GLOBAL SOLUTION FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS WITH A CLASS OF LARGE DATA IN $BMO^{-1}(\mathbb{R}^3)$

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Abstract. In this paper, we shall establish the global well-posedness, the space-time analyticity of the Navier-Stokes equations for a class of large periodic data $u_0 \in BMO^{-1}(\mathbb{R}^3)$. This improves the classical result of Koch & Tataru [10], for the global well-posedness with small initial data $u_0 \in BMO^{-1}(\mathbb{R}^n)$.

Keywords: Global regularity, Navier-Stokes Equations, Large data, Koch-Tataru solution.

1. Introduction

This paper is devoted to study the following incompressible Navier-Stokes equations with the periodic data on the domain $(x, t) \in \mathbb{R}^3 \times R^+$. 

\begin{equation}
\begin{cases}
U_t + U \cdot \nabla U - \Delta U + \nabla P = 0, & \text{in } \mathbb{R}^3 \times (0, +\infty), \\
\nabla \cdot U = 0, & \\
U(x, t)|_{t=0} = u_0(x).
\end{cases}
\end{equation}

At the very beginning, we shall recall some correlated research history about the incompressible Navier-Stokes equations. In [13], Leray proved the local well-posedness for strong solutions and for any finite square-integrable initial data there exists a (possibly not unique) global in time weak solution in $\mathbb{R}^n$. Moreover, for the case of two space dimensions, he proved in [14] the uniqueness of the weak solution. Subsequently, in the work of Fujita-Kato [6], they proved the local well-posedness for strong solutions to the Navier-Stokes equations in a scaling invariant space $H^{\frac{2}{p}}$. The scaling-invariance in the context of the Navier-Stokes equations is as follows. Define 

\begin{equation}
(U_\lambda, P_\lambda)(x, t) = (\lambda U(\lambda x, \lambda^2 t), \lambda^2 P(\lambda x, \lambda^2 t)),
\end{equation}

if the pair $(U, P)$ solves the incompressible Navier-Stokes equations, then $(U_\lambda(x, t), P_\lambda(x, t))$ is also a pair of solution to the incompressible Navier-Stokes equations with initial data $U_\lambda(x, 0) = \lambda u_0(\lambda x)$. The spaces which are invariant under such a scaling are also called critical spaces. The study of the Navier-Stokes equations in critical spaces was initiated by Kato [9] and continued by many authors, see [2, 8, 16] etc. In 2001 Koch-Tataru [10] proved the existence of solutions to Navier-Stokes equations in $\mathbb{R}^n$ when the

Date: September 29, 2018.

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initial data is in $\text{BMO}^{-1}$, see also \[12\]. Subsequently, the space and time analyticity of the Koch-Tataru solution have been studied by \[7, 5\] and \[15\]. The space $\text{BMO}^{-1}$ has a special role since it is the largest critical space for Navier-Stokes equations with well-posedness results available \[4\]. Hereafter, we call the solution presented by Koch & Tataru \[10\] as Koch-Tataru solution.

The results listed above are mostly concerned the system \(1.1\) with the initial data small enough. To prove whether global existence or finite time blow up for a large data is a famous open problem. Recently, in Lei-Lin-Zhou \[11\] the authors have proved the global well-posedness with a class of large data in the energy space which includes the Beltrami flow. This paper is to study the global well-posedness for the 3D incompressible Navier-Stokes equations with general initial data in the largest critical space $\text{BMO}^{-1}$.

Before stating the main results, we shall give some definitions and notations. Let

\[ Q(y_0, r) = B(y_0, r) \times (0, r^2) \]

be the space-time ball. For $(x, t) \in Q(y_0, r)$ means $x \in B(y_0, r)$ and $0 < t \leq r^2$, where $B(y_0, r) \subset \mathbb{R}^n$ is a $n$-dimensional space ball centered at $y_0 \in \mathbb{R}^n$ and radius $r$.

**Definition 1.1.** Let $f$ be a tempered distribution, $W$ be the solution of

\[ W_t - \Delta W = 0 \]

with initial data $f$. Denote

\[ [f]_{\text{BMO}(\mathbb{R}^n)} = \sup_{y_0 \in \mathbb{R}^n} \left( r^{-n} \int_{Q(y_0, r)} |\nabla W|^2 \, dtdy \right)^{\frac{1}{2}}, \]

we say the function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is in BMO if the semi-norm $[f]_{\text{BMO}}$ is finite.

If there exist $g_i \in \text{BMO}$ and $f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}$, denote

\[ [f]_{\text{BMO}^{-1}(\mathbb{R}^n)} = \inf \left\{ \|g_i\|_{\text{BMO}(\mathbb{R}^n)} \right\}, \]

we say $f \in \text{BMO}^{-1}$ if the above norm $[f]_{\text{BMO}^{-1}}$ is finite. It is easy to see that an equivalent definition for $\text{BMO}^{-1}$ is

\[ [f]_{\text{BMO}^{-1}(\mathbb{R}^n)} = \sup_{y_0 \in \mathbb{R}^n} \left( r^{-n} \int_{Q(y_0, r)} |W|^2 \, dtdy \right)^{\frac{1}{2}}. \]

Clearly the divergence of a vector field with components in BMO is in $\text{BMO}^{-1}$. See more details in Koch-Tataru \[10\] and Chap11 in \[12\].

Similarly, we define the space $\text{BMO}^{-2}$. If there exist $\bar{g}_i \in \text{BMO}^{-1}$ and $f = \sum_{i=1}^n \frac{\partial \bar{g}_i}{\partial x_i}$, denote

\[ [f]_{\text{BMO}^{-2}(\mathbb{R}^n)} = \inf \left\{ \|\bar{g}_i\|_{\text{BMO}^{-1}(\mathbb{R}^n)} \right\}, \]

we say $f \in \text{BMO}^{-2}$ if the above norm $[f]_{\text{BMO}^{-2}}$ is finite.

In Koch & Tataru \[10\], the following results have been presented:

**Theorem 1.2.** (Koch-Tataru\[10\]). If $\|u_0\|_{\text{BMO}^{-1}}$ is small enough, then the equation \(1.1\) admits a unique pair of global solution.

Our main result states as following:
Theorem 1.3. Let \( \lambda, 0 < b < 1 \) be two arbitrary given constants and \( M_0 \) be an arbitrary positive constant, suppose that the initial data \( u_0 \) of system (1.1) is a periodic function with

\[
\int_{T^3} u_0(y)dy = 0,
\]

and

\[
\|u_0\|_{BMO^{-1}(\mathbb{R}^3)} \leq M_0,
\]

then there exists a small constant \( \varepsilon = \varepsilon(b, M_0) \) which depends on \( M_0 \) and \( b \), but independent of \( \lambda \), such that the system (1.1) admits a unique pair of global periodic solutions provided

\[
\|\nabla \times u_0 - \lambda u_0\|_{BMO^{-2}(\mathbb{R}^3)} \leq \varepsilon(\lambda)^{-b},
\]

where \( \langle \lambda \rangle = \sqrt{1 + \lambda^2} \).

Here and hereafter, \( T^3 \) is a periodic domain, without loss of generality, in this paper we take \( T^3 = [-\pi, \pi]^3 \).

Furthermore, we also have the following results:

Theorem 1.4. Under the conditions of Theorem 1.3, then the solution presented by theorem 1.3 is analytic about the space and time.

Remark 1.5. We do not require \( M_0 \) to be small in our theorem, actually, it can be as large as you want. Up to our knowledge, it is the first large data results for the incompressible Navier-Stokes equations in \( BMO^{-1} \) space.

Remark 1.6. Our result generalizes the Koch and Tataru result in the space periodic case by taking \( \lambda = 0 \).

Remark 1.7. Our result implies the global nonlinear stability of Beltrami flow for the Navier-Stokes equations in the critical space \( BMO^{-1} \). Let \( u_0 = u_{01} + u_{02} \), with \( \int_{T^3} u_{01}(y)dy = \int_{T^3} u_{02}(y)dy = 0 \) and \( \nabla \cdot u_{01} = \nabla \cdot u_{02} = 0 \). We call \( u_{01} \) a Beltrami flow initial data if there exists a real number \( \lambda \) such that \( \nabla \times u_{01} = \lambda u_{01} \), then our Theorem implies that as long as \( u_{02} \) is small in \( BMO^{-1} \), we can have global existence.

This kind of data can be constructed as follows (without loss of generality, we consider the case \( \lambda > 0 \)), let

\[
u_{01} = \nabla \times u_{03} + (-\Delta)^{1/2} u_{03}
\]

with \( \nabla \cdot u_{03} = 0 \). Then, we get

\[
\nabla \times u_{01} = (-\Delta)^{1/2} u_{01}.
\]

If we take

\[
u_{03}(x) = \sum_{n \in \mathbb{Z}^3, |n| = \lambda} a_n e^{\sqrt{-1}n \cdot x}
\]

provided that \( \lambda \) is chosen such that the set \( n_1, n_2, n_3 \in \mathbb{Z}, n_1^2 + n_2^2 + n_3^2 = \lambda^2 \) is not empty. Then, we get \( (-\Delta)^{1/2} u_{01} = \lambda u_{01} \).
Moreover, we remark our result not only holds for small perturbation of a Beltrami flow. Actually, there exists large class of data that is not a small perturbation of Beltrami flow but still we can prove global existence, as the next Corollary demonstrates:

**Corollary 1.8.** Let $M_0$ be an arbitrary positive constant, suppose that the initial data $u_0$ of system (1.1) with

\[ u_0 = u_{01} + u_{02} \]

and $u_{01}, u_{02}$ are both periodic functions with

\[ \int_{T^3} u_{01}(y) dy = \int_{T^3} u_{02}(y) dy = 0, \quad \nabla \cdot u_{01} = \nabla \cdot u_{02} = 0, \]

as well as

\[ \|u_{01}\|_{BMO^{-1}(\mathbb{R}^3)} \leq M_0, \quad \|u_{02}\|_{BMO^{-1}(\mathbb{R}^3)} \leq \varepsilon_1. \]

Suppose that

\[ \phi_i \text{ are Beltrami type data satisfying} \]

\[ \nabla \times \phi_i = \lambda_i \phi_i, \quad \nabla \cdot \phi_i = 0 \]

with

\[ 1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_N \]

and there exists $0 < b < 1$, such that

\[ \lambda_N - \lambda_1 \leq \varepsilon \lambda_1^{1-b} \]

then there exists a small constant $\varepsilon(b, M_0)$ which depend on $b$ and $M_0$, such that the system (1.1) admits a unique global periodic solution provided that $\varepsilon \leq \varepsilon(b, M_0)$ and $\varepsilon_1 \leq \varepsilon(b, M_0) \lambda_1^{-b}(1 + \lambda_1)^{-1}$.

**Remark 1.9.** Take one more curl to $\phi_i$, we get

\[ -\Delta \phi_i = \nabla \times (\nabla \times \phi_i) = \lambda_i \nabla \times \phi_i = \lambda_i^2 \phi_i. \]

Thus, $\phi_i$ are the eigenfunction of the Laplacian and $\lambda_i^2$ are the eigenvalues of the Laplacian. Thus, (1.13) are noting but a Fourier series expansion.

Let us sketch a proof of This Corollary. We have

\[ \|\nabla \times u_0 - \lambda_1 u_0\|_{BMO^{-2}(\mathbb{R}^3)} \leq \|\nabla \times u_{01} - \lambda_1 u_{01}\|_{BMO^{-2}(\mathbb{R}^3)} + \|\nabla \times u_{02} - \lambda_1 u_{02}\|_{BMO^{-2}(\mathbb{R}^3)}. \]

Because the spectrum of $u_{01}$ is concentrated near $\lambda_1$, by Bernstein’s inequality, we have

\[ \|\nabla \times u_{01} - \lambda_1 u_{01}\|_{BMO^{-2}(\mathbb{R}^3)} \leq \lambda_1^{-1} \|\nabla \times u_{01} - \lambda_1 u_{01}\|_{BMO^{-1}(\mathbb{R}^3)} \]

\[ = \left\| \sum_{i=1}^{N} \frac{\lambda_i - \lambda_1}{\lambda_1} \phi_i \right\|_{BMO^{-1}(\mathbb{R}^3)} \leq M_0 \varepsilon \lambda_1^{-b}. \]
On the other hand, by Lemma 2.3, we have
\begin{align}
\|\nabla \times u_0 - \lambda_1 u_0\|_{BMO^{-2}(\mathbb{R}^3)} & \leq \|\nabla \times u_0\|_{BMO^{-2}(\mathbb{R}^3)} + \lambda_1 \|u_0\|_{BMO^{-2}(\mathbb{R}^3)} \\
& \leq \|u_0\|_{BMO^{-1}(\mathbb{R}^3)} + \lambda_1 \|u_0\|_{BMO^{-1}(\mathbb{R}^3)} \lesssim \varepsilon \lambda_1^{-b}.
\end{align}

Therefore, Corollary 1.8 follows.

Our paper is organized as follows: in the next section, we present some preliminaries. In section 3, we shall give some prepare work for the linear heat equation. The main theorem will be proved in section 4 (both Theorem 1.3 and Theorem 1.4). Throughout this paper, we sometimes use the notation $A \lesssim B$ as an equivalent to $A \leq CB$ with a uniform constant $C$.

2. Preliminaries

At the beginning, we recall some properties for the Leary projection operator $\mathbb{P}$ to divergence free vector fields, which is defined by its matrix valued Fourier multiplier $\hat{\mathbb{P}}(\xi) = \delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2}$. For any multi-indices $\alpha$, this symbol satisfies Mihlin-Hormander condition $\sup_{|\xi| \neq 0} |\xi|^\alpha |\hat{\mathbb{P}}(\xi)| \leq C$. Besides, we have the following pointwise bound (see [12] Proposition 11.1).

**Lemma 2.1.** Denote $e^{t\Delta}$ as the heat operator, $n$ is the space dimensions, and $\hat{\mathbb{P}}(x,t)$ is the kernel of $\nabla^k \mathbb{P} e^{t\Delta}$, then there holds
\begin{equation}
\hat{\mathbb{P}}(x,t) \lesssim C(k) \frac{1}{(\sqrt{t} + |x|)^{n+k+1}},
\end{equation}
where and $C(k)$ is a constant depending on $k$.

**Lemma 2.2.** Let $K(x,t) = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$, then there exists a polynomial $J^{k+2m}(\frac{x}{\sqrt{t}})$ with degree $k + 2m$, such that
\begin{equation}
\partial_t^m \nabla^k K(x,t) = \frac{1}{t^{m+\frac{1}{2}}} K(x,t) J^{k+2m}(\frac{x}{\sqrt{t}}).
\end{equation}

**Proof.** It can be proved by induction. \hfill \square

**Lemma 2.3.** under the assumption that $u_0$ is a periodic function and $\int_{T^3} u_0(y)dy = 0$, We have
\begin{equation}
\|u_0\|_{BMO^{-2}} \lesssim \|u_0\|_{BMO^{-1}}.
\end{equation}

**Proof.** Since $\int_{T^3} u_0(x + y)dy = 0$, we have
\begin{align}
\int_{T^3} u_0(x + y)dy & = \int_{T^3} (u_0(x) - u_0(x + y))dy \\
& = -\int_{T^3} \int_0^1 \frac{d}{ds} u_0(x + sy)dsdy = -\nabla_x \left( \int_0^1 \int_{T^3} u_0(x + sz) \cdot zdzds \right).
\end{align}
Thus, the conclusion follows \hfill \square

For completeness, we also give the following well-known equality:
Lemma 2.4. Let $\nabla \cdot b = 0$ and $\nabla \cdot h = 0$, then we have
\[ b \cdot \nabla h + h \cdot \nabla b = -b \times (\nabla \times h) - h \times (\nabla \times b) + \nabla (b \cdot h). \]

Lemma 2.5. Assume that
\[ \hat{F}(\xi) = \int_{\mathbb{R}^3} \frac{\hat{g}(\xi - \eta)\hat{h}(\eta)}{|\xi - \eta| + |\eta|} d\eta, \]
where $\hat{F}$ denotes the Fourier transform of $F$ etc. then for any $\frac{1}{2} < a < 1$, and $0 < \kappa < 1 - a$ there holds,
\[ \|F\|_{B^{a}_{\infty, \infty}} \leq \|g\|_{B^{a-1}_{\infty, \infty}} \|h\|_{B^{-a}_{\infty, \infty}} \]
where $\|F\|_{B^{a}_{\infty, \infty}} \equiv \sup_{\lambda} \|\mathcal{P}_\lambda F\|_{L^{\infty} \lambda^\kappa}$ with $\mathcal{P}_\lambda$ be the Littlewood-Paley decompositions.

Proof. We take a nonnegative smooth function $\psi$ with
\[ \text{Supp } \psi \subset \{ \frac{1}{4} \leq |\xi| \leq 4 \}. \]
Moreover, for all $\xi \in \mathbb{R}^3$, $0 \leq \psi \leq 1$ and
\[ \sum_{k \in \mathbb{Z}} \psi(2^{-k}\xi) = 1. \]
Write $g_\lambda = \mathcal{P}_\lambda g = \mathcal{F}^{-1}(\psi(\lambda^{-1}\xi)\hat{g})$ and $h_\mu = \mathcal{P}_\mu h = \mathcal{F}^{-1}(\psi(\mu^{-1}\xi)\hat{h})$, we have
\[ g = \sum_{\lambda = 2^j} g_\lambda, \]
and
\[ h = \sum_{\mu = 2^k} h_\mu. \]
Thus, we rewrite
\[ \hat{F}(\xi) = \sum_{\lambda = 2^j, \mu = 2^k} \int_{\mathbb{R}^3} \frac{\psi_1(\lambda^{-1}(\xi - \eta))\psi_1(\mu^{-1}\eta)\hat{g}_\lambda(\xi - \eta)\hat{h}_\mu(\eta)}{|\xi - \eta| + |\eta|} d\eta, \]
where $\psi_1 \in C_0^\infty$ such that $\psi_1(\xi) = 1$ in the support of $\psi$. Therefore
\[ F(x) = \sum_{\lambda = 2^j, \mu = 2^k} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \Phi_{\mu, \lambda}(x - y, x - z)g_\lambda(y)h_\mu(z)dydz \]
with
\[ \Phi_{\mu, \lambda}(\xi, \eta) = \frac{\psi_1(\lambda^{-1}\xi)\psi_1(\mu^{-1}\eta)}{|\xi| + |\eta|}. \]
We then split $F(x)$ as

$$
(2.14) \quad F(x) = \sum_{\lambda \leq 2^{-10} \mu} \int_{\mathbb{R}^3} \Phi_{\mu \lambda}(x - y, x - z) g_{\lambda}(y) h_{\mu}(z) dy dz \\
+ \sum_{\mu \leq 2^{-10} \lambda} \Phi_{\mu \lambda}(x - y, x - z) g_{\lambda}(y) h_{\mu}(z) dy dz \\
+ \sum_{2^{-10} \lambda \leq \mu \leq 2^{10} \lambda} \Phi_{\mu \lambda}(x - y, x - z) g_{\lambda}(y) h_{\mu}(z) dy dz \\
\triangleq F_1(x) + F_2(x) + F_3(x).
$$

Thus,

$$
(2.15) \quad \|F_1(x)\|_{B^s_{2, \infty}} \lesssim \sup_{\mu} \mu^\kappa \left( \sum_{\lambda \leq 2^{-10} \mu} \|\Phi_{\mu \lambda}\|_{L^1} \|g_{\lambda}\|_{L^\infty} \|h_{\mu}\|_{L^\infty} \right) \\
\lesssim \sup_{\mu} \mu^\kappa \left( \sum_{\lambda \leq 2^{-10} \mu} \|\Phi_{\mu \lambda}\|_{L^1} \lambda^{-\kappa+1-a} \mu^a \right) \|g\|_{B^{-1+a}_{2, \infty}} \|h\|_{B^a_{2, \infty}}.
$$

Noting that

$$
(2.16) \quad \|\Phi_{\mu \lambda}\|_{L^1} = \mu^{-1} \|\tilde{\Phi}_{\mu \lambda}\|_{L^1}
$$

where

$$
(2.17) \quad \tilde{\Phi}_{\mu \lambda} = \psi_{1}(\xi) \psi_1(\eta) \mu^{-\lambda} \lambda |\xi| + |\eta|.
$$

Obviously any differentiation of $\tilde{\Phi}_{\mu \lambda}$ is bounded, therefore $\tilde{\Phi}_{\mu \lambda}$ decays in any polynomial. Thus, $\|\tilde{\Phi}_{\mu \lambda}\|_{L^1} \leq C$, therefore we have

$$
\|F_1(x)\|_{B^s_{2, \infty}} \lesssim \sup_{\mu} \mu^\kappa \left( \sum_{\lambda \leq 2^{-10} \mu} \lambda^{-\kappa+1-a} \mu^a \right) \|g\|_{B^{-1+a}_{2, \infty}} \|h\|_{B^a_{2, \infty}}. 
$$

We can handle $F_2$ in a similar way, here we use the similar fact $\|\Phi_{\mu \lambda}\|_{L^1} = \lambda^{-1} \|\tilde{\Phi}_{\mu \lambda}\|_{L^1} \lesssim \lambda^{-1}$ when $\mu \leq 2^{-10} \lambda$.

To estimate $F_3$, noting there are finite terms for $2^{-10} \lambda \leq \mu \leq 2^{10} \lambda$ by a given $\mu$, therefore

$$
(2.18) \quad \|F_3(x)\|_{B^s_{2, \infty}} \lesssim \sup_{\mu_1} \mu_1^\kappa \left( \sum_{\mu \geq \mu_1} \sum_{2^{-10} \lambda \leq \mu \leq 2^{10} \lambda} \|\Phi_{\mu \lambda}\|_{L^1} \|g_{\lambda}\|_{L^\infty} \|h_{\mu}\|_{L^\infty} \right) \\
\lesssim \sup_{\mu_1} \mu_1^\kappa \left( \sum_{\mu \geq \mu_1} \sum_{2^{-10} \lambda \leq \mu \leq 2^{10} \lambda} \|\Phi_{\mu \lambda}\|_{L^1} \lambda^{-\kappa+1-a} \mu^a \right) \|g\|_{B^{-1+a}_{2, \infty}} \|h\|_{B^a_{2, \infty}} \\
\lesssim \sup_{\mu_1} \mu_1^\kappa \left( \mu^{-\kappa-a} \mu^a \right) \|g\|_{B^{-1+a}_{2, \infty}} \|h\|_{B^a_{2, \infty}} \lesssim \|g\|_{B^{-1+a}_{2, \infty}} \|h\|_{B^a_{2, \infty}}.
$$

\[\square\]
Corollary 2.6. Let \( F, g, h \) are all periodic functions in \( \mathbb{T}^3 \),

\[
g = \sum_{n} a_n e^{\sqrt{-1}n \cdot x}, \quad h = \sum_{n} b_n e^{\sqrt{-1}n \cdot x}, \quad F = C_n e^{\sqrt{-1}n \cdot x}
\]

and

\[
\int_{\mathbb{T}^3} g \, dx = \int_{\mathbb{T}^3} h \, dx = 0.
\]

If \( C_n = \sum_{j+k=n} \frac{a_j b_k}{|j| + |k|} \), then regard \( F, g, h \) as functions on \( \mathbb{R}^3 \), for \( \frac{1}{2} < a < 1 \) and \( 0 < \kappa < 1 - a \), we have

\[
\|F\|_{B_{2,\infty}^\kappa(\mathbb{R}^3)} \leq \|g\|_{B_{2,\infty}^{\kappa-\frac{1}{a}}(\mathbb{R}^3)} \|h\|_{B_{2,\infty}^{-\frac{1}{a}}(\mathbb{R}^3)}.
\]

Proof. We have for example

\[
F(p) = \sum_{n} a_n F_j \delta(-n + \xi),
\]

here and hereafter, we use the notations \( F \) and \( F^{-1} \) be the Fourier transform and its inversion respectively. Therefore, Corollary 2.6 follows from Lemma 2.5. □

3. SOME PREPARE WORK FOR LINEAR HEAT EQUATION

We introduce the following notations:

Definition 3.1. Let \( g \) be a function defined on \( \mathbb{R}^3 \times (0, T*) \) \( (0 < T* \leq +\infty) \), we say \( g \in X_{T*} \) if

\[
\|g\|_{X_{T*}} \triangleq \sup_{0 < t \leq T*} t^{\frac{1}{2}} \|g\|_{L^\infty(\mathbb{R}^3)} + \sup_{0 < r \leq \sqrt{T*}} \left( r^{-3} \int_{Q(y_0, r)} |g|^2 \, dy \, dt \right)^{\frac{1}{2}} < +\infty.
\]

We say \( g \in Z_{T*}^d \) if

\[
\|g\|_{Z_{T*}^d} \triangleq \sup_{0 < t \leq T*} t^{\frac{1-d}{2}} \|g\|_{L^\infty(\mathbb{R}^3)} + \sup_{0 < r \leq \sqrt{T*}} \left( r^{-(1+2d)} \int_{Q(y_0, r)} |g|^2 \, dy \, dt \right)^{\frac{1}{2}} < +\infty.
\]

We say \( g \in Y_{T*} \) if

\[
\|g\|_{Y_{T*}} \triangleq \sup_{0 < t \leq T*} t \|g\|_{L^\infty(\mathbb{R}^3)} + \sup_{0 < r \leq \sqrt{T*}} r^{-3} \int_{Q(y_0, r)} |g| \, dy \, dt < +\infty.
\]

To go ahead, we shall give some estimates for the following homogenous heat equation in \( \mathbb{R}^3 \):

\[
\begin{cases}
\partial_t u - \Delta u = 0, \\
u|_{t=0} = u_0, \quad \nabla \cdot u_0 = 0.
\end{cases}
\]

We have the following proposition:
Proposition 3.2. There exists a uniform constant $C_0$, such that
\begin{equation}
\|u\|_{X^{\frac{3}{2} \times}} \leq C_0\|u_0\|_{BMO^{-1}}.
\end{equation}

Proof. We write the solution of the linear equation as
\begin{equation}
u = S(t)u_0,
\end{equation}
where $S(t)$ is the heat flow. Then we have:
\begin{align}
t^\frac{7}{2}\|u\|_{L^\infty(\mathbb{R}^3)} &= 4t^{-\frac{7}{2}}\left\| \int_0^t S(t-\tau)(S(\tau)u_0)d\tau \right\|_{L^\infty(\mathbb{R}^3)} \\
&\leq t^{-\frac{7}{2}}\left\| \int_0^t \left( \int_{\mathbb{R}^3} \frac{1}{\sqrt{t-\tau}} e^{-\frac{(y-y')^2}{4(t-\tau)}} (e^{r^2\Delta u_0})^2 dy'd\tau \right)^{1/2} d\tau \right\|_{L^\infty(\mathbb{R}^3)} \\
&\leq t^{-\frac{7}{2}}\left\| \left( \sum_{q=0}^{\infty} \frac{1}{\sqrt{t}} \int_{\mathbb{R}^3} \left( e^{r^2\Delta u_0} \right)^2 dy'd\tau \right)^{1/2} \right\|_{L^\infty(\mathbb{R}^3)} \\
&\leq \sup_{y \in \mathbb{R}^3} \left( \frac{1}{\sqrt{t}} \int_0^t \int_{B(y,\sqrt{t})} (e^{r^2\Delta u_0})^2 dy'd\tau \right)^{1/2} \lesssim \|u_0\|_{BMO^{-1}(\mathbb{R}^3)}.
\end{align}

By using the definition $\sup_{y_0 \in \mathbb{R}^3} (r^{-\frac{3}{2}}\int_{Q(y_0,r)} u^2 dy dt)^{1/2} = \|u_0\|_{BMO^{-1}}$. Thus, we finished the proof of our Theorem.

\[\square\]

Proposition 3.3. Let $u$ be the solution of system (3.4), then there exists a uniform constant $C_0 > 0$, such that
\begin{equation}
\left( \int_0^{+\infty} \|u(t)\|_{L^2}^2 dt \right)^{\frac{1}{2}} \leq C_0\|u_0\|_{B_{5,2}}.
\end{equation}

Proof. See the proof of Lemma 1.5.1 in Chemin \[\square\].

Proposition 3.4. For any $0 < d < 1$, there exists a positive constant $C_0 = C_0(d)$, such that
\begin{equation}
\|(-\Delta)^{-d/2}u\|_{Z^{d}_{7/\ast}} \lesssim C_0\|u_0\|_{BMO^{-1}}.
\end{equation}

Proof. For the simplicity of exposition, we take $d = \frac{1}{2}$, the general case is the same. Similar to the proof of our previous Proposition, we can get
\begin{align}
t^\frac{7}{2}\|(-\Delta)^{-1/4}u\|_{L^\infty(\mathbb{R}^3)} \\
&\lesssim \sup_{y \in \mathbb{R}^3} \left( \frac{1}{\sqrt{t}} \int_0^t \int_{B(y,\sqrt{t})} ((-\Delta)^{-1/4}u(\tau,\tilde{y}))^2 dy'd\tau \right)^{1/2}.
\end{align}

Thus, it remains to prove
\begin{equation}
\sup_{y_0 \in \mathbb{R}^3} \left( r^{-4}\int_{Q(y_0,r)} ((-\Delta)^{-1/4}u)^2 dy dt \right)^{1/2} \lesssim \|u_0\|_{BMO^{-1}(\mathbb{R}^3)}.
\end{equation}
By definition, we have

\[(3.12) \quad \|(-\Delta)^{-1/4}u\|_{L^\infty(\mathbb{R}^3)} = 4r^{-2}\int_0^{2\pi} (-\Delta)^{-1/4} S(t-\tau)(S(\tau)u_0) d\tau\|_{L^\infty(\mathbb{R}^3)}.
\]

Denote \(K_1(x)\) be the kernel of the operator \((-\Delta)^{-1/4} S(t-\tau)\), then we have

\[(3.13) \quad |K_1(x)| = \left|\mathcal{F}^{-1}\left(\mathcal{F}\left((-\Delta)^{-1/4} S(t-\tau)\right)\right)\right| = \left|\int_{\mathbb{R}^3} e^{i\xi \cdot x} |\xi|^{-1/2} e^{-\xi^2 t} d\xi\right| = \left|\int_{\mathbb{R}^3} e^{-\frac{|\xi|^2}{4t}} \int_{\mathbb{R}^3} e^{-|\xi|^2} |\xi|^2 d\xi\right| \leq \frac{t^{1/4}}{\sqrt{t}} e^{-\frac{|\xi|^2}{4t}}.
\]

Then by using (3.13) and similar to (3.7), we have

\[(3.14) \quad \|(-\Delta)^{-1/4}u\| = 4r^{-2}\int_0^{2\pi} (-\Delta)^{-1/4} S(t-\tau)(S(\tau)u_0) d\tau\]

\[\leq r^{-2}\int_0^{2\pi} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K_1(x-y) \frac{1}{2} (\int_{\mathbb{R}^3} K_1(x-y)(e^{\tau \Delta} u_0)^2 d^3y)^{1/2} d^3x \right) d^3y d\tau \]

\[\leq r^{-1+\frac{1}{2}} \left( \int_0^{2\pi} \int_{\mathbb{R}^3} \frac{1}{\sqrt{1-t^2}} e^{-\frac{|x-y|^2}{4(1-t^2)}} (e^{\tau \Delta} u_0)^2 d^3y d\tau \right)^{1/2} \|u_0\|_{BMO^{-1}(\mathbb{R}^3)}.
\]

From (3.14) and noting (3.11), our proposition follows. \(\Box\)

**Corollary 3.5.** under the assumption that \(u_0\) is a periodic function and \(\int_{\mathbb{R}^3} u_0(y) dy = 0\), then for \(0 < d < 1\), we have

\[(3.15) \quad \|(-\Delta)^{-d/2} \nabla \times u_0\|_{BMO^{-1}} \leq \|u_0\|_{BMO^{-1}}.
\]

**Proof.** Without loss of generality, we take \(d = \frac{1}{2}\), the general case is the same.

Denote the matrix function \(K_2(x)\) be the kernel of the operator \(e^{\Delta}(-\Delta)^{-3/4} \nabla \times \).
We need to estimate the bound of \((K_2)_{ij}(x), i, j = 1, \cdots, 3\).

Firstly, we have get the matrix Fourier multiplier of the operator \(e^{\Delta}(-\Delta)^{-3/4} \nabla \times\) as

\[(3.16) \quad e^{-\xi^2 t} |\xi|^{-1/2} \begin{pmatrix} 0 & -i\xi_3 & i\xi_2 \\ i\xi_2 & 0 & -i\xi_1 \\ -i\xi_2 & i\xi_1 & 0 \end{pmatrix} = e^{-\xi^2 t} |\xi|^{-1/2} A(\xi).
\]
Then for $i, j = 1, \cdots, 3$, we have

$$
\|(K_2)_{ij}(x)\| = \left|F^{-1}e^{ix\xi}e^{-\xi^2 t}|\xi|^{-\frac{3}{2}}A_{ij}(\xi)\right| = \left|\int_{\mathbb{R}^3}e^{ix\xi}e^{-\xi^2 t}|\xi|^{-\frac{3}{2}}A_{ij}(\xi)d\xi\right|
\lesssim \frac{t^{1/4}}{\sqrt{t}}e^{-\frac{4}{9t}}\int_{\mathbb{R}^2}\int_0^\infty e^{-\sqrt{7}\xi_1^2}d\xi_1 \sqrt{7}\xi_1^{-\frac{3}{2}}A(\sqrt{7}\xi_1)d(\sqrt{7}\xi_1)\left|\nabla\xi A(\sqrt{7}\xi_1)d(\sqrt{7}\xi_1)\right| \lesssim \frac{t^{1/4}}{\sqrt{t}}e^{-\frac{4}{9t}}.
$$

Denote the matrix function $K_3(x)$ be the kernel of the operator $\nabla(-\Delta)^{-3/4}\nabla \times e^{t\Delta}$, for $i, j = 1, \cdots, 3$, we have

$$
\|(K_3)(x)\| = \left|F^{-1}e^{ix\xi}e^{-\xi^2 t}|\xi|^{-\frac{3}{2}}B(\xi)\right| = \left|\int_{\mathbb{R}^3}e^{ix\xi}e^{-\xi^2 t}|\xi|^{-\frac{3}{2}}B(\xi)d\xi\right|
\lesssim \frac{t^{1/4}}{\sqrt{t}}e^{-\frac{4}{9t}}\int_{\mathbb{R}^2}\int_0^\infty e^{-\sqrt{7}\xi_1^2}d\xi_1 \sqrt{7}\xi_1^{-\frac{3}{2}}B(\sqrt{7}\xi_1)d(\sqrt{7}\xi_1)\lesssim \frac{t^{1/4}}{\sqrt{t}}e^{-\frac{4}{9t}},
$$

where we denote $B(\xi) = \frac{1}{|\xi|} \otimes A(\xi)$.

Similarly to (3.11), we have

$$
\|(-\Delta)^{-3/4}\nabla \times e^{t\Delta}u_0\|
= 4r^{-2}\int_{\frac{2\pi}{\sqrt{t}}}^{2\pi} (-\Delta)^{-3/4}\nabla \times S(t-\tau)(S(\tau)u_0)d\tau
\lesssim r^{-2}\int_{\frac{2\pi}{\sqrt{t}}}^{2\pi} \left(\int_{\mathbb{R}^3}K_2(x-y)d\gamma\right)^{1/2}\left(\int_{\mathbb{R}^3}K_2(x-y)(e^{t\Delta}u_0)^2 d\gamma\right)^{1/2}d\tau
\lesssim r^{-1+\frac{1}{2}}\left(\int_{\frac{2\pi}{\sqrt{t}}}^{2\pi} \int_{\mathbb{R}^3} \frac{1}{\sqrt{t-\tau}}e^{-\frac{(y-y')^2}{4(t-\tau)}}(e^{t\Delta}u_0)^2 d\gamma d\tau\right)^{1/2}
\lesssim r^{-\frac{3}{2}}\|u_0\|_{BMO^{-1}(\mathbb{R}^3)}.
$$

Let $u_0 = (\nabla \times)^{-1}u_0$, then we also have

$$
\|(-\Delta)^{-3/4}\nabla \times e^{t\Delta}u_0\|
= 4r^{-2}\int_{\frac{2\pi}{\sqrt{t}}}^{2\pi} \nabla(-\Delta)^{-3/4}\nabla \times S(t-\tau)(S(\tau)u_0)d\tau
\lesssim r^{-2-\frac{1}{2}}\int_{\frac{2\pi}{\sqrt{t}}}^{2\pi} \left(\int_{\mathbb{R}^3}K_3(x-y)(e^{t\Delta}u_0)^2 d\gamma\right)^{1/2}d\tau
\lesssim r^{-1-\frac{1}{2}}\left(\int_{\frac{2\pi}{\sqrt{t}}}^{2\pi} \int_{\mathbb{R}^3} \frac{1}{\sqrt{t-\tau}}e^{-\frac{(y-y')^2}{4(t-\tau)}}(e^{t\Delta}u_0)^2 d\gamma d\tau\right)^{1/2}
\lesssim r^{-\frac{3}{2}}\|u_0\|_{BMO^{-2}(\mathbb{R}^3)}.
$$

Take inf with respect to $u_0$, we get

$$
\|(-\Delta)^{-3/4}\nabla \times e^{t\Delta}u_0\| \lesssim r^{-\frac{3}{2}}\|u_0\|_{BMO^{-2}(\mathbb{R}^3)},
$$

where $u_0 \in BMO^{-2}(\mathbb{R}^3)$.
Combing (3.19) and (3.21), we have

\[(3.22) \quad \left| (\Delta)^{-3/4} \nabla \times e^{t \Delta} u_0 \right| \leq r^{-1} \sup_{BMO^{-1}(\mathbb{R}^3)} \left\| u_0 \right\|_{BMO^{-2}(\mathbb{R}^3)} \leq r^{-1} \sup_{BMO^{-1}(\mathbb{R}^3)} \left\| u_0 \right\|_{BMO^{-1}(\mathbb{R}^3)}.
\]

The last inequality given above was from lemma 2.3.

By definition

\[(3.23) \quad \left\| (\Delta)^{-3/4} \nabla \times u_0 BMO^{-1} \right\| = \sup_{y_0 \in \mathbb{R}^3} \left( r^{-3} \int_{Q(y_0, r)} \left| (\Delta)^{-3/4} \nabla \times e^{t \Delta} u_0 \right|^2 dydt \right)^{1/2},
\]

then the conclusion follows from (3.21).

The following proposition appear in Koch and Taturu, for the convenience of the reader, we include a slightly different proof here.

**Proposition 3.6.** Let $G_1$ be a tensor, and $V_1$ be the solution of the following system,

\[
\begin{aligned}
&V_{1t} - \Delta V_1 + \nabla P_1 = \nabla \cdot G_1, \\
&\nabla \cdot V_1 = 0, \\
&t = 0: V_1 = 0,
\end{aligned}
\]

then we have

\[(3.24) \quad \left\| V_1 \right\|_{X_{T*}} \leq \left\| G_1 \right\|_{Y_{T*}}.
\]

**Proof.** First, we rewrite the system (3.24) as an integral equation

\[(3.25) \quad V_1(t) = \int_0^t S(t - \tau) \mathbb{P} \nabla \cdot G_1 d\tau.
\]

When $0 \leq \tau \leq t/2$, by using Lemma 2.1, we get:

\[(3.26) \quad t^{n/2} \int_0^t S(t - \tau) \mathbb{P} \nabla \cdot G_1 d\tau \leq t^{n/2} \int_0^t S(t - \tau) \mathbb{P} \nabla \cdot G_1 d\tau \leq \frac{1}{\sqrt{t}} \int_0^{t/2} \int_{\mathbb{R}^n} G_1 d\hat{y} dr \leq \frac{1}{\sqrt{t}} \int_0^{t/2} \int_{\mathbb{R}^n} \left| \frac{G_1}{(1 + q)^{n+1}} \right| d\hat{y} dr \leq \frac{1}{\sqrt{t}} \int_0^t \int_{B(n \sqrt{t})} \left| G_1 \right| d\hat{y} dr \leq \frac{1}{\sqrt{t}} \int_0^t \int_{B(n \sqrt{t})} \left| G_1 \right| d\hat{y} dr \leq \frac{1}{\sqrt{t}} \int_0^t \int_{B(n \sqrt{t})} \left| G_1 \right| d\hat{y} dr.
\]

If $\frac{t}{2} \leq \tau \leq t$, we get

\[(3.27) \quad \left\| S(t - \tau) \mathbb{P} \nabla \cdot G_1 \right\| \leq \left| \int_{\mathbb{R}^n} \frac{1}{\sqrt{t - \tau + |y - \hat{y}|^{n+1}}} d\hat{y} \right| \left| G_1 \right| \leq \frac{1}{\sqrt{t - \tau}}.
\]
Therefore we have
\begin{align}
& t^{\frac{1}{2}} \left\| \int_{1/2}^{t} S(t - \tau) P \nabla G_1 d\tau \right\|_{L^\infty} \\
& \lesssim t^{\frac{1}{2}} \int_{1/2}^{t} \frac{1}{\sqrt{t - \tau}} d\tau \left\| G_1 \right\|_{L^\infty} \lesssim t \left\| G_1 \right\|_{L^\infty}.
\end{align}

We still need to estimate the term \( \sup_{y_0 \in \mathbb{R}^n, r > 0} \frac{1}{r} \int_0^r \int_{B(y_0, r)} |V_1(s, y)|^2 dy ds \). For any given \( y_0 \in \mathbb{R}^n \) and \( r > 0 \), take a smooth cut-off function
\begin{equation}
\chi \left( \frac{y}{r} \right) = \begin{cases} 
1, & |y - y_0| \leq 3r; \\
0, & |y - y_0| \geq 5r.
\end{cases}
\end{equation}

Then, it is sufficient to estimate the following term
\begin{equation}
I = \sup_{y_0 \in \mathbb{R}^n, r > 0} \left( r^{-n} \int_{Q(y_0, r)} \left( \nabla \int_0^t S(t - \tau) P \chi G_1 d\tau \right)^2 dy dt \right)^{\frac{1}{2}} \\
+ \sup_{y_0 \in \mathbb{R}^n, r > 0} \left( r^{-n} \int_{Q(y_0, r)} \left( \nabla \int_0^t S(t - \tau) P (1 - \chi) G_1 d\tau \right)^2 dy dt \right)^{\frac{1}{2}} = I_1 + I_2.
\end{equation}

To deal with \( I_1 \), we will drop the projector \( P \), which is a bounded operator in \( L^2 \) and which commutes with \( S(t - \tau) \), \( \nabla \) and the integral about \( t \). We set up a heat function
\begin{equation}
W_1 - \Delta W = \chi G_1, \quad W_1|_{t=0} = 0.
\end{equation}

Then \( I_1 = \sup_{y_0 \in \mathbb{R}^n, r > 0} \left( r^{-n} \left\| \nabla W_1 \right\|_{L^2(Q(y_0, r))}^2 \right)^{\frac{1}{2}} \). We get
\begin{equation}
\left\| \nabla W_1 \right\|_{L^2(Q(y_0, r))}^2 \lesssim \int_0^r \int_{\mathbb{R}^n} (\nabla W)^2 dy d\tau \\
\lesssim \int_0^r \int_{\mathbb{R}^n} |\chi G_1 W| dy d\tau.
\end{equation}

For the term \( \int_0^r \int_{\mathbb{R}^n} |\chi G_1 W| dy d\tau \), recalling that \( \chi = \chi(y/r) \) and \( t \leq r^2 \), we have
\begin{equation}
\int_0^r \int_{\mathbb{R}^n} |\chi G_1 W| dy d\tau \lesssim \left\| W \right\|_{L^\infty_{Q(y_0, 5r)}} \int_0^r \int_{\mathbb{R}^n} |\chi G_1| dy d\tau,
\end{equation}

where
\begin{equation}
\left\| W \right\|_{L^\infty_{Q(y_0, 5r)}} = \left\| \int_0^{\tau/2} S(\tau - s) \chi G_1 ds \right\|_{L^\infty_{Q(y_0, 5r)}} + \left\| \int_{\tau/2}^\tau S(\tau - s) \chi G_1 ds \right\|_{L^\infty_{Q(y_0, 5r)}}.
\end{equation}

We shall study the above inequality separately.
By Lemma 2.2, we have

\begin{equation}
\| \int_0^{\tau/2} S(\tau - s) G_1 ds \|_{L^\infty_{(y_0, 5r)}} \\
\leq \frac{1}{\sqrt{\tau}} \int_0^{\tau/2} \sum_{q=0}^{n} \int_0^{\tau/2} e^{-q^2} G_1 d\tilde{y} ds \|_{L^\infty_{(y_0, 5r)}} \\
\leq \sup_{y \in \mathbb{R}^n, 0 < \tau \leq T^*} \frac{1}{\sqrt{\tau}} \int_0^{\tau} \int_{B(y, \sqrt{\tau})} |G_1| d\tilde{y} ds \leq \| G_1 \|_{Y_{T^*}}.
\end{equation}

Recalling the definition of the cut-off function \( \chi \), we have

\begin{equation}
\| \int_0^{\tau} \int_0^{\tau/2} \frac{1}{\sqrt{\tau - s}} e^{-\frac{(y - \tilde{y})^2}{4(\tau - s)}} \chi G_1 d\tilde{y} ds \|_{L^\infty_{(y_0, 5r)}} \\
\leq \int_0^{\tau} \int_0^{\tau/2} \frac{1}{\sqrt{\tau - s}} e^{-\frac{(y - \tilde{y})^2}{4(\tau - s)}} |\tilde{y} - \tau| G_1 d\tilde{y} ds \|_{L^\infty_{(y_0, 5r)}} \\
\leq \| G_1 \|_{Y_{T^*}} \int_0^{\tau} \frac{1}{\sqrt{\tau/2}} ds \leq \| G_1 \|_{Y_{T^*}}.
\end{equation}

To finish the estimate of \( I_1 \), from (3.37), we still need to estimate \( r^{-n} \int_0^r \int_{\mathbb{R}^n} |\chi G_1| dyd\tau \).

\begin{equation}
r^{-n} \int_0^r \int_{\mathbb{R}^n} |\chi G_1| dyd\tau \leq r^{-n} \int_0^r \int_{B(y_0, 5r)} |G_1| dyd\tau \leq \| G_1 \|_{Y_{T^*}}.
\end{equation}

Combining (3.33), (3.38), we get

\begin{equation}
I_1 \leq \| G_1 \|_{Y_{T^*}}.
\end{equation}

As to the term \( I_2 \), we have

\begin{equation}
I_2^2 \leq \sup_{0 < r \leq \sqrt{T^*}} \int_0^t S(t - \tau) P(1 - \chi) G_1 d\tau \|_{L^\infty_{(y_0, r)}}.
\end{equation}

When \( 0 \leq \tau \leq t \leq (r/2)^2 \), noting the cut-off function \( 1 - \chi \) and Lemma 2.1, we have

\begin{equation}
\sup_{y_0 \in \mathbb{R}^n, 0 < r \leq \sqrt{T^*}} \int_0^t S(t - \tau) P(1 - \chi) G_1 d\tau \|_{L^\infty_{(y_0, r)}} \\
\leq \sup_{y_0 \in \mathbb{R}^n, 0 < r \leq \sqrt{T^*}} \int_0^t \frac{1 - \chi(\tilde{y})}{(\sqrt{t - \tau} + |y - \tilde{y}|)^{n+1}} |G_1| d\tilde{y} d\tau \|_{L^\infty_{(y_0, r)}} \\
\leq \sup_{y_0 \in \mathbb{R}^n, 0 < r \leq \sqrt{T^*}} \int_0^t \sum_{q=1}^{\infty} \int_0^{(q+1)r} |G_1| (qr)^{n+1} d\tilde{y} d\tau \|_{L^\infty_{(y_0, r)}} \\
\leq \sup_{y_0 \in \mathbb{R}^n, 0 < r \leq \sqrt{T^*}} \left( \frac{1}{r^n} \int_0^{r} \int_{B(y, r)} |G_1| d\tilde{y} d\tau \right)^2 \leq \| G_1 \|_{Y_{T^*}}.
\end{equation}

For the remaining part \( (r/2)^2 < t < r^2 \), we shall divide into two parts to estimate.
(i): when \((r/2)^2 < t < r^2\) and \(0 \leq \tau \leq t/2\), by Lemma 2.1, we have

\[
(3.42) \quad \sup_{y_0 \in \mathbb{R}^n, 0 < r \leq R} \left( r^2 \right)^{t/2} \int_0^{t/2} S(t - \tau) \mathbb{P}(1 - \chi) G_1 d\tau \|L^\infty(Q(y_0, r)) \leq \left( r^2 \right)^{t/2} \int_0^{t/2} \frac{1}{\sqrt{t - \tau + |y|^n + 1}} |G_1(\tau, y)| dy d\tau \|L^\infty(Q(y_0, r)) \leq \sup_{y_0 \in \mathbb{R}^n, 0 < r \leq R} \left( r^2 \right)^{t/2} \int_0^{t/2} \frac{1}{\sqrt{t - \tau + |y|^n + 1}} |G_1(\tau, y)| dy d\tau \|L^\infty(Q(y_0, r)) \leq \|G_1\|_{Y^*_T}^2.
\]

(ii): when \((r/2)^2 < t < r^2\) and \(t/2 \leq \tau \leq t\), we have

\[
(3.43) \quad |r(1 - \chi)G_1| \leq \sqrt{r} |G_1| \leq \frac{1}{\sqrt{r}} \|G_1\|_{Y^*_T}^2.
\]

Then by (3.43) and Lemma 2.2, we get

\[
(3.44) \quad \sup_{y_0, r > 0} \left( r^2 \right)^{t/2} \int_0^{t/2} S(t - \tau) \mathbb{P}(1 - \chi) G_1 d\tau \|L^\infty(Q(y_0, r)) \leq \|G_1\|_{Y^*_T}^2 \left( \int_0^{t/2} \frac{1}{\sqrt{t - \tau + |y|^n + 1}} dy d\tau \right)^2 \leq \|G_1\|_{Y^*_T}^2 \left( \int_0^{t/2} \frac{1}{\sqrt{t - \tau + |y|^n + 1}} dy d\tau \right)^2 \leq \|G_1\|_{Y^*_T}^2.
\]

Then from (3.42) - (3.44), we can get

\[
(3.45) \quad I_2 \leq \|G_1\|_{Y^*_T}^2.
\]

**Corollary 3.7.** Let \(G_1\) be a tensor, and \(V_1\) be the solution of the system (3.24), then for any \(0 < a < 1\), we have

\[
(3.46) \quad \sup_{0 < t < T} t^{\frac{2a}{4}} \|(-\Delta)^{-a/2} V_1(t, \cdot)\|_{L^\infty} \leq \|G_1\|_{Y^*_T}.
\]

**Proof.** Denote \(K_4(x)\) and \(K_5(x)\) be the kernels of the operator \(e^{t\Delta}(-\Delta)^{-a/2}\) and \(e^{t\Delta}(-\Delta)^{-a/2}\). Similarly to (3.13) and (3.16), we have

\[
(3.47) \quad |K_4(x)| = |\mathcal{F}^{-1}(e^{-\xi^2} |\xi|^{-a})| \leq \frac{t^{a/2}}{\sqrt{t}} e^{-\frac{x^2}{4t}}, \quad \text{and} \quad |K_5(x)| \leq \frac{t^{a/2}}{\sqrt{t}} e^{-\frac{x^2}{4t}}.
\]
By using (3.24), we have

\[(3.48)\]
\[
\left\| (-\Delta)^{-\frac{3}{4}} V_1 \right\|_{L^\infty(\mathbb{R}^3)} 
\lesssim \left\| \int_0^t \mathcal{E} e(t-\tau) \left( -\Delta \right)^{-\frac{3}{4}} \| \mathbf{G}_1 \|_{L^\infty(\mathbb{R}^3)} \right\| d\tau.
\]

By using Proposition 3.5 and (3.37), when $0 < \tau < \frac{t}{2}$ we have

\[(3.49)\]
\[
\left\| \int_0^t e^{(t-\tau)\Delta} \left( -\Delta \right)^{-\frac{3}{4}} \mathcal{E} \cdot \mathbf{G}_1 d\tau d\tau \right\|_{L^\infty(\mathbb{R}^3)} 
= \left\| \int_0^t \mathcal{E} \cdot e^{(t-\tau)\Delta} \left( -\Delta \right)^{-\frac{3}{4}} \mathcal{E} \cdot \mathbf{G}_1 d\tau \right\|_{L^\infty(\mathbb{R}^3)} 
\lesssim \sup_{y_0 \in \mathbb{R}^3} \left\| \frac{1}{t-\tau} \int_0^t \mathcal{E} e^{(t-\tau)\Delta} \left( -\Delta \right)^{-\frac{3}{4}} \mathcal{E} \cdot \mathbf{G}_1 d\tau \right\|_{L^\infty(\mathbb{R}^3)} 
\lesssim \sup_{y_0 \in \mathbb{R}^3} \left\| \frac{1}{t-\tau} \int_0^t \mathcal{E} \cdot \mathbf{G}_1 d\tau \right\|_{L^\infty(\mathbb{R}^3)} 
\lesssim t^{\frac{\nu+1}{4}} \| \mathbf{G}_1 \|_{Y_T}.
\]

When $t/2 < \tau < t$, by using Lemma 2.1, we have

\[(3.50)\]
\[
\left\| \int_0^t e^{(t-\tau)\Delta} \left( -\Delta \right)^{-\frac{3}{4}} \mathcal{E} \cdot \mathbf{G}_1 d\tau d\tau \right\|_{L^\infty(\mathbb{R}^3)} 
= \left\| \int_0^t \mathcal{E} \cdot e^{(t-\tau)\Delta} \left( -\Delta \right)^{-\frac{3}{4}} \mathcal{E} \cdot \mathbf{G}_1 d\tau \right\|_{L^\infty(\mathbb{R}^3)} 
\lesssim \int_0^t \left\| \frac{1}{t-\tau} \int_0^t \mathcal{E} \cdot \mathbf{G}_1 d\tau \right\|_{L^\infty(\mathbb{R}^3)} 
\lesssim \int_0^t \left\| \frac{1}{t-\tau} \int_0^t \mathcal{E} \cdot \mathbf{G}_1 d\tau \right\|_{L^\infty(\mathbb{R}^3)} 
\lesssim \int_0^t \left\| \mathcal{E} \cdot \mathbf{G}_1 \right\|_{L^\infty(\mathbb{R}^3)} d\tau \lesssim \| \mathbf{G}_1 \|_{L^\infty(\mathbb{R}^3)} t^{\frac{\nu+1}{2}} \lesssim t^{\frac{\nu+1}{4}} \| \mathbf{G}_1 \|_{Y_T}.
\]

Combining (3.49) and (3.50), the Proposition 3.6 was proved.

\[\square\]

3.1. The Time and Space Analyticity. We give the following corollaries beforehand.

**Corollary 3.8.** For any integers $m, k \geq 0$, let $u$ be the solution of system (3.4), then we have

\[(3.51)\]
\[
\left\| t^{\frac{k+1}{2}} + m \nabla^m \nabla^k u \right\|_{X_{T^*}} \lesssim \left\| u_0 \right\|_{BMO^{-1}}.
\]
Therefore, it is sufficient to estimate the term \( \sup_{y_0 \in \mathbb{R}^3, r > 0} (r^{-n} \int_{Q(y_0, r)} (t^\frac{k+1}{4} + m) \nabla^k u^2) dy dt)^{1/2} \). Here, if we estimate it directly, then we will have a non-integrable factor. By using \( \hat{\nabla} u = \hat{\nabla}^{-1} \hat{\nabla} u = \Delta^m u \), we get

\begin{equation}
(3.52) \quad t^\frac{k+1}{4} + m \| \hat{\nabla}^m \nabla^k u \|_{L^\infty} = 4 t^\frac{k+1}{4} + m \int_0^\infty \hat{\nabla}^m \nabla^k S(t-\tau) (e^\tau \Delta u_0) d\tau \leq t^\frac{k-1}{4} + m \left( \int_{\mathbb{R}^3} \frac{J^{k+2m}(\frac{y}{\sqrt{t}})}{\sqrt{t-\tau}^{3}} e^{-\frac{(y-q)^2}{4(t-\tau)}} (e^\tau \Delta u_0)^2 dy \int_{\mathbb{R}^3} \frac{1}{\sqrt{t-\tau}^{k+2m}} d\tau \right)^{1/2} \leq C_0 \| u_0 \|_{BMO^{-1}}.
\end{equation}

We still need to estimate the term \( \sup_{y_0 \in \mathbb{R}^3, r > 0} (r^{-n} \int_{Q(y_0, r)} (t^\frac{k+1}{4} + m) \nabla^k u^2) dy dt)^{1/2} \). Here, if we estimate it directly, then we will have a non-integrable factor. By using \( \hat{\nabla} u = \hat{\nabla}^{-1} \hat{\nabla} u = \Delta^m u \), we get

\begin{equation}
(3.53) \quad \sup_{y_0 \in \mathbb{R}^3, r > 0} (r^{-n} \int_{Q(y_0, r)} (t^\frac{k+1}{4} + m) \nabla^k u^2) dy dt)^{1/2} \leq \sup_{y_0 \in \mathbb{R}^3, r > 0} (r^{-n} \int_{Q(y_0, r)} (t^\frac{k+1}{4} + m) \nabla^k u^2) dy dt)^{1/2}.
\end{equation}

Therefore, it is sufficient to estimate the term \( \sup_{y_0 \in \mathbb{R}^3, r > 0} (r^{-n} \int_{Q(y_0, r)} (t^\frac{k+1}{4} + m) \nabla^k u^2) dy dt)^{1/2} \) for any integer \( k > 0 \).

\begin{equation}
(3.54) \quad \sup_{y_0 \in \mathbb{R}^3, r > 0} (r^{-n} \int_{Q(y_0, r)} (t^\frac{k+1}{4} + m) \nabla^k u^2) dy dt)^{1/2} \leq \sup_{y_0 \in \mathbb{R}^3, r > 0} \left( \int_{Q(y_0, r)} (t^\frac{k+1}{4} + m) \nabla^k S(t-\tau) \Delta u_0 d\tau \right)^{1/2} \leq C_0 \| u_0 \|_{BMO^{-1}}.
\end{equation}

In the last inequality above, we have used (3.52).

\[ \square \]

**Corollary 3.9.** For any integers \( m, k \geq 0 \) any real constant \( \alpha \), let \( u \) be the solution of system (3.3), then we have

\begin{equation}
(3.55) \quad \| t^{\frac{k+1}{4} + m} \hat{\nabla}^m \nabla^k u \|_{L^\infty} \leq \| u_0 \|_{B_{-\alpha}^{m, k}}.
\end{equation}
Proof. This result can be verified by replacing $u$ as $\nabla \cdot u$, and then combing the process of Corollary 3.8 and the Theorem 2.2.3 in Danchin [4]. See also P125 in Lemarie-Rieusset [12]. □

Similarly to Proposition 3.4, we also have

**Corollary 3.10.** For any positive integers $m$ and $k$, let $u$ be the solution of system (3.4), then we have

$$
\|t^{\frac{5}{2}+m} \partial_t^m \nabla^k (-\Delta)^{-1/4} u\|_{Z_T} \leq C_0 \|u_0\|_{BMO}^{-1}.
$$

**Corollary 3.11.** Let $V_1$ and $G_1$ as given by Proposition 3.5, for any integers $M,K \geq 0$ we have

$$
\|t^{\frac{5}{2}+m} \partial_t^m \nabla^k V_1\|_{X_T} \leq \|G_1\|_{Y_T}.
$$

**Proof.** This results can be verified similar to Corollary 5.1. See also in [5]. □

**Corollary 3.12.** Let Let $V_2$ and $G_2$ as given by Corollary 3.6, then for any positive integers $m$ and $k$, we have

$$
\|t^{\frac{5}{2}+k} \partial_t^k \nabla^m V_2\|_{X_T} \leq \|t^{\frac{5}{2}+k} \partial_t^k \nabla^m G_2\|_{Y_T}.
$$

4. **Proof of Theorem 1.3 and Theorem 1.4**

We first consider the eigenvalue problem

$$
\begin{align*}
\nabla \times S &= \mu S \\
\nabla \cdot S &= 0.
\end{align*}
$$

Suppose that $S$ is a periodic function on $T^3$ such that

$$
\int_{T^3} S(x) dx = 0,
$$

then it is easy to see that in the space of divergence free vector field

$$(\nabla \times)^{-1} = (-\Delta)^{-1} \nabla \times.$$ 

So $(-\Delta)^{-1}$ is a compact operator, by the theory of eigenvalues of compact operators, we conclude that (4.1) has eigenvalues $\{\lambda_j\}_{j \in \mathbb{Z}}$ such that

$$
\cdots < \lambda_{-N} < \cdots < \lambda_{-1} < 0 < \lambda_1 < \cdots < \lambda_N < \cdots
$$

Therefore, we can write

$$
u_0 = \sum_{j=-\infty}^{+\infty} S_j
$$

where $S_j$ are the eigenfunctions with eigenvalues $\lambda_j$.

Take one more differentiation of (4.1), we see that

$$
-\Delta S = \mu^2 S.
$$

So $\mu^2$ is an eigenvalue of $-\Delta$ and $S$ is an eigenvector of $-\Delta$. Thus, (4.1) is nothing but a Fourier series expansion.
We are now ready to prove our theorem. Without loss of generality, we assume \( \lambda \geq 0 \), otherwise, a reflection \( x \mapsto -x \) reduce to this case.

Let
\[
(4.6) \quad u_{0+} = \sum_{j=1}^{\infty} S_j, \quad u_{0-} = \sum_{j=-\infty}^{-1} S_j
\]
then it is easy to see that
\[
(4.7) \quad u_{0+} = \frac{(u_0 + (-\Delta)^{-\frac{1}{2}} \nabla \times u_0)}{2},
\]
\[
(4.8) \quad u_{0-} = \frac{(u_0 - (-\Delta)^{-\frac{1}{2}} \nabla \times u_0)}{2},
\]
\[
(4.9) \quad (\sqrt{-\Delta} + \lambda)u_{0-} = \sum_{j=-\infty}^{-1} (-\lambda_j + \lambda)S_j = -\nabla \times u_{0-} + \lambda u_{0-}.
\]

Therefore, there holds
\[
(4.10) \quad \|u_{0-}\|_{BMO^{-1}} \leq \|(\sqrt{-\Delta} + \lambda)u_{0-}\|_{BMO^{-2}}
\]
\[
= \|\nabla \times u_{0-} - \lambda u_{0-}\|_{BMO^{-2}} \leq \|\nabla \times u_0 - \lambda u_0\|_{BMO^{-2}} \lesssim \varepsilon \langle \lambda \rangle^{-b}.
\]
In a similar way, we also have
\[
(4.11) \quad \|(\sqrt{-\Delta} - \lambda)u_{0+}\|_{BMO^{-2}}
\]
\[
= \|\nabla \times u_{0+} - \lambda u_{0+}\|_{BMO^{-2}} \leq \|\nabla \times u_0 - \lambda u_0\|_{BMO^{-2}} \lesssim \varepsilon \langle \lambda \rangle^{-b}.
\]

When \( 1 \leq \lambda_j \leq \lambda/2 \) or \( \lambda_j \geq 2\lambda \), the operator \( \sqrt{-\Delta} - \lambda \) is invertible. Denote
\[
(4.12) \quad u_{1+} = \sum_{1 \leq \lambda_j \leq \lambda/2} S_j + \sum_{\lambda_j \geq 2\lambda} S_j,
\]
\[
(4.13) \quad u_{2+} = \sum_{\lambda/2 < \lambda_j < 2\lambda} S_j.
\]
Therefore we get in a similar way that
\[
(4.14) \quad \|u_{1+}\|_{BMO^{-1}} \lesssim \varepsilon \langle \lambda \rangle^{-b}.
\]

Denote
\[
(4.15) \quad u_{01} = u_0 - u_{2+} = u_{0-} + u_{1+},
\]
and combining (4.10) and (4.14), we have
\[
(4.16) \quad \|u_{01}\|_{BMO^{-1}} \lesssim \varepsilon \langle \lambda \rangle^{-b}.
\]

Let
\[
(4.17) \quad U = u + v,
\]
with
\[
(4.18) \quad v = e^{t\Delta}u_{2+}.
\]
Then we rewrite the system (1.1) as the following system with small data:

$$
\begin{align*}
\begin{cases}
  v_t + (v \cdot \nabla)v + (u \cdot \nabla)v + (v \cdot \nabla)u + (u \cdot \nabla)u + \nabla P = \Delta v, \\
  t = 0 : v = u_{01}.
\end{cases}
\end{align*}
$$

4.1. **Step 1: Short time existence.** The purpose of the short time existence result is just to break the scaling. We shall study our problem by the classical fixed point argument.

First, we introduce the following space:

**Definition 4.1.** Let $T_1 > 0$ be a given constant and $\varepsilon_0$ be a small constant, we say $f \in \mathcal{E}_{T_1,\varepsilon_0}$ if the following holds:

(i): $f(t,x)$ is a periodic function of $x$ on $T^3$;

(ii): $\nabla \cdot f = 0$, $\|f\|_{X_{T_1}} \leq \varepsilon_0 < \lambda >^{-b}$.

We define $v = \tilde{v}$ by solving the following linear equation with $\tilde{v} \in \mathcal{E}_{T_1,\varepsilon_0}$

$$
\begin{align*}
\begin{cases}
  v_t - \Delta v + \nabla P = -((u + \tilde{v}) \cdot \nabla)(u + \tilde{v}), \\
  t = 0 : v = u_{01}.
\end{cases}
\end{align*}
$$

Let $g = u + \lambda \Delta^{-1}(\nabla \times u)$, then

$$
\nabla \times u - \lambda u = \nabla \times g,
$$

and

$$
u = g - \lambda \Delta^{-1}(\nabla \times u).
$$

By Proposition 3.2 and Proposition 3.5, we get

$$
\|v\|_{X_{T_1}} \lesssim \|(u + \tilde{v}) \otimes (u + \tilde{v})\|_{Y_{T_1}} + \|u_0 - \nabla \times u\|_{BMO^{-1}} \\
\lesssim \left(\|u\|_{X_{T_1}} + \|\tilde{v}\|_{X_{T_1}}\right)^2 + \varepsilon(\lambda)^{-b} \lesssim \left(\|u\|_{X_{T_1}} + \varepsilon_0(\lambda)^{-b}\right)^2 + \varepsilon(\lambda)^{-b}.
$$

By Proposition 3.3 and Corollary 3.4, we have

$$
\|u\|_{X_{T_1}} \lesssim \|g\|_{X_{T_1}} + \lambda \|\Delta^{-1}\nabla \times u\|_{X_{T_1}} \\
\lesssim \|\nabla \times u_0 - \lambda u_0\|_{BMO^{-2}} + \lambda T_1^{\frac{1}{4}}\|\sqrt{-\Delta} \Delta^{-1}\nabla \times u_0\|_{BMO^{-1}} \\
\lesssim \varepsilon(\lambda)^{-b} + \lambda T_1^{\frac{1}{4}}M_0.
$$

We take $\varepsilon_0 = C(M_0)^{-1}\varepsilon$ and take $T_1$ as

$$
T_1 = C(M_0)\varepsilon^2(\lambda)^{-2b-4}
$$

where the constant $C(M_0)$ is independent of $\lambda$ and $\varepsilon$ and $C(M_0)$ is choosing suitably small.

Then by using (4.23) and (4.25) we get

$$
\|v\|_{X_{T_1}} \leq \varepsilon_0(\lambda)^{-b}.
$$
By a similar process, we take \( v_1 \) be the corresponding solution to (4.19) where \( \tilde{v}_1 \in E_{T_1, \epsilon_0} \), and \( v_2 \) be the solution with \( \tilde{v}_2 \in E_{T_1, \epsilon_0} \). Denote

\[
(4.27) \quad \tilde{v} = \tilde{v}_1 - \tilde{v}_2, \quad \bar{v} = v_1 - v_2, \quad \tilde{q} = P_1 - P_2,
\]

then there hold

\[
(4.28) \quad \tilde{v}_t - \Delta \tilde{v} + \nabla \tilde{q} = (\bar{v}_2 \cdot \nabla)\tilde{v} + (\bar{v} \cdot \nabla)\tilde{v}_1 + (u \cdot \nabla \bar{v}) + (\bar{v} \cdot \nabla)u.
\]

Therefore, we get

\[
(4.29) \quad \| \tilde{v} \|_{X_{T_1}} \lesssim \| \tilde{v}_2 \|_{X_{T_1}} + \| \tilde{v}_1 \|_{X_{T_1}} + \| u \|_{X_{T_1}}.
\]

Since \( \epsilon_0 \) is small enough, then

\[
(4.30) \quad \| \tilde{v}_2 \|_{X_{T_1}} + \| \tilde{v}_1 \|_{X_{T_1}} + \| u \|_{X_{T_1}} \ll 1.
\]

Combining (4.29) and (4.30), we conclude that the mapping is a contraction.

Our next goal is to prove the space and time analyticity of the solution.

**Definition 4.2.** Let \( g \) be a function defined on \( \mathbb{R}^n \times [0, T_1) \), for any integers \( M, K \geq 0 \), we say \( g \in X^{M, K}_{T_1} \), if

\[
(4.31) \quad \| g \|_{X^{M, K}_{T_1}} = \max \left\{ \sum_{m=0}^M \sum_{k=0}^K \sup_{t} \left( \int |t|^{\frac{k+1}{2}} + m \| \tilde{\partial}_t^m \tilde{\nabla}^k g \|_{L^p(\mathbb{R}^3)} \right) + \sup_{y_0 \in \mathbb{R}^3, r > 0} \left( r^{-3} \int Q(y_0, r) |t|^{\frac{3}{2}} + m \| \tilde{\partial}_t^m \tilde{\nabla}^k g \|^2 dt \right) \right\} < +\infty.
\]

Subsequently, we give the following function spaces

**Definition 4.3.** Let \( T > 0 \) be a given constant and \( \epsilon_0 \) as in Theorem 1.3, for any integers \( M, K > 0 \) we say \( f \in E^{M, K}_{T, \epsilon_0} \), if the following holds:

(i): \( f(x, t) \) is a space periodic function on \( \mathbb{T}^3 \);
(ii): \( \nabla \cdot f = 0 \), and \( \| f(x, t) \|_{X^{M, K}_{T_1}} \leq \epsilon_0 < \lambda >^{-b} \).

Repeat the above process, we can show that \( v \in E^{M, K}_{T_1, \epsilon_0} \).

4.2. **Step 2: Local in time existence.** Now since \( v \) is analytic at time \( T_1 \), so it can be extended to some time interval \([T_1, T_2]\). We shall give an estimate of \( T_2 \). For that purpose, we only need to give an apriori estimate for the \( \| v(t) \|_{L^p} \) on the time interval \([T_1, T_2]\).

By Proposition 3.3 and Corollary 3.6, we get, for \( 0 < \alpha < 1 \),

\[
(4.32) \quad t^{\frac{1-\alpha}{2}} \| (-\Delta)^{-\frac{\alpha}{2}} v(t, \cdot) \|_{L^p} \lesssim \epsilon \langle \lambda \rangle^{-b}.
\]

Moreover, take \( t = T_1 \), we get

\[
\| (-\Delta)^{-\frac{\alpha}{2}} v(T_1, \cdot) \|_{L^p} \lesssim \epsilon \langle \lambda \rangle^{-b} \frac{T_1^{-\frac{1-\alpha}{2}}}{2} \lesssim \epsilon \langle \lambda \rangle^{-b} \langle \lambda \rangle^{b(1-a)} \epsilon^{-(1-a)}.
\]
Taking \( 1 - a = \frac{b}{1 + \tau} \), we get
\[
||(-\Delta)^{-\frac{\alpha}{2}} v(T_1, \cdot) ||_{L^\infty} \leq C_0(M_0) e^\alpha
\]

By Lemma 2.4, we rewrite (4.19) as
\[
\begin{aligned}
&\left\{ v_t + (v \cdot \nabla)v + (u \cdot \nabla)v + (v \cdot \nabla)u + \nabla \left( P + \frac{|u|^2}{2} \right) - \Delta v = u \times (\nabla \times u), \\
&\quad \text{in } \Omega, \quad \text{for } t \in (0, \infty),
\end{aligned}
\]

By the divergence free condition and Duhamel's formula, we get
\[
(4.35) \quad v(t) = e^{\Delta (t - T_1)} v(T_1) + \int_{T_1}^t \tilde{P}(t - \tau, \cdot) * (u \times (\nabla \times u))(\tau) d\tau
\]
\[
\quad + \int_{T_1}^t \nabla \tilde{P}(t - \tau) * (v \otimes v + v \otimes u + u \otimes v)(\tau) d\tau
\]
\[
= I + II + III.
\]

Where \( \tilde{P} \) is defined in Lemma 2.1.

Noting (4.33) it follows from Corollary 3.9 (in which we take \( m = k = 0, \alpha = a \)) that
\[
(4.36) \quad \| I \|_{L^\infty} \leq (t - T_1)^{-\frac{\alpha}{2}} \| v(T_1, \cdot) \|_{B^\alpha_{\infty, \infty}} \leq (t - T_1)^{-\frac{\alpha}{2}} e^\alpha
\]

Besides, we have
\[
\begin{aligned}
&\quad u \times (\nabla \times u) = u \times (\nabla \times u - \lambda u)
\end{aligned}
\]
\[
\begin{aligned}
&= \sum_{\lambda/2 < \lambda_j < 2\lambda} e^{-\lambda_j^2 t} S_j \times \sum_{\lambda/2 < \lambda_k < 2\lambda} e^{-\lambda_k^2 t} (\lambda_k - \lambda) S_k
\end{aligned}
\]
\[
\begin{aligned}
&= \sum_{\lambda/2 < \lambda_j, \lambda_k < 2\lambda} e^{-\lambda_j^2 t + \lambda_k^2 t} (\lambda_k - \lambda) S_j \times S_k
\end{aligned}
\]
\[
\begin{aligned}
&= \sum_{\lambda/2 < \lambda_j, \lambda_k < 2\lambda} e^{-\lambda_j^2 t + \lambda_k^2 t} \lambda_k - \lambda \frac{\lambda_j + \lambda_k}{\lambda_j \lambda_k} [(\nabla \times S_j) \times S_k + S_j \times (\nabla \times S_k)]
\end{aligned}
\]
\[
\begin{aligned}
&= \nabla \left( \sum_{\lambda/2 < \lambda_j, \lambda_k < 2\lambda} e^{-(\lambda_j^2 + \lambda_k^2) t} \lambda_k - \lambda \frac{\lambda_j + \lambda_k}{\lambda_j \lambda_k} S_j \cdot S_k \right)
\end{aligned}
\]
\[
\begin{aligned}
&+ \nabla \cdot \left[ \sum_{\lambda/2 < \lambda_j, \lambda_k < 2\lambda} e^{-(\lambda_j^2 + \lambda_k^2) t} \frac{1}{\lambda_j \lambda_k} (S_j \otimes (\nabla \times S_k - \lambda S_k) + (\nabla \times S_k - \lambda S_k) \otimes S_j) \right].
\end{aligned}
\]

Then it follows from Corollary 3.9 (in which we take \( k = m = 0 \), \( \alpha = 1 - \kappa \) ) that
\[
(4.37) \quad \| II \|_{L^\infty} \leq \int_{T_1}^t (t - \tau)^{-\frac{\alpha}{1 - \kappa}} \| F(\tau) \|_{B^\alpha_{\infty, \infty}} d\tau
\]

here, we take \( 0 < \kappa < 1 - a \) and
\[
\begin{aligned}
F(\tau) &= \sum_{\lambda/2 < \lambda_j, \lambda_k < 2\lambda} e^{-(\lambda_j^2 + \lambda_k^2) t} \frac{1}{\lambda_j \lambda_k} (S_j \otimes (\nabla \times S_k - \lambda S_k) + (\nabla \times S_k - \lambda S_k) \otimes S_j).
\end{aligned}
\]
On the other hand, we have

\[
(4.38) \quad \|III\|_{L^\infty} \leq \int_{T_1}^t \|\nabla \tilde{P}(t-\tau)\|_{L^1} \| (v \otimes v + v \otimes u + u \otimes v)(\tau)\|_{L^\infty} d\tau
\] 
\[
\leq \int_{T_1}^t (t-\tau)^{-\frac{1}{2}} \| (v \otimes v + v \otimes u + u \otimes v)(\tau)\|_{L^\infty} d\tau.
\]

Summarizing, we get

\[
(4.39) \quad \|v(t,\cdot)\|_{L^\infty} \leq C_0 (t-T_1)^{-\frac{3}{2}} \varepsilon + C_0 \int_{T_1}^t (t-\tau)^{-\frac{3}{2}} \|v(\tau,\cdot)\|_{L^\infty}^2 d\tau
\]
\[
+ C_0 \int_{T_1}^t (t-\tau)^{-1/2} \|v(\tau,\cdot)\|_{L^\infty} \|u(\tau,\cdot)\|_{L^\infty} d\tau + C_0 \int_{T_1}^t (t-\tau)^{-\frac{1+\alpha}{2}} \|F(\tau)\|_{B_{2,\infty}^\alpha} d\tau.
\]

By Corollary 2.6, we get

\[
\|F(\tau,\cdot)\|_{B_{2,\infty}^\alpha} \leq \|\nabla \times u - \lambda u\|_{B_{2,\infty}^{\alpha-1}} \|u\|_{B_{2,\infty}^\alpha}
\]
\[
\leq \tau^{-\frac{1+\alpha}{2}} \|\nabla \times u_0 - \lambda u_0\|_{B_{2,\infty}^{\alpha-1}}\|u_0\|_{B_{2,\infty}^1}
\]
\[
\leq (\tau - T_1)^{-\frac{1+\alpha}{2}} \|\nabla \times u_0 - \lambda u_0\|_{B_{2,\infty}^{\alpha-1}}\|u_0\|_{B_{2,\infty}^1}
\]
\[
\leq (\tau - T_1)^{-\frac{1+\alpha}{2}} C(M_0) \varepsilon^\alpha.
\]

Therefore, there holds

\[
\int_{T_1}^t (t-\tau)^{-\frac{1+\alpha}{2}} \|F(\tau)\|_{B_{2,\infty}^\alpha} d\tau
\]
\[
\leq \varepsilon^\alpha \int_{T_1}^t (t-\tau)^{-\frac{1+\alpha}{2}} (\tau - T_1)^{-\frac{1+\alpha}{2}} d\tau
\]
\[
\leq \varepsilon^\alpha (t - T_1)^{-\frac{1+\alpha}{2}} \int_{T_1}^{T_1 + \frac{t}{t - T_1}} (\tau - T_1)^{-\frac{1+\alpha}{2}} d\tau
\]
\[
+ \varepsilon^\alpha (t - T_1)^{-\frac{1+\alpha}{2}} \int_{T_1 + \frac{t}{t - T_1}}^t (t-\tau)^{-\frac{1+\alpha}{2}} d\tau \leq \varepsilon^\alpha (t - T_1)^{-\frac{1}{2}}.
\]

We also have

\[
\int_{T_1}^t (t-\tau)^{-1/2} \|v(\tau,\cdot)\|_{L^\infty} \|u(\tau,\cdot)\|_{L^\infty} d\tau
\]
\[
\leq C_0 \int_{T_1 + (1-\delta)(t-T_1)}^t (t-\tau)^{-\frac{1}{2}} \|u(\tau,\cdot)\|_{L^\infty} \|v(\tau,\cdot)\|_{L^\infty} d\tau
\]
\[
+ C(\delta)(t - T_1)^{-\frac{1}{2}} \int_{T_1}^{T_1 + (1-\delta)(t-T_1)} \|u(\tau,\cdot)\|_{L^\infty} \|v(\tau,\cdot)\|_{L^\infty} d\tau
\]
\[
\leq C_1 M_0 (t - T_1)^{-\frac{1}{2}} \sup_t (t - T_1)^{\frac{1}{2}} \|v(\tau,\cdot)\|_{L^\infty} \int_{T_1 + (1-\delta)(t-T_1)}^t (t - \tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} d\tau
\]
\[
+ C(\delta)(t - T_1)^{-\frac{1}{2}} \left( \int_{T_1}^{T_1 + (1-\delta)(t-T_1)} \|u(\tau,\cdot)\|_{L^\infty}^2 \|v(\tau,\cdot)\|_{L^\infty}^2 d\tau \right)^{1/2}.
\]
By let \((\tau - T_1) = \theta(t - T_1)\), we get
\[
\int_{T_1 + (1-\delta)(t-T_1)}^{t} (t - \tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} d\tau \leq \int_{1-\delta}^{1} (1 - \theta)^{-\frac{1}{2}} \theta^{-\frac{1}{2}} ds
\]
which can be small if \(\delta\) is made sufficiently small. We take
\[
2C_1M_0 \int_{T_1 + (1-\delta)(t-T_1)}^{t} (t - \tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} d\tau \leq 1/2.
\]

On the other hand, it is easy to see
\[
\int_{T_1}^{t} (t - \tau)^{-\frac{1}{2}} \tau^{-a} d\tau \leq \left( \int_{T_1}^{t} (t - \tau)^{-\frac{1}{2}} \tau^{-a} d\tau \right) \sup_{T_1 \leq \tau \leq t} \tau^a |v(\tau)|_{L^\infty}^2.
\]

Moreover, there also holds
\[
\int_{T_1}^{t} (t - \tau)^{-\frac{1}{2}} \tau^{-a} d\tau \leq \left( \int_{T_1}^{t} (t - \tau)^{-\frac{1}{2}} \tau^{-a} d\tau \right) + (t - T_1)^{-a} \int_{T_1 + \frac{\tau - T_1}{T_1}}^{t} (t - \tau)^{-\frac{1}{2}} d\tau \leq (t - T_1)^{\frac{1}{2} - a}.
\]

Thus, we get
\[
\sup_{T_1 \leq t \leq t} (t_1 - T_1) \frac{1}{2} \|v(t_1, \cdot)\|_{L^\infty} \leq \varepsilon^a + (t - T_1)^{\frac{1-a}{2}} \left( \sup_\tau \left( (\tau - T_1)^{\frac{a}{2}} \|v(\tau, \cdot)\|_{L^\infty} \right) \right)^2 + \left( \int_{T_1}^{t} \|u(\tau, \cdot)\|_{L^2}^2 \left( (\tau - T_1)^{\frac{a}{2}} \|v(\tau, \cdot)\|_{L^\infty} \right)^2 d\tau \right)^{\frac{1}{2}}.
\]

Take \(T_2 = T_1 + 1\), for \(T_1 \leq t \leq T_2\), we get
\[
\sup_{T_1 \leq t \leq t} (t_1 - T_1) \frac{1}{2} \|v(t_1, \cdot)\|_{L^\infty} \leq 2C_2 \left[ \varepsilon^a + \left( \int_{T_1}^{t} \|u(\tau, \cdot)\|_{L^2}^2 \left( (\tau - T_1)^{\frac{a}{2}} \|v(\tau, \cdot)\|_{L^\infty} \right)^2 d\tau \right)^{\frac{1}{2}} \right]
\]
provided that
\[
C_2 \sup_{T_1 \leq t} (t_1 - T_1) \frac{1}{2} \|v(t_1, \cdot)\|_{L^\infty} \leq \frac{1}{2}.
\]

Furthermore, we have
\[
\left( \int_{0}^{t} \|u(\tau, \cdot)\|_{L^\infty}^2 d\tau \right)^{\frac{1}{2}} \leq \|u_2^+\|_{B^{-1}_{x,2}}.
\]
However, \(u_2^+\) has only finite piece of the littlewood-Paley decomposition, thus
\[
\|u_2^+\|_{B^{-1}_{x,2}} \leq \|u_2^+\|_{B^{-1}_{x,2}} \leq \|u_0\|_{BMO^{-1}}.
\]

Then from (4.44) and the Gronwall’s inequality, we have
\[
\sup_{T_1 \leq t \leq T_2} (t - T_1) \frac{1}{2} \|v(t, \cdot)\|_{L^\infty} \leq 4C_2 \varepsilon^a.
\]
provided that $\varepsilon$ is sufficiently small. This shows that the assumption (4.15) is reasonable. Therefore, we can extend our solution to time $T_2$.

By a similar argument, we can get

$$\sup_{T_2 < t \leq T_2} (t - T_2) \frac{2}{3} + \frac{2}{5} |c_{m}^{k}v(t, \cdot)|_{L^{\infty}} \leq C_{m,k}\varepsilon^{\alpha}$$

for any $m \geq 0$, $k \geq 0$, which implies

$$|\nabla^{k}v(T_2, \cdot)|_{L^{\infty}} \leq C(k, M_0)\varepsilon^{\alpha}.$$  

4.3. Step 3: Global existence. We now consider the solution on a fixed periodic $x \in [-\pi, \pi]^3$. To prove global existence on the time interval $[T_2, +\infty)$, we only need to give an apriori $H^1$ bound of the solution $v$. We multiply the equation (4.34) by $\Delta v$ and integration by parts to get

$$\frac{1}{2} \|\nabla v(t, \cdot)\|_{L^2}^2 + \int_{T_2}^{t} \|\Delta v(\tau, \cdot)\|_{L^2}^2 d\tau \leq \frac{1}{2} \|\nabla v(T_2, \cdot)\|_{L^2}^2 + \int_{T_2}^{t} \|\Delta v(\tau, \cdot)\|_{L^2}^2 + \|v \cdot \nabla v + u \cdot \nabla v + v \cdot \nabla u\|_{L^2}^2(\tau) d\tau,$$

and

$$\|v \cdot \nabla v + u \cdot \nabla v + v \cdot \nabla u\|_{L^2} \leq \left( \|v\|_{L^\infty} + \|u\|_{L^\infty} \right) \|\nabla v\|_{L^2} + \|\Delta v\|_{L^2} + \|\nabla u\|_{L^2} \|v\|_{H^1} + \|\nabla v\|_{L^2} \|\nabla u\|_{L^2} \|v\|_{H^1}.$$  

By Poincare’s inequality, we get

$$\|v \cdot \nabla v + u \cdot \nabla v + v \cdot \nabla u\|_{L^2} \leq \|\nabla v\|_{L^2}^2 \|\Delta v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \|\nabla v\|_{L^2} \|\nabla u\|_{L^2}.$$  

Therefore, we get

$$\frac{1}{2} \|\nabla v(t, \cdot)\|_{L^2}^2 + \int_{T_2}^{t} \|\Delta v(\tau, \cdot)\|_{L^2}^2 d\tau \leq \varepsilon^{2\alpha} + \frac{1}{2} \int_{T_2}^{t} \|\Delta v(\tau, \cdot)\|_{L^2}^2 d\tau + \int_{T_2}^{t} \|\nabla v(\tau, \cdot)\|_{L^2}^2 \|\nabla v\|_{L^2} d\tau + \int_{T_2}^{t} \|\nabla v\|_{L^2} \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \|\nabla u\|_{L^2}^2 d\tau.$$
We have
\[
\int_{T_2}^T \left\| \nabla u \right\|_{L^2(T^2)}^4 + \left\| u \right\|_{L^\infty(T^2)}^2 d\tau \\
\lesssim \int_{T_2}^T \left( \tau^{-4} \left\| u_0 \right\|_{BMO^{-1}} + \tau^{-2} \left\| u_0 \right\|_{BMO^{-2}} \right) d\tau \\
\leq CM_0 (T_2^{-3} + T_2^{-1}) \leq C_3(M_0).
\]

Therefore, by a bootstrap argument, it follows from Gronwall’s inequality that
\[
\left\| \nabla v(t, \cdot) \right\|_{L^2}^2 + \int_{T_2}^T \left\| \Delta v(t, \cdot) \right\|_{L^2}^2 d\tau \leq C_3(M_0)\epsilon^{2\alpha}.
\]

This shows that the solution is global.

ACKNOWLEDGEMENT

Yi Du was supported by NSFC (grant No. 11471126). Yi Zhou was supported by Key Laboratory of Mathematics for Nonlinear Sciences (Fudan University), Ministry of Education of China, P.R.China. Shanghai Key Laboratory for Contemporary Applied Mathematics, School of Mathematical Sciences, Fudan University, P.R. China, NSFC (grants No. 11421061), 973 program (grant No. 2013CB834100) and 111 project.

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