Exact solution
to the Landau-Lifshitz equation
in a constant electromagnetic field

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Abstract

We are interested in the motion of a classical charge acted upon an external constant electromagnetic field where the back reaction of the particle’s own field is taken into account. The Landau-Lifshitz approximation to the Lorentz-Abraham-Dirac equation is solved exactly and in closed form. It is shown that the ultrarelativistic limit of the Landau-Lifshitz equation for a radiating charge is the equation for eigenvalues and eigenvectors of the external electromagnetic field tensor.

1 Introduction

There has been considerable interest in the classical motion of a radiating charge in strong electromagnetic field which is stimulated by various astrophysical situations such as stellar jets, cosmic rays, radiation from pulsars, etc. The other region of applicability is the behavior of charged particles in man-made devices, such as Penning traps and accelerators. There has been considerable interest in dynamics of a charged particle exposed to an ultraintense laser field [1, 2]. The radiation damping plays dominant role in ultrarelativistic regime.

The most prominent and widely accepted equation of motion of a point charged particle acted upon an external force as well as its own electromagnetic field is the Lorentz-Abraham-Dirac (LAD) equation [3]:

\[ a^\mu = \frac{1}{m} f^\mu_{\text{ext}} + \tau_0 \left[ \dot{a}^\mu - (a \cdot a) u^\mu \right]. \] (1.1)

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The factor by which the Abraham radiation reaction 4-vector is scaled,
\[ \tau_0 = \frac{2e^2}{3mc^3}, \] (1.2)
is a constant with dimension of time, whose numerical value for the electron is
\[ \tau_0 = 6.24 \cdot 10^{-24} \text{s}. \] Because of the third time derivative, the equation is beyond the Newtonian class of equations of motions which completely specify the trajectory of a particle once the initial position and velocity are given. LAD possesses pathological solutions [4, 5, 6], such as runaway solution (when acceleration grows exponentially with time) and preacceleration (when acceleration begins to increase prior the time at which the external force switches on).

Using the successive iteration, Landau and Lifshitz [7, §76] (LL) replace the square of acceleration by the negative scalar product \( (u \cdot \dot{a}) \) and substitute \( m^{-1} \frac{df_{\mu \nu}^{\mu}}{d\tau} \) for the derivative of particle’s acceleration:
\[ a^\mu = \frac{1}{m} f_{\mu \nu}^{\mu \nu} + \tau_0 \left( \delta^\mu_\nu + u^\mu u_\nu \right) \frac{1}{m} \frac{df_{\mu \nu}^\mu}{d\tau}. \] (1.3)
The LL equation is of Newtonian class which avoids the nonphysical solutions of the LAD [8] (see also Refs. [9] and [10]). The rigorous derivation of the complete first-order correction to the Lorentz force equation in the external field was recently given [11] using perturbation theory. For a structureless point particle, Eq. (1.3) has been obtained.

There are several analytic solutions to the Landau-Lifshitz equation. Ares de Parga and Mares [12] give the solution to the LL equation for the radiating charge in static and spatially uniform electromagnetic field. Rivera and Villarroel [13] find out the solutions to LL equation for two circling charges confined by specific electromagnetic field at the opposite ends of diameter. Rajeev [14] solves the non-relativistic approximation of the LL equation for a charged particle moving in a Coulomb potential. Di Piazza [15] gives the exact solution to the LL equation for a charge acted upon by a plane wave of arbitrary shape and polarization. Similar problem has been considered by Hadad et al. in Ref. [1] devoting to theoretical studies of radiation emission from free electrons in very intense laser pulses.

Ares de Parga and Mares [12] present the solution to Eq. (1.3) in the form \( u^\mu(\tau) = \eta(\tau) f^\mu(\tau) \) where \( u^\mu \) are components of particle’s 4-velocity, scalar function \( \eta(\tau) \) represents the damping effect, and 4-vector \( f^\mu(\tau) = \exp(\alpha(\tau)) f_L^\mu(\tau) \) is proportional to a solution \( f_L^\mu(\tau) \) of the Lorentz force equation. Because exponential factor, the expression differs from Chen’s solution [16, eq.(11)] to the “truncated” LL equation [16, eqs.(9),(10)] where the intermediate term is omitted.

In this paper, we study the Ares de Parga and Mares solution [12, eq. (12)]. In the present paper we use the Heaviside-Lorentz system of units and the mostly
plus Minkowski metrics $g_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$, and the equation looks as follows

$$u^\mu(\tau) = \frac{f^\mu(\tau)}{\sqrt{1 - 2\tau_0 \frac{e^2}{m^2} \int_0^\tau ds f_\nu(s) F_\nu \alpha F_{\alpha\beta} f^\beta(s)}}, \quad (1.4)$$

where

$$f^\mu(\tau) = \exp\left[\left(\frac{e}{m} F^\mu_\alpha + \tau_0 \frac{e^2}{m^2} F^\nu_\alpha F^\nu_\beta\right)\tau\right] f^\alpha(0). \quad (1.5)$$

The exponential function of the combination of the linear and squared applied field can be presented as series in powers of the electromagnetic field tensor $F$. To sum up the series we follow the algebraic approach sketched in Ref. [10]. According to the Hamilton-Cayley theorem, the Faraday tensor (2.10) satisfies its own characteristic equation

$$F^4 + SF^2 - \frac{1}{4}P^2 I = 0, \quad (1.6)$$

where $P$ and $S$ are invariants of this tensor given by eqs. (2.5). Hence, any power of $F$ may be reduced to a linear combination of the unit matrix $I$, $F$ itself, its square $F^2$, and its cube $F^3$. The goal of this paper is to present the formal solution (1.4) as a linear combination of $n$-tuples of $F$ no higher than cubic.

The paper is organized as follows. In Section 2, we review the basic algebraic properties of the electromagnetic field tensor such as its eigenvalues and eigenvectors. Following Ref. [17], we construct two projection operators which allow us to split the Minkowski space into two sets of mutually orthogonal planes. (The plane $(x^0 x^3)$ and its orthogonal $(x^1 x^2)$ offer a good illustration of this splitting.) In Section 3, we apply the projection technique for solving of the LL equation for a radiating charge in a static and spatially uniform electromagnetic field. We split the equation into two equations described the orbits in planes mentioned above. The equations are completely uncoupled to each other and we solve them separately. In Section 4, we study three main examples: motion in the field of magnetic type, in the field of electric type, and in the crossed field. In Section 5, we summarize the main ideas and results.

## 2 Fradkin’s projection operators

At each point of Minkowski space $\mathbb{M}_4$, the electromagnetic field is determined by the Faraday 2-form $\hat{F}$ and its Hodge dual 2-form $^*\hat{F}$. If the Minkowski rectangular coordinates are adapted, the state of electromagnetic field $\hat{F} = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta$ at point $x \in \mathbb{M}_4$ is specified by three components $(E^1, E^2, E^3)$ of electric field and three components $(B^1, B^2, B^3)$ of magnetic field:

$$(F_{\alpha\beta}) = \begin{pmatrix}
0 & -E^1 & -E^2 & -E^3 \\
E^1 & 0 & B^3 & -B^2 \\
E^2 & -B^3 & 0 & B^1 \\
E^3 & B^2 & -B^1 & 0
\end{pmatrix}. \quad (2.1)$$
Its Hodge dual $\hat{F} = \frac{1}{2} \hat{\omega}(\mathbf{F})$ is a 2-form, whose components are

$$( ^*F_{\mu\nu} ) = \begin{pmatrix}
0 & B^1 & B^2 & B^3 \\
-B^1 & 0 & E^3 & -E^2 \\
-B^2 & -E^3 & 0 & E^1 \\
-B^3 & E^2 & -E^1 & 0
\end{pmatrix}. \tag{2.2}$$

Before we introduce the projection operators providing by the electromagnetic field tensor (2.1), we recall its basic algebraic properties.

### 2.1 Invariants

The matrices (2.1) and (2.2) describe not only the electromagnetic field. They bear the imprint of the inertial frame which is used to determine the components of electric and magnetic fields. There exist two scalar functions which do not depend on the choice of basis. To construct the invariants of the electromagnetic field, we apply the Hodge star operator to 4-forms $\hat{F} \wedge \hat{F}$ and $\hat{F} \wedge ^*\hat{F}$. The manipulation results in two 0-forms

$$\mathcal{P} = ^* (\hat{F} \wedge \hat{F}) = \frac{1}{2} F_{\mu\nu}^* F^{\mu\nu}, \tag{2.3}$$

$$\mathcal{S} = ^* (\hat{F} \wedge ^*\hat{F}) = \frac{1}{2} F_{\mu\nu} F^{\mu\nu}. \tag{2.4}$$

For practical considerations, it is simple and convenient to express $\mathcal{P}$ and $\mathcal{S}$ in terms of three-vectors $\mathbf{E}$ and $\mathbf{B}$:

$$\mathcal{P} = 2 (\mathbf{E} \cdot \mathbf{B}), \quad \mathcal{S} = \mathbf{B}^2 - \mathbf{E}^2. \tag{2.5}$$

What are the consequences of the fact that the invariants “overlook” any transformations of basis? Electromagnetic fields are divided into classes according to whether $\mathcal{P} = 0$ or $\mathcal{P} \neq 0$ as well as whether $\mathbf{E}^2$ or $\mathbf{B}^2$ is greater. If the electric field and the magnetic field are mutually orthogonal in a given inertial frame, they are orthogonal in any other frame of reference. Indeed, despite a change of basis caused by a Poincaré transformation, the value of scalar product of three-vectors $\mathbf{E}$ and $\mathbf{B}$ remains zero. The other invariant is the difference between squared magnitudes of magnetic and electric fields. So, if they are equal to each other in a given Lorentz frame, they are equal in any other frame of reference. The commonly used and widely accepted classification is as follows:

(A) $\mathcal{P} = 0, \quad \mathcal{S} \neq 0$.

Since the sign of $\mathcal{S}$ is invariant, there are two possibilities:
(a) If $S < 0$, then in all reference frames the magnitude of electric field is greater than the magnitude of magnetic field. The field is said to be of electric type.

(b) If $S > 0$, then in all reference frames the magnitude of magnetic field is greater than that of electric field. The field is said to be of magnetic type.

(B) $\mathcal{P} \neq 0$, $S \neq 0$. Such a field belongs to the set of crossed fields.

(C) $\mathcal{P} = 0$, $S = 0$.

If both the invariants $S$ and $\mathcal{P}$ are equal to zero, we deal with the null field.

2.2 Eigenvalues and eigenvectors

Eigenvalues and eigenvectors of the electromagnetic field tensor are crucially important parts of the projection technique [17]. The eigenvalues of \((1)\) tensor $F^{\mu\beta} = g^{\mu\alpha}F_{\alpha\beta}$ are the roots of the fourth-degree polynomial:

$$\eta^4 + S\eta^2 - \frac{1}{4}\mathcal{P}^2 = 0,$$

(2.6)

where $S$ and $\mathcal{P}$ are invariants (2.5). The set of eigenvalues consists of two pairs, one real and the other imaginary

$$\{\eta\} = \{+b, -b, +ia, -ia\},$$

(2.7)

where $a$ and $b$ are the scalar functions of invariants:

$$a = \sqrt{\frac{1}{2} \left( S + \sqrt{S^2 + \mathcal{P}^2} \right)},$$

(2.8)

$$b = \sqrt{\frac{1}{2} \left( -S + \sqrt{S^2 + \mathcal{P}^2} \right)}.$$ 

(2.9)

If $\mathcal{P} = 0$, the set (2.7) contains doubly degenerate zero eigenvalue

- for a field of electric type ($S < 0$): $\{\eta\} = \{+\sqrt{-S}, -\sqrt{-S}, 0, 0\}$;
- for a field of magnetic type ($S > 0$): $\{\eta\} = \{0, 0, +i\sqrt{S}, -i\sqrt{S}\}$.

A null electromagnetic field (both the invariants $\mathcal{P} = 0$ and $S = 0$) is characterized by the quadruply degenerate zero eigenvalue.

The eigenvector of the square matrix

$$(F^\alpha_{\, \nu}) = \begin{pmatrix}
0 & E^1 & E^2 & E^3 \\
E^1 & 0 & B^3 & -B^2 \\
E^2 & -B^3 & 0 & B^1 \\
E^3 & B^2 & -B^1 & 0
\end{pmatrix},$$

(2.10)
is the non-zero vector, say \( g \), that after being transformed by the matrix remains parallel to the original vector

\[ F^\mu \alpha g^\alpha = \eta(\alpha)g^\mu. \]  

(2.11)

The factor \( \eta(\alpha) \) by which the eigenvector is scaled is a particular element of the eigenvalue set (2.7). If \( g \) is an eigenvector of the field tensor with eigenvalue \( \eta(\alpha) \), then the multiplication of \( g \) on any number is also an eigenvector of \( F \) with the same eigenvalue. All the eigenvectors of the Faraday tensor are four-vectors of null lengths.

2.2.1 Projection operators

According to the Hamilton-Cayley theorem, the Faraday tensor (2.10) satisfies characteristic equation (1.6). This matrix polynomial can be presented as the product of four factors [17, eq. (25)]

\[ (F - \eta(\alpha)I)(F + \eta(\alpha)I)\left( F - \frac{iab}{\eta(\alpha)}I \right)\left( F + \frac{iab}{\eta(\alpha)}I \right) = 0, \]  

(2.12)

where \( \eta(\alpha) \) is a particular element of the eigenvalue set (2.7). The equation suggests the form of the projection operator producing eigenvector with a given eigenvalue \( \eta(\alpha) \). Since the eigenvector satisfies the equation \( F - \eta(\alpha)I = 0 \), the desired matrix operator should be proportional to the product of the remaining matrices [16, eq. 26]

\[ P_{(\alpha)} = \prod_{\kappa \neq \alpha}^4 \left( \frac{F - \eta(\kappa)I}{\eta(\alpha) - \eta(\kappa)} \right). \]  

(2.13)

Fradkin [17] rewrites the characteristic equation (2.12) in the form

\[ (F^2 - b^2I)(F^2 + a^2I) = 0, \]  

(2.14)

and fabricates two operators

\[ O^{(a)} = -\frac{F^2 - b^2I}{a^2 + b^2}, \]  

(2.15)

\[ O^{(b)} = \frac{F^2 + a^2I}{a^2 + b^2}, \]  

(2.16)

which satisfy the projection algebra

\[ O^{(a)}O^{(a)} = O^{(a)}, \]  

\[ O^{(b)}O^{(b)} = O^{(b)}, \]  

\[ O^{(a)} + O^{(b)} = I, \]  

(2.17)
and
\[
O^{(a)} O^{(b)} = O^{(b)} O^{(a)} = 0. \tag{2.18}
\]
The operator \(O^{(a)}\) when acting on an arbitrary four-vector, say \(g\), produces the four-vector
\[
g^{(a)} = O^{(a)} g,
\]
which is a linear combination of eigenvectors associated with the imaginary eigenvalues \(+ia\) and \(-ia\). Indeed, the operator
\[
P_{(+a)} = \frac{F + iaI}{2ia} O^{(a)}, \tag{2.19}
\]
transforms the 4-vector \(g\) into eigenvector \(g_{(+a)}\) with eigenvalue \(+ia\) while
\[
P_{(-a)} = -\frac{F - iaI}{2ia} O^{(a)}, \tag{2.20}
\]
produces eigenvector \(g_{(-a)}\) with negative imaginary eigenvalue \(-ia\). Since \(P_{(+a)} + P_{(-a)} = O^{(a)}\), the projective operator \(\tag{2.15}\) separates two-dimensional subspace \(M_{x}^{(a)} \subset T_x M_4\) which is spanned by “imaginary” eigenvectors \(g_{(+a)}\) and \(g_{(-a)}\).

Analogously, \(O^{(b)}\) builds two-dimensional subspace \(M_{x}^{(b)} \subset T_x M_4\) spanned by “real” eigenvectors \(g_{(+b)} = P_{(+b)} g\) and \(g_{(-b)} = P_{(-b)} g\) where
\[
P_{(+b)} = \frac{F + bI}{2b} O^{(b)}, \quad P_{(-b)} = -\frac{F - bI}{2b} O^{(b)}. \tag{2.21}
\]
It is obviously that their sum is the projective operator \(\tag{2.16}\).

Since eq. \(\tag{2.18}\), the four-vectors \(g^{(a)} = O^{(a)} g\) and \(g^{(b)} = O^{(b)} g\) are orthogonal to each other:
\[
(g^{(a)} \cdot g^{(b)}) = 0.
\]
At each point \(x \in M_4\) a vector from the tangent space \(T_x M_4\) can be presented as a sum of vectors from \(M_x^{(a)}\) and \(M_x^{(b)}\).

## 3 Charge in static uniform electromagnetic field

In this Section we apply the Fradkin’s operators \(\tag{2.15}\) and \(\tag{2.16}\) to the Landau-Lifshitz equation for a radiating charge in a constant electromagnetic field. If the field strengths which constitute the tensor \(\tag{2.10}\) are static and spatially uniform, its eigenvectors do not change with points of Minkowski space. The subspaces spanned by these eigenvectors constitute foliation of \(M_4\) by two-dimensional planes. Traveling from point to point, we construct the coordinate grid from
these planes which covers all the flat spacetime. The particle’s world line may be decomposed into two orbits in these mutually orthogonal two-dimensional sheets.

The Landau-Lifshitz equation where the proper time derivative of the 4-acceleration is replaced by the derivative of the contraction of tensor \(2.10\) and particle’s 4-velocity

\[
\frac{1}{m} \frac{df_{\text{ext}}^\mu}{d\tau} = \frac{e}{m} F_{\alpha}^\mu a^\alpha = \frac{e^2}{m^2} F_{\alpha}^\mu F^\alpha_{\beta} u^\beta. \tag{3.1}
\]

is the so-called Herrera equation \([12, \text{eq. (4)}]\)

\[
\dot{u}^\mu = \frac{e}{m} F_{\alpha}^\mu u^\alpha + \tau_0 (\delta^\mu_\nu + u^\mu u_\nu) \frac{e^2}{m^2} F^\nu_\alpha F^\alpha_{\beta} u^\beta. \tag{3.2}
\]

At first, we decompose the particle’s four-velocity \(u\) into two mutually orthogonal four-vectors

\[
u^\alpha_{(a)} = O^{(a)} u^\mu \quad \text{and} \quad \nu^\alpha_{(b)} = O^{(b)} u^\mu. \tag{3.3}
\]

Since \(u = u_{(a)} + u_{(b)}\) and \((u \cdot u) = -1\), then \(u^2_{(a)} + u^2_{(b)} = -1\). The projection operators \((2.15)\) and \((2.16)\) commute with differential operator \(d/d\tau\) (for the constant field only) as well as with the field tensor \(F\) (for any field). To manipulate with the product \(F \cdot F\) we use the relations

\[
F^2 O^{(a)} = O^{(a)} F^2 \quad \text{and} \quad F^2 O^{(b)} = O^{(b)} F^2 = -a^2 O^{(a)}, \quad \text{and} \quad b^2 O^{(b)}, \tag{3.4}
\]

which can be readily derived from the characteristic equation \((2.14)\). So, acted on the expression \((3.1)\) by operators \(O^{(a)}\) and \(O^{(b)}\), one after another, we obtain

\[
O^{(a)} \frac{1}{m} \frac{df_{\text{ext}}^\mu}{d\tau} = -\omega_a^2 u^\mu_{(a)}, \quad \text{and} \quad O^{(b)} \frac{1}{m} \frac{df_{\text{ext}}^\mu}{d\tau} = \lambda_b^2 u^\mu_{(b)},
\]

where

\[
\omega_a = \frac{e}{m} a, \quad \lambda_b = \frac{e}{m} b,
\]

and constants \(a\) and \(b\) are given by the expressions \((2.8)\) and \((2.9)\), respectively. It is easy to show that contraction of 4-vector \((3.1)\) with particle’s 4-velocity is as follows:

\[
\frac{1}{m} \frac{df_{\text{ext}}^\mu}{d\tau} u_{\mu} = -\omega_a^2 u^2_{(a)} + \lambda_b^2 u^2_{(b)}; \tag{3.5}
\]

where \(u^2_{(a)}\) and \(u^2_{(b)}\) are the squared norms of 4-velocity projections.
Inserting these in Eq. (3.2), we split it into two second-order differential equations

\[
\begin{align*}
\frac{du_a^\mu}{d\tau} &= \frac{e}{m} F^\mu_\alpha u^\alpha_{(a)} - \omega_0 \left[1 + u^2_{(a)}\right] u^\mu_{(a)}, \\
\frac{du_b^\mu}{d\tau} &= \frac{e}{m} F^\mu_\alpha u^\alpha_{(b)} + \omega_0 \left[1 + u^2_{(b)}\right] u^\mu_{(b)},
\end{align*}
\]

(3.6) \quad (3.7)

where constant

\[
\omega_0 = \tau_0 \left(\omega_a^2 + \lambda_b^2\right)
\]

determines the intensity of the radiation damping. We take into account that \(u^2_{(a)} + u^2_{(b)} = -1\).

The two projections of Eq. (3.2) are completely uncoupled with each other and we solve them separately. Following Eq. (11) in Ref. [16], we express the solutions in the form

\[
\begin{align*}
u^\mu_{(a)}(\tau) &= \sqrt{A(\tau)} f^\mu_{(a)}(\tau), \\
u^\mu_{(b)}(\tau) &= \sqrt{-B(\tau)} f^\mu_{(b)}(\tau),
\end{align*}
\]

(3.9)

where 4-vectors \(f_{(a)}\) and \(f_{(b)}\) are normalized to +1 and −1, respectively. Inserting \(u_{(a)}\) and \(u_{(b)}\) in Eqs. (3.6) and (3.7), respectively, we derive the simple equations on the factors \(A\) and \(B\)

\[
\begin{align*}
\frac{1}{2} \frac{dA}{d\tau} &= -\omega_0 [1 + A] A, \\
\frac{1}{2} \frac{dB}{d\tau} &= \omega_0 [1 + B] B.
\end{align*}
\]

(3.10)

These ordinary differential equations are supplemented with initial conditions

\[
\begin{align*}
A(0) &= u^2_{(a)}(0), & B(0) &= u^2_{(b)}(0), \\
&= \alpha, & = -\beta.
\end{align*}
\]

The positive constants \(\alpha\) and \(\beta\) are related by the condition \(\beta = \alpha + 1\).

Equations (3.10) can be solved in a simple way and we do not go into detail. Contrary to the unperturbed Lorentz-force equation, the squared norms \(u^2_{(a)} := A\) and \(u^2_{(b)} := B\) of four-velocity projections change with time:

\[
\begin{align*}
u^2_{(a)}(\tau) &= \frac{\alpha}{\beta e^{2\omega_0 \tau} - \alpha}, \\
\nu^2_{(b)}(\tau) &= \frac{\beta}{\beta - \alpha e^{-2\omega_0 \tau}}.
\end{align*}
\]

(3.11) \quad (3.12)

It is worth noting that the authors [12, Eqs.(18)-(21)] derive the same expressions in Applications where the pure magnetic field and pure electric field have been considered.
Inserting Eqs. (3.9) into Eqs. (3.6) and (3.7), we see that the 4-vectors $f^\mu_{(a)}$ and $f^\mu_{(b)}$ satisfy the unperturbed Lorentz-force equation

$$\frac{df^\mu_{(a)}}{d\tau} = \frac{e}{m} F^\mu_{\alpha a} f^\alpha_{(a)}, \quad \frac{df^\mu_{(b)}}{d\tau} = \frac{e}{m} F^\mu_{\alpha b} f^\alpha_{(b)}.$$  

The sum of the two equations is the equation on the total 4-vector $f = f_{(a)} + f_{(b)}$

$$\frac{df^\mu}{d\tau} = \frac{e}{m} F^\mu_{\alpha} f^\alpha.$$  

In comparison to the Ares de Parga and Mares solution (1.5), the solution to this equation simplifies

$$f^\mu(\tau) = \exp\left(\frac{e}{m} F^\mu_{\alpha} \tau\right) f^\alpha(0). \quad (3.13)$$

The reason of the clear distinction is that the particle’s 4-velocity is the sum of two projections of 4-vector $f(\tau)$, $f_{(a)}(\tau)$, and $f_{(b)}(\tau)$, scaled by two quite different factors, $\sqrt{A}$ and $\sqrt{-B}$, (see Eqs. (3.9)). While in the the Ares de Parga and Mares scheme the 4-velocity is supposed to be proportional to $f$ itself.

Adapting the Rosen’s result [18, eq.(1.8)] for a Hermitian matrix to the Faraday matrix (2.10), Shen [16, Appendix B] presents the action of exponential matrix operator as the combination of a complete set of eigenvalues (2.7) and its appropriate projection operators, one for each eigenvalue

$$f(\tau) = \exp\left(\frac{e}{m} F_{\tau} \right) f(0)$$

$$= \sum_{\alpha=1}^{4} \exp\left(\frac{e}{m} \eta_{(\alpha)} \tau\right) P_{(\alpha)} f(0). \quad (3.14)$$

According to eqs. (3.9), the “a”-projection of particle’s 4-velocity is proportional to the “a”-projection of auxiliary 4-vector $f(\tau)$

$$u_{(a)}(\tau) = \sqrt{A} f_{(a)}(\tau)$$
$$= \sqrt{A} \left(e^{i\alpha_{a} \tau} P_{(+a)} + e^{-i\alpha_{a} \tau} P_{(-a)}\right) f(0),$$

while $u_{(b)}$ is proportional to $f_{(b)}$:

$$u_{(b)}(\tau) = \sqrt{-B} f_{(b)}(\tau)$$
$$= \sqrt{-B} \left(e^{\lambda_{a} \tau} P_{(+b)} + e^{-\lambda_{a} \tau} P_{(-b)}\right) f(0).$$

Because of $A(\tau)\big|_{\tau=0} = \alpha$ and $B(\tau)\big|_{\tau=0} = -\beta$, the initial values $f^\alpha_{(a)}(0) = u^\alpha_{(a)}(0)/\sqrt{\alpha}$ and $f^\alpha_{(b)}(0) = u^\alpha_{(b)}(0)/\sqrt{\beta}$. 

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Substituting the right-hand sides of eqs. (2.19)-(2.21) for the projection operators \( P_{(o)} \), we finally obtain the solutions to the Landau–Lifshitz equation (3.2):

\[
\begin{align*}
    u^\mu_{(a)}(\tau) &= \frac{1}{\sqrt{\beta e^{2\omega_0\tau} - \alpha}} \left[ u^\mu_{(a)}(0) \cos(\omega_a \tau) + a^{-1} F^\mu_{\alpha} u^\alpha_{(a)}(0) \sin(\omega_a \tau) \right], \\
    u^\mu_{(b)}(\tau) &= \frac{1}{\sqrt{\beta - \alpha e^{-2\omega_0\tau}}} \left[ u^\mu_{(b)}(0) \cosh(\lambda_b \tau) + b^{-1} F^\mu_{\alpha} u^\alpha_{(b)}(0) \sinh(\lambda_b \tau) \right].
\end{align*}
\]

(3.15)

The radiation reaction free solutions in between the square brackets coincide with the Fradkin’s solutions [17, eqs.(3.8),(3.9)] to the Lorentz force equation for static and spatially uniform electromagnetic field.

The squared norm (3.11) of \( O^{(a)} \)-projection of particle’s four-velocity decreases exponentially as the time parameter increases. It describes the decaying circular-like orbit in plane \( M^{(a)} \). We see that the radiation damping suppresses the particle’s oscillation.

The squared norm (3.12) of \( O^{(b)} \)-projection of particle’s four-velocity tends to \(-1\) as the time parameter increases. Contrary to oscillatory motion, the self force almost does not influence hyperbolic motion. In specific case of purely electric field \((\omega_a = 0)\), one can choose \( u^\mu_{(a)}(0) = 0\). In this case, the constant \( \alpha = 0 \) and the factor before the squared brackets in eq. (3.15) becomes 1. The radiation reaction force on a charged particle vanishes when the particle accelerates uniformly [19].

The \( O^{(b)} \)-projection of particle’s 4-velocity does not change under the influence of the self force even if an external field is very powerful. The greater is the argument \( \lambda_b \tau \) of hyperbolic functions, the closer are their magnitudes:

\[
\sinh(\lambda_b \tau) \approx \pm \cosh(\lambda_b \tau).
\]

So, if \( \lambda_b < 0 \) (negative charge), then \( O^{(b)} \)-projection of four-velocity (3.15) approaches the null 4-vector

\[
\begin{align*}
    u^\mu_{(b)}(\tau) &\approx \frac{1}{\sqrt{\beta - \alpha e^{-2\omega_0\tau}}} \left[ u^\mu_{(b)}(0) \cosh(\lambda_b \tau) - b^{-1} F^\mu_{\alpha} u^\alpha_{(b)}(0) \cosh(\lambda_b \tau) \right] \\
    &= CP_{(-b)} u^\alpha(0),
\end{align*}
\]

where

\[
C = 2 \frac{e^{\omega_0\tau}}{\sqrt{\beta e^{2\omega_0\tau} - \alpha}} \cosh(\lambda_b \tau),
\]

and \( P_{(-b)} \) is the projection operator (2.21) which produces eigenvector with negative real value \(-b\). If \( \lambda_b > 0 \) (positive charge), then \( O^{(b)} \)-projection of four-velocity (3.15) tends to eigenvector with positive real value:

\[
\begin{align*}
    u^\mu_{(b)}(\tau) &\approx \frac{1}{\sqrt{\beta - \alpha e^{-2\omega_0\tau}}} \left[ u^\mu_{(b)}(0) \cosh(\lambda_b \tau) + b^{-1} F^\mu_{\alpha} u^\alpha_{(b)}(0) \cosh(\lambda_b \tau) \right] \\
    &= CP_{(+b)} u^\alpha(0).
\end{align*}
\]
When the time increases, the orbit in plane $\mathcal{M}^{(b)}$ approaches the null ray along one of the “real” eigenvectors of the electromagnetic field tensor. Therefore, the equation for eigenvalues and eigenvectors of the (external) electromagnetic field tensor is the ultrarelativistic limit of the LL equation. Moreover, this is true not only for a constant field, but for a field of arbitrary configuration [20, Chap.9].

### 3.1 World line

For completeness, we give the expressions for the particle’s world line

$$\zeta : \mathbb{R} \rightarrow \mathbb{M}_4$$

$$\tau \mapsto (z^\alpha(\tau))$$ (3.16)

when the initial velocity is $u(0) = u_{(a)}(0) + u_{(b)}(0)$ and the initial position $z(0) = z_{(a)}(0) + z_{(b)}(0)$. Zeroth component $z^0(\tau)$ gives the laboratory time $t$.

Integrating Eqs. (3.15) we obtain the two different projections of the position 4-vector. For future convenience, we change the variable $x = \omega_0 \tau$ and introduce new parameter $\lambda = \sqrt{\alpha/\beta} < 1$


to derive the “a”-projection, we express the time-dependent amplitude under the integral signs in Eq. (3.17) as hypergeometric series [24, Eq.(15.1.8)]:

$$\lambda e^{-x} \sqrt{1 - (\lambda e^{-x})^2} = \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2} + n\right)}{\Gamma\left(\frac{1}{2}\right)n!} (\lambda e^{-x})^{2n+1}. \quad (3.19)$$

To perform calculations we use the formulae:

$$\int_{0}^{x} dt e^{qt} \cos(\omega t) = \frac{e^{qx} \sin(\omega x + \phi) - \sin \phi}{\sqrt{\omega^2 + q^2}},$$

$$\int_{0}^{x} dt e^{qt} \sin(\omega t) = -\frac{e^{qx} \cos(\omega x + \phi) - \cos \phi}{\sqrt{\omega^2 + q^2}},$$
where phase $\phi$ is given by
\[
\cos \phi = \frac{\omega}{\sqrt{\omega^2 + q^2}}, \quad \sin \phi = \frac{q}{\sqrt{\omega^2 + q^2}}.
\]
Inserting these into eq. (3.17), we derive the orbit in the plane $\mathcal{M}^{(a)}$
\[
z_{(a)}^\mu(\tau) = z_{(a)}^\mu(0) + \frac{1}{\sqrt{\alpha}} \left[ u_{(a)}^\mu(0) I_s - b^{-1} F_{\alpha}^\mu u_{(a)}^\alpha(0) I_c \right],
\] where symbols $I_s$ and $I_c$ denote the series
\[
I_s = \sum_{n=0}^{\infty} a_n \left[ e^{-(2n+1)\omega_0 \tau} \sin(\omega_a \tau + \phi_n) - \sin \phi_n \right],
\]
\[
I_c = \sum_{n=0}^{\infty} a_n \left[ e^{-(2n+1)\omega_0 \tau} \cos(\omega_a \tau + \phi_n) - \cos \phi_n \right],
\]
with coefficients
\[
a_n = \frac{\Gamma(\frac{1}{2} + n)}{\Gamma(\frac{1}{2} + n) \sqrt{\omega_a^2 + [(2n + 1)\omega_0]^2}}. \tag{3.21}
\]
Thus, the “a”-projection behaves like ensemble of damped harmonic oscillators with common frequency $\omega_a$ and different phases
\[
\cos \phi_n = \frac{\omega_a}{\sqrt{\omega_a^2 + [(2n + 1)\omega_0]^2}} \quad \text{and} \quad \sin \phi_n = -\frac{(2n + 1)\omega_0}{\sqrt{\omega_a^2 + [(2n + 1)\omega_0]^2}}. \tag{3.22}
\]
In the plane $\mathcal{M}^{(a)}$, the charged particle spirals inward, i.e., moves in a continuous curve that gets nearer to central point.

Now, integrating equation (3.18) we obtain the “b”-projection of position 4-vector. We expand the the time-dependent amplitude as hypergeometric series (3.19) and apply the formulae
\[
\int_0^x dt \ e^{\lambda t} \cosh(\lambda t) = \frac{e^{\lambda x} \sinh(\lambda x - \psi) + \sinh \psi}{\sqrt{\lambda^2 - q^2}},
\]
\[
\int_0^x dt \ e^{\lambda t} \sinh(\lambda t) = \frac{e^{\lambda x} \cosh(\lambda x - \psi) - \cosh \psi}{\sqrt{\lambda^2 - q^2}},
\]
where phase $\psi$ is given by
\[
\cosh \psi = \frac{\lambda}{\sqrt{\lambda^2 - q^2}}, \quad \sinh \psi = \frac{q}{\sqrt{\lambda^2 - q^2}}.
\]
The result is
\[
z_{(b)}^\mu(\tau) = z_{(b)}^\mu(0) + \frac{1}{\sqrt{\beta}} \left[ u_{(b)}^\mu(0) J_s + b^{-1} F_{\alpha}^\mu u_{(b)}^\alpha(0) J_c \right], \tag{3.23}
\]
where \( J_s \) and \( J_c \) are time-dependent series

\[
J_s = \sum_{n=0}^{\infty} b_n \left[ e^{-2n\omega_0\tau} \sinh(\lambda_b \tau - \psi_n) + \sinh \psi_n \right],
\]

\[
J_c = \sum_{n=0}^{\infty} b_n \left[ e^{-2n\omega_0\tau} \cosh(\lambda_b \tau - \psi_n) - \cosh \psi_n \right],
\]

with coefficients

\[
b_n = \frac{\Gamma\left(\frac{1}{2} + n\right)}{\Gamma\left(\frac{1}{2}\right)n!} \frac{\lambda_b^{2n}}{\sqrt{\lambda_b^2 - [2n\omega_0]^2}},
\]

(3.24)

\( n \)-th shift of the hyperbolic phase is given by

\[
cosh \psi_n = \frac{\lambda_b}{\sqrt{\lambda_b^2 - [2n\omega_0]^2}}, \quad \sinh \psi_n = -\frac{2n\omega_0}{\sqrt{\lambda_b^2 - [2n\omega_0]^2}}.
\]

A charged particle acted upon by a static spatially uniform electric field moves along hyperbola characterized by an inverse time constant \( e|E|/m \); a hyperbolic phase is determined by an initial velocity. Therefore, the “b”-projection looks like ensemble of identical charges with different (discrete) initial velocities moving in constant electric field. In the plane \( \mathcal{M}^{(b)} \), the charged particle exhibits an exponential proper time behavior: it asymptotically approaches the null ray aligned along one of two “real” eigenvectors of the electromagnetic field tensor.

Finally, using the relation \( z(\tau) = z_{(a)}(\tau) + z_{(b)}(\tau) \), the two preceding projections may be combined to give the total position 4-vector. To simplify future calculations, we present the product relations [17, Eqs.(2.5),(2.6)]

\[
F \cdot F = (b^2 - a^2)I + ^*F \cdot ^*F,
\]

(3.25)

\[
F \cdot ^*F = ^*F \cdot F = -\kappa ab I,
\]

(3.26)

where \( \kappa = \text{sign} \mathcal{P} \) and “dot” denotes the matrix product, e.g., \( F_{\mu \alpha} F^{\alpha \beta} \). After some algebra one can also deduce [17] eqs.(2.7)

\[
F \cdot F \cdot F = (b^2 - a^2)F - \kappa ab^*F.
\]

(3.27)

The Hamilton-Cayley equation (1.6) is not necessary in this calculations because the fourth power of \( F \) does not appear in Eqs. (3.20) and (3.23).

### 3.2 Radiation emission

The rate of energy-momentum loss due to radiation is given by the Abraham-Heaviside formula

\[
\frac{dP^\mu}{d\tau} = -\frac{2e^2}{3} (a \cdot a) u^\mu.
\]

(3.28)
Particle’s 4-velocity which satisfies the Lorentz-Abraham-Dirac equation (1.1) and its proper-time derivative as 4-acceleration should be substituted. According to Teitelboim’s careful examination [25], this is the energy-momentum carried by electromagnetic waves that detach themselves from the charge and lead an independent existence.

The zeroth component of 4-vector (3.28) is the rate of emitted energy which can be detected by distant devices. According to [1, eq. (35)], the rate with respect to laboratory time is of true physical sense:

$$\frac{dP^0}{dt} = -\frac{2e^2}{3}(a \cdot a).$$

(3.29)

To calculate it, we use the solution (3.15) to the Landau-Lifshitz approximation of the Lorentz-Abraham-Dirac equation. The square of its proper-time derivative is as follows:

$$(a \cdot a) = a^2_{(a)} + a^2_{(b)} = \lambda^2_0 + \frac{(\lambda e^{-\omega_0 \tau})^2}{1 - (\lambda e^{-\omega_0 \tau})^2} (\lambda^2_0 + \omega^2_0) + \frac{(\lambda e^{-\omega_0 \tau})^2}{1 - (\lambda e^{-\omega_0 \tau})^2} \omega^2_0,$$

(3.30)

where $\lambda = \sqrt{\alpha/\beta} < 1$. Finally, with a degree of accuracy sufficient for our purposes

$$\frac{dP^0}{dt} = -\frac{2e^2}{3} \lambda^2_0 - m\omega_0 \frac{(\lambda e^{-\omega_0 \tau})^2}{1 - (\lambda e^{-\omega_0 \tau})^2},$$

(3.31)

where $\omega_0$ is a constant defined by Eq. (3.8).

We see that the intensity of radiation strongly depends on time. (Strictly speaking, the proper time in the right-hand side of Eq. (3.31) should be substituted for the laboratory time. Since $t = z^0(\tau)$, the function $\tau(t)$ is inverse to the zeroth coordinate function $z^0(\tau)$ which is given by eqs. (3.20) and (3.23). It is of great importance that both the proper time and laboratory time are monotonic increasing quantities.) Initially, the synchrotron radiation and the longitudinal radiation present in almost equal quantities. But the radiation damping suppresses the “magnetic” rotation, so that the longitudinal radiation (this caused by uniformly accelerated charge) only survives. This is in line the main result by Fulton and Rohrlich [19, Eq.(3.8)]. So far as the synchrotron radiation is concerned, Shen [16] also states that it decreases with time as well as the influence of the radiation damping force.

4 Applications

We now discuss our main examples of dynamics of a radiating charge traveling in constant electromagnetic fields of magnetic type and of electric type as well
as in the crossed field. We demonstrate how the projection technique works and compare the results with these known in literature [12, 22, 23, 21, 10].

Assume that in the laboratory frame the state of the electromagnetic field at point \( x \in \mathcal{M}_4 \) is specified by three components \( (E^1, E^2, E^3) \) of electric field \( E \) and three components \( (B^1, B^2, B^3) \) of magnetic field \( B \). If the inertial frame moves with a velocity \( \mathbf{v} \) with respect to laboratory one, then the new strengths become

\[
E' = u^0 E - \frac{(u\mathbf{E})u}{u^0 + 1} + [u \times \mathbf{B}],
\]

\[
B' = u^0 B - \frac{(u\mathbf{B})u}{u^0 + 1} - [u \times \mathbf{E}].
\]

\( u^0 \) is zeroth component of normalized four-velocity \( u = (\gamma, \gamma v^1, \gamma v^2, \gamma v^3) \) which relates the inertial frames. Under a spatial rotation, the matrices \( (2.1) \) and \( (2.2) \) transform in such a way that electric field \( E \) and magnetic field \( B \) transform as three-vectors.

### 4.1 Motion in a field of magnetic type

If invariants \( P = 0 \) and \( \mathcal{S} > 0 \), then in all reference frames the magnitude of magnetic field is greater than that of electric field. The Lorentz transformation which is determined by 4-velocity

\[
u^0 = \frac{|\mathbf{B}|}{\sqrt{\mathcal{S}}}, \quad u = \frac{[\mathbf{E} \times \mathbf{B}]}{|\mathbf{B}| \sqrt{\mathcal{S}}}
\]

emphasizes the privileged inertial frame where the electric field \( E' = 0 \). The primed magnetic field \( (4.2) \)

\[
B' = \sqrt{\mathcal{S}} \mathbf{n}_B
\]

is the only one which “survives” in the privileged inertial frame. Here \( n_B = \mathbf{B}/|\mathbf{B}| \) is the unit vector in \( B \)-direction. If we align the \( z \)-axis along \( B' \), the electromagnetic field tensor \( (2.10) \) takes the form

\[
(F^\alpha_\nu) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & a & 0 \\
0 & -a & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

where \( a = \sqrt{\mathcal{S}} \) is the magnitude of imaginary eigenvalue. The set of eigenvalues consists of doubly degenerate zero and pair of imaginary ones, \( +ia \) and \( -ia \) (see Sec. \( 2.1 \)).

Taking into account that \( b = 0 \) we derive the projection operators

\[
O^{(a)} = \frac{-F^2}{a^2}
\]
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\tag{4.5}
\]

and

\[
O^{(b)} = \frac{F^2 + a^2 I}{a^2} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\tag{4.6}
\]

The “a projection” of particle’s 4-velocity has coordinates \((0, u^1, u^2, 0)\) while \(u^{(b)} = (u^0, 0, 0, u^3)\). Operators (4.5) and (4.6) split the tangent space \(T_x M_4\) into two mutually orthogonal subspaces \(M^{(a)}_x = \{v \in T_x M_4 : v^0 = 0, v^3 = 0\}\) and \(M^{(b)}_x = \{v \in T_x M_4 : v^1 = 0, v^2 = 0\}\). Traveling from point to point, we cover the affine space \(M_4\) by coordinate grid composed from mutually orthogonal planes of two dimensions. The projection operators

\[
P_{(+a)} = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & -i & 0 \\
0 & i & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\tag{4.7}
\]

and

\[
P_{(-a)} = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & i & 0 \\
0 & -i & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\tag{4.8}
\]

produce the eigenvectors of the field tensor (4.4) which correspond to eigenvalues \(+ia\) and \(-ia\), respectively.

The contractions of the field tensor (4.4) and velocity projections \(u^{(a)}\) and \(u^{(b)}\) are as follows:

\[
a^{-1} F^\mu_{\alpha} u^{\alpha}_{(a)} = (0, u^2, -u^1, 0), \quad F^\mu_{\alpha} u^{\alpha}_{(b)} = (0, 0, 0, 0).
\]

The solution (3.15) of the Herrera equation simplifies in this specific case

\[
u^0(\tau) = \frac{\sqrt{\beta} \cosh \chi_0}{\sqrt{\beta - \alpha e^{-2\omega_0 \tau}}},
\]

\[
u^1(\tau) = \frac{\sqrt{\alpha} \cos(\omega_0 \tau + \varphi_0)}{\sqrt{\beta e^{2\omega_0 \tau} - \alpha}},
\]

17
\[ u^2(\tau) = -\frac{\sqrt{\alpha} \sin(\omega_0 \tau + \varphi_0)}{\sqrt{\beta e^{2\omega_0 \tau} - \alpha}}, \]
\[ u^3(\tau) = \frac{\sqrt{\beta} \sinh \chi_0}{\sqrt{\beta - \alpha e^{-2\omega_0 \tau}}}, \]

where both the initial hyperbolic phase
\[ \cosh \chi_0 = \frac{u^0(0)}{\sqrt{\beta}}, \quad \sinh \chi_0 = \frac{u^3(0)}{\sqrt{\beta}}, \]

and the initial trigonometric phase
\[ \cos \varphi_0 = \frac{u^1(0)}{\sqrt{\alpha}}, \quad \sin \varphi_0 = -\frac{u^2(0)}{\sqrt{\alpha}}, \]

are defined by the components of the initial 4-velocity. Denoting \( \beta = \gamma_0^2 \) and \( \alpha = \gamma_0^2 - 1 \) we are sure that the result is in accord with that obtained in [12, Eqs. (19),(20)].

In series of papers [21, 22, 23], Herrera studied the behavior of radiating charge acted upon by a uniform magnetic field \( \mathbf{H} = (0, 0, H) \). The Lorentz-Abraham-Dirac equation (1.1) is chosen as the equation of motion of this charge. The solution is presented as series in powers of dimensionless small parameter
\[ \lambda = \frac{2e^2}{3mc^3} \frac{eH}{m}. \]

(The units are Gaussian.) The equation (3.2) is the first order approximation in Herrera’s approach. (Zeroth approximation gives the Lorentz force equation with no radiation.) The effect of radiation damping to the first power of the expansion parameter is illustrated by Eqs. (22) and (23) in [22] which are just the expressions (4.9) for the components of particle’s 4-velocity.

In Ref. [23, eq.(5)] the relation between the proper time \( \tau \) and the laboratory time \( t \) is found. In the first order degree of accuracy, Herrera’s function \( Z(t) \) is proportional to \( t \). The relation is presented in the form of the Lorentz factor
\[ \gamma(t) = \cosh \chi_0 \frac{\sqrt{\beta} + 1 + (\sqrt{\beta} - 1)e^{-2\omega_0' t}}{\sqrt{\beta} + 1 - (\sqrt{\beta} - 1)e^{-2\omega_0' t}}, \]

where
\[ \omega_0' = \frac{\omega_0}{\cosh \chi_0}, \]

(see also [10, eq. (9.20)]). Inverting the relation \( u^0(\tau) = \gamma(t) \) where the zeroth component is written in the first line of eqs. (4.9), we obtain
\[ \tau(t) = \frac{1}{\omega_0} \ln \left[ \cosh(\omega_0' t) + \beta^{-1/2} \sinh(\omega_0' t) \right]. \]
Substituting this into Eq. (4.14), we obtain the rate of radiated energy with respect to the laboratory time

\[ \frac{dP^0}{dt} = -\omega_0 \frac{\alpha}{\left[ \sqrt{\beta} \cosh(\omega_0' t) + \sinh(\omega_0' t) \right]^2}, \]

The intensity of radiation decreases with time exponentially.

Our next task is to derive the coordinate functions. Having integrated the zeroth component of 4-velocity, we obtain the zeroth coordinate

\[ z^0(\tau) = z^0(0) + \frac{u^0(0)}{\sqrt{\beta \omega_0}} \ln \sqrt{\beta \omega_0} + \sqrt{\beta e^{2\omega_0 \tau} - \alpha \sqrt{\beta + 1}}, \]

which gives the laboratory time \( t = z^0(\tau) \). We choose the constant of integration such that the laboratory watch and the particle’s watch show exactly the same time at the beginning of particle’s motion

\[ t = \frac{1}{\omega_0} \ln \frac{\sqrt{\beta e^{\omega_0 \tau} + \sqrt{\beta e^{2\omega_0 \tau} - \alpha}}}{\sqrt{\beta + 1}}. \]

The longitudinal motion has the form \( z^3(t) = z^3(0) + \tanh \chi_0 t \) in the laboratory time parametrization.

In the plane which is orthogonal to the magnetic field vector, the charge moves on the decaying spiral

\[
\begin{align*}
z^1(\tau) &= z^1(0) + \sum_{n=0}^{\infty} a_n \left[ e^{-(2n+1)\omega_0 \tau} \sin(\omega_a \tau + \varphi_0 + \phi_n) - \sin(\varphi_0 + \phi_n) \right], \\
z^2(\tau) &= z^2(0) + \sum_{n=0}^{\infty} a_n \left[ e^{-(2n+1)\omega_0 \tau} \cos(\omega_a \tau + \varphi_0 + \phi_n) - \cos(\varphi_0 + \phi_n) \right],
\end{align*}
\]

where \( \phi_n \) is given by Eqs. (3.22) and the phase’s shift \( \varphi_0 \) is defined by Eq. (4.11).

Let us assess how quickly the charge gets nearer to the central point with coordinates

\[
\begin{align*}
z_1^c &= z^1(0) - \sum_{n=0}^{\infty} a_n \sin(\varphi_0 + \phi_n), \\
z_2^c &= z^2(0) - \sum_{n=0}^{\infty} a_n \cos(\varphi_0 + \phi_n).
\end{align*}
\]

To sum up the series in Eqs. (4.17), we neglect the shifts \( \phi_n \) and the second term under the square root in the denominator of \( a_n \). In this approximation, the
$n$-dependent terms constitute the hypergeometric series. After some algebra, we obtain

$$z^1(\tau) = z^1(0) + \frac{\sqrt{\alpha} \sin(\omega_a \tau + \varphi_0)}{\omega_a \sqrt{\beta e^{2\omega_0 \tau} - \alpha}}, \quad z^2(\tau) = z^2(0) + \frac{\sqrt{\alpha} \cos(\omega_a \tau + \varphi_0)}{\omega_a \sqrt{\beta e^{2\omega_0 \tau} - \alpha}}.$$  

Substituting the laboratory time for the proper time in the expression for the decreasing radius

$$r(t) = \sqrt{[z^1(\tau) - z^1(0)]^2 + [z^2(\tau) - z^2(0)]^2}$$  

of the spiral, we obtain the formula

$$r(t) = \frac{\sqrt{\alpha}}{\omega_a} \frac{1}{\sqrt{\beta e^{2\omega_0 \tau} - \alpha}} \frac{1}{\gamma_0 \omega_a \left[ \cosh(\omega'_0 t) + \sqrt{\beta} \sinh(\omega'_0 t) \right]}$$  

which was first obtained by Spohn [10, eqs. (9.24)-(9.25)]. In fact, the radius decreases even more rapidly because we substitute the larger amplitude $\omega_a^{-1}$ for the smaller one $[\omega_a^2 + ((2n + 1)\omega_0]^2]^{-1/2}$ in the each coefficient of series (3.21).

### 4.2 Motion in a field of electric type

If invariants $P = 0$ and $S < 0$, then in all reference frames the magnitude of electric field is greater than that of magnetic field. The Lorentz transformation which is determined by 4-velocity

$$u^0 = \frac{|E|}{\sqrt{-S}}, \quad u = \frac{|E \times B|}{|E| \sqrt{-S}}$$  

emphasizes the privileged inertial frame where the magnetic field $B' = 0$. The primed electric field (4.11)

$$E' = \sqrt{-S} n_E$$  

is the only one which “survives” in the privileged inertial frame. Here $n_E = E/|E|$ is the unit vector in $E$-direction. If we align the $z$-axis along $E'$, the electromagnetic field tensor (2.10) takes the form

$$\left( F^\alpha_{\nu} \right) = \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \end{pmatrix},$$  

where $b = \sqrt{-S}$ is the magnitude of real eigenvalue. The set of eigenvalues consists of doubly degenerate zero and pair of real ones, $+b$ and $-b$ (see Sec. 2.4).
Fradkin’s operators $O^{(a)}$ and $O^{(b)}$ have the form (4.5) and (4.6), respectively. They produce the planes $\mathcal{M}^{(a)}$ and $\mathcal{M}^{(b)}$ which has been described in preceding Paragraph. The projection operators

$$P_{(+b)} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

(4.21)
and

$$P_{(-b)} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. $$

(4.22)
produce the eigenvectors of the field tensor (4.20) which correspond to eigenvalues $+b$ and $-b$, respectively.

The contractions of this tensor and velocity projections $u^{(a)} = (0, u^1, u^2, 0)$ and $u^{(b)} = (u^0, 0, 0, u^3)$ are as follows:

$$F^\mu_{\alpha} u^\alpha_{(a)} = (0, 0, 0, 0), \quad b^{-1}F^\mu_{\alpha} u^\alpha_{(b)} = (u^3, 0, 0, u^0).$$

Substituting this into Eqs. (3.15), we arrive at

$$u^0(\tau) = \frac{\sqrt{\beta} \cosh(\lambda_b \tau + \chi_0)}{\sqrt{\beta - \alpha e^{-2\omega_0 \tau}}},$$

(4.23)
$$u^1(\tau) = \frac{\sqrt{\alpha} \cos \varphi_0}{\sqrt{\beta e^{2\omega_0 \tau} - \alpha}},$$

(4.24)
$$u^2(\tau) = \frac{-\sqrt{\alpha} \sin \varphi_0}{\sqrt{\beta e^{2\omega_0 \tau} - \alpha}},$$

(4.25)
$$u^3(\tau) = \frac{\sqrt{\beta} \sinh(\lambda_b \tau + \chi_0)}{\sqrt{\beta - \alpha e^{-2\omega_0 \tau}}}.$$  

(4.26)

If the initial values $u^1(0) = 0$ and $u^2(0) = 0$ in the privileged reference frame determined by Eqs. (4.19), then the constants $\alpha = 0$ and $\beta = 1$. The simplified version of the expressions (4.23)–(4.26)

$$u^0(\tau) = \cosh(\lambda_b \tau + \chi_0),$$

$$u^1(\tau) = 0,$$

$$u^2(\tau) = 0,$$

$$u^3(\tau) = \sinh(\lambda_b \tau + \chi_0),$$

(4.27)
has been discussed in detail by Fulton and Rohrlich [19]. It is worth noting that the velocity components define the exact solution to the Lorentz-Abraham-Dirac
equation. In the absence of any transverse disturbances, the constant electric field supplies the amount of energy and momentum exactly equal to the energy and momentum losses due to radiation.

If the transverse components of initial 4-velocity do not vanish in the privileged inertial frame, the position functions \( z^0(\tau) \) and \( z^3(\tau) \) are given by the series

\[
\begin{align*}
    z^0(\tau) &= z^0(0) + \sum_{n=0}^{\infty} b_n \left[ e^{-2n\omega_0 \tau} \sinh(\lambda_b \tau - \psi_n + \chi_0) - \sinh(-\psi_n + \chi_0) \right], \\
    z^3(\tau) &= z^3(0) + \sum_{n=0}^{\infty} b_n \left[ e^{-2n\omega_0 \tau} \cosh(\lambda_b \tau - \psi_n + \chi_0) - \cosh(-\psi_n + \chi_0) \right],
\end{align*}
\]

(4.28)

where the hyperbolic phase shift \( \chi_0 \) is determined by Eq. (4.10) and coefficients \( b_n \) are defined by Eq. (3.24). The orbit in the transverse plane \( M^{(a)} \) is given by the coordinate functions

\[
\begin{align*}
    z^1(\tau) &= z^1(0) + \frac{\cos \varphi_0}{\omega_0} \left[ \arccos(\lambda e^{-\omega_0 \tau}) - \arccos \lambda \right], \\
    z^2(\tau) &= z^2(0) - \frac{\sin \varphi_0}{\omega_0} \left[ \arccos(\lambda e^{-\omega_0 \tau}) - \arccos \lambda \right],
\end{align*}
\]

(4.29)

which we derive by means of integration of the right-hand sides of Eqs. (4.24) and (4.25). If the proper time parameter increases, the transverse orbit tends to the point with coordinates

\[
\begin{align*}
    z^1_\infty &= z^1(0) + \frac{\cos \varphi_0}{\omega_0} \left[ \frac{\pi}{2} - \arccos \lambda \right], \\
    z^2_\infty &= z^2(0) - \frac{\sin \varphi_0}{\omega_0} \left[ \frac{\pi}{2} - \arccos \lambda \right],
\end{align*}
\]

at which the components \( u^1_\infty = 0 \) and \( u^2_\infty = 0 \). The orbit in the longitudinal plane \( M^{(b)} \) approaches the hyperbola.

### 4.3 Motion in a crossed field

If both the invariants \( S \neq 0 \) and \( P \neq 0 \), there exists the inertial frame in which the electric and magnetic fields are collinear. In terms of their magnitudes \( a = |B'| \) and \( b = |E'| \), the invariants (2.5) look as follows:

\[
P = 2\kappa ab, \quad S = a^2 - b^2.
\]

The sign parameter \( \kappa = +1 \) if \( E' \) and \( B' \) are in the same direction while \( \kappa = -1 \) if these vectors are directed oppositely. The real and positive solutions of this system of two algebraic equations are just the scalars (2.8) and (2.9).
The Lorentz transformation to privileged inertial frame is determined by three-velocity
\[ u = \frac{[E \times B]}{\sqrt{d(a^2 + b^2)}}. \]  
(4.30)

where
\[ d = B^2 + b^2 = E^2 + a^2 = \frac{1}{2} \left[ B^2 + E^2 + \sqrt{S^2 + T^2} \right]. \]

Inserting this in Eqs. (4.1) and (4.2), we obtain the “primed” fields

\[ E' = \frac{b^2 E + \kappa a b B}{\sqrt{d(a^2 + b^2)}} \]
\[ B' = \frac{a^2 B + \kappa a E}{\sqrt{d(a^2 + b^2)}} \]

Recall that \( E \) and \( B \) are electric and magnetic field strengths in a laboratory reference frame. It is easy to show that \( \kappa a E' = b B' \).

In the reference frame where the “primed” electric field is directed along \( Oz \)-axis, the field tensor (2.10) takes the form:

\[ (F^\alpha_\nu) = \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & \kappa a & 0 \\ 0 & -\kappa a & 0 & 0 \\ b & 0 & 0 & 0 \end{pmatrix}. \]  
(4.31)

Fradkin’s operators \( O^{(a)} \) and \( O^{(b)} \) are still given by Eqs. (4.5) and (4.6), respectively. The generalizations of projection operators (4.7) and (4.8)

\[ P_{(+a)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i\kappa & 0 \\ 0 & i\kappa & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]  
(4.32)

and

\[ P_{(-a)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & i\kappa & 0 \\ 0 & -i\kappa & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]  
(4.33)

produce the eigenvectors of the field tensor (4.31) which correspond to eigenvalues \(+ia\) and \(-ia\), respectively. The operators (4.21) and (4.22) generate “real” eigenvectors.
The contractions of tensor (4.31) and velocity projections \( u(a) = (0, u^1, u^2, 0) \) and \( u(b) = (u^0, 0, 0, u^3) \) are as follows:

\[
a^{-1} F^\mu_\alpha u^\alpha_{(a)} = (0, \kappa u^2, -\kappa u^1, 0), \quad b^{-1} F^\mu_\alpha u^\alpha_{(b)} = (u^3, 0, 0, u^0).
\]

Hence the 4-velocity of the charged particle has the components

\[
\begin{align*}
    u^0(\tau) &= \frac{\sqrt{\beta}}{\sqrt{\beta - \alpha e^{-2\omega_0 \tau}}} \cosh(\lambda_0 \tau + \chi_0), \\
    u^1(\tau) &= \frac{\sqrt{\alpha}}{\sqrt{\beta e^{2\omega_0 \tau} - \alpha}} \cos(\omega_0 \tau + \kappa \varphi_0), \\
    u^2(\tau) &= -\frac{\sqrt{\alpha}}{\sqrt{\beta e^{2\omega_0 \tau} - \alpha}} \sin(\omega_0 \tau + \kappa \varphi_0), \\
    u^3(\tau) &= \frac{\sqrt{\beta}}{\sqrt{\beta - \alpha e^{-2\omega_0 \tau}}} \sinh(\lambda_0 \tau + \chi_0),
\end{align*}
\]

where initial phases \( \varphi_0 \) and \( \chi_0 \) are given by Eqs. (4.11) and (4.10), respectively. Corresponding coordinate functions which determine the orbit in plane \( M^{(a)} \) are given by series (4.17), and the orbit in plane \( M^{(b)} \) are defined by series (4.28).

### 5 Conclusions

The projection technique developed in this paper allows us to “visualize” the formal solution (1.4) of the LL equation for a radiating charge in a constant electromagnetic field. The method can be applied to all types of fields, excepting ones of the null type (see Sec. 2.1).

The charge’s world line is the combination of two orbits which “live” in two mutually orthogonal planes of two dimensions. The eigenvectors of the pair of imaginary eigenvalues, \(+ia\) and \(-ia\) span the plane \( M^{(a)} \). The orbit in \( M^{(a)} \) is a continuous curve that curves around and gets nearer to the “end” point to which the charge approximately approaches. The orbit in the plane \( M^{(b)} \) spanned by eigenvectors of the pair of real eigenvalues, \(+b\) and \(-b\), is the slightly modified hyperbola.

The radiation damping suppresses the particle’s rotation in \( M^{(a)} \) while the radiation reaction force almost does not influence on the form of orbit in \( M^{(b)} \). The world line is the helix spiralled inward, i.e., a continuous curve that curves and rises around the line aligned with one of “real” eigenvectors of the electromagnetic field tensor. The radius of decaying orbit shrinks under the emission of synchrotron radiation.

Let us evaluate the rapidity of radiation damping. The radius of the circular-like orbit decreases with time exponentially. The argument of the exponential function is proportional to the constant \( \omega_0 = \tau_0 (\omega_a^2 + \lambda_0^2) \) which defines
the intensity of emission of energy. The stronger the electromagnetic field, the greater is \( \omega_0 \), and the shorter is the (proper) time interval \( \Delta \tau \) during which the argument \( -\omega_0 \Delta \tau \) of exponential function reaches \( -1 \). (This interval is necessary to make the magnitude of the “a projection” of particle’s 4-velocity smaller in \( e \approx 2.71828 \) times.) So, if an electron moves in Earth’s magnetic field \( (|\vec{B}| \approx 3 \cdot 10^{-5} T), \Delta \tau \approx 181 \) years. In a magnetic field of typical refrigerator magnet \( (|\vec{B}| \approx 5 \cdot 10^{-3} T), \) the time interval is \( \Delta \tau \approx 57 \) hours.

Much more stronger magnetic field \( (|\vec{B}| = 5 \div 10 T) \) is used in the so-called Penning trap \([26]\) where a combination of a strong homogeneous magnetic field and quadrupole electrostatic potential is applied to a charged particle to confine it during a very long time. (During the experiments described in Ref. \([27]\), a single electron was trapped continuously for more than 10 months.) So, if an electron moves within a processing chamber of a Penning trap, then the interval \( \Delta \tau \approx 0.143 \div 0.051 \) s. The precision measurement \([28]\) \((B = 6 T) \) results that “The average excitation energy decreases exponentially as a function of the delay time with a time constant of 0.27 \pm 0.04 s.” Radiation damping is the dominant mechanism of reducing of energy of the cyclotron motion of an electron in a Penning trap \([26]\).

Extremely powerful magnetic field \( (|\vec{B}| \approx 10^6 \div 10^8 T) \) is in a neighborhood of pulsar. (Pulsar is a rapidly rotating neutron star producing radiation and radio waves in regular amounts.) In a pulsar’s magnetosphere, the oscillations of electrons (positrons) are suppressed almost instantly: \( \Delta \tau \approx 5.16 ps \div 0.52 fs \). The suppression of electron’s oscillations caused by the radiation damping was first used by Rylov in series of papers \([29, 30, 31]\) devoted to the study of plasma’s behavior in a pulsar’s magnetosphere. The author considers the equation for eigenvalue and eigenvectors of the electromagnetic field tensor as ultrarelativistic limit of the Landau-Lifshitz equation.

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