A MULTIVARIATE CLT FOR BOUNDED DECOMPOSABLE RANDOM VECTORS WITH THE BEST KNOWN RATE

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Abstract

We prove a multivariate central limit theorem with explicit error bound on a non-smooth function distance for sums of bounded decomposable \(d\)-dimensional random vectors. The decomposition structure is similar to that of Barbour, Karonki and Ruciński (1989) and is more general than the local dependence structure considered in Chen and Shao (2004). The error bound is of the order \(d^{1\frac{1}{4}}n^{-\frac{1}{2}}\), where \(d\) is the dimension and \(n\) is the number of summands. The dependence on \(d\), namely \(d^{1\frac{1}{4}}\), is the best known dependence even for sums of independent and identically distributed random vectors, and the dependence on \(n\), namely \(n^{-\frac{1}{2}}\), is optimal. We apply our main result to a random graph example.

Key words and phrases: Stein’s method; multivariate normal approximation; non-smooth function distance; rate of convergence; random graph counting

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1 INTRODUCTION

Let \(W = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i\) be a standardized sum of independent and identically distributed \(d\)-dimensional random vectors such that \(\mathbb{E}X_1 = 0\), \(\text{Cov}(X_1) = \mathbb{I}_d\) and \(\mathbb{E}|X_1|^3 < \infty\) where \(\mathbb{I}_d\) denotes the \(d\)-dimensional identity matrix and \(|\cdot|\) denotes the Euclidean norm of a vector. Bentkus [5] proved the following bound on a non-smooth function distance between the distribution of \(W\) and the standard \(d\)-dimensional Gaussian distribution:

\[
\sup_{A \in \mathcal{A}} |\mathbb{P}(W \in A) - \mathbb{P}(Z \in A)| \leq C' \mathbb{E}|X_1|^3 d^{\frac{1}{4}} n^{-\frac{1}{2}}
\]  

where \(\mathcal{A}\) denotes the collection of all the convex sets in \(\mathbb{R}^d\), \(Z\) is a \(d\)-dimensional standard Gaussian vector, and \(C'\) is an absolute constant.

In this paper, we aim to prove a multivariate central limit theorem with an error bound having the same order of magnitude in terms of \(d\) and \(n\) for the same non-smooth function distance as in (1.1), but for general bounded dependent random vectors. The dependence structure we consider is similar to that of Barbour, Karonki and Ruciński [4] and is more general than the local

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dependence structure considered in Chen and Shao \[10\]. The approach we use is the recursive approach in Stein’s method for multivariate normal approximation.

Stein’s method was introduced by Stein \[18\] for normal approximation and has become a powerful tool in proving distributional approximations. We refer to Barbour and Chen \[3\] for an introduction to Stein’s method. Stein’s method for multivariate normal approximation was first studied in Götze \[13\]. Along with an inductive argument for sums of independent random vectors, he proved an error bound whose dependence on the dimension is $d^{3/2}$ for the distance in (1.1). The dependence on the dimension was improved to $d^{3/4}$ in Bentkus \[6\] by using the same inductive approach and a Lindeberg-type argument. Using the recursive approach in Stein’s method, Rinott and Rotar \[17\] proved a multivariate normal approximation result for sums of bounded random vectors that allow for a certain decomposition. The error bound they obtained for the distance in (1.1) is typically of the order $O_d(n^{-3/2} \log n)$ with unspecified dependence on $d$ and an additional logarithmic factor. Recently, Fang and Röllin \[11\] proved a multivariate normal approximation result under the general framework of Stein coupling \(\text{(cf. Chen and Röllin} \[9\])}. For sums of locally dependent bounded random vectors, their bound for the distance in (1.1) is typically of the order $d^{7/4} n^{-1/2}$. Compare with the existing literature on bounding the non-smooth function distance in (1.1) for multivariate normal approximation for sums of bounded random vectors, our new result not only considers a decomposition structure more general than local dependence, but also obtains an error bound typically of the same order as for sums of independent and identically distributed random vectors in (1.1). Stein’s method has also been used to prove smooth function distances for multivariate normal approximation under various dependence structures, see, for example, Goldstein and Rinott \[12\], Raić \[15\], Chatterjee and Meckes \[7\] and Reinert and Röllin \[16\].

Many problems in random graph counting satisfy the decomposition structure considered in this paper. See, for example, Barbour, Karoński and Ruciński \[4\], Janson and Nowicki \[14\], Avram and Bertsimas \[1\], and Rinott and Rotar \[17\]. We will study an example from Rinott and Rotar \[17\] and show that the error bound obtained by our main theorem is better than that by Rinott and Rotar \[17\].

This paper is organized as follows. In the next section, we state our main result. In Section 3, we study a random graph counting problem. In Section 4, we prove our main result. Throughout this article, $|\cdot|$ denotes the Euclidean norm of a vector or the cardinality of a set, and $I_d$ denotes the $d$-dimensional identity matrix. Define $[n] := \{1, \ldots, n\}$ and $X_N := \sum_{i \in N} X_i$ for an index set $N$.

2 MAIN RESULTS

For a sum of standardized $d$-dimensional random vectors $W = \sum_{i=1}^n X_i$ with certain decomposition structure, we aim to bound the quantity

$$d_c(\mathcal{L}(W), \mathcal{L}(Z)) = \sup_{A \in \mathcal{A}} |\mathbb{P}(W \in A) - \mathbb{P}(Z \in A)|,$$

where $Z$ has standard $d$-dimensional Gaussian distribution and $\mathcal{A}$ denotes the collection of all the convex sets in $\mathbb{R}^d$. The following is our main theorem.
**Theorem 2.1.** Let \( W = \sum_{i=1}^{n} X_i \) be a sum of \( d \)-dimensional random vectors such that \( \mathbb{E} X_i = 0 \) and \( \text{Cov}(W) = \mathbb{I}_d \). Suppose \( W \) can be decomposed as follows:

\[
\forall i \in [n], \exists i \in N_i \subset [n] \text{ such that } W - X_{N_i} \text{ is independent of } X_i; \tag{2.2}
\]

\[
\forall i \in [n], j \in N_i, \exists N_i \subset N_{ij} \subset [n] \text{ such that } W - X_{N_{ij}} \text{ is independent of } \{X_i, X_j\}; \tag{2.3}
\]

\[
\forall i \in [n], j \in N_i, k \in N_{ij}, \exists N_{ij} \subset N_{ijk} \subset [n] \text{ such that } W - X_{N_{ijk}} \text{ is independent of } \{X_i, X_j, X_k\}. \tag{2.4}
\]

Suppose further that for each \( i \in [n], j \in N_i \) and \( k \in N_{ij} \),

\[
|X_i| \leq \beta, \ |N_i| \leq n_1, \ |N_{ij}| \leq n_2, \ |N_{ijk}| \leq n_3. \tag{2.5}
\]

Then there is a universal constant \( C \) such that

\[
d_c(\mathcal{L}(W), \mathcal{L}(Z)) \leq Cd^{1/4}n\beta^3n_1(n_2 + \frac{n_3}{d}), \tag{2.6}
\]

where \( d_c \) is defined as in (2.1) and \( Z \) is a \( d \)-dimensional standard Gaussian random vector.

**Remark 2.2.** Under the conditions of Theorem 2.1 but with \( \text{Cov}(W) = \Sigma \), by considering \( \Sigma^{-1/2}W \), we have

\[
d_c(\mathcal{L}(W), \mathcal{L}(\Sigma^{1/2}Z)) \leq Cd^{1/4}n\beta^3n_1(n_2 + \frac{n_3}{d}), \tag{2.6}
\]

where \( \|\Sigma^{-1/2}\| \) is the operator norm of \( \Sigma^{-1/2} \).

**Remark 2.3.** The decomposition (2.2) and (2.3) is the same as that of Barbour, Karoński and Ruciński [4], and as observed there, is more general than the local dependence structure studied in Chen [8] and later in Chen and Shao [10]. Condition (2.4) is a natural extension of the decomposition of Barbour, Karoński and Ruciński [4]. We need this extra condition to obtain the bound in (2.6).

**Remark 2.4.** As mentioned at the beginning of the Introduction, the dependence on the dimension \( d \) in (2.6) is the best known dependence even for sums of independent and identically distributed random vectors. In typical applications, \( \beta \) is of the order \( O(n^{-1/2}) \). From the decomposition (2.2)–(2.4), the neighborhood sizes \( n_2 \) and \( n_3 \) are typically of the same order as \( n_1 \). On the other hand, in the local dependence structure studied in Chen [8] and Chen and Shao [10], quantities corresponding to \( n_2 \) and \( n_3 \) are typically of the order \( O(n_1^2) \) and \( O(n_1^3) \) respectively. Rinott and Rotar [17] proved a similar bound as in (2.6) for a different decomposition structure. Their bound does not have explicit dependence on \( d \) and has an additional \( \log n \) term. Recently, Fang and Röllin [11] proved a general multivariate central limit theorem for the non-smooth function distance under the framework of Stein coupling (cf. Chen and Röllin [9]) with boundedness conditions. Although their bound is more widely applicable, it does not yield optimal dependence on \( d \) and does not directly apply to the decomposition structure considered in this paper.
3 AN APPLICATION

Let \( n \geq 2, m \geq 1, d \geq 2 \) be positive integers. Consider a regular graph with \( n \) vertices and vertex degree \( m \). Let \( N = nm/2 \) be the total number of edges. We color each vertex independently by one of colors \( c_i \) with probability \( \pi_i \) for \( i \in \{1, \ldots, d\} \), where \( \sum_{i=1}^{d} \pi_i = 1 \). For \( i \in \{1, \ldots, d\} \), let \( W_i \) be the number of edges connecting vertices both of color \( c_i \). Formally,

\[
W_i = \sum_{j=1}^{N} X_{ji},
\]

where \( X_{ji} \) is the indicator of the event that the edge \( j \) connects two vertices both of color \( c_i \). Let

\[
W = (W_1, \ldots, W_d)^t.
\]

Let \( \lambda = \mathbb{E}W, \Sigma = \text{Cov}(W) \). It is known that

\[
\lambda = (N \pi_1^2, \ldots, N \pi_d^2)^t,
\]

and (cf. (3.1) of Rinott and Rotar [17])

\[
\text{Var}(W_i) = N \pi_i^2(1 - \pi_i^2) + 2N(m - 1)(\pi_i^3 - \pi_i^4),
\]

\[
\text{Cov}(W_i, W_j) = -N(2m - 1)\pi_i^2 \pi_j^2, \text{ for } i \neq j.
\]

We prove the following bound on the non-smooth function distance between the standardized distribution of \( W \) and the standard \( d \)-dimensional Gaussian distribution.

**Proposition 3.1.** Let \( W \) be defined as in (3.1). Let \( \lambda \) and \( \Sigma \) be the mean and covariance matrix of \( W \) respectively. We have

\[
d_c(\mathcal{L}(\Sigma^{-1/2}(W - \lambda)), \mathcal{L}(Z)) \leq C d^{7/4} m^{3/2} L^3 n^{-1/2}
\]

where \( d_c \) is defined as in (2.1), \( Z \) is a \( d \)-dimensional standard Gaussian vector, \( C \) is an absolute constant, and

\[
L = \left[ \min_{1 \leq i \leq d} \{ \pi_i^2(1 - \pi_i) \} \right]^{-1/2}.
\]

**Remark 3.2.** Rinott and Rotar [17] proved an upper bound for the left-hand side of (3.2) as follows (cf. (3.2) of Rinott and Rotar [17]):

\[
d_c(\mathcal{L}(\Sigma^{-1/2}(W - \lambda)), \mathcal{L}(Z)) \leq c_d m^{3/2} L^3 (|\log L| + \log n) n^{-1/2},
\]

where \( c_d \) is an unspecified constant depending on \( d \). Comparing with (3.3), our bound in (3.2) has explicit dependence on \( d \) and does not have the logarithmic terms. We remark that we can only get \( m^{5/2} \) instead of \( m^{3/2} \) in (3.3) by following the argument of Rinott and Rotar [17]. If this was indeed a typo of Rinott and Rotar [17], our bound (3.2) also improves the dependence on \( m \).
Proof of Proposition 3.1. Recall that
\[ \Sigma^{-1/2}(W - \lambda) = \sum_{j=1}^{N} \Sigma^{-1/2} \left( (X_{j1}, \ldots, X_{jd})^t - (\pi_1^2, \ldots, \pi_d^2)^t \right) =: \sum_{j=1}^{N} \xi_j. \]
Observe that at most one of \( \{X_{j1}, \ldots, X_{jd}\} \) can be non-zero. Together with \( \sum_{i=1}^{d} \pi_i = 1 \) and the fact that the maximal absolute value of \( \Sigma^{-1/2} \) is bounded by \( N^{-1/2}L \) (cf. page 339 of Rinott and Rotar [17]), we have
\[ |\xi_j| \leq 2d^{1/2}N^{-1/2}L. \]
Moreover, the summation \( \sum_{j=1}^{N} \xi_j \) can be easily seen to satisfy the decomposition structure (2.2)–(2.4) with
\[ |N_i| \leq 2m, \quad |N_{ij}| \leq 3m, \quad |N_{ijk}| \leq 4m. \]
By applying Theorem 2.1, we obtain the bound (3.2). \( \square \)

4 PROOF OF MAIN THEOREM

For given test function \( h \), we consider the Stein equation
\[ \Delta f(w) - w^t \nabla f(w) = h(w) - \mathbb{E}h(Z), \quad w \in \mathbb{R}^d, \]
where \( \Delta \) denotes the Laplacian operator and \( \nabla \) the gradient operator. If \( h \) is not continuous (like the indicator function of a convex set), then \( f \) is not smooth enough to apply Taylor expansion to the necessary degree, so more refined techniques are necessary.

We follow the smoothing technique of Bentkus [3]. Recall that \( \mathcal{A} \) is the collection of all the convex sets in \( \mathbb{R}^d \). For \( A \in \mathcal{A} \), let \( h_A(x) = I_A(x) \), and define the smoothed function
\[ h_{A,\varepsilon}(w) = \psi \left( \frac{\text{dist}(w, A)}{\varepsilon} \right), \]
where \( \text{dist}(w, A) = \inf_{v \in A} |w - v| \) and
\[ \psi(x) = \begin{cases} 1, & x < 0, \\ 1 - 2x^2, & 0 \leq x < \frac{1}{2}, \\ 2(1 - x)^2, & \frac{1}{2} \leq x < 1, \\ 0, & 1 \leq x. \end{cases} \]
Define also
\[ A^\varepsilon = \{ x \in \mathbb{R}^d : \text{dist}(x, A) \leq \varepsilon \}, \quad A^{-\varepsilon} = \{ x \in A : \text{dist}(x, \mathbb{R}^d \setminus A) > \varepsilon \} \]
(note that in general \( (A^{-\varepsilon})^\varepsilon \neq A \).

We will use the following lemmas in the proof of Theorem 2.1.
Lemma 4.1 (Lemma 2.3 of Bentkus [3]). The function \( h_{A, \varepsilon} \) as defined above has the following properties:

(i) \( h_{A, \varepsilon}(w) = 1 \) for all \( w \in A \),
(ii) \( h_{A, \varepsilon}(w) = 0 \) for all \( w \in \mathbb{R}^d \setminus A^\varepsilon \),
(iii) \( 0 \leq h_{A, \varepsilon}(w) \leq 1 \) for all \( w \in A^\varepsilon \setminus A \),
(iv) \( |\nabla h_{A, \varepsilon}(w)| \leq 2 \varepsilon^{-1} \) for all \( w \in \mathbb{R}^d \),
(v) \( |\nabla h_{A, \varepsilon}(v) - \nabla h_{A, \varepsilon}(w)| \leq 8|v - w|\varepsilon^{-2} \) for all \( v, w \in \mathbb{R}^d \).

Lemma 4.2 (Ball [2], Bentkus [3]). We have

\[ \sup_{A \in \mathcal{A}} \max \{ \mathbb{P}(Z \in A^\varepsilon \setminus A), \mathbb{P}(Z \in A \setminus A^-\varepsilon) \} \leq 4d^{1/4} \varepsilon, \]  

and the dependence on \( d \) in (4.9) is optimal.

Lemma 4.3 (Lemma 4.2 of Fang and Röllin [11]). For any \( d \)-dimensional random vector \( W \),

\[ d_c(\mathcal{L}(W), \mathcal{L}(Z)) \leq 4d^{1/4} \varepsilon + \sup_{A \in \mathcal{A}} \mathbb{E} h_{A, \varepsilon}(W) - \mathbb{E} h_{A, \varepsilon}(Z). \]  

Lemma 4.4 (Lemma 4.3 of Fang and Röllin [11]). For each map \( a : \{1, \ldots, d\}^k \to \mathbb{R} \), we have

\[ \int_{\mathbb{R}^d} \left( \sum_{i_1, \ldots, i_k = 1}^d a(i_1, \ldots, i_k) \frac{\varphi_{i_1, \ldots, i_k}(z)}{\varphi(z)} \right)^2 \varphi(z) dz \leq k! \sum_{i_1, \ldots, i_k = 1}^d (a(i_1, \ldots, i_k))^2, \]  

where \( \varphi(z) \) is the density of \( d \)-dimensional standard normal distribution and

\[ \varphi_{i_1, \ldots, i_k}(z) = \partial^k \varphi(z) / (\partial z_{i_1} \cdots \partial z_{i_k}). \]

Fix now \( \varepsilon \) and a convex \( A \subset \mathbb{R}^d \). It can be verified directly that with

\[ g_{A, \varepsilon}(w, \tau) = -\frac{1}{2(1 - \tau)} \int_{\mathbb{R}^d} \left( h_{A, \varepsilon}(\sqrt{1 - \tau} w + \sqrt{\tau} z) - \mathbb{E} h_{A, \varepsilon}(Z) \right) \varphi(z) dz, \]  

a solution to (1.1) is (cf. Götze [13])

\[ f_{A, \varepsilon}(w) = \int_0^1 g_{A, \varepsilon}(w, \tau) d\tau. \]  

In what follows, we keep the dependence on \( A \) and \( \varepsilon \) implicit and write \( g = g_{A, \varepsilon}, f = f_{A, \varepsilon} \) and \( h = h_{A, \varepsilon} \). For real-valued functions on \( \mathbb{R}^d \) we will write \( f_r(x) \) for \( \partial f(x)/\partial x_r, f_{rs}(x) \) for \( \partial^2 f(x)/\partial x_r \partial x_s \) and so forth. Also we write \( g_r(w, \tau) = \partial g(w, \tau) / \partial w_r \) and so on. Moreover, let \( \nabla g(w, \tau) = (g_1(w, \tau), \ldots, g_d(w, \tau))^t \) and let \( \Delta g(w, \tau) = \sum_{r=1}^d g_r(w, \tau) \).

Using this notation and the integration by parts formula, we have for \( 1 \leq r, s, t \leq d \) that

\[ g_{rs}(w, \tau) = -\frac{1}{2\tau} \int_{\mathbb{R}^d} h(\sqrt{1 - \tau} w + \sqrt{\tau} z) \varphi_{rs}(z) dz \]

\[ = \frac{1}{2\sqrt{\tau}} \int_{\mathbb{R}^d} h_s(\sqrt{1 - \tau} w + \sqrt{\tau} z) \varphi_r(z) dz \]  

(4.14)
and
\[ g_{rst}(w, \tau) = \frac{\sqrt{1 - \tau}}{2\tau^{3/2}} \int_{R^d} h(\sqrt{1 - \tau}w + \sqrt{\tau}z) \varphi_{rst}(z) dz \]
\[ = \frac{\sqrt{1 - \tau}}{2\sqrt{\tau}} \int_{R^d} h_{jk}(\sqrt{1 - \tau}w + \sqrt{\tau}z) \varphi_{r}(z) dz. \] (4.15)

**Proof of Theorem 2.1** Fix \( A \in A \) and \( \varepsilon > 0 \) (to be chosen later) and let \( f = f_{A,\varepsilon} \) be the solution to the Stein equation (4.1) with respect to \( h) = h_{A,\varepsilon} \) as defined by (4.2). Let\[ \kappa := d_{\varepsilon}(\mathcal{L}(W), \mathcal{L}(Z)). \] (4.16)
To avoid confusion, we will always use \( r, s, t \) to index the components of \( d \)-dimensional vectors.
Define \[ W_i := W - X_{Ni}, \quad W_{ij} := W - X_{N_{ij}}, \quad W_{ijk} := W - X_{N_{ijk}}. \]
By assumption (2.2) and \( \mathbb{E}X_i = 0 \), we have for \( g(w, \tau) = g_{A,\varepsilon}(w, \tau) \) in (4.12),
\[ - \mathbb{E}W^t \nabla g(W, \tau) = - \sum_{r=1}^{d} \mathbb{E}(W)_r g_r(W, \tau) \]
\[ = - \sum_{r=1}^{d} \sum_{i=1}^{n} \mathbb{E}(X_i)_r g_r(W, \tau) = \sum_{r=1}^{d} \sum_{i=1}^{n} \mathbb{E}(X_i)_r [g_r(W_i, \tau) - g_r(W, \tau)] \]
By assumption (2.2) and \( \text{Cov}(W) = \mathbb{I}_d \), we have \( \sum_{i=1}^{n} \sum_{j \in N_i} \mathbb{E}(X_i)_r (X_j)_s = \delta_{rs} \) where \( \delta \) denotes the Kronecker delta. This implies
\[ \mathbb{E} \Delta g(W, \tau) = \mathbb{E} \sum_{r,s=1}^{d} \sum_{i=1}^{n} \sum_{j \in N_i} (\mathbb{E}(X_i)_r (X_j)_s) g_{rs}(W, \tau). \]
Adding and subtracting the corresponding terms, and using (2.2), we have
\[ \mathbb{E} \{ \Delta g(W, \tau) - W^t \nabla g(W, \tau) \} \]
\[ = \mathbb{E} \sum_{r=1}^{d} \sum_{i=1}^{n} (X_i)_r [g_r(W_i, \tau) - g_r(W, \tau)] + \sum_{s=1}^{d} \sum_{j \in N_i} (X_j)_s g_{rs}(W, \tau) \]
\[ + \mathbb{E} \sum_{r,s=1}^{d} \sum_{i=1}^{n} \sum_{j \in N_i} (\mathbb{E}(X_i)_r (X_j)_s - (X_i)_r(X_j)_s) [g_{rs}(W, \tau) - g_{rs}(W_{ij}, \tau)] \]
\[ =: R_1(\tau) + R_2(\tau). \]
By (4.1) and (4.13),
\[ \mathbb{E} h(W) - \mathbb{E} h(Z) = \int_{0}^{1} (R_1(\tau) + R_2(\tau)) d\tau. \] (4.17)
In the following we will first give an upper bound for \( \int_{0}^{1} R_1(\tau) d\tau \), and then argue that an upper bound for \( \int_{0}^{1} R_2(\tau) d\tau \) can be derived similarly. To estimate \( \int_{0}^{1} R_1(\tau) d\tau \), we consider the cases
\( \varepsilon^2 < \tau < 1 \) and \( 0 < \tau < \varepsilon^2 \) separately. For the first case, we use the first expression of \( g_{rs}(w, \tau) \) in (4.1), Taylor’s expansion and the integration by parts formula, and get

\[
\int_{\varepsilon^2}^{1} R_1(\tau) d\tau = \mathbb{E} \sum_{r,s,t=1}^{d} \sum_{i,j,k,N_i} \int_{\varepsilon^2}^{1} \frac{1}{2\tau} \int_{\mathbb{R}^d} [h(\sqrt{1-\tau^2} W - UX_i) + \sqrt{\tau} z) - h(\sqrt{1-\tau^2} W + \sqrt{\tau} z)] \\
\times (X_i)_r(X_j)_s\varphi_{rs}(z) dz d\tau \\
= \mathbb{E} \sum_{r,s,t=1}^{d} \sum_{i,j,k,N_i} \int_{\varepsilon^2}^{1} \frac{1-\tau}{2\tau^{3/2}} \int_{\mathbb{R}^d} h(\sqrt{1-\tau^2} W + \sqrt{\tau} z) - h(\sqrt{1-\tau^2} UV X_i) \\
\times U(X_i)_r(X_j)_s(X_k)_t\varphi_{rst}(z) dz d\tau 
\]

where \( U \) and \( V \) are independent random variables distributed uniformly on \([0, 1]\). By writing

\[ h(\sqrt{1-\tau^2} W + \sqrt{\tau} z - \sqrt{1-\tau} UV X_i) \]

as a telescoping sum, and using the independence assumption (2.3), we have

\[
\int_{\varepsilon^2}^{1} R_1(\tau) d\tau = R_{1,1} + R_{1,2} + R_{1,3} + R_{1,4} 
\]

where

\[
R_{1,1} = \mathbb{E} \sum_{r,s,t=1}^{d} \sum_{i,j,k,N_i} \int_{\varepsilon^2}^{1} \frac{1-\tau}{2\tau^{3/2}} \int_{\mathbb{R}^d} [h(\sqrt{1-\tau^2} W + \sqrt{\tau} z - \sqrt{1-\tau} UV X_i) \\
- h(\sqrt{1-\tau^2} W + \sqrt{\tau} z)] U(X_i)_r(X_j)_s(X_k)_t\varphi_{rst}(z) dz d\tau, 
\]

\[
R_{1,2} = \mathbb{E} \sum_{r,s,t=1}^{d} \sum_{i,j,k,N_i} \int_{\varepsilon^2}^{1} \frac{1-\tau}{2\tau^{3/2}} \int_{\mathbb{R}^d} \mathbb{E} h(\sqrt{1-\tau^2} W + \sqrt{\tau} z) \\
- \mathbb{E} h(\sqrt{1-\tau^2} W + \sqrt{\tau} z)] U(X_i)_r(X_j)_s(X_k)_t\varphi_{rst}(z) dz d\tau, 
\]

\[
R_{1,3} = \mathbb{E} \sum_{r,s,t=1}^{d} \sum_{i,j,k,N_i} \int_{\varepsilon^2}^{1} \frac{1-\tau}{2\tau^{3/2}} \int_{\mathbb{R}^d} \mathbb{E} h(\sqrt{1-\tau^2} W + \sqrt{\tau} z) \\
- \mathbb{E} h(\sqrt{1-\tau^2} W + \sqrt{\tau} z)] U(X_i)_r(X_j)_s(X_k)_t\varphi_{rst}(z) dz d\tau, 
\]

\[
R_{1,4} = \mathbb{E} \sum_{r,s,t=1}^{d} \sum_{i,j,k,N_i} \int_{\varepsilon^2}^{1} \frac{1-\tau}{2\tau^{3/2}} \int_{\mathbb{R}^d} [h(\sqrt{1-\tau^2} W + \sqrt{\tau} z) \\
- h(\sqrt{1-\tau^2} W + \sqrt{\tau} z)] U(X_i)_r(X_j)_s(X_k)_t\varphi_{rst}(z) dz d\tau, 
\]

where \( Z \) is an independent \( d \)-dimensional standard Gaussian random vector.

By the properties of \( h \) in (4.4) and (4.5), the boundedness condition (2.5) and the independence assumption (2.3),

\[
|R_{1,1}| \leq \mathbb{E} \sum_{r,s,t=1}^{d} \sum_{i,j,k,N_i} \int_{\varepsilon^2}^{1} \frac{1-\tau}{4\tau^{3/2}} \int_{\mathbb{R}^d} I(\text{dist}(\sqrt{1-\tau^2} W + \sqrt{\tau} z, A^c \setminus A) \leq \sqrt{1-\tau} n_3 \beta) \\
\times |(X_i)_r(X_j)_s(X_k)_t\varphi_{rst}(z)| dz d\tau. 
\]

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By the boundedness condition (2.5), the definition of \( \kappa \) in (4.16) and (4.9),
\[
\mathbb{E}(\text{dist}(\sqrt{1 - \tau W_{ijk}} + \sqrt{\tau z}, A^c \setminus A) \leq \sqrt{1 - \tau n_3 \beta}) \\
\leq \mathbb{E}(\text{dist}(\sqrt{1 - \tau W} + \sqrt{\tau z}, A^c \setminus A) \leq 2\sqrt{1 - \tau n_3 \beta}) \\
\leq 4d^{1/4}(\varepsilon \sqrt{1 - \tau} + 4n_3 \beta) + 2\kappa.
\]
(4.18)

By the Cauchy-Schwartz inequality, (4.11) and the boundedness condition (2.5),
\[
\int_{\mathbb{R}^d} \sum_{r,s,t=1}^d |(X_i)_r(X_j)_s(X_k)_t \varphi_{rst}(z)| dz \leq \sqrt{6} \beta^3.
\]

Therefore,
\[
|R_{1,1}| \leq Cn \beta^3 n_1^2 \varepsilon^{1/2} (d^{1/4}(\varepsilon + n_3 \beta) + \kappa)
\]
where we used \( \int_{\varepsilon}^1 \frac{1}{\sqrt{1 - \tau}} d\tau \leq C \frac{1}{\varepsilon^2} \). Here and in the remaining of this proof, \( C \) denotes absolute constants which may differ from line to line.

By the same argument, \( |R_{1,2}| \) has the same upper bound as \( |R_{1,1}| \).

By the properties of \( h \) in (4.4) and (4.5), the definition of \( \kappa \) in (4.16), and (4.9),
\[
\mathbb{E}[h(\sqrt{1 - \tau W} + \sqrt{\tau z}) - h(\sqrt{1 - \tau Z} + \sqrt{\tau z})] \\
\leq \mathbb{E}[I(\sqrt{1 - \tau W} + \sqrt{\tau z} \in A^c) - I(\sqrt{1 - \tau Z} + \sqrt{\tau z} \in A)] \\
= \mathbb{E}[I(\sqrt{1 - \tau W} + \sqrt{\tau z} \in A^c) - I(\sqrt{1 - \tau Z} + \sqrt{\tau z} \in A^c) + I(\sqrt{1 - \tau Z} + \sqrt{\tau z} \in A^c \setminus A)] \\
\leq \kappa + 4d^{1/4} \frac{\varepsilon}{\sqrt{1 - \tau}}.
\]
By the same lower bound and a similar argument as for \( R_{1,1} \), we can bound \( R_{1,3} \) by
\[
|R_{1,3}| \leq Cn \beta^3 n_2^2 \frac{1}{\varepsilon} (d^{1/4}(\kappa + \varepsilon)).
\]

Using the first expression of \( g_{rst}(w, s) \) in (4.15), we have
\[
R_{1,4} = \mathbb{E} \sum_{r,s,t=1}^d \sum_{i=1}^n \sum_{j,k \in N_i} \int_{\varepsilon}^1 \int_\varepsilon^1 U \mathbb{E}((X_i)_r(X_j)_s(X_k)_t)g_{rst}(Z, \tau) dz d\tau.
\]

Observe that
\[
\mathbb{E} g(Z + w, \tau) = -\frac{1}{2(1 - \tau)} \mathbb{E} h(\sqrt{1 - \tau} (Z + w) + \sqrt{\tau} z) \varphi(z) dz \\
= -\frac{1}{2(1 - \tau)} \mathbb{E} h(\sqrt{1 - \tau w} + z) \varphi(z) dz + \frac{1}{2(1 - \tau)} \mathbb{E} h(Z) \\
= -\frac{1}{2(1 - \tau)} \mathbb{E} h(\sqrt{1 - \tau} w) \varphi(x - \sqrt{1 - \tau} w) dx + \frac{1}{2(1 - \tau)} \mathbb{E} h(Z).
\]
Differentiating and evaluating at \( w = 0 \), we obtain
\[
\mathbb{E} g_{rst}(Z, \tau) = \frac{\sqrt{1 - \tau}}{2} \int_{\mathbb{R}^d} h(x) \varphi_{rst}(x) dx.
\]
Now with (4.11) and (2.5),
\[
|R_{1,4}| \leq C n^3 n_1^2.
\]
For the case \( 0 < \tau \leq \varepsilon^2 \), we use the second expression of \( g_{rs}(w, \tau) \) in (4.14) and the Taylor expansion
\[
\int_0^{\varepsilon^2} R_1(\tau) d\tau = \mathbb{E} \sum_{r,s,t=1}^d \sum_{i,j,k \in N_i} \int_0^{\varepsilon^2} \frac{\sqrt{1 - \tau}}{2\sqrt{\tau}} \int_{\mathbb{R}^d} h_{st}(\sqrt{1 - \tau} W + \sqrt{\tau} z) d\tau
\]
\[
= \mathbb{E} \sum_{r,s,t=1}^d \sum_{i,j,k \in N_i} \int_0^{\varepsilon^2} \frac{\sqrt{1 - \tau}}{2\sqrt{\tau}} \int_{\mathbb{R}^d} h_{st}(\sqrt{1 - \tau} W + \sqrt{\tau} z) d\tau
\]
\[
\times U(X_i)_r(X_j)_s(X_k)_t \varphi_r(z) dz d\tau,
\]
where we recall \( U \) and \( V \) are independent random variables distributed uniformly on [0, 1]. By (4.2), (4.3), (4.8), and (2.5),
\[
\left| \int_0^{\varepsilon^2} R_1(\tau) d\tau \right| \leq \frac{8}{\varepsilon^2} \beta^2 n_1^3 \mathbb{E} \sum_{r=1}^d \sum_{i=1}^n \int_0^{\varepsilon^2} \frac{\sqrt{1 - \tau}}{2\sqrt{\tau}} \int_{\mathbb{R}^d} 1(\text{dist}(\sqrt{1 - \tau} W_i + \sqrt{\tau} z, A^\varepsilon \setminus A) \leq \sqrt{1 - \tau} n_1^3 \beta)
\]
\[
\times U|(|(X_i)_r \varphi_r(z)| dz d\tau.
\]
Similarly as in (4.13),
\[
\mathbb{E} I(\text{dist}(\sqrt{1 - \tau} W_i + \sqrt{\tau} z, A^\varepsilon \setminus A) \leq \sqrt{1 - \tau} n_1^3 \beta)
\]
\[
\leq \mathbb{E} I(\text{dist}(\sqrt{1 - \tau} W + \sqrt{\tau} z, A^\varepsilon \setminus A) \leq 2 \sqrt{1 - \tau} n_1^3 \beta)
\]
\[
\leq 4 d^{1/4} \left( \frac{\varepsilon}{\sqrt{1 - \tau}} + 4 n_1^3 \beta \right) + 2 \kappa.
\]
Together with (2.2), (2.5) and (4.11),
\[
\left| \int_0^{\varepsilon^2} R_1(\tau) d\tau \right| \leq C n^3 n_1^2 \left[ d^{1/4} (\varepsilon + n_1 \beta) + \kappa \right].
\]
Therefore,
\[
\left| \int_0^1 R_1(\tau) d\tau \right| \leq C n^3 n_1^2 \left[ d^{1/4} (\varepsilon + n_3 \beta) + \kappa \right]. \tag{4.19}
\]
Now we turn to bounding $|\int_0^1 R_2(s) ds|$. Similarly as for $R_1(s)$,

$$
\int_0^1 R_2(\tau) d\tau = -\mathbb{E} \sum_{r,s,t=1}^{d} \sum_{i=1}^{n} \sum_{j \in N_i} \sum_{k \in N_{ij}} \int_0^1 \left( h(\sqrt{1-\tau}W + \sqrt{\tau}z - \sqrt{1-\tau}UX_{N_{ij}}) 
\right.
\left. \times (X_k)_i \left[ (X_i)_r (X_j)_s - \mathbb{E} (X_i)_r (X_j)_s \right] \varphi_{rst}(z) d\tau \right)
- \mathbb{E} \sum_{r,s,t=1}^{d} \sum_{i=1}^{n} \sum_{j \in N_i} \sum_{k \in N_{ij}} \int_0^1 \left( \frac{-1}{2} \frac{\sqrt{1-\tau}}{\sqrt{\tau}} h_{st}(\sqrt{1-\tau}W + \sqrt{\tau}z - \sqrt{1-\tau}UX_{N_{ij}}) 
\right.
\left. \times (X_k)_i \left[ (X_i)_r (X_j)_s - \mathbb{E} (X_i)_r (X_j)_s \right] \varphi_{r}(z) d\tau \right)
$$

where $U$ is a independent random variable distributed uniformly on $[0, 1]$. By the same argument in bounding $|\int_0^1 R_1(\tau) d\tau|$, we have

$$
\left| \int_0^1 R_2(\tau) d\tau \right| \leq C n^3 n_1 n_2 \frac{1}{\varepsilon} [d^{1/4} (\varepsilon + n_3 \beta) + \kappa]. \tag{4.20}
$$

By (4.10), (4.17), (4.19) and (4.20),

$$
\kappa \leq 4 d^{1/4} \varepsilon + C n^3 n_1 n_2 \frac{1}{\varepsilon} [d^{1/4} (\varepsilon + n_3 \beta) + \kappa]. \tag{4.21}
$$

Choosing $\varepsilon = 2 C n^3 n_1 n_2$ with the same $C$ as in (4.21), solving the recursive inequality (4.21) and observing that $d \leq n \beta^2 n_1$ from $\text{Cov}(W) = I_d$ and (2.5) yield the final bound (2.6).

\[ \square \]

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