AN INDUCTIVE METHOD FOR OI-MODULES

WEE LIANG GAN AND LIPING LI

Abstract. In this paper we introduce an inductive method to study OI-modules presented in finite degrees, where OI is a skeleton of the category of finitely totally ordered sets and strictly increasing maps. As an application, we obtain an explicit upper bound for the Castelnuovo-Mumford regularity of OI-modules.

1. Introduction

1.1. Motivation. Let \( F : \mathcal{C} \to \mathcal{D} \) be a covariant (resp., contravariant). In the situation that objects in \( \mathcal{D} \) are equipped with a homology (resp., cohomology) theory (for instances, \( \mathcal{D} \) are the categories of topological spaces, manifolds, algebras, groups, etc.), the composite of \( F \) and the homology functors \( H_\bullet(-; R) : \mathcal{D} \to R\text{-Mod} \) over a coefficient ring \( R \) is a representation of \( \mathcal{C} \), and it can be used to simultaneously explore the homology groups of a collection of objects in \( \mathcal{D} \) parameterized by the object set of \( \mathcal{C} \). This strategy recently forms the central theme of representation stability theory introduced by Church and Farb in [4]. They and quite a few authors have systematically studied the representation theoretic properties of the category \( FI \) of finite sets and injections, and applied them to explore stability patterns of (co)homological groups of many interesting examples such as configuration spaces, congruence subgroups, mapping class groups, etc; see for instances [1, 2, 3, 5, 6, 7, 8, 12, 13, 14, 18].

Another important combinatorial category appearing in representation stability theory is the category OI, whose objects are \([n]\) for \( n \in \mathbb{N} \), and morphisms are strictly increasing maps. The category OI is closely related to semisimplicial category (or called category \( \Delta_+ \) in literature) of nonempty finite totally ordered sets and strictly increasing maps, which is familiar to topologists as it has been used to define semisimplicial objects. Recently, some authors begin to consider representation theory of category OI and its applications in the study of homology of groups, and establish the following results: every finitely generated OI-module over a commutative Noetherian ring is Noetherian, and its Castelnuovo-Mumford regularity is finite; the Hilbert function of a finitely generated OI-module over a field is eventually polynomial; see for instances [7, 17, 19]. These results can be deduced from an inductive method introduced by the authors in [7, 8]. However, compared to the fruitfulness of representation theory of FI, many aspects of the structures of OI-modules are still mysterious. In particular, as far as the authors know, quantitative results about OI-modules such as upper bounds of regularity are still missing, which, as have been shown in the representation theory of FI, are essential to bound stable ranges of (co)homology groups; see [1, 3, 9, 14].

The main goal of this paper is to introduce another inductive method for OI-modules with two obvious advantages compared to the previous one in [7]: it works for arbitrary OI-modules rather than finitely generated OI-modules over commutative Noetherian rings; and it can deduce some quantitative results such as explicit upper bounds of Castelnuovo-Mumford regularity and the natural number from which the eventually polynomial growth property of Hilbert functions starts. This inductive method is based on a key combinatorial proposition (see Proposition [8]) described in Section 3. Although similar results and proofs have been figured out by the authors for FI-modules and VI-modules (where VI is the category of finite dimensional vector spaces over a finite field and linear injections) in [9, 11], we shall point out that the combinatorial structure of OI seems more
complicate because of the lack of transitivity; that is, the endomorphism group of objects \( x \) in \( \mathbf{O} \) are all trivial, and hence do not act transitively on the morphism sets ending at \( x \). Therefore, the proof of this combinatorial proposition is very delicate, and the upper bounds we obtained in this paper is far less optimal compared to the upper bounds for \( \mathbf{FI} \)-modules (see [1, Theorem A] and [11, Theorem 1.3]) and \( \mathbf{VI} \)-modules (see [9, Theorem 1.1]).

1.2. Notations. In this paper we let \( \mathbb{N} \) be the set of non-negative integers and \( \mathbb{N}_+ \) be the set of positive integers. For any \( n \in \mathbb{N} \), denote by \([n]\) the set \( \{1, \ldots, n\} \); in particular, \([0]\) = \( \emptyset \) by convention. A map \( \alpha : [m] \to [n] \) is increasing if \( \alpha(1) < \cdots < \alpha(m) \). Let \( \mathbf{O} \) be the category whose objects are \([n]\) where \( n = 0, 1, \ldots \), and whose morphisms are the increasing maps. Note that \( \mathbf{O} \) is equivalent to the category of totally ordered finite sets and order-preserving injective maps.

Fix a commutative ring \( k \). By an \( \mathbf{O} \)-module, we mean a covariant functor from \( \mathbf{O} \) to the category of \( k \)-modules. Denote by \( \mathbf{O} \)-\text{-Mod} the category of \( \mathbf{O} \)-modules. This is an abelian category with enough projective objects. In particular, \( \mathbf{O} \)-modules of the form \( k^{\mathbf{O}}([n],-) \) (denoted by \( M(n) \) later) are projective. For an \( \mathbf{O} \)-module \( V \) and \( n \in \mathbb{N} \), we write \( V_n \) for \( V([n]) \). If \( V \) is nonzero, its degree \( \deg V \) is defined to be \( \sup \{ n \mid V_n \neq 0 \} \); otherwise, we set its degree to be \(-1\).

For any \( d \in \mathbb{N} \), we write \( V_{\leq d} \) for the smallest \( \mathbf{O} \)-submodule of \( V \) containing \( V_n \)'s with \( n < d \). Define an \( \mathbf{O} \)-submodule \( U \) of \( V \) by
\[
U_d = (V_{\leq d})_d \quad \text{for every } d.
\]
Define a functor \( \mathbf{P}_0^{\mathbf{O}} : \mathbf{O} \)-\text{-Mod} \to \mathbf{O} \)-\text{-Mod} by
\[
\mathbf{P}_0^{\mathbf{O}}(V) = V/U.
\]
The functor \( \mathbf{P}_0^{\mathbf{O}} \) is right-exact; let \( \mathbf{P}_i^{\mathbf{O}} \) be its \( i \)-th left derived functor, and set
\[
t_i(V) = \deg \mathbf{P}_i^{\mathbf{O}}(V).
\]
We call \( t_0(V) \) the generation degree of \( V \), \( t_1(V) \) the relation degree, and
\[
\text{prd } V = \max \{ t_0(V), t_1(V) \}
\]
the presentation degree of \( V \). We say that \( V \) is generated in finite degrees if \( t_0(V) \) is finite, and \( V \) is presented in finite degrees if the presentation degree of \( V \) finite. The regularity \( \text{reg}(V) \) of \( V \) is defined by
\[
\text{reg}(V) = \sup \{ t_i(V) - i \mid i \in \mathbb{N} \}.
\]
Remark. In literature \( \mathbf{P}_i^{\mathbf{O}}(V) \) is called the \( i \)-th homology group of \( V \), and the functor \( \mathbf{P}_i^{\mathbf{O}} \) is interpreted by the more traditional Tor functor via introducing the notion category algebras. In this paper we do not take this approach. For details, please refer to [7].

1.3. Self-embedding and shift functor. We now define a self-embedding functor \( \sigma : \mathbf{O} \to \mathbf{O} \) as follows. For each object \([n]\) of \( \mathbf{O} \), let
\[
\sigma([n]) = [n + 1].
\]
For each morphism \( \alpha : [m] \to [n] \) of \( \mathbf{O} \), define \( \sigma(\alpha) : [m + 1] \to [n + 1] \) by\[
\sigma(\alpha)(h) = \begin{cases} 1 & \text{if } h = 1, \\ \alpha(h - 1) + 1 & \text{if } h > 1. \end{cases}
\]
The functor \( \sigma \) induces a shift functor \( \Sigma : \mathbf{O} \)-\text{-Mod} \to \mathbf{O} \)-\text{-Mod} by defining
\[
\Sigma V = V \circ \sigma \quad \text{for every } V \in \mathbf{O} \text{-Mod}.
\]
Note that for every \( n \in \mathbb{N} \), one has: \((\Sigma V)_n = V_{n+1}\). For every \( r \in \mathbb{N} \), we write \( \Sigma^r \) for \( \underbrace{\Sigma \circ \cdots \circ \Sigma}_{r} \).

For each \( n \in \mathbb{N} \), we write \( \iota : [n] \to [n + 1] \) for the morphism of \( \mathbf{O} \) defined by
\[
\iota(h) = h + 1 \quad \text{for every } h \in [n].
\]
For any $\text{OI}$-module $V$, there is a natural $\text{OI}$-module homomorphism $V \to \Sigma V$ defined by

$$V_n \to (\Sigma V)_n, \quad v \mapsto \nu v, \quad \text{for every } n \in \mathbb{N}.$$ 

Let $\kappa V$ and $\Delta V$ be, respectively, the kernel and cokernel of $V \to \Sigma V$.

1.4. Main results. We now state our main results. The first theorem, though seems too technical, actually lays the foundation for us to develop an inductive method similar to that described in [7].

**Theorem 1.** Let $V$ be an $\text{OI}$-module presented in finite degrees, $d = t_0(V)$, and $r$ be any integer such that $r \geq \text{prd}(V)$. Define

$$\nabla = \frac{\Sigma^r V}{(\Sigma^r V)_{\leq d}}.$$

Then $\kappa \nabla = 0$.

Based on the above result, we develop a formal inductive method which allows us to verify representation theoretic properties of $\text{OI}$-modules presented in finite degrees in a convenient way. Please check Definition [17] for precise meanings of terminologies in the theorem.

**Theorem 2.** Let $(P)$ be a property of some $\text{OI}$-modules and suppose that the zero module has property $(P)$. Then every $\text{OI}$-module presented in finite degrees has property $(P)$ if and only if $(P)$ is glueable, $\Sigma$-dominant, and $\Delta$-predominant.

We shall point out that this theorem is definitely not a superficial extension of [7, Theorem 1.8] from the category of finitely generated $\text{OI}$-modules to the category of $\text{OI}$-modules presented in finite degrees. Actually, the former one heavily relies on the Noetherian property of finitely generated $\text{OI}$-modules over commutative Noetherian rings, while the second one is based on a completely novel machinery working for all $\text{OI}$-modules.

If we apply $\Sigma^n$ to an $\text{OI}$-module $V$ presented in finite degrees, it may not become a semi-induced module for $n \gg 0$. This is a big difference between $\text{FI}$-modules and $\text{OI}$-modules. However, we can still get a weaker stability phenomenon called filtration stability.

**Theorem 3.** Let $V$ be an $\text{OI}$-module presented in finite degrees. Then there exist a finite collection of $\text{OI}$-modules $\text{F}_V = \{V^1, \ldots, V^n\}$ and an integer $N \in \mathbb{N}$ such that for every $n \geq N$, there is a finite filtration on $\Sigma^n V$ with the property that each successive quotient is isomorphic to a member $V^i \in \text{F}_V$. Moreover, $t_0(V^i) \leq t_0(V)$ for $i \in [s]$.

It has been shown by the authors in [7] that a finitely generated $\text{OI}$-module over a commutative Noetherian ring has finite regularity. Theorem 2 allows us to establish the finiteness of regularity of $\text{OI}$-modules presented in finite degrees. Furthermore, combining the quantitative result in Theorem 4, we can obtain a simple (but far from optimal) upper bound of regularity for all $\text{OI}$-modules.

**Theorem 4.** For any nonzero $\text{OI}$-module $V$, one has:

$$\text{reg}(V) \leq 2^{t_0(V)} \text{prd}(V).$$

Note that if $V$ is the zero $\text{OI}$-module, then $\text{reg}(V) = -1$. Thus in the above theorem we require $V$ to be nonzero. We also point out that the theorem holds trivially if $t_0(V)$ or $t_1(V)$ is infinite. As an immediate corollary, we deduce that the category of $\text{OI}$-modules presented in finite degrees is an abelian subcategory of $\text{OI}$-$\text{Mod}$, so we can do homological algebra safely in it.

[7, Corollary 1.13] asserts that for a finitely generated $\text{OI}$-module $V$ over a field, there exists a positive integer $N_V$ such that its Hilbert function eventually coincides with a rational polynomial for $n \geq N_V$. The following theorem provides a bound for $N_V$.

**Theorem 5.** Let $V$ be a finitely generated $\text{OI}$-module over a field $k$. Then there exists a rational polynomial $P$ such that $\dim_k V_n = P(n)$ whenever

$$n \geq 2^{t_0(V)} \text{prd}(V).$$

Moreover, the degree of $P$ is at most $t_0(V)$. 

AN INDUCTIVE METHOD FOR OI-MODULES 3
1.5. Applications in other areas. The category $\text{OI}$ has close relations to many structures in other areas such as commutative algebra and algebraic topology. Recently, Güntürkün and Snowden studied the representation theory of the increasing monoid in [10]. We recall the following notations:

- $J$ is the monoid of increasing injections $\sigma : \mathbb{N}_+ \to \mathbb{N}_+$ such that there exists a certain $l \in \mathbb{N}$ satisfying $\sigma(n) = n + l$ for $n \gg 0$; see [10] Subsection 2.1.
- Let $k$ be a field, $k(1)$ be the graded vector space with $k$ in degree 1 and 0 in other degrees, and $A$ be the shuffle algebra $\text{Sym}_{\mathbb{N}}(k(1))$; see [10] Subsection 3.7.
- The semisimplicial category $\Delta_+$ of finite non-empty totally ordered sets and strictly increasing maps.

By [10] Remark 3.3, Propositions 3.7, 3.8, 3.10, we have the following equivalences:

- $J\text{-grMod} \cong \text{OI-Mod} \oplus k\text{-Mod}$, where $J\text{-grMod}$ is the category of graded $J$-modules; see [10] Subsection 3.3;
- $A\text{-Mod} \cong \text{OI-Mod}$ when $k$ is a field;
- the category of co-semisimplicial $k$-modules (which are covariant functors from $\Delta_+$ to the category $k\text{-Mod}$) is equivalent to the full subcategory of $\text{OI-Mod}$ consisting of those $\text{OI}$-modules $V$ with $V_0 = 0$.

Thus our main theorems apply to all these module categories (note that $J\text{-grMod}$ is essentially equivalent to $\text{OI-Mod}$). In particular, Theorem 4 gives an answer to the question raised in [10] Subsection 1.4.2.

1.6. Organization. The paper is organized as follows. In Section 2 we describe some preliminary results about the category $\text{OI}$ and its representations. In Section 3 we prove a key combinatorial proposition. Main theorems and some corollaries are proved in Section 4. In the last section we raise some questions which might be of certain interest to the reader.

2. Preliminaries

In this section we list some preliminary results on $\text{OI}$-modules. These results have appeared in literature, and were proved for general categories (including $\text{OI}$ as an example) equipped with shift functors satisfying certain axioms. For details, please refer to [7].

For any $m \in \mathbb{N}$, let $M(m)$ be the $\text{OI}$-module that takes each $[n]$ to the free $k$-module on the set of increasing maps from $[m]$ to $[n]$. If $\beta : [s] \to [n]$ is any morphism of $\text{OI}$, the induced map $M(m)_s \to M(m)_n$ is defined by

$$\sum_{\alpha : [m] \to [s]} c_{\alpha} \alpha \mapsto \sum_{\alpha : [m] \to [s]} c_{\alpha} \beta \alpha$$

where $c_{\alpha}$ are coefficients in $k$ and $\alpha$ runs over the increasing maps $[m] \to [s]$. Note that $t_0(M(m)) = m$. It is easy to check that $M(m)$ is a projective $\text{OI}$-module, so one has $t_i(M(m)) = -1$ for every $i \geq 1$. Furthermore, for any $\text{OI}$-module $V$, there exists a surjective homomorphism $F \to V$ where $F = \bigoplus_{j \in J} M(d_j)$ for some $d_j \in \mathbb{N}$ such that $t_0(F) = t_0(V)$.

Let $m, r \in \mathbb{N}$. Let $E \subset [r]$ and suppose $|E| \leq m$. Write $E = \{e_1, \ldots, e_\ell\}$ where $e_1 < \cdots < e_\ell$. For any increasing map $\alpha : [m-\ell] \to [n]$, define an increasing map $\alpha_E : [m] \to [n+r]$ by

$$\alpha_E(h) = \begin{cases} e_h & \text{if } h \leq \ell, \\ \alpha(h-\ell) + r & \text{if } h > \ell. \end{cases}$$

Define an $\text{OI}$-module homomorphism $\theta_E : M(m-\ell) \to \Sigma^r M(m)$ by

$$M(m-\ell)_n \to (\Sigma^r M(m))_n, \quad \alpha \mapsto \alpha_E$$
for every \( n \in \mathbb{N} \) and increasing map \( \alpha : [m - \ell] \to [n] \). Note that if \( r = 1 \) and \( E = \emptyset \), then \( \theta_E \) is the natural map \( M(m) \to \Sigma M(m) \).

**Lemma 6.** For any \( m, r \in \mathbb{N} \), there is a natural isomorphism

\[
\theta : \bigoplus_{\ell=0}^{m} \bigoplus_{E \subseteq [r] \mid |E| = \ell} M(m - \ell) \to \Sigma^r M(m).
\]

**Proof.** Let \( \theta \) be the homomorphism whose restriction to the direct summand \( M(m - \ell) \) indexed by \( E \) is \( \theta_E \). It is easy to check that \( \theta \) is an isomorphism. \( \square \)

It follows from Lemma 6 that for any \( \text{OI} \)-module \( V \) and \( r \in \mathbb{N} \), one has:

\[
t_0(\Sigma^r V) \leq t_0(V);
\]

if \( V \) is nonzero, then one has:

\[
t_0(\Delta V) \leq t_0(V) - 1,
\]

and the equality of the second formula holds whenever \( V \) is nonzero. The reader may refer to [7, Lemma 2.2] for details. These inequalities also hold for regularity. That is:

**Lemma 7.** Let \( V \) be an \( \text{OI} \)-module. Then one has:

\[
\mathrm{reg}(V) \leq \mathrm{reg}(\Sigma V) + 1.
\]

If \( \kappa V = 0 \), then one has:

\[
\mathrm{reg}(V) \leq \mathrm{reg}(\Delta V) + 1.
\]

**Proof.** See [7, Corollary 5.3] and [7, Corollary 5.6]. \( \square \)

3. A Key Proposition

In this section we prove a combinatorial proposition, which plays the central role for the proof of Theorem I. For this purpose we introduce a few notations.

Let \( V \) be an \( \text{OI} \)-module such that \( \max\{t_0(V), t_1(V)\} < \infty, d = t_0(V), \) and let \( r \) be any integer \( \geq \max\{t_0(V), t_1(V)\} \). Then there exists a surjective homomorphism \( F \to V \) where

\[
F = \bigoplus_{j \in J} M(d_j)
\]

for some indexing set \( J \) and \( d_j \in \mathbb{N} \) such that \( d_j \leq d \) for every \( j \in J \). Let \( W \) be the kernel of \( F \to V \). Then

\[
t_0(W) \leq \max\{t_0(V), t_1(V)\} \leq r.
\]

Let

\[
I = \{i \in J \mid d_i = d\}, \quad P = \bigoplus_{i \in I} M(d_i) = \bigoplus_{i \in I} M(d).
\]

By Lemma 6 there is a natural decomposition

\[
\Sigma^r F \cong P \oplus Q
\]

where \( Q \) is a direct sum of \( \text{OI} \)-modules of the form \( M(m) \) such that \( m < d \). Let

\[
\eta : \Sigma^r F \to P
\]

be the projection with kernel \( Q \), and

\[
\widehat{W} = \eta(\Sigma^r W) \subset P.
\]

For a morphism \( \alpha : [d] \to [s] \) in \( \text{OI} \), we let \( \ell = \alpha(1) \), and define

\[
\widehat{\alpha} : [d] \to [s - \ell + 1], \quad \widehat{\alpha}(h) = \alpha(h) - \ell + 1 \text{ for } h = 1, \ldots, d.
\]
It is always true that $\hat{\alpha}(1) = 1$. Since every element in $F_s = \bigoplus_{j \in J} M(d_j)_s$ is a linear combination of morphisms starting at a certain object $[d_j]$ with $j \in J$ and ending at the object $[s]$, for $w \in W_s \subset F_s$, we can write

$$w = \sum_{j \in J} \sum_{\alpha_j : [d_j] \to [s]} c_{j,\alpha} \alpha_j,$$

where $c_{j,\alpha}$ are coefficients in $k$ and $\alpha_j$ runs over all morphisms from $[d_j]$ to $[s]$. If $j \in I$, then $d_j = d$ by definition, so in this case we simply write $\alpha$ and $\hat{\alpha}$ rather than $\alpha_j$ and $\hat{\alpha}_j$ in the above expression. For each $\ell = 1, \ldots, s - d + 1$, we let

$$\hat{w}_\ell = \sum_{i \in I} \sum_{\alpha_i : [d] \to [s]} c_{i,\alpha} \hat{\alpha} \in P_{s-\ell+1}.$$  

In particular, if $d > s$, then $\hat{w}_\ell = 0$.

**Proposition 8.** The $	extbf{OI}$-module $\hat{W}$ is generated by the collection of elements $\hat{w}_\ell$ for all $w \in W_s$ such that $s \leq t_0(W)$ and for all $\ell = 1, \ldots, s - d + 1$.

**Proof.** Take any $w \in W_s$ where $s \leq t_0(W)$ and write it in the form of (1). For any $n \in \mathbb{N}$, we consider an increasing map $\beta : [s] \to [r + n]$. Then

$$\beta w = \sum_{j \in J} \sum_{\alpha_j : [d_j] \to [s]} c_{j,\alpha} \beta \alpha_j \in F_{r+n}.$$  

Note that:

- If $j \notin I$, then $\beta \alpha_j \in Q_n \subset (\Sigma^r F)_n = F_{r+n}$, so $\eta(\beta \alpha_j) = 0$.
- If $j \in I$ and $\beta \alpha_j(1) \leq r$, then $\beta \alpha_j \in Q_n \subset (\Sigma^r F)_n = F_{r+n}$, so $\eta(\beta \alpha_j) = 0$.

Therefore

$$\eta(\beta w) = \sum_{i \in I} \sum_{\alpha_i : [d] \to [s]} c_{i,\alpha} \eta(\beta \alpha) \in P_n.$$  

To show that $\hat{w}_\ell \in \hat{W}$ for each $\ell = 1, \ldots, s - d + 1$, we do a downward-induction on $\ell$. Let $\beta : [s] \to [s + r - \ell + 1]$ be the map defined by

$$\beta(h) = h + r - \ell + 1 \quad \text{for each } h.$$  

So $\beta$ will map $P_s$ to $P_{s+r-\ell+1}$. Note that for any $\alpha : [d] \to [s]$, we have:

$$\beta(\alpha(1)) = \begin{cases} \alpha(1) + r - \ell + 1 & \text{if } \alpha(1) < \ell, \\ = r + 1 & \text{if } \alpha(1) = \ell, \\ > r + 1 & \text{if } \alpha(1) > \ell. \end{cases}$$  

So

$$\eta(\beta w) = \sum_{i \in I} \sum_{\alpha_i : [d] \to [s]} c_{i,\alpha} \eta(\beta \alpha)$$  

$$= \hat{w}_\ell + \iota \hat{w}_{\ell+1} + \iota^2 (\hat{w}_{\ell+2}) + \cdots.$$  

By the downward-induction hypothesis, $\iota \hat{w}_{\ell+1}, \iota^2 (\hat{w}_{\ell+2}), \ldots$ are in $\hat{W}$, so it follows that $\hat{w}_\ell$ is also in $\hat{W}$. 

It remains to show that for any increasing map \( \beta : [s] \rightarrow [r+n] \), the element \( \eta(\beta w) \) is contained in the submodule of \( P \) generated by the collection of \( \hat{w}_t \). Note that

\[
\eta(\beta w) = \sum_{i \in I} \sum_{\alpha : [d] \rightarrow [s]} c_i,\alpha \eta(\beta \alpha) \\
= \sum_{\{\ell | \beta(\ell) > r\}} \sum_{i \in I} \sum_{\alpha : [d] \rightarrow [s]} c_i,\alpha \eta(\beta \alpha) \in P_n.
\]

It suffices to check that for each \( \ell \) such that \( \beta(\ell) > r \), the element

\[
\sum_{i \in I} \sum_{\alpha : [d] \rightarrow [s]} c_i,\alpha \gamma(\beta \alpha)
\]

is of the form \( \gamma(\hat{w}_t) \) for some \( \gamma : [s - \ell + 1] \rightarrow [n] \). To this end, define the map

\[
\gamma : [s - \ell + 1] \rightarrow [n]
\]

by

\[
\gamma(h) = \beta(\ell + h - 1) - r \quad \text{for each } h = 1, \ldots, s - \ell + 1.
\]

Then for each \( h = 1, \ldots, d \), we have:

\[
\gamma(\hat{\alpha}(h)) = \gamma(\alpha(h) - \ell + 1) = \beta(\alpha(h)) - r;
\]

this implies \( \gamma(\hat{\alpha}) = \eta(\beta \alpha) \). Hence,

\[
\gamma(\hat{w}_t) = \sum_{i \in I} \sum_{\alpha : [d] \rightarrow [s]} c_i,\alpha \gamma(\hat{\alpha}) = \sum_{i \in I} \sum_{\alpha : [d] \rightarrow [s]} c_i,\alpha \eta(\beta \alpha),
\]

as claimed. \( \square \)

4. Proofs of Main Theorems

In this section we keep the notation of the preceding, and prove the main theorems stated in the introduction.

4.1. A proof of Theorem 1 In this subsection we prove the following theorem.

**Theorem 9.** Let \( V \) be an OI-module presented in finite degrees, \( d = t_0(V) \), and \( r \) be any integer such that \( r \geq \text{prd}(V) \). Define

\[
\overline{V} = \frac{\Sigma^r V}{(\Sigma^r V)_{<d}}.
\]

Then \( \kappa \overline{V} = 0 \).

**Proof.** Since the functor \( \Sigma^r \) is exact, we have the short exact sequence

\[
0 \rightarrow \Sigma^r W \rightarrow \Sigma^r F \rightarrow \Sigma^r V \rightarrow 0.
\]

Recall the natural decomposition \( \Sigma^r F \cong P \oplus Q \). The restriction of the map \( \Sigma^r F \rightarrow \Sigma^r V \) to \( Q \) gives a surjective homomorphism \( Q \rightarrow (\Sigma^r V)_{<d} \). We get the following commuting diagram whose rows and columns are exact:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & Q & \longrightarrow & \Sigma^r F & \longrightarrow & P & \longrightarrow & 0 \\
\| & & \| & & \| & & \| & & \| \\
0 & \longrightarrow & (\Sigma^r V)_{<d} & \longrightarrow & \Sigma^r V & \longrightarrow & \overline{V} & \longrightarrow & 0 \\
\| & & \| & & \| & & \| & & \| \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]
By the snake lemma, the kernel of \( P \to V \) is \( \hat{W} \). Therefore \( V \) is isomorphic to \( P/\hat{W} \).

We now prove that \( \kappa V = 0 \). Suppose \( v \in P_n \) for some \( n \in \mathbb{N} \) and \( \nu v \in \hat{W}_{n+1} \). We need to show that \( v \in \hat{W}_n \).

By Proposition 8 we can write

\[
\nu v = \sum_{k=1}^{p} a_k \beta_k \omega_k \in \hat{W}_{n+1} \subset P_{n+1}
\]

where \( a_k \) are coefficients in \( k \), each \( \omega_k \in \hat{W}_{s_k} \) is an element of the form:

\[
\sum_{i \in I} \sum_{\alpha[\alpha(1)=1]} c_{i,\alpha} \alpha \in P_{s_k}
\]

for some coefficients \( c_{i,\alpha} \in k \), and \( \beta_k : [s_k] \to [n+1] \). We write this as:

\[
\nu v = \sum_{\{k | \beta_k(1)=1\}} a_k \beta_k \omega_k + \sum_{\{k | \beta_k(1)>1\}} a_k \beta_k \omega_k.
\]

Since \( \iota(1) > 1 \), we must have:

\[
\nu v = \sum_{\{k | \beta_k(1)>1\}} a_k \beta_k \omega_k.
\]

Now for each \( \beta_k \) such that \( \beta_k(1) > 1 \), define \( \beta'_k : [s_k] \to [n] \) by \( \beta'_k(h) = \beta_k(h) - 1 \) for each \( h \), so that \( \iota \beta'_k = \beta_k \). Then

\[
\nu v = \iota \left( \sum_{\{k | \beta_k(1)>1\}} a_k \beta'_k \omega_k \right).
\]

But \( \iota : P_n \to P_{n+1} \) is injective, so

\[
v = \sum_{\{k | \beta_k(1)>1\}} a_k \beta'_k \omega_k \in \hat{W}_n.
\]

This concludes the proof of Theorem 1. \( \square \)

The importance of this theorem lies in the following fact. Inductive methods such as the one described in [7] have played a prominent role in representation stability theory. To apply these methods, in general the first step is to convert an arbitrary module \( V \) into a closely related torsion free module \( \nabla \) such that \( \kappa \nabla = 0 \). For finitely generated \( \text{OI} \)-modules over commutative Noetherian rings, the authors have described a finite procedure to get such a \( \nabla \) in [7]. However, the finiteness of this procedure highly depends on the Noetherian property of finitely generated modules, and hence it can not extend to the more general framework of arbitrary \( \text{OI} \)-modules \( V \) presented in finite degrees. The above theorem provides a redemption for this failure. To avoid confusion, we remind the reader that the module \( V_{\text{reg}} \) defined in [7, Section 3] does not coincide with \( \nabla \) in this paper. Here is an example:

**Example 10.** Let \( V = M(1) \oplus M(0) \). Then \( t_0(V) = 1 \) and \( t_1(V) = -1 \). For any \( r \geq 1 \),

\[
\Sigma^r V = M(1) \oplus M(0)^{\oplus r+1} = V \oplus M(0)^{\oplus r}.
\]

Consequently,

\[
\nabla = \frac{\Sigma^r V}{(\Sigma^r V)_{<1}} = \frac{M(1) \oplus M(0)^{\oplus r+1}}{M(0)^{\oplus r+1}} = M(1).
\]

However, one has

\[
V_{\text{reg}} = V = M(1) \oplus M(0).
\]
For an $FI$-module $V$ presented finite degrees, after applying the shift functor $\Sigma$ (for $FI$-modules) $r$ times with $r \geq t_0(V) + t_1(V)$, one gets a semi-induced module, and in particular $\kappa \Sigma^r V = 0$; see for instances [16, Theorem 2.6], [11 Corollary 3.3], or [18 Theorem C]. However, for $OI$-modules, this result does not hold. The following example is provided by Eric Ramos:

**Example 11.** Let $V$ be the following $OI$-module: $V_0 = 0$, and $V_n = k$ for $n \geq 1$. For a morphism $\alpha : [m] \to [n]$ with $m \geq 1$, one defines $V(\alpha) : V_n \to V_m$ to be the identity map if $\alpha(m) = n$ and $V(\alpha)$ to be the zero map otherwise. The reader can check that $V$ is indeed an $OI$-module with $t_0(V) = 1$ and $t_1(V) = 2$. A direct computation shows $\Sigma^r V = V \oplus U$ for any $r \geq 1$, where $U_0 = k$ and $U_n = 0$ for all $n \geq 1$, and $\overline{V} = 0$. Therefore, $\Sigma^r V$ is not a semi-induced module for any $r \in \mathbb{N}$.

### 4.2. An upper bound of regularity

In this subsection we use Theorem 1 and an inductive method to establish Theorem 2. This method is base on the following two short exact sequences:

$$
0 \to (\Sigma^r V)_{<d} \to \Sigma^r V \to \overline{V} \to 0,
0 \to \overline{V} \to \Sigma \overline{V} \to \Delta \overline{V} \to 0.
$$

**Lemma 12.** Let $r$ be an integer $\geq \max\{t_0(V), t_1(V)\}$. We have:

$$
t_0(\Delta \overline{V}) \leq d - 1 \text{ and } t_1(\Delta \overline{V}) \leq r - 1.
$$

**Proof.** To check the first inequality, one looks at the first short exact sequence and note that

$$
t_0(\overline{V}) \leq t_0(\Sigma^r V) \leq t_0(V) = d,
$$

so the conclusion holds; see the paragraph before Lemma 7.

Now we turn to the second inequality. Applying the snake Lemma to the commutative diagram in the proof of Theorem 1 we get a short exact sequence

$$
0 \to U \to \Sigma^r W \to \hat{W} \to 0.
$$

Consequently, one gets

$$
t_0(\hat{W}) \leq t_0(\Sigma^r W) \leq t_0(W) \leq r.
$$

Furthermore, since $\Delta$ is right exact, applying it to the short exact sequence

$$
0 \to \hat{W} \to P \to \overline{V} \to 0
$$

we obtain another short exact sequence

$$
0 \to C \to \Delta P \to \Delta \overline{V} \to 0
$$

where $C$ is a quotient module of $\Delta \hat{W}$. Consequently, one has

$$
t_1(\Delta \overline{V}) \leq t_0(C) \leq t_0(\Delta \hat{W}) \leq t_0(\hat{W}) - 1 \leq r - 1
$$

as claimed. \[\square\]

To prove the upper bound for regularity of $OI$-modules presented in finite degrees, for each $d \in \mathbb{N}$ we introduce an auxiliary function $C_d : \mathbb{Z} \to \mathbb{Z}$ by the initial condition $C_0(r) = r$ and the recursive relation

$$
C_d(r) = C_{d-1}(C_{d-1}(r-1) + 3) + r.
$$

These functions are increasing with respect to $d$ and $r$, and have simple lower and upper bounds.

**Lemma 13.** Suppose $r \geq d \geq 0$.

(a) One has:

$$
C_d(r) \geq r,
C_d(r + 1) > C_d(r).
$$

(b) If $d \geq 1$, then

$$
C_d(r) > C_{d-1}(r).
$$
(c) Suppose $r \geq d \geq 0$. Then one has:
\[ C_d(r) \leq 2^{2^d} r. \]

Proof. (a) We use induction on $d$. When $d = 0$, the inequalities are obvious. Suppose $d \geq 1$. Then
\[ C_{d-1}(r - 1) \geq r - 1. \]
So we have:
\[
C_d(r) = C_{d-1}(C_{d-1}(r - 1) + 3) + r \\
\geq C_{d-1}((r - 1) + 3) + r \\
\geq r - 1 + 3 + r \\
\geq r,
\]
and
\[
C_d(r + 1) = C_{d-1}(C_{d-1}(r) + 3) + r + 1 \\
> C_{d-1}(C_{d-1}(r - 1) + 3) + r \\
= C_d(r).
\]
(b) Using (a), we have:
\[
C_d(r) = C_{d-1}(C_{d-1}(r - 1) + 3) + r \\
\geq C_{d-1}(r - 1 + 3) + r \\
> C_{d-1}(r).
\]
(c) Note that $C_0(r) = r$ and $C_1(r) = 2r + 2$. Let us prove that:
\[ C_d(r) \leq (2^{2^d} - 1)r \quad \text{for } r \geq d \geq 2. \]
We have:
\[
C_2(r) = C_1(C_1(r - 1) + 3) + r \\
= 2(2(r - 1) + 2 + 3) + 2 + r \\
= 5r + 8 \\
\leq 15r.
\]
We use induction for $d > 2$. By the induction hypothesis and the conclusion of Parts (a) and (b), we have:
\[
C_d(r) \leq (2^{2^d-1} - 1)(2^{2^d-1} - 1)(r - 1) + 3) + r \\
= (2^{2^d-1} - 1)(2^{2^d-1} - 1)(2^{2^d-1} - 1)(r - 2^{2^d-1} + 4) + r \\
\leq (2^{2^d-1})(2^{2^d-1}r - r) + r \\
= 2^d r - 2^{2^d-1}r + r \\
\leq (2^{2^d} - 1)r.
\]

Now we are ready to prove Theorem 14 by an induction on $d = t_0(V)$.

Theorem 14. For any nonzero $\mathcal{O}_I$-module $V$, one has:
\[ \text{reg}(V) \leq 2^{t_0(V)} \text{prd}(V). \]
Proof. Let $V$ be a nonzero $\text{OI}$-module. If $\text{prd}(V) = \infty$, the statement of Theorem 4 is trivial, so assume that $\text{prd}(V) < \infty$. Let $d = t_0(V)$ and $r$ be an integer such that $r \geq \text{prd}(V)$. We use an induction on $d$ to show that

$$\text{reg}(V) \leq C_d(r).$$

This inequality clearly implies the conclusion of the theorem.

Suppose $d = 0$. Then

$$t_0(\Delta V) \leq -1 \implies \Delta V = 0 \implies \text{reg}(\Delta V) = -1.$$

By Theorem 1 and Lemma 7, we have: $\text{reg}(V) \leq 0$. Since $(\Sigma^r V)_{< 0} = 0$, we have: $\Sigma^r V \cong V$, so $\text{reg}(\Sigma^r V) \leq 0$. By Lemma 7 it follows that $\text{reg} V \leq r = C_0(r)$.

Suppose $d \geq 1$. We shall use Lemma 13 several times below without further mention. By the induction hypothesis, we have:

$$\text{reg}(\Delta V) \leq C_{d-1}(r - 1).$$

By Theorem 1 and Lemma 7, we have:

$$\text{reg}(V) \leq C_{d-1}(r - 1) + 1,$$

so

$$t_2(V) \leq C_{d-1}(r - 1) + 3.$$

Therefore, from the short exact sequence

$$0 \to (\Sigma^r V)_{< d} \to \Sigma^r V \to V \to 0$$

we deduce

$$t_1((\Sigma^r V)_{< d}) \leq \max\{t_1(\Sigma^r V), t_2(V)\} \leq \max\{t_0(\Sigma^r W), t_2(V)\} \leq \max\{r, C_{d-1}(r - 1) + 3\} \leq C_{d-1}(r - 1) + 3.$$

By the induction hypothesis, we have:

$$\text{reg}((\Sigma^r V)_{< d}) \leq C_{d-1}(C_{d-1}(r - 1) + 3).$$

Hence,

$$\text{reg}(\Sigma^r V) \leq \max\{\text{reg}((\Sigma^r V)_{< d}), \text{reg}(V)\} \leq C_{d-1}(C_{d-1}(r - 1) + 3).$$

Therefore, by Lemma 7

$$\text{reg}(V) \leq C_{d-1}(C_{d-1}(r - 1) + 3) + r = C_d(r).$$

□

Remark. As far as we know, this theorem provides the first explicit upper bound for regularity of $\text{OI}$-modules. However, we believe that it is far from optimal. For example, take $n \in \mathbb{N}$, and let $V$ be an $\text{OI}$-module such that $V_n = k$ and $V_i = 0$ for $i \neq n$. A direct computation shows $\text{reg}(V) = n$, but the above theorem only asserts $\text{reg}(V) \leq 2^n(n + 1)$.

The careful reader can see that in the proof we have to use $\text{reg}(V)$ to bound $t_2(V)$, which significantly amplifies the final upper bound of $\text{reg}(V)$. If a more optimal upper bound for $t_2(V)$ becomes available as the case of $\text{FI}$-modules (see the proof of [11, Theorem 2.4] or $\text{VI}$-modules (see the proof of [9, Theorem 3.2]), then the conclusion of this theorem can be tremendously improved.

By the following corollary, we can do homological algebra safely in the category of $\text{OI}$-modules presented in finite degrees.

Corollary 15. The category of $\text{OI}$-modules presented in finite degrees is abelian.
Proof. The proof of this result is a routinely homological check. Let \( \phi : U \rightarrow V \) be a morphism in this category. It suffices to show that \( \text{Ker} \phi \) and \( \text{coKer} \phi \) also lie in it. Breaking this morphism into two short exact sequences

\[
\begin{align*}
0 & \rightarrow \text{Ker} \phi \rightarrow U \rightarrow \text{Im} \phi \rightarrow 0, \\
0 & \rightarrow \text{Im} \phi \rightarrow V \rightarrow \text{coKer} \phi \rightarrow 0,
\end{align*}
\]

one can check that all terms in them are presented in finite degrees. \( \square \)

For an \( \text{OI} \)-module \( V \) and any \( n \in \mathbb{N} \), we define a submodule \( \tau_n V \) by letting \( (\tau_n V)_i = 0 \) for \( i < n \) and \( (\tau_n V)_i = V_i \) for \( i \geq n \). The next corollary, which says that \( \tau_r V \) has a generalized Koszul property, is an immediate aftermath of [9, Theorem 5.6] and Theorem 4.

**Corollary 16.** If \( V \) is presented in finite degrees and \( r \geq 2^{|t_0(V)|} \) prd(V), then for any \( i \in \mathbb{N} \), \( H_i(\tau_r V) \) either is 0, or is generated by its value on the object \([r + i]\).

### 4.3. Inductive machinery.

An important consequence of Corollary 15 is to allow us to extend the inductive machinery introduced in [7] from the category of finitely generated \( \text{OI} \)-modules over Noetherian coefficient rings to the category of \( \text{OI} \)-modules presented in finite degrees. Let us recall [7, Definition 4].

**Definition 17.** Suppose that \( \mathcal{T} \) is a subcategory of \( \text{OI} \)-Mod and \( F : \mathcal{T} \rightarrow \mathcal{T} \) is a functor. We say that a property \( (P) \) of some \( \text{OI} \)-modules is:

- **glueable on \( \mathcal{T} \)** if, for every short exact sequence \( 0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0 \) in \( \mathcal{T} \):
  
  \( U \) and \( W \) has property \( (P) \) \( \implies \) \( V \) has property \( (P) \);

- **\( F \)-dominant on \( \mathcal{T} \)** if, for every \( V \in \mathcal{T} \):
  
  \( FV \) has property \( (P) \) \( \implies \) \( V \) has property \( (P) \);

- **\( F \)-predominant on \( \mathcal{T} \)** if, for every \( V \in \mathcal{T} \):
  
  \( FV \) has property \( (P) \) and \( \kappa V = 0 \) \( \implies \) \( V \) has property \( (P) \).

The following theorem provides a convenient way to check qualitative representation theoretic properties of \( \text{OI} \)-modules presented in finite degrees.

**Theorem 18.** Let \( (P) \) be a property of some \( \text{OI} \)-modules and suppose that the zero module has property \( (P) \). Then every \( \text{OI} \)-module presented in finite degrees has property \( (P) \) if and only if \( (P) \) is glueable, \( \Sigma \)-dominant, and \( \Delta \)-predominant.

**Proof.** This theorem actually formalizes the strategy we used to show Theorem 4. One direction is trivial, so we show the other one.

Firstly, since \( t_0(\Delta V) < t_0(V) \leq t_0(V) \), the induction hypothesis guarantees \( \Delta V \) has property \( (P) \). Since \( (P) \) is \( \Delta \)-predominant and \( \kappa V = 0 \), \( V \) has property \( (P) \). Similarly, \( (\Sigma' V)_{<d} \) has property \( (P) \). Since \( (P) \) is glueable, by the short exact sequence

\[
0 \rightarrow (\Sigma' V)_{<d} \rightarrow \Sigma' V \rightarrow V \rightarrow 0
\]

we conclude that \( \Sigma' V \) has property \( (P) \). But \( (P) \) is \( \Sigma \)-predominant, so \( V \) has property \( (P) \) as well. \( \square \)

For instances, let \( (P) \) be the property of having finite regularity. Then Lemma 7 asserts that \( (P) \) is \( \Sigma \)-dominant and \( \Delta \)-predominant. It is easy to see that \( (P) \) is glueable. Therefore, by the above theorem we know that every \( \text{OI} \) presented in finite degrees has finite regularity.
4.4. Filtration stability. If we apply $\Sigma^n$ to an $\OI$-module $V$ presented in finite degrees, it may not become a semi-induced module for $n \gg 0$, as explained in Example 11. However, we can still get a weaker stability result. That is, there is a finite set of $\OI$-modules such that each $\Sigma^n V$ has a finite filtration whose successive quotients lie in this set.

**Theorem 19.** Let $V$ be an $\OI$-module presented in finite degrees. Then there exist a finite collection of $\OI$-modules $\mathcal{F}_V = \{V^1, \ldots, V^s\}$ and an integer $N \in \mathbb{N}$ such that for every $n \geq N$, there is a finite filtration on $\Sigma^n V$ with the property that each successive quotient is isomorphic to a member $V^i \in \mathcal{F}_V$. Moreover, $t_0(V_i) \leq t_0(V)$ for $i \in [s]$.

**Proof.** Let (P) be the property addressed in the theorem. We show that (P) is glueable, $\Sigma$-dominant, and $\Delta$-predominant. But by carefully checking the proof of Theorem 2 we find that the glueable condition can be replaced by the following weaker condition:

(w) In the short exact sequence

$$0 \rightarrow (\Sigma^r V)_{<d} \rightarrow \Sigma^r V \rightarrow \Sigma^r V \rightarrow 0,$$

if the first and the third terms satisfy (P), so does the middle term. But this is clearly true. Indeed, if $\Sigma^l (\Sigma^r V)_{<d}$ and $\Sigma^m \Sigma^r V$ satisfy (P) for $l \geq N_1$ and $m \geq N_2$. Then take $N = \max\{N_1, N_2\}$ and let

$$\mathcal{F}_V^\ast = \mathcal{F}_{(\Sigma^r V)_{<d}} \cup \mathcal{F}_{\Sigma^r V}. $$

Note that the generation degree of each module in $\mathcal{F}_{(\Sigma^r V)_{<d}}$ (resp., $\mathcal{F}_{\Sigma^r V}$) is at most $d - 1$ (resp., $d$). We conclude that $\Sigma^n V$ satisfies (P) for $n \geq N$.

If $\Sigma^l (\Sigma^r V)$ satisfies (P) for $l \geq N$, then clearly $\Sigma^n V$ has property (P) for $n \geq N + 1$ by letting $\mathcal{F}_V = \mathcal{F}_{\Sigma^r V}$; that is, (P) is $\Sigma$-dominant.

Now suppose that $kV = 0$, or equivalently the natural map $V \rightarrow \Sigma V$ is injective, and $\Sigma^n (\Delta V)$ satisfies property (P) when $n \geq N$ for a certain $N \in \mathbb{N}$ and $\mathcal{F}_{\Delta V}$. We apply $\Sigma^n$ to get the exact sequence

$$0 \rightarrow \Sigma^n V \rightarrow \Sigma^{n+1} V \rightarrow \Sigma^n (\Delta V) \rightarrow 0.$$ 

We define

$$\mathcal{F}_V = \mathcal{F}_{\Delta V} \cup \{\Sigma^n V\}.$$ 

Note that the generation degree of every member in $\mathcal{F}_V$ is at most $t_0(V)$. Furthermore, $\Sigma^{n+1} V$ has a filtration whose successive quotients all lie in $\mathcal{F}_V$. In the next step, we have

$$0 \rightarrow \Sigma^{n+1} V \rightarrow \Sigma^{n+2} V \rightarrow \Sigma^{n+1} (\Delta V) \rightarrow 0.$$ 

Combining the filtration for $\Sigma^{n+1} V$ and the filtration for $\Sigma^{n+1} (\Delta V)$, we deduce that $\Sigma^{n+2} V$ has a filtration whose successive quotients all lie in $\mathcal{F}_V$. By an induction on $n$, we conclude that $\Sigma^n V$ satisfies property (P) for $n \geq N$. That is, the property (P) is $\Delta$-predominant. $\square$

4.5. Hilbert functions. In this subsection we study the Hilbert function of finitely generated $\OI$-modules $V$ when $k$ is a field. It is already known that these functions are eventually polynomial. The following theorem tells us where this phenomenon begins.

**Theorem 20.** Let $V$ be a finitely generated $\OI$-module over a field $k$. Then there exists a rational polynomial $P$ such that $\dim_k V_n = P(n)$ whenever

$$n \geq 2^{2t_0(V)} \text{ prd}(V).$$

Moreover, the degree of $P$ is at most $t_0(V)$.

**Proof.** The proof is almost the same as that of Theorem 11. Let $d = t_0(V)$ and $r = \text{prd}(V)$. The conclusion holds trivially for $d = -1$ (by convention, we suppose that the degree of the zero polynomial is $-1$), so we suppose that $d \geq 0$. It suffices to show the conclusion for $n \geq C_d(r)$. We use an induction on $d$. 

For $d = 0$, in the proof of Theorem 4 we know $\Sigma^r V \cong \overline{V}$. But $\overline{V} \cong \Sigma \overline{V}$ since $\Delta \overline{V} = 0$. This happens if and only if the Hilbert function of $\Sigma^r V$ is a constant function. Equivalently, the conclusion holds for $n \geq r = C_0(r)$.

Now suppose that $d \geq 1$. By the induction hypothesis, there is a rational polynomial $Q$ with degree at most $d' = t_0((\Sigma^r V)_{<d}) \leq d - 1$ such that

$$\dim_k((\Sigma^r V)_{<d})_n = Q(n) \quad \text{for } n \geq C_{d'}(r'),$$

where

$$r' = \max\{t_0((\Sigma^r V)_{<d}), t_1((\Sigma^r V)_{<d})\} \leq \max\{d - 1, C_{d-1}(r - 1) + 3\} = C_{d-1}(r - 1) + 3.$$

and the second inequality is shown in the proof of Theorem 4. Therefore,

$$\dim_k((\Sigma^r V)_{<d})_n = Q(n) \quad \text{for } n \geq C_{d-1}(C_{d-1}(r - 1) + 3).$$

Consider the short exact sequence

$$0 \to \overline{V} \to \Sigma \overline{V} \to \Delta \overline{V} \to 0.$$

Note that $t_0(\Delta \overline{V}) \leq d - 1$ and $t_1(\Delta \overline{V}) \leq r - 1$ by Lemma 12. By an analogue argument, there is a rational polynomial $T$ with degree at most $t_0(\Delta \overline{V}) \leq d - 1$ such that

$$\dim_k(\Delta \overline{V})_n = T(n) \quad \text{for } n \geq C_{d-1}(r - 1).$$

But we know

$$\dim_k(\Delta \overline{V})_n = \dim_k \overline{V}_{n+1} - \dim_k \overline{V}_n.$$

Consequently, the functions $n \mapsto \dim_k \overline{V}_n$ coincides with a polynomial with degree at most $d$ for $n \geq C_{d-1}(r - 1) + 1$.

By the short exact sequence

$$0 \to (\Sigma^r V)_{<d} \to \Sigma^r V \to \overline{V} \to 0,$$

we know that the function $n \mapsto \dim_k((\Sigma^r V)_n)$ is a polynomial with degree at most $d$ for $n \geq C_{d-1}(C_{d-1}(r - 1) + 3)$. This is equivalent to saying that the function

$$n \mapsto \dim_k \Sigma^r V_n, \quad n \geq C_{d-1}(C_{d-1}(r - 1) + 3) + r = C_d(r)$$

coincides with a rational polynomial with degree at most $d$. \hfill \Box

5. Further questions

There are still quite a lot of questions to be answered for a satisfactory understanding on the complete picture of $\text{O}I$-modules. Here we list a few question, which we believe deserve further research.

Question 21. Let $k$ be an arbitrary commutative ring. Classify $\text{O}I$-modules presented in finite degrees with finite projective dimension.

Question 22. Let $k$ be an arbitrary commutative ring. It is already known that $\text{H}^i_{\text{O}I}(V) = 0$ for $i \geq 1$ whenever $V$ is a semi-induced $\text{O}I$-module; see [8, Proposition 5.3]. Does the converse statement also hold? If the answer is negative, classify (or characterize) $\text{O}I$-modules whose positive homology groups vanish.

Question 23. Develop a torsion theory and a local cohomology theory in the category of $\text{O}I$-modules presented in finite degrees, as the second author and Ramos did for $\text{F}I$-modules in [12].

These questions have been answered for $\text{F}I$-modules via using the following crucial fact: for $\text{F}I$-modules, the functors $\Sigma$ and $\Delta$ commutes; that is, $\Sigma \circ \Delta \cong \Delta \circ \Sigma$. Unfortunately, this does not hold any longer for $\text{O}I$-modules. We also remark that when $k$ is a field, G"{u}nt"{u}rk"{u}n and Snowden provided answers of the first and third questions for graded modules of the increasing monoid in [10].
As Church, Miller, Nagpal and Reinhold did in [5] for FI-modules, for an OI-module $V$ presented in finite degrees, we define its stable degree to be

$$\text{std}(V) = \min\{t_0(\Sigma^n V) \mid n \in \mathbb{N}\}.$$  

Since $t_0(\Sigma V) \leq t_0(V) < \infty$, we know that std($V$) is finite. Furthermore, there exists a number $N \in \mathbb{N}$ such that

$$t_0(\Sigma^N V) = t_0(\Sigma^{N+1} V) = \ldots.$$  

Our next question is:

**Question 24.** Let $V$ be an OI-module presented in finite degrees. Describe an upper bound for $N$ in terms of $t_0(V)$ and $t_1(V)$ such that $t_0(\Sigma^n V) = \text{std}(V)$ for $n \geq N$.

Theorem 8 also raises a few interesting questions.

**Question 25.** Let $V$ be an OI-module presented in finite degrees and use the notation in Theorem 8.

1. Describe the OI-modules in the finite set $\mathcal{F}_V$ (although $\mathcal{F}_V$ might not be unique); or at least estimate the cardinality of $\mathcal{F}_V$ in terms of some intrinsic invariants of $V$.
2. Describe an upper bound for $N$ such that for $n \geq N$, $\Sigma^n V$ has a filtration for which all successive quotients lie in $\mathcal{F}_V$.
3. Describe the asymptotic behavior of multiplicities $c_i(n)$ such that a filtration of $\Sigma^n V$ for $n \gg 0$ contains exactly $c_i(n)$ copies of successive quotients isomorphic to $V^i \in \mathcal{F}_V$.

**References**

[1] Church, Thomas; Ellenberg, Jordan S. Homology of FI-modules. Geom. Topol. 21 (2017), no. 4, 2373–2418. arXiv:1506.01022v2.

[2] Church, Thomas; Ellenberg, Jordan S.; Farb, Benson. FI-modules and stability for representations of symmetric groups. Duke Math. J. 164 (2015), no. 9, 1833–1910. arXiv:1204.4533v4.

[3] Church, Thomas; Ellenberg, Jordan S.; Farb, Benson; Nagpal, Rohit. FI-modules over Noetherian rings. Geom. Topol. 18 (2014), no. 5, 2951–2984. arXiv:1210.1854v2.

[4] Church, Thomas; Farb, Benson. Representation theory and homological stability. Adv. Math., (2013), 250-314. arXiv:1008.1368v3.

[5] Church, Thomas; Miller, Jeremy; Nagpal, Rohit; Reinhold, Jens. Linear and quadratic ranges in representation stability. Adv. Math. 333 (2018), 1-40. arXiv:1706.03845v1.

[6] Gan, Wee Liang. A long exact sequence for homology of FI-modules. New York J. Math. 22 (2016), 1487-1502. arXiv:1602.08873v3.

[7] Gan, Wee Liang; Li, Liping. An inductive machinery for representations of categories with shift functors. Trans. Amer. Math. Soc. 371 (2019), 8513-8534. arXiv:1610.09081.

[8] Gan, Wee Liang; Li, Liping. Asymptotic behavior of representations of graded categories with inductive functors. J. Pure Appl. Algebra 223 (2019), 188-217. arXiv:1705.00882v5.

[9] Gan, Wee Liang; Li, Liping. Bounds on homological invariants of VI-modules. to appear in Mich. Math. J. arXiv:1710.10233v2.

[10] Güntürkün, Sema; Snowden, Andrew. The representation theory of the increasing monoid. Preprint. arXiv:1812.12042v1.

[11] Li, Liping. Upper bounds of homological invariants of $\text{FI}_G$-modules. Arch. Math. (Basel) 107 (2016), no. 3, 201-211. arXiv:1512.05879v3.

[12] Li, Liping; Ramos, Eric. Depth and the local cohomology of $\text{FI}_G$-modules. Adv. Math. 329 (2018), 704-741. arXiv:1602.04405v3.

[13] Li, Liping; Yu, Nina. Filtrations and homological degrees of FI-modules. J. Algebra 472 (2017), 369-398. arXiv:1511.02977v3.

[14] Miller, Jeremy; Wilson, Jennifer C. H. FI-hyperhomology and ordered configuration spaces. Preprint. arXiv:1903.02722v.

[15] Nagpal, Rohit. VI-modules in non-describing characteristic, part I. Preprint. arXiv:1709.07509v1.

[16] Nagpal, Rohit; Snowden, Andrew. Periodicity in the cohomology of symmetric groups via divided powers. Proc. Lond. Math. Soc. 116 (2018), 1244-1268. arXiv:1705.10028v2.

[17] Putman, Andrew; Sam, Steven V.; Snowden, Andrew. Stability in the homology of unipotent groups. Preprint. arXiv:1711.11080v4.
[18] Ramos, Eric. Homological invariants of FI-modules and FI_G-modules. J. Algebra 502 (2018), 163-195. arXiv:1511.03964v3.

[19] Sam, Steven V.; Snowden, Andrew. Gröbner methods for representations of combinatorial categories. J. Amer. Math. Soc. 30 (2017), 159203. arXiv:1409.1670v3.

Department of Mathematics, University of California, Riverside, CA 92521, USA
E-mail address: wlgan@ucr.edu

LCSM (Ministry of Education), School of Mathematics and Statistics, Hunan Normal University, Changsha, Hunan 410081, China
E-mail address: lipingli@hunnu.edu.cn