ON THE LOCAL BOUNDARY SMOOTHNESS OF AN ANALYTIC FUNCTION AND ITS MODULUS IN SEVERAL DIMENSIONS.

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Abstract. Local boundary smoothness of an analytic function $f$ on the unit ball of $\mathbb{C}^n$ is compared to the smoothness of its modulus. We prove that different conditions imposed on the zeros of $f$ imply different drops of the smoothness. We also show that some of the drops are the best possible.

1. Introduction

The following theorem was first published in the paper [9].

**Theorem A.** (Carleson–Jacobs–Havin–Shamoyan) Let $\alpha$ be a real number such that $\alpha \in (0, 1)$. Suppose that $f : \mathbb{D} \to \mathbb{C}$ is an outer function continuous on $\mathbb{D}$ which has an $\alpha$–Hölder modulus on the boundary circle $\Gamma$ of the open unit disc $\mathbb{D}$ and which is also continuous up to $\Gamma$. Then the function $f$ itself is $\alpha/2$–Hölder on $\Gamma$.

We refer the reader to the books [1] and [4] for the definition of an outer function on the unit disc.

In fact, Theorem A holds for all indexes $\alpha \in \mathbb{R}_+$. It seems that this result was first proved by L.Carleson. Nevertheless, the only published proof of this theorem is the one contained in the book [11]. We refer the reader to the paper [8] for a more detailed history of the subject.

The following local version of Theorem A was proved in the paper [8].

**Theorem B.** (Kislyakov–Vasin–Medvedev) Let $\alpha \in (0, 2)$. Suppose that $f : \mathbb{D} \to \mathbb{C}$ is an analytic function without zeros inside $\mathbb{D}$ which is $\alpha$–Hölder at some point $\xi$ which belongs to the boundary circle $\Gamma$. Suppose also that $\int_\Gamma |\log |f||^p < \infty$ for some $p > 1$. Then for all intervals $I$, containing the point $\xi$ the mean oscillation measuring smoothness $\nu(f, I)$ enjoys the following property:

$$\nu(f, I) := \inf_{a \in \mathbb{C}} \frac{1}{|I|} \int_I |f(z) - a| \, d\sigma(z) \lesssim C_{l(I)}^{\frac{\alpha}{2-q}},$$

where $q$ is the Hölder conjugate of $p : 1/p + 1/q = 1$.

Properties of the mean oscillation $\nu$ and its connections with the local and global Hölder and Lipschitz smoothness classes are discussed in the papers [13] and [8].

Several remarks are in order. First, we would like to draw the reader’s attention to the fact that Theorem A was used by J.Brennann in his paper [3], where with
help of this result he characterized planar domains on which any analytic function admits polynomial approximation in the $L^p$ metric. Another application of the global Carleson–Jacobs–Havin–Shamoyan theorem was found in the paper [2], where the authors use it in order to classify cyclic subspaces of the harmonic Dirichlet spaces. We also mention the paper [3] by Mashregh and Shabankhah, where the Carleson–Jacobs–Havin–Shamoyan was used to compare zero sets and uniqueness sets of functions in Dirichlet spaces. One more remark is that Theorem A was cited in papers [7] and [6]. Alas, it is yet unclear whether the local Theorem B has any of the mentioned applications. Nevertheless Theorem B provides a significant improvement of Theorem A and moreover these local estimates imply the global ones with the real (and not the mean integral) Lipschitz regularity. To illustrate this, we mention the following fact, proved in [13]: if there is a uniform bound of the mean oscillation of a function $f$ on some interval, then $f$ is Lipschitz on this interval.

This article is an honest attempt to treat the multidimensional case in the local problem. Before presenting the main results of this paper let us recall two definitions that are going to be very important for us in what follows.

**Definition 1.** The **Cauchy kernel** for the unit ball $B^n$ is the function defined as

$$C(z, \xi) = \left( \frac{1}{(1 - \langle z, \xi \rangle)^n} \right).$$

The function $\int_{\mathbb{S}^n} C(z, \xi) f(\xi)d\sigma(\xi)$ is going to be often called by us the **“convolution”** of $f$ with the Cauchy kernel.

**Definition 2.** A function $f : B^n \rightarrow \mathbb{C}$ is **called outer**, if for all $z \in B^n$ one has,

$$f(z) = \exp\left[ \int_{\mathbb{S}^n} (2C(z, \xi) - 1) \Re(\log f(\xi))d\sigma(\xi) \right].$$

The following theorems are the main results of our paper.

**Theorem 1.** Let $\alpha \in (0, 1)$. Suppose that $f : B^n \rightarrow \mathbb{C}$ is an analytic function without zeros inside $B^n$, continuous up to the boundary $n$–dimensional unit sphere $S^n$ such that for all $t \in S^n$ one has $|\phi(t) - \phi(1)| \leq C_0 d(t, 1)^\alpha$. Then for all non–isotropic balls $Q$ containing the point 1 the mean oscillation $\nu(f, Q)$ enjoys the following property:

$$\nu(f, Q) \leq Cl(Q)^{\alpha}.$$

**Remark 1.** If an estimate of the type $\nu(f, Q) \leq Cl(Q)^{\beta}$ holds for all cubes $Q$ containing some point $\xi \in S^n$ with some $\beta > 0$ we will sometimes say that $f$ is $\beta$-Lipschitz “in average” at $\xi$.

**Theorem 2.** Let $\alpha \in (0, 1)$. Suppose that $f : B^n \rightarrow \mathbb{C}$ is an outer function such that for all $t \in S^n$ one has $|\phi(t) - \phi(1)| \leq C_0 d(t, 1)^\alpha$, where $\phi := |f|$, $1 := (1, 0, \ldots, 0)$, $d(u, v) := |1 - \langle u, v \rangle|^{1/2}$ is the non–isotropic metric on the $n$–dimensional unit sphere $S^n$. Suppose also that $B_p := \int_{\mathbb{S}^n} |\log \phi|^p < \infty$ for some $p > 1$. Then for all non–isotropic balls $Q$ containing the point 1 the mean oscillation $\nu(f, Q)$
measuring smoothness enjoys the following property:

\[
\nu(f, Q) \left( := \inf_{a \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(z) - a| \, d\sigma(z) \right) \leq C \ell(Q)^{\frac{n-1}{n}} \left( \equiv C \ell(Q)^{\frac{n-\alpha}{n}} \right),
\]

where \( \ell(Q) \) is the radius of the cube \( Q \) and \( q \) is the Hölder conjugate of \( p \).

The exponent \( \frac{p}{p+n} \) in Theorem 2 is sharp, which is proved in the following theorem.

**Theorem 3.** Let \( p \in (n, +\infty) \). Then for each \( \delta > 0 \) there exists an outer function \( f_0 : \mathbb{B}^n \to \mathbb{C} \), such that \( \log |f_0| \in L^p(\mathbb{S}^n) \), \( |f_0| \in \text{Lip}_\alpha(1) \), for all non-isotropic balls \( Q \), containing \( 1 \), there holds

\[
\nu(f_0, Q) \leq l(Q)^{\frac{n-\alpha}{n}},
\]

where \( l(Q) \) is the radius of \( Q \), and satisfying

\[
f_0 \notin \text{Lip}_{\frac{np}{p+n+\delta}}(1).
\]

We explain the difference between the first two results. In the second theorem, our method yields nothing for \( p = 1 \) unless \( n = 1 \). However, if we formally put \( p = 1 \) there for comparison with the assumptions of Theorem 1, we arrive at a restriction on the surface integral. Namely, we impose the condition on the surface integral of the modulus of the logarithm of our function \( \phi \) whereas in the second one we demand boundness of the integrals of all slice functions, which is an (a priori) stronger condition (contact [12] for more details). We also remark that in the second case the “slice” condition holds automatically once we know that the function \( f \) has no zeros in \( \mathbb{B}^n \) and is both analytic in \( \mathbb{B}^n \) and continuous on the unit sphere \( \mathbb{S}^n \). Another observation is that if \( n = 1 \) then there is almost no difference between these two conditions and both results coincide (modulo some details) with Theorem C of the paper [8]. We also refer the reader to the paper [10], where the global problem in the case of the unit ball was considered.

As it seems to the author, it is plausible that there are versions of these theorems that hold true in a more general setting, namely in the context of the holomorphic functions defined on more general domains in \( \mathbb{C}^n \). The author does not know weather the theorems proved here hold if one considers \( \alpha \) strictly bigger than one in those. Neither does he know if the strong \( \alpha \)–Hölder condition can be substituted with a weaker “average”–type one. The author plans to prove both these generalizations in the nearest future.

2. Proof of Theorem 2

We start the proof with the following (technical) result, which we, nevertheless, call a Theorem by the reason of some nontrivial (at least in our opinion) estimates included in its proof.

**Theorem 4.** The following estimates hold under the conditions of Theorem

\[
(1) \text{ For all non–isotropic balls } Q \subset \mathbb{S}^n \text{ containing the point } 1 \nu(f, Q) \leq C \ell(Q)^{\alpha} + 2\phi(1).
\]
(2) If \( 1 \geqslant \phi(1) > 0 \) then for all non-isotropic balls \( Q \subset S^n \) containing the point \( 1 \) and such that \( l(Q) \leqslant (\phi(1)/C_0)^{1/\alpha} \) we have

\[
\nu(f, Q) \leqslant C_l(Q)^\alpha + C_l(Q)^2 \frac{l(Q)^{2}}{\phi(1)\left(2n+2 - \frac{2n}{\alpha}\right)^{-1}},
\]

where the constant \( C \) depends only on \( n, C_0 \) and \( B_p \).

**Proof:** Since \( f \) is an outer function we are allowed to write the following representation:

\[
f(z) = \exp \left[ \int_{S^n} \left( 2C(z, \xi) - 1 \right) \text{Re}(f(\xi)) d\sigma(\xi) \right] = \exp \left[ \int_{S^n} \left( 2C(z, \xi) - 1 \right) \log |f(\xi)| d\sigma(\xi) \right],
\]

where \( C \) is the Cauchy kernel for the unit sphere in the \( n \) dimensional complex space. Hence

\[
f(z) = \exp \left[ \int_{S^n} \left( \text{Re}(2C(z, \xi) - 1) + i \text{Im}(2C(z, \xi) - 1) \right) \log |f(\xi)| d\sigma(\xi) \right] = \phi(z) \exp \left[ i \int_{S^n} \text{Im}(2C(z, \xi) - 1) \log \phi(\xi) d\sigma(\xi) \right].
\]

Next we estimate \( \nu(f, Q) \) for some ball such that \( 1 \in Q \subset S^n \). First we choose \( a := \phi(1)e^{ic_0} \), for some positive constant \( c_0 \). From here we see that

\[
\nu(f, Q) \leqslant \frac{1}{|Q|} \int_{Q} |\phi - \phi(1)| + \frac{\phi(1)}{|Q|} \int_{Q} \left| \exp \left( i \int_{S^n} \text{Im}(2C(z, \xi) - 1) \log \phi(\xi) d\sigma(\xi) \right) - \exp(ic_0) \right| \leqslant C_l(Q)^\alpha + 2\phi(1),
\]

so the first claim of Theorem 4 follows. Next we set

\[
c_0 := \int_{S^n \setminus 2Q} \text{Im}(2C(1, \xi) - 1) (\log \phi(\xi) - \log \phi(1)) d\sigma(\xi).
\]

Hence, thanks to the fact that the integral of the function \( \log \phi(1) (2C(1, \xi) - 1) \) over the unit sphere equals zero, we have

\[
\nu(f, Q) \leqslant \frac{1}{|Q|} \int_{Q} |\phi - \phi(1)| + A,
\]
where we abbreviate
\[
A := \left[ \frac{\phi(1)}{|Q|} \int_Q \int_{S^n} \chi_{2Q} \cdot \text{Im} \left( 2C(z, \xi) - 1 \right) \cdot \left( \log \phi(\xi) - \log \phi(1) \right) d\sigma(\xi) d\sigma(z) + \int_Q \int_{S^n \setminus 2Q} \left( \text{Im} \left( 2C(z, \xi) - 1 \right) - \text{Im} \left( 2C(1, \xi) - 1 \right) \right) \left( \log \phi(\xi) - \log \phi(1) \right) d\sigma(\xi) d\sigma(z) \right] = C_1 + D.
\]

Note that for all \( \xi \in Q \),
\[
\phi(\xi) - \phi(1) = C_0 |\xi - 1|^\alpha \leq C_0 l(Q)^\alpha \leq \phi(1),
\]
where the last inequality here follows from the conditions imposed on \( Q \). Hence \( \phi(\xi) \leq 2\phi(1) \). Referring to this and to the trivial inequality
\[
|\ln(\mu) - \ln(\eta)| \leq |\mu - \eta|/\min(\mu, \eta),
\]
which is valid for all \( \mu, \eta > 0 \), we infer that
\[
|\log \phi(\xi) - \log \phi(1)| \leq \frac{C}{\phi(1)}|\phi(\xi) - \phi(1)|.
\]
This fact along with the \( L^2 \)-boundedness of the singular integral represented by the convolution with the Cauchy kernel, yields the usual trivial bound for \( C_1 \):
\[
C_1 \leq \left( \frac{\phi(1)}{|Q|} \int_Q \left( \int_{S^n} \chi_{2Q} \cdot \text{Im} \left( 2C(z, \xi) - 1 \right) \cdot \left( \log \phi(\xi) - \log \phi(1) \right) d\sigma(\xi) \right)^2 d\sigma(z) \right)^{\frac{1}{2}} \leq \left( \frac{\phi(1)}{|Q|} \int_Q \left( \int_{S^n} \left| \log \phi(\xi) - \log \phi(1) \right|^2 d\sigma(\xi) \right) \right)^{\frac{1}{2}} \leq C l(Q)^\alpha.
\]

Next we concentrate on the term \( D \). We decompose the unit \( n \)-sphere in the following way \( S^n = \bigcup_{j=1}^m \Omega_j \), where \( \Omega_j := \{ z \in S^n : d(z, 1) \in (2^j l(Q), 2^{j+1} l(Q)) \} \) (here \( m \) is the smallest natural number, such that \( 2^{m+1} Q \supset S \) ). Let us now use this decomposition in the estimate of the term \( D \):
\[
D \leq \frac{C \phi(1)}{|Q|} \int_Q \sum_{j=1}^m \int_{\Omega_j} \left| \log \phi(\xi) - \log \phi(1) \right| \cdot |\tilde{C}(z, \xi) - \tilde{C}(1, \xi)| d\sigma(\xi) d\sigma(z),
\]
where \( \tilde{C}(z, \xi) \) is the imaginary part of the Cauchy kernel. We further decompose each of the sets \( \Omega_j \) into two as follows: \( E_j := \{ \xi \in \Omega_j : \phi(\xi) \geq \phi(1)/2 \} \) and \( F_j := \Omega_j \setminus E_j \). For each \( j \) between 1 and \( m \) on the set \( E_j \) the following estimate holds
\[
|\log \phi(\xi) - \log \phi(1)| \leq \frac{C}{\phi(1)} (2^j l(Q))^\alpha.
\]
On the other hand, since \( \phi(1) \leq 1 \) we readily get for all \( \xi \in F_j \) the following chain of inequalities
\[
|\log \phi(\xi) - \log \phi(1)| = - \log \phi(\xi) + \log \phi(1) \leq \log \frac{1}{\phi(\xi)}.
\]
Since $E_j = \emptyset$ once $j \leq k$, where $1 \leq k \leq m$ is the biggest number such that $2^k l(Q) \leq (\phi(1)/C_0)^{1/\alpha}$, we infer that

$$D \leq \frac{C \phi(1)}{|Q|} \sum_{j=1}^{m} \frac{(2^j l(Q))^\alpha}{\phi(1)} \int_{E_j} |\tilde{C}(z, \xi) - \tilde{C}(1, \xi)| d\sigma(\xi)$$

$$+ \sum_{j=k+1}^{m} \int \log \frac{1}{\phi(\xi)} |\tilde{C}(z, \xi) - \tilde{C}(1, \xi)| d\sigma(\xi) \sigma(z) =: D_1 + D_2.$$

We estimate $D_1$ and $D_2$ separately. But before that, let us recall without a proof one lemma borrowed from [12]. This lemma establishes a usual bound of the kernel of a singular integral operator (in our case it is the function $\tilde{C}(z, \xi)$).

**Lemma 1.** For each $\delta > 0$ there holds

$$|\tilde{C}(z, \xi) - \tilde{C}(u, v)| \leq C(d(\xi, v)^\delta + d(z, u)^\delta)/(d(\xi, z) + d(u, v))^{2n+\delta}.$$

The term $D_1$ is a piece of cake:

$$(1) \quad D_1 \leq \frac{C \phi(1)}{|Q|} \sum_{j=1}^{m} \frac{l(Q)^\alpha}{\phi(1)} \sum_{j=1}^{m} \frac{2^{j}\alpha l(Q)|\Omega_j|}{(2^j l(Q))^{2n+1}} \leq C l(Q)^\alpha.$$

We continue the proof with the estimate of the term $D_2$. We use the Holder inequality and Lemma [1]

$$(2) \quad D_2 \leq \frac{C \phi(1)}{|Q|} \left[ \sum_{j=k+1}^{m} ||\phi||_{L^p(\mathbb{R}^n)} ||\tilde{C}(z, \cdot) - \tilde{C}(1, \cdot)||_{L^q(\Omega_j)} \right] \leq \frac{C \phi(1)}{|Q|} \sum_{j=k+1}^{m} \frac{l(Q)^2}{(2^j l(Q))^{2n+2 - \frac{2n}{q}}} \leq C \frac{l(Q)^2}{\phi(1)^{\frac{1}{\alpha}(2n+2 - \frac{2n}{q}) - 1}},$$

The inequalities (1) and (2) now yield

$$\nu(f, Q) \leq Cl(Q)^\alpha + C \frac{l(Q)^2}{\phi(1)^{\frac{1}{\alpha}(2n+2 - \frac{2n}{q}) - 1}},$$

and the theorem follows. □

Next we proceed to the proof of Theorem [2]

**Proof:** Let $Q$ be a non–isotropic ball such that $l(Q)^\gamma \geq C \phi(1)$ for some $0 < \gamma \leq \alpha$. Then from the first claim of Theorem 4 we infer that $\nu(f, Q) \leq C(l(Q)^\alpha + l(Q)^\gamma)$. On the other hand, if $l(Q)^\gamma \leq C \phi(1)$ for the very same $\gamma$, then the second claim of Theorem 4 gives the following estimate

$$\nu(f, Q) \leq C \left( l(Q)^\alpha + l(Q)^{2-\gamma(\frac{1}{\alpha}(2n+2 - \frac{2n}{q}) - 1)} \right).$$

Comparing these inequalities we obtain the following equation

$$\gamma = 2 - \frac{\gamma}{\alpha} \left( 2n + 2 - \frac{2n}{q} \right) + \gamma,$$

from where we deduce that $\gamma = \alpha p/(n + p)$. □
3. Proof of Theorem 3

Let us now prove that the exponent $p/(p+n)$ is the best possible in Theorem 2. 

**Proof:** We precede the proof with one technical lemma.

**Lemma 2.** Let $\epsilon > 0$, and let $\varphi : T \to \mathbb{R}_+$ be a function such that

$$
\log \varphi \in L^\frac{n+\epsilon}{n}(T).
$$

Define a function $f_0 : \mathbb{S}^n \to \mathbb{C}$ by the following formula: $f_0(z_1, \ldots, z_n) = g(z_1)$, where $g : \mathbb{D} \to \mathbb{C}$ is given by

$$
g(z) = \exp \left[ \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \cdot \log \varphi (e^{i\theta}) \, d\theta \right].
$$

Then $f_0$ satisfies $\log |f_0| \in L^p(S^n)$.

**Proof:** Take a point $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{S}^n$, where $\zeta_1 = r \cdot e^{i\beta}$. Let us write a formula for the function $\log |f|$ using the definitions of the functions $f_0$ and $g$:

$$
\log |f_0(\zeta_1, \ldots, \zeta_n)| = \log |g(\zeta_1)| = 
\log \left( \exp \left[ \int_0^{2\pi} \text{Re} \frac{e^{i\theta} + \zeta_1}{e^{i\theta} - \zeta_1} \cdot \log \varphi (e^{i\theta}) \, d\theta \right] \right) = 
\int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos (\beta - \theta)} \cdot \log \varphi (e^{i\theta}) \, d\theta = \log \varphi * P_r(\beta).
$$

Hence, thanks to the formula number 1.4.5 from the book of W. Rudin [12] (we mean the formula for the integral of a function of fewer variables), we infer that:

$$
\int_{\mathbb{S}^n} \left| \log |f_0(\zeta)| \right|^p d\sigma(\zeta) = \int_0^{2\pi} \left( 1 - r^2 \right)^{n-2} \cdot r \cdot |\log \varphi * P_r(\beta)|^p d\beta dr = 
\int_0^1 \left( 1 - r^2 \right)^{n-2} \cdot r \cdot \|\log \varphi * P_r\|_{L^p(T)}^p dr \leq \ldots.
$$

Next, the Young inequality yields

$$
\ldots \leq \int_0^1 \left( 1 - r^2 \right)^{n-2} \cdot \|\log \varphi\|_{L^\infty(T)}^p \cdot \|P_r\|_{L^q(T)}^p dr,
$$

where $q$ is the solution of the following equation:

$$
1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{n+\epsilon}.
$$
It is left to estimate the norm of the Poisson kernel $||P_r||_{L^q(T)}$ for $r \in (0, 1)$ and $q \geq 1$. We first treat the case when $r \in (0, 1/2)$, which turns to be a piece of cake:

$$||P_r||_{L^q(T)}^q = \int_0^{2\pi} \frac{(1 - r^2)^q}{(1 + r^2 - 2r \cos \theta)^q} \, d\theta \lesssim \int_0^{2\pi} \frac{(1 - r)^q}{(1 - r^2)^q} \, d\theta = \frac{1}{(1 - r)^q} \cdot \frac{1}{1 - r} \lesssim \frac{2}{(1 - r)^q - 1}.$$  

Henceforth we assume that $r \in [1/2, 1)$. We are going to use the fact that $\theta \in [0, 2\pi]$, $1 - \cos \theta \geq C_0 \cdot \theta^2$ for some universal constant $C_0 > 0$:

$$||P_r||_{L^q(T)}^q = \int_0^{2\pi} \frac{(1 - r^2)^q}{(1 + r^2 - 2r \cos \theta)^q} = \int_0^{2\pi} \frac{(1 - r^2)^q}{((1 - r)^2 + 2r(1 - \cos \theta))^q} \leq \int_0^{2\pi} \frac{(1 - r)^q}{((1 - r)^2 + 2rC_0 \theta^2)^q} \, d\theta$$

$$= \int_A \frac{(1 - r)^q}{((1 - r)^2 + 2rC_0 \theta^2)^q} + \int_B \frac{(1 - r)^q}{((1 - r)^2 + 2rC_0 \theta^2)^q},$$

where $A$ and $B$ stand for the sets

$$A = \{\theta \in [0, 2\pi) : (1 - r)^2 \geq 2C_0 \theta^2\}$$

and

$$B = \{\theta \in [0, 2\pi) : (1 - r)^2 < 2C_0 \theta^2\}$$

correspondingly. Note that $A \cap B = \emptyset$ and also that $A \cup B = [0, 2\pi)$. Moreover there trivially holds $|A| \lesssim (1 - r)$. Hence we infer that

$$||P_r||_{L^q(T)}^q \lesssim \int_A \frac{(1 - r)^q}{(1 - r)^2q} \, d\theta + \int_B \frac{(1 - r)^q}{\theta^2q} \, d\theta \lesssim \frac{1}{(1 - r)^q - 1}.$$

Let us use the inequality that we have just derived along with $\mathbb{3}$ and write

$$\int_{S^n} |\log |f_0(\zeta)|^p|d\sigma(\zeta) \lesssim \int_0^{1} (1 - r)^{n-2} \cdot \left( \frac{1}{(1 - r)^q - 1} \right)^{\frac{q}{p}} \, dr.$$

Next we are going to prove that $p_1 := (n - 2) - p(q - 1)/q$ is strictly larger than $-1$. The definition of the number $q$ yields that

$$p_1 = (n - 2) - p\left( \frac{q - 1}{q} \right) = (n - 2) - p\left( \frac{1}{p/n + \varepsilon} - \frac{1}{p} \right) = (n - 2) - \frac{np - p - n\varepsilon}{p + n\varepsilon} = \frac{n^2\varepsilon - p - n\varepsilon}{p + n\varepsilon} = \frac{n^2\varepsilon}{p + n\varepsilon} - 1 > -1,$$

and the lemma follows. \qed
Denote $p_2 = p/n + \varepsilon$. Since the one dimensional bound $p/(p+1)$ is the best possible (see [14]), there exists a one-dimensional function $\varphi$ satisfying $\log \varphi \in L^{p_2}(\mathbb{T})$, $|\varphi| \in \text{Lip}_\alpha(1)$, and $\varphi \in \text{Lip}_{p_2/(p_2+1)}(1)$ “in average”, and such that the corresponding outer function $O_\varphi$ does not belong to the space $L^1_{\text{Lip}_\alpha}(p_2+1)$ for each $\sigma > 0$. Let us construct according to the method described in the lemma functions $f_0$ and $g$ (note that the construction there yields that $g = O_\varphi$). Then it is obvious that $|f_0| \in \text{Lip}_\alpha(1)$, and by lemma we infer that $\log |f_0| \in L^p(S^n)$. Hence, according to Theorem 2, $f_0 \not\in \text{Lip}_{p_2/(p_2+1)}(1)$ “in average”. It follows from the choice of the function $\varphi$ that

$$f_0 \not\in \text{Lip}_{p_2/(p_2+1)+\sigma}(1)$$

for each $\sigma > 0$. On the other hand,

$$\frac{p_2\alpha}{p_2+1} + \sigma = \left(\frac{p_2}{p_2+1} + \varepsilon\right) + \sigma = \frac{p\alpha + \varepsilon\alpha n}{p + \varepsilon n + n} + \sigma = \frac{\alpha p}{p + n} + \tilde{\varepsilon},$$

where $\tilde{\varepsilon} \to 0$ once $\varepsilon$ and $\sigma$ tend to zero. Taking $\varepsilon$ and $\sigma$ sufficiently small, we infer that there exists a function $f_0$, satisfying

$$f_0 \not\in \text{Lip}_{p_2/(p_2+1)+\delta}(1),$$

(wher $\delta$ is exactly the same as in the formulation), and the theorem follows. □

4. PROOF OF THEOREM [1]

First of all, since the function $f$ is continuous up to the boundary we infer that

$$\sup_{\xi \in S^n} \int_{\mathbb{T}} |\log |f(\xi\lambda)|| |d\lambda| \leq B_0 < \infty.$$  

We are acting in a way similar to that of the first section. We first prove a technical theorem that will help us to obtain the desired bound on the mean oscillation $\nu(f, Q)$.

**Theorem 5.** The following estimate holds under the conditions of Theorem 7.

1. For all balls $Q \subset S^n$ in the nonisotropic metric containing the point $\mathbb{1}$

$$\nu(f, Q) \leq C l(Q)^{\alpha} + \phi(\mathbb{1}),$$

where $l(Q)$ is the radius of the ball $Q$.

2. If $1 \geq \phi(\mathbb{1}) > 0$ then for all non–isotropic balls $Q \subset S^n$ containing the point $\mathbb{1}$ and such that $l(Q) \leq (\phi(\mathbb{1})/C_0)^{1/\alpha}$ we have

$$\nu(f, Q) \leq C l(Q)^{\alpha} + C \frac{l(Q)^2}{\phi(\mathbb{1})^{1/\alpha - 1}},$$

where the constant $C$ depends only on $n, C_0$ and $B_0$.
Proof: With no loss of generality, we suppose that $f(0)$ is a real number (for the general case follows from the observation that the function $g(z) = f(z) + \overline{f(0)}$ enjoys $g(0) \in \mathbb{R}$). Since $f$ is an analytic function without zeros, we are allowed to write the following representation for the functions $f_r(\xi) := f(r\xi)$, $r < 1$:

$$f_r(z) = \exp \left[ \int_{S^n} (2C(z, \xi) - 1) \text{Re}(\log f_r(\xi)) d\sigma(\xi) \right] = \exp \left[ \int_{S^n} (2C(z, \xi) - 1) \log |f_r(\xi)| d\sigma(\xi) \right],$$

where $C$ is the Cauchy kernel for the unit sphere in the $n$ dimensional complex space. Hence

$$f_r(z) = \exp \left[ \int_{S^n} \left( \text{Re}(2C(z, \xi) - 1) + i \text{Im}(2C(z, \xi) - 1) \right) \log |f_r(\xi)| d\sigma(\xi) \right] = \phi_r(z) \exp \left[ i \int_{S^n} \text{Im}(2C(z, \xi) - 1) \log |\phi_r(\xi)| d\sigma(\xi) \right] =: \phi_r(z) e^{iG(z)},$$

where we write $\phi_r(\xi) := \phi(r\xi) = |f(r\xi)|$ for sake of brevity. Next we estimate $\nu(f, Q)$ for some non–isotropic ball $Q$ such that $1 \in Q \subset S^n$. First we choose $a := \phi(\mathbb{1}) e^{i\alpha_0}$ for some positive constant $c_0$. Note that since $f$ is continuous at any point $\xi$ of the boundary sphere, we infer that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $1 - \delta < r < 1$, then $|\phi_r(\xi) - \phi(\xi)| \leq \varepsilon$ and $|f_r(\xi) - f(\xi)| \leq \varepsilon$. From now on we consider only these $r$’s. From here we see that

$$\nu(f, Q) \leq \frac{1}{|Q|} \int_Q |f(z) - \phi(\mathbb{1}) e^{i\alpha_0}| \leq \frac{1}{|Q|} \int_Q |f(z) - f_r(z)| + \frac{1}{|Q|} \int_Q |\phi_r(z) e^{iG(z)} - \phi(\mathbb{1}) e^{i\alpha_0}| \leq \varepsilon + \frac{1}{|Q|} \int_Q |(\phi_r(z) - \phi(\mathbb{1})) e^{iG(z)}| + \frac{1}{|Q|} \phi(\mathbb{1}) |e^{iG(z)} - e^{i\alpha_0}| \leq \varepsilon + C l(Q)^n + 2\phi(\mathbb{1}),$$

so the first claim of Theorem 1 follows. Next we set

$$c_0 := \int_{S^n \setminus 2Q} \text{Im} (2C(\mathbb{1}, \xi) - 1) (\log \phi_r(\xi) - \log \phi(\mathbb{1})) d\sigma(\xi).$$

Hence, thanks to the fact that the integral of the function $\log \phi(\mathbb{1}) (2C(\mathbb{1}, \xi) - 1)$ over the unit sphere equals zero, we have

$$\nu(f, Q) \leq \frac{1}{|Q|} \int_Q |\phi - \phi(\mathbb{1})| + A + \varepsilon,$$
where we abbreviate
\[
A := \frac{\phi(1)}{|Q|} \left[ \int_Q \int_{S^n} \chi_{2Q} \cdot \text{Im} \left( 2C(z, \xi) - 1 \right) \cdot \left( \log \phi_r(\xi) - \log \phi(1) \right) \, \frac{\partial \phi_r(\xi)}{\partial z} \, d\sigma(\xi) \, d\sigma(z) + \int_Q \int_{S^n \setminus 2Q} \left( \text{Im} \left( 2C(z, \xi) - 1 \right) - \text{Im} \left( 2C(1, \xi) - 1 \right) \right) \cdot \left( \log \phi_r(\xi) - \log \phi(1) \right) \, d\sigma(\xi) \, d\sigma(z) \right],
\]

Thanks to the boundedness of the singular integral represented by the convolution with the Cauchy kernel, we have the usual trivial bound for \( C_1 \):
\[
C_1^2 \leq \frac{\phi(1)^2}{|Q|} \int_Q \left( \int_{S^n} \chi_{2Q} \cdot \text{Im} \left( 2C(z, \xi) - 1 \right) \cdot \left( \log \phi_r(\xi) - \log \phi(1) \right) \, d\sigma(\xi) \right)^2 \, d\sigma(z) \leq \frac{1}{|Q|} \int_Q \left( \log \phi_r(\xi) - \log \phi(1) \right)^2 \, d\sigma(\xi) \leq \alpha L(Q)^{2\alpha} + \varepsilon,
\]
where the last inequality follows from the fact that the function \( \phi \) is \( \alpha \)-Lipschitz at the point \( 1 \) and from the conditions imposed on \( r \).

Next we estimate the term \( D_r \). We introduce the following decomposition of the unit \( n \)-sphere: \( S^n = \bigcup_{j=1}^m \Omega_j \), where \( \Omega_j := \{ z \in S^n : d(z, 1) \in (2^j l(Q), 2^{j+1} l(Q)) \} \) (here \( m \) is the smallest natural number, such that \( 2^{m+1} Q > S \)). We further use this decomposition in the estimate of the term \( D \):
\[
D \leq C \phi(1)^2 \int_Q \sum_{j=1}^m \int_{\Omega_j} \left| \log \phi_r(\xi) - \log \phi(1) \right| \cdot \left| \tilde{C}(z, \xi) - \tilde{C}(1, \xi) \right| \, d\sigma(\xi) \, d\sigma(z),
\]
where \( \tilde{C}(z, \xi) \) is the imaginary part of the Cauchy kernel. Next, we decompose each of the sets \( \Omega_j \) into two as follows: \( E_j := \{ \xi \in \Omega_j : \phi_r(\xi) \geq \phi(1)/2 \} \) and \( F_j := \Omega_j \setminus E_j \). For each \( j \) between \( 1 \) and \( m \) the following estimate holds on the set \( E_j \):
\[
\left| \log \phi_r(\xi) - \log \phi(1) \right| \leq \frac{2}{\phi(1)} \left| \phi_r(\xi) - \phi(1) \right| \leq \varepsilon + \frac{C}{\phi(1)} \left( 2^j l(Q) \right)^{\alpha}.
\]
On the other hand, if \( \xi \in F_j \) then one has the following inequality
\[
\left| \log \phi_r(\xi) - \log \phi(1) \right| \leq \log \frac{1}{\phi_r(\xi)}.
\]
Since \( F_j = \emptyset \) once \( j < k \), where \( 1 \leq k \leq m \) is the biggest number such that \( 2^k l(Q) \leq (\phi(1)/C_0)^{1/\alpha} \) we infer that
\[
D \leq C \phi(1)^2 \int_Q \sum_{j=1}^m \left( \frac{2^j l(Q)}{\phi(1)} \right)^{\alpha} \int_{F_j} \left| \tilde{C}(z, \xi) - \tilde{C}(1, \xi) \right| \, d\sigma(\xi) \, d\sigma(z) =: D_1 + D_2 + C \varepsilon.
\]
We estimate $D_1$ and $D_2$ separately. The term $D_1$ is a piece of cake:

\begin{equation}
D_1 \leq C \frac{\phi(1)}{|Q|} |Q| \left[ \frac{l(Q)^\alpha}{\phi(1)} \sum_{j=1}^{m} \frac{2^{jn}l(Q)|\Omega_j|}{(2l(Q))^{2n+1}} \right] \leq Cl(Q)^\alpha.
\end{equation}

We finally proceed to the term $D_2$. First, it follows from the definitions of the functions $f_r$ and from the inequality (4) that for all $\xi \in \mathbb{S}^n, r < 1$ and $\rho \in [0, 2\pi)$ one has

\begin{equation}
\int_{-\rho}^{\rho} \left| \log |f_r(\xi e^{i\theta}, \ldots, \xi_e e^{i\theta})| \right| d\theta \leq B_0.
\end{equation}

We define sets $Q_j$ as $Q_j = \{ z \in \mathbb{S}^n : d(z, \bar{1}) \leq 2^j l(Q) \}$ and choose $\rho := 2^j l(Q)$ for some $j \in \mathbb{N}$. Integration of the last line with respect to the variable $\xi$ over the set $Q_j$ gives

\begin{equation}
CB_0 \rho^{2n} \geq \int_{Q_j} \int_{-\rho/2}^{\rho/2} \left| \log |f_r(\xi e^{i\theta}, \ldots, \xi_e e^{i\theta})| \right| d\theta d\sigma(\xi) \geq \int_{-\rho/2}^{\rho/2} \int_{Q_j} \left| \log |f_r(\xi e^{i\theta}, \ldots, \xi_e e^{i\theta})| \right| d\sigma(\xi) d\theta.
\end{equation}

For each $\theta \in (-\rho/2, \rho/2)$ and each $\xi \in Q_j$ we define a vector $z$ as $z = (z_1, \ldots, z_n)$, where $z_j := \xi_j e^{i\theta}$. We further define a function $F_\theta$ by the following formula $F_\theta(\xi) = z$. We finally proceed to the term $\rho/2$ separately. The term $2$ gives

\begin{equation}
CB_0 \rho^{2n} \geq \int_{-\rho/2}^{\rho/2} \int_{Q_j} \left| \log |f_r(z)| |e^{i\theta}|^n \right| d\sigma(\xi) d\theta = \int_{-\rho/2}^{\rho/2} \int_{-\rho/2}^{\rho/2} \left| \log |f_r(z)| \right| d\sigma(\xi) d\theta.
\end{equation}

We claim that $Q_j/2 \subset F_\theta(Q_j)$. To prove this, let us pick a point $\xi \in Q_j/2$. In order to show that $\xi \in F_\theta(Q_j)$ it is sufficient to prove that $\xi e^{-i\theta} \in Q_j$ (for in the latter case we can write $\xi = (\xi e^{-i\theta}) e^{i\theta}$). We check that $|1 - \xi_1 e^{-i\theta}| \leq \rho$ with the help of the triangle inequality:

\begin{equation}
|1 - \xi_1 e^{-i\theta}| \leq |1 - \xi_1| + |\xi_1| |1 - e^{-i\theta}| \leq \frac{\rho}{2} + |\theta| \leq \rho,
\end{equation}

and our claim follows. The line (8) now gives

\begin{equation}
\int_{Q_j} \left| \log |f_r(z)| \right| d\sigma(z) \leq CB_0 \rho^{2n-1} \leq CB_0 (2^{j-1} l(Q))^{2n-1}.
\end{equation}
Thanks to the inequality (9), we are now ready to finish off the desired bound of the term $D_2$:

$$D_2 \leq \frac{C\phi(1)}{|Q|} \int_Q \sum_{j=k+1}^{m} \log \frac{1}{\phi_r(\xi)} |\tilde{C}(z, \xi) - \tilde{C}(1, \xi)|d\sigma(\xi)d\sigma(z) \leq \frac{CB_0\phi(1)}{|Q|} \sum_{j=k+1}^{m} (2l(Q))^{2n-1} \frac{l(Q)}{(2l(Q))^{2n+1}} \leq \frac{l(Q)}{\phi(1)^{\frac{2}{n}-1}},$$

Theorem 5 will now follow from the inequalities (9), (6) and (5) simply by letting $\varepsilon$ tend to zero.

Thanks to the result that we have just obtained, Theorem 2 can now be proved exactly in the same way as Theorem 1.

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