Attitude Determination and Estimation using Vector Observations: Review, Challenges and Comparative Results

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Abstract

This paper concerns the problem of attitude determination and estimation. The early applications considered algebraic methods of attitude determination. Attitude determination algorithms were supplanted by the Gaussian attitude estimation filters (which continue to be widely used in commercial applications). However, the sensitivity of the Gaussian attitude filter to the measurement noise prompted the introduction of the nonlinear attitude filters which account for the nonlinear nature of the attitude dynamics problem and allow for a simpler filter derivation. This paper presents a survey of several types of attitude determination and estimation algorithms. Each category is detailed and illustrated with literature examples in both continuous and discrete form. A comparison between these algorithms is demonstrated in terms of transient and steady-state error through simulation results. The comparison is supplemented by statistical analysis of the error-related mean, infinity norm, and standard deviation of each algorithm in the steady-state.

Keywords: Comparative Study, Attitude, Determination, Estimation, Filter, Adaptive Filter, Gaussian Filter, Nonlinear Filter, Overview, Rodrigues Vector, Special Orthogonal Group, Unit-quaternion, Angle-axis, Deterministic, Stochastic, Continuous, Discrete.

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1. Introduction

Automated and semi-automated robotic applications such as unmanned aerial vehicles (UAVs), autonomous underwater vehicles (AUVs), ground vehicles, satellites, radars and others can be controlled to rotate successfully in the three dimensional (3D) space if the orientation of the rigid-body is accurately known. However, the true orientation of a rigid-body, generally referred to as attitude, cannot be extracted directly. Alternatively, the attitude can be determined using

1) a set of measurements available in the body-frame and
2) known observations in the inertial-frame.

In general, measurement units are corrupted with unknown bias and noise components. However, the quality of measurement units has a significant impact on the level of noise and bias components attached to the measurements. The measurement units can be broadly divided into two categories:

1) high-cost or high-quality measurement units and
2) low-cost or low-quality inertial measurement units (IMUs).

There are three main approaches to establishing the attitude:

1) algebraic determination algorithms,
2) vector-based filter dynamics, and
3) filter dynamics that mimic the true nature of the attitude dynamics problem.

As such, attitude determination or estimation problem is a fundamental sub-task in the majority of robotic applications. The accurate knowledge of the attitude is indispensable for the control process of most robotic applications. This is especially true for the applications that require fast maneuvering. Lack of accurate attitude information may result into an unstable control process. In this paper, the terms “filter” and “estimator” are equivalent and will be used interchangeably. Also, the term “attitude”, “orientation” and “rotational matrix” are equivalent and will be used interchangeably. The main goals of this paper are as follows:

1) introducing the attitude dynamics problem,
2) providing the assumptions necessary for the attitude determination and estimation problem,
3) presenting a brief survey of different types of attitude determination algorithms and attitude filters,
4) demonstrating several types of filter design in both continuous and discrete form as well as attitude determination algorithms,
5) comparing the results between different categories of attitude determination and estimation algorithms.

The paper is organized as follows: Section 2 contains abbreviations, math and attitude notations, math identities and attitude preliminaries. The attitude problem, inertial-frame observations, body-frame measurements and basic assumptions are outlined in Section 3. Section 4 gives a brief overview of attitude determination algorithms and presents a detailed description of the three most common algorithms. Section 4 explains the structure of Gaussian attitude filters and discusses two main algorithms of Gaussian attitude filters. Section 6 describes the structure of nonlinear attitude filters as well as different types of nonlinear attitude filters. Comparative results between the different categories of attitude determination algorithms and filters are given in Section 7. Finally, Section 8 summarizes the work.
2. Notation and Preliminaries

Table 1 lists the abbreviations used throughout the paper. Table 2 contains the important math notation used throughout the paper. Table 3 provides some important attitude-related definitions and notation.

| Abbreviation | Definition |
|--------------|------------|
| UAVs | Unmanned aerial vehicles |
| AUVs | autonomous underwater vehicles |
| IMU | Inertial measurement unit |
| QUEST | Quaternion estimator |
| SVD | Singular value decomposition |
| TRIAD | Triaxial attitude determination |
| KF | Kalman filter |
| EKF | Extended Kalman filter |
| MEKF | Multiplicative extended Kalman filter |
| GAMEF | Geometric Approximate Minimum-Energy Filter |
| NDAF | Nonlinear deterministic attitude filter |
| CG-NDAF | Constant gain NDAF |
| CGD-NDAF | Constant gain direct NDAF |
| CGSd-NDAF | Constant gain semi-direct NDAF |
| AG-NDAF | Adaptive gain NDAF |
| GP-NDAF | Guaranteed performance NDAF |
| GPSd-NDAF | Guaranteed performance semi-direct NDAF |
| GPD-NDAF | Guaranteed performance direct NDAF |
| NSAF | Nonlinear stochastic attitude filter |
| AGI-NSAF | Adaptive gain Ito NSAF |
| AGS-NSAF | Adaptive gain Stratonovich NSAF |
| GP-NSAF | Guaranteed performance NSAF |
| GPSd-NSAF | Guaranteed performance semi-direct NSAF |
| GPD-NSAF | Guaranteed performance direct NSAF |
| Symbol | Description |
|--------|-------------|
| $\mathbb{N}$ | The set of integer numbers |
| $\mathbb{R}_+$ | The set of nonnegative real numbers |
| $\mathbb{R}^n$ | Real $n$-dimensional vector |
| $\mathbb{R}^{n \times m}$ | Real $n \times m$ dimensional matrix |
| $\|\cdot\|$ | Euclidean norm, for $x \in \mathbb{R}^n$, $\|x\| = \sqrt{x^\top x}$ |
| $\mathbb{S}^2$ | Two-sphere, $\mathbb{S}^2 = \{ x = [x_1, x_2, x_3] \top \in \mathbb{R}^3 \mid \|x\| = 1 \}$ |
| $\mathbb{S}^3$ | 3-sphere, $\mathbb{S}^3 = \{ x \in \mathbb{R}^4 \mid \|x\| = 1 \}$ |
| $\top$ | Transpose of a component |
| $\times$ | Cross multiplication |
| $[\cdot]_\times$ | Skew-symmetric of a matrix |
| $\mathbf{I}_n$ | Identity matrix with dimension $n$-by-$n$ |
| $\det (\cdot)$ | Determinant of a component |
| $\text{Tr}\{\cdot\}$ | Trace of a component |
| $\exp (\cdot)$ | Exponential value of a component |
| $\lambda (\cdot)$ | A group of eigenvalues of a matrix |
| $\Lambda (\cdot)$ | Minimum eigenvalue of a matrix |
| $\mathbb{E}\{\cdot\}$ | Expected value of a component |
| $\mathbb{P}\{\cdot\}$ | Probability of a component |
Table 3: Attitude Notation

| Symbol | Description |
|--------|-------------|
| \{I\} | Inertial-frame of reference |
| \{B\} | Body-frame of reference |
| SO(3) | Special Orthogonal Group |
| so(3) | The space of 3 × 3 skew-symmetric matrices, and Lie-algebra of SO(3) |
| \(\mathcal{P}_a\) | Anti-symmetric projection operator |
| \(\mathcal{P}_s\) | Symmetric projection operator |
| \(R\) | True attitude/Rotational matrix/Orientation of a rigid-body, \(R \in SO(3)\) |
| \(\Omega\) | Angular velocity vector with \(\Omega = [\Omega_x, \Omega_y, \Omega_z]^\top \in \mathbb{R}^3\) |
| \(\Omega_m\) | Angular velocity measurement vector |
| \(b\) | The bias associated with \(\Omega_m\), \(b \in \mathbb{R}^3\) |
| \(\omega\) | The noise associated with \(\Omega_m\), \(\omega \in \mathbb{R}^3\) |
| \(Q_{\omega}\) | Diagonal covariance matrix of the noise \(\omega\) |
| \(\sigma\) | Upper bound of \(Q_{\omega}\), \(\sigma \in \mathbb{R}^3\) |
| \(v_i^I\) | The \(i\)th vector in the inertial-frame, \(v_i^I \in \mathbb{R}^3\) |
| \(v_i^B\) | The \(i\)th vector in the body-frame, \(v_i^B \in \mathbb{R}^3\) |
| \(\hat{v}_i^B\) | The true value of the \(i\)th vector in the body-frame, \(\hat{v}_i^B \in \mathbb{R}^3\) |
| \(b_i^B\) | The \(i\)th bias component of \(v_i^B\), \(b_i^B \in \mathbb{R}^3\) |
| \(\omega_i^B\) | The \(i\)th noise component of \(v_i^B\), \(\omega_i^B \in \mathbb{R}^3\) |
| \(s_i\) | The confidence level of \(i\)th measurement, \(s_i \in \mathbb{R}_+\) |
| \(\nu_i^I\) | Normalized value of \(v_i^I\), \(\nu_i^I \in \mathbb{R}^3\) |
| \(\nu_i^B\) | Normalized value of \(v_i^B\), \(\nu_i^B \in \mathbb{R}^3\) |
| \(\hat{\nu}_i^B\) | Normalized value of \(\hat{v}_i^B\), \(\hat{\nu}_i^B \in \mathbb{R}^3\) |
| \(R_Q\) | Attitude representation obtained using unit-quaternion vector, \(R_Q \in SO(3)\) |
| \(Q\) | True unit-quaternion vector, \(Q = [q_0, q]^\top \in \mathbb{S}^3\) |
| \(Q^*\) | Complex conjugate of unit-quaternion, \(Q^* \in \mathbb{S}^3\) |
| \(\circ\) | Multiplication operator of two unit-quaternion vectors |
| \(R_a\) | Attitude representation obtained using angle-axis parameterization, \(R_a \in SO(3)\) |
| \(\alpha\) | Angle of rotation, \(\alpha \in \mathbb{R}\) |
| \(u\) | Unit vector, \(u = [u_1, u_2, u_3]^\top \in \mathbb{S}^2\) |
| \(R_\rho\) | Attitude representation obtained using Rodriguez vector, \(R_\rho \in SO(3)\) |
| \(\rho\) | Rodriguez vector, \(\rho = [\rho_1, \rho_2, \rho_3]^\top \in \mathbb{R}^3\) |
| \(R_y\) | Reconstructed attitude, \(R_y \in SO(3)\) |
| \(||R||_I\) | Normalized Euclidean distance of \(R \in SO(3)\) |
\( Q_y \): Reconstructed unit-quaternion, \( Q_y \in S^3 \)
\( Q_{\text{opt}} \): Optimal unit-quaternion, \( Q_{\text{opt}} \in S^3 \)
\( \hat{R} \): Estimate of the true attitude, \( \hat{R} \in \text{SO} (3) \)
\( \hat{Q} \): Estimate of the true unit-quaternion, \( \hat{Q} \in S^3 \)
\( \hat{b} \): Estimate of the true bias, \( \hat{b} \in \mathbb{R}^3 \)
\( \hat{\sigma} \): Estimate of \( \sigma \), \( \hat{\sigma} \in \mathbb{R}^3 \)
\( \tilde{R} \): Attitude error, \( \tilde{R} \in \text{SO} (3) \)
\( \tilde{\rho} \): Rodriguez vector error, \( \tilde{\rho} \in \mathbb{R}^3 \)
\( \tilde{\alpha} \): Angle of rotation error, \( \tilde{\alpha} \in \mathbb{R} \)
\( \tilde{b} \): Bias error, \( \tilde{b} \in \mathbb{R}^3 \)
\( \tilde{\sigma} \): Upper bound covariance error, \( \tilde{\sigma} \in \mathbb{R}^3 \)
\( E \): Unconstrained error or transformed error, \( E \in \mathbb{R}^3 \)
\( \xi \): Prescribed performance measure function, \( \xi \in \mathbb{R} \)
\( \xi_0 \): Initial value of \( \xi \) (upper bound), \( \xi_0 \in \mathbb{R} \)
\( \xi_\infty \): Steady-state value of \( \xi \) (lower bound), \( \xi_\infty \in \mathbb{R} \)
\( \ell \): Convergence factor of \( \xi \) from \( \xi_0 \) to \( \xi_\infty \), \( \ell \in \mathbb{R} \)

Let \( \text{SO} (3) \) denote the Special Orthogonal Group. The relative orientation of a rigid-body in the body-frame \( \{B\} \) with respect to the inertial frame \( \{I\} \) is referred to as attitude or a rotational matrix \( R \) and is given by:

\[
\text{SO} (3) := \left\{ R \in \mathbb{R}^{3 \times 3} \middle| R^T R = RR^T = I_3, \det (R) = 1 \right\}
\]

with \( \det (\cdot) \) denoting a determinant of a matrix. The Lie-algebra related to \( \text{SO} (3) \) is denoted by \( \mathfrak{s}o (3) \) and is defined by

\[
\mathfrak{s}o (3) := \left\{ \mathcal{X} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \middle| \mathcal{X}^T = -\mathcal{X} \right\}
\]

where \( \mathcal{X} \in \mathbb{R}^{3 \times 3} \) is a skew-symmetric matrix. The map \( [\cdot]_\times : \mathbb{R}^3 \rightarrow \mathfrak{s}o (3) \) is given by

\[
[\mathcal{X}]_\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \quad (1)
\]

For \( x, y \in \mathbb{R}^3 \), one has

\[
[\mathcal{X}]_\times y = x \times y
\]

where \( \times \) is a cross product of the two given vectors. The mapping of a skew-symmetric matrix \( [\cdot]_\times \) to vector form is defined by a vex operator \( \text{vex} : \mathfrak{s}o (3) \rightarrow \mathbb{R}^3 \) such that

\[
\text{vex} (\mathcal{X}) = x
\]

with \( x \in \mathbb{R}^3 \) and \( \mathcal{X} \in \mathfrak{s}o (3) \) as defined in (1). Let \( \mathcal{P}_a \) be the anti-symmetric projection operator on the Lie-algebra \( \mathfrak{s}o (3) \) [2]. The related mapping is given by \( \mathcal{P}_a : \mathbb{R}^{3 \times 3} \rightarrow \mathfrak{s}o (3) \)

\[
\mathcal{P}_a (\mathcal{Y}) = \frac{1}{2} (\mathcal{Y} - \mathcal{Y}^T) \in \mathfrak{s}o (3), \quad \mathcal{Y} \in \mathbb{R}^{3 \times 3} \quad (2)
\]
The symmetric projection operator in the space of a square matrix is given by

\[
P_s(Y) = \frac{1}{2} (Y + Y^\top), \quad Y \in \mathbb{R}^{3 \times 3}
\]

(3)

The normalized Euclidean distance of a rotational matrix \( R \in SO(3) \) can be represented as follows

\[
||R||_I := \frac{1}{4} \text{Tr} \{I_3 - R\}
\]

(4)

where \( \text{Tr} \{ \cdot \} \) denotes a trace of the matrix and \( ||R||_I \in [0, 1] \). The following identities will prove useful in the subsequent derivations:

\[
||a||^2 = \text{Tr} \left\{ a a^\top \right\}
\]

(5)

\[
[a \times b]_\times = b a^\top - a b^\top, \quad a, b \in \mathbb{R}^3
\]

(6)

\[
[R a]_\times = R [a]_\times R^\top, \quad R \in SO(3), a \in \mathbb{R}^3
\]

(7)

\[
[a]^2_\times = -||a||^2 I_3 + a a^\top, \quad a \in \mathbb{R}^3
\]

(8)

\[
A, B = A B - B A, \quad A, B \in \mathbb{R}^{3 \times 3}
\]

(9)

\[
\text{Tr} \{[A, B]\} = \text{Tr} \{A B - B A\} = 0, \quad A, B \in \mathbb{R}^{3 \times 3}
\]

(10)

\[
\text{Tr} \{B [a]_\times\} = 0, \quad B = B^\top \in \mathbb{R}^{3 \times 3}, a \in \mathbb{R}^3
\]

(11)

\[
\text{Tr} \{A [a]_\times\} = \text{Tr} \{\mathcal{P}_a (A) [a]_\times\} = -2 \text{vex} (\mathcal{P}_a (A))^\top a, \quad A \in \mathbb{R}^{3 \times 3}, a \in \mathbb{R}^3
\]

(12)

\[
B [a]_\times + [a]_\times B = \text{Tr} \{B\} [a]_\times - [B a]_\times, \quad B = B^\top \in \mathbb{R}^{3 \times 3}, a \in \mathbb{R}^3
\]

(13)

The unit-quaternion is defined by

\[
Q = \begin{bmatrix} q_0 \\ q \end{bmatrix} \in S^3
\]

where \( q_0 \in \mathbb{R} \) and \( q = [q_1, q_2, q_3]^\top \in \mathbb{R}^3 \) such that

\[
S^3 = \left\{ Q \in \mathbb{R}^4 \mid ||Q|| = 1 \right\}
\]

(14)

Let \( Q = [q_0, q]^\top \in S^3 \). Hence, \( Q^* = Q^{-1} \in S^3 \) can be defined as follows

\[
Q^* = Q^{-1} = \begin{bmatrix} q_0 \\ -q \end{bmatrix} \in S^3
\]

(15)

where \( Q^* \) and \( Q^{-1} \) are a complex conjugate and an inverse of the unit-quaternion, respectively. For any \( Q_1, Q_2 \in S^3 \), the quaternion product between \( Q_1 \) and \( Q_2 \) can be found in the following manner

\[
Q_3 = Q_1 \odot Q_2 = \begin{bmatrix} q_{01} \\ q_1 \end{bmatrix} \odot \begin{bmatrix} q_{02} \\ q_2 \end{bmatrix}
\]

\[
= \begin{bmatrix} q_{01} q_{02} - q_1^\top q_2 \\ q_{01} q_2 q_1 + q_{02} q_1 + [q_1]_\times q_2 \end{bmatrix} \in S^3
\]

(16)

where \( q_{01}, q_{02} \in \mathbb{R} \) and \( q_1, q_2 \in \mathbb{R}^3 \). The coordinates of a moving frame can be defined with respect to the reference frame:

\[
\mathcal{R}_Q(Q) = \left( q_0^2 - ||q||^2 \right) I_3 + 2 q q^\top + 2 q_0 [q]_\times
\]

\[
= I_3 + 2 q_0 [q]_\times + 2 [q]^2_\times
\]

(17)
The attitude of a rigid-body can be obtained given a unit-axis \( u \in \mathbb{R}^3 \) and an angle of rotation \( \alpha \in \mathbb{R} \) in the 2-sphere \( S^2 \) [3, 4]

\[
\mathcal{R}_u (\alpha, u) = \exp \left( -\alpha [u]_\times \right) = I_3 + \sin (\alpha) [u]_\times + (1 - \cos (\alpha)) [u]_\times^2
\]  

(18)

Also, the attitude can be established using Rodriguez parameters vector \( \rho = [\rho_1, \rho_2, \rho_3]^\top \in \mathbb{R}^3 \) such that related map from vector form to \( \text{SO}(3) \) is

\[
\mathcal{R}_\rho = \frac{1}{1 + \|\rho\|^2} \left( \left( 1 - \|\rho\|^2 \right) I_3 + 2\rho\rho^\top + 2[\rho]_\times \right)
\]  

(19)

A more thorough overview of attitude mapping, important properties and helpful notes can be found in [4].

3. Attitude Dynamics and Measurements

Let \( R \in \text{SO}(3) \) denote the attitude (rotational matrix), which describes the relative orientation of the moving rigid-body in the body-frame \( \{B\} \) with respect to the fixed inertial-frame \( \{I\} \) as illustrated in Figure 1.

![Figure 1: The orientation of a 3D rigid-body in body-frame relative to inertial-frame [1].](image)

The attitude can be extracted through \( n \)-known non-collinear vectors in the inertial-frame and their measurements done relative to the coordinate system fixed to the rigid-body. For simplicity, let the superscripts \( I \) and \( B \) indicate that a vector is associated with the inertial-frame and body-frame, respectively. Let \( v^I_i \in \mathbb{R}^3 \) be a known vector in the inertial-frame which is measured in the coordinate system fixed to the rigid-body such that

\[
v^B_i = R^\top v^I_i + b^B_i + \omega^B_i \in \mathbb{R}^3
\]  

(20)

where \( b^B_i \in \mathbb{R}^3 \) stands for the bias component, and \( \omega^B_i \in \mathbb{R}^3 \) denotes the noise component attached to the \( i \)th body-frame measurement for all \( i = 1, 2, \ldots, n \). The measurement in (20) represents output of a typical measurement unit attached to a moving body. However, the values of \( b^B_i \) and \( \omega^B_i \) are heavily dependent on the quality of the measurement unit. Define the following two sets

\[
v^I = [v^I_1, v^I_2, \ldots, v^I_n] \in \mathbb{R}^{3 \times n}
v^B = [v^B_1, v^B_2, \ldots, v^B_n] \in \mathbb{R}^{3 \times n}
\]  

(21)
Remark 1. The attitude can be extracted given the availability of at least two known non-collinear observations in the inertial-frame and their measurements in the body-frame (The sets in (21) are at least of rank two). In case when \( n = 2 \), the third inertial-frame and body-frame vectors can be obtained by the cross product such that \( v_I^3 = v_I^1 \times v_I^2 \) and \( v_B^3 = v_B^1 \times v_B^2 \), respectively, which ensures non-collinearity of the vectors \( v_I^1, v_I^2, \) and \( v_I^3 \) as well as \( v_B^1, v_B^2, \) and \( v_B^3 \).

The dynamics of the true attitude are described by

\[
\dot{R} = R[\Omega]_x
\]

where \( \Omega \in \mathbb{R}^3 \) is the true value of angular velocity. Angular velocity of a moving body can be measured by the rate gyros, and its typical measurement is equivalent to

\[
\Omega_m = \Omega + b + \omega \in \{B\}
\]

where \( b \) and \( \omega \) denote the bias and noise components, respectively, attached to the measurement of angular velocity for all \( b, \omega \in \mathbb{R}^3 \). A low-cost module of an inertial measurement unit may consist of three different measuring subunits:

i) 3-axis magnetometers which can be represented by

\[
v^B_1 = R^\top v^I_1 + b^B_1 + \omega^B_1
\]

with \( v^I_1 \) being the earth-magnetic field, and \( b^B_1 \) and \( \omega^B_1 \) being the additive unknown bias and noise components, respectively.

ii) 3-axis accelerometers that can be represented by

\[
v^B_2 = R^\top \left( \dot{V} - v^I_2 \right) + b^B_2 + \omega^B_2
\]

with \( v^I_2 := [0, 0, 9.8]^\top \approx [0, 0, 9.8]^\top \) being the gravitational acceleration field defined in \( \{I\} \), \( \dot{V} \in \mathbb{R}^3 \) denoting the linear acceleration in \( \{I\} \), and \( b^B_2 \) and \( \omega^B_2 \) being the unknown bias and noise components added during the measurement process, respectively. At low frequency, \( ||v^B_2|| >> ||\dot{V}|| \) which allow one to obtain

\[
v^B_2 \approx -R^\top v^I_2 + b^B_2 + \omega^B_2
\]

iii) 3-axis rate gyros record the angular velocity measurement which can be denoted by \( \Omega_m \) as defined in (23).

Attitude determination or estimation may be utilized through normalized values of the vectorial measurements. The \( i \)th inertial-frame and body-frame vectors in (20) are normalized in the following manner:

\[
\psi^I_i = \frac{v^I_i}{||v^I_i||}, \quad \psi^B_i = \frac{v^B_i}{||v^B_i||}, \quad \forall i = 1, 2, \ldots, n
\]

As such, the sets of normalized values presented below get utilized by the attitude determination or estimation algorithms:

\[
\psi^I = \begin{bmatrix} \psi^I_1, \psi^I_2, \ldots, \psi^I_n \end{bmatrix} \in \mathbb{R}^{3 \times n}
\]

\[
\psi^B = \begin{bmatrix} \psi^B_1, \psi^B_2, \ldots, \psi^B_n \end{bmatrix} \in \mathbb{R}^{3 \times n}
\]

The exact integration of (22) is equivalent to

\[
R[k + 1] = R[k] \exp \left( [\Omega[k]]_x \Delta t \right)
\]
where $\Delta t$ is a small time sample and $[k]$ associated with a variable refers to its value at the $k$th sample for $k \in \mathbb{N}$.

On the other side, the vectorial measurements in (20), can be written in terms of unit-quaternion as

$$
\begin{bmatrix}
0 \\
v_i^B
\end{bmatrix} = Q^{-1} \odot \begin{bmatrix}
0 \\
v_i^I
\end{bmatrix} \odot Q + \begin{bmatrix}
0 \\
b_i^B
\end{bmatrix} + \begin{bmatrix}
0 \\
\omega_i^B
\end{bmatrix} \in \mathbb{R}^4
$$

(27)

where $Q \in \mathbb{S}^3$. In the same spirit, attitude dynamics can be redefined in terms of unit-quaternion as

$$
\dot{Q} = \frac{1}{2} \Gamma (\Omega) Q = \frac{1}{2} \begin{bmatrix}
0 & -\Omega^T \\
\Omega & -[\Omega]_x
\end{bmatrix} Q
$$

(28)

The exact integration of (28) results into

$$
Q[k+1] = \exp \left( \frac{1}{2} \Gamma (\Omega [k]) \Delta t \right) Q[k]
$$

(29)

For a comprehensive overview of attitude parameterization, mapping and related useful properties visit [4].

**Definition 1.** Consider a forward invariant unstable set $U_s \subseteq SO(3)$ defined by

$$
U_s = \{ R \in SO(3) | Tr \{ R \} = -1 \}
$$

(30)

where $Tr \{ R \} = -1$ only at one of the following three orientations

$$
\begin{align*}
R &= \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix} \\
R &= \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix} \\
R &= \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\end{align*}
$$

(31)

Directly substituting $R$ value in (31) with its definition in (4), it becomes apparent that $Tr \{ R \} = -1$ implies that $||R||_I = 1$.

**Assumption 1.** (Uniform boundedness of unknown bias $b$ in (23)) Let vector $b$ belong to a given compact set $\Delta_b$ where $b \in \Delta_b \subset \mathbb{R}^3$, and let $b$ be upper bounded by a scalar $\Gamma_b$ such that $||\Delta_b|| \leq \Gamma_b < \infty$.

**Assumption 2.** (Uniform boundedness of unknown noise $\omega$ in (23)) Let vector $\omega$ belong to a given compact set $\Delta_\omega$, where $\omega \in \Delta_\omega \subset \mathbb{R}^3$, and let $\omega$ be upper bounded by a scalar $\Gamma_\omega$ such that $||\Delta_\omega|| \leq \Gamma_\omega < \infty$.

4. Attitude Determination

As previously mentioned, attitude determination or estimation is an essential sub-task in most robotics and control applications. The attitude can be determined using a set of vector measurements made in body-frame and their observations in the inertial-frame as it acts as a linear transformation from one frame to the other [1, 5]. Attitude determination, in contrast to attitude estimation, takes an algebraic approach to
attitude reconstruction. Every attitude determination and estimation algorithm holds minimization of the cost function as its main objective. Wahba’s Problem presents an example of such a cost function \[6\]:

$$J(R) = \frac{1}{2} \sum_{i=1}^{n} s_i \left\| v_i^B - R^T v_i^I \right\|^2$$  \hspace{1cm} (32)

where \(s_i \in \mathbb{R}_+\) is the confidence level of the \(i\)th sensor measurement and at the same time it is a non-negative weight. The work proposed by \[6\] was purely algebraic. Over the following decades, a considerable effort was made in developing attitude determination algorithms based on a set of simultaneous inertial and body-frame vectors, for instance \[7, 8, 9, 10, 11, 12, 13, 14\]. All the algorithms in \[7, 8, 9, 10, 11, 12, 13, 14\] are applicable if and only if the statement in Remark 1 is met. In the subsections that follow, three common algebraic attitude determination algorithms are detailed, namely, triaxial attitude determination (TRIAD) \[15\], quaternion estimator (QUEST) \[9\], and singular value decomposition (SVD) \[10\].

4.1. Attitude Determination using TRIAD Algorithm

TRIaxial Attitude Determination (TRIAD) algorithm is one of the earliest and the simplest methods of attitude determination \[15\]. The TRIAD algorithm has been commonly used as a tool of attitude determination for almost two decades from the date invented until its replacement by more advanced algorithms. The underlining assumption of the TRIAD algorithm is the availability of two non-collinear vector observations at each time instant. Also, these vectors have to be non-collinear. The implementation of the TRIAD algorithm can be summarized in the following three steps:

\[
\begin{align*}
\text{Step1)} & \quad \text{normalization} \\
v_1^I &= v_1^I / \|v_1^I\|, \quad v_2^I = v_2^I / \|v_2^I\| \\
v_1^B &= v_1^B / \|v_1^B\|, \quad v_2^B = v_2^B / \|v_2^B\| \\
\text{Step2)} & \quad \text{collect three non-collinear vectors} \\
\tilde{v}_1^I &= \tilde{v}_1^I, \quad \tilde{v}_1^B = \tilde{v}_1^B \\
\tilde{v}_2^I &= \tilde{v}_1^I \times \tilde{v}_2^I, \quad \tilde{v}_2^B = \tilde{v}_1^B \times \tilde{v}_2^B \\
\tilde{v}_3^I &= \tilde{v}_1^I \times \tilde{v}_3^I, \quad \tilde{v}_3^B = \tilde{v}_1^B \times \tilde{v}_3^B \\
\text{Step3)} & \quad \text{obtain } R_y \in SO(3) \\
R_y &= \begin{bmatrix} \tilde{v}_1^B & \tilde{v}_2^B & \tilde{v}_3^B \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \tilde{v}_1^I & \tilde{v}_2^I & \tilde{v}_3^I \end{bmatrix} \begin{bmatrix} \tilde{v}_1^B & \tilde{v}_2^B & \tilde{v}_3^B \end{bmatrix} \\
&= \begin{bmatrix} v_1^I & v_2^I & v_3^I \end{bmatrix} \begin{bmatrix} v_1^B & v_2^B & v_3^B \end{bmatrix}
\end{align*}
\]

where \(R_y \in SO(3)\) denotes a reconstructed attitude. The aim of the algorithm is to drive \(R_y \rightarrow R\). Later, a series of modifications of the basic TRIAD algorithm \[15\] were proposed, such as the symmetric TRIAD algorithm \[13\] and optimal TRIAD algorithm \[14\]. The main shortcoming of the TRIAD algorithm in \[15\] is that by design it can use only two non-collinear observations. However, in spite of the above-mentioned drawback, TRIAD is a pioneer algorithm that served as a doorway to the more advanced methods and promoted the growth of the attitude estimation and determination research. As such, it is my believe that the earliest version of the TRIAD algorithm is brilliant in its simplicity and can be considered a predecessor of all the algorithms proposed after.
4.2. Attitude Determination using QUEST

TRIAD was displaced by QUaternion ESTimator (QUEST) [9], since QUEST allowed for attitude determination when two or more non-collinear observations are available ($n \geq 2$), in consistence with Remark 1. QUEST algorithm is able to find an optimal solution to Wahba’s problem [6] in (32) given $n$ observations. QUEST algorithm is a modification of its precursor, Davenport $q$-method [7], which provided an early solution to Wahba’s problem [6]. Let us represent the attitude with respect to the true unit-quaternion $Q = [q_0, q^\top] \in S^3$ [3, 4]

$$R_Q = \left(q_0^2 - \|q\|^2\right) I_3 + 2qq^\top + 2q_0 [q]_\times$$

as defined in (17). Consider the following weighting scheme

$$w_i = s_i / \sum_{i=1}^n s_i$$

with

$$B = \sum_{i=1}^n w_i \nu_i^B \left(\nu_i^T\right)^\top$$

(35)

Since the attitude in (34) represents a homogeneous quadratic function with respect to $Q$, one may obtain

$$\text{Tr} \{R_Q B^\top\} = Q^\top M Q$$

where $M$ is a symmetric matrix equivalent to

$$M = \begin{bmatrix}
\text{Tr} \{B\} & \left(\sum_{i=1}^n w_i \nu_i^B \times \nu_i^T\right)^\top \\
\sum_{i=1}^n w_i \nu_i^B \times \nu_i^T & B + B^\top - \text{Tr} \{B\} I_3 
\end{bmatrix}$$

(36)

Thus, it can be shown that the optimal unit-quaternion $Q_{opt} \in S^3$ satisfies

$$MQ_{opt} = \max \{\lambda (M)\} Q_{opt}$$

Accordingly, the complete QUEST algorithm can be given as follows:

$$\begin{align*}
\frac{w_i}{\sum_{i=1}^n s_i} & = s_i \\
B & = \sum_{i=1}^n w_i \nu_i^B \left(\nu_i^T\right)^\top \\
S & = B + B^\top \\
z & = \begin{bmatrix} B_{23} - B_{32} \\ B_{31} - B_{13} \\ B_{12} - B_{21} \end{bmatrix} = \sum_{i=1}^n w_i \nu_i^B \times \nu_i^T \\
M & = \begin{bmatrix}
\text{Tr} \{B\} & z^T \\
z & S - \text{Tr} \{B\} I_3 
\end{bmatrix} \\
\lambda_{max} & = \max \{\lambda (M)\} \\
\beta_1 & = \lambda_{max}^2 - \text{Tr} \{B\}^2 + \text{Tr} \{\text{adj} (S)\} \\
\beta_2 & = \lambda_{max} - \text{Tr} \{B\} \\
x_0 & = \det ((\lambda_{max} + \text{Tr} \{B\}) I_3 - S) \\
x & = (\beta_1 I_3 + \beta_2 S + S^2) z \\
Q_y & = \frac{1}{\sqrt{x_0^2 + \|x\|^2}} \begin{bmatrix} x_0 \\ x \end{bmatrix}
\end{align*}$$

(37)
where $Q_y \left[ q_{0y}, q_y^\top \right]^\top \in S^3$ denotes a reconstructed unit-quaternion with $q_{0y} \in \mathbb{R}$ and $q_y \in \mathbb{R}^3$, $\lambda(\mathcal{M})$ represents a set of eigenvalues of matrix $\mathcal{M}$, and $\lambda_{\text{max}}$ stands for the maximum value of $\lambda(\mathcal{M})$. Also, $\text{adj}(S)$ is an adjoint or adjugate of the square matrix $S$. It is obvious that the QUEST algorithm aims to drive $Q_y \to Q$. Up to the current moment QUEST remains one of the most widely used algorithms for solving Wahba’s problem [11].

4.3. Attitude Determination using SVD

Singular Value Decomposition (SVD) is another commonly used method of attitude determination. In consistence with Remark 1, it is able to use two or more non-collinear observations ($n \geq 2$). Considering the loss function in [16] and using SVD algorithm, attitude can be determined through the following series of steps [10]:

\begin{align*}
\text{Step 1)} & \text{ normalize weights } \\
& w_i = s_i / \sum_{i=1}^n s_i, \quad i = 1, 2, \ldots, n \\
\text{Step 2)} & \text{ minimize loss function } \\
& J(R) = 1 - \sum_{i=1}^n w_i \left( v_i^R \right)^\top R^\top v_i^I \\
& = 1 - \text{Tr} \left\{ R^\top B^I \right\} \\
& \text{where } \\
& B = \sum_{i=1}^n w_i v_i^B \left( v_i^I \right)^\top = USV^\top \\
\text{Step 3)} & \text{ solve for } U, S, \text{ and } V \text{ using SVD } \\
\text{Step 4)} & \text{ obtain } U_+ \text{ and } V_+ \\
& U_+ = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \text{det}(U) \end{bmatrix} \\
& V_+ = V \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \text{det}(V) \end{bmatrix} \\
\text{Step 5)} & \text{ obtain } R_y \in SO(3) \\
& R_y = V_+ U_+^\top \\
\end{align*}

where $R_y \in SO(3)$ denotes a reconstructed attitude and SVD aims to drive $R_y \to R$. The solution obtained by SVD is equivalent to the solution proposed in [16], with the only difference being the necessity to compute the SVD. In fact, SVD is one of the most robust numerical algorithms [17].

TRIAD, SVD, and QUEST algorithms outlined above along with [7, 8, 9, 10, 11, 12, 13, 14] are among several other algebraic algorithms proposed for attitude determination. Body-frame vector measurement are uncertain and are subject to significant bias and noise, which is not accounted for by the above-mentioned algebraic algorithms. In spite of their simplicity, the category of attitude determination algorithms in [7, 8, 9, 10, 11, 12, 13, 14] produce poor results in comparison with Gaussian and nonlinear attitude filters as will be illustrated in Section 7. Therefore, the attitude observation problem is best addressed using Gaussian and nonlinear filters.
5. Gaussian Attitude Filters

Define \( \Omega = [\Omega_x, \Omega_y, \Omega_z]^T \in \mathbb{R}^3 \) and let \( Q = [q_0, q^T] \in \mathbb{S}^3 \) denote a unit-quaternion vector that satisfies (14). Referring to the notation above define the following set of equations:

\[
\begin{align*}
\hat{\Omega} &= \begin{bmatrix} 0 \\ \Omega \end{bmatrix} \\
\Gamma(\Omega) &= \begin{bmatrix} 0 & -\Omega^T \\ \Omega & -[\Omega]_s \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_x & -\Omega_y & -\Omega_z \\ \Omega_x & 0 & \Omega_z & -\Omega_y \\ \Omega_y & -\Omega_x & 0 & \Omega_z \\ \Omega_z & \Omega_y & -\Omega_x & 0 \end{bmatrix} \\
\Xi(Q) &= \begin{bmatrix} -q^T \\
q_0 I_3 + [q]_x \end{bmatrix}
\end{align*}
\]

Recall the true attitude dynamics in unit-quaternion form \( \dot{Q} = \frac{1}{2} \Gamma(\Omega) Q \) in (28). Let \( \hat{Q} = [\hat{q}_0, \hat{q}^T]^T \in \mathbb{S}^3 \) denote the estimate of the true unit-quaternion vector \( Q \), where \( \hat{q}_0 \in \mathbb{R} \) and \( \hat{q} \in \mathbb{R}^3 \). Gaussian attitude filters aim to drive \( \hat{Q} \to Q \). The general design of a Gaussian attitude filter can be described with respect to unit-quaternion vector as follows:

\[
\dot{\hat{Q}} = \frac{1}{2} \Gamma(\widehat{\Omega}) \hat{Q}, \quad \hat{Q} \in \mathbb{S}^3 \text{ and } \widehat{\Omega} \in \mathbb{R}^3
\]

where \( \widehat{\Omega} \in \mathbb{R}^3 \) is to be designed subsequently. Gaussian filters can be easily utilized for the moving vehicles given the availability of:

- two or more non-collinear vectorial measurements in accordance with Remark 1 as well as,
- a rate gyroscope measurement (\( \Omega_m \)).

Let us modify the angular velocity measurements in (23) by adding a noise term:

\[
\Omega_m = \Omega + b + Q_\omega \omega
\]

with \( Q_\omega \in \mathbb{R}^{3 \times 3} \) being a nonzero diagonal weighting matrix associated with the angular velocity measurements whose covariance is \( \bar{Q}_\omega = Q_\omega Q_\omega^T \). Consider the bias \( b \) attached to angular velocity measurements to be unknown and slowly time-varying such that

\[
b = Q_b v(t)
\]

where \( Q_b \in \mathbb{R}^{3 \times 3} \) is a nonzero diagonal weighting matrix and \( Q_b = Q_b Q_b^T \). Slightly modifying the true body-frame measurements defined in (27) we obtain

\[
\begin{bmatrix} 0 \\ v_i^B \end{bmatrix} = Q^{-1} \circ \begin{bmatrix} 0 \\ v_i^T \end{bmatrix} \circ Q + \begin{bmatrix} 0 \\ Q_{v(i)\omega_i}^B \end{bmatrix}
\]

where \( Q_{v(i)} \in \mathbb{R}^{3 \times 3} \) is a nonzero diagonal weighting matrix such that the covariance associated with the body-frame measurements is \( \bar{Q}_{v(i)} = Q_{v(i)} Q_{v(i)}^T \) for all \( i = 1, 2, \ldots, n \). It can be noticed that \( \bar{Q}_\omega, \bar{Q}_b, \) and \( \bar{Q}_{v(i)} \) are positive definite matrices.

Over the past few decades, several Gaussian attitude filters have been proposed with the aim of improving the estimation process. The majority of the attitude filters within the Gaussian family formulate the attitude problem with respect to unit-quaternion [1]. The benefit of using unit-quaternion is the fact that it provides a nonsingular solution to the attitude parameterization. However, its main drawback is non-uniqueness in representation [3, 4]. The unit-quaternion attitude dynamics offer three main advantages, namely the dynamics in (28) are characterized by
1) vector form representation,
2) linearity, and
3) dependence on the quaternion state.

In consistence with the fact that the orientation of a rigid-body in the 3-dimensional space can be described by a 4-dimensional vector, the covariance matrix associated with noise has dimensions $4 \times 4$ and a rank of 3. One of the earliest attitude filters is the extended Kalman filter (EKF) proposed in [18]. EKF was followed by several Gaussian filters before the novel Kalman filter (KF) proposed in [19] which outperformed its predecessor. Multiplicative extended Kalman filter, which is a modification of EKF, is the state-of-the-art technology and an industry standard in the area of attitude estimation [20, 21, 22]. The other members of the Gaussian filter family include a modification of the EKF an invariant extended Kalman filter (IEKF); right IEKF which models the error in the inertial-frame [23]; left IEKF that is analogous to MEKF; and a Geometric approximate minimum energy filter (GAMEF) [21] that is developed based on the Mortensen’s approach [24]. When comparing the aforementioned Gaussian attitude filter, the following points should be taken into consideration [1, 25]:

1) KF, EKF, IEKF, and MEKF are quaternion-based, while GAMEF is developed on SO (3).
2) KF, EKF, and IEKF are based on optimal minimum-energy which is first order, while MEKF and GAMEF are based on optimal minimum-energy which is second order.
3) KF, EKF, and IEKF require less computational cost when compared to MEKF and GAMEF.
4) MEKF and GAMEF demonstrate better tracking performance when compared to KF, EKF, and IEKF.

Unscented Kalman filter (UKF) follows the Gaussian assumptions and has a structure analogous to KF. The only difference is that UKF uses a set of sigma points to improve the probability distribution [26, 27, 1, 25]. In comparison, one can find that

1) UKF outperforms KF and EKF in terms of tracking performance.
2) UKF requires more computational cost than both KF and EKF.
3) The use of sigma might add complexity to the estimation process.

Particle filters (PFs), despite being classified as stochastic filters, do not follow the Gaussian assumption [28, 29]. In comparison, it can noted that [1, 25]

1) PFs outperform UKF in terms of tracking performance.
2) PFs computational cost is higher than UKF.
3) PFs are not an optimal fit for small scale vehicles.
4) PFs do not have a clear measure of how close the obtained solution is to the optimal one.

In this Section, three of the most common continuous Gaussian attitude filters are presented, namely KF, MEKF and GAMEF. The discrete form of KF, MEKF and GAMEF can be found in the Appendix.
5.1. Kalman Filter

The normalized vectors of the inertial-frame observations and body-frame measurements defined in unit-quaternion form in (40) are as follows:

\[ \nu^I_i = \frac{\nu^I_i}{||\nu^I_i||}, \quad \nu^B_i = \frac{\nu^B_i}{||\nu^B_i||}, \quad \forall i = 1, 2, \ldots, n \]

Define the true body-frame vector and its normalized values, respectively, by

\[ \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = Q^{-1} \circ \begin{bmatrix} 0 \\ \nu^I_i \end{bmatrix} \circ Q \\ \dot{\nu}^B_i = \frac{\nu^B_i}{||\nu^B_i||}, \quad \forall i = 1, 2, \ldots, n \end{cases} \]

One could rewrite (41) as

\[ Q \circ \begin{bmatrix} 0 \\ \dot{\nu}^B_i \end{bmatrix} = \begin{bmatrix} 0 \\ \nu^I_i \end{bmatrix} \circ Q \]

\[ \begin{bmatrix} 0 \\ \dot{\nu}^B_i - (\dot{\nu}^B_i)^\top \end{bmatrix} = \begin{bmatrix} 0 \\ \nu^I_i \end{bmatrix} - (\nu^I_i)^\top \]

Consequently,

\[ \dot{\gamma} = 0_{4 \times 1} = \sum^n_i \left[ \begin{bmatrix} 0 \\ \dot{\nu}^B_i - \nu^I_i \end{bmatrix} - (\dot{\nu}^B_i - \nu^I_i)^\top \right] \]

where \( \dot{\gamma} \) denotes an ideal output signal. Accordingly, the true attitude problem can be represented as a linear time-variant state-space problem

\[ \begin{cases} \dot{Q} = \frac{1}{2} \Gamma (\Omega) Q \\ \dot{\gamma} = 0_{4 \times 1} \end{cases} \]

Unfortunately, the measuring unit cannot provide the true body-frame vector (\( \dot{\nu}^B_i \)). From (40), it can be found that

\[ Q \circ \begin{bmatrix} 0 \\ \nu^I_i \end{bmatrix} = \begin{bmatrix} 0 \\ \nu^I_i \end{bmatrix} \circ Q + \Gamma \left( Q_{v(i)} \omega^B_i \right) Q \]

that is

\[ \gamma = \sum^n_i \left[ \begin{bmatrix} 0 \\ \dot{\nu}^B_i - \nu^I_i \end{bmatrix} - (\dot{\nu}^B_i - \nu^I_i)^\top \right] \right] Q + \sum^n_i \Gamma \left( Q_{v(i)} \omega^B_i \right) \]

\[ = \frac{1}{2} \sum^n_i \xi(Q) Q_{v(i)} \omega^B_i \]

where \( Q_{v(i)} \omega^B_i \) is to be readjusted after normalization. For \( \Omega_m = \Omega + Q_{\omega} \omega \), the attitude problem becomes

\[ \begin{cases} \dot{Q} = \frac{1}{2} \Gamma (\Omega_m - Q_{\omega} \omega) Q \\ \dot{\gamma} = \frac{1}{2} \sum^n_i \xi(Q) Q_{v(i)} \omega^B_i \end{cases} \]

The basic Kalman filter of the problem in (43) and the novel Kalman filter proposed in [19] in their discrete form can be found in the Appendix.
5.2. Multiplicative Extended Kalman Filter

The MEKF and GAMEF are second order filters driven with respect to a cost function. For \( J = J(t; X_0, \omega_{\|0,t\}, v_{\|0,t}, \omega_i^B_{\|0,t}) \), consider the following cost function [21]:

\[
J = \frac{1}{2} \text{Tr} \left\{ (I_3 - R(0)) K_{R(0)}^\top (I_3 - R(0))^\top \right\} + \frac{1}{2} \int_0^T \left( \omega^\top \omega + v^\top v + \sum_{i=1}^n (\omega_i^B)^\top \omega_i^B \right) d\tau
\]

The optimal control problem of the cost function above can be approached in terms of the pre-Hamiltonian \( (H^-) \). Next, let us define a value function that is subject to minimization

\[
V(R,t) = \min_{\omega_{\|0,t}} J
\]

Applying the principle of dynamic programming in [30] yields a Hamilton-Jacobi-Bellman (HJB) equation

\[
H - \frac{\partial}{\partial t} V(R,t) = 0
\]

Resorting to the Mortensen’s approach [24] allows to obtain an explicit, recursive solution. The complete steps of the MEKF and GAMEF derivation can be found in [31, 22] and [21], respectively.

Multiplicative extended Kalman filter (MEKF) [31] is a standard in the industry of recursive attitude filtering applications [20, 21, 22, 25]. The structure of MEKF is as follows [31, 22]

\[
\begin{align*}
\dot{\hat{Q}} &= \frac{1}{2} \Gamma (\Omega_m - \hat{b} + P_a W) \hat{Q} \\
W &= \sum_{i=1}^n \dot{\omega}_i^B \times Q_{\nu(i)}^{-1} (\dot{\nu}_i^B - \nu_i^B)
\end{align*}
\]

where \( \hat{Q} \in S^3 \) is an estimate of the true unit-quaternion, \( \odot \) is a quaternion multiplication operator, \( v_i^T \in \mathbb{R}^3 \) is the \( i \)th vectorial measurement in the inertial-frame, \( \dot{\omega}_i^B \in \mathbb{R}^3 \) is the \( i \)th body-frame vectorial estimate. Additionally,

\[
\begin{align*}
\dot{\hat{b}} &= P_c^T W \\
S &= \sum_{i=1}^n \dot{\omega}_i^B \times Q_{\nu(i)}^{-1} [\dot{\nu}_i^B]_x \\
P_a &= Q_{\omega} + 2 \mathcal{P} S \left( P_a [\Omega_m - \hat{b}]_x - P_c \right) - P_a S P_a \\
P_b &= Q_{\omega} - P_c S P_c \\
P_c &= -[\Omega_m - \hat{b}]_x P_c - P_a S P_c - P_b
\end{align*}
\]

with \( Q_{\nu(i)}, Q_{\omega}, Q_b \in \mathbb{R}^{3 \times 3} \) being covariance matrices, for all \( i = 1, 2, \ldots, n \).

5.3. Geometric Approximate Minimum-Energy Filter

GAMEF is one of the recent Gaussian attitude filters [21]. Its structure is similar to the MEKF and can be presented as follows [21]:

\[
\begin{align*}
\dot{\hat{Q}} &= \frac{1}{2} \Gamma (\Omega_m - \hat{b} + P_a W) \hat{Q} \\
W &= \sum_{i=1}^n \dot{\omega}_i^B \times Q_{\nu(i)}^{-1} (\dot{\nu}_i^B - \nu_i^B)
\end{align*}
\]
with $\hat{Q} \in S^3$ being the estimate of the true unit-quaternion, $\circ$ being a quaternion multiplication operator, $\upsilon_f^i \in \mathbb{R}^3$ being the $i$th vectorial measurement in the inertial-frame, and $\delta B^i \in \mathbb{R}^3$ being the $i$th body-frame vectorial estimate. Additionally

$$
\dot{\hat{b}} = P_c^T W \\
S = \sum_{i=1}^{n} [\delta B^i] \times \hat{Q}_{v(i)}^{-1} \times [\delta B^i] \times \\
C = \sum_{i=1}^{n} P_s \left( \hat{Q}_{v(i)} \left( \delta B^i - \upsilon^i_B \right) \left( \delta B^i \right)^T \right) \\
E = \text{Tr} \{C\} I_3 - C \\
\dot{P}_a = Q_{\omega} + 2P_s \left( P_a \left[ \Omega_m - \hat{b} - \frac{1}{2}P_a W \right] x - P_c \right) + P_a (E - S) P_a \\
\dot{P}_b = Q_b + P_c (E - S) P_c \\
\dot{P}_c = - \left[ \Omega_m - \hat{b} - \frac{1}{2}P_a W \right] x P_c + P_a (E - S) P_c - P_b
$$

where $Q_{v(i)}, Q_{\omega}, Q_b \in \mathbb{R}^{3 \times 3}$ are covariance matrices, for all $i = 1, 2, \ldots, n$.

6. Nonlinear Attitude Filters

This section presents different categories of nonlinear attitude filters in continuous form, while the discrete representation can be found in the Appendix. Recall the true attitude dynamics $\dot{R} = R [\Omega] \times$ in (22). Let $\hat{R} \in \text{SO}(3)$ denote the estimate of the true attitude $R$. The goal of nonlinear attitude filters is to drive $\hat{R} \rightarrow R$. Due to the fact that the true attitude dynamics

1) modeled on the Lie group of SO(3) and 
2) naturally nonlinear,

nonlinear attitude filter design generally has the following structure

$$
\dot{R} = \hat{R} \left[ \hat{\Omega} \right] \times, \quad \hat{R} \in \text{SO}(3) \quad \text{and} \quad \hat{\Omega} \in \mathbb{R}^3
$$

Such filter design (48) is a perfect fit for the attitude kinematics as it is modeled on the Lie group of SO(3) and accounts for their nonlinear nature. $\hat{\Omega}$ is to be defined in the subsequent subsection. The need for nonlinear attitude filters that would be robust against uncertainty in sensor measurements has grown dramatically over the past two decades, in particular with the advancement of low-cost IMUs technology [32, 33, 34, 35, 1, 36]. The nonlinear filter design presented above can be implemented given

- two or more non-collinear vectorial measurements in accordance with Remark 1, as well as
- a rate gyroscope measurement ($\Omega_m$).

The above-mentioned measurements can be obtained, for example, by a low-cost IMU module as explained in Section 3. It is worth noting that high quality sensors are not an optimal fit for small vehicles due to the fact that they are normally

1) large in size, 
2) heavy in weight, and 
3) expensive.

In contrast, a typical low-cost IMU module has the following three merits:

1) small size,
2) low weight, and
3) low price.

However, the main challenge of working with the low-cost IMU modules is the fact that they are subject to high levels of noise and bias components [1, 25]. First and higher orders of Gaussian attitude filters provide reasonable estimates if the rigid-body is equipped with high quality sensors. Whereas, if the rigid-body is fitted with a low-cost IMU module, first order Gaussian attitude filter produce poor results. Thus, in that case, the user has to resort to either a nonlinear attitude filter or a high order Gaussian attitude filter. Nonlinear attitude filters have the following three advantages:
1) better tracking performance,
2) simplicity of filter derivation, and
3) less computational power requirements

when compared with Gaussian attitude filters [1, 5, 32, 25, 20]. Therefore, nonlinear attitude filters have received considerable attention over the last few decades, for example [20, 32, 37, 34, 35, 36, 1, 5, 25]. The family of nonlinear attitude filters can be further subdivided into two distinct categories:

1) **Nonlinear deterministic attitude filters**, which consider the measurements of angular velocity in (23) to be corrupted with unknown constant bias such as
\[ \Omega_m = \Omega + b \in \mathcal{B}, \ (\omega = 0) \]

However, they disregard the noise attached to \( \Omega_m \) in both filter derivation and the stability analysis, for example [32, 37, 34, 35, 5].

2) **Nonlinear stochastic attitude filters**, that consider angular velocity measurements in (23) to be
\[ \Omega_m = \Omega + b + \omega \in \mathcal{B}, \ (\omega \neq 0) \]

This way, both the unknown bias and the unknown noise attached to \( \Omega_m \) are accounted for in the process of filter derivation and the stability analysis, for instance, [36, 1, 38, 39].

6.1. Error Criteria, Filter Structure and Setup

Let \( \hat{R} \in SO(3) \) be the estimate of the true body-fixed rotation matrix. Let the error from the body-fixed frame to the estimator frame be given as
\[ \hat{R} = R^\top \hat{R} \]  
(49)

Recall (48) and consider the estimate of the attitude dynamics to be defined as
\[ \dot{\hat{R}} = \hat{R} \left[ \Omega_m - b - k_w W \right] \times \]  
(50)

where \( \Omega_m \) is a gyro measurement as in (23), \( \hat{b} \) is an estimate of the true bias \( b \) associated with angular velocity measurement, and \( W \) is a correction factor. The design of \( \hat{b} \) and \( W \) will vary based on the type of filter, and therefore, will be defined separately in each of the following Subsections: 6.2, 6.3, 6.4, 6.5, and 6.6. It is worth noting that the structure of the filter dynamics in (50) or a little bit of variation \( (\dot{\hat{R}} = [\hat{\Omega}] \times \hat{R}) \) is common when designing a nonlinear attitude filter, for example [20, 32, 37, 34, 35, 36, 1, 5, 25]. Define the error between the true and the estimated bias as
\[ \bar{b} = b - \hat{b} \]  
(51)

The difference between various nonlinear filters consists mainly in the design of \( \hat{b} \) and the correction factor \( W \), which in turn depend on
6.1.1. Direct Filter Setup
From (20) and (24), recall that $v^I_i \in \{I\}$ and $v^B_i \in \{B\}$ for $i = 1, 2, \ldots, n$. Define

$$M^I = (M^I)^\top = \sum_{i=1}^{n} s_i v^I_i (v^I_i)^\top$$

$$M^B = (M^B)^\top = \sum_{i=1}^{n} s_i v^B_i (v^B_i)^\top$$

$$= R^\top M^I R$$

(52)

where $s_i > 0$ indicates the confidence level of the $i$th sensor measurement for all $i = 1, 2, \ldots, n$. Since $\|v^I_i\|^2 = \|v^B_i\|^2 = 1$ and in accordance with property (5), one obtains

$$\text{Tr} \left\{ M^B \right\} = \text{Tr} \left\{ M^I \right\} = \sum_{i=1}^{n} s_i$$

(53)

Define

$$\hat{v}_i^B = \hat{R}^\top v^I_i$$

(54)

such that $\hat{v}_i^B$ is the estimate of $v^B_i$ for all $i = 1, 2, \ldots, n$. From (52) and with the aid of the identity in (9), one obtains

$$M^B = \hat{R}^\top M^I R + R^\top M^I \hat{R}$$

$$= -[\Omega]_x R^\top M^I R + R^\top M^I [\Omega]_x$$

$$= [M^B, [\Omega]_x]$$

(55)

where $M^B = R^\top M^I R$ as in (52). Also, $M^I = 0_{3 \times 3}$ due to the fact that $v^I_i \in \{I\}$ denotes a fixed observation. The next stage is the introduction of the three auxiliary variables in terms of vectorial measurements, namely, vex $\left( \mathcal{P}_a \left( M^B \hat{R} \right) \right)$, $||M^B \hat{R}||_I$, and $\mathbf{Y} \left( M^B, \hat{R} \right)$. From identity (6), one finds

$$\left[ \sum_{i=1}^{n} \frac{s_i}{2} \hat{v}_i^B \times v^B_i \right] \times = \sum_{i=1}^{n} \frac{s_i}{2} \left( v^B_i (\hat{v}_i^B)^\top - \hat{v}_i^B (v^B_i)^\top \right)$$

$$= \frac{1}{2} R^\top M^I R \hat{R} - \frac{1}{2} \hat{R}^\top R^\top M^I R$$

$$= \mathcal{P}_a \left( M^B \hat{R} \right)$$

such that

$$\text{vex} \left( \mathcal{P}_a \left( M^B \hat{R} \right) \right) = \sum_{i=1}^{n} \frac{s_i}{2} \hat{v}_i^B \times v^B_i$$

(56)
where \( s_i > 0 \) for all \( i = 1, 2, \ldots, n \). In the light of (4), the normalized Euclidean distance of \( M^B \tilde{R} \) is equivalent to

\[
||M^B \tilde{R}||_I = \frac{1}{4} \text{Tr} \left\{ I_3 - M^B \tilde{R} \right\} \\
= \frac{1}{4} \text{Tr} \left\{ I_3 - \sum_{i=1}^{n} s_i v_i^B \left( \hat{\theta}_i^B \right)^\top \right\} \\
= \frac{1}{4} \sum_{i=1}^{n} s_i \left( 1 - \left( \hat{\theta}_i^B \right)^\top v_i^B \right)
\]

(57)

Let us introduce the following variable

\[
\Upsilon \left( M^B, \tilde{R} \right) = \text{Tr} \left\{ \left( M^B \right)^{-1} M^B \tilde{R} \right\} \\
= \text{Tr} \left\{ \left( \sum_{i=1}^{n} s_i v_i^B \left( \hat{\theta}_i^B \right)^\top \right)^{-1} \sum_{i=1}^{n} s_i v_i^B \left( \hat{\theta}_i^B \right)^\top \right\} 
\]

(58)

6.1.2. Stochastic Filter Setup

The nonlinear stochastic attitude filters presented in [36, 1, 25, 39] consider the angular velocity measurements to be

\[
\Omega_m = \Omega + b + \omega
\]

where \( \omega \) is a zero-mean Gaussian noise vector which is bounded and therefore follows Assumption 2. Due to the fact that the derivative of any Gaussian process results in a Gaussian process [40, 41, 42, 1], the vector \( \omega \) can be redefined as a function of Brownian motion process vector

\[
\omega = Q_\omega \frac{d\beta}{dt}
\]

(59)

where \( Q_\omega \in \mathbb{R}^{3 \times 3} \) is a nonnegative real matrix whose diagonal consists of unknown time-variant nonnegative components while the off-diagonal components are zeros or, more simply put,

\[
Q_\omega = \begin{bmatrix}
Q_{\omega(1,1)} & 0 & 0 \\
0 & Q_{\omega(2,2)} & 0 \\
0 & 0 & Q_{\omega(3,3)}
\end{bmatrix}
\]

The covariance of the noise vector \( \omega \) is given by \( Q_\omega^2 = Q_\omega Q_\omega^\top \). Also, the properties of the Brownian motion process are given as follows [40, 41, 42]

\[
\text{P} \{ \beta (0) = 0 \} = 1, \quad \mathbb{E} [d\beta/dt] = 0, \quad \mathbb{E} [\beta] = 0
\]

Nonlinear stochastic attitude filters aim to achieve adaptive stabilization for the case of unknown bias and unknown time-variant covariance matrix. Therefore, let us define a new variable \( \sigma \in \mathbb{R}^3 \) which denotes the upper bound of the covariance matrix \( Q_\omega^2 \) [36, 1, 25, 39]

\[
\sigma = \left[ \max \left\{ Q_{\omega(1,1)}^2 \right\}, \max \left\{ Q_{\omega(2,2)}^2 \right\}, \max \left\{ Q_{\omega(3,3)}^2 \right\} \right]^\top
\]

(60)

with \( \max \{ \cdot \} \) being the maximum value of a component. According to (60), \( \sigma \) is a constant vector that refers to the upper bound of the diagonal of the covariance matrix \( Q_\omega^2 \). Let \( \hat{\sigma} \in \mathbb{R}^3 \) denote the estimate of \( \sigma \), and define the error between \( \sigma \) and \( \hat{\sigma} \) by

\[
\tilde{\sigma} = \sigma - \hat{\sigma}
\]

(61)
6.1.3. Error Dynamics and Error Function Criteria

From (22) and (50), the dynamics of the error in (49) are equivalent to

\[
\dot{\hat{R}} = R^T \dot{\hat{R}} + \dot{\hat{R}}^T R
\]

\[
= R^T [\Omega + \dot{b} - k_w W] x + [\Omega] x R^T \dot{R}
\]

\[
= \hat{R} \dot{[\Omega] x} - [\Omega] x \dot{\hat{R}} + \dot{\hat{R}} [\dot{b} - k_w W] x
\]

\[
= \dot{\hat{R}} [\Omega] x + \dot{\hat{R}} [\dot{b} - k_w W] x
\]

\[
= \hat{R} [\Omega] x + \dot{\hat{R}} [\dot{b} - k_w W] x
\]

(62)

where \([\Omega] x = - [\Omega] x\) and the Lie bracket \([R, [\Omega] x] = [\Omega] x B - [\Omega] x \dot{R}\) as in (9).

In general terms, the most important component of designing a new nonlinear attitude filter is a careful selection of an error function. The attitude error function presented in [37] has been one of the most commonly used error function over the last few years. Multiple attempts have been made to improve the error function in [37] through minor modifications [32, 33, 35]. However, the performance did not see significant improvement. The critical weakness of the error function in [37, 32, 33, 35] consists in the slow convergence of attitude error, in particular when faced with large error in attitude initialization. A new form of an error function introduced in [34, 43, 1, 36] provides faster convergence of attitude error to the stable equilibrium point or to its close neighborhood. Nonetheless, the error functions proposed in [34, 43, 1, 36] offer no systematic convergence in transient and steady-state performance. In simple terms, the transient performance of the error function in [34, 43, 1, 36] does not follow predefined dynamically reducing boundaries of transient and steady-state error. Therefore, the prediction of transient and steady-state performance of attitude error in [37, 32, 33, 35, 34, 43, 1, 36] is almost impossible. Aiming to provide fast and guaranteed transient and steady-state performance, new solutions are proposed in [5, 39]. The solution offered in [5] is a nonlinear deterministic filter, while the solution in [39] is a nonlinear stochastic filter.

Before we proceed further, it is important to define \(b\) as an unknown constant bias bounded in accordance with Assumption 1. Similarly, \(\sigma\) is an unknown constant vector defined in (60) and bounded in consistent with Assumption 1 and 2. Let us introduce the following unstable set which is similar to Definition 1 and includes three unstable equilibrium points

\[
U = \{ \hat{R} (0) \in SO (3) | \text{Tr} \{ \hat{R} (0) \} = -1 \}
\]

(63)

6.2. Constant Gain Nonlinear Deterministic Attitude Filter

6.2.1. Semi-direct Filter

Consider the error function defined in [32]

\[
E_{cgs} = \frac{1}{4} \| I_3 - \hat{R} \|^2
\]

\[
= \frac{1}{4} \text{Tr} \{ (I_3 - \hat{R})^T (I_3 - \hat{R}) \}
\]

\[
= \frac{1}{2} \text{Tr} \{ I_3 - \hat{R} \}
\]

Consider the following constant gain semi-direct nonlinear deterministic attitude filter (CGSd-NDAF) [32]

\[
\left\{
\begin{array}{l}
\dot{\hat{R}} = \hat{R} \left[ \Omega_m - \dot{\hat{b}} - k_w W \right] x \\
W = \text{vex} \left( \mathcal{P}_a (\hat{R}) \right), \quad \hat{R} = R^T \hat{y} \\
\dot{\hat{b}} = \gamma W
\end{array}
\right.
\]

(64)
where \( \gamma, k_w \in \mathbb{R}_+ \) are positive constants, \( W \) is a correction factor, \( \hat{b} \) is the estimate of the true bias, and \( R_y \) is a reconstructed attitude obtained by one of the algorithms in \((33), (37), (38)\), or any other method of attitude determination. Consider the Lyapunov function candidate

\[
V = 2E_{cgd} + \frac{1}{\gamma} \|\hat{b}\|^2 = \text{Tr} \{ I_3 - \hat{R} \} + \frac{1}{\gamma} \hat{b}^\top \hat{b}
\]

provided that \( \hat{R}(0) \notin U_6 \) in \((63)\). Differentiating \( V \), considering \( \dot{\hat{R}} \) in \((62)\), and directly substituting \( W \) and \( \hat{b} \) with their definitions in \((64)\), one obtains

\[
\dot{V} = -\text{Tr} \left\{ \dot{\hat{R}} \left[ \Omega_m - \hat{b} - k_w W \right] \times \right\} - \frac{2}{\gamma} \hat{b}^\top \hat{b} = 2\text{vex} \left( \mathcal{P}_a \left( \dot{\hat{R}} \right) \right)^\top \left( \hat{b} - k_w W \right) - \frac{2}{\gamma} \hat{b}^\top \hat{b} = -2k_w \| \text{vex} \left( \mathcal{P}_a \left( \dot{\hat{R}} \right) \right) \|^2
\]

where \(-\text{Tr} \left\{ \dot{\hat{R}} \left[ \hat{b} \right] \times \right\} = 2\text{vex} \left( \mathcal{P}_a (\dot{R}) \right)^\top \hat{b} \) as defined in identity \((12)\). As stated by Barbalat’s lemma, \( \| \text{vex} \left( \mathcal{P}_a (\dot{R}) \right) \|^2 \) converges to zero and \( \hat{R} \rightarrow I_3 \) as \( t \rightarrow \infty \).

### 6.2.2. Direct Filter

From \((52)\) and \((54)\), consider the error function below \([32]\)

\[
E_{cgd} = \sum_{i=1}^{n} s_i \left( 1 - \text{Tr} \left\{ V_i^B \left( \hat{b}_i^B \right)^\top \right\} \right) = \sum_{i=1}^{n} s_i \left( 1 - \text{Tr} \left\{ R_i^\top v_i^B \left( v_i^B \right)^\top R_i \right\} \right) = \sum_{i=1}^{n} s_i - \text{Tr} \left\{ M_i^B \hat{R} \right\}
\]

Consider the following constant gain direct nonlinear deterministic attitude filter (CGD-NDAF) \([32]\)

\[
\begin{align*}
\dot{\hat{R}} &= \hat{R} \left[ \Omega_m - \hat{b} - k_w W \right] \times \\
W &= \text{vex} \left( \mathcal{P}_a (M_i^B \hat{R}) \right) \\
\hat{b} &= \gamma W
\end{align*}
\]

with \( \gamma, k_w \in \mathbb{R}_+ \) being positive constants, \( W \) being a correction factor, \( \hat{b} \) being an estimate of the true bias, and \( \text{vex} (\mathcal{P}_a (M_i^B \hat{R})) \) being obtained through vectorial measurements as in \((56)\). Define the following Lyapunov function candidate

\[
V = E_{cgd} + \frac{1}{\gamma} \|\hat{b}\|^2 = \sum_{i=1}^{n} s_i - \text{Tr} \left\{ M_i^B \hat{R} \right\} + \frac{1}{\gamma} \hat{b}^\top \hat{b}
\]

provided that \( \hat{R}(0) \notin U_6 \) in \((63)\). Differentiating \( V \), considering \( \dot{\hat{R}} \) in \((62)\), and directly substituting \( W \) and \( \hat{b} \) with their definitions in \((64)\), one finds

\[
\dot{V} = -\text{Tr} \left\{ \left[ M_i^B \hat{R}, [\Omega_m]_x \right] \right\} - \text{Tr} \left\{ M_i^B \hat{R} \left[ \hat{b} - k_w W \right] \times \right\} - \frac{2}{\gamma} \hat{b}^\top \hat{b} = 2\text{vex} \left( \mathcal{P}_a (M_i^B \hat{R}) \right)^\top \left( \hat{b} - k_w W \right) - \frac{2}{\gamma} \hat{b}^\top \hat{b} = -2k_w \| \text{vex} \left( \mathcal{P}_a (M_i^B \hat{R}) \right) \|^2
\]
where $\text{Tr} \{ [M^B\hat{R}, \Omega]_\times \} = 0$ as in identity (10) and $-\text{Tr} \{ M^B\hat{R} \hat{b} \} = 2 \text{vex} (\mathcal{P}_a (M^B\hat{R}))^T \hat{b}$ as defined in the identity in (12). Also, Barbalat’s lemma could be invoked to illustrate that $\|\text{vex}(\mathcal{P}_a(M^B\hat{R}))\|^2$ converges to zero and $\hat{R} \to I_3$ as $t \to \infty$.

### 6.3. Adaptive Gain Nonlinear Deterministic Attitude Filter

The nonlinear deterministic filter proposed in Subsection 6.2 is characterized by slow convergence of attitude error. Aiming to address this shortcoming, several solutions designed with attitude error-based adaptive gain have been proposed, for instance [35, 34, 1, 36, 5, 39]. This Subsection presents an adaptive gain nonlinear deterministic attitude filter (AG-NDAF) proposed in [34] which is semi-direct (requires attitude reconstruction). Consider the following error function

$$E_{ag} = \frac{1}{1 + \text{Tr} \{ \hat{R} \}} \text{vex} (\mathcal{P}_a (\hat{R})) \text{, } \hat{R} = R^y_\gamma \hat{R}$$

Based on the error function given above, the AG-NDAF is designed as follows

$$\begin{cases} \dot{\hat{R}} = \hat{R} \left[ \Omega_m - \hat{b} - k_w W \right]_\times \\ W = \frac{1}{1 + \text{Tr} \{ \hat{R} \}} \text{vex} (\mathcal{P}_a (\hat{R})) \text{, } \hat{R} = R^y_\gamma \hat{R} \\ \dot{\hat{b}} = \gamma W \end{cases} \quad (66)$$

where $\gamma, k_w \in \mathbb{R}_+$ are positive constants, $W$ is a correction factor, $\hat{b}$ is an estimate of the true bias, and $R^y_\gamma$ is the reconstructed attitude obtained by one of the algorithms in (33), (37), (38) or any other method of attitude determination. It can be easily noticed that $1 / (1 + \text{Tr} \{ \hat{R} \})$ is an adaptive gain whose value becomes increasingly aggressive as $\text{Tr} \{ \hat{R} \} \to -1$. Define the following Lyapunov function candidate

$$V = \ln (2) - \frac{1}{2} \ln (1 + \text{Tr} \{ \hat{R} \}) + \frac{1}{\gamma} \hat{b}^T \hat{b}$$

For $\hat{R} (0) \notin \mathcal{U}_5$ in (63), differentiating $V$, considering $\dot{\hat{R}}$ in (62), and directly substituting $W$ and $\dot{\hat{b}}$ in (66), one obtains

$$\dot{V} = - k_w \| E_{ag} \|^2$$

Hence, in the light of Barbalat’s lemma, $\| E_{ag} \|^2$ converges to zero and $\hat{R} \to I_3$ as $t \to \infty$.

### 6.4. Guaranteed Performance Nonlinear Deterministic Attitude Filter

The filter proposed in Subsection 6.3 tackles the weakness of problem convergence of attitude error. However, it is not characterized by guaranteed measures of transient and steady-state performance of attitude error convergence [5, 39]. This Subsection presents guaranteed performance nonlinear deterministic attitude filters (GP-NDAF) introduced in [5]. GP-NDAF achieves guaranteed performance though the following steps:

**Step 1:** Define an attitude error function, for example, in terms of normalized Euclidean distance

$$\|\hat{R} (t)\|_1 = \frac{1}{4} \text{Tr} \{ I_3 - \hat{R} \} \quad (67)$$

in accordance with (4). In order to achieve guaranteed measures of transient and steady-state performance of the error function in (67), it is necessary to constrain $\|\hat{R} (t)\|_1$ to initially start within a large set and reduce systematically and smoothly to settle within a narrow set. Thus, the next step is the definition of the dynamically reducing boundaries.
**Step 2):** Define a dynamic reducing boundaries as
\[ \xi(t) = (\xi_0 - \xi_\infty) \exp(-\ell t) + \xi_\infty \]
with \( \xi_0 = \xi(0) \) being the upper bound of the predefined large set, \( \xi_\infty \) being the upper bound of the narrow set, and \( \ell \) being a positive constant refers to the convergence rate of \( \xi(t) \). Next, \( ||\dot{R}(t)||_I \) should be defined as a function of the dynamically reducing boundaries \( \xi(t) \).

**Step 3):** Redefine the error function
\[ ||\tilde{R}(t)||_I = \xi(t) Z(\mathcal{E}) \]
such that \( Z(\mathcal{E}) \) is a smooth function to be defined, for instance
\[ Z(\mathcal{E}) = \bar{\delta} \exp(\mathcal{E}) - \delta \exp(-\mathcal{E}) \]
where \( \bar{\delta} \) and \( \delta \) are positive constants selected to satisfy \(-\delta < Z(\mathcal{E}) < \bar{\delta}\), for \( ||\tilde{R}(0)||_I \geq 0 \). Since the error \( ||\tilde{R}(t)||_I \) is constrained by \( \xi(t) \), let us define the unconstrained error \( \mathcal{E} \).

**Step 4):** Obtain the unconstrained error
\[ \mathcal{E} = \frac{1}{2} \ln \frac{\delta + ||\tilde{R}||_I/\xi}{\delta - ||\tilde{R}||_I/\xi} \]  
with the following unconstrained error dynamics
\[ \dot{\mathcal{E}} = \mu \left( \frac{d}{dt} ||\tilde{R}||_I - \frac{\delta}{\xi} ||\tilde{R}||_I \right) \]  
and
\[ \mu = \frac{1}{2} \frac{1}{\delta \xi + ||\tilde{R}||_I} + \frac{1}{2} \frac{1}{\delta \xi - ||\tilde{R}||_I} \]
From (62), one finds
\[ \frac{d}{dt} ||\tilde{R}||_I = \frac{1}{2} \text{vex}(\mathcal{P}_a(\tilde{R}))^\top (\tilde{b} - W) \]

### 6.4.1. Semi-direct Filter

Consider the following design of a guaranteed performance semi-direct nonlinear deterministic attitude filter (GPSd-NDAF) [5]

\[
\begin{align*}
\dot{\hat{R}} &= \hat{R} \left[ \Omega_m - \tilde{b} - W \right]
\times \\
W &= 2k_w \frac{\mathcal{E} - \xi/\bar{\xi}}{1 - ||\tilde{R}||_I} \text{vex}(\mathcal{P}_a(\tilde{R})) \\
\tilde{b} &= \frac{1}{2} \mu \mathcal{E} \text{vex}(\mathcal{P}_a(\tilde{R})) \\
\hat{R} &= R_y \tilde{R}
\end{align*}
\]

where \( \gamma, k_w \in \mathbb{R}_+ \) are positive constants, \( W \) is a correction factor, \( \tilde{b} \) is the estimate of the true bias, and \( R_y \) is a reconstructed attitude obtained by one of the algorithms in (33), (37), (38), or any other method of attitude determination. From (71), it becomes apparent that the term multiplied by \( \text{vex}(\mathcal{P}_a(\tilde{R})) \) becomes increasingly aggressive as \( ||\tilde{R}||_I \to +1 \). Moreover, it forces the observer to obey the predefined transient and steady-state measures. Consider the following Lyapunov function candidate
\[ V = \frac{1}{2} \mathcal{E}^2 + \frac{1}{2\gamma} \tilde{b}^\top \tilde{b} \]
for any \( \hat{R}(0) \not\in \mathcal{U}_k \) in (63). Differentiating \( V \), considering \( \dot{E} \) in (69), and directly substituting \( W \) and \( \hat{b} \) with their definitions in (71), it can be found that

\[
\dot{V} = -4k_w \|\hat{R}\|_I \mu^2 \varepsilon^2
\]

On the basis of Barbalat’s lemma, \( \dot{V} \to 0 \) as \( t \to \infty \). Also, according to Proposition 1 in [5], \( \|E\| \to 0 \) implies that \( \|\hat{R}\|_I \to 0 \) and vice versa. In addition, \( \mu \) is positive for all \( t \geq 0 \). As such, \( \hat{R} \to I_3 \) as \( t \to \infty \) with guaranteed measures of transient and steady-state performance [5].

6.4.2. Direct Filter

Let us modify the error function in (67) to

\[
\|M^B \hat{R}\|_I = \frac{1}{4} \text{Tr} \left\{ I_3 - M^B \hat{R} \right\}
\]

(72)

where \( \|M^B \hat{R}\|_I \) is defined in terms of vectorial measurements as in (57). Thus, with the aid of (62), the following equations can be easily obtained

\[
\begin{align*}
\mathcal{E} &= \frac{1}{2} \text{ln} \left( \frac{\|\dot{M}^B \hat{R}\|_I}{\|\dot{M}^B \hat{R}\|_I} \right) \\
\mu &= \frac{\|\dot{M}^B \hat{R}\|_I}{\|\dot{M}^B \hat{R}\|_I} + \frac{1}{2} \left( I - \frac{1}{\xi} \right) \|M^B \hat{R}\|_I \\
\dot{\mathcal{E}} &= \frac{1}{2} \text{vex} \left( \mathcal{P}_a \left( M^B \hat{R} \right) \right) \left( \hat{b} - W \right) \\
\dot{\mu} &= \mu \left( \frac{1}{2} + \frac{1}{\|M^B \hat{R}\|_I} \right)
\end{align*}
\]

(73)

Consider the following design of guaranteed performance direct nonlinear deterministic attitude filter (GPD-NDAF) [5]

\[
\begin{align*}
\dot{R} &= \dot{R} \left[ \Omega_w - \hat{b} - W \right] \\
\dot{\hat{b}} &= \frac{\gamma}{2} \mu \mathcal{E} \text{vex} \left( \mathcal{P}_a \left( M^B \hat{R} \right) \right) \\
W &= \frac{4}{\Delta \left( \text{Tr} \left\{ M^B \right\} I_3 - M^B \right)} \text{vex} \left( \mathcal{P}_a \left( M^B \hat{R} \right) \right)
\end{align*}
\]

(74)

with \( \gamma, k_w \in \mathbb{R}_+ \) being positive constants, \( W \) being a correction factor, \( \hat{b} \) being the estimate of the true bias, and \( \text{vex} \left( \mathcal{P}_a \left( M^B \hat{R} \right) \right) \) and \( \mathcal{Y}(M^B, \hat{R}) \) being obtained through vectorial measurements as in (56) and (58), respectively. Also, \( \Delta := \frac{1}{2} \left( \text{Tr} \left\{ M^B \right\} I_3 - M^B \right) \) and denotes the minimum eigenvalue of the matrix. From (74), it can be noticed that the term multiplied by \( \text{vex} \left( \mathcal{P}_a \left( M^B \hat{R} \right) \right) \) becomes increasingly aggressive as \( \|\hat{R}\|_I \to +1 \). In addition, the above-mentioned term forces the observer to follow the predefined measures of transient and steady-state. Define the following Lyapunov function candidate

\[
\dot{V} = \frac{1}{2} \varepsilon^2 + \frac{1}{2} \hat{b}^T \hat{b}
\]

for any \( \hat{R}(0) \not\in \mathcal{U}_k \) in (63). Differentiating \( V \), considering \( \dot{E} \) in (73), and directly substituting \( W \) and \( \hat{b} \) in (74), one obtains

\[
\dot{V} \leq -k_w \mu^2 \varepsilon^2 \left\| M^B \hat{R} \right\|_I
\]

Consistent with Barbalat’s lemma, \( \dot{V} \to 0 \) as \( t \to \infty \). Also, according to Proposition 1 in [5], \( \|E\| \to 0 \) signifies that \( \|\hat{R}\|_I \to 0 \) and vice versa. Moreover, \( \mu \) is positive for all \( t \geq 0 \). Thus, \( \hat{R} \to I_3 \) as \( t \to \infty \) with guaranteed measures of transient and steady-state performance [5].
6.5. Adaptive Gain Nonlinear Stochastic Attitude Filter

The filters introduced in this Subsection were first proposed in [1, 36]. Although they share the nonlinear structure of the filters in Subsections 6.2 and 6.3, their main advantage is the stochastic design. One of the stochastic filters is developed in the sense of Ito, while the other one is developed in the sense of Stratonovich. The work in [1] gives a comparison between Ito and Stratonovich in terms of

1) effectiveness of filtering out white and colored noise, and
2) computational cost.

6.5.1. Ito Filter

Define the noise attached to angular velocity measurements by $\omega = Q_\omega d\beta/dt$ as introduced in (59). Define $\Psi(\tilde{R}) = \text{vex}(P_\delta(\tilde{R}))$ and consider the design of adaptive gain Ito nonlinear stochastic attitude filter (AGI-NSAF)

$$
\begin{align*}
\dot{\hat{R}} &= \hat{R} [\Omega_m - \hat{b} - W]_x \\
\dot{\hat{b}} &= \gamma_1 ||\tilde{R}||_1 \Psi(\hat{R}) - \gamma_1 k_b \hat{b} \\
\dot{\hat{\sigma}} &= k_w \gamma_2 ||\tilde{R}||_1 D_\Psi \Psi(\hat{R}) - \gamma_2 k_b \hat{\sigma} \\
W &= k_w \frac{2-||\tilde{R}||_1}{1-||\tilde{R}||_1} \Psi(\hat{R}) + k_2 D_\Psi \hat{\sigma}
\end{align*}
$$

(75)

where $\gamma_1, \gamma_2, k_w, k_b, k_2 > 0$ are positive constants, $W$ is a correction factor, $\hat{b}$ is the estimate of the true bias, $\hat{\sigma}$ is the estimate of the true upper bound of the covariance $\sigma$, $D_\Psi = [\Psi(\tilde{R}), \Psi(\hat{R}), \Psi(\hat{R})]$, $\hat{R} = R\tilde{R}$, and $R\tilde{R}$ is a reconstructed attitude obtained by one of the algorithms in (33), (37), (38), or any other method of attitude determination. From (22), (23), (59), and (75), the attitude error dynamics of (49) can be written in an incremental form as follows

$$
\begin{align*}
d\tilde{R} &= \tilde{R} [\Omega - \tilde{R}^T \Omega + \hat{b} - W]_x dt + \tilde{R} [Q_\omega d\beta]_x
\end{align*}
$$

(76)

Consider the attitude representation with respect to Rodriguez vector (19). The true attitude dynamics in terms of Rodriguez vector in incremental form is [1, 3]

$$
d\rho = \frac{1}{2} \left( I_3 + [\rho]_x + \rho \rho^T \right) \Omega dt
$$

The error dynamics in (76) can be expressed in terms of Rodriguez error vector as

$$
d\hat{\rho} = \hat{f} dt + \hat{g} Q_\omega d\beta
$$

with

$$
\hat{g} = \frac{1}{2} \left( I_3 + [\hat{\rho}]_x + \hat{\rho} \hat{\rho}^T \right) \\
\hat{f} = \hat{g} \left( \Omega - R\hat{\rho}^T \Omega + \hat{b} - W \right)
$$

such that

$$
R\hat{\rho} = \frac{1}{1 + ||\hat{\rho}||^2} \left( (1 - ||\hat{\rho}||^2) I_3 + 2\hat{\rho} \hat{\rho}^T + 2 [\hat{\rho}]_x \right)
$$

For more information on attitude mapping visit [1, 25, 4]. In accordance with (Subsection IV.A [1]), the Lyapunov function candidate should be obtained as a function of $\hat{\rho}$ and it should be twice differential. Accordingly, consider the following Lyapunov function candidate
\[ V (\tilde{\rho}, \tilde{b}, \hat{\sigma}) = \left( \frac{\|\tilde{\rho}\|^2}{1 + \|\tilde{\rho}\|^2} \right)^2 + \frac{1}{2\gamma_1} \tilde{b}^T \tilde{b} + \frac{1}{2\gamma_2} \hat{\sigma}^T \hat{\sigma} \]  

(77)

The first and second partial derivatives of the equation above (77) with respect to \( \tilde{\rho} \) are

\[
\begin{align*}
V_{\tilde{\rho}} &= \frac{\partial V}{\partial \tilde{\rho}} = 4 \left( \frac{\|\tilde{\rho}\|^2}{1 + \|\tilde{\rho}\|^2} \right) \tilde{\rho} \\
V_{\tilde{\rho}^2} &= \frac{\partial^2 V}{\partial \tilde{\rho}^2} = 4 \left( \frac{\|\tilde{\rho}\|^2}{1 + \|\tilde{\rho}\|^2} \right)^2 \tilde{\rho} \end{align*}
\]

(78)

For any \( \tilde{R} \in U_\Delta \) in (63) and with direct substitution of \( W, \tilde{b}, \) and \( \hat{\sigma} \) in (75), one has

\[ \mathcal{L} V \leq - \lambda (H) V + c_2 \]

where

\[ H = \begin{bmatrix} 4k_w/\varepsilon & 0_3^T & 0_3 \gamma_1 k_3 \hat{\sigma} \\ 0_3 & 0_3 & 0_3 \gamma_2 k_2 \hat{\sigma} \end{bmatrix} \in \mathbb{R}^{2\times7} \]

such that

\[ 0 \leq E [V (t)] \leq V (0) \exp (-\lambda (H) t) + \frac{c_2}{\lambda (H)}, \forall t \geq 0 \]

with \( \lambda (H) \) being the minimum eigenvalue of \( H \). As such, the error vector \([\hat{\rho}^T, \tilde{b}^T, \hat{\sigma}^T]^T \in \mathbb{R}^9\) is semi-globally uniformly ultimately bounded [1]. The nonlinear stochastic filter proposed in this subsection has been presented in terms of vectorial measurements in [36].

6.5.2. **Stratonovich Filter**

Define \( \Upsilon (\tilde{R}) = \text{vec}(\mathcal{P}_d (\tilde{R})) \) and consider the following design of an adaptive gain Stratonovich nonlinear stochastic attitude filter (AGS-NSAF)

\[
\begin{align*}
\dot{\tilde{R}} &= \tilde{R} \left[ \Omega_{\tilde{R}} - \tilde{b} - \frac{1}{2} \frac{\text{diag}(\Upsilon (\tilde{R}))}{1 - ||\tilde{R}||_F} \hat{\sigma} - W \right] \\
\dot{\tilde{b}} &= \gamma_1 ||\tilde{R}||_F \Upsilon (\tilde{R}) - \gamma_1 k_3 \hat{\sigma} \\
\dot{\hat{\sigma}} &= \gamma_2 ||\tilde{R}||_F \left( k_w D_{\hat{\sigma}} + \frac{1}{2} \frac{\text{diag}(\Upsilon (\tilde{R}))}{1 - ||\tilde{R}||_F} \right) \Upsilon (\tilde{R}) - \gamma_2 k_v \hat{\sigma} \\
W &= k_w \frac{2 - ||\tilde{R}||_F}{1 - ||\tilde{R}||_F} \Upsilon (\tilde{R}) + k_2 D_{\hat{\sigma}} \hat{\sigma}
\end{align*}
\]

(79)

where \( \gamma_1, \gamma_2, k_w, k_2, k_v, k_r > 0 \) are positive constants, \( W \) is a correction factor, \( \tilde{b} \) is the estimate of the true bias, \( \hat{\sigma} \) is the estimate of the true upper bound of the covariance \( \sigma \), \( D_{\hat{\sigma}} = \text{diag}(\Upsilon (\tilde{R}), \Upsilon (\tilde{R}), \Upsilon (\tilde{R})) \), \( \hat{\sigma} = R_f^T \tilde{R} \), and \( R_f \) is a reconstructed attitude obtained by one of the algorithms in (33), (37), (38), or any other method of attitude determination. From (22), (23), (59), and (79), the attitude error dynamics of (49) can be written in an incremental form as follows

\[ d\tilde{R} = \tilde{R} \left[ \Omega - \tilde{R}^T \Omega + \tilde{b} - \frac{1}{2} \text{diag}(\tilde{\rho}) \hat{\sigma} - W \right] dt + \tilde{R} [Q_\omega d\beta] \]

(80)

The error dynamics in (80) can be expressed in terms of Rodriguez error vector as

\[ d\hat{\rho} = \tilde{F} dt + \tilde{g} Q_\omega d\beta \]
with

\[ \tilde{g} = \frac{1}{2} \left( I_3 + [\rho]_\times + \rho \tilde{\rho}^T \right) \]
\[ \tilde{F} = \tilde{g} \left( \Omega - \tilde{R}^T \Omega + b - \frac{1}{2} \text{diag}(\tilde{\rho}) \tilde{\sigma} - W \right) + \mathcal{W}(\tilde{\rho}) \]
\[ \mathcal{W}(\tilde{\rho}) = \frac{1}{4} \left( I_3 + [\tilde{\rho}]_\times + \tilde{\rho} \tilde{\rho}^T \right) Q_\omega \tilde{\rho} \]

where \( \mathcal{W}(\tilde{\rho}) \) is the Wong-Zakai factor \([1, 25]\). Consider the following Lyapunov function candidate

\[ V(\tilde{\rho}, \tilde{b}, \tilde{\sigma}) = \left( \frac{||\tilde{\rho}||^2}{1 + ||\tilde{\rho}||^2} \right)^2 + \frac{1}{2} \tilde{b}^T \tilde{b} + \frac{1}{2} \tilde{\sigma}^T \tilde{\sigma} \]

The first and second partial derivatives of the equation above with respect to \( \tilde{\rho} \) are similar to (78). For \( \dot{\tilde{R}}(0) \in \mathcal{U}_i \) in (63), and with directly substituting \( W, \dot{\hat{b}}, \) and \( \dot{\hat{\sigma}} \) in (79), one obtains

\[ \mathcal{L} V \leq -\Lambda(\mathcal{H}) V + c_2 \]

where

\[ \mathcal{H} = \begin{bmatrix} 4k_\omega / \varepsilon & 0_3 & 0_3 \gamma_1 k_\delta I_3 & 0_3^T \gamma_1 k_\delta I_3 & 0_3 \gamma_2 k_\sigma I_3 & 0_3^T \gamma_2 k_\sigma I_3 \end{bmatrix} \in \mathbb{R}^{7 \times 7} \]

such that

\[ 0 \leq \mathbb{E}[V(t)] \leq V(0) \exp(-\Lambda(\mathcal{H}) t) + \frac{c_2}{\Lambda(\mathcal{H})}, \forall t \geq 0 \]

where \( \Lambda(\mathcal{H}) \) is the minimum eigenvalue of matrix \( \mathcal{H} \). Thus, the error vector \( [\tilde{\rho}^T, \tilde{b}^T, \tilde{\sigma}^T]^T \in \mathbb{R}^9 \) is proven to be semi-globally uniformly ultimately bounded \([1]\).

6.6. Guaranteed Performance Nonlinear Stochastic Attitude Filter

The filters described in this Subsection were first proposed in [39]. Despite sharing the nonlinear structure of the filters in Subsection 6.4, their main advantage is the stochastic design. Both stochastic filters presented below are driven in the sense of Stratonovich.

6.6.1. Semi-direct Filter

Given \( \omega = Q \dot{\beta} / dt \) as defined in (59), the normalized Euclidean distance of attitude error dynamics in (67) can be rewritten in an incremental form as follows

\[ d||\tilde{R}||_I = \frac{1}{2} \text{vex} \left( \mathcal{P}_a \left( \tilde{R} \right) \right)^T \left( (\Omega_m - b) dt - Q_\omega \omega \right) \]

Consider (81) and recall the following set of equations

\[
\begin{align*}
\dot{E} &= \frac{1}{2} \ln \left( \frac{E}{2} + \sqrt{ \frac{E^2}{4} + \frac{\xi^2}{4} } \right) \\
\dot{\mu} &= \frac{\exp(2\xi) + \exp(-2\xi)}{8\delta} \\
\dot{d||\tilde{R}||_I} &= \frac{1}{2} \text{vex} \left( \mathcal{P}_a \left( \tilde{R} \right) \right)^T \left( (\Omega_m - b) dt - Q_\omega \omega \right) \\
\dot{dE} &= 2\mu \left( d||\tilde{R}||_I - \frac{\xi}{4} \right) dt
\end{align*}
\]
Define $\Psi(\hat{R}) = \text{vex}(\mathcal{P}_a(\hat{R}))$ and consider the following design of a guaranteed performance semi-direct nonlinear stochastic attitude filter (GPSd-NSAF) [39]

\[
\begin{align*}
\dot{\hat{R}} &= \hat{R} \left[ \Omega_m - \hat{b} - W \right], \\
\dot{\hat{b}} &= \gamma_1 (E + 1) \exp(E) \mu \Psi(\hat{R}), \\
\dot{\sigma} &= \gamma_2 (E + 2) \exp(E) \mu^2 \text{diag}(\Psi(\hat{R})) \Psi(\hat{R}), \\
W &= 2 \frac{\hat{b}_w (E + 1) \mu - \xi}{\gamma_2} \Psi(\hat{R})
\end{align*}
\]  

(83)

where $\gamma_1, \gamma_2, k_w \in \mathbb{R}_+$ are positive constants, $W$ is a correction factor, $\hat{b}$ is the estimate of the true bias, $\hat{\sigma}$ is the estimate of the true upper bound of the covariance $\sigma$, $\hat{R} = R_{\hat{y}} \hat{R}$, and $R_{\hat{y}}$ is a reconstructed attitude obtained by one of the algorithms in (33), (37), (38), or any other method of attitude determination. From (83), it can be observed that the term multiplied by $\Psi(\hat{R}) = \text{vex}(\mathcal{P}_a(\hat{R}))$ becomes increasingly aggressive as $||\hat{R}||_1 \to +1$. Additionally, the above-mentioned term forces the filter to obey the predefined transient and steady-state measures. When selecting the Lyapunov function candidate the two important considerations are: it should be a function of $E$ and it should be twice differentiable. In the light of the above considerations, let us define Lyapunov function candidate as follows:

\[
V(E, \hat{b}, \sigma) = E \exp(E) + \frac{1}{2 \gamma_1} ||\hat{b}||^2 + \frac{1}{\gamma_2} ||\hat{\sigma}||^2
\]  

(84)

The first and second partial derivatives of the equation above (84) with respect to $E$ are

\[
\begin{align*}
V_E &= \frac{\partial V}{\partial E} = (E + 1) \exp(E), \\
V_{EE} &= \frac{\partial^2 V}{\partial E^2} = (E + 2) \exp(E)
\end{align*}
\]

(85)

For any $\hat{R}(0) \notin \mathcal{U}_s$ in (63), considering $E$ in (82), and directly substituting $W, \hat{b},$ and $\hat{\sigma}$ with their definitions in (83), one obtains

\[
\mathcal{L}V \leq -4 \hat{b}_w \xi \mu^2 (E + 1)^2 \exp(E) - \exp(-E) \exp(E)
\]

According to the fact that $\mathcal{L}V$ is bounded and $V$ is radially unbounded for any $\hat{R}(0) \notin \mathcal{U}_s$ and $E(0) \in \mathbb{R}$, it can be concluded that a unique strong solution to the stochastic system in (81) exists with a probability of one [40]. Thus, $E(t)$ is regulated asymptotically to the origin in probability of 1 for all $\hat{R}(0) \notin \mathcal{U}_s$ and $E(0) \in \mathbb{R}$ implying that $P\{\lim_{t \to \infty} \hat{R} = I_3\} = 1$ for all $\hat{R}(0) \notin \mathcal{U}_s$ and $E(0) \in \mathbb{R}$ [39].

6.6.2. Direct Filter

Consider modifying the error function in (67) to

\[
||M^B \tilde{R}||_I = \frac{1}{4} \text{Tr} \left\{ I_3 - M^B \tilde{R} \right\}
\]

(86)

where $||M^B \tilde{R}||_I$ is given with respect to vectorial measurements as in (57). Hence, the error function in (86) can be expressed in an incremental form as follows

\[
d||M^B \tilde{R}||_I = \frac{1}{2} \text{vex} \left( \mathcal{P}_a \left( M^B \tilde{R} \right) \right) ^\top \left( (\Omega_m - b) \, dt - Q_v \, d\beta \right)
\]

(87)

Accordingly, one can arrive at the following set of equations:

\[
\begin{align*}
\xi &= \frac{1}{2} \ln \frac{\dot{\xi} + ||M^B \tilde{R}||_I \, / \xi}{\exp(-2\xi) + \exp(-2\xi) + 2} \\
\mu &= \frac{2 \xi}{8 \xi^3} \\
d||M^B \tilde{R}||_I &= \frac{1}{2} \text{vex} \left( \mathcal{P}_a \left( M^B \tilde{R} \right) \right) ^\top \left( (\Omega_m - b) \, dt - Q_v \, d\beta \right) \\
dE &= 2 \xi \left( d||M^B \tilde{R}||_I - \frac{\xi}{2} ||M^B \tilde{R}||_I dt \right)
\end{align*}
\]

(88)
Define $\Psi(M^B\hat{R}) = \text{vex}(\mathcal{P}_a(M^B\hat{R}))$ and consider the following design of a guaranteed performance direct nonlinear stochastic attitude filter (GPD-NSAF) \[5\]

\[
\begin{align*}
\dot{\hat{R}} &= \hat{R} \left[ \Omega_m - \hat{b} - W \right], \\
\dot{\hat{b}} &= \gamma_1 (\mathcal{E} + 1) \exp(\mathcal{E}) \mu \Psi(M^B\hat{R}) \\
\dot{\hat{\sigma}} &= \gamma_2 (\mathcal{E} + 2) \exp(\mathcal{E}) \mu^2 \text{diag} \left( \Psi(M^B\hat{R}) \right) \hat{\sigma} + \frac{k_w \mu^2 \gamma}{2} \exp(\mathcal{E}) \\
W &= 2 \mathcal{E} + 2 \mathcal{E} + 1 \mu^2 \text{diag} \left( \Psi(M^B\hat{R}) \right) \hat{b} + 4 \frac{\lambda}{\lambda + 1} \Psi(M^B\hat{R})
\end{align*}
\]

with $\gamma_1, \gamma_2, k_w \in \mathbb{R}^+$ being positive constants, $W$ being a correction factor, $\hat{b}$ being the estimate of the true bias, $\hat{\sigma}$ being the estimate of the true upper bound of the covariance, and $\text{vex}(\mathcal{P}_a(M^B\hat{R}))$ and $\Psi(M^B\hat{R})$ being obtained through vectorial measurements as in (56) and (58), respectively. Also, $\lambda := \frac{\text{Tr} \{ M^B \} \mathbb{I}_3 - M^B }{2}$ denotes the minimum eigenvalue. Consider the below Lyapunov function candidate

\[ V(\mathcal{E}, \hat{b}, \hat{\sigma}) = \mathcal{E} \exp(\mathcal{E}) + \frac{1}{2\gamma_1} ||\hat{b}||^2 + \frac{1}{2\gamma_2} ||\hat{\sigma}||^2 \]

The first and second partial derivatives of the equation above are similar to (85). For any $\hat{R}(0) \not\in \mathcal{U}_0$ in (63), considering $\dot{\mathcal{E}}$ in (82), and directly substituting $W, \dot{\hat{b}},$ and $\dot{\hat{\sigma}}$ in (89), one has

\[ \mathcal{L}V \leq -k_w \gamma^2 (\mathcal{E} + 1)^2 \exp(\mathcal{E}) - \exp(-\mathcal{E}) \exp(\mathcal{E}) \]

Since $\mathcal{L}V$ is bounded and $V$ is radially unbounded for any $\hat{R}(0) \not\in \mathcal{U}_0$ and $\mathcal{E}(0) \in \mathbb{R}$, there exists a unique strong solution to the stochastic system in (87) with a probability of one \[40\]. Therefore, $\mathcal{E}(t)$ is regulated asymptotically to the origin in probability for all $\hat{R}(0) \not\in \mathcal{U}_0$ and $\mathcal{E}(0) \in \mathbb{R}$ which, in turn, implies that $\mathbb{P}\{\lim_{t \to \infty} \hat{R} = I_3\} = 1$ for all $\hat{R}(0) \not\in \mathcal{U}_0$ and $\mathcal{E}(0) \in \mathbb{R} \[39\].

7. Simulation and Comparative Results

7.1. Continuous Attitude Filters

This Subsection provides comparative results in accordance with Table 4.

| Category                  | Type                                      |
|---------------------------|-------------------------------------------|
| Attitude Determination    | TRIAD (Equation (33)), QUEST (Equation (37)) and SVD (Equation (38)) |
| Gaussian Attitude Filters | MEKF (Equation (44)) and GAMEF (Equation (46)) |
| Nonlinear Attitude Filters | CG-NDAF (Equation (64) and (65)), AG-NDAF (Equation (66)), GP-NDAF (Equation (71) and (74)), AG-NSAF (Equation (75) and (79)), and GP-NSAF (Equation (83) and (89)) |

According to the discussion given in the above Sections, AG-NDAF, GP-NDAF, AG-NSAF, and GP-NSAF are adaptively tuned. Thus, to ensure fair comparison, different scenarios have been considered for
CG-NDAF, MEKF, and GAMEF. Since KF demonstrates reasonable performance only if the sensor measurements are free of high level uncertainties, KF is not included in the comparison. The algorithms are implemented and the results are obtained using MATLAB®. The filter performance will be tested on angular velocity and body-frame vectorial measurements subject to

1) constant bias, and

2) noise that is normally distributed with a zero mean and a nonzero standard deviation (STD).

7.1.1. True Values and Measurements

Consider the attitude dynamics in equation (22), true angular velocity input signal (\( \Omega \)), and initial attitude (\( R(0) \)) to be given as

\[
\begin{align*}
\dot{R} &= R[\Omega] \times \\
\Omega &= \begin{bmatrix} \sin(0.4t) \\ 0.4 \sin(0.7t + \frac{\pi}{4}) \\ 0.4 \sin(0.3t + \frac{\pi}{2}) \end{bmatrix} \text{ (rad/sec)} \\
R(0) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
T &= 30 \text{ sec, (Total simulation time)}
\end{align*}
\]

Let the measurements of the true angular velocity (\( \Omega_m \)) be corrupted with unknown random noise and constant bias such that

\[
\begin{align*}
\Omega_m &= \Omega + b + \omega \\
b &= [-0.1, 0.1, 0.05]^T \\
\omega &= \mathcal{N} \sim (0, 0.2)
\end{align*}
\]

where \( \omega = \mathcal{N} \sim (0, 0.2) \) is a short-hand notation indicating that the noise (\( \omega \)) is normally distributed with zero mean (\( \mathbb{E}[\omega] = 0 \)) and STD = 0.2. To implement \( \omega = \mathcal{N} \sim (0, 0.2) \) at instant \( t \) in MATLAB use the following command: \( \omega(t) = 0.2 \times \text{randn}(3,1) \). Consider the following two non-collinear inertial-frame vectors

\[
\begin{align*}
v_1^I &= [1, -1, 1]^T \\
v_2^I &= [0, 0, 1]^T
\end{align*}
\]

The associated body-frame measurements are obtained as follows

\[
\begin{align*}
v_1^B &= R^T v_1^I + b_1^B + \omega_1^B \\
v_2^B &= R^T v_2^I + b_2^B + \omega_2^B
\end{align*}
\]

with

\[
\begin{align*}
b_1^B &= [0.13, -0.13, 0.13]^T \\
b_2^B &= [0, 0, 0.13]^T \\
\omega_1^B &= \mathcal{N} \sim (0, 0.13) \\
\omega_2^B &= \mathcal{N} \sim (0, 0.13)
\end{align*}
\]

The third inertial-frame and body-frame vectors are obtained as a cross product as follows

\[
\begin{align*}
v_3^I &= v_1^I \times v_2^I \\
v_3^B &= v_1^B \times v_2^B
\end{align*}
\]
Next step is normalization performed according to (24):

$$
v_i^T = \frac{v_i^T}{||v_i^T||}, \quad v_i^B = \frac{v_i^B}{||v_i^B||}, \quad \forall i = 1, 2, 3
$$

(92)

Consider the measurements of angular velocity in (90), the body-frame measurements in (91), and the normalized values of the body-frame measurements in (92). The true angular velocity and the normalized values of body-frame vectors are plotted against angular velocity measurements and the normalized values of body-frame vectorial measurements in Figure 2, respectively. It can be observed in Figure 2 that high values of noise and bias components corrupted the measurement process of the three categories of attitude determination and estimation algorithms listed in Table 4.

![Figure 2: Angular velocity and body-frame vectors: Measured and true.](image)

7.1.2. Initialization and Design Parameters

For the semi-direct filters in (64), (66), (71), and (83), $R_y$ is reconstructed with the aid of SVD in (38). For Gaussian and nonlinear attitude filters, the initial attitude estimate is given with respect to angle-
axis parameterization in (18) such that

\[
\begin{align*}
\hat{R}(0) &= R_\alpha (\alpha, u/\|u\|) \\
\alpha &= 178 \text{ (deg)} \\
u &= [8, 7, 4]^T
\end{align*}
\]

or more simply put

\[
\hat{R}(0) = \begin{bmatrix}
-0.0074 & 0.8557 & 0.5175 \\
0.8802 & -0.2399 & 0.4094 \\
0.4745 & 0.4586 & -0.7514
\end{bmatrix}
\]

where \(\|\hat{R}(0)\|_I = 0.9997\) initiated very close to the unstable equilibria (+1). Initial estimates used for all filters are as follows

\[
\hat{b}(0) = [0, 0, 0]^T \\
\hat{c}(0) = [0, 0, 0]^T
\]

The design parameters of the filters are summarized in Table 5. Since MEKF, GAMEF, and CG-NDAF are not characterized with adaptive gains, three cases of the design parameters are considered for each of the above-mentioned filters to ensure fair comparison. The comparison between the filtering methods in this section examines the transient and steady-state performance of the attitude error in terms of

1) normalized Euclidean distance of the attitude error

\[
\|\hat{R}\|_I = \frac{1}{4} \text{Tr} \left\{ I_3 - R^\top \hat{R} \right\}
\]

2) the error in rotation angle about the unit axis[4]

\[
\tilde{\alpha} = \cos^{-1} \left( \frac{\text{Tr}(R^\top \hat{R}) - 1}{2} \right)
\]

In all the simulations, the output values of \(\|\hat{R}\|_I\) and \(\tilde{\alpha}\) are recorded every 0.01 seconds with the infinity norm \(\|\|\| \|_\infty := \max_t (\|\hat{R}(t)\|_I)\) and \(\|\tilde{\alpha}\|_\infty := \max_t |\tilde{\alpha}(t)|\).

7.1.3. Attitude Determination Results

Figure 3 illustrates high sensitivity of algebraic attitude determination algorithms to bias and noise present in measurements. The poor performance observed in Figure 3 is reinforced by the oscillatory behavior of the constructed Euler angles when compared to the true Euler angles depicted in Figure 4. Table 6 containing statistical results of the mean, STD and \(\|\cdot\|_\infty\) of \(\|\hat{R}\|_I\) and \(\tilde{\alpha}\) provides additional evidence of the poor performance of the algebraic attitude determination algorithms: TRIAD, QUEST, and SVD when faced with biased and noisy measurements.
Table 5: Design parameters

| Filter  | Design parameters                                                                 |
|---------|-----------------------------------------------------------------------------------|
| MEKF    | Case 1: $\bar{Q}_{v(i)} = I_3$, $\bar{Q}_{\omega} = I_3$, and $\bar{Q}_{b} = I_3$ |
|         | Case 2: $\bar{Q}_{v(i)} = 0.1I_3$, $\bar{Q}_{\omega} = 10I_3$, and $\bar{Q}_{b} = 10I_3$ |
|         | Case 3: $\bar{Q}_{v(i)} = 0.01I_3$, $\bar{Q}_{\omega} = 100I_3$, and $\bar{Q}_{b} = 100I_3$ |
| GAMEF   | Case 1: $\bar{Q}_{v(i)} = I_3$, $\bar{Q}_{\omega} = I_3$, and $\bar{Q}_{b} = I_3$ |
|         | Case 2: $\bar{Q}_{v(i)} = 0.1I_3$, $\bar{Q}_{\omega} = 10I_3$, and $\bar{Q}_{b} = 10I_3$ |
|         | Case 3: $\bar{Q}_{v(i)} = 0.01I_3$, $\bar{Q}_{\omega} = 100I_3$, and $\bar{Q}_{b} = 100I_3$ |
| CG-NDAF | Case 1: $k_w = 1$                                                                   |
|         | Case 2: $k_w = 10$                                                                  |
|         | Case 3: $k_w = 100$                                                                 |
| AG-NDAF | $k_w = 8$                                                                           |
| GP-NDAF | $k_w = 2$, $\delta = 1.7$, $\sigma = 1.7$, $\zeta_0 = 0.08$, $\ell = 4$ and $\gamma = 1$ |
| AG-NSAF | $\gamma_1 = 1$, $\gamma_2 = 1$, $k_b = 0.01$, $k_v = 0.01$, $k_w = 2$, $k_2 = 0.5$ and $\varepsilon = 0.1$ |
| GP-NSAF | $k_w = 2$, $\delta = 1.7$, $\sigma = 1.7$, $\zeta_0 = 0.08$, $\ell = 4$, $\gamma_1 = 1$ and $\gamma_2 = 0.1$ |

Figure 3: Tracking error of TRIAD, QUEST and SVD: $||R||_I$ and $\bar{\alpha}$. 

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Table 6: Statistical analysis of $\|\tilde{R}\|_I$ and $\tilde{\alpha}$ of TRIAD, QUEST and SVD.

| Filter | Mean ($\|\tilde{R}\|_I$) | STD ($\|\tilde{R}\|_I$) | $\|\|\tilde{R}\|_I\|_{\infty}$ | Mean ($\tilde{\alpha}$) | STD ($\tilde{\alpha}$) | $\|\tilde{\alpha}\|_{\infty}$ |
|--------|--------------------------|--------------------------|-------------------------------|--------------------------|--------------------------|-------------------------------|
| TRIAD  | 0.0120                    | 0.0124                    | 0.1728                        | 11.2999                  | 5.6085                   | 49.130                        |
| QUEST  | 0.0124                    | 0.0123                    | 0.1685                        | 11.5080                  | 5.5839                   | 48.464                        |
| SVD    | 0.0124                    | 0.0123                    | 0.1685                        | 11.5080                  | 5.5839                   | 48.464                        |

7.1.4. Gaussian and Nonlinear Attitude Filters Results

Figure 5 and 6 demonstrate the superiority of Gaussian attitude filters over the determination algorithms in terms of tracking performance. It can be noticed that the design parameters in Case 1 and Case 2 of MEKF and GAMEF provide slower tracking performance with less oscillatory behavior in the steady-state. In contrast, Case 3 of MEKF and GAMEF offers faster tracking performance with higher oscillation in the steady-state. This can be confirmed through the statistical results listed in Table 7. However, MEKF requires less computational power in comparison with GAMEF. Figure 7 and 8 illustrate faster tracking performance of CG-NDAF (Case 3), AG-NDAF, GP-NDAF, AG-NSAF and GP-NSAF, in comparison with CG-NDAF (Case 1) and CG-NDAF (Case 2). Despite fast tracking performance, the main weakness of CG-NDAF (Case 3) shows unstable behavior. Also, CG-NDAF (Case 1), CG-NDAF (Case 2), AG-NDAF and AG-NSAF cannot demonstrate guaranteed measures of transient and steady-state error. It becomes apparent that the only two filters that have the advantage of guaranteed performance of transient and steady-state error are GP-NDAF and GP-NSAF. The side-by-side statistical comparison of the nonlinear attitude filters in Figure 7 and 8 can be found in Table 8.
Figure 5: Tracking error ($||\tilde{R}||$) of Gaussian attitude filters: MEKF and GAMEF.
Figure 6: Tracking error ($\tilde{\alpha}$) of Gaussian attitude filters: MEKF and GAMEF.
Figure 7: Tracking error ($||\tilde{R}||_1$) of nonlinear attitude filters.
Figure 8: Tracking error ($\hat{\alpha}$) of nonlinear attitude filters.
Table 7: Statistical analysis of Gaussian attitude filters (continuous time) of \( \hat{R}_I \) and \( \hat{\alpha} \) steady-state performance: MEKF and GAMEF.

| Filter          | Mean (\( ||\hat{R}_I|| || \)) | STD (\( ||\hat{R}_I|| || \)) | Mean (\( ||\hat{R}_I|| _\infty \)) | STD (\( ||\hat{\alpha}|| _\infty \)) | Transient | Overall |
|-----------------|---------------------------------|---------------------------------|------------------------------------|-----------------------------------|-----------|---------|
| MEKF (Case1)    | 0.0034                          | 0.0016                          | 0.0089                             | 6.4955                            | 1.6980    | 10.8532 | Very slow | Stable  |
| MEKF (Case2)    | 0.0035                          | 0.0021                          | 0.0150                             | 6.4816                            | 2.0886    | 14.0934 | Slow      | Stable  |
| MEKF (Case3)    | 0.0075                          | 0.0067                          | 0.0161                             | 9.0868                            | 4.0262    | 28.7459 | Fast      | Stable  |
| GAMEF (Case1)   | 0.0034                          | 0.0016                          | 0.0081                             | 6.4402                            | 1.6575    | 10.3092 | Very slow | Stable  |
| GAMEF (Case2)   | 0.0035                          | 0.0021                          | 0.0152                             | 6.4843                            | 2.0912    | 14.1452 | Slow      | Stable  |
| GAMEF (Case3)   | 0.0075                          | 0.0068                          | 0.0152                             | 9.1070                            | 4.0463    | 28.9409 | Fast      | Stable  |

Table 8: Statistical analysis of nonlinear attitude filters (continuous time) of \( \hat{R}_I \) and \( \hat{\alpha} \) steady-state performance.

| Filter          | Mean (\( ||\hat{R}_I|| || \)) | STD (\( ||\hat{R}_I|| || \)) | Mean (\( ||\hat{R}_I|| _\infty \)) | STD (\( ||\hat{\alpha}|| _\infty \)) | Transient | Overall |
|-----------------|---------------------------------|---------------------------------|------------------------------------|-----------------------------------|-----------|---------|
| CGSd-NDAF (Case1) | 0.0046                         | 0.0033                          | 0.0170                             | 7.3376                            | 2.6319    | 14.9764 | Very slow | Stable  |
| CGSd-NDAF (Case2) | 0.0033                         | 0.0019                          | 0.0131                             | 6.3162                            | 1.9338    | 13.1318 | Slow      | Stable  |
| CGSd-NDAF (Case3) | 0.1076                         | 0.0215                          | 0.1576                             | 38.1047                           | 4.0626    | 46.7679 | Fast      | Unstable|
| CGD-NDAF (Case1)  | 0.0035                         | 0.0013                          | 0.0064                             | 6.6564                            | 1.4671    | 9.1931  | Very slow | Stable  |
| CGD-NDAF (Case2)  | 0.0035                         | 0.0021                          | 0.0130                             | 6.4735                            | 2.0619    | 13.1090 | Slow      | Stable  |
| CGD-NDAF (Case3)  | 0.3459                         | 0.0375                          | 0.3852                             | 71.9788                           | 4.6348    | 76.7300 | Fast      | Unstable|
| AG-NDAF          | 0.0037                         | 0.0018                          | 0.0089                             | 6.7044                            | 1.8282    | 10.8265 | Fast      | Stable  |
| GPSd-NDAF        | 0.0033                         | 0.0017                          | 0.0094                             | 6.3198                            | 1.7286    | 11.1217 | Guaranteed | Stable  |
| GPD-NDAF         | 0.0030                         | 0.0014                          | 0.0091                             | 6.0537                            | 1.5424    | 10.9516 | Guaranteed | Stable  |
| AGI-NSAF         | 0.0032                         | 0.0017                          | 0.0089                             | 6.2420                            | 1.8141    | 10.8308 | Fast      | Stable  |
| AGS-NSAF         | 0.0032                         | 0.0017                          | 0.0090                             | 6.2356                            | 1.8140    | 10.8576 | Fast      | Stable  |
| GPSd-NSAF        | 0.0033                         | 0.0022                          | 0.0191                             | 6.2854                            | 2.0817    | 15.9870 | Guaranteed | Stable  |
| GPD-NSAF         | 0.0032                         | 0.0015                          | 0.009                              | 6.3104                            | 1.5301    | 10.9040 | Guaranteed | Stable  |

7.1.5. Discrete Nonlinear Filters Results

This part contains a brief comparison between nonlinear discrete attitude filters whose detailed descriptions can be found in the Appendix. The filters to be discussed are CG-NDAF (Equation (102) and (103)), AG-NDAF (Equation (104)), GP-NDAF (Equation (105) and (107)), and GP-NSAF (Equation (109) and (111)). The sampling time \( \Delta t \) is set to 0.01 seconds. Consider the measurement of the true angular velocity to be given similar to (90). Also, let the body-frame measurements be as in (91) and their normalized values as in (92). Figure 2 shows the true angular velocity and the normalized values of body-frame vectors plotted against angular velocity measurements and the normalized values of body-frame vectorial measurements, respectively. Figure 2 illustrates high values of noise and bias components corrupting the measurement process. As illustrated in Figure 9, CGD-NDAF as well as CGSd-NDAF showed stable performance with slower transient tracking response of \( ||\hat{R}_k|| || \) and \( \hat{\alpha}_k \) for Case 1 and 2. However, For Case 3 CGD-NDAF showed fast transient response with poor values of steady state error of \( ||\hat{R}_k|| || \) and \( \hat{\alpha}_k \). AG-NDAF demonstrated fast tracking performance with more oscillatory response in the steady-state. GPSd-NDAF and GPD-NDAF displayed fast transient response with stable performance in the steady-state. Similarly, GPSd-NSAF and GPD-NSAF exhibited fast tracking performance with less oscillation in the steady-state.
Figure 9: Tracking error of nonlinear discrete attitude filters: $||\hat{R}[k]||$ and $\hat{\alpha}[k]$. 
8. Conclusion

In conclusion, let us briefly summarize the history of development of the attitude determination and estimation methods over the past few decades. TRIAD algorithm is one of the earliest and simplest methods of attitude determination for two given simultaneous observations. SVD and QUEST displaced TRIAD and became more popular methods of attitude determinations as they allow for the case of two or more simultaneous observations. The family of Kalman filters was a pioneer of providing a reasonable estimate of the true attitude, in particular the multiplicative extended Kalman filter (MEKF). Nonlinear attitude filters were proposed to mimic the nonlinear nature of the attitude dynamics and to provide better results than Gaussian attitude filters. In fact, among other advantages over the Gaussian attitude filter, nonlinear attitude filters are simpler in derivation and require less computational power. A brief survey of attitude determination algorithms, Gaussian attitude filters, and nonlinear attitude filters is presented in this paper. The output performance of each category is illustrated through the simulation results for the purposes of validation and comparison.

Appendix

DISCRETE: GAUSSIAN AND NONLINEAR ATTITUDE FILTERS

The Appendix contains the discrete designs of Gaussian attitude filters (KF, MEKF and GAMEF) and nonlinear attitude filters (CG-NDAF, AG-NDAF, GP-NDAF, AG-NSAF and GP-NSAF) presented in Section 5 and 6. \( \Delta t \) denotes the sampling time which is assumed to be sufficiently small. Also, for any \( x \in \mathbb{R}^{n \times m} \), \( x[k] \) refers to the value of \( x \) at sample \( k \).

8.1. Discrete KF

For \( \Omega_m = \Omega + Q_\omega \omega \), recall the attitude problem in (43)

\[
\begin{align*}
Q & = \frac{1}{2} \Gamma (\Omega_m - Q_\omega \omega) Q \\
Y & = \frac{1}{2} \sum_i \Xi (Q) Q_{v(i)} \omega_i^B
\end{align*}
\]

For simplicity, let \( \omega_i^B = \omega_i^B[k] \), \( \omega = \omega[k] \), \( \Omega = \Omega[k] \), \( \Omega_m = \Omega_m[k] \), \( v_i^B = v_i^B[k] \) and \( v_i^T = v_i^T[k] \). The discrete form of (43) is as follows:

\[
\begin{align*}
Q[k + 1] &= \exp \left( \frac{1}{2} \Gamma (\Omega_m[k]) \Delta t \right) Q[k] - \frac{1}{2} \Xi (Q[k]) Q_\omega \omega[k] \Delta t \\
Y[k] &= \begin{bmatrix}
0 \\
\frac{1}{2} \Xi (Q[k]) Q_{v(i)} \omega_i^B
\end{bmatrix} \begin{bmatrix}
\omega[k] \\
v_i^B - v_i^T
\end{bmatrix}
\]

One can easily obtain the covariance by

\[
\begin{align*}
Q_e[k] &= \mathbb{E} \left[ (Q_\omega \omega[k]) (Q_\omega \omega[k])^\top \right] = Q_{\omega \omega}^2 \\
Q_q[k] &= \left( \frac{\Delta t}{2} \right)^2 \Xi (Q[k]) Q_{\omega \omega}^2 \Xi (Q[k])^\top \\
\mathcal{R}_e &= \mathbb{E} \left[ \sum_i (Q_{v(i)} \omega_i^B) (Q_{v(i)} \omega_i^B)^\top \right] = \sum_i Q_{\omega(i)}^2 \\
\mathcal{R}_q[k] &= \Xi (Q[k]) \mathcal{R}_e \Xi (Q[k])^\top
\end{align*}
\]
The discrete form of a basic attitude KF can be represented in two steps. The prediction step:
\[
\begin{align*}
\dot{\hat{q}}[0] &= \begin{bmatrix} 1 & 0_{1 \times 3} \end{bmatrix}^T \\
\Psi[k] &= \exp \left( \frac{1}{2} \Gamma \left( \Omega_m[k] \right) \Delta t \right) \\
\dot{Q}[k+1|k] &= \Psi[k] \dot{Q}[k] \\
Q_q[k] &= \left( \frac{\alpha}{2} \right)^2 (\dot{Q}[k]) Q_e[k] \Xi (\dot{Q}[k])^T \\
P[k+1|k] &= \Psi[k] P[k] \Psi[k]^T + Q_q[k]
\end{align*}
\] (93)

and the correction step:
\[
\begin{align*}
\mathcal{H}[k] &= \begin{bmatrix} 0 \\ (v^b - v^T)^T \\ [v^b + v^T] \end{bmatrix} \\
R_q[k+1] &= \frac{1}{2} \Xi \left( \dot{Q}[k = 1|k] \right) R_q \Xi \left( \dot{Q}[k+1|k] \right)^T + \alpha I_4 \\
S[k+1|k] &= \mathcal{H}[k] P[k+1|k] \mathcal{H}[k]^T + R_q[k+1] \\
K[k+1] &= P[k+1|k] \mathcal{H}[k]^T S[k+1|k]^{-1} \\
\dot{Q}[k+1] &= (I_4 - K[k+1] \mathcal{H}[k]) \dot{Q}[k+1|k] \\
P[k+1] &= (I_4 - K[k+1] \mathcal{H}[k]) P[k+1|k] (I_4 - K[k+1] \mathcal{H}[k])^T + K[k+1] R_q[k+1] K[k+1]^T \\
\dot{Q}[k+1] &= \dot{Q}[k+1] / \|\dot{Q}[k+1]\|
\end{align*}
\] (94)

where \(\alpha\) is a small positive constant. The basic attitude Kalman filter in (93) and (94) can be modified to account for bias compensation [19]. The modified attitude Kalman filter proposed in [19] is given in the following two steps. Prediction step:
\[
\begin{align*}
\varepsilon[0] &= \begin{bmatrix} 1 & 0_{1 \times 6} \end{bmatrix}^T \\
\varepsilon[k] &= \begin{bmatrix} \dot{\hat{q}}[k] \\ \tilde{b}[k] \end{bmatrix}^T \\
\dot{\Omega}[k] &= \Omega_m[k] - \tilde{b}[k] \\
\Psi[k] &= \exp \left( \frac{1}{2} \Gamma \left( \dot{\Omega}[k] \right) \Delta t \right) \\
\varepsilon[k+1|k] &= \begin{bmatrix} \Psi[k] \\ 0_{3 \times 4} \end{bmatrix} 0_{4 \times 3} \\
\Psi[k] &= \Psi[k] - \frac{\Delta t}{2} \Xi (\dot{Q}[k]) \\
M[k] &= \dot{Q}[k] Q[k]^T + P_{Q[k]} \\
P_{\varepsilon}[k] &= \left( \sigma_q^2 + \sigma_\varepsilon^2 \Delta t \right) \left( \text{Tr} \left( \dot{M}[k] \right) I_4 - \dot{M}[k] \right) 0_{4 \times 3} \\
P[k+1|k] &= \Psi[k] P[k] \Psi[k]^T + P_{\varepsilon}[k]
\end{align*}
\] (95)
Correction step:

\[
\begin{align*}
\mathcal{H} [k] &= \begin{bmatrix}
0 \\
(\nu^B [k] - \nu^F [k])^T \\
(\nu^B [k] + \nu^F [k])_x
\end{bmatrix} \\
\tilde{\mathcal{H}} [k] &= \left[ \mathcal{H} [k] \quad 0_{4 \times 3} \right] \\
\tilde{\mathbf{M}} [k+1 | k] &= \tilde{Q} [k+1 | k] \tilde{Q} [k+1 | k]^T + P_Q [k+1 | k] \\
P_v [k+1] &= \frac{1}{T} \mathbf{Tr} \left\{ \mathbf{M} [k+1 | k] \right\} \mathbf{I}_4 - \tilde{\mathbf{M}} [k+1 | k] - \Gamma (\nu^B) \tilde{\mathbf{M}} [k+1 | k] \Gamma (\nu^B)^T \\
S [k+1 | k] &= \mathcal{H} [k] P_Q [k+1 | k] \mathcal{H} [k]^T + P_v [k+1] \\
K [k+1] &= P [k+1 | k] \tilde{\mathcal{H}} [k]^T S [k+1 | k]^{-1} \\
\hat{x} [k+1] &= (\mathbf{I}_7 - K [k+1] \tilde{\mathcal{H}} [k]) \hat{x} [k+1 | k] \\
P [k+1] &= (\mathbf{I}_7 - K [k+1] \tilde{\mathcal{H}} [k]) P [k+1 | k] (\mathbf{I}_7 - K [k+1] \tilde{\mathcal{H}} [k])^T \\
\tilde{Q} [k+1] &= \tilde{Q} [k+1] / \| \tilde{Q} [k+1] \| \\
\end{align*}
\]

Go to prediction step Equation (95)

where \( Q_\omega = \text{diag} \{ \epsilon_1, \epsilon_2, \epsilon_3 \} \), \( Q_c [k] = \eta \mathbf{I}_3 \), and \( R_c [k] = \epsilon \mathbf{I}_3 \) with \( \eta \) and \( \epsilon \) being positive constants.

8.2. Discrete MEKF

The discrete form of MEKF in (44) and (45) is as follows:

\[
\begin{align*}
\begin{bmatrix}
0 \\
\tilde{\phi}^B [k]
\end{bmatrix} &= \tilde{Q} [k]^{-1} \odot \begin{bmatrix}
0 \\
\nu^F [k]
\end{bmatrix} \odot \tilde{Q} [k] \\
\tilde{Q} [k+1] &= \exp \left\{ \frac{1}{T} \mathbf{I} \Gamma \left( \Omega_m [k] - \hat{b} [k] + P_a [k] W [k] \right) \Delta t \right\} \tilde{Q} [k] \\
W [k] &= \sum_{i=1}^{n} \tilde{\phi}^B [k] \times Q^{-1} \left( \tilde{\phi}^B [k] - \nu^B [k] \right)
\end{align*}
\]

with

\[
\begin{align*}
\hat{b} [k+1] &= \hat{b} [k] + P_a^T [k] W [k] \Delta t \\
S [k] &= \sum_{i=1}^{n} \left[ \tilde{\phi}^B [k] \right]_x \tilde{Q}^{-1} \left( \tilde{\phi}^B [k] \right) \\
P_a [k+1] &= P_a [k] + \left( Q_\omega + 2P_S \left( P_a \left[ \Omega_m [k] - \hat{b} [k] \right]_x - P_c [k] \right) - P_a [k] S [k] P_a [k] \right) \Delta t \\
P_b [k+1] &= P_b [k] + \left( Q_b - P_c [k] S [k] P_a [k] \right) \Delta t \\
P_c [k+1] &= P_c [k] - \left( \left[ \Omega_m [k] - \hat{b} [k] \right]_x P_a [k] + P_a [k] S [k] P_c [k] + P_b [k] \right) \Delta t
\end{align*}
\]

8.3. Discrete GAMEF

The discrete form of GAMEF in (46) and (47) is as follows:

\[
\begin{align*}
\begin{bmatrix}
0 \\
\tilde{\phi}^B [k]
\end{bmatrix} &= \tilde{Q} [k]^{-1} \odot \begin{bmatrix}
0 \\
\nu^F [k]
\end{bmatrix} \odot \tilde{Q} [k] \\
\tilde{Q} [k+1] &= \exp \left\{ \frac{1}{T} \mathbf{I} \Gamma \left( \Omega_m [k] - \hat{b} [k] + P_a [k] W [k] \right) \Delta t \right\} \tilde{Q} [k] \\
W [k] &= \sum_{i=1}^{n} \tilde{\phi}^B [k] \times Q^{-1} \left( \tilde{\phi}^B [k] - \nu^B [k] \right)
\end{align*}
\]
where
\[
\begin{align*}
\dot{b} [k + 1] &= \dot{b} [k] + P_s^T [k] W [k] \Delta t \\
S [k] &= \sum_{i=1}^n [\alpha_i^B [k]] \times Q_{\alpha(i)}^{-1} [\alpha_i^B [k]] \\
C [k] &= \sum_{i=1}^n \mathcal{P}_s \left( Q_{\alpha(i)}^{-1} (\alpha_i^B [k] - \alpha_i^B [k]) (\alpha_i^B [k])^\top \right) \\
E [k] &= \text{Tr} \{ C [k] \} I_3 - C [k] \\
P_a [k + 1] &= \left( 2 \mathcal{P}_s \left( P_b [k] \left[ \Omega_m [k] - \dot{b} [k] - \frac{1}{2} P_a [k] W [k] \right] \times -P_c [k] \right) + P_a [k] (E [k] - S [k]) P_a [k] \right) \Delta t \\
P_b [k + 1] &= P_b [k] + \left( Q_\omega [k] + P_c [k] (E [k] - S [k]) P_c [k] \right) \Delta t \\
P_c [k + 1] &= P_c [k] - \left( \left[ \Omega_m [k] - \dot{b} [k] - \frac{1}{2} P_a [k] W [k] \right] \times P_c [k] - P_c [k] (E [k] - S [k]) P_c [k] + P_b [k] \right) \Delta t
\end{align*}
\]

8.4. Discrete CG-NDAF
Before we introduce the nonlinear filters in discrete form, let us recall (56), (57), and (58) and present them in sampling form

\[
\begin{align*}
\text{vex} \left( \mathcal{P}_a (M^B [k] \tilde{R} [k]) \right) &= \sum_{i=1}^n s_i \alpha_i^B [k] \times \alpha_i^B [k] \\
||M^B [k] \tilde{R} [k]||_1 &= \frac{1}{2} \sum_{i=1}^n s_i \left( 1 - (\alpha_i^B [k])^\top \alpha_i^B [k] \right) \\
Y (M^B [k], \tilde{R} [k]) &= \text{Tr} \left\{ \left( \sum_{i=1}^n s_i \alpha_i^B [k] \left( \alpha_i^B [k] \right)^\top \right)^{-1} \sum_{i=1}^n s_i \alpha_i^B [k] \left( \alpha_i^B [k] \right)^\top \right\}
\end{align*}
\]

8.4.1. Semi-direct Filter
The discrete form of CGSd-NDAF in (64) is as follows:
\[
\begin{align*}
\dot{R} [k + 1] &= \dot{R} [k] \exp \left( \left[ \Omega_m [k] - \dot{b} [k] - k_a W [k] \right] \times \Delta t \right) \\
W &= \text{vex} \left( \mathcal{P}_a (\tilde{R} [k]) \right), \quad \tilde{R} [k] = R_y^\top [k] \tilde{R} [k] \\
\dot{b} [k + 1] &= \dot{b} [k] + \gamma W [k] \Delta t
\end{align*}
\]

8.4.2. Direct Filter
The discrete form of CGD-NDAF in (65) is as follows:
\[
\begin{align*}
\dot{R} [k + 1] &= \dot{R} [k] \exp \left( \left[ \Omega_m [k] - \dot{b} [k] - k_a W [k] \right] \times \Delta t \right) \\
W [k] &= \text{vex} \left( \mathcal{P}_a (M^B [k] \tilde{R} [k]) \right) \\
\dot{b} [k + 1] &= \dot{b} [k] + \gamma W [k] \Delta t
\end{align*}
\]

with \( \text{vex} (\mathcal{P}_a (M^B [k] \tilde{R} [k])) \) being obtained through vectorial measurements as in (101).

8.5. Discrete AG-NDAF
The discrete form of AG-NDAF in (66) is as follows:
\[
\begin{align*}
\dot{R} [k + 1] &= \dot{R} [k] \exp \left( \left[ \Omega_m [k] - \dot{b} [k] - k_a W [k] \right] \times \Delta t \right) \\
W &= \frac{1}{\text{Tr} R_y [k]} \text{vex} \left( \mathcal{P}_a (R [k]) \right), \quad R [k] = R_y^\top [k] \tilde{R} [k] \\
\dot{b} [k + 1] &= \dot{b} [k] + \gamma W [k] \Delta t
\end{align*}
\]

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8.6. Discrete GP-NDAF

8.6.1. Semi-direct Filter

The discrete form of GPSd-NDAF in (71) is as follows:

\[
\begin{align*}
\hat{R} [k + 1] &= \hat{R} [k] \exp \left( \left[ \Omega_m [k] - \hat{b} [k] - k_w W [k] \right] \times \Delta t \right) \\
W [k] &= 2 \frac{k_w \mu k \xi [k] - \xi A / |\xi|}{1 - |\xi|} \text{vex} (P_a (\hat{R} [k])) \quad (105)
\end{align*}
\]

where

\[
\begin{align*}
\xi [k] &= (\xi_0 - \xi_\infty) \exp (-\ell k) + \xi_\infty \\
\xi_d [k] &= \frac{\xi [k] - \xi [k - 1]}{\Delta t} \\
\xi [k] &= \frac{1}{2} \ln \frac{B + |\xi [k]|/|\xi|}{\Delta t} \\
\mu [k] &= \frac{1}{2} \frac{1}{\xi^2 [k] + |\xi [k]|/|\xi|}
\end{align*}
\]

8.6.2. Direct Filter

The discrete form of GPD-NDAF in (74) is as follows:

\[
\begin{align*}
\hat{R} [k + 1] &= \hat{R} [k] \exp \left( \left[ \Omega_m [k] - \hat{b} [k] - k_w W [k] \right] \times \Delta t \right) \\
W [k] &= \frac{k_w \mu k \xi [k] - \xi A / |\xi|}{1 - |\xi|} \text{vex} (P_a (M^\mathcal{B} [k] \hat{R} [k])) \\
\hat{b} [k + 1] &= \hat{b} [k] + \frac{\mu k}{2} \xi \text{vex} (P_a (M^\mathcal{B} [k] \hat{R} [k])) \Delta t \\
\end{align*}
\]

where

\[
\begin{align*}
\xi [k] &= (\xi_0 - \xi_\infty) \exp (-\ell k) + \xi_\infty \\
\xi_d [k] &= \frac{\xi [k] - \xi [k - 1]}{\Delta t} \\
\xi [k] &= \frac{1}{2} \ln \frac{B + |\xi [k]|/|\xi|}{\Delta t} \\
\mu [k] &= \frac{1}{2} \frac{1}{\xi^2 [k] + |\xi [k]|/|\xi|}
\end{align*}
\]

8.7. Discrete GP-NSAF

8.7.1. Semi-direct Filter

The discrete form of GPSd-NSAF in (83) is as follows:

\[
\begin{align*}
\hat{R} [k + 1] &= \hat{R} [k] \exp \left( \left[ \Omega_m [k] - \hat{b} [k] - W [k] \right] \times \Delta t \right) \\
W [k] &= 2 \frac{k_w \mu k \xi [k] - \xi A / |\xi|}{1 - |\xi|} \text{vex} (P_a (\hat{R} [k])) \\
\hat{b} [k + 1] &= \hat{b} [k] + \gamma_1 (\xi [k] + 1) \exp (\xi [k]) \mu [k] \text{vex} (\hat{R} [k]) \Delta t \\
\hat{\sigma} [k + 1] &= \hat{\sigma} [k] + \gamma_2 \xi [k] (\xi [k] + 2) \exp (\xi [k]) \mu^2 [k] \text{vex} (\hat{R} [k]) \Delta t \\
\end{align*}
\]

where
where $\Psi(R) = \text{vex}(\mathcal{P}_a(R))$.

### 8.7.2. Direct Filter

The discrete form of GPD-NSAF in (89) is as follows:

\[
\begin{align*}
\dot{\hat{R}}[k+1] &= \hat{R}[k] \exp \left( [\Omega_m[k] - \hat{b}[k] - W[k]] \Delta t \right) \\
W[k] &= 2 \hat{\xi}[k+1] \mu[k] \text{diag} \left( \Psi(M^B[k] \hat{R}[k]) \right) \hat{\sigma}[k] + \frac{4}{\delta} \frac{k_{2\mu} \mu[k] \hat{\xi}[k] - \hat{\xi}[k]}{1 + \Psi(M^B[k], \hat{R}[k])} \Psi(M^B[k] \hat{R}[k]) \\
\hat{b}[k+1] &= \hat{b}[k] + \gamma_1 \mu[k] (\hat{\xi}[k] + 1) \exp (\hat{\xi}[k]) \Psi(M^B[k] \hat{R}[k]) \Delta t \\
\hat{\sigma}[k+1] &= \hat{\sigma}[k] + \gamma_2 (\hat{\xi}[k] + 2) \exp (\hat{\xi}[k]) \mu^2[k] \text{diag} \left( \Psi(M^B[k] \hat{R}[k]) \right) \Psi(M^B[k] \hat{R}[k]) \Delta t
\end{align*}
\]

where

\[
\begin{align*}
\hat{\xi}[k] &= (\hat{\xi}_0 - \xi_\infty) \exp (-\ell k) + \xi_\infty \\
\hat{\xi}_d[k] &= \frac{\hat{\xi}[k] - \xi[k-1]}{\xi[k] - \xi[k-1]} \\
\hat{\xi}[k] &= \frac{1}{2} \ln \frac{\hat{\xi}[k] + \xi[k]}{\hat{\xi}[k] - \xi[k]} \\
\hat{\mu}[k] &= \exp(2\hat{\xi}[k]) + \exp(-2\hat{\xi}[k]) + \frac{2}{\delta} \exp(\hat{\xi}[k]) \Psi(M^B[k] \hat{R}[k] \hat{R}[k]) \Delta t
\end{align*}
\]

with $\Psi(M^B\hat{R}) = \text{vex}(\mathcal{P}_a(M^B\hat{R}))$, $\text{vex} \left( \mathcal{P}_a (M^B[k] \hat{R}[k]) \right)$, $||M^B[k] \hat{R}[k]||$, and $Y(M^B[k], \hat{R}[k])$ being obtained through vectorial measurements as in (101).

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