‘Congruent Partitions’ of Polygons
– a Short Introduction

-R. Nandakumar
(nandacumar@gmail.com)

Abstract: We introduce the problem of partitioning 2D regions (usually convex polygonal regions) into mutually congruent pieces.

1. Introducing the Problem

This article is an elaboration on [1] where this problem was originally stated. It is essentially a collection of inter-related questions and a few partial answers.

We recall a definition from basic Euclidean geometry:

Two planar regions are congruent if one can be made to perfectly coincide with the other by translation, rotation or reflection (in 2D, flipping over).

The Basic Problem: To partition a given polygonal region $P$ into $N$ mutually congruent pieces (or ‘tiles’) so that the fraction of the area of $P$ not covered by the union of the pieces is as small as possible. $N$ is a finite positive integer. Each tile should have finitely many sides.

Some Definitions: A partition which leaves out the least area from $P$ is an optimal congruent partition for that $N$. If a congruent partition leaves no area of $P$ unused, then it is called a perfect congruent partition.

If for a given input polygon and $N$, the optimal congruent partition is not unique, we are interested in finding the one in which the pieces have least complexity (least number of edges) – such a partition may be called the simplest optimal congruent partition.

Remark: In the following discussions, the region to be partitioned $P$ and the tiles are convex, unless mentioned otherwise.

2. Preliminary Results and Questions

1. It is known that there exist even very simple polygons which do not allow perfect congruent partitioning (no left over) into pieces of finite complexity for any $N$.

Reference [2] proves that a convex quadrilateral with angles linearly independent over the rationals (for example, the 4 angles could be, in degrees, $\{\alpha_1=180/\sqrt{5}, \alpha_2=180/\sqrt{7}, \alpha_3=180/\sqrt{11}, \alpha_4=360-\alpha_1-\alpha_2-\alpha_3\}$) does not allow a perfect congruent partition for any $N$. 
Remark: [2] actually tries to prove a somewhat stronger result: even pieces with same sets of values for their angles (the sequence of the angles in different pieces could be different, as could be their sizes) fail to fully fill out the specified quadrilateral.

2. It is almost certain that if P convex, the simplest tile that gives the optimal congruent partition for some N need not be convex as well.

Consider the above example due to Friedman [3]. From a 3x7 rectangle, we chop off an isosceles triangle of area 1/2 from one corner, leaving a convex pentagon of area 20.5. For N=5, this pentagon allows a layout of 5 L-tetrominos (a non-convex shape of area 4) which leaves out only 0.5 units of area. There appears to be no 'better' congruent partitioning for N=5 with convex tiles.

3. A Claim: Given a convex P and any N; if P allows a perfect congruent partition of itself into N non-convex pieces each with finitely many sides, then P also allows a perfect congruent partition into N convex pieces with finitely many sides.

This claim holds for N=2. For N=3 and beyond, things are not clear.

A convex region, which allows perfect congruent partition only with non-convex tiles with finitely many sides for some specific N would counter this claim. On the other hand, if the above claim is true in general, that would mark a qualitative difference between perfect and non-perfect congruent partitions. It also puts strong constraints on the complexity of the simplest piece which achieves a perfect congruent partition of any convex polygon (see appendix below) and hence could help design a practical algorithm to check for the existence of perfect congruent partitions for a given convex polygon.

Question: Among all perfect congruent partitions of any given convex P, is the partition that minimizes the sum of perimeters of pieces, a partition into convex tiles?

4. Upper-bounding the area of the input P that gets left-over: The most basic question in this direction could be the following:

What is the shape of a convex region (with boundary not necessarily formed by straight edges) P such that if it is optimally congruent partitioned into 2 convex pieces, the fraction of the area of P that is left over is a maximum?

In other words, If P and the tiles are convex, can there be bounds on how sub-optimal a convex congruent partition into 2 pieces (in general, N pieces) can be?

5. Non-convex tiles: Question: if the tiles are allowed to be non-convex and arbitrarily complex, can we always achieve, for any N, a 'near-perfect' congruent partition – a congruent partition that is arbitrarily close to being a perfect congruent partition? Such a partition makes the fraction of P’s area that goes waste tend to 0. For instance, the N tiles
could be tightly packed spirals radiating from a suitable core region point in the interior; the 'pitch' of each spiral going to zero (causing the spiral to wind around the 'focal region' infinitely many times).

6. **The 3D version of the problem:** If we have 3D regions rather than polygons to be partitioned, is it true that any 3D region can indeed be partitioned into N mutually congruent connected regions for any N with no leftover or the leftover tending to 0 – with the pieces arbitrarily complex and also be densely entangled with one another (a similar scenario is discussed in [4], where an apple is shown to be eaten by 2 species of thin worms with the species avoiding one another – resulting in 2 separate connected regions of infinite complexity). In 2D (item 5 above) where there is no scope for entanglement of pieces, such a partition would be more difficult. Indeed, entanglement of regions could be a feature that could set perfect congruent partitions in higher dimensions apart from 2D.

We could also consider the 3D version of the claim in item 3 above. Consider a 3D region R that does not allow perfect congruent partition into N convex 3D regions. As suggested above, one may well be able to use entanglement to achieve a congruent partition of it into ‘infinitely fibrous’ pieces that is perfect. But is there such an R which cannot have perfect congruent partitions into N convex 3D regions but can be perfect congruent partitioned into N non-convex pieces of finite complexity?

7. **Deciding whether a given polygon P (not necessarily convex) allows a perfect congruent partition into N pieces of finite complexity** is another variant of our problem. The case N=2 has been solved in [4]. For larger N, things appear uncertain, even for convex P.

A possibility: If a convex polygon allows perfect congruent partition into N convex pieces, then a layout could always be created with only a few types of basic topology for any N – a straight chain, star, etc.. If that is indeed the case, the decision of whether a given convex polygon allows perfect congruent partition with N convex pieces, can be achieved by considering a few simple and enumerable cases.

8. Rather than break the input polygon P into congruent pieces, we could try to find a piece such that N congruent copies of it cover the whole of P with the least area of the tiles going waste (parts of the tiles could extend beyond of the boundary of P or may be allowed to overlap among themselves).

9. Instead of a set of N mutually congruent pieces, we could ask about partitioning a given region into N mutually identical sets of pieces (with each set required to have some finite number of pieces).

A simple fact: any triangle can be cut into 4 mutually congruent triangles. And any polygon can be divided into triangles; any m-gon will give m-2 triangles. This implies: if N is 4, we can triangulate any (m-vertex) input polygon P and further divide each resulting triangle into 4 so as to achieve a partition into 4 identical tile-sets
each with m-2 triangles. This is easily generalizable to N = any perfect square. No bit of P goes waste.

10. If we restrict the definition of ‘congruence’ of the pieces to only translation and rotation (this will mean a polygon and its mirror image are not necessarily congruent), how will the problem change?

3. Acknowledgements

The author is grateful to Swami Sarvottamananda (Shreesh Mj) and Br. Swathy Prabhu for discussions.

4. References

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Appendix

Here we derive a simple result on the complexity (number of edges) of tiles; the arguments follow [2].

Claim: If a given convex polygonal polygonal region P allows a perfect congruent partition for any given n with n convex tiles, the complexity of the tile is limited by that of P itself, with no dependence on n.

Proof:

Some notation:

- p – the number of vertices on the full convex polygon P.
- n - the number of mutually congruent convex tiles which perfectly partition the full polygon.
- k - the number of vertices on each tile.
- r - the number of vertices in the layout of tiles which lie on the boundary of the full polygon P, which are not the vertices of P itself (call these the boundary vertices in the layout).
- m- the number of vertices in the layout which lie in the interior of the full polygon (call these the internal vertices in the layout)

We obtain the following relations:
\[ nk \geq 3m + 2r + p \quad (1) \]

\[ n(k - 2) = 2m + r + p - 2 \quad (2) \]

Note: Relation (1) follows from the fact that when tiles are convex, every internal vertex in the layout has at least 3 tiles meeting there; further, at every boundary vertex, at least 2 tiles meet. \( nk \) is the total number of vertices for all the tiles taken together. Equation (2) follows from equating two sums of the angles, measured in units of \( \pi \); at each internal vertex in the layout the sum of angles meeting is \( 2\pi \) and at every boundary vertex, \( \pi \).

(1) and (2) together yield the inequality:

\[ (6 - k) n \geq r - p + 6. \quad (3) \]

We now consider the possibility of the complexity of the tile - its number of edges \( k \), being larger than the complexity of the full polygon \( p \).

So, we set \( k = p + \alpha \) where \( \alpha \) is a +ve integer.

With this substitution, (3) yields, \( 6 - p - \alpha \geq r/n + 6/n - p/n. \)

\[ => p \ (n-1) / n \leq 6 - 6/n - \alpha - r/n. \quad (4) \]

Now, the expression on the right side of (4) cannot be greater than 5 (because, by definition, \( \alpha \geq 1 \)). Since \((n-1)/n \) cannot be less than than \( \frac{1}{2} \) for positive integer \( n \), we get \( p \leq 10 \) for any value of \( n \).

Thus, if the convex tile is to have exactly 1 edge more than \( P \), the complexity of \( P \) and hence, the tile are limited by 10. And if the tile is to have a greater excess of edges \( \alpha \), the right side of (4) has a still lower limit than 5 so \( p \) and \( k \) are limited to 8 and 10 respectively.

**Conclusion:** if a tile in a convex congruent partition is to be more complex than a convex \( P \), neither of them can be very complex. Of course, if \( \alpha \) can be negative (if \( P \) is allowed to be more complex than the tile), \( k \) can be arbitrarily large - but less than \( p \), the complexity of \( P \).

**Note:** for very small values of \( p \) and \( n \), we sometimes have tiles more complex than \( P \) itself: for example, if \( P \) is an equilateral triangle and \( n = 3 \), we can have a perfect congruent partition with 3 quadrilaterals \((k=4 \text{ and } p=3)\).