Cosmological Brane Perturbations

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Two approaches to the study of cosmological perturbations in the brane-world scenario are compared: the first uses the 5D equations directly whereas the second approach projects them onto the 4D brane and then uses the effective 4D equations.

1 Introduction

At the end of the last millennium it was realized that the traditional Kaluza-Klein approach is not the only possibility of dimensional reduction. An alternative approach was developed, the brane-world scenario. In this scenario the standard matter fields are constrained to a lower dimensional hypersurface, or brane, which is embedded in a higher dimensional spacetime, or bulk. The gravitational field is not restricted to the brane and permeates the bulk as well as the brane.

One of the goals of current brane-world research is to contrast theoretical predictions with observations. In standard 4D cosmology one distinctive feature of the early universe models is the spectrum of density perturbation these models predict. In higher-dimensional models of the universe the evolution of cosmological scalar perturbations and the possible observational consequences these models have is still an open issue and a lot of work is currently devoted to calculating the spectrum of density perturbations in the brane-world scenario.

In this note we shall restrict the discussion to scalar perturbations, that is to perturbations that transform like scalars on 3-spaces of constant time. Tensor perturbations are discussed in reference and vector perturbations e.g. in.

In the next section we will briefly outline the simple brane-world model we are investigating. For a more thorough introduction to the brane-world scenario see for example the contribution

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of D. Langlois to these proceedings. In Sections 3 and 4 we shall describe two different approaches to solve the cosmological perturbation equations that arise from the brane-world model. We finish with a brief conclusion.

2 The Brane-world

We will assume that our 4-dimensional world is described by a 3-brane embedded in a 5-dimensional bulk. The equations of motion in the brane-world are given by Einstein’s equations

\[ G_{AB} = \kappa_5^2 (-\Lambda g_{AB} + T_{AB}) , \]  

where \( G_{AB} \) is the 5D Einstein tensor, \( g_{AB} \) is the 5D metric, \( \Lambda \) is the bulk cosmological constant, and \( T_{AB} \) is the energy momentum tensor of the bulk fields. The standard matter fields that live on the brane, enter the picture only through the Israel junction conditions: these junction conditions relate the matter on the brane, described by the energy momentum tensor \( T_{\mu \nu} \) and the brane tension \( \lambda \), to the extrinsic curvature \( K_{\mu \nu} \) of the brane, and are given by

\[ K_{\mu \nu} = \frac{\kappa_5^2}{2} \left( T_{\mu \nu} - \frac{1}{3} g_{\mu \nu} (T - \lambda) \right) , \]

where \( g_{\mu \nu} \) is the 4D metric on the brane and we assumed that the brane is located at a fixed point in the \( Z_2 \) symmetric bulk.

To solve the Einstein equations subject to the junction conditions Eq. (2) we have two options: either to use the 5D equations directly or to project everything onto the brane and then use the ensuing 4D equations. We shall describe the latter in the next section and the former in Section 4.

3 The Projective 4D Approach

For a 4D observer it is natural to ask, what he or she is able to observe living in a 5D brane-world but being restricted to the 4D brane. Using the Gauss and the Codazzi equations Shiromizu et al. showed that the 5D Einstein equations can be projected onto the brane to give the effective four-dimensional Einstein equations

\[ ^{(4)}G_{\mu \nu} + \Lambda_4 g_{\mu \nu} = \kappa_4^2 T_{\mu \nu} + \kappa_4^2 \Pi_{\mu \nu} - E_{\mu \nu} , \]

where the effective cosmological constant on the brane is \( \Lambda_4 = \frac{1}{2} \Lambda_5 + \frac{\kappa_4^2}{12} \lambda^2 \), the 4D coupling constant is related to the 5D coupling constant by \( \kappa_4^2 = \frac{\kappa_5^4}{6} \lambda \) and \( E_{\mu \nu} \) is the projected 5D Weyl tensor, which describes the non-local effect of the gravitational field. It is defined as

\[ E_{\mu \nu} \equiv C^E_{\ AFB} n^A \ g_{\mu}^F g_{\nu}^B , \]

where \( n_A \) is the normal vector to the brane. The projected Weyl tensor \( E_{\mu \nu} \) acts like an imperfect radiation fluid with anisotropic stress. The quadratic energy momentum tensor \( \Pi_{\mu \nu} \) is given by

\[ \Pi_{\mu \nu} = -\frac{1}{4} T_{\mu \alpha} T_{\nu}^\alpha + \frac{1}{12} TT_{\mu \nu} + \frac{1}{8} g_{\mu \nu} T_{\alpha \beta} T^{\alpha \beta} - \frac{1}{24} g_{\mu \nu} T^2 . \]

Although we started with the 5D Einstein equations we get 4D effective equations which are independent of the evolution of the bulk spacetime, being given entirely in terms of quantities defined on, or near, the brane.

\[^b\text{Note, that in this section we have set the bulk energy momentum tensor } T_{AB} = 0, \text{ since life is difficult enough without it. Here, the only energy in the bulk is the 5D cosmological constant.}\]
Unfortunately this leaves terms which are not completely determined by the local dynamics on the brane. As in the 4D case we would like to define a quantity that can source the large scale CMB anisotropies. Conservation of energy on the brane allows us to construct a curvature perturbation on uniform density hypersurfaces, $\zeta$, which is conserved on large scales for adiabatic matter perturbations. In a similar vain we can construct a curvature perturbation on uniform Weyl-fluid energy density hypersurfaces, $\zeta_{\text{Weyl}}$, since the Weyl-fluid energy density is also conserved to linear order. However, the quantity that sources the CMB anisotropies is the total curvature perturbation $\zeta_{\text{total}}$ and its evolution depends not only on $\zeta$ and $\zeta_{\text{Weyl}}$ but also on terms which can not be calculated using quantities defined solely on the brane.

Nonetheless the dynamics and effective gravity on the brane can be interpreted, and often most easily understood, in terms of the effective four-dimensional Einstein equations.

4 Using the Full 5D Equations

So instead of using the projective approach described in the last section we could use the full five-dimensional equations of motion $\Box g$. This approach has enjoyed considerable attention. For concreteness let us start with the metric

$$g_{AB} = \begin{pmatrix}
-n^2(1 + 2A) & a^2B_{i,j} & nA_y \\
a^2B_{i,j} & a^2[(1 + 2R)\delta_{ij} + 2E_{ij}] & a^2B_{y,i} \\
nA_y & a^2B_{y,i} & b^2(1 + 2A_{yy})
\end{pmatrix}. \tag{6}$$

Here $n$, $a$, and $b$ are scale factors and functions of coordinate time $t$ and the extra dimension $y$, and $A, B, E, R, B_y, A_y,$ and $A_{yy}$ are the scalar metric perturbations and functions of $x^A$. Although it is tedious, with the help of a computer algebra package it is possible to write down the full 5D equations of motion.

One problem in solving the equations of motion is the sheer size of the equations: Whereas the perturbed part of for example the 0-0 component of the 4D Einstein tensor has four terms in an arbitrary gauge, the same component has 14 terms for the 5D metric given in Eq. (6).

Another, even more unpleasant problem is as follows: In standard 4D cosmology we Fourier decompose the perturbations on 3-spaces of constant time, which reduces the equations of motion to a system of ordinary differential equations in $t$. In 5D, in order to be able to relate the results of the calculations to observations, we decompose the metric also on spatial 3-spaces into Fourier modes. But here we get a system of coupled partial differential equations in $t$ and $y$. As one can imagine, this makes finding general solutions extremely difficult.

Nevertheless progress has been made. A lot of work has been devoted to develop the formalism, which is by now quite well understood. In order to be able to rewrite the problem in terms of ordinary differential equations quite often a simplified background is used. For example if we choose the background to be static Minkowski space, we assume that the scale factors reduce to $n = b = a$ where $a = a(y)$ and also any other background quantities are $y$-dependent only. Although this is quite a severe simplification, it allows for considerable progress since the equations of motion are now more likely to decouple, see e.g. in the particular case of a dilatonic brane-world model.

5 Conclusion

The projective approach, described above in Section gives useful physical insights into the physics of the brane-world scenario. Unfortunately it does not give, in general, a closed system of equations. We therefore advocate the use of the full 5D equations, as described in the last section, as the way forward. More analytical work needs to be done to solve the evolution equations in a general setting, probably accompanied by numerical efforts.
Progress so far has been slow and painful for cosmological brane perturbations, but it is still early days . . .

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