Deformation Quantization of Fermi Fields

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Abstract

Deformation quantization for any Grassmann scalar free field is described via the Weyl-Wigner-Moyal formalism. The Stratonovich-Weyl quantizer, the Moyal $\star$-product and the Wigner functional are obtained by extending the formalism proposed recently in [35] to the fermionic systems of infinite number of degrees of freedom. In particular, this formalism is applied to quantize the Dirac free field. It is observed that the use of suitable oscillator variables facilitates considerably the procedure. The Stratonovich-Weyl quantizer, the Moyal $\star$-product, the Wigner functional, the normal ordering operator, and finally, the Dirac propagator have been found with the use of these variables.

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I. INTRODUCTION

Deformation quantization is commonly regarded as an alternative approach to quantization. In principle, it can be implemented into any classical system of particles, fields, strings or string fields. This is based on the philosophy of deforming suitable standard mathematical structures. Such a philosophy was introduced in 1978 by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [1] (for a recent account, see [2, 3]). The deformation quantization formalism has a firm mathematical basis. However, its application to quantize an arbitrary physical system has still great challenges (see the third reference from [3]).

The canonical realization of deformation quantization is done with the use of the Weyl-Wigner-Moyal (WWM) formalism introduced originally in Refs. [4], which establishes a one-one correspondence between the operator algebra (on certain Hilbert space) and the algebra of symbols of operators via the so called WWM correspondence. The product of operators is mapped into an associative and noncommutative product of the respective symbols called the $\star$-product. The simplest example is here the famous Moyal $\star$-product. Quantum theory in terms of deformation quantization has been intensively studied for physical systems with a finite number of degrees of freedom.

Recently, much more interesting and difficult case of classical fields (with an infinite number of degrees of freedom) has been also considered by some authors. For example, deformation quantization of scalar field is done in Refs. [5, 6, 7, 8, 9]. In some of these references it has been shown that the $\star$-product can be modified to absorb the divergences of the field theory with the $\lambda\phi^4$-interaction. This modification is accomplished by a $\lambda$ dependent redefinition of the $\star$-product consistent with a cocycle condition. Free electromagnetic field in deformation quantization within the Coulomb gauge has been studied in [10]. Deformation quantization of gravitational field as a constrained system has been discussed in [11, 12]. Then the linearized gravitational field from the point of view of deformation quantization has been described in [13]. It is worth while to note that also perturbation quantum field theory can be given the form of deformation quantization formalism [14, 15, 16]. Moreover, as has been shown in [17] deformation quantization theory can be applied to quantize the classical bosonic strings in the light-cone gauge. On the other hand, string theory uses also the deformation quantization formalism (WWM-formalism) to describe the spacetime worldvolume of a D-brane. The presence of a non-zero constant $B$-field on the worldvolume deforms the product of functions (or classical fields) on the D-brane into a Moyal $\star$-product, and the ordinary effective field theory on the D-brane turns into a noncommutative field theory with this Moyal $\star$-product [18].

Most of the cases studied by the deformation quantization for systems with a finite number of degrees of freedom deal with bosonic variables. However, the analysis of some classical physical systems requires the description of fermionic degrees of freedom which involve Grassmann variables. These systems have been discussed in the literature for several years [19, 20, 21, 22, 23, 24]. (Some extension of the WWM
formalism to infinite degrees of freedom for fermions can also be found in [25, 26, 27]). The canonical quantization of the fermionic systems by using the Grassmann variables has been studied in all detail in [28]. Very recently some authors started to apply the machinery of deformation quantization to quantize the classical fermionic systems [29, 30, 31, 32, 33, 34]. In particular, in [33, 35] it has been shown concretely how the deformation quantization program can be carried over to specific physical fermionic systems. The same techniques have been applied recently for the noncommutative superspace [36].

In the present paper we study the canonical approach to deformation quantization for fermionic fields by employing the traditional Weyl-Wigner-Moyal correspondence [37, 38, 39]. To this end we extend the results obtained in the previous paper [35] in the context of fermionic systems with a finite number of degrees of freedom. We deal with the Dirac free field and reproduce several results about its quantization and its Green’s functions.

Our paper is organized as follows. Sec. II is devoted to deformation quantization of a generic Fermi field. First, we construct the Stratonovich-Weyl (SW) quantizer for this Fermi field. Using a modified notion of the trace of an operator one finds that the properties of the fermionic SW quantizer have a similar form as the corresponding properties in the bosonic case. Then the Moyal ⋆-product and the Wigner functional are found. Finally, the normal ordering for the generic Fermi field is briefly discussed. In Sec. III the deformation quantization formalism developed in Sec. II is applied to the case of the Dirac free field. Using the oscillator variables we were able to simplify the considerations. In particular, with the use of these variables we have found the Wigner functional corresponding to the ground state and also the Wigner functional of an arbitrary excited state. Finally, in this section the Dirac propagator within the deformation quantization is computed. Some conclusions and final remarks in Sec. IV close the paper.

II. DEFORMATION QUANTIZATION OF GRASSMANN SCALAR FIELD

Consider a scalar Grassmann field on the Minkowski spacetime $M^{d+1}$ of signature $(-, -, \ldots, -, +)$. By a Grassmann scalar field we will understand a smooth function $\Theta$ over $M^{d+1}$ which takes values in the field of (anti-commuting) Grassmann numbers $\mathbb{G}$, i.e., $\Theta$ is the map $\Theta : M^{d+1} \to \mathbb{G}$. Canonical variables of this classical Grassmann field will be denoted by $\Theta(\vec{x}, t)$ and $\pi_{\Theta}(\vec{x}, t)$ with $(\vec{x}, t) \in M^{d+1} = \mathbb{R}^d \times \mathbb{R}$.

We deal with fields at the instant $t = 0$ and we denote $\Theta(\vec{x}, 0) \equiv \Theta(\vec{x})$ and $\pi_{\Theta}(\vec{x}, 0) \equiv \pi_{\Theta}(\vec{x})$. It is worth while to mention that some of the functional formulas and their manipulations are formal. It is also important to notice that since we will deal with Grassmann variables all the computations and results obtained in the present paper are valid under the specified conventions and ordering of factors given
in this section. In the present section we study the deformation quantization of the Grassmann fields, including: the Stratonovich-Weyl quantizer, the Moyal $\ast$-product, the Wigner functional \[37, 38, 39\], and the normal ordering. In this paper we follow the conventions and notation used in Ref. \[35\].

A. The Stratonovich-Weyl Quantizer

Let $F[\pi_{\Theta}, \Theta]$ be a functional on the phase space $\mathcal{Z}_G \equiv \{(\pi_{\Theta}, \Theta)\}$ and let $\tilde{F}[\lambda, \mu]$ be its Fourier transform

\[
\tilde{F}[\lambda, \mu] = \int F[\pi_{\Theta}, \Theta] \exp \left\{ -i \int d^d x \left( \pi_{\Theta}(\vec{x}) \cdot \lambda(\vec{x}) + \Theta(\vec{x}) \cdot \mu(\vec{x}) \right) \right\} \prod d\pi_{\Theta} d\Theta,
\]

where $\lambda(\vec{x})$ and $\mu(\vec{x})$ are the Fourier transformed Grassmann fields corresponding to $\pi_{\Theta}(\vec{x})$ and $\Theta(\vec{x})$ respectively, $\pi_{\Theta}(\vec{x}) \cdot \lambda(\vec{x}) = \sum_{\alpha=1}^{N} \pi_{\Theta}(\vec{x})\lambda_\alpha(\vec{x})$ and the functional measure $\int d\pi_{\Theta} d\Theta := \prod_{\vec{x},\alpha} d\pi_{\Theta}(\vec{x}) d\Theta_\alpha(\vec{x})$. By the analogy to quantum mechanics we can define the Weyl quantization rule as follows

\[
\tilde{F} = W(F[\pi_{\Theta}, \Theta]) := \int \tilde{F}[\lambda, \mu] \exp \left\{ i \int d^d x \left( \pi_{\Theta}(\vec{x}) \cdot \lambda(\vec{x}) + \Theta(\vec{x}) \cdot \mu(\vec{x}) \right) \right\} \prod d\lambda d\mu,
\]

with $\pi_{\Theta}$ and $\Theta$ being the field operators given by $\pi_{\Theta}(\vec{x})|\pi_{\Theta}\rangle = \pi_{\Theta}(\vec{x})|\pi_{\Theta}\rangle$ and $\Theta(\vec{x})|\Theta\rangle = \Theta(\vec{x})|\Theta\rangle$.

The coherent state $|\Theta\rangle$ can be defined in terms of the vacuum state $|0\rangle$ \[35, 40\]

\[
|\Theta\rangle = \exp \left\{ -\frac{i}{\hbar} \int d^d x \pi_{\Theta}(\vec{x}) \cdot \Theta(\vec{x}) \right\} |0\rangle,
\]

with $|\Theta\rangle \in \mathcal{F}_F$, where $\mathcal{F}_F = \mathcal{H}_F \oplus \mathcal{H}_F \otimes \mathcal{H}_F \oplus \cdots$ is the Fock space and $\mathcal{H}_F$ is a Hilbert space of fermions. By using Eq. (3) it is easy to see that the functional state satisfies the following property

\[
\exp \left\{ -\frac{i}{\hbar} \int d^d x \pi_{\Theta}(\vec{x}) \xi(\vec{x}) \right\} |\Theta\rangle = |\Theta + \xi\rangle.
\]

It is also known that $(|\Theta\rangle)^* \neq |\Theta|$, \[35, 40\]. The dual Fock space $\mathcal{F}_F^*$ is constructed from the dual vacuum state $\langle 0|$ as follows

\[
\langle \Theta | = \langle 0 | \prod_{\vec{x},\alpha} \hat{\Theta}_\alpha(\vec{x}) \exp \left\{ -\frac{i}{\hbar} \int d^d x \Theta(\vec{x}) \cdot \pi_{\Theta}(\vec{x}) \right\}
\]

\[
= \langle 0 | \prod_{\vec{x},\alpha} \hat{\Theta}_\alpha(\vec{x}) \exp \left\{ \frac{i}{\hbar} \int d^d x \pi_{\Theta}(\vec{x}) \cdot \Theta(\vec{x}) \right\}.
\]

This bra satisfies $\langle \Theta | \hat{\Theta}_\alpha(\vec{x}) = \langle \Theta | \Theta_\alpha(\vec{x})$ and

\[
\langle \Theta' | \Theta \rangle = \prod_{\vec{x},\alpha} \left( \Theta_\alpha(\vec{x}) - \Theta'_\alpha(\vec{x}) \right) = \delta(\Theta - \Theta').
\]

Analogously for the momentum representation we define

\[
|\pi_{\Theta}\rangle = \exp \left\{ -\frac{i}{\hbar} \int d^d x \hat{\Theta}(\vec{x}) \cdot \pi_{\Theta}(\vec{x}) \right\} \prod_{\vec{x},\alpha} \pi_{\Theta}(\vec{x}) |0\rangle,
\]

\[
\langle \pi_{\Theta} | = \langle 0 | \exp \left\{ \frac{i}{\hbar} \int d^d x \hat{\Theta}(\vec{x}) \cdot \pi_{\Theta}(\vec{x}) \right\}.
\]
Then one gets
\[ \langle \pi'_\Theta | \pi_\Theta \rangle = \prod_{\vec{x}, \alpha} \left( \pi'_{\Theta,\alpha}(\vec{x}) - \pi_{\Theta,\alpha}(\vec{x}) \right) =: \delta(\pi'_\Theta - \pi_\Theta). \] (8)

From the above relations it is easy to compute
\[ \langle \pi_\Theta | \Theta \rangle = \exp \left\{ -\frac{i}{\hbar} \int d^d x \pi_\Theta(\vec{x}) \cdot \Theta(\vec{x}) \right\}, \quad \langle \Theta | \pi_\Theta \rangle = (i^\infty \hbar)^\infty \exp \left\{ \frac{i}{\hbar} \int d^d x \pi_\Theta(\vec{x}) \cdot \Theta(\vec{x}) \right\}. \] (9)

With the use of (9) one can obtain the following completeness relations
\[ \int |\pi_\Theta\rangle (-1)^\infty \mathcal{D} \pi_\Theta \langle \pi_\Theta| = \hat{1}, \quad \int |\Theta\rangle \mathcal{D} \Theta \langle \Theta| = \hat{1}, \] (10)

where \( \mathcal{D} \Theta := \prod_{\vec{x}, \alpha} d\theta_{\alpha}(\vec{x}), \mathcal{D} \pi_\Theta := \prod_{\vec{x}, \alpha} d\pi_{\Theta,\alpha}(\vec{x}) \) and \( \prod_{\vec{x}, \alpha} \) means that the differentials are ordered oppositely to the ordering of the Grassmann variables in the integrated function.

Return to our description of the Weyl correspondence. Substituting (1) into (2) one gets
\[ \hat{F} = W(F[\pi_\Theta, \Theta]) = \int F[\pi_\Theta, \Theta] \widehat{\Omega}[\pi_\Theta, \Theta] \prod d\pi_\Theta d\Theta, \] (11)

where \( \widehat{\Omega} \) is the Stratonovich-Weyl quantizer (see Refs. [35, 37])
\[ \widehat{\Omega}[\pi_\Theta, \Theta] = \int \exp \left\{ -i \int d^d x \left( \pi_\Theta(\vec{x}) \cdot \lambda(\vec{x}) + \Theta(\vec{x}) \cdot \mu(\vec{x}) \right) \right\} \exp \left\{ i \int d^d x \left( \pi_\Theta(\vec{x}) \cdot \lambda(\vec{x}) + \Theta(\vec{x}) \cdot \mu(\vec{x}) \right) \right\} \prod d\lambda d\mu. \] (12)

Making use of the well known Campbell-Baker-Hausdorff formula for Grassmann variables in the appropriate ordering, the commutation rules for \( \pi_\Theta \) and \( \Theta \) and the relations (4) and (10), it is possible to reexpress \( \widehat{\Omega}[\pi_\Theta, \Theta] \) in a very useful form
\[ \widehat{\Omega}[\pi_\Theta, \Theta] = (-i)^\infty \int \mathcal{D} \mu \exp \left\{ -i \int d^d x \Theta(\vec{x}) \cdot \mu(\vec{x}) \right\} \left| \pi_\Theta - \frac{\hbar \mu}{2} \right\rangle \left\langle \pi_\Theta + \frac{\hbar \mu}{2} \right| \]
\[ = (i)^\infty \int \mathcal{D} \lambda \exp \left\{ -i \int d^d x \pi_\Theta(\vec{x}) \cdot \lambda(\vec{x}) \right\} \left| \Theta - \frac{\hbar \lambda}{2} \right\rangle \left\langle \Theta + \frac{\hbar \lambda}{2} \right|. \] (13)

Now let us define the "trace" as follows
\[ \text{tr}\{\hat{\Theta}\} := (i\hbar)^{-\infty} \int \mathcal{D} \Theta \langle \Theta | \hat{\Theta} | \Theta \rangle \]
\[ = (i\hbar)^{-\infty} \int \mathcal{D} \pi_\Theta \langle \pi_\Theta | \hat{\Theta} | \pi_\Theta \rangle, \]

for any operator \( \hat{\Theta} \) (compare with [35]).

With this definition one can check also that the Stratonovich-Weyl operator satisfies the following properties
\[ \text{tr}\{\widehat{\Omega}[\pi_\Theta, \Theta]\} = 1, \] (14)
\[ \text{tr}\left\{ \widehat{\Omega}[\pi_\Theta, \Theta] \widehat{\Omega}[\pi'_\Theta, \Theta'] \right\} = \prod_{\vec{x}, \alpha} \left( \Theta(\vec{x}) - \Theta'(\vec{x}) \right) \left( \pi_{\Theta,\alpha}(\vec{x}) - \pi'_{\Theta,\alpha}(\vec{x}) \right) =: \delta(\Theta - \Theta', \pi_\Theta - \pi'_\Theta). \] (15)
Finally, multiplying (11) by $\hat{\Omega}[^{\pi_\Theta, \Theta}]$ and taking into account the property (15) one easily gets the fundamental relation
\[ F[^{\pi_\Theta, \Theta}] = \text{tr} \left\{ \hat{\Omega}[^{\pi_\Theta, \Theta}] \hat{F} \right\}. \] (16)

B. The Moyal $\star$-Product

We are at the position to define the Moyal $\star$-product in field theory involving Grassmann scalar fields. Let $F = F[^{\pi_\Theta, \Theta}]$ and $G = G[^{\pi_\Theta, \Theta}]$ be some functionals on $Z_G$ that correspond to the field operators $\hat{F}$ and $\hat{G}$ respectively, i.e. $F[^{\pi_\Theta, \Theta}] = W^{-1}(\hat{F}) = \text{tr} \left\{ \hat{\Omega}[^{\pi_\Theta, \Theta}] \hat{F} \right\}$ and $G[^{\pi_\Theta, \Theta}] = W^{-1}(\hat{G}) = \text{tr} \left\{ \hat{\Omega}[^{\pi_\Theta, \Theta}] \hat{G} \right\}$. The functional which corresponds to the operator product $\hat{F} \hat{G}$ will be denoted by $(F \star G)[^{\pi_\Theta, \Theta}]$. So we have
\[ (F \star G)[^{\pi_\Theta, \Theta}] := W^{-1}(\hat{F} \hat{G}) = \text{tr} \left\{ \hat{\Omega}[^{\pi_\Theta, \Theta}] \hat{F} \hat{G} \right\}. \] (17)

Using Eqs. (11) and (17) and performing some simple but rather lengthy calculations one gets
\[ (F \star G)[^{\pi_\Theta, \Theta}] = \left( \frac{i\hbar}{2} \right)^{2\infty} \int F[^{\pi_\Theta', \Theta'}] G[^{\pi''_\Theta, \Theta''}] \times \exp \left\{ -\frac{2i\hbar}{\hbar} \int d^dx \left( \pi_\Theta(\Theta' - \Theta'') + \pi'_\Theta(\Theta'' - \Theta) + \pi''_\Theta(\Theta - \Theta') \right) \right\} \prod d\pi'_\Theta d\Theta' \prod d\pi''_\Theta d\Theta''. \] (18)

Introduce new variables: $\Psi' = \Theta' - \Theta$, $\Psi'' = \Theta'' - \Theta$, $\Pi' = \pi'_\Theta - \pi_\Theta$, $\Pi'' = \pi''_\Theta - \pi_\Theta$. Using the expansion of $F[^{\pi_\Theta', \Theta'}] = F[^{\pi_\Theta + \Pi', \Theta + \Psi'}]$ and $G[^{\pi''_\Theta, \Theta''}] = G[^{\pi_\Theta + \Pi'', \Theta + \Psi''}]$ into the Taylor series and performing some manipulations we obtain
\[ (F \star G)[^{\pi_\Theta, \Theta}] = F[^{\pi_\Theta, \Theta}] \exp \left\{ \frac{i\hbar}{2} \overset{\rightarrow}{P}_G \right\} G[^{\pi_\Theta, \Theta}], \] (19)

with
\[ \overset{\rightarrow}{P}_G := \int d^dx \left( \frac{\overset{\rightarrow}{\delta} \overset{\rightarrow}{\delta}}{\delta \Theta(\bar{x}) \delta \pi_\Theta(\bar{x})} + \frac{\overset{\right}{\delta} \overset{\right}{\delta}}{\delta \pi_\Theta(\bar{x}) \delta \Theta(\bar{x})} \right), \] (20)

where $\overset{\rightarrow}{\delta}$ and $\overset{\right}{\delta}$ are the right and left functional derivatives, respectively. The last formula corresponds exactly to the Poisson bracket for two functionals $F$ and $G$ which is given by
\[ F \overset{\rightarrow}{P}_G G := \{ F[^{\pi_\Theta, \Theta}], G[^{\pi_\Theta, \Theta}] \}_P \]
\[ = (-1)^{\epsilon_F} \int d^dx \left\{ \frac{\delta F[^{\pi_\Theta, \Theta}]}{\delta \Theta(\bar{x})} \frac{\delta G[^{\pi_\Theta, \Theta}]}{\delta \pi_\Theta(\bar{x})} + \frac{\delta F[^{\pi_\Theta, \Theta}]}{\delta \pi_\Theta(\bar{x})} \frac{\delta G[^{\pi_\Theta, \Theta}]}{\delta \Theta(\bar{x})} \right\}, \] (21)

with $\epsilon_F = 0, 1$ depending on whether the corresponding functional $F$ is even or odd, respectively. This Poisson bracket has been reported in the literature for many years \[20, 21, 22, 23, 29, 30, 31, 32, 33, 34, 35\].

The associated symplectic form
\[ \omega_G = \int d^d \delta \Theta(\bar{x}) \wedge \delta \pi_\Theta(\bar{x}), \] (22)

provides $Z_G$ with the structure of a symplectic supermanifold.
C. The Wigner Functional

If \( \hat{\rho} \) is the density operator of a quantum state then the functional \( \rho_w[\pi, \Theta] \) defined by

\[
\rho_w[\pi, \Theta] = \text{tr}\left( \hat{\Omega}[\pi, \Theta] \hat{\rho} \right) = 2^{-\infty} \int \mathcal{D}\eta \exp \left\{ -\frac{2i}{\hbar} \int d^d x \pi(x) \cdot \eta(x) \right\} \langle \Theta + \eta | \hat{\rho} | \Theta - \eta \rangle
\]

is called the Wigner functional corresponding to this state. In particular, the Wigner functional \( \rho_w[\pi, \Theta] \) corresponding to the quantum state \( \hat{\rho} = |\Phi\rangle\langle \Phi| \), where \( \langle \Phi\rangle := (|\Phi\rangle)^\dagger \), is given by \[4, 8, 35, 38, 39\]

\[
\rho_w[\pi, \Theta] = 2^{-\infty} \int \mathcal{D}\eta \exp \left\{ -\frac{2i}{\hbar} \int d^d x \pi(x) \cdot \eta(x) \right\} \Phi[\Theta + \eta] \Phi^\dagger[\Theta - \eta],
\]

where \( \Phi[\Theta + \eta] = \langle \Theta + \eta | \Phi \rangle \) and \( \Phi^\dagger[\Theta - \eta] = \langle \Phi^\dagger | \Theta - \eta \rangle \).

D. Normal Ordering

Let \( F[\pi, \Theta] \) be a functional on \( \mathbb{Z}_G \). We define

\[
\hat{F}_N := \hat{\mathcal{N}} F,
\]

The operator \( \hat{\mathcal{N}} \) is the relativistic field generalization of the respective operator given in \[35\] by the formula (64), and in the present case it reads

\[
\hat{\mathcal{N}} := \exp \left\{ \frac{1}{2} \sum_{i=1}^N \int d^d p f(E_{\vec{p}}) \left( \frac{\delta^2}{\delta b_i(\vec{p}) \delta b_i^*(\vec{p})} - \frac{\delta^2}{\delta d_i(\vec{p}) \delta d_i^*(\vec{p})} \right) \right\},
\]

where \( f(E_{\vec{p}}) \) is a function to be determined, \( b_i(\vec{p}), b_i^*(\vec{p}) \) stand for the \( W^{-1} \) images of the annihilation or creation operators (respectively) for particles, and \( d_i(\vec{p}), d_i^*(\vec{p}) \) denote the \( W^{-1} \) images of annihilation or creation operators for antiparticles.

Then the Berezin-Wick or normal quantization is given by

\[
\hat{F}_N = \int F_N \hat{\Omega}[\pi, \Theta] \prod d\pi d\Theta,
\]

and the Weyl mapping of \( F_N \) gives the normal ordering of the Weyl image of \( F_N \) i.e.

\[
: \hat{F} := W(F) \overset{df}{=} W(F_N) \overset{df}{=} W_N(F).
\]

In the next section the normal ordering in the case of the free Dirac field will be considered in more detail.

III. DEFORMATION QUANTIZATION OF THE DIRAC FREE FIELD

The aim of this section is to provide an example of the application of the deformation quantization of Grassmann fields to the Dirac free field. In addition we compute the propagator of the Dirac field in this context.
In the next subsection we briefly survey the Dirac field in order to introduce the notation and conventions that we use in this paper. We stress the uses of the oscillator variables $b^*$ and $b$ which allowed us to perform the construction [41].

A. The Dirac Free Field

In this section we discuss the Dirac free field $\psi(x)$ over Minkowski spacetime $M^{3+1} = \mathbb{R}^3 \times \mathbb{R}$ with the signature $(-, -, -, +)$ and $x = (\vec{x}, t) \in M$. The action is given by

$$I_D[\psi] = \int d^3x dt \mathcal{L}_D = \int d^3x d\vec{\psi}(\vec{x}, t) \cdot \hbar c (i \not{\partial} - \frac{mc}{\hbar}) \psi(\vec{x}, t),$$

where $\not{\partial} = \gamma^\mu \partial_\mu$, $\gamma^\mu$ are the Dirac matrices ($\mu = 0, \ldots, 3$), $\vec{\psi}(x) \equiv \psi^\dagger(x) \gamma^0$, $m$ is the mass parameter, $c$ is the speed of light and $\hbar$ the Planck constant. Thus, the field $\psi(x)$ fulfills the Dirac equation

$$(i \not{\partial} - \frac{mc}{\hbar}) \psi(\vec{x}, t) = 0.$$ (28)

Its conjugate momentum is given by $\pi_\psi(\vec{x}, t) = \frac{-\partial F}{\partial \dot{\psi}} = i\hbar \dot{\psi}(\vec{x}, t)$, where $\dot{\psi}(\vec{x}, t) \equiv \frac{\partial \psi(\vec{x}, t)}{\partial t}$ [46]. Then the Hamiltonian reads

$$H_D[\pi_\psi, \psi] = \int d^3x \dot{\psi}(\vec{x}, t) \cdot \hbar c (-i\gamma^j \partial_j + \frac{mc}{\hbar}) \psi(\vec{x}, t)$$

$$= \int d^3x \psi^\dagger(\vec{x}, t) \cdot i\hbar \frac{\partial}{\partial t} \psi(\vec{x}, t)$$

$$= \int d^3x \pi_\psi(\vec{x}, t) \cdot \frac{\partial}{\partial t} \psi(\vec{x}, t),$$ (30)

where as usual $\pi_\psi \cdot \partial_t \psi \equiv \sum_{\alpha=1}^4 \pi_{\psi_\alpha} \partial_t \psi_\alpha$, and $\gamma^i = \beta \alpha^i$, $\gamma^0 = \beta$, with $i = 1, 2, 3$. Note that we use the Weyl (or chiral) representation of the Dirac matrices $\gamma^\mu$, $\mu = 0, 1, 2, 3$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3$$ (31)

with $\sigma_j$ being Pauli’s matrices. The Dirac matrices have the following properties: $\gamma^j = -\gamma^j$, $\gamma^0 = \gamma^0$, $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$.

According to the definition of the Poisson bracket for Grassmann fields given by (21), the Poisson bracket corresponding to the Dirac free field takes the form

$$\{F, G\}_P = F \mathring{\mathcal{P}} G$$

where

$$\mathring{\mathcal{P}} = \int d^3x \left( \frac{\delta}{\delta \psi(\vec{x}, t)} \frac{\delta}{\delta \psi^\dagger(\vec{x}, t)} + \frac{\delta}{\delta \psi^\dagger(\vec{x}, t)} \frac{\delta}{\delta \psi(\vec{x}, t)} \right)$$

$$= -\frac{i}{\hbar} \int d^3x \left( \frac{\delta}{\delta \dot{\psi}(\vec{x}, t)} \frac{\delta}{\delta \dot{\psi}^\dagger(\vec{x}, t)} + \frac{\delta}{\delta \dot{\psi}^\dagger(\vec{x}, t)} \frac{\delta}{\delta \dot{\psi}(\vec{x}, t)} \right),$$ (32)
and \( \frac{\delta}{\delta \psi(x,t)} \frac{\delta}{\delta \pi(x,t)} \equiv \sum_{\alpha=1}^{4} \frac{\delta}{\delta \psi_{\alpha}(x,t)} \frac{\delta}{\delta \pi_{\alpha}(x,t)} \), etc.

Consequently, for \( \psi_{\alpha}(x,t) \) and \( \pi_{\psi_{\alpha}}(x,t) \) one gets

\[
\{ \psi_{\alpha}(x,t), \pi_{\psi_{\beta}}(\tilde{y},t) \}_P = \delta(x-\tilde{y})\delta_{\alpha\beta},
\]

\[
\{ \psi_{\alpha}(x,t), \psi_{\beta}(\tilde{y},t) \}_P = 0, \quad \{ \pi_{\psi_{\beta}}(x,t), \pi_{\psi_{\beta}}(\tilde{y},t) \}_P = 0. \tag{33}
\]

(Remember that \( \alpha = 1, 2, 3, 4 \) runs over the components of the Dirac spinor).

Usually quantization is done by the substitution \( \{\cdot,\cdot\}_P \mapsto \frac{1}{\hbar}\{\cdot,\cdot\}_+ \). Thus we obtain

\[
[\hat{\psi}_{\alpha}(x,t), \hat{\psi}_{\beta}(\tilde{y},t)]_+ = [\hat{\pi}_{\psi_{\alpha}}(x,t), \hat{\pi}_{\psi_{\beta}}(\tilde{y},t)]_+ = 0,
\]

\[
[\hat{\psi}_{\alpha}(x,t), \hat{\pi}_{\psi_{\beta}}(\tilde{y},t)]_+ = i\hbar\delta_{\alpha\beta}\delta(x-\tilde{y}), \tag{34}
\]

\[
[\hat{\psi}_{\alpha}^\dagger(x,t), \hat{\pi}_{\psi_{\beta}}^\dagger(\tilde{y},t)]_+ = \delta_{\alpha\beta}\delta(x-\tilde{y}).
\]

The field variables \( \psi_{\alpha}(x,t) \) and \( \pi_{\psi_{\alpha}}(x,t) \) can be expanded into the plane waves

\[
\psi_{\alpha}(x,t) = \sum_{i=1}^{2} \int \frac{d^3p}{(2\pi\hbar)^3} \frac{mc^2}{E_{\vec{p}}} \left[ b_i(\vec{p},t)u^i_{\alpha}(\vec{p})e^{i\vec{p}\cdot\vec{x}/\hbar} + d_i^*(\vec{p},t)v^i_{\alpha}(\vec{p})e^{-i\vec{p}\cdot\vec{x}/\hbar} \right], \tag{35}
\]

\[
\pi_{\psi_{\alpha}}(x,t) = i\hbar\psi_{\alpha}^\dagger(x,t) = \sum_{i=1}^{2} \int \frac{d^3p}{(2\pi\hbar)^3} \frac{ihmc^2}{E_{\vec{p}}} \left[ b_i^*(\vec{p},t)u^i_{\alpha}(\vec{p})e^{-i\vec{p}\cdot\vec{x}/\hbar} + d_i(\vec{p},t)v^i_{\alpha}(\vec{p})e^{i\vec{p}\cdot\vec{x}/\hbar} \right], \tag{36}
\]

where \( b_i(\vec{p},t) = b_i(\vec{p}) \exp(-iE_{\vec{p}}t/\hbar), d_i^*(\vec{p},t) = d_i^*(\vec{p}) \exp(iE_{\vec{p}}t/\hbar) \) and \( E_{\vec{p}} = \sqrt{c^2p^2 + m^2c^4} \). Here the index \( i = 1, 2 \) stands for the spin degrees of freedom. Therefore, \( u^i \) and \( v^i \) are solutions of the Dirac equation of momentum \( p \) with positive or negative energies, respectively

\[
(g' - mc)u^i = 0, \quad (g' + mc)v^i = 0, \tag{37}
\]

with \( g' = \gamma^\mu p_\mu \); and they are restricted to satisfy the following relations

\[
u^i(\vec{p})u^j(\vec{p}) = v^i(\vec{p})v^j(\vec{p}) = \frac{E_{\vec{p}}}{mc^2}\delta_{ij}, \quad \nabla^i(\vec{p})u^j(\vec{p}) = 0. \tag{38}
\]

Classically Dirac fields are described by Grassmann variables since they are functions of the anti-commuting Grassmann variables \( b_i(\vec{p}), b_i^*(\vec{p}), d_i(\vec{p}) \) and \( d_i^*(\vec{p}) \). Poisson brackets for these variables are

\[
\{ b_i(\vec{p}), b_j^*(\vec{p}') \}_P = \{ d_i(\vec{p}), d_j^*(\vec{p}') \}_P = \frac{i}{\hbar}(2\pi\hbar)^3 \left( \frac{E_{\vec{p}}}{mc^2} \right) \delta_{ij}\delta(\vec{p} - \vec{p}'). \tag{39}
\]

**Oscillator Variables**

The variables \( b \) and \( d \) are quite asymmetric and it is not easy to express these variables in terms of the field variables \( \psi \) and \( \pi_{\psi} \). To carry out the deformation quantization in the simplest possible way we introduce the variables \( b(\vec{p},t,r) \) and \( b^*(\vec{p},t,r) \) which are related to the variables \( b \) and \( d \) as follows

\[
b(\vec{p},t,r) = (2\pi\hbar)^{-3/2} \sqrt{\frac{mc^2}{E_{\vec{p}}}} b_r(\vec{p},t), \quad \text{with } r = 1, 2,
\]
and
\[ b(\vec{p}, t, r) = (2\pi \hbar)^{-3/2} \sqrt{\frac{mc^2}{E_\vec{p}}} d_{r-2}^*(-\vec{p}, t), \quad \text{with } r = 3, 4; \]
and \( b^* \) is determined by the complex conjugate of these equations.

These variables suggest to redefine the solutions to the Dirac equation in the following form
\[ w_\alpha(\vec{p}, r) = u_\alpha^r(\vec{p}), \quad \text{for } r = 1, 2, \]
\[ w_\alpha(\vec{p}, r) = u_\alpha^{r-2}(-\vec{p}), \quad \text{for } r = 3, 4. \] (40)

From (38) one gets the relations
\[ w^\dagger(\vec{p}, r) w(\vec{p}, r') = \frac{E_\vec{p}}{mc^2} \delta_{rr'}, \]
\[ \bar{w}(\varepsilon \vec{p}, r) w(\varepsilon' \vec{p}, r') = \varepsilon_r \delta_{rr'}, \]
\[ \sum_{r=1}^{4} w_\alpha(\vec{p}, r) w_\alpha^*(\vec{p}, r) = \frac{E_\vec{p}}{mc^2} \delta_{\alpha\alpha'}. \] (41)

In terms of these variables Eqs. (35) and (36) read
\[ \psi_\alpha(\vec{x}, t) = \sum_{r=1}^{4} \int \frac{d^3p}{(2\pi \hbar)^3/2} \sqrt{\frac{mc^2}{E_\vec{p}}} b(\vec{p}, t, r) \bar{w}_\alpha(\vec{p}, r) \exp \left( i \vec{p} \cdot \vec{x}/\hbar \right), \] (42)
\[ \pi_{\psi_\alpha}(\vec{x}, t) = i\hbar \bar{\psi}_\alpha^*(\vec{x}, t) = \sum_{r=1}^{4} \int \frac{d^3p}{(2\pi \hbar)^3/2} i\hbar \sqrt{\frac{mc^2}{E_\vec{p}}} b^*(\vec{p}, t, r) \bar{w}_\alpha(\vec{p}, r) \exp \left( -i \vec{p} \cdot \vec{x}/\hbar \right), \] (43)
where \( b(\vec{p}, t, r) = b(\vec{p}, r) \exp \left\{ -i\varepsilon_r E_\vec{p} t/\hbar \right\} \) with \( \varepsilon_r = 1 \) for \( r = 1, 2 \) and \( \varepsilon_r = -1 \) for \( r = 3, 4 \).

Substituting Eqs. (42) and (43) into (33) we find the Poisson brackets for the variables \( b \) and \( b^* \)
\[ \{ b(\vec{p}, r), b^*(\vec{p}', r') \}_P = -\frac{i}{\hbar} \delta(\vec{p} - \vec{p}') \delta_{rr'}, \]
\[ \{ b(\vec{p}, r), b(\vec{p}', r') \}_P = 0, \quad \{ b^*(\vec{p}, r), b^*(\vec{p}', r') \}_P = 0. \] (44)

Multiplying Eq. (42) by \( w_\beta(\vec{p}', r') \exp \left( -i \vec{p}' \cdot \vec{x}/\hbar \right) \) and integrating over \( \mathbb{R}^3 \) we get
\[ \int d^3x \ w_\beta(\vec{p}', r') \psi_\alpha(\vec{x}, t) \exp \left( -i \vec{p}' \cdot \vec{x}/\hbar \right) \]
\[ = \sum_{r=1}^{4} \int \frac{d^3x}{(2\pi \hbar)^3/2} \sqrt{\frac{mc^2}{E_\vec{p}}} w_\beta(\vec{p}', r') b(\vec{p}, t, r) \bar{w}_\alpha(\vec{p}, r) \exp \left( i(\vec{p} - \vec{p}') \cdot \vec{x}/\hbar \right). \]

Using now the relations (41), after some computations we obtain
\[ b(\vec{p}, t, r) = \sqrt{\frac{mc^2}{E_\vec{p}}} \sum_\alpha \int \frac{d^3x}{(2\pi \hbar)^3/2} \ w_\alpha(\vec{p}, r) \psi_\alpha(\vec{x}, t) \exp \left( -i \vec{p} \cdot \vec{x}/\hbar \right), \] (45)
\[ b^*(\vec{p}, t, r) = \sqrt{\frac{mc^2}{E_\vec{p}}} \sum_\alpha \int \frac{d^3x}{(2\pi \hbar)^3/2} \ \psi_\alpha^*(\vec{x}, t) w_\alpha(\vec{p}, r) \exp \left( i \vec{p} \cdot \vec{x}/\hbar \right). \] (46)
Thus the Grassmann variables $b$ and $b^*$, determine precisely the canonical conjugate variables which we will use to describe the Dirac field in the WWM formalism.

Substituting (42) and (43) into the Hamiltonian (30) yields

$$H_D[b^*, b] = \sum_{r=1}^{4} \int d^3 p \varepsilon_r E_p b^*(\vec{p}, r) b(\vec{p}, r),$$

(47)

or in terms of the standard variables,

$$H_D[b, b^*, d, d^*] = \sum_{i=1}^{2} \int \frac{d^3 p}{(2\pi\hbar)^3} mc^2 \left[ b_i^*(\vec{p}) b_i(\vec{p}) - d_i(\vec{p}) d_i^*(\vec{p}) \right].$$

(48)

B. The Stratonovich-Weyl Quantizer

The Weyl-Wigner-Moyal correspondence for the Dirac field in the field variables reads

$$\hat{F} = W(F[\pi_{\psi}, \psi]) = W(F[\psi^\dagger, \psi]) = (i\hbar)^{-\infty} \int F[\psi^\dagger, \psi] \Omega[\psi^\dagger, \psi] \prod d\psi^\dagger d\psi,$$

(49)

where

$$\Omega[\psi^\dagger, \psi] = (-i)^\infty \int D\mu \exp \left\{ -i \int d^3 x \psi(\vec{x}) \cdot \mu(\vec{x}) \right\} \exp \left\{ i \hbar \frac{\mu}{2} \right\} \exp \left\{ i \hbar \frac{\mu}{2} \right\}.$$

(50)

In terms of the oscillator variables we have

$$\hat{F} = W(F[b^*, b]) = (i\hbar)^{-\infty} \int \prod db^* db F[b^*, b] \hat{\Omega}[b^*, b],$$

(51)

and

$$\hat{\Omega}[b^*, b] = (-i)^\infty \int D\chi \exp \left\{ -i \sum_{r=1}^{4} \int d^3 p b(\vec{p}, r) \chi(\vec{p}, r) \right\} \exp \left\{ i \hbar \frac{\chi}{2} \right\} \exp \left\{ i \hbar \frac{\chi}{2} \right\}.$$

(52)

where $\chi$ and $\xi$ are Dirac spinors.

[Observe that here $\infty$ is in fact $4\infty$ as $r = 4$. Consequently, one can put 1 instead of $i\infty$ or $(-i)^\infty$.]

C. The Moyal $\ast$-Product

According to the WWM correspondence, the symbol of an operator $\hat{F}$ is given by

$$F[b^*, b] = W^{-1}(\hat{F}) = \text{tr}\left\{ \hat{\Omega}[b^*, b] \hat{F} \right\}.$$

(53)

The Moyal $\ast$-product in this case can be defined similarly as it has been done in (19). Let $F_1[b^*, b]$ and $F_2[b^*, b]$ be functionals over the Dirac phase space defined by: $Z_D = \{(\pi_{\psi_{\alpha}}(\vec{x}), \psi_{\alpha}(\vec{x}))_{\vec{x} \in \mathbb{R}^3}\} = \{\eta_{\alpha}(\vec{x}), \psi_{\alpha}(\vec{x})\}_{\vec{x} \in \mathbb{R}^3}$.
\{(ih\hat{b}^\dagger(\vec{p}, r), (b(\vec{p}, r)))_{r=1,\ldots,n}\}$ and let $\hat{F}_1$ and $\hat{F}_2$ be their corresponding operators. Then by a similar computation to that done in Sec. 113 we finally get

$$(F_1 \ast F_2)[b^*, b] = F_1[b^*, b] \exp \left( \frac{ih}{2} \hat{P}_D \right) F_2[b^*, b],$$

(54)

where

$$\hat{P}_D := -i \hbar \sum_{r=1}^{4} \int d^3 p \left( \frac{\delta}{\delta b(\vec{p}, r)} \frac{\delta}{\delta b^*(\vec{p}, r)} + \frac{\delta}{\delta b^*(\vec{p}, r)} \frac{\delta}{\delta b(\vec{p}, r)} \right),$$

(55)

is the Poisson operator. In terms of the standard Grassmann variables $b, b^*$, $d$ and $d^*$, it can be rewritten as

$$\hat{P}_D = -i \hbar (2\pi \hbar)^3 \sum_{i=1}^{2} \int d^3 p \left( E_{\vec{p}} \frac{\delta}{\delta b_i(\vec{p})} \frac{\delta}{\delta b_i^*(\vec{p})} + \frac{\delta}{\delta b_i^*(\vec{p})} \frac{\delta}{\delta b_i(\vec{p})} + \frac{\delta}{\delta d_i(\vec{p})} \frac{\delta}{\delta d_i^*(\vec{p})} + \frac{\delta}{\delta d_i^*(\vec{p})} \frac{\delta}{\delta d_i(\vec{p})} \right).$$

(56)

The operator $\hat{P}_D$ determines the Poisson bracket $\{F, G\}_P = F \hat{P}_D G$ defined by the symplectic structure

$$\omega_D = \int d^3 x \sum_{\alpha} \delta \psi_\alpha(\vec{x}) \wedge \delta \pi_\alpha(\vec{x}),$$

$$= i\hbar \int d^3 x \sum_{\alpha} \delta \psi_\alpha(\vec{x}) \wedge \delta \psi_\alpha^\dagger(\vec{x}).$$

(57)

This symplectic structure defines the symplectic supermanifold $(\mathcal{Z}_D, \omega_D)$ which represents the phase space. In terms of the original field variables $\psi_\alpha^\dagger, \psi_\alpha$ one has

$$\hat{P}_D := -i \hbar \sum_{\alpha} \int d^3 x \left( \frac{\delta}{\delta \psi_\alpha(\vec{x})} \frac{\delta}{\delta \psi_\alpha^\dagger(\vec{x})} + \frac{\delta}{\delta \psi_\alpha^\dagger(\vec{x})} \frac{\delta}{\delta \psi_\alpha(\vec{x})} \right).$$

(58)

D. The Wigner Functional

The Wigner functional for the Dirac free field is defined in analogy to the scalar field case. Let $\tilde{\rho}^{\text{phys}}$ be the density operator corresponding to the quantum physical state of the Dirac field. Then the Wigner functional corresponding to this state is given by

$$\rho_w[b^*, b] = \text{tr} \left\{ \hat{\Omega}[b^*, b] \tilde{\rho}^{\text{phys}} \right\},$$

$$= 2^{-\infty} \int \mathcal{D}\xi \exp \left\{ -2 \sum_{r=1}^{4} \int d^3 p \ b^*(\vec{p}, r) \xi(\vec{p}, r) \right\} \langle b - \xi | \tilde{\rho}^{\text{phys}} | b + \xi \rangle.$$  

(59)

In the case when $\tilde{\rho}^{\text{phys}} = |\Phi \rangle \langle \Phi |$ the above equation gives

$$\rho_w[b^*, b] = 2^{-\infty} \int \mathcal{D}\xi \exp \left\{ -2 \sum_{r=1}^{4} \int d^3 p \ b^*(\vec{p}, r) \xi(\vec{p}, r) \right\} \Phi[b - \xi | \Phi^\dagger | b + \xi],$$

(60)

where $\Phi[b] = \langle b | \Phi \rangle$ and $\Phi^\dagger[b] = \langle \Phi^\dagger | b \rangle$. 


Example: The Ground State

In the Schrödinger representation the ground state corresponding to the system described by the Dirac equation can be easily found. In order to do that we use the wave functional defined by $\Phi[\psi] = \langle \psi | \Phi \rangle$. Then the canonical conjugate momentum to $\psi$ is represented by $i\hbar \hat{p} \psi^\dagger = i\hbar \delta_{\psi} \delta \psi(\vec{x})$. Substituting this momentum into the Hamiltonian (30) we get the time independent Schrödinger equation

$$\int d^3x \left( h \frac{\delta}{\delta \psi(\vec{x})} \left( -i\alpha \cdot \vec{\nabla} + \beta \frac{mc}{\hbar} \right) \psi(\vec{x}) \right) \Phi[\psi] = E \Phi[\psi]. \quad (61)$$

It is an easy matter to show that Eqs. (42) and (43) at $t = 0$ can be rewritten in terms of respective operators in the Schrödinger representation as follows

$$\hat{\psi}(\vec{x}) = \sum_{r=1}^{4} \int \frac{d^3p}{(2\pi \hbar)^{3/2}} \sqrt{\frac{mc^2}{E_{\vec{p}}}} b(\vec{p}, r) w(\vec{p}, r) \exp \left( i\vec{p} \cdot \vec{x}/\hbar \right),$$

$$\hat{\psi}^\dagger(\vec{x}) = \frac{\delta}{\delta \psi}(\vec{x}) = \sum_{r=1}^{4} \int \frac{d^3p}{(2\pi \hbar)^{3/2}} \sqrt{\frac{mc^2}{E_{\vec{p}}}} \frac{\delta}{\delta b(\vec{p}, r)} b^\dagger(\vec{p}, r) w^\dagger(\vec{p}, r) \exp \left( -i\vec{p} \cdot \vec{x}/\hbar \right).$$

Inserting these formulae into (61) one gets

$$\sum_{r=1}^{4} \int d^3p \epsilon_r \frac{\delta}{\delta b(\vec{p}, r)} b(\vec{p}, r) \Phi[b] = E \Phi[b], \quad (62)$$

where $b(\vec{p}, r)$ is the eigenvalue of $\hat{b}(\vec{p}, r)$.

The stable vacuum $|0\rangle$ for the Dirac free field is the ground state in which all the negative energy states are filled. To be precise consider the case of having only one fermion state and remind that $b$ and $\hbar \frac{\delta}{\delta b}$ are the annihilation and creation operators of a single fermion. In addition, let $\Omega_0(b)$ and $\Omega_1(b)$ be the (one fermion) wave functions of the states $|0\rangle$ and $|1\rangle$, respectively. Hence one has

$$b \Omega_0(b) = 0, \quad \frac{\delta}{\delta b} \Omega_1(b) = 0,$$

$$b \Omega_1(b) = \Omega_0(b), \quad \frac{\delta}{\delta b} \Omega_0(b) = \Omega_1(b). \quad (63)$$

Since $b$ is a Grassmann variable it is easy to see that $\Omega_0(b) = b$ and $\Omega_1(b) = 1$. In order to be consistent with the structure of the stable vacuum the positive energy states ($r = 1, 2$) are determined by the function $b$, and therefore they are empty. While for the negative energy states ($r = 3, 4$), the function is 1, and consequently they are occupied.

The Hamiltonian is a sum over all momenta $p$'s and states $r$'s. This implies that the ground state is given by an infinite product

$$\Phi_0[b] = \eta \prod_{r=1}^{2} \prod_{\vec{p}} b(\vec{p}, r), \quad (64)$$
where \( \eta \) is a normalization factor. Thus the adjoint functional is given by

\[
\Phi_0^*[b] = \eta^* \prod_{r=3}^{4} \prod_{\vec{p}} b(\vec{p}, r).
\] (65)

Now we are at the position to compute the form of the Wigner functional of the ground state corresponding to the Dirac field. Substituting the wave functionals (64) and (65) into (60) and taking \( |\eta|^2 = 1 \) one finally gets

\[
\rho_{W_0}[b^*, b] = 2^{-\infty} \int D\xi \exp \left\{ -2 \sum_{r=1}^{4} \int d^3 p \ b^*(\vec{p}, r) \xi(\vec{p}, r) \right\} \Phi_0[b - \xi] \Phi_0^*[b + \xi].
\]

Recall that \( \xi \) is also a Dirac spinor and \( b(\vec{p}, r) + \varepsilon_r \xi \) is proportional to the Dirac delta function. Then the Wigner functional (including the normalization factor) for the ground state is given by

\[
\rho_{W_0}[b^*, b] = 2^{-\infty} \exp \left\{ -2 \int d^3 p \sum_{r=1}^{4} \varepsilon_r b^*(\vec{p}, r) b(\vec{p}, r) \right\}.
\] (67)

In terms of the standard variables it yields

\[
\rho_{W_0}[b, b^*, d, d^*] = 2^{-\infty} \exp \left\{ -2 \int \frac{d^3 p}{(2\pi \hbar)^3} \left( \frac{mc^2}{E_{\vec{p}}} \right) \sum_{i=1}^{2} \left( b_i^*(\vec{p}) b_i(\vec{p}) - d_i(\vec{p}) d_i^*(\vec{p}) \right) \right\}.
\] (68)

Remark: One can quickly show that the ground state Wigner functional \( \rho_{W_0}[b^*, b] \) can also be found within purely deformation quantization language from the following equations

\[
b(\vec{p}, r = 1, 2) \ast \rho_{W_0}[b^*, b] = 0 \quad \text{and} \quad b^*(\vec{p}, r = 3, 4) \ast \rho_{W_0}[b^*, b] = 0.
\]

**Wigner Functional for Excited States**

The Wigner functionals for excited states can be constructed as follows: If one wants to get the wave functional of an electron (of positive energy) of momentum \( \vec{p} \) and spin \( s = \pm \frac{1}{2} \), one have to remove from the ground state wave functional \( \Phi_0[b] \) a corresponding factor \( b(\vec{p}, r = 1, 2) \), where \( r = 1 \) corresponds to spin up and \( r = 2 \) to spin down, and substitute it by the factor 1. Thus, the wave functional for a state with one electron of momentum \( \vec{p} \) and spin up is: \( \Phi_{1e}[b] = \eta \prod_{r=1}^{2} \prod_{\vec{q}} b(\vec{q}, r) \), except the factor \( b(\vec{p}, 1) \). If you prefer to add a positron of momentum \( \vec{p} \) and spin \( s = \pm \frac{1}{2} \), it is necessary to annihilate an electron of negative energy. In order to do that one needs to multiply \( \Phi_0[b] \) by \( b(-\vec{p}, r = 4, 3) \), where \( r = 4 \) corresponds to the positron spin up and \( r = 3 \) to spin down (for details, see [41]). For example, the wave functional for the positron of momentum \( \vec{p} \) and spin up is given by: \( \Phi_{1p}[b] = \eta b(-\vec{p}, r = 4) \prod_{r=1}^{2} \prod_{\vec{q}} b(\vec{q}, r) \). The Wigner functionals corresponding to excited states can be
where

\[ \bar{\rho} = \hat{b}^\dagger(\vec{p}_1, r_1) \cdots \hat{b}^\dagger(\vec{p}_n, r_n) \hat{b}(-\vec{p}_1', r_1') \cdots \hat{b}(-\vec{p}_n', r_n') \]

\[ \times |0\rangle \langle 0| \hat{b}(-\vec{p}_n', r_n') \cdots \hat{b}(-\vec{p}_1', r_1') \hat{b}(\vec{p}_n, r_n) \cdots \hat{b}(\vec{p}_1, r_1). \]

(69)

Thus the Wigner functional corresponding to our excited quantum state reads

\[ \rho_W[b^*, b] = b^*(\vec{p}_1, r_1) \cdots b^*(\vec{p}_n, r_n) * b(-\vec{p}_1', r_1') \cdots b(-\vec{p}_n', r_n') * \rho_{W_0} \]

\[ * b^*(-\vec{p}_n', r_n') \cdots b^*(-\vec{p}_1', r_1') b(\vec{p}_n, r_n) \cdots b(\vec{p}_1, r_1). \]

(70)

where \( \rho_{W_0} \) is given by (67). Due to the form of the Moyal product (54) and (55) this Wigner functional can be rewritten in the following form

\[ \rho_W[b^*, b] = b^*(\vec{p}_1, r_1) \cdots b^*(\vec{p}_n, r_n) b(-\vec{p}_1', r_1') \cdots b(-\vec{p}_n', r_n') * \rho_{W_0} \]

\[ * b^*(-\vec{p}_n', r_n') \cdots b^*(-\vec{p}_1', r_1') b(\vec{p}_n, r_n) \cdots b(\vec{p}_1, r_1). \]

(71)

From this last formula one can find the Wigner functional for any excited state.

**Normal Ordering**

The normal ordering of the Dirac field operators can also be defined in the deformation quantization formalism. This can be done by using the operator \( \hat{N} \) acting on the phase space \( \mathcal{Z}_D \)

\[ \hat{N} := \exp \left\{ \frac{1}{2} \int d^3p \sum_{r=1}^{4} \frac{\varepsilon_r}{\delta b(\vec{p}, r) \delta^* b(\vec{p}, r)} \right\}. \]

(72)

In terms of the standard variables

\[ \hat{N} = \exp \left\{ \frac{1}{2}(2\pi \hbar)^3 \int d^3p \left( \frac{E_\vec{p}}{mc^2} \right) \sum_{i=1}^{2} \left( \frac{\delta^2}{\delta b_i(\vec{p}) \delta^* b_i(\vec{p})} - \frac{\delta^2}{\delta d_i(\vec{p}) \delta d_i(\vec{p})} \right) \right\}. \]

(73)

Let \( F[b^*, b] \) be a functional defined on the phase space \( \mathcal{Z}_D \). Then \( F_{\hat{N}} \) is defined as follows

\[ F_{\hat{N}}[b^*, b] = \hat{N} F[b^*, b]. \]

(74)

Consequently, the Weyl image of \( F_{\hat{N}}[b^*, b] \) gives the normal ordering of the Weyl image of \( F[b^*, b] \)

\[ \hat{F} := W(F[b^*, b]) : \overset{df}{=} W(F_{\hat{N}}[b^*, b]) \overset{df}{=} W_{\hat{N}}(F[b^*, b]). \]

(75)
Example: The Hamiltonian

The normal ordering of the Hamiltonian can be obtained by applying (7 5) with \( \hat{N} \) given by Eq. (72) to \( H_D[b^*, b] \) given by Eq. (47). Simple calculations yield

\[
H_{D_N} = H_D + 2 \int d^3p E_p \delta(0).
\] (76)

Then the normal ordered Hamiltonian operator reads

\[
: \hat{H}_D : = 2 \sum_{i=1}^2 \int \frac{d^3p}{(2\pi \hbar)^3} mc^2 \left( \hat{b}_i^*(p) \hat{b}_i(p) + \hat{d}_i^*(p) \hat{d}_i(p) \right).
\] (77)

E. Dirac Propagator

In order to compute the propagator of the Dirac field we need to find

\[
iS_F(x - y) = \langle 0 | \hat{\psi}_\alpha(x) \hat{\bar{\psi}}_\beta(y) | 0 \rangle \cdot \theta(t - t') - \langle 0 | \hat{\bar{\psi}}_\beta(y) \hat{\psi}_\alpha(x) | 0 \rangle \cdot \theta(t' - t). \] (78)

So we first compute the quantities \( \langle 0 | \hat{\psi}_\alpha(x) \hat{\bar{\psi}}_\beta(y) | 0 \rangle \) and \( \langle 0 | \hat{\bar{\psi}}_\beta(y) \hat{\psi}_\alpha(x) | 0 \rangle \). In terms of deformation quantization these expectation values are given by (compare with [10])

\[
\langle 0 | \hat{\psi}_\alpha(x) \hat{\bar{\psi}}_\beta(y) | 0 \rangle = \int \frac{d^3b^* db}{W_0[b^*, b]} \prod_{r=1}^2 w_\alpha(\vec{p}, r) \bar{w}_\beta(\vec{p}, r),
\] (79)

and the analogous formula for the second expectation value.

Carrying out the corresponding integrations and making use of the following relations

\[
\sum_{r=1}^2 w_\alpha(\vec{p}, r) \bar{w}_\beta(\vec{p}, r) = \frac{(p + mc)_{\alpha\beta}}{2mc}, \quad \sum_{r=3}^4 w_\alpha(\vec{p}, r) \bar{w}_\beta(\vec{p}, r) = \frac{(p - mc)_{\alpha\beta}}{2mc},
\]

after straightforward calculations we arrive at the results which are well known in quantum field theory

\[
\langle 0 | \hat{\psi}_\alpha(x) \hat{\bar{\psi}}_\beta(y) | 0 \rangle = \int \frac{d^3p}{(2\pi \hbar)^3} \frac{c(p + mc)_{\alpha\beta}}{2E_\beta} \exp \left( -ip \cdot (x - y)/\hbar \right); \] (80)

and

\[
\langle 0 | \hat{\bar{\psi}}_\beta(y) \hat{\psi}_\alpha(x) | 0 \rangle = \int \frac{d^3p}{(2\pi \hbar)^3} \frac{c(p - mc)_{\beta\alpha}}{2E_\beta} \exp \left( ip \cdot (x - y)/\hbar \right). \] (81)

The above formulas reproduce exactly the propagator of the Dirac field in the deformation quantization formalism.

IV. FINAL REMARKS

In this paper we have carried over the deformation quantization program via the Weyl-Wigner-Moyal formalism to fermionic fields. In a sense it is an extension of the results for a finite number of degrees of
freedom given in Ref. \[35\] to field theory. To see how our construction explicitly works we have applied it to the quantization of the Dirac free field. The quantization of this field is possible since the Dirac free field can be regarded as an infinite number of decoupled Fermi oscillators. Consequently, the prescription worked out in Refs. \[33, 35\] can be employed to each oscillator separately. The oscillator variables $b$ and $b^*$ greatly facilitates the procedure.

In the next step it is natural to consider a formalism which involves both bosonic and fermionic fields, and also their interactions in a similar way to the one given in Ref. \[35\] for the case of supersymmetric quantum mechanics. Then, gathering the results of Ref. \[10\] concerning deformation quantization of the electromagnetic field with those of the present paper (in addition of some other considerations), a version of QED in the context of deformation quantization can be quickly implemented \[42\].

We are aware that the quantization by deformation seems to be at most equivalent to those of canonical or path integral quantizations. However, there is some evidence that deformation quantization would be more general. For instance, recently it was found that one of the intriguing features of deformation quantization is well defined for spaces with orbifold and conical singularities \[43\]. The results obtained in the present paper would be important for the description of fermions on noncommutative orbifolds \[44, 45\]. This problem will be reported elsewhere.

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[46] The conjugate momentum $\tilde{\pi}_\psi$ of the field $\psi$ is defined by \cite{33,35}

$$\tilde{\pi}_{\psi\alpha} = \frac{\partial L}{\partial (\partial_t \psi_\alpha)} = -i\hbar \psi^\dagger_\alpha,$$

where the sign ($-$) appears as one consider $\bar{\psi}$ and $\psi$ as Grassmann variables. In field theory is usual to write down it as $\pi = i\hbar \psi^\dagger$, see for instance, \cite{40,41}. Therefore to be consistent with these references on quantum fields we choose $\psi$ and $\pi_\psi = i\hbar \psi^\dagger$, as our fundamental variables, instead of $\psi$ and $\tilde{\pi}_\psi$. 