Compactification of $\ast$-autonomous categories

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Abstract

We study the question when a $\ast$-autonomous (Mix-)category has a representation as a $\ast$-autonomous category of a compact one. We prove that necessary and sufficient condition is that weak distributivity maps are monic (or, equivalently epic). For a Mix-category, this condition is, in turn, equivalent to the requirement that Mix-maps be monic (or epic). We call categories satisfying this property torsion-free. An important side result is that torsion-free categories have canonical partial traces.

1 Introduction

$\ast$-Autonomous categories, monoidal categories with a particularly well-behaved duality, introduced by Barr [3] are known in logic and computer science literature as models of linear logic, but, of course, they deserve interest on their own as well. The best known and studied class of these is the class of compact (or compact closed) categories, in which duality preserves monoidal structure. The archetypical example is the category of finite-dimensional vector spaces, monoidal structure being the tensor product, and duality, the usual vector spaces duality.

In fact, many important $\ast$-autonomous categories have representations as subcategories of compact ones. In particular, in linear logic, a usual construction for building a non-degenerate model (compact categories, seen as models of linear logic, are degenerate) consists in some (often) ad hoc refinement of a given compact closed structure, which yields a new $\ast$-autonomous category, a subcategory of the initial compact one. The category of coherence spaces, which is the “original” model of linear logic, can be described in this way. Many other examples are considered in literature, see, say, [11].

So, at least from the academic point of view, it is interesting if we can characterize $\ast$-autonomous categories of such a form. In this paper we try to answer this question. That is, we find necessarily and sufficient conditions under which, for a given $\ast$-autonomous category, a compact envelope can be constructed. Here, we limit ourselves to discussing Mix-categories, a wide subclass.
of ∗-autonomous categories (see [5]), which are particularly easy to deal with, but our constructions and results apply, with slight technical modifications, to the general case as well; this will be discussed elsewhere.

The question of existence of a compact envelope for a general monoidal category has, in fact, been studied and answered in [14] and [2]. Embedding into the compact envelope depends on existence, on the given category, of a partial trace.

It is well known that compact categories are characterized by existence of categorical trace [12], a natural operation on morphisms, modeled after the usual linear operator trace in finite-dimensional vector spaces. Partial trace, introduced in [10] is a generalization of the ordinary (“total”) trace, which satisfies basically the same properties, but is not necessarily defined for all morphisms. In [14], [2] it is proven that partially traced categories are precisely monoidal subcategories of compact (i.e. totally traced) ones. A monoidal category embeds into a compact one, if and only if it has a partial trace.

It follows that if a ∗-autonomous category ∗-autonomously embeds into a compact one, it should have a partial trace (somehow well-interacting with the ∗-autonomous structure). Our construction of a compact envelope for a ∗-autonomous category (from a particular class) is based on a specific partial trace. What we do is, to a large extent, inspired by the partial trace representation theorem of [14], [2] and borrows some ideas and techniques from [2].

It is worth noting that, in logic and computer science literature, categorical trace (total or partial) is often used to model computation, being discussed in the context of Girard’s Geometry of Interaction [9] (and its various subsequent ramifications such as [1], [10]), which is closely tied with linear logic. Since linear logic corresponds to ∗-autonomous categories, it is therefore particularly interesting to find any general construction of a partial trace on such a category.

It turns out that possibility of embedding a ∗-autonomous category into a compact one depends on the properties of weak distributivity map, discussed in [7]. When the category is compact, all distributivity maps are isomorphisms. In a ∗-autonomous subcategory of a compact one, the distributivity maps, in general, are no longer invertible, but they must remain epic and monic. We find that this property is also sufficient: if a ∗-autonomous category has monic (or epic) weak distributivities, we can construct its compact envelope.

In particular, monic distributivity maps allow us defining a natural partial trace. This is already sufficient to embed the category into a compact one monoidally (preserving the monoidal structure), but, interestingly enough, such a monoidal embedding is not, in general ∗-autonomous (it does not preserve duality). In order to obtain a ∗-autonomous embedding, we extend the partial trace to a wider class of morphisms and define a more general trace-like operation, which we still call “trace”, abusing the terminology. With the extended trace, we build a compact envelope, using technique very similar to that of [2].

It follows that a ∗-autonomous category embeds ∗-autonomously into a compact one, if and only if it has monic (equivalently, epic) distributivity maps. For a Mix-category, this condition is, in turn, equivalent to the requirement that Mix-maps be monic (or epic). For the want of a better term, we call such
categories torsion-free. An important side-effect is that torsion-free categories have canonical partial traces.

We assume that the reader has some familiarity with monoidal categories, see [13]. In our notation for natural morphisms we often omit sub- and superscripts, when they are clear from the context. Most of routine diagram-chasing proofs are deferred till the Appendix.

2 Basics and definitions

Recall that a monoidal closed category is a monoidal category $K = (K, \otimes, 1)$ equipped with the internal homs functor $(\cdot, \cdot) \rightarrow (\cdot)$, contravariant in the first and covariant in the second entries, such that there is the bijection

$$K(A \otimes B, C) \cong K(A, B \rightarrow C),$$  \hspace{1cm} (1)$$
natural in $A, B, C$.

A $\ast$-autonomous category [3] is a symmetric monoidal category equipped with the dualizing object $\perp$, for which the natural map

$$i_A : A \rightarrow ((A \rightarrow \perp) \rightarrow \perp)$$
is an isomorphism for all $A$. Here the map $i_A$ comes from the identity map $id : A \rightarrow 1 \rightarrow A \rightarrow \perp$ under the series of bijections $K(A \rightarrow \perp, A \rightarrow \perp) \cong K(A \otimes (A \rightarrow \perp), \perp) \cong K(A, (A \rightarrow \perp) \rightarrow \perp)$ using the monoidal symmetry.

The dualizing object induces the contravariant involution $(\cdot)^\perp$, which we call duality, defined by

$$A^\perp = A \rightarrow \perp,$$
and the second symmetric monoidal structure $(\cdot)_{\perp}(\cdot)$, cotensor product, defined by

$$A_{\perp}B = (A^\perp \otimes B^\perp)^\perp,$$
with

$$A_{\perp} \perp \cong A.$$
The internal homs functor is naturally isomorphic to

$$A \rightarrow B \cong A^\perp_{\perp}B.$$(2)

Remark For simplifying computations, it is highly desirable to have strict equalities $A = A^{\perp\perp}$, rather than just isomorphisms. Fortunately, for the purposes of this paper we can always assume that this is the case as we discuss shortly.

In logic and computer science literature, $\ast$-autonomous categories are well-known as models of linear logic [4].

In this paper we restrict ourselves to a special class of $\ast$-autonomous categories, quite wide, but slightly easier to deal with than the general case.
Definition 1 A Mix-category is a $*$-autonomous category, equipped with the map
\[ m : \bot \rightarrow 1, \]
such that the following diagram commutes.

\[
\begin{array}{ccc}
\bot \otimes \bot & \xrightarrow{id \otimes m} & \bot \otimes 1 \\
\otimes & \xleftarrow{\bot \otimes \bot} & \otimes \\
1 \otimes \bot & \xrightarrow{id} & \bot \otimes \bot \otimes 1
\end{array}
\]

However the best known and, probably, best understood class of $*$-autonomous categories is that of compact (also called compact closed) ones, whose two monoidal structures are isomorphic.

Definition 2 A compact category is a $*$-autonomous category in which
\[ A \otimes B \cong A \oplus B \]
for all objects.

In a compact category we have
\[ \bot \cong 1, \ A \rightarrow B \cong A^\perp \otimes B. \]

A canonical example is the category of finite-dimensional vector spaces and linear maps. Basically, the compact closed structure is an abstraction of the monoidal closed structure of this category.

An important feature of a compact category is the categorical trace. For any morphism $\phi$ of the form
\[ \phi : A \otimes U \rightarrow B \otimes U \]
there exists the trace of $\phi$ over $U$, the morphism
\[ Tr_U^{A,B}(\phi) : A \rightarrow B. \]
The operation of trace is natural in various ways and satisfies certain properties, see [12] for details.

(Note that, in our notation, we use the subscript rather than the superscript for the traced object $U$. This seems to us more consistent with mathematical practice; in concrete examples, the trace is often defined in terms of integration or summation over the traced object, which appears in the subscript under the summation or the integration sign. Also, in speech, we say that we trace the morphism $\phi$ over, and not under $U$.)

It is well known that the existence of trace characterizes compact categories completely, in the sense that any compact category has a trace, and any category with a trace has canonical full embedding into a compact one [12]. Partial trace, introduced in [10], is a generalization of the ordinary ("total") trace satisfying basically the same properties, but not necessarily defined for all morphisms.
Typically, any monoidal subcategory $K$ of a compact $C$ has a partial trace. It is defined simply by restricting the canonical total trace of the ambient compact $C$ to morphisms of $K$, whenever the result is also in $K$. It has been proven \cite{14,2} that partially traced categories are precisely monoidal subcategories of compact (i.e. totally traced) ones.

In logic and computer science literature, categorical trace (total or partial) is often used to model computation, in particular cut-elimination, especially for linear logic. This is related to Girard’s program of Geometry of Interaction \cite{9} and its various subsequent ramifications, such as \cite{1,10}. Since linear logic corresponds to $\ast$-autonomous categories, it is particularly interesting to find any general construction of a partial trace on a $\ast$-autonomous category. In view of the representation theorem for partial traces, this question is related to embedding $\ast$-autonomous categories into compact ones.

2.1 Weak distributivities and torsion-free categories

Any monoidal closed category has the natural weak distributivity map

\[(A \multimap B) \otimes C \to A \multimap (B \otimes C),\]  

which arises from the equation

\[\text{ev}_{A,B} \otimes C : (A \multimap B) \otimes A \otimes C \to B \otimes C\]

under the monoidal symmetry and \cite{1}. Here the evaluation map

\[\text{ev}_{A,B} : (A \multimap B) \otimes A \to B\]  

is the image of $\text{id}_{A \multimap B}$ under bijection \cite{1}. In the $\ast$-autonomous case, the above mentioned weak distributivity map \cite{3}, has a number of versions, obtained using isomorphism \cite{2} and tensor or cotensor symmetries. In this paper we stick to the left distributivity map

\[\delta = \delta_{A,B,C} : A \otimes (B\varphi C) \to (A \otimes B)\varphi C\]

as default. However, the right distributivity map

\[\delta^R = \delta^R_{A,B,C} : (A\varphi B) \otimes C \to A\varphi (B \otimes C)\]

will also be used occasionally. Note that the two above families of maps are each other duals, up to a tensor symmetry.

We will be interested in categories for which the distributivity maps are monic, or, equivalently, epic. For the want of a better name we call such categories torsion-free.

**Definition 3** A $\ast$-autonomous category is torsion-free if its distributivity map is monic, equivalently, epic, for all objects.
Remark Obviously, it is not important which version of the distributivity map (left or right) to use in the definition. Also, since the left and right distributivities are each other duals, if one is monic, the other is epic.

Remark 2 The torsion-free property is non-trivial. Indeed, the category of locally compact abelian groups and continuous homomorphisms with the monoidal structure given by tensor product over $\mathbb{Z}$ is well-known as $\ast$-autonomous, the dualizing object $\perp = U(1)$ being the multiplicative group of complex numbers with unit modulus (the circle group). Duality is just the usual Pontriagin duality.

But $\perp \otimes \perp \cong \{0\}$ (since the circle group is both torsion and divisible), and it follows that $\delta_{\perp, \perp, \perp}$ is identically zero; in particular, it is not monic.

Note that, in this example, a crucial role is played by torsion of $U(1)$, which somewhat explains our choice of terminology.

When the category is compact, all versions of distributivity maps are isomorphisms. In particular, compact categories are torsion-free. Obviously, any $\ast$-autonomous subcategory of a compact one is torsion-free as well.

The last observation is important because many important $\ast$-autonomous categories have representations as subcategories of compact ones. One of the most obvious examples is the category of finite-dimensional vector spaces and linear maps of norm not greater than 1. The category of coherence spaces, important for linear logic [8], is a subcategory of the compact category of sets and relations. Many other $\ast$-autonomous categories considered for modeling linear logic have been constructed by a somewhat ad hoc refinement of a compact closed structure; this is a usual construction known under many versions and titles; see for example [11]. Thus a natural question arises: when a $\ast$-autonomous category is not a subcategory of a compact one?

We are going to answer this question by showing that Mix-categories that are $\ast$-autonomous subcategories of compact ones are precisely those which are torsion-free.

3 Notations, conventions and technical notes

Now we fix our notation and conventions and recall some basic categorical equations that will be used in the sequel.

To simplify computations, we will always assume that we are dealing with a strict category, where double duality and some other natural isomorphisms are identities.

Definition 4 A strict $\ast$-autonomous category is a $\ast$-autonomous category, whose monoidal associativity and unit isomorphisms as well as the double duality isomorphisms $i_A : A \cong A^{\perp\perp}$ are identities:

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C, \ A \otimes 1 = 1 \otimes A = A, \ A^{\perp\perp} = A.$$
We do not lose generality because of the following theorem.

**Theorem 1** [6] Any ∗-autonomous category is (strongly) equivalent, as a ∗-autonomous category, to a strict one. □

In a ∗-autonomous category we will stick to the following version of monoidal closed isomorphism [1]:

\[ \theta^{A,C}_B : \mathbf{K}(A \otimes B, C) \cong \mathbf{K}(A, C\wp B^\perp). \]  

(5)

Since we assume the involution (.)\(^\perp\) strict, a particular instance of the above isomorphism is

\[ \theta^{A,C}_B : \mathbf{K}(A \otimes B^\perp, C) \cong \mathbf{K}(A, C\wp B). \]  

(6)

We will use naturality of \(\theta\), so let us recall what does it mean explicitly.

Naturality in \(C\): for any \(\phi : C \to C'\) and \(f : A \otimes B \to C\) we have

\[ \theta_B(\phi \circ f) = (\phi \wp B^\perp) \circ \theta_B(f). \]

Naturality in \(A\): for any \(\psi : A' \to A\) and \(g : A \to C\wp B^\perp\) we have

\[ \theta_B^{-1}(g \circ \psi) = \theta_B^{-1}(g) \circ (\psi \otimes B). \]

In any monoidal closed category there exists *evaluation map* [4]. In the strict ∗-autonomous case we will extensively use its version \(\text{ev}_A : A \otimes A^\perp \to \perp\) obtained from the identity \(A = \perp \wp A\) under bijection [6].

The dual map \(\text{coev}_A : 1 \to A\wp A^\perp\) will also be used.

We denote the tensor and cotensor symmetries as

\[ \sigma_{A,B} : A \otimes B \to B \otimes A, \quad \tau_{A,B} : A\wp B \to B\wp A. \]

Note that these families of maps are each other duals.

**Note 1**

\[ \text{ev}_A = \text{ev}_{A^\perp} \circ \sigma_{A,A^\perp}, \quad \text{coev}_{A^\perp} = \tau_{A,A^\perp} \circ \text{coev}_A. \]

**Proof** in the Appendix. □

Iterating distributivities, combined with symmetries, we obtain a number of important maps, such as the following:

\[ \tau_{\wp \text{id}} \circ \delta_R \circ \text{id} \circ \tau \otimes \text{id} : (A\wp B) \otimes (C\wp D) \to (A \otimes C)\wp B\wp D. \]  

(7)

**Theorem 2** Any composition of distributivities and symmetries, resulting in a map of the form \((A\wp B) \otimes (C\wp D) \to (A \otimes C)\wp B\wp D\), results in [7].

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This follows from the defining diagrams for symmetric weakly distributive categories (*-autonomous categories form a subclass of those), see [7]. □

The following Lemma about distributivity map will be extensively used for proofs.

**Lemma 1** For any \( \phi : A \otimes B \rightarrow C \) and object \( X \) the following diagram commutes

\[
\begin{array}{ccc}
X \otimes A & \xrightarrow{\theta(X \otimes \phi)} & \delta(X,C,B) \\
X \otimes \theta \phi & \xrightarrow{\delta(X,C,B')} & (X \otimes (C \wp B)) \rightarrow (X \otimes C) \wp B\perp
\end{array}
\]

**Proof** in the Appendix. □

**Remark** Of course, an analogous statement can be formulated for the right distributivity map.

The following is an immediate consequence of isomorphism (5) and naturality of \( \theta \).

**Note 2** In a *-autonomous category, if a map \( \phi \) is epic (monic) then, for any object \( A \), maps \( A \otimes \phi \) and \( A \wp \phi \) are epic (monic). □

### 3.1 In the case of Mix-category

On a Mix-category there is a natural map

\[
\text{Mix}_{A,B} : A \otimes B \rightarrow A \wp B,
\]

defined as the following composition

\[
\begin{array}{ccc}
A \otimes (\perp \wp B) & \xrightarrow{\delta} & (A \otimes \perp) \wp B \\
& \xrightarrow{(A \otimes m) \wp B} & (A \otimes 1) \wp B \\
& \xrightarrow{\text{Mix}} & A \wp B.
\end{array}
\]

(8)

Observe that the map \( m : \perp = \perp \otimes 1 \rightarrow \perp \wp 1 = 1 \) is just a particular instance of Mix.

An equivalent definition is given by the following.

**Note 3**

\[
\text{Mix}_{A,B} = \theta_{B'} (A \otimes \tilde{e} \wp B).
\]

where \( \tilde{e} \wp = m \circ ev : B \otimes B\perp \rightarrow 1 \).

**Proof** in the Appendix. □
**Definition 5** A Mix-category is torsion-free, if it is a torsion-free $*$-autonomous category, and the canonical map $m : \bot \to 1$ is epic (or monic).

The distributivity and Mix-maps interact well. For example we have the following.

**Note 4** The diagrams below commute.

\[
\begin{array}{c}
A \otimes B \otimes C \xrightarrow{\text{Mix}} (A \otimes B) \Psi C \\
\downarrow \text{id} \otimes \text{Mix} \quad \delta \\
A \otimes (B \Psi C),
\end{array}
\]

\[
\begin{array}{c}
A \otimes B \otimes C \xrightarrow{\text{Mix}} A \Psi (B \otimes C) \\
\downarrow \text{Mix} \otimes \text{id} \quad \delta_R \\
(A \Psi B) \otimes C.
\end{array}
\]

**Proof** in the Appendix. \(\square\)

**Lemma 2** Mix-category \(K\) is torsion-free iff the Mix-map is epic (or monic).

**Proof** If \(K\) is torsion-free, then Mix is epic, since it is defined as a composition of distributivity maps and \(m\), which is tensored and then cotensored with identities. The latter map is epic by Note 2.

Assume now that Mix-map is epic. Since the map \(m : \bot \otimes 1 \to \bot \Psi 1\) is a particular instance of Mix, it is epic. The distributivity maps are epic because the first diagram in Note 4 commutes, and the vertical and horizontal arrows are epic (by Note 2). \(\square\)

### 4 Trace

A torsion-free Mix-category \(K\) has a natural operation of partial trace in the sense of [10, 14, 2]; this follows from the embedding theorem, which is the main result of the paper.

For our current purposes, however, we use a more general operation that we also call a (partial) trace. It is partially defined on morphisms of the form \(A \otimes U \to B \Psi U\).

Let \(\phi : A \otimes U \to B \Psi U\) be a morphism. Let

\[
\phi' = \theta^{-1}_{U \bot}(\phi) : A \otimes U \otimes U^\bot \to B.
\]

We have the natural map \(g_{A,U} : A \otimes U \otimes U^\bot \to A \otimes (U \Psi U^\bot), \ g = \text{id} \otimes \text{Mix}.\)
Definition 6 The trace $\text{Tr}^{A,B}_U(\phi)$ of $\phi$ over $U$ exists if the map $\phi'$ factors through $g$, i.e. for some $\psi : A \otimes (U \varphi U^\perp) \rightarrow B$ we have $\phi' = \psi \circ g$.

In this case
$$\text{Tr}^{A,B}_U(\phi) = \psi \circ (\text{id}_A \otimes \text{coev}_U) : A \rightarrow B,$$
as in the diagram below.

Note that, if $\psi$ in the above Definition exists, it is unique, because $g$ is epic by Note 2, so there is no ambiguity.

Remark A partial trace in the more usual sense of [10, 14, 2] for the morphism $\phi : A \otimes U \rightarrow B \otimes U$ can be defined as the trace in our sense applied to the composition $\text{Mix} \circ \phi$. It can be shown that this operation indeed satisfies all conditions of a partial trace as defined in [10, 14, 2]. We will not do this, because we do not need these conditions explicitly, whereas the fact that this is indeed a partial trace in the sense of [10, 14, 2] will follow from our embedding theorem that we prove closer to the end.

Obviously we can give an alternative definition of trace.

Note 5 For $\phi : A \otimes U \rightarrow B \varphi U$ the trace $\text{Tr}^{A,B}_U(\phi)$ can be equivalently defined by the following diagram. $\square$

The following is an analogue of the Yanking property of a usual trace on a monoidal category. Let the mixed symmetry $\tilde{\sigma}_{A,B} : A \otimes B \rightarrow B \varphi A$ be defined as $\tilde{\sigma} = \tau \circ \text{Mix} = \text{Mix} \circ \sigma$. 

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Theorem 3

\[ T_{U,U}^{U,U}(\tilde{\sigma}_{U,U}) = \text{id}_U. \]

Proof By Definition[6], the trace \( T_{U,U}^{U,U}(\tilde{\sigma}_{U,U}) \) exists, if \( \theta^{-1}(\tilde{\sigma}_{U,U}) \) factors through

\[ g_{U,U} : U \otimes U \otimes U^\perp \to U \otimes (U \otimes U^\perp), \quad g = U \otimes \text{Mix}_{U,U^\perp}. \]

This is indeed the case, as the following Lemma shows.

Lemma 3 The map \( \theta^{-1}(\tilde{\sigma}_{U,U}) \) factors as \( \theta^{-1}(\tilde{\sigma}_{U,U}) = \psi \circ g \), where

\[ \psi = (\text{ev}_{U \otimes U}) \circ \delta_{U,U^\perp} \circ (U \otimes \tau_{U,U^\perp}). \]

Proof in the Appendix. □

The rest follows from the following.

Lemma 4 The following diagram commutes

\[
\begin{array}{ccc}
U \otimes 1 & \overset{\text{id}}{\longrightarrow} & U^\perp \\
\downarrow \text{id} \otimes \text{coev}_U & & \downarrow \text{ev}_{U^\perp} \circ \text{id} \\
U \otimes (U \otimes U^\perp) & \overset{\text{id} \otimes \tau}{\longrightarrow} & (U \otimes U^\perp) \otimes U^\perp
\end{array}
\]

Proof in the Appendix. □

5 Loops and their congruences

For objects \( A, B \in K \) we define a loop \( p : A \rightarrow B \) as a tuple \( p = (\phi; U_1, \ldots, U_k) \), where \( k \in \mathbb{N}, U_i, i = 1, \ldots, k \), the hidden part, are objects of \( K \), and \( \phi \), the carrier, is a \( K \)-map \( \phi : A \otimes U_1 \otimes \ldots \otimes U_k \to B \otimes U_1 \otimes \ldots \otimes U_k \). The number \( k \) in the above definition can equal 0, in which case the hidden part is empty, and the corresponding morphism is just a \( K \)-morphism from \( A \) to \( B \). Thus, a \( K \)-morphism is identified as a loop with the empty hidden part.

Loops, more precisely their equivalence classes, will be morphisms in the compactification of \( K \). There are several operations on loops that we will use for its construction.

We introduce the following vector notation. We denote a tuple of objects as

\[ \vec{U} = (U_1, \ldots, U_n), \]

with conventions that

\[ A \otimes \vec{U} = A \otimes U_1 \otimes \ldots \otimes U_n, \quad A \otimes \vec{U} = A \otimes U_1 \otimes \ldots \otimes U_n, \]

\[ U^\perp = (U_1^\perp, \ldots, U_n^\perp) \]

and

\[ \tau_{A,\vec{U}} = \tau_{A,U_1 \otimes \ldots \otimes U_n}, \quad \sigma_{A,\vec{U}} = \sigma_{A,U_1 \otimes \ldots \otimes U_n}. \]
Also, if $\vec{V} = (V_1, \ldots, V_k)$, then
\[
\vec{V}\phi\vec{U} = V_1\phi \ldots \phi V_k\phi U_1 \phi \ldots \phi U_n, \quad \vec{V} \otimes \vec{U} = V_1 \otimes \ldots \otimes V_k \otimes U_1 \otimes \ldots \otimes U_n,
\]
and so on.

Here are operations on loops.

**Tensor product** For loops $p = (\phi; \vec{U}) : A \leftrightarrow B$, $q = (\psi; \vec{V}) : C \leftrightarrow D$, their tensor product
\[
p \otimes q : A \otimes C \leftrightarrow B \otimes D
\]
is the loop with the hidden part $(\vec{U}, \vec{V})$ and the carrier $\phi \otimes \psi$ defined by the composition
\[
\xymatrix{ A \otimes \vec{U} \otimes B \otimes \vec{V} \ar[r]^-\phi \otimes \psi & (A\phi\vec{U}) \otimes (B\psi\vec{V}) \ar[d]^\text{id} \otimes \sigma \otimes \text{id} \\
A \otimes B \otimes \vec{U} \otimes \vec{V} \ar[r]_-\phi \otimes \psi & (A \otimes B)\phi\vec{U}\psi\vec{V},}
\]
where the right vertical arrow is obtained as a composition of symmetries and distributivity maps (there is no ambiguity in its definition by Theorem 2).

Note that it follows from the same theorem that tensor product of loops is associative (remember that we work in a strict category).

**Dual** of the loop $p = (\phi; \vec{U}) : A \leftrightarrow B$ is the loop $p^\perp = (\phi^\perp; \vec{U}^\perp) : B^\perp \leftrightarrow A^\perp$.

**Cotensor product** is defined by tensor and duality. For loops $p = (\phi; \vec{U}) : A \leftrightarrow B$, $q = (\psi; \vec{V}) : C \leftrightarrow D$, their cotensor product $p \bowtie q : A\psi C \leftrightarrow B\phi D$ is the loop $(p^\perp \otimes q^\perp)^\perp$.

Note that for loops with empty hidden parts, i.e. usual $K$-morphisms the above are the usual operations on morphisms.

**Hiding** For the loop $p = (\phi; \vec{U}) : A \otimes V \leftrightarrow B\psi V$ we define the new loop
\[
\text{Hid}^{A,B}_V(p) = (\phi; V, \vec{U}) : A \leftrightarrow B.
\]

**Hidden symmetry** For the loop $p = (\phi; U_1, \ldots, U_k) : A \leftrightarrow B$ and a permutation $\alpha \in S_k$ we define the loop $\alpha p : A \leftrightarrow B$ by
\[
\alpha p = (\text{id} \otimes \tau_\alpha \circ \phi \otimes \text{id} \otimes \sigma_{\alpha^{-1}}; U_{\alpha(1)}, \ldots, U_{\alpha(n)}).
\]

**Hidden trace** This is a partially defined operation. For the loop
\[
p = (\phi; \vec{U}, V) : A \leftrightarrow B
\]
its hidden trace $Tr_V(p)$ over $V$ is defined if the morphism $\phi$ is traceable over $V$. Then
\[
Tr_Vp = (Tr_V(\phi); \vec{U}).
\]
When the tuple $\vec{U}$ is empty, the trace of $p$ is just a usual $K$-morphism.
We consider also iterated traces: $\text{Tr}_{V_1,\ldots,V_k}(p) = \text{Tr}_{V_1}(\ldots(\text{Tr}_{V_k}(p))\ldots)$, and use the obvious vector notation $\text{Tr}_{\vec{V}}(\cdot)$.

**Composition** For the loops $p = (\phi; \vec{U}) : A \leftrightarrow B$, $q = (\psi; \vec{V}) : B \leftrightarrow C$, their composition is the loop $q \circ p : A \leftrightarrow C$, with the hidden part $\vec{U}, \vec{V}$ and the carrier $\xi$ defined by the diagram

\[
\begin{array}{ccc}
(B \phi \vec{U}) \otimes \vec{V} & \rightarrow & (B \otimes \vec{V}) \phi \vec{U} \\
\phi \otimes \text{id} & & \quad \text{id} \psi \tau \\
A \otimes \vec{U} \otimes \vec{V} & \xrightarrow{\xi} & C \phi \vec{U} \phi \vec{V},
\end{array}
\]

where the upper horizontal arrow is obtained as a composition of symmetry and distributivity maps (there is no ambiguity in its definition by Theorem 2).

Note that it follows from the same Theorem and naturality of symmetries and distributivities that compositions of loops is associative.

We conclude with a couple of observations on the properties of hidden trace.

**Note 6** Hidden trace satisfies the following

(i) **Naturality:** for a loop $p : A \leftrightarrow B$ with the hidden part $(\vec{U}, \vec{V})$, and morphisms $\psi : B \rightarrow Y$, $\phi : X \rightarrow A$, if $\text{Tr}_{V}(p)$ exists, then

$$\text{Tr}_{V}(\psi \circ p \circ \phi) = \psi \circ \text{Tr}_{V}(p) \circ \phi.$$ 

(ii) **Strength:** for a loop $p : A \leftrightarrow B$ with the hidden part $(\vec{U}, \vec{V})$, and a morphism $\phi : X \rightarrow Y$, if $\text{Tr}_{V}(p)$ exists, then

$$\text{Tr}_{V}(\phi \otimes \phi) = \phi \otimes \text{Tr}_{V}(p).$$

**Proof** immediate from the definition of trace. □

**Note 7** If the trace $\text{Tr}_{V}(p)$ of a loop $p$ over $V$ exists, then

$$\text{Tr}_{V\perp}(p^\perp) = (\text{Tr}_{V}(p))^\perp.$$ 

**Proof** Follows from Note 6 □

## 6 Congruence and loop operations

We are going to define a certain equivalence relation on loops and see how it interacts with loop operations.

For any two objects $A, B$ and loops we define the *one-step congruence* equivalence relation $\sim$ on loops $A \leftrightarrow B$ by
(i) a loop is one-step congruent to its hidden trace;
(ii) loops, related by a hidden symmetry are one-step congruent.

Loop congruence is the equivalence relation, generated by the one-step congruence.

Note 8 If \( p_1 \sim p_2 \) then for any morphism \( \phi \) it holds that \( \phi \otimes p_1 \sim \phi \otimes p_2 \).

Proof If \( p_1 \) and \( p_2 \) are related by a hidden symmetry, the claim follows from naturality of weak distributivity map. Otherwise it follows from the Strength property of trace. \( \square \)

Note 9 For loops \( p_1, p_2 : A \leftrightarrow B \) and morphisms \( f : B \to Y, g : X \to A \), if \( p_1 \sim p_2 \) then \( f \circ p_1 \circ g \sim f \circ p_2 \circ g \).

Proof If the loops are related by a hidden symmetry, the claim is obvious. otherwise it follows from naturality of trace. \( \square \)

The two notes above imply

Lemma 5 Loop congruence is preserved by compositions and tensor products with morphisms. \( \square \)

The following is obvious.

Note 10 Loop congruence is preserved by hiding. \( \square \)

Now, tensor product of loops \( p \) and \( q \) is, in fact, nothing else than the tensor product of the carrier of \( p \) (considered as a loop with the empty hidden part, i.e. an ordinary morphism) with \( q \), followed by composition with a tensor symmetry on the left and a weak distributivity on the right and then by hiding. Since all these operations preserve loop congruence, it follows that tensor product with a loop preserves loop congruence.

Lemma 6 Tensor product of loops preserves loop congruence. \( \square \)

We have noted above that trace preserves duality. Hidden symmetry preserves duality as well, i.e., for a loop \( p \) and any permutation \( \alpha \) on its hidden part, we immediately see that

\[
(\alpha p)^\perp = \alpha^{-1} p^\perp.
\]

This, together with the preceding Lemma yields us the following.

Lemma 7 Duality and cotensor product of loops preserve loop congruence. \( \square \)

Finally, for loops \( p = (\phi; \vec{U}) : A \leftrightarrow B \) and \( q = (\psi; \vec{V}) : B \leftrightarrow C \), their composition is obtained from the loop \( \psi \circ \sigma_{\vec{V},B} \circ (id_{\vec{V}} \otimes p) \), by composing it with \( \sigma_{A,\vec{U}} \) on the left, then hiding \( \vec{V} \) and applying hidden symmetry (i.e. permuting \( \vec{U} \) and \( \vec{V} \)). Again, all operations involved preserve congruence, so composition with a loop preserves congruence as well.

Lemma 8 Loop congruence is preserved by composition of loops. \( \square \)
7 Compactification

From the above it follows that we can organise a well-defined monoidal category $\mathcal{C}(\mathcal{K})$ with duality $(\cdot)^\perp$ taking the same objects as in $\mathcal{K}$, with morphisms being equivalence classes of loops with respect to the loop congruence. We call this category the compactification of $\mathcal{K}$.

**Theorem 4** The category $\mathcal{C}(\mathcal{K})$ is $*$-autonomous. The functor $C : \mathcal{K} \to \mathcal{C}(\mathcal{K})$, identity on objects, and sending a morphism $\phi$ to the corresponding loop $(\phi,)$ with the empty hidden part, preserves $*$-autonomous structure.

**Proof** For the loop $p = (\phi, \vec{U}) : A \otimes B \rightarrow C$

we construct the new loop $p' : A \rightarrow C \, \Phi \, B^\perp$

with the carrier

$$\phi' = \theta_{\vec{U},\Phi} \, \theta_B (\theta_{\vec{U}}^{-1}(\phi) \circ (\text{id}_A \otimes \sigma_{\vec{U},\Phi}))$$

The assignment $p \mapsto p'$ is clearly a bijection between sets of loops, we need to check that it preserves loop congruence.

It is sufficient to note that this assignment preserves one-step congruence.

Indeed, if loops $p$ and $q$ are related by a hidden symmetry $\alpha$, then so are $p'$ and $q'$, as is easily seen from naturality of symmetries and $\theta$.

The case of hidden trace follows from the following.

**Lemma 9** In notations as above, assume that $p$ and $q$ are related by a hidden trace, i.e., $p = (\phi; \vec{U}, \vec{V}), q = \text{Tr}_V(p)$. Then $q' = \text{Tr}_V(p')$.

**Proof** in the Appendix. □

Finally, let us show that in the compactification $\mathcal{C}(\mathcal{K})$ the two monoidal structures become isomorphic, hence $\mathcal{C}(\mathcal{K})$ is compact.

**Lemma 12** In $\mathcal{C}(\mathcal{K})$ the Mix-map has inverse.
Proof The inverse of Mix : A ⊗ B → AφB is the loop

\[ \text{coMix}_{A,B} = (\text{Mix}_{A⊗B,AφB} \circ \sigma_{AφB,A⊗B} ; A,B) : AφB \leftrightarrow A ⊗ B. \]

It is sufficient to note that

\[ \text{coMix}_{A,B} \circ \text{Mix}_{A,B} = \text{Hid}_A(\tilde{\sigma}_{A,A}) \otimes \text{Hid}_B(\tilde{\sigma}_{B,B}). \]

This is established by simple diagram chasing, using repeated iteration of Note 4 together with Theorem 2. But by Theorem 3

\[ \text{Hid}_X(\tilde{\sigma}_{X,X}) \dashv \text{id}_X, \]

and, since tensor product of loops preserves congruence, we conclude that in \( C(K) \) it holds that \( \text{coMix} \circ \text{Mix} = \text{id} \). Then by duality \( \text{Mix} \circ \text{coMix} = \text{id} \) as well. □

We summarise this with the following

**Theorem 5** The category \( C(K) \) is compact, and the functor \( C : K \to C(K) \) is faithful and preserves the ∗-autonomous structure of \( K \). □

We have proven that any torsion-free Mix-category is strongly equivalent as a ∗-autonomous category to a ∗-autonomous subcategory of a compact one. Since, on the other hand, any compact category as well as any its ∗-autonomous Mix-subcategory is torsion-free, we get the conclusive theorem.

**Theorem 6** A Mix-category is strongly equivalent to a ∗-autonomous Mix-subcategory of a compact one iff it is torsion-free. □

Since monoidal subcategories of compact ones have a partial trace in the sense of \[14\], we get another important result as a corollary.

**Corollary 1** Any torsion-free Mix-category has a natural partial trace in the sense of \[14\]. □

8 Conclusion

We have proven that Mix-categories are ∗-autonomous subcategories of compact ones if and only if they are torsion-free. This can be seen as a partial classification result.

We restrict ourselves to Mix-categories, but the constructions of this paper apply, with a slight modifications to the general case as well; this will be discussed elsewhere. A more general question probably can be discussed: when a monoidal closed category is a monoidal closed subcategory of a compact one?

On the other hand, it may be interesting to understand the “opposite” side: what can be said about categories that are not torsion-free? How are the constructed? Is there any (partial) classification possible?
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A  Proofs

Proof of Note 1

The two equalities are each other duals, so it is sufficient to prove only the first one. The left-hand side of the first equality is $\theta^{-1}_{A^\perp}(\text{id}_A)$, whereas the right-hand side is $\theta^{-1}(i_A)$, where $i_A : A \to A^{\perp\perp}$ is the canonical isomorphism. But we assume the involution strict, $A^{\perp\perp} = A$, and $i_A = \text{id}_A$. □

Proof of Lemma 1

We have $\delta_{X,C,B^\perp} = \theta(X \otimes \text{ev}_{C,B})$, where $\text{ev}_{C,B} = \theta^{-1}(\text{id}_{C \infty B^\perp})$.

From the trivial commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\theta \phi} & A \\
\downarrow{\theta \phi} & & \downarrow{\theta \phi} \\
C \phi B^\perp & \xrightarrow{\text{id}} & C \phi B^\perp
\end{array}
\]

we get that the following diagram commutes

\[
\begin{array}{ccc}
A \otimes B & \xrightarrow{\theta \phi \otimes B} & (C \phi B^\perp) \otimes B \\
\downarrow{\phi} & & \downarrow{\phi} \\
(C \phi B^\perp) \otimes B & \xrightarrow{\text{ev}} & C
\end{array}
\]

by naturality of $\theta$.

Then the following commutes as well

\[
\begin{array}{ccc}
X \otimes A \otimes B & \xrightarrow{X \otimes \phi} & X \otimes \phi \\
\downarrow{X \otimes \phi} & & \downarrow{X \otimes \phi} \\
X \otimes (C \phi B^\perp) \otimes B & \xrightarrow{X \otimes \text{ev}} & X \otimes C.
\end{array}
\]

This implies the desired commutativity of the diagram in the formulation of the Lemma, by naturality of $\theta$. □

Proof of Note 3

From diagram 8 and Lemma 1 we have $\text{Mix}_{A,B} = ((A \otimes m) \circ B) \circ \theta_{B^\perp}(A \otimes \text{ev}_B)$, since the identity map (remember that we assume the category strict) $B = \perp \phi B$ is nothing but $\theta_{B^\perp}(\text{ev}_B)$. The rest follows from naturality of $\theta$. □

Proof of Note 4

Commutativity of the first diagram follows, for example, from Lemma 1 and Note 3. Commutativity of the second one can be established similarly, using, instead of left distributivity map, right distributivity map and its properties. □

Proof of Lemma 3
From Note 3 and naturality of \( \theta \) we have that for all objects \( X, Y \) the equality
\[
\theta^{-1}(\tilde{\sigma}_{X,Y}) = (Y \otimes \tilde{e}_X) \circ (\sigma_{X,Y} \otimes X^\perp). \tag{9}
\]

It follows from Lemma 1 that for the map
\[\phi : U \otimes U \otimes U^\perp \to (U \otimes U^\perp) \varphi U,\]
defined by the composition
\[
\begin{array}{c}
U \otimes U \otimes U^\perp \\
\downarrow \text{id} \otimes \sigma
\end{array}
\xrightarrow{\phi}
\begin{array}{c}
U \otimes (U^\perp \varphi U) \\
\downarrow \delta
\end{array}
\xrightarrow{\phi}
(U \otimes U^\perp) \varphi U,
\]
we have
\[\phi = \theta_{U^\perp}(U \otimes \theta^{-1}_{U^\perp}(\tilde{\sigma}_{U,U^\perp})).\]
By (9) we have the following commutative diagram.
\[
\begin{array}{c}
U \otimes U^\perp \otimes U^\perp \\
\downarrow \sigma \otimes \text{id}
\end{array}
\xrightarrow{\theta^{-1}_{U^\perp}}
\begin{array}{c}
U^\perp \otimes U \otimes U^\perp \\
\downarrow \text{id} \otimes \tilde{e} \otimes \text{ev}
\end{array}
\xrightarrow{\phi}
U^\perp
\]
Hence the following commutes as well.
\[
\begin{array}{c}
U \otimes U \otimes U^\perp \otimes U^\perp \\
\downarrow \text{id} \otimes \sigma \otimes \text{id}
\end{array}
\xrightarrow{\theta^{-1}_{U^\perp}}
\begin{array}{c}
U^\perp \otimes U \otimes U^\perp \\
\downarrow \text{id} \otimes \tilde{e} \otimes \text{ev}
\end{array}
\xrightarrow{\phi}
U \otimes U^\perp
\]
We are going to show that
\[ev_U \circ \theta^{-1}_{U^\perp}(\phi) = \theta^{-1}_{U^\perp}(a_U \circ \theta^{-1}_{U^\perp}(\tilde{\sigma}_{U,U})), \tag{11}\]
where \( a : U \to U^\perp \varphi U \) is the identity (remember that we assume our category strict). This will imply that
\[\theta^{-1}_{U^\perp}(\tilde{\sigma}_{U,U}) = a_U \circ \theta^{-1}_{U^\perp}(\tilde{\sigma}_{U,U}) = (ev_U \varphi U) \circ \phi \tag{12}\]
by naturality of \( \theta \).
Since \( \theta^{-1}_{U^\perp}(a_U) = ev_U \), it follows from naturality of \( \theta \) that the right-hand side in (11) is \( ev_U \circ (\theta^{-1}_{U^\perp}(\tilde{\sigma}_{U,U}) \otimes U^\perp) \). Further, from (9) we have
\[\theta^{-1}_{U^\perp}(\tilde{\sigma}_{U,U}) = (U \otimes \tilde{e}v_U) \circ (\sigma_{U,U} \otimes U^\perp).\]
It follows from the above, that equation (11) amounts to commutativity of the following diagram.

\[
\begin{array}{ccc}
U \otimes U \otimes U & & U \otimes U \otimes U \\
\downarrow \sigma_{U,U} \otimes \id & & \downarrow \sigma_{U,U} \otimes \id \\
U \otimes U \otimes U & & U \otimes U \otimes U \\
\downarrow \id \otimes \id \otimes \tilde{ev} & & \downarrow \id \otimes \id \otimes \tilde{ev} \\
U \otimes U & & U \otimes U \\
\downarrow \ev & & \downarrow \ev \\
\bot & & \bot
\end{array}
\]

The left side of the diagram corresponds to the left-hand side of (11), and the right side accordingly.

Now, the right hand side of (11) (or of the diagram) equals

\[\text{RHS} = \ev \circ (\tilde{ev} \otimes U \otimes U) \circ (U \otimes \sigma_{U,U} \otimes U),\]

as the following diagram shows:

\[
\begin{array}{ccc}
U \otimes U \otimes U & & U \otimes U \otimes U \\
\downarrow \sigma_{U,U} \otimes \id & & \downarrow \sigma_{U,U} \otimes \id \\
U \otimes U & & U \otimes U \\
\downarrow \id \otimes \tilde{ev} \otimes \id & & \downarrow \id \otimes \tilde{ev} \otimes \id \\
U \otimes U & & U \otimes U \\
\downarrow \ev & & \downarrow \ev \\
\bot & & \bot
\end{array}
\]

whereas the left-hand side is

\[\text{LHS} = \ev \circ (\tilde{ev} \otimes U \otimes U) \circ (U \otimes \sigma_{U,U} \otimes U).\]

Thus, to establish (11), it is sufficient to show that

\[\ev \circ (U \otimes U \otimes \tilde{ev}) = \ev \circ (\tilde{ev} \otimes U \otimes U),\]

which is the same as

\[\ev \otimes \tilde{ev} = \tilde{ev} \otimes \ev.\]

The latter equality is an immediate consequence of the basic equation for a Mix-category

\[\bot \otimes m = m \otimes \bot,\]

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and \([11]\), as well as \([12]\), is established.

Now combining definition of \(\phi\) with \([12]\), we get the commutative diagram

\[
\begin{align*}
U \otimes U & \otimes U^\perp \\
\downarrow \phi & \downarrow \theta^{-1}(\bar{\sigma}_{U,U}) \\
U & \to U \otimes (U^\perp \varphi U) \\
\downarrow \delta & \\
(U \otimes U^\perp) \varphi U & \to (U \otimes U^\perp) \varphi U,
\end{align*}
\]

and this proves the Lemma. □

**Proof of Lemma 4.**

By Note 1 this amounts to commutativity of

\[
\begin{align*}
U \otimes 1 & \otimes \perp \varphi U \\
\downarrow \text{id} & \downarrow \varphi \perp \\
U \otimes (U \varphi U) & \delta \to (U \otimes U^\perp) \varphi U.
\end{align*}
\]

By Lemma 1, if we fill in the diagonal arrow

\[
\begin{align*}
U \otimes 1 & \otimes \perp \varphi U \\
\downarrow \text{id} & \downarrow \varphi \perp \\
U \otimes (U \varphi U) & \delta \to (U \otimes U^\perp) \varphi U,
\end{align*}
\]

the lower triangle commutes. Then the upper triangle, by naturality of \(\theta\), corresponds to

\[
\begin{align*}
U \otimes U^\perp & \rightarrowsym \perp \\
\downarrow \text{id} & \downarrow \text{ev} \\
U \otimes U^\perp & \rightarrowsym \perp
\end{align*}
\]

and commutes as well. □

**Proof of Lemma 9.**

If the carrier of \(p\) is \(\phi\), then the carrier of \(q\) is \(Tr_V(\phi)\). By naturality of symmetries and by Definition \([8]\) of trace we have the commutative diagram for
The upper leg of the above is $\theta_B^{-1} \theta_{V^\perp}^{-1}(\phi')$, and, by naturality of $\theta_B$, the whole diagram corresponds to the following.

The lower diagonal arrow is precisely the carrier of $q'$, and on the other hand it is the trace $Tr_V(\phi')$ by Definition. The Lemma is proven. □