1 Introduction

A common method providing topological information of algebraic varieties is the consideration of fixed points under a torus action. For instance the Euler characteristic is already given by the Euler characteristic of their fixed point components. If we consider moduli spaces of stable quiver representations, we also often obtain interesting objects as fixed point components like indecomposable tree modules in the case of the Kronecker quiver, see [Wei3].

The main focus of this paper is on the Kronecker quiver. It is inasmuch particularly interesting as by use of the localization method we are able to prove parts of a conjecture of Michael Douglas [Dou] concerning the Euler characteristic of these moduli spaces. It says that for coprime dimension vectors \((d, e)\) the logarithm of the Euler characteristic continuously depends on the fraction \(\frac{e}{d}\). More specified this means that there exists a continuous function \(f\) such that for every coprime dimension vector \((d, e)\) there exists another dimension vector \((d_s, e_s)\) such that

\[
\ln \chi(M_{d_s+nd,e_s+ne}) = \lim_{n \to \infty} \frac{\chi(M_{d_s+nd,e_s+ne})}{nd + d_s}.
\]

In particular, the right side converges. In [Wei] a candidate for this function could be determined and it could be proved that under the assumption of continuity and the specification of the value at the point one the function is already uniquely determined.

We will see that the fixed points of moduli spaces for quivers without oriented cycles are exactly the stable representations of the universal abelian covering quiver. Actually, after suitable many localization steps the remaining torus fixed points are representations of the universal covering quiver. Even if continuity is still an open question, by use of the localization method and the just mentioned result we are able to calculate the value at the point one. Indeed, we are able to determine a formula for the Euler characteristic of the Kronecker moduli spaces for the dimension vectors \((d, d + 1)\). In particular, this shows the convergence at this point. Moreover, we are able to
prove the exponential growth implied by the conjecture. This also provides a lower bound for the function.

We will also show that the Euler characteristic for the dimension vectors \((d, (m-1)d)\) vanishes for \(d \geq 2\) because torus fixed points of this dimension type are always cyclic. Thereby the dimension type of a bipartite quiver is given by the sum of the dimensions of the sources and sinks respectively. Thus for this dimension type there do not exist stable representations of the universal covering quiver.

The investigation of the torus fixed point contains a detailed study of bipartite quivers. We will construct stable bipartite quivers of dimension type \((d_s, e_s) + n(d, e)\), i.e. quivers with a fixed dimension vector allowing at least one stable representation, by gluing certain bipartite quivers of dimension types \((d_s, e_s)\) and \((d, e)\). Thereby the dimension vector \((d_s, e_s)\) is uniquely determined by \((d, e)\).

Considering the localization in Kronecker moduli spaces we are able to prove another property concerning the Kronecker quiver. We can answer the question when there exist fixed point components containing infinitely many fixed points: in fact, there exist only finitely many torus fixed points for dimension vectors \((d, e)\) such that \(d = 1, 2\) or in the associated reflected cases.

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2 Generalities

Let \(k\) be an algebraically closed field.

**Definition 2.1** A quiver \(Q\) consists of a set of vertices \(Q_0\) and a set of arrows \(Q_1\) denoted by \(\alpha : i \to j\) for \(i, j \in Q_0\).

- A vertex \(q \in Q_0\) is called sink if there does not exist an arrow \(\alpha : q \to q' \in Q_1\).
- A vertex \(q \in Q_0\) is called source if there does not exist an arrow \(\alpha : q' \to q \in Q_1\).

A quiver is finite if \(Q_0\) and \(Q_1\) are finite.

A quiver is bipartite if \(Q_0 = I \cup J\) such that for all arrows \(\alpha : i \to j\) we have \(i \in I\) and \(j \in J\).

Define the abelian group

\[
\mathbb{Z}Q_0 = \bigoplus_{i \in Q_0} \mathbb{Z}i
\]
and its monoid of dimension vectors $\mathbb{N}Q_0$. A finite-dimensional $k$-representation of $Q$ is given by a tuple 

$$X = ((X_i)_{i \in Q_0}, (X_\alpha)_{\alpha \in Q_1} : X_i \to X_j)$$

of finite-dimensional $k$-vector spaces and $k$-linear maps between them. The dimension vector $\dim X \in \mathbb{N}Q_0$ of $X$ is defined by 

$$\dim X = \sum_{i \in Q_0} \dim_k X_i.$$

Let $d \in \mathbb{N}Q_0$ be a dimension vector. The variety $R_d(Q)$ of $k$-representations of $Q$ with dimension vector $d$ is defined as the affine $k$-vector space 

$$R_d(Q) = \bigoplus_{\alpha : i \to j} \text{Hom}_k(k^{d_i}, k^{d_j}).$$

The algebraic group 

$$G_d = \prod_{i \in I} \text{Gl}_{d_i}(k)$$

acts on $R_d(Q)$ via simultaneous base change, i.e. 

$$(g_i)_{i \in Q_0} \cdot (X_\alpha)_{\alpha \in Q_1} = (g_j X_\alpha g_i^{-1})_{\alpha : i \to j}.$$ 

The orbits are in bijection with the isomorphism classes of $k$-representations of $Q$ with dimension vector $d$.

In the space of $\mathbb{Z}$-linear functions $\text{Hom}_\mathbb{Z}(\mathbb{Z}Q_0, \mathbb{Z})$ we consider the basis given by the elements $i^*$ for $i \in Q_0$, i.e. $i^*(j) = \delta_{i,j}$ for $j \in Q_0$. Define 

$$\text{dim} := \sum_{i \in Q_0} i^*.$$ 

After choosing $\Theta \in \text{Hom}_\mathbb{Z}(\mathbb{Z}Q_0, \mathbb{Z})$, we define the slope function $\mu : \mathbb{N}Q_0 \to \mathbb{Q}$ via 

$$\mu(d) = \frac{\Theta(d)}{\text{dim}(d)}.$$ 

The slope $\mu(\dim X)$ of a representation $X$ of $Q$ is abbreviated to $\mu(X)$.

**Definition 2.2** A representation $X$ of $Q$ is semistable (resp. stable) if for all subrepresentations $U \subset X$ (resp. all proper subrepresentations $0 \neq U \subsetneq X$) the following holds: 

$$\mu(U) \leq \mu(X) \ (\text{resp. } \mu(U) < \mu(X)).$$
This definition is equivalent to that of A. King [King]. Let \( \hat{\Theta} \) again be a linear form. A representation \( X \) such that \( \hat{\Theta}(\dim X) = 0 \) is semistable (resp. stable) in the sense of King if and only if
\[
\hat{\Theta}(\dim U) \geq 0 \quad \text{and} \quad \hat{\Theta}(\dim U) > 0
\]
for all subrepresentations \( U \subset X \) (resp. all proper subrepresentations \( 0 \neq U \subsetneq X \)).

To see this define a linear form
\[
\Theta := \mu \cdot \dim - \hat{\Theta}
\]
and check that a representation \( X \) is semistable in the former sense if and only if it is \( \hat{\Theta} \)-semistable in the sense of King.

Denote the set of semistable (resp. stable) points by
\[
R^{ss}_{d}(Q) \quad \text{(resp.} \quad R^{s}_{d}(Q)\text{)}.
\]

In this situation we have the following theorem going back to Mumford’s GIT and which was proved by King, see [Mum], [King]:

**Theorem 2.3**  
1. The set of stable points \( R^{s}_{d}(Q) \) is an open subset of the set of semistable points \( R^{ss}_{d}(Q) \), which is an open subset of \( R^{\cdot}_{d}(Q) \).

2. There exists a categorical quotient \( M^{ss}_{d}(Q) := R^{ss}_{d}(Q)/G_{d} \). Moreover, \( M^{ss}_{d}(Q) \) is a projective variety.

3. There exists a geometric quotient \( M^{s}_{d}(Q) := R^{s}_{d}(Q)/G_{d} \), which is a smooth subvariety of \( M^{ss}_{d}(Q) \).

For a detailed description of the theory of quotients see [Muk]. We just briefly treat the construction.

We obtain the quotient \( M^{ss}_{d}(Q) \) called moduli space in what follows by defining a character \( \chi \) of \( G_{d} \) by
\[
\chi((g_{i})_{i \in I}) := \prod_{i \in I} \det(g_{i})^{\Theta(d) - \dim d \cdot \Theta_{i}},
\]
where \( \Theta \) is the linear form obtained from the previous consideration.

For an affine variety \( X \) the set of \( \chi^{-}\text{-semi-invariants} \) of weight \( \chi^{n} \) is defined by
\[
k[X]^{G, \chi^{n}} := \{ f \in k[X] \mid f(g \ast x) = \chi(g)^{n} \cdot f(x) \quad \forall g \in G, \forall x \in X \}.
\]

Furthermore, the ring of \( \chi^{-}\text{-semi-invariants} \) is given by
\[
k[X]^{G}_{\chi} := \bigoplus_{n=0}^{\infty} k[X]^{G, \chi^{n}}.
\]

Then we have
\[
M^{ss}_{d}(Q) = \Proj(k[X]^{G}_{\chi}),
\]
the projective spectrum of the ring of semi-invariants.
Remark 2.4

- Since there exists only one closed orbit, the affine quotient is just a point. Therefore we get $k|R_d(Q)|^G = k$. Thus the projective quotient has no affine component and is a projective variety.

- Since $R_d(Q)$ is an affine space and thus smooth, we get that the open subset of stable points is smooth. Thus, since the moduli space $M^s_d(Q)$ is an orbit space associated to the group action restricted to the stable points, it is smooth as well.

- The moduli space $M^s_s(Q)$ does not parametrize the semistable representations, but the polystable ones. Polystable representations are such representations which can be decomposed in stable ones of the same slope.

- If semistability and stability coincide, $M^s_s(Q)$ actually is a smooth projective variety. Obviously this is the case if $\mu(d) \neq \mu(e)$ for all $0 \neq e < d$. In this case the dimension vector $d$ is said to be $\Theta$-indivisible.

- Let $M^{ss}_d(Q) = M^s_d(Q)$. For a stable representation $X$ we have that its orbit is of maximal possible dimension. Since the scalar matrices act trivially on $R^d(Q)$, the isotropy group is at least one-dimensional. Thus for the dimension of the moduli space it follows that

$$\dim M^{ss}_d = 1 - \langle d, d \rangle.$$ 

Finally we point out some properties of (semi-)stable representations. These properties will be very useful at different points of this paper, for proofs see [HN].

Lemma 2.5 For a quiver $Q$ let $0 \rightarrow M \rightarrow X \rightarrow N \rightarrow 0$ be a short exact sequence of representations.

1. The following are equivalent:
   - $\mu(M) \leq \mu(X)$
   - $\mu(X) \leq \mu(N)$
   - $\mu(M) \leq \mu(N)$

2. The following holds:

$$\min(\mu(M), \mu(N)) \leq \mu(X) \leq \max(\mu(M), \mu(N)).$$

3. If $\mu(M) = \mu(X) = \mu(N)$, then $X$ is semistable if and only if $M$ and $N$ are semistable.
From the first property we immediately get that stable representations are indecomposable. As usual denote by $E_q$ the simple representation corresponding to the vertex $q$ defined by $X_q = \mathbb{C}$ and $X_{q'} = 0$ for all $q' \neq q$.

For a quiver $Q$ consider the matrix $A = (a_{i,j})_{i,j \in Q_0}$ with $a_{i,i} = 2$ and $-a_{i,j} = -a_{j,i}$ for $i \neq j$, in which $a_{i,j} = |\{\alpha \in Q_1 \mid \alpha : i \to j \vee \alpha : j \to i\}|$.

Fixed some $q \in Q_0$ define $r_q : \mathbb{Z}Q_0 \to \mathbb{Z}Q_0$ as

$$r_q(q') = q' - a_{q,q'}q.$$

We have the following theorem, see [BGP]:

**Theorem 2.6** Let $Q$ be a quiver and $q \in Q_0$ a fixed vertex. Let $q$ be a sink (resp. a source). Then there exists a functor

$$R_q^\pm : \text{mod} \ CQ \to \text{mod} \ CQ_q$$

with the following properties (if $q$ is a source, replace $+$ by $-$):

1. $R_q^+ (U \oplus U') = R_q^+ (U) \oplus R_q^+ (U')$.
2. Let $U$ be an indecomposable representation of $Q$.
   (a) If $U \cong E_q$, then $R_q^+ (E_q) = 0$.
   (b) If $U \not\cong E_q$, then $R_q^+ (U)$ is indecomposable with $R_q^+ R_q^- (U) \cong U$ and $\dim R_q^+ (U) = r_q(\dim(U))$.

In particular, we have: $\text{End} U \cong \text{End} R_q^+ (U)$.

3 **Localization in moduli spaces**

Analogously to [Rei3], in this section we translate the techniques of localization in moduli spaces of simple representation provided by [Rei1] into moduli spaces of stable representations. An explicit method to detect fixed points of these moduli spaces under a torus action is explained. These fixed points are stable representations of the universal abelian covering quiver.

3.1 **Torus fixed points**

Let $G$ be an algebraic group and $\chi : G \to \mathbb{C}^*$ be a character of $G$, i.e. a morphism of algebraic groups. Denote by $X(G)$ the set of all characters of $G$ with the group structure given in the obvious way. In the following the composition is written additively.

Let $V$ be a $\mathbb{C}$-vector space and $G$ a closed subgroup of $Gl(V)$ acting on $V$. For all characters $\chi \in X(G)$ define the semi-invariants of weight $\chi$ by

$$V_\chi = \{v \in V \mid g \cdot v = \chi(g)v \forall g \in G\}.$$
If \( \varphi : G \to Gl(V) \) is a rational representation, the definition can be transferred, i.e.

\[
V_\chi = \{ v \in V \mid \varphi(g)v = \chi(g)v \ \forall g \in G \}.
\]

For instance from \([Spr]\) we obtain:

**Lemma 3.1** Let \( G \) be a diagonalizable subgroup of \( Gl(V) \) and \( \varphi : G \to Gl(V) \) a rational representation. We have

\[
V = \bigoplus_{\chi \in X(G)} V_\chi.
\]

In the following we assume that the considered dimension vectors \( d \in \mathbb{N}Q_0 \) of a quiver \( Q \) are coprime. Thus \( M_d^\ast(Q) = M_d^\ast_d(Q) \).

Let further \( T := (\mathbb{C}^*)^{|Q_1|} \) be the \(|Q_1|\)-dimensional torus. It acts on \( R_d(Q) \) via

\[
((t_\alpha)_{\alpha \in Q_1}) \cdot ((X_\alpha)_{\alpha \in Q_1}) = (t_\alpha \cdot X_\alpha)_{\alpha \in Q_1}.
\]

Since the torus action commutes with the \( G_d \)-action, it induces a \( T \)-action on \( M_d^\ast(Q) \).

Since the scalar matrices act trivially on \( R_d(Q) \), the \( G_d \)-action factorises through the quotient

\[
PG_d := G_d / \mathbb{C}^*.
\]

Let \( X \in M_d^\ast(Q) \) be a fixed point under the torus action. Considering the algebraic group

\[
G := \{(g_i)_{i \in I}, t) \in PG_d \times T \mid t \cdot X = (g_i)_{i \in I} \ast X \}
\]

we get projections

\[
\begin{array}{c}
G \\
p_1 & \downarrow & p_2 \\
PG_d & \rightarrow & T
\end{array}
\]

with the following property:

**Lemma 3.2** Let \( X \) be a torus fixed point. The following holds:

1. The projection \( p_2 : G \to T \) is an isomorphism.

2. In particular, the projection \( p_1 : G \to PG_d \) induces a homomorphism of algebraic groups \( \varphi := p_1 \circ p_2^{-1} : T \to PG_d \) such that

\[
\varphi(t) \ast X = t \cdot X.
\]

*Proof.* Since \( X \in M_d^\ast(Q) \) is a fixed point, \( p_2 \) is surjective. Moreover, since \( X \) is stable, its orbit is of maximal possible dimension. Thus the isotropy group of \( X \) under the action of \( PG_d \) is trivial implying the injectivity.

The second part immediately follows from this.
We can choose a lift $\psi : T \rightarrow G_d$ for $\varphi$, which can be decomposed in $|Q_0|$ morphisms of algebraic groups $\psi_i : T \rightarrow GL_d$. Together with the projection $\pi : G_d \rightarrow PG_d$ we obtain the following commutative diagram:

\[
\begin{array}{c}
G_d \\
\downarrow \psi \\
T \\
\varphi \\
\downarrow \\
PG_d \\
\end{array}
\]

For instance the lift can be chosen by mapping $t$ to the tuple $(g_i)_{i \in I}$ satisfying $\det(g_i) = 1$ for all $i \in I$. We have the following property of a torus, see for example [Hum]:

**Lemma 3.3** Let $T$ be a $d$-dimensional torus. Then we have

$$X(T) \simeq \mathbb{Z}^d.$$  

Let $T = (\mathbb{C}^*)^d$ and let $\alpha_k \in X(T)$ be given by $\alpha(t_1, \ldots, t_d) = t_k$ for $1 \leq k \leq d$. Then we obtain the isomorphism by mapping $\alpha_k$ to $e_k$.

By considering the simultaneous eigenspace decomposition with respect to $\psi_i$, i.e.

$$V_i = \bigoplus_{\chi \in X(T)} V_{i,\chi},$$

we get the following important lemma:

**Lemma 3.4** Let $X = (X_\alpha)_{\alpha \in Q_1}$ a fixed point under the torus action. We have:

$$X_\alpha(V_{i,\chi}) \subseteq V_{j,\chi+e_\alpha} \text{ for all } \chi \in X(T), \alpha : i \rightarrow j.$$

**Proof.** Let $t = (t_\alpha)_{\alpha \in Q_1} \in T$ and $v \in V_{i,\chi}$. Then we have

$$\psi_j(t)X_\alpha(v) = \psi_j(t)X_\alpha\psi_i(t)^{-1}\psi_i(t)(v) = t_\alpha X_\alpha(t)(v) = (\chi + e_\alpha)(t)X_\alpha(v).$$

We define a quiver such that the fixed point components correspond to moduli spaces of this quiver with certain dimension vectors. In order to get a sufficient uniqueness of these dimension vectors we first investigate the modification of the lift $\psi$ and the choice of another representative of a fixed point $X$. 

\[\square\]
Remark 3.5

1. If we choose another lift \( \psi' \), one easily verifies that there exists a character \( \chi \in X(T) \) such that
   \[
   \psi = \chi \psi'.
   \]

2. If we have two representatives \( X \) and \( X' \) of a fixed point, there exists some \( g \in G_d \) such that
   \[
   X' = g \ast X.
   \]
   Thus if \( \varphi \) is a homomorphism belonging to \( X \), we have
   \[
   (g^{-1} \varphi(t)) \ast X' = t \cdot X'
   \]
   for all \( t \in T \).

By the preceding considerations we get:

Lemma 3.6 The following are equivalent:

- \( X \) is a fixed point.
- There exists a morphism of algebraic groups \( \varphi : T \to G_d \) such that
  \[
  (t_\alpha)_{\alpha \in Q_1} \cdot (X_\alpha)_{\alpha \in Q_1} = \varphi((t_\alpha)_{\alpha \in Q_1}) \ast (X_\alpha)_{\alpha \in Q_1}
  \]
  for all \( (t_\alpha)_{\alpha \in Q_1} \in T \).

Now we investigate the stability criterion for fixed points. We will see that it is enough to consider subspaces compatible with the weight space decomposition. This is important for the practicability of the introduced construction. Let \( X \) be a quiver representation. Define by \( \text{scss}(X) \) (strongly contradicting semistability) that subrepresentation \( Y \subsetneq X \) for which the following holds:

- \( \mu(Y) = \max \{ \mu(U) \mid U \subsetneq X \} \).
- \( \dim(Y) = \max \{ \dim(U) \mid U \subsetneq X, \mu(U) = \mu(Y) \} \).

Thus \( Y \) is of maximal dimension among the subrepresentations with maximal slope.

Lemma 3.7 Let \( X \) be a quiver representation. The subrepresentation \( \text{scss}(X) \) is uniquely determined.
Proof. Assume $U$ and $U'$ both satisfy the properties. We have $U + U' \subseteq X$.
Consider the short exact sequences

$$
0 \longrightarrow U \cap U' \longrightarrow U \oplus U' \longrightarrow U + U' \longrightarrow 0
$$

and

$$
0 \longrightarrow U \longrightarrow U \oplus U' \longrightarrow U' \longrightarrow 0.
$$

By assumption we have $\mu(U \cap U') \leq \mu(U) = \mu(U')$. From the second sequence together with Lemma 2.5 we obtain that $\mu(U) = \mu(U') = \mu(U \oplus U')$.

Again by Lemma 2.5 we get from $\mu(U \cap U') \leq \mu(U) = \mu(U \oplus U')$ that $\mu(U \oplus U') \leq \mu(U + U')$, hence $\mu(U) = \mu(U + U')$. Because of the maximality of the dimension of $U$, it follows that $\dim(U + U') = \dim(U) = \dim(U')$. But this means $U = U'$.

\[ \square \]

Lemma 3.8 Let $X$ be a fixed point of $Q$ with dimension vector $d$. Let

$$
X_i = \bigoplus_{\chi \in X(T)} X_{i,\chi}
$$

be the weight space decomposition with respect to the associated morphism $\varphi : T \to G_d$. Then the following are equivalent:

1. $X$ is semistable.

2. For all subrepresentations $U$, which are compatible with the weight space decomposition of $X$, i.e. $U_i = \bigoplus_{\chi \in X(T)} U_{i,\chi}$ for all $i \in Q_0$ where $U_{i,\chi} \subset X_{i,\chi}$, we have $\mu(U) \leq \mu(X)$.

Proof. One conclusion is clear. Thus let $X$ be a representation satisfying the second property. Moreover, let $U = \text{scs}(X)$ and consider

$$
\varphi(t)U := (\varphi_i(t_\alpha)_{\alpha \in Q_1}(U_i))_{i \in Q_0}
$$

for each $(t_\alpha)_{\alpha \in Q_1} \in T$.
Since $X$ is a fixed point, by Lemma 3.6 it follows that

$$
(t_\alpha)_{\alpha \in Q_1} : (X_\alpha)_{\alpha \in Q_1} = \varphi((t_\alpha)_{\alpha \in Q_1}) * (X_\alpha)_{\alpha \in Q_1}
$$

for all $(t_\alpha)_{\alpha \in Q_1} \in T$ and a morphism of algebraic groups $\varphi = (\varphi_i)_{i \in Q_0} : T \to PG_d$.
Hence for each arrow $\alpha : i \to j$ we obtain

$$
X_\alpha \varphi_i(t)U_i = \frac{1}{t_\alpha} \varphi_j(t)X_\alpha \varphi_i(t)^{-1} \varphi_i(t)U_i \subset \varphi_j(t)U_j
$$

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because $X_{\alpha}U_i \subset U_j$. Thus $\varphi(t)U$ is a subrepresentation of $X$. Since $\varphi_i(t)$ is invertible for all $i \in Q_0$, the dimension vectors of $U$ and $\varphi(t)U$ coincide. Because of the uniqueness of $\text{scss}(X)$ it follows that $\varphi(t)U = U$ for all $t \in T$. This is equivalent to $\varphi_i(t)U_i = U_i$ for all $t \in T$ and all $i \in Q_0$. This implies that $U = \text{scss}(X)$ is compatible with the weight space decomposition. Therefore, by assumption we have that $\mu(\text{scss}(X)) \leq \mu(X)$. Hence $X$ is semistable. Indeed, the slope of $\text{scss}(X)$ is maximal among the set of subrepresentations of $X$.

\[\square\]

Define the quiver $\hat{Q}$ by the vertex set

$$\hat{Q}_0 = Q_0 \times X(T)$$

and for each arrow $\alpha : i \to j$ and each character $\chi \in X(T)$ we have an arrow

$$(\alpha, \chi) : (i, \chi) \to (j, \chi + e_\alpha)$$

in $\hat{Q}_1$. This is the universal abelian covering quiver of $Q$.

Let $X$ be a fixed point of $Q$. Then define the corresponding dimension vector $\hat{d} \in \mathbb{N}\hat{Q}_0$ by

$$\hat{d}_{i,\chi} := \dim_{\mathbb{C}} V_{i,\chi}.$$ 

Obviously $X$ can be considered as representation of this quiver.

The stability condition for representations of this quiver is induced from $\Theta$, i.e. we define a linear form $\hat{\Theta} : \mathbb{Z}\hat{Q}_0 \to \mathbb{Z}$ such that

$$\hat{\Theta}_{i,\chi} = \Theta_i$$

for all $i \in Q_0$ and all $\chi \in X(T)$. Thus by Lemma 3.8 stable fixed points can be identified with stable representations of the just introduced quiver.

Next we show that such a representation corresponding to a fixed point $X$ is unique in a certain way. Following Remark 3.5 choosing another lift $\psi$ just changes the weights of the weight space decomposition by translation by the character $\chi$. This corresponds to a group action of $\mathbb{Z}Q_1$ on $\hat{Q}_0$ defined by

$$\mu \cdot (i, \chi) = (i, \chi + \mu).$$

Now this induces a group action on the set of dimension vectors $\mathbb{N}\hat{Q}_0$. Two dimension vectors contained in the same orbit are said to be equivalent in the following.

Also choosing another representative corresponds to an even translation of the weights and the dimensions of the weights spaces respectively. Thus we have in conclusion:

**Theorem 3.9** For all fixed points $X \in M^*_d(Q)^T$ there exists (up to equivalence) a unique dimension vector $\hat{d}$ for $\hat{Q}$ such that $X$ correspond to a representation of $\hat{Q}$ with dimension vector $\hat{d}$. 
3.2 Description of fixed points and bipartite quivers

Converse to the last section we construct an embedding of representations of the quiver $\hat{Q}$ into the fixed point set of $Q$. Therefore, fixing a representation of $\hat{Q}$ we construct a representation of $Q$ and show that the latter one is a fixed point.

Again consider the quiver $\hat{Q}$ and let $\hat{d}$ be a dimension vector such that the corresponding dimension vector $d$ of $Q$ is coprime. This means that $d_i$ with $i \in Q_0$ is given by

$$d_i = \sum_{\chi \in X(T)} \hat{d}_{i,\chi}.$$  

We call a dimension vector $\hat{d}$ satisfying this property compatible with $d$.

Let $V_{i,\chi}$ be vector spaces of dimension $\hat{d}_{i,\chi}$ for all $i \in Q_0$ and $\chi \in X(T)$. Consider the vector spaces

$$V_i := \bigoplus_{\chi \in X(T)} V_{i,\chi}.$$  

We obtain a representation of $Q$ by defining for given linear maps

$$X_{\alpha,\chi} : V_{i,\chi} \rightarrow V_{j,\chi + e_{\alpha}}$$  

linear maps

$$X_{\alpha} = \bigoplus_{\chi \in X(T)} X_{\alpha,\chi} : V_i \rightarrow V_j$$  

for all $\alpha : i \rightarrow j$.

This defines a linear map

$$P : R_{\hat{d}}(\hat{Q}) \rightarrow R_d(Q).$$  

Moreover, an embedding of $G_{\hat{d}}$ in $G_d$ arises from the decomposition of the vector spaces $V_i$. Since the linear map is equivariant under the group action of $G_{\hat{d}}$, the map $P$ induces a map

$$P : M_{\hat{d}}^{ss}(\hat{Q}) \rightarrow M_d^{ss}(Q)$$  

by use of the universal property of the quotient. Furthermore, define a map of algebraic groups $\psi = (\psi_i)_{i \in Q_0} : T \rightarrow G_d$ such that

$$\psi_i : T \rightarrow Gl(V_i)$$  

is defined by

$$\psi_i(t)v = \chi(t)v$$  

for all $t \in T$ and all $v \in V_{i,\chi}$.

This makes $\psi$ well-defined and by use of Lemma 3.6 we obtain a morphism
of algebraic groups $\varphi$ such that $P(X) = Y$ is a fixed point.  
Since the dimension vector $d$ is coprime, stability and semistability coincide.  
Thus stable representations of $\hat{Q}$ are mapped to stable representations of $Q$.

**Lemma 3.10** Let $X$ and $X'$ be representations of $\hat{Q}$ such that $P(X)$ and $P(X')$ are isomorphic. Then $X$ and $X'$ are already isomorphic.

**Proof.** Let $Y = P(X)$ and $Y' = P(X')$ and let  
\[ g = (g_i \in Gl(V_i))_{i \in Q_0} \]
be an isomorphism between $Y$ and $Y'$. We have  
\[ Y'_{\alpha}g_i = g_jY_{\alpha} \]
for all $\alpha : i \to j \in Q_1$. Since $Y$ is a fixed point, by Lemma 3.6 there exists a corresponding homomorphism of algebraic groups  
\[ \varphi = (\varphi_i)_{i \in Q_0} \]
and from this one can choose a lift  
\[ \psi = (\psi_i : T \to G_{d_i})_{i \in Q_0}. \]
Following Remark 3.5 we may assume that both fixed points induce the same lift. Indeed, by changing the lift the corresponding weight space decomposition does not change. For this lift, all $t \in T$ and all $\alpha : i \to j$ we have  
\[ \psi_j(t)Y'_{\alpha}\psi_i(t)^{-1}g_i = t_{\alpha} \cdot Y'_{\alpha}g_i \]
\[ = g_j(t_{\alpha} \cdot Y_{\alpha}) \]
\[ = g_j\psi_j(t)Y_{\alpha}\psi_i(t)^{-1}. \]
Therefore, this implies that  
\[ Y'_{\alpha}(\psi_i(t)^{-1}g_i\psi_i(t)) = (\psi_j(t)^{-1}g_j\psi_j(t))Y_{\alpha} \]
for all $\alpha \in Q_1$.  
Since $Y$ and $Y'$ are stable, the endomorphism ring only consists of scalars. Thus it follows  
\[ (\psi_i(t)^{-1}g_i\psi_i(t))_{i \in Q_0} = a \cdot g \]
for all $t \in T$ and an $a \in \mathbb{C}^*$. This defines a character $a \in X(T)$ such that we have  
\[ \psi(t)^{-1}g\psi(t) = a(t) \cdot g \]
for all $t \in T$.  
Each $g_i$ induces an isomorphism between the weight spaces $V_{i,\chi}$ and $V_{i,\chi+a}$. But since the vector spaces $V_i$ are finite dimensional, we already have $a = 0$. Hence we may understand $g$ as an isomorphism between $X$ and $X'$. 

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Every fixed point arises from such an embedding. Moreover, the images of these embeddings are pairwise disjoint so that we obtain the following concluding theorem:

**Theorem 3.11** The set of fixed points \( M^s_d(Q)^T \) is isomorphic to the disjoint union of moduli spaces

\[
\bigcup_{\hat{d}} M^s_{\hat{d}}(\hat{Q}),
\]

in which \( \hat{d} \) ranges over all equivalence classes of dimension vectors being compatible with \( d \).

### 3.3 Euler characteristic of moduli spaces

In this section we point out some basic properties of the Euler characteristic. For basics of algebraic topology see for instance [Lue]. Moreover, we discuss the conjecture of Michael Douglas concerning the asymptotic behaviour of the Euler characteristic of Kronecker moduli spaces.

#### 3.3.1 Definition and properties

Let \( X \) be a smooth projective variety over the complex numbers of dimension \( n \) and let \( H^i(X) \), \( i \in \mathbb{N}_0 \), be the \( i \)-th singular cohomology group with coefficients in \( \mathbb{C} \) which are \( \mathbb{C} \)-vector spaces satisfying \( H^i(X) = 0 \) if \( i > 2n \) as is known. Define

\[
h^i(X) = \dim_{\mathbb{C}} H^i(X).
\]

The Euler characteristic \( \chi \) of \( X \) is defined by

\[
\chi(X) = \sum_{k=0}^{2n} (-1)^k h^k(X).
\]

By the following theorem, for a proof see [CG] or [EM], it follows that the localization method is suitable to calculate the Euler characteristic of varieties.

**Theorem 3.12** Let \( X \) be a complex variety with a torus \( T \) acting on it. Let \( X^T \) be the fixed point set of \( X \) under this action. Then for the Euler characteristic we have

\[
\chi(X) = \chi(X^T).
\]

By use of Theorem 3.11 and because of the additivity of the Euler characteristic we obtain the following important result:
Theorem 3.13 Let $Q$ be a quiver with dimension vector $d$. Then for the Euler characteristic of the moduli space $M_d^s(Q)$ we have
\[ \chi(M_d^s(Q)) = \sum_d \chi(M_d^s(\hat{Q})), \]
where $\hat{Q}$ is the universal abelian covering quiver and $\hat{d}$ ranges over all equivalence classes being compatible with $d$.

For a quiver $Q$ and for a coprime dimension vector $d$ we again consider the moduli space of stable representations $M_d^s(Q)$. From [Rei2] we get that the odd cohomology vanishes. Moreover, from the Hard Lefschetz Theorem, see for instance [GH], we can conclude that
\[ h^k(M_d^s(Q)) \leq h^{k+2}(M_d^s(Q)) \]
for $k < n$ and
\[ h^k(M_d^s(Q)) \geq h^{k+2}(M_d^s(Q)) \]
for $k > n$ where $n$ is the dimension of $M_d^s(Q)$. Since we also have
\[ h^0(M_d^s(Q)) = h^{2n}(M_d^s(Q)) = 1, \]
we get the following result:

Corollary 3.14 For moduli spaces of stable representations of a quiver $Q$ with coprime dimension vector $d$ we have:
\[ \chi(M_d^s(Q)) \geq \dim(M_d^s(Q)) + 1. \]

3.4 Maps between universal quivers

Let $Q = (Q_0, Q_1)$ be a connected quiver without oriented cycles. Let $Q_1^{-1} = \{\alpha, \alpha^{-1} \mid \alpha \in Q_1\}$ where $\alpha^{-1}$ is the formal inverse of $\alpha$. We will write $\alpha^{-1} : j \to i$ for $\alpha : i \to j \in Q_1$. A path $p$ is a sequence $(i_1 | \alpha_1 \alpha_2 \ldots \alpha_n | i_{n+1})$ such that $\alpha_j : i_j \to i_{j+1} \in Q_1^{-1}$. Thereby, we have the equivalence generated by
\[ (i \mid \alpha \alpha^{-1} | i) \sim (i \parallel i). \]

In what follows, we always consider paths up to this equivalence. The set of words in $Q$ is generated by the arrows and their formal inverses, i.e. for a word $w$ we have $w = \alpha_1 \ldots \alpha_n$ where $\alpha_i \in Q_1^{-1}$. Denote the set of words of $Q$ by $W(Q)$. The universal covering quiver $\tilde{Q}$ of $Q$ is given by the vertex set
\[ \tilde{Q}_0 = \{(i, w) \mid i \in Q_0, w \in W(Q)\} \]
and the arrow set
\[ \tilde{Q}_1 = \{\alpha(i, w) : (i, w) \to (j, w\alpha) \mid \alpha : i \to j \in Q_1\}. \]
For a $\alpha \in Q_1^{-1}$ define

$$o(\alpha) = \begin{cases} 
1 & \text{if } \alpha \in Q_1 \\
-1 & \text{if } \alpha^{-1} \in Q_1.
\end{cases}$$

The universal abelian covering quiver $\hat{Q}$, see Section 3.1, is given by the vertex set

$$\hat{Q}_0 = Q_0 \times \mathbb{Z}Q_1 = \{(i, z_1) \mid i \in Q_0, z_1 \in \mathbb{Z}Q_1\}$$

and the arrow set

$$\hat{Q}_1 = \{\alpha_{z_1} : (i, z_1) \rightarrow (j, z_1 + e_\alpha) \mid \alpha : i \rightarrow j \in Q_1\}.$$ 

The $k$-th universal abelian covering quiver is recursively defined by the vertex set

$$\hat{Q}_k^0 = \hat{Q}_0^{k-1} \times \mathbb{Z}\hat{Q}_0^{k-1} = \{(i, (z_l)_{l=1}^{k}) \mid i \in Q_0, z_l \in \mathbb{Z}Q_1^{l-1}\}$$

and the arrow set

$$\hat{Q}_k^1 = \{\alpha_{(z_l)_{l=1}^{k}} : (i, (z_l)_{l=1}^{k}) \rightarrow (j, (z_l + e_{\alpha(z_l)_{l=1}^{k-1}})_{l=1}^{k}) \mid \alpha : i \rightarrow j \in Q_1, \alpha_{(z_l)_{l=1}^{k}} \in \hat{Q}_k^s \text{ for } s = 0, \ldots, k - 1\}.$$ 

where we define $\hat{Q}^0 = Q$ and $\alpha_{(z_l)_{l=1}^{0}} = \alpha$. Fixing a vertex $i \in Q_0$ we will always consider the connected components such that $(i, 1) \in \hat{Q}_0$ and $(i, 0, \ldots, 0) \in \hat{Q}^k_0$. We again denote this subquivers by $\hat{Q}$ and $\hat{Q}^k$. This is no restriction because we are only interested in stable and indecomposable representations respectively. Thus the support of the quiver has to be connected. Fix some vertex $(q, w) \in \hat{Q}_0$. Thus $w = (i \mid \alpha_1 \cdots \alpha_n \mid q)$ is a path in $\hat{Q}_1$ and we may assume that $\alpha_i \neq \alpha_{i+1}^{-1}$ for all $i = 1, \ldots, n - 1$. We call such a path reduced in what follows. Moreover, by $l(w) = n$ we define the length of the path $w$ and

$$h(i) := \frac{2i - 1 - o(\alpha_i)}{2}.$$ 

Define

$$z_1^i(w) = \sum_{i=1}^{l} o(\alpha_i) e_{\alpha_i}.$$ 

Furthermore, we recursively define

$$z_j^l(w) := \sum_{i=1}^{l} o(\alpha_i) e_{\alpha_i} \left(\prod_{s=1}^{h(i)} (e_{\alpha(s)})_{s=1}^{j-1} \right)$$

where $z_1^l(w) = 0$ if $l = 0$. Now we can define a map $f_k : \hat{Q} \rightarrow \hat{Q}_k$ by

$$f_k((q, w)) = (q, (z_j^l(w))_{j=1}^{k}).$$
and for some \( \alpha : (q_1, w) \to (q_2, w\alpha) \) we define
\[
f_k(\alpha) = \alpha_{(z_j^l(w))_{j=1,\ldots,k}} : (q_1, (z_j^l(w))_{j=1,\ldots,k}) \to (q_2, (z_j^{l(w\alpha)}(w\alpha))_{j=1,\ldots,k}).
\]
Obviously we have
\[
z_j^{l(w\alpha)}(w\alpha) = z_j^l(w) + e_{\alpha_{(z_j^l(w))_{j=1,\ldots,j-1}}}
\]
for \( j = 1, \ldots, k \).

\section*{Proposition 3.15}

1. The maps \( f_k \) are surjective for all \( k \).

2. For \( k \to \infty \) the map \( f_k \) is injective.

\textbf{Proof.} As already mentioned we may just consider the connected components such that \((i,1) \in \hat{Q}_0 \) and \((i,0,\ldots,0) \in \hat{Q}^k \) for some \( i \in Q_0 \). We first show that \( f_k \) is surjective. Thus let \((j, z_1, \ldots, z_k) \in Q_0^k \) such that there exists a reduced path \( w = ((i,0,\ldots,0) \mid \alpha_1 \ldots \alpha_n \mid (j, z_1, \ldots, z_k)) \).

We have
\[
z_t^l(w) = \sum_{i=1}^{l(w)} o(\alpha_i) e_{(\alpha_i)_{(z_s^h(w))_{s=1,\ldots,t-1}}}
\]
and thus \( f_k(j, w) = (j, z_1, \ldots, z_k) \).

Now consider \((q,1) \) and \((q, w) \) where \( w = \alpha_1 \ldots \alpha_{n_k} \neq 1 \) is a reduced path and \( q \) and \( n_k \) are minimal such that
\[
f_k((q,1)) = f_k((q, w)) = (q,0,\ldots,0).
\]
Then we claim that \( f_{k+1}((q, w)) \neq f_k(q,1) = (q,0,\ldots,0) \). Assume this is not the case. Then we have
\[
f_{k+1}((q, w)) = (q, (z_t^l(w))_{t=1,\ldots,k+1})
\]
with
\[
z_{k+1}^l(w) = \sum_{i=1}^{l(w)} o(\alpha_i) e_{(\alpha_i)_{(z_s^h(w))_{s=1,\ldots,k+1}}} = 0
\]
and \( l(w) = n_k \). Thus there exists some tuple \((i,i')\) with \( i \neq i' \) such that \( z_t^h(i)(w) = z_t^h(i')(w) \) for all \( l = 1, \ldots, k \). But since \( z_t^h(i)(w) - z_t^h(i')(w) = 0 \), this defines two vertices \((q', w_1)\) and \((q', w_1 w_2)\) such that \( f_k((q', w_1)) = f_k((q', w_1 w_2)) \) and \( l(w_2) < n_k \). But this is a contradiction and therefore we have \( n_{k+1} > n_k \) for all \( k \geq 0 \).

This already shows that \( f_\infty \) is injective.
Note that it is easy to check that we have an arrow between two vertices of the $k$-th universal abelian covering quiver if and only if we have one between two vertices of the universal covering quiver.

Let $T_k := (\mathbb{C}^\times)^{Q_k^{-1}}$. Define

$$M^\ast_d(Q)^{T,n} = (\ldots (M^\ast_d(Q)^{T_1}) \ldots)^{T_n}.$$ 

Using Theorem 3.11, we get the following:

**Theorem 3.16** For all dimension vectors $d$ there exists an $n \in \mathbb{N}_0$ such that we have

$$M^\ast_d(Q)^{T,n'} \cong \bigcup_{\tilde{d}} M^\ast_{\tilde{d}}(\tilde{Q})$$

for all $n' \geq n$ where $\tilde{d}$ ranges over all equivalence classes that are compatible with $d$.

Concerning the Euler characteristic of quiver moduli we get the following corollary:

**Corollary 3.17** Let $Q$ be a quiver with dimension vector $d$. Then for the Euler characteristic of the moduli space $M^\ast_d(Q)$ we have

$$\chi(M^\ast_d(Q)) = \sum_{\tilde{d}} \chi(M^\ast_{\tilde{d}}(\tilde{Q})),$$

where $\tilde{d}$ ranges over all equivalence classes being compatible with $d$.

Thus we may always assume that torus fixed points are given as representations of the universal covering quiver which has no cycles. Thus the representations theory simplifies in comparison with the universal abelian covering quiver.

**Remark 3.18**

- Note that the connected components of the universal covering quiver of the Kronecker quiver $K(m)$ is the infinite regular $m$-tree with an orientation. In particular every vertex is either source or sink and its neighbours are only sinks or sources.
4 Asymptotics and combinatorics of trees

The purpose of this section is to treat some aspects of combinatorics of trees. Fixing some properties we count the number of trees of this type, either exactly or at least asymptotically. This machinery will be used to count torus fixed points and fixed point components respectively. This gives rise to a lower bound for the number of fixed points and thus for the Euler characteristic of moduli spaces of the Kronecker quiver.

Let $a(x) = \sum_{n \geq 0} a_n x^n$ be a power series. In the following denote by $[x^n]a(x) := a_n$ with $n \geq 0$ its $n$-th coefficient.

**Definition 4.1** A graph $G$ consists of a non-empty finite set of points $V = V(G)$ together with a set $X = X(G)$ of unordered pairs of different points of $V$. Every pair $x = \{u, v\}$ is called an edge of $G$ and is denoted by $x = uv$. Two points $u, v \in V$ are called adjacent if there exists an edge $x = uv$. A walk in $G$ is an alternating sequence of points and edges $v_0, x_1, v_1, \ldots, v_{n-1}, x_n, v_n$ starting and terminating with points such that $x_i = v_{i-1}v_i$. Also denote a walk by $v_0 \ldots v_n$. A walk is called path if $x_i \neq x_j$ for $i \neq j$. A path is called closed or a cycle if $v_0 = v_n$ and open if not. A graph is called connected if there exists a path between each two points. A graph is called acyclic if it has no cycles.

**Definition 4.2** A tree is a connected acyclic graph. A rooted tree is a tree where a point is specified to be the root. A graph without cycles is called a forest, in particular the components are trees.

When restricting to trees the points are often called knots. For further details according to trees and their combinatorics see for example [HP] or [SF].

4.1 Simply generated trees

We discuss simply generated trees, which are related to so called localization quivers. They are quivers coming from torus fixed points. Simply generated trees were introduced by Meir and Moon, see [MM], and are constructed as follows:

Let $y(x) = \sum_{n \geq 0} y_n x^n$
be the generating function of a family $T$ of rooted trees. Moreover, let
\[ \phi(x) = \sum_{n \geq 0} \phi_n x^n \]
be a formal power series such that $\phi_n \geq 0$ for all $n \geq 0$ and also $\phi_0 > 0$ and $\phi_j > 0$ for a $j \geq 2$. If $y(x)$ satisfies the functional equation
\[ y(x) = x\phi(y(x)) \]
for such a power series $\phi$, we call $T$ a family of simply-generated trees. We also call a tree $T \in \mathcal{T}$ simply-generated. Note that a fixed tree may be contained in more than one family of simply-generated trees.

The weight $\omega(T)$ of a finite simply-generated tree $T \in \mathcal{T}$ is defined by
\[ \omega(T) = \prod_{j \geq 0} \phi^{D_j(T)} \]
where $D_j(T)$ is the number of knots with $j$ successors. Denote by $|T|$ the number of knots of a tree $T$. For the coefficients of the generating function we have
\[ y_n = \sum_{|T|=n} \omega(T). \]

For instance, if we define
\[ \phi(x) = 1 + 2x + x^2, \]
we obtain the family of binary trees. Here we take into account that we distinguish between left and right successors.

### 4.2 Lagrange inversion theorem

In this section we briefly discuss the Lagrange inversion theorem, which will become an important tool later.

**Theorem 4.3** Let $\phi(x) = \sum_{n \geq 0} \phi_n x^n$ be a power series such that $\phi(0) \neq 0$ and let $y(x)$ be a power series satisfying the functional equation
\[ y(x) = x\phi(y(x)). \]

Let $g(x)$ be another power series. Then $y(x)$ is invertible and for the coefficients of $g(y(x))$ we have
\[ [x^n]g(y(x)) = \frac{1}{n} [u^{n-1}] g'(u) \phi(u)^n \]
for all $n \geq 1$. Moreover, we have
\[ [x^n](y(x))^m = \frac{m}{n} [u^{n-m}] \phi(u)^n. \]
Note that this theorem is equivalent to the formulation of the Lagrange inversion theorem as usually stated in literature. For proofs and further details see for instance \cite{SF} or \cite{Drm}.

Next, we treat a special case which is important when counting localization quivers. In order to get a lower bound for the number of fixed points we use an asymptotic approximation arising from the next section.

Lemma 4.4 Let $\phi(x) = 1 + ax^b$ such that $y(x) = x\phi(y(x))$. Then we have

$$[x^n]y(x) = \frac{1}{n} \left( \frac{n}{n-1} \right) a^{n-1}$$

if $b|n - 1$ and $[x^n]y(x) = 0$ otherwise.

Proof. We have

$$\phi(x)^n = \sum_{k=0}^{n} \binom{n}{k} (ax^b)^k.$$

Thus we obtain for the $(n-1)$-th derivation

$$\phi^{(n-1)}(x)^n = \sum_{\substack{k=0, \ b k \geq n-1}}^{n} a^k b^k (b^k - 1) \ldots (b^k - (n-2)) \binom{n}{k} x^{b^k - (n-1)}.$$

By use of the Lagrange inversion theorem \ref{4.3} we get for the $n$-th Taylor coefficient that

$$[x^n]y(x) = \frac{1}{n} \left( \frac{n}{n-1} \right) a^{n-1}$$

if $b|n - 1$ and $[x^n]y(x) = 0$ otherwise.

\qed

Corollary 4.5 Let $m \geq 1$. We have

$$[x^n]y(x)^m = \frac{m}{n} \left( \frac{n}{n-m} \right) a^{n-m}$$

if $b|n - m$ and $n \geq m$ and $[x^n]y(x)^m = 0$ otherwise.

Let $a, b, m, n \in \mathbb{N}^+$. Define

$$\mathcal{A}_{a,b,m,n} := [x^n]y(x)^m$$

if $y(x)$ satisfies the functional equation $y(x) = x\phi(y(x))$ where $\phi(x) = 1 + ax^b$. Also define

$$\mathcal{A}_{a,b,n} := \mathcal{A}_{a,b,1,n}.$$
4.3 Asymptotic behaviour

From [Drm] we obtain the following important theorem:

**Theorem 4.6** Let \( F(x, y) \) be an analytic function in the variables \( x \) and \( y \) around \( x = y = 0 \) such that \( F(0, y) = 0 \) and such that the Taylor coefficients of \( F \) around 0 are real and non-negative. Then there exists an unique analytic solution \( y = y(x) \) of the functional equation

\[
y = F(x, y),
\]

which has non-negative Taylor coefficients around 0 and, moreover, \( y(0) = 0 \).

If the region of convergence of \( F(x, y) \) is large enough such that there exist positive solutions \( x = x_0 \) and \( y = y_0 \) of the system of functional equations given by

\[
y = F(x, y) \quad \text{and} \quad 1 = F_y(x, y)
\]
with \( F_x(x_0, y_0) \neq 0 \) and \( F_{yy}(x_0, y_0) \neq 0 \), then \( y(x) \) is analytic for \( |x| < x_0 \).

Moreover, there exist functions \( h(x) \) and \( g(x) \), which are analytic around \( x_0 \), such that

\[
y(x) = g(x) - h(x) \sqrt{1 - \frac{x}{x_0}}
\]
locally around \( x_0 \).

Then we have \( g(x_0) = y(x_0) \) and

\[
h(x_0) = \sqrt{\frac{2x_0 F_x(x_0, y_0)}{F_{yy}(x_0, y_0)}}.
\]

Furthermore, this provides a locally analytic continuation of \( y(x) \) for \( x - x_0 \neq 0 \).

If \( [x^n]y(x) > 0 \) for all \( n \geq n_0 \), we also have that \( x = x_0 \) is the only singularity of \( y(x) \) on the circle \( |x| = x_0 \). In conclusion, for \( [x^n]y(x) \) we get an asymptotic expansion of the form

\[
[x^n]y(x) = \sqrt{\frac{x_0 F_x(x_0, y_0)}{2\pi F_{yy}(x_0, y_0)}} x_0^n n^{-\frac{3}{2}} (1 + \mathcal{O}(n^{-1})).
\]

Using the notions of the preceding theorem we get the following corollary:

**Corollary 4.7** Let \( \phi(x) = 1 + ax^b \) and \( y(x) \) such that \( y(x) = x \phi(y(x)) \).

Then we have

\[
(x_0)^{-1} = ab \left( \frac{1}{(b - 1)a} \right)^{\frac{b-1}{b}}.
\]
Proof. By use of Theorem 4.6 we obtain

\[ y_0 = x_0(1 + ay_0^b) \quad \text{and} \quad 1 = x_0 (aby_0^{b-1}) \]

respectively. The corollary now follows from an easy calculation.

\[ \square \]

5 Localization in Kronecker moduli spaces

5.1 Kronecker moduli spaces

Since the main focus of the paper is on the \( m \)-Kronecker quiver with \( m \geq 3 \), we apply the introduced machinery to this case. In what follows let \( k = \mathbb{C} \). Thus we consider the quiver \( K(m) \) with two vertices and \( m \) arrows between them, i.e.:

\[
\begin{array}{c}
\bullet 1 \\
\alpha_1 \\
\vdots \\
\alpha_m \\
\bullet 2
\end{array}
\]

A representation of this quiver with dimension vector \((d, e)\) is given by two \( \mathbb{C} \)-vector spaces \( V \) and \( W \) of dimensions \( d \) and \( e \) and a \( m \)-tuple of linear maps

\[
(X_1, \ldots, X_m) \in \bigoplus_{i=1}^{m} \text{Hom}(V, W) = R_{d,e}(K(m)).
\]

The group \((\text{Gl}(V) \times \text{Gl}(W))\) acts on \( R_{d,e}(K(m)) \) via simultaneous base change. Since the scalar matrices act trivially, the group action factorises through the quotient \( P := (\text{Gl}(V) \times \text{Gl}(W))/\mathbb{C}^* \). For \( \Theta = (1, 0) \) the slope function \( \mu : \mathbb{N}^2 \to \mathbb{Q} \) is defined by

\[
\mu(d, e) := \frac{d}{d + e}.
\]

Thus we obtain the following criterion for the (semi-)stability of Kronecker representations:

**Lemma 5.1** A point \((X_1, \ldots, X_m) \in R_{d,e}(K(m)) \) is semistable (resp. stable) if and only if for all proper subspaces \( 0 \neq U \subseteq V \) the following holds:

\[
\dim \sum_{k=1}^{m} X_k(U) \geq \dim U \cdot \frac{e}{d} \quad (\text{resp.} \quad \dim \sum_{k=1}^{m} X_k(U) > \dim U \cdot \frac{e}{d}).
\]

Thus \( d \) and \( e \) being coprime implies that semistable points are already stable.
Definition 5.2 Let \( \gcd(d, e) = 1 \). The categorical quotient

\[
M_{m}^{d,e} = \text{Proj}(\mathbb{C}[\bigoplus_{i=1}^{m} \text{Hom}(V, W)])_{X}^{\text{Gl}_{d}(\mathbb{C}) \times \text{Gl}_{e}(\mathbb{C})}
\]

is called Kronecker moduli space.

Using standard methods from Algebraic Geometry, see [Sha1] and [Sha2], we get by use of Theorem 2.3 the following:

Corollary 5.3 Let \( d, e, m \in \mathbb{N} \) and \( m \geq 3 \) such that \( \gcd(d, e) = 1 \). The corresponding Kronecker moduli space \( M_{d,e}^{m} \) is a compact complex manifold. Furthermore, there exists a continuous map

\[
\Pi : R^{*}_{d,e}(K(m)) \to M_{d,e}^{m}
\]

such that the \( \Pi \)-fibres are exactly the orbits under the group action.

We state some helpful properties of the Kronecker moduli spaces:

Proposition 5.4 1. There exist isomorphisms of moduli spaces \( M_{d,e}^{m} \simeq M_{e,d}^{m} \) and \( M_{d,e}^{m} \simeq M_{m-d,e}^{m} \).

2. The dimension of the moduli spaces is given by

\[
\dim M_{d,e}^{m} = 1 - d^2 - e^2 + dem
\]

if \( M_{d,e}^{m} \neq \emptyset \).

3. Let \((d, e)\) be a root of \( K(m) \). We have \( M_{d,e}^{m} \neq \{pt\} \) if and only if

\[
\frac{m - \sqrt{m^2 - 4}}{2} \leq \frac{e}{d} \leq \frac{m + \sqrt{m^2 - 4}}{2}
\]

holds.

Proof. We obtain the first isomorphism by considering the map

\[
(A_{1},..., A_{m}) \to (A_{1}^{T},..., A_{m}^{T}).
\]

The second one is obtained via the reflection functor, see Theorem 2.6. The second part is a special case of the fifth part of Remark 2.4. If \( M_{d,e}^{m} \neq \{pt\} \) holds,

\[
\frac{m - \sqrt{m^2 - 4}}{2} \leq \frac{e}{d} \leq \frac{m + \sqrt{m^2 - 4}}{2}
\]

follows from the second part of the proposition.

The other direction is proved in [Rei2, Theorem 3.5].

\[\square\]

Note that the dimension vectors satisfying the inequality in the third part of the theorem are the imaginary roots. If the moduli space is a point, the dimension vector is a real root, i.e. a reflection of a dimension vector corresponding to the simple roots \((0,1)\) and \((1,0)\) respectively.
5.1.1 Conjecture about the asymptotic behaviour of the Euler characteristic

In this section we discuss a conjecture of Michael Douglas concerning the Euler characteristic of Kronecker moduli spaces and several consequences. Originally, Douglas formulated the conjecture in [Dou] as follows:

**Conjecture 5.5 (Douglas)** Fix \( r \in \mathbb{R}_+ \) and consider \((d,e) \in \mathbb{N}_+^2\) with \(\gcd(d,e) = 1\) and \(\frac{e}{d} \approx r\).

1. Then there exists a \( C_r \in \mathbb{R} \) such that for \(e,d \gg 0\) we have
   \[
   \frac{\ln(\chi(M_{m,d,e}^n))}{d} \approx C_r.
   \]

2. The function \( r \mapsto C_r \) is continuous.

Thus Douglas conjectures that \(\frac{\ln(\chi(M_{m,d,e}^n))}{d}\) and therefore the Euler characteristic is asymptotically already determined by the fraction \(\frac{e}{d}\). Moreover, the Euler characteristic depends continuously on it.

Let \(m_1 := \frac{m - \sqrt{m^2 - 4}}{2}\) and \(m_2 := \frac{m + \sqrt{m^2 - 4}}{2}\).

Based on Douglas conjecture we obtain from [Wei] the following precise formulation:

**Conjecture 5.6** Let \(m \geq 3, m \in \mathbb{N}\). There exists a continuous function \(f: [m_1, m_2] \subset \mathbb{R} \to \mathbb{R}\) such that the following holds:

For all \(r \in [m_1, m_2]\) and all \(\varepsilon > 0\) there exists a \(\delta > 0\) and a \(n \in \mathbb{N}\) such that for all \((d,e) \in \mathbb{N}^2\) with \(\gcd(d,e) = 1\), \(|r - e/d| < \delta\) and \(|d + e| > n\) we have

\[
|f(r) - \frac{\ln(\chi(M_{m,d,e}^n))}{d}| < \varepsilon.
\]

**Remark 5.7**

- We may also rephrase the conjecture as follows: there exists a continuous function \(f\) such that for every coprime dimension vector \((d,e)\) there exists a dimension vector \((d_0, e_0)\) such that

\[
f\left(\frac{e}{d}\right) = \lim_{n \to \infty} \frac{\ln(\chi(M_{d_0, e_0 + nd, e_0 + ne}^n))}{d_0 + nd}.
\]

In particular, the right hand side converges.
We discuss some consequences of the conjecture which are proved in [Wei]. For the remainder of this section we assume that the conjecture is true. Define
\[ K := (m - 1)^2 \ln((m - 1)^2) - (m^2 - 2m) \ln(m^2 - 2m). \]

**Theorem 5.8** The function \( f \) is given by
\[ f(r) = \frac{K}{\sqrt{m - 2}} \cdot \sqrt{r(m - r) - 1}. \]
In particular, the constant \( K \) is its value at the point \( r = 1 \). Moreover, we have that the Euler characteristic asymptotically only depends on the dimension of the moduli space:

**Corollary 5.9** The logarithm of the Euler characteristic \( \ln(\chi(M_{d,e}^m)) \) is asymptotically proportional to \( \sqrt{dem - d^2 - e^2} = \sqrt{\dim M_{d,e}^m - 1} \).

### 5.2 Localization quivers of the Kronecker quiver

In this section we again consider the generalized Kronecker quiver. We investigate the quivers arising from the localization method in detail. Furthermore, we characterise a huge class of fixed points growing exponentially with the dimension vector. From this we will obtain a lower bound of the Euler characteristic of Kronecker moduli spaces. In the next section we will also deal with some examples of dimension vectors, for which it is possible to determine all torus fixed points.

Let \((d, e) \in \mathbb{N}^2\) be a coprime dimension vector of the Kronecker quiver and let \( X = (X_1, \ldots, X_m) \in (M_{d,e}^m)^T \) be a fixed point. From the considerations of the third section we get a morphism of algebraic groups \( \varphi : T \to P = (Gl(V) \times Gl(W))/C^* \), for which we can choose a lift \( \psi : T \to Gl(V) \times Gl(W) \). It can be decomposed into two morphisms \( \psi_1 : T \to Gl(V) \) and \( \psi_2 : T \to Gl(W) \).

Let
\[ V = \bigoplus_{\chi \in X(T)} V_\chi \]
and
\[ W = \bigoplus_{\chi \in X(T)} W_\chi \]
be the simultaneous eigenspace decompositions with respect to \( \psi_1 \) and \( \psi_2 \) respectively. They satisfy
\[ X_k(V_\chi) \subseteq W_{\chi + \alpha_k} \]
for all $\chi \in X(T)$ and $k = 1, \ldots, m$.

The universal abelian covering quiver $\hat{K}(m)$ has vertices $(1, \chi)$ and $(2, \chi)$, where $\chi$ runs through all characters of $X(T)$ and arrows

$$(1, \chi) \to (2, \chi + e_k)$$

for each $k \in \{1, \ldots, m\}$ and each $\chi \in \mathbb{Z}^m$.

For every fixed point there exists a unique dimension vector $\hat{d}$ given by

$$d_{1,\chi} = \dim V_\chi \quad \text{and} \quad d_{2,\chi} = \dim W_\chi$$

for $(1, \chi), (2, \chi) \in \hat{K}(m)_0$.

The other way around consider $\hat{K}(m)$ with dimension vector $\hat{d}$. In the following let $Q_f$ be the subquiver having vertices $q \in \hat{Q}_0$ such that $\hat{d}_q \neq 0$ and arrows $\alpha : i \to j \in \hat{Q}_1$ such that $\hat{d}_i, \hat{d}_j \neq 0$.

Thus a stable representation of this quiver correspond to a torus fixed point with dimension vector $(d, e)$ where

$$d = \sum_{\chi \in X(T)} \hat{d}_{1,\chi}$$

and

$$e = \sum_{\chi \in X(T)} \hat{d}_{2,\chi}.$$ 

In what follows, we call the vector $(d, e)$ dimension type of the representation.

**Definition 5.10** A tuple consisting of a quiver and a dimension vector is called stable if there exists at least one stable representation for this quiver and dimension vector.

If it is clear which dimension vector we consider, we will simply call such a tuple stable quiver.

**Remark 5.11**

- The stability condition for representations of $\hat{K}(m)$ is induced by the original linear form $\Theta = (1, 0)$. It is given by

$$\mu(\hat{d}) = \frac{\sum_{\chi \in X(T)} \hat{d}_{1,\chi}}{\sum_{\chi \in X(T)} \hat{d}_{1,\chi} + \hat{d}_{2,\chi}}.$$

- Since $\hat{K}(m)_f$ is bipartite and in particular connected if it is stable, there exists an embedding $\lambda : (\hat{K}(m)_f)_0 \to \mathbb{Z}^m$ such that

$$\lambda(i, \chi) = \chi.$$

This embedding is unique up to addition of characters $\mu \in \mathbb{Z}^m$. 

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In the following let $I \cup J$ be the decomposition of the vertex set into sources and sinks. We may assume that they are elements of $\mathbb{Z}^m$.

Let $R \subset I \times J$ be the set of arrows. Then we have

$$(i, j) \in R \iff j = i + e_k$$

for $k \in \{1, \ldots, m\}$, where $e_1, \ldots, e_m$ denotes the standard basis of $\mathbb{Z}^m$.

This defines a map $c : R \to \{1, \ldots, m\}$ by setting $c(i, j) = k$ if $j = i + e_k$.

Obviously, the set $R$ and the function $c$ are already uniquely determined by the vertex set $I \cup J$. Nevertheless, they play an important role because they will describe different localization data for a given localization quiver.

Therefore, a fixed point $X$ determines a tuple $(I, J, \hat{d})$ which is unique up to translation by a vector $\mu \in \mathbb{Z}^m$. In what follows we always consider such tuples up to translation.

This leads to the following definition:

\textbf{Definition 5.12} Let $X$ be a fixed point. The associated set $(I, J, \hat{d})/\mathbb{Z}^m$ is called localization data of $X$.

\textit{In the following, the set of all localization data of the $m$-arrow Kronecker quiver with dimension vector $(d, e)$ is denoted by $\mathcal{L}_{d,e}^m$.}

First we point out some properties of the localization data of the quiver $\hat{K}(m)$. For a bipartite quiver with vertex set $I \cup J$ and dimension vector $d$ define the sets

$A_i := \{ j \in J \mid \alpha : i \to j \in Q_1, d_j \geq 1 \}$

and

$A_j := \{ i \in I \mid \alpha : i \to j \in Q_1, d_i \geq 1 \}$.

Furthermore, define $R_i = |A_i|$ and $R_j = |A_j|$. For a localization data we get the following conditions:

\textbf{Remark 5.13}

- Let $i \in I$ be a vertex such that $\dim(i) = 1$. Then we have for a stable quiver that $m \geq R_i > \frac{d}{7}$.
- For all $j \in J$ it holds that $R_j \leq m$.
- For all $(i, j), (i, j') \in R$ such that $j \neq j'$ we have $c(i, j') \neq c(i, j)$.
- Analogously, for $(i, j), (i', j) \in R$ we obtain $c(i, j) \neq c(i', j)$.

We call a colouring satisfying these conditions stable.

\textbf{Definition 5.14} A cycle in a localization data is a sequence of vertices $i_1, j_1, \ldots, j_n, i_{n+1} = i_1$ (resp. $j_1, i_1, \ldots, i_n, j_{n+1} = j_1$) such that

$(i_k, j_k) \in R$ (resp. $(i_k, j_{k+1}) \in R$) and $(i_{k+1}, j_k) \in R$ (resp. $(i_k, j_k) \in R$)

for $1 \leq k \leq n$. 

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Definition 5.15 A bipartite quiver is \( m \)-bipartite if we have for all sources \( i \in I \) and all sinks \( j \in J \) that
\[ R_i, R_j \leq m. \]

A \( m \)-bipartite quiver \( Q \) with dimension vector \( \hat{d} \) such that
\[ |\{ \alpha \in Q_1 \mid \alpha : i \to j \}| \leq 1 \]
for all pairs \( i, j \in I \cup J \) is called localization quiver for a dimension vector \( (d, e) \) if \( M^s_d(Q) \neq \emptyset \) and
\[ \sum_{i \in I} \hat{d}_i = d \text{ and } \sum_{j \in J} \hat{d}_j = e. \]

Because of Lemma 2.5 localization quivers have to be connected. Otherwise there would be an exact sequence contradicting the stability condition.

Remark 5.16

- In order to test a \( m \)-bipartite quiver with
\[ |\{ \alpha \in Q_1 \mid \alpha : i \to j \}| \leq 1 \]
for stability, we do not need to consider an explicit representation. We can rather consider an arbitrary representation \( X \) for this dimension vector satisfying for all \( j \in J \) and all subsets \( A'_j \subseteq A_j \) with \( R'_j := |A'_j| \) the following property:
\[ \dim(\bigcap_{i \in A'_j} X_\alpha(X_i)) = \max\{0, \sum_{i \in A'_j} \dim(X_\alpha(X_i)) - (R'_j - 1) \dim(X_j)\}. \]

Indeed, if we consider a bipartite quiver of the following form

\[
\begin{array}{c}
\bigoplus_{i=1}^{n_1} \langle \alpha_1 \rangle \\
\bigoplus_{i=1}^{n_2} \langle \alpha_2 \rangle \\
\vdots \\
\bigoplus_{i=1}^{n_t} \langle \alpha_t \rangle \\
\end{array}
\]

with \( n_i \leq n \) for all \( 1 \leq i \leq t \), there always exists a representation of this quiver such that for all tuples of linear maps \( X_{\alpha_{i_1}}, \ldots, X_{\alpha_{i_k}} \) with \( 1 \leq k \leq t \) and \( 1 \leq i_1 < i_2 < \ldots < i_k \leq t \) the dimension of the intersections of the images is minimal. One verifies the existence and the dimension formula by induction on the number of arrows.
Obviously, we obtain a localization quiver from every localization data. The other way around, given a localization quiver we can choose a map \( c \) satisfying the conditions of Remark 5.13 and colour the quiver in order to get a localization data.

**Remark 5.17**

- Fixing a localization quiver it may happen that different colourings of the arrows lead to different types of localization data. For instance, if we consider a localization quiver with a colouring \( c \) such that this colouring induces a weight space of weight \( \chi \) and one of weight \( \chi - e_k \), we have an arrow \( \alpha : \chi - e_k \rightarrow \chi \) and also a linear map
  \[
  X_{\alpha: \chi - e_k \rightarrow \chi} : V_{\chi - e_k} \rightarrow V_{\chi}.
  \]
  If this arrow did not appear in the localization quiver, we call such an arrow induced. Obviously, the dimension of the moduli space increases at least by one.

But Theorem 3.16 put things right. In particular, after suitable many localization steps the remaining torus fixed points are representations of the universal covering quivers which has no cycles.

**Lemma 5.18** Let \( Q \) be a localization quiver and \( c, c' \) stable colourings of the arrows. Then we have:

1. By colouring the arrows with \( c \) we obtain a localization data.

2. Fix \( c \) and \( c' \) such that \( c \) induces no cycles and \( c' \) induces at least one cycle. Moreover, let \( \dim(M_Q, c) \) and \( \dim(M_Q, c') \) be the dimensions of the resulting moduli spaces. We have
  \[
  \dim(M_Q, c) \leq \dim(M_Q, c').
  \]

**Proof.** Fixing a localization quiver and a stable colouring of the arrows we obtain a localization data. Every stable representation of \( Q \) induces a stable representation of \( \hat{K}(m) \), no matter if the colouring leads to cycles or not. Induced arrows let the dimension of the moduli space increase. Thus it remains to prove that the dimension of the moduli space increases if a colouring induces a cycle which does not come from an induced cycle.

Let \( j_1 = j_2 \), i.e. \( \dim(j_{1,2}) = d_{j_1} + d_{j_2} \). We have \( R_{j_1}, R_{j_2} \geq 1 \) in \( Q \). Define \( \dim A_j = \sum_{i \in A_j} \dim i \). Then we have for a colouring \( c' \) producing this cycle

\[
\dim(M_Q, c') = \dim(M_Q, c) + d_{j_1}^2 + d_{j_2}^2 - (d_{j_1} + d_{j_2})^2 - \dim A_{j_1} d_{j_1} - \dim A_{j_2} d_{j_2} + (\dim A_{j_1} + \dim A_{j_2})(d_{j_1} + d_{j_2}) \geq \dim(M_Q, c).
\]
Indeed, we have \( \dim A_{jk} \geq d_{jk} \) because of the stability of \( Q \). The case \( i_1 = i_2 \) is proved in the same way.

\[ \square \]

**Definition 5.19** A localization data is called localization data of type 1 if \( \dim(i) = 1 \) for all \( i \in (\hat{K}(m))_0 \).

A localization quiver \( Q \) is called localization quiver of type 1 if there exists a colouring \( c \) such that the induced localization data is of type 1.

In the following we do not always distinguish between localization quivers and localization data if no confusion arises.

### 5.3 Stability of bipartite quivers

In this section we investigate stable quivers arising from the localization method. Thus each stable colouring of the arrows yields some localization data. In particular, we study how to construct new localization quivers by glueing localization quivers of smaller dimension types.

Let \( Q = (I \cup J, Q_1) \) and \( Q' = (I' \cup J', Q'_1) \) be two bipartite quivers with \( j \in J, j' \in J' \). Define the bipartite quiver

\[ Q_{j,j'}(Q, Q') = (I \cup I' \cup J \setminus j \cup J' \setminus j' \cup j'', Q''_1) \]

such that \( \alpha : i \mapsto j_1 \in Q''_1 \) if and only if \( \alpha : i \mapsto j_1 \in Q_1 \) or \( \alpha : i \mapsto j_1 \in Q'_1 \) with \( j_1 \neq j, j' \) and \( \alpha : i \mapsto j'' \in Q''_1 \) if and only if \( \alpha : i \mapsto j_1 \in Q_1 \) or \( \alpha : i \mapsto j_1 \in Q'_1 \) such that \( j_1 = j \) or \( j_1 = j' \).

Thus the new quiver is generated by the former ones by identifying two vertices of the set of sinks of these quivers.

**Definition 5.20** The quiver \( Q_{j,j'}(Q, Q') \) is called the glueing quiver of \( Q \) and \( Q' \) and the vertices \( j, j'' \) the glueing vertices.

**Definition 5.21** Let \( Q \) be a bipartite quiver with sources \( I \). A subquiver of \( Q \) with sources \( I' \) is called boundary quiver if there exists precisely one \( i_0 \in I' \) such that \( |A_{i_0} \cap A_{I \setminus V}| = 1 \) and \( |A_i \cap A_{I \setminus V}| = 0 \) for all \( i \in I' \) with \( i \neq i_0 \). A boundary quiver is called proper boundary quiver if it does not contain any other boundary quiver.

This means that boundary quivers are such subquivers which only have one common sink with the remainder of the quiver.

In what follows we abbreviate the dimension of the image of a subspace \( U \) to \( d_U \). Thus if \( U = \bigoplus_{i \in I} U_i \), we define

\[ d_U := \sum_{\alpha : i \mapsto j} X_\alpha(U_i). \]
For a given dimension vector \((d,e)\) we now determine a unique dimension vector \((d_s,e_s)\) such that we are able to construct new stable quivers of dimension type \((d_s + kd, e_s + ke)\) by glueing quivers of the types \((d_s, e_s)\) and \(k(d, e)\).

Fixing some dimension vector \((d, e)\), we first show that there exists a dimension vector \((d_s, e_s)\) such that \(d_s \leq d\) and \(e_s \leq e\) satisfying the conditions

- \(\frac{e+e_s}{d+d_s}d < e + 1\)
- \(\frac{e-1}{d_s} \leq \frac{e}{d}\) if \(d \neq 1\) and \((e_s - 1)d = ed_s\) if \(d = 1\)
- \(\frac{e+e_s}{d+d_s}d' < \left\lceil \frac{e}{d} \right\rceil \forall d' < d\)
- \(\gcd(d + d_s, e + e_s) = 1\).

We refer to this conditions as gluing condition. The first property is equivalent to the following:

\[
\begin{align*}
de + de_s &< de + d + d_se + d_s \\
\iff de_s &< d + d_se + d_s \\
\iff d(e_s - 1) &< d_s(e + 1) \\
\iff \frac{e_s - 1}{d_s} &< \frac{e + 1}{d}.
\end{align*}
\]

The second one is equivalent to:

\[
\begin{align*}
ed + e_sd &\geq ed + ed_s \\
\iff \frac{e_s}{d_s} &> \frac{e}{d}.
\end{align*}
\]

Therefore, it suffices to prove the second and third property because the first one follows from the third one.

**Lemma 5.22** Let \((d, e) \in \mathbb{N}^2\) such that \(d \leq e\) and \(d, e\) are coprime. There exists a dimension vector \((d_s, e_s)\) satisfying the gluing condition. It is uniquely determined if we also assume that \(d_s \leq d\) and \(e_s \leq e\).

**Proof.** We first consider the special case \(d = 1\). It is easy to see that \((0,1)\) satisfies these properties for \((d,e) = (1,n)\) with \(n \in \mathbb{N}\).

If \(d \geq 2\), we already have \(e \geq 3\). Choose \(d_s \in \mathbb{N}\) minimal such that

\[d \mid 1 + ed_s.\]

This is possible because \(\gcd(d,e) = 1\) and, therefore, there exist \(\lambda', \mu'\) such that

\[\lambda'd = 1 - \mu'e.
\]

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If $\mu' > 0$, we have
\[
\lambda^2 d^2 = 1 - 2\mu' e + \mu'^2 e^2 = 1 + e(\mu'^2 e - 2\mu').
\]
Since $\mu'^2 e > 2\mu'$ for $e > 2$, we obtain the existence and in particular that $d_s \in \mathbb{N}$.

Define
\[
es_s = \frac{1 + e(d + d_s) - de}{d} = \frac{1 + d_s e}{d}.
\]
Because of the choice of $d_s$, we have $e_s \in \mathbb{N}$.

Moreover, we get
\[e(d + d_s) + d(e + e_s) = -ed - ed_s + de + d_s e + 1 = 1.\]

It follows that $\gcd(d + d_s, e + e_s) = 1$.

Now we get
\[
es_s \frac{e}{d} = \frac{1 + d_s e}{dd_s} > \frac{e}{d}
\]
and also
\[
es_s - \frac{1}{d_s} = \frac{d_s e - d + 1}{dd_s} < \frac{e}{d'}.
\]
Thus it remains to prove the fourth property. By an easy calculation we get
\[
\frac{e + e_s}{d + d_s} = \frac{e}{d} \left( \frac{ed + ed_s + 1}{ed + ed_s} \right) = \frac{1}{d} \left( 1 + \frac{1}{ed + ed_s} \right).
\]

Moreover, since
\[
[e d'] - \frac{e}{d} d' \geq \frac{1}{d}
\]
and
\[
\frac{d'}{ed + d_s e} < \frac{1}{d + d_s}
\]
for each $d' < d$, the existence of such a vector follows.

\[\square\]

In what follows, we call a vector $(d_s, e_s)$ satisfying these properties starting quiver for $(d, e)$. Now we show that $(d_s + kd, e_s + ke)$ with $k \geq 1$ also is a starting vector for the dimension vectors $(ld, le)$ with $l \geq 1$. Thus we can glue quivers of dimension type $(ld, le)$ on a stable quiver of type $(d_s + kd, e_s + ke)$ in order to obtain new stable quivers of dimension type $(d_s, e_s) + (k+l)(d, e)$.

**Corollary 5.23** Let $d$, $d_s$, $e$ and $e_s$ fulfil the glueing condition. Then we have $\gcd(d_s + kd, e_s + ke) = 1$ for all $k \geq 1$. 

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Proof. As before we also have

\[-e(kd + d_s) + d(ke + e_s) = 1\]

for arbitrary \(k \geq 1\).

\[\square\]

**Corollary 5.24** Let \(d, d_s, e, e_s\) satisfy the glueing condition and let \(k, l \in \mathbb{N}\). We have

1. \(\frac{e_s + ke}{d_s + kd} < \frac{le + 1}{d} \)

2. \(\frac{e_s + ke}{d_s + kd} > \frac{le}{d} \)

3. \(\frac{e_s + ke - 1}{d_s + kd} \leq \frac{le}{ld} \)

4. \(\frac{ke + e_s}{kd + d_s} < \left\lceil \frac{e}{d} \right\rceil d' < d \)

**Proof.** The second (resp. third) statement is equivalent to the second (resp. third) property of the dimension vector \((d + d_s, e + e_s)\) what can be seen easily. Now the first property again follows from the third one. Let \(k > 1\). Then we have

\[\frac{e_s + ke}{d_s + kd} \leq \frac{e + e_s}{d + d_s}\]

which is verified by an easy calculation. The fourth property follows from this.

\[\square\]

**Remark 5.25**

- If we want to decompose a dimension vector \((d, e)\) into

\[(d, e) = (d_s, e_s) + k(d', e')\]

for coprime \((d, e)\) such that \((d', e')\) and \((d_s, e_s)\) satisfy the glueing condition, we can proceed as follows:

fix \(e'\) minimal such that

\[e' \mid 1 + de'\]
and 
\[ d' = \frac{1 + e'd}{e}. \]

Now we compute \( d_s \) and \( e_s \) as before. It can be seen easily that these numbers satisfy the glueing condition. Indeed, one checks that 
\[ e - e_s = \frac{d - d_s}{d'}. \]

From this it follows that \( e' | e - e_s \) and \( d' | d - d_s \) because \( \gcd(d', e') = 1 \) and, trivially, \( e - e_s, d - d_s \in \mathbb{N} \) hold. Now define \( k = \frac{d - d_s}{d'}. \)

We need other properties of these natural numbers. By use of \( e_s d - ed_s = 1, \)

for \( 0 \leq k' \leq k \) we get 
\[ (ke + e_s)(k'd + d_s) + k - k' = (kd + d_s)(k'e + e_s). \]

For \( d_1 = k'd + d' \in \mathbb{N} \) with \( 0 \leq d' < d \) and \( 0 < d_1 \leq kd + d_s \) define a map 
\[ f(d_1) = \min\{n \in \mathbb{N} \mid \frac{(ke + e_s)d_1 + n}{kd + d_s} \in \mathbb{N}\}. \]

Note that \( f \) is injective because \( \gcd(d_s + kd, e_s + ke) = 1 \). Then we get the following lemma:

**Lemma 5.26** Let \( d_s, e_s, d, e \) fulfil the glueing condition. Then we have 
\[ (ke + e_s)(k'd + d_s) + k - k' = 0 \mod (kd + d_s) \]

for all \( k' \leq k \).

Let \( d_1 = k'd + d' \) with \( 0 \leq d' < d \). In particular, we have \( f(d_1) = k - k' \) if \( d' = d_s \) and thus \( f(d_1) \geq k + 1 \) if \( d' \neq d_s \).

Now we show how to get a stable quiver of dimension type \((d_s + (k+l)d, e_s + (k+l)e)\) by glueing a stable quiver of type \((d_s + kd, e_s + ke)\) and certain quivers of type \((ld, le + 1)\). At this, Corollary 5.24 assures that the considered subquivers satisfy the stability condition. Afterwards we prove that also direct summands of representations of the two subquivers do not contradict the stability condition what again follows by Corollary 5.24. We again point out Remark 5.16. Thus we do not consider specific representations, but those satisfying the properties mentioned in the remark.

Fix natural numbers \( d, e \) and \( m \) and let \( S_{ld,le+1}^m \) be the set of tuples consisting of a \( m \)-bipartite quiver of dimension type \((ld, le + 1)\) and a sink \( j \) satisfying the following properties:
There exists at most one arrow between two vertices.

After decreasing the dimension of the sink \( j \) by one, the resulting quiver is connected and semistable.

There exists a representation for the quiver such that for every \( d' \)-dimensional subspace \( U \) we have

\[
d_U > \frac{(k + l)e + e_s}{(k + l)d + d_s} d'.
\]

Let \( T_{d,e}^m \) the set of all stable \( m \)-bipartite quivers of dimension type \((d,e)\).

**Theorem 5.27** Let \( d, d_s, e, e_s \) fulfil the glueing condition and let \( k \in \mathbb{N} \). Let \( T^0 \in T_{d_s+kd,e_s+kd}^m \) and \((T^1, j_1) \in S_{ld,le+1}^m\). Moreover, let \( j_0 \) be a sink of \( T^0 \) such that \( R_{j_0} + R_{j_1} \leq m \). Then \( Q_{j_0,j_1}(T^0,T^1) \) with glueing vertex \( j_2 \) where \( \dim(j_2) := \dim(j_0) + \dim(j_1) - 1 \) is an element of \( T_{d_s+(k+l)d,e_s+(k+l)e}^m \).

**Proof.** For some subspace \( U \) of one of the two subquivers we denote by \( d_U \) the dimension of its image corresponding to its original quiver and by \( d'_U \) the dimension of its image corresponding to the glueing quiver.

First let \( U \) be a \( d' \)-dimensional subspace of \( T^1 \) such that \( d' < ld \). Then by definition we have

\[
(k + l)e + e_s < (k + l)d + d_s d_U = d'_U.
\]

If \( d' = ld \), the same inequality follows from \( d_U = le + 1 \) together with the first property of Corollary 5.24.

Since we also have

\[
e_s + ke > \frac{e_s + (k + l)e}{d_s + (k + l)d'}
\]

see the properties of the dimension vectors, the same follows for subspaces of the subquiver \( T^0 \).

It remains to prove that subspaces composed of subspaces of both subquivers fulfil the stability condition. Thus let \( U' \) and \( U'' \) be two subspaces of dimension \( 1 \leq d' \leq ld \) and \( 1 \leq d'' \leq kd + d_s \) such that we have proper inequality at least once.

Now it suffices to prove that

\[
d'_{U' \oplus U''} > \frac{le}{ld} d' + d'' \quad \Rightarrow \quad \frac{(k + l)e + e_s}{(k + l)d + d_s} (d' + d'')
\]

where the first inequality follows from the semistability of the quiver obtained from \( T^1 \) after decreasing the dimension of the vertex \( j_1 \) by one. This is equivalent to

\[
d_{U'' > \frac{(k + l)e + e_s}{(k + l)d + d_s} d'' + \frac{d'}{d((k + l)d + d_s)}}
\]
using \( e_s d - d_s e = 1 \).

By the preceding lemma together with the assumption we have

\[
d_{U''} \geq \frac{(k e + e_s)d'' + f(d'')}{kd + d_s}.
\]

First let \( d'' < kd + ds \). Assuming without lose of generality that \( d' = ld \), it remains to prove that

\[
ld'' + ((k + l)d + d_s)f(d'') > l(kd + d_s).
\]

But this is easily verified.

Finally, let \( d'' = kd + d_s \) and \( d' = l'd + d_1 < ld \) with \( 0 \leq d_1 < d \). We have

\[
\frac{(k + l)e + e_s(kd + d_s)}{(k + l)d + d_s} = ke + e_s - \frac{l}{(k + l)d + d_s}
\]

and again using \( e_s d - ed_s = 1 \). Thus it remains to prove

\[
\left\lfloor \frac{e}{d} (l'd + d_1) \right\rfloor = l'e + \left\lfloor \frac{e d_1}{d} \right\rfloor > \frac{(k + l)e + e_s}{(k + l)d + d_s} (l'd + d_1) - \frac{l}{(k + l)d + d_s}
\]

what follows from the fourth property of Corollary 5.24 together with \( l > l' \).

\[\square\]

If \( T^0 \) and \( T^1 \) satisfy the condition of the theorem we call \( T^0 \) starting quiver for \( T^1 \).

Next, we apply the result to specific quivers. Therefore, let \( T \in T_{d,e}^{m} \). Starting with this quiver, we construct new quivers \( \hat{T} \) of dimension type \((d, e + 1)\) in one of the following ways:

- Choose an \( i \in I \) such that \( R_i < m \) and define the new quiver by the vertex set \( \hat{T}_0 = T_0 \cup \{j\} \) and the arrow set \( \hat{T}_1 = T_1 \cup \{\alpha : i \to j\} \).
  Finally, let \( \dim(j) = 1 \).

- Choose a vertex \( j \in J \) with \( 1 < R_j < m \) and increase the dimension of the vertex by one.

- Choose a vertex \( j \in J \) such that

\[
\dim(j) < \sum_{i \in A_j} \dim(i)
\]

and increase the dimension of the vertex \( j \) by one.

Denote the set of the resulting quivers by \( \hat{T}_{d,e}^{m} \) and refer to \( j \) as modified vertex.
Corollary 5.28 Let \( d, d_s, e, e_s \) be as before and let \( k \in \mathbb{N} \). Moreover, let \( T^0 \in T^m_{d_s+kd, e_s+ke} \) and \( T^1 \in \hat{T}^m_{d, e} \) with modified vertex \( j \). Further let \( j_0 \) be a sink of \( T^0 \) such that \( R_{j_0} + R_{j_1} \leq m \). Then \( Q_{j_0, j_1}(T^0, T^1) \) with glueing vertex \( j \), where \( \dim(j) := \dim(j_0) + \dim(j_1) - 1 \), is an element of \( T_{d_s+(k+1)d, e_s+(k+1)e} \).

Proof. Let \( U \) be a \( d' \)-dimensional subspace of \( T^1 \). Since \( T^1 \) results from a stable quiver we have \( d' > \frac{e}{d} d' \). Moreover, by the fourth property of Corollary 5.24 it follows that \( (k+1)e + e_s (k+1)d + d_s < \lceil \frac{e}{d} d' \rceil \leq d_U \). If \( d' = d \), the same inequality follows from the first property together with

\[
\frac{(k+1)e + e_s d'}{(k+1)d + d_s} < \frac{e}{d} \leq d_U.
\]

Fixing a coprime dimension vector \((d, e)\) we now deal with the question how to construct a certain set of stable quivers. Therefore, we assign a set of stable quivers to tuple of natural numbers which is uniquely determined by the dimension vector, see also Example 5.30. These numbers correspond to the number of possible glueing vertices and possible colourings of the constructed quivers.

Fix a dimension vector \((d, e)\) and the corresponding starting vector \((d_s, e_s)\). Denote by \( T^{(d,e)}_{n_1} \) the set of stable quivers of dimension type \((d_s, e_s) + n_1(d, e)\) with \( n_1 \geq 1 \). As before let \( \hat{T}^{(d,e)}_{n_1} \) be the set which results by modifying a vertex \( j_1 \). Now we continue recursively: let \( S \in T^{(d,e)}_{n_k-1,...,n_1} \) and \( T \in \hat{T}^{(d,e)}_{n_k,...,n_1} \). Now let \( T^{(d,e)}_{1,n_k,...,n_1} \) be the set consisting of all quivers \( Q_{j_0,j_1}(S, T) \) such that \( R_{j_0} + R_{j_1} \leq m \). Moreover, let the dimension of the glueing vertex \( j \) be given by \( \dim(j) = \dim(j_0) + \dim(j_1) - 1 \). In general let \( T^{(d,e)}_{n_k+1,...,n_1} \) be the set of glueing quivers resulting from glueing a quiver \( S \in T^{(d,e)}_{n_k+1,...,n_1} \) and a quiver \( T \in T^{(d,e)}_{n_k,...,n_1} \) as described.

Corollary 5.29 The sets \( T^{(d,e)}_{n_k,...,n_1} \) only contain stable quivers.

Proof. It suffices to prove that these quivers satisfy the condition of Corollary 5.28. We assume that \( T^{(d,e)}_{n_k,...,n_1} \) only contains stable quivers. We have to prove that
\( \mathcal{T}_{n_{k+1}, \ldots, n_1} \) just consists of stable quivers for all \( n_{k+1} \geq 1 \). Therefore we show that the quivers in \( \mathcal{T}_{n_{k-1}, \ldots, n_1} \) are starting quivers for quivers in \( \mathcal{T}_{n_{k-1}, \ldots, n_1} \).

Let \((d^k, e^k)\) be the dimension vector corresponding to \( \mathcal{T}_{n_{k-1}, \ldots, n_1} \) and \((d^k_s, e^k_s)\) the one belonging to \( \mathcal{T}_{n_{k-1}, \ldots, n_1} \). It suffices to prove that

\[
(d^k_s + 1, e^k_s + 1) = (d^k_s, e^k_s) + (n_k - 1)(d^k, e^k)
\]

is the starting vector for

\[
(d^{k+1}, e^{k+1}) = (d^k_s, e^k_s) + n_k(d^k, e^k).
\]

Indeed, the quivers in \( \mathcal{T}_{n_{k-1}, \ldots, n_1} \) are obtained by the modification described in Corollary \( 5.28 \). But this is equivalent to

\[
e^{k+1}_s = \frac{1 + d^{k+1}_s e^{k+1}}{d^{k+1}}
\]

with the additional condition \( d^{k+1}_s \leq d^{k+1} \), see Lemma \( 5.22 \). The second property follows immediately, the first one is equivalent to

\[
e^k_s = \frac{1 + d^k e^k}{d^k},
\]

what follows by a direct calculation. Therefore, the claim follows by the induction hypothesis.

\[\square\]

**Example 5.30**

Let \((d_s, e_s) = (0, 1)\) and \((d, e) = (1, n - 1)\). Then we obtain the corresponding tuple of natural numbers \((n_k, \ldots, n_1)\) to a fixed dimension vector by proceeding as mentioned in Remark \( 5.25 \). More detailed we have \((d^k, e^k) = (d_s, e_s) + n_k(d^{k-1}, e^{k-1})\) and in this way we recursively obtain the whole tuple. The recursion terminates if \((d_s, e_s) = (0, 1)\).

For instance consider \((d, e) = (8, 13)\). The tuple of numbers is given by \((n_3, n_2, n_1) = (1, 2, 2)\) with \( n = 2 \). Thus we get

\[
(d, e) = (3, 5) + (5, 8) = (1, 2) + (2, 3) + ((1, 2) + 2(2, 3)) = (0, 1) + (1, 1) + (0, 1) + 2(1, 1) + ((0, 1) + (1, 1) + 2((0, 1) + 2(1, 1))).
\]

Initially, consider the localization quivers of the dimension types \((1, 2)\) and \((2, 3)\), i.e.

\[
\begin{array}{c}
1 \\
\downarrow \quad \downarrow \\
1
\end{array}
\]
and

![Diagram of quivers](image)

By use of Corollary 5.28 we obtain the following localization quivers of dimension type (3, 5) by glueing:

![Diagram of quivers](image)

Next, for instance we obtain the following localization quivers of type (5, 8) by glueing:

![Diagram of quivers](image)

Finally, for instance the following bipartite quiver of dimension type (8, 13) is stable by use of Corollary 5.28:

![Diagram of quivers](image)

### 5.4 A lower bound

Let $m \in \mathbb{N}$ with $m \geq 3$ again be the number of arrows of the Kronecker quiver. The aim of this section is to determine a lower bound for the Euler
characteristic of the moduli spaces of stable representations for coprime
dimension vectors. Therefore, we consider the cases \( e > (m-1)d \). The remain-
ing cases are obtained by the isomorphisms of the moduli spaces, see The-
orem 5.4. In the considered cases the moduli spaces are zero-dimensional.
Moreover, we will see that the recursive construction of the localization quiv-
ers simplifies.

Because of Theorem 3.16 we may assume that all torus fixed points are
representations of the universal covering quiver. Initially, consider the dimension vectors \((1, n-1)\) and \((1, n)\) with \(2 \leq n \leq m-1\) which are mapped to the dimension vectors \((n-1, m(n-1)-1)\) and
\((n, mn-1)\) by the mentioned isomorphisms. For \((1, n-1)\) there exists only
one localization quiver

\[
\begin{array}{ccccccc}
  & j_1 & & j_2 & & & \\
  i_1 & \rightarrow & j_1 & \rightarrow & j_2 & \rightarrow & \cdots \rightarrow j_{n-1}
\end{array}
\]

where \( \text{dim}(j_k) = \text{dim}(i_1) = 1 \) for all \(1 \leq k \leq n-1\). Analogously, we obtain
the unique localization quiver of dimension type \((1, n)\).

Consider the following localization quiver of dimension type \((n-1, m(n-1)-1)\) where \( \text{dim}(j) = n-2 \) and \( \text{dim}(j_{k,i}) = \text{dim}(i_k) = 1 \) otherwise:

\[
\begin{array}{ccccccc}
  & j & & j_1,1 & & & \\
  i_1 & \rightarrow & j_1,1 & \rightarrow & j_1,2 & \rightarrow & \cdots \rightarrow j_{n-1,1} \rightarrow j_{n-2,1} \rightarrow \cdots \rightarrow j_{1,m-1} \rightarrow j_1,n-1
\end{array}
\]

Again we analogously get the quiver of type \((n, mn-1)\).

**Remark 5.31**

- For the dimension vector \((n, mn-1)\), \(1 \leq n \leq m\), this is also the only
  localization quiver because obviously each one-dimensional subspace
  is forced to have a \(m\)-dimensional image. Moreover, because of the
  stability condition, we have for each subspace \(U\) of dimension \(d' < n\)
  that
  \[
  d_U > \frac{nm-1}{n}d'.
  \]
  Therefore, we have \(d_U \geq md'\) for all \(d' < n\). But, for any other quiver
  of this dimension type this condition is not satisfied.

- We also get this quivers by applying the reflection functor, see Theorem
  2.6.
By use of the procedure introduced in Section 5.3 we can glue these quiver together. Fix \( m \in \mathbb{N} \) and define \( Q^l \) by

\[
\begin{array}{cccccc}
& i_1 & \rightarrow & j_{1,1} & \rightarrow & j_{1,1} \\
\vdots & & \rightarrow & \vdots & & \vdots \\
& j_{1,m-1} & \rightarrow & \vdots & \rightarrow & j_{1,m-1} \\
\end{array}
\]

Let \( I \cup J \) the set of vertices and define \( J'_1 := J \setminus j_1 \). Let \( \dim(j_1) = l - 1 \) and let the other vertices be one-dimensional. Define the glueing quiver

\[
Q^{j_1,j_2} := Q_{j,j_2}(Q^{l_1}, Q^{l_2})
\]

with \( j \in J'_1 \). For the resulting quiver define \( \dim(j_2) = l_2 \) whereby the dimensions of the other vertices remain constant. For instance we obtain:

\[
\begin{array}{cccccc}
& i_1 & \rightarrow & j_{1,1} & \rightarrow & j_{1,1} \\
\vdots & & \rightarrow & \vdots & & \vdots \\
& j_{1,m-1} & \rightarrow & \vdots & \rightarrow & j_{1,m-1} \\
\end{array}
\]

We again consider the construction of Corollary 5.29. Let \((d, e) = (n_1(n - 1) + 1, n_1(m(n - 1) - 1) + m) = (1, m) + n_1(n - 1, m(n - 1) - 1)\). Then we obtain the cases

\[
\frac{m(n - 1) - 1}{n - 1} d' \leq e' \leq \frac{mn - 1}{n} d'.
\]

We only consider those quivers constructed in this section. Moreover, they are glued as explained in this section. Fixing \( n_1 \geq 1 \) we denote the resulting quivers by \( Q^n_{n_1} \). They obviously result from glueing \( n_1 \)-times a quiver of dimension type \((n - 1, m(n - 1) - 1)\). Call the glueing vertex corresponding to the first glueing initial glueing vertex. If \( j_1 \) is the initial glueing vertex, denote by \( \hat{Q}^n_{n_1} \) the set of quivers obtained by increasing the dimension of \( j_1 \) by one.

We now recursively define

\[
Q^n_{n_1,\ldots,n_1} = \{Q_{j,j_1}(S, T) \mid S \in Q^n_{n_1,\ldots,n_1}, T \in \hat{Q}^n_{n_1,\ldots,n_1}\},
\]

where \( j \in S_0 \) such that \( R_j = 1 \) and where \( j_1 \) is the initial glueing vertex of \( T \in Q^n_{n_1,\ldots,n_1} \). Furthermore, let \( Q^n_{0,\ldots,n_1} = Q^n_{n_1,\ldots,n_1} \). By Corollary 5.29
we know that every quiver $S \in Q^{n}_{n_{k+1} - 1, \ldots, n_{1}}$ is stable and that each of them satisfies the properties of the starting quiver for each $T \in \hat{Q}^{n}_{n_{k}, \ldots, n_{1}}$. Thus it follows that all of the quivers obtained by glueing are stable.

**Remark 5.32**

- If $(d', e')$ is given such that (1) holds, we can determine the corresponding tuple $(n_{k}, \ldots, n_{1})$ as described in Remark 5.25. There is an easier method to get this tuple by simply solving linear equations, see [Wei2].

Next, we determine the cardinality of these sets in order to obtain a lower bound for the Euler characteristic. The moduli spaces of the considered localization quivers are zero-dimensional, i.e. a point. And it is well-known that $\chi(\{pt\}) = 1$. Furthermore, by Theorem 5.4 we can assume that $n \geq \frac{m+1}{2}$. This is another advantage simplifying combinatorics. Indeed, because of this assumption it is just possible to glue one quiver on each vertex of dimension one. Otherwise, there would be no suitable coloring to obtain a localization data from the produced quiver because it is no subquiver of the regular $m$-tree.

Initially, consider the set $Q_{1}$ consisting of the quiver of dimension type $(n, mn - 1)$. After modifying a sink, considering the properties of Remark 5.13 and taking into account all symmetries and the fact that all quivers are glued as mentioned above, there exist

$$\binom{m-1}{n} \text{ possibilities}$$

to choose a coloring $c : R \mapsto \{1, \ldots, m\}$ where $R$ is the set of arrows. Each of the quivers has $n(m-1)$ knots, i.e. vertices $j \in J$ such that $R_{j} = 1$. Denote by $a_{n_{1}}^{n}$ the cardinality of $\hat{Q}^{n}_{n_{1}}$ in consideration of the different colorings. Furthermore, let $K_{n_{1}}^{n}$ the number of knots of these quivers which coincide for all quivers in this set. Using the notation of Section 44 we have

$$a_{n_{1}}^{n} = \binom{m-1}{n} A_{-1}^{(m-1), (n-1)(m-1), (n-1)(n-1), (n-1)(n-1), \ldots, (n-1)(m-1), n(m-1)}.$$ 

Moreover, we have

$$K_{n_{1}}^{n} = n(m-1) + (n_{1} - 1)(n - 1)(m - 1) - (n_{1} - 1).$$

Considering the construction we get the following lemma by an easy observation.
Lemma 5.33 Let \((n_{k+1}, \ldots, n_1) \in \mathbb{N}^{k+1}\).

1. The number of knots of the quivers in \(\hat{Q}_{n_{k+1}, \ldots, n_1}^n\) is given by
   \[
   K_{n_{k+1}, \ldots, n_1}^n = K_{n_{k-1}, \ldots, n_1}^n + n_{k+1}K_{n_{k-1}, \ldots, n_1}^n - n_{k+1}.
   \]

2. Moreover, we have
   \[
   a_{n_{k+1}, \ldots, n_1}^n = a_{n_{k-1}, \ldots, n_1}^n \cdot A_{a_{n_{k-1}, \ldots, n_1}^n, K_{n_{k-1}, \ldots, n_1}^n, K_{n_{k-1}, \ldots, n_1}^n - 1}.
   \]

Fixing a dimension vector, it suffices to determine the corresponding tuple of natural numbers in order to get a lower bound for the Euler characteristic. Given such a tuple define \(K_{d,e}^m := K_{n_{k+1}, \ldots, n_1}^n\) and \(a_{d,e}^m := a_{n_{k+1}, \ldots, n_1}^n\).

Consider the function
\[
\phi(x) = 1 + \frac{a_{d,e}^m}{d} x K_{d,e}^m.
\]
The generating function \(y(x)\) satisfies the functional equation
\[
y(x) = x \phi(y(x)).
\]

Since we are interested in some asymptotic value, which is independent of the number of starting knots, we can assume that there exists just one starting knot. Even the starting quiver only gives us a constant, which we may ignore.

For every coloured tree constructed like this we obtain some localization data by assigning the weight 0 to the source of the starting quiver. Thus it may happen that different trees define the same localization data. But, if \((d, e)\) is the considered dimension vector, the number of possible starting quivers is bounded by \(d\). Since
\[
\lim_{d \to \infty} \frac{\ln d}{d} = 0,
\]
we may disregard this as well when investigating the logarithmic asymptotic behaviour.

Define
\[
u_{d,e}^m := \frac{K_{d,e}^m}{d}.
\]

Theorem 5.34 Let \(e > (m - 1)d\). We have
\[
\lim_{n \to \infty} \frac{\ln \left( \chi(M_{d,e} + nd, e + nd) \right)}{d_s + nd} \geq \frac{1}{d} \left( \ln a_{d,e}^m + K_{d,e}^m \ln K_{d,e}^m - (K_{d,e}^m - 1) \ln (K_{d,e}^m - 1) \right).
\]

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Proof. Define
\[ F(x, y) = x\phi(y(x)). \]

By Theorem 4.6 we have
\[ [x^n]y(x) = \sqrt{\frac{x_0 F_x(x_0, y_0)}{2\pi F_y(x_0, y_0)}} x_0^{-n} n^{-\frac{3}{2}} (1 + O(n^{-1})). \]

Obviously, it suffices to consider \( x_0 \). By Corollary 4.7 we obtain
\[ (x_0)^{-1} = a_{d,e}^{m} K_{d,e}^{m} \left( \frac{1}{(K_{d,e}^{m} - 1)a_{d,e}^{m}} \right)^{\frac{K_{d,e}^{m} - 1}{K_{d,e}^{m}}}. \]

Then we get
\[ ((x_0)^{-1})^{nK_{d,e}} = (a_{d,e}^{m} K_{d,e}^{m})^{nu_{d,e}^{m}d} \left( \frac{1}{(K_{d,e}^{m} - 1)a_{d,e}^{m}} \right)^{nu_{d,e}^{m}d - n} \]
\[ = (a_{d,e}^{m} (K_{d,e}^{m} - 1))^{n} \left( \frac{K_{d,e}^{m}}{(K_{d,e}^{m} - 1)} \right)^{nu_{d,e}^{m}d}. \]

Hence we get that
\[
\ln(\chi(M_{d,e,n} + nd)) \geq \ln C + \frac{n \cdot \ln(a_{d,e}^{m} (K_{d,e}^{m} - 1))}{d_1 + nd} + \frac{nu_{d,e}^{m}d \cdot \ln(K_{d,e}^{m})}{d_1 + nd} - \frac{nu_{d,e}^{m}d \cdot \ln(K_{d,e}^{m} - 1)}{d_1 + nd}
\]
\[ =: L_{d,e,n}^{m} \]
for a constant \( C \in \mathbb{R} \). Thus it follows
\[
\lim_{n \to \infty} L_{d,e,n}^{m} = \frac{\ln(a_{d,e}^{m} (K_{d,e}^{m} - 1))}{d} + u_{d,e}^{m} (\ln K_{d,e}^{m} - \ln(K_{d,e}^{m} - 1)) = 1 \cdot (\ln a_{d,e}^{m} + K_{d,e}^{m} \ln K_{d,e}^{m} - (K_{d,e}^{m} - 1) \ln(K_{d,e}^{m} - 1)) =: L_{d,e}^{m}
\]
which proves the theorem.

By use of the isomorphisms of the moduli spaces we also get a lower bound for arbitrary \( d \) and \( e \).

Example 5.35
This example applies the introduced methods to the case \((d, e) = (5, 8)\) and \(m = 3\). For the starting dimension vector we get \((d_s, e_s) = (3, 5)\) and also

\[
\begin{array}{c}
s_{3,5} = \\
1 \Downarrow \quad 1 \\
1 \Downarrow \quad 1 \\
1 \Downarrow \quad 1 \\
1 \Downarrow \quad 1 \\
1 \Downarrow \quad 1
\end{array}
\]

For the quiver functions we get \(l_1 = 1\) and \(l_2 = 2\), i.e.:

\[
\hat{s}_{5,8} = (0, 1, 1).
\]

The reflected dimension vector is \((8, 19)\) and we obtain \(K_{5,8}^3 = 12\) and \(a_{5,8}^3 = 1664\). Thus in conclusion we have

\[
L_{5,8}^3 = \frac{1}{5} \ln (1664 \cdot \frac{12^{12}}{11^{11}}).
\]

6 Applications

6.1 The case \((3, 4)\)

In this section we consider the case \(d = 3\) and \(e = 4\) with \(m \geq 3\) in detail. Consider the localization quiver given by

\[
\begin{array}{c}
1 \Downarrow \quad 1 \\
i_1 \Downarrow \\
1 \Downarrow \quad 1 \\
i_2 \Downarrow \\
1 \Downarrow \quad 1 \\
i_3 \Downarrow \\
1 \Downarrow \quad 1 \\
i_4 \Downarrow \\
1 \Downarrow \quad 1 \\
i_5 \Downarrow \\
1 \Downarrow \quad 1 \\
i_6 \Downarrow \\
1 \Downarrow \quad 1
\end{array}
\]

Therefore, by colouring the arrows in the colours \({1, \ldots, m}\) satisfying the conditions of Remark 5.13 we obtain a localization data. In this case, the conditions are \(c(i_l) \neq c(i_{l+1})\) for \(1 \leq l \leq 5\). Each colouring is unique up to the symmetry of the symmetric group \(S_2\).

The colourings \((i, j, k, i, j, k)\) and \((i, j, k, i, j, i)\), such that \(i, j, k \in \{1, \ldots, m\}\) are pairwise disjoint, give rise to two cases, which we now consider in greater detail. In the first case we obtain

\[
\begin{array}{c}
1 \Downarrow \quad 2 \\
i \Downarrow \\
1 \Downarrow \quad 1 \\
j \Downarrow \\
1 \Downarrow \quad 1 \\
k \Downarrow \\
1 \Downarrow \quad 1 \\

\end{array}
\]
There is no new symmetry arising from this colouring. Furthermore, the moduli space is a point for this dimension vector. Note that the cycle breaks down after a second localization so that we get back the former quiver. The second special case is

\[
\begin{array}{c}
1 \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
1 \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
1 \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
1 \\
\end{array}
\]

The colouring induces an extra arrow and therefore another symmetry. In particular, the localization data is already determined by the choice of the colour of the free arrow, i.e. the one that does not appear in the cycle. But because of the extra arrow the moduli space is $\mathbb{P}^1$ so that the Euler characteristic is two.

Another possibility to see this is a second localization. Indeed, by considering the quiver without its colouring the fixed points are those representations satisfying $X_{i5} = 0$ or $X_{i7} = 0$ where $i_7$ is the extra arrow. Thus we again get back the original localization quiver by a second localization.

In conclusion we obtain that there are $\frac{m(m-1)^5}{|S_2|}$ possibilities to choose a colouring.

Further localization data are given by colourings of the following localization quiver:

\[
\begin{array}{c}
1 \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
1 \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
1 \\
\end{array}
\]

with the conditions $c(i_1) \neq c(i_2), c(i_3) \neq c(i_4), c(i_5) \neq c(i_6)$ and $c(i_2), c(i_3), c(i_5)$ pairwise disjoint. In consideration of the symmetries of $S_3$ we obtain $\frac{m(m-1)^3(m-2)}{|S_3|}$ possibilities.

We also get

\[
\begin{array}{c}
1 \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
1 \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
1 \\
\end{array}
\]
with the conditions \( c(i_1), c(i_2), c(i_3) \) pairwise disjoint and \( c(i_3) \neq c(i_4) \). Thus we get \( \frac{m(m-1)^4(m-2)}{|S_2|} \) possibilities.

If \( m \geq 4 \), we finally get the localization data coming from

\[
\begin{array}{c}
\text{i}_1 \\
\downarrow \\
3 \\
\downarrow \\
\text{i}_3 \\
\downarrow \\
\text{i}_4 \\
\downarrow \\
1 \\
\end{array}
\]

with the condition that the colours of all arrows are pairwise disjoint, hence \( \binom{m}{4} \) possibilities.

Since all fixed point components may be understood as points, for the Euler characteristic we have

\[
\chi(M^m_{3,4}) = \binom{m}{4} + \frac{m(m-1)^3(m-2)}{2} + \frac{m(m-1)^4(m-2)}{6} + \frac{m(m-1)^5}{2}.
\]

One easily verifies that this is the same result one obtains by the algorithm from [Rei2], i.e.:

\[
\chi(M^m_{3,4}) = \frac{1}{24} m(m-1)(4m^2 - 7m + 2)(4m^2 - 7m + 1).
\]

6.2 Euler characteristic in the case \((n, n)\)

**Lemma 6.1** Let \( m \geq 3 \). Every stable torus fixed point of the Kronecker quiver with \( m \) arrows and dimension vector \( (n, (m-1)n) \) has a cycle. Thus there exists a subspace \( U \) and maps \( f_1, \ldots, f_{2k} \in \{X_1, \ldots, X_m\} \) with \( f_i \neq f_{i+1} \) for \( 1 \leq i \leq 2l - 1 \) such that

\[
f_1 \circ \ldots \circ f^{-1}_{2k}(U) = U.
\]

**Remark 6.2**

- From the proof we even get the stronger result that a localization quiver of this dimension vector is forced to be cyclic. In particular, there exists no subquiver having just one common vertex with the remainder of the quiver.

**Proof.** Consider a subquiver of the form

\[
\begin{array}{c}
\text{j}_1 \\
\downarrow \\
\text{j}_2 \\
\downarrow \\
\vdots \\
\downarrow \\
\text{j}_m
\end{array}
\]

\[
\begin{array}{c}
\text{i} \\
\downarrow \\
\text{j}_m
\end{array}
\]

\[
\begin{array}{c}
\text{j}_1 \\
\downarrow \\
\text{j}_2 \\
\downarrow \\
\vdots \\
\downarrow \\
\text{j}_m
\end{array}
\]

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such that \( \dim(i) \leq d \). Because of the stability we have for the image dimension of a subspace \( i \), denoted by \( d_i \),

\[
d_i > \frac{(m-1)d}{d} \dim(i) = (m-1) \dim(i).
\]

We also have \( \dim(j_k) \geq \dim(i) \) for all \( k \). Indeed, if we had \( \dim(j_k) = l \) such that \( l < \dim(i) \), we could consider the \((\dim(i) - l)\)-subspace \( \ker(X_{\alpha;i-j_k}) \). It would have a \((\dim(i) - l)(m-1)\)-dimensional image, which obviously contradicts the stability condition.

Therefore, the subquiver is of dimension type \((\dim(i), e')\) with \( e' \geq m \dim(i) \). Moreover, the stability implies that each \( k \)-dimensional subspace has at least a \(((m-1)k + 1)\)-dimensional image.

Assume that the localization would not have a cycle. Thus in particular it has some proper boundary quiver, i.e. a subquiver, which has just one common vertex with the remainder of the whole quiver.

Fix such a quiver. It apparently has dimension type \((d_1, md_1)\). If we denote by \( b \) the dimension of the image of the remainder of the quiver, we get

\[
b \geq (m-1)(d - d_1) + 1.
\]

It follows

\[
(m-1)d = b + d_1 m - h \geq (m-1)(d - d_1) + 1 + d_1 m - h = (m-1)d + d_1 - h + 1
\]

where \( h \geq 1 \) is the dimension of the intersection of both subquivers. It follows \( h \geq d_1 + 1 \) what is obviously a contradiction.

\[
\square
\]

**Corollary 6.3** The Euler characteristic of the Kronecker moduli spaces with dimension vector \((n, n)\) vanishes if \( n \geq 2 \).

**Proof.** By the previous lemma we know that each representation of a localization quiver of dimension type \((n, (m-1)n)\) has a cycle. But because of Theorem 5.16 we can assume that fixed points of each Kronecker moduli space do not have cycles. Applying the previous result this means that for a fixed point there exists a subspace \( U \) such that its image is \((m-1) \dim(U)\)-dimensional. Hence there are no stable representations of the universal covering quiver of dimension type \((d, (m-1)d)\).

Because of the isomorphism between \( M^m_{n,(m-1)n} \) and \( M^m_{n,n} \), we in conclusion get

\[
\chi(M^m_{n,(m-1)n}) = \chi(M^m_{n,n}) = 0.
\]

\[
\square
\]
6.3 Finiteness of the fixed point set

In this section we investigate and answer a question posed in [Dre]. Namely for which dimension vectors is the set of fixed points finite and for which dimension vectors exists at least one \( n \)-dimensional fixed point component with \( n \geq 1 \). Again let \( \gcd(d, e) = 1 \).

**Theorem 6.4** Let \( d \geq 3 \), \( e \geq 4 \) and \( m \geq 3 \). Then there exist infinitely many torus fixed points.

**Proof.** Since the torus action is compatible with the isomorphisms, we may assume

\[
d < e < \frac{m}{2}d.
\]

Furthermore, let \( m' < \frac{m}{2} \in \mathbb{N} \) such that \((m' - 1)d < e < m'd \). By [Wei3] there exists a localization quiver \( s_{d,e}^m \) which consists of subquivers of the form \((1, m')\) and \((1, m' + 1)\) respectively. Since \( d \geq 3 \), in particular there exists a subquiver of the form

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\]

with \( s_1, s_2, s_3 \in \{m', m' + 1\} \). Fix by

\[
c(i_1, j_1, 1) = c(i_2, j_2, s_2) = c(i_3, j_3, s_3) = 1,
\]

\[
c(i_1, j_1, s_1) = c(i_3, j_3, 1) = 2
\]

and

\[
c(i_2, j_2, 1) = 3
\]

a colouring of the arrows. This colouring induces an extra arrow \((i_3, j_1, 1)\) such that \( c(i_3, j_1, 1) = 3 \). Hence the associated moduli space is at least one-dimensional implying that there are infinitely many torus fixed points.

\[\square\]
6.4 The case \((d,d+1)\)

In this section we investigate the function treated in Section 3.3 resulting from the conjecture of Douglas at the point 1. This means investigating the dimension vector \((d-1,d)\), which is equivalent to the dimension vector \((d,(m-1)d+1)\). The latter one is considered in the following. Because of Theorem 3.16 we may assume that localization data of this type are free of cycles. We will see that each localization data is of type one. In particular, all localization quivers consist of subquivers of dimension type \((1,m)\).

Lemma 6.5 All localization quivers are of type 1. In particular, we can assume that \(\dim(i) = \dim(j) = 1\) for all \(i \in I\) and \(j \in J\).

Proof. By use of Theorem 3.16 we can assume that a localization is free of cycles. Consider the subquiver

\[
\begin{array}{c}
\vdots \\
\jmath_2 \\
\jmath_1 \\
\jmath_m \\
i
\end{array}
\]

such that \(\dim(i) \leq d\). Because of the stability we get

\[
d_i > \frac{(m-1)d+1}{d} \dim(i) > (m-1) \dim(i).
\]

In particular, this holds if \(\dim(i) = 1\). Now we have \(\dim(j_k) \geq \dim(i)\) for all \(k\). Indeed, if we had \(\dim(j_k) = l\) such that \(l < \dim(i)\), we could consider the \((\dim(i)-l)\)-subspace \(\ker(f_{c(i,j_k)}(i))\) which would just have a \((\dim(i)-l)(m-1)\)-dimensional image. This contradicts the stability condition.

Therefore, the subquiver is of the form \((\dim(i), e')\) with \(e' \geq (\dim(i))\). Furthermore, because of the stability every \(k\)-dimensional subspace at least has a \(((m-1)k+1)\)-dimensional image.

If we fix a boundary quiver, which exists because the original quiver has no cycles, this subquiver just has one common vertex with the remainder of the quiver and is of dimension type \((d_1, md_1)\). Then for the image dimension of the remainder of the quiver \(b\) we have

\[
b \geq (m-1)(d - d_1) + 1.
\]

Thus we get

\[
(m-1)d+1 = b + d_1 m - h \geq (m-1)(d - d_1) + 1 + d_1 m - h = (m-1)d + d_1 - h + 1,
\]

where \(h \geq 1\) is the dimension of the intersection of the images of the two subquivers. Therefore, we have \(h \geq d_1\), i.e. \(h = d_1\).
We continue by proving that after removing this subquiver we get a stable quiver of dimension type \((d - d_1, (m - 1)(d - d_1) + 1)\). It suffices to prove stability because the subquiver just has a \(d_1\)-dimensional intersection with the remainder.

For an arbitrary subspace \(U\) with \(\dim U < d - d_1\) we have

\[
d_U > \frac{(m - 1)d + 1}{d} \dim U.
\]

Since \(\dim U < d - d_1\), we also have

\[
d_U > \frac{(m - 1)(d - d_1) + 1}{(d - d_1)} \dim U
\]

proving the claim in-between.

Thus we can proceed by induction on the number of subquivers in order to show that all localization quivers are of type one.

Consider the quiver consisting of one subquiver. Obviously, it is a stable quiver of type 1.

Assume that the quiver has \(n + 1\) subquivers. We may remove a boundary quiver so that we again get a stable quiver, which is of the requested type by induction hypothesis. But since the original quiver has no cycles, there exist at least two boundary quivers. Thus the assertion follows by applying the induction hypothesis to the respective subquivers after removing a boundary quiver.

\[\square\]

**Theorem 6.6** We have

\[
\chi(M_{d,d+1}^m) = \frac{m}{(d + 1)((m - 1)d + m)} \frac{(m - 1)^2 d + (m - 1)m}{d}.
\]

In particular, we also have

\[
\lim_{d \to \infty} \frac{\ln(\chi(M_{d,d+1}^m))}{d} = (m - 1)^2 \ln(m - 1) - (m^2 - 2m) \ln(m^2 - 2m).
\]

**Proof.** As shown previously, we may assume that all subquivers have vertex set

\[I \cup J = \{i_1, j_1, \ldots, j_m\}\]

and arrow set

\[R = \{(i_1, j_1), \ldots, (i_1, j_m)\} \]

with \(\dim(i_1) = \dim(j_k) = 1\). In particular, the moduli spaces of the considered quivers are zero-dimensional yielding that the Euler characteristic is one.

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By Remark 5.13 there exists exactly one possibility to choose a colouring $c$ taking into account the symmetries of $S_m$. Again by Remark 5.13 we can glue $k$ subquivers on each vertex $j_l$, $1 \leq l \leq m$, with $0 \leq k \leq (m - 1)$. But we have to take note of the symmetries of $S_k$. Assuming that there is only one starting knot let $y(x)$ the generating function of such quivers and consider

$$
\phi(x) = 1 + \frac{(m-1)}{|S_1|}x^{m-1} + \frac{(m-1)(m-2)}{|S_2|}x^{2(m-1)} + \ldots + \frac{\prod_{i=1}^{m-1}(m-i)}{|S_{m-1}|}x^{(m-1)(m-1)}.
$$

Then we have

$$
\phi(x) = \sum_{i=0}^{m-1} x^{i(m-1)} \binom{m-1}{i} = (1 + x^{m-1})^{m-1}.
$$

Because of Section 4.1 the generating function satisfies the functional equation $y(x) = x(\phi(y(x)))$. The generating function for all coloured localization quivers is obtained as follows: we start with the unique localization quiver of dimension type $(1, m)$ having $m$ knots. The resulting generating function is $y(x)^m$ and by applying the Lagrange inversion theorem we obtain that

$$
[x^n]y(x)^m = \frac{m}{n} u^{n-m} \phi(u)^n
= \frac{m}{n} \left( \frac{n(m-1)}{n-m} \right).
$$

If we assign the weight 0 to the sink of the starting quiver, every such quiver that has $(m-1)d + 1$ knots corresponds to a localization data of dimension type $(d, (m-1)d + 1)$. The other way around, we may assume that every localization data has some sink $i \in I$ with weight 0 what gives us $d$ choices. This means for every localization data we exactly get $d$ trees. Hence we get

$$
\chi(M_{d,(m-1)d+1}^m) = \frac{m}{d((m-1)d+1)} \left( \frac{(m-1)^2d + (m-1)}{d-1} \right)
= \frac{m}{d((m-1)(d-1) + m)} \left( \frac{(m-1)^2(d-1) + (m-1)m}{d-1} \right),
$$

Since $\chi(M_{d-1,d}^m) = \chi(M_{d,(m-1)d+1}^m)$, the assertion is proved.

The second part follows by applying Theorem 4.6. It may be left unconsidered that exactly $d$ trees define the same localization data. Indeed, obviously we have

$$
\lim_{d \to \infty} \frac{\ln d}{d} = 0.
$$
We can also assume that we just have one starting knot. Thus in addition to the functional equation \( y(x) = x(\phi(y(x))) \) we consider the functional equation
\[
1 = x(m - 1)^2(1 + y(x)^{m-1})^{m-2}y(x)^{m-2}.
\]
Moreover, we consider the equations
\[
y_0 = x_0(1 + y_0^{m-1})^{m-1}
\]
and
\[
1 = x_0(m - 1)^2(1 + y_0^{m-1})^{m-2}y_0^{m-2}.
\]
Then we have
\[
x_0 = \frac{1}{(m - 1)^2(1 + y_0^{m-1})^{m-2}y_0^{m-2}}
\]
which implies
\[
y_0 = \frac{(1 + y_0^{m-1})^{m-1}}{(m - 1)^2(1 + y_0^{m-1})^{m-2}y_0^{m-2}}.
\]
Thus we get
\[
(m - 1)^2y_0^{m-1} = 1 + y_0^{m-1}
\]
and therefore
\[
y_0^{m-1} = \frac{1}{(m - 1)^2 - 1}.
\]
Hence we have
\[
(x_0)^{-1} = (m - 1)^2 \left(\frac{(m - 1)^2}{(m - 1)^2 - 1}\right)^{m-2} \left(\frac{1}{(m - 1)^2 - 1}\right)^{\frac{m-2}{m-1}}
\]
\[
= (m - 1)^2(m-1) \left(\frac{1}{m^2 - 2m}\right)^{m-2+m-2} \left(\frac{m^2-2m}{m-1}\right)^{\frac{m^2-2m}{m-1}}.
\]
If the number of knots of the considered trees is \((m - 1)d + 1\), such a tree correspond to a localization quiver of dimension type \((d, (m - 1)d + 1)\) for all \(d \geq 1\). Since we consider the logarithm, we may discount the remaining factors of Theorem 5.13. Now Theorem 3.13 implies
\[
\lim_{d \to \infty} \frac{\ln \chi(M_{d,(m-1)d+1}^m)}{d} = \ln \left(\frac{(m - 1)^2(m-1)}{m^2 - 2m}\right)^{\frac{m^2-2m}{m-1}}\frac{(m-1)d+1}{d}
\]
\[
= (m - 1)^2 \ln(m - 1)^2 - (m^2 - 2m) \ln(m^2 - 2m).
\]
Because of the isomorphisms of moduli spaces, the assertion follows.
Remark 6.7

- By considering
  \[ f : \frac{e}{d} \mapsto \lim_{n \to \infty} \frac{\ln(\chi(M_m^{d_n+ne})))}{d}, \]
  we in particular get that
  \[ f(1) = (m - 1)^2 \ln(m - 1)^2 - (m^2 - 2m) \ln(m^2 - 2m) \]
as conjectured in [5.6]

6.5 Further examples and questions

In this section we discuss further examples and besides some problems and questions, which appear when dealing with localization in Kronecker moduli spaces. One fundamental question is how to determine all localization quivers and if it is perhaps enough to know all localization quivers of type 1.

6.5.1 Real roots

It is known that the real roots of the Kronecker quiver are reflections of the simple roots (0, 1) and (1, 0). In the case \( m = 3 \) we get the sequence

\[ (0, 1), (1, 3), (3, 8), (8, 21), (21, 55), \ldots. \]

The unique indecomposable representations for these dimension vectors are therefore given by the simple representation \( E_2 \) and its reflections under the reflection functor. In the next three cases we obtain:

\[ \text{Diagram of representations} \]

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Here the dots always represent one-dimensional subspaces. As shown in [FR], this gives rise to a decomposition of the odd Fibonacci numbers.

6.5.2 Open questions

As mentioned above, one could ask if it is possible to put all localization quivers down to the case of localization quivers of type one. For instance, when considering the localization quiver

we always assumed $j \neq k$. But if we consider the quiver

we could in a sense understand this quiver as the case $j = k$. But this raises another problem: we get additional conditions for $i$ and $j$ and moreover different symmetries. For instance, in the first case we have the symmetries of $S_2$. But in the second one we have the symmetries of $S_3$.

Another question is how to count or get all localization data (at least all of
type one). Unfortunately, by use of the method introduced in this chapter we do not get all localization quivers of type one. If it were possible to get all quivers of this type and if it could be shown that the other quivers come in a way from quivers of type one, one could probably prove the continuity. This would suffice to prove the existence of the conjectured function.

Finally, we give an example for a quiver of type one, which cannot be constructed by use of the methods of this chapter. Let \((d, e) = (7, 10) = (2, 3) + (5, 7) = (2, 3) + (2, 3) + (3, 4)\). Then we have \((d_s, e_s) = (2, 3)\). We have

\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}
\end{array}
\]

\[s_{2,3} =
\]

\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}
\end{array}
\]

\[\hat{s}_{2,3} =
\]

We get the quivers for the dimension vector \((3, 4)\) in the same way. But we do not get the following localization quiver with dimension vector \((9, 13) = (2, 3) + (7, 10)\) by sticking together the above ones:
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