SEMI-STABLE VECTOR BUNDLES ON FIBRED VARIETIES

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ABSTRACT. Let \( \pi: Y \rightarrow X \) be a surjective morphism between two irreducible, smooth complex projective varieties with \( \dim Y > \dim X > 0 \). We consider polarizations of the form \( L_c = L + c \cdot \pi^* A \) on \( Y \), with \( c > 0 \), where \( L, A \) are ample line bundles on \( Y, X \) respectively.

For \( c \) sufficiently large, we show that the restriction of a torsion free sheaf \( F \) on \( Y \) to the generic fibre \( \Phi \) is semi-stable as soon as \( F \) is \( L_c \)-semi-stable; conversely, if \( F \otimes O_\Phi \) is \( L \)-stable on \( \Phi \), then \( F \) is \( L_c \)-stable. We obtain explicit lower bounds for \( c \) satisfying these properties. Using this result, we discuss the construction of semi-stable vector bundles on Hirzebruch surfaces and on \( \mathbb{P}^2 \)-bundles over \( \mathbb{P}^1 \), and establish the irreducibility and the rationality of the corresponding moduli spaces.

INTRODUCTION

It is a non-trivial problem to explicitly exhibit (semi-)stable vector bundles in higher dimensions, and to study the geometric properties of the corresponding moduli spaces; these latter are mostly obtained as geometric invariant quotients of (large) quot schemes (see \cite{31,32,14,28,24}). Stable vector bundles of rank exceeding the dimension of the base, with large second Chern class are constructed in \cite{32}. Higher dimensional examples include the instanton bundles \cite{37}, which generalize the well-known ADHM construction \cite{3,5}. Also, the construction of instantons on \( \mathbb{P}^3 \) was extended in \cite{27} to Fano threefolds of index two, with cyclic Picard group.

This article attempts to develop yet another method of constructing (semi-)stable sheaves. We investigate the relationship between the (semi-)stability of a sheaf on the total space of a fibre bundle, and the (semi-)stability of its restriction to the generic fibre. This is different from the relative semi-stability concept in \cite{32}, where one requires that the restriction to each geometric fibre is semi-stable. Let \( \pi: Y \rightarrow X \) be a surjective morphism between two irreducible, smooth complex projective varieties, with \( d_Y := \dim Y > d_X := \dim X > 0 \). Such a \( \pi \) will be called a fibration. Let \( A \) be an ample line bundle on \( X \), and \( L \) be a big, semi-ample (that is some power is globally generated), and \( \pi \)-ample line bundle on \( Y \). For \( c > 0 \), we denote \( L_c := L + c \cdot \pi^* A \), and define the slope of a torsion free sheaf \( G \) on \( Y \) with respect to \( L_c, L, A \) by the formula

\[
\mu_{L_c}(G) := \frac{c_1(G)L_c A^{d_X - 1} L^d_Y - d_X - 1}{\text{rank}(G)}.
\]

The definition is inspired from \cite{28}, which considers semi-stability with respect to a collection of nef divisors. One can interpret \( \mu_{L_c} \) as the slope of the restriction of \( G \) to a general (movable) curve cut out by (multiples of) \( L_c, L, A \). This ties in with \cite{19}, where is argued that in higher dimensions one should consider ‘polarizations’ with respect to movable curves, rather than ample divisors. If \( X \) is a curve, the formula coincides, after replacing \( c \) by \( (d_Y - 1)c \), with the usual slope with respect to \( L_c \). Moreover, regardless of \( d_X \), \( (L_c, L, A) \)-semi-stability implies usual \( L_c \)-semi-stability, and, conversely, usual \( L_c \)-stability implies \( (L_c, L, A) \)-stability, for \( c \gg 0 \).

Theorem. Let \( F \) be a torsion free sheaf of rank \( r \) on \( Y \). Then there is a constant \( k_F \) such that the following hold:

(i) If \( F \) is \( L_c \)-(semi-)stable with \( c > k_F \), then the restriction of \( F \) to the generic fibre of \( \pi \) is semi-stable, and \( F \) is \( L_a \)-(semi-)stable for all \( a \geq c \).

(ii) If the restriction of \( F \) to the generic fibre of \( \pi \) is stable, then \( F \) is \( L_c \)-stable for all \( c > k_F \).

The same holds for principal \( G \)-bundles on \( Y \), for connected, reductive, linear algebraic groups \( G \).

2000 Mathematics Subject Classification. 14F05,14J60,14J26,14D20.
Thus any \( L_c \)-\((semi-)stable torsion free sheaf\) on \( Y \), with \( c \gg 0 \), determines a \textit{rational} map from \( X \) to the (course) moduli space \( \bar{\mathcal{M}}_{L_c} \) of \( \pi \)-relatively \((semi-)stable sheaves\) on \( Y \). Usually, this map \textit{does not extend} to \( X \), which is the main difference to \textit{loc. cit}. The result is a technical ingredient, a dimensional reduction, which is effective for varieties admitting fibrations onto lower dimensional varieties, such that one has \textit{a priori} knowledge about the \((semi-)stable sheaves\) on the generic fibre. Polarizations of the form \( L \), \( c \gg 0 \), have been considered in \cite{16,17} \((and\ \cite{11})\) for vector (respectively principal) bundles on elliptically fibred surfaces, and in \cite{18} on ruled surfaces.

The theorem is proved in section \( \S \) where we derive \textit{two distinct, explicit} lower bounds for the constant \( k_F \) above: they involve respectively the slope with respect to \( L \) \((see\ \cite{16})\,\text{and}\, the\ discriminant of} \( F \) \((see\ \cite{18})\. Our approach follows \cite{24} Theorem 5.3.2 and Remark 5.3.6, where the result is proved for surfaces, and \cite{28} Section 3, where are developed higher dimensional techniques. For elliptically fibred surfaces, the result appears in \cite{16} Section 2, \cite{17} Theorem 7.4.

Sections \( \S \) and \( \S \) illustrate the general principle with explicit examples. We study the moduli spaces of \((semi-)stable vector bundles\) on \textit{Hirzebruch surfaces}, and on \( \mathbb{P}^2\)-\textit{fibre bundles over} \( \mathbb{P}^1 \) respectively. Although each topic has its own intricacies, the underlying principle is the same: a \((semi-)stable vector bundle\) on a fibration is a family of \((semi-)stable vector bundles\) on the fibres. It is surprising that these topics are \textit{strongly connected}; for describing the geometry of vector bundles on \( \mathbb{P}^2\)-fibrations over \( \mathbb{P}^1 \), one needs to understand the case of Hirzebruch surfaces first. Thus, our approach yields a \textit{unified treatment}, and indeed generalizes \textit{several scattered subject matters}.

The \textit{former example}, that of \((semi-)stable vector bundles\) on Hirzebruch surfaces, was investigated in \cite{8,34,9}, where the authors describe the geometry of the corresponding moduli spaces. We also mention \cite{2}, for proving the non-emptyness and irreducibility of the moduli space of stable vector bundles with \( c_1 = 0 \) on a large class of rational surfaces, including the Hirzebruch surfaces. The last few years experienced a revived interest \cite{3} in constructing and understanding the properties of the moduli spaces of \((semi-)stable torsion free sheaves\) on Hirzebruch surfaces. Compared with the references above, we emphasize the \textit{brevity} and the \textit{detail} of our description of the geometry of the moduli space \( \bar{\mathcal{M}}_{L_c}^{st}(r; 0, n) \) of \( L_c\)-\((semi-)stable\) rank \( r \) torsion free sheaves on the Hirzebruch surface \( Y \), with \( c_1 = 0, c_2 = n \). Theorem \( \S \) reveals the existence of a stratification of \( \bar{\mathcal{M}}_{L_c}^{st}(r; 0, n) \) by locally closed strata, and the density of the stable vector bundles. Furthermore, we prove in \( \S \) and \( \S \) the existence of a surjective morphism onto \( \text{Hilb}_{\mathbb{P}^1}^n \cong \mathbb{P}^n \), the Hilbert scheme of \( n \) points on \( \mathbb{P}^1 \). For \( n = c_2 = 2 \), the existence of this morphism is obtained in \cite{8} by using monad theoretic techniques \cite{36}, but is defined only for vector bundles. This morphism is the key for proving:

\textbf{Theorem.} \textit{(see} \cite{8} \textit{and} \cite{24}) \( \bar{\mathcal{M}}_{L_c}^{st}(r; 0, n) \) \textit{is rational}, for any \( n \geq r \geq 2 \). \textit{Hence, for} \( \ell = 1 \), \textit{it follows that the moduli space} \( \mathcal{M}_{P^2}(r; 0, n) \) \textit{of stable rank} \( r \) \textit{vector bundles on} \( \mathbb{P}^2 \), with \( c_1 = 0, c_2 = n \) \textit{and} \( n \geq r \), \textit{is rational}.

The result should be compared with \cite{9}, where is proved that \( \bar{\mathcal{M}}_{L_c}^{st}(r; c_1, n) \) \textit{is rational} for any \( c_1 \), under the assumption that the discriminant \( 2rn - (r - 1)c_1^2 \) is very large, but without giving any bounds. The rationality of \( \mathcal{M}_{P^2}(r; c_1, c_2) \) has been intensely studied over the past decades; see \cite{3,33,20,30} for \( r = 2 \), \cite{20,29} for \( r = 3 \), \cite{15,11} for arbitrary \( r \). See also \cite{30} for a quiver-theoretical approach. Most of these references prove rationality under some arithmetical restrictions on \( r, c_1, c_2 \). In our approach, we \( (almost) \) \textit{explicitly exhibit a rational variety} which is birational to \( \mathcal{M}_{P^2}(r; 0, n) \).

Our \textit{second example} concerns \((semi-)stable vector bundles of arbitrary rank\), with Chern classes \( c_1 = 0, c_2 = n \cdot [O_n(1)]^2, c_3 = 0 \) on \( Y_{a,b} = \mathbb{P}(O_{P^1} \oplus O_{P^1(-a)} \oplus O_{P^1(-b)}) \). Here \( 0 \leq a \leq b \) are two integers, so \( \pi: Y_{a,b} \to \mathbb{P}^1 \) is a \( \mathbb{P}^2\)-\textit{fibre bundle} over \( \mathbb{P}^1 \). Moduli spaces of \textit{rank two vector bundles} on \( \mathbb{P}^n\)-\textit{bundles} over curves were studied in \cite{10}, and more generally on Fano fibrations in \cite{35}, using extensions of rank one sheaves; thus the method is strongly adapted to the rank two case. In \cite{54} we prove \( (as\ expected) \) that a \((semi-)stable vector bundle\) on \( Y_{a,b} \), satisfying some natural hypotheses, is the cohomology of a 1-parameter family of monads on \( \mathbb{P}^2 \). This \textit{generalizes} the construction of the \textit{instanton bundles} on \( \mathbb{P}^3 \) trivialized along a line \((see\ \cite{5,15})\).

The next step is to investigate the geometric properties of the corresponding moduli space.
**Theorem.** (see [3, 8].) The moduli space of semi-stable vector bundles on \( Y_{a,b} \) as above contains a non-empty ’main component’ which is irreducible, generically smooth, and rational.

The irreducibility of the full moduli space is a difficult issue, even in the case of \( Y_{0,1} \), the blow-up of \( \mathbb{P}^3 \) along a line (see [13, 44]). This leads us to single out the main component of the full moduli space, in a similar vein to [12]. Let us remark that on threefolds, unlike for surfaces, the semi-stability and the Riemann-Roch formula are not sufficient to address the generic smoothness. Concerning the rationality issue, the author of this article could find only the reference [10] Corollary 3.6] dealing with the rationality of certain moduli spaces of rank two vector bundles on higher dimensional varieties. We prove that the main component is birational to the moduli space of framed vector bundles on a reducible surface (a wedge) obtained by glueing a plane and a Hirzebruch surface along a line. Then our conclusion follows from the results obtained before. Apparently, this is new even in the much studied case of \( \mathbb{P}^3 \); the introduction of [12] mentions the rationality of the moduli space of instanton vector bundles, for \( c_2 = 2, 3, 5 \).

The results are stated for varieties defined over \( \mathbb{C} \). However, the usual base change arguments imply that they hold over any algebraically closed ground field of characteristic zero.

1. **Relative semi-stability for vector bundles**

Let \( \pi : Y \to X \) be a surjective morphism between irreducible, smooth, projective varieties with

\[ d_Y := \dim Y > d_X := \dim X > 0, \]

and denote by \( \Phi \) its generic fibre. We consider an ample line bundle \( \pi^* A \) and we let \( \text{NS}(Y) \subset G \) be the (unique) maximal, saturated subsheaf of \( F \) to the generic fibre \( \Phi \), given that \( \Phi \) is big and semi-ample on \( L \).

**Definition 1.1.** (i) For a torsion free sheaf \( G \) on \( Y \), we denote \( \xi_G := \frac{c_1(G)}{rk(G)} \in \text{NS}(Y)_\mathbb{Q} \), and define the slope of \( G \) with respect to \( L_c, L, A \) by the formula

\[ \mu_{L_c}(G) := \xi_G L_c A^{d_Y-1} L^{d_Y-d_X-1} = \xi_G \cdot (A^{d_X-1} L^{d_Y-d_X} + c A L^{d_Y-d_X}). \]  

(ii) The slope of a torsion free sheaf \( G' \) on the generic fibre \( \Phi \) is defined as

\[ \mu_{L,\Phi}(G') := \xi_{G'} \cdot A^{d_X} L^{d_Y-d_X-1}. \]

(iii) The torsion free sheaf \( F \) on \( Y \) is \( L_c \)-semi-stable if for all saturated subsheaves \( \mathcal{G} \subset F \) holds

\[ \mu_{L_c}(\mathcal{G}) \leq \mu_{L_c}(F). \]  

(iv) We say that \( F \) is \( \pi \)-relatively \( L \)-semi-stable if its restriction to the generic fibre of \( Y \) is \( (\pi \downarrow X) \)-stable with respect to \( L \otimes \mathcal{O}_X \).

**Notation 1.2.** Let \( G \) be a torsion free sheaf on \( Y \).

(i) Henceforth we denote by \( G_\Phi \) the restriction of \( G \) to the generic fibre of \( Y \to X \).

(ii) For any \( c > 0 \), we let \( G^{L_c-HN} \) be the maximal, saturated, \( L_c \)-de-semi-stabilizing subsheaf of \( \mathcal{G} \), that is the first term of its Harder-Narasimhan filtration with respect to \( L_c \). We remark that, since \( L \) is big and semi-ample on \( Y \), one can still define \( G^{L_c-HN} \), corresponding to \( c = 0 \), by a limiting argument (see [28, pp. 263]).

(iii) Let \( G^{L_{\pi-HN}} \) be the (unique) maximal, saturated subsheaf of \( \mathcal{G} \), whose restriction to the generic fibre \( \Phi \) is the first term of the Harder-Narasimhan filtration of \( G_\Phi \) with respect to \( L_\Phi \). It is defined as the sum of all the subsheaves \( G' \subset \mathcal{G} \) such that \( G'_\Phi = (G_\Phi)^{L_{\pi-HN}} \). \( G^{L_{\pi-HN}} \) is called the first term of the \( \pi \)-relative Harder-Narasimhan filtration of \( \mathcal{G} \) with respect to the relatively ample line bundle \( L \) (See [24, Section 2.3]).

(iv) To save space, instead of the exact sequence \( 0 \to A \to B \to C \to 0 \) we will write \( A \subset B \to C \).

For a torsion free sheaf \( F \) of rank \( r \) on \( Y \), we investigate the semi-stability of the restriction of \( F \) to the generic fibre \( \Phi \), given that \( F \) is \( L_c \)-semi-stable. We prove that \( \mu_{L_c} \)-semi-stability implies \( \pi \)-semi-stability, and conversely, \( \pi \)-relative stability implies \( \mu_{L_c} \)-stability, for \( c \) sufficiently
large. The technical issue is to determine lower bounds for the parameter \( c \), which guarantee these implications.

The \( L_c \)-stability of a sheaf is an open property for \( c > 0 \), independent of the relative semi-stability. One typically obtains different (semi-)stability conditions \([12]\), as the parameter \( c \geq 0 \) varies. The effect of the relative semi-stability is that of stabilizing the various concepts.

**Lemma 1.3.** (i) The set \( \{ c \in \mathbb{R} > 0 \mid \mathcal{F} \text{ is } L_c-(\text{semi-)stable} \} \) is an interval.
(ii) Assume that \( \mathcal{F} \) is \( L_c-(\text{semi-)stable} \), and relatively semi-stable. Then \( \mathcal{F} \) is \( L_{c+\varepsilon}-(\text{semi-)stable} \), for all \( \varepsilon > 0 \).

*Proof.* (i) Let \( a, b \in I \), and \( a < c < b \). Then \( c = (1-\lambda)a + \lambda b \) for some \( \lambda \in (0,1) \), and for any subsheaf \( \mathcal{G} \subset \mathcal{F} \), we have
\[
\mu_L(\mathcal{G}) = (1-\lambda) \cdot \mu_{La}(\mathcal{G}) + \lambda \cdot \mu_{Lb}(\mathcal{G}).
\]
(ii) Indeed, one has
\[
\mathcal{G} \subset \mathcal{F} \text{ is } L_{c+\varepsilon}-(\text{semi-)stable} \Rightarrow \mathcal{G} \subset \mathcal{F} \text{ is } L_c-(\text{semi-)stable}.
\]

Finally, let us remark that the definition \((1.1)\) of the slope differs from the usual one
\[
\mu_{Lc}^{\text{usual}}(\mathcal{G}) := \xi_\mathcal{G} L_c^{d_Y - 1} = \xi_\mathcal{G} \left[ L_c^{d_Y - 1} + \cdots + c^{d_X - 1} \frac{d_Y - 1}{d_X - 1} A^{d_X - 1} L_c^{d_Y - d_X} + c^{d_X} \frac{d_Y - 1}{d_X - 1} A^{d_X} L_c^{d_Y - d_X - 1} \right].
\]

By using our result, we can compare the two (semi-)stability concepts. The outcome is analogous to the relationship between the Gieseker and the (usual) slope (semi-)stability.

**Proposition 1.4.** \( L_c \)-stable \([1.1]\) \( \Rightarrow \) usual \( L_c \)-stable \([1.3]\), for \( c \gg 0 \);
\[
\text{usual } L_{c+\varepsilon} \text{-semi-stable } (1.3) \Rightarrow \text{L}_{c+\varepsilon} \text{-semi-stable } (1.1), \quad \text{for } c \gg 0.
\]
Consequently, the main theorem still holds for (usually) \( L_c \)-(semi-)stable sheaves.

*Proof.* View \([13]\) as a polynomial in the indeterminate \( c \), and observe that the two (rightmost) terms are, up to a scaling factor, precisely the slope \([1.1]\). Our main result provides the bounds (depending on the numerical data of \( \mathcal{F} \) only), necessary for proving the two implications. \( \square \)

If one is interested in the usual \( L_c \)-slope (semi-)stability, it is still possible to deduce effective bounds in this setting, albeit more involved. Below are a couple of examples.

(i) For \( d_X = 1 \), that is \( X \) is a curve, holds \( \mu_{Lc}^{\text{usual}}(\mathcal{G}) = \mu_{L(d_Y - 1) - 1}(\mathcal{G}) \), for any sheaf \( \mathcal{G} \) on \( Y \). Thus the constant \( k_c \) in the introduction gets replaced by \( k_c' := k_c/(d_Y - 1) \).
(ii) For \( d_Y = 2 \), that is \( X \) is a surface, holds \( L_c AL_c^{d_Y - 3} = AL_c^{d_Y - 2} + c A^2 L_c^{d_Y - 3} \), and \( L_c^{d_Y - 1} = c(d_Y - 1) \left[ \frac{1}{c(d_Y - 1)} A^{d_X - 1} L_c^{d_Y - 1} + AL_c^{d_Y - 2} + c \frac{d_Y - 2}{2} A^2 L_c^{d_Y - 3} \right] \).

If \( s := \mu_{Lc}^{\text{usual}}(\mathcal{G}) - \mu_{Lc}^{\text{usual}}(\mathcal{F}) \), where \( \mathcal{G} \) is the first term of the (usual) Harder-Narasimhan filtration of \( \mathcal{F} \) with respect to \( L \), then the main theorem holds for \( k'_c := \max \left\{ \frac{2k_c}{d_Y - 2}, \frac{d_X - 1}{d_Y - 1} \right\} \).

1.1. Relative semi-stability in terms of the slope of \( \mathcal{F} \). The following lemma is inspired from \([28\text{ pp. 263}]\).

**Lemma 1.5.** For \( c \) sufficiently large, the first term of the Harder-Narasimhan filtration of \( \mathcal{F} \) with respect to \( L_c \) is independent of \( c \). More precisely, it holds
\[
\mathcal{F}^{L_{c-HN}} = \mathcal{F}^{L_{\text{rel-HN}}}, \quad \forall c > a_F := r^2 \left( M_F - m_F \right)/A^{d_X}, \quad (1.4)
\]
with \( M_F := \mu_L(\mathcal{F}) \) and \( m_F := \mu_L(\mathcal{F}) \). In particular, if \( \mathcal{F} \) is \( L_{a+\varepsilon} \)-(semi-)stable for some \( a > a_F \), then \( \mathcal{F} \) is \( L_{c-\text{(semi-)stable}} \) for all \( c \geq a \).

*Proof.* The slope of a subsheaf \( \mathcal{G} \subset \mathcal{F} \) with respect to \( L_c \) is \( \mu_{L_c}(\mathcal{G}) = c \cdot \mu_L, \phi(\mathcal{G}) + \mu_L(\mathcal{G}) \). We endow the set \( S(\mathcal{F}) \) of all polynomials in \( c \) of the form \( \mu_{L_c}(\mathcal{G}) \) above, corresponding to some \( \mathcal{G} \subset \mathcal{F} \), with the lexicographic order (for which the indeterminate \( c \) is greater than \( 1 \)). The coefficients of the polynomials in \( S(\mathcal{F}) \) are bounded from above by \( \mu_{L, \phi}(\mathcal{F}) \) and \( \mu_{L, \phi}(\mathcal{F}) \) respectively, so \( S(\mathcal{F}) \) admits a maximal element \( S(\mathcal{F})_{\text{max}} \). Let us determine it. The coefficient of \( c \) is at most \( \mu_{L, \phi}(\mathcal{F}) \), and is attained for the subsheaves \( \mathcal{G} \subset \mathcal{F} \) such that \( \mu_{L, \phi}(\mathcal{G}) = \mu_{L, \phi}(\mathcal{F}) \). Then the maximal polynomial is:
\[
S(\mathcal{F})_{\text{max}} = c \cdot \mu_{L, \phi}(\mathcal{F}) + \max \left\{ \mu_L(\mathcal{G}) \mid \mathcal{G} \subset \mathcal{F} \text{ subsheaf}, \mu_{L, \phi}(\mathcal{G}) = \mu_{L, \phi}(\mathcal{F}) \right\}.
\]
Claim The maximum above equals \( \mu_L(\mathcal{F}^{L}\text{-rel-HN}) \). Indeed, let \( \mathcal{G} \) be such that \( \mu_{L, \phi}(\mathcal{G}^{L}\text{-rel-HN}) = \mu_{L, \phi}(\mathcal{F}^{L}\text{-rel-HN}) \), and \( \mu_L(\mathcal{G}) \) is maximal with this property. The \( L \)-slope of \( \mathcal{G} \) increases by taking its saturation (as \( L \) is semi-ample and big), so we may assume that \( \mathcal{G} \subset \mathcal{F} \) is saturated.

The uniqueness of the Harder-Narasimhan filtration of \( \mathcal{F}_\phi \) implies \( \mathcal{G}_\phi^{L}\text{-rel-HN} = \mathcal{F}^{L}\text{-rel-HN} \), so \( \mathcal{G} \subset \mathcal{F}^{L}\text{-rel-HN} \) by the maximality of \( \mathcal{F}^{L}\text{-rel-HN} \) (see [12]). Hence \( \mathcal{F}^{L}\text{-rel-HN}/\mathcal{G} \) is a torsion sheaf which vanishes over the generic fibre. Its support is a proper subscheme \( Z \subset Y \) such that \( \pi_*Z \subset X \) is also proper. It follows that

\[
\mu_L(\mathcal{F}^{L}\text{-rel-HN}) = \mu_L(\mathcal{G}) + \sum_{Z' \subset Z} \rho_{Z'} \cdot Z' A^{d_Y - 1} L^{d_Y - d_X}, \quad \text{with all } \rho_{Z'} > 0.
\]

But codim\(_X \pi_*Z' = 1 \) because \( \pi_*Z \subset X \) is proper, hence \( Z' A^{d_Y - 1} L^{d_Y - d_X} \) is strictly positive.

Overall, we proved that \( S(\mathcal{F})_{\max} = \mu_{L, \phi}(\mathcal{F}^{L}\text{-rel-HN}) + \mu_{L, \phi}(\mathcal{F}^{L}\text{-rel-HN}) \). To complete the proof, it is enough to show that if \( \mu_{L, \phi}(\mathcal{G}) < \text{lex} S(\mathcal{F})_{\max} \) (as a polynomials in \( c \)) for a subsheaf \( \mathcal{G} \subset \mathcal{F} \), then \( \mu_{L, \phi}(\mathcal{G}) < S(\mathcal{F})_{\max}(c) \) for all \( c > a_F \). There are two possibilities:

Case 1 \( \mu_{L, \phi}(\mathcal{F}^{L}\text{-rel-HN}) > \mu_{L, \phi}(\mathcal{G}) \Rightarrow \mu_{L, \phi}(\mathcal{F}^{L}\text{-rel-HN}) - \mu_{L, \phi}(\mathcal{G}) > A^d_x/r^x \).

Then follows:

\[
S(\mathcal{F})_{\max}(c) - \mu_{L, \phi}(\mathcal{G}) = c(\mu_{L, \phi}(\mathcal{F}^{L}\text{-rel-HN}) - \mu_{L, \phi}(\mathcal{G})) + (\mu_{L}(\mathcal{F}^{L}\text{-rel-HN}) - \mu_{L}(\mathcal{G}))
\geq \frac{c A^d_x}{r} + m_F - M_F = \frac{1}{r} \cdot [c A^d_x - r^2(M_F - m_F)] > 0.
\]

Case 2 \( \mu_{L, \phi}(\mathcal{F}^{L}\text{-rel-HN}) = \mu_{L, \phi}(\mathcal{G}) \), \( \mu_{L}(\mathcal{F}^{L}\text{-rel-HN}) > \mu_{L}(\mathcal{G}) \). Then holds \( S(\mathcal{F})_{\max}(c) - \mu_{L, \phi}(\mathcal{G}) > 0 \).

The last statement is a direct consequence of lemma [1.3].

**Theorem 1.6.** Let \( a_F \) be as in [1.4]. Then the following hold:

(i) If \( \mathcal{F} \) is \( L_a \)-semi-stable with \( a > a_F \), then \( \mathcal{F}_\phi \) is semi-stable.

(ii) If \( \mathcal{F}_\phi \) is stable, then \( \mathcal{F} \) is \( L_c \)-stable for all \( c > a_F \).

**Proof.** (i) The previous lemma implies that \( \mathcal{F} \) is \( L_c \)-semi-stable for all \( c \geq a \), and therefore \( \mathcal{F}^{L}\text{-rel-HN} = \mathcal{F}^{L}\text{-rel-HN} = \mathcal{F} \). Hence \( \mathcal{F}_\phi \) is indeed semi-stable.

(ii) Conversely, assume that \( \mathcal{F}_\phi \) is stable, so \( \mathcal{F}^{L}\text{-rel-HN} = \mathcal{F} \) and \( m_F = \mu_L(\mathcal{F}) \). If there is a destabilizing proper, saturated subsheaf \( \mathcal{G} \) of \( \mathcal{F} \), then

\[
\mu_L(\mathcal{F}) + c \cdot \mu_{L, \phi}(\mathcal{F}) = \mu_L(\mathcal{F}) \leq \mu_{L, \phi}(\mathcal{G}) = \mu_L(\mathcal{G}) + c \cdot \frac{\mu_{L, \phi}(\mathcal{G})}{M_F} \leq \mu_{L, \phi}(\mathcal{F}) - A^d_x/r(r-1).
\]

This contradicts the choice of \( c > r^2(M_F - m_F)/A^d_x \), so \( \mathcal{F} \) is \( L_c \)-stable. \( \square \)

1.2. Relative semi-stability in terms of the Chern classes of \( \mathcal{F} \). Here we derive a result analogous to theorem [1.6] above, with the difference that the lower bound for the parameter \( c \) is expressed in terms of the characteristic classes of \( \mathcal{F} \). For a torsion free sheaf \( \mathcal{G} \) on \( Y \), we denote \( \Delta(\mathcal{G}) = 2\text{rk}(\mathcal{G})c_2(\mathcal{G}) - (\text{rk}(\mathcal{G}) - 1)c_1^2(\mathcal{G}) \in H^4(Y; \mathbb{Q}) \) the discriminant of \( \mathcal{G} \). For shorthand, let

\[
[\gamma]_{AL} := \gamma \cdot A^{d_Y - 1} L^{d_Y - d_X - 1}, \quad \text{for all } \gamma \in H^4(Y; \mathbb{Q}).
\]

We consider the ‘light cone’

\[
K^+ := \{ \beta \in \text{NS}(Y)_\mathbb{Q} | [\beta^2]_{AL} > 0 \text{ and } [\beta \cdot D]_{AL} \geq 0, \text{ for all nef divisors } D \subset Y \},
\]

and we define

\[
C(\alpha) := \{ \beta \in K^+ | [\alpha \cdot \beta]_{AL} > 0 \}, \quad \forall \alpha \in \text{NS}(Y)_\mathbb{Q} \setminus \{0\}.
\]

**Proposition 1.7.** (i) Let \( \mathcal{F} \) be a torsion free sheaf on \( Y \) with \( c_1(\mathcal{F}) = 0 \).

(iia) If \( \mathcal{F} \) is not \( \pi \)-relatively semi-stable, then there is a proper saturated subsheaf \( \mathcal{G} \) of \( \mathcal{F} \) such that:

\[
\mu_{L, \phi}(\mathcal{G}) \geq \frac{A^d_x}{r},
\]

and either \( [\xi_\phi^2]_{AL} \geq 0 \), or \( 0 > [\xi_\phi^2]_{AL} \geq -\frac{2r}{r-1} \cdot [c_2(\mathcal{F})]_{AL} \).

(iib) If \( \mathcal{F} \) is not \( L_c \)-stable, then there is a proper saturated subsheaf \( \mathcal{G} \subset \mathcal{F} \) such that \( \mu_{L, \phi}(\mathcal{G}) \geq 0 \), and \( \xi_\phi \) satisfies one of (1.4).
(ii) The statements \((i_a)\) (respectively \((i_b)\)) still hold for \(c_1(\mathcal{F})\) arbitrary, with the modifications:
\[
\mu_{L,\Phi}(\mathcal{G}) \geq \mu_{L,\Phi}(\mathcal{F}) + \frac{\Delta^d_0}{r_1} \quad \text{(respectively, } \mu_{L,\Phi}(\mathcal{G}) \geq \mu_{L,\Phi}(\mathcal{F})). \tag{1.6'}
\]
and either \(\left[(\xi_\mathcal{G} - \xi_\mathcal{F})^2\right]_{AL} \geq 0, \) or \(0 > \left[(\xi_\mathcal{G} - \xi_\mathcal{F})^2\right]_{AL} \geq -\frac{\Delta(\mathcal{F})}{r_1}. \tag{1.7'}\)

The result is similar to the Bogomolov inequality \cite{24} Theorem 7.3.4 and \cite{28} Theorem 3.12], with the difference that here we simultaneously control the discriminant and the relative slope of the relatively de-semi-stabilizing subsheaf.

\textbf{Proof.} \((i_a)\) As \(\mathcal{F}_\Phi\) is not semi-stable, \(\mu_{L,\Phi}(\mathcal{F}) \geq \Delta_{\mathcal{F}}^d > 0.\) For shorthand we write \(\mathcal{F}' := \mathcal{F}_{rel-HN}\) and let \(\mathcal{F}'' := \mathcal{F}/\mathcal{F}'.\) If \(\left[\xi_\mathcal{F}'\right]_{AL} \geq 0,\) there is nothing to prove, so let us assume \(\left[\xi_\mathcal{F}'\right]_{AL} < 0.\) (Thus, in particular, \(\xi_\mathcal{F}' \notin -K^+\) and \(C(\xi_\mathcal{F}') \neq 0\).)

\textbf{Case 1} Assume that holds \(\left[\Delta(\mathcal{F}')\right]_{AL} \geq 0.\) The equality (see \cite{24} pp. 207))
\[
\left[\Delta(\mathcal{F}')\right]_{AL} \geq 0 \quad \text{and} \quad \left[\xi_\mathcal{F}'\right]_{AL} \geq -\frac{\left[\Delta(\mathcal{F}')\right]_{AL}}{r_1} \quad \text{implies that} \quad \left[\Delta(\mathcal{F}')\right]_{AL} > 0 \quad \text{and} \quad \left[\xi_\mathcal{F}'\right]_{AL} \geq -\frac{\left[\Delta(\mathcal{F}')\right]_{AL}}{r_1}. \tag{1.6''}
\]

\textbf{Case 2} Assume that holds \(\left[\Delta(\mathcal{F}')\right]_{AL} < 0.\) According to \cite{28} Theorem 3.12, there is a saturated subsheaf \(\mathcal{G}' \subset \mathcal{F}'\) such that \(\mathcal{G}' \geq \xi_\mathcal{F}' \in K^+.\) As \(\mathcal{G}' = \xi_\mathcal{F}' + \xi_\mathcal{F}',\) we deduce \(\mu_{L,\Phi}(\mathcal{G}') \geq 0,\) hence \(\mu_{L,\Phi}(\mathcal{G}') \geq \frac{\Delta_{\mathcal{G}'}^d}{r_1},\) and \(C(\mathcal{G}') \geq C(\xi_\mathcal{F}).\)

\textbf{Case 3} Assume that holds \(\left[\Delta(\mathcal{F}'')\right]_{AL} < 0.\) As before, there is a saturated subsheaf \(\mathcal{G}'' \subset \mathcal{F}''\) such that \(\xi_\mathcal{G}'' - \xi_\mathcal{F}' \in K^+.\) For \(\mathcal{G}'' := \operatorname{Ker}(\mathcal{F} \to \mathcal{F}'/\mathcal{G}'')\) holds \(\xi_\mathcal{G}'' = \xi_\mathcal{F}' \geq 0,\) with \(\rho' \geq 0 \geq 0\) (see \cite{24} pp. 206, equation (7.6)). Once again, this implies \(\mu_{L,\Phi}(\mathcal{G}'') \geq 0,\) hence \(\mu_{L,\Phi}(\mathcal{G}'') \geq \frac{\Delta_{\mathcal{G}''}^d}{r_1},\) and also \(C(\mathcal{G}'') \geq C(\xi_\mathcal{F}).\)

In both cases 2 and 3 we can replace \(\mathcal{F}'\) with another saturated subsheaf of \(\mathcal{F}\) which is still relatively de-semi-stabilizing (but not necessarily of maximal slope), and the corresponding cone \(\mathcal{L}_{10}^d\) is strictly larger. But this increasing sequence of cones stops because there are only finitely many possibilities for them. (See the proof of \cite{24} Theorem 7.3.3.\) Thus, after finitely many steps, we reach either the case 1, or the case \(\left[\xi_\mathcal{F}'\right]_{AL} \geq 0.\) The proof of \((i_b)\) is identical.

(ii) The proof is similar, except that one has to replace overall \(\xi_\mathcal{G}\) by \(\xi_\mathcal{G} - \xi_\mathcal{F} .\) (This is the explanation for the weaker bound \((1.6'').\) \)

Now we derive an inequality which relates the fibrewise and the absolute slope of a saturated sheaf on \(Y.\) The equality
\[
\left[\left(\xi \cdot A\right)_{AL} - \left[A \cdot L_c\right]_{AL}\right]_{AL} = 0 \tag{1.8}
\]
holds for any \(\xi \in NS(Y)_0\) and \(c \geq 0.\) As \(L\) on \(Y\) is semi-ample and \(A\) on \(X\) is ample, we can view this expression as the intersection product on a smooth (complete intersection) surface in \(Y,\) representing (a multiple of) the class \(A_{dx-1}L_{dy}^{dx-1},\) so the Hodge index theorem yields:
\[
0 \geq \left[\left(\xi \cdot A\right)_{AL} - \left[A \cdot L_c\right]_{AL}\right]^2_{AL} \Rightarrow 2 \left[A \cdot L_c\right]_{AL} \cdot \left(\xi \cdot A\right)_{AL} - \left[A \cdot L_c\right]_{AL} \geq \left[\xi \cdot A\right]_{AL}^2_{AL} + \left(L_c^2\right)_{AL} + \left(A_{dx}L_{dy}^{dx-1}\right)^2_{AL}, \tag{1.9}
\]
and the marked term above is
\[
(*) = A_{dx-1}L_{dy}^{dx-1}L_c^2 = A_{dx-1}L_{dy}^{dx-1+1} + 2c \cdot A_{dx}L_{dy}^{dx-1} = 2c \cdot A_{dx}L_{dy}^{dx-1}. \]

After dividing both sides of \((1.9)\) by \(A_{dx}L_{dy}^{dx-1},\) we deduce
\[
2\left[\xi \cdot A\right]_{AL} \cdot \left(\xi \cdot L_c\right)_{AL} \geq 2c \cdot \left[\xi \cdot A\right]_{AL}^2_{AL} + A_{dx}L_{dy}^{dx-1} \cdot \left[\xi^2\right]_{AL}. \tag{1.10}
\]

\textbf{Theorem 1.8.} (i) Assume \(c_1(\mathcal{F}) = 0,\) and let \(c_\mathcal{F} := r(r-1)\frac{A_{dx}L_{dy}^{dx-1}}{(A_{dx})^2} \cdot \left[c_2(\mathcal{F})\right]_{AL}.\) The following statements hold:
If $F$ is $L_a$-semi-stable with $a > c_F$, then $F_\Phi$ is semi-stable. In particular, if $F$ is $L_a$-
 Semi-stable, then it is $L_c$-semi-stable for all $c \geq a$.

(ii) If $F_\Phi$ is stable, then $F$ is $L_c$-stable for all $c > c_F$.

(i) For $c_1(F)$ arbitrary, (i), (ii) still hold for $c' := \frac{\tau^2(t-1)}{2} \cdot \frac{A^dX L^{dy-dx}}{(A^dX)^2} \cdot \Delta(F)_{AL}$.

Proof. (i) Indeed, assume that $F_\Phi$ is not semi-stable. Then there is a subsheaf $G$ of $F$ satisfying

$$2\mu_L(G) \cdot \mu_L(G) \geq 2a(A^dX)^2 + A^dX L^{dy-dx} \cdot \begin{bmatrix} 2 \end{bmatrix}_{AL},$$

and the right hand side is strictly positive: for $\begin{bmatrix} 2 \end{bmatrix}_{AL} \geq 0$ is clear, and otherwise $0 > \begin{bmatrix} 2 \end{bmatrix}_{AL} > -\frac{2}{r^2} [c_2(F)]_{AL}$. Thus $\mu_L(G) > 0$, which contradicts the $L_a$-semi-stability of $F$.

Conversely, if $F_\Phi$ is stable and $F$ is not $L_c$-stable over $Y$, there is a saturated subsheaf $G$ of $F$

such that $\mu_L(G) \geq 0$ and $\mu_L(G) < 0$. As before, the right hand side of the previous inequality

is strictly positive, so $\mu_L(G) < 0$, a contradiction.

(ii) Repeat the argument by using proposition 1.7(ii).

□

Remark 1.9. Recall that we required $L$ to be big, $\pi$-ample and semi-ample. Is possible to slightly weaken the bigness assumption, that is $L^{dy} > 0$. Proposition 1.7 hence also theorem 1.8 still hold for $L^{dy} = 0$ and $AL^{dy-1} > 0$. Indeed, in this case, both equations (1.8) and (1.9) hold for $L$ replaced by $L_\varepsilon$, with $\varepsilon > 0$ small, and (1.10) follows by a limiting argument.

1.3. Relative semi-stability for principal bundles. Our previous conclusions can be generalized to principal bundles with reductive structure groups. First we introduce the semi-stability concept with respect to $L_c, L, A$ (we call it $L_c$-(semi-)stability), analogous to (1.1).

Definition 1.10. (i) A principal $G$-bundle $\Omega$ on $Y$, where $G$ is a connected reductive linear group, is $L_c$-(semi-)stable if for any parabolic subgroup $P \subset G$ and any reduction $\sigma : U \to \Omega_u/P$ defined over an open subset $U \subset Y$ whose complement has co-dimension at least two in $Y$ holds

$$\deg_{L_c}(\sigma^*T_{\Omega_u/P}^{rel}) := c_1(\sigma^*T_{\Omega_u/P}^{rel}) : L_c A^{dX-1} L^{dy-dx-1} \geq 0, \quad (\geq)$$

where $T_{\Omega_u/P}^{rel}$ stands for the relative tangent bundle on $\Omega_u/P$.

(ii) The semi-stability of the restriction $\Omega_\Phi$ is defined with respect to $L_\Phi$, as usual.

Theorem 1.11. Let $G$ be a connected, reductive algebraic group, and $\Omega$ be a principal $G$-bundle on $Y$. There is a constant $c_\Omega$, such that the following hold:

(i) If $\Omega$ is $L_\Omega$-semi-stable, with $a > c_\Omega$, then its restriction $\Omega_\Phi$ is $L$-semi-stable.

(ii) If $\Omega_\Phi$ is $L$-stable, then $\Omega$ is $L_c$-stable, for all $c > c_\Omega$.

Proof. (i) The principal bundle $\Omega$ is semi-stable if and only if the vector bundle $\text{ad}(\Omega)$, induced by the adjoint representation $G \to \text{GL}(\text{End}(g))$ of $G$, is semi-stable (see [38 Corollary 3.18]).

(ii) Let us assume that $\Omega$ is not $L_c$-stable. Then there is an open subset $U \subset Y$, whose complement has co-dimension at least two in $Y$, and a reduction $(P, \sigma)$ of $\Omega$ over $U$, such that

$$\deg_{L_c}(\sigma^*T_{\Omega_u/P}^{rel}) + c \cdot c_1(\sigma^*T_{\Omega_u/P}^{rel}) A^{dX} L^{dy-dx-1} = \deg_{L_c}(\sigma^*T_{\Omega_u/P}^{rel}) \leq 0. \quad (1.12)$$

The restriction of $(P, \sigma)$ to the generic fibre still defines a reduction of $\Omega_\Phi$ over $U \cap \Phi$, and the complement of this latter in $\Phi$ has co-dimension at least two, too. The stability of $\Omega_\Phi$ implies

$$c_1(\sigma^*T_{\Omega_u/P}^{rel}) A^{dX} L^{dy-dx-1} = A^{dX} \deg_{L_\Phi}(\sigma^*T_{\Omega_u/P}^{rel}) \geq A^{dX} > 0.$$
2. APPLICATION: STABLE VECTOR BUNDLES ON HIRZEBRUCH SURFACES

For $\ell \geq 0$, the Hirzebruch surface $Y_\ell := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-\ell))$ is a $\mathbb{P}^1$-fibre bundle over $\mathbb{P}^1$. Let $\pi : Y_\ell \to \mathbb{P}^1$ be the projection, and $\mathcal{O}_a(1)$ the relatively ample line bundle. The ‘exceptional line’ $\Lambda = \mathcal{O}(\mathcal{O}_{\mathbb{P}^1} \oplus 0) \to Y_\ell$ has self-intersection $\Lambda^2 = -\ell$, $\mathcal{O}_a(1) = \ell \mathcal{O}_{\mathbb{P}^1}(1) + \mathcal{O}_Y(\Lambda)$, and the relative (and (absolute) canonical classes of $Y_\ell$ are respectively

$$
\kappa_{Y_\ell/\mathbb{P}^1} = \mathcal{O}_{\ell}(-2) + \ell \mathcal{O}_{\mathbb{P}^1}(1) \quad \text{and} \quad \kappa_{Y_\ell} = \mathcal{O}_{\ell}(-2) + (\ell - 2) \mathcal{O}_{\mathbb{P}^1}(1).
$$

(2.1)

The goal of this section is to study the geometry of the moduli space of polarizations sheaves on $Y_\ell$, of rank $r$, with $c_1(\mathcal{F}) = 0$ and $c_2(\mathcal{F}) = n$, which are semi-stable with respect to $L_c = \mathcal{O}_a(1) + c \cdot \mathcal{O}_{\mathbb{P}^1}(1)$. Our approach is similar to [13] Section 1, although in loc. cit. the authors consider polarizations $L_c$ with $0 < c \ll 1$ (while we consider $c \gg 0$).

2.1. Construction of semi-stable sheaves on $Y_\ell$.

**Lemma 2.1.** Let $\mathcal{F}$ be an $L_c$-semi-stable torsion free sheaf of rank $r$ on $Y_\ell$, with $c > r(r-1)n$, $c_1(\mathcal{F}) = 0$ and $c_2(\mathcal{F}) = n$. Then the following statements hold:

(i) We have $\Phi \cong \mathbb{P}^1$ and $\mathcal{F}_\Phi \cong \mathcal{O}_a^{\otimes r}$. If $\mathcal{F}$ is $L_c$-stable, then $h^1(Y_\ell, \mathcal{F}) = n - r$, so $n \geq r$.

(ii) The Chern character of the derived direct image $\pi_* \mathcal{F} = \pi_+ \mathcal{F} - R^1 \pi_* \mathcal{F}$ is

$$
\text{ch}(\pi_* \mathcal{F}) = \text{ch}_0(\pi_* \mathcal{F}) \oplus \text{ch}_1(\pi_* \mathcal{F}) = r - n.
$$

(2.2)

(iii) The natural homomorphism $f_* : \pi_+ \mathcal{F} \to \pi_* \mathcal{F}$ is injective, and $\text{det}(f_*)^\vee \in |\pi_* \mathcal{O}_{\mathbb{P}^1}(n\mathcal{F})|$ with $n \mathcal{F} \leq n$. We denote by $Z_{f_*} = \{ \sum m_i x_i \}$ the divisor of $\text{det}(f_*)^\vee$, with $x_i \in \mathbb{P}^1$, $m_i \geq 1$, and $\sum m_i = n \mathcal{F}$. The sheaf $R^1 \pi_* \mathcal{F}$ is torsion on $\mathbb{P}^1$, and $\text{deg} \mathcal{F} = n - \mathcal{F}$.

(iv) $\pi_* \mathcal{F}$ is locally free of rank $r$ on $\mathbb{P}^1$, so it splits:

$$
\pi_* \mathcal{F} \cong \bigoplus_{j=1}^p \mathcal{O}(-a_j)^{\otimes r_j}, \quad \text{with} \quad 0 \leq a_1 < \ldots < a_p \quad \text{and} \quad r_1 + \ldots + r_p = r, \quad a_1 r_1 + \ldots + a_p r_p = n \mathcal{F}.
$$

(2.3)

If $\Gamma(Y_\ell, \mathcal{F}) = 0$, then $a_j \geq 1$ for all $j$.

(v) $\pi_* \mathcal{F}(-\Lambda) = 0$ and $R^1 \pi_* \mathcal{F}(-\Lambda)$ is a torsion sheaf on $\mathbb{P}^1$, with $\text{deg} R^1 \pi_* \mathcal{F}(-\Lambda) = n$.

**Proof.** (i) By theorem [13], $\mathcal{F}_\Phi$ is semi-stable, so $\mathcal{F}_\Phi \cong \mathcal{O}_a^{\otimes r}$ because $\Phi \cong \mathbb{P}^1$. (For $\ell = 0$, see remark [13].) If $\mathcal{F}$ is stable, then $\Gamma(Y_\ell, \mathcal{F}) = H^2(Y_\ell, \mathcal{F}) = 0$, and $\dim H^1(\mathcal{F}) = n - r$ by the Riemann-Roch formula.

(ii)-(v) If $a_1 < 0$, then $\mathcal{O}_{\mathbb{P}^1} \subset \mathcal{F}(a_1)$, which contradicts the semi-stability of $\mathcal{F}$. Further, as $\mathcal{F}(\Lambda) \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\otimes r}$, we deduce $\Gamma(U, \pi_* \mathcal{F}(\Lambda) \cong 0$, for all open $U \subset \mathbb{P}^1$. The Grothendieck-Riemann-Roch theorem yields the formula for $\pi_*(\mathcal{F})$.

**Lemma 2.2.** The sheaf $\mathcal{Q}_{\mathcal{F}}$ defined by $0 \to \pi_+ \mathcal{F} \to \mathcal{F} \to \mathcal{Q}_{\mathcal{F}} \to 0$ has the following properties:

(i) Any local section $\sigma$ through a closed point $p \in \text{Supp}(\mathcal{Q}_{\mathcal{F}})$ is $\mathcal{Q}_{\mathcal{F},p}$-regular, that is the multiplication $\mathcal{Q}_{\mathcal{F},p} \to \mathcal{Q}_{\mathcal{F},p}$ by the defining equation of $\sigma$ is injective. Thus the homological dimension and the depth of $\mathcal{Q}_{\mathcal{F},p}$ are both equal one.

(ii) $\text{Supp}(\mathcal{Q}_{\mathcal{F}}) = \pi^{-1}(Z_{\mathcal{F}})$, and its Chern classes are $c_1(\mathcal{Q}_{\mathcal{F}}) = n \mathcal{F} \cdot [\mathcal{O}_{\mathbb{P}^1}(1)]$, $c_2(\mathcal{Q}_{\mathcal{F}}) = n$.

(iii) There is an exact sequence $0 \to \mathcal{Q}_{\mathcal{F}} \otimes \mathcal{O}_\Lambda \to R^1 \pi_* \mathcal{F}(-\Lambda) \to R^1 \pi_* \mathcal{Q}_{\mathcal{F}} \to 0$.

(iv) If $\text{det}(f_*)^\vee$ has simple zeros $x_1, \ldots, x_n \in \mathbb{P}^1$, then $\mathcal{Q}_{\mathcal{F}} = \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(-b_i)$, with $b_i \geq 1$ and $b_1 + \ldots + b_n = n$. (Thus, for $n \mathcal{F} = n$, we have $b_1 = \ldots = b_n = 1$.)

(v) An isomorphism $\mathcal{F} \xrightarrow{\theta} \mathcal{F'}$ induces the commutative diagram:

$$
\begin{array}{cccccc}
0 & \rightarrow & \pi_+ \mathcal{F} & \rightarrow & \mathcal{F} & \rightarrow & \mathcal{Q}_{\mathcal{F}} & \rightarrow & 0 \\
\pi_\mathcal{F} \xrightarrow{\theta} & \theta & \cong & \theta & \cong & \theta & \cong \\
0 & \rightarrow & \pi_+ \mathcal{F} & \rightarrow & \mathcal{F} & \rightarrow & \mathcal{Q}_{\mathcal{F}} & \rightarrow & 0.
\end{array}
$$
Proof. (i) We choose local coordinates $x, y$ around $p$, with $p = (0, 0)$ and $\pi$ given by $(x, y) \mapsto x$, and $\sigma(x) = (x, 0)$. (So, the local equation defining $\sigma$ is $y$.) As $\text{Supp}(Q_F) = \pi^{-1}(Z_F)$, there is $m > 0$ such that $Q_{F,F}$ is annihilated by $(x^m)$. Assume that the multiplication by $y$ is not injective. Then there is a non-trivial, zero-dimensional submodule of $Q_{F,F}$ annihilated by $I := \langle x^m, y \rangle \subset O_{Y_{i,F}}$, so $\pi^*L \subset F$ is non-saturated. The saturation $G \subset F$ of $\pi^*L$ has the property that $G := G/\pi^*L \subset Q_F$ is non-trivial, zero-dimensional, and
g\in \text{Ext}^1(Q', \pi^*L)\cong \text{Ext}^1(\pi^*L, \kappa_Y \otimes Q')^\vee = 0.
It follows $G = Q'/\pi^*L \subset F$, which contradicts that $F$ is torsion free. The second statement follows from the Auslander-Buchsbaum formula (see e.g. [24, Section 1.1]).

(ii)+(iii) The identities $\pi_*Q_F = 0$, $R^1\pi_*F \cong R^1\pi_*Q_F$, are immediate, and $\text{Coker}(\pi_*F \to F_L) = Q_F \otimes O_\Lambda$ follows by restricting to $\Lambda$. (Use that $\Lambda$ is $Q_F$-regular.) Now insert into $0 \to \pi_*F \to F_L \to R^1\pi_*F(\Lambda) \to R^1\pi_*F \to 0$ and obtain (iii).

(iv) Indeed, $Q_F$ is torsion free along $\pi^{-1}(x_i)$, so $Q_F = \bigoplus_{i=1}^n O_{\pi^{-1}(x_i)}(-b_i)$. Then $\pi_*Q_F = 0$ implies that all $b_i \geq 1$, and their sum is $c_2(Q_F) = n$.

(v) The statement is obvious. \hfill \Box

Lemma 2.3. The following equalities hold:

\begin{align*}
\text{ext}^0(Q_F, \pi^*L) &= \text{ext}^0(Q_F, F) = 0, \quad (Q_F \text{ is a torsion sheaf}), \\
\text{ext}^2(Q_F, \pi^*L) &= \text{ext}^2(Q_F, F) = \text{ext}^2(Q_F, Q_F) = 0, \\
\text{ext}^1(Q_F, \pi^*L) &= \text{ext}^1(Q_F, F) = r(n + n_F), \quad (\text{by the Riemann-Roch formula}).
\end{align*}

Proof. First we notice

\begin{align*}
\text{ext}^2(Q_F, \pi^*L) &= \text{ext}^0(\pi^*L, \kappa_Y \otimes Q_F)_{\text{Supp}(Q_F) = \pi^{-1}(Z_F)} = h^0(\mathcal{O}_{Y}(\Lambda - 2\Lambda) \otimes Q_F) = 0.
\end{align*}

The last vanishing holds because $\Lambda$ is $Q_F$-regular, so $\mathcal{O}_{Y}(\Lambda - 2\Lambda) \otimes Q_F \subset Q_F$. By applying the functors $\text{Hom}(Q_F, \cdot)$ and $\text{Hom}(\cdot, F)$ respectively to the sequence defining $Q_F$, we obtain:

\begin{align*}
&0 \to \frac{\text{Ext}^1(Q_F, \pi^*L)}{\text{End}(Q_F)} \to \text{Ext}^1(Q_F, F) \to \text{Ext}^1(Q_F, Q_F) \to 0 \to \text{Ext}^2(Q_F, F) \to \text{Ext}^2(Q_F, Q_F) \to 0, \quad (2.4) \\
\text{and} \quad &0 \to \frac{\text{End}(L_\mathcal{F})}{\text{End}(F)} \to \text{Ext}^1(Q_F, F) \to \text{Ext}^1(\pi^*L, F) \to \text{Ext}^2(Q_F, F) \to 0. \quad (2.5)
\end{align*}

We deduce that $\text{Ext}^2(Q_F, F) \cong \text{Ext}^2(Q_F, Q_F)$.

Claim \quad $\text{Ext}^2(Q_F, Q_F) = 0$.

Indeed, the (reduced) support of $Q_F$ is a finite, disjoint union of fibres of $\pi$, so we may assume that $Q_F$ is supported on the thickening of a single fibre $\pi^{-1}(o), \ o \in \mathbb{P}^1$. In this case the annihilator $\text{Ann}(Q_F) \supset \langle x^m \rangle$, for some $m \geq 1$, that is $Q_F$ is a $\mathbb{C}[x]/(x^m)$-module. For $m = 1$, $Q_F$ is a (torsion free) sheaf on $\pi^{-1}(o)$, so $\text{Ext}^2(Q_F, S) = \text{Ext}^2(S, Q_F) = 0$ for any $\mathbb{C}[x]/(x)$-module $S$. For the inductive step, notice that $\text{Ann}(xQ_F) \supset \langle x^{m-1} \rangle$, $\mathcal{T} := Q_F/xQ_F$ is a $\mathbb{C}[x]/(x)$-module, and we have the exact sequence $xQ_F \subset Q_F \to \mathcal{T}$. We deduce the exact sequences

\begin{align*}
\text{Ext}^2(T, xQ_F) & \to \text{Ext}^2(T, T) \\
\text{Ext}^2(Q_F, xQ_F) & \to \text{Ext}^2(Q_F, Q_F) \to \text{Ext}^2(Q_F, T) \\
\text{Ext}^2(xQ_F, xQ_F) & \to \text{Ext}^2(xQ_F, T),
\end{align*}

and apply the induction hypotheses. Similarly, $\text{Ext}^2(Q_F, S) = \text{Ext}^2(S, Q_F) = 0$ holds, for any $\mathbb{C}[x]/(x)$-module $S$. \hfill \Box

The previous lemmas show that any $L_e$-semi-stable $\mathcal{F}$ fits into an exact sequence

\begin{equation}
0 \to \pi^*L \xrightarrow{\imath} \mathcal{F} \xrightarrow{\pi} Q \to 0,
\end{equation}

with $L := \bigoplus_{j=1}^n \mathcal{O}(-a_j)\mathcal{O}^\vee, a_j \geq 1$, as in [23], and $Q$ as in [22]. The homomorphism $q$ is the (canonically defined) quotient map, while $f \in \text{Hom}(\pi^*L, \mathcal{F}) = \text{End}(L)$ is defined modulo $\text{Aut}(L)$. The
equivalence classes of sequences \( \sref{2.6} \) are parameterized by \( \text{Ext}^1(Q, \pi^*\mathbb{L}) \), where \( (f, q) \) is equivalent to \((\varphi f, q\varphi^{-1})\) for all \( \varphi \in \text{Aut}(\mathcal{F}) \). So there is an \( \text{Aut} (\mathbb{L}) / \text{Aut} (\mathcal{F}) \)-ambiguity in defining \( f \) in \( \sref{2.6} \).

**Remark 2.4.** \( \dim \text{End}(\mathbb{L}) \geq r^2 \) and \( \text{ext}^1(\pi^*\mathbb{L}, \mathcal{F}) \geq r(n - n_F) \). Equality holds (in both places) if and only if

\[
\mathbb{L} = \mathbb{L}_{n_F, r} := \mathcal{O}_p \oplus (a_1) \oplus \mathcal{O}_p \oplus (a_2) \oplus r \quad \text{with} \quad \begin{cases} a_1 = \lfloor \frac{n_F}{r} \rfloor, & r_1 = r + r \cdot \lfloor \frac{n_F}{r} \rfloor - n_F; \\ a_2 = \lfloor \frac{n_F}{r} \rfloor + 1, & r_2 = n - r \cdot \lfloor \frac{n_F}{r} \rfloor. \end{cases}
\]

(Here \( \lfloor \cdot \rfloor \) stands for the integral part. For \( n_F \) divisible by \( r \), the \( a_2 \)-term is missing.) Indeed, \( \text{ext}^0(\pi^*\mathbb{L}, \mathcal{F}) - \text{ext}^1(\pi^*\mathbb{L}, \mathcal{F}) = r^2 - r(n - n_F) \), and we notice that

\[
\dim \text{End} \left( \bigoplus_{j=1}^{p} \mathcal{O}(a_j) \right) = \sum_{i \geq 1} r_ir_i(a_i - a_i + 1) \geq \sum_i r_i^2 + 2 \sum r_i = (\sum r_i)^2 = r^2.
\]

Equality occurs precisely when the sequence \( a_1 < a_2 < \ldots \) contains at most two integers such that \( a_2 - a_1 = 1 \). In this case \( a_1, a_2, r_1, r_2 \) are as in \( \sref{2.7} \).

### 2.2. Basic properties of the moduli space of semi-stable sheaves on \( Y_t \).

**Lemma 2.5.** (i) The dimension of the locally closed subset

\[
M_{\mathbb{L}} := \{ \mathcal{F} \mid \mathcal{F} \text{ fits into } \sref{2.6} \} = \{ \mathcal{F} \mid \pi_* \mathcal{F} \cong \mathbb{L} \} \subset \mathcal{M}_{Y_t}^{L_r}(r; 0, n),
\]

is at most \( \text{ext}^1(\mathcal{F}, \mathcal{F}) - r(n - n_F) \).

(ii) \( \{ \mathcal{F} \in M_{\mathbb{L}} \mid \det(f)^* \text{ has simple zeros} \} \subset M_{\mathbb{L}} \) is dense. Thus the irreducible components of \( M_{\mathbb{L}} \) are \( M_{\mathbb{L}, b} := \{ \mathcal{F} \in M_{\mathbb{L}} \mid \mathcal{F}/\pi^*\mathbb{L} = Q \text{ of the form } \sref{2.6} \text{ iv} \} \), with \( b = (b_1, \ldots, b_{n_F}) \) as in \( \sref{2.7} \).

(iii) If \( r \geq 2 \), then the generic \( \mathcal{F} \in M_{\mathbb{L}, b} \) is locally free, for all \( (\mathbb{L}, b) \) as above. (This statement is false for \( r = 1 \), unless \( n = 0 \).)

**Proof.** (i) The infinitesimal deformations of \( \mathcal{F} \) induced by deformations of \( \pi^*\mathbb{L} \rightarrow \mathcal{F} \), where both \( \mathcal{F} \) and \( f \) vary, are given by \( \text{Im}(\text{Ext}^1(Q, \mathcal{F}) \rightarrow \text{Ext}^1(\mathcal{F}, \mathcal{F})) \cong \text{Ext}^1(Q, \mathcal{F})/\text{End}(\mathbb{L})/\text{End}(\mathcal{F}) \) and

\[
\begin{array}{ccc}
\text{Ext}^1(Q, \mathcal{F}) & \xrightarrow{d_{\mathbb{L}}} & \text{Ext}^1(\mathcal{F}, \mathcal{F}) \\
\text{Ext}^1(Q, \mathbb{L}) & \xrightarrow{d_{\mathbb{L}}} & \text{Ext}^1(\mathcal{F}, \mathbb{L})
\end{array}
\]

are the differentials of \( \mathcal{F} \rightarrow \mathcal{F} \) and \( \mathcal{F} \rightarrow \mathbb{L} \) respectively. (Notice that \( d_{\mathbb{L}} \) passes to the quotient, since \( \text{End}(\mathbb{L}) \subset \text{Ker}(d_{\mathbb{L}}) \).) The sequence \( \sref{2.6} \) implies

\[
\text{ext}^1(Q, \mathcal{F}) - \dim \left( \text{End}(\mathbb{L})/\text{End}(\mathcal{F}) \right) \leq \text{ext}^1(\mathcal{F}, \mathcal{F}) - r(n - n_F).
\]

(ii) **Step 1** Apply \( \text{Hom}(Q, \cdot) \) to \( \mathcal{F}(-\Lambda) \subset \mathcal{F} \rightarrow \mathcal{F}_\Lambda \), and deduce \( \text{Ext}^1(Q, \mathcal{F}) \rightarrow \text{Ext}^1(Q_\Lambda, \mathcal{F}_\Lambda) \) is surjective. Thus the infinitesimal deformations of \( \mathbb{L} \rightarrow \mathcal{F}_\Lambda \) on \( \Lambda \) come from infinitesimal deformations of \( \pi^*\mathbb{L} \rightarrow \mathcal{F} \) on \( Y_t \). (Notice that both deformations are unobstructed, as the corresponding \( \text{Ext}^2 \) groups vanish.)

**Step 2** Apply \( \text{Hom}(Q_\Lambda, \cdot) \) to \( \mathcal{L} \subset \mathcal{F}_\Lambda \rightarrow \mathcal{Q}_\Lambda \) (\( \Lambda \) is \( \mathcal{Q}_F \)-regular, by \( \sref{2.2} (i) \)), and deduce that \( \text{Ext}^1(Q_\Lambda, \Lambda) \rightarrow \text{Ext}^1(Q_\Lambda, \mathcal{Q}_\Lambda) \) is surjective. Thus, the deformations of \( Q_\Lambda \) on \( \Lambda \) (they are unobstructed) can be lifted to deformations of \( \mathbb{L} \rightarrow \mathcal{F}_\Lambda \).

**Step 3** \( Q_\Lambda \) is a sheaf of length \( n_F \) on \( \mathbb{F}^1 \), and its generic deformation is the structure sheaf of \( n_F \) distinct points on \( \mathbb{F}^1 \). By our previous discussion, this deformation is induced by a deformation \( \pi^*\mathbb{L} \rightarrow \mathcal{F}_\Lambda \) of \( \pi^*\mathbb{L} \rightarrow \mathcal{F} \) on \( Y_t \), and the quotient \( Q' \) is of the form \( \sref{2.2} \text{ iv} \).

---

1. \( \mathbb{L} \) is enlightening to outline the analytic proof of this statement for \( \mathcal{F} \) locally free. Let \( \mathcal{F} \) be the subjacent \( C^\infty \) vector bundle. (This is the same for all holomorphic deformations of \( \mathcal{F} \).) A holomorphic structure \( \mathcal{F}_\mathbb{C} \) in \( \mathbb{F}^1 \) is determined by \( \delta_\alpha : C^\infty(\mathbb{F}) \rightarrow \mathbb{F}^{0, 1}(\mathbb{F}) \) satisfying the Leibniz rule, and \( \delta_0^2 = 0 \). A deformation \( \mathcal{F}_\mathbb{C} = (\mathcal{F}, \delta_\alpha) \) of \( \mathcal{F}_\mathbb{C} \) is given by \( (\alpha, \phi) \in \mathbb{F}^{0, 1}(\text{End}(\mathbb{F})) \) such that \( \delta_\alpha \circ \phi = 0 \), that is \( \alpha \in H^1(\text{End}(\mathcal{F}_\mathbb{C})) \). A deformation \( \pi^*\mathbb{L} \rightarrow \mathcal{F}_\mathbb{C} \) is \( \mathcal{F} = (\mathcal{F}, \delta_\alpha) \) is given by \( (\alpha, \phi) \in \mathbb{F}^{0, 1}(\text{End}(\mathbb{F})) \) such that \( \delta_\alpha \circ \phi = 0 \). This is equivalent to saying that \( \alpha \in \text{Ker}(H^1(\text{End}(\mathcal{F}_\mathbb{C})) \rightarrow H^1(\text{Hom}(\pi^*\mathbb{L}, \mathcal{F}_\mathbb{C}))) \).

In this case, \( \phi \) is determined up to some \( \psi \in \text{End}(\mathbb{L}) \). However, these choices induce trivial deformations of \( \mathcal{F}_\mathbb{C} \).
(iii) By dualizing (2.6), the statement is equivalent to the surjectivity of the generic homomorphism
\[ \pi^*\mathcal{L} \cong \bigoplus_{i=1}^{n} \mathcal{O}_{\pi^{-1}(x_i)}(b_i). \]
This is true, since \( e \) factorizes
\[ \pi^*\mathcal{L} \twoheadrightarrow \bigoplus_{i=1}^{n} \pi^*\mathcal{L}_{\pi^{-1}(x_i)} \cong \bigoplus_{i=1}^{n} \mathcal{O}_{\pi^{-1}(x_i)} \to \bigoplus_{i=1}^{n} \mathcal{O}_{\pi^{-1}(x_i)}(b_i), \]
and each \( \mathcal{O}_{\pi^{-1}(x_i)}(b_i) \) can be generated by two sections \( (r \geq 2) \).

**Theorem 2.6.** Let \( \tilde{M}^{L_c}_{Y_i}(r; 0, n) \) be the moduli space of (equivalence classes of) \( L_c \)-semi-stable torsion free sheaves on \( Y_i \), with \( c > r(r - 1)n \). We denote by \( M^{L_c}_{Y_i}(r; 0, n)^{vb} \) the open subset corresponding to \( L_c \)-stable vector bundles. Then the following statements hold:

(i) \( \tilde{M}^{L_c}_{Y_i}(r; 0, n) \) is the union of the irreducible, locally closed strata \( M_{L_c b} \).
(ii) For any \( (L, b) \), the generic \( F \in M_{L_c b} \) satisfies:
   (a) \( \mathcal{F}_\Lambda \cong \mathcal{O}_\Lambda^\mathcal{O} \), \( \mathcal{F}_\Lambda \cong \mathcal{O}_\Lambda^\mathcal{O} \) for generic \( \lambda \in |\mathcal{O}(1)| \);
   (b) \( \mathcal{F} \) is locally free, for \( r \geq 2 \).

Conversely, any vector bundle \( F \) with this property is \( L_c \)-semi-stable.

(iii) \( M_{L_{n, r}} \) is the unique top dimensional, open stratum, which corresponds to sheaves \( F \) with \( \pi_* F \cong \mathbb{L}_{n, r} \).

(iv) The generic point \( F \in M_{L_{n, r}} \) satisfies \( \mathcal{Q}_F \cong \bigoplus_{i=1}^{n} \mathcal{O}_{\pi^{-1}(x_i)}(-1) \) with \( \{x_1, \ldots, x_n\} \subset \mathbb{P}^1 \)

Proof. (i)+(ii) Everything is proved in lemma 2.5 except that \( \mathcal{F}_\Lambda \) is trivializable. We know that the points \( \mathcal{F} \in M_{L_c b} \) with \( \mathcal{Q}_F \) of the form \( 2.2(1) \) are dense. For \( \mathcal{Q} \) of this form, the generic \( \mathcal{F}_\Lambda \in \text{Ext}^1(Q_\Lambda, L) \) is the trivial vector bundle on \( \Lambda \), and \( \text{Ext}^1(Q, \pi^*L) \to \text{Ext}^1(Q_\Lambda, L) \) is surjective. For proving that the generic \( F \in M_{L_c b} \) is trivializable along the generic \( \lambda \), we notice that \( \lambda \) is a flat deformation of \( (\lambda + \ell \text{ fibres of } \pi) \). But \( \mathcal{F}_\Lambda \) and \( \mathcal{F}_\Lambda \) are both trivializable, and the claim follows.

Conversely, if \( \mathcal{F}_\Lambda \) and \( \mathcal{F}_\Lambda \) are both trivial, then deg \( G \leq 0, \deg G \leq 0 \), for any saturated subsheaf \( G \subset F \), so \( F \) is \( L_c \)-semi-stable, indeed.

(iii) We should prove that the generic \( F \) is \( L_c \)-stable. Otherwise it admits a proper, stable subsheaf \( G \subset F \) such that deg \( G \) = deg \( F/G \) = 0, both \( G, F/G \) are torsion free, and \( F/G \) is semi-stable.

\( (G \) is the first term of the Jordan–Hölder filtration of \( F \).) For shorthand, we denote rk \( (G) = r \) and \( c_2(G) = n' \). As before, \( n' \geq r' \) and also:
\[
\chi(F) = \chi(G) + \chi(F/G) \Rightarrow n = c_2(F) = c_2(G) + c_2(F/G) = n' + c_2(F/G).
\]

Since \( G, F/G \) are semi-stable, the Bogomolov inequality 2.3 Theorem 3.4.1] implies \( 0 \leq n' \leq n \).

**Claim** The dimension of the infinitesimal deformations of \( F \) is strictly larger than that of \( G \):
\[ 2rn - r^2 + 1 > 2rn' - (r')^2 + 1 \iff (n - n')(n + n') > (n - n' - (r - r'))(n + n' - (r + r')). \]  
(2.9)

For the left hand side we used ext\( ^1(G, G) = 1 \) and ext\( ^2(G, G) = 0 \), as \( G \) is stable.) The latter inequality is indeed satisfied:
\[
(n - n')(n + n') \overset{(*)}{=} (n - n' - (r - r'))(n + n' - (r + r'))
\]

Concerning \((*)\): if \( n - n' - (r - r') \leq 0 \), then everything is fine, since in (2.9) the right hand side is negative. This proves the claim.

We obtained a contradiction: on one hand, the generic \( F \) is properly semi-stable, while, on the other hand, the possible de-semi-stabilizing subsheaves have strictly lower deformation space.

This proves that the generic \( F \) is indeed \( L_c \)-stable.

(iv) For any stable \( F \in \tilde{M}^{L_c}_{Y_i}(r; 0, n) \), the Riemann-Roch formula yields \( \chi(\text{End}(F)) = -2rn + r^2 \), while the stability of \( F \) implies \( h^0(\text{End}(F)) = 1, h^2(\text{End}(F)) = 0 \). Thus \( M^{L_c}_{Y_i}(r; 0, n)^{vb} \) is smooth, of dimension \( 2rn - r^2 + 1 \). On the other hand, all the strata \( M_{L_c} \) with \( L \neq \mathbb{L}_{n, r} \), are strictly lower dimensional.

**Theorem 2.7.** There is a well-defined surjective morphism
\[
h : \tilde{M}^{L_c}_{Y_i}(r; 0, n) \to \text{Hilb}_{\mathbb{P}^1}^n \cong \mathbb{P}^n, \quad F \mapsto \text{Supp} R^1\pi_*\mathcal{F}(\Lambda).
\]  
(2.10)
Its generic fibre is \((2rn - r^2 - n + 1)\)-dimensional, the quotient of an open set in \(\mathbb{A}^{2nr}\) by the linear action of a \((r^2 + n - 1)\)-dimensional group.

**Proof.** We saw that \(\deg_{\pi_{\ast}} R^1\pi_{\ast}\mathcal{F}(\Lambda) = n\), for all \(\mathcal{F}\), so \(h\) is well-defined set theoretically. Actually, there is a technical detail: \(\tilde{M}^L_Y(r; 0, n)\) parameterizes equivalence classes of \(L_c\)-semi-stable sheaves where \(\mathcal{F}\) and \(\mathcal{F}'\) are equivalent if their Jordan-Hölder factors are isomorphic. Thus we must prove that if \(\text{grad}^H(\mathcal{F}) \cong \text{grad}^H(\mathcal{F}')\), then \(R^1\pi_{\ast}\mathcal{F}(\Lambda) \cong R^1\pi_{\ast}\mathcal{F}'(\Lambda)\). For this, observe that if \(\mathcal{F}\) fits into \(0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0\), with \(\mathcal{F}_1, \mathcal{F}_2\) semi-stable (of degree zero), then \(0 \to R^1\pi_{\ast}\mathcal{F}_1(\Lambda) \to R^1\pi_{\ast}\mathcal{F}(\Lambda) \to R^1\pi_{\ast}\mathcal{F}_2(\Lambda) \to 0\). The conclusion follows now by induction on the length of the Jordan-Hölder filtration.

Returning to the theorem, we prove that \(h\) satisfies the functorial property of \(\text{Hilb}^n_{\mathbb{P}^1}\). For a scheme \(S\), we denote \(Y_S := S \times Y_{\ell}, \mathbb{P}^S := S \times \mathbb{P}^1, \pi_S := (\text{id}_S, \pi), \) etc., and \(\mathcal{O}(1) := \mathcal{O}_\pi(1) + \mathcal{O}_{\mathbb{P}^1}(1)\).

**Claim** Let \(\mathcal{G}\) be a torsion free sheaf on \(Y_S\), such that \(\pi_{\ast}\mathcal{G}_s = 0\), for all \(s \in S\) (thus \(\pi_{\ast}\mathcal{G} = 0\), too). Then the natural homomorphism \(\gamma_s : R^1\pi_{\ast}\mathcal{G} \otimes \mathcal{O}_{S,s} \to R^1\pi_{\ast}\mathcal{G}_s\) is an isomorphism, for all \(s \in S\), and the sheaf \(R^1\pi_{\ast}\mathcal{G}\) on \(\mathbb{P}^S\) is \(S\)-flat.

Since flatness is a local property, is enough to prove the statement for \(S = \text{Spec}(A)\), where \((A, \mathfrak{m})\) is a local ring, and \(s = \text{Spec}(A/\mathfrak{m}) \in S\).

Any \(\mathcal{G}\) admits a finite resolution \(\ldots \to \mathcal{L}_2 \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{G} \to 0\), with \(\mathcal{L}_j = \mathcal{O}(-c_j)^{\oplus m_{ij}}\). We prove the claim by induction on the length of the resolution. If the length is zero, that is \(\mathcal{G} = \mathcal{O}(-c)^{\oplus m}\) (notice that \(\pi_{\ast}\mathcal{G}_s = 0\) implies \(c > 0\)), then \(R^1\pi_{\ast}\mathcal{G}\) is locally free on \(\mathbb{P}^S\), and \(\gamma_s\) is an isomorphism.

For the inductive step, we fit \(\mathcal{G}\) into \(0 \to \mathcal{G}' \to \mathcal{L} \to \mathcal{G} \to 0\), where \(\mathcal{L} = \mathcal{O}(-c)^{\oplus m}, c > 0\), \(\mathcal{G}'\) admits a resolution of length one less, and also \(\pi_{\ast}\mathcal{G}'_s = 0\). By the hypothesis, \(\gamma'_s\) (for \(\mathcal{G}'\)) is an isomorphism, which immediately yields that \(\gamma_s\) (for \(\mathcal{G}\)) is an isomorphism too. Also, from the exact sequence \(0 \to R^1\pi_{\ast}\mathcal{G}' \to R^1\pi_{\ast}\mathcal{G} \to R^1\pi_{\ast}\mathcal{L} \to 0\), we deduce the equivalences:

\[
\begin{align*}
R^1\pi_{\ast}\mathcal{G} & \text{ is } A\text{-flat} \iff \text{Tor}^1_1 \left( R^1\pi_{\ast}\mathcal{G}, \frac{A}{\mathfrak{m}} \right) = 0 \iff R^1\pi_{\ast}\mathcal{G} \otimes_A \frac{A}{\mathfrak{m}} \text{ is } A\text{-flat} \iff R^1\pi_{\ast}\mathcal{L} \otimes_A \frac{A}{\mathfrak{m}} \cong R^1\pi_{\ast}\mathcal{G}' \cong R^1\pi_{\ast}\mathcal{G}_s.
\end{align*}
\]

The homomorphism on the right hand side is indeed injective, because \(\pi_{\ast}\mathcal{G}_s = 0\).

Now we apply the claim to our setting. For a torsion free sheaf \(\mathcal{F}\) on \(Y_S\) which is \(S\)-fibrewise \(L_c\)-semi-stable, we have \(\pi_{\ast}\mathcal{F}(\Lambda_S) = 0\), so \(R^1\pi_{\ast}\mathcal{F}(\Lambda_S)\) is \(S\)-flat. Hence \(h\) is a morphism, as desired. Its generic fibre is the quotient of an open subset of \(\text{Ext}^1 \left( \bigoplus_{i=1}^n \mathcal{O}_{\pi^{-1}(x_i)}(-1), \pi^*\mathbb{L}_{m,r} \right)\), which is \(2rn\)-dimensional, by the action of \((\text{Aut}(\mathbb{L}_{n,r}) \times (\mathbb{C}^*)^n) / \mathbb{C}^*\). Since \(h\) is dominant and \(\tilde{M}^L_Y(r; 0, n)\) is projective, we deduce that \(h\) is surjective. \(\square\)

### 2.3. Rationality issues.

We conclude this section with a self-contained proof of the rationality of \(\tilde{M}^L_Y(r; 0, n)\), and some applications. This result is proved in [2] for arbitrary \(c_1\), under the assumption that the discriminant \(\Delta(\mathcal{F})\) is very large. However, no explicit bounds are given.

**Theorem 2.8.** \(\tilde{M}^L_Y(r; 0, n)\) is a rational variety, for all \(n \geq r \geq 2\).

**Proof.** It is enough to prove that \(M^\circ := \{ \mathcal{F} \in M_{L_{n,r}} \mid \det(f) \text{ has simple zeros} \}\) is rational. Any \(\mathcal{F} \in M^\circ\) fits into

\[
0 \to \pi^*\mathbb{L} = \pi^* \left( \mathcal{O}_{\mathbb{P}^1}(-a_1)^{\oplus r_1} \oplus \mathcal{O}_{\mathbb{P}^2}(-a_1 - 1)^{\oplus r_2} \right) \to \mathcal{F} \to \bigoplus_{i=1}^n \mathcal{O}_{\pi^{-1}(x_i)}(-1) \to 0,
\]

with \(x := \{x_1, \ldots, x_n\} \subset \mathbb{P}^1\) pairwise distinct, and \(a_1, r_1, r_2\) given by (2.7). For given \(x\) these extensions are parameterized by

\[
\mathcal{E}_x := \bigoplus_{i=1}^n \mathbb{L}_{x_i} \otimes \mathcal{O}_{\pi^{-1}(x_i)}(1) = \left( \mathbb{L} \otimes \pi_\ast \mathcal{O}_{\pi}(1) \right) \otimes \mathcal{O}_x, \quad \dim \mathcal{E}_x = 2nr. \tag{2.11}
\]

This space is acted on by

\[
G_x := \left( \text{Aut}(\mathbb{L}) \times (\mathbb{C}^*)^n \right) / \mathbb{C}^* = \left( \text{Aut}(\mathbb{L}) \times \mathcal{O}_x^\ast \right) / \mathbb{C}^* \quad (\mathcal{O}_x^\ast \subset \mathcal{O}_x \text{ stands for the invertible elements}),
\]

so the fibre \(M_x^\circ = h^{-1}(x) \cap M^\circ\) is the quotient by \(G_x\) of an open subset of the affine space underlying \(\mathcal{E}_x\). The symbol ‘/’ will always stand for the quotient of some open subset.
In order to globalize this construction as $x \in \text{Hilb}^n_{\mathbb{P}^1}$ varies, we consider the diagram:

\[
\begin{array}{c}
\Xi \\
\xi \downarrow \phi \downarrow \psi \downarrow
\end{array}
\begin{array}{c}
Z \\
\zeta \downarrow \phi \downarrow \psi \downarrow
\end{array}
\begin{array}{c}
\text{Hilb}^n_{\mathbb{P}^1} \times \mathbb{P}^1 \\
\text{pr}_{\mathbb{P}^1} \downarrow \pi \downarrow
\end{array}
\begin{array}{c}
Y \ell
\end{array}
\]

(2.12)

Here $\mathfrak{S}_n$ stands for the group of permutations of $n$ elements. Since we are interested in birational properties, we will repeatedly restrict ourselves to appropriate open subsets; they will be denoted by $U \subset (\mathbb{P}^1)^n$ and $\mathcal{H} := U/\mathfrak{S}_n \subset \text{Hilb}^n_{\mathbb{P}^1}$. We start by restricting ourselves to the complement $U$ of the diagonals. (Thus $\mathfrak{S}_n$ acts freely, and $q$ is flat.) In algebraic terms, (2.12) reduces to

\[
\begin{array}{c}
\mathbb{C}[x_1, \ldots, x_n][z] \\
\mathbb{C}[s_1, \ldots, s_n][z]
\end{array}
\begin{array}{c}
\mathbb{C}[z] \\
\mathbb{C}[s_1, \ldots, s_n][z]
\end{array}
\begin{array}{c}
\mathbb{C}[x_1, \ldots, x_n] \\
\mathbb{C}[s_1, \ldots, s_n]
\end{array}
\]

where $s_1 = x_1 + \ldots + x_n, \ldots, s_n = x_1 \cdot \ldots \cdot x_n$ are the symmetric polynomials.

In this setting, (2.11) is the stalk at $\mathfrak{p}$ of

\[
\mathcal{E} := \zeta_* \left( \text{pr}^{\mathbb{P}^1}_* \left( \mathcal{O}_x \otimes \mathcal{O}_\pi(1) \right) \right) = \zeta_* \left( \text{pr}^{\mathbb{P}^1}_* \left( \mathcal{O}_x \otimes \left( \mathcal{O}_\mathbb{P}^1 \oplus \mathcal{O}_{\mathbb{P}^1}(\ell) \right) \right) \right),
\]

(2.14)

and the symmetry group acting on $\mathcal{E}$ is

\[
G := \left( \text{Aut}(\mathbb{L}) \times (\zeta_* \mathcal{O}_\mathcal{Z})^\times \right)/\mathbb{C}^*.
\]

(2.15)

Notice that $\zeta_* \mathcal{O}_\mathcal{Z}$ is a sheaf of algebras on $\mathcal{H}$, so it makes sense to consider the (multiplicative) subgroup of invertible elements $(\zeta_* \mathcal{O}_\mathcal{Z})^\times$.

We simplify $\mathcal{E}$ by shrinking $U$ and $\mathcal{H} = U/\mathfrak{S}_n$ further. Indeed, fix $\infty = (0, 1) \in \mathbb{P}^1$ and trivialize the various $\mathcal{O}_{\mathbb{P}^1}(a), a \in \mathbb{Z}$, appearing in (2.11) on the complement $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$. Moreover, we fix two general sections $\sigma_0, \sigma_1$ in $\mathcal{O}_\pi(1)$. Their zero loci intersect at $\ell$ points in $Y_\ell$, lying above $\{u_1, \ldots, u_\ell\} \subset \mathbb{A}^1$. For $x \neq u_j$, the restrictions $\sigma_{0,x}, \sigma_{1,x} \in \Gamma(\pi^{-1}(x), \mathcal{O}_\pi(1))$ yield a basis. Then take $U$ to consist of $(x_1, \ldots, x_n) \in \mathbb{A}^n \subset (\mathbb{P}^1)^n$ pairwise distinct, such that $x_i \neq u_j$ for $i = 1, \ldots, n, j = 1, \ldots, \ell$. One can trivialize $\mathcal{E}$ over $\mathcal{H} = U/\mathfrak{S}_n$ as follows:

\[
\begin{align*}
\mathcal{E} & = \mathcal{E}_\ell \\
\zeta_* \mathcal{O}_\mathcal{Z} & \cong \mathbb{C}[s_1, \ldots, s_n] + \ldots + z^{n-1}. \mathbb{C}[s_1, \ldots, s_n] \cong \mathcal{O}_\mathcal{H}. \\
\end{align*}
\]

(2.16)

The subscripts refer to the factors $\text{Aut}(\mathbb{L}), (\zeta_* \mathcal{O}_\mathcal{Z})^\times$ of $G$, respectively. Although they act simultaneously, $(\mathcal{O}_\mathcal{H}^\times \otimes \mathcal{C})^\times$ will be viewed (mainly) as an $\text{Aut}(\mathbb{L})$-module, while $(\mathcal{O}_\mathcal{H}^\times \otimes \mathcal{C})^\times$ will be (mainly) a $(\zeta_* \mathcal{O}_\mathcal{Z})^\times$-module. Let

\[
\begin{align*}
\mathcal{E} := \text{Spec}_{\text{ Hilb}^n_{\mathbb{P}^1}} (\text{Sym}^* \mathcal{E}^\vee) \cong \mathcal{H} \times \mathcal{H}^r \times \mathcal{H}^s \\
\end{align*}
\]

be the linear fibre space (quasi-projective variety) determined by $\mathcal{E}$. We write $\mathcal{L} = \left( \mathcal{C}^r \otimes \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{C}^s \otimes \mathcal{O}_{\mathbb{P}^1} \right) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$, and think off the elements of $\mathcal{L}_x$, for $x \in \mathbb{P}^1$, as column vectors with $r = r_1 + r_2$ entries. Then the elements of $\mathcal{E}_x, x \in \mathcal{H}$, can be represented as pairs of $r \times n$-matrices in the block form

\[
e = \left[
\begin{array}{cccc}
* & \cdots & \cdots & * \\
\mathbb{I}_{r_1 \times r_1} & \cdots & \cdots & \mathbb{I}_{r_2 \times r_2} \\
* & \cdots & \cdots & * \\
\mathbb{I}_{2 \times r_1} & \cdots & \cdots & \mathbb{I}_{2 \times r_2} \\
\mathbb{V}_{1 \times r_2} & \cdots & \cdots & \mathbb{V}_{1 \times r_2} \\
* & \cdots & \cdots & * \\
\mathbb{V}_{r_1 \times r_2} & \cdots & \cdots & \mathbb{V}_{r_1 \times r_2} \\
\mathbb{I}_{r_1 \times r_2} & \cdots & \cdots & \mathbb{I}_{r_1 \times r_2} \\
* & \cdots & \cdots & * \\
\mathbb{V}_{2 \times r_1} & \cdots & \cdots & \mathbb{V}_{2 \times r_1} \\
\end{array}\right]
\]

(2.17)

The strategy for proving the rationality of $M^0$ is to exhibit a subvariety (Luna slice) $S \subset E$ which is a locally trivial, linear fibre bundle over $\mathcal{H} \subset \text{Hilb}^n_{\mathbb{P}^1}$, and the restriction of $\mathcal{E} \to S$ is birational. (In the terminology of [39], Definition 2.9), $S$ will be a $(G, \mathcal{I})$-section of $E \to \mathcal{H}$.) The slice will be constructed by proving that the generic pair of matrices (2.17) admits a unique (suitable) canonical form.
The \( \text{Aut}(L) \)-action. Now we turn our attention to the \( \text{Aut}(L) \)-action on \( E_x \). First, we observe that the elements of \( \text{Aut}(L) \) can be represented schematically as follows:

\[
\text{Aut}(L) = \left\{ \begin{pmatrix} A & \mathbb{C}^r \oplus B \in \text{Gl}(r_1; \mathbb{C}) \\ 0 & \text{Hom}(\mathbb{C}^r, \mathbb{C}^r) \oplus \Gamma(\mathbb{O}_{p_1}(1)) \end{pmatrix} \right\},
\]

with \( \text{Hom}(\mathbb{C}^r, \mathbb{C}^r) \oplus \Gamma(\mathbb{O}_{p_1}(1)) = \{ H(z) = z^{(0)}H_0 + z^{(1)}H_1 \mid H_0, H_1 \in \text{Hom}(\mathbb{C}^r, \mathbb{C}^r) \} \).

Since we restricted ourselves to a subset of \( \mathbb{A}^1 \subset \mathbb{P}^1 \), we write \( H(z) = H_0 + zH_1 \). Let us consider

\[
e = \begin{bmatrix} u_0 \\ v_0 \\ u_1 \\ v_1 \\ \vdots \\ u_{n-1} \\ v_{n-1} \end{bmatrix} \in \mathbb{A}^{2n}_{\text{left}},
\]

where the columns refer to the splitting \( r = r_1 + r_2 \) in \( L \). Some calculations show that

\[
g = \begin{pmatrix} A \\ H_0 + zH_1 \\ B \end{pmatrix} \in \text{Aut}(L)
\]

acts on \( e \) as follows (\( v_{-1} = 0 \)):

\[
g \times e = \sum_{j=0}^{n-1} \hat{z}^j \begin{bmatrix} A_{uj} + H_0v_j + H_1(v_{j-1} + (-1)^{n-j-1}s_{n-j}v_{n-1}) \\ Bu_j \end{bmatrix}.
\]

(2.19)

In the block form (2.17), it reads:

\[
\begin{align}
g \times [I] &= A[u_0, \ldots, u_{r_1-1}] + H_0[v_0, \ldots, v_{r_1-1}] \\
&
+ H_1 \begin{pmatrix} (-1)^{n-1}s_{n-1} \cdot v_{n-1}, \ldots, v_{r_1-2} + (-1)^{n-r_1}s_{n-r_1+1} \cdot v_{n-1} \end{pmatrix} \\
&= A[I] + H_0[II] + H_1[II],
\end{align}
\]

\[
\begin{align}
g \times [III] &= A[u_{r_1}, \ldots, u_{r-1}] + H_0[v_{r_1}, \ldots, v_{r-1}] \\
&
+ H_1 \begin{pmatrix} v_{r_1-1} + (-1)^{n-r_1-1}s_{n-r_1} \cdot v_{n-1}, \ldots, v_{r-2} + (-1)^{n-r}s_{n-r+1} \cdot v_{n-1} \end{pmatrix} \\
&= A[III] + H_0[IV] + H_1[IV],
\end{align}
\]

\[
\begin{align}
g \times [IV] &= B[v_{r_1}, \ldots, v_{r-1}] = B[IV],
\end{align}
\]

\[
\begin{align}
g \times [V] &= A[u_r, \ldots, u_{r+r_2-1}] + H_0[v_r, \ldots, v_{r+r_2-1}] \\
&
+ H_1 \begin{pmatrix} v_{r-1} + (-1)^{n-r_1-1}s_{n-r} \cdot v_{n-1}, \ldots, v_{r+r_2-2} + (-1)^{n-r-r_2}s_{n-r-r_2+1} \cdot v_{n-1} \end{pmatrix} \\
&= A[V] + H_0[VI] + H_1[VI].
\end{align}
\]

The slice to the \( \text{Aut}(L) \)-action is obtained in several steps. (Recall that \( x, e \) are generic.)

- By using the \( \text{Gl}(r_2) \)-action, we may assume that \([IV]=\mathbb{I}_{r_2}\).
- We cancel [III]: just take \( g = \begin{pmatrix} \mathbb{I}_{r_1} & H(z) = -[III] \cdot [IV]^{-1} \\ 0 & \mathbb{I}_{r_2} \end{pmatrix} \).
- Also cancel [V] with an appropriate \( g = \begin{pmatrix} \mathbb{I}_{r_1} & H(z) = H_0 + zH_1 \\ 0 & \mathbb{I}_{r_2} \end{pmatrix} \), while keeping [III]= 0 and [IV]=\( \mathbb{I}_{r_2} \). Indeed, the equation \( g \times [III] = 0 \) yields \( H_0 = -H_1[IV][IV]^{-1} \), and then \( g \times [V] = 0 \) has a unique solution \( H_1 = [V] \cdot ([IV][IV]^{-1}[VI] - [VI])^{-1} \).
- Finally, by using \( g = \begin{pmatrix} A & \mathbb{I}_{r_1} & \mathbb{I}_{r_2} \end{pmatrix} \), we may assume \([I]=\mathbb{I}_{r_1}\), while keeping \([III]=0, [IV]=\mathbb{I}_{r_2}\).

Overall, by using the \( \text{Aut}(L) \)-action, the generic \( e \) (2.17) can be brought into the form

\[
\begin{pmatrix} \mathbb{I}_{r_1} & 0 & 0 & \cdots \\ \mathbb{I}_{r_2} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
\]

(2.20)

Claim Suppose \( x \) and \( e \) are generic. More precisely, the following matrices should be invertible:

\[
[I], \ [IV], \ [IV][IV]^{-1}[VI] - [VI] \]

These conditions are indeed generic: take e.g. \([I]=\mathbb{I}_{r_1}, [IV]=\mathbb{I}_{r_2}, [VI]=\text{diagonal matrix}, \) and \( v_{n-1} = 0 \).
If both \( e, g \times e \) are of the form (2.20), then \( g = 1 \).
\[
\begin{align*}
g \times [IV] & = \mathbb{1}_2, \quad g \times [III] = 0, \quad g \times [V] = 0 \Rightarrow H_0 = H_1 = 0, \\
g \times [I] & = \mathbb{1}_3.
\end{align*}
\]

Henceforth, we shrink \( \mathcal{H} \) to the open subset appearing in the claim above.

### 2.3.2. The (\( \zeta, \mathcal{O}_Z \))\(^*\)-action.

Now we turn our attention to the second action. Unfortunately \( T := (\zeta, \mathcal{O}_Z)\) is not a group, but rather a group scheme over \( \mathcal{H} \) with fibres isomorphic to \( (\mathbb{C}^*)^n \). (Notice that \( \mathbb{C}^* \) is still diagonally embedded in \( (\zeta, \mathcal{O}_Z)^* \).) We denote by
\[
F := \text{Spec}_{\text{Hilb}^n_{\mathbb{Q}}} (\text{Sym}^* \zeta, \mathcal{O}_Z) \cong \mathbb{H} \times \mathbb{A}^n
\]
the linear fibre space determined by \( \zeta, \mathcal{O}_Z \). Then \( T \) acts diagonally on \( F^2 := F \times_F F \), and the action on \( E \) consists in repeating \( r \) times the action on \( F^2 \).

The \( T \)-action on \( F \) is complicated in the trivialization (2.16), but is easy to understand the \( q^*T := (\xi, \mathcal{O}_X)\)\(^*\)-action on \( q^*F = \mathbb{U} \times \mathbb{A}^n \) in a different trivialization. Indeed,
\[
(\xi, \mathcal{O}_X)^* \cong (\mathbb{C}^*)^n, 
\]
and define \( \tilde{\mathbb{U}}'' := \mathbb{U} \times \mathbb{A}^n_{\text{left}} \times \mathbb{A}^n_{\text{right}} \subset q^*E = \mathbb{U} \times \mathbb{H} \). Clearly, the generic element of \( q^*E \) can be brought into such a form by using \( q^*T \), uniquely up to the diagonal \( \mathbb{C}^* \)-action. (Thus, \( \tilde{\mathbb{U}}'' \) is a slice for a \( (\mathbb{C}^*)^n/\mathbb{C}_{\text{diag}}\)-action.)

The next step is to descend \( \tilde{\mathbb{U}}'' \) to \( E \) itself. This is not immediate, since afterward we wish to take the slice (2.20) for the \( (\mathbb{S}_n \times \text{Aut}(\mathbb{L})) \)-action, but \( \tilde{\mathbb{U}}'' \) is not \( \text{Aut}(\mathbb{L}) \)-invariant. Fortunately, the no-name lemma (2.10) comes to the rescue. We consider the diagram
\[
\begin{array}{ccc}
\mathbb{U} \times \mathbb{A}^n_{\text{left}} \times \mathbb{A}^n_{\text{right}} & \xrightarrow{(\text{id}, \Theta)} & \mathbb{U} \times \mathbb{A}^n_{\text{left}} \times \mathbb{A}^n_{\text{right}} \\
\text{pr} \downarrow & & \text{pr} \downarrow \\
\mathbb{U} \times \mathbb{A}^n_{\text{left}} & \xrightarrow{\text{id}} & \mathbb{U} \times \mathbb{A}^n_{\text{right}} \\
\mathbb{U} \downarrow & & \mathbb{U} \downarrow \\
\mathbb{U} & & \mathbb{U}
\end{array}
\]

By our discussion at (2.3.1) the generic \( (\mathbb{S}_n \times \text{Aut}(\mathbb{L})) \)-stabilizer on \( \mathbb{U} \times \mathbb{A}^n_{\text{left}} \) is trivial. (Notice that \( \mathbb{S}_n \) acts trivially on \( \mathbb{A}^n_{\text{left}} \) because the trivialization (2.16) holds on \( \mathbb{H} \).) Then there is a \( (\mathbb{S}_n \times \text{Aut}(\mathbb{L})) \)-invariant open subset \( \tilde{\mathbb{O}} \subset \mathbb{U} \times \mathbb{A}^n_{\text{left}} \), and a birational pr-fibrewise linear map \( (\text{id}, \Theta) \) which is equivariant for the following actions:

- \( \mathbb{S}_n \) acts on \( \mathbb{A}^n_{\text{right}} \) by permuting the \( n \) copies of \( \mathbb{A}^n \);
- \( \text{Aut}(\mathbb{L}) \) acts on \( \mathbb{A}^n_{\text{right}} \) the same as on \( \mathbb{A}^n_{\text{left}} \). (Anyway, \( \text{Aut}(\mathbb{L}) \) acts diagonally on \( \mathbb{A}^{2rn} \).
- \( \mathbb{S}_n \times \text{Aut}(\mathbb{L}) \) acts trivially on \( \mathbb{A}^n_{\text{right}} \). The group \( q^*T \) acts both on the fibre and the base of \( \Theta \); it is a priori unclear whether \( \Theta \) is \( q^*T \)-invariant. (Although is likely that is possible to arrange this.) However, the \( q^*T \)-orbit of the generic point in \( \mathbb{U} \times \mathbb{A}^n_{\text{left}} \times \mathbb{A}^n_{\text{right}} \) intersects \( \tilde{\mathbb{U}}'' \cap \tilde{\Theta} \) along a unique \( \mathbb{C}^*_{\text{diag}} \)-orbit (a straight line). Indeed, the generic stabilizer is trivial, and the dimension of the \( q^*T \)-orbit of the \('bad locus' \( \mathbb{U} \times \mathbb{A}^n_{\text{left}} \setminus \tilde{\Theta} \) \( q^*T \)-invariant is at most \( \text{dim}(\mathbb{U} \times \mathbb{A}^n_{\text{left}} \setminus \tilde{\Theta}) + rn < \text{dim} \mathbb{U} + 2rn \). We consider the \('q^*T/\mathbb{C}_{\text{diag}}-slice' \)
\[\tilde{\mathbb{U}}'' := (\text{id}, \Theta)(\tilde{\mathbb{U}}'' \cap \tilde{\Theta}) \subset \tilde{\Theta} \times \mathbb{A}^n_{\text{right}}.\]

By the pr-linearity of \( (\text{id}, \Theta) \), \( \tilde{\mathbb{U}}'' \) is still a linear fibre over \( \tilde{\Theta} \), invariant under the \( \mathbb{S}_n \)-action (as \( \mathbb{U} \) is so). At this stage only, we take the quotient by \( \mathbb{S}_n \), and get the \('T/\mathbb{C}_{\text{diag}}-slice' \).
The intersection \( \Xi''_\theta \cap \mathfrak{S}_n \subset (\mathcal{O}/\mathfrak{S}_n) \times \mathbb{A}^n_\theta \subset E \). We denote \( O := \mathcal{O}/\mathfrak{S}_n \subset \mathcal{H} \times \mathbb{A}^n_\text{rel} \).

The essential property is that \( \text{Aut}(\mathcal{L}) = (\text{Aut}(\mathcal{L}) \times \mathbb{C}^\times_\text{diag})/\mathbb{C}^* \) acts on \( O \) as in 2.3.1 and by multiplication on the fibres of \( \text{pr}: O \times \mathbb{A}^n_\text{rel} \to O \).

2.3.3. The \( G \)-action. Now we assemble 2.3.1 and 2.3.2 to produce the \( G \)-slice on \( E \). We define

\[
S := \Xi''_\theta | O \cap (\mathcal{H} \times S') \subset E, \quad \text{where } S' \text{ consists of matrices of the form (2.20).} \tag{2.21}
\]

The intersection \( O \cap (\mathcal{H} \times S') \) is non-empty, because \( \text{Aut}(\mathcal{L}) : (\mathcal{H} \times S') \) is open. Clearly, \( S \subset E \) is \((n + 2 - 1)\)-co-dimensional, it is an open subset of a locally trivial fibration over some open \( \mathcal{H} \subset \text{Hilb}^n_{2} \). Our discussion shows that any the \( G \)-orbit of the generic intersects \( S \) at only one point. Indeed, the \( T \)-orbit intersects \( \Xi''_\theta \) along a \( \mathbb{C}^\times_\text{diag} \)-orbit, and we use the remaining \( \text{Aut}(\mathcal{L}) \)-action to move the point over \( O \cap (\mathcal{H} \times S') \).

\[
\text{Corollary 2.9. } M_p(r; 0, n) \text{ is a rational variety, for all } n \geq r \geq 2.
\]

For \( r = 2 \), the statement in proved in [30] (see also [20]). For arbitrary \( r \) and \( c_1 \), the best results are obtained in [9, 11, 45].

\[
\text{Proof. Let } \sigma : Y_1 \to \mathbb{P}^2 \text{ be the contraction of } \Lambda \text{ (equivalently, the blow-up of a point in } \mathbb{P}^2). \text{ According to (2.1.3) and 2.8, } \mathcal{M}^{L_\sigma}_Y(r; 0, n), \text{ where } \sigma \text{ is sufficiently large, is irreducible and rational, and there is an open subset of subset consisting of vector bundles } \mathcal{F} \text{ whose restrictions to both } \Lambda \subset Y_1 \text{ and the generic } \sigma \in \mathcal{O}_x(1) \text{ are trivializable. For any such } \mathcal{F}, \text{ the direct image } \sigma_* \mathcal{F} \text{ is locally free and semi-stable on } \mathbb{P}^2. \text{ This yields a rational map }
\]

\[
\sigma_* : \mathcal{M}^{L_\sigma}_Y(r; 0, n) \to \mathcal{M}_p(r; 0, n), \quad \mathcal{F} \mapsto \sigma_* \mathcal{F}.
\]

It is obviously injective on the open locus formed by \( \mathcal{F} \) as above. Moreover, \( \sigma_* \) is dominant, because \( \text{Ext}^1(\mathcal{F}, \mathcal{F}) \cong \text{Ext}^1(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}) \) for any \( \mathcal{F} \) which is trivializable along \( \Lambda \). Zariski’s main theorem implies that \( \sigma_* \) is birational. \( \square \)

Somewhat unexpectedly, moduli spaces of framed vector bundles will naturally occur in the next section.

\[
\text{Definition 2.10. [See e.g. [15, 6]] A framing of a sheaf } \mathcal{F} \text{ on } Y_1 \text{ along a reduced, irreducible curve } \lambda \subset Y_1 \text{ is an isomorphism } \mathcal{F}_\lambda \overset{\theta}{\to} \mathcal{O}^r_\lambda. \text{ Two framings of } \mathcal{F} \text{ along } \lambda \text{ are equivalent if there is an automorphism of } \mathcal{F} \text{ sending one framing into the other.}
\]

Thus, if \( \mathcal{F} \) is stable, two framings \( \theta, \theta' \) are equivalent if \( \theta' = c\theta \) for \( c \in \mathbb{C}^* \). We deduce that, in this latter case, the choice of a basis in \( \Gamma(\mathcal{F}_\lambda) \) (modulo \( \mathbb{C}^* \)) determines a framing of \( \mathcal{F} \) along \( \lambda \). Therefore, the possible framings of \( \mathcal{F} \) along \( \lambda \) is the PGL\((r)\)-orbit of a given framing. We are going to show how the slice \( S \) constructed above allows to do this for families of vector bundles. For shorthand, we let \( \mathcal{M}^{\mathcal{F}_b}_Y := \mathcal{M}^{L_\theta}_Y(r; 0, n) \), and denote by \( \mathcal{M}^{\mathcal{F}_b}_Y, \lambda \) the moduli space of \( \lambda \)-framed, \( L_\sigma \)-stable vector bundles on \( Y_1 \).

\[
\text{Corollary 2.11. Assume that the restriction to } \lambda \text{ of the generic } \mathcal{F} \in \mathcal{M}^{\mathcal{F}_b}_Y \text{ is trivializable. Then } \mathcal{M}^{\mathcal{F}_b}_Y, \lambda \text{ is birational to } \mathcal{M}^{\mathcal{F}_b}_Y \times \text{PGL}(r), \text{ so } \mathcal{M}^{\mathcal{F}_b}_Y, \lambda \text{ is an irreducible, } 2nr \text{-dimensional, rational variety.}
\]

\[
\text{Proof. Consider the sheaf } \mathcal{E} \text{ (2.14), and the corresponding linear fibre space } \tilde{\zeta} : E \to \mathcal{H}. \text{ We denote by } t \text{ the tautological section of } \zeta \mathcal{E} \text{ over } E. \text{ Also, notice that } \tilde{\mathcal{Z}} := E \times_{\mathcal{H}} \mathcal{Z} \times_{\mathbb{P}^1} Y \text{ is a subvariety of } E \times Y, \text{ since } \mathcal{Z} \subset \mathcal{H} \times \mathbb{P}^1. \text{ We consider the following composition of homomorphisms:}
\]

\[
\begin{align*}
\left( \pi \circ \text{pr}_{E \times Y} \right)^* \mathcal{O}_{\tilde{\mathcal{Z}}} & \overset{\text{pairing}}{\to} \left( \pi \circ \text{pr}_{E \times Y} \right)^* \pi_* \mathcal{O}_{\pi(1)} \otimes \mathcal{O}_{\tilde{\mathcal{Z}}} \\
\left( \text{pr}_{E \times Y} \right)^* \mathcal{O}_{\pi(1)} & \overset{\text{evaluation}}{\to} \left( \text{pr}_{E \times Y} \right)^* \mathcal{O}_{\pi(1)} \otimes \mathcal{O}_{\tilde{\mathcal{Z}}}.
\end{align*}
\]

At generic \( x \in \text{Hilb}^n_{\mathbb{P}^1}, \) \( e \in \mathcal{E}_x, \) \( u \) is

\[
\pi^* \mathcal{L}^r \overset{\text{left}}{\to} \bigoplus_{i=1}^n \mathcal{L}^{r} \overset{\cdot t}{\to} \bigoplus_{i=1}^n \Gamma(O_{\pi^{-1}(x_i)}(1)) \overset{\cdot t}{\to} \bigoplus_{i=1}^n \mathcal{O}_{\pi^{-1}(x_i)}(1).
\]
The pairing with \( t \) is surjective over the locus \( E' \) consisting of \( e \in E \), such that \( t(e) \) has linearly independent components at each \( x \in \zeta(e) \). (Obviously, \( E' \) is \( G \)-invariant.) Then \( \mathcal{F} := \text{Ker}(u)V_{(E' \cap S) \times Y} \) is a locally free sheaf over \( S' \times Y := (E' \times Y) \cap (S \times Y) \). (The intersection is non-empty, as \( G \cdot S \subset E \) is dense.) Since \( S \) is birational to \( M^b_{Y_t} \), \( \mathcal{F} \) is a universal sheaf over \( M' \times Y \), for some (non-empty) open subset \( M' \subset M^b_{Y_t} \).

After possibly shrinking \( M' \), we may assume that \( \mathcal{F}_\lambda \cong \mathcal{O}_\lambda^r \), for all \( \mathcal{F} \in M' \). Thus \( (\text{pr}_{M'} \mathcal{F}_\lambda) \mathcal{F} \) is locally free over \( M' \), and, after shrinking \( M' \) further, we may assume that this latter is trivializable over \( M' \). Then the moduli space of vector bundles \( \mathcal{F} \in M' \) which are \( \lambda \)-framed is isomorphic to \( M' \times \text{PGl}(r) \).

\[ \square \]

**Remark 2.12.** In [5, loc. cit.] we will encounter two types of framings:

(i) \( \lambda = \pi^{-1}(\ell) \) is a fibre of \( \pi \). Any \( \mathcal{F} \in M^b_{Y_t} \) fits into \( 0 \rightarrow \pi^* \pi_* \mathcal{F} \rightarrow \mathcal{F} \rightarrow \mathcal{Q}_\mathcal{F} \rightarrow 0 \), so \( \mathcal{F}_\lambda \cong \mathcal{O}_\lambda^r \), if \( \text{Supp}(\mathcal{Q}_\mathcal{F}) \) is disjoint of \( \lambda \).

(ii) The second kind of framings is along a general \( \lambda \in |O_\pi(1)| \). (In [5, section 6] we will need \( \ell = 1 \).) We already observed in (2.6(i)) that such a \( \lambda \)-framed vector bundle is automatically \( L_c \)-semi-stable.

As an immediate application, we deduce the following (apparently new) statement.

**Corollary 2.13.** The moduli spaces of framed sheaves with \( c_1 = 0 \) constructed in [5] are rational.

**Proof.** Use (2.11) and (2.12) \( \square \)

### 3. Application: Stable Vector Bundles on \( \mathbb{P}^2 \)-Bundles over \( \mathbb{P}^1 \)

Stable vector bundles on \( \mathbb{P}^2 \) are studied in [3, 23, 50]. Here we give a monad theoretic construction of stable vector bundles \( \mathcal{F} \) with \( c_1 = 0 \) on \( \mathbb{P}^2 \)-fibre bundles over \( \mathbb{P}^1 \). Let us remark that [10] studies the moduli space of rank two vector bundles on projective bundles over curves. If a vector bundle obtained this way has \( c_1 = 0 \), its \( c_2 \) is necessarily of the form \(-k^2u^2 + 2kluv\), with \( u, v \) as in (3.1) below and \( k, l \in \mathbb{Z} \). Thus our work brings novelities even in the rank two case.

For two integers \( 0 \leq a \leq b \), we consider \( Y = Y_{a,b} := \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-a) \oplus O_{\mathbb{P}^1}(-b)) \). The 3-fold \( Y \) admits the projection \( \pi : Y \rightarrow \mathbb{P}^1 \) with fibres isomorphic to \( \mathbb{P}^2 \). (We say that \( Y \) is a \( \mathbb{P}^2 \)-fibre bundle over \( \mathbb{P}^1 \).) Let \( \Phi = \mathbb{P}^2 \) be the generic fibre of \( \pi \). The relatively ample line bundle \( O_{\pi}(1) \) on \( Y \) is big, globally generated, \( \pi \)-ample, generated, \( \pi \)-ample (except for \( a = b = 0 \), when it satisfies [1.9]), and holds:

\[
\begin{align*}
H^2(Y; \mathcal{F}) &= \mathbb{Z} \cdot [O_{\pi}(1)] \oplus \mathbb{Z} \cdot [O_{\pi}(1)], \quad \text{with} \quad u^3 = (a + b)c - v^2v = a + b; \\
\kappa_\pi &= -3u + (a + b)v \quad \text{and} \quad \kappa_Y = -3u + (a + b - 2)v.
\end{align*}
\] (3.1)

Here \( \kappa_\pi \) and \( \kappa_Y \) stand for the relative and the (absolute) canonical class of \( Y \) respectively. The ‘exceptional line’ \( \Lambda := \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}) \subset Y \) has the property that \( O_\pi(1) \otimes O_\Lambda \cong O_\Lambda \).

#### 3.1. Review of the monad construction on \( \mathbb{P}^2 \)

In [5, section 6] it is proved that any stable vector bundle \( \mathcal{V} \) on \( \mathbb{P}^2 \) is the cohomology of a certain monad on \( \mathbb{P}^2 \). For completeness, we briefly recall some details. For a semi-stable rank \( r \) vector bundle \( \mathcal{V} \) on \( \Phi \), with \( c_1(\mathcal{V}) = 0 \) and \( c_2(\mathcal{V}) = n \), the following hold:

(i) \[
\begin{align*}
\Gamma(\Phi, \mathcal{V} \otimes O_\Phi(-j)) &= H^2(\Phi, \mathcal{V} \otimes O_\Phi(-j)) = 0 \\
\text{and} \quad \dim H^1(\Phi, \mathcal{V} \otimes O_\Phi(-j)) &= n,
\end{align*}
\] for \( j = 1, 2 \).

(ii) \[
\bigoplus_{l \geq 0} H^1(\Phi, \mathcal{V} \otimes O_\Phi(l - 1)) \text{ is generated by } H^1(\Phi, \mathcal{V} \otimes O_\Phi(-1)) \text{ over } H^0(\Phi, \mathcal{V} \otimes O_\Phi(l)).
\]

(iii) The identity in \( \text{End}(H^1(\Phi, \mathcal{V} \otimes O_\Phi(1)) \otimes \text{Ext}^1(H^1(\Phi, \mathcal{V} \otimes O_\Phi(-1)) \otimes O_\Phi(1), \mathcal{V})) \), defines the minimal \(-1\)-resolution of \( \mathcal{V} \) [5, Section 2]:

\[
0 \rightarrow \mathcal{V} \rightarrow Q_\mathcal{V} \rightarrow H^1(\Phi, \mathcal{V} \otimes O_\Phi(-1)) \otimes O_\Phi(1) \rightarrow 0.
\] (3.2)

\[ ^{3} \text{In [5, section 6] the authors assume that } \mathcal{V} \text{ is stable. However, one can easily check that the statements below are valid for } \mathcal{V} \text{ semi-stable. The reason is that, in loc. cit., the authors consider also minimal } -2\text{-resolutions, which indeed require the stability of } \mathcal{V}. \]
Similarly, we consider the minimal \(-1\)-resolution of \(\mathcal{V}\), and obtain the display of a (minimal) monad whose cohomology is \(\mathcal{V}\):

\[
\begin{array}{cccc}
H^1(\Phi, \mathcal{V}^\vee(-1))\otimes \mathcal{O}_\Phi(-1) & \to & K_\Phi & \to & \mathcal{V} \\
H^1(\Phi, \mathcal{V}^\vee(-1))\otimes \mathcal{O}_\Phi(-1) & \to & A_\Phi & \to & C_\Phi \\
& & B_\Phi & \to & H^1(\Phi, \mathcal{V}^\vee(-1)) \otimes \mathcal{O}_\Phi(1) & =: B_\Phi
\end{array}
\]

(3.3)

(iv) If \(\mathcal{V}\) on \(\Phi\) is stable, then \(h^1(\mathcal{V}) = n - r\), thus \(n \geq r\).

(v) The vector bundles whose restriction to a (straight) line \(\lambda \subset \Phi\) is isomorphic to \(\mathcal{O}_\lambda^{\oplus r}\) form an open and dense subset of the moduli space of semi-stable vector bundles on \(\Phi\), with \(c_1 = 0\), \(c_2 = n\). (See [23, Lemma 2.4.1] for the proof.)

3.2. The relative monad construction on \(Y_{a,b}\). Our main result is the following:

**Theorem 3.1.** Let \(\mathcal{F}\) be a rank \(r\) vector bundle on \(Y\) whose Chern classes are

\[
c_1(\mathcal{F}) = 0, \quad c_2(\mathcal{F}) = n \cdot u^2, \quad c_3(\mathcal{F}) = 0,
\]

and which is semi-stable with respect to \(L_c := \mathcal{O}_\pi(1) + c\mathcal{O}_\mathcal{P}(1)\), for \(c > r(r-1)n(a+b)\).

We assume that \(\mathcal{F}\) has the following properties:

(i) The restriction of \(\mathcal{F}\) to a line \(\lambda \cong \mathbb{P}^1\) in the generic fibre \(\Phi \cong \mathbb{P}^2\) is trivial. (3.5)

(ii) The restriction \(\mathcal{F}_\Lambda := \mathcal{F} \otimes \mathcal{O}_\Lambda\) to the exceptional line \(\Lambda\) is trivial. (3.6)

(iii) \(R^2\pi_* (\mathcal{F} \otimes \mathcal{O}_\pi(-2)) = 0\). (3.7)

\[H^1(Y, \mathcal{F} \otimes \mathcal{O}_\pi(-1) \otimes \pi^* \mathcal{O}_\mathcal{P}(1)) = 0,\]

(3.8)

\[H^1(Y, \mathcal{F} \otimes \mathcal{O}_\pi(-2) \otimes \pi^* \mathcal{O}_\mathcal{P}(a+b-1)) = 0.\]

(3.9)

Then \(\mathcal{F}\) can be written as the cohomology of a monad of the form

\[
\mathcal{O}_\pi(-1)^{\oplus n} \overset{A}{\to} \mathcal{O}_Y^{\oplus r+2n} \overset{B}{\to} \mathcal{O}_\pi(1)^{\oplus n}.
\]

(3.10)

If \(\mathcal{F}_\Phi\) is stable, then this monad is uniquely defined, up to the action of

\[G := \text{Gl}(n) \times \text{Gl}(r+2n) \times \text{Gl}(n) : (g_1, g_2) \times (A, B) := (gAg_1^{-1}, g_2Bg_2^{-1}).\]

(3.11)

The isotropy group of this action is \(\mathbb{C}^{*}\), diagonally embedded in \(G\).

**Remark 3.2.** Before starting the proof, we analyse the various conditions imposed on \(\mathcal{F}\).

- \(\mathcal{O}_\pi(\pm 1)\) is trivial along \(\Lambda\), thus (3.6) is necessary. Also, a simple diagram chasing in the display of (3.11) yields (3.7), (3.8), (3.9). Hence (3.6) – (3.9) must be imposed.

- (3.7) should be interpreted as a weak, \(\pi\)-relative semi-stability condition for \(\mathcal{F}\), because \(R^2\pi_* (\mathcal{F} \otimes \mathcal{O}_\pi(-2))\) is a torsion sheaf on \(\mathbb{P}^1\), anyway. Moreover, if \(\mathcal{F}\) is semi-stable on each fibre of \(\pi\), then (3.7) is automatically satisfied. However, as we explained in the introduction, we avoid this requirement in order to enlarge the frame of (3.11) [31, 32, 41].

- (3.8) is the only assumption imposed by technical reasons. (Is needed to control the middle term of the monad (3.10).) It should be viewed as a genericity condition for \(\mathcal{F}\). Indeed, the restriction to \(\Phi\) of any \(L_\lambda\)-semi-stable vector bundle on \(Y\) is \(\mathcal{O}_\Phi(1)\)-semi-stable; our previous discussion (point (v) above) states that most semi-stable vector bundles on \(\Phi\) are trivializable along \(\Lambda\). For \(r = 2\), the Grauert-Mülllich theorem (see [24, Chapter 3]) implies that (3.8) is automatically satisfied.

Throughout this section, for \(x \in \mathbb{P}^1\), we denote \(\phi_x := \pi^{-1}(x) \cong \mathbb{P}^2\). When (hopefully) no confusion is possible, we write \(\mathcal{F}(-1) := \mathcal{F} \otimes \mathcal{O}_\pi(-1)\), and similarly for \(\mathcal{F}'\), \(Q\), \(M\), etc. First we clarify the rationale for (3.5) and (3.9).

**Lemma 3.3.** Assume that \(\mathcal{F}\) on \(Y\) is semi-stable and satisfies (3.4), (3.7). Then hold:
Proof. (i) The restrictions of $\mathcal{F}, \mathcal{F}'$ to the generic fibre $\Phi$ are semi-stable of degree zero, so $\pi_\ast (\mathcal{F}(l)) = \pi_\ast (\mathcal{F}'(l)) = 0$, for $l = -2, -1$, because they are both torsion free sheaves.

A generic divisor $D \in |\mathcal{O}_1(1)|$ is isomorphic to the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{-1}(-a) \oplus \mathcal{O}_{-1}(-b))$. The push-forward by $\pi$ of $0 \to \mathcal{F}(-2) \to \mathcal{F}(-1) \to \mathcal{F}_D(-1) \to 0$ yields

$$R^2 \pi_\ast (\mathcal{F}(-2)) \to R^2 \pi_\ast (\mathcal{F}(-1)) \to R^2 \pi_\ast (\mathcal{F}_D(-1)).$$

The same argument shows $R^2 \pi_\ast (\mathcal{F}(k)) = 0$, for all $k \geq -2$.

By repeatedly applying the semi-continuity theorem [21, Ch. III, Theorem 12.11], we are going to prove that $R^2 \pi_\ast (\mathcal{F}'(l)) = 0$, for $l = -2, -1$. For shorthand, let $\mathcal{G} := \mathcal{F} \otimes \mathcal{O}_1(l)$.

$$(R^2 \pi_\ast \mathcal{G})_x \to H^2(\phi_x, \mathcal{G}_x)$$ is surjective because dim$_{P^1}Y = 2$, $R^2 \pi_\ast \mathcal{G} = 0$ is locally free,

$$0 \rightarrow H^2(\phi_x, \mathcal{G}_x) \rightarrow (R^1 \pi_\ast \mathcal{G})_x \rightarrow H^1(\phi_x, \mathcal{G}_x) \rightarrow 0$$

is isomorphism, $\forall x \in P^1$.

$$R^3 \pi_\ast \mathcal{G} = 0$$ is locally free because $\pi_\ast \mathcal{G} = R^2 \pi_\ast \mathcal{G} = 0$ is isomorphism, $\forall x \in P^1$.

For $l = -2, -1$, we deduce that $H^2(\phi_x, \mathcal{F}'(l)) \cong \Gamma(\phi_x, P^1(-3-l)) = 0$, $\forall x \in P^1$. Grauert’s criterion [21, Ch. III, Corollary 12.9] implies now that $R^2 \pi_\ast (\mathcal{F}'(l)) = 0$.

(ii) Since $\pi_\ast (\mathcal{F}(l)), R^2 \pi_\ast (\mathcal{F}(l))$ and $\pi_\ast (\mathcal{F}'(l)), R^2 \pi_\ast (\mathcal{F}'(l))$ vanish for $l = -2, -1$, it follows that $R^3 \pi_\ast (\mathcal{F}(l)), R^3 \pi_\ast (\mathcal{F}'(l))$ are locally free. Their rank and degree are given by the Grothendieck-Riemann-Roch formula.

(iii) The assumption (3.3) implies

$$0 = H^1(Y, \mathcal{F} \otimes \mathcal{O}_1(-1) \otimes \pi^* \mathcal{O}_{-1}(-1)) = \Gamma(\mathbb{P}^1, \mathcal{O}_{-1}(-1) \otimes R^3 \pi_\ast (\mathcal{F}(-1))),$$

so the locally free sheaf $R^3 \pi_\ast (\mathcal{F}(-1))$ decomposes into a direct sum of line bundles $\mathcal{O}_1(l)$, with $l \leq 0$. But their degrees add up to zero, so $R^3 \pi_\ast (\mathcal{F}(-1)) \cong \mathcal{O}_{-2}^\oplus$. Using Serre duality, we find

$$0 = H^2(Y, \mathcal{F}'(-2) \otimes \mathcal{O}_{-1}(a+b-1)) = H^1(\mathbb{P}^1, \mathcal{O}_{-1}(a+b-1) \otimes R^3 \pi_\ast (\mathcal{F}'(-2))).$$

A similar argument as before implies $R^3 \pi_\ast (\mathcal{F}'(-2)) \cong \mathcal{O}_{-2}^\oplus$. (3.14)

(iv) Let $L \cong \mathbb{P}^1$ be the intersection of two general divisors in $|\mathcal{O}_1(1)|$. The push-forward of the sequence $0 \to \mathcal{O}_2(-2) \to \mathcal{O}_2(-1) \oplus \mathcal{O}_2 \to \mathcal{O}_L \to 0$ tensored by $\mathcal{F}(l)$, with $l \geq 0$, yields $R^1 \pi_\ast (\mathcal{F}(l-1)) \otimes \mathcal{O}(1) \to R^1 \pi_\ast (\mathcal{F}(l))$ is surjective. The claim follows because $\text{Sym}^{l+1} \Gamma(Y, \mathcal{O}_1(1))$ generates $\pi_\ast \mathcal{O}_1(l + 1)$.

Now we consider the extensions

$$0 \rightarrow \mathcal{F} \rightarrow Q \rightarrow H^1(Y, \mathcal{F} \otimes \mathcal{O}_1(-1)) \otimes \mathcal{O}_1(1) \rightarrow 0, \quad \text{(3.12)}$$

$$0 \rightarrow H^1(Y, \mathcal{F} \otimes \mathcal{O}_1(-1) \otimes \mathcal{O}_1(1)) \rightarrow K \rightarrow \mathcal{F} \rightarrow 0 \quad \text{(3.13)}$$

corresponding respectively to the identity elements

$$1 \in \text{End}(H^1(Y, \mathcal{F} \otimes \mathcal{O}_1(-1))) \cong \text{Ext}^1(H^1(Y, \mathcal{F} \otimes \mathcal{O}_1(-1)) \otimes \mathcal{O}_1(1), \mathcal{F}),$$

$$1 \in \text{End}(H^1(Y, \mathcal{F} \otimes \mathcal{O}_1(-1))) \cong \text{Ext}^1(\mathcal{F}, H^1(Y, \mathcal{F} \otimes \mathcal{O}_1(-1) \otimes \mathcal{O}_1(1))). \quad \text{(3.14)}$$

(We remark that these extensions exist for any vector bundle $\mathcal{F}$.)
The rightmost arrow is injective, because the dual homomorphism

\[ H^1(F^\vee(-1)^\vee \otimes O_x(-1)) \xrightarrow{\mathcal{K}} F, \]

\[ H^1(F^\vee(-1)^\vee \otimes O_x(-1)) \xrightarrow{A} M \]

\[ H^1(F(-1)) \otimes O_x(1) \xrightarrow{B} H^1(F(-1)) \otimes O_x(1) \]

\[ = A \]

\[ = B \]

Step 4 The upper sequence yields

\[ \pi\text{-}\text{vanishing of } 0 = \pi \]

because \( \pi \) is the monad (3.3), because (3.12) and (3.13) restrict to the corresponding extensions (3.2) for \( \mathcal{V} := \mathcal{F} \otimes \mathcal{O}_\Phi \) and \( \mathcal{V}^\vee \) respectively.

Next we study the cohomological properties of the vector bundle \( \mathcal{M} \) appearing in (3.15).

Lemma 3.5. Let \( \mathcal{F} \) be a vector bundle on \( Y \) satisfying (3.3) – (3.9). Then holds:

\[ R^1\pi_*(\mathcal{M} \otimes O_\pi(l)) = 0, \forall l \in \mathbb{Z}. \]

Proof. The last column of (3.15) and lemma 3.3(iv) imply \( R^1\pi_*(\mathcal{Q}(l)) = 0, \forall l \geq -1 \). The middle horizontal sequence in (3.15) immediately yields the conclusion for \( l \geq -1 \).

The case \( l \leq -2 \) is treated in several steps. We consider \( D \in [O_x(1)] \) generic.

Step 1 For \( l \leq -1 \), the upper horizontal sequence yields \( \pi_*(\mathcal{K}_D(l)) \cong \pi_*(\mathcal{F}_D(l)) \). By (3.5), \( \mathcal{F}_D \) is trivial along the general fibre of \( D \rightarrow \mathbb{P}^1 \), so \( \pi_*(\mathcal{K}_D(l)) \cong \pi_*(\mathcal{F}_D(l)) = 0 \).

Step 2 For \( l \leq -2 \), the middle vertical exact sequence yields \( \pi_*(\mathcal{K}_D(l)) \cong \pi_*(\mathcal{M}_D(l)) \), so \( \pi_*(\mathcal{M}_D(l)) = 0 \).

Step 3 After twisting \( 0 \rightarrow O_\pi(-1) \rightarrow O_Y \rightarrow O_D \rightarrow 0 \) by \( \mathcal{M}(l) \), with \( l \leq -2 \), we deduce \( 0 = \pi_*(\mathcal{M}_D(l)) \rightarrow R^1\pi_*(\mathcal{M}(l-1)) \rightarrow R^1\pi_*(\mathcal{M}(l)), \forall l \leq -2 \). Hence it suffices to prove the vanishing of \( R^1\pi_*(\mathcal{M}(-2)) \).

Step 4 The upper sequence yields

\[ 0 \rightarrow R^1\pi_*(\mathcal{K}(-2)) \rightarrow R^1\pi_*(\mathcal{F}(-2)) \rightarrow H^1(Y, F^\vee(-1)^\vee \otimes R^2\pi_*O_x(-3)). \]

The rightmost arrow is injective, because the dual homomorphism

\[ H^1(Y, F^\vee(-1)^\vee \otimes O_p(a+b) \rightarrow R^1\pi_*(F^\vee(-1)^\vee \otimes O_p(a+b) \rightarrow R^1\pi_*(\mathcal{F}(-2)) \rightarrow R^2\pi_*(\mathcal{F}(-2)) = 0. \]

This is unclear for \( l \leq -3 \). We deduce that \( R^1\pi_*(\mathcal{K}(-2)) = 0 \).

Step 5 Finally, the middle vertical sequence implies \( R^1\pi_*(\mathcal{M}(-2)) = 0 \).

Lemma 3.6. Let \( \mathcal{F} \) be a vector bundle on \( Y \) satisfying (3.3) – (3.9). Then \( \pi_*(\mathcal{M} \otimes O_\pi(l)) \) and \( R^2\pi_*(\mathcal{M} \otimes O_\pi(l)) \) are locally free for all \( l \in \mathbb{Z} \).

Proof. For \( l \geq -1 \), the last column and the middle line of (3.15), imply that \( R^2\pi_*(\mathcal{M}(l)) = 0 \).

Since \( R^1\pi_*(\mathcal{M}(l)) = 0 \), it follows that \( \pi_*(\mathcal{M}(l)) \) is locally free.

On the other hand, for \( l \leq -2 \), the first line and the middle column in (3.15) imply that \( \pi_*(\mathcal{M}(l)) \) is locally free again.

Proof. (of theorem 3.1) All that remains to prove is that \( \mathcal{M} \cong O_Y^{\mathbb{P}^r+2n} \). We do this in two steps.

Step 1 First we prove that \( \mathcal{M} \cong \pi^*\pi_*\mathcal{M} \). Indeed, [21 Ch. III, Theorem 12.11] yields

\[ R^2\pi_*(\mathcal{M}(l))_x \rightarrow H^2(\phi_x, \mathcal{M}(l)) \]

is surjective.

\[ R^2\pi_*(\mathcal{M}(l))_x \rightarrow H^1(\phi_x, \mathcal{M}(l)) \]

is an isomorphism, \( \forall x \in \mathbb{P}^1 \),

thus \( H^1(\phi_x, \mathcal{M}(l)) = 0 \) for all \( x \in \mathbb{P}^1 \) and \( l \in \mathbb{Z} \). Horrocks’ criterion [3 Lemma 1, pp.334] implies that the restriction of \( \mathcal{M} \) to each fibre of \( \pi \) splits into a direct sum of line bundles.
The restriction of \( M_\pi \) to \( \Phi \) is the monad \( \Phi \) of \( M \). Thus \( M(-1) \) splits fiberwise, and its direct image under \( \pi \) vanishes. (It is simultaneously a torsion and torsion-free sheaf.) As before, \( \pi_*(M(-1)) \to \Gamma(\phi, M(-1)) \) is an isomorphism, so \( \pi_*(\phi, M(-1)) = 0 \) and the degrees of the direct summands of \( M_\phi \) are less or equal to zero. As the (total) degree of \( M \) is zero, we have \( M_\phi \cong O_{\mathbb{P}^1}^{\oplus n+2} \), for all \( x \in \mathbb{P}^1 \), so the natural homomorphism \( \pi^* \pi_* M \to M \) is an isomorphism.

Step 2 Let us denote \( S := \pi_* M \). The restriction of \( \pi_* M \) to the exceptional line \( \Lambda \cong \mathbb{P}^1 \) is a monad over \( \mathbb{P}^1 \), whose middle entry is \( S \) and the other entries are trivial vector bundles. It follows that \( S \) is itself trivial. This finishes the proof of the existence of the monad \( \Phi \).

Now we assume that \( \mathcal{F}_\Phi \) is stable. Then by [Remark, pp. 332] implies that the restriction of the monad to \( \Phi \), whose cohomology is \( \mathcal{F}_\Phi \), is uniquely defined up to the \( G \)-action. Thus the same statement holds for \( \mathcal{F} \). Furthermore, an element \( (g_1, g_2, g_3) \in G \) in the isotropy group of the action induces an automorphism of \( \mathcal{F} \). As \( \mathcal{F}_\Phi \) is stable, this automorphism is the multiplication by a scalar \( \varepsilon \), so \( (g_1, g_2, g_3) = (\varepsilon 1_1, \varepsilon 1_2, \varepsilon 1_3) \).

Let \( H \) be the affine space underlying \( \text{Hom}(\mathbb{C}^n, \mathbb{C}^{n+2}) \times \Gamma(O_x(1)) \), and define:

\[
\mathcal{V} := \{(A, B) \in H^2 \mid A \text{ injective}, B \text{ surjective}, BA = 0\}, \quad (3.16)
\]

\[
\mathcal{V} := \{(A, B) \in H^2 \mid A \text{ injective}, B \text{ surjective}, BA = 0\}, \quad (3.17)
\]

The group \( G = [Gl(n) \times Gl(r + 2n) \times Gl(n)]/\mathbb{C}^* \) acts on the affine variety \( \{(A, B) \in H^2 \mid BA = 0\} \), and \( \mathcal{V}, \mathcal{V} \) are \( G \)-invariant open subsets.

**Corollary 3.7.** For \( n \geq r \) and \( c > r(r - 1)n(a + b) \), the moduli space

\[
\mathcal{M}_Y^{\text{vb}} = \mathcal{M}_Y^{\text{vb}} = \mathcal{M}_Y^{\text{vb}}(r; 0, n[O_x(1)]^2, 0) \quad (3.18)
\]

of \( L \)-semi-stable vector bundles on \( Y_{a,b} \) satisfying \( (3.4) - (3.9) \) is the quotient of an open subset of \( \mathcal{V} \) by the action \( (3.11) \) of \( G \). For \( \mathcal{F} \in \mathcal{M}_Y^{\text{vb}} \), holds

\[
\chi(End(\mathcal{F})) = 1 - m, \quad \text{with } m := 2(1 + a + b)n r - r^2 + 1. \quad (3.18)
\]

In particular, if \( \mathcal{M}_Y^{\text{vb}} \) is non-empty, then its dimension is at least \( m \).

Proof. The condition \( BA = 0 \) imposes at most \( n^2 \cdot h^0(O_x(2)) \) conditions, so the dimension of \( \mathcal{M}_Y^{\text{vb}} \) is at least

\[
2(3 + a + b)n(r + 2n) - (6 + 4a + 4b)n^2 - [2n^2 + (r + 2n)^2 - 1] \quad (3.15)
\]

The Euler characteristic \( \chi(End(\mathcal{F})) \) is given by Riemann-Roch. □

The monad \( (3.11) \) still makes sense for \( (A, B) \in \mathcal{V} \). Its cohomology is a sheaf on \( Y \) which still satisfies \( (3.4) - (3.9) \). These objects naturally occur if one wishes to compactify \( \mathcal{M}_Y^{\text{vb}} \). It is unlikely, however, that by adding these sheaves one gets a projective (complete) variety. (If one thinks off the group action on \( \mathcal{V} \) in terms of quivers, one obtains a quiver with loops.)

***3.3. Geometric properties of \( \mathcal{M}_Y^{\text{vb}} \).*** At this point it is natural to ask whether \( \mathcal{M}_Y^{\text{vb}} \) is empty, irreducible, and has the expected dimension. The irreducibility is a complicated issue. If our benchmark is the case \( a = 0, b = 1, r = 2 \) (so \( Y_{0,1} \) is the blow-up of \( \mathbb{P}^3 \) along a line), the question reduces to the irreducibility of the moduli space of rank two mathematical instantons (see [42; 12] for details). The recent answer to this problem (see [43; 44]) involves impressive computations.

For this reason, our approach is similar to [42], namely we pin down a ‘main’ component of \( \mathcal{M}_Y^{\text{vb}} \) which has all the desired properties. For shorthand, we let \( O(k, l) := O_x(k) \otimes \pi^* O_{\mathbb{P}^1}(l) \). The following remark will be useful by the Riemann-Roch formula,

\[
\chi(S(-1, -1)) = 0, \quad \text{for any sheaf} \ S \text{ on } Y \text{ with } c_1 = 0, c_2 = cu^2. \quad (3.19)
\]

---

4. Indeed, the extensions \( (3.12), (3.13) \) satisfy \( (3.14) \).

5. The computations are unpleasing, and the author used MAPLE. For \( u, v \) as in \( (3.1) \) and \( rk(S) = s \), holds:
\[
\text{Td}(Y_{a,b}) = 1 + \frac{u(1+u)}{2} v + u^2 - \frac{1}{6}(a+b)^2 u^2, \quad \text{and} \quad \ch(S(-1, -1)) = s - s(u + v) + s + u^2 + \left( \frac{s+6}{2} \right) u^2 v.
\]
In particular, for $\mathcal{F} \in \tilde{M}_Y^{vb}$ and $\mathcal{S} = \text{End}(\mathcal{F})$, we deduce

$$H^1(\text{End}(\mathcal{F})(-1,-1)) = 0 \iff H^2(\text{End}(\mathcal{F})(-1,-1)) = 0.$$  

**Theorem 3.8.** Let $D \in |\mathcal{O}_x(1)|$ be a generic section, so $D \cong \mathbb{P}(|\mathcal{O}_x(1)\oplus\mathcal{O}_x(-b))$, and consider the following ‘main component’ of $\tilde{M}_Y^{vb}$:

$$\tilde{M} := \{\mathcal{F} \in \tilde{M}_Y^{vb} | H^1(\text{End}(\mathcal{F})(-1,-1)) = 0, \mathcal{F}_D \text{ is } L_c\text{-semi-stable}\}. \quad (3.20)$$

Then $\tilde{M}$ is non-empty, irreducible, generically smooth of the expected dimension, and the locus corresponding to the stable vector bundles is dense. Moreover, $\tilde{M}$ is a rational variety.

The proof of this statement is contained in the forthcoming lemmas.

For $\mathcal{F} \in \tilde{M}$ and general $D \in |\mathcal{O}_x(1)|$ and $P \in |\pi^*\mathcal{O}_x(1)|$, the restrictions $\mathcal{F}_D$ and $\mathcal{F}_P$ are semi-stable, and theorem 1.8 implies that $\mathcal{F}$ is trivializable along $\lambda := D \cap P$ (so $\mathcal{F}$ automatically satisfies the technical condition (3.5)).

**Definition 3.9.** For $D, P$ as above, let $\lambda := D \cap P$ and $\Delta := D \cup P$. We denote

$$\tilde{M}_{D,\lambda}^{vb} := \tilde{M}_{D,\lambda}^{L_c}(r;0,n(a+b))^{vb}$$

(respectively $\tilde{M}_{P,\lambda}^{vb}$) the moduli spaces of semi-stable vector bundles on $D$ (respectively on $P, \Delta$), framed along $\lambda$. (See remark 2.10 for the definition of a framing.)

Then the map which identifies (glues) the framings

$$\tilde{M}_{D,\lambda}^{vb} \times \tilde{M}_{P,\lambda}^{vb} \to \tilde{M}_{\Delta,\lambda}^{vb} \quad (3.21)$$

is an isomorphism (its inverse is the restriction to $D, P$), and $\text{PGL}(r)$ acts on $\tilde{M}_{\Delta,\lambda}^{vb}$ by changing the framing along $\lambda$. The quotient map for this action is the morphism which forgets the framing.

We denote $M_{\Delta,\lambda} := M_{\Delta,\lambda}^{vb}/\text{PGL}(r)$. A key role for understanding the geometry of $\tilde{M}$ is played by the rational map

$$\Theta : \tilde{M} \dashrightarrow M_{\Delta}^{vb}, \quad \mathcal{F} \mapsto \mathcal{F}_{\Delta}. \quad (3.22)$$

**Lemma 3.10.** Let $M_{\Delta,\lambda}^{vb}$ be the open locus of vector bundles whose restrictions to $D, P$ are stable. Then $M_{\Delta,\lambda}^{vb}$ is birational to $\tilde{M}_{D,\lambda}^{L_c}(r;0,n)^{vb} \times \tilde{M}_{P,\lambda}^{vb}$, thus $M_{\Delta,\lambda}^{vb}$ is a rational variety of dimension

$$\dim M_{\Delta,\lambda}^{vb} = 2(a+b)n + 2nr - (r^2 - 1) = m. \quad (3.23)$$

**Proof.** First, $D$ is isomorphic to the Hirzebruch surface $Y_\ell$, with $\ell = b-a$, so $M_{D,\lambda}^{vb}$ is birational to $\tilde{M}_{D,\lambda}^{L_c}(r;0,n)^{vb} \times \text{PGL}(r)$, according to corollary 2.14. Thus, by the definition, $M_{\Delta,\lambda}^{vb}$ is birational to $\tilde{M}_{D,\lambda}^{L_c}(r;0,n)^{vb} \times M_{P,\lambda}^{vb}$. Second, $M_{P,\lambda}^{vb}$ is irreducible because $P \cong \mathbb{P}^2$ (see [23, Theorem 2.2]), and is also rational, by 2.19.

3.3.1. **Differential properties of $\tilde{M}$.** We start by addressing the generic smoothness of $\tilde{M}$.

**Lemma 3.11.** For all $\mathcal{F} \in \tilde{M}$ holds:

(i) $H^2(\text{End}(\mathcal{F})) = 0$;

(ii) The differential of $\Theta$ at $\mathcal{F}$ is an isomorphism;

(iii) Each irreducible component has the expected dimension, the locus corresponding to stable bundles in dense, and $\Theta$ is generically finite onto $M_{\Delta,\lambda}^{vb}$.

**Proof.** (i) Since $\mathcal{F}_D$ and $\mathcal{F}_P$ are semi-stable, $\mathcal{E} := \text{End}(\mathcal{F})$ is the same, thus $\mathcal{E}_D, \mathcal{E}_P(-1,0)$ have vanishing $H^2$. Now we apply this in $\mathcal{E}(-1,-1) \subset \mathcal{E}(-1,0) \rightarrow \mathcal{E}_P(-1,0)$ and $\mathcal{E}(-1,-1) \subset \mathcal{E} \rightarrow \mathcal{E}_D$.

(ii) The differential of $\Theta$ at $\mathcal{F}$ is the restriction homomorphism $H^1(\mathcal{E}) \rightarrow H^1(\mathcal{E}_{\Delta})$. The exact sequence $\mathcal{E}(-1,-1) \subset \mathcal{E} \rightarrow \mathcal{E}_{\Delta}$ implies that this is indeed an isomorphism.

(iii) Since $\text{d}\Theta$ is an isomorphism, the restriction of $\Theta$ to each component of $\tilde{M}$ is dominant. But the stable vector bundles are dense in both $\tilde{M}_D^{vb}, \tilde{M}_P^{vb}$ (see theorem 2.10(iv)), and $\mathcal{F}$ is stable on $Y$, as soon as its restriction to $\mathcal{F}_{\Delta}$ is stable. (This also shows that $\Theta$ is well-defined at the generic point of each irreducible component of $\tilde{M}$.) In this case, $\text{ext}^1(\mathcal{F}, \mathcal{F}) = m$, so each component has the expected dimension. For the generic finiteness of $\Theta$, use (3.23).
3.3.2. Non-emptiness of $\mathcal{M}$. Here we give explicit examples of vector bundles satisfying the defining properties of $\mathcal{M}$.

Lemma 3.12. Assume $r \geq 3$. There is a non-empty component $\mathcal{M}_o \subset \mathcal{M}^b_Y$ such that the generic $\mathcal{F} \in \mathcal{M}_o$ has the following properties:

(i) Its restriction to the generic divisor in $|\mathcal{O}_x(1)|$ is semi-stable, the restriction to the line cut out by two generic divisors in $|\mathcal{O}_x(1)|$ is trivializable, and is semi-stable on all the fibres of $\pi$. Hence $\mathcal{F}$ is $L_o$-semi-stable on $Y$.

(ii) $H^1(\text{End}(\mathcal{F})(-1, -1)) = 0$.

Proof. Step 1 We consider two generic divisors $D, D' \in |\mathcal{O}_x(1)|$, and the (fat) line $L_o := nD \cap D'$ in $Y$. Its ideal sheaf admits the resolution $0 \to \mathcal{O}_x(-n-1) \to \mathcal{O}_x(-1) \oplus \mathcal{O}_x(-n) \to \mathcal{I}_{L_o} \to 0$, and determines the exact sequence

$$0 \to \mathcal{O}_x(-n-1) \to \mathcal{O}_x^{-1} \oplus \mathcal{O}_x(-1) \oplus \mathcal{O}_x(-n) \to \mathcal{F}_0 := \mathcal{O}_x^{-1} \oplus \mathcal{I}_{L_o} \to 0. \tag{3.24}$$

Then $\mathcal{F}_0$ is torsion free of rank $r$, with $c_1 = 0, c_2 = n \cdot |\mathcal{O}_x(1)|^2$. Its restriction to the generic intersection of two divisors in $|\mathcal{O}_x(1)|$ is trivializable, so $\mathcal{F}_0$ is $L_o$-semi-stable, for all $c > 0$.

Step 2 By deforming $0$ to $t \in \Gamma(\mathcal{O}_x(n+1))$ in (3.24), we obtain a flat family of sheaves (the Hilbert polynomial is constant) on $Y$

$$0 \to \mathcal{O}_x(-n-1) \to \mathcal{O}_x^{-1} \oplus \mathcal{O}_x(-1) \oplus \mathcal{O}_x(-n) \to \mathcal{F}_t \to 0 \tag{3.25}$$

Also, since $r \geq 3$, $(t, a)$ is pointwise injective for generic $t$, so $\mathcal{F}_t$ is locally free.

Claim For generic $t$, the vector bundle $\mathcal{F}_t$ defined by (3.25) has the desired properties.

- $\mathcal{F}_t$ satisfies (3.3) and (3.4), so the same holds for generic $t$.
- One may check that $\mathcal{F}_0$ satisfies (3.7), so the same holds for $\mathcal{F}_t$. Alternatively, $\mathcal{F}_t$ is $\pi$-fibrewise semi-stable, so (3.7) is automatically satisfied.
- (3.8) and (3.9) follow directly from (3.25).
- The three properties at (i) are open in flat families of torsion free sheaves, and they hold for $\mathcal{F}_0$. Thus the same holds for generic $t$.

- Let us verify (ii). (Incidentally, observe that $\text{Ext}^2(\mathcal{F}_0, \mathcal{F}_0) \neq 0$.) Since $\mathcal{F}_t$, so $\text{End}(\mathcal{F}_t)$ is semi-stable, we have $h^0(\text{End}(\mathcal{F}_t)(-1, -1)) = h^3(\text{End}(\mathcal{F}_t)(-1, -1)) = 0$. By (3.19), holds

$$h^1(\text{End}(\mathcal{F}_t)(-1, -1)) = 0 \iff h^2(\text{End}(\mathcal{F}_t)(-1, -1)) = 0.$$ 

This latter property is easier to check. For generic $t$, the dual of (3.25) yields

$$0 \to \text{End}(\mathcal{F}_t)(-1, -1) \to \mathcal{F}_t(-1, -1)^{-1} \oplus \mathcal{F}_t(0, -1) \oplus \mathcal{F}_t(n-1, -1) \to \mathcal{F}_t(n, -1) \to 0. \tag{3.26}$$

Now remark that (3.25) implies $H^1(\mathcal{F}_t(n, -1)) = 0$. Second, we claim that $H^2(\mathcal{F}_t(k, -1)) = 0$, for all $k \geq -1$. Indeed, the vanishing holds for $k = -1$, by (3.3). For the induction, let $L$ be the intersection of two generic divisors in $|\mathcal{O}_x(1)|$, twist the exact sequence $\mathcal{O}_x(-2) \subset \mathcal{O}_x(-1)^{\oplus 2} \to \mathcal{I}_L$ by $\mathcal{F}_t(k, -1)$, and use the semi-stability of $\mathcal{F}_t$.

We explicitly produced vector bundles satisfying the lemma. The properties are open in flat families, so there is a non-empty component $\mathcal{M}_o \subset \mathcal{M}^b_Y$ which (generically) satisfies all of them.

It remains to address the case of rank two vector bundles.

Lemma 3.13. For $r = 2$, there is a non-empty component $\mathcal{M}_o \subset \mathcal{M}$ whose generic point satisfies (3.12) (i),(ii).

Proof. Let $Z$ be the union of $n$ sections of $\pi : Y \to \mathbb{P}^1$, such that each $L \subset Z$ is the intersection of two generic divisors in $|\mathcal{O}_x(1)|$. The Hartshorne-Serre correspondence (see [22, 1]) yields a vector bundle $\mathcal{F}$ with $c_1 = 0, c_2 = |Z| \cdot |\mathcal{O}_x(1)|^2$, which fits into an exact sequence

$$0 \to \mathcal{O}_x(1) \to \mathcal{F} \to \mathcal{I}_Z(1) \to 0; \quad \mathcal{F}_Z \cong \mathcal{O}_Z^2. \tag{3.26}$$

The properties (i) are easy to verify. For the second statement, the dual of (3.26) yields

$$0 \to H^1(\mathcal{F}(-2, -1)) \to H^1(\text{End}(\mathcal{F})(-1, -1)) \to H^1(\mathcal{I}_Z \otimes \mathcal{F}(0, -1)) \to H^2(\mathcal{F}(-2, -1))...$$

We claim that $\phi$ is an isomorphism. Indeed, $\mathcal{F}$ is obtained by glueing the local Koszul resolutions of the components $\lambda$ of $Z$. Thus $\phi$ is the boundary map corresponding to the Koszul resolution
of (any) one of $L \subset Z$. Then $\mathcal{F}_Z \cong \mathcal{O}_Z^2$ implies $H^1(\mathcal{I}_Z \otimes \mathcal{F}(0, -1)) = H^1(\mathcal{I}_L \otimes \mathcal{F}(0, -1)) = H^1(\mathcal{F}(0, -1))$. Now use $\mathcal{F}(-2, -1) \subset \mathcal{F}(-1, -1)^2 \to \mathcal{I}_L \otimes \mathcal{F}(0, -1)$ to deduce that $H^1(\mathcal{I}_L \otimes \mathcal{F}(0, -1)) \to H^1(\mathcal{F}(-2, -1))$ is an isomorphism. □

3.3.3. Irreducibility of $\mathcal{M}$. We are going to prove that $\mathcal{M} = \mathcal{M}_o$, which yields the conclusion.

**Lemma 3.14.** For $\mathcal{F}_o \in \mathcal{M}_o$ and $\mathcal{F} \in \mathcal{M}$ arbitrary, holds $H^1(\text{Hom}(\mathcal{F}_o, \mathcal{F})(-1, -1)) = 0$.

**Proof.** First notice that $\mathcal{F}_o^o$ satisfies the hypotheses of the theorem [3.1] the conditions [3.8], [3.9] are satisfied by (4.19), and (4.7) holds because $\mathcal{F}_o$ is semi-stable on all the fibres of $\pi$. Thus $\mathcal{F}_o^o$ is the cohomology of a monad (4.19), and we denote by $\mathcal{Q}_o$ the corresponding entry in its display.

For $\mathcal{H} := \text{Hom}(\mathcal{F}_o, \mathcal{F})$, we prove that $H^1(\mathcal{H}(-1, -1)) = 0$. Since $\mathcal{F}$ is semi-stable, the exact sequence

$$0 \to \mathcal{H}(-1, -1) \to \mathcal{Q}_o \otimes \mathcal{F}(-1, -1) \to \mathcal{F}(0, -1)^n \to 0,$$

yields

$$0 = \Gamma(\mathcal{F}(0, -1)) \to H^1(\mathcal{H}(-1, -1)) \to H^1(\mathcal{Q}_o \otimes \mathcal{F}(-1, -1)) \to H^1(\mathcal{F}(0, -1))^n. \quad (3.27)$$

The conclusion follows as soon as we prove that the rightmost arrow is an isomorphism. For this, we must deduce $\mathcal{Q}_o$ better.

As $\mathcal{F}_{o, \lambda} \cong \mathcal{O}_\lambda^o$, the restriction to $\lambda$ of the monad defining $\mathcal{F}_o^o$ yields $\mathcal{O}_\lambda \subset \mathcal{Q}_{o, \lambda} \to \mathcal{O}_\lambda(1)^n$. This extension is necessarily trivial, so $\mathcal{Q}_{o, \lambda} \cong \mathcal{O}_\lambda^o \oplus \mathcal{O}_\lambda(1)^n$. We deduce that the sheaf homomorphism $s$ in the diagram (I) below is injective:

Now we further decompose $\mathcal{R}_o$; clearly, there is a decomposition $\mathcal{O}_Y^o = \mathcal{O}_Y^o \oplus \mathcal{O}_Y^n$, such that the diagram (II) has exact columns and rows; it determines the torsion sheaf $\mathcal{S}_o$.

In order to understand this latter, we proceed inductively: clearly, there is a decomposition $\mathcal{O}_Y^n = \mathcal{O}_Y \oplus \mathcal{O}_Y^{n-1}$ such that the diagram (III) has exact rows and columns. Here $D$ stands for a generic (the starting $\mathcal{F}_o$ is so) divisor in $|\mathcal{O}_\pi(1)|$. By repeating $n$ times this process, we deduce that $\mathcal{S}_o$ is a successive extension of $\mathcal{O}_D$, with $D_j \in |\mathcal{O}_\pi(1)|$ for $j = 1, \ldots, n$. Let $\mathcal{S}_o^{(\nu)}$ be the sheaf obtained by $\nu$ extensions, $\nu = 1, \ldots, n$.

By using that $H^1(\mathcal{F}(-2, -1)) = H^1(\mathcal{F}(-1, -1)) = H^2(\mathcal{F}(-1, -1)) = 0$ (see [3.3], we deduce the following implications, for arbitrary $D \in |\mathcal{O}_\pi(1)|$:

$$(I) \Rightarrow H^1(\mathcal{Q}_o \otimes \mathcal{F}(-1, -1)) \xrightarrow{\cong} H^1(\mathcal{R}_o \otimes \mathcal{F}(-1, -1))^n,$$

$$(II) \Rightarrow \Gamma(\mathcal{F}(-1, -1)^D) = H^2(\mathcal{F}(-1, -1)^D) = 0,$$

$$(III) \Rightarrow \mathcal{O}_o(-1) \subset \mathcal{O}_o \to \mathcal{O}_D \Rightarrow \Gamma(\mathcal{F}(-1, -1)^D)) \xrightarrow{\cong} H^2(\mathcal{F}(-1, -1)^D)) = 0.$$

$\mathcal{O}_o(-1) \subset \mathcal{O}_o \to \mathcal{O}_D \Rightarrow \Gamma(\mathcal{F}(-1, -1)^D) \xrightarrow{\cong} H^2(\mathcal{F}(-1, -1)^D) = 0.$

Overall, we deduce that $H^1(\mathcal{Q}_o \otimes \mathcal{F}(-1, -1)) \xrightarrow{\cong} H^1(\mathcal{S}_o^{(\nu)} \otimes \mathcal{F}(-1, -1)) \xrightarrow{\cong} H^1(\mathcal{S}_o^{(\nu-1)} \otimes \mathcal{F}(-1, -1)) \to 0$. Thus the rightmost arrow in (3.27) is an isomorphism, and consequently $H^1(\mathcal{H}(-1, -1)) = 0$. □

**Lemma 3.15.** $\mathcal{M}$ is irreducible.

**Proof.** We constructed the component $\mathcal{M}_o$, and assume $\mathcal{M}'$ is another component of $\mathcal{M}$. Since both $\mathcal{M}_o, \mathcal{M}'$ dominate $\mathcal{M}_{\Delta}$ (see [3.1]), we find general points $\mathcal{F}_o \in \mathcal{M}_o$ and $\mathcal{F} \in \mathcal{M}'$ with the properties:

- $\Theta(\mathcal{F}) = \Theta(\mathcal{F}_o)$, that is $\mathcal{F}_{\Delta} \cong \mathcal{F}_{o, \Delta}$. 


Both $F, F_0$ are stable (by the density of the stable locus).

$F_0$ satisfies the conditions

Then, for $H := \text{Hom}(F_0, F)$, the exact sequence

$$0 \rightarrow \Gamma(H) \rightarrow \Gamma(H_\Delta) \rightarrow H^1(H(\Delta)) \rightarrow \ldots.$$ 

has vanishing left hand side (as $H$ is semi-stable) and also right hand side (by lemma 3.14). Hence the isomorphism $F_{o, \Delta} \rightarrow F_\Delta$ can be extended to a homomorphism $F_o \rightarrow F$. Its determinant is a section of $\mathcal{O}_Y$, non-zero along $\Delta$, so $F_{o, \Delta}$ are isomorphic too. We deduce that $M_o = M'$, that is $M$ is irreducible, since they are irreducible components, and their general points coincide. \qed

3.3.4. Rationality of $M$.

Lemma 3.16. $\Theta : M \rightarrow M_{\Delta}^{vb}$ is generically injective and birational, so $M$ is a rational variety.

Proof. The same argument as in the proof of the previous lemma shows that $\Theta : M \rightarrow M_{\Delta}^{vb}$ is generically injective. Let $M^s \subset M_o$ the locus consisting of vector bundles whose restrictions to both $D, P$ are stable. Then $\Theta|_{M^s}$ is well-defined, generically injective, and dominant. Zariski’s main theorem implies that $\Theta$ is birational, so $M$ is a rational variety by 3.10. \qed

Remark 3.17. (i) In several cases (see [15, 9]) is more convenient to work with framed vector bundles (especially for the existence of universal families). We can reformulate the theorem by saying that the moduli space $M_\lambda$, consisting of semi-stable vector bundles $F \in M$ together with a framing along the line $\lambda$ (in contrast with the usual framings along divisors) is birational to $M_{\Delta, \lambda}$.

(ii) For $a = 0, b = 1$, one may easily check that $Y_0,1$ is isomorphic to the blow-up of $\mathbb{P}^3$ along a line, and theorem 3.1 reduces precisely to the monad construction [15, 15] of instantons on $\mathbb{P}^3$ trivialized along the line.

(iii) We conclude by noticing that theorem 3.1 yields also principal symplectic, respectively orthogonal bundles on $Y_{a, b}$. (Higher rank symplectic instanton bundles on $\mathbb{P}^3$ have been constructed recently in [7].) In this case, (3.5) and (3.9) are equivalent by the Riemann-Roch formula, so one should impose only the conditions (3.5) = (3.9). The outcome is that there is a non-degenerate (skew-)symmetric, bilinear form $b$ on $\mathbb{C}^{r+2n}$ (the middle term of (3.10)), such that the homomorphisms $A, B$ are dual to each other with respect to $b$, that is $B = A \cdot b$ (compare with [5, Section 4]). The monad condition $BA = 0$ translates into $A \cdot b \cdot A = 0$. The properties of the corresponding moduli spaces will be investigated in a future article.

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