Trivial 2-cocycles for invariants of mod $p$ homology spheres and Perron’s conjecture

Candidat: Ricard Riba Garcia

Director de la tesis: Dr. Wolfgang Pitsch
CERTIFICO que la present Memòria ha estat realitzada per en Ricard Riba Garcia, sota la meva supervisió i que constitueix la seva Tesi per a aspirar al grau de Doctor en Matemàtiques per la Universitat Autònoma de Barcelona. Bellaterra, 26 de febrer de 2020.

Dr. Wolfgang Pitsch
Sempre he pensat que quan un escriu un text amb passió i il·lusió, d’alguna manera els diferents moments que ha viscut en el transcurs de la escriptura queden immortalitzats entre les línies d’aquell text. Cada vegada que llegoix aquesta memòria puc sentir tots els bons moments que vaig viure quan la vaig escriure. Aquests agraïments només són un esbós de tota la gratitud que sento quan llegoix aquesta memòria.

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At the beginning of the 18th century, Leonhard Euler published a paper on the solution of the Königsberg bridge problem entitled: "Solutio problematis ad geometriam situs pertinentis", which translates into English as "The solution of a problem relating to the geometry of position". Perhaps, his work deserves to be considered as the beginnings of topology. In that paper, he demonstrated that it was impossible to find a route through the town of Königsberg (now Kaliningrad) that would cross each of its seven bridges exactly once. This result did not depend on the lengths of the bridges, nor on their distance from one another, but only on connectivity properties: which bridges connect to which islands or riverbanks. This problem in introductory mathematics called Seven Bridges of Königsberg led to the branch of mathematics known as graph theory.

The paper not only shows that the problem of crossing the seven bridges in a single journey is impossible, but generalises the problem to show that, in today’s notation,

A graph has a path traversing each edge exactly once if exactly two vertices have odd degree.

The next step in freeing mathematics from being a subject about measurement was also due to Leonhard Euler. In 1750 he wrote a letter to Christian Goldbach which, as well as commenting on a dispute Goldbach was having with a bookseller, gives Euler’s famous formula for a polyhedron:

\[ v - e + f = 2, \]

where \( v \) is the number of vertices of the polyhedron, \( e \) is the number of edges and \( f \) is the number of faces. Two years later, Leonhard Euler published details of his formula in two papers, the first admits that Euler cannot prove the result but the second gives a proof based on dissecting solids into tetrahedral slices. Following his ideas, at the begining of the 19th century, Antoine-Jean Lhuilier, who worked for most of his life on problems relating to Euler’s formula, published an important work
in which he noticed that Euler’s formula didn’t work for solids with holes in them. In particular, he proved that if a solid has \( g \) holes then
\[
v - e + f = 2 - 2g.
\]

This was the first known result on a topological invariant. In fact, this result was the first step to get the actual classification of compact connected surfaces. Then a natural question that arise from the classification of surfaces is if there exists a similar result for 3-manifolds. In particular, we concern about how to decide whenever two 3-manifolds are homeomorphic or not.

To deal with such problem, one tries to construct invariants of 3-manifolds, that is, a function which sends each 3-manifold to an element of an abelian group in such a way that if two 3-manifolds are homeomorphic then this function takes the same value on these 3-manifolds.

A first difficulty is to find a convenient way to describe 3-manifolds. A natural way to do this is using the "cutting and paste" techniques. These techniques consist on gluing several pieces to get a 3-manifold. When one uses these techniques one can either consider ”complicated” pieces and ”simple” gluing maps, like in Thurston’s decomposition, or ”simple” pieces and ”complicated” gluing maps. In the last case one finds the theory of Heegaard splittings, in which the pieces are two handlebodies and the glueing map is an element of the mapping class group. In this thesis we will use the Heegaard splittings theory.

Consider a standardly embedded surface \( \Sigma_g \) in the 3-sphere \( S^3 \). This surface separates \( S^3 \) in to two handlebodies of the same genus \( H_g, -H_g \). If we glue the boundaries of these handlebodies by an element \( f \) of the mapping class group of \( \Sigma_g \) we get another 3-manifold \( H_g \cup_f -H_g \). In fact, every 3-manifold can be obtained in this way. By technical reasons we consider \( \Sigma_{g,1} \) the surface \( \Sigma_g \) with a marked disk and \( M_{g,1} \) its mapping class group. The embedding \( \Sigma_g \hookrightarrow S^3 \) induces the following three natural subgroups of \( M_{g,1} \):

- \( A_{g,1} = \text{subgroup of restrictions of diffeomorphisms of the outer handlebody } -H_g \),
- \( B_{g,1} = \text{subgroup of restrictions of diffeomorphisms of the inner handlebody } H_g \),
- \( AB_{g,1} = \text{subgroup of restriction of diffeomorphisms of the 3-sphere } S^3 \).

Denote by \( V^3 \) the set of oriented diffeomorphism classes of compact, closed and oriented smooth 3-manifolds. The main advantage of using the Heegaard splittings theory instead of other theories, is the existence of the following bijection
\[
\lim_{g \to \infty} A_{g,1}|M_{g,1}/B_{g,1} \rightarrow V^3,
\]
\[
\phi \mapsto S^3_{\phi} = H_g \cup_{\gamma \phi} -H_g,
\]

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proved by J. Singer in 1933. This bijection allows us to translate topological problems into algebraic problems on $M_{g,1}$. In particular, let $S^3 \subset \Sigma^3$ be the subset of all integral homology 3-spheres, if we restrict the previous bijection to the Torelli group $T_{g,1}$, which is the subgroup formed by the elements that act trivially in the first homology group of the surface $\Sigma_{g,1}$, one has the following bijection:

$$\lim_{g \to \infty} A_{g,1} \rightarrow T_{g,1} \rightarrow \frac{B_{g,1}}{\sim} \cong S^3.$$ 

Moreover, in [38], W. Pitsch proved that the induced equivalence relation on $T_{g,1}$, which is given by: $\phi \sim \psi \iff \exists \zeta_a \in A_{g,1} \exists \zeta_b \in B_{g,1}$ such that $\zeta_a \phi \zeta_b = \psi$, can be rewritten as follows:

**Lemma.** Two maps $\phi, \psi \in M_{g,1}[d]$ are equivalent if and only if there exists a map $\mu \in AB_{g,1}$ and two maps $\xi_a \in A_{g,1}[d]$ and $\xi_b \in B_{g,1}[d]$ such that $\phi = \mu \xi_a \psi \xi_b \mu^{-1}$.

As a consequence, he obtained the following bijection:

$$\lim_{g \to \infty} (T_{A_{g,1}} \rightarrow T_{g,1} \rightarrow \frac{T_{B_{g,1}}}{\sim})_{AB_{g,1}} \cong S^3,$$

where $T_{A_{g,1}} = T_{g,1} \cap A_{g,1}$, $T_{B_{g,1}} = T_{g,1} \cap B_{g,1}$ and $(\_ )_{AB_{g,1}}$ are the $AB_{g,1}$-coinvariants.

This bijection allows us to define an invariant of integral homology 3-spheres $F : S^3 \rightarrow A$, as family of functions $\{F_g\}_g$ on the Torelli group with the following properties:

i) $F_{g+1}(x) = F_g(x)$ for every $x \in T_{g,1}$,

ii) $F_g(\xi_a x \xi_b) = F_g(x)$ for every $x \in T_{g,1}$, $\xi_a \in T_{A_{g,1}}$, $\xi_b \in T_{B_{g,1}}$,

iii) $F_g(\phi x \phi^{-1}) = F_g(x)$ for every $x \in T_{g,1}$, $\phi \in AB_{g,1}$.

In addition, if we consider the associated trivial 2-cocycles $\{C_g\}_g$, which measure the failure of the maps $\{F_g\}_g$ to be homomorphisms of groups, i.e.

$$C_g : T_{g,1} \times T_{g,1} \rightarrow A$$

$$(\phi, \psi) \mapsto F_g(\phi) + F_g(\psi) - F_g(\phi \psi),$$

then the family of 2-cocycles $\{C_g\}_g$ inherits the following properties:

1) The 2-cocycles $\{C_g\}_g$ are compatible with the stabilization map, i.e. the following diagram of maps commutes:

$$\begin{array}{ccc}
T_{g,1} \times T_{g,1} & \xrightarrow{\zeta} & T_{g+1,1} \times T_{g+1,1} \\
\downarrow & & \downarrow \\
A & \xrightarrow{C_g} & A
\end{array}$$
(2) The 2-cocycles \( \{C_g\}_g \) are invariant under conjugation by elements in \( \mathcal{AB}_{g,1} \), i.e. for every \( \phi \in \mathcal{AB}_{g,1} \),
\[
C_g(\phi - \phi^{-1}, \phi - \phi^{-1}) = C_g(-,-),
\]
(3) If \( \phi \in \mathcal{T}_A_{g,1} \) or \( \psi \in \mathcal{T}_B_{g,1} \) then \( C_g(\phi, \psi) = 0 \).

In 2008, W. Pitsch, using this setting, gave a new tool to get invariants with values in an abelian group without 2-torsion of integral homology 3-spheres as trivializations of 2-cocycles on \( \mathcal{T}_{g,1} \). Moreover, using this tool, he gave a new purely algebraic construction of the Casson’s invariant.

In this thesis we generalized this tool to invariants with values in an abelian group without restrictions getting the following result:

**Theorem A.** Let \( A \) be an abelian group and \( A_2 \) the subgroup of elements of order \( \leq 2 \). For each \( x \in A_2 \), a family of cocycles \( \{C_g\}_{g \geq 3} \) on the Torelli groups \( \mathcal{T}_{g,1}, g \geq 3 \), satisfying conditions (1)-(3) provides a compatible family of trivializations \( F_g + \mu^x_g : \mathcal{T}_{g,1} \rightarrow A \) that reassemble into an invariant of homology spheres
\[
\lim_{g \to \infty} F_g + \mu^x_g : S^3 \rightarrow A
\]
if and only if the following two conditions hold:

(i) The associated cohomology classes \( [C_g] \in H^2(\mathcal{T}_{g,1}; A) \) are trivial.

(ii) The associated torsors \( \rho(C_g) \in H^1(\mathcal{AB}_{g,1}, \text{Hom}(\mathcal{T}_{g,1}, A)) \) are trivial.

The main difference between this generalization and the tool given in [38] is the presence of the Rohlin invariant which accounts for the failure unicity in the construction of the invariants.

In the same year, B. Perron conjectured an invariant of \( \mathbb{Q} \)-homology spheres constructed from a Heegaard splitting with gluing map an element of the (mod \( p \)) Torelli group \( \mathcal{M}_{g,1}[p] \) with values on \( \mathbb{Z}/p \). Here we call the (mod \( p \)) Torelli group the kernel of the canonical map \( \mathcal{M}_{g,1} \rightarrow \text{Sp}_{2g}(\mathbb{Z}/p) \), which is product of the Torelli subgroup and the subgroup generated by \( p \)-powers of Dehn twists.

Such conjectured invariant consist on writing an element of the (mod \( p \)) Torelli group as a product of an element of the Torelli group and an element of the subgroup generated by \( p \)-powers of Dehn twists and taking the Casson’s invariant modulo \( p \) of the part of the Torelli group.

The main purpose of this thesis is to study the Perron’s conjecture.

In order to achieve our target, we first study the subclass of \( \mathbb{Q} \)-homology 3-spheres, which are the 3-manifolds whose first homology group is finite. It is well known that there is not a subgroup of \( \mathcal{M}_{g,1} \) that parametrizes all the \( \mathbb{Q} \)-homology 3-spheres. Nevertheless, the union of all (mod \( p \)) Torelli groups with \( p \) prime parametrizes such manifolds. Moreover we get the following criterion to know whenever a \( \mathbb{Q} \)-homology 3-sphere can be constructed as a Heegaard splitting with gluing map an element of (mod \( d \)) Torelli group.
Theorem B. Let $M$ be a $\mathbb{Q}$-homology 3-sphere and $n = |H_1(M;\mathbb{Z})|$. Then $M$ has a Heegaard splitting $\mathcal{H}_g \cup_{\phi} -\mathcal{H}_g$ of some genus $g$ with gluing map $f \in \mathcal{M}_{g,1}[d]$ with $d \geq 2$, if and only if $d$ divides $n - 1$ or $n + 1$.

Denote by $S^3[d]$ the set of $\mathbb{Q}$-homology 3-spheres which are homeomorphic to $\mathcal{H}_g \cup_{\phi} -\mathcal{H}_g$ for some $\phi \in \mathcal{M}_{g,1}[d]$. The above criterion tell us that, unlikely the case of integral homology 3-spheres and the Torelli group, in general, $S^3[d]$ does not coincide with the set of $\mathbb{Z}/d$-homology 3-spheres.

Nevertheless, we still have the following bijection:

$$
\lim_{g \to \infty} (\mathcal{A}_{g,1}[d]\backslash \mathcal{M}_{g,1}[d]/\mathcal{B}_{g,1}[d])_{\mathcal{A}B_{g,1}} \cong S^3[d],
$$

where $\mathcal{A}_{g,1}[d] = \mathcal{M}_{g,1}[d] \cap \mathcal{A}_{g,1}$ and $\mathcal{B}_{g,1}[d] = \mathcal{M}_{g,1}[d] \cap \mathcal{B}_{g,1}$.

Then using an analogous setting that we used to construct invariants of integral homology 3-spheres as a trivialization of 2-cocycles, for the $(\text{mod } d)$ Torelli group we get the following result:

Theorem C. Let $A$ an abelian group. For $g \geq 3$, $d \geq 3$ an odd integer and for $g \geq 5$, $d \geq 2$ an even integer such that $4 \nmid d$. For each $x \in A_d$, a family of 2-cocycles $\{C_g\}_{g \geq 3}$ on the $(\text{mod } d)$ Torelli groups $\mathcal{M}_{g,1}[d]$, with values in $A$, satisfying conditions analogous to (1)-(3) provides a compatible family of trivializations $F_g + \varphi_g^x : \mathcal{M}_{g,1}[d] \to A$ that reassembles into an invariant of $\mathbb{Q}$-homology spheres $S^3[d]$

$$
\lim_{g \to \infty} F_g + \varphi_g^x : S^3[d] \to A
$$

if and only if the following two conditions hold:

(i) The associated cohomology classes $[C_g] \in H^2(\mathcal{M}_{g,1}[d]; A)$ are trivial.

(ii) The associated torsors $\rho(C_g) \in H^1(\mathcal{A}B_{g,1}, \text{Hom}(\mathcal{M}_{g,1}[d], A))$ are trivial.

Unfortunately, in our attempt to use these tools to prove Perron’s conjecture, we realised that the conjecture is actually false. What is more, we get the following result:

Theorem D. Perron’s conjecture is false with an obstruction given by the non-vanishing first characteristic class of surface bundles reduced modulo $p$. 


Introduction

Structure of the thesis

As a starting point, in Chapter 1 we give the background that we will use throughout this thesis. In Chapter 2 we extend the tools given in [38] to give a construction of invariants with values on an abelian group without restrictions, from a suitable family of 2-cocycles on $T_{g,1}$, proving Theorem A. Moreover we give some interesting results about the Luft subgroup $L_{g,1}$ and the handlebody subgroup $B_{g,1}$.

In Chapter 3 we show that every Heegaard splitting with gluing map an element of the (mod $p$) Torelli group is a $\mathbb{Q}$-homology 3-sphere and that every $\mathbb{Q}$-homology 3-sphere has a Heegaard splitting with gluing map an element of the (mod $p$) Torelli group. In particular, we give a criterion to determine whenever a $\mathbb{Q}$-homology 3-sphere has a Heegaard splitting with gluing map an element of the (mod $d$) Torelli group, proving Theorem B.

In Chapter 4 we compare the automorphisms of $p$-nilpotent quotients of the free group given by the Stallings or Zassenhaus filtration of the free group. To be more precise, given $\Gamma$ a free group of finite rank and $\{\Gamma^k\}$, the Stallings or Zassenhaus $p$-central series and $N^k = \Gamma/N^k+1$. We show that there is a well-defined homomorphism $\psi_k : Aut(N^k+1) \rightarrow Aut(N^k)$ that fits into a non central extension

$$0 \rightarrow Hom(N_1^k, L^k+1) \rightarrow Aut(N^k) \rightarrow Aut(N^k) \rightarrow 1,$$

and we study its splitability.

In Chapter 5 we extend the results obtained in Chapter 2 to the case of the (mod $p$) Torelli group, getting a new tool to construct invariants of $\mathbb{Q}$-homology spheres as a trivializations of certain trivial 2-cocycles on the (mod $p$) Torelli group, proving Theorem C. Moreover, we get a new invariant of $\mathbb{Q}$-homology 3-spheres as a family of homomorphisms on the (mod $p$) Torelli group.

Finally, in Chapter 6 we give our obstruction to Perron’s conjecture, proving Theorem D.
In this Chapter we introduce basic background material that we will use throughout all this thesis.

1.1 Cohomology of groups and group extensions

In this section we give some definitions and elementary results about cohomology of groups and group extensions. For further information on these topics see [3] and [15].

1.1.1 The group algebra

Definition 1.1.1. The group algebra $\mathbb{Z}G$ is the free $\mathbb{Z}$-module with basis $G$ and with multiplication induced by the multiplication in the group $G$. Thus, elements are formal linear combinations

$$\sum_{g \in G} a_g g,$$

where $a_g \in \mathbb{Z}$, and where $a_g = 0$ for all but finitely many $g \in G$, and multiplication is given by

$$\sum_{g \in G} a_g g \sum_{h \in G} b_h h = \sum_{g,h \in G} (a_g b_h) g \cdot h.$$

With these definitions, the group algebra $\mathbb{Z}G$ becomes an associative ring with unit.

Invariants and coinvariants Let $M$ be a left $G$-module. The invariants of $M$ are the elements of the $\mathbb{Z}$-submodule

$$M^G = \{ m \in M \mid gm = m \text{ for all } g \in G \}.$$

In fact, $M^G$ is the largest submodule of $M$ on which $G$ acts trivially.
Chapter 1. Preliminaries

The coinvariants of $M$ are the elements of the quotient $\mathbb{Z}$-module

$$M_G = M/(gm - m \mid g \in G, m \in M).$$

Indeed, $M_G$ is the largest quotient of $M$ on which $G$ acts trivially.

**Left and right $G$-modules** Let $M$ be a non-trivial (left) $G$-module given by the action

$$G \times M \to M$$

$$(g, m) \mapsto gm,$$

since $G$ is a group and in particular every element $g \in G$ has an inverse $g^{-1} \in G$, then we can give to $M$ the structure of a non-trivial (right) $G$-module taking the action

$$M \times G \to M$$

$$(m, g) \mapsto g^{-1}m.$$ Analogously, if $M$ is a non-trivial (right) $G$-module then $M$ admits the structure of a non-trivial (left) $G$-module. Therefore we will not distinguish between left and right $G$-modules.

### 1.1.2 Homology and Cohomology of groups

We define the $n$-th cohomology group of $G$ with coefficients in the $G$-module $A$ by

$$H^n(G; A) = \text{Ext}_Z^n(G, A),$$

where $\mathbb{Z}$ is to be regarded as a trivial $G$-module. In particular $H^0(G; A) = A_G$.

The $n$-th homology group of $G$ with coefficients in the $G$-module $B$ is defined by

$$H_n(G, B) = \text{Tor}_n^{ZG}(B, \mathbb{Z}),$$

where again $\mathbb{Z}$ is to be regarded as trivial $G$-module. In particular, $H_0(G; B) = B^G$.

The way to compute such groups is taking a $G$-projective resolution $P$ of the trivial $G$-module $\mathbb{Z}$, form the complexes $\text{Hom}_{ZG}(P, A)$ and $B \otimes_{ZG} P$, and compute their homology.

In the next section we will give a standard procedure of constructing such a resolution $P$ from the group $G$. 
1.1.3 The bar resolution

We now describe a particular resolution $B \rightarrow Z$ of the trivial $ZG$-module $Z$, called the bar resolution or the standard resolution.

Let $B_n$ be the free $Z$-module with basis all $(n+1)$-tuples $(g_0, \ldots, g_n) \in G^{n+1}$. Then $B_n$ becomes a $ZG$-module via the diagonal action

$$g \cdot (g_0, \ldots, g_n) = (gg_0, \ldots, gg_n)$$

for $g \in G$. As a $ZG$-module, $B_n$ is free with basis all elements of the form

$$[g_1|\ldots|g_n] = (1, g_1, g_1g_2, \ldots, g_1g_2\ldots g_n)$$

where $g_1, \ldots, g_n \in G$. In particular, $B_0$ is free with basis $[\ ] = (1)$, so we may identify $B_0$ with the $ZG$-module $ZG$.

For $0 \leq i \leq n$, define the $i^{th}$ face map $d_i : B_n \rightarrow B_{n-1}$ to be the homomorphism of $Z$-modules determined by

$$d_i(g_0, \ldots, g_n) = (g_0, \ldots, \hat{g_i}, \ldots, g_n)$$

where the symbol $\hat{g_i}$ indicates that $g_i$ is to be omitted. Clearly, $d_i$ is also a homomorphism of $ZG$-modules. It is easy to verify that if $i < j$ then

$$d_i \circ d_j = d_{j-1} \circ d_i.$$

If we define $\hat{c}_n : B_n \rightarrow B_{n-1}$ by

$$\hat{c}_n = \sum_{i=0}^{n} (-1)^i d_i,$$

then the above relation implies that $\hat{c}_n \circ \hat{c}_n = 0$. Moreover $B_0/\text{Im } \hat{c}_1 = Z$, by a direct computation. Thus,

$$B_\bullet : \cdots \longrightarrow B_n \xrightarrow{\hat{c}_n} B_{n-1} \longrightarrow \cdots \longrightarrow B_1 \xrightarrow{\hat{c}_1} B_0 \longrightarrow 0$$

is a free resolution of the (left) $G$-module $Z$.

In terms of the basis $[g_1|\ldots|g_n]$ for $B_n$, the face maps $d_i$ take the form:

$$d_0[g_1|\ldots|g_n] = g_1[g_2|\ldots|g_n],$$
$$d_i[g_1|\ldots|g_n] = [g_1|\ldots|g_{i-1}|g_i|g_{i+1}|\ldots|g_n] \quad (0 < i < n),$$
$$d_n[g_1|\ldots|g_n] = [g_1|\ldots|g_{n-1}].$$
Thus, for instance
\[\partial_1 [g_1] = g_1 [] - [],\]
\[\partial_2 [g_1, g_2] = g_1 [g_2] - [g_1 g_2] + [g_2].\]

For any group \(G\) we can always take our resolution \(F\) to be the bar resolution. In this case we write \(C_*(G;M)\) for \(F \otimes_G M\) and \(C^*(G;M)\) for \(\text{Hom}_G(F,M)\). Thus an element of \(C_n(G;M)\) can be uniquely expressed as a finite sum of elements of the form \(m \otimes [g_1|\cdots|g_n]\), i.e., as a formal linear combination with coefficients in \(M\) of the symbols \([g_1|\cdots|g_n]\). The boundary operator \(\partial : C_n(G;M) \to C_{n-1}(G;M)\) is given by
\[\partial(m \otimes [g_1|\cdots|g_n]) = mg_1 \otimes [g_2|\cdots|g_n]\]
\[- m \otimes [g_1 g_2|\cdots|g_n] + \cdots + (-1)^n m \otimes [g_1|\cdots|g_{n-1}].\]

Similarly, an element of \(C^n(G,M)\) can be regarded as a function \(f : G^n \to M\), i.e., as a function of \(n\) variables from \(G\) to \(M\). The coboundary operator \(\delta : C^{n-1}(G;M) \to C^n(G;M)\) is given, up to sign by
\[(\delta f)(g_1, \ldots, g_n) = g_1 f(g_2, \ldots, g_n)\]
\[\quad - f(g_1 g_2, \ldots, g_n) + \cdots + (-1)^n f(g_1, \ldots, g_{n-1}).\]

### 1.1.4 Some results on Cohomology of Groups

**Lemma 1.1.1** (Center kills lemma (Lemma 5.4. of [5])). Let \(M\) be a (left) \(R[G]\)-module (\(R\) any commutative ring) and let \(\gamma \in G\) be a central element such that for some \(r \in R\), \(\gamma x = rx\) for all \(x \in M\). Then \((r-1)\) annihilates \(H_* (G;M)\).

**Definition 1.1.2** ([3], VI.7.). For \(M\) an abelian group, define \(M^* = \text{Hom}(M,\mathbb{Q}/\mathbb{Z})\). Let \(L \geq 2\) be an integer, observe that if \(M\) is a module over \(\mathbb{Z}/L\), then \(M^* \cong \text{Hom}(M,\mathbb{Z}/L)\).

**Lemma 1.1.2** ([3], VI. Proposition 7.1.). Let \(G\) be a group and let \(M\) be a \(G\)-module. Then there is a natural isomorphism \(H^k(G;M^*) \cong (H_k(G;M))^*\) for every \(k \geq 0\).

**Theorem 1.1.1** (K. Dekimpe, M. Hartl, S. Wauters [5]). For any group extension \(1 \to N \to G \to Q \to 1\) and any \(G\)-module \(M\), the Lyndon-Hochschild-Serre spectral sequence gives rise to an exact sequence:
\[
0 \longrightarrow H^1(Q;M^N) \xrightarrow{\text{inf}} H^1(G;M) \xrightarrow{\text{res}} H^1(N;M)^Q \xrightarrow{\text{tr}} H^2(Q;M^N)
\]
then there are natural exact sequences

\[ \inf H^2(G;M)_1 \xrightarrow{\rho} H^1(Q;H^1(N;M)) \xrightarrow{\lambda} H^3(Q;M^N), \]

where "inf" and "res" are respectively the inflation and restriction maps, and \( H^2(G;M)_1 \) is the kernel of the restriction map \( \text{res} : H^2(G,M) \to H^2(N,M) \).

**Theorem 1.1.2.** Let \( 0 \to L \to M \to N \to 0 \) a short exact sequence of \( G \)-modules, then there is a long exact sequence:

\[ 0 \to L^G \to M^G \to N^G \to H^1(G,L) \to H^1(G,M) \to H^1(G,N) \to H^2(G,L) \]

**Lemma 1.1.3** (Leedham \[24\]). If \( R \) is a commutative ring, \( A \) is an \( RG \)-module and \( n \geq 0 \) then there is a natural isomorphism

\[ \text{Ext}^n_{RG}(R,A) \cong \text{Ext}^n_{ZG}(Z,A). \]

**Proof.** If \( P \) is a projective \( ZG \)-module then \( P \otimes R \) is a projective \( RG \)-module as this is true for free modules. Let \( P \to Z \) be a projective resolution of \( Z \) as a \( ZG \)-module. Then \( P \otimes R \to R \) is a projective resolution of \( R \) as an \( RG \)-module since \( P \otimes R \to R \) is exact as its homology groups are \( \text{Tor}^n_P(Z,R) = 0 \). Since \( \text{Hom}_{RG}(P_n \otimes R,A) \) is naturally isomorphic to \( \text{Hom}_{ZG}(P_n,A) \) it follows that \( \text{Ext}^n_{ZG}(Z,A) \cong \text{Ext}^n_{RG}(R,A) \).

**Theorem 1.1.3** (Universal coefficients Theorem, Theorem 9.4.14 in \[24\]). Let \( G \) be a group, \( R \) a Dedekind domain considered as a trivial \( G \)-module, and \( A \) an \( RG \)-module with trivial \( G \)-action, then there are natural exact sequences

\[ 0 \to \text{Ext}^1_R(H_{n-1}(G;R),A) \to \text{Hom}_R(H_n(G;R),A) \to 0. \]

\[ 0 \to H_n(G;R) \otimes_R A \to H_n(G;A) \to \text{Tor}^1(R,H_{n-1}(G;R),A) \to 0. \]

which split, but not naturally.

**Remark 1.1.1.** The map \( \beta : H^n(G;A) \to \text{Hom}_R(H_n(G;R),A) \) of Theorem \[1.1.3\] is constructed as follows. By definition of the cohomology of groups we know that \( H^n(G;A) = \text{Ext}^n_{ZG}(Z,A) \) and by Lemma \[1.1.3\] have that \( \text{Ext}^n_{RG}(R,A) \cong \text{Ext}^n_{ZG}(Z,A) \).

Since the bar resolution \( C_*(G) \to Z \) is a projective resolution of \( Z \) as \( ZG \)-module. Then \( C_*(G) \otimes R \to R \) is a projective resolution of \( R \) as \( RG \)-module. Applying the contravariant functor \( \text{Hom}_{RG}(\cdot,A) \) to the projective resolution \( C_*(G) \otimes R \), we obtain the following chain complex

\[ \cdots \xrightarrow{\delta} \text{Hom}_{RG}(C_2(G) \otimes R,A) \xrightarrow{\delta} \text{Hom}_{RG}(C_1(G) \otimes R,A) \xrightarrow{\delta} \text{Hom}_{RG}(C_0(G) \otimes R,A) \xrightarrow{0}. \]
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Since $A$ is a trivial $RG$-module, we have the following canonic isomorphisms:

$$\text{Hom}_{RG}(C_n(G) \otimes R, A) = \text{Hom}_R(C_n(G) \otimes R, A)^G \cong \text{Hom}_R((C_n(G) \otimes R)_G, A) \cong \text{Hom}_R(C_n(G) \otimes \mathbb{Z}_G R, A).$$

Then the above chain complex becomes

$$\cdots \xrightarrow{\delta} \text{Hom}_R(C_2(G, R), A) \xrightarrow{\delta} \text{Hom}_R(C_1(G, R), A) \xrightarrow{\delta} \text{Hom}_R(C_0(G, R), A) \xleftarrow{0}.$$

Thus we have that

$$H^n(G; A) \cong H_n(\text{Hom}_R(C_*(G, R), A)). \quad (1.1.1)$$

Therefore $\beta : H^n(G; A) \to \text{Hom}_R(H_n(G; R), A)$ in Theorem (1.1.3) is given as follows. Let $z \in H^n(G; A)$ and $\tilde{z} \in H_n(\text{Hom}_R(C_*(G, R), A))$ the homology class that corresponds to $z$ by the isomorphism (1.1.1). Take a $n$-cycle $c \in \text{Hom}_{RG}(C_*(G, R), A)$, with associated homology class $\tilde{z}$. By passage to subquotients, $c$ induces an element of $\text{Hom}_R(H_n(G; R), A)$. Then $\beta(z)$ is defined as the evaluation of such $n$-cycle $c$, on $H_n(G; R)$.

**Theorem 1.1.4** (Theorem VI.8.1. in [15]). Let $1 \to N \to G \to Q \to 1$ be an exact sequence of groups. For $Q$-modules $A, B$, the following sequences are exact and natural

$$H_2(G; B) \to H_2(Q; B) \to B \otimes_Q N_{ab} \to H_1(G; B) \to H_1(Q; B) \to 0,$$

$$0 \to H^1(Q; A) \to H^1(G; A) \to \text{Hom}_Q(N_{ab}, A) \to H^2(Q; A) \to H^2(G; A).$$

Where natural means:

i) A commutative diagram of short exact sequences

$$\begin{array}{cccccc}
0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N' & \longrightarrow & G' & \longrightarrow & Q' & \longrightarrow & 1
\end{array} \quad (1.1.2)$$

induces the following commutative diagrams

$$\begin{array}{cccccc}
H_2(G; B) & \longrightarrow & H_2(Q; B) & \longrightarrow & B \otimes_Q N_{ab} & \longrightarrow & H_1(G; B) & \longrightarrow & H_1(Q; B) \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \quad (1.1.3) \\
H_2(G'; B) & \longrightarrow & H_2(Q'; B) & \longrightarrow & B \otimes_{Q'} N'_{ab} & \longrightarrow & H_1(G'; B) & \longrightarrow & H_1(Q'; B)
\end{array}$$
\begin{align}
H^1(Q';A) & \longrightarrow H^1(G';A) \longrightarrow Hom_Q(N'_{ab},A) \longrightarrow H^2(Q';A) \longrightarrow H^2(G';A) \tag{1.1.4} \\
H^1(Q;A) & \longrightarrow H^1(G;A) \longrightarrow Hom_Q(N_{ab},A) \longrightarrow H^2(Q;A) \longrightarrow H^2(G;A)
\end{align}

ii) A homomorphism of $Q$-modules $g : B \rightarrow B'$ induces the following commutative diagram
\[
\begin{array}{cccccc}
H_2(G;B) & \longrightarrow & H_2(Q;B) & \longrightarrow & B \otimes_Q N_{ab} & \longrightarrow H_1(G;B) & \longrightarrow H_1(Q;B) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_2(G;B') & \longrightarrow & H_2(Q;B') & \longrightarrow & B' \otimes_Q N_{ab} & \longrightarrow H_1(G;B') & \longrightarrow H_1(Q;B')
\end{array}
\tag{1.1.5}
\]

Similarly a homomorphism of $Q$-modules $f : A \rightarrow A'$ induces the following commutative diagram
\[
\begin{array}{cccccc}
H^1(Q;A) & \longrightarrow & H^1(G;A) & \longrightarrow & Hom_Q(N_{ab},A) & \longrightarrow H^2(Q;A) & \longrightarrow H^2(G;A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(Q;A') & \longrightarrow & H^1(G;A') & \longrightarrow & Hom_Q(N_{ab},A') & \longrightarrow H^2(Q;A') & \longrightarrow H^2(G;A')
\end{array}
\tag{1.1.6}
\]

Hopf formula modulo $q$  
In [7] Graham J. Ellis find a formula to compute the homology of groups with coefficients in $\mathbb{Z}/q$, with $q$ a positive integer. In particular, following the Exercise II.5.4 in [3] one can gives an explicit isomorphism which induces the Hopf modulo $q$ formula as follows.

Let $q$ be a positive integer and $G$ be a group with a finite representation $\{t_1, \ldots, t_n|r_1, \ldots, r_m\}$. We denote by $F$ the free group generated by a family of elements $T = \{t_1, \ldots, t_n\}$ and by $R$ the normal closure in $F$ of $\{r_1, \ldots, r_m\}$.

Consider the following exact sequence
\[
1 \longrightarrow R \overset{i}{\longrightarrow} F \overset{\overline{)}{}}{\longrightarrow} G \longrightarrow 1.
\]

where $\overline{)}$ is the quotient map corresponding to $R$.

For each $g \in G$ choose an element $s(g) \in F$ such that $\overline{s(g)} = g$. Given $g, h \in G$, write $s(g)s(h) = s(gh)r(g, h)$ where $r(g, h) \in R$. There is an abelian group homomorphism $C_2(G) \rightarrow R/[R, R]$ given by $[g, h] \mapsto r(g, h) \mod [R, R]$ and this induces an isomorphism
\[
\phi : H_2(G;\mathbb{Z}/q) \rightarrow \frac{R \cap ([F, F]F^q)}{[F, R]R^q}
\]
by passage to subquotients.
1.1.5 Group Extensions

In this section, we recall some basic results about group extensions, we make a summary of chapter IV in [3] and section VI.10 in [15].

An extension of a group $G$ by a group $N$ is a short exact sequence of groups

$$1 \to N \to E \to G \to 1 \quad(1.1.7)$$

A second extension $1 \to N \to E' \to G \to 1$ of $G$ by $N$ is said to be equivalent to $\text{(1.1.7)}$ if there is a homomorphism $E \to E'$ making the diagram

$$
\begin{array}{c}
1 \to N \to E \to G \to 1 \\
\downarrow \hspace{2cm} \downarrow \\
1 \to N \to E' \to G \to 1
\end{array}
$$

commute. Note that such a map is necessarily an isomorphism.

Consider the case where the kernel $N$ is an abelian group $A$ (written additively). A special feature of this case is that an extension

$$0 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1 \quad(1.1.8)$$

gives rise to an action of $G$ on $A$, making $A$ a $G$-module. For $E$ acts on $A$ by conjugation since $A$ is embedded as a normal subgroup of $E$; and the conjugation action of $A$ on itself is trivial, so there is an induced action of $E/A = G$ on $A$. Moreover observe that the $G$-action is trivial if and only if $i(A)$ is central in $E$. In the case that $i(A)$ is central in $E$, the extension $\text{(1.1.8)}$ is called a central extension.

**Split Extensions.** Fix a $G$-module $A$ and let

$$0 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1 \quad(1.1.9)$$

be an extension which gives rise to the given action of $G$ on $A$. We say that $\text{(1.1.9)}$ splits if there is a homomorphism $s: G \to E$ such that $\pi s = id_G$.

**Proposition 1.1.1.** The following two conditions on the extension $\text{(1.1.9)}$ are equivalent:

i) $\text{(1.1.9)}$ splits.

ii) $\text{(1.1.9)}$ is equivalent to the extension $0 \to A \to A \times G \to G \to 1$, where $A \times G$ is the semidirect product of $G$ and $A$ relative to the given action.
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Observe that Proposition (1.1.1) says that there is only one split extension $G$ by $A$ (up to equivalence) associated to the given action of $G$ on $A$. Nevertheless, there is an interesting "classification" problem involving split extensions: Given that an extension (1.1.9) splits, classify all possible splittings.

In case $G$ acts trivially on $A$, the group $E$ is isomorphic to the direct product $A \times G$ and then the splittings are obviously in 1-1 correspondence with homomorphisms $G \to A$. In the general case, splittings correspond to derivations (also called crossed homomorphisms). These are functions $d: G \to A$ satisfying

$$d(gh) = d(g) + g \cdot d(h)$$

for all $g, h \in G$.

Two splittings $s_1, s_2$ will be said to be $A$-conjugate if there is an element $a \in A$ such that $s_1(g) = i(a)s_2(g)i(a)^{-1}$ for all $g \in G$. Since $(a,1)(b,g)(a,1)^{-1} = (a + b, g \cdot a, g)$ in $A \times G$, this conjugacy relation becomes $d_1(g) = a + d_2(g) - g \cdot a$ in terms of the derivations $d_1, d_2$ corresponding $s_1, s_2$. Thus $d_1$ and $d_2$ correspond to $A$-conjugate splittings if and only if their difference $d_2 - d_1$ is a function $G \to A$ of the form $g \mapsto ga - a$ for some fixed $a \in A$. Such function is called a principal derivation.

Summarizing, the $A$-conjugacy classes of splittings of a split extension of $G$ by $A$ correspond to the elements of the quotient group $Der(G,A)/P(G,A)$, where $Der(G,A)$ is the abelian group of derivations $G \to A$ and $P(G,A)$ is the group of principal derivations. In addition, by Exercise 2 of III.1 in [3] one gets that the quotient $Der(G,A)/P(G,A)$ is canonically isomorphic to the first cohomology group $H^1(G,A)$. Therefore we have the following result:

**Proposition 1.1.2.** For any $G$-module $A$, the $A$-conjugacy classes of splittings of the split extension

$$0 \to A \to A \times G \to G \to 1$$

are in 1-1 correspondence with the elements of $H^1(G,A)$.

**Classification of extensions with abelian kernel.** Let $A$ be a fixed $G$-module. All extensions of $G$ by $A$ to be considered in this section will be assumed to give rise to the given action of $G$ on $A$. To analyse an extension

$$0 \to A \to E \xrightarrow{\pi} G \to 1,$$  (1.1.10)

we choose a set-theoretic cross-section of $\pi$, i.e., a function $s: G \to E$ such that $\pi s = id_G$.

If $s$ is a homomorphism, then the extension splits and by Proposition (1.1.1), we know that (1.1.10) is equivalent to $0 \to A \to A \times G \to G \to 1$. In the general case, however, there is a function $f: G \times G \to A$ which measures the failure of $s$ to be a homomorphism. Indeed, for any $g, h \in G$, the
elements \( s(gh) \) and \( s(g)s(h) \) of \( E \) both map to \( gh \) in \( G \), so they differ by an element of \( i(A) \). Thus we can define \( f \) by the equation:

\[
s(g)s(h) = i(f(g,h))s(gh). \tag{1.1.11}
\]

The function \( f \) is called the *factor set* associated to \((1.1.10)\) and \( s \).

The \( G \)-module structure on \( A \) and the factor set \( f \) classifies all extensions in the following way. Let \( E_f \) be the set \( A \times G \) with the group law: \((a, g)(b, h) = (a + gb + f(g,h), gh)\), then the original extension \((1.1.10)\) is equivalent to the extension \( 0 \rightarrow A \rightarrow E_f \rightarrow G \rightarrow 1 \). In particular to be \( E_f \) a group \( f \) has to satisfy the following identity:

\[
gf(h,k) - f(gh,k) + f(g,hk) - f(g,h) = 0
\]

for all \( g, h, k \in G \). As a consequence, \( f \) can be regarded as a 2-cocycle of the standard complex \( C^*(G, A) \) for computing \( H^*(G; A) \).

Notice that to get the factor set \( f \) we have chosen an arbitrary theoretic section \( s \). However, if we take two sections \( s_1, s_2 \) with associated factor sets \( f_1, f_2 \), then there exists a function \( d : G \rightarrow A \) such that

\[
f_2(g,h) = d(g) + gd(h) - d(gh) + f_1(g,h).
\]

Hence, changing the choice of the section in \((1.1.10)\) corresponds precisely to modifying the cocycle \( f \) in \( C^*(G, A) \) by a coboundary.

Therefore we get the following result:

**Proposition 1.1.3.** Let \( A \) be a \( G \)-module and let \( E(G, A) \) be the set of equivalence classes of extensions of \( G \) by \( A \) giving rise to the action of \( G \) on \( A \). Then there is a bijection \( E(G, A) \cong H^2(G, A) \).

Next we show another description of the bijection given in Proposition (1.1.3), due to U. Stammbach \[15\].

Denote by \([E^r_i] \), the class of \( E(G, A) \) containing the extension

\[
0 \longrightarrow A \overset{i}{\longrightarrow} E \overset{r}{\longrightarrow} G \longrightarrow 1. \tag{1.1.12}
\]

We now will define a map \( \Delta : E(G, A) \rightarrow H^2(G, A) \) as follows. Given an extension \((1.1.12)\), Theorem (1.1.4) yields the exact sequence

\[
0 \longrightarrow H^1(G; A) \longrightarrow H^1(E; A) \longrightarrow Hom_G(A, A) \overset{\theta_{E^r_i}}{\longrightarrow} H^2(G; A) \longrightarrow H^2(E; A). \tag{1.1.13}
\]
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Associate with the extension \( 1 \to A \to E \to G \to 1 \) the element

\[
\Delta[E'] = \theta_{E'}(id_A) \in H^2(G; A)
\]

The naturality of (1.1.13) immediately shows that \( \theta_{E'}(id_A) \in H^2(G; A) \) does not depend on the extension but only on its equivalence class in \( \mathcal{E}(G, A) \). Hence \( \Delta \) is well-defined,

\[
\Delta: \mathcal{E}(G, A) \to H^2(G, A).
\]

In VI.10. of [15], U. Stammbach showed moreover that \( \Delta \) is one-to-one.

Maps between extensions. Next we give some results about maps between two extensions induced by push-outs and pull-backs.

Proposition 1.1.4 (Exercice 1a, page 94 in [3]). Given an extension \( 0 \to A \to E \to G \to 1 \) and a group homomorphism \( \alpha: G' \to G \). There is an extension \( 0 \to A \to E' \to G' \to 1 \), characterized up to equivalence by the fact that it fits into a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow & & \downarrow \\
0 & \longrightarrow & A' \\
\end{array}
\quad
\begin{array}{ccc}
& i & \quad E \\
& e & \quad r \\
& & G \\
\alpha & & \downarrow \\
& & G' \\
\end{array}
\quad
\begin{array}{ccc}
& & 1 \\
\end{array}
\]

In fact \( (E'; r', e) \) is given by the pull-back of \( (\alpha, r) \).

As a consequence \( \alpha \) induces a map \( \alpha^*: \mathcal{E}(G, A) \to \mathcal{E}(G', A) \), which corresponds under the bijection of Proposition (1.1.3) to \( \alpha^*: H^2(G; A) \to H^2(G'; A) \).

Proposition 1.1.5 (Exercice 1b, page 94 in [3]). Given an extension \( 0 \to A \to E \to G \to 1 \) and a homomorphism \( f: A \to A' \) of \( G \)-modules, there is an extension \( 0 \to A' \to E' \to G \to 1 \), characterized up to equivalence by the fact that it fits into a commutative diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow & & \downarrow \\
0 & \longrightarrow & A' \\
\end{array}
\quad
\begin{array}{ccc}
& i & \quad E \\
& e & \quad r \\
& & G \\
\alpha & & \downarrow \\
& & G' \\
\end{array}
\quad
\begin{array}{ccc}
& & 1 \\
\end{array}
\]

In fact \( (E'; i', e) \) is given by the push-out of \( (f, i) \).

As a consequence \( f \) induces a map \( f_*: \mathcal{E}(G, A) \to \mathcal{E}(G, A') \), which corresponds under the bijection of Proposition (1.1.3) to \( f_*: H^2(G, A) \to H^2(G, A') \).
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**Corollary 1.1.1.** Let \( 1 \to A \to E \to G \to 1 \) be a central extension with associated cohomology class \( c \), let \( f : A \to A \) be a homomorphism of \( G \)-modules and \( \phi : G \to G \) a group homomorphism, such that \( f^*(c) = \phi^*(c) \) in \( H^2(G, A) \). Then there exists a homomorphism \( \Phi : E \to E \) such that the following diagram commutes.

\[
\begin{array}{cccccc}
0 & \to & A & \overset{i}{\to} & E & \overset{r}{\to} & G & \to & 1 \\
\downarrow{f} & & \downarrow{\Phi} & & \downarrow{\phi} & & \\
0 & \to & A & \overset{i}{\to} & E & \overset{r}{\to} & G & \to & 1.
\end{array}
\]

**Proof.** Given an extension \( 1 \to A \to E \to G \to 1 \), \( f : A \to A \) a homomorphism of \( G \)-modules and \( \phi : G \to G \) a group homomorphism, by Propositions (1.1.4), (1.1.5) we have the following commutative diagrams:

\[
\begin{array}{cccccc}
0 & \to & A & \overset{i}{\to} & E & \overset{r}{\to} & G & \to & 1 \\
\downarrow{f} & & \downarrow{\Phi} & & \downarrow{\phi} & & \\
0 & \to & A & \overset{i}{\to} & E' & \overset{r}{\to} & G & \to & 1.
\end{array}
\]

In addition, since \( f^*(c) = \phi^*(c) \) in \( H^2(G, A) \), using the bijection between \( \mathcal{E}(G, A) \) and \( H^2(G; A) \) given by Proposition (1.1.3), and Propositions (1.1.4), (1.1.5) we obtain the following commutative diagram

\[
\begin{array}{cccccc}
0 & \to & A & \overset{i}{\to} & E' & \overset{r}{\to} & G & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \overset{i}{\to} & E'' & \overset{r}{\to} & G & \to & 1.
\end{array}
\]

Finally reassembling the above three diagrams we get the result. \( \Box \)

**Lemma 1.1.4.** Let \( A \) be a fixed \( G \)-module,

\[
1 \to A \overset{i}{\to} E \overset{\pi}{\to} G \to 1
\]

a central extension with associated cohomology class \([c] \in H^2(G; A)\), and \( \pi^* : H^2(G; A) \to H^2(E; A) \) the induced morphism for \( \pi \). Then the cohomology class \([\pi^*(c)]\) is zero.

**Proof.** Let \([c] \in H^2(G; A)\), the cohomology class associated to

\[
1 \to A \overset{i}{\to} E \overset{\pi}{\to} G \to 1
\]
then we have the following commutative diagram

\[
\begin{array}{ccccc}
1 & \rightarrow & A & \xrightarrow{i} & \tilde{E} & \xrightarrow{\eta} & E & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & A & \xrightarrow{j} & E & \xrightarrow{\pi} & G & \rightarrow & 1
\end{array}
\]  

(1.1.14)

where the top central extension is associated to the cohomology class \([\pi^*(c)]\). But such cohomology class is given as follows. Let \(s\) a set theoretic section of \(\eta\), i.e. \(\eta \circ s = \text{id}\), then \([\pi^*(c)]\) is equal to the cohomology class of the 2-cocycle \(h\) defined by

\[
h([x|y]) = i^{-1}(s(x)s(y)s(xy)^{-1}).
\]

Now since \(j\) is injective, the diagram (1.1.14) is commutative and \(s\) is a theoretic section of \(\eta\), we have that

\[
0 = h([x|y]) \iff j(i^{-1}(s(x)s(y)s(xy)^{-1})) = 1 \iff \\
\iff (\eta \circ i)(i^{-1}(s(x)s(y)s(xy)^{-1})) = 1 \iff \\
\iff \eta(s(x)s(y)s(xy)^{-1})) = 0 \iff xy(xy)^{-1} = 1
\]

Therefore \([\pi^*(c)] \in H^2(E; A)\) is zero.

\[\square\]

**Stabilizing automorphisms of an extension**

**Definition 1.1.3** (Section 9.1.3. in [42]). An automorphism \(\varphi\) of a group \(E\) stabilizes an extension

\[
0 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1
\]

if the following diagram commutes:

\[
\begin{array}{ccccc}
0 & \rightarrow & A & \xrightarrow{i} & E & \xrightarrow{p} & G & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & A & \xrightarrow{i} & E & \xrightarrow{p} & G & \rightarrow & 1.
\end{array}
\]

The set of all stabilizing automorphisms of an extension of \(A\) by \(G\), where \(A\) is a \(G\)-module, is a group under composition and it is denoted by \(\text{Stab}(G, A)\). In addition, by Corollary 9.16 in [42],
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\( \text{Stab}(G, A) \) is isomorphic to the group of derivations \( \text{Der}(G, A) \) via the homomorphism

\[
\sigma : \text{Stab}(G, A) \to \text{Der}(G, A)
\]

\[
\varphi \mapsto (d : G \to A),
\]

where \( d(x) = \varphi(s(x)) - s(x) \) with \( s \) a section.

1.2 Geometric objects

1.2.1 Simple closed curves

We refer by a closed curve in a surface \( \Sigma \), a continuous map \( S^1 \to \Sigma \). We will usually identify a closed curve with its image in \( \Sigma \). A closed curve is called essential if it is not homotopic to a point, a puncture, or a boundary component.

A closed curve in \( \Sigma \) is simple if it is embedded, that is, if the map \( S^1 \to \Sigma \) is injective. Moreover, we say that \( \gamma \) is a bounding simple closed curve if its homology class in \( H_1(\Sigma) \) is zero, or equivalently, if \( \gamma \) separates \( \Sigma \) in two disjoint surfaces with boundary component \( \gamma \).

Any closed curve \( \alpha \) is homotopic to a smooth closed curve \( \alpha' \). What is more, if \( \alpha \) is simple, then \( \alpha' \) can be chosen to be simple.

Intersection number There are two natural ways to count the number of intersection points between two simple closed curves in an oriented surface: signed and unsigned. These correspond to the algebraic intersection number and geometric intersection number, respectively.

Let \( \alpha \) and \( \beta \) be a pair of transverse, oriented, simple closed curves in an oriented surface \( \Sigma \). The algebraic intersection number \( \tilde{i}(\alpha, \beta) \) is defined as the sum of the indices of the intersection points of \( \alpha \) and \( \beta \), where an intersection point is of index +1 when the orientation of the intersection agrees with the orientation of \( S \) and is −1 otherwise. This definition only depends on the homology classes of \( \alpha \) and \( \beta \).

The geometric intersection number between free homotopy classes \( a \) and \( b \) of simple closed curves in a surface \( S \) is defined to be the minimal number of intersection points between a representative curve in the class \( a \) and a representative curve in the class \( b \):

\[
i(a, b) = \min\{|\alpha \cap \beta| : \alpha \in a, \beta \in b\}.
\]

Isotopy for simple closed curves Two simple closed curves \( \alpha \) and \( \beta \) are isotopic if there is a smooth homotopy

\[
H : S^1 \times [0, 1] \to S
\]
from $\alpha$ to $\beta$ with the property that the closed curve $H(S^1 \times \{t\})$ is simple for each $t \in [0,1]$.

**Extension of isotopies** An isotopy of a surface $S$ is a smooth homotopy $H : S \times I \to S$ so that, for each $t \in [0,1]$, the map $H(S,t) : S \times \{t\} \to S$ is a homeomorphism. Given an isotopy between two simple closed curves in $S$, it will often be useful to promote this to an isotopy of $S$, which we call an *ambient isotopy* of $S$.

**Proposition 1.2.1.** Let $S$ be any surface. If $F : S^1 \times I \to S$ is a smooth isotopy of simple closed curves, then there is an isotopy $H : S \times I \to S$ so that $H|_{F(S^1 \times 0) \times I} = F$.

### 1.2.2 Handlebodies

**Definition 1.2.1** (Definition 1.7 in [21]). Let $B_1, \ldots, B_n$ be a collection of closed 3-balls and let $D_1, \ldots, D_m$, $D'_1, \ldots, D'_m$ be a collection of pairwise disjoint disks in $\cup \partial B_i$. For each $i \geq m$, let $\phi_i : D_i \to D'_i$ be a homeomorphism. Let $H$ be the result of gluing along $\phi_1$, then gluing along $\phi_2$, and so on. After the final gluing if $H$ is connected then $H$ is a handlebody.

**Definition 1.2.2.** The genus of a handlebody is the genus of its boundary $\partial H$.

![Figure 1.1: A Handlebody of genus 4](image)

**Definition 1.2.3.** A properly embedded disk $D \subset H$ is essential if its boundary does not bound a disk in $\partial H$.

**Definition 1.2.4** (Definition 2.1 in [21]). A collection $\{D_1, \ldots, D_m\}$ of properly embedded, essential disks is called a *system of disks* for $H$ if the complement of a regular neighbourhood of $\cup D_i$ is a collection of balls.
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Figure 1.2: A system of disks of a handlebody

**Lemma 1.2.1** (Lemma 2.2 in [21]). Every handlebody has a system of disks.

**Definition 1.2.5** (Definition 2.6 in [21]). A collection of disks in a handlebody is minimal if its complement is connected.

**Theorem 1.2.1** (Loop Theorem). Let $M$ be an 3-manifold with boundary, not necessarily compact or orientable, and let $S \subset M$ be a 2-sided surface. If the induced map $\pi_1 S \to \pi_1 M$ is not injective for some choice of base point in some component of $S$, then there is a disk $D^2 \subset M$ with $D^2 \cap S = \partial D^2$ a nontrivial circle in $S$.

**Proposition 1.2.2.** Let $\Sigma_g$ be a standardly embedded surface in $S^3$, and $\mathcal{H}_g$ the inner handlebody. Let $D_\gamma$ be an essential proper embedded disk in $\mathcal{H}_g$ with boundary a bounding simple closed curve $\gamma$. Then there exists a minimal system of disks $\{D_\beta\}_l$ that does not intersect $D_\gamma$.

**Proof.** Observe that the disk $D_\gamma$ gives us, respectively, two compatible decompositions of $\Sigma_g$ and $\mathcal{H}_g$ as:

$$\Sigma_g = \Sigma_h \#_\gamma \Sigma_{g-h}, \quad (1.2.1)$$

$$\mathcal{H}_g = V_r \#_{D_\gamma} V_{g-r}, \quad (1.2.2)$$

We want to show that $V_r$ and $V_{g-r}$ are handlebodies, obviously with $\partial V_r = \Sigma_h \cup D_\gamma$ and $\partial V_{g-r} = \Sigma_{g-h} \cup D_\gamma$. Take $S_h = \Sigma_h \cup D_\gamma$ and $S_{g-h} = \Sigma_{g-h} \cup D_\gamma$.

Observe that the decompositions (1.2.1), (1.2.2) induce analog decompositions in homotopy.

Fixed a point $x_0$ of the curve $\gamma$, using the Seifert van Kampen theorem we have that

$$\pi_1(\mathcal{H}_g, x_0) = \pi_1(V_r, x_0) *_{\pi_1(D_\gamma, x_0)} \pi_1(V_{g-r}, x_0) = \pi_1(V_r, x_0) * \pi_1(V_{g-r}, x_0).$$

Since $\pi_1(\mathcal{H}_g, x_0)$ is free, by Corollary 2.9. in [27], we have that $\pi_1(V_r, x_0)$ and $\pi_1(V_{g-r}, x_0)$ are free subgroups of $\pi_1(\mathcal{H}_g, x_0)$. Computing the abelianization, we have that they are free groups of rank $r$ and $g-r$ respectively.

Assume that $r \geq h$ (if $r \leq h$, an analogous proof holds switching $S_h$ and $S_{g-h}$).
1.2. Geometric objects

Now we prove that the inclusion $S_h \to V_r$ induces a surjective map $\pi_1(S_h) \to \pi_1(V_r)$ necessarily without trivial kernel. Observe that we have the following commutative diagram:

\[
\begin{array}{ccc}
\pi_1(S_h) & \longrightarrow & \pi_1(V_r) \\
\downarrow & & \downarrow \\
\pi_1(\Sigma_g) & \longrightarrow & \pi_1(H_g)
\end{array}
\]

where $\pi_1(\Sigma_g) \to \pi_1(H_g)$ is clearly surjective, and $\pi_1(H_g) \to \pi_1(V_r)$ is also surjective since every generator element of $\pi_1(V_r)$ is one of the generators of $\pi_1(H_g)$. Then $\pi_1(S_h) \to \pi_1(V_r)$ is surjective. Moreover this map can not be injective since $\pi_1(V_r)$ is free and $\pi_1(S_h)$ is not free and then it is not possible that $\pi_1(V_r) \cong \pi_1(S_h)$.

By Theorem (1.2.1), there exists a properly embedded disc $D_{\beta_1}$ in $V_r$, with $\partial D_{\beta_1} = \beta_1$ and $\beta_1$ not nullhomotopic in $S_h$, so in particular $D_{\beta_1}$ is essential.

Notice that, without lose of generality, we can assume that $\beta_1 \cap D_\gamma = \emptyset$. Since if $\beta_1$ intersects $D_\gamma$, then we can take an homotopy pushing $\beta_1$ out of $D_\gamma$.

Deleting a small open tubular neighbourhood $N(D_{\beta_1})$ of $D_{\beta_1}$ from $V_r$, such that it does not intersect $D_\gamma$, we obtain a 3-manifold $V_{r-1} = V_r - N(D_{\beta_1})$ with boundary. The neighbourhood $N(D_{\beta_1})$ is an interval-bundle over $D_{\beta_1}$, and since $D_{\beta_1}$ is orientable, $N(D_{\beta_1})$ is a product $D_{\beta_1} \times (-\varepsilon,\varepsilon)$. Denote by $D_{\beta_1}^+, D_{\beta_1}^-$ the disks $D_{\beta_1} \times \{-\varepsilon\}, D_{\beta_1} \times \{\varepsilon\}$ respectively. Then we have 3 embedded disks $D_{\beta_1}^+, D_{\beta_1}^-, D_\gamma$ in

$$S_{h-1} = \partial(V_r - N(D_{\beta_1})) = (S_h \setminus Int(C)) \cup D_{\beta_1}^+ \cup D_{\beta_1}^-,$$

where $C = N(D_{\beta_1})$.

We prove that $S_{h-1}$ is a surface of genus $h - 1$.

Recall that if $X$ is a topological space with $X = S' \cup S''$ where $S', S''$ are closed subsets then $\chi(X) = \chi(S') + \chi(S'') - \chi(S' \cap S'')$ (where $\chi$ denotes the Euler's characteristic).

In our case we have that

$$\chi(S_h) = \chi(S_h \setminus Int(C)) + \chi(C) - \chi((S_h \setminus Int(C)) \cap C)$$

$$= \chi(S_h \setminus Int(C)) + 0 - \chi(\partial D_{\beta_1}^- \cup \partial D_{\beta_1}^+)$$

$$= \chi(S_h \setminus Int(C)) - \chi(\partial D_{\beta_1}^-) - \chi(D_{\beta_1}^+)$$

$$= \chi(S_h \setminus Int(C)) - 0 - 0 = \chi(S_h \setminus Int(C))$$
and adding the disks $D_{\beta_1}^-, D_{\beta_1}^+$ to $S_h \setminus \text{Int}(C)$ we get that
\[
\chi(S_{h-1}) = \chi(S_h \setminus \text{Int}(C)) + \chi(D_{\beta_1}^+) + \chi(D_{\beta_1}^-) - \chi(\partial D_{\beta_1}^+) - \chi(\partial D_{\beta_1}^-)
= \chi(S_h \setminus \text{Int}(C)) + 1 + 1 - 0 - 0
= 2 - 2h + 2 = 2 - 2(h - 1).
\]

Then the new surface $S_{h-1}$ has genus $h - 1$.

Moreover, let be $p^-, p^+$ points of $D_{\beta_1}^-, D_{\beta_1}^+$ respectively, and $\epsilon \in N(D_{\beta_1})$ an arc with end points $p^-, p^+$. We can do a deformation retract from $V_r$ to $(V_r - N(D_{\beta_1})) \cup \epsilon$.

Since $S_{h-1}$ is arc-connected there exists another arc $\epsilon'$ in $S_{h-1}$ with end points $p^-, p^+$. So now we can do another deformation retract, from $(V_r - N(D_{\beta_1})) \cup \epsilon$ to $(V_r - N(D_{\beta_1})) \cup x_0 S^1$ sending $p^-$ to $p^+$ through $\epsilon'$. Thus,
\[
\pi_1(V_r - N(D_{\beta_1})) \simeq \pi_1(((V_r - N(D_{\beta_1})) \cup S^1) - (S^1 - \{p^+\})).
\]

Now taking $V_{r-1} = (V_r - N(D_{\beta_1}))$ and using the Seifert van Kampen theorem we get that
\[
\mathbb{F}_h = \pi_1(V_r, p^+) = \pi_1(V_{r-1} \cup_{p^+} S^1, p^+) = \pi_1(V_{r-1}, p^+) + \pi_1(S^1, p^+) = \pi_1(V_{r-1}, p^+) * \mathbb{F}_1.
\]

Since $\pi_1(V_r, p^+)$ is free, by Corollary 2.9 in [27], we have that $\pi_1(V_{r-1}, p^+)$ is a free subgroup of $\pi_1(V_r, p^+)$. Computing the abelianization, we have that this subgroup is a free group of rank $h - 1$.

Next, if we delete $N(D_{\beta_1})$ from $V_r$ we obtain an inclusion $S_{h-1} \hookrightarrow V_{r-1}$.

Repeating this argument $h$ times at the end we get a surface $S_0$ with genus zero, i.e. a smooth 2-sphere $S^2$ embedded in $S^3$. Applying Schöenflies theorem, we get that our embedded smooth 2-sphere $S^2$ is the boundary of two embedded smooth 3-balls in $S^3$. Then $V_{r-h}$ is one of these 3-balls, and so $\mathbb{F}_{r-h} = \pi_1(V_{r-h}) = 0$, i.e. $h = r$. Thus $V_h$ is a handlebody with system of disks given by $\{D_{\beta_i}\}_i = \{D_{\beta_1}, \ldots, D_{\beta_h}\}$ that is minimal by construction.

Using the same argument for the subsurface $S_{g-h}$, we obtain a handlebody $V_{g-r}$ with boundary $S_{g-h}$, and a minimal system of discs $\{D_{\beta_j}\}_j$ for $V_{g-k}$, such that $\{D_{\beta_j}\}_j$ do not intersect $D_\gamma, D_\delta$.

Then we can view $\mathcal{H}_g$ as a boundary connected sum of $V_k$ and $V_{g-k}$ given by:
\[
\mathcal{H}_g = V_k \#_{D_\gamma} V_{g-k} := (V_k \biguplus V_{g-k}) / D_\gamma
\]

Hence the union of the two families $\{D_{\beta_i}\}_i$ and $\{D_{\beta_j}\}_j$ gives us a new system of disks $\{D_{\beta_1'}\}_l$ for $\mathcal{H}_g$ such that $\{D_{\beta_i}\}_l$ don’t intersect $D_\gamma$. And by construction $\{D_{\beta_i}\}_l$, is a minimal disk system of $\mathcal{H}_g$. 

□
1.3 Mapping class group

Let \( \Sigma_{g,b} \) denote an oriented, connected surface of genus \( g \geq 0 \) with \( b \geq 0 \) disjoint open disks removed. Let \( \text{Homeo}^+(\Sigma_{g,b}, \partial \Sigma_{g,b}) \) denote the group of orientation-preserving homeomorphisms of \( \Sigma_{g,b} \) that restrict to the identity on \( \partial \Sigma_{g,b} \). We endow this group with the compact-open topology.

The mapping class group of \( \Sigma_{g,b} \), denoted \( \text{Mod}(\Sigma_{g,b}) \), is the group

\[
\text{Mod}(\Sigma_{g,b}) = \pi_0(\text{Homeo}^+(\Sigma_{g,b}, \partial \Sigma_{g,b})).
\]

Equivalently, \( \text{Mod}(\Sigma_{g,b}) \) is the group of isotopy classes of elements of \( \text{Homeo}^+(\Sigma_{g,b}, \partial \Sigma_{g,b}) \), where isotopies are required to fix the boundary pointwise. If \( \text{Homeo}_0(\Sigma_{g,b}, \partial \Sigma_{g,b}) \) denotes the connected component of the identity in \( \text{Homeo}^+(\Sigma_{g,b}, \partial \Sigma_{g,b}) \), then we can equivalently write

\[
\text{Mod}(\Sigma_{g,b}) = \text{Homeo}^+(\Sigma_{g,b}, \partial \Sigma_{g,b})/\text{Homeo}_0(\Sigma_{g,b}, \partial \Sigma_{g,b}).
\]

The elements of \( \text{Mod}(\Sigma_{g,b}) \) are called mapping classes.

There is not a unique definition of \( \text{Mod}(\Sigma_{g,b}) \). For example, we could consider diffeomorphisms instead of homeomorphisms, or homotopy classes instead of isotopy classes. However, by Section 1.4 in [8], these definitions would produce isomorphic groups, i.e. we have the following equivalent definitions of \( \text{Mod}(\Sigma_{g,b}) \):

\[
\text{Mod}(\Sigma_{g,b}) = \pi_0(\text{Homeo}^+(\Sigma_{g,b}, \partial \Sigma_{g,b})) \\
\approx \text{Homeo}^+(\Sigma_{g,b}, \partial \Sigma_{g,b})/\text{homotopy} \\
\approx \pi_0(\text{Diff}^+(\Sigma_{g,b}, \partial \Sigma_{g,b})),
\]

where \( \text{Diff}^+(\Sigma_{g,b}, \partial \Sigma_{g,b}) \) is the group of orientation-preserving diffeomorphisms of \( \Sigma_{g,b} \) that are the identity on the boundary.

1.3.1 Dehn twists

There is a particular type of mapping class called a Dehn twist. Dehn twists are the simplest infinite-order mapping classes in the sense that the Dehn twists play the role for mapping class groups that elementary matrices play for linear groups.

Twist map in annulus. Consider the annulus \( A = S^1 \times [0,1] \). To orient \( A \) we embed it in the \((\theta,r)\)-plane via the map \((\theta,t) \mapsto (\theta,t+1)\) and take the orientation induced by the standard orientation of the plane.
Let $T : A \rightarrow A$ be the twist map of $A$ given by the formula

$$T(\theta, t) = (\theta + 2\pi t, t).$$

The map $T$ is an orientation-preserving homeomorphism that fixes $\partial A$ pointwise. Note that instead of using $\theta + 2\pi t$ we could have used $\theta - 2\pi t$. Our choice is a left twist, while the other is a right twist.

**Dehn twist in a general surface.** Let $S$ be an arbitrary oriented surface and let $\alpha$ be a simple closed curve in $S$. Let $N$ be a regular neighbourhood of $\alpha$ and choose an orientation preserving homeomorphism $\phi : A \rightarrow N$. We obtain a homeomorphism $T_\alpha : S \rightarrow S$, called a Dehn twist about $\alpha$, as follows:

$$T_\alpha(x) = \begin{cases} 
\phi \circ T \circ \phi^{-1}(x) & \text{if } x \in N \\
x & \text{if } x \in S \setminus N.
\end{cases}$$

In other words, the instructions for $T_\alpha$ are ”perform the twist map $T$ on the annulus $N$ and fix every point outside of $N.”

The Dehn twist $T_\alpha$ depends on the choice of $N$ and the homeomorphism $\phi$. However, by the uniqueness of regular neighbourhoods, the isotopy class of $T_\alpha$ does not depend on either of these choices.

**The action on simple closed curves via surgery** We can understand $T_\alpha$ by examining its action on the isotopy classes of simple closed curves on $S$. If $b$ is an isotopy class with $i(a, b) = 0$, then $T_\alpha(b) = b$. In the case that $i(a, b) \neq 0$ the isotopy class $T_\alpha(b)$ is determined by the following rule: given particular representatives $\beta$ and $\alpha$ of $b$ and $a$, respectively, each segment of $\beta$ crossing $\alpha$ is replaced with a segment that turns left, follows $\alpha$ all the way around, and then turns right. Analogously, we can understand $T_\alpha^{-1}$ following the same rule described above switching ”left” by ”right”.

The reason that we can distinguish left from right is that the map $\phi$ used in the definition of $T_\alpha$ is taken to be orientation-preserving.
Properties of Dehn twists. Next we give some properties about Dehn twists. For the proofs of these properties see Section 3.3 in [8].

Let $f \in \text{Mod}(S)$ and $a, b$ isotopy classes of simple closed curves in $S$

i) $T_a = T_b \iff a = b$.

ii) $T_{f(a)} = fT_af^{-1}$.

iii) $f$ commutes with $T_a \iff f(a) = a$.

iv) $i(a, b) = 0 \iff T_a(b) = b \iff T_aT_b = T_bT_a$.

The Lantern relation The Lantern relation is a relation in $\text{Mod}(S)$ between seven Dehn twists all lying on a subsurface of $S$ homeomorphic to $S_{0,4}$, a sphere with four boundary components.

Proposition 1.3.1. Let $x, y, z, b_1, b_2, b_3$ and $b_4$ be simple closed curves in a surface $S$ that are arranged as the curves shown in Figure 1.5.

Figure 1.5: Lantern relation
Precisely this means that there is an orientation-preserving embedding $S_{0,4} \hookrightarrow S$ and that each of the above seven curves is the image of the curve with the same name in Figure 1.5. In $\text{Mod}(S)$ we have the relation
\[ T_x T_y T_z = T_{b_1} T_{b_2} T_{b_3} T_{b_4}. \]

1.3.2 Generators of the Mapping class group

**Theorem 1.3.1** (Theorem 4.1 in [8]). For $g \geq 0$, the mapping class group $\text{Mod}(S_g)$ is generated by finitely many Dehn twists about nonseparating simple closed curves.

In particular we have the following result.

**Theorem 1.3.2** (Theorem 4.14 in [8]). Let $S$ be either a closed surface or a surface with one boundary component and genus $g \geq 3$. Then the group $\text{Mod}(S)$ is generated by the Dehn twists about the $2g+1$ isotopy classes of nonseparating simple closed curves $c_0, \ldots, c_{2g}$ of Figure 1.6.

![Figure 1.6: Humphries generators](image)

1.3.3 The Symplectic representation

We first give the definition of symplectic matrix as well as the Symplectic group.

**Definition 1.3.1** (Section 1 in [34]). Consider the matrix $\Omega \in M_{2g \times 2g}(\mathbb{Z})$ given by
\[
\Omega = \begin{pmatrix}
0 & \text{Id}_g \\
-I\text{Id}_g & 0
\end{pmatrix}.
\]

If $M$ is a matrix satisfying $M\Omega M^t = \Omega$, then $M$ will be said to be *symplectic*. We define the Symplectic group as
\[ \text{Sp}_{2g}(\mathbb{Z}) = \{ M \in M_{2g \times 2g}(\mathbb{Z}) \mid M\Omega M^t = \Omega \}. \]

From the definitions, it is easy to verify that a matrix
\[
M = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]
is symplectic if and only if

\[ A D^t - B C^t = I d_g, \quad AB^t = B A^t, \quad CD^t = D C^t. \]

Next we define the Symplectic representation. Let \( S_g \) be a closed surface of genus \( g \) and \( Mod(S_g) \) its mapping class group. Recall that by definition, \( Mod(S_g) = \pi_0(Homeo^+(S_g)) \). Therefore we can view every mapping class of \( \phi \in Mod(S_g) \) as an element of \( Aut(\Gamma) \), where \( \Gamma = \pi_1(S_g) \) (the fundamental group of \( S_g \)). Moreover it is well known that the commutator subgroup \([\Gamma, \Gamma] \subset \Gamma\) is a characteristic subgroup, in other words, every element of \( Aut(\Gamma) \) preserves \([\Gamma, \Gamma]\).

Then taking \( \phi \in Aut(\Gamma) \) composed with the abelianization of \( \Gamma \) we get an element of \( Aut(H_1(S_g)) \) and as a consequence every element \( \phi \in Mod(S_g) \) induces an automorphism \( \phi_* : H_1(S_g; \mathbb{Z}) \to H_1(S_g; \mathbb{Z}) \).

More precisely, since \( H_1(S_g; \mathbb{Z}) \cong \mathbb{Z}^{2g} \), we get that \( \phi_* \in Aut(\mathbb{Z}^{2g}) = GL_{2g}(\mathbb{Z}) \). Therefore we have a linear representation \( \Psi : Mod(S_g) \to GL_{2g}(\mathbb{Z}) \).

Moreover, since the algebraic intersection number \( \widehat{i} : H_1(S_g; \mathbb{Z}) \times H_1(S_g; \mathbb{Z}) \to \mathbb{Z} \) on \( H_1(S_g; \mathbb{Z}) \) endows the vector space \( H_1(S_g; \mathbb{Z}) \) with a symplectic structure and this symplectic structure is preserved by the image of \( \Psi \), then \( \Psi \) is a representation \( \Psi : Mod(S_g) \to Sp_{2g}(\mathbb{Z}) \).

The homomorphism \( \Psi \) is called the symplectic representation of \( Mod(S_g) \).

**The action of a Dehn twist on homology** To understand \( \Psi \), first of all we need to understand what it does to Dehn twists. We have the following formula.

**Proposition 1.3.2.** Let \( a \) and \( b \) be isotopy classes of oriented simple closed curves in \( S_g \). For any \( k \geq 0 \), we have

\[ \Psi(T_b^k)([a]) = [a] + k \cdot \widehat{i}(a, b)[b]. \]

In particular from Proposition (1.3.2) we get that \( \Psi(T_a) = \Psi(T_{a'}) \iff [a] = [a'] \), and also that if \([a] = 0\), then \( \Psi(T_a) \) is the identity.

**Theorem 1.3.3** (Theorem 6.4 in [8]). The symplectic representation is surjective.
1.3.4 Heegaard splittings of 3-manifolds

Let $\Sigma_g$ an oriented surface of genus $g$ standardly embedded in the 3-sphere $S^3$. Denote by $\Sigma_{g,1}$ the complement of the interior of a small disc embedded in $\Sigma_g$. We fix a base point $x_0$ on the boundary of $\Sigma_{g,1}$.

![Figure 1.7: Standardly embedded $\Sigma_{g,1}$ in $S^3$](image)

Throughout this thesis we denote by $\mathcal{M}_{g,1}$ the following mapping class group

$$\mathcal{M}_{g,1} = \pi_0(\text{Diff}^+(\Sigma_{g,1}, \partial \Sigma_{g,1})).$$

**Handlebody subgroups.** Recall that our surface is standarly embedded in the oriented 3-sphere $S^3$. As such it determines two embedded handlebodies $S^3 = H_g \cup -H_g$. By the inner handlebody $H_g$ we will mean the one that is visible in Figure 1.7 and by the outer handlebody $-H_g$ we will mean the complementary handlebody. They are naturally pointed by $x_0 \in H_g \cap -H_g$. From these we get the following three natural subgroups of $\mathcal{M}_{g,1}$:

- $A_{g,1} = \text{subgroup of restrictions of diffeomorphisms of the outer handlebody } -H_g$,
- $B_{g,1} = \text{subgroup of restrictions of diffeomorphisms of the inner handlebody } H_g$,
- $AB_{g,1} = \text{subgroup of restriction of diffeomorphisms of the 3-sphere } S^3$.

Moreover, in [48], F. Waldhausen proved that in fact, $AB_{g,1} = A_{g,1} \cap B_{g,1}$.

**The stabilization map on the mapping class group.** Define the stabilization map on the mapping class group $\mathcal{M}_{g,1}$ as follows. Glue one of the boundary components of a two-holed torus on the boundary of $\Sigma_{g,1}$ to get $\Sigma_{g+1,1}$. Extending an element of $\mathcal{M}_{g,1}$ by the identity over the torus yields an injective homomorphism $\mathcal{M}_{g,1} \to \mathcal{M}_{g+1,1}$, this is the stabilization map. This map is compatible with the action on homology and is compatible with the definition of the subgroups $A_{g,1}$ and $B_{g,1}$.
Heggaard splittings of 3-manifolds

**Definition 1.3.2.** A *Heggaard splitting* for a 3-manifold $M$ is an ordered triple $(\Sigma, H_1, H_2)$ where $\Sigma$ is a closed surface embedded in $M$ and $H_1$ and $H_2$ are handlebodies embedded in $M$ such that $\partial H_1 = \Sigma = \partial H_2 = H_1 \cap H_2$ and $H_1 \cup H_2 = M$. The surface $\Sigma$ is called a *Heggaard surface*.

**Definition 1.3.3.** Let $H_1$ and $H_2$ be handlebodies of the same genus and $h : \partial H_1 \to \partial H_2$ a homeomorphism. Then the quotient of the disjoint union of the two handlebodies by the relation $x \sim y$ if $x \in \partial H_1$, $y \in \partial H_2$ and $h(x) = y$, is a closed 3-manifold. We denote by $M = H_1 \cup_h H_2$ such manifold.

Consider the following equivalence relation on $\mathcal{M}_{g,1}$:

$$\phi \sim \psi \iff \exists \zeta_a \in \mathcal{A}_{g,1} \exists \zeta_b \in \mathcal{B}_{g,1} \text{ such that } \zeta_a \phi \zeta_b = \psi \quad (1.3.1)$$

The equivalence relation (1.3.1) is compatible with the stabilization map.

Choose a map $\iota_g \in \mathcal{M}_{g,1}$ such that $S^3 = \mathcal{H}_g \cup \iota_g - \mathcal{H}_g$. It is also possible to choose the map $\iota_g$ to be compatible with the stabilization map $\iota_{g+1} |_{\sigma_g} = \iota_g$.

Denote by $\mathcal{V}^3$ the set of oriented diffeomorphism classes of compact, closed and oriented smooth 3-manifolds.

**Theorem 1.3.4** (J. Singer [46]). The following map is well defined and is bijective:

$$\lim_{g \to \infty} \mathcal{A}_{g,1} \backslash \mathcal{M}_{g,1} / \mathcal{B}_{g,1} \longrightarrow \mathcal{V}^3,$$

$$\phi \mapsto S^3_\phi = \mathcal{H}_g \cup \iota_g \phi - \mathcal{H}_g.$$

### 1.3.5 The Torelli group

The Torelli group $\mathcal{T}_{g,1}$ is the normal subgroup of the mapping class group $\mathcal{M}_{g,1}$ of those elements of $\mathcal{M}_{g,1}$ that act trivially on $H_1(\Sigma_{g,1}; \mathbb{Z})$. In other words, $\mathcal{T}_{g,1}$ is characterized by the following short exact sequence:

$$1 \longrightarrow \mathcal{T}_{g,1} \longrightarrow \mathcal{M}_{g,1} \longrightarrow \Psi \longrightarrow S_{p_{2g}}(\mathbb{Z}) \longrightarrow 1$$

**Example of elements of the Torelli group**

i) **Dehn twists about separating curves.** Each Dehn twist about a separating simple closed curve $\gamma$ in $\Sigma_{g,1}$ is an element of $\mathcal{T}_{g,1}$. This is because there exists a basis for $H_1(\Sigma_{g,1}; \mathbb{Z})$ where each element is represented by an oriented simple closed curve disjoint from $\gamma$. Since $T_\gamma$ fixes each of these curves, it fixes the corresponding homology classes and is hence an element of $\mathcal{T}_{g,1}$. 
Chapter 1. Preliminaries

The group generated by Dehn twists about separating simple closed curves in \( \Sigma_{g,1} \) is denoted by \( K_{g,1} \) and is known as the Johnson subgroup. In the following picture we give an example of a separating simple closed curve \( \gamma \).

![Figure 1.8: A separating simple closed curve](image)

ii) **Bounding pair maps.** A bounding pair in a surface is a pair of disjoint, homologous, nonseparating simple closed curves. A bounding pair map, (abbreviated, BP-map) is a mapping class of the form

\[
T_a T_b^{-1}
\]

where \( a \) and \( b \) form a bounding pair. Since \( a \) and \( b \) are homologous, by Proposition 6.3 in [8] the images of \( T_a \) and \( T_b \) in \( Sp_{2g}(\mathbb{Z}) \) are equal. Thus any bounding pair map is an element of \( T_{g,1} \). In the following picture we give two examples of a bounding pairs \( \{\eta, \eta'\} \) and \( \{\beta, \beta'\} \).

![Figure 1.9: Boundig pairs](image)

1.3.6 Heegaard splittings of homology 3-spheres

**Definition 1.3.4.** A 3-manifold \( X \) is an integral homology 3-sphere if

\[
H_*(X; \mathbb{Z}) \cong H_*(S^3; \mathbb{Z}).
\]

Throughout this thesis we will refer to such manifolds simply as homology 3-spheres.

**Example 1.3.1** (Section 9.D. in [41]). The most important example of a homology 3-sphere is the **Poincaré sphere** which is the 3-manifold given by a full dodecahedron with opposite faces identified with a twist of \( \frac{\pi}{5} \). The importance of this 3-manifold comes from the fact that it was the first
example of an homology 3-sphere which is not an homotopy 3-sphere, i.e. its homotopy groups do not coincide with the homotopy groups of the 3-sphere.

Denote by $S^3 \subset V^3$ the subset of integral homology 3-spheres.

**Theorem 1.3.5** (S. Morita [30]). The following map is well defined and is bijective:

$$\lim_{g \to \infty} A_{g,1} \cap T_{g,1} / B_{g,1} \longrightarrow S^3,$$

$$\phi \mapsto S^3_\phi = \mathcal{H}_g \cup \iota_{g,\phi} - \mathcal{H}_g.$$

Then we can get any element of $S^3$ as $H_g \cup \iota_{g,\phi} - H_g$ for a suitable $g$ and $\phi \in T_{g,1}$.

From the group-theoretical point of view, the induced equivalence relation on $T_{g,1}$, which is given by:

$$\phi \sim \psi \iff \exists \zeta_a \in A_{g,1} \exists \zeta_b \in B_{g,1} \text{ such that } \zeta_a \phi \zeta_b = \psi, \quad (1.3.2)$$

is a quite unsatisfactory, since it looks like, but is not, a double coset relation on $T_{g,1}$. However, by Lemma (1.3.1), due to W. Pitsch [38], we know that this relation is the composite of a double coset relation in $T_{g,1}$ and a conjugancy-induced equivalence relation.

**Definition 1.3.5.** We define the following subgroups of $T_{g,1}$:

$$T A_{g,1} = T_{g,1} \cap A_{g,1}, \quad T B_{g,1} = T_{g,1} \cap B_{g,1}, \quad T A B_{g,1} = T_{g,1} \cap A B_{g,1}.$$

**Lemma 1.3.1** (W. Pitsch [38]). Two maps $\phi, \psi \in T_{g,1}$ are equivalent if and only if there exists a map $\mu \in A B_{g,1}$ and two maps $\zeta_a \in T A_{g,1}$ and $\zeta_b \in T B_{g,1}$ such that $\phi = \mu \zeta_a \psi \zeta_b \mu^{-1}$.

To summarize, we have the following bijection:

$$\lim_{g \to \infty} (T A_{g,1} \cap T B_{g,1}) / A B_{g,1} \longrightarrow S^3,$$

$$\phi \mapsto S^3_\phi = \mathcal{H}_g \cup \iota_{g,\phi} - \mathcal{H}_g. \quad (1.3.3)$$

### 1.3.7 Homology actions

The isotopy class of the curves $\alpha_i, \beta_i$, for $1 \leq i \leq g$, are free generators of the free group $\pi_1(\Sigma_{g,1}, x_0)$, (see figure below). The first homology group of the surface $H := H_1(\Sigma_g, \mathbb{Z})$ is endowed via Poincaré duality with a natural symplectic intersection form $\omega: \wedge^2 H \to \mathbb{Z}$. The homology classes $\{A_i, B_i\}_i$ of the above curves freely generate the abelian group $H \simeq \mathbb{Z}^{2g}$ and define two transverse Lagrangians $A$ and $B$ in $H$. Throughout this thesis we fix a basis of $H$ given by $\langle A_1, \ldots, A_g, B_1, \ldots, B_g \rangle$ and a symplectic intersection form $\omega$ given by $\omega(B_i, A_i) = 1 = -\omega(A_i, B_i)$ for every $i$ and zero otherwise,
i.e. $Sp \omega \cong Sp_{2g}(\mathbb{Z})$.

According to Griffith [11], the subgroup $B_{g,1}$ (resp. $A_{g,1}$) is characterised by the fact that its action on $\pi_1(\Sigma_g, x_0)$ preserves the normal subgroup generated by the curves $\beta_1, \ldots, \beta_g$ (resp. $\alpha_1, \ldots, \alpha_g$). As a consequence the action on homology of $B_{g,1}$ (resp. $A_{g,1}$) preserves the Lagrangian $B$ (resp. $A$).

**Notation 1.3.1.** Throughout this thesis, given two groups $N, H$ and an action of $H$ on $N$, we will denote by $H \ltimes N$ the semidirect product of $H$ and $N$.

If one writes the matrices of the symplectic group $Sp_{2g}(\mathbb{Z})$ as blocks according to the decomposition $H = A \oplus B$, then the image of $B_{g,1} \to Sp_{2g}(\mathbb{Z})$ is contained in the subgroup $Sp_{B_{2g}}(\mathbb{Z})$ of matrices of the form:

$$\begin{pmatrix} G_1 & 0 \\ M & G_2 \end{pmatrix}.$$

Such matrices are symplectic if and only if $G_2 = ^tG_1^{-1}$ and $^tG_1^{-1}M$ is symmetric. As a consequence we have an isomorphism:

$$\phi^B : Sp^B_{2g}(\mathbb{Z}) \to GL_g(\mathbb{Z}) \ltimes_B S_g(\mathbb{Z}),$$

$$\begin{pmatrix} G & 0 \\ M & ^tG^{-1} \end{pmatrix} \mapsto (G, ^tGM).$$

(1.3.4)

Here $S_g(\mathbb{Z})$ denotes the symmetric group of $g \times g$ matrices over the integers; the composition on the semi-direct product is given by the rule

$$(G,S)(H,T) = (GH, ^tHS + T).$$

Analogously, the image of $A_{g,1} \to Sp_{2g}(\mathbb{Z})$ is contained in the subgroup $Sp^A_{2g}(\mathbb{Z})$ of matrices of the form:

$$\begin{pmatrix} H_1 & N \\ 0 & H_2 \end{pmatrix}.$$

Such matrices are symplectic if and only if $H_2 = ^tH_1^{-1}$ and $^tH_2N$ is symmetric. Similarly, we
have an isomorphism:
\[
\phi^A : Sp^A_{2g}(\mathbb{Z}) \longrightarrow GL_g(\mathbb{Z}) \ltimes_A S_g(\mathbb{Z}),
\]
\[
\begin{pmatrix}
H & N \\
0 & tH^{-1}
\end{pmatrix} \longmapsto (H, H^{-1}N).
\tag{1.3.5}
\]

Here the composition on the semi-direct product is given by the rule
\[
(G, S)(H, T) = (GH, tH^{-1}SH^{-1} + T).
\]

Finally, the image of $AB_{g,1} \to Sp_{2g}(\mathbb{Z})$ is contained in the subgroup $Sp^{AB}_{2g}(\mathbb{Z})$ of matrices of the form:
\[
\begin{pmatrix}
G_1 & 0 \\
0 & G_2
\end{pmatrix}.
\]

Such matrices are symplectic if and only if $G_2 = tG_1^{-1}$. As a consequence have an isomorphism:
\[
\phi^{AB} : Sp^{AB}_{2g}(\mathbb{Z}) \longrightarrow GL_g(\mathbb{Z}),
\]
\[
\begin{pmatrix}
G & 0 \\
0 & tG^{-1}
\end{pmatrix} \longmapsto G.
\tag{1.3.6}
\]

Checking on generators of $B_{g,1}$ (see [47]) we get:

**Lemma 1.3.2.** There is a short exact sequence of groups:
\[
1 \longrightarrow \mathcal{T}B_{g,1} \longrightarrow B_{g,1} \xrightarrow{\phi^B \circ \Psi} GL_g(\mathbb{Z}) \ltimes S_g(\mathbb{Z}) \longrightarrow 1
\]

An analogous statement holds for $A_{g,1}$ replacing the lagrangian $B$ by $A$.

Similarly, checking on generators of $AB_{g,1}$ we have the following result

**Lemma 1.3.3.** There is a short exact sequence of groups:
\[
1 \longrightarrow \mathcal{T}AB_{g,1} \longrightarrow AB_{g,1} \xrightarrow{\phi^{AB} \circ \Psi} GL_g(\mathbb{Z}) \longrightarrow 1
\]

### 1.3.8 The Johnson homomorphism

Computing the action of the Torelli group on the second nilpotent quotient of $\pi_1(\Sigma_{g,1}, x_0)$ Johnson defines a morphism of groups known as the first Johnson homomorphism:

\[
\tau : T_{g,1} \longrightarrow \bigwedge^3 H.
\]
Notice that the mapping class group $\mathcal{M}_{g,1}$ acts naturally by conjugation on $\mathcal{T}_{g,1}$ and acts also on $\wedge^3 H$ via its natural action on homology. In [16], [18], [19] Johnson proves that

**Proposition 1.3.3.** The map $\tau$ is $\mathcal{M}_{g,1}$-equivariant with respect to the above actions. Up to a finite dimensional $\mathbb{Z}/2$-vector space $\wedge^3 H$ is the abelianization of the Torelli group: any homomorphism $\mathcal{T}_{g,1} \to A$ where $A$ is an abelian group without 2-torsion factors uniquely through $\tau$. 
CHAPTER 2

Trivial cocycles and invariants on the Torelli group

In [38], W. Pitsch gave a tool to construct invariants of homology 3-spheres, with values in any abelian group without 2-torsion, from a family of trivial 2-cocycles on $\mathcal{T}_{g,1}$. In this chapter we generalize the results of [38], to include any abelian group without restrictions.

The main difficulty to generalize such results comes from the fact that if we consider an abelian group with 2-torsion, then the abelianization of $\mathcal{T}_{g,1}$ is given by $\wedge^3 H \oplus T$, where $T$ is a 2-torsion group, and $\text{Hom}(\mathcal{T}_{g,1}, A)^{\text{AB}_{g,1}}$ is not zero because, unlike the case of an abelian group without 2-torsion, in this case $T$ has not to be sent to zero.

We show that the elements $\text{Hom}(\mathcal{T}_{g,1}, A)^{\text{AB}_{g,1}}$ are given by multiples of the Rohlin invariant. As a consequence, any two invariants constructed from the same family of 2-cocycles differ by an element of $\text{Hom}(\mathcal{T}_{g,1}, A)^{\text{AB}_{g,1}}$, and hence the Rohlin invariant explains the failure of unicity in this case.

In the first section we review some basic definitions and properties about the Boolean algebra and the Birman-Craggs-Johnson-homomorphism. In the second section we recall the definition of a contractible bounding pair twist, the Luft group and we exhibit some interesting results about the handlebody subgroup $\mathcal{B}_{g,1}$ and the Luft-Torelli group $\mathcal{LTB}_{g,1}$. Finally, in the last two sections we give the aforementioned generalization.

2.1 The Boolean algebra and the BCJ-homomorphism

**Definition 2.1.1** (Boolean algebra). The Boolean polynomial algebra $\mathbb{B} = \mathbb{B}_{g,1}$ is a $\mathbb{Z}/2$-algebra (with unit 1) with a generator $\overline{x}$ for each $x \in H_1(\Sigma_{g,1}, \mathbb{Z}/2)$ and subject to the relations:

(a) $\overline{x + y} = \overline{x} + \overline{y} + x \cdot y$, where $x \cdot y$ is the mod 2 intersection number,
(b) \(x^2 = x\).

The relation (b) implies that \(p^2 = p\) for any \(p \in \mathcal{B}\) and also that if \(\{e_i \mid i \in \{1, \ldots, 2g\}\}\) is a basis for \(H_1(\Sigma_{g,1}, \mathbb{Z}/2)\) then the set of all monomials \(e_{i_1}e_{i_2} \cdots e_{i_r}\) with \(0 \leq r \leq 2g, 1 \leq i_1 < i_2 < \ldots < i_r \leq 2g\) is a \(\mathbb{Z}/2\)-basis for \(\mathcal{B}\). Denote by \(\mathcal{B}_k = \mathcal{B}_{g,1}^k\) the subspace generated by all monomials of "degree" \(\leq k\).

In [17], D. Johnson constructed a surjective homomorphism \(\sigma : \mathcal{T}_{g,1} \to \mathcal{B}_3\), called the Birman-Craggs-Johnson homomorphism (abbreviated BCJ-homomorphism), which may be described as follows.

Consider the \(\mathbb{Z}/2\)-basis of \(\mathcal{B}_3\) given by

\[\{1, \overline{A}_i, \overline{B}_i, \overline{A}_i\overline{A}_j, \overline{B}_i\overline{B}_j, \overline{A}_i\overline{A}_j\overline{A}_k, \overline{B}_i\overline{B}_j\overline{B}_k, \overline{A}_i\overline{B}_j\overline{A}_k, \overline{A}_i\overline{B}_j\overline{B}_k, \overline{A}_i\overline{A}_j\overline{B}_k, \overline{A}_i\overline{B}_i\overline{A}_j\overline{A}_k, \overline{A}_i\overline{B}_i\overline{B}_j\overline{A}_k\} ,\]

where \(i, j, k \in \{1, \ldots, 2g\}\) are pairwise distinct, and consider the curves depicted in the following figure:

![Diagram](image)

The BCJ-homomorphism is given on a BP-map \(T_\beta T_{\beta'}^{-1}\) by

\[\sigma(T_\beta T_{\beta'}^{-1}) = \sum_{i=1}^{k} C_i D_i (E + 1) ,\]

where \(E\) is the homology class of \(\beta\), and \(\{C_i, D_i\}\) is a symplectic basis of a subsurface \(\Sigma_{k,1}\) of \(\Sigma_{g,1}\) with boundary component \(\gamma\), such that \(\gamma \cup \beta \cup \beta'\) is the boundary of a subsurface with genus zero in \(\Sigma_{g,1}\).

The BCJ-homomorphism is given on a Dehn twist about a bounding simple closed curve \(\gamma\) of genus \(k\), by:

\[\sigma(T_\gamma) = \sum_{i=1}^{k} C_i D_i ,\]

where \(\{C_i, D_i\}\) is the symplectic basis of the subsurface of genus \(k\) bounded by \(\gamma\).
By Lemma 13 in [17], \( \sigma \) is an \( \mathcal{M}_{g,1} \)-equivariant map, in other words, for \( f \in \mathcal{M}_{g,1} \), \( h \in \mathcal{T}_{g,1} \) we have that \( \sigma(fhf^{-1}) = \hat{f} \cdot \sigma(h) \), where \( \hat{f} \) denotes the image of the map \( f \) under the symplectic representation mod 2, and the action of \( \hat{f} \) on \( \mathfrak{B}_3 \) is induced from the action of \( \hat{f} \) on \( H_1(\Sigma_{g,1}, \mathbb{Z}/2) \), i.e. \( \hat{f} \cdot (Z_iZ_jZ_k) = \hat{f}Z_i\hat{f}Z_j\hat{f}Z_k \), where \( Z_i, Z_j, Z_k \) are any three elements of \( H_1(\Sigma_{g,1}, \mathbb{Z}/2) \).

2.2 The Luft group and CBP-twists

Denote by \( L_{g,1} \) the kernel of the map \( B_{g,1} \to Aut(\pi_1(H_g)) \). We call it the Luft group. It was identified by Luft in [25] as the "Twist group" of the handlebody \( H_g \). Denote \( \mathcal{LT} B_{g,1} \) the intersection \( L \cap \mathcal{T} B_{g,1} \), and \( IA \) the kernel of the natural map \( Aut(\pi_1(H_g)) \to GL_g(\mathbb{Z}) \).

**Proposition 2.2.1.** There is a short exact sequence

\[
1 \longrightarrow \mathcal{LT} B_{g,1} \longrightarrow L_{g,1} \xrightarrow{\phi B \circ \Psi} S_g(\mathbb{Z}) \longrightarrow 1.
\] (2.2.1)

**Proof.** We first prove that \( \phi B \circ \Psi : L_{g,1} \to S_g(\mathbb{Z}) \) is well defined, i.e. \( \phi B \circ \Psi(L_{g,1}) \subset S_g(\mathbb{Z}) \). Recall that \( L_{g,1} = Ker(B_{g,1} \to Aut(\pi_1(H_g))) \). As a consequence, if \( x \in L_{g,1}, \Psi(x) = (Id_00) \) and then \( \phi B \circ \Psi(L_{g,1}) \subset \{Id\} \times S_g(\mathbb{Z}) \cong S_g(\mathbb{Z}) \).

Next we prove that \( \phi B \circ \Psi : L_{g,1} \to S_g(\mathbb{Z}) \) is surjective with kernel \( \mathcal{LT} B_{g,1} \).

Recall that \( S_g(\mathbb{Z}) \) is generated by the following family of matrices:

\[
\{E_{ii} \mid 1 \leq i \leq g\} \cup \{E_{ij} + E_{ji} \mid 1 \leq i < j \leq g\},
\]

where \( E_{ij} \) denotes the matrix with 1 at position \((i,j)\) and 0’s elsewhere. Thus it is enough to find a preimage for each \( E_{ii} \) and \( E_{ij} \). Consider \( \beta_i, \beta_j \) and \( \gamma_{ij} \) the curves depicted in the following figure:

![Figure 2.1: Contractible simple closed curves](image)

Notice that the curves \( \beta_k, \gamma_{ij} \) are contractible in \( H_g \) and as a consequence \( T_{\beta_k}, T_{\gamma_{ij}} \) are elements.
of $\mathcal{L}_{g,1}$. In addition, we have that
\[
\Psi(T_{\beta_k}^{-1})(a_l) = a_l - \bar{\gamma}(a_l, b_k)b_k = \begin{cases} 
 a_k + b_k & \text{if } l = k \\
 a_l & \text{if } l \neq k,
\end{cases}
\]
\[
\Psi(T_{\beta_k}^{-1})(b_l) = b_l,
\]
\[
\Psi(T_{\gamma_{ij}})(a_l) = a_l + \bar{\gamma}(a_l, [\gamma_{ij}])(\gamma_{ij}) = \begin{cases} 
 a_i - b_l + b_j & \text{if } l = i \\
 a_j + b_l - b_j & \text{if } l = j \\
 a_l & \text{if } l \neq i, j,
\end{cases}
\]
\[
\Psi(T_{\gamma_{ij}})(b_l) = b_l.
\]
Thus we get that
\[
\Psi(T_{\beta_k}^{-1}) = \begin{pmatrix} Id & 0 \\
 E_{kk} & Id \end{pmatrix}, \quad \Psi(T_{\gamma_{ij}}) = \begin{pmatrix} Id & 0 \\
 -E_{ii} - E_{jj} + E_{ij} + E_{ji} & Id \end{pmatrix}.
\]
As a consequence,
\[
\Psi(T_{\beta_k}^{-1}T_{\gamma_{ij}}T_{\beta_j}^{-1}) = \begin{pmatrix} Id & 0 \\
 E_{ij} + E_{ji} & Id \end{pmatrix}.
\]
Therefore $\phi^B \circ \Psi : \mathcal{L}_{g,1} \to S_g(\mathbb{Z})$ is surjective. In addition, notice that
\[
\text{Ker}(\phi^B \circ \Psi : \mathcal{L}_{g,1} \to S_g(\mathbb{Z})) = \text{Ker}(\mathcal{B}_{g,1} \to GL_{g}(\mathbb{Z}) \times S_g(\mathbb{Z})) \cap \mathcal{L}_{g,1} = \mathcal{T}B_{g,1} \cap \mathcal{L}_{g,1} = \mathcal{L}T\mathcal{B}_{g,1}.
\]
Therefore we get the desired short exact sequence. \hfill \Box

**Proposition 2.2.2.** For every $h \in \mathcal{B}_{g,1}$, there exist elements $l \in \mathcal{L}_{g,1}$, $f \in \mathcal{AB}_{g,1}$ and $\xi_b \in \mathcal{T}B_{g,1}$ such that $h = \xi_b fl$, i.e.
\[
\mathcal{B}_{g,1} = \mathcal{T}B_{g,1} \cdot \mathcal{AB}_{g,1} \cdot \mathcal{L}_{g,1}.
\]

**Proof.** Recall that by Lemma (1.3.2), we have the following short exact sequence:
\[
1 \longrightarrow \mathcal{T}B_{g,1} \longrightarrow \mathcal{B}_{g,1} \longrightarrow GL_{g}(\mathbb{Z}) \times S_g(\mathbb{Z}) \longrightarrow 1 \quad \text{(2.2.2)}
\]
Moreover, by Lemma (1.3.3) and Proposition (2.2.1) we know that $\phi^B \circ \Psi(\mathcal{AB}_{g,1}) = GL_{g}(\mathbb{Z})$ and $\phi^B \circ \Psi(\mathcal{L}_{g,1}) = S_g(\mathbb{Z})$, respectively.

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2.2. The Luft group and CBP-twists

As a consequence, if \( h \in \mathcal{B}_{g,1} \) then there exist \( f \in \mathcal{A}\mathcal{B}_{g,1}, \ l \in \mathcal{L}_{g,1} \) such that
\[
\Psi(hl^{-1}f^{-1}) = Id,
\]
and by the short exact sequence (2.2.2), we get that there exists an element \( \xi_b \in \mathcal{T}\mathcal{B}_{g,1} \) such that \( \xi_b = hl^{-1}f^{-1} \), so \( h = \xi_bfl \).

**Definition 2.2.1** (Section 4 in [38]). A contractible bounding pair map, abbreviated CBP-twist, is a map of the form \( T_{\beta}T_{\beta'}^{-1} \), where \( \beta \) and \( \beta' \) are two homologous non-isotopic and disjoint simple closed curves on \( \Sigma_{g,1} \) such that each one bounds a properly embedded disc in \( \mathcal{H}_g \).

**Proposition 2.2.3** (Theorem 9 in [38]). The Luft-Torelli group \( \mathcal{LT}\mathcal{B}_{g,1} \) is generated by CBP-twists.

**Lemma 2.2.1.** Let \( T_{\beta}T_{\beta'}^{-1} \) be a CBP-twist of genus \( k \), and let \( D_{\beta}, D_{\beta'} \) be two essential proper embedded discs in \( \mathcal{H}_g \) with boundaries \( \beta, \beta' \) respectively. Then there exist \( g-1 \) essential proper embedded discs \( D_{\beta_1}, \ldots, D_{\beta_{g-1}} \) in \( \mathcal{H}_g \), with boundaries \( \beta_1, \ldots, \beta_{g-1} \) respectively, such that
\[
\text{Int}(\mathcal{H}_g) - N\left(D_{\beta} \cup D_{\beta'} \cup D_{\beta_1} \cup \ldots \cup D_{\beta_{g-1}}\right)
\]
is the disjoint union of two open 3-balls.

**Proof.** Let \( D_{\beta}, D_{\beta'} \) be essential proper embedded discs with boundaries \( \beta, \beta' \) respectively. Since our embedding of \( \Sigma_{g,1} \) into \( S^3 \) is standard, there exists a simple closed curve \( \alpha \) in \( \Sigma_{g,1} \) which bounds a properly embedded disc \( D_{\alpha} \) in \( \mathcal{H}_g \) (the outer handlebody) and which intersects each of the curves \( \beta \) and \( \beta' \) in exactly one point. Consider a regular neighbourhood of the union \( D_{\beta} \cup D_{\alpha} \cup D_{\beta'} \). It is a 3-ball, whose intersection with \( \Sigma_{g,1} \) is the disjoint union of two bounding simple closed curves, which are in \( \mathcal{LT}\mathcal{B}_{g,1} \) by construction. Applying Proposition (1.2.2) to these curves we get the result. \( \Box \)

**Lemma 2.2.2** (Lema 2.9 in [21]). Let \( H, H' \) be handlebodies and let \( \{D_1, \ldots, D_m\}, \{D'_1, \ldots, D'_m\} \) be systems of disks for \( H, H' \), respectively. Assume that there is a homeomorphism \( \phi: \partial H \to \partial H' \) such that for each \( i \), \( \phi(\partial D_i) = \partial D'_i \). Then there is a homeomorphism \( \psi: H \to H' \) such that \( \psi|_{\partial H} = \phi \).

**Proposition 2.2.4.** \( \mathcal{B}_{g,1} \) acts transitively on CBP-twists of a given genus.

**Proof.** Let \( T_\zeta T_{\zeta'}^{-1}, T_{\beta}T_{\beta'}^{-1} \) be CBP-twists of genus \( k \) on \( \Sigma_{g,1} \). We prove that there is an element \( \psi \in \mathcal{B}_{g,1} \) such that \( \psi(\beta) = \zeta, \ \psi(\beta') = \zeta' \) and as a consequence we will get that the following equality:
\[
\psi(T_{\beta}T_{\beta'}^{-1})\psi^{-1} = T_{\psi(\beta)}T_{\psi(\beta')^{-1}} = T_{\zeta}\zeta'.
\]
Given \( T_{\beta}T_{\beta'}^{-1}, T_{\zeta}\zeta'^{-1} \) CBP-twists of genus \( k \), by Lemma (2.2.1), we know that there exist \( g-1 \) essential proper embedded discs \( D_{\beta_1}, \ldots, D_{\beta_{g-1}} \) in \( \mathcal{H}_g \), with boundaries \( \beta_1, \ldots, \beta_{g-1} \) respectively,
such that

$$\text{Int}(\mathcal{H}_g) - N \left( D_\beta \cup D_{\beta'} \cup D_{\beta_1} \cup \ldots \cup D_{\beta_{g-1}} \right)$$

is the disjoint union of two open 3-balls, and there exist $g - 1$ essential proper embedded discs $D_{\zeta_1}, \ldots, D_{\zeta_{g-1}}$ in $\mathcal{H}_g$, with boundaries $\zeta_1, \ldots, \zeta_{g-1}$ respectively, such that

$$\text{Int}(\mathcal{H}_g) - N \left( D_{\zeta} \cup D_{\zeta'} \cup D_{\zeta_1} \cup \ldots \cup D_{\zeta_{g-1}} \right)$$

is the disjoint union of two open 3-balls.

Observe that in particular, $\{D_\beta, D_{\beta'}, D_{\beta_1}, \ldots, D_{\beta_{g-1}}\}$ and $\{D_{\zeta}, D_{\zeta'}, D_{\zeta_1}, \ldots, D_{\zeta_{g-1}}\}$ are two systems of disks for $\mathcal{H}_g$.

Since $T_\zeta T_{\zeta'}^{-1}, T_{\beta'} T_{\beta}^{-1}$ are BP-maps of the same genus, by the change of coordinates principle, there exists a homeomorphism $\phi$ from $\Sigma_{g,1}$ to $\Sigma_{g,1}$ sending $\{\beta, \beta', \beta_1, \ldots, \beta_{g-1}\}$ to $\{\zeta, \zeta', \zeta_1, \ldots, \zeta_{g-1}\}$ respectively.

By Lemma (2.2.2) we get that there exists a homeomorphism $\psi$ from $\mathcal{H}_g$ to $\mathcal{H}_g$ such that $\psi |_{\partial \mathcal{H}_g} = \phi$. Therefore $\phi$ extends to $\mathcal{H}_{g,1}$ i.e. $\phi \in B_{g,1}$.

**Proposition 2.2.5.** Every CBP-twist of genus $k$ is a product of $k$ CBP-twists of genus 1.

**Proof.** Let $T_{\beta} T_{\beta'}^{-1}$ be a CBP-twist of genus $k$. Consider the following simple closed curves in the standardly embedded surface $\Sigma_{g,1}$:

![Diagram of curves](image)

Observe that $T_{\zeta} T_{\zeta'}^{-1}$ is a CBP-twist of genus $k$ and for $i = 0, \ldots, k - 1$, $T_{\zeta_i} T_{\zeta_{i+1}}^{-1}$ are CBP-twists of genus 1, where $\zeta_0 = \zeta$, $\zeta_k = \zeta'$. By Proposition (2.2.4) we know that there is an element $h \in B_{g,1}$ such that $T_{\beta} T_{\beta'}^{-1} = hT_{\zeta} T_{\zeta'}^{-1} h^{-1}$. Therefore,

$$T_{\beta} T_{\beta'}^{-1} = hT_{\zeta} T_{\zeta'}^{-1} h^{-1} = (hT_{\zeta_0} T_{\zeta_1}^{-1} h^{-1})(hT_{\zeta_1} T_{\zeta_2}^{-1} h^{-1}) \cdots (hT_{\zeta_{k-1}} T_{\zeta_k}^{-1} h^{-1}) = (T_{h(\zeta_0)} T_{h(\zeta_1)}^{-1})(T_{h(\zeta_1)} T_{h(\zeta_2)}^{-1}) \cdots (T_{h(\zeta_{k-1})} T_{h(\zeta_k)}^{-1}).$$

Since $\{T_{\zeta_i} T_{\zeta_{i+1}}^{-1}\}$ are CBP-twists of genus 1 and $h \in B_{g,1}$ then $\{T_{h(\zeta_i)} T_{h(\zeta_{i+1})}^{-1}\}$ are also CBP-twists of genus 1, as desired. 

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2.3. From invariants to trivial cocycles

Remark 2.2.1. A posteriori we found that, in [35], G. Omori had obtained independently Proposition (2.2.5).

2.3 From invariants to trivial cocycles

Let $A$ be an abelian group. Denote by $A_2$ the subgroup of 2-torsion elements.

Consider an $A$-valuated invariant of homology 3-spheres $F: S^3 \to A$.

Precomposing with the canonical maps $T_{g,1} \to \lim_{g \to \infty} T_{g,1}/\sim \to S^3$ we get a family of maps $\{F_g\}_g$ with $F_g: T_{g,1} \to A$ satisfying the following properties:

i) $F_{g+1}(x) = F_g(x)$ for every $x \in T_{g,1},$

ii) $F_g(\xi_ax\xi_b) = F_g(x)$ for every $x \in T_{g,1}, \xi_a \in TA_{g,1}, \xi_b \in TB_{g,1},$

iii) $F_g(\phi x \phi^{-1}) = F_g(x)$ for every $x \in T_{g,1}, \phi \in AB_{g,1}.$

Since the stabilization maps are injective, the map $F_g$ determines by restriction all maps $F_{g'}$ for $g' < g$. We avoid the peculiarities of the first Torelli groups by restricting ourselves to $g \geq 3$. We also consider the associated trivial 2-cocycles $\{C_g\}_g$, which measure the failure of the maps $\{F_g\}_g$ to be homomorphisms of groups, i.e.

$$C_g: T_{g,1} \times T_{g,1} \to A,$$

$$(\phi, \psi) \mapsto F_g(\phi) + F_g(\psi) - F_g(\phi\psi).$$

Since $F$ is an invariant, the family of maps $\{F_g\}_g$ satisfies the properties i), ii), iii) and as a consequence, the family of 2-cocycles $\{C_g\}_g$ inherits the following properties:

(1) The 2-cocycles $\{C_g\}_g$ are compatible with the stabilization map, i.e. the following diagram of maps commutes:

```
\begin{array}{ccc}
T_{g,1} \times T_{g,1} & \xrightarrow{C_g} & T_{g+1,1} \times T_{g+1,1} \\
\downarrow{C_g} \quad & & \downarrow{C_{g+1}} \\
A & \quad & A
\end{array}
```

(2) The 2-cocycles $\{C_g\}_g$ are invariant under conjugation by elements in $AB_{g,1}$, i.e. for every $\phi \in AB_{g,1}$,

$$C_g(\phi - \phi^{-1}, \phi - \phi^{-1}) = C_g(-,-).$$
(3) If \( \phi \in \mathcal{T}A_{g,1} \) or \( \psi \in \mathcal{T}B_{g,1} \) then \( C_g(\phi, \psi) = 0 \).

In general there are many families of maps \( \{ F_g \} \) satisfying the properties i) - iii) that induce the same family of trivial 2-cocycles \( \{ C_g \} \).

Recall that any two trivializations of a given trivial 2-cocycle on \( T_{g,1} \) differ by an element of the group \( \text{Hom}(T_{g,1}, A) \). Then, given two families of maps \( \{ F_g \}, \{ F_g' \} \) satisfying the properties i) - iii), we have that \( \{ F_g - F_g' \} \) is a family of homomorphisms satisfying the same properties. As a consequence, the number of families \( \{ F_g \} \) satisfying the properties i) - iii) that induce the same family of trivial 2-cocycles \( \{ C_g \} \), coincides with the number of homomorphisms in \( \text{Hom}(T_{g,1}, A)AB_{g,1} \) compatible with the stabilization map, that are trivial over \( \mathcal{T}A_{g,1} \) and \( \mathcal{T}B_{g,1} \). We devote the rest of this section to compute and study such homomorphisms. In order to achieve our target, we first give three algebraic lemmas. Throughout this chapter, given an element \( x \in A \) of order 2, set \( \varepsilon^x : \mathbb{Z}/2 \to A \) the homomorphism which sends 1 to \( x \).

**Lemma 2.3.1.** For \( g \geq 4 \), let \( \pi : \mathfrak{B}_3 \to \mathfrak{B}_0 = \mathbb{Z}/2 \) be the canonical projection. Then we have the following isomorphism:

\[
\Upsilon : A_2 \longrightarrow \text{Hom}(\mathfrak{B}^3_{g,1}, A)^{GL_g(\mathbb{Z})}
\]

\[
x \longmapsto \varphi^x_g = \varepsilon^x \circ \pi.
\]

**Proof.** First of all notice that \( \Upsilon \) is well defined because \( \varepsilon^x \) and the canonical projection \( \pi : \mathfrak{B}_3 \to \mathfrak{B}_0 \) are \( GL_g(\mathbb{Z}) \)-invariant. Moreover, it is clear that \( \Upsilon \) is injective. Next we show that \( \Upsilon \) is surjective.

Let \( \varphi_g \in \text{Hom}(\mathfrak{B}^3_{g,1}, A)^{GL_g(\mathbb{Z})} \), we prove that \( \varphi_g = \varphi^x_g \) for some \( x \in A_2 \).

Consider \( f \) the matrix \( \begin{pmatrix} G & 0 \\ 0 & I_{G^{-1}} \end{pmatrix} \) with \( G = (1)(2)(3) \in \mathfrak{S}_g \subset GL_g(\mathbb{Z}) \) for \( i, j, k \) pairwise different. Then, for \( Z_l = A_l \) or \( B_l \) for each \( l \), we have the following equalities

\[
\varphi_g(Z_1) = \varphi_g(f(Z_1)) = \varphi_g(Z_1), \tag{2.3.1}
\]

\[
\varphi_g(Z_1Z_2) = \varphi_g(f(Z_1)f(Z_2)) = \varphi_g(Z_1Z_2), \tag{2.3.2}
\]

\[
\varphi_g(A_1B_1) = \varphi_g(f(A_1)f(B_1)) = \varphi_g(A_1B_1), \tag{2.3.3}
\]

\[
\varphi_g(Z_1Z_2Z_3) = \varphi_g(f(Z_1)f(Z_2)f(Z_3)) = \varphi_g(Z_1Z_2Z_3), \tag{2.3.4}
\]

\[
\varphi_g(A_1B_1Z_2) = \varphi_g(f(A_1)f(B_1)f(Z_2)) = \varphi_g(A_1B_1Z_2). \tag{2.3.5}
\]

Next, consider \( f \) the matrix \( \begin{pmatrix} G & 0 \\ 0 & I_{G^{-1}} \end{pmatrix} \) with \( G \in GL_g(\mathbb{Z}) \) the matrix with 1’s at the diagonal and position \( (2,1) \) and 0’s elsewhere. Then we have the following equalities

\[
\varphi_g(A_1) = \varphi_g(f(A_1)) = \varphi_g(A_1 + A_2) = \varphi_g(A_1) + \varphi_g(A_2), \tag{2.3.6}
\]

hence, \( \varphi_g(A_2) = 0 \).
As a consequence of relations (2.3.4), (2.3.15), (2.3.16) we have that
\[ \varphi_g(B_2) = \varphi_g(f(B_2)) = \varphi_g(B_1 + B_2) = \varphi_g(B_1) + \varphi_g(B_2), \] (2.3.7)
hence, \( \varphi_g(B_1) = 0. \)

As a consequence of relations (2.3.1), (2.3.6), (2.3.7), we get that
\[ \varphi_g(Z_i) = 0 \text{ for all } i. \] (2.3.8)

In addition,
\[ \varphi_g(A_1Z_3) = \varphi_g(f(A_1)f(Z_3)) = \varphi_g((A_1 + A_2)Z_3) = \] (2.3.9)
\[ = \varphi_g(A_1Z_3) + \varphi_g(A_2Z_3), \text{ hence, } \varphi_g(A_2Z_3) = 0. \]

As a consequence of relations (2.3.2), (2.3.9), (2.3.10), we get that
\[ \varphi_g(Z_iZ_j) = 0 \text{ for all } i, j \text{ with } i \neq j. \] (2.3.11)

Besides,
\[ \varphi_g(A_3B_3A_1) = \varphi_g(f(A_3)f(B_3)f(A_1)) = \varphi_g(A_3B_3(A_1 + A_2)) = \] (2.3.12)
\[ = \varphi_g(A_3B_3A_1) + \varphi_g(A_3B_3A_2), \text{ hence, } \varphi_g(A_3B_3A_2) = 0. \]

As a consequence of relations (2.3.5), (2.3.12), (2.3.13) we get that
\[ \varphi_g(A_iB_iZ_j) = 0 \text{ for all } i, j \text{ with } i \neq j. \] (2.3.14)

Moreover,
\[ \varphi_g(A_1Z_3Z_4) = \varphi_g(f(A_1)f(Z_3)f(Z_4)) = \varphi_g((A_1 + A_2)Z_3Z_4) = \] (2.3.15)
\[ = \varphi_g(A_1Z_3Z_4) + \varphi_g(A_2Z_3Z_4), \text{ hence, } \varphi_g(A_2Z_3Z_4) = 0. \]

As a consequence of relations (2.3.4), (2.3.15), (2.3.16) we have that
\[ \varphi_g(Z_iZ_jZ_k) = 0 \text{ for all } i, j, k \text{ with } i \neq j \neq k. \]
Furthermore,
\[ \varphi_g(A_1B_1B_2) = \varphi_g(f(A_1)f(B_1)f(B_2)) = \varphi_g((A_1 + A_2)B_1(B_1 + B_2)) = \]
\[ = \varphi_g(A_1B_1) + \varphi_g(A_1B_1B_2) + \varphi_g(A_2B_1) + \varphi_g(A_2B_1B_2) \]  \hspace{1cm} (2.3.17)

Thus, \( \varphi_g(A_1B_1) = -\varphi_g(A_2B_1) - \varphi_g(A_2B_1B_2) \).

As a consequence of relations (2.3.11), (2.3.14), (2.3.17), we get that
\[ \varphi_g(A_iB_i) = 0 \text{ for all } i. \]

Thus \( \varphi_g \) is zero on all basis elements of \( \mathfrak{B}_3 \) except possibly on \( \bar{1} \), which can only be sent to an element of order \( \leq 2 \) in \( A \).

\begin{proof}
First of all we show that \( \Xi \) is well-defined, i.e. \( \varphi_g^{x_1,x_2} \) is \( GL_g(\mathbb{Z}) \)-invariant.

Let \( x_1 \) be a 2-torsion element of \( A \). Consider \( \varphi_g^{x_1,0} \) the homomorphism given by the composition of the projection map \( \mathfrak{B}_2 \to \mathfrak{B}_0 = \mathbb{Z}/2 \) and \( \varepsilon^{x_1} : \mathbb{Z}/2 \to A \). Notice that \( \varphi_g^{x_1,0} \) is a \( GL_g(\mathbb{Z}) \)-invariant homomorphism.

Let \( x_2 \) be a 2-torsion element of \( A \). Consider \( \varphi_g^{0,x_2} \in Hom(\mathfrak{B}_2^g, A) \) the homomorphism given by
\[ \varphi_g^{0,x_2}(Z_iZ_j) = \omega(Z_i,Z_j)x_2, \quad \varphi_g^{0,x_2}(\bar{Z}_iZ_i) = \omega(Z_i,Z_i)x_2 = 0 \text{ and } \varphi_g^{0,x_2}(\bar{1}) = 0. \]

Notice that the action of \( GL_g(\mathbb{Z}) \) on \( \bar{1} \) is trivial. On the other hand, since \( \omega \) is \( Sp_{2g} \)-invariant, we have that for any matrix \( f \) of the form \( \left( \begin{array}{cc} G & 0 \\ 0 & G^{-1} \end{array} \right) \), the following equality holds
\[ \varphi_g^{0,x_2}(f.Z_iZ_j) = \varphi_g^{0,x_2}(f(Z_i)f(Z_j)) = \omega(f(Z_i),f(Z_j))x_2 = \omega(Z_i,Z_j)x_2 = \varphi_g^{0,x_2}(Z_iZ_j). \]

Hence, \( \varphi_g^{0,x_2} \) is a \( GL_g(\mathbb{Z}) \)-invariant homomorphism. Therefore, \( \varphi_g^{x_1,x_2} = \varphi_g^{x_1,0} + \varphi_g^{0,x_2} \) is \( GL_g(\mathbb{Z}) \)-invariant. Moreover, it is clear that \( \Xi \) is injective. Next we prove that \( \Xi \) is surjective. Let \( \varphi_g \in Hom(\mathfrak{B}_2^g, A)^{GL_g(\mathbb{Z})} \), we show that \( \varphi_g = \varphi_g^{x_1,x_2} \) for some \( x_1, x_2 \in A_2 \).

Following the proof of Lemma (2.3.1), in particular, by the equalities (2.3.1), (2.3.2), (2.3.3), (2.3.6), (2.3.7), (2.3.9), (2.3.10) of Lemma (2.3.1), we have that
\[ \varphi_g(Z_i) = 0 \quad \varphi_g(Z_iZ_j) = 0 \quad \varphi_g(A_1B_1) = \varphi_g(A_1B_1) \text{ for all } i,j \text{ with } i \neq j. \]
Hence, the elements of $\text{Hom}(\mathfrak{B}^2_{g,1}, A)^{GL_g(\mathbb{Z})}$ are completely determined by the images of $\Gamma$ and $A_1B_1$. 

\textbf{Lemma 2.3.3.} For $g \geq 4$, the group $\text{Hom}(\wedge^3 H, A)^{GL_g(\mathbb{Z})}$ is zero.

\textbf{Proof.} Let $f$ be an element of $\text{Hom}(\wedge^3 H, A)^{GL_g(\mathbb{Z})}$.

- Consider the element $\phi = (G^i_0, 0) \in S_{p_2g}(\mathbb{Z})$ with $G = (1, i)(2, j)(3, k) \in G$. Then:

$$
\phi \cdot f(c_1 \wedge c_2 \wedge c_3) = f(\phi \cdot c_1 \wedge c_2 \wedge c_3) = f(c_1 \wedge c_j \wedge c_k),
$$

$$
\phi \cdot f(c_1 \wedge a_2 \wedge b_2) = f(\phi \cdot c_1 \wedge a_2 \wedge b_2) = f(c_1 \wedge a_j \wedge b_j).
$$

Thus, every element of $\text{Hom}(\wedge^3 H, A)^{GL_g(\mathbb{Z})}$ is determined by the images of the elements $c_1 \wedge c_2 \wedge c_3, c_1 \wedge a_2 \wedge b_2$ with $c_i = a_i$ or $b_i$.

- Consider the element $\phi = (G^i_0, 0) \in S_{p_2g}(\mathbb{Z})$ with $G \in GL_g(\mathbb{Z})$ the matrix with 1’s at the diagonal and position $(3, 4)$, and 0’s at the other positions. Then we get that

$$
\phi \cdot f(c_1 \wedge c_2 \wedge a_4) = f(\phi \cdot c_1 \wedge c_2 \wedge a_4) = f(c_1 \wedge c_2 \wedge a_3 + c_1 \wedge c_2 \wedge a_4)
$$

$$
= f(c_1 \wedge c_2 \wedge a_3) + f(c_1 \wedge c_2 \wedge a_4)
$$

Thus, $f(c_1 \wedge c_2 \wedge a_3) = 0$. Analogously, we have that $f(c_1 \wedge c_2 \wedge b_3) = 0$.

- Consider the element $\phi = (G^i_0, 0) \in S_{p_2g}(\mathbb{Z})$ with $G \in GL_g(\mathbb{Z})$ the matrix with 1’s at the diagonal and position $(1, 3)$, and 0’s at the other positions. Then we get that

$$
\phi \cdot f(a_3 \wedge a_2 \wedge a_2) = f(\phi \cdot a_3 \wedge a_2 \wedge a_2) = f(a_1 \wedge a_2 \wedge b_2 + a_3 \wedge a_2 \wedge b_2)
$$

$$
= f(a_1 \wedge a_2 \wedge b_2) + f(a_3 \wedge a_2 \wedge b_2)
$$

Thus, $f(a_1 \wedge a_2 \wedge b_2) = 0$. Analogously, $f(b_1 \wedge a_2 \wedge b_2) = 0$.

Therefore we get the desired result. 

\textbf{Remark 2.3.1.} Using the same arguments in proof of Lemma 2.3.3, we also have that given an integer $d \geq 2$, for any $g \geq 4$ $\text{Hom}(\wedge^3 H_d, A)^{GL_g(\mathbb{Z})}$ is zero.

Next, we give a lemma which ensures us that every $AB_{g,1}$-invariant homomorphism has to be zero on $T_{A_{g,1}}, T_{B_{g,1}}$.

\textbf{Lemma 2.3.4.} For $g \geq 3$, every $AB_{g,1}$-invariant homomorphism

$$
\varphi_g: TB_{g,1} \to A \quad \text{and} \quad \varphi_g: TA_{g,1} \to A
$$

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has to be zero.

Proof. We only give the proof for $\mathcal{T}B_{g,1}$ the other case is similar.

Let $\varphi_g \in \text{Hom}(\mathcal{T}B_{g,1}, A)^{AB_{g,1}}$. Consider the following short exact sequence:

$$1 \longrightarrow \mathcal{L}T B_{g,1} \longrightarrow \mathcal{T}B_{g,1} \longrightarrow IA \longrightarrow 1. \quad (2.3.18)$$

We prove that

I) $\varphi_g$ factors through $IA := \ker(Aut(\pi_1(\mathcal{H}_g)) \rightarrow GL_g(\mathbb{Z}))$.

II) the morphism $\varphi_g : IA \rightarrow A$ is trivial.

I) By the short exact sequence (2.3.18), $\varphi_g : \mathcal{T}B_{g,1} \rightarrow A$ factors through $IA$ if and only if $\varphi_g$ is zero over $\mathcal{L}T B_{g,1}$. Moreover, by Propositions (2.2.3) and (2.2.5), we have that $\mathcal{L}T B_{g,1}$ is generated by CBP-twists of genus 1. Thus it is enough to show that $\varphi_g$ is zero on all CBP-twists of genus 1.

Next, we divide the proof into three steps:

1) $\varphi_g$ takes the same value on all CBP-twists of genus 1.

2) $\varphi_g$ is zero on all CBP-twists of genus 2.

3) $\varphi_g$ is zero on all CBP-twists of genus 1.

1) Consider the following simple closed curves in the standardly embedded surface $\Sigma_{g,1}$:

![Diagram of simple closed curves]

Observe that $T_\beta T_{\beta'}^{-1}$ is a CBP-twist of genus 1.

By Proposition (2.2.4) we know that for every CBP-twist of genus 1, $T_\nu T_{\nu'}^{-1}$, on $\Sigma_{g,1}$ there exists an element $h \in B_{g,1}$ such that $T_\nu T_{\nu'}^{-1} = h T_\beta T_{\beta'}^{-1} h^{-1}$.

Besides, by Proposition (2.2.2), we have that there exist elements $l \in \mathcal{L}_{g,1}$, $f \in AB_{g,1}$ and $\xi_b \in \mathcal{T}B_{g,1}$ such that $h = \xi_b f l$. 

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Thus, since \( \phi \) is a \( A_{B_{g,1}} \)-invariant homomorphism, we get that

\[
\phi_g(T_\nu T_{\nu'}^{-1}) = \phi_g(hT_{\beta} T_{\beta'}^{-1} h^{-1}) = \phi_g(\xi_h fT_{\beta} T_{\beta'}^{-1} f^{-1} \xi_h^{-1}) = \\
= \phi_g(\xi_h) + \phi_g(fT_{\beta} T_{\beta'}^{-1} f^{-1}) + \phi_g(\xi_h^{-1}) = \\
= \phi_g(fT_{\beta} T_{\beta'}^{-1} f^{-1}) = \phi_g(\lambda T_{\beta} T_{\beta'}^{-1} l^{-1})
\]

Therefore there exists an element \( l \in L_{g,1} \) such that

\[
\phi_g(T_\nu T_{\nu'}^{-1}) = \phi_g(lT_{\beta} T_{\beta'}^{-1} l^{-1}).
\] (2.3.19)

Recall that \( S_g(\mathbb{Z}) \) is generated by the following family of matrices:

\[
\{E_{ii} \mid 1 \leq i \leq g\} \cup \{E_{ij} + E_{ji} \mid 1 \leq i < j \leq g\}
\]

where \( E_{ij} \) denotes the matrix with 1 at position \((i, j)\) and 0's elsewhere.

Take the theoretic section \( s: S_g(\mathbb{Z}) \to L_{g,1} \) of \( \Psi : L_{g,1} \to S_g(\mathbb{Z}) \) given by

\[
s(E_{ii}) = T_{\beta_i}, \quad s(E_{ij} + E_{ji}) = \begin{cases} 
T_{\beta_i}^{-1} T_{\gamma_{ij}} T_{\beta_j}^{-1} & \text{for } i = 1 \\
T_{\beta_i}^{-1} T_{\gamma_{ij}'} T_{\beta_j}^{-1} & \text{for } i \geq 2,
\end{cases}
\]

where \( \beta_i, \gamma_{ij}, \gamma_{ij}' \) are given in the following picture:

![Diagram of \( \beta_i, \gamma_{ij}, \gamma_{ij}' \) elements]

By the short exact sequence (2.2.1), we know that given an element \( l \in L_{g,1} \) there exists an element \( \xi_b \in LTB_{g,1} \) such that \( l = \xi_b s(\Psi(l)) \). Thus, by the equality (2.3.19),

\[
\phi_g(T_\nu T_{\nu'}^{-1}) = \phi_g(\lambda T_{\beta} T_{\beta'}^{-1} l^{-1}) = \phi_g(\xi_b s(\Psi(l)) T_{\beta} T_{\beta'}^{-1} s(\Psi(l))^{-1} \xi_b^{-1}) = \\
= \phi_g(s(\Psi(l)) T_{\beta} T_{\beta'}^{-1} s(\Psi(l))^{-1}).
\] (2.3.20)

Now observe that \( s(\Psi(l)) \) is a product of the following elements

\[
\{T_{\gamma_{ij}} \mid j \geq 2\}, \quad \{T_{\gamma_{ij}'} \mid 2 \leq i < j\}, \quad \{T_{\beta_i} \mid 1 \leq i \leq g\}.
\]
Recall that by the properties of Dehn twists we know that for every $a, b$ isotopy classes of simple closed curves on $\Sigma_{g,1}$,$$
exists \quad T_a T_b = T_b T_a \iff i(a,b) = 0.$$Then, since the curves$$\gamma_{12}, \quad \{\gamma'_{ij} \mid 2 \leq i < j\}, \quad \{\beta_i \mid 1 \leq i \leq g\}, \quad (2.3.21)$$are disjoint with the curves$$\beta, \quad \beta', \quad \{\gamma_{1j} \mid j \geq 3\}, \quad (2.3.22)$$the geometric intersection number between a curve of the family $(2.3.21)$ and a curve of the family $(2.3.22)$ is zero. Therefore, the elements of the family of Dehn twists$$T_{\gamma_{12}}, \quad \{T_{\gamma'_{ij}} \mid 2 \leq i < j\}, \quad \{T_{\beta_i} \mid 1 \leq i \leq g\}, \quad (2.3.23)$$commutes with the elements of the family of Dehn twists$$T_{\beta}, \quad T_{\beta'}, \quad \{T_{\gamma_{1j}} \mid j \geq 3\}. \quad (2.3.24)$$Furthermore, the elements of the family $\{T_{\gamma_{1j}} \mid j \geq 3\}$ commute between them because the curves of the family $\{\gamma_{1j} \mid j \geq 3\}$ are pairwise disjoint. Therefore,$$s(\Psi(l))T_{\beta}T_{\beta'}^{-1}s(\Psi(l))^{-1} = (T^{x_3}_{\gamma_{13}} \cdots T^{x_g}_{\gamma_{1g}})T_{\beta}T_{\beta'}^{-1}(T^{x_3}_{\gamma_{13}} \cdots T^{x_g}_{\gamma_{1g}})^{-1}, \quad (2.3.25)$$where $x_3, \ldots, x_g \in \mathbb{Z}$, and $\beta, \beta', \{\gamma_{1j} \mid 3 \leq j \geq g\}$ are the curves given in the following picture:

As a consequence of equalities $(2.3.20)$, $(2.3.25)$ we get that$$\varphi_g(T_{\beta}T_{\beta'}^{-1}) = \varphi_g((T^{x_3}_{\gamma_{13}} \cdots T^{x_g}_{\gamma_{1g}})T_{\beta}T_{\beta'}^{-1}(T^{x_3}_{\gamma_{13}} \cdots T^{x_g}_{\gamma_{1g}})^{-1}) = \varphi_g(T_{\beta}T_{\beta'}^{-1}(T^{x_3}_{\gamma_{13}} \cdots T^{x_g}_{\gamma_{1g}})(\beta')). \quad (2.3.26)$
Next, we prove that

$$\varphi_g(T_{\beta}T_{1 \gamma_1}^{-1}(\beta')) = \varphi_g(T_{\beta}T_{1 \gamma_1}^{-1}) + \sum_{i=1}^{g} x_i(\varphi_g(T_{\beta}T_{1 \gamma_1}^{-1}) - \varphi_g(T_{\beta}T_{1 \gamma_1}^{-1}, \beta')).$$

Consider the curves \(\{\gamma_{1j}, \gamma_{1j}' \mid 3 \leq j \leq g\}\) given in the following picture:

Notice that \(T_{\gamma_{1i}}^{-1} \in \mathcal{A}B_{g,1}\) for \(3 \leq i \leq g\).

Fix an integer \(j\) with \(3 \leq j \leq g\). Define \(\beta'' = (T_{\gamma_{1(i+1)}}^{-1} \cdots T_{\gamma_{1g}}^{-1})(\beta')\) for \(j \leq g - 1\) and \(\beta'' = \beta'\). Then the following equality holds

$$T_{\beta}T_{1 \gamma_{1j}}^{-1}(\beta'') = (T_{\beta}T_{1 \gamma_{1j}}^{-1})(T_{\beta}T_{1 \gamma_{1j}}^{-1})(T_{1 \gamma_{1j}}^{-1}(\beta)T_{1 \gamma_{1j}}^{-1}(\beta'')).$$

Since \(\varphi_g\) is an \(\mathcal{A}B_{g,1}\)-invariant homomorphism, we have that

$$\varphi_g(T_{\beta}T_{1 \gamma_{1j}}^{-1}(\beta'')) = \varphi_g(T_{\beta}T_{1 \gamma_{1j}}^{-1}) + \varphi_g(T_{\beta}T_{1 \gamma_{1j}}^{-1}(\beta)) + \varphi_g(T_{\beta}T_{1 \gamma_{1j}}^{-1}(\beta)T_{1 \gamma_{1j}}^{-1}(\beta'')) =$$

$$= \varphi_g(T_{\beta}T_{1 \gamma_{1j}}^{-1}) + \varphi_g(T_{\beta}T_{1 \gamma_{1j}}^{-1})T_{1 \gamma_{1j}}^{-1}(\beta)T_{1 \gamma_{1j}}^{-1}(\beta'') + \varphi_g(T_{\beta}T_{1 \gamma_{1j}}^{-1})T_{1 \gamma_{1j}}^{-1}(\beta)T_{1 \gamma_{1j}}^{-1}(\beta'')^{-1}) +$$

$$+ \varphi_g((T_{\gamma_{1j}}^{-1}T_{1 \gamma_{1j}}^{-1})(\beta)T_{1 \gamma_{1j}}^{-1}(\beta'')(T_{1 \gamma_{1j}}^{-1}(\beta'))^{-1}) =$$

$$= \varphi_g(T_{\beta}T_{1 \gamma_{1j}}^{-1}) + \varphi_g(T_{\beta}T_{1 \gamma_{1j}}^{-1}) + \varphi_g(T_{\beta}T_{1 \gamma_{1j}}^{-1}(\beta')).$$

Applying equality \([2.3.27]\) from \(k = x_j\) to \(k = 1\), we get that

$$\varphi_g(T_{\beta}T_{1 \gamma_{1j}}^{-1}(\beta'')) =$$

$$= x_j \varphi_g(T_{\beta}T_{1 \gamma_{1j}}^{-1}) + x_j \varphi_g(T_{\beta}T_{1 \gamma_{1j}}^{-1}(\beta')T_{1 \gamma_{1j}}^{-1}) + \varphi_g(T_{\beta}T_{1 \gamma_{1j}}^{-1}(\beta'')) =$$

$$= x_j(\varphi_g(T_{\beta}T_{1 \gamma_{1j}}^{-1}) - \varphi_g(T_{\beta}T_{1 \gamma_{1j}}^{-1}(\beta'))) + \varphi_g(T_{\beta}T_{1 \gamma_{1j}}^{-1}(\beta')).$$

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Then, applying recursively the equality (2.3.28) from \( j = 3 \) to \( j = g \), we obtain the following formula:

\[
\varphi_g(T_{\beta} T^{-1}_{(T_{\gamma_{13}} \cdots T_{\gamma_{1s}}) (\beta')}) = \varphi_g(T_{\beta} T^{-1}_{\beta'}) + \sum_{i=1}^{g} x_i (\varphi_g(T_{\beta} T^{-1}_{\beta'}) - \varphi_g(T_{\beta} T^{-1}_{T_{\gamma_{1i}} (\beta')})).
\] (2.3.29)

Next we prove that for \( 3 \leq k \leq g \),

\[
\varphi_g(T_{\beta} T^{-1}_{\gamma_{1k} (\beta')}) = \varphi_g(T_{\beta} T^{-1}_{T_{\gamma_{13}} (\beta')}).
\]

Consider the element \( f_k \in \mathcal{A}B_{g,1} \) given by the half twist of the shaded ball depicted in the following figure, that exchanges the holes 3 and \( k \).

Observe that \( f_k \) leaves \( \beta, \beta' \) invariant and sends \( \gamma_{1k} \) to \( \gamma_{13} \). Since \( \varphi_g \) is \( \mathcal{A}B_{g,1} \)-invariant, for \( 3 \leq k \leq g \), we have that

\[
\varphi_g(T_{\beta} T^{-1}_{\gamma_{1k} (\beta')}) = \varphi_g(T_{\gamma_{13}} T_{\beta} T^{-1}_{\beta'} T_{\gamma_{1k}}) = \varphi_g(T_{\beta} T^{-1}_{\beta'} T_{\gamma_{1k}} f_k T_{\gamma_{13}} f_k^{-1}) = \varphi_g(T_{\gamma_{13}} T_{\beta} T^{-1}_{\beta'} T_{\gamma_{13}} f_k T_{\gamma_{13}} f_k^{-1}) = \varphi_g(T_{\beta} T^{-1}_{T_{\gamma_{13}} (\beta')}).
\] (2.3.30)

Therefore it is enough to show that

\[
\varphi_g(T_{\beta} T^{-1}_{\beta'}) = \varphi_g(T_{\beta} T^{-1}_{T_{\gamma_{13}} (\beta')}).
\]

Since \( \beta_1, \beta_3 \) are disjoint with \( \beta, \beta', \gamma_{13} \) then

\[
T_{\beta} T^{-1}_{T_{\gamma_{13}} (\beta')} = T_{\gamma_{13}} T_{\beta} T_{\gamma_{13}}^{-1} T_{\beta'} T_{\gamma_{13}}^{-1} = (T_{\beta_1} T_{\gamma_{13}}^{-1} T_{\beta_3}^{-1}) T_{\beta} T_{\gamma_{13}}^{-1} (T_{\beta_1} T_{\gamma_{13}}^{-1} T_{\beta_3}^{-1})^{-1}.
\]

Now take \( f \in \mathcal{A}B_{g,1} \) given by \( f = (T_{\alpha_3} T_{\eta_{34}} T_{\beta_4})^{-1} \), where \( \alpha_3, \eta_{34}, \beta_4 \) are the curves on \( \Sigma_{g,1} \) given in the following picture:
2.3. From invariants to trivial cocycles

Since $\alpha_3$, $\eta_{34}$, $\beta_4$ do not intersect either of $\beta$, $\beta'$ then $f$ commutes with $T_{\beta}T_{\beta'}^{-1}$. Thus,

$$fT_{\gamma_{13}}T_{\beta}^{-1}T_{\gamma_{13}}T_{\beta}^{-1} = (f(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3}^{-1}f^{-1})T_{\beta}^{-1}T_{\beta'}^{-1}(f(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3}^{-1}f^{-1})^{-1})^{-1}.$$

Observe that

$$\Psi(f) = \begin{pmatrix} G & 0 \\ 0 & tG^{-1} \end{pmatrix} \quad \Psi(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3}^{-1}f^{-1}) = \begin{pmatrix} Id & 0 \\ M & Id \end{pmatrix},$$

with

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus

$$\Psi(f(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3}^{-1}f^{-1})) = \begin{pmatrix} Id & 0 \\ tG^{-1}MG^{-1} & Id \end{pmatrix}$$

with

$$tG^{-1}MG^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\Psi(f(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3}^{-1}f^{-1})) = \Psi((T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3}^{-1})(T_{\beta_1}^{-1}T_{\gamma_{14}}T_{\beta_4}^{-1})).$$

As a consequence, by the short exact sequence (2.2.1), there exists an element $\xi_b \in \mathcal{L}TB_{g,1}$ such
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that

\[ f(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3}^{-1})^{-1} = \xi_b(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3}^{-1})(T_{\beta_1}^{-1}T_{\gamma_{14}}T_{\beta_4}^{-1}). \]  

(2.3.31)

Since \( \varphi_g \) is an \( \mathcal{A}B_{g,1} \)-invariant homomorphism, \( T_{\beta_1}, T_{\beta_3}, T_{\beta_4} \) commute with \( T_{\gamma_{13}}, T_{\gamma_{14}}, T_{\beta}, T_{\beta'} \), and \( f \) commutes with \( T_{\beta}, T_{\beta'} \), by (2.3.31) we get that

\[ \varphi_g(T_{\gamma_{13}}T_{\beta}T_{\beta'}^{-1}T_{\gamma_{12}}) = \varphi_g((T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3}^{-1})T_{\beta}T_{\beta'}^{-1}(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3}^{-1})^{-1}) = \varphi_g((f(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3}^{-1})f^{-1})T_{\beta}T_{\beta'}^{-1}(f(T_{\beta_1}^{-1}T_{\gamma_{13}}T_{\beta_3}^{-1})f^{-1})^{-1}) = \varphi_g(\xi_b(T_{\gamma_{13}}T_{\gamma_{14}})T_{\beta}T_{\beta'}^{-1}(T_{\gamma_{13}}T_{\gamma_{14}})^{-1}\xi_b^{-1}) = \varphi_g(\xi_b) + \varphi_g((T_{\gamma_{13}}T_{\gamma_{14}})T_{\beta}T_{\beta'}^{-1}(T_{\gamma_{13}}T_{\gamma_{14}})^{-1}) - \varphi_g(\xi_b) = \varphi_g((T_{\gamma_{13}}T_{\gamma_{14}})T_{\beta}T_{\beta'}^{-1}(T_{\gamma_{13}}T_{\gamma_{14}})^{-1}) = \varphi_g(T_{\beta}T_{\gamma_{13}}T_{\gamma_{14}}(\beta')). \]  

(2.3.32)

Notice that

\[ T_{\beta}T_{\gamma_{13}}T_{\gamma_{14}}(\beta') = (T_{\beta}T_{\gamma_{14}}^{-1}(\beta'))(T_{\gamma_{13}}T_{\gamma_{14}}^{-1}(\beta'))(T_{\gamma_{13}}T_{\gamma_{14}}^{-1}(\beta'))(T_{\gamma_{13}}T_{\gamma_{14}}^{-1}(\beta')). \]

Since \( \varphi_g \) is a \( \mathcal{A}B_{g,1} \)-invariant homomorphism and \( f_4, T_{\gamma_{13}}T_{\gamma_{14}}^{-1} \in \mathcal{A}B_{g,1} \), we have that

\[ \varphi_g(T_{\gamma_{13}}T_{\gamma_{14}}^{-1}(\beta')) = \varphi_g(T_{\beta}T_{\gamma_{14}}^{-1}(\beta')) + \varphi_g(T_{\gamma_{13}}T_{\gamma_{14}}^{-1}(\beta')) + \varphi_g(T_{\gamma_{13}}T_{\gamma_{14}}^{-1}(\beta')) + \varphi_g(T_{\gamma_{13}}T_{\gamma_{14}}^{-1}(\beta')). \]  

(2.3.33)

Hence, by equalities (2.3.32), (2.3.33), we get that

\[ \varphi_g(T_{\beta}T_{\gamma_{13}}^{-1}(\beta')) = \varphi_g(T_{\beta}T_{\gamma_{14}}^{-1}(\beta')) - \varphi_g(T_{\beta}T_{\gamma_{13}}^{-1}(\beta')) + \varphi_g(T_{\beta}T_{\gamma_{14}}^{-1}(\beta')). \]

Then \( \varphi_g(T_{\beta}T_{\gamma_{13}}^{-1}(\beta')) = \varphi_g(T_{\beta}T_{\gamma_{14}}^{-1}(\beta')) \), and by (2.3.30) we get that

\[ \varphi_g(T_{\beta}T_{\gamma_{13}}^{-1}(\beta')) = \varphi_g(T_{\beta}T_{\gamma_{14}}^{-1}(\beta')). \]  

(2.3.34)
Finally, by equalities (2.3.29), (2.3.34), (2.3.26), we get that
\[ \varphi_g(T_\nu T_{\nu'}^{-1}) = \varphi_g(T_{\beta} T_{\beta'}^{-1}), \]
i.e. \( \varphi_g \) takes the same value on all CBP-twists of genus 1.

2) We prove that \( \varphi_g \) is zero on all CBP-twists of genus 2 using 1).

By Proposition (2.2.5) we know that every CBP-twist of genus 2 is a product of two CBP-twists of genus 1, what is more, by the proof of Proposition (2.2.5), we know that given \( T_\varepsilon T_{\varepsilon'}^{-1} \) a CBP-twist of genus 2, there exists a curve \( \zeta \) such that \( T_\varepsilon T_{\zeta}^{-1} \) and \( T_{\zeta} T_{\varepsilon'}^{-1} \) are CBP-twists of genus 1.

Thus we have that
\[ \varphi_g(T_\varepsilon T_{\zeta}^{-1}) = \varphi_g(T_\varepsilon T_{\zeta}^{-1}) + \varphi_g(T_{\zeta} T_{\varepsilon'}^{-1}) \]
\[ = \varphi_g(T_\varepsilon T_{\zeta}^{-1}) - \varphi_g(T_{\varepsilon'} T_{\zeta}^{-1}), \]
and since, by 1), we know that \( \varphi_g(T_\varepsilon T_{\zeta}^{-1}) = \varphi_g(T_{\varepsilon'} T_{\zeta}^{-1}) \), then
\[ \varphi_g(T_\varepsilon T_{\zeta}^{-1}) = 0, \]
i.e. \( \varphi_g \) is zero on all CBP-twists of genus 2.

3) We prove that \( \varphi_g \) is zero on all CBP-twists of genus 1 using 1), 2) and the lantern relation.

Consider the following curves in the standardly embedded surface \( \Sigma_{g,1} \):

Observe that for \( i = 1, 2, 3 \), \( T_{\beta_i} T_{\beta'_i}^{-1} \) are CBP-twists of genus 1, \( T_{\zeta_i} T_{\zeta'_i}^{-1} \) are CBP-twists of genus 2,
we get the following equalities:

\[(T_{\beta_2} T_{\beta_2}^{-1})(T_{\beta_2'} T_{\beta_2'}^{-1})(T_{\beta_3} T_{\beta_3}^{-1}) = T_{\gamma},\]

\[(T_{\zeta_1} T_{\zeta_1}^{-1})(T_{\zeta_2} T_{\zeta_2}^{-1})(T_{\zeta_3} T_{\zeta_3}^{-1}) = T_{\gamma}.\]

Thus,

\[T_{\beta_3} T_{\beta_3}^{-1} = (T_{\beta_1} T_{\beta_1}^{-1})(T_{\beta_2} T_{\beta_2}^{-1})(T_{\beta_3} T_{\beta_3}^{-1})(T_{\zeta_1} T_{\zeta_1}^{-1})(T_{\zeta_2} T_{\zeta_2}^{-1})(T_{\zeta_3} T_{\zeta_3}^{-1}).\]

In general, if \(T_{\nu} T_{\nu'}^{-1}\) a CBP-twist of genus 1, by Proposition [2.2.4], there exists an element \(h \in B_{g,1}\) such that \(T_{\nu} T_{\nu'}^{-1} = h T_{\beta_3} T_{\beta_3}^{-1} h^{-1}\).

Then

\[T_{\nu} T_{\nu'}^{-1} = (h T_{\beta_1} T_{\beta_1}^{-1} h^{-1})(h T_{\beta_2} T_{\beta_2}^{-1} h^{-1})(T_{\zeta_1} T_{\zeta_1}^{-1})(h T_{\beta_3} T_{\beta_3}^{-1} h^{-1})\]

\[= (T_{h(\beta_1)} T_{h(\beta_1)}^{-1})(T_{h(\beta_2)} T_{h(\beta_2)}^{-1})(T_{h(\zeta_1)} T_{h(\zeta_1)}^{-1})(T_{h(\beta_3)} T_{h(\beta_3)}^{-1} h^{-1})(T_{h(\zeta_2)} T_{h(\zeta_2)}^{-1} h^{-1})(T_{h(\zeta_3)} T_{h(\zeta_3)}^{-1} h^{-1}).\]

Since \(T_{\zeta_1} T_{\zeta_1}^{-1}, T_{\zeta_2} T_{\zeta_2}^{-1}, T_{\zeta_3} T_{\zeta_3}^{-1}\) are CBP-twists of genus 2 and \(h \in B_{g,1}\) then \(T_{h(\zeta_1)} T_{h(\zeta_1)}^{-1}, T_{h(\zeta_2)} T_{h(\zeta_2)}^{-1}, T_{h(\zeta_3)} T_{h(\zeta_3)}^{-1}\) also are CBP-twists of genus 2. Thus, by 2), we get that

\[\varphi_g(T_{\nu} T_{\nu'}^{-1}) = \varphi_g(T_{h(\beta_1)} T_{h(\beta_1)}^{-1})(T_{h(\beta_2)} T_{h(\beta_2)}^{-1})(T_{h(\beta_3)} T_{h(\beta_3)}^{-1} h^{-1})(T_{h(\zeta_2)} T_{h(\zeta_2)}^{-1} h^{-1})(T_{h(\zeta_3)} T_{h(\zeta_3)}^{-1} h^{-1}).\]

and since, by 1), we know that \(\varphi_g(T_{h(\beta_1)} T_{h(\beta_1)}^{-1}) = \varphi_g(T_{h(\beta_2)} T_{h(\beta_2)}^{-1})\), then

\[\varphi_g(T_{\nu} T_{\nu'}^{-1}) = 0,\]

i.e. \(\varphi_g\) is zero on all CBP-twists.
II) As a consequence of I), \( \varphi_g \) factors through IA. As the action on the fundamental group of the inner handlebody \( H_g \) induces a surjective map \( AB_{g,1} \to Aut(\pi_1(H_g)) \), we can view \( \varphi_g \) as an \( Aut(\pi_1(H_g)) \)-invariant map \( \varphi_g : IA \to A \). Let \( \alpha_1, \ldots, \alpha_g \) denote the generators of \( \pi_1(H_g) \). According to Magnus \[27\], the group IA is normally generated as a subgroup of \( Aut(\pi_1(H_g)) \) by the automorphism \( K_{12} \) given by \( K_{12}(\alpha_1) = \alpha_2 \alpha_1 \alpha_2^{-1} \) and \( K_{12}(\alpha_i) = \alpha_i \) for \( i \geq 2 \). By invariance, \( \varphi_g \) is determined by its value on \( K_{12} \). So it is enough to show that \( \varphi_g(K_{12}) = 0 \).

Consider the automorphism \( f \in Aut(\pi_1(H_g)) \) given by \( f(\alpha_3) = \alpha_3 \alpha_2 \) and \( f(\alpha_i) = \alpha_i \) for \( i \neq 3 \), with inverse \( f^{-1} \in Aut(\pi_1(H_g)) \) given by \( f^{-1}(\alpha_3) = \alpha_3 \alpha_2^{-1} \) and \( f^{-1}(\alpha_i) = \alpha_i \) for \( i \neq 3 \).

Take the element \( K_{13} \in Aut(\pi_1(H_g)) \) given by

\[
K_{13}(\alpha_1) = \alpha_3 \alpha_1 \alpha_3^{-1} \quad \text{and} \quad K_{13}(\alpha_i) = \alpha_i \quad \text{for} \quad i \geq 2.
\]

Clearly \( K_{13} \) is an element of IA. Observe that

\[
fK_{13}f^{-1}(\alpha_1) = \alpha_3 \alpha_2 \alpha_1 \alpha_2^{-1} \alpha_3^{-1} \quad \text{and} \quad fK_{13}f^{-1}(\alpha_i) = \alpha_i \quad \text{for} \quad i \geq 2.
\]

Then \( fK_{13}f^{-1} = K_{12}K_{13} \), and since \( \varphi_g \) is an \( Aut(\pi_1(H_g)) \)-invariant map, we have that

\[
\varphi_g(K_{13}) = \varphi_g(fK_{13}f^{-1}) = \varphi_g(K_{12}K_{13}) = \varphi_g(K_{12}) + \varphi_g(K_{13}).
\]

Therefore \( \varphi_g(K_{12}) = 0 \), as desired.

Now we are ready to compute the \( AB_{g,1} \)-invariant homomorphisms on the Torelli group.

**Lemma 2.3.5.** For \( g \geq 4 \), there is an isomorphism

\[
\Lambda : A_2 \rightarrow Hom(T_{g,1}, A)^{AB_{g,1}}
\]

\[
x \mapsto \mu^g_x = \varphi^g_x \circ \sigma
\]

where \( \varphi^x \) is the map defined in Lemma \( [2.3.1] \).

**Proof.** First of all, notice that \( \Lambda \) is well defined because the BCJ-homomorphism \( \sigma \) is \( Sp_2(Z) \)-equivariant, \( \varphi^x \) is \( AB_{g,1} \)-invariant and then \( \mu^g_x \) is also \( AB_{g,1} \)-invariant. Moreover, it is clear that \( \Lambda \) is injective. Next we will prove that \( \Lambda \) is surjective.

From the work of D. Johnson (see [19]), we know that there is a short exact sequence

\[
1 \rightarrow \mathbb{B}_2 \xrightarrow{i} H_1(T_{g,1}; \mathbb{Z}) \xrightarrow{\tau} \wedge^3 H \rightarrow 1.
\]

Taking the 5-term exact sequence associated to the above short exact sequence we get the exact
sequence
\[
0 \longrightarrow \text{Hom}(\Lambda^3 H, A) \xrightarrow{\text{inf}} \text{Hom}(H_1(\mathcal{T}, \mathcal{Z}), A) \xrightarrow{\text{res}} \text{Hom}(\mathfrak{G}_2, A).
\]

Taking $GL_g(\mathbb{Z})$-invariants we get another exact sequence
\[
0 \xrightarrow{\text{res}} \text{Hom}(\Lambda^3 H, A)^{GL_g(\mathbb{Z})} \xrightarrow{\text{res}} \text{Hom}(H_1(\mathcal{T}, \mathcal{Z}), A)^{GL_g(\mathbb{Z})} \xrightarrow{\text{res}} \text{Hom}(\mathfrak{G}_2, A)^{GL_g(\mathbb{Z})}.
\]

By Lemma (2.3.3), $\text{Hom}(\Lambda^3 H, A)^{GL_g(\mathbb{Z})} = 0$, for $g \geq 4$. As a consequence, we have an injection
\[
\text{Hom}(H_1(\mathcal{T}, \mathcal{Z}), A)^{GL_g(\mathbb{Z})} \hookrightarrow \text{Hom}(\mathfrak{G}_2, A)^{GL_g(\mathbb{Z})}.
\]

By Lemma (2.3.2), all elements of $\text{Hom}(\mathfrak{G}_2, A)^{GL_g(\mathbb{Z})}$ are $\varphi^{x_1, x_2}$ with $x_1, x_2 \in A_2$. Next we check which elements $\varphi^{x_1, x_2}$ can be extended to $\text{Hom}(H_1(\mathcal{T}, \mathcal{Z}), A)^{GL_g(\mathbb{Z})}$.

Suppose that for some $g \geq 4$ there exists an element $h_g \in \text{Hom}(H_1(\mathcal{T}, \mathcal{Z}), A)^{GL_g(\mathbb{Z})}$ which restricted on $\mathfrak{G}_2$ coincides with $\varphi^{x_1, x_2}$, for some $x_1, x_2 \in A_2$. By Lemma (2.3.4), $h_g$ has to be zero on $\mathcal{T} \mathcal{B}_{g,1}$, in particular on $KB_{g,1} = K_{g,1} \cap \mathcal{B}_{g,1}$.

Then, if we consider the element $T_\gamma \in K_{g,1}$ with $\gamma$ depicted in the following figure

![Diagram](image)

we have that $T_\gamma \in \mathcal{T} \mathcal{B}_{g,1}$, and by definition of the BCJ-homomorphism, $\sigma(T_\gamma) = A_1 B_1$. As a consequence, if $\varphi^{x_1, x_2}$ can be extended to $\text{Hom}(H_1(\mathcal{T}, \mathcal{Z}), A)^{GL_g(\mathbb{Z})}$, then $\varphi^{x_1, x_2}$ has to be zero on $A_1 B_1$, and so $x_2$ has to be zero. Finally, notice that $\varphi^{x, 0} \in \text{Hom}(\mathfrak{G}_2, A)^{GL_g(\mathbb{Z})}$ is the restriction of $\mu_\gamma^x \in \text{Hom}(H_1(\mathcal{T}, \mathcal{Z}), A)^{GL_g(\mathbb{Z})}$ to $\mathfrak{G}_2$.

\[\square\]

Next we show that the family of homomorphisms $\{\mu_g^x\}_g$ that we have given in Lemma (2.3.5) reassemble into an invariant of homology spheres and we identify such invariant.

**Lemma 2.3.6.** The homomorphisms $\{\mu_g^x\}_g$, defined in Lemma (2.3.5), are compatible with the stabilization map.

**Proof.** By definition, $\sigma : \mathcal{T}_{g,1} \rightarrow \mathfrak{G}_3$ and $\varphi_g^x : \mathfrak{G}_3 \rightarrow A$, are compatible with the stabilization map. As a consequence, the compositions of this maps, $\{\mu_g^x\}_g$, are compatible with the stabilization map. \[\square\]

By Lemmas (2.3.4), (2.3.6), the $\mathcal{A} \mathcal{B}_{g,1}$-invariant homomorphisms $\{\mu_g^x\}_g$, in Lemma (2.3.5), are
compatible with the stabilization map and zero on $\mathcal{T} A_{g,1}, \mathcal{T} B_{g,1}$. Then, by bijection (1.3.3), the family of homomorphism $\{\mu^x_g\}_g$ reassemble into an invariant of homology spheres.

**Lemma 2.3.7.** The homomorphisms $\{\mu^x_g\}_g$, defined in Lemma (2.3.5), take the value $x$ on the Poincaré sphere.

*Proof.* By Example 9.G.2. in [41], we know that the Poincaré sphere is obtained as a 1 Dehn surgery along the right hand trefoil knot. By Lemmas V.1.1, V.1.2, V.1.3 in [41], we have that this Dehn surgery is equivalent to a Heegaard splitting given by the Dehn twist about a right hand trefoil knot.

Next, we follow the construction of the Heegaard splitting given in the proofs of Lemmas V.1.1, V.1.2, V.1.3 in [41]. Take a Seifert surface $S$ of the right hand trefoil knot and thicken this surface obtaining a handlebody $S \times [0,1]$. Consider the boundary of $S \times [0,1]$ and the knot $K = \partial S \times \{1\}$, which is the right hand trefoil knot $K$ in $\partial S \times [0,1]$. Taking this Heegaard surface with the Dehn twist about $K$ we get the Heegaard model

![Heegaard splitting diagram](image)

which corresponds to the 1 Dehn surgery of right hand trefoil knot.

We show that the image of $T_K$ by $\mu^x_g$ is $x$. We first compute the image of $T_K$ by the BCJ-morphism $\sigma$ and then we apply $\phi^x_g$. To compute $\sigma(T_K)$, we write $T_K$ as a product of Dehn twists on unknotted simple closed curves.

Taking the product of left Dehn twists about the simple closed curves $a_1$ and $a_2$, we have that $T_{a_1}^{-1}T_{a_2}^{-1}(K) = K'$ as in the following figure.

![Dehn twist product](image)

Taking the left Dehn twist about the simple closed curve $c$ depicted in the following figure, we have that $T_c^{-1}(K') = \gamma$. 59
Then we have that the product of Dhen twists $T := T_c^{-1}T_{a_1}^{-1}T_{a_2}^{-1}$ takes $K$ to $\gamma$. Thus, $T^{-1}$ takes $\gamma$ to $K$. Hence

$$\sigma(T_K) = \sigma(T^{-1}T_{\gamma}T) = \Psi(T^{-1})\sigma(T_{\gamma}).$$

Consider the subsurface on the left of our curve $\gamma$ (if we take the other subsurface the same argument works), Observe that $A_1, B_1$ is a symplectic basis of the homology of this subsurface. So $\sigma(T_{\gamma}) = \overline{A_1B_1}$.

Next, we compute $\Psi(T^{-1})$. Since $\Psi$ is a homomorphism we have that

$$\Psi(T^{-1}) = \Psi(T_{a_2}T_{a_1}T_{c}) = \Psi(T_{a_2})\Psi(T_{a_1})\Psi(T_{c}).$$

By definition the images of $T_c, T_{a_1}$ and $T_{a_2}$ by $\Psi : M_g \to Sp(\mathbb{Z})$, are given by

$$\Psi(T_{a_1}) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Psi(T_{a_2}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Psi(T_{c}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}.$$ 

Hence

$$\Psi(T^{-1}) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}. $$
Thus, 
\[ \sigma(T_K) = \Psi(T^{-1})\sigma(T_\gamma) = \Psi(T^{-1})(A_1)\Psi(T^{-1})(B_1) \]
\[ = (A_2 + B_1 + B_2)(A_1 + B_1) \]
\[ = (\overline{A_2} + \overline{B_1} + \overline{B_2} + \mathbb{I})(\overline{A_1} + \overline{B_1} + \mathbb{I}) \]
\[ = A_2A_1 + B_1A_1 + B_2A_1 + A_1 + A_2B_1 + B_1B_1 + B_2B_1 + B_1 + A_2 + B_1 + B_2 + \mathbb{I} \]
\[ = A_2A_1 + B_1A_1 + B_2A_1 + A_1 + A_2B_1 + B_1 + B_2B_1 + A_2 + B_2 + \mathbb{I}. \]

Therefore, by definition of \( \varphi^x \), we have that \( \mu_g^x(T_K) = \varphi^x \circ \sigma(T_K) = x \), as desired. \hfill \square

**Proposition 2.3.1.** The invariant \( \mu_g^x \) coincides with the Rohlin invariant \( R_g \) composed with the homomorphism \( \varepsilon^x \).

**Proof.** In [20], D. Johnson proved that the Rohlin invariant induces a family of homomorphisms \( \{ R_g \}_g \) with \( R_g \in \text{Hom}(T_{g,1};\mathbb{Z}/2)^{AB_g,1} \). Since, by Lemma (2.3.5), there is only one non-zero element in \( \text{Hom}(T_{g,1};\mathbb{Z}/2)^{AB_g,1} \), we have that \( \mu_g^1 \) and \( R_g \) must coincide. Therefore, \( \varepsilon^x \circ \mu_g^1 \) and \( \varepsilon^x \circ R_g \) must coincide too. \hfill \square

### 2.4 From trivial cocycles to invariants

Conversely, what are the conditions for a family of trivial 2-cocycles \( C_g \) on \( T_{g,1} \) satisfying properties (1)-(3) to actually provide an invariant?

Firstly we need check the existence of an \( AB_{g,1} \)-invariant trivialization of each \( C_g \). This is a cohomological problem. Denote by \( Q_{C_g} \) the set of all trivializations of the cocycle \( C_g \):

\[ Q_{C_g} = \{ q : T_{g,1} \to A \mid q(\phi) + q(\psi) - q(\phi\psi) = C_g(\phi, \psi) \}. \]

Recall that any two trivializations of a given 2-cocycle differ by an element of \( \text{Hom}(T_{g,1}, A) \). As the cocycle \( C_g \) is invariant under conjugation by \( AB_{g,1} \) this latter group acts on \( Q_{C_g} \) via its conjugation action on the Torelli group. Explicitly, if \( \phi \in AB_{g,1} \) and \( q \in Q_{C_g} \) then \( \phi \cdot q(\eta) = q(\phi^{-1}\eta\phi) \). This action confers the set \( Q_{C_g} \) the structure of an affine set over the abelian group \( \text{Hom}(T_{g,1}, A) \). Choose an arbitrary element \( q \in Q_{C_g} \) and define a map as follows

\[ \rho_q : AB_{g,1} \longrightarrow \text{Hom}(T_{g,1}, A) \]
\[ \phi \mapsto \phi \cdot q - q. \]

A direct computation shows that \( \rho_q \) is a derivation, i.e. \( \rho_q(\phi\psi) = \phi \cdot \rho_q(\psi) + \rho_q(\psi) \), and the
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difference $\rho_q - \rho_{q'}$ for two elements in $Q_{C_g}$ is a principal derivation. Therefore we have a well-defined cohomology class

$$\rho(C_g) \in H^1(AB_{g,1}; Hom(T_{g,1}, A))$$
called the torsor of the cocycle $C_g$.

By construction, if the action of $AB_{g,1}$ on $Q_{C_g}$ has a fixed point, the class $\rho(C_g)$ is trivial. Conversely, if $\rho(C_g)$ is trivial, then for any $q \in Q_{C_g}$ the map $\rho_q$ is a principal derivation, i.e. there exists $m_q \in Hom(T_{g,1}, A)$ such that

$$\forall \phi \in AB_{g,1} \quad \rho_q(\phi) = \phi \cdot m_q - m_q.$$  

In particular the element $q - m_q \in Q_{C_g}$ is fixed under the action of $AB_{g,1}$, since

$$\phi \cdot (q - m_q) = \phi \cdot q - \phi \cdot m_q = (\rho_q(\phi) + q) - (\rho_q(\phi) + m_q) = q - m_q$$

So we have proved:

**Proposition 2.4.1.** The natural action of $AB_{g,1}$ on $Q_{C_g}$ admits a fixed point if and only if the associated torsor $\rho(C_g)$ is trivial.

Suppose that for every $g \geq 4$ there is a fixed point $q_g$ of $Q_{C_g}$ for the action of $AB_{g,1}$ on $Q_{C_g}$. Since every pair of $AB_{g,1}$-invariant trivialization differ by a $AB_{g,1}$-invariant homomorphism, by Lemma [2.3.5], for every $g \geq 3$ the fixed points are $q_g + \mu_g^x$ with $x \in A_2$.

By Lemma [2.3.6], all elements of $Hom(T_{g,1}, A)^{AB_{g,1}}$ are compatible with the stabilization map. Then, given two different fixed points $q_g, q'_g$ of $Q_{C_g}$ for the action of $AB_{g,1}$, we have that

$$q_g|_{T_{g-1,1}} - q'_g|_{T_{g-1,1}} = (q_g - q'_g)|_{T_{g-1,1}} = \mu_g^x|_{T_{g-1,1}} = \mu_g^x.$$  

Therefore the restriction of the trivializations of $Q_{C_g}$ to $T_{g-1,1}$ give us a bijection between the fixed points of $Q_{C_g}$ for the action of $AB_{g,1}$ and the fixed points of $Q_{C_{g-1}}$ for the action of $AB_{g-1,1}$.

Therefore, given an $AB_{g,1}$-invariant trivialization $q_g$, for each $x \in A_2$, we get a well-defined map

$$q + \mu^x = \lim_{g \to \infty} q_g + \mu^x \quad \lim_{g \to \infty} T_{g,1} \to A.$$  

These are the only candidates to be $A$-valued invariants of homology spheres with associated family of 2-cocycles $\{C_g\}_g$. For these maps to be invariants, since they are already $AB_{g,1}$-invariant, we only have to prove that they are constant on the double cosets $TA_{g,1}/T_{g,1}/TB_{g,1}$. From property
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(3) of our cocycle we have that \( \forall \phi \in T_{g,1}, \forall \psi_a \in TA_{g,1} \) and \( \forall \psi_b \in TB_{g,1} \),

\[
(q_g + \mu^x_g)(\phi) - (q_g + \mu^x_g)(\phi \psi_b) = -(q_g + \mu^x_g)(\psi_b),
(q_g + \mu^x_g)(\phi) - (q_g + \mu^x_g)(\psi_a \phi) = -(q_g + \mu^x_g)(\psi_a).
\]

Thus in particular, taking \( \phi = \psi_a, \psi_b \) in above equations, we get that \( q_g + \mu^x_g \) with \( x \in A_2 \), are homomorphisms on \( TA_{g,1}, TB_{g,1} \). Then, by Lemma (2.3.4), we get that \( q_g + \mu^x_g \) is trivial on \( TA_{g,1} \) and \( TB_{g,1} \).

Therefore, by equalities (2.4.1), we obtain that \( q_g + \mu^x_g \) with \( x \in A_2 \) are constant on the double cosets \( TA_{g,1}/Tg_1/TB_{g,1} \).

Summarizing, we get the following result:

**Theorem 2.4.1.** Let \( A \) be an abelian group and \( A_2 \) the subgroup of 2-torsion elements. For each \( x \in A_2 \), a family of cocycles \( \{C_g\}_{g \geq 3} \) on the Torelli groups \( T_{g,1}, g \geq 3 \), satisfying conditions (1)-(3) provides a compatible family of trivializations \( F_g + \mu^x_g : T_{g,1} \to A \) that reassemble into an invariant of homology spheres

\[
\lim_{g \to \infty} F_g + \mu^x_g : S^3 \to A
\]

if and only if the following two conditions hold:

(i) The associated cohomology classes \([C_g] \in H^2(T_{g,1}; A)\) are trivial.

(ii) The associated torsors \( \rho(C_g) \in H^1(AB_{g,1}, \text{Hom}(T_{g,1}, A)) \) are trivial.

Notice that Proposition (2.3.1) tell us that the homomorphisms \( \mu^x_g \) factor through the Rohlin invariant \( R_g \). Therefore the existence of the Rohlin invariant makes to fail the unicity in the construction of invariants from a family of 2-cocycles in Theorem (2.4.1).
Chapter 2. Trivial cocycles and invariants on the Torelli group
CHAPTER 3

The mod \( d \) Torelli group and homology 3-spheres

In this chapter, we study the relation between the mod \( d \) Torelli group, the \( \mathbb{Z}/d \)-homology 3-spheres and the \( \mathbb{Q} \)-homology 3-spheres. To be more precise, our aim is to show that given an integer \( d \geq 2 \), a Heegaard splitting with gluing map an element of the (mod \( d \)) Torelli group is a \( \mathbb{Z}/d \)-homology 3-sphere, and so a \( \mathbb{Q} \)-homology 3-sphere, and that every \( \mathbb{Q} \)-homology 3-sphere can be obtained as a Heegaard splitting with a gluing map an element of the (mod \( p \)) Torelli group. This last result is a direct consequence of Proposition 6 in [36], in which B. Perron stated that given a \( \mathbb{Q} \)-homology 3-sphere \( M^3 \) with \( n = |H_1(M^3; \mathbb{Z})| \), if \( d \mid n-1 \), then \( M^3 \) can be obtained as a Heegaard splitting with gluing map an element of the (mod \( d \)) Torelli group. Unfortunately, the proof of such result is not available in the literature. In this chapter, we prove a generalization of this result in which we consider \( d \mid n \pm 1 \) instead of \( d \mid n - 1 \). We point out that, as we will see, the divisibility condition is indispensable for such result.

As a starting point, in the first two sections, we give some basic definitions and properties about the symplectic representation modulo \( d \) as well as the (mod \( d \)) Torelli group. In the third section, we study the action of the subgroups \( A_{g,1}, B_{g,1}, AB_{g,1} \) on \( H_d := H_1(\Sigma_{g,1}; \mathbb{Z}/d) \). The main difference between the action on \( H_d \) and the action on \( H := H_1(\Sigma_{g,1}; \mathbb{Z}) \), is the fact that, if \( \Psi_d \) denotes the symplectic representation modulo \( d \), then \( \Psi_d(B_{g,1}) \) does not correspond to the subgroup of \( Sp_{2g}(\mathbb{Z}/d) \) formed by matrices of the form \( \left( \begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix} \right) \), which is isomorphic to \( GL_g(\mathbb{Z}/d) \rtimes S_g(\mathbb{Z}/d) \). But for \( d \) a prime number \( \Psi_d(B_{g,1}) \) is isomorphic to \( SL_2(\mathbb{Z}/d) \rtimes S_g(\mathbb{Z}/d) \). For \( d \) not prime, such isomorphism does not hold. However, there is an stable version that give us an isomorphism \( \Psi_d(B_{\infty,1}) \cong SL_2^*(\mathbb{Z}/d) \rtimes S(\mathbb{Z}/d) \).

Finally, in the last section, we give a criterion to know whenever a \( \mathbb{Q} \)-homology 3-sphere is homeomorphic to a Heegaard splitting with gluing map an element of (mod \( d \)) Torelli group. As an application of this criterion, if we denote by \( \mathcal{S}^3[d] \) the set of \( \mathbb{Q} \)-homology 3-spheres which are
homeomorphic to $\mathcal{H}_g \cup_{\psi} \mathcal{H}_g$ for some $\phi \in \mathcal{M}_{g,1}[d]$, unlike the case of the Torelli group and homology 3-spheres, we show that, in general, the set of $\mathbb{Z}/d$-homology 3-spheres do not coincide with $S^3[d]$. In addition, analogously to the case of the integral homology 3-spheres and the Torelli group, we show that there is the following bijection: $\lim_{g \to \infty} (\mathcal{A}_{g,1}[d]\mathcal{M}_{g,1}[d]/\mathcal{B}_{g,1}[d])_{\mathcal{A}B_{g,1}} \cong S^3[d].$

### 3.1 The Symplectic representation modulo $d$

**Definition 3.1.1** (Section 1 in [34]). Let $d \geq 2$ be an integer and $M \in \mathbb{M}_n(\mathbb{Z})$ be a matrix satisfying $M \Omega M^t \equiv \Omega \pmod{d}$, then $M$ will be said to be symplectic modulo $d$. We define the Symplectic group modulo $d$ as

$$Sp_{2g}(\mathbb{Z}/d) = \{M \in M_{2g \times 2g}(\mathbb{Z}/d) \mid M \Omega M^t = \Omega\}.$$ 

As in the case of the symplectic matrices, from the definitions, it is easy to verify that a matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is symplectic modulo $n$ if and only if

$$AD^t - BC^t = Id_g, \quad AB^t = BA^t, \quad CD^t = DC^t.$$

We define the Symplectic representation modulo $d$ as the composition

$$\Psi_d: \mathcal{M}_{g,1} \xrightarrow{\Psi} Sp_{2g}(\mathbb{Z}) \xrightarrow{r_d} Sp_{2g}(\mathbb{Z}/d).$$

Notice that Symplectic representation modulo $d$ is surjective because the Symplectic representation $\Psi$ is surjective and, by Theorem 1 in [34], $r_d$ is also surjective.

### 3.2 The mod $d$ Torelli group

The (mod $d$) Torelli group $\mathcal{M}_{g,1}[d]$ is the normal subgroup of the mapping class group $\mathcal{M}_{g,1}$ of those elements of $\mathcal{M}_{g,1}$ that act trivially on $H_1(\Sigma_{g,1}; \mathbb{Z}/d)$. In other words, $\mathcal{M}_{g,1}[d]$ is characterized by the following short exact sequence:

$$1 \longrightarrow \mathcal{M}_{g,1}[d] \longrightarrow \mathcal{M}_{g,1} \xrightarrow{\Psi_d} Sp_{2g}(\mathbb{Z}/d) \longrightarrow 1$$

Let $D_{g,1}[d]$ be the subgroup of $\mathcal{M}_{g,1}$ generated by the $d^{th}$-powers of Dehn twists. The following proposition and corollary announced in [36] by B. Perron and proved by J. Cooper in [4] enlightens
the structure of $\mathcal{M}_{g,1}[d]$.

**Proposition 3.2.1.** The mod $d$ Torelli group $\mathcal{M}_{g,1}[d]$ is the normal subgroup of $\mathcal{M}_{g,1}$ generated by the Torelli group $\mathcal{T}_{g,1}$ and the $d^{th}$-powers of Dehn twists.

Since a conjugate of a Dehn twist is a Dehn twist, one gets:

**Corollary 3.2.1.** Every element $\varphi \in \mathcal{M}_{g,1}[d]$ can be written as: $\varphi = f \circ m$ with $f \in \mathcal{T}_{g,1}$ and $m \in D_{g,1}[d]$.

**Remark 3.2.1.** In [4], J. Cooper give a proof of Proposition (3.2.1) and Corollary (3.2.1) for the case of $d$ prime. Nevertheless, following the same proof one gets the same result for the case of any integer $d$.

### 3.3 Homology actions modulo $d$

If one writes the matrices of the symplectic group modulo $d$ $Sp_{2g}(\mathbb{Z}/d)$ as blocks according to the decomposition $H_d = A_d \oplus B_d$, then the image of $B_{g,1} \to Sp_{2g}(\mathbb{Z}/d)$ is contained in the subgroup $Sp^B_{2g}(\mathbb{Z}/d)$ of matrices of the form: $$\begin{pmatrix} G_1 & 0 \\ M & G_2 \end{pmatrix}.$$ 

Such matrices are symplectic modulo $d$ if and only if $G_2 = ^tG_1^{-1}$ and $^tG_1^{-1}M$ is symmetric. As a consequence, we have an isomorphism:

$$\phi^B_d : Sp^B_{2g}(\mathbb{Z}/d) \to GL_g(\mathbb{Z}/d) \times_B S_g(\mathbb{Z}/d),$$

$$\begin{pmatrix} G & 0 \\ M & ^tG^{-1} \end{pmatrix} \mapsto (G, ^tGM).$$

Here $S_g(\mathbb{Z}/d)$ denotes the symmetric group of $g \times g$ matrices over the $\mathbb{Z}/d$. The composition on the semi-direct product is given by the rule $(G, S)(H, T) = (GH, ^tHSH + T)$.

Analogously, the image of $A_{g,1} \to Sp_{2g}(\mathbb{Z}/d)$ is contained in the subgroup $Sp^A_{2g}(\mathbb{Z}/d)$ of matrices of the form: $$\begin{pmatrix} H_1 & N \\ 0 & H_2 \end{pmatrix}.$$ 

Such matrices are symplectic modulo $d$ if and only if $H_2 = ^tH_1^{-1}$ and $^tH_2N$ is symmetric. As a consequence, we have an isomorphism:

$$\phi^A_d : Sp^A_{2g}(\mathbb{Z}/d) \to GL_g(\mathbb{Z}/d) \times_A S_g(\mathbb{Z}/d),$$

$$\begin{pmatrix} H & N \\ 0 & ^tH^{-1} \end{pmatrix} \mapsto (G, ^tGM).$$
where the composition on the semi-direct product is given by the rule

\[(G, S)(H, T) = (GH, tH^{-1}SH^{-1} + T).\]

Similarly, the image of \(AB_{g,1} \to Sp_{2g}(\mathbb{Z}/d)\) is contained in the subgroup \(Sp_{2g}^{AB}(\mathbb{Z}/d)\) of matrices of the form:

\[
\begin{pmatrix}
G_1 & 0 \\
0 & G_2
\end{pmatrix}.
\]

Such matrices are symplectic modulo \(d\) if and only if \(G_2 = tG_1^{-1}\). As a consequence, we have an isomorphism:

\[
\phi_d^{AB} : Sp_{2g}^{AB}(\mathbb{Z}/d) \to GL_g(\mathbb{Z}/d),
\]

\[
\begin{pmatrix}
G & 0 \\
0 & tG^{-1}
\end{pmatrix} \mapsto G.
\]

**Notation 3.3.1.** Denote by \(r'_d : GL_g(\mathbb{Z}) \to GL_g(\mathbb{Z}/d)\), \(r''_d : S_g(\mathbb{Z}) \to S_g(\mathbb{Z}/d)\) the respective reductions modulo \(d\).

**Definition 3.3.1.** Let \(d\) a positive integer, we define the following groups:

\[
SL_g^+(\mathbb{Z}/d) = \{ A \in M_{g \times g}(\mathbb{Z}/d) \mid \det(A) = \pm 1 \},
\]

\[
SL_g(\mathbb{Z},d) = \text{Ker}(r'_d : SL_g(\mathbb{Z}) \to SL_g(\mathbb{Z}/d)),
\]

\[
S_g(d\mathbb{Z}) = \text{Ker}(r''_d : S_g(\mathbb{Z}) \to S_g(\mathbb{Z}/d)).
\]

**Notation 3.3.2.** Let \(R\) be a division ring, \(i, j\) distinct integers between 1 and \(g\) and \(u \in R\). Denote by \(e_{ij}(u) \in SL_g(R)\) the matrix with entry \(u\) in the \((i, j)\) position, ones on the diagonal and zeros elsewhere. Such matrices are called elementary matrices of rank \(g\). We denote by \(E_g(R)\) the subgroup of \(SL_g(R)\) generated by elementary matrices of rank \(g\). We denote by \(D_g\) the diagonal matrix of rank \(g\), with a \(-1\) at position \((1,1)\) and \(1\)'s at positions \((i,i)\) with \(2 \leq i \leq g\).

**Theorem 3.3.1** (Theorem in [10]). Let \(R\) be a division ring. The group \(SL_g(R)\) coincides with \(E_g(R)\).

**Lemma 3.3.1.** Let \(p\) be a prime number, there is a short exact sequence of groups

\[
1 \longrightarrow SL_g(\mathbb{Z},p) \longrightarrow GL_g(\mathbb{Z}) \longrightarrow r'_p : SL_g^+(\mathbb{Z}/p) \longrightarrow 1.
\]

**Proof.** First of all, we show that \(r'_p : GL_g(\mathbb{Z}) \to SL_g^+(\mathbb{Z}/p)\) is surjective. By Theorem (3.3.1) \(SL_g(\mathbb{Z}/p) = E_g(\mathbb{Z}/p)\). Observe that if \(A \in SL_g^+(\mathbb{Z}/p)\) with \(\det(A) = -1\) then \(\det(D_gA) = 1\), i.e.
Let \( D_gA \in SL_g(\mathbb{Z}/p) \). Therefore \( SL_g^*(\mathbb{Z}/p) \) is generated by \( \{ D_g, e_{ij}(u) \mid u \in \mathbb{Z}/p \} \). Since \( r'_p(D_g) = D_g \) and \( r'_p(E_g(\mathbb{Z})) = E_g(\mathbb{Z}/p) \), we have that \( r'_p \) is surjective.

Next, notice that we have a pull-back diagram

\[
\begin{array}{ccc}
1 & \longrightarrow & \text{Ker}(r'_p) \\
\downarrow & & \downarrow \\
SL_g(\mathbb{Z}) & \longrightarrow & SL_g(\mathbb{Z}/p) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \text{Ker}(r''_p) \\
\downarrow & & \downarrow \\
GL_g(\mathbb{Z}) & \longrightarrow & SL_g^*(\mathbb{Z}/p) \\
\downarrow & & \downarrow \\
1 & & 1.
\end{array}
\]

As a consequence \( \text{Ker}(r'_p) \cong SL_g(\mathbb{Z}, p) \), as desired. \( \square \)

**Lemma 3.3.2** (Lemma 1 in [34]). Let \( d \) be an integer, there is a short exact sequence of groups

\[
1 \longrightarrow S_g(d\mathbb{Z}) \longrightarrow S_g(\mathbb{Z}) \longrightarrow S_g(\mathbb{Z}/d) \longrightarrow 1.
\]

**Lemma 3.3.3.** Let \( p \) be a prime number, there is a short exact sequence of groups

\[
1 \longrightarrow SL_g(\mathbb{Z}, p) \times S_g(p\mathbb{Z}) \longrightarrow GL_g(\mathbb{Z}) \times S_g(\mathbb{Z}) \longrightarrow SL_g^*(\mathbb{Z}/p) \times S_g(\mathbb{Z}/p) \longrightarrow 1.
\]

**Proof.** First of all we show that \( r'_d \times r''_d \) is a homomorphism.

Let \( G, H \in GL_g(\mathbb{Z}) \), \( S, T \in S_g(\mathbb{Z}) \), then we have that the following equality holds

\[
(r'_p \times r''_p)((G, S)(H, T)) = (r'_p \times r''_p)((GH, HSH^t + T)) \\
= (r'_p(GH), r''_p(HSH^t + T)) \\
= (GH, HSH^t + T) \\
= (G, S)(H, T) \\
= (r'_p \times r''_p)(G, S)(r'_p \times r''_p)(H, T).
\]

Notice that, by Lemmas (3.3.1), (3.3.2), \( \text{Ker}(r'_p) = SL_g(\mathbb{Z}, p) \), \( \text{Ker}(r''_p) = S_g(p\mathbb{Z}) \). Then

\[
\text{Ker}(r'_p \times r''_p) = \text{Ker}(r'_p) \times \text{Ker}(r''_p) = SL_g(\mathbb{Z}, p) \times S_g(\mathbb{Z}).
\]

Finally, we have that \( r'_p \times r''_p \) is surjective because \( r'_p, r''_p \) are surjective too. \( \square \)

**Definition 3.3.2.** We define the following subgroups of \( M_{g,1}[d] \):

\[
A_{g,1}[d] = M_{g,1}[d] \cap A_{g,1}, \quad B_{g,1}[d] = M_{g,1}[d] \cap B_{g,1}, \quad AB_{g,1}[d] = M_{g,1}[d] \cap AB_{g,1}.
\]
Lemma 3.3.4. Let \( p \) be a prime number, there are short exact sequence of groups

\[
\begin{align*}
1 \longrightarrow B_{g,1}[p] & \longrightarrow B_{g,1}[\phi^R_p \circ \Psi_p] \longrightarrow S L_g^+(\mathbb{Z}/p) \ltimes B S_g(\mathbb{Z}/p) \longrightarrow 1, \\
1 \longrightarrow A_{g,1}[p] & \longrightarrow A_{g,1}[\phi^A_p \circ \Psi_p] \longrightarrow S L_g^+(\mathbb{Z}/p) \ltimes A S_g(\mathbb{Z}/p) \longrightarrow 1, \\
1 \longrightarrow A B_{g,1}[p] & \longrightarrow A B_{g,1}[\phi^A B_p \circ \Psi_p] \longrightarrow S L_g^+(\mathbb{Z}/p) \longrightarrow 1.
\end{align*}
\]

Proof. We only give the proof for the first short exact sequence (the proof for the other two short exact sequences is analogous). Observe that by definition of \( B_{g,1}[p] \), we have that

\[\text{Ker}(\phi^R_p \circ \Psi_p : B_{g,1} \rightarrow S L_g^+(\mathbb{Z}/p) \ltimes S g(\mathbb{Z}/p)) = B_{g,1} \cap M_{g,1}[p] = B_{g,1}[p].\]

On the other hand, by Lemmas (1.3.2), (3.3.3) we have that

\[(\phi^R_p \circ \Psi_p)(B_{g,1}) = (\phi^R_p \circ r_p \circ \Psi)(B_{g,1}) = ((r_p' \times r_p'') \circ \phi^R \circ \Psi)(B_{g,1}) = (r_p' \times r_p')(G L_g(\mathbb{Z}) \ltimes S g(\mathbb{Z})) = S L_g^+(\mathbb{Z}/p) \ltimes S g(\mathbb{Z}/p),\]

as desired.

Notation 3.3.3. Throughout this thesis we set:

\[
\begin{align*}
S p^{A^+}_{2g}(\mathbb{Z}/d) &= (\phi^A)^{-1}(S L_g^+(\mathbb{Z}/d) \ltimes A S g(\mathbb{Z}/d)), \\
S p^{B^+}_{2g}(\mathbb{Z}/d) &= (\phi^B)^{-1}(S L_g^+(\mathbb{Z}/d) \ltimes B S g(\mathbb{Z}/d)).
\end{align*}
\]

We point out that Lemmas (3.3.1), (3.3.3), (3.3.4) are not true switching the prime \( p \) for a no prime integer \( d \). This comes from the fact that \( S L_g(\mathbb{Z}) \rightarrow S L_g(\mathbb{Z}/d) \) is surjective if and only if \( d \) is prime, which is based on the fact that \( E g(\mathbb{Z}/d) = S L_g(\mathbb{Z}/d) \) if and only if \( d \) is prime. Nevertheless, as we will show next, taking the direct limit of above families of groups one gets the analogous results for the stable case with an integer \( d \).

Direct systems and direct limits Let \( \langle I, \leq \rangle \) be a directed set. Let \( \{ A_i \mid i \in I \} \) be a family of objects indexed by \( I \) and \( f_{ij} : A_i \rightarrow A_j \) be a homomorphism for all \( i \leq j \) with the following properties:

i) \( f_{ii} \) is the identity of \( A_i \),

ii) \( f_{ik} = f_{jk} \circ f_{ij} \) for all \( i \leq j \leq k \).

Then the pair \( \langle A_i, f_{ij} \rangle \) is called a direct system over \( I \).
The direct limit of the direct system \((A_i, f_{ij})\) is denoted by \(\lim \to A_i\) and is defined as follows. Its underlying set is the disjoint union of the \(A_i\)'s modulo a certain equivalence relation \(\sim\):

\[
\lim \to A_i = \bigsqcup_i A_i / \sim.
\]

Here, if \(x_i \in A_i\) and \(x_j \in A_j\), then \(x_i \sim x_j\) if and only if there is some \(k \in I\) with \(i \leq k\) and \(j \leq k\) and such that \(f_{ik}(x_i) = f_{jk}(x_j)\).

**The stable General linear group and symmetric group.** Let \(R\) be a commutative ring with unit. Consider the General linear groups \(\{GL_n(R)\}_n\) and the homomorphism:

\[
i_{n,m} : GL_n(R) \to GL_m(R),
\]

which sends \(A \in GL_n(R)\) to

\[
\begin{pmatrix}
A & 0 \\
0 & Id_{m-n}
\end{pmatrix}.
\]

Then \(\{GL_n(R), i_{n,m}\}\) is a direct system.

Similarly, we can consider the symmetric groups \(\{S_n(R)\}_n\) and the homomorphism:

\[
j_{n,m} : S_n(R) \to S_m(R),
\]

which sends \(A \in S_n(R)\) to

\[
\begin{pmatrix}
A & 0 \\
0 & 0
\end{pmatrix}.
\]

Then \(\{S_n(R), j_{n,m}\}\) is a direct system.

Therefore, taking the direct limit of the above direct systems, we obtain the following groups:

\[
GL(R) = \lim \to GL_g(R), \quad SL(R) = \lim \to SL_g(R), \quad E(R) = \lim \to E_g(R),
\]

\[
SL^x(R) = \lim \to SL_g^x(R), \quad S(R) = \lim \to S_g(R).
\]

Throughout this section we denote by \(R^x\) the group of units of \(R\), i.e. the set of invertible elements of \(R\), and we set:

Observe that we have the following short exact sequence of groups:

\[
1 \to SL(R) \to GL(R) \to R^x \to 1.
\]
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Taking the quotient by $E(R)$, the above short exact sequence becomes

$$1 \rightarrow SK_1(R) \rightarrow K_1(R) \rightarrow R^* \rightarrow 1,$$

where

$$SK_1(R) = \frac{SL(R)}{E(R)} \quad \text{and} \quad K_1(R) = \frac{GL(R)}{E(R)}.$$  

These groups are well known in $K$-theory (see for instance [49], [28]).

**Definition 3.3.3** (Sections 19, 20 in [22]). A commutative ring $R$ is a local ring if it has a unique maximal ideal. A commutative ring $R$ is a semilocal ring if it only has a finite number of maximal ideals. In particular every commutative local ring is a commutative semilocal ring.

**Proposition 3.3.1** (Lemma III.1.4 in [49]). Let $R$ be a commutative semilocal ring, then

$$K_1(R) \cong R^* \quad \text{and} \quad SK_1(R) = 1.$$  

Notice that, for every $d \geq 2$, $\mathbb{Z}/d$ is a semilocal ring because $\mathbb{Z}/d$ is finite. Therefore, by Proposition (3.3.1), we get the following result:

**Corollary 3.3.1.** Let $d \geq 2$ be an integer, $SK_1(\mathbb{Z}/d) = 1$, i.e. $SL(\mathbb{Z}/d) = E(\mathbb{Z}/d)$.

**Remark 3.3.1** (Remark III.1.2.3 in [49]). If $I$ is an ideal of $R$, each homomorphism $E_n(R) \rightarrow E_n(R/I)$ is onto, because the generators $e_{ij}(r)$ of $E_n(R)$ map onto the generators $e_{ij}(\bar{r})$ of $E_n(R/I)$.

**Lemma 3.3.5.** Let $d \geq 2$ be an integer, we have a short exact sequence of groups

$$1 \rightarrow SL(\mathbb{Z}, d) \rightarrow GL(\mathbb{Z}) \rightarrow SL^*(\mathbb{Z}/d) \rightarrow 1,$$

where $r'_d$ is given by modulo $d$ reduction.

**Proof.** Fix an integer $g$. Consider $A \in SL^*(\mathbb{Z}/d)$. Then $D_g A \in SL_g(\mathbb{Z}/d)$. By Corollary (3.3.1) we have that there exists an integer $h$ and elementary matrices $e_k \in E_h(\mathbb{Z}/d)$ such that the following equality holds.

$$D_g A = i_{h,g} \left( \prod_{k=1}^{r} e_k \right).$$

Since $i_{h,g}(D_h) = D_g$ and $D_h = D_h^{-1}$, we have that

$$A = i_{h,g} \left( D_h \prod_{k=1}^{r} e_k \right).$$
Notice that $r'_d(D_h) = D_h$. Moreover, by Remark (3.3.1), we have that $r'_d(E_h(\mathbb{Z})) = E_h(\mathbb{Z}/d)$. Therefore $r'_p$ is surjective.

**Corollary 3.3.2.** Let $d \geq 2$ be an integer, there is a short exact sequence of groups

\[ 1 \longrightarrow S(d\mathbb{Z}) \longrightarrow S(\mathbb{Z}) \xrightarrow{r''_d} S(\mathbb{Z}/d) \longrightarrow 1, \]

where $r''_d$ is given by the reduction modulo $d$.

**Lemma 3.3.6.** Let $d \geq 2$ an integer, there is a short exact sequence of groups:

\[ 1 \longrightarrow SL(\mathbb{Z}, d) \times S(d\mathbb{Z}) \longrightarrow GL(\mathbb{Z}) \times S(\mathbb{Z}) \xrightarrow{r'_d \times r''_d} SL^*(\mathbb{Z}/d) \times S(\mathbb{Z}/d) \longrightarrow 1, \]

where $r'_d : GL(\mathbb{Z}) \to SL^*(\mathbb{Z}/d)$, $r''_d : S(\mathbb{Z}) \to S(\mathbb{Z}/d)$ are the reduction modulo $d$.

**Proof.** This result is a direct consequence of Lemma (3.3.5) and Corollary (3.3.2). □

**The stable Mapping class group.** Consider the Mapping class groups $\{M_{g,1}\}_g$ and the stabilization map $i_{g,h} : M_{g,1} \to M_{h,1}$. Then we have a direct system $\langle M_{g,1}, i_{g,h} \rangle$. Define

\[ M_{\infty,1} = \lim M_{g,1}. \]

Since $A_{g,1}$, $B_{g,1}$, $AB_{g,1}$ are subgroups of $M_{g,1}$, then we have the following natural subgroups of $M_{\infty,1}$:

\[ A_{\infty,1} = \lim A_{g,1}, \quad B_{\infty,1} = \lim B_{g,1}, \quad AB_{\infty,1} = \lim AB_{g,1}, \]

\[ A_{\infty,1}[d] = \lim A_{g,1}[d], \quad B_{\infty,1}[d] = \lim B_{g,1}[d], \quad AB_{\infty,1}[d] = \lim AB_{g,1}[d]. \]

Moreover, if we consider the direct limit of the short exact sequences in Lemmas (1.3.2) and (1.3.3) we get the following result.

**Lemma 3.3.7.** There are short exact sequences of groups

\[ 1 \longrightarrow \mathcal{T}B_{\infty,1} \longrightarrow B_{\infty,1} \xrightarrow{\Psi} GL(\mathbb{Z}) \times S(\mathbb{Z}) \longrightarrow 1, \]

\[ 1 \longrightarrow \mathcal{T}A_{\infty,1} \longrightarrow A_{\infty,1} \xrightarrow{\Psi} GL(\mathbb{Z}) \times S(\mathbb{Z}) \longrightarrow 1, \]

\[ 1 \longrightarrow \mathcal{T}AB_{\infty,1} \longrightarrow AB_{\infty,1} \xrightarrow{\Psi} GL(\mathbb{Z}) \longrightarrow 1. \]
Lemma 3.3.8. Let $d \geq 2$ an integer, there are short exact sequences of groups

$$
1 \longrightarrow B_{\infty,1}[d] \longrightarrow B_{\infty,1} \xrightarrow{\Psi_d} SL^+(\mathbb{Z}/d) \rtimes S(\mathbb{Z}/d) \longrightarrow 1,
$$

$$
1 \longrightarrow A_{\infty,1}[d] \longrightarrow A_{\infty,1} \xrightarrow{\Psi_d} SL^+(\mathbb{Z}/d) \rtimes S(\mathbb{Z}/d) \longrightarrow 1,
$$

$$
1 \longrightarrow AB_{\infty,1}[d] \longrightarrow AB_{\infty,1} \xrightarrow{\Psi_d} SL^+(\mathbb{Z}/d) \longrightarrow 1.
$$

Proof. We only give the proof for the first short exact sequence (the proof for the other two short exact sequences is analogous). By definition of $B_{g,1}[d]$, we have an exact sequence

$$
1 \longrightarrow B_{g,1}[d] \longrightarrow B_{g,1} \xrightarrow{\Psi_d} SL_g^+/(\mathbb{Z}/d) \rtimes S_g(\mathbb{Z}/d).
$$

Then taking the direct limit, the above exact sequences becomes

$$
1 \longrightarrow B_{\infty,1}[d] \longrightarrow B_{\infty,1} \xrightarrow{\Psi_d} SL^+(\mathbb{Z}/d) \rtimes S(\mathbb{Z}/d),
$$

and by Lemmas (3.3.7), (3.3.6) we have that $\Psi_d : B_{\infty,1} \longrightarrow SL^+(\mathbb{Z}/d) \rtimes S(\mathbb{Z}/d)$ is surjective, as desired.

3.4 Heegaard splittings of homology 3-spheres

In this section we show what are the relation between (mod $d$)-Torelli groups, $\mathbb{Q}$-homology 3-spheres and $\mathbb{Z}/d$-homology 3-spheres.

Definition 3.4.1. Let $R = \mathbb{Q}$ or $\mathbb{Z}/d$. A 3-manifold $X$ is a $R$-homology 3-sphere if

$$H_*(X; R) \cong H_*(S^3; R).$$

Remark 3.4.1. By Universal coefficients Theorem and the Poincaré duality, an orientable compact connected 3-manifold $M$ is a $\mathbb{Q}$-homology 3-sphere if and only if $H_1(M; \mathbb{Z})$ is finite. As consequence, every $\mathbb{Z}/d$-homology 3-sphere $M$ is also a $\mathbb{Q}$-homology 3-sphere, because $H_1(M; \mathbb{Z}/d) = 0$ implies that $H_1(M; \mathbb{Z})$ has no free part, i.e. $H_1(M; \mathbb{Z})$ is finite.

Example 3.4.1 (Section 9.B in [1]). Examples of $\mathbb{Q}$-homology 3-sphere are the lens spaces, which are defined as follows. Consider two solid tori $V_1$ and $V_2$. Fix longitude and meridian generators $a_1$ and $b_1$ for $\pi_1(\partial V_1)$, for instance
Consider an element $h \in \text{Homeo}^+(\partial V_1)$ such that $h_*(b_2) = pa_1 + qb_1$, where $p$ and $q$ are coprime integers.

The 3-manifold obtained gluing the boundaries of $V_1$ and $V_2$ by the homeomorphism $h$ is called the lens space of type $(p, q)$ and denoted traditionally by $L(p, q)$. In other words, a 3-manifold is a lens space if and only if it contains a solid torus, the closure of whose complement is also a solid torus. Moreover we have the following classification result:

**Theorem 3.4.1** (Remark 9.B.7 in [41]). Two lens spaces $L(p_1, q_1)$, $L(p_2, q_2)$ are homeomorphic if and only if $p_1 = \pm p_2$ and $q_1 \equiv \pm q_2^\pm 1 \pmod{p_1}$.

Next we show that a Heegaard splitting with gluing map an element of $(\mod d)$ Torelli group is $\mathbb{Z}/d$-homology 3-sphere and so a $\mathbb{Q}$-homology 3-sphere.

**Proposition 3.4.1.** Let $S^3 = \mathcal{H}_g \cup_{\iota_g} -\mathcal{H}_g$. If we twist this glueing by an arbitrary map $\phi \in \mathcal{M}_g, 1[d]$ we get a $\mathbb{Z}/d$-homology sphere $S^3_\phi = \mathcal{H}_g \cup_{\iota_g \phi} -\mathcal{H}_g$.

**Proof.** In order to prove this statement we compute the homology groups of $\mathcal{H}_g \cup_{\iota_g \phi} -\mathcal{H}_g$ using a Mayer-Vietoris sequence.

Consider the following Mayer-Vietoris sequence:

$$
H_3(\mathcal{H}_g; \mathbb{Z}/d) \oplus H_3(-\mathcal{H}_g; \mathbb{Z}/d) \longrightarrow H_3(M; \mathbb{Z}/d) \longrightarrow H_2(\Sigma_g; \mathbb{Z}/d) \longrightarrow H_2(\mathcal{H}_g; \mathbb{Z}/d) \oplus H_2(-\mathcal{H}_g; \mathbb{Z}/d).
$$

(3.4.1)

Observe that a handlebody $\mathcal{H}_g$ is homotopy equivalent to a wedge of circles $\vee_g S^1$, so

$$
H_3(\mathcal{H}_g; \mathbb{Z}/d) = H_3(-\mathcal{H}_g; \mathbb{Z}/d) = 0 \quad \text{and} \quad H_2(\mathcal{H}_g; \mathbb{Z}/d) = H_2(-\mathcal{H}_g; \mathbb{Z}/d) = 0.
$$

Thus the exact sequence (3.4.1) becomes

$$
\cdots \longrightarrow 0 \longrightarrow H_3(M; \mathbb{Z}/d) \longrightarrow H_2(\Sigma_g; \mathbb{Z}/d) \longrightarrow 0 \longrightarrow \cdots.
$$

Because $\Sigma_g$ is orientable $H_2(\Sigma_g; \mathbb{Z}/d) \cong \mathbb{Z}/d$. Hence $H_3(M; \mathbb{Z}/d) \cong H_2(\Sigma_g; \mathbb{Z}/d) = \mathbb{Z}/d$. 75
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Writing more terms of the Mayer-Vietoris sequence we get

$$H_2(H_g;\mathbb{Z}/d) \oplus H_2(-H_g;\mathbb{Z}/d) \longrightarrow H_2(M;\mathbb{Z}/d) \longrightarrow H_1(\Sigma g;\mathbb{Z}/d) \longrightarrow H_1(H_g;\mathbb{Z}/d) \oplus H_1(-H_g;\mathbb{Z}/d).$$

(3.4.2)

Since $H_2(H_g;\mathbb{Z}/d) = H_2(-H_g;\mathbb{Z}/d) = 0$, the exact sequence [3.4.2] becomes

$$0 \longrightarrow H_2(M;\mathbb{Z}/d) \longrightarrow H_1(\Sigma g;\mathbb{Z}/d) \longrightarrow H_1(H_g;\mathbb{Z}/d) \oplus H_1(-H_g;\mathbb{Z}/d).$$

Thus $H_2(M;\mathbb{Z}/d) = \ker(\varphi: H_1(\Sigma g;\mathbb{Z}/d) \longrightarrow H_1(H_g;\mathbb{Z}/d) \oplus H_1(-H_g;\mathbb{Z}/d))$.

Recall that the basis of $H_1(\Sigma g)$ is given by the homology classes $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$, the basis of $H_1(H_g)$ is given by the homology classes $\{a_1, \ldots, a_g\}$ and the basis of $H_1(-H_g)$ is given by the homology classes $\{-a_1, \ldots, -a_g\}$.

The map $\varphi$ is given by $\varphi = (incl. \circ \iota_g \circ \phi)$, where $\iota_g$ denotes the map that switch $a_i$ and $-b_i$ for all $i$, and $incl.$ denotes the map induced by the natural inclusion of $\Sigma g$ in $H_g$, i.e. the map that sends all $b_i$’s to zero. Recall that $\phi \in M_{g,1}[d]$, i.e. $\Psi_d(\phi) = Id$. So the induced map $\varphi$ in homology with coefficients $\mathbb{Z}/d$ has the following matrix form:

$$\varphi = \begin{pmatrix} Id_g & 0 \\ 0 & Id_g \end{pmatrix} \in M_{2g \times 2g}(\mathbb{Z}/d).$$

Thus $\ker(\varphi) = 0$, i.e. $H_2(M;\mathbb{Z}/d) = 0$.

Now, using the Mayer-Vietoris for reduced homology, since $\varphi = Id_{2g}$ i.e. $\varphi$ is an isomorphism, we get the following exact sequence:

$$0 \longrightarrow H_1(M;\mathbb{Z}/d) \longrightarrow \tilde{H}_0(\Sigma g;\mathbb{Z}/d) \longrightarrow \tilde{H}_0(H_g;\mathbb{Z}/d) \oplus \tilde{H}_0(-H_g;\mathbb{Z}/d) \longrightarrow \tilde{H}_0(M;\mathbb{Z}/d) \longrightarrow 0.$$

Since $H_0(\Sigma g;\mathbb{Z}/d) = H_0(H_g;\mathbb{Z}/d) = H_0(-H_g;\mathbb{Z}/d) = \mathbb{Z}/d$, then

$$\tilde{H}_0(\Sigma g;\mathbb{Z}/d) = \tilde{H}_0(H_g;\mathbb{Z}/d) = \tilde{H}_0(-H_g;\mathbb{Z}/d) = 0.$$

Thus, by above exact sequence,

$$H_1(M;\mathbb{Z}/d) = 0 \quad \text{and} \quad H_0(M;\mathbb{Z}/d) = \tilde{H}_0(M;\mathbb{Z}/d) \oplus \mathbb{Z}/d = \mathbb{Z}/d.$$

Therefore $M$ is a $\mathbb{Z}/d$-homology 3-sphere. \hfill \Box
Remark 3.4.2. As a direct consequence of Proposition 3.4.1 we get that the Heegaard splitting $S^3_\phi = H_g \cup_{\phi} H_g$ with $\phi \in M_{g,1}[d]$ is a $\mathbb{Q}$-homology 3-sphere.

We now look for the converse: when is a $\mathbb{Q}$-homology 3-sphere constructed from a Heegaard splitting with gluing map an element of $M_{g,1}[d]$?

Denote by $S^3(d) \subset V^3$ the subset of $\mathbb{Z}/d$-homology 3-spheres and by $S^3[d] \subset S^3(d)$ the set of $\mathbb{Z}/d$-homology 3-spheres which are homeomorphic to $H_g \cup_{\phi} H_g$ for some $\phi \in M_{g,1}[d]$.

Theorem 3.4.2 (Theorem 368 in [9]). Let $A \in M_{m \times n}(\mathbb{Z})$ be given. There exist matrices $U \in GL_m(\mathbb{Z})$, $V \in GL_n(\mathbb{Z})$ and a diagonal matrix $S \in M_{m \times n}(\mathbb{Z})$ such that $A = USV$, the diagonal entries of $S$ are $d_1, d_2, \ldots, d_r, 0, \ldots, 0$, each $d_i$ is a positive integer, and $d_i | d_{i+1}$ for $i = 1, 2, \ldots, r - 1$.

The diagonal matrix $S$ is called the Smith normal form of $A$. The diagonal entries $d_1, d_2, \ldots, d_r$, of the Smith normal form of $A$ are called the elementary divisors (or sometimes the invariant factors) of $A$.

Lemma 3.4.1. Let $M = H_g \cup_{f} H_g$ with $f \in M_{g,1}$ be a $\mathbb{Q}$-homology 3-sphere, $n = |H_1(M; \mathbb{Z})|$ and $\Psi(f) = (E \ F \ G \ H)$. Then $n = |\text{det}(H)|$.

Proof. Consider the Mayer-Vietoris sequence for the reduced homology of $H_g \cup_{f} H_g$:

$$H_1(\Sigma g; \mathbb{Z}) \xrightarrow{\varphi} H_1(H_g; \mathbb{Z}) \oplus H_1(-H_g; \mathbb{Z}) \rightarrow H_1(M; \mathbb{Z}) \rightarrow 0. \quad (3.4.3)$$

Then $H_1(M; \mathbb{Z}) \cong \text{Coker}(\varphi)$. A direct inspection shows that if $\Psi(f) = (E \ F \ G \ H)$, we have that $\varphi = (T \ O \ G \ H)$. Hence,

$$H_1(M; \mathbb{Z}) \cong \text{Coker}(H).$$

Next, by the Theory of invariant factors (see Theorem 3.4.2), we know that there exist matrices $U, V \in GL_{2g}(\mathbb{Z})$ such that

$$UHV = \begin{pmatrix} d_1 & \cdots & d_k & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & d_k & \cdots & 0 \end{pmatrix}.$$ 

As a consequence,

$$\text{Coker}(H) \cong \mathbb{Z}/d_1 \times \mathbb{Z}/d_2 \times \cdots \mathbb{Z}/d_k \times \mathbb{Z}^{g-k}.$$

Since $M$ is a $\mathbb{Q}$-homology sphere, by Universal coefficients Theorem, we have that $0 = H_1(M; \mathbb{Q}) \cong H_1(M; \mathbb{Z}) \otimes \mathbb{Q} \cong \mathbb{Q}^{g-k}$. Then $k = g$.

Hence,

$$\text{det}(H) = \pm \text{det}(UHV) = \pm \prod_{i=1}^{2g} d_i \neq 0.$$
Moreover, we have that
\[ n = |H_1(M; \mathbb{Z})| = \prod_{i=1}^{2g} d_i \neq 0. \]
Therefore we get that \( n = |\text{det}(H)| \), as desired.

Now we are ready to study whenever a \( \mathbb{Q} \)-homology 3-sphere is homeomorphic to a Heegaard splitting with gluing map an element of (mod \( d \)) Torelli group. The following result is inspired in Proposition 6 in [36] due to B. Perron.

**Theorem 3.4.3.** Let \( M \) be a \( \mathbb{Q} \)-homology 3-sphere and \( n = |H_1(M; \mathbb{Z})| \). Then \( M \) has a Heegaard splitting \( \mathcal{H}_g \cup_{\psi, f} \mathcal{H}_g \) of some genus \( g \) with gluing map \( f \in \mathcal{M}_{g,1}[d] \) with \( d \geq 2 \), if and only if \( d \) divides \( n-1 \) or \( n+1 \).

**Proof.** We first prove that if \( M \) is a \( \mathbb{Q} \)-homology 3-sphere with \( n = |H_1(M; \mathbb{Z})| \) and \( d \geq 2 \) divides \( n-1 \) or \( n+1 \), then \( M \in \mathcal{S}_3[d] \).

By Theorem (1.3.4), there exists an element \( f \in \mathcal{M}_{g,1} \) such that \( M \) is homeomorphic to \( \mathcal{H}_g \cup_{\psi, f} \mathcal{H}_g \).

By definition of \( \mathcal{M}_{g,1}[d] \) we have the short exact sequence
\[
1 \rightarrow \mathcal{M}_{g,1}[d] \rightarrow \mathcal{M}_{g,1} \xrightarrow{\Psi_d} \text{Sp}_{2g}(\mathbb{Z}/d) \rightarrow 1. \tag{3.4.4}
\]
Consider the image of \( f \) by \( \Psi \):
\[
\Psi(f) = \begin{pmatrix} E & F \\ G & H \end{pmatrix}.
\]
We show that there exist matrices \( X \in \text{Sp}_{2g}(\mathbb{Z}/d), Y \in \text{Sp}_{2g}(\mathbb{Z}/d) \), such that
\[
X \Psi_d(f) Y = \text{Id}. \tag{3.4.5}
\]
Then, by Lemma (3.3.8), we will get that there exist an integer \( h \geq g \) and elements \( \xi_a \in \mathcal{A}_{h,1}, \xi_b \in \mathcal{B}_{h,1}, \tilde{f} \in \mathcal{M}_{h,1} \) such that \( \Psi_d(\xi_a) = i_{g,h}(X), \Psi_d(\xi_b) = i_{g,h}(Y), \tilde{f} = i_{g,h}(f) \) and
\[
\Psi_d(\xi_a \tilde{f} \xi_b) = \text{Id}.
\]
As a consequence, by the short exact sequence (3.4.4) and Theorem (1.3.4), we will get that \( f \) is equivalent to an element of \( \mathcal{M}_{h,1}[d] \).

In order to construct such matrices \( X, Y \), we first show that \( H \in \text{SL}_g^+(\mathbb{Z}/d) \).

By hypothesis, \( d \mid n-1 \) or \( d \mid n+1 \), i.e. \( n \equiv \pm 1 \) (mod \( d \)). Moreover, by Lemma (3.4.1), we know that \( n = |\text{det}(H)| \). Therefore, \( \text{det}(H) \equiv \pm 1 \) (mod \( d \)), i.e. \( H \in \text{SL}_g^+(\mathbb{Z}/d) \).

Next we proceed to construct the aforementioned matrices \( X, Y \).
Consider the matrix
\[
X = \begin{pmatrix}
Id & A \\
0 & Id
\end{pmatrix},
\]
with \( A = -FH^{-1} \in M_{g \times g}(\mathbb{Z}/d) \).

We prove that \( X \) is an element of \( Sp^{A+}_{2g}(\mathbb{Z}/d) \).

Since \( \Psi_d(f) \in Sp_{2g}(\mathbb{Z}/d) \), by definition, \( H^tF \) is symmetric and as a consequence the following equality holds
\[
A = -FH^{-1} = -(H^t)^{-1}H^tFH^{-1} = -(H^{-1})^tF = -A^t.
\]

Then \( A \in S_g(\mathbb{Z}/d) \), and therefore \( X \in Sp^{A+}_{2g}(\mathbb{Z}/d) \).

Next, observe that we have that
\[
\begin{pmatrix}
Id & A \\
0 & Id
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix} = \begin{pmatrix}
E + AG & F + AH \\
G & H
\end{pmatrix} = \begin{pmatrix}
E + AG & 0 \\
G & H
\end{pmatrix},
\]
which is an element of \( Sp^B_{2g}(\mathbb{Z}/d) \).

In fact, since \( H \in SL_g^+(\mathbb{Z}/d) \), we know that \( \begin{pmatrix} E + AG & 0 \\ G & H \end{pmatrix} \in Sp^B_{2g}(\mathbb{Z}/d) \).

Therefore, setting
\[
Y = \begin{pmatrix} E + AG & 0 \\ G & H \end{pmatrix}^{-1} \in Sp^B_{2g}(\mathbb{Z}/d),
\]
we get the result.

Next we show that the condition \( d \) divides \( n + 1 \) or \( n - 1 \) is a necessary condition for \( M \) to be in \( S^3[d] \). Let \( M \in S^3[d] \). By Theorem (1.3.4), there exists an element \( f \in M_{g,1} \) such that \( M \) is homeomorphic to \( H \cup_{i,g} f - H_g \). In addition, since \( M \in S^3[d] \), there exists \( h \geq g \) for which the class \( \tilde{f} = i_{g,h}(f) \) in \( A_{h,1} \backslash M_{h,1} / B_{h,1} \) contains an element in \( M_{h,1}[d] \). Then there exist \( \xi_a \in A_{h,1}, \xi_b \in B_{h,1} \) and \( \varphi \in M_{h,1}[d] \) such that
\[
\xi_a \varphi \xi_b = \tilde{f}.
\]

Thus,
\[
\Psi_d(\tilde{f}) = \Psi_d(\xi_a) \Psi_d(\varphi) \Psi_d(\xi_b) = \Psi_d(\xi_a) \Psi_d(\xi_b).
\]

Set
\[
\Psi_d(\xi_a) = \begin{pmatrix} E & A \\ 0 & tE^{-1} \end{pmatrix} \in Sp^{A+}_{2h}(\mathbb{Z}/d), \quad \Psi_d(\xi_b) = \begin{pmatrix} F & 0 \\ B & tF^{-1} \end{pmatrix} \in Sp^B_{2h}(\mathbb{Z}/d).
\]

Then
\[
\Psi_d(\tilde{f}) = \Psi_d(\xi_a) \Psi_d(\xi_b) = \begin{pmatrix} E & A \\ 0 & tE^{-1} \end{pmatrix} \begin{pmatrix} F & 0 \\ B & tF^{-1} \end{pmatrix} = \begin{pmatrix} EF + AB & A^tF^{-1} \\ tE^{-1}B & tE^{-1}F^{-1} \end{pmatrix}.
\]
Since $E, F \in SL_k(\mathbb{Z}/d)$ and by Lemma (3.4.1) $n = |\det(tE^{-1}tF^{-1})|$, then we have that
\[ n \equiv \pm \det(tE^{-1}tF^{-1}) \equiv \pm 1 \pmod{d}, \]
as desired. \qed

As a consequence of Theorem (3.4.3), we get that the family of groups $\{M_{g,1}[d]\}_{g,d}$ is enough to produce all $\mathbb{Q}$-homology spheres.

Notice that Theorem (3.4.3) gives us a criterion to know whenever a $\mathbb{Q}$-homology sphere can be formed as a Heegaard splitting with gluing map an element of the $(\text{mod } d)$ Torelli group. Next, using this criterion, we study whenever the set of $\mathbb{Z}/d$-homology 3-spheres, $S^3(d)$, and the set of $\mathbb{Z}/d$-homology 3-spheres which are constructed as a Heegaard splitting with gluing map an element of the mod $d$ Torelli group, $S^3[d]$, coincide. As we will see, unlike the integral case, these two sets do not coincide in general.

**Lemma 3.4.2.** Let $d$ be a positive integer, the invertible elements of $\mathbb{Z}/d$ are contained in $\{1, -1\}$ if and only if $d = 2, 3, 4, 6$.

**Proof.** Let $d \geq 2$ be an integer. Recall that an element of $\mathbb{Z}/d$ is invertible if and only if $gcd(x, d) = 1$. As a consequence, given a prime number $p$ and $n \in \mathbb{N}$, $|(\mathbb{Z}/p^n)^\times| = (p - 1)p^{n-1}$. Let $d = p_1^{n_1} \cdots p_r^{n_r}$, we know that $(\mathbb{Z}/d)^\times = (\mathbb{Z}/p_1^{n_1})^\times \times \cdots \times (\mathbb{Z}/p_r^{n_r})^\times$, so
\[ |(\mathbb{Z}/d)^\times| = \prod_{i=1}^r (p_i - 1)p_i^{n_i-1}. \]
Observe that if for some $i, j$ with $j \neq i$, $p_i \geq 5$, or $p_i = 3$ and $n_i \geq 2$, or $p_i = 2$ and $n_i \geq 3$, or $p_i = 2$, $n_i = 2$ and $p_j = 3$, then $|(\mathbb{Z}/d)^\times| \geq 3$. On the other hand, we directly check that $|(\mathbb{Z}/d)^\times| \leq 2$ for $d = 2, 3, 4, 6$. \qed

**Proposition 3.4.2.** Given an integer $d = 5$ or $d \geq 7$. The set of $\mathbb{Z}/d$-homology 3-spheres, $S^3(d)$, and the set of $\mathbb{Z}/d$-homology 3-spheres which are constructed as a Heegaard splitting with gluing map an element of the mod $d$ Torelli group, $S^3[d]$, do not coincide.

**Proof.** We construct a $\mathbb{Q}$-homology 3-sphere $M \in S^3(d)$, which does not belong to $S^3[d]$. By Lemma (3.4.2) we know that, for $d = 5$ and for every $d \geq 7$, there exists an element $u \in (\mathbb{Z}/d)^\times$ with $u$ different from $\pm 1$. Fix such an element $u \in (\mathbb{Z}/d)^\times$.

Consider the matrix $\left( \begin{smallmatrix} G & 0 \\ 0 & G^{-1} \end{smallmatrix} \right) \in Sp_{2g}(\mathbb{Z}/d)$, with $G \in GL_g(\mathbb{Z}/d)$ given by
\[ G = \begin{pmatrix} u & 0 \\ 0 & Id \end{pmatrix}, \]

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3.4. Heegaard splittings of homology 3-spheres

Since \( \Psi_d \) is surjective, we know that there is an element \( f \in \mathcal{M}_{g,1} \) satisfying that

\[
\Psi_d(f) = \begin{pmatrix} G & 0 \\ 0 & tG^{-1} \end{pmatrix}.
\]

Consider the 3-manifold \( M \) given by the Heegaard splitting \( \mathcal{H}_g \cup_{\psi} \mathcal{H}_g \).

Following the argument of the proof of Proposition (3.4.1) with \( f \) defined as before, one gets that \( M \) is a \( \mathbb{Z}/d \)-homology 3-sphere, i.e. \( M \in S^3(d) \).

On the other hand, by Lemma (3.4.1), we have that \( \pm n = \pm |H_1(M; \mathbb{Z})| = \det(tG^{-1}) = u^{-1} \neq \pm 1 \). Then, by Theorem (3.4.3), \( M \) is not in \( S^3[d] \).

**Proposition 3.4.3.** Let \( d = 2, 3, 4, 6 \). If \( M \) is a \( \mathbb{Z}/d \)-homology 3-sphere, then \( M \) has a Heegaard splitting \( \mathcal{H}_g \cup_{\psi} \mathcal{H}_g \) of some genus \( g \) with gluing map \( f \in \mathcal{M}_{g,1}[d] \).

**Proof.** Let \( d = 2, 3, 4, 6 \), \( M \) a \( \mathbb{Z}/d \)-homology 3-sphere and \( n = |H_1(M; \mathbb{Z})| \). We show that \( n \equiv \pm 1 \) (mod \( d \)) and by Theorem (3.4.3) we will get the result.

Since \( M \) is a \( \mathbb{Z}/d \)-homology 3-sphere, we have that \( gcd(d, n) = 1 \). Then, by Bezout’s identity, there exist integers \( \lambda_1, \lambda_2 \) such that \( \lambda_1 d + \lambda_2 n = 1 \). As a consequence, \( \lambda_2 n \equiv 1 \) (mod \( d \)). Hence, \( n \in (\mathbb{Z}/d)^\times \). Finally, by Lemma (3.4.2), one gets that \( |(\mathbb{Z}/d)^\times| \leq 2 \) for \( d = 2, 3, 4, 6 \), as desired.

As a consequence of Proposition (3.4.1) and Theorem (1.3.4) we get the following result:

**Theorem 3.4.4.** The following map is well defined and bijective:

\[
\lim_{g \to \infty} A_{g,1}[d]/B_{g,1} \longrightarrow S^3[d],
\]

\[
\phi \mapsto S^3[\phi] = \mathcal{H}_g \cup_{\psi \phi} \mathcal{H}_g.
\]

From the group-theoretical point of view, the induced equivalence relation on \( \mathcal{M}_{g,1}[d] \), which is given by:

\[
\phi \sim \psi \iff \exists \zeta_a \in A_{g,1} \exists \zeta_b \in B_{g,1} \text{ such that } \zeta_a \phi \zeta_b = \psi,
\]

is quite unsatisfactory. However, as in the integral case, Lemma 4 in [38], we can rewrite this equivalence relation as follows:

**Lemma 3.4.3.** Two maps \( \phi, \psi \in \mathcal{M}_{g,1}[d] \) are equivalent if and only if there exists a map \( \mu \in AB_{g,1} \) and two maps \( \xi_a \in A_{g,1}[d] \) and \( \xi_b \in B_{g,1}[d] \) such that \( \phi = \mu \xi_a \psi \xi_b \mu^{-1} \).

**Proof.** The proof of this Lemma is analogous to the proof of Lemma 4 in [38] due to W. Pitsch. The "if" part of the Lemma is trivial. Conversely, assume that \( \psi = \xi_a \phi \xi_b \), where \( \psi, \phi \in \mathcal{M}_{g,1}[d] \).
Applying the symplectic representation modulo $d \Psi_d$ to this equality we get

$$Id = \Psi_d(\xi_a) \Psi_d(\xi_b) \pmod{d}.$$  

By section (1.3.7), $\Psi(\xi_b)$ is of the form

$$\begin{pmatrix} G & 0 \\ M & tG^{-1} \end{pmatrix} \quad \text{with } G \in GL_g(\mathbb{Z}) \text{ and } ^tGM \in S_g(\mathbb{Z}).$$

Similarly, $\Psi(\xi_a)$ is of the form

$$\begin{pmatrix} H & N \\ 0 & tH^{-1} \end{pmatrix} \quad \text{with } H \in GL_g(\mathbb{Z}) \text{ and } H^{-1}N \in S_g(\mathbb{Z}).$$

Therefore,

$$\begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix} \equiv \begin{pmatrix} H & N \\ 0 & tH^{-1} \end{pmatrix} \begin{pmatrix} G & 0 \\ M & tG^{-1} \end{pmatrix} = \begin{pmatrix} HG + NM & N^tG^{-1} \\ tH^{-1}M & t(HG)^{-1} \end{pmatrix} \pmod{d}.$$  

Thus $N \equiv 0 \equiv M \pmod{d}$ and $G \equiv H^{-1} \pmod{d}.$

By Lemma (1.3.3), we can choose a map $\mu \in AB_{g,1}$ such that

$$\Psi(\mu) = \begin{pmatrix} H & 0 \\ 0 & tH^{-1} \end{pmatrix}$$

Since $N \equiv 0 \equiv M \pmod{d}$ and $G \equiv H^{-1} \pmod{d}$, we have that

$$\Psi_d(\mu^{-1}\xi_a) = \begin{pmatrix} H^{-1} & 0 \\ 0 & tH \end{pmatrix} \begin{pmatrix} H & N \\ 0 & tH^{-1} \end{pmatrix} = \begin{pmatrix} Id & H^{-1}N \\ 0 & Id \end{pmatrix} \equiv Id \pmod{d}.$$  

$$\Psi_d(\xi_b\mu) = \begin{pmatrix} G & 0 \\ M & tG^{-1} \end{pmatrix} \begin{pmatrix} H & 0 \\ 0 & tH^{-1} \end{pmatrix} = \begin{pmatrix} GH & 0 \\ MH & t(GH)^{-1} \end{pmatrix} \equiv Id \pmod{d}.$$  

Therefore we get

$$\psi = \mu \circ (\mu^{-1}\xi_a) \phi(\xi_b\mu) \circ \mu^{-1},$$

where $(\mu^{-1}\xi_a) \in A_{g,1}[d]$, $(\xi_b\mu) \in B_{g,1}[d]$ and $\mu \in AB_{g,1}$ as desired.
To summarize, we get the following bijective map:

\[
\lim_{g \to \infty} (A_{g,1}[d] \setminus M_{g,1}[d] / B_{g,1}[d])_{A B_{g,1}} \to S^3[d],
\]

\[
\phi \mapsto S^3_{\phi} = H_g \cup \iota_g \phi - H_g.
\]

(3.4.7)

The bijection (3.4.7) will play an important role in this thesis. As we will see in the following sections, it allows us to relate the structure of invariants of \(S^3[d]\), with the algebraic structure of the group \(M_{g,1}[d]\).

As we have seen in this chapter, for \(d \neq 2,3,4,6\) the set of \(\mathbb{Z}/d\)-homology 3-spheres, \(S^3(d)\), and the set of \(\mathbb{Z}/d\)-homology 3-spheres which are constructed as a Heegaard splitting with gluing map an element of the mod \(d\) Torelli group, \(S^3[d]\), do not coincide. Then a natural question which arise from this chapter is:

What is the difference between \(S^3(d)\) and \(S^3[d]\) for \(d \neq 2,3,4,6\)?
Chapter 3. The mod $d$ Torelli group and homology 3-spheres
CHAPTER 4

Automorphisms of descending mod-p central series

Let $\Gamma$ be a free group of finite rank and \{\Gamma_k^*\} be the Stallings or Zassenhaus $p$-central series and $N_k^* = \Gamma/\Gamma_{k+1}^*$. In this chapter, we compare the automorphisms of the group $N_k^*$. To be more precise, we show that there is a well-defined homomorphism $\psi_k : Aut(N_{k+1}^*) \to Aut(N_k^*)$ that, using the theory of versal extensions mod $p$, which is a mod $p$ version of the versal extensions given in [37], fits into a non central extension

$$0 \longrightarrow \text{Hom}(N_1^*, L_{k+1}^*) \longrightarrow Aut(N_{k+1}^*) \overset{\psi_k}{\longrightarrow} Aut(N_k^*) \longrightarrow 1.$$ (4.0.1)

In addition, we show that such extension does not split for $k \geq 2$ and we study its splitability for $k = 1$ using the computations of $H^2(SL_n(\mathbb{Z}/p); sl_n(\mathbb{Z}/p))$.

As a starting point, in the first section, we recall some definitions about central series and mod $p$ central series with some examples. Moreover we give the notion of $p$-nilpotent group. In the second section, we give the notion of a versal extension modulo $p$ and we develop some theory about it. In the third section, we give some technical results that we will need in the last section. Finally, in the last section, we study the extensions (4.0.1).

4.1 On $p$-central series and $p$-nilpotent groups

Definition 4.1.1. A central series is a sequence of subgroups

$$\{1\} \triangleleft A_n \triangleleft \ldots \triangleleft A_2 \triangleleft A_1 = G$$

such that $[A_j, G] \subseteq A_{j+1}$. Equivalently, $A_j/A_{j+1}$ is central in $G/A_{j+1}$.  

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Definition 4.1.2. We say that a central series \( \{G_k\}_k \) is the fastest descending series respect to a property \( \mathcal{P} \) if for every central series \( \{H_k\}_k \) satisfying \( \mathcal{P} \), we have that \( G_k \leq H_k \) for every \( k \).

Definition 4.1.3. Two series \( \{G_i\} \) and \( \{H_i\} \) are cofinal if for every natural number \( n \) there exists a natural number \( m \) such that \( G_m < H_n \), and for every natural number \( l \) there is some natural number \( k \) such that \( H_k < G_l \).

Definition 4.1.4 (Lower central series). The lower central series \( \{G_k\}_k \) of a group \( G \) is defined recursively by \( G_1 = G \) and \( G_k = [G, G_{k-1}] \).

Next we give the "mod \( p \) version" of central series.

Definition 4.1.5. A \( p \)-central series is a central series whose successive quotients \( A_i/A_{i+1} \) are \( p \)-elementary abelian groups, i.e. abelian groups in which every nontrivial element has order \( p \).

There are many \( p \)-central series, but the most important are the Zassenhaus mod-\( p \) central series and the Stallings mod-\( p \) central series which are defined as follows:

Definition 4.1.6 (Zassenhaus). The Zassenhaus mod-\( p \) central series \( \{G^Z_k\} \) for \( G \) is defined by the rule:

\[
G^Z_k = \prod_{i|p^j \geq k} (G_i)^{p^j},
\]

where \( G_i \) denotes the \( i^{th} \) term of the lower central series of \( G \).

This series is the fastest descending series with respect to the properties:

- \([G^Z_k, G^Z_l] < G^Z_{k+l} \),
- \((G^Z_k)^p < G^Z_{p^k} \).

Definition 4.1.7 (Stallings). The Stallings mod-\( p \) central series (also known as the lower \( p \)-central series) \( \{G^S_k\} \) for \( G \) is defined recursively by setting

\[
G^S_1 = G \quad \text{and} \quad G^S_k = [G, G^S_{k-1}] (G^S_{k-1})^p.
\]

This series is the fastest descending series with respect to the properties:

- \([G^S_k, G^S_l] < G^S_{k+l} \),
- \((G^S_k)^p < G^S_{k+1} \).

Even though the aforementioned mod-\( p \) central series are distinct, we have the following result, which will allow us to compare them.
Proposition 4.1.1 (Proposition 2.6 in [4]). Given a prime number $p$. The Stallings and Zassenhaus mod-$p$ central series are cofinal. More precisely, for every positive integer $l$,

$$G^Z_p < G^S_l < G^Z_l.$$ 

Remark 4.1.1. In [4] J. Cooper proved Proposition (4.1.1) for $G$ a free group. The same proof gives the result for any group $G$.

Throughout this chapter we denote by $\Gamma$ a free group of finite rank and we set:

$$N_k = \Gamma/\Gamma_{k+1}, \quad L_k = \Gamma_k/\Gamma_{k+1},$$

$$\tilde{N}_k = \frac{\Gamma}{[\Gamma, \Gamma_k](\Gamma_k)^p}, \quad \tilde{L}_k = \frac{\Gamma_k}{[\Gamma, \Gamma_k](\Gamma_k)^p},$$

where $\bullet = S$ or $Z$.

Definition 4.1.8. A group $G$ is $p$-nilpotent if the corresponding lower $p$-central series has finite length. Notice that the groups $N_k^\bullet$ with $\bullet = Z, S$ and $k$ a positive integer, are $p$-nilpotent groups and, in particular, $p$-groups.

Definition 4.1.9. Let $G$ be a group and $d(G)$ the minimal cardinality of a generating set of $G$. The Frattini subgroup $\Phi(G)$ is defined as the intersection of the set of all maximal subgroups of $G$.

Proposition 4.1.2 (Proposition 1.2.4 in [24]). Let $G$ be a finite $p$-group. Then the Frattini subgroup $\Phi(G) = [G, G]G^p$ and $d(G)$ coincides with the rank of the abelian $p$-elementary group $G/\Phi(G)$.

4.2 Versal extensions modulo $p$

Fix a prime $p$. Let $G$ be a group such that $H_1(G; \mathbb{Z}/p)$ is a free $\mathbb{Z}/p$-module. Applying the Universal coefficients Theorem to the group $G$ and the $\mathbb{Z}/p$-module $H_2(G; \mathbb{Z}/p)$ with trivial $G$-action, since $H_1(G; \mathbb{Z}/p)$ is a free $\mathbb{Z}/p$-module, we have an induced natural isomorphism:

$$\eta: H^2(G; H_2(G; \mathbb{Z}/p)) \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}/p}(H_2(G; \mathbb{Z}/p), H_2(G; \mathbb{Z}/p)).$$

(4.2.1)

Definition 4.2.1 (Section 9.5 in [24]). Let $G$ be a group. A $p$-covering of $G$ is an extension of the form

$$0 \rightarrow H_2(G; \mathbb{Z}/p) \rightarrow E \rightarrow G \rightarrow 0.$$ 

In this case we say that $E$ is a $p$-covering group of $G$. 

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Denote by $v_p \in H^2(G; H_2(G; \mathbb{Z}/p))$ the preimage of the identity by the isomorphism (4.2.1) and by

$$V_p : 0 \longrightarrow H_2(G; \mathbb{Z}/p) \longrightarrow E \longrightarrow G \longrightarrow 0,$$

a central extension with associated cohomology class $v_p$.

**Definition 4.2.2.** A central extension $0 \longrightarrow H_2(G; \mathbb{Z}/p) \longrightarrow E' \longrightarrow G \longrightarrow 0$ is called a *versal* extension modulo $p$ if it is equivalent to $V_p$.

**Lemma 4.2.1.** Let $v_p \in H^2(G; H_2(G; \mathbb{Z}/p))$ denote the cohomology class given by the preimage of the identity by the isomorphism $\eta$ of (4.2.1). Then for each $\phi \in \text{Aut}(G)$, the following equality holds

$$(H_2(\phi; \mathbb{Z}/p))^*(v_p) = \phi^*(v_p).$$

Here $H_2(\phi; \mathbb{Z}/p)$ denotes the element of $\text{Aut}(H_2(\phi; \mathbb{Z}/p))$ induced by $\phi$.

**Proof.** Fix an element $\phi \in \text{Aut}(G)$, by naturality of the isomorphism $\eta$, we have that

$$\eta(\phi^*(v_p)) = (H_2(\phi; \mathbb{Z}/p))^*(id) = H_2(\phi; \mathbb{Z}/p),$$

$$\eta((H_2(\phi; \mathbb{Z}/p))^*(v_p)) = (H_2(\phi; \mathbb{Z}/p))^*(id) = H_2(\phi; \mathbb{Z}/p).$$

As a consequence we get the following result:

**Corollary 4.2.1.** Let $G$ be a group and $0 \rightarrow H_2(G; \mathbb{Z}/p) \rightarrow E \rightarrow G \rightarrow 1$ a versal extension modulo $p$. For every element $\phi \in \text{Aut}(G)$, there exists an element $\Phi \in \text{Aut}(E)$ such that the following diagram commutes

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H_2(G; \mathbb{Z}/p) & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\
& & \downarrow H_2(\phi; \mathbb{Z}/p) & & \Phi & & \phi & & \\
0 & \longrightarrow & H_2(G; \mathbb{Z}/p) & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1.
\end{array}
$$

(4.2.2)

Therefore every element $\phi \in \text{Aut}(G)$ is induced by an element $\Phi \in \text{Aut}(E)$.

**Proof.** Consider a versal extension modulo $p$

$$0 \longrightarrow H_2(G; \mathbb{Z}/p) \longrightarrow E \longrightarrow G \longrightarrow 1.$$

with associated cohomology class $v_p$. By Lemma (4.2.1) we know that for every $\phi \in \text{Aut}(G)$, the following equality holds $(H_2(\phi; \mathbb{Z}/p))^*(v_p) = \phi^*(v_p)$. Then, by Corollary (1.1.1), there exists an
element $\Phi \in Hom(E, E)$ making the following diagram commutative

\[
\begin{array}{cccccc}
0 & \rightarrow & H_2(G; \mathbb{Z}/p) & \rightarrow & E & \rightarrow & G & \rightarrow & 1 \\
& & \downarrow_{H_2(\phi; \mathbb{Z}/p)} & & \Phi & & \phi \\
0 & \rightarrow & H_2(G; \mathbb{Z}/p) & \rightarrow & E & \rightarrow & G & \rightarrow & 1.
\end{array}
\]

Finally, the 5-lemma implies that $\Phi$ is an automorphism of $E$ that lifts $\phi$ as desired. $\square$

4.2.1 Versal extensions modulo $p$ for $p$-groups

If $G$ is a $p$-group, the modulo $p$ Hopf formula becomes simpler. In this case, let $1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$ be a presentation of $G$ with $d(F) = d(G)$. Then $R \leq [F, F]F^p$, so $H_2(G; \mathbb{Z}/p) \cong R/[R, F]R^p$ and $F/[R, F]R^p$ is a $p$-covering group of $G$.

**Definition 4.2.3** (Definition 9.5.12 in [24]). We say that a $p$-covering of $G$

\[
0 \rightarrow H_2(G; \mathbb{Z}/p) \rightarrow P \rightarrow G \rightarrow 0
\]

is the universal $p$-covering of $G$ and $P$ is the universal $p$-covering group of $G$ (up to canonical isomorphisms) if for any other central extension

\[
0 \rightarrow A \rightarrow E \rightarrow G \rightarrow 0
\]

with $A$ a $p$-elementary abelian group satisfying that $A \leq \Phi(E)$, there is a surjection of $P$ onto $E$ that induces a surjection of $H_2(G; \mathbb{Z}/p)$ onto $A$, and the natural isomorphism from $P/H_2(G; \mathbb{Z}/p)$ onto $E/A$.

**Proposition 4.2.1** (Proposition 9.5.13 in [24]). The group $P = F/[R, F]R^p$ is the universal $p$-covering group of $G$.

**Proposition 4.2.2.** The central extension

\[
0 \rightarrow H_2(G; \mathbb{Z}/p) \rightarrow P \rightarrow G \rightarrow 1, \tag{4.2.3}
\]

where $P$ is the universal $p$-covering group of $G$, is a versal extension mod $p$.

**Proof.** Let $c \in H^2(G; R/[F, R]R^p)$ the cohomology class associated to the central extension

\[
0 \rightarrow [R, F]R^p \rightarrow [F, R]R^p \rightarrow G \rightarrow 1. \tag{4.2.4}
\]
Using the Universal coefficients Theorem we get the natural isomorphism

\[ \tilde{\eta} : H^2 \left( G; \frac{R}{[R,F]R^p} \right) \rightarrow Hom \left( H_2(G;\mathbb{Z}/p); \frac{R}{[R,F]R^p} \right), \tag{4.2.5} \]

that takes \( c \) to the homomorphism \( h \) given by the evaluation on \( H_2(G;\mathbb{Z}/p) \) of the 2-cocycle \( r([g,h]) = s(g)s(h)s(gh)^{-1} \) where \( s \) is a theoretic section of \( \text{(4.2.4)} \). Notice that, in fact, the homomorphism \( h \) is the modulo \( p \) Hopf isomorphism.

Since \( h \) is an isomorphism and \( \tilde{\eta} \) is natural, applying \( h^{-1} \) to \( \text{(4.2.5)} \) we get a natural isomorphism

\[ \eta : H^2(G;H_2(G;\mathbb{Z}/p)) \rightarrow Hom(H_2(G;\mathbb{Z}/p);H_2(G;\mathbb{Z}/p)), \]

which takes \( h^{-1}(c) \) to the identity as desired.

\[ \square \]

### 4.3 On central extensions and characteristic subgroups

Consider the following group extensions

\[ 0 \rightarrow L_{k+1}^\bullet \xrightarrow{i} N_{k+1}^\bullet \xrightarrow{\pi_k^\bullet} N_k^\bullet \rightarrow 1, \]
\[ 0 \rightarrow \overline{L}_{k+1}^\bullet \xrightarrow{i} \overline{N}_{k+1}^\bullet \xrightarrow{\overline{\pi}_k^\bullet} N_k^\bullet \rightarrow 1, \tag{4.3.1} \]

where \( i \) is the natural inclusion and \( \pi_k^\bullet, \overline{\pi}_k^\bullet \) are the quotient map corresponding to \( \Gamma_{k+1}^\bullet \).

**Proposition 4.3.1.** The above group extensions \( \text{(4.3.1)} \) are central.

**Proof.** Let \( N \) be either \( \Gamma_{k+2}^\bullet \) or \( \{ \Gamma, \Gamma_{k+1}^\bullet \}(\Gamma_{k+1}^\bullet)^p \). By construction, \( N \triangleleft \Gamma_{k+1}^\bullet \triangleleft \Gamma \), moreover \( [\Gamma, \Gamma_{k+1}^\bullet] \subset N \). Then a direct computation, using Hall identities shows that

\[ \left[ \frac{\Gamma}{N}, \frac{\Gamma_{k+1}^\bullet}{N} \right] \subset [\Gamma, \Gamma_{k+1}^\bullet] \mod N. \]

\[ \square \]

**Definition 4.3.1.** A subgroup \( H \) of a group \( G \) is a characteristic subgroup of \( G \) if every automorphism of \( G \) maps the subgroup \( H \) to itself.

**Proposition 4.3.2.** Let \( G \) be a group. The groups \( G_k^\bullet \) are characteristic subgroups of \( G \).

**Proof.** We begin with the case of the Stallings series. We proceed by induction on \( k \). The base case \( k = 1 \) holds because by definition, \( G_1^S = G \). Now assume that the Proposition holds for \( k \) and we prove it for \( k + 1 \).
4.3. On central extensions and characteristic subgroups

Let $f \in \text{Aut}(G)$ and $x \in G^S_{k+1}$. We prove that $f(x) \in G^S_{k+1}$. Recall that by definition, $G^S_{k+1} = [G, G^S_k](G^S_k)^p$. Thus $x$ is a product of elements of $[G, G^S_k](G^S_k)^p$. Since $f$ is a homomorphism, it is enough to prove that $[G, G^S_k]$ and $(G^S_k)^p$ are characteristic subgroups of $G$.

Let $y \in G$, $z \in G^S_k$. By induction hypothesis we have that $f(y) \in G$, $f(z) \in G^S_k$. Therefore

$$f([y,z]) = [f(y), f(z)] \in [G, G^S_k] \quad \text{and} \quad f(y^p) = f(z^p) \in (G^S_k)^p.$$ 

For the case of the Zassenhaus series, recall that by definition $G^Z_k = \prod_{ip^j \geq k}(G_i)^{p^j}$. Thus every element of $G^Z_k$ is a product of elements of the form $x_i^{p^j} \in G^Z_i$ with $x_i \in G_i$ and $ip^j \geq k$. Since $f$ is a homomorphism, it is enough to prove that $f(x_i^{p^j}) \in G^Z_k$ with $x_i \in G_i$ and $ip^j \geq k$.

It is well known that $G_i$ is a characteristic subgroup of $G$. Moreover, by the properties of the Zassenhaus series we know that $G_i \leq G^Z_i$ and $(G^Z_i)^{p^j} \leq G^Z_{ip^j} \leq G^Z_k$. Therefore we have that

$$f(x_i^{p^j}) = f(x_i)^{p^j} \in (G^Z_i)^{p^j} \leq G^Z_k.$$ 

\[ \square \]

**Proposition 4.3.3.** For every $k \geq 1$, there are isomorphisms

$$\mathcal{L}_k^S \cong (N^S_k)_k, \quad \mathcal{L}_k^Z \cong (N^Z_k)_k, \quad \mathcal{L}_k^Z \cong (N^Z_k)_k, \quad \frac{\Gamma_{k+1}^Z}{[\Gamma, \Gamma^Z_k](\Gamma^Z_k)^p} \cong (N^Z_k)_k. \quad (4.3.2)$$

**Proof.** First we prove the last three isomorphisms of (4.3.2).

Let $N$ be either $\Gamma_{k+1}^Z$ or $[\Gamma, \Gamma^Z_k](\Gamma^Z_k)^p$ and $l = k$ or $k+1$. We have the following epimorphism:

$$f : \prod_{ip^j \geq l} (\Gamma_i)^{p^j} \rightarrow \prod_{ip^j \geq l} \left( \left( \frac{\Gamma}{N} \right)_i \right)^{p^j}.$$ 

The kernel of $f$ is $N \cap \prod_{ip^j \geq l}(\Gamma_i)^{p^j} = N \cap \Gamma^Z_l$. Moreover, by the properties of Zassenhaus and Stallings series, we know that $N \subseteq \Gamma^Z_l$. Therefore, the first isomorphism Theorem of groups give us an isomorphism

$$\frac{\Gamma^Z_l}{N} \cong \prod_{ip^j \geq l} \left( \left( \frac{\Gamma}{N} \right)_i \right)^{p^j}.$$ 

Then we get the last three isomorphism of (4.3.2).

Next we prove that $\mathcal{L}_k^S \cong (N^S_k)_k$. For every $l \leq k$, consider the following epimorphism:

$$f_l : \Gamma^S_l \rightarrow \left( \frac{\Gamma}{\Gamma^S_{k+1}} \right)_l^S.$$ 

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We show that $f_l$ induces an isomorphism

$$\frac{\Gamma^S_l}{\Gamma^S_{k+1}} \cong \left( \frac{\Gamma}{\Gamma^S_{k+1}} \right)^S_l,$$

for $l \leq k$. We proceed by induction on $l$.

For $l = 1$, the result is clear by definition of the Stallings series. Assume that $f_{l-1}$ induces an isomorphism $f_{l-1}^* : \frac{\Gamma^S_{l-1}}{\Gamma^S_{k+1}} \cong \left( \frac{\Gamma}{\Gamma^S_{k+1}} \right)^S_{l-1}$.

We prove that $f_l$ induces an isomorphism (4.3.3). Observe that $f_l$ is given by an epimorphism

$$f_l : [\Gamma, \Gamma^S_{l-1}](\Gamma^S_{l-1})^p \to \left[ \frac{\Gamma}{\Gamma^S_{k+1}}, \frac{\Gamma^S_{l-1}}{\Gamma^S_{k+1}} \right] \left( \frac{\Gamma^S_{l-1}}{\Gamma^S_{k+1}} \right)^p$$

with kernel $\Gamma^S_{k+1} \cap [\Gamma, \Gamma^S_{l-1}](\Gamma^S_{l-1})^p = \Gamma^S_{k+1} \cap \Gamma^S_{l-1} = \Gamma^S_{k+1}$, because $l \leq k + 1$. Therefore, the first isomorphism theorem of groups give us the desired isomorphism.

As a consequence of Propositions (4.3.3), applying Proposition (4.3.2) to $N_k^\ast, \tilde{N}_k^\ast$ we get the following result.

**Corollary 4.3.1.** The groups $L_k^\ast, \tilde{L}_k^\ast, \frac{\Gamma^S_{k+1}}{[\Gamma, \Gamma^S_{k+1}]}(\Gamma^S_{k+1})^p$ are respectively characteristic subgroups of $N_k^\ast, \tilde{N}_k^\ast$.

**Proposition 4.3.4.** For any prime number $p$,

$$H_2(N_k^\ast; \mathbb{Z}/p) \cong \frac{\Gamma^S_{k+1}}{[\Gamma, \Gamma^S_{k+1}]}(\Gamma^S_{k+1})^p = \tilde{L}_{k+1}^\ast.$$

**Proof.** Applying the Hopf formula modulo $p$ to the presentation of $N_k^\ast$ given by

$$1 \to \Gamma_{k+1}^\ast \to \Gamma \to N_k^\ast \to 1,$$

since $\Gamma_{k+1}^\ast \subset [\Gamma, \Gamma] \Gamma^p$ we get that $H_2(N_k^\ast; \mathbb{Z}/p) \cong \frac{\Gamma^S_{k+1}}{[\Gamma, \Gamma^S_{k+1}]}(\Gamma^S_{k+1})^p$. 

### 4.4 Extensions of Automorphisms

Let $\Gamma$ be a free group of finite rank and $\{\Gamma^k\}$ be the Stallings or Zassenhaus $p$-central series. Since $L_{k+1}^\ast$ is a characteristic subgroup of $N_{k+1}^\ast$, there is a well defined homomorphism $\psi_k^\ast : Aut(N_{k+1}^\ast) \to$
4.4. Extensions of Automorphisms

$Aut(N_k^*)$. The main goal of this section is to show that there is a short exact sequence

$$0 \longrightarrow \text{Hom}(N_1^*, L_{k+1}^*) \overset{i}{\longrightarrow} \text{Aut}(N_{k+1}^*) \overset{\psi_k}{\longrightarrow} \text{Aut}(N_k^*) \longrightarrow 1. \quad (4.4.1)$$

To achieve our goal, we first prove that there is a short exact sequence

$$0 \longrightarrow \text{Hom}(N_1^*, H_2(N_k^*; \mathbb{Z}/p)) \overset{i}{\longrightarrow} \text{Aut}(N_{k+1}^*) \overset{\psi_k}{\longrightarrow} \text{Aut}(N_k^*) \longrightarrow 1. \quad (4.4.2)$$

Then we get the short exact sequence (4.4.1) as a push-out of this short exact sequence.

**Remark 4.4.1.** Notice that the short exact sequences (4.4.1), (4.4.2) coincide for $\bullet = S$ and differ for $\bullet = Z$.

Consider the group $\overline{N}_{k+1}^*$, which, by Proposition (4.2.1), is the universal p-covering group of $N_k^*$. Since $L_{k+1}^*$ is a characteristic subgroup of $\overline{N}_{k+1}^*$, there is a well defined homomorphism $\overline{\psi}_k^* : Aut(\overline{N}_{k+1}^*) \rightarrow Aut(N_k^*)$. Next we prove that $\overline{\psi}_k^*$ is an epimorphism and that $\text{Ker}(\overline{\psi}_k^*)$ is isomorphic to $\text{Hom}(N_1^*, H_2(N_k^*; \mathbb{Z}/p))$.

**The map $\overline{\psi}_k^*$ is an epimorphism.** Applying Corollary (4.2.1) to the universal p-covering group $\overline{N}_{k+1}^*$ of $N_k^*$, we get that $\overline{\psi}_k^* : Aut(\overline{N}_{k+1}^*) \rightarrow Aut(N_k^*)$ is an epimorphism.

**The Kernel of $\overline{\psi}_k^*$ is isomorphic to $\text{Hom}(N_1^*, H_2(N_k^*; \mathbb{Z}/p))$.**

**Proposition 4.4.1.** The kernel of the homomorphism $\overline{\psi}_k^* : Aut(\overline{N}_{k+1}^*) \rightarrow Aut(N_k^*)$ is isomorphic to the group $\text{Stab}(N_k^*, H_2(N_k^*; \mathbb{Z}/p))$, i.e. the group of stabilizing automorphisms of the extension $0 \rightarrow H_2(N_k^*; \mathbb{Z}/p) \rightarrow \overline{N}_{k+1}^* \rightarrow N_k^* \rightarrow 1$.

**Proof.** Consider the versal extension modulo $p$

$$0 \longrightarrow H_2(N_k^*; \mathbb{Z}/p) \longrightarrow \overline{N}_{k+1}^* \longrightarrow N_k^* \longrightarrow 1. \quad (4.4.3)$$

Denote $\nu_p \in H^2(N_k^*; H_2(N_k^*; \mathbb{Z}/p))$ its associated cohomology class. Let $\Phi \in \text{Ker}(\overline{\psi}_k^*)$. Then $\Phi$ induces a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(N_k^*; \mathbb{Z}/p) & \longrightarrow & \overline{N}_{k+1}^* & \longrightarrow & N_k^* & \longrightarrow & 1 \\
& & \downarrow{\psi} & & \downarrow{\Phi} & & \downarrow & & \\
0 & \longrightarrow & H_2(N_k^*; \mathbb{Z}/p) & \longrightarrow & \overline{N}_{k+1}^* & \longrightarrow & N_k^* & \longrightarrow & 1.
\end{array} \quad (4.4.4)
$$

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and this implies that $\psi_*(v_p) = v_p$. Applying the natural isomorphism $\eta$ of (4.2.1) to the above equality, we obtain that $id = \eta(v_p) = \psi_*\eta(v_p) = \psi_*(id) = \psi$. Therefore $\Phi$ stabilize the extension (4.4.3). Hence, $Ker(\tilde{\psi}^k) \cong Stab(N^*_k, H_2(N^*_k; \mathbb{Z}/p))$.

Moreover, we have the following isomorphisms:

$$Stab(N^*_k, H_2(N^*_k; \mathbb{Z}/p)) \cong Der(N^*_k, H_2(N^*_k; \mathbb{Z}/p)) = Hom(N^*_k, H_2(N^*_k; \mathbb{Z}/p)) = \cong Hom(N^*_1, H_2(N^*_k; \mathbb{Z}/p)).$$

Therefore we get the following result:

**Proposition 4.4.2.** We have an exact sequence of groups

$$0 \longrightarrow Hom(N^*_1, H_2(N^*_k; \mathbb{Z}/p)) \longrightarrow Aut\overline{N^*_k} \overrightarrow{\psi^k} Aut N^*_k \longrightarrow 1.$$

As a consequence of Proposition (4.3.4), using the modulo $p$ Hopf isomorphism, we get the following result:

**Corollary 4.4.1.** We have an exact sequence of groups

$$0 \longrightarrow Hom(N^*_1, L^*_k) \longrightarrow Aut\overline{N^*_k} \overrightarrow{\psi^k} Aut N^*_k \longrightarrow 1,$$

where $i$ is defined as $i(f) = (\gamma \mapsto f([\gamma])\gamma)$.

As a consequence of Corollary (4.4.1), we get the following result:

**Proposition 4.4.3.** We have an exact sequence of groups

$$0 \longrightarrow Hom(N^*_1, L^*_k) \longrightarrow Aut\overline{N^*_k} \overrightarrow{\psi^k} Aut N^*_k \longrightarrow 1,$$

where $i$ is defined as $i(f) = (\gamma \mapsto f([\gamma])\gamma)$.

**Proof.** Notice that we have a push-out diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & Hom(N^*_1, L^*_k) \\
\downarrow q & & \downarrow \psi^k \\
0 & \longrightarrow & Hom(N^*_1, L^*_k) \\
& & \overrightarrow{i} \\
& & Aut\overline{N^*_k} \\
& & Aut N^*_k \\
& & 1
\end{array}
$$

where $q$ is the quotient respect $\Gamma^{k+2}_{k+1}/[\Gamma^{k+2}_{k+1}(0)_{k+1}]^p$. 

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Remark 4.4.2. Despite the fact that the extensions of Proposition (4.4.3) for \( \bullet = S \) and \( \bullet = Z \) are different, notice that \( \Gamma^Z < \Gamma^S < \Gamma^Z \). Then, one can show that there is a well defined epimorphism \( \text{Aut}(N^S_k) \to \text{Aut}(N^Z_k) \).

4.4.1 On centrality

In this section we prove that the extension of Proposition (4.4.3) is not central. In particular, we prove that the action of \( \text{Aut}(N^\bullet_{k+1}) \) on \( \text{Hom}(N^\bullet_1, L^\bullet_{k+1}) \) factors through \( \text{Aut}(N^\bullet_1) \).

Denote by \( IA^p(N^\bullet_k) \) formed by the elements which act trivially on the first homology (with coefficients in \( \mathbb{Z}/p \)) of \( N^\bullet_k \). In other words, \( IA^p(N^\bullet_k) \) is the kernel of the surjection \( N^\bullet_k \to N^\bullet_1 \).

We have the following short exact sequence

\[
1 \longrightarrow IA^p(N^\bullet_k) \longrightarrow Aut N^\bullet_k \longrightarrow Aut N^\bullet_1 \longrightarrow 1. \tag{4.4.5}
\]

If we restrict the short exact sequence of Proposition (4.4.2) to \( IA^p(N^\bullet_{k+1}) \) we get the following short exact sequence

\[
0 \longrightarrow \text{Hom}(H_p, L^\bullet_{k+1}) \overset{i}{\longrightarrow} IA^p(N^\bullet_{k+1}) \overset{\psi^k}{\longrightarrow} IA^p(N^\bullet_k) \longrightarrow 1. \tag{4.4.6}
\]

Next we prove that the short exact sequence \( (4.4.6) \) is a central extension for \( \bullet = S, Z \). We first give some preliminary results.

**Definition 4.4.1** (complex commutator [12]). Let \( P_1, P_2, \ldots, P_r \) be any \( r \) elements of a group \( G \). We shall define by induction what we mean by a *complex commutator of weight \( w \) in the components* \( P_1, P_2, \ldots, P_r \). The complex commutators of weight 1 are the elements \( P_1, P_2, \ldots, P_r \) themselves. Supposing the complex commutators of all weights less than \( w \) have already been defined, then those of weight \( w \) consist of all the expressions of the form \([S, T]\), where \( S \) and \( T \) are any complex commutators of weights \( w_1 \) and \( w_2 \) in the components \( P_1, P_2, \ldots, P_r \) respectively, such that \( w_1 + w_2 = w \).

The weight of a complex commutator is, of course, always relative to the choice of components; these must be specified before the weight can be determined. For example, \([[P, Q], R]] \) is of weight 3 in the three components \( P, Q \) and \( R \); but of weight 2 in the two components \([P, Q]\) and \( R \).

**Theorem 4.4.1** (P. Hall, Theorem 3.2. in [12]). If \( p \) is a prime, \( \alpha \) is a positive integer, \( P \) and \( Q \) are any two elements of a group \( G \), and

\[ R_1, R_2, \ldots, R_i, \ldots \quad (R_1 = P, \ R_2 = Q) \]

are the various formally distinct complex commutators of \( P \) and \( Q \) arranged in order of increasing
weights, then integers \(n_1, n_2, \ldots, n_i, \ldots\) can be found \((n_1 = n_2 = p^\alpha)\) such that
\[
(PQ)^{p^\alpha} = R_1^{n_1}R_2^{n_2} \cdots R_i^{n_i} \cdots
\]
and if the weight \(w_i\) of \(R_i\) in \(P\) and \(Q\) satisfies the inequality \(w_i < p^\beta \leq p^\alpha\), then \(n_i\) is divisible by \(p^{\alpha - \beta + 1}\).

**Lemma 4.4.1** (Three Subgroups Lemma). Let \(A, B\) and \(C\) be subgroups of a group \(G\). If \(N \triangleleft G\) is normal subgroup such that \([A, [B, C]]\) and \([B, [C, A]]\) are contained in \(N\) then \([C, [A, B]]\) is also contained in \(N\).

**Definition 4.4.2.** (Holomorph group) Let \(G\) be a group. The Holomorph group of \(G\) is defined as the semidirect product
\[
\text{Hol}(G) = G \rtimes \text{Aut}(G),
\]
where the multiplication is given by \((g_1, f_1)(g_2, f_2) = (g_1f_1(g_2), f_1f_2)\).

**Remark 4.4.3.** Consider the natural inclusion homomorphisms
\[
\iota_1 : G \rightarrow G \rtimes \text{Aut}(G), \quad \iota_2 : \text{Aut}(G) \rightarrow G \rtimes \text{Aut}(G).
\]
\[
g \mapsto (g, \text{id}) \quad f \mapsto (1, f)
\]
Using these homomorphisms, we can view any subgroup of \(G\) and any subgroup of \(\text{Aut}(G)\) as subgroups of \(\text{Hol}(G)\), and any normal subgroup of \(G\) as a normal subgroup of \(\text{Hol}(G)\).

**Notation 4.4.1.** Let \(x \in G\), \(f \in \text{Aut}(G)\), \(H \triangleleft G\) and \(K \triangleleft \text{Aut}(G)\). Throughout this chapter we denote by \([f, x] \in G\) the element \([f, x] = f(x)x^{-1}\), and by \([K, H]\) the subgroup given by \([K, H] = \{[f, x] \in G \mid f \in K, x \in H\}\).

**Remark 4.4.4.** Given \(x \in G\) and \(f \in \text{Aut}(G)\). The notation \([f, x] = f(x)x^{-1}\) makes sense because computing the commutator \([\iota_2(f), \iota_1(x)] \in \text{Hol}(G)\) one gets that
\[
[\iota_2(f), \iota_1(x)] = [(1, f), (x, \text{id})] = (1, f)(x, \text{id})(1, f^{-1})(x^{-1}, \text{id}) =
= (f(x), f)(1, f^{-1})(x^{-1}, \text{id}) = (f(x), \text{id})(x^{-1}, \text{id}) =
= (f(x)x^{-1}, \text{id}) = \iota_1(f(x)x^{-1}),
\]
which corresponds, by the natural inclusion homomorphism \(\iota_1\), to \(f(x)x^{-1} \in G\).

**Lemma 4.4.2** (J. Cooper). Given a prime \(p\). If \(f \in \mathcal{I}_{g,1}(k)\) and \(x \in \Gamma^*_1\) then \(f(x)x^{-1} \in \Gamma^*_{k+l}\). Equivalently, \([\mathcal{I}_{g,1}(k), \Gamma^*_1] < \Gamma^*_{k+l}\).
Thus every element of $G$.

We begin with the case of the Stallings series. Recall that by definition, the Stallings series.

**Proof.**

Given a prime $p$. If $f \in IA^+_k(G)$ and $x \in G^+_l$, then $f(x)x^{-1} \in G^+_k$. Equivalently, $[IA^+_k(G), G^+_l] < G^+_k$.

**Remark 4.4.5.** In the proof of Lemma 4.4.2, J. Cooper asserts that the result for the case of the Zassenhaus filtration follows using the same argument that he used for the Stallings filtration. Reviewing his proof we found that it is not clear how to use the same argument to prove the statement for the Zassenhaus filtration. However, using the Hall-Petrescu identity, we will give an argument to get the result. Indeed, we realised that Lemma 4.4.2 is more general.

Next we will state a generalization of Lemma (4.4.2), whose proof follows the original one by J. Cooper.

**Definition 4.4.3.** Denote by $IA^+_k(G)$ the elements of $Aut(G)$ that act trivially on $G_k/k+1$, that is,

$$IA^+_k(G) = \{f \in Aut(G) \mid f(x)x^{-1} \in G^+_k \text{ for all } x \in G\}.$$  

Throughout this section we denote $IA^+_k(G)$ by $IA^p(G)$, with $\bullet = S$ or $Z$.

**Lemma 4.4.3.** Given a prime $p$. If $f \in IA^+_k(G)$ and $x \in G^+_l$, then $f(x)x^{-1} \in G^+_k$. Equivalently, $[IA^+_k(G), G^+_l] < G^+_k$.

**Proof.** We begin with the case of the Stallings series. Recall that by definition, $G^S_{l+1} = [G, G^S_l](G^S_l)^p$. Thus every element of $G^S_{l+1}$ is a product of elements of the form $[x, y] \in G^S_{l+1}$, $z^p \in G^S_{l+1}$, where $x \in G$ and $y, z \in G^S_l$. So first of all we will prove the statement for such elements and later for any product of them.

We proceed by induction on $l$. The base case $l = 1$ follows from the definition of $IA^S_k(G)$. Now assume that the lemma holds for $l$. We will prove that lemma holds for $l + 1$.

Consider elements of the form $[x, y] \in G^S_{l+1}$ and $z^p \in G^S_{l+1}$, where $x \in G$ and $y, z \in G^S_l$.

We first show that $f([x, y])[x, y]^{-1} \in G^S_{k+l+1}$ for $f \in IA^S_k(G)$. The main idea of this proof originally comes from \[2\]. First note that

$$f([x, y])[x, y]^{-1} = [f, [x, y]] \in [IA^S_k(G), [G, G^S_l]].$$

The idea is to apply the Three Subgroup Lemma for the subgroups $IA^S_k(G)$, $G^S_l$ of $Hol(G)$. Observe that by induction and the definition of the Stallings series,

$$[[IA^S_k(G), G^S_l], G] < [G^S_{k+l}, G] < G^S_{k+l+1},$$

$$[[IA^S_k(G), G^S_l], G^S_l] < [G^S_{k+l}, G^S_l] < G^S_{k+l+1}.$$  

Moreover, since $G^S_{k+l+1}$ is a normal subgroup of $G$, we can view $G^S_{k+l+1}$ as a normal subgroup of $Hol(G)$. Therefore the Three Subgroup Lemma implies that

$$[IA^S_k(G), [G, G^S_l]] < G^S_{k+l+1}.$$
Next we show that \( f(z^p)z^{-p} \in G_{k+l+1}^S \). First of all we prove that

\[
 f(z^p)z^{-p} \equiv (f(z)z^{-1})^p \mod G_{k+l+1}^S.
\]

Observe that the following equality holds

\[
 f(z^p)z^{-p} = [f, z^p] = [f, z][f, z]^2 \ldots [f, z]^{z_p-1}.
\]

Note that by induction, \([f, z] \in G_{k+l+1}^S\) and by normality, \([f, z]^{z^i} \in G_{k+l+1}^S\) for each \( i = 1, \ldots, p - 1 \). Furthermore, \([ [f, z], z^i ] \in G_{k+l+1}^S\) so that \([f, z] \equiv [f, z]^{z^i} \mod G_{k+l+1}^S\) and finally \( f(z^p)z^{-p} \equiv (f(z)z^{-1})^p \mod G_{k+l+1}^S\).

Now we note that by induction, \((f(z)z^{-1})^p \in (G_{k+l+1}^S)^p\), and by the properties of the Stallings series, \((G_{k+l}^S)^p \in G_{k+l+1}^S\). Therefore, \( f(z^p)z^{-p} \in G_{k+l+1}^S\).

Finally, we prove the statement for products of elements of \( G_{l+1}^S \). Let \( f \in IA_k^S(G) \) and \( \omega_i \in G_{l+1}^S \). Using the fact that \( f(\omega_i)\omega_i^{-1} \in G_{k+l+1}^S \) for all \( i \), we have:

\[
 f \left( \prod_{i=1}^{n} \omega_i \right) \left( \prod_{i=1}^{n} \omega_i \right)^{-1} = f(\omega_1) \cdots f(\omega_{n-1}) f(\omega_n) \omega_n^{-1} \omega_{n-1}^{-1} \cdots \omega_1 \\
 \equiv f(\omega_1) \cdots f(\omega_{n-1}) \omega_{n-1}^{-1} \cdots \omega_1^{-1} \mod G_{k+l+1}^S \\
 \equiv f(\omega_1) \cdots f(\omega_{n-2}) \omega_{n-2}^{-1} \cdots \omega_1^{-1} \mod G_{k+l+1}^S \\
 \equiv \cdots \equiv f(\omega_1) f(\omega_2) \omega_2^{-1} \omega_1^{-1} \equiv f(\omega_1) \omega_1 \equiv 1 \mod G_{k+l+1}^S.
\]

Therefore,

\[
 f \left( \prod_{i=1}^{n} \omega_i \right) \left( \prod_{i=1}^{n} \omega_i \right)^{-1} \in G_{k+l+1}^S.
\]

For the case of the Zassenhaus series, recall that by definition \( G_{l}^Z = \prod_{i=1}^{l} (G_i)^{p^i} \). Thus every element of \( G_l^Z \) is a product of elements of the form \( x_i^{p^i} \in G_l^Z \) with \( x_i \in G_i \) and \( ip^i \geq l \). So first of all we will prove the statement for such elements and later for any product of them.

Let \( f \in I_k^Z(G) \) and \( x_i^{p^i} \in G_l^Z \) with \( x_i \in G_i \), i.e. \( ip^i \geq l \). We want to prove that \( f(x_i^{p^i})x_i^{-p^i} \in G_l^Z \). Observe that if \( ip^i \geq l + 1 \) then \( x_i^{p^i} \in G_{l+1}^Z \) and as a consequence \( f(x_i^{p^i})x_i^{-p^i} \in G_{l+1}^Z \). Thus we can assume that \( ip^i = l \). Next, the idea is to prove the statement for the elements \( x_i \in G_l \), and use an inductive argument on \( j \) to prove the general result.

We first prove that \( f(x_i)x_i^{-1} \in G_{l+k}^Z \). By induction on \( l \). The base case \( x_1 \) follows from the definition of \( IA_k^Z(G) \). Now assume that the lemma holds for \( x_l \), and prove that the Lemma holds for \( x_{l+1} \). As in the case of Stallings, applying the Three Subgroup Lemma for the subgroups \( IA_k^Z(G), G, G_l^Z < Hol(G) \), we have that \([IA_k^Z(G), [G, G_l^Z]] < G_{k+l+1}^Z\).
Observe that
\[ f(x_i^{p_i})x_i^{-p_i} = f(x_i^{p_{i-1}})x_i^{-p_i}. \]
Let \( q = ip^{j-1} \), take \( x_q = x_i^{p_i} \). Observe that since \( x_q \in G_q \), by induction hypothesis, \( f(x_q)x_q^{-1} \in G_{q+k}^Z \).
Then there exists an element \( y_q+k \in G_{q+k}^Z \) such that \( f(x_q) = x_qy_{q+k} \).
Next we show that
\[ (x_qy_{q+k})^p \equiv x_q^p \pmod{G_{l+k}^Z}. \]

By Theorem 4.4.1, we have that
\[ (x_qy_{q+k})^p = R_1^{n_1}R_2^{n_2} \cdots R_i^{n_i} \]
where
\[ R_1, R_2, \ldots, R_i, \ldots \quad (R_1 = x_q, \quad R_2 = y_{q+k}) \]
are the various formally distinct complex commutators of \( x_q \) and \( y_{q+k} \), arranged in increasing weights order, and \( n_1, n_2, \ldots, n_i, \ldots \) positive integers such that \( n_1 = n_2 = p \) and if the weight \( w_i \) of \( R_i \) in \( x_q \) and \( y_{q+k} \) satisfies that \( w_i < p \) then \( n_i \) is divisible by \( p \).

Next we prove that \( R_i^{n_i} \in G_{l+k}^Z \) for \( i \geq 2 \).

- **For** \( i = 2 \), we know that \( R_2 = y_{q+k} \) and \( n_2 = p \). Since \( y_{q+k} \in G_{q+k}^Z \), we have that
  \[ y_q^p \in (G_q^Z)^p \leq G_{p(q+k)}^Z \leq G_{pq+k}^Z = G_{l+k}^Z. \]

- **For** \( i \geq 3 \), as \( R_i \) are complex commutators of weight \( w_i \) in the two components \( x_k, y_{q+k} \), we have that \( w_i \geq 2 \). As a consequence, at least one component of \( R_i \) has to be \( y_{q+k} \), because if it is not the case then \( R_i \) has to be \( 1 \).

If the weight \( w_i \) of \( R_i \) in \( x_k \) and \( y_{q+k} \) satisfies that \( 2 \leq w_i < p \), since at least one component of \( R_i \) has to be \( y_{q+k} \), we get that \( R_i \in G_{\omega(-1)q+(q+k)}^Z = G_{\omega q+k}^Z \). Moreover, in the case \( 2 \leq w_i < p \), we have that \( p \mid n_i \) and then
\[ R_i^{n_i} \in (G_{\omega(q+k)}^Z)^{n_i} \leq (G_{\omega(q+k)}^Z)^p \leq G_{p(q+k)}^Z \leq G_{pq+k}^Z = G_{l+k}^Z. \]

On the other hand, if the weight \( w_i \) of \( R_i \) in \( x_q \) and \( y_{q+k} \) satisfies that \( w_i \geq p \), since at least one component of \( R_i \) has to be \( y_{q+k} \), then \( R_i \in G_{\omega(-1)q+(q+k)}^Z = G_{\omega q+k}^Z \leq G_{pq+k}^Z \). As a consequence,
\[ R_i^{n_i} \in G_{pq+k}^Z. \]

Therefore, \( (x_qy_{q+k})^p \equiv x_q^p \pmod{G_{l+k}^Z} \), i.e.
\[ f(x_i^{p_j})x_i^{-p_j} \in G_{l+k}^Z. \]
Finally, we prove the statement for products. The same argument of the case of Stallings works here, giving that if \( f \in IA^S_k(G) \) and \( \omega_i \in G^Z_{l+k} \), then
\[
f \left( \prod_{i=1}^{n} \omega_i \right) \left( \prod_{i=1}^{n} \omega_i \right)^{-1} \in G^S_{l+k},
\]

As a direct consequence of Lemma \((4.4.3)\), using ideas of S. Andreadakis (see Theorem 1.1 in [2]), we get the following result.

**Corollary 4.4.2.** For any two elements \( \varphi \in IA^*_k(G) \) and \( \psi \in IA^*_l(G) \), the commutator \([\varphi, \psi]\) is contained in \( IA^*_{k+l}(G) \). Equivalently, \([IA^*_k(G), IA^*_l(G)] < IA^*_{k+l}(G)\).

**Proof.** Notice that \( G, IA^*_k(G), IA^*_l(G) \) are subgroups of \( Hol(G) \). By Lemma \((4.4.3)\),
\[
[IA^*_k(G), [IA^*_k(G), G]] < [IA^*_k(G), G^*_{k+1}] < G^*_{k+l+1},
\]
\[
[IA^*_l(G), [IA^*_l(G), G]] < [IA^*_l(G), G^*_{l+1}] < G^*_{k+l+1}.
\]

Then, by the Three Subgroups Lemma, \([G, [IA^*_k(G), IA^*_l(G)]] < G^*_{k+l+1} \). Hence,
\[
[[IA^*_k(G), IA^*_l(G)], G] < G^*_{k+l+1}.
\]

As a consequence, for every \( \gamma \in G, \varphi \in IA^*_k(G) \) and \( \psi \in IA^*_l(G) \), we have that
\[
[\varphi, \psi](\gamma) \gamma^{-1} \in G^*_{k+l+1},
\]
which, by definition, means that \([\varphi, \psi] \in IA^*_{k+l}(G)\), as desired. \( \square \)

Next, we show that, for a free group of finite rank, Lemma \((4.4.3)\) and Corollary \((4.4.2)\) give us the most efficient bound, in the sense that
\[
[IA^*_k(\Gamma), \Gamma^*_{l}] < \Gamma^*_{k+l}, \quad [IA^*_k(\Gamma), IA^*_l(\Gamma)] < IA^*_{k+l}(\Gamma),
\]
\[
[IA^*_k(\Gamma), \Gamma^*_l] \nsubseteq \Gamma^*_{k+l+1}, \quad [IA^*_k(\Gamma), IA^*_l(\Gamma)] \nsubseteq IA^*_{k+l+1}(\Gamma).
\]

**Proposition 4.4.4.** Let \( \Gamma \) be a free group of finite rank \( n > 1 \). Then
\[
i) \ [IA^*_k(\Gamma), \Gamma^*_l] \nsubseteq \Gamma^*_{k+l+1},
\]
\[
ii) \ [IA^*_k(\Gamma), IA^*_l(\Gamma)] \nsubseteq IA^*_{k+l+1}(\Gamma).
\]
4.4. Extensions of Automorphisms

Proof. Consider the free group $\Gamma = \langle x_1, \ldots, x_n \rangle$. Let $\gamma_1 = [x_1, [x_2, [x_1, [x_2, \ldots]]]]$, a commutator of length $k$ alternating $x_1$, $x_2$, and $\gamma_2 = [x_1, [x_2, [x_1, [x_2, \ldots]]]]$ a commutator of length $l$ alternating $x_1$, $x_2$.

i) Take $f \in IA_k^* (\Gamma)$ the inner automorphism given by the conjugation by $\gamma_1$. Notice that $f \in IA_k^* (\Gamma)$ because for every $x \in \Gamma$,

$$f(x) = \gamma_1 x \gamma_1^{-1} = [\gamma_1, x] x = x \pmod{\Gamma_{k+1}}.$$  

Then we have that $[f, \gamma_2] \in [IA_k^* (\Gamma), IA_l^* (\Gamma)]$ and

$$[f, \gamma_2] = f(\gamma_2) \gamma_2^{-1} = \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1} = [\gamma_1, \gamma_2] \in \Gamma_{k+l},$$

which does not belong to $\Gamma_{k+l+1} < \Gamma_{k+l+1} \Gamma^p$ because $[\gamma_1, \gamma_2]$ is neither a power $p$ element nor an element of $\Gamma_{k+l+1}$ because it can not be written as a product of at least $2(k+l) + 2$ generators.

ii) Take $\varphi \in IA_k^* (\Gamma)$, $\psi \in IA_l^* (\Gamma)$ the inner automorphisms respectively given by the conjugation by $\gamma_1$ and $\gamma_2$. Then we have that $[\varphi, \psi] \in [IA_k^* (\Gamma), IA_l^* (\Gamma)]$ and

$$[\varphi, \psi](x_1) x_1^{-1} = [\gamma_1, \gamma_2] x_1 [\gamma_1, \gamma_2]^{-1} x_1^{-1} = [[\gamma_1, \gamma_2], x_1] \in \Gamma_{k+l+1},$$

which, as in i), does not belong to $\Gamma_{k+l+2}$. Therefore, $[\varphi, \psi] \notin IA_{k+l+1}^* (\Gamma)$. □

As a direct consequence we have the following result:

**Corollary 4.4.3.** The group $[IA_k^* (N_n^*), (N_n^*)^*]$ with $l + k = n$ and $[IA_l^* (N_n^*), IA_l^*(N_n^*)]$ with $l + k = n - 1$, are not trivial.

**Proof.** We use the same notation of proof of (4.4.4). Take the $f \in IA_k^* (\Gamma)$ and $x \in \Gamma_l^*$ as in the proof of Proposition (4.4.4). These elements induce elements $\overline{f} \in IA_k^* (\Gamma)$ and $\overline{x} \in \Gamma_l^*$. Moreover, by Proposition (4.4.4), we have that

$$[\overline{f}, \overline{x}] = [\overline{f}, \overline{x}] = [\gamma_1, \gamma_2] \neq 1.$$  

Therefore there is an element in $[IA_k^* (N_n^*), (N_n^*)^*]$ which is not trivial.

Take $\varphi \in IA_k^* (\Gamma)$, $\psi \in IA_l^* (\Gamma)$, and $x_1 \in \Gamma$. These elements induce elements $\overline{\varphi} \in IA_k^* (N_n^*)$, $\overline{\psi} \in IA_l^* (N_n^*)$, and $\overline{x}_1 \in N_n^*$. Moreover, by Proposition (4.4.4), we have that

$$[[\overline{\varphi}, \overline{\psi}], \overline{x}_1] = [[\overline{\varphi}, \overline{\psi}], \overline{x}_1] = [[\gamma_1, \gamma_2], x_1] \neq 1.$$  

Therefore there is an element in $[IA_k^* (N_n^*), IA_l^* (N_n^*)]$ which is not trivial. □
Proposition 4.4.5. The natural action of $\text{Aut}(N^*_k)$ on $\text{Hom}(N^*_1, L^*_k)$ factors through $\text{Aut}(N^*_1)$.

Proof. In virtue of Proposition [4.4.3], the natural action of $\text{Aut}(N^*_k)$ on $\text{Hom}(N^*_1, L^*_k)$ is given by

$$
\text{Aut}(N^*_k) \times \text{Hom}(N^*_1, L^*_k) \rightarrow \text{Hom}(N^*_1, L^*_k)
$$

$$(h, f) \mapsto (x \mapsto h(f(h^{-1}x))),$$

where $h^{-1}x$ is the action of $h^{-1} \in \text{Aut}(N^*_k)$ on $x \in N^*_1$ via the surjection $\text{Aut} N^*_k \rightarrow \text{Aut} N^*_1$.

Moreover, by Proposition (4.3.3) and Lemma (4.4.3), we know that if $h \in IA^p(N^*_k)$ and $y \in L^*_k$, then $h(y) = y$. Therefore the action of $\text{Aut} N^*_k$ on $L^*_k$ factors through $\text{Aut}(N^*_1)$ via the surjection $\text{Aut} N^*_k \rightarrow \text{Aut} N^*_1$ and as a consequence we get the result.

As a direct consequence of Proposition (4.4.5), we have the following result:

Corollary 4.4.4. Let $\Gamma$ be a free group of finite rank $n > 1$. The extension

$$
0 \longrightarrow \text{Hom}(N^*_1, L^*_k) \xrightarrow{i} IA^p(N^*_k) \xrightarrow{\pi} IA^p(N^*_1) \longrightarrow 1
$$

is central.

Proposition 4.4.6. Let $\Gamma$ be a free group of finite rank $n > 1$. The extension

$$
0 \longrightarrow \text{Hom}(N^*_1, L^*_k) \xrightarrow{i} \text{Aut} N^*_k \xrightarrow{\psi^*} \text{Hom}(N^*_k) \longrightarrow 1
$$

is not central.

Proof. We provide a counterexample.

Consider $\Gamma = \{x_1, \ldots, x_n\}$ and $x = [x_1, [x_2, [x_1, [x_2, \ldots]]]]$ a commutator of length $k + 1$. Denote $[x_1]$ the class of $x_1$ in $N^*_1$, and $\overline{x}$ the class of $x$ in $L^*_k$. Observe that $N^*_1$ is a $\mathbb{F}_p$-vector space with basis $\{[x_1], \ldots, [x_n]\}$. Let $f \in \text{Hom}(N^*_1, L^*_k)$ be the homomorphism defined on the basis of $N^*_1$ by $f([x_1]) = \overline{x}$ and $f([x_i]) = 0$ for $1 < i \leq n$.

Consider $h = (x_1x_2) \in \mathcal{S}_n \subset \text{Aut} \Gamma$. Since $\Gamma^*_k$ is a characteristic subgroup of $\Gamma$, $h$ induces an element $\overline{h} \in \text{Aut} N^*_k$. Then we have that

$$
f([x_1]) = \overline{x} \quad \text{and} \quad \overline{h}(f((\overline{h})^{-1}[x_1])) = \overline{h}(f([x_2])) = \overline{h}(1) = 1.
$$

Therefore the extension is not central.
4.4. Extensions of Automorphims

4.4.2 Splitting the extensions

In this section, our goal is to study the splitability of the extension

\[ 0 \longrightarrow \text{Hom}(N_{k+1}^\bullet, L_{k+1}^\bullet) \overset{i}{\longrightarrow} \text{Aut} N_{k+1}^\bullet \overset{\psi}{\longrightarrow} \text{Aut} N_k^\bullet \longrightarrow 1. \]

We first consider the case \( k = 1 \). To make notations lighter, we will denote by \( H_p \) the group \( N_1^\mathbb{Z} = N_1^Z = \Gamma / \Gamma p \Gamma p \).

**Notation 4.4.2.** Let \( p \) be a prime number and \( n \geq 2 \) an integer, we denote by \( \text{gl}_n(\mathbb{Z}/p) \) the additive group of matrices \( n \times n \) with coefficients in \( \mathbb{Z}/p \), and by \( \text{sl}_n(\mathbb{Z}/p) \) the subgroup of \( \text{gl}_n(\mathbb{Z}/p) \) formed by the matrices of trace zero.

**Theorem 4.4.2** (Theorem 7 in [43]). Let \( p \) be a prime number and \( n \geq 2 \) an integer. The extension

\[ 0 \longrightarrow \text{sl}_n(\mathbb{Z}/p) \longrightarrow \text{SL}_n(\mathbb{Z}/p^2) \overset{r_p}{\longrightarrow} \text{SL}_n(\mathbb{Z}/p) \longrightarrow 1, \]

only splits for \((p, n) = (3, 2)\) and \((2, 3)\).

**Corollary 4.4.5.** Let \( p \) be a prime number and \( n \geq 2 \) an integer. The extension

\[ 0 \longrightarrow \text{gl}_n(\mathbb{Z}/p) \longrightarrow \text{GL}_n(\mathbb{Z}/p^2) \overset{r_p}{\longrightarrow} \text{GL}_n(\mathbb{Z}/p) \longrightarrow 1, \quad (4.4.7) \]

only splits for \((p, n) = (3, 2), (2, 2)\) and \((2, 3)\).

**Proof.** We first prove that the extension does not split for \((p, n) \neq (3, 2), (2, 2)\) and \((2, 3)\). Set

\[ \text{SL}_n^p(\mathbb{Z}/p^2) = \{ A \in \text{GL}_n(\mathbb{Z}/p^2) \mid \text{det}(A) \equiv 1 \pmod{p} \}. \]

Notice that we have a pull-back diagram

\[ 0 \longrightarrow \text{gl}_n(\mathbb{Z}/p) \longrightarrow \text{SL}_n^p(\mathbb{Z}/p^2) \overset{r_p}{\longrightarrow} \text{SL}_n(\mathbb{Z}/p) \longrightarrow 1 \]

\[ 0 \longrightarrow \text{gl}_n(\mathbb{Z}/p) \longrightarrow \text{GL}_n(\mathbb{Z}/p^2) \overset{r_p}{\longrightarrow} \text{GL}_n(\mathbb{Z}/p) \longrightarrow 1. \]

Then it is enough to show that the top extension in above diagram does not split.

Next notice that we have a push-out diagram

\[ 0 \longrightarrow \text{sl}_n(\mathbb{Z}/p) \longrightarrow \text{SL}_n(\mathbb{Z}/p^2) \overset{r_p}{\longrightarrow} \text{SL}_n(\mathbb{Z}/p) \longrightarrow 1 \]

\[ 0 \longrightarrow \text{gl}_n(\mathbb{Z}/p) \longrightarrow \text{SL}_n^p(\mathbb{Z}/p^2) \overset{r_p}{\longrightarrow} \text{SL}_n(\mathbb{Z}/p) \longrightarrow 1. \]

(4.4.8)
which induce a map \( i_* : H^2(SL_n(\mathbb{Z}/p); \mathfrak{sl}_n(\mathbb{Z}/p)) \rightarrow H^2(SL_n(\mathbb{Z}/p); \mathfrak{gl}_n(\mathbb{Z}/p)) \).

By Theorem (4.4.2) the top extension in above diagram does not split for \((p, n) \neq (3, 2), (2, 3)\).

Next we show that \( i_* \) is injective for \((p, n) \neq (2, 2), (3, 2)\), and as a consequence, we will get that the extension (4.4.7) does not split for \((p, n) \neq (3, 2), (2, 2), (2, 3)\).

Consider the short exact sequence

\[
0 \rightarrow \mathfrak{sl}_n(\mathbb{Z}/p) \rightarrow \mathfrak{gl}_n(\mathbb{Z}/p) \xrightarrow{\text{tr}} \mathbb{Z}/p \rightarrow 1,
\]

where \( \text{tr} \) is given by the matrix trace. The long chomology sequence for \( SL_n(\mathbb{Z}/p) \) with values in above short exact sequence, give us an exact sequence

\[
H^1(SL_n(\mathbb{Z}/p); \mathbb{Z}/p) \rightarrow H^2(SL_n(\mathbb{Z}/p); \mathfrak{sl}_n(\mathbb{Z}/p)) \xrightarrow{i_*} H^2(SL_n(\mathbb{Z}/p); \mathfrak{gl}_n(\mathbb{Z}/p)).
\]

By proof of Theorem 7 in [43], \( H^1(SL_n(\mathbb{Z}/p); \mathbb{Z}/p) = 0 \) for \((p, n) \neq (2, 2), (3, 2)\). As a consequence, \( i_* \) is injective for \((p, n) \neq (2, 2), (3, 2)\).

Next we prove that the extension (4.4.7) splits for \((p, n) = (3, 2), (2, 2) \) and \((2, 3)\).

By Proposition 4.5 in [44], we know that \( H^2(GL_2(\mathbb{Z}/p); \mathfrak{gl}_2(\mathbb{Z}/p)) = 0 \) for \( p = 2, 3 \). Therefore the extension (4.4.7) splits for \((p, n) = (3, 2) \) and \((2, 2)\).

For the case \((p, n) = (2, 3)\), consider the push-out diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{} & \mathfrak{sl}_3(\mathbb{Z}/2) \\
\downarrow \quad i & & \downarrow \quad r_2 \\
0 & \xrightarrow{} & SL_3(\mathbb{Z}/2) \quad \text{SL}_3(\mathbb{Z}/2) \rightarrow 1
\end{array}
\]

By Theorem (4.4.2) we know that the top extension of this commutative diagram splits. Therefore, the bottom extension in above diagram splits too. Notice that \( SL_3(\mathbb{Z}/2) = GL_3(\mathbb{Z}/2) \) and \( SL_3(\mathbb{Z}/2) = SL_3(\mathbb{Z}/2) \). Hence, the extension (4.4.7) splits for \((p, n) = (2, 3)\).

**Definition 4.4.4.** Let \( k \) be an arbitrary commutative ring, let \( V \) be a \( k \)-module, and let \( T^q(V) = V \otimes \cdots \otimes V \) (\( q \) copies of \( V \)), where \( \otimes = \otimes_k \). We denote by \( J^q(V) \) the quotient of \( T^q(V) \) by the \( k \)-submodule generated by the elements \( v_1 \otimes \cdots \otimes v_q \) such that \( v_i = v_{i+1} \) for some \( i \).

**Proposition 4.4.7.** Let \( p \) be a prime number and \( \Gamma \) a free group of rank \( n \). The extension

\[
0 \rightarrow \text{Hom}(N^\bullet_1, L^\bullet_2) \xrightarrow{i} \text{Aut} N^\bullet_2 \stackrel{\psi^\bullet_1}{\rightarrow} \text{Aut} N^\bullet_1 \rightarrow 1,
\]

only splits for \( \bullet = \mathbb{Z} \) with \( p \) odd and for \( \bullet = S \) with \( (p, n) = (3, 2), (2, 2), (2, 3) \).
Remark 4.4.6. Notice that for \( p = 2 \), the extensions (4.4.10) with \( \bullet = Z \), \( \bullet = S \) coincide, because in this case, \( \Gamma_i^S = \Gamma_i^Z \) for \( i = 1, 2, 3 \).

**Proof. For \( \bullet = Z \) and \( p \) an odd prime.** In this case, we have that

\[
L^Z_2 \cong \bigwedge^2 N_1^Z = \bigwedge^2 H_p, \quad \text{Aut } N_1^Z = \text{Aut } H_p \cong GL_n(\mathbb{Z}/p).
\]

Then, the extension (4.4.10) becomes

\[
0 \longrightarrow \text{Hom}(H_p, \bigwedge^2 H_p) \xrightarrow{i} \text{Aut } N_2^Z \xrightarrow{\psi^Z_1} GL_n(\mathbb{Z}/p) \longrightarrow 1.
\]

Notice that \(-Id\) is an element of the center of \( GL_n(\mathbb{Z}/p) \), which acts on \( \text{Hom}(H_p, \bigwedge^2 H_p) \) by the multiplication of \(-1\). Then, by the Center kills Lemma,

\[
H_2(GL_n(\mathbb{Z}/p); \text{Hom}(H_p, \bigwedge^2 H_p)) = 0.
\]

As a consequence, by Lemma [1.1.2], we have that

\[
H^2(GL_n(\mathbb{Z}/p); \text{Hom}(H_p, \bigwedge^2 H_p)) \cong H^2(GL_n(\mathbb{Z}/p); \text{Hom}(H_p, \bigwedge^2 H_p)^*) \cong (H^2(GL_n(\mathbb{Z}/p); \text{Hom}(H_p, \bigwedge^2 H_p))^*)^* = 0.
\]

Therefore the extension (4.4.10) splits.

For \( \bullet = S \) and \( p \) prime with \((p, n) \neq (3, 2), (2, 2), (2, 3)\). In this case, we have that

\[
\text{Aut } N_1^S = \text{Aut } H_p \cong GL_n(\mathbb{Z}/p).
\]

Then, the extension (4.4.10) becomes

\[
0 \longrightarrow \text{Hom}(H_p, L^S_2) \xrightarrow{i} \text{Aut } N_2^S \xrightarrow{\psi^S_1} GL_n(\mathbb{Z}/p) \longrightarrow 1.
\]

Set \( \overline{N}_2^S = \frac{\Gamma_2}{\gamma_2 \Gamma_2}, \overline{L}_2^S = \frac{L^S_2}{\gamma_2 \Gamma_2} \). Observe that \( \frac{\Gamma_2}{\gamma_2 \Gamma_2} \) is a characteristic group of \( \overline{N}_2^S \). Then there is a well defined homomorphism \( \overline{q} : \text{Aut } N_2^S \rightarrow \text{Aut } \overline{N}_2^S \) and there is a push-out diagram

\[
\begin{array}{c}
0 \longrightarrow \text{Hom}(H_p, \overline{L}_2^S) \xrightarrow{i} \text{Aut } \overline{N}_2^S \xrightarrow{\psi^S_1} GL_n(\mathbb{Z}/p) \longrightarrow 1 \\
\downarrow \overline{q} \quad \overline{q} \\
0 \longrightarrow \text{Hom}(H_p, \overline{L}_2^S) \xrightarrow{i} \text{Aut } \overline{N}_2^S \xrightarrow{\psi^S_1} GL_n(\mathbb{Z}/p) \longrightarrow 1,
\end{array}
\]

\( \text{105} \)
where \( q, \overline{q} \) are induced by the quotient map respect to \( \frac{\Gamma_2}{\Gamma_2 \Gamma_2} \). Next, notice that

\[
\text{Hom}(H_p, L^S_2) \cong \mathfrak{gl}_n(\mathbb{Z}/p), \quad \text{Aut} \ N^S_2 \cong GL_n(\mathbb{Z}/p^2).
\]

Thus, the bottom row of diagram (4.4.12) becomes

\[
0 \rightarrow \mathfrak{gl}_n(\mathbb{Z}/p) \rightarrow GL_n(\mathbb{Z}/p^2) \rightarrow GL_n(\mathbb{Z}/p) \rightarrow 1.
\]

By Corollary (4.4.5), the above extension does not split for \((p, n) \neq (3, 2), (2, 2), (3, 2)\). Then, by push-out diagram (4.4.12), we have that the extension (4.4.11) does not split for \((p, n) \neq (3, 2), (2, 2), (2, 3)\).

For \( \bullet = S \) with \((p, n) = (3, 2), (2, 2)\). In this case the extension (4.4.10) becomes

\[
0 \rightarrow \text{Hom}(H_p, L^S_2) \rightarrow \text{Aut} N^S_2 \rightarrow GL_2(\mathbb{Z}/p) \rightarrow 1. \quad (4.4.13)
\]

By Universal coefficients Theorem and Hopf formula we have that

\[
L^S_2 = H_2(N^S_1, \mathbb{Z}/p) \cong H_2(N^S_1) \otimes \mathbb{Z}/p \oplus \text{Tor}(H_1(N^S_1); \mathbb{Z}/p) \cong \wedge^2 H_p \oplus H_p,
\]

as \( GL_2(\mathbb{Z}/p) \)-modules. As a consequence, we have an isomorphism

\[
\text{Hom}(H_p, L^S_2) \cong \text{Hom}(H_p, \wedge^2 H_p) \oplus \text{Hom}(H_p, H_p),
\]

also as \( GL_2(\mathbb{Z}/p) \)-modules. Then we have the following isomorphism in cohomology:

\[
H^2(GL_2(\mathbb{Z}/p); \text{Hom}(H_p, L^S_2)) \cong H^2(GL_2(\mathbb{Z}/p); \text{Hom}(H_p, H_p)) \oplus H^2(GL_2(\mathbb{Z}/p); \text{Hom}(H_p, \wedge^2 H_p)).
\]

By Proposition 4.5 in [44], we know that \( H^2(GL_2(\mathbb{Z}/p); \text{Hom}(H_p, H_p)) = 0 \) for \( p = 2, 3 \).

For \((p, n) = (3, 2)\), by the Center kills lemma, we have that

\[
H^2(GL_2(\mathbb{Z}/3); \text{Hom}(H_3, \wedge^2 H_3)) = 0.
\]

For \((p, n) = (2, 2)\), the groups \( Sp_2(\mathbb{Z}/2) \), \( SL_2(\mathbb{Z}/2) \) coincide and so we have an isomorphism \( \wedge^2 H_2 \cong \mathbb{Z}/2 \) of \( SL_2(\mathbb{Z}/2) \)-modules given by the symplectic intersection form \( \omega \), which induces the following isomorphisms of \( SL_2(\mathbb{Z}/2) \)-modules

\[
\text{Hom}(H_2, \wedge^2 H_2) \cong \text{Hom}(H_2, \mathbb{Z}/2) \cong H_2.
\]

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Moreover, by Proposition 4.4 in [44], we know that \( H^2(SL_2(\mathbb{Z}/2); H_2) = 0. \)

Therefore \( H^2(GL_2(\mathbb{Z}/p); Hom(H_p, L^S_2)) = 0 \) for \( p = 2, 3 \) and so the extension (4.4.13) splits for \((p, n) = (3, 2), (2, 2)\).

For \( \bullet = S \) with \((p, n) = (2, 3)\). Analogously to the above case, the isomorphism

\[ q_1 \oplus q_2 : Hom(H_2, L^S_2) \to Hom(H_2, \bigwedge^2 H_2) \oplus Hom(H_2, H_2) \]

induces the following isomorphism in cohomology:

\[ H^2(GL_3(\mathbb{Z}/2); Hom(H_2, L^S_2)) \cong H^2(GL_3(\mathbb{Z}/2); Hom(H_2, H_2)) \oplus H^2(GL_3(\mathbb{Z}/2); Hom(H_2, \bigwedge^2 H_2)). \]

(4.4.14)

Notice that we have the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & Hom(H_2, gl_3(\mathbb{Z}/2)) & \overset{i}{\longrightarrow} & GL_3(\mathbb{Z}/4) & \longrightarrow & GL_3(\mathbb{Z}/2) & \longrightarrow & 1 \\
& & q_2 & \downarrow & \bar{q}_2 & & & & \\
0 & \longrightarrow & Hom(H_2, L^S_2) & \overset{i}{\longrightarrow} & Aut N^S_2 & \longrightarrow & GL_3(\mathbb{Z}/2) & \longrightarrow & 1 \\
& & q_1 & \downarrow & \bar{q}_1 & & & & \\
0 & \longrightarrow & Hom(H_2, \bigwedge^2 H_2) & \overset{i}{\longrightarrow} & Aut \Gamma \bigwedge^2 \Gamma & \longrightarrow & GL_3(\mathbb{Z}/2) & \longrightarrow & 1.
\end{array}
\]

(4.4.15)

By Corollary (4.4.5) the bottom extension in diagram (4.4.15) splits. Then, by the isomorphism (4.4.14), the middle extension in diagram (4.4.15) splits if and only if the bottom extension in diagram (4.4.15) splits.

Next, observe that we have the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & Hom(H_4, \bigwedge^2 H_4) & \overset{i}{\longrightarrow} & Aut \Gamma \bigwedge^2 \Gamma & \longrightarrow & GL_3(\mathbb{Z}/4) & \longrightarrow & 1 \\
& & \downarrow q & & \downarrow \bar{q} & & & & \\
0 & \longrightarrow & Hom(H_2, \bigwedge^2 H_2) & \overset{i}{\longrightarrow} & Aut \Gamma \bigwedge^2 \Gamma & \longrightarrow & GL_3(\mathbb{Z}/2) & \longrightarrow & 1.
\end{array}
\]

(4.4.16)

By the Center kills lemma, the top extension in the above diagram splits. and, by Corollary (4.4.5), the homomorphism \( GL_3(\mathbb{Z}/4) \to GL_3(\mathbb{Z}/2) \) also splits. As a consequence, the bottom extension in the above commutative diagram splits too, which implies the result.
Proposition 4.4.8. Let $\Gamma$ be a free group of finite rank $n > 1$. The extension

$$0 \longrightarrow \text{Hom}(N_1^*, L_{k+1}) \overset{i}{\longrightarrow} IA^p(N_{k+1}^*) \overset{\psi_k^*}{\longrightarrow} IA^p(N_k^*) \longrightarrow 1,$$

does not split for $k \geq 2$.

Proof. Applying Theorem (1.1.4) to the central extension

$$0 \longrightarrow \Hom(N_1^*, L_{k+1}) \overset{i}{\longrightarrow} IA^p(N_{k+1}^*) \overset{\psi_k^*}{\longrightarrow} IA^p(N_k^*) \longrightarrow 1,$$  \hspace{1cm} (4.4.17)

one gets the exact sequence

$$\begin{align*}
&\Hom(IA^p(N_1^*); \Hom(N_1^*, L_{k+1})) \overset{\text{res}}{\longrightarrow} \Hom(\Hom(N_1^*, L_{k+1}), \Hom(N_1^*, L_{k+1})) \overset{\delta}{\longrightarrow}
\end{align*}$$  \hspace{1cm} (4.4.18)

Then the cohomology class associated to the extension (4.4.17) is given by $\delta(id)$.

Suppose that the central extension (4.4.17) splits, i.e. $\delta(id) = 0$. Since (4.4.18) is exact, there would be an element $f \in \Hom(IA^p(N_1^*); \Hom(N_1^*, L_{k+1}))$ such that $\text{res}(f) = id$.

Notice that for every element $x \in [IA^p(N_{k+1}^*), IA^p(N_k^*)]$, one has that $f(x) = 0$ because $\Hom(N_1^*, L_{k+1})$ is abelian. On the other hand, by Corollary (4.4.3),

$$[IA^p(N_{k+1}^*), IA^p_{k-1}(N_{k+1}^*)] \neq 1.$$

In addition, by Corollary (4.4.2),

$$\psi_k^*([IA^p(N_{k+1}^*), IA^p_{k-1}(N_{k+1}^*)]) = [IA^p(N_k^*), IA^p_{k-1}(N_k^*)] \leq IA^p_k(N_k^*) = 1.$$

Then, by short exact sequence (4.4.6), $[IA^p(N_{k+1}^*), IA^p_{k-1}(N_{k+1}^*)] \hookrightarrow \Hom(N_1^*, L_{k+1})$.

As a consequence, $\text{res}(f)$ can not be the identity. \hfill $\square$

Corollary 4.4.6. Let $\Gamma$ be a free group of finite rank $n > 1$. The extension

$$0 \longrightarrow \Hom(N_1^*, L_{k+1}) \overset{i}{\longrightarrow} \text{Aut} N_{k+1}^* \overset{\psi_k^*}{\longrightarrow} \text{Aut} N_k^* \longrightarrow 1$$

does not split for $k \geq 2$.  

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Proof. Observe that we have the following pull-back diagram:

\[
\begin{array}{ccc}
0 & \overset{0}{\longrightarrow} & \text{Hom}(N^\bullet_1, L^\bullet_{k+1}) \overset{i}{\longrightarrow} \text{IA}^p(N^\bullet_{k+1}) \overset{\psi^\bullet_k}{\longrightarrow} \text{IA}^p(N^\bullet_k) \longrightarrow 1 \\
\downarrow & & \downarrow \\
0 & \overset{0}{\longrightarrow} & \text{Hom}(N^\bullet_1, L^\bullet_{k+1}) \overset{i}{\longrightarrow} \text{Aut} N^\bullet_{k+1} \overset{\psi^\bullet_k}{\longrightarrow} \text{Aut} N^\bullet_k \longrightarrow 1.
\end{array}
\]

By Proposition (4.4.8), the top extension of this diagram does not split. Therefore the bottom extension this diagram does not split. \(\Box\)
CHAPTER 5

Trivial cocycles and invariants on the (mod d) Torelli group

In this chapter, following the same strategy of Chapter 2, we give a tool to construct invariants of $\mathbb{Q}$-homology spheres that have a Heegaard splitting with gluing map an element of the (mod d) Torelli group, from a family of trivial 2-cocycles on the (mod d) Torelli group. As in Chapter 2, such a construction does not give a unique invariant. This is because there exists an invariant, which is not an extension of the Rohlin invariant, that makes to fail the unicity of the construction.

As a starting point, in the first section we give some results about a mod $p$ version of the Johnson homomorphism. In the second and third sections we exhibit the construction of invariants from a family of trivial 2-cocycles on the (mod d) Torelli group. Finally, in the last section we give a way to construct a family of 2-cocycles satisfying the hypothesis of our construction as a pull-back of bilinear forms along the mod $p$ Johnson homomorphisms.

5.1 Johnson homomorphisms mod p

Let $\Gamma$ be the fundamental group of $\Sigma_{g,1}$ and $\Gamma_k^*$ the $k$-term of one of the mod-$p$ central series defined in Section 4.1. The natural action of the mapping class group $\mathcal{M}_{g,1}$ on $\Gamma$ induces one on the $p$-group $N_k^* = \Gamma/\Gamma_{k+1}^*$. Therefore we have a representation

$$\rho_k^*: \mathcal{M}_{g,1} \to Aut N_k^*.$$

Denote by $\mathcal{I}_{g,1}^* (k) \subset \mathcal{M}_{g,1}$ the kernel of $\rho_k^*: \mathcal{M}_{g,1} \to Aut N_k^*$. For instance,

$$\mathcal{I}_{g,1}^S (1) = \mathcal{I}_{g,1}^Z (1) = \mathcal{M}_{g,1}[p].$$
In [4] J. Cooper defined the following maps

\[ \tau_k^\bullet : \mathcal{T}_{g,1}(k) \rightarrow \text{Hom}(H_p, \mathcal{L}_{k+1}^\bullet) \]
\[ f \mapsto (x \mapsto f(x)x^{-1}) \]

and proved that these are well defined homomorphisms. These homomorphisms are called the Zassenhaus mod-p Johnson homomorphisms for \( \bullet = Z \), and Stallings mod-p Johnson homomorphisms for \( \bullet = S \). In general, for \( \bullet = Z \) or \( S \), we call these maps the mod-p Johnson homomorphisms.

In fact, we show that these homomorphisms arise in a more natural way.

**Proposition 5.1.1.** The mod-p Johnson homomorphisms \( \tau_k^\bullet \) are the restriction of \( \rho_{k+1}^\bullet \) to \( \mathcal{T}_{g,1}^\bullet \).

**Proof.** Consider the following diagram:

\[
\begin{array}{c}
0 \longrightarrow \mathcal{T}_{g,1}(k) \longrightarrow \mathcal{M}_{g,1} \xrightarrow{\rho_k^\bullet} \text{Aut} N_k^\bullet \longrightarrow 1 \\
\downarrow \tau_k^\bullet \quad \quad \quad \downarrow \rho_{k+1}^\bullet \\
0 \longrightarrow \text{Hom}(H_p, \mathcal{L}_{k+1}^\bullet) \xrightarrow{i} \text{Aut} N_{k+1}^\bullet \xrightarrow{\pi} \text{Aut} N_k^\bullet \longrightarrow 1
\end{array}
\]

If \( f \in \mathcal{T}_{g,1}(k) \), a trivial computation show that show that \( i(\tau_k^\bullet(f)) = \rho_{k+1}^\bullet(f) \).

**Lemma 5.1.1** (Lemma 5.10 in [4]). Let \( \gamma \in \Sigma_{g,1} \) be a simple closed curve and \( y \in \Gamma \) be a based curve isotopic to \( \gamma \). Let \( x \in \Gamma \) with \( |x \cap y| = n \). Then

\[ T^n_\gamma(x)x^{-1} = \prod_{i=1}^n z_i^{\epsilon(p_i)p_i}z_i^{-1} \]

for some \( z_i \in \Gamma \) (where \( \epsilon(p_i) = \pm 1 \) depending on the sign of the intersection between \( x \) and \( y \) at \( p_i \)).

As a consequence of this result, following the proof of Lemma 5.11 in [4], we have the following proposition:

**Proposition 5.1.2.** Let \( p \) be an odd prime number and \( D_{g,1}[p] \) the subgroup of \( \mathcal{M}_{g,1}[p] \) generated by \( p \)-powers of Dehn twists. Then \( D_{g,1}[p] \subset \mathcal{I}_{g,1}^Z(p-1) \), i.e. \( D_{g,1}[p] \) is in the kernel of \( \tau_k^Z \) for every \( k \leq p-1 \).

**Proof.** We proceed by induction. We know that \( D_{g,1}[p] \subset \mathcal{I}_{g,1}(1) = \mathcal{M}_{g,1}[p] \). Now suppose that \( D_{g,1}[p] \subset \mathcal{I}_{g,1}^Z(k) \) for \( k \leq p-1 \) and we will prove that \( D_{g,1}[p] \subset \mathcal{I}_{g,1}^Z(k+1) \). Recall that by the properties of Zassenhaus filtration, we know that \( [\Gamma^Z_k, \Gamma_l] < \Gamma^Z_{k+l} \) and observe that by definition of \( \Gamma^Z_l \), for \( l \leq p \) we have that \( \Gamma^p \subset \Gamma^Z_l \), so we obtain that \( [\Gamma^p, \Gamma] < \Gamma^Z_l \) for every \( l \leq p \). Then by definition
of $\tau^Z_k$, we know that given an element $f \in I^Z_{g,1}(k)$,

$$\tau^Z_k(f) = (x \mapsto f(x)x^{-1} \mod \Gamma^Z_{k+1})$$

Taking $f = T^p_1$, since $[\Gamma^p, \Gamma] \leq \Gamma^Z_{k+1}$ and $\Gamma^p \leq \Gamma^Z_{k+1}$, we have that

$$f(x)x^{-1} = \prod_{i=1}^n z_i^y(x_i)^{p_i} z_i^{-1} \equiv y^p \equiv 0 \mod \Gamma^Z_{k+1},$$

where $k = \sum_{i=1}^n e(p_i)$. Thus $f \in \text{Ker}(\tau^Z_k) \subset \text{Ker}(\rho^Z_{k+1})$, i.e. $f \in I^Z_{g,1}(k + 1)$. □

**Remark 5.1.1.** Observe that $I^1_{g,1}(k) = \text{Ker}(\rho^1_k)$ and thus $I^1_{g,1}(k + 1) \subset I^1_{g,1}(k)$. So $\text{Ker}(\rho^1_{k+1})$ is equal to the kernel of $\rho^1_{k+1}$ restricted to $I^1_{g,1}(k)$, but by definition of $\tau^Z_k$, we know that this is the kernel of $\tau^Z_k$. Therefore $\text{Ker}(\rho^1_{k+1}) = \text{Ker}(\tau^1_k)$.

## 5.2 From invariants to trivial cocycles

Let $A$ be an abelian group. Denote by $A_d$ the subgroup formed by $d$-torsion elements.

Consider an $A$-valuated invariant of $\mathbb{Z}/d$-homology spheres

$$F : S^3[d] \to A.$$ Precomposing with the canonical maps $M_{g,1}[d] \to \lim_{g \to \infty} M_{g,1}[d]/\sim \to S^3[d]$ we get a family of maps $\{F_g\}_g$ with $F_g : M_{g,1}[d] \to A$ satisfying the following properties:

i) $F_{g+1}(x) = F_g(x)$ for every $x \in M_{g,1}[d]$,

ii) $F_g(\xi_a, \xi_b) = F_g(x)$ for every $x \in M_{g,1}[d]$, $\xi_a \in A_{g,1}[d]$, $\xi_b \in B_{g,1}[d]$,

iii) $F_g(\phi x \phi^{-1}) = F_g(x)$ for every $x \in M_{g,1}[d]$, $\phi \in A B_{g,1}$.

We avoid the peculiarities of the first Torelli groups by restricting ourselves to $g \geq 3$. If we consider the associated trivial 2-cocycles $\{C_g\}_g$, i.e.

$$C_g : M_{g,1}[d] \times M_{g,1}[d] \to A,$$

$$(\phi, \psi) \mapsto F_g(\phi) + F_g(\psi) - F_g(\phi \psi),$$

then the family of 2-cocycles $\{C_g\}_g$ inherits the following properties:

(1) The 2-cocycles $\{C_g\}_g$ are compatible with the stabilization map,

(2) The 2-cocycles $\{C_g\}_g$ are invariant under conjugation by elements in $A B_{g,1}$,
3) If $\phi \in A_{g,1}[d]$ or $\psi \in B_{g,1}[d]$ then $C_g(\phi, \psi) = 0$.

In general there are many families of maps $\{F_g\}_g$ satisfying the properties i) - iii) that induce the same family of trivial 2-cocycles $\{C_g\}_g$. Next, we concern about the number of such families.

Notice that given two families of maps $\{F_g\}_g$, $\{F'_g\}_g$ satisfying the properties i) - iii), we have that $\{F_g - F'_g\}_g$ is a family of homomorphisms satisfying the same conditions. As a consequence, the number of families $\{F_g\}_g$ satisfying the properties i) - iii) that induce the same family of trivial 2-cocycles $\{C_g\}_g$, coincides with the number of homomorphisms in $Hom(M_{g,1}[p], A)^{AB_{g,1}}$ compatible with the stabilization map, and that are trivial over $A_{g,1}[p]$ and $B_{g,1}[p]$. We devote the rest of this section to compute and study such homomorphisms.

**Proposition 5.2.1.** Let $A$ be an abelian group. For $g \geq 3$, $d \geq 3$ an odd integer and for $g \geq 5$, $d \geq 2$ an even integer such that $4 \nmid d$, the pull-back along $\Psi : M_{g,1}[d] \rightarrow Sp_{2g}(\mathbb{Z}, d)$ induces an isomorphism

$$Hom(M_{g,1}[d], A)^{AB_{g,1}} \cong Hom(Sp_{2g}(\mathbb{Z}, d), A)^{GL_g(\mathbb{Z})}.$$ 

**Proof.** Restricting the symplectic representation of the mapping class group to $M_{g,1}[d]$ we have a short exact sequence

$$1 \rightarrow \mathcal{T}_{g,1} \rightarrow M_{g,1}[d] \xrightarrow{\Psi} Sp_{2g}(\mathbb{Z}, d) \rightarrow 1.$$ 

Applying the 5-term sequence associated to the above short exact sequence, we get an exact sequence

$$H_1(\mathcal{T}_{g,1}; \mathbb{Z})Sp_{2g}(\mathbb{Z}, d) \xrightarrow{j} H_1(M_{g,1}[d]; \mathbb{Z}) \rightarrow H_1(Sp_{2g}(\mathbb{Z}, d); \mathbb{Z}) \rightarrow 1.$$ 

In Lemma 6.4. in [40], A. Putman proved that the homomorphism $j$ in (5.2.2) is injective for $g \geq 3$, $d \geq 3$ an odd integer and that for $g \geq 5$, $d \geq 2$ an even integer with $4 \nmid d$, the kernel of $j$ in (5.2.2) is $\mathbb{Z}/2$. Next we distinguish both cases.

**For $g \geq 3$ and $d \geq 3$ an odd integer,** we have the following commutative diagram of exact sequences:

$$
\begin{array}{ccc}
1 & \rightarrow & \mathcal{T}_{g,1} \\
\downarrow & & \downarrow i \\
1 & \rightarrow & M_{g,1}[d]
\end{array}
\quad
\begin{array}{ccc}
1 & \rightarrow & \mathcal{T}_{g,1} \\
\downarrow & & \downarrow i \\
1 & \rightarrow & M_{g,1}[d]
\end{array}
\quad
\begin{array}{ccc}
\Psi & \rightarrow & Sp_{2g}(\mathbb{Z}, d) \\
\downarrow & & \downarrow \\
\Psi & \rightarrow & Sp_{2g}(\mathbb{Z}, d) \\
\downarrow & & \downarrow \\
1 & \rightarrow & 1
\end{array}
\quad
\begin{array}{ccc}
\Psi & \rightarrow & Sp_{2g}(\mathbb{Z}, d) \\
\downarrow & & \downarrow \\
\Psi & \rightarrow & Sp_{2g}(\mathbb{Z}, d) \\
\downarrow & & \downarrow \\
1 & \rightarrow & 1
\end{array}
\quad
\begin{array}{ccc}
H_1(\mathcal{T}_{g,1}; \mathbb{Z})Sp_{2g}(\mathbb{Z}, d) & \xrightarrow{j} & H_1(M_{g,1}[d]; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H_1(\mathcal{T}_{g,1}; \mathbb{Z})Sp_{2g}(\mathbb{Z}, d) & \xrightarrow{j} & H_1(M_{g,1}[d]; \mathbb{Z}) \\
\downarrow & & \downarrow \\
1 & \rightarrow & 1
\end{array}
\quad
\begin{array}{ccc}
H_1(\mathcal{T}_{g,1}; \mathbb{Z})Sp_{2g}(\mathbb{Z}, d) & \xrightarrow{j} & H_1(M_{g,1}[d]; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H_1(\mathcal{T}_{g,1}; \mathbb{Z})Sp_{2g}(\mathbb{Z}, d) & \xrightarrow{j} & H_1(M_{g,1}[d]; \mathbb{Z}) \\
\downarrow & & \downarrow \\
1 & \rightarrow & 1
\end{array}
\quad
\begin{array}{ccc}
H_1(\mathcal{T}_{g,1}; \mathbb{Z})Sp_{2g}(\mathbb{Z}, d) & \xrightarrow{j} & H_1(M_{g,1}[d]; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H_1(\mathcal{T}_{g,1}; \mathbb{Z})Sp_{2g}(\mathbb{Z}, d) & \xrightarrow{j} & H_1(M_{g,1}[d]; \mathbb{Z}) \\
\downarrow & & \downarrow \\
1 & \rightarrow & 1
\end{array}
\quad
\begin{array}{ccc}
H_1(\mathcal{T}_{g,1}; \mathbb{Z})Sp_{2g}(\mathbb{Z}, d) & \xrightarrow{j} & H_1(M_{g,1}[d]; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H_1(\mathcal{T}_{g,1}; \mathbb{Z})Sp_{2g}(\mathbb{Z}, d) & \xrightarrow{j} & H_1(M_{g,1}[d]; \mathbb{Z}) \\
\downarrow & & \downarrow \\
1 & \rightarrow & 1
\end{array}
$$

By Proposition 6.6 in [40], we have an isomorphism $H_1(\mathcal{T}_{g,1}; \mathbb{Z})Sp_{2g}(\mathbb{Z}, d) \cong \wedge^3 H_d$, induced by the
first Zassenhaus mod- \( d \) Johnson homomorphism \( \tau^Z_1 \). Then we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{T}_{g,1} & \xrightarrow{i} & \mathcal{M}_{g,1}[d] \\
\downarrow{\tau^Z_1} & & \downarrow{H_1} \\
\wedge^3 H_d & \xrightarrow{j} & H_1(\mathcal{M}_{g,1}[d]; \mathbb{Z}).
\end{array}
\] (5.2.3)

Consider an element \( f \in Hom(\mathcal{M}_{g,1}[d], A)^{AB_{g,1}} \). Since \( A \) is an abelian group, by commutativity of diagram [5.2.3], the restriction of \( f \) to \( \mathcal{T}_{g,1} \) factors through \( \tau^Z_1 \). Since \( \tau^Z_1 \) is an \( \mathcal{M}_{g,1} \)-equivariant epimorphism, we have that the restriction of \( f \) to \( \mathcal{T}_{g,1} \) is a pull-back of an element of \( Hom(\wedge^3 H_d, A)^{GL_g(\mathbb{Z})} \) along \( \tau^Z_1 \). But, by Lemma [2.3.3], \( Hom(\wedge^3 H_d, A)^{GL_g(\mathbb{Z})} = 0 \).

As a consequence, the restriction of \( f \) to \( \mathcal{T}_{g,1} \), is zero and therefore \( f \) factors through the symplectic representation, i.e. we have an isomorphism

\[
Hom(\mathcal{M}_{g,1}[d], A)^{AB_{g,1}} \cong Hom(Sp_{2g}(\mathbb{Z}, d), A)^{GL_g(\mathbb{Z})}.
\]

For \( g \geq 5 \) and \( d \geq 2 \) an even integer such that \( 4 \nmid d \), we have the following commutative diagrams of exact sequences:

\[
\begin{array}{cccccc}
1 & \rightarrow & \mathcal{T}_{g,1} & \rightarrow & \mathcal{M}_{g,1}[d] & \rightarrow & \psi Sp_{2g}(\mathbb{Z}, d) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \mathbb{Z}/2 & \rightarrow & H_1(\mathcal{T}_{g,1}; \mathbb{Z})_{Sp_{2g}(\mathbb{Z}, d)} & \rightarrow & \psi H_1(\mathcal{M}_{g,1}[d]; \mathbb{Z}) & \rightarrow & H_1(\mathcal{M}_{g,1}[d]; \mathbb{Z}) & \rightarrow & 1, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \mathcal{K}_{g,1} & \rightarrow & \mathcal{T}_{g,1} & \rightarrow & \wedge^3 H & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \mathbb{B}_2 & \rightarrow & \mathcal{H}_1(\mathcal{T}_{g,1}; \mathbb{Z})_{Sp_{2g}(\mathbb{Z}, d)} & \rightarrow & \wedge^3 H_d & \rightarrow & 1.
\end{array}
\]

Reassembling these diagrams we get another commutative diagram

\[
\begin{array}{ccc}
\mathcal{K}_{g,1} & \rightarrow & \mathcal{T}_{g,1} \\
\downarrow& & \downarrow \\
\mathbb{B}_2 & \xrightarrow{i} & H_1(\mathcal{T}_{g,1}; \mathbb{Z})_{Sp_{2g}(\mathbb{Z}, d)} \\
\downarrow& & \downarrow \\
& & \wedge^3 H_d
\end{array}
\] (5.2.4)

Let \( f \in Hom(\mathcal{M}_{g,1}[d], A)^{AB_{g,1}} \). We show that \( f \) restricted to \( \mathcal{T}_{g,1} \) is zero.

Let \( \bar{f} \in Hom(\mathcal{T}_{g,1}, A)^{AB_{g,1}} \) be the restriction of \( f \) to \( \mathcal{T}_{g,1} \). By Lemma [2.3.5], we have that
\( \bar{f} = \varphi^x \circ \sigma \), where \( \varphi^x \) is zero on all \( \mathcal{B}_3 \) except possibly on \( \mathbf{T} \). Since \( A \) is abelian, \( \bar{f} \) restricted on \( \mathcal{K}_{g,1} \) factors through \( j \circ i \circ \sigma \).

On the other hand, by Theorem 7.8. in [40], we know that \( (j \circ i)(\mathbf{T}) = 0 \). As a consequence, \( \varphi^x_3(\mathbf{T}) = 0 \) and so \( f \) restricted to \( \mathcal{T}_{g,1} \) is zero. Therefore every element of \( \text{Hom}(\mathcal{M}_{g,1}[d], A)^{AB_{g,1}} \) factors through \( Sp_{2g}(\mathbb{Z}, d) \), i.e. we have an isomorphism

\[
\text{Hom}(\mathcal{M}_{g,1}[d], A)^{AB_{g,1}} \cong \text{Hom}(Sp_{2g}(\mathbb{Z}, d), A)^{GL_g(\mathbb{Z})}.
\]

\[\tag{P5.2.2}\]

**Proposition 5.2.2.** Let \( A \) be an abelian group and \( g \geq 3 \), for every integer \( d \geq 2 \) we have the following isomorphism

\[
\text{Hom}(Sp_{2g}(\mathbb{Z}, d), A)^{GL_g(\mathbb{Z})} \cong \text{Hom}(sp_{2g}(\mathbb{Z}/d), A)^{GL_g(\mathbb{Z})}.
\]

**Proof.** Denote by \( sp_{2g}(\mathbb{Z}/d) \) the additive group of \( 2g \times 2g \) matrices \( M \) with entries in \( \mathbb{Z}/d \) satisfying \( M^T \Omega + \Omega M = 0 \). In section 1 in [23], R. Lee and R.H. Szczarba showed that the homomorphism

\[
\text{abel} : Sp_{2g}(\mathbb{Z}, d) \rightarrow sp_{2g}(\mathbb{Z}/d)
\]

\[
Id_{2g} + dA \mapsto A \pmod{d}
\]

is the abelianization. Moreover, a direct computation shows that this homomorphism is \( Sp_{2g}(\mathbb{Z}) \)-equivariant. In Corollary of Lemmas 2.1, 2.2, 2.3 in [23], R. Lee and R.H. Szczarba proved that, for any prime \( d \), there is the following short exact sequence

\[
1 \rightarrow Sp_{2g}(\mathbb{Z}, d^2) \rightarrow Sp_{2g}(\mathbb{Z}, d) \xrightarrow{\text{abel}} sp_{2g}(\mathbb{Z}/d) \rightarrow 1. \tag{5.2.5}
\]

In fact, following their proof, the same short exact sequence holds for any integer \( d \geq 2 \). Next, we distinguish the case of \( d \) odd and the case of \( d \) even.

**The case \( d \) odd.** In [36], [45], [39], B. Perron, M. Sato and A. Putman respectively proved independently that for any \( g \geq 3 \) and an odd integer \( d \geq 3 \),

\[
[Sp_{2g}(\mathbb{Z}, d), Sp_{2g}(\mathbb{Z}, d)] = Sp_{2g}(\mathbb{Z}, d^2).
\]

Then, by the short exact sequence (5.2.5), since \( A \) is an abelian group, every homomorphism of \( \text{Hom}(Sp_{2g}(\mathbb{Z}, d), A)^{GL_g(\mathbb{Z})} \) factors through the homomorphism \( \text{abel} \), i.e. we have the following isomorphism

\[
\text{Hom}(Sp_{2g}(\mathbb{Z}, d), A)^{GL_g(\mathbb{Z})} \cong \text{Hom}(sp_{2g}(\mathbb{Z}/d), A)^{GL_g(\mathbb{Z})},
\]

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given by the pull-back of the map \textit{abel} and where \(GL_g(\mathbb{Z})\) acts on \(\mathfrak{sp}_2g(\mathbb{Z}/d)\) by conjugation.

The case \(d\) even. Consider the following normal subgroup of \(Sp_{2g}(\mathbb{Z})\):

\[
Sp_{2g}(\mathbb{Z}, d, 2d) = \{ A \in Sp_{2g}(\mathbb{Z}, d) \mid A = Id_{2g} + dA', A'_{g+i,i} \equiv A'_{i,g+i} \equiv 0 \mod 2 \forall i \}.
\]

In Proposition 10.1 in [45], M. Sato proved that for every even integer \(d \geq 2\),

\[
Sp_{2g}(\mathbb{Z}, d^2, 2d^2) = [Sp_{2g}(\mathbb{Z}, d), Sp_{2g}(\mathbb{Z}, d)].
\]

As a consequence, in (10.3) in [45], he obtained the following short exact sequence

\[
1 \longrightarrow \frac{Sp_{2g}(\mathbb{Z}, d^2)}{Sp_{2g}(\mathbb{Z}, d^2, 2d^2)} \longrightarrow \frac{Sp_{2g}(\mathbb{Z}, d)}{Sp_{2g}(\mathbb{Z}, d^2, 2d^2)} \overset{\text{abel}}{\longrightarrow} \mathfrak{sp}_{2g}(\mathbb{Z}/d) \longrightarrow 1.
\]

Since \(A\) is abelian, we have that

\[
Hom(\frac{Sp_{2g}(\mathbb{Z}, d)}{Sp_{2g}(\mathbb{Z}, d^2, 2d^2)}, A)^{GL_g(\mathbb{Z})} \cong Hom\left(\frac{Sp_{2g}(\mathbb{Z}, d)}{Sp_{2g}(\mathbb{Z}, d^2, 2d^2)}, A\right)^{GL_g(\mathbb{Z})}.
\]

We prove that every element \(\varphi_g \in Hom\left(\frac{Sp_{2g}(\mathbb{Z}, d)}{Sp_{2g}(\mathbb{Z}, d^2, 2d^2)}, A\right)^{GL_g(\mathbb{Z})}\) factors through \(\mathfrak{sp}_{2g}(\mathbb{Z}/d)\), i.e. \(\varphi_g\) is zero on \(\frac{Sp_{2g}(\mathbb{Z}, d^2)}{Sp_{2g}(\mathbb{Z}, d^2, 2d^2)}\).

In section 10 in [45], M. Sato proved that \(\frac{Sp_{2g}(\mathbb{Z}, d^2)}{Sp_{2g}(\mathbb{Z}, d^2, 2d^2)}\) is generated by

\[
\{(Id_{2g} + d^2E_{i,g+i}),(Id_{2g} + d^2E_{g+i,i})\}_{i=1}^{g}.
\]

Then it is enough to check that \(\varphi_g\) is zero on all elements of (5.2.6).

Take \(f\) the matrix \(\begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix}\) with \(G = (1, i) \in \mathfrak{S}_g\). Since \(\varphi_g\) is \(GL_g(\mathbb{Z})\)-invariant, we have that

\[
\varphi_g(Id_{2g} + d^2E_{i,g+i}) = \varphi_g(f(Id_{2g} + d^2E_{i,g+i})f^{-1}) = \varphi_g(Id_{2g} + d^2GE_{i,g+i}G^t) = \varphi_g(Id_{2g} + d^2E_{1,g+1}).
\]

(5.2.7)

Analogously, \(\varphi_g(Id_{2g} + d^2E_{g+i,i}) = \varphi_g(Id_{2g} + d^2E_{g+1,1})\).

Now, take \(f\) the matrix \(\begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix}\) with \(G \in GL_g(\mathbb{Z})\) the matrix with a 1 at position (2,1), 1’s at
the diagonal and 0’s at the other positions. Then we have that
\[ \varphi_g(Id_{2g} + d^2E_{1,g+1}) = \varphi_g(f(Id_{2g} + d^2E_{1,g+1})f^{-1}) = \]
\[ = \varphi_g(Id_{2g} + d^2GE_{1,g+1}G^t) = \]
\[ = \varphi_g(Id_{2g} + d^2(E_{1,g+1} + E_{2,g+2} + (E_{1,g+1} + E_{2,g+1}))) = \]
\[ = \varphi_g(Id_{2g} + d^2E_{1,g+1}) + \varphi_g(Id_{2g} + d^2E_{2,g+2}) + \]
\[ + \varphi_g(Id_{2g} + d^2(E_{1,g+2} + E_{2,g+1}))) \]

But \( Id_{2g} + d^2(E_{1,g+2} + E_{2,g+1}) \in Sp_{2g}(\mathbb{Z}, d^2, 2d^2) \), i.e. \( \varphi_g(Id_{2g} + d^2(E_{1,g+2} + E_{2,g+1})) = 0 \). Therefore, by relation \([5.2.7]\), we get that
\[ \varphi_g(Id_{2g} + d^2E_{1,g+1}) = 2\varphi_g(Id_{2g} + d^2E_{1,g+1}) \]
i.e.
\[ \varphi_g(Id_{2g} + d^2E_{1,g+1}) = 0. \]

Analogously \( \varphi_g(Id_{2g} + d^2E_{g+1,1}) = 0 \). Therefore we get the desired result. \( \square \)

**Notation 5.2.1.** Throughout this chapter, we denote by \( \pi_g \in Hom(sp_{2g}(\mathbb{Z}/d), gl_g(\mathbb{Z}/d)) \) the homomorphism given by
\[ \pi_g : sp_{2g}(\mathbb{Z}/d) \to gl_g(\mathbb{Z}/d) \]
\[ \left( \begin{array}{cc} \alpha & \beta \\ \gamma & -\alpha^t \end{array} \right) \mapsto \alpha. \]

**Lemma 5.2.1.** Let \( g \geq 3, d \geq 2 \) be integers. There is an isomorphism
\[ \Theta : A_d \longrightarrow Hom(sp_{2g}(\mathbb{Z}/d), A)^{GL_g(\mathbb{Z})} \]
\[ x \longmapsto (\overline{\varphi}_g : X \mapsto tr(\pi_g(X))x) \]

**Proof.** First of all we show that \( \Theta \) is well-defined, i.e. \( \overline{\varphi}_g \in Hom(sp_{2g}(\mathbb{Z}/d), A)^{GL_g(\mathbb{Z})}. \)

Let \( X' = \left( \begin{array}{cc} \alpha' & \beta' \\ \gamma & -\alpha^t \end{array} \right) \) and \( X = \left( \begin{array}{cc} \alpha & \beta \\ \gamma & -\alpha^t \end{array} \right) \), we have that
\[ \overline{\varphi}_g(X + X') = tr(\alpha + \alpha')x = tr(\alpha)x + tr(\alpha')x = \overline{\varphi}_g(X) + \overline{\varphi}_g(X'). \]

In addition, it is well known that in general \( tr(AB) = tr(BA) \) for any matrices \( A, B \in M_{g \times g}(\mathbb{Z}). \)

Then, for every matrix \( f \in Sp_{2g}(\mathbb{Z}) \) of the form \( \left( \begin{array}{cc} G & 0 \\ 0 & C^{-1} \end{array} \right) \) with \( G \in GL_g(\mathbb{Z}) \), we have that
\[ \overline{\varphi}_g(fXf^{-1}) = tr(G\alpha G^{-1})x = tr(\alpha)x = \overline{\varphi}_g(X). \]
Therefore, $\overline{\varphi}^n_g \in \text{Hom}(\mathfrak{sp}_{2g}(\mathbb{Z}/d), A)^{GL_g(\mathbb{Z})}$. Moreover, it is clear that $\Theta$ is injective.

Next we show that $\Theta$ is surjective. Recall that, by definition of $\mathfrak{sp}_{2g}(\mathbb{Z}/d)$, the elements of $\mathfrak{sp}_{2g}(\mathbb{Z}/d)$ are matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^t \end{pmatrix}$ where $\alpha, \beta, \gamma$ are matrices of size $g \times g$ with $\beta, \gamma$ symmetric matrices.

For $1 \leq i, j \leq 2g$, let $E_{i,j}$ denote the matrix with a 1 at position $(i, j)$ and 0's elsewhere. Define

$$S = \{ A_{i,j} | 1 \leq i, j \leq g \} \cup \{ B_i, B_i' | 1 \leq i \leq g \} \cup \{ C_{i,j}, C_{i,j}' | 1 \leq i < j \leq g \} \subseteq \mathfrak{sp}_{2g}(\mathbb{Z}/p),$$

where

$$A_{i,j} = E_{i,j} - E_{g+j,g+i}, \quad C_{i,j} = E_{g+i,j} + E_{g+j,i}, \quad C_{i,j}' = E_{i,g+j} + E_{j,g+i},$$

$$B_i = E_{g+i,i}, \quad B_i' = E_{i,g+i}.$$

By the form of the elements of $\mathfrak{sp}_{2g}(\mathbb{Z}/d)$, it is clear that $S$ is a basis for $\mathfrak{sp}_{2g}(\mathbb{Z}/d)$.

Moreover, given an element $X = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^t \end{pmatrix} \in \mathfrak{sp}_{2g}(\mathbb{Z}/d)$ and $f \in S_{2g}(\mathbb{Z})$ a matrix of the form $\begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix}$ with $G \in GL_g(\mathbb{Z})$, we have that

$$fxf^{-1} = \begin{pmatrix} G & 0 \\ 0 & tG^{-1} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^t \end{pmatrix} \begin{pmatrix} G^{-1} & 0 \\ 0 & G^t \end{pmatrix}$$

$$= \begin{pmatrix} G\alpha & G\beta \\ tG^{-1}\gamma & -tG^{-1}\alpha^t \end{pmatrix} \begin{pmatrix} G^{-1} & 0 \\ 0 & G^t \end{pmatrix}$$

$$= \begin{pmatrix} G\alpha G^{-1} & G\beta G^t \\ tG^{-1}\gamma G^{-1} & -(G\alpha G^{-1})^t \end{pmatrix}.$$
\[ \varphi_g(B_i) = \varphi_g(fB_if^{-1}) = \varphi_g(fE_{g+i,i}f^{-1}) = \varphi_g(E_{g+1,i}) = \varphi_g(B_1), \quad (5.2.12) \]
\[ \varphi_g(B'_i) = \varphi_g(fB'_if^{-1}) = \varphi_g(fE_{i,g+i}f^{-1}) = \varphi_g(E_{1,g+1}) = \varphi_g(B'_1). \quad (5.2.13) \]

- Take \( f \) the matrix \( \begin{pmatrix} G & 0 \\ 0 & t^{G^{-1}} \end{pmatrix} \) with \( G \in GL_g(\mathbb{Z}) \) the matrix with 1's at positions (1, 2), (2, 1) and \((i, i) \neq (2, 2) \ \forall i\) and 0's at the other positions.

Observe that

\[ fB'_1f^{-1} = fE_{1,g+1}f^{-1} = \begin{pmatrix} G & 0 \\ 0 & t^{-G} \end{pmatrix} \begin{pmatrix} 0 & E_{1,1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix} = \begin{pmatrix} 0 & GE_{1,1}G' \end{pmatrix}, \]

where

\[ GE_{1,1}G' = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = E_{1,1} + E_{2,2} + (E_{1,2} + E_{2,1}). \]

Thus \( fB'_1f^{-1} = E_{1,g+1} + E_{2,g+2} + (E_{1,g+2} + E_{2,g+1}) = B'_1 + B'_2 + C'_{1,2} \). Therefore,

\[ \varphi_g(B'_1) = \varphi_g(fB'_1f^{-1}) = \varphi_g(B'_1 + B'_2 + C'_{1,2}) = \varphi_g(B'_1) + \varphi_g(B'_2) + \varphi_g(C'_{1,2}), \]

i.e.

\[ \varphi_g(B'_2) = -\varphi_g(C'_{1,2}). \quad (5.2.14) \]

Similarly, we have that

\[ f^{-1}B_1f = f^{-1}E_{g+1,1}f = \begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} E_{1,1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & t^{-G} \end{pmatrix} = \begin{pmatrix} G'E_{1,1}G & 0 \\ 0 & 0 \end{pmatrix}, \]

where

\[ G'E_{1,1}G = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = E_{1,1} + E_{2,2} + (E_{1,2} + E_{2,1}). \]
Thus $f^{-1}B_1f = E_{g+1,1} + E_{g+2,2} + (E_{g+1,2} + E_{g+2,1}) = B_1 + B_2 + C_{1,2}$. Therefore,

$$\varphi_g(B_1) = \varphi_g(f^{-1}B_1f) = \varphi_g(B_1 + B_2 + C_{1,2}) = \varphi_g(B_1) + \varphi_g(B_2) + \varphi_g(C_{1,2}),$$

i.e.

$$\varphi_g(B_2) = -\varphi_g(C_{1,2}).$$  \hspace{2cm} (5.2.15)

- Take $f$ the matrix \( \begin{pmatrix} G & 0 \\ 0 & t_G \end{pmatrix} \) with $G \in GL_3(\mathbb{Z})$ the matrix with 1's at the diagonal and positions $(2,1),(3,1)$, and 0's elsewhere.

Observe that

$$fB_1'f^{-1} = fE_1g + f^{-1} = \begin{pmatrix} G & 0 \\ 0 & t_G^{-1} \end{pmatrix} \begin{pmatrix} 0 & E_{1,1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} G^{-1} & 0 \\ 0 & G^t \end{pmatrix} = \begin{pmatrix} 0 & GE_1G^t \end{pmatrix},$$

where

$$GE_1G^t = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & Id \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & Id \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$fB_1'f^{-1} = B_1' + B_2' + B_3' + C_{1,2}' + C_{2,3}' + C_{1,3}'.$$

Therefore,

$$\varphi_g(B_1') = \varphi_g(fB_1'f^{-1}) = \varphi_g(B_1') + \varphi_g(B_2') + \varphi_g(B_3') + \varphi_g(C_{1,2}') + \varphi_g(C_{2,3}') + \varphi_g(C_{1,3}'),$$

and by relations (5.2.13), (5.2.14), (5.2.10), we get that

$$\varphi_g(B_1') = 3\varphi_g(B_1') - 3\varphi_g(B_1') = 0.$$

Moreover by relations (5.2.13), (5.2.14), (5.2.10), we also get that

$$0 = \varphi_g(B_1') = \varphi_g(B_1') = \varphi_g(C_{1,2}').$$

Similarly, take $f$ the matrix \( \begin{pmatrix} G & 0 \\ 0 & t_G \end{pmatrix} \) with $G^t \in GL_3(\mathbb{Z})$ the matrix with 1's at the diagonal
and positions (2, 1), (3, 1), and 0’s at the other positions. Then we have that

\[
f^{-1}B_1f = f^{-1}E_{g+1,1}f = \begin{pmatrix} G^{-1} & 0 \\ 0 & G' \end{pmatrix} \begin{pmatrix} 0 & 0 \\ E_{1,1} & 0 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & \Gamma G^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ G' E_{1,1} G & 0 \end{pmatrix},
\]

where

\[
G' E_{1,1} G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_d \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_d \end{pmatrix} =
\]

\[
= E_{1,1} + E_{2,2} + E_{3,3} + (E_{1,2} + E_{2,1}) + (E_{2,3} + E_{3,2}) + (E_{1,3} + E_{3,1}).
\]

Thus,

\[
f B_1 f^{-1} = B_1 + B_2 + B_3 + C_{1,2} + C_{2,3} + C_{1,3}.
\]

Therefore,

\[
\varphi_g(B_1) = \varphi_g(f B_1 f^{-1}) = \varphi_g(B_1) + \varphi_g(B_2) + \varphi_g(B_3) + \varphi_g(C_{1,2}) + \varphi_g(C_{2,3}) + \varphi_g(C_{1,3}),
\]

and by relations (5.2.12), (5.2.15), (5.2.9), we have that

\[
\varphi_g(B_1) = 3\varphi_g(B_1) - 3\varphi_g(B_1) = 0.
\]

Moreover by relations (5.2.12), (5.2.15), (5.2.9), we have that

\[
0 = \varphi_g(B_1) = \varphi_g(B_1) = \varphi_g(C_{1,j}).
\]  \hfill (5.2.16)

• Take \( f \) the matrix \( \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} \) with \( G \in GL_g(\mathbb{Z}) \) the matrix with a 1’s at the diagonal and position (1, 2), and 0’s at the other positions.

Observe that

\[
f A_{2,1} f^{-1} = f(E_{2,1} - E_{g+1,g+2}) f^{-1} = \begin{pmatrix} G & 0 \\ 0 & \Gamma G^{-1} \end{pmatrix} \begin{pmatrix} E_{2,1} & 0 \\ 0 & -E_{1,2} \end{pmatrix} \begin{pmatrix} G^{-1} & 0 \\ 0 & G' \end{pmatrix}
\]

\[
= \begin{pmatrix} GE_{2,1} G^{-1} & 0 \\ 0 & -(GE_{2,1} G^{-1})^t \end{pmatrix},
\]
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where

\[ GE_2,1 G^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & Id \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & Id \end{pmatrix} \]

\[ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Id \end{pmatrix} \]

\[ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

\[ = E_{1,1} - E_{2,2} + E_{1,2} - E_{2,2}. \]

Thus

\[ f A_{2,1} f^{-1} = E_{1,1} - E_{2,2} - E_{1,2} + E_{2,1} - E_{g+1, g+1} + E_{g+2, g+2} + E_{g+2, g+1} - E_{g+1, g+2} \]

\[ = A_{1,1} - A_{2,2} + A_{1,2} - A_{2,1}. \]

Therefore,

\[ \varphi_g(A_{2,1}) = \varphi_g(f A_{2,1} f^{-1}) = \varphi_g(A_{1,1} - A_{2,2} + A_{1,2} - A_{2,1}) \]

\[ = \varphi_g(A_{1,1}) - \varphi_g(A_{2,2}) + \varphi_g(A_{1,2}) - \varphi_g(A_{2,1}), \]

and by relations (5.2.8), (5.2.11) we obtain that

\[ \varphi_g(A_{2,1}) = 0, \] (5.2.17)

and by relation (5.2.11) we get that \( \varphi_g(A_{i,j}) = 0 \) for all \( 1 \leq i < j \leq g \).

Summarizing, the unique elements of the basis \( S \), whose image by \( \varphi_g \) are not necessarily zero, are the elements \( \{ A_{i,i} \mid 1 \leq i \leq g \} \). By the relation (5.2.8) we have that \( \varphi_g(A_{i,i}) = \varphi_g(A_{1,1}) \). Thus, any homomorphism \( \varphi_g \in Hom(\text{sp}_2 g(\mathbb{Z}/d), A)GL_2(\mathbb{Z}) \) only depends on the image of \( A_{1,1} \). In particular, for all \( X = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in \text{sp}_2 g(\mathbb{Z}/d) \), we have that

\[ \varphi_g(X) = tr(\alpha) \varphi_g(A_{1,1}), \]

where \( tr(\alpha) \) denotes the trace of the matrix \( \alpha \).

Furthermore, since every element of \( \text{sp}_2 g(\mathbb{Z}/d) \) is a \( d \)-torsion element and \( \varphi_g \) is a homomorphism, then \( \varphi_g(A_{1,1}) \) is also a \( d \)-torsion element. Therefore, \( \varphi_g(A_{1,1}) \) may be any element of \( A_d \). \( \square \)

As a direct consequence of Propositions (5.2.1), (5.2.2) and Lemma (5.2.1) we have the following result:
Lemma 5.2.2. For \( g \geq 3, \ d \geq 3 \) an odd integer and for \( g \geq 5, \ d \geq 2 \) an even integer such that \( 4 \nmid d \), there is an isomorphism

\[
A_d \rightarrow Hom(M_{g,1}[d], A)^{AB_{g,1}}
\]

\[
x \mapsto \varphi_x^x = \varphi_x^x \circ \text{abel} \circ \Psi,
\]

where \( \varphi_x^x \) is defined in Lemma (5.2.1).

Next, we show that every element of \( Hom(M_{g,1}[d], A)^{AB_{g,1}} \) is zero on \( A_{g,1}[d], B_{g,1}[d] \). We first need to compute \( \Psi(A_{g,1}[d]), \Psi(B_{g,1}[d]) \). We also compute \( \Psi(AB_{g,1}[d]) \), for sake of completeness.

Lemma 5.2.3. There are short exact sequences of groups:

\[
\begin{align*}
1 & \rightarrow TB_{g,1} \rightarrow B_{g,1}[d] \xrightarrow{\phi^B \circ \Psi} SL_g(\mathbb{Z}, d) \times S_g(d\mathbb{Z}) \rightarrow 1, \\
1 & \rightarrow TA_{g,1} \rightarrow A_{g,1}[d] \xrightarrow{\phi^A \circ \Psi} SL_g(\mathbb{Z}, d) \times S_g(d\mathbb{Z}) \rightarrow 1, \\
1 & \rightarrow T(AB)_{g,1} \rightarrow (AB)_{g,1}[d] \xrightarrow{\phi^{AB} \circ \Psi} SL_g(\mathbb{Z}, d) \rightarrow 1.
\end{align*}
\]

Proof. We only prove the first short exact sequence (the proof for the other two short exact sequences is analogous). Observe that we have the following commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
1 \\
1 \\
1
\end{array} & \begin{array}{c}
\longrightarrow
\longrightarrow
\longrightarrow
\end{array} & \begin{array}{c}
TB_{g,1} \\
B_{g,1}[d] \\
B_{g,1}
\end{array} & \begin{array}{c}
\xrightarrow{\phi^B \circ \Psi} \\
\xrightarrow{\phi^B \circ \Psi} \\
\phi^B \circ \Psi
\end{array} & \begin{array}{c}
SL_g(\mathbb{Z}, d) \times S_g(d\mathbb{Z}) \\
GL_g(\mathbb{Z}) \times S_g(\mathbb{Z}) \\
GL_g(\mathbb{Z}/d) \times S_g(\mathbb{Z}/d)
\end{array} & 1 \\
\begin{array}{c}
\longrightarrow
\longrightarrow
\longrightarrow
\end{array} & \begin{array}{c}
TB_{g,1} \\
B_{g,1}[d] \\
B_{g,1}
\end{array} & \begin{array}{c}
\xrightarrow{\phi^B \circ \Psi} \\
\xrightarrow{\phi^B \circ \Psi} \\
\phi^B \circ \Psi
\end{array} & \begin{array}{c}
SL_g(\mathbb{Z}, d) \times S_g(d\mathbb{Z}) \\
GL_g(\mathbb{Z}) \times S_g(\mathbb{Z}) \\
GL_g(\mathbb{Z}/d) \times S_g(\mathbb{Z}/d)
\end{array} & 1
\end{array}
\]

where \( i \) denotes the natural inclusion. We prove that the 1st row is a short exact sequence.

We first show that the kernel of \( \phi^B \circ \Psi \) restricted to \( B_{g,1}[d] \) is \( TB_{g,1} \).

Since \( TB_{g,1} \subset Ker(\Psi) \), we have that

\[
TB_{g,1} \subset Ker((\phi^B \circ \Psi) : B_{g,1}[d] \rightarrow SL_g(\mathbb{Z}, d) \times S_g(d\mathbb{Z})) \subset B_{g,1} \cap T_{g,1} = TB_{g,1}.
\]

Next, we prove that \((\phi \circ \Psi) : B_{g,1}[d] \rightarrow SL_g(\mathbb{Z}, d) \times S_g(d\mathbb{Z})\) is surjective. Let \( x \in SL_g(\mathbb{Z}, d) \times S_g(d\mathbb{Z}) \).
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\( S_g(d\mathbb{Z}) \), then \( i(x) \in GL_g(\mathbb{Z}) \times S_g(\mathbb{Z}) \). By Lemma (1.3.2) we know that \( \phi \circ \Psi : B_{g,1} \rightarrow GL_g(\mathbb{Z}) \times S_g(\mathbb{Z}) \) is surjective and as a consequence there exists an element \( \overline{x} \in B_{g,1} \) such that \( (\phi \circ \Psi)(\overline{x}) = i(x) \). By the above commutative diagram and Lemma (3.3.3), we know that the 3rd column is exact and we have the following equality:

\[
(\phi_d \circ \Psi_d)(\overline{x}) = (r'_d \times r''_d)(i(x)) = 0.
\]

Thus \( (\phi_d \circ \Psi_d)(\overline{x}) = 0 \). Since by Lemma (3.3.4) the 2nd column is exact then we get that \( \overline{x} \in B_{g,1}[d] \). Therefore by the above commutative diagram we know that \( i((\phi \circ \Psi)(\overline{x})) = i(x) \) and since \( i \) is injective we get that \( (\phi \circ \Psi)(\overline{x}) = x \).

Now we are ready to prove the following result:

**Lemma 5.2.4.** Let \( A \) be an abelian group. Fix an integer \( d \geq 2 \), then for any \( g \geq 3 \) all \( AB_{g,1} \)-invariant homomorphisms

\[
\varphi_g : B_{g,1}[d] \rightarrow A \text{ and } \varphi_g : A_{g,1}[d] \rightarrow A,
\]

have to be zero.

**Proof.** We only prove the result for \( B_{g,1}[d] \), since the other case is similar.

Let \( \varphi_g : B_{g,1}[d] \rightarrow A \) be a \( AB_{g,1} \)-invariant homomorphism. By Lemma (2.3.4) we know that the restriction of \( \varphi_g \) to \( T B_{g,1} \) is zero. Then, by the short exact sequence in Lemma (5.2.3), \( \varphi_g \) factors through \( SL_g(\mathbb{Z},d) \times B S_g(d\mathbb{Z}) \). Thus we have an isomorphism

\[
\text{Hom}(B_{g,1}[d], A)^{AB_{g,1}} \cong \text{Hom}(SL_g(\mathbb{Z},d) \times B S_g(d\mathbb{Z}), A)^{GL_g(\mathbb{Z})}.
\]

Observe that we have the short exact sequence

\[
1 \rightarrow S_g(d\mathbb{Z}) \rightarrow SL_g(\mathbb{Z},d) \times B S_g(d\mathbb{Z}) \rightarrow SL_g(\mathbb{Z},d) \rightarrow 1.
\]

By analogous relations to the relations (5.2.9), (5.2.12), (5.2.15), (5.2.16) in the proof of Lemma (5.2.2), we have that \( \text{Hom}(S_g(\mathbb{Z},d), A)^{GL_g(\mathbb{Z})} = 0 \). Thus we have an isomorphism

\[
\text{Hom}(SL_g(\mathbb{Z},d) \times B S_g(d\mathbb{Z}), A)^{GL_g(\mathbb{Z})} \cong \text{Hom}(SL_g(\mathbb{Z},d), A)^{GL_g(\mathbb{Z})}.
\]

In [23], Lee and Szczarba showed that for \( g \geq 3 \) and any prime \( p \), \( H_1(SL_g(\mathbb{Z},p)) \cong sl_g(\mathbb{Z}/p) \), as modules over \( SL_g(\mathbb{Z}/p) \). Actually, following the same proof the same result holds for any integer \( d \). Then, since \( A \) is an abelian group, we have that every homomorphism \( \text{Hom}(SL_g(\mathbb{Z},d), A)^{GL_g(\mathbb{Z})} \),
factors through $\mathfrak{sl}(\mathbb{Z}/d)$, i.e. we have an isomorphism

$$\text{Hom}(\text{SL}_g(\mathbb{Z}, d), A)^{\text{GL}_g(\mathbb{Z})} \cong \text{Hom}(\mathfrak{sl}(\mathbb{Z}/d), A)^{\text{GL}_g(\mathbb{Z})}.$$  

Finally, by relations (5.2.8), (5.2.17), (5.2.11) in proof of Lemma (5.2.2), we get that every element of $\text{Hom}(\mathfrak{sl}(\mathbb{Z}/d), A)^{\text{GL}_g(\mathbb{Z})}$ have to be zero, obtaining the desired result. 

**Lemma 5.2.5.** The homomorphisms $\varphi_x^g$ defined in Lemma (5.2.2) are compatible with the stabilization map.

**Proof.** Recall that the homomorphisms $\varphi_x^g$, for $T \in M_{g,1}[d]$, are defined as follows

$$\varphi_x^g(T) = \text{tr}(\pi_g \circ \text{abel} \circ \Psi(T)) x.$$  

Let $\text{abel}(\Psi(T)) = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha' \end{pmatrix} \in \mathfrak{sp}_{2g}(\mathbb{Z}/d)$. Then if we take $T$ as an element of $M_{g+1,1}[d]$, we have that

$$\text{abel}(\Psi(T)) = \begin{pmatrix} \alpha' & \beta' \\ \gamma & -\alpha' \end{pmatrix} \in \mathfrak{sp}_{2(g+1)}(\mathbb{Z}/d),$$  

where $\alpha' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Hence, $\varphi_x^g(T) = \varphi_{x+1}^g(T)$ for every $T \in M_{g,1}[d]$. 

**Remark 5.2.1.** By Lemmas (5.2.4), (5.2.5), the $\mathcal{A}\mathcal{B}_{g,1}$-invariant homomorphisms $\{\varphi_x^g\}_g$, in Lemma (5.2.2), are compatible with the stabilization map and zero on $\mathcal{A}_{g,1}[d], \mathcal{B}_{g,1}[d]$. Then, by bijection (3.4.7), we get that, for each $x \in A_d$, the family of homomorphisms $\{\varphi_x^g\}_g$ of Lemma (5.2.2) reassemble into an invariant of $S^3[d]$.

A natural question that arise from this section is what 3-manifolds distinguish the invariant $\varphi^x$. Next we answer this question for lens spaces. Given a lens space $L(p,q)$, since Lens spaces are $\mathbb{Q}$-homology 3-spheres, by Theorem (3.4.3), we know that there exists an integer $d \geq 2$ for which $L(p,q) \in S^3[d]$. Thus, given an integer $d \geq 2$, we first need to determine which lens spaces are in $S^3[d]$.

**Proposition 5.2.3.** A lens space $L(p,q)$ is in $S^3[d]$ if and only if $p \equiv \pm 1(\text{mod } d)$.

**Proof.** Given a lens space $L(p,q)$, it is well known that its homology groups are:

$$H_k(L(p,q); \mathbb{Z}) = \begin{cases} 
\mathbb{Z}, & \text{for } k = 0, 3, \\
\mathbb{Z}/p, & \text{for } k = 1, \\
0, & \text{otherwise}. 
\end{cases}$$

Then $|H_1(L(p,q); \mathbb{Z})| = p$. By Theorem (3.4.3), $L(p,q) \in S^3[d]$ if and only if $p \equiv \pm 1(\text{mod } d)$. 

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As a consequence, all lens spaces in $S^3[d]$ are of the form $L(\pm 1 + dk, q)$ with $k, q \in \mathbb{Z}$. Next we compute the values of the invariant $\varphi^x$ on these lens spaces. By the classification of lens spaces we know that $L(1 + dk, q) \cong L(-1 - dk, q)$. Then, it is enough to compute the value of the invariant $\varphi^x$ on the lens spaces $L(-1 + dk, q)$.

Given a lens space $L(-1 + dk, q) \in S^3[d]$, by definition of lens spaces, such space has a Heegaard splitting of genus 1. Following the proof of Theorem (3.4.3), we have that there exists an element $f_d \in M_{1,1}[d]$ such that $L(-1 + dk, q) \cong H_1 \cup f_d - H_1$. Then, the lens space $L(-1 + dk, q)$ is homeomorphic to $L(-1 + dk, -dl)$ for suitable integers $k, l, m, r$, with

$$\Psi(f_d) = \left( \begin{array}{cc} 1 + dr & dm \\ dl & 1 - dk \end{array} \right) \in Sp_2(\mathbb{Z}).$$

Since $Sp_2(\mathbb{Z}) = SL_2(\mathbb{Z})$, $det(\Psi(f_d)) = 1$ and reducing modulo $d^2$ we have that

$$1 = det(\Psi(f_d)) = 1 + d(r - k) + d^2(-rk + lm) \equiv 1 + d(r - k) \quad (\text{mod} \ d^2).$$

Thus, $k \equiv r \quad (\text{mod} \ d)$. Then,

$$\varphi^x(L(-1 + pk, -pl)) = \varphi_1^x(f_d) = tr(\pi_1 \circ \text{abel} \circ \Psi(f_d))x = rx \equiv kx.$$

Therefore we get the following result:

**Proposition 5.2.4.** The homomorphisms $\{\varphi^x_g\}_g \in Hom(M_{g,1}[d], A)$, defined in Lemma (5.2.2), take the value $kx$ on lens spaces $L(-1 + dk, q)$ and $-kx$ on lens spaces $L(1 + dk, q)$.

### 5.3 From trivial cocycles to invariants

Conversely, what are the conditions for a family of trivial 2-cocycles $C_g$ on $M_{g,1}[d]$ satisfying properties (1)-(3) to actually provide an invariant? Here we follow the same arguments used in Section 2.4 taking $M_{g,1}[d]$ instead of $T_{g,1}$.

Firstly we need to check the existence of an $AB_{g,1}$-invariant trivialization of each $C_g$. Denote by $Q_{C_g}$ the set of all trivializations of the cocycle $C_g$:

$$Q_{C_g} = \{ q : M_{g,1}[d] \rightarrow A \mid q(\phi) + q(\psi) - q(\phi \psi) = C_g(\phi, \psi) \}.$$

The group $AB_{g,1}$ acts on $Q_g$ via its conjugation action on $M_{g,1}[d]$. This action confers the set $Q_{C_g}$ the structure of an affine set over the abelian group $Hom(M_{g,1}[d], A)$. On the other hand, choosing
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an arbitrary element \(q \in \mathcal{Q}_C \) the map defined as follows

\[
\rho_q : \mathcal{A}\mathcal{B}_{g,1} \to \text{Hom} (\mathcal{M}_{g,1}[d], A)
\]

\[
\phi \mapsto \phi \cdot q - q,
\]

induces a well-defined cohomology class \(\rho(C_g) \in H^1(\mathcal{A}\mathcal{B}_{g,1}; \text{Hom} (\mathcal{M}_{g,1}[d], A))\), called the torsor of the cocycle \(C_g\), and we have the following result:

**Proposition 5.3.1.** The natural action of \(\mathcal{A}\mathcal{B}_{g,1}\) on \(\mathcal{Q}_C\) admits a fixed point if and only if the associated torsor \(\rho(C_g)\) is trivial.

Suppose that for every \(g \geq 3\) there is a fixed point \(q_g \) of \(\mathcal{Q}_C\) for the action of \(\mathcal{A}\mathcal{B}_{g,1}\) on \(\mathcal{Q}_C\). Since every pair of \(\mathcal{A}\mathcal{B}_{g,1}\)-invariant trivialization differ by a \(\mathcal{A}\mathcal{B}_{g,1}\)-invariant homomorphism, by Lemma \(5.2.2\), for every \(g \geq 3\) the fixed points are \(q_g + \varphi_g^x\) with \(x \in A_d\).

By Lemma \(5.2.5\), all elements of \(\text{Hom} (\mathcal{M}_{g,1}[d], A)\)\(\mathcal{A}\mathcal{B}_{g,1}\) are compatible with the stabilization map. Then, given two different fixed points \(q_g, q'_g\) of \(\mathcal{Q}_C\) for the action of \(\mathcal{A}\mathcal{B}_{g,1}\), we have that

\[
q_g|\mathcal{M}_{g-1,1}[d] - q'_g|\mathcal{M}_{g-1,1}[d] = (q_g - q'_g)|\mathcal{M}_{g-1,1}[d] = \varphi_g^x|\mathcal{M}_{g-1,1}[d] = \varphi_g^{x-1}.
\]

Therefore the restriction of the trivializations of \(\mathcal{Q}_C\) to \(\mathcal{M}_{g-1,1}[d]\), give us a bijection between the fixed points of \(\mathcal{Q}_C\) for the action of \(\mathcal{A}\mathcal{B}_{g,1}\) and the fixed points of \(\mathcal{Q}_{C_{g-1}}\) for the action of \(\mathcal{A}\mathcal{B}_{g-1,1}\).

Therefore, given an \(\mathcal{A}\mathcal{B}_{g,1}\)-invariant trivialization \(q_g\), for each \(x \in A_d\) we get a well-defined map

\[
q + \varphi_g^x = \lim_{g \to \infty} q_g + \varphi_g^x : \lim_{g \to \infty} \mathcal{M}_{g,1}[d] \to A.
\]

These are the only candidates to be \(A\)-valued invariants of \(\mathbb{Z}/d\)-homology spheres with associated family of 2-cocycles \(\{C_g\}_g\). For these maps to be invariants, since they are already \(\mathcal{A}\mathcal{B}_{g,1}\)-invariant, we only have to prove that they are constant on the double cosets \(\mathcal{A}_{g,1}[d]\)\(\mathcal{M}_{g,1}[d]\)\(\mathcal{B}_{g,1}[d]\). From property (3) of our cocycle we have that \(\forall \phi \in \mathcal{M}_{g,1}[d], \forall \psi_a \in \mathcal{A}_{g,1}[d] \) and \(\forall \psi_b \in \mathcal{B}_{g,1}[d]\),

\[
(q_g + \varphi_g^x)(\phi) - (q_g + \varphi_g^x)(\phi \psi_b) = -(q_g + \varphi_g^x)(\psi_b),
\]

\[
(q_g + \varphi_g^x)(\phi) - (q_g + \varphi_g^x)(\psi_a \phi) = -(q_g + \varphi_g^x)(\psi_a).
\]

Thus in particular, taking \(\phi = \psi_a, \psi_b\) in above equations, we have that \(q_g + \varphi_g^x\) with \(x \in A_d\), are homomorphisms on \(\mathcal{A}_{g,1}[d], \mathcal{B}_{g,1}[d]\). Then, by Lemma \(5.2.4\) we have that \(q_g + \varphi_g^x\) are trivial on \(\mathcal{A}_{g,1}[d]\) and \(\mathcal{B}_{g,1}[d]\).

Therefore, by equalities \(5.3.1\), we get that \(q_g + \varphi_g^x\) with \(x \in A_d\) are constant on the double cosets \(\mathcal{A}_{g,1}[d]\)\(\mathcal{M}_{g,1}[d]\)\(\mathcal{B}_{g,1}[d]\).

Summarizing, we get the following result:
Theorem 5.3.1. Let $A$ an abelian group. For $g \geq 3$, $d \geq 3$ an odd integer and for $g \geq 5$, $d \geq 2$ an even integer such that $4 \nmid d$. For each $x \in A$, a family of 2-cocycles $(C_g)_{g \geq 3}$ on the (mod $d$) Torelli groups $M_{g,1}[d]$, with values in $A$, satisfying conditions (1)-(3) provides a compatible family of trivializations $F_g + \varphi_g^x : M_{g,1}[d] \to A$ that reassembles into an invariant of $Q$-homology spheres $S^3[d]$

$$\lim_{g \to \infty} F_g + \varphi_g^x : S^3[d] \to A$$

if and only if the following two conditions hold:

(i) The associated cohomology classes $[C_g] \in H^2(M_{g,1}[d]; A)$ are trivial.

(ii) The associated torsors $\rho(C_g) \in H^1(AB_{g,1}, Hom(M_{g,1}[d], A))$ are trivial.

5.4 Pull-back of 2-cocycles over abelian groups

In general, it is not easy to construct a family of 2-cocycles $\{C_g\}_{g \geq 3}$ satisfying the hypothesis of Theorem (5.3.1). The idea to construct such family of 2-cocycles $(C_g)_{g \geq 3}$, inspired on [38], is the following: Consider a $AB_{g,1}$-equivariant map $f$ from $M_{g,1}[p]$ to a certain module $V$. Then, we construct a family of bilinear forms $\{B_g\}_{g \geq 3}$ on the module $V$, (which are naturally 2-cocycles on $V$), in such a way the pull-back of this bilinear forms along $f$ will be the desired family of 2-cocycles on $M_{g,1}[p]$. For the purposes of this thesis it is enough to give the construction for the first Zassenhaus mod $p$ Johnson homomorphism $\tau_{Z,1}$, defined in Section (5.1), and bilinear forms on $\Lambda^3 H_p$.

Consider the first Zassenhaus mod $p$ Johnson homomorphism $\tau_{1}^{Z}$ and a family of bilinear forms $(B_g)_{g \geq 3}$, defined on $\Lambda^3 H_p$ satisfying the following properties:

(1') The 2-cocycles $\{B_g\}_{g}$ are compatible with the stabilization map.

(2') For every $\phi = \left( \begin{array}{cc} G & 0 \\ 0 & G^{-1} \end{array} \right) \in Sp_{2g}(\mathbb{Z})$ with $G \in GL_g(\mathbb{Z})$, $B_g(\phi - \phi^{-1}, \phi - \phi^{-1}) = B_g(\phi, \phi)$.

(3') If $\phi \in \tau_{1}^{Z}(A_{g,1}[d])$ or $\psi \in \tau_{1}^{Z}(B_{g,1}[d])$ then $B_g(\phi, \psi) = 0$.

Since $\tau_{1}^{Z}$ is an $A_{g,1}$-equivariant homomorphism and compatible with the stabilization map, the pull-back of this family of 2-cocycles along $\tau_{1}^{Z}$ give us a family of 2-cocycles on $M_{g,1}[p]$ satisfying the properties (1)-(3) in Section (5.2).

Then we devote this section to find families of 2-cocycles on $\Lambda^3 H_p$ satisfying the properties (1')-(3'). To find such families, we first compute the image of $A_{g,1}[d], B_{g,1}[d]$ under $\tau_{1}^{Z}$.
5.4.1 The extension of Johnson’s homomorphism

We first construct, in a natural way, a crossed homomorphism \( k_p : \mathcal{M}_{g,1} \to \wedge^3 H_p \) which extends the Johnson homomorphism \( \tau_1 \) modulo \( p \). By definition of the Zassenhaus filtration, we have a commutative diagram

\[
\begin{array}{cccccccc}
1 & \longrightarrow & T_{g,1} & \longrightarrow & \mathcal{M}_{g,1} & \xrightarrow{\psi} & Sp_{2g}(\mathbb{Z}) & \longrightarrow & 1 \\
\downarrow{\tau_1^Z} & & \downarrow{\rho_2^Z} & & \downarrow{\psi_1} & & \downarrow{r_p} & & \\
0 & \longrightarrow & \wedge^3 H_p & \longrightarrow & \rho_2^Z(\mathcal{M}_{g,1}) & \longrightarrow & Sp_{2g}(\mathbb{Z}/p) & \longrightarrow & 1
\end{array}
\]

(5.4.1)

By Lemma (1.1.2) and the fact that \( \wedge^3 H_p \) is an \( \mathbb{F}_p \)-vector space, we have isomorphisms

\[
H^*(Sp_{2g}(\mathbb{Z}/p);\wedge^3 H_p) \cong H^*(Sp_{2g}(\mathbb{Z}/p);(\wedge^3 H_p)^*) \cong (H_*\langle Sp_{2g}(\mathbb{Z}/p);\wedge^3 H_p \rangle)^*.
\]

Since \(-Id \in Sp_{2g}(\mathbb{Z}/p)\) acts on \( \wedge^3 H_p \) by the multiplication of \(-1\), by the Center kills Lemma we have that \( H_*(Sp_{2g}(\mathbb{Z}/p);\wedge^3 H_p) = 0 \). As a consequence,

\[
H^*(Sp_{2g}(\mathbb{Z}/p);\wedge^3 H_p) = 0.
\]

Then the bottom extension of diagram (5.4.1) splits with only one \( \wedge^3 H_p \)-conjugacy class of splittings. By Proposition (1.1.1), we have an isomorphism

\[
f : \rho_2^Z(\mathcal{M}_{g,1}) \longrightarrow \wedge^3 H_p \times Sp_{2g}(\mathbb{Z}).
\]

Recall that the operation on \( \wedge^3 H_p \times Sp_{2g}(\mathbb{Z}/p) \) is given by \((a,g) \cdot (b,h) = (a + g \cdot b, gh)\). Then, if we take \( \pi : \wedge^3 H_p \times Sp_{2g}(\mathbb{Z}/p) \to \wedge^3 H_p \) the projection on the first component, we get a derivation, and as a consequence we have that \( k_p = (\pi \circ f \circ \rho_2^Z) \) is a crossed homomorphism

\[
k_p : \mathcal{M}_{g,1} \longrightarrow \wedge^3 H_p.
\]

Observe that in fact \( \pi \) is a retraction of the bottom extension of (5.4.1), then using the commutative diagram (5.4.1) we have that \( k_p \) restricted to \( T_{g,1} \) is the image of the Johnson homomorphism reduced modulo \( p \). Thus, we have found an extension of the Johnson’s homomorphism modulo \( p \) to the whole mapping class group \( \mathcal{M}_{g,1} \).

Moreover, if we consider the crossed homomorphism \( k_p \in H^1(\mathcal{M}_{g,1};\wedge^3 H_p) \) restricted to \( \mathcal{M}_{g,1}[p] \) we get an extension of the Johnson’s homomorphism modulo \( p \) to \( \mathcal{M}_{g,1}[p] \). Notice that the first Zassenhaus mod \( p \) Johnson homomorphism \( \tau_1^Z \) is also an extension of the Johnson’s homomorphism modulo \( p \) to \( \mathcal{M}_{g,1}[p] \).
Next we concern about the unicity of such extensions.

**Proposition 5.4.1.** For any odd prime $p$ and $g \geq 4$, there are isomorphisms

\[ H^1(\mathcal{M}_{g,1}; \wedge^3 H_p) \cong H^1(\mathcal{M}_{g,1}[p]; \wedge^3 H_p)^{Sp_2g(\mathbb{Z}/p)} \cong H^1(\mathcal{T}_{g,1}; \wedge^3 H_p)^{Sp_2g(\mathbb{Z}/p)} \cong \text{Hom}(\wedge^3 H_p, \wedge^3 H_p)^{Sp_2g(\mathbb{Z}/p)} \cong (\mathbb{Z}/p)^2. \]

**Proof.** Consider the 5-term exact sequence associated to the short exact sequence

\[ 1 \longrightarrow \mathcal{M}_{g,1}[p] \longrightarrow \mathcal{M}_{g,1} \longrightarrow Sp_2g(\mathbb{Z}/p) \longrightarrow 1 \]

and the $Sp_2g(\mathbb{Z}/p)$-module $\wedge^3 H_p$. Then, we get the exact sequence

\[ 0 \longrightarrow H^1(Sp_2g(\mathbb{Z}/p); \wedge^3 H_p) \xrightarrow{inf} H^1(\mathcal{M}_{g,1}; \wedge^3 H_p) \xrightarrow{res} H^1(\mathcal{M}_{g,1}[p]; \wedge^3 H_p)^{Sp_2g(\mathbb{Z}/p)}. \]

By the Center kills Lemma, since $-Id$ acts on $\wedge^3 H_p$ as the multiplication by $-1$, we have that $H^1(Sp_2g(\mathbb{Z}/p); \wedge^3 H_p) = 0$. Therefore we have the injection

\[ res : H^1(\mathcal{M}_{g,1}; \wedge^3 H_p) \rightarrow H^1(\mathcal{M}_{g,1}[p]; \wedge^3 H_p)^{Sp_2g(\mathbb{Z}/p)}. \quad (5.4.2) \]

Next, consider the 5-term exact sequence associated to the short exact sequence

\[ 1 \longrightarrow \mathcal{T}_{g,1} \longrightarrow \mathcal{M}_{g,1}[p] \longrightarrow Sp_2g(\mathbb{Z}/p) \longrightarrow 1 \]

and the $Sp_2g(\mathbb{Z}/p)$-module $\wedge^3 H_p$. Then we get the exact sequence

\[ 0 \longrightarrow H^1(Sp_2g(\mathbb{Z}/p); \wedge^3 H_p) \xrightarrow{inf} H^1(\mathcal{M}_{g,1}[p]; \wedge^3 H_p) \xrightarrow{res} H^1(\mathcal{T}_{g,1}; \wedge^3 H_p). \]

Taking $Sp_2g(\mathbb{Z}/p)$-invariants, we get another exact sequence

\[ 0 \rightarrow H^1(Sp_2g(\mathbb{Z}/p); \wedge^3 H_p)^{Sp_2g(\mathbb{Z}/p)} \rightarrow H^1(\mathcal{M}_{g,1}[p]; \wedge^3 H_p)^{Sp_2g(\mathbb{Z}/p)} \rightarrow H^1(\mathcal{T}_{g,1}; \wedge^3 H_p)^{Sp_2g(\mathbb{Z}/p)}. \]

By the Universal coefficients Theorem and Proposition (5.2.2) we have that

\[ H^1(Sp_2g(\mathbb{Z}/p); \wedge^3 H_p)^{Sp_2g(\mathbb{Z}/p)} \cong \text{Hom}(Sp_2g(\mathbb{Z}/p); \wedge^3 H_p)^{Sp_2g(\mathbb{Z}/p)}. \]
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Notice that, since $-Id \in Sp_{2g}(\mathbb{Z}/p)$ acts trivially on $sp_{2g}(\mathbb{Z}/p)$ and as $-1$ on $\Lambda^3 H_p$, then

$$Hom(sp_{2g}(\mathbb{Z}/p), \Lambda^3 H_p)^{Sp_{2g}(\mathbb{Z}/p)} = 0.$$ 

Therefore we have the injection

$$res : H^1(\mathcal{M}_{g,1}[p]; \Lambda^3 H_p) \to H^1(\mathcal{T}_{g,1}; \Lambda^3 H_p)^{Sp_{2g}(\mathbb{Z}/p)}. \quad (5.4.3)$$

Moreover, by the Universal coefficients Theorem and Theorem 6.19 in [8], we have that the first Johnson homomorphism modulo $p$ induces an isomorphism

$$H^1(\mathcal{T}_{g,1}; \Lambda^3 H_p)^{Sp_{2g}(\mathbb{Z}/p)} \cong Hom(\Lambda^3 H_p, \Lambda^3 H_p)^{Sp_{2g}(\mathbb{Z}/p)}. \quad (5.4.4)$$

Next, we compute $Hom(\Lambda^3 H_p, \Lambda^3 H_p)^{Sp_{2g}(\mathbb{Z})}$. Let $c_i = a_i$ or $b_i$ for every $i \in \{1, \ldots, g\}$, and $f \in Hom(\Lambda^3 H_p, \Lambda^3 H_p)^{Sp_{2g}(\mathbb{Z})}$.

- Consider the element $\phi = \left( \begin{array}{cc} G & 0 \\ 0 & -Id \end{array} \right) \in Sp_{2g}(\mathbb{Z})$ with $G = (1, i)(2, j)(3, k) \in S_g$. Then we have that

$$\phi \cdot f(c_1 \land c_2 \land c_3) = f(\phi \cdot c_1 \land c_2 \land c_3) = f(c_1 \land c_j \land c_k),$$

$$\phi \cdot f(c_1 \land a_2 \land b_2) = f(\phi \cdot c_1 \land a_2 \land b_2) = f(c_1 \land a_j \land b_2).$$

Thus, every element of $Hom(\Lambda^3 H_p, \Lambda^3 H_p)^{Sp_{2g}(\mathbb{Z})}$ is determined by the images of the elements $c_1 \land c_2 \land c_3$, $c_1 \land a_2 \land b_2$ with $c_i = a_i$ or $b_i$.

- Consider the element $\phi = \left( \begin{array}{cc} 0 & Id \\ -Id & 0 \end{array} \right) \in Sp_{2g}(\mathbb{Z})$. Then we have that

$$\phi \cdot f(a_1 \land a_2 \land a_3) = f(\phi \cdot a_1 \land a_2 \land a_3) = f(-b_1 \land b_2 \land b_3) = -f(b_1 \land b_2 \land b_3),$$

$$\phi \cdot f(a_1 \land a_2 \land b_3) = f(\phi \cdot a_1 \land a_2 \land b_3) = f(b_1 \land b_2 \land a_3),$$

$$\phi \cdot f(a_1 \land a_2 \land b_2) = f(\phi \cdot a_1 \land a_2 \land b_2) = f(b_1 \land b_2 \land a_2).$$

- Consider the element $\phi \in Sp_{2g}(\mathbb{Z})$, the matrix with 1’s at the diagonal and position $(3, g+3)$, and 0’s elsewhere. Then we have that

$$\phi \cdot f(a_1 \land a_2 \land b_3) = f(\phi \cdot a_1 \land a_2 \land b_3) = f(a_1 \land a_2 \land b_3 + a_1 \land a_2 \land a_3) =$$

$$f(a_1 \land a_2 \land b_3) + f(a_1 \land a_2 \land a_3).$$

So, $f(a_1 \land a_2 \land a_3) = \phi \cdot f(a_1 \land a_2 \land b_3) - f(a_1 \land a_2 \land b_3)$. 

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5.4. The extension of Johnson’s homomorphism modulo $p$

- Consider the element $\phi = \left( \begin{array}{cc} G & 0 \\ 0 & G^{-1} \end{array} \right) \in Sp_{2g}(Z)$, with $G^{-1} \in GL_{2g}(Z)$ the matrix with 1’s at positions (3, 2) and at the diagonal, and 0’s elsewhere. Then we have that

$$
\phi \cdot f(a_1 \wedge a_2 \wedge b_2) = f(\phi \cdot a_1 \wedge a_2 \wedge b_2) = f(a_1 \wedge a_2 \wedge b_2 + a_1 \wedge a_2 \wedge b_3) = 
$$

$$
= f(a_1 \wedge a_2 \wedge b_2) + f(a_1 \wedge a_2 \wedge b_3).
$$

So, $f(a_1 \wedge a_2 \wedge b_3) = \phi \cdot f(a_1 \wedge a_2 \wedge b_2) - f(a_1 \wedge a_2 \wedge b_2)$.

Hence, every element of $Hom(A^3 H_p, A^3 H_p)_{Sp_{2g}(Z)}$ is determined by the image of the element $a_1 \wedge a_2 \wedge b_2$. Next we study the possible images of this element. Set

$$
f(a_1 \wedge a_2 \wedge b_2) = \sum_{i<j<k} m(c_i, c_j, c_k) c_i \wedge c_j \wedge c_k + \sum_{i<s} m(c_i, a_s, b_s) c_i \wedge a_s \wedge b_s,
$$

with $m(c_i, c_j, c_k) \in \{1, \ldots, p-1\}$. Let $l \neq 1$. Consider the element $\phi = \left( \begin{array}{cc} G & 0 \\ 0 & G^{-1} \end{array} \right) \in Sp_{2g}(Z)$, with $G \in GL_{2g}(Z)$ the matrix with 1 at position $(l, l)$ and 1’s at the positions $(r, r)$ with $r \neq l$. Then for $l = g$, we have that

$$
\sum_{i<j<k} m(c_i, c_j, c_k) c_i \wedge c_j \wedge c_k + \sum_{i<s} m(c_i, a_s, b_s) c_i \wedge a_s \wedge b_s = f(a_1 \wedge a_2 \wedge b_2) = f(\phi \cdot a_1 \wedge a_2 \wedge b_2) = \phi \cdot f(a_1 \wedge a_2 \wedge b_2) =
$$

$$
= \sum_{i<j<k<g} m(c_i, c_j, c_k) c_i \wedge c_j \wedge c_k + \sum_{i<s} m(c_i, a_s, b_s) c_i \wedge a_s \wedge b_s + 
$$

$$
- \sum_{i<j<g} m(c_i, c_j, c_g) c_i \wedge c_j \wedge c_g - \sum_s m(c_g, a_s, b_s) c_g \wedge a_s \wedge b_s.
$$

Hence $m(c_i, c_j, c_g) = 0$, for all $i, j$ such that $i < j < g$, and $m(c_g, a_s, b_s) = 0$ for all $s$. Then,

$$
f(a_1 \wedge a_2 \wedge b_2) = \sum_{i<j<k<g} m(c_i, c_j, c_k) c_i \wedge c_j \wedge c_k + \sum_{i<s} m(c_i, a_s, b_s) c_i \wedge a_s \wedge b_s.
$$

Repeating the same argument from $l = g - 1$ to $l = 2$ we get that $m(c_i, c_j, c_k) = 0$, for all $i, j, k$ such that $i < j < k$, and $m(c_i, a_s, b_s) = 0$ for all $i, s$ with $i \neq 1$. Then

$$
f(a_1 \wedge a_2 \wedge b_2) = \sum_j m(c_1, a_j, b_j) c_1 \wedge a_j \wedge b_j.
$$

Next, consider the element $\phi \in Sp_{2g}(Z)$ the matrix with 1’s at the diagonal and position $(1, g + 1)$.
Then we have that
\[
\sum_j m(c_1, a_j, b_j)c_1 \wedge a_j \wedge b_j = f(a_1 \wedge a_2 \wedge b_2) = \phi \cdot f(a_1 \wedge a_2 \wedge b_2) = \sum_j m(c_1, a_j, b_j)c_1 \wedge a_j \wedge b_j + \sum_j m(b_1, a_j, b_j)a_1 \wedge a_j \wedge b_j.
\]
Thus \(m(b_1, a_j, b_j) = 0\) for all \(j\), and we have that
\[
f(a_1 \wedge a_2 \wedge b_2) = \sum_j m(a_1, a_j, b_j)a_1 \wedge a_j \wedge b_j.
\]
For a fixed \(k \neq 1, 2\), consider the element \(\phi = \left( \begin{smallmatrix} G & 0 \\ 0 & G^{-1} \end{smallmatrix} \right) \in Sp_{2g}(\mathbb{Z})\), with \(G \in M_{2g}(\mathbb{Z})\) the matrix with 1's at the positions \((k + 1, k), (l, l)\). Then we have that
\[
\sum_j m(a_1, a_j, b_j)a_1 \wedge a_j \wedge b_j = f(a_1 \wedge a_2 \wedge b_2) = \phi \cdot f(a_1 \wedge a_2 \wedge b_2) = \phi \cdot f(a_1 \wedge a_2 \wedge b_2)
\]
\[
= m(a_1, a_k, b_k)a_1 \wedge a_{k+1} \wedge b_k - m(a_1, a_{k+1}, b_{k+1})a_1 \wedge a_{k+1} \wedge b_k + \sum_j m(a_1, a_j, b_j)a_1 \wedge a_j \wedge b_j.
\]
Thus \(m(a_1, a_k, b_k) = m(a_1, a_{k+1}, b_{k+1})\), for every \(k \neq 1, 2\). Hence,
\[
f(a_1 \wedge a_2 \wedge b_2) = m_1 a_1 \wedge a_2 \wedge b_2 + m_2 \sum_{j=3}^9 a_1 \wedge a_j \wedge b_j,
\]
with \(m_1, m_2 \in \mathbb{Z}/p\). Therefore \(\text{Hom}(\wedge^3 H_p, \wedge^3 H_p)_{Sp_{2g}(\mathbb{Z})}\) at most has \(p^2\) elements.

Observe that the identity is an element of \(\text{Hom}(\wedge^3 H_p, \wedge^3 H_p)_{Sp_{2g}(\mathbb{Z})}\). In addition, if we consider the \(Sp_{2g}(\mathbb{Z}/p)\)-equivariant maps
\[
C : \bigwedge^3 H_p \rightarrow H_p, \quad u : H_p \rightarrow \bigwedge^3 H_p,
\]
defined by
\[
C(a \wedge b \wedge c) = 2[\omega(b, c)a + \omega(c, a)b + \omega(a, b)c],
\]
\[
u(x) = x \wedge \left( \sum_{i=1}^9 b_i \wedge a_i \right).
\]
Then \(r = \frac{1}{2}u \circ C\) is an endomorphism of \(\wedge^3 H_p\) which is \(Sp_{2g}(\mathbb{Z}/p)\)-equivariant because \(\omega\) is \(Sp_{2g}(\mathbb{Z}/p)\)-
invariant and $\sum_{i=1}^{\rho} b_i \wedge a_i$ is a fixed point by the action of $Sp_{2g}(\mathbb{Z}/p)$. In addition we have that

$$r(a_1 \wedge a_2 \wedge b_2) = \frac{1}{2} u(C(a_1 \wedge a_2 \wedge b_2)) = -\sum_{j=2}^{\rho} a_1 \wedge a_j \wedge b_j.$$  

As a consequence, the homomorphism $r$ and the identity are linearly independent and generates $\text{Hom}(\wedge^3 H_p, \wedge^3 H_p)^{Sp_{2g}(\mathbb{Z}/p)}$. Therefore,

$$\text{Hom}(\wedge^3 H_p, \wedge^3 H_p)^{Sp_{2g}(\mathbb{Z}/p)} \cong (\mathbb{Z}/p)^2.$$  

Finally, observe that $Id \circ k_p, (u \circ C) \circ k_p$ are elements of $H^1(\mathcal{M}_{g,1}; \wedge^3 H_p)$, whose restriction to $T_{g,1}$ induce the homomorphisms $Id, (u \circ C) \in \text{Hom}(\wedge^3 H_p, \wedge^3 H_p)^{Sp_{2g}(\mathbb{Z}/p)}$ respectively. Therefore, by the injections (5.4.2), (5.4.3) and the isomorphism (5.4.4), we get the desired isomorphisms. □

As a consequence, the extension of the Johnson’s homomorphism modulo $p$ to $\mathcal{M}_{g,1}$ is unique up to principal derivations and the extension of the Johnson’s homomorphism modulo $p$ to $\mathcal{M}_{g,1}[p]$ is also unique. Therefore, the homomorphism $\tau^Z_1 : \mathcal{M}_{g,1}[p] \rightarrow \wedge^3 H_p$ coincides with the restriction of $k_p$ to $\mathcal{M}_{g,1}[p]$.

**Images of $A_{g,1}[p], B_{g,1}[p]$ under $\tau^Z_1$.** Next, we focus on computing the image of $A_{g,1}[p], B_{g,1}[p]$ under $\tau^Z_1$. First of all we need to compute the abelianization of $Sp_{2g}^B(\mathbb{Z}, p)$ and $Sp_{2g}^A(\mathbb{Z}, p)$.

**Definition 5.4.1.** For $1 \leq i, j \leq n$, denote by $E^g_{i,j}(r)$ the $n \times n$ matrix with an $r$ at position $(i, j)$ and 0’s elsewhere. Similarly, denote by $SE^g_{i,j}$ the $n \times n$ matrix with an $r$ at positions $(i, j)$ and $(j, i)$ and 0’s elsewhere.

**Definition 5.4.2.** For $1 \leq i, j \leq g$, denote by $\gamma^g_{i,j}(r)$ the matrix $\begin{pmatrix} I_{d_g} & 0 \\ SE^g_{i,j}(r) & I_{d_g} \end{pmatrix}$, by $Y^g_{i,j}(r)$ the matrix $\begin{pmatrix} I_{d_g} & 0 \\ 0 & I_{d_g} \end{pmatrix}$, and by $Z^g_{i,j}(r)$ the matrix $\begin{pmatrix} I_{d_g} + E^g_{i,j}(r) & 0 \\ 0 & I_{d_g} - E^g_{i,j}(r) \end{pmatrix}$.

The following Proposition is a direct consequence of results in [34] due to M. Newman and J.R. Smart, and [23] due to R. Lee and R.H. Szczarba.

**Lemma 5.4.1.** Let $g \geq 3$, and $p$ an odd prime, the following sequence is a short exact sequence.

$$1 \longrightarrow Sp_{2g}^B(\mathbb{Z}, p^2) \longrightarrow Sp_{2g}^B(\mathbb{Z}, p) \xrightarrow{\text{abel}} \mathfrak{sl}_{2g}(\mathbb{Z}/p) \oplus S_g(\mathbb{Z}/p) \longrightarrow 1$$

**Proof.** Consider the homomorphism $\text{abel} : Sp_{2g}(\mathbb{Z}, p) \rightarrow \mathfrak{sp}_{2g}(\mathbb{Z}/p)$. In [23], R. Lee and R.H. Szczarba proved that this map induces the following short exact sequence:

$$1 \longrightarrow Sp_{2g}(\mathbb{Z}, p^2) \longrightarrow Sp_{2g}(\mathbb{Z}, p) \xrightarrow{\text{abel}} \mathfrak{sp}_{2g}(\mathbb{Z}/p) \longrightarrow 1.$$
Observe that for any element \( \left( \begin{array}{c} G \\ M & tG^{-1} \end{array} \right) \in Sp_{2g}(\mathbb{Z},p) \) i.e. \( G \in SL_g(\mathbb{Z},p) \), \( ^tGM \in S_g(p\mathbb{Z}) \), we have that

\[
\left( \begin{array}{c} G \\ M & tG^{-1} \end{array} \right) = \left( \begin{array}{c} G \\ 0 & 0 \\ 0 & tG^{-1} \end{array} \right) \left( \begin{array}{c} Id \\ 0 \\ tGM & Id \end{array} \right).
\]

By [23] we get that \((G - Id)/p \) mod \( p \) is an element of \( \mathfrak{sl}_g(\mathbb{Z}/p) \), and clearly \(^tGM/p \) mod \( p \) is an element of \( S_g(\mathbb{Z}/p) \). So we get that

\[
\text{abel}(Sp_{2g}^B(\mathbb{Z},p)) \subset \mathfrak{sl}_g(\mathbb{Z}/p) \oplus S_g(\mathbb{Z}/p).
\]

On the other hand, in [23], R. Lee and R.H. Szczarba proves that the map \( \text{abel} : SL_g[p] \to \mathfrak{sl}_g(\mathbb{Z}/p) \) is surjective, and in [34] M. Newman and J.R. Smart proves that the map \( S_g(p\mathbb{Z}) \to S_g(\mathbb{Z}/p) \) given by \( N \mapsto N/p \) mod \( p \) is also surjective, then it is clear that \( \text{abel} : Sp_{2g}^B(\mathbb{Z},p) \to \mathfrak{sl}_g(\mathbb{Z}/p) \oplus S_g(\mathbb{Z}/p) \) is surjective. Finally, the kernel of this map is given by \( Sp_{2g}(\mathbb{Z},p^2) \cap Sp_{2g}^B(\mathbb{Z},p) \) and this is by definition \( Sp_{2g}^B(\mathbb{Z},p^2) \). Therefore we get the desired short exact sequence.

**Lemma 5.4.2.** Let \( g \geq 3 \), and \( p \) an odd prime,

\[
H_1(Sp_{2g}^B(\mathbb{Z},p); \mathbb{Z}/p) = \mathfrak{sl}_g(\mathbb{Z}/p) \oplus S_g(\mathbb{Z}/p), \quad H_1(Sp_{2g}^A(\mathbb{Z},p); \mathbb{Z}/p) = \mathfrak{sl}_g(\mathbb{Z}/p) \oplus S_g(\mathbb{Z}/p).
\]

**Proof.** We only prove the result for \( Sp_{2g}^B(\mathbb{Z},p) \) (the proof for \( Sp_{2g}^A(\mathbb{Z},p) \) is analogous). We show that \( [Sp_{2g}^B(\mathbb{Z},p), Sp_{2g}^B(\mathbb{Z},p)] = Sp_{2g}^B(\mathbb{Z},p^2) \). Then, by Lemma (5.4.1), we will get the result. Since \( \mathfrak{sl}_g(\mathbb{Z}/p) \oplus S_g(\mathbb{Z}/p) \) is abelian, \([Sp_{2g}^B(\mathbb{Z},p), Sp_{2g}^B(\mathbb{Z},p)] \subset Sp_{2g}^B(\mathbb{Z},p^2) \). Next we show that

\[
Sp_{2g}^B(\mathbb{Z},p^2) \subset [Sp_{2g}^B(\mathbb{Z},p), Sp_{2g}^B(\mathbb{Z},p)].
\]

Observe that for any element \( \left( \begin{array}{c} G \\ M & tG^{-1} \end{array} \right) \in Sp_{2g}^B(\mathbb{Z},p^2) \) i.e. \( G \in SL_g(\mathbb{Z},p^2) \), \( ^tGM \in S_g(p^2\mathbb{Z}) \) we have that

\[
\left( \begin{array}{c} G \\ M & tG^{-1} \end{array} \right) = \left( \begin{array}{c} G \\ 0 & 0 \\ 0 & tG^{-1} \end{array} \right) \left( \begin{array}{c} Id \\ 0 \\ tGM & Id \end{array} \right).
\]

In [23], R. Lee and R. H. Szczarba proved that

\[
SL_g(\mathbb{Z},p^2) = [SL_g(\mathbb{Z},p), SL_g(\mathbb{Z},p)].
\]

Then, for every \( G \in SL_g(\mathbb{Z},p^2) \), there exists a family of elements \( \{G_{i_1}, G_{i_2}\}_{i_1, i_2} \) of \( SL_g(\mathbb{Z},p) \) such
On the other hand, we have that any element of the form \( \begin{pmatrix} G & 0 \\ 0 & tG^{-1} \end{pmatrix} \) with \( H \in S_g(p^2 \mathbb{Z}) \) is a product of elements of the family \( \{ X_{i,j}^g(p^2) \}_{i,j} \), and moreover we have that

\[
X_{i,j}^g(p^2) = [X_{i,j}^g(p), \mathbb{Z}_{i,j}^g(p)] \subset \{ \text{Sp}_{2g}^B(\mathbb{Z}, p), \text{Sp}_{2g}^B(\mathbb{Z}, p) \}.
\]

Thus we get that \( \begin{pmatrix} G & 0 \\ 0 & tG^{-1} \end{pmatrix} \in \{ \text{Sp}_{2g}^B(\mathbb{Z}, p), \text{Sp}_{2g}^B(\mathbb{Z}, p) \} \). Therefore,

\[
\text{Sp}_{2g}^B(\mathbb{Z}, p^2) = \{ \text{Sp}_{2g}^B(\mathbb{Z}, p), \text{Sp}_{2g}^B(\mathbb{Z}, p) \},
\]

as desired. \(\Box\)

**Proposition 5.4.2.** The map \( \text{res} : H^1(\mathcal{B}_{g,1}[p]; \wedge^3 H_p)^{AB_{g,1}} \to H^1(\mathcal{T \mathcal{B}_{g,1}; \wedge^3 H_p})^{AB_{g,1}} \) is injective.

**Proof.** Taking the 5-term sequence associated to the short exact sequence

\[
1 \longrightarrow \mathcal{T \mathcal{B}_{g,1}} \longrightarrow \mathcal{B}_{g,1}[p] \longrightarrow \text{Sp}_{2g}^B(\mathbb{Z}, p) \longrightarrow 1,
\]

and the \( \mathcal{B}_{g,1}[p] \)-module \( \wedge^3 H_p \) with the trivial action, we get the exact sequence

\[
0 \longrightarrow H^1(\text{Sp}_{2g}^B(\mathbb{Z}, p); \wedge^3 H_p) \overset{\text{inf}}{\longrightarrow} H^1(\mathcal{B}_{g,1}[p]; \wedge^3 H_p) \overset{\text{res}}{\longrightarrow} H^1(\mathcal{T \mathcal{B}_{g,1}; \wedge^3 H_p}).
\]

By Universal coefficients Theorem and Lemma \(5.4.2\), we have that

\[
H^1(\text{Sp}_{2g}^B(\mathbb{Z}, p); \wedge^3 H_p) = \text{Hom}(\mathfrak{sl}_g(\mathbb{Z}/p) \oplus S_g(\mathbb{Z}/p), \wedge^3 H_p),
\]

and taking \( AB_{g,1} \)-invariants, we get the exact sequence

\[
0 \longrightarrow \text{Hom}(\mathfrak{sl}_g(\mathbb{Z}/p) \oplus S_g(\mathbb{Z}/p), \wedge^3 H_p)^{AB_{g,1}} \overset{\text{inf}}{\longrightarrow} H^1(\mathcal{B}_{g,1}[p]; \wedge^3 H_p)^{AB_{g,1}} \overset{\text{res}}{\longrightarrow} H^1(\mathcal{T \mathcal{B}_{g,1}; \wedge^3 H_p})^{AB_{g,1}}.
\]

Since \( \Psi(AB_{g,1}) \cong SL_g^*(\mathbb{Z}/p) \), there is an element \( f \) such that \( \Psi(f) = -Id \), which acts trivially on \( \mathfrak{sl}_g(\mathbb{Z}/p) \oplus S_g(\mathbb{Z}/p) \) and as \(-1\) on \( \wedge^3 H_p \). Hence, \( \text{Hom}(\mathfrak{sl}_g(\mathbb{Z}/p) \oplus S_g(\mathbb{Z}/p), \wedge^3 H_p)^{AB_{g,1}} = 0 \). \(\Box\)
Chapter 5. Trivial cocycles and invariants on the (mod \(d\)) Torelli group

Now we are ready to compute the images of \(A_{9,1}[p]\) and \(B_{9,1}[p]\) under \(\tau_1^Z\).

Recall that we have a decomposition \(H = A \oplus B\), this induces the decomposition \(\wedge^3 H = \wedge^3 A \oplus B \wedge (\wedge^2 A) \oplus A \wedge (\wedge^2 B) \oplus \wedge^3 B\). Set \(W_A = \wedge^3 A\), \(W_B = \wedge^3 B\) and \(W_{AB} = B \wedge (\wedge^2 A) \oplus A \wedge (\wedge^2 B)\). The Johnson homomorphism computes the action of the Torelli group on the second nilpotent quotient of the fundamental group of \(\Sigma_{g,1}\). Computing on specific elements one can check that (see [30])

**Lemma 5.4.3.** The image of \(\mathcal{T}A_{9,1}\) and \(\mathcal{T}B_{9,1}\) under \(\tau_1\) in \(\wedge^3 H\) are respectively

\[
W_A \oplus W_{AB} \quad \text{and} \quad W_B \oplus W_{AB}.
\]

Similarly as before, we have a decomposition \(H_p = A_p \oplus B_p\), and this induces the decomposition \(\wedge^3 H_p = \wedge^3 A_p \oplus B_p \wedge (\wedge^2 A_p) \oplus A_p \wedge (\wedge^2 B_p) \oplus \wedge^3 B_p\). Set \(W^p_A = \wedge^3 A_p\), \(W^p_B = \wedge^3 B_p\) and \(W^p_{AB} = B_p \wedge (\wedge^2 A_p) \oplus A_p \wedge (\wedge^2 B_p)\). We get the following lemma analog to (5.4.3).

**Lemma 5.4.4.** The image of \(A_{9,1}[p]\) and \(B_{9,1}[p]\) under \(\tau_1^Z\) in \(\wedge^3 H_p\) are respectively

\[
W^p_A \oplus W^p_{AB} \quad \text{and} \quad W^p_B \oplus W^p_{AB}.
\]

**Proof.** We only do the proof for \(B_{9,1}[p]\). For \(A_{9,1}[p]\) the argument is analogous. Observe that we have a commutative diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & \mathcal{T}B_{9,1} & \longrightarrow & B_{9,1} & \longrightarrow & \Psi \longrightarrow & \wedge^3 S^p_{2g}(\mathbb{Z}) & \longrightarrow & 1 \\
\downarrow & & \tau_1^Z & & \downarrow & & \rho^Z_2(B_{9,1}) & & \rho^Z_2 & \longrightarrow & \wedge^3 S^p_{2g}(\mathbb{Z}/p) & \longrightarrow & 1 \\
0 & \longrightarrow & W^p_B \oplus W^p_{AB} & \longrightarrow & \rho^Z_2(B_{9,1}) & \longrightarrow & S^p_{2g}(\mathbb{Z}/p) & \longrightarrow & 1.
\end{array}
\]

By Center kills Lemma we have that \(H^*(\wedge^3 S^p_{2g}(\mathbb{Z}/p); W^p_B \oplus W^p_{AB}) = 0\). Then the bottom row of the diagram (5.4.5) splits with only one \(W^p_B \oplus W^p_{AB}\)-equivariant homomorphism: \(k_B : B_{9,1} \longrightarrow W^p_B \oplus W^p_{AB}\). Composing \(k_B\) with the natural inclusion \(W^p_B \oplus W^p_{AB} \hookrightarrow \wedge^3 H_p\) we get an \(B_{9,1}\)-equivariant homomorphism:

\[
k_B : B_{9,1} \longrightarrow \wedge^3 H_p.
\]
and by the commutative diagram \([5.4.5]\), and the fact that \(\pi\) is a retraction, we get that \(k_B\) restricted to \(\mathcal{T}B_{g,1}\) is the Johnson homomorphism modulo \(p\).

On the other hand, if we take \(k_p : \mathcal{M}_{g,1}[p] \to \wedge^3 H_p\) restricted to \(\mathcal{B}_{g,1}[p]\) a priori we get another \(\mathcal{B}_{g,1}\)-equivariant homomorphism, which is the Johnson homomorphism modulo \(p\) on \(\mathcal{T}B_{g,1}\), but by Proposition \([5.4.2]\) there is only one \(\mathcal{B}_{g,1}\)-equivariant homomorphism that coincides with the Johnson homomorphism modulo \(p\) on \(\mathcal{T}B_{g,1}\). Hence, the restriction of \(k_p\) to \(\mathcal{B}_{g,1}[p]\) and \(k_B\) have to be equal and so \(k_p(\mathcal{B}_{g,1}[p]) = W_B^p \oplus W_{AB}^p\).

\[\square\]

### 5.4.2 Pull-back of 2-cocycles

For each \(g\), the intersection form on homology induces a bilinear form \(\omega : A \otimes B \to \mathbb{Z}/p\). This in turn induces the bilinear forms \(J_g : W_A^p \otimes W_B^p \to \mathbb{Z}/p\) and \(J^t_g : W_B^p \otimes W_A^p \to \mathbb{Z}/p\) that we extend by 0 to degenerate bilinear forms on \(\wedge^3 H_p = W_A^p \oplus W_{AB}^p \oplus W_B^p\). Written as matrices according to the decomposition \(\wedge^3 H_p = W_A^p \oplus W_{AB}^p \oplus W_B^p\) these are:

\[
J_g \equiv \begin{pmatrix} 0 & 0 & Id \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J^t_g \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Id & 0 & 0 \end{pmatrix}
\]

Notice that bilinear forms are naturally 2-cocycles on abelian groups.

**Proposition 5.4.3.** For each \(g \geq 3\), the 2-cocycle \(J^t_g\) is the unique cocycle (up to a multiplicative constant) on \(\wedge^3 H_p\) whose pull-back along \(k_p\) on the Torelli group modulo \(p\), \(\mathcal{M}_{g,1}[p]\) satisfies conditions (2) and (3). Moreover once we have fixed a common multiplicative constant the family of pull-backed cocycles satisfies also (1).

**Proof.** Fix an integer \(n \in \mathbb{Z}/p\). It is obvious from the definition and from Lemma \([5.4.4]\) that the family \(k_p^n(nJ^t_g)\) satisfies (1), (2) and (3).

Let \(B\) denote an arbitrary 2-cocycle on \(\wedge^3 H_p\) whose pull-back on \(\mathcal{M}_{g,1}[p]\) along \(\tau^Z\) satisfies (2) and (3). Write each element \(w \in \wedge^3 H_p\) as \(w = w_a + w_{ab} + w_b\) according to the decomposition \(W_A^p \oplus W_{AB}^p \oplus W_B^p\).

The cocycle relation together with condition (3) and Lemma \([5.4.4]\) imply that

\[
\forall v, w \in \wedge^3 H_p, \quad B(v, w) = B(v, w_a)
\]
We first prove that $B$ is bilinear. For the linearity on the first variable compute

\[
B(u + v, w) = B(u_b + v_b, w_a)
\]

\[
= B(v_b, w_a) + B(u, v_b + w_a) - B(u_b, v)
\]

\[
= B(u_b, w_a) + B(v, w_a)
\]

\[
= B(u, w) + B(v, w).
\]

where in the second equality we used the cocycle relation. A similar proof holds for the linearity on the second variable.

The equivariance of $k_p$ and the condition (3) and Lemma (5.4.4) implies that our bilinear form $B(x, y)$ should be zero for $y \in \wedge^3 H_p$, $x \in W^p_A \oplus W^p_{AB}$ and $x \in \wedge^3 H_p$, $y \in W^p_B \oplus W^p_{AB}$. By the equivariance properties of $k_p$ we have the following relations:

- Let $i, j, k, l, m, n \in \{1, \ldots g\}$ with $i, j, k$ pairwise distinct, $l, m, k$ pairwise distinct and $n \neq i, j, k, l$. Take $f$ the matrix $\begin{pmatrix} 0 & G & 0 \\ 0 & 0 & 0 \\ G & 0 & -1 \end{pmatrix}$ with $G \in \mathcal{S}_g \subset GL_g(\mathbb{Z})$ for $i \neq j$. Then we have that

\[
B(b_i \wedge b_j \wedge b_k, a_l \wedge a_m \wedge a_k) = B(f.(b_i \wedge b_j \wedge b_k), f.(a_l \wedge a_m \wedge a_k)) = B(b_i \wedge b_j \wedge b_k, a_l \wedge a_m \wedge a_k + a_l \wedge a_m \wedge a_n) = B(b_i \wedge b_j \wedge b_k, a_l \wedge a_m \wedge a_k) + B(b_i \wedge b_j \wedge b_k, a_l \wedge a_m \wedge a_n)
\]

Thus $B(b_i \wedge b_j \wedge b_k, a_l \wedge a_m \wedge a_n) = 0$, where $i, j$ not necessarily equal to $l, m$.

- Take $f$ the matrix $\begin{pmatrix} G & 0 & 0 \\ 0 & 0 & 0 \\ 0 & G & -1 \end{pmatrix}$ with $G = (1i)(2j)(3k) \in \mathcal{S}_g \subset GL_g(\mathbb{Z})$ for $i \neq j \neq k$. Then we have that

\[
B(b_1 \wedge b_2 \wedge b_3, a_1 \wedge a_2 \wedge a_3) = B(f.(b_1 \wedge b_2 \wedge b_3), f.(a_1 \wedge a_2 \wedge a_3)) = B(b_1 \wedge b_2 \wedge b_3, a_i \wedge a_j \wedge a_k)
\]

Thus $B$ must be $nJ_g^1$ for some $n \in \mathbb{Z}/p$. 

CHAPTER 6

Obstruction to Perron’s conjecture

In this chapter, our aim is to give an obstruction to the Perron’s conjecture. The main ideas of this chapter are inspired in the computations in [33] due to S. Morita.

As a starting point, in the first section, we state the Perron’s conjecture. In the second section, in order to study the Perron’s conjecture, we make some cohomological computations with coefficients in \( \mathbb{Z}/p \), which are similar to the cohomological computations with rational coefficients done in [33] by S. Morita. Finally, in the last section, we will use the computations of the second section to give an obstruction to Perron’s conjecture. As we will show in the last section, such obstruction comes from the fact that if the conjectured Perron’s invariant is well-defined, then the cohomology class of the associated 2-cocycle is equal to the restriction of the first characteristic class of surface bundles \( e_1 \in H^2(M_{g,1}; \mathbb{Z}) \), defined in [29], to \( M_{g,1}[p] \) reduced modulo \( p \). But such cohomology class is not zero.

Throughout this chapter we denote by \( \lambda \) the Casson’s invariant, and by \( S_3^g \) the Heegaard splitting \( \mathcal{H}_g \cup_{\psi f} -\mathcal{H}_g \) with \( f \in M_{g,1} \).

6.1 Perron’s conjecture

In [36] B. Perron stated the following conjectures:

Conjecture 6.1.1. Let \( g \geq 3 \) and \( p > 3 \) be a prime number. For every \( f \in T_{g,1} \cap D_{g,1}[p] \), we have that \( \lambda(S_3^f) \equiv 0 \pmod{p} \).

Conjecture 6.1.2. Let \( g \geq 3, p > 3 \) be a prime number and \( S_3^\varphi \in S^3[p] \) with \( \varphi = f \circ m \), where \( f \in T_{g,1} \), \( m \in D_{g,1}[p] \). Then the map \( \gamma_p : M_{g,1}[p] \to \mathbb{Z}/p \) given by

\[
\gamma_p(\varphi) = \lambda(S_3^f) \pmod{p}
\]

is a well defined invariant of \( S^3[p] \).
Remark 6.1.1. In particular in [36] B. Perron states that the conjecture (6.1.1) implies the conjecture (6.1.2). In addition, he asserts that we can reformulate the conjectures (6.1.1), (6.1.2) for all integer coprime with 6 instead of a prime $p > 3$. Unfortunately, the proof of such results are not available in the literature.

6.2 Preliminary results

Before to prove the obstruction’s result to the Perron’s conjecture, we need to do some cohomological computations with $\mathbb{Z}/p$ coefficients. We devote this section to do such computations, some of them quite similar to the cohomological computations with rational coefficients that S. Morita have done in [33].

Proposition 6.2.1. For $g \geq 4$ and $p$ an odd prime, the restriction maps

$$
\text{res} : H^2(Sp_2g(\mathbb{Z}); \mathbb{Z}/p) \to H^2(Sp_2g(\mathbb{Z},p); \mathbb{Z}/p),
$$

$$
\text{res} : H^2(M_{g,1}; \mathbb{Z}/p) \to H^2(M_{g,1}[p]; \mathbb{Z}/p),
$$

are injective homomorphisms.

Proof. Consider the short exact sequence

$$
1 \longrightarrow Sp_2g(\mathbb{Z},p) \longrightarrow Sp_2g(\mathbb{Z}) \overset{r_p}{\longrightarrow} Sp_2g(\mathbb{Z}/p) \longrightarrow 1,
$$

where the surjectivity of $r_p$ was been proved by M. Newmann and J. R. Smart in Theorem 1 in [34]. Taking the 5-term exact sequence associated to the above short exact sequence we get the exact sequence

$$
0 \longrightarrow H^1(Sp_2g(\mathbb{Z}/p); \mathbb{Z}/p) \overset{\text{inf}}{\longrightarrow} H^1(Sp_2g(\mathbb{Z}); \mathbb{Z}/p) \overset{\text{res}}{\longrightarrow} H^1(Sp_2g(\mathbb{Z},p); \mathbb{Z}/p) \overset{\text{tr}}{\longrightarrow}
$$

$$
H^2(Sp_2g(\mathbb{Z}/p); \mathbb{Z}/p) \longrightarrow H^2(Sp_2g(\mathbb{Z}); \mathbb{Z}/p) \longrightarrow H^1(Sp_2g(\mathbb{Z}; p); H^1(Sp_2g(\mathbb{Z},p); \mathbb{Z}/p)).
$$

By Lemma 3.7 and Theorem 3.8 in [40], we know that $H_2(Sp_2g(\mathbb{Z}); \mathbb{Z}) = 0$, $H_1(Sp_2g(\mathbb{Z}); \mathbb{Z}) = 0$. Then, by Universal coefficients Theorem, we have that

$$
H^2(Sp(\mathbb{Z}/p); \mathbb{Z}/p) \cong \text{Hom}(H_2(Sp(\mathbb{Z}/p); \mathbb{Z}), \mathbb{Z}/p) \oplus \text{Ext}^1_\mathbb{Z}(H_1(Sp(\mathbb{Z}/p); \mathbb{Z}), \mathbb{Z}/p)
$$

$$
= \text{Hom}(0, \mathbb{Z}/p) \oplus \text{Ext}^1_\mathbb{Z}(0, \mathbb{Z}/p) = 0.
$$

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Moreover, by Universal coefficients Theorem we have that

\[ H^1(\text{Sp}_{2g}(\mathbb{Z}/p); \mathbb{Z}/p) = \text{Hom}(H_1(\text{Sp}_{2g}(\mathbb{Z}/p), \mathbb{Z}/p) \oplus \text{Ext}_1^1(H_0(\text{Sp}_{2g}(\mathbb{Z}/p); \mathbb{Z}), \mathbb{Z}/p) = (\text{sp}_{2g}(\mathbb{Z}/p))^* \oplus \text{Ext}_1^1(\mathbb{Z}, \mathbb{Z}/p) = (\text{sp}_{2g}(\mathbb{Z}/p))^*. \]

Then \( H^1(\text{Sp}_{2g}(\mathbb{Z}/p); H^1(\text{Sp}_{2g}(\mathbb{Z}/p); \mathbb{Z}/p)) = H^1(\text{Sp}_{2g}(\mathbb{Z}/p); (\text{sp}_{2g}(\mathbb{Z}/p))^*) \).

By Lemma (1.1.2) we have that

\[ H^1(\text{Sp}_{2g}(\mathbb{Z}/p); (\text{sp}_{2g}(\mathbb{Z}/p))^*) \cong (H_1(\text{Sp}_{2g}(\mathbb{Z}/p); \text{sp}_{2g}(\mathbb{Z}/p)))^*. \]

By Theorem G in [40], for \( g \geq 3 \) and \( L \geq 2 \) such that \( 4 \nmid L \),

\[ H_1(\text{Sp}_{2g}(\mathbb{Z}/L); \text{sp}_{2g}(\mathbb{Z}/L)) = 0. \]

Thus \( H^1(\text{Sp}_{2g}(\mathbb{Z}/p); (\text{sp}_{2g}(\mathbb{Z}/p))^*) = 0 \), and as a consequence we get that

\[ H^2(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Z}/p)_1 = 0. \]

Therefore we get an injective homomorphism

\[ \text{res}: H^2(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Z}/p) \to H^2(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Z}/p). \]

Consider the commutative diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & \mathcal{T}_{g,1} & \longrightarrow & \mathcal{M}_{g,1}[p] & \longrightarrow & \text{Sp}_{2g}(\mathbb{Z}, p) & \longrightarrow & 1 \\
| & | & | & | & | & | & | \\
1 & \longrightarrow & \mathcal{T}_{g,1} & \longrightarrow & \mathcal{M}_{g,1} & \longrightarrow & \text{Sp}_{2g}(\mathbb{Z}) & \longrightarrow & 1,
\end{array}
\]

where the rows are short exact sequences. By naturality of the 5-term exact sequence, we get the commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^1(\text{Sp}_{2g}(\mathbb{Z}); \mathbb{Z}/p) & \overset{\text{inf}}{\longrightarrow} & H^1(\mathcal{M}_{g,1}; \mathbb{Z}/p) & \overset{\text{res}}{\longrightarrow} & H^1(\mathcal{T}_{g,1}; \mathbb{Z}/p)^{\text{Sp}_{2g}(\mathbb{Z})} \\
| & \downarrow & \uparrow & & \downarrow & \uparrow & \downarrow \\
0 & \longrightarrow & H^1(\text{Sp}_{2g}(\mathbb{Z}, p); \mathbb{Z}/p) & \overset{\text{inf}}{\longrightarrow} & H^1(\mathcal{M}_{g,1}[p]; \mathbb{Z}/p) & \overset{\text{res}}{\longrightarrow} & H^1(\mathcal{T}_{g,1}; \mathbb{Z}/p)^{\text{Sp}_{2g}(\mathbb{Z}, p)}
\end{array}
\]

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\[ \xymatrix{ H^2(Sp_{2g}(\mathbb{Z});\mathbb{Z}/p) \ar[r]^\inf \ar[d]_{\text{res}} & H^2(M_{g,1};\mathbb{Z}/p) \ar[d]^\text{res} \\
H^2(Sp_{2g}(\mathbb{Z},p);\mathbb{Z}/p) \ar[r]^\inf & H^2(M_{g,1}[p];\mathbb{Z}/p). } \]

Next we prove that \( \inf : H^2(Sp_{2g}(\mathbb{Z},p);\mathbb{Z}/p) \to H^2(M_{g,1}[p];\mathbb{Z}/p) \) is injective. By exactness of the 5-term exact sequence, it is enough to prove that the homomorphism \( \text{res} : H^1(M_{g,1}[p],\mathbb{Z}/p) \to H^1(T_{g,1},\mathbb{Z}/p) \) is surjective.

By Universal coefficients Theorem we have isomorphisms

\[
\begin{align*}
H^1(M_{g,1}[p],\mathbb{Z}/p) &\cong Hom(H_1(M_{g,1}[p]),\mathbb{Z}/p), \\
H^1(T_{g,1},\mathbb{Z}/p) &\cong Hom(H_1(T_{g,1}),\mathbb{Z}/p)Sp_{2g}(\mathbb{Z},p) \cong Hom(\bigwedge^3 H_p,\mathbb{Z}/p), \\
H^1(Sp_{2g}(\mathbb{Z},p);\mathbb{Z}/p) &\cong Hom(H_1(Sp_{2g}(\mathbb{Z},p)),\mathbb{Z}/p) \cong Hom(sp_{2g}(\mathbb{Z}/p),\mathbb{Z}/p).
\end{align*}
\]

By Theorem 5.12 in [4] we have a split extension of \( \mathbb{Z}/p \)-modules

\[
0 \longrightarrow \bigwedge^3 H_p \longrightarrow H_1(M_{g,1}[p]) \longrightarrow sp_{2g}(\mathbb{Z}/p) \longrightarrow 0.
\]

Then, applying the contravariant functor \( Hom(\ ,\mathbb{Z}/p) \), we get a short exact sequence of \( \mathbb{Z}/p \)-modules

\[
0 \longrightarrow Hom(sp_{2g}(\mathbb{Z}/p),\mathbb{Z}/p) \longrightarrow Hom(H_1(M_{g,1}[p]),\mathbb{Z}/p) \longrightarrow Hom(\bigwedge^3 H_p,\mathbb{Z}/p) \longrightarrow 0.
\]

Therefore the homomorphism \( \text{res} : H^1(M_{g,1}[p],\mathbb{Z}/p) \to H^1(T_{g,1},\mathbb{Z}/p) \) is surjective.

Next we show that \( \inf : H^2(Sp_{2g}(\mathbb{Z});\mathbb{Z}/p) \to H^2(M_{g,1};\mathbb{Z}/p) \) is an isomorphism. By Universal coefficients Theorem we have that

\[
H^1(T_{g,1},\mathbb{Z}/p)Sp_{2g}(\mathbb{Z}) \cong Hom(H_1(T_{g,1}),\mathbb{Z}/p)Sp_{2g}(\mathbb{Z}) \cong Hom(\bigwedge^3 H_p,\mathbb{Z}/p)^{Sp_{2g}(\mathbb{Z})} = 0.
\]

Thus, by exactness of the 5-term exact sequence, the map \( \inf : H^2(Sp_{2g}(\mathbb{Z});\mathbb{Z}/p) \to H^2(M_{g,1};\mathbb{Z}/p) \) is injective. Moreover, by the Universal coefficients Theorem, Theorem 5.1 in [40] and Theorem 5.8 in [8], we get that \( H^2(Sp_{2g}(\mathbb{Z});\mathbb{Z}/p) \cong \mathbb{Z}/p \) and \( H^2(M_{g,1};\mathbb{Z}/p) \cong \mathbb{Z}/p \). Hence the map \( \inf \) is an isomorphism.

As a consequence, we have the following commutative diagram:

\[
\begin{array}{ccc}
H^2(Sp_{2g}(\mathbb{Z});\mathbb{Z}/p) & \xrightarrow{\inf} & H^2(M_{g,1};\mathbb{Z}/p) \\
\downarrow \text{res} & & \downarrow \text{res} \\
H^2(Sp_{2g}(\mathbb{Z},p);\mathbb{Z}/p) & \xrightarrow{\inf} & H^2(M_{g,1}[p];\mathbb{Z}/p).
\end{array}
\]

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Therefore, the homomorphism

\[ \text{res}: H^2(M_{g,1}; \mathbb{Z}/p) \to H^2(M_{g,1}[p]; \mathbb{Z}/p) \]

is also injective.

**Definition 6.2.1.** Denote by \( S^2(\Lambda^2 H_p) \) the submodule of \( \Lambda^2 H_p \otimes \Lambda^2 H_p \) generated by all elements of the form:

i) \( (a \wedge b) \otimes (a \wedge b) \), also denoted by \( (a \wedge b)^{\otimes 2} \),

ii) \( (a \wedge b) \otimes (c \wedge d) + (c \wedge d) \otimes (a \wedge b) \), also denoted by \( a \wedge b \leftrightarrow c \wedge d \).

(where \( a, b, c, d \in H_p \)).

**Remark 6.2.1.** Observe that in Definition 6.2.1, if \( p \) is an odd prime, the elements of the form

ii) generate \( S^2(\Lambda^2 H_p) \) since \( (a \wedge b)^{\otimes 2} = (p + 1)(a \wedge b)^{\otimes 2} = \frac{p+1}{2}(a \wedge b \leftrightarrow a \wedge b) \)

In Theorem 3.1 in [31], S. Morita considered the 2-cocycle associated to following central extension

\[
0 \longrightarrow \text{Im}(\tau_2) \longrightarrow \rho_3(T_{g,1}) \xrightarrow{\tau_1} \Lambda^3 H \longrightarrow 1
\] (6.2.2)

and he determined the explicit corresponding map of \( \text{Hom}(\Lambda^2(\Lambda^3 H); \text{Im}(\tau_2)) \) as \( \pi \circ \chi \) where \( \chi \in \text{Hom}(\Lambda^2(\Lambda^3 H); S^2(\Lambda^2 H)) \) is given by

\[
\chi(\xi \wedge \eta) = -(a \cdot d)b \wedge c \leftrightarrow e \wedge f - (a \cdot e)b \wedge c \leftrightarrow f \wedge d - (a \cdot f)b \wedge c \leftrightarrow d \wedge e
\]

\[
- (b \cdot d)c \wedge a \leftrightarrow e \wedge f - (b \cdot e)c \wedge a \leftrightarrow f \wedge d - (b \cdot f)c \wedge a \leftrightarrow d \wedge e
\]

\[
- (c \cdot d)a \wedge b \leftrightarrow e \wedge f - (c \cdot e)a \wedge b \leftrightarrow f \wedge d - (c \cdot f)a \wedge b \leftrightarrow d \wedge e,
\]

with \( \xi = a \wedge b \wedge c, \eta = d \wedge e \wedge f \in \Lambda^3 H \) \((a, b, c, d, e, f \in H)\). And \( \pi \) is the following natural projection map of \( \text{Hom}(S^2(\Lambda^2 H), H \otimes \mathcal{L}_3) \) given by

\[
\pi(a \wedge b \leftrightarrow c \wedge d) = a \otimes [b, [c, d]] - b \otimes [a, [c, d]] + c \otimes [d, [a, b]] - d \otimes [c, [a, b]].
\]

We will denote by \( \chi_p \in \text{Hom}(\Lambda^2(\Lambda^3 H_p); \text{Im}(\tau_{2,p})) \) and \( \pi_p \in \text{Hom}(S^2(\Lambda^2 H_p), H_p \otimes \mathcal{L}_{3,p}) \) the reduction modulo \( p \) of the respective functions \( \chi \) and \( \pi \).

As a direct consequence of Section 5 in [31], we have that

**Proposition 6.2.2.** The kernel of \( \pi_p: S^2(\Lambda^2 H_p) \to H_p \otimes \mathcal{L}_{3,p} \) is generated by the elements of the form

\[
a \wedge b \leftrightarrow c \wedge d - a \wedge c \leftrightarrow b \wedge d + a \wedge d \leftrightarrow b \wedge c.
\]
Proposition 6.2.3. Let \( p \) be an odd prime, then \( \text{Hom}(S^2(\wedge^2 H_p), \mathbb{Z}/p)^{AB_{p,1}} \cong (\mathbb{Z}/p)^3 \) with a basis

\[
\begin{align*}
    d_1(a \wedge b \leftrightarrow c \wedge d) &= \omega(a, b)\omega(c, d), \\
    d_2(a \wedge b \leftrightarrow c \wedge d) &= \omega(a, c)\omega(b, d) - \omega(a, d)\omega(b, c), \\
    d_3(a \wedge b \leftrightarrow c \wedge d) &= \overline{\omega}(a, c)\overline{\omega}(b, d) - \overline{\omega}(a, d)\overline{\omega}(b, c),
\end{align*}
\]

where \( \omega \) is the intersection form and \( \overline{\omega} \) is the form associated to the matrix \( \left( \begin{array}{cc} 0 & Id \\ Id & 0 \end{array} \right) \).

Proof. We first show that every element \( f \in \text{Hom}(S^2(\wedge^2 H_p), \mathbb{Z}/p)^{AB_{p,1}} \) is completely determined by its value on the following elements:

\[ \{b_1 \wedge b_2 \leftrightarrow a_1 \wedge a_2, \ b_1 \wedge a_2 \leftrightarrow a_1 \wedge b_2, \ a_1 \wedge b_1 \leftrightarrow a_2 \wedge b_2\}. \]

- Let \( i, j, k, l \) be pairwise distinct sub-indexes. Consider the element \( \phi = \left( \begin{array}{cc} G & 0 \\ 0 & \iota_G^{-1} \end{array} \right) \in \text{Sp}_{2g}(\mathbb{Z}) \) with \( G = (1, i)(2, j)(3, k)(4, l) \in \mathfrak{S}_g \). Then we have that

\[
    f(c_i \wedge c_j' \leftrightarrow d_k \wedge d_l') = f(\phi \cdot (c_i \wedge c_j' \leftrightarrow d_k \wedge d_l')) = f(c_1 \wedge c_2' \leftrightarrow d_3 \wedge d_4').
\]

Consider the element \( \phi = \left( \begin{array}{cc} G & 0 \\ 0 & \iota_G^{-1} \end{array} \right) \in \text{Sp}_{2g}(\mathbb{Z}) \) with \( G = (1, 2) \in \mathfrak{S}_g \). Then we have that

\[
    f(c_1 \wedge c_2 \leftrightarrow d_3 \wedge d_4') = f(\phi \cdot (c_1 \wedge c_2 \leftrightarrow d_3 \wedge d_4')) = f(c_2 \wedge c_1 \leftrightarrow d_3 \wedge d_4') = - f(c_1 \wedge c_2 \leftrightarrow d_3 \wedge d_4').
\]

Thus \( f(c_1 \wedge c_2 \leftrightarrow d_3 \wedge d_4') = 0 \).

Consider the element \( \phi = \left( \begin{array}{cc} G & 0 \\ 0 & \iota_G^{-1} \end{array} \right) \in \text{Sp}_{2g}(\mathbb{Z}) \) with \( G \in GL_g(\mathbb{Z}) \), with 1’s at the diagonal and position \((2, 1)\), and 0’s elsewhere. Then we have that

\[
    f(a_1 \wedge b_1 \leftrightarrow d_3 \wedge d_4') = f(\phi \cdot (a_1 \wedge b_1 \leftrightarrow d_3 \wedge d_4')) = f(a_1 \wedge b_1 \leftrightarrow d_3 \wedge d_4') + f(a_2 \wedge b_1 \leftrightarrow d_3 \wedge d_4').
\]

Thus \( f(a_2 \wedge b_1 \leftrightarrow d_3 \wedge d_4') = 0 \).

Consider the element \( \phi = \left( \begin{array}{cc} G & 0 \\ 0 & \iota_G^{-1} \end{array} \right) \in \text{Sp}_{2g}(\mathbb{Z}) \) with \( G = (1, 2) \in \mathfrak{S}_g \). Then we get that

\[
    f(a_1 \wedge b_2 \leftrightarrow d_3 \wedge d_4') = f(\phi \cdot (a_1 \wedge b_2 \leftrightarrow d_3 \wedge d_4')) = f(a_2 \wedge b_1 \leftrightarrow d_3 \wedge d_4') = 0.
\]

Therefore \( f(c_i \wedge c_j' \leftrightarrow d_k \wedge d_l') = 0 \) with \( i, j, k, l \) pairwise distinct.

- Suppose that there are two sub-indexes equal and the other are different. Then we have to
study the image of \( f \in \text{Hom}(S^2(\Lambda^2 H_p), \mathbb{Z}/p)^{\text{AB}_{g,1}} \) on the elements of the form:

\[
\begin{align*}
c_i \wedge c'_i & \leftrightarrow d_j \wedge d'_k, \\
c_i \wedge c'_j & \leftrightarrow d_i \wedge d'_i, \\
c_j \wedge c'_i & \leftrightarrow d_i \wedge d'_i, \\
c_j \wedge c'_j & \leftrightarrow d_k \wedge d'_i.
\end{align*}
\]

But observe that

\[
c_i \wedge c'_i \leftrightarrow d_j \wedge d'_k = \leftrightarrow c_i \wedge c'_i,
\]

\[
c_i \wedge c'_j \leftrightarrow d_i \wedge d'_i = -c_i \wedge c_j \leftrightarrow d'_k \wedge d_i = c_j \wedge c_i \leftrightarrow d'_k \wedge d_i = -c'_j \wedge c_i \leftrightarrow d_i \wedge d'_k.
\]

Thus, it is enough to study image of \( f \in \text{Hom}(S^2(\Lambda^2 H_p), \mathbb{Z}/p)^{\text{AB}_{g,1}} \) on the elements of the form:

\[
c_i \wedge c'_i \leftrightarrow d_j \wedge d'_k, \quad c_i \wedge c'_j \leftrightarrow d_i \wedge d'_i.
\]

Consider the element \( \phi = \left( \begin{smallmatrix} G & 0 \\ 0 & G^{-1} \end{smallmatrix} \right) \in \text{Sp}_{2g}(\mathbb{Z}) \) with \( G = (1i)(2j)(3k) \in \mathfrak{S}_g \). Then we get that

\[
\begin{align*}
f(c_i \wedge c'_i \leftrightarrow d_j \wedge d'_k) &= f\left( \phi \cdot (c_i \wedge c'_i \leftrightarrow d_j \wedge d'_k) \right) = f(c_i \wedge c'_i \leftrightarrow d_2 \wedge d'_3), \\
f(c_i \wedge c'_j \leftrightarrow d_i \wedge d'_i) &= f\left( \phi \cdot (c_i \wedge c'_j \leftrightarrow d_i \wedge d'_i) \right) = f(c_i \wedge c'_2 \leftrightarrow d_1 \wedge d'_3).
\end{align*}
\]

Consider the element \( \phi = \left( \begin{smallmatrix} G & 0 \\ 0 & G^{-1} \end{smallmatrix} \right) \in \text{Sp}_{2g}(\mathbb{Z}) \) with \( G \in \text{GL}_g(\mathbb{Z}) \), with 1’s at the diagonal and position (2,3), and 0’s elsewhere. Then we have that

\[
\begin{align*}
f(c_1 \wedge c'_i \leftrightarrow a_3 \wedge b_3) &= f\left( \phi \cdot (c_1 \wedge c'_i \leftrightarrow a_3 \wedge b_3) \right) = f(c_1 \wedge c'_i \leftrightarrow (a_2 + a_3) \wedge b_3) = f(c_1 \wedge c'_i \leftrightarrow a_2 \wedge b_3) + f(c_1 \wedge c'_i \leftrightarrow a_3 \wedge b_3), \\
f(c_1 \wedge a_3 \leftrightarrow d_1 \wedge b_3) &= f\left( \phi \cdot (c_1 \wedge a_3 \leftrightarrow d_1 \wedge b_3) \right) = f(c_1 \wedge (a_2 + a_3) \leftrightarrow d_1 \wedge b_3) = f(c_1 \wedge a_2 \leftrightarrow d_1 \wedge b_3) + f(c_1 \wedge a_3 \leftrightarrow d_1 \wedge b_3).
\end{align*}
\]

Thus,

\[
f(c_1 \wedge c'_i \leftrightarrow a_2 \wedge b_3) = 0, \quad f(c_1 \wedge a_2 \leftrightarrow d_1 \wedge b_3) = 0.
\]

Taking the element \( \phi = \left( \begin{smallmatrix} G & 0 \\ 0 & G^{-1} \end{smallmatrix} \right) \in \text{Sp}_{2g}(\mathbb{Z}) \) with \( G = (23) \in \mathfrak{S}_g \) we also get that

\[
f(c_1 \wedge c'_i \leftrightarrow b_2 \wedge a_3) = 0, \quad f(c_1 \wedge b_2 \leftrightarrow d_1 \wedge a_3) = 0.
\]
Moreover taking the same element we have that

\[ f(c_1 \land c'_1 \leftrightarrow d_2 \land d_3) = f(\phi \cdot (c_1 \land c'_1 \leftrightarrow d_2 \land d_3)) = f(c_1 \land c'_1 \leftrightarrow d_3 \land d_2) = -f(c_1 \land c'_1 \leftrightarrow d_2 \land d_3). \]

Thus \( f(c_1 \land c'_1 \leftrightarrow d_2 \land d_3) = 0. \) Therefore \( f(c_1 \land c'_1 \leftrightarrow d_2 \land d'_3) = 0. \)

Then, it is enough to check the value of \( f(c_1 \land a_2 \leftrightarrow d_1 \land a_3), \) and \( f(c_1 \land b_2 \leftrightarrow d_1 \land b_3). \)

Consider the element \( \phi = \left( \begin{array}{cc} G & 0 \\ 0 & tG^{-1} \end{array} \right) \in Sp_{2g}(\mathbb{Z}) \) with \( G \in GL_g(\mathbb{Z}), \) with 1’s at the diagonal and position (2, 3), and 0’s elsewhere. Then we have that

\[ f(c_1 \land a_3 \leftrightarrow c'_1 \land a_3) = f(\phi \cdot (c_1 \land a_3 \leftrightarrow c'_1 \land a_3)) = f(c_1 \land (a_2 + a_3) \leftrightarrow c'_1 \land (a_2 + a_3)) = f(c_1 \land a_2 \leftrightarrow c'_1 \land a_2) + f(c_1 \land a_2 \leftrightarrow c'_1 \land a_3) + f(c_1 \land a_3 \leftrightarrow c'_1 \land a_2) + f(c_1 \land a_3 \leftrightarrow c'_1 \land a_3). \]

Thus \( f(c_1 \land a_2 \leftrightarrow c'_1 \land a_2) = -f(c_1 \land a_2 \leftrightarrow c'_1 \land a_3) - f(c_1 \land a_3 \leftrightarrow c'_1 \land a_2). \)

Now using the element \( \phi = \left( \begin{array}{cc} G & 0 \\ 0 & tG^{-1} \end{array} \right) \in Sp_{2g}(\mathbb{Z}) \) with \( G = (2, 3) \in \mathfrak{S}_g \) we have that \( f(c_1 \land a_2 \leftrightarrow c'_1 \land a_3) = f(c_1 \land a_3 \leftrightarrow c'_1 \land a_2). \) And as a consequence we get:

\[ f(c_1 \land a_2 \leftrightarrow c'_1 \land a_2) = -2f(c_1 \land a_2 \leftrightarrow c'_1 \land a_3). \]

Consider the element \( \phi = \left( \begin{array}{cc} G & 0 \\ 0 & tG^{-1} \end{array} \right) \in Sp_{2g}(\mathbb{Z}) \) with \( G \in GL_g(\mathbb{Z}), \) with 1’s at the diagonal and position (2, 3), and 0’s elsewhere. Then we get that

\[ f(c_1 \land a_2 \leftrightarrow c'_1 \land a_3) = f(c_1 \land a_2 \leftrightarrow c'_1 \land (a_2 + a_3)). \]

Thus \( f(c_1 \land a_2 \leftrightarrow c'_1 \land a_2) = 0, \) and therefore \( f(c_1 \land a_2 \leftrightarrow c'_1 \land a_3) = 0. \) Similarly switching \( G \) by \( tG^{-1} \) and the \( a \)’s by \( b \)’s, we also get that \( f(c_1 \land b_2 \leftrightarrow c'_1 \land b_3) = 0. \)

- Suppose that three indexes of \( i, j, k, l \) are equal and the other one is different. Then taking a suitable element \( \phi = \left( \begin{array}{cc} G & 0 \\ 0 & tG^{-1} \end{array} \right) \in Sp_{2g}(\mathbb{Z}) \) with \( G \in \mathfrak{S}_g, \) we get that

\[ f(c_i \land c'_i \leftrightarrow d_i \land d'_j) = f(c_1 \land c'_1 \leftrightarrow d_1 \land d'_2). \]

Consider the element \( \phi = \left( \begin{array}{cc} G & 0 \\ 0 & tG^{-1} \end{array} \right) \in Sp_{2g}(\mathbb{Z}) \) with \( G \in GL_g(\mathbb{Z}) \) with 1’s at the diagonal and
that an element $f$ of the following elements:

Thus $f(c_1 \land c'_1 \leftrightarrow d_1 \land d'_2) = 0$.

Summarizing, an element $f \in Hom(S^2(\wedge^2 H_p), \mathbb{Z}/p)^{AB_{h_1}}$ is completely determined by the image of $f$ on the following elements:

In particular if we take the element $\phi = (\begin{smallmatrix} G & 0 \\ 0 & G^{-1} \end{smallmatrix}) \in Sp_{2g}(\mathbb{Z})$ with $G = (1, i)(2, j) \in S_g$. Then we get that an element $f \in Hom(S^2(\wedge^2 H_p), \mathbb{Z}/p)^{AB_{h_1}}$ is completely determined by the image of $f$ on the following elements:

- Now take the element $\phi = (\begin{smallmatrix} G & 0 \\ 0 & G^{-1} \end{smallmatrix}) \in Sp_{2g}(\mathbb{Z})$ with $G \in GL_g(\mathbb{Z})$, with 1’s at the diagonal and 0’s elsewhere. Then we get that

since by above relations we know that $f(c_1 \land c'_3 \leftrightarrow c'_1 \land a_3) = 0$. Thus we get that

Analogously, we get that $f(c_1 \land b_2 \leftrightarrow c'_1 \land b_2) = 0$. Therefore the only elements of the form $c_1 \land c'_2 \leftrightarrow d_1 \land d'_2$ with $f(c_1 \land c'_2 \leftrightarrow d_1 \land d'_2) = 0$ are

$b_1 \land b_2 \leftrightarrow a_1 \land a_2$, $b_1 \land a_2 \leftrightarrow a_1 \land b_2$. 
Chapter 6. Obstruction to Perron’s conjecture

- Now observe that the elements of the form $c_1 \land c_1' \leftrightarrow d_2 \land d_2'$ are:

$$a_1 \land b_1 \leftrightarrow a_2 \land b_2, \quad b_1 \land a_1 \leftrightarrow a_2 \land b_2,$$
$$a_1 \land b_1 \leftrightarrow b_2 \land a_2, \quad b_1 \land a_1 \leftrightarrow b_2 \land a_2.$$  

Moreover we have that

$$f(a_1 \land b_1 \leftrightarrow a_2 \land b_2) = -f(b_1 \land a_1 \leftrightarrow a_2 \land b_2) =$$
$$= -f(a_1 \land b_1 \leftrightarrow b_2 \land a_2) = f(b_1 \land a_1 \leftrightarrow b_2 \land a_2).$$

- Now observe that the elements of the form $c_1 \land c_1' \leftrightarrow d_1 \land d_1'$ are:

$$a_1 \land b_1 \leftrightarrow a_1 \land b_1, \quad a_1 \land b_1 \leftrightarrow b_1 \land a_1.$$  

Moreover we have that $f(a_1 \land b_1 \leftrightarrow a_1 \land b_1) = -f(a_1 \land b_1 \leftrightarrow b_1 \land a_1)$.

Consider the element $\phi = \left( \begin{smallmatrix} G & 0 \\ 0 & G^{-1} \end{smallmatrix} \right) \in Sp_{2g}(\mathbb{Z})$ with $G \in GL_g(\mathbb{Z})$, with 1’s at the diagonal and position (1,2), and 0’s elsewhere. By the previous relations, we have that

$$0 = f(a_1 \land b_2 \leftrightarrow a_1 \land b_2) = f(\phi \cdot (a_1 \land b_2 \leftrightarrow a_1 \land b_2)) =$$
$$= f((a_1 + a_2) \land (-b_1 + b_2) \leftrightarrow (a_1 + a_2) \land (-b_1 + b_2)) =$$
$$= f(a_1 \land b_1 \leftrightarrow a_1 \land b_1) - f(a_1 \land b_1 \leftrightarrow a_2 \land b_2) +$$
$$- f(a_1 \land b_2 \leftrightarrow a_2 \land b_1) - f(a_2 \land b_1 \leftrightarrow a_1 \land b_2) +$$
$$- f(a_2 \land b_2 \leftrightarrow a_1 \land b_1) + f(a_2 \land b_2 \leftrightarrow a_2 \land b_2) =$$
$$= f(a_1 \land b_1 \leftrightarrow a_1 \land b_1) + f(a_2 \land b_2 \leftrightarrow a_2 \land b_2) +$$
$$- 2f(a_1 \land b_1 \leftrightarrow a_2 \land b_2) - 2f(a_1 \land b_2 \leftrightarrow a_2 \land b_1).$$

Thus, by the previous relations and the fact that 2 is invertible in $\mathbb{Z}/p$, we get that

$$f(a_1 \land b_1 \leftrightarrow a_1 \land b_1) = f(a_1 \land b_1 \leftrightarrow a_2 \land b_2) + f(a_1 \land b_2 \leftrightarrow a_2 \land b_1).$$

Therefore every element $f \in Hom(S^2(\Lambda^2 H_p), \mathbb{Z}/p)^{AB_{p,1}}$ is completely determined by its image on the following elements:

$$b_1 \land b_2 \leftrightarrow a_1 \land a_2, \quad b_1 \land a_2 \leftrightarrow a_1 \land b_2,$$
$$a_1 \land b_1 \leftrightarrow a_2 \land b_2.$$
Next consider the following maps:

\[ d_1(a \wedge b \leftrightarrow c \wedge d) = \omega(a, b)\omega(c, d), \]
\[ d_2(a \wedge b \leftrightarrow c \wedge d) = \omega(a, c)\omega(b, d) - \omega(a, d)\omega(b, c), \]
\[ d_3(a \wedge b \leftrightarrow c \wedge d) = \varpi(a, c)\varpi(b, d) - \varpi(a, d)\varpi(b, c), \]

where \( \omega \) is the intersection form and \( \varpi \) is the form associated to the matrix \( \begin{pmatrix} 0 & Id \end{pmatrix} \). Notice that these maps are \( \mathcal{A}\mathcal{B}_{g,1} \)-invariant homomorphisms. Moreover, observe that

\[ d_1(b_1 \wedge b_2 \leftrightarrow a_1 \wedge a_2) = \omega(b_1, b_2)\omega(a_1, a_2) = 0, \]
\[ d_1(b_1 \wedge a_2 \leftrightarrow a_1 \wedge b_2) = \omega(b_1, a_2)\omega(a_1, b_2) = 0, \]
\[ d_1(a_1 \wedge b_1 \leftrightarrow a_2 \wedge b_2) = \omega(a_1, b_1)\omega(a_2, b_2) = 1. \]

\[ d_2(b_1 \wedge b_2 \leftrightarrow a_1 \wedge a_2) = \omega(b_1, a_1)\omega(b_2, a_2) - \omega(b_1, a_2)\omega(b_2, a_1) = 1, \]
\[ d_2(b_1 \wedge a_2 \leftrightarrow a_1 \wedge b_2) = \omega(b_1, a_1)\omega(a_2, b_2) - \omega(b_1, b_2)\omega(a_2, a_1) = -1, \]
\[ d_2(a_1 \wedge b_1 \leftrightarrow a_2 \wedge b_2) = \omega(a_1, b_1)\omega(b_1, b_2) - \omega(a_1, b_2)\omega(b_1, a_2) = 0. \]

\[ d_3(b_1 \wedge b_2 \leftrightarrow a_1 \wedge a_2) = \varpi(b_1, a_1)\varpi(b_2, a_2) - \varpi(b_1, a_2)\varpi(b_2, a_1) = 1, \]
\[ d_3(b_1 \wedge a_2 \leftrightarrow a_1 \wedge b_2) = \varpi(b_1, a_1)\varpi(a_2, b_2) - \varpi(b_1, b_2)\varpi(a_2, a_1) = 1, \]
\[ d_3(a_1 \wedge b_1 \leftrightarrow a_2 \wedge b_2) = \varpi(a_1, a_2)\varpi(b_1, b_2) - \varpi(a_1, b_2)\varpi(b_1, a_2) = 0. \]

Therefore \( d_1, d_2, d_3 \) are linearly independent \( \mathcal{A}\mathcal{B}_{g,1} \)-invariant homomorphisms. Hence,

\[ \text{Hom}(S^2(\bigwedge^2 H_p), \mathbb{Z}/p)_{\mathcal{A}\mathcal{B}_{g,1}} \cong (\mathbb{Z}/p)^3, \]

and \( \{d_1, d_2, d_3\} \) is a basis of this \( \mathbb{F}_p \)-vector space. \( \square \)

**Proposition 6.2.4.** The homomorphisms \( d_3 \) and \( 2d_1 + d_2 \) of Proposition [6.2.3] factor through the homomorphism \( \pi_p \).

**Proof.** We show that \( d_3 \) is zero on \( \text{Ker}(\pi_p) \), and so it factors through \( \pi_p \). By Proposition [6.2.2], it is enough to check that \( d_3 \) is zero on the elements of the form

\[ a \wedge b \leftrightarrow c \wedge d - a \wedge c \leftrightarrow b \wedge d + a \wedge d \leftrightarrow b \wedge c. \]
Therefore \( \varpi \) is symmetric,

\[
d_3(a \wedge b \leftrightarrow c \wedge d - a \wedge c \leftrightarrow b \wedge d + a \wedge d \leftrightarrow b \wedge c) = \\
= d_3(a \wedge b \leftrightarrow c \wedge d) - d_3(a \wedge c \leftrightarrow b \wedge d) + d_3(a \wedge d \leftrightarrow b \wedge c) = \\
= \varpi(a, c)\varpi(b, d) - \varpi(a, d)\varpi(b, c) - \varpi(a, b)\varpi(c, d) + \\
+ \varpi(a, d)\varpi(c, b) + \varpi(a, b)\varpi(d, c) - \varpi(a, c)\varpi(d, b) = 0.
\]

In addition we have that, since \( \omega \) is skew-symmetric,

\[
(2d_1 + d_2)(a \wedge b \leftrightarrow c \wedge d - a \wedge c \leftrightarrow b \wedge d + a \wedge d \leftrightarrow b \wedge c) = \\
= (2d_1 + d_2)(a \wedge b \leftrightarrow c \wedge d) - (2d_1 + d_2)(a \wedge c \leftrightarrow b \wedge d) + (2d_1 + d_2)(a \wedge d \leftrightarrow b \wedge c) = \\
= 2\omega(a, b)\omega(c, d) + \omega(a, c)\omega(b, d) - \omega(a, d)\omega(b, c) + \\
- 2\omega(a, c)\omega(b, d) - \omega(a, b)\omega(c, d) + \omega(a, d)\omega(c, b) + \\
+ 2\omega(a, d)\omega(b, c) + \omega(a, b)\omega(d, c) - \omega(a, c)\omega(d, b) = 0.
\]

Therefore \( d_3 \) and \( 2d_1 + d_2 \) factor through \( \pi_p \).

**Proposition 6.2.5.** The homomorphisms \( d_1, d_2 \) do not factor through \( \pi_p \).

**Proof.** Consider the element of \( \text{Ker}(\pi_p) \),

\[
a_1 \wedge b_1 \leftrightarrow a_2 \wedge b_2 - a_1 \wedge a_2 \leftrightarrow b_1 \wedge b_2 + a_1 \wedge b_2 \leftrightarrow b_1 \wedge a_2.
\]

Observe that

\[
d_1(a_1 \wedge b_1 \leftrightarrow a_2 \wedge b_2 - a_1 \wedge a_2 \leftrightarrow b_1 \wedge b_2 + a_1 \wedge b_2 \leftrightarrow b_1 \wedge a_2) = 1 \neq 0,
\]

\[
d_2(a_1 \wedge b_1 \leftrightarrow a_2 \wedge b_2 - a_1 \wedge a_2 \leftrightarrow b_1 \wedge b_2 + a_1 \wedge b_2 \leftrightarrow b_1 \wedge a_2) = -2 \neq 0.
\]

**Proposition 6.2.6.** For any odd prime \( p \) and \( g \geq 3 \), there is an isomorphism

\[
\text{Hom}(\bigwedge^2(\bigwedge^3 H_p), \mathbb{Z}/p)^{GL_s(\mathbb{Z})} \cong (\mathbb{Z}/p)^3,
\]

and every element is uniquely determined by its values on

\[
(a_1 \wedge a_2 \wedge a_3) \wedge (b_1 \wedge b_2 \wedge b_3), \quad (a_1 \wedge a_2 \wedge b_2) \wedge (b_1 \wedge a_2 \wedge b_2),
\]

\[
(a_1 \wedge a_2 \wedge b_2) \wedge (b_1 \wedge a_3 \wedge b_3).
\]
Moreover it has a basis formed by bilinear maps: $Q_g$, $\Theta_g$, $(J_g^1 - J_g)$, where $\Theta_g$, $Q_g$ are defined on the basis of $\wedge^3 H_p$ as

$$
\Theta_g(c_i \wedge c_j \wedge c_k \otimes c'_l \wedge c'_j \wedge c'_k) = \sum_{\sigma \in S_3} (-1)^{\tau(\sigma)} (\omega(c_i, c'_{\sigma(i)} \wedge \omega(c_j, c'_{\sigma(j)}) \wedge \omega(c_k, c'_{\sigma(k)}),$$

$$Q_g(c_i \wedge c_j \wedge c_k \otimes c'_l \wedge c'_j \wedge c'_k) = \omega(C(c_i \wedge c_j \wedge c_k), C(c'_l \wedge c'_j \wedge c'_k),$$

where $\tau(\sigma)$ denotes the sign of the permutation $\sigma$.

**Proof.** Suppose that $B_g$ is an element of $Hom(\wedge^3 H_p \wedge \wedge^3 H_p; \mathbb{Z}/p)_{GL_g(\mathbb{Z})}$. Then $B_g$ satisfies the following equalities:

- Let $i, j, k, l, m, n \in \{1, \ldots, g\}$ with $i \neq j \neq k$. Consider the element $\phi = (G^0, 0, iG^{-1}) \in S_{2g}(\mathbb{Z})$ with $G = (1, i)(2, j)(3, k) \in S_{2g}$. Then we get that

$$B((c_i \wedge c_j \wedge c_k) \wedge (c'_l \wedge c'_m \wedge c'_n)) = B(\phi \cdot (c_i \wedge c_j \wedge c_k) \wedge \phi \cdot (c'_l \wedge c'_m \wedge c'_n)) =$$

$$= B((c_1 \wedge c_2 \wedge c_3) \wedge (c'_{G(l)} \wedge c'_{G(m)} \wedge c'_{G(n)}),$$

$$B((c_i \wedge a_j \wedge b_j) \wedge (c'_l \wedge c'_m \wedge c'_n)) = B(\phi \cdot (c_i \wedge a_j \wedge b_j) \wedge \phi \cdot (c'_l \wedge c'_m \wedge c'_n)) =$$

$$= B((c_1 \wedge a_2 \wedge b_2) \wedge (c'_{G(l)} \wedge c'_{G(m)} \wedge c'_{G(n)}).$$

- Suppose that $l \neq 1, 2, 3$, and $l \neq n \neq m$. Now take the element $\phi = (G^0, 0, iG^{-1}) \in S_{2g}(\mathbb{Z})$, with $G \in GL_g(\mathbb{Z})$ the matrix with a $-1$ at position $(l,l)$ and 1's at positions $(i, i)$ for $i \neq l$, and 0's elsewhere. Then we get that

$$B((c_1 \wedge c_2 \wedge c_3) \wedge (c'_l \wedge c'_m \wedge c'_n)) = B(\phi \cdot (c_1 \wedge c_2 \wedge c_3) \wedge \phi \cdot (c'_l \wedge c'_m \wedge c'_n)) =$$

$$= B((c_1 \wedge c_2 \wedge c_3) \wedge (-c'_l \wedge c'_m \wedge c'_n)) =$$

$$= -B((c_1 \wedge c_2 \wedge c_3) \wedge (c'_l \wedge c'_m \wedge c'_n)).$$

Observe that this argument also works for $m, n$ instead of $l$. So we get that

$$B((c_1 \wedge c_2 \wedge c_3) \wedge (c'_l \wedge c'_m \wedge c'_n)) = 0$$

for all $l, m, n \in \{1, \ldots, g\}$, such that $l \neq n \neq m$, with $l$ or $n$ or $m$ different of $1, 2, 3$. 153
Now suppose that \( l \neq n \neq m \). With out lose of generality we can suppose that \( l \neq 1 \), then
\[
B((c_1 \land a_2 \land b_2) \land (c'_1 \land c'_m \land c'_n)) = B(\phi \cdot (c_1 \land a_2 \land b_2) \land \phi \cdot (c'_1 \land c'_m \land c'_n)) = \\
= B((c_1 \land a_2 \land b_2) \land (-c'_1 \land c'_m \land c'_n)) = \\
= -B((c_1 \land a_2 \land b_2) \land (c'_1 \land c'_m \land c'_n))
\]
Thus \( B((c_1 \land a_2 \land b_2) \land (c'_1 \land c'_m \land c'_n)) = 0 \) for any \( l, m, n \in \{1, \ldots, g\} \) such that \( l \neq n \neq m \).

Now suppose that \( l \neq 1 \), then
\[
B((c_1 \land a_2 \land b_2) \land (c'_1 \land a_m \land b_m)) = B(\phi \cdot (c_1 \land a_2 \land b_2) \land \phi \cdot (c'_1 \land a_m \land b_m)) = \\
= B((c_1 \land a_2 \land b_2) \land (-c'_1 \land a_m \land b_m)) = \\
= -B((c_1 \land a_2 \land b_2) \land (c'_1 \land a_m \land b_m))
\]
Thus \( B((c_1 \land a_2 \land b_2) \land (c'_1 \land a_m \land b_m)) = 0 \) for any \( l, m \in \{1, \ldots, g\} \) such that \( 1 \neq l, l \neq m \).

- Suppose that \( l \neq m \), and with out lose of generality we can assume that \( 1 \neq l, m \). Now take the element \( \phi = \left( \begin{smallmatrix} G & 0 \\ 0 & G^{-1} \end{smallmatrix} \right) \in Sp_{2g}(\mathbb{Z}) \), with \( G \in M_{2g}(\mathbb{Z}) \) the matrix with a \(-1\) at position \((1, 1)\) and 1’s at positions \((i, i)\) for \( i \in \{2, \ldots, g\} \), and 0’s elsewhere. Then we get that
\[
B((c_1 \land c_2 \land c_3) \land (c'_1 \land a_m \land b_m)) = B(\phi \cdot (c_1 \land c_2 \land c_3) \land \phi \cdot (c'_1 \land a_m \land b_m)) = \\
= B((-c_1 \land c_2 \land c_3) \land (c'_1 \land a_m \land b_m)) = \\
= -B((c_1 \land c_2 \land c_3) \land (c'_1 \land a_m \land b_m))
\]
Thus \( B((c_1 \land c_2 \land c_3) \land (c'_1 \land a_m \land b_m)) = 0 \) for all \( l, m, n \in \{1, \ldots, g\} \).

- Let \( k \in \{1, \ldots, g\} \) such that \( k \neq 2 \). Consider the element \( \phi = \left( \begin{smallmatrix} G & 0 \\ 0 & G^{-1} \end{smallmatrix} \right) \in Sp_{2g}(\mathbb{Z}) \) with \( G = \left( \begin{smallmatrix} 3 \end{smallmatrix} \right) \in \mathcal{S}_g \). Then we get that
\[
B((c_1 \land a_2 \land b_2) \land (c'_1 \land a_k \land b_k)) = B(\phi \cdot (c_1 \land a_2 \land b_2) \land \phi \cdot (c'_1 \land a_k \land b_k)) = \\
= B((c_1 \land a_2 \land b_2) \land (c'_1 \land a_3 \land b_3)).
\]

Summarizing, every element \( B \in Hom(\wedge^2(\Lambda^3 H_p), \mathbb{Z}/p)^{\text{GL}_g(\mathbb{Z})} \) is completely determined by its image on the following elements:
\[
(c_1 \land c_2 \land c_3) \land (c'_1 \land c'_2 \land c'_3), \quad (c_1 \land a_2 \land b_2) \land (c'_1 \land a_2 \land b_2), \\
(c_1 \land a_2 \land b_2) \land (c'_1 \land a_3 \land b_3).
\]

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Observe that $B((c_1 \land a_2 \land b_2) \land (c'_1 \land a_2 \land b_2)) = -B((c'_1 \land a_2 \land b_2) \land (c_1 \land a_2 \land b_2))$. Thus
\[
B((c_1 \land a_2 \land b_2) \land (c_1 \land a_2 \land b_2)) = 0 \text{ for } c_1 = a_1 \text{ or } b_1,
\]
\[
B((a_1 \land a_2 \land b_2) \land (b_1 \land a_2 \land b_2)) = -B((b_1 \land a_2 \land b_2) \land (a_1 \land a_2 \land b_2)).
\]

Consider the element $\phi = \left( \begin{smallmatrix} G & 0 \\ 0 & G^{-1} \end{smallmatrix} \right) \in Sp_{2g}(\mathbb{Z})$ with $G = (2, 3) \in \mathbb{G}_g$. Then we get that
\[
B((c_1 \land a_2 \land b_2) \land (c'_1 \land a_2 \land b_2)) = B(\phi \cdot (c_1 \land a_2 \land b_2) \land \phi \cdot (c'_1 \land a_2 \land b_2)) =
\]
\[
B((c_1 \land a_3 \land b_3) \land (c'_1 \land a_2 \land b_2)) = -B((c'_1 \land a_2 \land b_2) \land (c_1 \land a_3 \land b_3))
\]

Thus
\[
B((c_1 \land a_2 \land b_2) \land (c_1 \land a_3 \land b_3)) = 0 \text{ for } c_1 = a_1 \text{ or } b_1,
\]
\[
B((a_1 \land a_2 \land b_2) \land (b_1 \land a_3 \land b_3)) = -B((b_1 \land a_2 \land b_2) \land (a_1 \land a_3 \land b_3)).
\]

Observe that $B((c_1 \land c_2 \land c_3) \land (c_1 \land c_2 \land c_3)) = -B((c_1 \land c_2 \land c_3) \land (c_1 \land c_2 \land c_3))$. Thus
\[
B((c_1 \land c_2 \land c_3) \land (c_1 \land c_2 \land c_3)) = 0 \text{ for } c_i = a_i \text{ or } b_i,
\]
\[
B((a_1 \land a_2 \land a_3) \land (b_1 \land b_2 \land b_3)) = -B((b_1 \land b_2 \land b_3) \land (a_1 \land a_2 \land a_3)),
\]
\[
B((a_1 \land b_2 \land a_3) \land (b_1 \land a_2 \land a_3)) = -B((b_1 \land a_2 \land a_3) \land (a_1 \land b_2 \land a_3)),
\]

Now suppose that there exist elements $i \neq j \in \{1, 2, 3\}$ such that $c_i = c'_j$ and $c_j \neq c'_j$. Taking the element $\phi = \left( \begin{smallmatrix} G & 0 \\ 0 & G^{-1} \end{smallmatrix} \right) \in Sp_{2g}(\mathbb{Z})$ with $G = (1, i)(2, j)$ Then without lose of generality we can suppose $c_1 = c'_1$ and $c_2 \neq c'_2$. Consider the element $\phi = \left( \begin{smallmatrix} G & 0 \\ 0 & G^{-1} \end{smallmatrix} \right) \in Sp_{2g}(\mathbb{Z})$ with $G \in \mathbb{G}_g$ the matrix with a $-1$ at position $(2, 1)$, 1’s at the diagonal and 0’s elsewhere. Then we get that
\[
B((a_1 \land a_2 \land c_3) \land (a_1 \land b_2 \land c'_3)) = B(\phi \cdot (a_1 \land a_2 \land c_3) \land \phi \cdot (a_1 \land b_2 \land c'_3)) =
\]
\[
B((a_1 \land a_2 \land c_3) \land ((a_1 - a_2) \land (b_2 + b_1) \land c'_3))
\]

Therefore we get that
\[
B((a_1 \land a_2 \land c_3) \land (a_1 \land b_1 \land c'_3)) = B((a_1 \land a_2 \land c_3) \land (a_2 \land b_2 \land c'_3))
\]
\[
+ B((a_1 \land a_2 \land c_3) \land (a_2 \land b_1 \land c'_3))
\]

But the above relations we get that $B((a_1 \land a_2 \land c_3) \land (a_2 \land b_1 \land c'_3)) = 0$

• Now taking the element $\phi = \left( \begin{smallmatrix} G & 0 \\ 0 & G^{-1} \end{smallmatrix} \right) \in Sp_{2g}(\mathbb{Z})$, with $G \in GL_g(\mathbb{Z})$ the matrix with 1’s at the
diagonal and position \((3, 2)\), and 0's elsewhere. Then we get that

\[
0 = B((a_1 \land a_2 \land b_2) \land (b_1 \land a_2 \land b_3)) = \\
B((a_1 \land (a_2 + a_3) \land b_2) \land (b_1 \land (a_2 + a_3) \land (b_3 - b_2))) = \\
B((a_1 \land a_2 \land b_2) \land (b_1 \land a_2 \land b_3)) - B((a_1 \land a_2 \land b_2) \land (b_1 \land a_2 \land b_2)) + \\
B((a_1 \land a_2 \land b_2) \land (b_1 \land a_2 \land b_3)) - B((a_1 \land a_2 \land b_2) \land (b_1 \land a_2 \land b_2)) + \\
B((a_1 \land a_2 \land b_2) \land (b_1 \land a_2 \land b_3)) - B((a_1 \land a_2 \land b_2) \land (b_1 \land a_2 \land b_2)) = \\
= - B((a_1 \land a_2 \land b_2) \land (b_1 \land a_2 \land b_2)) + B((a_1 \land a_2 \land b_2) \land (b_1 \land a_3 \land b_3)) + \\
B((a_1 \land a_3 \land b_2) \land (b_1 \land a_2 \land b_3)).
\]

Thus

\[
B((a_1 \land b_2 \land a_3) \land (b_1 \land a_2 \land b_3)) = \\
= - B((a_1 \land a_2 \land b_2) \land (b_1 \land a_2 \land b_2)) + B((a_1 \land a_2 \land b_2) \land (b_1 \land a_3 \land b_3)).
\]

Summarizing we have that, every element \(B \in \text{Hom}(\wedge^2(\Lambda^3 H_p), \mathbb{Z}/p)_{GL_g(\mathbb{Z})}\) is completely determined by its image on the following elements:

\[
(a_1 \land a_2 \land a_3) \land (b_1 \land b_2 \land b_3), \quad (a_1 \land a_2 \land b_2) \land (b_1 \land a_2 \land b_2), \\
(a_1 \land a_2 \land b_2) \land (b_1 \land a_3 \land b_3).
\]

Observe that \(Q_g, \Theta_g, (J_g^t - J_g)\) are elements of \(\text{Hom}(\wedge^2(\Lambda^3 H_p), \mathbb{Z}/p)_{GL_g(\mathbb{Z})}\) and in addition these elements are linearly independent because

\[
\begin{align*}
(J_g^t - J_g)((a_1 \land a_2 \land a_3) \land (b_1 \land b_2 \land b_3)) &= -1, \\
(J_g^t - J_g)((a_1 \land a_2 \land b_2) \land (b_1 \land a_2 \land b_2)) &= 0, \\
(J_g^t - J_g)((a_1 \land a_2 \land b_2) \land (b_1 \land a_3 \land b_3)) &= 0.
\end{align*}
\]

\[
\begin{align*}
Q_g((a_1 \land a_2 \land a_3) \land (b_1 \land b_2 \land b_3)) &= 0, \\
Q_g((a_1 \land a_2 \land b_2) \land (b_1 \land a_2 \land b_2)) &= -4, \\
Q_g((a_1 \land a_2 \land b_2) \land (b_1 \land a_3 \land b_3)) &= -4.
\end{align*}
\]
\[ \Theta_g((a_1 \wedge a_2 \wedge a_3) \wedge (b_1 \wedge b_2 \wedge b_3)) = -1, \]
\[ \Theta_g((a_1 \wedge a_2 \wedge b_2) \wedge (b_1 \wedge a_2 \wedge b_2)) = -1, \]
\[ \Theta_g((a_1 \wedge a_2 \wedge b_2) \wedge (b_1 \wedge a_3 \wedge b_3)) = 0. \]

\[ \square \]

**Corollary 6.2.1.** For any odd prime \( p \) and \( g \geq 4 \), there is an isomorphism
\[ \text{Hom}(\bigwedge^2(\bigwedge^3 H_p); \mathbb{Z}/p)^{Sp_2g(\mathbb{Z}/p)} \cong (\mathbb{Z}/p)^2. \]
Moreover it has a basis formed by the bilinear maps \( \Theta_g, Q_g \).

**Proof.** By Proposition (6.2.6) we know that \( \text{Hom}(\bigwedge^2(\bigwedge^3 H_p); \mathbb{Z}/p)^{GL_6(\mathbb{Z})} \) has a basis given by \((J^t_g - J_g), \Theta_g, Q_g\).

By definition, it is clear that \( \Theta_g, Q_g \) are \( Sp_2g(\mathbb{Z}) \)-invariant. Next we show that \((J^t_g - J_g)\) is not \( Sp_2g(\mathbb{Z}) \)-invariant.

Suppose that \((J^t_g - J_g)\) was \( Sp_2g(\mathbb{Z}) \)-invariant and consider the element \( \phi \in Sp_2g(\mathbb{Z}) \) the matrix with 1’s at the diagonal and position \((1, g + 1)\), and 0’s elsewhere. Then we would have that
\[ 0 = (J^t_g - J_g)((a_1 \wedge a_2 \wedge a_3) \wedge (b_1 \wedge b_2 \wedge b_3)) = \]
\[ = (J^t_g - J_g)(\phi \cdot (b_1 \wedge a_2 \wedge a_3) \wedge \phi \cdot (b_1 \wedge b_2 \wedge b_3)) \]
\[ = (J^t_g - J_g)(((a_1 + b_1) \wedge a_2 \wedge a_3) \wedge ((a_1 + b_1) \wedge b_2 \wedge b_3)) = \]
\[ = (J^t_g - J_g)((a_1 \wedge a_2 \wedge a_3) \wedge (b_1 \wedge b_2 \wedge b_3)) = -1, \]
which is a contradiction. Therefore, \( \text{Hom}(\bigwedge^2(\bigwedge^3 H_p); \mathbb{Z}/p)^{Sp_2g(\mathbb{Z}/p)} \cong \mathbb{Z}/p^2 \) with a basis given by the elements \( \Theta_g, Q_g \), as desired.

\[ \square \]

**Corollary 6.2.2.** For any odd prime \( p \) and \( g \geq 4 \), there are isomorphisms
\[ H^2(\bigwedge^3 H_p; \mathbb{Z}/p)^{GL_6(\mathbb{Z}/p)} \cong (\mathbb{Z}/p)^3, \quad H^2(\bigwedge^3 H_p; \mathbb{Z}/p)^{Sp_2g(\mathbb{Z}/p)} \cong (\mathbb{Z}/p)^2. \]

**Proof.** Since \( \bigwedge^3 H_p \) is abelian, the Universal Coefficients Theorem gives us a short exact sequence
\[ 0 \longrightarrow \text{Ext}^1(\mathbb{Z})(\bigwedge^3 H_p); \mathbb{Z}/p) \longrightarrow H^2(\bigwedge^3 H_p; \mathbb{Z}/p) \longrightarrow \text{Hom}(\bigwedge^2(\bigwedge^3 H_p); \mathbb{Z}/p) \longrightarrow 1. \]

A direct computation shows that \( \text{Ext}^1(\mathbb{Z})(\bigwedge^3 H_p); \mathbb{Z}/p) = \text{Hom}(\bigwedge^3 H_p, \mathbb{Z}/p) \). Then the above short
Chapter 6. Obstruction to Perron’s conjecture

The obstruction to Perron’s conjecture

Now we are ready to give an obstruction to Perron’s conjecture. We split the argument in the following two parts:

1. First of all we show that the triviality of the 2-cocycle \((\tau^Z_1)^*\) is a necessary condition for Conjecture (6.1.2) to be true.

2. Secondly, we show that, up to a non zero multiplicative constant modulo \(p\), the cohomology class of \((\tau^Z_1)^*(J^p_g)\) in \(H^2(\mathcal{M}_{g,1}[p];\mathbb{Z}/p)\) is equal to the restriction of the first characteristic

exact sequence becomes

\[
0 \longrightarrow \text{Hom}(\wedge^3 H_p, \mathbb{Z}/p) \longrightarrow H^2(\wedge^3 H_p; \mathbb{Z}/p) \longrightarrow \text{Hom}(\wedge^2(\wedge^3 H_p); \mathbb{Z}/p) \longrightarrow 1.
\]

Taking \(GL_g(\mathbb{Z}/p)\)-invariants and \(Sp_{2g}(\mathbb{Z})\)-invariants in the above short exact sequence, we get exact sequences

\[
0 \rightarrow \text{Hom}(\wedge^3 H_p, \mathbb{Z}/p)^{GL_g(\mathbb{Z}/p)} \rightarrow H^2(\wedge^3 H_p; \mathbb{Z}/p)^{GL_g(\mathbb{Z}/p)} \\
\rightarrow \text{Hom}(\wedge^2(\wedge^3 H_p); \mathbb{Z}/p)^{GL_g(\mathbb{Z}/p)} \rightarrow H^1(GL_g(\mathbb{Z}/p); (\wedge^3 H_p)^*),
\]

\[
0 \rightarrow \text{Hom}(\wedge^3 H_p, \mathbb{Z}/p)^{Sp_{2g}(\mathbb{Z}/p)} \rightarrow H^2(\wedge^3 H_p; \mathbb{Z}/p)^{Sp_{2g}(\mathbb{Z}/p)} \\
\rightarrow \text{Hom}(\wedge^2(\wedge^3 H_p); \mathbb{Z}/p)^{Sp_{2g}(\mathbb{Z}/p)} \rightarrow H^1(Sp_{2g}(\mathbb{Z}/p); (\wedge^3 H_p)^*).
\]

Observe that taking the action of \(-Id\), we get that

\[
\text{Hom}(\wedge^3 H_p, \mathbb{Z}/p)^{GL_g(\mathbb{Z}/p)} = \text{Hom}(\wedge^3 H_p, \mathbb{Z}/p)^{Sp_{2g}(\mathbb{Z}/p)} = 0.
\]

In addition, by the Center kills Lemma, since \(-Id\) acts on \((\wedge^3 H_p)^*\) by the multiplication by \(-1\), we get that

\[
H^1(GL_g(\mathbb{Z}/p); (\wedge^3 H_p)^*) = H^1(Sp_{2g}(\mathbb{Z}/p); (\wedge^3 H_p)^*) = 0.
\]

Therefore we obtain isomorphisms

\[
H^2(\wedge^3 H_p; \mathbb{Z}/p)^{GL_g(\mathbb{Z}/p)} \cong \text{Hom}(\wedge^2(\wedge^3 H_p); \mathbb{Z}/p)^{GL_g(\mathbb{Z}/p)},
\]

\[
H^2(\wedge^3 H_p; \mathbb{Z}/p)^{Sp_{2g}(\mathbb{Z}/p)} \cong \text{Hom}(\wedge^2(\wedge^3 H_p); \mathbb{Z}/p)^{Sp_{2g}(\mathbb{Z}/p)}.
\]

Finally, by Proposition (6.2.6) and Corollary (6.2.1), we get the desired result. \(\square\)

6.3 The obstruction to Perron’s conjecture

Now we are ready to give an obstruction to Perron’s conjecture. We split the argument in the following two parts:

1. First of all we show that the triviality of the 2-cocycle \((\tau^Z_1)^*\) is a necessary condition for Conjecture (6.1.2) to be true.

2. Secondly, we show that, up to a non zero multiplicative constant modulo \(p\), the cohomology class of \((\tau^Z_1)^*(J^p_g)\) in \(H^2(\mathcal{M}_{g,1}[p];\mathbb{Z}/p)\) is equal to the restriction of the first characteristic
class of surface bundles $e_1 \in H^2(M_{g,1};\mathbb{Z})$, defined in [29], to $M_{g,1}[p]$ reduced modulo $p$.

As a consequence, we will get that the 2-cocycle $(\tau_1^Z)^*(-2J_g^I)$ on $M_{g,1}[p]$ is not trivial. Therefore we will get an obstruction to the Perron’s conjecture.

1) We show that the triviality of $(\tau_1^Z)^*(-2J_g^I)$ is a necessary condition for the Perron’s conjecture to be true.

**Proposition 6.3.1.** If the Conjecture (6.1.2) is true, i.e. $\gamma_p$ is an invariant of $\mathcal{S}^3[p]$, then $(\tau_1^Z)^*(-2J_g^I)$ must be trivial with trivialization $\gamma_p$.

**Proof.** We show that given $\varphi_1, \varphi_2 \in M_{g,1}[p]$ such that $\varphi_1 = f_1 \circ m_1$, and $\varphi_2 = f_2 \circ m_2$ with $f_i \in T_{g,1}$ and $m_i \in D_{g,1}[p]$ for $i = 1, 2$, then the following equality holds

$$(\tau_1^Z)^*(-2J_g^I)(\varphi_1, \varphi_2) = \gamma_p(\varphi_1) + \gamma_p(\varphi_2) - \gamma_p(\varphi_1 \varphi_2)$$

Observe that

$$(\tau_1^Z)^*(-2J_g^I)(\varphi_1, \varphi_2) = (\tau_1^Z)^*(-2J_g^I)(f_1m_1, f_2m_2) = -2J_g^I(\tau_1^Z(f_1m_1), \tau_1^Z(f_2m_2)) = -2J_g^I(\tau_1^Z(f_1), \tau_1^Z(f_2)) = -2J_g^I(\tau_1, \tau_1) \mod p,$$

$$\gamma_p(\varphi_1) + \gamma_p(\varphi_2) - \gamma_p(\varphi_1 \varphi_2) = \gamma_p(f_1m_1) + \gamma_p(f_2m_2) - \gamma_p(f_1m_1f_2m_2) =$$

$$= \gamma_p(f_1) + \gamma_p(f_2) - \gamma_p(f_1f_2(f_2^{-1}m_1f_1)m_2) =$$

$$= \gamma_p(f_1) + \gamma_p(f_2) - \gamma_p(f_1f_2).$$

On the other hand, in [38], W. Pitsch proved that $\tau_1^*(1)$ is a trivial 2-cocycle with trivialization the Casson invariant. In other words, he proved that for every $f_1, f_2 \in T_{g,1}$ the following equality holds

$$-2J_g^I(\tau_1, \tau_1) = \lambda(f_1) + \lambda(f_2) - \lambda(f_1f_2).$$

Therefore

$$-2J_g^I(\tau_1, \tau_1) = \lambda(f_1) + \lambda(f_2) - \lambda(f_1f_2) \mod p,$$

and as a consequence,

$$(\tau_1^Z)^*(-2J_g^I)(\varphi_1, \varphi_2) = \gamma_p(\varphi_1) + \gamma_p(\varphi_2) - \gamma_p(\varphi_1 \varphi_2).$$

□

2) We show that the 2-cocycle $(\tau_1^Z)^*(J_g^I)$ is not trivial in $H^2(M_{g,1}[p];\mathbb{Z}/p)$. First of all we
rewrite the 2-cocycle \((\tau_1^Z)^*(J_g^t)\) as

\[
(\tau_1^Z)^*(J_g^t) = (\tau_1^Z)^*\left(J_g^t + \frac{1}{48}Q_g\right) - \frac{1}{48}(\tau_1^Z)^*(Q_g).
\]

We will see the following two facts:

i) The cohomology class of \((\tau_1^Z)^*(J_g^t + \frac{1}{48}Q_g)\) is zero in \(H^2(\mathcal{M}_{g,1}[p];\mathbb{Z}/p)\).

ii) The cohomology class of \((\tau_1^Z)^*(Q_g)\) is the restriction of the first characteristic class of surface bundles \(e_1 \in H^2(\mathcal{M}_{g,1};\mathbb{Z})\) to \(\mathcal{M}_{g,1}[p]\) reduced modulo \(p\). Therefore, the cohomology class of \((\tau_1^Z)^*(Q_g)\) is not zero in \(H^2(\mathcal{M}_{g,1}[p];\mathbb{Z}/p)\).

i) Consider the short exact sequence

\[
0 \longrightarrow \mathcal{I}_{g,1}^Z(2) \longrightarrow \mathcal{M}_{g,1}[p] \overset{\tau_1^Z}{\longrightarrow} \wedge^3 H_p \longrightarrow 0.
\]

Recall that \(\mathcal{I}_{g,1}^Z(k) = \text{Ker}(\rho_k^Z)\) and \(\rho_k^Z\) restricted to \(\mathcal{I}_{g,1}^Z(k)\) is \(\tau_k^Z\).

Since \(\mathcal{I}_{g,1}^Z(3) \subset \mathcal{I}_{g,1}^Z(2) \subset \mathcal{M}_{g,1}[p]\), then \(\text{Im}(\tau_2^Z) \cong \frac{\mathcal{I}_{g,1}^Z(2)}{\mathcal{I}_{g,1}^Z(3)}\) and \(\rho_3^Z(\mathcal{M}_{g,1}[p]) \cong \frac{\mathcal{M}_{g,1}[p]}{\mathcal{I}_{g,1}^Z(3)}\). Therefore, taking the quotient by \(\mathcal{I}_{g,1}^Z(3)\) in the above short exact sequence, we get another short exact sequence

\[
0 \longrightarrow \text{Im}(\tau_2^Z) \longrightarrow \rho_3^Z(\mathcal{M}_{g,1}[p]) \overset{\psi_2^Z}{\longrightarrow} \wedge^3 H_p \longrightarrow 0,
\]

which is central by Corollary 4.4.4. By Section V.3 in [3], we know that there is an associated 2-cocycle \(c \in H^2(\wedge^3 H_p;\text{Im}(\tau_2^Z))\) for the central extension (6.3.1) which is defined by \(c(x, y) = s(x)s(y)s(xy)^{-1}\) where \(s: \wedge^3 H_p \rightarrow \rho_2^Z(\mathcal{M}_{g,1}[p])\) is a set theoretic section. Let \(f = -6d_3 - 2d_2 - 4d_1\), by Proposition 6.2.4, we know that \(f\) factors through \(\pi_p\). Denote by \(\overline{f}\) the homomorphism satisfying that \(f = \pi_p \circ \overline{f}\).

Consider the short exact sequence obtained in the proof of Corollary 6.2.2:

\[
0 \longrightarrow \text{Hom}(\wedge^3 H_p;\mathbb{Z}/p) \overset{1}{\longrightarrow} H^2(\wedge^3 H_p;\mathbb{Z}/p) \overset{\theta}{\longrightarrow} \text{Hom}(\wedge^2(\wedge^3 H_p);\mathbb{Z}/p) \longrightarrow 1.
\]

Next we show that \([J_g^t + \frac{1}{48}Q_g] = [\overline{f}_s(c)]\).

Proposition 6.3.2. The following equality holds \(\theta(J_g^t + \frac{1}{48}Q_g) = \theta(\frac{1}{24}\overline{f}_s(c))\).

Proof. Observe that \(\theta\) and \(\overline{f}_s\) commutes so we have that \(\theta(\overline{f}_s(c)) = \overline{f}_s(\theta(c))\). Now we show that \(\theta(c) = \pi_p \circ \chi_p\). Recall that by definition of \(c\) we know that \(c(x, y) = s(x)s(y)s(xy)^{-1}\) where \(s\) is a set theoretic section of the central extension (6.3.1). So we have that

\[
\theta(c)(x \wedge y) = c([x|y]) - c([y|x]) = [s(x), s(y)].
\]
Since \( s(x), s(y) \in \rho_3^Z(\mathcal{M}_{g,1}[p]) \), we know that there exist elements \( \xi, \eta \in \mathcal{M}_{g,1}[p] \) such that \( \rho_3^Z(\xi) = x, \rho_3^Z(\eta) = y \), in particular, since \( \mathcal{M}_{g,1}[p] = T_{g,1}D_{g,1}[p] \), by Proposition (5.1.2) without lose of generality we can suppose that \( \xi, \eta \in T_{g,1} \). So \( \theta(c)(x \wedge y) = \rho_3^Z([\xi, \eta]) \). But we know that \([T_{g,1}, T_{g,1}] < K_{g,1} \), then \( \rho_3^Z([\xi, \eta]) = \tau_2^Z([\xi, \eta]) = \tau_2([\xi, \eta]) \) ( mod \( p \) ). By Theorem 3.1 in [31], we know that \( \tau_2([\xi, \eta]) = \pi(\chi(\tau_1(\xi) \wedge \tau_1(\eta))) = \pi(\chi(x \wedge y)) \), and by definition we have that \( \pi(\chi(x \wedge y)) \) mod \( p \) is equal to \( \pi_p(\chi_p(x \wedge y)) \). Thus \( \theta(\frac{1}{24}\overline{f}_s(c)) = \frac{1}{24}\overline{f}_s(\pi_p \circ \chi_p) \).

Moreover we have that \( \overline{f}, \pi_p \) and \( \chi_p \) are \( AB_{g,1} \)-equivariant, so in particular \( \overline{f}_s(\pi_p \circ \chi_p) \) is an element of \( \text{Hom}(\Lambda^2(\Lambda^3 H_p), \mathbb{Z}/p)AB_{g,1} \). On the other hand we have that \( \theta(J_g^t + \frac{1}{24}Q_g) = J_g^t - J_g + \frac{1}{24}Q_g \) and this is also an element of \( \text{Hom}(\Lambda^2(\Lambda^3 H_p), \mathbb{Z}/p)AB_{g,1} \). So applying Proposition (6.2.6) we get that it is enough to check that these two elements take the same values on the following elements

\[
(a_1 \wedge a_2 \wedge a_3) \wedge (b_1 \wedge b_2 \wedge b_3), \quad (a_1 \wedge a_2 \wedge b_2) \wedge (b_1 \wedge a_2 \wedge b_2),
\]

\[
(a_1 \wedge a_2 \wedge b_2) \wedge (b_1 \wedge a_3 \wedge b_3).
\]

One can check that:

\[
\frac{1}{24}\overline{f}(\pi_p \circ \chi_p)((a_1 \wedge a_2 \wedge a_3) \wedge (b_1 \wedge b_2 \wedge b_3)) = \frac{1}{24}\overline{f}(\pi_p(a_2 \wedge a_3 \leftrightarrow b_2 \wedge b_3 + a_3 \wedge a_1 \leftrightarrow b_3 \wedge b_1 + a_1 \wedge a_2 \leftrightarrow b_1 \wedge b_2) = \frac{1}{8}f(a_1 \wedge a_2 \leftrightarrow b_1 \wedge b_2) = \frac{1}{8}(-6 - 2 + 0) = -1,
\]

\[
\frac{1}{24}\overline{f}(\pi_p \circ \chi_p)((a_1 \wedge a_2 \wedge b_2) \wedge (b_1 \wedge a_2 \wedge b_2)) = \frac{1}{24}\overline{f}(\pi_p(a_2 \wedge b_2 \leftrightarrow a_2 \wedge b_2 + b_2 \wedge a_1 \leftrightarrow b_1 \wedge a_2 - a_1 \wedge a_2 \leftrightarrow b_2 \wedge b_1) = \frac{1}{24}[f(a_2 \wedge b_2 \leftrightarrow a_2 \wedge b_2) + f(b_2 \wedge a_1 \leftrightarrow b_1 \wedge a_2) - f(a_1 \wedge a_2 \leftrightarrow b_2 \wedge b_1)] = \frac{1}{24}[(6 - 2 - 4) + (6 - 2 + 0) - (6 + 2 + 0)] = -\frac{1}{6},
\]

\[
\frac{1}{24}\overline{f}(\pi_p \circ \chi_p)((a_1 \wedge a_2 \wedge b_2) \wedge (b_1 \wedge a_3 \wedge b_3)) = \frac{1}{24}f(a_2 \wedge b_2 \leftrightarrow a_2 \wedge b_3) = -\frac{1}{6},
\]

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Chapter 6. Obstruction to Perron’s conjecture

In his result, the isomorphism \( \nu \) may be described as follows. Write each element \( E \) of \( \text{Ext}(T,G) \) as a short exact sequence

\[
0 \longrightarrow G \longrightarrow B \xrightarrow{\mu} T \longrightarrow 0
\]

The corresponding homomorphism \( \nu E : T \to G/pG \) is then given as follows: to each \( t \in T \) choose a representative \( b(t) \in B \) with \( \mu b(t) = 0 \) and then set \( (\nu E)(t) = pb(t) + pG \). Now we apply this in our case. Let \( G_g = J_g^t + \frac{1}{48}Q_g - \overline{f}_s(c) \) we take the central abelian extension associated to \( G_g \):

\[
0 \longrightarrow \mathbb{Z}/p \xrightarrow{j} \mathbb{Z}/p \times_{G_g} \wedge^3 H_p \xrightarrow{\mu} \wedge^3 H_p \longrightarrow 0
\]

Then the corresponding homomorphism \( h : \wedge^3 H_p \to \mathbb{Z}/p \) is given by \( h(x) = j^{-1}((0, x)^p) \).

**Proposition 6.3.3.** The homomorphism \( h \in \text{Hom}(\wedge^3 H_p, \mathbb{Z}/p) \) described above is zero.

**Proof.** If we expand the expression \((0, x)^p\), since \( J_g^t + \frac{1}{48}Q_g \) is a bilinear map, we get that

\[
j^{-1}((0, x)^p) = j^{-1}((G_g(x, x) + G_g(2x, x) + \cdots + G_g((p-1)x, x), 0))
\]

\[
= \sum_{i=1}^{p-1} f_s(c)(ix, x) + \sum_{i=1}^{p-1} (J_g^t - \frac{1}{48}Q_g)(ix, x)
\]

\[
= \sum_{i=1}^{p-1} f_s(c)(ix, x) + \sum_{i=1}^{p-1} i(J_g^t - \frac{1}{48}Q_g)(x, x)
\]

\[
= \sum_{i=1}^{p-1} f_s(c)(ix, x) = f(\sum_{i=1}^{p-1} c(ix, x)).
\]
Thus, it is enough to prove that $\sum_{i=1}^{p-1} c(ix, x)$ is zero. Recall that $c$ is the 2-cocycle associated to the central extension

$$
\begin{array}{c}
0 \longrightarrow \text{Im}(\tau_Z) \longrightarrow \rho_3^Z(\mathcal{M}_{g,1}[p]) \longrightarrow \wedge^3 H_p \longrightarrow 1.
\end{array}
$$

Observe that if we take $(0, x) \in \text{Im}(\tau_Z) \times_c \wedge^3 H_p$ and we expand the $p^{th}$-power of $(0, x)$ we get that

$$(0, x)^p = (\sum_{i=1}^{p-1} c(ix, x), 0).$$

Hence, it is enough to show that $(0, x)^p \in \text{Im}(\tau_Z) \times_c \wedge^3 H_p$ is zero. Since $[c]$ is the cohomology class associated to the central extension \([6.3.1]\), there is an isomorphism $\varphi : \text{Im}(\tau_Z) \times_c \wedge^3 H_p \cong \rho_3^Z(\mathcal{M}_{g,1}[p])$, so there is an element $\tilde{x} \in \rho_3^Z(\mathcal{M}_{g,1}[p])$ such that $(0, x)^p = \tilde{x}^p$. As a consequence, there is an element $\xi \in \mathcal{M}_{g,1}[p]$ such that $\varphi(\rho_3^Z(\xi^p)) = (0, x)^p$. Moreover, by Proposition \([5.1.2]\), we know that $\rho_3^Z(D_{g,1}[p]) = \tau_Z^2(D_{g,1}[p]) = 0$. Therefore, without lose of generality, we can assume that $\xi \in \mathcal{T}_{g,1}$.

Now we show that $\rho_3^Z(\xi^p) = 0$. Since $\xi \in \mathcal{T}_{g,1}$, we can express $\xi^p$ as a product of BP-maps, i.e.

$$
\xi^p = ((T_{\beta_1}T_{\beta_1}^{-1})T_{\beta_2}T_{\beta_2}^{-1})\cdots(T_{\beta_l}T_{\beta_l}^{-1})^p.
$$

To prove that $\rho_3^Z(\xi^p) = 0$ we proceed by induction on the number $l$ of BP-maps.

First of all, observe that given $f, g \in \mathcal{T}_{g,1}$ we have that

$$
(fg)^p = (fg)^{p-1}gf[f^{-1}, g^{-1}] = (fg)^{p-2}fg^2f[f^{-1}, g^{-1}]
= (fg)^{p-2}g^2f[f^{-1}, g^{-2}]f[f^{-1}, g^{-1}]
= \vdots
= g^p[\prod_{k=1}^{p-3}f[f^{-1}, g^{-p+k-1}]]f^2[f^{-1}, g^{-2}][f^{-1}, g^{-1}]f^{-1}, g^{-1}]
= g^p[\prod_{k=2}^{p-1}f[f^{-1}, g^{-p+k-1}]]f^2[f^{-1}, g^{-2}][f^{-1}, g^{-1}]f^{-1}, g^{-1}]
= g^p f^p[\prod_{k=2}^{p-1}f^{-1}, g^{-p+k-1}]f^2[f^{-1}, g^{-2}][f^{-1}, g^{-1}]f^{-1}, g^{-1}].
$$

Now take $l = 1$. In this case, since $\tau_Z^2(D_{g,1}[p]) = 0$, we have that

$$
\rho_3^Z((T_{\beta_1}T_{\beta_1}^{-1})^p) = \rho_3^Z(T_{\beta_1}^{\psi_1}T_{\beta_1}^{\psi_1^{-1}}) = \tau_Z^2(T_{\beta_1}^{\psi_1}T_{\beta_1}^{\psi_1^{-1}}) = 0.
$$
Chapter 6. Obstruction to Perron’s conjecture

Suppose that the assertion is true for \( l - 1 \) and we will prove it for \( l \). So assume that

\[
\xi = (T_{\beta_1} T_{\beta_1}^{-1})(T_{\beta_2} T_{\beta_2}^{-1}) \cdots (T_{\beta_l} T_{\beta_l}^{-1}) = fg,
\]

with \( f = T_{\beta_1} T_{\beta_1}^{-1} \) and \( g = (T_{\beta_2} T_{\beta_2}^{-1}) \cdots (T_{\beta_l} T_{\beta_l}^{-1}) \). Now using the formula (6.3.2) and the induction’s hypothesis we have that

\[
\rho Z_3((fg)^p) = \rho Z_3((f^{-1}, g^p)) \prod_{k=2}^p \rho Z_3([f^{-1}, g^{k-p-2}], f^{k-p-1}) \rho Z_3([f^{-1}, g^{k-p-2}])
\]

But, by construction, we know that \( \tau Z_2([\xi, \eta]) = (\pi_p \circ \chi_p)(\tau Z_1(\xi) \land \tau Z_1(\eta)) \). Then,

\[
\rho Z_3((fg)^p) = \sum_{k=1}^p (p + 1 - k)(\pi_p \circ \chi_p)(\tau Z_1(f) \land \tau Z_1(g)) = \left(\sum_{k=1}^p k\right)(\pi_p \circ \chi_p)(\tau Z_1(f) \land \tau Z_1(g)) = 0 \pmod{p}
\]

Thus \( \rho Z_3(\xi^p) = 0 \), and therefore the homomorphism \( h \) is zero.

Therefore we get that \( [J_g^t + \frac{1}{38} Q_g] = [\bar{J}_s(c)] \) and as a consequence \( [(\tau Z_1^*)^* (J_g^t + \frac{1}{38} Q_g)] = [(\tau Z_1^*)^* (\bar{J}_s(c))] \).

Thus it is enough to prove the following proposition

**Proposition 6.3.4.** The cohomology class of the 2-cocycle \((\tau Z_1^*)^* (\bar{J}_s(c))\) is zero.

**Proof.** Recall that we have the following central extension

\[
0 \rightarrow Im(\tau Z_2) \rightarrow \rho Z_3(\mathcal{M}_{g,1}[p]) \rightarrow \Lambda^3 H_p \rightarrow 1.
\] (6.3.3)

Notice that \((\psi Z_2)^* \) commutes with \( f \). Moreover, on \( \mathcal{M}_{g,1}[p] \), \( \psi Z_2 \circ \rho Z_3^* = \tau Z_1^* \). Then we have a commu-
6.3. The obstruction to Perron’s conjecture

tative diagram

\[
\begin{array}{ccc}
H^2(\wedge^3 H_p; Im(\tau_2^Z)) & \xrightarrow{(\varphi_2^Z)^*} & H^2(\rho_3^Z(\mathcal{M}_{g,1}[p]); Im(\tau_2^Z)) \\
\downarrow f_* & & \downarrow f_* \\
H^2(\wedge^3 H_p; \mathbb{Z}/p) & \xrightarrow{(\varphi_2^Z)^*} & H^2(\rho_3^Z(\mathcal{M}_{g,1}[p]); \mathbb{Z}/p) \\
\downarrow (\tau_1^Z)^* & & \downarrow (\rho_3^Z)^* \\
H^2(\mathcal{M}_{g,1}[p]; \mathbb{Z}/p) & & H^2(\mathcal{M}_{g,1}[p]; \mathbb{Z}/p)
\end{array}
\] (6.3.4)

As a consequence, we have that \([(\tau_1^Z)^*(\bar{\tau}_s(c))] = [(\rho_3^Z)^*(\varphi_2^Z)^*(c)]\). On the other hand, applying Lemma (1.1.4) to the central extension (6.3.3) we get that \([(\varphi_2^Z)^*(c)]\) is zero. Therefore \([(\tau_1^Z)^*(\bar{\tau}_s(c))]\) is zero.

\[\square\]

\[\text{ii) We show that the cohomology class of the 2-cocycle } (\tau_1^Z)^*(Q_g) \text{ is not zero.}\]

In [32], S. Morita gave a crossed homomorphism \(k : \mathcal{M}_{g,1} \to \frac{1}{2} \wedge^3 H\) (denoted by \(\bar{k}\) in [32]) that extends the first Johnson homomorphism \(\tau_1 : \mathcal{T}_{g,1} \to \wedge^3 H\). Moreover, in [30], he defined a 2-cocycle \(\varsigma_g\) on \(\mathcal{M}_{g,1}\) (denoted by \(c\) in [30]) given by

\[
\varsigma_g(\phi, \psi) = \omega((k \circ C)(\phi), (k \circ C)(\psi^{-1}))
\]

where \(\omega\) is the intersection form and \(C\) is the contraction map.

Now, if we restrict the 2-cocycle \(\varsigma_g\) on \(\mathcal{M}_{g,1}[p]\) and take values in \(\mathbb{Z}/p\) we have the 2-cocycle

\[
\omega((\tau_1^Z \circ C)(\phi), (\tau_1^Z \circ C)(\psi^{-1})) = \omega((\tau_1^Z \circ C)(\phi), -(\tau_1^Z \circ C)(\psi)) = -\omega((\tau_1^Z \circ C)(\phi), (\tau_1^Z \circ C)(\psi)) = -(\tau_1^Z)^*(Q_g)(\phi, \psi).
\]

Because \(k\) restricted to \(\mathcal{M}_{g,1}[p]\) with values in \(\mathbb{Z}/p\) is also an extension of the first Johnson homomorphism modulo \(p\) and by Section (5.4.1) we know that this coincide with \(\tau_1^Z\).

In [14] J. Harer proved that the second cohomology group \(H^2(\mathcal{M}_{g,1}; \mathbb{Z})\) of \(\mathcal{M}_{g,1}\) is an infinite cyclic group (for \(g \geq 3\)) generated by the first Chern class \(c_1 \in H^2(\mathcal{M}_{g,1}; \mathbb{Z})\).

Now let \(e_1 \in H^2(\mathcal{M}_{g,1}; \mathbb{Z})\) the first characteristic class of surface bundles defined in [29]. S. Morita proved that

\[e_1 = 12c_1,\]

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and that $ς_g$ is a 2-cocycle which represents $e_1$. Therefore, the image of $ς_g$ is $12\mathbb{Z}$ and so, for $p > 3$, the cohomology class of $ς_g$ reduced modulo $p$ is not zero. As a consequence, since the restriction map $res : H^2(\mathcal{M}_{g,1}; \mathbb{Z}/p) \to H^2(\mathcal{M}_{g,1}[p]; \mathbb{Z}/p)$ is injective, we have that the cohomology class of $(\tau_1^Z)^*(Q_g)$ is not zero.
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