Spectra of RSOS Soliton Theories

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ABSTRACT

We study here the spectrum of soliton scattering theories based on interaction round the face lattice models. We take for the admissibility condition the fusion rules of each of the simple Lie algebras. It is found that the mass spectrum is given by that of the corresponding Toda theory, or, that the mass ratios of the different kinks in the model are described by the Perron–Frobenius vector of the Cartan matrix. The scalar part of the soliton amplitude is shown to be identical with the minimal part of the corresponding Toda amplitude.

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The factorization property of integrable field theories in two dimensions [1] provided an excellent way to study their $S$ matrices. The constraints arising from factorization are in many cases enough to determine them. In ref. [2] the $S$ matrices of $SU(N)$ RSOS theory were determined using the Boltzmann weights of the corresponding lattice models. Our purpose here is to extend this to other Lie algebras, and to determine especially the mass spectra of the solitons, continuing the work of ref. [3].

As an ingredient, we use the Boltzmann weights found in ref. [4]. Most of which, however, were not proven to satisfy the YBE and are only conjectured to do so. Part of our aim is to see whether a consistent soliton theory arises, which is a highly non–trivial check on these Boltzmann weights.

The following expression for the Boltzmann weights of the fusion interaction round the face (IRF) lattice model was found in ref. [4]. The model is denoted by IRF$(\mathcal{O}, x, x)$, where $\mathcal{O}$ is some rational conformal field theory, (RCFT), and $x$ is a primary field in it. The solution to the Young Baxter equation (YBE) associated to this lattice model is

$$X_i(u) = \sum_{a=0}^{n-1} f_a(u) P^a_i,$$

where

$$f_a(u) = \prod_{r=0}^{n-a-2} \sin(\zeta_r + u) \prod_{r=n-a-1}^{n-2} \sin(\zeta_r - u),$$

where

$$\zeta_a = \frac{\pi}{2} [\Delta(\phi_{a+1}) - \Delta(\phi_a)],$$

where $\phi_i$ are the fields appearing in the operator product of $x$ with respect to itself, arranged in increasing dimensions,

$$x \cdot x = \sum_{i=0}^{n-1} \phi_i,$$

and $\Delta(\phi)$ is the conformal dimension of $\phi$. The $P^a_i$ are the eigenvectors of the
braiding matrices of the RCFT, braided with the field $x$, and projected to the $a$
intermediate field in the $t$ channel (see ref. [4], for detail). The $X_i(u)$ so defined
obey the Young Baxter equation

$$X_i(u)X_{i+1}(u+v)X_{i}(v) = X_{i+1}(v)X_{i}(u+v)X_{i+1}(u). \quad (5)$$

Actually, the fact that $X_i(u)$ so defined obeys the Young Baxter equation eq. (5)
has been proved only for $n = 2$ [4], mainly, and is conjectured that it is so in
general.

Part of the motivation here in exploring this solution is to examine whether this
conjecture is plausible. This we do by studying soliton theories based on the anzats
eq (1), calculating their mass spectrum and comparing it to known systems. In a
nutshell, let $G$ stand for the WZW RCFT associated to the Lie group $G$, and let
$x$ be the fundamental representation. Then it is found here that the soliton theory
based on $X_i(u)$ has a mass spectrum which is identical to the classical masses of
the $G$ Toda theory.

The soliton scattering amplitude associated to $\text{IRF}(\mathcal{O}, x, x)$ is given by [4],

$$S_{c d}^{a b} (\theta) = F(\theta) \left( \frac{S_{a,0} S_{c,0}}{S_{b,0} S_{d,0}} \right)^{\theta/2} \frac{F(\theta)}{F(-\theta)} \left( \frac{b}{c} \frac{a}{d} \lambda \theta \right). \quad (6)$$

Here $S_{c d}^{a b} (\theta)$ is the scattering matrix of the kinks $K_{a b} + K_{b c} \rightarrow K_{a d} + K_{d c}$, at the
relative rapidity $i\pi \theta$, where $K_{a b}$ is the kink interpolating the $a$ and $b$ vacua. $S_{a b}$
is the torus modular matrix and $w \left( \frac{a}{c} \frac{b}{d} \bigg| u \right)$ is the Boltzmann weight associated
to $X_i(u)$,

$$w \left( \frac{a}{c} \frac{b}{d} \bigg| u \right) = \langle a c d | X_2(u) | a b d \rangle. \quad (7)$$

$F(\theta)$ obeys,

$$F(\theta)F(-\theta) = \prod_a \sin(\zeta_a - \lambda \theta)^{-1} \sin(\zeta_a + \lambda \theta)^{-1} = \rho(\theta)\rho(-\theta), \quad (8)$$

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along with

$$F(\theta) = F(1 - \theta),$$

(9)

if \(x\) is a real primary field, \(\bar{x} = x\), or otherwise

$$F(1 - \theta)F(1 + \theta) = \prod_a \sin(\tilde{\zeta}_a - \lambda \theta)^{-1} \sin(\tilde{\zeta} + \lambda \theta)^{-1} = \bar{\rho}(\theta)\bar{\rho}(-\theta),$$

(10)

where \(\tilde{\zeta}_a = \pi[\Delta(\tilde{\phi}_{a+1}) - \Delta(\tilde{\phi}_a)]\), and \(\tilde{\phi}_a\) are the primary fields appearing in the operator product

$$x \cdot \bar{x} = \sum_{a=0}^{m} \tilde{\phi}_a,$$

(11)

arranged according to increasing dimensions. \(\lambda = \tilde{\zeta}_1\) is the crossing parameter. The function \(F(\theta)\) is determined by eqs. (8,10) up to an ambiguity, called c.d.d. ambiguity, i.e., \(\tilde{F}(\theta)\) solves eqs. (8,10), if \(F(\theta)\) does, \(Q(\theta)\) is any holomorphic function, and

$$\tilde{F}(\theta) = F(\theta) \frac{Q(\theta)Q(1 - \theta)}{Q(-\theta)Q(1 + \theta)}.$$  

(12)

However, the assumption of minimality of the solution can be justified in most cases: \(F(\theta)\) is the solution of this equations with the minimal number of poles and zeros on the physical regime \(0 \leq \theta < 1\).

The bound states of the two kinks \(x\) appear as poles in the scattering amplitude for the rapidities \(0 < \theta < 1\). These, in turn are determined by the function \(F(\theta)\). We shall assume, owing to minimality that these are exactly the poles of \(\rho(\theta)\) or \(\bar{\rho}(1 - \theta)\). In other words, \(F(\theta)\) has poles at

$$\theta = \frac{\zeta_i}{\lambda}, \text{ or } \theta_i = 1 - \frac{\zeta_i}{\lambda}.$$  

(13)

We can now compute the bound states spectrum, under the above assumptions, for any RCFT and any primary field \(x\). Importantly, this does not require the
explicit knowledge of the Boltzmann weights, eq. (1). Our purpose is to check whether this leads to a sensible soliton scattering theory. We shall see that this is indeed so, by examining the different algebras, case by case.

\( \text{A}_n \) case. Denote by \( \Lambda_m \) the \( m \)'th fundamental weight of the Lie algebra \( \text{A}_n \), \( m = 1, 2, \ldots, n \). Consider the model \( \text{IRF}(\text{A}_n, \Lambda_m, \Lambda_m) \). We need to compute the parameters \( \zeta_m \) and \( \bar{\zeta}_m \). Thus, consider the two tensor products,

\[
[\Lambda_m] \times [\Lambda_m] = \sum_{r=0}^{m} [\Lambda_{2m-r} + \Lambda_r],
\]

\[
[\Lambda_m] \times [\bar{\Lambda}_m] = \sum_{r=0}^{m} [\Lambda_r + \bar{\Lambda}_r],
\]

where we arranged the representations on the right hand side according to increasing conformal dimensions. Using the dimension formula for WZW primary fields,

\[
\Delta = \frac{\lambda(\lambda + 2\rho)}{k + g},
\]

where \( \lambda \) is the highest weight, \( \rho \) is half the sum of positive roots, \( g \) is the dual Coxeter number and \( k \) is the level, we find

\[
\zeta_r = \frac{\pi}{2} \left( \Delta_{[\Lambda_{r+1} + \Lambda_{2m-r-1}]} - \Delta_{[\Lambda_r + \Lambda_{2m-r}]} \right) = \frac{\pi}{2} \frac{2(m-r)}{k+n+1},
\]

\[
\bar{\zeta}_r = \frac{\pi}{2} \left( \Delta_{[\Lambda_{r+1} + \bar{\Lambda}_{r+1}]} - \Delta_{[\Lambda_r + \bar{\Lambda}_r]} \right) = \frac{\pi}{2} \frac{n+1-2r}{k+n+1}.
\]

The crossing parameter is \( \lambda = \bar{\zeta}_1 = \frac{\pi(n+1)}{2(k+n+1)} \). Thus, the poles in the physical regime of \( F(\theta) \) are located at \( \zeta_r/\lambda \) and \( 1 - \bar{\zeta}_r/\lambda \) and are given by,

\[
\frac{2(r+1)}{n+1}, \quad r = 0, 1, \ldots, m-1,
\]
and
\[ \frac{2r}{n + 1} \quad \text{for } r = 1, 2, \ldots, m - 1. \]  
(20)

Denote by
\[ f_\alpha(\theta) = \frac{\sin[\frac{\pi}{2}(\alpha + \theta)]}{\sin[\frac{\pi}{2}(\alpha - \theta)]}. \]  
(21)

Then \( F(\theta) \) can be written as,
\[ F(\theta) = A_m \tilde{F}(\theta), \]  
(22)

where \( \tilde{F}(\theta) \) has no zeros or poles in the physical domain, and is the unique such solution of eqs. (8,10) with this property. \( A_m \) is given by
\[ A_m = f_{2/(n+1)}(\theta)^2 f_{4/(n+1)}(\theta)^2 \cdots f_{2(m-1)/(n+1)}(\theta)^2 f_{2m/(n+1)}(\theta). \]  
(23)

Note, that \( A_m \) is exactly the scattering amplitude for two \([m]\) particles in the diagonal \( A_n \) Koberle-Swieca [5] amplitude. This is essentially also the scattering matrix of the affine \( A_n \) Toda theory. Thus also the mass spectrum of the particles are identical. The mass of the \( \Lambda_m \) kink, as well as the \([m]\) particle in the Toda theory are
\[ M_m = \sin \left( \frac{\pi m}{n + 1} \right), \quad m = 1, 2, \ldots, n. \]  
(24)

The unique simple pole in \( A_m(\theta) \), at \( \theta = 2m/(n + 1) \) corresponds to the unique bound states in this channel, \([m] + [m] \rightarrow [2m]\). In general a pole at \( 0 < \theta_p < 1 \) corresponds to a bound states mass of
\[ M_B = 2M \cos \left( \frac{\pi \theta_p}{2} \right). \]  
(25)

We, thus, conclude that up to the ‘Z factor’ \( \tilde{F}(\theta) \), the scalar part of the \( A_m \) kink theory is identical to the Koberle–Swieca \( A_n \) amplitude, with the \([m]\) particle identified with the \( \Lambda_m \) kink.
Another example of an $A_n$ RSOS scattering theory is obtained by taking the lattice model $\text{IRF}(A_n, x, x)$, where $x$ is the representation with the highest weight $x = [m\Lambda_1]$. We can again calculate by the same method the crossing parameters. The relevant tensor products are

\[
[m\Lambda_1] \times [m\Lambda_1] = [m\Lambda_2] + [(m - 1)\Lambda_2 + 2\Lambda_1] + \ldots,
\]

\[
[m\Lambda_1] \times [m\bar{\Lambda}_1] = [0] + [\Lambda_1 + \bar{\Lambda}_1] + \ldots,
\]

where we indicated only the representations that are relevant for the poles in the physical sheet. We now find the parameters, by calculating the conformal dimensions,

\[
\lambda = \bar{\zeta}_0 = \frac{\pi}{2} \frac{n + 1}{k + n + 1}, \quad \zeta_0 = \frac{\pi}{2} \frac{2}{k + n + 1}.
\]

Thus, the unique pole on the physical sheet is at $\theta_p = 2/(n + 1)$. It follows that again the amplitude can be written as

\[
F(\theta) = A_1(\theta)\bar{F}(\theta),
\]

where $A_1(\theta)$ has all the zeros and poles in the physical sheet, and

\[
A_1(\theta) = f_{2/(n+1)}(\theta),
\]

Similarly on could consider the kink based on $\text{IRF}(A_n, x, x)$, where $x = [m\Lambda_s]$ is the representation with the highest weight $m\Lambda_s$. We find, in the same manner, the amplitude $A_s$. We conclude that independently of $m$, this scattering theory has the Koberle–Swieca mass spectrum, eq. (24), with $m$ identified with $s$, and that the scalar amplitudes are given precisely by the Koberle–Swieca scattering amplitudes. This implies also that the interaction among the representations is identical to that of the Koberle–Swieca.
An interesting question is which integrable theory is described by these amplitudes. For $m = 1$ it has been conjectured that this is the RCFT $\frac{SU(n+1)_1 \times SU(n+1)_k}{SU(n+1)_{k+1}}$, perturbed by the field $\phi^0_{ad}[2]$. However, we see that for any $m$ we get the same masses, and consequently, the same spins of the integrals of motion. Thus, it remains a mystery what is the corresponding perturbed RCFT.

$D_m$ case. Let us turn now to the calculation of the poles for the $D_m$ scattering theory. Consider the model IRF$(D_m, s, s)$ where $s$ stands for the spinor representation. We choose a basis in which the simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $i = 1, 2, \ldots, m-1$, $\alpha_m = \epsilon_i + \epsilon_{i+1}$, and $\epsilon_i$ form an orthonormal set of unit vectors. The spinor weight is $\lambda = (\epsilon_1 + \epsilon_2 + \ldots + \epsilon_m)/2$. The relevant tensor products are now, for even $m$,

$$s \times s = \sum_{r=0}^{m/2} [\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{2r}], \quad (31)$$

and $s$ is a real representation. For odd $m$ we find,

$$s \times s = \sum_{r=0}^{(m-1)/2} [\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{2r-1}], \quad (32)$$

$$s \times \bar{s} = \sum_{r=0}^{(m-1)/2} [\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{2r}]. \quad (33)$$

Denote by $\lambda_r = [\epsilon_1 + \epsilon_2 + \ldots + \epsilon_r]$. It is easy to compute the Casimir $c_r = \lambda_r(\lambda_r + 2\rho) = r(2m - r)$. For even $m$ we then find the poles of $F(\theta)$ at

$$p_r = \frac{\zeta_r}{\lambda} = \frac{c_{r+2} - c_r}{c_2 - c_0} = \frac{m - 1 - r}{m - 1}. \quad (34)$$

In addition, there are crossing channel poles at $1 - p_r = r/(m - 1)$. We conclude that the scalar $s - s$ scattering amplitude is given by,

$$S_{ss}(\theta) = \prod_{r=1}^{m-2} f_{r/(m-1)}(\theta) \cdot \tilde{F}(\theta), \quad (35)$$

where, as usual $\tilde{F}(\theta)$ is a Z-factor with no poles or zeros in the physical sheet.
For odd $m$ we find the poles at

$$\frac{\zeta_r}{\lambda} = \frac{c_{r+2} - c_r}{c_2 - c_0} = \frac{m - 1 - r}{m - 1}, \quad \text{for } r = 1, 3, \ldots, m - 2,$$

(36)

along with poles from the cross channel $s \times \bar{s}$ at

$$1 - \frac{\tilde{\zeta}_r}{\tilde{\lambda}} = 1 - \frac{c_{r+2} - c_r}{c_2 - c_0} = \frac{r}{m - 1} \quad \text{for } r = 2, 4, \ldots, m - 2.$$  

(37)

It follows that, again, the $s - s$ amplitude is

$$S_{ss}(\theta) = \prod_{r=1}^{m-2} f_{r/(m-1)}(\theta).$$  

(38)

The soliton scattering amplitude, eq. (38) is identical to the $s - s$ scattering amplitudes of the $D_m$ diagonal $S$ matrix, which in turn, is the same, up to $Z$ factors, as the scattering amplitudes of the affine $D_m$ Toda theory. The entire set of amplitudes, which is in correspondence with the fundamental weights of $D_m$, is given by the RSOS models based on $\text{IRF}(D_m, \lambda_a, \lambda_a)$, where $\lambda_a$ is the $a$’th fundamental weight, and can be found by bootstrapping the $s - s$ amplitude, or the vector amplitude, $\text{IRF}(D_m, v, v)$, which was treated in ref. [3]. It follows that the isospin singlet part of the amplitudes is given precisely by the $D_m$ diagonal amplitudes, which are also in correspondence with the fundamental weights of $D_m$, and the spectrum is identical, thus, to that of the $D_m$ diagonal theory,

$$m_a = 2 \sin \frac{\pi a}{2(n-1)}, \quad a = 1, 2, \ldots, n - 2, \quad M_s = M_{\bar{s}} = 1.$$  

(39)

$E_6$ case. Here we consider the model $\text{IRF}(E_6, [27], [27])$, where [27] is the 27th dimensional representation of $E_6$. Here, we need to consider the tensor products,

$$[27] \times [27] = [27] + [351] + [351],$$

$$[27] \times [27] = [1] + [78] + [650].$$  

(40)

where again, we labeled the representations by their dimension. The second Casimirs of the representations [1], [27], [78], [351], [351], and [650], are, respec-
tively, 0, 26/3, 12, 50/3, 56/3, and 18. It follows that the crossing parameters are

$$\zeta_1 = \frac{\pi}{2} \frac{8}{k + 12}, \quad \zeta_2 = \frac{\pi}{2} \frac{2}{k + 12}, \quad \lambda = \bar{\zeta}_1 = \frac{12}{2} \frac{1}{k + 12},$$

$$\bar{\zeta}_2 = \frac{\pi}{2} \frac{6}{k + 12}. \quad (41)$$

The poles are thus at $\zeta_i/\lambda$, $i = 1, 2$, and $1 - \bar{\zeta}_2/\lambda$, and

$$p_1 = \frac{\zeta_1}{\lambda} = \frac{2}{3}, \quad p_2 = \frac{\zeta_2}{\lambda} = \frac{1}{6}, \quad p_3 = 1 - \frac{\bar{\zeta}_2}{\lambda} = \frac{1}{2}. \quad (42)$$

Thus, the scalar amplitude is,

$$S_{27,27}(\theta) = f_{1/6}(\theta)f_{1/2}(\theta)f_{2/3}(\theta), \quad (43)$$

which is identical to the diagonal fundamental $E_6$ amplitude, from which all the other amplitudes can be found by bootstrap. It follows that the scalar part of the $E_6$ amplitudes is given, as for the other simply laced algebras, by the diagonal $E_6$ amplitudes, and thus, also, the mass spectrum is identical to that of the $E_6$ diagonal amplitude.

$E_7$ case. Here we consider the model $\text{IRF}(E_7, 56, 56)$. The relevant tensor product is,

$$[56] \times [56] = [1] + [133] + [1539] + [1463], \quad (44)$$

where we arranged the representations by increasing conformal dimension. The second Casimirs of the representations, [1], [133], [1539] and [1463], are, respectively, 0, 18, 28, and 30. Thus, the crossing parameters are

$$\lambda = \zeta_1 = \frac{\pi}{2} \frac{18}{k + 18}, \quad \zeta_2 = \frac{\pi}{2} \frac{10}{k + 18}, \quad \zeta_3 = \frac{\pi}{2} \frac{2}{k + 18}, \quad (45)$$

with poles at $p_2 = \frac{\zeta_2}{\lambda} = \frac{5}{9}$ and $p_3 = \frac{\zeta_3}{\lambda} = \frac{1}{9}$, to which we need to add the cross poles at $1 - p_2 = \frac{4}{9}$ and $1 - p_3 = \frac{8}{9}$, since $E_7$ is a real group. Thus, the scalar
amplitude is

\[ S_{6,6} = f_{1/9}(\theta)f_{4/9}(\theta)f_{5/9}(\theta)f_{8/9}(\theta), \]

(46)

which is identical to the fundamental \( E_7 \) diagonal amplitude. Thus, by bootstrap, all the scalar amplitudes are given by the diagonal \( E_7 \) ones, along with the mass spectrum.

\( E_8 \) case. Here we consider the model IRF(\( E_8 \), [248], [248]), where [248] is the adjoint representation. We have the tensor product,

\[ [248] \times [248] = [1] + [248] + [3875] + [30380] + [27000]. \]

(47)

The second Casimirs of the representations on the r.h.s. of eq. (47) are, respectively, 0, 30, 48, 60, 62. We thus find the crossing parameters,

\[ \lambda = \zeta_1 = \frac{\pi}{2} \cdot \frac{30}{k + 30}, \quad \zeta_2 = \frac{\pi}{2} \cdot \frac{18}{k + 30}, \]

\[ \zeta_3 = \frac{\pi}{2} \cdot \frac{12}{k + 30}, \quad \zeta_4 = \frac{\pi}{2} \cdot \frac{2}{k + 30}. \]

(48)

It follows that the poles are at \( p_2 = \zeta_2/\lambda = \frac{3}{5} \), \( p_3 = \zeta_3/\lambda = \frac{2}{5} \), and \( p_4 = \zeta_4/\lambda = \frac{1}{15} \).

These are exactly the poles appearing in the diagonal \( E_8 \) amplitude

\[ S_{11}(\theta) = f_{1/15}(\theta)f_{14/15}(\theta)f_{2/5}(\theta)f_{3/5}(\theta)f_{1/3}(\theta)f_{2/3}(\theta), \]

(49)

with the exception of the self interaction pole at \( \theta = 1/3 \) (and its crossing pole at \( 1 - \theta \)). The reason for this missing pole is not entirely clear, and might mean a breakdown in our assumption that the IRF Boltzmann weight is given by eq. (1), or that there is really a difference with respect to the \( E_8 \) diagonal theory. Note that, the mass ratios are still given by the \( E_8 \) diagonal values.
This exhausts the simply laced algebras. We see that in all the cases the assumption of the IRF Boltzmann weight eq. (1) appears to be consistent, and that we find a sensible spectrum. In all the cases the mass spectrum is identical to that of the corresponding affine Toda theory, while the scalar part of the scattering matrix is identical to the corresponding diagonal scattering theory. Note however that some problems arise for $E_8$ because of the self interaction pole.

We turn now to the non–simply laced cases. We find essentially the same result, that the mass spectrum is that of the classical affine Toda theory, while the scalar $S$ matrices agree with the conjectured classical Toda $S$ matrices refs. [6, 7, 8]. However, these $S$ matrices are known not to be consistent at the quantum level, requiring additional particles ref. [8, 7, 9]. It is, similarly, unclear, thus, whether the RSOS amplitudes are consistent without additional particles.

$G_2$ case. Let us consider now the case of $G_2$. For the model $\text{IRF}(G_2, [7], [7])$, we find the tensor product,

$$[7] \times [7] = [1] + [7] + [14] + [27].$$

(50)

The Casimirs of the representations, [1], [7], [14], and [27] are respectively, 0, 2, 6 and $14/3$. Thus the crossing parameters are,

$$\lambda = \zeta_1 = \frac{\pi}{2} \frac{2}{k+4}, \quad \zeta_2 = \zeta_1, \quad \zeta_3 = \frac{\pi}{2} \frac{2}{3(k+4)}.$$  

(51)

We thus have a pole at $p_3 = \frac{\zeta_3}{\lambda} = \frac{1}{3}$. Adding a crossing pole at $1 - p_3$ we arrive at the amplitude,

$$S_{11}(\theta) = f_{1/3}(\theta)f_{2/3}(\theta).$$

(52)

The pole corresponds to a bound state at $M_2 = 2M_1 \cos(\pi/6) = \sqrt{3}M_1$. This is exactly the mass ratio of the $G_2$ Toda theory. The amplitude, eq. (52), has been suggested to describe the light particle in the $G_2$ Toda theory. (We took the cube root of the amplitudes of ref. [8, 7]. This does not affect the mass spectrum or the bootstrap.)
\textbf{\textit{F}_4 \textit{ case.}} For the model IRF(\textit{F}_4, [26], [26]) we find the tensor product,

\begin{equation}
[26] \times [26] = [1] + [26] + [52] + [273] + [324].
\end{equation}

The Casimirs of the representations on the r.h.s. are 0, 6, 9, 12 and 13, respectively. The crossing parameters are

\begin{align*}
\lambda &= \zeta_1 = \frac{\pi}{2k+9}, & \zeta_2 &= \frac{\pi}{2k+9}, \\
\zeta_3 &= \frac{3}{2k+9}, & \zeta_4 &= \frac{1}{2k+9}.
\end{align*}

(54)

We thus have poles at $p_2 = \zeta_2/\lambda = \frac{1}{2}$, $p_3 = \zeta_3/\lambda = \frac{1}{2}$ and $p_4 = \zeta_4/\lambda = \frac{1}{6}$. Thus the amplitude becomes,

\begin{equation}
S_{11}(\theta) = f_{1/6}(\theta)f_{1/2}(\theta)^2f_{5/6}(\theta).
\end{equation}

(55)

When comparing with affine Toda results, we miss a self interaction pole at $\theta = 1/3, 2/3$. The Toda amplitude is

\begin{equation}
S_{11}(\theta) = f_{1/6}(\theta)f_{1/3}(\theta)f_{1/2}(\theta)^2f_{2/3}(\theta)f_{5/6}(\theta).
\end{equation}

(56)

As in the \textit{E}_8 case, the meaning of this missing pole is unclear. It might either signify a breakdown of the anzats for the Boltzmann weight, eq. (1), or, a small difference with respect to the conjectured Toda amplitude. We cannot just add the self interaction pole, since this will lead to problems in the bootstrap of the full amplitude (see the following).

\textbf{\textit{B}_n \textit{ case.}} Consider the model IRF(\textit{B}_n, s, s), where \textit{s} stands for the spinor representation. Denote the simple roots of \textit{B}_n by $\alpha_r = \epsilon_r - \epsilon_{r+1}$, $r = 1, 2, \ldots, n-1$,
\( \alpha_n = \epsilon_n \), where the \( \epsilon_i \) form an orthonormal set of vectors. Then, the spinor highest weight is,

\[
\mathbf{s} = \frac{\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n}{2},
\]

and we have the tensor product,

\[
\mathbf{s} \times \mathbf{s} = \sum_{m=0}^{n} [\epsilon_1 + \epsilon_2 + \ldots + \epsilon_m].
\]

For the Casimirs we find,

\[
C_{[\epsilon_1+\epsilon_2+\ldots+\epsilon_m]} = \frac{1}{2} m(2n + 1 - m).
\]

Thus the crossing parameters are

\[
\lambda = \zeta_1, \quad \zeta_m = \frac{\pi}{2} \cdot \frac{n - m + 1}{2n - 1 + k}, \quad m = 1, 2, \ldots, n.
\]

The poles are thus located at

\[
p_m = \frac{\zeta_m}{\lambda} = \frac{n - m + 1}{n}, \quad m = 2, 3, \ldots, n.
\]

Note, however that to these poles we need to add the cross channel poles located at \( 1 - p_m \), which so happen to coincide exactly with the original set of poles \( p_m \). Thus the full scalar amplitude becomes,

\[
S_{ss}(\theta) = \prod_{m=1}^{n-1} f_{m/n}(\theta)^2.
\]

Note the appearance of the perplexing double poles as a result of the coincidence of the direct and cross channel poles. In fact, exactly the same amplitude, \( S_{ss} \) appears in the \( B_n \) affine Toda theory. The perplexing double poles create a problem there, as well. One way to solve the problem was suggested \([8, 7, 9]\) is to add a fermion to the theory. The double poles then arise naturally from fermion loops. Presumably, the same kind of solution is needed here, namely, we are missing a fermion particle in the theory. More study of this is required.
Since all the other amplitude appear as a bootstrap of the fundamental \( S_{ss} \), we find an agreement between all the amplitudes and those of affine Toda theory. The model \( \text{IRF}(B_n, \lambda_m, \lambda_m) \), where \( \lambda_m \) is the \( m \)th fundamental weight, corresponds to the scattering of the \( m \)th Toda particle, or the \( \lambda_m \) kink, whose mass is given by

\[
M_m = 2 \sin\left(\frac{\pi m}{2n}\right), \quad m = 1, 2, \ldots, n-1, \quad M_s = 1.
\] (63)

The vector amplitude was considered directly in ref. [3] and, again, a perfect agreement with affine Toda theory is found.

\( C_n \) case. Here we consider the model \( \text{IRF}(C_n, [\lambda_1], [\lambda_1]) \), where \( \lambda_m \) is the \( m \)th fundamental weight (‘vector’)\(^*\). The relevant tensor product is

\[
[\lambda_1]_2^2 = [0] + [\lambda_2] + [2\lambda_1],
\] (64)

where we denoted the representations by their highest weight. The conformal dimensions of the fields are,

\[
\Delta_{[2\lambda_1]} = \frac{n+1}{k+n+1}, \quad \Delta_{[\lambda_2]} = \frac{n}{k+n+1},
\] (65)

and the crossing parameters become,

\[
\lambda = \zeta_1 = \frac{\pi \Delta_{[\lambda_2]}}{2} = \frac{\pi n}{2(k+n+1)}, \quad \zeta_1 = \frac{\pi}{2}(\Delta_{[2\lambda_1]} - \Delta_{[\lambda_2]}) = \frac{\pi}{2(k+n+1)}.
\] (66)

Thus, there is a unique pole at \( p_1 = \zeta_1 / \lambda = \frac{1}{n} \), along with the cross channel pole at \( 1 - p_1 = \frac{n-1}{n} \). The scalar amplitude thus becomes,

\[
S_{11}(\theta) = f_{1/n}(\theta)f_{1-1/n}(\theta).
\] (67)

This is exactly the fundamental amplitude of the \( C_n \) Toda theory from which all the other amplitudes can be obtained by bootstrap. The mass spectrum is also the

\* This model was considered also in ref. [3]. An error was made there in the comparison with the \( C_n \) model of ref. [10]. However, as we see here, the agreement with affine Toda still holds perfectly.
same as the Toda one, with the identification of the $m$ particle with the $\lambda_m$ kink,

$$M_m = 2 \sin \left( \frac{\pi m}{2n} \right), \quad m = 1, 2, \ldots, n. \quad (68)$$

Again, a double pole problem arises in some amplitudes which can be solved, in the Toda case, by adding additional two fermions ref. [8, 7, 9]. As the same problem arises here, presumably, the analogous particles need to be added.

This concludes all the algebras. We find, in all cases, an agreement with the spectrum of the corresponding affine Toda theory. The number of kinks is equal to the rank of the algebras. Their masses are given by the eigenvector with maximal eigenvalue (Perron Frobenius) vector of the Cartan matrix. The kinks are in one to one correspondence with the simple weights of the algebra. The admissibility condition with respect to the $\lambda$ kink, where $\lambda$ is a fundamental weight, is given by fusion with respect to $\lambda$, i.e., $a \sim b$ iff $a\lambda = b + \ldots$, where $a$ and $b$ are any weights. The scattering amplitude of two kinks contains a factor which is identical to the affine Toda scattering amplitude, which is responsible for all the poles in the physical sheet.

We wish to study now the bootstrap of the full RSOS amplitudes. Consistency requires that under bootstrap the set of amplitudes will be closed. We wish to judge if this is the case. Consider then the kink scattering process,

$$K_{\alpha,\beta} + K_{\beta,\gamma} \rightarrow K_{\alpha,\delta} + K_{\delta,\gamma}, \quad (69)$$

where $K_{\alpha,\beta}$ denotes the $K$ kink interpolating between the $\alpha$ and $\beta$ vacua. The scattering amplitude for this process is denoted by $S^{\beta,\gamma}_{\alpha,\delta}(\theta)$, where $i\pi\theta$ is the relative rapidity. The poles in $S$ on the physical sheet, $0 < \theta < 1$ correspond to bound state kinks. The coupling to the bound state, $g_{\alpha\beta\gamma}$, is given by,

$$g_{\alpha\beta\gamma}g_{\alpha\delta\gamma} = \pi \left. \text{Res} \right|_{\theta=\theta_0} S^{\beta\gamma}_{\alpha,\delta}(\theta), \quad (70)$$

where $0 < \theta_0 < 1$ is the location of the pole.
Denote the bound states kink by $B$. Then the scattering matrix of $B$ and the original kink, $K$, which is denoted by $H_{\alpha \eta}^{\gamma \delta}$, is given by the bootstrap equation, ref. [1]

$$g_{\eta \epsilon \delta}H_{\alpha \eta}^{\gamma \delta} = \sum_{\mu} g_{\alpha \mu \gamma} S_{\mu \epsilon}^{\gamma \delta} (\theta + \theta_p/2) S_{\alpha \eta}^{\mu \epsilon} (\theta - \theta_p/2).$$

(71)

Recall the anzats for the IRF scattering theory, eq. (6),

$$S_{cd}^{ab}(\theta) = F(\theta) \left( \frac{S_{a,0} S_{d,0}}{S_{b,0} S_{c,0}} \right)^{-\theta/2} \sum_{\alpha=1}^{n} f^{\alpha}(\theta) P^{\alpha} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right),$$

(72)

where the functions $f^{\alpha}(\theta)$ are given by eq. (2).

Consider now the poles of $F(\theta)$ which would lead to poles in the full $S$ matrix. These are located at the points $0 < \theta_l < 1$, where $\theta_l = \zeta_l/\lambda$, along with possible poles from the cross channel. Now, from the expression for $f^{a}(\theta)$, eq. (2), we find the important property, $f^{a}(\theta_l) = 0$, for $a = l, l+1, \ldots, n-1$. Thus the coupling to the bound states in the channel with the pole $\theta_l$ becomes,

$$\frac{1}{\pi} g_{\alpha \beta \gamma}^{2} = \left[ \text{Res} F(\theta) \right] \left[ \frac{S_{\beta,0}^{2}}{S_{\alpha,0} S_{\gamma,0}} \right]^{-\theta_l/2} \sum_{\alpha=0}^{l-1} f^{\alpha}(\theta_l) P^{\alpha} \left( \begin{array}{cc} \beta & \gamma \\ \alpha & \beta \end{array} \right).$$

(73)

Note, importantly, that only projection operators, $P^{a}$, with $a < l$ contribute to the coupling. However, $P^{a}$ corresponds in RCFT to the projection of the braiding matrix on the intermediate $t$ channel involving the field $\phi_{a}$. Thus, $P^{a} \left( \begin{array}{cc} \beta & \gamma \\ \alpha & \beta \end{array} \right) \neq 0$ only if $\alpha \times \phi_{a} = \gamma + \ldots$, where the product is in the fusion rules sense. It follows that the bound state $B$ obeys the admissibility condition, $B_{\alpha,\gamma} \neq 0$ only if $\alpha t = \gamma + \ldots$ where $t = \phi_{0}, \phi_{1}, \ldots, \phi_{l-1}$.

Thus, the admissibility condition for the IRF model obtained from the bootstrap equation, for the field $B$, is given by fusion with respect to the field, $\phi = \phi_{0} + \phi_{1} + \ldots + \phi_{l-1}$, where these are the fields appearing in the OPE, $x^{2} = \sum_{a} \phi_{a}$, arranged according to increasing conformal dimensions. For the examples at hands it
can be observed that the admissibility condition is equivalent to fusion with respect to the field \( \phi_{l-1} \), since \( \alpha \phi_r = \gamma + \ldots \) with \( r < l - 1 \), implies that \( \alpha \phi_{l-1} = \gamma + \ldots \). Thus, we conclude that the bound states \( B \), \( S \) matrix, is described by the model \( \text{IRF}(\mathcal{O}, \phi_{l-1}, \phi_{l-1}) \), and, importantly, is also a fusion IRF model. In summary, the bound states of the kink \( x \) with itself at the pole \( \zeta_l/\lambda \) is described by the field \( \phi_{l-1} \) appearing in the operator product of \( x \) with itself.

Let us turn now to examples.

(1) \( SU(n+1) \). Consider the scattering process of \( x = \Lambda_m \). We found above the unique pole at

\[
\theta_m = \frac{\zeta_0}{\lambda} = \frac{2m}{n+1},
\]

(74)

where \( \zeta_0 \) corresponds to the tensor product \( x^2 = \Lambda_{2m} + \ldots \). Since \( f_1(\theta_m) = f_2(\theta_m) = \ldots = 0 \), as explained above, it follows that the bound state \( S \) matrix corresponds to the model \( \text{IRF}(\Lambda_n, \Lambda_{2m}, \Lambda_{2m}) \). This is in agreement with the Toda picture, since, as explained above, the \( \Lambda_r \) representation is in correspondence with the \( r \) Toda particle, and the process in the Toda theory is \( m + m \rightarrow 2m \). Thus, the bound state representation, for consistency, has to be \( \Lambda_{2m} \). This, agrees with the direct calculation using the bootstrap equation.

(2) \( D_m \). For even \( m \) we have \( s = \bar{s} \) and the scattering of two spinor kinks is real. We have the OPE, eq. (31),

\[
s^2 = \sum_{r=0}^{m/2} [\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{2r}],
\]

(75)

The poles are located (see above) at

\[
p_r = \frac{m - 1 - 2r}{m - 1}, \quad r = 1, 2, \ldots, \frac{m}{2} - 1.
\]

(76)

The \( p_r \) pole gives the representation \( x = [\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{2r}] = \Lambda_{2r} \) as \( f^{a+1}(p_r) = 0 \) for \( a > r \). Thus, the bound states corresponds to the RSOS theory \( \text{IRF}(D_m, x, x) \).
The mass of this particle is,

\[
\frac{M_B}{M_s} = 2 \cos\left(\frac{\pi r}{2}\right) = 2 \sin\left(\frac{\pi r}{m - 1}\right),
\]

which is exactly the mass of the \(\Lambda_{2r}\) particle in the Toda theory. Thus, we get the correct representation according to the correspondence with Toda theory. For \(D_m\), odd \(m\), the calculation is similar. Again, we find the correct representations in the bootstrap.

It is similarly straightforward to see that for the Lie algebras \(B_n, C_n, E_6, E_7\) and \(G_2\), the fundamental amplitudes bootstrap correctly, and indeed the correct representations arise. We encounter problems, however, for the algebras \(F_4\) and \(E_8\). We already saw that, in these cases, we miss a self interaction pole at \(\theta = 2/3\). Furthermore, the other poles do not bootstrap correctly, i.e. wrong representations arise. We conclude that some modification of the anzats eq. (1) is required in these cases. This is the most logical conclusion, since the RSOS soliton picture should persist. Curiously, all the other poles are found to be in the correct place, so even for these problematic cases the anzats, eq. (1), seems ‘almost’ right.

The quantum field theory described by the algebra \(G\) RSOS scattering theory, with the simple weights for the admissibility condition, was conjectured to be the coset theory \(\frac{G_k \times G_1}{G_{k+1}}\) [2, 3], perturbed by the field \(\Phi_{\text{ad}}^{0,0}\). Evidence for this is that these perturbed cosets have the correct spins for the integrals of motions, which are the exponents of the Lie algebra modulo the Coxeter number. Further in ref. [11], these cosets were mapped directly to Toda theories, and it was argued, for \(A_n\), that the mass spectrum is the same as the corresponding Toda theory. This is additional evidence for this identification.

Finally, the anzats eq. (1) seems to be consistent in almost all the cases we studied. We find the correct poles and the correct bootstrap structure. This is a highly constrained check which lends considerable support for this anzats.

We hope that this work will be of benefit in the understanding of solvable lattice model and the related RSOS scattering theories.
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