Numerical integration for high order pyramidal finite elements

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Abstract

We examine the effect of numerical integration on the convergence of high order pyramidal finite element methods. Functions that are not smooth on each element are indispensable to the construction of pyramidal interpolants so the conventional treatment of numerical integration, which requires that the finite element approximation space is piecewise polynomial, cannot be applied. We develop an analysis that allows the finite element approximation space to include non-smooth functions and show that, despite this complication, conventional rules of thumb can still be used to select appropriate quadrature methods on pyramids. Along the way, we present a new family of high order pyramidal finite elements for each of the spaces of the de Rham complex.

Keywords: finite elements, quadrature, pyramid

1. Introduction

In prior work, [16, 17], we have presented a family of high-order finite element approximation spaces on a pyramidal element. Pyramidal finite elements are used in applications as “glue” in heterogeneous meshes containing hexahedra, tetrahedra and prisms. Various constructions of high order pyramidal elements have been proposed [7, 23, 12, 11, 22, 16]. A useful summary of the approaches taken for $H^1$-conforming elements is given by Bergot et al. [3], who also provide some motivating numerical results for the performance of methods based on meshes containing pyramidal elements. If they are to be used to implement stable mixed methods, such elements should also satisfy a commuting diagram property.

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In addition to our work, elements satisfying this property were constructed by Zaglmayr based on the theory of local exact sequences, [22], and are summarised in [9].

Our aim here is to study the errors due to numerical integration on arbitrarily high order pyramidal finite elements that approximate each of the spaces of the de Rham complex. Numerical quadrature is an important component of matrix assembly, and work in the use of quadrature schemes for finite element methods has been recently focussed on issues of efficiency and fast implementation, see e.g. [?] . The classical analysis of the effect of quadrature, see, e.g. [6, 5], has the lesser objective, nicely summed up in [6], of "[giving] sufficient conditions on the quadrature scheme which insures that the effect of the numerical integration does not decrease [the] order of convergence."

The approximation spaces we presented in [16] were shown to include complete sets of polynomials in [17] and so, at first glance, one might expect the classical arguments should hold in the case of the pyramidal finite elements as well. Somewhat to our surprise, this was not the case: see Example 8. Our exclusive focus in this paper, therefore, is a careful analysis of the errors introduced by quadrature when pyramidal finite elements are used.

A prototypical (linear) problem associated with the weak form of a PDE is of the form:

For $a : V \times V \to \mathbb{R}$ and $f \in V'$, find $u \in V$ such that: $a(u, v) = f(v) \quad \forall v \in V$, \hspace{1cm} (1)

where $V$ is a normed space of functions on a domain, $\Omega \subset \mathbb{R}^n$, $a(\cdot, \cdot)$ is a bilinear form, and $f(\cdot)$ is a linear functional on $V$. One approximation strategy is then to partition $\Omega$ using a triangulation, and replace $V$ by a (finite dimensional) finite element approximation subspace, $V_h$, where $h$ is a parameter associated with the size of the mesh. One then obtains a numerical approximation $u_h \in V_h$ to the true solution $u \in V$. A typical result is that the approximate solution converges to the true solution at some rate, $O(h^k)$ where the order of convergence, $k$, depends on the degree of largest complete space of polynomials used in the finite element approximation space.

In general, the bilinear form, $a(\cdot, \cdot)$ and the right hand side $f(\cdot)$ are evaluated using numerical integration rules. These are additional sources of errors in the approximate solution.

In this paper we will show that the quadratures described as conical product formulae by Stroud [19] are an appropriate choice for our pyramidal elements, in particular, that the $n$th order quadrature rule can be used for the integration of bilinear forms involving $n$th order elements without decreasing the order of convergence. The main challenge arises from the fact that the classical theory is only applicable to finite elements with approximation spaces consisting purely of polynomials, but pyramidal elements necessarily include functions other than polynomials, specifically rational functions with denominators which have roots on the boundary of the pyramid [16, 21, 20]. In contrast to the claim in [3], we show that the importance of these rational functions in constructing interpolants means that it is
not possible to achieve global estimates of the consistency error by summing element-wise estimates that only deal with polynomials.

Section 2 introduces a framework that will allow us to unify our analysis for discrete approximations to each of the spaces of the de Rham complex. We also recall the definitions of the approximation spaces for the elements in [16] and the quadrature rules given in [19].

In section 3 we show that the conical product formulae are exact for products of all pairs of functions from the approximation spaces, including the non-polynomials. The intuition from the classical theory would be that this is all that is required. However, as discussed, in section 4 we show that the reasoning behind this intuition is insufficient when functions other than polynomials are present. To overcome this, we derive a generalisation of the standard Bramble-Hilbert argument. In section 5 we present new families of approximation spaces that allow us to take advantage of this generalisation (and which can be used to construct new pyramidal finite elements in their own right). Finally, we pull everything together in section 6 and show that Stroud’s quadrature rules satisfy the desired property for both the new and original families of elements.

2. Definitions

2.1. Differential forms

We will want to make general statements that apply to approximations to each of the spaces of the de Rham complex. It is natural and increasingly popular to use differential forms and the exterior calculus in such a discussion [1, 2]. However, we will be dealing with pyramids, which are set firmly in three dimensions, and so will endeavour to keep things as concrete as possible. Consequently, at the expense of a few extra preliminaries, we will be able to keep much of the notation familiar to users of vector calculus.

Let $\Omega \subset \mathbb{R}^n$ and define $\Lambda^{(s)}(\Omega)$ as the space of differential $s$-forms on $\Omega$. A point, $x \in \Omega$, has coordinates $(x^i)_{i=0...n}$ and a given $u \in \Lambda^{(s)}(\Omega)$ can be expressed in terms of its components, $u = \sum u_\alpha dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_s}$ where each $u_\alpha \in C^\infty(\Omega)$ and the multi-indices, $\alpha = \alpha_1 \cdots \alpha_s$, run over the set, $\Upsilon_s$, of all increasing sequences, $\{1...s\} \rightarrow \{1...n\}$.

Define $\Theta^{(s)}(\Omega)$ to be the space of all (covariant) tensors, $A : \Lambda^{(s)}(\Omega) \times \Lambda^{(s)}(\Omega) \rightarrow C^\infty(\Omega)$ that can be defined in terms of the pointwise representation,

$$A(u, v)(x) := A^{\alpha\beta}(x)u_\alpha(x)v_\beta(x) \quad \forall u, v \in \Lambda^{(s)}(\Omega),$$

where we are using the Einstein summation convention, $A^{\alpha\beta}u_\alpha v_\beta := \sum_{\alpha, \beta \in \Upsilon_s} A^{\alpha\beta}u_\alpha v_\beta$. We will insist that $A^{\alpha\beta}$ is anti-symmetric in the first $s$ and second $s$ components, which makes the representation unique.

A tensor, $A \in \Theta^{(s)}(\Omega)$ induces a bilinear form on $\Lambda^{(s)}(\Omega)$:

$$(u, v)_A, \Omega = \int_\Omega A^{\alpha\beta}(x)u_\alpha(x)v_\beta(x) dx.$$
Let $\mathcal{T}$ be a partition of $\Omega$ where every $K \in \mathcal{T}$ is the image of a simple reference domain, $\hat{K} \subset \mathbb{R}^n$, under a diffeomorphism $\phi_K : \hat{K} \to K$. On each $K$, the reference coordinates, $\hat{x} = (x^i)_{i=0\ldots n}$ of any point $x \in K$, are given by $\hat{x} = \phi_K^{-1}(x)$. Given $u \in \Lambda^{(s)}(K)$, the reference coordinate system induces a new set of components $u_{\hat{a}}$. Differential forms are contravariant, so the components transform as:

$$u_{\hat{a}} = \sum_{a \in I_s} \frac{\partial x^{a_1}}{\partial x^{\hat{a}_1}} \cdots \frac{\partial x^{a_s}}{\partial x^{\hat{a}_s}} u_a. \quad (3)$$

The components of a covariant tensor, $A \in \Theta^{(s)}(\Omega)$ transform as:

$$A^{\hat{\alpha} \hat{\beta}} = \sum_{\alpha, \beta \in I_s} \frac{\partial x^{\hat{\alpha}_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x^{\hat{\alpha}_s}}{\partial x^{\alpha_s}} \frac{\partial x^{\hat{\beta}_1}}{\partial x^{\beta_1}} \cdots \frac{\partial x^{\hat{\beta}_s}}{\partial x^{\beta_s}} A^{\alpha \beta}. \quad (4)$$

Note that $(u(x), v(x))_{A(x)} = A^{\alpha \beta}(x) u_{\alpha}(x) v_{\beta}(x) = A^{\hat{\alpha} \hat{\beta}}(\hat{x}) u_{\hat{a}}(\hat{x}) v_{\hat{b}}(\hat{x})$ is just a 0-form and we have the change of variables formula on each element, $K$:

$$(u, v)_{A, K} = \int_K A^{\alpha \beta} u_{\alpha} v_{\beta} dx = \int_K A^{\hat{\alpha} \hat{\beta}} u_{\hat{a}} v_{\hat{b}} det(D\phi_K) d\hat{x}, \quad (5)$$

where $D\phi_K$ is the Jacobian of $\phi_K$ and $det(D\phi_K)$ is the determinant of the Jacobian.

When $n = 2$ and $n = 3$, it is conventional to think of differential forms in terms of proxy fields. The spaces $\Lambda^{(0)}(\Omega)$ and $\Lambda^{(n)}(\Omega)$ are always isomorphic to the scalar field, $C^\infty(\Omega)$. When $n = 3$, the spaces $\Lambda^{(1)}(\Omega)$ and $\Lambda^{(2)}(\Omega)$ are isomorphic to the vector field, $(C^\infty(\Omega))^3$. For $u \in \Lambda^{(s)}(\Omega)$, we denote the components of the proxy field as $u_i$ for $i \in I_s = \{1, \ldots, \binom{n}{s}\}$. The isomorphisms for the vector fields are given by

$$u \in \Lambda^{(1)}(\Omega) \mapsto \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad u \in \Lambda^{(2)}(\Omega) \mapsto \begin{pmatrix} u_{23} \\ -u_{13} \\ u_{12} \end{pmatrix}. \quad (6)$$

With these identifications, the exterior derivatives, $d : \Lambda^{(s)}(\Omega) \to \Lambda^{(s+1)}(\Omega)$ for $s = 0, 1, 2$ become the familiar grad, curl and div.

As with the differential forms, we will decorate the subscripts (and superscripts) of proxies with symbols to indicate the coordinate system that is being used to determine the components of the proxy fields. Given some $u \in \Lambda^{(s)}(\Omega)$, $u_{\nu}$ is the $i$th component of its proxy in the coordinate system $x' = (x'^1, x'^2, x'^3)$. We will also write $u' = (u_{\nu})_{i \in I_s}$ to indicate all the components of the vector (or scalar) field.

For a coordinate change, $x = \phi(x')$, the weights appearing in the contravariant and covariant transformation rules, (3) and (4), can be written as the entries of a $\binom{3}{s} \times \binom{3}{s}$
matrix, $w^{(s)}_{\phi}$. We choose to let $w^{(s)}_{\phi}$ to be the weight in the covariant transformation so that, for $u \in \Lambda^{(s)}(\Omega)$

$$
\sum_{i' \in \mathcal{I}_s} \left( w^{(s)}_{\phi} \right)_{i,i'} u_{i'} = u_i \quad \forall i \in \mathcal{I}_s.
$$

The weights can be calculated in terms of the Jacobian, $D\phi$:

$$
w^{(0)}_{\phi} = 1, \quad w^{(1)}_{\phi} = D\phi^{-1}, \quad w^{(2)}_{\phi} = \det(D\phi^{-1})D\phi, \quad w^{(3)}_{\phi} = \det(D\phi^{-1}).
$$

The exterior derivative is an intrinsic property of any manifold. This means that it is independent of coordinates; equivalently, the exterior derivative commutes with coordinate transformation.

The use of a reference coordinate system is a familiar concept, and shape functions for finite elements on simplices are often defined in terms of barycentric coordinates. One may think of a scalar or vector field as a proxy to a differential form. In this setting, the use of the reference coordinate system to study a shape function is analogous to mapping the differential form to a reference element using a pullback.

2.2. Sobolev spaces

Let the Sobolev semi-norms $|\cdot|_{W^{k,p}(\Omega)}$ and $|\cdot|_{H^{k}(\Omega)} = |\cdot|_{W^{k,2}(\Omega)}$ have their standard meanings. Define semi-norms and norms for any $u \in \Lambda^{(s)}(\Omega)$ as

$$
|u|_{k,\Omega}^2 := \sum_{i \in \mathcal{I}_s} |u_i|_{H^{k}(\Omega)}^2, \quad \|u\|_{k,\Omega}^2 := \sum_{r=0}^{k} |u|_{r,\Omega}^2.
$$

The Sobolev spaces, $H^{r}\Lambda^{(s)}(\Omega)$ and $\mathcal{H}^{(s),r}(\Omega)$ are then defined as the completion of $\Lambda^{(s)}(\Omega)$ in the norms $\|u\|_{r,\Omega}$ and $\|u\|_{\mathcal{H}^{(s),r}(\Omega)} = \|u\|_{r,\Omega}^2 + \|\partial u\|_{r,\Omega}^2$ respectively.

As a short-hand, we will write $\mathcal{H}^{(s)}(\Omega) = \mathcal{H}^{(s),0}(\Omega)$. The spaces of proxy fields corresponding to $\mathcal{H}^{(0)}(\Omega)$, $\mathcal{H}^{(1)}(\Omega)$, $\mathcal{H}^{(2)}(\Omega)$ and $\mathcal{H}^{(3)}(\Omega)$ are the familiar $H^{1}(\Omega)$, $H(\text{curl},\Omega)$, $H(\text{div},\Omega)$ and $L^2(\Omega)$.

Note\(^3\) that $H^{r+1}\Lambda^{(s)}(\Omega) \subseteq \mathcal{H}^{(s),r}(\Omega)$ and in particular $\mathcal{H}^{(0),r}(\Omega) = H^{r+1}\Lambda^{(0)}(\Omega) \cong H^{r+1}(\Omega)$, and $\mathcal{H}^{(n),r}(\Omega) = H^n\Lambda^{(n)}(\Omega) \cong H^r(\Omega)$.

For $A \in \Theta^{(s)}(\Omega)$, we similarly define

$$
|A|_{k,\infty,\Omega}^2 := \sum_{i,j \in \mathcal{I}_s} |A_{ij}|_{W^{k,\infty}(\Omega)}^2, \quad \|A\|_{k,\infty,\Omega}^2 := \sum_{r=0}^{k} |A|_{r,\infty,\Omega}^2
$$

\(^3\)When $r = 0$, this is the observation that $H^{1}(\Omega)^3 \subset H(\text{curl},\Omega)$ and $H^{1}(\Omega)^3 \subset H(\text{div},\Omega)$. 

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and define $W^{r,\infty}\Theta(s)(\Omega)$ to be the completion of $\Theta(s)(\Omega)$ in $\|\cdot\|_{r,\infty,\Omega}$.

For a given $K$ and $u \in \Lambda(s)(K)$ and $A \in \Theta(s)(K)$, define the reference semi-norms

$$|u|_{k,K} := \sum_{i \in I_s} |u_i|_{H^k(\tilde{K})}, \quad |A|_{k,\infty,\tilde{K}} := \sum_{i,j \in I_s} |A_{ij}|_{W^{k,\infty}(\tilde{K})}.$$

Suppose that $(\mathcal{T}_h)_{h>0}$ is a family of shape-regular partitions of $\Omega$, where every $K \in \mathcal{T}_h$ is affine equivalent to $\tilde{K}$ and each $\phi_K$ satisfies

$$\|D\phi_K\| \leq h \quad \text{and} \quad \|D\phi_K^{-1}\| \leq \frac{\rho}{h} \quad (8)$$

for some $\rho \geq 1$. For any $u \in \mathcal{H}^{(s),k}(K)$ and $A \in W^{k,\infty}\Theta(s)(K)$, we have the inequalities

$$\frac{1}{Cp^{k+s}} \frac{h^{k+s}}{\det(D\phi_K)^{1/2}} |u|_{k,K} \leq |u|_{k,\tilde{K}} \leq C \frac{h^{k+s}}{\det(D\phi_K)^{1/2}} |u|_{k,K} \quad (9)$$

$$\frac{1}{Cp^k} h^{k-2s} |A|_{k,\infty,\tilde{K}} \leq |A|_{k,\infty,K} \leq C p^2 h^{k-2s} |A|_{k,\infty,K} \quad (10)$$

for some constant $C = C(k,n)$ which is independent of $h$. These can be deduced from the standard scaling argument for Sobolev semi-norms of functions (see, for example, [6]) combined with the transformation rules, (3) and (4) and the observation that (8) implies that $\frac{\partial x^{\alpha_i}}{\partial \tilde{x}^j} \leq h$ and $\frac{\partial x^{\alpha_j}}{\partial \tilde{x}^i} \leq \frac{\rho}{h}$ for all $i,j$.

2.3. Pyramidal elements

From now on we will assume $\Omega \subset \mathbb{R}^3$. To contain the proliferation of indices, we will use the notation $(\xi,\eta,\zeta)$ for the reference coordinates $(x^1,x^2,x^3)$. The reference domain is defined as the pyramid:

$$\tilde{K} = \{ (\xi,\eta,\zeta) \mid 0 \leq \zeta \leq 1, 0 \leq \xi, \eta \leq \zeta \}.$$

We have chosen to restrict our analysis to affine maps $\phi_K$ so each element of the mesh, $K \in \mathcal{T}_h$, will be a parallelogram-based pyramid. We will refer to a general such $K$, as an affine pyramid. Note that this restriction is for the sake of exposition - in practice, the necessity and utility of pyramidal elements is evident in meshes comprised of tetrahedral, prismatic and hexahedral elements as well. In this case, a given element of the mesh is obtained via a mapping from one of three different reference elements. Our accounting of quadrature errors is based on local estimates, and the elements on the pyramid are conforming. Therefore, the extension of the present arguments to mixed meshes is straightforward.

4These are the norms induced by the metric in which the reference coordinates are orthonormal. They are used in the scaling argument in section 6.
As in [16] we will also use the infinite pyramid,

\[ K_\infty = \{(x, y, z) \mid 0 \leq x, y \leq 1, 0 \leq z \leq \infty \}, \]

as a tool to help analyse and define the pyramidal elements. The finite and infinite pyramids may be identified using the projective mapping,

\[
\phi : K_\infty \rightarrow \hat{K} \quad (11) \\
\phi : (x, y, z) \mapsto \left( \frac{x}{1+z}, \frac{y}{1+z}, \frac{z}{1+z} \right), \quad (12)
\]

which can be thought of as a change of coordinates and so, for any element, \( K \), induces the infinite pyramid coordinate system\(^5\) defined as \( \tilde{x} = \phi^{-1} \hat{x} \). We shall usually write \( \tilde{x} = (x, y, z) \).

The corresponding weights in the change of coordinates transformation rule can be calculated explicitly:

\[
w^{(0)}_\phi = 1, \quad (13a) \\
w^{(1)}_\phi = D\phi^{-1} = (1+z)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1+z \end{pmatrix}, \quad (13b) \\
w^{(2)}_\phi = det(D\phi^{-1})D\phi = (1+z)^2\begin{pmatrix} 1+z & 0 & -x \\ 0 & 1+z & -y \\ 0 & 0 & 1 \end{pmatrix}, \quad (13c) \\
w^{(3)}_\phi = det(D\phi^{-1}) = (1+z)^4. \quad (13d)
\]

The approximation spaces for the finite elements presented in [16] are defined on the infinite pyramid using \( k \)-weighted tensor product polynomials, \( Q^{l,m,n}_k[x, y, z] \), which are tensor product spaces of polynomials, \( Q^{l,m,n}[x, y, z] \), multiplied by a weight \( \frac{1}{(1+z)^k} \). That is, \( Q^{l,m,n}_k \) is spanned by the set\(^6\)

\[ \left\{ \frac{x^ay^bzc}{(1+z)^k}, \ 0 \leq a \leq l, 0 \leq b \leq m, 0 \leq c \leq n \right\}. \]

For each family of elements on the infinite pyramid, an underlying approximation space is defined for each order, \( k \geq 1. \)

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\(^5\)so-called because \( z \rightarrow \infty \) as \((\xi, \eta, \zeta) \rightarrow (0, 0, 1)\) at the top of the pyramid.

\(^6\)If \( l, m \) or \( n \) is negative then \( Q^{l,m,n}_k = \{0\} \).
• $H^1$-conforming element underlying space:

$$ \mathcal{U}_k^{(0)} = Q_k^{k,k,k-1} \oplus \text{span} \left\{ \frac{z^k}{(1 + z)^k} \right\}. \quad (14a) $$

• $H(\text{curl})$-conforming element underlying space:

$$ \mathcal{U}_k^{(1)} = Q_{k+1}^{k-1,k,k-1} \times Q_{k+1}^{k,k-1,k-1} \times Q_{k+1}^{k,k,2} \oplus \left\{ \frac{z^{k-1}}{(1 + z)^{k+1}} \begin{pmatrix} \frac{\partial r}{\partial x} \\ \frac{\partial r}{\partial y} \\ -r \end{pmatrix}, \quad r \in Q^{k,k}[x,y] \right\} \quad (14b) $$

• $H(\text{div})$-conforming element space:

$$ \mathcal{U}_k^{(2)} = Q_{k+2}^{k,k-1,k-2} \times Q_{k+2}^{k-1,k,k-2} \times Q_{k+2}^{k-1,k-1,k-1} \oplus \frac{z^{k-1}}{(1 + z)^{k+2}} \begin{pmatrix} 0 \\ 2s \\ s \end{pmatrix} \oplus \frac{z^{k-1}}{(1 + z)^{k+2}} \begin{pmatrix} 2t \\ 0 \\ t \end{pmatrix}, \quad (14c) $$

where $s(x,y) \in Q^{k-1,k}[x,y]$, $t(x,y) \in Q^{k,k-1}[x,y]$.

• $L^2$-conforming element underlying space:

$$ \mathcal{U}_k^{(3)} = Q_{k+3}^{k,k,k-1,k-1}. \quad (14d) $$

For an element defined on a pyramid, $K$, the underlying approximation space, $\mathcal{U}_k^{(s)}(K)$ is defined as the space containing all the $s$-forms whose components induced by the infinite pyramid coordinate system lie in $\mathcal{U}_k^{(s)}$:

$$ \mathcal{U}_k^{(s)}(K) = \left\{ u \in \Lambda^{(s)}(K) : (u_i)_{i \in I_s} \in \mathcal{U}_k^{(s)} \right\}. \quad (15) $$

By inspection, it can be seen that the exterior derivative $d : \mathcal{U}_k^{(s)}(K) \to \mathcal{U}_k^{(s+1)}(K)$ is well defined, and so, since $d$ is independent of coordinates, the exterior derivative on the spaces on each element,

$$ d : \mathcal{U}_k^{(s)}(K) \to \mathcal{U}_k^{(s+1)}(K) \quad (16) $$

is also well defined.

A full explanation of these spaces is provided in [16]. Some motivation may be seen from the following lemma.

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7We are using coordinate transformations here, but in [16], the underlying spaces are defined as the pullbacks, $\overline{\mathcal{U}}_k^{(s)}(\hat{K}) = \{ (\phi^{-1})^* v : v \in \mathcal{U}_k^{(s)} \}$ and $\overline{\mathcal{U}}_k^{(s)}(K) = \{ \phi_k * v : v \in \overline{\mathcal{U}}_k^{(s)}(\hat{K}) \}$. 

8
Lemma 1. For a given $K$ and $s \in \{0, 1, 2, 3\}$ let $u \in \mathcal{U}^{(s)}_k(K)$. Each component $u_i$ (where $\hat{i} \in \mathcal{I}_s$) of $u$ in the reference coordinate system satisfies

$$u_i \circ \phi \in Q^{k,k,k}_k.$$  \hspace{1cm} (17)

This means that

$$\mathcal{U}^{(s)}_k(K) \subset H^{(s)}(K).$$  \hspace{1cm} (18)

Proof. The relationship between the representations of $u$ in the reference and infinite pyramid coordinate systems is given by equation (6): $\hat{u} \circ \phi = w^{(s)}_\phi \tilde{u}$, where the weights, $w^{(s)}_\phi$, are given by (13a)-(13d). To establish (17), each $s \in \{0, 1, 2, 3\}$ needs to be dealt with as a separate case.

When $s = 0$, the weight, $w^{(0)}_\phi = 1$ and it is clear from (14a) that $\mathcal{U}^{(0)}_k(K) \subset Q^{k,k,k}_k$. When $s = 1$, inspection of (14b) reveals that $\mathcal{U}^{(1)}_k(K) \subset Q^{k-1,k,k}_k \times Q^{k,k-1}_k \times Q^{k,k,k-1}_k$. The weight, $w^{(1)}_\phi = (1 + z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1 + z \end{pmatrix}$, so $w^{(1)}_\phi \tilde{u} \in Q^{k,k,k}_k \times Q^{k,k,k}_k \times Q^{k,k,k}_k$. The cases $s = 2$ and $s = 3$ follow similarly.

Since $Q^{k,k,k}_k \subset L^\infty(K_\infty)$ each $u_i \circ \phi$ is bounded on $K_\infty$, which means that $u_i$ is bounded on $\hat{K}$ and therefore $u_i$ is bounded on $K$. Hence $\|u\|_{0,K}$ is finite. By (16), $du \in \mathcal{U}^{(s+1)}_k(K)$, so $\|du\|_{0,K}$ is finite too and $u \in H^{(s)}(K)$. \hfill $\square$

In order to construct pyramidal elements that are compatible with neighbouring tetrahedral (and hence polynomial) elements, subspaces of the underlying approximation spaces, $\mathcal{U}^{(s)}_k(K)$ are identified that contain only those functions whose traces on the triangular faces of the pyramid are contained in the trace space of the corresponding tetrahedral element. These approximation spaces are denoted $\mathcal{U}^{(s)}_{k,0}(K)$. We shall denote by $\mathcal{U}^{(s)}_{k,0}(K)$ the subspaces of $\mathcal{U}^{(s)}_k(K)$ with zero boundary traces. We also recall a key result allowing a Helmholtz decomposition of these spaces, which were proved in [16].

Theorem 2. Let $\mathcal{U}^{(s)}_{k,0}(K)$ be as defined above. Then the following decompositions hold:

1. $\mathcal{U}^{(1)}_{k,0}(K) = \text{grad}\mathcal{U}^{(0)}_{0,0}(K) \oplus \mathcal{U}^{(1)}_{0,\text{curl}}(K)$, where $\mathcal{U}^{(1)}_{0,\text{curl}}(K) := \left\{ v | v = (\phi^{-1})^*u, u \in \mathcal{U}^{(1)}_{0,\text{curl}}(K_\infty) \right\}$, and where $\mathcal{U}^{(1)}_{0,\text{curl}}(K_\infty) \subset \mathcal{U}^{(1)}_{0}(K_\infty)$

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The trace spaces for the tetrahedral Lagrange, Nedelec edge and Nedelec face elements are given in [14]. By construction, the traces of the underlying spaces on the quadrilateral face of the pyramid already match those of the corresponding hexahedral elements.
consists of functions $u$ of the form

$$u = \begin{pmatrix} y(1-y)zq_1 \\ x(1-x)zq_2 \\ x(1-x)y(1-y)p \end{pmatrix},$$

(19)

where $q_1 \in Q_{k+1}^{k-1,k-2,k-2}$, $q_2 \in Q_{k+1}^{k-2,k-1,k-2}$, $p \in Q_{k+1}^{k-2,k-2,k-2}[x,y]$. 

2. $\mathcal{U}_0^{(2)}(K) = \text{curl}\mathcal{U}_0^{(1)}(K) \oplus \mathcal{U}_0^{(2)}(K)$

where $\mathcal{U}_0^{(2)}(K) := \{ v|u = (\phi^{-1})^*(u), u \in \mathcal{U}_0^{(2)}(K_\infty) \}$ and where

$$\mathcal{U}_0^{(2)}(K_\infty) := \text{span}\left\{ \frac{z^k-1}{(1+z)^{k+2}} \begin{pmatrix} r_y + 2t \\ r_x + 2s \\ (1+z)(r_{xy} + s_y + t_x) \end{pmatrix} \right\} \oplus \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ z\chi_3 \end{pmatrix} \right\}$$

(20)

and where $r(x,y) = x(1-x)y(1-y)p(x,y), p \in Q_{k+1}^{k-2,k-2}, t = x(1-x)\tilde{t}, \tilde{t} \in P_{k-2}(x), s = y(1-y)\tilde{s}, \tilde{s} \in P_{k-2}(y)$, and $\chi_3 \in Q_{k+2}^{k-1,k-1,k-2}$.

3. $\text{div}\mathcal{U}_0^{(2)}(K) \oplus \mathbb{R} = \mathcal{U}^{(3)}(K)$.

Each approximation space $\mathcal{U}_k^{(s)}(K)$ is equipped with a set $\Sigma^{(s)}(K)$ of degrees of freedom that induce a linear interpolation operator,

$$\Pi_{k,K}^{(s)} : \mathcal{H}^{s,1/2+\epsilon}(K) \to \mathcal{U}_k^{(s)}(K), \quad \epsilon > 0,$$

so that $m(u) = m(\Pi_{k,K}^{(s)} u) \quad \forall m \in \Sigma^{(s)}(K),$

(21)

which completes the definition of the finite elements. The necessity of the extra $1/2 + \epsilon$ regularity can be seen as a consequence of taking point evaluations at the vertices of the pyramid for the $s = 0$ elements. It is necessary for both the projection based interpolants of [8] and the more explicit construction given in [16].

We now need to define a global interpolant. Given a mesh comprised of pyramidal elements, $\mathcal{T}_h$, for $\Omega$, we can assemble a global approximation space for $\mathcal{H}^{(s)}(\Omega)$,

$$\mathcal{V}_h^{(s)} = \{ v \in \mathcal{H}^{(s)}(\Omega) : v|_K \in \mathcal{U}_k^{(s)}(K) \ \forall K \in \mathcal{T}_h \}.$$  

(22)

Again, we stress that the restriction that all $K \in \mathcal{T}_h$ be mapped from a reference pyramid is for purposes of exposition. Indeed, the approximation spaces $\mathcal{U}_k^{(s)}(K)$ and the degrees of freedom $\Sigma^{(s)}(K)$ were designed to ensure these elements are conforming in a mesh consisting of tetrahedral and hexahedral elements. The element-wise interpolation operators respect traces on the boundary of the pyramid, i.e.

$$\text{tr} \ u|_{\partial K} = 0 \Rightarrow \text{tr} \ \Pi_{k,K}^{(s)} u|_{\partial K} = 0,$$

so we define a bounded global interpolation operator $\Pi_h^{(s)} : \mathcal{H}^{(s,1/2+\epsilon)}(\Omega) \to \mathcal{V}_h^{(s)}$ by $$(\Pi_h^{(s)} u)|_K := \Pi_{k,K}^{(s)}(u|_K) \text{ for all } K \in \mathcal{T}_h.$$
2.4. Conical product rule

Quadrature rules on the pyramid can be deduced as special cases of the conical product rule presented by Stroud [19, 13]. Stroud defines the quadrature scheme for any continuous function, \( f \in C(\hat{K}) \),

\[
S(f) := \sum_{i,j,l} f(\xi_i (1 - \zeta_l), \xi_j (1 - \zeta_l), z_l) \lambda_i \lambda_j \mu_l. \tag{23}
\]

He shows that given \( n \geq 0 \), a sufficient condition for \( S(p) = \int_{\hat{K}} p \, d\hat{x} \) for any polynomial, \( p \in P^n(\hat{x}) \), is that the two one-dimensional quadrature schemes given by the points \( \xi_i \) and \( \zeta_l \) with respective weights \( \lambda_i \) and \( \mu_l \) satisfy

\[
\sum_i \lambda_i g(\xi_i) = \int_0^1 g(x) \, dx \quad \forall g \in P^n, \tag{24}
\]

\[
\sum_i \mu_i h(\zeta_i) = \int_0^1 (1 - z)^2 h(z) \, dz \quad \forall h \in P^n. \tag{25}
\]

The \( k + 1 \) point Gauss-Legendre quadrature rule can be used to generate \( \xi_i \) and \( \lambda_i \) that make (24) exact for polynomials of degree \( 2k + 1 \). The \( k + 1 \) point Gauss-Jacobi scheme based on the Jacobi polynomial\(^9\), \( P_{k+1}^{(2,0)} \), generates \( \zeta_i \) and \( \mu_i \) that make (25) exact for polynomials of degree \( 2k + 1 \). We denote the quadrature scheme for \( \hat{K} \) based on (23) that uses these points and weights as \( S_{k,\hat{K}} \). The error,

\[
E_{k,\hat{K}}(f) := S_{k,\hat{K}}(f) - \int_{\hat{K}} f(\hat{x}) \, d\hat{x}.
\]

will be zero when \( f \in P^{2k+1} \).

When \( f \in C(K) \), where \( K \) is a pyramid equipped with a change of coordinates \( \phi_K : \hat{K} \to K \), we can define the quadrature and error functionals:

\[
S_{k,K}(f) := S_{k,\hat{K}} \left( |D\phi_K| \hat{f} \right) \sim \int_K f(x) \, dx, \tag{26}
\]

\[
E_{k,K}(f) := E_{k,\hat{K}} \left( |D\phi_K| \hat{f} \right) = S_{k,K}(f) - \int_K f(x) \, dx, \tag{27}
\]

where \( \hat{f} = f \circ \phi_K \), i.e. the expression of \( f \) in the reference coordinate system, \( \hat{x} \).

\(^9\)The Jacobi polynomials, \( P_{n}^{(a,b)}(s) \), \( n \geq 0 \), are typically defined on the interval \([-1,1]\). Under the change of variables, \( s = 2t - 1 \), they orthogonal with respect to the weight \( (1 - t)^a t^b \) on the interval \([0,1]\).
3. Pyramidal approximation spaces and quadrature

Now let's look at the effect of the conical product rules on our approximation spaces. If \( u \) and \( v \) are polynomials of degree \( k \) then their product, \( uv \in P^{2k} \) so from Stroud’s work, we know that \( S_{k,K}(uv) = \int_K uv \). In this section, we shall prove the stronger result:

**Theorem 3.** Let \( K \) be an affine pyramid; fix \( k \geq 1 \); let \( s \in \{0, 1, 2, 3\} \) and let \( A \in \Theta^s(K) \) be a constant tensor field. Then for any \( u, v \in U^s_k(K) \), the quadrature scheme \( S_{k,K} \) exactly evaluates the product, \( (u,v)_{A,K} \), i.e.

\[
S_{k,K}(A^{ij}u_iv_j) = (u,v)_{A,K}.
\]

To do this, we first need to understand exactly which functions our quadrature scheme integrates exactly.

**Lemma 4.** Suppose that \( f \) is a function defined on a pyramid, \( K \), and that the representation of \( f \) in the infinite pyramid coordinate system, \( \tilde{f} = f \circ \phi_K \circ \phi \), lies in the space \( Q^{2k+1,2k+1,2k+1} \). Then the quadrature scheme, \( S_{k,K} \) is exact for \( f \):

\[
S_{k,K}(f) = \int_K f dx
\]

**Proof.** It suffices to consider functions \( p \) with a representation in the infinite pyramid coordinate system:

\[
\tilde{p}(x, y, z) = \frac{x^a y^b}{(1 + z)^c} \quad 0 \leq a, b, c \leq 2k + 1,
\]

since these monomials span the space \( Q^{2k+1,2k+1,2k+1} \). In finite reference coordinates, \( p \) has the form \( \tilde{p}(\xi, \eta, \zeta) = \xi^a \eta^b (1 - \zeta)^{c-a-b} \), and so, using (23):

\[
S_{k,K}(p) = S_k(det(D\phi_K)p) = det(D\phi_K) \sum_{i,j,l} \xi_i^a (1 - \zeta)^a \eta_j^b (1 - \zeta)^b \lambda_i \lambda_j \mu_l \sum_{l} \mu_l (1 - \zeta)^c \]

\[
= det(D\phi_K) \int_0^1 s^a ds \int_0^1 t^b dt \int_0^1 (1 - \zeta)^{c+2} d\zeta.
\]

The last step is justified because each of the sums is a quadrature rule applied to a polynomial of degree \( \leq 2k + 1 \) and so we can apply (24) and (25). Apply the change of variables
\[ \xi = (1 - \zeta)s \quad \text{and} \quad \eta = (1 - \zeta)t \quad \text{to obtain:} \]

\[ S_{k,K}(p) = \det(D\phi_K) \int_0^1 \int_0^{1-\zeta} \int_0^{1-\zeta} (1 - \zeta)^{c-a-b} \xi^a \eta^b \, d\xi d\eta d\zeta \]
\[ = \det(D\phi_K) \int_K \hat{p}(\xi, \eta, \zeta) d\hat{x} \]
\[ = \int_K pdx. \]

We can now prove Theorem 3 where, in fact, we will only need Lemma 4 to be true for \( \tilde{f} \in Q_{2k}^{2k,2k} \), which is a subspace of \( Q_{2k+1}^{2k+1,2k+1,2k+1} \).

**Proof of Theorem 3.** Let \( u, v \in U_k^{(s)}(K) \). \( A \in \Theta^{(s)}(K) \) is a constant and so, by the first part of Lemma 1, in infinite reference coordinates, the function \( A(u,v) \) satisfies:

\[ A^{ij}u_jv_i \in Q_{2k}^{2k,2k} \]

Hence by Lemma 4,

\[ S_{k,K}(A^{ij}u_jv_i) = \int_K A^{ij}u_jv_i = (u,v)_{A,K}. \]

Observe that for the spaces \( U_k^{(3)}(K) \), the integrand, \( A^{ij}u_jv_i \in Q_{2k-2}^{2k-2,2k-2,2k-2} \), so we could in fact use the scheme \( S_{k-1} \).

**4. Numerical integration and convergence**

In this section, we will show where the classical theory of the effect of quadrature breaks down for pyramid elements and derive a generalisation of the Bramble Hilbert Lemma that we will use to resolve the problem.

Let \( a : \mathcal{H}^{(s)}(\Omega) \times \mathcal{H}^{(s)}(\Omega) \rightarrow \mathbb{R} \) be an elliptic bilinear form and let \( V \subset \mathcal{H}^{(s)}(\Omega) \) be chosen so that the problem of finding \( u \in V \) such that

\[ a(u,v) = f(v) \quad \forall v \in V \]  

(28)

has a unique solution for any linear functional, \( f \in V' \). A discrete version of this problem is to find \( u_h \in V_h \) such that

\[ a_h(u_h,v) = f(v) \quad \forall v \in V_h, \]  

(29)

13
where $V_h$ is an approximating subspace of $V$ and $a_h$ approximates $a$ using numerical integration\textsuperscript{10}. When $V_h$ is assembled using polynomial elements of degree $k$, the analysis of the effect of the numerical integration is classical; good expositions may be found in [6, 5].

For an example, take an elliptic bilinear form $a : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$, defined as

\[ a(u, v) = \int_\Omega A(du, dv) \, dx \tag{30} \]

where $A \in W^{k,\infty}(\Omega)$ and is uniformly positive definite. Assume that $V_h \subset H^1_0(\Omega)$ is some approximation space assembled using $k$th order polynomial finite elements and that there exists a numerical integration rule, $S_{h,k,\Omega}(\cdot)$ which satisfies $S_{h,k,\Omega}(\partial_i u \partial_j v) = \int_\Omega (\partial_i u \partial_j v)$ for any $i$ and $j$ and all pairs of functions $u, v \in V_h$. Let $a_h(u, v) = S_{h,k,\Omega}(A(du, dv))$. It is shown in [6, page 179] that the solution of (29) will satisfy the error estimate:

\[ \|u - u_h\|_1 \leq C h^k(|u|_{k+1} + \|A\|_{k,\infty}\|u\|_{k+1}). \]

This result is contingent on an estimate of the consistency error:

\[ \sup_{w_h \in V_h} \frac{|a(\Pi_h u, w_h) - a_h(\Pi_h u, w_h)|}{\|w_h\|_1} \leq C h^k \|A\|_{k,\infty}\|u\|_{k+1}, \tag{31} \]

where $\Pi_h : H^1_0(\Omega) \to V_h$ is an interpolation operator. The constant $C = C(\Omega, k)$ is independent of $h$.

More generally, an analysis for mixed problems can be found in [10]. The conclusion is the same: in order to preserve an $O(h^k)$ approximation error, each bilinear form must satisfy an $O(h^k)$ consistency error estimate.

The key ingredient in the proof of the consistency error estimate, (31) is a local estimate:

\textbf{Theorem 5} (See [6], Theorem 4.1.2). Given a simplex, $K \in \mathcal{T}_h$, assume that the quadrature rule is exact for $P^{2k-2}$. That is, for any polynomial $\psi \in P^{2k-2}(K)$, the quadrature error, $E_K(\psi) = 0$. Then there exists a constant $C$ independent of $K$ and $h$ such that

\[ \forall A \in W^{k,\infty}(K), \quad \forall p, q \in P^k(K) \]

\[ |E_K(A(dp, dq))| \leq C h^k \|A\|_{k,\infty,K} \|dp\|_{k-1,K} \|dq\|_{0,K}. \]

This theorem is proved by combining a scaling argument with the following famous result from [4].

\textsuperscript{10}We choose not to consider the effect of approximating $f(\cdot)$ by some $f_h(\cdot)$ using numerical integration because it is no different on the pyramid than for other elements. Error estimates may be obtained by applying the standard argument and using Theorem 3.
Theorem 6 (Bramble-Hilbert Lemma). Let $\Omega \subset \mathbb{R}^n$ be open with Lipschitz-continuous boundary. For some integer $k \geq 0$ and $p \in [0, \infty]$ let the linear functional, $f : W^{k+1,p}(\Omega) \to \mathbb{R}$ have the property that $\forall \psi \in P^k(\Omega)$, $f(\psi) = 0$. Then there exists a constant $C(\Omega)$ such that

$$\forall v \in W^{k+1,p}(\Omega), \quad |f(v)| \leq C(\Omega)\|f\|_{W^{k+1,p}(\Omega)'} \|v\|_{k+1,p,\Omega}$$

where $\|\cdot\|_{W^{k+1,p}(\Omega)'}$ is the operator norm.

In our more general framework, we may conjecture that an analogous statement to Theorem 5 is

Conjecture 7. Let $K \in \mathcal{T}_h$ be a pyramid. Let $s \in \{0, 1, 2, 3\}$ and $A \in W^{k,\infty} \Theta(s)(K)$. Then

$$\forall v, w \in U_k^{(s)}(K)$$

$$|E_{K,k}(A(v, w))| \leq Ch^k\|A\|_{k,\infty,K} \|v\|_{k-1,K} \|w\|_{0,K}$$

This conjecture is false. The problem is that, unlike the situation for purely polynomial spaces, we cannot differentiate basis functions arbitrarily. We do not have the inclusion, $U_k^{(s)}(K) \subset H^{(s),k-1}(K)$ for $k \geq 3$. The following example is illustrative:

Example 8. Take the $U_k^{(0)}(K)$ shape function associated with the base vertex, $(1,1,0)$:

$$v(\xi, \eta, \zeta) = \frac{\xi\eta}{1-\zeta}. \quad (34)$$

The $L^2$ norm of its third partial $\zeta$-derivative,

$$\int_K \left( \frac{\partial^3 v}{\partial \zeta^3} \right)^2 d\hat{x} = \int_0^1 \int_0^{1-\zeta} \int_0^{1-\zeta} \left( \frac{-6\xi\eta}{(1-\zeta)^4} \right)^2 d\xi d\eta d\zeta$$

$$= \int_0^1 \frac{9}{(1-\zeta)^2} d\zeta, \quad (35)$$

is infinite.

This means that a direct application of the argument in [6, section 4.1] would fail when we attempt to use the Bramble-Hilbert lemma (Theorem 6) to obtain the estimate

$$\left| \Pi_{k,\hat{K}}^{(s)} u \right|_{r,\hat{K}} \leq C |u|_{r,\hat{K}} \quad \forall r \in \{0, \ldots, k\}. \quad (37)$$

An attempt is made to avoid this problem in [3] by using the additional projector $\pi_r : H^{r+1}(K) \to P^r$ satisfying

$$\forall p \in P^r(K) \quad \pi_r p = p.$$
on each element, $K$. This allows element-wise estimates to be established. Unfortunately, there is no conforming interpolant onto element-wise polynomials for pyramidal elements (see [16] or [21]). In particular, there will be discontinuities at the element boundaries, which means that $\|u - \pi_r u\|_{1,\Omega}$ cannot be bounded. The alternative interpretation of $\pi_r$ as a global projection onto polynomials would not allow the element-wise estimates to be obtained.

Our solution starts with the observation that not all of the members of each $\mathcal{U}_k^{(s)}(K)$ behave as badly as the function $v$ defined in (34). There are subspaces of polynomials and of rational functions that can be differentiated more times before blowing up. For example, we will see in the proof of Lemma 17 that $v(\xi,\eta,\zeta)\xi^r \in H^{r+2}(\hat{K})$. So, we start by developing an analogue of (37) that, loosely speaking, allows us to retain as much regularity as possible.

**Theorem 9.** Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. Fix $\alpha \geq 0$ and let $k \geq \alpha$ be an integer. Suppose that:

- $R^k \subset H^\alpha(\Omega)$ is a finite dimensional space which includes all polynomials of degree $k$;
- $\Pi : H^\alpha(\Omega) \to R^k$ is a bounded linear projection;
- There exist $V_r \subset H^r(\Omega)$ for each $r \in \{0,\ldots,k\}$ such that we can decompose
  $R^k = V_0 \oplus \cdots \oplus V_k$.

Meaning that for a given $u \in H^k(\Omega)$, the interpolant, $\Pi u \in R^k$, may be decomposed into unique functions, $v_r \in V_r$,

$$\Pi u = v_0 + \cdots + v_k.$$ 

Then we have the following estimates for some of the functions, $v_r$:

- For each $r$ satisfying $\alpha \leq r \leq k$:
  $$|v_r|_r \leq C |u|_r.$$  
(38)

- If, additionally, $\tilde{P}_r \subset V_r$, where the space $\tilde{P}_r$ consists of polynomials of homogeneous degree, $r$, then for each $r$ satisfying $\alpha \leq r + 1 \leq k$:
  $$|v_r|_r \leq C |u|_{r+1} + |u|_r.$$  
(39)

**Proof.** For a given $r \geq \alpha$, write $W_r = V_r \cup \tilde{P}_r$. $W_r \subset R^k$ so we can let $\Psi_r : R^k \to W_r$ be any surjective linear projection. $\Psi_r$ is a linear map between finite spaces, so the operator $(I - \Psi_r \circ \Pi) : H^r(\Omega) \to W_r \subset H^r(\Omega)$ is bounded. Also, since both $\Psi_r$ and $\Pi$ are projections,
and \( P_r^{-1} \subset P^k \subset R^k \) we see that \( P_r^{-1} \subset \ker(I - \Psi_r \circ \Pi) \). The Bramble-Hilbert lemma gives
\[
\|(I - \Psi_r \circ \Pi)u\|_r \leq C |u|_r.
\]
By the definition of \( W_r \), we have \((\Psi_r \circ \Pi)u = v_r + p\) for some \( p \in P_r^{-1} \). So \(|u - v_r - p|_r \leq C |u|_r\), which implies
\[
|v_r|_r \leq C |u|_r + |u|_r + |p|_r = (C + 1) |u|_r.
\]
The proof of (39) follows a similar argument. The operator \((I - \Psi_r \circ \Pi) : H^{r+1}(\Omega) \to W_r \subset H^r(\Omega)\) is bounded because \( r + 1 \leq k \). The additional condition, \( \tilde{P}^r \subset V_r \), means that \( P^r \subset W_r \) and so \( P^r \subset \ker(I - \Psi_r \circ \Pi) \).

5. A new family of pyramidal approximation spaces

As identified in [3], the space \( U_k^{(0)} \) is sub-optimal in that there exist smaller spaces which contain the same complete space of polynomials and which are compatible with neighbouring tetrahedral and hexahedral elements. Here we will identify subspaces, \( R_k^{(s)}(K) \), of each of the original approximation spaces, \( U_k^{(s)}(K) \) that can be used to construct finite elements with the same approximation and compatibility properties and that still satisfy a commuting diagram property. This would be an interesting exercise in its own right but within the context of this paper we shall see that the importance of these spaces is that they support a decomposition in the manner of Theorem 9 whose components still have enough “room” for us to apply a Bramble-Hilbert type argument in Lemma 19.

We start the construction of these spaces in the infinite pyramid coordinate system using spaces of \( k \)-weighted polynomials, \( Q_k^{[l,m]} \), which we define in terms of basis functions \( \frac{x^a y^b}{(1+z)^c} \) where \( a, b \) and \( c \) are non-negative integers.
\[
Q_k^{[l,m]} = \text{span} \left\{ \frac{x^a y^b}{(1+z)^c} : c \leq k, a \leq c + l - k, \ b \leq c + m - k \right\}.
\] (40)
These spaces can be characterised via a decomposition into spaces of exactly \( r \)-weighted polynomials,
\[
Q_k^{[l,m]} = \bigoplus_{r=0}^{k} Q_r^{r+l-k, r+m-k, 0}.
\] (41)
It is also helpful to observe that \( \frac{x^a y^b}{(1+z)^c} \mapsto \xi^a \eta^b (1 - \zeta)^{c-a-b} \) under the coordinate transformation, \((\eta, \xi, \zeta) = \phi(x, y, z)\) given by (11). So if the representation in the infinite pyramid coordinate system of some polynomial \( f(\hat{x}) \) is \( \tilde{f} \in Q_k^{[l,m]} \) then \( f \) is at most degree \( k \) in \((\xi, \eta, \zeta)\) and at most degree \( l \) and \( m \) in \((\xi, \zeta)\) and \((\eta, \zeta)\) respectively.
Now define the spaces $\mathcal{R}^{(s)}_k$ as

\[
\begin{align*}
\mathcal{R}^{(0)}_k &= Q^{[k,k]}_k, \\
\mathcal{R}^{(1)}_k &= \left( Q^{[k-1,k]}_{k+1} \times Q^{[k,k-1]}_{k+1} \times \{0\} \right) \oplus \{ \nabla u : u \in Q^{[k,k]}_k \}, \\
\mathcal{R}^{(2)}_k &= \left( \{0\} \times \{0\} \times Q^{[k-1,k-1]}_{k+2} \right) \oplus \left\{ \nabla \times u : u \in \left( Q^{[k-1,k]}_{k+1} \times Q^{[k,k-1]}_{k+1} \times \{0\} \right) \right\}, \\
\mathcal{R}^{(3)}_k &= Q^{[k-1,k-1]}_{k+3}.
\end{align*}
\]

The decomposition in the definitions means that with the identification made as in Section 2.1, the exterior derivatives, $d : \mathcal{R}^{(s)}_k \to \mathcal{R}^{(s+1)}_k$ (precisely, the grad, curl and div operators) are well defined. The gradient is injective on $Q^{[k,k]}_k/\mathbb{R}$; the curl is injective on $\left( Q^{[k-1,k]}_{k+1} \times Q^{[k,k-1]}_{k+1} \times \{0\} \right)$ and the divergence is a bijection from $\left( \{0\} \times \{0\} \times Q^{[k-1,k-1]}_{k+2} \right)$ to $Q^{[k-1,k-1]}_{k+3}$, so the sequence,

\[
\mathbb{R} \longrightarrow \mathcal{R}^{(0)}_k \overset{\nabla}{\longrightarrow} \mathcal{R}^{(1)}_k \overset{\nabla \times}{\longrightarrow} \mathcal{R}^{(2)}_k \overset{\nabla}{\longrightarrow} \mathcal{R}^{(3)}_k \longrightarrow 0
\]

is exact. The following three lemmas relate these spaces to the $\mathcal{U}^{(s)}_k$ spaces. To avoid the proofs distracting from our main argument we have postponed them to Appendix A.

**Lemma 10.** The spaces $\mathcal{R}^{(s)}_k$ are subspaces of the $\mathcal{U}^{(s)}_k$:

\[
\mathcal{R}^{(s)}_k \subseteq \mathcal{U}^{(s)}_k \quad \forall s \in \{0, 1, 2, 3\}.
\]

Indeed, for $k \geq 2$ the $\mathcal{R}^{(s)}_k$ are strict subsets of the $\mathcal{U}^{(s)}_k$.

**Definition 11.** For a given $s \in \{0, 1, 2, 3\}$ and $k \geq 0$, we define the approximation space\(^{11}\) on a pyramid, $K$, as those differential forms whose infinite coordinate representation lie in $\mathcal{R}^{(s)}_k$:

\[
\mathcal{R}^{(s)}_k(K) = \left\{ u \in \Lambda^{(s)}(K) : (u_i) \in \mathcal{R}^{(s)}_k \right\}.
\]

These spaces still contain all the polynomials that were shown to be present ([17]) in the spaces $\mathcal{U}^{(s)}_k(K)$. Specifically:

\(^{11}\)c.f. the original approximation spaces, (15)
Lemma 12. If $K$ is an affine (i.e. parallelogram-based) pyramid then, for $k \geq 1$,
\[ P^k \subset \mathcal{R}_k^{(0)}(K) \]
\[ \left( P^{k-1} \right)^{(s)} \subset \mathcal{R}_k^{(s)}(K) \quad s \in \{ 1, 2, 3 \} \]

Just as with the original spaces, $U_k^{(s)}(K)$, the new spaces are compatible with Nedelec’s elements, which were first outlined in [15].

Lemma 13. Let $K$ be a pyramid. For each $s \in \{ 0, 1, 2 \}$ there is a trace operator that takes elements of $\mathcal{H}^{(s)}(K)$ to some distribution on the boundary, $\partial K$. The image of $\mathcal{R}_k^{(s)}(K)$ under this operator consists of all traces of elements of $\mathcal{H}^{(s)}(K)$ whose restriction to each triangular or quadrilateral face of $K$ is the trace of a corresponding $k$th order Lagrange, edge and face approximation function on a neighbouring tetrahedron or hexahedron.

Note that the original approximation spaces, $U_k^{(s)}(K)$, were defined by explicitly identifying the subsets of underlying spaces, $\overline{U}_k^{(s)}(K)$ which had such polynomial trace spaces. For the $\mathcal{R}_k^{(s)}(K)$, the polynomial trace property is inherent and this additional step is not required.

It is the demand that we can match polynomial traces on all faces simultaneously in Lemma 13 which creates the need for rational functions in our spaces. For example, there is no polynomial whose trace is the lowest order bubble on one triangular face and zero on all other faces.

From Lemmas 10 and 13 we see that:

**Corollary 14.** The new approximation spaces are subspaces of the original approximation spaces.

\[ \mathcal{R}_k^{(s)} \subseteq U_k^{(s)} \quad s \in \{ 0, 1, 2, 3 \} \]

We can reuse the interpolation operators from the old spaces, (21), to create interpolation operators for the new spaces. Since the trace spaces of $\mathcal{R}_k^{(s)}$ are the same as $U_k^{(s)}$, we just need to define projections $\Xi_k^{(s)} : U_k^{(s)}(K) \rightarrow \mathcal{R}_k^{(s)}(K)$ that do not change the trace data. We denote the subspace of consisting of all shape-functions in $\mathcal{R}_k^{(s)}(K)$ with zero trace as $\mathcal{R}_k^{(s),0}(K)$.

**Definition 15.** For $u \in U_k^{(s)}(K)$, define $\Xi_k^{(s)} : U_k^{(s)}(K) \rightarrow \mathcal{R}_k^{(s)}(K)$ as

\[ \Xi_k^{(s)} u := v_u + w_u \]
where \( v_u \in \mathcal{R}_k^{(s)}(K) \) is some function satisfying \( v_u|_{\partial K} = u|_{\partial K} \) and \( w_u \in \mathcal{R}_{k,0}^{(s)}(K) \) is the minimizer of the functional \( v \to \|d(u - v_u - v)\|_0 \) over the admissible set \( \mathcal{A}_{k,K}^{(s)} \), defined as:

\[
\begin{align*}
\mathcal{A}_{k,K}^{(0)} &:= \mathcal{R}_{k,0}^{(0)}(K) \\
\mathcal{A}_{k,K}^{(s)} &:= \left\{ v \in \mathcal{R}_{k,0}^{(s)}(K) : (v,dw) = 0 \ \forall w \in \mathcal{R}_{k,0}^{(s-1)}(K) \right\}, \quad s = 1, 2, 3. 
\end{align*}
\]

Lemma 13 means that the trace spaces of \( \mathcal{R}_k^{(s)}(K) \) and \( \mathcal{U}_k^{(s)}(K) \) are identical, so it is always possible to find an extension, \( v_u \). The spaces \( \mathcal{A}_{k,K}^{(s)} \) are non-empty because they always contain the zero-element so there always exists a minimiser, \( w_u \). The uniqueness of \( w_u \) (for a given choice of \( v_u \)) can be established using a Friedrichs-type inequality and it is then clear that \( \Xi_{k,K}^{(s)}u \) is independent of the choice of \( v_u \).

In fact, the operators \( \Xi_{k,K}^{(s)}u \) are just the projection-based interpolants of \( \mathcal{U}_k^{(s)}(K) \) onto \( \mathcal{R}_k^{(s)}(K) \). More details of projection-based interpolation can be found in [8], which also establishes the important commutativity property: \( \Xi_{k,K}^{(s+1)} \circ d = d \circ \Xi_{k,K}^{(s)} \).

Now define the maps \( \Phi_{k,K}^{(s)} : \mathcal{H}^{(s),1/2+\epsilon}(\Omega) \to \mathcal{R}_k^{(s)}(K) \) as

\[
\Phi_{k,K}^{(s)} = \Xi_{k,K}^{(s)} \circ \Pi_{k,K}^{(s)}. 
\]

Since both \( \Xi_{k,K}^{(s)} \) and \( \Pi_{k,K}^{(s)} \) commute with \( d \), so does \( \Phi_{k,K}^{(s)} \). In fact, if \( \Pi_{k,K}^{(s)} \) is a projection based interpolant, then so is \( \Phi_{k,K}^{(s)} \).

Of course, defining an interpolation operator is equivalent to defining degrees of freedom. The above construction implies that the internal degrees for the new elements, \( \mathcal{R}_k^{(s)} \) are analogous to those defined for \( \mathcal{U}_k^{(s)}(K) \) in [17]. Whereas the old elements used bases for Helmholtz decompositions of \( \mathcal{U}_k^{(s),0}(K) \) as test functions for the degrees; the new elements require Helmholtz decompositions of \( \mathcal{R}_k^{(s)}(K) \) which can be readily determined from the full-space Helmholtz decomposition implied in the definitions, (42a)-(42d). The external degrees of freedom for both sets of elements are identical.

As with (22), for a given \( k \), we can assemble a global approximation space,

\[
\mathcal{S}_h^{(s)} = \{ v \in \mathcal{H}^s(\Omega) : v|_K \in \mathcal{R}_k^{(s)}(K) \ \forall K \in \mathcal{T}_h \} 
\]

and define a global bounded interpolation operator \( \Phi_h^{(s)} : \mathcal{H}^{(s),1/2+\epsilon}(\Omega) \to \mathcal{S}_h^{(s)} \) by \( (\Phi_h^{(s)}u)|_K = \Phi_{k,K}^{(s)}(u|_K) \) for all \( K \in \mathcal{T}_h \).

Now that we have defined the new spaces, we shall present a decomposition that is compatible with Theorem 9.
Definition 16. Given a pyramid, \( K \) and \( s \in \{0, 1, 2, 3\} \) define, for each \( r \geq 0 \), the subspace of all the \( s \)-forms in \( \mathcal{R}^{(s)}_{k}(K) \) whose components are exactly \( r \)-weighted when composed with \( \phi : K_{\infty} \to \hat{K} \).

\[
\mathcal{X}^{(s)}_{r,k}(K) = \left\{ v \in \mathcal{R}^{(s)}_{k}(K) : v_{i} \circ \phi \in Q_{r}^{r+1,r+1,0} \right\}.
\]

Note that although the domain of \( v_{i} \circ \phi \) is \( K_{\infty} \), the condition is on the components in the reference coordinate system, \( v_{i} \), rather than the infinite pyramid coordinate system \( v_{i} \).

In effect, what we are saying is that each \( \mathcal{X}^{(s)}_{r,k}(K) \) is spanned by \( s \)-forms whose components have the form

\[
e(\xi, \eta, \zeta) = \xi^{a}\eta^{b}(1 - \zeta)^{r-a-b}
\]

where \( a, b \leq r + 1 \).

Lemma 17. For an affine pyramid, \( K \) and for each \( s \in \{0, 1, 2, 3\} \) and \( k \geq 1 \), each of the spaces \( \mathcal{X}^{(s)}_{r,k}(K) \) satisfy the criterion for \( V_{r} \) from Theorem 9. In fact,

\[
\mathcal{X}^{(s)}_{r,k}(K) \subset H^{r+1}A^{(s)}(K).
\]

Additionally, the semi-norm \( |\cdot|_{r,K} \) is actually a norm on each space \( \mathcal{X}^{(s)}_{r,k}(K) \).

Proof. Let \( u \in \mathcal{X}^{(s)}_{r,k}(K) \). Each \( u_{i} \) can be written in terms of functions, \( e(\xi, \eta, \zeta) = \xi^{a}\eta^{b}(1 - \zeta)^{r-a-b} \). When \( a + b > r \), these will be rational functions with a singularity at \( \zeta = 1 \). We need to understand their differentiability on the finite pyramid. Let \( \gamma = (\gamma_{1}, \gamma_{2}, \gamma_{3}) \) be a multi-index. The partial derivative,

\[
\frac{\partial^{\gamma}e}{\partial \tilde{\xi}^\gamma} = C \xi^{a-\gamma_{1}}\eta^{b-\gamma_{2}}(1 - \zeta)^{r-b-a-\gamma_{3}}
\]

where \( C = C(\gamma, a, b, r) \) is a (possibly zero) constant dependent only on \( \gamma, a, b \) and \( r \). Hence

\[
\int_{K} \left( \frac{\partial^{\gamma}e}{\partial \tilde{\xi}^\gamma} \right)^{2} = C \int_{0}^{1} \int_{0}^{1-\zeta} \int_{0}^{1-\zeta} \xi^{2a-2\gamma_{1}}\eta^{2b-2\gamma_{2}}(1 - \zeta)^{2r-2b-2a-2\gamma_{3}} \, d\xi \, d\eta \, d\zeta
\]

where \( C = C(\gamma, a, b, r) \) is a (possibly zero) constant dependent only on \( \gamma, a, b \) and \( r \). Hence

\[
\int_{K} \left( \frac{\partial^{\gamma}e}{\partial \tilde{\xi}^\gamma} \right)^{2} = C \int_{0}^{1} (1 - \zeta)^{2(r+1-\gamma_{1}-\gamma_{2}-\gamma_{3})} \, d\zeta
\]

This integral is finite if \( r + 1 - |\gamma| > -1/2 \), so \( e \in H^{r+3/2-\epsilon}_{r+3/2-\epsilon}(K) \). By affine equivalence of \( K \) and \( \hat{K} \), \( u \in H^{r+3/2-\epsilon}_{r+3/2-\epsilon}(K) \subset H^{r+1}(K) \).

Finally, (47) shows that each \( e(\xi, \eta, \zeta) \) is either a rational function, or a polynomial of degree exactly \( r \), so \( |e|_{r,K} \neq 0 \). Hence \( |u|_{r,K} \neq 0 \) and \( |\cdot|_{r,K} \) is a semi-norm on \( \mathcal{X}^{(s)}_{r,k}(K) \).
Lemma 18. For an affine pyramid, $K$ and for each $s \in \{0, 1, 2, 3\}$ and $k \geq 1$, each of the spaces $\mathcal{R}_k^{(s)}(K)$ may be decomposed:

$$\mathcal{R}_k^{(s)}(K) = \mathcal{X}_0^{(s)}(K) \oplus \cdots \oplus \mathcal{X}_k^{(s)}(K)$$

Proof. The decomposition (41) makes the claim look plausible. The details are left to Appendix A.

6. The effect of numerical integration on the pyramid

We are now ready to assemble all this machinery to prove a version of Theorem 5 for pyramidal finite elements. The first step is to establish an error estimate for each of the spaces in the decompositions in terms of the reference norms. Recall that in (26) we defined $S_{k,K}(\cdot)$ as the quadrature scheme which is exact on $P_{2k+1}^{r+1}$ on the pyramid, $K$, and that we call the error functional for this scheme $E_{k,K}(\cdot)$. We will also use the pointwise representation, $A(u,v) = A_{ij} u_i v_j$ given in (2).

Lemma 19. For any $s \in \{0, 1, 2, 3\}$ and an affine pyramid, $K$, let $v \in \mathcal{X}_r^{(s)}(K)$, $w \in \mathcal{R}_k^{(s)}(K)$ and $A \in W_{k+1,\infty}^r(\Theta^{(s)}(K))$. Then the error in the evaluation of the bilinear form, $(v,w)_{A,K}$ using the scheme $S_{k,K}(\cdot)$ can be bounded in terms of the reference (semi-)norms

$$|E_{k,K}(A(v,w))| \leq C \det D\phi_K |A|_{k-r+1,\infty,\hat{K}} |\hat{v}|_{r,\hat{K}} \|\hat{w}\|_{0,\hat{K}}$$

where $C = C(k)$ is a constant that depends only on $k$.

Proof. We can transform the error functional onto the reference pyramid using (27).

$$E_{k,K}(A(v,w)) = E_{k,K}(A_{ij} v_i w_j) = E_{k,\hat{K}} \left( \det(D\phi_K) A_{ij} \hat{v}_i \hat{w}_j \right) = \det(D\phi_K) E_{k,\hat{K}} \left( A_{ij} \hat{v}_i \hat{w}_j \right).$$

We are able to take $|D\phi_K|$ outside the integral because $\phi_K$ is affine. Define the linear functional $G \in W_{k-r+1,\infty}^{k-r+1,\infty} \Theta^{(s)}(\hat{K})'$ as

$$G(B) = E_{k,\hat{K}} \left( B_{ij} \hat{v}_i \hat{w}_j \right) \forall B \in W_{k-r+1,\infty}^{k-r+1,\infty} \Theta^{(s)}(\hat{K}).$$

Since $S_k(\cdot)$ takes point values of its argument,

$$|G(B)| \leq C \|B_{ij} \hat{v}_i \hat{w}_j\|_{0,\hat{K}} \leq C \|B\|_{k-r+1,\infty,\hat{K}} \|\hat{v}\|_{0,\hat{K}} \|\hat{w}\|_{0,\hat{K}}.$$

Furthermore, all norms are equivalent on the finite dimensional spaces, $\mathcal{X}_{r,k}^{(s)}(\hat{K})$ and $\mathcal{R}_k^{(s)}(\hat{K})$, and, by the last part of Lemma 17, $|\cdot|_{r,\hat{K}}$ is a norm for $\mathcal{X}_{r,k}^{(s)}$. So $G$ is continuous...
and \( \|G\| \leq C \|\tilde{v}\|_{r,K} \|\tilde{w}\|_{0,K} \). All of the equivalences of norms are done on the reference pyramid, so the constant, \( C \) depends only on \( k \) (in particular, it does not depend on \( K \)).

From the definition of \( \mathcal{A}_{r,k}^{(s)} \), we know that each \( v_i \circ \phi \in Q_{r+1,r+1,0}^{k} \) and by Lemma 1 and Corollary 14, \( w_j \circ \phi \in Q_{k,k,k}^{k} \) for each \( j \in \mathcal{L} \). Now suppose that \( B \) is polynomial of degree \( k-r \), i.e. each component, \( B_{i,j} \in \mathcal{P}^{k-r} \) for each \( i,j \in \mathcal{L} \). Then \( B_{i,j} \circ \phi \in Q_{k-r,k-r}^{k-r} \). We can assemble these facts to see that

\[
\left( B_{i,j} v_i w_j \right) \circ \phi = \left( B_{i,j} \circ \phi \right) \left( v_i \circ \phi \right) \left( w_j \circ \phi \right) \in Q_{2k+1}^{2k+1,2k+1,2k+1}.
\]

So, by Lemma 4, the quadrature error, \( E_{k,K}(B_{i,j} v_i w_j) = 0 \). Therefore, \( \mathcal{P}^{k-r} \subset \ker G \) and we can apply Theorem 6 (the Bramble-Hilbert Lemma) to obtain

\[
|G(A)| \leq C |A|_{k-r+1,\infty,K} |v|_{r,K} \|w\|_{0,K} \quad \forall A \in \mathcal{W}^{k-r+1,\infty}(\hat{K})
\]

For some constant \( C = C(k) \). Substituting (52) and (51) gives the desired result.

We can now apply a scaling argument to get an element-wise estimate on the quadrature error. Recall that we defined the interpolation operator, \( \Phi_{k,K}^{(s)} : \mathcal{H}^{(s),1/2+\epsilon}(K) \rightarrow \mathcal{R}_{k}^{(s)}(K) \) in (45).

**Lemma 20.** Let \( K \) be an affine pyramid satisfying the shape-regularity condition, (8), for some \( \rho \geq 1 \). Fix \( s \in \{0,1,2,3\} \) and take \( k \geq 2 \). Then

\[
\forall u \in H^k \mathcal{A}^{(s)}(K), w \in \mathcal{R}_{k}^{(s)}(K) \text{ and } A \in \mathcal{W}^{k+1,\infty}(K) \quad \forall u \in \mathcal{W}^{k+1,\infty}(K)
\]

\[
|E_{k,K}(A(\Phi_{k,K}^{(s)} u, w))| \leq \left( Ch^{k+1} \right) \|A\|_{k+1,\infty,K} \|u\|_{k,K} \|w\|_{0,K}
\]

where \( C = C(k) \) a constant dependent only on \( k \), and \( 0 < h < C \).

**Proof.** Use the decomposition given in Lemma 18 to write

\[
\Phi_{k,K}^{(s)} u = v_0 + \cdots + v_k \text{ where } v_r \in \mathcal{A}_{r,k}^{(s)}(K).
\]

By Lemma 19, we know that for each \( r \in \{0 \ldots k\} \),

\[
|E_{k,K}(A(v_r, w))| \leq C |D\Phi_K| |A|_{k-r+1,\infty,K} |v_r|_{r,K} \|w\|_{0,K}.
\]

The interpolation operator is bounded on \( \mathcal{H}^{(s),1/2+\epsilon}(K) \) which is a subset of \( H^{3/2+\epsilon}(K) \) so Theorem 9 is applicable with \( \alpha > 3/2 \). Pick some \( \alpha \in (3/2, 2] \) so that when \( r \geq 2 \) we can use the first estimate, (38), to obtain:

\[
|E_{k,K}(A(v_r, w))| \leq C |D\Phi_K| |A|_{k-r+1,\infty,K} |v_r|_{r,K} \|\hat{w}\|_{0,K}.
\]
Now apply the inequalities (9) and (10) to the semi-norms (and norm) on the right-hand side to obtain

\[
|E_{k,K}(A(v, w))| \leq C |D\phi_K| h^{k-r+1-2s} \rho^{2s} |A|_{k-r+1,\infty,K} \frac{h^{r+s}}{|D\phi_K|^{1/2}} |u|_{r,K} \frac{h^s}{|D\phi_K|^{1/2}} \|w\|_{0,K}
\]

where the generic constant, \( C \) still depends only on \( k \).

When \( r = 1 \), we can similarly apply the second estimate from Theorem 9 given in (39) to obtain:

\[
|E_{k,K}(A(v_1, w))| \leq C h^{k+1} |A|_{k,\infty,K} \left( |u|_{1,K} + h |u|_{2,K} \right) \|w\|_{0,K}.
\]  

(57)

For \( r = 0 \), note that \( \|v_0\|_{0,K} \leq C \|u\|_{3/2+\epsilon,K} \leq C \left( |u|_{0,K} + |u|_{1,K} + |u|_{2,K} \right) \), so

\[
|E_{k,K}(A(v_0, w))| \leq C h^{k+1} |A|_{k+1,\infty,K} \left( |u|_{0,K} + h |u|_{1,K} + h^2 |u|_{2,K} \right) \|w\|_{0,K}.
\]  

(58)

Summing over the \( v_r \), we obtain (54).

Summing these errors over each element gives an estimate for the global consistency error due to the numerical integration (we shall ignore the \( O(h^{k+2}) \) terms). Recall that in (46) we defined the global approximation space, \( S_h^{(s)} \subset H^{(s)}(\Omega) \).

**Theorem 21.** Let \( s \in \{0, 1, 2, 3\} \), \( k \geq 2 \) and assume that \( S_h^{(s)} \) is constructed using a shape regular mesh, \( T_h \) and finite elements, \( R^{(s)}_k(K) \) for each \( K \in T_h \). Let \( A \in W^{k+1,\infty}(\Theta^{(s)}(\Omega)) \) and \( u \in H^{(s),k}(\Omega) \). Then the interpolant \( \Phi_h u \in S_h^{(s)} \) satisfies

\[
\sup_{w_h \in S_h^{(s)}} \left| \left( \Phi_h u, w_h \right)_{A,\Omega} - \left( \Phi_h^{(s)} u, w_h \right)_{A,h,k,K} \right| \leq C h^{k+1} \|A\|_{k+1,\infty,\Omega} \|u\|_{k,\Omega}
\]

where we define \((v, w)_{A,h,k,K} := \sum_{K \in T_h} S_{K,k}(A(v, w))\). Here \( C > 0 \) is a constant which only depends on \( k \), and \( 0 < h < C \).

**Proof.** Let \( w_h \in S_h^{(s)} \).

\[
\left| \left( \Phi_h^{(s)} u, w_h \right)_{A,\Omega} - \left( \Phi_h^{(s)} u, w_h \right)_{A,h,k,K} \right| \leq C \sum_{K \in T_h} E_{k,K}(A(\Phi_h^{(s)} u, w_h))
\]

\[
\leq C h^{k+1} \sum_{K \in T_h} \|A\|_{k+1,\infty,K} \|u\|_{k,K} \|w_h\|_{0,K}
\]

\[
\leq C h^{k+1} \|A\|_{k+1,\infty,\Omega} \left( \sum_{K \in T_h} \|u\|_{k,K}^2 \right)^{1/2} \left( \sum_{K \in T_h} \|w_h\|_{0,K}^2 \right)^{1/2}
\]

\[
\leq C h^{k+1} \|A\|_{k+1,\infty,\Omega} \|u\|_{k,\Omega} \|w_h\|_{0,\Omega}
\]

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Dividing through by $\|w_h\|_{0,\Omega}$ gives the result.

In the proof of Lemma 19, the important condition for $w$ was that $w_j \circ \phi \in \mathcal{Q}_k^{k,k}$. So, by Lemma 1, we could equally well have taken $w \in \mathcal{U}_k^{(s)}(K)$. Furthermore, $\mathcal{S}^{(s)}_h \subset \mathcal{V}^{(s)}_h$ means that $\Phi^{(s)}_h u \in \mathcal{V}^{(s)}_h$. Hence we have a consistency error estimate for the global approximation spaces $\mathcal{V}^{(s)}_h$ based on the original elements:

**Corollary 22.** Under the same assumptions as Theorem 21, let $\mathcal{V}^{(s)}_h$ be constructed using finite elements based on the approximation spaces, $\mathcal{U}_k^{(s)}(K)$. Then the interpolant $\Phi^{(s)}_h u$ satisfies

$$\sup_{w_h \in \mathcal{V}^{(s)}_h} \frac{|(\Phi^{(s)}_h u, w_h)_{A,\Omega} - (\Phi^{(s)}_h u, w_h)_{A,h,k,\Omega}|}{\|w_h\|_0} \leq C h^{k+1} \|A\|_{k+1,\infty,\Omega} \|u\|_{k,\Omega}.$$

The error estimate may be applied to more general bilinear forms because of the commutativity $d \circ \Pi^{(s)}_h = \Pi^{(s+1)}_h \circ d$. For example, the consistency error for the elliptic bilinear form, (28), is

$$\sup_{v \in \mathcal{S}^{(0)}_h} \frac{|a(\Phi^{(0)}_h u, v) - a_h(\Phi^{(0)}_h u, v)|}{\|v\|_1} \leq \sup_{v \in \mathcal{S}^{(0)}_h} \frac{(d\Phi^{(0)}_h u, dv)_{A,\Omega} - (d\Phi^{(0)}_h u, dv)_{A,h,k,\Omega}}{\|dv\|_0} \leq \sup_{w \in \mathcal{S}^{(1)}_h} \frac{(\Phi^{(1)}_h du, w)_{A,\Omega} - (\Phi^{(1)}_h du, w)_{A,h,k,\Omega}}{\|w\|_0} \leq C h^{k+1} \|A\|_{k+1,\infty,\Omega} \|du\|_{k,\Omega} \leq C h^{k+1} \|A\|_{k+1,\infty,\Omega} \|u\|_{k+1,\Omega}.$$

A final note: as with the classical theory, the error estimates decay like $O(h^{k+1})$ but these are emphatically not $hp$-estimates. The degree, $k$ enters into the constants in several places, which is to be expected from arguments that rely on the Bramble-Hilbert Lemma.

**7. Conclusion**

The conventional finite element wisdom is that a $k$th order method requires a $k$th order quadrature scheme. We have shown that this is still true for some high order pyramidal finite elements, but that the non-polynomial nature of pyramidal elements requires some unconventional reasoning to justify the wisdom.

In the process, we have demonstrated new descriptions of families of high order finite elements for the de Rham complex that satisfy an exact sequence property. We will examine
these elements in more detail in future work, but a couple of notes are worth recording here.

- The approximation spaces for the first family in the sequence, $R_k^{(0)}(K)$ are the same as Zaglmayr’s elements, as described in [9], and which [3] describes as optimal with respect to their dimension and compatibility with neighbouring elements.

- Lemma 12 shows that the $R_k^{(s)}(K)$ spaces contain polynomials corresponding to the tetrahedron of the first type. Zaglmayr has constructed pyramidal elements containing polynomials corresponding to both types of tetrahedron, but only those corresponding to the second type are presented in [9]. It would, clearly, be interesting to compare our $R_k^{(s)}(K)$ spaces with the construction for the first type.

A. Properties of the new approximation spaces, $R_k^{(s)}$

In this appendix, we have collected proofs of various Lemmas in Section 5.

Proof of Lemma 10. The inclusions

$$Q_n^{[l,m]} \subseteq \left( Q_n^{l,m,\min\{l,m\}-1} + Q_n^{0,0,\min\{l,m\}} \right) \subseteq Q_n^{l,m,\min\{l,m\}}. \quad (59)$$

can be verified from the definition, (40). By the first inclusion, $Q_k^{[k,k]} \subseteq Q_k^{k,k,k-1} + Q_k^{0,0,k}$, which gives the $s = 0$ case: $R_k^{(0)} \subseteq U_k^{(0)}$.

The $s = 0$ result implies $\nabla R_k^{(0)} \subseteq \nabla U_k^{(0)}$. Thus, since $\nabla U_k^{(0)} \subset U_k^{(1)}$, we have $\nabla Q_k^{[k,k]} \subset U_k^{(1)}$, which establishes the result for the second space in the decomposition for $R_k^{(1)}$, given in (42b). To deal with the first space in this decomposition, apply (59) and the definition of $U_k^{(1)}$ given in (14b), to obtain

$$\left( Q_{k+1}^{[k-1,k]} \times Q_{k+1}^{[k,k-1]} \times \{0\} \right) \subseteq \left( Q_{k+1}^{k-1,k-1} \times Q_{k+1}^{k-1,k} \times \{0\} \right) \subset U_k^{(1)}.$$

The $s = 2$ case may be established similarly. The space $R_k^{(2)}$ is defined via a decomposition into two spaces, (42c). The second space in this decomposition can be seen to be a subset of $U_k^{(2)}$ by taking curls of the $s = 1$ result. The first space is dealt with by applying (59) directly to the definitions.

Another application of (59) gives $R_k^{(3)} = Q_{k+3}^{[k-1,k-1]} \subseteq Q_{k+3}^{k-1,k-1} = U_k^{(3)}$. \qed

Proof of Lemma 12. Since $P^k$ is preserved by affine transformation, we can work in the reference coordinate system, $\hat{x}$. Recall the components of the proxy representation of some $u \in \Lambda^{(s)}(K)$ in this coordinate system are denoted $u_i$, where $i \in I_s$. We will need to show that if all the components, $u_i \in P^k$ (or, for $s = 1, 2, 3$, $P^{k-1}$) then $u \in R_k^{(s)}(K)$. This is
equivalent to showing \( \tilde{u} \in \mathcal{R}^{(s)}_k \), which we will do using the transformation rule, (6), along with the explicit weights associated with the coordinate change \( \phi : K_\infty \to \hat{K} \) given in (13a)-(13d).

We start with the case \( s = 0 \). Let \( \hat{u} \in \Lambda^{(0)}(K) \) be any polynomial, \( \hat{u}(\xi, \eta, \zeta) = \xi^a \eta^b (1 - \zeta)^c \) where \( a + b + c \leq k \). Then

\[
\tilde{u} = \left( w_\phi^{(0)} \right)^{-1} \hat{u} \circ \phi = \frac{x^a y^b}{(1 + z)^a+b+c} \in Q^{[k,k]}_k = \mathcal{R}^{(0)}_k.
\]

Similarly, for \( s = 3 \), take \( \hat{u} \in \Lambda^{(3)}(K) \) as \( \hat{u}(\xi, \eta, \zeta) = \xi^a \eta^b (1 - \zeta)^c \) for \( a + b + c \leq k - 1 \). Then

\[
\tilde{u} = \frac{x^a y^b}{(1 + z)^a+b+c+4} \in Q^{[k-1,k-1]}_k = \mathcal{R}^{(3)}_k.
\]

The \( s = 1 \) case involves a little more work. Let \( u \in \Lambda^{(1)}(K) \) have polynomial components, \( u_1 \in P^{k-1} \). We can find \( q \in \Lambda^{(0)}(K) \) with representation \( \hat{q} \in P^k \) such that \( v = u - \nabla q \) has third component (in reference coordinates), \( v_3 = 0 \). By the result for \( s = 0 \), \( q \in \mathcal{R}^{(0)}(K) \), and so (by (42b)) \( \nabla q \in \mathcal{R}^{(1)}(K) \). We need to show that \( v \in \mathcal{R}^{(1)}(\hat{K}) \). Both \( v_1 \) and \( v_2 \) are in \( P^{k-1} \). Suppose first that \( v_1 = \xi^a \eta^b (1 - \zeta)^c \) where \( m := a + b + c \leq k - 1 \) and \( v_2 = 0 \).

\[
\tilde{v} = \left( w_\phi^{(1)} \right)^{-1} \tilde{v} \circ \phi = \frac{1}{(1 + z)^2} \begin{pmatrix} 1 + z & 0 & 0 \\ 0 & 1 + z & 0 \\ -x & -y & 1 \end{pmatrix} \begin{pmatrix} x^a y^b/(1 + z)^m \\ 0 \\ 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} x^a y^b/(1 + z)^{m+2} \\ -x^a y^b/(1 + z)^{m+2} \\ 0 \end{pmatrix} = \frac{1}{(1 + z)^{m+1}} \begin{pmatrix} 1 - \frac{a+1}{m+1} x^a y^b \\ -b/m+1 x^{a+1} y^{b-1} \\ 0 \end{pmatrix} + \frac{1}{m+1} \nabla x^{a+1} y^{b}/(1 + z)^{m+1}.
\]

Compare this last expression with the definition, (42b), to determine that \( \tilde{v} \in \mathcal{R}^{(1)}_k \). Note that when \( a = m \) (which includes the case \( a = k - 1 \)), the first term vanishes, because \( b = 0 \) and \( 1 - \frac{a+1}{m+1} = 0 \). An identical calculation establishes the same result when \( v_1 = 0 \) and \( v_2 = \xi^a \eta^b (1 - \zeta)^c \).

For \( s = 2 \), the change of coordinates formula for \( u \in \Lambda^{(2)}(K) \) is

\[
\tilde{u} = \left( w_\phi^{(2)} \right)^{-1} \tilde{u} \circ \phi = \frac{1}{(1 + z)^3} \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 + z \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \circ \phi
\]

\[ (60) \]

\[ ^{12} \text{In other words, } (\xi^a, 0, 0)^T \text{ is an exact 1-form.} \]
Suppose that \( u_3 = \xi^a \eta^b (1 - \zeta)^c \) with \( m := a + b + c \leq k - 1 \). Apply (60) to see that the contribution to \( \tilde{u} \) is \( (x^a y^b (1+z)^{m+\frac{3}{2}}, 0, 0)^t \). Let \( p = \frac{1}{m+2} x^a y^b \in Q_{k+1}^{[k-1,k-1]} \) and observe that \( \frac{x^a y^b}{(1+z)^{m+\frac{3}{2}}} = -\frac{\partial p}{\partial x} \) and \( \frac{\partial p}{\partial x} = \frac{a}{m+2} x^a y^b \in Q_{k+1}^{[k-1,k-1]} \) (the case \( b = m \) implies that \( a = 0 \) and therefore \( \frac{\partial p}{\partial x} = 0 \), so the final inequality in (40) is not violated). Hence

\[
\left( w^{(2)}_\phi \right)^{-1} \begin{pmatrix} \xi^a \eta^b (1 - \zeta)^c \\ 0 \\ 0 \end{pmatrix} \circ \phi = \nabla \times \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -\frac{\partial p}{\partial x} \end{pmatrix} \in \mathcal{R}^{(2)}_k.
\]

Polynomials in the second component can be dealt with similarly. When \( u_3 = \xi^a \eta^b (1 - \zeta)^c \), the contribution to \( \tilde{u} \) is \( (x^{a+1} y^b (1+z)^{m+\frac{3}{2}}, x^a y^{b+1} (1+z)^{m+\frac{3}{2}}, x^a y^b (1+z)^{m+\frac{3}{2}})^t \). Hence

\[
\left( w^{(2)}_\phi \right)^{-1} \begin{pmatrix} 0 \\ 0 \\ \xi^a \eta^b (1 - \zeta)^c \end{pmatrix} \circ \phi = \nabla \times \frac{1}{m+2} \begin{pmatrix} 0 \\ \frac{x^{a+1} y^b (1+z)^{m+\frac{3}{2}}}{(1+z)^{m+\frac{3}{2}}} \\ \frac{x^a y^{b+1} (1+z)^{m+\frac{3}{2}}}{(1+z)^{m+\frac{3}{2}}} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{x^a y^b (1+z)^{m+\frac{3}{2}}}{(1+z)^{m+\frac{3}{2}}} \end{pmatrix} \in \mathcal{R}^{(2)}_k.
\]

Note that \( \frac{x^a y^b}{(1+z)^{m+\frac{3}{2}}} \in Q_{k}^{[k-1,k-1]} \) unless \( a = m \) or \( b = m \), but in these cases, \( \left( 1 - \frac{a+b+2}{m+2} \right) = 0 \).\(^{13}\)

**Proof of Lemma 13.** An alternative, but less self-contained, way of stating this Lemma would be to claim that the trace spaces of the \( \mathcal{R}^{(s)}_k(K) \) elements are identical to those of the original \( U^{(s)}_k(K) \) elements, which satisfy exactly the same compatibility property. Consequently, the strategy and tools from [17] may be reapplied in an identical fashion. We will therefore just provide a sketch of how this may be done.

First we need to show that the restrictions of the traces of the \( \mathcal{R}^{(s)}_k(K) \) functions to each face lie in the trace spaces of the corresponding tetrahedral or hexahedral approximation spaces. Secondly we need to show that any valid trace can be achieved by some member of \( \mathcal{R}^{(s)}_k(K) \).

For the first step, convenient definitions of the tetrahedral and hexahedral spaces may be found in [14] and the traces of these spaces are identified explicitly in [17]. It is just a matter of exhaustive checking to determine that the inclusion holds. As an illustration, observe that members of the \( \mathcal{R}^{(0)}_k \) which are non-zero on the face \( y = 0 \) of the infinite pyramid can be expressed in terms of monomials \( \frac{x^a}{(1+z)^{m+\frac{3}{2}}} \), where \( a + c \leq k \), which map to \( \xi^a \zeta^{k-a-c} \), which will span all polynomials of degree \( k \) on the face \( \eta = 0 \) of the finite pyramid, which is precisely the trace space of the \( k \)th order Lagrange tetrahedron.

\(^{13}\)Just as earth-shattering, this is the observation that \((0,0,\xi^a)^t\) and \((0,0,\eta^b)^t\) are exact 2-forms.
The second step is equivalent to requiring that the combined external degrees of freedom inherited from the tetrahedra and hexahedra across all the vertices, edges and faces of the pyramid be dual to the trace spaces on the pyramid. So it can be proved by demonstrating a linearly independent set of pyramidal shape functions with non-zero traces that is the same size as the set of external degrees of freedom. This task can be made more manageable by instead showing that it is possible to achieve the lowest order bubble on each face, edge and vertex of the pyramid that is zero on every other face, edge or vertex, respectively. (N.B. In [17], for completeness we actually present an example of all the bubbles, not just the lowest order). Happily, the sets of shape functions presented for the \( U^{(s)}_k(K) \) in tables 2, 3 and 4 of [17] also suffice for the \( R^{(s)}_k(K) \).

\[ \square \]

**Proof of Lemma 18.** Each \( \mathcal{X}^{(s)}_{r,k} \) is a subset of \( R^{(s)}_k \), so

\[ \mathcal{X}^{(s)}_{0,k}(K) \oplus \cdots \oplus \mathcal{X}^{(s)}_{k,k}(K) \subset R^{(s)}_k(K) \]

For the reverse inclusion, we will deal with each \( s \in \{0,1,2,3\} \), in turn. For every \( s \in \{0,1,2,3\} \), the transformation rule, (6), gives \( \tilde{u} \circ \phi = w^{(s)}_\phi \tilde{u} \).

For 0-forms, the weight in the change of coordinates formula \( w^{(0)}_\phi \) is equal to 1 so any \( u \in R^{(s)}_k(K) \) satisfies \( \tilde{u} \circ \phi = \tilde{u} \in R^{(0)}_k = Q^{[k,k]}_k \). The decomposition, (41) gives

\[ Q^{[k,k]}_k = Q^{0,0,0}_0 \oplus \cdots \oplus Q^{k,k,0}_k \]

which is a subset of \( Q^{1,1,0}_0 \oplus \cdots Q^{k+1,k+1,0}_0 \) so \( u \in \mathcal{X}^{(0)}_{0,k}(K) \oplus \cdots \oplus \mathcal{X}^{(s)}_{k,k}(K) \).

For the cases \( s = 1 \) and \( s = 2 \), we will consider a basis for \( R^{(1)}_k(K) \) and show that each element, \( u \), of the basis is a member of \( \mathcal{X}^{(1)}_{r,k}(K) \) for some \( r \in \{0 \ldots k\} \), which amounts to showing that each \( u \circ \phi \in Q^{r+1,r+1,0}_r \).

From the definition given in (42c) it’s natural to consider three cases for an element of a basis for \( R^{(1)}_k(K) \). First suppose that \( \tilde{u} \in \left(Q^{[k-1,k]}_{k+1} \times 0 \times 0\right) \) with \( u_1 = \frac{x^ay^b}{(1+z)^c} \).

From the definition of \( Q^{[k-1,k]}_{k+1} \) we see that \( 0 \leq a \leq c - 2 \) and \( 0 \leq b \leq c - 1 \) and \( 2 \leq c \leq k + 1 \). Then \( w^{(1)}_\phi \tilde{u} = \left( \frac{x^ay^b}{(1+z)^c}, 0, \frac{x^{a+1}y^b}{(1+z)^{c-1}} \right)^t \) and so each \( u_i \in Q^{a+1,k,0}_{c-1} \subset Q^{r+1,r,0}_r \) where \( r = c - 1 \in \{1 \ldots k\} \). The second case is when \( \tilde{u} \in \left(0 \times Q^{[k-1,k]}_{k+1} \right) \times 0 \) and the reasoning is identical to the first. Finally suppose that \( \tilde{u} = \nabla p \) where \( p = \frac{x^ay^b}{(1+z)^c} \in Q^{[k,k]}_k \).

When \( c = 0, p = 1 \) and \( \nabla p = 0 \). So we can take \( c \geq 1 \) and see that each entry of

\[ w^{(1)}_\phi \tilde{u} = \left( \begin{array}{c} a \frac{x^{a-1}y^b}{(1+z)^c} \\ b \frac{x^ay^{b-1}}{(1+z)^{c-1}} \\ (a + b - c) \frac{x^ay^b}{(1+z)^{c-1}} \end{array} \right) \]

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is in $Q_{r+1,r+1,0}^{k+1}$ for some $r \in \{0 \ldots k\}$.

When $u \in \mathcal{R}_k^{(2)}(K)$, lets start with the case $\tilde{u} \in \left(0 \times 0 \times Q_{k+2}^{[k-1,k-1]}\right)$ and write $u_3 = \frac{x^ay^b}{(1+z)c}$. Again, it is simple to check that each of the entries in the vector $w_\phi^{(2)}\tilde{u} = \begin{pmatrix} -x^ay^b \frac{(1+z)c}{b^a y^{b+1}} & x^ay^b \frac{(1+z)c}{b^a y^{b+1}} \end{pmatrix}^T$ is in $Q_{r+1,r+1,0}^{k+1}$ for some $r \in \{0 \ldots k\}$. Now suppose that $\tilde{u} = \nabla \times \tilde{v}$ where $\tilde{v} \in \left(Q_{k+1}^{[k-1,k]} \times 0 \times 0\right)$ with $v_1 = \frac{x^ay^b}{(1+z)c}$. From the $s=1$ case, we know that $c \geq 2$ and so its straightforward to verify that each of the entries in

$$w_\phi^{(2)}\tilde{u} = (1+z)^2 \begin{pmatrix} 1+z & 0 & -x \\ 0 & 1+z & -y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{bx^ay^b}{(1+z)c^2} \\ \frac{bx^ay^b}{(1+z)c^2} \end{pmatrix} = \begin{pmatrix} \frac{-bx^ay^b}{(1+z)c^2} \\ -\frac{cx^ay^b}{(1+z)c^2} + \frac{-bx^ay^b}{(1+z)c^2} \\ \frac{bx^ay^b}{(1+z)c^2} \end{pmatrix}$$

are in $Q_{r+1,r+1,0}^{k+1}$ for some $r \in \{0 \ldots k\}$. The argument for $\tilde{u} = \nabla \times \tilde{v}$ with $\tilde{v} \in \left(Q_{k+1}^{[k-1,k-1]} \times 0 \times 0\right)$ is the same.

Finally, $u \in \mathcal{R}_k^{(3)}(K)$ means that $\tilde{u} \in Q_{k+3}^{[k-1,k-1]}$. The weight $w_\phi^{(3)} = \frac{1}{(1+z)^2}$ so $\hat{\tilde{u}} = \frac{\tilde{u}}{(1+z)^2} \in Q_{k+1}^{[k-1,k-1]}$ and the reasoning is the same as the 0-form case. 

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