Relative entropies, suitable weak solutions, and weak-strong uniqueness for the compressible Navier-Stokes system

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Abstract

We introduce the notion of relative entropy for the weak solutions to the compressible Navier-Stokes system. In particular, we show that any finite energy weak solution satisfies a relative entropy inequality with respect to any couple of smooth functions satisfying relevant boundary conditions. As a corollary, we establish the weak-strong uniqueness property in the class of finite energy weak solutions, extending thus the classical result of Prodi and Serrin to the class of compressible fluid flows.

1 Introduction

The method of relative entropy has been successfully applied to partial differential equations of different types. Relative entropies are non-negative quantities that provide a kind of distance between two solutions of the same problem, one of which typically enjoys some extra regularity properties. Carillo et al. [1] exploited entropy dissipation, expressed by means of the relative entropy with respect to a stationary
solution, in order to analyze the long-time behavior of certain quasilinear parabolic equations. Saint-Raymond [21] uses the relative entropy method to study the incompressible Euler limit of the Boltzmann equation. Other applications of the method can be found in Grenier [11], Masmoudi [16], Ukai [24], Wang and Jiang [25], among others.

Germain [10] introduced a class of (weak) solutions to the compressible Navier-Stokes system satisfying a relative entropy inequality with respect to a (hypothetical) strong solution of the same problem, and established the weak-strong uniqueness property within this class. Unfortunately, existence of solutions belonging to this class, where, in particular, the density possesses a spatial gradient in a suitable Lebesgue space, is not known. In [7], we introduced the concept of suitable weak solution for the compressible Navier-Stokes system, satisfying a general relative entropy inequality with respect to any sufficiently regular pair of functions. To be more specific, consider the fluid density \( \rho = \rho(t,x) \), together with the velocity field \( u = u(t,x) \), \( t \in \mathbb{R}, x \in \Omega \subset \mathbb{R}^3 \), the time evolution of which is governed by the Navier-Stokes system:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \text{div}_x (\rho u) &= 0, \\
\frac{\partial \rho}{\partial t} (\rho u) + \text{div}_x (\rho u \otimes u) + \nabla_x p(\rho) &= \text{div}_x S(\nabla_x u) + \rho f, \\
S &= \mu \left( \nabla_x u + \nabla_x u - \frac{2}{3} \text{div}_x u I \right) + \eta \text{div}_x u I, \quad \mu > 0, \quad \eta \geq 0,
\end{align*}
\]

supplemented with suitable boundary conditions, say,

\[ u|_{\partial \Omega} = 0. \]

If the domain \( \Omega \) is unbounded, we prescribe the far-field behavior:

\[ \rho \to \overline{\rho}, \quad u \to 0 \quad \text{as} \quad |x| \to \infty, \]

where \( \overline{\rho} \geq 0. \)

Relative entropy \( E \left( [\rho, u] \Big| [r, U] \right) \) with respect to \([r, U] \) is defined as

\[
E \left( [\rho, u] \Big| [r, U] \right) = \int_{\Omega} \left( \frac{1}{2} \rho |u - U|^2 + H(\rho) - H'(\rho) (\rho - r) - H(r) \right) \, dx,
\]

where

\[ H(\rho) = \rho \int_{\rho}^{\overline{\rho}} \frac{p(z)}{z^2} \, dz. \]

Following [7], we say that \( \rho, u \) is a suitable weak solution to problem (1.1)-(1.5) if equations (1.1-1.3) are satisfied in a weak sense, and, in addition to (1.1-1.4), the following (relative) energy inequality

\[
E \left( [\rho, u] \Big| [r, U] \right)(\tau) + \int_{0}^{\tau} \int_{\Omega} \left( S(\nabla_x u) - S(\nabla_x U) \right) : \left( \nabla_x u - \nabla_x U \right) \, dx \, dt \leq E \left( [\rho_0, u_0] \Big| [r(0,\cdot), U(0,\cdot)] \right) + \int_{0}^{\tau} \mathcal{R}(\rho, u, r, U) \, dt
\]

holds for a.a. \( \tau > 0 \), where

\[ \rho_0 = \rho(0,\cdot), \quad u_0 = u(0,\cdot), \]
and the remainder $\mathcal{R}$ reads

$$\mathcal{R}(\rho, u, r, U) \equiv \int_{\Omega} \rho \left( \partial_t U + u \nabla_x U \right) \cdot (U - u) \, dx \quad (1.9)$$

$$+ \int_{\Omega} S(\nabla_x U) : \nabla_x (U - u) \, dx + \int_{\Omega} \rho f \cdot (u - U) \, dx$$

$$+ \int_{\Omega} ((r - \rho) \partial_r H'(r) + \nabla_r H'(r) \cdot (rU - \rho u)) \, dx - \int_{\Omega} \div_x U \left( p(\rho) - p(r) \right) \, dx.$$  

Here, the functions $r, U$ are arbitrary smooth, $r$ strictly positive, and $U$ satisfying the no-slip boundary conditions (1.4). It is easy to check that (1.8) is satisfied as an equality as soon as the solution $\rho, u$ is smooth enough.

As shown in [7, Theorem 3.1], the Navier-Stokes system (1.1 - 1.5) admits global-in-time suitable weak solutions for any finite energy initial data. Moreover, the relative energy inequality (1.8) can be used to show that suitable weak solutions comply with the weak-strong uniqueness principle, meaning, a weak and strong solution emanating from the same initial data coincide as long as the latter exists. This can be seen by taking the strong solution as the “test” functions $r, U$ in the relative entropy inequality (1.8). Besides, a number of other interesting properties of the suitable weak solutions can be deduced, see [7, Section 4].

For the particular choice $r = \overline{\rho}, U = 0$, the relative energy inequality (1.8) reduces to the standard energy inequality

$$\mathcal{E}[\rho, u](\tau) + \int_{0}^{\tau} \int_{\Omega} S(\nabla_x u) : \nabla_u u \, dx \, dt \leq \mathcal{E}[\rho_0, u_0] + \int_{0}^{\tau} \int_{\Omega} \rho f \cdot u \, dx \, dt \text{ for a.a. } \tau > 0, \quad (1.10)$$

$$\mathcal{E}[\rho, u] = \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + H(\rho) - H'(\overline{\rho}) \left( \rho - \overline{\rho} \right) \right) \, dx.$$  

The weak solutions of the Navier-Stokes system satisfying, in addition, the energy inequality (1.10) are usually termed finite energy weak solutions, or, rather incorrectly, turbulent solutions in the sense of Leray’s original work [14].

Our goal in this paper is to show that any finite energy weak solution is in fact a suitable weak solution, in other words, the standard energy inequality (1.10) implies the relative energy inequality (1.8). In particular, the weak-strong uniqueness property as well as other results shown in [7] hold for the seemingly larger class of finite energy solutions. This observation extends easily to other types of boundary conditions and to a large class of domains. This kind of result can be viewed as an extension of the seminal work of Prodi [20] and Serrin [22] (see also Germain [9] for more recent results) to the compressible Navier-Stokes system. We provide an ultimate answer to the weak-strong uniqueness problem intimately related to the fundamental questions of the well-posedness for the compressible Navier-Stokes equations addressed by several authors, Desjardin [4], Germain [10], Hoff [12], [13], among others.

The paper is organized as follows. In Section 2, we provide an exact definition of finite energy weak solutions and state the main result. Section 3 is devoted to the proof of the main theorem and to possible extensions. Applications are discussed in Section 4.
2 Main results

For the sake of simplicity, we assume that the pressure \( p = p(\rho) \) is a continuously differentiable function of the density such that

\[
p \in C[0, \infty) \cap C^2(0, \infty), \quad p(0) = 0, \quad p'(\rho) > 0 \quad \text{for all} \quad \rho > 0, \quad \lim_{\rho \to \infty} \frac{p'(\rho)}{\rho^{\gamma-1}} = a > 0 \quad \text{for a certain} \quad \gamma > 3/2. \quad (2.1)
\]

Moreover, if \( \rho = 0 \), we suppose that \( p \) becomes asymptotically small for \( \rho \to 0 \) so that the function \( H \) defined in (1.7) is finite for any \( \rho > 0 \).

2.1 Finite energy weak solutions to the Navier-Stokes system

Definition 2.1

We shall say that \( \rho, u \) is a finite energy weak solution to the Navier-Stokes system (1.1 - 1.5) emanating from the initial data \( \rho_0, u_0 \) if

- \( \rho - \overline{\rho} \in L^\infty(0, T; L^2 + L^\gamma(\Omega)), \rho \geq 0 \quad \text{a.a. in} \quad (0, T) \times \Omega); \quad (2.2)\)
- \( u \in L^2(0, T; D_0^{1, 2}(\Omega; R^3)); \quad (2.3)\)
- \( \rho u \in L^\infty(0, T; L^2 + L^{2\gamma/(\gamma+1)}(\Omega; R^3)); \quad (2.4)\)
- \( p \in L^1_{\text{loc}}([0, T] \times \Omega); \quad (2.5)\)

- \( (\rho - \overline{\rho}) \in C_{\text{weak}}([0, T]; L^2 + L^\gamma(\Omega)) \) and the integral identity
  \[
  \int_\Omega \rho(\tau, \cdot) \varphi(\tau, \cdot) \, dx - \int_\Omega \rho_0 \varphi(0, \cdot) \, dx = \int_0^T \int_\Omega \left( \rho \partial_t \varphi + \rho u \cdot \nabla_x \varphi \right) \, dx \, dt \quad (2.6)\]

  holds for any \( \varphi \in C^\infty_c([0, T] \times \overline{\Omega}); \)

- \( \rho u \in C_{\text{weak}}([0, T]; L^2 + L^{2\gamma/(\gamma+1)}(\Omega; R^3)) \) and the integral identity
  \[
  \int_\Omega \rho u(\tau, \cdot) \cdot \varphi(\tau, \cdot) \, dx - \int_\Omega \rho_0 u_0 \cdot \varphi(0, \cdot) \, dx \quad (2.7)\]

  \[
  = \int_0^T \int_\Omega \left( \rho u \cdot \partial_t \varphi + (\rho u \otimes u) : \nabla_x \varphi + p(\rho) \text{div}_x \varphi - S(\nabla_x u) : \nabla_x \varphi + \rho f \cdot \varphi \right) \, dx \, dt
  \]

  is satisfied for any \( \varphi \in C^\infty_c([0, T] \times \Omega; R^3); \)

- the energy inequality
  \[
  \int_\Omega \left( \frac{1}{2} |u|^2 + H(\rho) - H'(\overline{\rho})(\rho - \overline{\rho}) - H(\overline{\rho}) \right)(\tau, \cdot) \, dx + \int_0^T \int_\Omega S(\nabla_x u) : \nabla_x u \, dx \, dt \quad (2.8)\]

  \[
  \leq \int_\Omega \left( \frac{1}{2} |u_0|^2 + H(\rho_0) - H'(\overline{\rho})(\rho_0 - \overline{\rho}) - H(\overline{\rho}) \right) \, dx + \int_0^T \int_\Omega \rho f \cdot u \, dx \, dt
  \]

  holds for a.a. \( \tau \in [0, T] \).
**Remark 2.1** We recall that the space \( D^{1,2}_0(\Omega) \) is defined as a completion of \( C_\infty^\infty(\Omega) \) with respect to the \( L^2 \)-norm of the gradient. In accordance with Sobolev’s inequality, \( D^{1,2}_0(\Omega) \subset L^6(\Omega) \), see Galdi [8].

**Remark 2.2** In (2.8), we tacitly assume that the initial data are chosen in such a way that the first integral on the right hand side is finite.

### 2.2 Finite energy weak solutions satisfy the relative energy inequality

Our main result reads as follows:

**Theorem 2.1** Let \( \Omega \subset \mathbb{R}^3 \) be a domain. Suppose that the pressure \( p \) satisfies hypothesis (2.1),
\[
f \in L^\infty(0, T; L^1 \cap L^\infty(\Omega; \mathbb{R}^3)),
\]
and that \( \varpi \geq 0 \). Let \( \varrho, u \) be a finite energy weak solution to the Navier-Stokes system (1.1 - 1.5) in the sense specified in Section 2.1.

Then \( \varrho, u \) satisfy the relative energy inequality (1.8) for any \( U \in C_\infty^\infty([0, T] \times \Omega; \mathbb{R}^3) \), and \( r > 0, r - \varpi \in C_\infty^\infty([0, T] \times \overline{\Omega}) \).

The proof and several extensions of Theorem 2.1 are presented in Section 3. Applications will be discussed in Section 4.

### 3 Proof of the main result

#### 3.1 Proof of Theorem 2.1

Take \( U \) as a test function in the momentum equation (2.7) to obtain
\[
\int_\Omega \varrho u(\tau, \cdot) \cdot U(\tau, \cdot) \, dx = \int_\Omega \varrho_0 u_0 \cdot U(0, \cdot) \, dx
\]
(3.1)

Similarly, we can use the scalar quantity \( \frac{1}{2}|U|^2 \) as a test function in (2.6):
\[
\int_\Omega \frac{1}{2} \varrho(\tau, \cdot)|U|^2(\tau, \cdot) \, dx = \int_\Omega \frac{1}{2} \varrho_0 |U(0, \cdot)|^2 \, dx + \int_0^\tau \int_\Omega \left( \varrho U \cdot \partial_t U + \varrho u \cdot \nabla_x U \cdot U \right) \, dx \, dt. \quad (3.2)
\]

Finally, we test (2.6) on \( H'(r) - H'(\varpi) \) to get
\[
\int_\Omega \varrho(\tau, \cdot) \left( H'(r)(\tau, \cdot) - H'(\varpi) \right) \, dx = \int_\Omega \varrho_0 \left( H'(r)(0, \cdot) - H'(\varpi) \right) \, dx
\]
(3.3)
\[
+ \int_0^\tau \int_\Omega \left( \varrho \partial_t H'(r) + \varrho u \cdot \nabla_x H'(r) \right) \, dx \, dt.
\]
Summing up relations (3.1 - 3.3) with the energy inequality (2.8), we infer that
\[
\int_{\Omega} \left( \frac{1}{2} \rho |u - U|^2 + H(\rho) - \left( H'(r)\rho - H'(|\rho|) \right) \right) (\tau, \cdot) \, dx + \int_0^T \int_{\Omega} \left( S(\nabla_x u) - S(\nabla_x U) \right) : \left( \nabla_x u - \nabla_x U \right) \, dx \, dt
\]
\[
= \int_{\Omega} \left( \frac{1}{2} \rho_0 |u_0 - U(0, \cdot)|^2 + H(\rho_0) - \left( H'(r(0, \cdot))\rho_0 - H'(|\rho_0|) \right) \right) \, dx + \int_0^T \int_{\Omega} \rho (\partial_t U + \rho u \cdot \nabla_x U) \cdot (U - u) \, dx \, dt
\]
\[
+ \int_0^T \int_{\Omega} S(\nabla_x U) : \nabla_x (U - u) \, dx + \int_{\Omega} \rho \cdot (u - U) \, dx \, dt - \int_0^T \int_{\Omega} p(r) \, dx \, dt.
\]
Realizing that
\[
H'(r) r - H(r) - H'(|\rho|) \rho = p(r) - p(|\rho|),
\]
we compute
\[
\int_{\Omega} \left( p(r) - p(|\rho|) \right) (\tau, \cdot) \, dx - \int_{\Omega} \left( p(r) - p(|\rho|) \right) (0, \cdot) \, dx = \int_0^T \int_{\Omega} \partial_t p(r) \, dx \, dt;
\]
whence, by virtue of the identity
\[
\int_{\Omega} \left( r \partial_t H'(r) + r \nabla_x H'(r) \cdot U + p(r) \nabla_x U \right) \, dx = \int_{\Omega} \partial_t p(r) \, dx,
\]
relation (3.4) implies (1.8). Theorem 2.1 has been proved. Note that (3.5) relies on the fact that \( U \cdot n |_{\partial \Omega} = 0 \).

3.2 Possible extensions
The conclusion of Theorem 2.1 can be extended in several directions. Here, we shortly discuss the problem of an alternative choice of boundary conditions as well as the class of admissible test functions \( r, U \).

3.2.1 General slip boundary conditions with friction
Similar result can be obtained provided the no-slip boundary condition (1.4) is replaced by the slip boundary conditions with friction (Navier’s boundary condition)
\[
u \cdot n = 0, \quad (S(\nabla_x u)n)_{\text{tan}} + \beta u_{\text{tan}} = 0 \text{ on } (0, T) \times \partial \Omega,
\]
where \( \beta \geq 0 \) and \( v_{\text{tan}} |_{\partial \Omega} = (v - (v \cdot n)n) |_{\partial \Omega} \) denotes the tangential component of a vector field \( v \) at the boundary. Note that the so-called complete slip boundary conditions correspond to the particular situation \( \beta = 0 \).

The definition of finite energy weak solutions is similar to Section 2.1 with the following modifications:
• the spatial domain $\Omega$ possesses a Lipschitz boundary, where (2.3) is replaced by the requirement $u \in L^2(0, T; D^{1,2}_0(\Omega; R^3))$, with

$$D^{1,2}_0(\Omega; R^3) = \left\{ v \in L^6_{\text{loc}}(\Omega; R^3) \mid \nabla_x v \in L^2(\Omega; R^{3 \times 3}), v \cdot n|_{\partial \Omega} = 0 \right\};$$

• the pressure satisfies

$$p(\rho) \in L^1_{\text{loc}}([0, T] \times \Omega) \quad (3.7)$$

instead of (2.5);

• the weak formulation of the momentum equation (2.6) has to be replaced by

$$\int_0^T \int_{\Omega} \left( \rho u \cdot \partial_t \varphi + \rho (u \otimes u) : \nabla_x \varphi + p(\rho) \text{div} \varphi \right) \, dx \, dt \quad (3.8)$$

$$- \int_0^T \int_{\Omega} S(\nabla_x u) : \nabla_x \varphi \, dx \, dt - \beta \int_0^T \int_{\partial \Omega} u \cdot \varphi \, dS \, dt \quad = - \int_0^T \int_{\Omega} \rho f \cdot \varphi \, dx \, dt + \int_{\Omega} (\rho u(\tau) \cdot \varphi(\tau, \cdot)) \, dx - \int_{\Omega} \rho_0 u_0 \cdot \varphi(0, \cdot) \, dx$$

for all $\tau \in [0, T]$, for any test function $\varphi \in C^\infty_c([0, T] \times \Omega; R^3)$, $\varphi \cdot n = 0$ on $[0, T] \times \partial \Omega$;

• energy inequality (2.7) is replaced by

$$\int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + H(\rho) - H'(\bar{\rho})(\rho - \bar{\rho}) - H(\bar{\rho}) \right) (\tau, \cdot) \, dx \quad (3.9)$$

$$+ \int_0^T \int_{\Omega} S(\nabla_x u) : \nabla_x u \, dx \, dt + \beta \int_0^T \int_{\partial \Omega} |u|^2 \, dS \, dt \quad \leq \int_0^T \int_{\Omega} \rho f \cdot u \, dx \, dt + \int_{\Omega} \left( \frac{1}{2} \rho_0 |u_0|^2 + H(\rho_0) - H'(\bar{\rho})(\rho_0 - \bar{\rho}) - H(\bar{\rho}) \right)(\tau, \cdot) \, dx$$

for a.a. $\tau \in (0, T)$.

In this case, the conclusion of Theorem 2.1 remains valid for any couple $(r, \mathbf{U})$ such that

$$r - \bar{\rho} \in C^\infty_c([0, T] \times \Omega), \quad \mathbf{U} \in C^\infty_c([0, T] \times \Omega; R^3), \quad \mathbf{U} \cdot n|_{\partial \Omega} = 0 \quad (3.10)$$

with the relative entropy inequality that reads

$$\mathcal{E}([\rho, \mathbf{u}] | [r, \mathbf{U}]) (\tau, \cdot) \quad (3.11)$$

$$+ \int_0^T \int_{\Omega} |S(\nabla_x \mathbf{u} - \nabla_x \mathbf{U})| : \nabla_x (\mathbf{u} - \mathbf{U}) \, dx \, dt + \beta \int_0^T \int_{\partial \Omega} |\mathbf{u} - \mathbf{U}|^2 \, dS \, dt \quad \leq \mathcal{E}([\rho_0, \mathbf{u}_0] | [r(0), \mathbf{U}(0)]) (\tau) + \int_0^T \mathcal{R} (\rho, \mathbf{u}, r, \mathbf{U}) \, dt \quad \text{for a.a. } \tau \in (0, T),$$

where

$$\mathcal{R} (\rho, \mathbf{u}, r, \mathbf{U}) = \int_{\Omega} \rho f \cdot (\mathbf{u} - \mathbf{U}) \, dx - \beta \int_0^T \int_{\partial \Omega} \mathbf{u} \cdot (\mathbf{u} - \mathbf{U}) \, dS \, dt \quad (3.12)$$

$$+ \int_{\Omega} \left( \rho \partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \right) \cdot (\mathbf{U} - \mathbf{u}) - S(\nabla_x \mathbf{U}) : \nabla_x (\mathbf{u} - \mathbf{U}) \right) \, dx$$

$$+ \int_{\Omega} \left( (r - \rho) \partial_t H'(r) + \nabla_x H'(r) \cdot (r \mathbf{U} - \rho \mathbf{u}) - \text{div}_x \mathbf{U} \left( p(\rho) - p(r) \right) \right) \, dx.$$
3.2.2 Extending the admissible class of test functions

Using density arguments we can extend considerably the class of test functions $r, U$ appearing in the relative energy inequality (1.8) resp. (3.11). Indeed:

- For the left hand side (1.8) resp. (3.11) to be well defined, the functions $r, U$ must belong at least to the class
  \[ r - \bar{r} \in C_{\text{weak}}([0,T]; L^2 + L^7(\Omega)), \]
  \[ U \in L^2(0,T; W^{1,2}(\Omega; R^3)). \]

- A short inspection (1.9) resp. (3.12) implies that the integrals are well-defined if, at least,
  \[ \partial_t U \in L^2(0,T; L^3 \cap L^{6/((5\gamma-6)-1)}(\Omega; R^3)) + L^1(0,T; L^{4/3} \cap L^{2\gamma/(\gamma-1)}(\Omega; R^3)), \]
  \[ \nabla_x U \in L^\infty(0,T; L^6 \cap L^{3\gamma/(2\gamma-3)}(\Omega; R^{3\times3})) + L^2(0,T; L^{12/7} \cap L^{6\gamma/(\gamma-3)}(\Omega, R^{3\times3})) \]
  \[ + L^1(0,T; L^\infty(\Omega; R^3)), \]
  \[ \text{div} U \in L^1(0,T; L^2(\Omega)), \]
  \[ \nabla x H'(r) \in L^\infty(0,T; L^3 \cap L^{6\gamma/(5\gamma-6)}(\Omega; R^3)) + L^1(0,T; L^{4/3} \cap L^{2\gamma/(\gamma-1)}(\Omega; R^3)). \]

- The function $r$ must be bounded below away from zero, and
  \[ \partial_t H'(r) \in L^1(0,T; L^{7/(\gamma-1)} \cap L^2(\Omega)), \]
  \[ \nabla_x H'(r) \in L^2(0,T; L^3 \cap L^{6\gamma/(5\gamma-6)}(\Omega; R^3)) + L^1(0,T; L^{4/3} \cap L^{2\gamma/(\gamma-1)}(\Omega, R^3)). \]

- Finally, the vector field $U$ has to satisfy
  \[ U|_{\partial\Omega} = 0 \text{ in the case of boundary conditions (1.4)}, \]
  \[ U \cdot n|_{\partial\Omega} = 0 \text{ in the case of boundary conditions (3.6)}. \]

Consequently, Theorem 2.1 is valid even if we replace the hypotheses on smoothness and integrability of the test functions $(r, U)$ by weaker hypotheses, namely (3.13–3.20).

In particular, $r, U$ may be another (strong) solution emanating from the same initial data $\bar{r}_0, u_0$. Specific examples will be discussed in the forthcoming section.

4 Applications

In this section, we show how Theorem 2.1 can be applied in order to establish weak-strong uniqueness property for the compressible Navier-Stokes system in the class of finite energy weak solutions in bounded and unbounded domains. Other applications can be found in [7].
4.1 Weak-strong uniqueness on bounded domains

4.1.1 No-slip boundary conditions

To begin, observe that any finite energy weak solution \( \rho, u \) of the compressible Navier-Stokes system (1.1) in \((0, T) \times \Omega\), where \( \Omega \) is a bounded domain, belongs to the class

\[
\rho \in C_{\text{weak}}([0, T]; L^2(\Omega)), \quad u \in C_{\text{weak}}([0, T]; W^{1,2}_0(\Omega; \mathbb{R}^3)),
\]

and, by virtue of the energy inequality (2.8),

\[
p(\rho) \in L^\infty(0, T; L^1(\Omega)).
\]

Moreover, it is easy to check that

\[
H(\rho) - H'(r)(\rho - r) - H(r) \geq c(r) \begin{cases} (\rho - r)^2 & \text{for } r/2 < \rho < 2r, \\ (1 + \rho^\gamma) & \text{otherwise} \end{cases},
\]

where \( c(r) \) is uniformly bounded for \( r \) belonging to compact sets in \((0, \infty)\).

Finally, note that, since the total mass is a conserved quantity on a bounded domain, we can take \( \overline{\rho} \) in (1.7) so that

\[
\int_\Omega (\rho - \overline{\rho}) \, dx = 0.
\]

The rather obvious leading idea of the proof of weak-strong uniqueness is to take \( r = \tilde{\rho}, \quad U = \tilde{\mathbf{u}} \) in the relative energy inequality (1.8), where \( \tilde{\rho}, \tilde{\mathbf{u}} \) is a (hypothetical) regular solution, originating from the same initial data. The following formal computations will require certain smoothness of \( \tilde{\rho}, \tilde{\mathbf{u}} \) specified in the concluding theorem. Moreover, we assume that \( \tilde{\rho} \) is bounded below away from zero on the whole compact time interval \([0, T]\).

Our goal is to examine all terms in the remainder (1.9) and to show they can be “absorbed” by the left-hand side of (1.3) by means of a Gronwall type argument.

1. We rewrite

\[
\int_\Omega \rho \left( \partial_t \tilde{\mathbf{u}} + \mathbf{u} \cdot \nabla_x \tilde{\mathbf{u}} \right) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \, dx = \int_\Omega \rho \left( \partial_t \tilde{\mathbf{u}} + \mathbf{u} \cdot \nabla_x \tilde{\mathbf{u}} \right) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \, dx + \int_\Omega \rho (\mathbf{u} - \tilde{\mathbf{u}}) \cdot \nabla_x \tilde{\mathbf{u}} \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \, dx.
\]

Seeing that

\[
\partial_t \tilde{\mathbf{u}} + \mathbf{u} \cdot \nabla_x \tilde{\mathbf{u}} = \frac{1}{\tilde{\rho}} \text{div}_x S(\nabla_x \tilde{\mathbf{u}}) + \mathbf{f} - \nabla_x H'(\tilde{\rho}),
\]

we go back to (1.9) to obtain

\[
\mathcal{R}(\rho, u, \tilde{\rho}, \tilde{\mathbf{u}}) = \int_\Omega \rho (\mathbf{u} - \tilde{\mathbf{u}}) \cdot \nabla_x \tilde{\mathbf{u}} \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \, dx + \int_\Omega \frac{1}{\tilde{\rho}} (\rho - \tilde{\rho}) \text{div}_x S(\nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \, dx
\]

\[
+ \int_\Omega (\tilde{\rho} - \rho) \left( \partial_t H'(\tilde{\rho}) + \nabla_x H'(\tilde{\rho}) \cdot \tilde{\mathbf{u}} \right) \, dx - \int_\Omega \text{div}_x \tilde{\mathbf{u}} \left( p(\rho) - p(\tilde{\rho}) \right) \, dx.
\]
2. Computing

\[(\bar{\rho} - \rho)(\partial_t H'(\bar{\rho}) + \nabla_x H'(\bar{\rho}) \cdot \bar{u}) = -\text{div}_x \bar{u}(\rho - \bar{\rho})p'(\bar{\rho}),\]

we may infer that

\[
\int_\Omega (\bar{\rho} - \rho)(\partial_t H'(\bar{\rho}) + \nabla_x H'(\bar{\rho}) \cdot \bar{u}) \, dx - \int_\Omega \text{div}_x \bar{u}(p(\rho) - p(\bar{\rho})) \, dx
\]

\[
= - \int_\Omega \text{div}_x \bar{u}\left(p(\rho) - p'(\bar{\rho})(\rho - \bar{\rho}) - p(\bar{\rho})\right) \, dx;
\]

whence

\[
R(\rho, u, \bar{\rho}, \bar{u}) = \int_\Omega \rho(u - \bar{u}) \cdot \nabla_x \bar{u} \cdot (\bar{u} - u) \, dx - \int_\Omega \text{div}_x \bar{u}\left(p(\rho) - p'(\bar{\rho})(\rho - \bar{\rho}) - p(\bar{\rho})\right) \, dx \tag{4.2}
\]

\[
+ \int_\Omega \frac{1}{\bar{\rho}}(\rho - \bar{\rho}) \, \text{div}_x S(\nabla_x \bar{u}) \cdot (\bar{u} - u) \, dx.
\]

3. In view of (4.1), we have

\[
\left| \int_\Omega \rho(u - \bar{u}) \cdot \nabla_x \bar{u} \cdot (\bar{u} - u) \, dx - \int_\Omega \text{div}_x \bar{u}\left(p(\rho) - p'(\bar{\rho})(\rho - \bar{\rho}) - p(\bar{\rho})\right) \, dx \right| \tag{4.3}
\]

\[
\leq c\|\nabla_x \bar{u}\|_{L^\infty(\Omega; R^d)} E\left(\left[\rho, u\right]\left[\bar{\rho}, \bar{u}\right]\right),
\]

provided

\[
0 < \inf_{[0,T] \times \Omega} \bar{\rho} \leq \bar{\rho}(t, x) \leq \sup_{[0,T] \times \Omega} \bar{\rho} < \infty. \tag{4.4}
\]

4. Finally, we write

\[
\int_\Omega \frac{1}{\bar{\rho}}(\rho - \bar{\rho}) \, \text{div}_x S(\nabla_x \bar{u}) \cdot (\bar{u} - u) \, dx
\]

\[
= \int_{\{\bar{\rho}/2 < \rho < 2\bar{\rho}\}} \frac{1}{\bar{\rho}}(\rho - \bar{\rho}) \, \text{div}_x S(\nabla_x \bar{u}) \cdot (\bar{u} - u) \, dx
\]

\[
+ \int_{\{\rho \leq \bar{\rho}/2\}} \frac{1}{\bar{\rho}}(\rho - \bar{\rho}) \, \text{div}_x S(\nabla_x \bar{u}) \cdot (\bar{u} - u) \, dx + \int_{\{\rho \geq 2\bar{\rho}\}} \frac{1}{\bar{\rho}}(\rho - \bar{\rho}) \, \text{div}_x S(\nabla_x \bar{u}) \cdot (\bar{u} - u) \, dx,
\]

where, by virtue of Hölder’s inequality,

\[
\left| \int_{\{\bar{\rho}/2 < \rho < 2\bar{\rho}\}} \frac{1}{\bar{\rho}}(\rho - \bar{\rho}) \, \text{div}_x S(\nabla_x \bar{u}) \cdot (\bar{u} - u) \, dx \right| \tag{4.5}
\]

\[
\leq c(\delta) \left\| \frac{1}{\bar{\rho}} \, \text{div}_x S(\nabla_x \bar{u}) \right\|_{L^1(\Omega; R^d)}^2 \int_{\{\bar{\rho}/2 < \rho < 2\bar{\rho}\}} (\rho - \bar{\rho})^2 \, dx + \delta \|\bar{u} - u\|_{L^2(\Omega; R^d)}^2
\]

for any $\delta > 0$. 

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Furthermore, in accordance with $[4.1]$, we get
\[
\int_{\{\tilde{\vartheta} \leq \vartheta < 2\tilde{\vartheta}\}} (q - \tilde{q})^2 \, dx \leq c\mathcal{E}\left([\vartheta, u]\big|\tilde{\vartheta}, \tilde{u}\right),
\]
(4.6)
while, by virtue of Sobolev’s inequality and Korn-type inequality (see e.g. Dain [3])
\[
\|z\|_{L^2(\Omega; \mathbb{R}^3)} \leq c\|\mathbb{S}(\nabla z)\|_{L^2(\Omega; \mathbb{R}^3)}, \quad z \in W^{1,2}(\Omega; \mathbb{R}^3),
\]
(4.7)
we have
\[
\|\tilde{u} - u\|_{L^6(\Omega; \mathbb{R}^3)}^2 \leq c\|\nabla x u - \nabla x \tilde{u}\|_{L^2(\Omega; \mathbb{R}^3)}^2 \leq c\|\mathbb{S}(\nabla x u - \nabla x \tilde{u})\|_{L^2(\Omega; \mathbb{R}^3)}^2.
\]
(4.8)
Therefore,
\[
\left|\int_{\{0 \leq \vartheta \leq \tilde{\vartheta}/2\}} \frac{1}{\vartheta} (q - \tilde{q}) \text{div}_x \mathbb{S}(\nabla x \tilde{u}) \cdot (\tilde{u} - u) \, dx\right| \leq c(\delta) \left\|\text{div}_x \mathbb{S}(\nabla x \tilde{u})\right\|_{L^2(\Omega; \mathbb{R}^3)}^2 \mathcal{E}\left([\vartheta, u]\big|\tilde{\vartheta}, \tilde{u}\right) + \delta\|\mathbb{S}(\nabla x u - \nabla x \tilde{u})\|_{L^2(\Omega; \mathbb{R}^3)}^2
\]
for any $\delta > 0$.
Next we realize that
\[
\mathcal{E}(\vartheta, \tilde{\vartheta} | \tilde{\vartheta}, \tilde{u}) \in L^\infty(0, T)
\]
and that
\[
\|\vartheta\|_{L^\gamma(\{\vartheta > 2\tilde{\vartheta}\})} \leq c\mathcal{E}(\vartheta, \tilde{\vartheta} | \tilde{\vartheta}, \tilde{u})^{1/\gamma}, \quad \|\vartheta^{\gamma/2}\|_{L^2(\{\vartheta > 2\tilde{\vartheta}\})} \leq c\left[\mathcal{E}(\vartheta, \tilde{\vartheta} | \tilde{\vartheta}, \tilde{u})\right]^{1/2}.
\]
Using these facts, we deduce
\[
\left|\int_{\{\vartheta \geq 2\tilde{\vartheta}\}} \frac{1}{\vartheta} (q - \tilde{q}) \text{div}_x \mathbb{S}(\nabla x \tilde{u}) \cdot (\tilde{u} - u) \, dx\right| \leq c\|\mathbb{S}(\nabla x u - \nabla x \tilde{u})\|_{L^2(\Omega; \mathbb{R}^3)}^2 \int_{\{\vartheta \geq 2\tilde{\vartheta}\}} \left|\text{div}_x \mathbb{S}(\nabla x \tilde{u})\right| (\tilde{u} - u) \, dx \leq \delta\|\mathbb{S}(\nabla x u - \nabla x \tilde{u})\|_{L^2(\Omega; \mathbb{R}^3)}^2 + c(\delta)\|\text{div}_x \mathbb{S}(\nabla x \tilde{u})\|_{L^2(\Omega; \mathbb{R}^3)}^2 \mathcal{E}(\vartheta, \tilde{\vartheta} | \tilde{\vartheta}, \tilde{u}), \quad q = \frac{6\gamma}{5\gamma - 6}.
\]
(4.9)
Summing up relations $[4.2]$ - $[4.9]$ we conclude that the relative entropy inequality, applied to $r = \tilde{\vartheta}$, $U = \tilde{u}$, yields the desired conclusion
\[
\mathcal{E}\left([\vartheta, u]\big|\tilde{\vartheta}, \tilde{u}\right)(\tau) \leq \int_0^\tau h(t)\mathcal{E}\left([\vartheta, u]\big|\tilde{\vartheta}, \tilde{u}\right)(t) \, dt, \quad \text{with } h \in L^1(0, T),
\]
(4.10)
provided $\tilde{\vartheta}$ satisfies $[4.4]$, and
\[
\nabla x \tilde{u} \in L^1(0, T; L^\infty(\Omega; \mathbb{R}^3)) \cap L^2(0, T; L^2(\Omega; \mathbb{R}^3)), \quad \text{div}_x \mathbb{S}(\nabla x \tilde{u}) \in L^2(0, T; L^3(\Omega; \mathbb{R}^3)),
\]
(4.11)
with
\[ q = \frac{6\gamma}{5\gamma - 6}. \]

We have shown the following result:

**Theorem 4.1** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded Lipschitz domain, let the pressure \( p \) satisfy hypothesis (2.1), and let
\[ f \in L^1(0, T; L^{2\gamma/(\gamma-1)}(\Omega; \mathbb{R}^3)). \]
Assume that \( \tilde{\rho}, \tilde{u} \) is a finite energy weak solution to the Navier-Stokes system (1.1 - 1.4) in \( (0, T) \times \Omega \), specified in Section 2.1. Let \( \hat{\rho}, \hat{u} \) be a (strong) solution of the same problem belonging to the class \( \hat{\rho} \leq \inf_{(0, T) \times \Omega} \hat{\rho} \leq \sup_{(0, T) \times \Omega} \hat{\rho} < \infty \), emanating from the same initial data.
Then
\[ \rho = \hat{\rho}, \quad u = \hat{u} \text{ in } (0, T) \times \Omega. \]

**Remark 4.1** We need \( \Omega \) to be at least Lipschitz to guarantee the \( W^{1,p} \) extension property, with the associated embedding relations.

**Remark 4.2** The reader will have noticed that the regularity properties required for \( \hat{\rho}, \hat{u} \) in Theorem 4.1 are in fact stronger than (4.11). The reason is that all integrands appearing in the relative energy inequality (1.8) must be well defined.

**Remark 4.3** Existence of finite energy weak solutions was shown in [6] for general (finite energy) data and without any restriction on imposed on smoothness of \( \partial \Omega \).

**Remark 4.4** Local-in-time existence of strong solutions belonging to the regularity class specified in Theorem 4.1 was proved by Sun, Wang, and Zhang [23], under natural restrictions imposed on the initial data.

### 4.1.2 Navier boundary conditions with friction

Theorem 4.1 holds in the case of Navier’s boundary condition (3.6). The proof remains basically without changes; the standard Korn-type inequality (4.1) has to be replaced by a more sophisticated one, namely
\[ \|v\|_{W^{1,2}((\Omega, \mathbb{R}^3))} \leq c(M, K, p) \left( \|\nabla_x v\|_{L^2((\Omega, \mathbb{R}^3))}^2 + \|R \mathbf{v}\|_{L^p(\Omega)} \right) \]
(4.12)
for any \( v \in W^{1,2}(\Omega; \mathbb{R}^3), \quad R \geq 0, \quad M \leq \int_\Omega R dx, \quad \|R\|_{L^p(\Omega)} \leq K, \)
where \( M, K > 0, \ p > 1 \) (see [5] Theorem 10.17). It is employed in estimate (4.8) with \( v = u - \bar{u} \) and \( R = \varrho \).
4.2 Weak strong uniqueness on unbounded domains

4.2.1 No-slip boundary conditions

If the Navier-Stokes system is considered on an unbounded domain \( \Omega \), the far-field behavior \( (1.5) \) must be specified. Here, we assume that \( \bar{\rho} > 0 \) so that the density \( \tilde{\rho} \) of the (hypothetical) strong solution may be bounded below away from zero. Moreover, the finite energy weak solutions necessarily belong to the class:

\[
\begin{align*}
\rho - \bar{\rho} \in L^\infty(0, T; L^2 + L^\gamma(\Omega)), & \quad p(\rho) - p(\bar{\rho}) \in L^\infty(0, T; L^2 + L^1(\Omega)), \\
u \in L^2(0, T; W^{1,2}_0(\Omega; R^3)), & \quad \rho \nu \in L^\infty(0, T; L^2 + L^{2\gamma/(\gamma+1)}(\Omega; R^3)).
\end{align*}
\] (4.13) (4.14)

An appropriate modification of Theorem 4.1 for unbounded domains reads:

**Theorem 4.2** Let \( \Omega \subset R^3 \) be an unbounded domain with a uniformly Lipschitz boundary, let the pressure \( p \) satisfy hypothesis \( (2.1) \), and let \( f \in L^1(0, T; L^1 \cap L^\infty(\Omega; R^3)) \). Assume that \( \rho, u \) is a finite energy weak solution to the Navier-Stokes system \( (1.1 - 1.4) \) in \( (0, T) \times \Omega \), specified in Section 2.1, satisfying the far-field boundary conditions \( (1.5) \), with \( \bar{\rho} > 0 \). Let \( \tilde{\rho}, \tilde{u} \) be a (strong) solution of the same problem belonging to the class

\[
\begin{align*}
\rho - \bar{\rho} \in L^\infty(0, T; L^2 + L^\gamma(\Omega)), & \quad p(\rho) - p(\bar{\rho}) \in L^\infty(0, T; L^2 + L^1(\Omega)), \\
u \in L^2(0, T; W^{1,2}_0(\Omega; R^3)), & \quad \rho \nu \in L^\infty(0, T; L^2 + L^{2\gamma/(\gamma+1)}(\Omega; R^3))
\end{align*}
\] (4.13) (4.14)

emanating from the same initial data, and satisfying the energy inequality \( (1.10) \).

Then \( \rho = \tilde{\rho}, \ u = \tilde{u} \) in \( (0, T) \times \Omega \).

**Remark 4.5** The uniformly Lipschitz boundary \( \partial \Omega \) guarantees the \( W^{1,p} \) extension property as well as validity of Korn’s inequality \( (4.7) \).

**Remark 4.6** Since the strong solution satisfies the energy (in)equality \( (1.10) \), it automatically belongs to the regularity class \( (4.13), (4.14) \).

**Remark 4.7** Existence of finite energy weak solutions for certain classes of unbounded domains was shown in \[19\], see also Lions \[15\].

**Remark 4.8** The reader may consult the nowadays classical papers by Matsumura and Nishida \[17\], \[18\] for the existence of strong solutions, more recent results can be found in Cho, Choe and Kim \[2\], and in the references cited therein.
4.2.2 Navier boundary conditions

Theorem 4.2 remains valid also for the Navier boundary conditions. We have however suppose that on the considered unbounded domain a sort of Korn type inequality holds, for example

\[
\|v\|_{W^{1,2}(\Omega;\mathbb{R}^3)}^2 \leq c(|V|) \left( \|\mathcal{S}(\nabla_x v)\|_{L^2(\Omega;\mathbb{R}^3)}^2 + \int_{\Omega \setminus V} |v|^2 \, dx \right),
\]

(4.15)

for any \( v \in W^{1,2}(\Omega), |V| < \infty. \)

Such inequality is known to hold in a half space, an exterior domain, a cylinder, a plane slab, to name only a few.

Since

\[
\left| \{|\varrho - \bar{\varrho}| \geq \bar{\varrho}/2\} \right| < \infty,
\]

inequality (4.15) implies the validity of (4.12) with \( v = u - \bar{u} \) and \( R = \varrho \). This inequality has to replace the standard Korn’s inequality (4.7) in estimate (4.8). Other arguments in the proof remain without changes.

References

[1] J. Carrillo, A. Jüngel, P.A. Markowich, G. Toscani, and A. Unterreiter. Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities. Monatshefte Math., 133:1–82, 2001.

[2] Y. Cho, H.J. Choe, and H. Kim. Unique solvability of the initial boundary value problems for compressible viscous fluids. J. Math. Pures. Appl., 83:243–275, 2004.

[3] S. Dain. Generalized Korn’s inequality and conformal Killing vectors. Calc. Var. Partial Differential Equations, 25:535–540, 2006.

[4] B. Desjardins. Regularity of weak solutions of the compressible isentropic Navier-Stokes equations. Commun. Partial Differential Equations, 22:977–1008, 1997.

[5] E. Feireisl and A. Novotný. Singular limits in thermodynamics of viscous fluids. Birkhauser, Basel, 2009.

[6] E. Feireisl, A. Novotný, and H. Petzeltová. On the domain dependence of solutions to the compressible Navier-Stokes equations of a barotropic fluid. Math. Meth. Appl. Sci., 25:1045–1073, 2002.

[7] E. Feireisl, A. Novotný, and Y. Sun. Suitable weak solutions to the Navier–Stokes equations of compressible viscous fluids. Indiana Univ. Math. J., 2011. To appear.

[8] G. P. Galdi. An introduction to the mathematical theory of the Navier - Stokes equations, I. Springer-Verlag, New York, 1994.

[9] P. Germain. Multipliers, paramultipliers, and weak-strong uniqueness for the Navier-Stokes equations. J. Differential Equations, 226:373–428, 2006.
[10] P. Germain. Weak-strong uniqueness for the isentropic compressible Navier-Stokes system. *J. Math. Fluid Mech.*, 2010. Published online.

[11] E. Grenier. Oscillatory perturbations of the Navier-Stokes equations. *J. Math. Pures Appl. (9)*, 76(6):477–498, 1997.

[12] D. Hoff. Compressible flow in a half-space with Navier boundary conditions. *J. Math. Fluid Mech.*, 7(3):315–338, 2005.

[13] D. Hoff. Uniqueness of weak solutions of the Navier-Stokes equations of multidimensional, compressible flow. *SIAM J. Math. Anal.*, 37(6):1742–1760 (electronic), 2006.

[14] J. Leray. Sur le mouvement d’un liquide visqueux emplissant l’espace. *Acta Math.*, 63:193–248, 1934.

[15] P.-L. Lions. *Mathematical topics in fluid dynamics, Vol.2, Compressible models*. Oxford Science Publication, Oxford, 1998.

[16] N. Masmoudi. Incompressible inviscid limit of the compressible Navier–Stokes system. *Ann. Inst. H. Poincaré, Anal. non linéraire*, 18:199–224, 2001.

[17] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.*, 20:67–104, 1980.

[18] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of compressible and heat conductive fluids. *Comm. Math. Phys.*, 89:445–464, 1983.

[19] A. Novotný and I. Straškraba. *Introduction to the mathematical theory of compressible flow*. Oxford University Press, Oxford, 2004.

[20] G. Prodi. Un teorema di unicità per le equazioni di Navier-Stokes. *Ann. Mat. Pura Appl.*, 48:173–182, 1959.

[21] L. Saint-Raymond. Hydrodynamic limits: some improvements of the relative entropy method. *Annal. I.H.Poincaré - AN*, 26:705–744, 2009.

[22] J. Serrin. The initial value problem for the Navier-stokes equations. *University of Wisconsin Press*, 9:69, 1963.

[23] Y. Sun, C. Wang, and Z. Zhang. A Beale-Kato-Majda blow-up criterion for the 3-D compressible Navies-Stokes equations. *Arch. Rational Mech. Anal.*, 2011. To appear.

[24] S. Ukai. The incompressible limit and the initial layer of the compressible Euler equation. *J. Math. Kyoto Univ.*, 26(2):323–331, 1986.

[25] S. Wang and S. Jiang. The convergence of the Navier-Stokes-Poisson system to the incompressible Euler equations. *Comm. Partial Differential Equations*, 31(4-6):571–591, 2006.