NOTE ON AFFINE GAGLIARDO-NIRENBERG INEQUALITIES

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ABSTRACT. This note proves sharp affine Gagliardo-Nirenberg inequalities which are stronger than all known sharp Euclidean Gagliardo-Nirenberg inequalities and imply the affine $L^p$–Sobolev inequalities. The logarithmic version of affine $L^p$–Sobolev inequalities is verified. Moreover, An alternative proof of the affine Moser-Trudinger and Morrey-Sobolev inequalities is given. The main tools are the equimeasurability of rearrangements and the strengthened version of the classical Pólya-Szegő principle.

1. INTRODUCTION

In this note, we prove sharp affine Gagliardo-Nirenberg inequalities. These inequalities generalize the sharp affine $L^p$–Sobolev inequalities

\begin{equation}
C_{p,n} \|f\|_{L_{\frac{n}{n-p}}^p(\mathbb{R}^n)} \leq \mathcal{E}_p(f) \quad \text{for } f \in W^{1,p}(\mathbb{R}^n), 1 \leq p < n,
\end{equation}

established by Lutwak, Yang and Zhang [33] for $1 < p < n$ and Zhang [45] for $p = 1$. Here $W^{1,p}(\mathbb{R}^n)$ is the usual Sobolev space defined as the set of functions $f \in L^p(\mathbb{R}^n)$ with weak derivative $\nabla f \in L^p(\mathbb{R}^n)$. $\mathcal{E}_p(f)$ is the $L^p$ affine energy of $f$ defined as

\[
\mathcal{E}_p(f) = c_{n,p} \left( \int_{S^{n-1}} \|D_v f\|_{L^p(\mathbb{R}^n)}^n dv \right)^{\frac{1}{n}} = c_{n,p} \left( \int_{S^{n-1}} \left( \int_{\mathbb{R}^n} |v \cdot \nabla f(x)|^p dx \right)^{\frac{n}{n/p}} dv \right)^{\frac{1}{n/p}}.
\]

The constant $c_{n,p} = \left( \frac{n\omega_n \omega_{n-1}}{2(\omega_{n+p-2})} \right)^{1/p} (n\omega_n)^{1/n}$ with $\omega_n$ being the $n$–dimensional volume enclosed by the unit sphere $S^{n-1}$. For each $v \in S^{n-1}$, $\|D_v f\|_{L^p(\mathbb{R}^n)}$ is the $L^p(\mathbb{R}^n)$ norm of the directional derivative $D_v f$ of $f$ along $v$.

Inequality (1.1) is stronger than the classical $L^p$–Sobolev inequalities

\begin{equation}
C_{p,n} \|f\|_{L_{\frac{n}{n-p}}^p(\mathbb{R}^n)} \leq \|\nabla f\|_{L^p(\mathbb{R}^n)} \quad \text{for } f \in W^{1,p}(\mathbb{R}^n), 1 \leq p < n,
\end{equation}

see Aubin [4] and Talenti [42] for $1 < p < n$, Federer and Fleming [17] and Maz’ya [37] for $p = 1$. This can be seen from

\begin{equation}
\mathcal{E}_p(f) \leq \|\nabla f\|_{L^p(\mathbb{R}^n)}
\end{equation}

due to Pólya-Szegő principle.

\[2000 \textbf{Mathematics Subject Classification.} \text{ Primary 46E35; 46E30.}\]

\textbf{Key words and phrases.} Sobolev spaces; Gagliardo-Nirenberg Inequalities; Sharp constant; Rearrangements; Pólya-Szegő principle.

Project supported in part by Natural Science and Engineering Research Council of Canada.
inequality and Morrey-Sobolev inequality are counterparts of (1.2) for $p = n$ and $p > n$, respectively. The first one, see Moser [38], means that there exists $m_n = \sup_{\phi} \int_0^\infty e^{(\phi(t))^{n'}} dt$ such that

$$
\frac{1}{\text{sprt} f} \int_{\mathbb{R}^n} e^{(m_n^{1/n} |f(x)|/\|\nabla f\|_n)^{n'}} dx \leq m_n
$$

for every $f \in W^{1,n} (\mathbb{R}^n)$ with $0 < |\text{spet} f| := \{|x \in \mathbb{R}^n : f(x) \neq 0\} < \infty$ and $n' = \frac{n}{n-1}$. Moreover, Carleson and Chang in [7] proved that extremals do existence for (1.4). Here $|A|$ is the Lebesgue measure of $A \subset \mathbb{R}^n$. For $p > n$, the Morrey-Sobolev inequality states that

$$
\|f\|_{L^\infty (\mathbb{R}^n)} \leq b_{n,p} |\text{spet} f|^{\frac{1}{p}-\frac{1}{q}} \|\nabla f\|_{L^p (\mathbb{R}^n)}
$$

for every $f \in W^{1,p} (\mathbb{R}^n)$ with $|\text{spet} f| < \infty$.

As a variant of the classical $L^p -$Sobolev inequality (1.2), the Euclidean Gagliardo-Nirenberg /Nash’s inequality states that

$$
\|f\|_{L^q (\mathbb{R}^n)} \leq C_{n,s,p,q} \|\nabla f\|_{L^p (\mathbb{R}^n)} \|f\|_{L^s (\mathbb{R}^n)}^{1-\theta}
$$

for $n \geq 1$, suitable constants $p, q, s$ and $\theta$. The Euclidean Gagliardo-Nirenberg /Nash’s inequality has been studied intensively and been applied in analysis and partial differential equations. See, for example, Nirenberg [39], Gagliardo [18], Cordero-Erausquin, Nazaret and Villani [11], Del Pino and Dolbeault [12]-[15], Del Pino, Dolbeault and Gentil [16], Carlen and Loss [6], Agué [1]-[3].

Inequalities (1.4) and (1.5) were also strengthened by the affine Moser-Trudinger inequality and affine Morrey sobolev inequality (see Cinachi, Lutwak, Yang and Zhang [11]), respectively. The main aim of this paper is to establish the following sharp affine Gagliardo-Nirenberg inequality was studied by Lutwak, Yang and Zhang in [39] with the restriction $s = p\frac{p-1}{p-1}$.

In this paper, we will remove this restriction. Below, we will denote $D^{p,q} (\mathbb{R}^n)$ as the completion of the space of smooth compactly supported functions $f$ on $\mathbb{R}^n$ for the norm $\|f\|_{p,q} = \|\nabla f\|_{L^p (\mathbb{R}^n)} + \|f\|_{L^q (\mathbb{R}^n)}$.

**Theorem 1.1.** Let $n, p, q$ and $s$ be such that

$$
1 < p < n \quad \text{and} \quad 1 \leq q < s < p^* = \frac{np}{n-p} \quad \text{if} \quad n > 1.
$$

Then the $L^p$ affine Gagliardo-Nirenberg inequality

$$
\|f\|_{L^q (\mathbb{R}^n)} \leq K_{\text{opt}} (C_p (f))^{\theta} \|f\|_{L^s (\mathbb{R}^n)}^{1-\theta}
$$

holds with $\theta = \frac{np(s-q)}{q(p-n)s(p-n)}$, and the sharp constant $K_{\text{opt}} > 0$ is explicitly given by

$$
K_{\text{opt}} = \left[ \frac{C(n,p,q,s)}{E(u_\infty)} \right]^{\frac{n(p+q-s-q)}{2(n-p)(n-q)}}.
$$

Here

$$
C(n,p,q,s) = \frac{\alpha + \beta}{(q\alpha)^{\frac{1}{q}}(p\beta)^{\frac{1}{p}}} , \alpha = np - s(n-p), \beta = n(s-q)
$$

$u_\infty$ is the minimizer of the variational problem

$$
\inf \left\{ E(u) = \frac{1}{p} \int_{\mathbb{R}^n} |\nabla u|^p dx + \frac{1}{q} \int_{\mathbb{R}^n} |u|^q dx : u \in D^{p,q} (\mathbb{R}^n), \|u\|_{L^s (\mathbb{R}^n)} = 1 \right\}.
$$
Moreover,
\begin{equation}
(f_{\sigma,x_0} = C u_{\infty}(A(x-x_0)))
\end{equation}
are optimal functions in inequality (1.7), for arbitrary \( C \neq 0 \), \( x_0 \in \mathbb{R}^n \) and \( A \in GL(n) \).

**Remark 1.2.** (i) For the proof of existence of a minimizer to problem (1.8), see, for example, Del-pino Dolbeault [13].

(ii) Under the assumption of Theorem 1.1, (1.7) implies the \( L^p \) Gagliardo-Nirenberg inequality, see Agueh [2]-[3]
\begin{equation}
\|f\|_{L^q(\mathbb{R}^n)} \leq K_{opt} \|\nabla f\|_{L^p(\mathbb{R}^n)}^{\theta} \|f\|_{L^n(\mathbb{R}^n)}^{1-\theta}, \forall f \in D^{n,q}(\mathbb{R}^n).
\end{equation}

Moreover, \( f_{\sigma,x_0} = C u_{\infty}(\sigma(x-x_0)) \) are optimal functions in inequality (1.7), for arbitrary \( C \neq 0 \).

(iii) If \( q = 1 \) and \( p = s \), from (1.7), we can get the affine \( L^p \) Nash’s inequality
\begin{equation}
\left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1+\frac{p}{n(p-1)}} \leq (K_{opt})^{p+\frac{p^2}{n(p-1)}} (E_p(f))^p \left( \int_{\mathbb{R}^n} |f(x)| dx \right)^{\frac{n^2}{np-1}}
\end{equation}
for \( 1 < p < n \) if \( n > 1 \).

Theorem 1.1 implies the following sharp affine Gagliardo-Nirenberg inequalities stronger than the Euclidean ones in [13].

**Corollary 1.3.** Let \( 1 < p < n \), \( p < q \leq \frac{d(n-1)}{n-p} \). Then for all \( f \in D^{n,q}(\mathbb{R}^n) \), we have
\begin{equation}
\|f\|_{L^q(\mathbb{R}^n)} \leq C_2 (E_p(f))^\theta \|f\|_{L^n(\mathbb{R}^n)}^{1-\theta}.
\end{equation}

Here \( s = p \frac{q-1}{p-1} \) and
\begin{equation}
\theta = \frac{(q-p)n}{(q-1)(np-(n-p)q)}
\end{equation}
and with \( \delta = np - q(n-p) > 0 \), the optimal constant \( C_2 \) takes the form
\begin{equation}
C_2 = \left( \frac{q-p}{p\sqrt{\pi}} \right)^\theta \left( \frac{pq}{n(q-p)} \right)^\frac{\delta}{pq} \left( \frac{\Gamma \left( \frac{q-1}{q-p} \right) \Gamma \left( \frac{n}{2} + 1 \right)}{\Gamma \left( \frac{p-1}{p} \right)} \right)^\frac{\delta}{q-p}.
\end{equation}

Equality holds in (1.11) if and only if for some \( \alpha \in \mathbb{R} \), \( \beta > 0 \), \( \tau \in \mathbb{R}^n \),
\begin{equation}
f(x) = \alpha \left( 1 + \beta |A(x-\tau)|^\frac{p-1}{q-p} \right)^{-\frac{p-1}{q-p}} \forall x \in \mathbb{R}^n
\end{equation}
with \( A \in GL(n) \).

**Remark 1.4.** (i) When \( q = p \frac{n-1}{n-p} \), \( \theta = 1 \) and \( s = \frac{np}{n-p} \). Thus inequality (1.11) implies the sharp affine \( L^p \)-Sobolev inequality.

(ii) Inequality (1.11) was proved by Lutwak, Yang and Zhang in [36] where the authors applied the optimal \( L^p \) Sobolev norm problems and \( L^p \) Petty projection inequality (see Gardner [19], Schneider [40] and Thompson [43] for \( p = 1 \), Lutwak, Yang and Zhang [32] for \( p > 1 \)).

Similarly, for \( q < p < n \), we can obtain the following results.
Corollary 1.5. Let $1 < p < n$, $1 < q < p$. Then for all $f \in D_{p,r}(\mathbb{R}^n)$, we have
\begin{equation}
\|f\|_{L^q(\mathbb{R}^n)} \leq C_3 (\mathcal{E}_p(f))^{\theta} \|f\|_{L^r(\mathbb{R}^n)}^{1-\theta}.
\end{equation}
Here $r = p_{\frac{q-1}{p-1}}$ and
\[\theta = \frac{q(p-n)}{q(n(n-q) + p(n-1))}\]
and with $\delta = np - q(n - p) > 0$, the optimal constant $C_3$ takes the form
\[C_3 = \left(\frac{p-q}{p-q}\right)^{\theta} \left(\frac{pq}{n(p-q)}\right)^{\frac{p}{\delta}} \left(\frac{\Gamma\left(\frac{p-1}{p-\delta}\right) + 1}{\Gamma\left(\frac{n-\delta}{p-\delta}\right) + 1}\right)^{\frac{\pi}{q-p}}\]
If $q > 2 - \frac{1}{p}$, equality holds in (1.11) if and only if for some $\alpha \in \mathbb{R}$, $\beta > 0$, $\pi \in \mathbb{R}^n$,
\begin{equation}
f(x) = \alpha \left(1 - \beta|A(x-\pi)|^{\frac{p-1}{p}}\right)^{\frac{1}{c-p}} \forall x \in \mathbb{R}^n
\end{equation}
with $A \in GL(n)$.

We get the following logarithmic version of (1.11).

Proposition 1.6. For any $f \in W^{1,p}(\mathbb{R}^n)$ with $1 < p < n$ and $\int_{\mathbb{R}^n} |f(x)|^p dx = 1$, we have
\begin{equation}
\int_{\mathbb{R}^n} |f(x)|^p \log |f(x)| dx \leq \frac{n}{p^2} \log (\mathcal{E}_p(f))^p.
\end{equation}
Here the optimal constant $C_4$ is defined by
\begin{equation}
C_4 = \frac{p}{n} \left(1 - \frac{1}{e}\right)^{p-1} \left[\frac{\Gamma\left(\frac{p-1}{p}\right) + 1}{\Gamma\left(n\frac{p-1}{p} + 1\right)}\right]^\frac{\pi}{p}.
\end{equation}
Inequality in (1.13) is optimal and equality holds if and only if for some $\sigma > 0$ and $\pi \in \mathbb{R}^n$,
\begin{equation}
f(x) = \pi^{\frac{n-1}{p}} \sigma^{-\frac{n-1}{p}} \frac{\Gamma\left(\frac{p-1}{p}\right) + 1}{\Gamma\left(n\frac{p-1}{p} + 1\right)} e^{-\frac{1}{2}|A(x-\pi)|^{\frac{p-1}{p}}} \forall x \in \mathbb{R}^n
\end{equation}
with $A \in GL(n)$.

Remark 1.7. Inequality (1.15) generalizes the sharp Euclidean $L^p$–Sobolev logarithmic equality since $\mathcal{E}_p(f) \leq \|\nabla f\|_{L^p(\mathbb{R}^n)}$. Meanwhile, it can also been viewed as the limiting case $r = p = q$ of inequality (1.11). For more details about Euclidean $L^p$–Sobolev logarithmic equality, see Weissler [44] and Groos [21], Del Pino and Dolbeault [12], Gentil [20] and the reference therein.

We give an alternative proof of the affine Moser-Trudinger and Morrey-Sobolev inequalities established by Cianchi, Lutwak, Yang and Zhang in [10].

Proposition 1.8. Suppose $n > 1$. Then for every $f \in W^{1,n}(\mathbb{R}^n)$ with $0 < |\text{supp}(f)| < \infty$,
\begin{equation}
\frac{1}{|\text{supp}(f)|} \int_{\text{supp}(f)} \exp\left(n\omega_n |f(x)|^{\frac{n}{\mathcal{E}_n(f)}}\right)^{n'} dx \leq m_n
\end{equation}
with \( m_n = \sup_{\phi} \int_0^\infty e^{(\phi(t))' - t} dt \). The constant \( n\omega_n^{1/n} \) is the best one in the sense that (1.18) would fail if \( n\omega_n^{1/n} \) is replaced by a larger one.

**Proposition 1.9.** If \( p < n \), then for every \( f \in W^{1,p}(\mathbb{R}^n) \) with \( \text{sprt}(f) \) \( < \infty \),

\[
\|f\|_{L^\infty(\mathbb{R}^n)} \leq b_{n,p} |\text{sprt}(f)|^{\frac{1}{p} - \frac{1}{n}} \mathcal{E}_p(f).
\]

Equality holds in (1.19) if and only if

\[
f(x) = a \left( 1 - |A(x - x_0)|^{\frac{2}{n}} \right) \text{ for some } a \in \mathbb{R}, x_0 \in \mathbb{R}^n, \text{ and } A \in GL(n).
\]

Here \( \text{“} + \text{”} \) denotes the “positive part”.

Cianchi, Lutwak, Yang and Zhang, in [10], proved inequality (1.18) by showing that

\[
m_n = \sup \frac{1}{a} \int_0^a \exp(n\omega_n^{1/n}\phi(s))^{\frac{1}{n}} ds
\]

and inequality (1.19) by the strengthened version of the classical Pólya-Szegő principle, the local absolute continuity of the decreasing rearrangement of \( f \) and the Hölder inequality. Here, we will prove inequalities (1.18) and (1.19) directly by the observation that sphere rearrangements of functions may give us better estimates for (affine) Sobolev type inequalities.

The rest of this paper is organized as follows: In Section 2, we recall some basic properties of rearrangements of functions and the strengthened version of the classical Pólya-Szegő principle. In Section 3, we prove Propositions 1.3-1.9.

### 2. Strengthened Version of the Classical Pólya-Szegő Principle

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) with

(2.1)
\[
|\{x \in \mathbb{R}^n : |f(x)| > t\}| < \infty \quad \text{for } t > 0.
\]

The distribution function \( m_f(t) \) of \( f \) is defined as

\[
m_f(t) = |\{x \in \mathbb{R}^n : |f(x)| > t\}|, \quad \text{for } t \geq 0.
\]

Functions having the same distribution function are referred to be equidistributed or equimeasurable. On the other hand, equidistributed functions are said to be rearrangements of each other. The decreasing rearrangement \( f^* \) of function \( f \) is defined as

\[
f^*(s) = \sup\{t \geq 0 : m_f(t) > s\} \quad \text{for } s \geq 0.
\]

The spherical symmetric rearrangement \( f^* : \mathbb{R}^n \rightarrow [0, \infty] \) is defined as

\[
f^*(x) = f^*(\omega_n|x|^n) \quad \text{for } x \in \mathbb{R}^n.
\]

Clearly, \( f, f^* \) and \( f^* \) are equidistributed functions. In fact, we have

\[
m_f = m_{f^*} = m_{f^*}.
\]

(2.2)
\[
|\text{sprt}(f)| = |\text{sprt}(f^*)| = |\text{sprt}(f^*)|,
\]

(2.3)
\[
\|f\|_{L^\infty(\mathbb{R}^n)} = f^*(0) = \|f^*\|_{L^\infty},
\]

and

(2.4)
\[
\int_{\mathbb{R}^n} \Phi(|f(x)|) dx = \int_0^\infty \Phi(f^*(s)) ds = \int_{\mathbb{R}^n} \Phi(f^*(x)) dx
\]
for every continuous increasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$. Thus, we have

$$\int_{\mathbb{R}^n} |f(x)|^p \, dx = \int_0^\infty (f^*(s))^p \, ds = \int_{\mathbb{R}^n} (f^*(x))^p \, dx$$

for every $p \geq 1$, and so Lebesgue norms will be invariant under the operations of decreasing rearrangement and of spherically symmetric rearrangement.

The classical Pólya-Szegő principle means that if $f$ with (2.1), is weakly differentiable in $\mathbb{R}^n$ and $|\nabla f| \in L^p(\mathbb{R}^n)$ for $p \in [1, \infty]$, then $f^*$ is locally absolutely continuous in $(0, +\infty)$, $f^*$ is weakly differentiable in $\mathbb{R}^n$ and

$$\|\nabla f^*\|_{L^p(\mathbb{R}^n)} \leq \|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

See, for example, Kawohl [25]-[24], Sperner [41], Talenti [42], Brothers and Ziemer [5], Hilden [22]. Inequality (2.6) is a powerful tool to many problems in physics and mathematics. On the other hand, several variants of inequality (2.6) have been established and applied intensively, see, for example, Kawohl [24]. Especially, Lutwak, Yang and Zhang in [32], Cianchi, Lutwak, Yang and Zhang in [10] proved the following strengthened affine version of inequality (2.6).

**Lemma 2.1.** [32] [10] Suppose $1 < n$ and $1 \leq p$. If $f \in W^{1,p}(\mathbb{R}^n)$, then $f^* \in W^{1,p}(\mathbb{R}^n)$,

$$\mathcal{E}_p(f^*) \leq \mathcal{E}_p(f)$$

and

$$\|\nabla f^*\| = \mathcal{E}_p(f^*).$$

**Remark 2.2.** We can see that both (2.7) and (2.8) are true for $f \in D^{p,q}(\mathbb{R}^n)$.

Inequality (2.6) is particular significant for the authors in [45], [33] and [10] to proved the affine $L^p$–Sobolev, affine Moser-Trudinger and affine Morrey-Sobolev inequalities. In this note, we will see that inequality (2.6) implies the affine Gagliardo-Nirenberg inequalities.

The proof of Lemma 2.1 depends on $L^p$ Brunn-Minkowski theory of convex bodies (see, for example, Chen [8], Chou and Wang [9], Hu, Ma and Shen [23], Ludwig [26]-[27], Lutwak [28]-[29], Lutwak and Oliker [30], Lutwak, Yang and Zhang [32]-[36]). In [10], Lutwak, Yang and Zhang proved Lemma 2.1 by applying the similar rearrangement argument used to prove the Euclidean Sobolev inequality. They solved a family of $L^p$ Minkowski problem (see, Lutwak, Yang and Zhang [35]) to reduce the estimates for $L^p$ gradient integrals to the estimates for $L^p$ mixed volumes of convex bodies. Thus they can replace the classical Euclidean isoperimetric inequality by the affine $L^p$ isoperimetric inequality (see, Lutwak, Yang and Zhang [32]). For the details of Lemma 2.1, we refer the interested reader to Lutwak, Yang and Zhang [10, Theorem 2.1].

**3. PROOF OF THE MAIN RESULTS**

**3.1. Proof of Theorem 1.1.** The symmetrization inequality (2.7) and inequality (2.5) are crucial for the proof of Theorem 1.1.

For any $f \in D^{p,q}(\mathbb{R}^n)$, inequality (2.5) implies that

$$\|f\|_{L^q(\mathbb{R}^n)} = \|f^*\|_{L^q(\mathbb{R}^n)}.$$
The classical Pólya-Szegö principle and inequality \((2.5)\) tell us that \(f^* \in D^{p,q}(\mathbb{R}^n)\). Thus, we can apply the sharp Euclidean Gagliardo-Nirenberg inequality \((1.10)\) (see [3, Theorem 2.1]) to \(f^*\) and get
\[
\|f^*\|_{L^p(\mathbb{R}^n)}\|f^*\|^{\theta-1}_{L^q(\mathbb{R}^n)} \leq C_2\|\nabla f^*\|_{L^p(\mathbb{R}^n)}^\theta.
\]
Lemma 2.1 and Remark 2.2 imply
\[
\|f\|_{L^p(\mathbb{R}^n)}\|f\|^{\theta-1}_{L^q(\mathbb{R}^n)} = \|f^*\|_{L^p(\mathbb{R}^n)}\|f^*\|^{\theta-1}_{L^q(\mathbb{R}^n)} 
\leq K_{opt}\|\nabla f^*\|_{L^p(\mathbb{R}^n)}^\theta 
= K_{opt}(\mathcal{E}_p(f^*))^\theta 
\leq K_{opt}(\mathcal{E}_p(f))^\theta.
\]
Thus, we get
\[
\|f\|_{L^p(\mathbb{R}^n)} \leq K_{opt}(\mathcal{E}_p(f))^\theta \|f\|^{1-\theta}_{L^q(\mathbb{R}^n)}.
\]
On the other hand, since
\[
(3.1) \quad \|f\|_{L^p(\mathbb{R}^n)} \leq K_{opt}(\mathcal{E}_p(f))^\theta \|f\|^{1-\theta}_{L^q(\mathbb{R}^n)} \leq K_{opt}\|\nabla f\|_{L^p(\mathbb{R}^n)}^\theta \|f\|^{1-\theta}_{L^q(\mathbb{R}^n)},
\]
the extremal for sharp Gagliardo-Nirenberg inequality \((1.10)\) is an extremal of \((3.1)\). It is easy to see that inequality \((1.11)\) is an affine inequality, thus composing the extremal functions of inequality \((1.10)\) with an element from \(GL(n)\) will also give an extremal for the affine Gagliardo-Nirenberg inequality \((1.7)\). Thus, the function given by \((1.9)\) is the extremal of inequality \((1.7)\).

3.2. Proof of Proposition 1.6. We combine the symmetrization inequality \((2.7)\) and inequality \((2.4)\) to prove Theorem 1.6. Since \(G(t) = t^p \log t : [0, \infty) \rightarrow [0, \infty)\) is continuous increasing, inequality \((2.4)\) verifies
\[
\int_{\mathbb{R}^n} |f(x)|^p \log |f(x)| \, dx = \int_{\mathbb{R}^n} |f^*(x)|^p \log |f^*(x)| \, dx.
\]
Lemma 2.1 verifies \(f^* \in W^{1,p}(\mathbb{R}^n)\) and inequality \((2.5)\) implies \(\|f^*\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} = 1\). Thus, we can apply the sharp Euclidean \(L^p\)-Sobolev logarithmic inequality (see Del Pino Dolbeaut [12, Theorem 1.1]) to \(f^*\) and obtain
\[
\int_{\mathbb{R}^n} |f^*(x)|^p \log |f^*(x)| \, dx \leq \frac{n}{p^2} \log \left( C_4 \|\nabla f^*\|_{L^p(\mathbb{R}^n)}^p \right).
\]
Similar to the proof of Theorem 1.3, Lemma 2.1 gives us
\[
\int_{\mathbb{R}^n} |f(x)|^p \log |f(x)| \, dx = \int_{\mathbb{R}^n} |f^*(x)|^p \log |f^*(x)| \, dx 
\leq \frac{n}{p^2} \log \left( C_4 \|\nabla f^*\|_{L^p(\mathbb{R}^n)}^p \right) 
= \frac{n}{p^2} \log \left( C_4 (\mathcal{E}_p(f^*))^p \right) 
\leq \frac{n}{p^2} \log \left( C_4 (\mathcal{E}_p(f))^p \right).
\]
Thus,
\[
\int_{\mathbb{R}^n} |f(x)|^p \log |f(x)| \, dx \leq \frac{n}{p^2} \log \left( C_4 (\mathcal{E}_p(f))^p \right).
\]
The function given by (1.17) is an extremal function inequality (1.15) since it is also an extremal function of the sharp Euclidean \(L^p\)-Sobolev inequality and
\[
\int_{\mathbb{R}^n} |f(x)|^p \log |f(x)| \, dx \leq \frac{n}{p^2} \log (C_A^p(f))^p \leq \frac{n}{p^2} \log (\|\nabla f\|_{L^p(\mathbb{R}^n)})^p.
\]

3.3. **Proof of Proposition 1.8** Under the assumption of Proposition 1.8 the sharp Moser-Trudinger inequality (1.4) holds. It follows from (1.3) and Lemma 2.7 that
\[
\|\nabla f^*\|_{L^n(\mathbb{R}^n)} = \mathcal{E}_n(f^*) \leq \mathcal{E}_n(f) \leq \|\nabla f\|_{L^n(\mathbb{R}^n)}.
\]

Then we get
\[
\leq \frac{1}{|\text{supp}(f)|} \int_{|\text{supp}(f)|} \exp\left( n\omega_{1/n}^{1/n} \frac{|f(x)|}{\mathcal{E}_n(f)} \right) \, dx
\]
\[
\leq \frac{1}{|\text{supp}(f)|} \int_{|\text{supp}(f)|} \exp\left( n\omega_{1/n}^{1/n} \frac{|f(x)|}{\mathcal{E}_n(f^*)} \right) \, dx
\]
\[
= \frac{1}{|\text{supp}(f^*)|} \int_{|\text{supp}(f^*)|} \exp\left( n\omega_{1/n}^{1/n} \frac{|f^*(x)|}{\mathcal{E}_n(f^*)} \right) \, dx
\]
\[
\leq m_n
\]
with the last inequality using (1.4). This implies that (1.18) holds. Since
\[
\leq \frac{1}{|\text{supp}(f)|} \int_{|\text{supp}(f)|} \exp\left( n\omega_{1/n}^{1/n} \frac{|f(x)|}{\|\nabla f\|_{L^n(\mathbb{R}^n)}} \right) \, dx
\]
\[
\leq \frac{1}{|\text{supp}(f)|} \int_{|\text{supp}(f)|} \exp\left( n\omega_{1/n}^{1/n} \frac{|f(x)|}{\mathcal{E}_n(f)} \right) \, dx \leq m_n
\]
and extremal functions for (1.4) exist, we see that \(f(Ax)\) is an extremal function of (1.18) for every extremal function \(f\) for (1.4) and \(A \in GL(n)\). On the other hand, to see the sharpness of \(n\omega_{1/n}^{1/n}\), we assume that (1.18) is true for some \(\beta > n\omega_{1/n}^{1/n}\) and any \(f \in W^{1,n}(\mathbb{R}^n)\) with \(0 < |\text{supp}(f)| < \infty\). Then we have \(f^* \in W^{1,n}(\mathbb{R}^n)\) and
\[
\leq \frac{1}{|\text{supp}(f^*)|} \int_{|\text{supp}(f^*)|} \exp\left( \beta \frac{|f^*(x)|}{\mathcal{E}_n(f^*)} \right) \, dx
\]
\[
= \frac{1}{|\text{supp}(f^*)|} \int_{|\text{supp}(f^*)|} \exp\left( \beta \frac{|f^*(x)|}{\|\nabla f^*\|_{L^n(\mathbb{R}^n)}} \right) \, dx
\]
\[
\leq m_n.
\]
The last inequality contradicts with the sharpness of \(n\omega_{1/n}^{1/n}\) in (1.4). This finishes the proof of Proposition 1.8.

3.4. **Proof of Proposition 1.9** Assume that \(f \in W^{1,p}\) with \(|\text{sprt} f| < \infty\). Then, from the classical Pólya-Szego principle, we know that \(f^* \in W^{1,p}(\mathbb{R}^n)\). On the other hand, equality (2.2) implies that \(|\text{sprt} f^*| < \infty\). Thus, for \(f^*\), we can apply the classical Morrey-Sobolev inequality and get
\[
\|f^*\|_{L^\infty(\mathbb{R}^n)} \leq b_{n,p}|\text{sprt}(f^*)|^{\frac{1}{p}} \|\nabla f\|_{L^p(\mathbb{R}^n)}.
\]
Equality (2.3) and Lemma 2.1 imply that
\[
\|f\|_{L^\infty(\mathbb{R}^n)} = \|f^*\|_{L^\infty(\mathbb{R}^n)} \leq b_{n,p} \text{sprt}(f^*)^{\frac{1}{n} - \frac{1}{p}} \|\nabla f^*\|_{L^p(\mathbb{R}^n)}
\]
\[
= b_{n,p} \text{sprt}(f^*)^{\frac{1}{n} - \frac{1}{p}} E_p(f^*)
\]
The verifying of extremal functions is obviously since the affine invariance of (1.19).

Acknowledgements. The author would like to thank Professor Jie Xiao for all kind encouragement.

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