Filtering problems with exponential criteria for
general Gaussian signals

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Abstract

The explicit solution of the discrete time filtering problems with
exponential criteria for a general Gaussian signal is obtained through
an approach based on a conditional Cameron-Martin type formula.
This key formula is derived for conditional expectations of exponen-
tials of some quadratic forms of Gaussian sequences. The formula
involves conditional expectations and conditional covariances in some
auxiliary optimal risk-neutral filtering problem which is used in the
proof. Closed form recursions of Volterra type for these ingredients
are provided. Particular cases for which the results can be further
elaborated are investigated.

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1 Introduction

The linear exponential Gaussian (LEG for short) filtering problem, i.e., with an exponential cost criteria (see the definition (6) below), and the so called risk-sensitive (RS for short) filtering problem (see [3] and the statement (36) below) have been given a great deal of interest over the last decades. Numerous results have been already reported in specific models, specially around Markov models, but, as far as we know, without exhibiting the relationship between these two problems. See, e.g., Whittle [16]-[17], Speyer et al. [15], Elliott et al. [2], [5], [7] and Bensoussan and van Schuppen [1] for contributions on this subject and related LEG and RS control problems. Therein the notion of “information state” has been introduced without any clear probabilistic meaning for auxiliary processes which are involved, even in the Gauss-Markov case. Moreover, the method proposed in [2] does not work in a non Markovian situation. In our paper [10], we have solved the LEG and RS filtering problems for general Gaussian signal processes in continuous time and in the particular setting where the functional in the exponential is a singular quadratic functional. Moreover we have proved that actually in this case the solutions coincide. In our paper [11] we have solved the LEG and RS filtering problems for Gauss-Markov processes but with a nonsingular quadratic functional in the exponential. In this setting we have proposed an example to show that the solutions may be different. On the other hand, the general solution for the optimal risk-neutral linear filtering problem and a Cameron-Martin type formula for general Gaussian sequences have been obtained in [9]. It seems natural to use the approach proposed in [9] and [10] to derive the solution of the LEG and RS filtering problems for general Gaussian signals in discrete time setting, to precise their link and also to give a probabilistic interpretation for the ingredients of the “information state”.

In the present paper we are interested in the explicit solution of the Linear Exponential Gaussian (LEG) and Risk Sensitive (RS) filtering problems for general Gaussian signals. Namely we deal with a signal-observation model
\((X_t, Y_t)_{t \geq 1}\), where the signal \(X = (X_t)_{t \geq 1}\) is an arbitrary Gaussian sequence with mean \(m = (m_t, t \geq 1)\) and covariance \(K = (K(t, s), t \geq 1, s \geq 1)\), i.e.,

\[
\mathbb{E}X_t = m_t, \quad \mathbb{E}(X_t - m_t)(X_s - m_s) = K(t, s), \quad t \geq 1, \ s \geq 1,
\]

and, for some sequence \(A = (A_t, t \geq 1)\) of the real numbers, the observation process \(Y = (Y_t, t \geq 1)\) is given by

\[
Y_t = A_t X_t + \varepsilon_t,
\]

where \(\varepsilon = (\varepsilon_t)_{t \geq 1}\) is a sequence of i.i.d. \(\mathcal{N}(0, 1)\) random variables and \(\varepsilon\) and \(X\) are independent.

Suppose that only \(Y\) is observed and for a given real number \(\mu\) and a fixed sequence \((Q_t)_{t \geq 1}\) of nonnegative real numbers, one wishes to minimize with respect to \(h : h_t \in \mathcal{Y}_t, t \geq 1\) the quantity:

\[
\mathbb{E} \mu \exp \left\{ \frac{\mu}{2} \sum_{t=1}^{T} (X_t - h_t)^2 Q_t \right\},
\]

where \((\mathcal{Y}_t)\) is the natural filtration of \(Y\), i.e., \(\mathcal{Y}_t = \sigma(\{Y_u, 1 \leq u \leq t\})\) and \(h_t \in \mathcal{Y}_t\) means that \(h_t\) is \(\mathcal{Y}_t\)-measurable.

Note that, according to the sign of the real parameter \(\mu\), there are two different cases for this linear exponential Gaussian (LEG) filtering problem (the terminology is taken from the linear exponential Gaussian optimal control problem):

- \(\mu < 0\), called risk-prefering filtering problem,
- \(\mu > 0\), called the risk-averse filtering problem.

It is well known (see, e.g., [15] for the Markov case) that the solution to this problem is not the conditional expectation of \(X_t\) given the \(\sigma\)-field \(\mathcal{Y}_t\). Our first aim is to show that the solution can be completely explicit: the characteristics of the optimal solution are obtained as the solution of a closed form system of Volterra type equations which actually reduce to the equations known also for the RS setting when the signal process \(X\) is Gauss-Markov (see, e.g., [14]). Our second aim is to give the probabilistic interpretation of this optimal solution in terms of an auxiliary risk-neutral filtering problem. Actually, we extend the filtering approach initiated in [9] and [10] for one-dimensional processes, to obtain a conditional Cameron-Martin type formula
for the *conditional Laplace transform* of a quadratic functional of the involved process. Namely, we give an explicit representation for the random variable

\[ I_T = \mathbb{E}\left( \exp\left\{ \frac{\mu}{2} \sum_{s=1}^{T} (X_s - h_s)^2 Q_s \right\} / Y_T \right), \]  

\( (3) \)

where \( h_s \in Y_s, s \geq 1 \).

The paper is organized as follows. In Section 2 we derive the solution of the LEG filtering problem: explicit recursive equations, involving the covariance function of the filtered process, are obtained. In particular, in Section 2.1 an appropriate auxiliary risk-neutral filtering problem is matched to that of deriving the key Cameron-Martin type formula. The solution of this auxiliary filtering problem is discussed in Section 2.2. In Section 3 we investigate some specific cases where the results can be further elaborated. In Section 4 we discuss the relationship between LEG and RS filtering problems. Section 5 is devoted to the interpretation for the ingredients of the “information state”. Finally, Sections 6 and 7 are devoted to a more general case, namely when the particular structure of the observation sequences \((Y_t)_{t \geq 1}\) is not specified.

## 2 Solution of the LEG filtering problem

Let us introduce the following condition \((C_\mu)\):

\((C_\mu)\) the equation

\[ \overline{\gamma}(t, s) = K(t, s) - \sum_{l=1}^{s-1} \overline{\gamma}(t, l)\overline{\gamma}(s, l) \frac{S_l}{1 + S_l\overline{\gamma}_l}, \quad S_l = A_l^2 - \mu Q_l \]  

\( (4) \)

has a unique and bounded solution on \( \{(t, s) : 1 \leq s \leq t \leq T\} \), such that \( \overline{\gamma}_l = \overline{\gamma}(l, l) \geq 0, l \geq 1 \) and moreover

\[ 1 + S_l\overline{\gamma}_l > 0, l \geq 1. \]

**Remark 1.** Notice that for all \( \mu \) negative the condition \((C_\mu)\) is satisfied and if \( \mu \) is positive, the condition \((C_\mu)\) is satisfied for \( \mu \) sufficiently small, for example, those such that for any \( t \leq T A_t^2 - \mu Q_t \) is nonnegative.

The first result is the following
**Theorem 1.** Suppose that the condition \((C_\mu)\) is satisfied. Let \((\overline{h}_t)_{t \geq 1}\) be the solution of the following equation:

\[
\overline{h}_t = m_t + \sum_{l=1}^{t} A_l \overline{\gamma}(t, l)(Y_l - A_l \overline{h}_l),
\]

where \(\overline{\gamma} = (\overline{\gamma}(t, s), 1 \leq s \leq t \leq T)\) is the unique solution of equation (4). Then \((\overline{h}_t)_{t \geq 1}\) is the solution of the LEG filtering problem, i.e.,

\[
\overline{h}_t = \arg\min_{h_t: h_t \in Y_t, t \geq 1} \mathbb{E} \mu \exp \left\{ \frac{\mu}{2} \sum_{t=1}^{T} (X_t - h_t)^2 Q_t \right\}. \tag{6}
\]

Moreover, the corresponding optimal risk is given by

\[
\mathbb{E} \mu \exp \left\{ \frac{\mu}{2} \sum_{t=1}^{T} (X_t - \overline{h}_t)^2 Q_t \right\} = \mu \prod_{t=1}^{T} \left[ \frac{1 + S_t \gamma_t}{1 + A_t^2 \gamma_t} \right]^{-1/2}.
\]

Theorem 1 is a direct consequence of results of Section 2.1. Its proof will be given at the end of Section 2.1.

**Remark 2.**

- Note that equation (5) is really recursive equation and it can be rewritten in the equivalent form:

\[
\overline{h}_t = \frac{1}{1 + A_t^2 \gamma_t} \left[ m_t + \sum_{l=1}^{t-1} A_l \overline{\gamma}(t, l)(Y_l - A_l \overline{h}_l) + A_l \overline{\gamma}_l Y_l \right],
\]

- It is worth emphasizing that taking \(\mu = 0\) in equation (4), one gets through equation (3) the solution \(\overline{h}\) of the risk-neutral filtering problem of the signal \(X\) given the observation \(Y\), i.e., \(\overline{h}_t = \mathbb{E}(X_t/ Y_t)\) (see, e.g., [9]).

### 2.1 Conditional version of a Cameron-Martin formula

The proof of Theorem 1 is based on the conditional version of the Cameron–Martin formula which provides the conditional expectation \(\mathcal{I}_t\) defined by (3). Let

\[
J_t = \exp \left\{ -\frac{1}{2} \sum_{s=1}^{t} (X_s - h_s)^2 Q_s \right\}. \tag{7}
\]
Then \( I_t = \pi_t(J_t) \), where for any random variable \( \eta \) such that \( \mathbb{E}|\eta| < +\infty \), the notation \( \pi_t(\eta) \) is used for the conditional expectation of \( \eta \) given the \( \sigma \)-field \( \mathcal{Y}_t = \sigma\{Y_s, 1 \leq s \leq t\} \),

\[
\pi_t(\eta) = \mathbb{E}(\eta/\mathcal{Y}_t).
\]

**Proposition 2.** Suppose that the condition \((C_\mu)\) is satisfied. Let \((\overline{\gamma}(t, s), 1 \leq s \leq t \leq T)\) be the solution of equation (3) and \((Z^h_t, t \geq 1)\) be the solution of the following equation

\[
Z^h_t = m_t - \sum_{l=1}^{t-1} \overline{\gamma}(t, l) \frac{\mu Q_t}{1 + S_l^{-1}} (h_l - Z^h_l) + \sum_{l=1}^{t-1} \overline{\gamma}(t, l) \frac{A_l}{1 + S_l^{-1}} (Y_t - A_t Z^h_l). \tag{8}
\]

Then the following representation of the random variable \( I_T \) defined by (3) holds for any \( T \geq 1 \):

\[
I_T = \prod_{t=1}^{T} \left[ \frac{1 + S_t^{-1}}{1 + A_t^{-1}} \right]^{-1/2} \times \exp \left\{ \frac{\mu Q_t}{2} \frac{1 + A_t^{-1} \overline{\gamma}_t}{1 + S_t^{-1}} \times \left[ h_t - \frac{Z^h_t + A_t^{-1} Y_t}{1 + A_t^{-1} \overline{\gamma}_t} \right]^2 \right\} \times \mathcal{M}_T,
\]

where \((\mathcal{M}_T)_{t \geq 1}\) is a martingale defined by:

\[
\mathcal{M}_T = \prod_{t=1}^{T} \left[ \frac{(1 + A_t^2 \overline{\gamma}_t)}{1 + A_t^{-1} \overline{\gamma}_t} \right]^{1/2} \times \exp \left\{ \frac{A_t}{1 + A_t^{-1} \overline{\gamma}_t} (Z^h_t - \pi_{t-1}(X_t)) \nu_t - \frac{1}{2} \frac{A_t^2}{1 + A_t^{-1} \overline{\gamma}_t} (Z^h_t - \pi_{t-1}(X_t))^2 - \frac{1}{2} \frac{A_t^2 (\gamma_t - \overline{\gamma}_t) \cdot \nu_t^2}{(1 + A_t^{-1} \overline{\gamma}_t)(1 + A_t^2 \overline{\gamma}_t)} \right\}, \tag{9}
\]

in terms of the innovation sequence \((\nu_t)_{t \geq 1}\):

\[
\nu_t = Y_t - A_t \pi_{t-1}(X_t); \quad \pi_{t-1}(X_t) = \mathbb{E}(X_t/\mathcal{Y}_{t-1}),
\]

and of the variances of one-step prediction errors \((\gamma_t)_{t \geq 1}\):

\[
\gamma_t = \mathbb{E}(X_t - \pi_{t-1}(X_t))^2.
\]

**Remark 3.**

1. The probabilistic interpretation of the auxiliary processes \((Z^h_t)\) and \((\overline{\gamma}_t)_{t \geq 1}\) appearing in the Proposition 2 will be clarified below.

2. Proposition 2 reduces to the ordinary Cameron-Martin type formula (cf. Theorem 1 [9]) for \( h \equiv 0 \) when \( A_t = 0, l \geq 1 \) and hence \( X \) and \( Y \) are independent.
Proof of Proposition 2. We will prove Proposition 2 for \( \mu < 0 \), namely \( \mu = -1 \). Then we can replace \( Q \) by \( -\mu Q \) and the statement of Proposition 2 is still valid because of the analytical properties of the involved functions.

The proof of Proposition 2 for \( \mu = -1 \) will be separated into two steps.

I. (Actually it is the discrete time analog for the general filtering theorem.) Since \( h_t \in \mathcal{Y}_t, t \geq 1 \), in the proof we can suppose that \( h \) is a deterministic function. First of all, we claim that for \( J_t \), defined by (7)

\[
\pi_t(J_t) = \left. \pi_{t-1}(J_t, \beta_t^{y_t}) \right|_{y=y_t} \quad (10)
\]

where \( \beta_t^{y_t} = \exp(A_t X_t \varepsilon_t - \frac{1}{2} A_t^2 X_t^2) \).

Indeed, let us introduce the new probability measure \( \hat{P} \), defined by

\[
\frac{d\hat{P}}{dP} = \exp(-A_t X_t \varepsilon_t - \frac{1}{2} A_t^2 X_t^2).
\]

The classical Bayes formula gives that

\[
\pi_t(J_t) = \left. \frac{\hat{\pi}_t(J_t \exp(A_t X_t \varepsilon_t + \frac{1}{2} A_t^2 X_t^2))}{\hat{\pi}_t(\exp(A_t X_t \varepsilon_t + \frac{1}{2} A_t^2 X_t^2))} \right|_{y=y_t} = \frac{\hat{\pi}_t(J_t \exp(A_t X_t Y_t - \frac{1}{2} A_t^2 X_t^2))}{\hat{\pi}_t(\exp(A_t X_t Y_t - \frac{1}{2} A_t^2 X_t^2))},
\]

where \( \hat{\pi}_t(\cdot) \) denotes a conditional expectation with respect to \( \mathcal{Y}_t \) under \( \hat{P} \). Note that under \( \hat{P} \) the distribution of \( (X_s, Y_r)_{s \leq t, r \leq t-1} \) is the same as under \( P \) and \( Y_t \) is a \( \mathcal{N}(0, 1) \) random variable independent of \( (X_s, Y_r)_{s \leq t, r \leq t-1} \).

To understand this point it is sufficient to write the following equality for
the mutual characteristic function with arbitrary real numbers \((\alpha_j, \lambda_j)\):

\[
\hat{E} \exp \left\{ i \sum_{j=1}^{t} \alpha_j X_j + i \sum_{j=1}^{t} \lambda_j Y_j \right\} =
\]

\[
= \mathbb{E} \exp \left\{ i \sum_{j=1}^{t} \alpha_j X_j + i \sum_{j=1}^{t-1} \lambda_j Y_{j+1} + i \lambda_t Y_t - A_t X_t \varepsilon_t - \frac{1}{2} A_t^2 X_t^2 \right\} =
\]

\[
= \mathbb{E} \left( \mathbb{E} \exp \left\{ i \sum_{j=1}^{t} \alpha_j X_j + i \sum_{j=1}^{t-1} \lambda_j Y_{j+1} + i \lambda_t Y_t - A_t X_t \varepsilon_t - \frac{1}{2} A_t^2 X_t^2 \right\} \right| \mathcal{X}_t \right) =
\]

\[
= \mathbb{E} \exp \left\{ i \sum_{j=1}^{t} \alpha_j X_j + i \sum_{j=1}^{t-1} \lambda_j Y_{j+1} + i \lambda_t Y_t - A_t X_t \varepsilon_t - \frac{1}{2} A_t^2 X_t^2 + \frac{1}{2} (i \lambda_t - A_t X_t)^2 \right\} =
\]

\[
e^{-\frac{1}{2} \lambda_t^2 X_t^2} \mathbb{E} \exp \left\{ i \sum_{j=1}^{t} \alpha_j X_j + i \sum_{j=1}^{t-1} \lambda_j Y_{j+1} \right\},
\]

where \(\mathcal{X}_t\) is the \(\sigma\)-field \(\mathcal{X}_t = \sigma(\{X_s, 1 \leq s \leq t\})\). Hence,

\[
\hat{\pi}_t(J_t \exp(A_t X_t Y_t - \frac{1}{2} A_t^2 X_t^2)) =
\]

\[
= \pi_{t-1}(J_t \exp(A_t X_t y - \frac{1}{2} A_t^2 X_t^2))|_{y=Y_t} = \]

\[
= \pi_{t-1}(J_t \beta^y_t)|_{y=Y_t}.
\]

Similarly,

\[
\hat{\pi}_t \left( \exp(A_t X_t y - \frac{1}{2} A_t^2 X_t^2) \right) = \pi_{t-1}(\beta^y_t)|_{y=Y_t},
\]

and hence \((\ref{eq10})\) holds.

**II.** In the second step we will calculate the ratio \(\frac{\mathcal{I}_t}{\mathcal{I}_{t-1}}\) which, due to \((\ref{eq10})\)
can be rewritten as

\[
\frac{\mathcal{I}_t}{\mathcal{I}_{t-1}} = \frac{\pi_t(J_t)}{\pi_{t-1}(J_{t-1})} = \frac{\pi_{t-1}(J_t \beta^y_t)}{\pi_{t-1}(J_{t-1}) \pi_{t-1}(\beta^y_t)|_{y=Y_t}}.
\]

For this aim similarly to what we proposed in \((\ref{eq9})\) and \((\ref{eq10})\) we introduce
the auxiliary processes \((Y^2_t)_{t\geq 1}\) and \((\xi_t)_{t\geq 1}\). Let \(\varepsilon = (\varepsilon_t)_{t\geq 1}\) be a sequence of
i.i.d. \(\mathcal{N}(0, 1)\) random variables independent of \(X\) and define \((Y^2_t, \xi_t)_{t\geq 1}\) by:

\[
Y^2_t = Q_t (X_t - h_t) + \sqrt{Q_t} \varepsilon_t,
\]

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\[
\xi_t = \sum_{s=1}^{t} (X_s - h_s)Y_s^2. \tag{13}
\]

Now the following equality holds:

\[
\frac{\pi_{t-1}(J_t \beta_t^y)}{\pi_{t-1}(J_{t-1})} \bigg|_{y=Y_t} = \frac{\pi_{t-1}(\exp\{-\frac{1}{2}Q_t(X_t - h_t)^2 - \xi_{t-1}\} \beta_t^y)}{\pi_{t-1}(\exp(-\xi_{t-1}))} \bigg|_{y=Y_t},
\]

where \( \bar{\pi}(\cdot) \) stands for a conditional expectation w.r.t. to the \( \sigma \)-field \( \mathcal{Y}_t = \sigma(S_s^2, s \leq t) \) under the initial measure \( \Pi \).

Again the proof of this equality is based on the Bayes formula. Namely, let \( \tilde{\Pi} \) be the new probability measure defined by

\[
\frac{d\tilde{\Pi}}{d\Pi} = \rho_{t-1} = \exp \left\{ -\frac{1}{2} \sum_{s=1}^{t-1} Q_s(X_s - h_s)^2 - \sum_{1}^{t-1} \sqrt{Q_s(X_s - h_s)\xi_s} \right\}. \tag{14}
\]

Then \( J_t \rho_{t-1} = \exp\{-\xi_{t-1} - \frac{1}{2}Q_t(X_t - h_t)^2\} \) and \( J_{t-1} \rho_{t-1} = \exp\{-\xi_{t-1}\}. \)

Thus

\[
\pi_{t-1}(\exp(-\xi_t - \frac{1}{2}Q_t(X_t - h_t)^2) \beta_t^y) \bigg|_{y=Y_t} = \frac{\mathbb{E}(J_t \beta_t^y \rho_{t-1}/\mathcal{Y}_{t-1})}{\mathbb{E}(\rho_{t-1}/\mathcal{Y}_{t-1})} \bigg|_{y=Y_t} = \frac{\mathbb{E}(J_t \beta_t^y \rho_{t-1}/\mathcal{Y}_{t-1})}{\mathbb{E}(J_{t-1} \rho_{t-1}/\mathcal{Y}_{t-1})} \bigg|_{y=Y_t} = \frac{\pi_{t-1}(J_t \beta_t^y)}{\pi_{t-1}(J_{t-1})} \bigg|_{y=Y_t},
\]

where the last equality holds because under the probability measure \( \tilde{\Pi} \) the distribution of \( (X_s, Y_s)_{s \leq t} \) is the same as under the initial measure \( \Pi \) and \( (X_s, Y_s)_{s \leq t-1} \) is independent of \( (Y_s^2)_{s \leq t-1} \).

Finally we have proved the following:

\[
\frac{\pi_{t}(J_t)}{\pi_{t-1}(J_{t-1})} = \frac{\pi_{t-1}(\exp\{-\xi_{t-1} + A_t X_t y - \frac{1}{2}Q_t(X_t - h_t)^2 - \frac{1}{2}A_t^2X_t^2\})}{\pi_{t-1}(\exp(-\xi_{t-1}))\pi_{t-1}(\beta_t^y)} \bigg|_{y=Y_t}. \tag{15}
\]

At this point we will use the conditionally Gaussian properties of \( (X_t, \xi_{t-1}) \) w.r.t. \( \mathcal{Y}_{t-1} \) and Lemma 11.6 \cite{12} which says that for a Gaussian pair \( (U, V) \) with mean values \( m_u, m_v \), variances \( \gamma_u, \gamma_v \) and covariance \( \gamma_{uv} \)
\[ \mathbb{E} \exp \left\{ -\frac{1}{2} DU^2 + \lambda_1 U - \lambda_2 V \right\} = (1 + D\gamma_u)^{-1/2} \times \exp \left\{ -\lambda_2 m_V + \frac{\lambda_2^2}{2} \gamma_V - \frac{1}{2} \cdot \frac{D}{1 + D\gamma_u} (m_V - \lambda_2 \gamma_{uv})^2 + \frac{\lambda_2^2 \gamma_V + 2\lambda_1 (m_V - \lambda_2 \gamma_{uv})}{2(1 + D\gamma_u)} \right\} \], (16)

for any real numbers \( \lambda_1, \lambda_2 \) and \( D \geq 0 \). Indeed, in (15) we will apply this formula to \((U, V) = (X_t, \xi_{t-1})\) given \( \bar{\gamma}_{t-1} \) with

\[ D = S_t = Q_t + A_t^2, \quad \lambda_2 = 1, \quad \lambda_1 = A_t y + Q_t h_t, \]

in the numerator and \( D = \lambda_1 = 1, \quad \lambda_2 = 1 \) in the first factor of the denominator and again to \((U, V) = (X_t, \xi_{t-1})\) given \( Y_{t-1} \) with

\[ D = A_t^2, \quad \lambda_2 = 0, \quad \lambda_1 = A_t y, \]

in the second factor of the denominator.

Collecting the terms as coefficients for \( h_t^2 \) and \( h_t \), we obtain that

\[
\frac{I_t}{I_{t-1}} = \frac{(1 + S_t \bar{\gamma}_t)^{-1/2}}{(1 + A_t^2 \gamma_t)^{-1/2}} \cdot \exp \left\{ -\frac{Q_t}{2} \frac{1 + A_t^2 \gamma_t}{1 + S_t \bar{\gamma}_t} \times \left[ h_t - \frac{Z_t^h + A_t \gamma_t Y_t}{1 + A_t \gamma_t} \right]^2 \right\} \times \exp \left\{ -\frac{A_t^2 (Z_t^h)^2 - A_t^2 \gamma_t Y_t^2}{2(1 + A_t^2 \gamma_t)} + \frac{Y_t Z_t^h A_t}{1 + A_t^2 \gamma_t} + \frac{1}{2} \cdot \frac{A_t^2 \gamma_t (X_t) - 2 A_t \gamma_t (X_t) Y_t - A_t^2 Y_t^2 \gamma_t}{1 + A_t^2 \gamma_t} \right\},
\]

where \( Z_t^h = \bar{\gamma}_{t-1} (X_t) - \bar{\gamma}_{X_t} (t) \) with

\[
\bar{\gamma}_{X_t} (t) = \mathbb{E}[(X_t - \bar{\gamma}_{t-1} (X_t))(\xi_{t-1} - \bar{\gamma}_{t-1} (\xi_{t-1}))/\bar{\gamma}_{t-1}], \quad t \geq 2; \quad \bar{\gamma}_{X_t} (1) = 0. \quad (17)
\]

To finish the proof we just replace \( Y_t \) by \( \nu_t + A_t \pi_{t-1} (X_t) \). Thus in the last exponential term we find:

\[
\exp \left\{ -\frac{\nu_t^2 A_t^2 (\gamma_t - \bar{\gamma}_t)}{2(1 + A_t^2 \gamma_t)(1 + A_t^2 \gamma_t)} + \frac{Z_t^h - \pi_{t-1} (X_t)}{1 + A_t^2 \gamma_t} A_t \nu_t - \frac{1}{2} \cdot \frac{A_t^2}{1 + A_t^2 \gamma_t} (Z_t^h - \pi_{t-1} (X_t))^2 \right\},
\]

which gives the Proposition.
Remark 4. 1. Note that now the probabilistic interpretation of the ingredients $\gamma_t$ and $Z^h_t$ is clarified for negative $\mu$. Namely, $\gamma_t = \mathbb{E}(X_t - \pi_{t-1}(X_t))^2$, and $Z^h_t = \pi_{t-1}(X_t) - \gamma_{X_t}(t)$, but when $\mu$ is positive, there is no such connection anymore.

2. Observe that actually $\pi_{t-1}(X_t)$ and $\gamma_{X_t}(t)$ are $\bar{Y}_t$-measurable, but the difference $Z^h_t = \pi_{t-1}(X_t) - \gamma_{X_t}(t)$ is $Y_t$-measurable.

Proof of Theorem 1 The statement of Theorem 1 is the direct consequence of Proposition 2.1. Indeed, we claim that the following chain of inequalities holds for any $h : h_t \in \mathcal{Y}_t, t \geq 1$:

$$
\mathbb{E} \mu \exp \left\{ \frac{\mu}{2} \sum_{t=1}^{T} (X_t - h_t)^2 Q_t \right\}
= \mathbb{E} \left[ \mathbb{E} \mu \left( \exp \left\{ \frac{\mu}{2} \sum_{t=1}^{T} (X_t - h_t)^2 Q_t \right\} \bigm/ \mathcal{Y}_T \right) \right]
= \mu \mathbb{E} \prod_{t=1}^{T} \left[ \frac{1 + S_t \gamma_{t}}{1 + A_t^2 \gamma_{t}} \right]^{-1/2} \times \exp \left\{ \frac{\mu}{2} Q_t \frac{1 + A_t^2 \gamma_{t}}{1 + S_t \gamma_{t}} \times \left[ h_t - \frac{Z^h_t + A_t \gamma_{Y_t}}{1 + A_t^2 \gamma_{t}} \right]^2 \right\} \times \mathcal{M}_T,
$$

$$(a) \geq \prod_{t=1}^{T} \left[ \frac{1 + S_t \gamma_{t}}{1 + A_t^2 \gamma_{t}} \right]^{-1/2} \mu \mathbb{E} \mathcal{M}_T
= (b) \mu \prod_{t=1}^{T} \left[ \frac{1 + S_t \gamma_{t}}{1 + A_t^2 \gamma_{t}} \right]^{-1/2} \mu \mathbb{E} \mathcal{M}_T.
$$

Of course under condition $(C_\mu)$, since the term in the last line is finite, it is sufficient to consider the case:

$$
\mathbb{E} \mu \exp \left\{ \frac{\mu}{2} \sum_{t=1}^{T} (X_t - h_t)^2 Q_t \right\} < \infty,
$$

which gives the first equality. Inequality $(a)$ follows directly from Proposition 2.1. Equality $(b)$ is a direct consequence of $(16)$ which gives that $\mathbb{E} \mathcal{M}_T = 1$. Now, to obtain the lower bound we must take

$$
\pi_t = \frac{Z^h_t + A_t \gamma_{Y_t}}{1 + A_t^2 \gamma_{t}}, t \geq 1,
$$

11
or equivalently

\[ \hat{h}_t = Z_h^\pi + \frac{A_t \gamma_t}{1 + A_t^2 \gamma_t} (Y_t - A_t Z_h^\pi), \quad t \geq 1, \]

where \( Z^h \) is the solution of equation (8), which means that

\[ Z^\pi_t = m_t + \sum_{l=1}^{t-1} \frac{\gamma(t,l) A_l}{1 + A_l^2 \gamma_l} [Y_l - A_l Z^\pi_l], \]

and hence

\[ \hat{h}_t = m_t + \sum_{l=1}^{t} \frac{\gamma(t,l) A_l}{1 + A_l^2 \gamma_l} [Y_l - A_l Z^\pi_l] = m_t + \sum_{l=1}^{t} A_l \gamma(t,l) (Y_l - A_l \hat{h}_t). \]

Thus \( \tilde{h} \) is the unique solution of equation (5). Finally for \( \bar{h} \) the lower bound is attained.

**Remark 5.**

1. It is worth emphasizing that the process \( \tilde{Z}^h_t = Z_h^\pi + \frac{A_t \gamma_t Y_t}{1 + A_t^2 \gamma_t} \) is the solution of the following recursive equation:

\[ \tilde{Z}^h_t = m_t - \sum_{l=1}^{t-1} \frac{\gamma(t,l) A_l}{1 + A_l^2 \gamma_l} (h_l - \tilde{Z}^h_l) + \sum_{l=1}^{t} \frac{\gamma(t,l) A_l (Y_l - A_l \tilde{Z}^h_l)}{1 + A_l^2 \gamma_l}, \quad (19) \]

and hence the equality \( \bar{h}_t = \tilde{Z}^\pi_t \) implies immediately the equation (8) for \( \bar{h} \). This process \( \tilde{Z}^h \) also has a probabilistic interpretation as well as \( \gamma_t = \frac{\gamma_t}{1 + A_t^2 \gamma_t} \). This interpretation will be given in Section 5.

### 2.2 Solution of the auxiliary filtering problems

Here, for an arbitrary Gaussian sequence \( X \), we deal with the one-step prediction and filtering problems of the signals \( X \) and \( \xi \) given by (13) respectively from the observation of \( \bar{Y} = (Y, Y^2) \) defined in (1) and (12). Actually, we follow the ideas proposed in our paper [9]. Recall that the solutions can be reduced to equations for the conditional moments. The following statement provides the equations for the characteristics which give the solution of the prediction problem and the equation for the other quantity \( \bar{\pi}_{t-1}(X_t) - \bar{\gamma}_{X,\xi}(t) \) appearing in Proposition 2 for \( \mu = -1 \).
Theorem 3. The conditional mean \( \pi_{t-1}(X_t) \) and the variance of the one-step prediction error \( \gamma_t = \mathbb{E}[X_t - \pi_{t-1}(X_t)]^2 \) are given by the equations

\[
\pi_{t-1}(X_t) = m_t + \sum_{s=1}^{t-1} \frac{\gamma(t,s)}{1 + (A_s^2 + Q_s)\gamma_s} [A_s(Y_s - A_s\pi_{s-1}(X_s)) + Q_s(Y_s^2 - Q_s(\pi_{s-1}(X_s) - h_s))] , \quad t \geq 1 , \quad (20)
\]

\[
\gamma_t = \gamma(t,t) , \quad t \geq 1 . \quad (21)
\]

where \( \gamma = (\gamma(t,s), 1 \leq s \leq t) \) is the unique solution of equation (4). Moreover, with \( \gamma_{Xt}(t) \) defined by (17), the difference \( \pi_{t-1}(X_t) - \gamma_{Xt}(t) \) is the solution \( Z_t^h \) of equation (8).

Proof Note that since \( h_t \in \mathcal{Y}_t \) and the joint distribution of \( (X_r, Y_s, Y_s^2 + Q_s h_s) \) for any \( r, s \) is Gaussian, we can apply the Note following Theorem 13.1 in [13]. For any \( k \leq t \) we can write

\[
\left\{ \begin{array}{l}
\pi_k(X_t) = \pi_{k-1}(X_t) + [\text{cov}(X_t, \nu_k)]' \text{var}(\nu_k)^{-1} \nu_k , \\
\pi_0(X_t) = m_t , 
\end{array} \right. \quad (22)
\]

where

\[
\nu_k = Y_k - \mathbb{E}(Y_k/\mathcal{Y}_{k-1}) = \begin{pmatrix} Y_k - A_k \pi_{k-1}(X_k) \\ Y_k^2 + Q_k h_k - Q_k \pi_{k-1}(X_k) \end{pmatrix}
\]

is the innovation with covariance matrices

\[
\text{var}(\nu_k) = \begin{pmatrix} 1 + A_k^2 \gamma_k & A_k Q_k \gamma_k \\ A_k Q_k \gamma_k & Q_k + Q_k^2 \gamma_k \end{pmatrix} , \quad (23)
\]

and

\[
\text{cov}(X_t, \nu_k) = \gamma(t,k) \begin{pmatrix} A_k \\ Q_k \end{pmatrix} , \quad (24)
\]

with

\[
\gamma(t,k) = \mathbb{E}(X_t - \pi_{k-1}(X_t))(X_k - \pi_{k-1}(X_k)) . \quad (25)
\]
By the definition (25), we see for \( k = t \) that the variance \( \gamma_t \) is given by (21).

Now, equality (22) implies

\[
\pi_k(X_t) = m_t + \sum_{l=1}^{k} \gamma(t, l) \left( A_t, Q_t \right) (\text{var} \nu_l)^{-1} \nu_l =
\]

\[
= m_t + \sum_{s=1}^{k} \frac{\gamma(t, s)}{1 + (A_s^2 + Q_s)\gamma_t} \left[ A_s(Y_s - A_s\pi_{s-1}(X_s)) + Q_s(Y_s^2 - Q_s(\pi_{s-1}(X_s) - h_s)) \right],
\]

and putting \( k = t - 1 \) we get nothing but equation (20). Concerning the solution of the one-step prediction problem, it just remains to show that the covariance \( \gamma(t, s) \) satisfies equation (4).

Let us define

\[
\delta_X(t, l) = X_t - \pi_l(X_t).
\]

According to (22) we can write

\[
\delta_X(t, l) = \delta_X(t, l - 1) - \gamma(t, l) \left( A_t, Q_t \right) (\text{var} \nu_l)^{-1} \nu_l,
\]

and so

\[
\mathbb{E} \delta_X(t_1, l) \delta_X(t_2, l) = \mathbb{E} \delta_X(t_1, l - 1) \delta_X(t_2, l - 1) - \gamma(t_1, l) \gamma(t_2, l) \left( A_t \right)^\prime \text{var}(\nu_l)^{-1} \left( A_t \right),
\]

or

\[
\mathbb{E} \delta_X(t^1, l) \delta_X(t^2, l) = \mathbb{E} \delta_X(t^1, 0) \delta_X(t^2, 0) - \gamma(t, r) \gamma(t, r) \frac{A_r^2 + Q_r}{1 + (A_r^2 + Q_r)\gamma_t}.
\]

Taking \( t^1 = t, t^2 = s, l = s - 1 \) in (27), it is readily seen that equation (3) holds for \( \gamma(t, s) \).

Now we analyze the difference \( \pi_{t-1}(X_t) - \gamma_{X_t}(t) \). Using the representation \( \xi_t = \sum_{s=1}^{t} (X_s - h_s)Y_s^2 \) we can rewrite \( \pi_{t-1}(\xi_{t-1}) \) in the following form
\[ \pi_{t-1}(\xi_{t-1}) = \sum_{s=1}^{t-1} \left( \pi_{t-1}(X_s) - h_s \right) Y_s^2, \]

which implies that

\[ \xi_{t-1} - \pi_{t-1}(\xi_{t-1}) = \sum_{s=1}^{t-1} (X_s - \pi_{t-1}(X_s)) Y_s^2. \]

So we have

\[ \gamma_{X_t}(t) = \sum_{s=1}^{t-1} \pi_{t-1}[(X_s - \pi_{t-1}(X_s))(X_t - \pi_{t-1}(X_t))] Y_s^2 = \sum_{s=1}^{t-1} \mathbb{E}(X_s - \pi_{t-1}(X_s))(X_t - \pi_{t-1}(X_t)) Y_s^2 = \sum_{s=1}^{t-1} \tilde{\gamma}(t, s) Y_s^2, \] (28)

where

\[ \tilde{\gamma}(t, s) = \mathbb{E}(X_s - \pi_{t-1}(X_s))(X_t - \pi_{t-1}(X_t)) = \gamma(s, t). \] (29)

Using the definitions (28) and (29) we can write

\[ \tilde{\gamma}(t, s) - \gamma(t, s) = - \mathbb{E} X_t (\pi_{t-1}(X_s) - \pi_{t-1}(X_s)). \]

Again, applying the Note following Theorem 13.1 in [13], we can write also

\[ \pi_t(X_r) = \pi_{t-1}(X_r) + \gamma(t, l) \left( A_l \ Q_l \right) (\text{var} \, \varpi_l)^{-1} \varpi_l. \]

This means that

\[ \pi_{t-1}(X_r) - \pi_{r-1}(X_r) = \sum_{l=r}^{t-1} \gamma(t, l) \left( A_l \ Q_l \right) (\text{var} \, \varpi_l)^{-1} \varpi_l, \]

or equivalently

\[ \pi_{t-1}(X_r) - \pi_{r-1}(X_r) = \sum_{l=r}^{t-1} \tilde{\gamma}(l, t) \left( A_l \ Q_l \right) (\text{var} \, \varpi_l)^{-1} \varpi_l. \]
Then, multiplying by $X_t$ and taking expectations in both sides, we get

$$
\mathbb{E}X_t(\pi_{t-1}(X_r) - \pi_{r-1}(X_r)) = \sum_{l=r}^{t-1} \bar{\gamma}(l, r) \left( \begin{array}{c} A_l \\ Q_l \end{array} \right) \left( \text{var} \nu_l \right)^{-1} \text{cov}(X_t, \nu_l) = \\
= \sum_{l=s}^{t-1} \bar{\gamma}(l, s) \bar{\gamma}(t, l) \frac{A_l^2 + Q_l}{1 + (A_l^2 + Q_l)\bar{\gamma}_l}.
$$

Hence we have proved the following relation

$$
\bar{\gamma}(t, s) - \bar{\gamma}(t, s) = - \sum_{l=s}^{t-1} \bar{\gamma}(l, s) \bar{\gamma}(t, l) \frac{A_l^2 + Q_l}{1 + (A_l^2 + Q_l)\bar{\gamma}_l}.
$$

(30)

Now we can show that the difference $Z^h_t = \pi_{t-1}(X_t) - \pi_{X_t}(t)$ satisfies the equation (8). Using (26) and (28), we can write

$$
Z^h_t = m_t + \sum_{l=1}^{t-1} \bar{\gamma}(t, l) \left( \begin{array}{c} A_l \\ Q_l \end{array} \right) \left( \text{var} \nu_l \right)^{-1} \nu_l - \sum_{s=1}^{t-1} \bar{\gamma}(t, s) Y_s^2 = \\
= m_t + \sum_{l=1}^{t-1} \frac{A_l \bar{\gamma}(t, l)}{1 + (A_l^2 + Q_l)\bar{\gamma}_l} (Y_l - A_l\bar{\pi}_{l-1}(X_l)) + \\
+ \sum_{l=1}^{t-1} \frac{\bar{\gamma}(t, l)}{1 + (A_l^2 + Q_l)\bar{\gamma}_l} (Y_l^2 - Q_l(\pi_{l-1}(X_l) - h_l)) - \sum_{l=1}^{t-1} \bar{\gamma}(t, l) Y_l^2 = \\
= m_t + \sum_{l=1}^{t-1} \frac{A_l \bar{\gamma}(t, l)}{1 + (A_l^2 + Q_l)\bar{\gamma}_l} Y_l + \sum_{l=1}^{t-1} \frac{\bar{\gamma}(t, l)}{1 + (A_l^2 + Q_l)\bar{\gamma}_l} Q_l h_l - \\
- \sum_{l=1}^{t-1} \bar{\gamma}(t, l) \frac{A_l^2 + Q_l}{1 + (A_l^2 + Q_l)\bar{\gamma}_l} \bar{\pi}_{l-1}(X_l) + \\
+ \sum_{l=1}^{t-1} \frac{\bar{\gamma}(t, l)}{1 + (A_l^2 + Q_l)\bar{\gamma}_l} - \bar{\gamma}(t, l) Y_l^2. \quad (31)
$$
Now we can rewrite the last term in (31) using the equality (30). We have

\[
\sum_{l=1}^{t-1} \frac{\gamma(t, l)}{1 + (A_t^2 + Q_l)\bar{\gamma}_l} - \tilde{\gamma}(t, l) Y_l^2 = \sum_{l=1}^{t-1} \gamma(t, l) \frac{1}{1 + (A_t^2 + Q_l)\bar{\gamma}_l} - 1)Y_l^2 + \\
+ \sum_{l=1}^{t-1} \sum_{r=l}^{t-1} \gamma(t, r) \tilde{\gamma}(r, l) \frac{A_r^2 + Q_r}{1 + (A_r^2 + Q_r)\bar{\gamma}_r} Y_r^2 = \\
= \sum_{r=1}^{t-1} \gamma(t, r) \sum_{l=1}^{r-1} \tilde{\gamma}(r, l) Y_l^2 \frac{A_r^2 + Q_r}{1 + (A_r^2 + Q_r)\bar{\gamma}_r} = \\
= \sum_{r=1}^{t-1} \gamma(t, r) \gamma_{x\xi}(r) \frac{A_r^2 + Q_r}{1 + (A_r^2 + Q_r)\bar{\gamma}_r},
\]

where in the last step we have used equality (28).

Finally (31)-(32) imply:

\[
Z_t^h = m_t + \sum_{l=1}^{t-1} \frac{A_l \gamma(t, l)}{1 + (A_t^2 + Q_l)\bar{\gamma}_l} Y_l + \sum_{l=1}^{t-1} \frac{Q_l \gamma(t, l)}{1 + (A_t^2 + Q_l)\bar{\gamma}_l} h_t - \\
- \sum_{l=1}^{t-1} \gamma(t, l) \frac{A_l^2 + Q_l}{1 + (A_l^2 + Q_l)\bar{\gamma}_l} [\gamma_{x\xi}(l) - \gamma_{x\xi}(l)] = \\
m_t + \sum_{l=1}^{t-1} \frac{A_l \gamma(t, l)}{1 + (A_t^2 + Q_l)\bar{\gamma}_l} Y_l + \sum_{l=1}^{t-1} \frac{Q_l \gamma(t, l)}{1 + (A_t^2 + Q_l)\bar{\gamma}_l} h_t - \\
- \sum_{l=1}^{t-1} \gamma(t, l) \frac{A_l^2 + Q_l}{1 + (A_l^2 + Q_l)\bar{\gamma}_l} Z_t^h,
\]

which is nothing else but equation (8) with \( \mu = -1 \).

### 3 Particular cases and applications

Here we deal with some specific cases where the results can be further elaborated. For two examples we can apply directly Theorem 1 and moreover the special structure of the covariances allows to simplify the answer.
3.1 LEG filtering of Gauss-Markov sequences

In this part we concentrate on the case of a Gaussian AR(1) process $X$, i.e., a Gauss-Markov process driven by

$$X_t = a_t X_{t-1} + D_t^{1/2} \tilde{\varepsilon}_t, \; t \geq 1; \; X_0 = x,$$

where $(\tilde{\varepsilon}_t, \; t = 1, 2, \ldots)$ is a sequence of i.i.d. standard Gaussian random variables and $(D_t, \; t \geq 1)$ is a (deterministic) sequence of real numbers such that $D_t \geq 0$ for $t \geq 1$. In this setting, it is easy to check that the mean and covariance functions of $X$ are given by

$$m_t = \left[ \prod_{u=1}^{t} a_u \right] x = \Lambda_t x; \; K(t, s) = \left[ \prod_{u=s+1}^{t} a_u \right] k_s = \frac{\Lambda_t}{\Lambda_s} k_s, \; 1 \leq s \leq t,$$

where $\Lambda_t = \prod_{u=1}^{t} a_u$ and

$$k_t = a_t^2 k_{t-1} + D_t, \; t \geq 1, \; k_0 = 0.$$

Suppose that the following the Riccati type equation

$$\gamma_s = D_s + \frac{a_s^2 \gamma_{s-1}}{1 + (A_{s-1}^2 - \mu Q_s) \gamma_{s-1}}, \; s \geq 1, \; \gamma_0 = 0,$$

has a unique nonnegative solution.

From the classical filtering theory it is well-known that (for $\mu < 0$) $\gamma_s$ is nothing but the variance of the error of the one-step prediction problem of the signal $X$ given by the auxiliary observation $\overline{Y}$ defined by equations (11) and (12). Then, it is readily seen that the function $\overline{\gamma}(t, s)$, where $\overline{\gamma}(t, s) = \frac{\Lambda_t}{\Lambda_s} \gamma_s$ is the solution of equation (13) and that moreover equation (13) for the solution $\overline{h}$ of the LEG filtering problem (6) can be reduced to the following one:

$$\overline{h}_t = \frac{a_t}{1 + A_t \overline{\gamma}_t} \overline{h}_{t-1} + \frac{A_t \overline{\gamma}_t}{1 + A_t \overline{\gamma}_t} Y_t, \; t \geq 1, \; \overline{h}_0 = x,$$

or, equivalently:

$$\overline{h}_t = a_t \overline{h}_{t-1} + \frac{A_t \overline{\gamma}_t}{1 + A_t \overline{\gamma}_t} [Y_t - a_t A_t \overline{h}_{t-1}], \; t \geq 1, \; \overline{h}_0 = x.$$
Actually equation (35) can also be obtained directly from the general filtering theory (for \( \mu = -1 \) and replacing \( Q \) by \(-\mu Q\)). For arbitrary \((h_t \in Y_t, t \geq 1)\) the Note following Theorem 13.1 in \cite{13} gives the equation for \( Z^h \):

\[
Z_t^h = a_t Z_{t-1}^h + a_t \gamma_t \frac{Q_{t-1}}{1 + S_{t-1} \gamma_t} [h_{t-1} - Z_{t-1}^h] + a_t \gamma_t \frac{A_{t-1}}{1 + S_{t-1} \gamma_t} [Y_{t-1} - A_{t-1} Z_{t-1}^h], \quad t \geq 1, \quad Z_0^h = x.
\]

Hence, again the solution \( \overline{h}_t = \frac{Z_t^h + A_t \gamma_t Y_t}{1 + A_t \gamma_t}, \quad t \geq 1 \), of the LEG filtering problem \( \text{(6)} \) is given by \( \text{(35)} \).

Let us emphasize that these equations are nothing but those given in Speyer et al. \cite{15}.

It is interesting to note that in the case \( a_t = 0 \) (i.i.d. signal) the solution of the LEG filtering problem is nothing else but the solution of the risk neutral filtering problem i.e. \( \overline{h}_t = \pi_t(X_t) \).

### 3.2 LEG filtering of moving averages of order 1

Here we consider the case of a MA(1) process, i.e., a non Markovian process \( X \) defined by

\[
X_t = \tilde{\varepsilon}_t + \lambda \tilde{\varepsilon}_{t-1}; \quad t \geq 1,
\]

where \((\tilde{\varepsilon}_0, \tilde{\varepsilon}_1, \ldots)\) is a sequence of i.i.d. standard Gaussian variables and \( \lambda \) is a real number. Of course \( X \) is centered and has the covariance function \( K(t, s) = 1 + \lambda^2 \) if \( s = t \), \( \lambda \) if \( s = t - 1 \) and 0 if \( s < t - 1 \). In order to solve equation \( \text{(4)} \) we can take

\[
\overline{\gamma}(t, s) = 0, \quad s < t - 1; \quad \overline{\gamma}(t, t - 1) = \lambda, \quad t \geq 1,
\]

and \( \overline{\gamma}(t, t) = \overline{\gamma}_t \) where \( \overline{\gamma}_t \) is the solution of the equation:

\[
\overline{\gamma}_t = 1 + \lambda^2 - \lambda \frac{A_{t-1}^2 - \mu Q_{t-1}}{1 + (A_{t-1}^2 - \mu Q_{t-1}) \overline{\gamma}_t}, \quad t \geq 1; \quad \overline{\gamma}_0 = 1 + \lambda^2,
\]

provided that this equation has a unique nonnegative solution.

Moreover equation \( \text{(5)} \) for the solution \( \overline{h} \) of the LEG filtering problem \( \text{(6)} \) can be reduced to the following one:

\[
\overline{h}_t = \lambda \frac{A_{t-1}}{1 + A_t^2 \overline{\gamma}_t} [Y_{t-1} - A_{t-1} \overline{h}_{t-1}] + \frac{A_t \overline{\gamma}_t}{1 + A_t^2 \overline{\gamma}_t} Y_t, \quad t \geq 1, \quad \overline{h}_0 = 0.
\]
Again, it is interesting to note that for \( \lambda = 0 \) (i.i.d. signal) the solution of LEG filtering problem is nothing else but the solution of the risk neutral filtering problem i.e. \( \overline{h}_t = \pi_t(X_t) \).

4 LEG and RS filtering problems

Here, at first we show that actually the LEG and RS filtering problems have the same solution. Then we give an example which shows that in a more general context similar problems may have different solutions.

4.1 Equivalence of LEG and RS filtering problems

Let \( \hat{h} = (\overline{h}_s)_{s \geq 1} \) be the solution of the LEG filtering problem (6) given by equation (5). For any fixed \( t \leq T \), let us denote by \( \hat{g}_t : \)

\[
\hat{g}_t = \arg \min_{g \in \mathcal{Y}_t} \mathbb{E} \left[ \mu \exp \left\{ \frac{\mu}{2} (X_t - g)^2 Q_t + \frac{\mu}{2} \sum_{s=1}^{t-1} (X_s - \overline{h}(s))^2 Q_s \right\} \right] / \mathcal{Y}_t,
\]

where \( g \in \mathcal{Y}_t \) means that \( g \) is a \( \mathcal{Y}_t \)-measurable variable. It follows directly from Proposition 2 that, provided that \( 1 + S_t \gamma_t > 0 \), the equality \( \hat{g}_t = \frac{\overline{h}_t + A_t \gamma_t Y_t}{1 + A_t^2 \gamma_t} \), \( t \geq 1 \) holds. Since it was noted in the proof of Theorem 1 that \( h_t = \frac{\overline{h}_t + A_t \gamma_t Y_t}{1 + A_t^2 \gamma_t} \), \( t \geq 1 \), hence we have also \( \hat{g}_t = \overline{h}_t \). It means that for \( t \geq 1 \) the solution \( \overline{h} \) of the LEG filtering problem satisfies the following recursive equation:

\[
\hat{g}_t = \arg \min_{g \in \mathcal{Y}_t} \mathbb{E} \left[ \mu \exp \left\{ \frac{\mu}{2} (X_t - g)^2 Q_t + \frac{\mu}{2} \sum_{s=1}^{t-1} (X_s - \overline{h}(s))^2 Q_s \right\} \right] / \mathcal{Y}_t. \quad (36)
\]

Indeed, in the literature, the recursion (36) is the basic definition of the so-called risk-sensitive (RS) filtering problem which was introduced in [6]. Therefore we have also proved the following statement

**Theorem 4.** Assume that the condition \( (C_\mu) \) is satisfied. Let \( \overline{h} = (\overline{h}_t)_{t \geq 1} \) be the unique solution of equation (5), i.e., \( \overline{h} \) is the solution of the LEG filtering problem (6). Then \( \overline{h} \) is the solution of the RS filtering problem (36).
4.2 Discrepancy between LEG and RS type filtering problems: an example

Actually, we did not find in the literature any trace of the discussion about the relationship between the LEG filtering problem (6) and the RS filtering problem (36) even in a Gauss-Markov case. As a complement to our observation that these two problems have the same solution, we propose an example to show that in a bit more general setting, two similar problems may have different solutions.

For given positive symmetric deterministic $2 \times 2$ matrices $\Lambda_s$, $1 \leq s \leq T$, let us set $\Phi_t(h) = (X_t h_t) \Lambda_t \left( \begin{array}{c} X_t \\ h_t \end{array} \right)$. We can define $\bar{h}_t \in \mathcal{Y}_t$, $t \geq 1$ as a solution of a LEG type filtering problem:

$$\bar{h}_t = \arg \min_{h_t \in \mathcal{Y}_t, t \geq 1} \mathbb{E} \left[ \mu \exp \left\{ \frac{\mu}{2} \sum_{t=1}^{T} \Phi_s(h) \right\} \right]. \quad (37)$$

We can also define $\hat{h}$ as the solution of the following recursive equation (RS type filtering problem):

$$\hat{h}_t = \arg \min_{g \in \mathcal{Y}_t} \mathbb{E} \left[ \mu \exp \left\{ \frac{\mu}{2} \Phi_t(g) + \frac{\mu}{2} \sum_{s=1}^{t-1} \Phi_s(h) \right\} \middle/ \mathcal{Y}_t \right]. \quad (38)$$

The question which we discuss now is the following: does the equality $\bar{h} = \hat{h}$ hold?

As we have just proved, the answer is positive for singular matrices $\Lambda$, namely, when $\Lambda_{11} = \Lambda_{22} = -\Lambda_{12} = Q$. But in the general situation the answer may be negative. Actually it is sufficient to consider the following example: $\Lambda = \left( \begin{array}{cc} 2 & -1 \\ -1 & 1 \end{array} \right)$, $A_t = 1$, $\mu = -1$ and $X_t = X_{t-1} + \tilde{\varepsilon}_t$, where $(\tilde{\varepsilon}_t, t = 1, 2, \ldots)$ is a sequence of i.i.d. standard Gaussian random variables. Even in this Markov case $\bar{h} \neq \bar{h}$. More explicitly let us introduce the new probability measure $\hat{\mathbb{P}}$:

$$\frac{d \hat{\mathbb{P}}}{d \mathbb{P}} = \frac{\exp \left[ -\frac{1}{2} \sum_{i=1}^{T} X_i^2 \right]}{\mathbb{E} \exp \left[ -\frac{1}{2} \sum_{i=1}^{T} X_i^2 \right]}. \quad (21)$$
One can check that with respect to \( \hat{\mathcal{P}} \) the observation model \((X_t, Y_t)_{t \geq 1}\) can be written in the following form:

\[
X_t = a_t X_{t-1} + D_t^\frac{1}{2} \hat{\varepsilon}_t, \quad t \geq 1; \quad X_0 = x, \quad Y_t = X_t + \varepsilon_t,
\]

where \((\hat{\varepsilon}_t)_{t \geq 1}\) is a sequence of i.i.d. standard Gaussian random variables independent of the sequence \(\varepsilon\),

\[
a_t = D_t = \frac{1}{1 + \Gamma(T, t)},
\]

and \(\Gamma(T, \cdot)\) is the solution of the backward Riccati equation

\[
\Gamma(T, t) = 1 + \frac{\Gamma(T, t+1)}{1 + \Gamma(T, t+1)}, \quad \Gamma(T, T) = 0.
\]

It can be checked that

\[
\Gamma(T, t) = 10 \frac{\lambda^T - \lambda^t}{(1 - \sqrt{5})\lambda^T - (1 + \sqrt{5})\lambda^t}, \quad \lambda = \frac{(3 - \sqrt{5})}{(3 + \sqrt{5})}.
\]

Indeed to explain this change of the observation model it is sufficient to calculate the conditional characteristic function:

\[
\hat{\mathbb{E}} \left[ \exp(i \lambda X_t) / \mathcal{X}_{t-1} \right] = \frac{\mathbb{E} \left[ \exp \left( i \lambda X_t - \frac{1}{2} \sum_{i=1}^{T} X_i^2 \right) \right] / \mathcal{X}_{t-1}}{\mathbb{E} \left[ \exp \left( - \frac{1}{2} \sum_{i=1}^{T} X_i^2 \right) \right] / \mathcal{X}_{t-1}},
\]

where \(\mathcal{X}_{t-1}\) is the \(\sigma\)-field \(\mathcal{X}_{t-1} = \sigma(\{X_s, 1 \leq s \leq t - 1\})\). But it follows directly from the equation (19)-(20) in [9] and from (10) that

\[
\hat{\mathbb{E}} \left[ \exp(i \lambda X_t) / \mathcal{X}_{t-1} \right] = \exp \left\{ \frac{i \lambda}{1 + \Gamma(T, t)} X_{t-1} - \frac{\lambda^2}{2(1 + \Gamma(T, t))} \right\}.
\]

Since the density \(\frac{d\hat{\mathcal{P}}}{d\mathcal{P}}\) does not depend on \(h\), the initial LEG filtering problem (37) can be rewritten as:

\[
\overline{h} = \arg \min_{h_t \in \mathcal{Y}_t, 1 \leq t \leq T} \hat{\mathbb{E}} \left[ - \exp \left\{ - \frac{1}{2} \sum_{1}^{T} (X_s - h_s)^2 \right\} \right].
\]
Hence we can apply Theorem 4 or in particular (34) and (35). Clearly, $\hat{h}$ depends on $T$ and $\hat{h}$ does not depend on $T$ by the definition. A bit more explicitly we have for example for $t = 1$: $\hat{h}_1 = \frac{1 + \Gamma(T, 1)}{2 + \Gamma(T, 1)} Y_1$ and obviously $\pi_1(X_1) = \frac{1}{1 + \gamma_1} = \frac{1}{4} Y_1$ and clearly they are different.

## 5 Information state, interpretation

In this section we discuss the probabilistic interpretation of the ingredients of the “information state” which was introduced in the context of RS filtering and LEG control problems. By the definition, the “information state” contains all the information needed to describe the solution of the concerned optimization problem. In particular it takes into account the cost function but not only estimates of the signal and it should give the total information about the model states available in the measurement.

### Risk-Sensitive Filtering

In the context of the RS filtering problem the definition of the information state can be found for example in [7]. It is the density $\lambda_t$, with respect to the Lebesgue measure, of the non normalized random measure $\omega_t$:

$$\omega_t(dx) = \mathbb{E} \left[ \mathbb{I}(X_t \in dx) \exp \left\{ \frac{\mu}{2} \sum_{s=1}^{t-1} (X_s - h(s))^2 Q_s \right\} \right]/\mathcal{Y}_t, \quad (39)$$

where $h_t \in \mathcal{Y}_t$, $t \geq 1$ and the observation $Y_t$ is defined by the equation (11).

In a classical Gauss-Markov setting, an explicit representation of $\lambda_t$ can be obtained as the solution of some recurrence equation (see, e.g., [3]). We claim that for a general Gaussian signal $X$ the density $\lambda_t$ satisfies the following equality:

$$\lambda_t(x) = \frac{1}{\sqrt{2\pi \gamma_t}} \exp \left\{ \frac{-(x - \hat{Z}_h^t)^2}{2\gamma_t} \right\} \times$$

$$\prod_{r=1}^{t-1} \left[ \frac{1 + S_r \gamma_r}{1 + A_r^2 \gamma_r} \right]^{-1/2} \times \exp \left\{ \frac{\mu}{2} Q_r \frac{1 + A_r^2 \gamma_r}{1 + S_r \gamma_r} \times \left[ h_r - \hat{Z}_r^h \right]^2 \right\} \times \mathcal{M}_t, \quad (40)$$

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where \( \tilde{Z}_t^h = \frac{Z_t^h + A_t \tilde{\gamma}_t Y_t}{1 + A_t^2 \tilde{\gamma}_t} \) is the solution of the equation (19), \( \tilde{\gamma}_t = \frac{-\tilde{\gamma}_t}{1 + A_t^2 \tilde{\gamma}_t} \), \( \tilde{\gamma}, Z_t^h \) are the solutions of equations (4) and (8) respectively and the martingale \( (\mathcal{M}_t)_{t \geq 1} \) is defined by (50).

Indeed, to prove (40) it is sufficient to write the following:

\[
\omega_t(dx) = \frac{\mathbb{E}[\mathbb{1}(X_t \in dx) \exp(-\xi_{t-1})/\mathcal{Y}_{t,t-1}]}{\mathbb{E}[\exp(-\xi_{t-1})/\mathcal{Y}_{t,t-1}]} \left[ \exp \left\{ \frac{\mu}{2} \sum_{s=1}^{t-1} (X_s - h(s))^2 Q_s \right\} / \mathcal{Y}_t \right],
\]

where \( \sigma \)-field \( \mathcal{Y}_{t,t-1} = \sigma(\{(Y_s, Y_r^2), 1 \leq s \leq t, 1 \leq r \leq t-1\}) \). Again, conditionally Gaussian properties of the pair \((X, \xi)\) imply that

\[
\mathbb{E} \left[ \mathbb{1}(X_t \in dx) \exp \{ -\xi_t \} / \mathcal{Y}_{t,t-1} \right] = [2\pi \tilde{\gamma}_t]^{-\frac{1}{2}}
\]

\[
\times \exp \left\{ -\frac{1}{2} (x - \tilde{Z}_t^h)^2 \tilde{\gamma}_{t-1}^{-1} \right\} \ dx,
\]

where \( \tilde{Z}_t^h = \mathbb{E}[X_t/\mathcal{Y}_{t,t-1}] - \mathbb{E}[(X_t - \mathbb{E}[X_t/\mathcal{Y}_{t,t-1}])((\xi_{t-1} - \tilde{\pi}_{t-1}(\xi_{t-1}))/\mathcal{Y}_{t,t-1}] \) and \( \tilde{\gamma}_t = \mathbb{E}[(X_t - \mathbb{E}[X_t/\mathcal{Y}_{t,t-1}])^2]. \) Now the desired equality (41) follows directly from Proposition 2.

It is worth emphasizing that (for negative \( \mu \)) now we know the probabilistic interpretation of the involved processes \( (Z_t^h, \tilde{Z}_t^h, \tilde{\gamma}, \tilde{\gamma}) \). Actually we have proved that \( Z_t^h \) is the difference \( \tilde{\pi}_t(X) - \tilde{\gamma}_{Xt}(t) \) and \( \tilde{\gamma} \) is nothing but the covariance of the filtering error of \( X \) in view of auxiliary observations \( \tilde{Y} \).

For the pair \( (\tilde{Z}_t^h, \tilde{\gamma}) \) we have the same relations but with respect to the \( \sigma \)-field \( \mathcal{Y}_{t,t-1} = \sigma(\{(Y_s, Y_r^2), 1 \leq s \leq t, 1 \leq r \leq t-1\}) \). Of course, after a simple integration of \( \lambda_t \), formula (40) gives Proposition 2.1 and therefore the solution of the LEG and RS filtering problems. Let us also observe that the relations

\[
\tilde{Z}_t^h = \frac{Z_t^h + A_t \gamma_t Y_t}{1 + A_t^2 \gamma_t}, \quad \tilde{\gamma}_t = \frac{-\gamma_t}{1 + A_t^2 \gamma_t}
\]

which were announced in Remark 5 follow from the Note following Theorem 13.1 in [13].

**Linear Exponential Gaussian Control**

In the context of the LEG control problem for a partially observed process, the information state is also defined (see, e.g., [7]) as the density \( \lambda_t \), with respect to the Lebesgue measure, of the non normalized random measure \( \omega_t \):

\[
\omega_t(dx) = \mathbb{E} \left[ \mathbb{1}(X_t \in dx) \exp \left\{ \frac{\mu}{2} \sum_{s=1}^{t-1} X_s^2 Q_s \right\} / \mathcal{Y}_t \right],
\]

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where $X$ is the controlled state governed by the equation:

$$X_t = a_t X_{t-1} + b_t u_t + \tilde{\epsilon}_t, \; t \geq 1; \; X_0 = 0,$$

($\tilde{\epsilon}_t)_{t \geq 1}$ is a sequence of i.i.d. standard Gaussian variables and $u_t \in Y_{t-1}$ corresponding to the available observation $Y$ defined by the equation (1).

By the same way that we have just explained, for the conditionally Gaussian pair $(X, Y)$, one can check that the density $\lambda_t$ satisfies the following equality:

$$\lambda_t(x) = \frac{1}{\sqrt{2\pi \tilde{\gamma}_t}} \exp \left\{ -\frac{(x - \tilde{Z}_t)^2}{2\tilde{\gamma}_t} \right\} \times \prod_{r=1}^{t-1} \left[ \frac{1 + S_r \tilde{\gamma}_r}{1 + A_r^2 \tilde{\gamma}_r} \right]^{-1/2} \times \exp \left\{ \frac{\mu}{2} \sum_{s=1}^{T} \bar{Q}_s \frac{1 + A_r^2 \tilde{\gamma}_r}{1 + S_r \tilde{\gamma}_r} \times \tilde{Z}_s^2 \right\} \times \mathcal{M}_t,$$  \hspace{1cm} (45)

where $\tilde{\gamma}_t = \gamma + \frac{S_t}{1 + A_t^2 \gamma}$, $\gamma$ is the solutions of equation (4), the martingale $(\mathcal{M}_t)_{t \geq 1}$ is defined by (50) and $\tilde{Z}$ is the solution of the equation

$$\tilde{Z}_t = \frac{a_t}{1 + S_t \tilde{\gamma}_t} \tilde{Z}_{t-1} + \frac{b_t}{1 + S_t \tilde{\gamma}_t} u_t + \tilde{\gamma}_t A_t Y_t.$$  \hspace{1cm} (46)

Actually it is the equation for the difference $\tilde{Z} = \pi_{t,t-1}(X) - \bar{\gamma}_t X(t, t - 1)$, where the conditional expectations are taken with respect to the auxiliary observation process $\bar{Y}$ defined by the equations (1) and (12) with $h = 0$.

Equality (45) gives the possibility to rewrite the cost function in terms of the completely observable process $\tilde{Z}$, namely:

$$\mathbb{E} \left[ \exp \left\{ \frac{\mu}{2} \sum_{s=1}^{T} X_s^2 Q_s \right\} \right] = \mathbb{E} \left[ \mathbb{E} \left[ \exp \left\{ \frac{\mu}{2} \sum_{s=1}^{T} X_s^2 Q_s \right\} \right] / \mathcal{Y}_{T} \right]$$

$$= \prod_{r=1}^{t-1} \left[ \frac{1 + S_r \bar{\gamma}_r}{1 + A_r^2 \bar{\gamma}_r} \right]^{-1/2} \mathbb{E} \left[ \exp \left\{ \frac{\mu}{2} \sum_{s=1}^{T} \bar{Q}_s \tilde{Z}_s^2 \right\} \right] \times \mathcal{M}_T$$

$$= \prod_{r=1}^{t-1} \left[ \frac{1 + S_r \bar{\gamma}_r}{1 + A_r^2 \bar{\gamma}_r} \right]^{-1/2} \mathbb{E} \left[ \exp \left\{ \frac{\mu}{2} \sum_{s=1}^{T} \bar{Q}_s \tilde{Z}_s^2 \right\} \right],$$

where $\bar{Q}_r = Q_r \frac{1 + A_r^2 \bar{\gamma}_r}{1 + S_r \bar{\gamma}_r}$ and $\mathbb{E}$ stands for an expectation with respect to the new measure $\mathbb{P}$ such that:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{M}_T.$$  \hspace{1cm} (25)
With respect to this new measure the solution of equation (46) can be represented as

\[ ˜Z_t = a_t \frac{1 + A_t^2 \bar{\gamma}_t}{1 + S_t \bar{\gamma}_t} \bar{Z}_{t-1} + b_t \frac{1 + A_t^2 \bar{\gamma}_t}{1 + S_t \bar{\gamma}_t} u_t + \frac{\bar{\gamma}_t A_t}{1 + A_r^2 \gamma_r} \bar{\varepsilon}_t, \]  

(47)

where \((\bar{\varepsilon}_t)_{t \geq 1}\) is a new sequence of i.i.d. standard Gaussian variables. Thus, the new process \(\bar{Z}\) plays the role of the completely observed controlled state (see [1] and [7]).

Now we emphasize that the probabilistic interpretation of the “information state” \(\bar{Z}\), used in [7] is nothing but \(\bar{Z}_t = \bar{\pi}_{t,t-1}(X) - \bar{\gamma}_{Xt}(t)\), where the conditional expectations are taken with respect to the auxiliary observation process \(\bar{Y}\) defined by the equations (1) and (12) with \(h = 0\). Also, \(\bar{\gamma}\) is the conditional covariance of \(X\).

6 Complementary part - More general case

In this section we analyze LEG and RS filtering problems in a more general contexts when we do not suppose a special structure of the observation sequence \((Y_t)_{t \geq 1}\). We suppose only that the process \((X_t, Y_t)_{t \geq 1}\) is Gaussian (even conditionally Gaussian). Our goal is to reduce LEG (RS) filtering problems to an auxiliary risk-neutral filtering problem. First of all we fix \(\mu = -1\) and we will find the probabilistic interpretation of the solution. After to find the solution for \(\mu \neq -1\) we shall have only to replace \(Q\) by \(-\mu Q\) in the answer. So, let \((Y_t^2, \xi_t)\) be defined by equations (12) - (13) and let us denote by

\[ \bar{Z}_t^h = \mathbb{E}[X_t/\bar{Y}_{t,t-1}] - \mathbb{E}[(X_t - \mathbb{E}[X_t/\bar{Y}_{t,t-1}])(\xi_{t-1} - \bar{\pi}_{t-1}(\xi_{t-1}))/\bar{Y}_{t,t-1}], \] 

(48)

\[ \bar{\gamma}_t = \mathbb{E}[(X_t - \mathbb{E}[X_t/\bar{Y}_{t,t-1}])(\xi_{t-1} - \bar{\pi}_{t-1}(\xi_{t-1}))/\bar{Y}_{t,t-1}], \] 

(49)

where \(\bar{Y}_{t,t-1}\) is the \(\sigma\)-field \(\bar{Y}_{t,t-1} = \sigma(\{Y_s, Y_r^2, 1 \leq s \leq t, 1 \leq r \leq t - 1\})\). Again, let \(J_t = \exp\left\{-\frac{1}{2} \sum_{s=1}^{t} (X_s - h_s)^2 Q_s\right\}\) and let us denote by \(\mathcal{I}_t\) the conditional expectation \(\mathcal{I}_t = \pi_t(J_t)\), or

\[ \mathcal{I}_t = \mathbb{E}\left( \exp\left\{-\frac{1}{2} \sum_{s=1}^{t} (X_s - h_s)^2 Q_s\right\} / \mathcal{Y}_t \right), \]

where \(h_s \in \mathcal{Y}_s, s \geq 1\). We claim the following generalization of Proposition 2.
Proposition 5. The following equality holds for any $T \geq 1$:

\[
I_T = \prod_{t=1}^{T} [1 + Q_t \tilde{\gamma}_t]^{1/2} \times \exp \left\{ -\frac{1}{2} \frac{Q_t}{1 + Q_t \tilde{\gamma}_t} \times \left[ h_t - \tilde{Z}_t^h \right] \right\} \times M_T,
\]

where $(M_T)_{T \geq 1}$ is a martingale defined by:

\[
M_T = \prod_{t=1}^{T} \left[ \frac{\sigma_t^2}{\sigma_t^2} \right]^{1/2} \exp \left\{ \frac{1}{2\sigma_t^2} (Y_t - \pi_{t-1}(Y_t))^2 - \frac{1}{2\bar{\sigma}_t^2} (Y_t - \bar{V}_t)^2 \right\}, \quad (50)
\]

where \( \sigma_t^2 = \mathbb{E}(Y_t - \pi_{t-1}(Y_t))^2 \), \( \bar{\sigma}_t^2 = \mathbb{E}(Y_t - \bar{\pi}_{t-1}(Y_t))^2 \),
\[
\bar{V}_t = \bar{\pi}_{t-1}(Y_t) - \gamma \bar{Y}(t), \quad \gamma \bar{Y}(t) = \mathbb{E}[(Y_t - \bar{\pi}_{t-1}(Y_t)) (\xi_{t-1} - \bar{\pi}_{t-1}(\xi_{t-1}))/\bar{Y}_{t-1}].
\]

Proof To prove Proposition 5 let us again calculate the ratio

\[
I_t/I_{t-1} = \frac{\pi_t(J_t)}{\pi_{t-1}(J_{t-1})} = \frac{\pi_t(J_t)}{\pi_t(J_{t-1})} \frac{\pi_t(J_{t-1})}{\pi_{t-1}(J_{t-1})} = \frac{\pi_t(J_t)}{\pi_{t-1}(J_{t-1})} \frac{M_t}{M_{t-1}}
\]

with a martingale $M_t$ such that:

\[
M_t = \prod_{s=1}^{t} \frac{\pi_s(J_{s-1})}{\pi_{s-1}(J_{s-1})}. \quad (51)
\]

The same arguments that we used in the proof of Proposition 2 show that

\[
\frac{\pi_t(J_t)}{\pi_t(J_{t-1})} = \frac{\pi_{t-1}(\exp\{-\frac{1}{2} Q_t (X_t - h_t)^2 - \xi_{t-1}\})}{\pi_{t-1}(\exp(-\xi_{t-1}))} = (1 + Q_t \tilde{\gamma}(t))^{-1/2} \exp \left\{ -\frac{1}{2} \frac{Q_t}{1 + Q_t \tilde{\gamma}(t)} (\tilde{Z}_t^h - h_t)^2 \right\}.
\]

To finish the proof we turn to the representation of the martingale $M_t$ defined by (51). First of all we claim that

\[
\frac{M_t}{M_{t-1}} = \frac{\bar{\pi}_{t-1}(\mathbb{I}(Y_t \in dy))}{\pi_{t-1}(\mathbb{I}(Y_t \in dy))} \bigg|_{y=Y_t}, \quad (52)
\]
where $\tilde{\pi}$ stands for the conditional expectation with respect to the measure $\tilde{\mathbb{P}}$ such that $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{M}_T$. Indeed, it is the direct consequence of the classical Bayes formula

$$
\tilde{\pi}_{t-1}(\mathbb{I}(Y_t \in dy)) = \frac{\pi_{t-1}(\mathbb{I}(Y_t \in dy)\mathcal{M}_T)}{\pi_{t-1}(\mathcal{M}_{t-1})} = \pi_{t-1}(\mathbb{I}(Y_t \in dy)\mathcal{M}_t).
$$

To finish the proof it is sufficient to note that representations (51) and (52) imply that

$$
\frac{\mathcal{M}_t}{\mathcal{M}_{t-1}} = \frac{\pi_t(J_{t-1})}{\pi_{t-1}(J_{t-1})} = \frac{\pi_{t-1}(\mathbb{I}(Y_t \in dy))}{\pi_{t-1}(\mathbb{I}(Y_t \in dy))}_{y=Y_t} = \frac{\pi_{t-1}(\mathbb{I}(Y_t \in dy)J_{t-1})}{\pi_{t-1}(J_{t-1})\pi_{t-1}(\mathbb{I}(Y_t \in dy))}_{y=Y_t}.
$$

Again, we can use the same arguments that we used in the proof of Proposition 2

$$
\frac{\pi_{t-1}(\mathbb{I}(Y_t \in dy)J_{t-1})}{\pi_{t-1}(J_{t-1})} = \frac{\pi_{t-1}(\mathbb{I}(Y_t \in dy)\exp\{-\xi_{t-1}\})}{\pi_{t-1}(\exp\{-\xi_{t-1}\})} = \frac{1}{\sqrt{2\pi\bar{\sigma}^2}} \exp\left(-\frac{(Y_t - \bar{V}_t)^2}{2\bar{\sigma}^2}\right).
$$

A direct consequence of Proposition 5 is the following statement:

**Corollary 6.** Let $\tilde{h}$ be the solution of LEG (and RS) filtering problem (6) (and (36)). Then the following equality holds for any $t \geq 1$:

$$
\tilde{h}_t = \tilde{Z}_t^{\tilde{h}}.
$$

## 7 Particular cases - again

### 7.1 Markov type observations

Here we turn to the case when the observations $(Y_t)_{t \geq 1}$ are conditionally independent given $X$. More precisely, we deal with a signal-observation model
\((X_t, Y_t)_{t \geq 1}\), where the signal \(X = (X_t)_{t \geq 1}, X_t \in \mathbb{R}^n\) is an arbitrary Gaussian sequence with mean vector \(m = (m_t, t \geq 1)\) and covariance matrix \(K = (K(t, s), t \geq 1, s \geq 1)\), i.e.,

\[
\mathbb{E}X_t = m_t, \quad \mathbb{E}(X_t - m_t)(X_s - m_s)' = K(t, s), t \geq 1, s \geq 1,
\]

The observation process \(Y = (Y_t, t \geq 1)\) is given by

\[
Y_t = A_t X_t + \varepsilon_t, \quad (53)
\]

for some sequence \(A = (A_t, t \geq 1)\) of \(m \times n\) matrices, where \(\varepsilon = (\varepsilon_t)_{t \geq 1}\) is a sequence of i.i.d. \(\mathcal{N}(0, Id)\) random variables and \(\varepsilon\) and \(X\) are independent. In this case we can write the multidimensional analogue of the equation (19), which is nothing else but the dynamic equation for the process \(\tilde{Z}_h\) defined by (48). We obtain:

\[
\tilde{Z}_h^t = m_t + \sum_{l=1}^{t-1} \gamma(t, l)[Id + \gamma_l(A_l' A_l - \mu Q_l)]^{-1} \mu Q_l(h_l - \tilde{Z}_h^l) + \sum_{l=1}^{t} \gamma(t, l) A_l' (Y_l - A_l \tilde{Z}_h^l),
\]

where the matrix \(\gamma(t, l)\) satisfies the following equation (which is the multidimensional analog of the equation (41)):

\[
\gamma(t, s) = K(t, s) - \sum_{l=1}^{s-1} \gamma(t, l) \bar{A}_l'[Id + \bar{A}_l \bar{A}_l']^{-1} \bar{A}_l' \gamma(s, l), \quad (54)
\]

where \(\bar{A}_l = \begin{pmatrix} A_l \\ -\mu Q_l \end{pmatrix}\).

Now the solution of the LEG (and RS) filtering problem \(\bar{h}\) is nothing else but:

\[
\bar{h}_t = m_t + \sum_{l=1}^{t} \gamma(t, l) A_l' (Y_l - A_l \bar{h}_l).
\]

### 7.2 Markov type observations, correlated signal and observation noises

Let us drop the assumption that \(X\) and \(\varepsilon\) in the observation equation (53) are independent. Denote by \(K_{X,\varepsilon}(t, s)\) the covariance matrix of the signal and the observation noise, i.e.,

\[
\mathbb{E}(X_t - m_t)\varepsilon_s' = K_{X,\varepsilon}(t, s), \quad t \geq 1, s \geq 1.
\]
It can be checked that the following slight modification of the previous statement holds.

Let the matrix $\gamma(t, l)$ be the unique solution of the following equation

$$
\gamma(t, s) = K(t, s) - \sum_{l=1}^{s-1} [\gamma(t, l) \bar{A}_l + K_{X\varepsilon}(t, l)] \\
[Id + \bar{A}_l \bar{A}_l' + \bar{A}_l K_{X\varepsilon}(l, l) + K_{X\varepsilon}(l, l)' \bar{A}_l']^{-1} \\
[\bar{A}_l \bar{\gamma}(s, l) + K'_{X\varepsilon}(s, l)], \quad (55)
$$

with $\bar{A}_l = \begin{pmatrix} A_l \\ -\mu Q_l \end{pmatrix}$, $\bar{K}_{X\varepsilon}(t, l) = (K_{X\varepsilon}(t, l) \ 0)$.

Then the solution of the LEG (and RS) filtering problem $\bar{h}$ satisfies the following equation

$$
\bar{h}_t = m_t + \sum_{l=1}^{t} [K_{X\varepsilon}(t, l) + \gamma(t, l) \bar{A}_l'] [Id + \bar{A}_l K_{X\varepsilon}(l, l)]^{-1} (Y_l - \bar{A}_l \bar{h}_l). \quad (56)
$$

### 7.3 Observations containing Moving Averages of order 1

Now we consider the case of a MA(1) type process, i.e., the following signal-observation model:

$$
X_t = \tilde{\varepsilon}_t + \lambda \tilde{\varepsilon}_{t-1}; t \geq 1, \\
Y_t = \alpha_t X_t + \varepsilon_t + \beta \varepsilon_{t-1}; t \geq 1,
$$

where $(\varepsilon_t, \tilde{\varepsilon}_t)_{t \geq 0}$ is a sequence of i.i.d. Gaussian variables and $\lambda$ and $\beta$ are real numbers.

Let us denote by $A_t$ the row $\bar{A}_t = (\alpha_t \ \beta)$ and by $\bar{X}_t$ the vector $\bar{X}_t = \begin{pmatrix} X_t \\ \varepsilon_{t-1} \end{pmatrix}$. Of course $\bar{X}$ is centered, has the covariance matrix $K(t, s) = (1 + \lambda^2)1(s = t - 1) + \lambda 1(s = t) \quad 0 \\
0 \quad 1(s = t)$ and the covariance between $\bar{X}$ and $\varepsilon$ is $K_{X\varepsilon}(t, s) = \begin{pmatrix} 0 \\ 1(s = t - 1) \end{pmatrix}$.

The solution $\gamma$ to (55) then can be found as:

$$
\gamma(t, s) = 0, \ s < t - 1; \quad \gamma(t, t - 1) = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \ t \geq 1,
$$
and \( \overline{\gamma}(t, t) = \overline{\gamma}_t \) where \( \overline{\gamma}_t \) is the solution of the equation:

\[
\overline{\gamma}_t = \left( \begin{array}{cc}
1 + \lambda^2 & 0 \\
0 & 1 
\end{array} \right) + \\
\left( \begin{array}{cc}
\lambda \alpha_{t-1} & -\lambda \mu Q_{t-1} \\
0 & 1
\end{array} \right) \left[ I_d + \left( \begin{array}{cc}
\alpha_{t-1} & \beta \\
-\mu Q_{t-1} & 0
\end{array} \right) \overline{\gamma}_{t-1} \left( \begin{array}{cc}
\alpha_{t-1} & -\mu Q_{t-1} \\
\beta & 0
\end{array} \right) \right]^{-1} \\
\times \left( \begin{array}{cc}
\lambda \alpha_{t-1} & 0 \\
-\lambda \mu Q_{t-1} & 1
\end{array} \right), \\
t \geq 1; \quad \overline{\gamma}_0 = 0.
\]

provided that this equation has the unique nonnegative definite solution.

Moreover, equation (56) for the solution \( \overline{h} \) of the LEG filtering problem can be reduced to the following one:

\[
\overline{h}_t = \Lambda_t^{-1} \left( \begin{array}{c}
\lambda \alpha_t \\
1 + \beta
\end{array} \right) [Y_{t-1} - A_{t-1} \overline{h}_{t-1}] + \Lambda_t^{-1} \overline{\gamma}_t \left( \begin{array}{c}
\alpha_t \\
\beta
\end{array} \right) Y_t, \\
t \geq 1, \quad \overline{h}_0 = 0,
\]

with \( \Lambda_t = I_d + \overline{\gamma}_t A_t' A_t \).

### 7.4 Observations containing Gaussian AR(1) process

In this part we concentrate on the case of a Gaussian AR(1) type process \( Y \), i.e.,

\[
Y_t = \alpha_t X_t + \varepsilon_t, \quad t \geq 1; \quad Y_0 = 0, \quad (57)
\]

where

\[
\varepsilon_t = b \varepsilon_{t-1} + \tilde{\varepsilon}_t,
\]

and \( (\tilde{\varepsilon}_t, \ t = 1, 2, \ldots) \) is a sequence of i.i.d. standard Gaussian random variables independent of \( X \). We also suppose that the signal \( X \) is a Gaussian AR(1) process, i.e.,

\[
Y_t = \alpha_t X_t + \epsilon_t, \quad t \geq 1; \quad X_0 = 0, \quad (58)
\]

and also \( (\epsilon_t, \ t = 1, 2, \ldots) \) is a sequence of i.i.d. standard Gaussian random variables. Proceeding as in Sections 3.1 and 7.3 we can write the dynamic equation for the solution of LEG and RS filtering problems \( \overline{h} \). Namely, \( \overline{h} \) is the first component \( \overline{h}_1 \) of the solution of the following recursive equation:

\[
\overline{h}_t = \overline{\gamma}_t A_t' [Y_t - A_t \overline{h}_t] + \left( \begin{array}{cc}
\alpha_t & 0 \\
0 & b
\end{array} \right) \overline{h}_{t-1} + \left( \begin{array}{c}
0 \\
1
\end{array} \right) \left[ Y_{t-1} - A_{t-1} \overline{h}_{t-1} \right], \\
t \geq 1, \quad \overline{h}_0 = 0.
\]
where $A_t = (\alpha_t \beta)$ and $\gamma_t$ is the unique nonnegative defined solution of the Ricatti equation:

$$
\gamma_t = \begin{pmatrix} a_t & 0 \\ 0 & b \end{pmatrix} \gamma_{t-1} \begin{pmatrix} a_t & 0 \\ 0 & b \end{pmatrix} - \left[ \begin{pmatrix} a_t & 0 \\ 0 & b \end{pmatrix} \gamma_{t-1} \bar{A}_{s-1} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right]
\times \left[ I_d + \bar{A}_{t-1} \bar{\gamma}_{t-1} \bar{A}'_{s-1} \right]^{-1} \left[ \bar{A}_{t-1} \bar{\gamma}_{t-1} \begin{pmatrix} a_t & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right],
$$

with $\bar{A}_t = \begin{pmatrix} \alpha_t & b \\ -\mu Q_t & 0 \end{pmatrix}$.

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