Analytic dilation for Laplacians on manifolds with corners of codimension 2

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Abstract

The analytic dilation method was originally used in the context of many body Schrödinger operators. In this paper we adapt it to the context of compatible Laplacians on complete manifolds with corners of codimension two. As in the original setting of application we show that the method allows us to: First, meromorphically extend the matrix elements associated to analytic vectors. Second, to prove absence of singular spectrum. Third, to find a discrete set that contains the accumulation points of the pure point spectrum, and finally, it provides a theory of quantum resonances. Apart from these results, we win also a deeper understanding of the essential spectrum of compatible Laplacians on complete manifolds with corners of codimension 2.

1 Introduction

In the spectral analysis of self-adjoint operators a fundamental problem is to show whether or not the singular spectrum exists and identify the set of accumulation points of the pure point spectrum. In this paper we solve this problem for compatible Laplacians on complete manifolds with corners of codimension two by adapting the analytic dilation method to this setting.

The method of analytic dilation was originally applied to $N$-particle Schrödinger operators, a classic reference in that setting is [11]. It has also been applied to the black-box perturbations of the Euclidean Laplacian in the series of papers [24] [25] [26] [27]. In the paper [1] it is used to study Laplacians on hyperbolic manifolds. The analytic dilation has also been applied to the study of the spectral and scattering theory of quantum wave guides and Dirichlet boundary domains, see e.g. [9] [17]. It has also been applied to arbitrary symmetric spaces of noncompact types in the papers [18] [19] [20].
In each of these settings new ideas and new methods carry out. In this paper we develop the analytic dilation method for Laplacians on complete manifolds with corners of codimension 2.

Let us begin recalling the geometric setting in which our results will be stated. Let $X_0$ be a compact manifold with boundary $M$. We say that $X_0$ has a corner of codimension 2 if:

i) There exists a hypersurface $Y$ of $M$ which divides $M$ in two manifolds with boundary $M_1$ and $M_2$, i.e. $M = M_1 \cup M_2$ and $Y = M_1 \cap M_2$.

ii) $X_0$ is endowed with a Riemannian metric $g$ that is a product metric on small neighborhoods of the $M_i$’s and the corner $Y$.

![Figure 1. Compact manifold with corner of codimension 2](image)

We construct from $X_0$ a complete manifold $X$ by attaching $([0, \infty) \times M_1)$, $([0, \infty) \times M_2)$ and filling the rest with $([0, \infty) \times [0, \infty) \times Y)$. As a set,

$$X := X_0 \cup ([0, \infty) \times M_1) \cup ([0, \infty) \times M_2) \cup ([0, \infty) \times [0, \infty) \times Y),$$

(1)

and it has the natural differential structure and Riemannian metric that are compatible with the product structures at the boundary of $X_0$. The manifold $X$ has associated a natural exhaustion given by:

$$X_T := X_0 \cup ([0, T] \times M_1) \cup ([0, T] \times M_2) \cup ([0, T])^2 \times Y).$$

(2)

![Figure 2. $X_T$, element of the exhaustion of $X$.](image)
For each $T \in [0, \infty)$, $X$ has two submanifolds with cylindrical ends, namely $M_i \times \{T\} \cup (Y \times \{T\}) \times [0, \infty)$, for $i = 1, 2$. We denote these manifolds by $Z_i$.

Let us now consider the operator $\Delta := d^* d : C^\infty_c(X) \rightarrow L^2(X)$, in local coordinates:

$$\Delta = \frac{-1}{\sqrt{|\det(g)|}} \frac{\partial}{\partial x_i} \left( g^{ij} \frac{\partial}{\partial x_j} \sqrt{|\det(g)|} \right). \tag{3}$$

The operator $\Delta$ is essentially self-adjoint and we denote by $H$ its self-adjoint extension. In section 3 we shall consider compatible Laplacians.

For $i = 1, 2$, since $Z_i$ is a complete manifold, the Laplacian $\Delta : C^\infty_c(Z_i) \rightarrow L^2(Z_i)$ is essentially self-adjoint; we denote by $H^{(i)}$ its self-adjoint extension. Similarly $\Delta : C^\infty(Y) \rightarrow L^2(Y)$ is essentially self-adjoint, and we denote its self-adjoint extension by $H^{(3)}$. The analytic dilation of a many-body Schrödinger operator depends on the analytic dilation of its subsystem Hamiltonians. In a similar way the analytic dilation of $H$ is described in terms of the spectral theory of the operators $H^{(1)}$, $H^{(2)}$ and $H^{(3)}$.

For $\theta > 0$, the operator $U_{i, \theta} : L^2(Z_i) \rightarrow L^2(Z_i)$ is essentially the dilation operator by $\theta + 1$ up to a compact set. More precisely:

$$U_{i, \theta} f(x) = \begin{cases} f(x) & \text{for } x \in M_i, \\ (\theta + 1)^{1/2} f((\theta + 1)u, y) & \text{for } x = (u, y) \in [0, \infty) \times Y \end{cases} \tag{4}$$

and for $u$ big enough,
and $U_{i,θ}f$ is extended to the whole $Z_i$ in such a way that it sends $C_c^∞(Z_i)$ into $C_c^∞(Z_i)$, and it becomes a unitary operator on $L^2(Z_i)$. Details will be worked out in section 2. Similarly, the operators $U_θ : L^2(X) → L^2(X)$ are defined by

$$U_θf(x) = \begin{cases} f(x) & \text{for } x ∈ X_0, \\ (θ + 1)^{1/2}U_{i,θ}f((θ + 1)u_i, z_i) & \text{for } x = (u_i, z_i) ∈ [0, ∞) × Z_i \\ \text{and for } u_i \text{ big enough.} \end{cases}$$

Again $U_θf$ is extended to the whole $X$ in such a way that, for $f ∈ C_c^∞(X)$, $U_θf ∈ C_c^∞(X)$, and $U_θ$ becomes a unitary operator in $L^2(X)$. Details will be given in section 3.1.

For $θ ∈ [0, ∞)$, define $H_θ := U_θHU_θ^{-1}$, a closed operator with domain

$$\mathcal{H}^2(X) := \{ f ∈ L^2(X) : \Delta_{dist}f ∈ L^2(X), \}$$

the second Sobolev space associated to $(X, g)$. We define the set:

$$Γ := \{ \theta := θ_0 + iθ_1 ∈ \mathbb{C} : θ_0 > 0, θ_0 ≥ |θ_1| ∏ θ(θ)^2 < 1/2 \}.$$

We will extend the family $H_θ$ from $[0, ∞)$ to $Γ$.

![Figure 4. Sketch of the region Γ](image)

We prove:
**Theorem 1** The family \((H_\theta)_{\theta \in [0, \infty)}\) extends to an holomorphic family for \(\theta \in \Gamma\), which satisfies:

1) \(H_\theta\) is a closed operator with domain \(\mathcal{W}_2(X)\) \(\theta \in \Gamma\).

2) For \(\varphi \in \mathcal{W}_2(X)\) the map \(\theta \mapsto H_\theta \varphi\) is holomorphic in \(\Gamma\).

An holomorphic family of operators satisfying (1) and (2) will be called a **holomorphic family of type A**. This theorem is proved using the analogous result that the family \(\{H^{(i)}_\theta\}_{\theta \in [0, \infty)}\) extends to a holomorphic family of type A in \(\Gamma\), where \(H^{(i)}_\theta\) denotes the closed operator associated to 

\[U_{i,\theta} \Delta_{Z_i} U_{i,\theta}^{-1}\]

with domain

\[\mathcal{W}_2(Z_i) := \{f \in L^2(Z_i) : \Delta_{\text{dist}}(f) \in L^2(Z_i)\}\],

the second Sobolev space associated to \((Z_i, g_i)\).

The families \(H_\theta\) and \(H^{(i)}_\theta\) extend to sets larger than \(\Gamma\), but \(\Gamma\) is enough for our outlined goals. In particular, we choose the domain \(\Gamma\) because for \(\theta \in \Gamma\) we can prove that \(H^{(i)}_\theta\) is \(m\)-sectorial (see section 2.7). We define

\[\theta' := \frac{1}{(\theta + 1)^2}\].

The parameter \(\theta'\) is very important in the description of the essential spectrum of \(H_\theta\) as we can see in the next theorem that will be proved in section 3.3.

**Theorem 2** For \(\theta \in \Gamma\),

\[\sigma_{\text{ess}}(H_\theta) = \bigcup_{\mu \in \sigma(H^{(3)})} (\mu + \theta'[0, \infty)) \cup \bigcup_{\lambda_1 \in \sigma_{\text{pp}}(H^{(1)}, \theta)} (\lambda_1 + \theta'[0, \infty)) \cup \bigcup_{\lambda_2 \in \sigma_{\text{pp}}(H^{(2)}, \theta)} (\lambda_2 + \theta'[0, \infty))\].

In section 3.4, we associate to \((U_\theta)_{\theta \in [0, \infty)}\) a set \(\mathcal{V} \subset \mathcal{W}_2(X)\) that satisfies:

i) \(\mathcal{V}\) is dense in \(L^2(X)\).

ii) for \(\varphi \in \mathcal{V}\), \(U_\theta \varphi\) is defined for all \(\theta \in \Gamma\).
iii) \( U_\theta \mathcal{V} \) is dense in \( L^2(X) \) for all \( \theta \in \Gamma \).

The elements of a subset of \( \mathscr{H}_2(X) \) which satisfies i) and ii) will be called **analytic vectors**. We denote by \( \Lambda \) the left-hand plane, more explicitly:

\[
\Lambda := \{(x, y) \in \mathbb{C} : x < 0\}.
\]

(11)

We denote by \( R(\lambda) \) the resolvent of \( H \) and by \( R(\lambda) \) the resolvent of \( H_\theta \). Using the general analytic dilation theory of Aguilar-Balslev-Combes (see [1]) we describe the nature of the spectrum of \( H \). This theory is based on:

i) The knowledge of the essential spectrum of \( H_\theta \), provided by theorem [2]

ii) The following equation, that is consequence of the unitarity of \( U_\theta \),

\[
\langle R(\lambda) f, g \rangle_{L^2(X)} = \langle R(\lambda, \theta) U_\theta f, U_\theta g \rangle_{L^2(X)},
\]

(12)

for \( f, g \in \mathcal{V} \) and \( \theta \in [0, \infty) \).

Since the right-hand side of (12) is defined for \( \lambda \in \Lambda \) and \( \theta \in \Gamma \), (12) provides a meromorphic extension of \( \lambda \mapsto \langle R(\lambda) f, g \rangle_{L^2(X)} \) from \( \Lambda \) to \( \mathbb{C} - \sigma(H_\theta) \). From this, we deduce the following theorem.

**Theorem 3** 1) For \( f, g \in \mathcal{A} \) the function \( \lambda \mapsto \langle R(\lambda) f, g \rangle_{L^2(X)} \) extends from \( \Lambda \) to \( \mathbb{C} - \sigma(H_\theta) \).

2) For all \( \theta \in \Gamma \), \( H_\theta \) has no singular spectrum.

3) The accumulation points of \( \sigma_{pp}(H) \) are contained in \( \{\infty\} \cup \sigma(H^{(3)}) \cup \cup_{i=1}^2 \sigma_{pp}(H^{(i)}) \).

The meromorphic extension of the resolvent entries can be used to extend generalized eigenfunctions that describe natural wave operators whose image is the complete set of absolute continuous states associated to \( H \). These results are worked out in detail in [4] [5] [6].

In appendix A based on [7], we geometrically refine the notion of singular spectrum introducing what we call boundary Weyl sequences; we use such refinement to compute the essential spectrum in section 3.3. In appendix B we give a brief introduction to sectorial operators in such a way that we can enunciate the Ichinose lemma (see theorem [18] in appendix B). Given two closed operators \( A \) and \( B \) acting on Hilbert spaces \( \mathscr{H}_1 \) and \( \mathscr{H}_2 \), the Ichinose
lemma gives sufficient conditions for having $\sigma(A \otimes 1 + 1 \otimes B) = \sigma(A) + \sigma(B)$. It is important for our results because, intuitively, the operator $H_\theta$ is of the form $A \otimes 1 + 1 \otimes B$ at infinity.

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2 Analytic dilation on complete manifolds with cylindrical end

In this section we generalize the method of analytic dilation to complete manifolds with cylindrical end. The results of this section are consequence of [16] and they are expected from the results of analytic dilation in wave guides (see [9] [17]). Because of that, and since the approach in the proof of the main results of this section is similar to the approach in section 3 we give most of the results without proof. The interested reader is referred to [6] or [16] for further details.

The analytic dilation on complete manifolds with cylindrical end will be important in section 3 because the analytic dilation of $H$ is described in terms of the analytic dilations of $H^{(1)}$ and $H^{(2)}$, compatible Laplacians on $Z_1$ and $Z_2$. In fact, one of our main results, theorem 2 shows that the essential spectrum of $H_\theta$ is described in terms of the pure point spectrum of $H^{(1)}_\theta$ and $H^{(2)}_\theta$, the dilated Laplacians associated to $H^{(1)}$ and $H^{(2)}$.

2.1 Manifolds with cylindrical end and their compatible Laplacians

Let $Z_0$ be a compact Riemannian manifold with boundary $Y := \partial Z_0$. We say that $Z_0$ is a **compact manifold with cylindrical end** if there exists a neighborhood, $Y \times (-\epsilon, 0]$, of the boundary $Y$ such that Riemannian metric of $Z_0$ is a product metric i. e. a metric of the form $g_Y + du \otimes du$ where $g_Y$ is a Riemannian metric on $Y$ and $u$ is the variable on $(-\epsilon, 0]$. 
We make from $Z_0$ a complete manifold $Z$ by attaching the infinite cylinder $Y \times [0, \infty)$ to $Z_0$. We have then:

$$Z := Z_0 \cup_Y (Y \times [0, \infty)), \quad (13)$$

where we are identifying the boundary of $Z_0$ with $Y \times \{0\}$. We extend the smooth structure and the Riemannian metric naturally. The manifold $Z$ is called **complete manifold with cylindrical end**. It looks as follows:

![Figure 6. Complete manifold with cylindrical end.](image)

Let $E$ be a vector bundle over $Z$ with an Hermitian metric. We assume that there exists $E'$ an Hermitian vector bundle over $Y$ such that $E|_{Y \times [0, \infty)}$ is the pull back of $E'$ by the projection $\pi : Y \times \mathbb{R}_+ \to Y$. We suppose that the Hermitian metric of $E$ is the pullback of the Hermitian metric of $E'$. Let $\Delta$ be a generalized Laplacian on $Z$, i.e. $\sigma_2(\Delta)(z, \xi) = |\xi|^2_{g_z}$. We assume furthermore that on $Y \times [0, \infty)$

$$\Delta = -\frac{\partial^2}{\partial u^2} + \Delta_Y, \quad (14)$$

where $\Delta_Y$ is a generalized Laplacian acting on $C^\infty(Y, E')$. In fact, we will denote by $\Delta_Y$ the operator acting on distributions and the self-adjoint operator induced by $(\Delta_Y, C^\infty(Y, E'))$.

A Laplacian satisfying the previous assumptions is called a **compatible Laplacian**.
2.2 The definition of $U_\theta$

Let $0 < K < R$ and $\varphi \in C^\infty(\mathbb{R})$ with $\varphi' \geq 0$ such that:

$$
\varphi(u) := \begin{cases} 
0 & \text{for } 0 < u < K \\
1 & \text{for } R < u < \infty.
\end{cases}
$$

(15)

Let $\theta \in [0, \infty)$, define the function:

$$
\psi_\theta(u) := (\varphi(u)\theta + 1)u = \varphi(u)u\theta + u,
$$

(16)

for $u \in \mathbb{R}_+$. Observe that

$$
\psi_\theta(u) = \begin{cases} 
u & u < K \\
(\theta + 1)u & u > R.
\end{cases}
$$

We calculate the first derivatives of $\psi_\theta$ with respect to $u$:

$$
\psi'_\theta(u) := \frac{\partial}{\partial u}(\psi_\theta)(u) = \varphi'(u)u\theta + \varphi(u)\theta + 1.
$$

(17)

$$
\psi''_\theta(u) := \frac{\partial^2}{\partial u^2} \psi_\theta(u) = \varphi''(u)u\theta + 2\varphi'(u)\theta.
$$

(18)

and

$$
\psi'''_\theta(u) := \frac{\partial^3}{\partial u^3} \psi_\theta(u) = \varphi'''(u)u\theta + 3\varphi''(u)\theta.
$$

(19)

We define $U_\theta : L^2(Z, E) \to L^2(Z, E)$:

$$
U_\theta f(z) = \begin{cases} 
f(z) & \text{for } z \in Z_0 \\
f(y, \psi_\theta(u))\psi_\theta'(u)^{1/2} & \text{for } z = (y, u) \in Y \times \mathbb{R}_+.
\end{cases}
$$

(20)

Observe that, for $\theta > 0$, the function $\psi_\theta$ is invertible (because $\psi'_\theta(u) \geq 1$ for $u \geq 0$). We will denote its inverse by $\alpha_\theta$.

For $\theta \in \mathbb{R}_+$ a natural inverse of $U_\theta$ is given by:

$$
U_\theta^{-1} f(z) := \begin{cases} 
f(z_0), & \text{for } z = z_0 \in Z_0. \\
(f(y, \alpha_\theta(u))\psi_\theta'(\alpha_\theta(u))^{-1/2} & \text{for } (y, u) \in Y \times \mathbb{R}_+.
\end{cases}
$$

(21)

We observe that for $f \in C^{\infty}(Z, E)$, $U_\theta f$ belongs to $C^{\infty}(Z, E)$ and, if $f \in C_c^{\infty}(Z, E)$, then $U_\theta f \in C_c^{\infty}(Z, E)$. It is easy to see:

**Proposition 1** For $\theta \in (0, \infty)$, $U_\theta$ induces a unitary operator acting on $L^2(Z, E)$.

In the next section we extend the family of operators $(U_\theta \Delta U_\theta^{-1})_{\theta \in (0, \infty)}$ to parameters $\theta \in \Gamma$ (see [7]).
2.3 The family $\Delta_\theta$

Calculating explicitly $U_\theta \frac{\partial^2}{\partial u^2} U_\theta^{-1} f(y, u)$, for $(y, u) \times Y \times \mathbb{R}_+$, we obtain:

$$
\Delta_\theta f(y, u) = \Delta_Y f(y, u) - \frac{\partial^2}{\partial u^2} f(y, u)(\alpha'_\theta(\psi_\theta(u)))^2
- \frac{\partial}{\partial u} f(y, u)(\psi_\theta(u))^2 (\alpha'_{\psi_\theta}(u)) + \frac{\partial}{\partial u} f(y, u)(\psi_\theta(u)^{-1}(\alpha'_{\psi_\theta}(u)))^2
- \frac{3}{4} f(y, u)^2 (\psi_\theta(u)^2) (\alpha'_{\psi_\theta}(\psi_\theta(u)))^2 + \frac{1}{2} f(y, u)^2 (\alpha'_{\psi_\theta}(\psi_\theta(u)))^2
+ \frac{1}{2} f(y, u)^2 (\alpha'_{\psi_\theta}(\psi_\theta(u)))^2.
$$

(22)

Observe that, on $Y \times \mathbb{R}_+$, we have:

$$
\Delta_\theta = a_2(\theta, u) \frac{\partial^2}{\partial u^2} + a_1(\theta, u) \frac{\partial}{\partial u} + a_0(\theta, u) + \Delta_Y,
$$

(23)

where $a_2(\theta, u), a_1(\theta, u)$ and $a_0(\theta, u)$ are given by

$$
a_2(\theta, u) := (\alpha'_\theta(\psi_\theta(u)))^2 = \frac{-1}{(\psi'_\theta(u))^2};
$$

$$
a_1(\theta, u) := -\alpha''_\theta(\psi_\theta(u)) + (\psi'_\theta(u))^{-1}(\alpha'_{\psi_\theta}(\psi_\theta(u)))^2;
$$

$$
a_0(\theta, u) := \frac{1}{2} \psi'_\theta(u)^{-1}(\alpha''_{\psi_\theta}(\psi_\theta(u)))^2 + \frac{1}{2} \psi'_\theta(u)^{-1}(\alpha''_{\psi_\theta}(\psi_\theta(u)))^2
+ \frac{3}{4} \psi'_\theta(u)^{-3/2}(\alpha''_{\psi_\theta}(\psi_\theta(u)))^2.
$$

(24)

Notice that, for all $u \in \mathbb{R}_+$, $a_2(\theta, u), a_1(\theta, u)$ and $a_0(\theta, u)$ are well defined and holomorphic for $Re(\theta) \geq 0$. We remark also that

$$
a_k(\theta, \cdot) \in C^\infty(\mathbb{R}_+),
$$

for $k = 0, 1, 2$ and $Re(\theta) > 0$. We will continue denoting $a_2(\theta, u), a_1(\theta, u)$ and $a_0(\theta, u)$, the coefficients of $\frac{\partial^2}{\partial u^2}, \frac{\partial}{\partial u}$ and $Id$, respectively, for the operator $\Delta_\theta$ localized in $Y \times \mathbb{R}_+$.

From now on, given $\theta \in \mathbb{C} - (-\infty, 0)$, we define $\theta'$ by

$$
\theta' := \frac{1}{(\theta + 1)^2}.
$$

(25)

The parameter $\theta'$ will appear naturally in the description of $\sigma_{ess}(\Delta_\theta)$ (see equation (32)). The next proposition follows easily from (22).
**Proposition 2** Let \( f \in C^\infty(Z, E) \). For \( \text{Re}(\theta) \geq 0 \), the formula for \( \Delta_\theta \) reduces for \((y, u) \in Y \times (0, K)\) to:

\[
\Delta_\theta f(u, y) = -\frac{\partial^2}{\partial u^2} f(u, y) + \Delta_Y f(u, y); \tag{26}
\]

and, for \((y, u) \in Y \times (R, \infty)\) to:

\[
\Delta_\theta f(u, y) = -\theta' \frac{\partial^2}{\partial u^2} f(u, y) + \Delta_Y f(u, y). \tag{27}
\]

The next proposition is a technical tool that can be deduced from (24).

**Proposition 3** Let \( a_0(\theta, u), a_1(\theta, u) \) and \( a_2(\theta, u) \) be given by (24). If \(|\theta| < N \) and \( \text{Re}(\theta) \geq 0 \), then there exists a \( C(N) \in \mathbb{R}_+ \) independent of \( \theta \) and \( u \in \mathbb{R}_+ \) such that for \( i = 0, 1, 2 \):

\[
|a_i(\theta, u)| \leq C(N), \tag{28}
\]

and,

\[
|\frac{\partial}{\partial \theta_i} (a_i)(\theta, u)| \leq C(N), \tag{29}
\]

where \( \theta := \theta_1 + i\theta_2 \).

We recall some definitions and results on manifolds with bounded geometry and their natural vector bundles, references for them are \([10]\) and \([23]\). A Riemannian manifold \( M \) has **bounded geometry** if its injectivity radius is positive and if all the derivatives of the curvature tensor, in the geodesic coordinates, are uniformly bounded. Let \( D \) be a linear differential operator in \( \text{Diff}(E, F) \), \( E \) and \( F \) vector bundles over \( M \) with bounded covariant derivatives \( \nabla^E \) and \( \nabla^F \) over \( M \), a manifold with bounded geometry. \( D \) has **bounded coefficients** if, in geodesic coordinates and in any synchronous maps, all the derivatives are uniformly bounded. It is easy to see that the manifold \( Z \) given by (13) has bounded geometry and the compatible Laplacians in \( \text{Diff}(E) \) have bounded coefficients. A differential operator \( D \) of order \( m \) is a **uniformly elliptic operator**, if the following inequality holds:

\[
|\sigma_m(D)^{-1}(z, \xi)|^{-1} \leq |\xi|^{-m}, \tag{30}
\]

for \( z \in Z \) and \( \xi \in T_z Z \). If \( A \) is a second order differential uniformly elliptic operator, then the norm \( f \mapsto ||f||_{L^2(Z, E)} + ||Af||_{L^2(Z, E)} \) on \( C_c^\infty(Z, E) \) is equivalent to the norm \( f \mapsto ||f|| + ||\Delta(f)|| \). In general, if \( A \) is a uniformly elliptic differential operator of order \( m \) acting on a manifold with bounded
geometry, then the norms \( f \mapsto \|f\| + \|A(f)\| \) and the \( m \)-Sobolev norm \( \|\cdot\|_m \), defined naturally using geodesic coordinates and synchronous frames, are equivalent. Since \( Z \) is a complete manifold and \( \Delta : C_c^\infty(Z,E) \to L^2(Z,E) \) is uniformly elliptic, then \( \Delta \) is essentially self-adjoint (see [23]). Abusing of the notation we denote by \( \Delta \) the differential operator acting on distributions and the self-adjoint operator itself. Denote by \( W^2_2(Z,E) \) the closure of \( C_c^\infty(Z,E) \) with respect to the norm \( \|f\|_2 := \|f\| + \|\Delta f\| \) for \( f \in C_c^\infty(Z,E) \). We call \( W^2_2(Z,E) \) the second Sobolev space.

Using the theory of manifolds with bounded geometry sketched above it is straight to prove the next theorem.

**Theorem 4** The family \( (\Delta_\theta)_{\theta \in \mathbb{R}_+} \) extends to an analytic family of type A for \( \text{Re}(\theta) > 0 \) i.e.

i) \( \Delta_\theta \) are closed operators with \( \text{Dom}(\Delta_\theta) \) independent of \( \theta \). More precisely, 
\[
\text{Dom}(\Delta_\theta) = W_2(Z,E).
\]

ii) For every \( f \in W_2(Z,E) \) the map \( \theta \mapsto \Delta_\theta f \) is analytic for \( \text{Re}(\theta) > 0 \).

### 2.4 The essential spectrum of \( \Delta_\theta \)

Recall that, given a closed operator \( A \), the pure point spectrum, discrete spectrum, and essential spectrum are the sets given by

\[
\sigma_{pp}(A) := \{ \lambda \in \mathbb{C} : \text{is an eigenvalue of } A \}, \\
\sigma_d(A) := \{ \lambda \in \mathbb{C} : \lambda \text{ is an isolated eigenvalue of } A \text{ of finite algebraic multiplicity} \}, \\
\sigma_{ess}(A) := \sigma(A) - \sigma_d(A),
\]

respectively. Our next goal is to prove the equality:

\[
\sigma_{ess}(\Delta_\theta) = \bigcup_{i=0}^{\infty} (\mu_i + \theta'[0,\infty)), \tag{32}
\]

where \( \sigma(\Delta_Y) := \{ \mu_i \}_{i=0}^{\infty} \). The first step towards (32) is to prove:

\[
N_{ess}(\Delta_\theta) = \bigcup_{i=0}^{\infty} (\mu_i + \theta'[0,\infty)), \tag{33}
\]

where \( N_{ess} \) is the set defined in appendix A, definition 2. Theorem 12 and equation (33) imply

\[
\sigma_{ess}(\Delta_\theta) = N_{ess}(\Delta_\theta), \tag{34}
\]
and hence (32).

The proof of equation (33) is based on the manipulation of singular sequences (see definition 1 in appendix A). In [6] we prove that singular sequences associated to $\Delta_i^\theta$, the closed operator associated to $-\theta' \frac{\partial^2}{\partial u^2} + \mu_i$ with Dirichlet boundary conditions, induce singular sequences associated $\Delta_\theta$; and the other way around, singular sequences associated to $\Delta_\theta$ induce singular sequences associated $\Delta_i^\theta$ for some $i$. A fundamental tool for formalizing these ideas is the Rellich theorem. As we said in the introduction we do not give details here since a similar approach will be used in section 3.3.

**Theorem 5** 1) Let $\text{Re}(\theta) \geq 0$ and let $(g_n)_{n \in \mathbb{N}}$ be an orthonormal singular sequence associated to $\lambda$ and $-\theta' \frac{\partial^2}{\partial u^2} + \mu_i$. Then, there exists a subsequence of $h_n := (\kappa g_n \phi_i) ||\kappa g_n \phi_i||^{-1}$ that is a singular sequence associated to the operator $\Delta_\theta$.

2) Let $\text{Re}(\theta) \geq 0$ and let $g_n$ be an orthonormal singular sequence associated to the operator $\Delta_\theta$ and the value $\lambda$. Then, there exists $i \in \mathbb{N}$ and a subsequence $s$ of $\mathbb{N}$ such that the function $u \mapsto \langle \kappa g_{s(n)}(u, \cdot), \phi_i \rangle_{L^2(\gamma)}$ in $C^\infty(\mathbb{R}^+)$, is a singular sequence of $\Delta_i^\theta$ and the value $\lambda$.

From theorem 5 follows that:

$$N_{\text{ess}}(\Delta_\theta) = \bigcup_{i=0}^{\infty} (\mu_i + \theta'[0, \infty)), \quad (35)$$

which, together with theorem 12 imply that

$$\sigma_{\text{ess}}(\Delta_\theta) = N_{\text{ess}}(\Delta_\theta). \quad (36)$$

### 2.5 The analytic vectors of $U_\theta$

In this section we construct a subset, $\mathcal{V} \subset L^2(Z, E)$ such that

i) $\mathcal{V}$ is a dense subset of $L^2(Z, E)$.

ii) For $f \in \mathcal{V}$ the function $\theta \mapsto U_\theta f \in L^2(Z, E)$ makes sense for $\theta \in \mathbb{C}$, $\text{Re}(\theta) > 0$.

iii) $U_\theta \mathcal{V}$ is dense in $L^2(Z, E)$ for $\text{Re}(\theta) > 0$. 

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Recall that we denote by \((\phi_i, \mu_i)_{i=1}^\infty\) a spectral resolution of the operator \(\Delta_Y\). Let \(\kappa \in C^\infty(\mathbb{R}_+)\) be a function satisfying \(0 \leq \kappa \leq 1\), \(\kappa' \geq 0\) and

\[
\kappa(u) = \begin{cases} 1 & K \leq u < \infty, \\ 0 & 1 < u \leq K - 1. \end{cases}
\]  

(37)

We extend \(\kappa\) to \(Y \times \mathbb{R}_+\) defining \(\kappa(y, u) := \kappa(u)\) for \((y, u) \in Y \times \mathbb{R}_+\). Making \(\kappa\) equal to 0 out of its support in \(Y \times \mathbb{R}_+\), we extend \(\kappa\) to \(Z\).

Define the set \(P\) of elements \(h \in L^2(Y \times \mathbb{R}_+, E)\) such that \(h\) has a Fourier expansion of the form \(h(y, u) = 1/u^2 \sum_{i=0}^\infty p_i(1/u)\phi_i(u)\), where \(p_i(x) \in \mathbb{C}[x]\).

Define:

\[
\mathcal{V} := \{(1 - \kappa)g + \kappa h : g \in L^2(Z, E) \text{ and } h \in \mathcal{P}\}.
\]  

(38)

i) and iii) are consequence of the Stone-Weierstrass theorem.

### 2.6 Consequences of Aguilar-Balslev-Combes theory

In this section we provide the description of \(\sigma_d(\Delta_\theta)\) and \(\sigma_{pp}(\Delta_\theta)\) that the analytic dilation provides. Most of the results that we compile in the next theorem are consequences of the Aguilar-Balslev-Combes theory as explained in the book [13].

**Theorem 6** We have:

a) The set of non-threshold eigenvalues\(^1\) of \(\Delta\) is equal to \(\sigma_d(\Delta_\theta) \cap \mathbb{R}\), for all \(\theta \in \Gamma - \mathbb{R}_+\). Moreover, given a non-threshold eigenvalue \(\lambda_0 \in \sigma(\Delta)\), the eigenspace \(E_{\lambda_0}(\Delta)\), associated to \(\Delta\) and \(\lambda_0\), has finite dimension bounded by the degree of the pole \(\lambda_0\) of the map \(\lambda \mapsto R(\lambda, \theta)\). This algebraic multiplicity is independent of \(\theta \in \Gamma - \mathbb{R}_+\).

b) Fix \(\theta \in \Gamma\). For \(f, g \in \mathcal{V}\) the function

\[
\lambda \mapsto \langle R(\lambda)f, g \rangle_{L^2(Z, E)}
\]

has a meromorphic continuation from \(\Lambda\) to \(\mathbb{C} - (\sigma_{ess}(\Delta_\theta) \cup \sigma_d(\Delta_\theta))\), where \(\sigma_{ess}(\Delta_\theta)\) is the set computed in section 2.4.

c) \(\Delta\) has no singular spectrum.

---

\(^1\)The set of thresholds of \(\Delta\), \(\tau(\Delta)\), is by definition \(\sigma(\Delta_Y)\)
d) Let \( \theta_1, \theta_2 \in \Gamma \) be such that \( \text{arg}(\theta'_1) \geq \text{arg}(\theta'_0) \) for \( 0 < \text{arg}(\theta'_0) < \pi/2 \), we have:

\[
\sigma_d(\Delta_{\theta_0}) = \sigma_d(\Delta_{\theta_1}) \cap \sigma_d(\Delta_{\theta_0}).
\]

(39)

e) Non-thresholds eigenvalues of \( \Delta \) are isolated (with respect to the eigenvalues of \( \Delta \)) and may only accumulate on the set of thresholds \( \mathbb{R} \) or at \( \infty \).

f) If the lowest eigenvalue, \( \mu_0 \), of \( \Delta_Y \) is larger than 0 then \( \sigma_d(\Delta) \) is a discrete subset of \([0, \mu_0)\). The unique possible accumulation point of \( \sigma_d(\Delta) \) is \( \gamma_0 \). If \( \mu_0 = 0 \), then \( \sigma_d(\Delta) = \emptyset \); in other words, all eigenvalues are embedded in the continuous spectrum.

The next proposition follows from the definition of essential spectrum and (32).

**Proposition 4**

i) If \( \lambda \in \sigma_{pp}(\Delta_{\theta}) \) and \( \lambda \notin \sigma_{ess}(\Delta_{\theta}) \), then \( \lambda \) is an isolated eigenvalue of finite multiplicity.

ii) For \( \text{Re}(\theta) > 0 \), \( \sigma_{pp}(\Delta_{\theta}) \) accumulates in \( \sigma_{ess}(\Delta_{\theta}) \). In particular, the real part of the pure point spectrum of \( \sigma_d(\Delta) \) accumulates only in \( \sigma(\Delta_Y) \).

Next we show that the unique possible accumulation point of \( \sigma_{pp}(\Delta) \) is \( \infty \).

For that we use the following theorem (see [8], pag. 352).

**Theorem 7** [8] If \( N(\lambda) \) denotes the number of eigenvalues of \( \Delta \) which are less than \( \lambda \), then one has

\[
N(\lambda) \leq C\lambda^{m-1/2}.
\]

(40)

In [8] the previous theorem is proved only for the Laplacian \( \Delta \) acting on functions, but it generalizes easily to our context. As we have previously said, theorem 7 implies the following corollary.

**Corollary 1** The unique possible accumulation point of \( \sigma_{pp}(\Delta) \) is \( \infty \).

We define the set of resonances of \( \Delta \) in the following way:

\[
\mathcal{R}_\theta(\Delta) = \{ \lambda \in \sigma_d(\Delta_{\theta}) : \lambda \notin \sigma_{pp}(\Delta) \}.
\]

(41)

The parameter \( \theta \) is simply uncovering new pure point spectrum in the sense of the following proposition, that is a consequence of the uniqueness of the meromorphic extension of \( \lambda \mapsto \langle R(\lambda)f, g \rangle_{L^2(Z,E)} \), for \( f,g \in \mathcal{Y} \).

\[\text{footnote}{\text{We prove in corollary \[1\] that, in fact, \( \infty \) is the unique possible accumulation point.}}\]
Proposition 5 Suppose \( \theta_1, \theta_0 \in \Gamma \) and \( 0 < \arg(\theta_0) < \arg(\theta_1) < \pi/2 \). Then
\[
\mathcal{R}_{\theta_0}(\Delta) \subset \mathcal{R}_{\theta_1}(\Delta).
\] (42)

There is an analogue version of the previous proposition for \( \theta_1, \theta_0 \in \Gamma \) and \( 0 > \arg(\theta_0) > \arg(\theta_1) > -\pi/2 \).

We recall some facts about the analytic extension of the resolvent \( R(\lambda) \) of \( \Delta \). First, we introduce some notation. Let \( \Sigma \) be the Riemann surface on which the functions \( \sqrt{z - \mu_i} \) are defined. Observe that \( \Sigma \) is a \( \omega \)-covering of \( \mathbb{C} \), with ramification points \( \{\mu_i : i \in \mathbb{N}\} \). Denote:
\[
L^2_\delta(Z, E) := \{ \varphi : Z \to E : \text{measurable section s.t. } \int_0^\infty \int_Y |\varphi|^2 e^{2\delta u} d\text{vol}(y) du < \infty \}.
\] (43)

In [12] (see also [14]) the resolvent is extended as a function of \( \lambda \in \Sigma \) taking values in the bounded operators from \( L^2_{-\delta}(Z, E) \) to \( L^2_\delta(Z, E) \). The next theorem provides more information about the resonances of \( \Delta \):

Theorem 8 ([14], theorem 3.26) Suppose that \( \lambda \in \mathcal{R}_\theta(\Delta) \). Then:

1) Suppose \( \text{Im}(\lambda) \neq 0 \). Then, if \( \lambda \notin \bigcup_{i \in \mathbb{N}} \mu_i + \theta'[0, \infty) \) then \( \lambda \in \sigma_d(\Delta_\theta) \).
   Under these conditions, if \( 0 < \arg(\theta') < \pi/2 \), then \( \text{Im}(\lambda) > 0 \); if \( 0 > \arg(\theta') > -\pi/2 \), then \( \text{Im}(\lambda) < 0 \).

2) There are not real resonances different than the set of thresholds \( \tau(\Delta_\theta) = \sigma(\Delta_Y) \). If \( \lambda = \mu_i \) for some \( i \in \mathbb{N} \), then the resolvent, as a function from \( L^2_\delta(Z, E) \) to \( L^2_\delta(Z, E) \), has a pole of at most second order. In fact, \( \mu_i \) is a pole of second order always that it is a \( L^2 \)-eigenvalue of \( \Delta \); in this case, the leading part of the Laurent expansion of \( R(\lambda) \) at \( \mu_i \) is the orthogonal projection in the \( L^2 \)-eigenspace space \( E_{\mu_i} \).

As a consequence of the previous theorem, we have the following corollary that completes the description of the spectrum of \( \Delta \) of theorem 6.

Corollary 2 The real resonances of \( \Delta \) are contained in \( \sigma(\Delta_Y) \).
2.7 \( \Delta_\theta \) are \( m \)-sectorial

The theory of \( m \)-sectorial operators and forms that we use in this section is described in appendix B. Our goal is to show that the operators \( \Delta_\theta \), for \( \theta \in \Gamma \) (see (7)), are \( m \)-sectorial (see definition 7). This result will be important when we calculate \( \sigma_{ess}(H_\theta) \).

Let \( \eta_0 \in C^\infty(\mathbb{R}_+) \) be a positive real function such that \( \eta_0(u) = 1 \) for \( u < 1 \), \( \eta_0(u) = 0 \) for \( u \in [K - 1, \infty) \), and \( \eta'_0(u) \leq 0 \) for \( u \in [1, K - 1] \), where we are considering \( K > 2 \). Let \( \eta_1 := 1 - \eta_0 \). Both \( \eta_0 \) and \( \eta_1 \) induce functions on \( Y \times \mathbb{R}_+ \), defining \( \eta_k(z,y) := \eta_k(u) \) for \( k = 0, 1 \); making \( \eta_k \) equal to 0 where it is not defined, we can extend it to all of \( Z \). In this way we think \( \eta_0 \) and \( \eta_1 \) as functions in \( C^\infty(Z) \).

**Proposition 6** For \( \text{Re}(\theta) > 0 \) there exist a \( \gamma \geq 0 \) such that

\[
\text{Re}(\langle a_0(\theta, u)f, \eta_1 f \rangle_{L^2(Z,E)} + \gamma \langle f, f \rangle_{L^2(Z,E)}) \geq |\text{Im}(\langle a_0(\theta, u)f, \eta_1 f \rangle_{L^2(Z,E)})| \tag{44}
\]

for all \( f \in L^2(Z,E) \).

**Proof:**
We observe that, by definition, \( \eta_0 + \eta_1 = 1 \). Let \( \gamma \geq 0 \), then, for all \( z \in Z \):

\[
\text{Re}(\langle a_0(\theta, u)f, \eta_1 f \rangle(z) + \gamma \langle f, f \rangle(z)) - |\text{Im}(\langle a_0(\theta, u)f, \eta_1 f \rangle(z))|
\geq \eta_1(z)\langle f, f \rangle(z)\{\text{Re}(a_0(\theta, u)) - |\text{Im}(a_0(\theta, u))| + \gamma\} + \gamma \eta_0(f, f)(z). \tag{45}
\]

Notice that in the previous calculations the inner product denote the Hermitian product in the fiber \( E_z \). Since, for all \( z \in Z \) \( \eta_1(z)\langle f, f \rangle(z) \) and \( \gamma \eta_0(z)\langle f, f \rangle(z) \) are both equal or larger than 0, then it is enough to prove that there exist \( \gamma > 0 \) such that

\[
\text{Re}(a_0(\theta, u)) - |\text{Im}(a_0(\theta, u))| + \gamma \geq 0 \tag{46}
\]

for all \( u \in \mathbb{R}_+ \). This is true because \( \{a(\theta, u) : u \in \mathbb{R}_+ \} \) is a compact subset of \( \mathbb{R}^2 \) (for \( \theta \) fixed), any compact subset of \( \mathbb{R}^2 \) is inside a cube \( [-n, n]^2 \), and we can always find a \( N \) such that \( [N - n, n + N]^2 \) is inside a cone, with slope 1, included in a right-half-plane. \( \Box \)

Let us now turn back to the set \( \Gamma \) defined in (7). The next theorem shows that the operators \( \Delta_\theta \) are \( m \)-sectorial for \( \theta \in \Gamma \). We will use this fact when we compute the essential spectrum of \( H_\theta \), in section 3.3.
Theorem 9 For $\theta \in \Gamma$ there exists a $\gamma(\theta) \in \mathbb{R}_+$ such that the form with domain $\mathcal{H}_1(Z,E)$ defined by $f \mapsto \langle \Delta_{\theta} f, f \rangle_{L^2(Z,E)} + \gamma(\theta) \langle f, f \rangle_{L^2(Z,E)}$ is $m$-sectorial.

Proof:
Let us prove that there exist $k > 0$ and $\gamma \in \mathbb{R}$ such that for all $f \in \mathcal{H}_2(Z,E)$:

$$Re \left( \langle \Delta_{\theta} f, f \rangle_{L^2(Z,E)} + \gamma \langle f, f \rangle_{L^2(Z,E)} \right) \geq k |Im \left( \langle \Delta_{\theta} f, f \rangle_{L^2(Z,E)} \right)|. \quad (47)$$

We observe that having the previous inequality, the theorems 14 and 15 imply that the bilinear form $f \mapsto \langle \Delta_{\theta} f, f \rangle_{L^2(Z,E)} + \gamma \langle f, f \rangle_{L^2(Z,E)}$ is strictly $m$-sectorial, and hence the form defined by $\Delta_{\theta}$ is $m$-sectorial.

Next we prove the inequality (47). We use proposition 2 to see that

$$\langle \eta_0 \Delta_{\theta}(f), f \rangle_{L^2(Z,E)} = \langle \frac{\partial}{\partial u}(f), \frac{\partial}{\partial u}(\eta_0 f) \rangle_{L^2(Z,E)} + \langle \nabla(f), \eta_0 \nabla(f) \rangle_{L^2(Z,E)}, \quad (48)$$

for $f \in \mathcal{H}_1(Z,E)$. Now let $a_2(\theta, u) \frac{\partial^2}{\partial u^2} + a_1(\theta, u) \frac{\partial}{\partial u} + a_0(\theta, u) + \Delta_Y$ be the local expression of the operator $\Delta_{\theta}$ (see equation (21)). We have:

$$\langle \eta_1 \Delta_{\theta}(f), f \rangle_{L^2(Z,E)} = -\langle \frac{\partial}{\partial u}(f), \frac{\partial}{\partial u}(\bar{\eta}_2 \eta_1 f) \rangle_{L^2(Z,E)} - \langle \frac{\partial}{\partial u}(f), (\bar{\eta}_2 \eta_1) \frac{\partial}{\partial u}(f) \rangle_{L^2(Z,E)} + \langle \frac{\partial}{\partial u}(f), \bar{\eta}_1 \eta_1 f \rangle_{L^2(Z,E)} + \langle f, \eta_0 \eta_1 f \rangle_{L^2(Z,E)} + \langle \eta_1 \Delta_Y(f), f \rangle_{L^2(Z,E)}. \quad (49)$$

Using (48) and (49) we find

$$\langle \eta_0 \Delta_{\theta}(f), f \rangle_{L^2(Z,E)} + \langle \eta_1 \Delta_{\theta}(f), f \rangle_{L^2(Z,E)}$$

$$= \langle \frac{\partial}{\partial u}(f), \left( -\frac{\partial}{\partial u}(\eta_0) - \frac{\partial}{\partial u}(\bar{\eta}_2 \eta_1) + \bar{\eta}_1 \eta_1 \right) f \rangle_{L^2(Z,E)}$$

$$+ \langle \frac{\partial}{\partial u}(f), \bar{\eta}_1 \eta_1 f \rangle_{L^2(Z,E)} + \langle f, \eta_0 \eta_1 f \rangle_{L^2(Z,E)} + \langle \eta_1 \Delta_Y(f), f \rangle_{L^2(Z,E)}. \quad (50)$$

Since

$$\langle \nabla(f), \eta_0 \nabla(f) \rangle_{L^2(Z,E)} = \int_{Z_0} \langle \nabla(f), \eta_0 \nabla(f) \rangle_{(z)}dz$$

$$+ \int_{Y \times [0,\infty)} \langle \nabla_Y(f), \eta_0 \nabla_Y(f) \rangle_{(z)}dz$$

$$- \int_{Y \times [0,\infty)} \langle \frac{\partial}{\partial u}(f), \eta_0 \frac{\partial}{\partial u}(f) \rangle_{(z)}dz, \quad (51)$$
we have that the term
\[ s(f) := \int_{Z_0} \langle \nabla f, \eta_0 \nabla f \rangle(z) \, dz + \int_{Y \times [0, \infty)} \langle \nabla_Y(f), \eta_0 \nabla_Y(f) \rangle(z) \, dz \]
\[ + \langle \eta_1 \Delta_Y(f), f \rangle_{L^2(Z, E)} \]
is greater or equal than 0.

We define the bilinear form \( h(\theta) \) by
\[ h(\theta)(f) := \left\langle \frac{\partial}{\partial u}(f), \left( -\frac{\partial}{\partial u}(\eta_0) - \frac{\partial}{\partial u}(\overline{\alpha}(\theta)\eta_1) + \overline{\beta}(\theta)\eta_1 \right) f \right\rangle_{L^2(Z, E)} \]
\[ - \left\langle \frac{\partial}{\partial u}(f), (\overline{\alpha}(\theta)\eta_1) \frac{\partial}{\partial u}(f) \right\rangle_{L^2(Z, E)} \]
\[ + \int_{Y \times [0, \infty)} \left\langle \frac{\partial}{\partial u}(f), \eta_0 \frac{\partial}{\partial u}(f) \right\rangle(z) \, dz. \] (53)

Since \( \langle \Delta_\theta(f), f \rangle_{L^2(Z, E)} = h(\theta)(f) + s(f) + (f, \overline{\alpha}(\theta)\eta_1 f)_{L^2(Z, E)}, s(f) \geq 0, \) theorem 16, proposition 6 and the definition of \( h(\theta) \), to finish the proof of (47), it only remains to prove that there exist \( \gamma > 0 \) and \( k > 0 \) that satisfy
\[ Re(h(\theta)f) + \gamma \langle f, f \rangle_{L^2(Z, E)} \geq k |Im(h(\theta)f)|. \] (54)

We observe that \( -\frac{\partial}{\partial u}(\eta_0) - \frac{\partial}{\partial u}(\overline{\alpha}(\theta)\eta_1) + \overline{\beta}(\theta)\eta_1 \) has support on \([0, \infty)\) (as a function of \( u, \theta \) fixed) and it is bounded there by a constant \( C \) (see proposition 6). Then:
\[ Re(h(\theta)(f)) - k |Im(h(\theta)(f))| \geq \]
\[ \int_{Y \times [0, \infty)} \{(-Re(a_2(\theta)) + k|Im(a_2(\theta))|)\eta_1 + \eta_0\} \left\langle \frac{\partial}{\partial u}f, \frac{\partial}{\partial u}f \right\rangle(z) \, dz \]
\[ - C \int_{Y \times [0, \infty)} \left\langle \frac{\partial}{\partial u}f, |f| \right\rangle(z) \, dz. \]

Notice that \( Re(a_2)\eta_1 = 1/2 Re(a_2)\eta_1 + 1/2 Re(a_2) - 1/2 Re(a_2)\eta_0. \) Using the fact that, for all \( \epsilon \in \mathbb{R}, \)
\[ \epsilon^2|u|^2 + 1/4\epsilon^2|v|^2 \geq |u||v|, \]
we have for all \( k \in \mathbb{R}_+ \):

\[
Re(h(\theta)(f)) - k|Im(h(\theta)(f))| \geq
\]

\[
\int_{Y \times [0, \infty)} (-1/2Re(a_2(\theta)) + k|Im(a_2(\theta))|)\eta_1(\frac{\partial}{\partial u}(f), \frac{\partial}{\partial u}(f))(z)dz
\]

\[
+ \int_{Y \times [0, \infty)} (-1/2Re(a_2(u, \theta)) - C \epsilon^2)\eta_0(\frac{\partial}{\partial u}(f), \frac{\partial}{\partial u}(f))(z)dz
\]

\[
+ \int_{Y \times [0, \infty)} (1 + 1/2Re(a_2(u, \theta)))\eta_0(\frac{\partial}{\partial u}(f), \frac{\partial}{\partial u}(f))(z)dz
\]

\[
- C/(4\epsilon^2) \int_{Y \times [0, \infty]} \langle f, f \rangle(z)dz.
\]

(55)

Recall that \( a_2(\theta, u) = -\frac{1}{\psi_\theta(u)^2} \). We observe that \( \psi_\theta \) is bounded and \( \psi_\theta'(u) = B(u)\theta + 1 \) where \( B(u) := \varphi'(u)u + \varphi(u) \geq 0 \) for all \( u \in [0, \infty) \). For \( \theta \in \Gamma, Im(\theta)^2 < 1/2 \), and \( (B(u)Re(\theta) + 1)^2 \geq 1 \), then:

\[
(B(u)Re(\theta) + 1)^2 - Im(\theta)^2 \geq 1 - Im(\theta)^2 > 1/2.
\]

(56)

Thus, there exists a constant \( C_0 > 0 \) such that

\[
- Re(a_2(u, \theta)) = \frac{(B(u)Re(\theta) + 1)^2 - Im(\theta)^2}{|\psi_\theta'(u)|^4}
\]

\[
\geq \frac{1}{2 \max\{u \in [0, \infty) : |\psi_\theta'(u)|^4\}} > C_0 > 0
\]

(57)

for all \( u \in \mathbb{R}_+ \). Hence, we can find \( \epsilon \) such that

\[
(-1/2Re(a_2(\theta, u)) - C \epsilon^2) > 0.
\]

(58)

Now we show that, for all \( \theta \in \Gamma \), there exists \( k \) such that \( Re(\psi_\theta'(u)^2) - k|Im(\psi_\theta'(u)^2)| \geq 0 \), for all \( u \in \mathbb{R}_+ \). We observe that \( \psi_\theta'(u)^2 = B(u)^2\theta^2 + 2B(u)\theta + 1 \). Suppose that \( \theta := \theta_0 + i\theta_1 \), for \( \theta_0 \) and \( \theta_1 \) real numbers. We denote by

\[
M := \max\{B(u) : u \in \mathbb{R}_+\} = \max\{\varphi(u)u + \varphi'(u) : u \in \mathbb{R}_+\}.
\]

(59)

From the definition of \( \Gamma \) in (17) it follows that, for \( \theta \in \Gamma, \theta_0^2 - |\theta_1|^2 \geq 0 \), then:

\[
Re(\psi_\theta'(u)^2) - k|Im(\psi_\theta'(u)^2)| \geq 1 - k\{2M^2|\theta_0\theta_1| + 2M|\theta_1|\}.
\]

(60)
The previous calculations show that, for fixed $\theta \in \Gamma$, we can always find a $k$ such that

$$\text{Re}(\psi'_\theta(u)^2) - k|\text{Im}(\psi'_\theta(u)^2)| \geq 0,$$

for all $u \in \mathbb{R}_+$. Finally, we observe that:

$$1 + 1/2 \text{Re}(a_2(\theta, u)) = 1 - 1/2 \frac{\text{Re}(\psi'^2_\theta(u))}{|\psi'_\theta(u)|^4} \geq 0,$$

because $2|\psi'_\theta(u)|^4 - \text{Re}(\psi'^2_\theta(u)) \geq 0$. This last inequality is true because, for all $a \in \mathbb{C}$, $2|a|^2 \geq \text{Re}(\overline{a})$.

Using (55), (58), (61) and (62), we can conclude that there exists $K_0 \geq 0$ such that

$$\text{Re}(h(\theta)(f)) - k|\text{Im}(h(\theta)(f))| \geq K_0 - C/(4\epsilon^2) \int_Y \langle f, f \rangle_{L^2(Z, E)} dz. \quad (63)$$

Finally, we can take $\gamma$ large enough to have:

$$\text{Re}(h(\theta)(f)) - k|\text{Im}(h(\theta)(f))| + \gamma \langle f, f \rangle_{L^2(Z, E)} \geq 0, \quad (64)$$

what finishes the proof of (54), and with it the proof of the theorem.$\Box$

### 3 Analytic dilation on complete manifolds with corners of codimension 2

Let $X$ be a complete manifold with corners of codimension 2 and let $E$ be an Hermitian vector bundle over $X$. Let $\Delta$ be a generalized Laplacian acting on $C^\infty(X, E)$. We say that $\Delta$ is a **compatible Laplacian** over $X$ if it satisfies the following properties:

1. There exist Hermitian vector bundle $E_i$ over $Z_i$ such that $E|_{[0, \infty) \times Z_i}$ is the pull-back of $E_i$ ($i = 1, 2$). We suppose also that the Hermitian metric of $E$ is the pullback of the Hermitian metric of $E_i$. In addition, on $[0, \infty) \times Z_i$ we have

$$\Delta = -\frac{\partial^2}{\partial u_k^2} + \Delta_{Z_i}, \quad (65)$$

where $\Delta_{Z_i}$ is a compatible Laplacian acting on $C^\infty(Z_i, E_i)$. 

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ii) There exists Hermitian vector bundle $S$ over $Y$ such that $E|_{(0,\infty)^2 \times Y}$ is the pull-back of $S$, and on $[0,\infty)^2 \times Y$ we have,

$$\Delta = -\frac{\partial^2}{\partial u_1^2} - \frac{\partial^2}{\partial u_2^2} + \Delta_Y$$

(66)

where $\Delta_Y$ is a generalized Laplacian acting on $C^\infty(Y,S)$.

Since $X$ is a manifold with bounded geometry and the vector bundle $E$ has bounded Hermitian metric and bounded connection, $\Delta : C^\infty_c(X,E) \to L^2(X,E)$ is essentially self-adjoint (see [23]). We denote by $H$ its self-adjoint extension. For $i = 1, 2$, $\Delta_{Z_i} : C^\infty_c(Z_i, E_i) \to L^2(Z_i, E_i)$ is also essentially self-adjoint and we denote its self adjoint extension by $H^{(i)}$. Let $b_i$ be the self-adjoint extension of $-\frac{\partial^2}{\partial u_i^2} : C^\infty_c([0,\infty)) \to L^2([0,\infty))$ obtained with Dirichlet boundary conditions. We denote $H_i$ the self-adjoint operator $-b_1 \otimes 1 + 1 \otimes H^{(i)}$ acting on $L^2(\mathbb{R}) \otimes L^2(Z_i, E_i)$. Similarly $H^{(3)}$ denotes the self-adjoint operator associated to the essentially self-adjoint operator $\Delta_Y : C^\infty_c(Y,S) \to L^2(Y,S)$, and we denote by $H_3$, the self-adjoint operator $H_3 := -b_1 \otimes 1 \otimes 1 - 1 \otimes b_2 \otimes 1 + 1 \otimes 1 \otimes H^{(3)}$ acting on $L^2([0,\infty)) \otimes L^2([0,\infty)) \otimes L^2(Y)$.

The operators $H_i$ are called channel operators for $i = 1, 2$ and $3$. The operators $H_1$ and $H_2$ have a free channel of dimension 1 (associated to $b_1$ and $b_2$ respectively), $H_3$ is channel operator with a free channel of dimension 2 (associated to $-b_1 \otimes 1 \otimes 1 - 1 \otimes b_2 \otimes 1$). In some parts of this text we abuse of the notation and denote by $H$ the Laplacian acting on distributions.

3.1 The definition of $U_\theta$ for $\theta \in [0,\infty)$

For $i = 1, 2$ and $\theta \in [0,\infty)$, $U_{i,\theta} : L^2(Z_i, E_i) \to L^2(Z_i, E_i)$ denotes an analytic dilation operator associated to the Laplacian $H^{(i)}$. The unitary operator $U_{i,\theta}$ was described in section 2.2. In this section we denote $H^{(i),\theta} := U_{i,\theta} H^{(i)} U_{i,\theta}^{-1}$. 

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In the next definition we use the exhaustion defined in (2). Let $\theta \in [0, \infty)$:

$$U_\theta f(x) := \begin{cases} 
  f(x_0) & \text{for } x = x_0 \in X_0, \\
  f(m_i, \psi_\theta(u_i))\psi_\theta^{1/2}(u_i) & \text{for } x = (m_i, u_i) \in M_i \times [0, \infty), i = 1, 2, \\
  f(y, \psi_\theta(u_1), \psi_\theta(u_2))\psi_\theta^{1/2}(u_1)\psi_\theta^{1/2}(u_2) & \text{for } x = (y, u_1, u_2) \in Y \times [0, \infty) \times [0, \infty).
\end{cases}$$  

(67)

The following proposition follows from the definition of $U_\theta$.

**Proposition 7** Let $\theta \in [0, \infty)$ and $f \in C_c^\infty(X, E)$,

i) $U_\theta f \in C_c^\infty(X, E)$.

ii) $U_\theta$ extends to a unitary operator in $L^2(X, E)$.

The inverse of $U_\theta$ is given by:

$$U_\theta^{-1} f(x) := \begin{cases} 
  f(x_0) & \text{for } x = x_0 \in X_0, \\
  f(m_i, \alpha_\theta(u_i))\psi_\theta^{1/2}(\alpha_\theta(u_i)) & \text{for } x = (m_i, u_i) \in M_i \times [0, \infty), i = 1, 2, \\
  f(y, \alpha_\theta(u_1), \alpha_\theta(u_2))\psi_\theta^{1/2}(\alpha_\theta(u_1))\psi_\theta^{1/2}(\alpha_\theta(u_2)) & \text{for } x = (y, u_1, u_2) \in Y \times [0, \infty)^2.
\end{cases}$$  

(68)

One can check the following proposition.

**Proposition 8** For $\theta \in [0, \infty)$ and $i = 1, 2$, if $f \in C_c^\infty(X, E)$ and $(z_i, u_i) \in [0, \infty) \times Z_i$, then:

i) $U_\theta f(u_i, z_i) = (U_{i, \theta} f)(\psi_\theta(u_i), z_i)\psi_\theta^{1/2}(u_i)$.

ii) $U_\theta^{-1} f(u_i, z_i) = U_{i, \theta}^{-1} f(\alpha_\theta(u_i), z_i)\psi_\theta^{1/2}(\alpha_\theta(u_i))$. 

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3.2 The family $H_\theta$ for $\theta \in \Gamma$

For $\theta \in [0, \infty)$, we define the operator $H_\theta := U_\theta H U_\theta^{-1}$. By direct calculation, one can prove:

**Proposition 9** For $\theta \in [0, \infty)$ and $i = 1, 2$, if $f \in C^\infty_c(X, E)$, then for all $(z_i, u) \in Z_i \times [0, \infty)$

\[
H_\theta f(u, z_i) = H^{(i), \theta} f(u, z_i) - (\frac{\partial^2}{\partial u^2} f)(u, z_i) \alpha'_\theta (\psi_\theta(u))^2 \\
- (\frac{\partial}{\partial u} f)(u, z_i) \alpha''_\theta (\psi_\theta(u))
\]

\[
+ (\frac{\partial}{\partial u} f)(u, z_i) \psi'_\theta(u) (u) \alpha'(\psi_\theta(u))^2 \\
+ 3/4 f(u, z_i) \psi'_\theta(u) (u) - 3/2 \psi''_\theta(u)^2 \alpha'(\psi_\theta(u))^2 \\
+ 1/2 f(u, z_i) \psi'_\theta(u) (u) \alpha'(\psi_\theta(u))^2 \\
+ 1/2 f(u, z_i) \psi'_\theta(u) (u) \alpha'_\theta (\psi_\theta(u)).
\]

(69)

In particular, if $u_i > R$, we have:

\[
H_\theta f(u_i, z_i) = -\theta \frac{\partial^2}{\partial u^2_i} f(u_i, z_i) + H^{(i), \theta} f(u_i, z_i),
\]

(70)

and, if $u_i < K$,

\[
H_\theta f(u_i, z_i) = -\frac{\partial^2}{\partial u^2_i} f(u_i, z_i) + H^{(i)} f(u_i, z_i).
\]

(71)

Similarly, the next proposition describes the operator $H_\theta$ on $Y \times [0, \infty)^2$:

**Proposition 10** For $\theta \in [0, \infty)$, for all $f \in C^\infty_c(X, E)$ and for all $(y, u_1, u_2) \in Y \times [0, \infty)^2$,

\[
H_\theta f(y, u_1, u_2) = H^{(3)} f(y, u_1, u_2) - \sum_{i=1}^2 \{(\frac{\partial^2}{\partial u_i^2} f)(y, u_1, u_2) \alpha'_\theta (\psi_\theta(u_i))
\]

\[
- (\frac{\partial}{\partial u} f)(y, u_1, u_2) \alpha''_\theta (\psi_\theta(u_i))
\]

\[
+ (\frac{\partial}{\partial u} f)(y, u_1, u_2) \psi'_\theta(u_i) (u) \alpha'_\theta (\psi_\theta(u_i))^2 \\
- 3/4 f(y, u_1, u_2) \psi'_\theta(u_i) (u) - 3/2 \psi''_\theta(u)^2 \alpha'_\theta (\psi_\theta(u_i))^2 \\
+ 1/2 f(y, u_1, u_2) \psi'_\theta(u_i) (u) \alpha'_\theta (\psi_\theta(u_i))^2 \\
+ 1/2 f(y, u_1, u_2) \psi'_\theta(u_i) (u) \alpha'_\theta (\psi_\theta(u_i)).
\]

(72)
In particular, if \( u_1, u_2 > R \), we have:

\[
H_\theta f(y, u_1, u_2) = -\theta' \frac{\partial^2}{\partial u_1^2} f(y, u_1, u_2) - \theta' \frac{\partial^2}{\partial u_2^2} f(y, u_1, u_2) + H^{(3)} f(y, u_1, u_2).
\]

(73)

an if \( u_1, u_2 < K \) we have:

\[
H_\theta f(y, u_1, u_2) = -\frac{\partial^2}{\partial u_1^2} f(y, u_1, u_2) - \frac{\partial^2}{\partial u_2^2} f(y, u_1, u_2) + H^{(3)} f(y, u_1, u_2).
\]

(74)

We observe that, for all \( f \in C^\infty_c(X, E) \) and \((y, u_1, u_2) \in Y \times [0, \infty)^2\), we can write:

\[
H_\theta(f)(y, u_1, u_2) = \sum_{i=1}^2 \{ a_2(\theta, u_i) \frac{\partial^2}{\partial u_i^2} f(y, u_1, u_2) + a_1(\theta, u_i) \frac{\partial}{\partial u_i} f(y, u_1, u_2) + a_0(\theta, u_i) f(y, u_1, u_2) \}
\]

(75)

where the functions \((\theta, u) \mapsto a_i(\theta, u)\) were defined in (24).

Our next goal is to prove that \((H_\theta)_{\theta \in \Gamma}\) is an holomorphic family of type A. We use the general theory of uniformly elliptic operators on manifolds with bounded geometry, as explained for example in [23].

**Proposition 11** For all \( \theta \in \Gamma \), the operator \( H_\theta \) is an uniformly elliptic operator, that is,

\[
|\sigma_2(H_\theta)^{-1}(x, \xi)|^{-1} \leq |\xi|^{-2},
\]

(76)

for all \( x \in X \) and \( \xi \in T_x X \).

As a consequence of proposition 11 we have the next corollary.

**Corollary 3** The operators \( H_\theta \) are closed operators in the domain \( \mathcal{D}(X, E) \).

Given \( f \in \text{Dom}(H) \) and \( g \in L^2(X, E) \), we will prove that the function \( \theta \mapsto \langle H_\theta f, g \rangle_{L^2(X, E)} \) is holomorphic for \( \theta \in \Gamma \), accordingly we consider the following partition of unity of \( X \). Let \( \eta \in C^\infty_c(\mathbb{R}) \) be such that \( \eta(u) = 1 \) for \( u \leq K - 2 \) and \( \eta(u) = 0 \) for \( u \geq K - 1 \). Let \( \kappa := 1 - \eta \), we define the following functions with their natural extensions to the whole \( X \). Let \((z_i, u_i) \in Z_i \times [0, \infty)\), then:

\[
\kappa_i(z_i, u_i) := \kappa(u_i), \quad \eta_i(z_i, u_i) := 1 - \kappa_i.
\]

We observe that \( \eta_i + \kappa_i = 1 \). In particular, we have that \((\eta_1 + \kappa_1)(\eta_2 + \kappa_2) = 1 \). We study the functions \( \theta \mapsto \kappa_1 \kappa_2 H_\theta \) and \( \theta \mapsto \eta \kappa H_\theta \). The next proposition is a technical tool.

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Proposition 12 1) Given $f \in \text{Dom}(H)$ and $g \in L^2(X,E)$, there exists $h \in L^1(X)$ such that:
\[
| < \kappa_1 \kappa_2 H_\theta f, g > (x) | \leq h(x), \tag{77}
\]
and
\[
| \frac{\partial}{\partial \theta} ( < \kappa_1 \kappa_2 H_\theta f, g > (x) ) | \leq h(x), \tag{78}
\]
for $\theta$ in any compact subset of $\Gamma$.

2) For $i, j \in \{1, 2\}$, $i \neq j$, and $\theta$ in a compact subset of $\Gamma$ given $f \in \text{Dom}(H)$ and $g \in L^2(X,E)$, there exists $h \in L^1(X)$ such that:
\[
| < \kappa_i \eta_j H_\theta f, g > (x) | \leq h(x), \tag{79}
\]
and
\[
| \frac{d}{d \theta} ( < \kappa_i \eta_j H_\theta f, g > (x) ) | \leq h(x). \tag{80}
\]

As a consequence of the previous proposition we can apply well known theorems (see [2], page 89) to put partial derivatives inside of the respective integrals. This and the Cauchy-Riemann equations imply the following theorem.

Theorem 10 Given $f \in \text{Dom}(H)$ and $g \in L^2(X,E)$, the function $\theta \mapsto \langle H_\theta f, g \rangle_{L^2(X,E)}$ is holomorphic.

Observe that theorem II and corollary III prove theorem I in the introduction.

3.3 The essential spectrum of $H_\theta$

The goal of this section is to prove theorem II in the introduction. Let us consider the set:
\[
\mathcal{F}_\theta := \left( \bigcup_{i=1}^{2} \bigcup_{\lambda \in \sigma_{pp}(H^{(i)})} \lambda + \theta'[0, \infty) \right) \cup \left( \bigcup_{\mu \in \sigma(H^{(3)})} \mu + \theta'[0, \infty) \right) \cup \left( \bigcup_{i=1}^{2} \bigcup_{\gamma \in \mathcal{A}(H^{(i)}, \theta)} \gamma + \theta'[0, \infty) \right), \tag{81}
\]
where $\mathcal{R}(H^{(i), \theta})$ is defined by:

$$
\mathcal{R}(H^{(i), \theta}) := \{ \lambda \in \sigma_{pp}(H^{(i), \theta}) : \lambda \notin \sigma_{pp}(H^{(i)}) \}. \tag{82}
$$

The elements of $\mathcal{R}(H^{(i), \theta})$ shall be called **resonances of $H^{(i), \theta}$**. We observe that the set $\mathcal{R}(H^{(i), \theta})$ is independent of $\theta$ in the sense that if $\arg(\theta_1') \geq \arg(\theta_2')$ then $\sigma_{pp}(H^{(i), \theta_1'}) \subset \sigma_{pp}(H^{(i), \theta_2'})$ (see item d), theorem [4]. The next proposition follows easily from the definition of $\mathcal{F}_\theta$.

**Proposition 13** For $\theta \in \Gamma$, the following equation holds:

$$
\mathcal{F}_\theta = \left( \bigcup_{i=1}^{2} \bigcup_{\lambda \in \sigma_{pp}(H^{(i), \theta})} \lambda + \theta'[0, \infty) \right) \cup \left( \bigcup_{\mu \in \sigma(H^{(3)})} \mu + \theta'[0, \infty) \right). \tag{83}
$$

By the previous proposition theorem [2] is equivalent to $\sigma_{ess}(H_\theta) = \mathcal{F}_\theta$ for $\theta \in \Gamma$. We use the results of appendix [A] about the sets $N_\infty(A)$ (see definition [3]), $N_{ess}(A)$ (see definition [2]) and $\sigma_{ess}(A)$ to show that $\sigma_{ess}(H_\theta) = \mathcal{F}_\theta$. The proof has two basic steps:

i) To prove that $N_\infty(H_\theta) = \mathcal{F}_\theta$.

ii) To prove that there exists $\eta_0^d \in C_c^\infty(X)$ such that $\text{supp}(\eta_0^d) \subset X_d$, and for all $f \in C_c^\infty(X, E)$,

$$
|||H_\theta, \eta_0^d|||_{L^2(X, E)} \leq \epsilon(d)(|||H_\theta f|||_{L^2(X, E)} + |||f|||_{L^2(X, E)})
$$

with $\epsilon(d) \to 0$ as $d \to \infty$.

Using ii) we see that $H_\theta$ satisfies the conditions for applying theorem [13] and we get $N_{ess}(H_\theta) = N_\infty(H_\theta)$. Using i) and part iii) of theorem [12] in appendix [A] we prove $\sigma_{ess}(H_\theta) = \mathcal{F}_\theta$.

### 3.3.1 The equality $N_\infty(H_\theta) = \mathcal{F}_\theta$

The boundary Weyl sequences (abbreviately bWs), defined in definition [3] will play a very important role in this section. Let $\mu \in \sigma(H^{(3)})$ and $\lambda \in \mu + \theta'[0, \infty)$. We observe that we can apply theorems [12] and [13] to the operators $\theta' \frac{\partial^2}{\partial u_1^2}$ and $\theta' \frac{\partial^2}{\partial u_2^2} + \mu$. Then there exist a bWs, $(p_n)$, associated to 0 and the operator $\theta' \frac{\partial^2}{\partial u_1^2}$, and a bWs, $(q_n)$, associated to $\lambda$ and the operator $\theta' \frac{\partial^2}{\partial u_2^2} + \mu$. Let $\varphi \in C^\infty(Y, S)$ be a normal eigenfunction of $H^{(3)}$ associated to the eigenvalue $\mu$. 27
Proposition 14 Let \( g_n \in C_\infty^c(Y \times [0, \infty)^2, E) \) be defined by \( g_n(u_1, u_2, y) = p_n(u_1)q_n(u_2)\varphi(y) \). Then \( (g_n) \) induces a boundary Weyl sequence for \( \lambda \) and the operator \( H_\theta \).

Proof:
It is easy to check \( g_n = C_\infty^c(X, E) \), \( ||g_n|| = 1 \) and that, for all \( K > 0 \), there exists a \( N \) such that, for all \( n > N \), \( suppg_n \cap X_K = \emptyset \), where \( X_K \) is defined in (2). We observe that:

\[
|| \left( \theta' \frac{\partial^2}{\partial u_1^2} + \theta' \frac{\partial^2}{\partial u_2^2} + H^{(3)} - \lambda \right) g_n || \leq \\
C(||\theta' \frac{\partial^2}{\partial u_1^2} p_n|| + ||(\theta' \frac{\partial^2}{\partial u_2^2} + \mu - \lambda) q_n||).
\]

(84)

Since \( (p_n) \) is a bWs of \( \theta' \frac{\partial^2}{\partial u_1^2} \) and the value 0, and \( (q_n) \) is a bWs of \( -\theta' \frac{\partial^2}{\partial u_2^2} + \mu \) and the value \( \lambda \), the last two terms of (84) tend to 0. We have proved that \( \lim_{n \to \infty} ||(H_\theta - \lambda) g_n|| = 0 \). \( \square \)

Now let \( \gamma \in \sigma_{pp}(H^{(i)},\theta) \) and \( \lambda \in \gamma + \theta'[0, \infty) \). Let \( \varphi \in C^\infty(Z_i, E_i) \) be a normal \( L^2 \)-eigenfunction of \( H^{(i)} \theta \) with eigenvalue \( \gamma \). Let \( \eta \in C^\infty(\mathbb{R}) \) such that \( \eta(u) = 1 \), for \( u \leq 1 \); \( \eta(u) = 0 \), for \( u > 2 \); and \( \eta'(u) \leq 0 \). Define \( \eta_n(u) := \eta(\frac{u}{n}) \). Let \( f_n \) be a bWs associated to the operator \( -\theta' \frac{\partial^2}{\partial u_1^2} + \eta_n \) and 0.

Proposition 15 We denote \( i, j \in \{1,2\} \) such that \( i \neq j \). Let \( g_n \in C^\infty(Z_i \times [0, \infty), E) \) be defined by \( g_n(u_1, u_2, z_i) := \|\eta_n(u_j) f_n(u_j) \varphi(z_i)\| \|\eta_n(u_j) f_n(u_j) \varphi(z_i)\| \). Then, \( (g_n) \) induces a boundary Weyl sequence associated to \( H_\theta \) and the value \( \lambda \).

Proof:
It is easy to check \( g_n \in C_\infty^c(X, E) \), \( ||g_n|| = 1 \) and that for all \( K \in \mathbb{N} \) there exists an \( N \) such that for all \( n > N \) \( suppg_n \cap X_K = \emptyset \). Since \( \lim_{n \to \infty} \|\eta_n(u_j) f_n(u_j) \varphi(z_i)\|_{L^2(X, E)} = 1 \), it is enough to prove:

\[
\lim_{n \to \infty} ||(\theta' \frac{\partial^2}{\partial u_1^2} + H^{(i), \theta}) g_n || = 0.
\]

We observe that:

\[
(H^{(i), \theta} - \gamma)(\eta_n(u_j) \varphi(z_i)) = - \theta' \frac{\partial^2}{\partial u_1^2}(\eta_n) \varphi - 2 \theta' \frac{\partial}{\partial u_j}(\eta_n) \frac{\partial}{\partial u_j}(\varphi).
\]

(85)

\( ^3 \) With this notation \( u_j \) is the real variable in the cylinder of \( Z_i \).

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Since \( \eta_n(u_j) = \eta(u_j/n) \), then \( \frac{\partial}{\partial u_j} (\eta_n) (u_j) = \frac{1}{n} \frac{\partial}{\partial u_j} (\eta)(u_j/n) \) and \( \frac{\partial^2}{\partial u_j^2} (\eta_n) (u_j) = \frac{1}{n^2} \frac{\partial^2}{\partial u_j^2} (\eta)(u_j/n) \). Hence, using equation (85)

\[
||| (\theta' \frac{\partial^2}{\partial u_i^2} + H^{(i),\theta} - \gamma) g_n ||| \leq C(A_n + B_n + C_n),
\]

where

\[
A_n := ||\eta_n(u_j) \varphi(z_i) (\theta' \frac{\partial^2}{\partial u_i^2}) f_n(u_i)||;
\]

\[
B_n := || \left( \theta' \frac{\partial^2}{\partial u_j^2} (\eta_n)(u_j) \right) \varphi(z_i) f_n(u_i)|| = || \left( \theta' \frac{\partial^2}{\partial u_j^2} (\eta_n)(u_j) \right) \varphi(z_i)||; \quad (86)
\]

\[
C_n := || \left( \theta' \frac{\partial}{\partial u_j} (\eta_n)(u_j) \right) \varphi(z_i) f_n(u_i)|| = || \left( \theta' \frac{\partial}{\partial u_j} (\eta_n)(u_j) \right) \varphi(z_i)||.
\]

For \( A_n \), we have:

\[
A_n \leq ||(\theta' \frac{\partial^2}{\partial u_i^2}) f_n(u_i)|| \to 0. \quad (87)
\]

For \( B_n \), we estimate:

\[
B_n \leq C(\theta) \frac{1}{n^2} \left( \int_{Z_i} |(\frac{\partial^2}{\partial u_j^2} \eta)(\frac{u_j}{n}) \varphi(z_i)|^2 dz_i \right)^{1/2} \leq \frac{1}{n^2} ||\varphi|| \to 0; \quad (88)
\]

and, finally, for \( C_n \):

\[
C_n \leq C(\theta) \frac{1}{n} \left( \int_{Z_i} |(\frac{\partial}{\partial u_j} \eta)(\frac{u_j}{n}) \varphi(z_i)|^2 dz_i \right)^{1/2} \leq \frac{1}{n} ||\varphi|| \to 0. \quad (89)
\]

Propositions 14 and 15 prove that \( \mathcal{F}_\theta \subset N_\infty(H_\theta) \). Now we are going to prove the other inclusion. We recall that we denote by \( b_i \) the self-adjoint operator in \( L^2([0, \infty)) \) obtained from \( \frac{\partial^2}{\partial u_i^2} \) with Dirichlet boundary conditions. We denote by \( H_{i,\theta} \) the closed operator \( -\theta' 1 \otimes b_i + H^{(i),\theta} \otimes 1 \) acting on \( L^2(Z_i \times [0, \infty), E) = L^2(Z_i, E_i) \otimes L^2([0, \infty)). \)

**Proposition 16** For \( i = 1, 2, \) and \( \theta \in \Gamma: \)

i) \( \sigma_{ess}(H_{i,\theta}) = N_\infty(H_{i,\theta}). \)

ii) \( N_\infty(H_{i,\theta}) \) is given by:

\[
N_\infty(H_{i,\theta}) = \left( \bigcup_{\lambda \in \sigma_{pp}(H^{(i),\theta})} \lambda + \theta'[0, \infty) \right) \cup \left( \bigcup_{\mu \in \sigma(H^{(3)})} \mu + \theta'[0, \infty) \right). \quad (90)
\]
Proof:
In the same way that we proved \( \mathcal{F}_\theta \subset N_\infty(H_\theta) \), using propositions 14 and 15 we can prove

\[
\left( \bigcup_{\lambda \in \sigma_{pp}(H^{(i),\theta})} \lambda + \theta'[0, \infty) \right) \cup \left( \bigcup_{\mu \in \sigma(H^{i})} \mu + \theta'[0, \infty) \right) \subset N_\infty(H_{i,\theta}). \tag{91}
\]

We denote by \( \mathcal{F}_{i,\theta} \) the right-side of (90). Then, (91) implies \( \mathcal{F}_{i,\theta} \subset N_\infty(H_{i,\theta}) \), and by proposition 21 \( \mathcal{F}_{i,\theta} \subset N_\infty(H_{i,\theta}) \subset \sigma_{ess}(H_{i,\theta}) \). We know that the operator \( H^{(i),\theta} \) is \( m \)-sectorial (theorem 9), so we can apply Ichinose lemma (see theorem 15) in the next computations:

\[
\sigma(H_{i,\theta}) = \sigma(H^{(i),\theta}) + \sigma\left(-\theta' \frac{\partial^2}{\partial u_i^2}\right)
\]

\[
= \left( \theta'[0, \infty) + \sigma_{pp}(H^{(i),\theta}) \right) \cup \left( \sigma(H^{(3)}) + \theta'[0, \infty) \right). \tag{92}
\]

The above equation implies \( N_\infty(H_{i,\theta}) \subset \mathcal{F}_{i,\theta} \) and \( \sigma_{ess}(H_{i,\theta}) \subset \mathcal{F}_{i,\theta} \), that together with (91) implies \( N_\infty(H_{i,\theta}) = \mathcal{F}_{i,\theta} = \sigma_{ess}(H_{i,\theta}) \). We have proven the proposition. \( \square \)

Next we prove \( N_\infty(H_\theta) \subset \mathcal{F}_\theta \). Let \( \lambda \in N_\infty(H_\theta) \) and let \( f_n \in C_c^\infty(X, E) \) be a bWs associated to the operator \( H_\theta \) and \( \lambda \). In order to prove \( N_\infty(H_\theta) \subset \mathcal{F}_\theta \) we will construct, using \( f_n \), a bWs associated to \( \lambda \) and one of the operators \( H_{i,\theta} \). We define \( \kappa_n := 1 - \eta_n \).

**Proposition 17** There exists \( c > 0 \) and a subsequence \( s \) of \( \mathbb{N} \) such that

\[
||\kappa_n(u_1)f_s(n)||_{L^2(X,E)} \geq c > 0 \quad \text{or} \quad ||\kappa_n(u_2)f_s(n)||_{L^2(X,E)} \geq c > 0. \tag{93}
\]

**Proof:**
Suppose \( ||\kappa_n(u_1)f_n||_{L^2(X,E)} \to 0 \) and \( ||\kappa_n(u_2)f_n||_{L^2(X,E)} \to 0 \). We can choose \( s \) such that \( \chi_n f_s(n) = f_s(n) \) where \( \chi_n \) denotes the characteristic function of \( X - X_{n,2+1} \). Since \( \chi_n^2 \leq \kappa_n(u_1)^2 + \kappa_n(u_1)^2 \), then

\[
1 = ||\chi_n f_s(n)||_{L^2(X,E)}^2 \leq ||\kappa_n(u_1)f_s(n)||_{L^2(X,E)}^2 + ||\kappa_n(u_2)f_s(n)||_{L^2(X,E)}^2, \tag{94}
\]

which is a contradiction. \( \square \)

The previous proposition allows us to suppose that \( 0 < c < ||\kappa_n(u_1)f_s(n)||_{L^2(X,E)} \).
Proposition 18 Denote by $g_n$ the function in $C^\infty(X, E)$, defined by $g_n := \frac{1}{\|\kappa_n(u_1)f_s(n)\|_{L^2(X, E)}}\kappa_n(u_1)f_s(n)$. Then, $g_n$ induces a boundary Weyl sequence associated to $H_{1, \theta}$ and $\lambda$.

Proof:
It is easy to check that $\|g_n\|_{L^2(X, E)} = 1$ and, for all $T > 0$, there exists $N \in \mathbb{R}$ such that $\forall n \geq N$, $supp g_n \cap X_T = \emptyset$. Denoting $\kappa(u) := 1 - \eta(u)$, we define $\kappa_n(u_1) := \kappa\left(\frac{u_1}{n}\right)$, then $\frac{\partial}{\partial u_1}(\kappa_n)(u_1) = \frac{1}{n} \frac{\partial \kappa}{\partial u_1}(\frac{u_1}{n})$, and $\frac{\partial^2}{\partial u_1^2}(\kappa_n)(u_1) = \frac{1}{n^2} \frac{\partial^2 \kappa}{\partial u_1^2}(\frac{u_1}{n})$. Hence:

$$\| (H_{1, \theta} - \lambda) (\kappa_n(u_1)f_s(n)) \|_{L^2(X, E)} \leq A_n + B_n + C_n \rightarrow 0,$$

where

$$A_n^2 := 4\left(\frac{\partial}{\partial u_1}(\kappa_n)(u_1)\frac{\partial}{\partial u_1}(f_s(n))\right)^2_{L^2(X, E)}$$

$$\leq \frac{C}{n^2} \int_{Z_2 \times [0, \infty)} |\frac{\partial}{\partial u_1}(\kappa)(\frac{u_1}{n})\frac{\partial}{\partial u_1}(f_s(n))|^2 dvol(x)$$

$$\leq \frac{C}{n^2} \|f_s(n)\|^2 \leq C\|f_s(n)\|^2 \cdot \frac{1}{n^2} \leq C \frac{1}{n^2} \rightarrow 0.$$

In the last inequalities we use that $\|\frac{\partial}{\partial u_1}(f_s(n))\| \leq C\|f_s(n)\|_2$, which follows from the theory of bounded differential operators on manifolds with bounded geometry (see [23, 10]). We use also $\|f_n\|_2 \leq C$, which follows from $\|f_n\| = 1$, $\lim_{n \to \infty} \|(H_\theta - \lambda)f_n\| = 0$ and since $H_\theta$ is uniformly elliptic. For $B_n$, we have:

$$B_n^2 := \left|\frac{\partial^2}{\partial u_1^2}(\kappa_n)(u_1)\frac{\partial}{\partial u_1}(f_s(n))\right|^2 \leq C \int_{Z_2 \times [0, \infty)} \left|\frac{\partial^2}{\partial u_1^2}(\kappa_n)(u_1)\frac{\partial}{\partial u_1}(f_s(n))\right|^2 dvol(x)$$

$$\leq C \frac{1}{n^4} \|f_s(n)\|^2 = C \frac{1}{n^4} \rightarrow 0.$$

Finally for $C_n$, we have

$$C_n := \|\kappa_n(u_1) (H_{1, \theta} - \lambda) f_s(n)\| \leq C\| (H_{1, \theta} - \lambda) f_s(n)\| \rightarrow 0. \square$$

Next we prove step ii) of our proof of $\sigma_{ess}(H_\theta) = \mathcal{F}_\theta$. Let $\eta \in C^\infty(\mathbb{R})$ such that $\eta(u) = 1$ for $u \leq 1$, $\eta(u) = 0$ for $u > 2$ and $\eta'(u) \leq 0$. Denote $\eta_n(u) := \eta\left(\frac{1}{n}\right)$, and define

$$\eta_n^{(d)}(u_1, u_2, y) := \eta_d(u_1)\eta_d(u_2).$$

(96)
Proposition 19 For all \( f \in C_c^\infty(X,E) \),

\[
\| [H_\theta, \eta_0^{(d)}] f \|_{L^2(X,E)} \leq \epsilon(d)( \| H_\theta f \|_{L^2(X,E)} + \| f \|_{L^2(X,E)}) \tag{97}
\]

with \( \lim_{d \to \infty} \epsilon(d) = 0 \).

**Proof:**

Let \( i, j \in \{1, 2\} \) and \( i \neq j \). We observe that

\[
H_\theta(\eta_0^{(d)} f) = \theta' \sum_{i=1}^1 \{2 \eta_d(u_j) \frac{\partial}{\partial u_i}(\eta_d)(u_i) \frac{\partial}{\partial u_i}(f) + \eta_d(u_j) \frac{\partial^2}{\partial u_i^2}(\eta_d)(u_i) f \} \tag{98}
\]

Hence,

\[
\| [H_\theta, \eta_0^{(d)}] f \|_{L^2(X,E)} \leq |\theta'| \cdot \sum_{i=1}^1 \{2 \| \eta_d(u_j) \frac{\partial}{\partial u_i}(\eta_d)(u_i) \frac{\partial}{\partial u_i}(f) \|_{L^2(X,E)} + \| \eta_d(u_j) \frac{\partial^2}{\partial u_i^2}(\eta_d)(u_i) f \|_{L^2(X,E)} \} \tag{99}
\]

By definition of \( \eta_0^{(d)} \), \( \frac{\partial}{\partial u_1}(\eta_0^{(d)})(u_1, u_2, y) = \frac{1}{d} \frac{\partial}{\partial u_1}(\eta)(u_1/d) \eta_d(u_2) \). Thus,

\[
\| \eta_d(u_j) \frac{\partial}{\partial u_i}(\eta_d(u_i)) \frac{\partial}{\partial u_i}(f) \|_{L^2(X,E)} \leq \int_X |\eta_d(u_j) \frac{\partial}{\partial u_i}(\eta_d)(u_i) \frac{\partial}{\partial u_i}(f) |^2 dx \\
\leq \frac{1}{d^2} \int_{Y \times [0,\infty)^2} \left| \frac{\partial}{\partial u_1}(\eta)(u_1/d) \eta_d(u_2) \frac{\partial}{\partial u_i}(f) \right|^2 dx \\
\leq \frac{C}{d^2}(\| f \|_{L^2(X,E)} + \| H_\theta f \|_{L^2(X,E)})^2 \to 0. \tag{100}
\]

In the last inequality we use that \( \frac{\partial}{\partial u_1}(\eta)(u_1/d) \eta_d(u_2) \frac{\partial}{\partial u_i}(f) \) is a bounded differential operator of degree 1 (hence a continuous operator from \( \mathcal{W}_2(X,E) \) to \( L^2(X,E) \)), and the fact that the norm \( f \mapsto \| f \|_{L^2(X,E)} + \| H_\theta f \|_{L^2(X,E)} \) is equivalent to \( \| \cdot \|_2 \) from the theory of uniformly elliptic operators acting on sections of vector bundles on manifolds with bounded geometry (see [23]).

We finish the proof of the proposition with the following calculation:

\[
\| \eta_d(u_j) \frac{\partial^2}{\partial u_i^2}(\eta_d)(u_i) f \|_{L^2(X,E)} \leq 1/d^4 \int_{Y \times [0,\infty)^2} |f|^2 dx \to 0. \tag{101}
\]
3.4 Analytic vectors

In this section we construct a subset $\mathcal{V}$ of $L^2(X,E)$ such that

i) $\mathcal{V}$ is a dense subset of $L^2(X,E)$.

ii) For $f \in \mathcal{V}$ the function $\theta \mapsto U_\theta f \in L^2(X,E)$ is well defined for $\theta \in \mathbb{C}$, $\Re(\theta) > 0$.

iii) $U_\theta \mathcal{V}$ is dense in $L^2(X,E)$ for $\Re(\theta) > 0$.

Let $\mathcal{V}_\iota$ be the analytic vectors associated to $U_{\iota,\theta}$ (see equation (38)). Let $\eta \in C^\infty([0, \infty))$ be such that $\eta' \geq 0$ and

$$\eta(u) = \begin{cases} 1 & K < u < \infty, \\ 0 & 0 < u \leq K - 1. \end{cases}$$

For $i = 1, 2$, define $\eta_i(z_i, u_i) := \eta(u_i)$ and extend them to $X$. Denote $\kappa := 1 - \eta_1 - \eta_2$. Define

$$\mathcal{V} := \{ (\kappa g + \sum_{i=1}^{2} \eta_i \frac{1}{u_i^2} p_i(\frac{1}{u_i}) f_i(z_i)) : g \in L^2(X,E), p_i(x) \in \mathbb{C}[x] \text{ and } f_i \in \mathcal{V}_\iota \}.$$

Then i), ii) and iii) can be deduced from the properties of $\mathcal{V}_\iota$ given in section 2.5.

3.5 Consequences of Aguilar-Balslev-Combes theory

In section 3.3 we calculated the essential spectrum of $H_\theta$. The following theorem is a consequence of theorem 3.3 and the existence of the analytic vectors satisfying i), ii) and iii) of section 3.4. It can be proved using the general ideas of Aguilar-Balslev-Combes theory as explained in [13].

**Theorem 11** We have:

a) The set of non-threshold eigenvalues of $H$ is equal to $\sigma_d(H_\theta) \cap \mathbb{R}$, for all $\theta \in \Gamma - [0, \infty)$. Moreover, given a non-threshold eigenvalue $\lambda_0$, the eigenspace $E_{\lambda_0}(H)$, associated to $H$ and $\lambda_0$, has finite dimension bounded by the degree of the pole $\lambda_0$ of the map $\lambda \mapsto R(\lambda, \theta)$. This algebraic multiplicity is independent of $\theta \in \Gamma - [0, \infty)$.

---

4The set of thresholds of $H$, $\tau(H)$, is equal to $\sigma(H^3) \cup \bigcup_{i=1}^{\kappa} \sigma_{p_i}(H^{(i), \theta})$. 33
b) Fix $\theta \in \Gamma$. For $f, g \in \mathcal{V}$ the function

$$\lambda \mapsto (R(\lambda) f, g)_{L^2(X, E)}$$

has a meromorphic continuation from $\Lambda$ to $\mathbb{C} - (\sigma_{\text{ess}}(H_\theta) \cup \sigma_{\text{pp}}(H_\theta))$, where $\sigma_{\text{ess}}(H_\theta)$ was calculated in theorem ??.

c) $H$ has no singular spectrum.

d) Let $\theta_1, \theta_2 \in \Gamma$ be such that $\arg(\theta'_1) \geq \arg(\theta'_0)$ for $0 < \arg(\theta'_i) < \pi/2$, we have:

$$\sigma_d(H_{\theta_0}) = \sigma_d(H_{\theta_1}) \cap \sigma_d(H_{\theta_0}). \quad (103)$$

e) Non-thresholds eigenvalues of $H$ are isolated (with respect to the eigenvalues of $H$) and may only accumulate on the set of thresholds or at $\infty$.

f) If the lowest threshold, $\gamma_0$, is larger than 0, then $\sigma_d(H)$ is a discrete subset of $[0, \gamma_0)$. In this case, the unique possible accumulation point of $\sigma_d(H)$ is $\gamma_0$. If $\gamma_0 = 0$, then $\sigma_d(H) = \emptyset$, in other words all eigenvalues are embedded in the continuous spectrum.

At the moment we do not know if there is a compatible Laplacian having embedded eigenvalues. It seems that the natural conjecture is that generically there are no embedded eigenvalues. Similarly, we do not know if it is possible to find a generalized Laplacian that has embedded eigenvalues accumulating at one of the thresholds. We believe that it is possible to prove that $\sigma_{\text{pp}}(H_\theta)$ can accumulate only from below in $\tau(H)$, and we will address this question in a forthcoming paper. In particular, this would imply that 0 is not an accumulation point of $\sigma_{\text{pp}}(H_\theta)$.

The next proposition is easy to prove from the definition of essential spectrum and from theorem ??.

**Proposition 20** ([4]) i) If $\lambda \in \sigma_{\text{pp}}(H_\theta)$ and $\lambda \notin \sigma_{\text{ess}}(H_\theta)$, then $\lambda$ an isolated eigenvalue of finite multiplicity.

ii) For $\text{Re}(\theta) > 0$, the accumulation points of $\sigma_{\text{pp}}(H_\theta)$ are contained in $\sigma_{\text{ess}}(H_\theta)$. In particular, the real part of the pure point spectrum of $H_\theta$ can accumulate only in $\tau(H)$.

In [4], using the results of this section, we define generalized eigenfunctions associated to $L^2$-eigenfunctions of $H^{(1)}$, $H^{(2)}$ and $H^{(3)}$. The wave packets
associated to these generalized eigenfunctions describe completely the absolutely continuous spectrum, $L^2_{ac}(X,E)$, associated to $H$. In fact, in [5] is proven the asymptotic completeness of waves operators associated to $H_1$, $H_2$ and $H_3$; the generalized eigenfunctions of [4] describe such wave operators.

A Geometric spectral analysis of $\sigma_{ess}$

In this appendix we remark that the results of section 3 of the paper [7] are also valid in the context of generalized Laplacians on complete manifolds with corners of codimension 2. In fact the proofs of the theorems in this appendix are essentially the same that those given in [7], and because of that we refer readers to that paper or to [6].

We begin by recalling the definition of singular sequences associated to a closed operator $A$:

**Definition 1** A sequence $(f_n)_{n \in \mathbb{N}} \subset \text{Dom}(A)$ is a singular sequence for $A$ associated to the value $\lambda \in \mathbb{C}$ if and only if

i) $||f_n|| = 1$ and $(f_n)_{n \in \mathbb{N}}$ has no $L^2$-convergent subsequence.

ii) $\lim_{n \to \infty} ||(A - \lambda)f_n|| = 0$.

In this section we distinguish between different types of singular sequences of a geometric operator and we describe some relations between them. They define different subsets of the essential spectrum defined in (31).

Let $A : \text{Dom}(A) \subset L^2(X) \to L^2(X)$ be a closed operator.

**Definition 2** Define the set $N_{ess}(A)$ of $\lambda \in \mathbb{C}$ such that there exists a sequence $(u_n)_{n \in \mathbb{N}} \subset \text{Dom}(A)$ such that $||u_n|| = 1$, $u_n \to 0$ (weakly) and $||(\lambda - A)u_n|| \to 0$.

Observe that if $\lambda \in N_{ess}(A)$, then the sequence $(u_n)_{n \in \mathbb{N}}$ associated to $\lambda$ is a singular sequence in the sense of definition (1).

Now we define another important class of singular sequences $N_\infty(A)$.

**Definition 3** (7, page 10) Let $N_\infty(A)$ be the set of $\lambda \in \mathbb{C}$ such that there exists a sequence $(u_n)_{n \in \mathbb{N}} \subset C^\infty_c(X)$ with

i) $||u_n|| = 1$, 

35
\( ii) \ (A - \lambda)u_n \to 0, \)

\( iii) \) for every compact subset \( K \subset M \) there exists \( N \in \mathbb{N} \) such that for \( n > N, \) \( \text{supp}(u_n) \cap K = \emptyset. \)

We will call the sequence \( u_n \) a **boundary Weyl sequence** (abbr. bWs).

Observe that if \( \lambda \in N_\infty(A) \) and \( (u_n)_{n \in \mathbb{N}} \) is a sequence as in definition \( 3 \), then \( (u_n)_{n \in \mathbb{N}} \) is a singular sequence associated to \( A \) and the value \( \lambda \).

**Proposition 21** ([7], page 9)

\( i) \ N_\text{ess}(A) \subset \sigma_\text{ess}(A). \)

\( ii) \ N_\text{ess}(A) \) is closed.

The following theorem gives conditions for the equality of \( \sigma_\text{ess}(A) \) and \( N_\text{ess}(A) \).

**Theorem 12** ([7], theorem 3.1) (Weyl’s criterion for \( \sigma_\text{ess}(A) \))

Let \( A \) be a closed operator on a Hilbert space \( \mathcal{H} \) with non-empty resolvent set. Then:

\( i) \ N_\text{ess}(A) \subset \sigma_\text{ess}(A). \)

\( ii) \) The boundary of \( \sigma_\text{ess}(A) \) is contained in \( N_\text{ess}(A). \)

\( iii) \ N_\text{ess}(A) = \sigma_\text{ess}(A) \) if and only if each connected component of the complement of \( N_\text{ess}(A) \) contains a point of \( \rho(A). \)

The next theorem gives conditions for the equality of \( N_\infty(A) \) and \( N_\text{ess}(A) \).

We will use the notation \( X_0 \) and \( X_d \) for the manifolds defined in (2) for \( T = 0 \) and \( T = d \) respectively.

**Theorem 13** ([7], theorem 3.2)

Let \( A \) be a closed operator on \( L^2(X) \) with non-empty resolvent set, having \( C^\infty_c(X) \) as a core. Let \( \eta_0 \in C^\infty_c(X) \) such that \( \eta_0(x) = 1 \) for \( x \in X_0 \). Let \( \eta_0^d \in C^\infty_c(X) \) such that \( \eta_0^d(x) = 1 \) for \( x \in X_d \), and \( 0 \leq \eta_0^d(x) \leq 1 \) for \( x \in X \). Suppose \( \forall d, \eta_0^d(z - A)^{-1} \) is compact for some \( z \in \rho(A) \), and that for all \( u \in C^\infty_c(X), \)

\[
||[A, \eta_0^d]u|| \leq \epsilon(d)(||Au|| + ||u||),
\]

(104)

with \( \epsilon(d) \to 0 \) as \( d \to \infty \). Then \( N_\infty(A) = N_\text{ess}(A). \)
B Ichinose lemma

In this appendix we recall some definitions and we formulate the Ichinose lemma (theorem 18). The following definitions follow [21] and [15], we refer there for a deeper study of the topic. Let $q$ be a bilinear form on a Hilbert space $H$ with domain $Q(q)$.

**Definition 4** We say that $q$ is **closed** if and only if always that a sequence $\varphi_n \in Q(q)$ converges $\varphi$ in the norm topology, and

$$\lim_{n,m \to \infty} q(\varphi_n - \varphi_m, \varphi_n - \varphi_m) = 0,$$

then, we have $\varphi \in Q(q)$ and $q(\varphi_n - \varphi, \varphi_n - \varphi) \to 0$.

**Definition 5** A quadratic form $q$ is **sectorial** if there exists $\theta$, $0 < \theta < \pi/2$ with $|\arg(q(\varphi, \varphi))| \leq \theta$ for all $\varphi \in Q(q)$.

**Definition 6** A quadratic form $q$ is called **strictly m-accretive** if it is both closed and sectorial.

**Definition 7** A form $q$ is called **strictly m-sectorial** if there are complex numbers $z$ and $e^{i\alpha}$, with $\alpha$ real, so that $e^{i\alpha}q + z$ is strictly $m$-accretive. The operator $T$ associated to $q$ is also called **strictly m-sectorial**.

Observe that in order to prove that $q$ is strictly $m$-sectorial it is enough to show that there exists $\gamma \in \mathbb{R}$ and $k \in \mathbb{R}_+$ such that for all $f \in Q(q)$

$$k Re(q(f)) - |Im(q(f))| \geq \gamma (f, f).$$

(105)

Every closed operator $T$ defines a dense form $q(T)$ by

$$q(t)(\varphi, \psi) := (\varphi, T\psi),$$

(106)

for $\varphi, \psi \in D(T)$.

**Definition 8** An operator $T$ is **sectorial** if there is a $\theta$, $0 < \theta < \pi/2$ such that its numerical range, $\Theta(T)$, is a subset of a sector $\{z \in \mathbb{C} : |\arg(z)| \leq \theta\}$.

The following theorems are important in section 2.7.

**Theorem 14** ([15], page 318) A sectorial operator $T$ is form closable, that is, the form $q(T)$ defined by (106) has an extension that is closed in the sense of definition 4.
Theorem 15 ([15], page 316) Let \( \tilde{q} \) be the closure of a densely defined form \( q \). The numerical range \( \Theta(q) \) of \( q \) is a dense subset of the numerical range \( \Theta(\tilde{q}) \) of \( \tilde{q} \).

The next theorem is also used in section 2.7.

Theorem 16 ([15], page 319) Let \( q_1, \ldots, q_s \) be sectorial forms in \( \mathcal{H} \) and let \( q := q_1 + \cdots + q_s \) [with \( D(q) := D(q_1) \cap \cdots \cap D(q_s) \)]. Then \( q \) is sectorial. If all \( q_j \) are closed, so is \( q \). If all the \( q_j \) are closable so is \( q \) and \( \tilde{q} \subset \tilde{q}_1 + \cdots + \tilde{q}_s \).

The following theorem naturally associates to strictly \( m \)-accretive quadratic forms a unique operator \( T \).

Theorem 17 ([21], page 281) Let \( q \) be a strictly \( m \)-accretive quadratic form with domain \( Q(q) \). Then there is a unique operator \( T \) on \( \mathcal{H} \) such that:

a) \( T \) is closed.

b) \( D(T) \subset Q(q) \) and if \( \varphi, \psi \in D(T) \), then \( q(\varphi, \psi) = (\varphi, T\psi) \). Further, \( D(T) \) is a form core for \( q \).

c) \( D(T^*) \subset Q(q) \) and if \( \varphi, \psi \in D(T) \), then \( q(\varphi, \psi) = (T^* \varphi, \psi) \). Further, \( D(T^*) \) is a form core for \( q \).

From this theorem we can define.

Definition 9 A closed operator \( T \) is called strictly \( m \)-sectorial operator if there exists \( q \) strictly \( m \)-sectorial such that \( q \) and \( T \) satisfy properties a), b) c) of the above theorem.

Now we can formulate the Ichinose lemma.

Theorem 18 ([22], page 183) (Ichinose’s lemma) Let \( \overline{S}_{\omega, \varphi, \theta} \) denote the sector \( \{ z | \varphi - \theta \leq \arg(z - \omega) \leq \varphi + \theta \; : \; \theta \geq \pi/2 \} \). Let \( A \) and \( B \) be strictly \( m \)-sectorial operators on Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) with sectors \( \overline{S}_{\omega_1, \varphi, \theta_1} \) and \( \overline{S}_{\omega_2, \varphi, \theta_2} \) (same \( \varphi \)). Let \( C \) denote the closure of \( A \otimes I + I \otimes B \) on \( D(A) \otimes D(B) \). Then \( C \) is a strictly \( m \)-sectorial operator with sector \( \overline{S}_{\omega_1 + \omega_2, \varphi, \min\{\theta_1, \theta_2\}} \) and \( \sigma(C) = \sigma(A) + \sigma(B) \).
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