The $\alpha^2(Z\alpha)^4m$ contributions to the Lamb shift and the fine structure in light muonic atoms

Evgeny Yu. Korzinin
D. I. Mendeleev Institute for Metrology, St.Petersburg, 190005, Russia

Vladimir G. Ivanov
Pulkovo Observatory, St.Petersburg, 196140, Russia

Savely G. Karshenboim
Max-Planck-Institut für Quantenoptik, Garching, 85748, Germany and
Pulkovo Observatory, St.Petersburg, 196140, Russia

Corrections to energy levels in light muonic atoms are investigated in order $\alpha^2(Z\alpha)^4m$. We pay attention to corrections which are specific for muonic atoms and include the electron vacuum polarization loop. In particular, we calculate relativistic and relativistic-recoil two-loop electron vacuum polarization contributions. The results are obtained for the levels with $n = 1, 2$ and in particular for the Lamb shift ($2p_{1/2} - 2s_{1/2}$) and fine-structure intervals ($2p_{3/2} - 2p_{1/2}$) in muonic hydrogen, deuterium, and muonic helium ions.

PACS numbers: 12.20.-m, 31.30.J-, 36.10.Gv, 32.10.Fn

I. INTRODUCTION

Precision studies of light muonic atoms allow a determination of nuclear structure with an accuracy not accessible otherwise. A recent result of the CREMA collaboration on two lines for the Lamb shift muonic hydrogen [1] [2], their current evaluation of the Lamb shift in muonic deuterium and their project on muonic helium Lamb shift necessitate a clarification of the related theory.

The problem becomes of special importance due to a discrepancy of the value of the proton radius [1] derived from the results on hydrogen and deuterium spectroscopy (see, e.g., [2]) and from electron-proton scattering (see, e.g., [3]). The situation is reviewed, e.g., in [4, 5].

A comprehensive compilation of the present theoretical situation on muonic-hydrogen Lamb shift can be found in recent overviews [1, 6–9] (see, also, [10–13]).

A theoretical expression for the Lamb shift in muonic hydrogen comprises a number of terms of a few clearly distinguishable types. Indeed, there are pure QED corrections and corrections which involve proton structure. The QED corrections may be of the same type as in ordinary hydrogen and those need only a rescaling with a substitution of the electron mass for the muon one. (Since the muon-proton mass ratio is about 1/9, while the electron-proton mass ratio is about 1/2000, one has to remember, indeed, higher importance of the recoil corrections in muonic hydrogen, as well as various reduced-mass effects.) A review on the Lamb shift in ordinary hydrogen can be found in [4, 13, 14].

In addition to those rescaled terms, there is a number of specific muonic-hydrogen contributions, which are summarized in Table I. They come from Feynman diagrams with closed electron loops.

| Term       | $\Delta E(2p_{1/2} - 2s_{1/2})$ [meV] |
|------------|-----------------------------------|
| $\mu\mathrm{H}$ | $\mu\mathrm{D}$ | $\mu^4\mathrm{He}$ | $\mu^4\mathrm{He}$ |
| eVP1       | 205.02612 | 227.65645 | 1642.3954 | 1666.2940 |
| eVP2       | 1.65885  | 1.83804  | 13.0843  | 13.2769  |
| eVP3       | 0.00752  | 0.00842(7)| 0.073(3)* | 0.074(3) |
| LbL        | $-0.00089(2)$ | $-0.00996(2)$ | $-0.0134(6)^*$ | $-0.0136(6)$ |

TABLE I: Specific contributions to the Lamb shift $\Delta E(2p_{1/2} - 2s_{1/2})$ in light muonic atoms up to the order $\alpha^2m$: hydrogen, deuterium, helium-3 and helium-4 ions. The results concern one-loop, two-loop, and three-loop eVP contributions and well as the contribution of the light-by-light scattering block (Fig. 1). The results marked with asterisk are obtained in this paper.

The results obtained up to date for muonic hydrogen include contributions of the one-loop, two-loop [10] and three-loop [12, 13] electronic vacuum polarization (eVP) as well as various contributions of the electronic block of the light-by-light scattering (LbL) [14, 20]. Except for the one-loop eVP contributions, the results are available only for the leading terms. For the one-loop contribution additionally to the leading non-relativistic term [21, 22], also a relativistic non-recoil [10, 12, 23, 24] and recoil [25, 26] terms are known. The results are summarized in Table I. The $n$-loop results are complete in a sense that they include all possible contributions of the related order with $n'$-eVP potentials ($n' \leq n$) and their iterations. E.g., the eVP2 result in Table I consists of a contribution of the Källen-Sabry potential and of a double-iteration term with the Uehling potential.
Most of the results mentioned are calculated in the leading non-relativistic approximation and thus do not contribute to the fine structure. The only correction among them, relevant for the fine structure, is the one-loop relativistic contribution \[10, 12, 23, 24\].

The two-loop eVP corrections, as mentioned, are known only in the leading order, which is \(\alpha^2(Z\alpha)^2m\), where \(Z\) is the nuclear charge and \(m\) is the muon mass, and here we consider relativistic corrections to them. They are of the order of \(\alpha^2(Z\alpha)^4m\). In muonic atoms a ratio of the muon and nuclear mass is small, but not very small and in particular in muonic hydrogen \(m/M \sim 0.1\). That means that any more or less accurate calculation should also involve recoil corrections. Here we consider them exactly in \(m/M\), which is the ratio of the muon and nuclear masses.

While the main purpose of this paper is to calculate two-loop relativistic and relativistic recoil eVP contributions, we also analyze all other sources of corrections of order of \(\alpha^2(Z\alpha)^4m\) and \(\alpha^2(Z\alpha)^2m^2/M\).

In principle, some of the \(\alpha^2(Z\alpha)^4m\) contributions can appear from the higher-order LbL contributions. The leading LbL term, presented in Table II, includes the Wichmann-Kroll contribution (Fig. 1a) in order \(\alpha (Z\alpha)^3m\) \[13, 19, 20, 27–31\], the virtual-Delbrück-scattering contribution (Fig. 1d) \[12, 19, 20, 28, 29, 32, 33\] and the third contribution, which does not have a specific ‘common’ name (Fig. 1).

\[\begin{align*}
\mu & \quad (a) \\
e & \quad (b) \\
\mu & \quad (c)
\end{align*}\]

FIG. 1: Leading-order diagrams, which include the electronic light-by-light scattering block. The horizontal double line is for the reduced Green function of a muon at the Coulomb field.

Higher-order corrections due to the addition of a radiative correction to the electron loop or the eVP to either line will add an extra factor of \(\alpha\). However, the LbL term is so uncertain that such a correction should be below uncertainty. Besides, it is rather substantially smaller than the two-loop eVP \(\alpha^2(Z\alpha)^4m\) contribution studied in this paper.

II. TWO-LOOP EVP RELATIVISTIC RECOIL CONTRIBUTION

A calculation of eVP non-relativistic contributions to the energy levels of a two-body muonic atom can be performed in terms of the non-relativistic perturbation theory (NRPT). The only potentials in such a calculation are the Coulomb and eVP potentials. While the Coulomb problem is considered non-perturbatively, all the eVP potentials (see Fig. 2) are considered as a perturbation. Non-relativistic two-loop \[10\] and three-loop \[15, 16\] eVP terms were found some time ago within such an NRPT framework.

\[\begin{align*}
\Delta E &= \langle \Psi_{nl} | \delta V | \Psi_{nl} \rangle + \langle \Psi_{nl} | \delta V G'_{nl} \delta V | \Psi_{nl} \rangle \\
&\quad + \langle \Psi_{nl} | \delta V G'_{nl} \delta V G'_{nl} \delta V | \Psi_{nl} \rangle \\
&\quad - \langle \Psi_{nl} | \delta V | \Psi_{nl} \rangle \times \langle \Psi_{nl} | \delta V G'_{nl} G'_{nl} \delta V | \Psi_{nl} \rangle.
\end{align*}\]

where \(\Psi_{nl}\) is the non-relativistic Coulomb wave function of the \(nl\)-state in hydrogenic atom (see, e.g., \[30\]), \(n\) is the principal quantum number, \(l\) is the orbital quantum number. Here, \(G'_{nl}\) stands for the nonrelativistic reduced Coulomb Green function.

The expression is valid for any central potential. In case of the spin-orbit and spin-spin interactions the identity needs some corrections. The interaction of the muon
spin and orbit with the nuclear spin is neglected, since it vanishes after we average over the hyperfine structure. When necessary, the hyperfine effects can be studied separately.

As for the spin-orbit interaction, we have to apply the wave functions the radial part of which is the same as that of \( \Psi_{nl} \), while the angular and spin part is chosen to realize the physical basis with eigen states of the muon angular momentum and its projections. Indeed, the matrix elements for the energy do not depend on the projection.

Strictly speaking, the contributions to the perturbation of the non-relativistic Hamiltonian, denoted as \( \delta V \), are not necessary potentials, since they may include momentum (see below) and thus be non-local. That does not change the equations and for simplicity we still use for then a term ‘effective potentials’.

For the one-loop eVP contribution the derivation of the NRPT equations was done in detail in [26]. A proper choice of the gauge of the photon propagator \( D_{\mu \nu} \) allows to avoid retardation effects in the \( D_{00} \) component of the one-photon exchange and neglect those effects in the \( D_{ij} \) component, since the retardation effects produce there only corrections in the higher order in \( (Z\alpha) \). Meanwhile the two-photon exchange contributions lead to \( (Z\alpha)^2 m^2/M \) terms only. Thus, the application of the NRPT approach to calculate \( \alpha(Z\alpha)^4 m \) exactly in \( m/M \) is validated.

The evaluation is based on the eVP correction to the photon propagator, which is proportional to the dispersion integral (see [26] for details)

\[
D_{\mu \nu}^{\text{VP}}(k) \propto \int_0^1 dv \rho_e(v) \frac{1}{k^2 - \lambda^2},
\]

where the dispersion parameter serves as an effective photon mass

\[
\lambda^2 = \frac{4m^2}{1 - v^2},
\]

and the dispersion function \( \rho_e \) depends on the contribution we are to study. In particular, for the one-loop eVP calculation the dispersion density is

\[
\rho_1(v) = \frac{3}{\pi} \frac{v^2(1 - v^2/3)}{1 - v^2}.
\]

The effective potentials at order \( \alpha^0 \) are determined for a Coulomb-bound two-body system by the standard Breit equation [37, 38]

\[
V_{Br}(r) = - \left( \frac{1}{m^3} + \frac{1}{M^3} \right) \frac{P^4}{8} + \frac{Z\alpha}{8} \left( \frac{1}{m^2} + \frac{1}{M^2} \right) 4\pi \delta^3(r) + \frac{Z\alpha}{4m^2} \left( \frac{1}{r^3} \frac{\mathbf{L} \cdot \sigma}{r^3} + \frac{2}{2mM} 4\pi \delta^3(r) \right) + \frac{Z\alpha}{2mM} \left[ \frac{1}{r^3} \mathbf{L}^2 - \frac{P^2}{r^2} \frac{1}{r} - \frac{1}{r} P^2 \right],
\]

and considered as a perturbation of the unperturbed problem of the non-relativistic Schrödinger equation with the Coulomb potential

\[
V_C(r) = - \frac{Z\alpha}{r}.
\]

Here \( M \) stands for the nuclear mass, \( m \) is for the muon mass \( m_\mu \), \( s = 1/2 \) and \( \mathbf{L} \) and \( \mathbf{P} \) are spin and orbital moments of muon, \( \mathbf{p} \) is the momentum operator and the relativistic units in which \( c = \hbar = 1 \) are applied. Here \( Z \) is the nuclear change and \( M \) is the nuclear mass and the final expression is valid for the nuclear spin 1/2, assuming that we average over the nuclear spin (i.e. over the hyperfine structure).

Those in order \( \alpha^1 \) are [38]

\[
V_{Br}^{\text{VP}}(r) = \left( \frac{1}{8m^2} + \frac{1}{8M^2} \right) \nabla^2 V_U + \left( \frac{1}{4m^2} + \frac{1}{2mM} \right) \frac{V_U' \mathbf{L} \cdot \sigma}{r} + \frac{1}{2mM} \nabla^2 \left[ V_U - \frac{1}{4} (rV_U')' \right] + \frac{1}{2mM} \left( \frac{V_U'}{r} \mathbf{L}^2 + \frac{P^2}{2}(V_U - rV_U') \right) + (V_U - rV_U') \frac{P^2}{2},
\]

where \( V_U \) is the Uehling potential

\[
V_U(r) = -Z\alpha \int_0^1 dv \rho_1(v) e^{-\lambda r}/r.
\]

Graphically, the related effective potential is presented in Fig. 3 and the diagrams for the calculation of the relativistic recoil corrections in order \( \alpha(Z\alpha)^4 m \) (exactly in \( m/M \)) are depicted in Fig. 4.

![FIG. 3: Effective potentials for the NRPT calculation of the relativistic recoil one-loop eVP contribution.](image-url)

The relativistic recoil eVP correction of order \( \alpha(Z\alpha)^4 m \) originates from terms of the first and second order of NRPT [11] with the effective potential defined in Eqs. (5) and (7). To generalize the result and calculate relativistic recoil two-loop eVP corrections, we have to calculate terms of the second (see Fig. 5) and third (see Fig. 5) order with the same potentials and the first and the second order with the effective two-loop potential (see Fig. 5a and b). The latter can be easily
obtained from the related one-loop potentials \( 7 \) by a substitution of the two-loop eVP dispersion density for the one-loop one.

The two-loop eVP dispersion function for the reducible part is \( 34, 35 \)

\[
\rho_{11}(v) = -\frac{1}{9} \left( \frac{\alpha}{\pi} \right)^2 \frac{v^2 (1 - v^2 / 3)}{1 - v^2} \times \left\{ 16 - 6v^2 + 3v(3 - v^2) \ln \left( \frac{1 - v}{1 + v} \right) \right\}, \tag{9}
\]

and for the irreducible one it takes the form \( 34, 41 \)

\[
\rho_{2}(v) = \frac{2}{3} \left( \frac{\alpha}{\pi} \right)^2 \frac{v}{1 - v^2} \times \left\{ (3 - v^2)(1 + v^2) \right\}
\]

\[
\left[ \text{Li}_2 \left( \frac{1 - v}{1 + v} \right) + 2\text{Li}_2 \left( \frac{1 - v}{1 + v} \right) \right]
+ \ln \left( \frac{1 + v}{1 - v} \right) \left( \frac{3}{2} \ln \left( \frac{1 + v}{2} \right) - \ln(v) \right)
+ \left( \frac{11}{16} (3 - v^2)(1 + v^2) + \frac{1}{4} v^4 \right) \ln \left( \frac{1 + v}{1 - v} \right)
+ \frac{3}{8} v(3 - v^2) \ln \left( \frac{1 - v^2}{4} \right) - 2v(3 - v^2) \ln(v)
+ \frac{3}{8} v(3 - v^2) \right\}, \tag{10}
\]

where \( \text{Li}_2 \) is the Euler dilogarithm \( 42 \).

The evaluation of the contributions in Figs. \( 5 \) and \( b \) of the reducible and irreducible parts of the two-loop eVP is similar to the related one-loop eVP contributions in Fig. \( 4 \) and immediately leads to a result. The non-recoil results of order \( \alpha^2(Z\alpha)^4 m_e c^2 \) for muonic hydrogen are summarized in Table \( 11 \) while the recoil corrections are presented in Table \( 11 \).

The evaluation of contributions related to Fig. \( 5 \), which presents terms of the third order of NRPT \( 11 \), is somewhat more complicated. The related calculation involves integrations with the radial parts of the reduced Green function of the non-relativistic Coulomb problem.

We use two representations of the reduced Coulomb Green function \( G^{\prime}_{nl} \), which allow us to provide a cross-check of our calculations. The most fruitful is a representation for the Coulomb Green function developed in \( 43 \). The expressions we applied for the radial part of the reduced Coulomb Green functions are \( 44, 45 \) (see also \( 10 \))

\[
\tilde{G}_{1s}(r, r') = 4Z\alpha m_e^2 \exp \left( \frac{z_> + z_<}{2} \right) \left\{ \frac{1}{z_>} + \frac{1}{z_<} \right. \\
\left. + \frac{7}{2} - \frac{z_> + z_<}{2} \right\} + \text{Ei}(z_<) - 2C \\
- \ln(z_>z_<) - \frac{v_{z_<}}{z_<}, \tag{11}
\]
\[ \tilde{G}_{nl}(r, r') = \frac{n^2}{(Z\alpha)^2 m_r} \left\{ \sum_{k, l, m} \frac{k}{k-n} R_{kl}(n; r) R_{kl}(n; r') + \frac{2}{3} R_n(n; r) R_n(n; r') + r R_n(n; r) R_n(n; r') + r' R_n(n; r') R_n(n; r) \right\}, \quad (14) \]

and \( R_n(n; r) \) stands for the radial part of the standard wave function of the non-relativistic Coulomb problem (see, e.g., [36]).

An evaluation of the relativistic corrections to the Hamiltonian \([46]\) and \([47]\) involves various differentiations and we consider them in the Appendix. The Laplacian of the Uehling potential is considered in App. A, the differentiation of the Green function in Sturmian representation is discussed in App. B and the differentiation procedure applied to the Green function with \( r_{\mu} \) is summarized in App. C. Such a special treatment of derivatives allows us to simplify the evaluation. The final results for relativistic non-recoil and recoil eVP contributions in order \( \alpha^2(Z\alpha)^4 m \) for the low-lying states in muonic hydrogen are summarized in Tables II and III.

As for the other light muonic atoms, our results in order \( \alpha^2(Z\alpha)^4 m \) and \( \alpha^2(Z\alpha)^4 m_r c^2 (m_r/M) \) are presented in Tables IV and V.

| Atom | \( \Delta E(nl_j) \) |
|------|-------------------|
| \( 1s \) | \( 2s \) | \( 2p_{1/2} \) | \( 2p_{3/2} \) |
| \( \mu H \) | -1.07 | -1.071 | -0.3383 | -0.00370 |
| \( \mu D \) | -1.13 | -0.180 | -0.0370 | -0.00415 |
| \( \mu^3 He \) | -2.21 | -0.347 | -0.113 | -0.0163 |
| \( \mu^4 He \) | -2.22 | -0.350 | -0.115 | -0.0165 |

TABLE IV: Relativistic eVP corrections (Fig. 5) for the low-lying levels in muonic hydrogen in the external field approximation. The result is for the Schrödinger problem with the reduced mass. The units are \((\alpha/\pi)^2 (Z\alpha)^4 m_r c^2 \).
The evaluation based on the Breit-type approach allows to obtain recoil effects in order $\alpha^2(Z\alpha)^4m$ exactly in $m/M$, and we have done here such a calculation. However, we have also performed another evaluation, applying an alternative technique, which allows terms linear in $m/M$ only. The details will be published elsewhere [48]. The results obtained within these two approaches are consistent. We thus consider our results on the relativistic recoil two-loop corrections as well established.

### III. CONCLUSIONS

Indeed, the most interesting are not the shifts of energy of any level by itself, but rather two intervals, namely, the Lamb-shift ($2p_{1/2} - 2s_{1/2}$) and the fine-structure ($2p_{3/2} - 2p_{1/2}$) intervals. The results for the two-loop eVP contributions (including the previously known leading term of order $\alpha^2(Z\alpha)^2m$ [10]) are summarized in Tables V and VI.

The results for different muonic atoms are obtained by the same method, however, following [13, 50], the so-called Zitterbewegung term is not included for the muonic deuterium and helium-4 ion (cf. [23, 26]).

| Atom | $\Delta E(n_l)$ [meV] |
|------|----------------------|
|      | 1s  | 2s  | $2p_{1/2}$ | $2p_{3/2}$ |
| $\mu H$   | 1.05 | 0.176 | 0.00501 | 0.00197 |
| $\mu D$   | 1.13 | 0.191 | 0.00453 | 0.00278 |
| $\mu^3$He | 1.39 | 0.276 | 0.00896 | 0.00544 |
| $\mu^4$He | 1.49 | 0.280 | 0.00852 | 0.00581 |

| Atom | $\Delta E(2p_{3/2} - 2p_{1/2})$ [meV] |
|------|----------------------------------|
|      | $\mu H$ | $\mu D$ | $\mu^3$He | $\mu^4$He |
| $\alpha^2(Z\alpha)^2m$ | 0.0000438 | 0.000502 | 0.00242 | 0.00247 |
| $\alpha^2(Z\alpha)^4m^*$ | $-4.5 \times 10^{-7}$ | $-1.4 \times 10^{-7}$ | $-3.2 \times 10^{-6}$ | $-1.9 \times 10^{-6}$ |
| Total | 0.0000433 | 0.000501 | 0.00242 | 0.00247 |

TABLE V: Relativistic recoil corrections in order $\alpha^2(Z\alpha)^4m$ (Fig. 3 for the low-lying levels in muonic hydrogen. The units are $(\alpha/\pi)^2(Z\alpha)^4m^2(M/m)$.

TABLE VI: The second-order eVP contributions to the Lamb shift $2p_{1/2} - 2s_{1/2}$ in light muonic atoms. The units are meV. The results marked with asterisk are obtained in this paper.

TABLE VII: The second-order eVP contributions to the fine-structure interval $2p_{3/2} - 2p_{1/2}$ in light muonic atoms. The units are meV. The results marked with asterisk are obtained in this paper.

It is interesting to compare the obtained above two-loop eVP relativistic contributions with other contributions of the same order, i.e. of order $\alpha^2(Z\alpha)^4m$. To conclude let us briefly overview such contributions.

Indeed, first of all there are rescaled contributions of the electronic Lamb shift which are well known (see, e.g., [4, 13, 14]). Additional specific contributions to the Lamb shift in muonic atoms in order $\alpha^2(Z\alpha)^4m$ are presented in Fig. 6.

As we mention in the introduction, one may also consider radiative corrections to the block of the light-by-light scattering, which modify the Wichmann-Kroll potential, and various Uehling corrections to the leading Wichmann-Kroll contribution. Since the leading Wichmann-Kroll contributions is very small and the uncertainty of the complete light-by-light scattering contribution is not small, we expect that the $\alpha$ corrections to the leading Wichmann-Kroll contribution are negligible and below that uncertainty. The related diagrams are not presented in Fig. 6. All the others are. They are split into several classes.

We remind that there are non-specific contributions of order $\alpha(Z\alpha)^4m$ which are obtained by the rescaling [4, 13, 14]. Meantime certain corrections to them with an additional factor of $\alpha$ are already specific. The typical diagrams are depicted in Fig. 6, a and b.

The $b$-type contributions are due to Uehling corrections to the contribution of the muon VP. A similar contribution comes from the Uehling correction to the hadronic vacuum polarization contribution (see Fig. 6).

We note that rescaling should include a substitution of the mass (electron $\rightarrow$ muon), but it keeps the same expression in terms of $\alpha$ and $Z\alpha$. Technically, that means that we keep the same expressions for the radiative corrections and in particular for the anomalous magnetic moment of the muon. The effects which contribute to the difference in the values of the anomalous magnetic moments, $a_e$ and $a_\mu$, should be considered separately. The contributions to $a_\mu - a_e$ appear in order $\alpha^2$ and most important of them are due to electronic or hadronic VP
(on various contributions to the anomalous magnetic moment of muon see [52, 53]. The characteristic diagrams are presented in Fig. 6d and e. Those diagrams are also responsible for a specific contribution to the slope of the Dirac form factor in order $\alpha^2$ and thus for the related contribution to the Lamb shift.

The related contributions are summarized in Tables VII and IX. Most of the contributions have been known before.

The type-a contributions to the Lamb shift and fine structure were considered in [54]. The b contributions were found in [55] (see also [13]) and [9] (see also [6, 17, 39]).

![FIG. 6: Diagrams for various classes of specific corrections, contributing at order $\alpha^2 (Z\alpha)^4 m$. $h$ stands for hadronic vacuum polarization](image)

| Atom | $\mu H$ | $\mu D$ | $\mu^3 He$ | $\mu^4 He$ |
|------|---------|---------|------------|------------|
| (a)  | 0.0105  | 0.0127  | 0.066      | 0.624      |
| (d)  | 0.0893  | 0.0991  | 1.64       | 1.67       |
| (e)  | 0.0010(1)| 0.0012(1)| 0.019(2)   | 0.020(2)   |
| eVP2 | 0.0433  | 0.0501  | 2.42       | 2.47       |
| Total| 0.144   | 0.164   | 4.69       | 4.78       |

**TABLE VIII:** Various $\alpha^2 (Z\alpha)^4 m$ contributions the the Lamb shift $(2p_{1/2} - 2s_{1/2})$ in light muonic atoms. The units are $\mu eV$. Notation follows Fig. 6. The $eVP2$ term is the correction of order $\alpha^2 (Z\alpha)^4 m$ of Table VII. The results marked with asterisk are obtained in this paper.

| Atom | $\mu H$ | $\mu D$ | $\mu^3 He$ | $\mu^4 He$ |
|------|---------|---------|------------|------------|
| (a)  | 0.0010  | 0.0127  | 0.066      | 0.624      |
| (d)  | 0.0893  | 0.0991  | 1.64       | 1.67       |
| (e)  | 0.0010(1)| 0.0012(1)| 0.019(2)   | 0.020(2)   |
| eVP2 | 0.0433  | 0.0501  | 2.42       | 2.47       |
| Total| 0.144   | 0.164   | 4.69       | 4.78       |

**TABLE IX:** Various $\alpha^2 (Z\alpha)^4 m$ contributions to the fine-structure interval $2p_{3/2} - 2p_{1/2}$ in light muonic atoms. The units are $\mu eV$. Notation follows Fig. 6. The $eVP2$ term is the correction of order $\alpha^2 (Z\alpha)^4 m$ of Table VII. The results marked with asterisk are obtained in this paper.

The contribution, which involves the hadronic vacuum polarization in the Coulomb photon, (Fig. 6f) is calculated in this paper. In particular, we found

$$\Delta E(2s, \mu H) = \frac{\alpha}{\pi} 3.2248... \Delta E_{had}^{(0)} (2s, \mu H),$$

where $\Delta E_{had}^{(0)} (\mu H)$ is the leading hadronic contribution, considered in the appendix (Sect. D).

The contributions $d$ and $e$ are considered for muonic hydrogen in [10, 13, 56, 57]. In particular, there is a result of [60] for the $e$ contributions. We have recalculated it and our result is different from that in [60]. The details of our calculations as well as a comparison with the earlier result is presented in Appendix E.

The corrections in Tables VII and IX are leading non-relativistic corrections in $Z\alpha$ the corresponding order. They are calculated by means of the non-relativistic atomic physics, i.e. the related wave functions and Coulomb Green functions are non-relativistic. That means that all recoil effects are covered by the reduced
mass. We note, that the internal integration of the radiative loops for the anomalous magnetic moment and the slope of the Dirac form factor are relativistic. In principle, additionally to those diagrams one has to take into account diagrams similar to those in Fig. 4a, d and e which are radiative corrections to the nuclear line. However, they are incorporated into the proton form factors and should be considered separately.

The complete result of the $a^2(Z\alpha)^4 m$ (see Table VIII) is comparable with the theoretical and experimental uncertainty for the Lamb shift in muonic hydrogen and has to be taken into account.

Acknowledgments

This work was supported in part by DFG under grant GZ: HA 1457/7-2 and RFBR under grant # 12-02-31741. A part of the work was done during a stay of VGI and EYK at the Max-Planck-Institut für Quantenoptik, and they are grateful to it for its warm hospitality.

Appendix A: Differentiation of the Uehling potential

Calculation of the Laplacian of the Uehling and Källen-Sabry potentials involves singularities and may cost certain troubles.

Following [10, 33, 52], we apply the identity
\[ \nabla^2 V(r) = \int_0^\infty dv\rho(v) \left( 4\pi\delta^3(r) - \frac{\lambda^2}{r} e^{-\lambda r} \right). \]

Appendix B: Differentiation for Sturmian basis functions

While calculating the third order of the NRPT we have to deal with $\Psi_{nlm} \mathbf{p}^4 \tilde{G}$, $\mathbf{p}^4 \tilde{G}$, $\Psi_{nlm} \mathbf{p}^2 \tilde{G}$, and $\tilde{G} A \mathbf{p}^3 \tilde{G}$, where $\Psi_{nlm}$ is the hydrogen wave function and $A$ is an operator, diagonal in coordinate space. All these expressions require only a calculation of $\mathbf{p}^2 \tilde{G}$ in closed form, since for $\mathbf{p}^1$ we can consider one $\mathbf{p}^2$ as acting on the right, while the other as acting on the left.

Differentiation of the Coulomb wave function is obvious
\[ \mathbf{p}^2 \Psi_{nlm}(r) = 2m_r \left( \frac{Z\alpha}{r} + E_n \right) \Psi_{nlm}(r). \]  
A Sturmian basis function
\[ \Phi_{klm}(n, r) = R_{kl}(n; r) Y_{lm}(\Omega), \]
is a solution of the related Sturm-Liouville problem, differential equation for which can be rewritten as
\[ \mathbf{p}^2 \Phi_{klm}(n, r) = 2m_r \left( \frac{Z\alpha}{n} \frac{r}{r} + E_n \right) \Phi_{klm}(n, r). \]  
These identities allow to carry out any differentiation of the Green function, presented in terms of the Sturmian basis, required for the calculation in the third order of NRPT.

Appendix C: Differentiation of the Green function

The representation of the Green function in terms of $r_s$ and $r_c$ has certain advantages, however its differentiation is somewhat complicated. To avoid it we used a trick described below. It may be applied to any representation of the Green function.

First we note that the NRPT expression was previously applied in a certain order of the expansion. Having in mind a calculation of operator $\mathbf{p}^4/m_r^2$, we consider now not a single term of the required order $a^2$, but the sum of all the terms up the second order in $a$. The sum of all the terms, which include $\mathbf{p}^4/m_r^2$, is
\[ \Delta E_4 = \langle \Psi_{nl} | \frac{\mathbf{p}^4}{m_r^2} | \Psi_{nl} \rangle \]
\[ + \langle \Psi_{nl} | \frac{\mathbf{p}^4}{m_r^2} G_{nl} V_U | \Psi_{nl} \rangle + \langle \Psi_{nl} | V_U G_{nl} \frac{\mathbf{p}^4}{m_r^2} | \Psi_{nl} \rangle \]
\[ + \langle \Psi_{nl} | \frac{\mathbf{p}^4}{m_r^2} G_{nl} V_U G_{nl} V_U | \Psi_{nl} \rangle \]
\[ - \langle \Psi_{nl} | V_U G_{nl} V_U | \Psi_{nl} \rangle \cdot \langle \Psi_{nl} | \frac{\mathbf{p}^4}{m_r^2} G_{nl} G_{nl} V_U | \Psi_{nl} \rangle \]
\[ + \langle \Psi_{nl} | V_U G_{nl} V_U G_{nl} \frac{\mathbf{p}^4}{m_r^2} | \Psi_{nl} \rangle \]
\[ - \langle \Psi_{nl} | V_U G_{nl} G_{nl} \frac{\mathbf{p}^4}{m_r^2} | \Psi_{nl} \rangle \cdot \langle \Psi_{nl} | V_U G_{nl} G_{nl} V_U | \Psi_{nl} \rangle. \]  
To find $\Delta E_4$ one can consider a solution of the Coulomb-Uehling problem
\[ \left( \frac{\mathbf{p}^2}{2m_r} + V_C + V_U \right) | \Psi_{nl}^{CU} \rangle = E_{CU} | \Psi_{nl}^{CU} \rangle. \]  
The energy and wave function can be presented in terms of series
\[ E_{CU} = E^{(0)} + \alpha E^{(1)} + \alpha^2 E^{(2)} + \ldots \]  
and
\[ \Psi_{nl}^{CU} = \Psi_{nl}^{(0)} + \alpha \Psi_{nl}^{(1)} + \alpha^2 \Psi_{nl}^{(2)} + \ldots. \]  
Indeed, we can find $E_{CU}$ and $\Psi_{nl}^{CU}$ only using a perturbation theory with the related leading terms that are the result of solving a pure Coulomb problem.
Meantime, we note that \( \Delta E_4 \) has in these terms a simple form

\[
\Delta E_4 = \langle \Psi_{nl}^{CU} | \frac{P_r^3}{m_r^2} | \Psi_{nl}^{CU} \rangle ,
\]

which after applying identity \( \text{C2} \) can we re-written as

\[
\Delta E_4 = 4 \langle \Psi_{nl}^{CU} | (E_{CU} - V_C - V_U)^2 | \Psi_{nl}^{CU} \rangle .
\]

To obtain \( \Delta E_4 \) one still has to apply the perturbative expressions for \( E_{CU} \) and \( m_r \), however, the further evaluation does not include any derivatives anymore.

A calculation of contributions of \( \Delta p \) can be done similarly.

**Appendix D: Leading contribution of the hadronic vacuum polarization**

In the leading order the hadronic vacuum polarization contribution (see Fig. 7) is determined by the value of polarizability at zero momentum transfer

\[
\begin{array}{c}
\frac{\mathcal{P}_{\text{hadr}}(-k^2)}{-k^4} |_{k^2=0} = - \int_{(2m_s)^2}^{\infty} ds \rho_{\text{hadr}}(s) \frac{s}{s},
\end{array}
\]

where the dispersion density function can be directly obtained from experiment by measuring, e.g., the cross section of \( e^+e^- \) annihilation into hadrons. The leading contribution has roughly order \( \alpha(Z\alpha)^4m \), but it is additionally suppressed by a factor \( 4m_s^2/m^2 \). It was calculated previously for a number of occasions \[52\] (see \[61\] for details). Here we recalculate it.

\[
\begin{array}{c}
\text{FIG. 7: The leading hadronic VP contribution to the Lamb shift}
\end{array}
\]

For the calculation we use a model of the dispersion function applied in \[62\]. Indeed, we have to update parameters \[53\] for the hadronic resonances. Following \[62\], we estimate the uncertainty at the level of few percent. The result of the hadronic vacuum polarization contribution for the \( ns \) state is

\[
\Delta E_{\text{hadr}}^{(0)}(nl) = -0.169(16) \frac{\alpha(Z\alpha)^4 m^3}{\pi n^3} m_s^2 \delta_{00},
\]

which corresponds to 0.0106(11) meV for the \( 2p_{1/2}-2s_{1/2} \) splitting in muonic hydrogen, which is consistent with the previous calculations \[53\] \[60\].

The dominant contribution, which is roughly two thirds of the total one, comes from the pion contributions, which is sufficiently described by the \( \rho \)-meson (see, e.g., \[64\]). The situation is very similar to that for the hadronic contribution to the anomalous magnetic moment of a muon and for the muonium hyperfine splitting.

If necessary, the leading term can be calculated with accuracy comparable with that for the anomalous magnetic moment of muon (see, e.g., \[53\] \[65\]) or the muonium hyperfine interval (see, e.g., \[66\]). That should provide uncertainty below 1%. However, to calculate higher-order effects, related to diagrams in Fig. 6 and e, the model considered here is sufficient.

**Appendix E: Muon electric form factor with insertion of the hadronic vacuum polarization**

The insertion of the hadronic vacuum polarization into the muon vertex (see Fig. 8) on the mass shell affect both Dirac \( (F_1) \) and Pauli \( (F_2) \) form factors. Those induce the contributions to the energy. The former is determined by the slope of the Dirac form factor \( \partial F_1(q^2)/\partial q^2 \) at \( q = 0 \), and the latter is determined by the value \( F_2(0) \), which is the anomalous magnetic moment of the muon.

While we agree with \[60\] on the calculation of the \( F_2 \) contribution, we do not agree on the \( F_1 \) contribution. Any vacuum polarization contribution into the slope can be described by integrating the Dirac form factor with a non-zero photon mass \( \sqrt{s} \) with a dispersion density function. The slope is of the form (cf. (11.3.25) in \[67\])

\[
\begin{array}{c}
\partial_q^2 F_1(q^2) |_{q^2=0} = \frac{\alpha}{2\pi m^2} \int_0^1 dz \left( 1 - \frac{z^3}{3D} \right) + \left( 1 - \frac{z}{2} \right) \frac{6}{D^2} \left( 1 - 4z + z^2 \right),
\end{array}
\]

where

\[
D = (1 - z)^2 + z \frac{s}{m^2} .
\]

This expression does not agree with \[60\]. Actually in each reference of \(60\) a different expression for the slope is presented and our does not agree with any of them.

To check \( \text{E1} \) and alternative expressions from \[60\] we performed several tests. First, we reproduced the well-known infrared logarithm in the Dirac form factor with \( s \to 0 \). Only one of three expressions in \[60\] reproduced it. Next, we considered a contribution of insertion of the muon VP into the muon vertex. It is indeed well known and we reproduced the known result \[68\] from \( \text{E1} \), but not from the expressions in \[60\]. Our expression \( \text{E1} \) is consistent with (11.3.25) in \[67\].

After those checks we calculated the contribution into the slope of the Dirac form factor from diagrams in Fig. 8 using the model of the hadronic VP density presented in Appendix F. Our result is presented in Table VIII. It disagrees with results published in \[60\] as
well as with those obtained by us from their expressions for the slope of the Dirac form factor. We believe we have performed a sufficient number of tests to rely on our results.

[1] R. Pohl, A. Antognini, F. Nez, F.D. Amaro, F. Biraben, J.M.R. Cardoso, D.S. Covita, A. Dax, S. Dhawan, L.M.P. Fernandes, A. Giesen, T. Graf, T.W. Hänsch, P. Indelicato, L. Julien, Cheng-Yang Kao, P. Knowles, E.-O. Le Bigot, Yi-Wei Liu, J.A.M. Lopes, L. Ludhova, C.M.B. Monteiro, F. Mulhauser, T. Nebel, P. Rabinowitz, J.M.F. dos Santos, L.A. Schaller, K. Schuhmann, C. Schwob, D. Taquq, J.F.C.A. Veloso and F. Kottmann, Nature (London) 466, 213 (2010);
A. Antognini, F. Nez, K. Schuhmann, F.D. Amaro, F. Biraben, J.M.R. Cardoso, D.S. Covita, A. Dax, S. Dhawan, M. Diepold, L.M.P. Fernandes, A. Giesen, A.L. Gouvea, T. Graf, T.W. Hänsch, P. Indelicato, L. Julien, Cheng-Yang Kao, P. Knowles, F. Kottmann, E.-O. Le Bigot, Yi-Wei Liu, J.A.M. Lopes, L. Ludhova, C.M.B. Monteiro, F. Mulhauser, T. Nebel, P. Rabinowitz, J.M.F. dos Santos, L.A. Schaller, C. Schwob, D. Taquq, J.F.C.A. Veloso, J. Vogelsang, R. Pohl, Science, 339, 417 (2013).

[2] R. de Beauvoir, F. Nez, B. Cagnac, F. Biraben, D. Touahri, L. Hilico, O. Acerf, A. Clairon and J. J. Zondy, Phys. Rev. Lett. 78, 440 (1997);
C. Schwob, L. Jozefowski, B. de Beauvoir, L. Hilico, F. Nez, L. Julien and F. Biraben, Phys. Rev. Lett. 82, 4960 (1999); 86, 4193 (2001) (Errata).

[3] J.C. Bernauer, P. Achenbach, C. Ayerbe Gayoso, R. Böhm, D. Bosnar, P. Indelicato, F. Nez, L. Julien and F. Biraben, Phys. Rev. Lett. 105, 242001 (2010);

[4] P.J. Mohr, B.N. Taylor, and D.B. Newell, Rev. Mod. Phys. 84, 1527 (2012).

[5] S.G. Karshenboim, Annalen der Physik 525, 472 (2013).

[6] A. Antognini, F. Kottmann, F. Biraben, P. Indelicato, F. Nez, R. Pohl, Ann. Phys. 331, 127 (2013).

[7] U.D. Jentschura, Ann.Phys.(N.Y.) 326, 500 (2011).

[8] E. Borie, Ann. Phys. 327, 733 (2012).

[9] P. Indelicato, Phys. Rev. A 87, 022501 (2013).

[10] K. Pachucki, Phys. Rev. A 53, 2092 (1996).

[11] K. Pachucki, Phys. Rev. A 60, 3599 (1999).

[12] E. Borie, Phys. Rev. A 71, 032505 (2008).

[13] M.I. Eides, H. Groth, and V.A. Shelyuto, Theory of Light Hyperonic Bound States (Springer, Berlin-Heidelberg-New York, 2007).

[14] S.G. Karshenboim, Phys. Rep. 422, 1 (2005).

[15] T. Kinoshita and M. Nio, Phys. Rev. Lett. 82, 3240 (1999).

[16] T. Kinoshita and M. Nio, Phys. Rev. 60, 053008 (1999).

[17] V.G. Ivanov, E.Yu. Korzinin, and S.G. Karshenboim, Phys. Rev. D 80, 027702 (2009). See, also: eprint arXiv:0905.4471.

[18] T. Kinoshita and M. Nio, Phys. Rev. Lett. 103, 079901 (2009) (errata).

[19] S.G. Karshenboim, V.G. Ivanov, E.Yu. Korzinin, and V.A. Shelyuto, Phys. Rev. A 81, 060501 (2010).

[20] S.G. Karshenboim, E.Yu. Korzinin, V.G. Ivanov and V.A. Shelyuto, Pis’ma v ZhETF 92, 9 (2010) [in Russian]; JETP Letters 92, 8 (2010).

[21] A.B. Mickelwait and H.C. Corben, Phys. Rev. 96, 1145 (1954);
G.E. Pustovalov, Zh. Eksp. Teor. Fiz. 32, 1519 (1957) [in Russian]; Sov. Phys. JETP 5, 1234 (1957);
D.D. Ivanenko and G.E. Pustovalov, Usp. Fiz. Nauk 61, 27 (1957) [in Russian]; Adv. Phys. Sci. 61, 1943 (1957).

[22] S.G. Karshenboim, V.G. Ivanov, and E.Yu. Korzinin, Eur. Phys. J. D 39, 351 (2006).

[23] E.Yu. Korzinin, V.G. Ivanov, and S.G. Karshenboim, Eur. Phys. J. D 41, 1 (2007).

[24] S.G. Karshenboim, Can. J. Phys. 76, 169 (1998);
S.G. Karshenboim, JETP 89, 850 (1999);
see for corrections S.G. Karshenboim, V.G. Ivanov, V.M. Shabaev, Can. J. Phys. 79, 81 (2001);
S.G. Karshenboim, V.G. Ivanov, V.M. Shabaev, JETP 93, 477 (2001).

[25] U.D. Jentschura, Phys. Rev. A 84, 012505 (2011).

[26] S.G. Karshenboim, V.G. Ivanov, and E.Yu. Korzinin, Phys. Rev. A 85, 032509 (2012).

[27] M.I. Eides, H. Groth and V.A. Shelyuto, Phys. Rep. 342, 63 (2001).

[28] E. Borie, Phys. Rev. A 72, 052511 (2005).

[29] E. Borie and G.A. Rinker, Phys. Rev. A 18, 324 (1978).

[30] P. Vogel, At. Data Nucl. Data Tables 14, 599 (1974).

[31] K. Blomquist, Nucl. Phys. B 48, 95 (1972).

[32] E. Borie and G.A. Rinker, Rev. Mod. Phys. 54, 67 (1982).

[33] E. Borie, Nucl. Phys. A 267, 485 (1976).

[34] G. Källen and A. Sabry, Kgl. Dan. Vidensk. Selsk. Mat.-Fys. Medd., 29 (1955) No.17.

[35] A. Veitia and K. Pachucki, Phys. Rev. A 69, 042501 (2004).

[36] L.D. Landau, E.M. Lifshitz. Quantum Mechanics: Non-Relativistic Theory, Vol. 3 (3rd ed.). Pergamon Press, 1977.

[37] W.A. Barker and F.N. Glover, Phys. Rev. 99, 317 (1955).

[38] V.B. Berestetskii, E.M. Lifshitz, and L.P. Pitaevskii, Quantum electrodynamics, Pergamon, Oxford (1992).

[39] S.G. Karshenboim, E.Yu. Korzinin, and V.G. Ivanov, JETP Letters 88, 641 (2008).

[40] J. Schwinger. Particles, Sources, and Fields, Vol. 2. Perseus Book, Reading, 1998.

[41] M.I. Eides, S.G. Karshenboim, and V.A. Shelyuto, Phys. Lett. B 229, 285 (1989); S.G. Karshenboim, V.A. Shelyuto, and M.I. Eides. Sov. J. Nucl. Phys., 50, 1015 (1989).

[42] L. Lewin. Dilogarithms and associated functions, Macdonald, London, 1958.

[43] L. Hostler, J. Math. Phys. 5, 591 (1964).

[44] H.F. Hameka, J. Chem. Phys. 47, 2728 (1967); Erratum. 48, 4810 (1968).

[45] L.C. Hostler, Phys. Rev. 178, 126313 (1969).

[46] V.G. Ivanov and S.G. Karshenboim, JETP 82, 656 (1996); E., 84, 209 (1997).

[47] P.P. Pavinsky and A.I. Sherstyuk, Vestnik Len. Gos.
Univ. 22, 11 (1968) in Russian;  
L. Hostler, J. Math. Phys. 11, 2966 (1970);  
S.V. Khristenko and S.I. Vechnikin, Opt. Spektrosk. 31, 503 (1971) in Russian;  
S.A. Zapryagaev, N.L. Manakov, and L.P. Rapoport, Yad. Fiz. 15, 508 (1972) [Sov. J. Nucl. Phys. 15, 282 (1972)];  
S.V. Khristenko, Teor. Mat. Fiz. 22 35 (1975) [Theor. and Math. Phys., 22 21 (1975)].  
[48] S.G. Karshenboim, E.Yu. Korzinin, and V.G. Ivanov, arXiv:1311.5822.  
[49] D.A. Owen, Phys. Rev. D 42, 3534 (1990); 46, 4782(E) (1992); Found. Phys. 24, 273 (1994);  
M. Halpert and D.A. Owen, J. Phys. G 20, 51 (1994).  
[50] K. Pachucki and S. G. Karshenboim, J. Phys. B 28, L221 (1995).  
[51] A. P. Martyzhenko, Yad. Fiz. 71, 126 (2008) [Phys. At. Nucl. 71, 125 (2008)].  
[52] K. Melnikov and A. Vainshtein, Theory of the Muon Anomalous Magnetic Moment. Springer Tracts Mod. Phys. 216 (Springer, Berlin, Heidelberg, 2006).  
[53] F. Jegerlehner, The Anomalous Magnetic Moment of the Muon. Springer Tracts Mod. Phys. 222 (Springer, Berlin, Heidelberg, 2008).  
[54] U.D. Jentschura and B.J. Wundt, Eur. Phys. J. D 65 357 (2011).  
[55] E. Borie, Helv. Physica Acta 48, 671 (1975).  
[56] E. Borie and G. A. Rinker, Rev. Mod. Phys. 54, 67 (1982).  
[57] R. Barbieri, M. Caffo, and E. Remiddi, Lett. Nuovo Cimento 7, 60 (1973).  
[58] A.P. Martyzhenko, ZhETF 133, 794 (2008) [in Russian];  
JETP 106, 690 (2008).  
[59] J.L. Friar, J. Martorell, and D.W.L. Sprung, Phys. Rev. A 59, 4061 (1999).  
[60] R.N. Faustov and A.P. Martyzhenko, EPJdirect C6, 1 (1999);  
Yad. Phys. 64, 1358 (2001) [Phys. Atom. Nuclei 64, 1282 (2001)].  
[61] V. N. Folomeshkina, Yad. Fiz. 19, 1157 (1974) [Sov. J. Nucl. Phys. 19, 592 (1974)];  
M. K. Sundaresan and P. J. S. Watson, Phys. Rev. D 11, 230 (1975);  
V. P. Gerdt, A. Karimkhodzhaev, and R. N. Faustov, Proc. of the Int. Workshop on High Energy Phys. and Quantum Filed Theory, 1978, p. 289;  
E. Borie, Z. Phys. A 302, 187 (1981).  
[62] S.G. Karshenboim, V.G. Ivanov, U.D. Jentschura, and G. Soff, JETP 86 226 (1998).  
[63] J. Beringer et al. (Particle Data Group), Phys. Rev. D86, 010001 (2012).  
[64] G.I. Gounaris and J.J. Sakurai, Phys. Rev. Lett. 21 244 (1968).  
[65] M. Davier, A. Hoecker, B. Malaccu, C. Z. Yuan, and Z. Zhang, Eur. Phys. J. C 66, 1 (2010).  
[66] A. Czarnecki, S.I. Eidelman, and S.G. Karshenboim, JETP 106, 690 (2008).