The Saturation Number for the length of Degree Monotone Paths

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Abstract

A degree monotone path in a graph \( G \) is a path \( P \) such that the sequence of degrees of the vertices in the order in which they appear on \( P \) is monotonic. The length of the longest degree monotone path in \( G \) is denoted by \( mp(G) \). This parameter, inspired by the well-known Erdős-Szekeres theorem, has been studied by the authors in two earlier papers. Here we consider a saturation problem for the parameter \( mp(G) \). We call \( G \) saturated if, for every edge \( e \) added to \( G \), \( mp(G + e) > mp(G) \), and we define \( h(n, k) \) to be the least possible number of edges in a saturated graph \( G \) on \( n \) vertices with \( mp(G) < k \), while \( mp(G + e) \geq k \) for every new edge \( e \).

We obtain linear lower and upper bounds for \( h(n, k) \), we determine exactly the values of \( h(n, k) \) for \( k = 3 \) and \( 4 \), and we present constructions of saturated graphs.

1 Introduction

Given a graph \( G \), a degree monotone path is a path \( v_1v_2\ldots v_k \) such that \( \text{deg}(v_1) \leq \text{deg}(v_2) \leq \ldots \leq \text{deg}(v_k) \) or \( \text{deg}(v_1) \geq \text{deg}(v_2) \geq \ldots \geq \text{deg}(v_k) \). This notion, inspired by the well-known Erdős-Szekeres Theorem [7, 9], was introduced in [6] under the name of uphill and downhill path in relation to domination problems, also studied in [4, 5, 11]. In [6], the study of the parameter \( mp(G) \), which denotes the length of the longest degree monotone path in \( G \), was specifically suggested. This parameter was studied by the authors in [2, 3], and among many results obtained, the parameter \( f(n, k) = \max\{|E(G)| : |V(G)| = n, mp(G) < k\} \) was also defined. It was shown that this is closely related to the Turan numbers \( t(n, k) = \max\{|E(G)| : |V(G)| = n, G \text{ contains no copy of } K_k\} \).

A general form of the Turan numbers with respect to a graph \( H \) is \( t(n, H) = \max\{|E(G)| : |V(G)| = n, G \text{ contains no copy of } H\} \). The study of Turan num-
bers is undoubtedly considered as one of the fundamental problems in extremal graph and hypergraph theory [1].

The Turan number has a counter-part known as the saturation number with respect to a given graph $H$, defined as

$$
\text{sat}(n, H) = \min \{|E(G)| : |V(G)| = n, G \text{ contains no copy of } H, \text{ but } G + e \text{ contains } H \text{ for any edge added to } G}\}
$$

Tuza and Kaszonyi in [12] launched a systematic study of $\text{sat}(n, H)$ following an earlier result by Erdos, Hajnal and Moon [8] who proved that $\text{sat}(n, K_k) = \binom{k-2}{2} + (k-2)(n-k+2)$ with a unique graph attaining this bound, namely $K_{k-2} + K_{n-k+2}$. For the current paper, it is worth noting that $\text{sat}(n, P_k)$ (or $\text{sat}(n, k)$ for short) is known [12] for every $k$ for $n$ sufficiently large with respect to $k$, and in particular for $n$ large enough, $\text{sat}(n, k) = n(1 - c(k))$. For a survey and recent information about saturation, see [10].

In this spirit, we call a graph $G$ saturated if $mp(G + e) > mp(G)$, for all new edges $e$ joining non-adjacent vertices in $G$. If it happens that $mp(G + e) \geq k$ for all new edges $e$ we sometimes refer to the saturated graph $G$ as $k$-saturated. By convention we say that $K_m$ is $k$-saturated for $m \leq k - 1$. Then we define

$$
h(n, k) = \min \{|E(G)| : |V(G)| = n, G \text{ is } k\text{-saturated}\}.
$$

In Section 2, we prove linear lower and upper bounds for this parameter. In Section 3, we provide exact determination of $h(n, k)$ for $k = 3, 4$. In Section 4 we present several open problems concerning $h(n, k)$ for $k \geq 5$ as well as several other problems and conjectures.

## 2 General Lower and Upper bounds

### 2.1 Lower bounds

We begin by showing that $\text{sat}(n, k)$ is a lower bound for $h(n, k)$.

**Proposition 2.1.**

For $k \geq 2$, $h(n, k) \geq \text{sat}(n, k)$.

**Proof.** Clearly, if $G$ is a graph realising $\text{sat}(n, P_k) = \text{sat}(n, k)$, this means that $G$ does not contain a copy of $P_k$, and hence no degree monotone path of length $k$. But $G + e$ contains $P_k$, but not necessarily a degree monotone path of length $k$. Hence $h(n, k) \geq \text{sat}(n, k)$.

Recall that for fixed $k$ and large $n$, $\text{sat}(n, k) = n(1 - c(k)) < n$. We now strengthen Proposition 2.1 to show that for $k \geq 4$, $h(n, k) \geq n$. We first prove a lemma, and subsequently a corollary, which will then be used in the main proof.

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Lemma 2.2. Let $G$ be a connected graph with a vertex $u$ of degree 1 and a vertex $v$ of maximum degree $\Delta \geq 2$ which are not adjacent. Then $mp(G + uv) \leq mp(G)$, namely $G$ is not saturated.

Proof. Let $H = G + uv$ and let $P$ be a path in $H$ which realizes $mp(H)$. Let $u^*$ and $v^*$ be the vertices $u$ and $v$ as they appear in $H$.

If $\Delta = 2$, then clearly $G$ is a path on $k \geq 4$ vertices and $mp(G) = k - 1$, and if we take $u$ to be the first vertex of the path, and $v$ to be the $(k - 1)^{th}$ vertex, then $mp(H) = k - 1 = mp(G)$.

So we may assume $\Delta \geq 3$. Now if $u^*$ and $v^*$ are not on $P$, then $P$ is degree monotone in $G$ and hence $mp(H) \leq mp(G)$. If $v^*$ is on $P$ but $u^*$ is not then $v^*$ must be the last vertex on $P$, and hence, in $G$, the path $P$ with $v^*$ replaced by $v$ is also degree monotone in $G$ and $mp(H) \leq mp(G)$. Similarly, if $u^*$ is on $P$ but $v^*$ is not, then $u^*$ must be the first vertex on $P$, since clearly $u^*$ cannot be in the “middle” of the path as then the next vertex on $P$ must be $v^*$, which is not on $P$. Then the path $P$ in $G$ with $u^*$ replaced by $u$ is also degree monotone in $G$ and again $mp(H) \leq mp(G)$. If $u^*$ is the last vertex on the path then clearly $P$ is not maximal, as $P \cup \{v^*\}$ has a neighbour $w$ which must be the last vertex on $P$, contradicting maximality of $P$.

So the only remaining case to consider is when $u^*$ and $v^*$ are both on $P$. Then clearly $v^*$ must be the last vertex on $P$. If $u^*$ is the first vertex, then either $P = u^*v^*$ and $mp(H) = 2 \leq mp(G)$, or the path $P$ is degree monotone in $G$ too. If $u^*$ is not the first vertex, then the next vertex on $P$ must be $v^*$ which is the last vertex. Hence, in this case, all predecessors of $u^*$ on $P$ must have degree at most 2. But if the first vertex $y$ in $P$ has degree 1, then, in $G$, the path $y \ldots u$ is disconnected from the rest of $G$, which is impossible. Therefore $deg(y) = 2$ and $y$ has a neighbour $w$ which must have degree greater than 2 (note that $w$ may be equal to $v^*$ but cannot be any other vertex on $P$). But then, the path $u \ldots yw$ is degree monotone in $G$ and is of the same length as $P$, and hence $mp(H) \leq mp(G)$.

Lemma 2.2 is best possible with respect to the adjacency condition between minimum degrees and maximum degrees because, if the minimum degree is greater than 1, and a vertex $u$ of minimum degree is not adjacent to vertex $v$, then $mp(G + uv)$ may be larger than $mp(G)$. As an example, consider graph $G_n$ made up of the cycle $C_{2n}$, $n \geq 3$, with vertices labelled $v_1, v_2, \ldots, v_{2n}$, and a vertex $w$ connected to vertices $v_1, v_3, v_5, \ldots, v_{2n-1}$. Thus $w$ has degree $\Delta = n$ and $\delta = 2$, and $mp(G_n) = 3$. The vertices of degree 2 are not connected to $w$, but connecting any such vertex to $w$ by an edge $e$ gives $mp(G_n + e) = 5$. In fact, these graphs are 5-saturated even though they have non-adjacent vertices of maximum degree $\Delta \geq 3$ and minimum degree $\delta = 2$.

Corollary 2.3. Let $T$ be a tree on $n \geq 3$ vertices. Then $T$ is saturated for a degree monotone path if and only if $T = K_{1,n-1}$.

Proof. Suppose first $mp(T) \geq 3$. Then clearly $T$ is a not a star, hence there is a leaf not connected to a vertex of maximum degree and by Lemma 2.2 $T$ is not saturated.
So suppose $mp(T) = 2$. If not all leaves are adjacent to the same vertex of maximum degree then again by Lemma 2.2, $T$ is not saturated. Hence $T$ must be a star $K_{1,n-1}$.

Indeed $K_{1,n-1}$ is saturated and $mp(K_{1,n-1}) = 2$ while $mp(K_{1,n-1} + e) = 3$ for every edge $e \notin E(K_{1,n-1})$.

**Theorem 2.4.** For $n \geq 3$ and $k \geq 4$, $h(n,k) \geq n$.

**Proof.** We may assume that $n \geq k$ for otherwise, trivially, $K_n$ is saturated having $\binom{n}{2} \geq n$ edges for $n \geq 3$.

So let $G$ be a graph on $n \geq k$ vertices realizing $h(n,k)$, $k \geq 4$. If $G$ is connected then by Corollary 2.3, $G$ is not a tree hence $|E(G)| \geq n$ as required.

So we may assume that $G$ is not connected, and let $G_1, G_2, \ldots G_t$ be the connected components of $G$. Again, by Corollary 2.3, we infer that every component on at least three vertices is not a tree and hence must have at least $|V(G_j)|$ edges.

If there are two components $G_i$ and $G_j$ on at most two vertices, adding an edge joining these two components does not create a degree monotone path of length 4 or more, contradicting the fact that $G$ is saturated.

If there is just one component on at most two vertices, then one can connect one vertex of this component to a vertex of maximum degree in another component, and again no degree monotone path of length four or more is created, contradicting the fact that $G$ is saturated.

Hence

$$|E(G)| = \sum_{i=1}^{t} |E(G_i)| \geq \sum_{i=1}^{t} |V(G_i)| = n,$$

and therefore $h(n,k) \geq n$ for $n \geq 3$ and $k \geq 4$. □

### 2.2 Upper bounds

We now give a linear upper bound for $h(n,k)$. We consider separately $k$ odd and $k$ even.

We first recall the definition of the Cartesian product $G \square H$ for two graphs $G$ and $H$. The vertex set of the product is $V(G) \times V(H)$. Two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if either $u_1$ and $u_2$ are adjacent in $G$ and $v_1 = v_2$, or $v_1, v_2$ are adjacent in $H$ and $u_1 = u_2$.

**Theorem 2.5.** If $k \geq 3$ is an odd integer, then $h(n,k) \leq \frac{n(3k-1)}{12}$ for $n = 0 \pmod{\frac{3(k-1)}{2}}$.

**Proof.** Consider the graph $G = P_3 \square K_t$, for $k \geq 3$ odd and $t = \frac{k-1}{2}$. Clearly $|V(G)| = 3k-1$ and $|E(G)| = \frac{3(k-1)(k-3)}{8} + \frac{2(k-1)}{2} = \frac{(k-1)(3k-1)}{8}$. For $k = 3$ (so $t = 1$) this is simply $P_3$ and $mp(P_3) = 2$ while for $k = 5$ (so $t = 2$), this gives the graph $G = P_3 \square K_2$, which is $C_6$ plus one edge joining two antipodal vertices and clearly $mp(G) = 4$.

We now show that this graph, which has $mp(G) = k - 1$, is saturated. In $G = P_3 \square K_t$, let the top $t$ vertices be $u_1, \ldots, u_t$, all having degree $t$, the middle
vertices $v_1, \ldots, v_t$ all having degree $t + 1$, and the bottom vertices $w_1, \ldots, w_t$ all having degree $t$. It is clear that $mpG = 2t = k - 1$, taking for example the path $u_1 \ldots u_tv_1 \ldots v_1$. Because of the symmetry of $G$, we only need to check the addition of the edges $u_1v_2, v_1w_2$ and $u_1w_1$.

- If the edge $u_1v_2$ is added, then the path $w_1 \ldots w_tv_1 \ldots v_1u_1v_2$ has exactly $t + t - 2 + 3 = 2t + 1 = k$ vertices.
- If the edge $v_1w_2$ is added, then the path $u_1 \ldots u_tv_1 \ldots v_2w_2v_1$ has exactly $t + t - 1 + 2 = 2t + 1 = k$ vertices.
- If the edge $u_1w_1$ is added, then the path $u_2 \ldots u_tv_1 \ldots v_1w_1u_1$ has exactly $t - 1 + t + 2 = 2t + 1 = k$ vertices.

Hence $G$ is saturated with $mp(G) = k - 1$.

We now consider two disjoint copies of $G$, $G_1$ and $G_2$. We label this graph $2G$ and show that this graph is also saturated. Again labelling the vertices of $G$ as above, by the symmetry of $G$ we only need to consider the addition of the edges joining $u_t$ in $G_1$ to $u_1$ in $G_2$, $u_t$ in $G_1$ to $v_1$ in $G_2$, and $v_t$ in $G_1$ to $v_1$ in $G_2$:

- If the edge joining $u_t$ in $G_1$ to $u_1$ in $G_2$ is added, then the path $u_1 \ldots u_tv_1 \ldots v_1u_t$ in $G_1$ followed by $u_1v_1 \ldots v_1$ in $G_2$ has exactly $t + t + 1 = 2t + 1 = k$ vertices.
- If the edge joining $u_t$ in $G_1$ to $v_1$ in $G_2$ is added, then the path $u_1 \ldots v_tv_1u_t$ in $G_1$ followed by $v_1 \ldots v_t$ in $G_2$ has exactly $t + t = 2t + 1 = k$ vertices.
- If the edge joining $v_t$ in $G_1$ to $v_1$ in $G_2$, is added, then the path $u_1 \ldots u_1v_1 \ldots v_t$ in $G_1$ followed by $v_1 \ldots v_1$ in $G_2$ has exactly $2t + 1 = k$ vertices.

Hence $2G$ is saturated, and clearly this also applies to $p \geq 3$ disjoint copies of $G$, $pG$. Now $pG$ has $n = p^{3(k - 1)}$ vertices and $p^{(k - 1)(3k - 1)}8$ edges. Hence, for $n \equiv 0 \pmod{\frac{(3k - 1)}{2}}$, the number of edges is $\frac{n(3k - 1)}{12}$, as stated.

**Lemma 2.6.** Let $G$ be a saturated graph with $mp(G) = k$. Consider the graph $H = G + v$, where $v$ is a new vertex connected to all the vertices of $G$. Then $mp(H) = k + 1$, and $H$ is saturated.

**Proof.** Consider the graph $H$. Then $deg(v) = |V(G)|$ and $v$ has maximum degree. So any degree monotone path in $G$ can be extended in $H$ by including vertex $v$, and hence $mp(H) = mp(G) + 1 = k + 1$.

Now since $G$ is saturated, adding any edge $e$ creates a degree monotone path of length $k + 1$, and hence, adding the same edge in $H$ creates a path of length $k + 2$. The only edges which can be added in $H$ are those that can be added in $G$, and hence $H$ is saturated with $mp(H) = k + 1$ as required.

This lemma, together with Theorem 2.2, leads to the following result:
Theorem 2.7. For \( k \geq 4 \) and \( k = 0 \) (mod 2), \( h(n, k) \leq \frac{n(3k+8)(k-2)}{4(3k-4)} \) for \( n = 0 \) (mod \( \frac{3k-4}{2} \)).

Proof. In Theorem 2.5 we proved that \( G = P_3 \Box K_t \), where \( t = \frac{j-1}{2} \) has \( mp(G) = j - 1 \) and is saturated for \( j \geq 3 \) and \( j \) odd. Now by Lemma 2.6, \( H = G + v \) has \( mp(H) = j + 1 \) (even) and is saturated. Then \( H \) has \( \frac{3(j+1)}{2} \) vertices and \( \frac{(j-1)(3j-1)}{8} + \frac{3(j-1)}{2} = \frac{(3j+11)(j-1)}{8} \) edges. Now let \( k = j + 1 \), and hence we have \( \frac{3k-4}{2} \) vertices and \( \frac{(3k+8)(k-2)}{2} \) edges.

We now consider two disjoint copies of \( H \), \( H_1 \) and \( H_2 \) and call this graph \( 2H \). We need only consider edges which involve the new vertex of degree \( \frac{3k-2}{2} \), which has the largest degree, as other edges have the same effect as they have in \( 2G \). If we connect the vertex of degree \( \frac{3k-2}{2} \) in \( H_1 \) to that of the same degree in \( H_2 \), we can take a path of length \( k - 1 \) in \( H_1 \) ending with the vertex of maximum degree and then move to the vertex in \( H_2 \), giving a path of length \( k \). If we connect the vertex of degree \( \frac{3k-2}{2} \) in \( H_1 \) to one of degree \( \frac{3k}{2} \) in \( H_2 \), then we take a path of length \( k - 1 \) in \( H_2 \) ending with the vertex connected to the vertex in \( H_1 \), and then move to this vertex in \( H_1 \) to give a degree monotone path of length \( k \). Finally, if we connect the vertex of degree \( \frac{3k-2}{2} \) in \( H_1 \) to one of degree \( \frac{k+1}{2} \) in \( H_2 \), then we can take a degree monotone path in \( H_2 \) of length \( k - 1 \) ending on the vertex connected to \( H_2 \), and then the vertex in \( H_2 \) to give a degree monotone path of length \( k \) in \( 2H \).

Hence \( 2H \) is saturated and this also applies to \( p \geq 3 \) disjoint copies of \( H \), \( pH \). This graph has \( n = p\frac{3k-4}{2} \) vertices and \( p\frac{(3k+8)(k-2)}{8} \) edges. Hence for \( n = 0 \) (mod \( \frac{3k-4}{2} \)), the number of edges is \( \frac{n(3k+8)(k-2)}{4(3k-4)} \) as stated. \( \square \)

We next show, as an example, how to extend the results given in Theorems 2.5 and 2.7, to the case where \( \alpha \neq 0 \) (mod \( f(k) \)), where \( f(k) \) is the modulus given in these theorems. We will demonstrate it in the case \( k = 5 \).

Proposition 2.8. For \( n \geq 8 \), \( h(n, 5) \leq \frac{7n+c(n \mod 6)}{6} \), where \( c(n \mod 6) = \{0, 35, 16, 27, 8, 28\} \) for \( n \) (mod 6) = 0, 1, 2, 3, 4, 5 respectively.

Proof. Consider the graphs \( G = P_3 \Box K_2 \), \( H = K_5 - e \) for \( e \in E(K_5) \) and \( K_4 \), which are saturated for \( k = 5 \) and clearly \( mp(G) = mp(H) = mp(K_4) = 4 \). Every integer \( n \geq 8 \) can be represented in the form \( 6x + 5y + 4z \) with \( x, y, z \) non-negative integers. Hence \( x \) copies of \( G \), \( y \) copies of \( H \) and \( z \) copies of \( K_4 \) produce graphs for every \( n \geq 8 \). It is easy to check that any graph made up of two vertex disjoint copies of any combination of \( G \), \( H \) and \( K_4 \) is also saturated, and hence any combination of vertex disjoint copies of these graphs is saturated.

Hence any graph made up of a disjoint combination of any number of these three graphs is saturated.

For \( n = 0 \) (mod 6), the result follows immediately by substituting for \( k = 5 \) in Theorem 2.5.

For \( n = 1 \) (mod 6), we take the graph made up of \( \frac{n-13}{6} \) copies of \( G \), two copies \( K_4 \) and one copy of \( H \). The graph thus obtained is saturated and has \( \frac{7(n-13)}{6} + 12 + 9 = \frac{7n+35}{6} \) edges.
For \( n = 2 \pmod{6} \), we take the graph made up of \( \frac{n-8}{6} \) copies of \( G \) and two copies \( K_4 \). The graph thus obtained is saturated and has \( \frac{7(n-8)}{6} + 12 = \frac{7n+16}{6} \) edges.

For \( n = 3 \pmod{6} \), we take the graph made up of \( \frac{n-9}{6} \) copies of \( G \), one copy of \( K_4 \) and one copy of \( H \). The graph thus obtained is saturated and has \( \frac{7(n-9)}{6} + 6 + 9 = \frac{7n+27}{6} \) edges.

For \( n = 4 \pmod{6} \), we take the graph made up of \( \frac{n-4}{6} \) copies of \( G \) and one copy of \( K_4 \). The graph thus obtained is saturated and has \( \frac{7(n-4)}{6} + 6 = \frac{7n+8}{6} \) edges.

For \( n = 5 \pmod{6} \), we take the graph made up of \( \frac{n-5}{6} \) copies of \( G \) and one copy of \( H \). The graph thus obtained is saturated and has \( \frac{7(n-5)}{6} + 9 = \frac{7n+28}{6} \) edges.

Note: Applying the technique demonstrated in Proposition 2.8, we can extend Theorems 2.5 and 2.7 to cover all \( n \geq (k-1)(k-2) \), and we state it rather crudely as follows:

1. For \( k \geq 3, k = 1 \pmod{2} \) and \( n \geq (k-1)(k-2) \), \( h(n, k) \leq \frac{n(3k-1)}{12} + O(k^2) \).
2. For \( k \geq 4, k = 0 \pmod{2} \) and \( n \geq (k-1)(k-2) \), \( h(n, k) \leq \frac{n(3k+8)(k-2)}{4(3k-4)} + O(k^2) \).

3 Determination of \( h(n, k) \) for \( k = 2, 3, 4 \).

We first determine the exact value of \( h(n, 2) \) and \( h(n, 3) \).

**Proposition 3.1.**

1. \( h(n, 2) = 0 \).
2. \( h(n, 3) = \frac{n}{2} \) for \( n = 0 \pmod{2} \), while \( h(n, 3) = \frac{n+1}{2} \) for \( n = 1 \pmod{2} \).

**Proof.**
1. \( mp(G) = 1 \) if and only if \( G \) is a graph with no edges, and any edge we add gives \( mp(G + e) = 2 \).
2. By Proposition 2.1, \( h(n, 3) \geq sat(n, 3) = \lfloor \frac{n}{2} \rfloor \). Consider \( n = 0 \pmod{2} \). Let \( G \) be made up of \( \frac{n}{2} \) copies of \( K_2 \). This is the only graph which achieves \( sat(n, 3) \). Clearly \( mp(G) = 2 \), and adding any edge will create a copy of \( P_4 \) so \( mp(G + e) = 3 \).

Now if \( n = 1 \pmod{2} \), the graph \( G \) made up of \( \lfloor \frac{n}{2} \rfloor \) copies of \( K_2 \) and one copy of \( K_1 \) achieves \( sat(n, 3) \), and is the only such graph. Again \( mp(G) = 2 \).

If we add an edge joining two vertices from disjoint copies of \( K_2 \) then we get a copy of \( P_4 \) and \( mp(G + e) = 3 \); however, if we add a vertex joining a vertex from \( K_2 \) to the vertex in \( K_1 \), this gives a copy of \( P_3 \), and \( mp(G + e) = 2 \), hence \( h(n, 3) \geq sat(n, 3) + 1 \).

Consider the graph \( G \) made up of \( \lfloor \frac{n-3}{2} \rfloor \) copies of \( K_2 \), and a single copy of \( P_3 \). Again it is clear that \( mp(G) = 2 \). Adding an edge joining two vertices
from disjoint copies of $K_2$ then we get a copy of $P_4$ and $mp(G + e) = 3$, while adding an edge joining a vertex from $K_2$ to one in $P_3$ gives $mp(G + e) = 4$. The number of edges in this graph is $\frac{n + 1}{2} = sat(n, 3) + 1$ as stated. 

We now determine the exact value of $h(n, 4)$. For this we need another lemma:

**Lemma 3.2.** Let $G$ be a saturated connected graph with $\lvert E(G) \rvert \leq \lvert V(G) \rvert$ and $2 \leq mp(G) \leq 3$. Then

1. If $mp(G) = 2$ then $G = K_{1,\Delta}$ and for $\Delta \geq 2$, $mp(G + e) = 3, \forall e \notin E(G)$.

2. If $mp(G) = 3$ then $G = K_3$ which is saturated by definition.

**Proof.** Let $G$ be such a graph. Then since $\lvert E(G) \rvert \leq \lvert V(G) \rvert$, $G$ is either a tree or is unicyclic.

If $G$ is a tree then either all leaves are adjacent to the same vertex which has maximum degree, that is $G = K_{1,\Delta}$. Then $mp(G) = 2$ and, in case $\Delta \geq 2$, adding any edge between two leaves $u$ and $v$ gives $mp(G + uv) = 3$. If $G$ is a tree but not $K_{1,\Delta}$, then there is a leaf $u$ and a vertex $v$ of maximum degree which are not adjacent, and hence by Lemma 2.2 $G$ is not saturated.

So suppose $G$ is unicyclic. Then it cannot be a simple cycle $C_n$ on $n \geq 4$ vertices as otherwise $mp(C_n) = n \geq 4$. Observe that $C_3 = K_3$ is saturated by definition.

So $G$ is unicyclic with at least one leaf if the cycle has at least four vertices.

Suppose $mp(G) = 2$. If there are at least two vertices on the cycle which have branches attached, then on one of these branches (including the vertex on the cycle) there must be a vertex of maximum degree, and on the other branch there must be a leaf not connected to this vertex of maximum degree, and hence by Lemma 2.2 $G$ is not saturated.

So there is precisely one vertex on the cycle with degree greater than two, which means that $mp(G) > 2$, a contradiction.

So now suppose $mp(G) = 3$. If there are at least two vertices on the cycle which have branches attached, then on one of these branches (including the vertex on the cycle) there must be a vertex of maximum degree, and on the other branch there must be a leaf not connected to this vertex of maximum degree, and hence by Lemma 2.2 $G$ is not saturated.

So there is precisely one vertex on the cycle with degree greater than two, and if the cycle has at least four vertices, $mp(G) \geq 4$, a contradiction.

So it remains to consider the cycle $K_3$ with exactly one vertex $x$ with degree greater than two.

Suppose the vertex $x$ has $p$ leaves and $q$ branches with $p, q \geq 0$. We consider several cases:

1. If $p \geq 2$, we connect two leaves to get $H$ with $mp(H) = mp(G) = 3$, and $G$ is not saturated. Hence $p \leq 1$.

2. If $p = 1$ and $q \geq 1$, then either $x$ is of maximum degree $\Delta \geq 3$, and there is a leaf not connected to $x$, so by Lemma 2.2 $G$ is not saturated, or there
is a vertex of maximum degree in one of these branches, so the leaf at \( x \) is not connected to the vertex of maximum degree and again by Lemma 2.2 \( G \) is not saturated.

3. If \( p = 1 \) and \( q = 0 \), then \( G \) is \( K_3 \) with a leaf attached and clearly it is not saturated.

4. If \( p = 0 \) and \( q \geq 2 \), then either \( x \) is of maximum degree \( \Delta \geq 3 \) and there is a leaf in the branch not connected to \( x \), so by Lemma 2.2 \( G \) is not saturated, or there is a vertex of maximum degree in one of these branches, so the leaf at \( x \) is not connected to the vertex of maximum degree and again by Lemma 2.2 \( G \) is not saturated.

5. If \( p = 0 \) and \( q = 1 \), then \( \deg(x) = 3 \). Let \( z \) be the neighbour of \( x \) in this branch. If \( \deg(z) \geq 3 \) then \( mp(G) \geq 4 \), a contradiction. Hence \( \deg(z) = 2 \), and let \( w \) be the neighbour of \( z \).

If \( \deg(w) = 1 \) then \( x \) has maximum degree, \( w \) is not connected to \( x \) and by Lemma 2.2 \( G \) is not saturated. So \( \deg(w) \geq 2 \). We consider two cases:

**Case 1:** \( \deg(w) = 2 \).

Let \( u \) be the neighbour of \( w \). If \( \deg(u) \leq 2 \), then we have \( uwzv \) a degree monotone path of length four. So \( \deg(u) \geq 3 \).

If \( \deg(u) > 3 \) then if the edge \( xw \) is added, \( mp(G + xw) = 3 \) and \( G \) is not saturated. Hence \( \deg(u) = 3 \). Let \( s \) and \( y \) be the neighbours of \( u \). If either \( s \) or \( y \) have degree at least three, we have \( zwux \) or \( zwuy \) degree monotone paths of length four, a contradiction. So both \( s \) and \( y \) have degree at most two.

If either \( s \) or \( y \) is a leaf, say \( s \), then either \( \Delta = 3 \) and \( s \) is leaf is not connected to \( x \), so by Lemma 2.2 \( G \) is not saturated, or \( \Delta \geq 4 \) and is realized by a vertex \( r \) say on the branch at \( y \). Again \( s \) is a leaf not adjacent to \( r \) and by lemma 2.2 \( G \) is not saturated.

So \( \deg(s) = \deg(y) = 2 \), and either the maximum degree \( \Delta = 3 \), and there is a leaf not adjacent to \( x \), so by Lemma 2.2 \( G \) is not saturated, or \( \Delta \geq 4 \) and is realized by a vertex \( r \) on a branch at some \( x_i \), \( i \neq j \). Then \( x_j \) is a leaf not adjacent to \( r \) and by Lemma 2.2 \( G \) is not saturated.

**Case 2:** \( \deg(w) = t \geq 3 \).

Let \( x_1, \ldots, x_t \) be the neighbors of \( w \). If for some \( j \), \( \deg(x_j) = 1 \), then either \( \Delta = 3 \) and \( x_j \) is not connected to \( x \), so by Lemma 2.2 \( G \) is not saturated, or \( \Delta \geq 4 \) and is realized by a vertex \( r \) on a branch at some \( x_i \), \( i \neq j \). Then \( x_j \) is a leaf not adjacent to \( r \) and by Lemma 2.2 \( G \) is not saturated.

So \( \deg(x_j) \geq 2 \) for \( j = 1, \ldots, t \). Now if \( \Delta = 3 \) then a leaf on one these branches starting at \( x_1, \ldots, x_j \) is not connected to \( x \) and by Lemma 2.2 \( G \) is not saturated. Otherwise \( \Delta \geq 4 \) and a vertex \( r \) of maximum degree
appears on the branch starting at say $x_j$. Then a leaf on any other branch is not connected to $r$ and by Lemma 2.2 $G$ is not saturated.

Hence $G = K_3$ is the only saturated graph with $|E(G)| \leq |V(G)|$ and $mp(G) = 3$.

**Theorem 3.3.** For $n = 0 \pmod{3}$, $h(n, 4) = n$ while for $n = 1, 2 \pmod{3}$, $h(n, 4) = n + 1$.

**Proof.** We first prove the upperbound for $h(n, 4)$. Consider the following cases:

1. Assume $n = 0 \pmod{3}$. Let $G$ be made up of $\frac{n}{3}$ copies of $K_3$, then clearly $mp(G) = 3$. Any edge we add gives a degree monotone path of length 4. So $G$ is saturated and hence $h(n, 4) \leq n$, for $n = 0 \pmod{3}$.

2. Assume $n = 1 \pmod{3}$ . Let $G$ be made up of $\frac{n-1}{3}$ copies of $K_3$ and a copy of $K_4 - e$, $e \in E(K_4)$. Clearly $mp(G) = 3$ and it is easy to see that $mp(G + e) \geq 4$. So $G$ is saturated and hence $h(n, 4) \leq n + 1$, for $n = 1 \pmod{3}$.

3. Assume $n = 2 \pmod{3}$ . Let $G$ be made up of $\frac{n-5}{3}$ copies of $K_3$ and two copies of $K_3$ with a common vertex. Clearly $mp(G) = 3$ and it is easy to see that $mp(G + e) \geq 4$. So $G$ is saturated and hence $h(n, 4) \leq n + 1$, for $n = 2 \pmod{3}$.

Now to the lower bound. Suppose $G$ is a graph on $n \geq 3$ vertices realising $h(n, 4)$. If $G$ is connected then by Lemma 3.2 either $G$ is $K_3$ or $|E(G)| \geq n + 1$. Hence we may assume that $G$ is not connected, and let $G_1, G_2, \ldots, G_t$ be the connected components of $G$.

Again, by Lemma 3.2 every component $G_j$ on at least 3 vertices is either $K_3$ or contains at least $|V(G_j)| + 1$ edges.

If there are at least two components say $G_i$ and $G_j$ on at most two vertices each, then we can just add an edge between a vertex in $G_i$ and one in $G_j$ without creating a degree monotone path of length more than 3, contradicting the fact that $G$ is saturated.

Lastly if there is just one component $G_j$ on at most two vertices, then if we connect a vertex in this component to a vertex $v$ of maximum degree in another component of $G$, then clearly no degree monotone path of length 4 or more is created, once again contradicting that $G$ is saturated.

Hence all components of $G$ have at least 3 vertices. If there are at least two components which are not $K_3$ then $|E(G)| \geq n + 2$, and this is not optimal by the constructions above. If there is just one component which is not $K_3$, then $|E(G)| \geq n + 1$ and so for $n = 1, 2 \pmod{3}$, $h(n, 4) \geq n + 1$ proving the constructions above are optimal.

Finally, if all components are $K_3$, then $|E(G)| = n$, proving $h(n, 4) = n$ for $n = 0 \pmod{3}$.

\end{proof}
4 Concluding Remarks and Open Problems

Several open problems have arised during our work on this paper. We list some of the more interesting ones:

- The major role played in this paper by Lemma 2.2 and its consequences suggest:
  Problem 1: Find further structural conditions (along the lines indicated in Lemma 2.2) indicating that a graph \( G \) is not saturated.

- In Corollary 2.3 we characterise saturated trees. In a previous paper [2] we characterised saturated graphs with \( mp(G) = 2 \). This leads to the following:
  Problem 2: Characterise \( k \)-saturated graphs for other families of graphs such as maximal outerplanar graphs, maximal planar graphs, regular graphs, etc.

  Problem 3: Characterise saturated graphs with \( mp(G) = 3 \).

- The parameter \( mp(G) \) can be very sensitive to edge-addition and edge-deletion, as shown in [3]. Also Theorem 2.5 gives \( h(n, 7) \leq \frac{4n}{7} \) for \( n = 0 \) (mod 9) while Theorem 2.7 gives \( h(n, 6) \leq \frac{13n}{7} \) for \( n = 0 \) (mod 7). These facts suggest the following monotonicity problem:
  Problem 4: Is it true that, at least for \( n \) large enough, depending on \( k \), and for \( k \geq 2 \), \( h(n, k + 1) \geq h(n, k) \)?

- If true, this will have the immediate implication that the construction for \( h(n, 6) \) is not optimal and that in fact \( h(n, 6) \leq \frac{5n(1+o(1))}{3} \) by the above upper bound for \( h(n, 7) \).

- The upper bound constructions given in Theorem 2.5 and Theorem 2.7 are probably not optimal.
  Problem 5: Improve upon the upper bounds obtained in Theorems 2.5 and 2.7.

- The lower bound given in Theorem 2.4 proved to be sharp in the case \( k = 4 \).
  Problem 6: Improve upon the lower bound \( h(n, k) \geq n \) for \( k \geq 5 \).

- In Proposition 2.8 we have shown that \( h(n, 5) \leq \frac{7n}{6} + c(n \text{ (mod 6)}) \).
  Problem 7: Determine \( h(n, 5) \) exactly. In particular, is it true that \( h(n, 5) = \frac{7n(1+o(1))}{6} \).

- Lastly recall that \( sat(n,k) = n(1-c(k)) < n \) for every large \( k \) and \( n \).
  Problem 8: Is it true that \( h(n, k) \leq cn \) for some constant \( c \) independent of \( k \).
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