Measuring quantum state overlaps of traveling optical fields

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Abstract

We propose a detection scheme for measuring the overlap of the quantum state of a weakly excited traveling-field mode with a desired reference quantum state, by successive mixing the signal mode with modes prepared in coherent states and performing photon-number measurements in an array of beam splitters. To illustrate the scheme, we discuss the measurement of the quantum phase and the detection of Schrödinger-cat-like states.
1 Introduction

The problem of measuring the overlap of an unknown signal-field quantum state with a desired quantum state has been of increasing interest in quantum state measurement. Typical setups that have been considered are photon chopping and related photodetection schemes for measuring overlaps of the signal-field quantum state with Fock states [1, 2], heterodyning for measuring overlaps of the signal-field quantum state with coherent states [3], and balanced and unbalanced homodyning, respectively, for measuring overlaps of the signal-field quantum state with field-strength states [4] and displaced Fock states [5]. It is therefore natural to ask the question of whether or not a universal setup can be constructed which can measure, in principle, arbitrary quantum mechanical overlaps.

Surprisingly the answer is yes. Extending earlier work on quantum state engineering [6], in this paper we show that arbitrary overlaps may be measured by feeding coherent states into a beam splitter array and performing zero and one photon measurements. Although in praxis the method cannot replace standard schemes such as homodyning for measuring field-strength distributions, it is noteworthy that it offers novel possibilities of measuring quantities whose direct detection has not been realized so far, particularly in a quantum regime. A well-known example is the problem of direct measurement of the quantum phase, where the overlaps of the signal quantum state with London phase states must be recorded.

The paper is organized as follows. In Section 2 the underlying formalism is outlined and the basic formulas are given. Section 3 shows how an arbitrary overlap measurement may be performed and discusses its efficiency. To illustrate the scheme, in Sections 4 and 5 the measurement of the quantum phase statistics and the statistics of Schrödinger-cat-like states are discussed respectively. A summary and some concluding remarks are given in Section 6.

2 Conditional quantum state transformation

Let us consider the setup outlined in Fig. 1. A signal-field mode prepared in a quantum state $\hat{\rho}_m$ passes an array of $N$ beam splitters at which it is mixed with modes prepared in coherent states $|\alpha_1\rangle, \ldots, |\alpha_N\rangle$, and photodetectors $D_1, \ldots, D_{N+1}$ measure photon numbers (clicks) of the outcoupled modes. For the sake of simplicity, we assume that all beam splitters have the same transmittance $T$ and reflectance $R$. Further, we restrict our attention to perfect photodetection.

We derive the transformation of the input quantum state step by step and begin with the first beam splitter and the associated photon-number measurement. When the signal mode prepared in a quantum state $\hat{\rho}_m$ and a mode prepared in a coherent state $|\alpha_1\rangle$ are combined and the detector $D_1$ registers a photon, then
\( \hat{\rho}_{\text{in}} \) is transformed according to
\[
\hat{\rho}_{\text{out1}} = \frac{\hat{Y}_1 \hat{\rho}_{\text{in}} \hat{Y}_1^\dagger}{p(1, 1)},
\]
where
\[
p(1, 1) = \text{Tr}(\hat{Y}_1 \hat{\rho}_{\text{in}} \hat{Y}_1^\dagger)
\]
is the probability of registering a photon, and
\[
\hat{Y}_i = -R^* \hat{D}(\alpha_i) T^* \hat{a} \hat{D} \left( -\frac{T^*}{R^*} \alpha_i \right)
\]
[\( \hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) \)] is the nonunitary transformation operator for the conditional quantum-state transmission through the \( i \)th beam splitter [7]. The output mode prepared in the quantum state \( \hat{\rho}_{\text{out}} \) now enters the second beam splitter and is mixed with a mode prepared in a coherent state \( |\alpha_2\rangle \). If the detector \( D_2 \) associated with the second beam splitter also registers a photon, the transformed quantum state becomes
\[
\hat{\rho}_{\text{out2}} = \frac{\hat{Y}_2 \hat{\rho}_{\text{out1}} \hat{Y}_2^\dagger}{p(2, 1|1, 1)}
\]
where
\[
p(2, 1|1, 1) = \text{Tr}(\hat{Y}_2 \hat{\rho}_{\text{out1}} \hat{Y}_2^\dagger)
\]
is the probability of registering the second photon conditioned by the detection of the first photon. Using Eq. (1), Eq. (4) can be rewritten as
\[
\hat{\rho}_{\text{out2}} = \frac{\hat{Y}_2 \hat{Y}_1 \hat{\rho}_{\text{in}} \hat{Y}_1^\dagger \hat{Y}_2^\dagger}{p(1, 1; 2, 1)}
\]
were
\[
p(1, 1; 2, 1) = \text{Tr}(\hat{Y}_2 \hat{Y}_1 \hat{\rho}_{\text{in}} \hat{Y}_1^\dagger \hat{Y}_2^\dagger) = p(2, 1|1, 1)p(1, 1)
\]
is the joint probability that the detectors \( D_1 \) and \( D_2 \) register a photon each. Repeating the procedure \( N \) times, we find that the overall output quantum state is
\[
\hat{\rho}_{\text{outN}} = \frac{\hat{Y} \hat{\rho}_{\text{in}} \hat{Y}^\dagger}{p(1, 1; 2, 1; \ldots; N, 1)}
\]
where
\[
p(1, 1; 2, 1; \ldots; N, 1) = \text{Tr}(\hat{Y} \hat{\rho}_{\text{in}} \hat{Y}^\dagger)
\]
is the joint probability that all the detectors register a photon each. Obviously, the overall nonunitary transformation operator \( \hat{Y} \) is given by the product of the operators \( \hat{Y}_i \),
\[
\hat{Y} = \hat{Y}_N \cdots \hat{Y}_2 \hat{Y}_1.
\]
3 Measuring arbitrary overlaps

Next let us consider a photon number measurement performed on the transformed
signal quantum state \( \hat{\rho}_{out,N} \) (detector \( D_{N+1} \) in Fig. I). The probability of detect-
ing no photons is \( \langle 0 | \hat{\rho}_{out,N} | 0 \rangle \). Equivalently, it is the probability of detecting no
photons in the \( (N+1) \)th measurement conditioned by the detection of a photon
in each of the \( N \) preceding measurements,

\[
p(N + 1, 0 | 1, 1; 2, 1; \ldots; N, 1) = \langle 0 | \hat{\rho}_{out,N} | 0 \rangle.
\]  (11)

Combining Eqs. (8) and (11), we easily see that

\[
p(1, 1; 2, 1; \ldots; N, 1; N + 1, 0) = \langle 0 | \hat{Y}_n \hat{\rho}_{in} \hat{Y}^\dagger | 0 \rangle
\]  (12)

is the joint probability that each of the detectors \( D_1, \ldots, D_N \) registers a photon
and the detector \( D_{N+1} \) does not register a photon.

Let

\[
\langle \Psi | \hat{\rho}_{in} | \Psi \rangle = \text{Tr} (\hat{\rho}_{in} | \Psi \rangle \langle \Psi |)
\]  (13)

be an overlap that is desired to be measured. When the values of the \( N \) coherent
amplitudes \( \alpha_i \) are chosen such that

\[
\frac{\hat{Y}^\dagger | 0 \rangle \langle 0 | \hat{Y}}{\| \hat{Y}^\dagger | 0 \rangle \| ^2} = | \Psi \rangle \langle \Psi |
\]  (14)

\((\| \Phi \| = \sqrt{\langle \Phi | \Phi \rangle})\), then the joint probability \( p(1, 1; 2, 1; \ldots; N, 1; N + 1, 0) \) be-
comes proportional to \( \langle \Psi | \hat{\rho}_{in} | \Psi \rangle \), as is easily seen from Eq. (12),

\[
\langle \Psi | \hat{\rho}_{in} | \Psi \rangle = \frac{p(1, 1; 2, 1; \ldots; N, 1; N + 1, 0)}{\| \hat{Y}^\dagger | 0 \rangle \| ^2}.
\]  (15)

In particular, performing the measurements for a set of states \( | \Psi_l \rangle \) which can
be used to define a positive operator valued measure (POVM) \( \Pi_l = | \Psi_l \rangle \langle \Psi_l |, l = 1, 2, \ldots \), directly yields the statistics of the quantity that is behind the POVM.

We now show that Eq. (14) can always be satisfied if the state \( | \Psi \rangle \) is a finite
superposition of Fock states,

\[
| \Psi \rangle = \sum_{n=0}^{N} | n \rangle \langle n | \Psi \rangle.
\]  (16)

Note that the expansion of any physical state in the Fock basis can always be
approximated to any desired degree of accuracy by truncating it at \( N \) if \( N \) is
suitably large. The state \( | \Psi \rangle \) in Eq. (14) is completely determined, e.g., by the
\( N \) zeros of its \( Q \)-function, i.e., the \( N \) solutions \( \beta_1, \ldots, \beta_N \) of the equation

\[
\langle \Psi | \beta \rangle = 0,
\]  (17)
In particular, when the signal coincides with the state because of
\[ |\Psi\rangle = \frac{\langle N|\Psi\rangle}{\sqrt{N!}} \prod_{k=1}^{N} (\hat{a}_k - \beta_k^\dagger) |0\rangle. \] (18)

In order to compare \(|\Psi\rangle\) with \(\hat{Y}^\dagger|0\rangle\), we use Eqs. (3) and (14) and obtain
\[ \hat{Y}^\dagger|0\rangle = e^{i\xi} R^N \hat{D} \left( \frac{T^*}{R^*} \alpha \right) \hat{a}^\dagger T^* \hat{D} \left( \frac{T^* \alpha_2 - \alpha_1}{R^*} \right) \hat{a}^\dagger T^* \hat{D} \left( \frac{T^* \alpha_3 - \alpha_2}{R^*} \right) \]
\[ \times \hat{a}^\dagger T^* \cdots \hat{D} \left( \frac{T^* \alpha_N - \alpha_{N-1}}{R^*} \right) \hat{a}^\dagger T^* \hat{D} \left( -\frac{\alpha_N}{R^*} \right) |0\rangle, \] (19)
where \(e^{i\xi}\) is an irrelevant phase factor. Now we rearrange the operator order such that the photon creation operators are on the left of the exponential operators, applying the rules \(\hat{D}(\beta)\hat{a}^\dagger = (\hat{a}^\dagger - \beta^*\hat{D}(\beta)\) and \(T^*\hat{a}^\dagger = T^*\hat{a}^\dagger T^*\). After some calculation we derive
\[ \hat{Y}^\dagger|0\rangle = \frac{e^{i\xi} R^N}{T^{N(1-N)/2}} \exp \left( -\frac{1}{2} \sum_{k=1}^{N} |\alpha_k|^2 \right) \prod_{k=1}^{N} \left( \hat{a}^\dagger - \frac{T^*}{R^*} \sum_{l=1}^{k} \frac{T^* \alpha_l - \alpha_{l-1}}{T^*} \right) |0\rangle \] (20)
\((\alpha_0 = 0)\). From a comparison of Eq. (20) with Eq. (15) it is seen that when the parameters \(\alpha_k\) and \(\beta_k\), \(k = 1, \ldots, N\), are related to each other as
\[ \beta_k = \frac{T}{R^*} \sum_{l=1}^{k} \frac{T^* \alpha_l - \alpha_{l-1}}{T^*}, \] (21)
or equivalently,
\[ \alpha_k = \frac{R^*}{T(T^*)^k} \sum_{l=1}^{k} |T|^2 (\beta_l - \beta_{l-1}) \] (22)
\((\beta_0 = 0)\), then Eq. (14) is satisfied, i.e., the desired overlap is observed.

From Eq. (13) it is seen that the sought overlap \(\langle \hat{\Psi}_0 | \hat{\Psi}_n | \Psi \rangle\) is determined by the measured joint probability \(p(1,1;2,1;\ldots;N,1;N+1,0)\) up to a factor of \(||\hat{Y}^\dagger|0\rangle||^2\) which may be viewed, in a sense, as a measure of the efficiency of the detection scheme. Insertion of Eqs. (21) and (18) in Eq. (14) gives
\[ ||\hat{Y}^\dagger|0\rangle||^2 = \frac{N!}{|\langle N|\Psi\rangle|^2} |R|^{2N} |T|^{N(N-1)} \exp \left( -\sum_{k=1}^{N} |\alpha_k|^2 \right). \] (23)

In particular, when the signal coincides with the state \(|\Psi\rangle\), i.e., \(\hat{\Psi}_0 = |\Psi\rangle \langle \Psi|\), then Eq. (13) reduces to \(||\hat{Y}^\dagger|0\rangle||^2 = p(1,1;2,1;\ldots;N,1;N+1,0)\). Obviously, \(||\hat{Y}^\dagger|0\rangle||^2\) is the joint probability of detecting one photon in the output channels 1, \ldots, N and no photon in the \((N+1)\)th output channel in the case when the signal is prepared just in the state with which the overlap is desired to be measured. It therefore follows that \(0 \leq ||\hat{Y}^\dagger|0\rangle||^2 \leq 1\). With increasing value of \(N\)
the value of $\|\hat{Y}^\dagger|0\rangle\|^2$ decreases rapidly in general and, accordingly, the number of recorded events must be increased in order to preserve accuracy. Note that $p(1, 1; 2, 1; \ldots; N, 1; N + 1, 0) \leq \|\hat{Y}^\dagger|0\rangle\|^2$, because of $\langle\Psi|\hat{\rho}_{in}|\Psi\rangle \leq 1$ in Eq. (15). So far we have considered equal beam splitters which, for chosen state $|\Psi\rangle$, do not realize the maximally attainable efficiency in general. Clearly, the beam splitter parameters can individually specified such that the efficiency is maximized, depending on the state $|\Psi\rangle$.

4 Quantum phase statistics

In order to illustrate the scheme, let us first consider the problem of measuring the canonical phase statistics, i.e., the problem of measuring overlaps of a signal-mode quantum state with London phase states. So far, the most direct method for measuring the canonical phase has been direct sampling in balanced homodyning of the exponential moments of the canonical phase, i.e, the Fourier components of the canonical phase distribution [8]. To measure the canonical phase statistics directly, it was proposed to combine the signal mode with a reference mode prepared such that the measured output becomes proportional to the overlap between the signal quantum state and phase states [9]. However, the method also called projection synthesis requires the reference mode to be prepared in highly involved nonclassical states.

On the contrary, our method only uses coherent states, without need to explicitly excite nonclassical states, so that it can be applied employing nowadays available techniques. Since any (physical) quantum state can be truncated at some photon number $N$ in the Fock basis, it is sufficient to consider the overlap with truncated London phase states,

$$|\varphi; N\rangle = \frac{1}{\sqrt{N+1}} \sum_{n=0}^{N} e^{in\varphi} |n\rangle.$$  \hspace{1cm} (24)

The efficiency $\|\hat{Y}^\dagger|0\rangle\|^2$, Eq. (23), for measuring overlaps with these states is plotted in Fig. 2 for $N = 1, \ldots, 8$. It is seen that the efficiency rapidly decreases with increasing $N$. Hence, the signal field should contain only few photons in practice. This is just the most interesting case for studying the quantum phase, otherwise the phase behaves nearly classically.

In the simple case when the signal-mode quantum state reduces to a (coherent) superposition of the vacuum state $|0\rangle$ and a one-photon Fock state $|1\rangle$, we have

$$|z\rangle = \frac{1}{\sqrt{1+|z|^2}}(|0\rangle + z|1\rangle).$$ \hspace{1cm} (25)

The overlap between the signal-mode quantum state $\hat{\rho}_{in} = |z\rangle\langle z|$ and a state $|\Psi\rangle$
\(|\varphi; N\rangle\) then reads
\[
\langle \Psi | \hat{\Phi}_{\text{in}} | \Psi \rangle = |\langle z | \varphi; N \rangle|^2 = \frac{1 + |z|^2 + 2|z|\cos(\varphi - \psi)}{(N + 1)(1 + |z|^2)}
\] (26)
where \(z = |z|e^{i\psi}\). Measuring the overlap for all phases \(\varphi\) yields the canonical phase distribution
\[
\text{prob } \varphi = \frac{N + 1}{2\pi} \langle \Psi | \hat{\Phi}_{\text{in}} | \Psi \rangle.
\] (27)
Note that the right-hand side in Eq. (27) is independent of \(N\), so that only one beam splitter in Fig. 1 is required. The canonical phase distribution is then given, according to Eq. (13), by the measured two-event joint probabilities \(p(1, 1; 2, 0)\),
\[
\text{prob } \varphi = \frac{1}{\pi} \frac{p(1, 1; 2, 0)}{\| \hat{Y}^\dagger | 0 \rangle \|^2}.
\] (28)
From Eq. (22) it is easily seen that the values of the coherent-state amplitude must be chosen such that \(\alpha = -(R^*/T^*)e^{i\varphi}\), and from Eq. (23) it follows that
\[
\| \hat{Y}^\dagger | 0 \rangle \|^2 = 2|R|^2 e^{-|R/T|^2}.
\] (29)
We see that \(\| \hat{Y}^\dagger | 0 \rangle \|^2\) attains at \(|T|^2 = 0.62\) a maximum of \(\| \hat{Y}^\dagger | 0 \rangle \|^2_{\text{max}} = 0.41\). Obviously, in the limit when \(|z| \to 0\) then the canonical phase of the vacuum is detected.

The method can also be used to measure the phase distributions that correspond to the Hermitian cosine- and sine-phase operators. In particular, the method allows one, for the first time, to measure the nontrivial cosine- and sine-phase distributions of the vacuum directly. The (truncated) cosine- and sine-phase states are superpositions of two (truncated) London phase states each (for details, see [10]),
\[
|\varphi, \chi; N\rangle = C(\varphi; N) \left[ e^{i\varphi} |\chi + \varphi; N\rangle - e^{-i\varphi} |\chi - \varphi; N\rangle \right],
\] (30)
with \(C(\varphi; N) = -i\{2 - 2\sin(N + 1)\varphi \cos(N + 2)\varphi/[(N + 1)\sin \varphi]\}^{-1/2}\). In particular, \(|\varphi, \chi = 0; N\rangle = |\cos \varphi; N\rangle\) and \(|\varphi, \chi = \pi/2; N\rangle = |\sin(\pi/2 - \varphi); N\rangle\) are the (truncated) Susskind-Glogower cosine- and sine-phase states, respectively. The overlap between the signal-mode quantum state \(\hat{\Phi}_{\text{in}} = |z\rangle \langle z|\) and a state \(|\Psi\rangle = |\varphi, \chi; N\rangle\) reads
\[
\langle \Psi | \hat{\Phi}_{\text{in}} | \Psi \rangle = |\langle z | \varphi, \chi; N \rangle|^2 = \frac{4|C(\varphi; N)|^2}{(N + 1)(1 + |z|^2)}
\times \left[ \sin^2 \varphi + 2|z|\cos(\psi - \chi) \sin \varphi \sin(2\varphi) + |z|^2 \sin^2 2\varphi \right].
\] (31)
Measuring this overlap for all phases \(\varphi\) yields, for chosen \(\chi\), the corresponding Susskind-Glogower trigonometric phase distribution
\[
\text{prob } \varphi |_\chi = \frac{N + 1}{2\pi |C(\varphi; N)|^2} \langle \Psi | \hat{\Phi}_{\text{in}} | \Psi \rangle.
\] (32)
Again, the right-hand side in Eq. (32) is independent of \( N \), so that it is sufficient to measure the joint probabilities for \( N=1 \), i.e., \( p(1, 1; 2, 0) \). From Eq. (22) the values of the coherent-state amplitude are then found to be \( \alpha = -[R^*/(2T^* \cos \varphi)] e^{i\chi} \), and Eq. (23) yields the efficiency

\[
\| \hat{Y}^\dagger |0\rangle \|^2 = \frac{1 - \cos \varphi \cos(3\varphi)}{\sin^2(2\varphi)} |R|^2 e^{-|R/(2T^* \cos \varphi)|^2} \tag{33}
\]

which is plotted in Fig. 3. Combining Eqs. (13), (32), and (33), we obtain

\[
\text{prob} \varphi |\chi = 2 \sin^2(2\varphi) \pi |R|^2 e^{-|R/(2T^* \cos \varphi)|^2} \tag{34}
\]

\( \parallel \hat{Y}^\dagger |0\rangle \parallel^2 \) becomes maximal for

\[
|T|^2 = \sqrt{1 + 16 \cos^2 \varphi} - 1 \frac{8 \cos^2 \varphi}{\cos^2 \varphi} \tag{35}
\]

[cf. Fig. 4], and hence \( 0.36 < \| \hat{Y}^\dagger |0\rangle \|_\text{max} < 0.52 \) for \( \varphi \) within a \( \pi \) interval. Note that \( \| \hat{Y}^\dagger |0\rangle \|^2 \) does not depend on \( \chi \). In the limit when \( |z| \to 0 \), then the nonuniformly distributed Susskind-Glogower trigonometric phase of the vacuum is detected.

### 5 Schrödinger-cat state statistics

Let us return to the general scheme in Fig. 1. When some of the zeros \( \beta_k^* \) in Eq. (18) are equal,

\[
|\Psi\rangle = \frac{\langle N|\Psi\rangle}{\sqrt{N!}} \prod_{l=1}^{M} (\hat{a}^\dagger - \beta_l^*)^d_l |0\rangle, \tag{36}
\]

\( (M < N) \), then the number of detectors can be reduced to \( M + 1 \), and the overlap is given by \( p(1, d_1; \ldots; M, d_M; M + 1, 0) \). It can be easily proved that the nonunitary transformation operator \( \hat{Y} \) now reads

\[
\hat{Y} = \hat{Y}_M \ldots \hat{Y}_2 \hat{Y}_1, \tag{37}
\]

where \( \hat{Y}_i \) is given by Eq. (3), with \( -R^* \hat{a}/\sqrt{d_i!} \) in place of \( -R^* \hat{a} \), and the overlap \( \langle \Psi | \hat{\xi}_n | \Psi \rangle \) is given by

\[
\langle \Psi | \hat{\xi}_n | \Psi \rangle = \frac{p(1, d_1; 2, d_2; \ldots; M, d_M; M + 1, 0)}{\| \hat{Y}^\dagger |0\rangle \|^2} \tag{38}
\]

in place of Eq. (14).

An interesting example is the measurement of the overlap of a signal-mode quantum state with quantum states

\[
|\Psi_n(\beta)\rangle = \mathcal{N}^{-1/2} [ (\hat{a}^\dagger)^2 - (\beta^*)^2 ]^n |0\rangle. \tag{39}
\]
Applying standard formulas \([11]\), the normalization factor \(\mathcal{N}\) can be expressed in terms of the generalized hypergeometric function \(1\, F_2(a, b, c; z)\),

\[
\mathcal{N} = \sum_{k=0}^{n} \binom{n}{k}^2 \frac{(2k)!}{|\beta|^{4(k-n)}} = 4^n n! \sqrt{\pi} \Gamma(n+\frac{1}{2}) 1\, F_2(-n, \frac{1}{2}-n, 1, \frac{1}{4}|\beta|^4) .
\]  

(40)

For \(n = |\beta|^2\) and increasing \(|\beta|^2\) the states \(|\Psi_n(\beta)\rangle\) behave like Schrödinger-cat states

\[
|\Psi(\beta)\rangle = \frac{e^{i|\beta|^2(\pi-2\varphi_\beta)}}{\sqrt{2}} (|i\beta\rangle + | -i\beta\rangle),
\]

(41)
as can be seen from the overlap \(\langle\Psi_n(\beta)|\Psi(\beta)\rangle^2\) for \(|\beta|^2 = n\),

\[
\langle\Psi_n(\beta)|\Psi(\beta)\rangle^2 = 2 \left(\frac{4}{e}\right)^n \left(\sum_{k=0}^{n} p_k f_k\right)^{-1},
\]

(42)

where

\[
p_k = \frac{1}{2^n} \binom{n}{k}, \quad f_k = 2^n \binom{n}{k} \frac{(2k)!}{n^2 k^2}.
\]

(43)

From the asymptotic behavior of \(p_k\) it can be seen that for sufficiently large \(n\) the sum in Eq. (12) can be replaced by \(f_2\). Applying the Stirling formula to \(f_2\) then yields \(\langle\Psi_n(\beta)|\Psi(\beta)\rangle^2 \to 1\) if \(n \to \infty\) in Eq. (12). Note that the phase factor \(\exp[i|\beta|^2(\pi-2\varphi_\beta)]\) in Eq. (11) follows directly from \(\langle i\beta|\Psi_n(\beta)\rangle\).

It is worth noting that \(|\Psi_{n=|\beta|^2}(\beta)\rangle\) is already a good approximation of \(|\Psi(\beta)\rangle\) for relatively small values of \(|\beta|^2\), e.g., \(\langle\Psi_{n=|\beta|^2}(\beta)|\Psi(\beta)\rangle^2 > 0.95\) for \(|\beta|^2 \geq 3\). Hence, measuring of the overlap of a signal-mode quantum state with a state \(|\Psi_{n=|\beta|^2}(\beta)\rangle\) corresponds, in good approximation, to measuring the overlap with a Schrödinger-cat state \(|\Psi(\beta)\rangle\). The two-beam-splitter scheme that realizes the measurement is shown in Fig. 3. The efficiency of the measurement is given by

\[
\|\hat{Y}^+|0\rangle\|^2 = \frac{\mathcal{N} |R^2 T|^{2n}}{n!^2} \exp\left\{ - \left| \frac{R \beta}{T} \right|^2 \left[ 1 + |T|^{-2} \left( 1 - 2|T|^2 \right)^2 \right]\right\}.
\]

(44)

Plots of \(\|\hat{Y}^+|0\rangle\|^2\) are shown in Fig. 3 for various values of \(n\). The scheme can easily be extended in order to measure the overlap of \(\hat{g}_\text{in}\) with an arbitrary superposition of two coherent states, \(|\Psi(\beta_1, \beta_2)\rangle = 2^{-1/2} (|\beta_1\rangle + |\beta_2\rangle)\). For this purpose the quantum state of the signal mode must be coherently displaced, \(\hat{g}_\text{in} \to \hat{D}^\dagger(\gamma) \hat{g}_\text{in} \hat{D}(\gamma)\), before measuring the overlap with \(|\Psi(\beta)\rangle\). Choosing the parameters such that \(\beta = (\beta_1 - \beta_2)/(2i)\) and \(\gamma = (\beta_1 + \beta_2)/2\), it is easily seen that

\[
\langle \Psi(\beta_1, \beta_2)|\hat{g}_\text{in}|\Psi(\beta_1, \beta_2)\rangle = \langle \Psi(\beta)|\hat{D}^\dagger(\gamma) \hat{g}_\text{in} \hat{D}(\gamma)|\Psi(\beta)\rangle.
\]

(45)

Note that a displacement may be achieved by mixing the signal with a mode prepared in a strong coherent state using a highly transmitting beam splitter (see, e.g., \([13]\)).
6 Conclusion

We have presented a scheme for the direct measurement of the overlaps of an unknown signal quantum state with arbitrary reference quantum states. It is based on superimposing the signal mode and modes prepared in coherent states and detecting specific coincident events in the photon statistics of the outgoing modes. The advantage of the scheme is that the measurements can be performed without any explicit preparation of the states the signal-mode quantum state is projected onto. The disadvantage is that the relative frequency of the desired events can be very small in general, which is typical of conditional measurement schemes. The applicability of the method is therefore expected to be restricted to measuring overlaps in which only a few photons are involved, i.e., it is the highly nonclassical area around the quantum vacuum which is covered by the method.

In particular, the method may be used for measuring quantum overlaps for which a direct, dynamical measurement method has not been available so far. In this context, we have considered the measurement of the overlaps of a signal-mode quantum state with canonical phase states and related cosine- and sine-phase states, particularly near the quantum vacuum, and calculated the detection probabilities. Further, we have considered the problem of measurement of the overlaps of a signal-mode quantum state with states that are superpositions of two coherent states. Recently it has been shown that from such overlaps the quantum state of the signal mode can be inferred [14]. In principle, these overlaps could be measured, as proposed in [14], in heterodyne detection, however with the reference mode being prepared in a superposition of two coherent states in place of a single coherent state – a rather difficult problem which has not been solved so far in practice.

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Figure 1: Measurement of the overlap $\langle \Psi | \hat{\rho}_{\text{in}} | \Psi \rangle$ of a signal-quantum state $\hat{\rho}_{\text{in}}$ with a given state $| \Psi \rangle$ by successive mixing with modes prepared in appropriately chosen coherent states $| \alpha_1 \rangle$ to $| \alpha_N \rangle$ at beam splitters and measuring the relative frequency of the event of detecting simultaneously 1 photon with the photodetectors $D_1$ to $D_N$ and 0 photons with $D_{N+1}$.
Figure 2: Efficiency $\|\hat{Y}^\dagger|0\rangle\|^2$, Eq. (23), for measuring the overlap of a signal-mode quantum state with truncated London phase states $|\varphi; N\rangle$, Eq. (24), as a function of the absolute value of the beam-splitter transmittance $|T|$ for $\varphi = 0$, the value of $N$ being varied from $N = 1$ (top curve) to $N = 8$ (bottom curve).
Figure 3: Efficiency $\|\hat{Y}^\dagger|0\rangle\|^2$, Eq. (33), for measuring the overlap of a signal-mode quantum state with a state $|\varphi,\chi;1\rangle$, Eq. (30), as a function of the phase $\varphi$ and the absolute value of the beam-splitter transmittance $|T|$. 
Figure 4: Maximal efficiency $\|\hat{Y}^\dagger|0\rangle\|_{\text{max}}^2$ as a function of the phase $\varphi$ as obtained with Eq. (33) if $|T|$ is chosen according to Eq. (35).
Figure 5: Measurement of the overlap $\langle \Psi_n(\beta)|\hat{\psi}_{\text{in}}|\Psi_n(\beta)\rangle$ of a signal-quantum state $\hat{\psi}_{\text{in}}$ with a state $|\Psi_n(\beta)\rangle$, Eq. (39), by mixing the signal mode with two modes prepared in coherent states at two beam splitters and measuring the relative frequency of the event of detecting simultaneously $n$ photons with the photodetectors $D_1$ and $D_2$ and 0 photons with $D_3$. 

$n$ clicks

$D_1$

$\frac{R^*}{T^*}\beta$

$n$ clicks

$D_2$

$|R^*\frac{1-2|T|^2}{T^*2}\beta\rangle$

$\hat{\psi}_{\text{in}}$

$D_3$

0 clicks
Figure 6: Efficiency $\|\hat{Y}^\dagger|0\rangle\|^2$, Eq. (23), for measuring the overlap of a signal-mode quantum state with Schrödinger-cat-like states $|\Psi_n=|\beta|^2(\beta)\rangle$, Eq. (39), as a function of the absolute value of the beam-splitter transmittance $|T|$, the value of $n$ being varied from $n = 1$ (top curve) to $n = 8$ (bottom curve).