The Master Ward Identity

M. Dütsch* and F.-M. Boas†
Institut für Theoretische Physik
Universität Göttingen
Bunsenstrasse 9
D-37073 Göttingen, Germany
duetsch@theorie.physik.uni-goettingen.de
boas.franz-marc@bcg.com

Abstract

In the framework of perturbative quantum field theory (QFT) we propose a new, universal (re)normalization condition (called 'master Ward identity') which expresses the symmetries of the underlying classical theory. It implies for example the field equations, energy-momentum, charge- and ghost-number conservation, renormalized equal-time commutation relations and BRST-symmetry.

It seems that the master Ward identity can nearly always be satisfied, the only exceptions we know are the usual anomalies. We prove the compatibility of the master Ward identity with the other (re)normalization conditions of causal perturbation theory, and for pure massive theories we show that the 'central solution' of Epstein and Glaser fulfills the master Ward identity, if the UV-scaling behavior of its individual terms is not relatively lowered.

Application of the master Ward identity to the BRST-current of non-Abelian gauge theories generates an identity (called 'master BRST-identity') which contains the information which is needed for a local construction of the algebra of observables, i.e. the elimination of the unphysical fields and the construction of physical states in the presence of an adiabatically switched off interaction.

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†The second author was involved in the treatment of BRST-symmetry.
Introduction

A perturbative interacting quantum field theory is usually constructed in terms of time ordered products (‘$T$-products’) $T(W_1, ..., W_n)(x_1, ..., x_n)$ of Wick polynomials $W_1(x_1), ..., W_n(x_n)$ of free fields. The $T$-products are ill-defined for coinciding points because they are (operator-valued) distributions. In the framework of the inductive construction of Bogoliubov [4] and Epstein/Glaser [19] (‘causal perturbation theory’) this can be formulated as follows: the $T$-products of $n$-factors are known by induction as operator-valued distributions up to the total diagonal $D_n \overset{def}{=} \{(x_1, ..., x_n) \mid x_1 = ... = x_n\}$. The problem of renormalization is
located in the extension of the \( T \)-products to \( D_n \), for every \( n \). This extension is always possible, but it is non-unique. The freedom is restricted by normalization conditions. They require that symmetries which are present outside \( D_n \) are maintained in the extension and that, for each term in the \( T \)-product, the order of the singularity at \( D_n \) is not increased by the extension. (The latter implies that an interaction with mass dimension \( \leq 4 \) yields a renormalizable theory (by power counting)). Epstein/Glaser \[13\] (see also \[3\]) give a general formula \((73)\) for the extension to \( D_n \) which satisfies the last requirement, but the other normalization conditions are not taken into account. So, the main problem of perturbative renormalization is to prove that there is an extension which fulfills all normalization conditions. In the framework of algebraic renormalization the corresponding problem is treated by means of the 'quantum action principle' \[27\], \[26\], \[31\], which states that the variation of Green’s functions (under a change of coordinates, a variation of the fields or a variation of a parameter) is equal to the insertion of a (local or space-time integrated) composite field operator. Recently a local algebraic operator formulation of certain cases of the quantum action principle has been given by using causal perturbation theory, and the connection to our normalization conditions has been clarified \[8\].

The master Ward identity (we will use the abbreviation 'MWI') is a universal normalization condition supplementing the obvious ones. It is an explicit expression for

\[
\partial_x^\nu T(W, W_1, ..., W_n)(x, x_1, ..., x_n) - T(\partial^\nu W, W_1, ..., W_n)(x, x_1, ..., x_n).
\]

Generally this difference cannot vanish for the following reason\[1\]: the Wick polynomials are built up from free fields, whereas the \( T \)-products are the building stones of the perturbative interacting fields \[4\]. However, the field equations of free and interacting fields are different.

Computing the difference \((1)\) by means of the Feynman rules and the normalization condition \((N3)\) (see sect. 2.2), it can be expressed solely by terms which contain the difference \( \partial_x^\nu \left( \Omega, T(\phi, \chi)(x, x_1)\Omega \right) - \Omega, T(\partial^\nu \phi, \chi)(x, x_1)\Omega \) of Feynman propagators, where \( \Omega \) is the Fock vacuum and \( \phi, \chi \) are free fields. The MWI requires that this structure is preserved in the process of renormalization (sect. 2). For tree diagrams this is automatically satisfied, but for loop diagrams it is a hard task to show that there exists a normalization which fulfills the MWI and the other normalization conditions (sect. 3). Unfortunately there are a few examples where this is impossible. However, the only obstructions we know are the usual, well-known anomalies of perturbative QFT (sect. 5).

The master Ward identity expresses all symmetries which can be traced back to the field equations in classical field theory\[2\]. In particular we will demonstrate that the MWI

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1In particular this argument, the name 'master Ward identity' and the application of the MWI to the computation of a rigorous substitute for the equal-time commutator of interacting fields \[4\] are due to Klaus Fredenhagen.

2In \[4\] we extensively work out the MWI in classical field theory. There the MWI can be formulated non-perturbatively: it is a consequence of the field equations and the fact that classical fields may be multiplied point-wise. Hence, the classical MWI holds always true. (This, together with the fact that in the perturbative expansion of classical fields solely tree
- the field equations for the interacting fields (sect. 4.1),
- conservation of the energy-momentum tensor (sect. 5.2),
- charge conservation in the presence of spinor fields (sect. 4.2),
- ghost number conservation in the presence of fermionic ghost fields (sect. 4.2), and
- the master BRST-identity (sect. 4.4-5), which contains the full information of BRST-symmetry [3] for massless and massive gauge fields.

The field equations, conservation of the energy-momentum tensor, charge and ghost number conservation have already been proved by using other methods of renormalization (see e.g. [1] for the field field equation and [28] for the energy-momentum tensor, both are based on BPHZ-renormalization) or/and in the framework of causal perturbation theory [10], [12], [29], [7]. Using the normal products of Zimmermann [41], Lowenstein has proved that it is allowed to take a partial derivative out of a Green’s function, if the degree of BPHZ-subtraction is lowered by one, see appendix B of [28]. However, we are not aware of a formulation of the MWI in its full generality in any method of renormalization.

Also the master BRST-identity is new to our knowledge. It is the answer to the obvious question: what results for

\[ [Q_0, T(W_1, ..., W_n)(x_1, ..., x_n)]_\mp \]  

(2)

if the MWI is satisfied? Thereby, \( W_1, ..., W_n \) are arbitrary Wick monomials, \( Q_0 \) is the generator of the BRST-transformation of the free fields and \( [\cdot, \cdot]_\mp \) means the \( \mathbb{Z}_2 \)-graded (with respect to the ghost number) commutator. There have been other approaches to formulate BRST-symmetry in the framework of causal perturbation theory. In particular the 'perturbative gauge invariance' of [11], [15], and [35], which was further developed by [28], [21], and [18], suffices for a consistent construction of the \( S \)-matrix in the adiabatic limit, provided this limit exists. However, this assumption holds certainly not true in massless non-Abelian gauge theories, it seems that the confinement is out of the reach of perturbation theory. In massive non-Abelian gauge theories the instability of physical particles (\( W \)- and \( Z \)-bosons, muon and tau etc.) is an obstacle for an \( S \)-matrix description with adiabatic limit. Our way out is to construct the observables locally (i.e. with the interaction adiabatically switched off, sect. 4.5), as we have done it for QED [3]. For our operator formalism the BRST-charge operator of Kugo and Ojima [24] seems to be the adequate tool to define the BRST-transformation. But in contrast to this reference we do not perform the adiabatic limit and, hence, avoid the infrared divergences. The mentioned perturbative gauge invariance [11], [15] does not suffice for our local construction of observables in non-Abelian (massless or massive) gauge theories. But we show that ghost number conservation and the master BRST-identity contain all information which is needed for this construction. In particular we will see that the master BRST-identity implies the perturbative gauge invariance of [11], [15] (diagrams appear, agrees with the triviality of the quantum MWI for tree diagrams.) The classical formulation of the MWI shows that there is a close connection to the Schwinger-Dyson equations.
(and even the generalization proposed in [18], which is called 'generalized (free perturbative operator) gauge invariance' in [16]).

In spite of all these important implications of the MWI, it is difficult to give a direct physical interpretation of this identity (in its full generality) or to formulate the symmetry which is expressed by it. We give two partial answers:

- In classical field theory the MWI can be understood as the most general identity which can be obtained from the field equations and the fact that classical fields may be multiplied point-wise and factorize: \((AB)_L(x) = A_L(x)B_L(x)\) (where \(A, B\) are field polynomials and \(A_L, B_L\) are the corresponding fields to the interaction \(L\), see [9]). But quantum fields may not be multiplied point-wise (because they are distributions) and, hence, the quantum MWI contains much more information than only the field equations.

- A particular case of the MWI is a formula for \(\partial_{x_0} T(A_L(x)B_L(y)) - T(\partial_{x_0} A_L(x)B_L(y))\), where \(A_L, B_L\) are interacting quantum fields to the interaction \(L\) (i.e. non-local formal power series in free fields) and \(T(...)\) means time ordering in \(x\) and \(y\). This difference can be interpreted as a rigorous substitute for the equal-time commutator of \(A_L\) and \(B_L\) defining heuristically \(\tilde{T}(A_L(x)B_L(y)) \equiv \Theta(x^0 - y^0)A_L(x)B_L(y) + \Theta(y^0 - x^0)B_L(y)A_L(x)\) we in fact obtain

\[
\partial_{x_0} \tilde{T}(A_L(x)B_L(y)) - \tilde{T}(\partial_{x_0} A_L(x)B_L(y)) = \delta(x_0 - y_0)[A_L(x), B_L(y)]. \tag{3}
\]

However, \(\tilde{T}(...)\) is problematic: \(\Theta(x^0 - y^0)A(x)B(y)\) exists if \(A\) and \(B\) are free fields, but this does not hold for \(A\) and \(B\) being Wick polynomials, and for interacting fields the situation is even worse. In addition \(\tilde{T}(...)\) is non-covariant: for a Lorentz covariant \(T\)-product (denoted by \(T(\ldots)\) in the following) and a free scalar field \(\phi\) we must have the relation \(\partial_{\mu} T(\phi(x)\partial_{\nu} \phi(y)) - T(\partial_{\mu} \phi(x)\partial_{\nu} \phi(y)) = Cg_{\mu\nu}\delta(x - y)\) (where \(C\) is an undetermined constant), which is obviously not satisfied by \(\tilde{T}(\ldots)\). But fortunately there is the possibility that the non-covariant terms (i.e. the terms coming from \(\tilde{T}(\ldots) - T(\ldots)\)) cancel out with other unwanted terms. This indeed happens for the (interacting) quark currents \(j_{aL}^\mu(t, \vec{x})\): the (heuristic) equal-time commutator \([j_{aL}^0(t, \vec{x}), j_{aL}^k(t, \vec{y})]\), \(k = 1, 2, 3\) has an 'anomalous' term, the Schwinger term:

\[
[j_{aL}^0(t, \vec{x}), j_{aL}^k(t, \vec{y})] = if_{abc}j_{bL}^k(t, \vec{x})\delta^{(3)}(\vec{x} - \vec{y}) + \sum_{l=1}^3 S_{a0}^{kl}\delta^{(3)}(\vec{x} - \vec{y}), \tag{4}
\]

where \(S_{a0}^{kl} \in \mathbb{C}\) is constant. In formula 11-89 of [23] it is postulated that the non-covariant terms of \(\tilde{T}(\ldots)\) are compensated by the Schwinger terms:

\[
\partial_{\mu} T(j_{aL}^\mu(x)j_{bL}^\nu(y)) - T(\partial_{\mu} j_{aL}^\mu(x)j_{bL}^\nu(y)) = if_{abc}j_{cL}^\nu(x)\delta^{(4)}(x - y). \tag{5}
\]

We will show that this identity is in fact a consequence of the MWI (sect. 4.3).

We return to the crucial question whether the MWI can be satisfied in agreement with all other normalization conditions. The compatibility with the other
normalization conditions can be proved generally (sect. 3.2). If all fields are massive there is a distinguished normalization of the \(T\)-products, the so-called central solution \([19]\). We prove that the central solutions fulfill the MWI (and all other normalization conditions) if the UV-scaling behaviour of its individual terms is not relatively lowered (sect. 3.3). This assumption holds mostly true. However, e.g. for the axial and pseudo-scalar triangle-diagram \([89]\) it is violated, and this makes possible the appearance of the axial anomaly.

2 Formulation of the master Ward identity

2.1 The symbolical algebra with internal and external derivative

Let \(\{\phi^{(k)} \mid k = 1, ..., M\}\) be the free quantum fields in terms of which the model is defined. We assume that this set is closed with respect to taking the adjoint operator. In the larger set \(\Phi_{\text{def}} = \{\partial^a \phi^{(k)} \mid k = 1, ..., M, a \in \mathbb{N}_0^4\}\) (\(\partial^a\) is a partial derivative of arbitrary order) we neglect the free field equations and write \(\Phi\) as a sequence \((\phi_l)_{l \in \mathbb{N}}\). To each \(\phi_l\) we associate a symbol \(\varphi_l \equiv \text{sym}(\phi_l)\), \(l \in \mathbb{N}\). Let \(\mathcal{P}\) be the unital, Abelian \(*\)-algebra generated by these symbols. Thereby the symbols corresponding to a free quantum field and to its partial derivatives are linearly independent. The \(*\)-operation in \(\mathcal{P}\) corresponds to taking the adjoint of the free field operators: \(\varphi_l^* \equiv \text{sym}(\phi_l)^* \overset{\text{def}}{=} \text{sym}(\phi_l)\). We define an internal derivative \(\partial^a : \mathcal{P} \to \mathcal{P}\) by \(\partial^a \varphi_l \equiv \partial^a \text{sym}(\phi_l) \overset{\text{def}}{=} \text{sym}(\partial^a \phi_l)\) and the requirements that \(\partial^a\) is linear and a derivation. Now we divide \(\mathcal{P}\) by the ideal \(J\) which is generated by the free field equations (with respect to the internal derivative) and denote the resulting unital, Abelian \(*\)-algebra by \(\mathcal{P}_0\),

\[\mathcal{P}_0 \overset{\text{def}}{=} \mathcal{P}/J.\] (6)

Let \(\pi\) be the projection \(\pi : \mathcal{P} \to \mathcal{P}_0 : A \to A + J\). Internal derivatives in \(\mathcal{P}_0\) are defined by \(\partial^a \pi(A) \overset{\text{def}}{=} \pi(\partial^a A)\), and in this sense the free field equations are valid in \(\mathcal{P}_0\).

In addition we introduce an external derivative \(\tilde{\partial}^a\) on \(\mathcal{P}_0\) which generates new symbols \(\tilde{\partial}^a A (A \in \mathcal{P}_0, a \in \mathbb{N}_0^4\), i.e. \(\tilde{\partial}^a\) means a higher external derivative of order \(|a| = a_0 + a_1 + a_2 + a_3\) and is required to be linear and a derivation. In particular we set \(\tilde{\partial}^a 1 \overset{\text{def}}{=} 0, \forall a \neq 0\). The Abelian, unital \(*\)-algebra (anticommuting in the case of Fermi fields) which is generated by these new symbols is denoted by \(\mathcal{P}_0\):

\[\mathcal{P}_0 \overset{\text{def}}{=} \bigvee \{\tilde{\partial}^a A \mid A \in \mathcal{P}_0, a \in \mathbb{N}_0^4\}.\] (7)

\(^4\)In the case of Fermi fields the symbols anticommute and we call them ‘fermionic symbols’.

\(^5\)Note that this definition is independent from the choice of the representative \(A\).

\(^6\)This external derivative has nothing to do with the exterior derivative of differential geometry.
Next we extend the external and internal derivatives and the *-operation to maps \( \tilde{P}_0 \to \tilde{P}_0 \). For the former two we set
\[
\partial^{(b+a)}A \overset{\text{def}}{=} \tilde{\partial}^{(b+a)}A, \quad \tilde{\partial}^b A \overset{\text{def}}{=} \tilde{\partial}^{(b)}A, \quad \forall A \in \mathcal{P}_0,
\]
and require that \( \tilde{\partial}^\mu \) and \( \partial^\mu \) are linear and derivations. The *-operation is extended by (\( \tilde{\partial}^a A \))\(^*\) \( \overset{\text{def}}{=} \tilde{\partial}^a (A^*) \) (\( A \in \mathcal{P}_0 \)) and by requiring the usual algebraic relations: anti-linearity, \( (BC)^* = C^*B^* \) and \( B^** = B \), \( \forall B, C \in \tilde{\mathcal{P}}_0 \).

Finally we introduce the space
\[
\mathcal{D}(\mathbb{R}^4, \tilde{\mathcal{P}}_0) \cong \tilde{\mathcal{P}}_0 \otimes \mathcal{D}(\mathbb{R}^4).
\]
(9)

The internal and external derivatives are defined on this space as the operators \( \partial^\mu \otimes 1 \) and \( \tilde{\partial}^\mu \otimes 1 \).

Remark: There exists a surjective algebra *-homomorphism \( \tilde{\sigma} : \tilde{\mathcal{P}}_0 \to \mathcal{P} \). This becomes clear from the formalism developed in [9]. Namely, we prove in appendix A of [9] that there exists a map \( \sigma : \mathcal{P}_0 \to \mathcal{P} \) (i.e. ‘from free fields to fields’) with the properties:
(i) \( \pi \circ \sigma = 1 \).
(ii) \( \sigma \) is an algebra *-homomorphism, i.e. \( \sigma(AB) = \sigma(A)\sigma(B) \) and \( \sigma(A^*) = \sigma(A)^* \).
(iii) The Lorentz transformation commutes with \( \sigma \pi \).
(iv) \( \sigma \pi (\mathcal{P}_1) \subset \mathcal{P}_1 \), where \( \mathcal{P}_1 \) is the sub vector space of \( \mathcal{P} \) with basis \( (\varphi_i)_1 \), i.e. the ‘one-factor symbols’.
(v) \( \sigma \) does not increase the mass dimension of the fields, i.e. \( \sigma \pi (B) \) is a sum of terms with mass dimension \( \leq \dim (B) \). In particular we find \( \sigma \pi (\varphi) = \varphi \), if \( \varphi \in \mathcal{P}_1 \) corresponds to a free field without any derivative.
(vi) \( \\{ \partial^\mu \sigma(A) | A \in \mathcal{P}_0, a \in \mathbb{N}_0 \} = \mathcal{P} \).

We now extend \( \sigma \) to a map \( \tilde{\sigma} : \tilde{\mathcal{P}}_0 \to \mathcal{P} \) by setting
\[
\tilde{\sigma}(\tilde{\partial}^a A) \overset{\text{def}}{=} \partial^a \sigma(A), \quad A \in \mathcal{P}_0,
\]
(10)
and requiring that \( \tilde{\sigma} \) is an algebra *-homomorphism. The property (vi) means that \( \tilde{\sigma} \) is surjective. However, \( \tilde{\sigma} \) is not injective. This follows from the following simple example: let \( \varphi \in \mathcal{P}_1 \) correspond to a free Klein Gordon field (without any derivative). Usually it holds \( \sigma \pi (\partial^\mu \varphi) = \partial^\mu \varphi \). Then, \( \tilde{\sigma}(\partial^\mu \pi \varphi) = \partial^\mu \varphi = \tilde{\sigma}(\partial^\mu \pi \varphi) \). This example does not appear if one introduces an additional field \( \varphi^\mu \) which replaces \( \partial^\mu \varphi \) (see sects. 2.3 and 3.2 of [1]), but \( \tilde{\sigma} \) is not injective in that case, too.

\(^7\)A fermionic \( V \in \tilde{\mathcal{P}}_0 \) is paired (to \( V \otimes f \)) with a Grassmann-valued test function \( f \), see e.g. appendix D of [8].
2.2 Inductive construction of time ordered products, basic normalization conditions (N0)-(N3)

The time-ordered product $T_n$ (also called 'T-product') is a linear, symmetrical map from $\mathcal{D}(\mathbb{R}^4, \mathcal{P}_0)^{\otimes n}$ into the (unbounded) operators on the Fock space of the free quantum fields. In particular linearity implies $T_n((\partial_\nu V^\nu) \otimes \ldots) = 0$ if $\partial_\nu V^\nu$ vanishes due to the free field equations. All $T$-products $T_n(f_1 \otimes \ldots \otimes f_n)$, $f_j \in \mathcal{D}(\mathbb{R}^4, \mathcal{P}_0)$, $n \in \mathbb{N}$, have the same domain $\mathcal{D}$ which is a dense subspace of the Fock space and which is invariant under all $T$-products. Physicists use mostly 'unsmeared $T$-products', which are defined by

$$\int dx_1 \ldots dx_n T_n(V_1, \ldots, V_n)(x_1, \ldots, x_n)g_1(x_1) \ldots g_n(x_n) \overset{\text{def}}{=} T_n(V_1 g_1 \otimes \ldots \otimes V_n g_n),$$

where $g_1, \ldots, g_n \in \mathcal{D}(\mathbb{R}^4)$, $V_1, \ldots, V_n \in \mathcal{P}_0$. More precisely $(V_1, \ldots, V_n) \mapsto T_n(V_1, \ldots, V_n)$ is a linear and symmetrical map from $(\mathcal{P}_0)^{\otimes n}$ into the operator-valued distributions.

Let $\mathcal{P}_0 \ni V = \prod_k \delta^{(k)}(\phi_{jk})$ (where $\phi_{jk} \in \Phi$, $\forall k$). Then we define $T_1$ by

$$T_1(V g) \overset{\text{def}}{=} \int dx : \prod_k \delta^{(k)}(\phi_{jk}) (x)g(x), \quad T_1(1g) \overset{\text{def}}{=} \int dx g(x), \quad g \in \mathcal{D}(\mathbb{R}^4),$$

and by linearity, where the double dots mean normal ordering of the free field operators. We point out that $T_1$ is not injective, because $T_1((\partial^\nu V) W g) = T_1((\partial^\nu V) W g), W \in \mathcal{P}_0$. However, $T_1$ is injective if it is restricted to $\mathcal{D}(\mathbb{R}^4, \mathcal{P}_0)$.

The $T$-products are required to satisfy causal factorization:

(Causality) \hspace{1em} $T_n(f_1 \otimes \ldots \otimes f_n) = T_k(f_1 \otimes \ldots \otimes f_k)T_{n-k}(f_{k+1} \otimes \ldots \otimes f_n)$ \hspace{1em} \hspace{1em} \hspace{1em} if \hspace{1em} \hspace{1em} (\text{supp } f_1 \cup \ldots \cup \text{supp } f_k) \cap (\text{supp } f_{k+1} \cup \ldots \cup \text{supp } f_n) + \mathcal{V}_- = \emptyset, \hspace{1em} (13)$

where $\mathcal{V}_-$ is the closed backward light cone in Minkowski space. Causality enables us to construct inductively the $T$-products of higher orders $n \geq 2$: if the time ordered products of less than $n$ factors are everywhere defined, the time ordered product of $n$ factors is uniquely determined up to the total diagonal $D_n \overset{\text{def}}{=} \{(x_1, \ldots, x_n) \mid x_1 = \ldots = x_n\}$. Thus renormalization amounts to an extension, for every $n$, of time ordered products to $D_n$. This extension is always possible, but it is non-unique. It can be done such that the following normalization conditions hold. (Note that these conditions are automatically fulfilled on $\mathcal{D}(\mathbb{R}^{4n} \setminus D_n)$ due to the inductive procedure and causal factorization.)

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8To distinguish the symmetry of $T_n$ from other symmetries we sometimes call it 'permutation symmetry'.

9In $[$1$]$ and $[$4$]$ the arguments of $T_n$ are elements of $\mathcal{D}(\mathbb{R}^4, \mathcal{P})^{\otimes n}$. The map $\tilde{T} \overset{\text{def}}{=} [\overline{\mathbf{T}}]$ connects the two formalisms.

10This is the reason for the name 'time ordered product'.

8
• **Poincare covariance**: Let $U$ be a unitary positive energy representation of the Poincare group $P_+^\uparrow$ in Fock space. $U$ induces an automorphic action $\alpha$ of $P_+^\uparrow$ on $D(\mathbb{R}^4, \mathcal{P}_0)$ by the definition

$$T_1(\alpha_L(f)) \overset{\text{def}}{=} \text{Ad} U(L)(T_1(f)), \quad \forall f \in D(\mathbb{R}^4, \mathcal{P}_0), \quad L \in P_+^\uparrow,$$  \hfill (14)

because $T_1$ is injective on this subspace. We extend $\alpha_L$ to $D(\mathbb{R}^4, \tilde{\mathcal{P}}_0)$ by the prescription that $(\tilde{\partial}^m \otimes 1)f$ transforms in the same way as $(\partial^m \otimes 1)f$, $m \in \mathbb{N}_0$ (where $\partial^m$ (resp.) denotes the $m$-th power of the gradient $\partial$ (resp.)).

More precisely let $f = \sum_i V_i \otimes g_i$, $V_i \in \mathcal{P}_0$, $g_i \in D(\mathbb{R}^4)$. From (14) we know the transformation of $(\partial^m \otimes 1)f$, which can be written in the form

$$\alpha_{(\Lambda,a)}(\partial^m \otimes 1)f = \sum_{i,j} (\partial^m V)_i \otimes D(\Lambda)_{ji} g_{(\Lambda,a)} j, \quad L \equiv (\Lambda,a) \in P_+^\uparrow, \quad m \in \mathbb{N}_0,$$

\hfill (15)

where $g_{(\Lambda,a)}(x) \overset{\text{def}}{=} g(\Lambda^{-1}(x-a))$. Then we define

$$\alpha_{(\Lambda,a)}(\tilde{\partial}^m \otimes 1)f = \sum_{i,j} (\tilde{\partial}^m V)_i \otimes D(\Lambda)_{ji} g_{(\Lambda,a)} j, \quad (\Lambda,a) \in P_+^\uparrow, \quad m \in \mathbb{N}_0.$$  \hfill (16)

One easily verifies $\alpha_{L_1 L_2} = \alpha_{L_1} \alpha_{L_2}$ and that equation (14) holds true for the extended $\alpha_L$, i.e. for all $f \in D(\mathbb{R}^4, \tilde{\mathcal{P}}_0)$. The normalization condition expressing the Poincare covariance of the time ordered products reads

$$(\text{N1}) \quad \text{Ad} U(L)(T(f_1 \otimes ... \otimes f_n)) = T(\alpha_L(f_1) \otimes ... \otimes \alpha_L(f_n)), \quad L \in P_+^\uparrow.$$  

For pure massive theories the so-called 'central solution/extension' (see [19] and sect. 3.3) is Poincare covariant. For theories with massless fields the existence of a Poincare covariant extension has been proved (in [39] and in the second paper of [11]) by tracing it back to a cohomological problem; an explicit solution has been given in [6].

• **Unitarity**: To explain what we mean by 'unitarity' we introduce the $S$-matrix (as a formal power series) which is the generating functional of the $T$-products

$$S(f) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} T_n(f \otimes ... \otimes f), \quad f \in D(\mathbb{R}^4, \tilde{\mathcal{P}}_0).$$  \hfill (17)

Since the zeroth order term does not vanish, it has a unique inverse in the sense of formal power series

$$S(f)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \tilde{T}_n(f \otimes ... \otimes f),$$  \hfill (18)

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where the 'anti-chronological products' \( \bar{T}(\ldots) \) can be expressed in terms of the time ordered products

\[
\bar{T}_n(f_1 \otimes \ldots \otimes f_n) \overset{\text{def}}{=} \sum_{P \in \mathcal{P}(\{1, \ldots, n\})} (-1)^{|P|+n} \prod_{p \in P} T_{|p|}(\otimes_{j \in p} f_j).
\]  

(Here \( \mathcal{P}(\{1, \ldots, n\}) \) is the set of all ordered partitions of \( \{1, \ldots, n\} \), \(|P|\) is the number of subsets in \( P \) and \(|p|\) is the number of elements of \( p \)). The reason for the word 'anti-chronological' is that the \( \bar{T} \)-products satisfy anti-causal factorization, which means \( (13) \) with reversed order of the factors on the r.h.s. Unitarity of the \( S \)-matrix is expressed by

\[
S(f^+) = S(f^*)^{-1}
\]  

(\( ^+ \) means the adjoint on \( D, \ (\phi, B\psi) = (B^+\phi, \psi) \), \( \phi, \psi \in D \).) Hence, for the \( T \)-products we require the normalization condition

\[
(T2) \quad T_n(f_1 \otimes \ldots \otimes f_n)^+ = \bar{T}_n(f_1^* \otimes \ldots \otimes f_n^*),
\]

which can easily be satisfied by symmetrizing an arbitrary normalized \( T \)-product (see \([19]\)).

**Relation to \( T \)-products of sub-polynomials:** Let \( \mathcal{G} \subset \mathcal{P}_0 \) be a linearly independent set of generators of \( \mathcal{P}_0 \), i.e. \( \mathcal{G} \) is a (vector space) basis of \( \pi \mathcal{P}_1 \) (see the Remark in sect. 2.1 for the definition of \( \mathcal{P}_1 \)). Then \( \tilde{\mathcal{G}} \overset{\text{def}}{=} \{ \tilde{\partial}^a \varphi \mid \varphi \in \mathcal{G}, a \in \mathbb{N}_0 \} \) is a set of linearly independent generators of \( \tilde{\mathcal{P}}_0 \). We define the commutator 'function' \( \Delta_{\varphi,\chi} \) by

\[
i \int dx \, dy \, h(x)g(y)\Delta_{\varphi,\chi}(x-y) \overset{\text{def}}{=} [T_1(\varphi h), T_1(\chi g)], \quad \varphi, \chi \in \tilde{\mathcal{G}}.
\]  

(21)

Every \( V \in \mathcal{P}_0 \) can uniquely be written as a polynomial in the generators \( \mathcal{G} \). By partial differentiation in this sense we obtain a 'sub-polynomial' \( \frac{\partial V}{\partial \varphi}, \forall \varphi \in \mathcal{G} \). For \( f(x) = \sum_i V_i f_i(x), \ f_i \in D(\mathbb{R}^4), \ V_i \in \mathcal{P}_0, \) we set

\[
\frac{\partial f}{\partial \varphi} \overset{\text{def}}{=} \sum_i \frac{\partial V_i}{\partial \varphi} f_i(x).
\]

(22)

For \( \psi \in \tilde{\mathcal{G}} \) we analogously define \( \frac{\partial}{\partial \psi} \) to be a linear derivation \( \tilde{\mathcal{P}}_0 \rightarrow \tilde{\mathcal{P}}_0 \) \((\mathcal{D}(\mathbb{R}^4, \tilde{\mathcal{P}}_0) \rightarrow \mathcal{D}(\mathbb{R}^4, \tilde{\mathcal{P}}_0) \) resp.) with

\[
\frac{\partial (\tilde{\partial}^a \chi)}{\partial (\tilde{\partial}^b \varphi)} = \delta_{a,b} \frac{\partial \chi}{\partial \varphi} = \delta_{a,b} \delta_{\chi;\varphi}, \quad \chi, \varphi \in \tilde{\mathcal{G}}.
\]

(23)

Generally we call \( W \in \tilde{\mathcal{P}}_0 \) a subpolynomial of \( V \in \tilde{\mathcal{P}}_0 \) iff it is of the form

\[
W = \frac{\partial^k V}{\partial \varphi_{i_1} \ldots \partial \varphi_{i_k}}
\]

for some \( k \in \mathbb{N}_0, \ \varphi_{i_1}, \ldots, \varphi_{i_k} \in \tilde{\mathcal{G}} \). The derivation property of

\[11\]If \( \varphi \in \mathcal{G} \) is a fermionic symbol, the derivative \( \frac{\partial}{\partial \varphi} \) is a graded derivation.
the commutator $[\cdot, T_1(\chi g)]$ implies

$$
[T_1(f), T_1(\chi g)] = i \sum_{\psi \in \hat{G}} T_1 \left( \frac{\partial f}{\partial \psi} \Delta_{\psi, \chi} * g \right), \quad \forall f \in D(\mathbb{R}^4, \hat{P}_0), \ \chi \in \hat{G}, \ g \in D(\mathbb{R}^4),
$$

(24)

where $*$ means convolution.

We now generalize the normalization condition (N3) of [7] to the present framework: we require

$$(N3)\quad [T_n(f_1 \otimes \cdots \otimes f_n), T_1(\chi g)] =$$

$$i \sum_{l=1}^{n} \sum_{\psi \in \hat{G}} T_n(f_1 \otimes \cdots \otimes \frac{\partial f_l}{\partial \psi} \Delta_{\psi, \chi} * g \otimes \cdots \otimes f_n)$$

(25)

where $f_1, \ldots, f_n \in D(\mathbb{R}^4, \hat{P}_0), \ \chi \in \hat{G}$. The r.h.s. is well-defined because $\Delta_{\psi, \chi} * g$ is a smooth function.

We point out that the defining properties of the $T$-products given so far (linearity, symmetry, causality, (N1), (N2) and (N3)) are purely algebraic conditions, they are independent from the choice of a state. In the realization of the $T$-products as operators in Fock space, (N3) becomes equivalent to the translation of the 'causal Wick expansion' of Epstein and Glaser$^{12}$ into our formalism, see [19] sect. 4.

- **Scaling degree:** (N3) gives the relation to time ordered products of sub-polynomials. Once these are known (in an inductive procedure), only the C-number part of the $T$-product (which is equal to the Fock vacuum expectation value of the $T$-product) has to be fixed. Due to translation invariance this scalar distribution depends only on the relative coordinates. Hence, the extension of the (operator valued) $T$-product to $D_n$ is reduced to the extension of a C-number distribution $t_0 \in D'(\mathbb{R}^{4(n-1)} \setminus \{0\})$ to $t \in D'(\mathbb{R}^{4(n-1)})$. (We call $t$ an extension of $t_0$ if $t(f) = t_0(f), \ \forall f \in D(\mathbb{R}^{4(n-1)} \setminus \{0\})$). The singularity of $t_0(y)$ and $t(y)$ at $y = 0$ is classified in terms of Steinmann’s scaling degree $^{37}$

$$sd(t) \overset{\text{def}}{=} \inf \{\delta \in \mathbb{R}, \ \lim_{\lambda \downarrow 0} \lambda^\delta t(\lambda x) = 0\}. \quad (26)$$

Note

$$sd(\partial^a t) \leq sd(t) + |a| \quad \text{and} \quad sd(\partial^a \delta^{(m)}) = m + |a|, \quad (27)$$

where $\delta^{(m)}$ denotes the $m$-dimensional $\delta$-distribution. By definition $sd(t_0) \leq sd(t)$, and the possible extensions are restricted by requiring

$$(N0)\quad sd(t_0) = sd(t). \quad (28)$$

$^{12}$Epstein/Glaser do not use this name, but it appears e.g. in [4].

11
Then the extension is unique for $sd(t_0) < 4(n - 1)$, and in the general case there remains the freedom to add derivatives of the $\delta$-distribution up to order $(sd(t_0) - 4(n - 1))$. In formula:

$$t(y) + \sum_{|a| \leq sd(t_0) - 4(n - 1)} C_a \partial^a \delta(y)$$

is the general solution, where $t$ is a special extension, and the constants $C_a$ are restricted by $(N1)$, $(N2)$, permutation symmetries and the normalization conditions $(N)$ (normalization of time ordered products of symbols with external derivative) and $(N)$ (MWI) below. For an interaction $L$ with UV-dimension $\text{dim}(L) \leq 4$ the requirement $(28)$ implies renormalizability by power counting, i.e. the number of indeterminate constants $C_a$ in $(T_n((gL)^{\otimes n}))_n$ ($g \in D(\mathbb{R}^4)$) does not increase by going over to higher perturbative orders $n$.

In the seminal paper Epstein and Glaser prove that there exists an extension to $D_n$ which fulfills the normalization conditions $(N0)$-$(N3)$, but they say only few about further symmetries which should be maintained in the extension, e.g. the field equations or gauge invariance. The MWI is a universal normalization condition which summarizes the request for most of this 'further symmetries'.

### 2.3 Normalization of time-ordered products of symbols with external derivative

The aim of this subsection is to fix the normalization of time-ordered products of symbols with external derivative(s) in terms of time-ordered products without external derivative. This fixation is a necessary ingredient of the formulation of the MWI, because $T$-products of symbols with external derivatives unavoidably appear in the MWI. Heuristically the external derivative is a derivative which acts after having done the time-ordered contractions of the corresponding symbols (free fields resp.), e.g.

$$T_{n+1}((\partial^\nu V)g \otimes W_1 f_1 \otimes ... \otimes W_n f_n) =$$

$$\int dx_1...dx_n \ g(x)f_1(x_1)...f_n(x_n)\partial^\nu_x T(V,W_1, ..., W_n)(x,x_1,...,x_n) \equiv$$

$$-T_{n+1}(V\partial^\nu g \otimes W_1 f_1 \otimes ... \otimes W_n f_n),$$

where $V,W_1,...,W_n \in P_0$. However, there are other time ordered products involving factors with external derivatives such as $(\partial^\nu V)W$ which cannot be defined in this way in terms of time ordered products of factors without any external derivative. Hence we proceed in an alternative, recursive way: we give an explicit expression for the difference

$$T_{n+1}((\partial^\nu V)Wg \otimes W_1 f_1 \otimes ... \otimes W_n f_n) - T_{n+1}((\partial^\nu V)Wg \otimes W_1 f_1 \otimes ... \otimes W_n f_n)$$

\[\text{Here we neglect that Epstein and Glaser work with a different formalism, in particular they do not have the external derivative } \partial.\]
where $V, W, W_1, \ldots, W_n \in \tilde{\mathcal{P}}_0$. For this purpose we introduce some notations: by means of the Feynman propagator $\Delta^{F}_{\chi, \psi}$

$$
\int dx \; dy \; f(x)g(y)i\Delta^{F}_{\chi, \psi}(x-y) \overset{\text{def}}{=} \langle \Omega, T_2(\chi f \otimes \psi g)\Omega \rangle, \quad \chi, \psi \in \tilde{\mathcal{G}}, \; f, g \in D(\mathbb{R}^4),
$$

(32)

(where $\Omega$ denotes the Fock vacuum) we define

$$
\delta^\mu_{\chi, \psi} \overset{\text{def}}{=} \partial^\mu \Delta^{F}_{\chi, \psi} - \Delta^{F}_{\partial^\mu \chi, \psi}, \quad \chi, \psi \in \tilde{\mathcal{G}}.
$$

(33)

Inserting the causal factorization of $T_2(...)(x,y)$ for $x \neq y$ we find that $\delta^\mu$ is a local distribution, hence it has the form

$$
\delta^\mu_{\chi, \psi}(z) = \sum_{a \in \mathbb{N}^4_0} C^\mu_{\chi, \psi; a} \partial^a \delta(z),
$$

(34)

where the $C^\mu_{\chi, \psi; a} \in \mathbb{C}$ are constant numbers. Then we define

$$
\Delta^\mu_{\chi, \psi} : D(\mathbb{R}^4, \tilde{\mathcal{P}}_0)^{\times 2} \longrightarrow D(\mathbb{R}^4, \tilde{\mathcal{P}}_0)
$$

$$
\Delta^\mu_{\chi, \psi}(Vg, Wf) = \sum_{a} C^\mu_{\chi, \psi; a} (-1)^{|a|} \sum_{0 \leq b \leq a} \frac{a!}{b!(a-b)!} (\partial^b V)(\partial^{a-b} g)f
$$

(35)

where $|a| = a_0 + a_1 + a_2 + a_3$ and $a! \equiv \prod_{\mu} a_{\mu}!$. This formula is motivated by the identity

$$
\int dx \; \partial^a \delta(x-y) V(x) g(x) W(y) f(y) = \sum_{0 \leq b \leq a} \frac{(-1)^{|a|}}{b!(a-b)!} (\partial^b V)(y) W(y)(\partial^{a-b} g)(y)f(y),
$$

(36)

where $V(x)$ and $W(y)$ are here Wick polynomials (cf. (12)). The subtle point in the definition (35) is that the derivative on $V$ on the r.h.s. is an external one. This results from the derivation of the MWI in classical field theory [9]. And, if the derivative on $V$ would be an internal one, we would get wrong results, e.g. for the BRST-transformation of the interacting gauge field in non-Abelian models [187]. Note that $\Delta^\mu_{\chi, \psi}$ is not invariant with respect to the exchange of its arguments.

\footnote{On the heuristic level of the Feynman rules this can be understood as follows (for simplicity we assume $|a| = 1$): one shifts the derivative $\partial^b$ from the difference (33) of Feynman propagators $\delta^\mu \sim \partial^b(x-y)$ to $V(x)$, however the (time-ordered) contractions of the legs of $V$ are already performed, i.e. $\partial^b V$ must be an external derivative. Thereby the term $VW(\partial g)f$ is the boundary term.}

13
We are now going to compute the difference \( \sum \) on a heuristic level according to our prescription that the external derivative acts after contracting. Let \( V = \prod_{k=1}^{m} \varphi_k \), \( W = \prod_{k=m+1}^{n} \varphi_k \), \( \varphi_k \in \mathcal{G} \), \( \varphi_k = \partial^{\phi_k} \text{sym}(\varphi_k) \). We consider the sum of diagrams in which \( \phi_1, ..., \phi_l \) (\( l \leq m \)) and \( \phi_{m+1}, ..., \phi_q \) (\( q \leq p \)) are contracted and \( \phi_{l+1}, ..., \phi_m, \phi_{q+1}, ..., \phi_p \) are not. By means of the Feynman rules and the normalization condition \( \text{(N3)} \) we compute the contribution of this sum of diagrams to the first \( T \)-product in \( \text{(31)} \):

\[
T_{n+1}((\partial \!^\nu V)W_g \otimes W_1 f_1 \otimes \ldots) = \sum_{r_1, \ldots, r_{l+q}} i^{l+q} \\
\left[ \frac{\partial \!^\nu}{\partial x_1} (\Delta^{F}_{\varphi_1}, (x - x_{r_1}) \ldots \Delta^{F}_{\varphi_l}, (x - x_{r_l}) \Delta^{F}_{\varphi_{m+1}}, (x - x_{r_{m+1}}) \ldots \Delta^{F}_{\varphi_q}, (x - x_{r_{m+q}}) \right] \\
: T(...) (x_1, \ldots, x_n) \prod_{k=l+1}^{m} \partial^{\phi_k} \phi_k(x) \cdot \prod_{k=q+1}^{p} \partial^{\phi_k} \phi_k(x) : \\
\Delta^{F}_{\varphi_1}, (x - x_{r_1}) \ldots \Delta^{F}_{\varphi_l}, (x - x_{r_l}) \Delta^{F}_{\varphi_{m+1}}, (x - x_{r_{m+1}}) \ldots \Delta^{F}_{\varphi_q}, (x - x_{r_{m+q}}) \right] \\
: T(...) (x_1, \ldots, x_n) \partial \!^\nu \left( \prod_{k=l+1}^{m} \partial^{\phi_k} \phi_k(x) \cdot \prod_{k=q+1}^{p} \partial^{\phi_k} \phi_k(x) \right) : + \ldots, \quad (37)
\]

where the double dots simply mean that the \( \phi_k(x), k = l + 1, \ldots, m, q + 1, \ldots, p \) are not contracted. (Note that normal ordering is defined for monomials only, not for polynomials.) With \( \text{(15)} \) we obtain the following heuristic result for the difference \( \text{(31)} \):

\[
\sum_{r_1, \ldots, r_{l+q}} i^{l+q} \sum_{t=1}^{l} \Delta^{F}_{\varphi_1}, (x - x_{r_1}) \ldots \Delta^{F}_{\varphi_t}, (x - x_{r_t}) \ldots \Delta^{F}_{\varphi_{m+1}}, (x - x_{r_{m+1}}) \ldots \Delta^{F}_{\varphi_q}, (x - x_{r_{m+q}}) \\
T(...) (x_1, \ldots, x_n) \prod_{k=l+1}^{m} \partial^{\phi_k} \phi_k(x) \cdot \prod_{k=q+1}^{p} \partial^{\phi_k} \phi_k(x) : + \ldots. \quad (38)
\]

We now require that this structure is maintained in the process of renormalization:

\[
\text{(N)} \quad T_{n+1}((\partial \!^\nu V)W_g \otimes W_1 f_1 \otimes \ldots \otimes W_n f_n) = \\
T_{n+1}((\partial \!^\nu V)W_g \otimes W_1 f_1 \otimes \ldots \otimes W_n f_n) \\
+ i \sum_{m=1}^{n} \sum_{\chi, \psi \in \mathcal{G}} (\pm) T_n \left( \Delta^{\nu}_{\chi, \psi} \frac{\partial V}{\partial \chi} W_g \frac{\partial W_m}{\partial \psi} f_m \right) \\
\otimes W_1 f_1 \otimes \ldots \otimes W_n f_n
\]

\[\text{In this calculation the indices} \ k \text{ of} \ \varphi_k \text{ and} \ \phi_k \text{ have nothing to do with the ones introduced in sect. 2.1.}\]

\[\text{The following implication of} \ \text{(N3) is used here: the Feynman propagators} \ \Delta^{F}_{\varphi_j, \chi} \text{ which appear in} \ \text{(31)} \text{ depend on} \ (\varphi_j, \chi) \text{ only. This means that for} \ (\varphi_j, \chi) = (\varphi_l, \psi) \text{ the undetermined parameters} \ \text{in} \ \Delta^{F}_{\varphi_j, \chi} \text{ and} \ \Delta^{F}_{\varphi_l, \psi} \text{ have the same values. Note additionally} \ \Delta^{F}_{\varphi_j, \chi} = \pm \Delta^{F}_{\varphi_j, \psi} \text{ due to} \ \text{(32).}\]

15 In this calculation the indices \( k \) of \( \varphi_k \) and \( \phi_k \) have nothing to do with the ones introduced in sect. 2.1.

16 The following implication of \( \text{(N3)} \) is used here: the Feynman propagators \( \Delta^{F}_{\varphi_j, \chi} \) which appear in \( \text{(31)} \) depend on \( (\varphi_j, \chi) \) only. This means that for \( (\varphi_j, \chi) = (\varphi_l, \psi) \) the undetermined parameters \( 298 \) in \( \Delta^{F}_{\varphi_j, \chi} \) and \( \Delta^{F}_{\varphi_l, \psi} \) have the same values. Note additionally \( \Delta^{F}_{\varphi_j, \chi} = \pm \Delta^{F}_{\varphi_j, \psi} \) due to \( \text{(32).} \)
where \( V, W, W_1, \ldots, W_n \in \tilde{P}_0 \), the sign \((\pm)\) comes from permutations of Fermi operators and \( \tilde{m} \) means that the corresponding factor is omitted. We now assume that \((N)\) holds true to lower orders \(\leq n\). Then, due to causal factorization of time ordered products, we conclude that the condition \((N)\) is satisfied for \(\text{supp} (g \otimes f_1 \otimes \ldots \otimes f_n) \cap D_{n+1} = \emptyset\). Hence \((N)\) is in fact a normalization condition. It can be satisfied by taking \((N)\) as the definition of the normalization of \(T_{n+1}(\partial^\nu V) g \otimes W g \otimes W f_1 \otimes \ldots \otimes W f_n\). There is only one non-trivial step in this procedure: the compatibility with \((N3)\). This is shown in sect. 3.1.

In models with anomalies, i.e. terms which violate the MWI (see the next subsection), the normalization condition \((N)\) will be modified: in order that \((30)\) holds true these anomalies must be taken into account in the difference \((31)\), they give an additional contribution to the r.h.s. of \((N)\) (cf. sect. 5).

In particular the normalization condition \((N)\) implies

\[
\Delta^F_{\partial^a \varphi_j, \partial^b \varphi_l} = (-1)^{|b|} \partial^a \partial^b \Delta^F_{\varphi_j, \varphi_l}, \quad \varphi_j, \varphi_l \in \mathcal{G},
\]

and hence

\[
\delta^\mu_{\partial^a \varphi_j, \partial^b \varphi_l} = (-1)^{|b|} \partial^a \partial^b \delta^\mu_{\varphi_j, \varphi_l}, \quad \varphi_j, \varphi_l \in \mathcal{G}.
\]

By repeated application of \((N)\) and \(\langle \Omega, T_1(Uh)\Omega \rangle = 0\) for \(U \in \tilde{P}_0 \neq \lambda 1, \lambda \in \mathbb{C}\), one finds

\[
\langle \Omega, T_2\left(\prod_{k=1}^r \partial^a \varphi_{j_k}\right) g \otimes \left(\prod_{k=1}^r \partial^b \varphi_{l_k}\right) f \rangle \Omega = \langle \Omega, T_2\left(\prod_{k=1}^r \partial^a \varphi_{j_k}\right) g \otimes \left(\prod_{k=1}^r \partial^b \varphi_{l_k}\right) f \rangle \Omega
\]

for \(r > 1\), where \(\varphi_m \in \mathcal{G} \quad \forall m, a^{(k)} \equiv (a^{(k)}_0, a^{(k)}_1, a^{(k)}_2, a^{(k)}_3)\) and similar for \(b^{(k)}\).

2.4 The master Ward identity

The MWI is an explicit formula for the difference

\[
\partial^\nu_x T(V, W_1, \ldots, W_n)(x, x_1, \ldots, x_n) - T(\partial^\nu V, W_1, \ldots, W_n)(x, x_1, \ldots, x_n),
\]

where \(V, W_1, \ldots, W_n \in \mathcal{P}_0\). It may be regarded as the postulate that the recursive definition \((N)\) reproduces, in the case \(W = 1\) and \(V, W_1, \ldots, W_n \in \mathcal{P}_0\), the direct definition \((31)\) (see the Remark below). However, this is a very technical and indirect way to the MWI. We found it by the following, intuitive procedure: the result of the Feynman rules for the difference \((42)\) is obtained from \((38)\) by choosing \(\varphi_k \in \mathcal{G}, \forall k\), and putting \(W = 1\) (i.e. \(p = q = m\)). The MWI requires that renormalization is done in such a way that this heuristic result is essentially
preserved:

\[
\begin{align*}
- T_{n+1}(V \partial^\nu g \otimes W_1 f_1 \otimes ... \otimes W_n f_n) &= \\
T_{n+1}((\partial^\nu V)g \otimes W_1 f_1 \otimes ... \otimes W_n f_n) \\
+i \sum_{m=1}^n \sum_{\chi, \psi \in \mathbb{G}} (\pm) T_n \left( \Delta^\nu_{\chi, \psi} \left( \frac{\partial V}{\partial \chi} g, \frac{\partial W_m}{\partial \psi} f_m \right) \right) \\
\otimes W_1 f_1 \otimes ... \hat{m}... \otimes W_n f_n 
\end{align*}
\]

where \( V, W_1, ..., W_n \in P_0 \) (not \( \tilde{P}_0 \)), the sign \((\pm)\) is due to permutations of Fermi operators and \( \hat{m} \) means that the corresponding factor is omitted. We recall that \( \Delta^\mu \) contains external derivatives. To give the correct formula for the difference (42) one needs the external derivative or an equivalent formalism (for a latter see (3) and the Remark at the end of sect. 2.1). Similarly to (\( \tilde{N} \)) the MWI (N) presupposes the normalization condition (N3), because (N3) is used in (37)-(38).

Remark: Instead of requiring (\( \tilde{N} \)) and (N), one can take (34) and (\( \tilde{N} \)) as the primary normalization conditions, because the latter two imply (N). This alternative and more compact formulation is the straightforward way to formulate the quantum MWI [1], when departing from classical field theory. However, the advantage of the present procedure is that it explicitly distinguishes the 'weak' normalization condition (N) (which only defines the normalization of the time ordered products with external derivatives) from the 'hard' one (N) (which expresses deep symmetries). This distinction plays an important role in our (incomplete) proof of the MWI (sect. 3).

We now assume that (N) holds true to lower orders \( \leq n \). Then, due to causal factorization of time ordered products, we conclude that the condition (N) is satisfied for \( \text{supp}(g \otimes f_1 \otimes ... \otimes f_n) \cap D_{n+1} = \emptyset \). Hence (N) is in fact a normalization condition. The compatibility with (N0)-(N2) is trivial and the compatibility with (N3) is proved in sect. 3.2. The hard question is whether (N) can be satisfied by choosing suitable normalizations (which are compatible with the other normalization conditions). The answer depends on the model. We will see that the MWI implies that there is no axial anomaly and no trace anomaly of the energy momentum tensor. Hence it must be impossible to fulfil the MWI in these cases. Generally we call any term that violates the MWI (and cannot be removed by an admissible, finite renormalization of the \( T \)-products) an anomaly.

If there is at most one contraction between \( V \) and \( W_1, ..., W_n \) (i.e. we have \( l = 0 \) or \( l = 1 \) and of course \( p = q = m \) in (38)) the expression (38) is well-defined and (re)normalization can be done such that (38) gives the contribution of these diagrams to the difference (12). In other words one can fulfil the MWI (N) for these 'tree-like' diagrams. The anomalies must come from 'loop-like' diagrams. In sect. 5 we give a more general formulation of the MWI which takes anomalies into account.
3 Steps towards a proof of the master Ward identity

We have to show that there exists a normalization of the $T$-products which satisfies $(\tilde{N})$, $(N)$ and also $(N_0)-(N_3)$. The compatibility of $(\tilde{N})$ and $(N)$ with $(N_0)-(N_2)$ is obvious, but the compatibility with $(N_3)$ requires some work which is done in the next two subsections. The proof of $(\tilde{N})$ is then easily completed (sect. 3.1).

But a general proof of $(N)$ is impossible, since it is well-known that there exist anomalies in certain models. If solely massive fields appear and if the scaling degrees $\mathfrak{sd}(t_1) + 1 = \mathfrak{sd}(t_2)$. We say that the scaling degree of $t_1$ (or $t_2$ resp.) is relatively lowered if $\mathfrak{sd}(t_1) < \mathfrak{sd}(t_2)$ or $\mathfrak{sd}(t_2) < \mathfrak{sd}(t_1) + 1$ (resp.).

A relative pre-factor $m^a$ ($m = \text{mass}$, $a > 0$) indicates a relatively lowered scaling degree. Let $\partial_\mu t_1^* = m^a t_2$ and we assume that $t_1$ and $t_2$ contain no global factor $b \not\in \mathbb{R} \setminus \{0\}$.

To simplify the formulas we restrict this section to bosonic fields, the inclusion of fermionic fields is obvious.

3.1 Proof of $(\tilde{N})$

The nontrivial part in the proof of $(\tilde{N})$ is the compatibility with $(N_3)$. The keys to show this and the compatibility of $(N)$ with $(N_3)$ (see sect. 3.2) are the following two Lemmas:

**Lemma 1:** Let $V \in \mathcal{P}_0$, $\varphi, \psi \in \hat{\mathcal{G}}$ and $f, h \in \mathcal{D}(\mathbb{R}^4)$. Then the following identities hold true within $\mathcal{D}(\mathbb{R}^4, \mathcal{P}_0)$:

\[
f \sum_{\varphi \in \hat{\mathcal{G}}} \frac{\partial}{\partial \varphi} \Delta_{\varphi, \psi} \ast h = f \sum_{\varphi \in \hat{\mathcal{G}}} \sum_{0 \leq b \leq a} \frac{a!}{b!(a-b)!} \frac{\partial^b}{\partial \varphi} \Delta_{\varphi, \psi} \ast h, \quad (44)
\]

\[
f \sum_{\varphi \in \hat{\mathcal{G}}} \frac{\partial}{\partial \varphi} \Delta_{\varphi, \psi} \ast h = f \sum_{\varphi \in \hat{\mathcal{G}}} \sum_{0 \leq b \leq a} \frac{a!}{b!(a-b)!} \frac{\partial^b}{\partial \varphi} \Delta_{\varphi, \psi} \ast h, \quad (45)
\]

where again $a! \equiv \prod_{\mu} a_{\mu}$. 

**Proof:** We first prove (44) for $|a| = 1$, i.e.

\[
f \sum_{\varphi \in \hat{\mathcal{G}}} \frac{\partial}{\partial \varphi} \Delta_{\varphi, \psi} \ast h = f \sum_{\varphi \in \hat{\mathcal{G}}} \frac{\partial^b}{\partial \varphi} \Delta_{\varphi, \psi} \ast h + \frac{\partial}{\partial \varphi} \Delta_{\varphi, \psi} \ast h]. \quad (46)
\]

It suffices to consider the case in which $V$ is a monomial. The proof goes by induction on the degree of this monomial. The case $V = 1$ is trivial. Let

\[17\]
\[ V = \chi W, \ \chi \in \hat{G}, \ W \in \hat{P}_0. \] By assumption \( W \) satisfies (46). Inserting now \( V = \chi W \) into (46) and using this assumption most terms cancel and it remains to show

\[ f \sum_{\varphi \in \hat{G}} \frac{\partial (\partial^\mu \chi)}{\partial \varphi} W \Delta_{\varphi, \psi} * h = f \sum_{\varphi \in \hat{G}} \frac{\partial \chi}{\partial \varphi} W \partial^\mu \Delta_{\varphi, \psi} * h. \]  

(47)

The l.h.s. is equal to \( fW \Delta_{\partial^\mu \chi, \psi} * h \) and the r.h.s. to \( fW \partial^\mu \Delta_{\chi, \psi} * h \). Obviously these two expressions agree.

To prove (44) for arbitrary \(|a|\) we proceed by induction on \(|a|\):

\[ f \sum_{\varphi \in \hat{G}} \frac{\partial (\partial^\mu \partial^a V)}{\partial \varphi} \Delta_{\varphi, \psi} * h = \]

\[ f \sum_{\varphi \in \hat{G}} \left[ \left( \partial^\mu \partial^a V \right) \partial \Delta_{\varphi, \psi} * h + \frac{\partial (\partial^a V)}{\partial \varphi} \partial^\mu \Delta_{\varphi, \psi} * h \right] = \]

\[ f \sum_{\varphi \in \hat{G}} \sum_{0 \leq b \leq a} \frac{a!}{b!} (a - b)!(\partial^b \partial^a V) \partial \Delta_{\varphi, \psi} * h \]  

\[ + \left( \partial^a \partial^b V \right) \partial^\mu \Delta_{\varphi, \psi} * h, \]  

(48)

where \( e_\mu = (0, ..., 1, ..., 0) \) with 1 at the \( \mu \)-th position. First we have used (46) (with \( V \) replaced by \( \partial^a V \)) and in the second equality sign we have inserted (44) (which is the inductive assumption) and \( \partial^\mu \otimes 1 \) applied to (44) (cf. (4)).

The proof of the second identity (45) is completely similar. One simply has to replace the internal derivatives \( \partial^a \otimes 1 \) by external ones \( \partial^a \otimes 1 \). In particular the validity of the equation corresponding to (47) relies on \( \Delta_{\partial^\mu \chi, \psi} = \partial^\mu \Delta_{\chi, \psi} \).

By means of Lemma 1 we will prove

**Lemma 2:** Let \( V, W \in \mathcal{P}_0, \chi, \psi, \kappa \in \hat{G} \) and \( f, g, h \in \mathcal{D}(\mathbb{R}^4) \). Then

\[ \sum_{\varphi \in \hat{G}} \frac{\partial \Delta^\mu_{\varphi, \psi}(V, W f)}{\partial \varphi} \Delta_{\varphi, \kappa} * h = \]

\[ \sum_{\varphi \in \hat{G}} \left[ \Delta^\mu_{\chi, \psi}(\partial V, W f, \partial \varphi) \Delta_{\varphi, \kappa} * h, W f \right] + \Delta^\mu_{\chi, \psi}(V, \partial W, \partial \varphi) \Delta_{\varphi, \kappa} * h]. \]  

(49)

**Proof:** Using the explicit form (38) for \( \Delta^\mu \) the l.h.s. of (49) is equal to

\[ \sum_{a} \sum_{\varphi \in \hat{G}} C^\mu_{\chi, \psi, a} (-1)^{|a|} \sum_{0 \leq b \leq a} \frac{a!}{b!(a - b)!} \partial^\mu (\partial^{a-b} V) \partial b W(\partial^b g) f \]

\[ + (\partial^{a-b} V) \frac{\partial W}{\partial \varphi} (\partial^b g) f] \Delta_{\varphi, \kappa} * h. \]  

(50)
Again by means of (43) the r.h.s. of (49) can be written as

\[ \sum_n \sum_{\varphi \in \mathcal{G}} C^\mu_{\chi, \psi ; a} (-1)^{|\alpha|} \frac{a!}{\alpha_!} \frac{b!}{\beta!} \sum_{0 \leq \beta \leq a} (\partial (a-b) g) W (\partial (b-c) \Delta_{\varphi, \kappa} \ast h) f + (\partial (a-b) \ast h) f (\partial (b-c) \Delta_{\varphi, \kappa} \ast h)]. \quad (51) \]

Due to (45) the expressions (50) and (51) agree.

We now come to the proof of (35). We show that the commutators of the l.h.s. and of the r.h.s. of (49) can be written as

\[ \sum_{\varphi \in \mathcal{G}} i \sum_{\varphi \in \mathcal{G}} \left[ T_{n+1} \left( \frac{\partial (a-b) g}{\partial \varphi} W + (\partial (b-c) \Delta_{\varphi, \kappa} \ast h) \right) \otimes W_1 f_1 \otimes \ldots \right] \]

To compute the commutator of the r.h.s. of (35) with (35) we use again (N3) and in addition Lemma 2. We obtain

\[ i \sum_{\varphi \in \mathcal{G}} \left[ T_{n+1} \left( \frac{\partial (a-b) g}{\partial \varphi} W + (\partial (b-c) \Delta_{\varphi, \kappa} \ast h) \right) \otimes W_1 f_1 \otimes \ldots \right] \]

Due to (N) for subpolynomials we have the following equations:

(secondary term in (52)) = (secondary term in (54)) + (secondary term in (56))
and \( \langle \hat{N} \rangle = \langle \hat{N}^{(1)} \rangle + \langle \hat{N}^{(2)} \rangle + \langle \hat{N}^{(3)} \rangle \).

To get the equality of \( \langle \hat{N}^{(2)} \rangle \) and \( \langle \hat{N}^{(3)} \rangle \) it remains to show:

(first term in \( \langle \hat{N}^{(2)} \rangle \)) = (first term in \( \langle \hat{N}^{(3)} \rangle \)).

To verify this we insert (46) with \( \partial^\mu \otimes 1 \) replaced by \( \partial^\mu \otimes 1 \) into the first term in \( \langle \hat{N}^{(2)} \rangle \) and the original (46) into the first term in \( \langle \hat{N}^{(3)} \rangle \). The remaining terms in (\( \ast \)) cancel by means of (\( \hat{N} \)) for subpolynomials.

From the just now proved result we conclude that (\( \hat{N} \)) can be violated by a C-number only:

\[
T_{n+1}((\partial^\nu V)WG \otimes W_1f_1 \otimes ...) - T_{n+1}((\partial^\nu V)WG \otimes W_1f_1 \otimes ...) \\
- i \sum_{m=1}^{n} \sum_{\chi, \psi \in \mathcal{G}} T_n\left( \Delta^\nu_{\chi, \psi} \left( \frac{\partial V}{\partial \chi} W_g, \frac{\partial W_m}{\partial \psi} f_m \right) \otimes W_1f_1 \otimes ...m... \right) = \\
\langle \Omega | T_{n+1}((\partial^\nu V)WG \otimes W_1f_1 \otimes ...) | \Omega \rangle - \langle \Omega | T_{n+1}((\partial^\nu V)WG \otimes W_1f_1 \otimes ...) | \Omega \rangle \\
- i \sum_{m=1}^{n} \sum_{\chi, \psi \in \mathcal{G}} \langle \Omega | T_n \left( \Delta^\nu_{\chi, \psi} \left( \frac{\partial V}{\partial \chi} W_g, \frac{\partial W_m}{\partial \psi} f_m \right) \otimes W_1f_1 \otimes ...m... \right) | \Omega \rangle \\
\overset{\text{def}}{=} \tilde{a}(g, f_1, ..., f_n). \quad (59)
\]

Due to causal factorization of the \( T \)-products and the validity of (\( \hat{N} \)) to lower orders \( \leq n \), the possible violation \( \tilde{a}(g, f_1, ..., f_n) \) of (\( \hat{N} \)) must be local

\[
\tilde{a}(g, f_1, ..., f_n) = \int dx dx_1 ... dx_n \sum_{|n|=0} C_n \partial^n \delta(x_1 - x, ..., x_n - x) g(x)f_1(x_1)...f_n(x_n),
\]

with unknown constants \( C_n \) and

\[
\omega \overset{\text{def}}{=} \text{sd}(\langle \Omega | T_{n+1}((\partial^\nu V)W, W_1, ..., W_n) | \Omega \rangle) - 4n. \quad (61)
\]

After the finite renormalization

\[
\langle \Omega | T_{n+1}((\partial^\nu V)WG \otimes W_1f_1 \otimes ...) | \Omega \rangle \\
\rightarrow \\
\langle \Omega | T_{n+1}((\partial^\nu V)WG \otimes W_1f_1 \otimes ...) | \Omega \rangle - \tilde{a}(g, f_1, ...) \quad (62)
\]

(\( \hat{N} \)) holds. By construction (in particular (61)) this renormalization respects (N0). From the definition (59) of \( \tilde{a}(g, f_1, ...) \) we see that (62) maintains (N1), (N2) and the permutation symmetry of \( \langle \Omega | T_{n+1}((\partial^\nu V)W, W_1, ...) | \Omega \rangle \).

However, in general (62) violates (N3), namely in the cases in which

\( T_{n+1}((\partial^\nu V)W, W_1, ...) \) appears on the r.h.s. of (N3).\[6\]

So we everywhere repair (N3) by a chain of finite renormalizations of \( T \)-products of order \( n + 1 \) with

---

\[6\] The cases of (N3) in which \( T_{n+1}((\partial^\nu V)W, W_1, ...) \) appears on the l.h.s. remain true, because only the C-number part of \( T_{n+1}((\partial^\nu V)W, W_1, ...) \) gets changed.
polynomial degree $d > |V| + |W| + |W_1| + ... + |W_n|$. It is obvious that this can be done such that (N0), (N1) and (N2) are preserved. The validity of (N) up to order $n + 1$ and polynomial degree $|V| + |W| + |W_1| + ... + |W_n|$ is not touched by these renormalizations. So the inductive step is finished. □

In other words, the compatibility of the renormalization (62) with (N3) follows from the following general observation: (N3) determines the operator-valued map $T_n$ completely in terms of the $\mathbb{C}$-valued map $\langle \Omega | T_n(\cdot) | \Omega \rangle : D(\mathbb{R}^4, \tilde{P}_0)^{\otimes n} \rightarrow \mathbb{C}$. However, (N3) does not give any relation among the vacuum expectation values of the $T$-products, they may be arbitrarily given. Hence, renormalizations of the vacuum expectation values of the $T$-products are not in conflict with (N3). We will use this second way of argumentation in the following.

### 3.2 Compatibility of the master Ward identity with (N3)

We start with $T$-products which fulfil (N0), (N1), (N2), (N3) and (N) to all orders. We use the same double induction as in the preceding subsection: we assume that (N) holds to lower orders $\leq n$ and for $T_{n+1}$ restricted to the elements of $D(\mathbb{R}^4, \tilde{P}_0)^{\otimes n+1}$ with a lower polynomial degree.

By means of (N3) we are going to prove that the commutators of the l.h.s. and of the r.h.s. of (N) with $T_1(\kappa h)$ are equal. For the l.h.s. it results

$$- i \sum_{\varphi \in \mathbb{G}} \left[ T_{n+1} \left( \frac{\partial V}{\partial \varphi} (\Delta_{\varphi, \kappa} * h) \otimes W_1 f_1 \otimes ... \right) \right]$$

and

$$+ \sum_{l=1}^{n} T_{n+1} \left( V \frac{\partial V}{\partial \varphi} g \otimes W_1 f_1 \otimes ... \otimes \frac{\partial W_l}{\partial \varphi} f_l (\Delta_{\varphi, \kappa} * h) \otimes ... \right).$$

By using again (N3) and in addition Lemma 2 we compute the commutator of the r.h.s. and obtain

$$i \sum_{\varphi \in \mathbb{G}} \left\{ T_{n+1} \left( \frac{\partial (\partial^\nu V)}{\partial \varphi} g (\Delta_{\varphi, \kappa} * h) \otimes W_1 f_1 \otimes ... \right) \right\}$$

and

$$+ i \sum_{m=1}^{n} \sum_{\chi, \psi \in \mathbb{G}} \left( T_n \left[ \Delta_{\chi, \psi}^\nu \left( \frac{\partial^2 V}{\partial \varphi \partial \psi} g (\Delta_{\varphi, \kappa} * h), \frac{\partial W_m}{\partial \psi} f_m \right) \right] \otimes W_1 f_1 \otimes ... \right)$$

and

$$+ \left( \Delta_{\chi, \psi}^\nu \left( \frac{\partial V}{\partial \chi} g, \frac{\partial^2 W_m}{\partial \varphi \partial \psi} f_m (\Delta_{\varphi, \kappa} * h) \right) \right) \otimes W_1 f_1 \otimes ... \right)$$

and

$$+ \sum_{l (l \neq m)} T_n \left( \Delta_{\chi, \psi}^\nu \left( \frac{\partial V}{\partial \chi} g, \frac{\partial W_m}{\partial \psi} f_m \right) \otimes W_1 f_1 \otimes ... \right)$$

and

$$... \otimes \frac{\partial W_l}{\partial \varphi} f_l (\Delta_{\varphi, \kappa} * h) \otimes ... \right) \right\}.$$
The validity of \((N)\) for sub-polynomials implies
\[
(N) = (\tilde{N}) + ([\tilde{N}]).
\]
It remains to prove
\[
(N) = (\tilde{N}) + ([\tilde{N}]).
\]
(\text{**})

After inserting \([\tilde{N}]\) into \([\tilde{N}]\) this equation (\text{**}) takes the form of (\text{N}) for some sub-polynomials, which holds true by the inductive assumption.

As in the case of \((\tilde{N})\) \([\tilde{N}]-[\tilde{N}]\), we conclude that the operator identity (N) can be violated by a local C-number \(a(g, f_1, ..., f_n)\) only:
\[
a(g, f_1, ..., f_n) \overset{\text{def}}{=} \langle \Omega| T_{n+1}((\partial^\nu V) g \otimes W_1 f_1 \otimes ...)\rangle + \langle \Omega| T_{n+1}((\partial^\nu V) g \otimes W_1 f_1 \otimes ...)\rangle
\]
\[
+ i \sum_{m=1}^n \sum_{\chi, \psi \in G} \langle \Omega| T_n(\Delta^\nu_{\chi, \psi}(g, \frac{\partial W_m}{\partial \nu} f_m) \otimes W_1 f_1 \otimes \ldots m\ldots)\rangle\Omega. \tag{70}
\]

The aim is now to remove \(a(g, f_1, ..., f_n)\) by finite renormalizations of the vacuum expectation values \(\langle \Omega| T(...)|\Omega\rangle\) on the r.h.s. Such renormalizations are not in conflict with (N0)-(N3) (see the end of the preceding subsection). So we have proved the compatibility of \(N\) with (N3).

We discuss the possibilities to remove \(a(g, f_1, ..., f_n)\): (A) The finite renormalization
\[
\langle \Omega| T_{n+1}((\partial^\nu V) g \otimes W_1 f_1 \otimes ...)\rangle \rightarrow \langle \Omega| T_{n+1}((\partial^\nu V) g \otimes W_1 f_1 \otimes ...)\rangle - a(g, f_1, ...), \tag{71}
\]
does this job and is compatible with (N0)-(N3), \((\tilde{N})\) and permutation symmetry. However, this procedure works only if \(\partial^\nu V \neq 0\) and if \(\text{sd}((\Omega| T_{n+1}((\partial^\nu V) g \otimes W_1, ..., W_n)|\Omega)) = \text{sd}((\Omega| T_{n+1}(V, W_1, ..., W_n)|\Omega)) + 1\) in case that the latter does not hold one gets in conflict with (N0). This happens e.g. for the axial and pseudo-scalar triangle-diagram, see \([29]\). In many important applications of the MWI \(V\) corresponds to a conserved current (i.e. \(\partial^\nu V = 0\)), for example \(V = \bar{\psi} Q \gamma_\nu \psi\), or if \(V\) is the free ghost current (cf. sect. 4.2 for both) or the free BRST-current (sect. 4.4).

(B) If \((\tilde{N})\) does not work one tries to satisfy (N) by renormalizing also \(\langle \Omega| T_{n+1}(V, W_1, ..., W_n)|\Omega\rangle\) and eventually \(\langle \Omega| T_{n}(\Delta^\nu_{\chi, \psi}(...), W_1)|\Omega\rangle\). This method does not ensure success. In detail one proceeds in the following way:

(B1) \(a(g, f_1, ...)\) has the form \([\tilde{N}]\) with \(\omega \overset{\text{def}}{=} \text{sd}((\Omega| T_{n+1}(V, W_1, ..., W_n)|\Omega)) + 1 - 4n\). Using symmetry properties (e.g. Poincare covariance, permutation symmetries) of the r.h.s. of \((\tilde{N})\) the constants \(C_a\) (we use the notation of \([\tilde{N}]\)) can be strongly restricted.

(B2) One works out the freedoms of normalization \([\tilde{N}]\) of \(\langle \Omega| T_{n+1}(V, W_1, ..., W_n)|\Omega\rangle\), \(\langle \Omega| T_{n+1}((\partial^\nu V), W_1, ..., W_n)|\Omega\rangle\) and eventually \(\langle \Omega| T_n(\Delta^\nu(...), W_1)|\Omega\rangle\) (the second is only available if \(\partial^\nu V \neq 0\)) which respect (N0)-(N2), \((\tilde{N})\) and permutation symmetry. Renormalizations of \(\langle \Omega| T_n(\Delta^\nu(...), W_1, ...)\rangle\) are also restricted by
3.3 Proof of the master Ward identity for solely massive fields and not relatively lowered scaling degree

We return to the end of sect. 2.2 and set

\[ \omega(t_0) \eqdef \text{sd}(t_0) - 4(n - 1). \] (72)

A possible extension \( t \in D'(\mathbb{R}^{4(n-1)}) \) of \( t_0 \in D'(\mathbb{R}^{4(n-1)} \setminus \{0\}) \) which respects (N0) is given by (cf. [3], [30])

\[ (t^w, h) \eqdef (t_0, h^w), \quad \forall h \in D(\mathbb{R}^{4(n-1)}) \] (73)

where

\[ h^w(x) \eqdef h(x) - w(x) \sum_{|a| = 0}^{\omega(t_0)} \frac{x^a}{a!} (\partial^a h)(0), \] (74)

and \( w \in D(\mathbb{R}^{4(n-1)}) \) is such that there exists a neighbourhood \( \mathcal{U} \) of \( 0 \in \mathbb{R}^{4(n-1)} \) with \( w|_{\mathcal{U}} \equiv 1 \). A change of \( w \) alters the normalization of \( t^w \). For \( \omega(t_0) < 0 \) we have \( h^w = h \) in agreement with the fact that the extension is unique in that case. Because there is no Lorentz invariant \( w \in D(\mathbb{R}^{4(n-1)}) \), the extension \( t^w \) is not Lorentz covariant (in general) and one has to perform a finite renormalization [23] to restore this symmetry (see the second papers of [3] and [11], as well as [3]). To avoid this one is tempted to choose \( w \equiv 1 \). But \( h^w \equiv 1 \) is not a test-function. However, if all fields are massive the infrared behavior is harmless and Epstein and Glaser [9] have shown that one may indeed choose \( w \equiv 1 \) in this case. The extension \( t^{(c)} \eqdef t^{(w \equiv 1)} \) is called ‘central solution’ (or better ‘central extension’ in our framework) and it was pointed out that it preserves nearly all symmetries [19], [13], [34], [30].

\[ \text{Epstein and Glaser have proved that one may choose } w \equiv 1 \text{ for the method of distribution splitting. In this footnote we show how their result applies to our extension procedure (73). From Epstein and Glaser [9] we know } t_0 \in \mathcal{S}'(\mathbb{R}^{4(n-1)} \setminus \{0\}) \text{ and hence } t^w \in \mathcal{S}'(\mathbb{R}^{4(n-1)}), \text{ so we may use Fourier transformation. Epstein and Glaser have proved that in the massive} \]
We are now going to show that the central extensions fulfil the MWI provided the scaling degree is not relatively lowered for the individual, contributing C-number distributions (a precise explanation of the latter expression is given below in (84), (85)). We define \( \tilde{t}^{(c)} \) and \( t_{b, \chi, \psi}^{(c)} \) to be central extensions:

\[
\int dx_1 \ldots dx_n \tilde{t}^{(c)}(x_1 - x, \ldots, x_n - x)g(x)f_1(x_1) \ldots f_n(x_n) = \langle \Omega | T_{n+1}(\{0\} \otimes W_1 f_1 \otimes \ldots \otimes W_n f_n) | \Omega^{(c)} \rangle
\]

\[
\int dx_1 \ldots dx_n t^{(c)}(x_1 - x, \ldots, x_n - x)g(x)f_1(x_1) \ldots f_n(x_n) = \langle \Omega | T_{n+1}(W g \otimes W_1 f_1 \otimes \ldots \otimes W_n f_n) | \Omega^{(c)} \rangle
\]

\[
\sum_b \int dx_1 \ldots dx_n t_{b, \chi, \psi}^{(c)}(x_1 - x_m, \ldots, x_n - x_m) \cdot \langle \Omega | T_n \left( \sum_{a=0}^{\infty} \frac{\partial^a}{\partial x^a} t_{b, \chi, \psi}^{(c)} \right) | \Omega^{(c)} \rangle
\]

where we have taken the definition \[ 35 \] of \( \Delta^\mu \) into account\[ 21 \]. The corresponding non-extended distributions are

\[
t_0^c, \quad t_0^c \in \mathcal{D}'(\mathbb{R}^4 \setminus \{0\}) \quad \text{and} \quad t_{b, \chi, \psi}^c \in \mathcal{D}'(\mathbb{R}^4(n-1) \setminus \{0\}).
\]

\[
\langle \Omega | T_n \left( \sum_{a=0}^{\infty} \frac{\partial^a}{\partial x^a} \tilde{t}_0^c \right) | \Omega^{(c)} \rangle
\]

The Fourier transformation of \( t^{(u)} \) reads\[ 76 \]

\[
\tilde{t}^{(u)}(p) = \tilde{t}_0(p) - \sum_{|a|=0}^{\omega(t_0)} \frac{p^a}{a!} \partial^a \tilde{t}_0(0).
\]

Note \( \tilde{t}_0 w = (2\pi)^{-1} \hat{w} \tilde{t}_0 \hat{w} \in \mathcal{C}^\infty \), i.e. \( \partial^a \tilde{t}_0(0) \) exists. Using the definition \[ 75 \] of the central extension we now find

\[
\tilde{t}^{(c)}(p) = \tilde{t}^{(u)}(p) - \sum_{|a|=0}^{\omega(t_0)} \frac{p^a}{a!} \partial^a \tilde{t}^{(u)}(0) = \tilde{t}_0(p) - \sum_{|a|=0}^{\omega(t_0)} \frac{p^a}{a!} \partial^a \tilde{t}_0(0).
\]

We see that we may set \( w \equiv 1 \) in \[ 76 \] and hence also in \[ 77 \], and that this choice is the central extension \[ 73 \].

\[ 21 \text{From } 75 \text{ we see that } \langle \Omega | T_n \left( \Delta^\mu \{g \otimes \cdots \otimes f_n \} \right) | \Omega^{(c)} \rangle \text{ is of the form} \]

\[
\sum_b \tilde{t}_b (f_1 \otimes \cdots \otimes \partial^a g \otimes f_m \otimes \cdots \otimes f_n) = \sum_b \int dx_1 \ldots dx_n \tilde{t}_b (x_1 - x_m, \ldots, x_n - x_m) \partial^a g (x) f_1(x_1) \ldots f_n(x_n),
\]

\[
\tilde{t}_b \in \mathcal{D}'(\mathbb{R}^4), \quad t_b \in \mathcal{D}'(\mathbb{R}^4(\mathbb{R}^4))
\]

21
In the preceding subsection we have learnt that the validity of (N3) reduces the proof of the MWI to the vacuum sector (see (70)). So we only have to show
\[ -\partial^\nu t^{(c)}(y_1, \ldots, y_n) = \bar{t}^{\nu(c)}(y_1, \ldots, y_n) + i \sum_{m=1}^{n} \sum_{\chi, \psi \in \mathcal{G}} t^{\nu(c)}_{b; \chi, \psi}(y_1 - y_m, \ldots, y_n - y_m) \partial^b \delta(y_m), \]  
where
\[ \partial^\nu \overset{\text{def}}{=} \partial_1^\nu + \ldots + \partial_n^\nu. \]  
By causal factorization and induction we know that this equation is fulfilled by the corresponding non-extended distributions (82). Setting \( y \equiv (y_1, \ldots, y_n) \) we obtain
\[ -\left( \partial^\nu t^{(c)}(y), h(y) \right) = \left( t_0(y), [\partial^\nu h(y) - \sum_{|a|=0}^{\omega(t_0)} \frac{y^a}{a!} (\partial^a \partial^\nu h)(0)] \right) = \left( t_0(y), \partial^\nu [h(y) - \sum_{|a|=0}^{\omega(t_0)} \frac{y^a}{a!} (\partial^a h)(0)] \right) = \left( \bar{t}^{\nu}_0(y), [h(y) - \sum_{|a|=0}^{\omega(t_0)+1} \frac{y^a}{a!} (\partial^a h)(0)] \right) + i \sum_{m=1}^{n} \sum_{\chi, \psi \in \mathcal{G}} t^{\nu}_{b; \chi, \psi; 0}(y_1 - y_m, \ldots, y_n - y_m) \partial^b \delta(y_m), [h(y) - \sum_{|a|=0}^{\omega(t_0)+1} \frac{y^a}{a!} (\partial^a h)(0)]. \]  
If the scaling degree is not relatively lowered, more precisely if
\[ \omega(\bar{t}^{\nu}_0) = \omega(t_0) + 1 \]  
and
\[ \omega(t^{\nu}_{b; \chi, \psi; 0}) = \omega(t_0) + 1 - |b|, \quad \forall b, \quad \forall \chi, \psi \in \mathcal{G}, \]  
then the terms in the final expression of (83) are the central extensions. For the \( \bar{t}^{\nu}_0 \)-term this is obvious. To verify this statement for the \( t^{\nu}_{b; \chi, \psi; 0} \)-terms (we omit the indices \( \nu, \chi, \psi \) in the following) it suffices to consider the term \( m = n \) and test-functions of the form \( h(y_1, \ldots, y_n) = h_1(z_1, \ldots, z_{n-1})h_2(z_n) \) where
\[ z \equiv (z_1, \ldots, z_n) = (y_1 - y_n, \ldots, y_{n-1} - y_n, y_n) \overset{\text{def}}{=} Ay, \quad A \in SL(n, \mathbb{R}). \]
Then we have

\[ h(y) - \sum_{|a| = 0}^{\omega(t_{b,0}) + |b|} \frac{y^a}{a!} (\partial^a h)(0) = \]

\[ (h_1 \otimes h_2)(z) - \sum_{|a| = 0}^{\omega(t_{b,0}) + |b|} \frac{y^a}{a!} ((A^T \partial)^a (h_1 \otimes h_2))(0) = \]

\[ (h_1 \otimes h_2)(z) - \sum_{|a| = 0}^{\omega(t_{b,0}) + |b|} \bar{z}^a (\partial^a (h_1 \otimes h_2))(0) \]

where \(A^T\) denotes the transposed matrix and we have used \(y^a \cdot (A^T \partial)^a = (Ay)^a \cdot \partial^a\). We set \(a = (a, a_n)\) and \(z \equiv (\bar{z}, z_n)\). Then the last term in (83) can be transformed in the following way:

\[
\left( t_{b,c}(y_1 - y_n, \ldots, y_{n-1} - y_n) \partial^b \delta(y_n), \partial^c (h(y)) - \sum_{|a| = 0}^{\omega(t_{b,0}) + |b|} \frac{y^a}{a!} (\partial^a h)(0) \right)_{y} =
\]

\[
\left( t_{b,0}(\bar{z}) \partial^b \delta(z_n), [h_1(\bar{z}) h_2(z_n)] - \sum_{|a| = 0}^{\omega(t_{b,0}) + |b|} \frac{\bar{z}^a}{a!} (\partial^a h_1)(0)(\partial^a h_2)(0) \right)_{\bar{z}} =
\]

\[
(-1)^{|b|} (\partial^b h_2)(0) \left( t_{b,0}(\bar{z}), [h_1(\bar{z}) - \sum_{|a| = 0}^{\omega(t_{b,0}) + |b|} \frac{\bar{z}^a}{a!} (\partial^a h_1)(0)] \right)_{\bar{z}} =
\]

\[
\left( t_{b}^{(c)}(\bar{z}) \partial^c \delta(z_n), h_1(\bar{z}) h_2(z_n) \right)_{\bar{z}} =
\]

\[
\left( t_{b}^{(c)}(y_1 - y_n, \ldots, y_{n-1} - y_n) \partial^c \delta(y_n), h(y) \right)_{y}. \tag{88}\]

Summing up we find the assertion (83) if (86) and (87) hold true, otherwise we have over-subtracted extensions. Note that this proof works also for \(\partial^\mu w = 0\) and hence \(\partial^\mu \delta = 0\). Obviously this method fails for extensions \(t^{(w)}\) with \(w \in \mathcal{D}(\mathbb{R}^{4(n-1)})\), because additional terms \(\sim \partial^\mu w\) appear in (83).

In case of the axial anomaly we set \(j_\mu^A \overset{\text{def}}{=} \bar{\psi} \gamma^\mu \gamma^5 \psi\), \(j_\mu^\mu \overset{\text{def}}{=} \bar{\psi} \gamma^\mu \psi\) and \(j_\tau^\tau \overset{\text{def}}{=} \bar{\psi} \gamma^\tau \psi\), and have

\[
\tilde{t}_\mu^{(c)}(x, x_1, x_2) = (\Omega | T_3(j_\mu^A, j_\lambda^A, j_\tau^A)(x, x_1, x_2) | \Omega)^{(c)},
\]

\[
\tilde{\nu}_\mu^{(c)}(x, x_1, x_2) = 2 m g^\mu \nu (\Omega | T_3(j_\nu^\nu, j_\lambda^\lambda, j_\tau^\tau)(x, x_1, x_2) | \Omega)^{(c)} \tag{89}\]

for the AVV-triangle diagram. The corresponding distributions for the AAA-triangle are obtained by replacing \(j_\lambda, j_\tau\) by \(j_\lambda^A, j_\tau^A\). All \(t_{b,0}\)-distributions vanish.

\footnote{For the method of distribution splitting the central solution in momentum space can be obtained by a dispersion integral \(\tilde{t}_{b,c}\). In [9] this dispersion integral has been used to prove gauge invariance of QED. The present proof \(\tilde{t}_{b,c}\) is a kind of \(x\)-space version of that procedure, which yields a more general result. In addition, it has the advantage that it is not necessary to treat the cases of different external legs individually.}
One finds $\omega(t) = 1$ and $\bar{\omega}(t) = 0 < \omega(t) + 1$. Hence, the present proof does not apply.

4 Applications of the master Ward identity

The main success of the MWI are its many, important and far-reaching consequences.

4.1 Field equation

Let us consider the pair $(\varphi, \chi)$ of symbols (corresponding to massive or massless free fields which fulfill the Klein-Gordon or wave equation) that is studied in appendix A and let $W_1, \ldots, W_n \in \mathcal{P}_0$. We assume that $W_1, \ldots, W_n$ contain only zeroth and first (internal) derivatives of $\chi$. By applying twice the MWI and using the explicit expressions (304)-(308) for $\delta \mu$ we obtain

$$T_{n+1}(\varphi(g + m^2) \otimes W_1 f_1 \otimes \ldots \otimes W_n f_n) =$$

$$-iT_{n+1}(\partial \varphi) \partial \mu g \otimes W_1 f_1 \otimes \ldots \otimes W_n f_n + m^2 T_{n+1}(\varphi g \otimes W_1 f_1 \otimes \ldots \otimes W_n f_n)$$

$$-i \sum_{m=1}^{n} \sum_{\psi \in G} (\pm) T_n \left( \Delta_{\mu \nu \psi} (\partial \mu g, \partial \psi f_m) \otimes W_1 f_1 \otimes \ldots \hat{m} \ldots \otimes W_n f_n \right)$$

$$+iT_n \left( \frac{\partial W_m}{\partial \chi} (\varphi, g \partial W_m \partial \psi f_m) \otimes W_1 f_1 \otimes \ldots \hat{m} \ldots \otimes W_n f_n \right). \quad (90)$$

This is the normalization condition (N4) of [7] and [3]. It is equivalent to

$$T_{n+1}(\varphi g \otimes W_1 f_1 \otimes \ldots \otimes W_n f_n) =$$

$$i \sum_{l=1}^{n} \sum_{\psi \in G} T_n \left( W_1 f_1 \otimes \ldots \otimes \frac{\partial W_l}{\partial \psi} \Delta_{\psi, \varphi} f l \otimes g \otimes \ldots \otimes W_n f_n \right) + \ldots, \quad (91)$$

where the dots stand for the terms in which $\varphi g$ is not contracted. We see from this formula (91) that the normalization condition (N4) can always be satisfied without getting in conflict with (N0)-(N3), even if anomalies are present. Note

\footnote{According to power counting one expects $\omega(t) = 1$, but the $\omega = 1$-terms are proportional to the spinor trace $\text{tr}(\gamma^5 p_1 \gamma^\mu \gamma^5 p_2 \gamma^\nu \gamma^\tau p_3 \gamma^\rho) = 0$.}

\footnote{For simplicity we choose $\epsilon = 1$ in (296).}
that the final result (on the r.h.s. in (90)) is independent from the normalization constant $C$ which appears in the intermediate formula. This must be so, because the Feynman propagators $\Delta_{\psi,\phi}^F$ in (91) do not contain this constant.

Generalizing Bogoliubov’s idea [4], we define the interacting field $\Lambda gL$ belonging to $\Lambda \in \mathcal{D}(\mathbb{R}^4, \tilde{\mathcal{P}}_0)$ and to the interaction $L \in \mathcal{P}_0$ in terms of the $T$-products by

$$
\Lambda gL \overset{\text{def}}{=} S(gL)^{-1} \frac{d}{d\lambda} \big|_{\lambda=0} S(gL + \lambda \Lambda) = \sum_{n=0}^{\infty} \frac{i^n}{n!} R_{n+1}((Lg)^{\otimes n}; \Lambda) = T_1(\Lambda) + \mathcal{O}(g),
$$

(92)

where the 'totally retarded products' $R_{n+1}(\ldots)$ (also called 'R-product') are defined by

$$
R_{n+1}(\Lambda_1 \otimes \ldots \otimes \Lambda_n; \Lambda) \overset{\text{def}}{=} \sum_{I \subset \{1, \ldots, n\}} (-1)^{|I|} \tilde{T}(\otimes_{i \in I} \Lambda_i) \tilde{T}((\otimes_{j \in I^c} \Lambda_j) \otimes \Lambda)
$$

(93)

and we have used (17) and (18). Similarly to the $S$-matrix (17), the interacting fields are formal power series. In the particular case $\Lambda = Wf, W \in \mathcal{P}_0, f \in \mathcal{D}(\mathbb{R}^4)$ we write $WgL(f)$ instead of $(Wf)gL$. Following [7] the condition (90) can easily be translated into an identity for $R_{n+1}(W_1f_1 \otimes \ldots \otimes W_n f_n; \phi(\Box + m^2) f)$.

The latter implies the field equation

$$
(\Box + m^2)\varphi gL = -g \left( \frac{\partial L}{\partial \chi} \right)_{gL} + \partial^\mu \left[ g \left( \frac{\partial L}{\partial (\partial_\mu \chi)} \right)_{gL} \right],
$$

(94)

where $g$ is a test function.

The calculation (94) can be carried over to external derivatives by using (N) instead of (N). More precisely let $W, W_1, \ldots, W_n \in \mathcal{P}_0$ and let us assume that $W_1, \ldots, W_n$ contain only zeroth and first (internal) derivatives of $\chi$. With that we obtain

$$
T_{n+1}(\Box + m^2)\varphi) W g \otimes W_1 f_1 \otimes \ldots \otimes W_n f_n =
\sum_{m=1}^{n} (\pm) T_n \left( W \frac{\partial W_m}{\partial \chi} g f_m \otimes W_1 f_1 \otimes \ldots \hat{m} \ldots \otimes W_n f_n \right)
+i \sum_{m=1}^{n} (\pm) T_n \left( W \frac{\partial W_m}{\partial (\partial_\mu \chi)} (\partial_\mu g) f_m \otimes W_1 f_1 \otimes \ldots \hat{m} \ldots \otimes W_n f_n \right)
+i \sum_{m=1}^{n} (\pm) T_n \left( (\partial_\mu W) \frac{\partial W_m}{\partial (\partial_\mu \chi)} g f_m \otimes W_1 f_1 \otimes \ldots \hat{m} \ldots \otimes W_n f_n \right)
$$

(95)

by proceeding analogously to (94), i.e. we have twice applied (N). In the special case that no derivatives of $\chi$ are present, the last two terms on the r.h.s. vanish.
4.2 Charge- and ghost-number conservation

We consider massive or massless spinors $\psi, \bar{\psi} \in \mathcal{P}_0$ fulfilling the Dirac equation and in particular the matter current $j_\mu \equiv \bar{\psi} \gamma_\mu \psi$ (which is conserved). We assume $W_1, ..., W_n \in \mathcal{P}_0$ and that no derivatives of $\psi$ and $\bar{\psi}$ are present. Charge conservation is expressed by the following Ward identity (N5) (charge) which is an immediate consequence of the master Ward identity (N)

$$-T_{n+1}(j_\mu \partial^\mu g \otimes W_1 f_1 \otimes ... \otimes W_n f_n) =$$

$$i \sum_{m=1}^n \left[ (\pm) T_n \left( \Delta^\mu_{\gamma_\mu, \bar{\psi}, \psi} \left( \frac{\partial j_\mu}{\partial (\gamma_\mu \psi)} g, \frac{\partial W_m}{\partial \bar{\psi}} f_m \right) \otimes W_1 f_1 \otimes ... \otimes W_n f_n \right] + (\pm) T_n \left( \Delta^\mu_{\bar{\psi}, \psi, \gamma_\mu} \left( \frac{\partial j_\mu}{\partial (\psi \gamma_\mu)} g, \frac{\partial W_m}{\partial \bar{\psi}} f_m \right) \otimes W_1 f_1 \otimes ... \otimes W_n f_n \right) \right] =$$

$$\sum_{m=1}^n T_n \left( W_1 f_1 \otimes ... \otimes (\bar{\psi} \frac{\partial W_m}{\partial \bar{\psi}} - \psi \frac{\partial W_m}{\partial \mu}) g f_m \otimes ... \otimes W_n f_n \right). \quad (96)$$

In the second step we have used the formulas (312)-(313) for $\delta^\mu$. Each monomial $W$ is an eigenvector of the operator $(\bar{\psi} \frac{\partial}{\partial \bar{\psi}} - \psi \frac{\partial}{\partial \mu})$ with eigenvalue: (number of $\bar{\psi}$ in $W$) minus (number of $\psi$ in $W$), which we call 'spinor charge'. That this Ward identity can be satisfied by choosing suitable normalizations which are compatible with (N0)-(N4) has been proved in [7] for the case that $W_1, ..., W_n$ are sub-monomials of the QED-interaction $L = A^\mu \bar{\psi} \gamma_\mu \psi$.

We turn to models which contain pairs $(\tilde{u}_a, u_a)$ of massive or massless, scalar, but fermionic ghost fields, e.g. non-Abelian gauge theories (see appendix A for the anti-commutators and Feynman propagators of the free ghost fields $\tilde{u}_a, u_a \in \mathcal{P}_0$.) The free ghost current

$$k^\mu = i \sum_a [u_a \partial^\mu \tilde{u}_a - \partial^\mu u_a \tilde{u}_a] \quad (97)$$

is conserved, because $u_a, \tilde{u}_a$ satisfy the Klein-Gordon or wave equation. Let $W_1, ..., W_n \in \mathcal{P}_0$ and we assume that only zeroth and first (internal) derivatives of $u_a$ and $\tilde{u}_a$ appear in $W_1, ..., W_n$. Similarly to (96) the MWI (N) implies the following Ward identity (N5) (ghost):

$$-T_{n+1}(k^\mu \partial^\mu h \otimes W_1 f_1 \otimes ... \otimes W_n f_n) =$$

$$\sum_{m=1}^n T_n \left( W_1 f_1 \otimes ... \otimes \left[ \left( u_a \frac{\partial W_m}{\partial u_a} - C_{u_a} \partial^\mu u_a \frac{\partial W_m}{\partial (\partial^\mu u_a)} \right) h f_m - \tilde{u}_a \frac{\partial W_m}{\partial \tilde{u}_a} + C_{u_a} \partial^\mu \tilde{u}_a \frac{\partial W_m}{\partial (\partial^\mu \tilde{u}_a)} \right] \right) =$$

$$+(1 + C_{u_a}) \left( (\tilde{\partial}^\mu u_a) \frac{\partial W_m}{\partial (\partial^\mu u_a)} h f_m + u_a \frac{\partial W_m}{\partial (\partial^\mu u_a)} (\partial^\mu h) f_m + (\tilde{\partial}^\mu \tilde{u}_a) \frac{\partial W_m}{\partial (\partial^\mu \tilde{u}_a)} (\partial^\mu h) f_m \right) \otimes ... \otimes W_n f_n, \quad (98)$$

29
where the normalization constant $C$ appearing in (305), (308) is specified by a lower index $u_a$. Every monomial $W$ is an eigenvector of the operator
\[
\Theta_g \equiv u_a \frac{\partial}{\partial u_a} + (\partial^\mu u_a) \frac{\partial}{\partial (\partial^\mu u_a)} - \bar{u}_a \frac{\partial}{\partial \bar{u}_a} - (\partial^\mu \bar{u}_a) \frac{\partial}{\partial (\partial^\mu \bar{u}_a)}
\]
and the eigenvalue is the ghost number $g(W)$:
\[
\Theta_g W = g(W) W, \quad g(W) \in \mathbb{Z}.
\] (100)
The identity (98) expresses ghost number conservation correctly if and only if
\[
C_{u_a} = -1, \quad \forall a.
\] (101)
With this normalization (N5) (ghost) takes the form
\[
-T_{n+1}(k_\mu \partial^\mu h \otimes W_1 f_1 \otimes \ldots \otimes W_n f_n) = \sum_{m=1}^{n} g(W_m) T_n \left( W_1 f_1 \otimes \ldots \otimes W_m f_m \otimes \ldots \otimes W_n f_n \right)
\] (102)
for monomials $W_1, \ldots, W_m \in \mathcal{P}_0$. That the normalization condition (N5) (ghost) (with $C_{u_a} = -1$) has common solutions with (N0)-(N4) has been proved in [3] by using the method of [7] appendix B. (A slight restriction on $W_1, \ldots, W_n$ is used in that proof).

Remark: The (free) ghost charge $Q_g$ is defined by
\[
Q_g \equiv \int_{x^0 = \text{const.}} d^3 x T_1(k^0(x)).
\] (103)
(N5) (ghost) implies the identity
\[
[Q_g, T_n \left( W_1 f_1 \otimes \ldots \otimes W_n f_n \right) \right) = \left( \sum_{m=1}^{n} g(W_m) \right) T_n \left( W_1 f_1 \otimes \ldots \otimes W_n f_n \right)
\] (104)
as can be seen by a suitable choice of the test-function $h$ in (102). For the details of this conclusion as well as for the existence of $Q_g$ see the corresponding procedure (118)-(121) for the free BRST-current.

4.3 Non-Abelian matter currents

The aim of this subsection is to derive the identity (3) from the MWI. Let
\[
j_a^\mu \equiv \bar{\psi}_\alpha^\mu \lambda_\alpha^\beta \frac{\lambda_a^\beta}{2} \psi_\beta
\] (105)
(we use matrix notation for the spinor structure) and
\[
[\lambda_a, \lambda_b] = 2i f_{abc} \lambda_c,
\] (106)
where \((f_{abc})_{a,b,c}\) are the structure constants of some Lie algebra. We assume that the masses of the spinor fields are colour independent

\[
(i\gamma^\mu \partial^\mu - m)\psi_\alpha = 0, \quad \forall \alpha, \quad (107)
\]

which implies

\[
\partial_\mu j_\mu^a = 0. \quad (108)
\]

We denote by \((A_a)_a\) the gauge fields and by \((u_a, \tilde{u}_a)_a\) the corresponding fermionic ghost fields, and consider an interaction of the form

\[
\mathcal{L} = j_\mu^a A_{a\mu} + \mathcal{L}_1(A, u, \tilde{u}), \quad (109)
\]

where \(\mathcal{L}_1(A, u, \tilde{u})\) is a polynomial in the symbols \(A, u, \tilde{u}\) and internal derivatives thereof. QCD fits in this framework: the quark fields \(\psi_\alpha\) are in the fundamental representation of \(SU(3)\).

To apply the MWI we need

\[
i\sum_{\chi, \varphi} \Delta^\mu_{\chi,\varphi} \left( \frac{\partial j_{\mu}^a f}{\partial x} \psi_{\gamma} \right) = \frac{f}{4} \psi_{\gamma} \left[ \lambda_a \, \lambda_b \right] \psi = i f h f_{abc} j_{\nu}^c \psi_{\gamma} \quad (110)
\]

\((112)\) and \((113)\) are used), and by contracting with \(A_{b\nu}\) we obtain

\[
-R_{n+1}((g\mathcal{L})^{\otimes n}; j_\mu^a \partial_\mu f) = in R_{n+1}((g\mathcal{L})^{\otimes(n-1)}; f_{abc} A_{b\nu} j_\nu^c f g), \quad (111)
\]

and hence

\[
-j_\mu^a g_L(\partial_\mu f) = (f_{abc} A_{b\nu} j_\nu^c) g_L(f g), \quad (112)
\]

which corresponds to the covariant conservation of the interacting classical current.

To formulate \((5)\) we need the time-ordered product \(T_{g\mathcal{L}}(W_1 f_1 \otimes ... \otimes W_m f_m)\) of the interacting fields \(W_1 g_L(f_1), ... , W_m g_L(f_m)\), which is defined by generalizing \((92)\) (cf. \((4), (19)\))

\[
T_{g\mathcal{L}}(W_1 f_1 \otimes ... \otimes W_m f_m) \overset{\text{def}}{=} S(g\mathcal{L}) \frac{d^n}{i^n d\lambda_1 ... d\lambda_m} |_{\lambda_1 = ... = \lambda_m = 0} S(g\mathcal{L} + \sum_{i=1}^{m} \lambda_i W_i f_i) =
\]

\[
\sum_{n=0}^{\infty} \frac{n^n}{n!} R_{n,m}((g\mathcal{L})^{\otimes n}; W_1 f_1 \otimes ... \otimes W_m f_m) \quad (113)
\]

with\(^{25}\)

\[
R_{n,m}(g_1 V_1 \otimes ... \otimes g_n V_n; W_1 f_1 \otimes ... \otimes W_m f_m) \overset{\text{def}}{=} \sum_{I \subseteq \{1, ..., n\}} (-1)^{|I|} T(\otimes_{i \in I} g_i V_i) T((\otimes_{j \in I^c} g_j V_j) \otimes (\otimes_{k=1}^{m} f_k W_k)) \quad (114)
\]

\(^{25}\)The connection to the notation \((9)\) reads: \(R_{n,1} \equiv R_{n+1} \).
By using (108) and (110) the MWI yields
\[-R_{n,2}( (g\mathcal{L})^{\otimes n}; j^\mu_a \partial_\mu f \otimes j^\nu_b h) = R_{n,1}( (g\mathcal{L})^{\otimes n}; i f_{abc} j^\rho_c f g \otimes j^\nu_b h) + i n R_{n-1,2}( (g\mathcal{L})^{\otimes (n-1)}; f_{a\tau d} \partial_\tau j^\tau_d f g \otimes j^\nu_b h) \] 

(115)

which gives
\[-T_g\mathcal{L}(j^\mu_a \partial_\mu f \otimes j^\nu_b h) = i f_{abc} j^\rho_c g\mathcal{L}(f h) - T_g\mathcal{L}(f_{a\tau d} \partial_\tau j^\tau_d f g \otimes j^\nu_b h). \] 

(116)

Due to (112) this is the formulation of (3) in the framework of causal perturbation theory. In the simple case that the gauge fields $A_a$ are external fields (which implies $L_1(A, u, \tilde{u}) \equiv 0$) and the spinor fields are massive ($m > 0$), the proof of sect. 3.3 applies, i.e. the central extensions fulfil the MWI. (Note that no factor $m$ appears in (111) and (115), which indicates that the scaling degree is not lowered.)

4.4 The master BRST-identity

We consider free gauge fields $A^\mu_a$, $a = 1, \ldots, N$, with mass $m_a \geq 0$ in Feynman gauge and the corresponding free ghost fields $\tilde{u}_a, u_a$ with the same mass $m_a$. For each fixed value of $a$ and $\mu$ the field $A^\mu_a$ is quantized as a real scalar field satisfying the Klein-Gordon or wave equation, i.e. in the formalism of appendix A we set $\varphi = A^\mu_a = \chi$, $\epsilon = 1$. The free ghost fields fulfil the same algebraic relations as in sect. 4.2 and in appendix A. For each massive gauge field $A^\mu_a$, $m_a > 0$, we introduce a free, real scalar field $\phi_a$ with the same mass $m_a$, which is quantized with a minus sign in the commutator, i.e. we have $\varphi = \phi_a = \chi$, $\epsilon = -1$ in the formalism of appendix A. (For the Fock space representation of these free fields see e.g. [35].) There is no obstacle to include spinor fields in our treatment of BRST-symmetry (sects. 4.4 and 4.5), see [3], the last paper of of [11], [15] and [21].

The free BRST-current (cf. [25], [13])
\[ j^\mu = \sum_a [ (\partial_\tau A^\tau_a + m_a \phi_a) \partial^\mu u_a - \partial^\mu (\partial_\rho A^\rho_a + m_a \phi_a) u_a ] \] 

(117)

is conserved, because $\partial_\tau A^\tau_a$, $u_a$ and $\phi_a$ fulfill the Klein-Gordon equation with the same mass $m_a$. We will see that the corresponding charge
\[ Q_0 \overset{\text{def}}{=} \int_{x^0 = \text{const.}} d^3 x T_1(j^0)(x), \] 

(118)

is the generator of the BRST-transformation of the free fields and Wick monomials. $Q_0$ is nilpotent,

\[ 2Q_0^2 = [Q_0, Q_0] = 0, \] 

(119)

because $[(\partial_\tau A^\tau_a + m_a \phi_a), (\partial_\rho A^\rho_b + m_b \phi_b)] = 0$. Without the scalar fields $\phi_a$ the charge $Q_0$ would not be nilpotent, if some gauge fields are massive. So, a main
purpose of the the scalar fields $\phi_a$ is to restore the nilpotency of $Q_0$. (For a
rigorous definition of $Q_0$, with 4-dimensional smearing with a test function and
taking a suitable limit, see [7] where a method of Requardt [33] is used.)

To obtain the master BRST-identity (i.e. the (anti)commutator of $Q_0$ with
arbitrary $T$-products [2]) we will compute

$$T_{n+1}(j_\mu \partial ^\mu g \otimes W_1 f_1 \otimes ... \otimes W_n f_n), \quad W_1, ..., W_n \in \mathcal{P}_0,$$

(120)

by means of the MWI (N). Thereby we assume that $W_1, ..., W_n$ have an even or
odd ghost number (no mixture). From this result we shall get $[Q_0, T(W_1, ..., W_n)]_\pi$
in the following way: let $\mathcal{O}$ be an open double cone with supp $f_j \subset \mathcal{O}$, $\forall j = 1, ..., n$. Following [7] (appendix B) we choose $g$ to be equal to 1 on a neighbour-
hood of $\mathcal{O}$ and decompose $\partial ^\mu g = b^\mu - a^\mu$ such that supp $a^\mu \cap (\nabla_+ + \mathcal{O}) = \emptyset$ and suppb $b^\mu \cap (\nabla_+ + \mathcal{O}) = \emptyset$. Then we apply causal factorization of the $T$-products:

$$-T_{n+1}(j_\mu (\partial ^\mu g) \otimes W_1 f_1 \otimes ... \otimes W_n f_n) =$$

$$T_1(j_\mu a^\mu)T_n(W_1 f_1 \otimes ... \otimes W_n f_n) \mp T_n(W_1 f_1 \otimes ... \otimes W_n f_n)T_1(j_\mu b^\mu) =$$

$$[T_1(j_\mu a^\mu), T_n(W_1 f_1 \otimes ... \otimes W_n f_n)]_\pi \mp T_n(W_1 f_1 \otimes ... \otimes W_n f_n)T_1(j_\mu \partial ^\mu g).$$

(121)

The last term on the r.h.s. vanishes because of $\partial ^\mu j_\mu = 0$. Since $T_n(W_1 f_1 \otimes ...
\otimes W_n f_n)$ is localized in $\mathcal{O}$, we may vary $a^\mu$ in the spatial complement of $\mathcal{O}$
without affecting $[T_1(j_\mu a^\mu), T_n(W_1 f_1 \otimes ...)]_\pi$. In this way and by using $\partial ^\mu j_\mu = 0$
we find

$$[T_1(j_\mu a^\mu), T_n(W_1 f_1 \otimes ... \otimes W_n f_n)]_\pi = [Q_0, T_n(W_1 f_1 \otimes ... \otimes W_n f_n)]_\pi$$

(122)

(see [7], appendix B for details of this conclusion).

We start the computation of (120) with the simplest case: $n = 1$. We
assume that the symbols in $W$ carry at most a first (internal) derivative (no
higher derivatives) and give the calculation in detail

$$-T_2(j_\mu (\partial ^\mu g) \otimes W f) = i \sum_{\chi, \psi \in \Phi} T_1\left(\Delta^\mu_{\chi, \psi}\left(\frac{\partial j_\mu}{\partial \chi} g, \frac{\partial W}{\partial \psi} f\right)\right).$$

(123)

The explicit results for the $\Delta^\mu$ with a non-vanishing contribution are listed in
appendix B. Thereby $C_{A_a}$ ($C_{1A_a}$ resp.), $C_{\phi_a}$ and $C_{\hat{u}_a}$ mean the normalization
constants $C(C_1$ resp.) in the cases $\varphi = A^\mu_a = \chi, \varphi = \phi_a = \chi$ and $\varphi = \hat{u}_a, \chi = u_a$.
In the present context they may depend on the colour index $a$. Inserting (113)-(120) into (123) we obtain

$$-T_2(j_\mu (\partial ^\mu g) \otimes W f) = T_1\left(s_0(W) g f \right.$$}

$$+ \ldots (\partial ^\mu g) f + \ldots (\partial ^\mu \partial ^\rho g) f + \ldots (\Box g) f\right)$$

(124)
by means of $T_1((\tilde{\partial}^a V \tilde{W} g) = T_1((\partial^a V) W g)$, where
\[
s_0(W) \overset{\text{def}}{=} \frac{\partial W}{\partial A_a^0} + (\partial^a \phi_a) \frac{\partial W}{\partial (\partial^a A_a^0)} - (\partial_r A_r^a + m_a \phi_a) \frac{\partial W}{\partial \phi_a} - (\partial_r (\partial_r A_r^a + m_a \phi_a)) \frac{\partial W}{\partial (\partial_r \phi_a)} + m_a u_a \frac{\partial W}{\partial \phi_a} + m_a (\partial_r u_a) \frac{\partial W}{\partial (\partial_r \phi_a)}.
\]
(125)
(The terms which are not written out depend on the normalization constants $C_{A_a}, C_{\phi_a}, C_{u_a}$ and $C_{A_{1\alpha}}$.) Using $gf = f$ and $(\partial^a g) = 0, \forall|a| \geq 1$ we end up with
\[
[Q_0, T_1(W f)] = T_1(s_0(W) f),
\]
(126)
where we have the anti-commutator iff $W$ has an odd ghost number. The normalization constants $C_{A_a}, C_{\phi_a}, C_{u_a}$ and $C_{A_{1\alpha}}$ have dropped out on the r.h.s., as it must be since they do not appear on the l.h.s. of (120). The result (126) is the well-known free BRST-transformation of a Wick polynomial $T_1(W) \rightarrow T_1(s_0(W))$ (cf. [11]) which we have obtained here with quite a lot of calculations. Note that in our framework $s_0$ is a derivation $s_0 : \mathcal{P}_0 \rightarrow \mathcal{P}_0$.

However, the advantage of the present method is that it can be used to compute commutators of $Q_0$ with $T$-products of higher orders. For $n = 2$ in (120) we obtain
\[
-T_3(j_\mu(\partial^a g) \otimes W_1 f_1 \otimes W_2 f_2) = i \sum_{\chi, \psi \in \mathcal{G}} T_2 \left( \Delta^\mu_{\chi, \psi} \frac{\partial j_\mu}{\partial \chi} g, \frac{\partial W_j}{\partial \psi} f_1 \otimes W_2 f_2 \right) + (\pm) \left( [W_1, f_1] \leftrightarrow [W_2, f_2] \right)
\]
(127)
where $(\pm)$ is still a sign coming from permutations of Fermi operators. We insert the expressions (315)-(326) for the various $\Delta^\mu_1$. For given $f_1, f_2$ we then choose $g$ as in (121), hence $gf_j = f_j$ and $(\partial^a g) f_j = 0, \forall|a| \geq 1$. It results
\[
[Q_0, T_2(W_1 f_1 \otimes W_2 f_2)] = i \left[ T_2 \left( \left( \frac{1}{4}(\partial^a u_a) + \frac{3}{4}(\partial^a u_a) \right) \frac{\partial W_1}{\partial A_a^0} ight) + (\pm) \left( [W_1, f_1] \leftrightarrow [W_2, f_2] \right) \right]
\]
(128)
To simplify this expression we insert the value \( C_{u_\alpha} = -1 \) which is required from ghost number conservation \([38]-[41]\). By means of (\( \tilde{\partial} \)) we replace the external derivatives by internal ones

\[
[Q_0, T_2(W_1 f_1 \otimes W_2 f_2)]_\mp = \left[ T_2\left(s_0(W_1) f_1 \otimes W_2 f_2\right) + T_1\left(G^{(1)}(W_1 f_1, W_2 f_2)\right)\right] + (\pm)\left[(W_1, f_1) \leftrightarrow (W_2, f_2)\right],
\]

where

\[
G^{(1)}(W_1 f_1, W_2 f_2) \equiv -\sum_{\nu \in \Omega} \left[ \frac{3}{4} \Delta_{u_\alpha, \nu}^\mu \left( \frac{\partial W_1}{\partial A_\alpha^\mu} f_1, \frac{\partial W_2}{\partial \psi} f_2 \right) + C_{u_\alpha} \Delta_{\partial_\nu u_\alpha, \nu}^\mu \left( \frac{\partial W_1}{\partial A_\alpha^\mu} f_1, \frac{\partial W_2}{\partial \psi} f_2 \right) - \frac{1}{2} + 2C_{1 A_\nu} \Delta_{\partial_\nu u_\alpha, \nu}^\mu \left( \frac{\partial W_1}{\partial (\partial_\nu A_\alpha^\mu)} f_1, \frac{\partial W_2}{\partial \psi} f_2 \right) + \frac{1}{2} - 2C_{1 A_\nu} \Delta_{\partial_\nu u_\alpha, \nu}^\mu \left( \frac{\partial W_1}{\partial (\partial_\nu A_\alpha^\mu)} f_1, \frac{\partial W_2}{\partial \psi} f_2 \right) + \frac{1}{2} + 2C_{1 A_\nu} \Delta_{\partial_\nu u_\alpha, \nu}^\mu \left( \frac{\partial W_1}{\partial (\partial_\nu A_\alpha^\mu)} f_1, \frac{\partial W_2}{\partial \psi} f_2 \right) + \frac{1}{2} - C_{1 A_\nu} \Delta_{\partial_\nu u_\alpha, \nu}^\mu \left( \frac{\partial W_1}{\partial (\partial_\nu A_\alpha^\mu)} f_1, \frac{\partial W_2}{\partial \psi} f_2 \right) + m_{a_0} (1 + C_{\phi_0} \Delta_{\partial_\nu u_\alpha, \nu}^\mu \left( \frac{\partial W_1}{\partial (\partial_\nu A_\alpha^\mu)} f_1, \frac{\partial W_2}{\partial \psi} f_2 \right)\right].
\]

Note that \( G^{(1)}(\cdot, \cdot) \) is not invariant with respect to the exchange of the two arguments. Now we assume that \( s_0(W_\mu) \) is a divergence, i.e. that there exists a (Lorentz) vector \((W'_\mu)_\nu = 0, \ldots, 3, W'_\mu \in \mathcal{P}_0\) with

\[
s_0(W_\mu) = i \partial_\nu W'_\mu, \quad j = 1, 2.
\]

By means of the MWI (\( N \)) we shift this derivative to the test-function

\[
[Q_0, T_2(W_1 f_1 \otimes W_2 f_2)]_\mp = \left[ -i T_2\left(W'_\mu \partial_\nu f_1 \otimes W_2 f_2\right) + T_1\left(G((W_1, W'_\nu) f_1, W_2 f_2)\right)\right] + (\pm)\left[(W_1, W'_\nu, f_1) \leftrightarrow (W_2, W'_2, f_2)\right],
\]

where

\[
G((W_1, W'_\nu) f_1, W_2 f_2) = G^{(1)}(W_1 f_1, W_2 f_2) + G^{(2)}(W'_1 f_1, W_2 f_2),
\]

with

\[
G^{(2)}(W'_1 f_1, W_2 f_2) \equiv \sum_{\chi, \psi \in \mathcal{G}} \Delta_{\chi, \psi}^\nu \left( \frac{\partial W'_1}{\partial \chi} f_1, \frac{\partial W_2}{\partial \psi} f_2 \right).
\]
If we only know that $s_0(W_1)$ is a divergence, our final result reads

$$[Q_0, T_2(W_1 f_1 \otimes W_2 f_2)] =$$

$$-iT_2(W_1' \partial^\nu f_1 \otimes W_2 f_2) + T_1(G((W_1, W_1' f_1, W_2 f_2))$$

$$+ (\pm) \left[ T_2(s_0(W_2 f_2 \otimes W_1 f_1) + T_1(G(1)(W_2 f_2, W_1 f_1)) \right]$$  \hspace{1cm} (135)$$

instead of \[132\).

The \((n = 2)-calculation generalizes to higher orders \(n \geq 2\) in a straightforward way: let \(W_1, \ldots, W_k, V_1, \ldots, V_{n-k} \in P_0\) with \(s_0(W_j) = i\partial^\nu W_{j'}\nu, \forall j = 1, \ldots, k\) and \(f_j, h_i \in D(\mathbb{R}^4)\). For simplicity we assume that each polynomial \(W_1, \ldots, W_k, V_1, \ldots, V_{n-k}\) has an even ghost number, otherwise some additional, obvious signs appear in the following formula. Setting \(m \overset{\text{def}}{=} n - k\) we obtain

$$[Q_0, T_n(W_1 f_1 \otimes \ldots \otimes W_k f_k \otimes V_1 h_1 \otimes \ldots \otimes V_m h_m)] =$$

$$= -i \sum_{l=1}^k T_n(W_1 f_1 \otimes \ldots \otimes W_l' f_l \partial f_1 \otimes \ldots \otimes W_k f_k \otimes V_1 h_1 \otimes \ldots) $$

$$+ \sum_{l=1}^m T_n(W_1 f_1 \otimes \ldots \otimes V_1 h_1 \otimes \ldots \otimes s_0(V_l) h_l \otimes \ldots \otimes V_m h_m)$$

$$+ \sum_{l,r=1}^m \left[ T_{n-1}(G((W_l, W_l' f_l, W_r f_r) \otimes W_1 f_1 \otimes \ldots \hat{f} \ldots \otimes W_k f_k \otimes V_1 h_1 \otimes \ldots) $$

$$+ T_{n-1}(G(1)(V_r h_r, W_1 f_1) \otimes W_1 f_1 \otimes \ldots \hat{h} \ldots \otimes V_1 h_1 \otimes \ldots) \right]$$

$$+ \sum_{l\neq r} T_{n-1}(G(1)(V_l h_l, V_r h_r) \otimes W_1 f_1 \otimes \ldots \otimes V_1 h_1 \otimes \ldots \hat{f} \ldots \otimes V_m h_m),$$

(136)

where \(\hat{f}\) or \(\hat{h}\) means that the corresponding factor is omitted. We call this equation the ‘master BRST-identity’. It is a consequence of the master Ward identity. Hence, the master BRST-identity is also a normalization condition. We point out that the \(G\)-terms (i.e. the terms in the last four lines) are explicitly known. We are not aware of any reference which gives a general formula for these terms, as it is done here.

So far we have not spoken about the interaction \(L \equiv L_0\); the master BRST-identity is a condition on \(T\)-products of arbitrary factors. Now we require that the interaction is \(s_0\)-invariant in some sense. The requirement \(s_0 L_0 = 0\) is too restrictive, it is not satisfied for physically relevant models. So we impose the weaker condition that \(s_0 L_0\) is a divergence:

$$s_0 L_0 = i\partial^\nu L_{1\nu}.$$  \hspace{1cm} (137)
The requirements that
(a) the master BRST-identity becomes particularly simple, and
(b) can be satisfied to all orders
for $T$-products involving the interaction are good criterions (among others) to restrict $\mathcal{L}_0$ further. We will make (a) explicit by the formula
\[
G((\mathcal{L}_0, \mathcal{L}_1)f, \mathcal{L}_0g) + (f \leftrightarrow g) = 0.
\] (138)

(b) means that anomalies (in the master BRST-identity) may not occur or must cancel. It is a hard job to work this out. For example it is well-known that in weak interactions the axial anomalies cancel only if the numbers of generations for leptons and quarks agree.

For an interaction $\mathcal{L}$ fulfilling (137) and (138) the validity of the master BRST-identity for $[Q_0, T_n(\mathcal{L}, ..., \mathcal{L})] \forall n \in \mathbb{N}$ implies
\[
[Q_0, S] = 0,
\] (139)

where $S$ is the $S$-matrix in the adiabatic limit,
\[
S \overset{\text{def}}{=} \lim_{g \to 1} S(g\mathcal{L}),
\] (140)

provided this limit exists \[20\]. Hence, $S$ induces a well-defined operator on the physical Hilbert space $\mathcal{H}_{\text{phys}} = \ker Q_0 \cap \text{ran } Q_0$, which is unitary if $\mathcal{L}_0^* = \mathcal{L}_0$ and (N2) is satisfied \[20\], \[15\], \[16\], \[21\].

Having determined the interaction by using (137), (138) and other (quite obvious) requirements, we will show that the validity of the master BRST-identity and of the ghost number conservation (N5) (ghost) suffices for a local construction of observables in non-Abelian gauge theories. This is a generalization of the corresponding construction for QED in \[7\]. In particular we will obtain an explicit formula for the computation of the nonlinear term in the BRST-transformation of an arbitrary interacting field.

### 4.5 Local construction of observables in gauge theories

For massive gauge fields the procedure is more involved. So we first treat massless gauge fields and afterwards give the modifications for the massive case.

#### 4.5.1 Massless gauge fields: determination of the interaction

Since we are considering solely massless fields ($m_a = 0$, $\forall a$), the scalar fields $\phi_a$ are superfluous. So we set $\phi_a \equiv 0$, $\forall a$.

First we determine the interaction $\mathcal{L}_0$ by the following requirements (cf. \[8\], \[18\], \[21\], \[16\] and \[35\]):

(A) There exist $\mathcal{L}_j \in (P_0)^J$, $j = 0, 1, ..., M$ which satisfy the ladder equations
\[
s_0 \mathcal{L}_{j+1}^{\mu_1, ..., \mu_j} = i \partial_{\mu_{j+1}} \mathcal{L}_{j+1}^{\mu_1, ..., \mu_j, \mu_{j+1}}, \quad j = 0, 1, ..., M - 1, \quad s_0 \mathcal{L}_M = 0
\] (141)
(B) \( \mathcal{L}_j \) is a polynomial in the gauge field \( A_\mu^a \) and in the fermionic ghost fields \( u_a, \bar{u}_a, a = 1, \ldots, N \), and internal derivatives of these symbols; each monomial has at least three factors.

(C) \( \mathcal{L}_j \) has UV-dimension \( \leq 4 \).

(D) \( \mathcal{L}_j \) is a Lorentz tensor of rank \( j \).

(E) \( \mathcal{L}_j \) has ghost number \( j \):

\[
g(\mathcal{L}_j) = j \quad \tag{142}
\]

(cf. \( \mathbf{100} \)). Thereby we take into account that \( s_0 \) increases the ghost number by 1. We conclude that the ladder \( \mathbf{141} \) stops at \( M \leq 3 \) for trilinear terms.

(F) unitarity (for \( \mathcal{L}_0 \) only): \( \mathcal{L}_0 = \mathcal{L}_a \).

Following \( \mathbf{38} \) we make the most general ansatz for \( \mathcal{L}_j, j = 0, 1, 2, 3 \) which satisfies (B)-(F) and insert it into \( \mathbf{141} \). The calculation excludes quadrilinear terms in \( \mathcal{L}_0 \). Using \( F^{\nu\mu} \overset{\text{def}}{=} \partial^\nu A^\mu - \partial^\mu A^\nu \) the most general solution for \( \mathcal{L}_0 \) reads

\[
\mathcal{L}_0 = g_0 f_{abc} \left[ \frac{1}{2} A_{a\mu} A_{bc} F^{\nu\mu} - u_a \partial^\nu \bar{u}_b A_{c\mu} \right] - i s_0 K_1 + \partial_\nu K_2^\nu, \quad \tag{143}
\]

where \( f_{abc} \) must be totally antisymmetric and \( g_0 \in \mathbb{R} \) is a constant. This implies that the colour index takes at least \( N \geq 3 \) values. The \( K_j \) are trilinear polynomials with ghost number \( (j - 2) \). We assume that the colour tensor in \( K_j, j = 1, 2 \) is also totally antisymmetric (i.e. \( K_j = h_{abc} \varphi_1^{(1)} \varphi_2^{(2)} \varphi_3^{(3)} \)) with a totally antisymmetric \( h^j \). Then one finds

\[
K_1 = g_0 h_{abc}^1 u_a \bar{u}_b \bar{u}_c, \quad K_2^\nu = g_0 h_{abc}^2 A_\mu^a u_b \bar{u}_c. \quad \tag{144}
\]

The most general solutions for \( \mathcal{L}_j, j \geq 1 \) contain trilinear terms only. Assuming again that solely totally antisymmetric colour tensors appear, we obtain

\[
\mathcal{L}_j^\nu = g_0 f_{abc} [A_{a\mu} u_b F^{\nu\mu} - \frac{1}{2} u_a u_b \partial^\nu \bar{u}_c] - i s_0 K_1^\nu + g_0 h_{abc}^3 \partial_\mu (u_a A^\nu_\mu A^\rho_c),
\]

\[
\mathcal{L}_2^{\nu\mu} = g_0 \frac{1}{2} f_{abc} u_a u_b F^{\nu\mu} - i s_0 (h_{abc}^3 A_\mu^a A_\rho^c),
\]

\[
+ g_0 h_{abc}^4 [u_a u_b \partial^\nu A^\rho_c + 2 u_a \partial^\nu u_b A^\rho_c - g^{\nu\rho} (u_a u_b \partial_\lambda A^\lambda_c - 2 u_a \partial_\lambda u_b A^\lambda_c)]]
\]

\[
\mathcal{L}_3^{\mu\lambda} = g_0 h_{abc}^4 [g^{\nu\rho} u_a u_b \partial^\lambda u_c + g^{\nu\rho} u_a u_b \partial^\lambda u_c], \quad \tag{145}
\]

where \( h^3, h^4 \) are totally antisymmetric. Note that the divergence \( \partial_\mu \) of the \( h^j \)-term in \( \mathcal{L}_2^{\nu\mu} \) vanishes. To simplify the formulas we choose

\[
h_{abc}^3 = 0, \quad \forall j = 1, 2, 3, 4. \quad \tag{146}
\]

\( h^4 = 0 \) is equivalent to \( \mathcal{L}_2^{\nu\mu} = -\mathcal{L}_2^{\mu\nu} \) and also to \( \mathcal{L}_3 = 0 \).

The requirements (A)-(F) used so far do not involve \( T \)-products, they are of first order perturbation theory. We now restrict \( \mathcal{L}_0 \) further by \( \mathbf{38} \), which can be interpreted as a requirement for second order tree diagrams, see \( \mathbf{150} \).
More precisely, we will work with the generalization of (138) to the ladder (141): in order that the master BRST-identity (136) implies the important equation (generalized perturbative gauge invariance)\(^\text{26}\)

\[
[Q_0, T_n(\mathcal{L}_{j_1} f_1 \otimes \cdots \otimes \mathcal{L}_{j_n} f_n)]_\equiv = -i \sum_{i=1}^n (-1)^{j_1+\cdots+j_i-1} T_n(\mathcal{L}_{j_1} f_1 \otimes \cdots \otimes \mathcal{L}_{j_i} f_i \otimes \cdots \otimes \mathcal{L}_{j_n} f_n),
\]

we require

\[
G((\mathcal{L}_j, \mathcal{L}_{j+1}) f, \mathcal{L}_k g) + (-1)^j k G((\mathcal{L}_k, \mathcal{L}_{k+1}) g, \mathcal{L}_j f) = 0, \quad j, k = 0, 1, 2, 3,
\]

\(\forall f, g \in \mathcal{D}(\mathbb{R}^4)\), where we set \(\mathcal{L}_4 \equiv 0\). Or, with the simplification (146) (which will always be used in the following), \(j\) and \(k\) run only through the values \(j, k = 0, 1, 2\). In the present case of solely massless fields this requirement can be fulfilled. It restricts the interaction \(\mathcal{L}_0\) further and determines the normalization constant \(C_{A_u}\). Namely, using the simplification (146), one finds by explicit calculation that the requirement (148) holds true if and only if

\[
C_{A_u} = -\frac{1}{2}, \quad \forall a,
\]

and the \(f_{abc}\) fulfil the Jacobi identity \(^\text{38, 37}\). Hence, the \(f_{abc}\) are structure constants of some Lie algebra. The total antisymmetry of \(f_{abc}\) implies that this Lie algebra is isomorphic to a direct sum of Abelian and simple compact Lie

\(^{26}\)In [1] this identity is called the normalization condition (N6), in [16] it is called ‘generalized’ (free perturbative gauge invariance’. The importance and usefulness of this identity has also been pointed out in the earlier paper [18]. The particular case

\(^{37}\)The Jacobi identity and (140) are required even from the particular case

\(G((\mathcal{L}_0, \mathcal{L}_j) f, \mathcal{L}_k g) + (f \leftrightarrow g) = 0\). This was demonstrated in [38] by reversing the calculation in [11]. The computation of the l.h.s. of (148) is lengthy. The straightforward way uses the definitions (39) and (43) of \(G^{(1)}\) and \(G^{(2)}\). To shorten the calculation one may choose

\(C_{1_{A_u}} = 0 = C_{1_{u_{a}}}\), because the terms \(\sim C_{1_{A_u}} C_{1_{u_{a}}}\) must drop out. This follows from the fact that \(\mathcal{L}_0\) (143), \(\mathcal{L}_1\) and \(\mathcal{L}_2\) (144) do not contain symbols with second or higher derivatives and, hence, the r.h.s. in

\[
T_1 \left( G((\mathcal{L}_j, \mathcal{L}_{j+1}) f, \mathcal{L}_k g) + (-1)^{j+k} \{ (j, f) \leftrightarrow (k, g) \} \right) =
\]

\[
[Q_0, T_2(\mathcal{L}_j f \otimes \mathcal{L}_k g)]_{1-\text{legs}} + i \left( T_2(\mathcal{L}_{j+1} \partial_v f \otimes \mathcal{L}_k g) + (-1)^{j+k} \{ (j, f) \leftrightarrow (k, g) \} \right) \big|_{1-\text{legs}}
\]

(cf. (132), ..., 1-\text{legs}) expresses that we mean the terms with 4 free field operators only does not contain the constants \(C_{1_{A_u}}\) and \(C_{1_{u_{a}}}\) (according to the definition (39)). However, even with this simplification, it seems to be faster to compute the r.h.s. of (150) (by using the techniques of [11], [12], instead of the straightforward computation of the l.h.s. Thereby, the \(T\)-products of second order must fulfill the normalization condition (N3), because the MWI presupposes this condition. Note that the derivation of (150) uses the MWI for tree diagrams only and, hence, (150) holds surely true.
algebras, see e.g. [35]. We point out, that the Lie algebraic structure is not put in, it is a consequence of our requirements.

Remarks: (1) If we do not use the simplification (146), but assume that \( K_1 \) and \( K_2 \) are built up from the same colour tensor \( f_{abc} \) as the first two terms in (143),

\[
K_1 = -i\beta_1 g_0 f_{abc} u_a \tilde{u}_b \tilde{u}_c, \quad K_2^\mu = \beta_2 g_0 f_{abc} A_\mu^a u_b \tilde{u}_c, \quad \beta_1, \beta_2 \in \mathbb{R} \tag{151}
\]

(instead of (144)), we can determine the parameters \( \beta_1 \) and \( \beta_2 \) from the particular case \((j, k) = (0, 0)\) of the requirement (148); additionally to (149) and the Jacobi identity this condition yields

\[
(\beta_1, \beta_2) \in \{(0, 0), (\frac{1}{2}, 1), (-\frac{1}{2}, 0), (1, 1)\} \tag{152}
\]

by a generalization of the calculation (149), where \( C_{u_0} = -1 \) (101) is used. Note that the first two solutions in (152), and also the latter two, are obtained from each other by replacing \( u \) by \( \gamma \tilde{u} \) and \( \tilde{u} \) by \((-\frac{1}{\gamma} u)\) in \( \mathcal{L}_0, \gamma \in i\mathbb{R} \setminus \{0\} \) arbitrary.

(2) In [14] it was found how the quadrilinear interactions of the usual Lagrangian formalism appear in our framework. Namely, from (143), (151) and (N3) it results

\[
T_2(\mathcal{L}_0, \mathcal{L}_0)(x_1, x_2) = f_{a b r} f_{c d s} \cdot \left( \Delta^F_{\partial' A'} \partial' A'(x_1 - x_2) \right) \cdot A_{a \mu}(x_1) A_{b \nu}(x_1) A_{c \rho}(x_2) A_{d \tau}(x_2) : \]

\[
\left[ + (\beta_2 - 2\beta_1)^2 g_{\mu\rho} g_{\tau\rho} : u_a(x_1) \tilde{u}_b(x_1) u_c(x_2) \tilde{u}_d(x_2) : \right] + \left\{ \Delta^F_{\partial' A'} \partial' u_r, (x_1 - x_2) \beta_2 (\beta_2 - 1) : A_{a \mu}(x_1) u_b(x_1) A_{c \nu}(x_2) \tilde{u}_d(x_2) : \right\} + (x_1 \leftrightarrow x_2) \bigg) + ..., \tag{153}
\]

where the terms which are denoted by the dots have no contributions of the form

\[
\delta(x_1 - x_2) : B_1(x_1) B_2(x_1) B_3(x_2) B_4(x_2) : \tag{154}
\]

\((B_1, ..., B_4)\) are free field operators), they are disconnected terms, tree terms with propagators \( \Delta^F(x_1 - x_2) \) or \( \partial' \Delta^F(x_1 - x_2) \), or loop terms. The terms of the form (154) correspond to the quadrilinear terms of the interaction Lagrangian of the conventional theory. So, \( C_{u_0} = -\frac{1}{2} \) (149) yields the usual 4-gluon coupling (cf. the first paper of [11]) and, if \( \beta_2 - 2\beta_1 \neq 0, \) a 4-ghost coupling. However, there is no \( A_n A \tilde{u} - \partial' A' \) coupling coming from \( C_{u_0} = -1 \), because \( \beta_2 (\beta_2 - 1) = 0 \).

(3) By using (N4) (11) we obtain for the interacting \( F \)-field \( (F_{\mu \nu} \overset{\text{def}}{=} \partial' A' - \partial' A') \)

\[
F_{\mu \nu \lambda}(x) = \partial' A'_{\mu \nu \lambda}(x) - \partial' A'_{\mu \tau}(x) - 2 C_{A_\mu} g g(x) f_{abc} (A_\mu^a A'_\nu^b A'_\lambda^c x). \tag{155}
\]

We see that the nonlinear term is due to the non-vanishing of \( C_{A_\mu} \) and that it agrees with the usual nonlinear term iff \( C_{A_\mu} = -\frac{1}{2} \), in agreement with (149).
In sect. 4.5.3 we will see that our local construction of observables works also if one replaces (148) by the weaker condition (209) (with the specifications (210)-(211) and (216)-(217)), i.e. one allows to introduce additional 4-legs couplings 'by hand'. This relaxed version of (148) has solutions for arbitrary $(\beta_1, \beta_2) \in \mathbb{R}^2$ (at least in the case $(j, k) = (0, 0)$), see e.g. [18].

4.5.2 Massless gauge fields: local construction of observables

In [7] a general local construction of observables in gauge theories and of the physical Hilbert space (in which the observables are faithfully represented) is given. This construction relies on some assumptions which can be fulfilled in QED [7]. We are now going to generalize the latter result to the class of interactions we have selected in the preceding subsection, which includes non-Abelian gauge theories. Thereby we assume that ghost number conservation (N5) (ghost) and certain cases of the master BRST-identity (136) are satisfied.

As in [7] we start with the local algebra of interacting gauge and ghost fields

$$F(O) = \bigvee\{W_{gL}(f) \mid f \in D(O), W = A^\mu, u, \tilde{u}, \ldots\}$$  \hspace{1cm} (156)$$

(the dots stand for polynomials in $A^\mu$, $u$ and $\tilde{u}$), where $O$ is a double cone and $g(x) = 1$, $\forall x \in O$. In [3] the crucial observation has been made that a change of the switching function $g$ outside of $O$, transforms all interacting fields $\in F(O)$ by the same unitary transformation. Therefore, the algebraic properties of $F(O)$ are independent of the adiabatic limit $g(x) \to 1$, $\forall x$. Hence, we may avoid this limit, which saves us from infrared divergences. It seems that a consistent perturbative construction of massless non-Abelian gauge theories can be done only locally, i.e. without performing the adiabatic limit, due to the confinement.

The field algebra $F(O)$ contains unphysical fields. The central problem in gauge theories is to eliminate the latter, i.e. to select the observables, and, in a second step, to construct (physical) states on the algebra of observables. We proceed as in [7]: roughly speaking we will define the observables to be the BRST-invariant fields. Thereby, we will define the BRST-transformation $\tilde{s}$ as the ($\mathbb{Z}_2$-graded) commutator with the (modified) interacting BRST-charge $Q_{gL}$ of Kugo and Ojima [24]. But in contrast to this reference we do not perform the adiabatic limit. The latter causes the complication that $Q_{gL}$ does not agree with its zeroth order contribution $Q_0$: it is a non-trivial formal power series (cf. [7], [16]).

The current belonging to $Q_{gL}$ is the interacting BRST-current $\tilde{j}_gL$. From our experience made in QED [3] we know that $\tilde{j}_gL$ should have the following properties:

(a) to zeroth order it agrees with the free BRST current $j^\mu$ (117),

(b) it is conserved up to (first) derivatives of the coupling 'constant' $g$: $\partial_\mu \tilde{j}_gL(x) \sim$ An alternative proof of this fact, which applies also to classical field theory, is given in the second paper of [8].
Unfortunately, (b) does not hold true for the interacting field $j_{\mu L}$ \[12\]-\[13] where $j_{\mu L}$ is constructed in terms of $T$-products satisfying the MWI (N): from \[12\]-\[13] we get

$$-T_2(j^\mu \partial_\mu f \otimes L_0 g) = -iT_1(L_1^\mu f \partial_\nu g) + iT_1(M^\nu(\partial_\nu f)g),$$

(157)

where

$$M^\nu \equiv -L_1^\nu + (3C_1 A_a + \frac{1}{2} \partial_\mu u_a \frac{\partial L_0}{\partial (\partial_\mu A_{a\nu})}) + \frac{3}{4} u_a \frac{\partial L_0}{\partial A_{a\nu}}.$$  

(158)

However, the wanted conservation property can be achieved by a change of the invariance (147) we define $k^\mu (where $k^\mu$ is the (free) ghost current \[17\] and generalized perturbative gauge invariance \[147\] we define

$$\tilde{T}_{n+1}(j^\mu f \otimes L_{j_1} f_1 \otimes \ldots \otimes L_{j_n} f_n) = [Q_0, T_{n+1}(k^\mu f \otimes L_{j_1} f_1 \otimes \ldots \otimes L_{j_n} f_n)] +$$

$$+i \sum_{l=1}^n (-1)^{j_1+\ldots+j_{l-1}} T_{n+1}(k^\mu f \otimes L_{j_1} f_1 \otimes \ldots \otimes L_{j_{l+1}} f_l \otimes \ldots \otimes L_{j_n} f_n).$$  

(160)

By means of (159) and (147) we find that $\tilde{T}_{n+1}(j^\mu L_{j_1} \ldots L_{j_n})$ factorizes causally \[13\], e.g. $\tilde{T}_{n+1}(j, L, \ldots, \ldots) = \tilde{T}_{n+1}(j, L, \ldots) T_{n-1}(L, \ldots, \ldots)$. In addition it is symmetrical in $L_{j_1}, \ldots, L_{j_n}$ and fulfills the normalization conditions (N1), (N2) and (N0). Hence, $T_{n+1}(j^\mu L_{j_1} \ldots) \rightarrow \tilde{T}_{n+1}(j^\mu L_{j_1} \ldots)$ is an admissible finite renormalization of $T$-products (solely the extension to $D_{n+1}$ is changed), which however violates (N3) and the MWI (N). We point out that the requirement $\partial_\mu j^\mu_{\mu L}(x) \sim (\partial g)(x)$ is not compatible with (N3), which can be seen by an explicit calculation of the first order tree diagrams of $j^\mu_{\mu L}$.\[29\]

To compute the first order tree diagrams of $j^\mu_{\mu L}$ read:

$$j_{\mu L}^{(1)}|_{\text{tree}}(x) = \frac{1}{\hbar} \int d^4 x_1 g(x_1) \{-: \partial^\mu u_a(x) A_{a\mu b}(x_1) F^{\nu} \lambda(x_2) : f_{abc} \partial_\lambda D^{\text{ret}}(x-x_1) +$$

$$+ : u_a(x) A_{a\mu b}(x_1) F^{\nu} \lambda(x_1) : f_{abc} \partial_\lambda D^{\text{ret}}(x-x_1) + C_1 g^\mu \delta(x-x_1) \} +$$

$$+ : \partial^\mu u_a(x) u_b(x_1) \partial^\lambda u_c(x_1) : f_{abc} \partial_\lambda D^{\text{ret}}(x-x_1) -$$

$$- : u_a(x) u_b(x_1) \partial^\lambda u_c(x_1) : f_{abc} \partial_\lambda D^{\text{ret}}(x-x_1) + C_2 g^\mu \delta(x-x_1) \} +$$

$$+ : \partial^\mu A_{\nu b}(x) A_{a\mu b}(x_1) u_c(x_1) : f_{abc} \partial_\lambda D^{\text{ret}}(x-x_1) -$$

$$- : \partial_\nu A_{\nu b}(x) A_{a\mu b}(x_1) u_c(x_1) : f_{abc} \partial_\lambda D^{\text{ret}}(x-x_1) + C_3 g^\mu \delta(x-x_1) \}$$

(161) $D^{\text{ret}}$ is the retarded Green's function of the wave operator \[146\] and $C_1, C_2$ and $C_3$ are undetermined normalization constants. The requirement $\partial_\mu j^\mu_{\mu L}(x) \sim (\partial g)(x)$ fixes the latter uniquely:

$$C_1 = -1, \quad C_2 = -\frac{1}{2}, \quad C_3 = 0.$$  

(167)
divergence of (160) with respect to $j^\mu$ we first apply (N5) (N3) (ghost) (102) to all terms on the r.h.s. (where we use $g(L_j) = j$) and afterwards generalized gauge invariance (147). In this way we obtain

$$\tilde{T}_{n+1}(j^\mu \partial_\mu f \otimes L_j, f_1 \otimes ... \otimes L_j, f_n) =$$

$$i \sum_{i=1}^{n} (-1)^{j_1 + ... + j_i} T_n(L_{j_1}, f_1 \otimes ... \otimes L_{j_i+1}, f \partial_\mu f \otimes ... \otimes L_j, f_n)$$

Conversely, this current conservation identity (169) implies the generalized perturbative gauge invariance (147) by proceeding similarly to (121). We denote by $\tilde{R}_{n+1}(L,...,L,j)$ (where $L = L_0$) the $R$-product (93) which is constructed in terms of $T_k(L,...,L)$ and $\tilde{T}_{k+1}(j,L,...,L)$, $1 \leq k \leq n$. Then, the identity (169) implies

$$\tilde{R}_{n+1}((Lg)^{\otimes n}; j^\mu \partial_\mu f) = inR_n((Lg)^{\otimes n-1}; L^\nu f \partial_\nu g)$$

Analogously to (149) we define

$$\tilde{j}_{gL}^\mu(f) \equiv \sum_{n=0}^{\infty} \frac{i^n}{n!} \tilde{R}_{n+1}((Lg)^{\otimes n}; j^\mu f)$$

This interacting BRST-current has the wanted conservation property

$$\tilde{j}_{gL}^\mu(\partial_\mu f) = -L^\nu g_{L}(f \partial_\nu g)$$

which agrees precisely with the corresponding result for QED (formula (5.12) of the first paper of [7]). The difference $(\tilde{j}_{gL}^\mu - j_{gL}^\mu)$ (where $j_{gL}^\mu$ still denotes the interacting field constructed in terms of $\tilde{T}$-products satisfying also the MWI and (N3)) is immediately obtained by applying the master BRST-identity:

$$\tilde{j}_{gL}^\mu(f) - j_{gL}^\mu(f) = i(G^{(1)}(k^\mu f, Lg))_{gL} + i(G((L, L_1)g, k^\mu f))_{gL}$$

The interacting BRST-charge operator is now defined by

$$Q_{gL} \equiv \int d^4x \, h^\mu(x) \tilde{j}_{gL}(x)$$

where $L = L_0$ and $h^\mu$ is a suitable test function (see [7], and subsection 4.5.4). $Q_{gL}$ is a formal power series and the construction is such that the relations

$$Q_{gL} = Q^*_{gL}, \quad (Q_{gL})_0 = Q_0, \quad (Q_{gL})_0 = Q_0$$

If (N3) (or equivalently the causal Wick expansion [19], [5]) holds true, the propagators in (102) and (164) are both equal to

$$i[...]|(1)R_2(A_\lambda; \partial^\mu \partial_\nu A^\nu)(x_1;x)|\Omega).$$

But this contradicts $C_2 \neq C_3$ (167).
hold true (where \((\ldots)_0\) means the zeroth order) and that \(Q_{\mathcal{L}}\) is nilpotent
\[(Q_{\mathcal{L}})^2 = 0.\] (176)

The latter property is proved in subsection 4.5.4 by using current conservation (172) and generalized gauge invariance (147). We point out that the conservation of the BRST-current (172) and the construction of the nilpotent BRST-charge (174)-(176) use the master BRST-identity for \(T_n(L_0,\ldots,L_0)\) and \(T_n(L_1,L_0,\ldots,L_0)\) (\(\forall n \in \mathbb{N}\)) only.

The BRST-transformation \(\tilde{s}\) of the interacting fields \(W_{\mathcal{L}}(f), f \in \mathcal{D}(\mathcal{O})\), is then defined by the commutator with \(Q_{\mathcal{L}}\) (or anti-commutator if \(W\) has an odd ghost number)
\[
\tilde{s}(W_{\mathcal{L}}(f)) \overset{\text{def}}{=} [Q_{\mathcal{L}},W_{\mathcal{L}}(f)]_\mp, \quad f \in \mathcal{D}(\mathcal{O}).
\] (177)

We extend \(\tilde{s}\) to a graded derivation \(\mathcal{F}(\mathcal{O}) \to \mathcal{F}(\mathcal{O})\). The local observables are selected by the definition \[\mathcal{A}(\mathcal{O}) \overset{\text{def}}{=} \ker \tilde{s} / \text{ran} \tilde{s}.\] (178)

Following (178) we look for states on \(\mathcal{A}(\mathcal{O})\) which take values in \(\hat{\mathcal{C}}\) (by which we mean the formal power series with coefficients in \(\mathbb{C}\)). Thereby, \(a \in \hat{\mathcal{C}}\) is called positive, if there exists \(b \in \hat{\mathcal{C}}\) with \(a = b^*b\), where \(*\) means complex conjugation. In \[44\] it is shown that \(\mathcal{A}(\mathcal{O})\) can be naturally represented on the cohomology of \(Q_{\mathcal{L}}\)
\[
\mathcal{H}_{\text{phys}} \overset{\text{def}}{=} \ker Q_{\mathcal{L}} / \text{ran} Q_{\mathcal{L}}
\] (179)
and that the induced inner product on \(\mathcal{H}_{\text{phys}}\) is positive definite in the just mentioned sense. Hence \(\mathcal{H}_{\text{phys}}\) is a pre Hilbert space and its elements are interpreted as physical states.

The master BRST-identity (136) yields also an explicit formula for the BRST-transformation of the interacting fields \(W_{\mathcal{L}}(f)\) \(\in \mathcal{F}(\mathcal{O})\), by which we will see that the definition (177) agrees with the usual BRST-transformation. (In addition this ensures the existence of non-trivial observables.) For this purpose we note that the master BRST-identity and the requirement (148) imply the following relation for the \(R\)-products (93):
\[
\begin{align*}
[Q_0,R_{n+1}((L_0g)\otimes^n;Wf)]_\mp &= -inR_{n+1}((L_0g)\otimes^{(n-1)} \otimes L_1\partial_\nu Wf) - iR_{n+1}((L_0g)\otimes^n;W'\partial_\nu f) \\
&+nR_n ((L_0g)\otimes^{(n-1)};[G((L_0,L_1)g,Wf) + G((W,W')f,L_0g)])
\end{align*}
\] (180)
where we have assumed \(s_0(W) = i\partial_\nu W_{\mathcal{L}}\). In subsection 4.5.4 it will be shown that this identity implies the BRST-transformation formula
\[
\begin{align*}
\tilde{s}(W_{\mathcal{L}}(f)) &= [Q_{\mathcal{L}},W_{\mathcal{L}}(f)]_\mp = -iW'_{\mathcal{L}}(\partial_\nu f) \\
&+i(G((L,L_1)g,Wf) + G((W,W')f,Lg))_{\mathcal{L}}, \quad f \in \mathcal{D}(\mathcal{O}).
\end{align*}
\] (181)
where $\mathcal{L} = \mathcal{L}_0$. The term in the second line is the nonlinear part of the BRST-transformation. In case that $W$ is a single symbol we find that $[G((\mathcal{L}, \mathcal{L}_1)g, Wf) + G((W, W')f, Lg)]$ is quadratic in the symbols (because $\mathcal{L}$ and $\mathcal{L}_1$ are trilinear), in agreement with the usual BRST-transformation. (To prove it will be shown that the terms $[(Q_{g\mathcal{L}} - Q_0), W_{g\mathcal{L}}(f)]$ cancel out with the terms $-i\eta_{n+1} (L_{0g}^{\otimes(n-1)} \otimes L'_i(\partial \varphi g; Wf)$.)

If we do not assume that $s_0W$ is a divergence we end up with

$$\tilde{s}(W_{g\mathcal{L}}(f)) = (s_0 W)_{g\mathcal{L}}(f) + i(G^{(1)}(Wf, Lg) + G((\mathcal{L}, \mathcal{L}_1)g, Wf))_{g\mathcal{L}},$$

instead of (181). We choose $W = k^\mu$, compare with (173) and find

$$[Q_{g\mathcal{L}}, k^\mu_{g\mathcal{L}}] = \tilde{j}^\mu_{g\mathcal{L}}.$$

Introducing the interacting ghost charge

$$Q^\mu_{g\mathcal{L}} \overset{\text{def}}{=} \int d^4x h^\mu(x)k^\mu_{g\mathcal{L}}(x),$$

where $h^\mu \in \mathcal{D}(\mathbb{R}^4)$ is chosen in precisely the same way as in $Q_{g\mathcal{L}}$ (see sect. 4.5.4), it results

$$[Q_{g\mathcal{L}}, Q^\mu_{g\mathcal{L}}] = Q_{g\mathcal{L}}$$

as in [24].

**Examples:** In most of the following examples the computation of $G^{(1)}$ and $G^{(2)}$ gives less work than it seems, because only very few terms contribute. We use the values $C_{ua} = -1$ (101) and $C_{Aa} = \frac{1}{2}$ (149) without further mentioning it.

1. BRST-transformation of $A^\mu_{a\mathcal{L}}(h)$:

   $$G^{(1)}(\mathcal{L}_0g, A^\mu_{a\mathcal{L}}h) = 0, \quad G^{(1)}(A^\mu_{a\mathcal{L}}h, \mathcal{L}_0g) = -\frac{3}{4}g_0 f_{abc} A^\mu_{b\mathcal{L}} u_c h g,$$

   $$G^{(2)}(\mathcal{L}_1g, A^\mu_{a\mathcal{L}}h) = \frac{3}{4}g_0 f_{abc} A^\mu_{b\mathcal{L}} u_c h g, \quad G^{(2)}(g^{\mu\nu} u_a h, \mathcal{L}_0g) = g_0 f_{abc} A^\mu_{b\mathcal{L}} u_c h g.$$

   Therefore,

   $$\tilde{s}(A^\mu_{a\mathcal{L}}(h)) = -iu_{a\mathcal{L}}(\partial^\mu h) + ig_0(f_{abc} A^\mu_{b\mathcal{L}} u_c)_{g\mathcal{L}}(gh).$$

   Taking $g_{\text{supp}} h = 1$ into account the last term takes the usual form $ig_0(f_{abc} A^\mu_{b\mathcal{L}} u_c)_{g\mathcal{L}}(h)$. We see that $G^{(1)}(A^\mu_{a\mathcal{L}}h, \mathcal{L}_0g) \neq 0$ gives a non-vanishing contribution to the nonlinear term in (187). This shows that the distinction of internal and external derivatives and in particular the appearance of the external derivative in the definition of $\Delta^\mu$ (35) is crucial to obtain the correct BRST-transformation.

2. BRST-transformation of $u_a g\mathcal{L}(h)$:

   $$G^{(1)}(\mathcal{L}_0g, u_a h) = 0, \quad G^{(1)}(u_a h, \mathcal{L}_0g) = 0,$$

   $$G^{(2)}(\mathcal{L}_1g, u_a h) = -\frac{1}{2}g_0 f_{abc} u_b u_c g h, \quad G(0h, \mathcal{L}_0g) = 0.$$
Hence,

\[ \tilde{s}(u_a g \mathcal{L}(h)) = -\frac{i}{2} g_0 (f_{abc} u_b u_c) g \mathcal{L}(h). \]  

(198)

(3) BRST-transformation of \( \tilde{u}_a g \mathcal{L}(h) \):

\[
G^{(1)}(\mathcal{L}_0 g, \tilde{u}_a h) = 0, \quad G^{(1)}(\tilde{u}_a h, \mathcal{L}_0 g) = 0, \\
G^{(2)}(\mathcal{L}_1 g, \tilde{u}_a h) = 0, \quad G^{(2)}(-\partial_{\nu} A^\nu h, \mathcal{L}_0 g) = 0,
\]

where we have used \( g^{\mu\nu} \frac{\partial \mathcal{L}}{\partial A^{\mu\nu}} = 0 \). So we obtain

\[ \tilde{s}(\tilde{u}_a g \mathcal{L}(h)) = i A^\nu_a g \mathcal{L}(\partial_{\nu} h). \]  

(199)

(4) BRST-transformation of \( F_a^{\mu\nu} g \mathcal{L}(h) \):

\[
G^{(1)}(\mathcal{L}_0 g, F_a^{\mu\nu} h) = 0, \\
G^{(1)}(F_a^{\mu\nu} h, \mathcal{L}_0 g) = \left\{ \frac{1}{2} + 3C_{1A_a}, g_0 f_{abc} A^\nu_b u_c (\partial^\mu h) g - (\mu \leftrightarrow \nu), \\
G^{(2)}(\mathcal{L}_1 g, F_a^{\mu\nu} h) = \left\{ \frac{1}{2} + 3C_{1A_a}, g_0 f_{abc} [\hat{\partial}^\mu (A^\nu_b u_c) gh + A^\nu_b u_c (\partial^\mu g) h] \\
- (\mu \leftrightarrow \nu) + g_0 f_{abc} F_b^{\mu\nu} u_c gh, \quad G^{(2)}(0 h, \mathcal{L}_0 g) = 0. \right. \]
\]

(200)

Now we apply the relation (30) (which holds obviously also for the \( R \)-products; cf. the remark at the end of sect. 2.4)

\[ R_{n+1}(\mathcal{L}_0 g \otimes \ldots f_{abc} \hat{\partial}^\mu (A^\nu_b u_c) gh) = - R_{n+1}(\mathcal{L}_0 g \otimes \ldots f_{abc} A^\nu_b u_c \partial^\mu (g h)). \]  

(201)

Hence, inserting these formulas into (188), the terms \( \sim (\frac{1}{2} + 3C_{1A_a}) \) cancel and it remains

\[ \tilde{s}(F_a^{\mu\nu} g \mathcal{L}(h)) = i g_0 (f_{abc} F_b^{\mu\nu} u_c) g \mathcal{L}(h). \]  

(202)

(5) BRST-transformation of \( (\sum_a F_a^{\mu\nu} F_a^{\rho\tau}) g \mathcal{L}(h) \) (we do not write \( \sum_a \) but always perform this sum):

\[
G^{(1)}(\mathcal{L}_0 g, F_a^{\mu\nu} F_a^{\rho\tau} h) = 0, \\
G^{(1)}(F_a^{\mu\nu} F_a^{\rho\tau} h, \mathcal{L}_0 g) = \left\{ \frac{1}{2} + 3C_{1A_a}, g_0 f_{abc} [F_a^{\rho\tau} A^\nu_b u_c (\partial^\mu h) g + (\hat{\partial}^\mu F_a^{\rho\tau}) A^\nu_b u_c h g] \\
- (\mu \leftrightarrow \nu) \right\} + \{ (\mu, \nu) \leftrightarrow (\rho, \tau) \}, \\
G^{(2)}(\mathcal{L}_1 g, F_a^{\mu\nu} F_a^{\rho\tau} h) = \left\{ \frac{1}{2} + 3C_{1A_a}, g_0 f_{abc} [F_a^{\rho\tau} \hat{\partial}^\mu (A^\nu_b u_c) gh + F_a^{\rho\tau} A^\nu_b u_c (\partial^\mu g) h] \\
- (\mu \leftrightarrow \nu) + g_0 f_{abc} F_a^{\rho\tau} F_b^{\mu\nu} u_c gh \} + \{ (\mu, \nu) \leftrightarrow (\rho, \tau) \}, \quad G^{(2)}(0 h, \mathcal{L}_0 g) = 0. \]
\]

(203)

The term \( \sim FF_{\mu\nu} g \mathcal{L}(h) \) in \( G^{(2)}(\mathcal{L}_1 g, FF_h) \) drops out because \( f_{abc} \) is totally antisymmetric. Inserting these formulas into (181), the terms \( \sim (\frac{1}{2} + 3C_{1A_a}) \) cancel again due to (38). So we obtain

\[ \tilde{s}(F_a^{\mu\nu} F_a^{\rho\tau} g \mathcal{L}(h)) = 0 \]  

(204)
and, hence, the corresponding equivalence class (cf. (178)) is a non-trivial observable.

(6) Due to the requirement $G((L_j, L_{j+1})f, L_kg)+(-1)^j\{(j, f) \leftrightarrow (k, g)\} = 0$ (148) we can easily write down the BRST-transformation of $L \to \hbar$:

$$\begin{align*}
\tilde{s}(L_{0gL}(h)) &= -iL_{1gL}(\partial h), \\
\tilde{s}(L_{1gL}(h)) &= -iL_{2gL}(\partial h), \\
\tilde{s}(L_{2gL}(h)) &= 0. \\
\end{align*}$$

(197)

Remarks: (1) Having determined the interaction $L_0$ we can explicitly write down the interacting field equations by means of (194): with the simplification (146) they read

$$\begin{align*}
\Box A_{\mu a gL} &= g_0 f_{abc} \partial^\nu [g(A_{\mu b}a_{\nu c})gL] - \\
&- g_0 f_{abc} (A_\nu^b F_{\nu c}^a)gL + g_0 f_{abc} (u_d \partial_d \tilde{u}_c)gL, \\
\Box u_{agL} &= -g_0 f_{abc} [g(A_\nu^b u_c)gL], \\
\Box u_{agL} &= -g_0 f_{abc} (A_\nu^b \partial_\nu \tilde{u}_c)gL. \\
\end{align*}$$

(198-200)

They hold true everywhere, $g$ needs not to be constant. In the classical limit $\hbar \to 0$ interacting fields factorize, $(VW)_{gL}(x) = V_{gL}(x)W_{gL}(x)$ (see [9]), and hence (198), (200) go over into the usual Yang-Mills equations.

(2) One might think that the MWI agrees with the quantum Noether condition (QNC) (23) in the application to $\partial j T(j^\mu, \mathcal{L}, ..., \mathcal{L})$, where $j^\mu$ is a conserved current and $\mathcal{L}$ the interaction. But for $j$ being the BRST-current (117) we explicitly see that the two conditions are different and that the QNC is less general. Namely, the QNC in terms of interacting fields is precisely the particular case $j_1 = ... = j_n = 0$ of (169). This equation is sufficient for perturbative gauge invariance (147) for $j_1 = ... = j_n = 0$ and, if ghost number conservation holds true, it is also necessary. The QNC does not contain any information about the $G$-terms in the master BRST-identity (138). In addition we recall that (169) (and hence in particular the QNC in terms of interacting fields) is not compatible with the MWI (157) and (N3). Comparing (201) with (168) we find that this holds true also for the QNC in terms of $T$-products.

In general note that the QNC is formulated for $T(j^\mu, \mathcal{L}, ..., \mathcal{L})$ with $\partial_\mu j^\mu = 0$ and $\mathcal{L}$ the interaction, only, whereas the MWI deals with $T$-products of arbitrary factors. In particular $\partial^\nu W$ in [9] needs not to vanish and there are important applications with $\partial^\nu W \neq 0$, e.g. the field equation (sect. 4.1), non-Abelian matter currents (sect. 4.3), massive axial currents $j_A$ (sect. 5.1), etc.

Note that the QNC in terms of $T$-products (given in the first paper of [22]) does not agree with the QNC in terms of interacting fields (second paper): they normalize the interacting BRST-current differently. The second formulation fixes the BRST-current such that it is conserved up to terms $\sim \partial g$ where $g$ is the coupling 'constant', in particular it yields the values (160) for the constants $C_1, C_2$ and $C_3$ in (104)–(166). But the first formulation requires

$$C_1 = -1, \quad C_2 = 0, \quad C_3 = 1.$$
4.5.3 Massive gauge fields

To simplify the notations we consider the most simple non-Abelian model, namely three massive gauge fields, \( m_a > 0, a = 1, 2, 3 \) and no massless fields. However, as far as anomalies are absent and there is a solution for the below given requirements (A)-(F) and (209)-(211), (210)-(217) on the interaction \( \mathcal{L} \), our method applies also to general models with arbitrary numbers of massive and massless gauge fields and spinor fields. We will find the well-known result that with the fields \( A_\mu^a, u_a, \tilde{u}_a \) and \( \phi_a \) \( (a = 1, 2, 3) \) only, a consistent construction of the model is impossible, more precisely generalized perturbative gauge invariance \([147]\) for second order tree diagrams cannot be satisfied \([15]\). We will solve this problem in the usual way: besides the scalar fields \( \phi_a \) \( (a = 1, 2, 3) \), we introduce an additional real (free) scalar field \( H \), the 'Higgs field', with arbitrary mass \( m_H \geq 0 \), which is quantized according to

\[
(\Box + m_H^2)H = 0, \quad H^* = H, \quad \Delta_{H,H} = -D_{m_H} \quad (202)
\]

and \( H \) commutes with all other free fields.

To determine the interaction \( \mathcal{L}_0 \) we require the same properties (A)-(F) as in the massless case. The only modification is that \( \mathcal{L}_j \) is now a polynomial in \( A_\mu^a, u_a, \tilde{u}_a, \phi_a, a = 1, 2, 3 \) and \( H \) and internal derivatives of these symbols (again we solely admit monomials which have at least three factors). Proceeding as in the massless case we find the following particular solution of (A)-(F):

\[
\mathcal{L}_0 = g_0 \{ f_{abc} [A_{a\mu} A_{b\nu} \partial^\mu A_{c\nu}^\rho - u_a \partial^\mu \tilde{u}_b A_{c\mu}] \\
+ d_{abc} A_\rho^a \phi_b \partial_\mu \phi_c + e_{abc} A_\mu^a A_{b\mu} \phi_c + h_{abc} \tilde{u}_a u_b \phi_c \\
+ l_{ab} \left[ \frac{1}{m_b} (\partial H) A_{a\mu} \partial_\mu (\partial H) + A_{a\mu} \partial_\mu A_{b\mu} H \right] \\
- H \tilde{u}_a u_b - \frac{m_H^2}{2m_a m_b} H \phi_a \phi_b \} + p H^3 + t H^4, \quad (203)
\]

\[
\mathcal{L}_1' = g_0 \{ f_{abc} [A_{a\mu} u_b \mathcal{F}_{c\mu}^\nu - \frac{1}{2} u_a u_b \partial^\nu \tilde{u}_c] + 2 e_{abc} u_a A^{\nu}_{c\mu} \phi_c \\
+ d_{abc} [u_a \partial_\mu \phi_b \partial^\mu \phi_c + m_c A_{a\mu}^c \phi_b u_c] \\
+ l_{ab} \left[ \frac{1}{m_b} u_a (\partial^\nu H - H \partial^\nu \phi_b) + A_{a\mu} \phi_b H \right], \quad (204)
\]

\[
\mathcal{L}_2^{\nu\mu} = g_0 \frac{f_{abc}}{2} u_a u_b \mathcal{F}_{c\mu}^{\nu}, \quad (205)
\]

\[
\mathcal{L}_3 = 0, \quad (206)
\]

where \( f_{abc} \in \mathbb{R} \) is totally antisymmetric, \( l_{ab} \in \mathbb{R} \) is symmetric and

\[
d_{abc} = f_{abc} \frac{m_\mu^2 + m_\nu^2 - m_\rho^2}{2m_b m_c}, \quad e_{abc} = f_{abc} \frac{m_\mu^2 - m_\rho^2}{2m_c},
\]

\[
h_{abc} = f_{abc} \frac{m_\mu^2 + m_\nu^2 - m_\rho^2}{2m_c}, \quad p, t \in \mathbb{R}. \quad (207)
\]
The most general solution for $\mathcal{L}_0$ differs from the particular solution (203) by a coboundary $-is_0K_1$ and a divergence $\partial_\nu K^\nu_2$ as in (143). In addition one has the freedom to add terms with vanishing divergence to $\mathcal{L}_1$ and $\mathcal{L}_2$ as in (145). It is a peculiarity of the present model, that the total antisymmetry of $f_{abc}$ implies the Jacobi identity, so that we obtain

$$f_{abc} = \epsilon_{abc} (= \text{structure constant of } SU(2))$$

(208)

by absorbing a constant factor in $g_0$. So far we could set $l_{ab} = 0$, $p = 0$, $t = 0$, in other words the Higgs field $H$ is not needed to satisfy (A)-(F).

Now we come to an interesting complication of the massive case: the requirement (209)-(210) is satisfied we indeed obtain $\text{(209)-(210)}$ cannot be satisfied! To save generalized perturbative gauge invariance we need instead the following weaker condition: there exist $N_{j,k} \in \mathcal{P}_0$, $j, k = 0, 1, 2$ such that

$$G((\mathcal{L}_j, \mathcal{L}_{j+1}), f, \mathcal{L}_k g) + (-1)^{jk}G((\mathcal{L}_k, \mathcal{L}_{k+1})g, \mathcal{L}_j f) + s_0(N_{j,k})fg = -i[N_{j+1,k}(\partial_\nu f)g + (-1)^{jk}N_{j,k+1}(\partial_\nu g)]$$

(209)

where $N_{3,k} = 0 = N_{k,3}$, $k = 0, 1, 2$, and that the finite renormalization

$$T_2(\mathcal{L}_j f \otimes \mathcal{L}_k g) \to T_2^N(\mathcal{L}_j f \otimes \mathcal{L}_k g) \overset{\text{def}}{=} T_2(\mathcal{L}_j f \otimes \mathcal{L}_k g) + T_1(N_{j,k}fg)$$

(210)

maintains the permutation symmetry of $T_2$ and the normalization conditions (N1), (N2), (N0), and preserves the ghost number: $[Q_g, T_2^N(\mathcal{L}_j f \otimes \mathcal{L}_k g)] = (j + k)T_2^N(\mathcal{L}_j f \otimes \mathcal{L}_k g)$ where we take $g(\mathcal{L}_j) = j$ (143) into account (cf. (104)). This requirement can only be satisfied for $t = 0$ in (209). Hence, the $G$-terms in (209) are 4-legs terms and, therefore, we may restrict the $N_{j,k}$ to be 4-legs terms, too. In other words the $N_{j,k}$ are sums of monomials of degree four in $A_\mu^a, u_a, \bar{u}_a, \phi_a$, $a = 1, 2, 3$ and in $H$ (without any derivative), which are Lorentz tensors of rank $(j + k)$ and satisfy

$$g(N_{j,k}) = j + k, \quad N^*_j = -N_{j,0}, \quad N_{j,k} = (-1)^{jk}N_{j,k}$$

(211)

These properties imply

$$N_{2,1} = 0 = N_{1,2}, \quad N_{2,2} = 0, \quad N^\mu_1 = -N^{\mu\nu}_1$$

(212)

If (209)-(210) is satisfied we indeed obtain

$$[Q_0, T_2^N(\mathcal{L}_{j_1} f_1 \otimes \mathcal{L}_{j_2} f_2)] = -i\left(T_2^N(\mathcal{L}_{j_1+1} \partial_\nu f_1 \otimes \mathcal{L}_{j_2} f_2) + (-1)^{jk}T_2^N(\mathcal{L}_{j_1} f_1 \otimes \mathcal{L}_{j_2+1} \partial_\nu f_2)\right)$$

(213)

by using the master BRST-identity (132).

Turning to arbitrary orders $n \geq 3$ we look for a sequence of $T$-products $(T_n^N)_{n \in \mathbb{N}}$ (in the sense of sect. 2.2)

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- which satisfies the normalization conditions (N1), (N2) and (N0),
- which agrees as far as possible with the given sequence \( (T_n)_{n \in \mathbb{N}} \) that satisfies all normalization conditions (also (N3), (N) and the MWI (N)),
- and for which \( T^N_2(\mathcal{L}_j, \mathcal{L}_k) \) is connected with \( T^2(\mathcal{L}_j, \mathcal{L}_k) \) by (210) for all \( j, k \).

For this purpose let \( \mathcal{B} = \{ \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, B_1, B_2, ... \} \) be a (vector space) basis of \( \mathcal{P}_0 \). Due to causality \( (13) \) the renormalization terms \( T^N_1 (\eta_j, k, f, g) \) in (210) propagate to higher orders. More precisely we define

\[
T^N_{l+n}(B_{k_1} g_1 \otimes ... \otimes B_{k_l} g_l \otimes \mathcal{L}_{j_1} f_1 \otimes ... \otimes \mathcal{L}_{j_n} f_n) = \sum_{m=0}^{l} \sum_{\pi \in \mathcal{S}_n} \frac{1}{2^m m! (n-2m)!} \eta^{(j_1, ..., j_n)}(B_{k_1} g_1 \otimes ... \otimes B_{k_l} g_l \otimes N_{\pi(2m-1), j_n} \partial_{\pi(n-2m)} f_{\pi(2m-1)} f_{\pi(2m)} \otimes ... \otimes N_{\pi(2m+1), j_n} \partial_{\pi(n+2m+1)} f_{\pi(2m+1)} f_{\pi(2m+2)} \otimes ... \otimes \mathcal{L}_{\pi(n+1)} f_{\pi(n+1)}),
\]

(214)

where \( B_{k_1}, ..., B_{k_l}, g_1, ..., g_l \in \mathcal{B} \) and \( \eta^{(j_1, ..., j_n)} \) is the sign coming from the permutation of Fermi-operators in \( (\mathcal{L}_{j_1}, ..., \mathcal{L}_{j_n}) \) → \( (\mathcal{L}_{j_{e_1}}, ..., \mathcal{L}_{j_{e_n}}) \). We extend this definition to \( D(\mathbb{R}^4, \mathcal{P}_0) \otimes (l+n) \) by requiring linearity and (permutation) symmetry. Obviously this \( (T^N_n)_{n \in \mathbb{N}} \) solves our requirements. The formula (214) is a particular (simple) case of Theorem 3.1 in [32], which is a precise formulation of a formula given in [31]. For later purpose we mention that the \( T^N \)-products (214) satisfy (N5)(ghost), because the \( T \)-products do so. In particular we have

\[
T^N_{n+1}(k^\mu \partial_{\mu} g \otimes \mathcal{L}_{j_1} f_1 \otimes ... \otimes \mathcal{L}_{j_n} f_n) = \sum_{l=1}^{n} j_l T^N(\mathcal{L}_{j_1} f_1 \otimes ... \otimes \mathcal{L}_{j_l} f_l g \otimes ... \otimes \mathcal{L}_{j_n} f_n).
\]

(215)

But in general the \( T^N \)-products violate (N3) and the MWI (N).

To obtain generalized perturbative gauge invariance to orders \( n \geq 3 \) we additionally require

\[
G(\mathcal{L}_j, \mathcal{L}_{j+1} f, N_{k,l,t} g) + G^{(1)}(N_{k,l,t} g, \mathcal{L}_j f) = 0, \quad j, k, l \in \{0, 1, 2\}, \quad \forall f, g \in D(\mathbb{R}^4),
\]

(216)

and

\[
G^{(1)}(N_{k,l,t} f, N_{r,s,t} g) = 0, \quad k, l, r, s \in \{0, 1, 2\}, \quad \forall f, g \in D(\mathbb{R}^4).
\]

(217)

Then, applying the master BRST-identity (136) to \( [Q_0, T^N_n(\mathcal{L}_{j_i} f_1 \otimes ... \otimes \mathcal{L}_{j_n} f_n)]_\mp \) and taking (209), (216) and (217) into account, we find the wanted (modified) generalized perturbative gauge invariance [31]

\[
[Q_0, T^N_n(\mathcal{L}_{j_1} f_1 \otimes ... \otimes \mathcal{L}_{j_n} f_n)]_\mp = -t \sum_{l=1}^{n} (-1)^{j_1 + ... + j_{l-1}} T^N_n(\mathcal{L}_{j_l} f_1 \otimes ... \otimes \mathcal{L}_{j_{l+1}} \partial_{\nu} f_l \otimes ... \otimes \mathcal{L}_{j_n} f_n).
\]

(218)

\[\text{For } j_1 = ... = j_n = 0 \text{ this is the formulation of perturbative gauge invariance for massive fields in [15, 38] and [21].}\]
The particular case \( j = k = l = 0 \) of the requirements (209)-(210) and (216) has been worked out for general models in [15], [36] and [21]. We specialize these results to the present model:

1. The second order requirement (209)-(210) is very restrictive: its restriction to \( j = k = 0 \) is satisfied if and only if the following relations (a)-(e) hold:
   a) the masses agree:
   \[
   m \overset{\text{def}}{=} m_1 = m_2 = m_3. \tag{219}
   \]
   b) \( f_{abc} \) satisfies the Jacobi identity. (In our simple model this is already known (208), but in the general case the Jacobi identity is obtained only at this stage here, similar to the massless case.)
   c) The \( H \)-coupling parameters take the values
   \[
   l_{ab} = \frac{\kappa m}{2} \delta_{ab}, \quad t = 0, \tag{220}
   \]
   where \( \kappa \in \{-1, 1\} \) is an undetermined sign, and \( p \) is still free. In particular we see that the Higgs field (or another enlargement/modification of the model) is indispensable.
   d) The constants \( C_{A_a}, C_{\phi_a} \) and \( C_H \) have the values:
   \[
   C_{A_a} = -\frac{1}{2}, \quad C_{\phi_a} = -1, \quad \forall a, \quad C_H = -1. \tag{221}
   \]
   e) The polynomial \( N_{0,0} \) reads
   \[
   N_{0,0} = ig_0^2 \left\{ \frac{m_H^2}{16m^2} \left[ \sum_{a=1}^{3} (\phi_a^2)^2 + 2H^2 \sum_{a=1}^{3} (\phi_a^2) + \lambda H^4 \right] \delta(x_1 - x_2) \right\}, \tag{222}
   \]
   where \( \lambda \in \mathbb{R} \) is a constant which is undetermined so far.

2. The third order requirement (216) fixes the remaining free parameters \( p \) and \( \lambda \) (which are the parameters of the \( H \)-self-couplings): the particular case \( j = k = l = 0 \) of (216) holds true if and only if
   \[
   p = \frac{m_H^2}{4m}, \quad \lambda = -\frac{m_H^2}{16m^2}. \tag{223}
   \]

The \( C_A, C_u, C_{\phi^*} \) and \( C_H \)-terms in the tree-diagram part of \( T_2^N(L_0, L_0) \) and \( N_{0,0} \) correspond to the quadrilinear terms in the interaction Lagrangian of the conventional theory (the latter are also of order \( g_0^2 \)), cf. Remark (2) in sect. 4.5.1. With this identification our resulting interaction agrees precisely with the

\[\text{[For general models (210) is replaced by more complicated mass relations, see [15], [36].]}\]

\[\text{[We recall that } C_{u_a} = -1 \text{ has already been used in the derivation of the master BRST-identity.]}\]
the conventional Lagrangian) which belong to \(N\) to the contribution to the Euler-Lagrange equation of that quadratic terms. Here we have derived it in a completely different way (cf. \([13], [30], [21]\) and \([19]\)).

By inserting the explicit expressions \((222)\) into the definition \((130)\) of \(G^{(1)}(\ldots)\) we verify that the fourth order requirement \((217)\) holds true for \(k = l = r = s = 0\). We strongly presume that the requirements \((209)-(210), (216)\) and \((217)\) can be fulfilled for all values of \(j, k, l, r, s\) \(^34\). In the following we assume that this conjecture holds true and that the master BRST-identity is fulfilled. In particular we will use the modified generalized perturbative gauge invariance \((218)\).

From the time ordered products \((T^n_N)\) we obtain the corresponding anti-chronological products \((T^-_N)\) by \([19]\). In terms of the \(T^N\) - and \(T^-_N\)-products we construct the totally retarded products \((R^N_{n+1})_n\) \([15]\). The generating functional of the latter is the interacting field \(\Lambda^N_{g_L}\) or \(W^N_{g_L}(f)\) \([23]\). Similarly to the original \(R\)-products, the \(R^N\)-products have retarded support

\[
supp R^N_{n+1}(\ldots)(x_1,\ldots,x_n;x) \subset \{(y_1,\ldots,y_n,y)|y_1 \in y + \tilde{V}_-, \forall I\}. \quad (224)
\]

The proof of this support property uses only the causal or anti-causal factorization of the \(T^N\)- and \(T^-_N\)-products (see \([19]\)). The replacement \(\Lambda_{g_L} \rightarrow \Lambda^N_{g_L}(f)\) is a finite renormalization of the interacting field.

The field equation for \(\varphi^N_{g_L}\) (where \(\varphi \in P_0\) corresponds to a free field without any derivative) differs from the one of \(\varphi_{g_L}\): instead of \((94)\) the master Ward identity implies

\[
\left(\Box + m^2\right)\varphi^N_{g_L}(x) = -g(x)\left(\frac{\partial \mathcal{L}}{\partial \chi}\right)_{g_L}^N(x) + \left(-\frac{1}{2}(g(x))^2\left(\frac{\partial N_{0,0}}{\partial \chi}\right)_{g_L}^N(x) + \partial^\mu g(x)\left(\frac{\partial \mathcal{L}}{\partial (\partial^\mu \chi)}\right)_{g_L}^N(x)\right), \quad (225)
\]

where we use that \(N_{0,0}\) contains no derivatives. The additional term corresponds to the contribution to the Euler-Lagrange equation of that quadrilinear terms (in the conventional Lagrangian) which belong to \(N_{0,0}\). The contributions of the quadrilinear terms belonging to the \(C_A, C_u, C_\phi\)- and \(C_H\)-terms are contained in the first term on the r.h.s. of \((225)\). For example the contribution of the \(g_0^2 A^4\)-coupling (which belongs to the \(C_A\)-terms) is contained in the \(g_0 AF\)-term in \([19]\), because \(F_{g_L}\) \([15]\) has a nonlinear term \(\sim g_0(AA)_{g_L}\). The latter is indeed \(\sim C_A\) in our framework.

The construction of the interacting BRST-current \(\tilde{j}_{g_L}\) \((171)\) must be modified correspondingly. We define

\[
\tilde{T}^N_{n+1}(j^\mu f \otimes \mathcal{L}_{\mathcal{J}_1} f_1 \otimes \ldots \otimes \mathcal{L}_{\mathcal{J}_n} f_n) \overset{\text{def}}{=} [Q_0, T^N_{n+1}(k^\mu f \otimes \mathcal{L}_{\mathcal{J}_1} f_1 \otimes \ldots \otimes \mathcal{L}_{\mathcal{J}_n} f_n)]^\pm + i \sum_{i=1}^n (-1)^{j_1 + \ldots + j_{i-1}} T^N_{n+1}(k^\mu f \otimes \mathcal{L}_{\mathcal{J}_1} f_1 \otimes \ldots \otimes \mathcal{L}_{\mathcal{J}_i}^{N_{g_L}^i} \mathcal{J}_i f_i \otimes \ldots \otimes \mathcal{L}_{\mathcal{J}_n} f_n). \quad (226)
\]

\(^34\)Additionally we expect that these requirements determine \(N_{1,0}, N_{1,1}\) and \(N_{2,0}\) uniquely, similarly to \(N_{0,0}\).
\( \tilde{T}^N(j, \mathcal{L}_j, \ldots) \) has the same properties as \( \tilde{T}(j, \mathcal{L}_j, \ldots) \) \((160)\), in particular it satisfies causality \((13)\) and the normalization condition \((\text{NO})\). Hence, \( T(j, \mathcal{L}_j, \ldots) \to \tilde{T}^N(j, \mathcal{L}_j, \ldots) \) is a change of normalization. The divergence identity \((169)\) holds true also for \( (\tilde{T}^N, T_n) \), because \( T^N(k^\mu, \mathcal{L}_j, \ldots) \) fulfills \((\text{N5})(\text{ghost})\) \((213)\) and \( T^N(\mathcal{L}_j, \ldots) \) satisfies generalized perturbative gauge invariance \((218)\). Let \( \tilde{R}_{n+1}(\mathcal{L}_j, \ldots) \) be the \( R \)-product \((93)\) which is constructed in terms of \( T_k^N(\mathcal{L}_j, \ldots, \mathcal{L}) \) and \( \tilde{T}_k^N(j, \mathcal{L}_j, \ldots, \mathcal{L}) \), \( 1 \leq k \leq n \). Then we define \( \tilde{g}^N_{ \mu } \) similarly to \((171)\), replacing \( \tilde{R}_{n+1}(\ldots) \) by \( \tilde{T}^N_{n+1}(\ldots) \). Analogously to \((172)\), we then find that this interacting BRST-current is conserved up to terms \( \sim \partial g \), more precisely

\[
\tilde{\gamma}^N_{ \mu } = - \mathcal{L}^N_{1 \mu}(\partial \mu f).
\]

The interacting BRST-charge is now defined by

\[
Q^N_{ \mu } = \int d^4 x h^\mu(x) j^N_{ \mu } (x)
\]

instead of \((174)\). As in the massless case the construction can be done such that \( Q^N_{ \mu } \) fulfills \((173)\) and is nilpotent \((176)\) (see subsection 4.5.4).

We turn to the BRST-transformation of the interacting fields \( f \in \mathcal{F}(\mathcal{O}) \) \((156)\). Let \( \tilde{j}_0(W) = i\partial^\nu W^\nu \). The formula \((180)\) is violated by the non-vanishing of \( G((\mathcal{L}_0, \mathcal{L}_1)f, \mathcal{L}_0g) + (f \leftrightarrow g) \). It must be modified:

\[
\mathcal{L}_0^N = -i R_{n+1}^N ((\mathcal{L}_0)^\otimes_n; Wf) + n R_{n+1} \left( (\mathcal{L}_0)^\otimes_n; [G((\mathcal{L}_0, \mathcal{L}_1)f, Wf) + G((W, W')f, \mathcal{L}_0g)] \right),
\]

where for simplicity we assume that \( W' \in \mathcal{P}_0 \) does not contain any derivative of \( \phi \) or \( H.\) This assumption ensures

\[
G((W, W')f, N_{0,0}g) = 0, \quad \forall f, g \in \mathcal{D}(\mathbb{R}^4),
\]

as can be seen by inserting the explicit expression \((222)\) for \( N_{0,0} \) into the definition \((133)\) of \( G(\ldots) \). Moreover, one finds

\[
G(1)(N_{0,0}g, Wf) = 0, \quad \forall f, g \in \mathcal{D}(\mathbb{R}^4),
\]

by inserting \((222)\) into the definition \((130)\) of \( G(1) \). Analogously to the derivation of \((213)\), the proof of \((224)\) is a straightforward application of the master BRST-identity to \([Q_0, T_{n+1}^N(Wf \otimes (\mathcal{L}_0g)^\otimes_1)]\), which uses \((203), (216), (217), (230)\) and \((231)\). The resulting formula is then translated into an identity for the \( R \)-products \((93)\). As in the massless case, \((229)\) yields the BRST-transformation

\[
\delta(W^N_{g\mathcal{L}}(f)) = [Q^N_{ \mu } W^N_{g\mathcal{L}}(f)]_{ \mu } = -i W^N_{g\mathcal{L}}(\partial \mu f) + i[G((\mathcal{L}_0, \mathcal{L}_1)f, Wf) + G((W, W')f, \mathcal{L}_0g)]^N_{g\mathcal{L}}, \quad f \in \mathcal{D}(\mathcal{O})
\]

(see subsection 4.5.4).
4.5.4 The interacting BRST-charge

In this subsection we summarize the construction of the interacting BRST-charge given in [7]. With that we prove the properties (173) and the nilpotency. Finally we show that the identity (180) ((224) resp.) implies the BRST-transformation formula (181) ((229) resp.) for the interacting fields. We deal with massive gauge fields, however, the massless case is included: in all formulas it is allowed to set \( m = 0, \phi \equiv 0 \) and \( H \equiv 0 \) (which replaces \( T^N \) by \( T \) etc.).

We assume that the double cone \( O \) is centered at the origin. Let \( r \) be the diameter of \( O \). The question is, how to choose \( g \) and \( h^\mu \) such that \( Q_{gL}^N \) satisfies (173) and is nilpotent. As explained in [7] it is hard to avoid a volume divergence in \( \int d^4 x h^\mu(x) \tilde{\gamma}_{gL}^N(x) \), at least for massless fields. To get rid of this problem we proceed as in [7]: we embed our double cone \( O \) isometrically into the cylinder \( \mathbb{R} \times C_L \) (the first factor denotes the time axis), where \( C_L \) is a cube of length \( L, \quad L \gg r \), with metallic boundary conditions for each free gauge fields \( A_a (a = 1, \ldots, N) \), and Dirichlet boundary conditions for the free ghost fields \( u_a, \tilde{u}_a \) and the free bosonic scalar fields \( \phi_a \) and \( H \). If we choose the compactification length \( L \) big enough, the physical properties of the local algebra \( F(O) \) are unchanged.

Following [7] we choose the switching function \( g \) to fulfil

\[
g(x) = 1 \quad \forall x \in O \cup \{(x_0, \vec{x}) \mid |x_0| < \epsilon \} \quad (r \gg \epsilon > 0)
\]  

on \( \mathbb{R} \times C_L \) and to have compact support in timelike directions. In addition let \( h^\mu \) be such that

\[
h^\mu(x) \equiv \delta_0^\mu h(x_0), \quad \text{where} \quad h \in \mathcal{D}([\epsilon, \epsilon], \mathbb{R}), \quad \int dx_0 h(x_0) = 1.
\]

Then

\[
Q_{gL}^N \equiv \int dx_0 h(x_0) \int_{C_L} d^3 x \tilde{\gamma}_{gL}^N(x)
\]

is well-defined, because \( (x_0, \vec{x}) \rightarrow h(x_0) \) is an admissible test function on \( \mathbb{R} \times C_L \). \( (Q_{gL}^N)_0 = Q_0 \) holds true, since \( (\tilde{\gamma}_{gL}^N)_0 = j_\mu \) is conserved. From the conservation of the interacting current, \( \partial^\mu \tilde{j}_\mu^N_{gL}(x) = 0 \) for \( x \in [-\epsilon, \epsilon] \times C_L \), we conclude that \( Q_{gL}^N \) is independent from the choice of \( h \). By (N2) and by the fact that \( h \) and \( g \) are real-valued we obtain \( Q_{gL}^{N, *} = Q_{gL}^N \).

It remains to prove the nilpotency. For this purpose we first show

\[
Q_{gL}^N = Q_0 + L_{1gL}^N (H \partial_x g)
\]

where

\[
H(x) \equiv \tilde{H}(x_0) \equiv \int_{-\infty}^{x_0} dt [-h(t) + h(t - a)]
\]
and \( a \in \mathbb{R} \) is such that the support of \((x_0, \bar{x}) \to h(x_0 - a)\) is earlier than the support of \( g \):

\[
x_0 < y_0, \quad \forall x_0 \in \text{supp } h(\cdot - a) \land \forall y_0 \text{ with } \exists \bar{y} \in C_L \text{ with } (y_0, \bar{y}) \in \text{supp } g.
\]

(238)

In particular we will need

\[
H(y)\partial g(y) = \partial g(y), \quad \forall y \in ((\text{supp } H \cup \mathcal{O}) + \bar{V}_-), \quad \mathcal{O} \cap (\text{supp } (H\partial g) + \bar{V}_-) = \emptyset
\]

(239)

(240)

and

\[
\partial H(y)\partial g(y) = 0.
\]

(241)

**Proof of (236):** From our definitions we immediately obtain

\[
Q_{gL}^N = \int_{\mathbb{R} \times C_L} d^4x \tilde{\gamma}_{\mu gL}^N(x) [-\partial_\mu H(x) + g_{\mu 0} h(x_0 - a)].
\]

(242)

Due to (238) and the retarded support of the \( R \)-products (224) we have

\[
\text{supp } (\tilde{\gamma}_{\mu gL}^N - j_\mu) \cap \text{supp } ((x_0, \bar{x}) \to h(x_0 - a)) = \emptyset.
\]

(243)

So the contribution of \( g_{\mu 0} h(x_0 - a) \) to (242) is \( Q_0 \), and by inserting (227) into the \( \partial_\mu H \)-term we obtain the assertion (236).

The formula (236) manifestly shows that \( Q_{gL}^N \) converges to \( Q_0 \) in the adiabatic limit \( g(x) \to 1 \), \( \forall x \), provided this limit exists. For pure massive theories this limit exists indeed in the strong sense [20]. So, in the adiabatic limit of a pure massive gauge theory, one has the simplification that the BRST-cohomology is given in terms of \( Q_0 \) (cf. (139)-(140) and [16]).

In the following proofs we will use Proposition 2 of [7], which is a formula for the (anti-)commutator of two interacting fields:

\[
[W_{gL}^N(f), V_{gL}^N(h)]_\mp = R_{gL}^N(W, V)(f, h) \mp R_{gL}^N(V, W)(h, f),
\]

(244)

where

\[
R_{gL}^N(W, V)(f, h) \overset{\text{def}}{=} - \sum_{n=0}^{\infty} \frac{i^n}{n!} R_{n+2}^N ((gL)^{\otimes n} \otimes Wf; Vh).
\]

(245)

Due to the retarded support (224) of the \( R^N \)-products we call \( R_{gL}^N(W, V)(f, h) \) the retarded part (and \( \mp R_{gL}^N(V, W)(h, f) \) the advanced part) of \([W_{gL}^N(f), V_{gL}^N(h)]_\mp \).

---

36 This formula, the retarded support of the \( R \)-products (224) and some further, quite obvious requirements can be viewed as the defining properties of the retarded products. They determine a direct construction of the \( R_{n+1} \) (\( n \in \mathbb{N}_0 \)) by induction on \( n \). If wanted, the \( T \)-products can then be obtained by reversing (93), see [37], the second paper of [8] and [10].
Using (246) and (245) we obtain for the first one the identity (229) gives

\[ Q_0, R_{n+1}^N ((\mathcal{L}g)^{\otimes n}; \mathcal{L}_1^\mu f) \]_+ = \begin{align*}
& -inR_{n+1}^N ((\mathcal{L}g)^{\otimes (n-1)} \otimes \mathcal{L}_1^\rho \partial_\rho g; \mathcal{L}_1^\mu f) \\
& -iR_{n+1}^N ((\mathcal{L}g)^{\otimes n}; \mathcal{L}_2^\mu \partial_\rho f).
\end{align*}

(246)

Due to \( \mathcal{L}_2^\mu = -\mathcal{L}_2^\mu \) we may require

\[ T_1^N (\mathcal{L}, ..., \mathcal{L}, \mathcal{L}_2^\mu) = -T_1^N (\mathcal{L}, ..., \mathcal{L}, \mathcal{L}_2^\mu), \quad \forall \, \mu \in \mathbb{N}. \]

(247)

This is an additional normalization condition, which is compatible with the other normalization conditions. Similarly to (N2), it can be satisfied by antisymmetrizing in \( \mu \leftrightarrow \rho \) a \( T_1^N (\mathcal{L}, ..., \mathcal{L}, \mathcal{L}_2^\mu) \) which satisfies the other normalization conditions. The corresponding \( R_{n+1}^N (\mathcal{L}, ..., \mathcal{L}; \mathcal{L}_2^\mu) \) is then also antisymmetric in \( \mu \leftrightarrow \rho \) and, hence, we have \( \mathcal{L}_2^\mu = -\mathcal{L}_2^\mu \).

By means of (246) and \( Q_0^N = 0 \) we find

\[ 2(Q_{g^N})^2 = [Q_{g^N}, Q_{g^N}]_+ = 2[Q_0, \mathcal{L}_{1g^N}^N (H \partial_\mu g)]_+ + [\mathcal{L}_{1g^N} (H \partial_\mu g), \mathcal{L}_{1g^N}^N (H \partial_\mu g)]_+. \]

(248)

Using (246) and (245) we obtain

\[ 2[Q_0, \mathcal{L}_{1g^N}^N (H \partial_\mu g)]_+ = -2i\mathcal{L}_2^g \mathcal{L}^\mu (\partial_\rho (H \partial_\rho g)) - 2\mathcal{L}_{1g^N}^N (\mathcal{L}_{1g^N}^\nu (H \partial_\nu g), \mathcal{L}_{1g^N}^N (H \partial_\mu g))_+. \]

(249)

In the first term the \( (\partial_\rho H)(\partial_\rho g) \)-part vanishes by (241) and the \( H \partial_\rho g \)-part is zero because of (247). In the remaining second term we first take (243) into account and then apply (244)

\[ 2[Q_0, \mathcal{L}_{1g^N}^N (H \partial_\mu g)]_+ = -2\mathcal{L}_{1g^N}^N (\mathcal{L}_{1g^N}^\nu (H \partial_\nu g), \mathcal{L}_{1g^N}^N (H \partial_\mu g))_+ \]

(250)

Inserting this into (248) we see that \( Q_{g^N}^N \) is in fact nilpotent. \( \Box \)

Proof of the formula

\[ [Q_{g^N}, W_{g^N}^N (f)]_+ = -iW_{g^N}^N (\partial_\nu f) + i(G((\mathcal{L}, \mathcal{L}_1)g, Wf) + G((W, W')f, \mathcal{L}g))^N_{\mathcal{L}} \]

(251)

for the BRST-Transformation of the interacting fields: let \( f \in \mathcal{D}(O) \). According to (251) we have to compute two terms:

\[ [Q_{g^N}, W_{g^N}^N (f)]_+ = [Q_0, W_{g^N}^N (f)]_+ + [\mathcal{L}_{1g^N}^N (H \partial_\mu g), W_{g^N}^N (f)]_+. \]

(252)

For the first one the identity (229) gives

\[ [Q_0, W_{g^N}^N (f)]_+ = -iW_{g^N}^N (\partial_\nu f) + i(G((\mathcal{L}, \mathcal{L}_1)g, Wf) + G((W, W')f, \mathcal{L}g))^N_{\mathcal{L}} - R_{1g^N}^N (\mathcal{L}_{1g^N}^\nu, W)(\partial_\nu g, f), \]

(253)
where we have used (245).

We turn to the other term in (252) and apply (244):

\[
\left[ L^N_{\nu g}(H\partial_\nu g), W^N_{\nu g}(f) \right] = R^N_{\nu g}(L^\nu, W)(H\partial_\nu g, f) \mp R^N_{\nu g}(W, L^\nu)(f, H\partial_\nu g).
\] (254)

The second term vanishes due to the support properties (224) and (240). Because of (239) we may omit the factor \( H \) in the first term. Hence, we find that \( [L^N_{\nu g}(H\partial_\nu g), W^N_{\nu g}(f)] \) cancels out with the last term in (253) and it remains the assertion (251).

We summarize: the following conditions on an interaction \( L_0 \) are sufficient for our local construction of observables.

- \( L_0 \) fulfills the conditions (A)-(F) given at the beginning of sect. 4.5.1. or 4.5.3 resp..
- There exist sums \( N_{j,k} \in \mathcal{P}_0 \) of monomials of degree four which satisfy (209)-(211) and (210)-(217).
- There is no restriction coming from ghost number conservation (N5) (ghost). In the relevant cases, this normalization condition has common solutions with (N0)-(N3), see appendix B.1.
- Simultaneously with (N0)-(N3) and (N5) (ghost), the master BRST-identity can be fulfilled for

\[
[Q_0, T_n(L_0, ..., L_0)] \quad \text{and} \quad [Q_0, T_n(L_1, ..., L_0)]
\] (255)

to all orders \( n \in \mathbb{N} \).

The conditions mentioned so far suffice for the construction of the BRST-charge \( Q^N_{\nu g} \). In our derivation of the BRST-transformation formula (232) of \( W^N_{\nu g} \) (\( W \in \mathcal{P}_0 \) arbitrary) we have additionally used:

- the master BRST-identity for

\[
[Q_0, T_n(W, L_0, ..., L_0)], \quad \forall n \in \mathbb{N},
\] (256)

- the properties (237) and (231) of \( N_{0,0} \).

5 Taking anomalies into account

5.1 General procedure and axial anomaly

We recall that we understand by the expression 'anomaly' any term that violates the master Ward identity (N) and cannot be removed by an admissible finite renormalization of the \( T \)-products. The aim of this section is to take anomalies into account in the formulation of the MWI. This has consequences for the normalization condition (\( \tilde{N} \)): We want to normalize \( T((\partial^n V) \otimes W_1 f_1 \otimes ...) \) such that (30) holds true: in the special case \( W = 1, V, W_1, ..., W_n \in \mathcal{P}_0 \) the r.h.s. of (\( \tilde{N} \)) should agree with the r.h.s. of (N). Hence, we will take the anomalies into account also in (\( \tilde{N} \)).
We proceed inductively with respect to the order $n$. Since we are not aware of a general proof that second order loop diagrams (i.e. $n = 1$ in (N)) are anomaly-free\footnote{In our proof of charge conservation (NS) (charge) (which is given in appendix B of \cite{1}) vacuum polarization plays an exceptional role, even to second order: our general argumentation does not yield $\partial_{\nu}T_2(f_\mu(x),\nu(y)) = 0$, an explicit calculation was necessary.} we start with that case. We set
\begin{align*}
-a^{(2)_\nu}_{\nu,V,W}(g,f) & \overset{\text{def}}{=} T_2(V\partial^{\nu}g \otimes Wf) + T_2((\partial^{\nu}V)g \otimes Wf) \\
+i \sum_{\chi,\psi \in \mathcal{G}} (\pm)T_1 \left( \Delta^{\nu}_{\chi,\psi} \left( \frac{\partial V}{\partial \chi} g, \frac{\partial W}{\partial \psi} f \right) \right), & \quad V, W \in \tilde{\mathcal{P}}_0. \quad (257)
\end{align*}

For later purpose we let $V, W \in \tilde{\mathcal{P}}_0$ (not only $V, W \in \mathcal{P}_0$ as in the MWI (N)). Due to causal factorization of the $T$-products we know that $a^{(2)_\nu}_{\nu,V,W}(g,f)$ is local. Therefore, there exists a unique $b^{(2)_\nu}_{\nu,V,W}(g,f) \in D(\mathbb{R}^4, \mathcal{P}_0)$ with\footnote{Remember $T_1(D(\mathbb{R}^4, \mathcal{P}_0)) = T_1(D(\mathbb{R}^4, \tilde{\mathcal{P}}_0))$ and that $T_1$ is injective on $D(\mathbb{R}^5, \mathcal{P}_0)$.}
\begin{align*}
a^{(2)_\nu}_{\nu,V,W}(g,f) & = T_1 \left( b^{(2)_\nu}_{\nu,V,W}(g,f) \right). \quad (258)
\end{align*}

Let us now assume that we have already defined $a^{(m+1)}_{V,W_1,\ldots,W_m}$ and $b^{(m+1)}_{V,W_1,\ldots,W_m} \in D(\mathbb{R}^4, \mathcal{P}_0)$ for all $m < n$. We then set
\begin{align*}
-a^{(n+1)_\nu}_{\nu,V,W}(g,f_1,\ldots,f_n) & \overset{\text{def}}{=} T_{n+1}(V\partial^{\nu}g \otimes W_1f_1 \otimes \cdots \otimes W_nf_n) \\
+i \sum_{m=1}^{n} \sum_{\chi,\psi \in \mathcal{G}} (\pm)T_n \left( \Delta^{\nu}_{\chi,\psi} \left( \frac{\partial V}{\partial \chi} g, \frac{\partial W_m}{\partial \psi} f_m \right) \otimes W_1f_1 \otimes \cdots \hat{m}_i \cdots \otimes W_nf_n \right) \\
+ \sum_{k=1}^{n-1} \sum_{1 \leq m_1 < \cdots < m_k \leq n} (\pm)T_{n+1-k} \left( b^{(k+1)_\nu}_{\nu,V,W_{m_1},\ldots,W_{m_k}}(g,f_{m_1},\ldots,f_{m_k}) \right. \\
& \left. \otimes W_1f_1 \otimes \cdots \hat{m}_1 \cdots \hat{m}_k \cdots \otimes W_nf_n \right), \quad (259)
\end{align*}
where $V, W_1,\ldots,W_n \in \tilde{\mathcal{P}}_0$ and the possible signs $(\pm)$ come from the permutation of Fermi operators. By causal factorization and the definition of the $(b^{(k+1)}_{\nu,V,W})_{k<n}$ we conclude that $a^{(n+1)}_{\nu,V,W}(g,f_1,\ldots,f_n)$ is local and, hence, that there exists a unique $b^{(n+1)}_{\nu,V,W}(g,f_1,\ldots,f_n) \in D(\mathbb{R}^4, \mathcal{P}_0)$ with
\begin{align*}
a^{(n+1)}_{\nu,V,W}(g,f_1,\ldots,f_n) & = T_1 \left( b^{(n+1)}_{\nu,V,W}(g,f_1,\ldots,f_n) \right). \quad (260)
\end{align*}

Obviously
\begin{align*}
b^{(n+1)}_{\nu,V,W} : D(\mathbb{R}^4, \tilde{\mathcal{P}}_0)^{\otimes n+1} & \to D(\mathbb{R}^4, \mathcal{P}_0), \\
Vg \otimes W_1f_1 \otimes \cdots \otimes W_nf_n & \to b^{(n+1)}_{\nu,V,W}(g,f_1,\ldots,f_n) \quad (261)
\end{align*}
is linear and symmetrical in all factors except the first one. As a consequence of the normalization condition (N3) the $b_{V,W_1,...,W_n}^{(n+1)}$, $V,W_1,...,W_n \in \mathcal{P}_0$, $n$ fixed, are not independent. For example let $V$ be a sub-polynomial of $V'$ and $W_k$ a sub-polynomial of $W_k'$, $\forall k = 1,...,n$ ($V = V'$ and $W_k = W_k'$ is admitted). Then $b_{V,W_1,...,W_n}^{(n+1)} \neq 0$ implies $b_{V',W_1',...,W_n'}^{(n+1)} \neq 0$. However, the $b_{V,W_1,...,W_n}^{(n+1)}$ are independent for different $n$, because the violations of the MWI coming from sub-diagrams are taken into account in (259) by the terms \( \sum_{k=1}^{n-1} \sum (\pm) T_{n+1-k}(b_{V,W_1',...,W_n'}^{(k+1)}(\cdots) \otimes \cdots) \).

Obviously the $b^{(k)}$, (259)-260] depend on the normalization of the $T$-products. We assume that the latter fulfil (N0)-(N3) and (N) in the following modified form:

\[
(N) \quad T_{n+1}((\partial^\nu V)W g \otimes W_1 f_1 \otimes \cdots \otimes W_n f_n) =
T_{n+1}((\partial^\nu V)W g \otimes W_1 f_1 \otimes \cdots \otimes W_n f_n)
+i \sum_{m=1}^{n} \sum_{\nu \in \mathcal{G}} \sum_{s \in \mathcal{P}_0} \sum_{r \in \mathcal{P}_0} (\pm) T_n \left( \Delta_{\nu}^{\nu} \left( \frac{\partial^\nu}{\partial \nu} W g, \frac{\partial^m}{\partial \nu} f_m \right) \otimes W_1 f_1 \otimes \cdots \otimes W_n f_n \right)
+ \sum_{k=1}^{n} \sum_{m_1 < \cdots < m_k} (\pm) T_{n+1-k} \left( b_{V,W_1',...,W_n'}^{(k+1)} (g, f_{m_1}, ..., f_{m_k}) W \otimes W_1 f_1 \otimes \cdots \otimes W_n f_n \right),
\]

(262)

where $V,W,W_1,...,W_n \in \mathcal{P}_0$. Note that the sum over $k$ in the last term runs here up to $n$. Setting $W = 1$ and using the definition (259)-260, we in fact obtain (30) (even for $V,W_1,...,W_n \in \mathcal{P}_0$), which is the main reason for this modification of (N). In sect. 2 this implication relied on the validity of the MWI (N). This assumption is not needed here to get (30).

Also the modified (N) fixes the normalization of the $T$-products of symbols with external derivatives in terms of $T$-products without external derivatives, namely by the following recursive procedure: For a monomial $W = \prod_{r \in \mathcal{P}_0} \partial^{a^{(s)}} \varphi_r \in \mathcal{P}_0$, $\varphi_r \in \mathcal{P}_0$, $a^{(s)} \in (N_0)^4$, we define $|W| = \sum_s |a^{(s)}|$ where $|a^{(s)}| = a_0^{(s)} + \cdots + a_4^{(s)}$. Let the normalization of $T_n(W_1,...,W_n)$ with $|W_1| + \cdots + |W_n| = 0$ (i.e. $W_1,...,W_n \in \mathcal{P}_0$) be given for all $n \in \mathbb{N}$. Then the determination of the $b_{W_1,...,W_n}^{(n)}$ and of the normalization of $T_n(W_1,...,W_n)$ with $|W_1| + \cdots + |W_n| > 0$ goes in a double inductive way: one makes a first induction with respect to the order $n$ and for each fixed $n$ a second induction with respect to $|W_1| + \cdots + |W_n|$. More precisely let $T_l(W_1,...,W_l)$ and $b_{W_1,...,W_l}^{(l)}$ be given for all $l \leq n$ and $W_1,...,W_l \in \mathcal{P}_0$, and also for $l = n + 1$ if $|W_1| + \cdots + |W_{n+1}| < d$ ($d \in \mathbb{N}$). Then we determine by (N) the normalization of the $T_{n+1}(W_1,...,W_{n+1})$ with $|W_1| + \cdots + |W_{n+1}| = d$ (this

\footnote{To see this we consider the $(V',W_1',...,W_n')$-diagrams in which the additional factors of $V',W_1',...,W_n'$ are external legs. By amputating these external legs we obtain all $(V,W_1,...,W_n)$-diagrams. (N3) requires that the non-amputated and amputated diagrams are equally normalized.}
step does not take place for $d = 0$, because $T_{n+1}$ is given in that case. More precisely we use $\langle N \rangle$ for $|V| + |W| + |W_1| + ... + |W_n| = d - 1$. Note that all $b^{(k+1)}$ and all $T$-products which appear in this case on the r.h.s. of $\langle N \rangle$ are inductively given. Finally, from (257)-(260) we obtain the $b^{(n+1)}_{V,W_1,...,W_n}$ with $|V| + |W_1| + ... + |W_n| = d$. Again we point out that thereby all terms which appear on the r.h.s. of (259) are inductively known. Starting this procedure with restricted $T$-products $(T_n|D(\mathbb{R}^4,\mathcal{P}_0)\otimes\mathbb{R})_{n\in\mathbb{N}}$ which satisfy (NO)-(N3) (such $T$-products exist [4]) we end up with $T$-products which fulfil (NO)-(N3) and the modified $\langle N \rangle$.

To formulate the modified MWI we specialize to the case $V, W_1, ..., W_n \in \mathcal{P}_0$.

Let us consider the set $\mathcal{T}$ of all $T$-products $(T_n)_{n\in\mathbb{N}}$ which satisfy the requirements of sect. 2.2 (in particular causality and the normalization conditions (NO)-(N3)) and the modified $\langle N \rangle$. We now define

$$A((T_n)_{n\in\mathbb{N}}) \overset{\text{def}}{=} \langle b^{(n+1)} \rangle_{n\in\mathbb{N}}, \quad \forall(T_n)_{n\in\mathbb{T}},$$

where $\langle b^{(n+1)} \rangle$ is the restriction of $b^{(n+1)}$ (261) to $D(\mathbb{R}^4,\mathcal{P}_0)\otimes\mathbb{R}^{n+1}$. The image $A(\mathcal{T})$ of this map is model dependent and it is usually hard work to get information about $\langle N \rangle$. If the zero-sequence (i.e. $0 \overset{\text{def}}{=} (0,0,...)$) is an element of $A(\mathcal{T})$, which means that the model is anomaly-free, we are in the situation of sect. 2.4: the MWI is then the normalization condition which forbids all $(T_n)_{n\in\mathbb{N}}$ which are not an element of $A^{-1}(0)$. If $0 \notin A(\mathcal{T})$ we choose a suitable (usually as simple as possible) $b \in A(\mathcal{T})$ and the master Ward identity is then the normalization condition that solely sequences $(T_n)_{n\in\mathbb{N}} \in A^{-1}(b)$ are allowed.

We illustrate this by the example of the axial anomaly. Let $\mathcal{P}_\psi$ be the linear space which is generated by $\mathcal{L} \overset{\text{def}}{=} \mathcal{A}_\mu j^\mu, \mathcal{J}_A, \mathcal{J}_\tau$ (cf. (83)) and all sub-monomials thereof. According to Bardeen [4] the most simple $b \in A(\mathcal{T})$ reads

$$b^{(n+1)}_{V,W_1,...,W_n} = 0, \quad \forall n \neq 3, \quad V, W_1, ..., W_n \in \mathcal{P}_\psi,$$

$$b^{(3)}_{j_{A\nu},j^\nu,j_{\mu_1},j_{\mu_2}}(g,f_1,f_2) = CE_{\mu_1\mu_2\rho}\tilde{\psi}(\partial_\rho f_1)\partial_\tau f_2,$$

$$b^{(3)}_{\mathcal{J}_{A\nu}j^\nu\mathcal{L}}(g,f_1,f_2) = CE_{\mu_1\mu_2\rho}\tilde{\psi}(\partial_\rho f_1 A_{\mu_1})\partial_\tau f_2,$$

$$b^{(3)}_{\mathcal{J}_{A\nu}j^\nu\mathcal{L}}(g,f_1,f_2) = CE_{\mu_1\mu_2\rho}\tilde{\psi}(\partial_\rho f_1 A_{\mu_1})\partial_\tau f_2,$$

and $b^{(3)}_{V,W_1,W_2} = 0$ for all other $(V,W_1,W_2) \in (\mathcal{P}_\psi)^3$, where $C$ is a well-known, fixed, complex number. Then particular cases of the MWI read

$$T_{n+1}(j^\mu \partial_\mu f \otimes \mathcal{L}g_1 \otimes ... \otimes \mathcal{L}g_n) = 0,$$

$$T_{n+1}(j^\mu \partial_\mu f \otimes \mathcal{L}g_1 \otimes ... \otimes \mathcal{L}g_n) = 2m T_{n+1}(j^\mu \partial_\mu f \otimes \mathcal{L}g_1 \otimes ... \otimes \mathcal{L}g_n),$$

$$\sum_{1 \leq m_1 < m_2 \leq n} C E_{\mu_1\mu_2\rho} T_{n-1}(f \partial_\rho (g_{m_1} A_{\mu_1}) \partial_\tau (g_{m_2} A_{\mu_2}) \otimes \mathcal{L}g_1 \otimes ... \mathcal{L}g_n),$$

(265)
which imply
\[ j^\mu_g \partial_\mu f = 0, \]
\[ -j^\mu_A g \partial_\mu f = 2m j^\mu g \partial_\mu f - g^2 C \epsilon^{\mu_1 \mu_2 \rho \tau} (F_{\rho \mu_1} F_{\tau \mu_2}) g \mathcal{L}(f), \tag{266} \]
where we assume \( g(x) = g_0 = \text{const.}, \forall x \in \text{supp } f. \)

Non-vanishing anomalies \( b^{(m+1)} \mathcal{W}_1, \ldots, \mathcal{W}_m \) are not an obstacle to fulfill the normalization condition \((N4)\) and hence the field equation \((94)\) (see sect. 4.1), because \((21)\) still solves \((N4)\) \((20)\). But the axial anomaly appears as an additional term in the charge conservation \((N5)\) \((96)\) and in the generalized perturbative gauge invariance \((N6)\) \((147)\) and hence also in the master BRST-identity, if axial fermions are present. However, for the non-Abelian gauge models studied in sect. 4.5, we expect that the master BRST-identity can be satisfied in the relevant cases \((255)\) and \((256)\), and that therefore our local construction of observables works. But this remains to be proved.

5.2 Energy momentum tensor: conservation and trace anomaly

We follow the procedure in \((29)\). Classically the canonical energy momentum tensor is the Noether current belonging to translation invariance (in time and space). Turning to QFT we consider a real, free, scalar field \( \phi \) of mass \( m \geq 0 \). (In the formalism of appendix A we set \( \varphi \overset{\text{def}}{=} \chi \overset{\text{def}}{=} \phi \) and choose \( \epsilon = 1 \).) The free canonical energy momentum tensor reads
\[ \Theta^{\mu \nu}_{\text{can}} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} g^{\mu \nu} \partial_\rho \phi \partial_\rho \phi + \frac{1}{2} g^{\mu \nu} m^2 \phi^2, \tag{267} \]
and this tensor is conserved due to the Klein-Gordon equation: \( \partial_\mu \Theta^{\mu \nu}_{\text{can}} = 0. \)

Now we add an interaction of the form
\[ \mathcal{L} = \lambda \phi^4. \tag{268} \]

The interacting canonical energy momentum tensor is not simply the interacting field belonging to \( \Theta^{\mu \nu}_{\text{can}} \), it has an additional term
\[ \Theta^{\mu \nu}_{\text{can}} g \mathcal{L}(f) = \Theta^{\mu \nu}_{\text{can}} g \mathcal{L}(f) + g^{\mu \nu} \mathcal{L}_g \mathcal{L}(gf). \tag{269} \]

Let \( W_1, \ldots, W_n \) be polynomials in \( \phi \) (without any derivative). Applying twice the definition \((253)\), \((260)\) we obtain the relation
\[ T(W_1 g_1 \otimes \ldots \otimes W_n g_n \otimes \Theta^{\mu \nu}_{\text{can}} \partial_\mu f) = \]
\[ -i \sum_{k=1}^n T(W_1 g_1 \otimes \ldots \otimes (\partial^\nu W_k) g_k f \otimes \ldots \otimes W_n g_n) - \mathcal{A}_1(f, g_1, \ldots, g_n) = \]
\[ \mathcal{A}_1(f, g_1, \ldots, g_n) + i \sum_{k=1}^n \left( T(W_1 g_1 \otimes \ldots \otimes W_k \partial^\nu (g_k f) \otimes \ldots \otimes W_n g_n) + \mathcal{A}_2(k(f, g_1, \ldots, g_n) \right), \tag{270} \]
\[
A_1(f, g_1, \ldots, g_n) \overset{\text{def}}{=} \sum_{j=1}^{n} \sum_{m_1 < \ldots < m_j} T(b^{(j+1)}_{\phi \Theta, g_1 \ldots g_n}, W_{m_1} \ldots W_{m_j} (f, g_{m_1}, \ldots, g_{m_j})) \\
\otimes W_1 g_1 \otimes \ldots \hat{m}_1 \ldots \hat{m}_j \ldots \otimes W_n g_n,
\]
\[
A_{2,k}(f, g_1, \ldots, g_n) \overset{\text{def}}{=} \sum_{l=1}^{n-1} \sum_{m_1 < \ldots < m_l (m_j \neq k)} T(b^{(l+1)}_{W_{k_1} \ldots W_{m_1} \ldots W_{m_l} (g_{k_1} g_1, \ldots, g_{m_l})}) \\
\otimes W_1 g_1 \ldots \hat{m}_1 \ldots \hat{m}_l \ldots \otimes W_n g_n.
\]

In [29] it is shown that there exists a normalization (which is compatible with (N0) - (N3)) such that \(-A_1 + i \sum_{k=1}^{n} A_{2,k} = 0\). In the following we use this normalization. Then the identity (271) and (263) imply
\[
\Theta_{\mu \nu}^{\text{can}, g \mathcal{L}} (\partial_\mu f) = -\mathcal{L}_{g \mathcal{L}} ((\partial^\mu g) f).
\]

The energy momentum tensor is only conserved in space-time regions in which the coupling 'constant' \(g\) is constant, in agreement with the fact that translation invariance is broken by a non-constant \(g\).

Unfortunately the trace of the canonical energy momentum tensor does not vanish, even for free fields. Following [29] and references cited therein, we assume \(m = 0\) (and still \(\mathcal{L} = \lambda \phi^4\)) and introduce the improved energy momentum tensor \(^\text{class}\) in (interacting) classical field theory it is defined by
\[
\Theta_{\mu \nu}^{\text{class}, \phi} \overset{\text{def}}{=} \Theta_{\mu \nu}^{\text{class}} - \frac{1}{3} (\partial^\nu (\partial^\rho \phi^\mu \phi^\nu) - g^{\mu \nu} \partial^\rho (\partial^\rho \phi^\text{class})), \quad (273)
\]
where \(\Theta_{\mu \nu}^{\text{class}}\) is given by the same formulas (267)-(269) as in QFT. This improved tensor is conserved and traceless. The latter relies on the field equation.

Now we are going to construct the corresponding tensor in QFT. We apply the definition (254), (260) to \(T((\partial^\mu \phi^\rho \phi) f \otimes \ldots)\) and (N) (264) to \(T((\partial^\mu \phi^\rho \phi) f \otimes \ldots)\). So we obtain
\[
-T((\partial^\mu \phi^\rho \phi) f \otimes W_1 g_1 \otimes \ldots \otimes W_n g_n) = T((\partial^\mu \phi^\rho \phi) f \otimes W_1 g_1 \otimes \ldots \otimes W_n g_n) \\
+T((\partial^\mu \phi^\rho \phi) f \otimes W_1 g_1 \otimes \ldots \otimes W_n g_n) + \mathcal{A}_{W_1, \ldots, W_n}^{(n+1) \mu \nu} \Gamma_1 g_1, \ldots, g_n),
\]
where
\[
\mathcal{A}_{W_1, \ldots, W_n}^{(n+1) \mu \nu} \overset{\text{def}}{=} \sum_{k=1}^{n} \sum_{m_1 < \ldots < m_k} T(b^{(k+1)}_{\phi \phi^\mu \phi^\nu, W_{m_1} \ldots W_{m_k} (f, g_{m_1}, \ldots, g_{m_k})}) \\
\otimes W_1 g_1 \otimes \ldots \hat{m}_1 \ldots \hat{m}_k \ldots \otimes W_n g_n.
\]

\(^40\)So far no external derivatives are present. Hence, (N) plays no role.
\(^41\)In massive theories it is already at the classical level impossible to construct an energy momentum tensor \(\Theta_{\mu \nu}^{\text{class}}\) which is conserved and traceless. Because the corresponding dilatation current \(B_{\mu \nu}^{\text{class}} \equiv x_{\phi} \Theta_{\mu \nu}^{\text{class}}\) would be conserved, but a (non-vanishing) mass breaks dilatation invariance.

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Here we have normalized $T(\partial^\nu \phi, W_{m_1}, ..., W_{m_k})$ according to (91) (N4), which implies

$$b^{(k+1)}_{\mu \nu, \phi, W_{m_1}, ..., W_{m_k}} = 0$$  \hspace{1cm} (276)

(there are no anomalies for tree-like diagrams, cf. sect 2.4). Hence, there are no $T(b^{(k+1)}...)$-terms in the application of (N) to $T((\phi \partial^\mu \partial^\nu \phi) f \otimes W_1 g_1 \otimes ...)$.

In particular it follows

$$T((\phi \partial^\mu \partial^\nu \phi) f \otimes W_1 g_1 \otimes ... \otimes W_n g_n) = T((\phi \partial^\mu \partial^\nu \phi) f \otimes W_1 g_1 \otimes ... \otimes W_n g_n).$$  \hspace{1cm} (277)

from (N). For the interacting fields the identity (274) implies

$$-(\phi \partial^\nu \phi)_{gL}(\partial^\mu f) = (\phi \partial^\nu \phi)_{gL}(f) + (\phi \partial^\mu \phi) g_{L}(f) + A_{g}^{\mu \nu}(f),$$  \hspace{1cm} (278)

where

$$A_{g}^{\mu \nu}(f) \equiv \sum_{n=1}^{\infty} \sum_{r=1}^{n} \frac{(-1)^{n-r}}{r! (n-r)!} T((g_{L})^{\otimes (n-r)}) A^{(r+1)\mu \nu}_{L,...,L}(f, g, ..., g).$$  \hspace{1cm} (279)

Without further knowledge about $b^{(k+1)\mu}_{\phi \partial^\nu \phi, L,...,L}$ we cannot interpret $A_{g}^{\mu \nu}(f)$ as an interacting field and $A_{g}^{\mu \nu}(f)$ needs not to be symmetrical in $\mu \leftrightarrow \nu$.\footnote{Due to (N0) the anomaly $a^{(k+1)\mu}_{\phi \partial^\nu \phi, L,...,L} = T_1 b^{(k+1)\mu}_{\phi \partial^\nu \phi, L,...,L}$ has the form

$$a^{(k+1)\mu}_{\phi \partial^\nu \phi, L,...,L}(f, g, ..., g) = \int dx f(x) \int dy_1...dy_k g(y_1)...g(y_k)$$

$$\{ P^{\mu \nu}_{\mu_1,...,\mu_k} (\partial_{y_1}, ..., \partial_{y_k}) \delta(y_1 - x, ..., y_k - x)$$

$$\sum_{j \leq i_s} : \phi(y_j) \phi(y_i) : P^{\mu \nu}_{\mu_1,...,\mu_k} (\partial_{y_1}, ..., \partial_{y_k}) \delta(y_1 - x, ..., y_k - x)$$

$$+ \sum_{j \leq i \leq s} : \phi(y_j) \phi(y_i) \phi(y_s) : P^{\mu \nu}_{\mu_1,...,\mu_k} (\partial_{y_1}, ..., \partial_{y_k}) \delta(y_1 - x, ..., y_k - x) \},$$

where $P^{\mu \nu}_{\mu_1,...,\mu_k} (\partial_{y_1}, ..., \partial_{y_k})$ is a polynomial of degree $m$ in the partial derivatives $\partial_{y_1}, ..., \partial_{y_k}$ and the expression in the {...}-bracket is symmetrical under permutations of $y_1, ..., y_k$. $P^{\mu \nu}_{\mu_1,...,\mu_k} (\partial_{y_1}, ..., \partial_{y_k})$ is $\sim g^{\mu \nu}$ and, hence, symmetrical in $\mu \leftrightarrow \nu$. But, e.g. for $k = 2$ the terms $\partial_{y_1}^\mu \partial_{y_2}^\nu + \partial_{y_1}^\nu \partial_{y_2}^\mu \delta(y_1 - x, y_2, x)$ and $\{ : \phi^2(y_1) : \partial_{y_1}^\mu \partial_{y_2}^\nu + : \phi^2(y_2) : \partial_{y_1}^\nu \partial_{y_2}^\mu \delta(y_1 - x, y_2, x)$ have not this $(\mu \leftrightarrow \nu)$-symmetry and their contributions to $a^{(k+1)\mu}_{\phi \partial^\nu \phi, L,...,L}(\partial f, g, ..., g) - a^{(k+1)\nu}_{\phi \partial^\mu \phi, L,...,L}(\partial f, g, ..., g)$ and hence to $I_{gL}^{\mu \nu}(\partial f) = A_{g}^{\mu \nu}(\partial f) - A_{g}^{\nu \mu}(\partial f)$ do not vanish.}

Now we define the improvement tensor

$$I_{gL}^{\mu \nu}(f) \equiv -(\phi \partial^\nu \phi)_{gL}(\partial^\mu f) + g^{\mu \nu}(\phi \partial^\mu \phi)_{gL}(f)$$  \hspace{1cm} (280)
By using (281) we find that it is conserved up to anomalous terms (i.e. terms which violate the MWI)

$$I^{\mu\nu}_{gL}(\partial_{\mu}f) = A^{\mu\nu}_{g}(\partial_{\mu}f) - A^{\mu\nu}_{g}(\partial_{\mu}f). \tag{283}$$

To compute the trace, we first mention

$$T((\phi\partial^{\mu}\partial_{\mu}\phi)f \otimes W_{1}g_{1} \otimes ... \otimes W_{n}g_{n}) = 

i \sum_{m=1}^{n} T(W_{1}g_{1} \otimes ... \otimes \phi \partial W_{m} \otimes f g_{m} \otimes ... \otimes W_{n}g_{n}) \tag{284}$$

which is a consequence of (N) and (276). Therefore,

$$(\phi\partial^{\mu}\partial_{\mu}\phi)g_{L}(f) = -(\phi\partial L \partial \phi)g_{L}(fg) = -4Lg_{L}(fg). \tag{285}$$

From (282), (278) and (285) we obtain

$$\frac{1}{3} I^{\mu}_{gL}(f) = -(\partial^{\mu}\phi\partial_{\mu}\phi)g_{L}(f) + 4Lg_{L}(fg) - A^{\mu}_{\text{can}g_{L}}(f) - A^{\mu}_{g}(f). \tag{286}$$

The improved energy momentum tensor is defined analogously to (273), namely

$$\Theta^{\mu\nu}_{\text{imp}gL}(f) \overset{\text{def}}{=} \Theta^{\mu\nu}_{\text{can}gL}(f) - \frac{1}{3} I^{\mu\nu}_{gL}(f). \tag{287}$$

Our results (272), (283) and (286) yield that it is conserved and traceless up to anomalous terms:

$$\Theta^{\mu\nu}_{\text{imp}gL}(\partial_{\mu}f) = \Theta^{\mu\nu}_{\text{can}gL}(\partial_{\mu}f) - \frac{1}{3}(A^{\mu\nu}_{g}(\partial_{\mu}f) - A^{\mu\nu}_{g}(\partial_{\mu}f))$$

$$= -Lg_{L}((\partial^{\nu}g) f) - \frac{1}{3}(A^{\mu\nu}_{\text{can}g_{L}}(\partial_{\mu}f) - A^{\mu\nu}_{g}(\partial_{\mu}f)),$$

$$\Theta^{\mu}_{\text{imp}gL}(f) = A^{\mu}_{g}(f). \tag{288}$$

In the literature ([29] and references cited therein) it is shown that the anomalous terms can be removed by suitable normalization in one of the two equations in (288), but not simultaneously in both. Usually one puts the priority on the conservation and allows for a trace anomaly. The latter breaks the dilatation invariance and gives rise for anomalous dimensions of the interacting fields.

**Remark:** (1) We are going to show that the trace anomaly is of order $O(g^{2})$ for the interaction (268). We have to verify that (270), (274) and (284) can be fulfilled without any anomalous terms $A^{\cdot\cdot\cdot}$ to first order in $g$. Due to (N3) we have

$$T_{2}(\partial^{\alpha}\phi \partial^{\beta}\partial^{\gamma}\phi, \phi^{4})(x, y) = :\partial^{\alpha}\phi \partial^{\beta+c}\phi(x)\phi^{4}(y) : \tag{289}$$

$$+ 4(\Omega, T_{2}(\partial^{\alpha}\phi, \phi)(x, y)\Omega) : \partial^{b+c}\phi(x)\phi^{3}(y) : \tag{290}$$

$$+ 4(\Omega, T_{2}(\partial^{a}\phi, \phi)(x, y)\Omega) : \partial^{b+c}\phi(x)\phi^{3}(y) : \tag{291}$$

$$+ 6(\Omega, T_{2}(\partial^{a}\phi \partial^{b}\partial^{c}\phi, \phi^{2})(x, y)\Omega) : \phi^{2}(y) : \tag{292}$$

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For the tree diagrams (289), (290) and (291) the MWI holds true. An anomaly must come from the loop diagram (292), which is the two-legs sector. We define the normalization of \[\Omega, T_2(\phi \partial^\mu \partial^\nu \phi, \phi^2)\Omega\] by (274) with \[\mathcal{A}^{(2)}_{\mu \nu} \equiv 0\]. The \((\mu \leftrightarrow \nu)\)-symmetry (277) holds, because all tensors of rank two are \(\sim g_{\mu \nu}\) or \(\sim p^\mu p^\nu\), where \(p\) is the momentum belonging to the relative coordinate \((x - y)\). The \(T\)-products on the r.h. sides of (270) and (284) have four legs for \(n = 1\) and \(W_1 = L = \lambda \phi^4\). Hence, it remains to show that there exits a normalization such that

\[\partial_x^\nu \langle \Omega, T_2(\partial^\mu \phi \partial^\nu \phi, \phi^2)(x, y)\rangle = \frac{1}{2} \partial_x^\nu \langle \Omega, T_2(\partial^\rho \phi \partial^\rho \phi, \phi^2)(x, y)\rangle\]  

(293)

(which is (270)) and

\[\partial_x^\mu \langle \Omega, T_2(\phi \partial^\mu \phi, \phi^2)(x, y)\rangle = \langle \Omega, T_2(\partial^\mu \phi \partial^\mu \phi, \phi^2)(x, y)\rangle\]  

(294)

(which is (284)). An explicit calculation shows that this can in fact be done.\(^4\)

6 Conclusions

The justifications to require the master Ward identity (as a normalization condition for the time-ordered products) are the following facts:

- In the classical limit \(\hbar \to 0\) the MWI becomes an identity which holds always true.\(^3\)

- The MWI has many, far-reaching and important consequences (see sect. 4) which we would like to hold true in QFT.\(^4\)

- It seems that the MWI can nearly always be satisfied: it is compatible with the other normalization conditions (sect. 3), and many consequences of the MWI (e.g. the field equation, charge- and ghost-number conservation, conservation of the energy momentum tensor and perturbative gauge invariance (147) with \(j_1 = \cdots = j_n = 0\) for \(SU(N)\)-Yang-Mills theories) have already been proved in the literature (sect. 4). The only counter-examples we know are the usual anomalies of perturbative QFT.

7 Appendix A: Feynman propagators

Let \(\varphi, \chi \in \mathcal{G}\) be the symbols corresponding to two massive or massless free fields (without derivatives) with the same mass and which satisfy the Klein-Gordon or wave equation

\[\square + m^2 \varphi = 0,\quad \square + m^2 \chi = 0,\quad m \geq 0,\]  

(295)

\(^4\)The C-number distributions in (292) for \(b = 0\) and the relevant values of \(a\) and \(c\) have essentially been calculated in the second paper of [11] (sect. 2 and appendix C).

\(^3\)We discovered (or invented) the MWI by searching for a local construction of observables in non-Abelian quantum gauge theories. (In [7] this construction is given for QED). We succeeded provided several normalization conditions are fulfilled, see [3]. In order to prove that the latter have a common solution we looked for a universal formulation of these normalization conditions - and found the MWI.
and obey Bose or Fermi statistics. We assume that $T_i(\varphi g), \ g \in D(\mathbb{R}^4)$ (anti-)commutes with all free fields except $T_i(\chi h), \ h \in D(\mathbb{R}^4)$ and the same for $\varphi$ and $\chi$ exchanged. The non-vanishing (anti-)commutator is given by

$$\Delta_{\varphi, \chi} = \epsilon D_m,$$

(296)

where $D_m$ is the (massive or massless) Pauli-Jordan distribution to the mass $m$, $\epsilon$ is a sign which depends on $(\varphi, \chi)$ and we have extended the notation \[21\] to anti-commutators. For a bosonic real scalar field it is $\chi = \varphi$ and for a bosonic complex scalar field we have $\chi = \varphi + i \chi$. In case of the fermionic ghost fields of non-Abelian gauge theories $\varphi$ and $\chi$ must be different: $\varphi = \tilde{u}_a, \chi = u_a, \epsilon = 1$ where $a$ is the colour index. Alternatively one may also set $\varphi = u_a, \chi = \tilde{u}_a, \epsilon = -1$.

Spinor fields will be treated later.

According to our definition \[32\] of the Feynman propagators and the normalization condition \[N0\], $\Delta_{\partial^a \varphi, \partial^b \chi}^F$ contains undetermined local terms if and only if

$$\omega \overset{\text{def}}{=} \text{sd}(\Delta_{\partial^a \varphi, \partial^b \chi}^F) - 4 \equiv -2 + |a| + |b| \geq 0,$$

(297)

namely

$$\Delta_{\partial^a \varphi, \partial^b \chi}^F = \epsilon(-1)^{|b|}[\partial^a \partial^b D_m^F + \sum_0 \omega C_{c(a, b)}^F \delta^c_{\partial^a}],$$

(298)

where $D_m^F$ is the massive or massless Feynman propagator and the $C_{c(a, b)}^F \in \mathbb{C}$ are constants. We give an explicit list of the undetermined terms for the lowest values of $|a| + |b|$: \(\Delta_{\partial^a \varphi, \partial^b \chi}^F = -\epsilon([\partial^a \partial^b D_m^F + C g^{\mu \nu} \delta]),\)

(299)

$$\Delta_{\partial^a \varphi, \partial^b \chi}^F = \epsilon(\partial^a \partial^b D_m^F + \frac{1}{4} g^{\mu \nu} \delta),$$

(300)

$$-\Delta_{\partial^a \varphi, \partial^b \chi}^F = \Delta_{\partial^a \varphi, \partial^b \chi}^F = \epsilon(\partial^a \partial^b D_m^F + \frac{1}{4} g^{\mu \nu} \delta),$$

(301)

$$\Delta_{\partial^a \partial^b \partial^c \varphi, \partial^d \chi}^F = \Delta_{\partial^a \partial^b \partial^c \chi, \partial^d \varphi}^F = \epsilon(\partial^a \partial^b \partial^c \partial^d D_m^F - \frac{1}{4} g^{\mu \nu} \delta),$$

(302)

where we have taken account of Poincare covariance, symmetry with respect to exchange of Lorentz indices and

$$\Delta_{\partial^a \varphi, \partial^b \chi}^F = -m^2 \Delta_{\partial^a \varphi, \partial^b \chi}^F = \Delta_{\partial^a \varphi, \partial^b \chi}^F.$$  

(303)

With these formulas and $(\Box + m^2)D_m^F = \delta$ we compute $\delta_{\chi, \psi}^F \overset{\text{def}}{=} \partial^a \Delta_{\chi, \psi}^F - \Delta_{\partial^a \chi, \psi}^F$.  

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\[\delta_{\phi, \chi} = 0, \quad \delta_{\phi, \partial \nu \chi} = \epsilon C g^{\mu \nu} \delta, \quad \delta_{\partial \nu \phi, \chi} = \epsilon \delta, \quad \delta_{\partial \nu \phi, \partial \mu \chi} = \epsilon \delta, \quad \delta_{\partial \nu \phi, \partial \mu \partial \tau \chi} = \epsilon \left[ \frac{1}{2} + 2 C_1 \right] g^{\tau \nu} \partial_{\mu} \delta - \left( \frac{1}{2} + 2 C_1 \right) g^{\mu \nu} \partial_{\tau} \delta \right]. \]

For spinor fields with mass \( m \geq 0 \) obeying the Dirac equation we have

\[\Delta_{\psi, \overline{\psi}} = -(i \gamma^\mu \partial_{\mu} + m) D_m\]

and find

\[\delta_{\gamma \mu, \psi, \overline{\psi}} = -i \delta, \quad \delta_{\overline{\psi} \gamma_{\mu, \psi}} = -i \delta.\]

8 Appendix B: Explicit results for \( \Delta^\mu \) used in the application of the MWI to the BRST-current

Let \( j_\mu \) be the free BRST-current [17]. We assume that each symbol in \( W \in P_0 \) carries at most a first (internal) derivative (no higher derivatives). Then the following

\[\Delta_{\chi, \psi}^\mu \left( \frac{\partial j_\mu}{\partial \chi} \varrho, \frac{\partial W}{\partial \psi} f \right), \quad \chi, \psi \in \mathcal{G},\]
do not vanish:

\[ \chi = \partial_r A_a : \quad \Delta^\mu_{\partial_r A_a, A_a^\mu} \left( (\partial^{\mu} u_a) g, \frac{\partial W}{\partial A_b^\mu} \right) = \frac{1}{4} (\partial^{\mu} u_a) \frac{\partial W}{\partial A_b^\mu} g f, \]  

\( (C_A + \frac{1}{2} + 2C_A) \left( (\partial^\mu u_a) \frac{\partial W}{\partial (\partial_r A_a^\mu)} g f \right) \]

\[ + (\partial_r u_a) \frac{\partial W}{\partial (\partial_r A_a^\mu)} (\partial^\mu g) f \]

\[ - C_A \left( (\partial^\tau \partial^\nu u_a) \frac{\partial W}{\partial (\partial_r A_a^\nu)} g f \right) + (\partial^\mu u_a) \frac{\partial W}{\partial (\partial_r A_a^\mu)} (\partial^\nu g) f \]

\[ + (\partial^\mu u_a) \frac{\partial W}{\partial (\partial_r A_a^\mu)} (\partial^\nu g) f, \]  

\( \chi = \partial_r u_a : \quad \Delta^\mu_{\partial_r u_a, u_a} \left( (\partial_r A_a^\tau + m_a \phi_a) g, \frac{\partial W}{\partial u_b} f \right) = \]

\[ -(\partial_r A_a^\tau + m_a \phi_a) \frac{\partial W}{\partial u_b} g f, \]  

\[ \Delta^\mu_{\partial_r u_a, \partial_r \bar{u}_b} \left( (\partial_r A_a^\tau + m_a \phi_a) g, \frac{\partial W}{\partial (\partial_r \bar{u}_b)} f \right) = \]

\[ -(1 + C_A) \left( (\partial_r (\partial_r A_a^\tau + m_a \phi_a)) \frac{\partial W}{\partial (\partial_r u_a)} g f \right) \]

\[ + (\partial_r A_a^\tau + m_a \phi_a) \frac{\partial W}{\partial (\partial_r u_a)} (\partial_r g) f, \]  

\[ \chi = \partial_r A_a : \quad \Delta^\mu_{\partial_r A_a, A_a^\mu} \left( -u_a g, \frac{\partial W}{\partial A_b^\mu} f \right) = \]

\[ \frac{3}{4} [ (\partial^\nu u_a) \frac{\partial W}{\partial A_b^\nu} g f \] 

\[ + u_a \frac{\partial W}{\partial (\partial_r A_a^\nu)} (\partial^\nu g) f, \]  

\[ \Delta^\mu_{\partial_r A_a, \partial^\nu A_a^\mu} \left( -u_a g, \frac{\partial W}{\partial (\partial_r A_a^\nu)} f \right) = \left( \frac{1}{2} - C_A \right) \left( \frac{\partial W}{\partial (\partial_r A_a^\nu)} f \right) \]

\[ + u_a \frac{\partial W}{\partial (\partial_r A_a^\nu)} (\partial^\nu g) f \] 

\[ + (\partial_r u_a) \frac{\partial W}{\partial (\partial_r A_a^\nu)} (\partial^\nu g) f \]

\[ + 2(\partial_r u_a) \frac{\partial W}{\partial (\partial_r A_a^\nu)} (\partial^\nu g) f \] 

\[ + C_A m_a^2 u_a \frac{\partial W}{\partial (\partial_r A_a^\nu)} g f, \]  

\( \text{Eq. } 315, 316, 317, 318, 319, 320 \)

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\[ \chi = u_a : \quad \Delta^\mu_{u_a, \tilde{u}_b} \left( - (\partial_\mu (\partial_\tau A^\tau_a + m_a \phi_a)) g, \frac{\partial W}{\partial \tilde{u}_b} f \right) = 0, \quad (321) \]

\[ \Delta^\mu_{u_a, \partial_\nu \tilde{u}_b} \left( - (\partial_\mu (\partial_\tau A^\tau_a + m_a \phi_a)) g, \frac{\partial W}{\partial (\partial_\nu \tilde{u}_b)} f \right) = C_{u_a} (\partial_\nu (\partial_\tau A^\tau_a + m_a \phi_a)) \frac{\partial W}{\partial (\partial_\nu \tilde{u}_b)} g f, \quad (322) \]

\[ \chi = \phi_a : \quad \Delta^\mu_{\phi_a, \phi_b} \left( (\partial_\mu u_a) g, \frac{\partial W}{\partial \phi_b} f \right) = 0, \quad (323) \]

\[ \Delta^\mu_{\phi_a, \partial_\nu \phi_b} \left( (\partial_\mu u_a) g, \frac{\partial W}{\partial (\partial_\nu \phi_b)} f \right) = -C_{\phi_a} (\partial_\mu u_a) \frac{\partial W}{\partial (\partial_\nu \phi_a)} g f, (324) \]

\[ \chi = \partial_\mu \phi_a : \quad \Delta^\mu_{\partial_\mu \phi_a, \phi_b} \left( - u_a g, \frac{\partial W}{\partial \phi_b} f \right) = u_a \frac{\partial W}{\partial \phi_a} g f, \quad (325) \]

\[ \Delta^\mu_{\partial_\mu \phi_a, \partial_\nu \phi_b} \left( - u_a g, \frac{\partial W}{\partial (\partial_\nu \phi_b)} f \right) = (1 + C_{\phi_a}) \left( (\partial_\nu u_a) \frac{\partial W}{\partial (\partial_\nu \phi_a)} g f + u_a \frac{\partial W}{\partial (\partial_\nu \phi_a)} (\partial_\nu g) f \right), \quad (326) \]

where we have used the explicit expressions (304)-(310) for the \( \delta^\mu \) and the definition (35) of \( \Delta^\mu \).

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