\textbf{Q-LINEAR DEPENDENCE OF CERTAIN BESSEL MOMENTS}

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\textbf{ABSTRACT.} Let $I_0$ and $K_0$ be modified Bessel functions of the zeroth order. We use Vanhoove’s differential operators for Feynman integrals to derive upper bounds for dimensions of the $Q$-vector space spanned by certain sequences of Bessel moments

\[
\left\{ \int_0^\infty [I_0(t)]^a[K_0(t)]^b t^{2k+1} \, dt \mid k \in \mathbb{Z}_{\geq 0} \right\},
\]

where $a$ and $b$ are fixed non-negative integers. For $a \in \mathbb{Z} \cap [1, b)$, our upper bound for the $Q$-linear dimension is $\lfloor (a + b - 1)/2 \rfloor$, which improves the Borwein–Salvy bound $\lfloor (a + b + 1)/2 \rfloor$. Our new upper bound $\lfloor (a + b - 1)/2 \rfloor$ is not sharp for $a = 2, b = 6$, due to an exceptional $Q$-linear relation

\[
\int_0^\infty [I_0(t)]^2[K_0(t)]^6 t^2 \, dt = 72 \int_0^\infty [I_0(t)]^2[K_0(t)]^6 t^3 \, dt,
\]

which is provable by integrating modular forms.

\section{1. Introduction}

We define modified Bessel functions of the zeroth order as $I_0(t) = \frac{1}{\pi} \int_0^\pi e^{t \cos \theta} \, d\theta$ and $K_0(t) = \int_0^\infty e^{-t \cosh u} \, du$, where $t > 0$. For certain non-negative integers $a, b$ and $k$, the Bessel moments

\[
\text{IKM}(a, b; 2k + 1) := \int_0^\infty [I_0(t)]^a[K_0(t)]^b t^{2k+1} \, dt \tag{1.1}
\]

are interesting objects to both physicists and mathematicians. In the graphical language of physicists, they represent Feynman diagrams in 2-dimensional quantum field theory \cite{2}, and also contribute to the finite part of renormalized perturbative expansions in $(4 - \epsilon)$-dimensional quantum electrodynamics \cite{17, 18}. From the analytic perspective of mathematicians, every well-defined sequence of Bessel moments $[\text{IKM}(a, b; 2k + 1) | k \in \mathbb{Z}_{\geq 0}]$ is completely determined by the first few terms and a linear recursion \cite{6}, and certain Bessel moments are related to critical $L$-values attached to special modular forms \cite{9, 24}.

According to the theory of Borwein–Salvy \cite{3}, we have the following recursions for sequences of Bessel moments satisfying $a + b \in \{5, 6\}$ and $a \in \mathbb{Z} \cap [0, b)$:

\[
\sum_{j=0}^{3} (-1)^j p_{a+b,j}(j+k+1) \text{IKM}(a, b; 2j + k) = 0 \tag{1.2}
\]
From the recursions above, one might deduce upper bounds $\dim_{\mathbb{Q}} \text{span}_{\mathbb{Q}} \{\text{IKM}(a, b; 2k + 1) | k \in \mathbb{Z}_{\geq 0}\} \leq 3$ for these sequences of Bessel moments involving 5 or 6 Bessel factors in the integrands. However, for $a + b \in \{5, 6\}$ and non-vacuum diagrams satisfying $a \in \mathbb{Z} \cap [1, b)$, such upper bounds are not tight enough. It has been shown by Bailey–Borwein–Broadhurst–Glasser \cite[§5.10]{2} that $45 \text{IKM}(2, 3; 5) = 228 \text{IKM}(2, 3; 3) - 16 \text{IKM}(2, 3; 1)$, and several similar sum rules have been proposed and checked up to 1200 decimal places.

**Conjecture 1.1** (Bailey–Borwein–Broadhurst–Glasser \cite[§§5.1, 5.5, 6.1, 6.2, 6.4]{2}). The following integral identities are true:

\[
\begin{align*}
\int_0^\infty I_0(t)[K_0(t)]^4 t (16 - 228r^2 + 45t^4) \, dt &= 0, \quad \text{(1.4)} \\
\int_0^\infty [K_0(t)]^5 t (16 - 228r^2 + 45t^4) \, dt &= 24, \quad \text{(1.5)} \\
\int_0^\infty [I_0(t)]^2 [K_0(t)]^4 t (2 - 85r^2 + 72t^4) \, dt &= 0, \quad \text{(1.6)} \\
\int_0^\infty I_0(t)[K_0(t)]^5 t (2 - 85r^2 + 72t^4) \, dt &= 0, \quad \text{(1.7)} \\
\int_0^\infty [K_0(t)]^6 t (2 - 85r^2 + 72t^4) \, dt &= \frac{15}{2}. \quad \text{(1.8)}
\end{align*}
\]

In our recent work \cite[§3]{24}, we have verified (1.4) through explicit evaluations of $\text{IKM}(1, 4; 1)$, $\text{IKM}(1, 4; 3)$ and $\text{IKM}(1, 4; 5)$. In \cite[Lemma 3.4]{28}, we confirmed (1.6) and (1.7), using a special differential operator of fourth order; a similar service was performed on (1.5) and (1.8) in \cite[Proposition 5.3 and Lemma 5.8]{25}. In §2 of this paper, we give a unified proof of all the identities in Conjecture 1.1 along with a generalization to arbitrarily many Bessel factors, as described in the theorem below.

**Theorem 1.2** ($\mathbb{Q}$-linear dependence of certain Bessel moments). When $a \in \mathbb{Z} \cap [1, b), b \in \mathbb{Z}_{\geq 2}$, the set

\[
\{\text{IKM}(a, b; 2k + 1) | k \in \mathbb{Z} \cap [0, (a + b - 1)/2]\}
\]

is linearly dependent over $\mathbb{Q}$, and

\[
\dim_{\mathbb{Q}} \text{span}_{\mathbb{Q}} \{\text{IKM}(a, b; 2k + 1) | k \in \mathbb{Z}_{\geq 0}\} \leq [(a + b - 1)/2]. \quad \text{(1.9)}
\]

When $n \in \mathbb{Z}_{\geq 2}$, the set

\[
\{1\} \cup \{\text{IKM}(0, n; 2k + 1) | k \in \mathbb{Z} \cap [0, (n - 1)/2]\}
\]

is linearly dependent over $\mathbb{Q}$, and

\[
\dim_{\mathbb{Q}} \text{span}_{\mathbb{Q}} \{\text{IKM}(0, n; 2k + 1) | k \in \mathbb{Z}_{\geq 0}\} \leq [(n + 1)/2]. \quad \text{(1.10)}
\]

In the statements above, $[x]$ stands for the greatest integer less than or equal to $x$. 

Here, we point out that the inequality in (1.9′) follows immediately from the \( Q \)-linear dependence of the set in (1.9), because the Borwein–Salvy theory [6, Theorem 1.1] has already provided us with a linear recursion with non-vanishing integer coefficients, involving \( \lfloor (a + b + 3)/2 \rfloor \) consecutive terms in the corresponding sequence \( \{IKM(a, b; 2k + 1 | k \in \mathbb{Z}_{\geq 0}) \} \). Thus, our upper bound given in (1.9′) is exactly one dimension smaller than what is inferable from the Borwein–Salvy recurrence. Meanwhile, the inequality in (1.10′) gives the same upper bound on \( Q \)-linear dimension as the Borwein–Salvy recursion. What is new here is that the \( Q \)-linear basis can be constructed from a subset of \( \{1\} \cup \{IKM(0, n; 2k + 1 | k \in \mathbb{Z} \cap [0, (n - 1)/2]) \} \), rather than a subset of vacuum diagrams \( \{IKM(0, n; 2k + 1 | k \in \mathbb{Z} \cap [0, n - 3/2]) \} \). Recently, a more insightful interpretation of the dimension bounds in Theorem 1.2, based on de Rham cohomology classes, has been presented by Fresán–Sabbah–Yu [16, Remark 6.8].

The present author initially had thought that the bound in (1.9′) could no longer be improved, until a numerical counterexample \( IKM(2, 6; 1) = 72 IKM(2, 6; 3) \) suggested otherwise. Extending some modular techniques developed in [24], we will verify this surprising \( Q \)-linear dependence in §3 along with a few related results in the theorem below.

**Theorem 1.3** (Some exceptional sum rules). If we define

\[
\hat{IKM}(4, 4; 3) := \int_{0}^{\infty} [I_0(t)K_0(t)]^2 \left\{ [I_0(t)K_0(t)]^2 - \frac{1}{4t^2} \right\} t^3 dt \quad (1.11)
\]

as an “honorary Bessel moment”, then we have the following identities:

\[
\begin{align*}
IKM(4, 4; 1) - 72 \hat{IKM}(4, 4; 3) & = \frac{7 \log 2}{2}, \\
IKM(3, 5; 1) - 72 IKM(3, 5; 3) & = -\frac{5\pi^2}{12}, \\
IKM(2, 6; 1) - 72 IKM(2, 6; 3) & = 0, \\
IKM(1, 7; 1) - 72 IKM(1, 7; 3) & = \frac{7\pi^4}{48}.
\end{align*}
\]

Moreover, the set \( \{\pi, IKM(1, 7; 1), IKM(1, 7; 5), IKM(2, 6; 1), IKM(2, 6; 5)\} \) is algebraically dependent, under the constraint of a non-linear sum rule

\[
7\pi^4 IKM(2, 6; 1) - 6912[IKM(1, 7; 1) IKM(2, 6; 5) - IKM(1, 7; 5) IKM(2, 6; 1)] = \frac{45\pi^6}{16}. \quad (1.16)
\]

We note that the algebraic relations [over \( \mathbb{Q}(\pi, \log 2) \)] displayed in (1.12)–(1.16) are truly aberrant, in that they do not belong to the Broadhurst–Roberts ideal [30, §4.2] that presumably exhausts algebraic relations for on-shell Bessel moments.

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2. APPLICATIONS OF VANHOVE’S DIFFERENTIAL EQUATIONS TO ℚ-LINEAR DEPENDENCE

As in our recent work [25, §2], we introduce abbreviations for off-shell Feynman diagrams

$$\tilde{IKM}(a+1,b;m|u) := \int_0^\infty I_0(\sqrt{ut})[I_0(t)]^a[K_0(t)]^b t^m \mathrm{d}t,$$

(2.1)

$$\tilde{IKM}(a,b+1;m|u) := \int_0^\infty K_0(\sqrt{ut})[I_0(t)]^a[K_0(t)]^b t^m \mathrm{d}t,$$

(2.2)

whenever the non-negative integers \(a, b, m \in \mathbb{Z}_{\geq 0}\) ensure convergence of the integrals. For a smooth bivariate function \(f(t,u)\), we define \(D^0 f(t,u) = f(t,u)\), and \(D^{n+1} f(t,u) = \frac{\partial}{\partial u} D^n f(t,u)\) for all \(n \in \mathbb{Z}_{\geq 0}\). For convenience, we will also write \(D^n f(t,1)\) for evaluating \(D^n f(t,u)\) at \(u = 1\), and so forth.

Vanhoe’s operator \(\tilde{L}_n\) is an \(n\)th order holonomic differential operator in the variable \(u\), which acts on off-shell Feynman diagrams \(\tilde{IKM}(a,n+2-a;1|u), a \in \mathbb{Z} \cap [1,(n+2)/2)\) to produce constants. For example, Vanhoe’s third- and fourth-order operator can be written explicitly as follows [20, Table 1, \(n = 4\) and \(u = 5\):]

$$\tilde{L}_3 := u^2(u-4)(u-16)D^3 + 6u(u^2 - 15u + 32)D^2 + (7u^2 - 68u + 4)D^1 + (u-4)D^0,$$

(2.3)

$$\tilde{L}_4 := u^2(u-25)(u-9)(u-1)D^4 + 2u(5u^3 - 140u^2 + 777u - 450)D^3 + (25u^3 - 518u^2 + 1839u - 450)D^2 + (3u-5)(5u-57)D^1 + (u-5)D^0.$$

(2.4)

They satisfy the following differential equations of Vanhoe’s type [25, Lemmata 2.2 and 3.1]:

\[
\begin{align*}
\tilde{L}_3 \tilde{IKM}(2,3;1|u) &= 0, \quad \forall u \in (0,4); \\
\tilde{L}_3 \tilde{IKM}(1,4;1|u) &= -3, \quad \forall u \in (0,16); \\
\tilde{L}_3 \tilde{IKM}(1,4;1|u) &= \frac{3}{2}, \quad \forall u \in (0,\infty).
\end{align*}
\]

(2.5)

\[
\begin{align*}
\tilde{L}_4 \tilde{IKM}(2,4;1|u) &= 0, \quad \forall u \in (0,9); \\
\tilde{L}_4 \tilde{IKM}(1,5;1|u) &= -\frac{15}{2}, \quad \forall u \in (0,25); \\
\tilde{L}_4 \tilde{IKM}(1,5;1|u) &= \frac{3}{2}, \quad \forall u \in (0,\infty).
\end{align*}
\]

(2.6)

In what follows, we use Vanhoe’s differential equations to produce an algorithmic proof of Theorem [25], which also recovers Conjecture [11] as special cases.

**Theorem 2.1** (Sum rules for arbitrarily many Bessel factors). For each \(n \in \mathbb{Z}_{\geq 0}\), there exists a non-zero polynomial \(f_n(\xi) \in \frac{1}{n+4}\mathbb{Z}[\xi]\) whose degree does not exceed \(\lfloor(n+1)/2\rfloor\), such that the following inhomogeneous sum rule

$$\int_0^\infty [K_0(t)]^{n+2}f_n(t^2) \mathrm{d}t = (n+1)! \equiv \Gamma(n+2)$$

(2.7)

holds. Accordingly, the same polynomial applies to a homogeneous sum rule

$$\int_0^\infty [I_0(t)]^a[K_0(t)]^{n+2-a}f_n(t^2) \mathrm{d}t = 0,$$

(2.8)

where \(a \in \mathbb{Z} \cap [1,(n+2)/2)\). As a result, the statements about \(\mathbb{Q}\)-linear dependence in Theorem [25] are true.
Proof. We recall from [25, Lemma 4.2] that we have the following differential equations of Van- 
hove’s type:

\[
\begin{cases}
\tilde{L}_n \tilde{IKM}(1, n + 1, 1|u) = -\frac{(n+1)!}{2^n}, \\
\tilde{L}_n \tilde{IKM}(1, n + 1, 1|u) = \frac{n!}{2^n}, \\
\tilde{L}_n \tilde{IKM}(j, n + 2 - j, 1|u) = 0, & \forall j \in \mathbb{Z} \cap [2, \frac{n}{2} + 1], \\
\tilde{L}_n \tilde{IKM}(j, n + 2 - j, 1|u) = 0, & \forall j \in \mathbb{Z} \cap [2, \frac{n+1}{2}].
\end{cases}
\] (2.9)

which can be verified through the relations

\[
\begin{cases}
t\tilde{L}_n I_0(\sqrt{ut}) = \frac{(-1)^n}{2^n} I_{n+2}^*(t_0(\sqrt{ut})), \\
t\tilde{L}_n K_0(\sqrt{ut}) = \frac{(-1)^n}{2^n} I_{n+2}^* \frac{K_0(\sqrt{ut})}{t}.
\end{cases}
\] (2.10)

and integrations by parts in the variable \( t \). Here,

\[L_{n+2}^* g(t, u) = \sum_{k=0}^{n+2} (-1)^k \partial^k_t \left[ \lambda_{n+2,k}(t) g(t, u) \right],\] (2.11)

defines the formal adjoint to the Borwein–Salvy operator

\[L_{n+2} = \mathcal{L}_{n+2,n+2} = \sum_{k=0}^{n+2} \lambda_{n+2,k}(t) \frac{\partial^k}{\partial t^k},\] (2.12)

which in turn, is constructed by the Bronstein–Mulders–Weil algorithm [12, Theorem 1]:

\[
\begin{aligned}
\mathcal{L}_{n+2,0} &= \left( \frac{d}{dt} \right)^0, \\
\mathcal{L}_{n+2,1} &= \frac{d}{dt}, \\
\mathcal{L}_{n+2,k+1} &= \frac{d}{dt} \mathcal{L}_{n+2,k} - k(n+2-k) \mathcal{L}_{n+2,k-1}, & \forall k \in \mathbb{Z} \cap [1, n+1].
\end{aligned}
\] (2.13)

We also remind our readers of the fact that the Borwein–Salvy operator \( L_{n+2} \) is the \((n+1)\)st symmetric power of the Bessel differential operator, so that it annihilates every member in the set \(|I_0(t)|^n [K_0(t)]^{n+1} \in \mathbb{Z} \cap [0, n+1]|\).

As we specialize the procedure of integration by parts in [25, (4.24)] to \( u = 1 \), we have

\[
0 = \int_0^\infty \frac{I_0(t)}{t} L_{n+1}([K_0(t)]^n) \, dt = \int_0^\infty \frac{I_0(t)}{t} \mathcal{L}_{n+1,n+1}([K_0(t)]^n) \, dt \\
= \int_0^\infty \frac{I_0(t)}{t} \frac{d}{dt} \mathcal{L}_{n+1,n}([K_0(t)]^n) \, dt - n \int_0^\infty \frac{I_0(t)}{t} \mathcal{L}_{n+1,n-1}([K_0(t)]^n) \, dt \\
= -(-1)^n n! - \int_0^\infty \mathcal{L}_{n+1,n}([K_0(t)]^n) \frac{dI_0(t)}{dt} \, dt \\
- n \int_0^\infty \frac{I_0(t)}{t} \mathcal{L}_{n+1,n-1}([K_0(t)]^n) \, dt.
\] (2.14)

From [25, Lemma 4.2], we know that all subsequent integrations by parts will not result in any boundary contributions like \((-1)^n n!\). Without loss of generality, we assume that \( n \in \mathbb{Z}_{\geq 2} \), and carry
on the computations above a few steps further:

\[
(-1)^n n! = - \int_0^\infty I_1(t) \mathcal{L}_{n+1,n} [[K_0(t)]^n] \, dt - n \int_0^\infty t I_0(t) \mathcal{L}_{n+1,n-1} [[K_0(t)]^n] \, dt
\]

\[
= \int_0^\infty \left\{ \frac{d(t I_1(t))}{dt} - n t I_0(t) \right\} \mathcal{L}_{n+1,n-1} [[K_0(t)]^n] \, dt
\]

\[
+ 2(n-1) \int_0^\infty t^2 I_1(t) \mathcal{L}_{n+1,n-2} [[K_0(t)]^n] \, dt
\]

\[
= -(n-1) \int_0^\infty t I_0(t) \mathcal{L}_{n+1,n-1} [[K_0(t)]^n] \, dt
\]

\[
+ 2(n-1) \int_0^\infty t^2 I_1(t) \mathcal{L}_{n+1,n-2} [[K_0(t)]^n] \, dt.
\]

(2.15)

Arguing along this line, and exploiting the following identities for \( m \in \mathbb{Z}_{\geq 0} \):

\[
\frac{d}{dt} [t^{2m+1} I_1(t)] = 2mt^{2m} I_1(t) + t^{2m+1} I_0(t),
\]

(2.16)

\[
\frac{d}{dt} [t^{2m} I_0(t)] = t^{2m} I_1(t) + 2mt^{2m-1} I_0(t),
\]

(2.17)

we have

\[
(-1)^n n! = \int_0^\infty [K_0(t)]^n \mathcal{L}_{n+1}^* \frac{I_0(t)}{t} \, dt
\]

\[
= (n-1) \int_0^\infty [t A_{n,[n-1]/2} (t^2) - t^2 B_{n,[n-2]/2} (t^2)] I_1(t) [K_0(t)]^n \, dt
\]

(2.18)

where \( A_{n,[n-1]/2} \) (resp. \( B_{n,[n-2]/2} \)) is a polynomial with integer coefficients, whose degree does not exceed \( [n-1]/2 \) (resp. \( [(n-2)/2] \)).

By a similar procedure, with the following identities for \( m \in \mathbb{Z}_{\geq 0} \):

\[
\frac{d}{dt} [t^{2m+1} K_1(t)] = 2mt^{2m} K_1(t) - t^{2m+1} K_0(t),
\]

(2.19)

\[
\frac{d}{dt} [t^{2m} K_0(t)] = - t^{2m} K_1(t) + 2mt^{2m-1} K_0(t),
\]

(2.20)

we can deduce from [25, (4.26)] the following equation:

\[
-(1)^n (n-1)! = \int_0^\infty I_0(t) [K_0(t)]^{n-1} \mathcal{L}_{n+1}^* \frac{K_0(t)}{t} \, dt
\]

\[
= (n-1) \int_0^\infty [t A_{n,[n-1]/2} (t^2) - t^2 B_{n,[n-2]/2} (t^2)] K_1(t) I_0(t) [K_0(t)]^{n-1} \, dt.
\]

(2.21)

Bearing in mind the Wronskian relation \( I_0(t) K_1(t) + I_1(t) K_0(t) = \frac{1}{t} \), we may subtract (2.21) from (2.18), arriving at

\[
(-1)^n (n-2)! (n+1) = \int_0^\infty t B_{n,[n-2]/2} (t^2) [K_0(t)]^{n-1} \, dt.
\]

(2.22)

Choosing \( f_{n-3}(\xi) = \frac{(-1)^n}{n+1} B_{n,[n-2]/2} (\xi) \in \frac{1}{n+1} \mathbb{Z} [\xi] \), we can verify the first sentence in our theorem.

So far, we have verified the inhomogeneous sum rules for vacuum diagrams in any loop order. With the last two lines in (2.9), we can establish the homogeneous sum rules for non-vacuum diagrams in a similar vein, if not simpler.
Remark As a service to the quantum field community, we list our computations of $f_n(r^2), n \in \mathbb{Z} \cap [1, 10]$ in Table I. Clearly, the entries $f_3$ and $f_4$ allow us to verify all the identities declared in Conjecture 1.1. (It is our hope that, by working a little harder, one can perhaps show that $f_n(\xi) \in \mathbb{Z}[\xi]$ and $f_n(0) = 2^{n+1}$ for all $n \in \mathbb{Z}_{>0}$. However, we are not going to pursue in this direction, as it will not affect the qualitative structure of $\mathbb{Q}$-linear dependence.)

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
$n$ & $f_n(r^2)$ \\
\hline
1 & $4 - 3r^2$ \\
2 & $8(1 - 4r^2)$ \\
3 & $16 - 228r^2 + 45r^4$ \\
4 & $16(2 - 85r^2 + 72r^4)$ \\
5 & $64 - 7344r^2 + 17720r^4 - 1575r^6$ \\
6 & $128(1 - 291r^2 + 1662r^4 - 576r^6)$ \\
7 & $256 - 181056r^2 + 2199408r^4 - 1974168r^6 + 99225r^8$ \\
8 & $256(2 - 3335r^2 + 80370r^4 - 155256r^6 + 28800r^8)$ \\
9 & $1024 - 3936000r^2 + 179221616r^4 - 669169296r^6 + 304572636r^8 - 9823275r^{10}$ \\
10 & $2048(1 - 8708r^2 + 722853r^4 - 4861164r^6 + 4513680r^8 - 518400r^{10})$ \\
\hline
\end{tabular}
\caption{The first ten polynomials that satisfy the inhomogeneous sum rules (2.7)}
\end{table}

Remark Let $C = \frac{1}{240\sqrt{3\pi}} \Gamma\left(\frac{1}{15}\right) \Gamma\left(\frac{2}{15}\right) \Gamma\left(\frac{4}{15}\right) \Gamma\left(\frac{6}{15}\right)$ be the “Bologna constant” attributed to Broadhurst [8, 2] and Laporta [17]. The results $\text{IKM}(2, 3; 1) = \frac{\sqrt{15\pi}}{2} C$, $\text{IKM}(2, 3; 3) = \frac{\sqrt{15\pi}}{2} \left(\frac{2}{3}\right)^2 \left(13C + \frac{1}{10C}\right)$ and $\text{IKM}(2, 3; 5) = \frac{\sqrt{15\pi}}{2} \left(\frac{4}{15}\right)^3 \left(43C + \frac{19}{40C}\right)$ were confirmed by Bailey–Borwein–Broadhurst–Glasser [2, §5.10]. In [24, §2], we wrote a short proof for $\text{IKM}(1, 4; 1) = \pi^2 C$ based on Wick rotations, which simplified the arguments by Bloch–Kerr–Vanhove [5] and Samart [19]; we then verified the formulae $\text{IKM}(1, 4; 3) = \pi^2 \left(\frac{2}{3}\right)^2 \left(13C - \frac{1}{10C}\right)$ and $\text{IKM}(1, 4; 5) = \pi^2 \left(\frac{4}{15}\right)^3 \left(43C - \frac{19}{40C}\right)$ using Eichler integrals [24, §3]. In retrospect, we could have dispensed with the computations of Eichler integrals, in favor of some algebraic manipulations related to Vanhove’s operators. Indeed, in [25, §2], we demonstrated that the determinant

$$
\det \begin{pmatrix} \text{IKM}(1, 4; 1) & \text{IKM}(1, 4; 3) \\ \text{IKM}(2, 3; 1) & \text{IKM}(2, 3; 3) \end{pmatrix} = \frac{2\pi^3}{\sqrt{3^{3}5^{5}}} \tag{2.23}
$$

followed from factorizations of certain Wronskians related to $\bar{L}_3$, which would enable us to evaluate $\text{IKM}(1, 4; 3)$ algebraically, drawing on the knowledge of the other three matrix elements. The sum rule (1.4), provable by the Vanhove procedure outlined above, now provides us with an algebraic route towards the closed form of $\text{IKM}(1, 4; 5)$.

3. Hankel–Vanhove mechanism and exceptional sum rules

To prove Theorem 1.3, we will need to investigate certain modular forms on the Chan–Zudilin group $\Gamma_0(6)_3 = \langle \Gamma_0(6), W_3 \rangle$ [13], which is

$$
\Gamma_0(6) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}; ad - bc = 1; c \equiv 0 (\text{mod } 6) \right\} \tag{3.1}
$$
adjoining an involution $\hat{W}_3 := \frac{1}{\sqrt{3}} \left( \begin{smallmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & 1 & 1 \end{smallmatrix} \right)$. In the meantime, we will revisit our treatment of the 8-Bessel problems in [24, §5], with some extensions and simplifications.

3.1. Chan–Zudilin representations of Bessel moments and their Hankel fusions. As pointed out by Chan–Zudilin [13, (2.2)], the group $\Gamma_0(6)_+$ enjoys a Hauptmodul

$$X_{6,3}(z) := \left[ \frac{\eta(2z)\eta(6z)}{\eta(z)\eta(3z)} \right]^6, \quad z \in \mathcal{S} := \{ \tau \in \mathbb{C} | \text{Im} \tau > 0 \} , \quad (3.2)$$

expressible in terms of the Dedekind eta function $\eta(\tau) := e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2 \pi i n \tau})$, $\tau \in \mathcal{S}$. This Hauptmodul satisfies the following invariance properties:

$$X_{6,3}(\hat{\gamma}z) = X_{6,3}(z), \quad \forall z \in \mathcal{S}, \hat{\gamma} \in \Gamma_0(6), \quad (3.3)$$

$$X_{6,3}(\hat{W}_3z) = X_{6,3}(z), \quad \forall z \in \mathcal{S}, \quad (3.4)$$

where we have set $\hat{T}z := \frac{az+b}{cz+d}$ for $T = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$, by convention.

Moreover, in [13, (2.5)], Chan–Zudilin introduced a notation

$$Z_{6,3}(z) := \left[ \frac{\eta(z)\eta(3z)}{\eta(2z)\eta(6z)} \right]^4, \quad z \in \mathcal{S} \quad (3.5)$$

for a modular form of weight 2 on $\Gamma_0(6)_+$. It transforms as follows:

$$Z_{6,3}(\hat{\gamma}z) = (cz+d)^2Z_{6,3}(z), \quad \forall z \in \mathcal{S}, \hat{\gamma} = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(6), \quad (3.6)$$

$$Z_{6,3}(\hat{W}_3z) = -3(2z-1)^2Z_{6,3}(z), \quad \forall z \in \mathcal{S}. \quad (3.7)$$

The modular form $Z_{6,3}(z)$ and the Hauptmodul $X_{6,3}(z)$ are both useful in the evaluation of Bessel moments, as shown by the proposition below.

**Proposition 3.1** (Modular parametrization of Bessel moments). We have

$$\tilde{\text{IKM}}(2,3;1| - 64X_{6,3}(z)) := \int_0^{\infty} I_0 \left( \left[ \frac{2\eta(2z)\eta(6z)}{\eta(z)\eta(3z)} \right]^3 \frac{t}{i} \right) I_0(t)[K_0(t)]^3 t \, dt$$

$$= \frac{\pi^2}{16} Z_{6,3}(z), \quad (3.8)$$

$$\tilde{\text{IKM}}(2,3;1| - 64X_{6,3}(z)) := \int_0^{\infty} K_0 \left( \left[ \frac{2\eta(2z)\eta(6z)}{\eta(z)\eta(3z)} \right]^3 \frac{t}{i} \right) [I_0(t)K_0(t)]^2 t \, dt$$

$$= \frac{\pi^2}{96} [1 - 3(2z-1)^2]Z_{6,3}(z), \quad (3.9)$$

$$\tilde{\text{IKM}}(1,4;1| - 64X_{6,3}(z)) + 4\tilde{\text{IKM}}(1,4;1| - 64X_{6,3}(z))$$

$$= \int_0^{\infty} I_0 \left( \left[ \frac{2\eta(2z)\eta(6z)}{\eta(z)\eta(3z)} \right]^3 \frac{t}{i} \right) [K_0(t)]^4 t \, dt + 4 \int_0^{\infty} K_0 \left( \left[ \frac{2\eta(2z)\eta(6z)}{\eta(z)\eta(3z)} \right]^3 \frac{t}{i} \right) I_0(t)[K_0(t)]^3 t \, dt$$

$$= \frac{\pi^3}{8i} (2z-1)Z_{6,3}(z), \quad (3.10)$$
for $z = \frac{1}{2} + iy$, $y \geq \frac{1}{\sqrt{3}}$, where $u = -64X_{6,3}(z) \in (0, 4)$. For $u = -64X_{6,3}(z) \in [4, \infty)$, the relation in (3.9) remains a valid modular parametrization for the Bessel moment $\int_{0}^{\infty} K_{0}(\sqrt{ut})[I_{0}(t)K_{0}(t)]^{3}t\,dt$, where

$$
\begin{cases}
  u = -64X_{6,3}(z) \in [4, 16], & z = \frac{1}{2} + \frac{i\varphi}{\sqrt{3}}, \varphi \in [0, \frac{\pi}{3}], \\
  u = -64X_{6,3}(z) \in [16, \infty), & z = \frac{1}{6}(1 + e^{i\psi}), \psi \in [\frac{\pi}{3}, \pi).
\end{cases}
$$

(3.11)

Proof. Both (3.8) and (3.9) can be verified in three steps. First, using integration by parts and symmetric properties of Bessel differential operators [25, Lemma 4.2], one checks that $f(u) = \tilde{I}_{K}M(2, 3; 1|u)$ [or $f(u) = I_{K}M(2, 3; 1|u)$] satisfies a homogeneous differential equation

$$
u^{2}(u - 4)(u - 16)f'''(u) + 6u(u^{2} - 15u + 32)f''(u) + (7u^{2} - 68u + 64)f'(u) + (u - 4)f(u) = 0. \quad (3.12)$$

Second, one notes that every solution to such a homogeneous differential equation must assume the form $f(-64X_{6,3}(z)) = Z_{6,3}(z)(c_{0} + c_{1}z + c_{2}z^{2})$ for constants $c_{0}, c_{1}, c_{2} \in \mathbb{C}$ [24, Theorems 1 and 3]. Third, examining special values (including asymptotic behavior) of the function in question, one determines the constants $c_{0}, c_{1}, c_{2}$.

Here, the third step deserves further explanations.

For (3.8) (see also [24, (3.1.6)]), we have the following asymptotic behavior [2, (54)]:

$$
\lim_{z \to \frac{1}{2} + i\infty} \tilde{I}_{K}M(2, 3; 1| - 64X_{6,3}(z)) = \int_{0}^{\infty} I_{0}(t)[K_{0}(t)]^{3}t\,dt = \frac{\pi^{2}}{16}. \quad (3.13)
$$

Meanwhile, the relation $\lim_{z \to \frac{1}{2} + i\infty} Z_{6,3}(z) = 1$ follows from (3.5) and the definition of the Dedekind eta function as an infinite product. Thus one immediately determines $c_{0} = \frac{\pi^{2}}{16}, c_{1} = c_{2} = 0$.

For (3.9), we need two observations: (i) The function $\tilde{I}_{K}M(2, 3; 1| - 64X_{6,3}(z))$ analytically continues across the point $z = \frac{1}{2} + \frac{1}{2\sqrt{3}}$, and remains holomorphic in an open neighborhood of the ray $z = \frac{1}{2} + iy, y > 0$; and (ii) To remain compatible with the transformation laws in (3.4) and (3.7), we must have

$$
\tilde{I}_{K}M(2, 3; 1| - 64X_{6,3}(z)) := (k_{0}[1 - 3(2z - 1)^{2}] + k_{1}(2z - 1)]Z_{6,3}(z) \quad (3.14)
$$

on the ray $z = \frac{1}{2} + iy, y > 0$. To compute the constants $k_{0}, k_{1} \in \mathbb{C}$, we rely on two quick facts: (1) We have $\tilde{I}_{K}M(2, 3; 1|u) = \frac{1}{32} \log^{-2} \frac{u^{2}}{8} + O(\log u)$, as $u \to 0^{+}$ [28, (3.18)–(3.19)], so $k_{0} = \frac{\pi^{2}}{96}$; (2) We have $\tilde{I}_{K}M(2, 3; 1| - 64X_{6,3}(\frac{1}{2} + \frac{i\psi}{2\sqrt{3}})) = \tilde{I}_{K}M(2, 3; 1| - 64X_{6,3}(\frac{1}{2} + \frac{i\psi}{2\sqrt{3}})) = \tilde{I}_{K}M(2, 3; 1| - 64X_{6,3}(\frac{1}{2} + \frac{i\sqrt{3}}{2\sqrt{3}})) = \frac{\pi^{2}}{16}Z_{6,3}(\frac{1}{2} + \frac{i\sqrt{3}}{2\sqrt{3}})$, as can be inferred from [24, Table 1] and (3.8), so $k_{1} = 0$.

The proof of (3.10) (which corrects a misprint sign in [24, (5.1.33)]) is similar to that of (3.9). See [24, Proposition 5.1.4] for details.

Corollary 3.2 (Analytic continuations of Bessel moments). Let $J_{0}(x) := \frac{2}{\pi} \int_{0}^{\pi/2} \cos(x \cos \varphi) \, d\varphi \equiv I_{0}(ix), x \in \mathbb{R}$ and $Y_{0}(x) := -\frac{2}{\pi} \int_{0}^{\infty} \cos(x \cosh u) \, du, x \in (0, \infty)$ be Bessel functions of the zeroth order.
For $z/i > 0$, the following identities hold:

\[
\int_0^\infty J_0 \left( \left[ \frac{2\eta(2z)\eta(6z)}{\eta(z)\eta(3z)} \right]^3 t \right) I_0(t)[K_0(t)]^3 t \, dt = \frac{\pi^2}{16} Z_{6,3}(z), \tag{3.15}
\]

\[
\int_0^\infty J_0 \left( \left[ \frac{2\eta(2z)\eta(6z)}{\eta(z)\eta(3z)} \right]^3 t \right) [I_0(t)K_0(t)]^2 t \, dt = \frac{\pi z}{4i} Z_{6,3}(z), \tag{3.16}
\]

\[
\int_0^\infty Y_0 \left( \left[ \frac{2\eta(2z)\eta(6z)}{\eta(z)\eta(3z)} \right]^3 t \right) [I_0(t)K_0(t)]^2 t \, dt = \frac{\pi (z^2 + \frac{1}{6})}{4} Z_{6,3}(z). \tag{3.17}
\]

Furthermore, we have the following relation for $z/i > 0$:

\[
\int_0^\infty J_0 \left( \left[ \frac{2\eta(2z)\eta(6z)}{\eta(z)\eta(3z)} \right]^3 t \right) [K_0(t)]^4 t \, dt - 2\pi \int_0^\infty Y_0 \left( \left[ \frac{2\eta(2z)\eta(6z)}{\eta(z)\eta(3z)} \right]^3 t \right) I_0(t)[K_0(t)]^3 t \, dt
\]

\[= \frac{\pi^3 z}{4i} Z_{6,3}(z). \tag{3.18}\]

\[\text{Proof.}\] The definitions of $J_0(x), x \in \mathbb{R}$ and $Y_0(x), x \in (0, \infty)$ can be analytically continued to $J_0(z), z \in \mathbb{C}, Y_0(z), z \in \mathbb{C} \setminus (-\infty, 0]$, through which one can define the Hankel functions of zeroth order $H_0^{(1)}(z) = J_0(z) + iY_0(z), z \in \mathbb{C} \setminus (-\infty, 0]$ and $H_0^{(2)}(z) = J_0(z) - iY_0(z), z \in \mathbb{C} \setminus (-\infty, 0]$.

Thus, the relation $\frac{\pi}{2} K_0(y) = H_0^{(1)}(iy) = J_0(iy) + iY_0(iy), y > 0$ reveals (3.16) and (3.17) as natural consequences of (3.9).

One can check that (3.18) is an analytic continuation of (3.10), as in [24, Proposition 5.1.4]. \[\blacksquare\]

As in [9, 24, 29], we introduce the following cusp form of weight 6 and level 6:

\[
f_{6,6}(z) := \frac{[\eta(2z)\eta(6z)]^9}{[\eta(z)\eta(6z)]^3} + \frac{[\eta(z)\eta(6z)]^9}{[\eta(2z)\eta(3z)]^3} = \frac{[Z_{6,3}(z)]^2}{2\pi i} \frac{dX_{6,3}(z)}{dz}. \tag{3.19}\]

This cusp form featured prominently in Broadhurst’s conjectures [9, (142)–(146)] that related $\text{IKM}(a, 8-a; 1), a \in \{1, 2, 3, 4\}$ to special values of the $L$-function:

\[L(f_{6,6}, s) := \frac{1}{\Gamma(s)} \int_0^\infty f_{6,6}(iy)(2\pi y)^s \frac{dy}{y}. \tag{3.20}\]

In the next theorem, we recapitulate from [24, §5] our verification of Broadhurst’s conjectures, along with some key simplifications.

\textbf{Theorem 3.3} (Broadhurst representations for 8-Bessel problems). (a) We have

\[
\text{IKM}(4, 4; 1) = L(f_{6,6}, 3), \tag{3.21}
\]

\[
\text{IKM}(3, 5; 1) = \frac{\pi^2}{4} L(f_{6,6}, 2), \tag{3.22}
\]

\[
\text{IKM}(2, 6; 1) = \frac{\pi^4}{8} L(f_{6,6}, 1), \tag{3.23}
\]

\[
\text{IKM}(1, 7; 1) = \frac{\pi^4}{4} L(f_{6,6}, 2). \tag{3.24}
\]
(b) We have the following identity:

$$\pi^2 L(f_{6,1}) = 36L(f_{6,3}),$$

which entails

$$14\mathbf{IKM}(2, 6; 1) = 9\pi^2 \mathbf{IKM}(4, 4; 1).$$

Proof. (a) As in [24], we refer to the Parseval–Plancherel identity for Hankel transforms, namely

$$\int_0^\infty \left[ \int_0^\infty J_0(xt)F(t)\,dt \right] \left[ \int_0^\infty Y_0(x\tau)\tau\,d\tau \right] \,xdx = \int_0^\infty F(t)G(t)\,dt$$

as “Hankel fusion”. Applying Hankel fusions to (3.15) and (3.16), one can verify (3.21)–(3.23) (cf. [9, (143)–(145)], [24, (1.2.7)–(1.2.9)], [29, (23)–(25)]).

Moreover, applying the Hilbert cancelation formula [24, (4.2.19)]

$$\int_0^\infty \left[ \int_0^\infty J_0(xt)F(t)\,dt \right] \left[ \int_0^\infty Y_0(x\tau)\tau\,d\tau \right] \,xdx = 0$$

to $F(t) = I_0(t)[K_0(t)]^3$, one can deduce [24] (cf. [9, (146)], [24, (1.2.8)], [29, (24)]) from (3.15) and (3.18). (See [24, Lemma 4.2.4] for the connection between (3.28) and Hilbert transforms.) A comparison between (3.22) and (3.24) then leads us to a sum rule $\mathbf{IKM}(1, 7; 1) = \pi^2 \mathbf{IKM}(3, 5; 1)$ [9, (148)], which can also be verified by real-analytic properties of Hilbert transforms [26, Theorem 3.3], without invoking special $L$-values.

(b) To begin, we note that the Hilbert cancelation formula can be extended to

$$\int_0^\infty \left[ \int_0^\infty J_0(xt)F(t)\,dt \right] \left[ \int_0^\infty Y_0(x\tau)\tau\,d\tau \right] \,xdx + \int_0^\infty \left[ \int_0^\infty J_0(xt)G(t)\,dt \right] \left[ \int_0^\infty Y_0(x\tau)\tau\,d\tau \right] \,xdx = 0,$$

for suitably regular $F$ and $G$. Setting $F(t) = I_0(t)[K_0(t)]^3$ and $G(t) = [I_0(t)K_0(t)]^2$ in the equation above, while referring back to (3.15)–(3.18), we obtain

$$\frac{\pi^4i}{6} \int_0^\infty f_{6,6}(z)(1 + 18z^2)\,dz - \frac{1}{2\pi} \mathbf{IKM}(2, 6; 1) = 0.$$

Using (3.23), we can further deduce that

$$\int_0^\infty f_{6,6}(z)(7 + 72z^2)\,dz = 0,$$

which is our goal.

Remark Unlike our previous formulation of [24, Theorems 5.2.1 and 5.2.2], the proof above requires no contour deformations. Instead, it relies only on the properties of Hilbert transforms, and is essentially real-analytic. Thus, contrary to our statement in the closing paragraph of [26], the sum rule $14\mathbf{IKM}(2, 6; 1) = 9\pi^2 \mathbf{IKM}(4, 4; 1)$ is not beyond the reach of Hilbert transforms.

Remark From the work of Yun [22, Theorem 1.1.5], we know that the Fourier coefficients $a_p$ (for $p$ prime) in the weight-6 modular form $f_{6,6}(z) = \sum_{m=1}^\infty a_m e^{2\pi i mz}$ encode the information for the 8th symmetric power $\text{Kl}^8_2 = \text{Sym}^8(\text{Kl}_2)$ of the Kloosterman sum $\text{Kl}_2(F_p, a) = \sum_{x\in F_p^*} e^{2\pi i (x+a)/p}$.
To quantum field theory, see [24, Theorem 2.2.2] and [27, Theorem 4.1].

Prior to Broadhurst’s study [9], the idea of discretizing Bessel moments via Kloosterman sums appeared in Noam D. Elkies’ post on MathOverflow [14] about the integral \( \int_{0}^{\infty} x J_{0}(x) dx \). For the connections between Elkies’ question to quantum field theory, see [24, Theorem 2.2.2] and [27, Theorem 4.1].

Table II. A partial list of Vanhove operators \( \tilde{L}_{m} \) for \( u < 0 \), reformulated from [20, Table 1] so as to highlight the parity of each individual operator.

| \( n \) | \( \tilde{L}_{m} \) |
|---|---|
| 1 | \( \sqrt{u(u-4)}D^{1} \left[ \sqrt{u(u-4)}D^{0} \right] \) |
| 2 | \( D^{1} \left[ u(u-1)(u-9)D^{1} \right] + (u-3)D^{0} \) |
| 3 | \( D^{1} \left( \sqrt{u^{2}(u-4)(u-16)} \right) \right] + \sqrt{u(u-8)}D^{1} \left[ \sqrt{u(u-8)}D^{0} \right] \) |
| 4 | \( D^{2} \left[ u^{2}(u-1)(u-9)(u-25)D^{2} + D^{1} \left[ u^{5}u^{2} - 98u + 285D^{1} \right] + (u-5)D^{0} \right] \) |
| 5 | \( D^{2} \left[ \sqrt{u^{3}(u-4)(u-16)(u-36)}D^{2} \right] \) |

averaged over \( a \in \mathbb{F}_{p}^{\times} \). Treating \( \text{Kl}_{2}(\mathbb{F}_{p},a) \) as a finite-field analog of the Bessel function \( J_{0}(\sqrt{t}) := \frac{1}{2\pi i} \int_{|t|=1} e^{-\zeta} d\zeta \), Broadhurst [9] and Broadhurst–Roberts [11] studied \( \mathbb{F}_{p}^{\times} \)-averages of \( n \)-th symmetric powers \( \text{Kl}_{n}^{p} = \text{Sym}^{n}(\text{Kl}_{2}) \) as \( p \)-adic Bessel moments, leading to various conjectural evaluations of Feynman diagrams via \( L \)-functions for Kloosterman moments. Broadhurst and Roberts conjectured that these \( L \)-functions satisfy a general functional equation, which has been recently verified by Fresán–Sabbah–Yu [15, Theorems 1.2 and 1.3].

3.2. Vanhove reflections and modular cancelation formulae. As seen from Table III, the Vanhove operator \( \tilde{L}_{3} = -\tilde{L}_{3}^{*} \) is skew-symmetric (see [30, Proposition 2.3]) for a proof of the generic parity relation \( \tilde{L}_{m} = (-1)^{m}\tilde{L}_{m}^{*} \). If we define the commutator as \( [A, B] = AB - BA \), then we have

\[
\begin{align*}
\tilde{L}_{3} &= \frac{3 \log(-u) - 4 \log(4 - u) + \log(16 - u)}{192} \\
&= 3(uD^{2} + D^{1}) + \left[ \frac{2}{(u - 4)^{2}} + \frac{1}{3(u - 4)} + \frac{8}{(u - 16)^{2}} + \frac{2}{3(u - 16)} \right] D^{0}. \tag{3.32}
\end{align*}
\]

Now suppose that \( f, g \in \ker \tilde{L}_{3} \) and define \( \langle f, g \rangle := \int_{-\infty}^{0} f(u)g(u) du \), then we can evaluate the integral

\[
3\langle f, (uD^{2} + D^{1})g \rangle + \int_{-\infty}^{0} f(u)g(u) \left[ \frac{2}{(u - 4)^{2}} + \frac{1}{3(u - 4)} + \frac{8}{(u - 16)^{2}} + \frac{2}{3(u - 16)} \right] du \tag{3.33}
\]

by collecting all the boundary contributions from integration by parts. We will refer to this trick as a Vanhove reflection on the pair \( f, g \in \ker \tilde{L}_{3} \).

In what follows, we need some asymptotic analyses in the \( u \to 0^{-} \) and the \( u \to -\infty \) regimes (Lemma 3.4), before fully elucidating the boundary contributions in Vanhove reflections (Theorem 3.5).

1Prior to Broadhurst’s study [9], the idea of discretizing Bessel moments via Kloosterman sums appeared in Noam D. Elkies’ post on MathOverflow [14] about the integral \( \int_{0}^{\infty} xJ_{0}(x) dx \). For the connections between Elkies’ question to quantum field theory, see [24, Theorem 2.2.2] and [27, Theorem 4.1].
Lemma 3.4 (Asymptotic behavior of certain off-shell Bessel moments). The following relations hold true:

\[
\tilde{\text{IKM}}(2,3;1|u) = \begin{cases} \frac{\pi^2}{16} \left[ 1 + \frac{u}{16} + O(u^2) \right], & u \to 0, \\ -\frac{3\log^2(-\frac{1}{u})}{4u} \left[ 1 + O \left( \frac{1}{u} \right) \right], & u \to -\infty, \end{cases}
\]

(3.34)

\[
\tilde{\text{IKM}}(3,2;1|u) = \begin{cases} -\frac{\log(-\frac{1}{u})}{8} \left( 1 + \frac{u}{16} \right) + O(u), & u \to 0^-, \\ \frac{\log(-\frac{1}{u})}{u} \left[ 1 + O \left( \frac{1}{u} \right) \right], & u \to -\infty. \end{cases}
\]

(3.35)

As a result, for

\[
\ell(u) := \frac{3\log(-u) - 4\log(4 - u) + \log(16 - u)}{192},
\]

(3.36)

\[
\tilde{D}f(u) := \sqrt{u^2(u - 4)(u - 16)}D^1f(u),
\]

(3.37)

we have asymptotic expansions

\[
\tilde{D}^2[\ell(u)\tilde{\text{IKM}}(2,3;1|u)] = \begin{cases} O(u\log(-u)), & u \to 0^-; \\ o(1), & u \to -\infty, \end{cases}
\]

(3.38)

\[
\tilde{D}[\ell(u)\tilde{\text{IKM}}(2,3;1|u)] = \begin{cases} \frac{-\pi^2}{192} + O(u\log(-u)), & u \to 0^-; \\ o(1), & u \to -\infty, \end{cases}
\]

(3.39)

\[
\tilde{D}^2\tilde{\text{IKM}}(2,3;1|u) = \begin{cases} O(u), & u \to 0; \\ o(1), & u \to -\infty, \end{cases}
\]

(3.40)

\[
\tilde{D}\tilde{\text{IKM}}(2,3;1|u) = \begin{cases} O(u), & u \to 0; \\ o(1), & u \to -\infty, \end{cases}
\]

(3.41)

\[
\tilde{D}^2[\ell(u)\tilde{\text{IKM}}(3,2;1|u)] = \begin{cases} \frac{\log \left( \frac{-u^3}{2048} \right)}{96} + O(u\log^2(-u)), & u \to 0^-; \\ o(1), & u \to -\infty, \end{cases}
\]

(3.42)

\[
\tilde{D}[\ell(u)\tilde{\text{IKM}}(3,2;1|u)] = \begin{cases} \frac{3}{4} + O(u\log^2(-u)), & u \to 0^-; \\ o(1), & u \to -\infty, \end{cases}
\]

(3.43)

\[
\tilde{D}^2\tilde{\text{IKM}}(3,2;1|u) = \begin{cases} O(u\log(-u)), & u \to 0^-; \\ o(1), & u \to -\infty, \end{cases}
\]

(3.44)

\[
\tilde{D}\tilde{\text{IKM}}(3,2;1|u) = \begin{cases} 1 + O(u\log(-u)), & u \to 0^-; \\ o(1), & u \to -\infty, \end{cases}
\]

(3.45)

\[\text{Proof.} \] As \( z \to i\infty \), we have \( u = -64X_{6,3}(z) \to 0 \) and \([\text{cf. (3.8) and (3.15)}]\)

\[
\tilde{\text{IKM}}(2,3;1| - 64X_{6,3}(z)) = \frac{\pi^2}{16} [1 - 4q + O(q^2)]
\]

(3.46)

for \( q = e^{2\pi iz} \to 0 \). Meanwhile, we have the \( q \)-expansion \( u = -64q + O(q^2) \). This confirms the stated behavior of \( \tilde{\text{IKM}}(2,3;1|u), u \to 0 \).
Let \( \hat{W}_6 := \frac{1}{\sqrt{6} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) be the Chan–Zudilin involution \([13, \S 2]\) that sends \( z \) to \(- \frac{1}{6z}\). Using the modular transformation \( \eta(-1/\tau) = \sqrt{\tau/\eta(\tau)} \), one can verify that the modular function \( X_{6,3}(z) \) and the modular form \( Z_{6,3}(z) \) satisfy the following transformation laws:

\[
\begin{align*}
X_{6,3}(\hat{W}_6 z) &= \frac{1}{64X_{6,3}(z)}, \\
Z_{6,3}(\hat{W}_6 z) &= -48z^2Z_{6,3}(z)X_{6,3}(z),
\end{align*}
\]

for arbitrary \( z \in \mathbb{H} \). Thus, we can perform \( q \)-expansion on \( \tilde{\text{IKM}}(2, 3; 1|1/X_{6,3}(z)) = -3\pi^2 z^2 Z_{6,3}(z)X_{6,3}(z) \), \( z/i > 0 \) to deduce the leading order asymptotics of \( \tilde{\text{IKM}}(2, 3; 1|u), u \to -\infty \).

To verify \((3.35)\), check the \( q \)-expansions of \((3.34)\) and \( \tilde{\text{IKM}}(3, 2; 1|1/X_{6,3}(z)) = \frac{2\pi i}{z} Z_{6,3}(z)X_{6,3}(z), z/i > 0 \), while exploiting the relation \( e^{2\pi i z} = -\frac{u}{64} + O(u^2) \) in the \( u \to 0^- \) regime.

One can verify \((3.38)\)-(\(3.45)\) through straightforward computations. \( \blacksquare \)

**Theorem 3.5** (Exceptional sum rules via Vanhove reflections). If we define

\[
\begin{align*}
\varphi_{6,6}(z) &= f_{6,6}(z) \left\{ \frac{2}{[-64X_{6,3}(z)-4]^2} + \frac{1}{3[-64X_{6,3}(z)-4]} \right\}, \\
\chi_{6,6}(z) &= f_{6,6}(z) \left\{ \frac{8}{[-64X_{6,3}(z)-16]^2} + \frac{2}{3[-64X_{6,3}(z)-16]} \right\},
\end{align*}
\]

then we have

\[
\begin{align*}
\text{IKM}(2, 6; 3) &= -\frac{\pi^5}{3i} \int_0^\infty [\varphi_{6,6}(z) + \chi_{6,6}(z)] \, dz = \frac{\text{IKM}(2, 6; 1)}{72}, \\
-\frac{\pi^2}{192} + \text{IKM}(3, 5; 3) &= \frac{4\pi^4}{3} \int_0^\infty [\varphi_{6,6}(z) + \chi_{6,6}(z)] \, dz = \frac{\text{IKM}(3, 5; 1)}{72} + \frac{\pi^2}{1728}, \\
\frac{7\log 2}{144} + \text{IKM}(4, 4; 3) &= \frac{16\pi^3}{3} \int_0^\infty [\varphi_{6,6}(z) + \chi_{6,6}(z)] z^2 \, dz = \frac{\text{IKM}(4, 4; 1)}{72},
\end{align*}
\]

leading to

\[
\dim_{\mathbb{Q}} \text{span}_{\mathbb{Q}} |\text{IKM}(2, 6, 2k+1) \{k \in \mathbb{Z}_{\geq 0}\}| \leq 2.
\]

**Proof.** We recall from Table II that \( \bar{L}_3 = D^1 \bar{D}^2 + \sqrt{u(u-8)} D^1 \left[ \sqrt{u(u-8)} D^0 \right] \). Plugging

\[
\begin{align*}
f(u) = g(u) = \int_0^\infty J_0(\sqrt{-u} t) I_0(t) (K_0(t))^3 \, dt = \text{IKM}(2, 3; 1|u), \quad u < 0
\end{align*}
\]

into \((3.33)\), we have vanishing boundary contributions during integrations by parts:

\[
\begin{align*}
\langle f, \bar{L}_3 [\ell(u) g] \rangle &= \langle \hat{D} f, D^1 \hat{D} [\ell(u) g] \rangle - \langle \sqrt{u(u-8)} D^1 \left[ \sqrt{u(u-8)} f \right], \ell(u) g \rangle \\
\langle \hat{D}^2 f, D^1 \left[ \ell(u) g \right] \rangle &= \langle \sqrt{u(u-8)} D^1 \left[ \sqrt{u(u-8)} f \right], \ell(u) g \rangle \\
\langle \hat{L}_3 f, \ell(u) g \rangle &= 0,
\end{align*}
\]

so \((3.33)\) must be equal to

\[
\langle f, [\bar{L}_3, \ell(u) D^0] g \rangle = -\langle \bar{L}_3 f, \ell(u) g \rangle - \langle \ell(u) f, \bar{L}_3 g \rangle = 0.
\]

\( \blacksquare \)
Thus, a Vanhove reflection leads us to the first equality in (3.51), thanks to the Bessel differential equation \((uD^2 + D^1)\tilde{\text{IKM}}(2, 3; 1|u) = \frac{1}{4} \tilde{\text{IKM}}(2, 3; 3|u)\).

Performing a Vanhove reflection on \(f(u) = \tilde{\text{IKM}}(3, 2; 1|u), \) \(g(u) = \tilde{\text{IKM}}(2, 3; 1|u)\) [resp. \(f(u) = g(u) = \tilde{\text{IKM}}(3, 2; 1|u)\)], and carefully handling boundary contributions from integrations by parts, we can verify the first equality in (3.52) [resp. (3.53)]. Concretely speaking, when \(f(u) = \tilde{\text{IKM}}(3, 2; 1|u), \) \(g(u) = \tilde{\text{IKM}}(2, 3; 1|u)\), we have

\[
\langle f, \tilde{L}_3[\ell(u)g] \rangle = \frac{\pi^2}{128} - \langle \tilde{D}f, D^1\tilde{D}[\ell(u)g] \rangle - \left( \sqrt{u(u-8)}D^1\left[ \sqrt{u(u-8)}f \right], \ell(u)g \right)
\]

which entails [cf. the first equality in (3.52)]

\[
\frac{\pi^2}{128} = \langle f, \left[ \tilde{L}_3, \ell(u)D^0 \right]g \rangle = \frac{3 \tilde{\text{IKM}}(3, 5; 3)}{2} + \int_{-\infty}^0 f(u)g(u) \left[ \frac{2}{u-4} + \frac{1}{3(u-4)} + \frac{8}{(u-16)^2} + \frac{2}{3(u-16)} \right] du; \quad (3.59)
\]

when \(f(u) = g(u) = \tilde{\text{IKM}}(3, 2; 1|u)\), we consider \(\langle f_1, f_2 \rangle_\varepsilon = \int_{-\varepsilon}^{\varepsilon} f_1(u)f_2(u) du\) in the \(\varepsilon \to 0^+\) regime, so that we have

\[
\langle f, D^1\tilde{D}^2[\ell(u)g] \rangle_\varepsilon = \frac{\log \frac{\varepsilon}{64}}{32} - \langle \tilde{D}f, D^1\tilde{D}[\ell(u)g] \rangle_\varepsilon + O(\varepsilon \log \varepsilon)
\]

\[
\frac{\log \frac{\varepsilon}{64}}{32} - \frac{\log \frac{\varepsilon}{2048}}{96} + \langle \tilde{D}^2f, D^1[\ell(u)g] \rangle_\varepsilon + O(\varepsilon \log^2 \varepsilon)
\]

\[
- \frac{7 \log 2}{96} - \langle D^1\tilde{D}^2f, \ell(u)g \rangle_\varepsilon + O(\varepsilon \log^3 \varepsilon)
\]

as well as

\[
\left( f, \sqrt{u(u-8)}D^1\left[ \sqrt{u(u-8)}D^0 \right][\ell(u)g] \right) = \frac{\pi^2}{128} - \left( \sqrt{u(u-8)}D^1\left[ \sqrt{u(u-8)}f \right], \ell(u)g \right), \quad (3.61)
\]

which can be combined into \(-\frac{7 \log 2}{96} = \langle f, \left[ \tilde{L}_3, \ell(u)D^0 \right]g \rangle\). Here, to fully connect this to the first equality in (3.53), we also require the following observation

\[
(uD^2 + D^1)\tilde{\text{IKM}}(3, 2; 1|u) = \frac{1}{4} \int_0^{\infty} J_0(\sqrt{-ut}) \left\{ \left[ I_0(t)K_0(t) \right]^2 - \frac{1}{4t^2} \right\} t^3 dt. \quad (3.62)
\]

To verify the identity above, we need a variation on [28, (3.32)]:

\[
\lim_{T \to \infty} \int_{0^+ - iT}^{0^+ + iT} H_0^{(1)}(i\sqrt{-uz})[H_0^{(1)}(z)H_0^{(2)}(z)]^2 \, dz = 0, \quad (3.63a)
\]

\[
\lim_{T \to \infty} \int_{0^+ - iT}^{0^+ + iT} H_0^{(1)}(i\sqrt{-uz}) \left\{ [H_0^{(1)}(z)H_0^{(2)}(z)]^2 - \frac{4}{\pi^2 z^2} \right\} z^3 \, dz = 0, \quad (3.63b)
\]
where the contours close to the right. Spelling out the Hankel functions using the Bessel functions, we can turn the last pair of vanishing integrals into
\[
\int_0^\infty J_0(\sqrt{-u}t)[\pi I_0(t)K_0(t)]^2 t \, dt = \int_0^\infty J_0(\sqrt{-u}t)[K_0(t)]^{3} t \, dt - 2\pi \int_0^\infty Y_0(\sqrt{-u}t)I_0(t)[K_0(t)]^{3} t \, dt,
\]
for \( u < 0 \). Hitting the Bessel differential operator on the right hand side of (3.64a) [which is equivalent to a combination of (3.16) and (3.18)], we can deduce (3.62) from (3.64b).

Before verifying the second halves of (3.51)–(3.53), we transcribe the Chan–Zudilin base-change formulae [13, (3.3)–(3.5)] into the following identities
\[
2^{4}3^{4} \varphi_{6,6}(z) = \frac{[P_{-1/3}(1-2\alpha_3(z))]^{4}[1-2\alpha_3(z)] \partial \alpha_3(z)}{\pi i} \frac{\partial \alpha_3(z)}{\partial z},
\]
(3.65)
\[
\frac{3}{2} f_{6,6}(z) + 2^{4}3^{4} \chi_{6,6}(z) = - \frac{[P_{-1/3}(1-2\alpha_3(2z))]^{4}[1-2\alpha_3(2z)] \partial \alpha_3(2z)}{\pi i} \frac{\partial \alpha_3(2z)}{\partial z},
\]
(3.66)
for \( z/i > 0 \). Here, the Ramanujan cubic invariant (see [4, Chap. 33, §§2–8] and [1, Chap. 9])
\[
\alpha_3(z) := \left[ \frac{[\eta(z/3)]^3}{3[\eta(3z)]^3} + 1 \right]^{-3} = \left( \frac{[\eta(z)]^{12}}{27[\eta(3z)]^{12}} + 1 \right)^{-1}
\]
maps the positive \( \text{Im} z \)-axis bijectively to the open unit interval \((0, 1)\); the Legendre function of degree \(-1/3\) [3, p. 22, (1.6.28)]
\[
P_{-1/3}(\cos \theta) = \frac{2}{\pi} \int_0^\theta \frac{\cos \beta}{\sqrt{2(\cos \beta - \cos \theta)}} \, d\beta, \quad \theta \in (0, \pi)
\]
(3.68)
satisfies [4, Chap. 33, (5.24)]
\[
z = \frac{ iP_{-1/3}(2\alpha_3(z)-1)}{\sqrt{3}P_{-1/3}(1-2\alpha_3(z))}, \quad z/i > 0.
\]
(3.69)

In view of the relations above, we arrive at the second equality in (3.51), after canceling out the contributions from the right-hand sides of (3.65) and (3.66) as \( \int_{-1}^{1} x[P_{-1/3}(x)]^{4} \, d x - \int_{-1}^{1} x[P_{-1/3}(x)]^{4} \, d x = 0 \), and referring back to (3.21). Similarly, the second equality in (3.53) follows from a trivial identity \( \int_{-1}^{1} x[P_{-1/3}(x)]^{2}[P_{-1/3}(-x)]^{2} \, d x = 0 \) along with (3.21). To deduce the second equality in (3.52), we need both (3.22) and a closed-form evaluation \( \int_{-1}^{1} x[P_{-1/3}(x)]^{3}P_{-1/3}(-x) \, d x = -\frac{9{\sqrt{7}}}{4\pi^{2}}\frac{23}{28} \).

\textbf{Corollary 3.6} (Consequences of exceptional sum rules). The identities (1.15) and (1.16) are true.

\textbf{Proof.} We recall the following relations for Crandall numbers
\[
\begin{align*}
\pi^{2} \text{IKM}(3,5;1) - \text{IKM}(1,7;1) &= 0, \\
\pi^{2} \text{IKM}(3,5;3) - \text{IKM}(1,7;3) &= \frac{4}{27}, \\
\pi^{2} \text{IKM}(3,5;5) - \text{IKM}(1,7;5) &= \frac{4}{27}.
\end{align*}
\]
(3.70)
which had arisen from numerical experiments of Broadhurst–Mellit (see \([10, 7.10]\) or \([9, 149]\) in Conjecture 5) before being verified algebraically (see \([26, 3.2]\) or \([29, 3]\)). The first two
equations of these allow us to deduce (1.15) from (3.52).

The following determinant
\[
\det \begin{pmatrix}
\text{IKM}(1, 7; 1) & \text{IKM}(1, 7; 3) & \text{IKM}(1, 7; 5) \\
\text{IKM}(2, 6; 1) & \text{IKM}(2, 6; 3) & \text{IKM}(2, 6; 5) \\
\text{IKM}(3, 5; 1) & \text{IKM}(3, 5; 3) & \text{IKM}(3, 5; 5)
\end{pmatrix} = \frac{5\pi^8}{2^{19/3}} \tag{3.71}
\]

had been discovered numerically (see \([10, 7.11]\) or \([9, 163]\)) before a mathematical proof was
found \([25, 4]\). According to (3.70), the determinant above must be equal to
\[
\det \begin{pmatrix}
\text{IKM}(1, 7; 1) & \text{IKM}(1, 7; 3) & \text{IKM}(1, 7; 5) \\
\text{IKM}(2, 6; 1) & \text{IKM}(2, 6; 3) & \text{IKM}(2, 6; 5) \\
0 & \frac{\pi^2}{2^9} & \frac{\pi^2}{2^9}
\end{pmatrix}. \tag{3.72}
\]

Then, we subtract \(\frac{1}{12}\) times the first column from the second column, while referring to (1.14)–
(1.15), so as to equate the last determinant with
\[
\det \begin{pmatrix}
\text{IKM}(1, 7; 1) & \frac{7\pi^4}{3456} & \text{IKM}(1, 7; 5) \\
\text{IKM}(2, 6; 1) & 0 & \text{IKM}(2, 6; 5) \\
0 & \frac{\pi^2}{2^9} & \frac{\pi^2}{2^9}
\end{pmatrix}. \tag{3.73}
\]

Therefore, the non-linear sum rule (1.16) is true.

At present, we are not aware of any applications of the “honor ary Bessel moment” \(\text{IKM}(4, 4; 3)\)
to quantum field theory, but this quantity does play a rôle in the asymptotic analysis of a probability
density function, as we describe below.

**Corollary 3.7 (Asymptotic behavior of Kluyver’s \(p_7(x)\)).** Let \(p_n(x) = \int_0^\infty J_0(xt)[J_0(t)]^n xt \, dt\) be
Kluyver’s probability density for the distance \(x\) traveled by a rambler walking in the Euclidean
plane, who takes \(n\) steps of unit lengths, while aiming at uniformly distributed directions. We have
\[
\lim_{x \to 1^-} \left[ \frac{p_7(x)}{12\pi^2 x} + \frac{d^2}{dx^2} \frac{p_7(x)}{35 x} \right] = \frac{19L(f_{6, 6}, 1)}{648\pi^2} + \frac{7\log 2}{72\pi^4}. \tag{3.74}
\]

In other words, the following asymptotic expansion applies to \(x \to 1^-\):
\[
\frac{p_7(x)}{35 x} = \frac{L(f_{6, 6}, 1)}{9\pi^2} \left[ 1 - \frac{x - 1}{4} + \frac{19(x - 1)^2}{144} \right] - \frac{3(x - 1)^2}{32\pi^4} + \frac{(x - 1)^2}{16\pi^4} \log \frac{1 - x}{2^{11/9}} + o((x - 1)^2 \log(1 - x)). \tag{3.75}
\]

**Proof.** From \([27, 4.2]\), we know that
\[
\frac{p_7(x)}{35} = \frac{4}{\pi^6} \int_0^\infty I_0(xt)I_0(t)[K_0(t)]^6 xt \, dt - \frac{2}{\pi^4} \int_0^\infty I_0(xt)[I_0(t)]^3[K_0(t)]^4 xt \, dt
\]
holds for \(x \in [0, 1]\). (This corrects \([27, 4.6]\), where a factor of \(x\) went missing on the right-hand
side.) Thus, it follows that (see \([27, 4.1, 4.3]\); see also \([3.21, 3.23]\) and \([3.25]\) above)
\[
\frac{p_7(1)}{35} = \frac{4}{\pi^6} \text{IKM}(2, 6; 1) - \frac{2}{\pi^4} \text{IKM}(4, 4; 1) = \frac{L(f_{6, 6}, 1)}{9\pi^2}. \tag{3.77}
\]
Differentiating $p_7(x)/x = \int_0^\infty J_0(xt) [J_0(t)]^7 t \, dt$ under the integral sign and integrating by parts, we have
\[
\frac{d}{dx} \left|_{x=1} \right. \frac{p_7(x)}{x} = - \int_0^\infty J_1(t) [J_0(t)]^7 t^2 \, dt = -\frac{p_7(1)}{4},
\]
for $J_1(x) = - \frac{d J_0(x)}{dx}$.

Meanwhile, using the Bessel differential equation, we can turn (3.76) into
\[
\left( \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} \right) \frac{p_7(x)}{35x} = \frac{4}{\pi^6} \int_0^\infty I_0(xt) I_0(t) \{ K_0(t) \}^6 r^3 \, dt - \frac{2}{\pi^4} \int_0^\infty I_0(xt) I_0(t) \{ K_0(t) \}^3 \{ K_0(t) \}^4 r^3 \, dt,
\]
for $0 < x < 1$. As we juxtapose the last equation with the following identity (see [24, Lemma 4.1.1] or [27, (3.3)])
\[
p_3(x) = \frac{6}{\pi^2} \int_0^\infty I_0(xt) I_0(t) \{ K_0(t) \}^2 xt \, dt, \quad 0 \leq x < 1,
\]
we obtain
\[
\lim_{x \to 1^-} \left[ \frac{p_3(x)}{12\pi^2 x} + \frac{d^2}{dx^2} \frac{p_7(x)}{35x} \right] = \frac{4}{\pi^6} \text{IKM}(2,6;3) - \frac{2}{\pi^4} \text{IKM}(4,4;3).
\]
Thanks to (3.78), (1.12) and (1.14), this further reduces into
\[
\lim_{x \to 1^-} \left[ \frac{p_3(x)}{12\pi^2 x} + \frac{d^2}{dx^2} \frac{p_7(x)}{35x} \right] = \frac{p_7(1)}{140} = \frac{p_7(1)}{2520} + \frac{7 \log 2}{72\pi^4},
\]
hence our claim in (3.74).

On the other hand, through asymptotic analysis of the explicit formula for $p_3(x)$ [7, (4)], we see that
\[
\frac{p_3(x)}{12\pi^2 x} = \frac{1}{8\pi^4} \log \frac{4}{1-x} + O((x-1) \log(1-x)), \quad x \to 1^-.
\]
Therefore, we arrive at
\[
\lim_{x \to 1^-} \left[ -\frac{\log(1-x)}{8\pi^4} + \frac{d^2}{dx^2} \frac{p_7(x)}{35x} \right] = \frac{19 L(f_{6,6},1)}{648\pi^2} - \frac{11 \log 2}{72\pi^4},
\]
which is compatible with (3.75).

**Remark** Comparing the right-hand side of (3.76) with [30, Lemma 3.4(b)], and setting
\[
g(x,t) := I_0(xt) \{ K_0(t) \}^5 \left\{ [K_0(t)]^2 - 3[\pi I_0(t)]^2 \right\}^2
+ K_0(xt) I_0(t) \{ K_0(t) \}^4 \left\{ 7[K_0(t)]^2 - 5[\pi I_0(t)]^2 \right\},
\]
we see that
\[
\frac{p_7(x)}{35} + \frac{2\log(1-x)}{\pi^8} \int_0^\infty g(x,t) t \, dt
\]
equals a Taylor series expansion in powers of $(x-1)^{\geq 0}$, as $x \to 1^-$. Meanwhile, by [30, Lemma 3.4(b)], we have another Taylor expansion in the neighborhood of $x = 1$, with leading order behavior
\[
\int_0^\infty g(x,t) t \, dt = -\frac{\pi^4}{32} (x-1)^2 + O((x-1)^3).
\]
This explains the qualitative structure of the right-hand side in (3.75).
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