Higher dimensional CFTs as 2D conformally-equivariant topological field theories

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Abstract: Two and three-point functions of primary fields in four dimensional CFT have a simple space-time dependences factored out from the combinatoric structure which enumerates the fields and gives their couplings. This has led to the formulation of two dimensional topological field theories with $SO(4, 2)$ equivariance which are conjectured to be equivalent to higher dimensional conformal field theories. We review this CFT4/TFT2 construction in the simplest possible setting of a free scalar field, which gives an algebraic construction of the correlators in terms of an infinite dimensional $so(4, 2)$ equivariant algebra with finite dimensional subspaces at fixed scaling dimension. Crossing symmetry of the CFT4 is related to associativity of the algebra. This construction is then extended to describe perturbative CFT4, by making use of deformed co-products. Motivated by the Wilson-Fisher CFT we outline the construction of $U(so(d,2))$ equivariant TFT2 for non-integer $d$, in terms of diagram algebras and their representations.
1 Introduction

Conformal Field Theories (CFTs) in $d > 2$ dimensions have been an active topic of study in recent years. In part this activity has been stimulated by the AdS/CFT correspondence, originally stated as an equivalence between $\mathcal{N} = 4$ super Yang-Mills (SYM) theory on $R^{3,1}$ with $U(N)$ gauge group and 10 dimensional string theory\[1\]. A key question is to understand how the higher dimensional quantum gravity emerges from local CFT operators and their correlators. Another motivation has been to gain an understanding of exotic CFTs that do not have a conventional Lagrangian description. Good examples of these theories include the Argyres-Douglas fixed points in 4D \[2\] as well as the (0,2) theories in 6D \[3\]. In addition, promising tools with which to study higher dimensional CFTs have become available with the revival of the bootstrap program\[4\], which uses associativity of the operator product expansion (OPE) to determine the CFT data.

CFTs in $d = 2$ (CFT2) have been well studied since the 80’s. The primary stimulus for this activity is the worldsheet dynamics of strings in critical string theory, described by a CFT2 plus ghost system. These theories have a rich structure leading to a fruitful interaction between mathematics and physics \[5, 6\]. A central role is played by

- Infinite dimensional Lie algebras (the Virasoro algebra and current algebras) which control their spectrum and correlators.
- The representation theory of these algebras, extended by considerations of modular transformations of characters.
- Rational conformal field theories, with finitely many primary fields for these algebras.
- Vertex operator algebras, which provide mathematical constructions for field operators and for the OPEs.

An important observation is that the mathematics of CFT2s use two kinds of algebras. First, there are the symmetry algebras given by infinite dimensional Lie algebras (the Virasoro algebra, current algebra etc.). Secondly, there is the algebra of the quantum fields themselves, formalized through vertex operator algebras. This situation is analogous to constructions in non-commutative geometry where we have a fuzzy or quantum space which is an associative coordinate algebra \[7, 8, 9\], as well as a Hopf algebra acting as a symmetry of the quantum space. It is natural to expect a similar structure for CFTd, except that we have a finite dimensional symmetry algebra SO(d,2) replacing the Virasoro algebra (and its generalizations) and large multiplicities of irreducible representations (irreps) coming from the fields/quantum states. This expectation is, at least partly, motivated by the operator/state correspondence of radial quantization

$$\lim_{x \to 0} \mathcal{O}_a(x)|0\rangle = |\mathcal{O}_a\rangle$$ (1)
which is a general property of CFTs for any $d$. The AdS/CFT correspondence together with the operator state correspondence implies, for example, that string states in AdS$_5 \times$S$^5$ are in correspondence with operators in $\mathcal{N} = 4$ SYM. An understanding of the quantum states (and associated physics) in quantum gravity on AdS spacetimes requires a detailed understanding of the CFT operators and associated algebraic structures.

The $1/2$-BPS sector is an interesting sector of $\mathcal{N} = 4$ SYM where these ideas can be developed quite explicitly. On the AdS side of the duality, there is a rich spectrum of physical states including gravitons, strings, branes and non-trivial spacetime geometries. On the CFT side of the duality, this sector is constructed from a single complex matrix $Z$, transforming in the adjoint representation

$$Z \to UZU^\dagger \quad \text{(2)}$$

under the $U(N)$ gauge symmetry. The generic gauge invariant operator is a multi-trace operator. For example, the complete set of gauge invariant operators that can be constructed using three fields is given by

$$\text{Tr}Z^3 \quad \text{Tr}Z^2\text{Tr}Z \quad (\text{Tr}Z)^3 \quad \text{(3)}$$

These operators have degree 3 and are in correspondence with the partitions of 3

$$3 = 3 \quad 3 = 2 + 1 \quad 3 = 1 + 1 + 1 \quad \text{(4)}$$

In general, operators of degree $n$ correspond to partitions of $n$ and they can be constructed using permutations $\sigma \in S_n$ as follows

$$\mathcal{O}_\sigma(Z) = \sum_{\sigma(1),\ldots,\sigma(n)} Z_{\sigma(1)} \cdots Z_{\sigma(n)} \quad \text{(5)}$$

For example

$$\text{Tr}Z^3 = \sum_{i_1,i_2,i_3} Z_{i_1}^{i_1} Z_{i_2}^{i_2} Z_{i_3}^{i_3} = \sum_{i_1,i_2,i_3} Z_{\sigma(1)}^{i_1} Z_{\sigma(2)}^{i_2} Z_{\sigma(3)}^{i_3} \quad \text{(6)}$$

with $\sigma = (123)$. The mapping between permutations and gauge invariant operators is not one-to-one since

$$\mathcal{O}_\sigma(Z) = \mathcal{O}_{\gamma\sigma\gamma^{-1}}(Z) \quad \text{for all} \quad \gamma \in S_n \quad \text{(7)}$$

which implies that two permutations in the same conjugacy class define the same gauge invariant operator. This nicely explains why gauge invariant operators correspond to partitions of $n$. The two-point function of degree $n$ operators is derived, as usual, by using Wick’s theorem and the basic 2-point function.
\[ \langle Z_j(x_1)(Z_j^\dagger(x_2)) = \frac{\delta_i^j \delta_k^l}{(x_1 - x_2)^2} \]  

This allows us to express the correlator in terms of permutation group multiplications as follows [10]

\[ \langle O_{\sigma_1}(Z(x_1))O_{\sigma_2}(Z^\dagger(x_2)) \rangle = \frac{1}{((x_1 - x_2)^2)^n} \frac{n!}{|C_{p_1}| |C_{p_2}|} \times \sum \sum \sum \delta(\sigma_1 \sigma_2 \sigma_3) N^{C_{\sigma_3}} \]  

The combinatoric part of this answer is a quantity in a 2D topological field theory (TFT2). TFT2s are equivalent to Frobenius algebras. A Frobenius algebra is an associative algebra with a non-degenerate pairing. The algebra corresponding to the combinatoric TFT2 of the \( \frac{1}{2} \)-BPS sector is the centre of the group algebra of the symmetric group \( S_n \). The connection to TFT2 can be generalized beyond the \( \frac{1}{2} \)-BPS sector and it turns out that multi-matrix sectors of \( \mathcal{N} = 4 \) SYM are related to other Frobenius algebras, built from permutations or associated diagram algebras such as Brauer algebras [10, 11, 12, 13, 14, 15, 16]. For a review of these ideas, the reader can consult [17].

It is natural to ask if the space-time dependence of correlators in CFT4 (and CFT4 for \( d > 2 \)) can also be described using a TFT2/Frobenius algebra language. The paper [18] gives a positive answer for the case of a free 4D massless scalar field, along with the cases in which the scalar transforms in the fundamental or in the adjoint of a global symmetry. The construction uses an infinite dimensional associative algebra which reproduces free field correlators of arbitrary free field composites and is a representation of \( so(4, 2) \) or \( Uso(4, 2) \). This algebra has an \( so(4,2) \) invariant non-degenerate pairing. In the paper [19] the algebraic structures associated with this CFT4/TFT2 construction were used to develop novel counting formulae and construction algorithms for the primary fields of free CFT4. The paper [20] describes perturbative CFTs from this algebraic point of view (equivariant algebras). Concrete examples that are described include sectors of \( d = 4 \mathcal{N} = 4 \) SYM at weak coupling as well as the Wilson-Fischer CFT, defined in \( d = 4 - \epsilon \) using the \( \phi^4 \) interaction. Novel algebraic structures needed to accomplish this include a deformed co-product for \( Uso(4, 2) \), the role of indecomposable representations of \( Uso(4,2) \) and diagram algebras which generalize known diagram algebras appearing in the representation theory of \( Uso(4,2) \). This work has some overlap with the paper [23].

This paper is organized as follows: Section 2 reviews the CFT4/TFT2 construction in the simplest possible setting of a free scalar field. The result is a \( U(so(4,2)) \) equivariant TFT2 with the quantum field realized as a vertex operator. Section 3 describes perturbative CFT4 by making use of deformed
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co-products while Section 4 introduces diagram algebras and their representations, motivated by the Wilson-Fisher CFT.

2 CFT4/TFT2 Construction of the free scalar field

The axiomatic approach to TFT2 that we adopt associates geometrical objects to algebraic objects, following the standard discussions (see the original work [21] and textbooks such as [22]), with appropriate adaptations to account for the infinite-dimensionality of the state spaces. For example, a vector space $\mathcal{H}$ is associated with a circle

$$ \bigcap \rightarrow \mathcal{H} \quad (10) $$

while tensor products of $\mathcal{H}$ go to disjoint unions of circles

$$ \bigodot \rightarrow \mathcal{H} \otimes \mathcal{H} \quad (11) $$

Interpolating surfaces between circles (cobordisms) are associated with linear maps between the vector spaces. For example, the map $\delta : \mathcal{H} \rightarrow \mathcal{H}$ is represented as a cylinder

$$ \delta^{AB} = \quad (12) $$

which takes circle $A$ into circle $B$, while the non-degenerate pairing $\eta : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathbb{C}$

$$ \eta^{AB} = \quad (13) $$

takes two circles to nothing. The product $C : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$

$$ C^{ABD} = \quad (14) $$

takes two circles to a circle. In the language of category theory, the circles are objects and the interpolating surfaces (cobordisms) are morphisms in a geometrical category, while the vector spaces are objects, and the linear maps are morphisms in an algebraic category. The correspondence between geometrical objects and algebraic objects is a functor between the two categories. The existence of this functor requires that all relations on the geometrical side should be mirrored on the algebraic side. As an example, the statement that the pairing $\eta^{AB}$ is non-degenerate is expressed in terms of the inverse
pairing

$$\tilde{\eta}^{AB} = \begin{array}{c}
A \\
B
\end{array}$$

(15)

as the statement that $\eta$ and $\tilde{\eta}$ glue to give the cylinder

$$\eta_{AB}\tilde{\eta}^{BC} = \delta^C_A$$

(16)

where the gluing operation is implemented by summing over the circles to be glued. Using the product $C_{AB}^D$ and the pairing $\eta_{AB}$ we can define a new map

$$C_{ABD} = \eta_{DC}C_{AB}^D$$

(17)

which is the familiar relation between the CFT correlator ($C_{ABD}$) and the OPE ($C_{AB}^D$). Finally, associativity of the OPE is expressed as

$$C_{AB}^E C_{EC}^D = C_{BC}^E C_{EA}^D$$

(18)

To summarise, TFT2's correspond to commutative, associative, non-degenerate algebras known as Frobenius algebras. TFT2 with a global symmetry group $G$ is defined by [24]. Since this will play an important role in our construction, it is worth summarizing the essential features from [24] with one important modification of the discussion due to the infinite dimensionality of the state space

1. The state space is a representation of a group $G$ - which will be $SO(4,2)$ in our application.
2. The linear maps are $G$-equivariant linear maps.
3. The state space is infinite dimensional: amplitudes are defined for surfaces without handles. Consequently, this is a genus restricted TFT2.

The basic two point function in the CFT of a free scalar in four dimensions is
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\[ \langle \phi(x_1)\phi(x_2) \rangle = \frac{1}{(x_1 - x_2)^2} \]  

(19)

Correlators of composite operators are constructed using this contraction, according to Wick’s theorem. Thus, the first step in our construction is to understand the 2-point function of the elementary field in the TFT2 language. The \( so(4,2) \) symmetry of the CFT is our starting point. The Lie algebra is spanned by the dilatation operator \( D \) which generates dilatations, the momenta \( P_\mu \) which generate translations, the generators \( M_{\mu\nu} \) of \( so(4) \) rotations and the generators \( K_\mu \) of special conformal transformations. To carry out a radial quantization of the theory we choose a point, say the origin of Euclidean \( \mathbb{R}^4 \). The usual state operator map associates states with the scalar field and its descendents

\[
\begin{align*}
\lim_{x \to 0} \phi(x)|0\rangle &= v^+ \\
\lim_{x \to 0} \partial_\mu \phi(x)|0\rangle &= P_\mu v^+ \\
&\vdots
\end{align*}
\]

(20)

The state \( v^+ \) is the lowest energy state in a lowest-weight representation \( V_+ \) of \( so(4,2) \)

\[
\begin{align*}
Dv^+ &= v^+ \\
K_\mu v^+ &= 0 \\
M_{\mu\nu} v^+ &= 0
\end{align*}
\]

(21)

Higher energy states are generated by \( S^{\mu_1\mu_2\cdots\mu_l}_{l} P_{\mu_1} P_{\mu_2} \cdots P_{\mu_l} v^+ \), where \( S^{\mu_1\mu_2\cdots\mu_l}_{l} \) is a symmetric traceless tensor of \( so(4) \). The index \( l \) labels a basis of linearly independent symmetric traceless tensors. There is also a dual representation \( V_- \), which is a representation with negative scaling dimensions

\[
\begin{align*}
Dv^- &= -v^- \\
P_\mu v^- &= 0 \\
M_{\mu\nu} v^- &= 0
\end{align*}
\]

(22)

Other states in this representation are generated by acting with \( S^{\mu_1\mu_2\cdots\mu_l}_{l} K_{\mu_1} K_{\mu_2} \cdots K_{\mu_l} \). There is a \( \eta : V_+ \otimes V_- \to \mathbb{C} \), which is \( so(4,2) \) invariant

\[
\eta(\mathcal{L}_a v, w) + \eta(v, \mathcal{L}_a w) = 0
\]

(23)

After making the choice

\[
\eta(v^+, v^-) = 1
\]

(24)

the invariance conditions \([23] \) and the properties of the states \( v^+ \) and \( v^- \), determine \( \eta \). For example
\[ \eta(P^\mu v, K^\nu v) = \eta(P^\mu v^+, P^\mu K^\nu v^-) \\
= \eta(v^+, (-2D\delta_{\mu\nu} + 2M_{\mu\nu})v^-) = 2\delta_{\mu\nu} \quad (25) \]

Using invariance conditions one finds that \( \eta(P^\mu P^\mu v^+, v^-) \) is zero. Setting
\[ P^\mu P^\mu v^+ = 0, \]
which physically corresponds to imposing the equation of motion, identifies \( V_+ \) as a quotient of a bigger representation \( \tilde{V}_+ \). \( \tilde{V}_+ \) is spanned by
\[ P^\mu_1 \cdots P^\mu_l v^+ \quad (26) \]
i.e it is the vector space of polynomials in \( P^\mu \). This is an indecomposable representation. After we perform the quotient by the equation of motion, we recover the irreducible representation \( V_+ \). The quotient also ensures that \( \eta \) is non-degenerate i.e. that there are no null vectors. So we see that \( \eta \) is the structure we need for the construction of a TFT2 with \( \so(4,2) \) symmetry. It has both the non-degeneracy property and the invariance property. So there is an invariant in \( V_+ \otimes V_- \) and thus in \( V_- \otimes V_+ \), but not in \( V_+ \otimes V_+ \) or \( V_- \otimes V_- \). It is useful to introduce \( V = V_+ \oplus V_- \) and define \( \tilde{\eta} : V \otimes V \to \mathbb{C} \)
\[ \tilde{\eta} = \begin{pmatrix} 0 & \eta_{++} \\ \eta_{--} & 0 \end{pmatrix} \quad (27) \]

In \( V \) we have a state, corresponding to the quantum field, given by
\[ \Phi(x) = \frac{1}{\sqrt{2}}(e^{-ip^\mu x^\mu} + x^\mu x'^\nu e^{iK^\nu x'}v^-) \quad x'_\mu = \frac{x_\mu}{x^2} \quad (28) \]

and a calculation with the invariant pairing shows that
\[ \eta(\Phi(x_1), \Phi(x_2)) = \frac{1}{(x_1 - x_2)^2} \quad (29) \]

This is the basic free field 2-point function, now constructed from the invariant map \( \eta : V \otimes V \to \mathbb{C} \). To get all correlators, we must set up a state space, which knows about composite operators. The states obtained by the standard operator state correspondence from general local operators are of the form
\[ P^\mu_1 \cdots P^\mu_n \phi P^\nu_1 \cdots P^\nu_m \phi \cdots P^\tau_1 \cdots P^\tau_m \phi \quad (30) \]

Composite operators belonging to the \( n \) field sector correspond to states in which \( n \phi \) fields appears. Particular linear combinations of these states are primary fields, which are lowest weight states (annihilated by \( K^\mu \)) that generate irreducible representations (irreps) of \( \so(4,2) \) through the action of the raising operators \( (P^\mu) \). The list of primary fields in the \( n \)-field sector is obtained by decomposing the space
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\[ \text{Proj}_{S_n} \text{inv} (V_+^{\otimes n}) \equiv \text{Sym}^n (V_+) \]  

(31)

into SO(4,2) irreps. The symmetrization on the right hand side of (31) is needed because \( \phi \) is a boson. We can now introduce the state space \( \mathcal{H} \) of the TFT2, which we associate to a circle in TFT2. The state space consists of all possible primaries and their descendents

\[ \mathcal{H} = \bigoplus_{n=0}^{\infty} \text{Sym}^n (V) \]

(32)

where \( V = V_+ \oplus V_- \). This state space is big enough to accommodate all the composite operators and it admits an invariant pairing. The state space is small enough for the invariant pairing to be non-degenerate. The state space contains

\[ \Phi(x) \otimes \Phi(x) \otimes \Phi \cdots \otimes \Phi(x) \]

(33)

which is used to construct composite operators in the TFT2 set-up. By construction, the pairing \( \eta : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathbb{C} \) reproduces all 2-point functions of arbitrary composite operators. The construction is straight forward: recall \( \mathcal{H} \) is built from tensor products of \( V \), and we have already introduced an “elementary” \( \tilde{\eta} : V \otimes V \rightarrow \mathbb{C} \). The construction of the complete \( \eta \) map is built from products of the elementary \( \tilde{\eta} \), using Wick contraction sums, in the obvious way. As an example, for \( v_1, v_2, v_3, v_4 \in V \) we have

\[ \eta(v_1 \otimes v_2, v_3 \otimes v_4) = \tilde{\eta}(v_1, v_3) \tilde{\eta}(v_2, v_4) + \tilde{\eta}(v_1, v_4) \tilde{\eta}(v_2, v_3) \]

(34)

We complete the definition by setting

\[ \eta^{(n)}(v^{(m)}) \propto \delta^{mn} \]

(35)

where \( v^{(k)} \in \text{Sym}^k (V) \). This defines the pairing \( \eta_{AB} \) where \( A, B \) take values in the space \( \mathcal{H} \) given by the sum of all \( n \)-fold symmetric products of \( V = V_+ \oplus V_- \). Notice that the building blocks used in constructing \( \eta \) are invariant maps. The product of these invariant maps is also obviously invariant. We can also demonstrate that \( \eta \) is non-degenerate. The basic idea is that if you have a non-degenerate pairing \( V \otimes V \rightarrow \mathbb{C} \), it extends to a non-degenerate pairing on \( \mathcal{H} \otimes \mathcal{H} \rightarrow \mathbb{C} \), by using the sum over Wick patterns. Consequently, we have

\[ \eta_{AB} \tilde{\eta}^{BC} = \delta^C_A \]

(36)

This is the snake-cylinder equation, given in (16).

In a very similar way it is possible to define 3-point functions \( C_{ABC} \) and, in general higher point functions \( C_{ABC \cdots} \) using Wick pattern products of the basic \( \eta \)’s. By writing explicit formulae for these sums over Wick patterns, we can show that the associativity equations are satisfied. Consider equation
The $C_{ABC}$ give 3-point functions, while the $C_{ABD} = C_{ABD} \tilde{\eta}^{DC}$ give the OPE-coefficients. Through this connection, the associativity equations of the TFT2 are the crossing equations of CFT4, obtained by equating expressions for a 4-point correlator obtained by doing OPEs in two different ways. There an important property of the OPE in this language, easily illustrated by the product

$$\text{Sym}^2(V) \otimes \text{Sym}^2(V) \rightarrow \text{Sym}^4(V) \oplus \text{Sym}^2(V) \oplus \mathbb{C}$$

(37)

which corresponds to the free field OPE, which takes the schematic form

$$\phi^2(x)\phi^2(0) \rightarrow \phi^4 \oplus \phi^2 \oplus 1$$

(38)

This demonstrates that the presence of both $V_+$ and $V_-$ is needed if the TFT2 is to construct this OPE in representation theory.

The algebraic framework developed above allows us to exhibit novel ring structures in the state space of the TFT2. Further, this algebraic structure can profitably be used to give a construction of primary fields in free CFT4/CFT $d$ [19]. The state space in radial quantization, set up around $x = 0$, is (for $x' = 0$, we would keep $V_-$ instead)

$$\mathcal{H}_+ = \bigoplus_{n=0}^{\infty} \text{Sym}^n(V_+)$$

(39)

The irrep $V_+$ is isomorphic to a space of polynomials in variables $x_\mu$, quotiented by the ideal generated by $x_\mu x_\mu$. Taking a many-body physics view of $\mathcal{H}_+$, this is a quotient of a polynomial ring in $dn$ variables $x_I^\mu$. The construction of primaries, or equivalently, the problem of describing lowest weight states of irreducible representations in $\mathcal{H}_+$ is usefully done by recognizing the connection to a closely related problem about rings. It turns out that the construction of primary fields in $d$ dimensions, and the refined counting of these primaries, according to their scaling dimension $n$ and $so(d)$ irreps, is equivalent to studying a polynomial ring in variables the $X_A^\mu$ with $\mu \in \{1, \cdots, d\}$ and $A \in \{1, 2, \cdots, n-1\}$, under the constraints

$$A(1 - A^2) \sum_{\mu=1}^{d} X_\mu^A X_\mu^A + \sum_{B:B > A}^{d} \sum_{\mu=1}^{d} 2A(1 + A)X_\mu^A X_B^B$$

$$+ \sum_{B:B < A}^{d} \sum_{\mu=1}^{d} B(1 + B)X_B^B X_\mu^B = 0$$

(40)

for $1 \leq A \leq (n-1)$, and

$$\sum_{A=1}^{n-1} \sum_{\mu=1}^{d} X_\mu^A X_\mu^A = 0$$

(41)
The details of the derivation of these constraints are explained in \[19\].

### 3 Perturbative CFTs

Having explained the CFT4/TFT2 construction for the free scalar field, it is natural to ask about constructions for interacting theories. Towards this end, the first example theory we have in mind is the Wilson-Fischer (WF) fixed point, described by the Lagrangian

$$\int d^d x (\partial_\mu \phi \partial^\mu \phi + \frac{g}{4!} \phi^4)$$

(42)

Together with a continuation of the Feynman rules to \(d = 4 - \epsilon\) dimensions. Choosing the critical value of the coupling constant

$$g^* \sim \frac{16\pi^2}{3} \epsilon + O(\epsilon^2)$$

(43)

leads to a vanishing beta function and, consequently, a CFT. The fundamental field \(\phi\) as well as composite operators (given by polynomials in derivatives of \(\phi\)) have a modified dimension. Apart from the classical dimension, there is also an anomalous dimension, generated by loop corrections. The anomalous dimensions of the WF theory are captured by a dilatation operator. In particular, the one-loop corrections to the dimensions of composite operators

$$\partial^{k_1} \phi \partial^{k_2} \phi \cdots \partial^{k_L} \phi$$

(44)

are captured by a 2-body Hamiltonian

$$H = \sum_{i<j} \rho_{ij}(P_0)$$

(45)

where \(P_0\) is a projector to an irrep in \(V \otimes V\) with \(V\) the irrep of the scalar \(\phi\). At order \(\epsilon\), \(\phi\) has a vanishing anomalous dimension, while that of \(\phi^2\) is non-vanishing. A naive intuition informed by tensor products of representations would suggest that dimension of a composite operator is given by the sum of the dimensions of its constituents, but this is not correct. To obtain the correct dimension for \(\phi^2\) we need

$$D(v \otimes v) = (D \otimes 1 + 1 \otimes D)(v \otimes v) + \frac{\epsilon}{3} P_0(v \otimes v)$$

(46)

This motivates the definition of the deformed co-product

$$\Delta(D) = D \otimes 1 + 1 \otimes D + \frac{\epsilon}{3} P_0$$

(47)
This deformation is highly reminiscent of deformations we encounter in quantum groups. For example, in $U_q(su(2))$ we have

$$\Delta(J_+) = J_+ \otimes q^H + q^{-H} \otimes J_+$$

(48)

and

$$\Delta(L_a) \in \text{End}(V \otimes W) \quad \Delta_\epsilon(L_a) \in \text{End}(V \otimes W)$$

(49)

with

$$\Delta(L_a) = \Delta_0(L_a) + \epsilon \Delta_\epsilon(L_a)$$

$$\Delta_0(L_a) = L_a \otimes 1 + 1 \otimes L_a$$

(50)

such that

$$[L_a, L_b] = f_{abc} L_c$$

$$[\Delta(L_a), \Delta(L_b)] = f_{abc} \Delta(L_c)$$

(51)

At order $\epsilon$ we have worked out the deformation needed to explain the complete spectrum of one loop anomalous dimensions. The co-products for the complete set of generators are

$$\Delta(D) = D \otimes 1 + 1 \otimes D + \frac{\epsilon}{3} P_0$$

$$\Delta(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu$$

$$\Delta(K_\mu) = K_\mu \otimes 1 + 1 \otimes K_\mu - \frac{\epsilon}{3} P_0 \left( \frac{\partial}{\partial P_\mu} \otimes 1 + 1 \otimes \frac{\partial}{\partial P_\mu} \right) P_0$$

$$\Delta(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}$$

(52)

It is a straightforward exercise to verify that the above co-products are consistent with the commutation relations of $so(4, 2)$. For example, we have checked that

$$[\Delta(K_\mu), \Delta(P_\nu)] = 2\Delta(M_{\mu\nu}) - 2\delta_{\mu\nu} \Delta(D)$$

(53)

In performing this check and others like it, it is useful to note $\Delta_0(L_a)P_0 = P_0\Delta_0(L_a)$ and $P_0^2 = P_0$.

The planar $SU(2)$ sector in $\mathcal{N} = 4$ SYM is another interacting CFT that has an instructive TFT2/CFT4 construction. For this example there is no need to continue to $d = 4 - \epsilon$ dimensions. In this example too, deformed co-products are needed to reproduce the one loop spectrum of anomalous dimensions. Consider the three operators

$$O_z = \frac{1}{\sqrt{2N}} \text{Tr}(Z^2) \quad O_y = \frac{1}{\sqrt{2N}} \text{Tr}(Y^2)$$

$$O_{zy} = \frac{1}{\sqrt{3N^2}} \left( \text{Tr}(YZYZ) - \text{Tr}(Y^2Z^2) \right)$$

(54)
These operators are all eigenstates of the one loop dilatation operator. $O_{zy}$ has a non-zero anomalous dimension $\delta = \frac{3\lambda^4}{\pi^2}$ in the planar limit [26]. The anomalous dimensions of both $O_z$ and $O_y$ vanish. In the free theory, the dimensions add

$$\text{Dim}(O_z) + \text{Dim}(O_y) = \text{Dim}(O_{zy})$$  \hspace{1cm} (55)

At first order in the interaction this relationship is corrected as follows

$$\text{Dim}(O_z) + \text{Dim}(O_y) = \text{Dim}(O_{zy}) - \delta$$  \hspace{1cm} (56)

As the first step, consider $\mathfrak{so}(4,2)$ irrep generated by $O_z$. In the operator-state correspondence, the operator $O_z$ corresponds to a tower of operators

$$O_z(0) \rightarrow v_z$$

$$\partial_{\mu_1} O_z(0) \rightarrow P_{\mu_1} v_z$$

$$\partial_{\mu_1} \partial_{\mu_2} O_z(0) \rightarrow P_{\mu_1} P_{\mu_2} v_z$$

$$\vdots$$  \hspace{1cm} (57)

The states live in a representation $V_z$ of $\mathfrak{so}(4,2)$. The lowest weight state $v_z$ has the properties

$$Dv_z = 2v_z \quad M_{\mu\nu} v_z = 0 \quad K_{\mu} v_z = 0$$  \hspace{1cm} (58)

At dimension $(2 + k)$ we have states

$$V_k = \text{Span} \{ P_{\mu_1} \cdots P_{\mu_k} v_z \}$$  \hspace{1cm} (59)

The direct sum forms the $\mathfrak{so}(4;2)$ irrep $V^{(z)}$

$$V = \bigoplus_{k=0}^{\infty} V_k$$  \hspace{1cm} (60)

There is a similar representation $V_y$ built on the primary $O_y$. $V_z$ and $V_y$ are isomorphic representations of $\mathfrak{so}(4,2)$. We also need the representation $V_{zy}$, built on $O_{zy}$. This representation has lowest weight state $v_{zy}$ with properties

$$Dv_{zy} = (4 + \delta)v_{zy}$$

$$K_{\mu} v_{zy} = 0$$

$$M_{\mu\nu} v_{zy} = 0$$  \hspace{1cm} (61)

States at $D = 4 + \delta + k$ are obtained by acting with $k$ $P$’s.

Given the non-additivity of the anomalous dimensions, we cannot model the 3-point correlator with the standard action of the Lie algebra on $V_z \otimes V_y$. If we use the standard action, we would have
\[ \Delta_0(D)(v_z \otimes v_y) = (D \otimes 1 + 1 \otimes D)(v_z \otimes v_y) = 4v_z \otimes v_y \]  
(62)

whereas the dimension of \( v_{zy} \) is \( 4 + \delta \). The map \( f : v_{zy} \rightarrow v_z \otimes v_y \)

\[ \Delta_0(D)f(v_{zy}) = f\Delta_0(D)(v_{zy}) \]  
(63)

can be extended to an equivariant map \( V_{zy} \rightarrow V_z \otimes V_y \) at zero coupling, but cannot be so extended when we turn on \( \delta \) at non-zero coupling.

Let \( P_4 \) be the projector to \( V_4 \) - the so(4,2) representation with scalar lowest weight of dimension 4 - in the standard tensor product decomposition of \( V_2 \otimes V_2 \). We can define a deformed co-product

\[
\begin{align*}
\Delta(D) &= \Delta_0(D) + \delta P_4 \\
\Delta(P_\mu) &= \Delta_0(P_\mu) \\
\Delta(M_{\mu\nu}) &= \Delta_0(M_{\mu\nu}) \\
\Delta(K_\mu) &= \Delta_0(K_\mu) - \frac{\delta}{2} P_4 \Delta_0 \left( \frac{\partial}{\partial P_\mu} \right) P_4 
\end{align*}
\]  
(64)

With the so(4,2) action on \( V_z \otimes V_y \) given by

\[ L_a : v_1 \otimes v_2 \rightarrow \Delta(L_a)(v_1 \otimes v_2) \]  
(65)

and the so(4,2) action on \( V_{zy} \) which we will refer to as \( \rho_{zy} \), we can extend \( f \)

\[ f : V_{zy} \rightarrow V_z \otimes V_y \]  
(66)

such that

\[ f\rho_{zy}(L_a) = \Delta(L_a)f \]  
(67)

Using the map \( f \), we can construct the correlator as follows

\[ \eta((e^{-iP_{x_1}v_+^+} \otimes e^{-iP_{x_2}v_+^+}, (x_3^i)^2 f(e^{iK_i x_3^i v_{zy}^+})) \]  
(68)

The inner product \( g \) on \( V_z \otimes V_y \) is related by using the anti-automorphism on so(4,2) to the invariant pairing on

\[ \eta : (V_+ \otimes V_+) \otimes (V_- \otimes V_-) \rightarrow C \]  
(69)

### 4 \( d = 4 - \epsilon \) and diagram algebras

In our example of the Wilson-Fischer, we need to continue from \( d = 4 \) to \( d = 4 - \epsilon \) dimensions in order to obtain a non-trivial CFT. Analytically continued tensor rules, and in particular the rule \( \delta_{\mu}^\nu = 4 - \epsilon \), are needed to construct the stress tensor with the right properties. The state space \( V_+ \) used
in the free scalar field theory is a quotient of a space $\tilde{V}_+$ spanned by states of the form

$$\{ P_{\mu_1} \cdots P_{\mu_k} v \}$$

(70)

The quotient amounts to setting to zero $P_{\mu}P_{\mu}v$. The stress tensor

$$T_{\mu\nu} = \frac{1}{2}(P_{\mu}v \otimes P_{\nu}v + P_{\nu}v \otimes P_{\mu}v - \delta_{\mu\nu}P_{\tau}v \otimes P_{\tau}v)$$



$$- \frac{\alpha}{6} \Delta (P_{\mu}P_{\nu} - P^2 \delta_{\mu\nu})v \otimes v$$

(71)

is a state in $\tilde{V}_+ \otimes \tilde{V}_+$. The above state is conserved and traceless upon using the interacting equation of motion, along with

$$Dv = \left(1 - \frac{\epsilon}{2}\right)v \quad M_{\alpha\beta}v = 0 \quad \delta_{\mu\mu} = 4 - \epsilon$$

(72)

The positive part of the state space

$$\bigoplus_{n=0}^{\infty} \text{Sym}^n(V_+)$$

(73)

where $V_+ = \tilde{V}_+/\{P^2v\}$ is replaced by

$$\bigoplus_{n=0}^{\infty} \text{Sym}^n(\tilde{V}_+)$$

(74)

and we need to quotient by

$$P^2v - 4g^*v \otimes v \otimes v$$

(75)

Thus, understanding the interacting equations of motion in terms a quotient space, requires working with $\tilde{V}_+$ and its tensor powers. The quotient condition relates states in $V_+$ to $\tilde{V}_+^{\otimes 3}$. Notice that in both the free and interacting theories, we move from $V_+$ to $\tilde{V}_+$ by quotienting with the equation of motion.

To make sense of the rule $\delta_{\mu\mu} = 4 - \epsilon$, we need to construct a diagram algebra, much like the Brauer algebras. In our TFT2 setting, $Uso(d)$ (and $Uso(d,2)$) itself has to be made diagrammatic in order to give a TFT2 with conformal equivariance formulation of the perturbative correlators.

If we depict the product $M_{ij}M_{kl}$ in the universal enveloping algebra $Uso(d)$ by juxtaposing two boxes side to side, we can express

$$M_{ij}M_{kl} - M_{kl}M_{ij} = \delta_{jk}M_{il} + \delta_{il}M_{jk} - \delta_{jl}M_{ik} - \delta_{ik}M_{jl}$$

(76)

as a relation between diagrams as follows
To go from the diagrammatic relation to the equation in Uso(d), we attach the labels $i, j, k, l$ to the crosses starting with $i$ for the left-most cross and proceeding with $j, k, l$ as we go to the crosses towards the right. The anti-symmetry can be expressed diagrammatically as follows:

![Diagram](image)

The quadratic Casimir $M_{ij}M_{ij}$ is associated to the diagram shown below:

![Diagram](image)

We will define an infinite dimensional associative algebra over $\mathbb{C}$, denoted $\mathcal{F}$, abstracted from the generators $M_{ij}$ of Uso(d). An associative algebra is a vector space equipped with a product $m$

$$m : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$$

The vector space $\mathcal{F}$ is:

$$\mathcal{F} = \mathbb{C} \oplus \text{Span}_\mathbb{C}(M) \oplus \cdots$$

The $\cdots$ refers to subspaces which can be specified efficiently, using the oscillator construction of $Uso(d)$ and its interpretation in terms of equivariant maps and diagrams. The $d$-dimensional oscillator relations are:

$$[a_i^\dagger, a_j] = -\delta_{ij}$$

and the Lie algebra generators of so(d) can be written as:

$$M_{ij} = a_i^\dagger a_j - a_j^\dagger a_i$$

Think of this as specifying a state (which we can also call $M_{ij}$) using $V = \text{Span}(a, a^\dagger)$ and $W = \text{Span}(e_i : i \in 1, \cdots, d)$:

$$M_{ij} = a^\dagger \otimes e_i \otimes a \otimes e_j - a^\dagger \otimes e_j \otimes a \otimes e_i \in V \otimes W \otimes V \otimes W$$
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It is useful to write this as

$$M_{ij} = P^W_A \otimes W (a^\dagger \otimes e_i \otimes a \otimes e_j) \quad (85)$$

The number of these $M_{ij}$ is $d(d-1)/2$. This obstructs continuing $d$ to non-integer dimensions. In contrast to this, the space of equivariant maps

$$P_A(W \otimes W) \rightarrow P^W_A (V_+ \otimes W \otimes V_- \otimes W) \quad (86)$$

is a one-dimensional vector space (for $d > 4$) spanned by the map $M$ acting as

$$M : (e_{i_1} \otimes e_{i_2} - e_{i_2} \otimes e_{i_1}) \rightarrow (a^\dagger \otimes e_{i_1} \otimes a \otimes e_{i_2} - a^\dagger \otimes e_{i_2} \otimes a \otimes e_{i_1}) \quad (87)$$

More compactly, we can write

$$M : W_2 \rightarrow (VW)_2 \quad (88)$$

where $W_2 = P_A(W \otimes W)$ and

$$(VW)_2 = P^W_A (V_+ \otimes W \otimes V_- \otimes W) \quad (89)$$

Now, introduce the infinite dimensional associative algebra

$$\mathcal{F} = \oplus_{m,n=0} \mathcal{F}_{m,n} \quad (90)$$

with

$$\mathcal{F}_{m,n} = \text{Hom}_{so(d):d>2m+2n}(W_2^\otimes m, (VW)^\otimes n) \quad (91)$$

This algebra contains all the $M$-diagrams we drew earlier, and it includes diagram with arbitrarily large numbers of $M$-boxes. We define $U_*$ as a quotient of this space of diagrammatic maps by the commutation relation (77).

In order to understand the representation theory of $U_*$, which will be a diagrammatic analog of the representation theory of $\text{Uso}(d)$ at large $d$, we will start by interpreting the basic equation

$$[M_{ij}, a_k^\dagger] = \delta_{jk} a_k^\dagger - \delta_{ik} a_j^\dagger \quad (92)$$

which gives the action of $Uso(d)$ on the $d$-dimensional vector representation. By using labelled $M$-box diagrams, and associating to $a_k^\dagger$ a line joining a cross to a circle, the above equation becomes

$^1$ For $d = 4$ we can also use $\epsilon_{i_1 i_2 i_3 i_4}$ which gives another map, so we will use large $d$ in the appropriate places in our definitions to keep things as simple as possible.
Using the definitions from above,
\[ a^i = a^i \otimes e_i \in V_+ \otimes W \quad e_i \in W \]  

There is an \( \text{so}(d) \) equivariant map \( \rho \)
\[ \rho : W \rightarrow (V_+ \otimes W) \]

We can think of this as the map which attaches \( e_i \in W \) to \( a^i \) to produce \( a^i \). The map commutes with \( \text{so}(d) \). This leads to the definition of a vector space of diagrams
\[ V^* = \bigoplus_{n=0}^{\infty} \text{Hom}_{\text{so}(d); d \gg n} (W \otimes W, V_+ \otimes W) \]

We can define a diagrammatic inner product for \( V^* \) (which involves loops evaluating to \( d \)) and show that \( V^* \otimes V^* \) contains orthogonal subspaces corresponding to the symmetric-traceless, the trace, and the anti-symmetric. The proof proceeds by proving \( B_{n=2}(d) \) commutes with \( U_\text{so}(d) \) action on \( V^* \otimes V^* \). Much as \( B_2(d) \) commutes with \( U_\text{so}(d) \) on \( V_2 \otimes V_2 \), but in the above both the algebra and the representation space are spanned by diagrams. \( d \) appears upon evaluation of Casimirs, which involve loops evaluated as \( d \) - which can then be set to \( 4 - \epsilon \).

There are some rather natural conjectures we can formulate about \( (V^*) \). First, the action of \( U^* \) should commute with a known diagrammatic algebra, the Brauer algebra \( B_d(n) \), much as \( B_4(n) \) commutes with \( U_\text{so}(d) \) in \( V_2 \). Proving this conjecture would involve generalising arguments given in [20].

These are the first steps towards a fully diagrammatic Schur-Weyl duality where \( U^* \), with loop parameter \( d \), acts on \( V^* \) and is Schur-Weyl dual to \( B_d(n) \).

5 Summary and Outlook

Our key result [20] has been to define \( U_{\epsilon,2} \) acting on \( V^{* \otimes 2} \) as a generic \( d \) version of \( U_\text{so}(d,2) \) acting on \( V \). To summarise, we present evidence that perturbative CFT can be formulated in terms of \( U_\text{so}(d,2) \) (for theories in
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integer dimensions) or $U_{r,2}$ (for theories like Wilson Fischer), using familiar constructions in algebra/representation theory, namely

- indecomposable representations,
- deformed co-products and
- diagram algebras.

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