One plus two-body random matrix ensembles with parity:
Density of states and parity ratios

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Abstract

One plus two-body embedded Gaussian orthogonal ensemble of random matrices with parity [EGOE(1+2)-π] generated by a random two-body interaction (modeled by GOE in two particle spaces) in the presence of a mean-field, for spinless identical fermion systems, is defined, generalizing the two-body ensemble with parity analyzed by Papenbrock and Weidenmüller [Phys. Rev. C 78, 054305 (2008)], in terms of two mixing parameters and a gap between the positive (π = +) and negative (π = −) parity single particle (sp) states. Numerical calculations are used to demonstrate, using realistic values of the mixing parameters appropriate for some nuclei, that the EGOE(1+2)-π ensemble generates Gaussian form (with corrections) for fixed parity eigenvalue densities (i.e. state densities). The random matrix model also generates many features in parity ratios of state densities that are similar to those predicted by a method based on the Fermi-gas model for nuclei. We have also obtained, by applying the formulation due to Chang et al [Ann. Phys. (N.Y.) 66, 137 (1971)], a simple formula for the spectral variances defined over fixed-(m₁, m₂) spaces, where m₁ is the number of fermions in the +ve parity sp states and m₂ is the number of fermions in the −ve parity sp states. Similarly, using the binary correlation approximation, in the dilute limit, we have derived expressions for the lowest two shape parameters. The smoothed densities generated by the sum of fixed-(m₁, m₂) Gaussians with lowest two shape corrections describe the numerical results in many situations. The model also generates preponderance of +ve parity ground states for small values of the mixing parameters and this is a feature seen in nuclear shell model results.

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I. INTRODUCTION

Random matrix theory (RMT), starting with Wigner and Dyson’s Gaussian random ensembles [1, 2] introduced to describe neutron resonance data [3, 4], has emerged as a powerful statistical approach leading to paradigmatic models describing generic properties of complex systems [5–9]. Developments and applications of RMT in nuclear physics in last 30 years have been reviewed recently by Weidenmüller and collaborators [10, 11]. The Wigner-Dyson classical Gaussian orthogonal (GOE), unitary (GUE) and symplectic (GSE) ensembles are ensembles of multi-body interactions while the nuclear interparticle interactions are essentially two-body in nature. This together with nuclear shell model examples led to the introduction of random matrix ensembles generated by two-body interactions in 1970-1971 [12, 13]. These two-body ensembles are defined by representing the two-particle Hamiltonian by one of the classical ensembles and then the $m$ ($m > 2$) particle $H$-matrix is generated by the Hilbert space geometry. Thus the random matrix ensemble in the two-particle spaces is embedded in the $m$-particle $H$-matrix and therefore these ensembles are generically called embedded ensembles (EEs). Simplest of these ensembles is the embedded Gaussian orthogonal ensemble of random matrices generated by two-body interactions for spinless fermion (boson) systems, denoted by EGOE(2) [BEGOE(2); here ‘B’ stands for bosons]. In addition to the complexity generating two-body interaction, Hamiltonians for realistic systems such as nuclei consist of a mean-field one-body part. Then the appropriate random matrix ensembles are EE(1+2). The spinless fermion/boson EGEs (orthogonal and unitary versions) have been explored in detail from 70’s with a major revival from 1995 and it is now well understood that EGEs model many-body chaos or stochasticity exhibited by isolated finite interacting quantum systems [9, 14]. Besides the mean-field and the two-body character, realistic Hamiltonians also carry a variety of symmetries. In many applications of EGEs, generic properties of EGEs for spinless fermions are ‘assumed’ to extend to symmetry subspaces [15]. More importantly, there are several properties of real systems that require explicit inclusion of symmetries and they are defined by a variety of Lie algebras. This led to studies on EGEs with symmetries such as spin [16–20], spin-isospin $SU(4)$ [21], $J$-symmetry [22] and many others (see for example [23, 24]). In the present paper, we consider parity symmetry in EE as there are several nuclear structure quantities that require explicit inclusion of parity. Some of these are as follows.
Parity ratios of nuclear level densities is an important ingredient in nuclear astrophysical applications. Recently, a method based on non-interacting Fermi-gas model for proton-neutron systems has been developed and the parity ($\pi$) ratios as a function of excitation energy in large number of nuclei of astrophysical interest have been tabulated [25]. The method is based on the assumption that the probability to occupy $s$ out of $N$ given single particle (sp) states follow Poisson distribution in the dilute limit ($m \ll N, N \to \infty$ where $m$ is the number of particles). Then the ratio of the partition functions for the $+ve$ and $-ve$ parity states is given by the simple formula $Z_-/Z_+ = \tanh f$, where $f$ is average number of particles in the $+ve$ parity states. Starting with this, an iterative method is developed with inputs from the Fermi-Dirac distribution for occupancies including pairing effects and the Fermi-gas form for the total level density. In the examples studied in [25], parity ratios are found to equilibrate only around $5-10$ MeV excitation energy. However, ab-initio interacting particle theory for parity ratios is not yet available.

A closely related question is about the form of the density of states defined over spaces with fixed-$\pi$. In general, fixed-$\pi$ density of states can be written as a sum of appropriate partial densities. In the situation that the form of the partial densities is determined by a few parameters (as it is with a Gaussian or a Gaussian with one or two corrections), it is possible to derive a theory for these parameters and using these, one can construct fixed-$\pi$ density of states and calculate parity ratios. Such a theory with interactions in general follows from random matrix theory [15].

In addition to the questions related to fixed-$\pi$ density of states and parity ratios, there is also the important recognition in the past few years that random interactions generate regular structures [22, 26, 27]. It was shown in [28] that shell model for even-even nuclei gives preponderance of $+ve$ parity ground states. A parameter-free EGOE with parity has been defined and analyzed recently [29] to address the question of ‘preponderance of ground states with positive parity’ for systems with even number of fermions. They show that in the dilute limit, $+ve$ parity ground states appear with only 50% probability. Thus, a random matrix theory describing shell model results is not yet available.

With the success of the embedded random matrix ensembles (EE) [9, 14], one can argue that the EE generated by parity preserving random interaction may provide generic results for the three nuclear structure quantities mentioned above. For nuclei, the GOE versions of EE are relevant. Then, with a random (modeled by GOE) two-body interaction preserving
parity in the presence of a mean-field, we have embedded Gaussian orthogonal ensemble of one plus two-body interactions with parity [hereafter called EGOE(1+2)-π]. This model contains two mixing parameters and a gap between the +ve and −ve parity sp states and it goes much beyond the simpler model considered in [29]. In the random matrix model used in the present paper, proton-neutron degrees of freedom and angular momentum (J) are not considered. Let us add that in the present paper for the first time a random matrix theory for parity ratios is attempted. Now we will give a preview.

Section II gives the definition of EGOE(1+2)-π and a method for its construction. From the results known for EE for spinless fermion (boson) systems, for fermions (bosons) with spin and from shell model calculations [3, 14, 18], it is expected that the fixed-π state densities (more appropriately partial densities) approach Gaussian form in general. Therefore, exact propagation formulas for fixed-π energy centroids and spectral variances are derived and the results are given in Section III. Used here is the group theoretical formulation developed by Chang et al [30]. Similarly in Appendix B, given are the formulas for the ensemble averaged skewness \( \gamma_1(m_1, m_2) \) and excess \( \gamma_2(m_1, m_2) \) parameters for fixed-(\( m_1, m_2 \)) partial densities (with \( m_1 \) fermions distributed in \( N_+ \) number of +ve parity sp levels and similarly \( m_2 \) fermions in \( N_- \) number of −ve parity sp levels) and used is the binary correlation approximation method described in [31–34]. These will provide corrections to the Gaussian state densities. In Section IV, presented are the numerical results for (i) fixed-π state densities, (ii) parity ratios of state densities and (iii) probability for +ve parity ground states. Finally, Section V gives conclusions and future outlook.

II. EGOE(1+2)-π ENSEMBLE

Given \( N_+ \) number of positive parity sp states and similarly \( N_- \) number of negative parity sp states, let us assume, for simplicity, that the +ve and −ve parity states are degenerate and separated by energy \( \Delta \) (see Fig. 1). This defines the one-body part \( h(1) \) of the Hamiltonian \( H \) with \( N = N_+ + N_- \) sp states. The matrix for the two-body part \( V(2) \) of \( H \) [we assume \( H \) is (1+2)-body] will be a \( 3 \times 3 \) block matrix in two particle spaces as there are three possible ways to generate two particle states with definite parity: (i) both fermions in +ve parity states; (ii) both fermions in −ve parity states; (iii) one fermion in +ve and other fermion in −ve parity states. They will give the matrices \( A, B \) and \( C \) respectively in Fig. 1. For
parity preserving interactions only the states (i) and (ii) will be mixed and mixing matrix is $D$ in Fig. 1. Note that the matrices $A$, $B$ and $C$ are symmetric square matrices while $D$ is in general a rectangular mixing matrix. Consider $N$ sp states arranged such that the states 1 to $N_+$ have +ve parity and states $N_+ + 1$ to $N$ have −ve parity. Then the operator form of $H$ preserving parity is,

$$H = h(1) + V(2);$$

$$h(1) = \sum_{i=1}^{N_+} \epsilon_i^{(+)\dagger} n_i^{(+)} + \sum_{i=N_++1}^{N} \epsilon_i^{(-)\dagger} n_i^{(-)}; \quad \epsilon_i^{(+)\dagger} = 0, \quad \epsilon_i^{(-)\dagger} = \Delta,$$

$$V(2) = \sum_{i,j,k,l=1}^{N} \langle \nu_k \nu_l | V | \nu_i \nu_j \rangle a_k^{\dagger} a_l^{\dagger} a_j a_i$$

$$+ \sum_{i',j',k',\ell'=N_++1}^{N} \langle \nu_{i'} \nu_{j'} | V | \nu_{i'} \nu_{j'} \rangle a_{i'}^{\dagger} a_{j'}^{\dagger} a_{j'} a_{i'},$$

$$+ \sum_{i'',k''=1}^{N_+} \sum_{j'',\ell''=N_++1}^{N} \langle \nu_{i''} \nu_{k''} | V | \nu_{i''} \nu_{j''} \rangle a_{i''}^{\dagger} a_{k''}^{\dagger} a_{j''} a_{i''}$$

$$+ \sum_{P,Q=1}^{N_+} \sum_{R,S=N_++1}^{N} \left[ \langle \nu_P \nu_Q | V | \nu_R \nu_S \rangle a_P^{\dagger} a_Q^{\dagger} a_S a_R + \text{h.c.} \right].$$

In Eq. (1), $\nu_i$’s are sp states with $i = 1, 2, \ldots, N$ (the first $N_+$ states are +ve parity and remaining −ve parity). Similarly, $\langle \ldots | V | \ldots \rangle$ are the two-particle matrix elements, $n_i$ are number operators and $a_i^{\dagger}$ and $a_i$ are creation and annihilation operators respectively. Note that the four terms in the RHS of the expression for $V(2)$ in Eq. (1) correspond respectively to the matrices $A$, $B$, $C$ and $D$ shown in Fig. 1.

Many particle states for $m$ fermions in the $N$ sp states can be obtained by distributing
m_1 fermions in the +ve parity sp states (N_+ in number) and similarly, m_2 fermions in the -ve parity sp states (N_- in number) with m = m_1 + m_2. Let us denote each distribution of m_1 fermions in N_+ sp states by m_1 and similarly, m_2 for m_2 fermions in N_- sp states. Many particle basis defined by (m_1, m_2) with m_2 even will form the basis for +ve parity states and similarly, with m_2 odd for -ve parity states. In the (m_1, m_2) basis with m_2 even (or odd), the H matrix construction reduces to the matrix construction for spinless fermion systems. The method of construction for spinless fermion systems is well known [14] and therefore it is easy to construct the many particle H matrices in +ve and -ve parity spaces. The matrix dimensions d_+ for +ve parity and d_- for -ve parity spaces are given by,

\[ d_+ = \sum_{m_1, m_2 \text{ (even)}} \binom{N_+}{m_1} \binom{N_-}{m_2}, \quad d_- = \sum_{m_1, m_2 \text{ (odd)}} \binom{N_+}{m_1} \binom{N_-}{m_2}. \]

Some examples for the dimensions d_+ and d_- are given in Table I.

The EGOE(1+2)-\pi ensemble is defined by choosing the matrices A, B and C to be independent GOE’s with matrix elements variances v_a^2, v_b^2 and v_c^2 respectively. Similarly the matrix elements of the mixing D matrix are chosen to be independent (independent of A, B and C matrix elements) zero centered Gaussian variables with variance v_d^2. Without loss of generality we choose \Delta = 1 so that all the v’s are in \Delta units. This general EGOE(1+2)-\pi model will have too many parameters (v_a^2, v_b^2, v_c^2, v_d^2, N_+, N_-, m) and therefore it is necessary
TABLE I. Hamiltonian matrix dimensions $d_+$ and $d_-$ for various values of $(N_+, N_-, m)$.

| $N_+$ | $N_-$ | $m$ | $d_+$ | $d_-$ | $N_+$ | $N_-$ | $m$ | $d_+$ | $d_-$ |
|-------|-------|-----|-------|-------|-------|-------|-----|-------|-------|
| 6     | 6     | 6   | 452   | 472   | 8     | 8     | 4   | 924   | 896   |
| 7     | 5     | 6   | 462   | 462   | 5     | 2184  | 2184 |
| 7     | 7     | 5   | 1001  | 1001  | 6     | 3976  | 4032 |
|       |       |     | 1484  | 1519  | 10    | 4     | 900  | 920   |
|       |       |     | 1716  | 1716  | 5     | 2202  | 2166 |
| 8     | 6     | 5   | 1016  | 986   | 6     | 4036  | 3972 |
|       |       |     | 1499  | 1504  | 10    | 4     | 900  | 920   |
| 9     | 5     | 5   | 1011  | 911   | 5     | 2166  | 2202 |
|       |       |     | 1524  | 1479  | 6     | 4036  | 3972 |
| 5     | 10    | 4   | 665   | 700   | 9     | 6     | 9240 | 9324  |
|       |       |     | 1501  | 1502  | 10    | 8     | 9268 | 9296  |
|       |       |     |       |       | 10    | 10    | 5   | 7752  | 7752  |
|       |       |     |       |       |       |       |     | 6     | 19320 | 19440 |

Thus EGOE(1+2) we employ is

\[
A : \text{GOE}(0 : \tau^2), \quad B : \text{GOE}(0 : \tau^2), \quad C : \text{GOE}(0 : \tau^2), \quad D : \text{GOE}(0 : \alpha^2);
\]

\[A, B, C, D\] are independent GOE’s.

Note that the $D$ matrix is a GOE only in the sense that the matrix elements $D_{ij}$ are all independent zero centered Gaussian variables with variance $\alpha^2$. In the limit $\tau^2 \to \infty$ and $\alpha = \tau$, the model defined by Eqs. (1), (3) and (4) reduces to the simpler model analyzed in [29].

Before proceeding further, it is useful to mention that we are considering in this paper spinless fermion systems (with parity included) just as in the previous investigation [29]. It
is possible to extend the ensemble to nucleons in shell model \( j \)-orbits (including both \(+ve\) and \(-ve\) orbits such as for example \(sd\) and \(fp\)) and construct the ensemble in many-nucleon spaces with a given \( J^\pi \) or \( J^\pi T \) using a shell model code. However, such an attempt has not been made, just as in [29], as our focus is on parity. Also the ensemble for spinless systems will give the essential features due to parity and these can be used in later explorations using ensembles with \( J^\pi \) or \( J^\pi T \) which are more complicated numerically and more importantly from analytical point of view [3, 22]. In fact, due to the severe problems associated with analytical tractability, a variety of EGOE are being analyzed since 1995; see [9, 14, 22] for reviews. At this point it is also useful to mention that EGOE(1+2)’s are also called TBRE in literature; see Section 5.7 in [9] for clarifications on this nomenclature. As Brody et al state [3]: The most severe mathematical difficulties with TBRE are due to angular momentum constraints \ldots Another type of ensemble, \ldots much closer to being mathematical tractable abandons the \( J \) restrictions entirely \ldots an embedded GOE, or EGOE for short.

Starting with the EGOE(1+2)-\( \pi \) ensemble defined by Eqs. (1), (3) and (4), we have numerically constructed 100 (in some examples 200) members of the ensemble in many-particle \(+ve\) and \(-ve\) parity spaces with dimensions \( d_+ \) and \( d_- \) given by Eq. (2) for several values of \((N_+, N_-, m)\) and varying the parameters \( \tau \) and \( \alpha \). This means we have considered 100 realizations of EGOE(1+2)-\( \pi \) random matrices in \((N_+, N_-, m)\) spaces - we use the phrase ‘members’ throughout the paper instead of ‘realizations’ (other names used by some authors are ‘sets’, ‘samples’ and ‘trials’) as in all our previous papers. Before discussing the numerical calculations, we present the results for the energy centroids, variances and also the shape parameters (skewness and excess) defining the normalized fixed-\((m_1, m_2)\) partial densities

\[
\rho^{m_1,m_2}(E) = \langle \delta(H - E) \rangle^{m_1,m_2} = \frac{1}{d(m_1, m_2)} \sum_{\alpha,\beta} |C^{m_1,m_2,\alpha}_{E,\beta}|^2 ;
\]

\[
|m_1, m_2, \alpha\rangle = \sum_{\beta} C^{m_1,m_2,\alpha}_{E,\beta} |E, \beta\rangle ,
\]

where, \(\langle \ldots \rangle\) corresponds to average and \(\alpha\) and \(\beta\) are extra labels required to specify completely the states with a given \((m_1, m_2)\) and \(E\) respectively. Later we use the symbol \(\langle\langle \ldots \rangle\rangle\) that denotes the trace. These will allow us to understand some of the numerical results. Let us add that the fixed-\( \pi \) eigenvalue densities \(I_{\pm}(E)\) are sum of the appropriate partial densities as given by Eq. (16) ahead. Note that the densities \(I_{\pm}(E)\) are normalized to \(d_{\pm}\).
III. ENERGY CENTROIDS, VARIANCES, SKEWNESS AND EXCESS PARAMETERS FOR FIXED-$(m_1,m_2)$ PARTIAL DENSITIES

Let us call the set of $+$-ve parity sp states as unitary orbit #1 and similarly the set of $-$-ve parity sp states as unitary orbit #2; see [15] for unitary orbits notation and significance. For convenience, from now on, we denote the sp states by the roman letters $(i,j,\ldots)$ and unitary orbits by greek letters $(\alpha,\beta,\ldots)$. Note that $\alpha = 1$ corresponds to the $+$-ve parity unitary orbit and $\alpha = 2$ corresponds to the $-$-ve parity unitary orbit (with this notation, $N_1 = N_+$ and $N_2 = N_-$. The sp states that belong to a unitary orbit $\alpha$ are denoted as $i_\alpha, j_\alpha, \ldots$. Propagation formulas for the energy centroids and variances of the partial densities $\rho^{m_1,m_2}(E)$ follow from the unitary decomposition of $V(2)$ with respect to the sub-algebra $U(N_+) \oplus U(N_-)$ contained in $U(N)$. Note that $(m_1,m_2)$ label the irreducible representations (irreps) of $U(N_+) \oplus U(N_-)$ and they all belong to the $U(N)$ irreps labeled by $m$. The $(m_1,m_2)$ are often called unitary configurations [15]. With respect to $U(N_+) \oplus U(N_-)$, the operator $V(2)$ decomposes into three parts $V(2) \rightarrow V[0] + V[1] + V[2]$. The $V[0]$ generates the energy centroids $\langle V \rangle^{m_1,m_2}$, $V[1]$ corresponds to the ‘algebraic’ mean-field generated by $V$ and $V[2]$ is the remaining irreducible two-body part. Extending the unitary decomposition for the situation with a single orbit for spinless fermions (given in Appendix A) and also using the detailed results in [30], we obtain the following formulas for the $V[\nu]$’s. The $V[0]$ is given by (with $\alpha = 1, 2$ and $\beta = 1, 2$)

\[
V[0] = \sum_{\alpha \geq \beta} \frac{\hat{n}_\alpha (\hat{n}_\beta - \delta_{\alpha\beta})}{(1 + \delta_{\alpha\beta})} V_{\alpha\beta} ;
\]

\[
V_{\alpha\alpha} = \binom{N_\alpha}{2}^{-1} \sum_{i > j} V_{ia,jia,j} ,
\]

\[
V_{\alpha\beta} = (N_\alpha N_\beta)^{-1} \sum_{i,j} V_{ia,jia,j} ; \quad \alpha \neq \beta .
\]
Then the traceless part \( \widetilde{V} = V - V^0 = V^{[1]} + V^{[2]} \) where \((\widetilde{V})_{ia\beta i\alpha \beta} = V_{ia\beta i\alpha \beta} - V_{\alpha \beta} \) and \((\widetilde{V})_{ijkl} = V_{ijkl} \) for all others. Now the \( V^{[1]} \) part is

\[
V^{[1]} = \sum_{\alpha, i, j} \tilde{\xi}_{ia\beta} a_{ia}^\dagger a_{ja};
\]

\[
\tilde{\xi}_{ia\beta} = \sum_{\beta} \frac{\hat{N}_{\beta} - \delta_{\alpha \beta}}{N_{\beta} - 2\delta_{\alpha \beta}} \xi_{ia\beta}(\beta), \quad \zeta_{ia\beta}(\beta) = \sum_{k_{\beta}} \tilde{V}_{k_{\beta}ia\beta\beta}.
\]  

(7)

Finally, the \( V^{[2]} \) part is as follows,

\[
V^{[2]} = \widetilde{V} - V^{[1]};
\]

\[
V^{[2]}_{ia\beta i\alpha \beta} = \tilde{V}_{ia\beta i\alpha \beta} - \left[ \frac{\xi_{ia\beta}(\beta)}{N_{\beta} - 2\delta_{\alpha \beta}} + \frac{\zeta_{i\beta j\beta}(\alpha)}{N_{\alpha} - 2\delta_{\alpha \beta}} \right],
\]

\[
V^{[2]}_{i\beta j\beta k_{\alpha} \alpha \beta} = \tilde{V}_{i\beta j\beta k_{\alpha} \alpha \beta} - \frac{\zeta_{i\beta j\beta}(\alpha)}{N_{\alpha} - 2\delta_{\alpha \beta}}; \quad i_{\beta} \neq j_{\beta},
\]

\[
V^{[2]}_{ijkl} = \tilde{V}_{ijkl} \text{ for all others}.
\]

(8)

Given Eqs. (6), (7) and (8), by intuition and using Eq. (A3), it is possible to write the propagation formulas for the energy centroids and variances of \( \rho_{m_1,m_2}^m(E) \). Note that these are essentially traces of \( H \) and \( H^2 \) over the space defined by the two-orbit configurations \((m_1, m_2)\); see Eqs. (9) and (10) ahead. A direct approach to write the propagation formulas for centroids and variances for a multi-orbit configuration was given in detail first by French and Ratcliff [35]. The formula for the variance given in [35] is cumbersome and it is realized later [30] that they can be made compact by applying group theory (see also [14, 15, 36]). We have adopted the group theoretical approach for the two-orbit averages and obtained formulas. Propagation formula for the fixed-\((m_1, m_2)\) energy centroids is

\[
E_c(m_1, m_2) = \langle H \rangle_{m_1,m_2} = m_2 \Delta + \sum_{\alpha \geq \beta} \frac{m_{\alpha}(m_{\beta} - \delta_{\alpha \beta})}{(1 + \delta_{\alpha \beta})} V_{\alpha \beta}.
\]

(9)
First term in Eq. (9) is generated by \( h(1) \) and it is simple because of the choice of the sp energies as shown in Fig. 1. Propagation formula for fixed-(\( m_1, m_2 \)) variances is,

\[
\sigma^2(m_1, m_2) = \langle H^2 \rangle^{m_1,m_2} - [\langle H \rangle^{m_1,m_2}]^2 
\]

\[
= \sum_{\alpha} \frac{m_\alpha (N_\alpha - m_\alpha)}{N_\alpha (N_\alpha - 1)} \sum_{i_\alpha,j_\alpha} [\xi_{i_\alpha,j_\alpha}(m_1, m_2)]^2 
\]

\[
+ \sum_{\alpha,\beta,\gamma,\delta} \frac{m_\alpha (m_\beta - \delta_{\alpha\beta}) (N_\gamma - m_\gamma) (N_\delta - m_\delta - \delta_{\gamma\delta})}{N_\alpha (N_\beta - \delta_{\alpha\beta}) (N_\gamma - \delta_{\gamma\alpha} - \delta_{\gamma\beta}) (N_\delta - \delta_{\delta\alpha} - \delta_{\delta\beta} - \delta_{\delta\gamma})} (X) ; 
\]

\[
\xi_{i_\alpha,j_\alpha}(m_1, m_2) = \sum_{\beta} \frac{m_\beta - \delta_{\alpha\beta}}{N_\beta - 2\delta_{\alpha\beta}} [\xi_{i_\alpha,j_\alpha}(\beta)]^2, 
\]

\[
X = \sum_{i_\alpha,j_\alpha} \left( V_{i_\alpha,j_\alpha} V_{k_\gamma, \ell_\delta} \right)^2 . 
\]

The ‘prime’ over summations in Eq. (10) implies that the summations are not free sums. Note that \((\alpha, \beta, \gamma, \delta)\) take values (1,1,1,1), (2,2,2,2), (1,2,1,2), (1,1,2,2) and (2,2,1,1). Similarly, in the sum over \((i_\alpha,j_\beta)\), \(i \leq j\) if \(\alpha = \beta\) and otherwise the sum is over all \(i\) and \(j\). Similarly, for \((k_\gamma, \ell_\delta)\). Using \(E_c(m_1, m_2)\) and \(\sigma^2(m_1, m_2)\), the fixed-parity energy centroids and spectral variances \(\text{[they define } I_\pm(E)\text{]}\) can be obtained as follows,

\[
E_c(m, \pm) = \langle H \rangle^{m,\pm} = \frac{1}{d_\pm} \sum_{m_1,m_2} d(m_1,m_2) E_c(m_1, m_2) , 
\]

\[
\sigma^2(m, \pm) = \langle H^2 \rangle^{m,\pm} - [\langle H \rangle^{m,\pm}]^2 ; 
\]

\[
\langle H^2 \rangle^{m,\pm} = \frac{1}{d_\pm} \sum_{m_1,m_2} d(m_1,m_2) \left[ \sigma^2(m_1, m_2) + E_c^2(m_1, m_2) \right]. 
\]

The ‘prime’ over summations in Eq. (11) implies that \(m_2\) is even(odd) for +ve(−ve) parity.

It should be pointed out that the formulas given by Eqs. (9), (10) and (11) are compact and easy to understand compared to Eqs. (10)-(14) of [29] and also those that follow from Eqs. (129) and (133) of [35] where unitary decomposition is not employed. We have verified Eqs. (9) and (10) by explicit construction of the \(H\) matrices in many examples. In principle, it is possible to obtain a formula for the ensemble averaged variances using Eq. (10); the ensemble averaged centroids derive only from \(h(1)\). Simple asymptotic formulas for ensemble averaged variances follow by neglecting the \(\delta\)-functions that appear in Eq. (10)
and replacing $\tilde{V}_{ijkl}^2$ by $\tau^2$ and $\alpha^2$ appropriately. Then the final formula for the ensemble averaged fixed-$(m_1, m_2)$ variances is,

$$\sigma^2(m_1, m_2) \approx m \left[ \sum_{\alpha=1}^{2} m_\alpha (N_\alpha - m_\alpha) \right] \tau^2$$

$$+ \left[ \binom{m_1}{2} \binom{\tilde{m}_1}{2} + \binom{m_2}{2} \binom{\tilde{m}_2}{2} + m_1 m_2 \tilde{m}_1 \tilde{m}_2 \right] \tau^2$$

$$+ \left[ \binom{m_1}{2} \binom{\tilde{m}_2}{2} + \binom{m_2}{2} \binom{\tilde{m}_1}{2} \right] \alpha^2.$$ (12)

Here, $\tilde{m}_1 = N_1 - m_1$ and $\tilde{m}_2 = N_2 - m_2$. The ‘overline’ in Eq. (12) denotes ensemble average. In Table II, we compare the results obtained from Eq. (12) with those obtained for various 100 member ensembles using Eq. (10) and the agreements are quite good. Therefore, in many practical applications, one can use Eq. (12).

The skewness and excess parameters $\gamma_1$ and $\gamma_2$ give information about the shape of the partial densities and they are close to zero implies Gaussian form. Formulas for the $M_r$, $r = 3, 4$ for a given one plus two-body Hamiltonian defined by Eq. (1) follow from the results given in [36–41] many years back. However, these formulas contain very large number of complicated terms (in particular for $M_4$) and carrying out analytically ensemble averaging is proved to be impractical (we are not aware if anyone was successful in the past). Some idea of the difficulty in carrying out simplifications can be seen from the attempt in [42]. An alternative is to program the exact formulas and evaluate the moments numerically for each member of EGOE(1+2)-π by considering say 500 members in two particle spaces. As pointed out by Terán and Johnson [43] in their most recent attempt, these calculations for the 4th moments are time consuming if not impractical. All the problems with the exact formulas have been emphasized in [15]. Because of these (in future with much faster computers it may be possible to use the exact formulas), we have adopted the binary correlation approximation, first used by Mon and French [31, 32] and later by French et al [33, 34], which is good in the dilute limit: $m_1, N_1, m_2, N_2 \to \infty$, $m/N_1 \to 0$ and $m/N_2 \to 0$, where $m$ is $m_1$ or $m_2$, for deriving formulas for the ensemble averaged $M_3$ and $M_4$. The final formulas are given in Appendix B and details of the derivations will be reported elsewhere [44]. The following results are inferred from the results in Appendix B.

It is seen from Eq. (B9), $\gamma_1(m_1, m_2)$ will be non-zero only when $\alpha \neq 0$ and the $\tau$
TABLE II. Ensemble averaged fixed-($m_1, m_2$) widths $\sigma(m_1, m_2)$ and the total spectral width $\sigma_t$ for different ($\tau, \alpha$) values. For each ($\tau, \alpha$), the $\sigma(m_1, m_2)$ are given in the table and they are obtained using the exact propagation formula Eq. (10) for each member of the ensemble. In all the calculations, ensembles with 100 members are employed. Numbers in the bracket are obtained by using the asymptotic formula given in Eq. (12). Last row for each ($N_+, N_-$) gives the corresponding $\sigma_t$ values. All the results are given for 6 particle systems and the dimensions $d(m_1, m_2)$ are also given in the table. See text for details.

| ($N_+, N_-$) | $m_1$ | $m_2$ | $d(m_1, m_2)$ | ($\tau, \alpha/\tau$) | (0.1, 0.5) | (0.1, 1.5) | (0.2, 0.5) | (0.2, 1.5) |
|--------------|------|------|----------------|----------------------|-----------|-----------|-----------|-----------|
| (8, 8)       | 0    | 6    | 28             | 1.36(1.39) 3.21(3.21) 2.73(2.77) 6.41(6.42) |
|              | 1    | 5    | 448            | 1.76(1.79) 2.70(2.72) 3.52(3.57) 5.41(5.44) |
|              | 2    | 4    | 1960           | 2.05(2.09) 2.48(2.50) 4.11(4.17) 4.96(5.01) |
|              | 3    | 3    | 3136           | 2.16(2.19) 2.42(2.45) 4.31(4.38) 4.84(4.90) |
|              | 4    | 2    | 1960           | 2.05(2.09) 2.48(2.50) 4.11(4.17) 4.95(5.01) |
|              | 5    | 1    | 448            | 1.76(1.79) 2.70(2.72) 3.52(3.57) 5.41(5.44) |
|              | 6    | 0    | 28             | 1.37(1.39) 3.21(3.21) 2.75(2.77) 6.42(6.42) |
|              |      |      |                | 2.29(2.32) 2.68(2.71) 4.24(4.30) 5.08(5.13) |
| (6, 10)      | 0    | 6    | 210            | 1.67(1.70) 2.70(2.72) 3.34(3.41) 5.41(5.44) |
|              | 1    | 5    | 1512           | 2.04(2.07) 2.48(2.51) 4.08(4.15) 4.97(5.02) |
|              | 2    | 4    | 3150           | 2.19(2.22) 2.41(2.44) 4.37(4.44) 4.82(4.88) |
|              | 3    | 3    | 2400           | 2.11(2.14) 2.43(2.46) 4.22(4.28) 4.86(4.91) |
|              | 4    | 2    | 675            | 1.84(1.87) 2.60(2.62) 3.67(3.73) 5.20(5.24) |
|              | 5    | 1    | 60             | 1.46(1.48) 3.06(3.06) 2.92(2.96) 6.12(6.13) |
|              | 6    | 0    | 1              | 1.30(1.30) 3.90(3.90) 2.60(2.60) 7.81(7.79) |
|              |      |      |                | 2.31(2.33) 2.65(2.67) 4.30(4.36) 5.02(5.07) |
| (10, 10)     | 0    | 6    | 210            | 1.97(2.01) 4.16(4.19) 3.95(4.01) 8.33(8.37) |
|              | 1    | 5    | 2520           | 2.44(2.49) 3.63(3.66) 4.90(4.98) 7.25(7.32) |
|              | 2    | 4    | 9450           | 2.76(2.81) 3.36(3.40) 5.53(5.61) 6.71(6.79) |
|              | 3    | 3    | 14400          | 2.87(2.92) 3.28(3.32) 5.74(5.83) 6.56(6.64) |
|              | 4    | 2    | 9450           | 2.76(2.81) 3.36(3.40) 5.53(5.61) 6.71(6.79) |
|              | 5    | 1    | 2520           | 2.44(2.49) 3.63(3.66) 4.90(4.98) 7.25(7.32) |
|              | 6    | 0    | 210            | 1.97(2.01) 4.16(4.19) 3.95(4.01) 8.33(8.37) |
|              |      |      |                | 2.95(2.99) 3.54(3.57) 5.62(5.70) 6.83(6.91) |
dependence is weak. Also, it is seen that for \( N_+ = N_- \), \( \gamma_1(m_1, m_2) = -\gamma_1(m_2, m_1) \). Similarly, Eq. (B10) shows that for \( N_+ = N_- \), \( \gamma_2(m_1, m_2) = \gamma_2(m_2, m_1) \). In the dilute limit, with some approximations as discussed after Eq. (B10), the expression for \( \gamma_2(m_2, m_1) \) is given by Eq. (B11). This shows that, for \( \alpha << \tau \) or \( \tau << \alpha \), the \( C_1 \) and \( C_2 \) in Eq. (B11) will be negligible and then, \( \gamma_2 \sim -4/m \) for \( m_1 = m_2 = m/2 \) and \( N_1 = N_2 = N \). This is same as the result for spinless fermion EGOE(2) \([31, 32]\) and shows that for a range of \((\tau, \alpha)\) values, \( \rho^{m_1,m_2}(E) \) will be close to Gaussian. Moreover, to the extent that Eq. (B11) applies, the density \( \rho^{m_1,m_2}(E) \) is a convolution of the densities generated by \( X(2) \) and \( D(2) \) operators. Let us add that the binary correlation results presented in Appendix B, with further extensions, will be useful in the study of partitioned EGOE discussed in \([14, 45]\).

IV. RESULTS AND DISCUSSION

In order to proceed with the numerical calculations, we need to have some idea of the range of the parameters \((\tau, \alpha, m/N_+, N_+/N_-)\). Towards this end, we have used realistic nuclear effective interactions in \( sdfp \) \([46]\) and \( fpg_{9/2} \) \([47]\) spaces and calculated the variances \( v_a^2, v_b^2, v_c^2, v_d^2 \) for these interactions. Note that it is easy to identify the matrices \( A, B, C \) and \( D \) given the interaction matrix elements \( \langle (j_1j_2)JT \mid V \mid (j_3j_4)JT \rangle \). To calculate the mean-squared matrix elements \( v^2 \)'s, we put the diagonal two-particle matrix elements to be zero and use the weight factor \((2J + 1)(2T + 1)\). Assuming that \( \Delta = 3 \) MeV and 5 MeV (these are reasonable values for \( A = 20 - 80 \) nuclei), we obtain \( \tau \sim 0.09 - 0.24 \) and \( \alpha \sim (0.9 - 1.3) \times \tau \). These deduced values of \( \alpha \) and \( \tau \) clearly point out that one has to go beyond the highly restricted ensemble employed in \([29]\) and it is necessary to consider the more general EGOE(1+2)-\( \pi \) defined in Section II. Similarly, for \( sdfp \) and \( fpg_{9/2} \) spaces \( N_+/N_- \sim 0.5 - 2.0 \). Finally, for nuclei with \( m \) number of valence nucleons (particles or holes) where \( sdfp \) or \( fpg_{9/2} \) spaces are appropriate, usually \( m \lesssim N_+ \) or \( N_- \), whichever is lower. Given these, we have selected the following examples: \((N_+, N_-, m) = (8,8,4), (8,8,5), (10,6,4), (10,6,5), (6,10,4), (6,10,5), (8,8,6), (6,6,6), (7,7,7) \) and \((7,7,6)\). To go beyond the matrix dimensions \( \sim 5000 \) with 100 members is not feasible at present with the HPC cluster that is used for all the calculations. Most of the discussion in this paper is restricted to \( N = N_+ + N_- = 16 \) and \( m << N \) as in this dilute limit it is possible to understand the ensemble results better. Following the nuclear examples mentioned above, we have chosen


\[ \tau = 0.05, 0.1, 0.2, 0.3 \text{ and } \alpha/\tau = 0.5, 1.0, 1.5. \]

We will make some comments on the results for other \((\tau, \alpha)\) values at appropriate places.

Now we will present the results for (i) the form of the +ve and −ve parity state densities \(I_+(E)\) and \(I_-(E)\) respectively, (ii) the parity ratios \(I_-(E)/I_+(E)\) vs \(E\) where \(E\) is the excitation energy of the system and (iii) the probability for +ve parity ground states generated by the EGOE(1+2)-π ensemble.

A. Gaussian form for fixed-π state densities

Using the method discussed in Sec. II, we have numerically constructed in +ve and −ve parity spaces EGOE(1+2)-π ensembles of random matrices consisting of 100 Hamiltonian matrices in large number of examples, i.e. for \((N_+, N_-, m)\) and \((\tau, \alpha)\) parameters mentioned above. Diagonalizing these matrices, ensemble averaged eigenvalue (state) densities,

\[ I_{\pm}(E) = \langle \langle \delta (H - E) \rangle \rangle^\pm, \]

are constructed. From now on, we drop the ‘overline’ symbol when there is no confusion. Results are shown for \((N_+, N_-, m) = (8, 8, 4), (8, 8, 5), (10, 6, 5)\) and \((6, 10, 5)\) for several values of \((\tau, \alpha)\) in Figs. 2, 3 and 4. To construct the fixed-parity eigenvalue densities, we first make the centroids \(E_c(m, \pm)\) of all the members of the ensemble to be zero and variances \(\sigma^2(m, \pm)\) to be unity, i.e. for each member we have the standardized eigenvalues \(\hat{E} = [E - E_c(m, \pm)]/\sigma(m, \pm)\). Then, combining all the \(\hat{E}\) and using a bin-size \(\Delta \hat{E} = 0.2\), histograms for \(I_{\pm}(E)\) are generated. It is seen that the state densities are multimodal for small \(\tau\) values and for \(\tau \geq 0.1\), they are unimodal and close to a Gaussian. Note that in our examples, \(\alpha = (0.5 - 1.5) \times \tau\).

For \(V(2) = 0\), the eigenvalue densities will be a sum of spikes at \(0, 2\Delta, 4\Delta, \ldots\) for +ve parity densities and similarly at \(\Delta, 3\Delta, 5\Delta, \ldots\) for −ve parity densities. As we switch on \(V(2)\), the spikes will spread due to the matrices \(A, B\) and \(C\) in Fig. 1 and mix due to the matrix \(D\). The variance \(\sigma^2(m_1, m_2)\) can be written as,

\[ \sigma^2(m_1, m_2) = \sigma^2(m_1, m_2 \rightarrow m_1, m_2) + \sigma^2(m_1, m_2 \rightarrow m_1 \pm 2, m_2 \mp 2). \]  

The internal variance \(\sigma^2(m_1, m_2 \rightarrow m_1, m_2)\) is due to \(A, B\) and \(C\) matrices and it receives contribution only from the \(\tau\) parameter. Similarly, the external variance \(\sigma^2(m_1, m_2 \rightarrow m_1 \pm 2, m_2 \mp 2)\)
FIG. 2. (Color online) Positive and negative parity state densities for various $(\tau, \alpha)$ values for $(N_+, N_-, m) = (8, 8, 4)$ system. See text for details.
FIG. 3. (Color online) Positive and negative parity state densities for various \((\tau, \alpha)\) values for \((N_+, N_-, m) = (8, 8, 5)\) system. Histograms are numerical ensemble results. The dashed (red) curve corresponds to Gaussian form for \(\rho_{m_1,m_2}(E)\) in Eq. (16) and similarly, solid (green) curve corresponds to Edgeworth corrected Gaussian form with \(\gamma_1(m_1,m_2)\) and \(\gamma_2(m_1,m_2)\) obtained using the results in Appendix B. See text for details.
$$\langle N_+, N_-, m \rangle = (10, 6, 5)$$

FIG. 4. (Color online) Positive and negative parity state densities for various \((\tau, \alpha)\) values for \((N_+, N_-, m) = (10, 6, 5)\) and \((6, 10, 5)\) systems. Histograms are numerical ensemble results. The dashed (red) curve corresponds to Gaussian form for \(\rho_{m_1, m_2}(E)\) in Eq. (16) and similarly, solid (green) curve corresponds to Edgeworth corrected Gaussian form with \(\gamma_1(m_1, m_2)\) and \(\gamma_2(m_1, m_2)\) obtained using the results in Appendix B. See text for details.
$m_1 \pm 2, m_2 \pm 2$) is due to the matrix $D$ and it receives contribution only from the $\alpha$ parameter. When we switch on $V(2)$, as the ensemble averaged centroids generated by $V(2)$ will be zero, the positions of the spikes will be largely unaltered. However, they will start spreading and mixing as $\tau$ and $\alpha$ increase. Therefore, the density will be multimodal with the modes well separated for very small ($\tau, \alpha$) values. Some examples for this are shown in Fig. 5. As $\tau$ and $\alpha$ start increasing from zero, the spikes spread and will start overlapping for $\sigma(m_1, m_2) \gtrsim \Delta$. This is the situation with $\tau = 0.05$ shown in Figs. 2, 3 and 4. However, as $\tau$ increases (with $\alpha \sim \tau$), the densities start becoming unimodal as seen from the $\tau = 0.1$ and 0.2 examples. Also, the $m$ dependence is not strong as seen from the Figs. 2, 3 and 4. Now we will discuss the comparison of the ensemble results with the smoothed densities constructed using $E_c(m_1, m_2), \sigma^2(m_1, m_2), \gamma_1(m_1, m_2)$ and $\gamma_2(m_1, m_2)$.

FIG. 5. (Color online) Positive and negative parity state densities for some small values of ($\tau, \alpha$). The $(N_+, N_-, m)$ values are given in the figures. See text for details.
$\tau=0.1, \alpha=0.5\tau$  $\tau=0.1, \alpha=1.5\tau$  $\tau=0.2, \alpha=0.5\tau$  $\tau=0.2, \alpha=1.5\tau$

$\left[E-E_c(m,+)\right]/\sigma(m,+)$

$\left[E-E_c(m,-)\right]/\sigma(m,-)$

$(N_+, N_- , m) = (8, 8, 6)$  $(N_+, N_- , m) = (6, 10, 6)$  $(N_+, N_- , m) = (10, 6, 6)$

FIG. 6. (Color online) Positive and negative parity state densities for various $(\tau, \alpha)$ values for $(N_+, N_-, m) = (8, 8, 6)$, $(6, 10, 6)$ and $(10, 6, 6)$ systems. Smoothed curves (solid red lines) are obtained using fixed-$(m_1, m_2)$ partial densities. See text for details.
As the particle numbers in the examples shown in Figs. 2, 3 and 4 are small, the excess parameter \( \gamma_2(m, \pi) \sim -0.7 \) to \(-0.8\) (skewness parameter \( \gamma_1(m, \pi) \sim 0 \) in all our examples). Therefore the densities are not very close to a Gaussian form. It has been well established that the ensemble averaged eigenvalue density takes Gaussian form in the case of spinless fermion (as well as boson) systems and also for the embedded ensembles extending to those with good quantum numbers; see [3, 9, 14, 18] and references therein. Thus, it can be anticipated that Gaussian form is generic for the state densities or more appropriately, for the partial densities \( \rho_{m_1,m_2}(E) \) generated by EGOE(1+2)-\(\pi\) for some range of \((\tau, \alpha)\) values.

Results for the fixed-\(\pi\) densities for \((N_+, N_-, m) = (8, 8, 6), (6, 10, 6)\) and \((10, 6, 6)\) systems are shown in Fig. 6. The smoothed +ve and –ve parity densities are a sum of the partial densities \( \rho_{m_1,m_2}(E) \),

\[
\rho_{\pm}(E) = \frac{1}{d_{\pm}} \sum_{m_1,m_2} d(m_1,m_2) \rho_{m_1,m_2}(E) . \tag{15}
\]

Note that the summation in Eq. (15) is over \( m_2 \) even for +ve parity density and similarly over \( m_2 \) odd for –ve parity density. Here \( \rho_{\pm}(E) \) as well as \( \rho_{m_1,m_2}(E) \) are normalized to unity. However, in practice, the densities normalized to dimensions are needed and they are denoted, as used earlier, by \( I_{\pm}(E) \) and \( I_{m_1,m_2}(E) \) respectively,

\[
I_{\pm}(E) = d_{\pm} \rho_{\pm}(E) = \sum_{m_1,m_2} I_{m_1,m_2}(E) ; \quad I_{m_1,m_2}(E) = d(m_1,m_2) \rho_{m_1,m_2}(E) . \tag{16}
\]

We employ the Edgeworth (ED) form that includes \( \gamma_1 \) and \( \gamma_2 \) corrections to the Gaussian partial densities \( \rho_{G_{m_1,m_2}}(E) \). Then

\[
\rho_{m_1,m_2}(E) \to \rho_{G_{m_1,m_2}}(E) \to \rho_{ED_{m_1,m_2}}(E)
\]

and in terms of the standardized variable \( \hat{E} \), the ED form is given by

\[
\eta_{ED}(\hat{E}) = \eta_{G}(\hat{E}) \left\{ 1 + \left[ \frac{\gamma_1}{6} H_{3}(\hat{E}) \right] + \left[ \frac{\gamma_2}{24} H_{4}(\hat{E}) + \frac{\gamma_2^2}{72} H_{6}(\hat{E}) \right] \right\} ; \tag{17}
\]

\[
\eta_{G}(\hat{E}) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{\hat{E}^2}{2} \right) .
\]

Here, \( H \) are Hermite polynomials: \( H_3(x) = x^3 - 3x \), \( H_4(x) = x^4 - 6x^2 + 3 \) and \( H_6(x) = x^6 - 15x^4 + 45x^2 - 15 \). Using Eqs. (15) and (17) with exact centroids and variances given by the propagation formulas in Section III and the binary correlation results for \( \gamma_1 \) and \( \gamma_2 \)
as given by the formulas in Appendix B, the smoothed +ve and −ve parity state densities are constructed. We put \( \eta_{ED}(\hat{E}) = 0 \) when \( \eta_{ED}(\hat{E}) < 0 \). It is clearly seen from Fig. 6 that the sum of partial densities, with the partial densities represented by ED corrected Gaussians, describe extremely well the exact fixed-\( \pi \) densities in these examples. Therefore, for the \( (\tau, \alpha) \) values in the range determined by nuclear \( sdpf \) and \( fpag/2 \) interactions, i.e. \( \tau \sim 0.1 - 0.3 \) and \( \alpha \sim 0.5\tau - 2\tau \), the partial densities can be well represented by ED corrected Gaussians and total densities are also close to ED corrected Gaussians. Unlike Fig. 6, densities in Figs. 2, 3 and 4 show, in many cases, strong departures form Gaussian form. Therefore, it is important to test how well Eq. (16) with ED corrected Gaussian for \( \rho_{m_1,m_2}(E) \) describes the numerical results for \( I_\pm(E) \). We show this comparison for all the densities in Figs. 3 and 4. It is clearly seen that the agreements with ED corrected Gaussians are good in all the cases. Therefore, the large deviations from the Gaussian form for \( I_\pm(E) \) arise mainly because of the distribution of the centroids [this involves dimensions of the \((m_1,m_2)\) configurations] of the partial densities involved. It is possible that the agreements in Figs. 3 and 4 may become more perfect if we employ, for the partial densities, some non-canonical forms defined by the first four moments as given for example in [48, 49]. However, as these forms are not derived using any random matrix ensemble, we haven’t used these for the partial densities in our present investigation. In conclusion, for the physically relevant range of \((\tau, \alpha)\) values, the propagation formulas for centroids and variances given by Eqs. (9) and (10) or alternatively with \( E_c(m_1,m_2) = m_2\Delta \) and Eq. (12) along with the EGOE(1+2)-\( \pi \) ensemble averaged \( \gamma_1(m_1,m_2) \) and \( \gamma_2(m_1,m_2) \) formulas (obtained using the binary correlation approximation as given in Appendix B) can be used to construct fixed-\( \pi \) state densities for larger \((N_+,N_-,m)\) systems.

**B. Parity ratios for state densities**

As stated in the beginning, parity ratio of state densities at a given excitation energy \( (E) \) is a quantity of considerable interest in nuclear structure. For the systems shown in Figs. 2, 3 and 4 and also for many other systems, we have studied the parity ratios and the results are shown in Figs. 7-10. As the parity ratios need to be calculated at a given value of excitation energy \( E \), we measure the eigenvalues in both +ve and −ve parity spaces with respect to the absolute ground state energy \( E_{gs} \) of the \( N = N_+ + N_- \) system. Thus, \( E_{gs} \) is defined by
taking all the $+\text{ve}$ and $-\text{ve}$ parity eigenvalues of all members of the ensemble and choosing the lowest of all these. The ground state energy can also be determined by averaging the $+\text{ve}$ and $-\text{ve}$ parity ground state energies over the ensemble and then the ground state energy is minimum of the two. It is seen that the results for parity ratios are essentially independent of the choice of $E_{gs}$ and thus we employ absolute ground state energy in our calculations. We use the ensemble averaged total ($+\text{ve}$ and $-\text{ve}$ eigenvalues combined) spectrum width $\sigma_t$ of the system for scaling. The total widths $\sigma_t$ can be calculated also by using $E_c(m_1,m_2)$ and $\sigma^2(m_1,m_2)$. Examples for $\sigma_t$ are shown in Table II and they are in good agreement with the results obtained using the simple formula given by Eq. (12). We use the variable $E = (E - E_{gs})/\sigma_t$ for calculating parity ratios. Starting with $E_{gs}$ and using a bin-size of $\Delta E = 0.2$, we have calculated the number of states $I_+(E)$ with $+\text{ve}$ parity and also the number of states $I_-(E)$ with $-\text{ve}$ parity in a given bin and then the ratio $I_-(E)/I_+(E)$ is the parity ratio. Note that the results in Figs. 7-10 are shown for $E = 0 - 3$ as the spectrum span is $\sim 5.5\sigma_t$. To go beyond the middle of the spectrum, for real nuclei, one has to include more sp levels (also a finer splitting of the $+\text{ve}$ and $-\text{ve}$ parity levels may be needed) and therefore, $N_+$ and $N_-$ change. Continuing with this, one obtains the Bethe form for nuclear level densities [15].

General observations from Figs. 7-10 are as follows. (i) The parity ratio $I_-(E)/I_+(E)$ will be zero up to an energy $E_0$. (ii) Then, it starts increasing and becomes larger than unity at an energy $E_m$. (iii) From here on, the parity ratio decreases and saturates quickly to unity from an energy $E_1$. In these examples, $E_0 \lesssim 0.4$, $E_m \sim 1$ and $E_1 \sim 1.5$. It is seen that the curves shift towards left as $\tau$ increases. Also the position of the peak shifts to much larger value of $E_m$ and equilibration gets delayed as $\alpha$ increases for a fixed $\tau$ value. Therefore for larger $\tau$, the energies $(E_0, E_m, E_1)$ are smaller compared to those for a smaller $\tau$. The three transition energies also depend on $(N_+, N_-, m)$. We have also verified, as shown in Fig. 9, that the general structure of the parity ratios will remain same even when we change $\Delta \rightarrow -\Delta$ (i.e. $-\text{ve}$ parity sp states below the $+\text{ve}$ parity sp states). For $(N_+, N_-, m) = (8, 8, 4)$ system, results for $\Delta = 1$ are given in Fig. 7 and they are almost same as the results with $\Delta = -1$ given in Fig. 9. The general structures (i)-(iii) are clearly seen in the numerical examples shown in [25] where a method based on the Fermi-gas model has been employed. If $\sigma_t \sim 6 - 8$ MeV, equilibration in parities is expected around $E \sim 8 - 10$ MeV and this is clearly seen in the examples in [25]. It is also seen from Fig.
that the equilibration is quite poor for very small values of \( \tau \) and therefore comparing with the results in [25], it can be argued that very small values of \( \tau \) are ruled out for nuclei. Hence, it is plausible to conclude that generic results for parity ratios can be derived using EGOE(1+2)-\( \pi \) with reasonably large \((\tau, \alpha)\) values. Let us add that the interpretations in [25] are based on the occupancies of the sp orbits while in the present work, they are in terms of \( \tau \) and \( \alpha \) parameters.

Using the smoothed \( I^{\pm}(E) \), constructed as discussed in Section IV A, smoothed forms for parity ratios are calculated as follows. Starting with the absolute ground state energy \( E_{gs} \) and using a bin-size of \( \Delta E = 0.2 \), +ve and −ve parity densities in a given energy bin are obtained and their ratio is the parity ratio at a given \( E \). We have chosen the examples where \( I_{+} \) and \( I_{-} \) are close to Gaussians. It is seen from Fig. 10 that the agreement with

FIG. 7. (Color online) Parity ratios for various \((\tau, \alpha)\) values for \((N_{+}, N_{-}, m) = (8, 8, 4)\) and \((10, 6, 4)\) systems. See text for details.
FIG. 8. (Color online) Parity ratios for various $(\tau, \alpha)$ values for $(N_+, N_-, m) = (6, 10, 4)$ and $(6, 10, 5)$ systems. See text for details.
exact results is good for $E \gtrsim 0.5$. However, for smaller $E$, to obtain a good agreement one should have a better prescription for determining the tail part of the $\rho^{m_1,m_2}(E)$ distributions. Developing the theory for this is beyond the scope of the present paper as this requires more complete analytical treatment of the ensemble.

C. Probability for +ve parity ground states

Papenbrock and Weidenmüller used the $\tau \to \infty$, $\alpha = \tau$ limit of EGOE(1+2)-π for several $(N_+,N_-,m)$ systems to study the probability ($R_+$) for +ve parity ground states over the
ensemble [29]. As stated before, this exercise was motivated by shell model results with random interaction giving preponderance of +ve parity ground states [28]. The numerical calculations in [29] showed considerable variation (18−84\%) in $R_+$. In addition, they gave a plausible proof that in the dilute limit $m \ll (N_+, N_-)$, $R_+$ will approach 50\%. Combining these, they argued that the observed preponderance of +ve parity ground states could be a finite size (finite $N_+, N_-, m$) effect. For the extended EGOE(1+2)-\pi considered in the present work, where the $\tau \to \infty$ and $\alpha = \tau$ restriction is relaxed, as we will discuss now, $R_+$ can reach 100\%.

For EGOE(1+2)-\pi with $\tau \sim 0$, clearly one will get $R_+ = 100\%$ (for even $m$ and $m \ll N_+, N_-$) and therefore it is of interest to study $R_+$ variation with $(\tau, \alpha)$. We have carried out calculations using a 200 member ensemble for $(N_+, N_-, m) = (6, 6, 6)$ and 100 member
FIG. 11. (Color online) Probability ($R_+$) for +ve parity ground states for various ($\tau, \alpha$) values and for various ($N_+, N_-, m$) systems. See text for details.

ensembles for (8, 8, 5), (6, 6, 6), (6, 10, 4) and (6, 10, 5) systems. In these calculations, we use $\alpha = \tau$ and $1.5\tau$. The results are shown in Fig. 11. For $\alpha = \tau$, the results are as follows. For $\tau \lesssim 0.04$, we have $R_+ \sim 100\%$ and then $R_+$ starts decreasing with some fluctuations between $\tau = 0.1$ and 0.2. The origin of these fluctuations is not clear. As $\tau > 1$ is not realistic, we have restricted the $R_+$ calculations to $\tau \leq 1$. We see from the figure that EGOE(1+2)-$\pi$ generates $R_+ \gtrsim 50\%$ for $\tau \leq 0.3$ independent of ($N_+, N_-, m$). Also, $R_+$
decreases much faster with \( \tau \) and reaches \( \sim 30\% \) for \( \tau = 0.5 \) for \((N_+, N_-, m) = (6, 6, 6)\).

For \( m < (N_+, N_-) \), the decrease in \( R_+ \) is slower. If we increase \( \alpha \), from the structure of the two-particle \( H \) matrix in Fig. 1, we can easily infer that the width of the lowest +ve parity \((m_1, m_2)\) unitary configuration becomes much larger compared to the lowest −ve parity unitary configuration (see Table II for examples). Therefore, with increasing \( \alpha \) we expect \( R_+ \) to increase and this is clearly seen in Fig. 11. Thus \( \alpha \gtrsim \tau \) is required for \( R_+ \) to be large. A quantitative description of \( R_+ \) requires the construction of +ve and −ve parity state densities more accurately in the tail region and the theory for this is not yet available.

V. CONCLUSIONS AND FUTURE OUTLOOK

In the present work, we have introduced a generalized EGOE(1+2)-\( \pi \) ensemble for identical fermions and its construction follows from EGOE(1+2) for spinless fermion systems. Using this generalized EE, we have not only studied \( R_+ \), as it was done by Papenbrock and Weidenmüller [29] using a simpler two-body ensemble with parity, but also studied the form of fixed-\( \pi \) state densities and parity ratios which are important nuclear structure quantities.

Numerical examples (see Figs. 2-4 and 6), with the range of the various parameters in the model fixed using realistic nuclear effective interactions, are used to show that the fixed-\( \pi \) state densities in finite dimensional spaces are of Gaussian form for sufficiently large values of the mixing parameters \((\tau, \alpha)\). The random matrix model also captures the essential features of parity ratios as seen in the method based on non-interacting Fermi-gas model reported in [25]. We also found preponderance of +ve parity ground states for \( \tau \lesssim 0.5 \) and \( \alpha \sim 1.5\tau \). In addition, for constructing fixed-\( \pi \) Gaussian densities we have derived an easy to understand propagation formula [see Eq. (10)] for the spectral variances of the partial densities \( \rho^{m_1, m_2}(E) \) that generate \( I_+ \) and \( I_- \). Similarly, for calculating the corrections to the Gaussian forms, formulas for skewness \( \gamma_1 \) and excess \( \gamma_2 \) of the partial densities \( \rho^{m_1, m_2}(E) \) are derived using the binary correlation approximation (see Appendix B for the formulas). The smoothed densities constructed using Edgeworth corrected Gaussians are shown to describe the numerical results for \( I_{\pm}(E) \) [for \((\tau, \alpha)\) values in the range defined by nuclear \( sd fp \) and \( fpg_{9/2} \) interactions - see beginning of Sec. IV] and also the parity ratios at energies away from the ground state. Numerical results presented for parity ratios at lower energies show that a better theory for the tails of the partial densities is needed (see Figs. 7-10). Thus,
the results in the present paper represent considerable progress in analyzing EGOE(1+2)-π ensemble going much beyond the analysis presented in [29].

Results in the present work are largely numerical and they clearly show that developing a complete analytical theory, going beyond the results presented in Sections III and Appendix B, for EGOE(1+2)-π is important. In future, it is important to investigate EGOE(1+2)-π for proton-neutron systems and then we will have four unitary orbits (two for protons and two for neutrons). In addition, by including non-degenerate +ve and −ve parity sp states the model could be applied to nuclei for predicting parity ratios. This extended EGOE(1+2)-π model with protons and neutrons occupying different sp states will be generated by a 10×10 block matrix for V(2) in two-particle spaces. Therefore, parametrization of this ensemble is more complex. Analysis of this extended EGOE(1+2)-π is for future.

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APPENDIX A

Let us consider a system of m fermions in N sp states with a (1+2)-body Hamiltonian $H = h(1) + V(2)$ where $h(1) = \sum_i \epsilon_i \hat{n}_i$ and $V(2)$ is defined by the two-body matrix elements $V_{ijkl} = \langle kl | V(2) | ij \rangle$. With respect to the $U(N)$ group, the two-body interaction $V(2)$ can be separated into scalar ($\nu = 0$), effective one-body ($\nu = 1$) and irreducible two-body ($\nu = 2$)
parts [14, 15, 30, 36],
\begin{align*}
V^{\nu=0} &= \frac{\hat{n}(\hat{n} - 1)}{2} \mathbf{\nabla} ; \quad \mathbf{\nabla} = \left( \begin{array}{c} N \\ 2 \end{array} \right)^{-1} \sum_{i<j} V_{ij} ; \\
V^{\nu=1} &= \frac{\hat{n} - 1}{N - 2} \sum_{i,j} \zeta_{i,j} a_{i}^{\dagger} a_{j} ; \quad \zeta_{i,j} = \left[ \sum_{k} V_{kk} \right] - \left[ \left( N \right)^{-1} \sum_{r,s} V_{rs} \right] \delta_{i,j} , \\
V^{\nu=2} &= \mathbf{V} - V^{\nu=0} - V^{\nu=1} \iff V_{ijkl}^{\nu=2} ; \\
V_{ijkl}^{\nu=2} &= V_{ijkl} - \mathbf{\nabla} - (N - 2)^{-1} (\zeta_{i,i} + \zeta_{j,j}) .
\end{align*}

(A1)

Similar to Eq. (A1), the \( h^{(1)} \) operator will have \( \nu = 0, 1 \) parts,
\begin{align*}
\hat{h}^{\nu=0} &= \tau \hat{n} , \quad \tau = (N)^{-1} \sum_{i} \epsilon_{i} , \\
\hat{h}^{\nu=1} &= \sum_{i} \epsilon_{i}^{1} \hat{n}_{i} , \quad \epsilon_{i}^{1} = \epsilon_{i} - \tau .
\end{align*}

(A2)

Then the propagation equations for the \( m \)-particle centroids and variances are [14, 15, 30, 36],
\begin{align*}
E_{c}(m) &= \langle H \rangle^{m} = m \tau + \left( m \right) \mathbf{\nabla} , \\
\sigma^{2}(m) &= \langle H^{2} \rangle^{m} - \langle E_{c}(m) \rangle^{2} \\
&= \frac{m(N - m)}{N(N - 1)} \sum_{i,j} \left\{ \epsilon_{i}^{1} \delta_{i,j} + \frac{m - 1}{N - 2} \zeta_{i,j} \right\}^{2} + \frac{m(m - 1)(N - m)(N - m - 1)}{N(N - 1)(N - 2)(N - 3)} \left\langle \langle (V^{\nu=2})^{2} \rangle \right\rangle^{2} .
\end{align*}

(A3)

APPENDIX B

For the EGOE(1+2)-\( \pi \) Hamiltonian defined in Eq. (1), we have \( H = h^{(1)} + V^{(2)} = h^{(1)} + X^{(2)} + D^{(2)} \) with \( X^{(2)} = A \oplus B \oplus C \) is the direct sum of the spreading matrices \( A, B \) and \( C \) and \( D^{(2)} = D + \tilde{D} \) is the off-diagonal mixing matrix as shown in Fig. 1. Here, \( \tilde{D} \) is the transpose of the matrix \( D \). With the sp energies defining the mean field \( h^{(1)} \) as given in Eq. (1), the first moment \( M_{1} \) of \( \rho^{m_{1},m_{2}}(E) \) is trivially,
\begin{align*}
M_{1}(m_{1}, m_{2}) &= \langle (h + V) \rangle^{m_{1}, m_{2}} = m_{2} ,
\end{align*}

(B1)
TABLE III. Exact results for skewness and excess parameters for fixed-\(\pi\) eigenvalue densities \(I_\pm(E)\) compared with the binary correlation results (in the table, called ‘Approx’). For exact results, we have used the eigenvalues obtained from EGOE(1+2)-\(\pi\) ensembles with 100 members. The binary correlation results are obtained using Eqs. (B1)-(B8) and extension of Eq. (11). See text for details.

| \((N_+, N_-, m)\) | \((\tau, \alpha/\tau)\) | \(\gamma_1(m, \pi)\) | \(\gamma_2(m, \pi)\) |
|------------------|------------------|------------------|------------------|
|                  | \(\pi = +\) | \(\pi = -\) | \(\pi = +\) | \(\pi = -\) | \(\pi = +\) | \(\pi = -\) |
| (8, 8, 5)        | (0.05, 0.5)     | 0.15            | −0.15           | 0.15           | −0.15        | −0.52          | −0.52          | −0.52          |
|                  | (0.05, 1.0)     | 0.16            | −0.16           | 0.16           | −0.16        | −0.50          | −0.50          | −0.50          |
|                  | (0.05, 1.5)     | 0.18            | −0.17           | 0.18           | −0.18        | −0.46          | −0.46          | −0.46          |
|                  | (0.2, 0.5)      | −0.03           | 0.03            | −0.03          | 0.03         | −0.71          | −0.71          | −0.71          |
|                  | (0.2, 1.0)      | −0.01           | 0.01            | −0.01          | 0.01         | −0.73          | −0.73          | −0.74          |
|                  | (0.2, 1.5)      | 0.02            | −0.02           | 0.02           | −0.02        | −0.72          | −0.72          | −0.73          |
| (10, 6, 5)       | (0.05, 0.5)     | −0.06           | 0.09            | −0.07          | 0.09         | −0.26          | −0.26          | −0.26          |
|                  | (0.05, 1.5)     | −0.04           | 0.15            | −0.05          | 0.15         | −0.01          | −0.86          | −0.01          |
|                  | (0.2, 0.5)      | 0.01            | −0.04           | 0.01           | −0.04        | −0.73          | −0.69          | −0.73          |
|                  | (0.2, 1.5)      | 0.01            | 0.02            | 0.01           | 0.02         | −0.69          | −0.75          | −0.70          |
| (6, 10, 5)       | (0.05, 0.5)     | −0.09           | 0.07            | −0.09          | 0.07         | −0.76          | −0.26          | −0.75          |
|                  | (0.05, 1.5)     | −0.15           | 0.05            | −0.15          | 0.05         | −0.86          | −0.01          | −0.86          |
|                  | (0.2, 0.5)      | 0.04            | −0.01           | 0.04           | −0.01        | −0.68          | −0.73          | −0.69          |
|                  | (0.2, 1.5)      | −0.02           | −0.01           | −0.02          | −0.01        | −0.75          | −0.69          | −0.75          |

as \(\langle h^r \rangle^{m_1, m_2} = \langle m_2 \rangle^r\) and \(\overline{\langle V \rangle}^{m_1, m_2} = 0\). By extending the binary correlation method to traces over two-orbit configurations, we have derived formulas for the second, third and fourth order traces giving \(M_\tau(m_1, m_2), \tau = 2 - 4\). It is important to mention that the presence of the mixing matrix \(D\) makes the derivations lengthy. Therefore, we give only the final formulas in this paper and discuss elsewhere the details of the derivations [44]. The
second moment $M_2$ is,

\[ M_2(m_1, m_2) = \langle (h + V)^2 \rangle_{m_1, m_2} = \langle h^2 \rangle_{m_1, m_2} + \langle V^2 \rangle_{m_1, m_2} \]
\[ = (m_2)^2 + \mathcal{X}(m_1, m_2) + \mathcal{D}(m_1, m_2) + \tilde{\mathcal{D}}(m_1, m_2); \]

\[ \mathcal{X}(m_1, m_2) = \langle X^2 \rangle_{m_1, m_2} = \tau^2 \sum_{i+j=2} \binom{\tilde{m}_1 + i}{i} \binom{m_1}{i} \binom{\tilde{m}_2 + j}{j} \binom{m_2}{j}, \]
\[ (B2) \]

\[ \mathcal{D}(m_1, m_2) = \langle D\tilde{D} \rangle_{m_1, m_2} = \alpha^2 \binom{m_1}{2} \binom{m_2}{2}, \]
\[ \tilde{\mathcal{D}}(m_1, m_2) = \langle \tilde{D}D \rangle_{m_1, m_2} = \alpha^2 \binom{\tilde{m}_1}{2} \binom{m_2}{2}. \]

Here, for brevity we have defined $\mathcal{X}(m_1, m_2) = \langle X^2 \rangle_{m_1, m_2}$, $\mathcal{D}(m_1, m_2) = \langle D\tilde{D} \rangle_{m_1, m_2}$ and $\tilde{\mathcal{D}}(m_1, m_2) = \langle \tilde{D}D \rangle_{m_1, m_2}$. Note that, Eq. (B2) gives the binary correlation formula for $\sigma^2(m_1, m_2)$ and it reduces to Eq. (12) as expected. Similarly, the third moment $M_3$ is

\[ M_3(m_1, m_2) = \langle (h + V)^3 \rangle_{m_1, m_2} = \langle h^3 \rangle_{m_1, m_2} + 2 \langle h \rangle_{m_1, m_2} \langle V \rangle_{m_1, m_2} \]
\[ + \langle XhX \rangle_{m_1, m_2} + \langle Dh\tilde{D} \rangle_{m_1, m_2} + \langle \tilde{D}hD \rangle_{m_1, m_2} \]
\[ = (m_2)^3 + 3(m_2) \mathcal{X}(m_1, m_2) + (3m_2 + 2)\mathcal{D}(m_1, m_2) + (3m_2 - 2)\tilde{\mathcal{D}}(m_1, m_2) \]
\[ (B3) \]

The formula for the fourth moment $M_4$ is,

\[ M_4(m_1, m_2) = \langle (h + V)^4 \rangle_{m_1, m_2} = \langle h^4 \rangle_{m_1, m_2} + 3 \langle h^2 \rangle_{m_1, m_2} \langle V^2 \rangle_{m_1, m_2} \]
\[ + \langle h^2 \rangle_{m_1, m_2} \langle X^2 \rangle_{m_1, m_2} + \langle Dh^2\tilde{D} \rangle_{m_1, m_2} + \langle \tilde{D}h^2D \rangle_{m_1, m_2} + 2 \langle hXhX \rangle_{m_1, m_2} \]
\[ + 2 \langle hDh\tilde{D} \rangle_{m_1, m_2} + 2 \langle h\tilde{D}hD \rangle_{m_1, m_2} + \langle V^4 \rangle_{m_1, m_2} \]
\[ (B4) \]

\[ = (m_2)^4 + 6(m_2)^2 \mathcal{X}(m_1, m_2) + [6(m_2)^2 + 8(m_2) + 4] \mathcal{D}(m_1, m_2) \]
\[ + [6(m_2)^2 - 8(m_2) + 4] \tilde{\mathcal{D}}(m_1, m_2) + \langle V^4 \rangle_{m_1, m_2} \]
The only unknown in Eq. (B4) is $\langle V_4 \rangle_{m_1,m_2}$ and the expression for this is complicated,

$$
\langle V_4 \rangle_{m_1,m_2} = \langle X_4 \rangle_{m_1,m_2} + 3 \langle X_2 \rangle_{m_1,m_2} \left\{ \langle D\tilde{D} \rangle_{m_1,m_2} + \langle \tilde{D}D \rangle_{m_1,m_2} \right\} + \langle DX^2\tilde{D} \rangle_{m_1,m_2} \\
+ \langle \tilde{D}X^2 \rangle_{m_1,m_2} + 2 \langle XDX \rangle_{m_1,m_2} + 2 \langle X\tilde{D}X \rangle_{m_1,m_2} + \langle (D + \tilde{D})^4 \rangle_{m_1,m_2}
$$

$$
= 2\chi(m_1,m_2)^2 + 3\chi(m_1,m_2)[\mathcal{D}(m_1,m_2) + \tilde{\mathcal{D}}(m_1,m_2)]
+ T_1(m_1,m_2) + T_2(m_1,m_2) + 2T_3(m_1,m_2) + T_4(m_1,m_2) .
$$

(B5)

where

$$
T_1(m_1,m_2) = \tau^4 \sum_{i+j=2, t+u=2} F(m_1, N_1, i, t) F(m_2, N_2, j, u) ;
$$

$$
F(m, N, k_1, k_2) = \sum_{s=0}^{k_2} \left( \frac{m-s}{k_2-s} \right)^2 \left( \frac{N-m+k_1-s}{k_1} \right) \left( \frac{m-s}{s} \right) \left( \frac{N-m}{s} \right) \left( \frac{m}{s} \right) \left( \frac{N+1}{s} \right) \times \frac{N-2s+1}{N-s+1} \left( \frac{N-s}{k_2} \right)^{-1} \left( \frac{k_2}{s} \right)^{-1} ,
$$

$$
T_2(m_1,m_2) = \mathcal{D}(m_1,m_2)\chi(m_1-2,m_2+2) + \tilde{\mathcal{D}}(m_1,m_2)\chi(m_1+2,m_2-2) ,
$$

$$
T_3(m_1,m_2) = \tau^2 \alpha^2 \sum_{i+j=2} \left[ \left( \frac{m_1-i}{2} \right) \left( \frac{\tilde{m}_2-j}{2} \right) + \left( \frac{\tilde{m}_1-i}{2} \right) \left( \frac{m_2-j}{2} \right) \right] \times \left( \frac{\tilde{m}_1+i}{i} \right) \left( \frac{m_1}{i} \right) \left( \frac{\tilde{m}_2+j}{j} \right) \left( \frac{m_2}{j} \right) ,
$$

$$
T_4(m_1,m_2) = [\mathcal{D}(m_1,m_2)]^2 + [\tilde{\mathcal{D}}(m_1,m_2)]^2
+ \mathcal{D}(m_1,m_2) \left[ 2 \mathcal{D}(m_1-2,m_2+2) + \tilde{\mathcal{D}}(m_1-2,m_2+2) \right]
+ \tilde{\mathcal{D}}(m_1,m_2) \left[ 2 \tilde{\mathcal{D}}(m_1+2,m_2-2) + \mathcal{D}(m_1+2,m_2-2) \right] + 4 \mathcal{D}(m_1,m_2)\tilde{\mathcal{D}}(m_1,m_2) .
$$

(B6)
Given the moments $M_r(m_1, m_2) = \langle H^r \rangle_{m_1, m_2}; r = 1 - 4$, the skewness and excess parameters $\gamma_1$ and $\gamma_2$ are as follows [50],

$$
\gamma_1(m_1, m_2) = \frac{k_3(m_1, m_2)}{[k_2(m_1, m_2)]^{3/2}}, \quad \gamma_2(m_1, m_2) = \frac{k_4(m_1, m_2)}{[k_2(m_1, m_2)]^2}, \quad (B7)
$$

where,

$$
k_2(m_1, m_2) = M_2(m_1, m_2) - M_1^2(m_1, m_2),
$$

$$
k_3(m_1, m_2) = M_3(m_1, m_2) - 3 M_2(m_1, m_2) M_1(m_1, m_2) + 2 M_1^3(m_1, m_2),
$$

$$
k_4(m_1, m_2) = M_4(m_1, m_2) - 4 M_3(m_1, m_2) M_1(m_1, m_2) - 3 M_2^2(m_1, m_2)
$$

$$
+ 12 M_2(m_1, m_2) M_1^2(m_1, m_2) - 6 M_1^4(m_1, m_2) . \quad (B8)
$$

After carrying out the simplifications using Eqs. (B1)-(B8), it is easily seen that,

$$
\gamma_1(m_1, m_2) = \frac{2 \left[ D(m_1, m_2) - \tilde{D}(m_1, m_2) \right]}{\left\{ D(m_1, m_2) + \tilde{D}(m_1, m_2) + X(m_1, m_2) \right\}^{3/2}} . \quad (B9)
$$

The expression for $\gamma_2$ is more complex,

$$
\gamma_2(m_1, m_2) = \left\{ \tilde{D}(m_1, m_2) + D(m_1, m_2) + X(m_1, m_2) \right\}^{-2}
\times [T_1(m_1, m_2) + T_2(m_1, m_2) + 2 T_3(m_1, m_2) + T_4(m_1, m_2)
+ \left[ \tilde{D}(m_1, m_2) + D(m_1, m_2) \right] [4 - X(m_1, m_2)] - 2 \left[ \tilde{D}(m_1, m_2) + D(m_1, m_2) \right]^2 - 1 . \quad (B10)
$$

With $T_1 \sim [X(m_1, m_2)]^2 + C_1(m_1, m_2)$, $T_2 = T_3 \sim X(m_1, m_2) [\tilde{D}(m_1, m_2) + D(m_1, m_2)]$ and $T_4 \sim 3 [\tilde{D}(m_1, m_2) + D(m_1, m_2)]^2 + C_2(m_1, m_2)$ which are good in the dilute limit ($|C_1|$ and $|C_2|$ will be close to zero), we have

$$
\gamma_2(m_1, m_2) = \frac{C_1(m_1, m_2) + C_2(m_1, m_2) + 4 \left[ \tilde{D}(m_1, m_2) + D(m_1, m_2) \right]}{\left\{ \tilde{D}(m_1, m_2) + D(m_1, m_2) + X(m_1, m_2) \right\}^2} . \quad (B11)
$$

Note that $C_1$ and $X$ depend only on $\tau$. Similarly, $C_2$ and ($\tilde{D}, D$) depend only on $\alpha$. The ($\tilde{D} + D$) term in the numerator will contribute to $\gamma_2(m_1, m_2)$ when $\tau = 0$ and $\alpha$ is very small. The approximation $T_2 = T_3 \sim X(\tilde{D} + D)$ is crucial in obtaining the numerator in Eq. (B11) with no cross-terms involving the $\alpha$ and $\tau$ parameters. With this, we have $k_4$ to be the sum of $k_4$'s coming from $X(2)$ and $D(2)$ matrices [note that, as mentioned before, $X(2) = A \oplus B \oplus C$ and $D(2) = D + \tilde{D}$].

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To test the accuracy of the formulas for $M_r$ given by Eqs. (B1)-(B6), the binary correlation results for $\gamma_1(m, \pm)$ and $\gamma_2(m, \pm)$ are compared with exact results obtained using the eigenvalues from EGOE(1+2)-\pi ensembles with 100 members for several values of $(N_+, N_-, m)$ and $(\tau, \alpha)$ parameters in Table III. Extension of Eq. (11) along with the results derived for $M_r(m_1, m_2)$ will give the binary correlation results for $\gamma_1(m, \pm)$ and $\gamma_2(m, \pm)$. It is clearly seen from the results in the Table that in all the examples considered, the binary correlation results are quite close to the exact results. Similar agreements are also seen in many other examples which are not shown in the table.

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