The spectrum of simplicial volume of non-compact manifolds

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Abstract
We show that, in dimension at least 4, the set of locally finite simplicial volumes of oriented connected open manifolds is $[0, \infty]$. Moreover, we consider the case of tame open manifolds and some low-dimensional examples.

Keywords
Simplicial volume · Non-compact manifolds

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1 Introduction

Simplicial volumes are invariants of manifolds defined in terms of the $\ell^1$-semi-norm on singular homology [9].

Definition 1.1 (simplicial volume) Let $M$ be an oriented connected $d$-manifold without boundary. Then the simplicial volume of $M$ is defined by

$$\|M\|_{lf} := \inf \{ |c|_1 \mid c \in C_d^{lf}(M; \mathbb{R}) \text{is a fundamental cycle of } M \},$$

where $C_d^{lf}$ denotes the locally finite singular chain complex. If $M$ is compact, then we also write $\|M\| := \|M\|_{lf}$. Using relative fundamental cycles, the notion of simplicial volume can be extended to oriented manifolds with boundary.

Simplicial volumes are related to negative curvature, volume estimates, and amenability [9]. In the present article, we focus on simplicial volumes of non-compact manifolds. Only few concrete results are known in this context: There are computations for certain locally symmetric spaces [3,12,15,16] as well as the general volume estimates [9], vanishing results [8,9], and finiteness results [9,14].
Let \( d \in \mathbb{N} \), let \( M(d) \) be the class of all oriented closed connected \( d \)-manifolds, and let \( M^\text{lf}(d) \) be the class of all oriented connected manifolds without boundary. Then we set

\[
SV(d) := \{ \| M \| \mid M \in M(d) \}
\]

and

\[
SV^\text{lf}(d) := \{ \| M \|^\text{lf} \mid M \in M^\text{lf}(d) \}.
\]

It is known that \( SV(d) \) is countable and that this set has no gap at 0 if \( d \geq 4 \):

**Theorem 1.2** [10, Theorem A] Let \( d \in \mathbb{N}_{\geq 4} \). Then \( SV(d) \) is dense in \( \mathbb{R}_{\geq 0} \) and \( 0 \in SV(d) \).

In contrast, if we allow non-compact manifolds, we can realise all non-negative real numbers:

**Theorem A** Let \( d \in \mathbb{N}_{\geq 4} \). Then \( SV^\text{lf}(d) = [0, \infty] \).

The proof uses the no-gap theorem Theorem 1.2 and a suitable connected sum construction.

**Theorem B** Let \( d \in \mathbb{N} \). Then the set \( SV^\text{lf}_{\text{tame}}(d) \subset [0, \infty] \) is countable. In particular, the set \([0, \infty] \setminus SV^\text{lf}_{\text{tame}}(d)\) is uncountable.

As an explicit example, we compute \( SV^\text{lf}(2) \) and \( SV^\text{lf}_{\text{tame}}(2) \) (Proposition 4.2) as well as \( SV^\text{lf}_{\text{tame}}(3) \) (Proposition 4.3). The case of non-tame 3-manifolds seems to be fairly tricky.

**Question 1.3** What is \( SV^\text{lf}(3) \)?

As \( SV(4) \subset SV^\text{lf}_{\text{tame}}(4) \), we know that \( SV^\text{lf}_{\text{tame}}(4) \) contains arbitrarily small transcendental numbers [11].

From a geometric point of view, the so-called Lipschitz simplicial volume is more suitable for Riemannian non-compact manifolds than the locally finite simplicial volume. It is therefore natural to ask the following:

**Question 1.4** Do Theorem A and Theorem B also hold for the Lipschitz simplicial volume of oriented connected open Riemannian manifolds?

**Organisation of this article**

Section 2 contains the proof of Theorem A. The proof of Theorem B is given in Sect. 3. The low-dimensional case is treated in Sect. 4.

**2 Proof of Theorem A**

Let \( d \in \mathbb{N}_{\geq 4} \) and let \( \alpha \in [0, \infty] \). Because \( SV(d) \) is dense in \( \mathbb{R}_{\geq 0} \) (Theorem 1.2), there exists a sequence \((\alpha_n)_{n \in \mathbb{N}} \) in \( SV(d) \) with \( \sum_{n=0}^{\infty} \alpha_n = \alpha \).

**2.1 Construction**

We first describe the construction of a corresponding oriented connected open manifold \( M \): For each \( n \in \mathbb{N} \), we choose an oriented closed connected \( d \)-manifold \( M_n \) with \( \| M_n \| = \alpha_n \). Moreover, for \( n > 0 \), we set

\[
W_n := M_n \setminus (B_{n,-}^\circ \cup B_{n,+}^\circ).
\]
where $B_{n,-} = i_{n,-}(D^d)$ and $B_{n,+} = i_{n,+}(D^d)$ are two disjointly embedded closed $d$-balls in $M_n$. Similarly, we set $W_0 := M_0 \setminus B_{0,+}^o$. Furthermore, we choose an orientation-reversing homeomorphism $f_n : S^{d-1} \rightarrow S^{d-1}$. We then consider the infinite “linear” connected sum manifold (Fig. 1)

$$M := M_0 \# M_1 \# M_2 \# \ldots$$

where $\sim$ is the equivalence relation generated by

$$i_{n+1,-}(x) \sim i_{n,+}(f_n(x))$$

for all $n \in \mathbb{N}$ and all $x \in S^{d-1} \subset D^d$; we denote the induced inclusion $W_n \rightarrow M$ by $i_n$. By construction, $M$ is connected and inherits an orientation from the $M_n$.

### 2.2 Computation of the simplicial volume

We will now verify that $\|M\|_{lf} = \alpha$:

**Claim 2.1** We have $\|M\|_{lf} \leq \alpha$.

**Proof** The proof is a straightforward adaption of the chain-level proof of sub-additivity of simplicial volume with respect to amenable glueings.

In particular, we will use the uniform boundary condition [19] and the equivalence theorem [2,9]:

**UBC** The chain complex $C_\ast(S^{d-1}; \mathbb{R})$ satisfies $(d-1)$-UBC, i.e., there is a constant $K$ such that: For each $c \in \text{im} \partial_d \subset C_{d-1}(S^{d-1}; \mathbb{R})$, there exists a chain $b \in C_d(S^{d-1}; \mathbb{R})$ with

$$\partial_d b = c \quad \text{and} \quad |b|_1 \leq K \cdot |c|_1.$$

**EQT** Let $N$ be an oriented closed connected $d$-manifold, let $B_1, \ldots, B_k$ be disjointly embedded $d$-balls in $N$, and let $W := N \setminus (B_1^o \cup \ldots \cup B_k^o)$. Moreover, let $\epsilon \in \mathbb{R}_{>0}$. Then

$$\|N\| = \inf \left\{ |z|_1 \mid z \in Z(W; \mathbb{R}), \ |\partial_d z|_1 \leq \epsilon \right\},$$

where $Z(W; \mathbb{R}) \subset C_d(W; \mathbb{R})$ denotes the set of all relative fundamental cycles of $W$.

Let $\epsilon \in \mathbb{R}_{>0}$. By EQT, for each $n \in \mathbb{N}$, there exists a relative fundamental cycle $z_n \in Z(W_n; \mathbb{R})$ with

$$|z_n|_1 \leq \alpha_n + \frac{1}{2^n} \cdot \epsilon \quad \text{and} \quad |\partial_d z_n|_1 \leq \frac{1}{2^n} \cdot \epsilon.$$
We now use UBC to construct a locally finite fundamental cycle of $M$ out of these relative cycles: For $n \in \mathbb{N}$, the boundary parts $C_{d-1}(i_n; \mathbb{R}) (\partial_d z_n |_{B_{n,+}})$ and $-C_{d-1}(i_{n+1}; \mathbb{R}) (\partial_d z_{n+1} |_{B_{n+1,-}})$ are fundamental cycles of the sphere $S^{d-1}$ (embedded via $i_n \circ i_{n,+}$ and $i_{n+1} \circ i_{n+1,-}$ into $M$), which implicitly uses the orientation-reversing homeomorphism $f_n$). By UBC, there exists a chain $b_n \in C_d(S^{d-1}; \mathbb{R})$ with
\[
\partial_d C_d(i_n \circ i_{n,+}; \mathbb{R})(b_n) = C_{d-1}(i_n; \mathbb{R}) (\partial_d z_n |_{B_{n,+}}) + C_{d-1}(i_{n+1}; \mathbb{R}) (\partial_d z_{n+1} |_{B_{n+1,-}})
\]
and
\[
|b_n| \leq K \left( \frac{1}{2n} + \frac{1}{2n+1} \right) \cdot \epsilon \leq K \left( \frac{1}{2n-1} \cdot \epsilon \right).
\]
A straightforward computation shows that
\[
c := \sum_{n=0}^\infty C_d(i_n; \mathbb{R}) (z_n - C_d(i_{n,+}; \mathbb{R})(b_n))
\]
is a locally finite $d$-cycle on $M$. Moreover, the local contribution on $W_0$ shows that $c$ is a locally finite fundamental cycle of $M$. By construction,
\[
|c|_1 \leq \sum_{n=0}^\infty (|z_n|_1 + |b_n|_1) \\
\leq \sum_{n=0}^\infty \left( \alpha_n + \frac{1}{2n} \cdot \epsilon + K \cdot \frac{1}{2n-1} \cdot \epsilon \right) \leq \sum_{n=0}^\infty \alpha_n + (2 + 4 \cdot K) \cdot \epsilon \\
= \alpha + (2 + 4 \cdot K) \cdot \epsilon.
\]
Thus, taking $\epsilon \to 0$, we obtain $\|M\|_{\text{lf}} \leq \alpha$. 

**Claim 2.2** We have $\|M\|_{\text{lf}} \geq \alpha$.

**Proof** Without loss of generality we may assume that $\|M\|_{\text{lf}}$ is finite. Let $c \in C_d^\text{lf}(M; \mathbb{R})$ be a locally finite fundamental cycle of $M$ with $|c|_1 < \infty$. For $n \in \mathbb{N}$, we consider the subchain $c_n := c |_{W(n)}$ of $c$, consisting of all simplices whose images touch $W(n) := \bigcup_{k=0}^n i_k(W_k) \subset M$. Because $c$ is locally finite, each $c_n$ is a finite singular chain and $|c_n|_1 \neq \infty$ is a monotonically increasing sequence with limit $|c|_1$.

Let $\epsilon \in \mathbb{R}_{\geq 0}$. Then there is an $n \in \mathbb{N}_{>0}$ that satisfies $|c - c_n|_1 \leq \epsilon$ and $\alpha - \sum_{k=0}^n \alpha_k \leq \epsilon$. Let
\[
p : M \to W(n)/i_n(B_{n,+}) =: W
\]
be the map that collapses everything beyond stage $n + 1$ to a single point $x$. Then $z := C_d(p; \mathbb{R})(c_n) \in C_d(W, \{x\}; \mathbb{R})$ is a relative cycle and
\[
|\partial_d z|_1 \leq |\partial_d c_n|_1 \leq |\partial_d(c - c_n)|_1 \leq (d + 1) \cdot |c - c_n|_1 \leq (d + 1) \cdot \epsilon.
\]
Because $d > 1$, there exists a chain $b \in C_d(\{x\}; \mathbb{R})$ with
\[
\partial_d b = \partial_d z \quad \text{and} \quad |b|_1 \leq |\partial_d z| \leq (d + 1) \cdot \epsilon.
\]
Then
\[
\zeta := z - b \in C_d(W; \mathbb{R})
\]
is a cycle on $W$; because $z$ and $\bar{z}$ have the same local contribution on $W_0$, the cycle $z$ is a fundamental cycle of the manifold

$$W \cong M_0 \# \cdots \# M_n.$$  

As $d > 2$, the construction of our chains and additivity of simplicial volume under connected sums [2,9] show that

$$|c|_1 \geq |c_n|_1 \geq |z|_1 \geq |\bar{z}|_1 - |b|_1$$

$$\geq \|W\| - (d + 1) \cdot \epsilon = \sum_{k=0}^n \|M_k\| - (d + 1) \cdot \epsilon$$

$$\geq \alpha - (d + 2) \cdot \epsilon.$$  

Thus, taking $\epsilon \to 0$, we obtain $|c|_1 \geq \alpha$; hence, $\|M\|_{lf} \geq \alpha$.  

This completes the proof of Theorem A. 

**Remark 2.3** (adding geometric structures) In fact, this argument can also be performed smoothly: The constructions leading to Theorem 1.2 can be carried out in the smooth setting. Therefore, we can choose the $(M_n)_{n \in \mathbb{N}}$ to be smooth and equip $M$ with a corresponding smooth structure. Moreover, we can endow these smooth pieces with Riemannian metrics. Scaling these Riemannian metrics appropriately shows that we can turn $M$ into a Riemannian manifold of finite volume.

### 3 Proof of Theorem B

In this section, we prove Theorem B, i.e., that the set of simplicial volumes of tame manifolds is countable.

**Definition 3.1** A manifold $M$ without boundary is tame if there exists a compact connected manifold $W$ with boundary such that $M$ is homeomorphic to $W^\circ := W \setminus \partial W$.

As in the closed case, our proof is based on a counting argument:

**Proposition 3.2** There are only countably many proper homotopy types of tame manifolds.

As we could not find a proof of this statement in the literature, we will give a complete proof in Sect. 3.1 below. Theorem B is a direct consequence of Proposition 3.2:

**Proof of Theorem B** The simplicial volume $\| \cdot \|_{lf}$ is invariant under proper homotopy equivalence (this can be shown as in the compact case). Therefore, the countability of $SV^{lf}(d)$ follows from the countability of the set of proper homotopy types of tame $d$-manifolds (Proposition 3.2).

**Remark 3.3** Let $d \in \mathbb{N}_{\geq 3}$. Then $\infty \in SV^{lf}_{tame}(d)$: Let $N$ be an oriented closed connected hyperbolic $(d - 1)$-manifold and let $M := N \times \mathbb{R}$. Then $M$ is tame (as interior of $N \times [0, 1]$) and $\|N\| > 0$ [9, Section 0.3] [23, Theorem 6.2]. Hence, by the finiteness criterion [9, p. 17] [14, Theorem 6.4], we obtain that $\|M\|_{lf} = \infty$.  


3.1 Counting tame manifolds

It remains to prove Proposition 3.2. We use the following observations:

**Definition 3.4 (models of tame manifolds)**

- A *model* of a tame manifold $M$ is a finite CW-pair $(X, A)$ (i.e., a finite CW-complex $X$ with a finite subcomplex $A$) that is homotopy equivalent (as pairs of spaces) to $(W, \partial W)$, where $W$ is a compact connected manifold with boundary whose interior is homeomorphic to $M$.
- Two models of tame manifolds are *equivalent* if they are homotopy equivalent as pairs of spaces.

**Lemma 3.5 (existence of models)** Let $W$ be a compact connected manifold. Then there exists a finite CW-pair $(X, A)$ such that $(W, \partial W)$ and $(X, A)$ are homotopy equivalent pairs of spaces.

*In particular:* Every tame manifold admits a model.

**Proof** It should be noted that we work with topological manifolds; hence, we cannot argue directly via triangulations. Of course, the main ingredient is the fact that every compact manifold is homotopy equivalent to a finite complex [13,22].

Hence, there exist finite CW-complexes $A$ and $Y$ with homotopy equivalences $f : A \to \partial W$ and $g : Y \to W$. Let $j := \overline{g} \circ i \circ f$, where $i : \partial W \hookrightarrow W$ is the inclusion and $\overline{g}$ is a homotopy inverse of $g$. By construction, the upper square in the diagram in Fig. 2 is homotopy commutative.

As next step, we replace $j : A \to Y$ by a homotopic map $jc : A \to Y$ that is cellular (second square in Fig. 2).

The mapping cylinder $Z$ of $jc$ has a finite CW-structure (as $jc$ is cellular) and the canonical map $p : Z \to Y$ allows to factor $jc$ into an inclusion $J$ of a subcomplex and the homotopy equivalence $p$ (third square in Fig. 2).

We thus obtain a homotopy commutative square

$$
\begin{array}{ccc}
\partial W & \xrightarrow{i} & W \\
\downarrow f & & \downarrow j \\
A & \xrightarrow{j} & Z
\end{array}
$$

where the vertical arrows are homotopy equivalences, the upper horizontal arrow is the inclusion, and the lower horizontal arrow is the inclusion of a subcomplex.

Using a homotopy between $i \circ f$ and $F \circ J$ and adding another cylinder to $Z$, we can replace $Z$ by a finite CW-complex $X$ (that still contains $A$ as subcomplex) to obtain a strictly commutative diagram

$$
\begin{array}{ccc}
\partial W & \xrightarrow{i} & W \\
\downarrow f \cong & & \downarrow \cong \\
A & \xrightarrow{\cong} & X
\end{array}
$$

whose vertical arrows are homotopy equivalences and whose horizontal arrows are inclusions.

Because the inclusions $\partial W \hookrightarrow W$ (as inclusion of the boundary of a compact topological manifold) and $A \hookrightarrow X$ (as inclusion of a subcomplex) are cofibrations, this already implies that the vertical arrows form a homotopy equivalence $(X, A) \to (W, \partial W)$ of pairs [18, Chapter 6.5].
Lemma 3.6 (equivalence of models) If $M$ and $N$ are tame manifolds with equivalent models, then $M$ and $N$ are properly homotopy equivalent.

Proof As $M$ and $N$ admit equivalent models, there exist compact connected manifolds $W$ and $V$ with boundary such that $M \cong W^\circ$ and $N \cong V^\circ$ and such that the pairs $(W, \partial W)$ and $(V, \partial V)$ are homotopy equivalent (by transitivity of homotopy equivalence of pairs of spaces). Let $(f, f_\partial) : (W, \partial W) \rightarrow (V, \partial V)$ and $(g, g_\partial) : (V, \partial V) \rightarrow (W, \partial W)$ be mutually homotopy inverse homotopy equivalences of pairs.

By the topological collar theorem [5,6], we have homeomorphisms

$$M \cong W \cup_{\partial W} (\partial W \times [0, \infty)),$$

$$N \cong V \cup_{\partial V} (\partial V \times [0, \infty)),$$

where the glueing occurs via the canonical inclusions $\partial W \hookrightarrow \partial W \times [0, \infty)$ and $\partial V \hookrightarrow \partial V \times [0, \infty)$ at parameter 0.

Then the maps $f$ and $f_\partial \times \text{id}_{[0, \infty)}$ glue to a well-defined proper continuous map $F : M \rightarrow N$ and the maps $g$ and $g_\partial \times \text{id}_{[0, \infty)}$ glue to a well-defined proper continuous map $G : N \rightarrow M$.

Moreover, the homotopy of pairs between $(f \circ g, f_\partial \circ g_\partial)$ and $(\text{id}_V, \text{id}_{\partial V})$ glues into a proper homotopy between $F \circ G$ and $\text{id}_M$. In the same way, there is a proper homotopy between $G \circ F$ and $\text{id}_N$. Hence, the spaces $M$ and $N$ are properly homotopy equivalent. □

Lemma 3.7 (countability of models) There exist only countably many equivalence classes of models.

Proof There are only countably many homotopy types of finite CW-complexes (because every finite CW-complex is homotopy equivalent to a finite simplicial complex). Moreover, every finite CW-complex has only finitely many subcomplexes. Therefore, there are only countably many homotopy types (of pairs of spaces) of finite CW-pairs. □

Proof of Proposition 3.2 We only need to combine Lemma 3.5, Lemma 3.6, and Lemma 3.7. □

4 Low dimensions

4.1 Dimension 2

We now compute the set of simplicial volumes of surfaces. We first consider the tame case:
Example 4.1 (tame surfaces) Let $W$ be an oriented compact connected surface with $g \in \mathbb{N}$ handles and $b \in \mathbb{N}$ boundary components. Then the proportionality principle for simplicial volume of hyperbolic manifolds [9, p. 11] (a thorough exposition is given, for instance, by Fujiwara and Manning [7, Appendix A]) gives

$$\|W\|^\text{lf} = \begin{cases} 
4 \cdot (g - 1) + 2 \cdot b & \text{if } g > 0 \\
2 \cdot b - 4 & \text{if } g = 0 \text{ and } b > 1 \\
0 & \text{if } g = 0 \text{ and } b \in \{0, 1\}.
\end{cases}$$

Proposition 4.2 We have $\text{SV}^\text{lf}(2) = 2 \cdot \mathbb{N} \cup \{\infty\}$ and $\text{SV}^\text{lf}_{\text{tame}}(2) = 2 \cdot \mathbb{N}$.

Proof We first prove $2 \cdot \mathbb{N} \subset \text{SV}^\text{lf}_{\text{tame}}(2) \subset \text{SV}^\text{lf}(2)$ and $\infty \in \text{SV}^\text{lf}(2)$, i.e., that all the given values may be realised: In view of Example 4.1, all even numbers occur as simplicial volume of some (possibly open) tame surface.

Let

$$M := T^2 \# T^2 \# T^2 \# \ldots$$

be an infinite “linear” connected sum of tori $T^2$. Collapsing $M$ to the first $g \in \mathbb{N}$ summands and an argument as in the proof of Claim 2.2 shows that

$$\|M\|^\text{lf} \geq \|\Sigma_g\| = 4 \cdot g - 4$$

for all $g \in \mathbb{N}_{\geq 1}$. Hence, $\|M\|^\text{lf} = \infty$.

It remains to show that $\text{SV}^\text{lf}(2) \subset 2 \cdot \mathbb{N} \cup \{\infty\}$: Let $M$ be an oriented connected (topological, separable, Hausdorff) 2-manifold without boundary. Then $M$ admits a smooth structure [20] and whence a proper smooth map $p : M \to \mathbb{R}$. Using suitable regular values of $p$, we can thus write $M$ as an ascending union

$$M = \bigcup_{n \in \mathbb{N}} M_n$$

of oriented connected compact submanifolds (possibly with boundary) $M_n$ that are nested via $M_0 \subset M_1 \subset \ldots$ Then one of the following cases occurs:

1. There exists an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{\geq N}$ the inclusion $M_n \hookrightarrow M_{n+1}$ is a homotopy equivalence.
2. For each $N \in \mathbb{N}$ there exists an $n \in \mathbb{N}_{\geq N}$ such that the inclusion $M_n \hookrightarrow M_{n+1}$ is not a homotopy equivalence.

In the first case, the classification of compact surfaces with boundary shows that $M$ is tame. Hence $\|M\|^\text{lf} \in 2 \cdot \mathbb{N}$ (Example 4.1).

In the second case, the manifold $M$ is not tame (which can, e.g., be derived from the classification of compact surfaces with boundary). We show that $\|M\|^\text{lf} = \infty$. To this end, we distinguish two cases:

a. The sequence $(h(M_n))_{n \in \mathbb{N}}$ is unbounded, where $h(\cdot)$ denotes the number of handles of the surface.

b. The sequence $(h(M_n))_{n \in \mathbb{N}}$ is bounded.

In the unbounded case, a collapsing argument (similar to the argument for $T^2 \# T^2 \# \ldots$ and Claim 2.2) shows that $\|M\|^\text{lf} = \infty$.

We claim that also in the bounded case we have $\|M\|^\text{lf} = \infty$: Shifting the sequence in such a way that all handles are collected in $M_0$, we may assume without loss of generality that
the sequence \((h(M_n))_{n \in \mathbb{N}}\) is constant. Thus, for each \(n \in \mathbb{N}\), the surface \(M_{n+1}\) is obtained from \(M_n\) by adding a finite disjoint union of disks and of spheres with finitely many (at least two) disks removed; we can reorganise this sequence in such a way that no disks are added. Hence, we may assume that \(M_n\) is a retract of \(M_{n+1}\) for each \(n \in \mathbb{N}\). Furthermore, because we are in case 2, the classification of compact surfaces shows (with the help of Example 4.1) that

\[
\lim_{n \to \infty} \|M_n\| = \infty.
\]

Let \(c \in C^\text{lf}_2(M; \mathbb{R})\) be a locally finite fundamental cycle of \(M\) and let \(n \in \mathbb{N}\). Because \(c\) is locally finite, there is a \(k \in \mathbb{N}\) such that \(c|_{M_n}\) is supported on \(M_{n+k}\); the restriction \(c|_{M_n}\) consists of all summands of \(c\) whose supports intersect with \(M_n\). Because \(M_n\) is a retract of \(M_{n+k}\), we obtain from \(c|_{M_n}\) a relative fundamental cycle \(c_n\) of \(M_n\) by pushing the chain \(c|_{M_n}\) to \(M_n\) via a retraction \(M_{n+k} \to M_n\). Therefore,

\[
|c|_1 \geq |c|_{M_n}|_1 \geq |c_n|_1 \geq \|M_n\|.
\]

Taking \(n \to \infty\) shows that \(|c|_1 = \infty\). Taking the infimum over all locally finite fundamental cycles \(c\) of \(M\) proves that \(\|M\|_{\text{lf}} = \infty\).

Moreover, Example 4.1 shows that \(\infty \notin SV_{\text{tame}}(2)\). \(\square\)

### 4.2 Dimension 3

The general case of non-compact 3-manifolds seems to be rather involved (as the structure of non-compact 3-manifolds can get fairly complicated). We can at least deal with the tame case:

**Proposition 4.3** We have \(SV_{\text{tame}}^\text{lf}(3) = SV(3) \cup \{\infty\}\).

**Proof** Clearly, \(SV(3) \subset SV_{\text{tame}}^\text{lf}(3)\) and \(\infty \in SV_{\text{tame}}^\text{lf}(3)\) (Remark 3.3).

Conversely, let \(W\) be an oriented compact connected 3-manifold and let \(M := W^\circ\). We distinguish the following cases:

- If at least one of the boundary components of \(W\) has genus at least 2, then the finiteness criterion [9, p. 17] [14, Theorem 6.4] shows that \(\|M\|_{\text{lf}} = \infty\).
- If the boundary of \(W\) consists only of spheres and tori, then we proceed as follows: In a first step, we fill in all spherical boundary components of \(W\) by 3-balls and thus obtain an oriented compact connected 3-manifold \(V\) all of whose boundary components are tori. In view of considerations on tame manifolds with amenable boundary [12] and glueing results for bounded cohomology [9] [2], we obtain that

\[
\|M\|_{\text{lf}} = \|W\| = \|V\|.
\]

By Kneser’s prime decomposition theorem [1, Theorem 1.2.1] and the additivity of (relative) simplicial volume with respect to connected sums [2,9] in dimension 3, we may assume that \(V\) is prime (i.e., admits no non-trivial decomposition as a connected sum). Moreover, because \(\|S^1 \times S^2\| = 0\), we may even assume that \(V\) is irreducible [1, p. 3].

By geometrisation [1, Theorem 1.7.6], then \(V\) admits a decomposition along finitely many incompressible tori into Seifert fibred manifolds (which have trivial simplicial volume [23, Corollary 6.5.3]) and hyperbolic pieces \(V_1, \ldots, V_k\). As the tori are incompressible,
we can now again apply additivity \([2,9]\) to conclude that

\[
\|V\| = \sum_{j=1}^{k} \|V_j\|.
\]

Let \(j \in \{1, \ldots, k\}\). Then the boundary components of \(V_j\) are \(\pi_1\)-injective tori (as the interior of \(V_j\) admits a complete hyperbolic metric of finite volume) \([4, \text{Proposition D.3.18}]\). Let \(S\) be a Seifert 3-manifold whose boundary is a \(\pi_1\)-injective torus (e.g., the knot complement of a non-trivial torus knot \([21, \text{Theorem 2}]\) \([17, \text{Lemma 4.4}]\)). Filling each boundary component of \(V_j\) with a copy of \(S\) results in an oriented closed connected 3-manifold \(N_j\), which satisfies (again, by additivity)

\[
\|N_j\| = \|V_j\| + 0 = \|V_j\|.
\]

Therefore, the oriented closed connected 3-manifold \(N := N_1 \# \cdots \# N_k\) satisfies

\[
\|N\| = \sum_{j=1}^{k} \|N_j\| = \sum_{j=1}^{k} \|V_j\| = \|V\|.
\]

In particular, \(\|M\|_{lf} = \|V\| = \|N\| \in SV(3)\).

\(\square\)

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