Two New Techniques to Synchronize Phase Planar Systems Using Discontinuous Feedback

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Abstract. We present two algorithms to synchronize second order systems that can be described by phase state variables, we call them phase planar systems. We use the master-slave configuration and the synchronization objective is to obtain identical synchronization between the master and slave systems in spite of the existence of external perturbations and parametric variations. We use discontinuous control techniques to design the coupling signal that produce sliding modes of first and second order. These discontinuous controllers render the closed loop system robust with respect to matched bounded disturbances and to terms produced by parametric variations. The performance of the proposed synchronization techniques are illustrated experimentally.

1. Introduction
Synchronization, in its most general interpretation, means correlated or corresponding in time behavior of two or more processes [1]. In some situations the synchronization is a natural phenomenon; however, in other cases we must add an interconnection or controller system to attain it, or to improve its transient characteristic. In this situation the synchronization became a control objective and it is called controlled synchronization [2]. In this sense, several control techniques have been used in the design of coupling signals to reach synchronization between two or more systems.

There are many applications of controlled synchronization, for example, the synchronization of chaotic systems for the development of private communications systems [3] and [4], and the synchronization of mechanical systems in production processes [5] and [6].

A common interconnection scheme is the so called master/slave. Here a system, denominated master, imposes its dynamics to the rest of the systems, called slave systems.

Many results on synchronization assume that the systems are identical and that parametric uncertainties and external disturbances do not exist. Some examples of the control techniques used in the design of the coupling signals under these conditions are the linear state feedback for the synchronization of chaotic systems [7] and PID controllers to synchronize mechanisms [6].

There exist several problems in the controlled synchronization that have not been solved completely, among them, we can mention the lack of robustness in the synchronization of systems under external disturbances and parametric uncertainties as well as the synchronization of systems of different nature.
There are some proposals to solve these problems, one of them is to use the sliding control technique to design the coupling signal. This technique is used in [8] for the synchronization of chaotic systems with uncertainties. An option to solve the problem of parametric uncertainties is to use of adaptive control, see for example [5] where a PID controller with an adaptation mechanism is proposed to synchronize mechanical systems.

An important class of planar systems are those represented by the following equation

\[ \dot{y} + f(t, y, \dot{y}) = u \]  

(1)

where \( y \in \mathbb{R} \) is the output, \( u \in \mathbb{R} \) is the input and \( f: \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R} \) is a function continuous in \( t, y \) and \( \dot{y} \); we call them phase planar systems. To this class of systems belong many real and didactic systems; for example, one degree of freedom (1DOF) Lagrangian and Hamiltonian systems, linear systems, some systems with discontinuities as mechanical systems with Coulomb friction, and some chaotic systems.

In this paper we propose two techniques to synchronize two phase planar systems based on discontinuous control techniques. The objective is to obtain asymptotic synchronization between two systems connected in a master/slave configuration, in spite of the presence of parametric uncertainties and bounded disturbances present in both systems.

The first procedure is based on a first order sliding control technique. Both systems do not need to be identical. We show that the control scheme is robust with respect to some parametric uncertainties and matched bounded disturbances.

In the second technique, we propose a coupling signal that eliminates not desired terms and adds a proportional, a derivative, and a discontinuous term of the synchronization error. The last term gives robustness to the closed loop system and produces a second order sliding mode. This technique guarantees asymptotic synchronization with exponential rate of convergence.

This paper will be outlined as follows. The statement of the problem is given in section 2, where the type of systems we consider in this work and the synchronization criterion are defined. In sections 3 and 4 we describe the design of the coupling signal using the first and second order sliding mode techniques, respectively. To illustrate the proposed synchronization technique, in section 5 we present experimental results. Finally, the conclusions are given in section 6.

2. The synchronization problem
In this section we present the synchronization problem for phase planar systems. Let us consider a master system given by

\[
\begin{bmatrix}
\dot{x}_{1,m} \\
\dot{x}_{2,m}
\end{bmatrix} = \begin{bmatrix}
x_{2,m} \\
f_m (x_m) + \gamma_m (t) + c_m u_m
\end{bmatrix},
\]

(2)

\[ y_m = x_{1,m}, \]  

(3)

where \( x_m = [x_{1,m}, x_{2,m}]^T \) is the state vector, \( f_m : \mathbb{R}^2 \rightarrow \mathbb{R} \) is a nonlinear function, \( \gamma_m (t) \) is a bounded perturbation that satisfies \( |\gamma_m (t)| \leq \mu_m \), where \( \mu_m \) is a constant. \( u_m \) and \( y_m \) are the input and the output of the system, respectively. We assume that the signal \( u_m \) is bounded and produces a bounded behavior of the system.

Similarly, the slave system is represented by

\[
\begin{bmatrix}
\dot{x}_{1,s} \\
\dot{x}_{2,s}
\end{bmatrix} = \begin{bmatrix}
x_{2,s} \\
f_s (x_s) + \gamma_s (t) + c_s (u_s + v)
\end{bmatrix},
\]

(4)

\[ y_s = x_{1,s}, \]  

(5)
where \( x_s = [x_{1,s}, x_{2,s}]^T \) is the state vector, \( \gamma_s(t) \) is an external bounded perturbation such that \( |\gamma_s(t)| \leq \mu_s \), where \( \mu_s \) is a constant, \( v \) is a coupling signal, and the other terms are defined similarly to the master system (2-3). Also consider a functional \( \Gamma(\cdot) \) that defines the synchronization objective given by

\[
\Gamma(x_m(t), x_s(t)) = \|x_m(t) - x_s(t)\|.
\]

**Definition 1.** The slave system (4-5) is called to be asymptotically synchronized with the master system (2-3), with respect to the functional \( \Gamma(\cdot) \), if

\[
limit_{t \to \infty} \Gamma(x_m(t), x_s(t)) = 0 \quad \forall \quad x_m(0), x_s(0) \in \mathbb{R}^2.
\]

**Definition 2.** The problem of controlled synchronization with respect to the functional \( \Gamma(\cdot) \) is to find a coupling signal \( v \), as a feedback function of the states \( x_m \) and \( x_s \), such that systems (2-3) and (4-5) satisfies the condition (6).

In this paper we focus on the design of a coupling signal \( v \) such that the synchronization problem given in definition 2 will be satisfied in spite of the existence of the uncertainty terms \( \gamma_m(t) \) and \( \gamma_s(t) \).

Let us define the error variables \( e_1 = x_{1,m} - x_{1,s}, e_2 = x_{2,m} - x_{2,s} \), and the error vector \( e = (e_1, e_2)^T \), whose dynamic is given by

\[
\begin{bmatrix}
\dot{e}_1 \\
\dot{e}_2
\end{bmatrix} =
\begin{bmatrix}
e_2 \\
 f_m(x_m) - f_s(x_m - e) + \epsilon(t) - c_s v
\end{bmatrix},
\]

where \( \epsilon(t) = \gamma_m(t) - \gamma_s(t) + c_m u_m - c_s u_s \). Note that this term is bounded, \( |\epsilon(t)| \leq \rho \).

Now the synchronization problem is transformed to design a coupling signal \( v \) such that the origin of system (7) will be an asymptotic stable equilibrium point.

### 3. Design of the coupling signal using first order sliding modes

In this section we propose a coupling signal based on first order sliding mode control technique.

Consider a discontinuity surface defined by

\[
\sigma = ae_1 + e_2,
\]

where \( a \) is a positive constant, and define a coupling signal \( v \) given by

\[
v = F(x_m, e) \text{ sign}(\sigma),
\]

where \( F(\cdot) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) is, in general, a piecewise smooth function and \( \text{sign}(\cdot) \) is the signum function. In the following subsections we present a way to design the surface \( \sigma \) and the coupling signal \( v \) so that the synchronization problem is solved.

#### 3.1. Sliding surface design

The sliding surface must be designed such that, when the trajectories arrived at the discontinuity surface defined by \( \sigma = 0 \), they must be directed to the origin of the state space. One way to define the behavior of system (7) when its trajectories slide in the surface \( \sigma = 0 \) is using the equivalent control approach.

For this case the equivalent control is given by
Substituting $v_{eq}$ into (7) gives

$$
\dot{e}_1 = e_2, \\
\dot{e}_2 = -ae_2.
$$

The last equation is a linear system decoupled from $e_1$ and has an equilibrium point exponentially stable if the constant $a > 0$ (then $e_2 \to 0$ as $t \to \infty$ and, for $\sigma = 0$, $e_1 \to 0$ too). Thus, the trajectories in the discontinuity surface will go to the origin of the error space.

### 3.2. Design of the coupling signal

Now, we will find the conditions on $v$ such that the trajectories out of the surface $\sigma = 0$ go to it in finite time.

Consider the criterion given in [9]: If

$$
\sigma \dot{\sigma} < 0 \quad \forall \sigma \neq 0, \quad t \geq 0,
$$

then the surface $\sigma = 0$ is a sliding surface.

In our case, we have the following

$$
\sigma \dot{\sigma} = \sigma (ae_2 + fm(x_m) - fs(x_m - e) + \epsilon(t)) - csF(x_m, e)|\sigma|.
$$

Let us suppose that $F(\cdot)$ has the form

$$
F(\cdot) = \frac{1}{c_s} h(\cdot)
$$

where

$$
h(\cdot) > a |e_2| + |fm(x_m)| + |fs(x_m - e)| + |\epsilon(t)|
$$

$\forall e, x_m$ and $\forall t \geq 0$; then, the synchronization problem is solved.

On the other hand, the condition about the convergence in finite time to the surface $\sigma = 0$ is also satisfied with the control input given by (8) and (9). We can prove the last statement applying the following criterion given in [9].

If

$$
\lim_{\sigma \to 0^-} \dot{\sigma} > 0, \quad \lim_{\sigma \to 0^+} \dot{\sigma} < 0
$$

then the convergence to the surface is in finite time.

For this case, both conditions are satisfied in global or local form, this depends on the function $h(\cdot)$.

### 4. Synchronization using second order sliding modes

Consider the error system (7) and a coupling signal $v$ given by

$$
v = \frac{1}{c_s} (k_p e_1 + k_e e_2 + \Psi(x_m, e) + k_c \text{sign}(e_1))
$$

where $k_p$ and $k_e$ are positive constants, $\Psi(x_m, e)$ is a function that, in general, eliminates not desired terms in (7) and $\text{sgn}(\cdot)$ is the signum function. The discontinuous term produces a
second order sliding mode [10] and gives, as we will show later, good robustness properties to the closed loop system.

If the master and slave systems have viscous friction and \( \Psi (x_m, e) \) does not depend on \( x_{m,2} \) and \( x_{s,2} \), then the coupling signal \( v \) may not depend on the velocities; the 1DOF Lagrangian systems are an example. For other cases, in general, we need an observer if the velocities are not measured.

Now, substituting (10) into (7), and assuming that the term \( \Psi (x_m, e) \) eliminates all nonlinear terms in (7), the closed loop system is given by

\[
\begin{bmatrix}
\dot{e}_1 \\
\dot{e}_2
\end{bmatrix} = \begin{bmatrix}
e_2 \\
-k_p e_1 - k_v e_2 + \epsilon (t) - k_c \text{sgn} (e_1)
\end{bmatrix}.
\]

(11)

Define a matrix \( A \) as

\[
A = \begin{bmatrix}
0 & 1 \\
-k_p & -k_v
\end{bmatrix},
\]

(12)

and a matrix \( P \)

\[
P = \begin{bmatrix}
p_{11} & p_{12} \\
p_{12} & p_{22}
\end{bmatrix},
\]

(13)

which is the solution of the Lyapunov equation \( A^T P + P A = -I \), where \( I \) is the identity matrix and \( \lambda_{\min} (P) \) and \( \lambda_{\max} (P) \) are its minimum and maximum eigenvalues respectively. Note that \( A \) is strictly Hurwitz \((k_p \text{ and } k_v \text{ are positive}); then \( P > 0 \). The condition on \( k_c \) such that the synchronization problem is solved is given by the following theorem.

**Theorem 1.** The slave system (4-3), with the coupling signal (10), will be asymptotically synchronized with the master system (2-5), that is (6) will be satisfied, if

\[
k_c > 2 \lambda_{\max} (P) \sqrt{\frac{\lambda_{\max} (P)}{\lambda_{\min} (P)}} \left( \frac{k_p \theta}{\theta} \right)
\]

(14)

for some positive constant \( \theta < 1 \).

**Proof.** Note that this result can be proved showing that trajectories of system (11) converge asymptotically to the origin for any bounded disturbance \( \epsilon (t) \).

The proof is divided in two parts; first we define the nominal system as (11) with \( \epsilon (t) = 0 \), and prove the stability of the origin using tools from variable structure systems \(^1\). After that, we find the condition on \( k_c \) such that the stability properties are maintained for the perturbed system.

The nominal system of (11) is defined as

\[
\begin{align*}
\dot{e}_1 &= e_2, \\
\dot{e}_2 &= -k_p e_1 - k_v e_2 - k_c \text{sgn} (e_1).
\end{align*}
\]

(15)

System (15) has two structures: \( S_1 \) for \( e_1 > 0 \),

\[
S_1 : \quad \begin{cases}
\dot{e}_1 = e_2, \\
\dot{e}_2 = -k_p e_1 - k_v e_2 - k_c,
\end{cases}
\]

and \( S_2 \) for \( e_1 < 0 \)

\[
S_2 : \quad \begin{cases}
\dot{e}_1 = e_2, \\
\dot{e}_2 = -k_p e_1 - k_v e_2 + k_c,
\end{cases}
\]

\(^1\) There exists a Lyapunov function to prove stability; however, the proof presented here will help us to show the robust characteristics of the system.
Each structure has a different equilibrium point; for $S_1$ it is $\sigma_{S_1} = (-k_c/k_p, 0)$ and for $S_2$ we have $\sigma_{S_2} = (k_c/k_p, 0)$.

Each equilibrium point is globally exponentially stable with the following Lyapunov functions; for $S_1$

$$V_{S_1} (e) = e^T Pe + 2e^T P\gamma + \left( \frac{k_c}{k_p} \right)^2 p_{11},$$

$$\dot{V}_{S_1} (e) = -e^T e - 2e^T \gamma - \left( \frac{k_c}{k_p} \right)^2,$$

and for $S_2$

$$V_{S_2} (e) = e^T Pe - 2e^T P\gamma + \left( \frac{k_c}{k_p} \right)^2 p_{11},$$

$$\dot{V}_{S_2} (e) = -e^T e + 2e^T \gamma - \left( \frac{k_c}{k_p} \right)^2,$$

where $\gamma = \left[ \frac{k_c}{k_p} \ 0 \right]^T$.

A direct application of the criterion given in [9] to prove the existence of sliding modes allows us to conclude that the discontinuity surface given by $\sigma = e_1 = 0$ is not a sliding surface. Note also that the solutions cross the line $e_1 = 0$ from quadrant II to quadrant I, and from quadrant IV to quadrant III.

Now consider the functions $V_{S_1} (e), V_{S_2} (e)$ and their time derivatives (17) and (19). Functions $V_{S_1} (e)$ and $V_{S_2} (e)$ intersect at the origin with a value $V_{S_i} (0) = (k_c/k_p)^2 p_{11},$ for $i = 1, 2$. Define two neighborhoods of the origin, $\Omega_\epsilon$ with radio $\epsilon > 0$, and $\Omega_\beta$ defined in the following form,

$$\Omega_\beta = \Omega_1 \cup \Omega_2,$$

$$\Omega_1 = \{ e \in \mathbb{R}^2 \mid e_1 \geq 0, V_{S_1} (e) \leq \beta \},$$

$$\Omega_2 = \{ e \in \mathbb{R}^2 \mid e_1 < 0, V_{S_2} (e) \leq \beta \},$$

where $\beta > (k_c/k_p)^2 p_{11}$. Finally, define a neighborhood $\Omega_\delta$ with radio $\delta < \epsilon (\delta$ can depend on $\epsilon$ and $\beta; \delta(\epsilon, \beta)$ such that $\Omega_\delta \subset \Omega_\beta$ (see Figure 1).

Define a countable index set $I = \{1, 2, \ldots\}$, these indices indicate the number of structure commutations of the system. Also define a set of times $T = \{t_1, t_2, \ldots, t_i, \ldots\}$. At these moments the structure commutations appear. We assume that $t_0 < t_1 < t_2$.

If $\| e (t_0) \| < \delta$ and $e (t_0) \in \Omega_k \subset \Omega_\beta$ for some $k = 1, 2$ (the $k$–th structure is active), then the first change of structure appears at time $t_1$, and because $V_{S_k} < 0$, we have $\| e (t_0) \| \geq \| e (t_1) \|$, then $V_{S_k} (e (t_0)) > V_{S_k} (e (t_1))$.

Now $e (t_1)$ is the initial condition for the next structure and, by construction of $V_{S_1}$, $V_{S_k} (e (t_1)) > V_{S_{k+1}} (e (t_1))$ by a factor $11 = p_{11} (k_c/k_p)$, we obtain this factor subtracting $V_{S_{k+1}} (e (t_1))$ to $V_{S_k} (e (t_1))$. The second commutation appears at time $t_2$; the system goes from $\Omega_{k+1}$ to $\Omega_k$, $\| e (t_1) \| \geq \| e (t_2) \|$, $V_{S_{k+1}} (e (t_1)) > V_{S_{k+1}} (e (t_2))$ and $V_{S_{k+1}} (e (t_2)) > V_{S_k} (e (t_2))$ and so on for all $t_i \in T$.

Then, we see that the sequences $W_1 = \{ V_{S_k} (t_1), V_{S_k} (t_3), \ldots \}$ and $W_2 = \{ V_{S_{k+1}} (t_2), V_{S_{k+1}} (t_4), \ldots \}$ are strictly decreasing and lower bounded, and converge to $(k_c/k_p)^2 p_{11}$. Also it is satisfied that $\| e (t_{i+1}) \| < \| e (t_i) \| < \ldots < \| e (t_0) \| < \delta < \epsilon \ \forall t > t_0, \forall i$.

For all $\epsilon > 0$ and $\beta > (k_c/k_p)^2 p_{11}$ we can find a number $\delta$ so that the trajectories initiating in $\Omega_\delta$ will remain within the vicinity $\Omega_\epsilon$ for all $t \geq t_0$. Therefore, the origin is stable in the Lyapunov sense.

To demonstrate asymptotic stability it is enough to notice that

$$\lim_{i \to \infty} V_{S_k} (t_i) = \lim_{i \to \infty} V_{S_{k+1}} (t_i) = \left( \frac{k_c}{k_p} \right)^2 p_{11},$$
this is the value that takes both Lyapunov functions at the origin; then
\[
\lim_{t \to \infty} e(t) = 0.
\]

To demonstrate exponential stability it is enough to notice that the solution in the interval time \([t_0, t_1]\) decreases in exponential form due to the exponential stability of the equilibrium point of that structure. When the structure is changed, the solution maintains the decreasing rate because both structures have the same matrix \(A\), therefore \(\|e(t)\|\) in the time intervals \([t_i, t_{i+1}]\) will be below an exponential function that dominates the solution in the first interval \([t_0, t_1]\); therefore the origin is an exponentially stable equilibrium point.

Finally, to demonstrate that this result is global it is enough to notice that each structure has a global exponentially stable equilibrium; therefore, all properties mentioned before will remain for all initial condition.

Now we analyze the perturbed system (11). Consider the structure \(S_1\) of the system (the analysis of the structure \(S_2\) is similar),
\[
\begin{align*}
\dot{e}_1 &= e_2, \\
\dot{e}_2 &= -k_p e_1 - k_v e_2 + \epsilon(t) - k_c,
\end{align*}
\]

and make the following change of variables \(z_1 = e_1 + k_c/k_p\) and \(z_2 = e_2\). The dynamics of system

\[\text{Figure 1. Definition of the neighborhoods on the state space.}\]
(11), in the new state space, is given by
\[ \dot{z}_1 = z_2, \]
\[ \dot{z}_2 = -k_p z_1 - k_v z_2 + \epsilon(t), \]
or in simplified form,
\[ \dot{z} = Az + g, \]
where \( g = \begin{bmatrix} 0 & \epsilon(t) \end{bmatrix}^T \).

Propose a Lyapunov function
\[ V(z) = z^T P z, \]
where the matrices \( A \) and \( P \) are defined by equations (12) and (13), respectively. The derivative of \( V \) is given by
\[ \dot{V}(z) = -z^T z + 2z^T P g \leq -\|z\|^2 + 2\lambda_{\text{max}}(P) \|z\| \rho. \]

Because \( k_p > 0 \) and \( k_v > 0 \), we can apply lemma 9.2 given in [11], and conclude that, for all \( \|z(t_0)\| > \mu \) the solution \( z(t) \) satisfies
\[ \|z(t)\| \leq k \exp(\gamma_h(t-t_0)) \|z(t_0)\| \]
\[ \forall t_0 \leq t < t_0 + t_f, \]
and
\[ \|z(t)\| \leq \mu \quad \forall t \geq t_0 + t_f, \]
where \( t_f \) is a finite time, and
\[ k = \sqrt{\frac{\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(P)}}, \]
\[ \gamma_h = \frac{(1-\theta)}{2\lambda_{\text{max}}(P)}, \]
\[ \mu = 2\lambda_{\text{max}}(P) \sqrt{\frac{\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(P)}} \theta, \]
for some \( \theta, 0 < \theta < 1 \) and \( \rho \) is maximum value of the disturbance term. This part shows that the ball of radius \( \mu \), with center located at \((-k_c/k_p, 0)\), is an attractor for structure \( S_1 \), denoted as \( B_{S_1} \).

Similarly, the trajectories of the structure \( S_2 \)
\[ \dot{e}_1 = e_2, \]
\[ \dot{e}_2 = -k_p e_1 - k_v e_2 + \epsilon(t) + k_c, \]
converge to the ball \( B_{S_2} \) of radius \( \mu \) and centered at \((k_c/k_p, 0)\).

Therefore, each structure of the perturbed system has an attractor (a ball) of radius \( \mu \), symmetrically located on the \( e_1 \)-axis and at a distance \( r = k_c/k_p \) from the origin. If this distance \( r \) is greater than \( \mu \), i.e. if
\[ k_c > 2\lambda_{\text{max}}(P) \sqrt{\frac{\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(P)}} \left( \frac{k_p \rho}{\theta} \right), \]
then the two attractor \( B_{S_1} \) and \( B_{S_2} \) do not intersect each other, and the behavior of the solution of the perturbed system will be qualitatively similar to the behavior of the nominal system, for which these attractors correspond to the equilibrium points \((\pm k_c/k_p, 0)\), that is, when \( \mu = 0 \). Therefore, the perturbed system converges to the origin in the same way than the nominal system.
5. Some examples

5.1. Synchronization of a simple pendulum and a Duffing oscillator with chaotic behavior using first order sliding modes

The Duffing circuit, shown in Figure 7, is the master system, with a model given by

\[
\dot{z}_{1,m} = z_{2,m}, \quad (21)
\]

\[
\dot{z}_{2,m} = 41.345z_{1,m} - 41.345z_{1,m}^3 - 1.6075z_{2,m} + 12.403\sin(6.43t),
\]

\[
y_m = z_{1,m},
\]

\[
\text{Figure 2. The Duffing circuit.}
\]

The slave system is a perturbed simple pendulum given by

\[
\dot{z}_{1,s} = z_{2,s}, \quad (22)
\]

\[
\dot{z}_{2,s} = -(0.0299 + \Delta_1)z_{2,s} - (0.005 + \Delta_2)\text{sign}\,(z_{2,s})
- (67.91 + \Delta_3)\sin\,(z_{1,s}) + (55.54 + \Delta_4)\,(\tau\,(t) + v\,(t)),
\]

\[
y_s = z_{1,s},
\]

where \(\Delta_i, \, i = 1, \ldots, 4,\) are possible parametric variations\(^2\). Now, we define the discontinuity surface \(S\) as

\[
S = \gamma e_1 + e_2, \quad (23)
\]

and an input control \(v\) given by

\[
v = (k_4 - \Delta_{4\text{max}}) F\,(e, z_m) \text{sign}\,(S)
= (a_1 |e_2| + a_2 |z_{1,m}| + a_3 |z_{1,m}^3| + a_4 |z_{2,m}|
+ a_5 |z_{2,m} - e_2| + a_6) \text{sign}\,(S). \quad (24)
\]

\(^2\) We consider variations of \(\pm20\%\) from the nominal values of the parameters of system (22).
We took the following control parameters: $a_1 = (\gamma/31.973)1.2$, $a_2 = 1.3$, $a_3 = 1.3$, $a_4 = 0.1$, $a_5 = 0.01$, $a_6 = 4.4$ and $\gamma = 30$.

In the experiment, the pendulum was emulated in a real time card. The behavior of the master and slave systems with no control input is shown in Figure 3. As we can see, there is not natural synchronization; their corresponding behavior is very different.

Figure 4 shows the results when the input control (24) is applied to the slave system. As we can see, in this case the error variables $e_1$ and $e_2$ have values near to zero but they do not stay at zero. Therefore, we can say that the systems are approximately synchronized [2]. The synchronization errors and the coupling signal are shown in Figure 5, where we can see the change on the synchronization errors when the coupling signal is applied, at 6.5 seconds in our experiment.

![Figure 3](image-url). Experimental results. Behavior of the Duffing circuit and the pendulum without coupling signal.

### 5.2. Synchronization of two simple pendulums

In this section, we show the application of the proposed synchronization technique using second order sliding modes on the synchronization of two simple pendulums. Because they are Lagrangian systems we do not need to use the velocity in the realization of the coupling signal; therefore, we can attain the synchronization objective without the use of an observer.

Consider two simple pendulums; the master is the mechanical system shown in figure 6 modeled by

\[
\begin{bmatrix}
\dot{x}_{1,m} \\
\dot{x}_{2,m}
\end{bmatrix} = 
\begin{bmatrix}
-k_1 \sin(x_{1,m}) - k_2 x_{2,m} - \gamma_m (t, x_m) + k_3 \tau_m
\end{bmatrix}
\]

The slave system is the electronic circuit shown in figure 7 that emulates the dynamics of the
Figure 4. Experimental results. Behavior of the Duffing circuit (master) and the pendulum (slave) with coupling signal.

Figure 5. Experimental results. Synchronization errors and coupling signal. The coupling signal is applied at 6.5 sec. (master: Duffing circuit, slave: pendulum).
mechanical pendulum, its model is given by
\[
\begin{bmatrix}
\dot{x}_{1,s} \\
\dot{x}_{2,s}
\end{bmatrix}
= 
\begin{bmatrix}
x_{2,s} \\
-k_1 \sin (x_{1,s}) - k_2 x_{2,s} - \gamma_s(t, x_s) + k_3 (\tau_s + v)
\end{bmatrix}
\]

Figure 6. Mechanical pendulum.

The nominal values of the parameters are: \( k_1 = 67.912 \), \( k_2 = 3.05007 \) and \( k_3 = 55.549 \). The dynamics of the synchronization error is given by
\[
\begin{bmatrix}
\dot{e}_1 \\
\dot{e}_2
\end{bmatrix}
= 
\begin{bmatrix}
e_2 \\
-k_2 e_2 + k_1 \sin (x_1 - e) - k_1 \sin (x_1) + \epsilon(t, e) - k_3 v
\end{bmatrix}
\]

where \( \epsilon(t, e) = \gamma_s(t, x_s) - \gamma_m(t, x_m) \) and we considered that \( \tau_m = \tau_s = A \sin (wt) \).

The coupling signal is
\[
v = \frac{1}{k_3} (k_1 (\sin (x_{1,m} - e) - \sin (x_{1,m})) + k_p e_1 + k_c \text{sgn}(e_1))
\]

We select \( k_p = 1 \) and \( k_c = 0.5 \).

Figure 8 shows the behavior of the master an slave outputs and synchronization position error when there is no coupling \((v = 0)\). As we can see, although their dynamics is similar, the synchronization error is large. Figure 9 shows the system outputs, the synchronization error and the coupling signal. We see that the position error is small (about 1% in the position in average), and the coupling signal is well inside of the physical bounds of the electronic driver \((\pm 10V)\).

6. Conclusions
In this paper we have proposed two new algorithm to synchronize phase planar systems. The condition to apply these algorithms is that the disturbances in both systems satisfy
the matching conditions. The main advantage of the proposed algorithms is that it can be applied to synchronize systems with different structure, parametric uncertainties, and bounded perturbations.

In theory, these algorithms guarantee zero error; however, due to high frequency components in the coupling signal, in practice the synchronization errors display small chattering and we obtain approximate synchronization. The magnitude of the chattering in the error state is, in general, directly related to the disturbance terms in the closed loop system. In our examples, the disturbance terms are delays in the real time card and parametric variations in the circuits, as well as modeling errors. However, in many applications these errors can be small enough such that the synchronization algorithm can be applied.

One restriction of these algorithms is that they need full knowledge of the state vector of both systems. When we do not have full access of these states, we need to design observers.

Figure 7. Circuit that emulate the dynamics of a simple pendulum.
Figure 8. Output of the master and slave systems and synchronization error when there is not a coupling signal.

Figure 9. Output of the master and slave systems, synchronization error and coupling signal.
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