9. Exponential maps and explicit formulas

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In this section we introduce an exponential homomorphism for the Milnor $K$-groups for a complete discrete valuation field of mixed characteristics.

In general, to work with the additive group is easier than with the multiplicative group, and the exponential map can be used to understand the structure of the multiplicative group by using that of the additive group. We would like to study the structure of $K_q(K)$ for a complete discrete valuation field $K$ of mixed characteristics in order to obtain arithmetic information of $K$. Note that the Milnor $K$-groups can be viewed as a generalization of the multiplicative group. Our exponential map reduces some problems in the Milnor $K$-groups to those of the differential modules $\Omega_{\mathcal{O}_K}$ which is relatively easier than the Milnor $K$-groups.

As an application, we study explicit formulas of certain type.

9.1. Notation and exponential homomorphisms

Let $K$ be a complete discrete valuation field of mixed characteristics $(0, p)$. Let $\mathcal{O}_K$ be the ring of integers, and $F$ be its the residue field. Denote by $\text{ord}_p: K^* \rightarrow \mathbb{Q}$ the additive valuation normalized by $\text{ord}_p(p) = 1$. For $\eta \in \mathcal{O}_K$ we have an exponential homomorphism

$$\exp_\eta: \mathcal{O}_K \rightarrow K^*, \quad a \mapsto \exp(\eta a) = \sum_{n=0}^{\infty} (\eta a)^n / n!$$

if $\text{ord}_p(\eta) > 1/(p - 1)$.

For $q > 0$ let $K_q(K)$ be the $q$ th Milnor $K$-group, and define $\widehat{K}_q(K)$ as the $p$-adic completion of $K_q(K)$, i.e.

$$\widehat{K}_q(K) = \lim_{\leftarrow} K_q(K) \otimes \mathbb{Z}/p^n.$$
For a ring $A$, we denote as usual by $\Omega^1_A$ the module of the absolute differentials, i.e. $\Omega^1_A = \Omega^1_{A/\mathbb{Z}}$. For a field $F$ of characteristic $p$ and a $p$-base $I$ of $F$, $\Omega^1_F$ is an $F$-vector space with basis $dt$ ($t \in I$). Let $K$ be as above, and consider the $p$-adic completion $\widehat{\Omega}^1_{O_K}$ of $\Omega^1_{O_K}$

$$\widehat{\Omega}^1_{O_K} = \lim_{\leftarrow} \Omega^1_{O_K} \otimes \mathbb{Z}/p^n.$$ 

We take a lifting $\bar{I}$ of a $p$-base $I$ of $F$, and take a prime element $\pi$ of $K$. Then, $\widehat{\Omega}^1_{O_K}$ is an $O_K$-module (topologically) generated by $d\pi$ and $dT$ ($T \in \bar{I}$) ([Ku1, Lemma 1.1]). If $I$ is finite, then $\widehat{\Omega}^1_{O_K}$ is generated by $d\pi$ and $dT$ ($T \in \bar{I}$) in the ordinary sense. Put

$$\widehat{\Omega}^g_{O_K} = \wedge^g \widehat{\Omega}^1_{O_K}.$$ 

**Theorem** ([Ku3]). Let $\eta \in K$ be an element such that $\text{ord}_p(\eta) \geq 2/(p-1)$. Then for $q > 0$ there exists a homomorphism

$$\exp^g(\eta) : \widehat{\Omega}^g_{O_K} \to \hat{K}_q(K)$$

such that

$$\exp^g(\eta)\left(\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{q-1}}{b_{q-1}}\right) = \{\exp(\eta a), b_1, \ldots, b_{q-1}\}$$

for any $a \in O_K$ and any $b_1, \ldots, b_{q-1} \in O^*_K$.

Note that we have no assumption on $F$ ($F$ may be imperfect). For $b_1, \ldots, b_{q-1} \in O_K$ we have

$$\exp^g(\eta)(a \cdot db_1 \wedge \cdots \wedge db_{q-1}) = \{\exp(\eta ab_1 \cdots \cdot b_{q-1}), b_1, \ldots, b_{q-1}\}.$$ 

### 9.2. Explicit formula of Sen

Let $K$ be a finite extension of $\mathbb{Q}_p$ and assume that a primitive $p^n$th root $\zeta_{p^n}$ is in $K$. Denote by $K_0$ the subfield of $K$ such that $K/K_0$ is totally ramified and $K_0/\mathbb{Q}_p$ is unramified. Let $\pi$ be a prime element of $O_K$, and $g(T)$ and $h(T) \in O_{K_0}[T]$ be polynomials such that $g(\pi) = \beta$ and $h(\pi) = \zeta_{p^n}$, respectively. Assume that $\alpha$ satisfies $\text{ord}_p(\alpha) \geq 2/(p-1)$ and $\beta \in O^*_K$. Then, Sen’s formula ([S]) is

$$(\alpha, \beta) = \zeta_{p^n}^c, \quad c = \frac{1}{p^n} \text{Tr}_{K/\mathbb{Q}_p}\left(\frac{\zeta_{p^n}}{h'(\pi)} \frac{g'(\pi)}{\beta} \log \alpha\right)$$

where $(\alpha, \beta)$ is the Hilbert symbol defined by $(\alpha, \beta) = \gamma^{-1} \Psi_K(\alpha)(\gamma)$ where $\gamma p^n = \beta$ and $\Psi_K$ is the reciprocity map.

The existence of our exponential homomorphism introduced in the previous subsection helps to provide a new proof of this formula by reducing it to Artin–Hasse’s
formula for \((\alpha, \zeta_p^n)\). In fact, put \(k = \mathbb{Q}_p(\zeta_p^n)\), and let \(\eta\) be an element of \(k\) such that \(\text{ord}_p(\eta) = 2/(p - 1)\). Then, the commutative diagram

\[
\begin{array}{ccc}
\hat{\Omega}^1_{\mathcal{O}_K} & \xrightarrow{\exp_\eta} & \hat{K}_2(K) \\
\text{Tr} \downarrow & & \downarrow N \\
\hat{\Omega}^1_{\mathcal{O}_k} & \xrightarrow{\exp_\eta} & \hat{K}_2(k)
\end{array}
\]

\((N: \hat{K}_2(K) \to \hat{K}_2(k)\) is the norm map of the Milnor \(K\)-groups, and \(\text{Tr}: \hat{\Omega}^1_{\mathcal{O}_K} \to \hat{\Omega}^1_{\mathcal{O}_k}\) is the trace map of differential modules) reduces the calculation of the Hilbert symbol of elements in \(K\) to that of the Hilbert symbol of elements in \(k\) (namely reduces the problem to Iwasawa’s formula [1]).

Further, since any element of \(\hat{\Omega}^1_{\mathcal{O}_k}\) can be written in the form \(ad\zeta_p^n/\zeta_p^n\), we can reduce the problem to the calculation of \((\alpha, \zeta_p^n)\).

In the same way, we can construct a formula of Sen’s type for a higher dimensional local field (see [Ku3]), using a commutative diagram

\[
\begin{array}{ccc}
\hat{\Omega}^q_{\mathcal{O}_K}^{\{T\}} & \xrightarrow{\exp_\eta} & \hat{K}_{q+1}(K^{\{T\}}) \\
\text{res} \downarrow & & \downarrow \text{res} \\
\hat{\Omega}^{q-1}_{\mathcal{O}_K} & \xrightarrow{\exp_\eta} & \hat{K}_q(K)
\end{array}
\]

where the right arrow is the residue homomorphism \(\{\alpha, T\} \mapsto \alpha\) in [Ka], and the left arrow is the residue homomorphism \(\omega dT/T \mapsto \omega\). The field \(K^{\{T\}}\) is defined in Example 3 of subsection 1.1 and \(\mathcal{O}_K^{\{T\}} = \mathcal{O}_K^{\{T\}}\).

### 9.3. Some open problems

**Problem 1.** Determine the kernel of \(\exp^{(q)}_p\) completely. Especially, in the case of a \(d\)-dimensional local field \(K\), the knowledge of the kernel of \(\exp^{(q)}_p\) will give a lot of information on the arithmetic of \(K\) by class field theory. Generally, one can show that

\[
pd\hat{\Omega}^{q-2}_{\mathcal{O}_K} \subset \ker(\exp^{(q)}_p: \hat{\Omega}^{q-1}_{\mathcal{O}_K} \to \hat{K}_q(K)).
\]

For example, if \(K\) is absolutely unramified (namely, \(p\) is a prime element of \(K\) ) and \(p > 2\), then \(pd\hat{\Omega}^{q-2}_{\mathcal{O}_K}\) coincides with the kernel of \(\exp^{(q)}_p\) ([Ku2]). But in general, this is not true. For example, if \(K = \mathbb{Q}_p^{\{T\}} (\sqrt[p]{T})\) and \(p > 2\), we can show that the kernel of \(\exp^{(2)}_p\) is generated by \(pd\mathcal{O}_K\) and the elements of the form \(\log(1 - x^p)dx/x\) for any \(x \in \mathcal{M}_K\) where \(\mathcal{M}_K\) is the maximal ideal of \(\mathcal{O}_K\).
Problem 2. Can one generalize our exponential map to some (formal) groups? For example, let $G$ be a $p$-divisible group over $K$ with $|K : \mathbb{Q}_p| < \infty$. Assume that the $[p^n]$-torsion points $\ker[p^n]$ of $G(K^{alg})$ are in $G(K)$. We define the Hilbert symbol $K^* \times G(K) \to \ker[p^n]$ by $(\alpha, \beta) = \Psi_K(\alpha)(\gamma) - G\gamma$ where $[p^n]\gamma = \beta$. Benois obtained an explicit formula ([B]) for this Hilbert symbol, which is a generalization of Sen’s formula. Can one define a map $\exp_G: \Omega_{Q_K}^1 \otimes \text{Lie}(G) \to K^* \times G(K)/\sim$ (some quotient of $K^* \times G(K)$) by which we can interpret Benois’s formula? We also remark that Fukaya recently obtained some generalization ([F]) of Benois’s formula for a higher dimensional local field.

References

[B] D. Benois, Périodes $p$–adiques et lois de réciprocité explicites, J. reine angew. Math. 493(1997), 115–151.

[F] T. Fukaya, Explicit reciprocity laws for $p$–divisible groups over higher dimensional local fields, preprint 1999.

[I] K. Iwasawa, On explicit formulas for the norm residue symbols, J. Math. Soc. Japan, 20(1968), 151–165.

[Ka] K. Kato, Residue homomorphisms in Milnor $K$-theory, in Galois groups and their representations, Adv. Studies in Pure Math. 2, Kinokuniya, Tokyo (1983), 153–172.

[Ku1] M. Kurihara, On two types of complete discrete valuation fields, Comp. Math., 63(1987), 237–257.

[Ku2] M. Kurihara, Abelian extensions of an absolutely unramified local field with general residue field, Invent. math., 93(1988), 451–480.

[Ku3] M. Kurihara, The exponential homomorphisms for the Milnor $K$-groups and an explicit reciprocity law, J. reine angew. Math., 498(1998), 201–221.

[S] S. Sen, On explicit reciprocity laws, J. reine angew. Math., 313(1980), 1–26.

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