Dichotomy Results for Fixed-Point Existence Problems for Boolean Dynamical Systems

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Abstract. A complete classification of the computational complexity of the fixed-point existence problem for Boolean dynamical systems, i.e., finite discrete dynamical systems over the domain \{0, 1\}, is presented. For function classes \(F\) and graph classes \(G\), an \((F, G)\)-system is a Boolean dynamical system such that all local transition functions lie in \(F\) and the underlying graph lies in \(G\). Let \(F\) be a class of Boolean functions which is closed under composition and let \(G\) be a class of graphs which is closed under taking minors. The following dichotomy theorems are shown: (1) If \(F\) contains the self-dual functions and \(G\) contains the planar graphs, then the fixed-point existence problem for \((F, G)\)-systems with local transition function given by truth-tables is NP-complete; otherwise, it is decidable in polynomial time. (2) If \(F\) contains the self-dual functions and \(G\) contains the graphs having vertex covers of size one, then the fixed-point existence problem for \((F, G)\)-systems with local transition function given by formulas or circuits is NP-complete; otherwise, it is decidable in polynomial time.

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1. Introduction

Background on complex systems. A complex system, in a mathematical sense, can be viewed as a collection of highly interdependent variables. A discrete dynamical system is a complex system where variables update their values in discrete time. Though the interdependencies among the variables might have quite simple descriptions on a local level, the overall global behavior of the systems can be as complicated as unpredictable or undecidable (see, e.g., [12, 20]). This phenomenon has been widely studied in the theory of cellular automata [31, 33] and its applications (see, e.g., [16, 21]).
Finite discrete dynamical systems are characterized by finite sets of variables which can take values from a finite domain. In essence (see, e.g., [9,10]), a finite discrete dynamical system (over a finite domain) consists of (1) a finite undirected graph, where vertices correspond to variables and edges correspond to an interdependence between the two connected variables, (2) a set of local transition functions, one for each vertex, that map values of variables depending on the values of all connected variables to new variable values, and (3) an update schedule that governs which variables are allowed to update their values in a certain time step. A formal definition is given in Section 2. Due to their structural simplicity and modeling flexibility, finite discrete dynamical systems are suitable for analyzing the behavior of real-world complex systems. In fact, the conception is motivated by analysis and simulation issues in traffic flow (see, e.g, [2,3]) and inter-domain routing [18]. It also has applications to two-species diffusion-reaction systems such as synchronous or asynchronous versions of the nearest neighbor coalescence reaction $A + A \rightarrow A$ on a lattice in the immobile reactant case [1].

A central problem in the study of discrete dynamical systems is the classification of systems according to the predictability of their behavior. In the finite setting, a certain behavioral pattern is considered predictable if it can be decided in polynomial time whether a given system will show the behavioral pattern [12]. In a rather strong sense, predictability and tractability are identified. It is not surprising that the reachability of patterns is, in general, an intractable problem, i.e., at least NP-hard (see, e.g., [7,17,27]). However, some restricted subclasses of finite discrete dynamical systems, i.e., systems given by restricted sets of local transition functions and network topologies, are known to possess easy-to-predict patterns (see, e.g., [5–7] and the discussion of related work below). For the purpose of analyzing and simulating real-world systems by finite discrete dynamical systems, it is highly desirable to have sharp boundaries between tractable and intractable cases.

A fundamental behavioral pattern for discrete dynamical systems are fixed points (homogeneous states, equilibria). A value assignment to the variables of a system is a fixed point if the values assigned to the variables are left unchanged if the system updates the values. A series of recent papers has been devoted to identification of finite systems with tractable/intractable fixed-point analyses [8, 28–30]. But although it is an old question how common intractability results are for discrete dynamical systems (see [32, Problem 19]), a precise characterization of the islands of tractability even for the fixed-point existence problem in the simplest case of the Boolean domain \{0, 1\} has remained an open problem (see [8]).

**Contributions of the paper.** We contribute to the problem of classifying Boolean (discrete) dynamical systems, i.e., finite discrete dynamical systems over the domain \{0, 1\}, with regard to the computational complexity of the fixed-point existence problem, i.e., decide whether a given system has a fixed point, in two ways.

A first contribution is the proposal of a general analysis framework for systems. We say that a Boolean dynamical system is an $(F,G)$-system if its local