Estimates for a class of oscillatory integrals and decay rates for wave-type equations

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Abstract. This paper investigates higher order wave-type equations of the form $\partial_t^2 u + P(D_x)u = 0$, where the symbol $P(\xi)$ is a real, non-degenerate elliptic polynomial of the order $m \geq 4$ on $\mathbb{R}^n$. Using methods from harmonic analysis, we first establish global pointwise time-space estimates for a class of oscillatory integrals that appear as the fundamental solutions to the Cauchy problem of such wave equations. These estimates are then used to establish (pointwise-in-time) $L^p - L^q$ estimates on the wave solution in terms of the initial conditions.

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1. Introduction

It is well known that the solution $u(t, x)$ of the Cauchy problem for the wave equation:

\[
\begin{cases}
\partial_t^2 u(t, x) - \Delta u(t, x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n \\
u(0, x) = u_0(x), \partial_t u(0, x) = u_1(x), & x \in \mathbb{R}^n
\end{cases}
\]

has the following form:

\[
u(t, x) = \mathcal{F}^{-1} \cos(|\xi|t) \mathcal{F}u_0 + \mathcal{F}^{-1} \frac{\sin(|\xi|t)}{|\xi|} \mathcal{F}u_1,
\]

(1.1)
where $\mathcal{F}$ (resp. $\mathcal{F}^{-1}$) denotes the Fourier transform (resp. its inverse). On the other hand,

$$u(t, x) = \mathcal{F}^{-1} \cos \left( \left[ 1 + |\xi|^2 \right]^{1/2} t \right) \mathcal{F} u_0 + \mathcal{F}^{-1} \frac{\sin \left( \left[ 1 + |\xi|^2 \right]^{1/2} t \right)}{\left( 1 + |\xi|^2 \right)^{1/2}} \mathcal{F} u_1$$

is the solution of the linear Klein-Gordon equation:

$$\begin{cases}
\partial_t^2 u(t, x) - \Delta u(t, x) + u(t, x) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n \\
u(0, x) = u_0(x), \ \partial_t u(0, x) = u_1(x), \quad x \in \mathbb{R}^n.
\end{cases}$$

If we use $P(\xi) = |\xi|^2$ and $P = 1 + |\xi|^2$, respectively in (1.1) and (1.2), then the above solutions read as follows:

$$u(t, x) = \mathcal{F}^{-1} \cos \left( P^{1/2}(\xi)t \right) \mathcal{F} u_0 + \mathcal{F}^{-1} \frac{\sin \left( P^{1/2}(\xi)t \right)}{P^{1/2}(\xi)} \mathcal{F} u_1$$

$$= \left( \mathcal{F}^{-1} \frac{e^{iP^{1/2}(\xi)t} + e^{-iP^{1/2}(\xi)t}}{2} \right) * u_0 + \left( \mathcal{F}^{-1} \frac{e^{iP^{1/2}(\xi)t} - e^{-iP^{1/2}(\xi)t}}{2iP^{1/2}(\xi)} \right) * u_1.$$  

(1.3)

The main focus of this paper is to derive pointwise estimates (both in $t$ and $x$) on the oscillatory integrals

$$I_1(t, x) := \int_{\mathbb{R}^n} e^{i<x, \xi> \pm utP^{1/2}(\xi)} d\xi$$

and

$$I_2(t, x) := \int_{\mathbb{R}^n} e^{i<x, \xi> \pm utP^{1/2}(\xi)} P^{-1/2}(\xi) d\xi$$

appearing in (1.3) – but for a larger class of symbols $P(\xi)$. From such estimates on the fundamental solution one can then derive solution properties, like its spatial decay at a fixed time, or decay/growth estimates of $\|u(t, .)\|_{L^q}$ in time.

For the classical wave and the Klein-Gordon equations, such pointwise-in-time $L^p$–$L^q$ decay estimates (i.e. estimates on $\|u(t, .)\|_{L^q}$ in terms of $\|u_0\|_{L^p}$ and $\|u_1\|_{L^p}$) can be found frequently in the literature [8, 29, 25, 26, 39]. It is also well known that such $L^p$–$L^q$ estimates allow to deduce the famous Strichartz inequalities, which are very useful for the analysis of nonlinear wave equations (see e.g. [16, 19, 38, 36, 40]). More generally, many similar Strichartz-type estimates (local and global in time, or with certain spatial weights) for second order hyperbolic equations have been established in the case of variable coefficients or on Riemannian manifolds. There, crucial analytic tools from microlocal analysis or spectral theory are employed (see e.g. [2], [7], [9], [18], [28], [33], [31], [41] and the references therein). We remark that these mentioned Strichartz-type estimates are for space-time-integrals, while our estimates are all pointwise in time.
In this paper, our main aim is to derive $L^p$–$L^q$ estimates for the following general wave-type equations:

\[
\begin{cases}
\partial_{tt} u(t,x) + P(-i\nabla)u(t,x) = 0,
(t,x) \in \mathbb{R} \times \mathbb{R}^n \\
u(0,x) = u_0(x), \partial_t u(0,x) = u_1(x), \quad x \in \mathbb{R}^n,
\end{cases}
\]

where $P(\xi)$ is a positive, real valued polynomial of higher (even) order $m \geq 4$ on $\mathbb{R}^n$. In order to derive $L^p$–$L^q$ estimates of the solution (1.3), it suffices to study pointwise estimates of the oscillatory integrals (1.4) and (1.5) associated to the general polynomial $P$. To this end, we need the following assumptions on $P(\xi)$:

(H1): $P : \mathbb{R}^n \to \mathbb{R}$ is a real elliptic inhomogeneous polynomial of even order $m \geq 4$ with $P(\xi) > 0$ for all $\xi \in \mathbb{R}^n$, and $n \geq 2$.

(H2): $P$ is non-degenerate, i.e. the determinant of the Hessian

\[
\det \left(\frac{\partial^2 P_m(\xi)}{\partial \xi_i \partial \xi_j}\right)_{n \times n} \neq 0 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\},
\]

where $P_m$ is the principal part of $P$.

It is well known that for elliptic polynomials $P$, condition (H2) is equivalent to the following condition (H2′) (see Lemma 2 in [12]).

(H2′): For any fixed $z \in S^{n-1}$ (the unit sphere of $\mathbb{R}^n$), the function $\psi(\omega) := < z, \omega > (P_m(\omega))^{-1/m}$, defined on $S^{n-1}$, is non-degenerate at its critical points. This means: If $d_\omega \psi$, the differential of $\psi$ at a point $\omega \in S^{n-1}$ vanishes, then $d^2_\omega \psi$, the second order differential of $\psi$ at this point is non-degenerate. Note that the non-degeneracy of $P$ is also equivalent to $\det(\partial_i \partial_j P(\xi))_{n \times n}$ being an elliptic polynomial of order $n(m-2)$.

Particular examples of such higher order wave-type equations have already been studied in several papers. For $P = 1 + |\xi|^4$ (linear beam equations of forth order), Levandosky [22] obtained $L^p$–$L^q$ estimates and space-time integrability estimates. He used them to study the local existence and the asymptotic behavior of solutions to the nonlinear equation with nonlinear terms growing like a certain power of $u$. Further, Levandosky and Strauss [23], Pausader [30, 31] established the scattering theory of the nonlinear beam equation with subcritical nonlinear terms for energy initial values. Even earlier, for $P = 1 + |\xi|^m$ (with $m \geq 4$ even), Pecher [32] studied $L^p$–$L^q$ estimates of such higher order wave equations and
also considered their application to nonlinear problems. Clearly, these polynomials are special cases satisfying our assumptions stated above. In the sequel, we shall deal with the general class in the form of the oscillatory integrals (1.4) and (1.5) under the assumptions (H1) and (H2). Comparing with the classical wave equation and Klein-Gordon equations, the fundamental solutions of higher order wave-type equations behave “better” in the dispersions relation and w.r.t. the gain of a certain decay in the space variable $x$. As a consequence, we can obtain a larger set of admissible $(1/p, 1/q)$-pairs such that the $L^p-L^q$ estimates hold (see §4).

Concerning dispersive estimates, our methods (mainly from harmonic analysis) and results are similar to those of various dispersive Schrödinger-type equations. And on this topic there exists a vast body of literature, see e.g. [1, 3, 4, 5, 6, 10, 11, 12, 14, 15, 21, 20, 26, 42, 43].

The oscillatory integrals (1.4) and (1.5) can initially be understood in the distributional sense. Based on the assumption that $P$ is elliptic, it is easy to see that $I_j(t,x); j = 1, 2$ are infinitely differentiable functions in the $x$ variable for every fixed $t \neq 0$ (e.g. see §1 of [12]). In this paper, we shall derive pointwise time-space estimates for the oscillatory integrals (1.4) and (1.5). Subsequently, such estimates are used to establish $L^p - L^q$ estimates for the wave solutions. Finally, we also remark that, based on these $L^p - L^q$ estimates, some applications to nonlinear problems can be expected, which will be investigated in a following paper.

This paper is organized as follows. In Section 2, we make some pretreatment to the oscillatory integrals (1.4) and (1.5), review the (polar coordinate transformation) method of Balabane et al. [1] and its extension by Cui [12]. In the core Section 3 we prove the pointwise time-space estimates on (1.4), (1.5) following the strategy from [21]. Finally, in §4 these estimates are applied to obtain $L^p - L^q$ estimates for solutions to higher order wave equations.

2. Preliminaries

We denote by $S^{n-1}$ the unit sphere in $\mathbb{R}^n$, and by $(\rho, \omega) \in [0, \infty) \times S^{n-1}$ the polar coordinates in $\mathbb{R}^n$. Throughout this paper, we assume that $P : \mathbb{R}^n \to \mathbb{R}$ satisfies the assumptions (H1) and (H2) (or (H2')). Hence, $P_m(\xi) > 0$ for $\xi \neq 0$. This
implies that there exists a large enough constant \( a > 0 \) with: For each fixed \( s \geq a \) and each fixed \( \omega \in S^{n-1} \), the equation \( P(\rho \omega) = s \) has a unique positive solution \( \rho = \rho(s, \omega) \in C^\infty([a, \infty) \times S^{n-1}) \). By Lemma 2 in [1], \( \rho \) can be decomposed as

\[
\rho(s, \omega) = s^{\frac{1}{m}} (P_m(\omega))^{-\frac{1}{m}} + \sigma(s, \omega),
\]

(2.1)

where \( \sigma \in S^0_1([a, \infty) \times S^{n-1}) \). This symbol class denotes functions in \( C^\infty([a, \infty) \times S^{n-1}) \) that satisfy the following condition (cf. [12, 37]): For every \( k \in \mathbb{N}_0 \) and every differential operator \( L_\omega \) on the sphere \( S^{n-1} \), there exists a constant \( C_{kL} \) such that

\[
|\partial^k_s L_\omega \sigma(s, \omega)| \leq C_{kL} (1 + s)^{-k}
\]

for \( s \geq a \) and \( \omega \in S^{n-1} \).

(2.2)

We now recall two lemmata (see [1, 12]) for the following phase function

\[
\phi(s, \omega) := s^{-\frac{1}{m}} \rho(s, \omega)(z, \omega) \quad \text{for} \quad s \geq a \quad \text{and} \quad \omega \in S^{n-1},
\]

with some fixed \( z \in S^{n-1} \). Clearly, \( \phi \in S^0_0([a, \infty) \times S^{n-1}) \). For every fixed \( z_0 \in S^{n-1} \) there exists a (sufficiently small) neighborhood \( U_{z_0} \subset S^{n-1} \) of \( z_0 \) such that the following lemmata hold uniformly in \( z \in U_{z_0} \) (i.e. the constants in Lemma 2.1, Lemma 2.2, and Lemma 2.3 are then independent of \( z \)). Therefore we do not write the variable \( z \) in the function \( \phi \).

**Lemma 2.1 (Lemma 4 of [12], Lemma 3 of [1]).** There exists a constant \( a_0 \geq a \) and an open cover \( \{ \Omega_0, \Omega_+, \Omega_- \} \) of \( S^{n-1} \) with \( \Omega_+ \cap \Omega_- = \emptyset \) such that it holds for \( s \geq a_0 \):

(a) The function \( \Omega_0 \ni \omega \mapsto \phi(s, \omega) \) has no critical points, and

\[
\|d_\omega \phi(s, \omega)\| \geq c > 0 \quad \text{for} \quad \omega \in \Omega_0,
\]

(2.3)

where the constant \( c \) is independent of \( s \).

(b) Each of the two functions \( \Omega_+ \ni \omega \mapsto \phi(s, \omega) \) has a unique critical point, which satisfies: \( \omega_\pm = \omega_\pm(s) \in C^\infty([a_0, \infty); \Omega_+^\prime) \) for some open subset \( \Omega_+^\prime \) with \( \Omega_+^\prime \subset \Omega_+ \), respectively. Furthermore,

\[
\|(d^2_\omega \phi(s, \omega))^{-1}\| \leq c_0 \quad \text{for} \quad \omega \in \Omega_\pm,
\]

(2.4)

where the constant \( c_0 \) is independent of \( s \). Moreover, \( \lim_{s \to \infty} \omega_\pm(s) \) exists and

\[
|\omega_\pm^{(k)}(s)| \leq c_k (1 + s)^{-k-\frac{1}{m}} \quad \text{for} \quad k \in \mathbb{N}.
\]
Lemma 2.2 (Lemma 6 of [12]). We define \( \phi_{\pm}(t, r, s) := st + rs^{\frac{2}{m}} \phi(s^2, \omega_{\pm}(s^2)) \) for \( t, r > 0 \), and \( s \geq a \). Then, there exist constants \( a_1 \geq \max(a_0, \sqrt{a}) \) and \( c_2 > c_1 > 0 \) such that we have for \( s \geq a_1 \), \( t > 0 \), and \( r > 0 \):

\[
\begin{align*}
  c_1 & \leq \pm \phi(s, \omega_{\pm}(s)) \leq c_2, \quad (2.5) \\
  \partial_s \phi_{\pm}(t, r, s) & \geq t + c_1 rs^{\frac{2}{m} - 1}, \quad (2.6) \\
  t - c_2 rs^{\frac{2}{m} - 1} & \leq \partial_s \phi_{\pm}(t, r, s) \leq t - c_1 rs^{\frac{2}{m} - 1}, \quad (2.7) \\
  c_1 rs^{\frac{2}{m} - 2} & \leq |\partial_s^2 \phi_{\pm}(t, r, s)| \leq c_2 rs^{\frac{2}{m} - 2}, \quad (2.8)
\end{align*}
\]

and

\[
|\partial_s^k \phi_{\pm}(t, r, s)| \leq c_2 rs^{\frac{2}{m} - k} \quad \text{for } k = 2, 3, \ldots \quad (2.9)
\]

With this preparation we are able to estimate the following oscillatory integral

\[
\Phi(\lambda, s) := \int_{S^{n-1}} e^{i \lambda \phi(s, \omega)} b(s, \omega) d\omega, \quad (2.10)
\]

where \( b(s, \omega) := s^{1 - \frac{n}{m}} \rho^{-1} \partial_s \rho \in S^0([a, \infty) \times S^{n-1}) \) and \( \lambda > 0 \). Let \( \varphi_+, \varphi_, \varphi_0 \) be a partition of unity of \( S^{n-1} \), subordinate to the open cover given in Lemma 2.1.

Then we decompose \( \Phi \) as

\[
\Phi(\lambda, s) = \Phi_+(\lambda, s) + \Phi_-(\lambda, s) + \Psi_0(\lambda, s),
\]

where

\[
\Phi_{\pm}(\lambda, s) := \int_{S^{n-1}} e^{i \lambda \phi(s, \omega)} b(s, \omega) \varphi_{\pm}(\omega) d\omega
\]

and

\[
\Psi_0(\lambda, s) := \int_{S^{n-1}} e^{i \lambda \phi(s, \omega)} b(s, \omega) \varphi_0(\omega) d\omega.
\]

By using the stationary phase method for \( \Psi_0 \), and Lemma 2.1 and [35] (Corollary 1.1.8, §1.2) for \( \Phi_{\pm} \), one obtains the following result.

Lemma 2.3. For \( \lambda > 0 \) and \( s > a_1 \) we have

\[
\Phi(\lambda, s) = \lambda^{-\frac{n+1}{2}} e^{i \lambda \phi(s, \omega_{\pm}(s))} \Psi_{\pm}(\lambda, s) + \lambda^{-\frac{n+1}{2}} e^{i \lambda \phi(s, \omega_{\pm}(s))} \Psi_{\mp}(\lambda, s) + \Psi_0(\lambda, s), \quad (2.11)
\]

where \( \Psi_{\pm}, \Psi_0 \in C^\infty((0, \infty) \times [a_0, \infty)) \) and

\[
|\partial_s^k \partial_s^l \Psi_{\pm}(\lambda, s)| \leq c_{k,j}(1 + \lambda)^{-k} s^{-j} \quad \text{for } k, j \in N_0, \quad (2.12)
\]

\[
|\partial_s^k \partial_s^l \Psi_0(\lambda, s)| \leq c_{k,j,l}(1 + \lambda)^{-l} s^{-j} \quad \text{for } k, j, l \in N_0. \quad (2.13)
\]
3. Estimates on the oscillatory integrals

In this section we establish pointwise time-space estimates of the oscillatory integrals \((1.4)\) and \((1.5)\). Like in \([21]\), we aim at simultaneous estimates in the time and spatial variables. This is a refinement of the analysis in \([12]\), where only spatial decay estimates of the oscillatory integrals are derived. With our refined analysis we are able to give here global-in-time estimates on the wave solution.

**Theorem 3.1.** Assume that the polynomial \(P\) satisfies the conditions \((H1)\) and \((H2)\) from \([17]\) and let \(n \geq m\). Then there exists a constant \(C > 0\) such that

\[
|I_2(t, x)| \leq \begin{cases} 
C|t|^{-\frac{n-1}{m_1}}(1 + |t|^{-\frac{1}{m_1}}|x|)^{-\mu}, & \text{for } 0 < |t| \leq 1, \\
C|t|^{-\frac{n}{m_1}}(1 + |t|^{-1}|x|)^{-\mu}, & \text{for } |t| \geq 1,
\end{cases}
\]

where \(m_1 := \frac{m}{2}\), \(\mu := \frac{m - 4n + 2m}{2(m - 2)} > 0\).

**Proof.** In the sequel, \(C\) denotes some generic (but not necessarily identical) positive constants, independent of \(t\), \(\xi\), \(x\), and so forth. Since the integrals \(I_2(t, x)\) and \(I_2(-t, x)\) are structurally identical, it suffices to estimate \(I_2(t, x)\) for \(t > 0\). We shall now analyze \(I_2\) for three different cases of its arguments, starting with the most delicate situation.

**Case (i):** \(t \geq 1\) and \(r := |x| \geq t\).

Choose \(\psi \in C^\infty(\mathbb{R})\) such that

\[
\psi(s) = \begin{cases} 
0, & \text{for } s \leq a_1 \\
1, & \text{for } s > 2a_1,
\end{cases}
\]

where \(a_1\) is given in Lemma \([22]\). We write

\[
I_2(t, x) = \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle \pm tP^{1/2}(\xi)} P^{-\frac{1}{2}}(\xi) \psi(P^{1/2}(\xi)) d\xi + \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle \pm tP^{1/2}(\xi)} P^{-\frac{1}{2}}(\xi)[1 - \psi(P^{1/2}(\xi))] d\xi =: I_{21}(t, x) + I_{22}(t, x).
\]

First we rewrite \(I_{22}\) as the Fourier transform of a measure, supported on the graph \(S := \{z = \pm P^{1/2}(\xi); \ \xi \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}\):

\[
I_{22}(t, x) = \int_{\mathbb{R}^{n+1}} e^{i\langle (x, \xi) \pm tz \rangle P^{-\frac{1}{2}}(\xi)[1 - \psi(P^{1/2}(\xi))] \delta(z \mp P(\xi)^{1/2})} d\xi dz. \tag{3.2}
\]

Since the polynomial \(P\) is of order \(m\), the supporting manifold of the above integrand is of type less or equal \(m\) (in the sense of \(\S\) VIII.3.2, \([37]\); see the Appendix
Then, Theorem 2 of § VIII.3 in [37] implies
\[
|I_{22}(t, x)| \leq C(1 + |t| + |x|)^{-\frac{1}{2}} \quad \forall t, x. \tag{3.3}
\]
This can be generalized: Since \( f(t, \xi) := e^{\pm \imath t P^{1/2} P^{-1/2}} |1 - \psi(P^{1/2})| \in C_c^\infty(\mathbb{R}^n) \) for every \( t > 0 \), we obtain by integration by parts
\[
I_{22}(t, x) = i \int_{\mathbb{R}^n} e^{i(x, \xi)} \frac{x}{|x|^2} \cdot \nabla f(t, \xi) d\xi.
\]
Proceeding recursively this implies (in the spirit of the Paley-Wiener-Schwartz theorem)
\[
|I_{22}(t, x)| \leq C_k t^k r^{-k} \quad \text{for } k \in \mathbb{N}_0, x \neq 0, t \geq 1, \tag{3.4}
\]
and hence also \( \forall k \geq 0 \). But proceeding as in (3.2) yields the improvement
\[
|I_{22}(t, x)| \leq C_k |t|^{-\frac{1}{2}} (1 + |t|^{-1}|x|)^{-\left((k + \frac{1}{2})\right)} \quad \text{for } |t| \geq 1, x \in \mathbb{R}^n, \forall k \geq 0. \tag{3.5}
\]
To estimate \( I_{21} \), we shall derive an \( \varepsilon \)-uniform estimate of its regularization
\[
J_\varepsilon(t, x) := \int_{\mathbb{R}^n} e^{-\varepsilon t P^{1/2} (\xi) + i(x, \xi) t P^{1/2} (\xi)} P^{-1/2} (\xi) \psi(P^{1/2} (\xi)) d\xi \quad \text{for } \varepsilon > 0. \tag{3.6}
\]
By the polar coordinate transform and the change of variables \((\rho, \omega) \rightarrow (s, \omega)\) such that \( \rho = \rho(s, \omega) \) (with \( P(\rho \omega) = s \)), we have
\[
J_\varepsilon(t, x) = \int_0^\infty \int_{S^{n-1}} e^{-\varepsilon t P^{1/2}(\rho \omega) + i(x, \rho \omega) t P^{1/2}(\rho \omega)} P^{-1/2}(\rho \omega) \psi(P^{1/2}(\rho \omega)) \rho^{n-1} d\omega d\rho
\]
\[
= \int_0^\infty \int_{S^{n-1}} e^{-\varepsilon \sqrt{s} \pm it \sqrt{s} + i r(s, \omega)} \psi(\sqrt{s}) s^{-1/2} \rho^{n-1} \partial_s \rho d\omega ds
\]
\[
= \int_0^\infty e^{-\varepsilon \sqrt{s} \pm it \sqrt{s}} s^{\frac{n-1}{2}} s^{-1/2} \psi(\sqrt{s}) \Phi(r, s) ds
\]
\[
= 2 \int_0^\infty e^{-\varepsilon \sqrt{s} \pm it \sqrt{s}} s^{\frac{n-1}{2}} \psi(\sqrt{s}) \Phi(r, s) ds,
\]
where \( z := x/|x| \) enters in the oscillatory integral \( \Phi \) from (2.10). For the transformation \( \rho \rightarrow s \) we used that \( \psi(s) = 0 \) on \([0, a_1]\) (see §2 in [12] for a more detailed discussion). Here and in the sequel we assume that the functions \( \Phi, \Phi_\pm, \phi_\pm, \Psi_\pm, \Psi_0 \) are smoothly extended to \([0, a]\), in order to write the \( s \)-integrals on \( \mathbb{R}_+ \). The precise form of this extension, however, will not matter – due to the cut-off function \( \psi \).

The main goal of this proof is to derive, for any \( z_0 \in S^{n-1} \), an \( \varepsilon \)-uniform estimate of the form \( |J_\varepsilon(t, x)| \leq Ct^{-\nu} r^{-\mu} \), with \( \nu := \frac{n-m}{m-2} \geq 0 \) (since \( n \geq m \)). Because of the Lemmata 2.1, 2.3 this estimate will hold uniformly on \( z = x/|x| \in \).
$U_{z_0}$ with a constant $C = C(z_0)$. Due to the compactness of $\mathbf{S}^{n-1}$, finitely many points $z_1, \ldots, z_N$ will suffice to yield a uniform estimate of $|J_{\varepsilon}(t, x)|$ on $\{r \geq t \geq 1\}$, using $C = \max_{j=1, \ldots, N} C(z_j)$. Here, we only consider the case of $e^{-\varepsilon \pm i \tau s}$; for $e^{-\varepsilon - i \tau s}$ the estimates are analogous.

Following Lemma 2.3 we decompose $J_{\varepsilon}$ as follows:

$$J_{\varepsilon}(t, x) = 2r^{-\frac{n-1}{2}} \int_0^\infty e^{-\varepsilon s + i \phi_+(t, r, s)} s^{\frac{n-1}{2} - 2} \psi(s) \Psi_+(rs^{\frac{m}{n}}, s^2) ds$$

$$+ 2r^{-\frac{n-1}{2}} \int_0^\infty e^{-\varepsilon s + i \phi_-(t, r, s)} s^{\frac{n-1}{2} - 2} \psi(s) \Psi_-(rs^{\frac{m}{n}}, s^2) ds$$

$$+ 2 \int_0^\infty e^{-\varepsilon s + i \tau s} s^{\frac{m}{n} - 2} \psi(s) \Psi_0(rs^{\frac{m}{n}}, s^2) ds$$

$$=: R^+_\varepsilon(t, x) + R^-_\varepsilon(t, x) + R^0_\varepsilon(t, x),$$

where $\phi_{\pm}$ is defined in Lemma 2.2.

We shall first estimate the integral $R^0_\varepsilon(t, x)$ and set $v_0(s) := s^{\frac{m}{n} - 2} \psi(s) \Psi_0(rs^{\frac{m}{n}}, s^2)$. By the Leibniz rule and (2.13), we have

$$|v_0^{(k)}(s)| \leq C(r s^{\frac{m}{n}})^{-l} s^{\frac{m}{n} - 2 - k}$$

for $l, k \in \mathbb{N}_0$,

where $r \geq 1$ and $s \geq a_1$. Choose $l \geq \mu \geq 0$ and $k \geq \nu \geq 0$. It thus follows by integration by parts that

$$|R^0_\varepsilon(t, x)| \leq Ct^{-k} \int_0^\infty (rs^{\frac{m}{n}})^{-l} s^{\frac{m}{n} - 2 - k} ds \leq Ct^{-k} r^{-l} \leq Ct^{-\nu} r^{-\mu}. \quad (3.8)$$

To estimate the integral $R^+_\varepsilon(t, x)$, for given $r \geq t \geq 1$, we set

$$\begin{cases} u_+(s) := -\varepsilon s + i \phi_+(t, r, s), \\
v_+(s) := s^{\frac{2m}{n} - 2} \psi(s) \Psi_+(rs^{\frac{m}{n}}, s^2) \end{cases}$$

for $s \geq 0$. Since $u'_+(s) \neq 0$ for $s \geq a_1$, we can define $D_s f := (gf)'$ for $f \in C^1(0, \infty)$, where $g := -1/u'_+$. It is not hard to show

$$D^j_s v_+ = \sum_{\alpha} c_\alpha g^{(\alpha_1)}(s) \cdots g^{(\alpha_j)}(s) v_+^{(\alpha_{j+1})} \quad \text{for } j \in \mathbb{N}, \quad (3.9)$$

where the sum runs over all $\alpha = (\alpha_1, \cdots, \alpha_{j+1}) \in \mathbb{N}_0^{j+1}$ such that $|\alpha| = j$ and $0 \leq \alpha_1 \leq \cdots \leq \alpha_j$. Since (2.6) and (2.9) imply, respectively, $|g(s)| \leq C r^{-1} s^{1 - \frac{m}{n}}$ and

$$|u_+^{(k)}(s)| \leq C r s^{\frac{m}{n} - k} \quad \text{for } k = 2, 3, \cdots,$$

we find by induction on $k$:

$$|g^{(k)}(s)| \leq C r^{-1} s^{1 - \frac{m}{n} - k} \quad \text{for } k \in \mathbb{N}_0,$$
which shall yield the spatial decay of $I_2$. To derive the time decay of $I_2$, we note that \((2.10)\) also implies $|g(s)| \leq t^{-1}$. Using this inequality for just one factor in $g^{(k)}$ we obtain:

$$|g^{(k)}(s)| \leq Ct^{-1}s^{-k} \quad \text{for } k \in \mathbb{N}_0.$$  \hfill (3.10)

The novel key step is now to interpolate these two inequalities, which will allow us to derive estimates also for large time. We have for any $\theta \in [0,1]$:

$$|g^{(k)}(s)| \leq Ct^{\theta-1}r^{-\theta}s^{\theta(1-\frac{2}{m})-k} \quad \text{for } k \in \mathbb{N}_0.$$  \hfill (3.11)

It thus follows from (3.9) – (3.11) that

$$|g^{(k)}(s)| \leq Ct^{\theta-1}r^{-\theta}s^{\theta(1-\frac{2}{m})-k} \quad \text{for } j \in \mathbb{N}_0,$$  \hfill (3.12)

where $D^j_1v_+ := v_+$. The particular choice $\theta = \frac{\mu}{n}$, $j = n$ yields

$$|D^j_1v_+(s)| \leq Ct^{\mu-n}r^{-\mu}s^{\frac{2n-2\mu+2}{m}}. \hfill (3.13)$$

Noting that $\mu - n < -\nu$, one gets by integration by parts

$$|R^+_\varepsilon(t,x)| = 2r^{-\frac{n-1}{2}} \int_0^\infty e^{u_+(D^j_1v_+)}ds \leq Ct^{\mu-n}r^{-\frac{2-n}{m}} \leq Ct^{-\nu}r^{-\mu}.$$

We now turn to the integral $R^-_\varepsilon(t,x)$. Here we put

$$\begin{align*}
\{u_+(s) := -\varepsilon s + i\phi_-(t,r,s), \\
v_-(s) := s^{\frac{n+1}{m}-2}\psi(s)\Psi_n(rs^{\frac{1}{m}},s^2)
\end{align*}$$

for $s \geq 0$. We shall denote $s_0 := (r/t)^{\frac{m}{m-2}}$, $c'_1 := (c_1/2)^{\frac{m}{m-2}}$, and $c'_2 := (2c_2)^{\frac{m}{m-2}}$, with $c_1$ and $c_2$ given in Lemma \([2.2]\). Now we decompose $R^-_\varepsilon$ as

$$R^-_\varepsilon(t,x) = 2r^{-\frac{n-1}{2}} \left\{ \int_0^{c'_1s_0} + \int_{c'_2s_0}^{\infty} e^{u_-(s)}v_-(s)ds \right\} =: R^-_{\varepsilon 1}(t,x) + R^-_{\varepsilon 2}(t,x) + R^-_{\varepsilon 3}(t,x).$$

This decomposition is motivated by the fact that the phase $\partial_s\phi_-(t,r,\cdot)$ is negative on $[0,c'_1s_0)$, positive on $[c'_2s_0,\infty)$, and is has exactly one zero on $[c'_1s_0,c'_2s_0]$ (cf. \([2.7], 2.8\)).

Integrating by parts we obtain

$$R^-_{\varepsilon 3}(t,x) = 2r^{-\frac{n-1}{2}} \left( e^{u_-(c'_2s_0)} \sum_{j=0}^{n-1} (D^j_1v_+)(c'_2s_0) + \int_{c'_2s_0}^\infty e^{u_-(D^j_1v_+)}ds \right).$$
Here and in the sequel, the differential operator $D_x f = (gf)'$ is considered with $g = -1/u'$. Since \((2.7)\) implies $|u'(s)| \geq c_2 r s^{\frac{2i}{m} - 1}$ for $s \geq c_2 s_0$, we find that $v_\nu(s)$ also satisfies \((3.12)\) and \((3.13)\) for $s \geq c_2 s_0$. If $c_2 s_0 \leq a_1$, then $(D^s_\nu(v))(c_2 s_0) = 0$ for $j = 0, \cdots, n - 1$ (note that $\psi \equiv 0$ on $[0, a_1]$). Integration by parts then yields

$$|R_{c_2}(t, x)| = \left|2r^{-\frac{n+1}{m}} \int_{a_1}^{\infty} e^{u_\nu(D^s_\nu(v))} ds\right| \leq Ct^{-\nu} r^{-\mu},$$

exactly as done for $R_{c_2}^+(t, x)$. If $c_2 s_0 > a_1$, then

$$|R_{c_2}(t, x)| \leq Cr^{-\frac{n+1}{m}} (r s_0^{\frac{2i}{m} - 1})^{-1} \sum_{j=0}^{n-1} r^{-j} s_0^{2 - j} \int_{c_2 s_0}^{\infty} r^{-n} s^{-n+m-1} ds \leq Cr^{-\frac{n+1}{m}} r^{-1} s_0^{\frac{2i}{m} - 1} \sum_{j=0}^{n-1} (r s_0^{\frac{2i}{m} - j} + r^{-n} s_0^{-n+m-1}).$$

Noting that $r \geq 1$, $s_0 > a_1/c_2$, and $t \geq 1$, it follows that

$$|R_{c_2}(t, x)| \leq Cr^{-\frac{n+1}{m}} s_0^{-n+m-1} = Ct^{-\nu} r^{-\mu} r^{-\frac{n}{2}} t^{-\frac{1}{2}} \leq Ct^{-\nu} r^{-\mu}.$$ 

Next we turn to $R_{c_2}^1(t, x)$, which is 0 for $c_1 s_0 < a_1$. If $c_1 s_0 \geq a_1$, we use $|u'(s)| \geq c_1 r s^{\frac{2i}{m} - 1}$ for $a_1 < s \leq c_1 s_0$. Then, a slight modification of the above method yields again $R_{c_2}^1(t, x) \leq Ct^{-\nu} r^{-\mu}$.

To estimate $R_{c_2}(t, x)$, it suffices to estimate the integral

$$R_{c_2}(t, x) = 2r^{-\frac{n+1}{m}} \int_{c_1 s_0}^{c_2 s_0} e^{i \phi_\nu(t, r, s) v_\nu(s)} ds = 2r^{-\frac{n+1}{m}} s_0 \int_{c_1}^{c_2} e^{i \phi_\nu(t, r, s_0 \tau) v_\nu(s_0 \tau)} d\tau,$$

where the interval of integration is now independent of the parameters $t, r$. We obtain from \((2.8)\) that

$$|\partial^2 \phi_\nu(t, r, s_0 \tau)| \geq c_1 r s_0^2 (s_0 \tau)^{\frac{2i}{m} - 2} \geq Cr s_0^{\frac{2i}{m}}$$

for $\tau \in [c_1, c_2]$. Since $v_\nu(s)$ also satisfies \((3.11)\), we obtain by using (a corollary of) the Van der Corput lemma (cf. \cite{37}, p. 334) (uniformly for
$\varepsilon > 0$ small enough):

\[ |R_{02}(t,x)| \leq Cr^{-\frac{n+1}{2}} s_0 (rs_0^2)^{-\frac{3}{2}} \left( |v_\cdot(T_0^t)| + \int_0^t |v_\cdot(T_0^\tau)|d\tau \right) \]

\[ \leq Cr^{-\frac{n+1}{2}} s_0 (rs_0^2)^{-\frac{3}{2}} s_0^{-\frac{n+1}{2}} \]

\[ = Ct^{-\nu} r^{-\mu}. \]

The dominated convergence theorem implies that $J_\varepsilon(t,\cdot)$ converges (as $\varepsilon \to 0$) uniformly for $x$ in compact subsets of $\{ x \in \mathbb{R}^n; |x| \geq 1 \}$. By summarizing the above estimates we have

\[ |I_{21}(t,x)| \leq Ct^{-\nu} |x|^{-\mu} \text{ for } |x| \geq t \geq 1, \]

and hence

\[ |I_{21}(t,x)| \leq Ct^{-\frac{1}{2}} (1 + t^{-1} |x|)^{-\mu} \leq Ct^{-\frac{1}{2}} (1 + t^{-1} |x|)^{-\mu} \text{ for } |x| \geq t \geq 1. \]

Combining this with the estimate (3.5) on $I_{22}$ (put $k = \mu - \frac{1}{m}$), we have

\[ |I_2(t,x)| \leq Ct^{-\frac{1}{2}} (1 + t^{-1} |x|)^{-\mu} \text{ for } |x| \geq t \geq 1. \]

Case (ii): $t \geq 1$ and $|x| \leq t$.

For $I_{21}$ we shall prove now that

\[ |I_{21}(t,x)| \leq C|t|^{-n/2} \text{ for } |t| \geq 1 \text{ and } |x| \leq |t|. \]

We proceed as in [21] and write the integral $I_{21}(t,x)$ as follows:

\[ I_{21}(t,x) = \int_{\mathbb{R}^n} e^{it(\pm P^{1/2}(\xi) + (x/t,\xi))} P^{-1/2}(\xi) \psi(P^{1/2}(\xi))d\xi \]

\[ = \int_{\mathbb{R}^n} e^{it\Phi(\xi,x,t)} P^{-1/2}(\xi) \psi(P^{1/2}(\xi))d\xi, \]

but we shall focus on the case $\Phi = P^{1/2}(\xi) + (x/t,\xi)$, and the other case is analogous.

Since $|x/t| \leq 1$, $P^{1/2}(\xi) \leq c_1 |\xi|^{m_1}$, and $|\nabla P(\xi)| \geq c_2 |\xi|^{m-1}$ for large $|\xi|$, the possible critical points satisfying

\[ \nabla_\xi \Phi(\xi,x,t) = \frac{\nabla P(\xi)}{2P^{1/2}(\xi)} + \frac{x}{t} = 0 \]

must be located in some bounded ball. In order to apply later the stationary phase principle, let $\Omega \subset \mathbb{R}^n$ be some open set such that $\text{supp} \psi(P^{1/2}) \subset \Omega$ and $|\nabla P^{1/2}(\xi)| \geq c|\xi|^{m_1-1}$ on $\Omega$. Note that the constant $a_1$ (from the definition of $\psi$
and Lemma 2.2 could be increased, if necessary, such that both of those conditions can hold. Then we decompose $\Omega$ into $\Omega_1 \cup \Omega_2$, where

$$\Omega_1 = \left\{ \xi \in \Omega : \left| \nabla P^{1/2}(\xi) + \frac{x}{t} \right| < \frac{1}{2} \left| \nabla P^{1/2}(\xi) \right| + 1 \right\}$$

and

$$\Omega_2 = \left\{ \xi \in \Omega : \left| \nabla P^{1/2}(\xi) + \frac{x}{t} \right| > \frac{1}{4} \left| \nabla P^{1/2}(\xi) \right| \right\}.$$  

Since $|\frac{x}{t}| \leq 1$ and $|\nabla P^{1/2}(\xi)| \to \infty$ as $|\xi| \to \infty$, $\Omega_1$ must be a bounded domain and includes all critical points of $\Phi$ inside $\Omega$. Now we choose smooth functions $\eta_1(\xi)$ and $\eta_2(\xi)$ such that $\text{supp} \, \eta_j \subset \Omega_j$ and $\eta_1(\xi) + \eta_2(\xi) = 1$ in $\Omega$. And we decompose $I_{21}$ as

$$I_{21}(t, x) = I_{211}(t, x) + I_{212}(t, x),$$

$$I_{21j}(t, x) := \int_{\mathbb{R}^n} e^{it\Phi(\xi, x, t)} \eta_j(\xi) P^{1/2}(\xi) \psi(P^{1/2}(\xi)) d\xi; \quad j = 1, 2.$$ 

To estimate $I_{211}$ we note that

$$\det(\partial_\xi, \partial_\xi, \Phi)_{n \times n}(\xi, x, t, \xi) = \det(\partial_\xi, \partial_\xi, P^{1/2})_{n \times n}(\xi).$$  

Lemma 5.3 (see the Appendix below) implies that the r.h.s. is nonzero on $\Omega$ (if necessary, we can increase the value of $a_1$ to satisfy the requirement), that is, the Hessian matrix is non-degenerate on $\Omega$. Moreover, $|\partial_\xi^\alpha \Phi| \leq C_\alpha$ on $\Omega_1$ for any multi-index $\alpha \in \mathbb{N}_0^n$. Hence we obtain by the stationary phase principle that

$$|I_{211}(t, x)| \leq C |t|^{-n/2}. \quad (3.15)$$

To estimate $I_{212}$, we shall use some cut-off in order to make the subsequent integrations by parts meaningful (cp. to the procedure in (3.6)). Using a smooth, compactly supported cut-off function $0 \leq \varphi \leq 1$ with $\varphi(0) = 1$, we shall derive an $\varepsilon$–uniform estimate (as $\varepsilon \to 0$) of

$$I_{212}(t, x) := \int_{\mathbb{R}^n} e^{it\Phi(\varepsilon \xi, x, t)} \eta_2(\xi) \varphi(\varepsilon \xi) P^{1/2}(\xi) \psi(P^{1/2}(\xi)) d\xi.$$ 

Note that $|\nabla_\xi \Phi| = |\nabla P^{1/2}(\xi) + \frac{\varepsilon}{t} | \geq \frac{1}{4} |\nabla P^{1/2}(\xi)| \geq c|\xi|^{m_1-1}$ for $\xi \in \Omega_2$ and $|\partial_\xi^\alpha \Phi| \leq C_\alpha |\xi|^{m_1-\alpha}$ for $|\alpha| \geq 2$. Now we define the operator $L$ by

$$L f := \frac{\langle \nabla_\xi \Phi, \nabla_\xi \rangle}{it|\nabla_\xi \Phi|^2} f.$$
Since $Le^{it\Phi} = e^{it\Phi}$, we obtain by $N$ iterated integrations by parts:

$$|I_{212}(t,x)| = \left| \int_{\mathbb{R}^n} e^{it\Phi(\xi,x,t)} (L^*)^N \left[ \varphi(\varepsilon \xi) \eta_2(\xi) P^{-1/2}(\xi) \psi(P^{1/2}(\xi)) \right] d\xi \right| \leq C_N |t|^{-N} \int_{\text{supp}(P^{1/2})} |\xi|^{-mN} d\xi \leq C'_N |t|^{-N},$$

where $N > n$ and $L^*$ is the adjoint operator of $L$. Combining the estimates (3.15) and (3.16) yields the claimed estimate $|I_{21}| \leq C|t|^{-n/2}$ for $|t| \geq 1$ and $|x| \leq |t|$.

Together with the estimate (3.15) (with $k = \mu - \frac{1}{m}$) on $I_{22}$ this yields

$$|I_{22}(t,x)| \leq C t^{-\frac{m}{2}} (1 + t^{-1}|x|)^{-\mu} \quad \text{for } t \geq 1 \text{ and } |x| \leq |t|. \quad (3.17)$$

Thus, combining the cases (i) and (ii), we conclude

$$|I_2(t,x)| \leq C t^{-\frac{m}{2}} (1 + t^{-1}|x|)^{-\mu} \quad \text{for } t \geq 1 \text{ and } x \in \mathbb{R}^n. \quad (3.18)$$

Case (iii): For $0 < t < 1$ and $x \in \mathbb{R}^n$ we shall use a standard scaling argument. We observe that

$$I_2(t,x) = \int_{\mathbb{R}^n} e^{i \langle (x,\xi) + tP^{1/2}(\xi) \rangle P^{-1/2}(\xi) d\xi = t^{-\frac{m}{2}} \int_{\mathbb{R}^n} e^{i \langle (x,\xi) + tP^{1/2}(\xi) \rangle P^{-1/2}(t^{-1/m} \xi) d\xi = t^{-\frac{m}{2} + 1} \int_{\mathbb{R}^n} e^{i \langle (t^{-1/m} x,\xi) + (t^2 P((t^2)^{-\frac{m}{2}}) \xi) \rangle} (t^2 P((t^2)^{-\frac{m}{2}}) \xi)^{-1/2} d\xi.$$

Let $P_t(\xi) := t^2 P(t^{-\frac{m}{2}} \xi)$, $\rho_t(s,\omega) := t^{\frac{m}{2}} \rho(t^{-\frac{m}{2}} \xi)$, and $\sigma_t(s,\omega) := t^{\frac{m}{2}} \sigma(t^{-\frac{m}{2}} \xi)$. Then (2.1) still holds when $P$, $\rho$, $\sigma$ are replaced, respectively, by $P_t$, $\rho_t$, $\sigma_t$. It is easy to check that $\sigma_t$ also satisfies (2.2) with the same constants $C_{kL}$. Hence, we can deduce from (3.18) (with $t = 1$) that

$$|I_2(t,x)| \leq C t^{-\frac{m}{2} + 1} (1 + t^{-1/m} |x|)^{-\mu}, \quad \text{for } t \in (0,1) \text{ and } x \in \mathbb{R}^n. \quad (3.19)$$

This completes the proof of the theorem. \[\square\]

Remark 3.2. If one checks the details of the proof for the cases (i) and (ii) above, one finds that the estimate for $I_2(t = 1, x)$ does not use the condition $n \geq m$. Therefore the estimate (3.19) of $I_2(t, x)$ for $0 < t < 1$ is also obtained by scaling without the restriction $n \geq m$. 

Similarly to the above proof of $I_2$, we obtain the following result for the oscillatory integral $I_1(x,t)$.

**Theorem 3.3.** Assume that the polynomial $P$ satisfies (H1) and (H2). Then

$$|I_1(t,x)| \leq \begin{cases} C|t|^{\frac{-m_1}{2}} (1 + |t|^{\frac{-1}{m_1}} |x|)^{-\frac{n(m-4)}{2(m-2)}}, & \text{for } 0 < |t| \leq 1, \\ C|t|^{\frac{1}{2}} (1 + |t|^{-1} |x|)^{-\frac{n(m-4)}{2(m-2)}}, & \text{for } |t| \geq 1. \end{cases} \quad (3.20)$$

Note that $I_1$ has the same structure as the oscillatory integral $I(t,x)$ in [21] for higher order Schrödinger equations, when replacing $P^{1/2}(\xi)$ from (1.4) by $P(\xi)$. Thus, Theorem 3.3 is closely related to Theorem 3.1 of [21] (when replacing $m_1$ by $m$). This similarity is also easily seen on the level of the considered evolution equations: The differential operator of our wave-type equation can be factored as

$$\partial_t + P(D_x) = [\partial_t + i\sqrt{P}(D_x)] [\partial_t - i\sqrt{P}(D_x)],$$

where each squared bracket corresponds to a time-dependent Schrödinger equation.

### 4. Decay/growth estimates for wave-type equations

Here we shall apply the Theorems 3.1, 3.3 to establish $L^p - L^q$ estimates for the solution of the following higher order wave-type equation:

$$\begin{cases} \partial_t u(t,x) + P(-i\nabla)u(t,x) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \phantom{\partial_t u(0,x) = u_1(x),} x \in \mathbb{R}^n. \end{cases}$$

As in (1.3), its solution is given by

$$u(t,x) = \mathcal{F}^{-1} \cos \left( P^{1/2}(\xi)t \right) \mathcal{F}u_0 + \mathcal{F}^{-1} \frac{\sin \left( P^{1/2}(\xi)t \right)}{P^{1/2}(\xi)} \mathcal{F}u_1 =: U(t,x) + V(t,x).$$

For any $\alpha \in \mathbb{R}$ we define the following set of admissible index pairs.

$$\Delta_\alpha := \{(p,q); \ (\frac{1}{p}, \frac{1}{q}) \ \text{lies in the closed quadrangle } ABCD\},$$

where $A = (\frac{1}{2}, 0)$, $B = (1, \frac{1}{q_0})$, $C = (1, 0)$, and $D = (\frac{1}{q_0}, 0)$ for $q_0 := \frac{n}{\mu_a}$, $\mu_a := \frac{mn - 4n + 2a}{2(m-2)}$, and $\frac{1}{q} + \frac{1}{q} = 1$. Moreover, we denote the Lorentz space by $L^{p,q}(\mathbb{R}^n)$ (see p.48 in [17]).
Theorem 4.1. Assume that the polynomial $P$ satisfies the conditions (H1) and (H2), and let $n \geq m$ (and hence $2 \leq q_m < \infty$). Then we have

$$
\|V(t, \cdot)\|_{L^q} \leq \begin{cases} 
C |t|^{\frac{m}{m+1} - \frac{1}{q} + 1} \|u_1\|_{L^p}, & \text{for } 0 < |t| \leq 1, \\
C |t|^{\frac{n}{2} - \frac{1}{p} - \frac{1}{m}} \|u_1\|_{L^p}, & \text{for } |t| \geq 1,
\end{cases} 
(4.1)
$$

where $(p, q) \in \triangle_m$, $m_1 := m/2$. Here, the pair of spaces $(L^p, L^q)$ has the following meaning:

$$(L^p, L^q) = \begin{cases} 
(L^1, L^{q_m, \infty}), & \text{if } (p, q) = (1, q_m), \\
(L^{q_m, 1}, L^\infty), & \text{if } (p, q) = (q_m, \infty), \\
(L^p, L^q), & \text{otherwise}.
\end{cases} 
(4.2)
$$

Proof. By the assumption (H1), one has

$$
\left| \sin \left(\frac{P^{1/2}(\xi)t}{P^{1/2}(\xi)}\right) \right| \leq \begin{cases} 
|t|, & \text{for } 0 < |t| \leq 1, \\
C, & \text{for } |t| \geq 1.
\end{cases}
$$

Then, the Plancherel theorem gives the result for the index point $A$:

$$
\|V(t, \cdot)\|_{L^2} \leq \begin{cases} 
|t| \|u_1\|_{L^2}, & \text{for } 0 < |t| \leq 1, \\
C \|u_1\|_{L^2}, & \text{for } |t| \geq 1.
\end{cases} 
(4.3)
$$

On the other hand, by Theorem 3.1 we have for each $t \neq 0$: $I_2(t, \cdot) \in L^q(\mathbb{R}^n)$ $\forall q > q_m$ and $I_2(t, \cdot) \in L^{q_m, \infty}(\mathbb{R}^n)$ (the weak $L^{q_m}$ space). Applying the (weak)
Young inequality (see p.22 in [17]) to the second term of (1.3) then implies
\[ \|V(t, \cdot)\|_{L^q} \leq \begin{cases} C|t|^\frac{n}{m} + 1 \|u_1\|_{L^1}, & \text{for } 0 < |t| \leq 1, \\ C|t|^{-\frac{m}{n}} \|u_1\|_{L^1}, & \text{for } |t| \geq 1. \end{cases} \] (4.4)

This proves the estimate for the points \((1, \frac{1}{q})\) on the edge \(CB\). Applying the Marcinkiewicz interpolation theorem (see p.56 in [17]) to (4.3) and (4.4) proves (4.1) for the points in the closed triangle \(ABC\). By duality, the estimate for the triangle \(ADC\) follows immediately from the result in the triangle \(ABC\) (note that the adjoint operator of \(I_2 * u_1\) has the same structure). To include the result for the index point \(D\), we remark that \(L^{q'_m, \infty} \subset (L^{q_m, \infty})^*\) (cf. [13]). This completes the proof of the theorem. \(\Box\)

Next we shall complement this result with a straightforward estimation of
\[ V(t, x) = F^{-1}Q(t, \xi) \mathcal{F}u_1, \quad Q(t, \xi) := \frac{\sin \left( \frac{P_1}{2}(\xi) t \right)}{P_1(\xi) t}. \]

To this end we define the index points \(E = (n + m, \frac{1}{2}), F = (\frac{1}{2}, n - m)\).

**Theorem 4.2.** Let the polynomial \(P\) satisfy (H1), and let \(n \geq m\). Then we have
\[ \|V(t, \cdot)\|_{L^q} \leq \begin{cases} C|t|^\frac{n}{m} + 1 \|u_1\|_{L^p}, & \text{for } 0 < |t| \leq 1, \\ C\|u_1\|_{L^p}, & \text{for } |t| \geq 1, \end{cases} \] (4.5)

where \((\frac{1}{p}, \frac{1}{q})\) lies in the closed triangle \(AEF\) and \(m_1 := m/2\). Here, we denote \(L^q := L^{q, \infty}\) if \(\frac{1}{q} = \frac{1}{p} - \frac{m}{2n}\). And \(L^q := L^q\), elsewise.

**Proof.** By the assumption (H1), we have \(|Q(t, \xi)| \leq C|\xi|^{-m_1}\). And hence, \(Q(t, \cdot) \in L^{2\infty, \infty}(\mathbb{R}^n)\). Since we assumed \(u_1 \in L^p\) for some \(\frac{1}{2} \leq \frac{1}{p} \leq \frac{n + m}{2n}\), we have \(\mathcal{F}u_1 \in L^{\tilde{p}}\).

And the Hölder inequality for Lorentz spaces (cf. [17]) implies
\[ Q(t, \xi) \mathcal{F}u_1 \in L^{\tilde{p}, \infty}, \quad \tilde{p} := \frac{p}{p - 1 + \frac{m}{2n}}. \]

The Hausdorff-Young inequality for Lorentz spaces (cf. [24]) then yields the result for the edge \(EF\) with \(\frac{1}{q} = \frac{1}{p} - \frac{m}{2n}:
\[ \|V(t, \cdot)\|_{L^{q, \infty}} \leq C\|u_1\|_{L^p}, \quad \forall t \in \mathbb{R}. \]

Applying the Marcinkiewicz interpolation theorem (with (4.3)) concludes the proof. \(\Box\)
Remark 4.3. 1. The short time behavior of $u$ in Th. 4.1 and Th. 4.2 coincides for the indices in $AEF \cap ABCD$. But for large time, the r.h.s. of (4.3) stays uniformly bounded, which is not always the case in (4.1).

2. A Marcinkiewicz interpolation between the edges $EF$ and $BC$ (plus a duality argument for $\frac{1}{q} < \frac{1}{p}′$) allows to extend the decay/growth estimate on $u$ to the closed hexagon $AEBCDF$. But since this follows exactly the above strategy, we do not give details here.

3. Theorem 4.2 actually also holds for $n < m$, but we skipped the statement for notational simplicity. If $m \in (n, 2n)$ one obtains a decay/growth estimate for the index pair $(\frac{1}{p}, \frac{1}{q})$ in the closed pentagon described by the five different endpoints: $(\frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}), (1, \frac{n-m}{n}), (\frac{m}{n}, 0), (\frac{1}{2}, 0)$. And for $m \geq 2n$ for the whole index square $\frac{1}{2} \leq \frac{1}{p} \leq 1, 0 \leq \frac{1}{q} \leq \frac{1}{2}$.

Now we turn to the estimate of $U$:

Theorem 4.4. Assume that the polynomial $P$ satisfies (H1) and (H2). Then we have

\[
\|U(t,x)\|_{L^q_{x}} \leq \begin{cases} 
C|t|^{\frac{n-m}{n} - \frac{1}{p} - \frac{1}{q}}\|u_0\|_{L^p_{x}}, & \text{for } 0 < |t| \leq 1, \\
C|t|^{\frac{n-m}{n} - \frac{1}{p} - \frac{1}{q}} \|u_0\|_{L^p_{x}}, & \text{for } |t| \geq 1,
\end{cases}
\]

(4.6)

where $(p,q) \in \triangle_0$ and $(L^p_{x}, L^q_{x})$ is defined in (4.2) (when replacing $q_m$ by $q_0$).

Using Th. 3.3, the proof of Th. 4.4 is very similar to Th. 4.1. So we omit the details here.

Remark 4.5. Let us briefly compare our results to the literature: While Theorem 2.3 of [32] only yields $L^p - L^{p'}$ estimates for the case $P = 1 + |\xi|^m$, our Th. 4.1 provides more general $L^p - L^q$ estimates. Moreover, our result applies to more general polynomials $P$.

5. Appendix: The type of a hypersurface

In § VIII.3.2 of [37] the type of a hypersurface $S := \{z = \Phi(\xi); \xi \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$ is defined as follows: The type $\tilde{m}(\xi_0)$ of $S$ at $\xi_0$ is the smallest integer $k \geq 2$, such that the matrix (or tensor) $(\partial^\alpha \Phi(\xi_0))_{|\alpha|=k}$ does not vanish. Then, the type of $S$ is $\tilde{m} := \max_{\xi_0 \in \mathbb{R}^n} \tilde{m}(\xi_0)$. 
Lemma 5.1. Let \( \deg P(\xi) = m \geq 4 \) and \( P(\xi) > 0 \) on \( \mathbb{R}^n \). Then, the type of \( S := \{ z = P^{1/2}(\xi) \} \) satisfies \( \tilde{m} \leq m \).

Proof. Assume that there exists a \( \xi_0 \in \mathbb{R}^n \) with \( \tilde{m}(\xi_0) > m \). Since \( P^{1/2} \) is smooth we have in a small neighborhood around \( \xi_0 \):

\[
P^{1/2}(\xi) = P^{1/2}(\xi_0) + (\xi - \xi_0) \cdot \nabla_{\xi} P^{1/2}(\xi_0) + O \left( |\xi - \xi_0|^\tilde{m}(\xi_0) \right),
\]

Hence,

\[
P(\xi) = P(\xi_0) + 2(\xi - \xi_0) \cdot \nabla_{\xi} P^{1/2}(\xi_0) P^{1/2}(\xi_0) + \left[ (\xi - \xi_0) \cdot \nabla_{\xi} P^{1/2}(\xi_0) \right]^2 + O \left( |\xi - \xi_0|^\tilde{m}(\xi_0) \right),
\]

which contradicts \( \deg P(\xi) = m \) with \( m \geq 4 \).

\( \square \)

Remark 5.2. 1. If we assume \( m = 2 \) in Lemma 5.1 we obtain \( \tilde{m} = 2 \).

2. In the example \( P(\xi) = 1 + 2|\xi|^2 + |\xi|^4 \) we have \( P^{1/2}(\xi) = 1 + |\xi|^2 \), and hence \( \tilde{m} = 2 < m \). But in general we can only conclude \( \tilde{m} \leq m \) for \( m \geq 4 \).

Lemma 5.3. Let the polynomial \( P \) on \( \mathbb{R}^n \) satisfy (H1) and (H2). Then

\[
\det \left( \frac{\partial^2 P^{1/2}(\xi)}{\partial \xi_i \partial \xi_j} \right)_{n \times n} \sim c \left( \frac{\xi}{|\xi|} \right)^{|\xi|^{n(\overline{d} - 2)}} \text{ for } |\xi| \text{ large},
\]

where \( c \) is a smooth function on the unit sphere of \( \mathbb{R}^n \), bounded away from 0.

Proof. Step 1:

With \( P_m \) denoting the principal part of \( P \), we define \( \phi(\xi) := P_m^{1/m}(\xi) \), which is positive for \( \xi \neq 0 \) and homogeneous of degree one. Now we consider its level-1-hypersurface

\[
\Sigma := \{ \xi \in \mathbb{R}^n ; \phi(\xi) = 1 \} \subset \mathbb{R}^n.
\]

Since \( P_m = \phi^m \) is non-degenerate by assumption (H2) (i.e. \( \det (\partial_i \partial_j \phi^m) \neq 0 \) for \( \xi \neq 0 \)), Proposition 4.2 from [10] implies that \( \Sigma \) is strictly convex and of type 2.

Applying again Proposition 4.2 (with \( \lambda = m/2 \)) implies

\[
\det \left( \frac{\partial^2 P_m^{1/2}(\xi)}{\partial \xi_i \partial \xi_j} \right)_{n \times n} \neq 0 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \tag{5.1}
\]

Step 2:

Now we decompose

\[
P^{1/2}(\xi) = P_m^{1/2}(\xi) + a(\xi),
\]
with \( a = \frac{P - P_m}{\sqrt{P + P_m}} \in S^{\frac{n}{2} - 1} \). Hence,

\[
\det \left( \frac{\partial^2 P_{1/2} (\xi)}{\partial \xi_i \partial \xi_j} \right) = \det \left( \frac{\partial^2 P_{m}^{1/2} (\xi)}{\partial \xi_i \partial \xi_j} \right) + Q(\xi),
\]

where the first term on the r.h.s. is \( O \left( |\xi|^n \left( \frac{m^2}{m^2 - 2} \right) \right) \) for \( \xi \) large, and the second term is of the order \( O \left( |\xi|^n \left( \frac{m^2}{m^2 - 2} - 1 \right) \right) \). The claim then follows from (5.1). \( \square \)

References

[1] M. Balabane and H. A. Emami-Rad, \( L^p \) estimates for Schrödinger evolution equations, Trans. Amer. Math. Soc. 292 (1985), 357-373.
[2] M. Ben-Artzi, Eigenfunction expansion and spacetime estimates for generators in divergence form, Rev. Math. Phy. 22 (2010), 1209-1240.
[3] M. Ben-Artzi, H. Koch and J.-C. Saut, Dispersion estimates for fourth order Schrödinger equations, CRAS Paris 330 (2000), 87-92.
[4] M. Ben-Artzi, H. Koch and J.-C. Saut, Dispersion estimates for third order equations in two dimensions, Comm. PDE 28 (2003), 1943-1974.
[5] M. Ben-Artzi and J.-C. Saut, Uniform estimates for a class of oscillatory integrals and applications, Diff. and Int. Eq. 12 (1999), 137-145.
[6] M. Ben-Artzi and F. Treves, Uniform estimates for a class of evolution equations, J. Funct. Anal. 120 (1994), 264-299.
[7] J.M Bouclet and N. Tzvetkov, On global Strichartz estimates for non-trapping metrics, J. Funct. Anal. 254 (2008), 1661-1682.
[8] P. Brenner, \( L_p - L_p \) estimates for the wave-equation, Math. Z. 145 (1975), 251-25.
[9] N. Burq, Global Strichartz estimates for non-trapping geometries: About an article by H. Smith and C. Sogge, Comm. PDE 28 (2003), 1675-1683.
[10] W. Chen, C. Miao, and X. Yao, Dispersive estimates with geometry of finite type, Comm. PDE 37 (2012) 479-510.
[11] S. Cui, Point-wise estimates for a class of oscillatory integrals and related \( L^p \) - \( L^q \) estimates, J. Fourier Anal. and Appl. 11 (2005), 441-457.
[12] S. Cui, Point-wise estimates for oscillatory integrals and related \( L^p \) - \( L^q \) estimates: Multi-dimensional case, J. Fourier Anal. and Appl. 12 (2006), 605-627.
[13] M. Cwikel, The dual of weak \( L^p \), Annales de l’institut Fourier 25 (1975), 81-126.
[14] Y. Ding and X. Yao, \( L^p \) - \( L^q \) estimates for dispersive equations and related applications, J. Math. Anal. and Appl. 356 (2009), 711-728.
[15] Y. Ding and X. Yao, \( H^p \) - \( H^q \) estimates for dispersive equations and related applications, J. Funct. Anal. 257 (2009), 2067-2087.
[16] J. Ginibre and G. Velo, Generalized Strichartz inequalities for the wave equation. J. Funct. Anal. 133 (1995), 50-68.
[17] L. Grafakos, Classical and modern Fourier analysis, Prentice Hall, New Jersey, 2003.
[18] L.V. Kapitanski, The Cauchy problem for a semilinear wave equation, Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov (LOMI) 181-182 (1990) 24-64, and 38-85.
[19] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. 120 (1998), 360-413.

[20] C.E. Kenig, G. Ponce, and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. 40 (1991), 33-69.

[21] J.M. Kim, A. Arnod, X. Yao, *Global estimates of fundamental solutions for higher-order Schrödinger equations*, To appear in Monatsh. Math. 2012.

[22] S. Levandosky, *Decay estimates for the fourth-order wave equations*, J. Differential Equations 143 (1998), 360-413.

[23] S. Levandosky and W. Strauss, *Time decay for the nonlinear beam equation*, Methods and Applications of Analysis 7 (2000), 783-798.

[24] L. Maligranda and L.E. Persson, *Inequalities and interpolation*, Collect. Math. 44 (1993), 181-199.

[25] B. Marshall, W. Strauss and S. Wainger, *$L^p - L^q$ estimates for the Klein-Gordon equation*, J. Math. Pure Appl. 59 (1980), 417-440.

[26] C. Miao, *Harmonic Analysis and its Application in Partial Differential Equations*, (in Chinese) Science Press, Beijing, 2006.

[27] A. Miyachi, *On some estimates for the wave equation in $L^p$ and $H^p$*, J. Fac. Sci. Univ. Tokyo, 27 (1980), 331-354.

[28] G. Mockenhaupt, A. Seeger, and C. Sogge. *Local smoothing of Fourier integral operators and Carleson-Sjölin estimates*, J. Amer. Math. Soc. 6 (1993), 65-130.

[29] C.S. Morawetz, *Time decay for the Klein-Gordon equation*, Proc. Roy. Soc. A 306 (1968), 291-296.

[30] B. Pausader, *Scattering and the Levandosky-Strauss conjecture for fourth order nonlinear wave equations*, J. Differential Equations 241 (2) (2007), 237-278.

[31] B. Pausader, *Scattering for the Beam equation in low dimensions*, Indiana Univ. Math. J. 59 (3) (2010), 791-822.

[32] H. Pecher, *$L^p$-Abschätzungen und klassische Lösungen für nicht-lineare Wellengleichungen*, Math. Z. 150 (1976), 159-183.

[33] H. Smith and C.D. Sogge, *On the critical semilinear wave equation outside convex obstacles*, J. Amer. Math. Soc. 8 (1995), 879-916.

[34] H. Smith and C.D. Sogge, *Global Strichartz estimates for nontrapping perturbations of the Laplacian*, Comm. PDE 25 (2000), 2171-2183.

[35] C.D. Sogge, *Fourier Integrals in Classical Analysis*, Cambridge Univ. Press, Cambridge, 1993.

[36] C.D. Sogge, *Lecture on Nonlinear Wave Equations*, International Press Publications, 1995.

[37] E. M. Stein, *Harmonic Analysis: Real Variable Method, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, New Jersey, 1993.

[38] W. A. Strauss, *Nonlinear wave equations*, CBMS Regional Conference Series in Mathematics, No.73, 1989.

[39] R. Strichartz, *A priori estimates for the wave equations and some applications*, J. Funct. Anal. 5 (1970), 218-235.

[40] T. Tao, *Nonlinear dispersive equations, local and global analysis*, CBMS Regional Conference Series in Mathematics, No.106, 2006.
[41] D. Tataru, *Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients III*, J. Amer. Math. Soc. 15 (2001), 385-423.

[42] X. Yao and Q. Zheng, *Oscillatory integrals and $L^p$ estimates for Schrödinger equations*, J. Diff. Eq. 244 (2008), 741-752.

[43] Q. Zheng, X. Yao and D. Fan, *Convex hypersurface and $L^p$ estimates for Schrödinger equations*, J. Funct. Anal. 208 (2004), 122-139.

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