MAGNETIC $Z_N$ SYMMETRY IN 2+1 DIMENSIONS.

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Abstract

This review describes the role of magnetic symmetry in 2+1 dimensional gauge theories. In confining theories without matter fields in fundamental representation the magnetic symmetry is spontaneously broken. Under some mild assumptions, the low-energy dynamics is determined universally by this spontaneous breaking phenomenon. The degrees of freedom in the effective theory are magnetic vortices. Their role in confining dynamics is similar to that played by pions and $\sigma$ in the chiral symmetry breaking dynamics. I give an explicit derivation of the effective theory in (2+1)-dimensional weakly coupled confining models and argue that it remains qualitatively the same in strongly coupled (2+1)-dimensional gluodynamics. Confinement in this effective theory is a very simple classical statement about the long range interaction between topological solitons, which follows (as a result of a simple direct classical calculation) from the structure of the effective Lagrangian. I show that if fundamentally charged dynamical fields are present the magnetic symmetry becomes local rather than global. The modifications to the effective low energy description in the case of heavy dynamical fundamental matter are discussed. This effective lagrangian naturally yields a bag like description of baryonic excitations. I also discuss the fate of the magnetic symmetry in gauge theories with the Chern-Simons term.

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1 Introduction

This review is devoted to detailed discussion of the magnetic $Z_N$ symmetry in 2+1 dimensional $SU(N)$ gauge theories. The main motivation behind it, is understanding confinement in gluodynamics in terms of universal properties determined by realization of global symmetries.

Understanding of the low energy structure of strongly interacting theories is a very difficult matter. There are instances however when we can understand main features of the low-energy dynamics without being able to solve the “microscopic” theory in detail. This happens whenever we are lucky enough to have a spontaneously broken global continuous symmetry. The Goldstone theorem assures us that such a spontaneous symmetry breaking results in the occurrence of massless particles. The latter are, of course, natural low-energy degrees of freedom. Moreover, the original symmetry must be manifest in the effective theory that governs the interactions between the Goldstone bosons, albeit its implementation is nonlinear. This severely constrains the interactions of the Goldstone bosons and, in fact, lends considerable predictive power to the effective theory.

The classic example of this type is the spontaneous chiral symmetry breaking which results in the the low-energy physics being dominated by pions which are described by an effective chiral Lagrangian. In real life the chiral symmetry is broken explicitly, the pions are massive. However, the effects due to the small explicit breaking are easily accommodated in the chiral Lagrangian. The great thing about the chiral Lagrangian, of course, is that it is universal. It does not care what exact dynamics is responsible for the symmetry breaking, what are the degrees of freedom of the “microscopic” theory or any other fine details. All you need to know is that there was a symmetry, and this symmetry was spontaneously broken.

Can some universal considerations of a similar kind determine the structure of low-energy theory in pure gluodynamics? The chiral symmetry is of no relevance here. Nevertheless, I will argue that the symmetry path is a very fruitful one. The main thesis is this: there is a global discrete symmetry in gluodynamics, which is spontaneously broken in the vacuum. In what follows this symmetry will be called the magnetic $Z_N$ symmetry. Although it is discreet and, therefore, does not have all the bliss of the Goldstone theorem, under some natural assumptions it does indeed determine the low-energy dynamics. In physical (3+1)-dimensional theory this symmetry is of a somewhat unusual type – its charge is not a volume integral, but rather a surface integral. The implications of such a symmetry for the low-energy Lagrangian have to be studied in more detail; this has not been done so far. However, one can go much further in 2+1 dimensions, where this symmetry is of a familiar garden variety. This review will, therefore, be devoted to (2+1)-dimensional non-Abelian theories. The structure of the paper is the following. In Sec. 2 I will discuss in detail the nature of the magnetic $Z_N$ symmetry and the explicit construction of both, the generator of the group and the local order parameter. In Sect. 3 I will show how this symmetry is realized in dif-
ferent phases of Abelian gauge theories. I will show that in the Coulomb phase this symmetry is spontaneously broken and in the Higgs one it is respected by the vacuum state. In the Abelian theories the magnetic symmetry is presented by a continuous U(1) group – its spontaneous breaking leads to the appearance of a massless particle, the photon. I will derive the low-energy effective Lagrangian that describes this symmetry breaking pattern and show that it exhibits logarithmic confinement in the Coulomb phase. In Sec. 4 I will discuss non Abelian confining theories. Here the magnetic symmetry is discrete, but it is also spontaneously broken as in the Coulomb phase of QED. In weakly interacting theories, like the Georgi-Glashow model, it is possible to derive an effective Lagrangian which follows from this symmetry breaking pattern. I will demonstrate that, due to the fact that the magnetic symmetry is discrete, the effective theory exhibits linear confinement, and that this confinement mechanism in the effective theory appears very simply at the classical level. I will give arguments that the basic structure of the effective Lagrangian as well as physics of confinement stay the same in the limit of pure gluodynamics. I will discuss similarities and differences between the confining properties of the weakly interacting non-Abelian theories and gluodynamics. In Sec. 5 I will discuss the modifications to the magnetic symmetry concept when fundamentally charged matter fields are present. It turns out that in theories of this kind the magnetic $Z_N$ symmetry is gauged. Consequently the low energy effective theories is a local $Z_N$ invariant model. It is however still quite useful. In particular this approach yields a distinctive low energy description of baryonic states as “bags” of deconfined phase which contain the fundamental quarks inside them. Sec. 6 is devoted to the discussion of $SU(N)$ gauge theories with the Chern-Simons term. I show that the magnetic symmetry is unbroken in these theories in the perturbative domain. It is however possible that it is spontaneously broken at low values of the Chern-Simons coefficient, and the theories are then confining. I end with a short discussion in Sec.7.

2 The magnetic symmetry in 2+1 dimensions

A while ago ’t Hooft gave an argument establishing that a non-Abelian $SU(N)$ gauge theory without charged fields in the fundamental representation possesses a global $Z_N$ symmetry.[1]

Consider first a theory with several adjoint Higgs fields, so that by varying parameters in the Higgs sector the $SU(N)$ gauge symmetry can be broken completely. In this phase the perturbative spectrum will contain the usual massive “gluons” and the Higgs particles. However, in addition to those, there will be heavy stable magnetic vortices. Those are analogs of Abrikosov-Nielsen-Olesen vortices in the superconductors; they must be stable by virtue of the following topological argument. All the fields
in the vortex configuration away from the vortex core are pure gauge,

$$H^\alpha(x) = U(x)h^\alpha, \quad A^\mu = iU\partial^\mu U^\dagger.$$  \hspace{1cm} (1)

Here the index $\alpha$ labels the scalar fields in the theory, $h^\alpha$ are the constant vacuum expectation values of these fields, and $U(x)$ is a unitary matrix. As one goes around the location of the vortex in space, the matrix $U$ winds nontrivially in the gauge group. This is possible, since the gauge group in the theory without fundamental fields is $SU(N)/Z_N$. It has a nonvanishing first homotopy group $\pi_1(SU(N)/Z_N) = Z_N$. Practically, this means that when one makes a full circle around the vortex location, $U$ does not return to the same $SU(N)$ group element $U_0$, but, rather, ends up at $\exp\{i2\pi N\}U_0$. Adjoint fields do not feel this type of discontinuity in $U$ and, therefore, the energy of such a configuration is finite. Since such a configuration can not be smoothly deformed into a trivial one, a single vortex is stable. Processes involving annihilation of $N$ such vortices in the vacuum are allowed since the $N$-vortex configurations are topologically trivial. One can, of course, find explicit vortex solutions once the Higgs potential is specified. As any other semiclassical solution at weak coupling, its energy is inversely proportional to the gauge coupling constant and is, therefore, very large. One is faced with the situation where the spectrum of the theory contains a stable particle even though its mass is much larger than the masses of many other particles (the gauge and Higgs bosons) and the phase space for its decay into these particles is enormous. The only possible reason for the existence of such a heavy stable particle is that it carries a conserved quantum number. The theory must, therefore, possess a global symmetry which is unbroken in the completely Higgsed phase. The symmetry group must be $Z_N$ since the number of vortices is only conserved modulo $N$. ’t Hooft dubbed this symmetry “topological.” I prefer to call it “magnetic” for reasons that will become apparent in a short while.

Now imagine smoothly changing the parameters in the Higgs sector so that the expectation values of the Higgs fields become smaller, and, finally, the theory undergoes a phase transition into the confining phase. One can further change the parameters so that the adjoint scalars become heavy and eventually decouple completely from the glue. This limiting process does not change topology of the gauge group and, therefore, does not change the symmetry content of the theory. One concludes that the pure Yang-Mills theory also possesses a $Z_N$ symmetry. Of course since the confining phase is separated from the completely Higgsed phase by a phase transition one may expect that the $Z_N$ symmetry in the confining phase is realized differently in the vacuum than in the completely “Higgsed” phase. In fact, the original paper by ’t Hooft, as well as subsequent work, convincingly argued that in the confining phase the $Z_N$ symmetry is spontaneously broken, the breaking being related to the confinement phenomenon. We will have more to say about this later.

The physical considerations given above can be put on a firmer formal basis. In particular, one can explicitly construct the $Z_N$ generator as
well as the order parameter associated with it, the operator that creates the magnetic vortex. We start this discussion by considering Abelian theories, where things are simpler and are under full control.

2.1 Abelian theories

In the U(1) case the homotopy group is \( Z \) and, therefore, the magnetic symmetry is U(1) rather than \( Z_N \). Identifying the relevant charge is, in fact, an absolutely straightforward task. It is nothing but the magnetic flux through the equal time plane, with the associated conserved current being the dual of the electromagnetic field strength

\[
\Phi = \int d^2 x B(x), \quad \partial \mu \tilde{F}_\mu = 0. \tag{2}
\]

The current conservation is insured by the Bianchi identity. It may come as a surprise that we are seriously considering a current whose conservation equation is an “identity.” However “identity” is a relative thing. The conservation equation is trivial only because we have written the components of the field strength tensor in terms of the vector potential. The introduction of the vector potential is none other than the potentiation of the conserved current \( \tilde{F}_\mu \), that is explicit solution of the conservation equation. In exactly the same way one can potentiate any conserved current, and such a potentiation will turn the pertinent conservation equation into identity. Thus \( \tilde{F}_\mu \) has exactly the same status as any other local conserved gauge invariant current, and should be treated as such.

Once we have the current and the charge, we also know the elements of the symmetry group. A group element of the U(1) magnetic symmetry group is

\[
W_\alpha(\infty) = \exp\{i\alpha \Phi\} \tag{3}
\]

for the arbitrary value of \( \alpha \). The notation \( W \) is not accidental here, since the group element is indeed a large spatial Wilson loop defined on a contour that encloses the whole system.

The question that might bother us is whether this group acts nontrivially on any of the local physical observables in our theory. The obvious gauge invariant observables like \( B \) and \( E \) commute with \( W \). However, there is another set of local gauge invariant observables in the theory which do indeed transform nontrivially under the action of \( W \). Consider, following ’t Hooft, the operator of the “singular gauge transformation,”

\[
V(x) = \exp \frac{i}{g} \int d^2 y \left[ \epsilon_{ij} \frac{(x - y)_i}{(x - y)_j^2} E_j(y) + \Theta(x - y) J_0(y) \right], \tag{4}
\]

where \( \Theta(x - y) \) is the polar angle function and \( J_0 \) is the electric charge density of whatever matter fields are present in the theory. The cut discontinuity in the function \( \Theta \) looks bothersome; but it is, in fact, not physical and completely harmless. The gauge function jumps across the discontinuity by \( 2\pi/g \), but since the only dynamical fields in the theory have
charges that are integer multiples of \( g \), the discontinuity is not observable. The cut can be chosen parallel to the horizontal axis. Using the Gauss’ law constraint this can be cast in a different form, which we find more convenient for our discussion,

\[
V(x) = \exp \frac{2\pi i}{g} \int_C dy^i \epsilon_{ij} E_i(y),
\]  

where the integration goes along the cut of the function \( \Theta \) which starts at the point \( x \) and goes to spatial infinity. In this form it is clear that the operator does not depend on where precisely one chooses the cut to lie. To see this, note that changing the position of the cut \( C \) to \( C' \) adds

\[
\frac{2\pi}{g} \int_S d^2 x \partial_i E^i
\]

to the phase (here \( S \) is the area bounded by \( C - C' \)). In the theory under consideration only charged particles where charges multiples of \( g \) are present. Therefore, the charge within any closed area is a multiple integer of the gauge coupling \( \int_S d^2 x \partial_i E^i = gn \) and the extra phase factor is always unity. The only point in space where the action of \( V(x) \) on any physical state is nontrivial is the point \( x \). Therefore, the field \( V(x) \) acts like any other local field. With a little more work one can prove not only that \( V(x) \) is a local field, in the sense that it commutes with any other local gauge invariant operators \( O(y), x \neq y \), but also that it is a \textit{bona fide} Lorentz scalar.

The physical meaning of the operator \( V \) is very simple. Calculating its commutator with the magnetic field \( B \) we find

\[
V(x)B(y)V^\dagger(x) = B(y) + \frac{2\pi}{g} \alpha s^2(x - y).
\]

Thus, \( V \) creates a pointlike magnetic vortex of flux \( 2\pi/g \). This commutator also tells us that

\[
W_\alpha^\dagger V(x)W_\alpha = e^{i\frac{2\pi \alpha}{g}} V(x),
\]

and, therefore, \( V \) is indeed a local eigenoperator of the magnetic U(1) symmetry group.

Equations (5) and (6) formalize the physical arguments of ‘t Hooft in the Abelian case. We have explicitly constructed the generator and the local order parameter of the magnetic symmetry.

### 2.2 Non-Abelian theories at weak coupling

Let us now move on and consider an analogous construction for the non-Abelian theories. Ultimately we are interested in pure Yang-Mills theory. However, it is quite illuminating to start with the theory with an adjoint Higgs field and take the decoupling limit later. For simplicity we discuss
the SU(2) gauge theory. Consider the Georgi-Glashow model – SU(2) gauge theory with the adjoint Higgs field,
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} (\mathcal{D}_\mu H^b)^2 + \tilde{\mu}^2 H^2 - \tilde{\lambda} (H^2)^2,
\]
where
\[
\mathcal{D}_\mu H^b = \partial_\mu H^a - gf^{abc} A_\mu^b H^c.
\]
At large and positive $\tilde{\mu}^2$ the model is weakly coupled. The SU(2) gauge symmetry is broken down to U(1) and the Higgs mechanism takes place. Two gauge bosons, $W^\pm$, acquire a mass, while the third one, the “photon,” remains massless to all orders in perturbation theory. The theory in this region of the parameter space very much resembles electrodynamics of the vector charged field. The Abelian construction can, therefore, be repeated. The SU(2) gauge invariant analog of the conserved dual field strength is
\[
\tilde{F}^\mu = \frac{1}{2} \epsilon^{\mu\nu\lambda} F^{a\lambda} n^a - \frac{1}{g} \epsilon^{\mu\nu\lambda} \epsilon^{abc} n_a (D_\nu n)^b (D_\lambda n)^c,
\]
where $n^a \equiv \frac{H^a}{|H|}$ is the unit vector in the direction of the Higgs field. Classically this current satisfies the conservation equation
\[
\partial^\mu \tilde{F}_\mu = 0.
\]
The easiest way to see this is to choose a unitary gauge of the form $H^a(x) = H(x) \delta^a_3$. In this gauge $\tilde{F}$ is equal to the Abelian part of the dual field strength in the third direction in the color space. Its conservation then follows from the Bianchi identity.

Thus, classically the theory has a conserved U(1) magnetic charge $\Phi = \int d^2 x \tilde{F}_0$, just like QED. However the unitary gauge can not be imposed at the points where $H$ vanishes, which necessarily happens in the core of the 't Hooft-Polyakov monopole. It is well-known, of course, that the monopoles are the most important nonperturbative configurations in this model. Their presence leads to a nonvanishing small photon mass, as well as confinement of the charged gauge bosons, with a tiny nonperturbative string tension. As far as the monopole effects on the magnetic flux are concerned, their presence leads to a quantum anomaly in the conservation equation. A result, only the discrete $Z_2$ subgroup of the transformation group generated by $\Phi$ remains unbroken in quantum theory. The detailed discussion of this anomaly, the residual $Z_2$ symmetry and their relation to monopoles is given in Ref. 15. The nonanomalous $Z_2$ magnetic symmetry transformation is generated by the operator
\[
U = \exp\{\frac{i}{2} \Phi\}.
\]
The order parameter for the magnetic $Z_2$ symmetry is constructed analogously to QED as a singular gauge transformation generated by the gauge invariant electric charge operator,
\[
J^\mu = \epsilon^{\mu\nu\lambda} \partial_\nu (\tilde{F}^{a\lambda} n^a), \quad Q = \int d^2 x J_0(x).
\]
Explicitly

\[
V(x) = \exp \left( \frac{i}{g} \int d^2 y \left[ \epsilon_{ij} \frac{(x - y)}{(x - y)^2} n^a(y) E_i^a(y) + \Theta(x - y) J_0(y) \right] \right)
\]

\[
= \exp \left( \frac{2\pi i}{g} \int_C dy^i \epsilon_{ij} n^a E_i^a(y) \right).
\]  

(14)

One can think of it as of a singular SU(2) gauge transformation with the field dependent gauge function

\[
\lambda^a(y) = \frac{1}{g} \Theta(x - y)n^a(y).
\]  

(15)

This field dependence of the gauge function ensures the gauge invariance of the operator \(V\). Just like in QED, it can be shown \([4, 9]\) that the operator \(V\) is a local scalar field. Again like in QED, the vortex operator \(V\) is a local eigenoperator of the Abelian magnetic field \(B(x) = \tilde{F}_0\).

\[
[V(x), B(y)] = -\frac{2\pi i}{g} V(x) \delta^2(x - y).
\]  

(16)

That is to say, when acting on a state it creates a pointlike magnetic vortex which carries a quantized unit of the magnetic flux.

The magnetic \(Z_2\) acts on the vortex field \(V\) as a phase rotation by \(\pi\)

\[
e^{i \Phi} V(x) e^{-i \Phi} = -V(x).
\]  

(17)

This is an explicit realization of the magnetic \(Z_2\) symmetry in the Georgi-Glashow model.

2.3 Gluodynamics

From the Georgi-Glashow model we can easily get to pure Yang-Mills theory. This is achieved by smoothly varying the \(\mu^2\) coefficient in the Lagrangian, so that it becomes negative and eventually arbitrarily large. In this limit the Higgs field has a large mass and, therefore, decouples leaving pure gluodynamics behind. It is well known \([5]\) that in this model the weakly coupled Higgs regime and the strongly coupled confining regime are not separated by a phase transition. The limit of pure Yang-Mills in this model is, therefore, smooth.

In the limit of pure Yang-Mills the expressions \([10], [11], \) and \([12]\) have to be taken with care. \([12]\) When the mass of the Higgs field is very large, the configurations that dominate the path integral of the theory are those with a very small value of the modulus of the Higgs field \(|H| \propto 1/M\). The modulus of the Higgs field, in turn, controls fluctuations of the unit vector \(n^a\), since the kinetic term for \(n\) in the Lagrangian is \(|H|^2 (D_a n)^2\). Thus, as the mass of the Higgs field increases, the fluctuations of \(n\) grow both, in amplitude and frequency, and the magnetic field operator \(B\) as defined in Eq. \([14]\) fluctuates wildly. Of course, this situation is not
unusual. It happens whenever one wants to consider (in the effective low-energy theory) an operator which explicitly depends on fast, high-energy variables. The standard way of dealing with it is to integrate over the fast variables. There could be two possible outcomes of this “integrating out” procedure. Either the operator in question becomes trivial (if it depends strongly on the fast variables), or its reduced version is well-defined and regular in the low-energy Hilbert space. The “magnetic field” operator $B$ in Eq. (10) is obviously of the first type. Since in the limit of pure Yang-Mills all orientations of $n^a$ are equally probable, integrating over the Higgs field at fixed $A_\mu$ will lead to vanishing $B$. However, what interests us is not so much the magnetic field but, rather, the generator of the magnetic $Z_2$ transformation. It is actually instructive to consider an operator that performs the $Z_2$ transformation not everywhere in space, but only inside a contour $C$

$$U_C = \exp \left\{ i \frac{g}{4} \int_S d^2 x B(x) \right\}, \quad (18)$$

with the area $S$ being bounded by $C$. In the limit of gluodynamics we are led to consider the operator

$$U_C = \lim_{H \to 0} \int Dn^a \exp \left\{ - |H|^2 (\bar{D}n_a)^2 \right\} \times \exp \left\{ i \frac{g}{4} \int_C d^2 x \left( \epsilon_{ij} F^n_{ij} n^a - \frac{1}{g} \epsilon^{abc} n_a (D_i n)^b (D_j n)^c \right) \right\}. \quad (19)$$

The weight for the $n$ integration is the kinetic term for the isovector $n^a$. As was noted before, the action does not depend on $n^a$ in the Yang-Mills limit since $H^2 \to 0$. However, the first term in Eq. (19) regulates the path integral and we keep it for this reason. This operator may look somewhat unfamiliar, at first sight. However, in a remarkable paper (see Ref. 18) Diakonov and Petrov showed that Eq. (19) is equal to the trace of the fundamental Wilson loop along the contour $C$.

$$U_C = W_C \equiv \text{Tr} \mathcal{P} \exp \left\{ i g \int_C d^2 x \epsilon_i A_i \right\}. \quad (20)$$

Taking the contour $C$ to run at infinity, we see that in gluodynamics the generator of the magnetic $Z_2$ symmetry is the fundamental spatial Wilson loop along the boundary of the spatial plane.$^1$

$^1$We note that Diakonov and Petrov had to introduce a regulator to define the path integral over $n$. The regulator they required was of precisely the same form as in Eq. (19).

$^2$There is a slight subtlety here that may be worth mentioning. The generator of a unitary transformation should be a unitary operator. On the other hand, the trace of the fundamental Wilson loop is not unitary. Therefore, one should strictly speaking consider, instead, a unitarized Wilson loop $\tilde{W} = \frac{W}{\sqrt{WW^\dagger}}$. However, the factor between the two operators $\sqrt{WW^\dagger}$ is an operator that is only sensitive to the field behavior at infinity. It commutes with all physical local operators $O(x)$ unless $x \to \infty$. In this aspect, it is very different from the Wilson loop itself, which has a nontrivial commutator with vortex operators $V(x)$ at all values of $x$. Since
Of course, one does not have to go through the exercise with the Georgi-Glashow model in order to show that the fundamental Wilson loop generates a symmetry. Instead one can directly consider the commutator

\[ [W, H] = \lim_{C \to \infty} \oint_C dx_i \text{Tr} \mathcal{P} E_i(x) \exp \left\{ i \oint_{C(x,x)} dy_i A_i(y) \right\} \to_{C \to \infty} 0. \] (21)

Here the integral in the exponential on the right-hand side starts and ends at the point of insertion of the electric field. For a finite contour \( C \) the commutator does not vanish only along the contour itself – it does not contain any bulk terms. Making the contour \( C \) go to infinity and assuming, as usual, that in the theory with a finite mass gap at infinity no physical modes are excited we conclude that the commutator of \( H \) with infinitely large Wilson loop vanishes.

Next we consider the vortex operator Eq. (14). Again, in order to find the limit of pure Yang-Mills we have to integrate this expression over the orientations of the unit vector \( n_a \). This integration is equivalent to averaging over the gauge group. Following Ref. 18 one can write \( n_a \) in terms of the SU(2) gauge transformation matrix \( \Omega \),

\[ \vec{n} = \frac{1}{2} \text{Tr} \Omega \tau^1 \tau^3. \] (22)

The vortex operator in the limit of pure gluodynamics then becomes

\[ \tilde{V}(x) = \int D\Omega \exp \frac{2\pi i}{g} \int_C dy_i \epsilon_{ij} \text{Tr} \Omega \epsilon_j \tau^1 \tau^3. \] (23)

This form makes it explicit that \( \tilde{V}(x) \) is defined as the gauge singlet part of the following, apparently non-gauge-invariant operator:

\[ V(x) = \exp \frac{2\pi i}{g} \int_C dy_i \epsilon_{ij} E^3_i(y). \] (24)

the correlators of all gauge invariant local fields in the pure Yang-Mills theory are massive and, therefore, short range, the operator \( \sqrt{W(C)} W^\dagger(C) \) at \( C \to \infty \) must be a constant operator on all finite-energy states. Therefore, the difference between \( W \) and \( \tilde{W} \) is a trivial constant factor and we will not bother with it in what follows.

3Note that the nonvanishing of the commutator at finite \( C \) is of exactly the same nature as for any other “conserved charge,” which is defined as an integral of local charge density

\[ Q = \lim_{C \to \infty} \int_{C \times C} d^2x \rho(x). \] (25)

The commutator of such a charge with the Hamiltonian also contains surface terms, since the charge density \( \rho \) itself never commutes with the Hamiltonian. Rather, the commutator is a total derivative. For a conserved charge, due to the continuity equation, this surface term is equal to the circulation of the spatial component of the current

\[ [Q, H] = \to_{C \to \infty} \oint dx_i j_i. \] (26)

Again, the vanishing of this term is the consequence of the vanishing of the physical fields at infinity in the theory with a mass gap. When the charge is not conserved, the commutator contains a bulk term, in addition to the surface term. It is the absence of such bulk terms that is a unique property of the conserved charge. The same conclusion is reached if, rather than considering the generator of the algebra, one considers the commutator of the group element for either continuous or discrete symmetry groups. Therefore, the commutator in Eq. (23) indeed tells us that \( W \) is a conserved operator.
The integration over $\Omega$ obviously projects out the gauge singlet part of $V$. In the present case, however, this projection is redundant. This is because even though $V$ itself is not gauge invariant, when acting on a physical state it transforms it into another physical state. By physical states we mean the states which satisfy Gauss’ constraint in pure Yang-Mills theory. This property of $V$ was noted by ’t Hooft.\cite{1} To show this let us consider $V(x)$ as defined in Eq. (24) and its gauge transform $V_\Omega = \Omega V\Omega^\dagger$ where $\Omega$ is an arbitrary nonsingular gauge transformation operator. The wave functional of any physical state depends only on gauge invariant characteristics of the vector potential, i.e. only on the values of the Wilson loops over all possible contours,

$$\Psi[A_i] = \Psi[[W(C)]]$$

Acting on this state by the operators $V$ and $V_\Omega$, respectively, we obtain

$$V|\Psi\rangle = \Psi_V[A_i] = \Psi[[VW(C)V^\dagger]],$$

$$V_\Omega|\Psi\rangle = \Psi_{V_\Omega}[A_i] = \Psi[[V_\Omega W(C)V_\Omega^\dagger]].$$

However, it is easy to see that the action of $V(x)$ and $V_\Omega(x)$ on the Wilson loop is identical – they both multiply it by a phase belonging to the center of the group if $x$ is inside $C$, and do nothing otherwise. Therefore,

$$V|\Psi\rangle = V_\Omega|\Psi\rangle$$

for any physical state $\Psi$. Thus, we have

$$\Omega V|\Psi\rangle = \Omega V\Omega^\dagger|\Psi\rangle = V|\Psi\rangle$$

where the first equality follows from the fact that any physical state is invariant under the action of any gauge transformation $\Omega$, while the second equality follows from Eq. (27). This equation is nothing but the assertion that the state $V|\Psi\rangle$ is physical, i.e. invariant under any nonsingular gauge transformation.

Thus, we have proved that when acting on a physical state the vortex operator creates another physical state. Given such an operator, the gauge invariant projection only affects its matrix elements between unphysical states. Since we are interested only in calculating correlators of $V$ between physical states, the gauge projection is redundant and we can freely use $V$ rather than $\tilde{V}$ to represent the vortex operator.

The formulae in this section can be straightforwardly generalized to the SU($N$) gauge theories. Once again one can start with the Georgi-Glashow-like model, where SU($N$) is Higgsed to U(1)($N-1$). The construction of the vortex operator and the generator of $Z_N$ in this case is very similar; necessary details are given in Ref. 15. Taking the mass of the Higgs field to infinity again projects the generator onto the trace of the fundamental Wilson loop. The vortex operator can be taken as

$$V(x) = \exp\left\{\frac{4\pi i}{gN} \int_C dy^i \epsilon_{ij} \text{Tr}(YE_i(y))\right\}$$

This is not a trivial statement, since a generic non-gauge-invariant operator has nonvanishing matrix elements between the physical and unphysical sectors.
where the hypercharge generator $Y$ is defined as

$$Y = \text{diag}(1, 1, ..., -(N - 1)),$$

and the electric field is taken in the matrix notation $E_i = \lambda^a E^a_i$. Here $\lambda^a$ stand for the SU($N$) generator matrices in the fundamental representation.

To summarize this section, we have established two important facts. First, SU($N$) gauge theories in 2+1 dimensions have a global $Z_N$ magnetic symmetry. The generator of this magnetic symmetry group is the fundamental Wilson loop around the spatial boundary of the system. Second, this symmetry has a local order parameter. This order parameter is a local gauge invariant scalar field which creates a magnetic vortex of fundamental flux.

The next question we should ask is whether this global symmetry is at all relevant for low-energy dynamics. In the next section we will show that this is indeed the case. We will calculate the expectation value of $V$ in confining and nonconfining situations and will show that confinement is rigidly related to the spontaneous breaking of the magnetic symmetry.

3 The vacuum realization of the magnetic symmetry, the effective Lagrangian and confinement. Abelian theory

Again, we start our discussion with the Abelian theory. Consider a U(1) gauge theory with scalar matter field

$$L = -\frac{1}{4} F^2 + |D_\mu \phi|^2 - M^2 |\phi|^2 - \lambda (\phi^* \phi)^2.$$  

(31)

Depending on the values of the coupling constant this theory can be either in the Coulomb phase with the massless photon and logarithmically confined charges, or in the Higgs phase, which is massive – all electric charges are screened.

3.1 Realizations of the magnetic symmetry

Let us start by calculating $\langle V \rangle$ in the Coulomb phase. This can be done using the standard weak coupling perturbation theory. The expectation value of $V$ is given by the following expression:

$$\langle V(x) \rangle = N^{-1} \lim_{T \to \infty} \int dA_0(0) e^{iTH} e^{2 \pi i \gamma \epsilon \epsilon} \int_C d\gamma^i \epsilon_{ij} E_i(x) e^{i \int A_0[\partial_i E_i - J_0]} e^{iTH} |0 \rangle.$$  

(32)

Here $N^{-1}$ is the normalization factor – the usual vacuum-to-vacuum amplitude, $|0 \rangle$ is the perturbative Fock vacuum and the integral over $A_0$ is the standard representation of the projection operator which projects the Fock vacuum $|0 \rangle$ onto the gauge invariant subspace which satisfies Gauss’
law. As usual, discretizing time, introducing resolution of identity at every time slice and integrating over $E_i$ in the phase space path integral, this expression can be rewritten as a path integral in the field space. The result is easy to understand – it is almost the same as for the vacuum-to-vacuum amplitude, except that at time $t = 0$ the spatial derivative of the scalar potential $A_0$ is shifted by a $c$-number field due to the presence of the vortex operator,

$$\langle V(x) \rangle = N^{-1} \int \mathcal{D}A_\mu \exp \left[ -\frac{1}{4} \int d^3y [\tilde{F}_\mu(y) - \tilde{f}_\mu(y - x)]^2 + L_{Higgs} \right].$$

(33)

The $c$-number field $\tilde{f}_\mu$ is the magnetic field of an infinitely thin magnetic vortex which terminates at the point $x$. One can view it as the Dirac string of a (three-dimensional Euclidean) magnetic monopole,

$$\tilde{f}_0 = \tilde{f}_2 = 0 \quad \tilde{f}_1(y) = \frac{2\pi}{g} \theta(y_1) \delta(y_2) \delta(y_3).$$

(34)

Thus, at weak coupling we have to find the solution of the classical equations of motion following from the action with the external source Eq. (33). The nature of this solution is clear: it is just the Dirac monopole. The action of this solution is infrared-finite, since the contribution of the Dirac string (which normally would be linearly infrared-divergent) is cancelled by the external source,

$$\langle V \rangle = \exp\{-S_{cl}\},$$

(35)

with

$$S_{cl} = \frac{\Lambda}{g^2}.$$  

(36)

Here $\Lambda$ is an ultraviolet cutoff which has to be introduced since the action of the pointlike monopole diverges in the ultraviolet. This ultraviolet divergence is benign since it can be eliminated by the multiplicative renormalization of the vortex operator $\tilde{f}_\mu$. The important point is that since there is no divergence in the infrared, the expectation value of $V$ is nonvanishing. Thus, we conclude that in the Coulomb phase of QED the magnetic symmetry is spontaneously broken.

The spontaneous breaking of a continuous symmetry must be accompanied by a massless Goldstone particle. Indeed, in QED such a particle exists – this is the massless photon. The matrix element of the magnetic current between the vacuum and the one-photon state is

$$\langle 0 | \tilde{F}_\mu | k_i \rangle = Z^{1/2}(0) k_\mu.$$

(37)

where $Z(0)$ is the on-shell photon wave function renormalization.

This is the standard form of the matrix element of a spontaneously broken current, with $Z(0)$ playing the role of $f_0^2$.

Let us now perform the same calculation in the Higgs phase. The path integral representation in Eq. (33) is still valid. However, the classical solution that dominates this path integral is now very different. Since
in the Higgs phase the photon has a nonzero mass $\mu$ the classical action of the three-dimensional monopole in the superconducting medium is linearly divergent in the infrared. Essentially, the magnetic flux that emanates from the monopole can not spread out in space (time) but is rather concentrated inside a flux tube of the thickness $1/\mu$ which starts at the location of the monopole and goes to a spatial boundary at infinity. The action of such a field configuration is proportional to the linear size of the system and diverges in the thermodynamic limit. As a result, the expectation value of $V$ in the Higgs phase vanishes,

$$\langle V \rangle = e^{-L} \rightarrow L \rightarrow \infty 0.$$  \hfill (38)$$

Similarly, the vortex field correlator is given in terms of the classical energy of a monopole-antimonopole pair in the superconductor. The Euclidean action of this configuration is proportional to the distance between the monopole and the antimonopole and, therefore, the correlator of the $V$ decays exponentially,

$$\langle V^*(x)V(y) \rangle \sim e^{-M_V|x-y|},$$  \hfill (39)$$

with $M_V \propto 1/g^2$ being the mass of the ANO vortex.

This simple calculation can be improved perturbatively. In the next-to-leading order one has to calculate the determinant of the Schrödinger operator of a particle in the field of a monopole, and this corrects the value of the mass $M_V$. Higher orders in perturbation theory can be calculated too; we will not pursue this calculation here.

The main lesson is that the expectation value of the order parameter vanishes in the Higgs phase; thus, the magnetic symmetry is unbroken.

### 3.2 The low-energy effective Lagrangian and the logarithmic confinement of electric charges

Thus we see that the Coulomb–Higgs phase transition can be described as that due to the restoration of the magnetic $U(1)$ symmetry, the pertinent local order parameter being the vortex operator $V$. It should then be true that the low-energy dynamics in the vicinity of the phase transition is described by an effective low-energy Lagrangian. For the $U(1)$ symmetry breaking such an effective Lagrangian can be immediately written as follows:

$$L = \partial_\mu V^* \partial^\mu V - \lambda(V^* V - \mu^2)^2.$$  \hfill (40)$$

Although this Lagrangian may seem a bit unfamiliar in the context of QED, a little thought convinces one that it indeed describes all the relevant light degrees of freedom of the theory. In the Coulomb phase, where $\langle V \rangle = \mu \neq 0$, the physical particles are interpolated by the phase and the radial part of $V$:

$$V(x) = \rho e^{i\chi}.$$  \hfill (41)$$

The phase $\chi$ is of course the massless Goldstone boson field, i.e. the photon. The fluctuation of the radial component $\rho - \mu$ is the lightest scalar
particle, which in this case is the lightest meson, or scalar positronium. In the Higgs phase, the field $V$ interpolates physical excitations – the ANO vortices. Of course, far from the phase transition the vortices are heavy and there are other, lighter excitations in the spectrum. The validity of this effective Lagrangian on the Higgs phase side is therefore limited to a narrow critical region where vortices are indeed the lightest particles.

The charged particles have not appeared in our discussion so far. Indeed in 2+1 dimensions the electrical charges are confined, and, therefore, we do not expect the charged fields to appear as basic degrees of freedom in the effective low-energy Lagrangian. However, our original purpose was precisely to understand the mechanism of confinement through studying the effective Lagrangian. Thus, if we are unable to identify the charged objects in this framework our program is doomed to failure. Fortunately it is not difficult to understand how the charged states are represented in the Lagrangian Eq. (40). The easiest way to do this is to identify the electric charge through the Maxwell equation,

$$J_\mu = \frac{1}{4} \epsilon_{\mu\nu\lambda} \partial_\nu \tilde{F}_\lambda.$$  \hfill (42)

The dual field strength $\tilde{F}_\mu$ is obviously proportional to the conserved U(1) current

$$\tilde{F}_\mu = \frac{2\pi}{g} i(V^* \partial V - h.c.).$$  \hfill (43)

The proportionality constant in this relation is dictated by the fact that in the Higgs phase the magnetic vortices that carry one unit of the U(1) charge, carry the magnetic flux of $2\pi/g$. The above two relations give

$$\frac{g}{\pi} J_\mu = i\epsilon_{\mu\nu\lambda} \partial_\nu (V^* \partial_\lambda V).$$  \hfill (44)

To calculate the electric charge we integrate the zeroth component of the current over the two-dimensional plane

$$\frac{g}{\pi} Q = \mu^2 \oint_{C \rightarrow \infty} dx_1 \partial_1 \chi.$$  \hfill (45)

Therefore, the electric charge is proportional to the winding number of the phase of the field $V$.

So, the charged states do appear in the low-energy description in a very natural way. A charged state is a soliton of $V$ with a nonzero winding number. This identification immediately tells us that the charged particles are logarithmically confined. Consider, for example, the minimal energy configuration in the sector with the unit winding number. This is a rotationally invariant hedgehog, Fig. 1, which, far from the soliton core, has the form

$$V(x) = \mu e^{i\theta(x)}.$$  \hfill (46)

Here $\theta(x)$ is the angle between the vector $x$ and one of the axes.
The self-energy of this configuration is logarithmically divergent in the infrared due to the contribution of the kinetic term
\[ E = \pi \mu^2 \ln(\lambda \mu^2 L^2). \] (47)
This is nothing but the electromagnetic self-energy of an electrically charged state associated with the logarithmic Coulomb potential in two spatial dimensions. Therefore, the logarithmic confinement of QED is indeed very easily and transparently seen at the level of the low-energy effective Lagrangian. In itself this is, perhaps, not such a big deal, since confinement in this model is a kinematic phenomenon: it is a direct consequence of the logarithmic behavior of the Coulomb potential. We will see later, however, that this low-energy picture also naturally generalizes to non-Abelian theories and easily accommodates linear confinement.

Before moving on to the non-Abelian theories, let me make two comments. First, in perturbative regime the couplings of the effective Lagrangian can be determined in terms of the couplings of the fundamental QED Lagrangian. To determine two constants in Eq. (40) one needs two matching conditions. One of them can be naturally taken as the coefficient of the infrared logarithm in the self-energy of a charged state. Matching Eq. (47) with the Coulomb self-energy gives
\[ \mu^2 = \frac{g^2}{8\pi^2}. \] (48)
The other coefficient is determined by requiring that the mass of the radial excitation \( \rho \) matches the mass of the scalar positronium, which to leading order in \( g \) is just \( 2M \). This condition gives
\[ \lambda = \frac{2\pi^2 M}{g^2}. \] (49)
The second comment is about the relation between the vortex operator as defined in Eq. (13) and the field \( V \) that enters the effective Lagrangian (40).
The vortex operator, as defined in Eq. (14) has a fixed length, whereas the field $V$ which enters the Lagrangian (40) is a conventional complex field. How should one understand that? First of all, at weak gauge coupling the quartic coupling in the dual Lagrangian is large, $\lambda \to \infty$. This condition freezes the radius of $V$ dynamically. In fact, even at a finite value of $\lambda$, if one is interested in the low-energy physics, the radial component is irrelevant as long as it is much heavier than the phase. Indeed, at weak gauge coupling the phase of $V$ (which interpolates the photon) is much lighter than all other excitations in the theory. Effectively at low energies Eq. (40) reduces to a nonlinear $\sigma$ model and one can identify the field $V$ entering Eq. (40) directly with the vortex operator of Eq. (14). However, it is well known that quantum mechanically the radial degree of freedom of the $\sigma$ model field is always resurrected. The spectrum of such theory always contains a scalar particle which can be combined with the phase into a variable-length complex field. The question is purely quantitative – how heavy is this scalar field relative to the phase?

Another way of expressing this is as follows. The fixed length field $V$ is defined at the scale of the ultraviolet cutoff in the original theory. To arrive at a low-energy effective Lagrangian one has to integrate over all quantum fluctuations down to some much lower energy scale. In the process of this integrating out, the field is “renormalized” and acquires a dynamical radial part. Then the mass of this radial part is equal to the mass of the lowest particle with the same quantum numbers in the original theory. That is why the parameters in Eq. (40) must be such that the mass of the radial part of $V$ would be equal to the mass of the lightest scalar positronium.

4 Non-Abelian theories

In this section I want to extend the construction of the low-energy effective Lagrangian discussed above to the non-Abelian theories and see how the realization of the magnetic symmetry in this Lagrangian is related to confinement. Before discussing specific models in detail, let me present a general argument which establishes that the spontaneous breaking of magnetic $Z_N$ in non-Abelian theories implies the area law behavior of the fundamental Wilson loop.

4.1 Broken $Z_N$ means confinement

As we discussed in the Sec. 2, the generator of the magnetic $Z_N$ in pure gluodynamics is the fundamental Wilson loop around the spatial boundary of the system. By the same token, the Wilson loop around a closed spatial contour $C$ generates the $Z_N$ transformation at the points inside the contour $C$. Let us imagine that the $Z_N$ symmetry is spontaneously broken in the vacuum and consider the expectation value of $W(C)$ in such a state $|0\rangle$. The expectation value $\langle 0|W(C)|0\rangle$ is nothing but the overlap of the vacuum state $|0\rangle$ and the state $|S\rangle$ which is obtained by acting on the
vacuum by $W(C)$, $|S⟩ = W(C)|0⟩$. If the symmetry is broken, the wave function $|0⟩$ depends explicitly on the degrees of freedom which are non-invariant under the symmetry transformation, and is peaked around some specific orientation of these variables in the group space. For simplicity let us think about all these non-invariant variables as being represented by the vortex field $V$. In the vacuum state the field $V$ has a nonvanishing vacuum expectation value (VEV) and points in some fixed direction in the internal space. On the other hand, in the state $|S⟩$ its direction in the internal space is different — rotated by $2\pi/N$ — at the points inside the area $S$ bounded by $C$, since at these points the field $V$ has been rotated by the action of $W(C)$. In the local theory with finite correlation length the overlap between the two states approximately factorizes in the product of the overlaps taken over the regions of space of linear dimension of the order of the correlation length $l$,

$$\langle 0|S⟩ = \Pi_x \langle 0_x|S_x⟩,$$  \hspace{1cm} (50)

where the label $x$ is the coordinate of the point in the center of a given small region of space. For $x$ outside the area $S$ the two states $|0_x⟩$ and $|S_x⟩$ are identical and, therefore, the overlap is unity. However, for $x$ inside $S$ the states are different and the overlap is a number $e^{-\gamma}$ smaller than unity. The number of such regions inside the area is obviously of the order of $S/l^2$ and, thus,

$$\langle W(C)⟩ = \exp\{-\gamma S\}.$$  \hspace{1cm} (51)

Therefore, in the broken phase the spatial Wilson loop has the area law behavior.

Now consider the unbroken phase. Again the average of $W(C)$ has the form of the overlap of two states which factorizes as in Eq. (50). Now, however, all observables non-invariant under $Z_N$ vanish in the vacuum. The action of the symmetry generator does not affect the state $|0⟩$. Hence, the state $|S⟩$ is locally exactly the same as the state $|0⟩$ except along the boundary $C$. Therefore, the only regions of space which contribute to the overlap are those which lay within one correlation length from the boundary. Thus,

$$\langle W(C)⟩ = \exp\{-\gamma P(C)\},$$  \hspace{1cm} (52)

where $P(C)$ is the perimeter of the curve $C$.

The crucial requirement for this argument to hold is the existence of a mass gap in the theory. If the theory contains massless excitations the factorization of the overlap does not hold, and so in principle even in the broken phase the Wilson loop can have a perimeter behavior. This indeed is the case in the Abelian theories.

We now turn to the discussion of low-energy effective theories. This will allow us to see an explicit realization of this general argument.

4.2 The Georgi-Glashow model

Let us again start with the Georgi-Glashow model. For simplicity all explicit calculations in this section will be performed for the $SU(2)$ gauge
theory. Generalization to the SU(N) group is not difficult and is discussed in Ref. 15.

Not much has to be done here to parallel the calculation of $\langle V \rangle$ of the previous section. The theory is weakly interacting, and all calculations are explicit. Choosing the unitary gauge $n^a = \delta^{a3}$, the perturbative calculation becomes essentially identical to that in the Coulomb phase of QED. The only difference is that the charged matter fields are vectors ($W_\mu^\pm$) rather than scalars ($\phi$), but this only enters at the level of the loop corrections. The nonperturbative monopole contributions are there, but they affect the value of $\langle V \rangle$ very little, since $\langle V \rangle \neq 0$ already in perturbation theory. Thus just like in the Coulomb phase of QED, $\langle V \rangle \neq 0$ and the magnetic symmetry is spontaneously broken. The real difference comes only when we ask ourselves what is the effective Lagrangian that describes the low-energy physics. Here the monopole contributions are crucial, since as we have seen before the U(1) magnetic symmetry of QED is explicitly (anomalously) broken by these contributions to $Z_2$. Therefore the effective Lagrangian must have extra terms which reduce the symmetry of the Eq. (40). The relevant effective Lagrangian is

$$L = \partial_\mu V^* \partial^\mu V - \lambda(V^*V - \mu^2)^2 + \zeta(V^2 + (V^*)^2).$$ (53)

The addition of this extra symmetry breaking term has an immediate effect on the mass of the “would be” photon – the phase of $V$. Expanding around $\langle V \rangle = \mu$ we see that the phase field now has a mass $m_{ph}^2 = 4\zeta$. This is consistent with the classical analysis by Polyakov – the monopole contributions turn the massless photon of QED into a massive (pseudo)scalar with an exponentially small mass, $m_{ph} \propto \exp(-4\pi M_W/g^2)$. As a matter of fact, for very weak coupling, when the modulus of $V$ can be considered as frozen, the Lagrangian (53) in terms of the phase $\chi$ reduces to Polyakov’s dual Lagrangian. The exact correspondence between the two is discussed in Ref. 15.

The explicit symmetry breaking causes a dramatic change in the topologically charged (soliton) sector. We know from Polyakov’s analysis that the charges in this model are confined by linear potential, as opposed to QED where confinement is logarithmic. In the effective Lagrangian description this is due to the explicit symmetry breaking term. The crucial point is that the vacuum of the theory is not infinitely degenerate $\langle V \rangle = e^{i\chi} \mu$ with arbitrary constant $\chi$, as in QED, but only doubly degenerate $\langle V \rangle = \pm \mu$. Thus, the lowest energy state in the nontrivial winding sector cannot be a hedgehog. In the hedgehog configuration the field $V$ at each point at spatial infinity points in a different direction in the internal space. This is OK if all these directions are minima of the potential. Then the total energy of the configuration comes from the kinetic term, and as we saw, is logarithmic. However, now the potential has only two minima. Thus, the hedgehog field is far from the vacuum everywhere in space. The energy of such a state diverges as the volume of the system, $E \propto g^2 m^2 L^2$. Clearly, to minimize the energy in a state with a nonzero winding, the system must be in one of the two vacuum states in as large
a region of space as possible. However, since the field has to wind when one goes around the position of a soliton even at an arbitrarily large distance, \( V \) cannot be aligned with the vacuum everywhere at infinity. The best bet for a system is, therefore, to choose a string-like configuration, Fig. 2. The phase of \( V(x) \) deviates from 0 (or \( \pi \)) only inside a strip of width \( d \sim 1/m_{ph} \) stretching from the location of the soliton (charge) to infinity. The energy of such a configuration diverges only linearly with the dimension \( L \). In fact, a back-of-the-envelope estimate with the effective Lagrangian \((53)\) gives the following energy of such a confining string:

\[
E \propto g^2 m_{ph} L .
\]

Clearly the energy of a soliton and an anti-soliton separated by a large distance \( R \) is \( E = \sigma R \), with the string tension \( \sigma \propto g^2 m_{ph} \).

This is a simple picture of confinement in the effective Lagrangian approach in the weakly coupled regime.

The preceding discussion pertains to confinement of “adjoint” color charges. So far we have been considering topological solitons with the unit winding, which corresponds to the charge of the massive \( W^\pm \) bosons, or “massive gluons." It should be noted that the notion of the adjoint string tension is not an absolute one. So far, our discussion neglected the fact that the solitons have a finite core energy and, therefore, in principle can be created in pairs from the vacuum. Thus, the soliton-antisoliton interaction at a distance \( R \) can be screened by creating such a pair, if \( R \) is big enough. The distance at which the string breaking occurs can be estimated from the energy balance between the energy stored in the string \( E_S = \sigma_{Ad} R \) and the core energy of the soliton-antisoliton pair, which is twice the mass of the \( W \)-boson, \( 2M_W \), in our model,

\[
g^2 m_{ph} R = 2M_W .
\]
The distance at which the string breaks is
\[ R_{\text{breaking}} \propto \frac{M_W}{g^2} \frac{1}{m_{ph}}. \] (55)
Since the width of the string is of the order of \( 1/m_{ph} \), and \( M_W \gg g^2 \) at
the weak coupling, the length of the string is indeed much greater than
its width. One can therefore sensibly talk about a well formed adjoint
confining string.

As opposed to the adjoint string tension, the concept of the funda-
mental string tension is sharply defined. This is because the theory does
not contain particles with fundamental charge and, thus, an external funda-
damental charge can not be screened. To discuss confinement of external
fundamental charges we have to learn how to deal with half-integer wind-
ings. Imagine adding some extra very heavy fields in the fundamental
representation to the Georgi-Glashow model. The quanta of these fields
will carry a half-integer “electric” charge \( \frac{1}{2} \) and will be confined with
a different string tension than \( W^\pm \). To calculate this string tension we
should consider the Abelian Wilson loop with a half-integer charge. We
will now do it in the effective theory framework.

Let us first consider a space-like Wilson loop. As discussed in the
previous section, this operator is closely related to the generator of the
magnetic \( Z_2 \). In fact, \( W(C) \) is nothing but the operator that performs
the \( Z_2 \) transformation inside the area bounded by the contour \( C \).

It is straightforward to write down an operator in the effective theory,
in terms of the field \( V \), that has the same property,
\[ W(C) = e^{i \pi \int_S d^2x P(x)}. \] (56)
Here \( S \) is the surface bounded by the contour \( C \), and \( P \) is the momentum
operator conjugate to the phase of \( V \). In terms of the radius and phase
of \( V \) the path integral representation for calculating the vacuum average
of this Wilson loop is
\[ \langle W(C) \rangle = \int DV \exp \left\{ i \int d^3x \rho^2 (\partial_\mu X - j_\mu^S)^2 + (\partial_\mu \rho)^2 - U(V) \right\}. \] (57)
where \( U(V) \) is the \( Z_2 \) invariant potential of Eq. (53). The external current
\( j_\mu^S(x) \) does not vanish only at points \( x \) which belong to the surface \( S \) and
is proportional to the unit normal \( n_\mu \) to the surface \( S \). Its magnitude is
such that when integrated in the direction of \( n \) it is equal to \( \pi \). These
properties are conveniently encoded in the following expression:
\[ \int_T dx_\mu j_\mu^S(x) = \pi n(T, C). \] (58)
Here \( T \) is an arbitrary closed contour, and \( n(T, C) \) is the linking number between two closed curves \( T \) and \( C \).

\footnote{Note that here we are dealing with the path integral representation, and thus the contour \( C \) and the surface \( S \) are embedded into a three dimensional Euclidean space. The linking number between two curves is also defined in three dimensions.}
Figure 3: The external current $j_\mu$ which creates a pair of static fundamental charges in the effective Lagrangian description.

This path integral representation follows immediately if we note that the conjugate momentum $P$ is $P(x) = 2\rho^2(x)\partial_0\chi(x)$ at some fixed time $t$. This accounts for the linear in $\partial_\mu \chi$ term in the exponential \( \text{(57)} \). The constant term $j^2$ arises due to the standard integration over the conjugate momenta in passing to the path integral representation.

The path integral representation was constructed for the spatial Wilson loops. However, the expression \( \text{(57)} \) is completely covariant, and in this form it is valid for time-like Wilson loops as well. It is important to note that, although the expression for the current depends on the surface $S$, the Wilson loop operator depends only on the contour $C$ that bounds this surface. A simple way to see this is to observe that a change of variables $\chi \rightarrow \chi + \pi$ in the volume bounded by $S + S'$ leads to the change $j_\mu^S \rightarrow j_\mu^{S'}$ in Eq. \( \text{(57)} \). The potential is not affected by this change since it is globally $\mathbb{Z}_2$ invariant. Therefore, the operators defined with $S$ and $S'$ are completely equivalent.

To calculate the energy of a pair of static fundamental charges at the points $A$ and $B$ we have to consider a time-like fundamental Wilson loop of an infinite time dimension. This corresponds to time-independent $j_\mu$ which does not vanish only along a spatial curve $G$ (in the equal-time cross section) connecting the two points and pointing in the direction normal to this curve, Fig.3. The shape of the curve itself does not matter, since changing the curve without changing its endpoints is equivalent to changing the surface $S$ in Eq. \( \text{(57)} \).

In the classical approximation the path integral \( \text{(57)} \) is dominated by a static configuration of $V$. To determine it we have to minimize the energy on the static configurations in the presence of the external current $j_\mu$. The qualitative features of the minimal energy solution are quite clear. The effect of the external current is to flip the phase of $V$ by $\pi$ across the curve $G$, as is expressed in Eq. \( \text{(58)} \). Any configuration that does not have this behavior will have the energy proportional to the length of $G$ and to the ultraviolet cutoff scale. Recall that the vacuum in the theory at hand is doubly degenerate. The sign change of $V$ transforms one vacuum configuration into another. Therefore, the presence of $j_\mu$ requires that on the opposite sides of the curve $G$, immediately adjacent to $G$, there should be different vacuum states. It is clear, however, that far away from $G$ in either direction the field should approach the same vacuum state,
otherwise the energy of the configuration diverges linearly in the infrared. The phase of $V$ has to make half a wind somewhere in space to return to the same vacuum state far below $G$, as the vacuum state that exists far above $G$. If the distance between $A$ and $B$ is much larger than the mass of the lightest particle in the theory, this is achieved by having a segment of a domain wall between the two vacua connecting the points $A$ and $B$. Clearly to minimize the energy, the domain wall must connect $A$ and $B$ along a straight line. The energy of such domain wall is proportional to its length, and, therefore, the Wilson loop has the area law behavior. The minimal energy solution is schematically depicted in Fig. 4.

We see that the string tension for the fundamental string is equal to the tension of the domain wall which separates the two vacua in the theory. This relation has been discussed a long time ago by 't Hooft. Parametrically, this string tension is clearly the same as the adjoint one,

$$\sigma_f \propto g^2 m_{ph},$$

although the proportionality constant is different. We will briefly discuss the relation between the adjoint and the fundamental string tensions in the next subsection.

Note that the fundamental string is an absolutely stable topological object in the $Z_2$ invariant theory: the domain wall. It can not break, if one makes the distance between the two charges larger. In the effective theory it is also obvious since there are no point-like (particle-like) objects in the theory on which the domain wall can terminate since there are no dynamical objects with half-integer winding number.
We interpreted this calculation as the calculation of the potential between the fundamental adjoint charges – the time-like Wilson loop. However, in the Euclidean formulation there is no difference between the time-like and space-like Wilson loops. Interpreted in this way the calculation becomes a technical illustration to the argument given in the previous subsection: space-like Wilson loops have the area law if the magnetic $Z_2$ symmetry is spontaneously broken.

4.3 Gluodynamics

In the weakly interacting case the effective low-energy Lagrangian can be derived as explained above in perturbation theory, combined with the dilute monopole gas approximation. The more interesting regime is certainly that of strong coupling, which is essentially pure Yang-Mills theory. The luxury one has in 2+1 dimensions is that the weak and strong coupling regimes are not separated by a phase transition. This means that whatever global symmetries the theory has, their realization must be the same in the weakly coupled and strongly coupled vacua. The existence of the $Z_N$ symmetry is an exact statement not related to the weak coupling limit. Therefore, it is natural to expect that this symmetry must be non-trivially represented in the effective low-energy Lagrangian. It is plausible then that the low-energy dynamics at strong coupling is described by the same effective Lagrangian which encodes the spontaneous magnetic symmetry breaking. Certainly, the values of the coupling constants will be different in two regimes, but the qualitative behavior should be similar.

Strictly speaking, this is an assumption rather than theorem. That is where the difference between continuous and discrete symmetries comes in. Were magnetic symmetry continuous (like in the Abelian case), its spontaneous breaking would unambiguously determine the structure of the effective Lagrangian, whether the theory was weakly or strongly coupled. With discrete symmetries this is not necessarily the case. It could happen that even though the symmetry is broken, the “pseudo-Goldstone” particle is so heavy that it decouples from the low energy dynamics. For this to happen, though, the symmetry breaking would have to occur at a very high energy scale. In gluodynamics this is very unlikely, since the theory has only one dynamical scale. In fact, as was discussed in the beginning of this section, the fundamental string tension determines the scale at which $Z_N$ is broken. Wilson loops of linear size $l \leq (\sigma)^{-1/2}$ do not distinguish between confining and nonconfining behavior, and hence between the broken and the unbroken $Z_N$. The scale of $Z_N$ breaking is $(\sigma)^{1/2}$ which is precisely a natural dynamical scale of QCD.

Therefore, generally speaking, we expect that the pseudo-Goldstone meson stays among the low-energy excitations not only in the weakly coupled limit but also in pure gluodynamics. If this is the case the degrees of freedom that enter the effective Lagrangian in the weakly coupled phase also interpolate actual low-energy physical states of the strong coupling regime. That is to say, the radial and phase components of the vortex field $V$ must correspond to the lightest glueballs of pure SU($N$) Yang
Mills theory. We can check whether this is the case by considering the lattice gauge theory data \[10\] on the spectrum. The radial part of \(V\) is obviously a scalar and has quantum numbers \(0^{++}\). The quantum numbers of the phase are easily determined from the definition (14). Those are \(0^{-+}\).

The spectrum of pure SU(\(N\)) Yang Mills theory in 2+1 dimensions was extensively studied recently \[10\] on the lattice. The two lightest glueballs for any \(N\) are found to have exactly these quantum numbers. The lightest excitation is scalar, while the next one is a charge-conjugation-odd pseudo-scalar meson with the ratio of the masses roughly \(m_p/m_s = 1.5\) for any \(N\).

The situation, therefore, is likely the following. The low-energy physics of the SU(2) gauge theory is always described by the effective Lagrangian (53). In the weak coupling regime the parameters are given in Eqs. (48) and (49). Here the pseudo-scalar particle is the lightest state in the spectrum while the scalar is the first excitation. The pseudo-scalar meson presents an “almost massless photon.” The scalar meson is the massive Higgs particle. Moving towards the strong coupling regime (decreasing the Higgs VEV) one increases the pseudo-scalar mass while reducing the scalar mass; the parameters of the effective Lagrangian change accordingly. The crossover between the weak and strong coupling regimes occurs roughly where the scalar and the pseudo-scalar become degenerate. At strong coupling the degrees of freedom in the effective Lagrangian are the two lightest glueballs. They are still collected in one complex field which nontrivially represents the exact \(Z_2\) symmetry of the theory (or \(Z_N\) for SU(\(N\)).

Of course the spectrum of pure Yang-Mills theory, apart from the scalar and the pseudo-scalar glueballs, contains many other massive glueball states and those are not separated by a large gap from the two lowest ones. Application of the effective Lagrangian in the strong coupling regime has to be considered as qualitative rather than quantitative. At this qualitative level though, linear confinement is an immediate property of the effective Lagrangians of this type.

The fact that the masses of the scalar and pseudo-scalar particles are interchanged in gluodynamics relative to the weakly coupled regime leads \[10\] to some interesting qualitative differences. In particular, the structure of the confining string and the interaction between the strings differ in some important ways. Let me briefly discuss this aspect.

In the weakly coupled regime the phase of \(V\) is much lighter than the radial part. A cartoon of the fundamental string in this situation is depicted in Fig. 3.

The radial part \(\rho\) being very heavy practically does not change inside the string. The value of \(\rho\) in the middle of the string can be estimated...
from the following simple argument. The width of the region where \( \rho \) varies from its vacuum value \( \mu \) to the value \( \rho_0 \) in the middle, is of the order of the inverse mass of \( \rho \). In terms of the energy per unit length this variation costs
\[
\sigma_\rho \sim M(\mu - \rho_0)^2 + x \frac{m^2}{M} \rho_0^2,
\]
where \( x \) is a dimensionless number of order unity. The first term is the contribution of the kinetic term of \( \rho \) while the second contribution comes from the interaction term between \( \rho \) and \( \chi \) due to the fact that the value of \( \chi \) in the middle of the string differs from its vacuum value. Our notations are such that \( m \) is the mass of the pseudo-scalar particle and \( M \) is the mass of the scalar one. Minimizing this with respect to \( \rho_0 \) we find
\[
\rho_0 = \mu (1 - x \frac{m^2}{M^2}).
\]
Thus, even in the middle of the string the difference in the value of \( \rho \) and its VEV is of the second order in the small ratio \( m/M \). Correspondingly, the contribution of the energy density of \( \rho \) to the total energy density is also very small,
\[
\sigma_\rho \sim \frac{m}{M} \mu^2.
\]
This is to be compared with the total tension of the string which is contributed mainly by the pseudo-scalar phase \( \chi \),
\[
\sigma_\chi \sim m\mu^2.
\]

We obtain this by estimating the kinetic energy of \( \chi \) on a configuration of width \( 1/m \) where \( \chi \) changes by an amount of order 1.

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7 The fact that the heavy radial field \( \rho \) practically does not contribute to the string tension is natural from the point of view of decoupling. In the limit of infinite mass \( \rho \) should decouple from the theory without changing its physical properties. It is very different, however, from the situation in superconductors. In the superconductor of the second kind, where the order parameter field is much heavier than the photon ( \( \kappa > \sqrt{2} \) ) the magnetic field and the order...
Let us now consider the domain wall (or fundamental string) in the opposite regime, when the mass of the scalar meson is much smaller than that of the pseudo-scalar one. Now, the profile of the fields in the wall is very different. A cartoon of this situation is given in Fig. 6.

We will use the same notations, denoting the mass of the pseudo-scalar by $m$ and the mass of the scalar by $M$, but now $m \gg M$. Let us again estimate the string tension and the contributions of the scalar and of a pseudo-scalar to it. The width of the region in which the variation of $\rho$ takes place is of the order of its inverse mass. An estimate of the energy density of the $\rho$ field is given by the contribution of the kinetic term

$$\sigma_{\rho} \sim M(\mu - \rho_0)^2.$$  \hfill (65)

The width of the region in which the phase $\chi$ varies is $\sim 1/m$. In this narrow strip the radial field $\rho$ is practically constant and is equal to $\rho_0$. Hence, the kinetic energy of $\chi$ contributes

$$\sigma_{\chi} \sim m\rho_0^2.$$  \hfill (66)

Minimizing the sum of the above two contributions with respect to $\rho_0$ we find

$$\rho_0 \sim \frac{M}{m} \mu \ll \mu.$$  \hfill (67)

parameter give contributions of the same order (up to logarithmic corrections $O(\log \kappa)$) to the energy of the Abrikosov vortex. This is the consequence of the fact that the order parameter itself is forced to vanish in the core of the vortex, and therefore even though it is heavy, its variation inside the vortex is large. An even more spectacular situation arises if we consider a domain wall between two vacuum states in which the heavy field has different values.[11] In this situation the contribution of the heavy field $\phi$ to the tension would be

$$\sigma_{\text{heavy}} = M(\Delta \phi)^2$$  \hfill (64)

where $\Delta \phi$ is the difference in the values of $\phi$ on both sides of the wall. For a fixed $\Delta \phi$ the energy density diverges when $\phi$ becomes heavy. In the present case this does not happen since the two vacua which are separated by the domain wall differ only in VEV of the light field $\chi$ and not the heavy field $\rho$. 

Figure 6: The structure of the string (domain wall) in the regime when the pseudo-scalar is heavier than the scalar, $m > M$. 

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28
Also
\[ \sigma_\chi \sim \frac{M}{m} M \mu^2, \quad \sigma_F = \sigma_\rho \sim M \mu^2. \]  

(68)

Now the radial field is very small in the core of the string. The energy density is contributed almost entirely by the scalar rather than by the pseudo-scalar field.

The extreme situation \( m \gg M \) is not realized in the non-Abelian gauge theory. From the lattice simulations we know that in actuality even in the case of pure Yang-Mills the ratio between the pseudo-scalar and scalar masses is about 1.5 -- not a very large number. The analysis of the previous paragraph does not reflect the situation in the strongly coupled regime of the theory. Rather we expect that the actual profile of the string is somewhere in between Fig. 5 and Fig. 6, although somewhat closer to Fig. 6. The widths of the string in terms of the scalar and pseudo-scalar fields are of the same order, although the scalar component is somewhat wider. The same is applicable to the contribution to the string tension. Both glueballs contribute, with the scalar contribution being somewhat larger.

The interaction between two domain walls (confining fundamental strings) in the two extreme regimes is also quite different. In the weakly coupled region we can disregard the variation of \( \rho \). For two widely separated parallel strings the interaction energy comes from the kinetic term of \( \chi \). This obviously leads to a repulsion, since for both strings in the interaction region the derivative of the phase is positive. On the other hand, if the pseudo-scalar is very heavy the main interaction at large separation is through the “exchange” of the scalar. This interaction is clearly attractive, since if the strings overlap, the region of the space where \( \rho \) is different from its value in the vacuum is reduced relative to the situation when the strings are far apart.

Therefore, the situation is very similar to that in superconductivity. The confining strings in the weakly coupled and strongly coupled regimes behave like the Abrikosov vortices in the superconductor of the second and first kind, respectively. This observation has an immediate implication for the string tension of the adjoint string. As we have discussed in the previous section the phase \( \chi \) changes from 0 to \( 2\pi \) inside the adjoint string. The adjoint string can be pictured as two fundamental strings running parallel to each other. In the weak coupling regime the two fundamental strings repel each other. Therefore, two fundamental strings within the adjoint string will not overlap, and the energy of the adjoint string is twice the energy of the fundamental one,
\[ \sigma_{\text{Adj}} = 2\sigma_F. \]  

(69)

---

\( ^{8} \) Again, this is in agreement with decoupling. The heavier field does not contribute to the energy, even though its values on the opposite sides of the wall differ by \( O(1) \). Its contribution to the energy is suppressed by the factor \( \rho_0^2 \) which is very small inside the wall.

\( ^{9} \) One also expects similar non-negligible contributions from higher mass glueballs which are not taken into account in our effective Lagrangian framework.
In the strongly coupled region the situation is quite different. The strings attract. It is clear that the contribution of $\rho$ to the energy will be minimized if they overlap completely. In this case the contribution of $\rho$ in the fundamental and adjoint strings will be roughly the same. There will still be a repulsion between the pseudo-scalar cores of the two fundamental strings, so presumably the core energy will be doubled. In this way we arrive at an estimate

$$\sigma_{\text{Adj}} = \sigma_F + O\left(\frac{M}{m}\sigma_F\right).$$

Again the situation is more complicated in the limit of pure Yang-Mills. The scalar is lighter, therefore the interaction at large distances is attractive. However, the pseudo-scalar core size and its contribution to the tension are not small. In other words, $M/m$ in Eq. (70) is a number of order one.

Thus, in broad terms the relation between the weakly and strongly coupled confining theories is similar to the relation of the superconductors of the second and first kind.

5 Fundamental Matter.

The discussion in the preceding sections pertained only to theories without matter fields in fundamental representation. Obviously it is very important to understand what becomes of the magnetic symmetry and the associated order parameter if fundamental matter is present. In this section we address this question.

For simplicity we will consider scalar quarks. The Lagrangian of the theory we are interested in is

$$L = -\frac{1}{4} \text{Tr} F^2 + |D_\mu \Phi|^2 - M^2 \Phi^\ast \Phi$$

where the scalar field $\Phi^\alpha$ transforms according to the fundamental representation of the $SU(N)$ color group.

5.1 $Z_N$ as a local symmetry.

First thing to note is that the fundamental Wilson loop still commutes with the Hamiltonian. This is obvious, since the extra terms in the Hamiltonian in the presence of the quarks do not involve electric field operator, but only vector potential. The Wilson loop commutes with the vector potential, and thus with the additional terms in the Hamiltonian. The theory thus still has the magnetic $Z_N$ symmetry.

This may seem somewhat surprising at first sight. We have shown above that spontaneous breaking of the magnetic $Z_N$ implies the area law for the Wilson loop, and conversely the perimeter law of $W$ implies unbroken $Z_N$. In the theory with fundamental charges the Wilson loop is known to have perimeter law due to breaking of the confining string.
at any finite value of the fundamental mass \( M \). We might then conclude that the magnetic \( Z_N \) is restored at any, arbitrarily large value of \( M \) but is broken at \( M \to \infty \). The common lore is that the \( Z_N \) breaking phase transition for \( N > 2 \) is first order, and this then implies a discontinuous behaviour of the theory in the infinite mass limit. This of course is completely counterintuitive and in fact plain wrong. The caveat in this line of reasoning is the following. The relation between the behaviour of the Wilson loop and the mode of the realization of the magnetic symmetry hinges crucially on the existence of a local order parameter of the magnetic \( Z_N \). In the absence of such an order parameter it is not true that the Wilson loop locally changes the quantum state inside the loop and this invalidates the whole argument. In particular in the absence of a local order parameter, the \( Z_N \) symmetry can be spontaneously broken but the Wilson loop can have a perimeter law.

In fact it is easy to see that for any finite \( M \) the magnetic \( Z_N \) does not have a local order parameter. The only candidate for such an order parameter is the vortex operator \( V(x) \) defined in eq.(29), since it has to be local also relative to the purely gluonic operators. However in the presence of fundamental quarks the operator \( V(x) \) is not local anymore. To see this consider the dependence of \( V_C(x) \) on the curve \( C \) which enters its definition. As before the operators \( V_C \) and \( V_{C'} \) are related by

\[
V_C(x) = V_{C'}(x) \exp\left(\frac{4\pi i}{gN} \int_S d^2x \text{Tr} \partial_i Y E^i \right)
\]

where \( S \) is the area bounded by \( C - C' \). As before, due to the Gauss’ law the integral in the exponential is equal to the total hypercharge in the area \( S \). However the hypercharge of fundamental quarks has eigenvalues \( \pm g/2 \). The extra phase factor is therefore not unity anymore but can rather take values \( \exp\{2\pi i/N\} \) depending on the state and the choice of the contour \( C \).

The status of the magnetic \( Z_N \) is thus quite different in a theory with fundamental quarks - it does not have a local order parameter. Nevertheless it is clear that at least as long as the mass \( M \) is large \( M/g^2 > 1 \), the relevant degrees of freedom for the effective infrared dynamics should still be the vortices \( V \) and the main factor which determines their dynamics should still be the magnetic \( Z_N \). At large \( M \) the dependence of \( V \) on the curve \( C \) is weak, since the probability of the vacuum fluctuations which involve fundamental charges is small. The probability of appearance of a virtual \( q\bar{q} \) pair separated by a distance \( l \) is suppressed by the exponential factor \( \exp\{-Ml\} \). The typical distance scale for the "glueball" physics is \( 1/g^2 \). Thus at these distances such fluctuations are unimportant and should not affect much the dynamics. Things are different however if one is interested also in the baryonic sector of the theory. The baryons are necessarily heavy and in order to be able to discuss their structure we must understand the main dynamical effects also at shorter distances.

Therefore our aim now is to understand what is the main effect of the non locality discussed above on the dynamics of magnetic vortices.
The situation we have just described - a symmetry without a local order parameter - is not exceptional in quantum field theory. This is precisely the property of the global part of any Abelian gauge group. Consider for example quantum electrodynamics. The global electric charge is of course a physical gauge invariant charge with the corresponding gauge invariant local charge density.

$$Q = \int d^2 x \rho$$  \hspace{1cm} (73)

Nevertheless there is no local operator that carries this charge. This is a direct consequence of the Gauss' law

$$\partial_i E_i = g \rho$$  \hspace{1cm} (74)

Any physical, gauge invariant operator that carries $Q$ must also carry the long range electric field, which can not fall off faster than a power of the distance. The gauge invariant QED Lagrangian is written in terms of "local" charged fields $\phi$. But appearances are deceptive: these fields are not gauge invariant, and therefore not physical. A gauge invariant charged field can be constructed from $\phi$ by multiplying it by a phase factor

$$\phi_{phys}(x) = \phi \exp \{ ig \int d^2 y \epsilon_i(x-y) A_i(y) \}$$  \hspace{1cm} (75)

with the c-number field $\epsilon_i$ satisfying

$$\partial_i \epsilon_i = \delta^2(x)$$  \hspace{1cm} (76)

For any $\epsilon_i$ satisfying this condition the operator $\phi_{phys}$ is gauge invariant, and therefore physical. It is however necessarily nonlocal. Different choices of $\epsilon_i$ define different gauge invariant operators and correspond to different gauge fixings. Thus for $\epsilon_i(x) = x_i/x^2$ the field $\phi_{phys}$ is the field $\phi$ in the Coulomb gauge, while $\epsilon_i(x) = \delta_i \delta(x_2) \theta(x_1)$ corresponds to the axial gauge $A_1 = 0$, and so on. Different definitions of $\phi_{phys}$ differ from each other by a phase factor, which is precisely the gauge ambiguity of the original field $\phi$.

The $U(1)$ gauge group is the most natural Abelian gauge symmetry to consider in continuum field theory. One can however also consider discrete groups like $\mathbb{Z}_N$. In this case again no local operator that carries the global $Z_N$ charge exists. The various gauge invariant charged operators are nonlocal and differ from each other by a local $Z_N$ - valued phase. These different operators again correspond to different gauge fixings of the local $Z_N$ group.

This is precisely the structure that emerged in our discussion in the earlier part of this section. We have

1. Global magnetic $Z_N$ symmetry generated by the fundamental Wilson loop.
2. The set of nonlocal vortex operators $V_C(x)$, which all carry the $Z_N$ charge and differ from each other by a $Z_N$ valued phase factors.
It is very suggestive therefore to think about \( V_{C}(x) \) as of different gauge fixed versions of a field charged under local \( Z_{N} \). This leads us to expect that the low energy theory we are after should be a \( Z_{N} \) gauge theory of the magnetic vortex field \( V \).

In fact coming back to the discussion in the beginning of this section, we see that from this vantage point it is obvious why the Wilson loop has a perimeter law, even if the global \( Z_{N} \) is broken spontaneously. The action of a Wilson loop of a finite size inside the contour is a gauge transformation. Thus in physical terms locally inside the contour the new state is the same as the old one and so the overlap between the two locally is unity. The only nontrivial contributions to the overlap come from the region close to the contour, thus giving the perimeter law.

We will now construct the effective low energy locally \( Z_{N} \) invariant theory, and discuss its relation to the original QCD Lagrangian. Although it is possible to give a more 'analytic' derivation of the effective Lagrangian, we prefer to write it down directly guided by the previous discussion. We will then explain the physical meaning of the various fields that appear in it.

5.2 Gauging the Wilson loop.

The easiest way to construct a \( Z_{N} \) gauge theory in the continuum is to consider a \( U(1) \) gauge theory with the Higgs field of charge \( N \) which has a large expectation value \([14]\). Consider therefore the following Lagrangian:

\[
L = -\frac{1}{4e_{2}^{2}}f_{\mu\nu}^{2} + |(\partial_{\mu} - i\frac{1}{N}b_{\mu})V|^{2} + |(\partial_{\mu} - ib_{\mu})U|^{2} - \lambda(V^{*}V - \mu^{2})^{2} - \xi(V^{*}U^{*} + V^{*}N U) - \tilde{\lambda}(U^{*}U - u^{2})^{2}.
\]

Here \( f_{\mu\nu} = \partial_{\mu}b_{\nu} - \partial_{\nu}b_{\mu} \). We take the parameters such that, \( \tilde{\lambda} >> \lambda \) and \( u^{2} >> \mu^{2} \).

The 'Higgs' field \( U \) has a large expectation value and breaks the \( U(1) \) gauge symmetry down to its \( Z_{N} \) subgroup \( V(x) \rightarrow \exp\{i\frac{2\pi n(x)}{N}\} V(x) \). Below the scale determined by the expectation value \( u \), the field \( U \) is practically frozen and its fluctuations are unimportant. In this regime the model indeed describes the locally \( Z_{N} \) invariant theory. The global part of the gauge group is our \( Z_{N} \) magnetic symmetry generated by the Wilson loop. The larger \( U(1) \) gauge structure at this point is just an auxiliary trick which enables us to write down a discrete gauge theory in continuous notations. We will however see later that it does in fact has a real physical meaning of its own and arises naturally in the effective theory.

Let us first discuss how this Lagrangian reduces to the effective Lagrangian of the pure Yang-Mills theory eq.(53) in the limit of the large quark mass. In this limit not only the field \( U \) must decouple, but also the gauge interactions of the field \( V \) must vanish. There is another consistency requirement. In the limit of zero gauge coupling \( e_{2}^{2} \rightarrow 0 \) the gauge \( U(1) \) symmetry becomes global \( U(1) \) and is broken due to non vanishing expectation value of \( U \). The spectrum therefore contains a massless Goldstone
boson. This Goldstone boson is of course the longitudinal component of the gauge field $b_\mu$. There is however no such massless particle in the pure gluodynamics nor in the effective Lagrangian eq.(53). This means that the couplings in eq.(77) should depend on the quark mass in such a way that the vector particles remains heavy for any finite $M$ and its mass goes to zero very sharply only in the limit when it is completely decoupled. In terms of the Goldstone boson couplings it means that $f_\pi$ must be larger than any scale relevant to the dynamics of the vortex field $V$. All these conditions can be met by choosing for example

$$e^2 = \frac{y}{M}, \quad u^2 = xM, \quad \xi = \frac{\zeta}{u} \quad (78)$$

With this choice the mass of the vector boson $m^2 = e^2u^2$ stays finite as $M \to \infty$ and can be arbitrarily large. The Goldstone boson in the decoupling limit has an infinite $f_\pi$ and is completely decoupled just as the "invisible axion".

How do the basic fields present in the effective Lagrangian arise in the fundamental theory eq.(71)? To understand this, let us start with considering the symmetries of the theory.

The Lagrangian eq.(77) has two global $U(1)$ symmetries. One is the global part of the local $U(1)$ with the conserved current $e j_\mu^T = \partial_\nu f_{\nu\mu}$. The other current, conserved by virtue of the homogeneous Maxwell equation is $\tilde{f}_\mu = \epsilon_{\mu\nu\lambda} f_{\nu\lambda}$. The two charges have quite different nature. The second one, the dual magnetic flux

$$\Phi_D = \int d^2 x \tilde{f}_0 \quad (79)$$

has a local order parameter. It can be constructed in a way similar to the vortex field in QED [3]-[4]. The global $U(1)$ gauge charge, on the other hand does not have a local order parameter, as discussed earlier.

Both these conserved currents also exist in the fundamental theory. It is fairly straightforward to identify them. The QCD Lagrangian has one obvious global $U(1)$ charge - the baryon number. This charge has local order parameters - gauge invariant baryon fields of QCD, and is therefore identified with the dual magnetic flux

$$\frac{1}{2\pi} \tilde{f}_\mu = J^B_\mu, \quad Q_B = \Phi_D \quad (80)$$

The second charge can be expressed in terms of the spatial current components of the first one

$$Q^T = \int d^2 x \tilde{f}_0^T = \int d^2 x \partial_i \left[ \frac{1}{e^2} \epsilon_{ij} \tilde{f}_j \right] \quad (81)$$

It is thus the vorticity associated with the baryon number current.

The vortex operator $V$ which appears in eq.(77) is not gauge invariant and is only physical after complete gauge fixing of the $U(1)$ gauge group. After such a gauge fixing (which amounts to multiplying $V$ by an operator
valued phase) the Gauss law requires that on the physical states the physical operator \( V \) carry the charge \( Q_T \). Due to the identification eqs.\([51,51]\) this leads to an interesting conclusion that any physical operator \( V \) in the effective theory creates a vortex of the original baryon number current.

This somewhat unexpected conclusion is in fact quite natural for an eigenoperator of the magnetic \( Z_N \). Consider the vortex operator \( V \) defined by eq.\((29)\). In QCD just like in pure gluodynamics, it has an alternative representation of an operator of the singular gauge transformation of the form eq.\((14)\).

\[
V_C(x) = \exp\left\{ \frac{2i}{gN} \int d^2 \epsilon_{ij} \frac{x_i - y_j}{(x - y)^2} \text{Tr}(YE_j(y)) + \theta(x - y)J^Y_0(y) \right\} \tag{82}
\]

The only difference is that now \( J^Y_0 \) is the hypercharge operator due to both, gluons and fundamental quarks

\[
J^Y_0 = ig \left[ \text{Tr}[A_i, E_i] + \gamma_\alpha (\Phi^* \Pi^\alpha - \Phi \Pi^\alpha) \right] \tag{83}
\]

Consider the action of this operator on a quark field \( \Phi \). The transformed quark field \( \Phi' = V_C^\dagger \Phi V_C \) is a gauge transform of \( \Phi \) everywhere except along the branch cut of the function \( \theta \). Across this cut the phase of \( \Phi' \) is discontinuous - it jumps by \( 2\pi/N \) for all color components of \( \Phi' \). Due to this discontinuity the baryon current - the global \( U(1) \) current of \( \Phi \) - does not vanish at points along the cut. Calculating explicitly the action of \( V \) on the baryon number current we find

\[
V_C^\dagger J^B_0(x)V_C = iV_C^\dagger (\Phi^* \partial_i \Phi - \partial_i \Phi^* \Phi) V_C = J^B_0(x) + \frac{2\pi}{N} n_i^C(x) \delta(x \in C) \Phi^* \Phi(x) \tag{84}
\]

where \( n_i^C(x) \) is a unit vector normal to the branch cut \( C \) at the point \( x \).

It is natural to define the local vorticity associated with the baryon number as

\[
\rho_T = i \epsilon_{ij} \partial_i \left[ \frac{(\Phi^* \partial_j \Phi - \partial_j \Phi^* \Phi)}{\Phi^* \Phi} \right] \tag{85}
\]

The vortex operator \( V \) therefore creates a vortex of baryon number current with fractional vorticity \( 2\pi/N \)

\[
V_C^\dagger(x) \rho_T(y) V_C(x) = \rho_T(y) + \frac{2\pi}{N} \delta^2(x - y) \tag{86}
\]

The operator \( U \) due to the Gauss' law also carries baryon vorticity. Since its gauge coupling is \( N \) times the coupling of \( V \), it creates one unit of vorticity.

This simple exercise also helps us to identify the value of the gauge coupling constant \( e^2 \) in the effective theory. Comparing eq.\((52)\) with eq.\((81)\) we find

\[
e^2 \propto \Phi^* \Phi \tag{87}
\]

The same relation is obtained by comparing the current algebra in the fundamental and the effective theories. The commutator of the baryon
charge density with the baryon current density in the fundamental theory is
\[
[J^B_0(x), J^B_i(y)] = i\Phi^* \Phi \partial_i \delta^2(x - y) \tag{88}
\]
In the effective theory using the canonical commutators that follow from eq.(77) and the identification eq.(80) we find
\[
[J^B_0(x), J^B_i(y)] = ie^2 \frac{1}{4\pi^2} \partial_i \delta^2(x - y) \tag{89}
\]
Again we deduce eq.(87). The operator on the right hand side of this equation in the effective theory is indeed a constant. The effective theory should be valid at long distances. In this regime in the leading order in the derivative expansion the operator $\Phi^* \Phi$ should be approximated by its expectation value. Taking into account fluctuations of $\Phi^* \Phi$ is tantamount to including higher derivative terms in the effective Lagrangian eq.(77).

At this, higher order in derivative expansion the gauge coupling constant would become a dynamical field. While this is perfectly legitimate, it is beyond our present framework.

When calculating the expectation value of $\Phi^* \Phi$ we should remember that it has to be calculated with the cutoff $\Lambda$ appropriate for the effective theory. This cutoff must be above the characteristic scale of the pure gluodynamics (determined by the string tension) but below the heavy quark mass. With this in mind we get
\[
<\Phi^* \Phi>_{\Lambda} = \int_0^\Lambda \frac{d^2 p}{8\pi^2} \frac{1}{(p^2 + M^2)^{1/2}} \propto \frac{\Lambda^2}{M} \tag{90}
\]
So that finally
\[
e^2 \propto \frac{\Lambda^2}{M} \tag{91}
\]
which is consistent with the expected scaling in the large mass limit eq.(78).

5.3 The baryon and the bag.
Having understood the origin of the fields and the symmetries in the effective theory, we would like to see how it encodes the qualitative features of the low energy QCD physics. Since we are considering a heavy quark theory, below the fundamental mass scale the spectrum should be the same as in pure gluodynamics. We have already seen that, in the infinite mass limit, the effective theory reduces to that of (53). Indeed, even at finite but large $M$, this is the case at low energies. Since the VEV of the field $U$ is large, we can impose the unitary gauge condition on it. In this unitary gauge the phase of $U$ disappears. The modulus of $U$ is very heavy, and so is the vector field $b_\mu$. Thus at low energies we recover the effective theory of pure glue sector. There is however a set of configurations, on which the unitary gauge can not be imposed. Those are configurations in which $U$ vanishes at some points in space. Indeed it is these configurations that are important for the baryonic sector. Recall that $U$ is a vortex of

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baryon number current. Thus, by duality one expects that the baryon charge is associated with the vortex configuration of the field $U$. In the core of the vortex the field $U$ has to vanish, and so the unitary gauge is not admissible. Thus in the baryon sector we can not think of $U$ as frozen at its expectation value and instead have to treat it as a dynamical field.

That the baryon does indeed carry vorticity of the field $U$, can be seen by the following simple argument. The baryon number is represented in the effective theory by the dual magnetic flux. The baryonic state must therefore be the dual magnetic vortex. Such a vortex in a nonsingular gauge has a vector potential of the form $b_i = \epsilon_{ij} \frac{x_j}{x^2}$. To have a finite energy it must be accompanied by the winding of the phases of both $V$ and $U$. Since $U$ carries $N$ times the charge of $V$, the only states that are allowed energetically are those that carry $N$ vortices of $U$.

This is natural in view of the ‘dual’ relation between the effective and the fundamental theories. The field $U$ is the ‘vortex’ dual to the fundamental quark. We thus expect that its elementary vortex would represent the fundamental quark itself, and so finite energy states must contain $N$ such elementary vortices. A single vortex must be confined. In the same sense, $V$ is ‘dual’ to the adjoint gluon field. The elementary vortex of $V$ is then, in a sense, the ‘constituent’ gluon. In fact, such single gluon should not exist as a finite energy state either, and we expect it also to be confined. The single vortex of $V$ should therefore bind either with the anti-vortex of the same type, or with $N$ vortices of $U$. The former type of state is a glueball, and exists in the pure glue theory $[9]$, while the latter type is a baryon.

Interestingly enough, this line of reasoning leads us to expect that the baryon must have a bag-like structure. Namely, the quarks are bound to the $V$ field vortex. Inside this vortex, the value of $V$ is small - it vanishes in the middle, and than rises quite slowly (relative to the scale of $1/M$) towards the edges. Recall that $V = 0$ corresponds to a non confining state $[9]$. The quarks are therefore sitting in the ‘perturbative’ region of space - where there are no confining forces. Only when they separate far from each other - into the region with non vanishing $V$, the linear potential pulls them inside again.

Let us look at this more carefully. Consider for simplicity the $Z_2$ symmetric case $N = 2$. The baryon is the dual vortex with dual magnetic flux $2\pi/e$. Far from the vortex core, the field configuration is pure gauge, with the phases of $V$ and $U$ following the vector potential:

$$b_i = \epsilon_{ij} \frac{x_j}{x^2}, \quad V(x) = ve^{ia(x)}, \quad U(x) = ue^{2ia(x)}. \quad (92)$$

The parameters of the model are such that the field $V$ is much lighter than both $U$ and $b_i$. Thus the size of the vortex core of $V$ is large - of the order of the inverse glueball mass. The two $U$-vortices which have a very small size core, sit inside this core. Since the length associated with the dual magnetic field is much smaller than the core size of the $V$ - vortex, the dual flux is concentrated on the $U$ - vortices. From the low energy point of view, the picture is that two point-like magnetic vortices sit inside
a soft core of a $V$ field vortex. The field configuration looks roughly as depicted on figure 7.

The magnetic flux is concentrated at the points $A$ and $B$. The phase of the field $U$ follows the variation of the vector potential very closely. The phase of $V$ is also trying to do that, but it can not quite follow it all the way, since on the line between the two vortices it would have to be discontinuous. The most important energy contribution (apart form the core energy of small vortices) comes therefore from the vicinity of this line. The phase of $V$ obviously has to interpolate across this line between the values 0 and $\pi$. Since the modulus of $V$ is not extremely rigid, it will be smaller along this line than in the immediate neighborhood. Both, the variation of the modulus and the phase of $V$ along the line connecting the small vortices contribute to the energy which is clearly linear in the distance $|A - B|$.

To study the structure of the baryon in the effective theory more quantitatively let us fix the gauge such that

$$U(x) = u \exp[i \theta(x)]$$

$$\theta(x) = \arctan\left(\frac{y}{x-a}\right) + \arctan\left(\frac{y}{x+a}\right).$$

This is a valid gauge choice for the configuration with two vortices of unit winding at the positions $(a, 0)$ and $(-a, 0)$. 

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Due to the conditions on the parameters of our model, the vector field $b_μ$ follows the phase of $U$ in the whole space. Thus

$$b_0 = 0, \quad b_j = - \left[ \epsilon_{jk} \frac{(x - x^{(l)})_k}{|x - x^{(l)}|^2} + \epsilon_{jk} \frac{(x - x^{(r)})_k}{|x - x^{(r)}|^2} \right]$$  \hspace{1cm} (94)

where: $x_1 \equiv x, x_2 \equiv y$, and $l, r$ are the positions of the left and right vortices, respectively: $x^{(l)} = (-a, 0), x^{(r)} = (a, 0)$.

There are two interesting limiting situations. The first is when the distance $a$ is larger than the dynamical distance scales $(\lambda u^2)^{-1/2}$ and $(\xi u^2)^{-1/2}$ which determine the masses of the glueballs. The second is the reverse, that is when the two vortices are sitting well inside the glueball correlation length. Let us look at them in turn.

When the distance between the vortices is large we expect the potential between them to be linear with the string tension calculated in the theory without the dynamical $U$ field. We will study the interquark potential in our low energy theory in the semiclassical approximation.

To find the minimal energy configuration, we have to solve the classical equations of motion for the field $V$ at a fixed configuration of $U$ and $b_i$ given by eqs.(93,94). Let us concentrate on the points which are close to the $x$-axis, with $|x| << a$. The main contribution to the energy comes from this region of space. In this region the vector potential $b_i$ vanishes, and the phase of the field $V$ is zero. Thus the equations of motion for the field $V$ are the same as in the pure gluodynamics. Also as long as we stick to this region, the configuration of $V$ is translationally invariant in the $x$ direction. What determines the energy then are the boundary conditions on the field $V$. In this configuration clearly the phase of the field $V$ is $\pi/2$ far above the $x$ axis and $-\pi/2$ far below the axis, fig.7. Thus both, the equations and the boundary conditions are precisely the same as for the domain wall separating the two degenerate vacua in the effective theory of pure gluodynamics ($e^2 = 0, \tilde{\lambda} \to \infty$). There is an extra contribution to the energy that comes from the region of space close to the points $A$ and $B$. But this energy does not depend on $a$ and is subleading for large $a$. The rest of the space does not contribute to the energy, since the field configuration there is pure gauge.

Thus as expected the energy in this regime is $E = a\sigma$, where $\sigma$ is the domain wall tension (fundamental string tension) calculated in pure gluodynamics $^4$.

It is also interesting to consider the opposite situation, when the distance between the quarks is smaller than the glueball correlation length. This is the regime in which we do not expect to see any stringy structure. Instead we can ask whether the lowest energy configuration has any resemblance to a bag. To study this question we take the limit $a \to 0$. In this case the phase of the field $V$ will follow the phase of $U$ in the whole space

$$\left( \frac{V}{V^*} \right)^2 = \frac{U}{U^*}. \hspace{1cm} (95)$$

Given this condition, only the variation of the radial component of $V$ has to be determined. Since the problem has rotational symmetry, the
equation of motion for the modulus $\rho$ becomes

$$-\frac{d^2 \rho}{dr^2} - 2\eta \rho + 2\lambda \rho^3 = 0.$$  \hspace{1cm} (96)

with $\eta = \lambda \mu^2 - \xi u$, which is assumed to be positive throughout this paper. The relevant boundary condition in this case is that at infinity $\rho$ approaches its vacuum value $\rho_{r \to \infty} \to v$ while in the vortex core it vanishes $\rho(0) = 0$. With these boundary conditions, eq. (96) has the familiar form of the $\varphi^4$ static kink equation in 1 + 1 dimensions with the solution

$$\rho(r) = v \frac{1 - e^{-\sqrt{v} \sqrt{\eta} r}}{1 + e^{-\sqrt{v} \sqrt{\eta} r}}.$$  \hspace{1cm} (97)

The energy of this solution is

$$E[V]_{a=0} = \lambda \int_0^\infty 2\pi drr(v^4 - \rho^4) = \frac{\pi^3 \eta}{12 \lambda} = \frac{\pi^3}{12} \left( \mu^2 - \frac{\xi u}{\lambda} \right).$$  \hspace{1cm} (98)

The picture is thus just as described in the previous subsection. Since the two quark state is accompanied by the vortex of the field $V$, the quarks effectively ‘dig a hole’ in the vacuum. In their immediate vicinity the modulus $\rho$ vanishes, and therefore there is a ‘bag’ of the nonconfining state. The radius of this bag is given by the mass of the scalar glueball $2\sqrt{\eta}$.

It is interesting to note that, although for large separation $a$ the energy of the string gets contributions from both, the scalar and the pseudoscalar glueballs (the modulus and the phase of $V$), the ‘bag constant’ is determined solely by the scalar glueball. For small $a$ the phase of $V$ is not excited and only $\rho$ deviates from the vacuum state inside the ‘bag’. This is consistent with the common lore that the inside of the bag is distinguished from the outside by the value of the $F^2$ condensate. In fact since $\rho$ has vacuum quantum numbers and interpolates in our effective theory the scalar glueball, it is naturally associated with the operator $F^2$ which has a large overlap with the scalar glueball in QCD.

The bag we are talking about here arises in a very different situation than in the usual bag model [15]. There the bag describes the structure of the baryon containing light quarks. The radius of the bag in this situation is determined by the balance of the vacuum pressure and the pressure due to the free motion of the light quarks inside, and in fact depends on the quark wave function. In our case the inside of the bag contains heavy quarks. Their kinetic energy is small, and we have treated them here as static. The radius of the bag thus is determined purely by the dynamics of the scalar glueball field and is not sensitive to the state of the heavy quarks. This is true for low lying excitations for which the radius of the quark state is smaller than the inverse glueball mass. When these two scales are comparable presumably the quark pressure will also be important and will play a role in the energy balance. Thus in this intermediate regime we expect the $V$-vortex to be similar to the bag in the usual bag model. For states of even larger size the potential between
quarks is linear with the fundamental string tension. The bag picture
should therefore go smoothly into the string picture.

Three space-time dimensions are unique in many ways. In particular
they allow massive gauge theories, where the mass of the gauge fields
does not arise due to the Higgs mechanism. We mean here the Chern-
Simons gauge theories. The discussion of the magnetic symmetry in 2+1
dimensions would not be complete if we did not discuss this type of models.
The next section is devoted to such a discussion.

6 Magnetic symmetry and the Chern-
Simons term.

The question of vortices in the Chern-Simons theories is interesting for
the following reason. The Yang-Mills-Chern-Simons (YMCS) theories are
completely massive, and so the Wilson loop is expected to have a perimeter
law behavior [13]. By the argument we have established earlier, it should
then follow that the magnetic symmetry is unbroken and the spectrum
should contain vortex states. This is an interesting proposition worth
investigating. This conclusion would not hold if the magnetic symmetry
had no local order parameter. If that were the case the symmetry can be
unbroken, $W$ may have perimeter law and there could still be no magnetic
vortex states. This is the main question we address in this section. We
want to examine more carefully the question of locality of the vortex
operator [17].

We start our discussion by considering a simpler theory - compact
electrodynamics with CS term. Conceptual questions here are similar but
the technical side is much simpler. We find that generically the theory
does indeed contain local vortex operators and a global discreet magnetic
symmetry which is unbroken. Nevertheless in the continuum limit there
are no magnetic vortices in the spectrum. The reason is that the energy
of such a vortex is logarithmically UV divergent. We find however that
with a particular scaling of the CS coefficient (logarithmically vanishing
when UV cutoff is removed) the energy of the vortex becomes finite. This
suggests that the theory may indeed have a phase with finite energy vortex
states and vanishing photon mass.

6.1 Compact QED with the Chern Simons term.

The Lagrangian of Abelian Chern Simons theory in the formal continuum
limit is

$$L = -\frac{1}{4g^2} F_{\mu \nu} F^{\mu \nu} + \frac{\kappa}{2} \epsilon^{\mu \nu \rho} \partial_\rho A_\mu A_\nu.$$  (99)

The gauge coupling $g^2$ has dimension of mass and $\kappa$ is dimensionless.
Equations of motion read as $\partial_\mu F^{\mu \nu} = \kappa g^2 \epsilon^{\mu \nu \rho} \partial_\rho A_\mu$. The mass of the
gauge particle is $M = \kappa g^2$. The canonical structure of the theory is
simplest in the Hamiltonian gauge, $A_0 = 0$. 

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The Hamiltonian is
\[ H = \frac{1}{2g^2}(E_i^2 + B^2) \] (100)

with canonical momenta related to the time derivatives of the fields by
\[ \Pi_i = -\frac{1}{g^2} A_i + \kappa \epsilon^{ij} A_j. \]

The gauge fields and canonical momenta form the canonical algebra, and the algebra involving the electric fields is
\[ [E_i(\vec{x}), E_j(\vec{y})] = -ig^2 \epsilon_{ij} \delta^2(\vec{x} - \vec{y}), \quad [A_i(\vec{x}), E_j(\vec{y})] = -ig^2 \delta_{ij} \delta^2(\vec{x} - \vec{y}) \] (101)

The Gauss law,
\[ \partial_i E_i - \kappa g^2 \epsilon^{ij} \partial_j A_i = 0 \] (102)
on a spatial plane generates time independent local gauge transformations. The elements of the local gauge group take the form
\[ U(\lambda) = \exp\left\{ \frac{1}{g^2} i \int d^2 x \lambda(x) \left( \partial_i E^i - g^2 \kappa \epsilon^{ij} \partial_i A_j \right) \right\} \] (103)
such that \( UAU^{-1} = A + d\lambda \). In the non-compact theory \( \lambda \in R \) should be a single-valued function such that the eigenvalue of the operator (103) on physical states is unity. Singular \( \lambda \)'s correspond to transformations which are in general nontrivial on the physical states.

Our interest however is in the compact theory. This means that magnetic vortices of flux 2\( \pi \) must be physically unobservable. As discussed in [21] this amounts to further restricting the physical Hilbert space to states which are trivial under the action of the vortex operator.

In other words certain large gauge transformations must act on the physical states trivially in the compact theory as opposed to the non-compact one. The compact gauge group therefore includes these singular gauge transformations in addition to the regular ones, which form the gauge group in the noncompact theory. Consider a multi-valued angle function \( \theta(x, x_0) \) which is singular at one point and has a discontinuity along a straight curve \( C(x_0) \) that starts at the point \( x_0 \) and goes to infinity. The operator of the gauge transformation with this singular gauge function creates a magnetic vortex. Its explicit form (after partial integration and dropping a boundary term owing to the fact that all gauge invariant fields decay at infinity) is
\[ \tilde{V}(x_0) = \exp\left\{ \frac{1}{g^2} i \int d^2 x \tilde{\partial}_i \theta(x - x_0) \left( E^i - g^2 \kappa \epsilon^{ij} \partial_j A_j \right) \right\} \] (104)

We have defined
\[ \tilde{\partial}_i \theta(x - x_0) = \partial_i \theta(x - x_0) - 2\pi \epsilon_{ij} c(x)_j \delta(x - C(x_0)) = \frac{\epsilon_{ij}(x - x_0)_j}{(x - x_0)^2} \] (105)
c\( (x)_j \) is a unit vector tangent to the curve \( C(x, x_0) \).
We may want to include $\tilde{V}$ into the compact gauge group. However to be part of the gauge group, it must commute (at least weakly) with other elements of the group. One can check explicitly that $\tilde{V}(x, x_0)$ does not commute with the elements of the noncompact group. To rectify this situation we define, following [21] a slightly modified operator

$$V(C, x_0) = \exp \frac{2\pi i g^2}{2} \varepsilon_{ij} \int d^2 x e(x) \delta(x - C(x, x_0)) E_i(x)$$

This operator is merely a “collection” of the electric fields which are perpendicular to the curve $C(x_0)$. More explicitly one can write it in the following form

$$V(C, x_0) = \exp \frac{2\pi i g^2}{2} \varepsilon_{ij} \int_C dl_i E_j(x)$$

Gauge invariance of this operator, $[V, U] = 0$, follows immediately. We also need to check the commutativity of $V(x)$ with $V(y)$. A straightforward calculation gives

$$V(C_0)V(C_1) = V(C_1)V(C_0)\exp \left\{ i8\pi^2 \kappa L(C_0, C_1) \right\}$$

Where $L(C_1, C_2) = \pm 1$ if the curves cross each other and $L = 0$ if they don’t. In order that $V$ be a Lorentz scalar the commutator should not depend on the curves $C_1$ and $C_2$. To guarantee this we need to set

$$4\pi \kappa = k \in \mathbb{Z}$$

We find therefore that the requirement of compactness quantizes the coefficient of the Chern Simons term very much like in the non-Abelian theory.

The commutator

$$[B(x), V^m(x_0)] = 2\pi m \delta^2(x - x_0) V^m(x_0), \quad m \in \mathbb{Z},$$

indicates that $[107]$ creates magnetic vortices of integer strength. Being gauge invariant this operator also creates an electric charge

$$Q = \frac{1}{g^2} \int_{\Sigma} d^2 x \partial_i E^i = \frac{k}{4\pi} \int_{\Sigma} d^2 x B = \frac{mk}{2}$$

Since $V$ has to be included in the gauge group, the magnetic flux and the electric charge created by it must be unobservable. Therefore the Hamiltonian of the theory must commute with $V$. The noncompact Hamiltonian eq.100 does not quite do the job. It should be modified but in such a way that in the continuum limit the same form is recovered for smooth fields. The modified Hamiltonian that satisfies these conditions has been suggested in [21]. Since the UV structure is important for our considerations, it is most usefully presented in the lattice notations

$$H_B = \frac{1}{a^3 g^2 n^2} \sum_x \left( 1 - \text{Re} \ e^{i n a^2 B(x)} \right), \quad H_E = \frac{m^2 g^2}{4\pi^2 n^2} \sum_x \left( 1 - \text{Re} \ e^{i \pi \frac{m}{na^2} \varepsilon_{ij} \partial_j E_i(x)} \right),$$

(112)
$a$ is the lattice spacing and $m, n \in \mathbb{Z}$ and $\hat{n}_j$ is the unit vector parallel to the link. The normalization of the electric and magnetic terms is such that in the naive continuum limit $a \to 0$ they reduce to $B^2$ and $E^2$ respectively.

Using the Gauss’ law one can see that if $2n = k$, the magnetic part $H_B$ becomes a combination of the vortex operators $V$. Therefore without loss of generality we assume $2n < k$. For $m = 1$ the electric part of the Hamiltonian is also a sum of a fundamental vortex and anti-vortex, and we take $m > 1$.

Now that we have the formulation of the compact CS QED we can ask about the locality properties of vortex operators. The operator $V$ we have considered so far is of no interest of itself, since it is trivial on all physical states. We thus have to look at the operators which create magnetic flux smaller than $2\pi$

\[ V_p(C, x_0) = \exp \left( \frac{2\pi i}{q^2} \epsilon_{ij} \int_C d_l E_j(x) \right), \quad p \in Q \quad (113) \]

Here $p$ is a rational number $p \in (0, 1)$. The question we are asking is, are there such values of $p$ for which $V_p$ is a gauge invariant local operator. The gauge invariance with respect to the noncompact gauge group is straightforward, since $V_p$ only depends on the electric field, and the electric field itself is gauge invariant. However $V_p$ should also commute with the "fundamental" vortex $V$, since $V$ is part of the gauge group. Therefore we have

\[ [V, V_p] = 0 \quad \implies kp = l \in \mathbb{Z} \quad \text{and} \quad l < k. \quad (114) \]

The condition of locality requires that $V_p(x)$ commute with each other at different points $x$ and $y$. This commutator also should be independent of the contour $C$ in the definition eq.(113).

\[ [V_p(x), V_p(y)] = 0 \quad \implies kp^2 = r \in \mathbb{Z} \quad (115) \]

Both equations (114) and (115) have to be satisfied for the existence of non-trivial local vortices. Whether it is possible or not to satisfy these equations clearly depends on the CS coefficient $k$. For example there are no solutions for $k = 2$ and $k = 3$ theories. For $k = 4$ we can choose $l = 2$ and this gives a vortex of vorticity $p = 1/2$. In general one can solve the constraints in the following way. Writing CS coefficient in terms of its prime factors, $k = q_1 q_2 q_3 \ldots q_m$, where all $q_i$ are not necessarily different, one has the following two conditions to satisfy

\[ q_1 q_2 q_3 \ldots q_m p = l, \quad q_1 q_2 q_3 \ldots q_m p^2 = r. \quad (116) \]

The first condition is solved if $p$ divides $k$,

\[ p = \frac{1}{q_1 q_2 q_3 \ldots q_i}, \quad \text{where} \quad i < m \quad (117) \]

Using this in the second condition one can see that the most general form of $k$ which allows vortices is

\[ \text{If} \quad k = t^2 z \quad \implies p = \frac{1}{t}, \quad t \geq 2 \quad \text{and} \quad t, z \in \mathbb{Z} \quad (118) \]
For example if \( k \) is a prime number there are no solutions. Generically it is easier to find a solution at large values of \( k \).

The above relations also show that should a solution exist, there is always a vortex of minimal vorticity. All the other local vortices are simple powers of this minimal vortex. For example for \( k = 36 \), the above conditions give three solutions (and their integer multiples), \( p = (1/2, 1/3, 1/6) \). Obviously \( "p= 1/6" \) is the minimal vortex. The minimal value of \( p = \frac{1}{w} \) determines the global magnetic symmetry group of the theory as \( Z_w \).

One last requirement that \( V_p \) should satisfy, is locality with respect to the energy density eq.(112). In obvious notation

\[
[h_E(x), V_p(y)] = 0, \quad x \neq y \implies \frac{kp}{m} \in Z
\] (119)

This can always be satisfied by choosing \( m = k \). To satisfy the other condition

\[
[h_B(x), V_p(y)] = 0 \implies \frac{np}{2} \in Z
\] (120)

we can take \( n = 2 \). Certainly one can define other Hamiltonians which will be compatible with the above conditions. We see therefore that for those values of \( k \) for which vortex operators are local with respect to each other we can always choose the Hamiltonian such that they are also local relative to the Hamiltonian density.

Thus we conclude that for many values of \( k \) local physical vortex operators exist. They are order parameters for a global \( Z_w \) magnetic symmetry. The value of \( w \) is determined by \( k \) through the solution of the equations for minimal \( p \). Thus the argument described above applies and, at least in the lattice theory there are vortex states. Calculating their energy in the lattice theory is not a simple matter. However the interesting question is whether these states survive in the continuum limit. That is to say, whether their energy stays finite as the lattice spacing approaches zero.

In the continuum limit for smooth configurations of the fields the theory is described by the Lagrangian eq.(99). However while solving continuum equations we may sometimes encounter field configurations with fast variations. For these configurations it is important to take into account the compactness of the theory. In particular consider the electric field created by the "minimal" vortex operator \( V_{1/w} \).

\[
[V(x), E_i(y)] = E_i(y) + e_i(x, y), \quad e_i(x, y) = \frac{1}{w} g^i \hat{n}(y) \delta(y - C(x, y))
\] (121)

where \( \hat{n}(y) \) is the vector tangential the curve \( C \) at the point \( y \). Since the operator \( V(x) \) is local, its only observable action in the compact theory is at the point \( x \). However if we just calculate the energy using the naive Hamiltonian eq.(100) we find infrared divergence proportional to the length of the curve \( C \). Clearly if faced with this type of configurations in continuum calculations we should subtract this infrared divergence by hand. Rather than do this we find it convenient to think about it in the following way. Let us split the general electric field configuration into a
smooth piece and a piece that contains arbitrary number of strings of the type of eq. (121)

\[ E^i = E^i_{\text{smooth}} + e^i \]  

and subtract the contribution of \( e^i \) in the Hamiltonian. The only remnant of \( e^i \) then is in the Gauss' law, since \( e^i \) of eq. (121) corresponds to a pointlike charge \( \frac{\kappa}{2w} \) at the point \( x \). Thus the smooth field \( E^i_{\text{smooth}} \) satisfies not the naive Gauss' law, but rather a modified one

\[ \partial_i E^i_{\text{smooth}} \big|_{\text{mod} \frac{\kappa^2}{2w}(x)} - \kappa g^2 \epsilon^{ij} \partial_i A_j = 0 \]  

In other words we can work entirely in terms of \( E_{\text{smooth}} \) if we remember that we may allow Gauss' law to be violated by the presence of pointlike charges of charge \( \kappa g^2 / w \). The appearance of \( w \) in this way is the only remnant of the compactness of the theory. In the following we will work in terms of the smooth fields but will drop the subscript \( \text{smooth} \) for brevity.

With this caveat in mind, to determine the energy of the magnetic vortex in the continuum limit we now should solve the continuum equations of motion. For a minimum vorticity solution \( 1/w \) following [22, 23] one can take the time independent symmetric ansatz,

\[ A_i(r) = \epsilon_{ij} g(r), \quad A_0(r) = h(r) \] 

The equations of motion read

\[ g''(r) - \frac{1}{r} g'(r) - r M h(r) = 0, \quad h''(r) + \frac{1}{r} h'(r) - \frac{M}{r} g'(r) = 0, \] 

where \( M = g^2 \kappa \). We are looking for the solutions with vorticity \( 1/w \). The magnetic field is \( B = -\frac{1}{r} g'(r) \) so we impose \( g(0) = 1/w \) and \( g(\infty) = 0 \) and we also demand that the fields decay exponentially at infinity. The solution under these conditions can be found as

\[ g(r) = \frac{Mr}{w} K_1(Mr), \quad B(r) = \frac{M^2}{w} K_0(Mr) \] 

\[ h(r) = -\frac{M}{w} K_0(Mr), \quad E^i = -\frac{\epsilon^{ij} M^2}{wr} K_1(Mr), \] 

where \( K_0, K_1 \) are the Bessel functions. The energy of this vortex follows as

\[ E = \frac{\pi}{g^2} \int_0^\infty r \, dr \left\{ \frac{dh}{dr} \right\}^2 + \frac{1}{r^2} \left( \frac{dg}{dr} \right)^2 \right\} = \frac{\pi \kappa^2 M}{w^2} \left( -\gamma_E + \ln 2 - \ln \frac{M}{\Lambda} \right) \] 

where \( \Lambda \) is the ultraviolet cutoff scale.

This result warrants several comments. First, we see that the energy of the vortex is IR finite. This is closely related to the fact that the locality of the operator \( V_{1/w} \) allowed us to "violate" the Gauss' law. Looking at the electric and magnetic fields in eqn (127) we indeed see that the naive Gauss' law is violated precisely by the amount allowed by eq. (123).

Second, the energy is UV divergent. Thus the magnetic vortices do not survive in the continuum limit as finite energy excitations. This of
course does not contradict our original argument. The $Z_w$ symmetry is unbroken in the vacuum and excitations carrying the quantum numbers of this symmetry are very heavy. The symmetry therefore seems completely irrelevant for the low energy dynamics. The curious thing though is that although the energy of the vortex is large, it is still much lower than the natural ultraviolet scale $\Lambda$. The vortices therefore are not genuine ultraviolet objects in the lattice theory, but rather occupy an intermediate scale between the UV scale $\Lambda$ and the IR scale $M$. In fact bringing $M$ down to zero makes vortices light. At $M = \mu \ln^{-1} \frac{1}{\mu}$ the energy of the vortex is finite.\(^{10}\) This behavior in fact is very reminiscent of vortices in the Higgs phase of the Abelian Higgs model. The mass of such a vortex at weak coupling is very large: $M \propto M_v^2 / g^2 \ln M_H / M_v$, where $M_v$ is the mass of the massive photon and $M_H$ is the mass of the Higgs particle. However as the photon mass decreases, that is as the theory approaches the phase transition line, vortices become light. On the phase transition itself they in fact become massless and condense in the Coulomb phase. It is not unlikely that similar phenomenon occurs in our model. As $\kappa$ decreases at fixed $g^2$ the photon becomes lighter, and the mass of the vortex also decreases. It could happen that at some value of $\kappa$ the vortices actually become massless and drive a phase transition into a phase with broken magnetic symmetry. A transition of precisely such type was conjectured to happen in the lattice model at $k = 8$ in \(^{24}\) and was seen in the variational calculation of \(^{21}\). Of course, within the naive continuum limit we consider here we are unable to see such behavior. However the fact that the vortices become light within the validity of the naive continuum limit is quite suggestive in this respect. Note that if the vortices indeed do condense, the magnetic $Z_w$ symmetry is spontaneously broken. By virtue of the argument given in Sec. 3 this means that the low $k$ phase is confining.

### 6.2 Non-Abelian theories.

We now extend this analysis to non-Abelian $SU(N)$ theories,

$$L = \frac{1}{2g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} - \kappa \epsilon^{\mu\nu\lambda} \text{tr} \left( A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right)$$  \hspace{1cm} (129)$$

CS coefficient has the well known quantization, $4\pi \kappa = k$ \(^{14}\). The classical equations of motion follow as

$$D_\mu F^{\mu\nu} - g^2 \kappa \epsilon^{\nu\lambda\mu} F_{\lambda\mu} = 0$$ \hspace{1cm} (130)$$

---

\(^{10}\)The reader may wonder why we are not bothered by the factor $\kappa$ in eq. (128). After all we saw that in the continuum limit naturally $\kappa \to \infty$. The point is that the flux of the minimal vortex generically scales as $w^2 \propto k$. For example if $k = x^2$ with some integer $x$, then clearly the minimal solution of the eq. (118) corresponds to $w = x$. Thus $k/w^2$ is finite in the continuum limit.
The canonical structure of this theory is similar to the Abelian case. In the $A_0 = 0$ gauge

$$\Pi^a_i = -\frac{1}{g^2} E^a_i + \frac{\kappa}{2} \epsilon^{ij} A_j^a, \quad \text{where} \quad E^a_i = \dot{A}^a_i.$$  \hfill (131)

The Hamiltonian is

$$H = \frac{1}{2g^2} \left( (E^a_i)^2 + (B^a)^2 \right), \quad \text{where} \quad B^a = \frac{1}{2} \epsilon_{ij} F_{ij}^a.$$  \hfill (132)

The canonical algebra is

$$[E^a_i(x), E^b_j(y)] = -i\delta^{ab} \kappa g^4 \epsilon_{ij} \delta^2(\vec{x} - \vec{y}), \quad [A^a_i(x), E^b_j(y)] = -ig^2 \delta^{ab} \delta_{ij} \delta^2(\vec{x} - \vec{y}).$$  \hfill (133)

In terms of the momenta the Gauss law is

$$(D_i \Pi_i)^a = -\frac{K}{2} \epsilon^{ij} \partial_j A_i^a.$$  \hfill (134)

In the non-Abelian YMCS theory, the large Wilson loop still commutes with the Hamiltonian. This is obvious in the Hamiltonian formalism, since the commutation relation between the vector potential and the chromoelectric field is unaffected by the presence of the CS term. The form of the Hamiltonian in terms of $A_i$ and $E_i$ is also the same as without the CS term. Since $W$ is a function of $A_i$ only, its commutator with the Hamiltonian is exactly the same as in the theory without CS. Therefore the fundamental Wilson loop still generates a symmetry.

Again our question is whether the theory admits local vortex operators. In the non-Abelian theory our choices are more limited than in compact QED. In the SU(N) Yang Mills theory, the only candidates for local operators are those that create quantized flux \cite{1,4}. The vortex operator in YM theory is \cite{4,9}:

$$V(x) = \exp \left\{ \frac{4\pi i}{g^2 N} \int_C dy^i \epsilon_{ij} \text{Tr}(YE_j(y)) \right\} \hfill (135)$$

where the hypercharge generator $Y$ is

$$Y = \text{diag}(1, 1, \ldots, -(N-1)).$$  \hfill (136)

We have proved in the earlier sections that in SU(N) YM theory, this operator despite its nonlocal and gauge non-invariant appearance is in fact a local, gauge invariant, Lorentz scalar field \cite{4,9}. The way it was constructed there was to require that it satisfies the 'tHooft algebra \cite{4} with the fundamental Wilson loop

$$V^\dagger(x)W(C)V(x) = \exp \left\{ \frac{2\pi i}{N} n(x, C) \right\} W(C)$$  \hfill (137)

with $n(x, C)$ being the linking number on the plane between the point $x$ and the closed curve $C$. 

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One can see that the operator in (135) is also the appropriate vortex operator when a Chern-Simons term is included for the gauge field. The commutation relation (137) is still satisfied by the expression (135). However an additional requirement is that \( V \) be gauge invariant, in the sense that the matrix elements of \( V \) between physical states (those that satisfy Gauss’ law) and non-physical states (non-singlet under gauge transformations) vanish. This means that when we calculate matrix elements of any number of operators \( V \) we thus have to check that it transforms a physical state into another physical state. The difference with the pure YM theory is that the wave function of a gauge invariant state does not depend only on the Wilson loops. The physical wave function should satisfy the following equation

\[
i(D_i \frac{\delta}{\delta A_i})^a \Psi[A] = \frac{\kappa}{2} \epsilon_{ij} \partial_i A_j^a \Psi[A]
\]

The general form of \( \Psi \) has been determined in [18] in terms of certain nonlinear variables. For our purposes we find it more convenient to work directly in terms of the vector potentials \( A_i \). Let us take \( \Psi \) in the form

\[
\Psi = \exp\{-iS\}
\]

The eikonal \( S \) satisfies a linear inhomogeneous equation

\[
D^{ab}_{ij} \frac{\delta}{\delta A_i^a} S[A] = \frac{\kappa}{2} \epsilon_{ij} \partial_i A_j^a
\]

The solution of the homogeneous equation is indeed any functional that depends on Wilson loops \( S_0[W] \). We can find a particular solution of the inhomogeneous equation using the following argument. \( S[A] \) must be a functional whose change under a standard gauge transformation of vector potentials \( \delta A_i = D_i \lambda \) is proportional to \( \frac{\kappa}{2} \int d^2 x \epsilon_{ij} \partial_i A_j^a \lambda^a \). Such a functional can be represented as a Chern Simons action on a space with a boundary. Let us introduce an additional coordinate \( \tau \in [-\infty, 1] \) and functions of three coordinates \( A_i(x, \tau) \) so that at the boundary \( \tau = 1 \), the value of these functions is equal to the value of the vector potentials in our theory \( A_i(x, \tau = 1) = A_i(x) \). Let us write the Chern Simons term (in the Weyl gauge) on this manifold

\[
S_{CS} = \int_{-\infty}^{1} d\tau \int d^2 x \epsilon_{ij} \dot{A}_i^a(x, \tau) \dot{A}_j^a(x, \tau)
\]

Under the \( \tau \) independent gauge transformation this action changes by a boundary term

\[
\delta S_{CS} = -\int d^2 x \epsilon_{ij} \lambda^a(x) \partial_i A_j^a(x, \tau = 1) = -\int d^2 x \epsilon_{ij} \lambda^a(x) \partial_i A_j^a(x)
\]

which is precisely of the form required to satisfy eq.(140). A particular solution of eq.(140) is therefore

\[
S_p = -\frac{\kappa}{2} S_{CS} = -\frac{\kappa}{2} \int_{-\infty}^{1} d\tau \int d^2 x \epsilon_{ij} \dot{A}_i^a(x, \tau) \dot{A}_j^a(x, \tau)
\]
The introduction of the extra coordinate $\tau$ and the expression eq.(141) is not at all unnatural. One should view this extra coordinate as parameterizing a curve in the field space. With this interpretation we have
\[ d\tau \dot{A}_i(x, \tau) = \delta A_i \] (144)
and
\[ \int_{-\infty}^{1} d\tau \int d^2 x \epsilon_{ij} A^i(x, \tau) \dot{A}_j(x, \tau) = \int_C \delta A_i \epsilon_{ij} A_j \] (145)
where the line integral is taken over the trajectory $C$ in the field space which ends at the point $\{A_i(x)\}$.

We have thus determined the general form of the wave function of a physical state in the YMCS theory to be
\[ \Psi[A] = \exp\left\{ i \kappa \int_{-\infty}^{1} d\tau \int d^2 x \epsilon_{ij} A^i(x, \tau) \dot{A}_j(x, \tau) \right\} \Psi_0[\bar{W}] \] (146)

Now it is straightforward to see how the vortex operator acts on it. Under the action of the vortex operator
\[ V(x) A^i(y, \tau) V^\dagger(x) = A_i(x, \tau) + 4\pi N \, \text{Tr} Y T^a \epsilon_{ij} \int dz_i \delta^2(z-C(x, y)) \] (147)
Remembering that $E^a_i = -g^2 \Pi_i^a + g^2 \kappa \epsilon_{ij} A^a_j$, we see that the change in the phase factor in the wave functional is exactly cancelled by the $A$-dependent term in the vortex operator eq.(29).

Thus
\[ V \Psi[A] = \exp\left\{ i \kappa \int_{-\infty}^{1} d\tau \int d^2 x \epsilon_{ij} A^i(x, \tau) \dot{A}_j(x, \tau) \right\} \Psi_0[V^\dagger W V] \] (149)
Clearly a gauge transformed vortex operator $V_\Omega$ has exactly the same action on the wave functional $\Psi[A]$,
\[ V \Psi = V_\Omega \Psi \] (150)
which establishes gauge invariance of $V$ in the same sense as in the YM theory.

We now can check the locality of the operator $V$ by calculating straightforwardly the relevant commutation relation. A simple calculation gives $[V(x), V(y)] = 0$. Thus the operators are local with respect to each other. When considering the locality with respect to the Hamiltonian density we are faced with the same ambiguity as in the Abelian theory. The electric part of naive continuum Hamiltonian is not local relative to $V$, since $E_i$ is shifted by the action of $V$ along the curve $C$. Just like in the Abelian case one should consider a properly regularized version of $H$ in order to be able to draw a definite conclusion. In the non-Abelian case such a regularized
Hamiltonian is not available. However in the Abelian case we saw that there is quite a lot of flexibility in defining such a regularized version. In particular we saw that whenever the vortex operators were local with respect to each other, we were always able to define the local Hamiltonian density. We expect that this situation persists in the non-Abelian theory too.

The situation in the continuum limit is again similar to the Abelian case. There are no finite energy solutions of the non-Abelian equations of motion which have finite vorticity. The only way to find such solutions would be again to relax the Gauss’ law constraint by allowing point like charges which correspond to the singular chromoelectric field created by $V$. However again those IR finite configurations will have UV logarithmically divergent energy. In fact taking Abelian ansatz the YM equations of motion reduce to those we considered in the previous section and thus lead to the same energy dependence on the UV cutoff. Strictly speaking this conclusion is only valid for large enough value of $k$, since for small $k$ quantum corrections to this classical analysis may be large. Thus again it is possible that at small $k$ the theory is in a different phase as suggested in [13].

To summarize, we have found that compact CS QED does admit local vortex operators for many values of the CS coefficient $k$. The energy of the vortex excitations however generically is logarithmically UV divergent in the continuum limit. With a particular scaling of the CS coefficient these vortices become light and might condense at small values of $k$. Our results for non-Abelian CSYM theory are similar. Local vortex operators exist, but the particles that carry vorticity are heavy in the continuum limit.

These results are broadly compatible with suggestions made in the literature that at low values of the Chern Simons coefficient the YMCS theory might undergo a phase transition. If this happens it is very likely that this other phase has a broken magnetic symmetry and is therefore confining. This is a very interesting possibility which seems worthwhile exploring.

7 Summary

In this review we have discussed various aspects of the magnetic symmetry in 2+1 dimensional gauge theories. This symmetry is sometimes called topological and is looked at as a somewhat esoteric concept. In fact, as we have discussed here, it is not topological at all, in the sense that its charge density (or group element per unit volume in the discrete case) is a gauge invariant local operator. Moreover in gauge theories without matter fields in the fundamental representation it has a local order parameter. It thus behaves in all respects as a normal nontopological symmetry. As such its mode of realization is rigidly linked to confining properties of the theory. Whenever this symmetry is spontaneously broken, the gauge theory is confining, while unbroken realization of magnetic symmetry signifies existence of massive gauge bosons.
We have constructed the low energy description of gauge theories based on the effective Lagrangian for the order parameter of the magnetic symmetry. This Lagrangian realises confinement in a very straightforward and simple way on the classical level.

In theories with fundamental matter, the magnetic symmetry is still present, but it does not have a local order parameter. As a result it is implemented as a local rather than global symmetry. We have shown how to implement it in a consistent way in the low energy effective Lagrangian in the case of heavy fundamental matter. Interestingly, the resulting effective Lagrangian provides a bag like description of baryons, as quarks surrounded by a "bag" of nonconfining vacuum.

In Chern-Simons theories we proved that the magnetic symmetry is also present and is unbroken in the vacuum. Magnetic vortices - the excitations carrying the magnetic charge are however infinitely heavy in the continuum limit, at least as long as the Chern-Simons coefficient is large. For small coefficient there is a possibility that the vortices become massless and condense, thus driving the theory into the confining phase.

In short, magnetic symmetry is a "trademark" of 2+1 dimensional gauge theories, and most qualitative features of their low energy dynamics is determined by the realization of this symmetry. It is hard to over emphasise the importance of symmetries, especially in strongly interacting theories, where a symmetry may be our only lead in understanding dynamics. Even more so in pure gauge theories, where the magnetic symmetry is the only global symmetry of which existence we know. Hopefully we will learn how to utilize it to the fullest and will thus gain more information about the strongly coupled regime of these theories.

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