On the Stability of (M theory) Stars
against Collapse: Role of Anisotropic Pressures

S. Kalyana Rama
Institute of Mathematical Sciences, C. I. T. Campus,
Tharamani, CHENNAI 600 113, India.
email: krama@imsc.res.in

ABSTRACT

Unitarity of evolution in gravitational collapses implies existence of macroscopic stable horizonless objects. With such objects in mind, we study the effects of anisotropy of pressures on the stability of stars. We consider stars in four or higher dimensions and also stars in M theory made up of (intersecting) branes. Taking the stars to be static, spherically symmetric and the equations of state to be linear, we study ‘singular solutions’ and the asymptotic perturbations around them. Oscillatory perturbations are likely to imply instability. We find that non oscillatory perturbations, which may imply stability, are possible if an appropriate amount of anisotropy is present. This result suggests that it may be possible to have stable horizonless objects in four or any higher dimensions, and that anisotropic pressures may play a crucial role in ensuring their stability.
1. Introduction

A sufficiently massive star in four dimensional spacetime is believed to be unstable against gravitational collapse. Depending on its mass and its evolutionary stage, the star may collapse to form a white dwarf or a neutron star or, if supermassive, form a black hole. White dwarfs or neutron stars may collapse further if they gain sufficient mass, for example, by accretion. Thus, all sufficiently massive objects are expected to collapse to ultimately form black holes [1]. The same is expected in higher dimensional spacetime also.

Assume that a black hole is formed in a collapse. It has a horizon, emits Hawking radiation, and is believed to evolve unitarily. Presence of horizon means that no information from inside the horizon is accessible to an outside observer. But unitary evolution means that inside information must become accessible to an outside observer at least after some time. Horizon must then cease to exist from this moment of accessibility. The black hole at this time should still be of macroscopic size so that information density within is at most of order Planckian scale, and not parametrically larger [2, 3]. And, the time when the horizon ceases to exist is expected to be the Page time which is of the order of evaporation time when about half the black hole has evaporated [4, 5, 6, 7].

Another scenario is also possible where black holes do not form at all. The gravitational collapse, that would have led to a black hole, will instead lead to an object which has no horizon as, for example, in Mathur’s fuzz ball proposal [8, 9, 10]. Unitary evolution in this scenario will proceed as for any system with large number of degrees of freedom.

Thus, whichever scenario proves to be correct, the unitarity of evolution implies that there must exist horizonless objects which form about half way through a black hole evaporation if a black hole has initially formed in a collapse, or right after a gravitational collapse in which a black hole might have been expected to form.

It must then be possible to construct such horizonless objects using appropriate sources just as, for example, one constructs neutron stars using Oppenheimer – Volkoff equations and appropriate equations of state. In the case of neutron stars, one has a very good knowledge about the nature and the properties of the constituents, namely protons, neutrons, quarks, gluons, et cetera. In the case of horizonless objects, which can be macroscopic,
such knowledge is absent at present. These horizonless objects may be the massive remnants suggested in [2] whose size and mass depend on the information contained; or they may be the ones which develop ‘firewalls’ as suggested in [11, 12, 13]; or they may be the ‘fuzz balls’ of Mathur’s proposal which have no horizon at anytime [8, 9, 10]. Recently, we have also argued for the existence of such horizonless fuzz ball like objects in any quantum theory of gravity where singularities are resolved and evolutions are unitary [14, 15, 16]. A variety of horizonless objects have been advocated in the past also from several points of view, see [17] – [28] for a sample of them.

Although nothing is known rigorously about the horizonless objects and their constituents, it is physically reasonable to expect them to have the following properties. (i) Their central densities are likely to be of the order of Planckian densities since the quantum gravity effects are expected to play an important role in creating them. (ii) Their sizes are likely to be of the order of Schwarzschild radii since they are expected to be similar to black holes, only without horizons. (iii) Being the end products of prior gravitational collapses and evolutions, they must be stable against any further gravitational collapse. (iv) Their constituents must have a large number of entropic degrees of freedom and, hence, thermodynamics must be applicable. Therefore these constituents, even if highly quantum in nature, may be modelled using density, pressures, and suitable equations of state. (v) These horizonless objects can be arbitrarily massive. Hence, at least qualitatively, the corresponding spacetime may be described by general relativity equations with appropriate energy momentum tensors and constituent equations of state.

In this paper, we consider static, spherically symmetric cases and focus on property (iii), namely the stability property. In the following, for the sake of brevity, we will refer to these horizonless objects as stars although they are not stars in a conventional sense. For example, their central densities are likely to be of the order of Planckian densities; and their sizes are likely to be of the order of Schwarzschild radii.

Sufficiently massive conventional stars have been found to be unstable against gravitational collapse. But, in his study of relativistic stars in arbitrary dimensions [29], see also [30], Chavanis found among other things that stars can be stable against collapse, albeit only in higher dimensions. To the best of our knowledge, this is the only work which has shown the possibility of arbitrarily massive stars being also stable; moreover, the radii of these stable
stars scale as Schwarzschild radii. ¹ Chavanis considered static spherically symmetric stars made up of a perfect fluid with a linear equation of state and enclosed them in a box. ² Following the methods of Chandrasekhar [34], he then obtained ‘singular solutions’ and analysed the asymptotic perturbations around them. He has shown in detail that the damped oscillatory, or monotonic non oscillatory, behaviour of these perturbations around singular solutions leads to the damped oscillatory, or monotonic non oscillatory, behaviour for the mass – central density profile of the star in the asymptotic region of large central density or large radius. He has shown further that equilibrium configurations become unstable beyond the first maximum in this profile, hence its oscillatory behaviour in the asymptotic regions will imply instability against collapse. Studying the effects of higher dimensions, he then found that the mass – central density profiles of the stars in eleven or higher dimensions are non oscillatory in both the asymptotic and non asymptotic regions, and imply stability.

If the number eleven above counts both compact and non compact directions then stability in, for example, four dimensional non compact spacetime may be obtained by having a eleven dimensional spacetime with seven compact dimensions, which is quite natural in the context of M theory. With this motivation, in [37], we generalised the work of Chavanis to include compact directions and multi component perfect fluids, the later also appearing quite naturally in string and M theory black branes. However, we found that, even in these generalised cases, eleven non compact dimensions are needed for stability. In particular, this implied that stars in four dimensions and those in M theory are unstable and that they will collapse if sufficiently massive.

Reviewing the underlying assumptions in [29, 37] and studying carefully some of the works on the horizonless objects, particularly [19] – [24], we realised that the pressures inside the stars along the noncompact spatial directions need not be isotropic. The assumption about isotropy may be unwarranted and restrictive: it is not required by spherical symmetry and,

¹Similar results were also obtained around the same time in [31, 32], but in asymptotically anti de Sitter spacetimes whose radii limit the masses of the stable stars.

²This box is needed since the radius of the star will be infinite otherwise [33]. As explained in [29], enclosing the star within a box prevents the evaporation of its constituents and makes its total mass finite. The radius and the mass may also be made finite by constructing a composite configuration consisting of a perfect fluid core and a crust of constant density or a gaseous envelope exerting a constant pressure, see [34, 35, 36].
in general, the pressures may be different along the radial and the transverse spherical directions of the non compact space. For conventional boson stars, anisotropic pressures arise naturally when scalar fields are present [38, 39]. See also the reviews [40, 41, 42, 43] and the recent paper [44]. For the horizonless objects, which are implied by unitarity in lieu of black holes but are referred to as stars here, there may be other mechanisms giving rise to anisotropic pressures. However, since not much is known about the constituents of such objects, we do not know any possible exact mechanisms.

In this paper, therefore, we simply assume that pressures in the stars may be anisotropic and study, a la Chavanis, the effects of such anisotropy on the stability of stars. Following closely the works in [29, 34, 37] and, generalising them now by including anisotropic cases, we study stars in higher dimensional spacetime which may have compact toroidal directions also. The stars are assumed to be static and spherically symmetric in the non compact space and to have suitable isometries along the compact directions. Taking a suitable ansatz for the metric, we write down the equations of motion. Then, as in [29, 34], we assume linear equations of state and obtain the ‘singular solutions’ and the asymptotic perturbations around them. The oscillatory or non oscillatory behaviour of these perturbations lead to corresponding behaviour for the mass – central density profile in the asymptotic regions. We assume that, as in Chavanis’ works [29], the equilibrium configurations become unstable beyond the first maximum in the mass – central density profile; hence that the damped oscillatory behaviour of this profile implies instability whereas monotonic non oscillatory behaviour throughout implies stability. Studying then the asymptotic perturbations around the ‘singular solutions’, we obtain the criteria under which the perturbations are non oscillatory. These will then be the necessary criteria for stability.

We perform the above analysis first for stars in spacetime with no compact directions; then include compact directions; then repeat the analysis for two examples of stars in M theory made up of stacks of (intersecting) two branes and five branes. The formulation presented in this paper may, however, be used to

\[3\] One must now show that the equilibrium configurations in the mass – central density profile become unstable beyond the first maximum. And, for the case where asymptotic behaviour is non oscillatory, one must show that this profile remains monotonic everywhere, including the non asymptotic region also. Showing these is beyond the scope of the present paper since it requires a knowledge of the nature and the properties of constituents of the horizonless objects, which is lacking at present.
study a variety of other examples also.

We find that, in an \((m + 2)\) dimensional non compact spacetime with \(m \geq 2\), non oscillatory perturbations are possible for any value of \(m\) if an appropriate amount of anisotropy is present, namely if

\[
\frac{\Pi - p}{\Pi} \geq \frac{9 - m}{4m}
\]

where \(\Pi\) and \(p\) are the pressures along the radial and the transverse spherical directions. Also, we find that presence of compact directions does not change this result. Similar result follows for stars in M theory also. It can now be seen that if \(m \geq 9\), namely if the non compact spacetime is eleven or higher dimensional, then the isotropic case (above ratio = 0) is included in the above range, thus reproducing the result in [29]. For lower values of \(m\), certain amount of anisotropy is needed to obtain non oscillatory perturbations; for example, for four dimensional spacetime, \(m = 2\) and the above ratio needs to be \(\geq \frac{7}{8}\).

This result suggests that it may be possible to have stable horizonless objects in four or any higher dimensions, and that anisotropic pressures may play a crucial role in ensuring their stability. Although much remains to be done, for example resolve the issues mentioned in footnote 3, it is worth emphasising that this is an important result because it bears on the horizonless objects which are implied by unitarity in lieu of black holes, and it points out a necessary ingredient for their stability. To actually construct such objects, however, requires detailed understanding of many issues such as the nature of the constituents and the physical mechanisms that may provide the required amount of anisotropy. We will discuss briefly these and other issues at the end of the paper.

This paper is organised as follows. In section 2, we first mention briefly the relevant aspects of [29] which are used here. We then briefly describe M theory stars and mention how the equations of state are obtained. In section 3, we set up our notations and conventions and present the equations of motion in a suitable form. In section 4, we give the details of the asymptotic analysis, which involve singular solutions and perturbations around them, and outline briefly the significance of perturbations. In section 5, we obtain singular solutions for stars in spacetime with no compact directions, study the perturbations around them, and obtain the criteria for their non oscillatory behaviour. In section 6, we repeat the analysis for two examples of stars in...
M theory: in one, stars are made up of a stack of two branes or five branes; in another, stars are made up of four stacks of intersecting branes, two stacks each of two branes and five branes. In section 7, we conclude with a brief summary and a discussion of some of the issues which require further study.

2. Some general remarks

In this section, we first mention briefly some aspects of [29] which are relevant here. For more details, see [29] and the references there. We then briefly describe M theory stars and mention how the equations of state are obtained.

Chavanis considers static spherically symmetric star with its constituents obeying linear equations of state. In cosmological contexts, it is a standard practice to use linear equations state. In the contexts of stars, Chavanis explains several situations where such equations of state arise and the reasons for using them. The star is then enclosed in a box of radius $r_*$ which prevents the evaporation of its constituents and makes its total mass finite. The general relativistic equations are then solved to obtain hydrostatic equilibrium configurations.

Considering a series of equilibria, the mass–central density profile is obtained and shown to have damped oscillatory behaviour. The hydrostatic equilibrium configurations become unstable beyond the first maximum in this profile. This is also shown to correspond to conditions for nonlinear dynamic stability. Thus, for a given volume, the star has a maximum mass above which it becomes unstable.

Following Chandrasekhar [34], singular solutions and the asymptotic perturbations around them are obtained. The singular solutions give the scaling relations between mass, radius, and other quantities of the stars. The damped oscillatory, or monotonic non oscillatory, behaviour of the perturbations are shown to be responsible for the corresponding behaviour for the mass–central density profile in the asymptotic region.

Chavanis obtains the mass–central density profile in the non asymptotic regions also. He shows that the profile in $D < D_{\text{crit}}$ dimensional space-time starts from origin, increases monotonically, reaches a maximum, then decreases, and oscillates with damped amplitudes around the constant value line given by the singular solutions; whereas, in $D \geq D_{\text{crit}}$ dimensional space-
time, the entire profile is monotonically increasing, and asymptotes to the constant value line given by the singular solutions. The critical dimension $D_{\text{crit}} \simeq 11$ and depends on the pressure to density ratio. See Figure 23 in [29].

If the mass–central density profile is oscillatory then the scaling behaviour given by the singular solutions is not stable since the stability is lost beyond the first maximum of the profile. The singular solutions are then of no relevance. The mass–radius relations and other quantities of stable stars must be obtained from detailed analysis of the equations in the non asymptotic regions before the first maximum.

If the entire mass–central density profile is monotonic and non oscillatory then the series of equilibria represented by its points are all stable. Then the singular solutions correspond to stable configurations and they can be used to obtain mass–radius relations and other quantities of stable stars in the limit of large central density or large radius.

In the case of horizonless objects which are implied by unitarity and which we study here, nothing is known rigorously about the nature and properties of their constituents. Hence a thorough analysis as in [29] is presently not possible. Nevertheless, it is possible to make some progress using the physically reasonable properties of such objects listed in the Introduction. We assume a linear equation of state for the constituents. The stiffest equation of state ($\text{pressure} = \text{density}$, $\text{sound speed} = \text{light speed}$) is of this type. Also, this linearity helps in obtaining analytically the singular solutions and the asymptotic perturbations around them, and in studying whether monotonic non oscillatory behaviour is possible. We enclose the star in a box of radius $r_*$ which prevents the evaporation of its constituents and makes its total mass finite. Then, as in [29], the damped oscillatory, or monotonic non oscillatory, behaviour of these perturbations can be shown to lead to the corresponding behaviour for the mass–central density profile of the star in the asymptotic region. However, we can not obtain this profile in the non asymptotic regions, nor study the stability properties around the maxima, without knowing the detailed properties of the constituents.

Detailed knowledge of the constituents of the horizonless objects is also needed in order to understand the fate of, for example, a collapsing massive neutron star which would have formed a black hole in the standard scenario. This knowledge is needed to understand what happens to neutrons, protons, et cetera in the quantum gravity regime where the singularities are assumed
to be resolved. This requires understanding the relation between quantum gravity theory and standard model particles.

M theory stars

In String/M theory, the entropy and Hawking radiation of a class of extremal and near extremal black holes have been understood rigorously in terms of various intersecting brane configurations and the low energy excitations on them. Mathur’s fuzz ball proposal also arises naturally in this context. The entropy and Hawking radiation of neutral or far from extremal black holes are not understood equally rigorously, but their explanations are likely to be in terms of intersecting brane antibranes configurations along the lines given in [45] – [54]. In the intersecting brane configurations, various stacks of branes wrap around compact toroidal directions, intersecting according to BPS rules whereby, in M theory, \(^4\) two stacks of five branes intersect along three common spatial directions; a stack each of two branes and five branes intersect along one common spatial direction; and two stacks of two branes intersect along zero common spatial direction. See [55] and the references therein. Given the understanding of near extremal black hole properties in terms of intersecting branes, and given the fuzz ball proposal, it is natural to expect that there must be stars in M theory made up of intersecting branes \(^5\) which, when sufficiently massive, will collapse and form

\[^4\]The brane configurations in string and M theories are equivalent and are related by chains of U duality operations involving dimensional reduction and upliftment between string and M theory, and the T and S dualities of the string theories. Hence, in the following, we will restrict ourselves to M theory. The corresponding string theory results are straightforward to obtain.

\[^5\]M theory stars which are considered here and are made up of, for example, two branes may be visualised as follows. The spacetime is eleven dimensional with two compact toroidal spatial directions. The non compact space is eight dimensional, described by polar coordinates \((r, \theta_1, \cdots, \theta_7)\). Thus there is a torus at every point of the non compact space. Stacks of two branes wrap around these tori such that there are a total of \(n(r)\) number of two branes per unit volume of the non compact space. This number density \(n(r) \neq 0\) for \(r < r_\ast\) and \(= 0\) for \(r \geq r_\ast\). This provides a picture of an M theory star of radius \(r_\ast\) made up of two branes. This picture is analogous to that of a four dimensional star of radius \(r_\ast\) where \((r, \theta_1, \theta_2)\) are the polar coordinates and \(n(r)\) is the particle density. The stack of branes wrapping the torus at a point is like a ‘particle’ in the non compact space. One may similarly visualise M theory stars made up of other intersecting brane configurations. A ‘particle’ in the non compact space now is the \(\mathcal{N}\) stacks of two and five branes wrapping the tori at a point, with necessary isometries, and intersecting according
either black holes having horizon or fuzz ball like objects having no horizon. Also, for the same entropic reasons which are explained in [56, 57, 58] where early universe was studied in string/M theory, we may assume that stars in M theory are made up of stacks of intersecting branes.

M theory stars may be studied using the formalism given in this paper. Assuming the spatial directions of the brane worldvolumes to be toroidal and assuming necessary isometries, the M theory brane configurations consisting of \( \mathcal{N} \) stacks of two and five branes which intersect according to BPS rules, can be described by \( \mathcal{N} \) seperately conserved energy momentum tensors \( T_{MN(I)} \), \( I = 1, 2, \cdots, \mathcal{N} \), with appropriate equations of state among their components. See [59, 60, 61] and the references therein. The equations of state may follow from an action; or they may be derived using the microscopic dynamics of the constituents which are far from being rigorously known; or they may simply be postulated as an ansatz.  

In M theory, equations describing black hole spacetimes follow from an eleven dimensional low energy effective action [45, 46, 55, 59, 60, 61]. In the context of an expanding universe, the equations of state for intersecting branes have been derived in certain approximations using microscopic dynamics of branes [56, 57]. In the context of an expanding universe, and also of stars, we used U duality symmetries and derived a relation among the components of the energy momentum tensor [58]. These relations, one each for each of the \( \mathcal{N} \) stacks in the intersecting brane configurations, follow as a consequence of U duality symmetries and, therefore, must always be valid independent of the details of the equations of state. The equations of state in [45, 46, 55, 59, 60, 61, 56, 57], which were obtained by other methods, all obey these U duality relations.

---

\(^{6}\)This situation is similar to that for, for example, charged black holes versus for stars made up of charged particles or for an expanding universe containing them: In the case of black holes, equations are obtained from an action. In the case of stars, or an expanding universe, one may use statistical mechanics to describe charges, anticharges, and the photons between them; or one may simply take, for example, a linear equation of state as an ansatz.

\(^{7}\)Using dimensional reduction and upliftment, T dualities, and S dualities, we obtained the U duality relations first in the cosmological context in [58] and then, in unpublished notes, for stars and black branes also. See the comments and the application of these U duality relations in [62].
The U duality relations need to be supplemented with the equations of state in, say $d$ dimensional, non compact part of the total eleven dimensional spacetime. These data on the equations of state are same as those needed for any expanding universes or for any stars in $d$ dimensional non compact spacetimes but now with no compact dimensions. See [58, 63, 64, 65, 37] for examples and more details, as well as for applications and non trivial consequences of these U duality relations for expanding universes and for stars in M theory.

Note that, in the context of M theory also, understanding the fate of, for example, a collapsing massive neutron star which would have formed a black hole in the standard scenario, requires understanding the relation between M theory and standard model particles.

3. Equations of motion

In this paper, we consider static cases which are spherically symmetric in higher dimensional spacetime. We will also consider such stars in eleven dimensional M theory. Although the formalism and the results in this paper are applicable to conventional stars also, our main interest here is in the horizonless objects and their stability properties. As explained in the Introduction, these objects are implied by unitarity in lieu of black holes and, for the sake of brevity, are also referred to as stars here. They are not stars in a conventional sense because, for example, their central densities are likely to be of the order of Planckian densities; and their sizes are likely to be of the order of Schwarzschild radii.

The spacetime is assumed to be $D = n_c + m + 2$ dimensional with $m \geq 2$, with $n_c$ dimensional compact toroidal space, and with $(m + 1)$ dimensional non compact space. The stars are assumed to be static and spherically symmetric in the non compact space, and to be made up of non interacting multicomponent fluids with linear equations of state. The M theory stars are eleven dimensional and are assumed to be made up of $N$ stacks of $M2$ and $M5$ branes, intersecting according to the BPS rules whereby two stacks of five branes intersect along three common spatial directions; a stack each of two branes and five branes intersect along one common spatial direction; and two stacks of two branes intersect along zero common spatial direction. Assuming the spatial directions of the brane worldvolumes to be toroidal
and assuming necessary isometries, the intersecting M theory branes can be described by \( \mathcal{N} \) separately conserved energy momentum tensors \( T_{MN(I)} \), \( I = 1, 2, \cdots, \mathcal{N} \), with appropriate equations of state among their components [55, 59, 60, 61, 56, 57, 58, 63, 64, 65].

Following closely the notations and conventions of our earlier work [37], we now write a suitable ansatz for the metric and obtain the equations of motion. Let \( x^M = (t, x^i, r, \theta^a) \) be the spacetime coordinates where \( x^i, \ i = 1, 2, \cdots, n_c \), describe the \( n_c \) dimensional toroidal space; and the radial and the spherical coordinates \( (r, \theta^a), \ a = 1, 2, \cdots, m \), describe the \( (m + 1) \) dimensional non compact space. In standard notation and with \( \kappa^2 = 8\pi G_D = 1 \), the equations of motion may be written as

\[
R_{MN} - \frac{1}{2} g_{MN} R = T_{MN} = \sum_I T_{MN(I)} \tag{1}
\]

\[
\sum_M \nabla_M T^{M}_{N(I)} = 0 \tag{2}
\]

where \( T_{MN} \) is the total energy momentum tensor of non interacting multicomponent fluids and \( T_{MN(I)} \) is the energy momentum tensor for the \( I^{th} \) component fluid. For stars in M theory, \( T_{MN} \) is the total energy momentum tensor for intersecting branes and \( T_{MN(I)} \) is the energy momentum tensor for the \( I^{th} \) stack of branes.

In the following, we consider static solutions which are spherically symmetric in the \( (m + 1) \) dimensional non compact space. We will study the singular solutions and the asymptotic perturbations around them in the limit of large \( r \) [34, 29]. The suitable ansatz for the line element \( ds \) is given by

\[
ds^2 = g_{MN} \, dx^M \, dx^N = -e^{2\lambda_0} \, dt^2 + \sum_i e^{2\lambda_i} (dx^i)^2 + e^{2\lambda} \, dr^2 + e^{2\sigma} \, d\Omega_m^2 \tag{3}
\]

where \( d\Omega_m \) is the standard line element on an \( m \) dimensional unit sphere. The energy momentum tensors \( T_{MN(I)} \) are assumed to be diagonal. These diagonal elements are denoted as

\[
\left( T^0_{0(I)}, T^i_{i(I)}, T^r_{r(I)}, T^a_{a(I)} \right) = (p_{0I}, p_{iI}, \Pi_I, p_{aI})
\]

where \( p_{0I} = -\rho_I \) and \( p_{aI} = p_I \) for all \( a \). The total energy momentum tensor is now given by \( T^M_N = diag \ (p_0, p_i, \Pi, p_a) \) where \( p_0 = -\rho, \ p_a = p \) for all
\[
\rho = \sum I_p^I, \quad p_i = \sum I_p^I, \quad \Pi = \sum I_{\Pi^I}, \quad p = \sum I_p^I.
\]

Define
\[
\alpha = (0, i, a), \quad \lambda^\alpha = (\lambda^0, \lambda^i, \lambda^a), \quad p_\alpha I = (p_0^I, p_i^I, p_a^I)
\]
where \(\lambda^a = \sigma\) for all \(a\). Also, define
\[
\Lambda = \sum \alpha \lambda^\alpha = \lambda^0 + \sum \lambda^i + m\sigma, \quad T_I = \sum M^I T^M_{M(I)} = \Pi_I + \sum \alpha p_\alpha I.
\]

For static solutions which are spherically symmetric in the \((m+1)\)-dimensional non-compact space, the fields \((\lambda^\alpha, \lambda)\) and \((p_\alpha I, \Pi_I)\) depend only on the coordinate \(r\). Using the above definitions and the metric given in equation (3), it follows straightforwardly that the equations of motion (1) and (2) now give
\[
\begin{align*}
(\Pi_I)_r &= -\Pi_I \Lambda_r + \sum \alpha p_\alpha I \lambda^\alpha_r \\
\Lambda_r^2 - \sum \alpha (\lambda^\alpha_r)^2 &= 2 \sum I \Pi_I e^{2\lambda} + m(m-1) e^{2\lambda-2\sigma} \quad (4) \\
\lambda^\alpha_{rr} + (\Lambda_r - \lambda_r) \lambda^\alpha_r &= \sum I \left( -p_\alpha I + \frac{T_I}{D-2} \right) e^{2\lambda} + \delta^{\alpha a} (m-1) e^{2\lambda-2\sigma} \quad (5)
\end{align*}
\]

where the subscripts \(r\) denote \(r\)-derivatives. We also define a function \(f(r)\) and a mass function \(M(r)\) by
\[
e^{2\lambda-2\sigma} = \frac{1}{r^2 f}, \quad f = 1 - \frac{M}{r^{m-1}}
\]
so that either of them may be traded for the function \(\lambda(r)\).

**Reduction to \(d = m + 2\) dimensions**

We will now dimensionally reduce on the \(n_c\) dimensional toroidal space from \(D\) dimensions to \(d = m + 2\) dimensions described by the \(x^\mu = (t, r, \theta^a)\) coordinates. Consider the \(D\) dimensional line element \(ds\) given by equation (3), and denote its \(d\) dimensional part as follows:
\[
ds_d^2 = g_{\mu\nu(d)} dx^\mu dx^\nu = -e^{2\lambda^0} dt^2 + e^{2\lambda} dr^2 + e^{2\sigma} d\Omega_m^2.
\]
Upon dimensional reduction, symbolically, we have

\[ S \sim \int d^D x \sqrt{-\bar{g}} \, R \sim \int d^d x \sqrt{-g_{(d)}} \, e^{\Lambda^c} \left( R_{(d)} + \cdots \right) \]

\[ \sim \int d^d x \sqrt{-\bar{\bar{g}}} \, (\bar{\bar{R}} + \cdots) \]

where \( \Lambda^c = \sum_i \lambda^i \) and \( \bar{\bar{g}}_{\mu\nu} = e^{2\bar{\bar{\lambda}}^c} \, g_{\mu\nu(d)} \) is the \( d \) dimensional Einstein frame metric. The corresponding line element \( \bar{\bar{d}}s_d \) then becomes

\[ \bar{\bar{d}}s_d^2 = \bar{\bar{g}}_{\mu\nu} \, dx^\mu \, dx^\nu = -e^{2\bar{\bar{\lambda}}^0} \, dt^2 + e^{2\bar{\bar{\lambda}}} \, dr^2 + e^{2\bar{\bar{\sigma}}} \, d\Omega_m^2 \]  

(7)

where

\[ \bar{\bar{\lambda}}^0 = \bar{\lambda}^0 + \frac{\Lambda^c}{m}, \quad \bar{\bar{\lambda}} = \bar{\lambda} + \frac{\Lambda^c}{m}, \]

(8)

and \( \bar{\bar{\lambda}}^a = \bar{\sigma} \) for all \( a \). Furthermore, one has

\[ \bar{\bar{\lambda}}^0 = \bar{\lambda}^0 + m \bar{\sigma} = \Lambda + \frac{\Lambda^c}{m}, \quad e^{2\bar{\bar{\lambda}} - 2\bar{\sigma}} = e^{2\lambda - 2\sigma} = \frac{1}{r^2 f}. \]

Note that \( \bar{\bar{\lambda}}^i \) and \( \lambda^i \) can be expressed easily in terms of each other \(^8\) and, hence, they are both equally convenient to work with.

Writing \( p_0 I = -\rho_I \), it now follows from equations (4) – (6) that

\[ (\Pi_I)_r = -\Pi_I \, \bar{\bar{\lambda}}_r - \rho_I \, \bar{\bar{\lambda}}^0_r + m \, p_I \, \bar{\bar{\sigma}}_r + \sum_i \left( p_{iI} - \frac{T_i}{m} \right) \lambda^i_r + \frac{2\Pi_I}{m} \, \Lambda^c_r \]  

(9)

\[ 2\bar{\bar{\lambda}}^0 \bar{\sigma}_r + (m - 1)(\bar{\bar{\sigma}})_r^2 = \frac{2}{m} \sum_I \Pi_I \, e^{2\lambda} + (m - 1) \, e^{2\lambda - 2\sigma} + \frac{B}{m} \]  

(10)

\[ \bar{\bar{\sigma}}_{rr} + (\bar{\bar{\lambda}}_r - \bar{\lambda}_r) \, \bar{\bar{\sigma}}_r = \sum_I \left( -p_I + \frac{T_i}{m} \right) \, e^{2\lambda} + (m - 1) \, e^{2\lambda - 2\sigma} \]  

(11)

\[ \bar{\bar{\lambda}}^0_{rr} + (\bar{\bar{\lambda}}_r - \bar{\lambda}_r) \, \bar{\bar{\lambda}}^0_r = \sum_I \left( p_i + \frac{T_i}{m} \right) \, e^{2\lambda} \]  

(12)

\[ \bar{\bar{\lambda}}^i_{rr} + (\bar{\bar{\lambda}}_r - \bar{\lambda}_r) \, \bar{\bar{\lambda}}^i_r = \sum_I \left( -p_{iI} + \frac{T_i}{m} \right) \, e^{2\lambda} \]  

(13)

\(^8\)For any \( a^\alpha \)s, let \( \bar{\bar{a}}^\alpha = a^\alpha + \frac{a^c}{m} \) where \( a^c = \sum_i a^i \). Then \( \bar{\bar{a}}^c = \sum_i \bar{\bar{a}}^i = (n_c + m) \frac{a^c}{m} \) and, hence, \( a^\alpha = \bar{\bar{a}}^\alpha - \frac{\bar{\bar{a}}^c}{n_c + m} \). Similarly for any \( b^\beta \)s. Also, \( \sum_i \bar{\bar{a}}^i b^i = \sum_i a^i b^i = \sum_i a^i b^i + \frac{a^c b^c}{m} \), which further implies that \( \sum_i \bar{\bar{a}}^i a^i > 0 \) if \( a^i \) do not all vanish.
where $T_I = \Pi_I - \rho_I + m p_I$ and $B = \sum_i \tilde{\lambda}^i_r \lambda^i_r$. Using the diffeomorphic freedom in defining the radial coordinate, we now set $e^{\bar{\sigma}} = r$. Equations (10) and (11) become

$$ r \tilde{\lambda}^0_r = \sum_I \frac{\Pi_I}{m} r^2 e^{2\lambda} + \frac{m - 1}{2} (e^{2\tilde{\lambda}} - 1) + \frac{r^2 B}{2m} \quad (14) $$

$$ r (\tilde{\lambda}^0_r - \tilde{\lambda}_r) = \sum_I \left( \frac{\Pi_I - \rho_I}{m} \right) r^2 e^{2\lambda} + (m - 1) (e^{2\tilde{\lambda}} - 1) \quad (15) $$

$$ \Rightarrow r \tilde{\lambda}_r = \sum_I \frac{\rho_I}{m} r^2 e^{2\lambda} - \frac{m - 1}{2} (e^{2\tilde{\lambda}} - 1) + \frac{r^2 B}{2m} \quad (16) $$

Since $e^{\bar{\sigma}} = r$, we also have

$$ \tilde{d}s^2 = -e^{2\bar{\lambda}^0} dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2_m \quad , \quad f = e^{2\bar{\lambda}} = 1 - \frac{M}{r^{m-1}}. $$

Thus, the line element $\tilde{d}s_d$ in the $d = m + 2$ dimensional Einstein frame takes the standard form. Therefore the functions appearing in it, e.g. the mass function $M$, may be interpreted in the standard way. For instance, $M_{ADM}$, the ADM mass of the star of radius $r_*$ is related to the mass function by $M(r_*) = \frac{16 \pi G}{m S_m} M_{ADM}$ where $S_m = \frac{2 \pi^{m+1}}{m (m + 1)}$ is the ‘area’ of an $m$ dimensional unit sphere and $V_{nc}$ is the coordinate volume of the $n_c$ dimensional toroidal space. Note that equation (16) and the relation $M(r) = r^{m-1} (1 - e^{-2\bar{\lambda}})$ now give

$$ M_r = \frac{r^m}{m} \left( 2 \sum_I \rho_I e^{-2\bar{\lambda}^c} + B e^{-2\bar{\lambda}} \right). \quad (17) $$

**Linear equations of state**

To solve equations (9) and (12) – (15), and to obtain solutions for the fields $(\lambda^a, \rho_{aI}, \Pi_I)$, one further requires equations of state which give $p_{aI}$ and $\Pi_I$ as functions of $\rho_I$. In cosmological contexts, it is a standard practice to use linear equations state. In the contexts of stars, Chavanis explains several situations where such equations of state arise and the reasons for using them [29]. In the case of horizonless objects which are implied by unitarity and
which are of main interest here, nothing is known rigorously about the nature and properties of their constituents. Hence, it is not presently possible to derive the equations of state from the underlying microscopic physics. In order to make progress, we will assume here linear equations of state. The stiffest equation of state (\(\text{pressure} = \text{density}, \ \text{sound speed} = \text{light speed}\)) is of this type. Also, the linearity helps in obtaining explicitly the singular solutions and the asymptotic perturbations around them, and in studying the stability properties.

Thus, let

\[
p_{\alpha I} = w_{\alpha}^I \rho_I , \quad \Pi_I = w_{\pi}^I \rho_I
\]  

where \(w_{\alpha}^I\) and \(w_{\pi}^I\) are constants, \(w_{0}^I = -1\) since \(p_{0I} = -\rho_I\), \(w_{a}^I = w^I\) since \(p_{aI} = p_I\) for all \(a\), and we assume that \(w_{\pi}^I > 0\). It is common to take the pressures inside the stars to be isotropic along the noncompact spatial directions, namely to take the pressure \(\Pi_I\) along the radial direction to be equal to the pressure \(p_I\) along the transverse spherical directions. However, the assumption about isotropy may be unwarranted and restrictive: it is not required by spherical symmetry and, in general, the pressures inside the stars may be different along the radial and the transverse spherical directions of the noncompact space. For conventional boson stars, anisotropic pressures arise naturally when scalar fields are present [38, 39]. See also the reviews [40, 41, 42, 43] and the recent paper [44]. We will assume here that such an anisotropy may be present and, hence, that \(\Pi_I \neq p_I\) in general. As a measure of this anisotropy, we define a dimensionless parameter \(\eta^I\) by

\[
\eta^I = \frac{m}{2} \left( \frac{\Pi_I - p_I}{\Pi_I} \right) = \frac{m}{2} \left( \frac{w_{\pi}^I - w^I}{w_{\pi}^I} \right)
\]

so that \(\eta^I = 0\) corresponds to the isotropic case. The factor of \(\frac{m}{2}\) is for convenience.

Since \(\mathcal{T}_I = \Pi_I - \rho_I + m \, p_I\), equations (18) give

\[
\left(-p_{\alpha I} + \frac{\mathcal{T}_I}{m}\right) = \tilde{c}^{\alpha I} \rho_I , \quad \tilde{c}^{\alpha I} = -w_{\alpha}^I + w^I + \frac{w_{\pi}^I - 1}{m} .
\]

Hence

\[
\tilde{c}^{0I} = 1 + w^I + \tilde{c}^I , \quad \tilde{c}^{iI} = -w_{i}^I + w^I + \tilde{c}^I , \quad \tilde{c}^{aI} = \tilde{c}^I = \frac{w_{\pi}^I - 1}{m} .
\]

(20)
The corresponding untilded coefficients are given by $c^{\alpha I} = \tilde{c}^{\alpha I} - \sum_{n=1}^{m} \tilde{c}^{\alpha I} n c^{\alpha I}$, see footnote 8. Let $\phi^I$ be given by

$$w^I \phi^I = -(1 + w^I) \tilde{\lambda}^0 - 2 \eta^I w^I \tilde{\sigma} - \sum_i c^{\alpha I} \tilde{\lambda}^i .$$

(21)

It then follows from equations (9) and (19), and from $e^{\tilde{\sigma}} = r$, that

$$\rho^I = \rho_{I0} e^{\phi^I} e^{\frac{2\tilde{\lambda}^0}{m}} , \quad r^2 \rho^I e^{2\lambda} = \rho_{I0} e^{\phi^I + 2\tilde{\lambda} + 2\tilde{\sigma}}$$

(22)

where $\rho_{I0} > 0$ is a constant. One also obtains, for any function $X(r(\tilde{\sigma}))$,

$$X_{\tilde{\sigma}} = r X_r , \quad X_{\tilde{\sigma}\tilde{\sigma}} = r^2 X_{rr} + r X_r$$

$$r^2 \left( X_{rr} + (\tilde{\Lambda}_r - \tilde{\lambda}_r) X_r \right) = X_{\tilde{\sigma}\tilde{\sigma}} + (\tilde{\chi} - \tilde{\lambda}_r) X_{\tilde{\sigma}}$$

where the subscripts $\tilde{\sigma}$ denote $\tilde{\sigma}$-derivatives and $\tilde{\chi} = \tilde{\Lambda} - \tilde{\sigma} = \tilde{\lambda}^0 + (m-1) \tilde{\sigma}$.

Equations (12) – (16), written in terms of $\tilde{\sigma}$, now become

$$\tilde{\lambda}_{\tilde{\sigma}\tilde{\sigma}}^0 + (\tilde{\chi} - \tilde{\lambda}_r) \tilde{\lambda}_{\tilde{\sigma}}^0 = \sum_I c^{\alpha I} \rho_{I0} e^{\phi^I + 2\tilde{\lambda} + 2\tilde{\sigma}}$$

(23)

$$\tilde{\lambda}_{\tilde{\sigma}\tilde{\sigma}}^i + (\tilde{\chi} - \tilde{\lambda}_r) \tilde{\lambda}_{\tilde{\sigma}}^i = \sum_I c^{\alpha I} \rho_{I0} e^{\phi^I + 2\tilde{\lambda} + 2\tilde{\sigma}}$$

(24)

$$\tilde{\lambda}_\tilde{\sigma}^0 = \frac{\sum_I w^I}{m} \rho_{I0} e^{\phi^I + 2\tilde{\lambda} + 2\tilde{\sigma}} + \frac{m-1}{2} \left( e^{2\tilde{\lambda}} - 1 \right) + \frac{r^2 B}{2m}$$

(25)

$$\tilde{\lambda}_\tilde{\sigma}^0 - \tilde{\lambda}_\tilde{\sigma} = \frac{\sum_I c^{\alpha I}}{m} \rho_{I0} e^{\phi^I + 2\tilde{\lambda} + 2\tilde{\sigma}} + (m-1) \left( e^{2\tilde{\lambda}} - 1 \right)$$

(26)

$$\Rightarrow \quad \tilde{\lambda}_\tilde{\sigma} = \frac{\sum_I \rho_{I0}}{m} e^{\phi^I + 2\tilde{\lambda} + 2\tilde{\sigma}} - \frac{m-1}{2} \left( e^{2\tilde{\lambda}} - 1 \right) + \frac{r^2 B}{2m}$$

(27)

where $r^2 B = \sum_i \tilde{\lambda}_\tilde{\sigma}^i \lambda_{\tilde{\sigma}}^i$. The above equations thus describe stars whose constituents obey the linear equations of state (18).

4. Asymptotic analysis : Singular solutions and perturbations
Consider the limit \( r = e^\tilde{\sigma} \to \infty \). Suitable ansätze for \( \tilde{\lambda}^0 \), \( \tilde{\lambda}^i \), and \( \tilde{\lambda} \) in this limit are given by

\[
\begin{align*}
\tilde{\lambda}^0 &= s^0 \tilde{\sigma} + \tilde{u}^0 \\
\tilde{\lambda}^i &= s^i \tilde{\sigma} + \tilde{u}^i \\
\tilde{\lambda} &= \tilde{\lambda}^0 + s \tilde{\sigma} + \tilde{u}
\end{align*}
\] (28)

where \( s^0, s^i, s, \) and \( \tilde{\lambda}^0 \) are constants and \( \tilde{u}^0, \tilde{u}^i, \) and \( \tilde{u} \) are functions of \( r \). Further constants \( \tilde{\lambda}^0_0 \) and \( \tilde{\lambda}^i_0 \) could have been added to \( \tilde{\lambda}^0 \) and \( \tilde{\lambda}^i \) also but, with no loss of generality, they have been set to zero. Writing

\[
\phi^I = q^I \tilde{\sigma} + y^I
\] (29)

it follows from equation (21) that \( q^I \) and \( y^I \) are given by

\[
\begin{align*}
w^I q^I &= -(1 + w^I) s^0 - 2 \eta^I w^I - \sum_i c^I \tilde{s}^i \\
w^I y^I &= -(1 + w^I) \tilde{u}^0 - \sum_i c^I \tilde{u}^i
\end{align*}
\] (30) (31)

Also, write \( r^2 B = \sum_i \tilde{\lambda}^i \lambda^i = B_0 + 2B_1 + B_2 \) where

\[
B_0 = \sum_i \tilde{s}^i s^i, \quad B_1 = \sum_i s^i \tilde{u}^i, \quad B_2 = \sum_i \tilde{u}^i \tilde{u}^i.
\]

In the limit \( r \to \infty \), the \( \tilde{\lambda}^0 \) and the \( \tilde{\sigma} \) terms in the above equations give the leading zeroth order asymptotic solutions. They give the singular solutions of [34, 29]. The functions \( \tilde{u}^i \)'s are treated as perturbations and are used to obtain the first order corrections to the leading asymptotic solutions. They give the perturbations around the singular solutions.

**Zeroth order : Singular solutions**

Consider the equations of motion (23) – (27) and expand them to zeroth and first order in the functions \( \tilde{u}^i \)'s. At zeroth order, equating the powers of \( r \) gives

\[
2 + q^I + 2\tilde{s} = \tilde{s} = 0
\] (32)
and, hence, \( q^I = -2 \). Now, equation (30) becomes

\[
2 \, w^I_\pi (1 - \eta^I) = (1 + w^I_\pi) \, \tilde{s}^0 + \sum_i c^{Ii} \, \tilde{s}^i .
\]  

(33)

Also, upto first order in the functions \( \tilde{u} \)'s, we have

\[
r^2 (\rho_I e^{2\lambda}) = \rho_{I0} e^{2\tilde{\lambda}_0 + y^I + 2\tilde{u}} = R_I (1 + y^I + 2\tilde{u} + \cdots)
\]

\[
e^{2\tilde{\lambda} - 1} = e^{2\tilde{\lambda}_0 + 2\tilde{u} - 1} = (e^{2\tilde{\lambda}_0} - 1) + e^{2\tilde{\lambda}_0} (2\tilde{u} + \cdots)
\]

\[
\tilde{x}_\sigma - \tilde{\lambda}_\sigma = \alpha + (\tilde{u}^0_\sigma - \tilde{u}_\sigma) , \quad \alpha = m - 1 + s^0 - \tilde{s}
\]

where \( R_I = \rho_{I0} e^{2\tilde{\lambda}_0} \). At zeroth order, equations (23) – (27) now give

\[
\alpha \, \tilde{s}^0 = \sum_i \tilde{c}^{0i} \, R_i , \quad \alpha \, \tilde{s}^i = \sum_i \tilde{c}^{Ii} \, R_i \quad (34)
\]

\[
\tilde{s}^0 = \sum_i \frac{w^I_\pi}{m} \, R_i + \frac{m - 1}{2} \left( e^{2\tilde{\lambda}_0} - 1 \right) + \frac{B_0}{2m} \quad (35)
\]

\[
\tilde{s}^0 - \tilde{s} = \sum_i \tilde{c}^I \, R_i + (m - 1) \left( e^{2\tilde{\lambda}_0} - 1 \right) \quad (36)
\]

\[
\tilde{s} = \sum_i \frac{R_i}{m} - \frac{m - 1}{2} \left( e^{2\tilde{\lambda}_0} - 1 \right) + \frac{B_0}{2m} \quad , \quad (37)
\]

and equation (36) gives

\[
\alpha = m - 1 + s^0 - \tilde{s} = \sum_i \tilde{c}^I \, R_i + (m - 1) \, e^{2\tilde{\lambda}_0} . \quad (38)
\]

Equations (33) – (38) constitute the equations of motion at the leading zeroth order in the functions \( \tilde{u} \)'s. They will give the singular solutions.

**First order : perturbations around singular solutions**

\footnote{In general, one should analyse equation (30) which determines \( q^I \) and thus the asymptotic behaviour of \( \rho_I \). For a given set of values for \( w^I_\pi \) and \( w^I \), some of the resulting \( q^I \)'s may lead to subdominant terms. Then the corresponding \( \rho_I \)'s become unimportant and effectively reduce \( N \). With no loss of generality, we are assuming that \( w^I_\pi \) and \( w^I \) are such that \( q^I = -2 \) for all \( I \), thus all \( \rho_I \)'s remain important and \( N \) remains unreduced.}
At first order in the \( \tilde{u} \)'s, equations (23) – (27) give

\[
\tilde{u}_{a}^{0} + \alpha \tilde{u}_{a}^{0} + \tilde{s}^{0} ( \tilde{u}_{a}^{0} - \tilde{u}_{a} ) = \sum_{I} c^{0I} R_{I} (y^{I} + 2\tilde{u})
\]
\[
\tilde{u}_{ii}^{0} + \alpha \tilde{u}_{ii}^{0} + \tilde{s}^{0} ( \tilde{u}_{ii}^{0} - \tilde{u}_{ii} ) = \sum_{I} \tilde{c}^{0I} R_{I} (y^{I} + 2\tilde{u})
\]

\[
\tilde{u}_{0}^{0} = \sum_{I} \frac{w_{I}^{0}}{m} R_{I} (y^{I} + 2\tilde{u}) + (m - 1) e^{2\tilde{\lambda}_{0}} \tilde{u} + \frac{B_{1}}{m}
\]
\[
\tilde{u}_{a}^{0} - \tilde{u}_{a} = \sum_{I} \tilde{c}^{I} R_{I} (y^{I} + 2\tilde{u}) + (m - 1) e^{2\tilde{\lambda}_{0}} (2\tilde{u})
\]
\[
\tilde{u}_{a} = \sum_{I} R_{I} (y^{I} + 2\tilde{u}) - (m - 1) e^{2\tilde{\lambda}_{0}} \tilde{u} + \frac{B_{1}}{m}.
\]

Using the zeroth order results for the \( \tilde{u} \)-terms in the right hand sides of the above equations, one obtains

\[
\tilde{u}_{0}^{0} = \sum_{I} \frac{w_{I}^{0}}{m} R_{I} y^{I} + \left( 2s^{0} + m - 1 + \frac{B_{0}}{m} \right) \tilde{u} + \frac{B_{1}}{m}
\]  
(39)
\[
\tilde{u}_{a}^{0} - \tilde{u}_{a} = \sum_{I} \tilde{c}^{I} R_{I} y^{I} + 2\alpha \tilde{u}  
\]  
(40)
\[
\tilde{u}_{a} = \sum_{I} \frac{R_{I}}{m} y^{I} - \left( m - 1 + \frac{B_{0}}{m} \right) \tilde{u} + \frac{B_{1}}{m} .
\]  
(41)

The \( \tilde{u}_{a}^{0} \) and the \( \tilde{u}_{a}^{i} \) equations now become

\[
\tilde{u}_{a}^{0} + \alpha \tilde{u}_{a}^{0} = \sum_{I} (\tilde{c}^{0I} - \tilde{s}^{0} \tilde{c}^{I}) R_{I} y^{I}
\]  
(42)
\[
\tilde{u}_{a}^{i} + \alpha \tilde{u}_{a}^{i} = \sum_{I} (\tilde{c}^{0I} - \tilde{s}^{i} \tilde{c}^{I}) R_{I} y^{I}
\]  
(43)

Equations (31) and (39) – (43) constitute the equations of motion at first order in the functions \( \tilde{u} \)'s. Their solutions give the perturbations around the singular solutions.

**Significance of the perturbations**

20
We now outline briefly the significance of the singular solutions and the perturbations around them. In section 2, we have briefly mentioned a few of these aspects. See [29] for complete details.

The singular solutions and the perturbations around them can be used, among other things, as indicators of the stability of stars. A star, with its constituents obeying linear equations of state, is enclosed in a box of radius \( r_\ast \) so as to prevent the evaporation of its constituents and to make its total mass finite. The mass of the star is then given, upto constant numerical factors, by the mass function \( M(r_\ast) \) evaluated at \( r_\ast \). The singular solutions give the scaling relations between mass, radius, and other quantities of the stars. The singular solutions are of no relevance when the stars are unstable but, when stable, these solutions give the scaling relations in the limit of large central density or large radius of the stable stars.

The behaviour, namely damped oscillatory or monotonic non oscillatory, of the perturbations around the singular solutions leads to the corresponding behaviour for the mass – central density profile in the asymptotic regions. Considering a series of equilibria, Chavanis obtains the mass – central density profile in the asymptotic and non asymptotic regions. He shows that equilibrium configurations become unstable beyond the first maximum in this profile. This is also shown to correspond to conditions for nonlinear dynamic stability. In those higher dimensional cases where the behaviour of the perturbations is monotonic non oscillatory, Chavanis obtains the mass – central density profile in the non asymptotic regions also and shows that the entire profile is monotonically increasing and asymptotes to the constant value line given by the singular solutions.

In this paper, assuming linear equations of state, we will obtain singular solutions and perturbations around them. Enclosing the star in a box of radius \( r_\ast \), the mass – central density profile and its behaviour in the asymptotic regions can also be obtained from the perturbations. However, in the case of horizonless objects of interest here, nothing is known rigorously about the nature and properties of their constituents. Hence, we are unable to carry out the analogs of the analyses mentioned in the previous paragraph. Namely, since the detailed properties of the constituents are not known, we are unable to obtain the profile in the non asymptotic regions, and to study the stability properties around the maxima.

We note below a few useful points.
The expressions $\frac{M(r)}{r^{m-1}} = 1 - e^{-2\tilde{\lambda}}$ and $\tilde{\lambda} = \tilde{\lambda}_0 + \tilde{u}$ imply that, in the limit of large $r$,

$$\frac{M(r)}{r^{m-1}} = 1 - e^{-2\tilde{\lambda}_0} + 2 e^{-2\tilde{\lambda}_0} \tilde{u} + \cdots. \quad (44)$$

The first two terms correspond to the singular solutions and the $\tilde{u}$ term describes the perturbations in the mass function. Similarly, $y^f$ describes the perturbations in the density $\rho^f$.

(2) The radius $r_*$ of a spherically symmetric star, defined to be given by $\Pi(r_*) = 0$, is infinite when it is made up of perfect fluids with linear equations of state, see [33] for a derivation. Hence, let the star be enclosed in a box of radius $r_*$ which will render its mass

$$M(r_*) \simeq \int_{r_*}^{r_*} dr \: M_r$$

finite where $M_r$ is given in equation (17). Then, following the analysis of [29], the perturbations around the singular solutions can be used to obtain the mass – central density profile of the star in the asymptotic limit of large central density or large radius.

Let $x_{ch} \propto \sqrt{\rho_c} \: r_*$ be a measure of central density $\rho_c$, and let $y_{ch} = \frac{M(r_*)}{r_*}$ be a measure of the mass of the star. By detailed analysis of the equations, and incorporating the properties of the constituents of the star, Chavanis obtains the mass – central density profile in both the asymptotic and non asymptotic regions and finds that: (i) As $x_{ch}$ increases from zero to $\infty$, $y_{ch}$ increases from zero to a (first) maximum $y_1$ at $x_1$, thereafter exhibits damped oscillations, asymptoting to a value $y_s$. (ii) The behaviour for large values of $x_{ch}$ can be seen from the singular solutions and the asymptotic perturbations around them. (iii) The solutions are unstable beyond the first maximum which is at $(x_{ch}, \: y_{ch}) = (x_1, \: y_1)$.

As a consequence, one has the following. For a given value of central density, the radius $r_*$ must be $< r_{1*}$ where $r_{1*}$ corresponds to $x_1$. The mass of the star must then be less than $(y_1 \: r_{1*}^{m-1})$. A more massive star will be unstable and will collapse.

(3) For $D = m + 2$ dimensional stars, $n_c = 0$ and $N = 1$ in our notation, Chavanis finds that if $m \geq m_{cr} \sim 9$ then $y_{ch}$ increases monotonically
from zero to $y_s$, effectively making $x_1$ infinite and $y_1 = y_s$. See Figure 23, and also Figures 20 – 22, in [29]. The asymptotic perturbations around the corresponding singular solutions exhibit monotonic behaviour with no oscillations.

As a consequence, one has the following. For $m \geq 9$, $x_1$ is effectively infinite. Then, for a given value of central density, $r_{1s}$ is infinite which makes the upper limit $(y_s r_{1s}^{m-1})$ on the mass of the star also infinite. Hence, a star can be arbitrarily massive and stable when $m \geq 9$. Also, in the limit of large central density or large radius, one obtains the scaling relation $M(r_{s}) \sim r_{s}^{m-1}$ which shows that the sizes of the large stable stars scale as their Schwarzschild radii.

In [37], we generalised this study to $D = n_c + m + 2$ dimensional stars, with $n_c$ toroidal directions, made up of $\mathcal{N} > 1$ number of perfect fluids where pressures are isotropic along the non compact spatial directions. Stars in M theory correspond to specific values of $n_c$ and $\mathcal{N}$. We found that, even in these generalised cases, $m \geq 9$ is required for stability.

5. Singular solutions and asymptotic perturbations

Taking the pressures inside the stars to be anisotropic along the radial and the spherical directions of the non compact space, and hence taking the anisotropy parameters $\eta^I \neq 0$, we now obtain singular solutions to the equations of motion and asymptotic perturbations around them, generalising those given in [34, 29, 37]. These solutions may be obtained for any general set of values for $n_c$, $m$, $\mathcal{N}$, $w^I_\alpha$ and $w^I_\pi$. However, such a generality is neither illuminating nor needed for our purposes here. Hence we will present only three cases which are illustrative and are also of direct interest.

In this section we will consider $(m + 2)$ dimensional stars made up of a single fluid. In the next section we will consider two examples of stars in M theory: In one, the stars are made up of a stack of $M2$ or $M5$ branes. In another, the stars are made up of four stacks of intersecting branes, two stacks each of $M2$ and $M5$ branes, see footnote 5.

$(m + 2)$ dimensional stars with $\mathcal{N} = 1$

Consider $(m + 2)$ dimensional stars made up of a single fluid. Then
The fluid is assumed to have anisotropic pressures in general. Its radial and the transverse spherical pressures, and the anisotropy parameter $\eta$, are given by

$$\Pi = w_\pi \rho, \quad p = w \rho, \quad \eta = \frac{m}{2} \left( \frac{w_\pi - w}{w_\pi} \right).$$

Also, $c^0 = 1 + w + c$ and $c^a = c = \frac{w_\pi - 1}{m}$, and hence

$$m \: c^0 = (m - 1) \: (1 + w_\pi) + 2 \: w_\pi \: (1 - \eta).$$

We now write down the leading order asymptotic solutions to the equations of motion and perturbations around them. We have $s = 0$ and $q = -2$. The zeroth order equations (33) – (38) then give the following relations.

$$s^0 \: = \: \frac{2 \: w_\pi \: (1 - \eta)}{1 + w_\pi} \quad \Rightarrow \quad \alpha = m - 1 + s^0 = \frac{m \: c^0}{1 + w_\pi}$$

$$R \: = \: \frac{\alpha \: s^0}{c^0} = \frac{2 \: m \: w_\pi \: (1 - \eta)}{(1 + w_\pi)^2}$$

$$(m - 1) \: e^{2\lambda_0} = \alpha - c \: R = \frac{D}{(1 + w_\pi)^2}$$

$$D \: = \: (m - 1) \: (1 + w_\pi)^2 + 4 \: w_\pi \: (1 - \eta).$$

The above expressions describe the singular solutions. For example, they give

$$e^{2\lambda_0} \simeq r^{2s^0}, \quad e^{2\lambda} \simeq 1 + \frac{4w_\pi (1 - \eta)}{(1 + w_\pi)^2},$$

$$\rho \simeq \frac{\rho_0}{r^2} = \frac{2m(m - 1)w_\pi(1 - \eta)}{r^2D}. $$

Note that, from $R \propto \rho_0 > 0$ and $w_\pi > 0$, it follows that $\eta < 1$ and, hence, that $mc^0$, $s^0$, $\alpha$, and $D$ are all positive. The mass function can be obtained from equation (44) and, using the above expression for $e^{2\lambda_0}$, it is given by

$$\frac{M(r)}{r^{m-1}} = \frac{2}{D} \left( 2 \: w_\pi \: (1 - \eta) + (m - 1) \: (1 + w_\pi)^2 \: u + \cdots \right). \quad (45)$$
The mass of the star \( M(r_\ast) \) enclosed in a box of large radius \( r_\ast \) is then given by

\[
M(r_\ast) \simeq \frac{4w_\pi (1 - \eta)}{D} r_\ast^{m-1}.
\]

Consider now the first order equations of motion (31) and (39) – (42). Their solutions will give the asymptotic perturbations around the singular solutions. Equation (31) gives, upon using the expressions for \( s^0 \) and \( R \),

\[
s^0 y = -2(1 - \eta) u^0, \quad R y = -2(1 - \eta) \alpha \frac{u^0}{c^0}.
\]

Equations (39) and (41) give

\[
\begin{align*}
 u^0_\sigma &= -s^0 u^0 + (m - 1 + 2s^0) u \\
 u_\sigma &= -\frac{s^0}{w_\pi} u^0 - (m - 1) u
\end{align*}
\]

from which it follows, after a little algebra, that both \( u^0 \) and \( u \) obey the same equation given by

\[
(*)_{\sigma\sigma} + \alpha (*)_\sigma + \frac{2(1 - \eta) D}{(1 + w_\pi)^2} (*) = 0 \tag{46}
\]

where \( (*) = u^0 \) or \( u \). It is straightforward to show that equation (42) also gives the above equation for \( u^0 \).

The solutions to equation (46) are of the form \( (*) \sim e^{k \sigma} \) where

\[
k = -\frac{\alpha \pm \sqrt{\Delta}}{2}, \quad \Delta = \alpha^2 - \frac{8(1 - \eta) D}{(1 + w_\pi)^2}.
\]

If \( \Delta < 0 \) then, in an obvious notation, \( k = -k_{re} \pm i k_{im} \) with \( k_{re} > 0 \). The solutions for \( (*) \) are then oscillatory and, since \( \sigma = \ln r \), they may be written as

\[
(*) = \frac{A_*}{r^{k_{re}}} \sin (k_{im} \ln r + B_*)
\]

where \( A_* \) and \( B_* \) are integration constants. If \( \Delta > 0 \) then \( \alpha > \sqrt{\Delta} \) since \( 1 - \eta > 0 \) and \( D > 0 \) and, in an obvious notation, \( k = -k_1 \pm k_2 \) with

25
\( k_1 > k_2 > 0 \). Hence the solutions for (\(*\)) are non oscillatory and they may be written as

\[
(\ast) = \frac{A_{\ast}}{r^{k_1 - k_2}} \left( 1 + \frac{B_{\ast}}{r^{2k_2}} \right).
\]

If \( \Delta = 0 \) then \( k_2 = 0 \) and the solutions are (\(*\)) = \( \frac{A_{\ast}}{r^{k_1}} \left( \ln r + B_{\ast} \right) \) and are non oscillatory. Considering the star enclosed in a box of radius \( r_{\ast} \), introducing Milne variables, and following the analysis of Chavanis given in [29], one can now obtain the mass – central density profile in the asymptotic limit of large central density or large radius. The asymptotic behaviour of this profile is similar to that of the perturbations: it is oscillatory if \( \Delta < 0 \), and is non oscillatory if \( \Delta \geq 0 \).

Comparing with the work of Chavanis in [29], we note that the solutions for (\(*\)) given above when \( \Delta < 0 \) and when \( \Delta \geq 0 \) will lead to the analogs of equations (174) and (175) in [29]. The resulting mass – central density profile for \( M(r_{\ast}) \) will lead to the analogs of the asymptotic parts of Figure 23 in [29]. The numerical and the analytical studies leading to the analogs of the non asymptotic initial parts of that Figure are beyond the scope of the present paper and hence, although important, are not attempted here. If one assumes that these non asymptotic initial parts remain qualitatively the same for the anisotropic case also, then one may conclude that a star can be arbitrarily massive and stable when \( \Delta \geq 0 \); and that the singular solutions describe its mass – radius relations in the limit of large central density or large radius.

We now study the conditions under which \( \Delta < 0 \) and \( \Delta \geq 0 \). Let

\[
\Delta = \frac{\hat{\Delta}}{(1+w_{\ast})^2}.
\]

It then follows that

\[
k = \frac{-m c^0 \pm \sqrt{\hat{\Delta}}}{2 (1+w_{\ast})}, \quad \hat{\Delta} = A w_{\pi}^2 + 2B w_{\pi} + C
\]

where, after some algebra, one obtains

\[
A = (m - 3 + 2\eta)^2
\]

\[
B = (m + 1 - 2\eta) (m - 9 + 8\eta)
\]

\[
C = (m - 1) (m - 9 + 8\eta)
\]

26
\[ AC - B^2 = 32 \left(1 - \eta^2\right) (m - \eta) (m - 9 + 8\eta). \]

Consider now the sign of \( \Delta \). It is same as that of \( \hat{\Delta} \). Using the quadratic expression for \( \hat{\Delta} \) given above, it can be seen that if \( AC - B^2 < 0 \) then \( \hat{\Delta} \) can be negative for a range of values for \( m, \eta, \) and \( w_\pi \). For example, if \( m = 2 \) and \( \eta = 0 \) then \( AC - B^2 < 0 \). Then \( \hat{\Delta} = w_\pi^2 - 42w_\pi - 7 \) and is negative, for example, for \( 0 < w_\pi < 1 \). If \( AC - B^2 \geq 0 \) then \( \hat{\Delta} \geq 0 \) always.

In the isotropic case, \( \eta = 0 \) and the above expressions reduce to those given in [29]. Then \( \hat{\Delta} \geq 0 \) if \( m \geq 9 \) and, depending on the value of \( w_\pi = w \), \( \hat{\Delta}(w) \geq 0 \), if \( m \geq m_{cr}(w) \sim 9 \). In the anisotropic case, \( \eta \) is non vanishing. Noting that \( \eta < 1 \) and \( m \geq 2 \), it follows from the above expressions that \( AC - B^2 \geq 0 \) if the values of \( \eta \) and \( \frac{w}{w_\pi} \) lie in the range given by

\[
\frac{9 - m}{8} \leq \eta < 1 \quad \longleftrightarrow \quad \frac{5m - 9}{4m} \geq \frac{w}{w_\pi} > \frac{m - 2}{m};
\]

then \( \hat{\Delta} \geq 0 \), hence \( \Delta \geq 0 \), and the solutions are non oscillatory. The dependence of these ranges on \( w_\pi \) can also be incorporated, but is superfluous for our present purposes of showing that anisotropy can lead to asymptotic non oscillatory solutions. Note that the isotropic case \( \eta = 0 \), equivalently \( w = w_\pi \), is included in the ranges given above only when \( m \geq 9 \). For lower values of \( m \), (for example, for \( m = 2 \) which corresponds to four dimensional spacetime) certain amount of anisotropy ( \( \eta \geq \frac{7}{8} \) ) is needed to obtain the non oscillatory behaviour of the asymptotic perturbations around the singular solutions.

6. Stars in M theory

In this section we will analyse stars in M theory. In section 2, we have described briefly M theory stars and how the necessary equations of state are obtained. We now proceed with the analysis.

The spacetime is eleven dimensional in M theory, having \( n_c \) dimensional compact toroidal space and \( (m + 2) \) dimensional non compact spacetime where \( n_c + m = 9 \). The M theory stars are taken to be made up of \( N \) stacks of \( M2 \) and \( M5 \) branes, intersecting according to the BPS rules. We take the spatial directions of the brane worldvolumes to be toroidal and
assume necessary isometries. These intersecting branes can be modelled by \( N \) number of separately conserved energy momentum tensors \([55, 59, 60, 61]\). Hence, the present formalism can be applied to the corresponding stars.

M theory has U duality symmetries. As shown in \([58]\), see also footnote 7, they lead to a relation among the components \((p_{\alpha I}, \Pi_I)\) of the energy momentum tensor \(T_{MN(I)}\) which is given by

\[
p_{\parallel I} = \Pi_I + p_{0I} + p_{\perp I} + m (p_I - p_{\perp I})
\]

where \(p_{\parallel I}\) and \(p_{\perp I}\) are the pressures along the directions that are parallel and transverse to the worldvolume of the \(I^{th}\) stack of branes. The above relation is a consequence of U duality symmetries and, therefore, must always be valid independent of the details of the equations of state. Also, since the sphere directions are transverse to the branes, it is natural to set \(p_I = p_{\perp I}\).

Note that, for stars in M theory, there is no compelling reason to take the pressures \(\Pi_I\) and \(\rho_I\) to be equal. Indeed, in the case of charged intersecting black branes in M theory, these pressures are not equal although the relation \(p_I = p_{\perp I}\) and the U duality relation above are obeyed. Hence we assume that, in general, \(\Pi_I \neq \rho_I\) for stars in M theory. Setting \(p_{\perp I} = p_I\) and \(p_{0I} = -\rho_I\), the U duality relation now becomes

\[
p_{\parallel I} = \Pi_I - \rho_I + p_I.
\]

Consider the linear equations of state given by (18). Let \(p_{\parallel I} = w_{\parallel I} \rho_I\) and \(p_{\perp I} = w_{\perp I} \rho_I\) where \(w_{\perp I} = w_I\) since \(p_{\perp I} = p_I\). The U duality relation then gives

\[
w_{\parallel I} = w_{\perp I} - 1 + w_I.
\]

Now consider the coefficients \(\tilde{c}^{\parallel I} = -w_{\parallel I} + w_I + \tilde{c}_I\) defined in equation (20). Note that \(w_{\parallel I} = w_I\) if \(i \in \parallel I\), namely if \(x^i\) is a worldvolume coordinate of the \(I^{th}\) stack of branes; otherwise, \(w_{\parallel I} = w_{\perp I}\). It then follows from \(w_{\perp I} = w_I\) and the U duality relation for \(w_{\parallel I}\) given above, that the corresponding coefficients \(\tilde{c}^{\parallel I}\) and \(\tilde{c}^{\perp I}\) are given by

\[
\tilde{c}^{\parallel I} = -w_{\parallel I} + 1 + \tilde{c}_I = (1 - m) \tilde{c}_I, \quad \tilde{c}^{\perp I} = \tilde{c}_I = \frac{w_I - 1}{m}.
\]

Hence, for any \(a^i\) with \(\tilde{a}^i = a^i + \frac{w_{\parallel I}}{m}\) and \(\alpha^c = \sum_i a^i\), it follows that

\[
\sum_i \tilde{c}^{\parallel I} a^i = \sum_i \tilde{c}^{\parallel I} a^i = \tilde{c}^{\parallel I} \alpha^c - m \tilde{c}^{\parallel I} \sum_{i \in \parallel I} a^i.
\]
In passing, we note that the relation \( \tilde{c}^{I} = (1 - m) \tilde{c}^{I} \) is same as that obtained in the corresponding isotropic cases studied in [37]. Consequently, the resulting equations for M theory stars will have very similar structure in both the isotropic and the anisotropic cases. With \( w^{I}_{\alpha} \) and \( \tilde{c}^{\alpha I} \) specified for intersecting branes, we now consider two examples of stars in M theory.

**M theory stars made up of M2 or M5 branes**

Consider stars in M theory made up of a stack of M2 or M5 branes. Then \( \mathcal{N} = 1 \), \( n_{c} = 2 \) or 5, and \( m = 7 \) or 4. The \( I \)–scripts on various quantities are now unnecessary and, hence, we omit them. Also, we will first write the solutions in a form applicable for \( \mathcal{N} = 1 \) and for any values of \( n_{c} \), \( m \), and \( \tilde{c}^{i} \); and then, at the end, specialise to the case of M2 or M5 brane stars.

We now write down the leading order asymptotic solutions to the equations of motion and perturbations around them. We have \( \tilde{s} = 0 \) and \( q = -2 \). The zeroth order equations (33) – (36) then give the following relations. They describe the analogs of singular solutions in this context.

\[
2 w_{\pi} (1 - \eta) = (1 + w_{\pi}) \tilde{s}^{0} + \sum_{i} c^{i} \tilde{s}^{i} \\
\alpha \tilde{s}^{0} = \tilde{c}^{0} R \ , \quad \alpha \tilde{s}^{i} = \tilde{c}^{i} R \\
\tilde{s}^{0} = \frac{w_{\pi}}{m} R + \frac{m - 1}{2} (e^{2\tilde{\lambda}_{0}} - 1) + \frac{B_{0}}{2m} \\
\tilde{s}^{0} = \tilde{c} R + (m - 1) \left( e^{2\tilde{\lambda}_{0}} - 1 \right). \]

Using these relations, one obtains

\[
R = \alpha \frac{\tilde{s}^{0}}{\tilde{c}^{0}} , \quad \tilde{s}^{i} = \tilde{c}^{i} \frac{\tilde{s}^{0}}{\tilde{c}^{0}} , \quad \tilde{s}^{0} = \frac{2 w_{\pi} (1 - \eta)}{(1 + w_{\pi}) (1 + \gamma)} \quad (52) 
\]

where \( \gamma = \frac{\sum \tilde{c}^{i} \tilde{c}^{i}}{(1 + w_{\pi}) \tilde{c}^{0}} \); and, after some algebra,

\[
(m - 1) e^{2\tilde{\lambda}_{0}} = \alpha - \tilde{c} R = \frac{\alpha}{1 + \gamma} \left( \frac{\mathcal{D}}{m \tilde{c}^{0} (1 + w_{\pi}) + \gamma} \right) \quad (53)
\]
where \( D = (m - 1) (1 + w_r)^2 + 4 w_r (1 - \eta) \). Note that, from \( R \propto \rho_0 > 0 \) and \( w_r > 0 \), it follows that \( \eta < 1 \); and, hence, that \( m \bar{c}^0, \gamma, \bar{s}^0, \alpha, \) and \( D \) are all positive. Note also that the effects of the compact toroidal space appear through the parameter \( \gamma \) alone.

Consider the first order equations of motion (31) and (39) – (43). Their solutions will give the asymptotic perturbations around the singular solutions. Upon using \( \bar{s}^i = \bar{c}^i \frac{\bar{u}^0_\sigma}{\bar{c}_0} \), equations (42) and (43) give

\[
F^i_{\bar{c}} + \alpha F^i_{\bar{\sigma}} = 0
\]

where \( F^i = \tilde{u}^i - \bar{c}^i \frac{\bar{u}^0_\sigma}{\bar{c}_0} \). Although it follows that \( F^i = F^i_1 e^{-\alpha \tilde{\sigma}} \) in general, we will set the integration constants \( F^i_1 \) to zero. This gives \( \tilde{u}^i = \bar{c}^i \frac{\bar{u}^0_\sigma}{\bar{c}_0} \), which we now use in the remaining equations (31) and (39) – (42).

Equation (31) gives, upon using equations (52) for \( \bar{s}^0_0 \) and \( R \),

\[
\tilde{u}^0_\sigma y = -2 (1 - \eta) \bar{u}^0, \quad R y = -2 (1 - \eta) \alpha \frac{\bar{u}^0_\sigma}{\bar{c}_0}.
\]

Equation (39) gives, after some manipulations,

\[
\tilde{u}^0_{\bar{\sigma}} = -(1 + \gamma) \bar{s}^0_0 \bar{u}^0 + (m - 1 + (2 + \gamma) \bar{s}^0_0) \tilde{u}.
\]

Equation (40) gives straightforwardly

\[
\tilde{u}^0_{\bar{\sigma}} - \tilde{u}^0_{\sigma} = -2 (1 - \eta) \alpha \bar{c} \frac{\bar{u}^0_\sigma}{\bar{c}_0} + 2 \alpha \tilde{u}.
\]

After a little algebra, it follows from these two equations that both \( u^0 \) and \( u \) obey the same equation which we write as

\[
(*)_{\bar{\sigma}} + \alpha (*)_\sigma + 2 (1 - \eta) (\alpha - \bar{c} R) (*) = 0 \quad (54)
\]

where \( (*) = u^0 \) or \( u \) and \( (\alpha - \bar{c} R) \) is given in equation (53). Equation (42) gives the equation for \( u^0 \) straightforwardly in the above form.

The solutions to equation (54) are of the form \( (*) \sim e^{k \sigma} \) where

\[
k = \frac{-\alpha \pm \sqrt{\Delta}}{2}, \quad \Delta = \alpha^2 - 8 (1 - \eta) (\alpha - \bar{c} R).
\]

---

\[1^0\] We used the expressions \( B_0 = \gamma (1 + w_r) \left( \frac{s^0}{\bar{c}} \right)^2, \quad B_1 = \gamma (1 + w_r) \left( \frac{s^0}{\bar{c}} \right) \bar{u}^0_\sigma \), and \( \frac{m \bar{c}^0}{1 + w_r} = \alpha + \gamma s^0 \), which can all be derived easily.
As explained in the case of \((m+2)\) dimensional star, it is important to study the sign of \(\Delta\) since it determines whether the solutions for \((\dagger)\) are oscillatory or not. This behaviour of the perturbations \((\dagger)\) will lead to corresponding asymptotic behaviour in the mass–central density profile of a star enclosed in a box. Writing

\[
\Delta = \frac{\alpha (\hat{\Delta} + \hat{\delta})}{m \tilde{c}^0 (1 + \pi^w) (1 + \gamma)},
\]

it can be shown after a long but straightforward algebra that \(\hat{\Delta}\) and \(\hat{\delta}\) are given by equation (47) and

\[
\hat{\delta} = m (m - 9 + 8\eta) \sum_i \tilde{c}^i c^i.
\]

(55)

Consider the sign of \(\Delta\). It is same as that of \((\hat{\Delta} + \hat{\delta})\). If \(\tilde{c}^i = 0\) for all \(i\) then \(\hat{\delta} = \gamma = 0\) and the earlier analysis given below equation (47) applies directly. Now let \(\tilde{c}^i\) do not all vanish. Then \(\sum_i \tilde{c}^i c^i > 0\), see footnote 8. It then follows that \(\hat{\delta} \geq 0\) if \(m - 9 + 8\eta \geq 0\). Hence, noting that \(\eta < 1\), one has \(\hat{\delta} \geq 0\) if the inequalities (48) are satisfied. Interestingly, this is precisely the condition that also ensures that \(\hat{\Delta} \geq 0\) but we do not know a simple reason, if any, for this coincidence. We thus have that \(\Delta \geq 0\) and the solutions are non oscillatory if the values of the anisotropy parameter \(\eta\) and, equivalently, of \(\frac{w}{w\pi}\) lie in the ranges given in equation (48). Also, note that the value \(\eta = 0\), equivalently \(w = w\pi\), is included in these ranges only when \(m \geq 9\). Otherwise certain amount of anisotropy is needed along the non compact spatial dimensions to obtain the non oscillatory behaviour of the perturbations around the leading order asymptotic solutions.

The solutions obtained above are for \(\mathcal{N} = 1\) and are applicable for any values of \(n_c\), \(m\), and \(\tilde{c}^i\). In particular, they are also applicable to stars in M theory made up of a stack of \(M2\) or \(M5\) branes for which \(n_c = 2\) or \(5\), \(m = 7\) or \(4\), and \(\tilde{c}^i = \tilde{c}^\parallel = (1 - m) \tilde{c}\) as given in equation (50). It then follows easily that

\[
\sum_i \tilde{c}^i c^i = \frac{n_c m}{n_c + m} (\tilde{c}^\parallel)^2 = A_{n_c} (w\pi - 1)^2
\]

where \(A_{n_c} = \frac{8}{7}\) for \(M2\) branes and \(= \frac{5}{4}\) for \(M5\) branes.

**M theory stars made up of \(\mathcal{N} = 4\), \(22'55'\) intersecting branes**
Consider stars in M theory made up of two stacks each of $M^2$ and $M^5$ branes intersecting according to the BPS rules whereby two stacks of five branes intersect along three common spatial directions; a stack each of two branes and five branes intersect along one common spatial direction; and two stacks of two branes intersect along zero common spatial direction. Let $22'55': (12, 34, 13567, 24567)$ denote the configuration of the intersecting branes and indicate the spatial worldvolume directions of the four stacks of branes. Now $N = 4, n_c = 7, \text{ and } m = 2$. Also, the coefficients $\tilde{c}^{I}$ are given by equations (50). Thus,

$$\tilde{c}^{\parallel I} = - \tilde{c}^{\perp I} = - \frac{w^I - 1}{2}.$$  

Note that equation (51) gives, with $S^c = \sum_i s^i$ and $U^c = \sum_i u^i$,

$$\sum_i c^I \tilde{s}^i = \sum_i \tilde{c}^{\parallel I} s^i = \tilde{c}^l S^c - m \tilde{c}^l \sum_{i \in \parallel I} s^i$$  

$$\sum_i \tilde{c}^{\parallel I} \tilde{u}^i = \sum_i \tilde{c}^{\perp I} u^i = \tilde{c}^l U^c - m \tilde{c}^l \sum_{i \in \parallel I} u^i.$$  

We now write down the leading order asymptotic solutions to the equations of motion and perturbations around them. They will describe the analogs of the singular solutions and the asymptotic perturbations around them. We have $\tilde{s} = 0$ and $q^I = -2$. Equation (33) can then be satisfied for all $I$ by choosing $w^I = w$ for all $I$, see footnote 9. Then $\eta^I = \eta$, $\tilde{c}^I = \tilde{c}$, and $\tilde{c}^{\parallel I} = \tilde{c}^{\perp I}$ for all $I$, but $\tilde{c}^{I}$ do depend on $I$ since $\tilde{c}^{I} = \tilde{c}^{\parallel I}$ if $i \in \parallel I$ and $\tilde{c}^{I} = \tilde{c}^{\perp I}$ otherwise. Now, using equation (57) for the $22'55': (12, 34, 13567, 24567)$ configuration, note that

$$\sum_I \left( \sum_i c^{I} \tilde{u}^i \right) = \tilde{c} U^c (N - 2m) = 0$$

since $N = 4$ and $m = 2$.  

11 Cancellations of this type do not happen for all intersecting brane configurations. It happens in the present case, and for the case where three stacks of two branes intersect (for which $N = 3, n_c = 6, \text{ and } m = 3$), and for the equivalent U dual versions of these two configurations. Similar cancellations happen for these two independent configurations in the cosmological context also. There, the cancellations may be understood as arising due to the balancing of contraction or expansion forces applied by the branes on the compact directions parallel or transverse to the worldvolume directions [63, 64, 65]. These cancellations are responsible for the stabilisation of the compact toroidal directions.
using \( \tilde{c} \parallel = - \tilde{c} \perp \) and denoting the \( X_I \)’s as \((X_2, X_2', X_5, X_5')\), such sums become

\[
\sum_I \tilde{c}^{1I} X_I = \tilde{c} \parallel (X_2 - X_2' + X_5 - X_5')
\]
\[
\sum_I \tilde{c}^{2I} X_I = \tilde{c} \parallel (X_2 - X_5 - X_5' + X_5')
\]
\[
\sum_I \tilde{c}^{3I} X_I = \tilde{c} \parallel (-X_2 + X_2' + X_5 - X_5')
\]
\[
\sum_I \tilde{c}^{4I} X_I = \tilde{c} \parallel (-X_2 + X_2' - X_5 + X_5')
\]
\[
\sum_I \tilde{c}^{5,6,7} \ I X_I = \tilde{c} \parallel (-X_2 - X_2' + X_5 + X_5') . \tag{59}
\]

Consider equation (33). Upon using equation (56), it gives

\[
\sum_{i \in I} s^i = \frac{(1 + w_\pi) \tilde{s}^0 - 2w_\pi (1 - \eta)}{m \tilde{c}} + \frac{S_c}{m},
\]

which implies that the sum \( \sum_{i \in I} s^i \) must be same for all \( I \). Thus, for the \( 22'55' : (12, 34, 13567, 24567) \) configuration, it follows that

\[
s^1 + s^2 = s^3 + s^4 = s^1 + s^3 + s^5 + s^6 + s^7 = s^2 + s^4 + s^5 + s^6 + s^7 \]
\[
\implies s^3 = s^2 , \quad s^4 = s^1 , \quad s^5 + s^6 + s^7 = 0 , \quad S_c = 2 (s^1 + s^2) .
\]

It can then be shown that \( \tilde{s}^i = s^i + \frac{S_c}{m} \) satisfy the relations

\[
\tilde{s}^3 = \tilde{s}^2 , \quad \tilde{s}^4 = \tilde{s}^1 , \quad \tilde{s}^5 + \tilde{s}^6 + \tilde{s}^7 = \tilde{s}^1 + \tilde{s}^2 . \tag{60}
\]

Consider equation (34) for \( \tilde{s}^i : \alpha \tilde{s}^i = \sum_I \tilde{c}^{1I} R_I \). Applying equations (59) now gives

\[
\alpha \tilde{s}^1 = \sum_I \tilde{c}^{1I} R_I = \tilde{c} \parallel (R_2 - R_2' + R_5 - R_5')
\]
\[
\alpha \tilde{s}^2 = \sum_I \tilde{c}^{2I} R_I = \tilde{c} \parallel (R_2 - R_2' - R_5 + R_5')
\]
\[
\alpha \tilde{s}^3 = \sum_I \tilde{c}^{3I} R_I = \tilde{c} \parallel (-R_2 + R_2' + R_5 - R_5')
\]
\[
\alpha \tilde{s}^4 = \sum_I \tilde{c}^{4I} R_I = \tilde{c} \parallel (-R_2 + R_2' - R_5 + R_5')
\]
\[
\alpha \tilde{s}^{5,6,7} = \sum_I \tilde{c}^{5,6,7} \ I R_I = \tilde{c} \parallel (-R_2 - R_2' + R_5 + R_5') .
\]

33
The three relations (60) on the \( \tilde{s}^i \) then imply that (see footnote 11)

\[
R_2 = R_2' = R_5 = R_5' = \frac{R}{4}, \quad R = \sum_I R_I,
\]

\[
\implies \tilde{s}^i = 0 \implies s^i = S^c = B_0 = B_1 = 0.
\]

Using equations (34) – (36), we then get the same zeroth order results as for the \((m + 2)\) dimensional stars where \(m = 2\) now. Namely, we get

\[
s^0 = \frac{2 \pi (1 - \eta)}{1 + \pi} \implies \alpha = m - 1 + s^0 = \frac{m \cdot c^0}{1 + \pi},
\]

\[
R = \frac{\alpha s^0}{c^0} = \frac{2 m \pi (1 - \eta)}{(1 + \pi)^2},
\]

\[
(m - 1) e^{2\lambda_0} = \alpha - \tilde{c} R = \frac{D}{(1 + \pi)^2}
\]

where \(D = (m - 1)(1 + \pi)^2 + 4 \pi (1 - \eta)\) and \(m = 2\).

Consider the first order equations of motion (31) and (39) – (43). Their solutions will give the asymptotic perturbations around the singular solutions. Equations (31) and (57) give

\[
w_\pi y^I = - (1 + \pi) \tilde{u}^0 - \tilde{c} U^c + m \tilde{c} \sum_{i \in I} u^i.
\]

Using \(\tilde{c}^I = \tilde{c}, \ c^0I = \tilde{c}^0,\) and \(B_0 = B_1 = 0\), the equations for \(\tilde{u}^0\) and \(\tilde{u}\) can be written as

\[
\tilde{u}^0 = \frac{\pi}{m} \sum_I R_I y^I + (2s^0 + m - 1) \tilde{u}
\]

\[
\tilde{u}^0 - \tilde{u} = \tilde{c} \sum_I R_I y^I + 2\alpha \tilde{u}
\]

\[
\tilde{u} = \frac{1}{m} \sum_I R_I y^I - (m - 1) \tilde{u}
\]

\[
\tilde{u}^0 + \alpha \tilde{u}^0 = (\tilde{c}^0 - s^0 \tilde{c}) \sum_I R_I y^I
\]

\[
\sum_I R_I y^I = - \frac{(1 + \pi)}{\pi} R \tilde{u}^0 = - 2 (1 - \eta) \alpha \tilde{u}^0 \frac{c^0}{c^0}.
\]
The above equation for $\sum R_I y^I$ follows from $R_I = R$, the expression for $y^I$, equation (58), and from the results obtained for $R$ at the zeroth order.

It is easy to see now that these equations for $\tilde{u}_0^i$ and $\tilde{u}_i$ are same as those for $u_0^i$ and $u$ in the case of the $(m+2)$ dimensional stars with $m = 2$. Hence, further analysis of these equations and the consequent results are also the same. In particular, $\tilde{u}_0^i$ and $\tilde{u}_i$ obey equation (46). And, their solutions are non oscillatory if the values of the anisotropy parameter $\eta$ and, equivalently, of $\frac{w}{w_\pi}$ lie in the ranges given in equation (48). Thus, since $m = 2$, certain amount of anisotropy, $\eta \geq \frac{7}{8}$, is needed to obtain the non oscillatory behaviour of the perturbations around the singular solutions in the asymptotic region.

Consider now the equations for $\tilde{u}_i^j$. They are not needed for present purposes but we analyse them for the sake of completeness. Since $\tilde{s}_i^j = 0$ and $R_I = \frac{R}{4}$, they are given by

$$\tilde{u}_i^j + \alpha \tilde{u}_i^j = \frac{R}{4} \sum_I \tilde{c}^I y^I .$$

(61)

Evaluating the sum $\sum_I \tilde{c}^I y^I$ using equations (59) gives

$$\sum_I \tilde{c}^I = 0 \implies \sum_I \tilde{c}^I y^I = \left( \frac{m \tilde{c}}{w_\pi} \right) \sum_I \tilde{c}^I \left( \sum_{j \in I} u^j \right) .$$

For $I : (2, 2', 5, 5')$, the sums $\sum_{j \in I} u^j$ are given by $(u^1 + u^2)$, $(u^3 + u^4)$, $(u^1 + u^3 + u^5 + u^6 + u^7)$, and $(u^2 + u^4 + u^5 + u^6 + u^7)$. Using equations (59) again gives

$$\sum_I \tilde{c}^I y^I = 2 \tilde{c}^I (u^1 - u^4)$$

$$\sum_I \tilde{c}^I y^I = 2 \tilde{c}^I (u^2 - u^3)$$

$$\sum_I \tilde{c}^I y^I = 2 \tilde{c}^I (u^3 - u^2)$$

$$\sum_I \tilde{c}^I y^I = 2 \tilde{c}^I (u^4 - u^1)$$

$$\sum_I \tilde{c}^I y^I = 2 \tilde{c}^I (u^5 + u^6 + u^7) .$$

(62)

\footnote{In [37], the factor $\frac{m \tilde{c}}{w_\pi}$ appearing below was omitted inadvertently. Consequently, the equation for $(*)_1$ given there is incorrect upto this factor in the last term. Equation (63) for $(*)_1$ given below is the correct one.}
It then follows, after some manipulations involving $u$'s, $\tilde{u}$'s, equations (61) and (62), that

\[(*)_{1}\tilde{\sigma} + \alpha(*)_{1}\tilde{\sigma} + \frac{2 R \tilde{c}^2}{w} (*)_{1} = 0\]  

(63)

where $(*_{1}) = (\tilde{u}^{1} - \tilde{u}^{4})$, $\tilde{u}^{2} - \tilde{u}^{3}$, and $(u^{5} + u^{6} + u^{7})$; and

\[(*)_{2}\tilde{\sigma} + \alpha(*)_{2}\tilde{\sigma} = 0\]  

(64)

where $(*_{2}) = (\tilde{u}^{1} + \tilde{u}^{4})$, $\tilde{u}^{2} + \tilde{u}^{3}$, $(\tilde{u}^{5} + \tilde{u}^{6} - 2\tilde{u}^{7})$, and $(\tilde{u}^{5} - 2\tilde{u}^{6} + \tilde{u}^{7})$.

### 7. Conclusion

We now conclude with a summary and a discussion of some of the issues that require further study.

A brief summary of the present paper is as follows. Unitarity of evolution in gravitational collapses implies that horizonless objects must exist which can be macroscopic and must be stable. In this paper, with such objects in mind, we studied the effects of anisotropy of pressures on the stability of stars. The stars are assumed to be static and spherically symmetric in the non compact space, to have suitable isometries along the compact directions, and to be made up of constituents with linear equations of state. Studying the singular solutions and asymptotic perturbations around them, we obtained the criteria for the perturbations to be non oscillatory.

We studied stars in four or higher dimensional spacetime with no compact directions, and also two examples of stars in M theory made up of stacks of (intersecting) two branes and five branes. A variety of other examples may also be studied using the present formulation. We find that non oscillatory perturbations around the singular solutions are possible if an appropriate amount of anisotropy is present. The details are given in the paper.

The behaviour of these perturbations lead to corresponding asymptotic behaviour in the mass – central density profile of a star enclosed in a box of radius $r_{*}$. The non oscillatory behaviour of the perturbations are likely to indicate stability. In that case, singular solutions will correspond to stable configurations, and give the mass – radius relation $M(r_{*}) \sim r_{*}^{m-1}$, in the limit of large central density or large radius.

Our results suggest that it may be possible to have stable horizonless objects in four or any higher dimensions, and that anisotropic pressures may
play a crucial role in ensuring their stability. Although much remains to be
done, it is worth emphasising that these are important results because they
bear on the horizonless objects which are implied by unitarity in lieu of black
holes, and they point out a necessary ingredient for their stability. To actually
construct such objects, however, requires detailed understanding of many
issues such as the nature of the constituents and the physical mechanisms
that may provide the required amount of anisotropy. We now discuss some
of these issues which may be studied further.

To show the stability of the equilibrium configurations given by the mass
– central density profile whose asymptotic behaviour is monotonic non oscil-
latory, one may follow Chavanis and show that the equilibrium configurations
in the mass – central density profile become unstable beyond the first maxi-
mum; then construct the entire profile and show that it remains monotonic
and increasing in both the non asymptotic and asymptotic regions. Although
desirable, we are unable do any of this since the detailed properties of the
constituents are not known which cause the required amount of anisotropy.
Proving these things may give valueable insights into the horizonless objects.

One may try to use scalar fields as in [38] to produce anisotropy and
thereby to construct a stable horizonless object. The stars constructed in
these works are unstable for a sufficiently high mass. This may be because
the scalar field potentials are not tailored to generate anisotropic linear equa-
tions of state. In the cosmological context, a linear equation of state can be
mimicked using a scalar field with an exponential potential. One may simi-
larly try to mimic anisotropic linear equations of state with scalar fields with
appropriate potentials and then study the resulting stars.

There is a vast body of works devoted to construction of anisotropic stars.
A small sample of them is given in [66] – [73]. Typically, in these works, it is
found that anisotropy affects stability properties, and that instability sets in
for a sufficiently high mass. The nature of their ansatzes for anisotropy is very
different from ours and, hence, there seems to be no discernible contradic-
tion between their results and ours.

There is, however, a distinct possibility that anisotropy of pressures as
found here is a necessary condition for stability but it may not be suffi-
cient; other ingredient(s) may also be needed. There are two reasons for
entertaining this possibility. First, in the works [19] – [24] on horizonless ob-
jects, anisotropy of the pressures was found to be an important ingredient;
but a positive cosmological constant, more generally matter with negative
pressures, residing in the inner region was also required in an essential way. Second, if there is a similarity between the singularities in gravitational collapses and in cosmological big bang/crunch evolutions then the mechanisms resolving these singularities may also be expected to be similar. Usually, such mechanisms involve new types of matter and/or interactions. For example, matter violating null energy conditions can cause a bounce and resolve big bang/crunch singularity. Then, going by the similarities, one may also expect similar ingredients to play a role in stabilising gravitational collapses. Thus it seems possible that other ingredient(s), besides anisotropy, may still be needed for constructing stable horizonless objects. Nevertheless, using the anisotropy criteria given here, one may try to construct such objects and see if, and which, further ingredients are needed.

At a technical level, although our formulations included multi component fluids, we only considered cases where $\mathcal{N} = 1$, or chose $w^I_\pi = w_\pi$ and $w^I = w$ for all $I$ when $\mathcal{N} = 4$. It may be worthwhile to investigate situations where more than one component become crucial and play a significant role. For example, is it possible that one component dominates the inner regions and another the outer regions but such that, together, they lead to stability against collapse?

References

[1] S. L. Shapiro and S. A. Teukolsky, 
Black holes, white dwarfs and neutron stars: the physics of compact objects, John Wiley & Sons (1983).

[2] S. B. Giddings, “Black holes and massive remnants,” 
Phys. Rev. D 46, 1347 (1992) [hep-th/9203059].

[3] S. Hossenfelder and L. Smolin, 
“Conservative solutions to the black hole information problem,” 
Phys. Rev. D 81, 064009 (2010) [arXiv:0901.3156 [gr-qc]].

[4] D. N. Page, “Average entropy of a subsystem,” 
Phys. Rev. Lett. 71, 1291 (1993) [gr-qc/9305007].
[5] D. N. Page, “Information in black hole radiation,” 
Phys. Rev. Lett. 71, 3743 (1993) [hep-th/9306083];

[6] D. N. Page, “Time Dependence of Hawking Radiation Entropy,” 
JCAP 09, 028 (2013) [arXiv:1301.4995 [hep-th]].

[7] Siddhartha Sen, “Average entropy of a subsystem,” 
Phys. Rev. Lett. 77, 1 (1996) [hep-th/9601132].

[8] S. D. Mathur, “The Fuzzball proposal for black holes: An Elementary review,” Fortsch. Phys. 53, 793 (2005) [hep-th/0502050].

[9] K. Skenderis and M. Taylor, “The fuzzball proposal for black holes,” 
Phys. Rept. 467, 117 (2008) [arXiv:0804.0552 [hep-th]].

[10] B. D. Chowdhury and A. Virmani, “Modave Lectures on Fuzzballs and Emission from the D1-D5 System,” arXiv:1001.1444 [hep-th].

[11] A. Almheiri, D. Marolf, J. Polchinski and J. Sully, 
“Black Holes: Complementarity or Firewalls?,” 
JHEP 02, 062 (2013) [arXiv:1207.3123 [hep-th]].

[12] S. L. Braunstein, “Black hole entropy as entropy of entanglement, or it’s curtains for the equivalence principle”, arXiv:0907.1190v1 [quant-ph], published as [13].

[13] S. L. Braunstein, S. Pirandola and K. Zyczkowski, 
“Better Late than Never: Information Retrieval from Black Holes,” 
Phys. Rev. Lett. 110, 101301 (2013) [arXiv:0907.1190 [quant-ph]].

[14] S. Kalyana Rama, “Remarks on Black Hole Evolution a la Firewalls and Fuzzballs,” arXiv:1211.5645 [hep-th].

[15] S. Kalyana Rama, “Massive Compact Objects in a Quantum Theory of Gravity,” arXiv:1409.3462 [gr-qc].

[16] S. Kalyana Rama, “Singularity Resolution + Unitary Evolution + Horizon = Firewall ?,” arXiv:1412.8629 [hep-th].
[17] S. Kalyana Rama and S. Ghosh, “Short distance repulsive gravity as a consequence of nontrivial PPN parameters Beta and gamma,” Phys. Lett. B 383, 31 (1996) [hep-th/9505167].

[18] A. Bagchi and S. Kalyana Rama, “Cosmology and static spherically symmetric solutions in D-dimensional scalar tensor theories: Some Novel features,” Phys. Rev. D 70, 104030 (2004) [gr-qc/0408030].

[19] G. Chapline, E. Hohlfeld, R. B. Laughlin and D. I. Santiago, “Quantum phase transitions and the breakdown of classical general relativity,” Philosophical Magazine B 81, 235 (2001) [gr-qc/0012094].

[20] R. B. Laughlin, “Emergent relativity,” Int. J. Mod. Phys. A 18, 831 (2003) [gr-qc/0302028].

[21] P. O. Mazur and E. Mottola, “Gravitational condensate stars: An alternative to black holes,” gr-qc/0109035.

[22] P. O. Mazur and E. Mottola, “Gravitational vacuum condensate stars,” Proc. Nat. Acad. Sci. 101, 9545 (2004) [gr-qc/0407075].

[23] M. Visser and D. L. Wiltshire, “Stable gravastars: An Alternative to black holes?,” Class. Quant. Grav. 21, 1135 (2004) [gr-qc/0310107].

[24] C. Cattoen, T. Faber and M. Visser, “Gravastars must have anisotropic pressures,” Class. Quant. Grav. 22, 4189 (2005) [gr-qc/0505137].

[25] G. Dvali and C. Gomez, “Black Hole’s Quantum N-Portrait,” Fortsch. Phys. 61, 742 (2013) [arXiv:1112.3359 [hep-th]].

[26] G. Dvali and C. Gomez, “Landau-Ginzburg Limit of Black Hole’s Quantum Portrait: Self Similarity and Critical Exponent,” Phys. Lett. B 716, 240 (2012) [arXiv:1203.3372 [hep-th]].

[27] G. Dvali and C. Gomez, “Black Holes as Critical Point of Quantum Phase Transition,” Eur. Phys. J. C 74, 2752 (2014) [arXiv:1207.4059 [hep-th]].

[28] D. N. Page, “Hyper-Entropic Gravitational Fireballs (Grireballs) with Firewalls,” JCAP 04, 037 (2013) [arXiv:1211.6734 [hep-th]].
[29] P. -H. Chavanis, “Relativistic stars with a linear equation of state: analogy with classical isothermal spheres and black holes,” Astron. Astrophys. 483, 673 (2008) [arXiv:0707.2292 [astro-ph]].

[30] P. -H. Chavanis, “Gravitational instability of finite isothermal spheres in general relativity. Analogy with neutron stars,” Astron. Astrophys. 381, 709 (2002) [astro-ph/0108230].

[31] V. Vaganov, “Self-gravitating radiation in AdS(d),” arXiv:0707.0864 [gr-qc].

[32] J. Hammersley, “A critical dimension for the stability of perfect fluid spheres of radiation”, Class. Quant. Grav. 25, 205010 (2008) [arXiv:0707.0961 [hep-th]].

[33] A. D. Rendall and B. G. Schmidt, “Existence and properties of spherically symmetric static fluid bodies with a given equation of state,” Class. Quant. Grav. 8, 985 (1991).

[34] S. Chandrasekhar, “A limiting case of relativistic equilibrium”, pg 185 – 199, Studies in Relativity (Papers in honour of J. L. Synge), edited by L. O’Raifeartaigh, Clarendon Press, Oxford (1972).

[35] C. W. Misner and H. S. Zapolsky, “High-Density Behavior and Dynamical Stability of Neutron Star Models,” Phys. Rev. Lett. 12, 635 (1964).

[36] S. Yabushita, “On the analogy between neutron star models and isothermal gas spheres and their general relativistic instability,” Mon. Not. Roy. Astron. Soc. 167, 95 (1974).

[37] S. Kalyana Rama, “Stars in M theory (made up of intersecting branes),” Phys. Rev. D 89, 084019 (2014) [arXiv:1312.7762 [hep-th]].

[38] M. Gleiser, “Stability of Boson Stars,” Phys. Rev. D 38, 2376 (1988) [Phys. Rev. D 39, 1258 (1989)].

[39] M. Gleiser and R. Watkins, “Gravitational Stability of Scalar Matter,” Nucl. Phys. B 319, 733 (1989).
[40] P. Jetzer, “Boson stars,” Phys. Rept. 220, 163 (1992).

[41] A. R. Liddle and M. S. Madsen, “The Structure and formation of boson stars,” Int. J. Mod. Phys. D 1, 101 (1992).

[42] F. E. Schunck and E. W. Mielke, “General relativistic boson stars,” Class. Quant. Grav. 20, R301 (2003) [arXiv:0801.0307 [astro-ph]].

[43] S. L. Liebling and C. Palenzuela, “Dynamical Boson Stars,” Living Rev. Rel. 15, 6 (2012) [arXiv:1202.5809 [gr-qc]].

[44] P. Boonserm, T. Ngampitipan and M. Visser, “Modelling anisotropic fluid spheres in general relativity,” arXiv:1501.07044 [gr-qc].

[45] G. T. Horowitz, J. M. Maldacena and A. Strominger, “Nonextremal black hole microstates and U duality,” Phys. Lett. B 383, 151 (1996) [hep-th/9603109].

[46] G. T. Horowitz, D. A. Lowe and J. M. Maldacena, “Statistical entropy of nonextremal four-dimensional black holes and U duality,” Phys. Rev. Lett. 77, 430 (1996) [hep-th/9603195].

[47] U. H. Danielsson, A. Guijosa and M. Kruczenski, “Brane anti-brane systems at finite temperature and the entropy of black branes,” JHEP 09, 011 (2001) [hep-th/0106201].

[48] U. H. Danielsson, A. Guijosa and M. Kruczenski, “Black brane entropy from brane - anti-brane systems,” Rev. Mex. Fis. 49S2, 61 (2003) [gr-qc/0204010].

[49] A. Guijosa, H. H. Hernandez Hernandez and H. A. Morales Teotl, “The Entropy of the rotating charged black three-brane from a brane anti-brane system,” JHEP 03, 069 (2004) [hep-th/0402158].

[50] O. Saremi and A. W. Peet, “Brane - anti-brane systems and the thermal life of neutral black holes,” Phys. Rev. D 70, 026008 (2004) [hep-th/0403170].

[51] O. Bergman and G. Lifschytz, “Schwarzschild black branes from unstable D-branes,” JHEP 04, 060 (2004) [hep-th/0403189].
[52] S. Kalyana Rama, “A Description of Schwarzschild black holes in terms of intersecting M-branes and anti-branes,” Phys. Lett. B 593, 227 (2004) [hep-th/0404026].

[53] G. Lifschytz, “Charged black holes from near extremal black holes,” JHEP 09, 009 (2004) [hep-th/0405042].

[54] S. Kalyana Rama and S. Siwach, “A Description of multi charged black holes in terms of branes and antibranes,” Phys. Lett. B 596, 221 (2004) [hep-th/0405084].

[55] M. Cvetic and A. A. Tseytlin, “Nonextreme black holes from nonextreme intersecting M-branes,” Nucl. Phys. B 478, 181 (1996) [hep-th/9606033].

[56] B. D. Chowdhury and S. D. Mathur, “Fractional Brane State in the Early Universe,” Class. Quant. Grav. 24, 2689 (2007) [hep-th/0611330].

[57] S. D. Mathur, “What is the state of the Early Universe?,” J. Phys. Conf. Ser. 140, 012009 (2008) [arXiv:0803.3727 [hep-th]].

[58] S. Kalyana Rama, “Consequences of U dualities for Intersecting Branes in the Universe,” Phys. Lett. B 656, 226 (2007) [arXiv:0707.1421 [hep-th]].

[59] R. Argurio, F. Englert and L. Houart, “Intersection rules for p-branes,” Phys. Lett. B 398, 61 (1997) [hep-th/9701042].

[60] N. Ohta, “Intersection rules for nonextreme p-branes,” Phys. Lett. B 403, 218 (1997) [hep-th/9702164].

[61] J. D. Edelstein, L. Tataru and R. Tatar, “Rules for localized overlappings and intersections of p-branes,” JHEP 06, 003 (1998) [hep-th/9801049].

[62] S. Kalyana Rama, “M-theory branes: U-duality properties and a class of new static solutions,” Phys. Rev. D 88, 044007 (2013) [arXiv:1304.6537 [hep-th]].
[63] S. Bhowmick, S. Digal and S. Kalyana Rama, “Stabilisation of Seven (Toroidal) Directions and Expansion of the remaining Three in an M theoretic Early Universe Model,” Phys. Rev. D 79, 101901 (2009) [arXiv:0810.4049 [hep-th]].

[64] S. Bhowmick and S. Kalyana Rama, “10 + 1 to 3 + 1 in an Early Universe with mutually BPS Intersecting Branes,” Phys. Rev. D 82, 083526 (2010) [arXiv:1007.0205 [hep-th]].

[65] S. Bhowmick, “Study of Early Universe in an M Theoretic Model,” PhD thesis, arXiv:1201.5712 [hep-th].

[66] K. Dev and M. Gleiser, “Anisotropic stars: Exact solutions,” Gen. Rel. Grav. 34, 1793 (2002) [astro-ph/0012265].

[67] K. Dev and M. Gleiser, “Anisotropic stars. 2. Stability,” Gen. Rel. Grav. 35, 1435 (2003) [gr-qc/0303077].

[68] K. Dev and M. Gleiser, “Anisotropic stars: Exact solutions and stability,” Int. J. Mod. Phys. D 13, 1389 (2004) [astro-ph/0401546].

[69] M. K. Mak and T. Harko, “Anisotropic stars in general relativity,” Proc. Roy. Soc. Lond. A 459, 393 (2003) [gr-qc/0110103].

[70] B. V. Ivanov, “Maximum bounds on the surface redshift of anisotropic stars,” Phys. Rev. D 65, 104011 (2002) [gr-qc/0201090].

[71] K. Lake, “Galactic potentials,” Phys. Rev. Lett. 92, 051101 (2004) [gr-qc/0302067].

[72] L. Herrera, J. Ospino and A. Di Prisco, “All static spherically symmetric anisotropic solutions of Einstein’s equations,” Phys. Rev. D 77, 027502 (2008) [arXiv:0712.0713 [gr-qc]].

[73] L. Herrera and W. Barreto, “General relativistic polytropes for anisotropic matter: The general formalism and applications,” Phys. Rev. D 88, 084022 (2013) [arXiv:1310.1114 [gr-qc]].