Bose-Fermi Degeneracy and Duality in Non-Supersymmetric Strings

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Abstract

Following Kachru, Kumar and Silverstein, we construct a set of non-supersymmetric Type II string models which have equal numbers of bosons and fermions at each mass level. The models are asymmetric $\mathbb{Z}_2 \otimes \mathbb{Z}_2'$ orbifolds. We demonstrate that this bose-fermi degeneracy feature implies that both the one-loop and the two-loop contributions to the cosmological constant vanish. We conjecture that the cosmological constant actually vanishes to all loops. We construct a strong-weak dual pair of models, both of which have bose-fermi degeneracy. This implies that at least some of the non-perturbative corrections to the cosmological constant are absent.

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I. INTRODUCTION

Bose-fermi degeneracy is well known to be a consequence of supersymmetry. However, it is not a feature one expects in a non-supersymmetric theory. So it is very interesting that, recently, Kachru, Kumar and Silverstein [1] constructed a 4-dimensional non-supersymmetric Type II string model (the KKS model) which has this bose-fermi degeneracy feature, i.e., the number of bosonic and fermionic degrees of freedom are equal at every mass level of the spectrum. In this paper, we construct a set of string models that also share this same feature.

The models are non-supersymmetric 4-dimensional Type II string models, constructed as asymmetric $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ orbifolds. Such models are easy to construct, using the free fermionic string model construction [2]. Starting from the 4-dimensional $\mathcal{N} = 8$ supersymmetric (i.e., $\mathcal{N} = (4,4)$) Type II model, we introduce a $\mathbb{Z}_2$ twist which breaks the supersymmetry to $\mathcal{N} = 2$ (i.e., $\mathcal{N} = (2,0)$). On the other hand, we can introduce a different $\mathbb{Z}_2'$ twist on the original $\mathcal{N} = 8$ model. Judicious choices of these two twists result in non-supersymmetric $\mathbb{Z}_2 \otimes \mathbb{Z}_2'$ orbifold models with zero one-loop partition functions. It turns out that there are two inequivalent types of $\mathbb{Z}_2'$ twist that satisfy the requirements. The first type of $\mathbb{Z}_2'$ twist breaks supersymmetry to $\mathcal{N} = 2$ (i.e., $\mathcal{N} = (0,2)$). The $\mathbb{Z}_2'$ twist in Ref. [1,3] belongs to this type. The other type of $\mathbb{Z}_2'$ twist breaks supersymmetry to $\mathcal{N} = 4$ (i.e., $\mathcal{N} = (0,4)$). In contrast to the model in Ref. [1] where the one-loop partition function vanishes because of the non-Abelian nature of the orbifold, the one-loop vanishing in the $\mathbb{Z}_2 \otimes \mathbb{Z}_2'$ orbifolds presented here is due to the fermionic zero mode. Another novel feature of this set of models is that there are massless twisted sector states. Despite the fact that the models are non-supersymmetric (as the gravitinos are all projected out), the twisted sector states seem to fall into appropriate supermultiplets.

The non-trivial question about these models is whether the cosmological constant remains zero at higher loops. Using an approach similar to that given in Ref [1], we demonstrate that the cosmological constant vanishes at two-loops (with the two-loop integrand vanishing point-wise in the moduli space). The structure of the models naturally leads to the conjecture that this property (i.e., vanishing multi-loop integrand) persists to all loops. We give a plausible argument for this conjecture.

Let us assume that the cosmological constant does vanish to all orders in the perturbation expansion. Each model actually represents a class of such models, since they have scalar fields that are moduli, i.e., the cosmological constant remains zero for different choices of the scalar field expectation values. The existence of a set of such models with this feature strongly suggests an underlying stringy symmetry that remains to be identified.

Some duality aspects of these types of models have been explored recently [3,4]. In particular, Kachru and Silverstein show that the KKS model is self-dual. Here we generalize their analysis to more intricate situations. Some care is necessary to distinguish seemingly similar models (with the same orbifold twists and the same counting of states in the spectra) which have different GSO projections (due to discrete torsions), and hence give rise to different spectra. A priori, they have different duality properties, reminiscent of the situation of Type IIA/IIB. This allows us to reproduce the self-duality property of the KKS model, and in addition, construct a new strong-weak dual (i.e., $U$-dual) pair, both of which have bose-fermi degeneracy. We argue that at least some of the non-perturbative corrections to
the cosmological constants are absent.

The existence of these models further suggests that it may not be too difficult to arrange realistic non-supersymmetric models with zero cosmological constant. In the realistic universe, the cosmological constant has to vanish only at an isolated point in the field space, which is a much more relaxed constraint. Even if the cosmological constant vanishes in perturbative expansion, it may still receive a non-perturbative contribution. At weak coupling, we expect at most an exponentially suppressed cosmological constant, which may be consistent with nature.

This paper is organized as follows. In Section II we review the rules of the free fermionic string model construction. We follow the notation of Ref. [5]. Readers who are familiar with orbifolds but not familiar with the fermionic string construction may skip this section, since we will give a translation between the two languages later on. In Section III we give the explicit constructions of the models. It is illuminating to show how the massless spectra emerge under the various asymmetric $Z_2$ twists. They are shown in the tables. In Section IV we show that one of the models constructed is self-dual. That model is similar to the KKS model. We also show that two of the models form a dual pair. In Section V we briefly discuss the issues of multi-loop contributions to the cosmological constant, and demonstrate the vanishing of the two-loop vacuum amplitude. We also give a heuristic argument for higher loops. The last section contains the summary and remarks.

II. PRELIMINARIES

In this section we review the rules of free fermionic string model construction. Readers who are familiar with the formulation may skip this section and go directly to Section III where the models are presented. Readers who are familiar with orbifolds but not familiar with the fermionic string construction may also skip this section, since a translation between the two languages is given in Section III. In what follows, we will concentrate on Type II string vacua with four non-compact space-time dimensions in the light-cone RNS formulation. In the free-fermionic construction, all of the world-sheet degrees of freedom corresponding to the compactified six space-like dimensions are fermionized. Moreover, these degrees of freedom are described via free world-sheet fermions, which are written in the complex basis, with various spin structures, that is, boundary conditions.

We have the following world-sheet degrees of freedom: one complex world-sheet boson $\phi^0$ corresponding to two real transverse space-time coordinates in the light-cone gauge; one complex fermion $\psi^0$ which is the world-sheet superpartner of $\phi^0$; three pairs of complex fermions $\chi^\ell$ and $\lambda^\ell$, $\ell = 1, 2, 3$, corresponding to fermionization of three complex (six real) world-sheet bosons $\phi^\ell$ describing the compactified dimensions; three complex world-sheet fermions $\psi^\ell$ which are superpartners of the world-sheet bosons $\phi^\ell$. Since we are discussing Type II strings, each of the above world-sheet degrees of freedom has a left- and right-moving component which we will denote by subscripts "L" and "R", respectively.

In the following discussion we will mostly deal with the world-sheet fermions $\psi^0, \psi^\ell, \chi^\ell, \lambda^\ell$. We will collectively refer to them as $\Psi^r$, $r = 0, \ldots, 9$, where $\Psi^0 \equiv \psi^0$, $\Psi^\ell \equiv \psi^\ell$, $\Psi^{1+\ell} \equiv \chi^\ell$, and $\Psi^{2+\ell} \equiv \lambda^\ell$. Similar notation will be used for the corresponding left- and right-moving components as well.
It is convenient to organize the string states into sectors labeled by the monodromies of the world-sheet degrees of freedom. Thus, consider the sector where

\[
\Psi_L^0(e^{2\pi i}) = \exp(-2\pi iv_{Li})\Psi_L^i(z),
\]

\[
\Psi_R^0(e^{-2\pi i}) = \exp(-2\pi iv_{Ri})\Psi_R^i(z). \tag{1}
\]

Note that \(\phi^0(e^{2\pi i}, e^{-2\pi i}) = \phi^0(z, \bar{z})\) since \(\phi^0\) corresponds to space-time coordinates. These monodromies can be combined into a single vector

\[
V_i = \left[ v_{Li}^0, v_{Li}^2, v_{Li}^3, v_{Li}^4, v_{Li}^5, v_{Li}^6, v_{Li}^7, v_{Li}^8, v_{Li}^9, v_{Ri}^0, v_{Ri}^1, v_{Ri}^2, v_{Ri}^3, v_{Ri}^4, v_{Ri}^5, v_{Ri}^6, v_{Ri}^8, v_{Ri}^9 \right]. \tag{2}
\]

The double vertical line separates the monodromies corresponding to the left- and right-moving components. Without loss of generality we can restrict the values of \(v_{Li}, v_{Ri}\) as follows: \(-\frac{1}{2} \leq v_{Li}, v_{Ri} < \frac{1}{2}\). Note that if a given monodromy is \(-\frac{1}{2}\) then the corresponding world-sheet degree of freedom is a complex Ramond fermion; if this monodromy is 0, then it is a Neveu-Schwarz complex fermion.

The monodromies \(V_i\) can be viewed as fields \(\Psi^r\) being periodic \(\Psi^r(e^{2\pi i}, e^{-2\pi i}) = \Psi^r(z, \bar{z})\) up to the identification \(\Psi^r \sim g(V_i)\Psi^r g^{-1}(V_i)\), where \(g(V_i)\) is an element of a finite discrete orbifold group \(\Gamma\). In this paper we will only consider Abelian orbifold groups \(\Gamma\). In fact, we will focus on orbifold groups which are direct products of \(\mathbb{Z}_2\) subgroups. In these cases all the \(V_i\) vectors only have elements taking values 0 or \(-\frac{1}{2}\).

The requirement of world-sheet supersymmetry is necessary to ensure space-time Lorentz invariance in the covariant gauge, which implies that the world-sheet supercurrents must have well defined monodromies. That is, for \(\ell = 1, 2, 3\):

\[
v_{Li}^\ell + v_{Li}^{\ell+1} + v_{Li}^{\ell+2} \equiv s_i \pmod{1},
\]

\[
v_{Ri}^\ell + v_{Ri}^{\ell+1} + v_{Ri}^{\ell+2} \equiv \overline{s}_i \pmod{1}. \tag{5}
\]

where \(s_i \equiv v_{Li}^0\) and \(\overline{s}_i \equiv v_{Ri}^0\) determine whether the corresponding space-time states are bosons or fermions: the NS-NS sectors with \(s_i = \overline{s}_i = 0\) as well as the R-R sectors with \(s_i = \overline{s}_i = -\frac{1}{2}\) give rise to space-time bosons; the NS-R sectors with \(s_i = 0, \overline{s}_i = -\frac{1}{2}\) as well as the R-NS sectors with \(s_i = -\frac{1}{2}, \overline{s}_i = 0\) give rise to space-time fermions.

The notation we have introduced proves convenient in describing the sectors of a given string model with an orbifold group \(\Gamma\). As we have already mentioned, we will confine our attention to the orbifold groups \(\Gamma \approx (\mathbb{Z}_2)^{2n}\) so that all the elements \(V_i^s\) \((s = 0, \ldots, 19)\) are either 0 or \(-\frac{1}{2}\). Here \(V_i^s = v_{Li}^s\) for \(s = r \in \{0, \ldots, 9\}\), and \(V_i^s = v_{Ri}^s\) for \(s = r + 10 = r + 10, \ldots, 19\). To describe all of the \(2^n\) elements of group \(\Gamma\), it is convenient to introduce a set of generating vectors \(\{V_i\}\) such that \(\alpha V = 0\) if and only if \(\alpha_i \equiv 0\). Here 0 is the null vector:

\[
0 = \left[ 0(000)(000)(000)|0(000)(000)(000) \right] \equiv \left[ 0(000)\right] \left[ 0(000) \right]. \tag{6}
\]

Also, \(\alpha V \equiv \sum_i \alpha_i V_i\) with the summation defined as \((V_i + V_j)^s = V_i^s + V_j^s\), and \(\alpha_i = 0, 1\). The overbar notation is defined as \(\overline{\alpha V} \equiv \alpha V - \Delta(\alpha)\), and the elements of \(\overline{\alpha V}\) satisfy \(-\frac{1}{2} \leq \overline{\alpha V} < \frac{1}{2}\), where \(\Delta(\alpha) \in \mathbb{Z}\). The elements \(g(\overline{\alpha V})\) of the group \(\Gamma\) are in one-to-one correspondence with the vectors \(\overline{\alpha V}\). It is the Abelian nature of \(\Gamma\) that allows for this correspondence by simply taking all possible linear combinations of the generating vectors \(V_i\).
Now we can identify the sectors of a given model. They are labeled by the vectors \( \alpha V \), and in a given sector \( \alpha V \) the monodromies of the string degrees of freedom are given by \( \Psi^r(ze^{2\pi i}, ze^{-2\pi i}) = g(\alpha V)\Psi^r(z, \overline{z})g^{-1}(\alpha V) \). The sectors with \( \alpha V^0 = \overline{\alpha V}^0 \) give rise to space-time bosons, whereas the sectors with \( \alpha V^0 \neq \overline{\alpha V}^0 \) give rise to space-time fermions.

In the next section we will show that the following vector must always be present among the generating vectors:

\[
V_0 = \left[ -\frac{1}{2} \left( -\frac{1}{2} - \frac{1}{2} - \frac{1}{2} \right) \right].
\] (7)

(The presence of this vector is required by one-loop modular invariance.) In fact, the \( \Gamma \approx \mathbb{Z}_2 \) orbifold model (which contains two sectors \( 0 \) and \( V_0 \)) is a non-supersymmetric theory without any fermions in its spectrum. To construct supersymmetric theories we must add additional vectors to the generating set \( \{V_i\} \). Consider adding two more vectors of the following form:

\[
V_1 = \left[ -\frac{1}{2} \left( -\frac{1}{2}00 \right) \right],
\] (8)

\[
V_2 = \left[ 0(000) \right] - \frac{1}{2} \left( -\frac{1}{2}00 \right)^3 .
\] (9)

The orbifold group now is \( \Gamma \approx (\mathbb{Z}_2)^3 \). Such a model (subject to the consistency conditions discussed in the subsequent sections) corresponds to a four dimensional Type IIA or Type IIB string theory (depending on certain phases entering the one-loop partition function - see below) with \( \mathcal{N} = 8 \) space-time supersymmetry.

As mentioned before, we will focus on orbifold groups that are direct products of \( \mathbb{Z}_2 \) subgroups. Hence the boundary conditions of the complex fermions \( \Psi^\ell \) can only take values 0 or 1/2. This implies that the worldsheet fermions do not need to be written in the complex basis (in contrast to other \( \mathbb{Z}_N \) orbifolds where the monodromies are complex and so a complex basis of the fermions is necessary). In cases where the worldsheet degrees of freedom contain real fermions, the generating set \( \{V_i\} \) must satisfy the cubic constraint [5]:

\[
4 \sum_{\ell \text{real}} V_i^\ell V_j^\ell V_k^\ell = 0 \pmod{1} \quad \text{for all } i, j, k.
\] (10)

This ensures that we can find a complex basis for any three generating vectors, even though there is no complex basis which is common for \textit{all} the generating vectors.

### A. One-Loop Modular Invariance

The \( g \)-loop scattering amplitudes must satisfy modular invariance. Let us start with the one-loop vacuum amplitude: a torus. Conformally inequivalent tori are labeled by a single complex modular parameter \( \tau = \tau_1 + i\tau_2 \). A torus has two non-contractable cycles \( a \) and \( b \) depicted as follows:
The identification for the $a$ cycle corresponds to considering states from the $g_a$ sector propagating in the loop. The identification for the $b$ cycle corresponds to inserting the $g_β$ twist into the path integral for the world-sheet fermions. Thus, the world-sheet fermionic contribution to the corresponding vacuum amplitude $Z_β$ is given by:

$$Z_β = \text{Tr} \left( \exp \left[ 2\pi i τ H_α - 2\pi i τ \mathcal{H}_α \right] g_β \right),$$

where $H_α$ and $\mathcal{H}_α$ are the Hamiltonians corresponding to the left- and right-moving fermions, and the trace is taken over all the states in the $g_α$ twisted sector. The character $Z_α^{g_β}$ is simply given by a product of characters $Z_{αV}^{g_β}$ for each world-sheet fermion:

$$Z_β = \prod_{s=0}^{19} Z_{αV}^{g_β},$$

where we have taken into account that the action of $g_β$ on any given state in the $g_α$ twisted sector amounts to simply multiplying it by a phase. This follows from the fact that here we are considering an Abelian orbifold. The individual characters $Z_{αV}^{g_β}$ are given by (up to overall phases):

$$Z_v^u = \left[ η(τ) \right]^{-1} \sum_{n ∈ \mathbb{Z}} \exp \left( iπ(n - v)^2 τ + 2πinu \right),$$

for a left-moving fermion with boundary conditions $v$ and $u$ around the $a$ and $b$ cycles, and by the complex conjugate of this expression for a right-handed fermion with the same boundary conditions. Here $η(τ)$ is the Dedekind $η$-function.

The world-sheet fermionic contribution to the one-loop partition function is obtained by summing over all possible boundary conditions $α$ and $β$ around the $a$ and $b$ cycles:

$$Z_1 = \sum_{α,β} C_β^{α} Z_β^{α},$$

where $C_β^{α}$ are certain phases which we have omitted in (13), so that $Z_β^{α}$ in (14) are computed using (12) with the characters $Z_{αV}^{g_β}$ defined as in (13) without the phases. Consistency requires $Z_1$ to be modular invariant, i.e., invariant under the modular transformations generated by $τ → τ + 1$ and $τ → -\frac{1}{τ}$.

Thus, let us consider the behavior of $Z_{αV}^{g_β} ≡ Z_β^{α}$ under the modular transformations:
\[
\tau \to -\frac{1}{\tau} : \ Z_{\alpha V, \beta V}^{\alpha V} \to \exp \left( 2\pi i \alpha V \cdot \beta V \right) Z_{\alpha V, \beta V}^{\alpha V}, \quad (15)
\]
\[
\tau \to \tau + 1 : \ Z_{\alpha V, \beta V}^{\alpha V} \to \exp \left( \pi i \alpha V \cdot \alpha V \right) Z_{\alpha V, \beta V}^{\alpha V} - \alpha V + V_0, \quad (16)
\]

where the dot product of two vectors is defined as follows:
\[
\alpha V \cdot \beta V \equiv \sum_{s=0}^{9} \alpha V_s \beta V_s - \sum_{s=10}^{19} \alpha V_s \beta V_s,
\]
that is, this dot product is defined with the Lorentzian signature \((+, -)^{10}\) with the plus and minus signs corresponding to the left- and right-moving components, respectively.

To ensure that \(Z_1\) is modular invariant, we must require that
\[
\tau \to -\frac{1}{\tau} : \ C_{\alpha V, \beta V}^{\alpha V} \exp \left( 2\pi i \alpha V \cdot \beta V \right) = C_{\beta V, -\alpha V}^{\beta V}, \quad (18)
\]
\[
\tau \to \tau + 1 : \ C_{\alpha V, \beta V}^{\alpha V} \exp \left( \pi i \alpha V \cdot \alpha V \right) = C_{\alpha V, \beta V}^{\alpha V} - \alpha V + V_0, \quad (19)
\]

B. Multi-loop Modular Invariance and Factorization

The requirements of multi-loop modular invariance and factorization impose additional constraints on these phases. For a given generating set of vectors \(\{V_i\}\), which must satisfy certain constraints, this allows to solve for these phases in terms of certain discrete constants.

The corresponding \(g\)-loop \(Z_g\) reads:
\[
Z_g = \sum_{\alpha_i, \beta_i} \zeta_g C_{\beta_1, \ldots, \beta_g}^{\alpha_1, \ldots, \alpha_g} Z_{\beta_1, \ldots, \beta_g}^{\alpha_1, \ldots, \alpha_g}, \quad (20)
\]
where \(\zeta_g(\tau)\) is a factor that includes time-like and logitudinal modes, (super-)ghost modes and picture-changing operator insertions. \(Z_{\beta_1, \ldots, \beta_g}^{\alpha_1, \ldots, \alpha_g}\) is given by a product of the contributions corresponding to individual left- and right-moving complex fermions. The characters for a left-moving fermion read:
\[
Z_{u_1, \ldots, u_g}^{\alpha_1, \ldots, \alpha_g} = D_g(\tau) \sum_{\{n_i\} \in \mathbb{Z}^g} \exp \left( i\pi (n_i - v_i)(n_j - v_j) \tau_{ij} + 2\pi i n_i u_i \right), \quad (21)
\]
where \(D_g(\tau)\) is a certain function of \(\tau_{ij}\) whose precise form is not going to be important in the following.

From (21) we see that the characters \(Z_{\beta_1, \ldots, \beta_g}^{\alpha_1, \ldots, \alpha_g}\) indeed factorize:
\[
Z_{\beta_1, \ldots, \beta_g}^{\alpha_1, \ldots, \alpha_g}(\tau, \overline{\tau}) \to \prod_i Z_{\beta_i}^{\alpha_i}(\tau_{ii}, \overline{\tau}) \quad (22)
\]

Thus, for \(Z_g\) to factorize we must require that the phases satisfy the following condition:
\[
C_{\beta_1, \ldots, \beta_g}^{\alpha_1, \ldots, \alpha_g} \equiv \prod_i C_{\beta_i}^{\alpha_i} \quad (23)
\]

Next, let us consider requirements that higher genus modular invariance imposes on the phases \(C_{\beta}^{\alpha}\). It fact, it is is sufficient to consider these constraints at \(g = 2\) as they are
powerful enough to solve for the phases $C_{\beta}^\alpha$. Once these phases are obtained using the genus-two constraints, it is straightforward to check that higher genus constraints are also satisfied. At genus two we have

$$Z_2 = \sum_{\alpha_i, \beta_i} \zeta_2 C_{\beta_1}^{\alpha_1} C_{\beta_2}^{\alpha_2} Z_{\beta_1, \beta_2}^{\alpha_1, \alpha_2}. \quad (24)$$

Now consider the $Sp(4, \mathbb{Z})$ modular transformation $\tau_{12} \rightarrow \tau_{12} + 1$. The corresponding constraint reads:

$$C_{\beta}^{\alpha V} C_{\gamma V}^{\delta} \exp \left(2\pi i \left[ \alpha V \cdot \gamma V + \alpha (s + \bar{s}) + \gamma (s + \bar{s}) \right] \right) = C_{\beta}^{\alpha V} C_{(\beta - \gamma) V}^{\alpha V}. \quad (25)$$

Here the phase $2\pi i \alpha V \cdot \gamma V$ comes from the modular transformation of the character $Z_g$ corresponding to $\Psi^r$. However, the phase $2\pi i [\alpha (s + \bar{s}) + \gamma (s + \bar{s})]$ comes from the modular transformation of $\zeta_2$.

The two-loop modular invariance constraints (25) together with the one-loop modular invariance constraints (18) plus (19) are restrictive enough to solve for the phases $C_{\beta}^{\alpha V}$. The most general solution reads:

$$C_{\beta}^{\alpha V} = \exp \left[2\pi i \beta \phi (\alpha) + \alpha (s + \bar{s}) \right], \quad (26)$$

where

$$\phi_{i}(\alpha) \equiv \sum_j k_{ij} \alpha_j + s_i + \bar{s}_j - V_i \cdot \alpha V, \quad (27)$$

and the structure constants $k_{ij} = 0, 1/2$ must satisfy the following constraints:

$$k_{ij} + k_{ji} - V_i \cdot V_j = 0 \pmod{1}, \quad (28)$$

$$k_{ii} + k_{i0} + s_i + \bar{s}_i - \frac{1}{2} V_i \cdot V_i = 0 \pmod{1}. \quad (29)$$

These constraints together with (28) and (27) are necessary and sufficient to guarantee multi-loop modular invariance. Here the factorized form of (23) is understood.

### III. NON-SUPERSYMMETRIC FREE FERMIONIC STRING MODELS AND ONE-LOOP COSMOLOGICAL CONSTANT

In this section, we present a class of non-supersymmetric string models constructed in the free fermionic framework. In spite of the fact that the models are non-supersymmetric, there is an equal number of bosons and fermions at each mass level. This implies that not just the one-loop contribution to the cosmological constant $\Lambda$ is zero, but that the one-loop amplitude vanishes point by point in the moduli-space of Riemann surfaces. In contrast to Ref. [1] in which the one-loop contribution vanishes because of the non-Abelian nature of the orbifold, the one-loop vanishing in this class of models is due to the fermionic zero modes.

Let us start with Type II string theory compactified on $T^6$. As discussed before, the vectors $V_1$ and $V_2$ are always present in the generating set $\{V_i\}$. Therefore, before orbifolding,
the model has $\mathcal{N} = (4, 4)$ supersymmetry. Let us denote the corresponding one-loop partition function by $Z_0$. Consider orbifolding Type II theory by two commuting $\mathbb{Z}_2$ elements $f$ and $g$, the one-loop partition function is given by

$$Z = \frac{1}{4} \sum_{k,l,m,n} Z(f^k g^l, f^m g^n)$$

(30)

where $f^k g^l$ and $f^m g^n$ are the twists in the $a$- and $b$-cycle respectively.

Suppose that the term $Z(f, g)$ vanishes, then by one-loop modular transformation, the terms $Z(f, f^m g)$, $Z(g, f^m g)$, $Z(f^k g, f)$ and $Z(fg^l, g)$ also vanish. The expression can then be simplified to:

$$Z = \frac{1}{2} Z_f + \frac{1}{2} Z_g + \frac{1}{2} Z_{fg} - \frac{1}{2} Z_0$$

(31)

where $Z_f$, $Z_g$ and $Z_{fg}$ are the one-loop partition functions of the $\mathbb{Z}_2$ orbifolds generated by $f$, $g$ and $fg$ respectively. Clearly, $Z_0$ is zero because of supersymmetry. If the individual $\mathbb{Z}_2$ orbifolds (generated by $f$, $g$ and $fg$ respectively) are supersymmetric, then the total one-loop partition function is identically zero. So the strategy is to find a pair of twists $f$ and $g$ with the above properties.

The generating vectors that are already present in the original $\mathcal{N} = 8$ model are given by:

$$V_0 = \left[ -\frac{1}{2} \left( -\frac{1}{2} - \frac{1}{2} - \frac{1}{2} \right) \right]$$

$$V_1 = \left[ -\frac{1}{2} \left( -\frac{1}{2} \right) \right]$$

$$V_2 = \left[ 0 \right]$$

(32) (33) (34)

The $\mathbb{Z}_2$ elements $f$ and $g$ can be represented by the additional generating vectors $V_3$ and $V_4$. The constraints from multi-loop modular invariance discussed in the previous section imply:

(i) \hspace{1cm} $V_i \cdot V_1 = 0, 1/2 \pmod{1}$, \hspace{1cm} $V_i \cdot V_2 = 0, 1/2 \pmod{1}$, \hspace{1cm} for $i=3,4$.

(ii) \hspace{1cm} $V_3^2, V_4^2 \in \mathbb{Z}$.

(iii) \hspace{1cm} $V_3 \cdot V_4 = 0, 1/2 \pmod{1}$.

Without loss of generality, we can take $s_i = \overline{s}_i = 0$. This is because $V_1$ and $V_2$ are among the generating vectors, and adding these vectors changes the spin statistics. The supercurrent constraint (Eq.(4)) and (i) restrict the form of $V_i = (v_{Li} \mid v_{Ri})$. Let us consider $v_{Li}$ for the moment (as the analysis for $v_{Ri}$ is completely parallel), it takes one of the following forms:

$$V_i \cdot V_1 = 0 : \hspace{1cm} v_{Li} = \left[ 0 \ (0 \ a \ a) (0 \ b \ b) (0 \ c \ c) \right] .$$

(35)

$$V_i \cdot V_1 = \frac{1}{2} : \hspace{1cm} v_{Li} = \left[ 0 \left( -\frac{1}{2} - \frac{1}{2} - \frac{1}{2} \right) \right] .$$

(36)
where \(a, b, c, d = 0\) or \(1/2\).

One can also include in the orbifold twist \((f\) and \(g)\) an action of \((-1)^{F_L}\) which acts with a \((-1)\) on all spacetime spinors coming from the left-moving degrees of freedom. This is equivalent to turning on a discrete torsion \([6]\) since in the left-moving Ramond sector the orbifold projection has the opposite sign from what it would have without the \((-1)^{F_L}\) action. The discrete torsion is determined by the choice of the structure constants \(k_{ii}\) which can take values \(0\) or \(1/2\). Choosing \(k_{1i} = 1/2\) is equivalent to including the \((-1)^{F_L}\) action in the orbifold twist. Similarly, one can include the \((-1)^{F_R}\) action in the orbifold twist by choosing \(k_{2i} = 1/2\).

In the case where \(V_i \cdot V_1 = 0\), the number of supersymmetries remains after orbifolding depends on the discrete torsion. The 4 gravitinos in the R-NS sector remain in the spectrum if \(k_{1i} = 0\) (hence \(N_L = 4\)) but are projected out if \(k_{1i} = 1/2\) (hence \(N_L = 0\)). For \(V_i \cdot V_1 = 1/2\), the number of supersymmetries broken by the orbifold twist is independent of the discrete torsion. The choice of \(k_{1i}\) simply determines which two gravitinos are projected out. The number of supersymmetries remains is \(N_L = 2\).

The individual \(Z_2\) orbifolds (generated by \(f\), \(g\) and \(fg\)) are supersymmetric whereas the resulting \(Z_2 \otimes Z'_2\) orbifold is non-supersymmetric. By counting the number of gravitinos, it is easy to see from Eq.(31) that there are two independent classes of models. In the first class of models, the orbifold twists \(f\), \(g\) and \(fg\) break \(N = (4, 4)\) supersymmetry to \(N = (2, 0), (0, 2)\) and \(2, 2)\) respectively. The model presented in Ref. [1] (and also its variant in Ref. [3]) belongs to this class. In the other class of models, the orbifold twists \(f\), \(g\) and \(fg\) break \(N = (4, 4)\) supersymmetry to \(N = (2, 0), (0, 4)\) and \(2, 0)\) respectively.

Before we consider the non-supersymmetric models in detail, let us briefly discuss the massless spectrum of the original \(N = 8\) supersymmetric model to set up our notation. There are 4 sectors that give rise to massless states: 0 sector \((i.e.,\) NS-NS sector) and \(V_1 + V_2\) sector \((i.e.,\) R-R sector) give rise to spacetime bosons, \(V_1\) sector \((i.e.,\) R-NS sector) and \(V_2\) sector \((i.e.,\) NS-R sector) give rise to spacetime fermions. The graviton \(G_{ij}\), the antisymmetric tensor \(B_{ij}\) and the dilaton \(\phi\) come from the NS-NS sector. In addition, there are \(U(1)^6 \otimes U(1)^6\) gauge bosons and 36 real scalars in this sector due to the compactification on \(T^6\). Therefore the NS-NS sector has 64 bosonic degrees of freedom. The R-R sector also provides \(U(1)^{16}\) gauge fields. They differ from that in the NS-NS sector in that they are obtained by tensoring the left- and right-moving spinor states. This gives rise to \(U(1)^{16}\) gauge bosons and 32 scalars, a total of 64 bosonic degrees of freedom. Because of the \(N_L = 4\) supersymmetry, there are 4 gravitinos in the R-NS sector. Together with the 28 spinors in this sector, they provide 64 fermionic degrees of freedom. The spectrum in the NS-R sector is similar to that in the R-NS sector \((with\ the\ left-\ and\ right-moving\ quantum\ numbers\ interchanged)\). Hence it also gives rise to \(N_L = 4\) fermionic degrees of freedom. The one-loop partition function \(Z_0\) vanishes because of supersymmetry. Therefore, the number of bosonic and fermionic degrees of freedom are equal at each mass level. In particular, we have seen that this is the case at the massless level.

Let us now turn to the non-supersymmetric models:

**A. Class I**

The general forms of \(V_3\) and \(V_4\) are given by:
generating vectors). It turns out there are two models in this class (plus their variations):

\[ V_3 = \left[ 0 \left( -\frac{1}{2} - \frac{1}{2} 0 \right)^2 (0 d_3 d_3) || 0 (0 a_3 a_3)(0 b_3 b_3)(0 c_3 c_3) \right], \]  
\[ V_4 = \left[ 0 (0 a_4 a_4)(0 b_4 b_4)(0 c_4 c_4) || 0 \left( -\frac{1}{2} - \frac{1}{2} 0 \right)^2 (0 d_4 d_4) \right]. \]  

The structure constants \( k_{23} = 1/2, k_{14} = 1/2 \), whereas the other \( k_{ij} \) are not fixed by the supersymmetries of the individual \( \mathbb{Z}_2 \) orbifolds (these \( k_{ij} \) determine which two of the gravitinos are projected out and which two are kept. Each choice corresponds to a different GSO projection). This implies that with the same orbifold twists, there exists more than one model. This point will be crucial later on when we discuss the issue of duality. Here, \( a_i, b_i, c_i \) and \( d_i \) are chosen so that \( V_i^2 \in \mathbb{Z} \) and \( V_3 \cdot V_4 = 0 \) or \( 1/2 \) (mod 1). The one-loop partition function of the \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) orbifold is zero if we demand the contribution of the one-loop diagram with \( f \) twist on the \( a \)-cycle and \( g \) twist on the \( b \)-cycle (for all space-time spin structure) to vanish. In other words,

\[ Z(V_3 + \sum_{i=0}^{2} \alpha_i V_i, V_4 + \sum_{i=0}^{2} \beta_i V_i) = 0 \]  

for all \( \alpha_i \) and \( \beta_i \). This can be achieved by demanding that at least one of the 20 complex worldsheet fermions (10 left-moving and 10 right-moving) has periodic boundary conditions in both the \( a \)-cycle and the \( b \)-cycle. In other words,

\[ V_3^a + \sum_{i=0}^{2} \alpha_i V_i^a = V_4^a + \sum_{i=0}^{2} \beta_i V_i^a = \frac{1}{2}, \]  

for all \( \alpha_i \) and \( \beta_i \). Then the one-loop diagram is proportional to \( \theta [1/2, 1/2] \) which vanishes because of the fermionic zero mode.

There are many choices of \( a_i, b_i, c_i \) and \( d_i \) which satisfy the constraints, but not all choices are independent (since \( V_0, V_1 \) and \( V_2 \) are always in the generating set, adding linear combinations of these vectors to \( V_3 \) and \( V_4 \) gives rise to a seemingly different but equivalent set of generating vectors). It turns out there are two models in this class (plus their variations):

- **Model IA**

\[ V_3 = \left[ 0 \left( -\frac{1}{2} - \frac{1}{2} 0 \right)^2 (0000) || 0 (0 - \frac{1}{2} - \frac{1}{2} 0)^2 (0000) \right], \]
\[ V_4 = \left[ 0 (0 - \frac{1}{2} - \frac{1}{2} 0)^2 (0000) || 0 \left( -\frac{1}{2} - \frac{1}{2} 0 \right)^2 (0000) \right]. \]

The twists \( f \) and \( g \) are left-right mirror of each other, i.e. \( v_{3L} = v_{4R} \) and \( v_{3R} = v_{4L} \). Therefore, the spectrum of the \( \mathbb{Z}_2 \) orbifold generated by \( f \) can be obtained from that of \( g \) by interchanging the left- and right-moving quantum numbers. In what follows, the \( \mathbb{Z}_2 \) orbifold refers to the orbifold generated by \( f \).

The untwisted NS-NS sector (0 sector) gives rise to the graviton \( G_{ij} \), the antisymmetric tensor \( B_{ij} \) and the dilaton \( \phi \); all of them survive both \( f \) and \( g \) projection. In addition, there are \( U(1)^6 \otimes U(1)^2 \) gauge bosons and 12 scalars in the \( \mathbb{Z}_2 \) orbifold. The \( g \) twist further projects out 4 of the \( U(1)^6 \) gauge bosons and 8 of the scalars. Therefore, the \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) orbifold has \( U(1)^2 \otimes U(1)^2 \) gauge bosons and 4 scalars.
The untwisted R-R sector ($V_1 + V_2$ sector) also gives rise to $U(1)$ gauge fields. They differ from those in the NS-NS sector in that they are obtained by tensoring the left- and right-moving spinor states. As a result, there are $U(1)^8$ gauge bosons and 16 scalars in the $\mathbb{Z}_2$ orbifold. The $g$ twist projects out half of the spectrum and so there are $U(1)^4$ gauge bosons and 8 scalars in the R-R sector of the non-supersymmetric model.

The $f$ twist breaks the $\mathcal{N} = (4,4)$ supersymmetry to $\mathcal{N} = (2,0)$. There are therefore 2 gravitinos from the R-NS sector ($V_1$ sector), but not in the NS-R sector ($V_2$ sector). In addition, there are some spinors in both $V_1$ and $V_2$ sector. Notice that the number of bosonic and fermionic degrees of freedom cancel among the untwisted sector states even in the non-supersymmetric $\mathbb{Z}_2 \otimes \mathbb{Z}'_2$ orbifold.

There are also massless twisted sector states. In the $\mathbb{Z}_2$ orbifold generated by $f$, the $f$ twisted NS-NS sector ($V_3$ sector) provides 64 scalars. Their fermionic superpartners come from the $f$ twisted R-NS sector ($V_3 + V_1$) since the left-moving supersymmetry is unbroken. Upon further projection by the $g$ twist, there are 32 scalars in the $V_3$ sector. Notice that even the left-moving gravitinos are projected out by $g$, the fermionic superpartners of the 32 scalars still come from $V_3 + V_1$ sector.

There are other massless twisted sector states in the non-supersymmetric model that are absent in the $\mathbb{Z}_2$ orbifold. They are the massless states coming from the $fg$ twisted sector. In Model IA, they come from the sectors $V_3 + V_4 + \alpha_1V_1 + \alpha_2V_2$ for $\alpha_i = 0, 1$. Again, there are equal number of bosons and fermions among those sectors. The massless spectrum of the model is summarized in Table [II]. Here we emphasize that Model IA refers to more than one model:

- (i) We leave the structure constants $k_{13}$ and $k_{24}$ unspecified. Different choices of $k_{13}$ and $k_{24}$ correspond to different GSO projections, and hence the internal quantum numbers of the states that are kept are different. The quantum numbers can be worked out easily by using the spectrum generating formula in Ref. [5]. However, the resulting spectra have the same counting of the states, i.e., the models have the same number of scalars, spinors and $U(1)$ gauge bosons from each sector. We will see in Section [V] that models with different $k_{13}$ and $k_{24}$ have different duals.

- (ii) The last $T^2$ are not touched by the $\mathbb{Z}_2$ twists $f$ and $g$. We can include shifts in this $T^2$ without affecting the supersymmetries of the individual $\mathbb{Z}_2$ orbifolds as well as the bose-fermi degeneracy feature of the final $\mathbb{Z}_2 \otimes \mathbb{Z}'_2$ orbifold. We will consider some variations by including shifts in the following. The massless spectra for the models with shifts are slightly different from the one without shift (some twisted sector states become massive). Nonetheless, including shifts will not affect our discussions of duality in Section [V].

Let us consider some variations of the above model. In the above, we have assumed that the worldsheet fermions can always be written in the complex basis. This is not necessary for worldsheet consistencies if only $\mathbb{Z}_2$ twists are involved. As mentioned before in Section [II] the complex fermions can be split into pairs of real fermions with different boundary conditions, as long as the cubic constraint (Eq. (10)) is satisfied. This opens up the possibilities for other models. Consider a variation of Model IA with the following generating vector:

$$V_3 = \left[ 0 \left( -\frac{1}{2} - \frac{1}{2} \right)^2 (000), (0 - \frac{1}{2} - \frac{1}{2})_r, |0 \left( 0 - \frac{1}{2} - \frac{1}{2} \right)^2 (000), (0 - \frac{1}{2} - \frac{1}{2})_r \right],$$

$$V_4 = \left[ 0 \left( 0 - \frac{1}{2} - \frac{1}{2} \right)^2 (0 - \frac{1}{2} - \frac{1}{2})_r, (000), |0 \left( -\frac{1}{2} - \frac{1}{2} - \frac{1}{2} \right)^2 (0 - \frac{1}{2} - \frac{1}{2})_r, (000) \right].$$

where the subscript $r$ indicates that the corresponding worldsheet fermions are real, the
worldsheet fermions are complex otherwise. Clearly, the cubic constraint is satisfied. This model is related to the model presented in Ref. [4]. This can be seen as follows. Since we have fermionized a real boson \( \phi \) into a pair of real fermions \( \psi_1 \) and \( \psi_2 \):

\[
\psi_1 + i \psi_2 = e^{i\phi}
\]  

(45)

The \( Z_2 \) twist on the fermions: \( \psi_1 \to -\psi_1 \) and \( \psi_2 \to -\psi_2 \) corresponds to a \( Z_2 \) shift on the boson: \( \phi \to \phi + \pi \). The \( Z_2 \) twist on the fermions: \( \psi_1 \to \psi_1 \) and \( \psi_2 \to -\psi_2 \) corresponds to a \( Z_2 \) twist on the boson: \( \phi \to -\phi \). In the orbifold language, the \( f \) and \( g \) twists can be written as:

\[
f = \left[ (-1^4, 1^2), (1^6), (0^4, 0, s), (s^4, 0, s), (-1)^F_R \right],
\]

(46)

\[
g = \left[ (1^6), (-1^4, 1^2), (s^4, s, 0), (0^4, s, 0), (-1)^F_L \right].
\]

(47)

Here, the element of the space group of the orbifold is denoted by

\[
[(\theta_L), (\theta_R), (v_L), (v_R), \Theta]
\]

(48)

where \( \theta_{L,R} \) are rotations by elements of \( SO(6) \), and \( v_{L,R} \) are shifts acting on the left- and right-moving degrees of freedom respectively. The eigenvalue of \( \Theta \) can be 1 or \( (-1)^F_{L,R} \) depending on whether there is discrete torsion.

Notice that the variation (corresponding to a shift on the last circle) does not affect the supersymmetries of the individual \( Z_2 \) orbifolds generated by \( f \), \( g \) and \( fg \), as well as the final non-supersymmetric \( Z_2 \otimes Z_2' \) orbifold (since the extra shift has no effect on the gravitinos). Furthermore, the requirement for vanishing one-loop cosmological constant (Eq. (40)) is preserved. The multi-loop analysis (see Section V) also remains unchanged by the modification.

The model has the same massless untwisted sector spectrum as Model IA. However, some of the twisted sector states become massive because of the extra shift. The massless twisted sector states come only from the following 4 sectors: \( V_3 + V_4 + V_0 + \sum_{i=1}^2 \alpha_i V_i \) where \( \alpha_i = 0, 1 \). The \( V_3 + V_4 + V_0 \) sector provides \( U(1)^8 \) gauge bosons and 16 real scalars, \( V_3 + V_4 + V_0 + V_1 + V_2 \) provides 32 scalars. The sectors \( V_3 + V_4 + V_0 + V_1 \) and \( V_3 + V_4 + V_0 + V_2 \) each provides 16 spinors. The number of bosonic and fermionic degrees of freedom are equal in the twisted sectors.

We can construct yet another variation of Model IA with the following generating vectors:

\[
V_3 = \left[ 0 \left( -\frac{1}{2} - \frac{1}{2} \right)^2 (000)_r, (0 - \frac{1}{2} - \frac{1}{2})_r | 0 \left( 0 - \frac{1}{2} - \frac{1}{2} \right)^2 (000)_r, (0 - \frac{1}{2} - \frac{1}{2})_r \right],
\]

(49)

\[
V_4 = \left[ 0 \left( 0 - \frac{1}{2} - \frac{1}{2} \right)^2 (000)_r, (0 - \frac{1}{2} - \frac{1}{2})_r | 0 \left( -\frac{1}{2} - \frac{1}{2} 0 \right)^2 (000)_r, (0 - \frac{1}{2} - \frac{1}{2})_r \right].
\]

(50)

which in the orbifold language corresponds to:

\[
f = \left[ (-1^4, 1^2), (1^6), (0^4, 0, s), (s^4, 0, s), (-1)^F_R \right],
\]

(51)

\[
g = \left[ (1^6), (-1^4, 1^2), (s^4, s, 0), (0^4, s, 0), (-1)^F_L \right].
\]

(52)

A similar model has been considered in Ref. [3]. Again, it has the same untwisted sector massless spectrum. The twisted sector massless states come from the sectors \( V_3 + V_4 + \sum_{i=1}^2 \alpha_i V_i \). The number of bosonic and fermionic degrees of freedom are equal in the twisted sectors.
In Model IA and two of its variations above, the radii of the last $T^2$ are not fixed by the asymmetric orbifold. Changing the radii does not change the bose-fermi degeneracy feature of the resulting $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ models, and hence they are free moduli. Starting from the model constructed above by the free fermionic approach, which fixes the radii to be $1$, we can freely change the complex and the Kähler moduli of the last $T^2$ while maintaining the bose-fermi degeneracy.

Clearly, many more variations of the above model with the bose-fermi degenerate feature can be constructed using real worldsheet fermions. It would be interesting to work out the models in detail, and perhaps classify all model with this feature.

• Model IB

\[
V_3 = \left[ 0 \left( -\frac{1}{2} - \frac{1}{2} 0 \right)^2 (000) || 0 \left( 0 - \frac{1}{2} - \frac{1}{2} \right)^2 (000) \right],
\]

\[
V_4 = \left[ 0 (000) \left( 0 - \frac{1}{2} - \frac{1}{2} \right)^2 || 0 (000) \left( -\frac{1}{2} - \frac{1}{2} 0 \right)^2 \right].
\]

The massless spectrum of the model is given in Table II. It is identical to that of Model IA except that the $f g$ twisted sector states come from the $V_3 + V_4 + V_0 + \alpha_1 V_1 + \alpha_2 V_2$ sectors.

Similarly, one can construct variations of the above model by using real worldsheet fermions. An interesting feature of this model is that all the radii of $T_6$ are fixed by the asymmetric orbifold (as compared with Model IA in which the radii of the last two component of $T_6$ are free) and yet gives rise to the same counting of the massless spectrum as Model IA.

B. Class II

The general forms of $V_3$ and $V_4$ are given by:

\[
V_3 = \left[ 0 \left( -\frac{1}{2} - \frac{1}{2} 0 \right)^2 (0 d_3 d_3) || 0 (a_3 a_3) (0 b_3 b_3) (0 c_3 c_3) \right],
\]

\[
V_4 = \left[ 0 (0 a_4 a_4) (0 b_4 b_4) (0 c_4 c_4) || 0 (a_4' a_4') (0 b_4' b_4') (0 c_4' c_4') \right].
\]

The structure constants $k_{23} = 1/2$, $k_{14} = 1/2$, $k_{24} = 0$, whereas the other $k_{ij}$ are not fixed by the supersymmetries of the individual $\mathbb{Z}_2$ orbifolds. The monodromies must be chosen such that $V_i^2 \in \mathbb{Z}$, $V_3 \cdot V_4 = 0$ or $1/2$ (mod 1) and Eq. (40) is satisfied. There are three models in this class. The massless spectra of these models are given in the tables. Here, we only list the generating vectors corresponding to the twists $f$ and $g$:

• Model IIA

\[
V_3 = \left[ 0 \left( -\frac{1}{2} - \frac{1}{2} 0 \right)^2 (000) || 0 \left( 0 - \frac{1}{2} - \frac{1}{2} \right)^2 (000) \right],
\]

\[
V_4 = \left[ 0 (0 - \frac{1}{2} - \frac{1}{2})^2 (000) || 0 (000)^3 \right].
\]

The massless spectrum of this model is given in Table III.

\[1\] We use the convention that $P_{L,R} = \frac{m}{2\pi} \pm nR$ for $m, n \in \mathbb{Z}$. 

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• Model IIB

\[ V_3 = \left[ 0 \left( -\frac{1}{2} - \frac{1}{2} 0 \right)^2(000) | 0 \left( 0 - \frac{1}{2} - \frac{1}{2} \right)^2(000) \right], \tag{59} \]
\[ V_4 = \left[ 0 (000)^2 | 0 (000)(0 - \frac{1}{2} - \frac{1}{2})^2 \right]. \tag{60} \]

The massless spectrum of this model is given in Table IV.

• Model IIC

\[ V_3 = \left[ 0 \left( -\frac{1}{2} - \frac{1}{2} 0 \right)^2(000) | 0 \left( 0 - \frac{1}{2} - \frac{1}{2} \right)^2(000) \right], \tag{61} \]
\[ V_4 = \left[ 0 (000)^2 | 0 (000)(0 - \frac{1}{2} - \frac{1}{2})^2 \right]. \tag{62} \]

The massless spectrum of this model is given in Table V.

Similar to the set of models in Class I, one can construct variations of the above models by using real worldsheet fermions. Notice that the structure constant \( k_{13} \) is unspecified, different choices of \( k_{13} \) correspond to different GSO projection, and hence different spectra. However, the number of scalars, spinors and \( U(1) \) gauge bosons from each sector are not altered.

If the \( N \)-loop vacuum amplitude vanishes, the one-, the two-, and the three-point \( N \)-loop amplitudes generically vanish as well. In particular, since the mass corrections come from the two-point \( N \)-loop amplitudes, the vanishing \( N \)-loop vacuum amplitude implies vanishing mass corrections. If the cosmological constant vanishes perturbatively, the tree level mass spectrum is not corrected perturbatively.

There are massless twisted sector states in this set of models (in both Class I and II). Despite the fact that the model is non-supersymmetric (since the gravitinos are projected out), the twisted sector states seem to fall into appropriate supermultiplets (depending on the number of supersymmetries that survive the orbifold twist of the sector under consideration). Whether the twisted sector states indeed assemble themselves into supermultiplets can be confirmed with an explicit calculation of the couplings between the various massless states. This may not be surprising since the gravitinos are projected out by the discrete torsions \( (-1)^F_L \) (associated with the \( g \) twist) and \( (-1)^F_R \) (associated with the \( f \) twist). If at least one of the discrete torsions is absent, then the model is supersymmetric. Each \( \mathbb{Z}_2 \) twist projects out half of the spectrum. The discrete torsion interchanges the role of the half which is projected out and the other half which is kept. The number of bosons as well as the number of fermions in each half of the spectrum are nonetheless the same. Hence the bose-fermi degeneracy in the non-supersymmetric model has a supersymmetric origin. Even though the 4-dimensional gravitinos \( \chi_a^i \) are all projected out with the particular choice of discrete torsion, the same number of fermionic degrees of freedom is restored by the spinors \( \chi_a^i \) that are originally projected out in the supersymmetric model. Upon decompactification of \( T^6 \), these fermions become part of the 10-dimensional gravitinos.

As pointed out in Ref [1,4] and the discussions in the next sections here, at least some of these models are expected to have exactly zero cosmological constants. The vanishing of the cosmological constant in these models has a natural explanation in string theory—it is simply a remnant of the 10-dimensional supersymmetry that defines Type II string
theory. In 4-dimensional supersymmetric string models, the remnant of the 10-dimensional supersymmetry lives in the spacetime sector, yielding lower spacetime supersymmetry. Here, the remnant of the 10-dimensional supersymmetry somehow lives inside the internal space, yielding non-supersymmetric models. The exact form of this remnant symmetry remains to be understood.

IV. DUALITY AND SELF-DUALITY

Finally we are ready to address the issue of duality of Model IA. As emphasized before, Model IA refers to more than one model since some of the discrete torsions which do not affect the counting of the spectrum (corresponding to the values of $k_{13}$ and $k_{24}$) are unspecified. However, these discrete torsions turn out to be crucial when we address the issue of duality because models with different discrete torsions have different duals. In particular, there are three independent choices: (i) Model IA$_i$: $k_{13} = 0, k_{24} = 1/2$; (ii) Model IA$_{ii}$: $k_{13} = 1/2, k_{24} = 0$; and (iii) Model IA$_{iii}$: $k_{13} = k_{24} = 0$. (The remaining choice $k_{13} = k_{24} = 1/2$ is related to Model IA$_{iii}$ by a reflection of left- and right-moving quantum numbers). Before we go into the discussion, let us summarize the final conclusion: Model IA$_i$ and IA$_{ii}$ are both self-dual (although the $f$ and $g$ twisted sectors are interchanged by duality in Model IA$_{ii}$). Model IA$_{iii}$ and Model IIA are a strong-weak dual (i.e., $U$-dual) pair. Let us first briefly review the relevant ingredients of the construction of Type II string dual pairs [7] and the particular application used in Ref [4]. We follow closely their notation.

A systematic construction of dual pairs of type II compactifications in $D = 4$ dimensions was discussed by Sen and Vafa [7]. The U-duality group of Type II compactifications on $T^4$ is $SO(5,5; \mathbb{Z})$, while the perturbatively obvious T-duality subgroup is $SO(4,4; \mathbb{Z})$. Consider two elements $h, \tilde{h} \in SO(4,4; \mathbb{Z})$, which are not conjugate in $SO(4,4; \mathbb{Z})$ but which are conjugate in $SO(5,5; \mathbb{Z})$:

$$ghg^{-1} = \tilde{h}, \quad g \in SO(5,5; \mathbb{Z}) \quad (63)$$

The particular $SO(5,5; \mathbb{Z})$ element $\mathcal{G}$ of interest is given in terms of the element $\sigma$ of the ten-dimensional $SL(2, \mathbb{Z})$ symmetry group of Type IIB which inverts the ten-dimensional axion/dilaton field and the T-duality element $\tau_{1234}$ that inverts the volume of the $T^4$:

$$\mathcal{G} = \sigma \cdot \tau_{1234} \cdot \sigma^{-1} \quad (64)$$

This maps fundamental strings (without winding on the $T^4$) to NS fivebranes wrapped on the $T^4$. The element $\mathcal{G}$ has the property that

$$\mathcal{G}h\mathcal{G}^{-1} \in SO(4,4; \mathbb{Z}) \quad (65)$$

for all $h \in SO(4,4; \mathbb{Z})$ [7].

Now, consider compactifying on an additional $T^2$, which we can take to be a product of two circles. We can orbifold the resulting compactifications by $h$ ($\tilde{h}$) acting on the $T^4$. By the adiabatic argument [8], the resulting models in four dimensions are still dual. In fact, the dual models that the adiabatic argument yields will be related by $S - T$ exchange:

$$\tilde{S} = T, \quad \tilde{T} = S \quad (66)$$
where $T$ is the Kähler modulus associated with the $T^2$ and $S$ is the axion-dilaton in four dimensions.

Consider an element $h$ of $SO(4,4;\mathbb{Z})$ which acts on $X_L^{1...4}, X_R^{1...4}$ as pairwise rotation, labeled by the four angles $(\theta_L, \phi_L, \theta_R, \phi_R)$. Then $\mathcal{G}$ conjugates $h$ to $\tilde{h}$ which acts on $X_L^{1...4}, X_R^{1...4}$ as $(\tilde{\theta}_L, \tilde{\phi}_L, \tilde{\theta}_R, \tilde{\phi}_R)$ where

\[
\begin{pmatrix}
\tilde{\theta}_L \\
\tilde{\phi}_L \\
\tilde{\theta}_R \\
\tilde{\phi}_R
\end{pmatrix} = \begin{pmatrix}
1/2 & -1/2 & 1/2 & -1/2 \\
-1/2 & 1/2 & 1/2 & -1/2 \\
1/2 & 1/2 & 1/2 & 1/2 \\
-1/2 & -1/2 & 1/2 & 1/2
\end{pmatrix} \begin{pmatrix}
\theta_L \\
\phi_L \\
\theta_R \\
\phi_R
\end{pmatrix}
\] (67)

is the equation that will yield the Type II duals of our orbifolds.

Now let us concentrate on the action on the first $T^4$. In the above notation, $V_3$ in Model IA (or the KKS model) can be represented as

\[
f = (\pi, \pm \pi, 2\pi, 0),
\] (68)

where the $\pi$ corresponds to a $\mathbb{Z}_2$ twist, $(-1)^F_R$ (i.e., the discrete torsion from the structure constant $k_{23} = 1/2$) is represented by a $2\pi$ rotation on right movers, and the choice $-\pi$ corresponds to a $\mathbb{Z}_2$ twist with a discrete torsion $(-1)^F_L$ (i.e., the discrete torsion from the structure constant $k_{13} = 1/2$). The 0 implies no twist but still there can be shifts. In both cases in Eq. (68), the left-moving supersymmetry is broken to $\mathcal{N} = 2$ (the discrete torsion determines which 2 of the gravitinos are kept). Therefore, in the previous section, we leave it unspecified. However, these two cases transform differently under the duality transformation Eq. (67), which we will now discuss.

For $f = (\pi, \pi, 2\pi, 0)$, from the action of Eq. (67), we see that $\tilde{f} = f$, implying that the $(2,0)$ model ($\mathcal{N}_L = 2$ and $\mathcal{N}_R = 0$) is self-dual. On the other hand, if $f = (\pi, -\pi, 2\pi, 0)$, then $\tilde{f} = (2\pi, 0, \pi, \pi)$. Therefore, the $(2,0)$ model is mapped to the $(0,2)$ model.

Similarly, one can consider the duality transformation of the other $\mathbb{Z}_2$ twists $g$ and $fg$. It turns out there are three independent cases for the resulting $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ orbifold:

- (i) $f = (\pi, \pi, 2\pi, 0)$, $g = (2\pi, 0, \pi, -\pi)$ and $fg = (-\pi, \pi, -\pi, -\pi)$.

Duality transformation Eq. (67) maps them to $\tilde{f} = f$, $\tilde{g} = g$ and $\tilde{fg} = fg$. The $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ orbifold is self-dual. This is the dual pair (KKS Model) considered in Ref. [4]. With the appropriate choices of $k_{ij}$ (i.e., $k_{13} = 0$ and $k_{24} = 1/2$), Model IA also belongs to this type.

Duality transformation Eq. (67) maps them to $\tilde{f} = f$, $\tilde{g} = g$ and $\tilde{fg} = fg$. The $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ orbifold is self-dual. This is the dual pair (KKS Model) considered in Ref. [4]. With the appropriate choices of $k_{ij}$ (i.e., $k_{13} = 0$ and $k_{24} = 1/2$), Model IA also belongs to this type.
Let us denote the corresponding model by Model IA$_i$.

- (ii) $f = (\pi, -\pi, 2\pi, 0)$, $g = (2\pi, 0, \pi, \pi)$ and $fg = (-\pi, -\pi, -\pi, \pi)$.

$N = (4, 4)$

\[
\begin{array}{ccc}
& f & g \\
N = (2, 0) & (2, 2) & (0, 2) \\
(0, 0) & & \\
& fg & \\
\end{array}
\]

\[
\begin{array}{ccc}
& \tilde{f} & \tilde{g} \\
N = (2, 0) & (2, 2) & (2, 0) \\
(0, 0) & & \\
& f \tilde{g} & \\
\end{array}
\]

Duality transformation maps them to $\tilde{f} = g$, $\tilde{g} = f$ and $f \tilde{g} = fg$. This corresponds to $k_{13} = 1/2$ and $k_{24} = 0$. Let us denote the corresponding $\mathbb{Z}_2 \otimes \mathbb{Z}_2'$ orbifold by Model IA$_{ii}$. The model is still self-dual, but the $f$ and $g$ twist sectors are interchanged.

- (iii) $f = (\pi, \pi, 2\pi, 0)$, $g = (2\pi, 0, \pi, \pi)$ and $fg = (-\pi, \pi, -\pi, \pi)$.

$N = (4, 4)$

\[
\begin{array}{ccc}
& f & g \\
N = (2, 0) & (2, 2) & (0, 2) \\
(0, 0) & & \\
& fg & \\
\end{array}
\]

\[
\begin{array}{ccc}
& \tilde{f} & \tilde{g} \\
N = (2, 0) & (0, 4) & (2, 0) \\
(0, 0) & & \\
& f \tilde{g} & \\
\end{array}
\]

Duality transformation maps them to $\tilde{f} = f$, $\tilde{g} = (\pi, -\pi, 2\pi, 0)$ and $f \tilde{g} = (2\pi, 0, 0, 0)$. This corresponds to $k_{13} = k_{24} = 0$. We denote the corresponding $\mathbb{Z}_2 \otimes \mathbb{Z}_2'$ orbifold by Model IA$_{iii}$. In this case, the dual pair is more non-trivial. In the dual model, the twists $\tilde{f}$, $\tilde{g}$ and $f \tilde{g}$ break supersymmetry to $\mathcal{N} = (2, 0)$, $\mathcal{N} = (2, 0)$ and $\mathcal{N} = (0, 4)$ respectively. Since duality maps the $(2, 2)$ model to the $(0, 4)$ model, the $U(1)$ gauge fields from the NS-NS sector in one model would be mapped to those in the R-R sector of its dual.

Notice that even though the duality transformation can map the individual $\mathbb{Z}_2$ orbifolds to orbifolds with different supersymmetries, the resulting model $\mathbb{Z}_2 \otimes \mathbb{Z}_2'$ orbifold is still non-supersymmetric. This can be seen from Eq. (31). In particular, consider the contribution of the gravitinos in the partition function:

\[
(\mathcal{N}_L, \mathcal{N}_R) = \frac{1}{2} (2, 0) + \frac{1}{2} (0, 2) + \frac{1}{2} (2, 2) - \frac{1}{2} (4, 4) = (0, 0)
\]

\[
\rightarrow \frac{1}{2} (2, 0) + \frac{1}{2} (0, 2) + \frac{1}{2} (2, 2) - \frac{1}{2} (4, 4) = (0, 0)
\]
Therefore, the dual model is also non-supersymmetric. Furthermore, if the dual model satisfies also the requirement Eq. (40), then the model and its dual both have vanishing one-loop cosmological constant. This requirement is trivially satisfied for cases (i) and (ii), but a priori, it may not be satisfied in case (iii). However, a closer examination of the structure clearly shows that the dual model also has bose-fermi degeneracy.

To find its dual, we first notice that the $T^2$ in Model IA is untouched, meaning that it is toroidally compactified. To find its dual, we need to have appropriate twists such that the same $T^2$ is untouched. A natural candidate for the dual of Model IA$_{iii}$ with the above mentioned properties is Mode IIA. In Model IIA, the generating vectors $V_3$, $V_4$ and $V_3 + V_4$ (corresponding to three different $\mathbb{Z}_2$ twists) break supersymmetry to $\mathcal{N} = (2, 0), (0, 4)$ and $(2, 0)$ respectively. We can identify $\tilde{f}$ with $V_3$, $\tilde{g}$ with $V_3 + V_4$ and $\tilde{f} \tilde{g}$ with $V_4$. As we have shown in the previous section, the requirement Eq. (40) is satisfied in Model IIA and so it has vanishing one-loop cosmological constant. Thus, the dual of Model IA$_{iii}$ is a different orbifold but yet exhibits the same perturbative vanishing of the cosmological constant as the the original model. The massless spectrum of Model IIA is given in Table III. Despite the fact that Model IA$_{iii}$ and Model IIA have the same counting of states, a careful examination of the spectrum (using the spectrum generating formula in Ref. [5]) indicates that they have different internal quantum numbers, and hence different spectra.

Let us summarize the duality relations between different models as follows:

\begin{center}
\begin{tikzpicture}
  \node (IA) at (0,0) {IA$_i$};
  \node (IAii) at (1,0) {IA$_{ii}$};
  \node (IAiii) at (2,0) {IA$_{iii}$};
  \node (IIA) at (3,0) {IIA};
  \draw[->] (IA) to [bend left = 45] (IAii);
  \draw[->] (IAii) to [bend left = 45] (IAiii);
  \draw[->] (IAiii) to [bend left = 45] (IIA);
\end{tikzpicture}
\end{center}

In both Model IA and Model IIA, the radii of the last $T^2$ are not fixed by the asymmetric orbifold. Changing the radii of the last $T^2$ does not affect the supersymmetries of the individual $\mathbb{Z}_2$ orbifolds as well as the bose-fermi degeneracy feature of the resulting $\mathbb{Z}_2 \otimes \mathbb{Z}_2'$ orbifold and hence they are free moduli. The string coupling $S$ and the Kähler modulus associated with the $T^2$, i.e., $T$, are exchanged in the dual model by Eq. (66). This allows us to argue that in various strongly coupled limits of the original model, there exists weakly coupled duals which exhibit the same perturbative cancellations as the original model:

- For $S \to \infty$ and arbitrary $T$, the original model is weakly coupled.
- For $T \to \infty$ and arbitrary $S$, one can go to the dual with $\tilde{S} = T \to \infty$ and $\tilde{T} = S$ which is weakly coupled and perturbative contributions cancel in a similar way as the original model.
- For $S \to 0$ and $T \to 0$, the original model is strongly coupled and at a small radius. One can $T$-dualize $T^2$ to get a model at strong coupling but with large radius $T \to \infty$. From the $S - T$ exchange (66), the dual model is weakly coupled $\tilde{S} \to \infty$ and at small radius.

One can further deduce from string duality that at least some of the non-perturbative corrections to the cosmological constant are absent. A priori, there can be non-perturbative
corrections to the cosmological constant for finite $S$. For instance, there may be non-zero contribution due to an unequal number (or masses) of bosonic and fermionic solitonic states coming from the NS fivebranes wrapped on $T^4$. Let us consider the limit when $T$ is large. In this limit, the dual model is weakly coupled ($\tilde{S} = T \to \infty$) and the perturbative contributions in the dual model vanish for arbitrary $\tilde{T} = S$ due to the bose-fermi degeneracy, while the non-perturbative contributions in the dual model are exponentially suppressed. Since Eq. (54) maps the NS fivebranes wrapped on $T^4$ to the fundamental strings (without winding on the $T^4$) in the dual model [3], the contribution from this type of solitonic states also obey bose-fermi degeneracy. As a result, the non-perturbative corrections due to this type of wrapped NS fivebranes are absent due to bose-fermi degeneracy.

It will be interesting to find out whether the IA model has exactly zero cosmological constant or not, that is, whether all its solitonic contributions obey bose-fermi degeneracy. As first pointed out by Harvey [3], duality is a powerful tool in addressing this issue. However, it is not clear that all duals are relevant to the above question. In particular, the heterotic dual [1] of a Type II model with bose-fermi degeneracy has a non-zero one-loop cosmological constant. Naively, this implies a non-vanishing dilaton potential, which is non-perturbative in the original Type II coupling. However, there is a barrier (hence a tachyon) as the dilaton expectation value interpolates between this particular dual pair. It is then not clear that, both away and around the barrier, the non-vanishing cosmological constant can be interpreted as coming from a set of solitonic states of the Type II model. If the dual pair is not connected by varying parameters in the non-perturbative string model, it may not shed light on the solitonic bose-fermi degeneracy issue.

A non-zero cosmological constant coming from the solitonic sector is exponentially suppressed in the coupling. Since the coupling in nature is rather small, a non-perturbative cosmological constant may be perfectly harmless phenomenologically.

Since Model IA$_i$, IA$_{ii}$ and IA$_{iii}$ have different duality properties under the map Eq. (54), it is interesting to understand further the relations between these models. Recall that these models differ only in the structure constants $k_{13}$ and $k_{24}$ which can take values 0 or 1/2. In the orbifold language, $k_{13} = 1/2$ corresponds to including $(-1)^{F_L}$ in the action of $f$. Similarly, $k_{24} = 1/2$ corresponds to including $(-1)^{F_R}$ in the action of $g$. Let us start with Model IA$_{iii}$ which has $k_{13} = k_{24} = 0$. Consider $T$-dualizing one of the circles in $T^2$, i.e. $R \to 1/R$. Since $T$-duality is a one-sided parity operation, the action on the left- and right-moving coordinates is given by $X_L \to X_L$ and $X_R \to -X_R$. The corresponding action on the worldsheet fermions (from worldsheet supersymmetry) is $(-1)^{F_R}$ (since it anti-commutes with $\psi_R$). In other words, Model IA$_i$ (with $k_{13} = 0$ and $k_{24} = 1/2$) can be obtained from Model IA$_{iii}$ by $T$-dualizing the action of $g$. Similarly, Model IA$_{ii}$ can be obtained from Model IA$_{iii}$ by first interchanging the left- and right-moving worldsheet degrees of freedom and then $T$-dualizing the action of $f$. As a result, the various models are then related by an intricate web of dualities:
where $U$ is the $U$-duality operator defined in Eq. (64), $T$ and $T'$ are two different $T$-duality operators.

It would be interesting to work out the duality properties of the other models (i.e. IIB, IIB and IIC) presented in this paper.

V. MULTI-LOOP AMPLITUDES

The question whether the cosmological constants of the models constructed above vanish beyond one-loop is clearly important. Unfortunately, we are not able to give a definitive answer to this question, due to the subtleties involved in the multi-loop calculations. We can summarize the situation as follows. The calculation of the two-loop cosmological constant of the above models is exactly parallel to that of the model in Ref. [1]. If we take the gauge choice in Ref. [1], it is straightforward to demonstrate the vanishing of the cosmological constant; in fact, the two-loop integrand vanishes point-wise in the moduli space. A naive generalization of the two-loop argument actually yields the point-wise vanishing of the multi-loop integrand in the vacuum amplitude. However, there is no proof of the validity of the particular gauge choice beyond two-loops. So, whether the cosmological constants of the these models vanish beyond one-loop hinges on the legitimacy of the particular gauge choice. Since we do not have any particularly intelligent observations to make on this point, our discussions here will be brief.

In evaluating the $g$-loop string vacuum amplitude, we are evaluating the amplitude associated with genus-$g$ Riemann surfaces. The amplitude has a holomorphic and a similar anti-holomorphic part, and we may consider them separately in the discussion. In the covariant formulation, it is necessary to insert $2(g-1)$ picture-changing operators, which correspond to the integration of the odd moduli and/or absorb the zero modes of the $\beta$ superghost [9]. Moving the positions $z_i$ of these insertion points changes the integrand in the vacuum amplitude by a total derivative. In this sense, we see the choice of $z_i$ as pure gauge choices. For arbitrary $z_i$, the modes along the longitudinal and the time directions are not cancelled by the (super-)ghost modes (except in the one-loop case, where there is no picture-changing operator insertion), i.e., their respective theta functions do not cancel. However, in the light-cone gauge, we have neither longitudinal/time nor ghost modes. So we expect that there must exist a choice of the positions $z_i$ so that the theta functions from the longitudinal/time modes cancel that from the ghost modes. The choice that is closest to this picture is the so called “unitary” gauge [10, 12],

$$\sum_{i=1}^{2g-1} z_i = 2\Delta$$  \hspace{1cm} (69)
where \( \Delta \) is a Riemann constant. For generic values of \( z_i \) satisfying this unitary gauge condition, the theta functions still do not exactly cancel. In the two-loop case, there is a \( z_1 - z_2 \) dependence in the integrand. Now let us look at the situation in the light-cone gauge. Recall that a picture-changing supercurrent must be inserted at each interaction vertex \([13]\). Since there are \( 2(g-1) \) vertices in the \( g \)-loop diagram, we see that such insertions are an intrinsic part of the multi-loop light-cone amplitudes, \( i.e., \) the \( z_1 - z_2 \) dependence in the two-loop case is not unexpected.

The consistency of the unitary gauge has been demonstrated in Ref \([11,12]\). In particular, modular invariance has been shown for generic choices of \( z_i \) satisfying the unitary gauge condition. In Ref \([1]\), the further choice is made for the two-loop case \([10,12]\)

\[
z_1 - z_2 = 0
\]  

This choice is perfectly reasonable. Since the cosmological constant does not depend on \( z_1 - z_2 \), and there is no integration over \( z_i \) in the evaluation of the cosmological constant, the \( z_1 - z_2 \) dependence must be a pure gauge choice. Different choices simply gives different total derivative terms, and the total derivative term in this gauge is simply zero. Now, under the Dehn twist, the phase factor \( 2\pi i [\alpha (s + \bar{s}) + \gamma (s + \bar{s})] \) in Eq. \((25)\) needed for modular invariance is apparently absent in this \( z_1 = z_2 \) gauge, \( i.e., \) modular invariance is obscure. To resolve this ambiguity, we shall employ the following point-split approach. To carry out the Dehn twist, we first split \( z_1 \) and \( z_2 \), do the modular transformation, and then take the limit \( z_1 = z_2 \) again. In this gauge choice, the cosmological constant of the non-supersymmetric string model constructed in Ref \([1]\) is shown to be zero (the two-loop integrand vanishes point-wise) in Ref. \([1]\) , and similarly for the models given in this paper. We shall demonstrate the point-wise vanishing of the two-loop integrand, using the above defined modular transformation.

We also give an argument for the vanishing of the higher loop vacuum amplitudes. The argument relies on the assumption that even for higher loops, we can always go to the unitary gauge (at least locally on the Riemann surface) in which the combined contribution of the ghost and longitudinal components is spin structure independent. This may be achieved with the special choice of \( z_i - z_{i+1} = 0 \). Again, the higher-loop integrand vanishes point-wise in the moduli space. This is the basis of our conjecture that the cosmological constant vanishes to all orders in the perturbation expansion.

**Vanishing of the Vacuum Amplitude**

In the following, we give an argument for the vanishing of the two-loop amplitude. For our considerations, we only need to worry about diagrams that involve both \( f \) and \( g \) twists. Other diagrams which involve only one type of \( \mathbb{Z}_2 \) twist do not break supersymmetry and hence give no contribution to the cosmological constant. One can show that by multi-loop modular transformation, any twist that breaks space-time supersymmetry can be brought to one of the canonical forms: \((f, g, 1, 1, \ldots, 1, 1)\) and \((f, 1, g, 1, 1, 1, \ldots, 1, 1)\) where the twists are around the \((a_1, b_1, \ldots, a_g, b_g)\) cycles. We will show in the following that for twists of the first type, the two-loop contribution vanishes (before summing over the spin structures) because of the fermionic zero modes – at least one of the 20 worldsheet fermions has odd
spin structure, whose theta function is identically zero. For twists of the second type, part of the two-loop contribution vanishes because of the fermionic zero modes, and the remaining contributions vanish (after summing over the spin structures of the first torus) due to the (2, 0) supersymmetry (the supersymmetry that remains after only the $f$ twist).

Let us first consider the twist of the first type, i.e., the $f$ and $g$ twists are around the $a$- and $b$-cycle respectively of only one of the tori. The worldsheet fermions can naturally be divided into four groups:

\[
\begin{align*}
(i) & \quad \Psi^i_L \quad \text{for } i = 0, 1, 4, 7 \\
(ii) & \quad \Psi^j_L \quad \text{for } j = 2, 3, 5, 6, 8, 9 \\
(iii) & \quad \Psi^i_R \quad \text{for } i = 0, 1, 4, 7 \\
(iv) & \quad \Psi^j_R \quad \text{for } j = 2, 3, 5, 6, 8, 9.
\end{align*}
\]

(71)

In each of the above groups, the worldsheet fermions have the same boundary conditions on the tori where the $f$ and $g$ twists do not act (since the only generating vectors are $V_0$, $V_1$, and $V_2$ on these tori). As a result, the fermions in the same group have the same $(g - 1)$-loop spin structure, i.e., they all have odd spin structure or even spin structure.

One the other hand, one can easily show that for at least one of the groups, the fermions in the same group must have opposite spin structure in the torus that both $f$ and $g$ act. The implication is that no matter what the $(g - 1)$-loop spin structure of this group of fermions is, we can always find fermions within the group such that the total spin structure is odd. Since the $g$-loop theta function for odd spin structure is zero, we conclude that the $g$-loop amplitudes of this type of diagrams vanish.

It remains to show that the contribution of the other type of twists is zero. To show this, let us consider $z_1 \neq z_2$ but $z_1 \to z_2$. In this limit, the combined contribution of the ghosts and longitudinal components does not have spin structure dependence. The $g$-loop amplitude with $f$ and $g$ twists around the $a$-cycles of two adjacent tori is given by

\[
\sum_{\{\alpha^i, \beta^j\}} C^{\alpha^1 \alpha^2 \cdots \alpha^g}_{\beta^1 \beta^2 \cdots \beta^g} Z^{\alpha^1 \alpha^2 \cdots \alpha^g}_{\beta^1 \beta^2 \cdots \beta^g}
\]

(72)

where $\alpha^1 = V_3 + \sum_{j=0}^2 \alpha^1_j V_j$, $\alpha^2 = V_4 + \sum_{j=0}^2 \alpha^2_j V_j$, $\alpha^i = \sum_{j=0}^2 \alpha^i_j V_j$ for $i = 3, 4, \ldots, g$ and $\beta^i = \sum_{j=0}^2 \beta^i_j V_j$ for $i = 1, 2, \ldots, g$.

Consider the two characters with the same $\alpha^i$ and $\beta^j$ except for $\alpha^1$:

- $\alpha^1 = V_3 + \alpha_1 V_1 + \alpha_2 V_2$
- $\alpha^1 = V_3 + V_0 + \alpha_1 V_1 + \alpha_2 V_2$

They have the same theta function dependence (as only the roles of $\Psi^1_L$ and $\Psi^0_L$ are interchanged). The corresponding phases $C^{\alpha^1 \alpha^2 \cdots \alpha^g}_{\beta^1 \beta^2 \cdots \beta^g}$ differ by a factor of $-e^{2\pi i (\beta^1_0 + \beta^1_1)}$. If $k_{00} = 0$ or $\beta^1_0 + \beta^1_1 = 0$ then the two terms in the $g$-loop amplitude have opposite signs and so they cancel. If $k_{00} = 1/2$ and $\beta^1_0 + \beta^1_1 = 1$, the two terms do not cancel. In this case, however, the fermions in group (i), i.e., $\Psi^i_L$, have periodic boundary conditions in the $b$-cycle of the first torus. On the other hand, around the $a$-cycle of the first torus, some of the fermions in group (i) have periodic boundary conditions and some have anti-periodic boundary conditions. As a result, there are fermions with odd and even spin structure in the first torus. Therefore, no matter what the spin structure of the $(g - 1)$-loop amplitude is, there are always fermions with $g$-loop odd spin structure, and hence the $g$-loop amplitude vanishes due to the fermionic zero mode.
VI. SUMMARY AND REMARKS

Duality of non-supersymmetric strings have been studied in the literature [14]. We see that the models with the bose-fermi degeneracy feature are under better control. Using free fermionic construction, we have constructed a set of non-supersymmetric string models with the property that the number of bosonic and fermionic degrees of freedom are equal at each mass level. Since the models are constructed as Abelian orbifolds (in contrast to the non-Abelian orbifold in Ref. [1]), explicit calculations of the couplings and $N$-point amplitudes would be easier to carry out.

The vanishing of the cosmological constant in the models has a natural explanation in string theory— it is simply a remnant of the 10-dimensional supersymmetry that defines Type II string theory. The exact form of this stringy symmetry remains to be identified. Understanding this symmetry would help in constructing more realistic non-supersymmetric models in which the cosmological constant vanishes only at an isolated point of the moduli space (instead of having free moduli as in the present case).

Another motivation for considering non-supersymmetric strings with vanishing cosmological constant is the recent interests in TeV scale superstings [15]. In this new scenario, gauge and gravitational couplings can unify at the string scale which can be as low as a TeV. This means the hierarchy problem is absent and so supersymmetry is no longer needed. (This is a relief, since dynamical supersymmetry breaking in string theory is poorly understood). It is therefore important to explore non-supersymmetric models with certain realistic features, for instance, vanishing (or very small) cosmological constant. However, a realistic model would require non-Abelian gauge group which is absent in perturbative Type II string theory. It would be interesting to work out the D-brane spectrum (which can give rise to non-Abelian gauge group) and see if the vanishing of the cosmological constant still persists in the presence of D-branes.

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| Sector | $\mathbb{Z}_2$ orbifold | $\mathbb{Z}'_2$ orbifold | $\mathbb{Z}_2 \otimes \mathbb{Z}'_2$ orbifold |
|--------|------------------|------------------|------------------|
| $0$    | $G_{ij}, B_{ij}, \phi$ $U(1)^6 \otimes U(1)^2$ 12 scalars | $G_{ij}, B_{ij}, \phi$ $U(1)^6 \otimes U(1)^6$ 12 scalars | $G_{ij}, B_{ij}, \phi$ $U(1)^2 \otimes U(1)^2$ 4 scalars |
| $V_1 + V_2$ | $U(1)^8$ 16 scalars | $U(1)^8$ 16 scalars | $U(1)^4$ 8 scalars |
| $V_1$   | 2 gravitinos 14 spinors | — 16 spinors | 8 spinors |
| $V_2$   | — 16 spinors | 2 gravitinos 14 spinors | 8 spinors |
| $V_3$   | 64 scalars | N/A | 32 scalars |
| $V_3 + V_1$ | 32 spinors | N/A | 16 spinors |
| $V_4$   | N/A | 64 scalars | 32 scalars |
| $V_4 + V_2$ | N/A | 32 spinors | 16 spinors |
| $V_3 + V_4$ | N/A | N/A | 16 scalars |
| $V_3 + V_4 + V_1 + V_2$ | N/A | N/A | $U(1)^4$ 8 scalars |
| $V_3 + V_4 + V_1$ | N/A | N/A | 8 spinors |
| $V_3 + V_4 + V_2$ | N/A | N/A | 8 spinors |

**TABLE I.** The massless spectrum of Model IA. The $\mathbb{Z}_2$, $\mathbb{Z}'_2$ orbifolds are supersymmetric and are generated by $f$ and $g$ respectively. The $\mathbb{Z}_2 \otimes \mathbb{Z}'_2$ orbifold is non-supersymmetric. Here, N/A applies to the sectors that are absent in the orbifold, and — indicates that the states are projected out.
| Sector   | $\mathbb{Z}_2$ orbifold | $\mathbb{Z}'_2$ orbifold | $\mathbb{Z}_2 \otimes \mathbb{Z}'_2$ orbifold |
|---------|-------------------------|---------------------------|-----------------------------------------------|
| 0       | $G_{ij}, B_{ij}, \phi$  | $G_{ij}, B_{ij}, \phi$   | $G_{ij}, B_{ij}, \phi$                       |
|         | $U(1)^6 \otimes U(1)^2$ | $U(1)^2 \otimes U(1)^6$  | $U(1)^2 \otimes U(1)^2$                      |
|         | 12 scalars              | 12 scalars                | 4 scalars                                     |
| $V_1 + V_2$ | $U(1)^8$               | $U(1)^8$                  | $U(1)^4$                                      |
|         | 16 scalars              | 16 scalars                | 8 scalars                                     |
| $V_1$   | 2 gravitinos            | —                         | —                                             |
|         | 14 spinors              | 16 spinors                | 8 spinors                                     |
| $V_2$   | —                       | 2 gravitinos              | —                                             |
|         | 16 spinors              | 14 spinors                | 8 spinors                                     |
| $V_3$   | 64 scalars              | N/A                       | 32 scalars                                    |
| $V_3 + V_1$ | 32 spinors            | N/A                       | 16 spinors                                    |
| $V_4$   | N/A                     | 64 scalars                | 32 scalars                                    |
| $V_4 + V_2$ | N/A                     | 32 spinors                | 16 spinors                                    |
| $V_3 + V_4 + V_0$ | N/A                  | N/A                       | $U(1)^4$                                      |
|         |                         |                           | 8 scalars                                     |
| $V_3 + V_4 + V_0 + V_1 + V_2$ | N/A              | N/A                       | 16 scalars                                    |
| $V_3 + V_4 + V_0 + V_1$ | N/A               | N/A                       | 8 spinors                                     |
| $V_3 + V_4 + V_0 + V_2$ | N/A              | N/A                       | 8 spinors                                     |

**TABLE II.** The massless spectrum of Model IB. The $\mathbb{Z}_2$, $\mathbb{Z}'_2$ orbifolds are supersymmetric and are generated by $f$ and $g$ respectively. The $\mathbb{Z}_2 \otimes \mathbb{Z}'_2$ orbifold is non-supersymmetric. Here, N/A applies to the sectors that are absent in the orbifold, and — indicates that the states are projected out.
| Sector | $\mathbb{Z}_2$ orbifold | $\mathbb{Z}'_2$ orbifold | $\mathbb{Z}_2 \otimes \mathbb{Z}'_2$ orbifold |
|--------|-----------------|-----------------|-----------------|
| 0      | $G_{ij}, B_{ij}, \phi$  
$U(1)^6 \otimes U(1)^2$  
12 scalars | $G_{ij}, B_{ij}, \phi$  
$U(1)^6 \otimes U(1)^6$  
36 scalars | $G_{ij}, B_{ij}, \phi$  
$U(1)^6 \otimes U(1)^2$  
12 scalars |
| $V_1 + V_2$ | $U(1)^8$  
16 scalars | — | — |
| $V_1$  | 2 gravitinos  
14 spinors | — | — |
| $V_2$  | —  
16 spinors | 4 gravitinos  
28 spinors | 16 spinors |
| $V_3$  | 64 scalars | N/A | 32 scalars |
| $V_3 + V_1$ | 32 spinors | N/A | 16 spinors |
| $V_4$  | N/A | $U(1)^8$  
48 scalars | $U(1)^4$  
24 scalars |
| $V_4 + V_2$ | N/A | 32 spinors | 16 spinors |
| $V_3 + V_4$ | N/A | N/A | 32 scalars |
| $V_3 + V_4 + V_1$ | N/A | N/A | 16 spinors |

TABLE III. The massless spectrum of Model IIA. The $\mathbb{Z}_2$, $\mathbb{Z}'_2$ orbifolds are supersymmetric and are generated by $f$ and $g$ respectively. The $\mathbb{Z}_2 \otimes \mathbb{Z}'_2$ orbifold is non-supersymmetric. Here, N/A applies to the sectors that are absent in the orbifold, and — indicates that the states are projected out.
\[
\begin{array}{|c|c|c|c|}
\hline
\text{Sector} & \mathbb{Z}_2 \text{ orbifold} & \mathbb{Z}_2' \text{ orbifold} & \mathbb{Z}_2 \otimes \mathbb{Z}_2' \text{ orbifold} \\
\hline
0 & G_{ij}, B_{ij}, \phi & G_{ij}, B_{ij}, \phi & G_{ij}, B_{ij}, \phi \\
& U(1)^6 \otimes U(1)^2 & U(1)^6 \otimes U(1)^2 & U(1)^6 \otimes U(1)^2 \\
& 12 \text{ scalars} & 36 \text{ scalars} & 12 \text{ scalars} \\
\hline
V_1 + V_2 & U(1)^8 & — & — \\
& 16 \text{ scalars} & — & — \\
\hline
V_1 & 2 \text{ gravitinos} & — & — \\
& 14 \text{ spinors} & — & — \\
\hline
V_2 & — & 4 \text{ gravitinos} & — \\
& 16 \text{ spinors} & 28 \text{ spinors} & 16 \text{ spinors} \\
\hline
V_3 & 64 \text{ scalars} & \text{N/A} & 32 \text{ scalars} \\
\hline
V_3 + V_1 & 32 \text{ spinors} & \text{N/A} & 16 \text{ spinors} \\
\hline
V_4 & \text{N/A} & — & — \\
\hline
V_4 + V_1 & \text{N/A} & — & — \\
\hline
V_3 + V_4 & \text{N/A} & \text{N/A} & 32 \text{ scalars} \\
\hline
V_3 + V_4 + V_1 & \text{N/A} & \text{N/A} & 16 \text{ spinors} \\
\hline
\end{array}
\]

TABLE IV. The massless spectrum of Model IIB. The $\mathbb{Z}_2$, $\mathbb{Z}_2'$ orbifolds are supersymmetric and are generated by $f$ and $g$ respectively. The $\mathbb{Z}_2 \otimes \mathbb{Z}_2'$ orbifold is non-supersymmetric. Here, N/A applies to the sectors that are absent in the orbifold, — indicates that the states are projected out.
| Sector | $\mathbb{Z}_2$ orbifold | $\mathbb{Z}'_2$ orbifold | $\mathbb{Z}_2 \otimes \mathbb{Z}'_2$ orbifold |
|--------|-------------------------|-------------------------|----------------------------------------|
| 0      | $G_{ij}, B_{ij}, \phi$  | $G_{ij}, B_{ij}, \phi$  | $G_{ij}, B_{ij}, \phi$                |
|        | $U(1)^6 \otimes U(1)^2$| $U(1)^6 \otimes U(1)^2$| $U(1)^6 \otimes U(1)^2$              |
|        | 12 scalars              | 36 scalars              | 12 scalars                            |
| $V_1 + V_2$ | $U(1)^8$              | —                       | —                                      |
|         | 16 scalars              | —                       | —                                      |
| $V_1$  | 2 gravitinos            | —                       | —                                      |
|         | 14 spinors              | —                       | —                                      |
| $V_2$  | —                       | 4 gravitinos            | —                                      |
|         | 16 spinors              | 28 spinors              | 16 spinors                            |
| $V_3$  | 64 scalars              | N/A                     | 32 scalars                            |
| $V_3 + V_1$ | 32 spinors            | N/A                     | 16 spinors                            |
| $V_4$  | N/A                     | —                       | —                                      |
| $V_4 + V_1$ | N/A                 | —                       | —                                      |
| $V_3 + V_4 + V_0 + V_1 + V_2$ | N/A                  | N/A                     | 32 scalars                            |
| $V_3 + V_4 + V_0 + V_2$ | N/A                  | N/A                     | 16 spinors                            |

**TABLE V.** The massless spectrum of Model IIC. The $\mathbb{Z}_2$, $\mathbb{Z}'_2$ orbifolds are supersymmetric and are generated by $f$ and $g$ respectively. The $\mathbb{Z}_2 \otimes \mathbb{Z}'_2$ orbifold is non-supersymmetric. Here, N/A applies to the sectors that are absent in the orbifold, — indicates that the states are projected out.
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