta(X_h) - i_{X_h} \circ i_\xi \circ d \eta),

= i_\xi dh = \xi(h). \quad (14)

Thus we see that the diffeomorphism generated by } X_h \text{ is indeed a contact transformation }

\mathcal{L}_{X_h} \eta = \xi(h) \eta. \quad (15)

Therefore, for } X_h \text{ to be a symmetry of the contact form } \eta, \text{ it is sufficient that } \xi(h) \text{ vanishes, i.e. when the contact Hamiltonian is a purely horizontal function.}

3.1. Legendre symmetry

Let us consider the contact Hamiltonian

\begin{align*}
  h_L &= \frac{1}{2} \sum_{i=1}^{m} \left( q_i^2 + p_i^2 \right), \quad (16)
\end{align*}

Its corresponding contact Hamiltonian vector field is

\begin{align*}
  X_{h_L} &= \sum_{i=1}^{m} \left[ \frac{1}{2} \left( q_i^2 + p_i^2 \right) \xi + q_i P^i - p_i Q_i \right], \quad (17)
\end{align*}

where we will use } i, j, k, \text{ etc. to distinguish the indices that take values on a subset of the coordinates, } i, j, k = 1, \ldots, m, \text{ where } m < n, \text{ from those that take values on the complete set of
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coordinates, \(a, b, c = 1, \ldots, n\). For the indices of the remaining coordinates we will use capital letters, \(I, J = m + 1, \ldots, n\). Note that, restricted to \(\mathcal{D}\), this is the generator of rotations in each contact plane, whose canonical basis is given by \(\{Q_a, P_a\}_{a=1}^n\). Indeed, its flow generates the 1-parameter family of diffeomorphisms \(\Phi_t : \mathcal{T} \rightarrow \mathcal{T}\)

\[
\Phi_t^m = \begin{cases} 
\tilde{w}(t) = w - \frac{1}{2} \sin(t) \sum_{i=1}^{m} \left( p_i^2 - q_i^2 \right) \cos(t) + 2 \sin(t) q^i p_i \\
\tilde{q}^i(t) = q^i \cos(t) - p_i \sin(t) \\
\tilde{p}_i(t) = q^i \sin(t) + p_i \cos(t),
\end{cases}
\]

whilst mapping the rest of the coordinate functions into themselves. In particular, we have that a \(\pi/2\)-rotation in each contact plane of \(\mathcal{D}\)

\[
\Phi_{\pi/2}^m = \begin{cases} 
\tilde{w}(\pi/2) = w - \sum_{i=1}^{m} q^i p_i \\
\tilde{q}^i(\pi/2) = -p_i \\
\tilde{p}_i(\pi/2) = q^i
\end{cases}
\]

corresponds to a partial change of symplectic polarization in \(\mathcal{D}\) and generates a partial Legendre transformation in \(\mathcal{T}\).

Since the contact Hamiltonian (16) is a purely horizontal function, it follows that the contact 1-form \(\eta\) is propagated along the flow of \(X_{h_L}\), that is

\[
\mathcal{L}_{X_{h_L}} \eta = 0.
\]

In particular, we have that

\[
\left[ \Phi_{\pi/2}^m \right]^* (\eta) = \eta,
\]

where \(\left[ \Phi_{\pi/2}^m \right]^* : T^* \mathcal{T} \rightarrow T^* \mathcal{T}\) is the pullback induced map by \(\Phi_{\pi/2}\). This is, the finite transformation is a symmetry of the contact structure. This is obvious in the sense that the contact 1-form has \(X_{h_L}\) as an infinitesimal symmetry. However, as we will shortly see, it is the discrete symmetries the ones which are relevant in the context of thermodynamics and, while infinitesimal symmetries imply the discrete case, the converse is not necessarily true.

It is also useful to have the expressions for the Lie derivatives of the basis vectors. The case of the Reeb vector field is straightforward

\[
\mathcal{L}_{X_{h_L}} \xi = [X_{h_L}, \xi] = -[\xi, X_{h_L}] = 0.
\]
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Then, for \( i, j = 1, \ldots, m \) we have

\[
\mathcal{L}_{X_{hL}} Q_i = \left[ \sum_{j=1}^{m} \frac{1}{2} \left( p_j^2 + q_j^2 \right) \xi + q^i P_j - p_j Q_j, Q_i \right]
\]

\[
= - \left[ Q_i, \sum_{j=1}^{m} \frac{1}{2} \left( p_j^2 + q_j^2 \right) \xi \right] - \left[ Q_i, \sum_{j=1}^{m} q^i P_j \right] + \left[ Q_i, \sum_{j=1}^{m} p_j Q_j \right]
\]

\[
= - \left( q^i \xi + \sum_{j=1}^{m} \frac{1}{2} \left( p_j^2 + q_j^2 \right) [Q_i, \xi] \right) - \left( P^i + q^i [Q_i, P^i] \right)
\]

\[
= - q^i \xi - P^i + q^i \xi
\]

\[
= - P^i,
\]

while for \( I = m + 1, \ldots, n \) it is trivial that

\[
\mathcal{L}_{X_{hL}} Q^I = 0.
\]

Similar calculations yield

\[
\mathcal{L}_{X_{hL}} P^i = Q_i,
\]

and

\[
\mathcal{L}_{X_{hL}} P^I = 0,
\]

where we have used the canonical commutation relations \(^{11}\) to obtain the results. In addition, the Lie derivative of the 1-form horizontal basis is given by

\[
\mathcal{L}_{X_{hL}} dq^i = i_{X_{hL}} dq^i + d \left[ i_{X_{hL}} dq^i \right]
\]

\[
= d \left[ dq^i \left( \sum_{j=1}^{m} \frac{1}{2} \left( p_j^2 + q_j^2 \right) \xi + q^i P_j - p_j Q_j \right) \right]
\]

\[
= - dp_i,
\]

and

\[
\mathcal{L}_{X_{hL}} dp_i = dq^i,
\]

while

\[
\mathcal{L}_{X_{hL}} dq^I = 0,
\]

and

\[
\mathcal{L}_{X_{hL}} dp_I = 0,
\]

for \( I = m + 1, \ldots, n \).
3.2. Polarization scalings

Consider now the contact Hamiltonian

$$ h_S = \sum_{a=1}^{n} q^a p_a. $$

In this case the contact Hamiltonian vector field is

$$ X_{h_S} = \sum_{a=1}^{n} (q^a p_a) \xi + \sum_{a=1}^{n} [p_a P^a - q^a Q_a]. $$

Again, this is clearly a contact symmetry [cf. equation (15)]

$$ \mathcal{L}_{X_{h_S}} \eta = 0, $$

and its flow generates the 1-parameter family of anisotropic scalings $\delta_t : \mathcal{T} \to \mathcal{T}$

$$ \delta_t = \begin{cases} 
\tilde{w}(t) = w \\
\tilde{q}^a(t) = q^a e^{-t} \\
\tilde{p}_a(t) = p_a e^t
\end{cases}. $$

In this case, the transformation acts on every coordinate of $\mathcal{D}$. In particular we have

$$ \delta^*_t(\eta) = \eta, $$

where $\delta^*_t T^* \mathcal{T} \to T^* \mathcal{T}$ is the pullback induced map by $\delta_t$. Note that, for $t > 0$ one of the polarizations expands while the other shrinks. Thus, we call each member of the 1-parameter family $\delta_t$ a polarization scaling.

Again, the action of the Lie derivative of the horizontal basis with respect to the generator $X_{h_S}$ is

$$ \mathcal{L}_{X_{h_S}} Q_a = \left[ \sum_{b=1}^{n} (q^b p_b) \xi + \sum_{b=1}^{n} (p_b P^b - q^b Q_b), Q_a \right] $$

$$ = - \left[ Q_a, \sum_{b=1}^{n} q^b p_b \xi \right] - \left[ Q_a, \sum_{b=1}^{n} p_b P^b \right] + \left[ Q_a, \sum_{b=1}^{n} q^b Q_b \right] $$

$$ = - p_a \xi - \sum_{b=1}^{n} q^b p_b [Q_a, \xi] - \sum_{b=1}^{n} p_b [Q_a, P^b] + Q_a + \sum_{b=1}^{n} q^b [Q_a, Q_b] $$

$$ = - p_a \xi + p_a \xi + Q_a $$

$$ \mathcal{L}_{X_{h_S}} Q_a = Q_a $$

and

$$ \mathcal{L}_{X_{h_S}} P^a = - P^a, $$

(36)
while the action on the dual 1-forms is given by

\[ \mathcal{L}_{X_{h,S}} dq^a = \mathcal{L}_{X_{h,L}} dq^a = d [dq^a + \sum_{b=1}^n q^p_p b\xi + \sum_{b=1}^n \left(p_b P_b^b - q^b Q_b\right)] \]

and

\[ \mathcal{L}_{X_{h,S}} dq^a = -dp_a. \tag{38} \]

Clearly, the generators of Legendre symmetries and polarization scalings do not commute. Indeed,

\[ [X_{h,S}, X_{h,L}] = \sum_{a=1}^n \left(q^p_p a\xi - p_a Q_a + p_a P_a^a\right) + \sum_{i=1}^m \left\{ \frac{1}{2} (p_i^2 + q^i Q_i) - p_i Q_i + q^i P^i\right\} \]

\[ = \sum_{i=1}^m \left\{ (p_i^2 - q^i)\xi - 2 (p_i Q_i + q^i P^i)\right\}. \tag{40} \]

4. Almost Contact and Almost para-contact Structures

An almost contact structure is a triplet \((\eta, \xi, \phi)\) consisting of a contact 1-form \(\eta\), its corresponding Reeb vector field \(\xi\) and an automorphism \(\phi: T\mathcal{T} \to T\mathcal{T}\) such that \(\phi^2 = \phi \circ \phi = -1 + \eta \otimes \xi\) with \(\phi(\xi) = 0\) and \(\eta \circ \phi = 0\). \tag{41}\]

Here, \(1\) represents the identity map on \(T\mathcal{T}\). In this sense, the map \(\phi\) corresponds to the extension of an almost complex structure on a symplectic manifold to the contact case.

Similarly, an almost para-contact structure is a map \(\varphi: T\mathcal{T} \to T\mathcal{T}\) such that \(\varphi^2 = \varphi \circ \varphi = 1 - \eta \otimes \xi\) with \(\varphi(\xi) = 0\) and \(\eta \circ \varphi = 0\). \tag{42}\]

In the canonical basis it is expressed as

\[ 1 = \eta \otimes \xi + \sum_{i=1}^n \left[dq^i \otimes Q_i + dp_i \otimes P^i\right] \]

from which the basis expressions for \(\phi^2\) and \(\varphi^2\) follow

\[ \phi^2 = -\sum_{a=1}^n \left[dq^a \otimes Q_a + dp_a \otimes P^a\right] \quad \text{while} \quad \varphi^2 = \sum_{a=1}^n \left[dq^a \otimes Q_a + dp_a \otimes P^a\right]. \tag{44} \]

As both transformations act non-trivially only on \(\mathcal{D}\), we can express \(\phi\) and \(\varphi\) as their corresponding actions on the generators of \(\mathcal{D}\).
4.1. Almost Contact Structure

Let us begin by examining the almost contact case. Given a chosen polarization and the canonical almost complex structure $J : D \to D$, $J = -1$ such that $J(Q_a) = -P_a$ and $J(P_a) = Q_a$, a possible form for $\phi$ is as a map exchanging the polarization for $D$, that is, $\phi(\xi) = 0$, $\phi(Q_a) = -P_a$ and $\phi(P_a) = Q_a$. (45)

Note that this is merely a $\pi/2$-rotation acting on the generators of the contact distribution and, indeed, successive applications of the transformation yield a rotation by $\pi$, namely $\phi[\phi[0]] = \phi[0] = 0$ (46) $\phi[\phi[-P_a]] = \phi[-Q_a]$ (47) $\phi[\phi[-P_a]] = \phi[Q_a] = -P_a$. (48)

Thus, we can express $\phi$ in terms of the basis (1)

$$\phi = \sum_{a=1}^{n} [dp_a \otimes Q_a - dq^a \otimes P^a],$$

which clearly satisfies (41), [cf. equations (46) - (48) with (44)].

Such form suggests that $\phi$ has rotational (Legendre) symmetry. Indeed, using (23) - (28) one directly obtains

$$\mathcal{L}_{X_{h_L}} \phi = \mathcal{L}_{X_{h_L}} \sum_{a=1}^{n} [dp_a \otimes Q^a - dq^a \otimes P^a]$$

$$= \sum_{a=1}^{n} \left[ \mathcal{L}_{X_{h_L}} (dp_a \otimes Q^a) - \mathcal{L}_{X_{h_L}} (dq^a \otimes Q_a) \right]$$

$$= \sum_{a=1}^{n} \left[ \mathcal{L}_{X_{h_L}} dp_a \otimes Q_a + dp_a \otimes \mathcal{L}_{X_{h_L}} Q_a - \mathcal{L}_{X_{h_L}} dq^a \otimes P^a - dq^a \otimes \mathcal{L}_{X_{h_L}} P^a \right]$$

$$= \sum_{a=1}^{n} [dq^a \otimes Q_a - dp_a \otimes P^a + dp_a \otimes P^a - dq^a \otimes Q_a]$$

$$= 0.$$ (50)

It is also easy to see that such a transformation does not exhibit scaling symmetry, that is

$$\mathcal{L}_{X_{h_S}} \phi = 2 \sum_{a=1}^{n} [dp_a \otimes Q_a + dq^a \otimes P^a].$$ (51)

4.2. Almost para-contact Structure

We here consider three immediately obvious ways of satisfying (42), namely, $\pi$-rotations $\varphi_\pi(\xi) = 0$, $\varphi_\pi(Q_a) = -Q_a$ and $\varphi_\pi(P_a) = -P_a$. (52)
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partial polarization reflections

$$\varphi_r(\xi) = 0, \quad \varphi_r(Q_a) = Q_a \quad \text{and} \quad \varphi_r(P^a) = -P^a,$$

and the composition of clockwise $\pi/2$-rotations and partial reflections, $\varphi_s = \varphi_r \circ \varphi_{\pi/2}$

$$\varphi_s(\xi) = 0, \quad \varphi_s(Q_a) = P^a \quad \text{and} \quad \varphi_s(P^a) = Q_a.$$  (54)

These three automorphisms satisfy (42), therefore each of the $\varphi$ is an almost para-contact structure

$$\varphi^2 = \varphi_r^2 = \varphi_s^2 = 1 - \eta \otimes \xi.$$  (55)

Their corresponding local expressions are

$$\varphi_s = -\sum_{a=1}^n [d q^a \otimes Q_a + d p_a \otimes P^a]$$  (56)

and

$$\varphi_r = \sum_{i=1}^n [d q^a \otimes Q_a - d p_a \otimes P^a],$$  (57)

and

$$\varphi_s = \sum_{i=1}^n [d q^a \otimes P^a + d p_a \otimes Q_a].$$  (58)

It is also straightforward to show that

$$\mathcal{L}_{X_{h_L}} \varphi_s = 0, \quad \text{and} \quad \mathcal{L}_{X_{h_S}} \varphi_s = 0,$$  (59)

and

$$\mathcal{L}_{X_{h_L}} \varphi_r = -2 \sum_{i=1}^n \left[ d p_i \otimes Q_i + d q^i \otimes P^i \right] \quad \text{and} \quad \mathcal{L}_{X_{h_S}} \varphi_r = 0,$$  (60)

whilst

$$\mathcal{L}_{X_{h_L}} \varphi_s = 2 \sum_{i=1}^m \left[ d q^i \otimes Q_i - d p_i \otimes P^i \right] \quad \text{and} \quad \mathcal{L}_{X_{h_S}} \varphi_s = -2 \sum_{a=1}^n [d q^a \otimes P^a - d p_a \otimes Q_a].$$  (61)

Nevertheless, it can be shown that

$$\mathcal{L}_{X_{h_L}} \circ \mathcal{L}_{X_{h_S}} \varphi_s = \mathcal{L}_{X_{h_S}} \circ \mathcal{L}_{X_{h_L}} \varphi_s = 0,$$  (62)

therefore, it is trivially satisfied that

$$\mathcal{L}_{[X_{h_L}, X_{h_S}]} \varphi_s = 0.$$  (63)

As one might have expected, $\varphi_\pi$ has rotation symmetry, and thus it is polarization (Legendre) invariant. Additionally, it also has polarization scaling invariance. The structure $\varphi_s$ is not polarization nor scaling invariant, however it is symmetric under the composition in any order of these two transformations. On the other hand, $\varphi_r$ is not propagated along the flow of the infinitesimal generator of the Legendre transformations. This is not surprising since $\varphi_r$ acts on $\mathcal{D}$ as a change in orientation, which is not obtained by any sequence of rotations. However, it does present polarization scaling symmetry. This latter structure plays a central role in the construction of metric tensors for the various proposals in geometric thermodynamics, as we will now show.

\[\dagger\] The composition in the reverse order is $\varphi_s = \varphi_{\pi/2} \circ \varphi_r = -\varphi_s$, therefore we will only analyze the automorphism $\varphi_s$. 

5. Associated metrics and geometric thermodynamics

Most studies in contact geometry build upon their corresponding counterpart in the symplectic case. Thus, it can be seen that almost contact structures (resp. almost para-contact structures) are extensions of almost complex structures. Thus, an Hermitian metric on an almost complex manifold satisfies

$$g(JX, JY) = g(X, Y),$$

(64)

where $X$ and $Y$ are vector fields on a symplectic manifold and $J$ is an almost complex structure, i.e. a $(1, 1)$-tensor field satisfying that $J^2 = J \circ J = -1$.

A similar structure arises in almost contact manifolds \[25\] (resp. almost para-contact manifolds \[28\]). We say that a metric tensor is a compatible metric if it satisfies \[25, 28\]

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad \text{(resp. } g(\varphi X, \varphi Y) = -[g(X, Y) - \eta(X)\eta(Y)]\text{)},$$

(65)

and $(\phi, \xi, \eta, g)$ is called an almost contact metric structure (resp. $(\varphi, \xi, \eta, g)$ is called an almost para-contact metric structure). If the metric for the almost contact (resp. para-contact) metric structure satisfies

$$g(X, \phi Y) = d\eta(X, Y) \quad \text{(resp. } g(X, \varphi Y) = d\eta(X, Y)),$$

(66)

it is said that $g$ is an associated metric and $(\phi, \xi, \eta, g)$ a contact metric structure (resp. $(\varphi, \xi, \eta, g)$ a para-contact metric structure).

Given an almost contact structure $(\eta, \xi, \phi)$ (resp. almost para-contact structure $(\eta, \xi, \varphi)$), a metric tensor can be constructed solely from these ingredients. Here, we consider metric tensors in a broader sense, that is, $(0, 2)$ non-degenerate and symmetric tensor fields, while relaxing the condition of begin positive definite. These metrics so constructed are, for the almost contact structure

$$g = \eta \otimes \eta + d\eta \circ (\phi \otimes \mathbb{1}),$$

(67)

and for the almost para-contact structure\[6\]

$$g = \eta \otimes \eta - d\eta \circ (\varphi \otimes \mathbb{1}),$$

(68)

where it must be understood that

$$(\phi \otimes \mathbb{1})(X, Y) = (\phi X, Y) \quad \text{(resp. } (\varphi \otimes \mathbb{1})(X, Y) = (\varphi X, Y)), $$

(69)

for $X, Y \in TT$, i.e. here $\otimes$ acts as a separator for $\phi$ (resp. $\varphi$) and $\mathbb{1}$ acting on $T_pT \times T_pT$ for every $p \in T$.

It must be noticed that $d\eta \circ (\phi \otimes \mathbb{1})$ (resp. $d\eta \circ (\varphi \otimes \mathbb{1})$) does not necessarily yields a metric for any almost contact (para-contact) structure, it must be a symmetric non-degenerate tensor on its restriction to $D$ to do so.

Then, it is sufficient for the tensor $d\eta \circ (\phi \otimes \mathbb{1})$ (resp. $d\eta \circ (\varphi \otimes \mathbb{1})$) to be symmetric and non-degenerate on $D$ to render the metric \[67\] (resp. \[68\]) a compatible metric with the almost contact (para-contact) structure, i.e., condition \[65\] is satisfied for $g$.

Likewise, it can also be shown that if metrics constructed as in \[67\] and \[68\] are compatible metrics, then they also are associated metrics.

\[\S\] The metric with a plus sign, $g = \eta \otimes \eta + d\eta \circ (\varphi \otimes \mathbb{1})$, is compatible if $d\eta \circ (\varphi \otimes \mathbb{1})$ is symmetric, but it is not an associated metric.
5.1. Associated metric to an Almost Contact Structure

The differential of the contact 1-form is a $(0,2)$ antisymmetric tensor field, then in (67) the use of a contact almost structure can be used to render this tensor symmetric. Let us consider the canonical almost contact structure presented in subsection 4.1. Its action on a vector field $X \in T T$ is given by

$$\phi X = \sum_{a=1}^{n} [dp_a \otimes Q_a - dq^a \otimes P^a] \left( X^w \xi + \sum_{b=1}^{n} [X^{Q^b} Q_b + X^{P^b} P_b] \right)$$

$$= \sum_{a=1}^{n} \left[ X^{P_a} Q_a - X^{Q_a} P_a \right], \quad (70)$$

and the combination

$$d \eta (\phi X, Y) = -\frac{1}{2} \sum_{a=1}^{n} [dp_a \otimes dq^a - dq^a \otimes dp_a] (\phi X, Y)$$

$$= -\frac{1}{2} \sum_{a=1}^{n} \left[ dp_a \left( -X^{Q_a} P_a \right) dq^a \left( Y^{Q_a} Q_a \right) - dq^a \left( X^{P_a} Q_a \right) dp_a \left( Y^{P_a} P_a \right) \right]$$

$$= \frac{1}{2} \sum_{a=1}^{n} \left[ X^{Q_a} Y^{Q_a} + X^{P_a} Y^{P_a} \right]$$

$$= \frac{1}{2} \sum_{a=1}^{n} [dq^a \otimes dq^a + dp_a \otimes dp_a] (X, Y) \quad (71)$$

is a symmetric and non-degenerate 2-rank tensor field restricted to $D$. Therefore, the metric (67) with $\phi$ given by (49) is compatible and associated.

In the local basis (67) with (49) is written as

$$g = \eta \otimes \eta + \frac{1}{2} \sum_{a=1}^{n} [dq^a \otimes dq^a + dp_a \otimes dp_a]. \quad (72)$$

Thus we see that the metric obtained from the canonical almost contact structure renders the frame defined from the Heisenberg group commutation relations (11), orthogonal. That is

$$g(\xi, Q_a) = g(\xi, P_a) = g(Q_a, P_b) = 0, \quad g(\xi, \xi) = 1 \quad (73)$$

and

$$g(Q_a, Q_b) = g(P_a, P_b) = \begin{cases} \frac{1}{2} & a = b \\ 0 & a \neq b \end{cases} \quad (74)$$

Albeit the structure of the metric resembles that of the Euclidean space, it is not the case. The metric is expressed in terms of the dual frame. In particular, a direct calculation of the Ricci curvature tensor yields

$$Ric = 2n (\eta \otimes \eta) - \sum_{a=1}^{n} [dq^a \otimes dq^a + dp_a \otimes dp_a]$$

$$= \lambda \eta \otimes \eta + \nu g, \quad (75)$$
where \( \lambda = 2n + 2 \) and \( \nu = -2 \). When condition (75) is satisfied, it is said that \((\eta, \xi, \phi, g)\) is an \( \eta \)-Einstein manifold.

It can be shown that the Lie derivative with respect to a vector field \( X \in T\mathcal{T} \) of (67) (resp. (68)) is

\[
\mathcal{L}_X g = \mathcal{L}_X (\eta \otimes \eta) + (\mathcal{L}_X d\eta) \circ (\phi \otimes \mathbb{1}) + d\eta \circ (\mathcal{L}_X \phi \otimes \mathbb{1}).
\] (76)

Therefore, the Lie derivative of \( g \) with respect to a contact hamiltonian vector field \( X_h \) of an horizontal hamiltonian \( \xi(h) = 0 \), reduces to

\[
\mathcal{L}_{X_h} g = d\eta \circ (\mathcal{L}_{X_h} \phi \otimes \mathbb{1}) \quad \text{(resp.} \quad \mathcal{L}_{X_h} g = d\eta \circ (\mathcal{L}_{X_h} \varphi \otimes \mathbb{1}) \text{)}.
\] (77)

Thus, for the transformations generated by \( X_h \) to be an isometry of the metrics (67) and (68) it is sufficient to have \( \mathcal{L}_{X_h} \phi = 0 \) and \( \mathcal{L}_{X_h} \varphi = 0 \), respectively. That is, the symmetry property of the almost contact (para-contact) structure is inherited to the metrics here considered.

Therefore, we have that (72) has Legendre transformations as symmetries, i.e.

\[
\mathcal{L}_{X_{h_{L}}} g = 0,
\] (78)

while it is clear that it does not have polarization scaling transformations as an isometry

\[
\mathcal{L}_{X_{h_{S}}} g = \sum_{a=1}^{n} \left[ dp_a \otimes dq_a - dq_a \otimes dp_a \right].
\] (79)

5.2. Associated metric to an almost para-contact Structure

In this subsection we consider the almost-para-contact structures explored in subsection 4.2, namely those related to \( \pi \)-rotations, to partial polarization reflections and to the composition of \( \pi/2 \)-rotations with the partial polarization reflections.

The action of \( \varphi_{\pi} \) on a vector field \( X \in T\mathcal{T} \) is given by

\[
\varphi_{\pi} X = - \sum_{a=1}^{n} \left( dq^a \otimes Q_a + dp_a \otimes P^a \right) \left( X^w \xi + \sum_{b=1}^{n} \left[ X^{Q_b} Q_b + X_{P_b} P^b \right] \right)
\]

\[
= - \sum_{a=1}^{n} \left[ X_{P_a} P^a + X^{Q_a} Q_a \right],
\] (80)

and the combination

\[
d\eta (\varphi_{\pi} X, Y) = - \frac{1}{2} \sum_{a=1}^{n} \left[ dp_a \otimes dq^a - dq^a \otimes dp_a \right] (\varphi_{\pi} X, Y)
\]

\[
= - \frac{1}{2} \sum_{a=1}^{n} \left[ dp_a (-X_{P_a} P^a) dq^a \left( Y^{Q_a} Q_a \right) - dq^a \left( -X^{Q_a} Q_a \right) dp_a (Y_{P_a} P^a) \right]
\]

\[
= \frac{1}{2} \sum_{a=1}^{n} \left[ X_{P_a} Y^{Q_a} - X^{Q_a} Y_{P_a} \right]
\]

\[
= \frac{1}{2} \sum_{a=1}^{n} \left[ dp_a \otimes dq^a - dq^a \otimes dp_a \right] (X, Y)
\] (81)
is antisymmetric. Then, it is not possible to construct a metric as in the almost contact case, instead this almost para-contact structure can be used, in analogy, to define a 2-rank covariant tensor field given by

$$\alpha_\pi = \eta \otimes \eta + dn_\eta \circ (\varphi_\pi \otimes 1) = \eta \otimes \eta - d\eta,$$

which in the local basis is written as

$$\alpha_\pi = \eta \otimes \eta + \frac{1}{2} \sum_{a=1}^{n} [dp_a \otimes dq^a - dq^a \otimes dp_a].$$

Since the tensor (82) is constructed from the almost-para-contact structure (52) with rotational and polarization scaling symmetries [cf. equation (59)], \(\alpha_\pi\) satisfies

$$\mathcal{L}_{X_{\kappa L}} \alpha_\pi = 0 \quad \text{and} \quad \mathcal{L}_{X_{\kappa S}} \alpha_\pi = 0.$$  

Similarly, the metric constructed from the almost para-contact structure with polarization reflection symmetry [cf. equation (53)] can be constructed by considering the action of \(\varphi_r\) on a vector field \(X \in T\mathcal{T}\),

$$\varphi_r X = \sum_{a=1}^{n} [dq^a \otimes Q_a - dp_a \otimes P^a] \left( X^{\underline{w} \underline{z}} + \sum_{b=1}^{n} [X^{Qb} Q_b + X^{Pb} P_b] \right)$$

$$= \sum_{a=1}^{n} [X^{Qa} Q_a - X^{Pa} P^a]$$

and, again, noting that

$$d\eta (\varphi_r X, Y) = -\frac{1}{2} \sum_{a=1}^{n} [dp_a \otimes dq^a - dq^a \otimes dp_a] (\varphi_r X, Y)$$

$$= -\frac{1}{2} \sum_{a=1}^{n} \left[ dp_a (-X^{Pa} P^a) dq^a (Y^{Qa} Q_a) - dq^a (X^{Qa} Q_a) dp_a (Y^{Pa} P^a) \right]$$

$$= \frac{1}{2} \sum_{a=1}^{n} [X^{Pa} Y^{Qa} + X^{Qa} Y^{Pa}]$$

$$= \frac{1}{2} \sum_{a=1}^{n} [dp_a \otimes dq^a + dq^a \otimes dp_a] (X, Y)$$

is symmetric and non-degenerate on \(\mathcal{D}\). Therefore, the metric on \(\mathcal{T}\) constructed as (68) with \(\varphi = \varphi_r\)

$$g_r = \eta \otimes \eta - d\eta \circ (\varphi_r \otimes 1)$$

is compatible and consequently associated too. Expressed in terms of the local basis takes the following form

$$g_r = \eta \otimes \eta - \frac{1}{2} \sum_{a=1}^{n} [dp_a \otimes dq^a + dq^a \otimes dp_a].$$
Thus, the metric $g_r$ does not render the frame (1) orthonormal, instead,

$$g_r(\xi, Q_a) = g_r(\xi, P_a) = 0, \quad g_r(\xi, \xi) = 1$$  \hspace{1cm} (89)

and

$$g_r(Q_a, Q_b) = g_r(P^a, P^b) = 0, \quad g_r(P^a, Q_b) = g_r(Q_b, P^a) = -\frac{1}{2} \delta^a_b.$$ \hspace{1cm} (90)

From (60) we deduce that the metric $g_r$ has scaling symmetry

$$\mathcal{L}_{X_{hs}} g_r = 0,$$ \hspace{1cm} (91)

but they fail to be propagated along the Legendre symmetry generator, i.e.

$$\mathcal{L}_{X_{hs}} g_r = -\sum_{i=1}^{m} \left[ dq^i \otimes dq^i - dp_i \otimes dp_i \right].$$ \hspace{1cm} (92)

For the metric constructed from $\varphi_s$ we follow the same steps as before. The action of $\varphi_s$ on $X \in TT$ is

$$\varphi_s X = \sum_{a=1}^{n} [dq^a \otimes P^a + dp_a \otimes Q_a] \left( X^w \xi + \sum_{b=1}^{n} \left[ X^Q b Q_b + X^P b P_b \right] \right),$$

$$= \sum_{a=1}^{n} \left[ X^Q a P^a + X^P a Q_a \right],$$ \hspace{1cm} (93)

and

$$d\eta(\varphi_s X, Y) = -\frac{1}{2} \sum_{a=1}^{n} [dp_a \otimes dq^a - dq^a \otimes dp_a] (\varphi_s X, Y)$$

$$= -\frac{1}{2} \sum_{a=1}^{n} \left[ dp_a(X^Q b P^b) dq^a(Y^Q b Q_b) - dq^a(X^P b Q_a) dp_a(Y^P b P_b) \right]$$

$$= -\frac{1}{2} \sum_{a=1}^{n} \left[ X^Q a Y^Q a - X^P a Y^P a \right]$$

$$= -\frac{1}{2} \sum_{a=1}^{n} [dq^a \otimes dq^a - dp_a \otimes dp_a] (X, Y).$$ \hspace{1cm} (94)

is a symmetric non-degenerate tensor on $\mathcal{D}$. Hence the metric constructed as (68) from $\varphi_s$ is a compatible and associated metric, which in terms of the local basis takes the following form

$$g_s = \eta \otimes \eta + \frac{1}{2} \sum_{a=1}^{n} [dq^a \otimes dq^a - dp_a \otimes dp_a].$$ \hspace{1cm} (95)

The frame (1) is pseudo-orthogonal with respect to $g_s$

$$g_s(\xi, Q_a) = g_s(\xi, P_a) = g_s(Q_a, P^b) = 0, \quad g_s(\xi, \xi) = 1$$ \hspace{1cm} (96)
and
\[ g_s(Q_a, Q_b) = \begin{cases} \frac{1}{2} & a = b \\ 0 & a \neq b \end{cases} \quad \text{and} \quad g_s(P_a, P_b) = \begin{cases} -\frac{1}{2} & a = b \\ 0 & a \neq b. \end{cases} \]  

(97)

In what refers to the Legendre and scaling transformations, give n the properties of \( \varphi_s \), these are not isometries for this metric, instead
\[ \mathcal{L}_{X_h} g_s = -\sum_{i=1}^m (dq^i \otimes dp_i + dp_i \otimes dq^i) \quad \text{and} \quad \mathcal{L}_{X_{h_a}} g_s = -\sum_{a=1}^n (dp_a \otimes dp_a + dq^a \otimes dq^a), \]

(98)

while it is symmetric for the composition of the transformations in any order
\[ \mathcal{L}_{X_{h_a}} \circ \mathcal{L}_{X_{h_b}} g_s = \mathcal{L}_{X_{h_b}} \circ \mathcal{L}_{X_{h_a}} g_s = 0 \quad \text{then} \quad \mathcal{L}_{[X_{h_a}, X_{h_b}]} g_s = 0, \]

(99)

just as \( \varphi_s \) is.

Equation (92) shows us that (87) is not Legendre invariant. The pullback of this metric onto its corresponding Legendre submanifold takes the form of a Hessian metric and plays a preponderant part in the geometric description of thermodynamics [11, 29]. It has been shown that such a metric carries information about the fluctuations around equilibrium [30]. This metric does not preserve, in general, its Hessian form under a change of thermodynamic potential [16] making the geometric description of fluctuations ensemble dependent [15]. In [31] it was investigated a group of transformations that leave Weinhold’s metric invariant, finding that a total Legendre transformation preserves the form of the metric while all the partial Legendre transformation do not belong to this group. Other attempts to find metrics for the space of thermodynamic equilibrium states which are invariant under Legendre transformations were conducted in a strongly dependent thermodynamic coordinate framework [19] yielding a set of metrics whose components are proportional to the components of the Hessian metrics. Recently, it was shown in [20, 32] that Legendre invariant metrics can be related physically to reparametrizations of the thermodynamic state variables. The set of metrics found in [20] have as a particular case those of [19]. Hence, it is physically relevant and mathematically consistent to generate Legendre invariant metrics from purely geometric structures.

In the next section we construct a family of such metrics from an automorphism which is a modification to the almost para-contact structure related to partial polarization reflections \( \varphi_r \).

This automorphism is not a almost para-contact structure but it will be shown that it satisfy some generalized version of (42).

6. Polarization independent metrics

Let \( \psi : \mathcal{L} \to \mathcal{T} \) be an embedding of \( \mathcal{L} \) into the contact manifold \( \mathcal{T} \) defined by the condition \( \psi^* \eta = 0 \) and \( \text{dim(\mathcal{L})} = n \). In geometric thermodynamics, the Legendre submanifold \( \mathcal{L} \) is referred as the space of equilibrium states since its defining condition can be identified with the first law of thermodynamics [cf. equation (3), above]. In addition, it is straightforward to equip such a sub-manifold with a Riemannian structure by considering the induced metric
\[ \psi^* g_r = -\frac{\partial^2 w}{\partial q^a \partial q^b} dq^a \otimes dq^b. \]

(100)

This is a widely used metric in the geometric description of thermodynamics, where the function \( w(q^a) \) is usually identified with the entropy or the internal energy of the thermodynamic system [11, 29].
On the contact manifold, different polarizations of the symplectic distribution correspond to distinct choices of thermodynamic potentials. In this section we aim to construct polarization independent metrics on \( T \), i.e. metrics which have Legendre transformations as isometries, subject to the condition that the components of its pullback onto its corresponding Legendre submanifold are proportional to the components of the Hessian of a thermodynamic potential. We have seen that the contact form is polarization independent, that translates in the thermodynamic language into the Legendre invariance of the description of equilibrium thermodynamics, namely, the Legendre invariance of the first law. It has been argued that the metric on the space of equilibrium states with components given by the negative Hessian of the entropy encloses the information of the second law and its positive definiteness accounts for the local stability conditions of the system \cite{30}. It is also well known that such a Hessian metric does not posses a symmetry under a general Legendre transformation, we can see precisely that from equation (92). As we will see it will be necessary to abandon the possibility of having a metric contact manifold. Nevertheless, a generalization of the notion of metric contact manifold can be proposed in terms of a modification of the almost para-contact structure \( \phi_r \) from which a metric can be constructed which allows the possibility of being Legendre invariant.

Let us consider an automorphism \( \varphi_\Omega \) constructed from the almost para-contact structure multiplying each component by a function \( \Omega_a(w,q,p) \), where \( q = (q^1, \ldots, q^n) \) and similarly for \( p = (p^1, \ldots, p^n) \)

\[
\varphi_\Omega = \sum_{a=1}^{n} \Omega_a(q^a \otimes Q_a - dp_a \otimes P^a).
\]

This tensor field is not a almost contact or para-contact structure, but satisfies

\[
\varphi_\Omega(\xi) = 0, \quad \varphi_\Omega(Q_a) = \Omega_a Q_a \quad \text{and} \quad \varphi_\Omega(P^a) = -\Omega_a P^a,
\]

and

\[
\varphi_\Omega^2(\xi) = 0, \quad \varphi_\Omega^2(Q_a) = \Omega_a^2 Q_a \quad \text{and} \quad \varphi_\Omega^2(P^a) = \Omega_a^2 P^a,
\]

which can be expressed as

\[
\varphi_\Omega^2 = \mathbb{I}_\Omega - \eta \otimes \xi,
\]

where

\[
\mathbb{I}_\Omega = \eta \otimes \xi + \sum_{a=1}^{n} \Omega_a (dq^a \otimes Q_a + dp_a \otimes P^a).
\]

As we have seen \( \varphi_r \) is invariant under polarization scaling but fails to be symmetric under Legendre transformations. Let us analyze the symmetry properties of \( \varphi_\Omega \) under these transformations. Let us consider the infinitesimal action of \( X_{h_s} \) on \( \varphi_\Omega \) first

\[
\mathcal{L}_{X_{h_s}} \varphi_\Omega = \sum_{a=1}^{n} \left( \mathcal{L}_{X_{h_s}} \Omega_a \right) (dq^a \otimes Q_a - dp_a \otimes P^a) + \Omega_a \mathcal{L}_{X_{h_s}} (dq^a \otimes Q_a - dp_a \otimes P^a)
\]

\[
= \sum_{a=1}^{n} \left( \mathcal{L}_{X_{h_s}} \Omega_a \right) (dq^a \otimes Q_a - dp_a \otimes P^a).
\]

Then, for \( \varphi_{\Omega_a} \) to be polarization scaling invariant it must be satisfied that

\[
\mathcal{L}_{X_{h_s}} \Omega_a = 0,
\]
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and from the expression for the Hamiltonian vector field generating such transformations (62) we obtain the following condition

\[ \sum_{b=1}^{n} \left( p_b \frac{\partial \Omega(a)}{\partial p_b} - q^b \frac{\partial \Omega(a)}{\partial q^b} \right) = 0. \]  

(108)

The solutions that make \( \varphi_\Omega \) polarization scaling invariant are a set of functions which have the general form

\[ \Omega(a) = \Omega(a) \left( w, q^b p^c, q^b p^c, q^b q^c \right), \]  

(109)

for all \( a = 1, \ldots, n \), where \( b \neq c \) is fixed at any value from 1 to \( n \) and \( c \) takes all the other values.

The metric constructed from this automorphism (101)

\[ g_\Omega = \eta \otimes \eta - \frac{1}{2} \sum_{a=1}^{n} \Omega(a) \left( dp_a \otimes dq^a + dq^a \otimes dp_a \right). \]  

(111)

These metrics are not compatible nor associated because \( \varphi_\Omega \) is not an almost contact or para-contact structure. Let us analyze the behavior of (111) under an infinitesimal Legendre transformation. Its Lie derivative with respect to \( X_{hL} \),

\[ \mathcal{L}_{X_{hL}} g_\Omega = -\frac{1}{2} \sum_{i=1}^{m} \left( \mathcal{L}_{X_{hL}} \Omega(i) \right) \left( dp_i \otimes dq^i + dq^i \otimes dp_i \right) + \Omega(i) \left( dq^i \otimes dq^i - dp_i \otimes dp_i \right), \]  

(112)

shows that the metric is not invariant under an infinitesimal strict contactomorphism as there are not functions \( \Omega(i) \) such that \( \mathcal{L}_{X_{hL}} g_\Omega = 0 \).

Nevertheless, in thermodynamics one is interested in the finite version of the strict contactomorphisms given by (19), those representing a partial change of the symplectic polarization in \( \mathcal{D} \) as \( \pi/2 \)-rotations of some polarization planes. For \( g_\Omega \) to be Legendre invariant, the functions \( \Omega(a) \) must be such that

\[ \left[ \Phi_{m}^{\pi/2} \right]^* g_\Omega = g_\Omega. \]  

(113)

Then, we can establish the conditions on the functions \( \Omega(i) \) as follows

\[ 0 = \left[ \Phi_{m}^{\pi/2} \right]^* g_\Omega - g_\Omega = \frac{1}{2} \sum_{i=1}^{m} \left( \left[ \Phi_{m}^{\pi/2} \right]^* \Omega(i) + \Omega(i) \right) \left( dq^i \otimes dp_i + dp_i \otimes dq^i \right) \]

\[ - \frac{1}{2} \sum_{I=m+1}^{n} \left( \left[ \Phi_{m}^{\pi/2} \right]^* \Omega(I) - \Omega(I) \right) \left( dq^I \otimes dp_I + dp_I \otimes dq^I \right). \]  

(114)

Therefore, for \( g_\Omega \) to be Legendre invariant it must be satisfied that under a Legendre transformation on the \( i = 1, \ldots, m \) directions

\[ \left[ \Phi_{m}^{\pi/2} \right]^* \Omega(i) = -\Omega(i) \]  

and \[ \left[ \Phi_{m}^{\pi/2} \right]^* \Omega(I) = \Omega(I), \]  

(115)
for each \( i = 1, \ldots, m \) and \( I = m + 1, \ldots, n \).

It is clear that the above conditions impose further restrictions on the functions than those in (109). Evidently, we must have \( \partial_a \Omega_{(a)} = 0 \). We are interested in having a metric that is invariant under all the Legendre transformations, that is, for any subdivision of the indices \( i \) and \( I \). Therefore an obvious choice for the functions \( \Omega_{(a)} \) which is a particular case of (109), and consequently invariant also under polarization scaling, is

\[
\Omega_{(a)} = \Omega_{(a)}(q^a p_a)
\]

for \( a = 1, \ldots n \). Hence, \( \Omega_{(a)} \) for each \( a \) is a function of only the product of the corresponding variables, e.g. \( \Omega_{(1)} = \Omega_{(1)}(q^1 p_1) \). These functions must also be odd

\[
\Omega_{(a)}(-q^a p_a) = -\Omega_{(a)}(q^a p_a).
\]

It is worth mentioning that there is another simple choice for \( \Omega_{(a)} \) that makes \( g_\Omega \) Legendre invariant, although not invariant under polarization scaling. It was explored in [20] from a different approach and considers the functions \( \Omega_{(a)} \) as the product of two functions. That is, \( \Omega_{(a)} = f_{(a)}(q^a)g_{(a)}(p_a) \) where \( f \) and \( g \) have the same functional form and are odd functions.

In general, given a contact structure and an associated metric, the explicit form of the almost para-contact structure can be obtained by the covariant derivative of the Reeb vector [28],

\[
\nabla \xi = \varphi - \varphi \kappa,
\]

where

\[
\kappa = \frac{1}{2} \mathcal{L}_\xi \varphi.
\]

The tensor field \( \kappa \) vanishes if \( \xi \) corresponds to a Killing vector of the associated metric. Therefore, the almost para-contact structure is defined by

\[
\nabla \xi = \varphi.
\]

Since the metric components of (111) does not depend of \( \omega \), \( \xi \) is indeed a Killing vector of \( g_\Omega \). The covariant derivative associated to (111), of the Reeb vector in the basis \( e^{(i)} = \{ \xi, \frac{\partial}{\partial p_a}, \frac{\partial}{\partial q_a} \} \) and dual basis \( \theta^{(a)} = \{ d\omega, dp_a, dq^a \} \) is

\[
\nabla \xi = \sum_{a,b=0}^{2n} \Gamma^a_{\omega b} e_{(a)} \otimes \theta^{(b)}
\]

\[
= \sum_a^n \left( \Gamma^{p_a}_{\omega p_a} \frac{\partial}{\partial p_a} \otimes dp_a + \Gamma^{q_a}_{\omega q_a} \frac{\partial}{\partial q^a} \otimes dq^a + \Gamma^{a}_{\omega q^a} \xi \otimes dq^a \right)
\]

\[
= -\sum_{a=1}^{n} \frac{1}{\Omega_{(a)}} \left[ dq^a \otimes Q_{(a)} - dp_a \otimes P^a \right]
\]

\[
= -\varphi_{\Omega}
\]

where

\[
\varphi_{\Omega} = \sum_{a=1}^{n} \frac{1}{\Omega_{(a)}} \left[ dq^a \otimes Q_{(a)} - dp_a \otimes P^a \right].
\]
This confirms that the automorphism (101) is not an almost para-contact structure as we have seen before. Indeed, \(-\varphi \bar{\Omega}\) is significantly different to \(\varphi \Omega\). We can construct a metric from the automorphism given by (122) as

\[
\bar{g}_\Omega = \eta \otimes \eta - d\eta \circ (\varphi \Omega \otimes \mathbb{1})
\]  

(123)

and, as we did before, it is now possible to calculate the covariant derivative of the Reeb vector, this time associated to the affine connection of (123) instead

\[
\bar{\nabla} \xi = \sum_{a,b=0}^{2n} \Gamma^a_{\omega b} e_{(a)} \otimes \theta^{(b)}
\]

\[
= \sum_{a=1}^{n} \left( \Gamma^p_{\omega p} \frac{\partial}{\partial p_a} \otimes dp_a + \Gamma^q_{\omega q} \frac{\partial}{\partial q_a} \otimes dq_a + \Gamma^\omega_{\omega q} \xi \otimes dq_a \right)
\]

\[
= \sum_{i=a}^{n} \Omega_{(a)} \left[ dq_a \otimes Q_{(a)} - dp_a \otimes P^a \right]
\]

\[
= -\varphi \Omega.
\]  

(124)

Thus, we observe that the covariant derivative of the Reeb vector with respect to the Levi-Civita connection of the metrics \(g_\Omega\) and \(\bar{g}_\Omega\) establish a sort of dual relation between the automorphisms \(\varphi \Omega\) and \(\varphi \bar{\Omega}\). Hence the introduction of these \(\Omega_{(a)}\) functions multiplying each component of \(\varphi_r\) changes the usual properties of the Reeb vector field.

We can conclude that in order to construct a Legendre invariant metric which induces a metric on the Legendre submanifold with components proportional to the Hessian of a function we must give up the contact metric structure. It is interesting that the modifications proposed allow us to introduce a further relationship between structures for a metric contact manifold. Indeed, these dual automorphisms satisfy

\[
\varphi \Omega \circ \varphi \bar{\Omega} = \varphi \bar{\Omega} \circ \varphi \Omega = \sum_{i=1}^{n} \left[ dq_i \otimes Q_i + dp_i \otimes P^i \right] = \mathbb{1} - \eta \otimes \xi.
\]  

(125)

Therefore, these modifications in (101) not only generate families of Legendre invariant metrics, they generate structures with similar, but more general, properties of almost para-contact structures.

7. Closing remarks

We have analyzed the construction of associated metrics using different almost contact and para-contact structures which where defined according to its symmetry properties under change of symplectic polarizations and polarization scalings. As it is already known, the associated metric constructed from the almost para-contact structure \((\mathcal{P}, \mathcal{C})\) plays an important role in the geometric description of thermodynamics as it gives a Hessian metric on the Legendre submanifold, which physically represents the space of equilibrium states. We showed that this metric \(g_r\) is not invariant under changes of polarization, namely, Legendre transformations, but it has scaling symmetry. This proves that it is not possible to construct an associated metric satisfying both properties, being Legendre invariant and inducing a Hessian metric into a Legendre submanifold. Although there are
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associated metrics to an almost contact structure (72) invariant under Legendre transformations, they do not satisfy the second requirement expressed above. We have seen that both – polarization and scaling invariance – cannot be simultaneous isometries of an associated metric.

It was our aim to construct a metric satisfying a relaxed version of the requirements mentioned above: to be Legendre invariant and to induce a metric on the Legendre manifold whose components are each proportional to the entries of a Hessian matrix. In order to achieve this, we have found that it is necessary to generalize the concept of almost para-contact structures and consequently the notion of a metric contact manifold. This generalization has been established in the last section and it can be understood as an anisotropic scaling of the almost para-contact structure (67). The generalized structures that we have found seem to have interesting properties which demand further exploration.

The Legendre invariant metrics presented here were constructed with the sole objective of minimally modifying the structures defining a contact metric manifold. Some other works had reached equivalent results using as a motivation some physical criteria [19,20,32] and from a different perspective. For instance, in [31] it was studied the group of transformations that leave the Hessian metrics invariant.

We leave for future work the analysis of the generalized structures found here and their possible consequences beyond the realm of thermodynamics. In particular, let us close this work by noting that a choice of symplectic polarization for the contact distribution is equivalent to an election of thermodynamic potential. In this sense, a Legendre transformation is nothing but a mere change of symplectic polarization in the context of contact geometry.

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