CONVERGENCE FOR THE FRACTIONAL $p$-LAPLACIAN AND ITS CORRESPONDING EXTENDED NIRENBERG PROBLEM

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Abstract. The main objective of this paper is to establish the convergence for the fractional $p$-Laplacian of nonnegative sequence of functions with $p > 2$. Further, we show the blow-up phenomena for solutions to the extended Nirenberg problem modeled by fractional $p$-Laplacian with the prescribed negative functions.

1. Introduction and main results

Let $n \geq 1$, $p \geq 2$ and $0 < \sigma < 1$. Define the fractional $p$-Laplacian $(-\Delta)^\sigma_p$ as follows:

$$(-\Delta)^\sigma_p u(x) = c_{n,\sigma p} \text{P.V.} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+\sigma p}} dy,$$

where $c_{n,\sigma p}$ is a positive constant and P.V. represents the Cauchy principal value. It is worth pointing out that $(-\Delta)^\sigma_p$ becomes the linear fractional Laplacian operator $(-\Delta)^\sigma$ if $p = 2$, while it is a nonlinear nonlocal operator if $p > 2$. The definition of $(-\Delta)^\sigma_p u$ is valid under the condition that $u \in C^{\sigma p+\alpha}_{\text{loc}}(\mathbb{R}^n) \cap L^{\sigma p}(\mathbb{R}^n)$ for some $\alpha > 0$, where $C^{\sigma p+\alpha}_{\text{loc}}(\mathbb{R}^n) := C^{\lfloor \sigma p + \alpha \rfloor}_{\text{loc}}[\sigma p + \alpha - \lfloor \sigma p + \alpha \rfloor]_{\text{loc}}$ with $\lfloor \sigma p + \alpha \rfloor$ denoting the integer part of $\sigma p + \alpha$.

Recently, Du, Jin, Xiong and Yang [5] derived the following fact:

"If $u_i \rightharpoonup u$ in $C^{\sigma p + \alpha}_{\text{loc}}$ as $i \to \infty$, and $\{(-\Delta)^\sigma u_i\}$ converges pointwisely,

then $(-\Delta)^\sigma u_i \rightharpoonup (-\Delta)^\sigma u - \theta$ for some $\theta \geq 0,"$

where the constant $\theta$ may be greater than zero, which is different from the classical Laplacian operator. However, the proof for the linear fractional Laplacian cannot be directly used to deal with the nonlinear case. So, in this paper we aim to overcome the nonlinear difficulty for the fractional $p$-Laplacian operator and prove that the above fact also holds for the nonlinear nonlocal operator $(-\Delta)^\sigma_p$ with $p > 2$. The principal result of this paper is stated as follows.

**Theorem 1.1.** Let $n \geq 1$, $p > 2$, $0 < \sigma < 1$ and $\alpha > 0$. Assume that a sequence of nonnegative functions $\{u_i\} \subset L^{\sigma p}(\mathbb{R}^n) \cap C^{\sigma p+\alpha}_{\text{loc}}(\mathbb{R}^n)$ converges in $C^{\sigma p+\alpha}_{\text{loc}}(\mathbb{R}^n)$ to a function $u \in L^{\sigma p}(\mathbb{R}^n)$, and $\{(-\Delta)^\sigma u_i\}$ converges pointwisely in $\mathbb{R}^n$. Then for any $x \in \mathbb{R}^n$,

$$\lim_{i \to \infty} (-\Delta)^\sigma_p u_i(x) = (-\Delta)^\sigma_p u(x) - \theta,$$

**Date:** March 2, 2022.
where $\theta$ is a nonnegative constant given by

$$
\theta = c_{n, \sigma_p} \lim_{R \to \infty} \lim_{i \to \infty} \int_{B_R} \frac{u_i^{p-1}(x)}{|x|^{n+\sigma_p}} dx.
$$

Proof of Theorem 1.1 For any fixed $x \in \mathbb{R}^n$ and $R >> |x| + 1$, let

$$
(-\Delta)_p^\sigma u(x) - (-\Delta)_p^\sigma u_i(x) = \Phi_i(x, R) + \Phi(x, R) + \Psi_i(x, R).
$$

In light of the fact that $u_i \to u$ in $C^{\sigma p + \alpha}(B_R(0))$, we obtain that for each $0 < \varepsilon < 1$, there exists an integer $N > 0$ such that for every $i > N$,

$$
\|u_i - u\|_{C^{\sigma p + \alpha}(B_R(0))} \leq \varepsilon^{\min\{p, p-2\}}, \quad \|u_i\|_{C^{\sigma p + \alpha}(B_R(0))} + 1.
$$

Define

$$
\Phi_i(x, R \setminus \varepsilon) := \Phi_i(x, R) - \Phi_i(x, \varepsilon), \quad \mathcal{M} := \|u\|_{C^{\sigma p + \alpha}(B_R(0))} + 1,
$$

where $\Phi_i(x, \varepsilon)$ denotes the integral in $\Phi_i(x, R)$ with the domain $B_R(0)$ replaced by $B_\varepsilon(x)$. Using (1.2), we deduce that for $x, y \in B_R(0)$, $i > N$,

$$
\left| |u(x) - u(y)|^{p-2}(u(x) - u(y)) - |u_i(x) - u_i(y)|^{p-2}(u_i(x) - u_i(y)) \right|
\leq |u(x) - u(y)|^{p-2}|(u - u_i)(x) - (u - u_i)(y)|
+ |u(x) - u(y)|^{p-2}|u_i(x) - u_i(y)|^{p-2} |u_i(x) - u_i(y)|
\leq C(p, \mathcal{M})\|u_i - u\|_{L^{\infty}(B_R(0))} \leq C(p, \mathcal{M})\varepsilon^p,
$$

which yields that

$$
|\Phi_i(x, R \setminus \varepsilon)| \leq C(p, \mathcal{M})\varepsilon^p \int_{B_R(x) \setminus B_{\varepsilon}(x)} \frac{dy}{|x - y|^{n+\sigma_p}} \leq C(p, n, \sigma, \mathcal{M})\varepsilon^{1-\sigma p}. \quad (1.3)
$$

On the other hand, if $\sigma p + \alpha \in (0, 1)$, then it follows from (1.2) that

$$
|\Phi_i(x, \varepsilon)| \leq C(p, \sigma, \alpha, \mathcal{M}) \int_{B_\varepsilon(x)} \frac{|x - y|^{(\sigma p + \alpha)(p-1)}}{|x - y|^{n+\sigma_p}} \leq C(p, n, \sigma, \alpha, \mathcal{M})\varepsilon^{(\sigma p + \alpha)(p-2) + \alpha}. \quad (1.4)
$$

When $\sigma p + \alpha \in (1, \infty)$, utilizing (1.2) again, it follows from Taylor expansion that

$$
\left| |u_i(x) - u_i(y)|^{p-2}(u_i(x) - u_i(y)) - |\nabla u_i(x)(x - y)|^{p-2} \nabla u_i(x)(x - y) \right|
\leq C(p, \sigma, \alpha, \mathcal{M}) \left( |\nabla u_i(x)(x - y)|^{p-2} + |x - y|^{\min\{2, \sigma p + \alpha\}(p-2)} \right)|x - y|^{\min\{2, \sigma p + \alpha\}}
\leq C(p, \sigma, \alpha, \mathcal{M})|x - y|^{\min\{p, (\sigma + 1)p + \alpha - 2\}.
$$

where we utilized the following element inequality

\[ ||a||^p - 2 - |b||^p - 2b| \leq C(p)|a - b| (|a|^{p-2} + |b|^{p-2}), \quad \text{for } a, b \in \mathbb{R}^n.\]

By the same argument, we have

\[ |u(x) - u(y)|^p - 2(u(x) - u(y)) - |\nabla u(x)(x - y)|^p - 2\nabla u(x)(x - y)| \leq C(p, \sigma, \alpha, \mathcal{M})|x - y|^{\min\{p, (\sigma + 1)p + \alpha - 2\}}. \]

Therefore, we obtain that if \( \sigma p + \alpha \in (1, \infty) \),

\[ |\Phi_i(x, \varepsilon)| \leq C(p, \sigma, \alpha, \mathcal{M}) \int_{B_1(x)} \frac{|x - y|^{\min\{p, (\sigma + 1)p + \alpha - 2\}}}{|x - y|^{n + \sigma p}} dy \leq C(p, n, \sigma, \alpha, \mathcal{M}) \varepsilon^{\min\{1 - \sigma)p, p + \alpha - 2\}}, \quad (1.5) \]

where we utilized the anti-symmetry of \( \nabla u(x)(x - y) \) and \( \nabla u_i(x)(x - y) \) with regard to the center \( x \). Consequently, combining (1.3)–(1.5), we deduce that for every \( i > N \),

\[ |\Phi_i(x, R)| \leq C(p, n, \sigma, \alpha, \mathcal{M}) \left\{ \begin{array}{ll} \varepsilon^{\min\{1 - \sigma)p, (\sigma + \alpha)(p - 2) + \alpha\}}, & \text{if } \sigma p + \alpha \in (0, 1], \\
\varepsilon^{\min\{1 - \sigma)p, p + \alpha - 2\}}, & \text{if } \sigma p + \alpha \in (1, \infty), \end{array} \right. \]

which implies that

\[ \lim_{i \to \infty} \Phi_i(x, R) = 0. \quad (1.6) \]

Note that \( \{(-\Delta)^p u_i\} \) is a pointwise convergent sequence, we then deduce from (1.1) and (1.6) that

\[ \lim_{i \to \infty} \Psi_i(x, R) \text{ exists and is finite.} \quad (1.7) \]

Since \( u \in \mathcal{L}_{\sigma p}(\mathbb{R}^n) \) and \( R >> |x| + 1 \), then

\[ \limsup_{R \to \infty} \int_{B_R(0)} \frac{|u(x) - u(y)|^{p-1}}{|x - y|^{n + \sigma p}} dy \leq \limsup_{R \to \infty} \frac{R}{R - |x|} \left( \frac{R}{R - |x|} \right)^{n + \sigma p} \int_{B_R(0)} \frac{C(p)(u^{p-1}(x) + u^{p-1}(y))}{|y|^{n + \sigma p}} dy = 0, \]

which yields that

\[ \lim_{R \to \infty} \Theta(x, R) = 0. \]

This, in combination with (1.1) and (1.6)–(1.7), leads to that \( \lim_{R \to \infty} \lim_{i \to \infty} \Psi_i(x, R) \) exists and is finite,

\[ (-\Delta)^p u(x) - \lim_{i \to \infty} (-\Delta)^p u_i(x) = \lim_{R \to \infty} \lim_{i \to \infty} \Psi_i(x, R). \quad (1.8) \]

Denote

\[ \mathcal{K}_1 := -u_i^{p-2}(y)u_i(x), \]
\[ \mathcal{K}_2 := \left( u_i^{p-2}(y) - |u_i(x) - u_i(y)|^{p-2} \right) u_i(x), \]
\[ \mathcal{K}_3 := -\left( u_i^{p-2}(y) - |u_i(x) - u_i(y)|^{p-2} \right) u_i(y), \]
\[ \Theta := |u_i(x) - u_i(y)|^{p-2}(u_i(x) - u_i(y)). \]
Then we have
\[ u_i^{p-1}(y) - \sum_{j=1}^{3} |K_j| \leq \Theta = u_i^{p-1}(y) + \sum_{j=1}^{3} K_j \leq u_i^{p-1}(y) + \sum_{j=2}^{3} |K_j|. \]  
(1.9)

For any given \( \varepsilon > 0 \), it follows from Young’s inequality that
\[ |K_1| \leq \varepsilon u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-1}(x). \]  
(1.10)

We now divide into three cases to estimate \( K_2 \) and \( K_3 \) in the following.

**Case 1.** Consider \( 2 < p \leq 3 \). Since
\[ u_i^{p-2}(y) \leq (|u_i(y) - u_i(x)| + u_i(x))^{p-2} \leq |u_i(y) - u_i(x)|^{p-2} + u_i^{p-2}(x), \]
\[ |u_i(y) - u_i(x)|^{p-2} \leq u_i^{p-2}(y) + u_i^{p-2}(x), \]
then
\[ |u_i^{p-2}(y) - |u_i(x) - u_i(y)|^{p-2}| \leq u_i^{p-2}(x). \]

Hence it follows from Young’s inequality that
\[ |K_2| \leq u_i^{p-1}(x), \quad |K_3| \leq \varepsilon u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-1}(x). \]  
(1.11)

Substituting (1.10) into (1.11), we derive
\[ (1 - 2\varepsilon) u_i^{p-1}(y) - \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-1}(x) \leq \Theta \leq (1 + \varepsilon) u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-1}(x). \]  
(1.12)

**Case 2.** Consider the case when \( p > 3 \) is an integer. From the binomial theorem and Young’s inequality, we have
\[ (a + b)^{p-2} = a^{p-2} + \sum_{j=1}^{p-2} \binom{p-2}{j} a^{p-2-j} b^j \leq \left(1 + \varepsilon\right)a^{p-2} + \varepsilon \left(1 + \varepsilon\right)a^{p-2} + C(p) \sum_{j=1}^{p-2} \varepsilon^{p-2-j} \]
\[ \leq \left(1 + \varepsilon\right)a^{p-2} + \frac{C(p)}{\varepsilon^{p-3}} b^{p-2}, \quad \text{for any } a, b \geq 0. \]  
(1.13)

Using (1.11), we deduce
\[ u_i^{p-2}(y) \leq (|u_i(y) - u_i(x)| + u_i(x))^{p-2} \leq (1 + \varepsilon)|u_i(x) - u_i(y)|^{p-2} + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x), \]
which implies that
\[ u_i^{p-2}(y) - |u_i(y) - u_i(x)|^{p-2} \leq \varepsilon|u_i(x) - u_i(y)|^{p-2} + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x) \]
\[ \leq \varepsilon(1 + \varepsilon) u_i^{p-2}(y) + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x). \]

Analogously,
\[ |u_i(y) - u_i(x)|^{p-2} - u_i^{p-2}(y) \leq \varepsilon u_i^{p-2}(y) + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x). \]

Hence, we have
\[ |u_i^{p-2}(y) - |u_i(y) - u_i(x)|^{p-2}| \leq \varepsilon(1 + \varepsilon) u_i^{p-2}(y) + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x). \]  
(1.14)
Utilizing (1.14) and Young’s inequality, we obtain
\[ |K_2| \leq \varepsilon (1 + \varepsilon) u_i^{p-2}(y) u_i(x) + \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-1}(x) \leq \varepsilon u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-1}(x), \]
\[ |K_3| \leq \varepsilon (1 + \varepsilon) u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-2}(x) u_i(y) \leq \varepsilon (2 + \varepsilon) u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-1}(x), \]
which, in combination with (1.13)–(1.14), gives that
\[ \Theta \leq (1 + 3\varepsilon + \varepsilon^2) u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-1}(x), \] (1.15)
\[ \Theta \geq (1 - 4\varepsilon - \varepsilon^2) u_i^{p-1}(y) - \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-1}(x). \] (1.16)

**Case 3.** Consider the case when \( p > 3 \) is not an integer. On one hand, making use of (1.13), we obtain
\[ |u_i(x) - u_i(y)|^{p-2} \leq (u_i(x) + u_i(y))^{[p-2]+(p-[p])} \]
\[ = \left( (1 + \varepsilon) u_i^{[p-2]}(y) + \frac{C(p)}{\varepsilon^{p-3}} u_i^{[p-2]}(x) \right) \left( u_i^{p-[p]}(y) + u_i^{p-[p]}(x) \right) \]
\[ = (1 + \varepsilon) u_i^{p-2}(y) + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x) \]
\[ + (1 + \varepsilon) u_i^{p-2}(y) u_i^{p-[p]}(x) + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x). \] (1.17)
From Young’s inequality, we deduce
\[ \frac{C(p)}{\varepsilon^{[p-3]}} u_i^{[p-2]}(y) u_i^{[p-2]}(x) \leq \varepsilon u_i^{p-2}(y) + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x), \] (1.18)
\[ (1 + \varepsilon) u_i^{[p-2]}(y) u_i^{p-[p]}(x) \leq \varepsilon u_i^{p-2}(y) + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x). \] (1.19)
Substituting (1.18)–(1.19) into (1.17), it follows that
\[ |u_i(x) - u_i(y)|^{p-2} - u_i^{p-2}(y) \leq 3\varepsilon u_i^{p-2}(y) + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x). \] (1.20)

On the other hand, using (1.18) again, we have
\[ u_i^{p-2}(y) \leq (|u_i(y) - u_i(x)| + u_i(x))^{p-2} + u_i^{p-[p]} \]
\[ \leq \left( (1 + \varepsilon) |u_i(x) - u_i(y)|^{p-[p]} + \frac{C(p)}{\varepsilon^{p-3}} u_i^{[p-2]}(x) \right) \]
\[ \cdot \left( |u_i(x) - u_i(y)|^{p-[p]} + u_i^{p-[p]}(x) \right) \]
\[ = (1 + \varepsilon) |u_i(x) - u_i(y)|^{p-2} + \frac{C(p)}{\varepsilon^{p-3}} u_i^{[p-2]}(x) |u_i(x) - u_i(y)|^{p-[p]} \]
\[ + (1 + \varepsilon) u_i^{p-[p]}(x) |u_i(x) - u_i(y)|^{p-2} + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x). \] (1.21)
It follows from Young’s inequality that
\[ \frac{C(p)}{\varepsilon^{p-3}} u_i^{[p-2]}(x) |u_i(x) - u_i(y)|^{p-[p]} \leq \varepsilon |u_i(x) - u_i(y)|^{p-2} + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x), \] (1.22)
\[ (1 + \varepsilon) u_i^{p-[p]}(x) |u_i(x) - u_i(y)|^{p-2} \leq \varepsilon |u_i(x) - u_i(y)|^{p-2} + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x). \] (1.23)
Combining (1.20)–(1.23), we deduce
\[ u_i^{p-2}(y) - |u_i(x) - u_i(y)|^{p-2} \leq \varepsilon |u_i(x) - u_i(y)|^{p-2} + \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-2}(x) \leq 3\varepsilon(1 + 3\varepsilon)u_i^{p-2}(y) + \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-2}(x). \]

This, together with (1.20) again, gives that
\[ \left| u_i^{p-2}(y) - |u_i(x) - u_i(y)|^{p-2} \right| \leq \varepsilon(1 + 3\varepsilon)u_i^{p-2}(y) + \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-2}(x). \] (1.24)

In light of (1.24), it follows from Young’s inequality that
\[ |K_2| \leq 3\varepsilon(1 + 3\varepsilon)u_i^{p-2}(y)u_i(x) + \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-1}(x) \leq \varepsilon u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-1}(x), \] (1.25)
\[ |K_3| \leq 3\varepsilon(1 + 3\varepsilon)u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-2}(x)u_i(y) \leq \varepsilon(4 + 9\varepsilon)u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-1}(x). \] (1.26)

Therefore, substituting (1.10) and (1.25) (1.26) into (1.9), we derive
\[ \Theta \leq (1 + 5\varepsilon + 9\varepsilon^2)u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-1}(x), \] (1.27)
\[ \Theta \geq (1 - 6\varepsilon - 9\varepsilon^2)u_i^{p-1}(y) - \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-1}(x). \] (1.28)

Observe that
\[ \lim_{R \to \infty} \lim_{i \to \infty} \int_{B_R(0)} \frac{u_i^{p-1}(x)}{|x - y|^{n+sp}} dy = \lim_{R \to \infty} \int_{B_R(0)} \frac{dy}{|x - y|^{n+sp}} = u^{p-1}(x) \lim_{R \to \infty} \int_{B_R(0)} \frac{dy}{|x - y|^{n+sp}} = 0. \] (1.29)

Since \( \lim_{R \to \infty} \lim_{i \to \infty} \Psi_i(x, R) \) exists and is finite, it follows from (1.12), (1.13)–(1.10) and (1.27) (1.29) that
\[ \lim_{R \to \infty} \lim_{i \to \infty} \Psi_i(x, R) \leq c_{n, sp}(1 + \varepsilon^{(1)}_p) \lim_{R \to \infty} \liminf_{i \to \infty} \int_{B_R(0)} \frac{u_i^{p-1}(y)}{|x - y|^{n+sp}} dy, \]
\[ \lim_{R \to \infty} \limsup_{i \to \infty} \Psi_i(x, R) \geq c_{n, sp}(1 - \varepsilon^{(2)}_p) \limsup_{R \to \infty} \limsup_{i \to \infty} \int_{B_R(0)} \frac{u_i^{p-1}(y)}{|x - y|^{n+sp}} dy, \]
where
\[
\varepsilon_p^{(1)} = \begin{cases} 
\varepsilon, & \text{if } 2 < p \leq 3, \\
\varepsilon(3 + \varepsilon), & \text{if } p > 3 \text{ is an integer}, \\
\varepsilon(5 + 9\varepsilon), & \text{if } p > 3 \text{ is not an integer}, 
\end{cases}
\]
\[
\varepsilon_p^{(2)} = \begin{cases} 
2\varepsilon, & \text{if } 2 < p \leq 3, \\
\varepsilon(4 + \varepsilon), & \text{if } p > 3 \text{ is an integer}, \\
3\varepsilon(2 + 3\varepsilon), & \text{if } p > 3 \text{ is not an integer}.
\end{cases}
\]

Due to the fact that \( R >> |x| + 1 \), we have
\[
\frac{(R - |x|)|y|}{R} \leq |y - x| \leq \frac{(R + |x|)|y|}{R}, \quad \text{for } y \in B_R^+(0).
\]

Hence, we deduce
\[
\lim_{R \to \infty} \lim_{i \to \infty} \Psi_i(x, R) \leq c_{n, \sigma p}(1 + \varepsilon_p^{(1)}) \lim_{R \to \infty} \liminf_{i \to \infty} \int_{B_R^+(0)} \frac{u_i^{p-1}(y)}{|y|^{n+\sigma p}} \, dy,
\]
\[
\lim_{R \to \infty} \lim_{i \to \infty} \Psi_i(x, R) \geq c_{n, \sigma p}(1 - \varepsilon_p^{(2)}) \lim_{R \to \infty} \limsup_{i \to \infty} \int_{B_R^+(0)} \frac{u_i^{p-1}(y)}{|y|^{n+\sigma p}} \, dy.
\]

By virtue of the arbitrariness of \( \varepsilon \) and \( \{u_i\} \) is nonnegative, we obtain
\[
\lim_{R \to \infty} \lim_{i \to \infty} \Psi_i(x, R) = c_{n, \sigma p} \lim_{R \to \infty} \lim_{i \to \infty} \int_{B_R^+(0)} \frac{u_i^{p-1}(y)}{|y|^{n+\sigma p}} \, dy \geq 0.
\]

This, together with (1.30), yields that Theorem 1.1 holds.

In order to show that the limit constant \( \theta \) captured in Theorem 1.1 may be positive, we consider a sequence of nonnegative functions in the following. Choose a smooth cut-off function \( \eta \) satisfying that
\[
\eta(t) \equiv 0 \text{ in } (-\infty, 0], \quad \eta(t) \equiv 1 \text{ in } [1, \infty), \quad \text{and } 0 \leq \eta(t) \leq 1 \text{ in } [0, 1].
\]

Then for any \( 0 < s < t \) and \( j \geq 1 \), define
\[
v_j(x) := j^{-s}w_j(R_j^{-1}x), \quad w_j(x) := \begin{cases} j^s + j^t \phi(x), & \text{in } B_6, \\
(1 - \psi(x))(j^s + j^t), & \text{in } B_6^c,
\end{cases}
\]
(1.31)

where \( \phi(x) = \eta(|x| - 3) \), and \( \psi(x) = \eta(|x| - 6) \), \( R_j = j^{\frac{(t-s)(p-1)}{sp}} \beta \)
\[
\beta := c_{n, \sigma p} \left( \int_{B_1 \setminus B_3} \frac{\phi^{p-1}(y)}{|y|^{n+\sigma p}} \, dy + \int_{B_1 \setminus B_4} \frac{dy}{|y|^{n+\sigma p}} + \int_{B_5^c} \frac{(1 - \psi(y))^{p-1}}{|y|^{n+\sigma p}} \, dy \right).
\]
(1.32)

Example 1.2. Let \( n \geq 1 \), \( p > 2 \) and \( 0 < \sigma < 1 \). If condition (1.31) holds, then we obtain that \( v_j \) converges to 1 in \( C^2_{\text{loc}}(\mathbb{R}^n) \), and
\[
\lim_{i \to \infty} (-\Delta)^{\sigma} p v_j(x) = -1.
\]

Remark 1.3. We here would like to point out that the examples constructed in Example 1.2 and Theorem 2.1 were first given in [5].
Proof. It is easily seen from (2.1) that \( v_j \in C_c^\infty(\mathbb{R}^n) \), \( v_j \geq 0 \) in \( \mathbb{R}^n \), \( v_j = 1 \) in \( B_{R_j} \), and \( \|v_j - 1\|_{C^{2,\alpha}} \to 0 \), as \( i \to \infty \). A direct computation gives that

\[
(-\Delta)_p^s v_j(x) = j^{-s(p-1)} R_j^{-\sigma_p} (-\Delta)_p^s w_j(R_j^{-1} x), \quad \text{for } x \in B_{R_j}, \quad (1.33)
\]

For any fixed \( x \in \mathbb{R}^n \), we have

\[
j^{-t(p-1)} (-\Delta)_p^s w_j(R_j^{-1} x) = -c_{n,\sigma_p} \int_{B_4 \setminus B_3} \frac{\phi^{p-1}(y)}{R_j^{1-n}|x-y|^{n+\sigma_p}} dy - c_{n,\sigma_p} \int_{B_{4j} \setminus B_4} \frac{\phi^{p-1}(y)}{R_j^{1-n}|x-y|^{n+\sigma_p}} dy
\]

\[
+ c_{n,\sigma_p} \int_{B_4} \frac{\psi(y)}{j^{1-\sigma}} \int_{B_{4j} \setminus B_4} \frac{\phi^{p-1}(y)}{R_j^{1-n}|x-y|^{n+\sigma_p}} dy
\]

\[
- c_{n,\sigma_p} \int_{B_4} \frac{(1 - \psi(y))}{j^{1-\sigma}} \int_{B_{4j} \setminus B_4} \frac{\phi^{p-1}(y)}{R_j^{1-n}|x-y|^{n+\sigma_p}} dy
\]

\[
\to -\beta, \quad \text{as } j \text{ goes to } \infty,
\]

where \( \beta \) is defined by (1.32). This, together with (1.33), gives that

\[
\lim_{j \to \infty} (-\Delta)_p^s v_j(x) = \beta^{-1} \lim_{j \to \infty} j^{-t(p-1)} (-\Delta)_p^s w_j(R_j^{-1} x) = -1, \quad \text{in } \mathbb{R}^n.
\]

The proof is complete. \( \square \)

2. Blow-up analysis for the extended fractional Nirenberg problem

The extended fractional Nirenberg problem is equivalent to investigating the following equation

\[
(-\Delta)_p^s u(x) = K(x) u^{q}(x), \quad \text{for } x \in \mathbb{R}^n, \quad (2.1)
\]

where \( p \geq 2 \) and \( q \in \mathbb{R} \). It has been shown in [5] that there arises blow-up phenomena for the linear fractional Laplacian due to the nonzero constant \( \theta \) captured in Theorem 1.1. Specially, for \( p = 2 \), the compactness of solutions to (2.1) will fail in the region where \( K \) is negative. In the following, we follow the proof of Theorem 1.3 in [5] and extend the result to the nonlinear case of \( p > 2 \). On the other hand, when \( K \) is positive, Jin, Li and Xiong [6,8] derived a priori estimates for the fractional equation (2.1) with \( p = 2 \).

While these above-mentioned works are related to the fractional Nirenberg problem, there is another direction of research to study the classical elliptic equation 

\[-\Delta u = K(x) u^p. \]

When \( n = 1, 2 \) and \( 1 < p < \infty \), or \( n \geq 3 \) and \( 1 < p < \frac{n+2}{n-2} \), \( p \) is called a subcritical Sobolev exponent, while it is the critical Sobolev exponent if \( n \geq 3 \) and \( p = \frac{n+2}{n-2} \). In particular, the elliptic equation in the case of critical Sobolev exponent corresponds to the Nirenberg problem, which is to seek a new metric conformal to the flat metric on \( \mathbb{R}^n \) so that its scalar curvature is \( K(x) \). Generally, it needs to establish priori estimates of the solutions for the purpose of obtaining the existence of solutions. We refer to [3,4] for the subcritical case. With regard to the critical case, see [1,9,11] for positive functions \( K \) and [2,10] for \( K \) changing signs, respectively.
Theorem 2.1. Assume that \( n \geq 1, p > 2, 0 < \sigma < 1, q \in \mathbb{R} \) and \( s > -\frac{\sigma p}{p-1} \). Then there exist two positive constants \( c_0 = c_0(n, \sigma, p, q, s) \) and \( C_0 = C_0(n, \sigma, p, q, s) \), a sequence of functions \( \{K_j\} \subset C^\infty(\mathbb{R}^n) \) satisfying

\[-C_0 \leq K_j(x) \leq -c_0, \quad c_0 \leq \|\nabla K_j(x)\| \leq C_0, \quad \|\nabla^2 K_j(x)\| \leq C_0, \quad \text{in } B_2,
\]

and a sequence of positive functions \( \{u_j\} \subset C^\infty(\mathbb{R}^n) \) such that

\[(-\Delta)_p^{\sigma} u_j(x) = K_j(x) u_j^{q(p-1)}(x), \quad \text{for } x \in \mathbb{R}^n, \quad |x|^s u_j(x) \to 1, \quad \text{as } |x| \to \infty,\]

and

\[
\min_{B_1} u_j \to \infty, \quad \text{as } j \to \infty.
\]

Proof. Let \( \eta \) and \( \phi \) be defined in (1.30) and (1.31). For \( q \in \mathbb{R} \) and \( s > -\frac{\sigma p}{p-1} \), let

\[u_j(x) = \begin{cases} j + j^{\eta} \phi(x), & \text{in } B_R, \\ (1 - \varphi(x))(j + j^{\eta}) + \varphi(x)|x|^{-s}, & \text{in } B_R^c, \end{cases}\]

where \( \varphi(x) = \eta(|x|-R) \) and \( R = R(n, p, \sigma, \sigma, s, j) > 9 \) is a sufficiently large constant to be determined later. Then \( u_j \in C^\infty(\mathbb{R}^n) \cap L_\sigma(\mathbb{R}^n) \) and \( u_j > 0 \) in \( \mathbb{R}^n \). Denote

\[K_j(x) := \frac{(-\Delta)_p^{\sigma} u_j(x)}{u_j^{q(p-1)}(x)}, \quad \text{in } \mathbb{R}^n.\]

Then \( K_j \in C^\infty(\mathbb{R}^n) \). Moreover, \( \{K_j\} \) satisfies the following properties: there exists four positive constants \( C_i := C_i(n, \sigma, p), i = 1, 2, 3, 4 \), such that for every \( j \geq 1 \),

\[(K1) \quad -C_1 \leq K_j(x) \leq -C_2, \quad \text{and } \sum_{i=1}^3 |\nabla^i K_j(x)| \leq C_3 \text{ in } B_2;\]

\[(K2) \quad \nabla^2 K_j(0) \leq -C_4 I_n, \quad \text{where } I_n \text{ denotes } n \times n \text{ identity matrix.}\]

We first prove (K1). Observe that

\[
c_{n, \sigma, p}^{-1} K_j(x) = -\int_{B_n \setminus B_1} \frac{\phi^{p-1}(y)}{|x-y|^{n+\sigma p}} \, dy - \int_{B_n \setminus B_1} \frac{dy}{|x-y|^{n+\sigma p}} + \int_{B_n^c} |A_\sigma(y)|^{p-2} A_\sigma(y) \, dy := \sum_{i=1}^3 J_i, \tag{2.2}
\]

where \( A_\sigma(y) := \varphi(y) - 1 + j^{-1-\eta} \varphi(y) - j^{-\eta} |y|^{-s} \varphi(y) \). For simplicity, let

\[
\gamma := \gamma(n, \sigma, p) = \int_{B_1^c} \frac{dy}{|y|^{n+\sigma p}} = \frac{|\mathbb{S}^{n-1}|}{\sigma p},
\]

\[
\tau := \tau(n, \sigma, p, s) = \int_{B_1^c} \frac{dy}{|y|^{n+\sigma p+s(p-1)}} = \frac{|\mathbb{S}^{n-1}|}{\sigma p + s(p-1)}.
\]

A straightforward computation yields that

\[0 \geq J_1 \geq -\int_{B_1(x)} \frac{dy}{|x-y|^{n+\sigma p}} = -\gamma,
\]

and

\[-\gamma \leq J_2 \leq -\int_{B_{R-2} \setminus B_0} \frac{dy}{|y|^{n+\sigma p}} = -(6^{-\sigma p} - (R-2)^{-\sigma p})\gamma.\]
Furthermore, after differentiating (2.2), it follows from a similar calculation that
\[ |J_3| \leq 2^{(\sigma+1)p+n-2} \int_{B_R^c} \frac{(1+j^{1-q})^{p-1}+j^{-q(p-1)}|y|^{-s(p-1)}}{|y|^{n+\sigma p}} dy \]
\[ = 2^{(\sigma+1)p+n-2} \gamma (1+j^{1-q})^{p-1}R^{-\sigma p} + 2^{(\sigma+1)p+n-2}j^{-q(p-1)}R^{-\sigma p-s(p-1)}. \]

For a sufficiently large \( R > 9 \), we have
\[ (R - 2)^{-\sigma p \gamma} + 2^{(\sigma+1)p+n-2}R^{-\sigma p} \left( \gamma (1+j^{1-q})^{p-1} + \tau j^{-q(p-1)}R^{-s(p-1)} \right) \leq \frac{\gamma6^{-\sigma p}}{2}, \]
which implies that
\[ -3c_n,\sigma \gamma \leq K_j(x) \leq -\frac{c_n,\sigma \gamma 6^{-\sigma p}}{2}, \quad \forall \ x \in B_2, \ j \geq 1. \]
Furthermore, after differentiating (2.2), it follows from a similar calculation that
\[ \sum_{i=1}^{3} |\nabla^i K_j(x)| \leq C(n,\sigma,p), \quad \text{for} \ x \in B_2, \ j \geq 1. \]

We proceed to verify property (K2). A simple calculation shows that for \( y \in B_1^c \),
\[ \partial_{x_k x_l}^2 \left( \frac{1}{|x-y|^{n+\sigma p}} \right)(0) = \frac{(n+\sigma p)[(n+\sigma p+2)y_k y_l - \delta_{kl}|y|^2]}{|y|^{n+\sigma p+4}}. \quad (2.3) \]

Since the integral domain is symmetric, then we see from (2.2), (2.3) that
\[ \partial_{x_k x_l}^2 K_j(0) = 0, \quad \text{for} \ k \neq l. \]
If \( k = l \), it follows from the radial symmetry of \( \phi \) and \( \varphi \) that
\[ [(n+\sigma p)c_n,\sigma \gamma]^{-1} \partial_{x_k x_k}^2 K_j(0) \]
\[ = \frac{\sigma p + 2}{n} \left( \int_{B_1^c} \frac{\phi^{p-1}(y)}{|y|^{n+\sigma p+2}} dy + \int_{B_R^c} \frac{|A_\varphi(y)|^{p-2}(j^{1-q} \varphi(y) - A_\varphi(y))}{|y|^{n+\sigma p+2}} dy \right) \]
\[ - \frac{\sigma p + 2}{n} \left( \int_{B_1^c \setminus B_1} \frac{dy}{|y|^{n+\sigma p+2}} - j^{-1-q} \int_{B_R^c} \frac{|A_\varphi(y)|^{p-2} \varphi(y)}{|y|^{n+\sigma p+2}} dy \right) \]
\[ \leq -|B_1| \left( 4^{-(\sigma p+2)} - R^{-(\sigma p+2)} - 3j^{1-q} R^{-(\sigma p+2)} \right) \]
\[ \leq -|B_1| 4^{-(\sigma p+3)}, \quad \text{for a sufficiently large} \ R > 9, \]
where we used the fact that \( |A_\varphi(y)|^{p-2} \varphi(y) \leq 3 \) in \( B_1^c \). That is, property (K2) holds.

From the radial symmetry of \( u_j \) with respect to the origin, we know that \( K_j \) is also radially symmetric. Then we have
\[ \nabla K_j(0) = 0, \]
which, together with (K1)–(K2), leads to that for \( j \geq 1, \)
\[ |\nabla K_j(x)| \geq c_1, \quad \text{in} \ B_{2\varepsilon_0}(4\varepsilon_0 e_1), \quad (2.4) \]
where \( e_1 = (1,0,\ldots,0) \in \mathbb{R}^n, \varepsilon_0 := \varepsilon_0(n,p,\sigma) \in (0,1/4) \) is a small constant and \( c_1 := c_1(n,p,\sigma) \) is a positive constant.

Define
\[ \tilde{u}_j(x) := \varepsilon_0^6 u_j(\varepsilon_0(x + 4e_1)), \quad \text{and} \quad \tilde{K}_j(x) := \varepsilon_0^{\sigma p-s(p-1)(q-1)} K_j(\varepsilon_0(x + 4e_1)). \]
Therefore,

\[ (-\Delta)^{\sigma} \bar{u}_j = \bar{K}_j(x) \bar{u}_j^{(p-1)}, \quad \text{for } x \in \mathbb{R}^n. \]

Then combining (K1) and (2.4), we obtain

\[-\bar{C} \leq \bar{K}_j(x) \leq -\bar{c}, \quad \bar{c} \leq |\nabla \bar{K}_j(x)| \leq \bar{C}, \quad \text{in } B_2,\]

where \( \bar{c} = \bar{c}(n, \sigma, p, q, s) \) and \( \bar{C} = \bar{C}(n, \sigma, p, q, s) \). Moreover, recalling the definition of \( u_j \), we have

\[ \lim_{|x| \to \infty} |x|^s \bar{u}_j = 1, \quad \text{and} \quad \min_{\mathbb{R}^s_+} \bar{u}_j = \epsilon_0^s j \to \infty, \quad \text{as } j \to \infty. \]

The proof is finished. \( \square \)

**Acknowledgements.** The author would like to thank Prof. C.X. Miao for his constant encouragement and useful discussions. The author was partially supported by CPSF (2021M700358).

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