ON HARMONIC $\nu$-BLOCH AND $\nu$-BLOCH-TYPE MAPPINGS

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Abstract. The aim of this paper is twofold. One is to introduce the class of harmonic $\nu$-Bloch-type mappings as a generalization of harmonic $\nu$-Bloch mappings and thereby we generalize some recent results of harmonic 1-Bloch-type mappings investigated recently by Efraimidis et al. [12]. The other is to investigate some subordination principles for harmonic Bloch mappings and then establish Bohr’s theorem for these mappings and in a general setting, in some cases.

1. Introduction

A significant part of function theory deals with univalent functions, function spaces such as Bloch spaces, Bohr’s phenomenon and their various generalizations. Several authors have contributed a lot to this development, and most importantly, in the area of planar harmonic mappings. For basic results about harmonic mappings, the reader may refer [8], the monograph of Duren [11] and the recent survey of some basic materials from [20]. Concerning classical Bloch spaces, see [3, 4, 10]. In recent years, Bohr’s phenomenon, its various generalizations including higher dimensional analogues and its harmonic analogues have been studied by various authors. For more details of the importance, background, development and results, we refer to the recent survey on this topic [2] and the references therein. The recent results on this topic for harmonic mappings may be obtained from [15, 16]. Our primary goal here is to continue to study harmonic Bloch-type mappings and as applications, we consider Bohr’s inequality in a general setting.

Throughout we consider complex-valued harmonic mappings in the open unit disk $D = \{ z : |z| < 1 \}: \Delta f = 0$. It is well known that every harmonic mapping in $D$ has a canonical decomposition $f = h + g$, where $h$ and $g$ are analytic functions with $g(0) = 0$. Thus, we may express $h$ and $g$ as

\[ h(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \]  

Moreover, the function $f = h + g$ is locally univalent and sense-preserving in $D$ if and only if its Jacobian $J_f = \| f_z \|^2 - \| f_{\overline{z}} \|^2 = |h'|^2 - |g'|^2 > 0$ in $D$ by Lewy’s theorem (see [17]), i.e., $|h'| > |g'|$ or $|\omega_f| < 1$ in $D$, where $\omega_f = g'/h'$ is the dilatation of $f$.

For a given $\nu \in (0, \infty)$, a harmonic mapping $f = h + g$ in $D$ is called a harmonic $\nu$-Bloch mapping if

\[ \beta_\nu(f) := \sup_{z \in D}(1 - |z|^2)^\nu(|h'(z)| + |g'(z)|) < \infty. \]
This defines a seminorm, and the space equipped with the norm
\[ ||f||_{B_H(\nu)} := |f(0)| + \beta_\nu(f) \]
is called the harmonic \( \nu \)-Bloch space, denoted by \( B_H(\nu) \). It is a Banach space. In particular the space \( B(\nu) \) defined by
\[ B(\nu) = \{ f = h + g \in B_H(\nu) : g \equiv 0 \} \]
forms a Banach space equipped with the norm \( ||f||_{B(\nu)} := |f(0)| + \beta_\nu(f) \). Clearly, \( f = h + g \in B_H(\nu) \) if and only if \( h, g \in B(\nu) \), since
\[ \max\{\beta_\nu(h), \beta_\nu(g)\} \leq \beta_\nu(f) \leq \beta_\nu(h) + \beta_\nu(g) \).

The harmonic \( \nu \)-Bloch space \( B_H(\nu) \) was introduced in [3], which was a generalization of \( B_H(1) \) that was studied by Colonna in [10] as a generalization of classical Bloch space \( B(1) \). One can refer to [3, 4, 5, 7, 19, 23] for information on \( B(1) \) and its extension. Motivated by results on analytic Bloch functions, Efraimidis et al. [12] introduced harmonic Bloch-type mappings, which coincide with the following harmonic \( 1 \)-Bloch-type mappings.

**Definition 1.** For a given \( \nu \in (0, \infty) \), a harmonic mapping \( f \) on \( \mathbb{D} \) is called a harmonic \( \nu \)-Bloch-type mapping if
\[ \beta_\nu^*(f) := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\nu \sqrt{|J_f(z)|} < \infty. \]
We write \( B_H^*(\nu) \) for the space of all such mappings and we call
\[ ||f||_{B_H^*(\nu)} := |f(0)| + \beta_\nu^*(f), \]
the pseudo-norm of \( f \).

In Section 2 we will see that \( B_H^*(\nu) \) is not a linear space for any \( \nu > 0 \). Because
\[ (1 - |z|^2)^\nu \sqrt{|J_f(z)|} \leq (1 - |z|^2)^\nu (|f_z(z)| + |f_{\overline{z}}(z)|), \quad z \in \mathbb{D},\]
it is clearly that \( B_H(\nu) \subset B_H^*(\nu) \) and thus, the space \( B_H^*(\nu) \) is a generalization of \( B_H(\nu) \). In addition, in the case of analytic functions \( f \), these spaces coincide and thus, we have
\[ ||f||_{B_H^*(\nu)} = ||f||_{B_H(\nu)} = ||f||_{B(\nu)}. \]

One of the aims of this article is to generalize some of the known results of harmonic \( \nu \)-Bloch mappings and \( \nu \)-Bloch-type mappings (especially, the results of [12]). The paper is divided into sections as follow: In Section 2 for the function spaces \( B_H(\nu) \) and \( B_H^*(\mu) \), we investigate its affine and linear invariance, and the inclusion relations under particular conditions. In Section 3 we find a connection between these function spaces and the space of uniformly locally univalent harmonic mappings. Moreover, some subordination principles concerning the spaces \( B_H(1) \) and \( B_H^*(1) \) are also investigated. In Section 4 we give the growth and coefficients estimates for sense-preserving mappings in \( B_H^*(\nu) \). Finally, as applications of our investigations, we determine the Bohr radius for functions in \( B(\nu) \), and \( p \)-Bohr radius for functions in \( B_H(\nu) \) and \( B_H^*(\nu) \) (sense-preserving) in Section 5.
2. AFFINE AND LINEAR INVARIANCE AND INCLUSION RELATIONS

Throughout the article, \( \nu \) is a constant in the interval \((0, \infty)\). We first discuss the affine and linear invariance of \( \mathcal{B}_H(\nu) \) and \( \mathcal{B}_H^*(\nu) \). Let \( L \) be a family of harmonic mappings defined in \( \mathbb{D} \). Then the family \( L \) is said to be affine invariant if \( A \circ f \in L \) for each \( f \in L \) and for all affine mappings \( A \) of the form \( A(z) = az + b \overline{z} \) \((a, b \in \mathbb{C})\). The family \( L \) is called linear invariant if for each \( f \in L \),

\[
f \circ \varphi \alpha \in L \quad \forall \varphi \alpha(z) = \frac{z + \alpha}{1 + \overline{\alpha}z} \in \text{Aut}(\mathbb{D}).
\]

**Proposition 1.**

1. Both \( \mathcal{B}_H(\nu) \) and \( \mathcal{B}_H^*(\nu) \) are affine invariant.
2. Each of \( \mathcal{B}(\nu) \), \( \mathcal{B}_H(\nu) \) and \( \mathcal{B}_H^*(\nu) \) is linear invariant.

**Proof.**

(1) Let \( f = h + \overline{g} \) and \( A(z) = az + b \overline{z} \) \((a, b \in \mathbb{C})\). Then

\[
A \circ f = ah + bg + \overline{ag} + \overline{bh}
\]

and thus,

\[
|(ah + bg)'| + |(\overline{ag} + \overline{bh})'| \leq (|a| + |b|)(|h'| + |g'|)
\]

and

\[
J_{A \circ f} = \left( (ah + bg)'\right)^2 - |(\overline{ag} + \overline{bh})'|^2 = (|a|^2 - |b|^2)J_f.
\]

The desired conclusion now easily follows.

(2) We only need to prove that \( \mathcal{B}_H^*(\nu) \) is linear invariant. For \( \varphi \alpha(z) \in \text{Aut}(\mathbb{D}) \), we have

\[
J_{f \circ \varphi \alpha}(z) = |\varphi \alpha'(z)|^2 J_f(\varphi \alpha(z)) \quad \text{and} \quad 1 - |\varphi \alpha(z)|^2 = |\varphi \alpha(z)|(1 - |z|^2),
\]

and obtain

\[
(1 - |z|^2)\nu \sqrt{|J_{f \circ \varphi \alpha}(z)|} = (1 - |z|^2)\nu |\varphi \alpha'(z)| \sqrt{|J_f(\varphi \alpha(z))|}
\]

\[
= \left( \frac{1 - |z|^2}{1 - |\varphi \alpha(z)|^2} \right)^{\nu-1} (1 - |\varphi \alpha(z)|^2)^\nu \sqrt{|J_f(\varphi \alpha(z))|}
\]

\[
\leq \left( \frac{1 + \overline{\alpha}z^2}{1 - |\alpha|^2} \right)^{\nu-1} \beta^*_\nu(f)
\]

\[
\leq \left( \frac{1 + |\alpha|}{1 - |\alpha|} \right)^{|\nu-1|} \beta^*_\nu(f), \quad z \in \mathbb{D}.
\]

Now it is obvious that if \( f \in \mathcal{B}_H^*(\nu) \), then \( f \circ \varphi \alpha \in \mathcal{B}_H^*(\nu) \) for each \( \varphi \alpha \in \text{Aut}(\mathbb{D}) \).

Similarly, \( \mathcal{B}(\nu) \) is linear invariant so that \( \mathcal{B}_H(\nu) \) is also linear invariant. \( \square \)

For each \( \nu > 0 \), although both \( \mathcal{B}(\nu) \) and \( \mathcal{B}_H(\nu) \) are Banach spaces, the following example shows that \( \mathcal{B}_H^*(\nu) \) is not a linear space. It also shows that some functions in \( \mathcal{B}_H^*(\nu) \) may grow arbitrarily fast. Therefore, to study certain properties of functions in \( \mathcal{B}_H^*(\nu) \) in what follows, we shall restrict harmonic mappings to be sense-preserving.

**Example 1.** Let \( f = h + \overline{g} \), where \( h(z) = (\mu - 1)^{-1}(1 - z)^{1-\mu} \) for some \( \mu > 2\nu + 1 \). Clearly, we have \( f(z) \) and the identity function \( z \) belong to \( \mathcal{B}_H^*(\nu) \) whereas \( F(z) = f(z) + z \) does not, since

\[
(1 - x^2)^{2\nu}|J_F(x)| = (1 + x)^{2\nu} \frac{2 + (1 - x)^{\mu}}{(1 - x)^{\mu - 2\nu}} \to \infty \quad \text{as} \quad (0, 1) \ni x \to 1^-.
\]
Next we deal with the inclusion relations $\mathcal{B}(\nu) \subset \mathcal{B}_H(\nu) \subseteq \mathcal{B}^*_H(\nu)$.

**Proposition 2.** Let $\mu$ and $\nu$ be two constants with $0 < \mu < \nu$. We have

1. $\mathcal{B}(\nu) \subset \mathcal{B}_H(\nu) \subset \mathcal{B}^*_H(\nu)$;
2. $\mathcal{B}(\mu) \subset \mathcal{B}(\nu)$, $\mathcal{B}_H(\mu) \subset \mathcal{B}_H(\nu)$ and $\mathcal{B}^*_H(\mu) \subset \mathcal{B}^*_H(\nu)$.

**Proof.** (1) It only needs to find a function $f_\nu \in \mathcal{B}^*_H(\nu) \setminus \mathcal{B}_H(\nu)$ for each $\nu > 0$. For the sake of the context later, we will prove that the one parameter family of functions $f_{\nu,t} \in \mathcal{B}^*_H(\nu) \setminus \mathcal{B}_H(\nu)$ for each $\nu > 0$, where

$$f_{\nu,t}(z) = h_{\nu}(z) + g_{\nu,t}(z), \quad t \in [0,1), \quad \nu > 0,$$

with

$$h_{\nu}(z) = \begin{cases} -\log(1 - z) & \text{for } \nu = 1/2, \\ (\nu - 1/2)^{-1} \left( (1 - z)^{1/2 - \nu} - 1 \right) & \text{for } \nu \neq 1/2, \end{cases}$$

and

$$g_{\nu,t}(z) = \begin{cases} -\log(1 - z) - (1 - t)z \frac{1}{1 - z} + (1 - t) \log(1 - z) & \text{for } \nu = 1/2, \\ (1 - z)^{1/2 - \nu} - 1 \frac{1}{\nu - 1/2} - (1 - t) \frac{(1 - z)^{3/2 - \nu} - 1}{\nu - 3/2} & \text{for } \nu \neq 1/2, 3/2. \end{cases}$$

Fix $t \in [0,1)$. A direct computation reveals that $f_{\nu,t}$ is sense-preserving in $\mathbb{D}$ with the dilatation $\omega_{f_{\nu,t}}(z) = t + (1 - t)z$ for each $\nu > 0$. Again, by computation, we have

$$(1 - |z|^2)^\nu \sqrt{|J_{f_{\nu,t}}(z)|} = \frac{(1 - |z|^2)^\nu}{|1 - z|^{\nu + 1/2}} \sqrt{1 - |\omega_{f_{\nu,t}}(z)|^2}$$

$$= (1 + |z|)\nu \frac{|1 - |z|^2|^\nu}{|1 - z|^{\nu + 1/2}} \sqrt{1 - |z|^2 - 2t \Re(\overline{(1 - z)}) - t^2 |1 - z|^2}$$

$$\leq 2^{\nu + 1/2} \sqrt{1 + t},$$

which gives $f_{\nu,t} \in \mathcal{B}^*_H(\nu)$ for each $\nu > 0$. Since

$$(1 - x^2)^\nu |h'_\nu(x)| = \frac{(1 + x)^\nu}{\sqrt{1 - x}} \to \infty \text{ as } (0,1) \ni x \to 1^-,$$

we obtain that for each $\nu > 0$, $h_{\nu} \notin \mathcal{B}_H(\nu)$ and thus, $f_{\nu,t} \notin \mathcal{B}_H(\nu)$.

(2) Let $0 < \mu < \nu$. Clearly, $\mathcal{B}(\mu) \subset \mathcal{B}(\nu)$, $\mathcal{B}_H(\mu) \subset \mathcal{B}_H(\nu)$ and $\mathcal{B}^*_H(\mu) \subset \mathcal{B}^*_H(\nu)$. Then the inclusions $\mathcal{B}(\mu) \subset \mathcal{B}(\nu)$ and $\mathcal{B}^*_H(\mu) \subset \mathcal{B}^*_H(\nu)$ obviously follow by (1) if we prove $\mathcal{B}(\mu) \subset \mathcal{B}(\nu)$. For this, we simply consider the function $f_\nu$ satisfying $f'_\nu(z) = (1 - z)^{-\nu}$, it is easy to see that $f_\nu \in \mathcal{B}(\nu) \setminus \mathcal{B}(\mu)$. This completes the proof. \(\square\)

It is natural to ask for the structure of the set $\mathcal{B}^*_H(\nu) \setminus \mathcal{B}_H(\nu)$.

**Proposition 3.** Let $f = h + \overline{g}$ be a harmonic mapping in $\mathbb{D}$. Then $f \in \mathcal{B}_H(\nu)$ if and only if $f \in \mathcal{B}^*_H(\nu)$ and either $h \in \mathcal{B}(\nu)$ or $g \in \mathcal{B}(\nu)$. We get

$$\mathcal{B}_H(\nu) \setminus \mathcal{B}_H(\nu) = \{ f = h + \overline{g} \in \mathcal{B}^*_H(\nu) : h \notin \mathcal{B}(\nu) \text{ and } g \notin \mathcal{B}(\nu) \}.$$
Proof. It suffices to observe that $|h'| \leq \sqrt{|J_f|} + |g'|$ and $|g'| \leq \sqrt{|J_f|} + |h'|$ for a harmonic mapping $f = h + \overline{g}$.

The following question arises.

**Problem 1.** Suppose that $f \in B^*_H(\nu)$. Does there exist a constant $c(\nu)$ depending only on \( \nu \) such that $f \in B_H(c(\nu))$?

In order to give an affirmative answer to this problem, we need some extra conditions based on the following observation for the function $f = h + \overline{g}$, where $h(z) = \exp((1 + z)/(1 - z))$. Clearly, $f \in B^*_H(\nu)$ for all $\nu > 0$. However, $f \notin B_H(\nu)$ for any $\nu > 0$, since $h \notin B(\nu)$, which can be deduced from

$$(1 - x^2)^{\nu} |h'(x)| = 2(1 + x)^{2\nu - 2} \left[\left(\frac{1 - x}{1 + x}\right)^{\nu - 2} e^{\frac{i\pi}{2}}\right] \to \infty \text{ as } (0, 1) \ni x \to 1^-.$$

**Proposition 4.** Let $f$ be a locally univalent harmonic mapping in $\mathbb{D}$. If $f \in B^*_H(\nu)$, then $f \in B_H(\nu + 1/2)$. Moreover, the constant $1/2$ is sharp for each $\nu > 0$.

Proof. Note that $f \in B_H(\nu)$ (resp. $B^*_H(\nu)$) if and only if $\overline{f} \in B_H(\nu)$ (resp. $B^*_H(\nu)$). Without loss of generality, we may thus assume that $f = h + \overline{g}$ is sense-preserving with the dilatation $\omega = \omega_f$ so that

$$g' = \omega h' \quad \text{and} \quad J_f = |h'|^2 (1 - |\omega|^2) \quad \text{or} \quad |h'| = \frac{\sqrt{J_f}}{1 - |\omega|^2}.$$

It follows (see [13, Corollary 1.3]) that

$$|\omega(z)| \leq \frac{|z| + |\omega(0)|}{1 + |\omega(0)||z|}, \quad z \in \mathbb{D}.$$

Now we suppose that $f \in B^*_H(\nu)$. Then we get

$$(1 - |z|^2)^{\nu} \sqrt{|J_f(z)|} \leq \beta^*_\nu(f) < \infty, \quad z \in \mathbb{D}.$$

Consequently, we have $|g'(z)| < |h'(z)|$ in $\mathbb{D}$, where

$$|h'(z)| = \sqrt{\frac{J_f(z)}{1 - |\omega(z)|^2}} \leq \frac{\beta^*_\nu(f)}{(1 - |z|^2)^\nu} \frac{1}{\sqrt{1 - |\omega(z)|^2}}.$$

Thus,

$$\leq \frac{\beta^*_\nu(f)}{(1 - |z|^2)^\nu} \left[1 - \left(\frac{|z| + |\omega(0)||z|}{1 + |\omega(0)||z|}\right)^2\right]^{-1/2}$$

$$= \frac{\beta^*_\nu(f)}{(1 - |z|^2)^\nu} \left[\frac{1 + |\omega(0)||z|}{\sqrt{(1 - |z|^2)(1 - |\omega(0)|^2)}}\right]$$

$$\leq \frac{\beta^*_\nu(f)}{(1 - |z|^2)^{\nu + \frac{1}{2}}} \sqrt{\frac{1 + |\omega(0)|}{1 - |\omega(0)|^2}}, \quad (4)$$

$$\leq \frac{\beta^*_\nu(f)}{(1 - |z|^2)^{\nu + 1}} \sqrt{\frac{1 + |\omega(0)|}{1 - |\omega(0)|}}. \quad (5)$$
which shows that $h$ (and hence $g$) belongs to $\mathcal{B}_H(\nu + 1/2)$. Hence, $f \in \mathcal{B}_H(\nu + 1/2)$.

To see that the constant $1/2$ is sharp for each $\nu > 0$, it suffices to check for the function $f_{\nu,0} = h_{\nu} + g_{\nu,0}$ defined by (2). From the proof of Proposition 2 the function $f_{\nu,0} \in \mathcal{B}_H^*$. On the other hand, it is easy to see that $h_{\nu} \in \mathcal{B}(\nu + 1/2)$, which implies $g_{\nu,0} \in \mathcal{B}(\nu + 1/2)$ and thus, $f_{\nu} \in \mathcal{B}_H(\nu + 1/2)$. However, we have that for any $0 < \varepsilon < \nu + 1/2$, $h_{\nu} \not\in \mathcal{B}(\varepsilon)$, which means $f_{\nu,0} \not\in \mathcal{B}_H(\varepsilon)$. We complete the proof. \hfill $\square$

3. Uniformly locally univalent and subordination principles

3.1. Connection with uniformly locally univalent harmonic mappings. Motivated by the characterization of Bloch space $\mathcal{B}(1)$ and the recent work of the authors [18] concerning equivalent conditions of uniformly locally univalent (briefly, ULU) harmonic mappings, we will show the connections among harmonic $\nu$-Bloch, $\nu$-Bloch-type mappings and ULU harmonic mappings.

We first introduce the notion and some properties of ULU harmonic mappings. A harmonic mapping $f = h + \overline{g}$ in $\mathbb{D}$ is called ULU if there exists a constant $\rho > 0$ such that $f$ is univalent on the hyperbolic disk

$$D_h(a, \rho) = \left\{ z \in \mathbb{D} : \left| \frac{z - a}{1 - \overline{a}z} \right| < \tanh \rho \right\},$$

of radius $\rho$, for every $a \in \mathbb{D}$. One of equivalent conditions of ULU is stated in terms of the pre-Schwarzian derivative or norm. Let $f$ be a locally univalent harmonic mapping in $\mathbb{D}$. The pre-Schwarzian derivative and the norm of $f$ are defined as [14] (see also [9])

$$P_f = (\log J_f)_z, \quad z \in \mathbb{D}, \quad \text{and} \quad ||P_f|| = \sup_{z \in \mathbb{D}} (1 - |z|^2)|P_f(z)|,$$

respectively. Clearly, the two definitions coincide with the corresponding definitions in the analytic case. Similar to the proof of [14] Theorem 7], the function $f = h + \overline{g}$ in $\mathbb{D}$ is ULU if and only if $||P_f|| < \infty$ (see also [18] Theorem 4.1]). Several equivalent conditions of ULU mappings can be found in these three papers and the references therein.

Now let’s restrict $f$ to be analytic in $\mathbb{D}$. It is well-known that $f \in \mathcal{B}(1)$ if and only if there exists a constant $c > 0$ and a univalent analytic function $F$ such that $f = c \log F'$ (see [19]). On the other hand, $f$ is ULU if and only if there exists a constant $c > 0$ and a univalent analytic function $F$ such that $f' = (F')^c$ (see [22] Theorem 2]). Thus, $f \in \mathcal{B}(1)$ if and only if there exists a ULU analytic function $F$ such that $f = \log F'$. Furthermore, a harmonic mapping $f = h + \overline{g}$ belongs to $\mathcal{B}_H(1)$ if and only if there exist two ULU analytic functions $H$ and $G$ such that $f = \log H' + \log G'$. A natural question is to ask: What about the characterization of $\mathcal{B}_H^*(1)$? The following theorem and example show some extraneous complexities of the structure of the space $\mathcal{B}_H^*(1)$, which are different from Example 4.

**Theorem 1.** Let $F = H + \overline{G}$ be sense-preserving and ULU in $\mathbb{D}$. Then for each $\varepsilon \in \overline{\mathbb{D}}$, the function $f_{\varepsilon} = h_{\varepsilon} + \overline{g_{\varepsilon}}$ belongs to $\mathcal{B}_H^*(1)$, where $h_{\varepsilon} = \log(H' + \varepsilon G')$ and $\omega = g_{\varepsilon}' / h_{\varepsilon}'$ is bounded in $\mathbb{D}$. 
Proof. Suppose that $F = H + \overline{G}$ is a sense-preserving and ULU in $\mathbb{D}$. It follows from [18, Theorem 4.1] that $\|P_{H+\varepsilon G}\| < \infty$ for all $\varepsilon \in \overline{\mathbb{D}}$. By assumption, for each $\varepsilon \in \mathbb{D}$, we have

$$(1 - |z|^2) \sqrt{|J_{f_{\varepsilon}}(z)|} \leq (1 - |z|^2)|h'_{\varepsilon}(z)|(1 + \sup_{z \in \mathbb{D}} |\omega(z)|)$$

$$= (1 - |z|^2) \frac{H'' + \varepsilon G''}{H' + \varepsilon G'}(1 + \sup_{z \in \mathbb{D}} |\omega(z)|)$$

$$\leq \|P_{H+\varepsilon G}\|(1 + \sup_{z \in \mathbb{D}} |\omega(z)|) < \infty$$

and the assertion follows.

□

Example 2. Consider the function $f = h + \overline{f}$ in $\mathbb{D}$ with the dilatation $\omega_f(z) = e^{i\theta}z$, where $h = \log H'$ and $H(z) = \exp \left( \sqrt{\frac{1 + z}{1 - z}} \right) =: \exp(q(z))$, $z \in \mathbb{D}$, and the principal branch of the square root is chosen such that $q(0) = 1$. We claim that $f \not\in \mathcal{B}_H(1) \setminus \mathcal{B}_H(1)$ and $H$ is locally univalent but not ULU in $\mathbb{D}$. To do this, straightforward computations give that

$$H'(z) = \frac{1}{(1 - z)^2} \sqrt{\frac{1 - z}{1 + z}} \exp \left( \sqrt{\frac{1 + z}{1 - z}} \right) \neq 0, \quad z \in \mathbb{D},$$

and

$$\|P_H\| = \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \frac{(1 + 2z)\sqrt{1 - z} + \sqrt{1 + z}}{(1 - z^2)\sqrt{1 - z}} \right| = \infty,$$

showing that $H$ is locally univalent but not ULU in $\mathbb{D}$. Again, elementary computations show that

$$(1 - |z|^2) \sqrt{|J_f(z)|} = (1 - |z|^2)|h'(z)|\sqrt{1 - |z|^2}$$

$$= (1 - |z|^2) \frac{(1 + 2z)\sqrt{1 - z} + \sqrt{1 + z}}{(1 - z^2)\sqrt{1 - z}} \sqrt{1 - |z|^2}$$

$$\leq \left| (1 + 2z)\sqrt{1 - z} + \sqrt{1 + z} \right| \sqrt{1 + |z|} < \infty, \quad z \in \mathbb{D},$$

which implies $f \in \mathcal{B}_H^*(1)$. Moreover, because $H$ is not ULU, we find that $h \not\in \mathcal{B}(1)$ and thus, $f \not\in \mathcal{B}_H(1)$. Hence we conclude that, $f \in \mathcal{B}_H^*(1) \setminus \mathcal{B}_H(1)$.

Although Theorem 1 is a generalization of [12, Theorem 2], we can't give a complement characterization of $\mathcal{B}_H^*(1)$, let alone to $\mathcal{B}_H^*(\nu)$ ($\nu > 0$). However, we will see that any ULU harmonic mapping is a $\nu$-Bloch-type mapping for some $\nu > 0$. Recall that $f$ is ULU if and only if $\|P_f\| < \infty$. In view of this, to describe our result more precisely, we define the set

$$\mathbb{B}_H(\nu) = \{ f : f \text{ is a locally univalent harmonic mapping in } \mathbb{D} \text{ with } \|P_f\| \leq \nu \}$$

and its subset $\mathbb{B}(\nu)$ of all analytic functions in $\mathbb{B}_H(\nu)$.

Theorem 2. For any $\nu > 0$, we have $\mathbb{B}_H(\nu) \subset \mathcal{B}_H^*(\nu/2)$. In particular, $\mathbb{B}(\nu) \subset \mathcal{B}(\nu/2)$. Moreover, these two inclusions are best possible.
Proof. Assume \( f \in \mathbb{B}_H(\nu) \) for some \( \nu > 0 \). Note that \( ||P_f|| = ||P_{\bar{f}}|| \). Without loss of generality, we may assume that \( f \) is sense-preserving. Then, because \( f_z(0) \neq 0 \), we may consider

\[
F(z) = \frac{f(z) - f(0)}{f_z(0)}.
\]

Then \( F \) is sense-preserving in \( \mathbb{D} \) with the normalization \( F(0) = f_z(0) - 1 = 0 \). We have \( ||P_F|| = ||P_f|| \) and thus, \( F \in \mathbb{B}_H(\nu) \). It follows from [18, Theorem 6.1] that

\[
J_F(z) \leq (1 - |F_z(0)|^2) \left( \frac{1 + |z|}{1 - |z|} \right)^{\nu}, \quad z \in \mathbb{D},
\]

which implies \( F \in \mathcal{B}_H^\nu(\nu/2) \). Since \( \mathcal{B}_H^\nu(\nu) \) preserves affine invariance for each \( \nu > 0 \), we get \( f \in \mathcal{B}_H^\nu(\nu/2) \). Clearly, \( \mathbb{B}_H(\nu) \subset \mathcal{B}_H^\nu(\nu/2) \) from Example [11]. The sharpness follows if we choose

\[
f(z) = f_\nu(z) = \int_0^z \left( \frac{1 + t}{1 - t} \right)^{\nu/2} dt + b_1 \int_0^z \left( \frac{1 + t}{1 - t} \right)^{\nu/2} dt, \quad z \in \mathbb{D},
\]

where \( |b_1| < 1 \). Indeed, it is easy to see that \( ||P_{f_\nu}|| = \nu \) and \( f_\nu \in \mathcal{B}_H^\nu(\nu/2) \) but \( f_\nu \notin \mathcal{B}_H^\nu(\varepsilon) \) for any \( 0 < \varepsilon < \nu/2 \).

If \( f \) is restricted to be analytic, then a similar proof shows that \( \mathbb{B}(\nu) \subset \mathcal{B}(\nu/2) \). The sharpness can be easily seen by considering the above function \( f_\nu \) with \( b_1 = 0 \). \( \Box 

3.2. Subordination principles. Every bounded analytic function in \( \mathbb{D} \) belongs to the (analytic) Bloch space \( B(1) \). This fact also holds for harmonic mappings (see [10, Theorem 3]) and for a simpler proof of it (using subordination), we refer to [6, Theorem A]. That is, if a harmonic mapping \( f \) is bounded in \( \mathbb{D} \), then \( f \) belongs to the (harmonic) Bloch space \( \mathcal{B}_H(1) \). Next we will investigate some subordination principles for some harmonic Bloch mappings.

Let \( \mathcal{A}_D \) denotes the class of analytic functions \( \phi : \mathbb{D} \to \mathbb{D} \) and \( \mathcal{A}_D^0 \) denotes the subclass of \( \mathcal{A}_D \) with the normalization \( \phi(0) = 0 \). In 2000, Schaubroeck [21] generalized the notion of subordination from analytic functions to harmonic mappings. Let \( f \) and \( F \) be two harmonic mappings in \( \mathbb{D} \). Then \( f \) is subordinate to \( F \), denoted by \( f \prec F \), if there is a function \( \phi \in \mathcal{A}_D^0 \) such that \( f = F \circ \phi \). We denote \( f \preceq F \) if there exists a function \( \phi \in \mathcal{A}_D \) such that \( f = F \circ \phi \). Clearly, if \( f \prec F \) then \( f \preceq F \).

Theorem 3. (Subordination principle) Let \( f \) and \( F \) be two harmonic mappings in \( \mathbb{D} \). If \( f \preceq F \) and \( F \in \mathcal{B}_H(1) \) (resp. \( \mathcal{B}_H^*(1) \)), then \( f \in \mathcal{B}_H(1) \) (resp. \( \mathcal{B}_H^*(1) \)). In particular, if \( f \preceq F \) and \( F \in \mathcal{B}(1) \), then \( f \in \mathcal{B}(1) \).

Proof. We just need to prove the case of \( \nu \)-Bloch-type mappings since the proof of the remaining cases are similar. Assume that \( f \preceq F \) and \( F \in \mathcal{B}_H^*(1) \). Then there exists a function \( \phi \in \mathcal{A}_D \) such that \( f = F \circ \phi \). We find that

\[
J_f(z) = J_F(\phi(z))|\phi'(z)|^2
\]
and by the Schwarz-Pick lemma, we get \((1 - |z|^2)|\phi'(z)| \leq 1 - |\phi(z)|^2\). Consequently,
\[
(1 - |z|^2)\sqrt{|J_f(z)|} = (1 - |z|^2)|\phi'(z)|\sqrt{|J_F(\phi(z))|}
\]
\[
\leq (1 - |\phi(z)|^2)\sqrt{|J_F(\phi(z))|} \leq \beta^*_1(F) < \infty, \quad z \in \mathbb{D},
\]
which clearly shows that \(f \in \mathcal{B}_H^1(1)\). \(\square\)

**Remark 1.** We remind that \(f = h + \overline{g} \in \mathcal{B}_H(1)\) does not mean that either \(h, g\) or \(f\) is bounded even if \(f\) is sense-preserving in \(\mathbb{D}\). For instance, consider
\[
f_1(z) = h(z) + g(z) = \log(1 - z) + \overline{z} + \log(1 - z) = \overline{z} + 2 \log |1 - z|.
\]
and
\[
f_2(z) = h(z) - g(z) = \log(1 - z) - \overline{z} + \log(1 - z) = -\overline{z} + 2i \arg(1 - z).
\]
Then it is easy to verify that \(f_1, f_2 \in \mathcal{B}_H^1(1)\), and both \(f_1\) and \(f_2\) are sense-preserving in \(\mathbb{D}\). However, except \(f_2\), neither \(h\), nor \(g\) nor \(f_1\) is bounded in \(\mathbb{D}\).

### 4. Growth and Coefficients Estimates

In this section, we investigate some growth and coefficients estimates for functions in \(\mathcal{B}_H^\nu(\nu)\). For corresponding results in the case of \(\mathcal{B}_H(\nu)\), the reader can refer to \([3, 23]\).

**Theorem 4.** Suppose that \(f = h + \overline{g} \in \mathcal{B}_H^\nu(\nu)\) is sense-preserving in \(\mathbb{D}\) with the dilatation \(\omega_f\), where \(h\) and \(g\) are given by (1). Then
\[
\max\{|h(z) - a_0|, |g(z)|\} \leq \beta^*_\nu(f) \sqrt{\frac{1 + |\omega_f(0)|}{1 - |\omega_f(0)|}} h_\nu(r), \quad |z| = r < 1,
\]
where \(h_\nu\) is defined by (3). The estimate is sharp in order of magnitude for each \(\nu > 1/2\). If \(\nu < 1/2\), then each of \(h, g, f\) is bounded in \(\mathbb{D}\).

**Proof.** Let \(|z| = r < 1\). Following the proof of Proposition 3 and (3), because \(f\) is sense-preserving, we have
\[
\max\{|h(z) - a_0|, |g(z)|\} = \max \left\{ \left| z \int_0^1 h'(t) dt \right|, \left| z \int_0^1 g'(t) dt \right| \right\}
\leq r \int_0^1 |h'(t) dt|
\leq \beta^*_\nu(f) \sqrt{\frac{1 + |\omega_f(0)|}{1 - |\omega_f(0)|}} \int_0^1 \frac{r}{(1 - r^2 t^2)^{\nu+1/2}} dt
\leq \beta^*_\nu(f) \sqrt{\frac{1 + |\omega_f(0)|}{1 - |\omega_f(0)|}} \int_0^1 \frac{r}{(1 - rt)^{\nu+1/2}} dt
= \beta^*_\nu(f) \sqrt{\frac{1 + |\omega_f(0)|}{1 - |\omega_f(0)|}} h_\nu(r).
\]

For each \(\nu > 1/2\), the sharpness of the order of magnitude can be seen from the functions \(f_{\nu,t} = h_\nu + \overline{g_{\nu,t}}\) defined by (2) for \(t \in [0, 1)\). Clearly, it is sharp for \(h_\nu\) from its
formulation. Fix \( t \in [0, 1) \). It is also sharp for the function \( g_{\nu,t} \), since for \( x \in (0, 1) \) and any \( \varepsilon > 0 \),

\[
(1 - x^2)^{\nu-1/2-\varepsilon} |g_{\nu,t}(x)| = \frac{1 + (1 - x)^{\nu-1/2}}{(1 - x)^\varepsilon} \left| \frac{1 - (1 - x)^{\nu-1/2}}{\nu - 1/2} - \frac{1 - t}{\nu - 3/2} \left[ (1 - x) - (1 - x)^{\nu-1/2} \right] \right| \to \infty
\]
as \( x \to 1^- \) when \( \nu > 1/2 \) but \( \nu \neq 3/2 \), and

\[
(1 - x^2)^{1-\varepsilon} |g_{3/2,t}(x)| = (1 + x)^{1-\varepsilon}/(1 - x)^\varepsilon |x + (1 - t)(1 - x) \log(1 - x)| \to \infty
\]
as \( x \to 1^- \).

If \( \nu < 1/2 \), then for \( |z| = r < 1 \) we have

\[
\max\{ |h(z) - a_0|, |g(z)|\} \leq \beta_\nu^*(f) \sqrt{\frac{1 + |\omega_f(0)|}{1 - |\omega_f(0)|}} (1/2 - \nu)^{-1} (1 - (1 - r)^{1/2-\nu})
\]

\[
= \beta_\nu^*(f) \sqrt{n} \leq \frac{\beta_\nu^*(f)}{\sqrt{1 - |\omega_f(0)|^2}},
\]

and

\[
\max\{ |a_n|, |b_n|\} \leq \frac{\beta_\nu^*(f)}{\sqrt{2\nu + 1}} \left( \frac{e}{2\nu + 1} \right)^{\nu+1/2} \sqrt{\frac{1 + |\omega_f(0)|}{1 - |\omega_f(0)|}} (n + 2\nu)^{-1/2}, \quad n \geq 2.
\]

**Proof.** The first inequality follows if we set \( z = 0 \) in (1). For the second inequality, we recall from (2) that

\[
|g'(z)|^2 < |h'(z)|^2 \leq \frac{1 + |\omega_f(0)|}{1 - |\omega_f(0)|} \frac{\beta_\nu^*(f)^2}{(1 - |z|^2)^{2\nu+1}}, \quad z \in \mathbb{D}.
\]

We integrate this inequality over the circle \( |z| = r \) and get

\[
\sum_{n=1}^{\infty} n^2 |b_n|^2 r^{2(n-1)} \leq \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2(n-1)} \leq \frac{1 + |\omega_f(0)|}{1 - |\omega_f(0)|} \frac{\beta_\nu^*(f)^2}{(1 - r^2)^{2\nu+1}}.
\]

Thus, for \( n \geq 2 \), we obtain

\[
\max\{ |a_n|, |b_n|\} \leq \frac{\beta_\nu^*(f)}{n} \sqrt{\frac{1 + |\omega_f(0)|}{1 - |\omega_f(0)|}} \frac{1}{r^{n-2}(1 - r)^{2\nu+1/2}}.
\]
It is a simple exercise to see that \( r^{1-n}(1 - r^2)^{-(\nu+1/2)} \) is maximized in \( r \in (0, 1) \) for \( r = \sqrt{\frac{n-1}{n+2\nu}} \). Consequently,

\[
\max \{|a_n|, |b_n|\} \leq \frac{\beta^*_\nu(f)}{n} \left( \frac{1 + |f(0,0)|}{1 - |f(0,0)|} \right)^{n/2-1/2} \left( \frac{n + 2\nu}{2\nu + 1} \right)^{\nu+1/2} \left( \frac{n + 2\nu}{2\nu + 1} \right)^{\nu+1/2},
\]

where

\[
\phi_\nu(x) = \left( 1 + \frac{2\nu + 1}{x - 1} \right)^{x/2} \left( 1 + \frac{2\nu}{x} \right), \quad x \geq 2.
\]

Next we prove that \( \phi_\nu \) is an increasing function of \( x \) to its limit \( e^{\nu+1/2} \) in \([2, \infty)\). Clearly, \( \phi_\nu(x) > 0 \) for all \( x \geq 2 \). For convenience, we let

\[
\Phi_\nu(x) = (\log \phi_\nu(x))' = \frac{\phi'_\nu(x)}{\phi_\nu(x)} = \frac{1}{2} \log \left( \frac{x + 2\nu}{x - 1} \right) - \frac{(2\nu + 1)x + 4\nu}{2(x + 2\nu)}.
\]

Differentiating with respect to \( x \) yields

\[
\Phi'_\nu(x) = -\frac{\psi_\nu(x)}{2x^2(x - 1)(x + 2\nu)^2}, \quad \psi_\nu(x) = (2\nu - 1)^2x^2 + 8(\nu - \nu^2)x + 8\nu^2.
\]

If \( \nu = 1/2 \), then \( \psi_\nu(x) = 2x + 2 \geq \psi_\nu(2) = 6 > 0 \) for all \( x \geq 2 \). If \( \nu \neq 1/2 \), then we obtain

\[
\psi'_\nu(x) = 2(2\nu - 1)^2x + 8(\nu - \nu^2) \geq \psi'_\nu(2) = 8(\nu - 1/2)^2 + 2 > 0 \quad \text{for all} \quad x \geq 2
\]

and thus, \( \psi_\nu(x) \geq \psi_\nu(2) = 4(2\nu^2 + 1) > 0 \) for all \( x \geq 2 \).

Hence, \( \Phi'_\nu(x) < 0 \) in \([2, \infty)\) so that \( \Phi_\nu(x) = \lim_{x \to \infty} \Phi_\nu(x) = 0 \) for all \( x \in [2, \infty) \). Therefore, we obtain \( \phi'_\nu(x) > 0 \) in \([2, \infty)\) and the proof is complete. \( \square \)

5. Bohr’s inequalities

One of the classical problems in the theory of analytic functions which inspire many researchers is to determine

\[
r_0 = \sup \left\{ r \in (0, 1) : M_f(r) := \sum_{n=0}^{\infty} |a_n|r^n \leq 1 \right\},
\]

where the supremum is taken over the class which consists of all functions of the form \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) that converges in \( \mathbb{D} \) and \( |f(z)| \leq 1 \) in \( \mathbb{D} \). It is well-known that \( r_0 = 1/3 \) and the number 1/3 is called the classical Bohr radius for the class of all analytic self-maps of the unit disk \( \mathbb{D} \). Many authors have discussed the Bohr radius and extended this notion to various settings which led to the introduction of Bohr’s phenomenon. As remarked in the introduction, we refer to [2, 15, 16] and the references therein for results on this topic. Moreover, in [15] the authors introduced the notion of \( p\)-Bohr radius for harmonic mappings which is defined as follows: Let \( f = h + g \) be a harmonic mapping in
where $h$ and $g$ have the form (1). For $p \geq 1$, the $p$-Bohr radius for $f$ is defined to be the largest value $r_p$ such that

$$|a_0| + \sum_{n=1}^{\infty} (|a_n|^p + |b_n|^p)^{1/p} r^n \leq 1 \text{ for } |z| = r \leq r_p.$$  

Clearly, all these radii coincide in the analytic case. The classical case $p = 1$ is considered first time in [1].

In this section, we determine the Bohr radius for analytic functions in $B(\nu)$ and $p$-Bohr radius for harmonic mappings in $B_H(\nu)$ and $B_H^*(\nu)$. The following results are generalizations of that of the results of Kayumov et al. [16, Section 4].

**Theorem 6.** Assume that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belongs to $B(\nu)$ and $||f||_{B(\nu)} \leq 1$. Then

$$\sum_{n=0}^{\infty} |a_n| r^n \leq 1$$

for $|z| = r \leq r(\nu) = \max\{r_1(\nu), r_2(k)\}$ when $\nu \in (k/2, (k+1)/2]$ for some $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Here $r_1(\nu)$ is the unique solution in $(0, 1)$ to the equation

$$6(1 - r^2)^{2\nu} - \pi^2 r^2 = 0 \quad (6)$$

and $r_2(k)$ is the unique solution in $(0, 1)$ to the equation

$$r F_k(r) - 1 + r = 0, \quad (7)$$

where

$$F_k(r) = \begin{cases} \sum_{n=1}^{\infty} \frac{r^n}{n_2}, & \text{for } k = 0, \\ \log \frac{1}{1 - r}, & \text{for } k = 1, \\ \frac{1}{k} \log \frac{1}{1 - r} + \frac{1}{k} \sum_{n=1}^{k-1} \frac{1}{n} \left( \frac{1}{(1 - r)^n} - 1 \right), & \text{for } k \geq 2. \end{cases}$$

Moreover, $r(\nu)$ can not be replaced by $r_3(\nu)$ when $\nu \geq 1$, where

$$r_3(1) = 0.624162, \quad \text{and } r_3(\nu) = \min \left\{ 0.624162, \sqrt{1 - 1/\nu - \sqrt{2\nu - 1}} \right\} \text{ for } \nu > 1.$$  

**Proof.** By hypothesis, we have $||f||_{B(\nu)} \leq 1$ which gives

$$\left| \sum_{n=1}^{\infty} n a_n z^{n-1} \right|^2 = |f'(z)|^2 \leq \frac{(1 - |a_0|)^2}{(1 - |z|^2)^{2\nu}}, \quad z \in \mathbb{D}.$$  

Integrating the inequality over the circle $|z| = r$ yields

$$\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2(n-1)} \leq \frac{(1 - |a_0|)^2}{(1 - r^2)^{2\nu}}. \quad (9)$$
By the classical Cauchy–Schwarz inequality, we obtain
\[
|a_0| + \sum_{n=1}^{\infty} |a_n|r^n \leq |a_0| + \sqrt{\sum_{n=1}^{\infty} n^2|a_n|^{2r^{2n}} \sum_{n=1}^{\infty} \frac{1}{n^2}} \\
\leq |a_0| + \frac{(1 - |a_0|)r}{(1 - r^{2})^\frac{\nu}{2}} \sqrt{\frac{\pi^2}{6}} \\
\leq 1 \text{ for } r \leq r_1(\nu),
\]
where \( r_1(\nu) \) is the unique solution in \((0,1)\) to the equation of (3). In fact, for each \( \nu \in (0, \infty) \), the function \( r/(1 - r^{2})^\nu \) increases from 0 to \( \infty \) in \([0,1)\).

On the other hand, if \( \nu \in \left(\frac{k}{2}, \frac{k+1}{2}\right] \) for some \( k \in \mathbb{N}_0 \), then it follows from (9) that
\[
\sum_{n=1}^{\infty} n^2|a_n|^{2r^{n-1}} \leq \frac{(1 - |a_0|)^2}{(1 - r)^{k+1}}.
\]
Integrating the above inequality twice (with respect to \( r \)) yields
\[
\sum_{n=1}^{\infty} |a_n|^{2r^n} \leq (1 - |a_0|)^2 F_k(r),
\]
where \( F_k(r) \) is defined by (8). Applying the Cauchy–Schwarz inequality again, we have
\[
|a_0| + \sum_{n=1}^{\infty} |a_n|r^n \leq |a_0| + \sqrt{\sum_{n=1}^{\infty} |a_n|^{2r^n} \sum_{n=1}^{\infty} r^n} \\
\leq |a_0| + (1 - |a_0|) \sqrt{\frac{F_k(r)r}{1 - r}} \\
\leq 1 \text{ for } r \leq r_2(k),
\]
where \( r_2(k) \) is the unique solution to the equation of (7). Note that both \( F_k(r) \) and \( r/(1 - r) \) are strictly increasing in \([0,1)\). Combining the two estimates yields the desired conclusion.

For the upper bound of \( r(\nu) \), since \( B(\nu) \supseteq B(1) \) for any \( \nu \geq 1 \), it follows from [16, Theorem 9] that \( r(\nu) \) can not be replaced by 0.624162 for \( \nu \geq 1 \). In addition, let’s consider the function
\[
f_\nu(z) = \frac{(1 - z^2)^{1-\nu} - 1}{2(\nu - 1)} = \sum_{n=1}^{\infty} a_{\nu,n} z^n, \quad z \in \mathbb{D}.
\]
A basic computation shows that \( f_\nu \in B(\nu) \) and \( ||f_\nu||_{B(\nu)} = 1 \) when \( \nu > 1 \). It is easy to see that all coefficient \( a_{\nu,n} \) are non-negative real number for each \( \nu > 1 \) and \( a_{\nu,n} = 0 \) for odd integer values of \( n \geq 1 \). For \( \nu > 1 \), we consider the following inequality
\[
\sum_{n=1}^{\infty} a_{\nu,n} r^n = \frac{(1 - r^{2})^{1-\nu} - 1}{2(\nu - 1)} \leq 1,
\]
provided \( r \leq \sqrt{1 - 1/ \nu - \sqrt{2\nu - 1}} \) and thus, the conclusion follows. \( \square \)
Integrating the inequality over the circle

Theorem 7. Suppose that \( f = h + g \in B_H(\nu) \), where \( h \) and \( g \) are given by (1). If \( \|f\|_{B_H(\nu)} \leq 1 \) and \( p \geq 1 \), then we have

\[
|a_0| + \sum_{n=1}^{\infty} (|a_n|^p + |b_n|^p)^{1/p} \leq 1
\]

for \( |z| = r \leq \max\{r_1(\nu,p), r_2(k,p)\} \) when \( \nu \in (k/2, (k+1)/2] \) for some \( k \in \mathbb{N}_0 \). Here \( r_1(\nu,p) \) is the unique solution in \((0,1)\) to the equation

\[
6(1-r^2)^{2\nu} - M_p r^2 = 0
\]

and \( r_2(k,p) \) is the unique solution in \((0,1)\) to the equation \( M_p r F_k(r) - 1 + r = 0 \), where \( M_p = \max\{2/(2p-1), 1\} \) and \( F_k(r) \) is defined by (5).

Proof. By assumption, we see that

\[
|h'(z)|^2 + |g'(z)|^2 \leq (|h'(z)| + |g'(z)|)^2 \leq \frac{(1 - |a_0|^2)}{(1 - |z|^2)^{2\nu}}, \quad z \in \mathbb{D}.
\]

Integrating the inequality over the circle \( |z| = r \) so we get

\[
\sum_{n=1}^{\infty} n^2(|a_n|^2 + |b_n|^2) \leq \frac{(1 - |a_0|^2)}{(1 - r^2)^{2\nu}}.
\]

Using the Cauchy-Schwarz inequality, we obtain

\[
|a_0| + \sum_{n=1}^{\infty} (|a_n|^p + |b_n|^p)^{1/p} \leq |a_0| + \sqrt{\left( \sum_{n=1}^{\infty} n^2(|a_n|^p + |b_n|^p)^{2/p} \right)^{1/2} \sum_{n=1}^{\infty} \frac{1}{n^2}} \leq |a_0| + \sqrt{M_p \sum_{n=1}^{\infty} n^2(|a_n|^2 + |b_n|^2) r^{2n}} \leq |a_0| + \sqrt{M_p \frac{(1 - |a_0|^2)r}{(1 - r^2)^{2\nu}} \left( \frac{\pi}{\sqrt{6}} \right)}
\]
which is less than or equal to 1 provided \( r \leq r_1(\nu, p) \), where \( r_1(\nu, p) \) is defined by (10). If \( \nu \in (k/2, (k + 1)/2] \) for some \( k \in \mathbb{N}_0 \), then we can combine the above proof with the corresponding proof of Theorem 7. The resulting discussion completes the proof. \(\square\)

Next we will study \( p \)-Bohr radius for functions in \( \mathcal{B}_H^\nu(\nu) \). Consider

\[
f(z) = h(z) + g(z) = \frac{1}{1 - z} + \left( \frac{z}{1 - z} \right)
\]

so that \( a_0 = 1 \) and \( a_n = b_n = 1 \) for \( n \geq 1 \). Clearly, \( f \in \mathcal{B}_H^\nu(\nu) \) and \( ||f||_{\mathcal{B}_H^\nu(\nu)} = |a_0| = 1 \) for any \( \nu > 0 \). However, we have

\[
|a_0| + \sum_{n=1}^{\infty} (|a_n|^p + |b_n|^p)^{1/p} r^n > |a_0| = ||f||_{\mathcal{B}_H^\nu(\nu)}
\]

for any \( r > 0 \). In this case, the \( p \)-Bohr radius for \( f \) is 0. This is the reason why we add the condition of sense-preserving in the following result.

**Theorem 8.** Suppose that \( f = h + \overline{g} \in \mathcal{B}_H^\nu(\nu) \) is a sense-preserving harmonic mapping, where \( h \) and \( g \) are given by (1). If \( ||f||_{\mathcal{B}_H^\nu(\nu)} \leq 1 \) and \( p \geq 1 \), then

\[
|a_0| + \sum_{n=1}^{\infty} (|a_n|^p + |b_n|^p)^{1/p} r^n \leq 1
\]

for \( |z| = r \leq \max\{r_1(\nu, p, |\omega_f(0)|), r_2(k, p, |\omega_f(0)|)\} \) when \( \nu \in (k/2, (k + 1)/2] \) for some \( k \in \mathbb{N}_0 \). Here \( r_1(\nu, p, |\omega_f(0)|) \) is the unique solution in \((0, 1)\) to the equation

\[
3(1 - |\omega_f(0)|)(1 - r^2)^{\nu + 1} - M_p r^2 (1 + |\omega_f(0)|) r^2 = 0
\]

and \( r_2(k, p, |\omega_f(0)|) \) is the unique solution in \((0, 1)\) to the equation

\[
2M_p (1 + |\omega_f(0)|) r F_k(r) - (1 - |\omega_f(0)|)(1 - r) = 0,
\]

where \( M_p = \max\{2^{2/p - 1}, 1\} \) and \( F_k(r) \) is defined by (8).

**Proof.** By hypothesis \( |g'(z)| < |h'(z)| \) and \( ||f||_{\mathcal{B}_H^\nu(\nu)} \leq 1 \) and thus, it follows from (5) that

\[
|h'(z)| \leq \frac{1 - |a_0|}{(1 - |z|^2)^{\nu + 1/2}} \sqrt{1 + |\omega_f(0)| - |\omega_f(0)|^2} - 1, \quad z \in \mathbb{D}.
\]

As in the proof of the previous theorem, we obtain that

\[
\sum_{n=1}^{\infty} n^2 (|a_n|^2 + |b_n|^2) r^n (\nu - 1) \leq 2 \frac{1 + |\omega_f(0)|}{1 - |\omega_f(0)|^2} (1 - |a_0|^2) \frac{(1 - r^2)^{2\nu + 1}}{1 - (1 - r^2)^{2\nu + 1}}.
\]

The remaining part of the proof is identical to Theorem 7 and thus, we omit the details. The proof is complete. \(\square\)

The dependence of \( |\omega_f(0)| \) about \( p \)-Bohr radius in Theorem 8 can be seen from the following example.
Example 3. For $t \in [1/2, 1)$, we consider the one parameter family of functions $f_t$ on $\mathbb{D}$ given by
\[
f_t(z) = h_t(z) + g_t(z) = \sum_{n=0}^{\infty} a_n(t) z^n + \sum_{n=1}^{\infty} b_n(t) z^n, \quad z \in \mathbb{D},
\]
where
\[
h_t(z) = 1 - 2\sqrt{t - t^2} + \frac{1}{2} \log \frac{1 + z}{1 - z} \quad \text{and} \quad g_t(z) = \frac{t - 1}{2} \log(1 - z^2) + \frac{t}{2} \log \frac{1 + z}{1 - z}.
\]
It is easy to see that each $f_t$ is sense-preserving in $\mathbb{D}$ with the dilatation $\omega_{f_t}(z) = (1-t)z + t$. We find that
\[
h_t'(z) = \frac{1}{1 - z^2}, \quad |g_t'(z)| = \left| \frac{(1-t)z + t}{1 - z^2} \right| \geq \frac{t - (1-t)|z|}{|1 - z^2|} \geq \frac{2t - 1}{|1 - z^2|}, \quad z \in \mathbb{D},
\]
and thus,
\[
(1 - |z|^2) \sqrt{J_{f_t}(z)} \leq (1 - |z|^2) \sqrt{\frac{1}{|1 - z^2|} - \frac{(2t - 1)^2}{|1 - z^2|^2}} \leq 2\sqrt{t - t^2}, \quad z \in \mathbb{D},
\]
which implies that $f_t \in B_{p,t}(1)$. Also, we observe that
\[
(1 - |x|^2) \sqrt{J_{f_t}(x)} \to 2\sqrt{t - t^2} \quad \text{as} \quad (-1, 0) \ni x \to -1^+,
\]
which infers $\beta_1^t(f_t) = 2\sqrt{t - t^2}$ and $\|f_t\|_{B_{p,t}(1)} = 1$. Clearly,
\[
|a_0(t)|^p + \sum_{n=1}^{\infty} |a_n(t)|^p + |b_n(t)|^{1/p} \geq |a_0(t)| = 1 - 2\sqrt{t - t^2}
\]
for any $r > 0$. Note that $|\omega_{f_t}(0)| = |g_t'(0)| = |h_t'(0)| = t$ and $1 - 2\sqrt{t - t^2} \to 1 = \|f_t\|_{B_{p,t}(1)}$ as $t \to 1^-$. This means that if $t$ approaches to $1^-$, then the $p$-Bohr radius for $f_t$ approaches to 0.

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