Mixed convection and thermally radiative hybrid nanofluid flow over a curved surface

Gohar¹, Tahir Saeed Khan¹, Imran Khan², Taza Gul³ and Muhammad Bilal³

Abstract
This study explored the Darcy Forchheimer flow of Casson hybrid nanofluid (NF) via a persistently stretching curved surface. The Darcy-Forchheimer contribution articulates viscous fluid flow in permeable media. Hybrid nanofluids are made from carbon nanotubes (CNTs) having a cylindrical shape and iron ferrite nanoparticulate. The main equations are reorganized into non-dimensional ODEs via similarity replacement. To accomplish the analytic simulation of modeled equations, the “Homotopy analysis approach” is performed. The effects of flow factors on velocity and energy profiles have been tabulated and discussed. It has been noticed that the integration of iron ferrite and CNT nanoparticulate in the base fluid to control the coolant level in industrial apparatus is quite useful. The energy field ascension effect is helpful for industrial uses since the energy fields exhibit favorable behavior against rising values of both type of nanomaterials. The Casson constraint’s rising values decline the hybrid NF motion.

Keywords
Heat and mass transmission, Casson fluid, heat absorption omission, hybrid nanofluid, Darcy Forchheimer, BVP h2.0 package

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Introduction
The fluid flow across a stretching texture has established an outstanding status due to its momentous character in several segments of industry and manufacturing, such as turning of fiber compression process, casting of fiber, paper production and plastic sheets extraction, etc. The flow across an enlarging surface was initially studied by Crane.¹ Sajid et al.² goes on to describe the fluid flow and determining that the curvature factor causes the boundary layer thickness to shrink. Gul et al.³ inspected the fluid flow of over a strained surface. The fluid flow with the implications of the magnetic flux across an extended curved surface was established by Imtiaz et al.⁴ The energy contour has been shown to be improved by the impact of the curvature constraint. Rosca and Pop⁵ assessed the fluid flow generated by expanding and contracting sheets. Saeed et al.⁶ conducted a thorough research of the Darcy hybrid NF flow via a continuously growing curved texture. The findings show that the hybrid NF improves heat allocation rate by employing MWCNTs, SWCNTs, and Fe₃O₄ nanocrystals. The hydrological importance of hybrid NF wave transmission over a curved surface...
heated stretched curved substrate with the impacts of a magnetic field was examined by Ali et al. Carbon nanotubes contained in a magnetite NF reduce local heat flow while increasing surface drag. The impact of radiation on the Casson fluid across an exponentially curved sheet was examined by Kumar et al. Reddy and Sreedevi addressed the effects of energy slips on the heat and mass transport properties of a Maxwell hybrid NF across a stretching and shrinking surface. As the number of the thermal relaxation content raise, the temperature scatterings of hybrid NFs decrease. Hussain et al. published the results of hybrid NF flow along a curved sheet. For high frequencies of the curving index, the energy conveyance performance in hybrid nanomaterials is more than that in nanofluids, according to the outcomes. Qian et al. evaluated the conducting flow of NF across an indefinitely curved stretched surface influenced heat transfer and radiation. Their findings were found to be in good agreement with those of prior research.

In current history, heat transfer in carbon nanofluids has piqued the interest of researchers from a range of sectors. Carbon nanotubes (CNTs) are cylinder shape carbon nanotubes having a basic chemical property and a carboxyl group production. CNTs are ideal for use as an aggregate in a base fluid because of their thermophysical, superior chemical, and mechanical features. Because of their small size, structure, layout, dimension, and toughness, they have several benefits over other nanocrystals. Haq et al. looked at the numerical simulations for conductive fluid employing carbon nanoscale across a broad area. Ahmadian et al. explored the 3D model of an inconsistent hybrid NF flow with solvent and momentum transfer produced by surface acceleration. The usage of hybrid nanocomposites is considered to have significantly improved the thermal characteristics of the carrier fluid. CNTs are more effective in the carrier fluid than other nanomaterials because of the C–C connection. Through non-covalent and covalent modification, CNTs NF may be further derivatized to obtain the desired outcome, which can then be employed in a variety of functions. Employing fluid flow across a curved surface, Saeed et al. investigated the hybrid NF using CNTs and iron oxide nanoparticles. Inside a square cavity, Sreedevi and Sudarsana Reddy and Sreedevi and Reddy performed a statistical analysis of energy transfer, flow, and entropy generating aspects of a hybrid NF. The passage of hybrid NF via a blood artery for medicine was observed by Alghamdi et al. Li et al. developed a fractional model for hybrid nanoliquid over a permeable whirling surface. Reddy et al. addressed the thermal slip factor effect on the temperature distribution of CNTs water-based NF across a spinning disk. The results show that when the nanoparticles concentration factor increases, the heat of the fluid rises. Furthermore, when nanomaterial of 0.05 volume fraction is suspended in the base solvent, thermal efficiency increases from 4.8% to 14.6%. The effect of a magnetic flux on the CNT NFs flow via a stirring absorbent medium was described by Akbar et al. Gul et al. evaluated and contrasted the properties of simple and hybrid nanomaterials moving on a spreading sheet. The turbulence of fluid flow may also be synchronized using a magnetic effect corresponding to their conclusions. Bilal et al. measured the Darcy Forchheimer hybrid NF flow using an inclined expandable cylinder. Carbon nanotubes (CNTs) and iron oxide Fe3O4 were studied separately as nanomaterials. The results show that hybrid NFs are the most effectual way to upsurge energy transmission and can also be used for cooling. Ahmed used nanofluids to depict temperature dispersion in a wavy-wall impermeable container. It was determined that raising the sheet’s waviness increases both the heat communication rate and the mass conduction rate. Reddy and Sreedevi and Sreedevi and Sudarsana Reddy compared the unstable and persistent MHD using Williamson and Tiwari-Das model. They presented NF flow through a wedge inhabited by CNTs of multiwalled kerosene oil. In both the unstable and steady scenarios, the rates of surface warming as the values of the nanoparticle volume fraction factor climb.

The inertia effect is accounted for by adding a squared element to the momentum equation, which is referred to as Forchheimer’s term. This novel notion was dubbed the “Forchheimer factor” by Muskat. To accurately portray real-world issues, non-Darcy effects must be included in convective transport analyses. The present mathematical model analyzed and modified the concept of Saba et al. and Xue to study the hybrid NF flow across an extending surface with heat and mass transmission. The CNTs and Fe3O4 nanoparticulate are utilized in the base solvent to boosts the energy and mass transition for industrial and biomedical purposes. The non-Newtonian Casson hybrid NF and heat absorption/generation term has been employed as an expansion in the current study.

Formulation

We considered the steady two-dimensional Darcy hybrid NF flow across an intensifying curved surface. The fluid flow is supposed to be a crossed an expanding surface having radius \( R \), as portrayed in Figure 1. The term \((r, s)\) is taken along \((u, v)\) direction. Here, \( U_v (s) = a e^{r/2} \), is the stretching velocity. Based on above presumption, the basic equations are formulated as:

\[
\frac{\partial}{\partial r} ((r + R) v) + R \frac{\partial u}{\partial s} = 0.
\]
The basic flow conditions are:

\[ \nu = 0, \; u = U_w(s) = ae^{\nu/L}, \; T = T_w + T_0e^{\nu/2L} = T_w, \]

\[ C = C_0e^{\nu/2L} + C_w = C_w, \; \text{at} \; r = 0, \]

\[ u \to 0, \; \frac{\partial u}{\partial r}, \; C \to C_w, \; T \to T_w, \; \text{at} \; r \to \infty \]

Here, \( K^* \) and \( F = \frac{C_0}{K^*} \), is the permeability and inertia factors.

The mathematical expression for the above experimental terms of hybrid NF are:

\[ \rho_{\text{hyf}} = (1 - (\phi_1 + \phi_2))\rho_f + \phi_1\rho_{C1} + \phi_2\rho_{C2}, \]

\[ \mu_{\text{hyf}} = \mu_f/(1 - (\phi_1 + \phi_2))^{2.5} \]

\[ (\rho C_p)_{\text{hyf}} = (1 - (\phi_1 + \phi_2))(\rho C_p)_{r} + \phi_1(\rho C_p)_{s1} + \phi_2(\rho C_p)_{s2}, \]

\[ k_{\text{hyf}} = k_{\text{r}}((k_{s1} + 2k_{\text{hyf}} - 2\sigma_1(k_{s1} - k_{s2}))/ \]

\[ (k_{s2} + 2k_{\text{hyf}} + \sigma_1(k_{s2} - k_{s1})) \]

\[ k_{\text{hyf}} = k_{\text{r}}((k_{s1} + 2k_{\text{hyf}} - 2\sigma_1(k_{s1} - k_{s2}))/ \]

\[ (k_{s2} + 2k_{\text{hyf}} + \sigma_1(k_{s2} - k_{s1})) \]

\[ \sigma_{\text{hyf}} = (1 - (\phi_1 + \phi_2))\sigma_f + \phi_1\sigma_{C1} + \phi_2\sigma_{C2}, \]

Here, \( \rho_{\text{hyf}} \) expose the density, \( \mu_{\text{hyf}} \) viscosity, \( \sigma_{\text{hyf}} \) electrical conductivity, and \( k_{\text{hyf}} \) thermal conductivity of hybrid NF. \( \phi_1 \) and \( \phi_2 \) is the volume friction parameters of CNTs and Fe_{3}O_{4}.

The transformation variables are\(^{31} \):

\[ \eta = \left( \frac{ae^{\nu/L}}{2u_{L}} \right)^{1/2} r, \]

\[ v = \frac{R}{r + R} \sqrt{\frac{ar_{L}e^{\nu/L}}{2L}} (f(\eta) + \eta f'(\eta)), \]

\[ u = U_w = ae^{\nu/L}f(\eta), \]

\[ p = \rho_f a e^{\nu/L}H(\eta), \; T = T_w + T_0e^{\nu/2L} \Theta(\eta), \]

\[ C = C_w + C_0e^{\nu/2L} \Omega(\eta). \]

By using equation (7), equations (2)–(6) yield

\[ H' = \left( \begin{array}{c} (\rho)_{\text{hyf}} \\ (\rho_f) \end{array} \right)^{1/2} \frac{1}{\eta + Kf'^2}, \]

\[ \left( 1 + \frac{1}{\beta} \right) \left( f'' - \frac{1}{\eta + K} f'' - \frac{1}{(\eta + K)^2} f'' - 2\lambda f' \right) \]

\[ = (1 - (\phi_1)^{2.5}(1 - \phi_2)^{2.5}) \left( \frac{(\rho)_{\text{hyf}}}{(\rho_f)} \right) \]

\[ \left( \frac{\eta + 2K}{(\eta + K)^2} K(f')^2 - \frac{K}{\eta + K} f'' - \frac{K}{(\eta + K)^2} + 2\lambda f'^2 \right) \]

\[ = (1 - (\phi_1)^{2.5}(1 - \phi_2)^{2.5}) \left( \frac{K}{\eta + K} (4H + \eta H) - (Gr\Theta + G\xi), \Theta(\eta) \right), \]

\[ \left( \frac{k_{\text{hyf}}}{k_f} + Rd \right) \left( \Phi'' - \frac{1}{\eta + K} \Phi'' + \frac{(\rho C_p)_{\text{hyf}}}{(\rho C_p_f)} \right) \]

\[ \Pr \left[ \frac{K}{\eta + K} (f\Theta' - \alpha\Phi') + \delta\Theta \right] = 0, \]

\[ (1 - (\phi_1)(1 - \phi_2)) \left( \Phi'' - \frac{1}{\eta + K} \Phi'' - k_1\Phi \right) \]

\[ + Sc \left( \frac{K}{\eta + K} f' \Phi' \right) = 0. \]
\[

t^m + \frac{2}{\eta + K} f^m - \frac{1}{(\eta + K)^2} f'' + \frac{1}{(\eta + K)^3} f''' - 2\lambda \left( f'' + \frac{1}{(\eta + K)} f' \right)
\]

\[
\left( \frac{1}{\eta + K} \right)
\]

\[
+ \frac{\rho_{bf}}{\rho_f} \left[ \frac{K}{(\eta + K)} f^{m+1} + \frac{K}{(\eta + K)^2} f'' + \frac{K}{(\eta + K)^3} f''' + \frac{3K}{(\eta + K)^4} f'''ight] 
+ \frac{1}{\eta + K} \left[ \frac{\beta_{bf}}{\beta_{tf}} (G_r \Theta) + \frac{\beta_{Chf}}{\beta_Cf} (G_c \Phi) \right] = 0.
\]

The transform conditions are:

\[
f = 0, \ f' = 1, \ \Phi = 1, \ \Theta = 1 \text{ at } \eta = 0,
\]

\[
f' \to 0, \ f'' \to 0, \ \Phi \to 0, \ \Theta \to 0 \text{ at } \eta \to \infty.
\]

Where, \( F_r \) is the Forchheimer term, \( k \) is the curvature parameters, \( \lambda \) is the porosity term, \( Gr \) is the thermal Grashop number, \( G_c \) is the mass Grashop number, \( Rd \) is the thermal radiation constant, and \( k_1 \) is the energy activation parameter respectively.

\[
Fr = \frac{C_h}{K^{1/2} \delta}, \quad Pr = \frac{v_f}{\alpha_f}, \quad \delta = \frac{2QL}{U_w(\rho e_p)}, \quad k = \left( \frac{a e^{L/L/2}}{2v_f L} \right),
\]

\[
\lambda = \frac{v_f}{K^{1/2} U_w}, \quad Sc = \frac{v_f}{D_f},
\]

\[
Gr = \frac{g \beta_f (T_w - T_0) L}{U_w^2}, \quad G_c = \frac{g \beta_Cf (C_w - C_n) L}{U_w^3}
\]

\[
Rd = \frac{16\sigma T_0^2}{3kk}, \quad k_1 = \frac{k_2}{a}.
\]

The physical interest quantities as:

\[
\frac{L}{S} \left( \frac{Re}{2} \right)^{1/2} N_{fr} = -\left( \frac{k_{bf}}{k_{tf}} + Rd \right) \Theta'(0), \quad \frac{L}{S} \left( \frac{Re}{2} \right)^{1/2} S_{ht} = -\Phi'(0),
\]

\[
\sqrt{\frac{Re}{2} C_f x} = \frac{1}{(1 - \phi_0)^{1/2} (1 - \phi_2)^{1/2}} \left( 1 + \frac{1}{\beta} \right) f''(0).
\]

Where local Reynolds number is

\[
Re_x = \frac{u_0 \alpha x}{v_f}.
\]

**Problem solution**

The HAM methodology has been employed to obtained set of ODEs as.\(^{32,33}\) The initial guesses are:

\[
f_0(\eta) = e^{-\eta} - e^{-2\eta}, \ \Theta_0(\eta) = e^{-\eta}, \ \Phi_0(\eta) = e^{-\eta}.
\]

The linear terms are

\[
\mathcal{L}_f(f) = f'' \text{ and } \mathcal{L}_\Theta(\Theta) = \Theta''.
\]

The expanded form of \( \mathcal{L}_f, \mathcal{L}_\Theta \) and \( \mathcal{L}_\Phi \) are:

\[
\mathcal{L}_f [\chi_1 + \chi_2 \eta + \chi_3 \eta^2 + \chi_4 \eta^3] = 0,
\]

\[
\mathcal{L}_\Phi [\chi_5 + \chi_6 \eta] = 0, \ \mathcal{L}_\Phi [\chi_7 + \chi_8 \eta] = 0.
\]

**OHAM convergence**

The convergence of OHAM technique is achieved as\(^{34,35}\).

\[
\varepsilon_f^m = \frac{1}{l + 1} \sum_{j=1}^{l} \left[ N_f \left( \sum_{j=1}^{m} f(\eta) \right) \right]^{2},
\]

\[
\varepsilon_\Theta^m = \frac{1}{l + 1} \sum_{j=1}^{l} \left[ N_\Theta \left( \sum_{k=1}^{m} \Theta(\eta), \sum_{j=1}^{m} \Theta(\eta) \right) \right]^{2},
\]

\[
\varepsilon_\Phi^m = \frac{1}{l + 1} \sum_{j=1}^{l} \left[ N_\Phi \left( \sum_{k=1}^{m} \Phi(\eta) \right) \right]^{2}.
\]

The sum of residual error is \( \varepsilon_m^e = \varepsilon_f^m + \varepsilon_\Theta^m + \varepsilon_\Phi^m \).

The thermophysical characteristics of solid material and fundamental fluids are displayed in Table 1. The consolidation of the OHAM approach has been determined up to the 30th repetition and is shown in Table 2. Table 3 compares and contrasts the current investigation with the available literature.

| Material       | \( \rho (\text{kg/m}^3) \) | \( C_p (\text{j/kgK}) \) | \( k (\text{W/mK}) \) |
|----------------|---------------------------|---------------------------|---------------------------|
| Pure water     | 997.1                     | 4179                      | 0.613                     |
| Fe\(_3\)O\(_4\) | 5200                      | 670                       | 6                         |
| SWCNTs         | 2600                      | 425                       | 6600                      |
| MWCNTs         | 1600                      | 796                       | 300                       |
Table 2. The residual errors.

| m  | $e^1_{m,SWCNTs}$ | $e^1_{m,MWCNTs}$ | $e^1_{m,Fe_2O_4}$ |
|----|------------------|------------------|------------------|
| 5  | $1.7168 \times 10^{-5}$ | $1.8479 \times 10^{-5}$ | $1.3257 \times 10^{-5}$ |
| 13 | $1.0223 \times 10^{-6}$ | $1.1354 \times 10^{-6}$ | $1.0831 \times 10^{-6}$ |
| 23 | $1.2599 \times 10^{-7}$ | $0.3698 \times 10^{-7}$ | $0.3489 \times 10^{-7}$ |
| 30 | $3.1578 \times 10^{-8}$ | $4.2669 \times 10^{-8}$ | $4.0464 \times 10^{-8}$ |

Table 3. The statistical outputs, when $\phi_1 = \phi_2$, $Fr = k = 0.6$, $\lambda = 0.2$.

|                        | Hayat et al. | Present  |
|------------------------|-------------|---------|
| $f''(0)$              | 0.735       | 0.7352130 |
| $-\Theta(0)$          | -1.375      | -1.3752410 |
| $-\Phi(0)$            | ............| -1.3620189 |

Figure 2. Error analysis of the HAM solution.

Results and discussion:

The purpose of this segment is to look forward to observe how the energy and velocity trajectories behave when the projected components are taken into account. Figure 1 depicts the flow arrangement. Figure 2 depicts the progress and error analysis of the OHAM approach.

Figure 3(a)–(f) highlighted the performance of velocity profile $f'(\eta)$ against the variation of mass Grashop number $Gc$, thermal Grashop number $Gr$, Forchheimer term $Fr$, the volume fraction of nanoparticles $\phi_1$ & $\phi_2$, curvature parameter $k$, and porosity parameter $\lambda$ respectively. Figure 3(a) and (b) revealed that the velocity field augments with the effect of mass and thermal Grashop number. Physically, the stretching velocity of curved surface declines upon the variation of mass and thermal Grashop number, which results in the declination of the velocity field. Figure 3(c) and (d) manifested that the fluid velocity $f'(\eta)$ consistently diminish with the influence of Forchheimer term $Fr$ and volume fraction of nanoparticles $\phi_1$ & $\phi_2$. The inclusion of nanomaterials in the water enhances the viscosity of the fluid, which then resists the flow field, that’s why such senior observed. Figure 3(e) and (f) shows that the upshot of curvature term $k$ improves the velocity field, while the effect of porosity parameter $\lambda$ declines it. Practically, the kinetic viscosity stretching surface enhances with the action of the absorbency factor, thus, such scene has been noticed in Figure 3(f).

Figure 4(a)–(d) elaborated the trend of energy profile $\Theta(\eta)$ against the variation of temperature exponent $A$, Forchheimer term $Fr$, curvature parameters $k$, and porosity term $\lambda$ respectively. Figure 4(a) and (b) magnify that the energy field deteriorates with the influence of energy exponent coefficient $A$ and elevates with the upshot of Forchheimer term $Fr$. Since the permeability of fluid diminishes by the act of the Forchheimer term, consequently, such a phenomenon has been observed. Figure 4(c) and (d) shows that the effects of curvature parameters $k$ and the local porosity constant boost the energy transition rate. Practically, the kinetic viscosity and length of the extending surface area enhances with the act of the porosity term, therefore, such a marvel has been perceived.

Figure 5(a)–(d) displays the behavior of energy profile $\Theta(\eta)$ against the variation of volume fraction of nanoparticles $\phi_1$ & $\phi_2$, Prandtl number $Pr$, and thermal radiation $Rd$ respectively. Figure 5(a) and (b) elaborated that the consequences of volume friction coefficients $\phi_{CNT}$ and $\phi_{Fe_2O_4}$, accelerating the energy propagation, while the influence of Prandtl number $Pr$ declines it. Water has a far larger specific heat capacity than iron and carbon nanostructures. When nanoparticles are added to water, their heat-absorbing ability is reduced, resulting in an excessive quantity of heat in the fluid. These variables contribute to the acceleration of the thermal energy transition. The thermal energy profile $\Theta(\eta)$ also elevated with flourishing values of the parameter $\delta$ and thermal radiation $Rd$ as shown in Figure 5(c) and (d).

Figure 6(a)–(d) reported the behavior of mass profile $\Phi(\eta)$ against the variation of volume fraction of nanoparticles, Schmidt number $Sc$, activation energy parameter $k_1$, and curvature parameters $k$. Figure 6(a) and (b) to elaborate that mass transmission declines with the variation of both type of nanoparticles and Schmidt number. On the other hand, the mass transition rate decreases with the effect of activation energy parameter $k_1$ and enhances with the curvature term.

Figure 7(a) and (b) illustrate the numerical outcomes for the Nusselt number $\frac{h}{\sqrt{\frac{Re}{C_f}}} N_u$. The surface drag force $\sqrt{\frac{Re}{C_f}}$ for carbon NF and $Fe_2O_4$ is declared via Figure 7(c) and (d). It’s been evidenced that as the curvature increases, the skin friction drops. Figure 7(e)
indicates the Sherwood number $\left( \frac{\alpha}{\eta} \right)^{\gamma/\delta} Sh_x$ as an enhancing factor of the Schmidt number.

The statistical data are presented through bar charts 7 for skin fraction, Nusselt number, and Sherwood number. Also, the percent wise enhancement is sketched. Bar chart 7(a, b) shows that $\lambda$ increases the skin friction as well as Nusselt number. Bar chart 7(c) shows that skin friction is decreased by increase in $Fr$. While it can be notice that from bar chart 7(e, f) how $\phi_1, \phi_2, \phi_3$ enhancing the heat transfer rate the percent wise data also sketched.

Figure 3. The behavior of velocity profile $f'(\eta)$ against the variation of: (a) mass Grashop number $Ge$, (b) thermal Grashop number $Gr$, (c) Forchheimer term $Fr$, (d) volume friction of nanoparticles $f_1$ & $f_2$, (e) curvature parameter $k$, and (f) porosity parameter $\lambda$.

Conclusion

In this problem, we analyzed the Darcy flow of Casson hybrid NF caused by an extending curved surface. The set of differential equations are analytically resolved by employing the “HAM” approach. The supremacy of NF in energy and mass communication in modern era are highlighted in the current mathematical model. The key results are as follows:

- This energy field ascension effect is helpful for industrial uses since the energy fields exhibit
Figure 4. The behavior of energy profile $\Theta(\eta)$ against the variation of: (a) temperature exponent $A$, (b) Forchheimer term $Fr$, (c) curvature parameters $k$, and (d) porosity term $\lambda$.

Figure 5. The behavior of energy profile $\Theta(\eta)$ against the variation of: (a) volume friction of nanoparticles $\phi_1$ & $\phi_2$, (b) Prandtl number $Pr$, (c) parameter $\delta$, and (d) thermal radiation $Rd$. 
Figure 6. The behavior of mass profile $\Phi(\eta)$ against the variation of: (a) volume friction of nanoparticles $\phi_1$ & $\phi_2$, (b) Schmidt number $Sc$, (c) activation energy parameter $k_1$, and (d) curvature parameters $k$.

Figure 7. (a) Nusselt number $\frac{\lambda}{\nu} \left( \frac{\eta}{\rho} \right)^2 Nu_x$ for. (b) Skin friction $\sqrt{\frac{\mu}{\nu}} C_{f}$ for $\lambda$. (c) Skin friction $\sqrt{\frac{\mu}{\nu}} C_{f}$ for $Fr$. (d) Sherwood number $\frac{\lambda}{\nu} \left( \frac{\eta}{\rho} \right)^2 Sh_x$ for $k$. (e) Nusselt friction $\frac{\lambda}{\nu} \left( \frac{\eta}{\rho} \right)^2 Nu_x$ for $\phi_1$, $\phi_2$, $\phi_3$. (f) Nusselt friction $\frac{\lambda}{\nu} \left( \frac{\eta}{\rho} \right)^2 Nu_x$ percentage wise.
favorable behavior against rising values of both type of nanomaterials.

- The Casson constraint’s rising values decline the hybrid NF motion.
- The integration of Fe₃O₄ and CNT nanocrystals in base solvent to control the coolant level in industrial apparatus is quite useful.
- The temperature profile displays a decreasing trend as the energy exponent coefficient $A$ escalate.
- Fluid velocity is intensified by enhancing the credit of curvature term $k$.
- When the Forchheimer coefficient is elevated, the velocity and energy profiles show the reverse trend.
- In comparison to ordinary fluids, hybrid NF is rapid agent for thermal energy transition.

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