Variance Reduced Stochastic Proximal Algorithm for AUC Maximization

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Abstract

Stochastic Gradient Descent has been widely studied with classification accuracy as a performance measure. However, these stochastic algorithms cannot be directly used when non-decomposable pairwise performance measures are used such as Area under the ROC curve (AUC) which is a common performance metric when the classes are imbalanced. There have been several algorithms proposed for optimizing AUC as a performance metric, and one of the recent being a stochastic proximal gradient algorithm (SPAM). But the downside of the stochastic methods is that they suffer from high variance leading to slower convergence. To combat this issue, several variance reduced methods have been proposed with faster convergence guarantees than vanilla stochastic gradient descent. Again, these variance reduced methods are not directly applicable when non-decomposable performance measures are used. In this paper, we develop a Variance Reduced Stochastic Proximal algorithm for AUC Maximization (VRSPAM) and perform a theoretical analysis as well as empirical analysis to show that our algorithm converges faster than SPAM which is the previous state-of-the-art for the AUC maximization problem.

1 Introduction

Classification accuracy is a commonly used performance measure to evaluate a classifier. However, this measure is not suitable in the presence of class imbalance i.e. when one class occurs much more frequently than the other class (Elkan 2001). To overcome this drawback, Area under the ROC curve (AUC) (Hanley and McNeil 1982; Bradley 1997; Fawcett 2006) is used as a standard metric for quantifying the performance of a classifier. AUC measures the ability of a family of classifiers to correctly rank a positive example with respect to a randomly selected negative example.

There have been several algorithms for AUC maximization in the batch setting, where all the training data is assumed to be available at the beginning (Rakotomamonjy and Cun 2004; Herschtal and Raskutti 2004; Zhang, Saha, and Vishwanathan 2012; Joachims 2005). However, this assumption is unrealistic in several cases, especially for streaming data analysis. Several online algorithms proposed for such settings where the per iteration complexity is low (Bottou and Cun 2004; Shalev-Shwartz and others 2012; Hazan and Kale 2012; Rakhlin, Shamir, and Sridharan 2011; Orabona 2014). Although online algorithms have been thoroughly explored for classification accuracy where the loss decomposes over individual examples, the case of maximizing AUC as performance measure has been looked at only recently (Zhao et al. 2011; Wang et al. 2012; Kar et al. 2013). The main challenge in the AUC maximization framework is that at each step the algorithm needs to pair the current datapoint with all previously seen datapoints leading to $O(td)$ space and time complexity at step $t$ where the dimension of the instance space is $d$. The problem was just slightly alleviated by the technique of buffering (Zhao et al. 2011; Wang et al. 2012; Kar et al. 2013) as the good generalization performance depends on having a large buffer size. Recently, (Palaniappan and Bach 2016) provided a primal dual algorithm by extending stochastic variance reduced algorithms (SVRG, SAGA) to handle non-decomposable losses or regularizers (in the form of convex-concave saddle point problem) and thereby provided linear convergence rate $O(\frac{1}{t})$. Although this can be applied to AUC optimization with the least-squared loss, their algorithm needs to assume strong convexity of both the primal and dual variables. Their algorithm also has expensive per-iteration complexity $O(n + d)$ where $n$ is the number of data points and $d$ is the dimension.

Recent works take a different approach by reformulating the AUC maximization problem with the least square loss. (Ying, Wen, and Lyu 2016) reformulated it as a saddle point problem and gave an algorithm which has a convergence rate of $O(\frac{1}{\sqrt{t}})$. However, they only consider smooth penalty terms such as Frobenius norm and their convergence rate is still sub-optimal to $O(\frac{1}{t})$ which is what stochastic gradient descent (SGD) achieves with classification accuracy as a performance measure. (Nzotole, Ying, and Lyu 2018) then proposed a stochastic proximal algorithm for AUC maximization which under assumptions of strong convexity can achieve a convergence rate of $O(\frac{1}{\sqrt{td}})$ and has per iteration complexity of $O(d)$ i.e. one data-point and is applicable to general non-smooth regularization terms. However, due to
the inherent variance of random sampling we need to pick a step size of \( O\left(\frac{1}{d}\right) \) for SGD which leads to a slower sub-linear convergence rate of \( O\left(\frac{1}{d}\right) \). Thus, for SGD we have a low per iteration complexity and slow convergence versus high per iteration complexity and fast convergence for full gradient descent. Thus, SGD might take long to get a good approximation of the solution of the optimization problem.

In the context of classification accuracy, to reduce the variance of SGD several popular methods have been proposed such as SAG (Roux, Schmidt, and Bach 2012), SDCA (Shalev-Shwartz and Zhang 2013), SVRG (Johnson and Zhang 2013). One issue with SAG and SDCA is that they require the storage of all the gradients and dual variables respectively. On the other hand, SVRG enjoys the same fast convergence rates as SDCA and SAG but has a much simpler analysis and does not require storage of gradients. This allows SVRG to be applicable in complex problems where the storage of all gradients would be infeasible, unlike, SAG and SDCA.

Since SVRG is applicable only for smooth strongly convex functions, several works have explored ways to tackle the presence of a regularizer term in addition to the average of several smooth component function term. Two simple strategies are to use the Proximal Full Gradient and the Proximal Stochastic Gradient method. While the Proximal Stochastic Gradient is much faster since it computes only the gradient of a single component function per iteration, it convergences much slower than the Proximal Full Gradient method. The proximal gradient methods can be viewed as a special case of splitting methods (Lions and Mercier 1979; Bauschke, Combettes, and others 2011; Tseng 2000; Beck and Teboulle 2008). However, both Proximal methods do not fully exploit the problem structure. Proximal SVRG (Xiao and Zhang 2014) is an extension of the SVRG (Johnson and Zhang 2013) technique and can be used whenever the objective function is composed of two terms- the first term is an average of smooth functions (decomposes across the individual instances) and the second term admits a simple proximal mapping. Prox-SVRG needs far fewer iterations to achieve the same approximation ratio than the proximal full and stochastic gradient descent methods. However, there is an important gap that has not been addressed yet — existing techniques that guarantee faster convergence by controlling the variance are not directly applicable to non-decomposable loss functions as in the problem of AUC optimization and this is the gap that we close in this paper.

In this paper, we present Variance Reduced Stochastic Proximal algorithm for AUC Maximization (VRSPAM). VRSPAM applies the standard SVRG variance reduction technique to the SPAM algorithm, which is a proximal stochastic gradient descent applied to a convex surrogate of the AUC maximization problem. We provide theoretical analysis for the VRSPAM algorithm showing that it achieves linear convergence rate with a fixed step size better than SPAM which has sub-linear convergence rate and constantly decreasing step size. Also, the theoretical analysis provided in the paper is much simpler as compared to the analysis of SPAM. We also perform numerical experiments to show that the VRSPAM algorithm converges faster than SPAM.

The organization of the remainder of the paper is as follows. In Section 2, we briefly state the AUC optimization problem and state the equivalent formulation that is necessary for our algorithmic analysis. In Section 3, we discuss our algorithm for faster AUC optimization with variance reduction and do a thorough convergence analysis of it in Section 4. In Section 5, we perform experiments on a suite of UCI datasets to show our proposed algorithm indeed converges faster than the state-of-the-art algorithms for AUC optimization. We conclude in Section 6 with some potential avenues for future research.

### 2 AUC formulation

The AUC score associated with a linear scoring function \( g(x) = w^T x \), is defined as the probability that the score of a randomly chosen positive example is higher than a randomly chosen negative example (Hanley and McNeil 1982; Clémençon et al. 2008) and is denoted by AUC(w). If \( z = (x, y) \) and \( z' = (x', y') \) is drawn independently from an unknown distribution \( Z = X \times \mathcal{Y}_0 \), then

\[
\text{AUC}(w) = Pr(w^T x \geq w^T x' | y = 1, y' = -1) = E[\delta_{w^T(x-x') \geq 0} | y = 1, y' = -1]\]

Since AUC(w) in the above form is not convex because of the \( 0-1 \) loss, it is a common practice to replace this by a convex surrogate loss. In this paper, we focus on the least square loss which is consistent unlike some other choices such as the hinge loss. The following is the objective for AUC maximization:

\[
\min_{w \in \mathbb{R}^d} (1-p) \mathbb{E}[(1-w^T(x-x'))^2 | y = 1, y' = -1] + \Omega(w) \tag{1}
\]

Let \( f(w) = p(1-p) \mathbb{E}[(1-w^T(x-x'))^2 | y = 1, y' = -1] \) such that the function in above minimization problem can be written as \( f(w) + \Omega(w) \). Here, \( p = Pr(y = +1) \) and \( 1-p = Pr(y = -1) \) are the class priors and \( \Omega \) is the convex regularizer. Throughout this paper we assume:

- \( \Omega \) is \( \beta \) strongly convex i.e. for any \( w, w' \in \mathbb{R}^d, \Omega(w) \geq \Omega(w') + \partial \Omega(w')^T (w-w') + \frac{\beta}{2} \|w-w'\|^2 \)
- \( \exists M \) such that \( \|x\| \leq M \forall x \in \mathcal{X} \).

In this paper we have used Frobenius norm \( \Omega(w) = \beta \|w\|^2 \) and Elastic Net \( \Omega(w) = \beta \|w\|^2 + \nu \|w\|_1 \) as the convex regularizers where \( \beta, \nu \) are regularization parameters.

The minimization problem in equation \( 1 \) can be reformulated such that stochastic gradient descent can be performed to find the optimum value. Below is the equivalent formulation from Theorem 1 in (Natele, Ying, and Lyu 2018):

\[
\min_{w,a,b}\max_{\zeta, z} \mathbb{E} [F(w,a,b,\zeta;z)] + \Omega(w) \]

where the expectation is with respect to \( z = (x,y) \) and

\[
F(w,a,b,\zeta;z) = (1-p)(w^T x - a)^2\mathbb{I}_{[y=1]} + p(w^T x - b)^2\mathbb{I}_{[y=-1]} + 2(1 + \zeta)w^T x (p\mathbb{I}_{[y=-1]} - (1-p)\mathbb{I}_{[y=1]}) - p(1-p)\zeta^2
\]
Thus, \( f(w) = \min_{a,b} \max_{\zeta \in R} \mathbb{E}[F(w,a,b,\zeta ; z)] \). (Natole, Ying, and Lyu 2018) also state that the optimal choices for \( a, b, \zeta \)
\[
\begin{align*}
    a(w) &= w^T \mathbb{E}[x|y = 1] \\
    b(w) &= w^T \mathbb{E}[x|y = -1] \\
    \zeta(w) &= w^T (\mathbb{E}[x'|y' = -1] - \mathbb{E}[x|y = 1])
\end{align*}
\]

An important thing to note here is that we differentiate the objective function only with respect to \( w \) (which is also the case for SPAM) and do not compute the gradient with respect to the other parameters which themselves depend on \( w \). This is the reason why existing methods cannot be applied directly — since the other parameters which also depend on \( w \) are then updated in closed form.

### 3 Method

The major issue that slows down convergence for SGD is the decay of the step size to 0 as the iteration increase. This is necessary for mitigating the effect of variance introduced by random sampling in SGD. We follow the method of Prox-SVRG closely on the reformulation of AUC to derive the proximal SVRG algorithm for AUC maximization given in Algorithm 1. We store a \( \hat{w} \) after every \( m \) Prox-SGD iterations that is progressively closer to the optimal \( w \) (essentially an estimate of \( w^* \)). Full gradient \( \hat{\mu} \) is computed whenever \( \hat{w} \) gets updated, i.e., after every \( m \) iterations Prox-SGD:
\[
\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} G(\hat{w}, z_i)
\]

\( \hat{\mu} \) is used to update next \( m \) gradients. Next \( m \) iterations are initialized by \( w_0 = \hat{w} \). For each iteration, we randomly pick \( i_t \in \{1,..,n\} \) and compute
\[
w_t = w_{t-1} - \eta v_t
\]
where \( v_t = G(w, z_{i_t}) - G(\hat{w}, z_{i_t}) + \hat{\mu} \) and \( G(w; z) = \frac{\partial f(w,a(w),b(w),\zeta(w);z)}{\partial w} \) and then the proximal step is taken
\[
w_t = \text{prox}_{\eta \Omega}(w_t)
\]
Notice that if we take expectation of \( G(\hat{w}, z_{i_{t-1}}) \) with respect to \( i_t \) we get \( \mathbb{E}[G(\hat{w}, z_{i_{t-1}})] = \hat{\mu} \). Now if we take expectation of \( v_t \) with respect to \( i_t \) conditioned on \( w_{t-1} \), we can get the following:
\[
\mathbb{E}[v_t] = \mathbb{E}[G(w, z_{i_{t-1}})] - \mathbb{E}[G(\hat{w}, z_{i_{t-1}}) + \hat{\mu}]
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} G(w_{t-1}, z_i)
\]
Hence the modified direction \( v_t \) is stochastic gradient of \( G \) at \( w_{t-1} \) — similar to \( G(w, z_{i_{t-1}}) \). However, the variance \( \mathbb{E}[v_t^2] \) can be much smaller than \( \mathbb{E}[(G(w_{t-1}, z_{i_{t-1}}) - \partial f(w_{t-1}))^2] \). We will show in section 4 that the following inequality holds
\[
\mathbb{E}[(G(w_t, z_{i_t}) - G(\hat{w}, z_{i_t}) + \hat{\mu} - \partial f(w^*))^2]\]
\[
\leq 2(8M^2)^2 \|w_t - w^*\|^2 + 2(8M^2)^2 \|\hat{w} - w^*\|
\]

**Algorithm 1** Proximal SVRG for AUC maximization

**INPUT** Constant step size \( \eta \) and update frequency \( m \)

**INITIALIZE** \( w_0 \)

**for** \( s = 1,2,.. \) **do**
\[
\hat{w} = \tilde{w}_{s-1}
\]
\[
\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} G(\hat{w}, z_i)
\]
\[
w_0 = w
\]

**for** \( t = 1,2,..,m \) **do**

Randomly pick \( i_t \in \{1,..,n\} \) and update weight
\[
w_t = w_{t-1} - \eta (G(w_{t-1}, z_{i_t}) - G(\hat{w}, z_{i_t}) + \hat{\mu})
\]
\[
w_t = \text{prox}_{\eta \Omega}(w_t)
\]
\[
w_s = w_m
\]

From above, when both \( w_t \) and \( \hat{w} \) converge to \( w^* \), the variance goes to 0. Therefore, by using a constant size we can achieve better convergence rate. Thus, this is a multi-stage scheme to explicitly reduce the variance of the modified proximal gradient.

### 4 Convergence Analysis

In this section we analyse the convergence rate of VRSPAM formally. We first define some lemmas which will be used for proving the Theorem 1 which is the main theorem proving the geometric convergence of Algorithm 1. First is the Lemma [1]from (Natole, Ying, and Lyu 2018) which states that \( \partial_w F(w_t, a(w_t), b(w_t), \alpha(w_t); z_t) \) is an unbiased estimator of the true gradient. As we are not calculating the true gradient in VRSPAM, we need the following Lemma to prove the convergence result.

**Lemma 1** ([Natole, Ying, and Lyu 2018]). Let \( w_t \) be given by VRSPAM in Algorithm 1. Then, we have
\[
\partial f(w_t) = \mathbb{E}_{z_t}[\partial_w F(w_t, a(w_t), b(w_t), \alpha(w_t); z_t)]
\]

This Lemma is directly applicable in VRSPAM since the proof of the Lemma hinges on the objective function formulation and not on the algorithm specifics.

The next lemma from (Natole, Ying, and Lyu 2018) provides an upper bound on the norm of difference of gradients at different time steps.

**Lemma 2** ([Natole, Ying, and Lyu 2018]). Let \( w_t \) be described as above. Then, we have
\[
\|G(w_t; z_t) - G(w_t; z_t)\| \leq 8M^2\|w_t - w_t\|
\]

**Proof.**
\[
\|G(w_t; z_t) - G(w_t; z_t)\| \leq 4M^2\|w_t - w^*\|1_{[w_t \neq -1]} + 4M^2(1 - p)\|w_t - w^*\|1_{[w_t = -1]} + 4M^2\|w_t - w^*\|1_{[w_t = -1]} + 4M^2\|w_t - w^*\|1_{[w_t = -1]}\|w_t - w^*\|
\]
\[
\leq 8M^2\|w_t - w_t\|
\]

The proof directly follows by writing out the difference and using the second assumption on the boundedness of \( z_t \).
We now present and prove a result that will be necessary in showing convergence in Theorem 1.

**Lemma 3.** Let $C = \frac{1+128M^4\eta^2}{(1+\eta\beta)^2}$ and $D = \frac{128M^4\eta^2}{(1+\eta\beta)^2}$, if $\eta \leq \frac{\beta}{128M^4}$, then $C^m + DC^m C^{-1} - 1 \leq 1$ holds true.

**Proof.** We start with:

\[
C^m + DC^m C^{-1} - 1 
\leq
\Rightarrow DC^m C^{-1} - 1 \leq 1 - C^m 
\Rightarrow D \leq \frac{1 - C}{C}
\]

Substituting values of $C$ and $D$ and using the condition that $D \leq 128M^4\eta^2$, we get

\[
128M^4\eta^2 \leq \frac{1 - \frac{1+128M^4\eta^2}{(1+\eta\beta)^2}}{1+\frac{128M^4\eta^2}{(1+\eta\beta)^2}}
\]

\[
128M^4\eta^2 \leq \frac{(1 + \eta\beta)^2 - 1 + 128M^4\eta^2}{1 + 128M^4\eta^2}
\]

\[
128M^4\eta^2 \leq (\eta\beta)^2 + 2\eta\beta - 128M^4\eta^2
\]

\[
128M^4\eta^2 \leq \eta\beta
\]

\[
\Rightarrow \eta \leq \frac{\beta}{128M^4}
\]

The following is the main theorem of this paper stating the convergence rate of Algorithm 1 and its analysis.

**Theorem 1.** Consider VRSPAM (Algorithm 1) and let $w^* = \arg\min_w f(w)$, if $\eta < \frac{\beta}{128M^4}$, then the following inequality holds true

\[
\alpha = C^m + DC^m C^{-1} - 1 < 1
\]

and we have the geometric convergence in expectation:

\[
E[\|w_n - w^*\|^2] \leq \alpha^n E[\|w_0 - w^*\|^2]
\]

For proving the above theorem, first we upper bound the variance of the gradient step and show that it approaches zero as $w_n$ approaches $w^*$.

**Bounding the variance**

Bound on the variance of modified gradient $v_k = G(w_t, z_t) - G(\bar{w}, z_t) + \mu$ is given by following theorem:

**Theorem 2.** Consider VRSPAM (Algorithm 1), then the variance of the $v_k$ is upper bounded as:

\[
E[\|v_k - \mathbb{E}[v_k]\|^2] \leq 2(8M^2)^2\|w_t - w^*\|^2 + 2(8M^2)^2\|\bar{w} - w^*\|
\]

**Proof.** Let the variance reduced update be denoted as $v_k = G(w_t, z_t) - G(\bar{w}, z_t) + \mu$. As we know $\mathbb{E}[v_k] = \partial f(w^*)$, the variance of $v_k$ can be written as below

\[
E[\|G(w_t, z_t) - G(\bar{w}, z_t) + \mu - \partial f(w^*)\|^2]
\]

\[
\leq 2 E[\|G(w_t, z_t) - G(w^*, z_t)\|^2]
\]

\[
+ 2 E[\|G(w^*, z_t) - G(\bar{w}, z_t) + \mu - \partial f(w^*)\|^2]
\]

Also, $E[\|G(w^*, z_t) - G(\bar{w}, z_t)\| = \partial f(\bar{w}) - \partial f(w^*)$ from Lemma 1 and using the property that $E[\|X - \mathbb{E}[X]\|^2] \leq E[X^2]$ we get

\[
E[\|G(w_t, z_t) - G(\bar{w}, z_t) + \mu - \partial f(w^*)\|^2]
\]

\[
\leq 2 E[\|G(w_t, z_t) - G(w^*, z_t)\|^2]
\]

\[
+ 2 E[\|G(w^*, z_t) - G(\bar{w}, z_t) + \mu - \partial f(w^*)\|^2]
\]

From Lemma 2 we have $\|G(w_t, z_t) - G(w^*, z_t)\| \leq 8M^2\|w_t - w^*\|^2$ and $\|G(w^*, z_t) - G(\bar{w}, z_t)\| \leq 8M^2\|\bar{w} - w^*\|^2$. Using this, we can upper bound the variance of gradient step as:

\[
E[\|G(w_t, z_t) - G(\bar{w}, z_t) + \mu - \partial f(w^*)\|^2]
\]

\[
\leq 2(8M^2)^2\|w_t - w^*\|^2 + 2(8M^2)^2\|\bar{w} - w^*\|
\]

We have the desired result.

At the convergence, $\bar{w} = w^*$ and $w_t = w^*$. Thus the variance of the updates are bounded and go to zero as the algorithm converges. Whereas in the case of SPAM algorithm, the variance of the gradient does not go to zero as it is a stochastic gradient descent based algorithm.

We now present the proof of Theorem 1.

**Proof of Theorem 1**

From the first order optimality condition, we can directly write

\[
w^* = \text{prox}_{\eta f}(w - \eta\partial f(w^*))
\]

Using the above we can write

\[
\|w_{t+1} - w^*\|^2
\]

\[
= \|\text{prox}_{\eta f}(w_{t+1}) - \text{prox}_{\eta f}(w^* - \eta\partial f(w^*))\|^2
\]

Using Proposition 23.11 from [Bauschke, Combettes, and others 2011], we have $\text{prox}_{\eta f}$ is $(1 + \eta\beta)$-cocoercive and for any $u$ and $w$ using Cauchy Schwartz we can get the following inequality

\[
\|\text{prox}_{\eta f}(u) - \text{prox}_{\eta f}(w)\| \leq \frac{1}{1 + \eta\beta}\|u - w\|
\]

From above we get

\[
\|w_{t+1} - w^*\|^2
\]

\[
\leq \frac{1}{(1 + \eta\beta)^2}\|w_{t+1} - (w^* - \eta\partial f(w^*))\|^2
\]

\[
\leq \frac{1}{(1 + \eta\beta)^2}\|w_t - w^*\| - \eta\|G(w_t, z_t) - G(\bar{w}, z_t) + \mu - \partial f(w^*)\|^2
\]
Taking expectation on both sides we get
\[
\mathbb{E} \|w_{t+1} - w^*\|^2 \leq \frac{1}{(1 + \eta_\beta)^2} (\eta^2 \mathbb{E}[\|G(w_t, z_i) - G(w^*, z_i)\|^2] + \|w_t - w^*\|^2 - 2\eta \mathbb{E}[\|w_t - w^*, G(w_t, z_i) - G(w_t, z_i) + \bar{\mu} - \bar{\nabla}f(w^*)\|])
\]
Now, we first bound the last term \( T = \mathbb{E}[\|w_t - w^*, G(w_t, z_i) - G(w^*, z_i) + \bar{\mu} - \bar{\nabla}f(w^*)\|] \) in equation \( 3 \) Using Lemma \( 1 \) we can write
\[
T = \mathbb{E}[\|w_t - w^*, E_{z_i}[G(w_t, z_i)] - E_{z_i}[G(w^*, z_i)] + \bar{\mu} - \bar{\nabla}f(w^*)\] 
= \mathbb{E}[\|w_t - w^*, E_{z_i}[G(w_t, z_i)] - \partial f(w^*)\] 
= \mathbb{E}[\|w_t - w^*, \partial f(w_t) - \partial f(w^*)\] 
\geq 0
\]
Now, \( \|w_{t+1} - w^*\|^2 \) can be bounded by using above bound and Theorem \( 2 \) as below
\[
\mathbb{E} \|w_{t+1} - w^*\|^2 \leq \frac{1}{(1 + \eta_\beta)^2} (\mathbb{E}[\|w_t - w^*\|^2] + 2(8M^2)^2 \mathbb{E}[\|w_t - w^*\|^2] + 2(8M^2)^2 \mathbb{E}[\|w_t - w^*\|^2] + \frac{1}{(1 + \eta_\beta)^2} \mathbb{E}[\|w_t - w^*\|^2] + \frac{128M^4\eta^2}{(1 + \eta_\beta)^2} \mathbb{E}[\|w - w^*\|^2])
\]
Let \( C = \frac{\sum 128M^4\eta^2}{(1 + \eta_\beta)^2} \) and \( D = \frac{128M^4\eta^2}{(1 + \eta_\beta)^2} \), then after \( m \) iterations \( w_T = w_s \) and \( w_0 = w_{s-1} \)
\[
\mathbb{E} \|w_s - w^*\|^2 \leq \frac{Cm}{C} (\mathbb{E} \|w_{s-1} - w^*\|^2 + \sum_{i=0}^{m-1} \frac{D}{C^i} \mathbb{E} \|w_{s-1} - w^*\|^2)
\]
\[
\leq \left( \frac{Cm}{C} + \sum_{i=0}^{m-1} \frac{DCm}{C^i} \right) \mathbb{E} \|w_{s-1} - w^*\|^2
\]
\[
\leq \left( \frac{Cm}{C} + DCm \right) \frac{1 - (1/C)^m}{1 - 1/C} \mathbb{E} \|w_{s-1} - w^*\|^2
\]
\[
\leq \left( \frac{Cm}{C} + DCm \right) \frac{1 - (1/C)^m}{1 - 1/C} \mathbb{E} \|w_{s-1} - w^*\|^2
\]
\[
\leq \frac{1}{C} \mathbb{E} \|w_{s-1} - w^*\|^2
\]
where \( \alpha = Cm + DCm \) is the decay parameter, and \( \alpha < 1 \) by using Lemma \( 1 \). After \( s \) steps in outer loop of Algorithm \( 1 \) we get \( \mathbb{E} \|w_i - w^*\|^2 \leq C^s \mathbb{E} \|w_0 - w^*\|^2 \) where \( \alpha < 1 \). Hence, we get geometric convergence of \( \alpha^s \) which is much stronger than the \( O(\frac{1}{s}) \) convergence obtained in \( \text{[Natole, Ying, and Lyu 2018]} \). In the next section we derive the time complexity of the algorithm and investigate dependence of \( \alpha \) on the problem parameters.

**Complexity analysis**
To get \( \mathbb{E} \|w_s - w^*\|^2 \leq \epsilon \), the number of iterations \( s \) required is
\[
s \geq \log \frac{1}{\alpha} \log \frac{\mathbb{E} \|w_s - w^*\|^2}{\epsilon}
\]
At each stage, the number of gradient evaluations are \( n + 2m \) where \( n \) is the number of samples and \( m \) is the iterations in the inner loop then the complexity is \( O(n + m)(\log(\frac{1}{\epsilon})) \). Algorithm \( 1 \) takes \( O(n + m)(\log(\frac{1}{\epsilon})) \) iterations to achieve accuracy of \( \epsilon \). Here, the complexity is dependent on \( M \) and \( \beta \) as \( m \) itself is dependent on \( M \) and \( \beta \).

Now we find the dependence of \( \alpha \) and \( m \) on \( M \) and \( \beta \). Let \( \eta = \frac{\theta \beta}{128M^4} \) where \( \theta < 1 \), then
\[
C = \frac{1 + 128M^4\eta^2}{1 + \eta_\beta^2}
\]
\[
< \frac{1 + \theta^2\beta^2}{1 + \frac{\theta^2\beta^2}{128M^4}}
\]
\[
< \frac{1 + \theta^2\beta^2}{1 + \frac{\theta^2\beta^2}{128M^4}}
\]
\[
= \frac{1}{1 + \frac{\theta^2\beta^2}{128M^4}}
\]
\[
= E
\]
therefore \( D = \theta(E - E^2) \) and \( DC < \theta E^2(1 - E) \), using the above equations we can simplify \( \alpha \) as
\[
\alpha = Cm + DCm \frac{1 - Cm}{1 - C}
\]
\[
< Cm + \theta E^2(1 - E) \frac{1 - Cm}{1 - C}
\]
\[
< Cm + \theta E^2(1 - Cm) \cdot \frac{1 - E}{1 - C} < 1
\]
\[
= \theta E^2 + Cm - \theta E^2 Cm
\]
In the above equation, only \( Cm - \theta E^2 Cm \) depends on \( m \), if we choose \( m \) to be sufficiently large then \( \alpha = \theta E^2 \). An important thing to note here is that \( \theta E < C < E \), now if we choose \( m \approx 2 \sqrt{\frac{\log \epsilon}{E}} \) then \( \alpha \approx 2\theta E^2 \) which is independent of \( m \). Thus the time complexity of the algorithm is \( O(n + \frac{\log \epsilon}{E}) \) when \( m = \Theta(\log \frac{1}{E}) \). As the order has inverse dependency on \( \log E = \log(\frac{128M^4}{1 + \theta^2\beta^2}) \), increase in \( M \) will result in increase in number of iterations i.e. as the maximum norm of training samples is increased, larger \( m \) is required to reach \( \epsilon \) accuracy.

Now we will compare the time complexity of our algorithm with SPAM algorithm. First, we find the time complexity of SPAM. We will use Theorem 3 from \( \text{[Natole, Ying, and Lyu 2018]} \) which states that SPAM achieves the following:
\[
\mathbb{E}[\|w_{T+1} - w^*\|] \leq \frac{T_0}{T} \mathbb{E}[\|w_0 - w^*\|] + C \log T T^{-1}
\]
where \( t_0 = \max \left( 2, \left\lfloor 1 + \left( \frac{128M^4 + \beta^2}{128M^4} \right)^2 \right\rfloor \right) \), \( T \) is the number of iterations and \( c \) is a constant. Through averaging scheme developed by (Lacoste-Julien, Schmidt, and Bach 2012) the following can be obtained:

\[
E[\|w_{t+1} - w^*\|] \leq \frac{t_0}{T} E[\|w_{t_0} - w^*\|] \tag{4}
\]

where \( E[\|w_{t_0} - w^*\|] \leq \frac{2\sigma^2}{\tilde{C}_{\beta,M}} + \exp \left( \frac{128M^4}{\tilde{C}_{\beta,M}} \right) = F, \)

\[
\tilde{C}_{\beta,M} = \left( 1 + \frac{128M^4}{\beta^2} \right)^2 \quad \text{and} \quad E[\|G(w^*; z) - \partial f(w^*)\|_2^2] = \sigma^2.
\]

Using equation 4, time complexity of SPAM algorithm can be written as \( O(\frac{t_0F}{\epsilon}) \) i.e. SPAM algorithm takes \( O(\frac{t_0F}{\epsilon}) \) iterations to achieve \( \epsilon \) accuracy. Thus, SPAM has lower per iteration complexity but slower convergence rate as compared to VRSPAM. Therefore, VRSPAM will take less time to get a good approximation of the solution.

5 Experiment

Here we empirically compare VRSPAM with other existing algorithms used for AUC maximization. We use two variants of our proposed algorithm depending on the regularizer used:

- **VRSPAM - \( L^2 \)**: \( \Omega(w) = \frac{\beta}{2} \|w\|^2 \) (Frobenius Norm Regularizer)
- **VRSPAM - \( NET \)**: \( \Omega(w) = \frac{\beta}{2} \|w\|^2 + \beta_1 \|w\|_1 \) (Elastic Net Regularizer (Zou and Hastie 2005)). The proximal step for elastic net is given as \( \arg\min_w \left\{ \frac{1}{2} \|w - \hat{w}_{t+1} - \eta \beta_1 \|_1 \right\} \). The proximal step for elastic net is given as \( \arg\min_w \left\{ \frac{1}{2} \|w - \hat{w}_{t+1} - \eta \beta_1 \|_1 \right\} \).

VRSPAM is compared with SPAM, SOLAM (Ying, Wen, and Lyu 2016) and one-pass AUC optimization algorithm (OPAUC) (Gao et al. 2013). SOLAM was modified to have the Frobenius Norm Regularizer (as in Natole, Ying,
The training set. All the code is implemented in MATLAB and the least square loss.

Table 2: Datasets across which we evaluate our algorithm

| N  | Name      | Instances | Features |
|----|-----------|-----------|----------|
| 1  | DIABETES  | 768       | 8        |
| 2  | GERMAN    | 1000      | 24       |
| 3  | SPLICE    | 3,175     | 60       |
| 4  | USPS      | 9,298     | 256      |
| 5  | LETTER    | 20,000    | 16       |
| 6  | A9A       | 32,561    | 123      |
| 7  | W8A       | 64,700    | 300      |
| 8  | MNIST     | 60,000    | 780      |
| 9  | ACOUSTIC  | 78,823    | 50       |
| 10 | IJCNN1    | 141,691   | 22       |

Table 1: AUC values (mean±std) comparison for different algorithms on test data

| N  | VRSPAM-$L^2$ | VRSPAM-NET | SPAM-$L^2$ | SPAM-NET | SOLAM | OPAUC |
|----|--------------|------------|------------|----------|-------|-------|
| 1  | .8299±.0323  | .8305±.0319| .8272±.0277| .8085±.0431| .8128±.0304| .8309±.0350|
| 2  | .7902±.0386  | .7845±.0398| .7942±.0388| .7937±.0386| .7778±.0373| .7978±.0347|
| 3  | .9640±.0156  | .9699±.0139| .9263±.0091| .9267±.0090| .9246±.0087| .9232±.0099|
| 4  | .8552±.0066  | .8549±.0059| .8542±.0388| .8537±.0386| .8395±.0061| .8114±.0065|
| 5  | .9834±.0023  | .9804±.0032| .9868±.0032| .9855±.0029| .9822±.0036| .9620±.0040|
| 6  | .9003±.0045  | .8981±.0046| .8998±.0046| .8980±.0047| .8966±.0043| .9002±.0047|
| 7  | .9876±.0008  | .9787±.0013| .9682±.0020| .9604±.0020| .9817±.0015| .9633±.0035|
| 8  | .9465±.0014  | .9351±.0014| .9254±.0025| .9132±.0026| .9118±.0029| .9242±.0021|
| 9  | .8093±.0033  | .8052±.0033| .8120±.0030| .8109±.0028| .8099±.0036| .8192±.0032|
| 10 | .9750±.0011  | .9745±.0002| .9174±.0024| .9155±.0024| .9129±.0030| .9269±.0021|

Note that, the initial weights of VRSPAM are set to be the output generated by SPAM after 1 iteration which is similar practice to (Johnson and Zhang 2013).

Table 1 summarizes the AUC evaluation for different algorithms. AUC values for SPAM-$L^2$, SPAM-NET, SOLAM and OPAUC were taken from (Natole, Ying, and Lyu 2018).

6 Conclusion

In this paper, we propose a variance reduced stochastic proximal algorithm for AUC maximization (VRSPAM). We theoretically analyze the proposed algorithm and derive a much faster convergence rate of $O(\alpha^2)$ where $\alpha < 1$ (linear convergence rate), improving upon state-of-the-art methods ((Natole, Ying, and Lyu 2018)) which have a convergence rate of $O(\frac{1}{\sqrt{T}})$ (sub-linear convergence rate), for strongly convex objective functions with per iteration complexity of one data-point. We gave a theoretical analysis of this and showed empirically VRSPAM converges faster than other methods for AUC maximization.

For future work, it is interesting to explore if other algorithms to accelerate SGD can be used in this setting and if they lead to even faster convergence. It is also interesting to apply the proposed methods in practice to non-decomposable performance measures other than AUC.

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