Gibbs phenomenon and the emergence of the steady-state in quantum transport

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Simulations are increasingly employing explicit reservoirs – internal, finite regions – to drive electronic or particle transport. This naturally occurs in simulations of transport via ultracold atomic gases. Whether the simulation is numerical or physical, these approaches rely on the rapid development of the steady state. We demonstrate that steady state formation is a manifestation of the Gibbs phenomenon well-known in signal processing and in truncated discrete Fourier expansions. Each particle separately develops into an individual steady state due to the spreading of its wave packet in energy. The rise to the steady state for an individual particle depends on the particle energy – and thus can be slow – and ringing oscillations appear due to filtering of the response through the electronic bandwidth. However, the rise to the total steady state – the one from all particles – is rapid, with timescale \( \pi / W \), where \( W \) is the bandwidth. Ringing oscillations are now also filtered through the bias window, and they decay with a higher power. The Gibbs constant – the overshoot of the first ring – can appear in the simulation error. These results shed light on the formation of the steady state and support the practical use of explicit reservoirs to simulate transport at the nanoscale or using ultracold atomic lattices.

There are numerous characterization tools to investigate steady-state electronic transport in molecular and nanoscale devices. At the theoretical level, scattering-based approaches – centered on the Landauer formula – and non-equilibrium Green’s functions are the norm to calculate steady-state current. These tools can highlight the role of electronic structure, electron-vibration interactions, quantum interference, etc. Increasingly, however, studies are seeking to understand, exploit, and probe dynamical effects on both slow and fast timescales, including the generation of topological matter via time-dependent fields.

An avenue to computationally study transient and dynamical phenomena is to include particle reservoirs explicitly in the simulation, essentially letting a “capacitor” discharge and drive current through a region of interest. This approach is of particular relevance to ultracold atomic, which by default have finite, closed reservoirs. The inclusion of relaxation can give a true steady state while still permitting the examination of transient/dynamical processes (including for thermal transport). This type of “open” system approach has a long history (see discussion in Ref. 51), including designs for time-dependent density functional theory (TD-DFT) and other techniques) generally do not give direct insight into the formation of the steady state and the factors controlling transient behavior.

Here, we employ a Kubo approach to study transients in closed, noninteracting fermionic systems. Even though perturbative, one can directly follow the emergence of the steady state. We demonstrate its application using a system set out of equilibrium by adding a link at time \( t = 0 \) between two initially disconnected regions \( \mathcal{L} \) and \( \mathcal{R} \). There is a density imbalance (black line) that creates a chemical potential drop \( \mu_\mathcal{L} - \mu_\mathcal{R} \) (alternatively, there can initially be a uniform potential and a bias simultaneously turns on when connecting the lattice). The current, \( I \), is the step response to the addition of the link (green) filtered by the electronic bandwidth and bias window.

Figure 1. Schematic of a lattice set out of equilibrium by adding a link at time \( t = 0 \) between two initially disconnected regions \( \mathcal{L} \) and \( \mathcal{R} \). There is a density imbalance (black line) that creates a chemical potential drop \( \mu_\mathcal{L} - \mu_\mathcal{R} \) (alternatively, there can initially be a uniform potential and a bias simultaneously turns on when connecting the lattice). The current, \( I \), is the step response to the addition of the link (green) filtered by the electronic bandwidth and bias window.

state arises, how oscillations decay, and how different frequency scales contribute to transport, as well as how to minimize simulation error. We expect that this approach will find further application in dynamical, many-body transport in both nanoscale and ultracold atomic systems, including diagnosing pathological numerical setups and increasing efficiency.

Before the lattices come into contact (i.e., for times \( t < 0 \) in Fig. 1), the Hamiltonian is

\[ H_0 = H_\mathcal{L} + H_\mathcal{R}, \]  

with

\[ H_\mathcal{L} = \sum_{k \in \mathcal{L}} \hbar \omega_k a_k^\dagger a_k, \quad H_\mathcal{R} = \sum_{k \in \mathcal{R}} \hbar \omega_k b_k^\dagger b_k \]

and \( a_k^\dagger \) (\( a_k \)) and \( b_k^\dagger \) (\( b_k \)) are the fermionic creation (annihilation) operators on the left (\( \mathcal{L} \)) and right (\( \mathcal{R} \)), respectively. These are noninteracting lattices with \( N_\mathcal{L}(\mathcal{R}) \) levels and frequencies \( \omega_k \). The initial state is one with a density imbalance, where the left region has particles up to the chemical potential \( \mu_\mathcal{L} \) and the right to \( \mu_\mathcal{R} \). This drives the current when the lattices come in contact. The inclusion of the chemical potential in this way makes the calculations nonperturbative in the bias.

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for $t \geq 0$. Here, $N_C$ is the number operator in the Heisenberg picture on the left, $dN_C/dt = -i[H_0, H']$, and the factor of 2 appears due to taking the imaginary component $\Im$ (i.e., not due to spin). Applying Eq. (4) to $a_k^\dagger b_{k'}(t)$ yields

$$\langle a_k^\dagger b_{k'}(t) \rangle = -v_{kk'}(n_k - n_{k'}) \frac{e^{it(\omega_k - \omega_{k'})} - 1}{\omega_k - \omega_{k'}}. \quad (6)$$

where $n_k$ are the initial particle occupancies and we use that $\langle a_k^\dagger b_{k'} \rangle_0 = 0$ for two initially disjoint lattices. The total current from this perturbative result is thus

$$I(t) = 2 \sum_{k,k'} v_{kk'}^2 (n_k - n_{k'}) \frac{\sin[(\omega_k - \omega_{k'})t]}{\omega_k - \omega_{k'}}. \quad (7)$$

So far we only assume that the two lattices are initially disconnected and have occupancies from their separate single-particle eigenstates.

Let’s first examine the current, $I_k(t)$, from a particle in state $k$ on the left going into an empty reservoir of bandwidth $W$ on the right. Setting $v_{kk'} = v/\sqrt{N_C N_R}$ i.e., a flat band – and taking $\sum_{k'} 1/N_R \rightarrow \int d\omega/W$, gives

$$I_k(t) = \frac{2v^2}{N_C W} \int_{-W/2}^{W/2} d\omega' \frac{\sin[(\omega_k - \omega')t]}{\omega_k - \omega'}. \quad (8)$$

When $t \rightarrow \infty$, the integrand approaches $\pi \delta(\omega_k - \omega')$. That is, within the perturbative expression, a steady state can form so long as there is a state of energy $\omega_k$ in $R$ (which, of course, is simply conservation of energy in the absence of inelastic processes and in the long-time limit). On the left, $N_C$ can be just 1 level (and since there is only one particle, it can be a fermion or a massive boson). However, since there is one particle giving a current $\approx 2v^2/\pi N_C W$, there is a decay time $T^* = \pi N_C W/2 v^2$. For times much shorter than this, the particle looks to be in a true steady state. This demonstrates that constructive contributions from many incoming particles are not necessary for steady state formation, but rather it is the spread of a single particle into many different states – its wave-like nature in energy space – that results in steady current. Since only a single particle is present, the steady state is just a linear increase in time of the probability for the particle to be in $R$, which would be possible to measure in cold atom lattices by repetition of the experiment many times.

For finite times, the integral in Eq. (8) is just

$$I_k(t) = \frac{2v^2}{N_C W} \left\{ \text{Si} [t(\omega_k + W/2)] - \text{Si} [t(\omega_k - W/2)] \right\}, \quad (9)$$

where $\text{Si}[\cdot]$ is the sine integral. The derivative $dI_k/dt|_{t=0}$ determines the rise to the steady state. For the single particle current this thus depends on the smaller of the two energies, $|\omega_k + W/2|$ or $|\omega_k - W/2|$. For instance, for $\omega_k = 0$, the initial (linear) rise occurs with slope $2v^2/\pi N_C$. Thus, the time to reach the steady state value, $2v^2/\pi N_C W$, is $\pi/|W|$, at which time the current begins oscillating. If $\omega_k$ on the left) approaches the band edge (on the right), then the steady state can take a long time to develop. In that case, there is a fast process – where one of the sine integral quickly rises – and a slow process – where the other rises with time $\sim 1/(W/2 - |\omega_k|)$. 

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**Figure 2.** Current through a weak link when density-imbalanced flat band lattices come into contact. The blue line (black squares) shows the Kubo (exact) result for $v = W \cdot 10^{-6}$, $\mu = W/10$, and $N_C(R) = 100$. The dotted red line is the steady state current and the dashed green line the rise to the steady state at time $\pi/W$. From this we see that the system is a quasi-steady state. The inset shows the forward, $I_L(n_k = 1$ and $n_{k'} = 0$ in Eq. (7)), and backward, $I_R(n_k = 0$ and $n_{k'} = 1$ in Eq. (7)) currents. These two currents are of significantly larger transients but they partially cancel, leaving more regular – but algebraically decaying – oscillations in $I$. A true steady state will form when $N \rightarrow \infty$ and then $t \rightarrow \infty$.

At $t = 0$, the perturbing Hamiltonian

$$H' = \sum_{k \in L, k' \in R} \hbar v_{kk'} \left( a_k^\dagger b_{k'} + b_k^\dagger a_{k'} \right) \quad (3)$$

connects the two lattices, as shown in Fig. 1. The strength of the connection is the total hopping frequency $v = \sqrt{\sum_{k,k'} v_{kk'}^2}$, which we will treat as a perturbation. We can relate this to a real-space model with contact at, e.g., one site via the identification $c_1 = \sum_k U_{kk} a_k$, $d_1 = \sum_{k,k'} V_{kk'} b_k$ and $v_{kk'} = vU_{kk}^* V_{kk'}$, giving the connection $\hbar v(c_1^\dagger d_1 + d_1^\dagger c_1)$. Here, the quantities $U$ and $V$ are the transformation matrices from energy- to real-space on the left and right lattices.

We will apply the Kubo formula

$$\langle O(t) \rangle = \langle O \rangle_0 - t \int_0^t dt' \langle \{ O(t) , H'(t') \} \rangle_0 \quad (4)$$

for the observable $O$, where $O(t) = e^{iH_0 t} O e^{-iH_0 t}$ is an operator in the interaction picture and $\langle O \rangle_0$ indicates an average with respect to the initial state. While our focus is on closed, finite systems, we will take the infinite system limit to make some expressions more transparent. This will not obscure their interpretation for finite systems.

The particle current from left to right is

$$I(t) = -\langle dN_L/dt \rangle = -2 \sum_{k,k'} v_{kk'} \Im \langle a_k^\dagger b_{k'}(t) \rangle \quad (5)$$

For \( \omega_k = 0 \), the rapid rise and “ringing oscillation” is none other than Gibbs phenomenon\(^{20,61}\) for the step function \( \text{sign}[t] \) sent through a low-pass frequency filter. The Fourier transform of \( \text{sign}[t] \) is \( i\sqrt{2/\pi}/\omega \). Filtering the frequencies outside of the bandwidth \([-W/2, W/2]\) and taking the inverse transform gives Eq. (8) up to a prefactor\(^{65}\). The oscillations are thus an inherent aspect of electronic transport. Moreover, the “overshoot” of the current — its first and maximum oscillation overtop the steady state value — is \( G \cdot 2v^2\pi/NW \), where \( G = 2S[\pi]/\pi - 1 = 0.1789 \ldots \) is the Gibbs constant. That is, the overshoot is about 18% higher than the steady state value. Regardless of the bandwidth, the magnitude of the overshoot and initial rise. Just like the individual particles at small scales appear in Eq. (12), e.g., \( \omega_k \approx \pi/\sqrt{\mu} \). As already mentioned, this can introduce a slow rise to the steady state when \( \omega_k \) approaches the band edges. We note that different spectral densities of the reservoirs and strong coupling will give different overshoot values. However, the physical process is universal, the signal is filtered through the bandwidth giving rise to ringing oscillations.

We now examine the total current in the presence of a chemical potential drop. Considering the flat band case and equal bandwidths in \( L \) and \( R \), the continuum limit of Eq. (7) gives

\[
I(t) = \int_{-W/2}^{W/2} d\omega \delta I(\omega, t),
\]

where the contribution to the current at frequency \( \omega \) in \( L \) is

\[
\delta I = \frac{2v^2}{W^2} \int_{-W/2}^{W/2} d\omega' \left[ n_L(\omega) - n_R(\omega') \right] \frac{\sin[(\omega - \omega')t]}{\omega - \omega'}.
\]

We now explicitly label the occupancies \( n_L(\tau) \). The steady state current is \( 2v^2\mu/W^2 \) for a chemical potential drop of \( \mu = \mu_L - \mu_R \). As expected, these equations show that, to highest order in \( v \), there is a one way flow from filled states on the left into empty states on the right lattice (for \( \mu_L > \mu_R \)). Indeed, as with Eq. (8), states at frequency \( \omega \) go into states \( \omega' = \omega \) as \( t \to \infty \), giving the standard bias window.

Taking \( \mu_L = \mu/2, \mu_R = -\mu/2 \), and performing the integrations at zero temperature (so \( n_L(\tau) = 0 \) or 1) yields

\[
I(t) = \frac{2v^2}{W^2} \left\{ \sum_{\pm} \pm(W \pm \mu) \text{Si} \left[ \frac{t}{2}(W \pm \mu) \right] \right. \\
\left. + \frac{4 \sin \left[ \frac{Wt}{2} \right] \sin \left[ \frac{\mu t}{2} \right]}{t} \right\}
\]

\[
\approx \frac{4v^2}{W^2} \text{Si}[Wt/2] \mu,
\]

where the second expression is for a small bias, showing exactly the same manifestation of Gibbs phenomenon as the individual particles at the Fermi level [discussed following Eq. (9)]. Figure 2 shows the Kubo result, Eq. (12), together with the exact result for a finite system, as well as the steady state value and initial rise. Just like the individual particles at small frequencies (compared to \( W \)), the total current rises to its steady state level at time \( \pi/W \). Unlike the individual particles, this result is nearly true even when a small frequency scale appears in Eq. (12), e.g., \( (W - \mu) \) for a chemical potential drop comparable to the bandwidth. As with the individual particle case, the component with the small frequency scale takes a longer time to reach its steady state. However, this...
component appears with a prefactor that is also the small frequency scale. Hence, while it takes some time to rise, it has a small contribution to the total current. As a separate note, the convergence to the infinite system limit is non-monotonic purely due to the discrete nature of the states and filling.  

We can also examine the contribution to the current from different frequency scales on the left, Eq. (11). All frequency scales contribute to the current for short times, see Fig. 3. but this contribution decays in magnitude with both frequency and time. By \( t = 4 \cdot 2\pi/W \), the contribution is small outside the bias window and, as time progresses, it takes on the form of the bias window, Fig. 3b (the contributions reflect the band structure/couplings and thus are flat for the flat band model). When solving problems numerically, one reduces continuum reservoir/environments into a finite, discrete number of components. The decay of the contribution with frequency (outside the bias window) suggests routes to alternative coarse-grainings in frequency to enhance the simulation efficiency, as done in Ref. 68. The influence of different frequency scales will ultimately depend on details of the model (e.g., the presence of interactions, etc.), but we expect that the Kubo approach will help reveal the errors incurred by various coarse-grainings. We leave this for future studies and instead focus on errors in estimating the steady-state value of the current.  

The rise time of the current is rapid, indicating that already for small system sizes and times one can get a reasonably accurate value of the steady-state current (in the model here, taking the first maximum as an estimate of the steady-state current would only give a relative error of \( G \), about 18%). The slowly (algebraically) decaying nature of oscillations, though, influence the accuracy of further simulation. From Eq. (12), the asymptotic decay of the current to its steady state is

\[
- \frac{1}{T^2} \left[ \cos \left( \frac{W_t}{2} \right) \sin \left( \frac{\mu t}{W} \right) - 8 \cos \left( \frac{\mu t}{W} \right) \sin \left( \frac{W t}{2} \right) \right],
\]

compared with

\[
- \frac{1}{t} \left[ 2 \rho^2 \frac{W \cos \left( \omega_k t \right) \cos \left( \frac{W t}{2} \right) + 2 \omega_k \sin \left( \omega_k t \right) \sin \left( \frac{W t}{2} \right) }{(W/2)^2 - \omega_k^2} \right],
\]

from Eq. (9) for a single particle going into an empty band. Both expressions are in the long-limit compared to all other timescales (namely, 1/\( \mu \) and 1/\( \omega_k \), as well as 1/\( W \)). In the case of an infinitesimal bias (1/\( \mu \) \( \to \) \( \infty \) before the long time limit), one also gets oscillations that decay as 1/\( t \) (specifically, \(-4u \cos(Wt/2)/Wt\), as with the single particle at \( \omega_k = 0 \)).

Two common strategies for estimating the steady state current from a closed, finite-sized simulation are to (1) just take the value of the current at the end of the simulation or (2) average the current over some region of time. Which strategy is better? It is clear that these approaches can sometimes serendipitously yield the exact – or nearly exact – current. Thus, we will either work with error envelopes, i.e., the smooth curve going through the set of maxima in the error versus time or with asymptotic forms for the error decay. Considering the relative error, \( 1 - I_{\text{sim}}/I_{\text{exact}} \) with \( I_{\text{sim}} \) the current from simulation and \( I_{\text{exact}} \) the exact current, strategy (1) gives

\[
\frac{8}{\pi \mu W T^2}
\]

for the error envelope, where \( T \) is the total simulation time and we have taken a small bias approximation after the \( T \to \infty \) limit to get the decay (if we had first taken the small bias approximation and then \( T \to \infty \), we would find 1/\( T \) decay in the oscillations and error). For (2), the steady state estimate is

\[
I_{\text{sim}} = \frac{1}{T - T_0} \int_{T_0}^{T} I(t) \, dt.
\]
Closed forms for the relative error follow from integrating this equation with $I(t)$ from Eq. (10). To simplify calculations, though, we can work with the small bias expression directly in the case of strategy (2), as the average in Eq. (17) will have a dominant error due to short time contributions and thus the error will decay as $1/T$ so long as $T_0$ is not too large (i.e., either $T \to \infty$ and then $\mu \to 0$, or the reverse, will do).

We note, surprisingly, that $T_0$ should be set at a extremum to obtain minimal error. Intuitively, consider taking $T_0$ such that $I(T_0) = I_{\text{exact}}$, which may seem to be the best tactic (obviously, $I_{\text{exact}}$ is not known except for test cases, but one can take, e.g., the average of the first maxima and first minima, or where the slope is maximal, as a reasonable approximation). Continuing to integrate (i.e., increase $T$) first captures a half oscillation of the current above (or below) before the half oscillation of the current below (or above) smooths out and corrects $I_{\text{sim}}$. This indicates that starting away from the oscillatory extrema is poor tactic. Figure 4(b) shows the relative error versus $T$ and $T_0$. As both times become large, the minimum error comes at integer multiples of $2\pi/W$ for $T_0$, for any value of $T$ (at short $T_0$ or $T$, the minimum error can be slightly offset from these values). If $T-T_0$ is a multiple of $\pi/W$ (i.e., a complete oscillation for the flat band model), then the simulation error does decrease when $T_0$ is an odd multiple of $\pi/W$ (i.e., between a minimum and maximum). However, this decrease is simply a saddle on the error manifold and moving $T_0$ toward the extrema (holding $T$ constant) decreases further the error. While one may not generally know the (many) intrinsic timescales that give oscillations in a many-body system, such systems display the same behavior as here\textsuperscript{45,48} (which shows the Gibbs phenomenon and rapid development of the steady state). Thus, we expect that the choice of $T_0$ as extrema is generally a good heuristic.

Returning to the question of how strategy (2) compares with (1), we show in Fig. 4(b) the decay of the error for different $T_0$, including the asymptotic decay (dashed lines). For instance, for $T_0 = 2\pi/W$, the relative error asymptotically decays as $2(2-\pi^2 G)/\pi WT$. Fig. 4(c) shows the coefficient of the decay versus $T_0$ for two values of $T$ and the asymptotic form (first $T \to \infty$ and then $T_0 \to \infty$)

$$\frac{8 \sin (T_0 W/2)}{\pi W^2 TT_0}. \quad (18)$$



In this limit, the error is at a minimum when $T_0$ is an integer multiple of $2\pi/W$. While this asymptotic form qualitatively captures what is happening even in a non-asymptotic regime (i.e., the form of the curve on a regular scale, which hides inaccuracies for small relative error), it misses an important component of the error decay. Comparing Eq. (18) with $T_0 = 2\pi/W$ to the actual decay, $2(2-\pi^2 G)/\pi WT$, for large $T$ but not large $T_0$, it is clear that the actual coefficient of the decay is due to early time behavior (and hence why Gibbs’ constant appears). It is the initial error in obtaining the steady-state current, an error which slowly decays away as $T$ increases in the integration, that plays the important role.

Thus, given that strategy (2) has error decaying as $1/T$ and (1) as $1/T^2$, it seems the latter should be better. This is certainly true for long simulations (all other issues aside, such as reservoir sizes, matrix product dimensions, etc.). However, in practice, large systems and times are inaccessible, i.e., simulations are typically in the range of 10 to 100 natural time units, see, e.g., Refs. 45 and 48. Thus, the coefficient of the decay matters. Since strategy (1) has higher error for small $T$, there is a crossing time when strategy (1) becomes better than (2). This crossing time is much greater than $100 \cdot 2\pi/W$ except for $T_0 = 0$, for which it comes at about $60 \cdot 2\pi/W$. Thus, averaging within a window (with $T_0$ at an extremum) is generally a better strategy. Indeed, we closely followed this procedure for many-body transport simulated with matrix product state\textsuperscript{61} albeit empirically determined. We emphasize that strategy matters, as even if the goal is only moderate accuracy (e.g., 1%), different strategies can mean orders of magnitude longer simulations. Moreover, the computational cost will scale even worse. An order of magnitude longer simulation requires an order of magnitude larger system, as the maximum simulation time is proportional to $N^{70}$. If the computational cost scales as $T N^p$, where $p \geq 2$, then a 10 times longer simulation will mean at least a 100 times the computational cost.

Despite being perturbative\textsuperscript{70}, we have shown that a Kubo approach elucidates the physics behind the development of the steady state and transient oscillations. These oscillations are none other than the Gibbs phenomenon due to the filtering of the current through the electronic bandwidth and bias window. Unlike the original context of the Gibbs phenomenon\textsuperscript{41,42} (and in filtering signals), the ringing oscillations are not “spurious,” but physical. For individual particles, the quasi-steady state is a manifestation of the wave-like nature of particles – the particle is in a superposition across spatial states, which enables it to form into a steady state on its own. However, for many particles, the current will near its steady state value in time $\pi/W$. This is why tensor network simulations of the current obtain reasonable results even for quite small simulations. We expect that the Kubo approach will assist in understanding other features of simulations, providing general guidance and informing new strategies for enhancing efficiency.

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The text also discusses the Kubo result, which captures the signal exactly while still being perturbative in the bias. For instance, the flat band model and, e.g., a 1D additional entanglement that gives an effective signal of interest. We note that the non-zero real component. This is not part of the signal of interest. We note that the \( v^2 \) contribution to the prefactor can be written as \( v^2 \theta(t) \), which properly removes the signal for \( t < 0 \).

The total coupling mode in the bias window determines the current magnitude. Each mode can be thought of taking up \( \Delta = W/N \) width of the spectral density. For even \( N \) and placing the modes uniformly across the total bandwidth \( -W/2 < \omega < W/2 \), the number of modes in the bias window is \( n = 2 \cdot \text{Floor}[N \mu/2W + 1/2] \) when the bias is applied symmetrically \( (\mu = -\mu) \). To obtain the proper steady state current, the fraction of the spectral density in the bias window must be equal to the bias as a fraction of the bandwidth, \( n/N = \mu/W \). This is a purely artificial constraint to get the right coupling strength (and hence current) for finite-size systems. As \( N \to \infty \), the corrections away from integer values decay as \( 1/N \). While these results are for the flat band model, the calculation can be done in other scenarios as well (more complex band structures and couplings, and potentially even out of the weak coupling limit by checking finite-size convergence of the Greens functions). The types of oscillations and decay discussed mathematically here give insight into contributions observed computationally in density functional theory calculation.

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If the coupling from state \( k \) to \( \mathbf{R} \) was given by the spectral function \( \gamma/2\pi \) (i.e., flat but unbounded when \( W \to \infty \)), then \( n_k(t) \) will be exponentially decaying even for short times.

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The contribution from the cosine component of the inverse Fourier transform is zero. However, working directly with Eq. (8), i.e., without taking the imaginary component, the integral of \( (\cos[t(\omega + \omega')] - 1)/(\omega' - \omega) \) yields a non-zero real component. This is not part of the signal of interest. We note that the \( v^2 \) contribution to the prefactor can be written as \( v^2 \theta(t) \), which properly removes the signal for \( t < 0 \).

The total coupling mode in the bias window determines the current magnitude. Each mode can be thought of taking up \( \Delta = W/N \) width of the spectral density. For even \( N \) and placing the modes uniformly across the total bandwidth \( -W/2 < \omega < W/2 \), the number of modes in the bias window is \( n = 2 \cdot \text{Floor}[N \mu/2W + 1/2] \) when the bias is applied symmetrically \( (\mu = -\mu) \). To obtain the proper steady state current, the fraction of the spectral density in the bias window must be equal to the bias as a fraction of the bandwidth, \( n/N = \mu/W \). After rearranging, \( N \mu/2W = n/2 \) or, in other words, \( N \mu/2W \) has to be an integer \((n)\) (even). When \( N \mu/2W \) is not an integer, the modes in the bias window have \( n/W/N \leq \mu \) and therefore whether they underestimate or overestimate the current. Note, as well, that \( N \mu/2W = \text{Integer} \) is a purely artificial constraint to get the right coupling strength (and hence current) for finite-size systems. As \( N \to \infty \), the corrections away from integer values decay as \( 1/N \). While these results are for the flat band model, the calculation can be done in other scenarios as well (more complex band structures and couplings, and potentially even out of the weak coupling limit by checking finite-size convergence of the Greens functions). The types of oscillations and decay discussed mathematically here give insight into contributions observed computationally in density functional theory calculation. The Kubo result captures this exactly while still being perturbative.