Gravastars in $f(R, T)$ gravity

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We propose a unique stellar model under the $f(R, T)$ gravity by using the conjecture of Mazur-Mottola [P. Mazur and E. Mottola, Report number: LA-UR-01-5067, P. Mazur and E. Mottola, Proc. Natl. Acad. Sci. USA 101, 9545 (2004),] which is known as gravastar and a viable alternative to the black hole as available in literature. This gravastar is described by the three different regions, viz., (I) Interior core region, (II) Intermediate thin shell, and (III) Exterior spherical region. The pressure within the interior region is equal to the constant negative matter density which provides a repulsive force over the thin spherical shell. This thin shell is assumed to be formed by a fluid of ultrarelativistic plasma and the pressure, which is directly proportional to the matter-energy density according to Zel’dovich’s conjecture of stiff fluid [Y.B. Zel’dovich, Mon. Not. R. Astron. Soc. 160, 1 (1972).], does counterbalance the repulsive force exerted by the interior core region.

The exterior spherical region is completely vacuum and assumed to be de Sitter spacetime which can be described by the Schwarzschild solution. Under this specification we find out a set of exact and singularity-free solution of the gravastar which presents several other physically valid features within the framework of alternative gravity.

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I. INTRODUCTION

Mazur and Mottola [1, 2] first ever proposed a model considering the gravitationally vacuum star (gravastar) as an alternative to the system of gravitational collapse, i.e., black hole. They generated a new type of solution by extending the idea of Bose-Einstein condensation in construction of the gravastar as a cold, dark, and compact object of interior de Sitter condensate phase.

The scenario of this gravastar can be envisaged as follows: the interior is surrounded by a thin shell of ultrarelativistic matter whereas the exterior region is completely vacuum and hence the Schwarzschild spacetime at the outside can be considered to fit for the system. The shell is assumed to be very thin with a finite width in the range $r_1 < r < r_2$, where $r_1 \equiv D$ and $r_2 \equiv D + \epsilon$ are the interior and exterior radii of the gravastar respectively under consideration. Therefore, we can represent the entire system of gravastar into three specific segments based on the equation of state (EOS) as follows: (I) Interior $(0 \leq r < r_1)$: $p = -\rho$, (II) Shell $(r_1 \leq r \leq r_2)$: $p = +\rho$, and (III) Exterior $(r_2 < r)$: $p = \rho = 0$.

We note that related to the gravastar there are lot of works available in the literature based on different mathematical as well as physical issues. However, these works are mainly carried out by several authors in the framework of Einstein’s general relativity [1–19]. Though it is well known that Einstein’s general relativity is a unique tool for uncovering many hidden mysteries of Nature, yet some observational evidences of the accelerating universe along with the existence of dark matter has imposed a theoretical challenge to this theory [20–26]. Therefore, several alternative theories have been proposed successively amongst which $f(R)$ gravity, $f(T)$ gravity, and $f(R, T)$ gravity have received more attention. In the present project our motivation is to study the gravastar under one of the alternative gravity theories, namely $f(R, T)$ gravity [27] and to observe different physical features of the object - their nontriviality as well as triviality. Actually our previously performed successful works on the initial phases of compact stars under alternative gravity [28–29] motivate us to exploit the alternative formalism to the case of the gravastar, a viable alternative to the ultimate stellar phase of a black hole.

It has been argued that among all other modified grav-
ity theories the $f(R, T)$ theory of gravity can be considered as a useful formulation which is based on the nonminimally curvature matter coupling. In the $f(R, T)$ theory of gravity \cite{27} the gravitational Lagrangian of the standard Einstein-Hilbert action is defined by an arbitrary function of the Ricci scalar $R$ and the trace of the energy-momentum tensor $T$. One can note that such a dependence on $T$ may come from the presence of an imperfect fluid or from the consideration of quantum effects. The application of $f(R, T)$ gravity theory to different cosmological \cite{31, 42} realm can be noted in the literature.

Among several astrophysical applications it is worthy of mentioning the Refs. \cite{43-52}. In their work \cite{43} Sharif et al. have studied the stability of collapsing spherical body of an isotropic fluid distribution considering the nonstatic spherically symmetric line element. A perturbation scheme has been used to find the collapse equation and the condition on the adiabatic index has been constructed for Newtonian and post-Newtonian eras for addressing instability problem by Noureen et al. \cite{44} whereas in another work \cite{45} Noureen et al. have investigated the range of instability under the $f(R, T)$ theory for an anisotropic background constrained by zero expansion. Also, by applying a perturbation scheme on the $f(R, T)$ field equations the evolution of a spherical star has been studied by Zubair et al. \cite{47}. Zubair et al. \cite{47} have analyzed the dynamics of gravitating sources along with axial symmetry under the $f(R, T)$ gravity. Some other relevant studies on the $f(R, T)$ theory of gravity can be observed in the following works \cite{48-50}. Under different physical motivations, Youssf et al. \cite{51} have explored the evolutionary behaviors of compact objects in the framework of $f(R, T)$ gravity theory with the help of structure scalars whereas they \cite{52} have investigated the irregularity factors for a self-gravitating spherical star evolving in the presence of imperfect fluid.

The outline of the present study is therefore as follows: In Sec. II the basic mathematical formalism of the $f(R, T)$ gravity theory has been provided as the background of the study. Thereafter in Sec. III we discuss the field equations and their solutions in $f(R, T)$ gravity considering the interior spacetime, exterior spacetime, and thin shell cases of the gravastars with their respective solutions. We provide the junction conditions, which are essential in connection to the three regions of the gravastar, in Sec. IV. Several physical properties of the models, viz. proper length, energy content, entropy and equation of state, are discussed in Sec. V. Some concluding remarks are provided in Sec. VI.

II. BASIC MATHEMATICAL FORMALISM OF THE $f(R, T)$ THEORY

The action of the $f(R, T)$ theory \cite{27} reads

$$S = \frac{1}{16\pi} \int d^4x f(R, T)\sqrt{-g} + \int d^4x L_m \sqrt{-g},$$  \hspace{1cm} (1)$$

where $f(R, T)$ is the function of the Ricci scalar $R$ and the trace of the energy-momentum tensor $T, L_m$ being the matter Lagrangian density, and $g$ is the determinant of the metric $g_{\mu\nu}$. Throughout the paper we assume the geometrical units $G = c = 1$.

Varying the action \cite{11} with respect to the metric $g_{\mu\nu}$, one can obtain the following field equations of $f(R, T)$ gravity:

$$f_R(R, T)R_{\mu\nu} - \frac{1}{2} f(R, T)g_{\mu\nu} + (g_{\mu\nu} \square - \nabla_{\mu} \nabla_{\nu})f(R, T) = 8\pi T_{\mu\nu} - f_T(R, T)T_{\mu\nu} - f_T(R, T)\Theta_{\mu\nu},$$  \hspace{1cm} (2)$$

where $f_R(R, T) = \partial f(R, T)/\partial R$, $f_T(R, T) = \partial f(R, T)/\partial T$, $\square \equiv \partial_{\mu}(\sqrt{-g} g^{\mu\nu}\partial_{\nu})/\sqrt{-g}$, $R_{\mu\nu}$ is the Ricci tensor, $\nabla_{\mu}$ provides the covariant derivative with respect to the symmetric connection associated to $g_{\mu\nu}$, $\Theta_{\mu\nu} = g^{\alpha\beta}\delta T_{\alpha\beta}/\delta g^{\mu\nu}$ and the stress-energy tensor is defined as $T_{\mu\nu} = g_{\mu\lambda}L_m - 2\partial L_m/\partial g^{\mu\nu}$.

The covariant divergence of \cite{2} reads \cite{53}

$$\nabla^{\mu}T_{\mu\nu} = \frac{8\pi}{f_T(R, T)}[(T_{\mu\nu} + \Theta_{\mu\nu})\nabla^{\mu}\ln f_T(R, T)] + \nabla^{\nu}\Theta_{\mu\nu} - (1/2)g_{\mu\nu}\nabla^{\mu}\nabla^{\nu}.$$

It is vivid from Eq. \cite{3} that the energy-momentum tensor is not conserved for the $f(R, T)$ theory of gravity unlike the general relativistic case.

In the present paper we assume the energy-momentum tensor to be that of a perfect fluid, i.e.,

$$T_{\mu\nu} = (p + \rho)u_{\mu}u_{\nu} - pg_{\mu\nu},$$  \hspace{1cm} (4)$$

with $u^{\mu}u_{\mu} = 1$ and $u^{\mu}\nabla_{\mu} = 0$. Besides these conditions we also have $L_m = -\rho$ and $\Theta_{\mu\nu} = -2T_{\mu\nu} - pg_{\mu\nu}$.

Following the proposition of Harko et al. \cite{27}, we take the functional form of $f(R, T)$ as $f(R, T) = R + 2\chi T$, with $\chi$ being a constant. One can note that this form has been extensively used to obtain many cosmological solutions in $f(R, T)$ gravity \cite{30, 32, 33, 41, 53}. By substituting the above form of $f(R, T)$ in \cite{2}, we get \cite{30, 31}

$$G_{\mu\nu} = 8\pi T_{\mu\nu} + \chi T g_{\mu\nu} + 2\chi(T_{\mu\nu} + pg_{\mu\nu}),$$

where $G_{\mu\nu}$ is the Einstein tensor.

One can easily get back to the result of general relativity just by setting $\chi = 0$ in the above Eq. \cite{5}. Moreover, for $f(R, T) = R + 2\chi T$, Eq. \cite{3} yields

$$\nabla^{\mu}T_{\mu\nu} = -\frac{2\chi}{(8\pi + 2\chi)}\left[\nabla^{\mu}(pg_{\mu\nu}) + \frac{1}{2}g_{\mu\nu}\nabla^{\mu}T\right].$$

Curiously, by substituting $\chi = 0$ in Eq. \cite{9} one can verify that the energy-momentum tensor is conserved as in the case of general relativity.

III. THE FIELD EQUATIONS AND THEIR SOLUTIONS IN $f(R, T)$ GRAVITY

For the spherically symmetric metric

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

(7)
one can find the nonzero components of the Einstein tensors as

\begin{equation}
G^0_0 = \frac{e^{-\lambda}}{r^2}(-1 + e^{\lambda} + \lambda' r),
\end{equation}

\begin{equation}
G^1_1 = \frac{e^{-\lambda}}{r^2}(-1 + e^{\lambda} - \nu' r),
\end{equation}

\begin{equation}
G^2_2 = G^3_3 = \frac{e^{-\lambda}}{r} \left[ 2(\lambda' - \nu') - (2\nu'' + \nu'^2 - \nu'\lambda') r \right],
\end{equation}

where primes stand for derivative with respect to the radial coordinate \( r \).

Substituting Eqs. (4), (8), (9), and (10) in Eq. (5) one can get

\begin{equation}
-1 + e^\lambda + \lambda' r = \Pi(r)[8\pi\rho + \chi(3\rho - p)],
\end{equation}

\begin{equation}
-1 + e^\lambda - \nu' r = \Pi(r)[-8\pi p + \chi(\rho - 3p)],
\end{equation}

\begin{equation}
\frac{r^2}{2}(\lambda' - \nu') - (2\nu'' + \nu'^2 - \nu'\lambda') \frac{r^2}{4} = \Pi(r)[-8\pi p + \chi(\rho - 3p)],
\end{equation}

with \( \Pi(r) \equiv r^2/e^{-\lambda} \).

Now, from the equation for the nonconservation of the energy-momentum tensor in \( f(R, T) \) theory \( \text{(3)} \) one can obtain

\begin{equation}
\frac{dp}{dr} + \frac{\nu' (\rho + p)}{2} + \frac{\chi}{2(4\pi + \chi)}(\nu' - \rho') = 0.
\end{equation}

If we consider the quantity \( m \) as the gravitational mass within the sphere of radius \( r \), then from Eq. (11) we can write

\begin{equation}
e^{-\lambda} = 1 - \frac{2m}{r} - \chi(\rho - \frac{p}{3})r^2.
\end{equation}

Again from Eqs. (12), (14), and (15) one can get the equation of hydrostatic equilibrium in \( f(R, T) \) theory as

\begin{equation}
p' = -(\rho + p) \left[ \frac{4\pi pr - \chi(\rho - 3p)r}{2} + \frac{1}{2} \frac{\chi}{2(4\pi + \chi)}(\nu' - \rho')^2 \right],
\end{equation}

\begin{equation}
[1 - \frac{2m}{r} - \chi(\rho - \frac{p}{3})r^2] \left[ 1 + \frac{\chi(1 - \frac{2m}{r})}{2(4\pi + \chi)} \right],
\end{equation}

considering the fact that the energy density \( \rho \) depends on the pressure \( p \) i.e. \( \rho = \rho(p) \).

Also, by letting \( \chi = 0 \) the standard form of the Tolman-Oppenheimer-Volkoff (TOV) equation can be retrieved as applicable in the case of general theory of relativity.

### A. Interior spacetime

Following the proposition of Mazur-Mottola [1, 2], let us assume the equation of state (EOS) for the interior region as

\begin{equation}
p = -\rho.
\end{equation}

The above EOS is a special form of \( p = \omega \rho \), with the EOS parameter \( \omega = -1 \) and is known as the dark energy equation of state.

Again using the above EOS, and from Eq. (14) one can obtain

\begin{equation}
\rho = \rho_0 \, \text{(constant)},
\end{equation}

and the pressure turns out to be

\begin{equation}
p = -\rho_0.
\end{equation}

Now, using Eqs. (11), (19) one gets the metric potential \( \lambda \) as

\begin{equation}
e^{-\lambda} = 1 - \frac{4(2\pi + \chi)\rho_0 r^2}{3} + \frac{A}{r},
\end{equation}

where \( A \) is an integration constant which is set to zero as the solution is regular at the center \( (r = 0) \). Hence we have

\begin{equation}
e^{-\lambda} = 1 - \frac{4(2\pi + \chi)\rho_0 r^2}{3}.
\end{equation}

Again, using Eqs. (11), (12), (18) and (19) one can get the following relation between the metric potentials \( \nu \) and \( \lambda \) as

\begin{equation}
e^\nu = Be^{-\lambda},
\end{equation}

where \( B \) is an integration constant. Here the spacetime metric is free from any central singularity.

Also the gravitational mass \( M(D) \) can be found out to be

\begin{equation}
M(D) = \int_0^{r_i = D} 4\pi r^2 \rho_0 dr = \frac{4}{3} \pi D^3 \rho_0.
\end{equation}

### B. Shell

Let us consider that the shell consists of ultrarelativistic fluid, obeying the EOS \( p = \rho \). Zel’dovich [55] conceived the idea of this fluid in connection to cold baryonic universe and is known as the stiff fluid. In the present context this may come from thermal excitations with negligible chemical potential or from conserved number density of gravitational quanta at zero temperature [1, 2]. This type of fluid has been extensively used by several authors to study various cosmological [54, 55] as well as astrophysical [58, 60] phenomena.
One can note that within the nonvacuum region, i.e., the shell it is very difficult to find solution of the field equations. However, it is possible to obtain an analytical solution within the framework of thin shell limit, i.e., $0 < e^{-\lambda} \ll 1$. Physically this means that when two spacetimes join together at a place (in our case the vacuum interior and the Schwarzschild exterior) the intermediate region must be a thin shell (see the Ref. [61]). Now in thin shell as $r \to 0$, any parameter which is function of $r$ is, in general, $\ll 1$. Under this approximation along with the above EOS as well as Eqs. (11), (12) and (13), one can find the following equations

$$\frac{de^{-\lambda}}{dr} = \frac{2}{r},$$  \hspace{1cm} (24)

$$\left(\frac{3}{2r} + \frac{\nu'}{4}\right)\frac{de^{-\lambda}}{dr} = \frac{1}{r^2}.$$  \hspace{1cm} (25)

Integrating Eq. (24) we get

$$e^{-\lambda} = 2 \ln r + C,$$  \hspace{1cm} (26)

where $C$ is an integration constant and range of $r$ is $D \leq r \leq D + \epsilon$. Under the condition $\epsilon \ll 1$, we get $C \ll 1$ as $e^{-\lambda} \ll 1$.

Also from Eqs. (24) and (25), one can get

$$e^\nu = Fr^{-4},$$  \hspace{1cm} (27)

where $F$ is an integration constant.

Also Eq. (14), along with the EOS $p = \rho = Hr^4$, yields

$$p = \rho = Hr^4,$$  \hspace{1cm} (28)

$H$ being a constant. As $\rho \propto r^4$, we can infer that the ultrarelativistic fluid within the shell is more dense at the outer boundary than the inner boundary.

C. Exterior spacetime

The exterior region obeying the EOS ($\rho = H = 0$) can be defined by the well-known static exterior Schwarzschild solution which is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right),$$  \hspace{1cm} (29)

where $M$ is the total mass of the gravitating system.

IV. JUNCTION CONDITION

It is already mentioned that the gravastar consists of three regions, i.e., interior region (I), shell (II), and exterior region (III). The interior region (I) is connected with the exterior region at the junction interface, i.e., at the shell. According to the Damour-Israel formalism [61, 62] there should be smooth matching between the regions I and III of the gravastar. The metric coefficients are continuous at the junction surface ($\Sigma$), i.e., at $r = D$, though their derivatives may not be continuous. However, one can determine the surface stress-energy $S_{ij}$ by using the above mentioned formalism.

Now, the intrinsic surface stress-energy tensor $S_{ij}$ is given by the Lanczos equation [61, 62] as

$$S_{ij} = \frac{1}{8\pi} (\kappa_{ij}^\nu - \delta_{ij}^\nu \kappa_k^\nu),$$  \hspace{1cm} (30)

where $\kappa_{ij} = K_{ij}^+ - K_{ij}^-$ provide the discontinuity in the second fundamental forms or extrinsic curvatures. Here the signs “+” and “−” correspond to the interior and the exterior regions respectively. Now, the second fundamental forms [67, 72] associated with the two sides of the shell are given by

$$K_{ij}^\pm = -n^\pm_\nu \left[ \frac{\partial^2 x_\nu}{\partial \xi^i \partial \xi^j} + \Gamma_{\rho\beta}^\nu \frac{\partial x_\rho}{\partial \xi^i} \frac{\partial x_\beta}{\partial \xi^j} \right] \bigg|_\Sigma,$$  \hspace{1cm} (31)

where $\xi^i$ are the intrinsic coordinates on the shell, $n^\pm_\nu$ are the unit normals to the surface $\Sigma$ and for the spherically symmetric static metric

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 (d\theta^2 + \sin^2\theta d\phi^2),$$  \hspace{1cm} (32)

$n^\pm_\nu$ can be written as

$$n^\pm_\nu = \pm g^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial f}{\partial x^\beta} \frac{1}{\sqrt{\left| g^{\nu\sigma} \frac{\partial f}{\partial x^\sigma} \right|}},$$  \hspace{1cm} (33)
with $n^\mu n_\mu = 1$.

Using the Lanczos equation we can get the surface stress-energy tensor as $S_{ij} = \text{diag}[\sigma, -\nu, -\nu, -\nu]$, where $\sigma$ is the surface energy density and $\nu$ is the surface pressure. The surface energy density ($\sigma$) and the surface pressure ($\nu$) can be respectively expressed as

$$\sigma = -\frac{1}{4\pi D} \left[ \sqrt{f} \right]_+^+,$$

$$\nu = -\frac{\sigma}{2} + \frac{1}{16\pi} \left[ \frac{f'}{\sqrt{f}} \right]_+^-.$$

So, by using the above two equations we obtain

$$\sigma = -\frac{1}{4\pi D} \left[ \sqrt{1 - \frac{2M}{D}} - \sqrt{1 - \frac{4(2\pi + \chi)\rho_0 D^2}{3}} \right],$$

$$\nu = \frac{1}{8\pi D} \left[ \frac{1}{\sqrt{1 - \frac{M}{D}}} - \frac{1 - \frac{8(2\pi + \chi)\rho_0 D^2}{3}}{\sqrt{1 - \frac{4(2\pi + \chi)\rho_0 D^2}{3}}} \right].$$

Also, the mass of the thin shell can be written as

$$m_s = 4\pi D^2 \sigma = D \left[ \sqrt{1 - \frac{4(2\pi + \chi)\rho_0 D^2}{3}} - \sqrt{1 - \frac{2M}{D}} \right].$$

Here $M$ is the total mass of the gravastar and it can be expressed in the following form

$$M = \frac{2(2\pi + \chi)\rho_0 D^3}{3} + m_s \sqrt{1 - \frac{4(2\pi + \chi)\rho_0 D^2}{3}} \frac{m_s^2}{2D}.$$

V. PHYSICAL FEATURES OF THE MODEL

A. Proper length of the shell

Let us consider that the stiff fluid shell is situated at the surface $r = D$ defining the phase boundary of region I. The proper thickness of the shell is assumed to be very small, i.e., $\epsilon \ll 1$. Thus the region III starts from the interface at $r = D + \epsilon$. So, the proper thickness between two interfaces, i.e., of the shell is determined as

$$\ell = \int_D^{D+\epsilon} \sqrt{e^\chi dr} = \int_D^{D+\epsilon} \frac{dr}{\sqrt{2 \ln r + C}}.$$ 

Integrating the above equation one can get

$$\ell = \left[ -\frac{\pi}{2eC} \right]^{D+\epsilon} \text{erf} \left( \frac{\ln \left( \frac{r}{\sqrt{2}} \right) - C}{\sqrt{2}} \right).$$

B. Energy content

In the interior region we consider the EOS in the form $p = -\rho$ which indicates the negative energy region confirming the repulsive nature of the interior region.

However, the energy within the shell turns out to be

$$\mathcal{E} = \int_D^{D+\epsilon} 4\pi \rho r^2 dr = \int_D^{D+\epsilon} 4\pi H r^6 dr = \frac{4\pi}{7} \left[ (D + \epsilon)^7 - D^7 \right].$$

Taking into account the thin shell approximation one may write the energy $\mathcal{E}$ up to the first order of $\epsilon$ $(\ll 1)$ as

$$\mathcal{E} \approx 4\pi \epsilon H D^6.$$

The above relation indicates that the energy of the shell is directly proportional to the $\epsilon$, i.e., the thickness of the shell.

C. Entropy

According to the prescription of Mazur and Mottola [1, 2] in the interior region I, the entropy density is zero which is consistent with a single condensate state. However, within the shell the entropy is given by

$$S = \int_D^{D+\epsilon} 4\pi r^2 s(r) \sqrt{e^\chi} dr,$$

where $s(r)$ is the entropy density for local temperature $T(r)$ and may be written as

$$s(r) = \frac{\alpha^2 k_B^2 T(r)}{4\pi \hbar^2} = \alpha \left( \frac{k_B}{\hbar} \right) \sqrt{\frac{p}{2\pi \hbar^2}}.$$
FIG. 3: Energy $\varepsilon$ (km) within the shell is plotted with respect to the thickness of the shell $\epsilon$ (km).

$\alpha$ being a dimensionless constant. We note that in the present work we assume the geometrical units, i.e., $G = c = 1$, and also in Planckian units $k_B = \hbar = 1$. So, the entropy density within the shell turns out to be

$$s(r) = \alpha \sqrt{\frac{p}{2\pi}}.$$  \hspace{1cm} (46)

Integrating the above equation we get

$$S = (8\pi H)^{\frac{1}{2}} \alpha \left[ -\left( -\frac{\pi}{10e5c}\right)^{\frac{1}{2}} \right] \times$$

$$erf \left\{ \frac{\sqrt{5[ln \left( \frac{1}{r^2} \right) - C]} }{\sqrt{2}} \right\} D^{+\epsilon}.$$  \hspace{1cm} (48)

D. Equation of state

The EOS, at $r = D$, as usual can be expressed in the following form

$$\nu = \omega(D)\sigma.$$  \hspace{1cm} (49)

Hence, by virtue of Eqs. (36) and (37) the equation of state parameter can explicitly be written as

$$\omega(D) = \frac{\left(1 - \frac{M}{D}\right)}{\sqrt{1 - \frac{M}{D}}} - \left\{ \frac{1 - \frac{s(2\pi + \chi)\rho_0 D^2}{3}}{\sqrt{1 - \frac{2M}{D}}} \right\}.$$  \hspace{1cm} (50)

For $\omega(D)$ to be real it requires $\frac{2M}{D} < 1$ as well as $\frac{4(2\pi + \chi)\rho_0 D^2}{3} < 1$. Moreover, if one expands the square-root terms in the numerator and the denominator of the expressions of Eq. (50) under the conditions $\frac{M}{D} \ll 1$ and $\frac{4(2\pi + \chi)\rho_0 D^2}{3} \ll 1$ in a binomial series and retains the terms up to the first order, then one can get

$$\omega(D) \approx \frac{3}{2\left[\frac{3M}{2(2\pi + \chi)\rho_0 D^2} - 1\right]}.$$  \hspace{1cm} (51)

Now, if one examines the above expression for $\omega(D)$ then two possibilities may emerge out: either $\omega(D)$ is positive if $\frac{M}{D^5} > \frac{2(2\pi + \chi)\rho_0}{3}$, or $\omega(D)$ is negative if $\frac{M}{D^5} < \frac{2(2\pi + \chi)\rho_0}{3}$.

VI. CONCLUSION

In the present work we have proposed a unique stellar model under the $f(R, T)$ gravity as originally conjectured by Mazur-Mottola \cite{11, 12} in the framework of general relativity. The stellar model which they termed as gravastar, may be considered to be a viable alternative to the black hole. To fulfill the criteria of a gravastar they described the spherically symmetric stellar system by the three different regions: interior core region, intermediate thin shell, and exterior spherical region with specific EOS for each of the region. Under this type of specification we have found out a set of exact and singularity-free solution of the gravitationally collapsing system which presents several interesting properties which are physically viable
within the framework of alternative gravity of the form \( f(R, T) \).

In studying the above mentioned structural form of a gravastar we have noted down several salient aspects of the solution set as can be described below:

1. **Pressure-density profile**: The pressure and density relationship \((p = \rho)\) of the ultrarelativistic fluid in the shell is shown with respect to the radial coordinate \(r\) in Fig. 1 which maintains a constant variation throughout the shell.

2. **Proper length**: The proper length \(\ell\) of the shell as plotted with respect to the thickness of the shell \(\epsilon\) (in Fig. 2) shows a gradual increasing profile.

3. **Energy content**: The energy of the shell is directly proportional to the thickness of the shell \(\epsilon\) (in Fig. 3).

4. **Entropy**: The entropy \(S\) within the shell has been plotted with respect to the thickness of the shell \(\epsilon\) (in Fig. 4). This plot shows a physically valid feature that the entropy is gradually increasing with respect to the thickness of the shell \(\epsilon\) and thus suggesting a maximum value on the surface of the gravastar.

5. **Equation of state**: For \(\omega(D)\) to be real it requires \(\frac{2\pi}{\rho} < 1\) as well as \(\frac{4(2\pi + \chi)\rho D^2}{3} < 1\). Moreover, under the conditions \(\frac{M}{D^3} < 1\) and \(\frac{4(2\pi + \chi)\rho D^2}{3} < 1\) upon expansion of the expressions for \(\omega(D)\) in a binomial series and retaining the terms up to the first order two possibilities have been emerged out: either \(\omega(D)\) is positive if \(\frac{M}{D^3} > \frac{2(2\pi + \chi)\rho D}{3}\), or \(\omega(D)\) is negative if \(\frac{M}{D^3} < \frac{2(2\pi + \chi)\rho D}{3}\).

Besides these important general features we have an overall observation regarding the model in \(f(R, T)\) gravity which is as follows: unlike Einstein’s general relativity there is an extra term involving \(\chi\) in the present model which has a definite role and makes the fundamental differences between the expressions in both the theories, as such vanishing of this coupling constant \(\chi\) provides a limiting case for getting back the results of general relativity (e.g. note the Ref. [19]). This aspect can be verified through a comparative case study between the present work and that of Ghosh et al. [73] under 4-dimensional background. In this sense \(f(R, T)\) gravity generates more generalized solutions for gravastar than general relativity.

One final comment: as a possible astrophysical implication of our results and tests to detect gravastars under \(f(R, T)\) gravity one may study their gravitational lensing effects as suggested by several authors, solely for gravastars [74] as well as for \(f(R, T)\) gravity [49]. According to the methodology of Kubo and Sakai one may adopt a spherical thin-shell model of a gravastar developed by Visser and Wiltshire [2], which connects interior de Sitter geometry and exterior Schwarzschild geometry. Now, assuming that its surface is optically transparent they calculate the image of a companion which rotates around the gravastar and find that some characteristic images appear, depending on whether the gravastar possess unstable circular orbits of photons (Model 1) or not (Model 2). For Model 2, Kubo and Sakai calculate the total luminosity change, which is called microlensing effects, where the maximal luminosity could be considerably larger than the black hole with the same mass. In future, if one study the similar effects under \(f(R, T)\) gravity, then one can compare the effects of modified gravity on the above mentioned tests with that of the results based on general theory of relativity.

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