ENHANCEMENT OF DAMAGED-IMAGE PREDICTION THROUGH CAHN-HILLIARD IMAGE INPAINTING

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Abstract. We assess the benefit of including an image inpainting filter before passing damaged images into a classification neural network. We follow Bertozzi et al. [IEEE 16 285-291 (2007)] and we employ an appropriately modified Cahn-Hilliard equation as an image inpainting filter which is solved numerically with a finite-volume scheme exhibiting reduced computational cost and the properties of energy stability and boundedness. The benchmark dataset employed here is the MNIST one, which consists of binary images of digits. We train a neural network based on dense layers with MNIST, and subsequently we contaminate the test set with damages of different types and intensities. We then compare the prediction accuracy of the neural network with and without applying the Cahn-Hilliard filter to the damaged images test. Our results quantify the significant improvement of damaged-image prediction due to applying the Cahn-Hilliard filter, which for specific damages can increase up to 50% and is advantageous for low to moderate damage.

1. Introduction

Image inpainting consists in filling damaged or missing areas of an image, with the ultimate objective of restoring it and making it appear as the true and original image. There are multiple applications of image inpainting, ranging from restoration of the missing areas of oil paintings and removal scratches in photographs to noisy MRI scans and blurred satellite images of the earth. Manual image inpainting techniques have been employed for many centuries by art conservators and professional restorers, but it was not until the turn of the 21st century that digital image inpainting models based on PDEs and variational methods were introduced [9,18,36]. These methods are usually referred to as non-texture, geometrical or structural inpainting since they focus on restoring the structural information in the inpainted domain such as edges, corners or curvatures. This is done by performing an image interpolation of the damaged areas based on the information collected from the surrounding environment only, leading to appealing images for the human vision system. On the contrary, texture inpainting is based on recovering global patterns of the image for the inpainted region [26], and a popular tool in this category is the exemplar-based inpainting methods [25,34]. Associated with these developments, a field that has gained a lot of traction in recent years is the so-called generative image inpainting, where deep learning based approaches have proven to be successful even for blind inpainting in which the inpainted region is not provided a priori [47–49]. In this work we focus on non-texture image inpainting methods based on PDEs, and we refer the reader to Ref. [38] for a general review of the topic.

There have been multiple PDE models for image inpainting proposed since the initial work of Bertalmio et al. [9] nearly 20 years ago. Their trailblazing model is able to propagate isotopes, i.e. contours of uniform grayscale image intensity, through the inpainted region, a common technique employed by museum artists in restoration. As it also turns out that the original model bears close connection to fluid dynamics through the Navier-Stokes equation with the image intensity function acting as the stream function [8]. Another fluid dynamic equation that has played a pivotal role in
image inpainting is the Cahn-Hilliard (CH) equation, initially proposed in [16] for phase separation in binary alloys. It has been employed in a wide spectrum of applications from wetting phenomena [4,37] and polymer science [22] to tumour growth [44]. This equation satisfies a gradient-flow structure using an $H^{-1}$ norm,

$$\frac{\partial \phi(x,t)}{\partial t} = \nabla \cdot \left( M(\phi) \nabla \delta F[\phi] \right),$$

where $\phi$ is an order parameter widely referred to as the phase field which for a binary system takes on the value $\phi = 1$ in one of the phases and $\phi = -1$ in the other, while varying smoothly in the interface region with a width of $O(\epsilon)$. $M(\phi)$ is the mobility obtained here from the one-sided model $M(\phi) = 1$ and $F[\phi]$ is the free energy satisfying

$$F[\phi] = \int_{\Omega} \left( H(\phi) + \frac{\epsilon^2}{2} |\nabla \phi|^2 \right) dx,$$

with the variation of the free energy given by

$$\frac{\delta F[\phi]}{\delta \phi} = H'(\phi) - \epsilon^2 \Delta \phi,$$

and $H(\phi)$ taken here as the Ginzburg-Landau double-well potential with the two wells corresponding to the two phases,

$$H(\phi) = \frac{1}{4} \left( \phi^2 - 1 \right)^2 \text{ for } \phi \in [-1,1]$$

(see e.g. [3,45] for discussions of the physical significance of the various terms of the CH equation).

The CH equation was firstly proposed in the context of image inpainting by [11]. Specifically, the authors adopted a modified CH equation for binary images with inpainting quality as accurate as the state-of-art inpainting models but with a much faster computational speed taking advantage of the efficient computational techniques already available for the CH equation [30,41]. Since then several authors have extended the applicability of the CH equation in the field of image inpainting, for instance by taking into account grayvalue images [15,21], nonsmooth potentials instead of the double-well potential (4) [13], and considering color image inpainting [20]. The modified CH equation in [11] introduces a fidelity term $\lambda(x)$ to avoid modifying the original image outside of the inpainted region $D$, and the CH equation in (1) becomes

$$\frac{\partial \phi(x,t)}{\partial t} = -\nabla^2 \left( \epsilon^2 \nabla^2 \phi - H'(\phi) \right) + \lambda(x) \left( \phi(x,t=0) - \phi \right),$$

where

$$\lambda(x) = \begin{cases} 0 & \text{if } x \in D, \\ \lambda_0 & \text{if } x \notin D, \end{cases}$$

and $\phi(x,t=0)$ refers to the original damaged image. The parameter $\epsilon$ plays a similar role as in the original CH equation, and here it is related to the interface between the two phases or colours presented in the image. The two parameters $\epsilon$ and $\lambda_0$ are essential to achieve an adequate image inpainting outcome, and it is usually necessary to iterate until finding appropriate tunings for their values, which typically depend on the image specifications.

As already alluded to above one of the main advantages of employing the CH equation for image inpainting is the myriad of fast and reliable numerical methods available for its solution. A pivotal contribution was the convex-splitting scheme developed by Eyre [30] which is unconditionally energy-stable by treating as implicit the convex terms of the free energy in (2), while keeping the concave terms explicit. The design of energy-stable and maximum-principle satisfying schemes for the CH equation has been a really active area of research [40], and several authors have proposed schemes based on finite differences [31,33,43], finite elements [7,29], finite volumes [27] or discontinuous Galerkin [23,35,42,46]. These schemes have also proven effective for degenerate mobilities or logarithmic potentials. Schemes satisfying the maximum principle condition for specific choices of free energy have also been constructed.
We refer the reader to [39] for a recent work discussing the state-of-the-art numerical techniques for nonlinear gradient flows.

In a recent effort [6] we constructed a robust semi-implicit finite-volume scheme for the CH equation that offers crucial advantages when applied to the field of image inpainting. Firstly, our scheme is based on a dimensional splitting approach, so that the cost of solving the CH equation for a \( N^d \times N^d \) image in \( d \) dimensions is reduced from \( \mathcal{O}(N^{d\gamma}) \) to \( \mathcal{O}(dN^{d+\gamma-1}) \), with \( 2 < \gamma < 3 \) (see [24] for details). This is already advantageous for a two-dimensional (2D) image, and the computational cost is further reduced for high-dimensional images, such as the ones for MRI [14] or X-ray computed tomography [32] in medical image analysis. Secondly, our scheme satisfies the discrete decay of the free energy for different choices of potentials (4) [13], and in addition we prove their boundedness for mobilities of the form \( M(\phi) = 1 - \phi^2 \). The combination of these properties and reduced computational cost, together with the versatility of finite volumes, make our scheme efficient and robust for the solution of the modified CH equation in (5) for a variety of applications in image inpainting.

The objective of this work is to show precisely the applicability of our numerical framework in [6] for a benchmark dataset of images in need of restoration through image inpainting. For this task we purposely add different types and intensities of damage to the popular MNIST dataset [28], and then apply image inpainting by solving the modified CH equation (5) with our finite-volume scheme in [6]. We also assess the improvement in pattern recognition accuracy of the restored MNIST images, and for this we construct a neural network for the task of classification. A key objective of our study is to quantify the benefits of including a CH filter before introducing a damaged image into a neural network. Our results demonstrate that accuracies in classification can increase up to 50\% for particular damages in the images, and, in general, applying the CH filter improves the accuracy prediction for a wide range of low to moderate image damage.

In section 2 we outline the methodology: in subsection 2.1 we adapt our finite-volume scheme in [6] for the modified CH equation in (5); in subsection 2.2 we recall the two-step method for image inpainting in [11]; in subsection 2.3 we detail the neural network architecture for the classification task; and lastly in subsection 2.4 we explain the structure of the integrated algorithm which takes a damaged image, applies a CH filter to it, and then classifies the image through a neural network. Subsequently in section 3 we present the results of the integrated algorithm applied to the MNIST dataset: in subsection 3.1 we begin by identifying appropriate tunings for the values of \( \epsilon \) and \( \lambda_0 \); in subsection 3.2 we present the different types of damage introduced into the MNIST testset of images; and finally in subsection 3.3 we quantify the improvement in accuracy of applying the CH filter to the damaged MNIST images before introducing them into the neural network. A discussion and final remarks are offered in section 4.

2. INTEGRATED ALGORITHM WITH IMAGE INPAINTING AND PATTERN RECOGNITION

We detail the construction of an integrated algorithm that firstly applies image inpainting and subsequently conducts pattern recognition for the restored image. In subsection 2.1 we begin by presenting the finite-volume scheme employed to solve the modified CH equation, based on the work in [6]. Then in subsection 2.2 we illustrate the two-step method for image inpainting, based on tuning the parameters \( \epsilon \) and \( \lambda_0 \) of the modified CH equation. In subsection 2.3 we present the neural network employed for pattern recognition, detailing its architecture and training parameters. Finally, in subsection 2.4 we gather all previous elements to formulate an integrated algorithm for prediction with an image inpainting filter.

2.1. 2D finite-volume scheme for the modified CH equation. We summarise the 2D finite-volume scheme constructed for the original CH equation in our previous work [6]. In this scheme the discrete free energy of the CH equation in (2) decays unconditionally under a certain CFL condition for the timestep, while ensuring the boundedness of the phase field when choosing physical mobilities of the form \( M(\phi) = 1 - \phi^2 \). The scheme can be straightforwardly extended to the modified CH equation in (5) proposed in Ref. [11] as we show here. We also remark that in [6] we detail how to turn the
scheme into a dimensional splitting one, with promising applicability in high-dimensional images such as medical ones. We refer the reader to [5] for further details about dimensional-splitting schemes.

We begin by dividing the computational domain \([0, L] \times [0, L]\) in \(N \times N\) cells \(C_{i,j} := [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]\), all with uniform size \(\Delta x \Delta y\) so that \(x_{i+1/2} - x_{i-1/2} = \Delta x\) and \(y_{j+1/2} - y_{j-1/2} = \Delta y\). In each of the cells we define the cell average \(\phi_i\) as

\[
\phi_{i,j}(t) = \frac{1}{\Delta x \Delta y} \int_{C_{i,j}} \phi(x, y, t) dx dy.
\]

The finite-volume scheme is derived by integrating the modified CH equation (5) over each of the cells \(C_{i,j}\) of the domain, leading to

\[
\frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} = -\frac{F_{i+\frac{1}{2},j}^{n+1} - F_{i-\frac{1}{2},j}^{n+1}}{\Delta x} - \frac{G_{i,j+\frac{1}{2}}^{n+1} - G_{i,j-\frac{1}{2}}^{n+1}}{\Delta y} + \lambda_{i,j}(\phi_{i,j}^0 - \phi_{i,j}^n),
\]

with \(\phi_{i,j}^0\) denoting the phase field of the initial damaged image to be inpainted, and \(\lambda_{i,j}\) being the discrete version of \(\lambda(x)\) in (6) satisfying

\[
\lambda_{i,j} = \begin{cases} 
0 & \text{if } (x_i, y_j) \in D, \\
\lambda_0 & \text{if } (x_i, y_j) \notin D,
\end{cases}
\]

with \(D\) being the inpainted domain.

The approximation of the fluxes at the boundaries follows an upwind and implicit approach inspired by the works of [5, 12, 17], satisfying

\[
F_{i+\frac{1}{2},j}^{n+1} = \left( u_{i+1/2,j}^{n+1} \right)^+ + \left( v_{i+1/2,j}^{n+1} \right)^-, \\
G_{i,j+\frac{1}{2}}^{n+1} = \left( u_{i,j+1/2}^{n+1} \right)^+ + \left( v_{i,j+1/2}^{n+1} \right)^-,
\]

where the velocities \(u_{i+1/2,j}^{n+1}\) and \(v_{i,j+1/2}^{n+1}\) are taken as

\[
u_{i+\frac{1}{2},j}^{n+1} = -\frac{\xi_{i+1,j} - \xi_{i,j}}{\Delta x}, \quad v_{i,j+\frac{1}{2}}^{n+1} = -\frac{\xi_{i,j+1} - \xi_{i,j}}{\Delta y},
\]

and the upwind approach in (10) accomplished by

\[
\begin{aligned}
\left( u_{i+1/2,j}^{n+1} \right)^+ &= \max(u_{i+1/2,j}^{n+1}, 0), & \left( u_{i+1/2,j}^{n+1} \right)^- &= \min(u_{i+1/2,j}^{n+1}, 0), \\
\left( v_{i,j+1/2}^{n+1} \right)^+ &= \max(v_{i,j+1/2}^{n+1}, 0), & \left( v_{i,j+1/2}^{n+1} \right)^- &= \min(v_{i,j+1/2}^{n+1}, 0).
\end{aligned}
\]

The discretized variation of the free energy \(\xi_{i,j}^{n+1}\) in (3) follows a semi-implicit scheme inspired by the ideas of [30, 41], where the so-called convexity splitting scheme is proposed to construct unconditional gradient-stable schemes (i.e. schemes that ensure the decay of the discrete version of the free energy in (2)). In our recent effort [6] we show that our finite-volume scheme decreases the discrete free energy of the CH equation if the contractive part of the potential, \(H_c(\rho)\), is taken as implicit; the expansive part of the potential, \(H_e(\rho)\), is taken as explicit; and the Laplacian is taken as an average between the explicit and the implicit second-order discretizations, so that

\[
\xi_{i,j}^{n+1} = H_c(\phi_{i,j}^{n+1}) - H_c(\phi_{i,j}^n) - \frac{\epsilon^2}{2} \left[ (\Delta \phi)_{i,j}^n + (\Delta \phi)_{i,j}^{n+1} \right],
\]

\[
H(\phi) = H_c(\phi) - H_e(\phi) = \frac{\phi^4}{4} - \frac{\phi^2}{2}.
\]
where the discrete two-dimensional approximation of the Laplacian \((\Delta \phi)_{j,k}\) is chosen to satisfy the second-order form

\[
(\Delta \phi)^n_{i,j} := \frac{\phi^n_{i+1,j} - 2\phi^n_{i,j} + \phi^n_{i-1,j}}{\Delta x^2} + \frac{\phi^n_{i,j+1} - 2\phi^n_{i,j} + \phi^n_{i,j-1}}{\Delta y^2},
\]

\[
(\Delta \phi)_{i,j}^{n+1} := \frac{\phi_{i+1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i-1,j}^{n+1}}{\Delta x^2} + \frac{\phi_{i,j+1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j-1}^{n+1}}{\Delta y^2}.
\]

The modified CH equation in (5) employs no-flux boundary conditions that are numerically implemented as,

\[
F^x_{i-\frac{1}{2},j} = 0 \text{ for } i = 1, \forall j; \quad F^x_{i+\frac{1}{2},j} = 0 \text{ for } i = N, \forall j;
\]

\[
F^y_{i,j-\frac{1}{2}} = 0 \text{ for } j = 1, \forall i; \quad F^y_{i,j+\frac{1}{2}} = 0 \text{ for } j = N, \forall i.
\]

2.2. Two-step method for the modified CH equation. We apply the two-step method to solve the modified CH equation in (5) proposed in [11]. It basically consists of dividing the image inpainting in two subsequent stages, so that in each one of them the finite-volume scheme in subsection 2.1 is implemented with different values of the parameter \(\epsilon\). The first stage consists of taking a large \(\epsilon\) to execute a large-scale topological reconnection of shapes, leading to images with diffused edges. Subsequently, and in order to sharpen the edges after the first stage, \(\epsilon\) is substantially reduced, and the final outcome becomes less blurry and diffused. We denote the corresponding values of \(\epsilon\) as \(\epsilon_1\) and \(\epsilon_2\).

Adequately tuning the two values of \(\epsilon\) in each stage, as well as \(\lambda\), is vital to complete a successful image inpainting. Those values have to be chosen empirically and depend on the dataset and type of damage, and in subsection 3.1 we conduct a study to select them. As explained there, the appropriate values for \(\epsilon\) are between 0.5 and 1.5 for MNIST-like images, while \(\lambda \in [1, 1000]\). The cell sizes are \(\Delta x = \Delta y = 1\). The reader can find the exact values employed after the analysis of subsection 3.1 in Table 2.

2.3. Neural network architecture for classification. The prediction of the label in the restored images is performed via a neural network constructed in TensorFlow [2]. Its architecture is defined taking into account that in this work we employ the MNIST dataset [28], which contains binary images of digits from 0 to 9 and has a resolution of 28×28 pixels. This is a benchmark dataset in the community and is the de facto “hello world” dataset of computer vision. There are consequently plenty of neural network architectures attaining extremely high accuracies for the MNIST dataset, and we refer the reader to the Kaggle competition of Digit Recognizer in [1] for examples of such architectures.

Here, however, our overarching objective aim is to quantify how the prediction of damaged images is enhanced once the CH filter is applied to the images beforehand. Hence we do not require a highly sophisticated neural network and a cutting-edge architecture as in computer vision: our images are not going to be exactly the same as in the training set due to the damage and the subsequent restoration. We then select a standard architecture for classification based on sequential dense layers. Such architecture is formed by:

1) A flatten layer that takes the 28×28 image input and turns it into an array with 784 elements. There are no weights to optimize in this layer.

2) A dense layer with 64 units and the ReLU activation function, defined as \(f(x) = \max\{0, x\}\). There are 784×64 weights to optimize in this layer, in addition to the bias term in each of the 64 units.

3) Another dense layer with 64 units and the ReLU activation function. There are 64×64 weights to optimize in this layer, in addition to the bias term in each of the 64 units.

4) A final dense layer with 10 units and the softmax activation function, which returns the normalized probability distribution for the 10 labels and satisfies \(\sigma(z_i) = \exp(z_i) / \sum_{j=1}^{10} \exp(z_j)\), with \(z = (z_1, \ldots, z_{10})\) being the output of the final dense layer with 10 units. There are 64×10 weights to optimize in this layer, in addition to the bias term in each of the 10 units.
For the training of this network we initially divide the original MNIST dataset in 60000 training images and 10000 testing images. Then we train the neural network for 10 epochs with the Adam optimizer, choosing as loss function the categorical crossentropy defined as

$$J(w) = \frac{1}{N} \sum_{i=1}^{N} [y_i \log(\hat{y}_i) + (1 - y_i) \log(1 - \hat{y}_i)],$$

with $w$ being the weights to optimize, $y_i$ each of the $N$ true labels of the training dataset, and $\hat{y}_i$ each of the $N$ predicted labels. After 10 epochs we get an accuracy for the training dataset of 99.02%, while for the test set the accuracy is 97.47%. Once the neural network is trained we keep the weights fixed for the comparison of damaged and restored images in section 3. A display of the neural network is depicted in Figure 1.

![Diagram showing the layers of the neural network of subsection 2.3.](image)

**Figure 1.** Diagram showing the layers of the neural network of subsection 2.3.

### 2.4. Integrated Algorithm.

The integrated algorithm proposed and tested in this work takes as input a damaged image, applies the CH filter based on subsection 2.1 and subsection 2.2 to restore it, and finally applies the already-trained neural network in subsection 2.3 to predict its label.

To show the applicability of this integrated algorithm we initially create damage in the images of the test set in the MNIST dataset [28]. After we apply the image inpainting to the damaged test images, we introduce the restored images in the neural network. At that point, and since we have the true labels of the test set, we can assess the attained accuracy in comparison to directly introducing the damaged images or the original images into the neural network. This procedure is conducted for multiple types of damage in section 3, and a schematic representation of all steps is depicted in Figure 2.

### 3. Application of the integrated algorithm to the MNIST dataset

Our focus here is testing the applicability of the integrated algorithm in subsection 2.4 to increase predictability in damaged images. In subsection 3.1 we start by analysing the impact of the parameters $\lambda_0$ and $\epsilon$ on the inpainting process, with the objective of calibrating them before employing the MNIST dataset. In subsection 3.2 we detail the types of damage that we insert into the MNIST dataset, and we also show the restored outcomes of applying the CH equation as an image inpainting filter. Finally, in subsection 3.3 we evaluate how the accuracy of the damaged images increases after applying the CH filter to them, for various types and degrees of damage.
3.1. Inpainting of a Crossline. We employ the crossline example in [11] to analyse the role of the parameters $\epsilon$ and $\lambda_0$ in the finite-volume scheme of subsection 2.1. These two parameters crucially determine the success of the image inpainting procedure, and consequently appropriate calibrations for the parameters must be chosen before running the scheme. The original crossline image is depicted in Figure 3a, and we add to it a grey damage in the center, as shown in Figure 3b. This image contains $50 \times 50$ pixels or cells, each with a size of $\Delta x = \Delta y = 1$. We apply the finite-volume scheme in subsection 2.1 to the damaged image in Figure 3b.

We first aim to determine $\lambda_0$ in (9) and we set $\epsilon = 1$ as an initial guess so that $\epsilon = \Delta x = \Delta y$. From (9), $\lambda_{i,j}$ is only nonzero for the predefined area of undamaged pixels. Indeed the term with $\lambda_{i,j}$ in the finite-volume scheme in (8) ensures that the undamaged pixels are not modified during the image inpainting, but for this $\lambda_0$ has to be sufficiently large to counterbalance the fluxes of the scheme. Bearing this in mind we run the numerical scheme with $\Delta t = 0.1$ and until the $L^1$ norm between successive states is lower than a certain tolerance fixed to be $10^{-4}$. Our simulation produces satisfactory results and does not break down for a range of $\lambda \in [1,1000]$. Table 1 shows that the computational time to reach the required tolerance decreases when increasing the value of $\lambda$. In addition, the $L^1$ norm between the final and initial state is also lower for greater $\lambda$. It is worth mentioning that different choices for $\Delta t$ yield different ranges of valid $\lambda$, given that (8) is a singularly perturbed problem for large $\lambda_0$. Hence, greater values of $\lambda$ are possible if $\Delta t$ is refined. In our case, with the choice of $\Delta t = 0.1$, our finite-volume scheme does not yield any result and breaks down for values of $\lambda \notin [1,1000]$.

We next consider the tuning of the parameter $\epsilon$ which in turn is related to the pixel size $\Delta x$ and $\Delta y$. For values of $\epsilon$ larger than the pixel size the outcome tends to be diffusive, while for smaller values the edges are sharpened. When applying the finite-volume scheme in (8) with $\lambda \in [1,1000]$ we obtain satisfactory results for $\epsilon \in [0.5,1.5]$, while for values outside this range the simulation breaks down.
Table 1. Comparison for different values of $\lambda$: computational time before reaching the tolerance and $L^1$ norm between the final and initial state.

| $\lambda$ | time | $L^1$ norm |
|-----------|------|------------|
| 1         | 489.8 | 61.2       |
| 10        | 489.7 | 60.8       |
| 100       | 484.8 | 60.8       |
| 1000      | 481.5 | 60.7       |

because of the singular nature of (8). As a consequence, for the two-step method in subsection 2.2 we first take the value $\epsilon_1 = 1.5$ for the large-scale topological reconnection of shapes, while for the second step we choose the value $\epsilon_2 = 0.5$ to sharpen the edges. The image inpainting of the damaged image in Figure 3b resulting from applying this choice of parameters is shown in Figure 3c.

![Initial crossline](image1.png) ![Damaged crossline](image2.png) ![Result from image inpainting](image3.png)

Figure 3. Image inpainting of a crossline, inspired by [11].

The final outcome after the image inpainting in Figure 3b is not the same as the original image in Figure 3a. The reason from this is explained in the work of [10], where multiple steady-state solutions of the modified CH equation were shown to exist. As the information under the inpainting region has been destroyed, there is no way of knowing that the steady state we obtain is less accurate than other viable solutions, in comparison to Figure 3a. For further details we refer the reader to [10], where a bifurcation analysis is carried out to show that the steady state may vary depending on the choices for $\epsilon$ and $\Delta x, \Delta y$.

3.2. Damage introduced in the MNIST dataset. Here we discuss the types of damage inserted into the MNIST test set, with the objective of subsequently applying the CH filter developed in subsection 2.1 for image inpainting. The varied damage employed aims to represent a mock case of damage that may be encountered in an image in need for restoration. As a result, we decide to employ two kinds of damage with different intensities: customized damage affecting particular regions of the image, and random damage selecting arbitrary pixels or horizontal lines in the image. The details of both are:

a) Customized damage: this type of damage is applied in four different fashions, as shown in Figure 4. The basic idea is to turn vertical or horizontal lines of pixels into a uniform grey intensity between black and white color. In Figure 4 we show the outcome of applying the CH filter to the damaged images. It can be seen that our model is able to recover the images from the different types of damage, albeit with varying degrees of success. For instance, the damage introduced in Figure 4c is a thick horizontal line which implied a considerable loss of information from the original image, compared to the other types of damage. As a result, the
inpainted image filter for this type of damage is not as effective as the other ones, as it can be seen from the inpaintings in Figure 4.

![Figure 4](image_url)

**Figure 4.** Customized damage applied to a particular sample of the MNIST dataset. (A)-(D): the sample with four different types of damage; (E)-(F): The outcome of applying image inpainting to the damaged samples.

b) Random damage: this second type of damage is inserted in a random fashion and with different levels of intensity. Two ways of randomly creating damage are considered: one makes use of randomly selecting whole horizontal rows of pixels, while the other is obtained by randomly selecting individual pixels. In addition, for both types of random damage we employ different levels of damage intensity, so that a higher percentage of the image contains damage if the intensity rises. This allows us to test how our image inpainting algorithm behaves with increasing levels of damage in the image. Examples of these damages are shown in Figure 5. Similarly to the case of customized damage, the higher the intensity of damage the more information is lost in the inpainting, as we can see for example in the case of 80% pixel damage in Figure 5. But despite of this, our image inpainting algorithm renders recognisable images even with relative high amounts of damage.

### Table 2.

| Parameters               | $\epsilon_1$ | $\epsilon_2$ | $\lambda$ |
|--------------------------|---------------|---------------|-----------|
| Customized damage        | 1.5           | 0.5           | 1000      |
| Random damage (Rows)     | 1.5           | 0.5           | 1000      |
| Random damage (Pixels)   | 1.5           | 0.5           | 9000      |

The parameter values of $\lambda$, $\epsilon_1$ and $\epsilon_2$ are gathered in Table 2, and follow the reasoning discussed in subsection 3.2. This choice of parameters in our finite-volume scheme in subsection 2.1 leads to an effective image inpainting algorithm capable of restoring images with damage of varied nature as
Figure 5. Examples of image inpainting for random damage in whole horizontal rows and in individual pixels. The column entitled “Damaged rows” marks the number of rows randomly selected for damage in the $28 \times 28$ images. The column entitled “Damaged pixels” marks the percentage of randomly damaged pixels over the whole image. For higher levels of damage intensity the inpaintings lose more information.

3.3. Pattern recognition for inpainted images. We now apply the neural network described in subsection 2.3 to predict labels of damaged images with and without image inpainting, with the aim
of quantifying the improvement of accuracy following the application of the CH filter. This study is completed for the different types and intensities of the noise depicted in subsection 3.2.

We begin by adding the types of damage in subsection 3.2 to the 10,000 samples of the MNIST test dataset. The next step is to apply the CH filter and two-step method to each one of them, while also saving copies of the test images with the damage. Eventually, for each type of damage we get two batches of 10,000 images: one still with the damage, and another one with image inpainting applied. Given that the neural network of subsection 2.3 is already trained with the 60,000 samples of the MNIST training dataset, we can directly compute the accuracy of each of the two batches. This way we are able to assess the improvement in accuracy thanks to applying image inpainting to restore the damage.

Here for the validation we just employ the accuracy metric, which is defined as follows

\[
\text{Accuracy} = \frac{\text{Number of correct predictions}}{\text{Number of total predictions}}.
\]

There are however many other metrics apart from the accuracy one that play a vital role in other classification problems: recall, precision, F1 score, true positive rate and so on. Here we believe that the accuracy metric is enough to draw conclusions about how the image inpainting is improving the predictions with respect to the damage images. This is due to the fact that the MNIST dataset is a balanced dataset, where there is generally no preference between false positives and false negatives.

The measure of improvement between the batch of samples with image inpainting and the damaged ones without it is computed as

\[
\text{Improvement} \equiv \frac{\text{Accuracy with CH filter} - \text{Accuracy without CH filter}}{\text{Accuracy without CH filter}},
\]

and it basically represents the percentage of improvement that results from adding the CH filter to the prediction process.

The results for all the types of damage under consideration are displayed in Table 3, Table 4 and Table 5. In Table 3 we gather the prediction accuracies for the four customized damages displayed in Figure 4, as well as the prediction accuracy for the unmodified MNIST test set, which for our neural network architecture is 0.97. We observe that for the types of more intense customized damage B and C the accuracy prediction for the damaged images without CH filter drops to 0.71 and 0.64, respectively. By applying the filter we find that the accuracy predictions can significantly escalate to 0.93 and 0.82, leading to improvements of 31% and 28% respectively. The other two types of customized damage A and D are not as pervasive as B and C, and as a result the accuracy predictions are high even without applying the CH filter.

| Customized damage | Without CH filter | With CH filter | Improvement |
|-------------------|-------------------|----------------|-------------|
| A                 | 0.84              | 0.96           | 14%         |
| B                 | 0.71              | 0.93           | 31%         |
| C                 | 0.64              | 0.82           | 28%         |
| D                 | 0.90              | 0.96           | 7%          |
| Initial test images | -                | 0.97           | -           |

Table 3. Accuracy for the test dataset of MNIST without and with the CH filter, for the customized damage in Figure 4. The improvement is computed following (16).

In Table 4 and Table 5 we test the accuracies for random damage with various levels of intensities. The objective here is to analyse how the CH filter responds when the damage occupies more and more space in the images, both for the case of rows or pixels, as displayed in Figure 5. In Table 4 we show the accuracies for a range of damaged rows between 6 and 26, bearing in mind that the dimensions of the MNIST images are 28 × 28. We observe that for low numbers of damaged rows the accuracy prediction even without the CH filter is high and it does not improve significantly by adding the filter.
But then the improvement surges until reaching a maximum value of 47% for 16 random damaged rows, where the prediction without CH filter is 0.55 and with CH filter 0.81. From larger numbers of damaged rows the accuracies drastically drop due to the large amount of information lost, and not even the image inpainting process is able to achieve decent accuracies. In the limit of damaged number of rows tending to 28 we observe that the accuracies are close to the ones of a dummy classifier with one out of ten chances of rightly guessing the label. In this limit there is no difference between adding the CH filter or not, and it turns out that the improvements are even negative.

| Damaged rows | Without CH filter | With CH filter | Improvement |
|--------------|------------------|----------------|-------------|
| 6            | 0.89             | 0.96           | 8%          |
| 8            | 0.82             | 0.93           | 13%         |
| 10           | 0.73             | 0.91           | 25%         |
| 12           | 0.66             | 0.87           | 32%         |
| 14           | 0.6              | 0.87           | 45%         |
| 16           | 0.55             | 0.81           | 47%         |
| 18           | 0.47             | 0.68           | 45%         |
| 20           | 0.40             | 0.48           | 20%         |
| 22           | 0.39             | 0.45           | 15%         |
| 24           | 0.33             | 0.26           | -21%        |
| 26           | 0.20             | 0.12           | -40%        |

Table 4. Accuracy for the test dataset of MNIST without and with the CH filter, for the case of random damage in rows. The improvement is computed following (16).

We observe a similar pattern for the case of random damaged pixels in Table 5 we observe a similar pattern. For low percentages of damaged pixels the improvement of adding the CH filter is negligible and the accuracies with and without the filter are quite high. As we increase the percentage of damaged pixels the improvement escalates until it reaches 45% for a scenario with 80% of the pixels randomly damaged. For this case the accuracy prediction without the filter is just 0.55, but thanks to the filter it significantly increases to a decent value of 0.8. For larger percentage of pixels the improvement and accuracies drop, and in the limit towards 100% of damaged pixels we get close to the accuracy of a dummy classifier. This is due to the large loss of information that the original images have suffered.

| Damaged pixels | Without CH filter | With CH filter | Improvement |
|----------------|-------------------|----------------|-------------|
| 30%            | 0.93              | 0.93           | 0%          |
| 40%            | 0.96              | 0.96           | 0%          |
| 50%            | 0.91              | 0.95           | 4%          |
| 60%            | 0.8               | 0.94           | 18%         |
| 70%            | 0.75              | 0.93           | 24%         |
| 80%            | 0.55              | 0.8            | 45%         |
| 90%            | 0.39              | 0.46           | 18%         |
| 92%            | 0.32              | 0.37           | 16%         |
| 94%            | 0.33              | 0.34           | 3%          |
| 96%            | 0.20              | 0.23           | 15%         |

Table 5. Accuracy for the test dataset of MNIST without and with the CH filter, for the case of random damage in pixels. The improvement is computed following (16).

In Figure 6 we depict some specific examples for which the label is only predicted correctly after applying the CH filter to the damaged image. These are just some particular samples out of the 10,000
images contained in the test dataset of MNIST, and for some of them the opposite effect can occur: that the label is correctly predicted for the damaged image and following the inpainting process it is predicted incorrectly. However, we have shown in Table 4 and Table 5 that overall the CH filter increases the global accuracy for images with low to moderate damage, and consequently we expect that scenarios such as the ones displayed in Figure 6 are much more common than the opposite ones.

| Initial Image | True label | Damaged image | $P_D$ | Inpainted image | $P_I$ |
|---------------|------------|---------------|-------|-----------------|-------|
| 4             | 4          | 6             | 4     |                 |
| 0             | 0          | 8             | 0     |                 |
| 5             | 5          | 4             | 5     |                 |

Figure 6. Particular examples of label predictions for MNIST samples with various types of damage: customized damage C, random damage in 16 rows and random damage in 70% of pixels. The label of the damaged image is wrongly predicted, while the label for the inpainted image is correctly predicted. $P_D$ and $P_I$ represent the label predictions of the damaged and inpainted images, respectively.

4. Discussion and conclusions

We have quantified the prediction improvement of employing a CH image inpainting filter to restore damaged images which are then passed into a neural network. We combined a finite-volume scheme with a neural network for pattern recognition to develop an integrated algorithm summing up the process of adding damage to the images and then predicting their label. Our results for the MNIST dataset suggest that, in general, the accuracy is improved for a wide range of low to moderate damages, while for some particular cases we reach improvements of up to 50%. We also provide the image inpainting outcome of multiple damage scenarios and the benefits of adding the CH filter to predict the label of the image are easily visible.

We believe that our results employing the MNIST dataset lay the foundations towards the application of image inpainting in more complex datasets. Here we have demonstrated the benefit of combining the fields of image inpainting with machine learning, and we believe that many applications can take advantage of it. For instance there are applications such as medical images from MRI or satellite observations where there is typically some inherent noise or damage involved and where there may be potential to employ tools from machine learning as was done here.
MNIST is one of the most well-known and convenient benchmark datasets. It is possible that applying our methodology to increasingly complex datasets might bring about new challenges. At the same time we have relied on two main assumptions about the damage and the images: the first one is that the images are binary, leading to the standard CH potential in (4) that has only two wells (i.e. one for each of the two colours); the second one is that the damage is not blind, meaning that the location of the damage is known. Performing image inpainting without these two assumptions becomes substantially more involved as already pointed out in [15, 20, 47]. We will be exploring these and related questions in future works.

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