A Structural Model with Jump Diffusion Processes

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Abstract

In this paper, we extend the framework of Leland’s 94 by examining corporate debt, equity and firm values with jump-diffusion processes. We choose two kinds of jumps such as the uniform and double exponential jumps to model the distribution of the log jump sizes. By this choice, we are able to derive closed-form results in both models for equity, debt and firm values. Analysis of credit spread, debt value and firm value has been done for three proposed models: diffusion process diffusion with uniform jump and diffusion with double exponential jumps. Our results have the same forms as those of Leland’s 94. However, in both of our models, the spreads are modified significantly in comparison with those of Leland due to jumps’ assumption.

Keywords: Default Barrier, Structural Model, Jump Diffusion, Double Exponential Jump, Uniform Jump, Perfectual American Put.

JEL Classification: G12, G13, G33

1 Introduction

Credit risk and credit derivatives have received an increasing interest from both researchers and professionals over the last decade. The three main principal credit risk modelling approaches are the structural approach, the intensity approach and the rating based approach. We are interested only in the structural approach, which is marked by a series of publications by Leland (1994a) and Leland and Toft (1996) who have considered the question of the firm’s optimal

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capital structure and its endogenous default barrier. Despite very appealing, interesting and intuitive ideas, the results of their models did not prove entirely satisfactory. Especially, the yield spread generated by their model did not match empirical data, due to the assumption that the underlying process follows a geometric Brownian motion (log normal distribution) with constant parameters, as in the Black Scholes framework (1973). Since then, many proposals modifying this model have been made, such as changing to the jump diffusion process to incorporate discontinuous paths, using stochastic volatility or stochastic interest rates, or even introducing transaction costs. Among these propositions, the presence of jumps in the asset price processes were further endorsed and supported by Bates (1996) and Jorion (1988) publications with empirical evidence, and by what was observed in the market. The jump diffusion processes are particularly helpful in correcting the two main empirical failures of the normal distribution: firstly, the large random fluctuations (discontinuous paths) such as crashes or external events and secondly, the non-normal features, such as the negative skewness and the leptokurtic feature.

The literature on modelling the jump diffusion processes or the Lévy processes for pricing options is exhaustive, such as Merton (1976), Bates (1971), Mordecki (1999, 2002), Kou and Wang (1999, 2000), Boyarchenko and Levendorskii (2002) and Chesney and Jeanblanc (2004). Discussions regarding the use of various processes such as variance gamma, Carr-German-Madan-Yor, hyperbolic, normal inverse Gaussian and Levy processes in general for option pricing and default risk can be found in Cont and Tankov (2003). However, little research has been devoted to the use of jump diffusion processes in the credit risk context.

In credit risk literature, for researches using jump diffusion processes to replace the geometric Brownian motion of the Black Scholes framework, we can refer to Merton (1976) and Zhou (1997) who have used a normal jump-diffusion. Hilberink and Rogers (2002) were the first authors to introduce a spectrally negative Lévy process into the framework of Leland’s models (1994a). However, the approach is restricted to negative jumps only, and this does not yield extensive results. Le Courtois and Quittard Pinon (2004) have recently extended Hilberink and Rogers’ work by using the Lévy stable, two-sided jump process, as the asset price process. Boyarchenko and Levendorskii (2002) proposed a structural model whose assets follow a regular Lévy process of exponential type. Their computation method is mainly based on the Wiener Hopft factorization. In their model, there are two approximate formulations wherein the firm’s assets, in extreme cases, are either very near or very far from the bankruptcy level. However, these formulations still appear too complicated to apply on a general basis because the explicit computation of the Wiener Hopft factorization is difficult. Eberlein and Özkan (2003) and Özkan (2002) have used the Lévy process for credit risk mostly in the intensity approach. Recent papers such as Cremers et al (2006) and Dong et al (2011) also used a structural approach with the jump-diffusion process and extensively analyzed the credit spread. Kyprianou and Surya (2007) also analyzed the endogenous default barrier in a jump-diffusion process.
In this paper, we present a structural model which is in line with Leland’s approach (1994), i.e., a perpetual coupon debt structure. While Leland used a geometric Brownian motion to model the firm’s assets, we propose here jump diffusion processes with two different kind of jump distributions. The first one is the double exponential density distribution which was proposed earlier by Kou (2002) and the second being the uniform density distribution which is new and is one of the original results in our paper. The motivations and reasons for these two kinds of jumps are based on two main criteria. The first one is a test of the influence of the amplitude of different jumps in the evaluation of the firm, debt and equity values, so more different jumps, the better. The two kinds of jumps proposed here include uniform and double exponential jumps to provide a better match to the characteristics of firms. The second criterion is the modelling which should give analytical or explicit solutions. As many researches have chosen the log normal jump which was first introduced by Merton (1976) or the bivariate jumps used by Gukhal (2001), it is interesting to fill the gap by using other kind of jumps such as the uniform jump and the double exponential jump.

While the uniform jump diffusion, from a mathematical standpoint, is the simplest and most intuitive jump process, its derivative formula is not straightforward. Furthermore, from a financial standpoint, if the firm is an insurance business whose claims come in as a jump process and its redemption is proportional to a uniform distribution, modelling the firm’s assets as a negative uniform jump diffusion seems quite appropriate.

The double exponential jump diffusion\(^\text{1}\), a special case of the Lévy processes, is used to model the firm’s assets process, presenting two interesting properties due to the exponential distribution. The first property is that the double exponential distribution which performs two-sided jumps, has the leptokurtic feature of the jump size that provides the peak and tails of the return distribution found in reality. The second property is that the double exponential distribution has a memoryless property which makes it easier to obtain analytical solutions for the calculation of expected values and the computation of the distribution of the first passage times. This memoryless property helps to solve the problem of overshoots (which occurs when a jump diffusion process \(V\) crosses a barrier level \(L\), as sometimes it hits the barrier level exactly \(V_{\tau_L}=L\) over the barrier), and to obtain analytic solutions.

More detailed explanations of the advantages of the double exponential jump diffusion model and its comparisons with other models such as the constant elasticity model (CEV), the normal jump diffusion model, the model based on \(t\)-distribution, the stochastic volatility model, the affine jump diffusion model, and the model based on Lévy process can be found in Kou (2002).

The recent empirical paper of Hanson and Zhu (2004) compared the fitting of three processes such as normal, uniform and double exponential jump diffusions, to the Standard and Poor’s 500 log return market data, given the same

\(^1\)The research paper by Dao and Jeanblanc also used double exponential jump diffusion process for another capital structure named "rollover" debt structure.
first moment and second central moments. They have shown very encouraging results for the use of jump diffusion in general and for the uniform jump in particular. Their main results are that the uniform jump has a superior qualitative performance since it produces genuine fat tails that are typical of market data, whereas the other two have exponentially thinner tails.

A close relationship exists between the option pricing literature and credit risk literature as the latter considers that all equity, debt and firm values can be written as options (or contingent claims) on the underlying firm’s assets. We find out that in our model of perpetual coupon-paying debt structure, the main tool (building block) we need is the perpetual American put. We base on the Leland (1994a) model, that is, the default barrier is determined endogenously by maximizing the equity value. Intuitively, we can explain why we find the formula of the perpetual American put when maximizing the equity value. When the shareholders choose the level of bankruptcy \( L \) that maximizes the value of their shares, they actually choose to sell the firm’s assets at the moment when it is no longer interesting for them to hold the firm’s assets. The holder of a perpetual American put, by definition has the same right to sell the assets at the moment of their choice.

In order to obtain a more realistic model, we also consider the tax benefit of coupon payments and the reorganization costs at default while studying additional factor usually observed in financial markets, that is the violation of the absolute priority rule (APR), which means that, at default time, the shareholders take a part of the firm’s value and do not respect the bondholders’ first priority. Empirical studies conducted by Franks and Torous (1989) and Eberhart et al (1990) show that APR is violated in about 50% of the cases. We suppose that the shareholders will take \( \gamma \) of the firm’s expected value, after the reorganization costs at default, and the bondholders will receive only \( 1 - \gamma \) of the residual expected value at default.

A comparative analysis between the three models (diffusion process, diffusion with uniform jumps and diffusion with double exponential jumps) on credit spread, debt, equity and firm value has been carried out. We find, in general, the debt, equity and firm values in our models have the same forms as those in Leland (1994a) which confirms that his model is robust. However, in our models, the yield spreads are higher than those in Leland’s due to the presence of jumps. The model of the negative uniform jump generates higher spread than the model of double exponential jump. The presence of jumps in the process of the firm’s assets makes the firm’s risk increase. This result means that a firm whose price is driven by such a process, has a higher risk from the investors or debt holders point of views, so leading to increase the yield spreads as a compensation. This result also corrects one of the weaknesses of the structural model that usually produces a lower level of yield spreads than found in the market.

The paper is organized in sections. Section 2 describes the debt equity structure of the firm as well as the general formulae of equity, debt, firm values. The fact that the perpetual American put is relevant to equity value is highlighted. Using this highlight, section 3 derives the values of the perpetual American put,
equity, debt and firm in a continuous geometric Brownian motion model. Section 4 presents the general jump diffusion model and the two models of jump diffusion processes for the firm’s assets process. Section 5 derives formulae of the perpetual American put and also equity, debt and firm values for each process. In section 6, comparative results on debt value, yield spreads and firm value among the three models (the continuous geometric Brownian motion model and the two discontinuous jump diffusion models) are presented. Section 7 presents the conclusion of this paper.

2 The Debt-Equity Structure of the Firm

We consider the structure of the firm in this paper is modelled as in Leland (1994a). We assume that the firm is partly financed by debt which is time independent2 or in infinite maturity. In other words, the firm promises to pay a perpetual coupon \( C \) \((C \geq 0)\) to debt holders until bankruptcy, which occurs at time \( \tau \) when the value of the firm’s assets \( V \) falls below a default barrier \( L \). In this case, a fraction \( \alpha \) of the value of the firm’s assets will be lost due to bankruptcy costs or to the reorganization. Suppose that the APR is not respected and the shareholders will receive \( \gamma \), \(0 \leq \gamma \leq 1\), the constant fraction of the residual asset value at default.

The value of the debt, with this structure, is composed of two parts. The first part is the discounted expected value of coupon payment \( C \) if there is no default. The second part is the discounted expected value at default after bankruptcy costs, i.e., \((1 - \hat{\alpha})(1 - \gamma)V_\tau\) if there is default. Denoting a risk neutral probability as \( Q \), the formula of the debt value can be written as follows:

\[
D = D(V, L) = E_Q \left[ \int_0^\tau Ce^{-ru} du + (1 - \hat{\alpha})(1 - \gamma)V_\tau e^{-r\tau} 1_{\tau<\infty} \right]
\]

Note that the \( 1_{\tau<\infty} \) designates the indicator function that takes the value 1 if the stopping time \( \tau \) occurs before infinity \( \infty \) and takes the value 0 if not.

As \( C \) is constant, the formula can be reduced to:

\[
D = \frac{C}{r} \left[ 1 - E_Q \left( e^{-rt} 1_{\tau<\infty} \right) \right] + E_Q \left[ (1 - \hat{\alpha})(1 - \gamma)V_\tau e^{-r\tau} 1_{\tau<\infty} \right]
\]

Note that \((1 - \hat{\alpha})(1 - \gamma) = 1 - \alpha\), where \( \alpha = (\hat{\alpha} + \gamma - \alpha\gamma) \) is the fraction of asset value lost at default from the debt holders’ point of view. It is important for the debt holders to consider the residual fraction of asset value at default, the default or bankruptcy costs (it is not relevant whether these costs are due to reorganization costs or due to shareholders’ power). Thus, from now on, we just use the notation \( \alpha \). With this notation, the final formula of the debt is as follows:

\[
D = \frac{C}{r} - E_Q \left[ e^{-r\tau} \left( \frac{C}{r} - (1 - \alpha)V_\tau \right) 1_{\tau<\infty} \right]
\]

2 This time-independent debt structure can be justified in reality when debt has sufficiently long maturity, hence the return of principal can be ignored.
where the first term is the present value of the expected cash flow $C$ if there is no default to infinity and the second term is the present value of the expected cash flows if default occurs.

The value of the firm is considered as composed of three terms. The first term is the value of the firm’s assets ($V$). The second one is the expected value of the tax benefit ($TB$), while the tax rate is $\theta$, i.e., $TB = \frac{\theta C}{r} (1 - e^{-r\tau}1_{\tau<\infty})$. The third term is the value of expected bankruptcy cost (reorganization and violation APR), as $BC = -\alpha V e^{-r\tau}1_{\tau<\infty}$. The value of the firm is therefore written as follows:

$$v = v(V, L) = V + EQ \left[ \frac{\theta C}{r} (1 - e^{-r\tau}1_{\tau<\infty}) - \alpha V e^{-r\tau}1_{\tau<\infty} \right]$$

$$= V + \frac{\theta C}{r} [1 - EQ (e^{-r\tau}1_{\tau<\infty})] - EQ [\alpha V e^{-r\tau}1_{\tau<\infty}]$$

$$= V + \frac{\theta C}{r} - EQ \left[ e^{-r\tau} \left( \frac{\theta C}{r} + \alpha V \right) 1_{\tau<\infty} \right]$$

(2)

As discussed in Miller (1977), in the presence of personal tax rate, the effective tax advantage of debt, $\theta$ is as follows:

$$\theta = \frac{(1 - \theta_c)(1 - \theta_s)}{1 - \theta_d}$$

where $\theta_c$ is the corporate tax rate, $\theta_s$ is the personal tax rate on equity income and $\theta_d$ is the tax rate on debt income.

The value of the equity which is equal to the firm value minus the debt value, can be thus expressed as:

$$E = E(V, L) = v(V, L) - D(V, L)$$

Rearranging the terms gives us the final formula of the equity value:

$$E = V - \frac{(1 - \theta) C}{r} + EQ \left[ e^{-r\tau} \left( \frac{(1 - \theta) C}{r} - V \right) 1_{\tau<\infty} \right]$$

(3)

Remark: all the three equations of the debt, firm, equity values, (1), (2), (3), respectively are not depending on the specification of the dynamics of the firm’s asset value $V$.

**The Equity Maximization as a Perpetual American Put**

Seen from the modern financial theory standpoint, the shareholders’ objective is to maximize their own value, meaning the market value of the shares. We take this usual assumption that the shareholders’ objective is to maximize the value of the equity $E$. This indicates that the shareholders will maximize the expected part value $EQ \left[ e^{-r\tau} \left( \frac{(1 - \theta) C}{r} - V \right) 1_{\tau<\infty} \right]$ by choosing the value of the bankruptcy level, $L$.

From the formula of the equity value, it can observed that with the introduction of a variable $K = (1 - \theta) \frac{C}{r}$, the problem of maximizing the equity value $E$ becomes:

$$\sup_{L} EQ \left[ e^{-r\tau L} \left( K - V_{\tau_L} \right) 1_{\tau_L<\infty} \right]$$

(4)
This formula is actually the formula of a perpetual American put (PAP) with strike $K$, which indeed is

$$P^A = \sup_{\tau} E_Q \left[ e^{-r\tau} (K - V_\tau)^+ 1_{\tau<\infty} \right]$$

Mordecki (2002) proved that the family of stopping times (in this case) can be reduced to the hitting times of a constant barrier, i.e.,

$$P^A = \sup_L E_Q \left[ e^{-r\tau_L} \left( K - V_{\tau_L} \right)^+ 1_{\tau_L<\infty} \right]$$

where

$$\tau_L = \inf \{ t \geq 0 : V_t \leq L \}$$

Intuitively, we can explain why we find the formula of the perpetual American put when maximizing the equity value. When the shareholders choose the level of bankruptcy $L$, that maximizes the value of their shares, they actually choose to sell the assets of the firm at the moment when it is no longer interesting for them to hold the firm’s assets. The holder of a perpetual American put has the same right to sell the assets at the moment of his choice.

With this remark, our problem is equivalent to finding the value of a perpetual American put so as to reach the final result. When we have this value, we can replace it in the formulae of equity, debt, and firm values to obtain the final results. In our model, we call the perpetual American put, a building block due to its leading role.

The dynamics of the firm’s assets, the underlying process will be specified in the next sections with geometric Brownian motion and the two particular jump diffusion models.

3 The Model of Geometric Brownian Motion Process

A Geometric Brownian Motion Model

Let us assume that the dynamics of the firm’s assets follow a particular diffusion process, a geometric Brownian motion, under the risk neutral probability measure $Q$ as:

$$\frac{dV_t}{V_t} = (r - \delta)dt + \sigma dW_t$$

(5)

where the riskfree interest rate $r$, the firm payout ratio $\delta$ and the volatility of the firm’s asset value $\sigma$ are supposed to be constant and $(W_t, t \geq 0)$ is a one dimensional standard Brownian motion. This equation (5) has the unique solution

$$V_t = V \exp \left[ \left( r - \delta - \frac{\sigma^2}{2} \right) t + \sigma W_t \right]$$

(6)

where $V$ is the initial firm’s asset value at time 0.
Under this model, the values of debt, firm and equity are again written as in equations (1), (2) and (3) respectively.

**The Perpetual American Put**

As we mentioned previously, in order to obtain the value the equity, debt and firm, we need to price the perpetual American put option:

\[ P^A(V) = \sup_L E_Q \left( e^{-r\tau_L} (K - V_{\tau_L})^+ 1_{\tau_L < \infty} \right) = \sup_L E_Q \left[ e^{-r\tau_L} (K - V_{\tau_L})^+ \right] \]

where \( \tau_L = \inf \{ t \geq 0 : V_t \leq L \} \), with \( L \) is the exercise boundary and \( V_t \) is the process given by equation (6) and \( K \) is the strike.

The random variable \( \tau_L \) is the first stopping time that a drifted Brownian motion (the drift is \( r - \delta - \frac{\sigma^2}{2} \)) crosses a barrier from above. So, \( \tau_L \) is finite only if \( r - \delta - \frac{\sigma^2}{2} < 0 \), if not, \( \tau_L \) is infinite with a non zero probability.

The second equality of (7) follows as we have for any \( \tau_L \):

\[ \sup_L E_Q \left[ e^{-r\tau_L} (K - V_{\tau_L})^+ 1_{\tau_L = \infty} \right] = \sup_L \left[ E_Q \left[ e^{-r\tau_L} (K - V_{\tau_L})^+ \right] - E_Q \left[ e^{-r\tau_L} (K - V_{\tau_L})^+ 1_{\tau_L = \infty} \right] \right] \]

and

\[ E_Q \left[ e^{-r\tau_L} (K - V_{\tau_L})^+ 1_{\tau_L = \infty} \right] = 0 \]

Proof. We have:

\[ E_Q \left[ e^{-r\tau_L} K 1_{\tau_L = \infty} \right] = 0 \] because the exponential vanishes

and \[ E_Q \left[ e^{-r\tau_L} V_{\tau_L} 1_{\tau_L = \infty} \right] = 0 \]

for the latter, we know that

\[ V_t = V e^{(r-\delta)t} e^{\sigma W_t - \frac{\sigma^2}{2} t} \]

so

\[ V_t e^{-rt} \leq V_t e^{-(r-\delta)t} = V e^{\sigma W_t - \frac{\sigma^2}{2} t} = V e^{t \left( \frac{\sigma W_t - \sigma^2}{2} \right)} \to 0 \text{ as } \frac{W_t}{t} \to 0 \]

Hence the second equality is always true in the geometric Brownian motion case.

The following results are well known (see in Karatzas and Shreve (1991) or in page 196 of Elliott and Kopp (1999), for a proof of an equivalent formula).

The value of the perpetual American put in a geometric Brownian motion model is given as:

\[ P^A(V) = \begin{cases} K - V & \text{if } 0 \leq V < L \\ (K - L) \left( \frac{L}{K} \right)^{\beta} & \text{if } V \geq L \end{cases} \]  

where the exercise boundary \( L \) is:

\[ L = K \frac{\beta}{\beta + 1} \]
and $-\beta$ is the negative root of the equation:
\[ \frac{1}{2} \sigma^2 \beta^2 + \left( r - \delta - \frac{1}{2} \sigma^2 \right) \beta = r \]  
(9)

i.e.,
\[ \beta = \frac{\left( r - \delta - \frac{1}{2} \sigma^2 \right) + \sqrt{(r - \delta - \frac{1}{2} \sigma^2)^2 + 2r \sigma^2}}{\sigma^2} \]  
(10)

Note that if $\delta = 0$, then $\beta = \frac{2r}{\sigma^2}$ (this $\beta$ is a frequently used root).

**The Equity, Debt and Firm Values**

Inserting the formula of (8) into the formula of the equity value (3), we obtain:
\[ E(V) = V - \frac{(1 - \theta) C}{r} + (K - L) \left( \frac{L}{V} \right)^{\beta} \]
\[ = V - \frac{(1 - \theta) C}{r} + \frac{(1 - \theta) C}{r} \frac{1}{1 + \beta} \left( \frac{(1 - \theta) C}{r} \right)^{\frac{\beta}{\beta + 1}} \left( \frac{1}{V} \right)^{\beta} \]

The second equation is obtained by replacing $K = \frac{1 - \theta C}{r}$ and $L$ by its value.

From this equation, the value of the equity value can be reduced as:
\[ E(V) = V - \frac{(1 - \theta) C}{r} \left( 1 - \left( \frac{C}{V} \right)^{\beta} k_e \right) \]  
(11)

where the constant $k_e$ is:
\[ k_e = \left[ \frac{1 - \theta}{r} \frac{\beta}{1 + \beta} \right]^{\beta} / 1 + \beta \]

This is exactly the equity value formula of Leland (1994a)\(^3\) which is obtained by solving an ordinary differential equation, where $k_e$ is equivalent to the parameter $m$ of his notation.

From the debt value in equation (1), the expected value part of the debt value is almost the same American perpetual put with strike $\hat{K} = \frac{C}{(1 - \alpha r)}$ instead of $K = \frac{(1 - \theta C)}{r}$. The exercise boundary is still $L$, as a result of the same perpetual American put.

So we obtain the value of the debt directly as:
\[ D(V) = \frac{C}{r} - (1 - \alpha) \left( \hat{K} - L \right) \left( \frac{L}{V} \right)^{\beta} \]
\[ = \frac{C}{r} - (1 - \alpha) \left( \frac{C}{r(1 - \alpha)} - \frac{(1 - \theta) C}{r} \frac{\beta}{\beta + 1} \right) \left( \frac{(1 - \theta) C}{r} \frac{\beta}{\beta + 1} \right)^{\frac{1 - \theta C}{r}} \left( \frac{1}{V} \right)^{\beta} \]
\[ = \frac{C}{r} \left[ 1 - \left( \frac{C}{V} \right)^{\beta} k_d \right] \]  
(12)

\(^3\)As Leland (1994) did not calculate the firm, debt, equity values with the parameter $\delta$, the payout ratio, his results and ours are different by the value of $\beta$. Remember that his $\beta$ is $2r/\sigma^2$ and our $\beta$ is as in (10).
where the constant $k_d$ is:

$$k_d = [1 + \beta - (1 - \alpha)(1 - \theta) \beta] k_e$$

This $k_d$ is equivalent with the parameter $k$ of Leland (1994a).

The firm value can be obtained in two ways. The first possibility is to apply the same technic used for the debt value, replacing the strike with its equivalent value.

The second one is to remember that the value of the firm is the total value of its debt and equity, so

$$v(V) = E(V) + D(V)$$

$$= V - \frac{(1 - \theta) C}{r} \left( 1 - \left( \frac{C}{V} \right)^{\beta} k_e \right) + \frac{C}{r} \left( 1 - \left( \frac{C}{V} \right)^{\beta} k_d \right)$$

$$= V + \frac{\theta C}{r} \left( 1 - \left( \frac{C}{V} \right)^{\beta} k_v \right)$$

(13)

where the constant $k_v$ is:

$$k_v = [1 + \beta + \alpha (1 - \theta) \beta/\theta] k_e$$

Remark: all the three equity, debt and firm values as presented in equations (11), (12) and (13) are power functions of the coupon $C$ and the firm’s asset initial value $V$.

Note that the notation $k_e$, $k_d$, $k_v$ are equivalent to the notation $m$, $k$, $h$ in the paper of Leland (1994a) but $\beta$ has different value.

4 The Model of Jump Diffusion Processes

4.1 A General Jump Diffusion Model

The dynamics of the firm’s assets follow a jump diffusion process, with two parts, a continuous part driven by a geometric Brownian motion and a jump part driven by a compound Poisson process. Under the objective probability measure $P$, the value of the firm’s assets is assumed to follow the dynamics described in Merton (1976), i.e.,

$$\frac{dV_t}{V_{t-}} = (\pi^P + r - \delta)dt + \sigma dW^P_t + d \left( \sum_{k=1}^{N^P_t} Z^P_k \right)$$

(14)

where the riskfree interest rate $r$, the firm payout ratio $\delta$ and the volatility of the firm’s asset value $\sigma$ are supposed to be constant. Here $\pi^P$ is the asset risk premium, $(W^P_t, t \geq 0)$ is a one dimensional standard Brownian motion, $(N^P_t, t \geq 0)$ is a Poisson process with a constant intensity rate $\lambda^P > 0$ and
\( \{Z_k^P, k \geq 0\} \) a sequence of independent identically distributed (i.i.d) random variables. All three sources of randomness, \( N^P, W^P \) and \( Z_k^P \)'s are assumed to be independent.

Remark: This ratio \( \delta \) is a proportional rate at which profit is distributed to investors (both shareholders and bondholders). Coupon is paid at rate \( c \) to bondholders, thus \( \delta_c = \delta V - c \) is the payout rate to stockholders. Note that this payout rate \( \delta_c \) declines as the firm’s asset value \( V \) declines and may become negative (i.e., new equity must be issued to meet bond requirements).

In this modelling of the firm’s assets as a jump diffusion process, there exist an infinite number of risk neutral probability measures (in other words, the market is not complete) under which the discounted value of the firm’s assets and its discounted derivative prices are martingales. It is shown in Kou (2002) and in Huang-Huang (2003) that, by using the rational expectations with a HARA type utility function, a particular risk neutral probability measure \( Q \) can be chosen.

Under a particular risk neutral probability measure \( Q \), the value of the firm’s assets can be written as:

\[
dV_t = (r - \delta)dt + \sigma dW_t + dM_t
\]

where \((W_t, t \geq 0)\) is a one dimensional standard Brownian motion under \( Q \). The process \((M_t, t \geq 0)\) is the compensated martingale of a compound Poisson process, i.e.,

\[
M_t = \sum_{k=1}^{N_t} Z_k - \lambda t E_Q [Z_1]
\]

where \((N_t, t \geq 0)\) is a Poisson process with a constant intensity rate \( \lambda > 0 \), the random variables \( \{Z_k, k \geq 0\} \) are i.i.d and \( E_Q [Z_1] < \infty \). All sources of randomness \( N, W \) and \( Z_k \)'s are assumed to be independent under \( Q \).

We consider the filtered probability space \((\Omega, \mathcal{F}, Q)\) where the filtration is defined as

\[
F_t = \sigma (W_s, N_s, 0 \leq s \leq t)
\]

Denote \((T_k, k \geq 1)\) the sequence of jump times for \( N \).

Since the jumps of the firm’s assets process satisfy \( \Delta V_{T_n} = V_{T_n} - V_{T_{n-}} = V_{T_{n-}} (Z_n + 1) \), the unique solution of equation (15) is

\[
V_t = V \exp \left( \left( r - \delta - \frac{\sigma^2}{2} - \lambda E[Z_1] \right) t + \sigma W_t \right) \prod_{k=1}^{N_t} (Z_k + 1)
\]

where \( V = e^x \) is the value at time 0 (the initial firm’s asset value).

In the case where the \( Z_k \)'s are valued in \([-1, \infty[ \), the firm’s asset random variable is positive and we can write its value by the exponential form:

\[
V_t = Ve^{X_t} \text{ where } X_t = \left( r - \delta - \frac{\sigma^2}{2} - \lambda E[Z_1] \right) t + \sigma W_t + \sum_{k=1}^{N_t} \ln (Z_k + 1)
\]

11
In order to simplify the formula and the modelling of jump sizes later on, we introduce the variables $Y_k = \ln (Z_k + 1)$. These variables are also i.i.d. random variables, hence we have the following formulae:

$$X_t = \left( r - \delta - \frac{\sigma^2}{2} - \lambda (E[e^{Y_1}] - 1) \right) t + \sigma W_t + \sum_{k=1}^{N_t} Y_k$$

where $\mu = r - \delta - \frac{\sigma^2}{2} - \lambda (E[e^{Y_1}] - 1)$

(18)

The process $X$ is a Lévy process, i.e., a process with stationary and independent increments. Moreover, assuming that $E[e^{\beta X_t}] < \infty$ (i.e., $E[e^{\beta Y_1}] < \infty$), the property of stationary and independent increments implies that there exists a Laplace exponent function. By definition, the Laplace exponent function is the function $G(\beta)$ such that, for any $t$ and for any $\beta$, we have

$$E[e^{\beta X_t}] = \exp \{ G(\beta) t \}$$

(19)

where:

$$G(\beta) = \frac{1}{2} \sigma^2 \beta^2 + \left( r - \delta - \frac{\sigma^2}{2} - \lambda (E[e^{Y_1}] - 1) \right) \beta + \lambda (E[e^{\beta Y_1}] - 1)$$

(20)

and

$$= \frac{1}{2} \sigma^2 \beta^2 + \mu \beta + \lambda (E[e^{\beta Y_1}] - 1)$$

(21)

The Laplace exponent function $G(\beta)$ can also be called the characteristic function. As the process $V$ is composed of a Brownian motion and of a compound Poisson process, then $G(\beta)$ is sum of the characteristic functions of a Brownian motion and of a compound Poisson process (as a result of the stationarity and independence of the increments).

Remark: The function $G(\beta)$ is a convex function of $\beta$ as this function is the sum of two convex functions $\frac{1}{2} \sigma^2 \beta^2 + \mu \beta$ and $\lambda (E[e^{\beta Y_1}] - 1)$. The function $\frac{1}{2} \sigma^2 \beta^2 + \mu \beta$ is convex, as the second derivative with respect to $\beta$ is strictly positive ($\sigma^2 > 0$). And the function $\lambda (E[e^{\beta Y_1}] - 1)$ is also convex as its second derivative $\lambda E[Y^2 e^{\beta Y_1}]$ is positive for any jump size $Y$.

Remark: If the process $X$ has no jumps (i.e., a drifted Brownian motion), the Laplace exponent function $G(\beta)$ is reduced to:

$$G(\beta) = \frac{1}{2} \sigma^2 \beta^2 + \left( r - \delta - \frac{\sigma^2}{2} \right) \beta$$

which is in turn exactly the right-hand side of the equation (9) in the geometric Brownian motion model.

Until now we have not yet specified the distribution of jump sizes $Z_k$ or of the log jump sizes $Y_k$. We focus next on two special kinds of jumps. In the first case, the log jump sizes $Y_k$ follow a double exponential distribution, while in the second case, they follow a spectrally negative uniform distribution.

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4 For further use, that from (20), we have $G(1) = r - \delta$. 

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4.2 The Double Exponential Jump Diffusion Model

We present here the Double Exponential Jump Diffusion Model, developed by Kou and Wang (2003) among others. Let us suppose that the jump sizes $Y_k$’s follow an asymmetric double exponential distribution with density:

$$f(y) = p\eta_1 e^{-\eta_1 y} \mathbb{1}_{y \geq 0} + q\eta_2 e^{\eta_2 y} \mathbb{1}_{y < 0}, \quad \eta_1 > 1, \eta_2 > 0$$  \hspace{1cm} (22)

where $p, q \geq 0$, $p + q = 1$, represent the probabilities of upward and downward jumps. We can also see this density distribution of $Y$ as $Y_d = \frac{1}{2} \xi^+ + \xi^-$, with probability $p - \xi^+ + q$, where $\xi^+$ and $\xi^-$ are exponential random variables with means $\frac{1}{\eta_1}$ and $\frac{1}{\eta_2}$, respectively, and the notation $\mathbb{1}$ signifies equal in distribution. The condition $\eta_1 > 1$ is used to ensure that the assets’ value has finite expectation, i.e., $E(V_t) < \infty$. This condition implies that the average upward jump cannot exceed 100% which is reasonable. Indeed, if $\eta_1 > 1$, then $E(e^{Y_1})$ is finite.

As the $Y_k$’s follow the double exponential distribution, $E(e^{\beta Y_1})$ exists when $\eta_1 - \beta > 0$ and $\eta_2 + \beta > 0$, i.e., $-\eta_2 < \beta < \eta_1$, and can be calculated as follows:

$$E(e^{\beta Y_1}) = \int_{-\infty}^{\infty} e^{\beta u} f(u) \, du$$

$$= \int_{0}^{\infty} e^{\beta u} q\eta_2 e^{\eta_2 u} \, du + \int_{-\infty}^{0} e^{\beta u} p\eta_1 e^{-\eta_1 u} \, du$$

So finally, one obtains

$$E(e^{\beta Y_1}) = \frac{p\eta_1}{\eta_1 - \beta} + \frac{q\eta_2}{\eta_2 + \beta}$$

In particular,

$$E(e^{Y_1}) = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1}$$

Then, the Laplace exponent function in the case of double exponential jump diffusion is:

$$G(\beta) = \frac{1}{2} \sigma^2 \beta^2 + \left( r - \delta - \frac{1}{2} \sigma^2 - \lambda \left( \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1 \right) \right) \beta$$  \hspace{1cm} (23)

$$+ \lambda \left( \frac{p\eta_1}{\eta_1 - \beta} + \frac{q\eta_2}{\eta_2 + \beta} - 1 \right)$$

The expectation $E(e^{\beta Y_1})$ is defined only for $-\eta_2 < \beta < \eta_1$, hence the Laplace exponent $G(\beta)$ is defined only for $-\eta_2 < \beta < \eta_1$ as (23). For ease of

---

5The double exponential distribution was first introduced by Laplace (1749-1827), and the symmetric double exponential distribution was called Laplace distribution. This double exponential distribution has been further used in mathematical psychology literature.
notation, we denote also by \( G(\beta) \) the function (23) defined on \([-\infty, -\eta_2] \cup [-\eta_2, \eta_1] \cup [\eta_1, +\infty[\].

The equation \( G(\beta) = \rho, \rho > 0 \) reduces to a polynomial equation with degree four, so the equation can have at most four real roots.

As shown in table 1, as \( G(-\infty) = +\infty \) and \( G(-\eta_2) = -\infty \), there is at least one real root on the interval \([-\infty, -\eta_2]\]. Similarly, on the interval \([\eta_1, +\infty[\), there is also at least one real root as \( G(\eta_1) = -\infty \) and \( G(+\infty) = +\infty \).

On the interval \([-\eta_2, \eta_1]\), the function \( G(\beta) \) is convex. Moreover, \( G(-\eta_2) = +\infty \) and \( G(\eta_1) = -\infty \), so \( G(\beta) = \rho, \rho > 0 \) has exactly two real roots, one in the interval \([-\eta_2, 0]\) and the other one in the interval \([0, \eta_1]\).

So \( G(\beta) = \rho, \rho > 0 \) has exactly four real roots, two positive and two negative.

| \( \beta \) | \(-\infty\) | \(-\eta_2\) | 0 | \( \eta_1\) | \(+\infty\) |
|---|---|---|---|---|---|
| \( G(\beta) \) | \(+\infty\) | \(-\infty\) | \(+\infty\) | 0 | \(+\infty\) |

The graph in figure 1 plots an example where the equation \( G(\beta) = \rho, \rho > 0 \) in the double exponential jump diffusion, has exactly four real roots: \( \beta_1, \rho, \beta_2, \rho, -\beta_3, \rho, -\beta_4, \rho \) where \( 0 < \beta_1, \rho < \eta_1 < \beta_2, \rho < +\infty \) and \( 0 < \beta_3, \rho < \eta_2 < \beta_4, \rho < +\infty \).

### 4.3 The Uniform Jump Diffusion Model

We now study the case where the log jump sizes \( Y_k \) follow a uniform distribution on the interval \([-2a, -a]\), i.e., a law with density:

\[
f(y) = \frac{1}{a} \mathbf{1}_{[-2a,-a]}(y) \quad \text{with} \quad a > 0
\]

The parameter \( a > 0 \) specifies the lower boundary \((-2a)\) and the upper boundary \((-a)\) of the uniform jump diffusion. To the best of our knowledge, this case has never been studied before.

The quantity \( E(e^{\beta Y_1}) \) is defined for any \( \beta \) as:

\[
E(e^{\beta Y_1}) = \int_{-2a}^{-a} e^{\beta u} \frac{1}{a} du = \frac{e^{-2\beta a} - e^{-\beta a}}{\beta}
\]

and in particular,

\[
E(e^{Y_1}) = \frac{e^{-2a} - e^{-a}}{a}
\]

The Laplace exponent function \( G(\beta) \) in the uniform jump diffusion is defined for any \( \beta \) as:

\[
G(\beta) = \frac{1}{2} \sigma^2 \beta^2 + \left( r - \frac{1}{2} \sigma^2 - \lambda \left( \frac{e^{-2a} - e^{-a}}{a} - 1 \right) \right) \beta + \lambda \left( \frac{e^{-2a} - e^{-a}}{a \beta} - 1 \right)
\]
As mentioned already in remark 1, the function $G(\beta)$ is convex. So the equation $G(\beta) = \rho$, $\rho > 0$ admits also at most two real roots. As $G(-\infty) = +\infty$, $G(+\infty) = +\infty$ and $G(0) = 0$, so $G(\beta) = \rho$, $\rho > 0$ has at least two real roots. Hence, $G(\beta) = \rho$, $\rho > 0$ has exactly two real roots, one negative and one positive.

The graph in figure 1 plots an example where the equation $G(\beta) = \rho$, $\rho > 0$ in the uniform jump diffusion, has exactly two real roots: $\beta_{1,\rho}, \beta_{2,\rho}$ where $\beta_{1,\rho} < 0 < \beta_{2,\rho}$.

For example with $r = 0.07$, $\delta = 0.06$, $\sigma = 0.2$, $\eta_1 = 50$, $\eta_2 = 33.33$, $p = 0.5$, $a = 0.1$, $\lambda = 3$, we can plot the curves of $G(\beta), \rho > 0$ in the two cases as:

Double Exponential Jump Diffusion  Uniform Jump Diffusion

| Graphic of $G(\beta) = \rho$ | Graphic of $G(\beta) = \rho$ |
|--------------------------------|--------------------------------|
| Figure 1                       |                                |

In the next section, we shall present results for the value of the perpetual American put and also respectively the values of equity, debt and firm in each of the two models.

5 The Perpetual American Put, Debt, Equity and Firm Values

5.1 The Double Exponential Jump Diffusion Model

The value a perpetual American put is:

$$P^A(V) = \sup_{\tau} E^Q \left( e^{-r\tau} (K - V_\tau)^+ 1_{\tau < \infty} \right) = \sup_L E^Q \left[ e^{-r\tau L} \left( K - V_{\tau L} \right)^+ 1_{\tau L < \infty} \right]$$

$$= \sup_l E^Q \left[ e^{-r\tau_l} \left( K - e^{\delta X_{\tau_l}} \right)^+ 1_{\tau_l < \infty} \right]$$
where the stopping time is defined as:
\[
\tau_L = \inf \{ t \geq 0 : V_t \leq L \} = \inf \{ t \geq 0 : e^{X_t} \leq L = e^{t_0} = e^{x_1} \}
\]
and \( Q \) is the risk neutral probability under which \((V_t, t \geq 0)\) and \((X_t, t \geq 0)\) follow (15) and (18) respectively.

Under the double exponential jump diffusion model, the value of the perpetual American put (see Kou and Wang (2003) [21]) is given by \( P_A(V) = u(x) \), where \( V = e^x \) and
\[
u(x) = \begin{cases} K - e^x & \text{if } x < l_0 \\ Ae^{-x_3} + Be^{-x_4} & \text{if } x \geq l_0 \end{cases}
\]
Here \( l_0 \), such that \( L = e^{l_0} \), is the exercise boundary; furthermore \( L \) is:
\[
L = K \eta_2 + 1 \frac{\beta_3}{\beta_3 - \beta_4} \frac{1}{\beta_4} \eta_2 > 0
\]
The coefficients \( A, B \) are defined in terms of \( K \) and \( L \) by
\[
A = L^{3, r} \frac{1 + \beta_4}{\beta_4 - \beta_3} \left[ \frac{\beta_4 - K}{1 + \beta_4} \right]
\]
\[
B = L^{3, r} \frac{1 + \beta_3}{\beta_3 - \beta_4} \left[ L - \frac{\beta_3 - K}{1 + \beta_3} \right]
\]
and \( K = \frac{\beta_3}{\beta_4} \) is the mean of downward jumps and \( -\beta_3, -\beta_4 \) are the only two negative roots of the Laplace exponent equation (23) \( G(\beta) = r \).

Remark: it is easy to check that \( A, B \) are strictly positive by writing that
\[
\frac{\beta_3}{\beta_4} K - L = K \frac{1}{1 + \beta_4} \frac{1}{1 + \beta_3} \eta_2 - \eta_2 > 0
\]
\[
L - \frac{\beta_3}{\beta_4} K = K \frac{1}{1 + \beta_3} \frac{1}{1 + \beta_4} \frac{\beta_4 - \eta_2}{\eta_2} > 0
\]
Finally, we can rewrite the coefficients \( A, B \) in terms of \( L \) and \( C \) by replacing \( K \) with \( \frac{(1-\theta)C}{r} \)
\[
A = L^{3, r} \frac{(1-\theta)C}{r} \frac{1}{1 + \beta_4} \frac{1}{1 + \beta_3} \eta_2 - \frac{1}{1 + \beta_4} \frac{1}{1 + \beta_3} \eta_2
\]
\[
B = L^{3, r} \frac{(1-\theta)C}{r} \frac{1}{1 + \beta_4} \frac{1}{1 + \beta_3} \eta_2 - \frac{1}{1 + \beta_4} \frac{1}{1 + \beta_3} \eta_2
\]
We can see that the coefficients \( A, B \) are increasing function of the exercise boundary \( L \) and the coupon \( C \) and decreasing function of the tax rate \( \theta \) and interest rate \( r \).

Inserting the formula of the perpetual American put (27) into the formulae of equity, debt and firm values, which are the equations (1), (2) and (3) in the case \( x \geq l_0 \) or \( V \geq L \), we obtain the values of equity, debt and firms values.
The value of the equity, in terms of barrier $L$ and initial assets’ value $V$ is equal to:

$$E(V) = V - \frac{(1 - \theta)C}{r}$$

$$+ \frac{(1 - \theta)C \beta_{3,r} - \beta_{3,r} \eta_{2}}{\beta_{4,r} - \beta_{3,r} \eta_{2} 1 + \beta_{3,r}} \left( \frac{L}{V} \right)^{\beta_{3,r}}$$

$$+ \frac{(1 - \theta)C \beta_{4,r} - \beta_{3,r} \eta_{2}}{\beta_{4,r} - \beta_{3,r} \eta_{2} 1 + \beta_{4,r}} \left( \frac{L}{V} \right)^{\beta_{4,r}}$$

(29)

The value of the equity, in terms of coupon $C$ and $V$, reduces to:

$$E(V) = V - \frac{(1 - \theta)C}{r} + (1 - \theta)C \left[ A_e \left( \frac{C}{V} \right)^{\beta_{3,r}} + B_e \left( \frac{C}{V} \right)^{\beta_{4,r}} \right]$$

(30)

and the coefficients $A_e, B_e$ are constants:

$$A_e = \frac{\eta_{2} - \beta_{3,r} \eta_{2}}{\beta_{4,r} - \beta_{3,r} \eta_{2} 1 + \beta_{3,r}} \left( \frac{1 - \theta) \eta_{2} + \frac{1}{r} \beta_{3,r} + \frac{1}{1 + \beta_{4,r}} \right)^{\beta_{3,r}}$$

$$B_e = \frac{\beta_{4,r} - \eta_{2} \beta_{3,r}}{\beta_{4,r} - \beta_{3,r} \eta_{2} 1 + \beta_{4,r}} \left( \frac{1 - \theta) \eta_{2} + \frac{1}{r} \beta_{3,r} + \frac{1}{1 + \beta_{4,r}} \right)^{\beta_{4,r}}$$

Based on the relationship between the coefficients $k_e, k_d, k_v$ and the value of equity, debt and firm, the value of the debt can be written as:

$$D(V) = C - \frac{C}{r} \left[ A_d \left( \frac{C}{V} \right)^{\beta_{3,r}} + B_d \left( \frac{C}{V} \right)^{\beta_{4,r}} \right]$$

(31)

Here the values of the constants $A_d, B_d$ are as follows:

$$A_d = \left[ 1 + \beta_{3,r} - (1 - \alpha) (1 - \theta) \beta_{3,r} \right] A_e$$

$$B_d = \left[ 1 + \beta_{4,r} - (1 - \alpha) (1 - \theta) \beta_{4,r} \right] B_e$$

The value of the firm is then the sum of both the debt value and the equity value, which in turn can be written as:

$$v(V) = E(V) + D(V)$$

$$= V + \frac{\theta C}{r} - \frac{\theta C}{r} \left[ A_v \left( \frac{C}{V} \right)^{\beta_{3,r}} + B_v \left( \frac{C}{V} \right)^{\beta_{4,r}} \right]$$

(32)

where the values $A_v, B_v$ are as follows:

$$A_v = \left[ 1 + \beta_{3,r} + \frac{\alpha (1 - \theta) \beta_{3,r}}{\theta} \right] A_e$$

$$B_v = \left[ 1 + \beta_{4,r} + \frac{\alpha (1 - \theta) \beta_{4,r}}{\theta} \right] B_e$$
The case \( x < l_0 \) (or \( V < L \)) can happen in the perpetual American put but it does not have an economical meaning for the firm. It states that the firm decides to borrow an important debt at time 0 (higher than the firm’s actual assets value) then it is forced to default immediately and as a consequence, it has to support a positive loss of \( \alpha V \). Hence, from a financial standpoint, it is not rational for a firm to make this decision.

Remark: it is straightforward to check that all the three equity, debt and firm values as presented in equations (30), (31) and (32) are power functions of the coupon \( C \) and initial assets’ value \( V \). Indeed, it is necessary to develop the debt value formula (31) by replacing \( L \) and \( K \) by their values as follows:

\[
D(C, V) = \frac{C}{r} - \frac{C}{r} A_d \left( \frac{A_2 + 1}{\eta_2} \frac{\beta_{3,r}}{1 + \beta_{3,r}} \frac{\beta_{4,r}}{1 + \beta_{4,r}} \left( 1 - \theta \right) \frac{C}{V} \right)^{\beta_{3,r}}
- \frac{C}{r} B_d \left( \frac{A_2 + 1}{\eta_2} \frac{\beta_{3,r}}{1 + \beta_{3,r}} \frac{\beta_{4,r}}{1 + \beta_{4,r}} \left( 1 - \theta \right) \frac{C}{V} \right)^{\beta_{4,r}}
\]

Obviously, the debt value \( D(V) \) is a concave function of coupon \( C \) and initial assets’ value \( V \).

5.2 The Uniform Jump Diffusion Model

Under the uniform jump diffusion model, we shall prove that the value of the perpetual American put is given by \( P^A (V) = u(x) \) where \( V = e^x \) and

\[
u(x) = \begin{cases} 
K - e^x & \text{if } x < l_0 \\
u_n(x) & \text{if } x \in I_n = [l_0 + na, l_0 + (n + 1)a], n \geq 0
\end{cases}
\]

Here \( a > 0 \) specifies the uniform boundary \([-2a, -a]\) and \( l_0 \), such that \( L = e^{l_0} = e^{x+l} \), is the exercise boundary; furthermore \( L \) is:

\[
L = K \frac{r \tilde{\beta} - 1}{\delta \tilde{\beta}}
\]

where \( \tilde{\beta} \) is the positive root of the Laplace exponent equation (26): \( G(\tilde{\beta}) = r \).

In the interval \( I_n, n \geq 0 \) the functions \( u_n(x) \) are defined as:

\[
u_n(x) = A_n(x) e^{\beta_1 x} + B_n(x) e^{\beta_2 x} + C_n e^x + D_n(x)
\]

where \( C_n \) is a constant, \( A_n(\cdot), B_n(\cdot), D_n(\cdot) \) are polynomial functions with degree \( n \) and \( \beta_1 \) (respectively \( \beta_2 \)) is the negative (respectively positive) root of

\[
\frac{1}{2} \sigma^2 \beta^2 + m \beta - (\lambda + r) = 0
\]

i.e.,

\[
\beta_1 = -\mu - \sqrt{\mu^2 + 2\sigma^2 \lambda} \quad \text{and} \quad \beta_2 = -\mu + \sqrt{\mu^2 + 2\sigma^2 \lambda}
\]
Note that $\beta_1 < -1$ and $\beta_2 > 1$.

The proofs of these formulae of the perpetual American put in the uniform jump diffusion model are given in the Technical Appendix of Dao's thesis (2005) or in the paper of Bellamy, Dao and Jeanblanc (2012). Indeed, in the Technical Appendix, the author establishes firstly the original results of the perpetual American put in a general spectrally negative jump diffusion model. The Technical Appendix also presents the formula in the spectrally negative uniform jump diffusion model. In general, the model of the spectrally negative jump diffusion process can be considered as constituting the third example of the proposed model in this paper, as the building block needed can be priced within this model. Generally speaking, the proposed model can be extended to any jump diffusion process for which we can obtain the price of a perpetual American put.

Inserting the formula of the perpetual American put into the formulae of equity, debt and firm values, i.e., the equations (3), (1) and (2) respectively in the case $x \geq l_0$ (or $V \geq L$) we obtain the value of the firm, of the equity, of the debt in the uniform jump diffusion case.

The value of the equity, in the case $V \geq L$, can be written as:

$$E(V) = V - \frac{(1-\theta)C}{r} + u_n(\ln V, K) \text{ for } V \in [e^{l_0+na}, e^{l_0+(n+1)a}]$$  \hspace{1cm} (35)

where $u_n(\ln V, K)$ is the same function $u_n(x)$ in (34) with strike $K = \frac{(1-\theta)C}{r}$.

The value of the debt is as follows:

$$D(V) = \frac{C}{r} - (1-\alpha)u_n(\ln V, \hat{K}) \text{ for } V \in [e^{l_0+na}, e^{l_0+(n+1)a}]$$  \hspace{1cm} (36)

where $u_n(\ln V, \hat{K})$ is the same function $u_n(x)$ in (34) with strike $\hat{K} = \frac{C}{r(1-\alpha)}$.

Hence, the value of the firm is:

$$v(V) = E(V) + D(V)$$

$$= V + \frac{\theta C}{r} + u_n(\ln V, K) - (1-\alpha)u_n(\ln V, \hat{K}) \text{ for } V \in [e^{l_0+na}, e^{l_0+(n+1)a}]$$  \hspace{1cm} (37)

Remark: it is not straight forward to see that all the three equity, debt and firm values as presented in equations (35), (36) and (37) are power functions of the coupon $C$. However, we can see that is a piecewise power form, partly from the convexity of the put function and partly from the plotting graphs in the next section. In general, one can explain this power form is due to the diffusion part assumption of the proposed models.

6 The Comparative Results

One of our objectives in this paper is to assess the influence of the amplitude of different jumps on the firm value, the debt value and the yield spreads.
We suggest a comparison between the model without jumps (the continuous diffusion model) and the model with jumps (negative jump and two-sided jump diffusion). More precisely, we analyze the debt value, the yield spreads and the firm value for the three models: the particular diffusion model, geometric Brownian motion, as in the Leland’s model (1994a), the Double Exponential jump diffusion model and the Uniform jump diffusion model (in short, we shall call them Leland, Double Exponential, Uniform, respectively). We also make the comparison of the results of these three models.

6.1 Methodology

In order to compare with Leland’s model, we take his input parameters. They are risk-free interest rate \( r = 7\% \), tax rate \( \tau = 0.35 \), bankruptcy cost \( \alpha = 0.5 \) (equivalent to our model, i.e., \( \hat{\alpha} \approx 0.4 \) and \( \gamma \approx 0.2 \)), firm’s assets value at time 0 is \( V = e^x = 100 \), and the coupon varies from \( C = (0, \ldots, 14) \). For the payout ratio, we take \( \delta = 1\% \). Like Leland did, we plot the Leland graphs with three different values of the firm’s volatility with \( \sigma = 20\% \) (the basis case), \( \sigma = 15\% \) and \( \sigma = 25\% \). Note that the formulae used for the Leland graphs are found in section 2.3 of this paper.

For the Double Exponential jump diffusion model, we take the same parameters as in Leland for the diffusion part and for the double exponential jumps part, we use the parameters proposed in Kou and Wang (2002). Therefore we use: risk-free interest rate \( r = 7\% \), tax rate \( \tau = 0.35 \), bankruptcy cost \( \alpha = 0.5 \), payout ratio \( \delta = 1\% \), firm’s assets at time 0 is \( V = 100 \), probability of upward jumps \( p = 0.3 \) (\( q = 0.7 \)), the mean of upward jumps \( 1/\eta_1 = 0.02 \), the mean of downward jumps \( 1/\eta_2 = 0.03 \), intensity of the jump process \( \lambda = 3 \), the volatility \( \sigma = 15\% \), 20\% and \( \sigma = 25\% \), and the coupon varies \( C = (0, \ldots, 14) \).

For the Uniform jump diffusion model, we take the same parameters as in Leland for the diffusion part and for the uniform jumps part, we choose our simulated value. Therefore we use: risk-free interest rate \( r = 7\% \), tax rate \( \tau = 0.35 \), bankruptcy cost \( \alpha = 0.5 \), payout ratio \( \delta = 1\% \), firm’s assets at time 0 is \( V = 100 \), intensity of the jump process \( \lambda = 3 \), jump size distribution parameter \( a = 0.07 \) (meaning that jumps sizes vary uniformly in the interval \([-2a, -a]\), i.e. \([-0.14, -0.07]\)), the volatility \( \sigma = 15\% \), 20\% and \( \sigma = 25\% \), and the coupon varies \( C = (0, \ldots, 14) \).

In the following, we shall compare the debt value, the yield spread value and the firm value as a function of the coupon and the leverage. Hence, we define here several parameters used in this analysis. The debt value as a function of the coupon is obtained directly from the formulation of the debt as \( D(V) \) or \( D(V,C) \).

The leverage is the proportion of borrowed money in the total assets of the firm. It measures the partition of the funds used by the firm between debt and equity. A firm that is highly leveraged may increase its bankruptcy probability and may also decrease its opportunities to find new lenders in the future. The leverage can increase the shareholders’ return on their investment and often there are tax benefits from the coupon payment associated with debt. The
leverage represents in fact, the percentage of firm’s assets financed by the debt. The leverage ratio is hereby defined by the proportion of debt value over firm value, i.e., 

\[ Lev = \frac{D(V, C)}{v(V, C)} \]

Here, we are in a perpetual coupon-paying debt model. As the coupon varies, the debt value varies, meaning that the percentage of debt over firm value varies, which implies a change of the leverage. The higher coupon level (in one sense) signifies the higher the leverage and vice versa. Thus, the coupon may be perceived as a proxy for the leverage. Debt value as a function of the coupon can be seen as pseudo debt value as a function of the leverage.

For the debt value as a function of the leverage’s graphs, we should first calculate the leverage level \( Lev \) as in the above formula and then plot the debt value as a function of this calculated leverage level.

The yield of perpetual coupon-paying bond, by definition, is the coupon rate divided by the market price, i.e., Coupon/Debt Value. The yield spread is the difference between the yield (the rate) that the firm has to pay to bondholders and the default-free interest rate. Therefore, the yield spread is defined as

\[ YS = \frac{C}{D(V, C)} - r \]

The yield spreads as a function of the coupon can be obtained directly from the above formula. And the yield spreads as a function of the leverage can be obtained using the same technique as that used for the debt value as a function of the leverage.

### 6.2 The Comparative Statics of the Debt Value

When we take \( V = 100 \), the normalization implies that the coupon level \( C \) (in monetary) also represents the coupon rate as a percentage of firm’s assets value, \( V \). Normally, we should observe that the debt value is a concave function of the coupon \( C \) and firm’s assets value \( V \).

- Leland

Leland-Figure 1 shows the relationship between the debt value and the coupon for varying firm volatility (sigma varies from 20%, 15%, 25%). The formula of debt value in the Leland model i.e., in the geometric Brownian motion, is as in equation (12).

It is obvious to see that an increase in coupon level generally increases the debt value. However, at high coupon level, the effect is inverses. At high coupon levels (junk bonds), we see the reversal among the three graphs, that is, the debt of firms with high risk is of higher value than that of firms with less risk. However, the peak (the maximum) of each curve indicates the maximum debt capacity, which corresponds to the leverage level.
Leland-Figure 2 repeats Leland-Figure 1, but with the leverage ratio \((D/v)\). The leverage level is calculated from the formula (12) and (13) as follows:

\[
\text{Lev} = \frac{D(V,C)}{v(V,C)} = \frac{C \left[ 1 - \left( \frac{C}{V} \right)^\beta \right] k_d}{V + \frac{2C}{\alpha C} \left[ 1 - \left( \frac{C}{V} \right)^\beta \right] k_v}
\]

We then plot the debt value (12) as a function of this leverage. We observe that the reversals seen in Leland-Figure 1 do appear here but less significative. This is because the leverage itself depends on the debt value.
Leland-Figure 1B shows the relationship between debt value and the coupon for varying bankruptcy costs (alpha varies from 50%, 100%, 0%) with the same parameters as in the Leland-Figure 1 graphs and the volatility at 20%. Note that the constant of the debt value $k_d$ is the function of $\alpha$, i.e.,

$$k_d = [1 + \beta - (1 - \alpha)(1 - \theta) \beta] k_e$$

and

$$k_e = \left[\frac{1 - \theta}{1 + \beta}\right]^\beta$$

It is normal to observe that larger bankruptcy costs decrease the value of debt and there are no reversals seen in this figure.

- Double Exponential

Double Exponential-Figure 1 presents the relationship between debt value and the coupon obtained by the formula (31) for different firm volatility (sigma varies from 20%, 15%, 25%) in the Double Exponential Jump Diffusion model. In Double Exponential- Figure 2 is the debt value as a function of the leverage. Again, the leverage is first calculated from the formula (31) and (32) as follows:

$$Lev = \frac{D(V, C)}{v(V, C)} = \frac{C v - \frac{C \theta}{V} \left[ A_d \left( \frac{C}{V} \right)^{\beta_3} + B_d \left( \frac{C}{V} \right)^{\beta_4} \right]}{V + \frac{\rho C}{V} \left[ A_v \left( \frac{C}{V} \right)^{\beta_3} + B_v \left( \frac{C}{V} \right)^{\beta_4} \right]}$$

We observe that the Double Exponential Jump Diffusion model takes the same result or the same form as the Leland’s curves. There are also reversals seen in Double Exponential-Figure 2 as in the Leland’s.
Note that the roots of $\beta_{3,r}$ and $\beta_{4,r}$ play almost a symmetric role in the general value formulae. However, we observe a greater contribution to values (for example, see in the debt value) by the coefficient of $\exp(l\beta_{3,r})$ than $\exp(l\beta_{4,r})$. More precisely, the coefficients $A_d (A_v, A_u)$ play a more important role than the coefficients $B_d (B_v, B_u)$. One possible explanation is that the root $\beta_{4,r}$ is almost as the same as the parameter $\eta_2$. We also note that it should take at least 10 digital places for the roots $\beta_{3,r}$ and $\beta_{4,r}$ in order to obtain a reasonable accuracy.

- **Uniform Jump Diffusion**

Uniform-Figure 1 presents the relationship between debt value and the coupon obtained by the formula (36) and Proposition ??, for different firm volatility with uniform distribution. In the Uniform-Figure 2 is the debt value as a function of the leverage. Here, the leverage is calculated from equation (36) and (37) as follows:

$$Lev = \frac{D(V, C)}{v(V, C)} = \frac{\frac{Q}{r} - (1 - \alpha) u_n \left(\ln V, \hat{K}\right)}{V + \frac{ac}{r} + u_n \left(\ln V, K\right) - (1 - \alpha) u_n \left(\ln V, \hat{K}\right)}$$

Note that we compute the value function of the perpetual American put $u_n \left(\ln V, \hat{K}\right)$ for $V \in \left[e^{\theta a + n a}, e^{\theta a + (n+1)a}\right]$ in exact value only for $n = -1, 0, 1, 2, 3$. 

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For $n \geq 4$, we use an interpolation technique to obtain the approximate value of the perpetual American put.

We also notice the peak of debt capacity in these two figures. However, no reversals can be seen in the Uniform-Figure 1 and Uniform-Figure 2, making them quite different in comparison to those of both the Leland and Double Exponential models. The hypothesized negative jump size may be the reason for this feature. The approximate method may also be the cause of this phenomenon. A net reduction in debt value is also noted in the uniform jump diffusion model in comparison with the other two.

In the all three models, we all observe that the debt value is a concave function of the coupon $C$ and the leverage.

### 6.3 The Comparative Statics of the Yield Spread

- **Leland**

From the equation (12) of the debt value, we obtain the yield spread of closed form as follows:

$$YS = \frac{C}{D(V)} - r = \frac{C}{\frac{C}{r} \left[ 1 - \left( \frac{C}{V} \right)^\alpha k_d \right]} - r = \frac{r \left( \frac{C}{V} \right)^\beta k_d}{1 - \left( \frac{C}{V} \right)^\beta k_d}$$
Leland-Figure 3 and Leland-Figure 4 illustrate yield spreads as a function of coupon level and leverage respectively, with three different levels of the firm’s volatility. Reversals happen in the two figures at high coupon level and high level of leverage. This indicates that with junk bonds and heavy leverage, yield spread may decline when firm’s risk increases.

- **Double Exponential**

  From the formula of the Debt Value for the Double Exponential Jump Diffusion (31), we have the yield spread as follows

  \[
  YS = \frac{C}{D(V)} - r = \frac{r \left[ A_d \left( \frac{C}{\tau} \right)^{\beta_4,r} + B_d \left( \frac{C}{\tau} \right)^{\beta_4,r} \right]}{1 - \left[ A_d \left( \frac{C}{\tau} \right)^{\beta_3,r} + B_d \left( \frac{C}{\tau} \right)^{\beta_4,r} \right]}
  \]

  Double Exponential-Figure 3 and Double Exponential-Figure 4 present the yield spreads against coupon level and leverage, as a function of the firm’s volatility. The phenomenon of reversals also happens here in the case of Double Exponential.
It can be observed that the yield spreads are much higher in the Double Exponential Jump Diffusion than in the Leland’s graphs equivalent Leland-Figure 3 and Leland-Figure 4. That may be due to the jump size assumption. In this case, with a volatility of 20%; it seems that the yield spreads are much higher than those with a volatility of 15% and a volatility of 25% at the low coupon level. The inverse is also observed at high coupon level. Among the three levels of volatility, the curve volatility 15% generates a very high yield spread at high coupon level, which confirms the reversal phenomenon and the behavior of junk bonds.

- Uniform

From the equation (36) of the debt value, we obtain the yield spread as follows:

\[ YS = \frac{C}{D(V)} - r = \frac{C}{\frac{C}{\tau} - (1 - \alpha) u_n \left( \ln V, \hat{K} \right)} - r \]

\[ = \frac{r (1 - \alpha) u_n \left( \ln V, \hat{K} \right)}{1 - r (1 - \alpha) u_n \left( \ln V, \hat{K} \right)} \text{ for } V \in \left[ e^{\ln a}, e^{\ln a + (n+1)a} \right] \]
Uniform-Figure 3 and Uniform-Figure 4 also present the yield spreads against coupon level and leverage, as a function of the firm’s volatility. The phenomenon of reversals does not happen in these two figures.

It can also be observed that the yield spreads are much higher in the Uniform jump diffusion than the Double Exponential jump diffusion. This may be caused by the difficulty in approximation of the formulae of debt in the Uniform jump model. Another possible explanation for the yield spread is that fact that we use negative jumps here.

6.4 The Comparative Statics of the Firm Value

- Leland

Leland-Figure 5 and Leland-Figure 6 plot the firm value against coupon level and leverage, respectively, as the firm’s risk varies. The Leland model’s firm value as a function of the coupon is the formula (13). We adopt the same technique used to plot the debt value as a function of the leverage, for the firm value as a function of the leverage. The firm value as a function of the leverage can be seen in each curve that has its peak, which means there exists an optimal leverage level and optimal firm’s value. Again the reversal behavior of total firm value for a high coupon level (junk bond) and a high leverage level can be observed. As these state, in the presence of bankruptcy costs and corporate taxes, total firm value may rise as firm’s risk increases.
Double Exponential

The Double Exponential model’s firm value as a function of the coupon is the formula (32) and from this formula, we plot the firm value as a function of the leverage.

Double Exponential-Figure 5 and Double Exponential-Figure 6 illustrate once again the firm value as a function of coupon level and leverage respectively, as the firm’s risk varies.

Again, each curve has its peak and the reversals can also be seen.
• Uniform

The Uniform model’s firm value as a function of the coupon is the formula (37) and from this formula we plot the firm value as a function of the leverage. Uniform-Figure 5 and Uniform-Figure 6 plot the firm value as a function of coupon level and leverage respectively, as the firm’s volatility varies. Once again we may see a peak in each curve. However, the reversals are not seen here. It can be noted that the same results were observed in the figures of debt values in Uniform jump model.
We also observe that in all three models, the firm values are also a concave function of the coupon $C$ (as shown in the graphs and by formulae) and a concave function of the firm’s assets value $V$ (as shown by formulae).

The optimal firm value is obtained at the peak of each curve of Leland, DoubleExponential and Uniform firm value’s curves. The capital structure can be seen directly in these graphs. For example, for the double exponential jump diffusion, the optimal capital structure is at the firm value of 130, with the coupon level of $C = 6.5$ and the leverage of $Lev = 60\%$ in the case of volatility $\sigma = 15\%$. In the other cases of volatilities, $\sigma = 20\%$ and $\sigma = 25\%$, we can observe the presence of the optimal capital structure. For the uniform jump diffusion process, the optimal capital structure is at the firm value of 115, with the coupon level of $C = 7$ and the leverage of $Lev = 70\%$ in the case of volatility $\sigma = 15\%$. Furthermore, we can observe that in all our graphs there is a unique leverage level maximizing the firm’s value, thus imposing the idea of an optimal financing structure, i.e., an optimal capital structure.

In the trade-off model, the optimal firm value indicates also the optimal leverage which balances between the advantage of tax shield and the costs of bankruptcy. The optimal leverage here is about $70\%$ at level of coupon 7. It is the level of coupon that a marginal increase in tax shield is equal to a marginal increase in bankruptcy costs (as the probability of default and financial distress increase when the coupon increases).
7 Conclusion

We propose a structural model of the perpetual coupon debt in a framework of two different jump diffusion processes, the uniform and the double exponential. We find the connection between the perpetual American put and the equity value. The perpetual American put is derived from the model of uniform jump diffusion and negative jump diffusion in general. The modelling of double exponential and uniform jump diffusion is first used in the structural approach of endogenous default barrier. Analysis of comparative results is done among three models (geometric Brownian motion, uniform jump diffusion and double exponential jump diffusion) on the debt value, the yield spreads and the firm value.

The influence of jumps on the debt, equity and firm values in a structural model with endogenous default barrier has been tested using the proposed approach. It confirms that the yield spreads are higher with the jump diffusion model. This corrects one of the weaknesses of the structural approach: that the yield spread is much lower than the observed level in the market, especially for junk bonds. One of the original results of adding jumps into the diffusion process while modelling the firm’s asset is that the time of default is no longer predictable.

The approach presented in our paper may be extended to other jump processes such as the Lévy stable, to account for strategic behavior such as debt negotiations or conflict between bondholders and shareholders. The same framework may also be used for other forms of time independent debt or for accounting for the tax benefit lost when the value of the firm is lower than at a certain level.

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