Expanding perfect fluid generalizations of the C-metric

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1. INTRODUCTION

The C-metric is a well-known exact solution of Einstein’s vacuum equation with zero cosmological constant. The static region of the corresponding spacetime was first described by Weyl\textsuperscript{2}. At about the same time Levi-Civita\textsuperscript{3} constructed its line element in closed form, arriving at essentially one cubic polynomial with two parameters as the metric structure function. The C-metric is a Petrov type $D$ solution for which at each spacetime point both Weyl principal null directions (PNDs) are geodesic, non-shearing, non-twisting but diverging; it thus belongs to the Robinson-Trautman class of solutions and was rediscovered as such\textsuperscript{4}. The label ‘C’ derives from the invariant classification of static degenerate Petrov type $D$ vacuum spacetimes by Ehlers and Kundt\textsuperscript{5}. The importance of this solution as summarized by Kinnersley and Walker\textsuperscript{6}, is threefold. First, the C-metric describes a spacetime with only two independent Killing vector fields (KVF s) which can be fully analyzed. Next, it is an ‘example of almost everything’, most notably it describes a radiative, locally asymptotically flat spacetime, whilst containing a static region. The C-metric is contained in the class of boost-rotation-symmetric spacetimes\textsuperscript{7,8}, which are the only axially symmetric, radiative and asymptotically flat spacetimes with two Killing vectors. Finally, the solution has a clear physical interpretation as the anisotropic gravitational field of two Schwarzschild black holes being uniformly accelerated in opposite directions by a cosmic string or strut, provided that $ma < 1/\sqrt{27}$, where the mass $m$ and acceleration $a$ are equivalents of the two essential parameters of Levi-Civita\textsuperscript{6,9} (see, however, the end of §2.3 for a comment).

Generalizations of the C-metric have been widely considered. Adding a cosmological constant $\Lambda$ is straightforward, and we will henceforth refer with ‘C-metric’ to such Einstein spaces. Incorporating electromagnetic charge $|q|^2 = c^2 + g^2$ is equally natural and leads to quartic structure functions\textsuperscript{8}. Recently, the question how to include rotation for the holes received a new answer\textsuperscript{10,11}, avoiding the NUT-like behavior of the previously considered ‘spinning C-metric’\textsuperscript{12,13}. All these generalizations fit in the well-established class $D$ of Petrov type $D$ Einstein-Maxwell solutions with a non-null electromagnetic field possessing geodesic and non-shearing null directions aligned with the PNDs\textsuperscript{14,15}, which reduces for zero electromagnetic field to the subclass $D_0$ of Petrov type $D$ Einstein spaces and which contains all well-known 4D black hole metrics. In fact, all $D$-metrics can be derived by performing ‘limiting contractions’\textsuperscript{16} from the most general member, the Plebianski-Demianski line element\textsuperscript{17}, which exhibits two quartic structure functions with six essential parameters $m, \alpha, |q|^2, \Lambda, NUT$ parameter $I\textsuperscript{18}$ and angular momentum $\alpha$. A physically comprehensive and simplified treatment can be found in\textsuperscript{19}, also surveying recent work in this direction.

In this paper we present a new family of Petrov type $D$, expanding and anisotropic perfect fluid (PF) generalizations of the C-metric. The direct motivation and background for this work is the following.

According to the Goldberg-Sachs theorem\textsuperscript{20} the two PNDs of any member of $D_0$ are precisely those null directions which are geodesic and non-shearing. Such a member is purely electric (PE, cf. appendix B) precisely when both PNDs, as well as the complex null directions orthogonal to them, are non-twisting (non-rotating or hypersurface-orthogonal (HO)). This is in particular the case for the C-metric. As we will show, it implies the existence of an umbilical synchronization (US), i.e., a non-shearing and non-rotating unit timelike vector field...
(tangent to a congruence of observers). The importance of USs in cosmology was stressed in [21]. If a congruence of observers measuring isotropic radiation admits orthogonal hypersurfaces, an US exists. Only small deviations from isotropy are seen in the cosmic microwave background, and scalar perturbations of a Friedmann-Lemaître-Robertson-Walker universe preserve the existence of an US [22]. In general, spacetimes admitting an US have zero magnetic part of the Weyl tensor wrt a tangent to their flow lines. He was able to generalize Stephani’s results on conformally flat PF solutions which are either of generalized Schwarzschild type or of generalized Friedmann type (so called Stephani universes) [24]. The type D solutions were integrated and invariantly partitioned, based on the direction of the gradient of the energy density relative to the PNdS and the flow vector at each point. Class I, characterized by the energy density being constant on the hypersurfaces orthogonal to the flow lines and thus the only class containing Einstein spaces as limit cases, was further subdivided using the gradient of $\Psi_2$ (cf. §2.2 for details). By solving the field equations, Barnes concludes that class ID, consisting of the anisotropic class I models, solely contains non-expanding solutions. Hence, these PF solutions would not be viable as a cosmological model. However, based on an integrability analysis of class I in the Geroch-Held-Penrose (GHP) formalism [27], we found that this conclusion cannot be valid and this led to a detailed reinvestigation.

In this article we construct the general line element of the full ID class, comprising both the known non-expanding perfect fluid models and the new expanding ones, and discuss some elementary properties. We want to stress the following point. The full class represents a PF generalization of the C-metric in the sense that the C-metric is contained as the Einstein space limit. The physical interpretation of this fact is however not established. This would require to exhibit this solution for small masses as a perturbation of a known PF solution, just as the C-metric interpretation of small accelerating black holes has been established in a flat or (anti-)de Sitter background [1] [28] [31].

However, the mathematical relation with the C-metric is useful. As already deduced in [1], the PF solution is, just as the C-metric, conformally related to the direct sum of two 2D metrics. The fact that one part is equal for the PF solution and the C-metric is helpful in the analysis, e.g. we will show that (a part of) the axis of symmetry can readily be identified as a conical singularity, analogous to the defect of the cosmic string present in the C-metric. The non-static spacetimes presented are exact perfect fluid solutions with only this symmetry, and the analysis appears to be within reach. For the expanding ID PF models both the matter density $\rho(t)$ and the expansion scalar $\theta(t)$ can be arbitrary functions. This freedom is displayed explicitly in the metric form, and makes the solutions more attractive as a cosmological model.

The paper is organized as follows. In section 2 we present the GHP approach to class I. We derive a closed set of equations, construct suitable scalar invariants, interpret the invariant subclassification of (1) and start the integration. At the end we provide alternative characterizations for the Einstein space members and identify their static regions and USs. In section 3 we finish the construction of the general ID line element in a transparent way, and correct the calculative error of [1] in the original approach. Then we deduce basic properties of the ID perfect fluid models. In section 4 we summarize the main results and indicate points of further research. The work greatly benefited from the use of the GHP formalism, which at the same time elucidates the deviation from the C-metric. In appendix A we provide a pragmatic survey of this formalism for the non-expert reader. In appendix B, finally, we present criteria for deciding when a Petrov type D spacetime admits a (rigid) US or is static.

**Notation.** For spacetimes $(M, g_{ab})$ we take $(+ + + -)$ as the metric signature and use geometrized units $8\pi G = c = 1$, where $G$ is the gravitational coupling constant and $c$ the speed of light. $\Lambda$ denotes the cosmological constant. We make consistent use of the abstract Latin index notation for tensor fields, as advocated in [32]. Round (square) brackets denote (anti-)symmetrization, $\eta_{abcd}$ is the spacetime alternating pseudo-tensor and $\nabla_a T_{ab...} (L_X T_{ab...})$ designates the Levi-Civita covariant derivative (Lie derivative wrt $X^a$) of the tensor field $T_{ab...}$. One has

$$d_a f = \nabla_a f, \quad d_a Y_a = \nabla_{[a} Y_{b]}$$

for the exterior derivative of a scalar field $f$, resp. one-form field $Y_a$, and

$$X(f) \equiv X^a d_a f$$

denotes the Leibniz action of a vector field $X^a$ on $f$; when $X^a$ is the $x^a$-coordinate vector field $\partial_{x^a}$ we write $\partial_x f$ or $f_x$, and a prime denotes ordinary derivation for functions of one variable, $f'(x) \equiv \partial_x f(x)$. However, we use index-free notation in line elements $d^2 x = g_{ij} dx^i dx^j$. The specific GHP notation is introduced in appendix A.

2. **GHP APPROACH TO CLASS I**

2.1. **Definition and integrability**

We consider Barnes’ class I [1], consisting of spacetimes $(M, g_{ab})$ with the following properties:
(i) the spacetime admits a unit timelike vector field \( u^a \) \((u^au_a = -1)\) which is non-shearing and non-rotating, i.e., its covariant derivative is of the form

\[
\nabla_b u_a = \partial_b u_a - \bar{u}_a u_b, \quad h_{ab} \equiv g_{ab} + u_a u_b,
\]

where the acceleration \( \bar{u}^a = u^b \nabla_b u^a \) and expansion rate \( \dot{h} = \nabla_a u^a \) are the remaining kinematic quantities of \( u^a \);

(ii) the Einstein tensor has the structure

\[
G_{ab} = S u_a u_b + pg_{ab} = w u_a u_b + ph_{ab},
\]

\[
D_a w \equiv h_b \nabla_b w = 0,
\]

i.e., the spacetime represents the gravitational field of either a perfect fluid with shearfree normal four-velocity \( u^a \), pressure \( p + \Lambda \) and spatially homogeneous energy density \( w - \Lambda \) (case \( S \equiv w + p \neq 0 \)) or a vacuum (Einstein space case \( S = 0 \), where \( w = -p \) may be identified with \( \Lambda \));

(iii) the Weyl tensor \( C_{abcd} \) is degenerate but non-zero, i.e., the spacetime is algebraically special but not conformally flat.

Choose null vector fields \( k^a \) and \( l^a \), subject to the normalization condition \( k^a l_a = -1 \), such that

\[
u^a = \frac{1}{\sqrt{2q}} \left( q k^a + l^a \right), \quad q > 0.
\]

Within the GHP formalism (cf. appendix A) based on the complex null tetrad \((k^a, l^a, m^a, \overline{m}^a)\), \( q \) is \((2,-2)\)-weighted and the conditions (i) and (ii) translate into

\[
π + \tau = q \overline{π} + q^{-1}ν, \quad λ = q \overline{τ}, \quad μ - \overline{π} = q(\overline{π} - ρ),
\]

\[
\nu' q - q \nu p = -2q(μ - q \overline{π}), \quad q \partial q = \partial' q = 0
\]

and

\[
Φ_{01} = Φ_{12} = Φ_{02} = 0, \quad Φ_{11} \equiv \frac{S}{S}, \quad Φ_{00} = \frac{S}{4q}, \quad Φ_{22} = \frac{q S}{4},
\]

\[
R \equiv 24π = w + 3p = 4w - 3S, \quad q \partial w = q' w = 0, \quad \nu' w - q \nu p w = 0,
\]

respectively. By virtue of condition (i) the magnetic part \( H_{ab} \equiv \frac{1}{2} \mu_{\nu c m n} C_{b c d n} u^c u^d \) of the Weyl tensor wrt \( u^a \) vanishes \([23]\). In combination with condition (iii) it follows that the Weyl tensor is purely electric wrt \( u^a \), \( E_{ab} \equiv C_{ab cd} u^d \neq 0 \), the Weyl-Petrov type is \( D \), and at each point \( u^a \) lies in the plane \( Σ \) spanned by the Weyl PNDs (cf. appendix B for a GHP proof of these well-known facts). Hence, choosing \( k^a \) and \( l^a \) along the PNDs, \((k^a, l^a, m^a, \overline{m}^a)\) is a Weyl principal null tetrad (WPNT) and we have

\[
Ψ_0 = Ψ_1 = Ψ_3 = Ψ_4 = 0, \quad Ψ = Ψ_2 = 0,
\]

Under the restrictions \((7)\) and \((11)\), the GHP Bianchi equations are given by \([A_{31}] - [A_{36}]\) and their prime duals. Combining these with the other equations in \([15] - [12]\) results in

\[
\kappa = 0, \quad ν = 0, \quad ζ = λ = 0, \quad S = 0, \quad \tau = -\tau, \quad \Psi = 3ρ, \quad \partial' Ψ = -3\tau, \quad \partial Ψ = -3π, \quad \partial' S = -q S, \quad \partial S = π S
\]

\[

\partial' w = q w = -\frac{3S(μ - q ρ)}{2}.
\]

With \((7) - (9)\) and \((11) - (13)\) the Ricci equations, given by \([A_{25}] - [A_{30}]\) and their prime duals, reduce to

\[
\partial μ = -ν' ρ
\]

\[
= -\partial' τ + \mu π + \tau S + \frac{π \Psi + w}{3} - \frac{S}{4}, \quad \partial' μ = 0
\]

\[
\partial ρ = ρ + \frac{S}{4q}, \quad \partial ρ = 0
\]

\[
\partial τ = ν' τ = 0, \quad ζ = 0
\]

\[
-\partial' π = -π' τ = \frac{H}{2}
\]

and the complex conjugates of \([24]\), while the commutator relations applied to a \((w_p, w_q)\)-weighted scalar \( η \) become

\[
\partial [P, P']\eta = (w_p + w_q) \left( τ - \frac{π}{2} + \frac{w}{6} - \frac{S}{4} \right) η, \quad \partial [0, ξ']\eta = (w_p - w_q) \left( -μ ρ + \frac{π}{2} - \frac{w}{6} \right) η
\]

\[
\partial [P, ξ']\eta = (-τ P + ρ ξ + w_q μ τ) η, \quad \partial [0, ξ']\eta = (-τ P + ρ ξ + w_p μ τ) η
\]

\[
\partial [P', ξ']\eta = (-τ P' + μ ξ + w_q μ τ) η, \quad \partial [0, ξ']\eta = (-τ P' + μ ξ + w_p μ τ) η
\]

Then the \([0, ξ'](τ), [0, ξ'](τ), [P, ξ'](τ)\) and \([P', ξ'](τ)\) commutator relations imply

\[
\partial H = 2π(Ψ + G), \quad ξ' H = 2τ(Ψ + G), \quad Ψ H = ρ(H + F), \quad Ψ' H = -μ(H + F), \quad τ H = -μ(H + F)
\]

where

\[
F \equiv 2τ, \quad G \equiv 2μ ρ + \frac{w}{3}
\]

One checks that the integrability conditions for the system \([0] - [33]\) of partial differential equations (PDEs) are identically satisfied, indicating that corresponding solutions exist. Those for which \( u^a \) is non-expanding additionally satisfy

\[
θ \sim μ - q \overline{π} = 0
\]
2.2. Metric structure and subclassification

The first, second and last parts of (13)-(14) precisely account for the hypersurface-orthogonality of $k^a$, $l^a$ and $m^a \leftrightarrow \bar{m}^a$, respectively. Thus real scalar fields $u$, $v$, (zero-weighted) and $U$, $V$ ((−1, −1)-resp. (1, 1)-weighted), and complex scalar fields $\zeta$ (zero-weighted) and $Z$ ((1, −1)-weighted) exist such that

$$d_a u = \frac{\Psi^{1/3}}{U} k_a, \quad d_a v = \frac{\Psi^{1/3}}{V} l_a, \quad d_a \zeta = \frac{\Psi^{1/3}}{Z} m_a.$$ (35)

By (16) this is equivalent to

$$P' u = -\Psi^{1/3} U, \quad d u = \bar{u} = \bar{\zeta}' = 0,$$ (36)

$$d v = -\Psi^{1/3} V, \quad d v = \bar{v} = \bar{\zeta}' = 0,$$ (37)

$$\bar{\zeta}' = \bar{\zeta}' = \bar{\zeta}' = \bar{\zeta}' = 0.$$ (38)

Hence, when we take these fields as coordinates, (38) imply that the zero-weighted fields $U$, $V$ and $Z$ only depend on $(u,v)$, resp. $(\zeta, \bar{\zeta})$, such that all class I metrics are conformally related to direct sums of metrics on two-spaces:

$$g_{ab} = \Psi^{-2/3} (g^1_{ab} \otimes g^2_{ab}),$$ (42)

$$g^1_{ab} = 2 \Psi^{1/2} m_{(a \bar{b})},$$ (43)

$$g^2_{ab} = -2 \Psi^{1/2} k_{(a \bar{b})}.$$ (44)

The line elements of $g^1_{ab}$ and $g^2_{ab}$ will be denoted by $d_{ab}$, resp. $d_{\Sigma}$. In the case where such a two-space is not of constant curvature, however, we will construct more suitable coordinates in the sequel. Inspired by the GHP manipulations of (33) for type D vacua [57], we start this construction by deducing suitable combinations of the scalar invariants $F$, $G$, $H$ and $\Psi$. From (13), (16) and (18-33) it is found that

$$d_a F = 3 \Psi^{1/3} \varphi \alpha_a, \quad d_a G = 3 \Psi^{1/3} \gamma \beta_a,$$ (45)

$$d_a \varphi = 2 \Psi^{1/3} x \alpha_a, \quad d_a \gamma = 2 \Psi^{1/3} y \beta_a,$$ (46)

$$d_a x = \Psi^{1/3} \alpha_a, \quad d_a y = \Psi^{1/3} \beta_a.$$ (47)

where

$$\alpha_a = \tau_m a + \tau \bar{m}_a, \quad \beta_a = \mu k_a - \rho \bar{m}_a$$ (48)

are invariantly-defined one-forms and

$$\varphi \equiv \frac{H + F}{3 \Psi^{1/3}}, \quad \gamma \equiv \frac{H + \Psi + F + 2G}{3 \Psi^{1/3}}.$$ (49)

Consequently, the scalar invariants

$$C \equiv 3(\varphi - x^2) = -H - 2 \Psi - G,$$ (50)

$$D \equiv -x^3 - Cy - D = -H^2 + 2 \Psi G - G$$ (51)

are constant (d$_a C = d_a D = 0$). From (35) and (36) it follows that $F$, $G$, $H$ and $\Psi$ are biunivocally related to $x$, $y$, $C$ and $D$, where

$$2 \tau \varphi = F = x^3 + Cy + D,$$ (52)

$$2 \mu \gamma = G = y^3 + Cy + D = -w^3,$$ (53)

$$2 \psi \gamma = H = 3x^2 + y^3 + Cy - D,$$ (54)

such that $x$ is constant. In combination with the last part of (43) and (19-22) imply

$$H = F = \varphi = 0, \quad \Psi - G = 3x^2 \Psi^{1/3},$$ (55)

$$C = -3x^2, \quad D = 2x^3,$$ (56)

$$\tau = 0 \Leftrightarrow \pi = 0 \Leftrightarrow x = const \Leftrightarrow \nabla \Psi \in \Sigma.$$ (57)

Barnes [1] partitioned class I according to the position of the gradient $\nabla \Psi$ relative to $\Sigma$ and $\Sigma^\perp$. This relates to the vanishing of the invariants $\tau \varphi = -\pi \mu$ or $\mu \rho$, maximal symmetry of $g^1_{ab}$ or $g^2_{ab}$ and spatial rotation or boost of $g_{ab}$, as follows. First assume $\tau = 0$. In this case (25) and the first parts of (33) and (19-22) imply

$$\tau = 0 \Leftrightarrow \pi = 0 \Leftrightarrow x = const \Leftrightarrow \nabla \Psi \in \Sigma.$$ (58)

The $[\bar{\zeta}, \bar{\zeta}]$ commutator relation applied to $\zeta$, $\bar{\zeta}$ and $Z$ imply $\partial Z = \partial Z = 0$ and $\partial \bar{Z} = 3 \Psi^{1/3} Z$. Herewith the Gaussian curvature of the two-space with metric $g^1_{ab}$ becomes

$$K^\perp = -(Z Z)^{-1} (\ln ZZ) = -3x.$$ (59)

where the dual of (33) was used in the calculation. In conjunction with the results of Goode and Wainwright [34], we conclude that (33) yields the class I solutions which are locally rotationally symmetric (LRS) of label II in the Stewart-Ellis classification [35], characterized by $g^1_{ab}$ having constant curvature $K^\perp = -3x$. As well known (see e.g. the appendix of (32)) the coordinates $\zeta$ and $\bar{\zeta}$ may
then be adapted such that \( Z^Z(\zeta, \overline{\zeta}) = (1 + K^\perp \zeta \overline{\zeta})^{-1} \) in \((43)\), or an alternative form may be taken:

\[
ds_1^2 = \frac{2d\zeta d\overline{\zeta}}{1 + K^\perp \zeta \overline{\zeta}} = Y_1^2(dx_1^2 + \cos(\sqrt{k_1} x_1)^2 dx_2^2),
\]

\[
K^\perp = k_1 Y_1^{-2}, \quad k_1 \in \{-1, 0, 1\}.
\]

Now assume \( \mu \rho = 0 \). It follows from \((20)-(23), (56)\), and the second parts of \((49)-(52)\) that

\[
S = \gamma - 1, \quad G = \frac{w}{3}, \quad H = 2 \Psi + \frac{C}{3} = 3y \sqrt{\Psi}^{2/3},
\]

\[
C = -3y^2, \quad D = -2y^3 - \frac{\Lambda}{3},
\]

such that \( y \) is constant. In combination with \((15)\) and the second parts of \((47)-(48)\) this implies

\[
\mu \rho = 0 \iff \mu = \rho = 0 \iff y = \text{const} \iff \nabla^a \Psi = \Sigma^\perp, \quad (62)
\]

By a similar reasoning as in the case \( \tau = 0 \) one concludes that \((52)\) yields the locally boost-isotropic Einstein spaces of Petrov type \( D \), characterized by \( g^\Sigma_{ab} \) having constant curvature

\[
K^\Sigma = -3y,
\]

such that in this case one may take \( UV(u, v) = (1 - K^\Sigma uv/2)^{-1} \) in \((43)\) and we have

\[
ds_2^2 = -\frac{2du dv}{1 - k^\perp uv} = Y^2(dx_3^2 - \cos(\sqrt{k_2} x_3)^2 dx_4^2),
\]

\[
K^\Sigma = k_2 Y_2^{-2}, \quad k_2 \in \{-1, 0, 1\}.
\]

With \((42)\) and \( ds_2^2 \) written in the second form, it is clear that

\[
\partial x^a = -\Psi^{-2/3} Y^2 \cos(\sqrt{k_2} x_3)^2 d^a x_4
\]

is a HO timelike Killing vector field.

Four subclasses of class I thus arise, which were labeled by Barnes as follows:

- IA : \( \tau = 0 = \mu \rho \)
- IB : \( \tau = 0 \neq \mu \rho \)
- IC : \( \tau = 0 = \mu \rho \neq 0 \)
- ID : \( \tau = 0 \neq \mu \rho \)

We proceed with the respective integrations. Notice that in the joint case \( \mu \tau = 0 \) one has

\[
2(\tau + \mu) = (x + y)^3 + K(x + y)^2 - \frac{w}{3},
\]

with \( K = K^\perp \) for \( \tau = 0 \) and \( K = K^\Sigma \) for \( \mu \rho = 0 \). When \( \tau \neq 0 \) or \( \mu \rho \neq 0 \) we may take \( x \), resp. \( y \) as a coordinate, where \((47)-(48)\) and \((52)\) imply

\[
(x + y)(\tau m_a - \mu \overline{m}_a) = d_a x,
\]

\[
(x + y)(\mu k_a - \rho \overline{k}_a) = d_a y.
\]

In view of \((42)-(44)\) and \((56)\) it then remains to determine suitable complementary coordinates for \( x \) in \( g^\Sigma_{ab} \) or \( y \) in \( g^\Sigma_{ab} \). For \( \tau \neq 0 \), Frobenius’ theorem and \((68)\) suggest to examine whether zero-weighted functions \( \phi \) and \( f \) exist such that

\[
\frac{i x + y}{2 \tau} (\tau m_a - \tau \overline{m}_a) = f d_a \phi.
\]

By \((A10)\) this amounts to calculating the integrability conditions of the system

\[
\frac{d}{d\tau} \phi = \tau \partial^\tau \phi = 0, \quad \tau \partial \phi = -\tau \partial^\tau \phi = i \frac{x + y}{2 f}
\]

which turn out to be

\[
\frac{d}{d\tau} f = \tau \partial^\tau f = 0, \quad \alpha \tau \nabla^a f = 0.
\]

These last equations have the trivial solution \( f = 1 \), for which a solution \( \phi \) of \((71)\) is determined up to an irrelevant constant. Herewith the invariantly-defined one-form on the left-hand side in \((70)\) is exact, and we take \( \phi \) as the coordinate complementary to \( x \). On solving \((68)\) and \((70)\) with \( f = 1 \) for \( m_a \) and \( \overline{m}_a \) and using \((65)\) we conclude from \((49)\) that

\[
ds_2^2 = \frac{dy^2}{2 \tau} + 2 \tau \partial \phi d^2 \phi, \quad 2 \tau \partial = x^3 + C x + D
\]

for classes IC and ID. Clearly, the metric solutions should be restricted to spacetime regions where \( x^2 + C x + D > 0 \) for consistency, while

\[
\partial \psi = \partial \partial^\tau \psi = 0, \quad \mu \tau \partial \psi = \rho \partial^\tau \psi = \frac{x + y}{2 y}
\]

but the integrability conditions of this system are now

\[
\partial g = \partial \partial^\tau g = 0, \quad \beta \nabla^a g = -g S \frac{\partial^2 + q^2 r^2}{q \mu \rho}
\]

So \( g = 1 \) is only a solution in the Einstein subcase \( S = 0 \), for which we then get

\[
ds_2^2 = \frac{dy^2}{2 \mu \rho} - 2 \mu \rho d\psi^2, \quad 2 \mu \rho = y^3 + C y - D - \frac{\Lambda}{3}
\]

from \((41), (54), (69)\) and \((75)\), with KVF

\[
\partial \psi^a = \frac{\mu k^a + \rho \overline{k}_a}{x + y} = -\frac{2 \mu \rho}{(x + y)^2} d^a \psi,
\]

which is timelike for \( \mu \rho > 0 \) and spacelike for \( \mu \rho < 0 \). In general, the second vector field in \((75)\) is always HO: the integrability conditions of \((70)\) are checked to be identically satisfied, such that solutions \( g \) and a corresponding solution \( \psi \) of \((75)\) exist. However, taking \( \psi \) as a complementary coordinate of \( y \) eventually leads to a very complicated system of coupled partial differential equations for \( g = g(y, \psi) \), which is impossible to solve explicitly. We shall remedy this in section \((51)\) but now discuss characterizing features of the Einstein space limit cases.
2.3. Characterizations of PE Petrov type $D$ Einstein spaces

Petrov type $D$ Einstein spaces constitute the class $D_0$ (cf. the introduction) and are all explicitly known. The line elements are obtained by putting the electromagnetic charge parameter $\Phi_0$ or $e^2 + g^2$ equal to zero in the $D$-metrics given by Debever et al. [14], resp. García [15]. These coordinate forms generalize and streamline those found by Kinnersley [37] in the $\Lambda = 0$ case.

Recently, a manifestly invariant treatment of $D_0$, making use of the GHP formalism, was presented [33]. Within GHP, $D_0$-metrics are characterized by the existence of a complex null tetrad wrt which (7)-(13) hold. According to the Goldberg-Sachs theorem [20], (13) holds just as $\Phi = 0$, $\pi = \tau = 0$, and the first equation of (26) are also valid in general. From these relations it follows that

$$C^{00}_+ : \tau = 0 = \mu \rho, \quad C^{00}_- : \tau \neq 0 = \mu \rho,$$

$$C^{0+}_+ : \tau = 0 \neq \mu \rho, \quad C^{0+}_- : \tau \neq 0 \neq \mu \rho.$$ (81)

$$\mu = 0 \Rightarrow \rho = 0 \text{ or } D_0 \neq 0 \neq \frac{\pi + \tau}{\pi}.$$ (82)

These coordinate forms generalize and streamline those found by Kinnersley in the $\Lambda = 0$ case.

Recently, a manifestly invariant treatment of $D_0$, making use of the GHP formalism, was presented [33]. Within GHP, $D_0$-metrics are characterized by the existence of a complex null tetrad wrt which (7)-(13) hold. According to the Goldberg-Sachs theorem [20], (13) holds just as $\Phi = 0$, $\pi = \tau = 0$, and the first equation of (26) are also valid in general. From these relations it follows that

$$C^{00}_+ : \tau = 0 = \mu \rho, \quad C^{00}_- : \tau \neq 0 = \mu \rho,$$

$$C^{0+}_+ : \tau = 0 \neq \mu \rho, \quad C^{0+}_- : \tau \neq 0 \neq \mu \rho.$$ (81)

These coordinate forms generalize and streamline those found by Kinnersley in the $\Lambda = 0$ case.

The result is in agreement with proposition B.1, which provides criteria for deciding when a Petrov type $D$ spacetime allows for an US, regardless of the structure of the energy-momentum tensor. The hypersurface-orthogonality (79) of the WPNT directions corresponds to criterion 5 and is actually equivalent to a one-degree freedom of USs. It is worth to mention that all LRS II spacetimes, i.e. those exhibiting (pseudo-) spherical or plane symmetry, share this property with the $D_0$- and $D$-metrics. On the other hand, certainly not all PE spacetimes admit an US. For instance, the Gödel solution is an LRS I PE perfect fluid of Petrov type $D$, described in GHP by (77)-(80) and

$$S/2 = w = p = -3V_2 = -2\mu \rho = \text{const.} > 0,$$ (85)

$$\pi = \tau = 0, \quad \mu = q \rho, \quad \pi = -\tau, \quad \phi = q \rho^2$$ (86)

$q > 0$ being annihilated by all weighted GHP-derivatives; hence the invariant $(\mu - \pi)(p - \pi) = 4q \rho \pi q$ appearing in criterion 2 of proposition B.1 is strictly negative, and it follows that the Gödel solution does not admit an US. As another example, the spatially-homogeneous $\Lambda = 0$ vacuum metrics

$$ds^2 = t^2 \rho, \quad dx^2 + t^2 \rho^2 dy^2 + t^2 \rho^2 dz^2 - dt^2,$$ (87)

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1, \quad p_1 p_2 p_3 \neq 0,$$ (88)

attributed to Kasner [39] are PE [58]. If the $p_i$ are all different, there is a complete group $G_3$ of isometries and the Petrov type is I. In this case $p_i$ is the up to reflection unique Weyl principal vector field and hence the only possible US-candidate; however, its shear tensor has the non-zero eigenvalues $(1/3 - p_i)/t$ and hence the spacetime does not admit an US. On the other hand, if two $p_i$’s are equal it follows that $p_2 = p_3 = -2p_1 = 2/3$ (without loss of generality). Then the line element represents a Petrov type $D$, non-stationary, plane-symmetric vacuum which, according to theorem 2.1, admits a one-degree freedom of USs (cf. the end of this section).

In theorems 3 of [1] it is claimed that all vacuum spacetimes admitting an US are static, which would generalize Trümper’s result [24] by including Petrov type $D$. However, this conclusion only holds when $\mu \rho \geq 0$. Indeed, a static member of $D_0$ necessarily admits a rigid (i.e. non-expanding) US, such that $\mu \rho \geq 0$, cf. [33]. Conversely, when $\mu \rho = 0$ or $\mu \rho > 0$ for a PE member, it
admits the HO timelike KVF \( \text{KVF} \) and is thus static. This is in agreement with proposition \( \text{III, 3} \) regarding \( \mu \rho = 0 \) criterion 6° tells that in fact all boost-isotropic spacetimes, with \( \pi = -\mathbf{w} \) wrt a WPN, are static, while for \( \mu \rho > 0 \) one checks that criterion 2° is satisfied by virtue of \( \text{I} - \text{III} \). In appendix B the freedom of the rigid USs and HO timelike KVF directions (static observers) in these cases is also specified, which is in accordance with a result by Wahlquist and Estabrook \( \text{[10]} \).

In summary we have:

**Theorem 2.2** A Petrov type \( D \) Einstein space is static if it admits a rigid US. This is precisely the case when the spacetime is PE and has a positive or zero scalar invariant \( \mu \rho \), being the product of the divergences of (non-twisting) Weyl principal null vectors \( k^a \) and \( k^a \) subject to \( k^a k_a = -1 \). For \( \mu \rho > 0 \) there is an up to reflection unique rigid US, defined from the geometry by \( \text{I} - \text{III} \) and \( q = \mu \rho / \pi \) and parallel to the HO timelike KVF direction. For \( \mu \rho = 0 \Rightarrow \mu = 0 \) (classes IA and IC) all USs are rigid USs and have a one-degree freedom, while the HO timelike KVF directions are parametrized by two constants.

For completeness we display standard coordinate forms of the PE Petrov type \( D \) Einstein space metrics, as recovered by \( \text{[42, 50]} \) and \( \text{[40, 44, 73, 77]} \).

\( C^{00} \) corresponds to \( \text{[20]} \) and \( \text{[61]} \). From \( \text{[57, 59, 61] and [63]} \) one deduces that

\[
K^\perp = -3x = -3y = K \Sigma, \quad \Psi_2 = -\frac{\Lambda}{3} = 4x^3 \neq 0.
\]

Rescaling \( \zeta \), \( u \) and \( v \) by a factor \( (2x)^{-1} \) one arrives at

\[
ds^2 = \frac{2d\zeta d\dot{\zeta}}{1 + \frac{2}{3}x} - \frac{2du dv}{1 - \frac{2}{3}x}, \quad 1 + \frac{\Lambda}{2}x^3 > 0, \quad \Lambda \neq 0.
\]

This represents the Einstein space limit \( \Phi_0 = 0 \) of Bertotti’s static and homogeneous electrowave family with cosmological constant \( \text{[11, 12]} \) exhibiting spatial rotation and boost isotropy (complete group \( G_2 \) of isometries). The \( \Lambda = 0 \) limit yields flat Minkowski spacetime.

\( C^0+ \) and \( C^0- \) correspond to \( \text{[20]} \) and \( \text{[67]} \), resp. \( \text{[61] and [63]} \). Making use of \( \text{[67]} \), replacing in the \( C^0+ \) (\( C^0- \)) case the coordinate \( y(x) \) by \( r = -(2m)^{1/3}/(x+y) \), rescaling the remaining coordinates by a factor \( (2m)^{-1/3} \) and writing \( Y \equiv (2m)^{1/3} > 0 \) one finds

\[
ds^2 = r^2 \left( d\xi^2 + \delta \cos(\sqrt{\delta} \xi) d\eta^2 \right) + \frac{dr^2}{g_k(r)} - \delta g_k(r) d\chi^2,
\]

\[
g_k(r) = k - \frac{2m}{r} - \frac{\Lambda}{3} r^2, \quad k = K(2m)^{1/3} \in \{-1, 0, 1\},
\]

with \( \delta = 1 \) for \( C^0+ \) (\( \delta = -1 \) for \( C^0- \)). These solutions have a complete group \( G_4 \) of isometries acting on spacelike (timelike) three-dimensional orbits, and for \( \Lambda = 0 \) correspond to Kinnersley’s case I (IV) with \( l = 0 \). The static region of \( C^0+ \) (\( g_k(r) > 0 \)) yields class A in the classification of static Petrov type \( D \) vacua by Ehlers and Kundt \( \text{[3]} \): \( C^0- \) is static everywhere and corresponds to class \( B \). Regarding \( C^0+ \), the subclass \( k = 1 \) reproduces after \( \xi \to \pi/2 - \xi \) the well-known forms of the spherically symmetric Schwarzschild-Kottler interior and exterior metrics \( \text{[13, 44]} \); the subclass \( k = \Lambda = 0, r \to 0 \) (\( g_k(r) < 0 \)) gives another form of the plane-symmetric Kasner metrics (cf. supra).

\( C^0+ \) corresponds to \( \text{[73 and 77]} \), which gives the line element

\[
ds^2 = \frac{1}{(x+y)^2} \left( \frac{dx^2}{f(x)} + \frac{d\phi^2}{g(y)} - g(y) d\psi^2 \right),
\]

\[
f(x) = x^3 + Cx + D > 0, \quad g(y) = -f(-y) - \frac{\Lambda}{3}.
\]

The KVF \( \partial_x \) and \( \partial_\psi \) generate the complete, abelian group \( G_2 \) of isometries. For \( \Lambda = 0 \), \( \text{[89]} \) is the form of the C-metric obtained by Levi-Civita and recovered by Ehlers and Kundt, and corresponds to Kinnersley’s case IIIA. It is generally assumed - and suggested in the original paper \( \text{[3]} \) - that the Kinnersley-Walker form

\[
ds^2 = \frac{1}{\alpha^2(\xi + \eta)^2} \left( \frac{d\xi^2}{h(\xi)} + \frac{d\eta^2}{k(\eta)} - k(\eta) d\psi^2 \right),
\]

\[
h(\xi) = 1 - \xi^2 - 2m\xi^3 > 0, \quad k(\eta) = -h(-\eta)
\]

equivalently describes the gravitational field of the \( \Lambda = 0 \) C-metric. However, this is not entirely correct. Equating the Lorentz invariants appearing in the right hand sides of the equations in \( \text{[60-62]} \), calculated for the metrics \( \text{[89 and 91]} \), yields

\[
x = -(2m)^{1/3} \left( \alpha \xi + \frac{1}{6m} \right), \quad y = -(2m)^{1/3} \left( \alpha \eta - \frac{1}{6m} \right),
\]

\[
C = -\frac{1}{3(2m)^{1/3}}, \quad D = \alpha^2 - \frac{1}{54m^2}.
\]

Hence \( \text{[91]} \) only covers the range \( C < 0, D > -2 \left(-\frac{1}{3}\right)^{2} \), whereas in general the constant scalar invariants \( C \) and \( D \) are allowed to take any real value. Yet, the cubic \( f(x) \) has discriminant \(-4C^3 - 27D^2\); thus it has three distinct real roots if and only if

\[
C < -3 \left( \frac{D}{2} \right)^{2} \iff C < 0, |D| < 2 \left(-\frac{C}{3}\right)^{2}.
\]

Thus \( \text{[91]} \) is compatible for this case, and by further rescaling \( \phi \) and \( \psi \) with a factor \( \alpha(2m)^{-1/3} \) one arrives at \( \text{[90]} \); \( \text{[92]} \) is equivalent with \( ma < 1/\sqrt{27} \), leading to the physical interpretation of two uniformly accelerating masses. Recently, Hong and Teo \( \text{[12]} \) introduced a normalized factored form for this situation, which greatly simplifies certain analyses of the C-metric. A further coordinate transformation can be made such that the Schwarzschild metric is comprised as the subspace \( a = 0 \). This was further exploited for the full D-class in \( \text{[18]} \).

Finally, we write down the equations which determine all USs for a member of \( C^0+ \), in the coordinates \( x \) and \( \psi \)
of (89). Let
\[ U^a = -\frac{x+y}{\sqrt{g(y)}} \partial_t a, \quad V^a = -(x+y)\sqrt{g(y)} \partial_y a \]
in the static region and
\[ U^a = -(x+y)\sqrt{g(y)} \partial_t a, \quad V^a = \frac{x+y}{\sqrt{-g(y)}} \partial_y a \]
in the non-static region, and gauge-fix \( k^a = (U^a + V^a)/\sqrt{(2)}, \ l^a = (U^a - V^a)/\sqrt{(2)} \). The unit timelike field (31) is an US if and only (36) holds; this translates to \( q = q(y, \psi) \) and
\[ g(y)(q \pm 1)q_y + (q \mp 1)(q_{\psi} + g'(y)q) = 0. \] (93)

Here and below the upper (lower) signs should be taken in the static (non-static) region. For solutions \( q = q(y) \), i.e. \( q_{\psi} = 0 \), direct integration of (93) yields
\[ g(y)(q \mp 1)^2 = E_{\pm} q(y), \] (94)
with \( E_+ > 0 \) and \( E_- < 0 \) constants of integration. Notice that in the static region the solution \( q(y) = 1 \) yields the unique static observer. In the case \( q_{\psi} \neq 0 \) the solutions get implicitly determined by an equation of the form \( \psi = \nu(y, q) \), on applying the method of characteristics for first-order PDEs (see e.g. (37)). In the subcase where \( C = D = \Lambda = 0 \), \( g(y) \) reduces to \( y^3 \) and this equation reads
\[ \psi = -\left(\frac{y(q \mp 1)^{3/2}}{y^{3/2}}\right)^2 \int (q \mp 1)\frac{dq}{3^q} + Z \left(\frac{y(q \mp 1)^{3/2}}{y^{3/2}}\right). \] (95)

Here \( Z \) is a free function of its argument, making the one-degree freedom of USs more explicit. Replacing (73) by (93) does not alter these equations, i.e., the above remains valid for \( C^0_{\perp} \). Then \( C = D = 0 \) is equivalent to \( x = K^1 = 0 \), cf. (67) and (59); for \( \Lambda = 0 \) the non-static region \( y \neq 0 \) corresponds to the plane-symmetric Kasner vacuum metrics, where \( y = -(3/2)^{-2/3} \) and a rescaling of the other coordinates recovers (57), the USs being determined by (24) and (55) with the lower sign.

3. PERFECT FLUID GENERALIZATIONS OF THE C-METRIC

3.1. Line element

We resume the integration of class I started at the end of § 23. We thereby focus on the subclass ID characterized by \( \tau \neq 0 \neq \mu \rho \). Let us first summarize what we did so far. We started off with the closed set (39)-(53) of first-order GHP equations in the seven (weighted) real variables \( \Psi, S, w, \mu, \rho, \sigma, \partial \tau \) and the complex variable \( \tau \). These variables are equivalent to two dimensionless spin and boost gauge fields, e.g. \( \tau/\Psi \) and \( \mu/\rho \), and seven real scalar invariants. The boost and spin gauge fields could serve to invariantly fix the tetrad – the ID members being therefore anisotropic – but can be further ignored. For the \( C^*- \)Einstein spaces, \( S = 0 \) and \( w = \Lambda = \text{const} \), and we remarked that \( q \) is not a part of the intrinsic describing set of variables. Hence we end up with four real scalar invariants in this subcase. These invariants are equivalent to the two constants \( C \) and \( D \) and two independent functions \( x \) and \( y \), which we took as coordinates and in terms of which, on adding two coordinates \( \phi \) and \( \psi \) related to the symmetries, the corresponding \( C^*- \)metric can be expressed. In the perfect fluid case \( S \neq 0 \), (83) gives the boost field \( q \) which, starting from an arbitrary gauge \((k^a, l^a)\), turns \( u^a \) given by (101) into the invariantly-defined fluid four-velocity. The four invariants and their use persist, just as the coordinate \( \phi \). However, the scalar invariants \( w \) and \( S \) are no longer constant and \( \psi \) is no longer a suitable coordinate. Thus we need one more scalar invariant for our description and one remaining coordinate complementary to \( y \).

For the first purpose it is natural to look at the kinematics of the fluid, which are fully determined by
\[ b \equiv 2\nabla(a_{\nu c})m^o_{\nu c \Psi} = \nabla(a_{\nu c})v^o_{\nu c} \frac{\theta}{3}, \] (96)
\[ \tilde{u}_\parallel = v_{a_{\nu c}^*}, \quad \tilde{u}_\perp = 2m(a_{\nu c})\tilde{w}_c. \] (97)

Here
\[ v^o = \frac{1}{\sqrt{2q}}(qk^a - l^a) \] (98)

is the intrinsic spacelike vector field, which determines at each point the up to reflection unique normalized vector orthogonal to \( u^a \) and lying in the PND plane \( \Sigma \), while \( \tilde{u}_\parallel \) and \( \tilde{u}_\perp \) are the component along \( v^o \), resp. projection onto \( \Sigma^\perp \) of the acceleration \( \dot{u}_a \). In analogy with (96) we define the invariant
\[ \tilde{b} \equiv 2\nabla(a_{\nu c})m^o_{\nu c \Psi}. \] (99)

The relation with GHP quantities is
\[ b = \frac{\mu - q\rho}{\sqrt{2q}}, \quad \tilde{b} = -\frac{\mu + q\rho}{\sqrt{2q}}, \] (100)
\[ \dot{u}_\parallel = (2q)^{-3/2}(b^q + q\rho b) = \frac{bq}{\sqrt{2q}} b, \] (101)
\[ -\dot{u}_\perp = \tau m_a + \tau m_a \equiv a_d = \frac{d_{a c}}{x + y}. \] (102)

Notice that (100) is equivalent to
\[ b u_a - \tilde{b} v_a = \mu k_a - \rho l_a \equiv \beta_a = \frac{d_{a c}}{x + y}. \] (103)

In combination with (63)-(64), (102) and (103) imply
\[ 2\tau\Psi = \dot{u}_\perp^a u^{a - a} = x^3 + C x + D, \] (104)
\[ 2\mu \rho = \tilde{b}^2 - b^2 = y^3 + C y - D - \frac{w}{3}. \] (105)
We choose \( b \) as the final describing invariant and use \( \tilde{b} \) and \( u_\parallel \) as auxiliary variables. In view of \((100)-(102)\) one deduces that the differential information for \( S, w \) and \( b \) comprised in \((10)-(33)\) is precisely
\[
D_a S = -S \dot{u}_a, \quad \text{(106)}
\]
\[
d_{ab} w = -u(w) u_a, \quad u(w) = -3b S, \quad \text{(107)}
\]
\[
d_a b = -u(b) u_a, \quad u(b) = -\sqrt{\tilde{b}} + \tilde{b}(u_\parallel - \tilde{b}) - \frac{S}{2}, \quad \text{(108)}
\]
\[
v(\tilde{b}) = -\frac{x + y}{2} (3y^2 + C). \quad \text{(109)}
\]

From \((109)\) it follows that \( \tilde{b} \) is non-constant, such that we may see the second part of \((108)\) as a definition of \( u_\parallel \). \((107)\) is nothing but the energy resp. momentum conservation equations for a perfect fluid subject to \( D_a w = 0 \).

The first part of \((108)\) confirms that \( D_a \tilde{b} = 0 \) \((23)\), whilst the second implies again that the expansion scalar does not vanish in general (cf. the end of §2.1 and below).

For the second purpose we rely on the hypersurface-orthogonality of \( u^a \) by assumption: zero-weighted real scalar fields \( t \) and \( I \) exist such that
\[
d_a t = I u_a. \quad \text{(110)}
\]

The integrability condition hereof is
\[
D_a I = -I \dot{u}_a = -I (u_\parallel^a + \dot{u}_\parallel v_a), \quad \text{(111)}
\]
which is equivalent to
\[
\partial I = \tau I, \quad v(I) = -\dot{u}_\parallel I. \quad \text{(112)}
\]

From \( \tilde{b} \neq 0 \), \((103)\) and \((110)\) it follows that \( t \) is functionally independent of \( y \) (and of \( x \), \( \phi \)) and we take it as the fourth coordinate. With the aid of \((110)-(111), \(106)\) and the first parts of \((107)\) and \((108)\) precisely tell that \( A \equiv \tau_\parallel \), \( w \) and \( b \) only depend on \( t \). Hence \( \tilde{b} = \tilde{b}(y, t) \) from \((105)\). On using \((102)-(103)\), the first part of \((112)\) is equivalent to \( J = J(y, t) \), where \( J \equiv \frac{\tilde{b}}{y} \). Eliminating \( \tilde{u}_\parallel \) between the second parts of \((108)\) and \((112)\), and using \( v(x + y) = -\tilde{b}(x + y) \) implied by \((104)-(105)\), yields
\[
\tilde{b}^2 v(J) = \tilde{b} J u(b) + A(x + y). \quad \text{(113)}
\]
Inverting \((103)\) and \((110)\) we get
\[
(x + y) u_a = b J d_a t, \quad (x + y) v_a = b J d_a t - \frac{d_a y}{b}, \quad \text{(114)}
\]
or dually
\[
-\frac{u^a}{x + y} = \frac{\partial^a}{b J} + b \partial_y a, \quad -\frac{v^a}{x + y} = \tilde{b} \partial_y a. \quad \text{(115)}
\]
Thus in the chosen coordinates \((113)\) reads
\[
\partial_y J(y, t) = \tilde{b}(y, t)^{-3} [b'(t) - A(t)]. \quad \text{(116)}
\]

From \((12)-(23)\), \( g_{ab} = (c_v v_b - u_a u_b)/(x + y)^2 \) and the only remaining equation \((107)\) we obtain the line element
\[
ds^2 = (x + y)^{-2} [(\dot{x}_a^2 + \dot{y}_b^2)], \quad \text{(117)}
\]
\[
ds^2_\parallel = \frac{dx^2}{2\tau} + 2\tau d\phi^2, \quad 2\tau = x^3 + Cx + D, \quad \text{(118)}
\]
\[
ds^2_\perp = \left( b J dt - \frac{dy}{b} \right)^2 - \left( \tilde{b} J dt \right)^2, \quad \text{(119)}
\]
where
\[
b = b(t), \quad w = w(t), \quad A \equiv \frac{\tilde{b} J S}{2(x + y)} = A(t), \quad \text{(120)}
\]
\[
w'(t) = 6b(t) A(t) \leftrightarrow d_a w = 6b A d_a t = 3b S u_a, \quad \text{(121)}
\]
\[
\tilde{b} = \tilde{b}(y, t) = \sqrt{y^3 + Cy - D + b(t)^2 - w(t)/3}, \quad \text{(122)}
\]
\[
J = J(y, t) = [b'(t) - A(t)] \int \frac{dy}{b(y, t)^2} + L(t), \quad \text{(123)}
\]
with \( L(t) \) a free function of integration. The solutions are defined and regular in the coordinate regions
\[
2\tau \equiv x^3 + Cx + D > 0, \quad \text{(124)}
\]
\[
\tilde{b}(y, t)^2 \equiv y^3 + Cy - D + b(t)^2 - w(t)/3 > 0. \quad \text{(125)}
\]

Notice that we nowhere used \( S \neq 0 \) explicitly in the above integration procedure. Therefore, the above line element describes the complete class ID, including the \( C^-\)-vacuum limits which correspond to \( w(t) = A \) and \( A(t) = 0 \), cf. \((21)\). In this case the coordinate transformation \( (t, y, x, \phi) \mapsto (\psi, y, x, \phi) \), which connects \((117)-(123)\) to the original form \((89)\), eliminates \( b(t) \) and \( L(t) \) and follows from \((4), (72), (85), (100)\) and \((114)\), giving
\[
d_a \psi = -\frac{x + y}{2\mu \rho} (\mu k_a + \rho a) = \frac{x + y}{2\mu \rho} \left( \tilde{b} u_a - b v_a \right) = J d_a t + \frac{b}{\tilde{b}} \frac{d_a y}{b(\tilde{b}^2 - b^2)}. \quad \text{(126)}
\]

Hence, \( \psi = \psi(y, t) \) and it is the solution of the consistent system
\[
\partial_y \psi = J, \quad \partial_y \psi = \frac{b}{\tilde{b}^2 - b^2}, \quad \text{(127)}
\]
the integrability condition hereof being precisely \((116)\) with \( A(t) = 0 \). The transformation is singular at degenerate roots of \( \tilde{b}^2 \) and at the union of the black hole and acceleration horizons \([15, 17] \tilde{b}^2 - b^2 = f(-y) + A/3 = 0 \), which separate the static from the non-static regions. Let us emphasize that the \( b(t) \)-freedom is essentially a freedom in the choice of coordinates. The form \((89)\) describes the full C-metric manifold; \( y \) can take any value, and the sign of \( f(-y) + \frac{A}{3} \) is positive in the static region and negative in the non-static region. In the form \((117)-(119)\) \( y \) is always spacelike and we have constructed \( t \) as a synchronized timelike coordinate corresponding to
an US $u^a$, with associated expansion rate $\theta(t) = 3b(t)$; for fixed $b(t)$ the range of $y$ is constrained by (123) and only this subregion of the manifold is described by the coordinates. E.g. (117)-(123) with $A(t) = 0, \psi(t) = \Lambda/3, b(t) = 0$ and $L(t) = 1$ (which formally reduces to (59) on putting $t = \psi$ additionally) only describes the static part of the C-metric, the vector field $u^a$ then lying along the unique HO timelike KVF direction. However, in the neighborhood of any point with coordinate label $y$, the metric can be described by (117)-(123), by choosing $b(t)^2 > f(-y) + \Lambda/3$.

We neither used $\tau \neq 0$. This implies that the line element of the complete class IB, characterized by $\mu \rho \neq 0 = \tau$ and constituted by all LRS II Einstein spaces and shear-free perfect fluids with $D_\rho \psi = 0$, is described by (117)-(123), with (118) replaced by (60). This class was first described by Kustaanheimo [18] and rediscovered by Barnes [1] both using different coordinates (see also (16.49), (16.51) in [16]).

Of course, the result (117)-(123) could have been obtained without referring to GHP calculus. Barnes [1] showed that the metric can be written in the form
\[
d s^2 = (x + Y)^2 \left( f^{-1} dx^2 + f d\phi^2 + dz^2 - e^{-2Z} dt^2 \right),
\]
with $f = f(x)$. Indeed, from (114), (116) and (122) it follows that $(x + y)v_\alpha$ is exact: $(x + y)v_\alpha = \delta_{\alpha z}; z$ is used as a coordinate instead of $y$, and one puts $\tilde{J}\tilde{b} \equiv e^z, Z = Z(y, t)$. Notice from (130) that now
\[
y = Y(z, t), \quad z = -\tilde{b}, \quad \theta = 3Y_y e^{-Z}.
\]
Let us directly attack the field equations in these coordinates, thereby correcting [1]. One can check that only four of the field equations are not identically satisfied (the indices 1 to 4 label the Weyl principal tetrad vectors naturally associated with (128)):
\[
G_{34} = 0 = -Y_{,zz} + Y_z Z_{,z},
\]
\[
G_{11} = G_{33} = 0 =
\]
\[
f'' - 2Y_{,zz} \left( x + Y \right) \left( Z_z^2 + Z_{zz} - f''/2 \right),
\]
\[
G_{11} = p =
\]
\[
2 \left( e^{-2Z} (Y_{tt} - Y_{zt}) - Y_t Z_{,z} - f''/2 \right) (x + Y)
\]
\[
+ \left( Z_{,zz} + Z_{zz}^2 \right) (x + Y)^2 + 3 \left( Y_z^2 + f - Y_z^2 e^{-2Z} \right),
\]
\[
G_{33} + G_{44} = S =
\]
\[
2 \left( x + Y \right) \left( Y_{zz} - Y_z Z_{,z} + e^{-2Z} (Y_{tt} - Y_{zt}) \right).
\]
Hence, if supplemented with $\theta \sim Y_t = 0$, these equations are the ones obtained in Barnes [1]; equation (130) $\equiv v(\theta) = 0$ was missed out, and both equations (132) and (133) differ from equations (4.23), resp. (4.24) in [1] by a term $2 (x + Y) Y_{tt} e^{-2Z}$. Thus, it is clear that with these differences a correct non-expanding solution can be found, but the analysis of expanding solutions will be incorrect.

Differentiating (132) twice wrt $x$ yields $d^4 f(x)/dx^4 = 0$, whence
\[
f(x) = ax^3 + bx^2 + cx + d.
\]
Substituting this in equation (132), and equating coefficients of powers of $x$, leads to
\[
Z_{,zz}(z, t) + Z_{,zz}(z, t)^2 = 3a Y(z, t) - b, \quad (135)
\]
\[
Y_{,zz}(z, t) = \frac{c}{2} - Y(z, t)b + \frac{3}{2}Y_{,zz}(z, t)^2. \quad (136)
\]
The solutions $Y(z, t)$ of the last equation are defined by
\[
\int^{Y(z, t)} \frac{dY}{\sqrt{ar^2 - br^2 + cr + f_1(t)}} - z + f_2(t) = 0, \quad (137)
\]
which can be solved for $z$ in terms of $Y$. This eventually suggests to transform coordinates from $(z, t)$ into $(y, t)$, with $y = Y(z, t)$. Rescaling and translating coordinates allows us to set $a = 1$ and $b = 0$. One can check that the remaining equations lead exactly to equations (116) and (121), recovering solution (117)-(123).

### 3.2. Properties

Consider the metric (117)-(123), for which we assume henceforth that it describes a perfect fluid ($A(t) \neq 0$). In contrast to the Einstein subcase, $u^a$ is now the unique invariantly-defined fluid velocity, and the expansion rate $\theta(t)$ and energy density $w(t) - \Lambda$ of the fluid are scalar invariants. Expressions for the pressure $p + \Lambda$ and the components $\dot{u}^{(a)}$ and $\ddot{u}^{(a)}$ of the acceleration follow from (102), (108)-(109), (115) and (120):
\[
p = \frac{2(x + y)A(t)}{(bJ)(y, t)} - w(t), \quad \dot{u}^{(a)} = -(x + y)f(x)\partial_x a,
\]
\[
\ddot{u} = \ddot{b}(y, t) - \frac{x + y}{b(y, t)} \left( 3y^2 + C \right) + \frac{b'(t) - A(t)}{(bJ)(y, t)}.
\]
The fluid is non-shearing and non-rotating, i.e. $u^a$ is an US. Because of (13)-(14) criterion 5 of proposition [3,1] is satisfied, such that there is a one-degree freedom of USs. These can be found by taking $q = 1$ in (4) and (9), hereby fixing the $(k^a, l^a)$-gauge geometrically, and solving (10) with $q$ replaced by $Q$, the USs then being $(Qk^a + l^a)/\sqrt{2Q}$. This yields $Q = Q(y, t)$ and
\[
\ddot{b}J[\ddot{b}(Q + 1) + b(Q - 1)]Q_y + (Q - 1)Q_t
\]
\[
= -2Q(Q - 1)\ddot{b}(bJ)_y
\]
\[
= \frac{Q(Q - 1)}{b} \left[ (3y^2 + C)\dot{b}J + 2(b' - A) \right].
\]
If the class is to be used as a cosmological model, it is interesting to discuss the intrinsic freedom. By (121) and (124) we have that $2A(t) \partial_4 t = Sl_{\alpha \beta} \text{and}$
\[
J(y, t) \partial_{4a} t = (x + y)u_a/b \text{are invariantly-defined one-forms}, \text{and hence so is } L(t) \partial_4 t \text{because of (123). It follows that}
\]
\[
\frac{1}{V(t)} \text{is a scalar invariant. Moreover, as } A(t) \partial_4 t \text{is exact we may remove the only remaining coordinate freedom on } t \text{by putting } A(t) = 1, \text{such that the conservation}
\]
of energy equation \( l_{21} \) can be considered as a definition \( \theta(t) = w'(t)/2 \). Hence, in this most general picture for \( S \neq 0 \), the scalar constants \( C, D \) and invariants \( l_{21}(t), w(t) \) characterize the model within the class. Notice that the presence of two invariantly-defined, distinguishing free functions could have been predicted, since after elimination of \( u_\parallel \), there are two scalar invariants \( u(b) \) and \( u(S) \) remaining unprescribed in the system of equations \( 116, 119 \).

In this fashion however, the physical implications remain obscure: it would be nice to have a free function, with a clear physical interpretation, instead of \( L/A \). Spacetimes with \( L(t) = 0 \) have \( w(t) \) as the only free function. If \( L(t) \neq 0 \), \( L(t) \) can alternatively be fixed to 1 by a t-coordinate transformation. In this case the metric structure functions display the expansion scalar, the energy density and the pressure (since \( 2A(t)d_A t = (w + p)u_a \); these are related by energy conservation \( 121 \)), where \( w(t) \) and \( A(t) \) can be chosen freely. Alternatively, one can subdivide further in \( \theta = 0 \) and \( \theta \neq 0 \). In the case \( \theta = 0 \), the energy density \( w - \Lambda \) is constant because of \( 121 \) and can be chosen freely, just as \( A(t) \). In the most interesting case \( \theta \neq 0 \), \( w(t) \) and \( \theta(t) \) can be chosen freely, determining \( S, u_a \) via \( 121 \).

Thus class ID provides a class of anisotropic cosmological models with arbitrary evolution of matter density and (non-zero) expansion.

Regarding all perfect fluid ID models admit at least one KVF \( \partial^a \phi \) given by \( 74 \), which at each point yields an invariantly-defined spacelike vector orthogonal to \( x^+ \) and lying in \( \Sigma^+ \). If \( \phi \) is chosen to be a periodic coordinate, with range given by \( [-\pi E, \pi E] \), the spacetime is cyclically symmetric. We will then refer to the region \( F \equiv f(x) = 0 \), where the norm of \( \partial^a \phi \) vanishes, as the axis of symmetry \( 17, 19 \). Finding the complete group of isometries and their nature is trivial in our approach. The functions \( x, y, w \) and \( L/A \) are invariant scalars, such that \( K^a d_A x = K^a d_A y = K^a d_A w = K^a d_A \frac{\dot{\phi}}{\phi} = 0 \) for any KVF \( K^a \). As the ID models are anisotropic, it follows that the complete isometry group is at most \( G_2 \), and if it is \( G_2 \), both \( w \) and \( L/A \) are constant. Conversely, when \( w \) and \( L/A \) are constant we have \( \theta \equiv 3b = 0 \) from \( 121 \), \( \bar{b} = \bar{b}(y) \) from \( 122 \) and \( J(y, t) = -A(t)F_2(y) \) from \( 123 \). By defining the time coordinate such that \( A(t) = 1 \) one sees from \( 117-123 \) that \( \partial^a \phi \) is a HO timelike KVF. We conclude that the ID perfect fluid models have at least one spacelike KVF \( \partial^a \phi \), which may be interpreted as the generator of cyclic symmetry. They admit a second independent KVF if and only if both scalar invariants \( w \) and \( L/A \) are constant, in which case the spacetimes are static and the complete group of isometries is abelian \( G_2 \), generated by \( \partial^a \phi \) and \( \partial^a \phi \).

Consider the case where \( f(x) \) has 3 real non-degenerate roots \( x_1, x_2 \) i.e. \( 72 \) holds. If \( x_1 < x_2 < x_3 \) then \( f(x) > 0 \) for all \( x \in [x_1, x_2] \). Furthermore, we let \( \phi \) be a periodic coordinate. The ratio between circumference and radius of a small circle around the axis, \( x = x_1 \) or \( x = x_2 \), is given by

\[
\lim_{x \to \frac{x_1}{x_2}} \frac{2\pi E \sqrt{f(x)}}{\sqrt{f^{-1}(x)} dx} = -\pi E (3x_{1/2}^2 + C) \quad (138)
\]

respectively

\[
\lim_{x \to \frac{x_1}{x_2}} \frac{2\pi E \sqrt{f(x)}}{\sqrt{f^{-1}(x)} dx} = \pi E (3x_1^2 + C). \quad (139)
\]

It is only possible to choose the parameter \( E \) such that the complete axis is regular, if \( 3x_1^2 + C = -(3x_2^2 + C) \). However, eliminating \( C \) and \( D \) between this equation and \( f(x_1) = f(x_2) = 0 \) implies \( x_1 = x_2 \). Consequently, if \( f(x) \) has three real non-degenerate roots, the spacetime contains a conical singularity. This echoes the properties of the C-metric \( 4, 15 \), and suggests the presence of a cosmic string.

4. CONCLUSIONS AND DISCUSSION

A new class of Petrov type D exact solutions of Einstein’s field equation in a perfect fluid with spatially homogeneous energy density has been presented. It consists of all anisotropic such fluids with shearfree normal four-velocity, and generalizes a previously found class to include non-zero expansion. The analysis and integration was rooted in the 2+2 structure of the metric and use of invariant quantities. This approach clarified the link with the vacuum C-metric limit, and certain properties of the vacuum case are inherited. However, the presence of the perfect fluid defines generically two extra invariants. For the expanding solutions, this translates into an evolution of energy density and expansion which can be chosen freely. This subclass contains only one (potentially cyclic) symmetry.

The viability of these solutions as a low-symmetry class of cosmological models is subject to further research. More in particular, it should be clarified whether a thermodynamic interpretation of the perfect fluid can be made - it is certainly not possible to prescribe a barotropic equation of state \( p = p(w) \). The relation with the C-metric also suggests to further examine the arising coordinate ranges, properties of horizons, and whether an interpretation as a perturbation for small masses of a known PF solution exists for certain members.

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Appendix A: Geroch-Held-Penrose (GHP) formalism

The GHP formalism[16,27] is a complex, scalar formalism, which is a ‘weighted’ version of the Newman-Penrose (NP) tetrad formalism. Use is made of a complex null tetrad \( (e^a, e^a, e^a, e^a) \equiv (m^a, \overline{m}^a, l^a, k^a) \), where

\[ k^a l_a = -1, \quad m^a \overline{m}_a = 1 \]  

(A1)

and all other inner products vanish. To put it in other words, at each point one takes a timelike plane, two vectors \( k^a \) and \( l^a \) lying along its real null directions, and two vectors \( m^a \) and \( \overline{m}^a \) lying along the complex conjugate null directions of the orthogonal spacelike plane, these pairs of vectors satisfying the normalization conditions \( (A1) \). We use the labels \( \hat{a}, \hat{b} \) etc. for the tetrad indices. The basic variables of the formalism are the spin coefficients \( (\Gamma_{\hat{a}\hat{b}c} \equiv e_{\hat{a}}^a \nabla_c (e_{\hat{b}})_a e_{\hat{c}}^c = -\Gamma_{\hat{b}\hat{a}c}^c) \)

\[ \kappa = \Gamma_{\hat{a}\hat{b}\hat{c}}, \quad \tau = \Gamma_{\hat{a}\hat{b}1}, \quad \sigma = \Gamma_{\hat{a}\hat{b}11}, \quad \rho = \Gamma_{\hat{a}\hat{b}12}, \quad \nu = \Gamma_{\hat{a}\hat{b}33}, \quad \pi = \Gamma_{\hat{a}\hat{b}34}, \quad \lambda = \Gamma_{\hat{a}\hat{b}35}, \quad \mu = \Gamma_{\hat{a}\hat{b}36} \]  

(A2)

the 9 independent components of the traceless part of the Ricci tensor \( S_{ab} = R_{ab} - \frac{1}{4} R g_{ab} \),

\[ \Phi_{\hat{a}\hat{b}} = \frac{1}{2} S_{\hat{a}\hat{b}} k^\alpha k^\beta, \quad \Phi_{\hat{a}\hat{b}\hat{c}} = \frac{1}{2} S_{\hat{a}\hat{b}\hat{c}} m^\alpha m^\beta, \quad \Phi_{\hat{a}\hat{b}\hat{c}\hat{d}} = \frac{1}{2} S_{\hat{a}\hat{b}\hat{c}\hat{d}} (m^\alpha l^\beta + m^\beta l^\alpha) \]  

(A4)

with \( \Phi_{\hat{a}\hat{b}} = \Phi_{\hat{b}\hat{a}} \), the multiple

\[ \Pi = \frac{R}{24} \]  

(A7)

of the Ricci scalar, and the 10 independent components of the Weyl tensor \( C_{\hat{a}\hat{b}\hat{c}\hat{d}} \),

\[ \Psi_0 = C_{\hat{a}\hat{b}\hat{c}\hat{d}} m^\alpha m^\beta k^\gamma k^\delta, \quad \Psi_4 = C_{\hat{a}\hat{b}\hat{c}\hat{d}} m^\alpha m^\beta \overline{m}^\gamma \overline{m}^\delta, \quad \Psi_1 = C_{\hat{a}\hat{b}\hat{c}\hat{d}} k^\alpha k^\beta m^\gamma m^\delta, \quad \Psi_3 = C_{\hat{a}\hat{b}\hat{c}\hat{d}} k^\alpha k^\beta \overline{m}^\gamma \overline{m}^\delta, \quad \Psi_2 = C_{\hat{a}\hat{b}\hat{c}\hat{d}} \overline{m}^\alpha \overline{m}^\beta m^\gamma m^\delta. \]  

(A8)

Changes of the tetrad leaving the null directions spanned by \( k^a, l^a, m^a \) and \( \overline{m}^a \) invariant, and at the same time preserving the normalization conditions \( (A1) \), consist of boosts

\[ k^a \rightarrow A k^a, \quad l^a \rightarrow A^{-1} l^a \]  

(11)

and spatial rotations

\[ m^a \rightarrow e^{i\theta} m^a. \]  

(A12)

Quantities transforming under \( (A11)-(A12) \) as

\[ \eta \rightarrow A^{-\frac{w_p + w_q}{2}} e^{\frac{w_p - w_q}{2} \theta} \eta \]  

(A13)

are called well-weighted of type \( (w_p, w_q) \) or \( (w_p, w_q) \)-weighted (zero-weighted in the case of type \( (0, 0) \)). They have boost-weight \( w_B(\eta) = \frac{w_p + w_q}{2} \) and spin-weight \( w_S(\eta) = \frac{w_p - w_q}{2} \). One can check that the GHP basic variables are well-weighted, their weights following from the definitions \( (A2) \),\( (A10) \) and \( (A11)-(A13) \) - see also equation \( (7.36) \) in [16]. E.g. \( w_B(\nu) = -2 \), \( w_S(\nu) = -1 \), implying \( \nu \) is of type \( (-3,-1) \). The following derivative operators are defined such that a well-weighted quantity \( \eta \) is transformed in a well-weighted quantity:

\[ D_a \eta = e_a(\eta) + \Gamma_{\hat{a}\hat{b}c} w_B(\eta) \eta + \Gamma_{\hat{a}12c} w_S(\eta) \eta \]  

(A14)

When \( \eta \) is of type \( (w_p, w_q) \) one can check that

\[ w_B(D_a \eta) = w_B(\eta) + \tilde{w}_B(\hat{a}), \quad w_S(D_A \eta) = w_S(\eta) + \tilde{w}_S(\hat{a}), \]  

where

\[ \tilde{w}_B(\hat{a}) = \begin{cases} 1, & \hat{a} = 4, \\ -1, & \hat{a} = 3, \\ 0, & \hat{a} = 1,2 \end{cases}, \quad \tilde{w}_S(\hat{a}) = \begin{cases} 1, & \hat{a} = 1 \\ -1, & \hat{a} = 2 \\ 0, & \hat{a} = 3,4 \end{cases} \]

One uses the notation

\[ \partial \equiv D_1, \quad \partial' \equiv D_2, \quad \partial'' \equiv D_3, \quad \partial'' \equiv D_4. \]  

(A15)

Notice that the differential of zero-weighted scalars \( f \) can be expressed as

\[ df = -\partial' f \partial l_a - \partial l_a + \partial' f \partial m_a + \partial f \partial \overline{m}_a \]

\[ = -\partial(f) k_a - k(f) l_a + \overline{m}(f) m_a + m(f) \overline{m}_a \]

\[ = -u(f) u_a + v(f) v_a + \overline{m}(f) m_a + m(f) \overline{m}_a \]  

(A17)

where \( u^a \) and \( v^a \) are related to \( k^a \) and \( l^a \) according to \( (A1) \), resp. \( (A3) \).

The basic (or ’structure’) equations of the GHP formalism are

(a) the commutator relations of the weighted derivatives, in the joint \( D_a \) notation given by

\[ [D_a, D_b] \eta = 2\Gamma_{\hat{a}\hat{b}}^\gamma D_\gamma \eta + w_B(\eta) \left( R_{3\hat{a}\hat{b}4} + 2\Gamma_3^\gamma [\hat{a}] \Gamma_4^\gamma [\hat{b}] \right) \eta \]

\[ + w_S(\eta) \left( R_{1\hat{a}\hat{b}2} + 2\Gamma_1^\gamma [\hat{a}] \Gamma_2^\gamma [\hat{b}] \right) \eta + \tilde{w}_B(\hat{a}) \left( R_{1\hat{a}2} \right) \eta - \tilde{w}_S(\hat{a}) \left( R_{2\hat{a}1} \right) \eta \]  

(b) 12 complex Ricci identities (or ’equations’), namely

\[ e_{\hat{a}}(\Gamma_{\hat{a}\hat{b}}) - e_{\hat{a}}(\Gamma_{\hat{a}\hat{b}}) = R_{\hat{a}\hat{b}\hat{c}\hat{d}} - 2\Gamma_{\hat{a}[\hat{c}] \Gamma_{\hat{d}]}^\hat{b}] - 2\Gamma_{\hat{a}[\hat{b}] \Gamma_{\hat{c}]}^\hat{d}] \]  

with \( [\hat{a}\hat{b}] = [14, 23] \) (the complex conjugates corresponding to \( [\hat{a}\hat{b}] = [24, 13] \));

(c) the Bianchi identities (or ’equations’)

\[ e_{\hat{a}}(R_{[\hat{a}\hat{b}][\hat{c}]} - e_{\hat{a}}(R_{[\hat{a}\hat{b}][\hat{c}]}) = -2R_{\hat{a}[\hat{b}] [\hat{d}] f} + \Gamma_{\hat{b}[\hat{c}] R_{\hat{d}]} f} - \Gamma_{\hat{b}[\hat{c}] R_{\hat{d}]} f} \]  

One can show that, after writing the directional derivatives \( e_{\hat{a}} \) in terms of the weighted derivatives \( D_a \), these basic equations (a)-(c) form a consistent, closed system.
of PDEs in the variables $\{A2\}-\{A10\}$ and with formal derivative operators $D_k \eta$. Compared to the NP formalism, the 6 complex Ricci identities which concern directional derivatives of the non-well-formed NP spin coefficients $\alpha, \beta, \gamma$ and $\epsilon$ (corresponding to $[\alpha\beta] = [12], [34]$) have been absorbed in the commutator relations (1). Explicitly, for a $(w_p, w_q)$-weighted scalar one gets

\[ [\mathcal{P}, \mathcal{P}'](\eta) = (\pi + \tau) \partial(\eta) + (\pi + \tau) \partial'(\eta) + (\kappa \nu - \pi \tau + \Phi_{11} - \Psi_2)w_p \eta + (\kappa \pi - \pi \tau + \Phi_{11} - \Psi_2)w_q \eta, \]  

(A19)

\[ [\mathcal{P}, \partial]\eta) = (\pi - \tau) \partial(\eta) + (\pi - \tau) \partial'(\eta) + (\kappa \sigma - \mu \rho - \Phi_{11} + \Psi_2)w_p \eta - (\kappa \lambda - \Phi_{01})w_q \eta, \]  

(A20)

\[ \mathcal{P} \leftrightarrow \mathcal{P}', \partial \leftrightarrow \partial', \]  

(A21)

together with the equations obtained by applying the complex conjugate and/or prime dual operation to (A21). This prime dual operation is generated by interchanging $k^a \leftrightarrow l^a$ and $m^a \leftrightarrow \bar{m}^a$, which comes down to

\[ \kappa \leftrightarrow -\nu, \tau \leftrightarrow -\pi, \sigma \leftrightarrow -\lambda, \rho \leftrightarrow -\mu, \]  

(A22)

\[ \Phi_{ij} \leftrightarrow \Phi_{-i-2-j}, \Psi_{ij} \leftrightarrow \Psi_{4-i}, \]  

(A23)

\[ \mathcal{P} \leftrightarrow \mathcal{P}', \partial \leftrightarrow \partial', \]  

(A24)

The interchange (A21) means that $(\mathcal{P}\eta)' = \mathcal{P}'\eta'$ etc., and is due to (A13) and

\[ w_B(\eta') = -w_B(\eta), \]  

\[ w_S(\eta') = -w_S(\eta), \]  

\[ i.e. w_p(\eta') = -w_p(\eta), \]  

\[ w_q(\eta') = -w_q(\eta). \]  

Regarding complex conjugation one has $\overline{\mathcal{P}}\eta = \mathcal{P}\eta, \overline{\partial}\eta = \partial\eta$ and

\[ w_B(\overline{\eta}) = w_B(\eta), w_S(\overline{\eta}) = w_S(\eta), \]  

\[ i.e. w_p(\overline{\eta}) = w_q(\eta), \]  

\[ w_q(\overline{\eta}) = w_p(\eta). \]  

Explicitly, the 12 complex Ricci identities read

\[ \partial = -\sigma = \rho = \Phi_{01} + \Psi_1, \]  

(A25)

\[ \mathcal{P}\sigma = -\Phi_{01} - \Psi_1, \]  

(A26)

\[ \mathcal{P}\sigma = -\Phi_{01} - \Psi_1, \]  

(A27)

\[ \mathcal{P}\sigma = -\Phi_{01} - \Psi_1, \]  

(A28)

\[ \mathcal{P}\sigma = -\Phi_{01} - \Psi_1, \]  

(A29)

\[ \mathcal{P}\sigma = -\Phi_{01} - \Psi_1, \]  

(A30)

\[ \]  

and their prime duals $\{A25\}-\{A30\}$. Finally, the Bianchi identities involve weighted derivatives of the Riemann tensor components. In full generality they are given in ref. [16], (7.32a-k), or [32], (4.12-36-41).

The formalism is especially suited for situations where two null directions are singled out by the geometry, such that $k^a$ and $l^a$ can be chosen along them. In particular, the Weyl tensor of a Petrov type $D$ spacetime has precisely two PNDs; choosing $k^a$ and $l^a$ along them is equivalent to condition (11), and a complex null tetrad realizing this condition is called a Weyl principal null tetrad (WPNT). When (7) and (11) are both satisfied, the Bianchi identities reduce to

\[ 0 = \sigma(2\Phi_{11} + 3\Psi_2) - 3\Phi_{00}, \]  

(A31)

\[ \mathcal{P}\Psi + \mathcal{P}\Phi_{00} + 2\mathcal{P}\Pi = \rho(2\Phi_{11} + 3\Psi_2) - 3\Phi_{00}, \]  

(A32)

\[ \mathcal{P}\Phi_{11} + \mathcal{P}\Phi_{00} + 2\mathcal{P}\Pi = 2(\rho + \pi)\Phi_{11} - (\mu + \pi)\Phi_{00}, \]  

(A33)

\[ \mathcal{P}\Phi_{11} + 2\mathcal{P}\Pi = \tau(2\Phi_{11} + 3\Psi_2) + 3\Phi_{00}, \]  

(A34)

\[ \mathcal{P}\Phi_{11} + 3\mathcal{P}\Pi = 2(\tau - \pi)\Phi_{11} - 3\Phi_{00} + \kappa\Phi_{22}, \]  

(A35)

\[ \mathcal{P}\Phi_{00} = \kappa(2\Phi_{11} - 3\Psi_2) - \Phi_{00}, \]  

(A36)

and their prime duals $\{A31\}-\{A36\}$.

In general, the GHP formalism may be used to find a class of solutions, defined by a particular set of properties. One first translates these properties in terms of GHP variables, yielding (algebraic or differential) constraints on the system of basic equations, then recloses the resulting extended system (integrability analysis) and finally describes the corresponding metrics in terms of coordinates (integration). These coordinates are four suitable, functionally independent zero-weighted scalars $f$ they may be combinations of (derivatives of) basic variables, appearing in the reclosed system $\mathcal{S}$ itself, or ‘external’ coordinates associated to HO vector fields due to Frobenius’ theorem. The geometric duals of the null tetrad vectors, and hence the metrics $\nu_{ab} = -2k_{(a}b_{b)} + 2m_{a}m_{b}$, are obtained by inverting (A16) for the chosen $f$’s. Eventually the remaining equations of $\mathcal{S}$ are written in terms of these coordinates and the resulting PDE’s are solved as far as possible. We refer to [51] or for enlightening discussions, and to e.g. [52] or this work for illustrations. In particular for Petrov type $D$ spacetimes, notice that zero-weighted combinations of WPNT spin coefficients and their weighted derivatives (e.g. $\mu\rho$ or $\partial\tau$) are scalar (Lorentz) invariants $x$, which are thus annihilated by any present KV $K^a, \mathcal{K}(x) = \mathcal{L}_K x = 0$. This facilitates the detection of KV’s. More generally, zero-weighted tensor fields $\mathcal{T}_{ab\cdots}$, algebraically constructed from the Riemann tensor, WPNT vectors and covariant derivatives thereof, are invariantly-defined by the geometry, and $\mathcal{L}_K \mathcal{T}_{ab\cdots} = 0$.

Appendix B: (Rigid) shearfree normality and staticity of Petrov type $D$ spacetimes

Consider (an open region of) a spacetime and a unit timelike vector field $u^a$ defined on it. Choose a null vector field $k^a$. At each point, $k^a$ and $u^a$ span a timelike plane $\Sigma$, the first null direction of which is spanned by $k^a$. Construct the null vector field $l^a$ by taking at each point the unique vector lying along the second null direction and satisfying $l^a k_a = -1$. Then $u^a$ is decomposed as in (41), where $q = A^2, A = -\sqrt{k^a u_a}$. The field $u^a$ defined in (45) determines at each point the up to
reflection unique unit spacelike vector lying in $\Sigma$ and orthogonal to $u^a$. The electric and magnetic parts of the Weyl tensor wrt $u^a$ can be decomposed as

$$E_{ab} \equiv C_{abcd}u^au^d = (\Psi_2 + \overline{\Psi}_2)[v_av_b - \overline{\epsilon}(a_m b_b)]$$

$$+ \left[ \frac{\Psi_3 + q\overline{\Psi}_3}{2q} m_a m_b + 2 \frac{\Psi_3 - q\overline{\Psi}_3}{\sqrt{2q}} m(a) b_b + c.c. \right] \quad \text{(B1)}$$

$$H_{ab} \equiv \frac{\eta_{acmn}}{2} \overline{C}^{acmn} b_d u^d = i(\Psi_2 - \overline{\Psi}_2)[v_av_b - \overline{\epsilon}(a_m b_b)]$$

$$+ i \left[ \frac{\Psi_4 - q\overline{\Psi}_4}{2q} m_a m_b + 2 \frac{\Psi_4 + q\overline{\Psi}_4}{\sqrt{2q}} m(a) b_b + c.c. \right] \quad \text{(B2)}$$

If $u^a$ exists such that $H_{ab} = 0$, the Weyl tensor is purely electric (PE) wrt $u^a$, the spacetime itself being also called PE. A criterion in terms of Weyl tensor concomitants, deciding whether this is the case, follows from the flow diagram 9.1 in [18] and theorem 1 in [53].

Suppose now that the spacetime admits a unit timelike vector field $u^a$ satisfying (11) - corresponding to an US, i.e. forming the tangent field of a shearfree and vorticity-free cloud of test particles. Within the GHP formalism based on $k^a$ and $l^a$ as introduced above, this is the case if and only if a (-2,-2)-weighted field $q$ exists satisfying (4)-(9). By virtue of these relations, the $[\overline{\Psi}, \overline{\Psi}](q)$ commutator relation yields (12), adding $2q(\left[\overline{A}^{26} - A^{26}\right])$ to the $[\overline{P}', q\overline{P}](q)$ commutator relation gives $\Psi_4 + q\overline{\Psi}_4 = 0$ and the combination $q\left[\overline{A}^{27} + A^{27}\right] + q\left[\overline{A}^{29} - A^{29}\right]$ produces $\Psi_4 - q\overline{\Psi}_4 = 0$. Hence $H_{ab} = 0$ from (12), and if we choose $k^a$ to be a (multiple) PND, $\Psi_0 = 0$ ($\Psi_3 = 0$), then also $l^a$ is a (multiple) PND, $\Psi_4 = 0$ ($\Psi_3 = 0$). Hence the spacetime must be either conformally flat (all $\Psi_i$ zero), and then USs are always admitted (see e.g. (6.15)) in [18], or purely electric (PE) and of Petrov type D or I, the Weyl tensor being PE wrt $u^a$. For Petrov type I, there are 4 distinct PNDs, and $u^a$ is the up to reflection unique timelike vector lying along the intersection of the planes spanned by two particular pairs of PNDs. For Petrov type D, $k^a$ and $l^a$ can be taken to be the multiple PNDs, and $u^a$ lies in the plane $\Sigma$ spanned by them.

Propositions 4 in [54] and 16 in [55] imply intrinsic, easily testable criteria for deciding when a Petrov type I spacetime admits an US, resp. is static. Here we present likewise criteria in the Petrov type D case. These criteria are invariant statements, in terms of GHP basic variables and weighted derivatives associated to an arbitrary WPNT. Given a Petrov type D spacetime in coordinates, the determination of the PNDs, and hence the WPNTs, is straightforward and can be performed covariantly. It then suffices to fix one WPNT and calculate the appearing spin-boost covariant expressions by using definitions (A2)-(A10) and (A11). For complex $(2k, 2k)$-weighted scalars ($k \in \mathbb{Z}$) $z = \text{Re}(z) + i\text{Im}(z)$ we mean with $z > 0$ ($z < 0$) that $z$ is real and strict positive (negative) in the sequel.

It turns out that, given (11)-(12), the integrability conditions of (5) are identically satisfied. Thus we find:

**Proposition B.1** A Petrov type D spacetime admits an US if and only if, wrt an arbitrary WPNT, $\Psi_4$ is real and one of the following sets of conditions holds:

1. $\sigma \neq 0$, the scalar invariant $\lambda > 0$, and $q_0 \equiv \lambda/c$ satisfies (5)-(6);
2. $\rho \neq 0$, the real scalar invariant $(\mu - \overline{\mu})(\rho - \overline{\rho}) > 0$, and $q_0 \equiv -(\mu - \overline{\mu})(\rho - \overline{\rho})$ satisfies (5)-(6);
3. $\lambda = \sigma = \mu - \overline{\mu} = \rho - \overline{\rho} = 0$, the scalar invariant $\kappa \nu \neq 0$ and one of the following situations occurs, where $q_0$ defined in each subcase satisfies (6) and where $b \equiv (\pi + \overline{\nu})/\kappa$, $c \equiv \nu/\kappa$:
   3a. Im$(b)$Im$(c) > 0$ and $q_0 \equiv \text{Im}(c)/\text{Im}(b)$ also satisfies $q_0^2 - \text{Re}(b)q_0 + \text{Re}(c) = 0$;
   3b. $b = \text{Re}(b)$, $c < 0$ and $q_0 \equiv (b + \sqrt{b^2 - 4c})/2$;
   3c. $b > 0$, $c > 0$, $b^2 \geq 4c$, and $q_0 \equiv (b + \sqrt{b^2 - 4c})/2$;
4. $\lambda = \sigma = \mu - \overline{\mu} = \rho - \overline{\rho} = 0$, and either
   4.1 $\kappa = 0 \neq \nu$, $(\pi + \overline{\nu}) > 0$, and $q_0 = \nu/(\pi + \overline{\nu})$ satisfies (6), or
   4.2 $\kappa = 0 \neq \nu$, $(\pi + \overline{\nu}) \kappa > 0$ and $q_0 = (\pi + \overline{\nu})/\kappa$ satisfies (6); and
5. the WPNT directions are HO, i.e., (12)-(14) holds.

The subdivision of case 3 stems from a straightforward analysis of the second equation of (5). In cases 1, 2, 3a, 3b and 4 there is a unique US, whereas there may be one or two USs in case 3c. Due to the number and nature of the equations (5) there is a one-degree freedom of USs in case 5, where the condition $\Psi_2 = 0$ can be dropped since it is implied by the imaginary part of (A26) + (A30) and (13)-(14). Important examples of spacetimes satisfying criterion 5 are the Petrov type D purely electric Einstein spaces and their 'electrovac' generalizations (see [38, 56] and [2, 23]) and all spacetimes with (pseudo-)spherical or planar symmetry (which constitute the LRS II Lorentzian spaces, see [55]). These examples all satisfy (6) on top of (12)-(14) and are further characterized by $\Phi_{00} = $ $\Phi_{22} = (\Phi_{11} = 0$, resp. $\pi = \sigma = 0 R = 0$ (cf. [24]).

The spacetime will admit a unit timelike vector field $u^a$ satisfying

$$u_{a,b} = -\overline{u}_a u_b,$$  \hspace{1cm}  (B3)

corresponding to a rigid US or modeling a rigid non-rotating cloud of test particles, if and only if a (-2,-2)-weighted field $q$ exists satisfying (5)-(6) and (34). Notice that, given (34), the third equation of (5) is identically satisfied. Hence we have

**Proposition B.2** A Petrov type D spacetime admits a rigid US if and only if, wrt an arbitrary WPNT, $\Psi_4$ is real and one of the following sets of conditions holds:
1'. condition 1 with the third equation of (5) replaced by (34).

2'. the scalar invariant \( \mu \rho > 0 \) and \( q_0 \equiv \mu / \rho \) satisfies (35)-(36).

3'-5'. conditions 3-5 with \( \mu - \pi = \rho - \pi = 0 \) replaced by \( \mu = \rho = 0 \).

In case 5', the spacetime possesses geodesic, shear-free and non-diverging PNDs (\( \kappa = \sigma = \rho = 0 \), \( \nu = \lambda = \mu = 0 \)) - thus belonging to Kundt’s class - and HO Weyl principal complex null directions (\( \lambda = \sigma = \pi + \tau = 0 \)), and admits a one-degree freedom of rigid USs.

The spacetime is static if and only if it admits a HO timelike KVF. An equivalent characterization was given by Ehlers and Kundt [5]: the spacetime is static if and only if a unit timelike vector field \( u^a \) exists for which shear, vorticity and expansion scalar vanish, i.e. (B5) holds, and for which the acceleration \( \ddot{u}^a \) is Fermi-propagated along the integral curves of \( u^a \):

\[
\ddot{u}_a u_b = 0. \tag{B4}
\]

The field \( u^a \) is then parallel to a (HO and timelike) KVF and identified with a congruence of static observers. By a extra condition (B4) is equivalent to (B6) reduces to (A28)\(^{'}\) yield \( \Phi_{12} + q \Phi_{01} = 0 \), (B7), the \( \{[\mathcal{P}, \mathcal{P}], [\mathcal{P}, \mathcal{O}] \} \) commutators applied to \( q \) yield

\[
\mathcal{P} \mathcal{P} q = -q \mathcal{P} \mathcal{P} q + (\mathcal{P} q)^2, \tag{B9}
\]

\[
\mathcal{O} \mathcal{P} q = \tau \mathcal{P} q, \quad \mathcal{O} \mathcal{P} q = \tau \mathcal{P} q. \tag{B10}
\]

The compatibility requirement of (B8)-(B10) with the commutator relations for \( \mathcal{P} q \) gives the single condition

\[
\mathcal{P} R + \mathcal{P} R = 0. \tag{B11}
\]

According to the Sach’s star dual [27] of the LRS criterion in [31], the subcase \( \mathcal{P} R \mathcal{P} R = 0 \) of (B11) precisely corresponds to a boost-isotropic spacetime with \( \pi + \tau = 0 \). From the above we conclude:

**Proposition B.3** A Petrov type D spacetime is static if and only if, wrt an arbitrary WPNT, one of the following sets of conditions holds:

1"-4". \( \Psi_2 \) is real, conditions 1'-4' hold and \( q_0 \) additionally satisfies (B5)- (B6).

5'a. condition 5' holds, the scalar invariant \( \Phi_{01} \Phi_{21} < 0 \) and \( q_0 \equiv -\Phi_{12}/\Phi_{01} \) satisfies (6) and (B8);

5'b. condition 5' holds, \( \Phi_{01} = \Phi_{21} = 0 \), the scalar invariant \( \Phi_{01} \mathcal{P} R \mathcal{P} R < 0 \) and \( q_0 \equiv -\Phi_{12}/\Phi_{01} \) satisfies (6) and (B5);

6'. the spacetime is (locally) boost-isotropic and \( \pi + \tau = 0 \).

The HO timelike KVF directions are parametrized by two constants in case 6' [60], are 1 or 2 in number in case 3'c and are unique in all other cases.

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[49] More precisely equations (48)-(53) of ref. [33]. There are some typos in these equations: the integers 3 and 9 should be omitted in (50) and (51)-(52), respectively, while there should be $9k$ instead of $k$ in (53).

[50] If one of the $p_i$'s is zero then so is a second one; this reproduces Minkowski.

[51] Strictly speaking, with the terminology of [16], the spacetime has in general only a cyclic symmetry, as the axis will be shown to be irregular, and consequently not part of the spacetime.

[52] The two constants result from the integrations of (B8)-(B10), giving $\beta_1q$, and consecutively of (6).