RAINFOREST SIMPLICES IN TRIANGULATIONS OF MANIFOLDS

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ABSTRACT. Given a coloration of the vertices of a triangulation of a manifold, we give homological conditions on the chromatic complexes under which it is possible to obtain a rainbow simplex.

1. INTRODUCTION AND PRELIMINARIES

Consider a simplicial complex $K$ which is a triangulation of an $n$-dimensional manifold and whose vertices are partitioned into $n + 1$ subsets $V_0, \ldots, V_n$. Following the spirit of the Sperner lemma, the purpose of this paper is to obtain conditions that allow us to ensure the existence of a rainbow simplex, that is, an $n$-simplex of $K$ with exactly one vertex in each $V_i$. In particular, we will be interested in giving homological conditions on the chromatic complexes $K_{\{i\}}$, where we denote by $K_{\{i\}}$ the subcomplex of $K$ generated by the vertices of $V_i$.

During this paper, we use reduced homology with coefficients in an arbitrary field and, if no confusion arises, we shall not distinguish between a simplicial complex and its topological realization. For instance, if $L$ is a subcomplex of the simplicial complex $K$, in this paper we shall denote by $K \setminus L$ the space $|K| \setminus |L|$.

Meshulam’s lemma [6, Proposition 1.6] and [5, Theorem 1.5] is a Sperner-lemma type result, dealing with colored simplicial complexes and rainbow simplices, and in which the classical boundary condition of the Sperner lemma is replaced by an acyclicity condition. It is an important result from topological combinatorics with several applications in combinatorics, such as the generalization of Edmonds’ intersection theorem by Aharoni and Berger [1] and many other results in which obtaining a system of distinct representatives is relevant, like for example, the Hall’s theorem for hypergraphs [2]. Following this spirit, Meunier and Montejano [3] generalized Meshulam’s lemma obtaining the following result which is the main tool in this paper to obtain rainbow simplices.

Consider a simplicial complex $K$ whose vertex set is partitioned into $V_0, \ldots, V_n$. For $S \subseteq \{0, 1, \ldots, n\}$, we denote by $K_S$ the subcomplex of $K$ induced by the vertices in $\bigcup_{i \in S} V_i$. Suppose that for every nonempty $S \subseteq \{0, 1, \ldots, n\}$,

\[ \tilde{H}_{|S|-2}(K_S) = 0. \]

Then there exists a rainbow simplex $\sigma$ in $K$. See [4, Theorem 4] and [3] for a proof.

We summarize below what we need about PL topology in this paper. See, for example, the book of Rourke and Sanderson [7].
Let \( \{U_1, U_2\} \) be a partition of the vertices \( V(K) \) of the simplicial complex \( K \) and let \( < U_1 > \) and \( < U_2 > \) be the subcomplexes of \( K \) induced by \( U_1 \) and \( U_2 \), respectively. Let \( N(< U_i >, K') \) be the derived neighborhoods of \( U_i \) in \( K \) as subcomplex of the first barycentric subdivision \( K', i = 1, 2 \). Hence:

- \( N(< U_i >, K') \) is a strong deformation retraction of \( < U_i >, i = 1, 2 \) and
- \( K \setminus < U_2 > \) is a strong deformation retraction of \( < U_1 > \).

2. **Rainbow simplices in triangulations of 2 and 3-manifolds**

Our first result deals with 3-colorations in triangulation of surfaces

**Theorem 1.** Consider a simplicial complex \( K \) which is a triangulation of a 2-dimensional manifold and whose vertices are partitioned into 3 nonempty subsets \( V_0, V_1, V_2 \). If for every \( i = 0, 1, 2 \),

\[
\overline{H}_1(K_{\{i\}}, K_{\{i\}} \cap \partial K) = 0,
\]

then \( K \) admits a rainbow triangle.

**Proof.** The condition \( \overline{H}_1(K_{\{i\}}, K_{\{i\}} \cap \partial K) = 0 \) implies that every component \( L \) of \( K_{\{i\}} \) is contractible and the intersection of \( L \) with the boundary of \( K \) is either empty or contractible. Of course, we may assume that \( K \) is connected. Let us assume first that \( K \) is the triangulation of a simply connected surface without boundary. In order to find a rainbow triangle in \( K \), by (1), it will be enough to prove that \( K_S \) is connected, for every subset \( S \subset \{0, 1, 2\} \) of size two. Assume \( S = \{1, 2\} \) and let \( p \) and \( q \) be two points in \( K_{\{1,2\}} \). By hypothesis, \( N(K_{\{0\}}, K') \) is a countable collection \( \{D_i\} \) of pairwise disjoint topological disk embedded in the surface \( K \). Let \( f_i: \mathbb{B}^2 \to D_i \) be homeomorphisms and let \( \{x_i = f_i(0)\} \) be the collection of centers of all these disks. Since \( K \) is connected, there is an arc \( \Gamma \) joining \( p \) and \( q \) in \( K \). Furthermore, by transversality, we may assume without loss of generality, that this arc \( \Gamma \) does not intersect the collection of centers \( \{x_i\} \). Moreover, use the radial structure of the disks \( \{D_i\} \), giving by the homeomorphisms \( \{f_i\} \), to push the arc \( \Gamma \) outside \( K_{\{0\}} \). Since \( K_{\{1,2\}} \) is a strong deformation retract of \( K \setminus K_{\{0\}} \), we can deform de arc \( \Gamma \) to an arc from \( p \) to \( q \) inside \( K_{\{1,2\}} \), those proving the connectivity of \( K_{\{1,2\}} \) as we wished.

Suppose now that \( K \) is the triangulation of a 2-dimensional non simply connected manifold. Taking the universal cover of this surface we obtain a simply connected simplicial complex \( \widetilde{K} \) that inherits from \( K \) a 3-coloration \( \widetilde{V}_0, \widetilde{V}_1, \widetilde{V}_2 \) on its vertices. That is, there is a simplicial map \( \pi: \widetilde{K} \to K \) which is a universal cover, where \( \widetilde{V}_i = \pi^{-1}(V_i), i = 0, 1, 2 \). Note that if \( L \) is contractible, hence \( \pi^{-1}(L) \) is a countable union of pairwise disjoint contractible subcomplexes of \( \widetilde{K} \). Therefore, the fact that every component \( L \) of \( K_{\{i\}} \) is contractible implies that every component of \( \widetilde{K}_{\{i\}} \) is contractible. Consequently, by the first part of the proof, there is a rainbow simplex in \( \widetilde{K} \) and since \( \pi \) sends colorful simplices of \( \widetilde{K} \) into colorful simplices of \( K \), we obtain our desired rainbow triangle.

The proof of the theorem for triangulations of surfaces with boundary is completely similar, except that if for a component \( L \) of \( K_{\{i\}} \) such that \( L \) and \( L \cap \partial K \) are nonempty contractible spaces, then we use a homeomorphism

\[
f_i: (\mathbb{B}^2 \cap \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}, [-1, 1] \times \{0\}) \to (N(L, K'), N(L \cap \partial K, \partial K'))
\]

in such a way that the center \( x_i = f_i(0) \) lies in \( \partial K \) the boundary of \( K \). □
For triangulations of 3-dimensional manifolds, we have the following theorem.

**Theorem 2.** Consider a simplicial complex $K$ which is a triangulation of a 3-dimensional manifold whose vertices are partitioned into 4 subsets $V_0, V_1, V_2, V_3$. Suppose that

1. $\tilde{H}_2(K) = 0$,
2. for $i = 0, \ldots, 3$, $K_{(i)}$ is contractible and the intersection of $K_{(i)}$ with the boundary of $K$ is either empty or contractible,
3. for every pair of integers $0 \leq i < j \leq 3$, there is a 1-dimensional simplex with one vertex in $V_i$ and the other in $V_j$.

then $K$ admits a rainbow tetrahedron.

**Proof.** As in the proof of Theorem 1, we may assume without loss of generality that $K$ is a triangulation of a connected, simply connected 3-dimensional manifold without boundary. Since $\tilde{H}_2(K) = 0$, in order to get a rainbow simplex of $K$ it is enough to prove that

- $\tilde{H}_1(K_S) = 0$, for every $S \subset \{0, 1, 2, 3\}$ of size 3, and
- $\tilde{H}_0(K_S) = 0$, for every $S \subset \{0, 1, 2, 3\}$ of size 2.

Indeed, we shall prove that for every $S \subset \{0, 1, 2, 3\}$ of size 3, $K_S$ is simply connected and for every $S \subset \{0, 1, 2, 3\}$ of size 2, $K_S$ is connected. Assume that $S = \{1, 2, 3\}$. Let $\alpha : S^1 \to K_{\{1,2,3\}}$ be a continuous map. Since $K$ is simply connected, there is a map $\beta : \mathbb{B}^2 \to K$ extending $\alpha$. Since $K_{\{0\}}$ is contractible, hence its derived neighborhood $N(K_{\{0\}}, K')$ is a 3-dimensional ball. Assume that for a parametrization $f : \mathbb{B}^3 \to N(K_{\{0\}}, K')$, the point $x = f(0)$ is its center. As in the proof of Theorem 1, by transversality, we may assume without loss of generality, that the point $x$ does not lie in $\beta(\mathbb{B}^2)$. Moreover, use the radial structure of the 3-ball $N(K_{\{0\}}, K')$, given by the homeomorphism $f$, to push $\beta(\mathbb{B}^2)$ outside $K_{\{0\}}$. Since $K_{\{1,2,3\}}$ is a strong deformation retract of $K \setminus K_{\{0\}}$, the map $\beta : \mathbb{B}^2 \to K$ is homotopic to a map whose image lies inside $K_{\{1,2,3\}}$ and of course extends the map $\alpha$. This proves that $K_{\{1,2,3\}}$ is simply connected. Finally, given two different integers $0 \leq i < j \leq m$, the connectivity of $K_{\{i,j\}}$ follows from the connectivity of $K_{\{i\}}$ and $K_{\{j\}}$ plus the existence of a 1-dimensional simplex with one vertex in $V_i$ and the other in $V_j$. This completes the proof of this theorem.

\[\square\]

3. **Rainbow simplices in triangulations of $n$-dimensional manifolds**

**Theorem 3.** Consider a simplicial complex $K$ which is a triangulation of a 4-dimensional closed manifold and whose vertices are partitioned into 5 subsets $V_0, \ldots, V_4$.

1. $\tilde{H}_2(K) = \tilde{H}_3(K) = 0$,
2. for every pair of integers $0 \leq i < j \leq 4$ there is a 1-dimensional simplex with one vertex in $V_i$ and the other in $V_j$,
3. for every $i = 0, \ldots, 4$, the subcomplex $K_{\{i\}}$ is contractible and has as a regular neighborhood a 4-ball,
4. for every $S \subset \{1, \ldots, 5\}$ of size 2, $K_S$ have as a regular neighborhood a handle body then $K$ admits a rainbow 4-simplex.

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Theorem 5. Consider a simplicial complex $K$ which is a triangulation of a $n$-dimensional closed manifold and whose vertices are partitioned into $n + 1$ subsets $V_0, \ldots, V_n$. If

1. $\widetilde{H}_2(K) = \ldots = \widetilde{H}_{n-1}(K) = 0$,
2. for every $S \subset \{0, \ldots, n\}$ of size $i + 1$, there is an $i$-dimensional complex $L^i$ such that $L^i$ is a strong deformation retract of $N(K_S, K')$, $0 \leq i \leq n - 2$,

then $K$ admits a rainbow $n$-simplex.

Finally, we shall use Alexander duality to prove the following theorem for colored triangulations of spheres

Theorem 4. Consider a simplicial complex $K$ which is a triangulation of a $n$-dimensional closed manifold and whose vertices are partitioned into $n + 1$ subsets $V_0, \ldots, V_n$. If

1. $\widetilde{H}_2(K) = \ldots = \widetilde{H}_{n-1}(K) = 0$,
2. for every $S \subset \{0, \ldots, n\}$ of size $i + 1$, there is an $i$-dimensional complex $L^i$ such that $L^i$ is a strong deformation retract of $N(K_S, K')$, $0 \leq i \leq n - 2$,

then $K$ admits a rainbow $n$-simplex.
Proof. Let $S \subset [m] = \{0, 1, \ldots, m\}$. Then $K - K_S$ has the homotopy type of $K_{[m] \setminus S}$. By Alexander duality, $\tilde{H}_{|S| - 2}(K_S) = \tilde{H}_{m+1-|S|}(K_{[m] \setminus S}) = 0$. Consequently, by (1), $K$ admits a rainbow $n$-simplex.

\[ \square \]

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