Non-perturbative membrane spin-orbit couplings in M/IIA theory

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Abstract

Membrane source-probe dynamics is investigated in the framework of the finite $N$-sector DLCQ $M$ theory compactified on a transverse two-torus for an arbitrary size of the longitudinal dimension. The non-perturbative two fermion terms in the effective action of the matrix theory, the (2+1)-dimensional supersymmetric Yang-Mills theory, that are related to the four derivative $F^4$ terms by the supersymmetry transformation are obtained, including the one-loop term and full instanton corrections. On the supergravity side, we compute the classical probe action up to two fermion terms based on the classical supermembrane formulation in an arbitrary curved background geometry produced by source membranes satisfying the BPS condition; two fermion terms correspond to the spin-orbit couplings for membranes. We find precise agreement between two approaches when the background space-time is chosen to be that of the DLCQ $M$ theory, which is asymptotically locally Anti-de Sitter.

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1 Introduction

By now considerable body of evidence toward the feasibility of the quantum description of $M$ theory via matrix theory [1, 2] has been accumulated. Especially within the framework of the discrete light-cone quantization (DLCQ), the explicit scattering calculations performed in matrix theory were successfully compared to the supergravity calculations; Becker, Becker, Polchinski, and Tseytlin considered the scattering between two $D$-particles ($M$-momentum) and showed that the matrix theory side calculation for the effective action precisely reproduces the eleven-dimensional supergravity side calculation up to two loops [3]. Similarly in the context of the membrane scatterings, especially for the weak coupling limit (the limit where the size of the longitudinal eleventh circle is small), the agreement between the two approaches was obtained by many authors [6]-[12]. Recalling that the focus of the most of these analysis has been the perturbative brane dynamics, what we attempt in this paper is a systematic study of the non-perturbative brane dynamics.

Our approach is based on two recent lines of developments. First, it was observed in [13, 14, 15] that the appropriate space-time background geometries for the description of the $N$-sector DLCQ $M$ theory compactified on a transverse $p$-torus ($p > 1$) are not asymptotically flat. For the membrane dynamics, that can be most easily studied within the context of the $M$ theory compactified on a transverse two-torus, the relevant background geometry is asymptotically locally Anti-de Sitter (AdS) type [14, 15]. In this paper, we study, in detail, the consequence to the effective action of these non-asymptotically flat background geometries. Second, initiated by Stern, Sethi and Paban [16, 17], it has been noted that the strong coupling dynamics and thus the effective action of supersymmetric gauge theories are strongly constrained by the supersymmetry. In the case when there are sixteen supercharges, the constraints are strong enough to uniquely determine the

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1In Ref. [3], an LSZ formalism for the scattering problems in the context of the eleven-dimensional $M$ theory was developed. Recently, the general two-body scattering perturbative dynamics in $M$ theory was systematically analyzed in Ref. [8] up to four fermion terms.

2For the $M$-momentum dynamics and the matrix quantum mechanics, as was first formally noted in [3] and clarified in [13], the background geometry is described by the (zero-mode part of) Aichelberg-Sexl type shockwave geometry [15]. This geometry is asymptotically flat in eleven dimensions and the time coordinate is asymptotically light-like.
full non-perturbative eight fermion terms of the effective action (up to an overall con-
stant) of the (2+1)-dimensional supersymmetric Yang-Mills (SYM) theory \[17\], which
are related to the bosonic four derivative $F^4$ terms by supersymmetry transformations.
The analysis presented in this paper gives an $M$-theory interpretation of the Stern-Sethi-
Paban’s work in terms of the $M$ theory in non-asymptotically flat background geometries
of Refs. \[14, 15\].

According to the arguments of Seiberg and Sen \[18\], the microscopic dynamics of
DLCQ $M$ theory compactified on a transverse two-torus is described by the (2+1)-
dimensional SYM theory (we loosely call it matrix theory throughout this paper). What
we find in this paper is; the DLCQ supergravity effective action for membrane dynamics
whose computation is based on Ref. \[14, 15\] precisely agrees with the matrix theory ef-
tective action calculated by the techniques based on Ref. \[16, 17\] for an arbitrary value
of the coupling constant (or the size of the longitudinal eleventh circle), consistent with
Ref. \[2\]. This precise agreement will be explicitly verified for the bosonic four derivative
$F^4$ terms and the two fermion terms related to them by supersymmetry in the effective
action independently computed in both approaches. It turns out that the two fermion
terms, now including full non-perturbative corrections, can be interpreted as spin-orbit
couplings for membranes. In the context of the matrix quantum mechanics versus su-
pergravity, the spin-orbit couplings for the D-particles were successfully computed and
positively compared in both approaches at the perturbative level \[19\]-\[24\]. Since we con-
sider an arbitrary value of the longitudinal eleventh circle size, our results apply equally
well to the IIA theory D2-branes, as well as to the eleven-dimensional $M$ membranes.
When we take the size of the eleventh circle to the infinity, the background geometry of
Refs. \[14, 15\] becomes an Anti-de Sitter space tensored with a fixed size seven sphere,
$AdS_4 \times S^7$. In this case, the duality conjecture of Maldacena \[25, 26\] relates the super-
gravity to the conformal phase (the infra-red limit) of the (2+1)-dimensional SYM theory.
Since we independently compute the effective action for each theory, our results constitute
a strong supporting evidence for the Maldacena conjecture.

The material presented in Sec. 2.1 and Sec. 2.2 is partially based on our results reported
in Ref. \[27\] on the membrane dynamics in the DLCQ $M$ theory.
2 Membrane dynamics: DLCQ $M$ theory compactified on a transverse two-torus

This section is organized as follows. In Sec. 2.1, we start from the calculation of the effective action for the spinless probe membrane moving in the background geometry of spinless source membranes in the DLCQ supergravity framework. Since the longitudinal eleventh direction is also compactified as dictated by the DLCQ prescription \cite{2}, our effective action includes all the contribution from the mirror membranes. This result is reshuffled by applying the Poisson resummation formula along the eleventh direction for further analysis. In Sec. 2.2, we show the correspondence between the DLCQ supergravity and matrix theory at the level of the bosonic effective action. Our logic is as follows; utilizing the sixteen supersymmetries of (2+1)-dimensional SYM theory, Stern, Sethi and Paban \cite{17} determined the exact eight fermion terms in the effective action. We start by recalling why their analysis works and, based on a supersymmetry argument, we calculate the exact four derivative bosonic effective action from their eight fermion terms. Thus determined effective action from the SYM theory is shown to be identical to the supergravity bosonic effective action computed in Sec. 2.1. Based on the same supersymmetry argument, we sketch how to recursively determine higher fermion terms from the bosonic four derivative terms. In particular, we explicitly obtain the two fermion terms including the one-loop and full instanton corrections\cite{4}. In Sec. 2.3, the DLCQ supergravity side meaning of the two-fermion terms of the matrix theory effective action is investigated. Instead of considering a spinless probe membrane, we consider the spinning probe membrane dynamics using the curved background supermembrane formalism of Ref. \cite{29}, that was further analyzed in Ref. \cite{30}, while for simplicity the source membranes are still kept spinless. This analysis is performed on a general curved background geometry produced by source membranes satisfying the BPS condition. The leading two fermion contribution of the spin effects to the effective action turns out to be the spin-orbit couplings; we explicitly determine the spin-orbit couplings for membranes. This two fermion effective action obtained from the purely supergravity side analysis is exactly

\cite{3}In Ref. \cite{28}, we determined, in the framework of supersymmetric quantum mechanics, all fermion one-loop exact terms via supersymmetric completion and worked out the corrected supersymmetry transformation due to the inclusion of the four derivative terms.
identical to the two fermion terms computed in matrix theory in Sec. 2.2 when we choose
the harmonic function of the DLCQ supergravity as in Sec. 2.1. Our analysis in Sec. 2.3
points toward the possibility that the effective action computed from both approaches
should agree for all fermion number terms.

2.1 Preliminary: Bosonic effective action from DLCQ supergravity analysis and the SYM theory basics

Following the arguments of Seiberg and Sen to take appropriate chains of $U$-dual trans-
formations, the background geometry of the $N$-sector DLCQ $M$ theory compactified on
a transverse two-torus is given by the following eleven-dimensional covering space metric

$$ds_{11}^2 = h^{-2/3}(-dt^2 + dx_8^2 + dx_9^2) + h^{1/3}(dx_1^2 + \cdots + dx_7^2 + dx_{11}^2),$$

where the covering space eleventh coordinate $x_{11}$ parameterizes a real line, with the peri-
odic identification via

$$x_{11} \simeq x_{11} + 2\pi R.$$  \hspace{1cm} (2)

The eleventh direction thus becomes a circle with a radius $R$. The $N$ coincident source
membranes wrap the torus that extends over the $x_8$ and $x_9$ directions. The eleven-
dimensional harmonic function $h$ is given by

$$h = \sum_{n=-\infty}^{\infty} \frac{\kappa N}{(r^2 + (x_{11} + 2\pi Rn)^2)^{3/2}},$$

where $\kappa$ is a dimensionful constant and $r^2 = x_1^2 + \cdots + x_7^2$ is an $SO(7)$ invariant. The
harmonic function $h$ contains the contribution from all mirror charges to respect the
periodicity under the lattice translation $x_{11} \rightarrow x_{11} + 2\pi R$.

In the limit of the vanishingly small $R$, we can replace the summation in Eq. (3) with
an integration and recover the near-horizon geometry of the $N$ $D2$-branes of the type IIA
supergravity. At the decompactification limit of the DLCQ $M$ theory, that corresponds
to the large $R$ limit, the eleventh direction becomes indistinguishable from other non-
compact directions ($x_1, \cdots, x_7$). In particular, the summation in the expression for $h$ gets
dominated by the $n = 0$ term, which has the manifest $SO(8)$ symmetry; the perpendic-
ular $SO(7)$ symmetry gets enhanced to the $SO(8)$ symmetry at the decompactification
limit. In terms of an $SO(8)$ invariant $\tilde{r}^2 = r^2 + x_{11}^2$, the harmonic function $h$ in this limit has a simple power law dependence on $\tilde{r}$ like $\tilde{r}^{-6}$. Since the transversal space metric $h^{1/3}$ scales as $\tilde{r}^{-2}$, we see that the background geometry precisely becomes $AdS_4 \times S^7$, where the seven-sphere $S^7$ has a constant size $[31]$. This is the limit where we have the large $N$ correspondence between the AdS supergravity and conformal field theory (CFT), in which the AdS supergravity and the CFT near the infrared fixed point, i.e., the conformal phase of the (2+1)-dimensional SYM theory, become a dual description to each other $[25]$.

In the context of the $\mathcal{N} = 8$, (2+1)-dimensional SYM theory, the moduli space of the Coulomb branch is described by $N$ abelian dual magnetic $\phi^8$ scalars and $7N$ scalars $\phi^i$, where $i = (1, ..., 7)$ is the vector index of the $SO(7)$ $R$-symmetry. The Yang-Mills coupling constant $g_{YM}^2$ has mass dimension one, and the values of the $N$ magnetic scalars should be periodically identified with a period proportional to $g_{YM}^2$. The moduli space is then $N$ symmetric product $S^N(R^7 \times S^1)$. In our supergravity set-up, we have $N$ identical source membranes, whose BPS solution space can be parametrized by $7N$ positions in the non-compact direction $(x_1, ..., x_7)$ and $N$ positions along the $M$ theory circle; we reserve the right to construct multi-center solutions from Eq. (3) without violating the BPS condition. At the origin of the SYM theory moduli space, i.e., in the case of the $N$ coincident source membranes as in Eq. (3), it is known that the SYM theory flows to an interacting $Spin(8)$ invariant theory in the infra-red limit $[32]$. Since the $g_{YM}^2$ has mass dimension one, the infra-red limit corresponds to the strong coupling limit. The arguments of Seiberg and Sen $[18]$ imply $g_{YM}^2 = g_s/l_s = M_p^3 R^2$, where $g_s$, $l_s$ and $M_p$ denote the string coupling, string scale and the eleven-dimensional Planck mass, respectively$[4]$. The strong coupling limit in the SYM theory consequently implies the decompactification limit $R \rightarrow \infty$ on the supergravity side. We have already seen from Eq. (11) that in the decompactification limit, the perpendicular symmetry of the background geometry enhances from $SO(7)$ to $SO(8)$, and the background geometry becomes $AdS_4 \times S^7$ for the $N$ coincident source membranes. This suggests the validity of the aforementioned duality between the infra-red, i.e., conformal, phase of the (2+1)-dimensional SYM and the $AdS_4$ supergravity.

One of the main themes of our paper, the correspondence between the matrix theory

\[^{4}\text{From Eq. (11), we find that } \phi^8 = x_{11}/l_s^2. \text{ Since the period of } x_{11} \text{ is proportional to } R, \text{ the period of } \phi^8 \text{ should be proportional to } R/l_s^2 = R^2 M_p^3 = g_{YM}^2, \text{ as mentioned before.} \]
and the supergravity on the asymptotically locally Anti-de Sitter background geometry is motivated by the consideration along the above line at least for the large $R$ limit. Our primary interest here, however, will be to study the case of the arbitrary values of $N$ and $R$ (thus $g_{YM}^2$) following the DLCQ prescription of Ref. [2]. On the supergravity side, the treatment of the finite $R$ is straightforward; we simply have to add all contributions from the mirror membranes to respect the lattice translation symmetry $x_{11} \to x_{11} + 2\pi R$, as we did in Eq. (3). However, on the SYM theory side, we expect considerable instanton corrections when $g_{YM}^2$ is not very small\footnote{For small $R$ and thus small $g_{YM}^2$, we are in the regime where we can use the IIA supergravity analysis. On the SYM theory side, perturbative calculations would be enough. The agreement between the perturbative one-loop SYM theory and the classical D2-brane dynamics was reported already in the literature [6, 7, 8] (modulo the $r$ independent $v^4$ term which vanishes in the DLCQ supergravity as shown in Eq. (13)).}. The generic contributions from instantons to the effective potential are exponential terms, while the harmonic function from the supergravity has power law dependence. The key observation to solve this apparent problem is to recall that the $(2+1)$-dimensional SYM theory is the dimensional reduction of the ten-dimensional SYM theory. We reshuffle the series summation of Eq. (3), that is the lattice translation along the eleventh circle, using the Poisson resummation formula:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi f(\phi) e^{2\pi im\phi}. \tag{4}$$

The resummation can be exactly performed to yield the following identity.

$$\sum_{n=-\infty}^{\infty} \frac{1}{(r^2 + (x_{11} + 2\pi R n)^2)^3} = \frac{1}{16R} \left[ \frac{3}{r^5} + \sum_{m=1}^{\infty} e^{-mr/R} \left( \frac{m^2}{R^2 r^3} + \frac{m}{R r^4} + \frac{3}{r^5} \right) 2 \cos(mx_{11}/R) \right] \tag{5}$$

$$= \frac{1}{16R} \left[ \frac{3}{r^5} + \sum_{m=1}^{\infty} \left( \frac{2}{\pi} \right)^{1/2} \frac{m^2}{R^5} \frac{1}{r^3} \left( \frac{R}{r} \right)^{5/2} K_{5/2}(mr/R) 2 \cos(mx_{11}/R) \right]. \tag{6}$$

In going from (5) to (6), we use the modified Bessel function $K_{\nu}$ with a half-integer $\nu$, which has the finite number of terms in an expansion \footnote{For small $R$ and thus small $g_{YM}^2$, we are in the regime where we can use the IIA supergravity analysis. On the SYM theory side, perturbative calculations would be enough. The agreement between the perturbative one-loop SYM theory and the classical D2-brane dynamics was reported already in the literature [6, 7, 8] (modulo the $r$ independent $v^4$ term which vanishes in the DLCQ supergravity as shown in Eq. (13)).}.

$$K_{j+1/2}(z) = \left( \frac{\pi}{2z} \right)^{1/2} e^{-z} \sum_{k=0}^{j} \frac{(j + k)!}{k!(j - k)!}(2z)^k. \tag{7}$$
Each term of (5) is the harmonic function of the eight dimensional space \((x_1, \cdots, x_7, x_{11})\) perpendicular to the source membranes. The first term of (6) is in fact the harmonic function of the seven dimensional \((x_1, \cdots, x_7)\) space. As such, it appears in the construction of the IIA supergravity D2-brane solutions. It vanishes when we act \((\partial_{11})^2\) and thus it is the contribution from the massless modes under the Kaluza-Klein dimensional reduction along the \(M\) theory circle. The remaining exponential terms are from the massive Kaluza-Klein modes; when we act \((\partial_{11})^2\) to the \(m\)-th term, we get the eigenvalue \(-m^2/R^2\). In Ref. [9], noting that \(2 \cos x = \exp(ix) + \exp(-ix)\), these remaining terms were interpreted as originating from the \(M\)-momentum transfer between the source and probe D2-branes. From the Yang-Mills theory point of view, the exponential terms look generically like the \(m\)-instanton contributions. The \(r^{-4}\) and \(r^{-5}\) terms of the \(m\)-th term in (6) represent the perturbative corrections in the \(m\)-instanton background. A priori, these perturbative corrections should continue to all orders of the coupling \(g_{YM}^2\). However, as we will show in Sec. 2.2, the constraints from the remaining sixteen supersymmetry based on the argument of Ref. [17] cut the contribution at the finite order.

We now consider the purely bosonic dynamics of a probe membrane, which is taken to span the \(x_8, x_9\) directions and is moving with a constant velocity \(v^\hat{i} = \partial_\hat{0}x^{\hat{i}}\) \((\hat{i} = 1, \cdots, 7)\) in a direction perpendicular to the probe and \(x_{11}\). The background geometry for the finite value of \(R\) has the \(SO(7)\) symmetry and the velocity is an \(SO(7)\) vector, consistent with the symmetry of the background geometry. The action for the probe membrane is

\[
S = T_2 \int d^3\zeta \left[ -\sqrt{-g} - \frac{1}{6} \epsilon^{ijk} \partial_i x^{\hat{i}} \partial_j x^{\hat{j}} \partial_k x^{\hat{k}} C_{\hat{i}\hat{j}\hat{k}} \right],
\]

where \(T_2\) is the membrane tension and \(C_{\hat{i}\hat{j}\hat{k}}\) is the three-form gauge field of the eleven-dimensional supergravity. Here \(i, j, k\) are the world-volume indices and the hatted indices represent the eleven-dimensional indices. The metric \(g_{ij}\) is the induced metric on the world-volume of the probe membrane given by

\[
g_{ij} = g_{\hat{i}\hat{j}} + \partial_{\hat{i}} x^{\hat{i}} \partial_{\hat{j}} x^{\hat{j}} g_{\hat{I}\hat{J}},
\]

where the indices \(\hat{I}, \hat{J}\) represent the directions perpendicular to the probe. We choose the static gauge where \(\partial_\hat{0}x^0 = \partial_\hat{I}x^\hat{I} = \partial_\hat{2}x^2 = 1\) and other derivatives of \(x^3\) with respect to \(x^\hat{j}\) are zero. We plug the metric Eq. (1) with the function \(h\) of Eq. (3) into the action \(S\) and
expand it in powers of the transverse velocity \( v \). The action \( S \) becomes

\[
S = \int d^3x \left[ \frac{1}{2} T_2 v^2 - V_2 + \mathcal{O}(v^3) \right],
\]  

(11)

where \( V_2 \) is the effective potential given by

\[
V_2 = \frac{1}{8} T_2 h(r, x_{11}) (v^2)^2
= \frac{N}{16 R M_p^3} (v^2)^2
\times \left[ \frac{3}{r^5} + \sum_{m=1}^{\infty} \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \frac{m^2 m^{1/2}}{R^5} \left( \frac{R}{r} \right)^{5/2} K_{5/2}(m r / R) 2 \cos(mx_{11}/R) \right].
\]  

(12)

Going to the last line, we use the fact that \( T_2 \kappa = 8 M_p^{-3} \) where \( M_p \) is the eleven-dimensional Planck scale \([9]\) and perform the Poisson resummation. It should be noted that the potential is valid for any value of \( R \). If \( R \) is very small (or \( r \gg R \)), the potential is approximated by

\[
V_2 \approx \frac{3N}{16 R M_p^3} \frac{(v^2)^2}{r^5} - \frac{N}{16 R^3 M_p^3} \frac{(v^2)^2}{r^3} e^{-r/R} 2 \cos(x_{11}/R).
\]  

(13)

The first term of Eq. (13) is the usual potential between two D2-branes in the ten-dimensional type IIA theory \([8]\) and the second term is the potential due to the effect of a single \( M \)-momentum transfer \([9]\). The approximate potential Eq. (13) shows a notable feature that there is no \( r \) independent \( v^4 \) term that appeared in \([8]\). Had we started from an asymptotically flat background geometry, that term will inevitably appear. In the large \( N \) limit, it is natural to drop the term as was done in, for example, \([8]\). In the DLCQ framework, however, this term is automatically absent \([12, 13]\). This feature is also present in the case of the exact potential, Eq. (12).

2.2 Matrix theory calculation of two fermion terms: supersymmetric completion

According to the prescriptions of Seiberg and Sen, the DLCQ \( M \)-theory on a transverse two-torus is described by a system of D2-branes wrapped on its \( T \)-dual two-torus \([18]\), which becomes very large when the original two-torus has a vanishingly small size. When the number of D2-branes is \( N \), the action for the system is just the (2+1)-dimensional
The effective potential between the source and the probe membranes is given by the effective potential of the SYM theory, and we compare the supergravity bosonic effective potential Eq. (12) to the bosonic effective potential of the SYM theory. We note that our supergravity side calculation is actually for the two-body dynamics of the source and the probe. From the gauge theory point of view, we do not give the vacuum expectation values to the scalars that represent the position of the $N$ source membranes, thereby making them localized at one transversal space-time point, corresponding to the origin of the SYM theory moduli space.

Since the metric of the moduli space of the (2+1)-dimensional SYM theory is flat, the quadratic effective action of the SYM theory can be straightforwardly written as:

$$\Gamma^{(0)} = \int d^3x \left( \frac{1}{2} u^i \dot{u}^i + \frac{i}{2} \psi \dot{\psi} \right) ,$$

where $u^i = \dot{\phi}^i = F_{0i}$, the $i$-th component of the electric field. The scalars $\phi^i$ ($i = 1, \cdots, 7$) are the seven scalars of the vector multiplet (thereby having the $SO(7)$ symmetry). Assigning an ordering $O(\partial_\mu) = 1$ and $O(\psi) = 1/2$, we note that $O(\Gamma^{(0)}) = 2$. The action (14) is invariant under the tree-level supersymmetry transformation:

$$\delta \phi^i = -i \epsilon \gamma^i \psi ,$$

$$\delta \psi = \epsilon \gamma^i \frac{d}{dt} \phi^i = u^i \epsilon \gamma^i ,$$

where we assign $O(\epsilon) = -1/2$. The general structure of the effective action $\Gamma^{(1)}$, which is of the order $O(\Gamma^{(1)}) = 4$, looks schematically like $[19, 20]

$$\Gamma^{(1)} = \int d^3x \left( f^{(0)} u^4 + f^{(2)} u^2 \left[ \psi^2 \right] + f^{(4)} u^2 \left[ \psi^4 \right] + f^{(6)} u \left[ \psi^6 \right] + f^{(8)} \left[ \psi^8 \right] \right) ,$$

where $[\psi^p]$ denotes a generic $p$ fermion structure, and $f^{(p)}$ represents the bosonic coefficient function of the corresponding $p$ fermion structure. Upon adding $\Gamma^{(1)}$ to the quadratic effective action $\Gamma^{(0)}$, the supersymmetry transformation law in Eq. (15) should be modified;

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For our later purpose, we do not write terms in action with $\partial_1 \phi^i$ and $\partial_2 \phi^i$. Similarly, except for the supersymmetric partner terms of the bosonic quadratic terms, we do not write fermion derivative terms. The spinors have 2 of $SO(2, 1)$ indices and it is always implicitly assumed that an appropriate $2 \times 2$ matrix is sandwiched between two fermions. We use 8 representation of $Spin(7)$, but sometimes we implicitly use 8, or 8, of $Spin(8)$. Essentially, we are considering the ‘center of mass’ dynamics of the probe membrane. As such, our presentation closely parallels the supersymmetric quantum mechanics of [14] and we follow their notation for the most part.

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$\text{U}(N)$ SYM theory.
we thus write the $\Gamma^{(1)}$-corrected supersymmetry transformation as

$$\delta \phi^i = -i \epsilon \gamma^i \psi + \epsilon N^i \psi,$$

$$\delta \psi = u^i \epsilon \gamma^i + M \epsilon .$$

We note that $O(N) = 2$ and $O(M) = 3$ and, therefore, we can schematically write

$$N^i = u^2 N^{i(0)} + u N^{i(2)} \left[ \psi^2 \right] + N^{i(4)} \left[ \psi^4 \right]$$

and

$$M = u^3 M^{(0)} + u^2 M^{(2)} \left[ \psi^2 \right] + u M^{(4)} \left[ \psi^4 \right] + M^{(6)} \left[ \psi^6 \right] .$$

The effective action $\Gamma^{(0)} + \Gamma^{(1)}$ should be invariant under the supersymmetry transformation Eq. (17) order by order. At the lowest order, $O = 2$, $\Gamma^{(0)}$ itself is invariant under Eq. (15). At the next order, $O = 4$, we have two contributions; one from the supersymmetry transformation of $\Gamma^{(0)}$ due to the corrections $N^i$ and $M$, and another from the variation of $\Gamma^{(1)}$ under Eq. (17). The variation $\delta(\Gamma^{(0)} + \Gamma^{(1)})$ contains one, three, five, seven and nine $\psi$ terms, and they have to separately vanish for the invariance of the effective action under supersymmetry transformations. Specifically, we have:

$$\delta_B(f^{(0)} u^4) + \delta_F(f^{(2)} u^3 \left[ \psi^2 \right]) + u^i \epsilon u^2 \epsilon \hat{N}^{i(0)} \psi + \frac{i}{2} u^3 \psi \hat{M}^{(0)} \epsilon = 0 ,$$

$$\delta_B(f^{(2)} u^3 \left[ \psi^2 \right]) + \delta_F(f^{(2)} u^2 \left[ \psi^2 \right]) + u^i \epsilon u^2 \epsilon \hat{N}^{i(2)} \left[ \psi^2 \right] + \frac{i}{2} u^2 \psi \hat{M}^{(2)} \left[ \psi^2 \right] \epsilon = 0 ,$$

$$\delta_B(f^{(4)} u^2 \left[ \psi^4 \right]) + \delta_F(f^{(4)} u \left[ \psi^4 \right]) + u^i \epsilon u^2 \epsilon \hat{N}^{i(4)} \left[ \psi^4 \right] \psi + \frac{i}{2} u \psi \hat{M}^{(4)} \left[ \psi^4 \right] \epsilon = 0 ,$$

$$\delta_B(f^{(6)} u \left[ \psi^6 \right]) + \delta_F(f^{(6)} \left[ \psi^6 \right]) + \frac{i}{2} u \psi \hat{M}^{(6)} \left[ \psi^6 \right] \epsilon = 0 ,$$

$$\delta_B(f^{(8)} \left[ \psi^8 \right]) = 0 ,$$

where $\delta_B$ and $\delta_F$ represent the supersymmetric variation of the bosonic fields and the fermionic fields, respectively. The key insight of Paban, Sethi and Stern [17] is that the eight fermion terms can be exactly computed via Eq. (24) up to an overall constant without the knowledge of other fermion number terms, $N^i$ and $M$; $[\psi^8]$ consists of terms with zero, two and four scalar structure. The scalar number $s (s = 0, 2, 4)$ represents the number of scalar fields contracted to the fermion structure. Generically, the effective
action $\Gamma^{(1)}$ of the SYM theory is the summation of a perturbative term and $m$-instanton terms. Hereafter, for the notational convenience, we call the perturbative term 0-instanton sector. The function $f^{(p)}$ consists of the instanton summation and we represent it as $f^{(p)} = \sum_{m=-\infty}^{\infty} f^{(p)}_m$ where $m$ is the instanton number. From [17], $f^{(8)}_{4,m}$ is given by

$$f^{(8)}_{4,m} = m^6 |m|^{1/2} \frac{1}{g_{YM}^{28}} \left( \frac{g_{YM}}{\phi} \right)^{13/2} K_{13/2}(|m|/g_{YM}^2) e^{i m \phi / g_{YM}^2} ,$$

up to an overall multiplicative constant, where the extra scalar $\phi^8$ is the dual magnetic scalar. Here, $f^{(8)}_{4,m}$ is the coefficient function of the four scalar structure term among $[\psi^8]$ in the $m$-instanton sector

$$\sum_{m=-\infty}^{\infty} f^{(8)}_{4,m} (\phi^i \phi^j \phi^k \phi^l T_{ijkl}) ,$$

where $T_{ijkl}$ is the eight fermion structure. The function $f^{(p)}_{q,m}$ denotes the bosonic coefficient function of the $q$ scalar structure term of the $m$-instanton sector in $p$ fermion terms. We remark that Eq. (25) gives the perturbative term, when we set $m = 0$, proportional to $\phi^{-13}$.

A crucial observation first made in Ref. [27] is that the bosonic zero fermion term $f^{(0)}_m$ can also be determined without the knowledge of $N^i$ and $M$, once the $f^{(8)}_{4,m}$ terms are determined. To explicitly see this, we pick out the maximum scalar structure term from each $2p$-fermion term of the effective action (16) ($p = 0, 1, 2, 3$):

$$f^{(0)}_m (u^2)^2 , f^{(2)}_{1,m} \phi_i u_j u^2 \left[ \psi^2 \right]_{ij} , f^{(4)}_{2,m} \phi_i \phi_j u_k u_l \left[ \psi^4 \right]_{ijkl} ,$$

$$f^{(6)}_{3,m} \phi_i \phi_j \phi_k u_l \left[ \psi^6 \right]_{ijkl} , f^{(8)}_{4,m} \phi_i \phi_j \phi_k \phi_l \left[ \psi^8 \right]_{ijkl} ,$$

Here, the functions $f^{(2p)}_{p,m}$ depend only on an $SO(7)$ invariant $\phi^2 = \phi_i \phi_i$. The supersymmetric variation of the fermion fields of (27) will contribute to one, three, five and seven fermion terms shown in Eqs. (20)–(23):

$$f^{(2)}_{1,m} \phi_i u_j u_k u^2 \left[ \psi \gamma^k \epsilon \right]_{ij} ,$$

$$f^{(4)}_{2,m} \phi_i \phi_j u_k u_l \left[ \psi \gamma^m \epsilon \right]_{ijkl} ,$$

$$f^{(6)}_{3,m} \phi_i \phi_j \phi_k u_l \left[ \psi \gamma^m \epsilon \right]_{ijkl} .$$
\[ f^{(8)}_{4,m} \phi_i \phi_j \phi_k \phi_l u^m [\gamma^m \epsilon]_{ijkl} , \]  
respectively. The supersymmetric variation of the bosonic coefficient functions of (27) 
gives the following contributions to one, three, five and seven fermion terms shown in 
Eqs. (20)-(23):

\[ \left( \frac{d}{\phi d\phi} f^{(0)}_m \right) \phi_i (u^2)^2 (\epsilon \gamma^i \psi) , \]  
(32)

\[ \left( \frac{d}{\phi d\phi} f^{(2)}_{1,m} \right) \phi_i \phi_j u_k u^2 (\epsilon \gamma^i \psi) \left[ \psi^2 \right]_{jk} , \]  
(33)

\[ \left( \frac{d}{\phi d\phi} f^{(4)}_{2,m} \right) \phi_i \phi_j \phi_k \phi_l u^m (\epsilon \gamma^i \psi) \left[ \psi^4 \right]_{jklm} , \]  
(34)

\[ \left( \frac{d}{\phi d\phi} f^{(6)}_{3,m} \right) \phi_i \phi_j \phi_k \phi_l \phi_m (\epsilon \gamma^i \psi) \left[ \psi^6 \right]_{jklm} . \]  
(35)

The supersymmetric variation of the bosonic fields \( \phi_i \)’s appearing in Eq. (27) will reduce 
the scalar number, and these terms are not shown in Eqs. (33)-(35) since they are no 
longer maximum scalar structure terms. At each \( p \)-fermion term, there are contributions 
from the maximum scalar number terms of \( N^{(p-1)} \) and \( M^{(p-1)} \). However due to the time 
derivative 
\[ \frac{d}{dt} = u^i \phi^i \frac{d}{\phi d\phi} , \]
these contributions always include \( u^i \phi^i \) factor. In contrast, in (32) the contribution is 
\( \phi^i (u^2)^2 \). In (33)-(35), recalling that the \( 2p \)-fermion structure is in general a \( p \)-copy product 
of \( \psi \gamma^{i_1 j_1} \psi \ldots \psi \gamma^{i_p j_p} \psi, \phi^n \) and \( u^p \) appearing there are always anti-symmetrized. Therefore, 
the contributions (A) from \( N^{(p-1)} \) and \( M^{(p-1)} \) do not mix with the contributions (B) from 
(32)-(35). Two linearly-independent contributions (A) and (B) should separately cancel 
the contributions from (28)-(31) in Eqs. (20)-(23). Thus, the function \( f^{(0)}_m \) is related to 
\( f^{(8)}_{4,m} \) by

\[ C_p \left( \frac{d}{\phi d\phi} \right) f^{(2p)}_{p,m} = f^{(2p+2)}_{p+1,m} \rightarrow k \left( \frac{d}{\phi d\phi} \right)^4 f^{(0)}_m = f^{(8)}_{4,m} , \]  
(36)

where \( C_p \) are the numbers determined by working out the spinor algebra and \( k = C_0 C_1 C_2 C_3 \). Noting [38]

\[ \left( \frac{d}{zdz} \right)^a (z^{-\nu} K_\nu(z)) = (-1)^a z^{-\nu-a} K_{\nu+a}(z) , \]  
(37)
we conclude

\[ f^{(0)}_m = C m^2 |m|^{1/2} \left( \frac{g_{YM}^2}{\phi} \right)^{5/2} K_{5/2}(|m|\phi/g_{YM}^2) e^{im\phi^8/g_{YM}^2} \]  

(38)

from Eq. (36), where \( C \) is an overall constant. When integrating Eq. (36), there are in general four constants of integration. All these contributions, however, do not contain exponential functions and, thus, comparing to the well-behaved perturbative results for the weak coupling limit calculations [9], they are all set to zero. The constant \( C \) cannot be determined by the argument so far, but the one-instanton calculation of Ref. [9] determines it to be \( C = N(2/\pi)^{1/2}g_{YM}^2/16 \). Thus, the bosonic effective action \( \Gamma^{(1)}_B \) from the SYM theory is

\[ \Gamma^{(1)}_B = \int d^3x \sum_{m=-\infty}^{\infty} f^{(0)}_m(u^2)^2 , \]  

(39)

including the full non-perturbative instanton corrections. Since \( \Gamma^{(1)}_B = -\int d^3x V_B \), the effective potential \( V_B \) is

\[ V_B = -\frac{N}{16} (u^2)^2 \times \left[ \frac{3}{\phi^6} + \sum_{m=1}^{\infty} \left( \frac{2}{\pi} \right)^{1/2} \frac{m^2 m^{1/2}}{g_{YM}^2} \left( \frac{g_{YM}^2}{\phi} \right)^{5/2} K_{5/2}(m\phi/g_{YM}^2) 2 \cos(m\phi^8/g_{YM}^2) \right] \]  

(40)

We note that Eq. (40) is exactly identical to Eq. (42) if we identify

\[ \phi_i = x_i/l_s^2, \phi^8 = x_{11}/l_s^2, u = v/l_s^2, \]  

(41)

and use \( g_{YM}^2 = g_s/l_s \). The string coupling constant \( g_s \) and the string length scale \( l_s \) are related to the \( M \) theory quantities by \( g_s = (RM_p)^{3/2} \) and \( l_s = (RM_p^3)^{-1/2} \).

We now turn to the case of two fermion terms in \( \Gamma^{(1)} \), which is usually interpreted as the spin-orbit interaction. Generally, we can write it down as

\[ \Gamma^{(1)}_{\text{spin-orbit}} = \int d^3x \sum_{m=-\infty}^{\infty} f^{(2)}_{1,m} u^2 \partial^i \psi^j (\psi^j \gamma^i \psi) . \]  

(42)

From Eq. (20), relating different scalar coefficient functions \( f^{(p)} \) and recalling our previous remarks in this section, the function \( f^{(2)}_{1,m} \) can be easily determined from the given purely
bosonic coefficient function $f_{m}^{(0)}$ in Eq. (38). Working out the simple spinor algebra in Eq. (28), $f_{1,m}^{(2)}$ is related to $f_{m}^{(0)}$ by

$$f_{1,m}^{(2)} = \frac{i}{2} \left( \frac{d}{\phi d\phi} \right) f_{m}^{(0)},$$

(43)

or in other words

$$\phi_{j} f_{1,m}^{(2)} = \frac{i}{2} \partial_{j} f_{m}^{(0)}.$$  

(44)

Written explicitly, the effective potential $V_{\text{spin-orbit}}$ from the two fermion terms that satisfies $\Gamma^{(1)}_{\text{spin-orbit}} = - \int d^{3} x V_{\text{spin-orbit}}^{(1)}$ is thus

$$V_{\text{spin-orbit}} = \frac{i N}{32} u^{2} u^{j} \phi^{j} (\psi^{\gamma i j} \psi)$$

$$\times \left[ \frac{15}{\phi^{7}} + \sum_{m=1}^{\infty} \left( \frac{2}{m} \right)^{1/2} \frac{m^{7/2}}{g_{\text{YM}}^{2}} \left( \frac{g_{\text{YM}}^{2}}{\phi} \right)^{7/2} K_{7/2}(m\phi/g_{\text{YM}}^{2}) 2 \cos(m\phi/g_{\text{YM}}^{2}) \right].$$

(45)

If we rewrite the $V_{\text{spin-orbit}}$ in terms of the $M$ theory quantities, it becomes

$$V_{\text{spin-orbit}} = \frac{i N}{32(R M^{3})^{3}} v^{2} v^{i} x^{j} (\psi^{\gamma i j} \psi)$$

$$\times \left[ \frac{15}{r^{7}} + \sum_{m=1}^{\infty} \left( \frac{2}{m} \right)^{1/2} \frac{m^{7/2}}{R^{7}} \left( \frac{R}{r} \right)^{7/2} K_{7/2}(mr/R) 2 \cos(mx_{11}/R) \right]$$

$$= \frac{i}{R^{2} M_{p}^{3}} \sum_{n=-\infty}^{\infty} \frac{v^{2} v^{i} x^{j} (\psi^{\gamma i j} \psi)}{r^{2} + (x_{11} + 2\pi R n)^{2}}.$$  

(46)

where we Poisson-resummed back the expression going from the first line to the second line.

### 2.3 Membrane spin-orbit coupling from supergravity: matrix theory-supergravity correspondence for two fermion terms

We now calculate membrane spin-orbit couplings from the classical supergravity side. For this purpose, we consider the dynamics of a spinning probe membrane moving in the background geometry produced by spinless source membranes. The BPS background fields produced by the source membranes are known to be determined by a harmonic function in IIA supergravity or in eleven-dimensional supergravity. A notable technical feature of
our calculation is that we perform the calculation for an arbitrary choice of the harmonic function in the eleven-dimensional supergravity. Thus, by linearly superposing all mirror brane contributions, which results from the compactification of the $M$ theory circle, our results are applicable to both type IIA D-membranes and $M$-membranes. This will be useful for the comparison to the matrix theory side calculations in Sec. 2.2, since we included full non-perturbative instanton corrections when computing the matrix theory side results. By appropriately choosing the constant of motion for the harmonic function, the spin-orbit couplings for the asymptotically flat and $SO(1,2) \times SO(8)$ invariant background geometry can be immediately written down from our analysis. For the precise agreement with the matrix theory side calculations, we need, however, a non-asymptotically flat background geometry that is asymptotically locally $AdS_4 \times S^7$.

In the superspace formalism with superspace coordinates $Z^M(\zeta) = (X^\mu(\zeta), \theta^\alpha(\zeta))$ as functions of the world-volume coordinates $\zeta^i$, the probe dynamics of the supermembranes in the eleven-dimensional supergravity is described by the following action

$$ S[Z(\zeta)] = T_2 \int d^3 \zeta \left[ -\sqrt{-g(Z(\zeta))} - \frac{1}{6} \epsilon^{ijk} \Pi^A_i \Pi^B_j \Pi^C_k B_{BCA}(Z(\zeta)) \right], $$

where the pull-back $\Pi^A_i$ of the supervielbein $E^A_M$ to the membrane world-volume satisfies $\Pi^A_i = \partial Z^M / \partial \zeta^i E^A_M$, and $B_{MNP}$ represents the anti-symmetric tensor gauge superfield. The induced metric $g_{ij}$ on the world-volume satisfies $g_{ij} = \Pi^r_i \Pi^s_j \eta_{rs}$, where $\eta_{rs}$ is the Lorentz invariant constant metric. For a given background geometry, we have to expand the action Eq. (47) to the quadratic terms in the Majorana spinor variable $\theta$, which represents the probe spin. From Ref. [30], we have the following explicit covariant expressions for the superfields in terms of the component fields up to the quadratic terms in the fermionic $\theta$:

$$ \Pi^r_i = \partial_i X^\mu (e^\nu_{\mu} - \frac{1}{4} \sqrt{-g} \Gamma^{rst} \theta \tilde{\omega}_{rst} + \sqrt{-g} T^T_{\mu} T_{\mu} \theta F_{\nu} \eta_{\rho \sigma} \lambda) + \sqrt{-g} \partial_i \theta + \cdots, \quad (48) $$

---

Our conventions for indices are as follows. We use $(\mu \nu \rho \cdots)$ for bosonic curved space indices and $(\alpha \beta \gamma \cdots)$ for fermionic curved space indices. We write these two indices collectively as $(MNP \cdots)$. Among the bosonic indices, the directions tangential to membranes will be denoted as $(ijk \cdots)$, and the directions perpendicular to the branes, $(IJK \cdots)$. Turning to the tangent space, we use $(rst \cdots)$ for bosonic tangent space indices and $(abc \cdots)$ for fermionic tangent space indices. Collectively these two indices will be written as $(ABC \cdots)$. Among the tangent space bosonic indices, $(ijk \cdots)$ represent the directions tangential to membranes, and $(IJK \cdots)$, the perpendicular directions. The bosonic world volume indices will be denoted as $(ijk \cdots)$. Our signature choice for the metric throughout this paper is $(-+++ \cdots)$, and the totally anti-symmetric three-form tensor satisfies $\epsilon_{012} = -\epsilon^{012} = +1$.  

---

15
\[-\frac{1}{6} \epsilon^{ijk} \Pi^A_i \Pi^B_j \Pi^C_k B_{CBA} = \frac{1}{6} dX^{\mu \nu \rho} \left[ C_{\mu \nu \rho} + \frac{3}{4} \bar{\theta} \Gamma_{rs} \Gamma_{\mu \nu} \theta \bar{\omega}^r_s - 3 \bar{\theta} \Gamma_{\mu \nu} T^{\sigma \lambda \kappa \tau} \theta \bar{F}_{\sigma \lambda \kappa \tau} \right] \]  

\[-\frac{1}{2} \epsilon^{ijk} \bar{\theta} \Gamma_{\mu \nu \rho} \partial_k \theta \partial_i X^\mu \partial_j X^\nu + \cdots . \]

The Dirac conjugate is defined as \( \bar{\theta} = i \theta^T \Gamma^0 \). Since we are considering spinless background geometries, the background gravitino field is set to zero. The spin connection and the four-form gauge field strength for the background geometry are denoted as \( \hat{\omega}_{\mu st} \) and 

\[ \bar{F}_{\mu \nu \rho \sigma} = 4 \partial_\mu C_{\nu \rho \sigma}, \]

respectively, where the bracket implies the antisymmetrization normalized to unity. The \( k \) eleven-dimensional gamma matrix products \( \Gamma^{r_1 \cdots r_k} \) are totally antisymmetrized (normalized to unity) with respect to all indices. The symbols \( T_{\mu}^{\nu \rho \sigma \lambda} \) and \( dX^{\mu \nu \rho} \) are defined as

\[ T_{\mu}^{\nu \rho \sigma \lambda} = \frac{1}{288} (\Gamma_{\mu}^{\nu \rho \sigma \lambda} - 8 \delta_{\mu}^{[\nu} \Gamma_{\rho \sigma \lambda]}), \]

\[ dX^{\mu \nu \rho} = \epsilon^{ijk} \partial_i X^\mu \partial_j X^\nu \partial_k X^\rho. \]

A spinless BPS background geometry produced by source membranes has the following metric and the gauge field

\[ ds^2 = h^{-2/3} \eta_{ij} dx^i dx^j + h^{1/3} \delta_{ij} dx^i dx^j, \]  

\[ C_{i j k} = -\frac{1}{h} \epsilon_{i j k}, \]

where \( h \) is a harmonic function defined on the transversal space to the source membranes. As far as the BPS condition is not violated, we can linearly-superpose the harmonic function from each source membrane. The metric \( (50) \) determines the non-vanishing vielbeins and spin connections as

\[ e^j_i = h^{-1/3} \delta^j_i , \quad e^j_i = h^{1/6} \delta^j_i \]  

\[ \hat{\omega}^i_{j k} = \frac{1}{3} h^{-2/3} \partial_j h , \quad \hat{\omega}^i_{j k} = -\frac{1}{6} h^{-2/3} \partial_j h , \]

\[ \hat{\omega}^i_{j k} = -\frac{1}{3} h^{-2/3} \partial_j h , \quad \hat{\omega}^i_{j k} = \frac{1}{6} h^{-2/3} \partial_j h . \]

We note that the repeated indices in Eq. \( (53) \) are not summed. For the description of the probe membrane, we use the static gauge where we set \( \partial_i X^j = \delta^j_i \). Due to the existence of
the \(\kappa\)-symmetry for the membrane action \[47\], the fermions \(\theta\) are constrained to satisfy the \(\kappa\)-symmetry gauge fixing condition

\[
(1 - \tilde{\Gamma})\theta = 0 , \tag{54}
\]

where \(\tilde{\Gamma} = \Gamma^{012}\). Paying attention to the center of mass motion of membranes, we set

\[
\partial_0 X^i = v^i , \quad \partial_1 X^i = 0 , \quad \partial_2 X^i = 0 . \tag{55}
\]

The static limit is when \(v^i = 0\). By plugging Eqs. \(52\)-\(55\) into the first term of Eq. \(47\), via Eq. \(48\), we obtain

\[
\int d^3\zeta - \sqrt{-g(Z(\zeta))} = \int -h^{-1}\sqrt{1 - hv^2}[1 + \frac{1}{2}h^{2/3}[2h^{-1/3}\tilde{\theta}\Gamma\partial_1\theta + 2h^{-1/3}\tilde{\theta}\Gamma^2\partial_2\theta] \tag{56}

- \frac{1}{1 - hv^2}(-h^{-4/3}v^j\partial_j h\tilde{\theta}\Gamma^0_i\tilde{\theta} - 2h^{-1/3}\tilde{\theta}\Gamma^0_i\partial_\theta + 2h^{1/6}v^j\tilde{\theta}\Gamma^0_i\partial_\theta)] + \ldots] d^3\zeta
\]

up to two fermion terms. Here \(v^2\) denotes \(v^2 \equiv \delta_{ij}v^i v^j\). Likewise, the second term of Eq. \(17\), via Eq. \(19\), is computed to be

\[
\int d^3\zeta - \frac{1}{6} \epsilon^{ijk} \Pi^A_i \Pi^B_j \Pi^C_k B_{CBA} = \int [h^{-1} - h^{-2/3}\tilde{\theta}(\Gamma^0_0\partial_0 + \Gamma^1\partial_1 + \Gamma^2\partial_2)\theta \tag{57}

+ \frac{1}{2}h^{-5/3}v^i\partial_j h\tilde{\theta}\Gamma^0_0\tilde{\theta} - 2h^{-1/3}\tilde{\theta}\Gamma^0_i\partial_\theta + 2h^{1/6}v^j\tilde{\theta}\Gamma^0_i\partial_\theta] + \ldots] d^3\zeta
\]

up to two fermion terms. In deriving Eqs. \(56\) and \(57\), we use the Majorana properties for the spinor \(\theta\) such as \(\tilde{\theta}\Gamma^{\tau_1\ldots\tau_k}\theta = 0\) for \(k = 1, 2, 5, 6, 9, 10\) and the \(\kappa\)-projection condition \(54\).

For the slow speed expansion, we introduce an ordering where \(O(v) = 1, O(\partial_1) = 1\) and \(O(\tilde{\theta}\theta) = 1\). Adding Eqs. \(56\) and \(57\) and retaining terms of order up to four, we obtain

\[
S = T_2 \int d^3\zeta (\mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \mathcal{L}^{(4)} \ldots) \tag{58}
\]

where

\[
\mathcal{L}^{(2)} = \frac{1}{2}v^2 - 2h^{-2/3}\tilde{\theta}(\Gamma^0_0\partial_0 + \Gamma^1\partial_1 + \Gamma^2\partial_2)\theta, \tag{59}
\]

\[
\mathcal{L}^{(3)} = h^{-1/6}v^i\tilde{\theta}(\Gamma^0_0\partial_0 + \Gamma^1\partial_1 + \Gamma^2\partial_2 - \Gamma^0_0\partial_0)\theta, \tag{60}
\]

\[
\mathcal{L}^{(4)} = \frac{1}{8}h(v^2)^2 - \frac{1}{4}h^{-2/3}v^i v^j \partial_j h\tilde{\theta}\Gamma^0_0\tilde{\theta} - 2h^{-1/3}v^i\tilde{\theta}(\Gamma^1\partial_1 + \Gamma^2\partial_2 - \Gamma^0_0\partial_0)\theta. \tag{61}
\]
Upon deleting all the two fermion terms, we recover the bosonic effective action of Sec. 2.1. We note that the static potential vanishes up to two fermion terms consistent with the analysis of [34], and the fermion terms other than the spin-orbit coupling term of $L^{(4)}$ all contain spinor field derivatives. Since $\theta$ is a Majorana spinor satisfying $f^2\bar{\theta}\tilde{\Gamma}^i\partial_i\theta = (f\bar{\theta})\Gamma^i\partial_i(f\theta)$ for an arbitrary scalar function $f$, the transformation of the spinor $\theta$ into $\psi$ via

$$\psi = 2h^{-1/3}\theta$$

brings the quadratic terms $L^{(2)}$ to the quadratic action (14) of the (2+1)-dimensional SYM theory with the standard normalization, recalling $(\Gamma^0)^2 = -1$. To compare the action Eq. (58) to the one derived in Sec. 2.2, we decompose $SO(1,10)$ spinor $\psi$ into $SO(1,2) \times Spin(7)$ (or $SO(1,2) \times Spin(8)$ in the decompactification limit $R = \infty$) by assigning it an $SO(1,2)$ index $\alpha$ and $Spin(7)$ index (or $Spin(8)$ index in the decompactification limit) $a$, $\psi_{\alpha a}$. Furthermore, as a simple background choice as before, we suppose $v^I$ and $\psi_{\alpha a}$ are a constant number and a constant spinor, respectively. Then, the fermion derivative terms drop out and we finally obtain

$$S = T_2 \int d^3\zeta \left( \frac{1}{2}v^2 + \frac{1}{8}h(v^2)^2 + \frac{i}{16}v^2v^I\partial_Ih\psi^\alpha_\alpha(\gamma^{IJ})_{ab}\psi_{ab} \right).$$

Up until now, our derivation is valid for an arbitrary harmonic function $h$. Choosing $h$ of Sec. 2.1 corresponding to the asymptotically locally $AdS_4$ background geometry, we find that the action (63) is identical to the matrix theory effective action $\Gamma^{(0)} + \Gamma^{(1)}_B + \Gamma^{(1)}_{\text{spin-orbit}}$ from Eqs. (14), (39) and (42).

### 3 Discussions

Our analysis in this paper suggests that the supersymmetry might be the key element for the agreement between the matrix theory and the supergravity. With sixteen supercharges, the $F^4$ term in the supersymmetric Yang-Mills theory effective action is strongly constrained to be determined up to an overall numerical factor, which can in turn be uniquely fixed by the known perturbative analysis of, for example, Ref. [9]. On the supergravity side, the bosonic background geometries are determined by the BPS equations. Once this background geometry is determined, the fermionic parts of the effective action
can also be determined by the supersymmetry. Therefore, considering our previous work [27] that showed the agreement of the bosonic effective action between the two approaches, it is not surprising to find a precise agreement for the spin-orbit coupling terms.

A pleasing feature of the effective action Eq. (63) is that as soon as we choose the background geometry satisfying the BPS ansatz (thereby requiring $h$ be a harmonic function), the classical fermionic action from supergravity immediately assumes the form of the fermionic terms generated by the supersymmetric completion of the bosonic four derivative $F^4$ terms of the SYM theory. Furthermore, for an arbitrary harmonic function $h$, the quadratic (free field) classical action $\mathcal{L}^{(2)}$, Eq. (59), looks as if it is a theory on a flat background geometry (including fermion term). This behavior is consistent with the flatness of the (2+1)-dimensional SYM theory moduli space. A similar behavior, in the context of the Yang-Mills quantum mechanics with sixteen supercharges, was observed for the quadratic supersymmetric Yang-Mills theory effective action [16], where the non-renormalization theorem for the terms was also proved.

The precise agreement between the matrix theory side description and supergravity was verified for an arbitrary value of the longitudinal eleventh circle size and for all distance $r$, consistent with the DLCQ procedure of Ref. [2], which was conjectured to be valid for the finite $N$ (for a fixed value of $p_\perp = N/R$, $N$ is proportional to $R$). The background metric that produces this agreement is that of the asymptotically locally $AdS_4$ metric. In the decompactification limit of the eleventh circle, this background geometry reduces to that of $AdS_4 \times S^7$. In this case, the harmonic function $h$ vanishes like $r^{-6}$ as one approaches the asymptotic infinity, unlike the asymptotically flat geometries where $h$ goes to one. It is amusing to note that, therefore, the relationship Eq. (62) between $\theta$ and $\psi$ is the multiplication by an infinitely large scale factor. This transformation is rather similar to the ‘removal of the pole contribution’ for the spinning fields in the treatment of the AdS/CFT correspondence, which yields the holographic identification of the bulk fields and the boundary fields up to conformal transformation [20].

There are several lines of generalizations to the analysis presented in this paper. One issue is the determination of the static potential between two membranes. Eight fermion terms of the (2+1)-dimensional SYM theory was, as noted before, already non-
perturbatively obtained in Ref. [17]. On the supergravity side, the full expansion up to all fermion terms of the superfield in terms of the component fields is available in Ref. [35], at least in the $AdS_4 \times S^7$ background geometry. It will be interesting to explicitly verify if the agreement between the strong coupling SYM theory and the membrane dynamics in $AdS_4$ supergravity holds for eight fermion terms and to test if, of the possible $256 \times 256$ membrane-membrane polarization states, only 256 states have the vanishing static potentials. Secondly, since we expect that the consideration of the spinless probe in the presence of a spinning source will produce the same answer to the one obtained here, due to the two-body nature of the source-probe dynamics, it will be interesting to do the explicit calculations of the bosonic probe action in the presence of a non-trivial gravitino field. In this case, as noted in [34], the non-vanishing gravitino field induces rotations in the background geometry. This was in fact an approach taken by [21] for the supergravity side analysis to determine the spin-orbit couplings for particle dynamics, which was in turn shown to be identical to that of the supersymmetric Yang-Mills quantum mechanics two fermion terms.

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References

[1] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, Phys. Rev. D 55, 5112 (1997).
[2] L. Susskind, hep-th/9704080.
[3] K. Becker, M. Becker, J. Polchinski and A. Tseytlin, Phys. Rev. D 56, 3174 (1997), hep-th/9706072.
[4] J. Plefka, M. Serone and A. Waldron, Phys. Rev. Lett. 81, 2866 (1998), hep-th/9806081.
[5] W. Taylor and M. Van Raamsdonk, hep-th/9812239.
[6] J. Maldacena, Nucl. Phys. Proc. Suppl. 68, 17 (1998), hep-th/9709099.
[7] A. A. Tseytlin, Nucl. Phys. Proc. Suppl. 68, 99 (1998), hep-th/9709123.

[8] G. Lifschytz and S. D. Mathur, Nucl. Phys. B499, 283 (1997), hep-th/9612087; O. Aharony and M. Berkooz, Nucl. Phys. B491, 184 (1997), hep-th/9611215.

[9] J. Polchinski and P. Pouliot, Phys. Rev. D 56, 6601 (1997), hep-th/9704029.

[10] N. Dorey, V. V. Khoze and M. P. Mattis, Nucl. Phys. B 502, 94 (1997), hep-th/9704197.

[11] E. Keski-Vakkuri and P. Kraus, Nucl. Phys. B 530, 137 (1998), hep-th/9804067.

[12] I. Chepelev and A. A. Tseytlin, Nucl. Phys. B 524, 69 (1998), hep-th/9801120.

[13] S. Hyun, Y. Kiem and H. Shin, Phys. Rev. D 57, 4856 (1998), hep-th/9712021.

[14] S. Hyun, Phys. Lett. B 441, 116 (1998), hep-th/9802026.

[15] S. Hyun and Y. Kiem, Phys. Rev. D 59, 026003 (1999), hep-th/9805136.

[16] S. Paban, S. Sethi and M. Stern, Nucl. Phys. B 534, 137 (1998), hep-th/9805018.

[17] S. Paban, S. Sethi and M. Stern, hep-th/9808119.

[18] N. Seiberg, Phys. Rev. Lett. 79, 3577 (1997), hep-th/9710009; A. Sen, Adv. Theor. Math. Phys. 2, 51 (1998), hep-th/9709220.

[19] J. A. Harvey, Nucl. Phys. Proc. Suppl. 68, 113 (1998), hep-th/9706039.

[20] J. F. Morales, C. A. Scrucca and M. Serone, Phys. Lett. B 417, 233 (1998), hep-th/9709063; Nucl. Phys. B 534, 223 (1998), hep-th/9801183.

[21] P. Kraus, Phys. Lett. B 419, 73 (1998), hep-th/9709199.

[22] I. N. McArthur, Nucl. Phys. B 534, 183 (1998), hep-th/9806082.

[23] M. Barrio, R. Helling and G. Polhemus, J. High Energy Phys. 05, 012 (1998), hep-th/9801189.
[24] J. F. Morales, J. Plefka, C. A. Scrucca, M. Serone and A. Waldron, hep-th/9812039.

[25] J. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998), hep-th/9711200.

[26] S. Gubser, I. Klebanov and A. Polyakov, Phys. Lett. B 428, 105 (1998), hep-th/9802109; E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998), hep-th/9802150.

[27] S. Hyun, Y. Kiem and H. Shin, Phys. Rev. D 59, 021901 (1999), Rapid Comm., hep-th/9808183.

[28] S. Hyun, Y. Kiem and H. Shin, hep-th/9903022.

[29] E. Bergshoeff, E. Sezgin and P.K. Townsend, Phys. Lett. B 189, 75 (1987); Ann. Phys. 185, 330 (1988).

[30] B. de Wit, K. Peeters and J. Plefka, Nucl. Phys. B 532, 99 (1998), hep-th/9803209.

[31] N. Itzhaki, J. Maldacena, J. Sonnenschein and S. Yankielowicz, Phys.Rev. D 58, 046004 (1998), hep-th/9802042.

[32] S. Sethi and L. Susskind, Phys. Lett. B 400, 265 (1997); T. Banks and N. Seiberg, Nucl. Phys. B 497, 41 (1997); N. Seiberg, Nucl. Phys. Proc. Suppl. 67, 158 (1998), hep-th/9705117.

[33] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed. (Academic Press, 1994).

[34] V. Balasubramanian, D. Kastor, J. Traschen and K.Z. Win, hep-th/9811037.

[35] B. de Wit, K. Peeters, J. Plefka and A. Sevrin, Phys. Lett. B 443, 153 (1998), hep-th/9808052.