CORRECTIONS TO: “A MURNAGHAN–NAKAYAMA RULE FOR VALUES OF UNIPOTENT CHARACTERS IN CLASSICAL GROUPS”

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Abstract. We settle a missing case in the proof of one of the main applications of our results in [Frank Lübeck and Gunter Malle, A Murnaghan–Nakayama rule for values of unipotent characters in classical groups, Represent. Theory 20 (2016), 139–161, MR3466537].

In [5] we derived a Murnaghan–Nakayama formula for the values of unipotent characters of finite classical groups. As one application, we showed that the first Cartan invariant $c_{11}^{(\ell)}$ of a finite simple classical group $G$ in non-defining characteristic $\ell$ is never equal to 2 if Sylow $\ell$-subgroups of $G$ are non-cyclic. But contrary to what was claimed in the first line of the proof of [5, Thm. 7.1], the same assertion for the case of defining characteristic does not follow from the work of Koshitani, Külshammer, and Sambale [2]. We thank Shigeo Koshitani and Jürgen Müller for pointing this out to us; see [3].

Here we give a short argument for the missing case. The argument also covers the analogous statement in [4, Prop. 6.3] pertaining to simple groups of exceptional Lie type. The crucial observation is that Sylow $p$-subgroups in groups of Lie type of characteristic $p$ are large. Except for a few small rank cases, we do not need any knowledge about the ordinary character table of the group $G$.

Note that if $c_{11}^{(p)} = 2$, then there exists $\chi \in \text{Irr}(G)$ such that $1_G + \chi$ is the character of a projective module; in particular, $\chi(g) = -1$ for all $p$-singular elements $g \in G$.

Lemma 1. Let $G$ be a finite group and $p$ a prime. Assume that $H \leq G$ is a subgroup such that $|G : H|$ is divisible by $p$, but $|G : H| < |G|^p$. Then $c_{11}^{(p)} > 2$.

Proof. The permutation character $\pi$ of $G$ on the cosets of $H$ has exactly one trivial constituent $1_G$. On the other hand, as $|G : H|$ is divisible by $p$, the modular reduction of the permutation representation has (at least) two trivial modular constituents: one in the socle, one in the head. Thus, the $p$-modular reduction of one of the non-trivial ordinary constituents $\chi$ of $\pi$ must contain the trivial $p$-modular character of $G$. By Brauer reciprocity, $\chi$ then occurs in the projective cover $\Psi$ of the trivial character. If we had $c_{11}^{(p)} = 2$, then necessarily $\Psi = 1_G + \chi$. Since the degree of any projective character is divisible by $|G|^p$, this is a contradiction to $|G|^p \leq \Psi(1) = 1_G(1) + \chi(1) \leq \pi(1) = |G : H|$. □
Proposition 2. Let $G$ be a finite simple group of Lie type in characteristic $p$. Then $c_{11}^{(p)} > 2$ unless $G = L_2(p)$.

Proof. If the Lie-rank of $G$ is not very small, then we can find a subgroup of $G$ which fulfills the condition of Lemma 1. In Table 1 we give a subgroup $H$ (see for example [6, §13] for the existence) of a suitable central extension $\hat{G}$ of $G$. From the order formulae for these groups (see for example [6, Tab. 24.1]) it is easy to check that $H \leq \hat{G}$ fulfills the condition of Lemma 1. Since the center of $\hat{G}$ is of order prime to $p$, the same holds for $H \leq G$ where $H$ is the image of $\hat{H}$ in $G$.

It remains to consider the groups of type $A_1$, $A_2$, $2A_2$, $3D_4$, $2B_2$, $3G_2$ and $2F_4$. For all of these the (generic) character tables are known and available in CHEVIE [1]. Here we look for all irreducible characters $\chi$ which have value $-1$ on all $p$-singular (non-semisimple) elements, so that $\chi + 1_G$ is a candidate for the 1-PIM of $G$. It is easy to see that, apart from two exceptions, all characters either: vanish on some $p$-singular element, or that a value on a non-trivial unipotent class is positive, or that the values on two non-trivial unipotent classes are different.

The two exceptions occur in type $A_1$, so $G = L_2(q)$, which has a family of characters of degree $q - 1$ with value $-1$ on all $p$-singular elements, and in type $A_2$, so $G = L_3(q)$, which has a family of characters of degree $q^3 - 1$ which also take value $-1$ on all $p$-singular elements. By [7, Thm. 5.2] the 1-PIM of $G = L_2(q)$ has degree $(2f - 1)q$, where $q = p^f$. Since the maximal character degree for $G$ is $q + 1$, this shows that $c_{11}^{(p)} > 2$ unless $f = 1$; that is, unless $q = p$ is prime, in which case the Sylow $p$-subgroups of $G$ are cyclic (and so indeed $c_{11}^{(p)} = 2$). Similarly, by [7, Thm. 5.3] the 1-PIM of $G = L_3(q)$, $q = p^f$, has degree $(12f - 6f + 1)q^3$, so none of the candidate $1_G + \chi$ mentioned above can be the 1-PIM. \hfill \square

Table 1. Large subgroups

| $G$  | $H$   | $G$  | $H$   |
|------|-------|------|-------|
| $\SL_4^\pm(q)$ | $\Sp_4(q)$ | $G_2(q)$ | $^2A_2(q)$ |
| $\SL_n^\pm(q)$ | $(n \geq 5)$ | $\GL_n^\pm(q)$ | $F_4(q)$ | $B_4(q)$ |
| $\Spin_{2n+1}(q)$ | $(n \geq 2)$ | $\Spin_{2n}(q)$ | $E_6^\pm(q)$ | $F_4(q)$ |
| $\Sp_{2n}(q)$ | $(n \geq 3)$ | $\Sp_{2n-2}(q) \Sp_{2}(q)$ | $E_7(q)$ | $E_6(q)$ |
| $\Spin_{2n}^\pm(q)$ | $(n \geq 4)$ | $\Spin_{2n-1}(q)$ | $E_8(q)$ | $E_7(q)$ |

Observe that the case $p = 2$ was already dealt with in [2, Thm. 4.5].

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