The fine gradings of $sl(3, C)$ and their symmetries\textsuperscript{1}

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Abstract

We describe the normalizers for all non-conjugate maximal Abelian subgroups of diagonalizable automorphisms of $sl(3, C)$ and show their relation to the symmetries of equations related to the graded contraction.

Introduction

Admissible gradings of a simple Lie algebra $L$ over the complex or real number field are basic structural properties of each $L$. Examples of exploitation of coarse gradings like $Z_2$ are easy to find in the physics literature. Here we are interested in the opposite extreme: the fine gradings of $L$ which only recently were described \textsuperscript{[8,9]}. Our aim is to point out the interesting symmetries of the fine gradings on the example of $sl(3, C)$.

Such symmetries can be used in the study of graded contractions of $L$. Indeed, they are the symmetries of the system of quadratic equations for the contraction parameters. Graded contractions are a systematic way of forming from $L$ a family of equidimensional Lie algebras which are not isomorphic to $L$. An insight into such parameter–dependent families of Lie algebras provides a group theoretical tool for investigating relations between different physical theories through their symmetries.

In this contribution we consider finite dimensional $L$ and announce specific results for $sl(3, C)$ only.

The decomposition $\Gamma : L = \bigoplus_{i \in I} L_i$ is called a grading if, for any pair of indices $i, j \in I$, there exists an index $k \in I$ such that $L_i L_j \subseteq L_k$. There are infinitely many gradings of a given Lie algebra. We do not need to distinguish those gradings which can be transformed into each other. More precisely, if $\Gamma : L = \bigoplus_{i \in I} L_i$ is a grading and $g$ is an automorphism of $L$, then $\tilde{\Gamma} : L = \bigoplus_{i \in I} g(L_i)$ is also a grading. Two such gradings are called equivalent.

A grading $\Gamma : L = \bigoplus_{i \in I} L_i$ is a refinement of the grading $\overline{\Gamma} : L = \bigoplus_{i \in J} L_i$ if for any $i \in I$ there exists $j \in J$ such that $L_i \subseteq L_j$. A grading which cannot be properly refined is called fine.

For construction of a grading of $L$, one can use any diagonalizable automorphism $g$ from $Aut L$. It is easy to see that decomposition of $L$ into eigenspaces of $g$ is a

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grading. Similarly, if one takes a set of commuting automorphisms from \( \text{Aut} L \) and decomposes \( L \) according to all automorphisms into their eigenspaces, a grading of \( L \) is obtained as well. In order to describe all inequivalent gradings, one has first to describe all fine gradings. Then any other grading can be found as a coarsening of a fine one (a process inverse to refinement of a grading). In [1], the crucial role of MAD–groups in the theory of fine gradings was shown. Here MAD–group stands for a maximal Abelian group of diagonalizable automorphisms of \( L \). It was proven in [1] that if \( L \) is a simple Lie algebra over \( C \), then a grading \( \Gamma \) of \( L \) is fine if and only if there exists a MAD-group \( G \) such that \( \Gamma \) is the decomposition into eigenspaces of automorphisms from \( G \). Moreover, two fine gradings \( \Gamma_1 \) and \( \Gamma_2 \) are equivalent, if and only if the corresponding MAD-groups \( G_1 \) and \( G_2 \) are conjugate in \( \text{Aut} L \).

The idea of graded contractions of Lie algebra was introduced in [2]. If \( \Gamma : L = \bigoplus_{i \in I} L_i \) is a grading, one can modify the commutation relation in \( L \) by introducing a set of parameters \( \varepsilon_{ij} = \varepsilon_{ji}, \ i, j \in I \) and redefining \( [x, y]_{\text{new}} := \varepsilon_{ij}[x, y] \) for any \( x \in L_i, y \in L_j \). In order to preserve the Jacobi identities, the parameters \( \varepsilon_{ij} \) must satisfy a system of quadratic equations [2–7]. The finer the grading is, the bigger system of equations has to be solved. If \( L \) is decomposed into \( k \) subspaces, then the system has, in general, \( (k+1)^2 \) variables \( \varepsilon_{ij} \).

For Lie algebras of low dimension, or for coarse gradings, the solution of the system of grading equations is not difficult to find. However it becomes rather laborious for higher cases [3,5,6,12]. The symmetries of the set of grading equations could turn out to be the tool for finding the solutions more efficiently.

As we have already said, any fine grading of a simple Lie algebra over \( C \) is the decomposition \( \Gamma \) of \( L \) into eigenspaces of automorphisms belonging to some MAD–group \( G \subset \text{Aut} L \). For any subspace \( L_i \) of the fine grading and any element \( g \in G \) we have \( g L_i = L_i \), i.e. elements of a MAD–group preserve each subspace of the grading corresponding to the MAD–group \( G \) (\( G \)-grading). Here we are interested in automorphisms which preserve the \( G \)-grading of \( L \) but not each of its subspaces separately. Clearly elements of the normalizer of the MAD–group in \( \text{Aut} L \) have this property. Recall that the normalizer of \( G \) is defined as follows: \( N(G) = \{ h \in \text{Aut} L \mid h^{-1}Gh \subset G \} \). Let \( h \in N(G) \) and \( L_i \) be a subspace of the fine grading \( \bigoplus_{i \in I} L_i \) corresponding to the MAD–group \( G \). Since \( h^{-1}fh \in G \) for arbitrarily chosen \( f \in G \), we can find \( g \in G \) such that \( h^{-1}fh = g \). Applying this automorphism to \( L_i \) and using that \( g L_i = L_i \), we obtain \( f(hL_i) = hL_i \), i.e. \( hL_i \) is an eigenspace for any automorphism \( f \in G \), which means that \( hL_i = L_j \) for some index \( j \in I \).

Any \( h \in N(G) \) generates a permutation \( \pi \) on grading indices \( \pi : I \mapsto I \), and thus \( h \) provides a substitution \( (\varepsilon_{ij}) \mapsto (\varepsilon_{\pi(i)\pi(j)}) \) under which the set of solutions of quadratic contraction equations is invariant.

### MAD–groups of \( sl(3, C) \) and their normalizers

The complete classification of MAD–groups for classical Lie algebras over \( C \) is found in [8]. Let us recall the particular case of the Lie algebra \( sl(3, C) \) we are interested in. The group of automorphisms \( \text{Aut} sl(3, C) \) consists of inner automorphisms.
Ad}_A X$ and the outer ones $Out_A X$:

\[
\begin{align*}
Ad_A X &:= A^{-1}XA, \quad A \in SL(3, C) = \{B \in C^{3 \times 3}, \det B = 1\}, \\
Out_A X &:= -(A^{-1}XA)^T, \quad A \in SL(3, C)
\end{align*}
\]

According to [8], $\text{Aut \, sl}(3, C)$ contains four non-conjugate MAD–groups and therefore four inequivalent fine gradings. Let us list the four grading groups:

\[G_1 = \{Ad_A \mid A = \text{diag}(\alpha_1, \alpha_2, \alpha_3), \alpha_i \in C^*\}\]

\[G_2 = \{Ad_A \mid A = \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3), \varepsilon_i = \pm 1\} \cup \{Out_A \mid A = \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3), \varepsilon_i = \pm 1\}\]

\[G_3 = \{Ad_A \mid A = \text{diag}(\varepsilon, \alpha, \alpha^{-1}), \varepsilon = \pm 1, \alpha \in C^*\} \cup \]

\[\begin{cases} 
Out_A \mid A = \begin{pmatrix} \varepsilon & 0 & 0 \\
0 & 0 & \alpha \\
0 & \alpha^{-1} & 0 \end{pmatrix}, \varepsilon = \pm 1, \alpha \in C^* \end{cases}\]

\[G_4 = \left\{\text{Ad}_{P^jQ^j} \mid Q = \begin{pmatrix} 0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \end{pmatrix}, P = \text{diag}(1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}), \ k, j = 0, 1, 2 \right\}\]

Note that $G_1$ and $G_4$ are MAD–groups formed by inner automorphisms only. The group $G_1$ is infinite; it is a maximal torus of $sl(3, C)$. The group $G_4$ is finite and consists of 9 elements; it is called the Pauli group in [10]. The groups $G_2$ and $G_3$ contain outer automorphisms. There are 8 elements in $G_2$. The group $G_3$ is infinite, containing a one–parameter subgroup of the torus. Using these MAD–groups we have decomposed $sl(3, C)$ and we have obtained the following fine gradings.

The fine grading corresponding to $G_1$ is the Cartan decomposition consisting of 7 subspaces: one of them is two-dimensional (the Cartan subalgebra), and 6 remaining subspaces are one-dimensional. We denote by $E_{ij} \in C^{3 \times 3}$ the matrix with 1 on the position $ij$ and zeros elsewhere, and by $\{E_{ij}\}_{\text{lin}}$ the linear envelope of $E_{ij}$, and we identify the grading subspaces by the corresponding roots as subscripts. The Cartan grading has the form:

\[\Gamma_1 : \text{sl}(3, C) = N_0 \oplus N_{\alpha_1} \oplus N_{\alpha_2} \oplus N_{\alpha_1+\alpha_2} \oplus N_{-\alpha_1-\alpha_2} \oplus N_{-\alpha_2} \oplus N_{-\alpha_1},\]

where $N_0 = \{\text{diag}(a, -a, b, -b) \mid a, b \in C\}$ and

\[
\begin{align*}
N_{\alpha_1} &= \{E_{12}\}_{\text{lin}}, \quad N_{-\alpha_1} = \{E_{21}\}_{\text{lin}}, \quad N_{\alpha_1+\alpha_2} = \{E_{13}\}_{\text{lin}}, \\
N_{-\alpha_1-\alpha_2} &= \{E_{31}\}_{\text{lin}}, \quad N_{\alpha_2} = \{E_{23}\}_{\text{lin}}, \quad N_{-\alpha_2} = \{E_{32}\}_{\text{lin}}.
\end{align*}
\]

Subspaces in the remaining gradings cannot be indexed by roots. Nevertheless, another nice choice of the index set $I$ is possible. Since for each pair of indices $i, j \in I$ there exists an index $k \in I$ such that $[L_i, L_j] \subseteq L_k$, we have a partially binary operation $(i, j) \mapsto k$ on the index set $I$. (If $[L_i, L_j] = \{0\}$, then $k$ can be
chosen arbitrarily.) It is proven in [1] that, for simple \( L \), the index set \( I \) with this operation can be embedded into an Abelian additive group \( G \). This enables to compute the commutation very easily because \([L_i, L_j] \subseteq L_{i+j}\).

For \( \mathcal{G}_1 \), \( G = Z_3 \times Z_3 \) or \( Z_7 \) [3].

The index set of the fine grading corresponding to the group \( \mathcal{G}_2 \) can be embedded into the Abelian group \( Z_2 \times Z_2 \times Z_2 \). This group has 8 elements, while the fine grading has 7 subspaces only: there is no subspace labeled by the neutral element \((0,0,0)\). The grading has the form:

\[
\Gamma_2 : \text{sl}(3, C) = K_{(0,0,1)} \oplus K_{(1,1,1)} \oplus K_{(1,0,1)} \oplus K_{(0,1,1)} \oplus K_{(1,1,0)} \oplus K_{(0,1,0)} \oplus K_{(1,0,0)} ,
\]

where \( K_{(0,0,1)} = \{ \text{diag}(a, b, -a - b) \mid a, b \in C \} \) and

\[
\begin{align*}
K_{(1,1,1)} &= \{ E_{21} + E_{12} \}_{\text{lin}}, & K_{(1,0,1)} &= \{ E_{31} + E_{13} \}_{\text{lin}}, & K_{(0,1,1)} &= \{ E_{23} + E_{32} \}_{\text{lin}}, \\
K_{(1,1,0)} &= \{ E_{21} - E_{12} \}_{\text{lin}}, & K_{(0,1,0)} &= \{ E_{23} - E_{32} \}_{\text{lin}}, & K_{(0,0,0)} &= \{ E_{31} - E_{13} \}_{\text{lin}}.
\end{align*}
\]

The grading corresponding to the \( \text{MAD–group} \text{ group} \mathcal{G}_3 \) has as its index set the group \( Z_8 \):

\[
\Gamma_3 : \text{sl}(3, C) = M_0 \oplus M_1 \oplus M_2 \oplus M_3 \oplus M_4 \oplus M_5 \oplus M_6 \oplus M_7 ,
\]

where

\[
\begin{align*}
M_0 &= \{ E_{22} - E_{33} \}_{\text{lin}}, & M_1 &= \{ E_{12} - E_{31} \}_{\text{lin}}, & M_2 &= \{ E_{23} \}_{\text{lin}}, \\
M_3 &= \{ E_{13} + E_{21} \}_{\text{lin}}, & M_4 &= \{ 2E_{11} - E_{22} - E_{33} \}_{\text{lin}}, & M_5 &= \{ E_{12} + E_{31} \}_{\text{lin}}, \\
M_6 &= \{ E_{32} \}_{\text{lin}}, & M_7 &= \{ E_{13} - E_{21} \}_{\text{lin}}.
\end{align*}
\]

The last fine grading corresponding to the \( \text{MAD–group} \text{ group} \mathcal{G}_4 \) can be briefly written in terms of generalized Pauli matrices \( P \) and \( Q \) introduced in the description of the \( \text{MAD–group} \text{ group} \) itself. Definition of generalized Pauli matrices in \( C^{n \times n} \) and their properties are found in [10], [11]. Now the index set is a subset of the group \( Z_3 \times Z_3 \).

Again, no subspace is labeled by the neutral element:

\[
\Gamma_4 : \text{sl}(3, C) = L_{(1,0)} \oplus L_{(2,0)} \oplus L_{(0,1)} \oplus L_{(0,2)} \oplus L_{(1,1)} \oplus L_{(1,2)} \oplus L_{(2,1)} \oplus L_{(2,2)} ,
\]

where

\[
\begin{align*}
L_{(1,0)} &= \{ P \}_{\text{lin}}, & L_{(2,0)} &= \{ P^2 \}_{\text{lin}}, & L_{(0,1)} &= \{ Q \}_{\text{lin}}, & L_{(0,2)} &= \{ Q^2 \}_{\text{lin}}, \\
L_{(1,1)} &= \{ PQ \}_{\text{lin}}, & L_{(2,1)} &= \{ P^2 Q \}_{\text{lin}}, & L_{(1,2)} &= \{ P Q^2 \}_{\text{lin}}, & L_{(2,2)} &= \{ P^2 Q^2 \}_{\text{lin}}.
\end{align*}
\]

Since any grading can be derived from some of the fine ones by collecting several subspaces together into subspaces of higher dimension, one can construct all coarser gradings, i.e. try all possibilities of coarser decompositions and check for each decomposition whether it is a grading or not. Let us note that the algebra \( \text{sl}(3, C) \) has 17 inequivalent gradings including 4 fine gradings and the coarsest one formed by only one space, namely the whole \( \text{sl}(3, C) \).
For any group $G$, the normalizer $N(G)$ contains $G$ as its normal subgroup. Hence one has the quotient group $N(G) / G$. Now two elements $h_1$ and $h_2$ of the normalizer $N(G)$ belong to the same coset, if there exists $g \in G$ such that $h_1 = h_2 g$. For the grading $\Gamma$ it means that the automorphisms $h_1$ and $h_2$ yield the same permutation of the subspaces of $\Gamma$. Therefore $N(G) / G$ is isomorphic to some subgroup of the symmetric group $S_n$, where $n$ is the cardinality of the index set $I$. An automorphism which preserves the grading maps a subspace $L_i$ to a subspace $L_j = gL_i$ of the same dimension. Particularly, for gradings $\Gamma_1$ and $\Gamma_2$, the two dimensional subspaces $N_0$ and $K(0,0,1)$ must be mapped onto themselves. Thus only permutations of the remaining subspaces are possible. There are $6! = 720$ of such permutations. The normalizers are in fact much smaller. The gradings $\Gamma_3$ and $\Gamma_4$ consist of one-dimensional subspaces only, hence there are $8!$ permutations of these subspaces. But again, only very few of this enormous number of permutations correspond to an automorphism, as will be seen below.

Let $B_1$ and $B_2$ denote the permutation matrices

$$
B_1 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
$$

and $B_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}$

and $H$ the diagonal matrix $H = diag(1, i, i)$.

- **The normalizer of the MAD–group $\mathcal{G}_1$** contains besides $\mathcal{G}_1$ the transposition $Out_I$ and the automorphisms $Ad_{B_i}$, where $B_i$ are permutation matrices. Inner automorphisms of $N(\mathcal{G}_1) / \mathcal{G}_1$ form the symmetry group $S_3$. It is easy to check that such automorphisms belong to the normalizer. To prove that any other element of the normalizer is only their composition, is only somewhat more laborious. The conclusion is:

$$
N(\mathcal{G}_1) / \mathcal{G}_1 \text{ has 12 elements and is generated by } Out_I, Ad_{B_1}, Ad_{B_2}.
$$

- **The normalizer of the MAD–group $\mathcal{G}_2$** contains besides $\mathcal{G}_2$ the automorphisms $Ad_{B_1}$, $Ad_{B_2}$ and $Ad_H$. We thus conclude:

$$
N(\mathcal{G}_2) / \mathcal{G}_2 \text{ has 18 elements and is generated by } Ad_{B_1}, Ad_{B_2}, Ad_H.
$$

- **The normalizer of the MAD–group $\mathcal{G}_3$** contains besides $\mathcal{G}_3$ the automorphisms $Ad_{B_2}$ and $Ad_H$. These commuting automorphisms are of order two (in the quotient group) and generate the normalizer. The conclusion is:

$$
N(\mathcal{G}_3) / \mathcal{G}_3 \text{ is isomorphic to } Z_2 \times Z_2, \text{ has 4 elements and is generated by } Ad_{B_2}, Ad_H.
$$

- **The normalizer of the MAD–group $\mathcal{G}_4$** needs for its description the Sylvester matrix $S$ and the diagonal matrix $D = diag(1, 1, e^{2\pi i/3})$. The Sylvester matrix is the symmetric matrix which transforms the generalized Pauli matrices $P$ and $Q$ into each
other, i.e. $SPS^{-1} = Q$. It is curious to note that the subgroup of $\mathcal{N}(G_4)$ formed by the inner automorphisms is isomorphic to the group $SL(2, Z_3)$ of order 24. Finally we can conclude:

$$\mathcal{N}(G_4)/G_4$$

has 48 elements and is generated by $Out_I, Ad_S, Ad_D$.

Unlike the cases of the groups $G_1, G_2$ and $G_3$, it is a nontrivial task to show that the normalizer of $G_4$ is generated by the three automorphisms given above. The detailed proof of this statement can be found in [11] and concerns MAD–groups generated by generalized Pauli matrices in the general case of $sl(n, C)$.

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