A NOTE ON KAWAGUCHI-SILVERMAN CONJECTURE

SICHEN LI AND YOHSUKE MATSUZAWA

ABSTRACT. We collect some results on endomorphisms on projective varieties related with the Kawaguchi-Silverman conjecture. We discuss certain condition on automorphism groups of projective varieties and positivity conditions on leading real eigendivisors of self-morphisms. We prove Kawaguchi-Silverman conjecture for endomorphisms on projective bundles on a smooth Fano variety of Picard number one. In the last section, we discuss endomorphisms and augmented base loci of their eigendivisors.

1. INTRODUCTION

Let $X$ be a smooth projective variety of dimension $n \geq 1$ defined over $\mathbb{Q}$. Let $f : X \dashrightarrow X$ be a dominant rational map, and $f^* : N^1(X)_\mathbb{R} \to N^1(X)_\mathbb{R}$ be the induced map, where $N^1(X)_\mathbb{R} := N^1(X) \otimes \mathbb{R}$ and $N^1(X)$ is the group of Cartier divisors modulo numerical equivalence. Let $\rho(T, V)$ denote the spectral radius of a linear transformation $T : V \to V$ of a real or complex vector space. Then the first dynamical degree of $f$ is the quantity

$$\delta_f := \lim_{m \to \infty} \rho((f^m)^*, N^1(X)_\mathbb{R})^{1/m}.$$ 

Alternatively, if we let $H$ be any ample divisor on $X$, then $\delta_f$ is also given by the formula

$$\delta_f = \lim_{n \to \infty} \left( (f^m)^* H \cdot H^{n-1} \right)^{1/m}.$$ 

By definition, it is easy to see that $\delta_{f^r} = (\delta_f)^r$. Dynamical degree is actually a birational invariant and therefore we can define it for dominant rational maps on possibly singular projective varieties by taking resolution of singularities. For basic properties of dynamical degree, see [Dan17, DF01, Tru15].

In [KS16a, KS16b], Kawaguchi and Silverman studied an analogous arithmetic degree, which we now describe. Assume that $X$ and $f$ are defined over $\mathbb{Q}$, and write $X(\mathbb{Q})_f$ for the set of points $x$ whose forward $f$-orbit

$$\mathcal{O}_f(x) = \{x, f(x), f^2(x), \cdots\}$$

is well defined (i.e. $f^m(x)$ is not contained in the indeterminacy locus of $f$ for all $m \geq 0$). Further, let

$$h_X : X(\mathbb{Q}) \to [0, \infty)$$

2010 Mathematics Subject Classification. 37P55, 08A35.
be a logarithmic Weil height function on $X$ associated with an ample divisor, and let $h^+_X = \max \{1, h_X\}$. The arithmetic degree of $f$ at $x \in X(\overline{\mathbb{Q}})_f$ is the quantity
\[
\alpha_f(x) = \lim_{n \to \infty} h^+_X(f^n(x))^{1/n},
\]
if the limit exists.

The following Kawaguchi-Silverman conjecture (KSC for short) asserts that for a dominant rational self-map $f : X \dashrightarrow X$ of a projective variety $X$ over $\overline{\mathbb{Q}}$, the arithmetic degree $\alpha_f(x)$ of any point $x$ with Zariski dense $f$-orbit is equal to the first dynamical degree $\delta_f$ of $f$.

**Conjecture 1.1 (KSC).** (cf. [KS16b, Conjecture 6]) Let $f : X \to X$ be a dominant rational self-map of a projective variety $X$ over $\overline{\mathbb{Q}}$, and let $x \in X(\overline{\mathbb{Q}})_f$. Then the following hold.

1. The limit defining $\alpha_f(x)$ exists.
2. If $O_f(x)$ is Zariski dense in $X$, then $\alpha_f(x) = \delta_f$.

**Remark 1.2.** There are many special cases that this conjecture is proved. For recent results, see for example [LS18, Mat19, MY19, MZ19].

**Remark 1.3.** If $X$ has positive Kodaira dimension, then any dominant rational map does not have any Zariski dense orbits. This follows from “finiteness of pluricanonical representation”. See [MSS18, Remark 1.2], [Uen75, Theorem 14.10] or [NZ09, Theorem A]. Therefore, Conjecture 1.1 (2) is meaningful only for projective varieties of nonpositive Kodaira dimension.

**Remark 1.4.** If $f$ is a morphism, then the existence of arithmetic degree is proved, i.e. Conjecture 1.1(1) is true [KS16a].

In this paper, we focus on endomorphisms on projective varieties and collect some results related to this conjecture. In section 2, we list some basic facts on KSC. In section 3, we show some reduction results on KSC. In section 4, we prove KSC for endomorphisms on projective bundles over a smooth Fano variety of Picard number one. In section 5, we give positivity conditions on leading eigendivisors that is enough to prove KSC. In section 6, we discuss the augmented base loci of leading eigendivisors. This section is less related with KSC.

Throughout this paper, the ground field is $\overline{\mathbb{Q}}$ unless otherwise stated.

### 2. Preliminaries

We gather some facts on height functions. See [BG07, HS00, Lan83] for the definition and basic properties of height functions. Here, we simply list some fundamental facts that will be used in this paper.

- $h_E$ is bounded below outside $\text{Supp } E$ for any effective Cartier divisor $E$. 
• \( h_{\sum a_iD_i} = \sum a_i h_{D_i} + O(1) \) where \( O(1) \) is a bounded function.

• Let \( \pi : X \to Y \) be a surjective morphism of normal projective varieties and \( B \) an \( \mathbb{R} \)-Cartier divisor on \( Y \). Then \( h_B(\pi(x)) = h_{\pi^*B}(x) + O(1) \) for any \( x \in X(\overline{\mathbb{Q}}) \).

We list several basic facts on KSC which will be used in the rest of the paper.

**Lemma 2.1.** (cf. [MZ19, Lemma 2.5]) Let \( \pi : X \to Y \) be a dominant rational map of projective varieties. Let \( f : X \to X \) and \( g : Y \to Y \) be surjective endomorphisms such that \( g \circ \pi = \pi \circ f \). Then the following hold.

1. Suppose \( \pi \) is generically finite. Then KSC holds for \( f \) if and only if KSC holds for \( g \).
2. Suppose \( \delta_f = \delta_g \) and KSC holds for \( g \). Then KSC holds for \( f \).

**Lemma 2.2.** (cf. [San16, lemma 3.2], [Sil17, Lemma 3.3]) Let \( f : X \to X \) and \( g : Y \to Y \) be two surjective morphisms of projective varieties. Suppose KSC holds for both \( f \) and \( g \). Then KSC holds for \( f \times g \).

**Lemma 2.3.** (cf. [Mat19, Remark 2.9]) Let \( f : X \to X \) be a surjective endomorphism on a projective variety \( X \). Let \( n \) be a positive integer. Then KSC for \( f \) is equivalent to KSC for \( f^n \).

A normal projective variety \( X \) is called \( Q \)-abelian if there is an abelian variety \( A \) and a finite surjective morphism \( A \to X \) which is quasi-étale, i.e. étale in codimension one.

**Theorem 2.4.** (cf. [MZ19, Theorem 2.8]) Let \( X \) be a \( Q \)-abelian variety. Then KSC holds for any surjective endomorphism of \( X \).

3. REDUCTIONS OF KSC

3.1. Automorphism groups. We refer to Kollár-Mori [KM98] for standard notions and terminologies in birational geometry. A normal projective variety \( X \) is called weak Calabi-Yau variety if \( X \) has at most canonical singularities, \( K_X \sim 0 \), and the augmented irregularity

\[
\tilde{q}(X) := \sup\{ h^1(Y, \mathcal{O}_Y) \mid Y \to X \text{ finite surjective quasi-étale} \}
\]

is zero.

**Proposition 3.1.** Let \( X \) be a normal projective variety with at most klt singularities of dimension \( n \geq 1 \). Then Conjecture 1.1 is true for all automorphisms of normal projective varieties with dimension at most \( n \) and \( K_X \) is numerically trivial if and only if Conjecture 1.1 is true for all automorphisms of weak Calabi-Yau varieties with dimension at most \( n \).

**Remark 3.2.** This is a generalization of [LS18, Corollary 1.4] to singular case.

**Remark 3.3.** Let \( f \) be an automorphism of a normal projective variety \( X \). There is a \( G \)-equivariant resolution \( \pi : X' \to X \) and an automorphism \( f' \) of \( X' \) such that \( \pi \circ f' = f \circ \pi \) (cf. [BM97, Theorem 13.2]). Therefore, KSC for \( f \) reduces to KSC for \( f' \), which is an
automorphism on a smooth projective variety. This makes problem easier sometime, but it is sometime better to work on singular variety because it might have better birational geometric properties.

We use the following Kawamata-Nakayama-Zhang’s weak decomposition theorem (cf. [Kaw85, NZ10], [HL19, Lemma 2.7]).

**Lemma 3.4.** Let $X$ be a normal projective variety with at most klt singularities such that $K_X \sim Q$, and $f$ an automorphism of $X$. Then there exist a morphism $\pi: \tilde{X} \to X$ from a normal projective variety $\tilde{X}$, an automorphism $\tilde{f}$ of $\tilde{X}$ such that the following conditions hold.

1. $\pi$ is finite surjective and étale in codimension one.
2. $\tilde{X}$ is isomorphic to the product variety $Z \times A$ for a weak Calabi–Yau variety $Z$ and an abelian variety $A$.
3. The dimension of $A$ equals the augmented irregularity $\tilde{q}(X)$ of $X$.
4. There are automorphisms $\tilde{f}_Z$ and $\tilde{f}_A$ of $Z$ and $A$, respectively, such that the following diagram commutes:

   $\begin{array}{ccc}
   X & \xrightarrow{\pi} & \tilde{X} \\
   f \downarrow & & \downarrow \tilde{f} \\
   X & \xrightarrow{\pi} & \tilde{X} \\
   \end{array} \cong \begin{array}{ccc}
   \cong & & \cong \\\n   Z \times A & \xrightarrow{\tilde{f}_Z \times \tilde{f}_A} & Z \times A \\
   \end{array}$

**Proof of Proposition 3.1.** This follows from abundance for numerically trivial canonical divisor, Lemma 3.4, Lemma 2.1(1), Lemma 2.2, and Theorem 2.4.

Certain conditions on automorphism groups of a variety make us possible to run MMP equivariantly and reduce KSC to that of special varieties. For a normal projective variety $X$ of dimension $n$ and a subgroup $G$ of the automorphism group $\text{Aut}(X)$ of $X$, consider the following condition:

$\text{Hyp}(n, r)$. $G \simeq \mathbb{Z}^r$ with $1 \leq r \leq n - 1$ and $G$ is of positive entropy, i.e., $\delta_g > 1$ for all $g \in G \setminus \{\text{id}\}$.

**Remark 3.5.** In [DS04], they prove that every commutative subgroup $G$ of $\text{Aut}(X)$ has rank at most $\dim X - 1$.

**Theorem 3.6.** Let $X$ be a normal projective variety of dimension $n$ with at most klt singularities and $G$ be a subgroup of $\text{Aut}(X)$. Then the following statements hold.

1. Suppose $X$ and $G$ satisfies $\text{Hyp}(n, n - 1)$ and $X$ is not rationally connected. Then Conjecture 1.1 is true for all automorphisms $g \in G$.
2. Suppose $K_X \equiv 0$ and $(X, G)$ satisfies $\text{Hyp}(n, n - 2)$. Then Conjecture 1.1 is true for all automorphisms $g \in G$ if Conjecture 1.1 is true for all automorphisms of weak Calabi-Yau varieties of dimension $n$. 

Proof of Theorem 3.6. (1) Since \( X \) and \( G \) satisfies \( \text{Hyp}(n, n - 1) \) and \( X \) is not rationally connected, then by [Zha16, Theorems 1.1 and 2.4] or [HL19, Theorem 1.1], after replacing \( G \) by a finite-index subgroup, \( X \) is \( G \)-equivariantly birational to a \( Q \)-abelian variety \( Y \). Thus statement follows from Lemma 2.1 and Theorem 2.4.

(2) Since \( K_X \equiv 0 \) and \( X \) and \( G \) satisfies \( \text{Hyp}(n, n - 2) \), then by [HL19, Theorem 1.2], after replacing \( G \) by a finite-index subgroup, there is a \( G \)-equivariant quasi-étale morphism \( Y \to X \), such that \( Y \) is \( G \)-equivariantly birational to either a weak Calabi–Yau variety, an abelian variety, or a product of a weak Calabi–Yau surface and an abelian variety. In the last case automorphisms on the product are split, i.e. they are products of automorphisms on each factor ([NZ10, Lemma 2.14]). As for endomorphisms on surfaces, Conjecture 1.1 is proved in [MZ19, Theorem 1.3]. Thus the statement follows from Lemmas 2.1, 2.2 and 3.4 and Theorem 2.4. \( \Box \)

3.2. Albanese morphisms. To prove KSC, we can assume that the albanese morphism is surjective due to the following proposition.

Proposition 3.7. Let \( f \) be a surjective endomorphism of a normal projective variety \( X \). If \( f \) has Zariski dense orbits, then the Albanese morphism \( \pi : X \to A \) is surjective.

Proof. Let \( Y = \pi(X) \). Suppose \( Y \neq A \). Let \( B \) be the identity component of the stabilizer \( \text{Stab}(Y) \) of \( Y \) in \( A \). Since \( Y \neq A \), we have \( B \neq A \). Then \( A/B \) is a positive dimensional abelian variety and the image \( Z \) of \( Y \) in \( A/B \) has finite stabilizer in \( A/B \). Then \( Z \) is of general type (cf. [Mor85, Theorem 3.7]). Note that \( \dim Z > 0 \) because it generates \( A/B \).

Now, by the universality of the albanese morphism, \( f \) induces an endomorphism \( T_a \circ g \) on \( A \), where \( T_a \) is the translation by an element \( a \in A \) and \( g \) is a group endomorphism of \( A \). Then for any \( b \in B \), we have

\[
Y + g(b) = (T_a \circ g)(Y) + g(b) = g(Y) + g(b) + a = g(Y + b) + a = g(Y) + a = Y.
\]

Therefore, we have \( g(B) \subset B \). Thus, \( g \) induces an group endomorphism \( \overline{g} \) on \( A/B \). Then \( f \) induces \( T_a \circ g \) on \( A \) and \( T_{\overline{a}} \circ \overline{g} \) on \( A/B \), where \( \overline{a} \) is the image of \( a \) in \( A/B \). Since \( Z \), which is the image of \( X \) in \( A/B \), is of general type, \( T_{\overline{a}} \circ \overline{g} \) restricted on \( Z \) is a finite order automorphism. This contradicts to the fact that \( f \) has Zariski dense orbits. \( \Box \)

4. Projective bundles

Proposition 4.1. Let \( Y \) be a smooth Fano variety over \( \overline{\mathbb{Q}} \) of Picard number one. Let \( X = \mathbb{P}_Y(\mathcal{E}) \) be a projective bundle over \( Y \). Then KSC holds for all surjective endomorphisms \( f : X \to X \).

Proof. Let \( \pi : X \to Y \) be the projection.
Step 1. Replacing $f$ with its iterate, we may assume that $f$ induces an endomorphism $g: Y \to Y$ such that $g \circ \pi = \pi \circ f$ (cf. discussion before Theorem 2 in [Am03]).

Step 2. Since the Picard number of $Y$ is one, $g$ is polarized and KSC holds for $g$. Therefore, we may assume $\delta_f > \delta_g$. If $\delta_g > 1$, then $f$ is an int-amplified endomorphism. Since $X$ is smooth rationally connected, KSC holds for $f$ by [MY19, Theorem 1.1].

Step 3. Suppose $\delta_f > \delta_g = 1$. We claim that $E$ is a direct sum of invertible sheaves, i.e. $E \cong \bigoplus_i L_i$ for some $L_i$. If $\dim Y = 1$, then $Y \cong \mathbb{P}^1$ and the claim follows from Grothendieck’s theorem. Suppose $\dim Y > 1$. By Kodaira vanishing and Serre duality, $H^1(Y, L) = 0$ for all invertible sheaves $L$ on $Y$ (we use the assumption that the Picard number of $Y$ is one). Also, $Y$ is simply connected. We can apply [AK17, Theorem 2] so that we get the claim.

Now, $g$ is an automorphism because $Y$ has Picard number one and $\delta_g = 1$. Since $Y$ is a smooth Fano, we have $\text{Pic} Y = \mathbb{Z}$. Therefore $g^* \text{acting on } \text{Pic} Y$ as identity, and we have $g^* E \cong \bigoplus_i g^* L_i \cong \bigoplus_i L_i \cong E$. Thus we get the following commutative diagram:

![Diagram](attachment:image.png)

where $h$ is the isomorphism induced by the isomorphism $E \cong g^* E$ and $F$ is the morphism induced by the universal property of fiber product. Since $f$ has degree larger than one on the fibers, so does $F$. By [Am03, Theorem 1] and the comment below it and simply connectedness of $Y$, we get $X \cong Y \times \mathbb{P}^r$ where $r + 1$ is the rank of $E$. By [San16, Theorem 4.6], and KSC for polarized endomorphisms, we are done. \[\square\]

5. Canonical Heights

**Proposition 5.1.** Let $X$ be a geometrically integral variety over a number field $K$ and suppose $X_K$ is $\mathbb{Q}$-factorial normal projective variety. Let $D$ be an $\mathbb{R}$-divisor on $X$. Suppose $D$ is $\mathbb{R}$-linearly equivalent to at least two effective $\mathbb{R}$-divisors, i.e. there exist effective $\mathbb{R}$-divisors $D'$ and $D''$ such that $D \sim_\mathbb{R} D' \sim_\mathbb{R} D''$ and $D' \neq D''$. Let $h_D$ be a Weil height function associated with $D$ and $B, d$ positive real numbers. Then the set

$$\left\{ P \in X(L) \middle| h_D(P) \leq B, K \subseteq L \subseteq \overline{K} \text{ is an intermediate field with } [L : K] \leq d \right\}$$

is not Zariski dense in $X_K$.

**Proof.** Step 1. There is an effective $\mathbb{R}$-divisor $\sum_{i=1}^r c_i F_i$ where $r \geq 1$ and $F_i$ are prime divisor such that $D \sim_\mathbb{R} \sum_{i=1}^r c_i F_i$, $c_1 \neq 0$, and $F_1$ has positive Iitaka dimension.
Let $D' = \sum a_i E_i$ and $D'' = \sum b_i E_i$ where $E_i$’s are prime divisors. Set $D_0 = \sum \min\{a_i, b_i\} E_i$. Then $D - D_0 \sim R D' - D_0 \sim R D'' - D_0$ and two effective divisors $D' - D_0$ and $D'' - D_0$ have no common components. By [BCHM10, Proposition 3.5.4], $D - D_0 \sim R M$ where $M$ is an effective $R$-divisor such that every component is movable.

Step 2. By step one, $h_D = \sum c_i h_{F_i} + O(1) \geq c_i h_{F_i} + O(1)$ on $(X_R \setminus \bigcup_{i \geq 2} \text{Supp } F_i)(\overline{K})$. Now the proposition follows from [Mat19, Proposition 3.5].

\[\square\]

**Proposition 5.2.** (cf. [Mat19, Proposition 3.6]) Let $X$ be a $\mathbb{Q}$-factorial normal projective variety and $f : X \rightarrow X$ be a surjective morphism with $\delta_f > 1$.

1. If there is an $R$-divisor $D$ which is $R$-linearly equivalent to at least two effective $R$-divisors such that $f^* D \sim R \delta_f D$, then KSC holds for $f$.

2. Suppose $f$ is an automorphism. Suppose further that there are $R$-divisors $D_+$ and $D_-$ such that $f^* D_+ \sim R \delta_f D_+$, $(f^{-1})^* D_- \sim R \delta_{f^{-1}} D_-$ and $D_+ + D_-$ is $R$-linearly equivalent to at least two effective $R$-divisors. Then KSC holds for $f$.

**Remark 5.3.** The condition in (2) is a generalization of property (B) in [LS18, Definition 3.7].

**Proof.** (1) Take a height function $h_D$ associated with $D$. Let $\hat{h}(x) = \lim_{n \rightarrow \infty} h_D(f^n(x))/\delta^n_f$ for all $x \in X(\overline{K})$. Then $\hat{h}$ satisfies:

- $\hat{h}(f(x)) = \delta_f \hat{h}(x)$;
- $\hat{h} = h_D + O(1)$;
- The set of points $x \in X(\overline{K})$ such that $\hat{h}(x) = 0$ and $x$ is defined over a single number field is not Zariski dense.

This implies that if $x \in X(\overline{K})$ has Zariski dense $f$-orbit, then $\hat{h}(x) > 0$ and $\alpha_f(x) = \delta_f$.

(2) As in (1), let

\[
\hat{h}_{D_+}(x) = \lim_{n \rightarrow \infty} \frac{h_{D_+}(f^n(x))}{\delta^n_f} \\
\hat{h}_{D_-}(x) = \lim_{n \rightarrow \infty} \frac{h_{D_-}(f^{-n}(x))}{\delta_{f^{-1}}^n}
\]

for all $x \in X(\overline{K})$. Then

- $\hat{h}_{D_+}(f(x)) = \delta_f \hat{h}_{D_+}(x)$, $\hat{h}_{D_-}(f(x)) = \delta_{f^{-1}} \hat{h}_{D_-}(x)$;
- $\hat{h}_{D_+} = h_{D_+} + O(1)$, $\hat{h}_{D_-} = h_{D_-} + O(1)$;
- The set of points $x \in X(\overline{K})$ on which $\hat{h}_{D_+} + \hat{h}_{D_-}$ is bounded and $x$ is defined over a single number field is not Zariski dense.

This implies that if $x \in X(\overline{K})$ has Zariski dense $f$-orbit, then $\hat{h}_{D_+}(x) > 0$ and $\alpha_f(x) = \delta_f$.

\[\square\]
6. Augmented Base Loci and Endomorphisms

The augmented base locus of an $\mathbb{R}$-Cartier divisor $D$ is defined as follows.

**Definition 6.1.** The augmented base locus of an $\mathbb{R}$-Cartier divisor $D$ on a normal projective variety $X$ is the Zariski closed subset

\[ B_+(D) := \bigcap \mathcal{B}(D - A), \]

where $\mathcal{B}(\cdot)$ stands for the stable base locus and the intersection is taken over all $\mathbb{R}$-Cartier ample divisors $A$ such that $D - A$ is $\mathbb{Q}$-Cartier.

The augmented base locus $B_+(D)$ is equal to the exceptional locus $E(D)$ if $D$ is nef (cf. [Bir17]).

**Proposition 6.2.** Let $f : X \rightarrow X$ be a surjective endomorphism and $D$ an $\mathbb{R}$-Cartier divisor on a normal projective variety $X$. If $f^* D \equiv dD$ with $d \in \mathbb{R}_{>0}$, then $f(B_+(D)) \subset B_+(D)$.

Moreover, if $D$ is nef, then $f^{-1}(B_+(D)) = B_+(D)$.

**Proof.** Suppose $x \notin f^{-1}(B_+(D))$. Then there is an $\mathbb{R}$-Cartier ample divisor $D$ on $X$ such that $D - A$ is $\mathbb{Q}$-Cartier and $f(x) \notin B(D - A)$. Then $x \notin B(f^* D - f^* A)$, and this implies $x \notin B_+(f^* D)$. Since the augmented base locus depends only on the numerical class of the divisor and it does not change by positive real multiplication of the divisor, we have $x \notin B_+(f^* D) = B_+(dD) = B_+(D)$. This shows the first statement.

Now assume $D$ is nef. Since the exceptional locus is compatible with pull-backs by finite surjective morphisms, we have

\[ f^{-1}(B_+(D)) = f^{-1}(E(D)) = E(f^* D) = B_+(f^* D) = B_+(dD) = B_+(D). \]

Let $X$ be a normal projective variety and $f : X \rightarrow X$ be a surjective endomorphism. Suppose there exists a nef and big $\mathbb{R}$-Cartier divisor $D$ such that $f^* D \equiv dD$ with $d \in \mathbb{R}_{>1}$. By [MZ18], $f$ is actually polarized and we can construct canonical heights associated with ampler divisors. But we may also consider the canonical height $\hat{h}_D(x) = \lim_{n \to \infty} h_D(f^n(x))/d^n$ associated with $D$ and this has the following property: If $x \in (X \setminus B_+(D))(\overline{\mathbb{Q}})$, then $x$ is $f$-preperiodic if and only if $\hat{h}_D(x) = 0$.

Let us give a purely geometric application of Proposition 6.2.

**Proposition 6.3.** Let $X$ be a $\mathbb{Q}$-factorial normal projective variety and $f : X \rightarrow X$ be a surjective endomorphism such that the map $f^* : N^1(X)_{\mathbb{R}} \rightarrow N^1(X)_{\mathbb{R}}$ is a scalar multiplication. Suppose further that $f$ has no non-trivial totally invariant closed subset (i.e. $f^{-1}(Z) = Z$ implies $Z = X$ or $Z = \emptyset$). Then the pseudo-effective cone of $X$ is equal to the nef cone of $X$: $\overline{\operatorname{Eff}}(X) = \operatorname{Nef}(X)$.
**Proof.** By Proposition 6.2 and the assumption, for every nef and big $\mathbb{R}$-Cartier divisor $D$ on $X$, we have $B_+(D) = \emptyset$. That is, every nef and big $\mathbb{R}$-Cartier divisor is ample. This implies all big $\mathbb{R}$-Cartier divisor is nef, and therefore ample. □

**Corollary 6.4.** Let $X$ be a smooth rationally connected variety and $f : X \to X$ be a polarized endomorphism (i.e. $f^*H \equiv dH$ for some ample $\mathbb{R}$-divisor $H$ and $d \in \mathbb{R}_{>1}$). Suppose $f$ has no non-trivial totally invariant closed subset. Then $X$ is a Fano variety with Picard number at most $\dim X$.

**Proof.** By [MZ18], if we replace $f$ with its iterate, $f^*$ acts as a scalar multiplication on $N_1^1(X)_{\mathbb{R}}$. By Proposition 6.3, we have $\text{Eff}(X) = \text{Nef}(X)$. By [Yos20, Corollary 1.4], $X$ is of Fano type, i.e. $-K_X - \Delta$ is ample for some effective $\mathbb{Q}$-divisor $\Delta$. Thus, we get $-K_X$ is ample.

Every $K_X$-negative contraction of $X$ is of fiber type because $f$ has no non-trivial totally invariant closed subset. Moreover $f$ induces a polarized endomorphism on the base of the contraction which has no non-trivial totally invariant closed subset. We can repeat this process and final output is a point (cf. [MZ18]). This implies that the Picard number of $X$ is less than or equal to $\dim X$. □

**Acknowledgments**

The first author was partially supported by the China Scholar Council ‘High-level university graduate program’. The second author is supported by JSPS Overseas Research Fellowship.

**References**

[Am03] Ekaterina Amerik, *On endomorphisms of projective bundles*, Manuscripta Math. 111 (2003), no. 1, 17–28. ↑

[AK17] Ekaterina Amerik and Alexandra Kuznetsova, *Endomorphisms of projective bundles over a certain class of varieties*, Bull. Korean Math. Soc. 54 (2017), no. 5, 1743–1755. ↑

[Bir17] Caucher Birkar, *The augmented base locus of real divisors over arbitrary fields*, Math. Ann. 368 (2017), no. 3-4, 905–921. ↑

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. 23(2) (2010), 405-468. ↑

[BM97] E. Bierstone and P. D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. 128 (1997), 207-302. ↑

[BG07] Enrico Bombieri, Walter Gubler, *Heights in Diophantine geometry*, Cambridge university press, 2007. ↑

[Dan17] Dang, N-B., *Degrees of iterates of rational maps on normal projective varieties*, arXiv:1701.07760. ↑

[DF01] Diller, J., Favre, C., *Dynamics of bimeromorphic maps of surfaces*, Amer. J. Math. 123 no. 6, 1135–1169, (2001). ↑

[DS04] Tien-Cuong Dinh and Nessim Sibony, *Groupes commutatifs d’automorphismes d’une variété kählérienne compacte*, Duke Math. J. 123 (2004), no. 2, 311–328. ↑
[HL19] Fei Hu and Sichen Li, Free abelian group actions on normal projective varieties: sub-maximal dynamical rank case, arXiv:1907.00229.↑, 4, 5

[HS00] Marc Hindry, Joseph H. Silverman, Diophantine geometry. An introduction, Graduate Text in Mathematics, no. 20, Springer-Verlag, New York, 2000. ↑

[Lan83] Serge Lang, Fundamentals of Diophantine Geometry, Springer-Verlag, New York, 1983. ↑

[LS18] John Leslie and Matthew Satriano, Canonical heights on hyper-Kähler varieties and the Kawaguchi-Silverman conjecture, to appear in Int. Math. Res. Notices, arXiv:1802.07388, ↑, 2, 3, 7

[Kaw85] Yujiro Kawamata, Minimal models and the Kodaira dimension of algebraic fiber spaces, J. Reine Angew. Math. 363 (1985), 1–46. ↑ 4

[KM98] János Kollár and Shigefumi Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. ↑ 3

[KS16a] Shu Kawaguchi and Joseph J. Silverman, Dynamical canonical heights for Jordan blocks, arithmetic degrees of orbits, and nef canonical heights on abelian varieties, Trans. Amer. Math. Soc. 386 (2016), no. 7, 5009-5035.↑, 1, 2

[KS16b] Shu Kawaguchi and Joseph J. Silverman, On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties, J. reine angew. Math. 713 (2016), 21-48. ↑ 1, 2

[Mat19] Yohsuke Matsuzawa, Kawaguchi-Silverman conjecture for endomorphisms on several classes of varieties, to appear in Adv. Math. arXiv:1902.06072.↑, 2, 3, 7

[MSS18] Yohsuke Matsuzawa, Kaoru Sano and Takahiro Shibata, Arithmetic degrees and dynamical degrees of endomorphisms on surfaces, Algebra Number Theory 12 (2018), no. 7, 1635-1657. ↑ 2

[MY19] Yohsuke Matsuzawa and Show Yoshikawa, Kawaguchi-Silverman conjecture for endomorphisms on rationally connected varieties admitting an int-amplified endomorphism, arXiv:1908.11537.↑, 2, 6

[MZ18] Sheng Meng and De-Qi Zhang, Building blocks of polarized endomorphisms of normal projective varieties, Adv. Math. 325 (2018), 243-273.↑, 8, 9

[MZ19] Sheng Meng and De-Qi Zhang, Kawaguchi-Silverman conjecture for surjective endomorphisms, arXiv:1908.01605.↑, 2, 3, 5

[Mor85] Shigefumi Mori, Classification of higher-dimensional varieties, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 269–331, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987. ↑ 5

[NZ09] Noboru Nakayama and De-Qi Zhang, Building blocks of étale endomorphisms of complex projective manifolds, Proc. Lond. Math. Soc. 99 (2009), 725-756. ↑ 2

[NZ10] Noboru Nakayama and De-Qi Zhang, Polarized endomorphisms of complex normal varieties, Math. Ann. 246 (2010), 991-1018. ↑, 4, 5

[San16] Kaoru Sano, Dynamical degree and arithmetic degree of endomorphisms on product varieties, to appear in Tohoku Math. J., arXiv:1604.04174.↑, 3, 6

[Sil17] Joseph H. Silverman, Arithmetic and dynamical degrees on abelian varieties, J. Théor. Nombres Bordeaux 29 (2017), no. 1, 151-167. ↑ 3

[Tru15] Truong, T. T., (Relative) dynamical degrees of rational maps over an algebraic closed field, arXiv:1501.01523v1.↑ 1

[Uen75] Kenji Ueno, Classification theory of algebraic varieties and compact complex spaces, Lecture Notes in Mathematics, vol. 439, Springer-Verlag, Berlin-New York, 1975, Notes written in collaboration with P. Cherenack. ↑ 2

[Yos20] Shou Yoshikawa, Structure of Fano fibrations of varieties admitting an int-amplified endomorphism, arXiv:2002.01257 ↑ 9
[Zha16] De-Qi Zhang, *n*-dimensional projective varieties with the action of an abelian group of rank \(n-1\), Trans. Amer. Math. Soc. **368** (2016), no. 12, 8849–8872.

School of Mathematical Sciences, Fudan University, Shanghai 200433, People’s Republic of China

*E-mail address*: lisichen123@foxmail.com

*URL*: https://www.researchgate.net/profile/Sichen_Li4

Department of Mathematics, Box 1917, Brown University, Providence, Rhode Island 02912, USA

*E-mail address*: matsuzawa@math.brown.edu