Symmetries and the Antibracket: The Batalin-Vilkovisky Method

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1 Introduction

The most general quantization prescription available today is provided by the Batalin-Vilkovisky (BV) method\cite{1,2}. It includes all the results of the BRST method, and also permits a systematic quantization of systems with an open algebra of constraints. Moreover it possesses a rich algebraic structure, that have been considered in the description of String Field Theory \cite{3,4}.

Because of the many advantages of the BV method, it is useful to understand more deeply the nature and reasons for the different prescriptions that appear there. This we have done in a series of papers\cite{5,6,7,8}: Impossing that the most general Schwinger -Dyson equations of a theory should come from a symmetry principle(SD-BRST symmetry), we have been able to derive the BV method from standard BRST Lagrangian quantization. In so doing, we have understood the meaning of the antifields, the appearence of the graded canonical structure generated by the antibracket and ultimately, the underlying reason for the existence of a master equation. From this perspective, it is quite natural to generalize the BV structure, first to non-abelian invariances of the path integral measure\cite{9}, later to the most general open algebra\cite{10,11}. 
In these lectures, we review these recent developments: In section II, we study the derivation of BV from Schwinger-Dyson-BRST symmetry, in section III, we consider some generalizations of the BV structure suggested by our approach. In section IV, we present a connection between the Poisson bracket and the antibracket and use it to relate the Nambu bracket with the generalized \( n \)-brackets of section III. Finally in section V, we draw some conclusions.

## 2 Derivation of the Batalin-Vilkovisky Method from Schwinger-Dyson BRST Symmetry

In this section, we review the derivation of the BV method from Schwinger-Dyson BRST symmetry done in \[5\].

Let us recapitulate the basic ingredients of the Batalin-Vilkovisky (BV) method \[1, 2\].

Begin with a set of fields \( \phi^A(x) \) of given Grassmann parity (statistics) \( \epsilon(\phi^A) = \epsilon_A \), and then introduce for each field a corresponding antifield \( \phi^*_A \) of opposite Grassmann parity \( \epsilon(\phi^*_A) = \epsilon_A + 1 \). The fields and antifields are taken to be canonically conjugate,

\[
(\phi^A, \phi^*_B) = \delta^A_B, \quad (\phi^A, \phi^B) = (\phi^*_A, \phi^*_B) = 0, \quad (1)
\]

within a certain graded bracket structure \((\cdot, \cdot)\), the antibracket:

\[
(F, G) = \frac{\delta'^F \delta'^G}{\delta \phi^A \delta \phi^*_A} - \frac{\delta'^F \delta'^G}{\delta \phi^*_A \delta \phi^A}. \quad (2)
\]

The subscripts \( l \) and \( r \) denote left and right differentiation, respectively. The summation over indices \( A \) includes an integration over continuous variables such as space-time \( x \), when required.

This antibracket is statistics-changing in the sense that

\[
\epsilon[(F, G)] = \epsilon(F) + \epsilon(G) + 1, \quad (3)
\]

and satisfies the following exchange relation:

\[
(F, G) = - (-1)^{\epsilon(F)+1}(\epsilon(G)+1)(G, F). \quad (4)
\]
Furthermore, one may verify that the antibracket acts as a derivation of the kind

\[(F, GH) = (F, G)H + (-1)^{(c(G)(c(F) + 1)} G(F, H)\]

\[(FG, H) = F(G, H) + (-1)^{(c(G)(c(H) + 1)} (F, H)G,\]

and satisfies a Jacobi identity of the form

\[(-1)^{(c(F) + 1)(c(H) + 1)} (F, (G, H)) + \text{cyclic perm.} = 0.\]

Some simple consequences of these relations are that \((F, F) = 0\) for any Grassmann odd \(F\), and \((F, (F, F)) = ((F, F), F) = 0\) for any \(F\).

The antifields \(\phi^*_A\) are also given definite ghost numbers \(gh(\phi^*_A)\), related to those of the fields \(\phi^A\):

\[gh(\phi^*_A) = -gh(\phi^A) - 1.\]

The absolute value of the ghost number can be fixed by requiring that the action carries ghost number zero.

The Batalin-Vilkovisky quantization prescription can now be formulated as follows. First solve the equation

\[\frac{1}{2} (W, W) = i\hbar \Delta W\]

where

\[\Delta = (-1)^{c_A+1} \frac{\delta^r}{\delta \phi^A} \frac{\delta^r}{\delta \phi^*_A}.\]

This \(W\) will be the “quantum action”, presumed expandable in powers of \(\hbar\):

\[W = S + \sum_{n=0}^{\infty} \hbar^n M_n,\]

and a boundary condition is that \(S\) in eq. \(\ref{eq:S}\) should coincide with the classical action when all antifields are removed, i.e., after setting \(\phi^*_A = 0\). One can solve for the additional \(M_n\)-terms through a recursive procedure, order-by-order in an \(\hbar\)-expansion. To lowest order in \(\hbar\) this is the Master Equation:

\[(S, S) = 0.\]
Similarly, one can view eq. (8) as the full “quantum Master Equation”.

A correct path integral prescription for the quantization of the classical theory $S[\phi^A, \phi^*_A = 0]$ is that one should find an appropriate “gauge fermion” $\Psi$ such that the partition function is given by

$$Z = \int [d\phi^A][d\phi^*_A] \delta(\phi^*_A - \frac{\delta^r \Psi}{\delta \phi^A}) \exp \left[ \frac{i}{\hbar} W \right].$$

This prescription guarantees gauge-independence of the $S$-matrix of the theory. The “extended action” $S \equiv S_{ext}[\phi^A, \phi^*_A]$ is a solution of the Master Equation (11), and can been given by an expansion in powers of antifields \[.\] After the elimination of the antifields by the $\delta$-function constraint in eq. (12), one can verify that the action is invariant under the usual BRST symmetry, which we will here denote by $\delta$.

In the following, we will uncover and derive in a simple manner the Batalin-Vilkovisky formalism starting from two basic ingredients: A) Standard BRST Lagrangian quantization, and B) The requirement that the most general Schwinger-Dyson equations of the full quantum theory follow from a symmetry principle. We shall throughout, unless otherwise stated, assume that when ultraviolet regularization is required, a suitable regulator which preserves the relevant BRST symmetry exists.

Schwinger-Dyson equations will thus play a crucial rôle in this analysis. The idea is to enlarge the usual BRST symmetry in precisely such a way that both the usual gauge-symmetry Ward Identities and the most general Schwinger-Dyson equations of the full quantum theory follow from a symmetry principle. We will throughout, unless otherwise stated, assume that when ultraviolet regularization is required, a suitable regulator which preserves the relevant BRST symmetry exists.

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2.1 No gauge Symmetries

Consider a quantum field theory based on an action $S[\phi^A]$ without any internal gauge symmetries. Such a quantum theory can be described by a path integral.

$$Z = \int [d\phi^A] \exp \left[ \frac{i}{\hbar} S[\phi^A] \right],$$

and the associated generating functional.

Equivalently, such a quantum field theory can be entirely described by the solution of the corresponding Schwinger-Dyson equations, once appropriate
boundary conditions have been imposed. At the path integral level they follow from invariances of the measure. Let us for simplicity consider the case of a flat measure which is invariant under arbitrary local shifts, \( \phi^A(x) \to \phi^A(x) + \varepsilon^A(x) \). We can gauge this symmetry by means of collective fields \( \varphi^A(x) \): Suppose we transform the original field as

\[
\phi^A(x) \to \phi^A(x) - \varphi^A(x),
\]

then the transformed action \( S[\phi^A - \varphi^A] \) is trivially invariant under the local gauge symmetry

\[
\delta \phi^A(x) = \Theta(x), \quad \delta \varphi^A(x) = \Theta(x),
\]

and the measure for \( \phi^A \) in eq. (13) is also invariant. The gauge invariant functions are of the form \( F[\phi^A - \varphi^A] \), for any \( F \).

We next integrate over the collective field in the transformed path integral, using the same flat measure. The integration is of course very formal since it will include the whole volume of the gauge group. To cure this problem, we gauge-fix in the standard BRST Lagrangian manner [9]. That is, we add to the transformed Lagrangian a BRST-exact term in such a way that the local gauge symmetry is broken. In this case an obvious BRST multiplet consists of a ghost-antighost pair \( c^A(x), \phi^*_A(x) \), and a Nakanishi-Lautrup field \( B_A(x) \):

\[
\begin{align*}
\delta \phi^A(x) &= c^A(x) \\
\delta \varphi^A(x) &= c^A(x) \\
\delta c^A(x) &= 0 \\
\delta \phi^*_A(x) &= B_A(x) \\
\delta B_A(x) &= 0.
\end{align*}
\]

No assumptions will be made as to whether \( \phi^A \) are of odd or even Grassmann parity. We assign the usual ghost numbers to the new fields,

\[
gh(c^A) = 1, \quad gh(\phi^*_A) = -1, \quad gh(B_A) = 0,
\]

and the operation \( \delta \) is statistics-changing. The rules for operating with \( \delta \) are given in the Appendix.

\[1\text{This situation is no different from usual path integral manipulations of gauge theories.}\]
Let us choose to gauge-fix the transformed action by adding to the Lagrangian a term of the form
\[
- \delta [\phi^*_A(x) \phi^A(x)] = (-1)^{\epsilon(A)+1} B_A(x) \varphi^A(x) - \phi^*_A(x) c^A(x). \tag{18}
\]
The partition function is now again well-defined:
\[
Z = \int [d\phi][d\phi^*][dc][dB] \exp \left[ \frac{i}{\hbar} \left( S[\phi - \varphi] - \int dx \{-1)^{\epsilon(A)} B_A(x) \varphi^A(x) + \phi^*_A(x) c^A(x) \} \right) \right]. \tag{19}
\]
Since the collective field has just been gauge fixed to zero, it may appear useful to integrate both it and the field \( B_A(x) \) out. We are then left with
\[
S_{ext} = S[\phi^A] - \int dx \phi^*_A(x) c^A(x), \tag{20}
\]
which obviously coincides with the original expression (13) apart from the trivially decoupled ghosts. But the remnant BRST symmetry is still non-trivial: We find it in the usual way by substituting for \( B_A(x) \) its equation of motion. This gives
\[
\begin{align*}
\delta \phi^A(x) &= c^A(x) \\
\delta c^A(x) &= 0 \\
\delta \phi^*_A(x) &= -\frac{\delta' S}{\delta \phi^A(x)}. \tag{21}
\end{align*}
\]
The functional measure is also invariant under this symmetry, according to our assumption about the measure for \( \phi^A \), and assuming a flat measure for \( \phi^*_A \) as well. The Ward Identities following from this symmetry are the sought-for Schwinger-Dyson equations:
\[
0 = \langle \delta \{ \phi^*_A(x) F[\phi^A] \} \rangle, \quad \text{where we have chosen } F \text{ to depend only on } \phi^A \text{ just to ensure that the whole object carries overall ghost number zero. After integrating over both ghosts } c^A \text{ and antighosts } \phi^*_A, \text{ this Ward Identity can be written}
\]
\[
\left\langle \frac{\delta' F}{\delta \phi^A(x)} + \left( \frac{i}{\hbar} \right) \frac{\delta' S}{\delta \phi^A(x)} F[\phi^A] \right\rangle = 0 \tag{22},
\]
that is, precisely the most general Schwinger-Dyson equations for this theory. The symmetry (21) can be viewed as the BRST Schwinger-Dyson algebra.

Consider now the equation that expresses BRST invariance of the extended action $S_{\text{ext}}$:

$$0 = \delta S_{\text{ext}} = \int dx \frac{\delta^r S_{\text{ext}}}{\delta \phi^A(x)} c^A(x) - \int dx \frac{\delta^r S_{\text{ext}}}{\delta \phi^*_A(x)} \delta \phi^A(x)$$

$$= \int dx \frac{\delta^r S_{\text{ext}}}{\delta \phi^A(x)} c^A(x) - \int dx \frac{\delta^r S_{\text{ext}}}{\delta \phi^*_A(x)} \delta S_{\text{ext}} .$$

(23)

In the last line we have used the fact that $S$ differs from $S_{\text{ext}}$ by a term independent of $\phi^A$. Using the notation of the antibracket (2), this is seen to correspond to a Master Equation of the form

$$\frac{1}{2}(S_{\text{ext}}, S_{\text{ext}}) = -\int dx \frac{\delta^r S_{\text{ext}}}{\delta \phi^A(x)} c^A(x)$$. (24)

The ghosts $c^A$ play the rôle of spectator fields in the antibracket. But their appearance on the r.h.s. of the Master Equation ensures that the solution $S_{\text{ext}}$ will contain these fields.

The extended action of Batalin and Vilkovisky does however not coincide with $S_{\text{ext}}$ as defined above. But suppose we integrate only over these ghosts $c^A(x)$, without integrating over the corresponding antighosts $\phi^*_A(x)$. Then the partition function reads

$$Z = \int [d\phi^A] [d\phi^*_A] \delta (\phi^*_A) \exp \left[ \frac{i}{\hbar} S[\phi^A] \right] .$$ (25)

What has happened to the BRST algebra? For the present case of a ghost field $c^A$ appearing linearly in the action before being integrated out, we can derive the correct substitution rule as follows. First, we should really phrase the question in a more precise manner. What we need to know is how to replace $c$ inside the path integral, i.e., inside Green functions. This will automatically give us the correct transformation rules for those fields that are not integrated out. Consider the identity

$$\int [dc] F(c^B(y)) \exp \left[ -\frac{i}{\hbar} \int dx \phi^*_A(x)c^A(x) \right] = F \left( \frac{i\hbar}{\delta \phi^*_B(y)} \right) \exp \left[ -\frac{i}{\hbar} \int dx \phi^*_A(x)c^A(x) \right] .$$ (26)
Eq. (26) teaches us that it is not enough to replace \(c\) by its equation of motion \((c(x) = 0)\); a “quantum correction” in the form of the operator \(\hbar \delta / \delta \phi^*\) must be added as well. The appearance of this operator is the final step towards unravelling the canonical structure in the formalism of Batalin and Vilkovisky. It also shows that even in this trivial case we have to include “quantum corrections” to BRST symmetries if we insist on integrating out only one ghost field, while keeping its antighost.

It is important that the operator \(\hbar \delta / \delta \phi^*\) in eq. (26) always acts on the integral (really a \(\delta\)-function) to its right.

We now make this replacement, having always in mind that it is only meaningful inside the path integral. For the BRST transformation itself we get, upon one partial integration

\[
\delta \phi^A(x) = i\hbar (-1)^{\epsilon_A} \frac{\delta^r}{\delta \phi^*_A(x)}
\]

\[
\delta \phi^*_A(x) = - \frac{\delta^r S}{\delta \phi^A(x)} .
\]  

(27)

We know from our derivation that this transformation leaves at least the combination of measure and action invariant. As a check, if we consider the same Ward Identity as above, based on \(0 = \langle \delta \{ \phi^*_A(x) F[\phi^A] \} \rangle\), we recover the Schwinger-Dyson equation (22).

The original \(S[\phi^A]\) can in this case be identified with the extended action of Batalin and Vilkovisky, and the antighost \(\phi^*_A\) is the antifield corresponding to \(\phi^A\). Because there are no internal gauge symmetries, the extended action turns out to be independent of the antifields. Although our \(S_{\text{ext}}\) of eq. (20) cannot be identified with the extended action of Batalin-Vilkovisky, the Master Equation derived in eq. (23) is of course very similar to their corresponding Master Equation. Writing eq.(23) in terms of the antibracket is a little forced. It is done in order to facilitate the comparison.

Finally, it only remains to be seen what has happened to this bracket structure after having integrated out the ghost. We will keep the same notation as before, so that in this case the “extended” action is trivially equal to the original action: \(S_{\text{ext}} = S[\phi^A]\). Let us then again consider the variation of an arbitrary functional \(G\), this time only a function of \(\phi^A\) and \(\phi^*_A\). Inside
the path integral (and only there!) we can represent the variation of $G$ as:

$$\delta G[\phi^A, \phi^*_A] = \int dx \frac{\delta^r G}{\delta \phi^A(x)} \left[ \frac{\delta^r S_{ext}}{\delta \phi^*_A(x)} + (\imath h)(-1)^{r_A} \frac{\delta^r}{\delta \phi^*_A(x)} \right] - \int dx \frac{\delta^r G}{\delta \phi^*_A(x)} \frac{\delta^r S_{ext}}{\delta \phi^A(x)} ,$$

(28)

where the derivative operator no longer acts on the $\delta$-function of $\phi^*_A$. We have kept the term proportional to $\delta S_{ext}/\delta \phi^*_A$, even though $S_{ext}$ in this simple case is independent of $\phi^*_A$ (and that term therefore vanishes). It comes from the partial integration with respect to the operator $\delta^r/\delta \phi^*_A$, and is there in general when $S_{ext}$ depends on $\phi^*_A$.

This equation precisely describes the “quantum deformation” of the classical BRST charge, as it occurs in the Batalin-Vilkovisky framework:

$$\delta G = (G, S_{ext}) - i\hbar \Delta G .$$

(29)

The BRST operator in this form is often denoted by $\sigma$.

We have here introduced

$$\Delta \equiv (-1)^{r_A+1} \frac{\delta^r \delta^r}{\delta \phi^*_A(x) \delta \phi^A(x)}$$

(30)

which is identical to the operator (I) of Batalin and Vilkovisky. Again, this term arises as a consequence of the partial integration which allows us to expose the operator $\delta/\delta \phi^*_A(x)$ that otherwise acts only on the $\delta$-functional $\delta(\phi^*_A)$. In other words, the $\delta$-function constraint on $\phi^*_A(x)$ is now considered as part of the functional measure for $\phi^*_A$.

Precisely which properties of the partially-integrated extended action are then responsible for the canonical structure behind the Batalin-Vilkovisky formalism? As we have seen, the crucial ingredients come from integrating out the Nakanishi-Lautrup fields $B_A$ and the ghosts $c^A$. Integrating out $B_A$ changes the BRST variation of the antighosts $\phi^*_A$ into $-\delta^r S_{ext}/\delta \phi^A$. This is an inevitable consequence of introducing collective fields as shifts of the original fields (and hence enforcing Schwinger-Dyson equations), and then gauge-fixing them to zero. But this only provides half of the canonical structure, making, loosely speaking, $\phi^A$ canonically conjugate to $\phi^*_A$, but not vice versa. The rest is provided by integrating over the ghosts $c^A$; at the linear level it changes the BRST variation of the fields $\phi^A$ themselves into $\delta^r S_{ext}/\delta \phi^*_A$. This
in turn is again an inevitable consequence of having introduced the collective fields as shifts, and then having gauge-fixed them to zero.\textsuperscript{2}

The latter operation is, however, complicated by the fact that the fields $\phi^*_A$ which are fixed in the process of integrating out $c^A$ are chosen to remain in the path integral. This makes it impossible to discard the “quantum correction” to $\delta S_{\text{ext}} / \delta \phi^*_A$. So, in fact, if one insists on keeping these antighosts $\phi^*_A$, now seen as canonically conjugate partners of $\phi^A$, the simple canonical structure is in this sense never truly realized.

We have seen these features only in what is the trivial case of no internal gauge symmetries. But as we shall show in the following section, they hold in greater generality.

### 2.2 Gauge Theories: Yang-Mills

Let us illustrate our approach with another simple example: Yang-Mills theory.

We start with the pure Yang-Mills action $S[A_\mu]$, and define a covariant derivative with respect to $A_\mu$ as

$$D_\mu^{(A)} \equiv \partial_\mu - [A_\mu, \cdot]. \quad (31)$$

Next, we introduce the collective field $a_\mu$ which will enforce Schwinger-Dyson equations as a result of the BRST algebra:

$$A_\mu(x) \rightarrow A_\mu(x) - a_\mu(x). \quad (32)$$

In comparison with the previous example, the only new aspect here is that the transformed action $S[A_\mu - a_\mu]$ actually has two independent gauge symmetries. Because of the redundancy introduced by the collective field, we can write the two symmetries in different ways. To make contact with the Batalin-Vilkovisky formalism, we will choose the following version,

$$\delta A_\mu(x) = \Theta_\mu(x)$$

\textsuperscript{2}Gauge-fixing the collective fields to zero implies linear couplings to the auxiliary fields and ghosts, respectively. This is one of the central properties of the extended action that leads to the canonical structure, and to the fact that the extended action $S_{\text{ext}}$ itself is the (classical) BRST generator. However, this does not exclude the possibility that different gauge fixings of the shift symmetries could produce a more general formalism.
\[ \delta a_\mu(x) = \Theta_\mu(x) - D_\mu^{(A-a)} \varepsilon(x), \]  

which also shows the need for being careful in defining what we mean by a covariant derivative. (The general principle is that we choose the original gauge symmetry of the original field to be carried entirely by the collective field; the transformation of the original gauge field is then always just a shift.) Although \( \Theta(x) \) includes arbitrary deformations, it only leaves the transformed field invariant, while of course the action is also invariant under Yang-Mills gauge transformations of this transformed field itself. Hence the need for including two independent gauge transformations.

We now gauge-fix these two gauge symmetries, one at a time, in the standard BRST manner. As a start, we introduce a suitable multiplet of ghosts and auxiliary fields. We need one Lorentz vector ghost \( \psi_\mu(x) \) for the shift symmetry of \( A_\mu \), and one Yang-Mills ghost \( c(x) \). These are of course Grassmann odd, and both carry the same ghost number

\[ gh(\psi_\mu) = gh(c) = 1. \]  

Next, we gauge-fix the shift symmetry of \( A_\mu \) by removing the collective field \( a_\mu \). This leads us to introduce a corresponding antighost \( A_\mu^*(x) \), Grassmann odd, and an auxiliary field \( b_\mu(x) \), Grassmann even. They have the usual ghost number assignments,

\[ gh(A_\mu^*) = -1, \quad gh(b_\mu) = 0, \]  

and we now have the nilpotent BRST algebra

\[
\begin{align*}
\delta A_\mu(x) &= \psi_\mu(x) \\
\delta a_\mu(x) &= \psi_\mu(x) - D_\mu^{(A-a)} c(x) \\
\delta c(x) &= -\frac{1}{2} [c(x), c(x)] \\
\delta \psi_\mu(x) &= 0 \\
\delta A_\mu^*(x) &= b_\mu(x) \\
\delta b_\mu(x) &= 0.
\end{align*}
\]  

Fixing \( a_\mu(x) \) to zero is achieved by adding a term

\[ -\delta A_\mu^*(x) a_\mu(x) = -b_\mu(x) a_\mu(x) - A_\mu^*(x) \{ \psi_\mu - D_\mu^{(A-a)} c(x) \} \]
to the Lagrangian. We shall follow the usual rule of only starting to integrate over pairs of ghost-antighosts in the partition function. With this rule we shall still keep $c(x)$ unintegrated (since we have not yet introduced its corresponding antighost), but we can now integrate over both $\psi_\mu(x)$ and $A^*_\mu(x)$. This leads to the following extended, but not yet fully gauge-fixed, action $S_{\text{ext}}$:

$$Z = \int [dA_\mu][da_\mu][d\psi_\mu][dA^*_\mu][db_\mu] \exp \left[ \frac{i}{\hbar} S_{\text{ext}} \right]$$

$$S_{\text{ext}} = S[A_\mu - a_\mu] - \int dx \{ b_\mu(x) a_\mu(x) + A^*_\mu(x) [\psi^\mu(x) - D^\mu_{(A-a)} c(x)] \} \tag{38}$$

This extended action is invariant under the BRST transformation (36). The full integration measure is also invariant. Of course, the expression above is still formal, since we have not yet gauge fixed ordinary Yang-Mills invariance.

Furthermore, we are eventually going to integrate over the ghost $c$, which already now appears in the extended action. If we insist that the Schwinger-Dyson equations involving this field, i.e., equations of the form

$$0 = \int [dc] \frac{\delta l}{\delta c(x)} \left[ F e^{i \delta \chi [\text{Action}]} \right] \tag{39}$$

are to be satisfied automatically (for reasonable choices of functionals $F$) by means of the full unbroken BRST algebra, we must introduce yet one more collective field. This new collective field, call it $\tilde{c}(x)$, is Grassmann odd, and has $gh(\tilde{c}) = 1$. Now shift the Yang-Mills ghost:

$$c(x) \to c(x) - \tilde{c}(x). \tag{40}$$

To fix the associated fermionic gauge symmetry, we introduce a new BRST multiplet of a ghost-antighost pair and an auxiliary field. We follow the same rule as before and let the transformation of the new collective field $\tilde{c}$ carry the (BRST) transformation of the original ghost $c$, viz.,

$$\delta c(x) = C(x)$$
$$\delta \tilde{c}(x) = C(x) + \frac{1}{2} [c(x) - \tilde{c}(x), c(x) - \tilde{c}(x)]$$
$$\delta C(x) = 0$$
$$\delta c^*(x) = B(x)$$
$$\delta B(x) = 0. \tag{41}$$
It follows from (41) that ghost number assignments should be as follows:

\[ gh(C) = 2, \quad gh(c^*) = -2, \quad gh(B) = -1, \quad (42) \]

and \( B \) is a fermionic Nakanishi-Lautrup field. Next, gauge-fix \( \tilde{c}(x) \) to zero by adding a term

\[ -\delta[c^*(x)\tilde{c}(x)] = B(x)\tilde{c}(x) - c^*(x)\{C(x) + \frac{1}{2}[c(x) - \tilde{c}(x), c(x) - \tilde{c}(x)]\} \quad (43) \]

to the Lagrangian. This gives us the fully extended action,

\[
S_{\text{ext}} = S[A_\mu - a_\mu] - \int dx \{b_\mu(x)a_\mu(x) + A_\mu^*(x)[\psi^\mu(x) - D_\mu^{(A-a)}(c(x) - \tilde{c}(x))] \\
- B(x)\tilde{c}(x) + c^*(x)(C(x) + \frac{1}{2}[c(x) - \tilde{c}(x), c(x) - \tilde{c}(x)])\},
\]

with the partition function so far being integrated over all fields appearing above, except for \( c \) (whose antighost \( \bar{c} \) still has to be introduced when we gauge-fix the original Yang-Mills symmetry).

The extended action and the functional measure is formally invariant under the following set of transformations:

\[
\begin{align*}
\delta A_\mu(x) &= \psi_\mu(x), & \delta \psi_\mu(x) &= 0, \\
\delta a_\mu(x) &= \psi_\mu(x) - D_\mu^{(A-a)}[c(x) - \tilde{c}(x)], & \delta c(x) &= C(x), \\
\delta A_\mu^*(x) &= b_\mu(x), & \delta b_\mu(x) &= 0, \\
\delta \tilde{c}(x) &= C(x) + \frac{1}{2}[c(x) - \tilde{c}(x), c(x) - \tilde{c}(x)], & \delta C(x) &= 0, \\
\delta c^*(x) &= B(x), & \delta B(x) &= 0.
\end{align*}
\]

As the notation indicates, the fields \( A_\mu^*(x) \) and \( c^*(x) \) can be identified with the Batalin-Vilkovisky antifields of \( A_\mu(x) \) and \( c(x) \), respectively. These antifields are the usual antighosts of the collective fields enforcing Schwinger-Dyson equations through shift symmetries.

Note that the general rule of assigning ghost number and Grassmann parity to the antifields,

\[
gh(\phi^*_A) = -(gh(\phi^A) + 1), \quad \epsilon(\phi^*_A) = \epsilon(\phi^A) + 1, \quad (45)
\]

arises in a completely straightforward manner. Here, it is a simple consequence of the fact that the BRST operator raises ghost number by one unit.
(and changes statistics), supplemented with the usual rule that antighosts have opposite ghost number of the ghosts.

The extended action (44) above has more fields than the extended action of Batalin and Vilkovisky, and the transformation laws of what we identify as antifields do not match those of ref. [1]. In the form we have presented it, the BRST symmetry is nilpotent also off-shell. If we integrate over the auxiliary fields $b_\mu$ and $B$ (and subsequently over $a_\mu$ and $\tilde{c}$) the BRST symmetry becomes nilpotent only on-shell. The extended action then takes the following form:

$$S_{ext} = S[A_\mu] - \int dx \{ A^*_\mu(x) [\psi_\mu(x) - D^\mu c(x)] + c^*(x) (C(x) + \frac{1}{2}[c(x), c(x)]) \}. \quad (46)$$

This differs from the extended action of Batalin and Vilkovisky by the terms involving the ghost fields $\psi_\mu(x)$ and $c^*(x)$. Comparing with the case of no gauge symmetries, this is exactly what we should expect. These ghost fields $\psi_\mu$ and $c^*$ ensure the correct Schwinger-Dyson equations for $A_\mu$ and $c$, respectively.

To find the corresponding BRST symmetry we use the equations of motion for the auxiliary fields $b_\mu$ and $B$, and use the $\delta$-function constraints on $a_\mu$ and $\tilde{c}$. This gives

$$\begin{align*}
\delta A_\mu(x) &= \psi_\mu(x) \\
\delta \psi_\mu(x) &= 0 \\
\delta c(x) &= C(x) \\
\delta C(x) &= 0 \\
\delta A^*_\mu(x) &= -\frac{\delta^r S_{ext}}{\delta A_\mu(x)} \\
\delta c^*(x) &= -\frac{\delta^r S_{ext}}{\delta c(x)}
\end{align*} \quad (47)$$

BRST invariance of the extended action (46) immediately implies that it satisfies a Master Equation we can write as

$$\int dx \frac{\delta^r S_{ext}}{\delta A^*_\mu(x)} \frac{\delta^r S_{ext}}{\delta A_\mu(x)} + \int dx \frac{\delta^r S_{ext}}{\delta c^*(x)} \frac{\delta^r S_{ext}}{\delta c(x)} = \int dx \frac{\delta^r S_{ext}}{\delta A_\mu(x)} \psi_\mu(x) + \int dx \frac{\delta^r S_{ext}}{\delta c(x)} C(x). \quad (48)$$

We are dropping the subscript on the covariant derivative since no confusion can arise at this point.
Note that this Master Equation precisely is of the form

$$\frac{1}{2}(S_{\text{ext}}^{\text{ext}}, S_{\text{ext}}^{\text{ext}}) = -\int dx \frac{\delta S_{\text{ext}}}{\delta \phi^A(x)} c^A(x), \quad (49)$$

with two ghost fields $c^A$ that are just $\psi_\mu$ and $C(x)$.

To finally make contact with the Batalin-Vilkovisky formalism, let us integrate out the ghost $\psi_\mu$. As in eq.(26), we shall use an identity of the form

$$\int [d\psi_\mu] F[\psi^\mu(y)] \exp \left[ -\frac{i}{\hbar} \int dx A_\mu^*(x) \{ \psi^\mu(x) - D_\mu c(x) \} \right]$$

$$= F \left[ D^\mu c(y) + (i\hbar) \frac{\delta l}{\delta A_\mu^*(y)} \right] \int [d\psi_\mu] \exp \left[ -\frac{i}{\hbar} \int dx A_\mu^*(x) \{ \psi^\mu(x) - D_\mu c(x) \} \right]$$

$$= \exp \left[ \frac{i}{\hbar} \int dx A_\mu^*(x) D^\mu c(x) \right] F \left[ (i\hbar) \frac{\delta l}{\delta A_\mu^*(y)} \right] \delta(A_\mu^*) \quad (50)$$

This shows that we should replace $\psi^\mu(x)$ by its equation of motion, plus the shown quantum correction of $O(\hbar)$ which then acts on the rest of the integral, or equivalently, by just the derivative operator (which then acts solely on the functional $\delta$-function). To get a useful representation of $\psi^\mu(y)$ we integrate the latter version of the identity by parts, thus letting the derivative operator act on everything except $\delta(A_\mu^*)$. This automatically brings down the equations of motion for $\psi^\mu$.

Having in this manner integrated out $\psi^\mu$ and $C$, the partition function reads

$$Z = \int [dA_\mu][dA_\mu^*][dc^*][d\phi^A]\delta(A_\mu^*) \delta(c^*) \exp \left[ \frac{i}{\hbar} S_{\text{ext}} \right]$$

$$S_{\text{ext}} = S[A_\mu] + \int dx \{ A_\mu^*(x) D^\mu c(x) - \frac{1}{2} c^*[c(x), c(x)] \} \quad (51)$$

and the classical BRST symmetry follows as discussed above by substituting only the equations of motion for $\psi_\mu$ and $C$:

$$\delta A_\mu(x) = \frac{\delta l}{\delta A_\mu^*(x)} = D_\mu c(x)$$

$$\delta c(x) = \frac{\delta l}{\delta c^*(x)} - \frac{1}{2} [c(x), c(x)]$$
\[
\begin{align*}
\delta A^\ast_\mu(x) &= -\frac{\delta!S_{\text{ext}}}{\delta A_\mu(x)} \\
\delta c^\ast(x) &= -\frac{\delta!S_{\text{ext}}}{\delta c(x)}.
\end{align*}
\]

(52)

This is the usual extended action of Batalin and Vilkovisky and the corresponding classical BRST symmetry. Of course, in the partition function the integrals over \(A^\ast_\mu\) and \(c^\ast\) are trivial. This is as it should be, because by integrating out these antighosts we should finally recover the starting measure and the still not gauge-fixed Yang-Mills action. What we have provided here is thus only a very precise functional derivation of the extended action. It shows that we can understand the extended action in the usual path integral framework, and that the integration measures for \(A^\ast_\mu\) and \(c^\ast\) (with the accompanying \(\delta\)-function constraints) are provided automatically.

Obviously, the extended action is not yet very useful from the point of view of ordinary BRST gauge fixing. To unravel some of the mechanisms behind the Batalin-Vilkovisky scheme, it is nevertheless advantageous to keep the antifields. In fact, it is even more useful to return to the formulation in eq. (46), where there is yet no split into a classical and a quantum part of the symmetry. Let us therefore take (46) as the starting action, and now just gauge-fix in a standard manner the Yang-Mills symmetry. Choosing, e.g., a covariant gauge, we therefore finally extend the BRST multiplet to include a Yang-Mills antighost \(\bar{c}\) and a Nakanishi-Lautrup scalar \(b\). For there to be no doubt, let us also note that these fields have

\[
gh(\bar{c}) = -1, \quad gh(b) = 0.
\]

(53)

The BRST transformations are the usual \(\delta \bar{c}(x) = b(x)\) and \(\delta b(x) = 0\). Gauge fixing to a covariant \(\alpha\)-gauge can be achieved by adding a term

\[
\delta[\bar{c}(x)\{\partial_\mu A^\mu(x) - \frac{1}{2\alpha}b(x)\}] = b(x)\partial_\mu A^\mu(x) + \bar{c}(x)\partial_\mu\psi^\mu(x) - \frac{1}{2\alpha}b(x)^2
\]

(54)
to the Lagrangian. The corresponding completely gauge-fixed extended action then reads

\[
S_{\text{ext}} = S[A_\mu] - \int dx \{A^\ast_\mu(x)[\psi^\mu(x) - D^\mu c(x)] + c^\ast(x)(C(x) + \frac{1}{2}[c(x), c(x)])
- b(x)\partial_\mu A^\mu(x) - \bar{c}(x)\partial_\mu\psi^\mu(x) + \frac{1}{2\alpha}b(x)^2\}.
\]

(55)
Now integrate out $\psi^\mu$ and $C$. The result is indeed a partition function of the form introduced by the formalism of Batalin and Vilkovisky:

$$Z = \int [dA^\mu] [dA^\ast] [d\bar{c}] [dc^\ast] [db] \delta(A^\ast_\mu + \partial_\mu \bar{c}) \delta(c^\ast) e^{\int S_{ext}}$$

$$S_{ext} = S[A_\mu] + \int dx \{ A^\ast_\mu(x) D^\mu c(x) - \frac{1}{2} c^\ast(x) [c(x), c(x)] + b(x) \partial_\mu A^\mu(x) - \frac{1}{2\alpha} b(x)^2 \}.$$ \hspace{1cm} (56)

Note that by adding the Yang-Mills gauge-fixing terms, the $\delta$-function constraint on the antifield $A^\ast_\mu$ has been shifted. Thus when doing the $A^\ast_\mu$-integral, we are in effect substituting not $A^\ast_\mu(x) = 0$ but

$$A^\ast_\mu(x) = -\partial_\mu \bar{c}(x) = \frac{\delta^\ast \Psi}{\delta A^\mu(x)}, \hspace{1cm} (57)$$

where $\Psi$ is defined as the term whose BRST variation is added to the action, $i.e.$, in this particular case,

$$\Psi = \int dx \{ \bar{c}(x)(\partial_\mu A_\mu(x) - \frac{1}{2\alpha} b(x)) \}.$$ \hspace{1cm} (58)

Upon doing the $A^\ast_\mu$ and $c^\ast$ integrals, one recovers the standard covariantly gauge-fixed Yang-Mills theory

$$S = S[A_\mu] + \int dx \{ \bar{c}(x) \partial^\mu D_\mu c(x) + b(x) \partial^\mu A_\mu(x) - \frac{1}{2\alpha} b(x)^2 \}.$$ \hspace{1cm} (59)

The identification (57) strongly suggests that one can see gauge fixing as a particular canonical transformation involving new fields (the antihost $\bar{c}$). But there are terms in eq. (59) (those involving $b(x)$) which do not immediately follow from this perspective. In the Batalin-Vilkovisky framework, this is resolved by noting that one can always add terms of new fields and antifields with trivial antibrackets. In this version of the gauge-fixing procedure, one returns to the (“minimally”) extended action of eq. (56) and extends it in a “non-minimal” way. In the Yang-Mills case, this includes an additional term in the action of the form

$$S_{nm} = \int dx c^\ast(x) b(x), \hspace{1cm} (60)$$
with $\bar{c}^*$ and $b$ having the same ghost number and Grassmann parity:

$$gh(\bar{c}^*) = gh(b) = 0 ; \quad \epsilon(\bar{c}^*) = \epsilon(b) = 0 .$$  \hspace{1cm} (61)

As the notation indicates, these new fields $\bar{c}^*$ and $b$ are indeed just the anti-field of $\bar{c}$, and the usual Nakanishi-Lautrup field, respectively.

Gauge fixing to the same gauge as above can then be achieved by the same gauge fermion (58) which now affects both $A_\mu^*$ and $\bar{c}^*$ within the antibracket. It can thus be seen as the canonical transformation that shifts $A_\mu^*$ and $\bar{c}^*$ from zero to

$$A_\mu^*(x) = \frac{\delta^* \Psi}{\delta A^\mu(x)} , \quad \bar{c}^*(x) = \frac{\delta^* \Psi}{\delta \bar{c}(x)} .$$  \hspace{1cm} (62)

Since $\Psi$ does not depend on the antifields, this canonical transformation leaves all fields $A_\mu, c$ and $\bar{c}$ unchanged.

Can we understand the non-minimally extended action from our point of view too? Consider the stage at which we introduce the antighost $\bar{c}$. This field does not yet appear in the action, but we can of course still introduce a corresponding collective “shift” field $\tilde{c}'$ for $\bar{c}$ as well. The corresponding BRST multiplet consists of a new “shift-antighost” $\lambda(x)$, an “anti-antighost” $\bar{c}^*(x)$, and the associated auxiliary field $B'(x)$:

$$\begin{align*}
d\bar{c}(x) &= \lambda(x) \\
d\tilde{c}'(x) &= \lambda(x) - b(x) \\
d\lambda(x) &= 0 \\
db(x) &= 0 \\
d\bar{c}'(x) &= B'(x) \\
dB'(x) &= 0 .
\end{align*}$$  \hspace{1cm} (63)

The assignments will then have to be exactly as in eq. (61), supplemented with $gh(B') = \epsilon(B') = 1$. We are again dealing with two symmetries, because the shifted field $\bar{c}(x) - \tilde{c}'(x)$ itself can still be shifted by the usual Nakanishi-Lautrup field. Let us now gauge-fix this large symmetry. We do it in the most simple manner by adding a term

$$- \delta[\bar{c}^*(x)\tilde{c}'(x)] = B'(x)\tilde{c}'(x) - \bar{c}'(x)(\lambda(x) - b(x))$$  \hspace{1cm} (64)

to the Lagrangian. The integrals over $B'$ and $\tilde{c}'$ are of course trivial and we are left with the non-minimally extended action for this theory plus, as
expected, the corresponding term with the new ghost $\lambda$. The final gauge-fixing of the Yang-Mills symmetry will now consist in adding, instead of eq. (54),

$$\delta[\bar{c}(x)\{\partial_\mu A_\mu(x) - \frac{1}{2\alpha}b(x)\}] = \lambda(x)\partial_\mu A_\mu(x) + \bar{c}(x)\partial_\mu \psi^\mu(x) - \frac{1}{2\alpha}\lambda(x)b(x) .$$

(65)

Before Yang-Mills gauge fixing, the integral over $\lambda(x)$ just gave a factor of $\delta(\bar{c}^*)$. After adding the gauge-fixing term, this is modified:

$$\delta(\bar{c}^*) \to \delta(\bar{c}^*(x) - \partial_\mu A_\mu(x) + \frac{1}{2\alpha}b(x)) .$$

(66)

Substituting this back into the extended action, we recover the result (59). Note that this indeed can be viewed as a canonical transformation within the antibracket. All the correct $\delta$-function constraints are provided by the collective fields and their ghosts. Since the functional $\Psi$ has been chosen to depend only on the fundamental fields, and not on the antifields, the fields $A_\mu, c$ and $\bar{c}$ are all left untouched by this canonical transformation. Extending the action from the minimal to the non-minimal case is equivalent to demanding that also Schwinger-Dyson equations for $\bar{c}(x)$ follow as Ward identities of the BRST symmetry. Since the antighost $\bar{c}$ remains in the path integral after gauge fixing, it would indeed be very unnatural not to demand that correct Schwinger-Dyson equations for this field follow as well. As shown, this requirement automatically leads to the non-minimally extended action.

As for the functional measure, we have stressed earlier that we always assume the existence of a suitable regulator that preserves the pertinent BRST symmetry. We can make this statement a little more explicit by detailing the required symmetries of the measure in this Yang-Mills case. Before integrating out any fields, the measures for $A_\mu, c$ and $\bar{c}$ should all be invariant under local shifts. For $A_\mu$ this corresponds to the usual euclidean measure (and it is very difficult to imagine this shift symmetry being broken by any reasonable regulator), while for $c$ and $\bar{c}$ this is consistent with the usual rules of Berezin integration. The measures for the three collective fields should in addition be invariant under what corresponds to usual Yang-Mills BRST transformations, a property that indeed holds formally. Finally, the measures for all antifields are only required to be invariant under local shifts. After having integrated out the auxiliary fields $B_A$, invariance of these
measures of the antifields is now non-trivial but one can check explicitly that it is formally satisfied. This should indeed be the case, because it is straightforward to check that the action remains invariant. Since at least the combination of measure and action must remain invariant after integrating out some of the fields, invariance of the measure is in this case formally guaranteed.

Let us finally point out that once the antighost $\bar{c}$ is being treated on equal footing with $A_\mu$ and $c$, a Master Equation of the form

$$\frac{1}{2}(S_{\text{ext}}, S_{\text{ext}}) = - \int dx \frac{\delta S_{\text{ext}}}{\delta \phi^A(x)} c^A(x)$$

(67)

now holds with $\phi^A$ denoting all the fields that finally remain in the path integral: $A_\mu, c$ and $\bar{c}$. Similarly, the BRST algebra becomes, upon integrating out the collective fields $\varphi^A$ and the auxiliary fields $B_A$, of the very simple form (21) we encountered already in the case of no internal gauge symmetries. The general case of arbitrary gauge symmetries is considered in [5].

2.3 Quantum Master Equations

We have seen from the collective field method that for closed irreducible gauge algebras we get an extended action that can be split into a part independent of the new ghosts $c^A$, and a simple quadratic term of the form $\phi^*_A c^A$. Let us, for reasons that will become evident shortly, denote the part which is independent of $c^A$ by $S^{(BV)}$, i.e.:

$$S_{\text{ext}}[\phi, \phi^*, c] = S^{(BV)}[\phi, \phi^*] - \phi^*_A c^A.$$  

(68)

This action is invariant under the transformations

$$\delta \phi^A = c^A$$

$$\delta c^A = 0$$

$$\delta \phi^*_A = - \frac{\delta S_{\text{ext}}}{\delta \phi^A}.$$  

(69)

Moreover, the functional measure is also formally guaranteed to be invariant in this case. It follows that in this case the Ward Identities of the kind $0 = \langle \delta[\phi^*_A F[\phi]] \rangle$ are the most general Schwinger-Dyson equations for the quantum theory defined by the classical action $S[\phi]$.  

20
But demanding that both the action $S_{\text{ext}}$ and the functional measure be invariant under the BRST Schwinger-Dyson symmetry above is not the most general condition. To derive the correct Ward Identities we only need that just the combination of action and measure is invariant. In this subsection we want to discuss the more general case in which the set of transformations (69) still generate a symmetry of the combination of measure and action, but not of each individually. If we insist on a solution of the form (68), then the other property that is required, $\langle c^A \phi_B^* \rangle = -i \hbar \delta_{AB}^A$, follows automatically.

It thus remains to be found under what conditions the combination of the action and the measure remain invariant under the BRST Schwinger-Dyson symmetry. With an action $S_{\text{ext}}$ of the form (68), we get

$$
\delta S_{\text{ext}} = \frac{\delta^r S^{(BV)}}{\delta \phi_A} c^A + \frac{\delta^r S^{(BV)}}{\delta \phi_A^*} \left( -\frac{\delta^l S_{\text{ext}}}{\delta \phi_A} - \frac{\delta^l (\phi_A^* c^A)}{\delta \phi_A} - \frac{\delta^l S_{\text{ext}}}{\delta \phi_A^*} - \frac{\delta^l (\phi_A^* c^A)}{\delta \phi_A^*} \right)
$$

$$
= \frac{\delta^r S^{(BV)}}{\delta \phi_A^*} c^A + \frac{\delta^r S^{(BV)}}{\delta \phi_A} \left( -\frac{\delta^l S^{(BV)}}{\delta \phi_A} - (-1)^{\epsilon_A+1} c^A \right)
$$

$$
= -\frac{\delta^r S^{(BV)}}{\delta \phi_A^*} \frac{\delta^l S^{(BV)}}{\delta \phi_A} = \frac{1}{2} \left( S^{(BV)}, S^{(BV)} \right).
$$

(70)

We will still assume that we are integrating over a flat euclidean measure for the fundamental field $\phi^A$. This measure is formally invariant under the transformation (69). However, for a corresponding flat euclidean measure for $\phi_A^*$, the Jacobian of the transformation (69) will in general be different from unity. As we already discussed above, the Jacobian equals

$$
J = 1 - \frac{\delta^r}{\delta \phi_A^*} \left( \frac{\delta^l S_{\text{ext}}}{\delta \phi_A}(x)^\mu \right) .
$$

(71)

Thus to demand that the combination of measure and action remains invariant, we must in general require that

$$
\frac{1}{2} (S_{\text{ext}}, S_{\text{ext}}) = -\frac{\delta^r S_{\text{ext}}}{\delta \phi_A} c^A + i \hbar \Delta S_{\text{ext}} ,
$$

(72)

which, assuming the form (68) – since we know that this is sufficient to guarantee the correct Schwinger-Dyson equations – reduces to the quantum Master Equation of Batalin and Vilkovisky:

$$
\frac{1}{2} (S^{(BV)}, S^{(BV)}) = i \hbar \Delta S^{(BV)} .
$$

(73)
Let us emphasize that this equation follows even before possible gauge fixings. It is required in order that the general Schwinger-Dyson equations for the fundamental fields are satisfied, and is not postulated on only the requirement that the final functional integral be independent of the gauge-fixing function. However, gauge independence of the functional integral upon the addition of a term of the form $\delta \Psi[\phi]$ now follows straightforwardly, since for a functional $\Psi$ that depends only on the fields $\phi$, we have $\delta^2 \Psi[\phi] = 0$.

### 2.4 Quantum BRST

In subsection A we noted that the usual BRST Schwinger-Dyson symmetry acquires a “quantum correction” if one insists on using the formalism where the new ghost fields $c^A$ have been integrated out of the path integral. As we saw already in the case of no gauge symmetries, this deforms the BRST operator:

$$\delta \rightarrow \sigma = \delta - i\hbar \Delta.$$  \hfill (74)

The notation is not entirely precise, because the operator $\delta$ on the right hand side of this equation of course equals the operator $\delta$ on the left hand side only modulo those changes incurred by integrating out the ghosts $c^A$. But we keep it like this to avoid complications in the notation. After having integrated out the ghosts $c^A$, the BRST operator $\delta$ will become identical to the variation within the antibracket.

Since this quantum deformation involves the same operator $\hbar \Delta$ that in certain specific cases may modify the classical Master Equation, one might be led to believe that these two issues are related, i.e., that the “quantum BRST” operator should only be applied when there are, or as a consequence of having) quantum corrections in the full gauge-fixed action. This is actually not the case, and we therefore find it useful to return briefly to the meaning of the quantum BRST operator, here denoted by $\sigma$.

Let us again choose the simplest solution to the Master Equation of the form (68). We emphasize that it is immaterial whether this extended action $S_{ext}$ satisfies the classical or quantum Master Equations. Since we are interested in seeing the effect of integrating out the ghosts $c^A$, consider, as in subsection A, the expectation value of the BRST variation of an arbitrary
The derivation given here corresponds to the path integral before gauge fixing, but it goes through in entirely the same manner in the gauge-fixed case. (The only difference is that the relevant \(\delta\)-function reads \(\delta(\phi^* - \delta \Psi/\delta \phi)\) instead of \(\delta(\phi^*)\); this does not affect the manipulations above).

The emergence of the “quantum correction” in the BRST operator is thus completely independent of the particular solution \(S^{(BV)}[\phi, \phi^*]\); it must always be included when one uses the formalism in which the ghosts \(c_A\) have been integrated out. The quantum BRST operator \(\sigma\) is unusual, because it appears only after functional manipulations inside the path integral.

Since by construction the partition function is invariant under \(\delta\) (when keeping the ghosts \(c_A\)) and \(\sigma\) (after having integrated out these ghosts), it follows that all expectation values involving these operators vanish:

\[
\langle \delta G[\phi, \phi^*] \rangle = 0 \quad \text{when keeping } c_A, \quad \langle \sigma G[\phi, \phi^*] \rangle = 0 \quad \text{when the } c_A \text{ have been integrated out.}
\]

This of course holds for the action as well:

\[
\langle \delta S_{\text{ext}} \rangle = 0 ; \quad \langle \sigma S^{(BV)} \rangle = 0 .
\]

The first of these equations is trivially satisfied when \(S_{\text{ext}}\) satisfies the classical Master Equation, because then the variation \(\delta S_{\text{ext}}\) itself vanishes. This equation is then only non-trivially satisfied when \(\Delta S_{\text{ext}} \neq 0\).
Since the two operations $\delta$ and $\sigma$ are equivalent in the precise sense given above, the same considerations should apply to the second equation. Indeed it does: When $S^{(BV)}$ satisfies the classical Master Equation, $\sigma S^{(BV)} = 0$ at the operator level, while that equation is satisfied only in terms of expectation values when $\Delta S^{(BV)} \neq 0$.

Note that when $\Delta S_{\text{ext}} \neq 0$ (or $\Delta S^{(BV)} \neq 0$), the quantum action is neither invariant under $\delta$ nor $\sigma$. The action precisely has to remain non-invariant in order to cancel the non-trivial contribution from the measure in that case. This is the origin of the factor $1/2$ difference between the quantum Master Equation

$$\frac{1}{2}(S^{(BV)}, S^{(BV)}) - i\hbar \Delta S^{(BV)} = 0 \quad (79)$$

and the operator $\sigma$ (when acting on $S^{(BV)}$):

$$\langle (S^{(BV)}, S^{(BV)}) - i\hbar \Delta S^{(BV)} \rangle = 0 . \quad (80)$$

The combination of these two equations yields the new identities

$$\langle \Delta S^{(BV)} \rangle = 0 , \quad \langle (S^{(BV)}, S^{(BV)}) \rangle = 0 \quad (81)$$

which can also be verified directly using the path integral.

The operator $\delta$ defines a BRST cohomology only on the subspace of fields $\phi^A$; it is only nilpotent on that subspace. The operator $\sigma$ is nilpotent in general: $\sigma^2 = 0$ (a consequence of having performed partial integrations in deriving it). However, the two operators share the same physical content.

3 Generalizations of the Batalin-Vilkovisky formalism

In order to see how the conventional antibracket formalism of Batalin and Vilkovisky can be generalized, it is important to have a fundamental principle from which this formalism can be derived. As has been discussed in the previous sections, this principle is that Schwinger-Dyson BRST symmetry must be imposed on the full path integral.

Schwinger-Dyson BRST symmetry can be derived from the local symmetries of the given path integral measure. When the measure is flat, the relevant
symmetry is that of local shifts, and the resulting Schwinger-Dyson BRST symmetry leads directly to a quantum Master Equation on the action $S$ which is exponentiated inside the path integral. This action depends on two new sets of ghosts and antighosts, $c^A$ and $\phi^*_A$ [5]. The conventional Batalin-Vilkovisky formalism for an action $S_{BV}$ follows if one substitutes $S[\phi, \phi^*, c] = S_{BV}[\phi, \phi^*] - \phi^*_A c^A$ and integrates out the ghosts $c^A$. The so-called “antifields” of the Batalin-Vilkovisky formalism are simply the Schwinger-Dyson BRST antighosts $\phi^*_A$ [5].

It is of interest to see what happens if one abandons the assumption of flat measures for the fields $\phi^A$, and if one does not restrict oneself to local transformations that leave the functional measure invariant. Some steps in this direction were recently taken in ref. [6]. One here exploits the reparametrization invariance encoded in the path integral by performing field transformations $\phi^A = g^A(\phi', a)$ depending on new fields $a_i$. It is natural to assume that these transformations form a group, or more precisely, a quasigroup [13]. The objects

$$u^A_i \equiv \left. \frac{\delta g^A}{\delta a^i} \right|_{a=0}$$

are gauge generators of this group. They satisfy

$$\frac{\delta^r u^A_i}{\delta \phi^B} u^B_j - (-1)^{\epsilon_i \epsilon_j} \frac{\delta^r u^A_j}{\delta \phi^B} u^B_i = -u^A_k U^k_{ij},$$

where the $U^k_{ij}$ are structure “coefficients” of the group. They are supernumbers with the property

$$U^k_{ij} = (-1)^{\epsilon_i \epsilon_j} U^k_{ji}. \quad (84)$$

In ref. [6], specializing to compact supergroups for which $(-1)^{\epsilon_i} U^i_{ij} = 0$, the following $\Delta$-operator was derived:

$$\Delta G \equiv (-1)^{\epsilon_i} \left[ \frac{\delta^r}{\delta \phi^A} \frac{\delta^r}{\delta \phi^*_i} G \right] u^A_i + \frac{1}{2} (-1)^{\epsilon_i + 1} \left[ \frac{\delta^r}{\delta \phi^*_j} \frac{\delta^r}{\delta \phi^*_i} G \right] \phi^*_k U^k_{ji}. \quad (85)$$

Footnote 4: See the 2nd reference in [6]. This is related to the covariant formulations of the antibracket formalism [12].
When the coefficients $U_{ij}^k$ are constant, this $\Delta$-operator is nilpotent: $\Delta^2 = 0$. As noted by Koszul [14], and rediscovered by Witten [15], one can define an antibracket $(F,G)$ by the rule

$$\Delta(FG) = F(\Delta G) + (-1)^{\epsilon_G}(\Delta F)G + (-1)^{\epsilon_G}(F,G).$$

Explicitly, for the case above, this leads to the following new antibracket [6]:

$$(F,G) \equiv (-1)^{\epsilon_G} \epsilon_{i} \left( \epsilon_{i} \epsilon_{A+1} \right) \frac{\delta^r F}{\delta \phi_i^*} u_i^A \frac{\delta^l G}{\delta \phi_i} - \frac{\delta^r F}{\delta \phi_i^*} u_i^A \frac{\delta^l G}{\delta \phi_i} + \frac{\delta^r F}{\delta \phi_i^*} c_k U_{ij}^k \frac{\delta^l G}{\delta \phi_j^*}$$

This antibracket is statistics-changing, $\epsilon((F,G)) = \epsilon(F) + \epsilon(G) + 1$, and has the following properties:

$$\begin{align*}
(F,G) &\equiv (-1)^{\epsilon_F \epsilon_G + \epsilon_F + \epsilon_G} (G,F), \\
(F,GH) &\equiv (F,G)H + (-1)^{\epsilon_G(\epsilon_F+1)} G(F,H), \\
(FG,H) &\equiv F(G,H) + (-1)^{\epsilon_G(\epsilon_H+1)} (F,H)G, \\
0 &\equiv (-1)^{\epsilon_F+1}(\epsilon_H+1)(F,(G,H)) + \text{cyclic perm.}.
\end{align*}$$

Furthermore,

$$\Delta(F,G) = (F,\Delta G) - (-1)^{\epsilon_G}(\Delta F,G).$$

The $\Delta$ given in eq. (85) is clearly a non-Abelian generalization of the conventional $\Delta$-operator of the Batalin-Vilkovisky formalism.

We shall now show how to extend this construction to the general case of non-linear and open algebras. Recently, interest in more complicated algebras such as strongly homotopy Lie algebras [16] has arisen in the context of string field theory [17].

Consider the quantized Hamiltonian BRST operator $\Omega$ for first-class constraints with an arbitrary, possibly open, gauge algebra [18]. Apart from a set of phase space operators $Q^i$ and $P_i$, introduce a ghost pair $\eta_i, \bar{p}_i$. They

\footnotetext{5}{For a comprehensive review of the classical Hamiltonian BRST formalism, see, e.g., ref. [19].}
have Grassmann parities $\epsilon(\eta^i) = \epsilon(P^i) = \epsilon(Q^i) + 1 \equiv \epsilon_i + 1$, and are canonically conjugate with respect to the usual graded commutator:

$$[\eta^i, P^j] = \eta^i P^j - (-1)^{(\epsilon_i + 1)(\epsilon_j + 1)} P^j \eta^i = i \delta^i_j . \tag{92}$$

In addition $[\eta^i, \eta^j] = [P_i, P_j] = 0$. The quantum mechanical BRST operator can then be written in the form of a $P \eta$ normal-ordered expansion in powers of the $P$’s [18]:

$$\Omega = G_i \eta^i + \sum_{n=1}^{\infty} P_{i_1} \cdots P_{i_n} U_{i_1 \cdots i_n} . \tag{93}$$

Here

$$U_{i_1 \cdots i_n} = \frac{(-1)^{\epsilon_{i_1 \cdots i_{n-1}}}}{(n+1)!} U_{j_1 \cdots j_{n+1}} \eta^{j_{n+1}} \cdots \eta^{j_1} , \tag{94}$$

and the sign factor is defined by:

$$\epsilon_{i_1 \cdots i_{n-1}} = \sum_{k=1}^{n-1} \sum_{l=1}^{k} \epsilon_{i_l} + \sum_{k=1}^{n} \sum_{l=1}^{k} \epsilon_{j_l} . \tag{95}$$

The $U_{j_1 \cdots j_{n+1}}$’s are generalized structure coefficients. For rank-1 theories the expansion ends with the 2nd term, involving the usual Lie algebra structure coefficients $U^k_{ij}$. The number of terms that must be included in the expansion of eq. (93) increases with the rank. By construction $\Omega^2 = 0$.

The functions $G_i$ appearing in eq. (93) are the constraints. In the quantum case they satisfy the constraint algebra

$$[G_i, G_j] = i G_k U^k_{ij} . \tag{96}$$

We choose these to be the ones associated with motion on the supergroup manifold defined by the transformation $\phi^A = g^A(\phi', a)$.

When considering representations of the (super) Heisenberg algebra [92], one normally chooses the operators to act to the right. Thus, in the ghost coordinate representation we could take

$$P_j = i(-1)^{\epsilon_j} \frac{\delta^j}{\delta \eta^j} . \tag{97}$$
and similarly in the ghost momentum representation we could take

\[ \eta^j = i \frac{\delta l}{\delta P_j}. \quad (98) \]

On the other hand, the most convenient representation of the constraint \( G_j \) is

\[ \tilde{G}_j = -i \frac{\delta r}{\delta \phi^A} u^A_j, \quad (99) \]

which involves a right-derivative acting to the left. Using eq. (83), \( \tilde{G}_j \) is seen to satisfy

\[ [\tilde{G}_i, \tilde{G}_j] = i \tilde{G}_k U^k_{ij}. \quad (100) \]

Since we wish \( \Omega \) of eq. (93) to act in a definite way, we choose representations of the (super) Heisenberg algebra (92) that involve operators acting to the left as well. These are

\[ \tilde{P}_j = i \frac{\delta r}{\delta \eta^j}, \quad (101) \]

in the ghost coordinate representation, and

\[ \tilde{\eta}_j = i (-1)^{\epsilon_j} \frac{\delta r}{\delta P_j}, \quad (102) \]

in the ghost momentum representation. Inserting these operators into eq. (93) will give the corresponding BRST operator \( \tilde{\Omega} \) acting to the left. We now identify the ghost momentum \( P_j \) with the Lagrangian antighost ("antifield") \( \phi^*_j \).

As a special case, consider the operator \( \tilde{\Omega} \) in the case of an ordinary rank-1 super Lie algebra for which \( (-1)^{\epsilon_i} U^i_{ij} = 0 \). In the ghost momentum representation \( \tilde{\Omega} \) takes the form

\[ \tilde{\Omega} = (-1)^{\epsilon_i} \frac{\delta r}{\delta \phi^A} u^A_i \frac{\delta r}{\delta \phi^*_i} - \frac{1}{2} (-1)^{\epsilon_i} \phi^*_k U^k_{ij} \frac{\delta r}{\delta \phi^*_j} \frac{\delta r}{\delta \phi^*_i}. \quad (103) \]
One notices that the $\Omega$ of the above equation coincides with our non-Abelian $\Delta$-operator of eq. (33). In detail:

$$\Delta F \equiv F\Omega.$$  \hspace{1cm} (104)

For a rank-0 algebra – the Abelian case – we get, with the same identification,

$$\Omega = (-1)^{\epsilon A} \frac{\delta^r}{\delta \phi^A} \frac{\delta^r}{\delta \phi_A^*}.$$  \hspace{1cm} (105)

The associated $\Delta$-operator, defined through eq. (104) is seen to agree with the $\Delta$ of the conventional Batalin-Vilkovisky formalism eq.(9).

We define the general $\Delta$-operator through the identification (104) and the complete expansion

$$\Omega = (-1)^i \frac{\delta^r}{\delta \phi^A} u_i^A \frac{\delta^r}{\delta \phi_i^*} + \sum_{n=1}^{\infty} \phi_{i_n}^* \cdots \phi_{i_1}^* U_{i_1 \cdots i_n}.$$  \hspace{1cm} (106)

Here

$$U_{i_1 \cdots i_n} = \frac{(-1)^{\epsilon_{i_1 \cdots i_n}}}{(n+1)!} (i)^{n+1} \frac{\delta^r}{\delta \phi_{j_1}^*} \cdots \frac{\delta^r}{\delta \phi_{j_n}^*}.$$  \hspace{1cm} (107)

By construction we then have $\Delta^2 = 0$.

It is quite remarkable that the above derivation, based on Hamiltonian BRST theory in the operator language, has a direct counterpart in the Lagrangian path integral. The two simplest cases, that of rank-0 and rank-1 algebras have been derived in detail in the Lagrangian formalism in ref. [6]. It is intriguing that completely different manipulations (integrating out the corresponding ghosts $c_i$, and partial integrations inside the functional integral) in the Lagrangian framework leads to these quantized Hamiltonian BRST operators. The rank-0 case, that of the conventional Batalin-Vilkovisky formalism, corresponds to the gauge generators

$$\hat{G}_A = -i \frac{\delta^r}{\delta \phi^A}.$$  \hspace{1cm} (108)
These are generators of translations: when the functional measure is flat, the Schwinger-Dyson BRST symmetry is generated by local translations. The non-Abelian generalizations correspond to imposing different symmetries as BRST symmetries in the path integral [6].

These non-Abelian BRST operators $\tilde{\Omega}$ can be Abelianized by canonical transformations involving the ghosts [20], but the significance of this in the present context is not clear. Since in general the number of “antifields” $\phi^*_i$ will differ from that of the fields $\phi^A$, it is obvious that $u^A_i$ in general will be non-invertible. Even when the number of antifields matches that of fields, the associated matrix $u^A_B$ may be non-invertible (“degenerate”).

Having the general $\Delta$-operator available, the next step consists in extracting the associated antibracket. By the definition (86), this antibracket measures the failure of $\Delta$ to be a derivation. When $\Delta$ is a second-order operator, the antibracket so defined will itself obey the derivation rule (89). For higher-order $\Delta$-operators this is no longer the case. The antibracket will then in all generality only obey the much weaker relation

$$(F, GH) = (F, G)H - (-1)^{e_G} F(G, H) + (-1)^{e_G} (FG, H).$$

(109)

The relation (111) also holds in all generality. When the $\Delta$-operator is of order three or higher, the antibracket defined by (86) will not only fail to be a derivation, but will also violate the Jacobi identity (10).

For higher-order $\Delta$-operators one can, as explained by Koszul [14], use the failure of the antibracket to be a derivation to define higher antibrackets. These are Grassmann-odd analogues of Nambu brackets [23, 24]. The construction is most conveniently done in an iterative procedure, starting with the $\Delta$-operator itself [14, 25]. To this end, introduce objects $\Phi^n_\Delta$ which are defined as follows:

$$\Phi^1_\Delta(A) = (-1)^{e_A} \Delta A$$

6 In the special case where $u^A_B$ is invertible, the transformation $\phi^*_i \to \phi^*_i (u^{-1})^B_A$ makes the corresponding $\Delta$-operator Abelian [11], but we are not interested in that case here. See also refs. [21, 22].

7 Note that our definitions differ slightly from ref. [14, 23] due to our $\Delta$-operators being based on right-derivatives, while those of ref. [14, 25] are based on left-derivatives.
\[
\begin{align*}
\Phi^2_\Delta(A, B) &= \Phi^1_\Delta(AB) - \Phi^1_\Delta(A)B - (-1)^{\epsilon_A} A\Phi^1_\Delta(B) \\
\Phi^3_\Delta(A, B, C) &= \Phi^2_\Delta(ABC) - \Phi^2_\Delta(A, B)C - (-1)^{\epsilon_B(\epsilon_A+1)} B\Phi^2_\Delta(A, C) \\
\Phi^{n+1}_\Delta(A_1, \ldots, A_{n+1}) &= \Phi^n_\Delta(A_1, \ldots, A_n A_{n+1}) - \Phi^n_\Delta(A_1, \ldots, A_n)A_{n+1} \\
&\quad - (-1)^{\epsilon_{A_n}(\epsilon_{A_1}+\cdots+\epsilon_{A_{n-1}}+1)} A_n \Phi^n_\Delta(A_1, \ldots, A_{n-1}, A(1\bar{1}))
\end{align*}
\]

The \(\Phi^n_\Delta\)'s define the higher antibrackets. For example, the usual antibracket is given by
\[
(A, B) \equiv (-1)^{\epsilon_A} \Phi^2_\Delta(A, B) .
\] (111)

The iterative procedure clearly stops at the first bracket that acts like a derivation. For example, the “three-antibracket” defined by \(\Phi^3_\Delta(A, B, C)\) directly measures the failure of \(\Phi^2_\Delta\) to act like a derivation. But more importantly, it also measures the failure of the usual antibracket to satisfy the graded Jacobi identity:
\[
\sum_{\text{cycl.}} (-1)^{(\epsilon_A+1)(\epsilon_C+1)}(A, (B, C)) \quad = \quad (-1)^{\epsilon_A(\epsilon_C+1)+\epsilon_B+\epsilon_C} \Phi^4_\Delta(A, B, C) \\
&\quad + \sum_{\text{cycl.}} (-1)^{\epsilon_A(\epsilon_C+1)+\epsilon_B+\epsilon_C} \Phi^3_\Delta(\Phi^1_\Delta(A), B(1\bar{1}))
\]

and so on for the higher brackets.

When there is an infinite number of higher antibrackets, the associated algebraic structure is analogous to a strongly homotopy Lie algebra \(L_\infty\). The \(L_1\) algebra is then given by the nilpotent \(\Delta\)-operator, the \(L_2\) algebra is given by \(\Delta\) and the usual antibracket, the \(L_3\) algebra by these two and the additional “three-antibracket”, etc. The set of higher antibrackets defined above seems natural in closed string field theory [17], the corresponding \(\Delta\)-operator being given by the string field BRST operator \(Q\).

Having constructed the \(\Delta\)-operator (and its associated hierarchy of antibrackets), it is natural to consider a quantum Master Equation of the form
\[
\Delta \exp \left[ \frac{i}{\hbar} S(\phi, \phi^*) \right] = 0 .
\] (113)
Using the properties of the $\Phi^n$'s defined above, we can write this Master Equation as a series in the higher antibrackets,

$$\sum_{k=1}^{\infty} \left( \frac{i}{\hbar} \right)^k \Phi^k(S, S, \ldots, S) \frac{k!}{(k-1)!} = 0 ,$$  \hspace{1cm} (114)

where each of the higher antibrackets $\Phi^k(S, S, \ldots, S)$ has $k$ entries. The series terminates at a finite order if the associated BRST operator terminates at a finite order. For example, in the Abelian case of shift symmetry the general equation (114) reduces to $i\hbar \Delta S - \frac{1}{2} (S, S) = 0$, the Master Equation of the conventional Batalin-Vilkovisky formalism.

A solution $S$ to the general Master Equation (114) is invariant under deformations

$$\delta S = \sum_{k=1}^{\infty} \left( \frac{i}{\hbar} \right)^{k-1} \Phi^k(\epsilon, S, S, \ldots, S) \frac{(k-1)!}{(k-1)!} ,$$  \hspace{1cm} (115)

where again each $\Phi^k$ has $k$ entries, and $\epsilon$ is Grassmann-odd. One can view this as the possibility of adding a BRST variation

$$\sigma \epsilon = \sum_{k=1}^{\infty} \left( \frac{i}{\hbar} \right)^{k-1} \Phi^k(\epsilon, S, S, \ldots, S) \frac{(k-1)!}{(k-1)!} ,$$  \hspace{1cm} (116)

to the action. Here $\sigma$ is the appropriately generalized “quantum BRST operator”\footnote{For finite order, a rearrangement in terms of increasing rather than decreasing orders of $\hbar$ may be more convenient.}. In the case of the Abelian shift symmetry, the above $\sigma$-operator becomes $\sigma \epsilon = \Delta \epsilon + (i/\hbar)(\epsilon, S)$, which precisely equals $((i\hbar)^{-1}$ times) the quantum BRST operator of the conventional Batalin-Vilkovisky formalism.

We note that the general Master Equation (114) and the BRST symmetry (115) has the same relation to closed string field theory \cite{17, 27} that the conventional Batalin-Vilkovisky Master Equation and BRST symmetry has to open string field theory \cite{15}. The rôle of the action $S$ is then played by the string field $\Psi$, and the Master Equation (114) is the analogue of the closed string field equations. The symmetry (115) is then the analogue of the closed string field theory gauge transformations.
The present definition of higher antibrackets suggests the existence of an analogous hierarchy of Grassmann-even brackets based on a supermanifold and a non-Abelian open algebra – a natural generalization of Poisson-Lie brackets. It is also interesting to investigate the Poisson brackets and Nambu brackets generated by the generalized antibrackets and suitable vector fields $V$ anticommuting with the generalized $\Delta$-operator (and in particular certain Hamiltonian vector fields within the antibrackets), as described in the case of the usual antibracket in ref. [26]. We explore this in the next section.

4 Poisson bracket and antibracket

In this section we want to make an explicit connection between Poisson bracket and the antibracket.

For this purpose, consider the canonical algebra:

\[
[x^i, x^j] = 0 \quad (117)
\]

\[
[p_i, p_j] = 0 \quad (118)
\]

\[
[x^i, p^j] = i\delta_{ij} \quad (119)
\]

Now consider the non-abelian antibracket corresponding to a Lie algebra (see second term of equation (85)):

\[
(A, B) = \frac{\partial_r A}{\partial z^*_i} U^k_{ij} \frac{\partial_u B}{\partial z^*_j} \quad (120)
\]

We choose $U^k_{ij}$ corresponding to the structure constants of the canonical algebra. For each generator of the algebra we include an antifield, $z^*_i$ ($z^*_0$ is the antifield associated to the generator 1).

Now we introduce the operator $d$ ("exterior derivative"). For functions of $z^i$ alone (i.e they do not depend on the antifields), it is:

\[
\begin{align*}
\frac{dA}{dz^*_i} &= A_{,i} z^*_i \\
\frac{dB}{dz^*_i} &= B_{,i} z^*_i
\end{align*}
\]

For the canonical algebra, we get:

\[
\begin{align*}
(dA, dB) &= \{A, B\} z^*_0 \\
\{A, B\} &= \frac{\partial A}{\partial x^i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial x^i}
\end{align*}
\]

33
We see that \( \{ , \} \) is the usual Poisson bracket.

### 4.1 Nambu bracket and generalizations

In [23, 24] it is considered a possible generalization of the canonical formalism of classical mechanics, where the evolution equation contains two (or more) Hamiltonians:

\[
\frac{dF}{dt} = [F, H_1, H_2] \quad (125)
\]

\[
[G_1, G_2, G_3] = \epsilon_{ijk} \frac{\partial G_1}{\partial x^i} \frac{\partial G_2}{\partial x^j} \frac{\partial G_3}{\partial x^k}, \quad i, j, k = 1, 2, 3 \quad (126)
\]

\([G_1, G_2, G_3]\) is called a Nambu bracket.

Now we show how to obtain it (and its generalizations) from the higher antibrackets of [7]

Introduce:

\[
\Delta = \lambda \epsilon_{ijk} \frac{\partial}{\partial x^*_i} \frac{\partial}{\partial x^*_j} \frac{\partial}{\partial x^*_k}, \quad i, j, k = 1, 2, 3 \quad (127)
\]

\[\Delta^2 = 0 \quad (128)\]

For the 3-bracket we obtain:

\[
\Phi_3(A, B, C) = 6 \lambda (-1)^{(B)} \epsilon_{ijk} \frac{\partial A}{\partial x^*_i} \frac{\partial B}{\partial x^*_j} \frac{\partial C}{\partial x^*_k} \quad (129)
\]

Now, as in the previous subsection, choose:

\[
dG_1 = G_1, i x^*_i \quad (130)
\]

\[
dG_2 = G_2, i x^*_i \quad (131)
\]

\[
dG_3 = G_3, i x^*_i \quad (132)
\]

the \(G_k\) are functions of \(x^j\) alone (they do not depend on the antifields).

Choosing \(\lambda\) appropriately, we get:

\[
\Phi_3(dG_1, dG_2, dG_3) = [G_1, G_2, G_3] \quad (133)
\]

It is clear that the identities satisfied by \(\Phi_3(A, B, C)\) imply identities for \([G_1, G_2, G_3]\) [24]. They, in turn, are a direct consequence of \(\Delta\) being nilpotent and of third order in the derivatives.

The most general Nambu bracket and its properties are obtained in a similar way, starting from a suitable nilpotent \(\Delta\)-operator and the identification eq.(132) and eq.(133).
5 Conclusions

In these lectures we have studied the Batalin-Vilkovisky(BV) method of quantization from the perspective of the BRST-Schwinger-Dyson symmetry. This has led us directly to a generalization of the algebraic structure behind BV, first to non-abelian Lie algebras and later on, to general open algebras. New $\Delta$ operators emerge quite naturally. From them, using the Koszul procedure, we have derived a tower of $n$-brackets, a master equation and a corresponding invariance of the master equation.

As a simple application of the new formalism, we have exhibit a connection between the non-abelian antibracket and the Poisson bracket. In addition to this, we have explained how to get the Nambu brackets from a suitable $\Delta$ operator combined with the Koszul procedure.

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Appendix

In this appendix we give some additional conventions, and list some useful identities.

The Leibniz rules for derivations of the left and right kind read

$$\frac{\delta^l(F \cdot G)}{\delta A} = \frac{\delta^l F}{\delta A} G + (-1)^{\epsilon_F \cdot \epsilon_A} F \frac{\delta^l G}{\delta A} \tag{134}$$

and

$$\frac{\delta^r(F \cdot G)}{\delta A} = F \frac{\delta^r G}{\delta A} + (-1)^{\epsilon_G \cdot \epsilon_A} \frac{\delta^r F}{\delta A} G , \tag{135}$$
where $A$ denotes a field (or antifield) of arbitrary Grassmann parity $\epsilon_A$. Similarly, $\epsilon_F$ and $\epsilon_G$ are the Grassmann parities of the functionals $F$ and $G$.

Actual variations, let us denote them by $\bar{\delta}$ in contrast to the BRST transformations $\delta$ of the paper, are defined as follows:

$$F[A + \bar{\delta}A] - F[A] \equiv \bar{\delta}F \equiv \bar{\delta}A \frac{\delta F}{\delta A} \equiv \delta^r F \frac{\delta A}{\delta A} \bar{\delta}A .$$ (136)

The commutation rule of two arbitrary fields is

$$A \cdot B = (-1)^{\epsilon_A \epsilon_B} B \cdot A ,$$ (137)

and for actual variations one has the simple rule that

$$\bar{\delta}(F \cdot G) = (\bar{\delta}F)G + F(\bar{\delta}G)$$ (138)

independent of the Grassmann parities $\epsilon_F$ and $\epsilon_G$. The rules (122) and (123) in conjunction lead to the useful identity

$$\delta^r F \frac{\delta A}{\delta A} = (-1)^{\epsilon_A (\epsilon_F + 1)} \delta^r F \frac{\delta A}{\delta A} .$$ (139)

The BRST variations we have worked with in this paper correspond to right derivation rules. This is of course not imposed upon us, but it is convenient if we wish to compare our expressions with those of Batalin and Vilkovisky. It follows from requiring the actual variations to be related to the BRST transformations by multiplication of an anticommuting parameter $\mu$ from the right. This then provides us with very helpful operational rules for the BRST transformations $\delta$. In particular,

$$\bar{\delta}F \equiv (\delta F)\mu = \delta^r F \frac{\delta A}{\delta A} \bar{\delta}A .$$ (140)

Now, since

$$\delta F \equiv \delta^r \delta F \frac{\delta A}{\delta \mu} ,$$ (141)

it follows that

$$\delta F = \frac{\delta^r F}{\delta A} \delta A .$$ (142)

From this it also follows directly that the BRST transformations act as right derivations:

$$\delta(F \cdot G) = F(\delta G) + (-1)^{\epsilon_G (\delta F)}G .$$ (143)

These are the basic rules that are needed for the manipulations in the main text.
References

[1] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. B**102** (1981) 27; Phys. Rev. D**28** (1983) 2567 [E: D**30** (1984) 508]; Nucl. Phys. B**234** (1984) 106; J. Math. Phys. **26** (1985) 172.

[2] M. Henneaux, Nucl. Phys. B (Proc. Suppl.) **18A** (1990) 47.
J.M.L. Fisch, Univ. Brux. preprint ULB TH2/90-01.
M. Henneaux and C. Teitelboim, "Quantization of Gauge Systems", Princeton University Press, Princeton, New Jersey (1992).

[3] W. Siegel, Phys. Lett. **151B** (1985) 391; 396.
C. Thorn, Nucl. Phys. B**287** (1987) 61.
M. Bochicchio, Phys. Lett. B**193** (1987) 31.
H. Hata, Nucl. Phys. B**329** (1990) 698.
B. Zwiebach, Nucl.Phys.B390:33-152,1993.
E. Witten, Phys.Rev.D46:5467-5473,1992.
H. Hata and B. Zwiebach,Ann.Phys.229:177-216,1994.

[4] E. Verlinde, Nucl. Phys. B**381** (1992) 141.

[5] J. Alfaro and P.H. Damgaard,Nucl. Phys. B404(1993)751.

[6] J. Alfaro and P.H. Damgaard, Nucl. Phys. B455(1995)409; Phys. Lett. B334(1994)369.

[7] J. Alfaro and P.H. Damgaard, Phys. Lett. B369(1996)289.

[8] K. Bering, P. H. Damgaard and J. Alfaro, Nucl.Phys.B478(1996)459.

[9] T. Kugo and S. Uehara, Nucl. Phys. B**197** (1982) 378.
L. Baulieu, Phys. Rep. **129** (1985) 1.

[10] J. Alfaro and P.H. Damgaard, Phys. Lett. B**222** (1989) 425.
J. Alfaro, P.H. Damgaard, J. Latorre and D. Montano, Phys. Lett. B**233** (1989) 153.

[11] J. Alfaro and P.H. Damgaard, Ann. Phys.(N.Y.) **202** (1990) 398.
[12] O.M. Khudaverdian, J. Math. Phys. 32 (1991) 1934.

O.M. Khudaverdian and A.P. Nersessian, Mod. Phys. Lett. A8 (1993) 2377.

A. Schwarz, Commun. Math. Phys. 155 (1993) 249; 158 (1993) 373.

I.A. Batalin and I.V. Tyutin, Int. J. Mod. Phys. A8 (1993) 2333; Mod. Phys. Lett. A8 (1993) 3673.

H. Hata and B. Zwiebach, Ann. Phys. (NY) 229 (1994) 177.

[13] I.A. Batalin, J. Math. Phys. 22 (1981) 1837.

[14] J.-L. Koszul, Astérisque, hors serie (1985) 257.

[15] E. Witten, Mod. Phys. Lett. A5 (1990) 487.

[16] T. Lada and J. Stasheff, Int. J. Theor. Phys. 32 (1993) 1087.

[17] E. Witten and B. Zwiebach, Nucl. Phys. B377 (1992) 55.

B. Zwiebach, Nucl. Phys. B390 (1993) 33.

[18] I.A. Batalin and E.S. Fradkin, Phys. Lett. 128B (1983) 303.

[19] M. Henneaux, Phys. Rep. 126 (1985) 1.

[20] I.A. Batalin and E.S. Fradkin, J. Math. Phys. 25 (1984) 2426.

[21] I.A. Batalin and I.V. Tyutin, Int. J. Mod. Phys. A9 (1994) 517.

[22] A. Nersessian, [hep-th/9305181]

[23] Y. Nambu, Phys. Rev. D7 (1973) 2405.

[24] L. Takhtajan, Commun. Math. Phys. 160 (1994) 295.

[25] F. Akman, [q-alg/9506027]

[26] A. Nersessian and P.H. Damgaard, Phys. Lett. B355 (1995) 150.

[27] T. Kugo and K. Suehiro, Nucl. Phys. B337 (1990) 434.