CHAOTIC BEHAVIOR OF RAPIDLY OSCILLATING LAGRANGIAN SYSTEMS

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Abstract. In the paper we prove that the Lagrangian system
\[ \ddot{q} = \alpha(\omega t)V'(q), \quad t \in \mathbb{R}, \; q \in \mathbb{R}^N, \]
has, for some classes of functions \( \alpha \), a chaotic behavior—more precisely the system has multi-bump solutions—for all \( \omega \) large. These classes of functions include some quasi-periodic and some limit-periodic ones, but not any periodic function.

We prove the result using global variational methods.

1. Introduction. Let us consider the Lagrangian system
\[ \ddot{q} = \alpha(\omega t)V'(q), \quad t \in \mathbb{R}, \; q \in \mathbb{R}^N, \]
where \( \omega > 0 \), \( \alpha \) is a non constant, strictly positive, almost-periodic function such that \( \alpha = \inf_{\mathbb{R}} \alpha > 0 \) and \( V \in C^2(\mathbb{R}^N, \mathbb{R}) \) satisfies
\begin{enumerate}
  \item[(V1)] \( V(x + \xi) = V(x) \) for all \( x \in \mathbb{R}^N \) and \( \xi \in \mathbb{Z}^N \);
  \item[(V2)] \( V(x) > V(0) = 0 \) for all \( x \in \mathbb{R}^N \setminus \mathbb{Z}^N \);
  \item[(V3)] there exists \( \nu > 0 \) such that \( \langle V''(0)x \mid x \rangle \geq \nu |x|^2 \) for all \( x \in \mathbb{R}^N \).
\end{enumerate}

Using variational methods it has been proved that (\( L_\omega \)) has, for all such \( \alpha \) and \( \omega > 0 \), infinitely many homoclinic solutions (see for example [1, 15, 3]). It is also known such a system has multibump solutions, and hence chaotic behavior, whenever a suitable nondegeneracy condition is satisfied (following ideas introduced by E. Séré in [16]). Such a condition has been shown to be satisfied for a given \( \alpha \) whenever \( \omega \) is small enough (see [8, 1, 2]), and generically in \( \alpha \) (see [2]). See also [7] for a case in which \( \alpha \) is the sum of a slowly oscillating part and a fast oscillating one.

The main purpose of this paper is to describe some examples of (class of) functions \( \alpha \) such that (\( L_\omega \)) has a multibump solution for all \( \omega > 0 \) large. We will give three examples: a quasi periodic one (subsection 4.1), an almost periodic one (subsection 4.2) and a limit periodic one (subsection 4.3).

To prove the result we will need (see section 4) that for all \( \omega > 0 \) one can split the function \( \alpha(\omega t) = \alpha_{\omega,1}(t) + \alpha_{\omega,2}(t) + \alpha_{\omega,3}(t) \) in such a way that \( \alpha_{\omega,1} \) is a nontrivial slowly oscillating function, \( \alpha_{\omega,2} \) is a fast oscillating function and \( \alpha_{\omega,3} \) is small compared to \( \alpha_{\omega,1} \). Here the time scale is given by the time it takes to the

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homoclinic solution of $\ddot{q} = \alpha V'(q)$ to go from a neighborhood of 0 to a neighborhood of $\xi$. For this reason we cannot handle the case in which $\alpha$ is a periodic function (since in this case there $\alpha_{\omega,1} \equiv 0$) or is a quasi-periodic function $\alpha(t) = \beta(\gamma t)$ where $\beta: \mathbb{T}^k \to \mathbb{R}$ is an analytic function (since in this case $\alpha_{\omega,3}$ is not sufficiently small compared to $\alpha_{\omega,1}$).

One of the motivations in studying such a problem lies in the fact that the system $(L_\omega)$ is equivalent to the following

$$\ddot{q} = \frac{1}{\omega^2} \alpha(t) V'(q)$$

(it is enough to perform the change of variable $q(t) \mapsto q(\omega t)$). This system can be seen as a small perturbation of the completely integrable system $\ddot{q} = 0$ on the torus $\mathbb{T}^N$. The result then shows that there are perturbations, however small, that change completely the dynamics of the system.

A particularly interesting case is the one in which $\alpha$ is a quasi-periodic function. Indeed in this case the problem is related with that of Arnold’s diffusion. Results on this kind of problem have been obtained by various Authors; let us recall here [4, 14, 12, 11, 10, 9]. Let us point out that all the results we are aware of work for equations of the form $\ddot{q} = \varepsilon \alpha(\omega t) V'(q)$ for small $\varepsilon$ and requires analyticity of $V$ in $q$.

2. Variational setting and preliminary results. In this first section we discuss some preliminary results which will be basic in the proof of our main Theorem. Even if most of these properties are well known (see for example [1]) we will give some of the proofs.

Given any $\pi > \alpha > 0$, we will denote by

$$\mathcal{A} = \{ \alpha \in C(\mathbb{R}; \mathbb{R}) \mid 0 < \alpha \leq \alpha(t) \leq \pi, \ \alpha \text{ almost periodic} \}$$

and we let, for $\alpha \in \mathcal{A}$ and $V$ which satisfy (V1)–(V3),

$$L_\alpha(t, q, \dot{q}) = \frac{1}{2} |\dot{q}|^2 + \alpha(t) V(q)$$

be the Lagrangian of the system

$$\ddot{q} = \alpha(t) V'(q). \quad (L_\alpha)$$

We also introduce the function space

$$E = \{ q \in H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^N) \mid \int_{\mathbb{R}} |\dot{q}(t)|^2 \, dt < +\infty \}$$

which becomes a Hilbert space with inner product

$$(x \mid y) = \langle x(0) \mid y(0) \rangle + \int_{\mathbb{R}} \langle \dot{x}(t) \mid \dot{y}(t) \rangle \, dt.$$ 

Remark 2.1. Let us remark that (V3) implies that there is are $r_0 \in \left(0, \frac{1}{b} \right)$ and $\tilde{\nu} > 0$ such that, for all $|y| \leq r_0$ and for all $x \in \mathbb{R}^N$

$$\langle V''(y)x \mid x \rangle \geq \frac{\nu}{2} |x|^2 \quad (2.1)$$

$$\langle V'(y) \mid y \rangle \geq \frac{\nu}{2} |y|^2 \quad (2.2)$$

$$|\langle V'(y) \mid x \rangle| \leq \tilde{\nu} |x| \quad (2.3)$$

$$\frac{\nu}{4} |y|^2 \leq V(y) \leq \tilde{\nu} |y|^2 \quad (2.4)$$
We then define, for all \( q \in E \),
\[
f_\alpha(q) = \int_{\mathbb{R}} L_\alpha(t,q(t),\dot{q}(t)) \, dt.
\]
We also define, for \( r \in (0,r_0) \),
\[
\mu_r = \inf\{ V(x) \mid x \notin B_r(\mathbb{Z}^N) \},
\]
where \( B_r(\mathbb{Z}^N) = \cup_{\xi \in \mathbb{Z}^N} B_r(\xi) \). By assumption (V2) we have that \( \mu_r > 0 \).

**Lemma 2.2.** Let \( r \in (0,r_0) \). If \( q \in E \) is such that \( q(t) \notin B_r(\mathbb{Z}^N) \) for all \( t \in [\frac{1}{4}, 1] \) then
\[
f_\alpha(q) \geq \int_{\frac{1}{4}}^T L_\alpha(t,q(t),\dot{q}(t)) \, dt \geq \frac{1}{2(\mu_r(T-\frac{1}{4}))} |q(T) - q(\frac{1}{4})|^2 + \mu_r(T-\frac{1}{4}),
\]
(2.6) follows since
\[
|q(T) - q(\frac{1}{4})| \leq \sqrt{\frac{2}{\mu_r}} |q(T) - q(\frac{1}{4})| + \frac{1}{\sqrt{2\mu_r}} f_\alpha(q),
\]
(2.7) for all \( \alpha \in A \).

**Proof.** (2.6) is an immediate consequence of the definition of \( \mu_r \).

By Lemma 2.2, for any \( r > 0 \), any \( q \in E \) with \( f_\alpha(q) < \infty \) must definitely enter in \( B_r(\mathbb{Z}^N) \). More precisely, one easily proves

**Lemma 2.3.** If \( q \in E \) is such that \( f_\alpha(q) < +\infty \), then \( q \in L^\infty(\mathbb{R};\mathbb{R}^N) \) and there exist \( \xi^- \), \( \xi^+ \in \mathbb{Z}^N \) such that \( q(t) \to \xi^\pm \) as \( t \to \pm\infty \).

We define, for all \( \xi \in \mathbb{Z}^N \) and \( \alpha \in A \)
\[
\Gamma_\xi = \{ q \in E \mid q(-\infty) = 0, \quad q(+\infty) = \xi \}, \quad c_\xi(\alpha) = \inf\{ f_\alpha(q) \mid q \in \Gamma_\xi \}.
\]
Note that from (2.7) for all \( \alpha \in A \) and \( q \in \Gamma_\xi \), we have
\[
f_\alpha(q) \geq \sqrt{\frac{2\mu_1/4}{2}} = \zeta.
\]
Hence, using Lemma 2.2 we obtain

**Lemma 2.4.** For all \( \xi \in \mathbb{Z}^N \setminus \{0\} \), \( \alpha \in A \) we have that \( c_\xi(\alpha) \geq \zeta \) and \( c_\xi(\alpha) \to +\infty \) as \( |\xi| \to +\infty \).

Following [6], among the \( \xi \in \mathbb{Z}^N \) it is possible to select a finite number of them which generate all of \( \mathbb{Z}^N \) and have additional properties, as stated in the following lemma.

**Lemma 2.5.** Let \( \alpha \in A \). There exist \( \eta_1,\ldots,\eta_m \in \mathbb{Z}^N \setminus \{0\} \) such that \( \mathbb{Z}^N = \{ \sum_{j=1}^m n_j \eta_j \mid n_j \in \mathbb{N} \} \) and if \( \xi_1,\xi_2 \in \mathbb{Z}^N \setminus \{0\} \) are such that \( \xi_1 + \xi_2 = \eta \in \{ \eta_1,\ldots,\eta_m \} \), then
\[
c_{\xi_1}(\alpha) + c_{\xi_2}(\alpha) \geq c_{\eta}(\alpha) + \zeta.
\]
(2.8)
Proof. It follows from Lemma 2.4 that there is \( \eta \in \mathbb{Z} \setminus \{0\} \) such that
\[
c_{\eta}(\alpha) = \min \{ c_{\xi}(\alpha) \mid \xi \in \mathbb{Z}^N \setminus \{0\} \}.
\]

Suppose that we have already chosen \{\( \eta_1, \ldots, \eta_{j-1} \)\}, and \( \{\sum_{\ell=1}^{j-1} n_{\ell} \eta_{\ell} \mid n_{\ell} \in \mathbb{N}\} \neq \mathbb{Z}^N \). Then we can find \( \eta_j \) such that
\[
c_{\eta_j}(\alpha) = \min \{ c_{\xi}(\alpha) \mid \xi \in \mathbb{Z}^N \setminus \{\sum_{\ell=1}^{j-1} n_{\ell} \eta_{\ell} \mid n_{\ell} \in \mathbb{N}\} \}.
\]
If \( \xi_1 + \xi_2 = \eta_j \), then \( \max \{ c_{\xi_1}(\alpha), c_{\xi_2}(\alpha) \} \geq c_{\eta_j}(\alpha) \), and we have that \( c_{\xi_1}(\alpha) + c_{\xi_2}(\alpha) \geq c_{\eta}(\alpha) \) and \( c_{\eta}(\alpha) \), so the lemma follows since, by Lemma 2.4, \( \xi \leq c_{\eta}(\alpha) \). \( \square \)

Even if the set \( \{\eta_1, \ldots, \eta_m\} \) may depend on \( \alpha \), the minimum levels \( c_{\eta}(\alpha), i = 1, \ldots, m \), are uniformly bounded from above by a constant \( \overline{\tau} \) independent from \( \alpha \in \mathcal{A} \). Precisely

**Lemma 2.6.** There exist \( \overline{\tau} > 0 \) such that for all \( \alpha \in \mathcal{A} \) we have \( c_{\eta}(\alpha) \leq \overline{\tau} \).

**Proof.** Let \( e_1, \ldots, e_N \) be the canonical base of \( \mathbb{R}^N \) and denote \( \xi_1 = e_1, \ldots, \xi_N = e_N \), \( \xi_{N+1} = -e_1, \ldots, \xi_{2N} = -e_N \). Then, \( \{\sum_{\ell=1}^{2N} n_{\ell} \xi_{\ell} \mid n_{\ell} \in \mathbb{N}\} = \mathbb{Z}^N \) and we set \( \overline{\tau} = \max\{c_{\xi_1}(\overline{\tau}), \ldots, c_{\xi_{2N}}(\overline{\tau})\} \). Letting \( \alpha \in \mathcal{A} \), by construction we have \( c_{\eta}(\alpha) \leq \cdots \leq c_{\eta}(\alpha) \) and to prove the lemma it is sufficient to show that \( c_{\eta}(\alpha) \leq \overline{\tau} \). To this aim note that \( \{\sum_{\ell=1}^{m-1} n_{\ell} \eta_{\ell} \mid n_{\ell} \in \mathbb{N}\} \neq \mathbb{Z}^N \) and so there exists \( \xi_t \notin \{\sum_{\ell=1}^{m-1} n_{\ell} \eta_{\ell} \mid n_{\ell} \in \mathbb{N}\} \). Then
\[
c_{\eta}(\alpha) = \min\{ c_{\xi}(\alpha) \mid \xi \in \mathbb{Z}^N \setminus \{\sum_{\ell=1}^{m-1} n_{\ell} \eta_{\ell} \mid n_{\ell} \in \mathbb{N}\} \} \leq c_{\xi}(\alpha) \leq c_{\xi}(\overline{\tau}) \leq \overline{\tau}.
\]

For all \( \delta \in (0, r_0) \), let
\[
\lambda_3 = \left(1 + \frac{1}{2} \overline{\tau}\right) \delta^2.
\]

The next Lemma, which follows Lemma 2.5, describes the concentration properties satisfied by the functions in \( \Gamma_{\eta} \cap \{f_\alpha \leq c_{\eta}(\alpha) + \lambda_3\} \), \( i \in \{1, \ldots, m\} \), when \( \delta \) is small enough.

**Lemma 2.7.** There exists \( \delta \in (0, r_0) \) such that for all \( \delta \in (0, \delta) \), there exists \( \lambda_3 > 0 \), for which, for all \( \alpha \in \mathcal{A} \), if \( q \in \Gamma_{\eta} \), satisfy \( f_\alpha(q) \leq c_{\eta}(\alpha) + \lambda_3 \) for some \( i \in \{1, \ldots, m\} \) then

(i) if \( q(t) \notin B_3(\mathbb{Z}^N) \) for all \( t \in [a, b] \) then \( b - a \leq L_3 \);
(ii) if \( |q(0)| = |q(b) - \eta| = \delta \), then \( |q(t)| < 2\sqrt{3} \) for all \( t \leq a \) and \( |q(t) - \eta| < 2\sqrt{3} \) for all \( t \geq b \);
(iii) \( |q(t)| - |q| > \delta \) for all \( \eta \in \mathbb{Z}^N \setminus \{\eta_\eta\} \) and \( t \in \mathbb{R} \).

**Proof.** Let us choose \( \delta \) such that:
\[
\lambda_3 = \left(1 + \frac{1}{2}\right) \delta^2;
\]
\[
\delta < \frac{1}{4} \min\{r_0^2, \frac{\sqrt{2} \overline{\tau}}{3(1 + 2\overline{\tau})}\};
\]
\[
\inf_{x \notin B_3(\mathbb{Z}^N)} V(x) = \inf_{|x| = \delta} V(x) \geq \frac{\nu}{4} \delta^2.
\]
Then for any $r \in (0, \bar{\delta})$ one has that
\[
\mu_r = \inf_{x \in \mathcal{B}_r(\mathbb{Z}^N)} V(x) = \inf_{r \leq |x| \leq r_0} V(x) \geq \frac{\nu r^2}{4}.
\]

Given $\delta \in (0, \bar{\delta})$, let $\rho_3 = 2\sqrt{\delta}$ and take $q \in \Gamma_{\eta_i}$ such that $f_\alpha(q) \leq c_{\eta_i}(\alpha) + \lambda_\delta$.

First of all we note that (i) follows from (2.6) and Lemma 2.6.

To prove (ii) let $t^- \in \mathbb{R}$ be such that $|q(t^-)| = \delta$ and let
\[
\tilde{q}(t) = \begin{cases} 
0 & \text{if } t < t^- - 1, \\
\text{linear} & \text{if } t^- - 1 \leq t \leq t^-, \\
q(t) & \text{if } t > t^-.
\end{cases}
\]

Then, $\tilde{q} \in \Gamma_{\eta_i}$ and so $f_\alpha(\tilde{q}) \geq c_{\eta_i}(\alpha)$. With a direct computation we obtain
\[
\int_{-\infty}^{t^-} L_\alpha(t, q(t), \dot{q}(t)) dt = f_\alpha(q) - f_\alpha(\tilde{q}) + \int_{t^-}^\infty L_\alpha(t, q(t), \dot{q}(t)) dt \leq 2\lambda_\delta.
\]

Assume now by contradiction that there exists $s < t^-$ such that $|q(s)| = \rho_3$. Then, since $\delta < \sqrt{\delta} < \rho_3 < r_0$, there exists $(s^-, s^+) \subset (s, t^-)$ such that $q(t) \in (\sqrt{\delta}, \rho_3)$ for all $t \in (s^-, s^+)$. By (2.4), for $\sqrt{\delta} < |x| < \rho_3$ we have $V(x) \geq \frac{\nu}{4}\delta$. Then, by (2.7) and the choice of $\delta$ we get
\[
\int_{-\infty}^{t^-} L_\alpha(t, q(t), \dot{q}(t)) dt \geq \int_{s^-}^{s^+} L_\alpha(t, q(t), \dot{q}(t)) dt \geq \sqrt{\frac{\nu}{2} - 3(1 + \sqrt{\delta})^2} = 3\lambda_\delta.
\]

a contradiction which proves that $|q(t)| \leq \rho_3$ for all $t < t^-$. Analogously one shows that if $|q(t^+) - \eta_i| = \delta$ then $|q(t) - \eta_i| < \rho_3$ for all $t > t^+$ and so (ii) follows.

To prove (iii) assume that there exists $t_0 \in \mathbb{R}$ and $\eta \in \mathbb{Z}^N \setminus \{0, \eta_i\}$ such that $|q(t_0) - \eta| \leq \delta$. We define
\[
q_1(t) = \begin{cases} 
q(t) & \text{if } t < t_0, \\
\text{linear} & \text{if } t_0 \leq t \leq t_0 + 1, \\
\eta & \text{if } t > t_0 + 1,
\end{cases} \quad q_2(t) = \begin{cases} 
\eta & \text{if } t < t_0 - 1, \\
\text{linear} & \text{if } t_0 - 1 \leq t \leq t_0, \\
q(t) & \text{if } t > t_0.
\end{cases}
\]

Noting that $q_1 \in \Gamma_{\eta_i}$ and $q_2 - \eta \in \Gamma_{\eta_i - \eta}$, we obtain
\[
\int_{-\infty}^{t_0} L_\alpha(t, q(t), \dot{q}(t)) dt \geq f_\alpha(q_1) - \lambda_\delta \geq c_{\eta_i}(\alpha) - \lambda_\delta
\]
and
\[
\int_{t_0}^{+\infty} L_\alpha(t, q(t), \dot{q}(t)) dt \geq f_\alpha(q_2 - \eta) - \lambda\delta \geq c_{\eta_i - \eta}(\alpha) - \lambda_\delta.
\]

Hence, since $\lambda_\delta < \lambda_\delta < \frac{\nu}{4}$, by (2.8) we obtain the contradiction
\[
f_\alpha(q) \geq c_{\eta_i}(\alpha) + c_{\eta_i - \eta}(\alpha) - 2\lambda_\delta \geq c_{\eta_i}(\alpha) + \frac{\nu}{4} - 2\lambda_\delta > c_{\eta_i}(\alpha) + 2\lambda_\delta.
\]

These concentration properties imply compactness of the minimizing sequences in $\Gamma_{\eta_i}$.

**Lemma 2.8.** Let $\alpha \in A$.

Suppose that $q_n \in \Gamma_{\eta_i}$, $f_\alpha(q_n) \to c_{\eta_i}(\alpha)$ and that there exist $\tau \in \mathbb{R}$ and $r \in (0, \bar{\delta})$ such that $|q_n(\tau)| = r$ for all $n \in \mathbb{N}$ or $|q_n(\tau) - \eta_i| = r$ for all $n$.

Then, there exists $q \in \Gamma_{\eta_i}$ such that, along a subsequence, $q_n \to q$ weakly in $H^1_{loc}(\mathbb{R}; \mathbb{R}^N)$ and $f_\alpha(q) = c_{\eta_i}(\alpha)$. 

□
Proof. We consider only the case \(|q_0(\tau)| = \tau\) for all \(n \in \mathbb{N}\), the other case being similar. We choose \(\delta \in (0, \tau)\) such that \(\rho_\delta \equiv 2\sqrt{\delta} \in (0, \tau)\). Since \(f_\alpha(q_0) \rightarrow c_n(\alpha)\) it is not restrictive to assume that \(f_\alpha(q) \leq c_n(\alpha) + \lambda_\delta\) for all \(n \in \mathbb{N}\). By Lemma 2.7, for any \(n \in \mathbb{N}\), we have

\[
\begin{align*}
(i) & \quad |q_n(t) - \eta| > \delta \text{ for all } \eta \in \mathbb{Z}^N \setminus \{0, \eta_i\} \text{ and } t \in \mathbb{R}; \\
(ii) & \quad \text{there exist } a_n \in (\tau - L_\delta, \tau) \text{ and } b_n \in (\tau, \tau + L_\delta) \text{ such that } |q_n(a_n)| = |q_n(b_n) - \eta| = \delta \text{ and } |q_n(t)| \leq \rho_\delta \text{ for all } t \leq a_n \text{ and } |q_n(t) - \eta| \leq \rho_\delta \text{ for all } t \geq b_n. 
\end{align*}
\]

By Lemma 2.2 and since \(f_\alpha(q_n) \rightarrow c_n(\alpha)\), we have that \((q_n)\) is bounded in \(H^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)\) and then, up to a subsequence, it converges weakly in \(H^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)\) and strongly in \(L^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)\) to some \(q \in E\). Since \(f_\alpha\) is plainly weakly lower semi-continuous in \(H^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)\), we have \(f_\alpha(q) \leq c_n(\alpha)\). Moreover, \(q\) satisfies

\[
\begin{align*}
(i) & \quad |q(t) - \eta| \geq \delta \text{ for all } \eta \in \mathbb{Z}^N \setminus \{0, \eta_i\} \text{ and } t \in \mathbb{R}; \\
(ii) & \quad |q(t)| \leq \rho_\delta \text{ for all } t \leq \tau - L_\delta \text{ and } |q(t) - \eta| \leq \rho_\delta \text{ for all } t \geq \tau + L_\delta.
\end{align*}
\]

Then, by Lemma 2.3, \(q \in \Gamma_{\eta_i}\) and \(f_\alpha(q) = c_n(\alpha)\).

Let us now state some consequence of the fact that \(\alpha \in \mathcal{A}\) is an almost-periodic function. We recall that

**Definition 2.9.** A continuous function \(\alpha: \mathbb{R} \rightarrow \mathbb{R}\) is an almost-periodic function if for every \(\varepsilon > 0\), there is \(T_\varepsilon > 0\) such that every interval \([a, a + T_\varepsilon] \subset \mathbb{R}\) contains \(\tau\) such that

\[|\alpha(t + \tau) - \alpha(t)| < \varepsilon\quad \text{for all } t \in \mathbb{R}.
\]

If this holds one say that \(\tau\) is an \(\varepsilon\)-periods of \(\alpha\) and that the \(\varepsilon\)-periods are \(T_\varepsilon\)-dense in \(\mathbb{R}\). See, for the more information on almost periodic functions, [5, 13].

The almost periodicity of the function \(\alpha\) implies that the functional \(f_\alpha\) has similar properties. This fact is exploited in next Lemma in which we construct infinitely many test functions which will be used in a cutting and pasting procedure in the proof of Theorem 3.4 below.

**Lemma 2.10.** For all \(\alpha \in \mathcal{A}\), \(\varepsilon > 0\) there is a \(M_0 > 0\) such that for all \(\eta_i, i = 1, \ldots, m\), and for all \([a, b] \subset \mathbb{R}\) with \(b - a > M_0\) we can find a function \(q \in \Gamma_{\eta_i}\) such that \(q(t) = 0\) for all \(t \leq a\), \(q(t) = \eta_i\) for all \(t \geq b\), \(f_\alpha(q) \leq c_n(\alpha) + \varepsilon\).

**Proof.** Given any \(\varepsilon > 0\), let \(\delta \in (0, \delta)\) be such that \(\lambda_\delta < \varepsilon\). Then, choose \(q_i \in \Gamma_{\eta_i}, i \in \{1, \ldots, m\}\), such that \(f_\alpha(q_i) \leq c_n(\alpha) + \lambda_\delta\).

By Lemma 2.7 for all \(i\) there is \([a_i, b_i]\), with \(b_i - a_i < L_\delta\), such that \(|q_i(a_i)| = \delta\) and \(|q_i(b_i) - \eta_i| = \delta\).

Then, letting

\[
\bar{q}_i(t) = \begin{cases}
0 & t \leq a_i - 1 \\
\text{linear} & a_i - 1 \leq t \leq a_i \\
q_i(t) & a_i \leq t \leq b_i \\
\text{linear} & b_i \leq t \leq b_i + 1 \\
\eta_i & b_i \leq t
\end{cases}
\]

we obtain \(\bar{q}_i \in \Gamma_{\eta_i}\) and \(f_\alpha(\bar{q}_i) \leq c_n(\alpha) + 2\lambda_\delta\).

Let now \(\varepsilon_1 > 0\) such that \(|f_\alpha(\bar{q}_i) - f_\alpha(\bar{q}_i(\cdot - \tau))| < \varepsilon\) for all \(i\) provided \(\tau\) is an \(\varepsilon_1\)-period of \(\alpha\). Let \(T\) be such that \(\varepsilon_1/2\)-periods are \(T\)-dense. Take \(M_0 = L_\delta + 2T + 2\) and let \((a, b) \subset \mathbb{R}\) be such that \(b - a \geq M_0\). Then we can find \(\tau_i \in [a_i - 1 - T, a_i - 1]\) and \(\tau \in [a, a + T], \varepsilon_1/2\)-periods for \(\alpha\). Letting \(\bar{q}_i(t) = \bar{q}_i(t + \tau_i - \tau)\), we obtain
Proof. It is a standard fact that such a minimum exist and that it is a solution of the problem. To show that $|q(t)|$ is sufficiently small, it is enough to remark that the “cost” (as measured by $\int_a^b L_\alpha(t, q(t), \dot{q}(t)) \, dt$) of going from $\zeta_1$ to $\partial B_{\tau}(0)$ back to $\zeta_2$ exceed, uniformly for $\alpha \in \mathcal{A}$, the cost of going from $\zeta_1$ to $0$ back to $\zeta_2$. Uniqueness then follows from the convexity of $L_\alpha(t, q, \dot{q})$ for $|q| \leq r_0$.

Lemma 2.11. Let $b > a$. Then there is $\delta_0 \in (0, \bar{\delta})$ such that for all $\delta \in (0, \delta_0]$, $\alpha \in \mathcal{A}$, $\zeta_1, \zeta_2 \in B_\delta(0)$ the minimization problem

$$\inf \left\{ \int_a^b L_\alpha(t, q(t), \dot{q}(t)) \, dt \mid q \in H^1((a, b); \mathbb{R}^N), \, q(a) = \zeta_1, \, q(b) = \zeta_2 \right\}$$

is achieved at a unique function $q$, which is a solution of $(L_\alpha)$ satisfying $|q(t)| \leq r_0$ for all $t \in [a, b]$.

Proof. It is a standard fact that such a minimum exist and that it is a solution of the problem. To show that $|q(t)| \leq r_0$ for all $t \in [a, b]$ we note that if $\delta_0$ is sufficiently small, it is enough to remark that the “cost” (as measured by $\int_a^b L_\alpha(t, q(t), \dot{q}(t)) \, dt$) of going from $\zeta_1$ to $\partial B_{\tau}(0)$ back to $\zeta_2$ exceed, uniformly for $\alpha \in \mathcal{A}$, the cost of going from $\zeta_1$ to $0$ back to $\zeta_2$. Uniqueness then follows from the convexity of $L_\alpha(t, q, \dot{q})$ for $|q| \leq r_0$.

Lemma 2.12. Let $q$ be a solution of $(L_\alpha)$ such that $|q(t)| \leq r_0$ for all $t \in [a, b]$. Assume that $b > a$.

Then, for all $t \in [a, b]$,

$$|q(t)|^2 \leq \frac{|q(a)|^2 - |q(b)|^2 e^{-\sqrt{\Delta} (b-a)}}{1 - e^{-2\sqrt{\Delta} (b-a)}} e^{-\sqrt{\Delta} (t-a)} + \frac{|q(b)|^2 - |q(a)|^2 e^{-\sqrt{\Delta} (b-a)}}{1 - e^{-2\sqrt{\Delta} (b-a)}} e^{\sqrt{\Delta} (t-b)}, \quad (2.9)$$

in particular

$$|q(t)|^2 < r_0 \quad \text{for all } t \in (a, b).$$

Proof. Take $\nu$ given by assumption (V3) and let

$$L(p) = -\ddot{p} + \alpha \nu p, \quad \forall p \in C^2(\mathbb{R}, \mathbb{R}),$$

and

$$w(t) = \frac{|q(a)|^2 - |q(b)|^2 e^{-\sqrt{\Delta} (t-a)}}{1 - e^{-2\sqrt{\Delta} (t-a)}} e^{-\sqrt{\Delta} (t-a)} + \frac{|q(b)|^2 - |q(a)|^2 e^{-\sqrt{\Delta} (b-a)}}{1 - e^{-2\sqrt{\Delta} (b-a)}} e^{\sqrt{\Delta} (t-b)}$$

where $\Delta = b - a$.

It is immediate to check that $w$ solves the boundary value problem

$$\begin{cases}
L(w) = 0 & \text{in } [a, b] \\
w(a) = |q(a)|^2 \\
w(b) = |q(b)|^2.
\end{cases} \quad (2.10)$$
Let \( q \) be a solution of \((L_\alpha)\) such that \(|q(t)| \leq r_0\) for all \( t \in [a, b] \). Then, using \((L_\alpha)\) and \((2.2)\), we deduce that
\[
L(|q(t)|^2) = -2\dot{q}(t)^2 - 2\langle q(t) | \dot{q}(t) \rangle + \alpha \nu |q(t)|^2 \\
\leq -2\langle q(t) | \alpha(t)V'(q(t)) \rangle + \alpha \nu |q(t)|^2 \\
\leq -2\alpha \langle V'(q(t)) | q(t) \rangle + \alpha \nu |q(t)|^2 \\
\leq -\alpha \nu |q(t)|^2 + \alpha \nu |q(t)|^2 = 0
\]
for all \( t \in [a, b] \). Then
\[
\begin{cases}
L(w - |q|^2) \geq 0 & \text{in } (a, b) \\
w(t) - |q|^2 = 0 & \text{for } t = a, b.
\end{cases}
\]
By the maximum principle we obtain
\[
|q(t)|^2 \leq w(t) \quad \text{for all } t \in [a, b].
\]

\textbf{Remark 2.13.} The same arguments used in the proof of Lemma 2.12 show that
\[
|q(t)|^2 \leq |q(a)|^2 e^{\frac{\alpha \nu}{2}(t-a)} \quad \text{for all } t \leq a
\]
whenever \( q \) is a solution of \((L_\alpha)\) such that \(|q(t)| \leq r_0\) for all \( t \leq a \), and that
\[
|q(t)|^2 \leq |q(b)|^2 e^{\frac{\alpha \nu}{2}(b-t)} \quad \text{for all } t \geq b
\]
whenever \( q \) is a solution of \((L_\alpha)\) such that \(|q(t)| \leq r_0\) for all \( t \geq b \).

By periodicity, analogous estimates hold for \( q(t) - \eta_i \) if \(|q(t) - \eta_i| \leq r_0\) for all \( t \in [a, b] \).

The following final two Lemmas give further properties of solutions of \((L_\alpha)\) which are minimal for \( f_\alpha \) in \( \Gamma_n \). Even if in general, when \( \alpha \) is almost periodic, one cannot say that these minimal solutions exist, these properties will be frequently used below in contradiction arguments.

\textbf{Lemma 2.14.} There exists \( \mathcal{V} > 0 \) such that for all \( \alpha \in \mathcal{A} \), if \( q \in \Gamma_n \) satisfies \( f_\alpha(q) = c_{\eta_i}(\alpha) \) some \( i \in \{1, \ldots, m\} \), then
\[
\int_{\mathbb{R}} V(q(t)) \, dt \geq \mathcal{V}.
\]

\textbf{Proof.} To prove \((2.13)\), one can find \([a, b] \subset \mathbb{R}\) such that \( q(t) \notin B_{r_0}(\mathbb{Z}_N) \) for all \( t \in [a, b] \) and such that \( \frac{1}{2} \leq |q(a) - q(b)| \). Then one simply remarks that
\[
\frac{1}{2} \leq |q(a) - q(b)| \leq \int_a^b |\dot{q}(t)| \, dt \\
\leq \sqrt{b-a} \left( \int_a^b |\dot{q}(t)|^2 \, dt \right)^{\frac{1}{2}} \leq \sqrt{b-a} \sqrt{2c_{\eta_i}(\alpha)}.
\]
Then \( b-a \geq \frac{1}{8c_{\eta_i}(\alpha)} \) and hence, by Lemma 2.6,
\[
\int_{\mathbb{R}} V(q(t)) \, dt \geq \int_a^b V(q(t)) \, dt \geq (b-a) \mu_{r_0} \geq \frac{\mu_{r_0}}{8c_{\eta_i}} \geq \frac{\mu_{r_0}}{8c}.
\]
Let $\delta_0$ be fixed as in Lemma 2.11 and denote $\rho_0 = 2\sqrt{\delta_0}$ and $L_0 = L_0$. Then we have

**Lemma 2.15.** For any $d > 0$ there exists $M(d) > 0$ such that for all $\alpha \in A$, if $q \in \Gamma_\eta$ satisfies $f_\alpha(q) = c_\eta(\alpha)$ for some $i \in \{1, \ldots, m\}$ and $|q(\tau)| = \delta_0$ or $|q(\tau) - \eta_i| = \delta_0$, then

$$\int_{|t-\tau| \geq M(d)} V(q(t)) \, dt < d \int_{\mathbb{R}} V(q(t)) \, dt.$$  

**Proof.** We will assume $|q(\tau)| = \delta_0$, the other case can be handled in the same way.

First we prove that for every $L > 0$,

$$\int_{\mathbb{R}} V(q(t)) \, dt \leq 2 \frac{\bar{\varrho} \delta_0^2}{\sqrt{2L}} e^{-\sqrt{\gamma L}}, \quad (2.14)$$

To this aim, note that since $q$ satisfy $f_\alpha(q) = c_\eta(\eta_i)$, it follows from Lemma 2.7 that $|q(t)| \leq \rho_0 < r_0$ for all $t \leq \tau$. Then, noting that $q$ is a solution of $(L_\alpha)$, by Remark 2.13, we obtain that $|q(t)| < \delta_0$ for all $t < \tau$. The same argument shows that $|q(t)| > \delta_0$ for all $t > \tau$. Therefore, again by Lemma 2.7, $|q(t) - \eta_i| < \delta_0$ for all $t > \tau + L_0$. Using Lemma 2.12 (see also Remark 2.13), we have

$$|q(t)|^2 \leq \delta_0^2 e^{\sqrt{\gamma L}(t-\tau)} \quad t \leq \tau$$

$$|q(t) - \eta_i|^2 \leq \delta_0^2 e^{\sqrt{\gamma L}(\tau + L_0 - t)} \quad t \geq \tau + L_0.$$  

Then, by (2.4)

$$V(q(t)) \leq \bar{\varrho} |q(t)|^2 \leq \bar{\varrho} \delta_0^2 e^{\sqrt{\gamma L}(t-\tau)} \quad t \leq \tau$$

$$V(q(t)) \leq \bar{\varrho} |q(t) - \eta_i|^2 \leq \bar{\varrho} \delta_0^2 e^{\sqrt{\gamma L}(\tau + L_0 - t)} \quad t \geq \tau + L_0.$$  

We deduce that

$$\int_{-\infty}^{\tau-L} V(q(t)) \, dt \leq \int_{\mathbb{R}} V(q(t)) \, dt \leq \bar{\varrho} \delta_0^2 \int_{-\infty}^{\tau-L} e^{\sqrt{\gamma L}(\tau-\tau)} \, dt = \frac{\bar{\varrho} \delta_0^2}{\sqrt{2L}} e^{-\sqrt{\gamma L}}$$

and

$$\int_{\tau+L_0+L}^{+\infty} V(q(t)) \, dt \leq \frac{\bar{\varrho} \delta_0^2}{\sqrt{2L}} e^{-\sqrt{\gamma L}}.$$  

Hence (2.14) follow.

Now, fixed any $d > 0$ let $L > 0$ be such that $2 \frac{\bar{\varrho} \delta_0^2}{\sqrt{2L}} e^{-\sqrt{\gamma L}} < dV$. Then, by Lemma 2.14, the lemma follows setting $M(d) = L + L_0$. \hfill $\square$

3. **Multibump solutions.** In this section we will show that under an additional assumption our system exhibits a chaotic behavior.

Let us fix $\delta_0$ as in Lemma 2.11. We say that $\alpha \in A$ satisfies $(\ast)$ if there is $\tau \in \mathbb{R}$ such that $q \in \Gamma_\eta$ and $|q(\tau)| = \delta_0$

or $|q(\tau) - \eta_i| = \delta_0$ implies that $f_\alpha(q) > c_\eta(\alpha)$, \hfill $(\ast)$

for some $i \in \{1, \ldots, m\}$.

**Remark 3.1.** Let us remark that, in order to prove results like the ones contained in Theorem 3.4, one could actually assume a little bit less then $(\ast)$. Indeed, as one can see for example in [1, 15], it is enough to assume that there is a $\tau \in \mathbb{R}$ such that $q \in \Gamma_\eta$ and $|q(\tau)| = \delta_0$ implies $f_\alpha(q) > c_\eta(\alpha)$.
Remark 3.2. Let us remark that \( f_\alpha \) has in \( \Gamma_{\eta_i} \) uncountably many minimizers (and hence \((L_{\alpha})\) has uncountably many solutions) whenever \((*)\) is not satisfied.

In order to state the result of this section, let us present some consequence of assumption \((*)\) and introduce some notation.

**Lemma 3.3.** Suppose \( \alpha \) satisfies \((*)\).

Then there is \( \lambda > 0 \), \( \varepsilon_\lambda > 0 \) such that for all \( \varepsilon \)-periods for \( \alpha \) and for all \( q \in \Gamma_{\eta_i} \) such that \( |q(\tau + T)| = \delta_0 \) or \( |q(\tau + T) - \eta_i| = \delta_0 \) we have that \( f_\alpha(q) \geq c_{\eta_i}(\alpha) + \lambda \).

**Proof.** Suppose not.

Then there is a sequence \( T_n \) of \( \frac{1}{n} \)-periods and functions \( q_n \in \Gamma_{\eta_i} \) such that
\[
|q_n(\tau + T_n)| = \delta_0 \quad \text{or} \quad |q_n(\tau + T_n) - \eta_i| = \delta_0 \quad \text{and} \quad f_\alpha(q_n) \leq c_{\eta_i}(\alpha) + \frac{\lambda}{n}.
\]

But then, setting \( \tilde{q}_n \) such that \( |\tilde{q}_n(\tau)| = \delta_0 \) or \( |\tilde{q}_n(\tau) - \eta_i| = \delta_0 \) and \( f_\alpha(\tilde{q}_n) \leq c_{\eta_i}(\alpha) + \frac{\lambda}{n} \), we have that for all \( \eta : (\tilde{q},(\tau - \eta)| = \delta_0 \), a contradiction with assumption \((*)\).

We now fix an integer \( k \). Let \( \lambda \) be given by Lemma 3.3, and take \( M_0 \) be as in Lemma 2.10 corresponding to \( \varepsilon = \frac{\lambda}{2} \). We choose numbers \( \tau_1, \ldots, \tau_{2k} \in \mathbb{R} \) in such a way that
\[
\begin{align*}
\tau_{j+1} - \tau_j &> M_0 \quad 1 \leq j < 2k \\
\text{\( \tau_j \) is an } \varepsilon_1 \text{-period of } \alpha \text{ for all } j = 1, \ldots, 2k,
\end{align*}
\]
where \( \varepsilon_1 \) is chosen in such a way that \( f_\alpha(q(-\tau_j)) \leq c_{\eta_i}(\alpha) + \frac{\lambda}{2} \) for all \( q \in \Gamma_{\eta_i} \) such that \( f_\alpha(q) = c_{\eta_i}(\alpha) \). We will also set \( \tau_0 = -\infty \), \( \tau_{2k+1} = +\infty \). Finally we set \( \mathcal{M}_k = \{ q \in E \mid q(-\infty) = 0, \quad |q(t) - \ell \eta_i| \leq \delta_0 \text{ for all } t \in [\tau_{2\ell} + \tau, \tau_{2\ell+1} + \tau], \ell = 0, \ldots, k \}
\)
and \( q(+\infty) = k \eta_i \).

We can now state the main result of this section:

**Theorem 3.4.** Assume \( V \) satisfies assumptions (V1), (V2) and (V3). Let \( \alpha \in \mathcal{A} \), and assume that \( \alpha \) satisfies \((*)\).

Then for all \( k \in \mathbb{N} \) and all choices of \( \tau_1, \ldots, \tau_{2k} \) satisfying (3.1) there is a solution \( q_k \) of \((L_{\alpha})\) in \( \mathcal{M}_k \).

**Proof.** The proof is simply based on a minimization argument. Indeed, let us consider the minimization problem
\[
m_k = \min\{ f_\alpha(q) \mid q \in \mathcal{M}_k \}.
\]
It is a standard fact that a minimum \( q \in \mathcal{M}_k \) exists and that is a solution of system \((L_{\alpha})\) for all \( t \) with the exception of the times \( t \in [\tau_{2\ell} + \tau, \tau_{2\ell+1} + \tau] \), for some \( \ell \in \{0, \ldots, k\} \), for which \( |q(t) - \ell \eta_i| = \delta_0 \). Using Lemma 2.11 and Lemma 2.12 we deduce that \( |q(t) - \ell \eta_i| = \delta_0 \) for some \( t \in [\tau_{2\ell} + \tau, \tau_{2\ell+1} - \tau] \) only if \( t = \tau_{2\ell} + \tau \) or \( t = \tau_{2\ell+1} + \tau \).

We will now assume that
\[
|q(\tau_{2\ell+1} + \tau) - \ell \eta_i| = \delta_0 \quad \text{or} \quad |q(\tau_{2\ell+2} + \tau) - (\ell + 1) \eta_i| = \delta_0,
\]
and show that we reach a contradiction with \((*)\).
First of all, let us consider \( \dot{q} \) as in Lemma 2.10 such that \( \dot{q} \in \Gamma_{\eta_i} \), \( \dot{q}(t) = 0 \) for all \( t \leq \tau_{2\ell+1} + \tau \), \( \dot{q}(t) = \eta_i \) for all \( t \geq \tau_{2\ell+2} + \tau \) and \( f_{\alpha}(\dot{q}) \leq c_{\eta_i} + \frac{\lambda}{2} \). We also let \( m_1 = \tau + \frac{(\tau_{2\ell+1} + \tau_{2\ell+2})}{2} \) and \( m_2 = \tau + \frac{(\tau_{2\ell+1} + \tau_{2\ell+2})}{2} \). Then we define

\[
\dot{q}(t) = \begin{cases}
q(t) & t \leq m_1 \\
\text{linear} & m_1 \leq t \leq m_1 + 1 \\
\ell \eta_i & m_1 + 1 \leq t \leq \tau_{2\ell+1} + \tau \\
(\ell + 1) \eta_i & \tau_{2\ell+1} + \tau \leq t \leq \tau_{2\ell+2} + \tau \\
\text{linear} & m_2 - 1 \leq t \leq m_2 \\
q(t) & t \geq m_2
\end{cases}
\]

Since \( \dot{q} \in \mathcal{M}_{\delta} \), we have that

\[
0 \leq f_{\alpha}(\dot{q}) - f_{\alpha}(q) = f_{\alpha}(\dot{q}) + \int_{m_1}^{m_1+1} L_{\alpha}(t, \dot{q}, \dot{\dot{q}}) dt + \int_{m_2-1}^{m_2} L_{\alpha}(t, \dot{q}, \dot{\dot{q}}) dt - \int_{m_1}^{m_2} L_{\alpha}(t, q, \dot{q}) dt
\]

and hence, eventually taking \( M_0 \) larger,

\[
\int_{m_1}^{m_2} L_{\alpha}(t, q, \dot{q}) dt \leq f(\dot{q}) + \int_{m_1}^{m_1+1} L_{\alpha}(t, \dot{q}, \dot{\dot{q}}) dt + \int_{m_2-1}^{m_2} L_{\alpha}(t, \dot{q}, \dot{\dot{q}}) dt \leq c_{\eta_i}(\alpha) + \frac{\lambda}{2}.
\]

Define now

\[
\dot{q}(t) = \begin{cases}
0 & t \leq m_1 \\
\text{linear} & m_1 \leq t \leq m_1 + 1 \\
q(t) - \ell \eta_i & m_1 + 1 \leq t \leq m_2 - 1 \\
\text{linear} & m_2 - 1 \leq t \leq m_2 \\
\eta_i & t \geq m_2
\end{cases}
\]

Then \( \dot{q} \in \Gamma_{\eta_i} \) and

\[
f_{\alpha}(\dot{q}) \leq \int_{m_1}^{m_2} L_{\alpha}(t, q, \dot{q}) dt + \frac{\lambda}{4} \leq c_{\eta_i}(\alpha) + \frac{3}{4} \lambda
\]

and

\[
|\dot{q}(\tau_{2\ell+1} + \tau)| = |q(\tau_{2\ell+1} + \tau) - \ell \eta_i| = \delta_0
\]

or

\[
|\dot{q}(\tau_{2\ell+2} + \tau)| = |q(\tau_{2\ell+2} + \tau) - (\ell + 1) \eta_i| = \delta_0,
\]

Then also \( \dot{q}_1(t) = \dot{q}(\cdot - \tau_{2\ell+1}) \) and \( \dot{q}_2(t) = \dot{q}(\cdot - \tau_{2\ell+2}) \in \Gamma_{\eta_i} \),

\[
|\dot{q}_1(\tau)| = \delta_0 \text{ or } |\dot{q}_2(\tau) - \eta_i| = \delta_0,
\]

and, since \( \tau_{2\ell+1} \) is an \( \varepsilon_1 \)-period, by the choice of \( \varepsilon_1 \) we obtain

\[
f_{\alpha}(\dot{q}_i) \leq f_{\alpha}(\dot{q}) + \frac{\lambda}{8} \leq c_{\eta_i}(\alpha) + \frac{7}{8} \lambda,
\]

a contradiction with Lemma 3.3 which proves the Theorem. \( \square \)
Remark 3.5. Let \((\tau_j)_{j \in \mathbb{N}} \subset \mathbb{R}\) be such that any \(\tau_j\) is an \(\varepsilon_1\)-period of \(\alpha\) with \(\tau_{j+1} - \tau_j > M_0\), for all \(j \in \mathbb{N}\).

For all \(k \in \mathbb{N}\), let \(q_k \in C^2(\mathbb{R}, \mathbb{R}^N)\) be the solution to \((L_\alpha)\) such that \(q_k \in \mathcal{M}_k\). Then, one can easily see that for any compact \(K \subset \mathbb{R}\) there exists a constant \(C = C(K) > 0\) such that
\[
\|q_k\|_{L^\infty(K)} \leq C, \quad \text{for all } k \in \mathbb{N}.
\]

Hence, since any \(q_k\) is a solution to \((L_\alpha)\), we obtain that for any compact \(K \subset \mathbb{R}\) there exists a constant \(\tilde{C} = \tilde{C}(K) > 0\) such that
\[
\|q_k\|_{L^\infty(K)} \leq \tilde{C}, \quad \text{for all } k \in \mathbb{N}.
\]

Using the Ascoli-Arzela Theorem, we obtain that along a subsequence, \((q_k)_{k \in \mathbb{N}}\) converges in \(C^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)\) to a solution \(q\) to \((L_\alpha)\), which verify
\[
|q(t) - \ell\eta_k| \leq \delta_0 \text{ for all } t \in [\tau_2t + \tau, \tau_2t + 1 + \tau], \; \ell \in \mathbb{N}.
\]

4. On the condition (*) In this section we discuss some situation in which condition (*) with \(\tau = 0\) is verified by system \((L_\omega)\) for any \(\omega > 0\). We study three different examples of almost periodic functions, a case in which \(\alpha\) is quasi periodic, a case in which it is limit periodic and a case in which it is only almost periodic.

In the three cases the argument of the proof has the same structure which we can roughly describe as follows. Fixed any \(\omega > 0\) we denote \(\alpha_\omega(t) = \alpha(\omega t)\) and we argue by contradiction, assuming that, for some \(i\), there exists \(q \in \Gamma_{\eta_i}\) such that \(f_{\alpha_\omega}(q) = c_{\eta_i}(\alpha_\omega)\) and \(|q(0)| = \delta_0\) or \(|q(0) - \eta_i| = \delta_0\). Then we show that we can split \(\alpha_\omega\) in the sum of three non negative functions \(\alpha_\omega = \alpha_1 + \alpha_2 + \alpha_3\) in such a way that there exist \(\delta > 0\) and \(\tau_0 \in \mathbb{R}\) for which the following conditions are verified
\[
\begin{align*}
(\alpha 1) & \int_{\mathbb{R}} [\alpha_1(t + \tau_0) - \alpha_1(t)]V(q(t)) \, dt < -\delta \int_{\mathbb{R}} V(q(t)) \, dt; \\
(\alpha 2) & \int_{\mathbb{R}} [\alpha_2(t + \tau_0) - \alpha_2(t)]V(q(t)) \, dt < \delta \int_{\mathbb{R}} V(q(t)) \, dt; \\
(\alpha 3) & \|\alpha_3\|_{\infty} < \frac{\delta}{4}.
\end{align*}
\]

We get a contradiction simply noting that \((\tau_0 \ast q)(t) \equiv q(t - \tau_0) \in \Gamma_{\eta_i}\) and then \(f_{\alpha_\omega}(\tau_0 \ast q) \geq c_{\eta_i}(\alpha_\omega) = f_{\alpha_\omega}(q)\), while by (\(\alpha 1\)), (\(\alpha 2\)) and (\(\alpha 3\))
\[
\begin{align*}
f_{\alpha_\omega}(\tau_0 \ast q) - f_{\alpha_\omega}(q) & = \int_{\mathbb{R}} [\alpha_1(t + \tau_0) - \alpha_1(t)]V(q(t)) \, dt \\
& \quad + \int_{\mathbb{R}} [\alpha_2(t + \tau_0) - \alpha_2(t)]V(q(t)) \, dt \\
& \quad + \int_{\mathbb{R}} [\alpha_3(t + \tau_0) - \alpha_3(t)]V(q(t)) \, dt \\
& \leq (\delta + \frac{\delta}{4} + \frac{\delta}{4}) \int_{\mathbb{R}} V(q(t)) \, dt < 0.
\end{align*}
\]

4.1 A quasi-periodic case. We recall that the function \(\alpha\) is a quasi-periodic function if \(\alpha(t) = F(\gamma_1 t, \gamma_2 t)\) for some function \(F(x, y)\) periodic both in \(x\) and \(y\) and for some constants \(\gamma_1, \gamma_2 \in \mathbb{R}\). Then
\[
F(x, y) = \sum_{k_1, k_2 \in \mathbb{Z}} c_{k_1, k_2} \exp(idx + k_2y)
\]
and
\[
\alpha(t) = F(\gamma_1 t, \gamma_2 t) = \sum_{k_1, k_2 \in \mathbb{Z}} c_{k_1, k_2} e^{i(k_1 \gamma_1 + k_2 \gamma_2)t}
\]

As in the paper [9], we take \(\gamma_1 = 1, \gamma_2 = \gamma = \frac{1 + \sqrt{5}}{2}\) and consider
\[
\alpha(t) = 1 + \sum_{k=1}^{\infty} a_k \cos(\omega_k t).
\]

where \((\omega_k)\) is a sequence in \(\{k_1 + \gamma k_2 \mid k_1, k_2 \in \mathbb{Z}\}\). Then our function is a quasi-periodic function. Let us remark immediately that our assumptions will imply that \(F\) is not an analytic function.

On the frequencies \(\omega_k\) we will assume that
\[
\omega_k = F_{\ell k} - \gamma F_{\ell k - 1}, \quad k \in \mathbb{N},
\]
where \(\ell \in \mathbb{N}\) and \(F_n\) is the \(n\) Fibonacci number (defined by \(F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}\) for all \(n \geq 1\)). Let us also recall that
\[
F_n = \frac{n^\gamma - (-1)^n \gamma^{-n}}{\sqrt{5}}
\]
so that as \(n \to \infty\)
\[
F_n - \gamma F_{n-1} = (-1)^n \frac{C_F}{F_{n-1}} + O(F_{n-1}^{-3}), \quad C_F = \frac{1}{\gamma + \gamma^{-1}},
\]
see [10]. As a consequence, we have that there exists \(\ell_1 \in \mathbb{N}\) (independent of \(k\)) such that
\[
|\omega_k| = \frac{C_F}{\gamma^{\ell k - 1}} + O(\gamma^{-3k+3}) \leq \frac{C_2}{\gamma^{\ell k - 1}} \tag{4.1}
\]
for all \(\ell \geq \ell_1\) and \(k \geq 1\).

**Theorem 4.1.** Suppose
\[
\alpha(t) = 1 + \sum_{k=1}^{\infty} a_k \cos(\omega_k t)
\]
where there exist \(0 < A \leq 1, B > 0\) and \(\beta > 1 + \frac{4B}{A}\) such that \(\frac{A}{\beta} \leq a_k \leq \frac{B}{\beta}\).

Then there is \(\ell_0 \in \mathbb{N}\) such that for all \(\ell \geq \ell_0\) and for all \(\omega > 0\) the function \(\alpha_\omega(t) = \alpha(\omega t)\) satisfies condition (*) with \(\tau = 0\).

**Proof.** To prove the Theorem we need some preliminary estimates.

Let us first note that from the assumption on \(a_k\) it follows that for any \(j \in \mathbb{N}\)
\[
\sum_{k=j}^{\infty} a_k \leq B \sum_{k=j}^{\infty} \beta^{-k} = B \frac{\beta^{-j}}{\beta - 1} \leq \frac{A}{4} \beta^{-j+1} \leq \frac{a_{j-1}}{4}
\]
In particular
\[
\sum_{k=1}^{\infty} a_k \leq \frac{A}{4} \leq \frac{1}{4}
\]
Hence for any \(\ell \in \mathbb{N}\) and \(\omega > 0\), we have that \(\frac{3}{4} \leq \alpha_\omega(t) \leq \frac{5}{4}\).

We will drop the dependence on \(\ell\), and denote \(f_\omega(q) = f_{\alpha_\omega}(q)\).

Arguing by contradiction, we will assume that for some \(i\) there is a function \(q_\omega \in \Gamma_{\eta_i}\) with \(|q_\omega(0)| = \delta_0\) or \(|q_\omega(0) - \eta_i| = \delta_0\) such that \(f_\omega(q_\omega) = c_{\eta_i}(\alpha_\omega)\).
Let \( M_1 \) be given by Lemma 2.15 with \( d = \frac{1}{10} \) and \( \tau = 0 \), i.e. be such that
\[
\int_{|t| \geq M_1} V(q(t)) \, dt < \frac{1}{10} \int_{\mathbb{R}} V(q(t)) \, dt.
\]
We then define \( \bar{k} \in \mathbb{N} \) (which depends on \( \omega \) and \( \ell \)) by
\[
\bar{k} = \min \{ k \in \mathbb{N} \mid \omega \omega^k M_1 < \frac{1}{8} \},
\]
and split \( \alpha \omega \) setting
\[
\begin{align*}
\alpha_1(t) &= a_{\bar{k}} \cos(\omega \omega^\bar{k} t) \\
\alpha_2(t) &= 1 + \sum_{k=1}^{\bar{k}-1} a_k \cos(\omega \omega^k t) \\
\alpha_3(t) &= \sum_{k=\bar{k}+1}^{\infty} a_k \cos(\omega \omega^k t)
\end{align*}
\]
We immediately have that
\[
\|\alpha_3\|_\infty \leq \sum_{k=\bar{k}+1}^{\infty} a_k < a_{\bar{k}} \frac{\ell}{4},
\]
and \((\alpha 3)\) is verified setting \( \delta = a_{\bar{k}} \). Note that the splitting and \( \delta \) depend on \( \omega \) and \( \ell \).

To deal with \( \alpha_1 \) and \( \alpha_2 \) we first observe that

**Lemma 4.2.** For all \( k \) and \( \ell \in \mathbb{N} \) we have that \( \omega^k \) and \( \omega^{k+1} \) are rationally independent.

**Proof.** Suppose there are \( i, j \in \mathbb{Z} \setminus \{0\} \) such that \( j \omega^k = i \omega^{k+1} \). Then
\[
j(F_{\ell k} - \gamma F_{\ell k-1}) = i(F_{\ell k+t} - \gamma F_{\ell k+t-1}).
\]
Since \( \gamma \) is irrational, necessarily
\[
jF_{\ell k} - iF_{\ell k+t} = 0, \quad jF_{\ell k-1} - iF_{\ell k+t-1} = 0
\]
and then, by definition of the Fibonacci’s sequence
\[
jF_{\ell k-j} - iF_{\ell k+t-j} = 0, \quad j = 2, \ldots, \ell k.
\]
In particular, for \( j = \ell k \),
\[
F_{\ell k} = 0,
\]
a contradiction if \( \ell \geq 1 \).

As a consequence of this lemma, for all \( \varepsilon \in (0, 1) \), using the Kronecker’s Theorem (see [13]), it is possible to find \( \tau_0 \in \mathbb{R} \) (which depends on \( \bar{k} \), \( \ell \) and \( \omega \)) such that
\[
|\cos(\omega \omega^k \tau_0) + 1| < \varepsilon, \quad |\cos(\omega \omega^k \tau_0) - 1| < \varepsilon, \\
|\sin(\omega \omega^k \tau_0)| < \varepsilon, \quad |\sin(\omega \omega^k \tau_0)| < \varepsilon.
\]
Then, for all \( |t| < M_1 \) we have \( |\omega \omega^k t| < \omega \omega^k M_1 < \frac{1}{8} \) and then
\[
\cos(\omega \omega^k (t + \tau_0)) - \cos(\omega \omega^k t) = \cos(\omega \omega^k t)(\cos(\omega \omega^k \tau_0) - 1) - \sin(\omega \omega^k t) \sin(\omega \omega^k \tau_0)
\leq (-2 + \varepsilon) \cos \frac{1}{8} + \varepsilon \sin \frac{1}{8} < -2 + \frac{1}{8}
\]
provided $\varepsilon$ is sufficiently small. Moreover, for all $t \in \mathbb{R}$,
\[
|\cos(\omega \omega_{k-1} (t + \tau_0)) - \cos(\omega \omega_{k-1} t)| \\
\leq |\cos(\omega \omega_{k-1} t)(\cos(\omega \omega_{k-1} \tau_0) - 1)| + |\sin(\omega \omega_{k-1} t)\sin(\omega \omega_{k-1} \tau_0)| \leq 2\varepsilon. \quad (4.2)
\]

As a consequence and using Lemma 2.15 we have that
\[
\int_{\mathbb{R}} [\alpha_1(t + \tau_0) - \alpha_1(t)]V(q_\omega(t)) \, dt = \int_{|t| \leq M_1} [\alpha_1(t + \tau_0) - \alpha_1(t)]V(q_\omega(t)) \, dt \\
+ \int_{|t| \geq M_1} [\alpha_1(t + \tau_0) - \alpha_1(t)]V(q_\omega(t)) \, dt \\
\leq a_k (-2 + \frac{1}{8}) \int_{|t| \leq M_1} V(q_\omega(t)) \, dt + 2a_k \int_{|t| \geq M_1} V(q_\omega(t)) \, dt \\
\leq a_k \left[\frac{9}{10} (-2 + \frac{1}{8}) + \frac{2}{10}\right] \int_{\mathbb{R}} V(q_\omega(t)) \, dt \\
\leq -a_k \int_{\mathbb{R}} V(q_\omega(t)) \, dt = -\delta \int_{\mathbb{R}} V(q_\omega(t)) \, dt,
\]
and the condition (a1) holds true. To finally show that also (a2) holds, let us prove the following

**Lemma 4.3.** There is $C_3 = C_3(V, \alpha, \sigma)$ such that, for all $t \in \mathbb{R}$, $\omega > 0$ and for all $q_\omega \in \Gamma$, such that $f_\omega(q_\omega) = c_\omega (\alpha_\omega)$ the following holds:
\[
\left|\int_{\mathbb{R}} \cos(\nu t)V(q_\omega(t)) \, dt\right| \leq C_3 \int_{\mathbb{R}} V(q_\omega(t)) \, dt \quad \text{for all } \nu > 0
\]

**Proof.** Using the fact that $q_\omega$ is a solution, a simple integration by part shows that:
\[
\int_{\mathbb{R}} \cos(\nu t)V(q_\omega(t)) \, dt = -\frac{1}{\nu} \int_{\mathbb{R}} \sin(\nu t)(V'(q_\omega(t)) \mid \dot{q}_\omega(t)) \, dt \\
= -\frac{1}{\nu^2} \int_{\mathbb{R}} \cos(\nu t)[V''(q_\omega(t))\dot{q}_\omega(t) \mid \ddot{q}_\omega(t)] - |V'(q_\omega(t))|^2 \alpha_\omega(t) \, dt
\]
Since $V''(x)$ is bounded and $|V'(x)|^2 \leq CV(x)$ for some positive $C$, one deduces that
\[
\left|\int_{\mathbb{R}} \cos(\nu t)V(q_\omega(t)) \, dt\right| \leq \frac{C}{\nu^2} \int_{\mathbb{R}} \frac{1}{2} |\dot{q}_\omega(t)|^2 + \alpha_\omega(t)V(q_\omega(t)) \, dt \\
= \frac{C}{\nu^2} c_\omega (\alpha_\omega) \leq \frac{C c_\omega (\alpha_\omega)}{\nu^2}
\]
and the claim follows by Lemmas 2.14 and 2.6. \qed

Now we remark that, thanks to (4.2),
\[
a_{k-1} \int_{\mathbb{R}} \left[\cos(\omega \omega_{k-1} (t + \tau_0)) - \cos(\omega \omega_{k-1} t)\right]V(q_\omega(t)) \, dt \leq 2a_{k-1} \varepsilon \int_{\mathbb{R}} V(q_\omega(t)) \, dt \\
\leq \frac{\delta}{8} \int_{\mathbb{R}} V(q_\omega(t)) \, dt
\]
provided $\varepsilon$ has been chosen small enough.
Then it follows from Lemma 4.3 that
\[
\left| \int_{\mathbb{R}} (\alpha_2(t + \tau_0) - \alpha_2(t)) V(q_\omega(t)) \, dt \right| \leq \\
\leq \sum_{k=1}^{\tau-2} a_k \left| \int_{\mathbb{R}} [\cos(\omega_k(t + \tau_0)) - \cos(\omega_k t)] V(q_\omega(t)) \, dt \right| \\
+ a_{\tau-1} \left| \int_{\mathbb{R}} [\cos(\omega_{\tau-1}(t + \tau_0)) - \cos(\omega_{\tau-1} t)] V(q_\omega(t)) \, dt \right|
\]
\[
\leq \sum_{k=1}^{\tau-2} 3a_k \frac{C_3}{\omega^2 \omega_k^2} \int_{\mathbb{R}} V(q_\omega(t)) \, dt + \delta \int_{\mathbb{R}} V(q_\omega(t)) \, dt
\]
\[
= \left( \sum_{k=1}^{\tau-2} 3a_k \frac{C_3}{\omega^2 \omega_k^2} + \delta \right) \int_{\mathbb{R}} V(q_\omega(t)) \, dt
\]
(4.3)

We want to show that the term \( \sum_{k=1}^{\tau-2} 3a_k C_3 \omega^{-2} \omega_k^{-2} \) is small for all \( \omega \) provided \( \ell \) is large enough. Some care is necessary since \( F \) depends on \( \omega \) and \( \ell \).

From the definition of \( F \) we know that \( \omega \omega_{\tau-1} M_1 < 1/8 \) and that \( \omega \omega_{\tau-1} M_1 \geq 1/8 \), so that \( \omega^{-1} \leq 8 \omega_{\tau-1} M_1 \) and, using (4.1),
\[
\sum_{k=1}^{\tau-2} 3a_k \frac{C_3}{\omega^2 \omega_k^2} \leq 3 \cdot 64 C_3 M_1^2 \omega_0^{-2} \sum_{k=1}^{\tau-2} \frac{a_k}{\omega_k^2} \leq \frac{C_4}{\gamma^{2\ell (\tau-1)}} \sum_{k=1}^{\tau-2} a_k \gamma^{2\ell k}
\]
\[
\leq \frac{BC_4}{\gamma^{2\ell (\tau-1)}} \sum_{k=1}^{\tau-2} \frac{\gamma^{2\ell k}}{\beta^k} = \frac{BC_4}{\beta^{\ell-1}} \theta^{1-\ell} \sum_{k=1}^{\tau-2} \theta^k = \frac{C_5}{\beta^k} \frac{1 - \theta^{2-k}}{\theta - 1}
\]
where \( \theta = \gamma^{2\ell} / \beta \to \infty \) as \( \ell \to \infty \). Then, there exists \( \ell_0 \geq \ell_1 \) (independent of \( \omega \) and \( k \) such that for all \( \ell \geq \ell_0 \) we have
\[
\frac{C_5}{\beta^k} \frac{1 - \theta^{2-k}}{\theta - 1} < A \frac{\delta}{\beta^k} \leq \frac{a_k}{\lambda} = \frac{\delta}{\lambda}
\]
and we deduce from (4.3) that
\[
\left| \int_{\mathbb{R}} (\alpha_2(t + \tau_0) - \alpha_2(t)) V(q_\omega(t)) \, dt \right| \leq \frac{\delta}{4} \int_{\mathbb{R}} V(q_\omega(t)) \, dt.
\]
Then, also assumption (\( \alpha_2 \)) is verified. As explained above, this gives the contradiction which proves Theorem 4.1.

4.2. **An almost-periodic case.** We now present a different situation, in which the function \( \alpha \) is an almost-periodic function (see definition 2.9) which is not quasi-periodic. Intuitively an almost-periodic function can be seen as a combination of harmonics having countably many rationally independent frequencies, in contrast to quasi-periodic ones (which have harmonics of only finitely many rationally independent frequencies) and periodic (which have harmonics of just one frequency). As for the quasi-periodic example, we just present here a rather simple situation just to illustrate our method.
Theorem 4.4. Suppose
\[\alpha(t) = \sum_{k=1}^{\infty} a_k b(\omega_k t)\]
where

(B1) \( b \in C(\mathbb{R}; \mathbb{R}) \) is \( T \)-periodic, \( \frac{1}{2} \leq b(t) \leq 2 \);
(B2) There is \( t_0 > 0 \) such that \( b \) is strictly decreasing in \([-t_0, t_0]\);
(B3) \( a_k > 0, \sum_{k=1}^{\infty} a_k < +\infty \);
(B4) \( \omega_k > \omega_{k+1} > 0, \omega_k \to 0 \);
(B5) every finite subset of \( \{ \omega_k \mid k \in \mathbb{N} \} \) is rationally independent.

Then, for all \( \omega > 0 \), \( \alpha_\omega(t) = \alpha(\omega t) \) satisfies (\*) with \( \tau = 0 \).

Proof. Let
\[ b_0 = \min_{t \in \left[-\frac{t_0}{2}, \frac{t_0}{2}\right]} b(t) - \max_{t \in \left[\frac{t_0}{2}, t_0\right]} b(t) > 0. \]
As in the proof of Theorem 4.1, we denote \( f_\omega(q) = \alpha_\omega(q) \). Arguing by contradiction, assume there is, for a given \( \omega > 0 \), a function \( q \in \Gamma_\eta \) such that \( f_\omega(q) = c_\eta \) and \( |q(0)| = \delta_0 \) or \( |q(0) - \eta| = \delta_0 \).

Let \( M_1 \) be fixed in Lemma 2.15 with \( d = \frac{b_0}{8} \) and \( \tau = 0 \) and note that since \( b_0 \leq \frac{3}{2} \) we have
\[
\int_{|t| \geq M_1} V(q(t)) \, dt < \frac{b_0}{8} \int_{\mathbb{R}} V(q(t)) \, dt,
\]
\[
\int_{|t| \leq M_1} V(q(t)) \, dt \geq \frac{1}{2} \int_{\mathbb{R}} V(q(t)) \, dt.
\]
Take \( \bar{k} \geq 1 \) such that
\[
\omega\omega_{\bar{k}} \leq \frac{t_0}{8M_1}
\]
and \( \bar{k} \geq \bar{k} \) such that
\[
\sum_{k=\bar{k}}^{\infty} a_k < \frac{a_{\bar{k}}b_0}{32}.
\]
Then, we consider the following splitting of \( \alpha_\omega \) as
\[
\alpha_1(t) = a_{\bar{k}} b(\omega_{\bar{k}} t)
\]
\[
\alpha_2(t) = \sum_{k=1}^{\bar{k}} a_k b(\omega_{k} t)
\]
\[
\alpha_3(t) = \sum_{k=\bar{k}+1}^{\infty} a_k b(\omega_{k} t).
\]
First we have
\[
\|\alpha_3\|_\infty \leq 2 \sum_{k=\bar{k}+1}^{\infty} a_k < \frac{1}{16}a_{\bar{k}}b_0
\]
and so (\alpha3) holds true letting \( \delta = \frac{a_{\bar{k}}b_0}{4} \).
By uniform continuity, let \( \varepsilon > 0 \) be such that
\[
|b(t) - b(s)| < \frac{b_0a_{\bar{k}}}{16 \sum_{k=1}^{\infty} a_k} \quad \forall |t - s| < \varepsilon.
\]
By Kronecker's Theorem we can find \( \tau_0 \in \mathbb{R} \) and \( n_1, \ldots, n_\bar{k} \in \mathbb{N} \) such that
\[
\omega_n \tau_0 = n_k T + \frac{3}{4} t_0
\]
\[
|\omega_n \tau_0 - n_k T| < \varepsilon, \quad k = 1, \ldots, \bar{k}, \ k \neq \bar{k}.
\]
So we can estimate
\[
I_1 = \int_{\mathbb{R}} [\alpha_1(t + \tau_0) - \alpha_1(t)] V(q(t)) \, dt
\]
\[
= a_\bar{k} \int_{\mathbb{R}} [b(\omega_{\bar{k}} \tau_0 + \omega_{\bar{k}} t) - b(\omega_{\bar{k}} t)] V(q(t)) \, dt
\]
\[
= a_\bar{k} \int_{\mathbb{R}} [b(\frac{3}{2} t_0 + \omega_{\bar{k}} t) - b(\omega_{\bar{k}} t)] V(q(t)) \, dt
\]
\[
= a_\bar{k} \int_{|t| \leq M_1} [b(\frac{3}{2} t_0 + \omega_{\bar{k}} t) - b(\omega_{\bar{k}} t)] V(q(t)) \, dt +
\]
\[
+ a_\bar{k} \int_{|t| \geq M_1} [b(\frac{3}{2} t_0 + \omega_{\bar{k}} t) - b(\omega_{\bar{k}} t)] V(q(t)) \, dt.
\]
Since for all \( |t| \leq M_1 \) we have that \( |\omega_{\bar{k}} t| \leq |\omega_{\bar{k}} M_1| \leq \frac{\varepsilon}{8} \), by the choice of \( M_1 \) we obtain
\[
I_1 \leq -a_\bar{k} b_0 \int_{|t| \leq M_1} V(q(t)) \, dt + 2a_\bar{k} \int_{|t| \geq M_1} V(q(t)) \, dt
\]
\[
\leq -\frac{a_\bar{k} b_0}{2} \int_{\mathbb{R}} V(q(t)) \, dt + \frac{a_\bar{k} b_0}{4} \int_{\mathbb{R}} V(q(t)) \, dt = -\frac{a_\bar{k} b_0}{4} \int_{\mathbb{R}} V(q(t)) \, dt
\]
so that (a1) holds. Moreover, by the choice of \( \tau_0 \) and (4.4), setting \( \varepsilon_k = \omega_{\bar{k}} \tau_0 - n_k T \), we have
\[
I_2 = \int_{\mathbb{R}} [\alpha_2(t + \tau_0) - \alpha_2(t)] V(q(t)) \, dt
\]
\[
= \sum_{k=1}^{\bar{k}} a_k \int_{\mathbb{R}} [b(\omega_k t + \omega_k \tau_0) - b(\omega_k t)] V(q(t)) \, dt
\]
\[
= \sum_{k=1}^{\bar{k}} a_k \int_{\mathbb{R}} [b(\omega_k t + \varepsilon_k) - b(\omega_k t)] V(q(t)) \, dt
\]
\[
\leq \sum_{k=1 \text{ or } k \neq \bar{k}}^{\bar{k}} \frac{b_0 a_k}{16} \sum_{\ell=1}^{\infty} \alpha_{\ell} a_k \int_{\mathbb{R}} V(q(t)) \, dt \leq \frac{1}{16} b_0 a_k \int_{\mathbb{R}} V(q(t)) \, dt = \frac{\delta}{4} \int_{\mathbb{R}} V(q(t))
\]
and also (a2) holds. This completes the proof of the Theorem.

4.3. A limit-periodic case. We give now a third example. We consider here a limit-periodic function \( \alpha \). By definition, a limit-periodic function is a continuous function \( \alpha: \mathbb{R} \rightarrow \mathbb{R} \) which is the uniform limit of periodic functions.

**Theorem 4.5.** Suppose
\[
\alpha(t) = \sum_{j=0}^{\infty} a_k b(\omega_k t)
\]
where

(C1) \( b \in C^1(\mathbb{R}; \mathbb{R}) \) is \( T \)-periodic, \( \frac{1}{2} \leq b(t) \leq 2 \);

(C2) \( b \) is even and strictly decreasing in \([0, \frac{T}{2}]\);

(C3) \( \sum_{k=0}^{\infty} a_k < +\infty \) and \( a_k \geq a_{k+1} > 0 \) for all \( k \in \mathbb{N} \);

(C4) \( \frac{\omega_k}{\omega_{k+1}} \) is an even integer for all \( k \in \mathbb{N} \).

Then, for all \( \omega > 0 \), \( \alpha_\omega(t) = \alpha(\omega t) \) satisfies (*) with \( \tau = 0 \).

Proof. By assumption (C2),

\[
b_0 = \min_{t \in [\frac{-T}{2}, \frac{T}{2}]} b(t) - \max_{t \in [\frac{T}{2}, \frac{3T}{2}]} b(t) > 0.
\]

As in the proof of Theorem 4.1, we denote \( f_\omega(q) = f_{\alpha_\omega}(q) \) and, by contradiction, assume that there is, for a given \( \tau \), \( \eta \), and \( |q(t)| = \delta_0 \) or \( |q(0)| = \delta_0 \).

Let \( M = \|b\|_\infty \) and let \( M_1 \) be fixed in Lemma 2.15 with \( d = \min\left\{ \frac{b_0}{16}, \frac{b_0}{8TM}, \frac{3}{2} \right\} \) and \( \tau = 0 \). Then,

\[
\int_{|t| \geq M_1} V(q(t)) \, dt < \min\left\{ \frac{b_0}{16}, \frac{b_0}{8TM} \right\} \int_{\mathbb{R}} V(q(t)) \, dt
\]

\[
\int_{|t| \leq M_1} V(q(t)) \, dt \geq \frac{1}{2} \int_{\mathbb{R}} V(q(t)) \, dt
\]

Noting that \( \omega_k \to 0 \), take \( k \) such that

\[
\omega_k \leq \frac{T}{8M_1}
\]

and split \( \alpha_\omega \) setting

\[
\alpha_1(t) = a_k b(\omega_k t)
\]

\[
\alpha_2(t) = \sum_{\substack{k=1 \backslash k \neq \bar{k}}}^{\infty} a_k b(\omega_k t)
\]

\[
\alpha_3(t) = 0.
\]

Let \( \tau_0 = \frac{T}{2\omega_k} \). Then

\[
I_1 = \int_{\mathbb{R}} [\alpha_1(t + \tau_0) - \alpha_1(t)] V(q(t)) \, dt
\]

\[
= a_k \int_{\mathbb{R}} [b(\omega_k t + \omega_k \tau_0) - b(\omega_k t)] V(q(t)) \, dt
\]

\[
= a_k \int_{\mathbb{R}} [b(\frac{T}{2} + \omega_k t) - b(\omega_k t)] V(q(t)) \, dt
\]

\[
= a_k \int_{|t| \leq M_1} [b(\frac{T}{2} + \omega_k t) - b(\omega_k t)] V(q(t)) \, dt +
\]

\[
a_k \int_{|t| \geq M_1} [b(\frac{T}{2} + \omega_k t) - b(\omega_k t)] V(q(t)) \, dt.
\]
Since for all $|t| \leq M_1$ we have that $|\omega \omega_k t| \leq |\omega \omega_k M_1| \leq \frac{T}{N}$, by the choice of $M_1$ we obtain

\[
I_1 \leq -a_k b_0 \int_{|t| \leq M_1} V(q(t)) \, dt + 4a_k \int_{|t| \geq M_1} V(q(t)) \, dt
\]

so that (α1) holds with $\delta = a_k \frac{b_0}{4}$.

Moreover, we have that

\[
I_2 = \int_{\mathbb{R}} [\alpha_2(t + \tau_0) - \alpha_2(t)] V(q(t)) \, dt
\]

\[
= \sum_{k=1}^{k-1} a_k \int_{\mathbb{R}} [b(\omega \omega_k T \omega \omega_k t + \omega \omega_k t) - b(\omega \omega_k t)] V(q(t)) \, dt + \sum_{k=k+1}^{\infty} a_k \int_{\mathbb{R}} [b(\omega \omega_k T \omega \omega_k t + \omega \omega_k t) - b(\omega \omega_k t)] V(q(t)) \, dt.
\]

From assumption (C4) it follows that for all $k = 1, \ldots, k-1$ there is an integer $n_k$ such that $2n_k \omega_k = \omega_k$, so that $\omega \omega_k T = T_n \omega_k = T_n$ and

\[
\sum_{k=1}^{k-1} a_k \int_{\mathbb{R}} [b(\omega \omega_k T \omega \omega_k t + \omega \omega_k t) - b(\omega \omega_k t)] V(q(t)) \, dt = 0
\]

Moreover for all $k > \tilde{k}$ and for all $|t| \leq M_1$ we have $\omega \omega_k |t| \leq \omega \omega_k M_1 \leq \frac{T}{N}$ and $\omega \omega_k \tau_0 = \frac{\omega \omega_k}{2n_k} T = \frac{T}{n_k}$ for a certain $n \in \mathbb{N}$. Then from assumption (C2) we also deduce that for all $k > \tilde{k}$ and for all $|t| \leq M_1$

\[
b(\omega \omega_k T \omega \omega_k t + \omega \omega_k t) - b(\omega \omega_k t) \leq \max_{|t| \leq M_1} b(\omega \omega_k t + \omega \omega_k \tau_0) - \min_{|t| \leq M_1} b(\omega \omega_k t)
\]

\[
= b(-\omega \omega_k M_1 + \omega \omega_k \tau_0) - b(\omega \omega_k M_1) \leq 0
\]

from which

\[
I_2 \leq \sum_{k=\tilde{k}+1}^{\infty} a_k \int_{|t| \geq M_1} [b(\omega \omega_k T \omega \omega_k t + \omega \omega_k t) - b(\omega \omega_k t)] V(q(t)) \, dt
\]

\[
\leq \sum_{k=\tilde{k}+1}^{\infty} M a_k \omega \omega_k \tau_0 \int_{|t| \geq M_1} V(q(t)) \, dt
\]

\[
= \left( \sum_{k=\tilde{k}+1}^{\infty} a_k \frac{\omega \omega_k}{2} \right) \frac{TM}{2} \int_{|t| \geq M_1} V(q(t)) \, dt.
\]

Finally remark that, by assumption (C3)

\[
\sum_{k=\tilde{k}+1}^{\infty} a_k \frac{\omega \omega_k}{2} \leq a_k \sum_{k=\tilde{k}+1}^{\infty} \frac{1}{2^{k-\tilde{k}}} \leq a_k
\]

so that, by the choice of $M_1$,

\[
I_2 \leq a_k \frac{TM}{2} \int_{|t| \geq M_1} V(q(t)) \, dt \leq \frac{a_k b_0}{16} \int_{\mathbb{R}} V(q(t)) \, dt = \frac{\delta}{4} \int_{\mathbb{R}} V(q(t)) \, dt
\]

and the theorem follows. \qed
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