Integrable systems associated with generalized Sklyanin algebra

Yu.Chernyakov, 1

Institute for Theoretical and Experimental Physics, Moscow.

Abstract

Using the point fusion procedure we obtain the new integrable systems from the Elliptic Schlesinger system (ESS). These new systems have the pole orders higher than one in the matrix of the Lax operator. Quadratic Poisson algebras on the phase space of the new systems generalize the Sklyanin algebras and have the graduated structure.

February 12, 2022

1 Introduction

The examples of the systems with Lax operators having the pole orders higher than one were considered in papers [1] and [2]. These systems were obtained by point fusion procedure called in algebra Inonu-Wigner (3) contraction. The main idea of this method consists of finding such decomposition of the variables, which gives us the pole order in the matrix of Lax operator being higher than one at some marked point. It implies the existence of Hamiltonian and Casimir numbers in this new system being the same as in the initial one. The main object of this procedure in the present paper is the Lax operator of the Elliptic Schlesinger system (ESS) which was considered in [4].

Historically ESS appeared as an approach to the decision of Riman’s problem about searching the differential equations with regular singularities and intended monodromy data: \( M_j, j = 1, \ldots, n \) (\( \Psi \to \Psi M_j \)), where \( \Psi \) is the solution of initial linear system. In [5] L. Schlesinger had considered the first order system of differential equations for \( n > 3 \) matrices \( S^j \) (\( j = 1, \ldots, n \)), depending on \( n \) points \( x_k \in \mathbb{C} \).

\[
\begin{align*}
\left\{ \begin{array}{l}
\partial_z + \sum_{j=1}^n \frac{S^j}{z-x_j} \Psi = 0 \\
\partial_k \Psi = 0
\end{array} \right.
\end{align*}
\]

and found the preserving conditions on matrix \( S^j \) with respect to the changing the point positions \( x_j \). He obtained the system of first order differential equations for \( n > 3 \) matrices \( S^j \) (\( j = 1, \ldots, n \)), depending on \( n \) points \( x_k \in \mathbb{C} \).

\[
\partial_k S^j = \frac{[S^k, S^j]}{x_k - x_j}, \ (k \neq j), \quad \partial_k = \partial_{x_k},
\]

\[
\partial_k S^k = -\sum_{j \neq k} \frac{[S^k, S^j]}{x_k - x_j}.
\]

This system is non-autonomous Hamiltonian system and has the Hamiltonian form with respect to the linear (Lie-Poisson) brackets on \( \mathfrak{sl}(N, \mathbb{C}) \). The Hamiltonian

\[
H_k = \sum_{j \neq k} \frac{\langle S^k S^j \rangle}{x_k - x_j} \quad (\langle \rangle = \text{tr})
\]

1 e-mail: chernyakov@itep.ru
defines the evolution with respect to the time $x_k$.

For two by two matrices and four marked points the Schlesinger system is equivalent to the Painlevé VI equation \((6), (4)\). In this case the position of three points can be fixed as \((0,1,\infty)\) while $x_4$ play the role of an independent variable.

If we replace $\mathbb{C}P^1$ by an elliptic curve, we define a similar system (the elliptic Schlesinger system (ESS)). In this case, in addition to the coordinates of the marked points, a new independent variable appears inevitably. It is the modular parameter of the curve, and thereby we have an additional new Hamiltonian. This system was introduced originally by Takasaki \([7]\).

His derivation is based on the quasi-classical limit of the quantum $SU(N)$ version of the XYZ model. In \([4]\) was obtained as symplectic quotient of the symplectic space of connections of principle bundles of degree one over the elliptic curves with $n$ marked points. This approach was previously developed in \([8]\).

Let us give a short description of ESS.

**Elliptic Schlesinger System.** Let us consider elliptic curve $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$ with the modular parameter $\tau$, $\Im(\tau) > 0$ and

$$D_n = (x_1, \ldots, x_n), \quad x_j \neq x_k, \quad x_k \in \Sigma_\tau$$

be the divisor of non-coincident points with the condition

$$\sum x_j \in (\mathbb{Z} + \tau \mathbb{Z}). \quad (1.3)$$

Consider the space $\mathcal{P}_{n,N}$ of $n$ copies of the Lie coalgebra $\mathfrak{g}^* \sim \text{sl}(N, \mathbb{C})^*$, related to the points of the divisor.

$$\mathcal{P}_{n,N} = \bigoplus_{j=1}^n \mathfrak{g}_j^*, \quad \mathfrak{g}_j^* = \{S^\alpha = \sum_{\alpha \in \mathbb{Z}_N^2} S^\alpha_\gamma T^\alpha\}, \quad (1.4)$$

where $T^\alpha$ is the basis element of $GL(N, \mathbb{C})$ (see Appendix B).

Introduce operators acting from $\mathcal{P}_{n,N}$ to the dual $\bigoplus_{j=1}^n \mathfrak{g}_j$.

$$\mathbf{I}_{kj} : \mathfrak{g}_k^* \rightarrow \mathfrak{g}_j^*, \quad S^\kappa_\gamma \mapsto (I_{kj})_\gamma S^\kappa_\gamma, \quad (I_{kj})_\gamma = \varphi_\gamma(x_j - x_k), \quad (1.5)$$

$$\mathbf{J}_{jj} : \mathfrak{g}_j^* \rightarrow \mathfrak{g}_j^*, \quad S^\kappa_\gamma \mapsto J_\gamma S^\kappa_\gamma, \quad J_\gamma = E_2(\gamma), \quad (1.6)$$

$$\mathbf{J}_{kj} : \mathfrak{g}_k^* \rightarrow \mathfrak{g}_j^*, \quad S^\kappa_\gamma \mapsto (J_{kj})_\gamma S^\kappa_\gamma, \quad (J_{kj})_\gamma = f_\gamma(x_j - x_k) \quad (1.7)$$

where $\varphi_\gamma(x)$, $E_2(\gamma)$, $f_\gamma(x)$ are defined in Appendix B.

The positions of the marked points $x_j \in D_n$ and the modular parameter $\tau$ are local coordinates in an open cell in the moduli space $\mathcal{M}_{1,n}$ of elliptic curves with $n$ marked points. They play the role of times.

**Definition 1.1.** The elliptic Schlesinger system (ESS) is the consistent dynamical system on $\mathcal{P}_{n,N}$ with independent variables from $\mathcal{M}_{1,n}$

$$\partial_j S^k = [\mathbf{I}_{kj}(S^j), S^k], \quad (k \neq j), \quad \partial_k = \partial_{x_k}, \quad (1.8)$$

$$\partial_k S^k = - \sum_{j \neq k} [\mathbf{I}_{jk}(S^j), S^k], \quad (1.9)$$

$$\partial_\tau S^j = \sum_{k \neq j} \frac{1}{2\pi i} [S^j, \mathbf{J}_{kj}(S^k)] + \frac{1}{4\pi i} [S^j, \mathbf{J}_{jj}(S^j)], \quad (1.10)$$

where the commutators are understand as the coadjoint action of $\mathfrak{g}_j$ on $\mathfrak{g}_j^*$. 

2
In the basis $t^\alpha \ (\alpha \in \hat{\mathbb{Z}}_N^{(2)})$ \((B.7)\) the ESS takes the form
\[
\partial_k S_j^\alpha = \sum_{\gamma \in \hat{\mathbb{Z}}_N^{(2)}} C(\gamma, \alpha) S_j^\alpha S_j^\gamma \varphi_\gamma(x_j - x_k), \quad (k \neq j),
\] (1.11)
\[
\partial_k S_k^\alpha = \sum_{\gamma \in \hat{\mathbb{Z}}_N^{(2)}} C(\gamma, \alpha) \sum_{j \neq k} S_j^\alpha S_k^\gamma \varphi_\alpha(x_k - x_j),
\] (1.12)
\[
\partial_\tau S_k^\alpha = \frac{1}{2\pi t} \sum_{\gamma \in \hat{\mathbb{Z}}_N^{(2)}} C(\alpha, \gamma) \left( \sum_{k \neq j} S_k^\alpha S_j^\gamma f_\gamma(x_k - x_j) + S_k^\alpha S_k^\gamma E_2(\gamma) \right).
\] (1.13)

**Remark 1.1.** In the rational limit ($3m\tau \to \infty$) \((1.11)\) and \((1.12)\) pass to the standard Schlesinger system \((1.1), (1.2)\) (see \((A.13)\)).

As in the rational case the ESS has some fundamental properties. The space $\mathcal{P}_{n,N}^1$ is Poisson with respect to the linear Lie-Poisson brackets on $\mathfrak{g}^*$
\[
\{S_j^\alpha, S_j^\beta\}_1 = \delta^{jk} C(\alpha, \beta) S_j^\alpha S_j^\beta
\] (1.14)

ESS is a non-autonomous Hamiltonian system on $\mathcal{P}_{n,N}$
\[
\partial_k S_j^\alpha = \{H_k, S_j^\alpha\}_1, \quad \partial_\tau S_j^\alpha = \{H_0, S_j^\alpha\}_1,
\] (1.15)
(1.16)
where
\[
H_k = -\sum_{j \neq k} (I_{kj}(S_k^j)S_j^k) = -\sum_{j \neq k} S_k^j S_j^k \varphi_\gamma(x_j - x_k),
\] (1.17)
\[
H_\tau = H_0 = -\frac{1}{2\pi t} \left( \sum_{k \neq j} \langle S_j^k J_{kj}(S_k^j) \rangle + \sum_j \langle S_j^j J_{jj}(S_j^j) \rangle \right)
= -\frac{1}{2\pi t} \left( \sum_{k \neq j} \sum_{\gamma \in \hat{\mathbb{Z}}_N^{(2)}} S_j^k S_j^\gamma f_{\gamma}(x_k - x_j) + \sum_j \sum_{\gamma \in \hat{\mathbb{Z}}_N^{(2)}} S_j^j S_j^\gamma E_2(\gamma) \right).
\] (1.18)

The brackets \((1.14)\) are degenerate. The symplectic leaves are $n$ copies of coadjoint orbits $\mathcal{O}_j \ (j = 1, \ldots, n)$ of $\text{SL}(N, \mathbb{C})$. Assume that all orbits are generic, and let $c^{\mu}(j)$ be corresponding Casimir functions of order $\mu \ (\mu = 2, \ldots, N)$. The phase space of ESS is
\[
\mathcal{R}_{n,N} \sim \mathcal{P}_{n,N}/\{c^{\mu}(j) = c^{\mu}(j)\}_0 \sim \prod \mathcal{O}_j,
\] (1.19)
\[
\dim \mathcal{R}_{n,N} = nN(N - 1).
\] (1.20)

The ESS can be considered as a system of interacting non-autonomous $\text{SL}(N, \mathbb{C})$ Euler-Arnold tops, where operators \((1.5), (1.6), (1.7)\) play the role of the inverse inertia tensors.
2 Classical integrable systems obtained from ESS

ESS in the case of two marked point and \(N = 2\). The Lax operator of ESS in the case of \(n\) marked points has the following form:

\[
L(z) = -\frac{1}{N} E_1(z) T_0 + \sum_{j=1}^{n} \sum_{\alpha \in \mathbb{Z}^{\otimes 2}_N} S^j_\alpha \varphi_\alpha(z - x_j) T_\gamma. \tag{2.21}
\]

where \(T_0\) and \(T_\alpha\) are basis elements of \(GL(N, \mathbb{C})\).

Let us consider the modified Lax operator \((4)\)

\[
L_{\text{group}}(z) = S_0 T_0 + \sum_j \left( S^j_0 E_1(z - x_j) T_0 + \sum_\alpha S^j_\alpha \varphi_\alpha(z - x_j) \right) T_\alpha, \tag{2.22}
\]

where we attribute to the marked points of the divisor \(D_n\) \(n\) copies of the \(GL(N, \mathbb{C})\)-valued elements

\[
x_j \rightarrow S^j_0 T_0 + S^j = \sum_{\alpha \in \mathbb{Z}^{\otimes 2}_N} S^j_\alpha T_\alpha,
\]

adding to this set a variable \(S_0 \in \mathbb{C}\). So, it defines

\[
P^+_{n,N} = \left\{ S_0, (S^j_0, S^j, j = 1, \ldots, n) \mid \sum_{j=1}^{n} S^j_0 = 0 \right\}.
\]

It is possible to obtain the equation of motion for ESS from the quadratic brackets on the space \(P^+_{n,N}\), extracting them from the classical exchange algebra

\[
\{ L_{\text{group}}(z), L_{\text{group}}(w) \} = \left[ r(z - w), L_{\text{group}}(z) \otimes L_{\text{group}}(w) \right]. \tag{2.23}
\]

where \(r\) is the classical Belavin-Drinfeld r-matrix.

Let us consider the first occurrence: \(N = 2\) and \(n = 2\) is a number of the marked points. Then the Lax operator takes the following form:

\[
L_{\text{group}}(z) = \left( S_0 + S^a_0 E_1(z - x_a) + S^b_0 E_1(z - x_b) \right) \sigma_0 + \sum_\alpha \left( S^a_\alpha \varphi_\alpha(z - x_a) + S^b_\alpha \varphi_\alpha(z - x_b) \right) \sigma_\alpha, \tag{2.24}
\]

\[
S^a_0 + S^b_0 = 0.
\]

Note, that for \(N = 2\) the basis \(T_\alpha\) is proportional to the basis of the Pauli matrices. The dimension of the phase space \(R_{2,2}\) is four. The Hamiltonians of this system are \(S_0\) and \(S^a_0\). The quadratic brackets:

\[
\partial_z S^a_\alpha = \frac{1}{2} \{ S^a_\alpha, S_0 \} = \tag{2.25}
\]

\[
= i \varepsilon_{\alpha\beta\gamma} (E_2(\gamma) - E_2(\beta)) S^b_\beta S^a_\gamma - i \varepsilon_{\alpha\beta\gamma} \varphi'_\gamma(x_{ab}) S^b_\beta S^a_\gamma + i \varepsilon_{\alpha\beta\gamma} \varphi'_\beta(x_{ab}) S^b_\beta S^a_\gamma,
\]

\[
\partial_{x_a} S^a_\alpha = - \partial_{x_b} S^a_\alpha = \frac{1}{2} \{ S^a_\alpha, S^b_0 \} = \tag{2.26}
\]

\[
= i \varepsilon_{\alpha\beta\gamma} \varphi_\beta(x_{ab}) S^b_\beta S^a_\gamma - i \varepsilon_{\alpha\beta\gamma} \varphi_\gamma(x_{ab}) S^b_\beta S^a_\gamma.
\]
System obtained via two point fusion. Let us fulfil the following coordinate transformation and decomposition of the variables \( S \): 

\[
x_b = x_a + \varepsilon, \quad S_0^0 = c_{a,0} S_0^0 + c_{a,1} S_1^0 \varepsilon^{-1}, \quad S_0^a = c_{a,a} S_0^0 + c_{a,1} S_0^1 \varepsilon^{-1},
\]

where \( c \) are some coefficients. Taking the limit \( \varepsilon \to 0 \) and putting some additional conditions on coefficients (the requirements of the absence of singularities) we get the new Lax operator \( L_{\text{fusion}}(z) \)

\[
L_{\text{fusion}}(z) = (S_0 + S_1^0 E_2(z)) \sigma_0 + \sum_{\alpha} \left( S_0^\alpha \phi_\alpha(z) + S_1^\alpha \phi'_\alpha(z) \right) \sigma_\alpha,
\]

where we put the position \( x_a \) equal to 0. The dimension of the phase space

\[
P_{f,2} = (S_0, S_1, S_0^0, S_1^a)
\]

is 8 and it is equal to the dimension of \( P^+_{n,N} \) before point fusion.

To find the exchange relations between the new variables \( S \) we consider the equation (2.23) and the new Lax operator (2.28). These exchange relations are the coefficients at the function products \( 1, E_2(z), \phi_\alpha(z), \phi'_\alpha(z) \) and \( 1, E_2(w), \phi_\alpha(w), \phi'_\alpha(w) \). So we get

**Proposition 2.1.** The space \( P_{f,2} \) is Poisson with respect to the quadratic brackets

\[
\{S_0, S_1^0\} = 0,
\]

\[
\begin{align*}
\{S_0^0, S_\beta^0\} &= 2i\varepsilon_{\alpha,\beta,\gamma} S_\beta^0 S_\gamma^0 - 2i\varepsilon_{\alpha,\beta,\gamma} E_2(\gamma) S_0^0 S_\gamma^0, \\
\{S_1^0, S_\beta^0\} &= 2i\varepsilon_{\alpha,\beta,\gamma} S_0^1 S_\gamma^0 - 2i\varepsilon_{\alpha,\beta,\gamma} E_2(\alpha) S_0^0 S_\gamma^1,
\end{align*}
\]

\[
\begin{align*}
\{S_0^1, S_\beta^1\} &= 2i\varepsilon_{\alpha,\beta,\gamma} S_0^1 S_\gamma^0, \\
\{S_1^1, S_\beta^1\} &= 2i\varepsilon_{\alpha,\beta,\gamma} (E_2(\gamma) - E_2(\beta)) S_0^0 S_\gamma^1 + 2i\varepsilon_{\alpha,\beta,\gamma} E_2(\alpha) (E_2(\gamma) - E_2(\beta)) S_1^0 S_\gamma^1, \\
\{S_0^1, S_\beta^0\} &= 2i\varepsilon_{\alpha,\beta,\gamma} E_2(\beta) S_0^0 S_\gamma^1 - 2i\varepsilon_{\alpha,\beta,\gamma} E_2(\gamma) S_1^0 S_\gamma^0, \\
\{S_1^1, S_\beta^0\} &= 2i\varepsilon_{\alpha,\beta,\gamma} E_2(\gamma) - E_2(\beta) S_1^1 S_\gamma^1, \\
\{S_0^1, S_\beta^1\} &= -2i\varepsilon_{\alpha,\beta,\gamma} S_\beta^1 S_\gamma^1 + 2i\varepsilon_{\alpha,\beta,\gamma} S_0^1 S_\gamma^0.
\end{align*}
\]

**Proof:**

Let us consider the classical Poisson brackets

\[
\{L(z), L(w)\} = [r(z - w), L(z) \otimes L(w)],
\]

where the Lax operator has the following form:

\[
L(z) = L_{\text{fusion}}(z) = (S_0 + S_1^0 E_2(z)) \sigma_0 + \sum_{\alpha} \left( S_0^\alpha \phi_\alpha(z) + S_1^\alpha \phi'_\alpha(z) \right) \sigma_\alpha,
\]

In the l.h.s. for the matrix element \( \sigma_\alpha \otimes \sigma_\beta \) in (A.24) we have the sum consisted of the following terms:

\[
\{S_0^0, S_\beta^0\} \phi_\alpha(z) \phi'_\beta(w), \quad \{S_0^1, S_\beta^1\} \phi_\alpha(z) \phi'_\beta(w).
\]
As a result of the expansion in function product for the matrix element \( \sigma \) and for the matrix element \( \sigma' \): 

\[
\{ S^1_\alpha, S^0_\beta \} \varphi'_\alpha(z) \varphi_\beta(w), \quad \{ S^1_\alpha, S^1_\beta \} \varphi'_\alpha(z) \varphi'_\beta(w),
\]

and for the matrix element \( \sigma \otimes I \): 

\[
\{ S^0_\alpha, S_0 \} \varphi_\alpha(z), \quad \{ S^0_\alpha, S^1_\beta \} \varphi_\alpha(z) E_2(w), \quad \{ S^1_\alpha, S_0 \} \varphi'_\alpha(z), \quad \{ S^1_\alpha, S^0_\beta \} \varphi'_\alpha(z) E_2(w).
\]

In the r.h.s. we have for the same matrix elements the sums consisted of the following terms: 

\[
2i\varepsilon_{\alpha\beta\gamma} S^0_\gamma S_0 (\varphi_\alpha(z - w) \varphi_\gamma(w) - \varphi_\beta(z - w) \varphi_\gamma(z)),
\]

\[
2i\varepsilon_{\alpha\beta\gamma} S^0_\gamma S^1_0 (\varphi_\alpha(z - w) \varphi_\gamma(w) E_2(z) - \varphi_\beta(z - w) \varphi_\gamma(z) E_2(w)),
\]

\[
2i\varepsilon_{\alpha\beta\gamma} S^1_\gamma S_0 \left( \varphi_\alpha(z - w) \varphi'_\gamma(w) - \varphi_\beta(z - w) \varphi'_\gamma(z) \right),
\]

\[
2i\varepsilon_{\alpha\beta\gamma} S^1_\gamma S^1_0 \left( \varphi_\alpha(z - w) \varphi'_\gamma(w) E_2(z) - \varphi_\beta(z - w) \varphi'_\gamma(z) E_2(w) \right)
\]

and 

\[
2i\varepsilon_{\alpha\beta\gamma} S^0_\gamma S^0_\beta (\varphi_\gamma(z - w) \varphi_\beta(z) \varphi_\gamma(w) - \varphi_\beta(z - w) \varphi_\gamma(z) \varphi_\beta(w)),
\]

\[
2i\varepsilon_{\alpha\beta\gamma} S^1_\gamma S^0_\beta \left( \varphi_\gamma(z - w) \varphi_\beta(z) \varphi'_\gamma(w) - \varphi_\beta(z - w) \varphi'_\gamma(z) \varphi_\beta(w) \right),
\]

\[
2i\varepsilon_{\alpha\beta\gamma} S^0_\gamma S^1_\beta \left( \varphi_\gamma(z - w) \varphi'_\beta(z) \varphi_\gamma(w) - \varphi_\beta(z - w) \varphi_\gamma(z) \varphi'_\beta(w) \right),
\]

\[
2i\varepsilon_{\alpha\beta\gamma} S^1_\gamma S^1_\beta \left( \varphi_\gamma(z - w) \varphi'_\beta(z) \varphi'_\gamma(w) - \varphi_\beta(z - w) \varphi'_\gamma(z) \varphi'_\beta(w) \right)
\]

As a result of the expansion in function product for the matrix element \( \sigma \otimes \sigma \) we get: 

\[
2i\varepsilon_{\alpha\beta\gamma} S^0_\gamma S_0 (\varphi_\alpha(z - w) \varphi_\gamma(w) - \varphi_\beta(z - w) \varphi_\gamma(z)) = 2i\varepsilon_{\alpha\beta\gamma} \varphi_\alpha(z) \varphi_\beta(w) S^0_\gamma S_0,
\]

\[
2i\varepsilon_{\alpha\beta\gamma} S^0_\gamma S^0_\beta (\varphi_\alpha(z - w) \varphi_\gamma(w) E_2(z) - \varphi_\beta(z - w) \varphi_\gamma(z) E_2(w)) =
\]

\[
= 2i\varepsilon_{\alpha\beta\gamma} \left( \varphi_\alpha(z) \varphi_\beta(w) E_2(\gamma) - \varphi'_\alpha(z) \varphi'_\beta(w) \right) S^0_\gamma S^0_\beta,
\]

\[
2i\varepsilon_{\alpha\beta\gamma} S^1_\gamma S^1_0 \left( \varphi_\alpha(z - w) \varphi'_\gamma(w) - \varphi_\beta(z - w) \varphi'_\gamma(z) \right) =
\]

\[
= 2i\varepsilon_{\alpha\beta\gamma} \left( \varphi'_\alpha(z) \varphi_\beta(w) + \varphi_\alpha(z) \varphi'_\beta(w) \right) S^1_\gamma S^1_0,
\]

and for the matrix element \( \sigma \otimes I \): 

\[
2i\varepsilon_{\alpha\beta\gamma} S^0_\gamma S^0_\beta \left( \varphi_\gamma(z - w) \varphi_\beta(z) \varphi_\gamma(w) - \varphi_\beta(z - w) \varphi_\gamma(z) \varphi_\beta(w) \right) =
\]

\[
= 2i\varepsilon_{\alpha\beta\gamma} \varphi_\alpha(z) (E_2(\gamma) - E_2(\beta)) S^0_\gamma S^0_\beta,
\]

\[
2i\varepsilon_{\alpha\beta\gamma} S^1_\gamma S^0_\beta \left( \varphi_\gamma(z - w) \varphi_\beta(z) \varphi'_\gamma(w) - \varphi_\beta(z - w) \varphi'_\gamma(z) \varphi_\beta(w) \right) =
\]

\[
= 2i\varepsilon_{\alpha\beta\gamma} \varphi_\alpha(z) (E_2(\gamma) - E_2(\beta)) S^1_\gamma S^0_\beta.
\]
\[= 2i\varepsilon_{\alpha\beta\gamma} \left( -\varphi'_\alpha(z)E_2(\beta) + \varphi'_\alpha(z)E_2(w) \right) S_1^1 S_0^0, \]
\[2i\varepsilon_{\alpha\beta\gamma} S_0^0 S_1^1 \left( \varphi_\gamma(z - w)\varphi'_\beta(z)\varphi_\gamma(w) - \varphi_\beta(z - w)\varphi_\gamma(z)\varphi'_\beta(w) \right) = \]
\[= 2i\varepsilon_{\alpha\beta\gamma} \left( \varphi'_\alpha(z)E_2(\gamma) - \varphi'_\alpha(z)E_2(w) \right) S_1^1 S_0^0, \]
\[2i\varepsilon_{\alpha\beta\gamma} S_0^1 S_1^1 \left( \varphi_\gamma(z - w)\varphi'_\beta(z)\varphi_\gamma(w) - \varphi_\beta(z - w)\varphi'_\gamma(z)\varphi'_\beta(w) \right) = \]
\[= 2i\varepsilon_{\alpha\beta\gamma} \left( -\varphi_\alpha(z)E_2(\alpha) \left( E_2(\beta) - E_2(\gamma) \right) + \varphi_\alpha(z)E_2(w) \left( E_2(\beta) - E_2(\gamma) \right) \right) S_1^1 S_1^1. \]

After regrouping the terms we get \((2.29), (2.30)\) and \((2.31)\) as the coefficients at the function products 1, \(E_2(z), \varphi_\alpha(z), \varphi'_\alpha(z)\) and \(1, E_2(w), \varphi_\alpha(w), \varphi'_\alpha(w)\). □

Let us write these exchange relations in the form of the tables (1) and (2):

| \[2i\varepsilon_{\alpha\beta\gamma}.\] | \(S_0^0 S_\gamma\) | \(S_0^1 S_\gamma\) | \(S_0^1 S_\gamma^0\) | \(S_0^1 S_\gamma^1\) |
|------------------------------|----------------|----------------|----------------|----------------|
| \(\{S_0^0 S_\beta^0\}\)     | +1            | 0             | -J_\gamma     | 0             |
| \(\{S_0^1 S_\beta^0\}\)     | 0             | +1            | 0             | -J_\alpha     |
| \(\{S_0^0 S_\beta^1\}\)     | 0             | +1            | 0             | -J_\beta      |
| \(\{S_0^1 S_\beta^1\}\)     | 0             | 0             | +1            | 0             |

and

| \[2i\varepsilon_{\alpha\beta\gamma}.\] | \(S_\beta^0 S_\gamma^0\) | \(S_\beta^0 S_\gamma^1\) | \(S_\beta^1 S_\gamma^0\) | \(S_\beta^1 S_\gamma^1\) |
|------------------------------|----------------|----------------|----------------|----------------|
| \(\{S_\beta^0 S_0\}\)       | +J_\alpha\beta | 0             | 0             | +J_\alpha J_\gamma\beta |
| \(\{S_\beta^1 S_0\}\)       | 0             | -J_\beta      | +J_\gamma    | 0             |
| \(\{S_\beta^0 S_1^1\}\)     | 0             | 0             | 0             | +J_\gamma\beta |
| \(\{S_\beta^1 S_1^1\}\)     | 0             | -1            | +1            | 0             |

Table 1. Exchange relations \(S_\alpha, S_\beta\)

Table 2. Exchange relations \(S_\alpha, S_0\)
where $J_\alpha = E_2(\alpha)$, $J_\gamma = E_2(\gamma) - E_2(\beta)$. The Jacobi identity for $\mathcal{P}_{f,2}$ follows from the classical Yang-Baxter equation for $r$ matrix. Note that the Poisson algebra $\mathcal{P}_{f,2}$ come to the Sklyanin algebra (9), if we put $S_0^1 = 0$, $S_1^1 = 0$.

The quadratic brackets are not degenerate on the orbits. To describe the system we define the Casimir functions and Hamiltonians. The equation for the spectral curve has the following form:

$$\det(L(z) - \lambda I) = 0,$$

$$\lambda^2 - Tr(L(z))\lambda + detL(z) = 0,$$

where $det$ is determinant. The coefficients $TrL(z)$ and $detL(z)$ of this equation define the Casimir functions and Hamiltonians of the system. $TrL(z)$ and $detL(z)$ are doubly periodic functions and they can be decomposed into the basis of Eisenstien functions.

$$\frac{1}{2} TrL(z) = S_0 + E_2(z)S_0^1,$$

$$detL(z) = C_0 + C_1E_1(z) + C_2E_2(z) + C_3E_2'(z) + C_4E_2''(z).$$

So, we have two Hamiltonians $S_0$, $S_1^0$ and four the Casimir functions ($C_1 = 0$):

$$C_0 = S_0^2 - 4\eta_0^2(S_0^1)^2 + E_2(\alpha)(S_0^0)^2 +$$

$$+ \left((E_2(\alpha))^2 - (E_1(\alpha))^2E_2(\alpha) + E_1(\alpha)E_2(\alpha) + \frac{1}{3}E_2''(\alpha)\right)(S_0^1)^2,$$

$$C_2 = S_0S_1^0 + 4\eta_1(S_1^0)^2 - (S_0^0)^2 + ((E_1(\alpha))^2 - E_2(\alpha))(S_0^1)^2,$$

$$C_3 = -S_0S_1^1,$$

$$C_4 = \frac{1}{6}\left( (S_0^1)^2 - (S_0^1)^2 \right).$$

To calculate the Casimir functions we use (A.7) - (A.10) of Appendix A. The dimension of the phase space $\mathcal{R}_{f,2}$ (symplectic leaves) is equal to 4.

$$\mathcal{R}_{f,2} \sim \mathcal{P}_{f,2}/\{C_i, i = 1, 4\}$$

In terms of the quadratic brackets the equations of motion have the following form:

$$\partial_{t_0} S_0^0 = \frac{1}{2}\{S_0^0, S_0\} =$$

$$= i\varepsilon_{\alpha\beta\gamma}(E_2(\gamma) - E_2(\beta))S_3^0S_0^0 + i\varepsilon_{\alpha\beta\gamma}E_2(\alpha)(E_2(\gamma) - E_2(\beta))S_3^1S_0^1,$$

$$\partial_{t_0} S_0^1 = \frac{1}{2}\{S_0^1, S_0\} = -i\varepsilon_{\alpha\beta\gamma}E_2(\beta)S_0^0S_0^1 + i\varepsilon_{\alpha\beta\gamma}E_2(\gamma)S_0^1S_0^0,$$

$$\partial_{t_1} S_0^0 = \frac{1}{2}\{S_0^0, S_0^1\} = i\varepsilon_{\alpha\beta\gamma}(E_2(\gamma) - E_2(\beta))S_3^1S_0^1,$$

$$\partial_{t_1} S_0^1 = \frac{1}{2}\{S_0^1, S_0^1\} = -i\varepsilon_{\alpha\beta\gamma}S_0^0S_0^1 + i\varepsilon_{\alpha\beta\gamma}S_0^1S_0^0.$$
Graduation and the systems via three point fusion. Let us do the following observation. One can see from the tables (3) and (4) that coefficients at $SS$ products have the certain dependence of the same products. Consider the bracket $\{S^0_\alpha, S^0_\beta\}$, we note that $S_0S^0_\gamma$ differs from $S^1_\gamma S^0_\gamma$ by one of the multipliers. The coefficients take the values $+1$ and $-J_\gamma$ correspondingly. For the bracket $\{S^0_\alpha, S^1_\beta\}$ we have the same situation: the product $S_0S^1_\gamma$ differs from $S^1_\gamma S^1_\delta$ one by the same multipliers $S_0$ and $S^1_\delta$. The coefficients take the values $+1$ and $-J_\delta$. Now the coefficient at $S^1_\delta S^1_\zeta$ depend on $\delta$. Taking into account $J_\delta = E_2(u) = -\frac{\partial}{\partial u}E_1(u)$, we see that each function $J$ has the degree of the operator $\frac{\partial}{\partial u}$ proportional to 1, but it takes the different value. It is possible to say that the function $+1$ has the degree of the operator $\frac{\partial}{\partial u}$ equal to 0. It is possible to see the regularization in the coefficient positions for the other brackets. So, we can define the notion of the graduation in the following manner. We consider the Lax operator $L_{fusion}(z)$ (2.28) and assign that the graduation of variables $S - \text{grad}(S) \equiv [S]$ is proportional to the modulus of the pole order of the function at $S$ in (2.28). It is possible to assume that in the expression of the each Poisson bracket all the terms have the equal graduation. Here we take into account that the graduation of the coefficients (which is proportional to the degree of the operator) is opposite in sign to the $SS$ products. The graduation of each term is the sum of the graduations:

$$[J_\beta S^1_\delta S^1_\zeta] = [J_\beta] + [S^1_\delta] + [S^1_\zeta].$$

From this approach the interesting fact is that the operation of taking Poisson bracket acquire the graduation too.

$$[\{S^0_\alpha, S^0_\beta\}] = [\{\}, \beta] + [S^0_\alpha] + [S^0_\beta].$$

Now let us compose the linear equations. In a simplest case of the Sclyanin algebra (9), if we put $S^1_\alpha$ and $S^0_\delta$ equal to 0, we get the following equations:

$$2b + [\{\}, \alpha] = a + b,
\quad a + b + [\{\}, \beta] = d + 2b$$

where we put $[S_0] = a, [S^0_\alpha] = b, [J_\alpha] = d$. We get the following important relation $2[\{\}, \delta] = d$. Note that we get it without using the value of graduations $a$ and $b$.

Let us consider now the three point fusion. In this case the Lax operator has the following form:

$$L_{fusion}(z) =
\quad = \left(S_0 + S^1_\delta E_2(z) + S^0_\delta E_2'(z)\right)\sigma_0 + \sum_\alpha \left(S^0_\alpha \varphi_\alpha(z) + S^1_\alpha \varphi_\alpha'(z) + S^0_\alpha \varphi_\alpha''(z)\right) \sigma_\alpha.$$

(2.36)

Analyzing tables (3) and (4) it is possible to conclude that the “structural blocks” for the coefficients at $SS$ are the product of the function $J$. Using the graduation we can write the presumable form of the exchange relations. Let us consider for example $\{S^0_\alpha, S_0\}$:

$$\{S^0_\alpha, S_0\} = 2i\varepsilon_{\alpha\beta\gamma} \left(J_\gamma S_0 S^0_\gamma + J_\alpha J_\beta S^1_\beta S^1_\gamma + c_{2,0} S^2_\beta S^0_\gamma + c_{0,2} S^0_\beta S^2_\gamma + c_{2,2} S^2_\beta S^2_\gamma\right).$$

(2.37)

Put $[S_0] = 0, [S^0_\alpha] = 1, [S^1_\beta] = 2, [S^1_\zeta] = 2, [S^0_\delta] = 3, [S^1_\gamma] = 3$ in accordance with notation above, we get $[J_\alpha] = -2$ and $[\{S^0_\alpha, S_0\}] = 0$. So there are all allowed terms in (2.37). For example it is not possible to write the term with $S^1_\delta S^0_\gamma$. Its graduation is equal to 5, and we cannot pick out the coefficient consisted of $J$-function product because the graduation of the whole term is not equal to 0 for each case. Compared the results of the explicit calculation ((Table 3),(Table 4)), we get the coefficients. The graduation shows the possible positions of the coefficients. Note
that the coefficients at $SS$ are the invariants of the transformations which do not change the Poisson brackets.

Table 3. Exchange relations $S_\alpha$, $S_\beta$

| $2i\varepsilon_{\alpha\beta\gamma}$ | $S_\alpha S_\gamma$ | $S_\beta S_\gamma$ | $S_\alpha S_\beta$ | $S_\beta S_\alpha$ | $S_\alpha^2 S_\gamma$ | $S_\beta^2 S_\gamma$ | $S_\alpha S_\beta^2$ | $S_\beta S_\alpha^2$ |
|-----------------------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| $\{S_\alpha^0, S_\beta^0\}$       | +1                  | 0                   | $-J_\gamma$         | 0                   | 0                   | 0                   | $-J_\gamma J_\alpha$ | $J_\gamma J_\beta$  |
| $\{S_\alpha^1, S_\beta^0\}$       | 0                   | +1                  | 0                   | $-J_\alpha$         | 0                   | $-J_\gamma$         | 0                   | $J_\gamma J_\alpha$ |
| $\{S_\alpha^0, S_\beta^1\}$       | 0                   | +1                  | 0                   | $-J_\beta$          | 0                   | $-J_\gamma$         | 0                   | $J_\gamma J_\beta$  |
| $\{S_\alpha^1, S_\beta^1\}$       | 0                   | 0                   | 1                   | 0                   | +1                  | 0                   | $(J_\alpha + J_\beta)$ | 0                   |
| $\{S_\alpha^0, S_\beta^2\}$       | 0                   | 0                   | 0                   | 0                   | +1                  | 0                   | 0                   | $-J_\alpha$         |
| $\{S_\alpha^1, S_\beta^2\}$       | 0                   | 0                   | 0                   | 0                   | 0                   | 1                   | 0                   | $J_\alpha$          |
| $\{S_\alpha^2, S_\beta^1\}$       | 0                   | 0                   | 0                   | 0                   | 0                   | 0                   | 1                   | $J_\alpha$          |
| $\{S_\alpha^2, S_\beta^2\}$       | 0                   | 0                   | 0                   | 0                   | 0                   | 0                   | $-\frac{1}{2}$      | $+\frac{1}{2}$      |

Table 4. Exchange relations $S_\alpha$, $S_0$

| $2i\varepsilon_{\alpha\beta\gamma}$ | $S_\alpha^0 S_\gamma^0$ | $S_\beta^0 S_\gamma^0$ | $S_\alpha^0 S_\gamma^1$ | $S_\beta^0 S_\gamma^1$ | $S_\alpha^1 S_\gamma^0$ | $S_\beta^1 S_\gamma^0$ | $S_\alpha^1 S_\gamma^1$ | $S_\beta^1 S_\gamma^1$ | $S_\alpha S_\gamma^0$ | $S_\beta S_\gamma^0$ |
|---------------------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| $\{S_\alpha^0, S_\gamma^0\}$          | $J_\beta$                | 0                        | 0                        | 0                        | $J_\alpha J_\beta$         | $-J_\alpha J_\beta$         | $-J_\alpha J_\beta$         | 0                        | 0                        | $(J_\beta J_\gamma - J_\alpha^2) J_\beta$ |
| $\{S_\alpha^1, S_\gamma^0\}$          | 0                        | $-J_\beta$               | $J_\gamma$               | 0                        | 0                        | 0                        | 0                        | $(J_\gamma J_\alpha) J_\beta$         | 0                        | $(J_\alpha + J_\alpha^2) J_\beta$ |
| $\{S_\alpha^0, S_\gamma^1\}$          | 0                        | 0                        | 0                        | $J_\gamma$               | 0                        | 0                        | 0                        | $(J_\alpha + J_\alpha^2) J_\beta$         | 0                        | $(J_\alpha + J_\alpha^2) J_\beta$ |
| $\{S_\alpha^1, S_\gamma^1\}$          | 0                        | 0                        | 0                        | 0                        | $J_\alpha$                | 0                        | 0                        | $-J_\alpha J_\beta$         | 0                        | $-J_\alpha J_\beta$         |
| $\{S_\alpha^0, S_\gamma^2\}$          | 0                        | 0                        | 0                        | 0                        | 0                        | 0                        | 0                        | $\frac{1}{2} J_\beta$         | $-\frac{1}{2} J_\beta$         | 0                        |
| $\{S_\alpha^1, S_\gamma^2\}$          | 0                        | 0                        | 0                        | 0                        | 0                        | 0                        | 0                        | $-J_\gamma^2$                | 0                        | $J_\gamma^2$                |
| $\{S_\alpha^2, S_\gamma^1\}$          | 0                        | 0                        | 0                        | 0                        | $-1$                     | $+1$                     | 0                        | 0                        | $-J_\gamma^2$                | 0                        |
| $\{S_\alpha^2, S_\gamma^2\}$          | 0                        | 0                        | 0                        | 0                        | 0                        | 0                        | $-\frac{1}{2}$         | $+\frac{1}{2}$         | 0                        | 0                        |

As the result we get the following
Proposition 2.2. The space $P_{f_3,2}$ is Poisson with respect to the corresponding quadratic brackets Table.3 and Table.4.

3 Conclusion.

Although the graduation clarifies the algebraic structure of the phase space by indicating the possible positions of the coefficients at SS product but it is not possible to write the explicit forms of these coefficients. The other question is about the representation of the coefficients by $J$-function product. The goal of the further researching can be simulate the structure of the phase space without using the calculation and abstracting from the elliptic dependence.

Acknowledgments. Author would like to thank A. Levin for fruitful discussions and M. Olshanetsky for the suggested theme, important discussion and attention during the writing of article. The work was partly supported by grants RFBR-06-02-17381, NSch-8065-2006.2, RFBR-06-01-92054-KE and by Federal Nuclear Energy Agency.

4 Appendix.

4.1 Appendix A. Elliptic functions

We assume that $q = \exp 2\pi i \tau$, where $\tau$ is the modular parameter of the elliptic curve $E_\tau$.

The basic element is the theta function:

$$\vartheta(z|\tau) = q^{\frac{1}{8}} \sum_{n \in \mathbb{Z}} (-1)^n e \left( \frac{1}{2} \pi(n+1) \tau + nz \right) = (e = \exp 2\pi i)$$ (A.1)

The Eisenstein functions

$$E_1(z|\tau) = \partial_z \log \vartheta(z|\tau), \quad E_1(z|\tau) \sim \frac{1}{z} - 2\eta_1 z,$$ (A.2)

where

$$\eta_1(\tau) = \frac{24 \eta'(\tau)}{2\pi i \eta(\tau)} , \quad \eta(\tau) = q^{\frac{1}{24}} \prod_{n>0} (1 - q^n) .$$ (A.3)

is the Dedekind function.

$$E_2(z|\tau) = -\partial_z E_1(z|\tau) = \partial_z^2 \log \vartheta(z|\tau), \quad E_2(z|\tau) \sim \frac{1}{z^2} + 2\eta_1 .$$ (A.4)

Relation to the Weierstrass functions

$$\zeta(z, \tau) = E_1(z, \tau) + 2\eta_1(\tau) z, \quad \wp(z, \tau) = E_2(z, \tau) - 2\eta_1(\tau) .$$ (A.5)

The highest Eisenstein functions

$$E_j(z) = \frac{(-1)^j}{(j-1)!} \partial^{j-2} E_2(z), \quad (j > 2).$$ (A.6)

$$\varphi(u, z) \varphi(-u, z) = E_2(z) - E_2(u) ,$$ (A.7)
\[ \phi(u, z)\phi'(-u, z) = + \left( E_1(u)E_2(u) + \frac{1}{2}E_2(u)' \right) - E_1(u)E_2(z) + \frac{1}{2}E_2'(z). \] (A.8)

\[ \phi'(u, z)\phi(-u, z) = - \left( E_1(u)E_2(u) + \frac{1}{2}E_2(u)' \right) + E_1(u)E_2'(z) + \frac{1}{2}E_2'(z). \] (A.9)

\[ \phi'(u, z)\phi'(-u, z) = \]
\[ = \left( -(E_2(u))^2 + (E_1(u))^2E_2(u) + E_1(u)E_2'(u) + \frac{1}{3}E_2(u)'' \right) - \]
\[ + \left( -(E_1(u))^2 + E_2(u) \right) E_2(z) + \frac{1}{6}E_2''(z). \] (A.10)

The next important function is

\[ \phi(u, z) = \frac{\vartheta(u + z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}. \] (A.11)

\[ \phi(u, z) = \phi(z, u), \quad \phi(-u, -z) = -\phi(u, z). \] (A.12)

It has a pole at \( z = 0 \) and

\[ \phi(u, z) = \frac{1}{z} + E_1(u) + \frac{z}{2}(E_1^2(u) - \varphi(u)) + \ldots. \] (A.13)

\[ \partial_u\phi(u, z) = \phi(u, z)(E_1(u + z) - E_1(u)). \] (A.14)

\[ \partial_z\phi(u, z) = \phi(u, z)(E_1(u + z) - E_1(z)). \] (A.15)

\[ \lim_{z \to 0} \partial_u\phi(u, z) = -E_2(u). \] (A.16)

**Heat equation**

\[ \partial_t\phi(u, w) - \frac{1}{2\pi i}\partial_u\partial_w\phi(u, w) = 0. \] (A.17)

**Quasi-periodicity**

\[ \vartheta(z + 1) = -\vartheta(z), \quad \vartheta(z + \tau) = -q^{-\frac{1}{2}}e^{-2\pi i z}\vartheta(z), \] (A.18)

\[ E_1(z + 1) = E_1(z), \quad E_1(z + \tau) = E_1(z) - 2\pi i, \] (A.19)

\[ E_2(z + 1) = E_2(z), \quad E_2(z + \tau) = E_2(z), \] (A.20)

\[ \phi(u, z + 1) = \phi(u, z), \quad \phi(u, z + \tau) = e^{-2\pi i u}\phi(u, z). \] (A.21)

\[ \partial_u\phi(u, z + 1) = \partial_u\phi(u, z), \quad \partial_u\phi(u, z + \tau) = e^{-2\pi i u}\partial_u\phi(u, z) - 2\pi i\phi(u, z). \] (A.22)

**The Fay three-section formula:**

\[ \phi(u_1, z_1)\phi(u_2, z_2) - \phi(u_1 + u_2, z_1)\phi(u_2, z_2 - z_1) - \phi(u_1 + u_2, z_2)\phi(u_1, z_1 - z_2) = 0. \] (A.23)

From (A.15) and (A.23) we have:
\[ \phi(u_1, z)\phi(u_2, z) = \phi(u_1 + u_2, z)(E_1(u_1) + E_1(u_2) - E_1(u_1 + u_2 + z) + E_1(z)). \]  

(A.24)

Particular cases of this formula are the functional equations

\[ \phi(u, z)\partial_v\phi(v, z) - \phi(v, z)\partial_u\phi(u, z) = (E_2(v) - E_2(u))\phi(u + v, z), \]  

(A.25)

\[ \phi(u, z_1)\phi(-u, z_2) = \phi(u, z_1 - z_2)(-E_1(z_1) + E_1(z_2) - E_1(u) + E_1(u + z_1 - z_2)) = \]  

\[ = \phi(u, z_1 - z_2)(-E_1(z_1) + E_1(z_2) + \partial_u\phi(u, z_2 - z_1)), \]

\[ \phi(u, z)\phi(-u, z) = E_2(z) - E_2(u). \]  

(A.27)

\[ \phi(v, z - w)\phi(u_1 - v, z)\phi(u_2 + v, w) - \phi(u_1 - u_2 - v, z - w)\phi(u_2 + v, z)\phi(u_1 - v, w) = \]  

\[ \phi(u_1, z)\phi(u_2, w)f(u_1, u_2, v), \]

where

\[ f(u_1, u_2, v) = E_1(v) - E_1(u_1 - u_2 - v) + E_1(u_1 - v) - E_1(u_2 + v). \]  

(A.29)

One can rewrite the last function as

\[ f(u_1, u_2, v) = -\frac{\partial' (0)\partial(u_1)\partial(u_2)\partial(u_2 - u_1 + 2v)}{\partial(u_1 - v)\partial(u_2 + v)\partial(u_2 - u_1 + v)\partial(v)}. \]  

(A.30)

Using [A.2], [A.4], [A.13] one can derive from [A.28] some important particular cases. One of them corresponding to \( v = u_1 \) (or \( v = -u_2 \)), is the Fay identity [A.23]. Another particular case comes from \( u_1 = 0 \) (or \( u_2 = u \)):

\[ \phi(v, z - w)\phi(-v, z)\phi(u + v, w) - \phi(-u - v, z - w)\phi(u + v, z)\phi(-v, w) = \]  

\[ \phi(u_1, z)(E_2(u + v) - E_2(v)). \]  

(A.31)

If \( u_2 \rightarrow -v \) then [A.28] in the first non-trivial order take the form for \( u_1 = \alpha, \ u_2 = \beta \)

\[ \phi(-\beta, z - w)E_1(w)\phi(\alpha + \beta, z) - \phi(\alpha, z - w)E_1(z)\phi(\alpha + \beta, w) = \]  

\[ \phi(\alpha, z)\phi(\beta, w)(E_1(\alpha) + E_1(\beta) - E_1(\alpha + \beta)). \]  

(A.32)

4.2 Appendix B. Lie algebra \( \mathfrak{sl}(N, \mathbb{C}) \), Group \( GL(N, \mathbb{C}) \) and elliptic functions

Introduce the notation

\[ \mathbf{e}_N(z) = \exp\left(\frac{2\pi i}{N}z\right) \]

and two matrices

\[ Q = \text{diag}(\mathbf{e}_N(1), \ldots, \mathbf{e}_N(m), \ldots, 1) \]  

(B.1)

\[ \Lambda = \delta_{j,j+1}, \quad (j = 1, \ldots, N, \text{ mod } N). \]  

(B.2)

Let

\[ \mathbb{Z}_N^{(2)} = (\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}), \]  

\[ \tilde{\mathbb{Z}}_N^{(2)} = \mathbb{Z}_N^{(2)} \setminus (0, 0) \]  

(B.3)

be the two-dimensional lattice of order \( N^2 \) and \( N^2 - 1 \) correspondingly. The matrices \( Q^{a_1}\Lambda^{a_2}, \ a = (a_1, a_2) \in \mathbb{Z}_N^{(2)} \) generate a basis in the group \( GL(N, \mathbb{C}) \), while \( Q^{a_1}\Lambda^{a_2}, \ a = (\alpha_1, \alpha_2) \in \tilde{\mathbb{Z}}_N^{(2)} \)
generate a basis in the Lie algebra \( \text{sl}(N, \mathbb{C}) \). More exactly, we introduce the following basis in \( \text{GL}(N, \mathbb{C}) \). Consider the projective representation of \( \mathbb{Z}_N^{(2)} \) in \( \text{GL}(N, \mathbb{C}) \)

\[
a \rightarrow T_a = \frac{N}{2\pi i} e_N \left( \frac{a_1 a_2}{2} \right) Q^{a_1} A^{a_2},
\]

(B.4)

\[
T_a T_b = \frac{N}{2\pi i} e_N \left( -\frac{a \times b}{2} \right) T_{a+b}, \quad (a \times b = a_1 b_2 - a_2 b_1).
\]

(B.5)

Here \( \frac{N}{2\pi i} e_N \left( -\frac{a \times b}{2} \right) \) is a non-trivial two-cocycle in \( H^2(\mathbb{Z}_N^{(2)}, \mathbb{Z}_{2N}) \). The matrices \( T_a, \alpha \in \tilde{\mathbb{Z}}_N^{(2)} \) generate a basis in \( \text{sl}(N, \mathbb{C}) \). It follows from (B.5) that

\[
[T_\alpha, T_\beta] = C(\alpha, \beta) T_{\alpha+\beta},
\]

(B.6)

where \( C(\alpha, \beta) = \frac{N}{i} \sin \frac{\pi}{N}(\alpha \times \beta) \) are the structure constants of \( \text{sl}(N, \mathbb{C}) \).

For \( N = 2 \) the basis \( T_\alpha \) is proportional to the basis of the Pauli matrices:

\[
T_{(1,0)} = \frac{1}{\pi i} \sigma_3, \quad T_{(0,1)} = \frac{1}{\pi i} \sigma_1, \quad T_{(1,1)} = \frac{1}{\pi i} \sigma_2.
\]

The Lie coalgebra \( \mathfrak{g}^* = \text{sl}(N, \mathbb{C}) \) has the dual basis

\[
\mathfrak{g}^* = \{ S = \sum_{Z \in \tilde{\mathbb{Z}}_N^{(2)}} S_Z t^Z \}, \quad t^Z = \frac{2\pi i}{N^2} T_{-Z}, \quad \langle T_a t^\beta \rangle = \delta_{\alpha \beta}.
\]

(B.7)

It follows from (B.6) that \( \mathfrak{g}^* \) is a Poisson space with the linear brackets

\[
\{ S_\alpha, S_\beta \} = C(\alpha, \beta) S_{\alpha+\beta}.
\]

(B.8)

The coadjoint action in these bases takes the form

\[
ad_{T_a} t^\beta = C(\alpha, \beta) t^{\alpha+\beta}.
\]

(B.9)

Let \( \check{\gamma} = \frac{2\gamma + 2\tau N}{N} \). Then introduce the following constants on \( \tilde{\mathbb{Z}}^{(2)} \):

\[
\vartheta(\gamma) = \vartheta(\frac{\gamma_1 + \gamma_2 \tau}{N}), \quad E_1(\gamma) = E_1(\frac{\gamma_1 + \gamma_2 \tau}{N}), \quad E_2(\gamma) = E_2(\frac{\gamma_1 + \gamma_2 \tau}{N}),
\]

(B.10)

\[
\varphi_\gamma(z) = \varphi(\check{\gamma}, z),
\]

(B.11)

\[
\varphi_\gamma(z) = e_N(\gamma z) \varphi_\gamma(z),
\]

(B.12)

\[
\varphi_{\gamma,\eta}(z) = e_N(\gamma z) \varphi(\eta + \frac{\gamma_1 + \gamma_2 \tau}{N}, z).
\]

(B.13)

They have the following quasi-periodicities

\[
\varphi_\gamma(z + 1) = e_N(\gamma_2) \varphi_\gamma(z), \quad \varphi_\gamma(z + \tau) = e_N(-\gamma_1) \varphi_\gamma(z),
\]

(B.14)

\[
\varphi_{\gamma,\eta}(z + 1) = e_N(\gamma_2) \varphi_{\gamma,\eta}(z), \quad \varphi_{\gamma,\eta}(z + \tau) = e_N(-\gamma_1 - \eta) \varphi_{\gamma,\eta}(z),
\]

(B.15)

The important formulas with \( \varphi_\alpha(z) \)

\[
\varphi_\gamma'(z) = -\varphi_\alpha(z) \varphi_\beta(z) = \varphi_\gamma(z)(E_1(\gamma + z) - E_1(z - E_1(\gamma))),
\]

(B.16)

\[
\varphi_\gamma''(z) = \varphi_\gamma(z)(E_1'(\alpha) + E_1'(\beta)) - 2 \varphi_\gamma(z) E_1'(z).
\]

(B.17)
References

[1] Yu. Chernyakov, *TMF*, **T 141** (2004) 1, 38-59;

[2] Musso F., Petrera M., Ragnisco O., *Jour. Nonlinear Math. Phys.*, **12 suppl.** 1, (2005) 482-498;

[3] E.Inonu, E.Wigner, On contraction of groups and their representations, *Proc. Nat. Acad. Sci.*, **39** (1953), 510-24;

[4] Yu.Chernyakov, A.M.Levin, M.Olshanetsky, A.Zotov, *Jour. of Phys.A*, **v39**, 39, (2006) 12083-12101;

[5] Schlesinger L., *J.Reine Angew.Math*, **141**, (1912) 96-145;

[6] A.Levin, M.Olshanetsky, A.Zotov, Painleve VI, Rigid Tops and Reflection Equation, math.QA/0508058, submitt. to *Comm. Math. Phys. ;*

[7] K.Takasaki, *Lett.Math.Phys.*, **44**, (1998) 143–156. hep-th/9711058;

[8] A.Levin, M.Olshanetsky, *Amer. Math. Soc. Transl. Ser. 2*, **191**, Amer. Math. Soc., Providence, RI, (1999), 223–262, hep-th/9709207;

[9] E.Sklyanin, *Func. Anal. Appl. Vol.16*, (1982) 283.