Hadronic Vacuum Polarization Contribution to the Muonium Hyperfine Splitting

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Abstract

We discuss hadronic effects in the muonium hyperfine structure and derive an expression for the hadronic contribution to the hfs interval in form of the one-dimensional integral of the cross section of $e^+e^-$ annihilation into hadrons. Higher-order hadronic contributions are also considered.

Key words: Muonium, Hyperfine structure, QED, Hadronic contribution
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1 Introduction and Results

Quantum Electrodynamics (QED) provides an opportunity to determine characteristics of various particles and simple atomic systems. However, any pure QED calculation is incomplete even in the case of a purely leptonic system because of a contribution of the strong interaction originating from hadronic intermediate states. For example, a contribution of the hadronic effects to the anomalous magnetic moment of the muon ($a_\mu$) is about 60 ppm of its QED value. This value is larger than both the uncertainty of the QED calculations (see e. g. [1]) and experiment [2]. The leading part of this correction (Fig. 1) can be presented in the form

$$\Delta a_\mu (\text{hadr}) = \frac{\alpha^2}{3\pi^2} \int \frac{ds}{s} K_a(s) R(s).$$

Here $\alpha$ is the fine structure constant and relativistic units in which $\hbar = c = 1$ will be used throughout the paper. Such a presentation is quite useful since it

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clearly separates the QED and hadronic parts. The QED part is known [3] in a closed analytic form

\[ K_a = -\left(\frac{s^2}{2m^4_\mu} - \frac{2s}{m^2_\mu} + 1\right) \frac{1}{r} \ln \frac{1 + r}{1 - r} + \left(\frac{s^2}{2m^4_\mu} - \frac{s}{m^2_\mu}\right) \ln \frac{s}{m^2_\mu} - \frac{s}{m^2_\mu} + \frac{1}{2}, \]

while the hadronic factor

\[ R(s) = \frac{\sigma(e^+e^- \rightarrow \gamma \rightarrow \text{hadrons})}{4\pi \alpha^2/3s} \quad (2) \]

can be determined either from direct measurements of the cross sections of \(e^+e^-\)-annihilation into hadrons at the energy \(E_{\text{c.m.}} = \sqrt{s}\) or from theoretical estimations. Here \(r = (1 - 4m^2_\mu/s)^{1/2}\).

In this paper we derive a similar expression for the hyperfine structure (hfs) interval in a two-body hydrogen-like atom with a point-like nucleus (Fig. 2). In particular, in the case of the ground state hfs in muonium the result is

\[ \Delta E_{VP} = \frac{2}{3} \frac{\alpha^2 m_e}{\pi^2 m_\mu} E_F \int \frac{ds}{s} K_{\text{Mu}}(s) R(s), \quad (3) \]

where

\[ K_{\text{Mu}}(s) = -\left(\frac{s}{4m^2_\mu} + 2\right) \frac{1}{r} \ln \frac{1 + r}{1 - r} + \left(\frac{s}{4m^2_\mu} + \frac{3}{2}\right) \ln \frac{s}{m^2_\mu} - \frac{1}{2}, \quad (4) \]

and \(E_F = 8\alpha^4 m^3_R/3m_e m_\mu\) is the leading contribution to the hyperfine structure (the so called Fermi energy) and \(m_R\) is the reduced mass.

Presently, the accuracy of the QED calculations [4–6] and of evaluations of the hadronic vacuum polarization contributions (see e.g. [7]) is sufficient for comparison with the experiment [8]. However, the expected progress in development of intensive muon sources for needs of particle physics offers an
opportunity to increase the statistics of muonic events and to provide a much better source of muonium \[9\]. In view of this increase of the experimental accuracy we need more precise theoretical calculations of various contributions and in particular of those for the hadronic vacuum polarization (Fig. \(2\)). The accuracy of such a hadronic calculation should establish the frontiers of any possible precision test of the bound state QED with muonium.

The expression (3) was obtained by us some time ago and the result appeared in a paper of one of us \[10\]. However, neither derivation nor discussion of the corrections to (3) was presented. Since a calculation based on this expression is now in progress \(6\), the derivation of (3) will be given here in detail. The accuracy of an incoming calculation \(6\) based on our expression (3) is of the 1%-level and we discuss here various corrections to (3) and in particular those due to the higher-order hadronic vacuum effects (Fig. \(3\)). Eventually we found that there is only one higher-order correction to (3) above the uncertainty level of the calculation in \(6\) and it increases the vacuum polarization contribution by roughly 3%.

![Diagram](image)

Fig. 3. Second-order hadronic corrections to the muonium hfs

2 General expression

We start the derivation with the two-photon exchange diagrams. Their contribution can be presented in the form (cf. \[11\])

\[
\Delta E_{Sc} = \frac{\alpha}{\pi} \frac{m_e m_\mu}{m_\mu^2 - m_e^2} E_F \cdot \left[ I(m_\mu) - I(m_e) \right], \quad \text{where} \quad I(m) = \int_0^\infty \frac{dk^2}{k^2} J(k, m)
\]

and

\[
J(k, m) = 2 \left( \frac{1}{k} \sqrt{k^2 + 4m^2} - 1 \right) - \frac{1}{4m^2} \left( k\sqrt{k^2 + 4m^2} - k^2 - 2m^2 \right). \quad (5)
\]
When calculating the integral over the euclidean momentum $k$ a divergency at low momentum should appear and it is necessary to rearrange the expressions. By introducing the vacuum polarization insertion in the skeleton integral (5), we reach our goal without any additional rearrangement.

We take the polarization into account substituting the photon propagator with a dispersion presentation of the polarization

$$
\frac{1}{k^2} \rightarrow \frac{\alpha}{\pi} \int \frac{ds \rho(s)}{k^2 + s}.
$$

The choice of the dispersion weight function $\rho(s)$ and the limits of the integration depend on what contribution is to be calculated. In the case of the hadronic vacuum polarization one finds $s \geq 4m_e^2 \gg m_e^2$, and $\rho(s) = R(s)/3s$ where $R(s)$ is defined in (2). Finally, we present the vacuum polarization contribution in the form

$$
\Delta E_{VP} = 2 \alpha^2 \frac{m_em_\mu}{m_\mu^2 - m_e^2} E_F \cdot \int ds \rho(s) \left[ I_{VP}(s, m_\mu) - I_{VP}(s, m_e) \right],
$$

where

$$
I_{VP}(s, m) = \int_0^\infty \frac{dk^2}{k^2 + s} J(k, m).
$$

After integrating over the momentum $k$ we find

$$
I_{VP}(s, m) = -2 \left[ 2 + \frac{s}{4m^2} \right] L(s/4m^2) + 2 \left[ \frac{3}{2} + \left( \frac{s}{4m^2} \right)^2 \right] \ln(s/m^2) - \frac{1}{2},
$$

where

$$
L(\tau) = \begin{cases} 
\frac{\sqrt{1-\tau^2}}{\tau} \tan^{-1} \frac{\sqrt{1-\tau^2}}{\tau}, & \tau < 1, \\
\frac{\sqrt{\tau^2-1}}{\tau} \ln \frac{\tau+\sqrt{\tau^2-1}}{\tau-\sqrt{\tau^2-1}}, & \tau \geq 1.
\end{cases}
$$

That is the most general expression for the hadronic contribution to the hyperfine splitting in a hydrogen-like system with a point-like nucleus. In the case of the hadronic contribution to the $hfs$ interval in muonium one can take advantage of $s > 4m^2_\mu \gg 4m_e^2$ and neglect a contribution of $m = m_e$

$$
\left[ \ldots \right]_{m = m_e} \approx -\frac{4m_e^2}{s} \left( \frac{9}{8} \ln \frac{s}{m_e^2} + \frac{15}{16} \right).
$$
Fig. 4. The QED kernels for the hadronic contributions to the muonium hyperfine splitting \( (K_{\text{Mu}}) \) and to the anomalous magnetic moment of muon \( (K_a) \) and their leading asymptotics, \( K_{\text{Mu}}^{(0)} = (18 \ln(s/m_\mu^2) + 15)m_\mu^2/4s \) and \( K_a^{(0)} = m_\mu^2/3s \). The kernels are shown in Fig. a with solid lines, while the asymptotics are with dashed lines. In Fig. b a result for \( (K_{\text{Mu}} - K_{\text{Mu}}^{(0)})/K_{\text{Mu}} \) is presented.

After simple transformations we arrive to the result (3). This expression is appropriate for calculations of any vacuum polarization contributions with \( s \leq 4m_\mu^2 \) such as the hadronic or \( \tau \)-leptonic contributions. The neglected term (9) does not exceed \( 10^{-3} \) of the main contribution for \( s \geq (2m_\pi)^2 \).

### 3 Asymptotics of the QED kernels \( K_{\text{Mu}}(s) \) at high energy

Since \( s \geq (2m_\pi)^2 > (2m_\mu)^2 \) the asymptotic behaviours of the QED kernel \( K_{\text{Mu}} \) at high values of \( s \) is of interest. We find

\[
K_{\text{Mu}} = \frac{4m_\mu^2}{s} \left[ \frac{9}{8} \ln \frac{s}{m_\mu^2} + \frac{15}{16} \right] + \left( \frac{4m_\mu^2}{s} \right)^2 \left[ \frac{5}{16} \ln \frac{s}{m_\mu^2} - \frac{17}{96} \right] + \ldots
\]  

(10)

In the case of the \( \rho \)-meson contribution, which is dominant (see Sect. 4), the parameter of expansion is \( 4m_\mu^2/m_\rho^2 \simeq 0.076 \) and the leading contribution at \( s = m_\rho^2 \) is 0.409, while the higher order terms are only 0.006.

This behaviour should be compared with the expansion of \( K_a(s) \) (see Fig. 4)

\[
K_a(s) \simeq \frac{1}{12} \frac{4m_\mu^2}{s} - \left( \frac{4m_\mu^2}{s} \right)^2 \left[ \frac{1}{16} \ln \frac{s}{m_\mu^2} - \frac{25}{192} \right] + \ldots
\]  

(11)
4 Estimation within a simplified narrow-pole model

The essential part of the hadronic vacuum polarization contribution comes from the $\rho$-meson-pole and it can be easily estimated (cf. [12]) from a simple approximation

$$\rho_{\text{pole}}(s) = \sum_{\text{res}} \frac{4\pi^2}{f_{\text{res}}^2} \delta(s - M_{\text{res}}^2) = \sum_{\text{res}} \frac{\Gamma(\text{res} \to e^+e^-)}{\alpha^2/3\pi M_{\text{res}}} \delta(s - M_{\text{res}}^2),$$  \hspace{1cm} (12)

with a sum over three light vector mesons dominating in $e^+e^-$ annihilation at low energies: $\rho$, $\omega$ and $\phi$. The partial width of a resonance decay into an electron-positron pair is a value determined experimentally. The results for the $\rho$-, $\omega$- and $\phi$-mesons within the simple pole approximation are summarized in Table 1 as well as their masses and electronic widths. This calculation is important to check if our expression is consistent with previous calculations of the hadronic contributions. The $\rho$-meson contribution is 65% of the hadronic contribution [6] and all three pole contributions deliver 76% of $\Delta\nu_{\text{hadr}}$ (or $0.18$ kHz or $4.0 \cdot 10^{-8}\nu_F$). The value of the $\rho$-meson contribution is in a good agreement with the $\pi\pi$ contribution in [6]. Due to that we study any correction to (3) with the help of these resonance contributions obtained from the simple pole approximation.

| Resonance | $\rho$-meson [MeV] | $\omega$-meson [MeV] | $\phi$-meson [MeV] |
|-----------|-------------------|---------------------|-------------------|
| $M_{\text{res}}$ | 769(1) | 783 | 1019 |
| $\Gamma(\text{res} \to e^+e^-)$ [keV] | 6.77(32) | 0.60(2) | 1.32(5) |
| $\Delta\nu_{\text{res}}$ [kHz] | 0.151(7) | 0.013 | 0.014 |
| $\Delta\nu_{\text{res}}/\Delta\nu_{\text{hadr}}$ | 65(3)% | 5.5(2)% | 6.0(2)% |
| $K_{\text{Mu}}$ | 0.415 | 0.404 | 0.262 |
| $K_{\text{Mu}}^{(0)}$ | 0.409 | 0.398 | 0.260 |

Table 1
Vector meson resonances, their parameters and contributions to the hyperfine structure. The fractional value in respect the to the complete hadronic contribution is given for $\Delta\nu_{\text{hadr}} = 0.233(3)$ kHz [6]. The parameters are taken from [13]

5 Higher-Order Hadronic Contributions

To estimate the higher-order hadronic contribution in order $\alpha^2(Z\alpha)$ we simply note that only one set of diagrams leads to a contribution enhanced by a logarithmic factor $\ln(s/m_e^2)$ (Fig. 3a). These diagrams contain the electronic
vacuum polarization insertion and in the logarithmic approximation can be easily calculated for the \( \rho \)-meson contribution:

\[
\Delta E(\text{log}) = 3 \cdot \frac{\alpha}{3\pi} \cdot \ln \frac{m^2_{\rho}}{(2m_e)^2} \cdot \Delta E(\text{leading contribution}) \tag{13}
\]

The same estimation can be reproduced for two other resonances in Table 1. About 80\% of the hadronic contribution comes from these three resonances, while the rest is essentially related to the high-\( s \) background. It it easy to check that in the case of the background contribution and other resonances the logarithmic approximation is valid and we confirm the logarithmic approximation (cf. (13)) for the complete result, which is about 3\% of the leading contribution. We expect that a non-logarithmic part of the contributions (Fig. 3) is below 1\%-level.

A similar estimation is misleading in the case of the anomalous magnetic moment of the muon [14], where the contribution related to insertion of the electronic vacuum polarization into leading hadronic diagram Fig. 1 does not dominate among higher-order hadronic contributions. We believe that it is caused by a special structure of the kernel (11), which has a small numerical coefficient and no logarithmic enhancement at high \( s \). That is not a case for the muonium hfs (10) and we finally estimate the higher order hadronic contribution as \((3 \pm 1)\%\) of the leading hadronic term, or 0.007(2) kHz.

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