Infrared renormalon in the supersymmetric $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$

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2/8/2019

In the leading order of the large $N$ approximation, we study the renormalon ambiguity in the gluon (or more appropriately, photon) condensate in the two-dimensional supersymmetric $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$ with the $\mathbb{Z}_N$ twisted boundary conditions. In our large $N$ limit, the combination $\Lambda R$, where $\Lambda$ is the dynamical scale and $R$ is the $S^1$ radius, is kept fixed ($\Lambda R \ll 1$). We extract the perturbative part from the large $N$ expression of the gluon condensate and obtain the corresponding Borel transform $B(u)$. For $\mathbb{R} \times S^1$, we find that the Borel singularity at $u = 2$, which exists in the system on the un-compactified $\mathbb{R}^2$ and corresponds to twice the minimal bion action, disappears. Instead, an unfamiliar renormalon singularity emerges at $u = 3/2$ for the compactified space $\mathbb{R} \times S^1$. The semi-classical interpretation of this peculiar singularity is not clear because $u = 3/2$ is not dividable by the minimal bion action. It appears that our observation for the system on $\mathbb{R} \times S^1$ prompts reconsideration on the semi-classical bion picture of the infrared renormalon.

Subject Index: B06, B16, B32, B34, B35
1. Introduction

In Refs. [1–4], it was claimed that the ambiguity in perturbation theory caused by the infrared (IR) renormalon [5, 6]—a single Feynman diagram whose amplitude grows factorially as the function of the order—is canceled by the ambiguity associated with a semi-classical object called the (neutral) bion—a pair of the fractional instanton [7–12] and the fractional anti-instanton. This cancellation mechanism between perturbation theory and a semi-classical object is analogous to the cancellation between the ambiguity caused by the proliferation of the number of Feynman diagrams [13, 14] (see also Ref. [15] for a review) and the ambiguity associated with the instanton–anti-instanton pair [16, 17]. Such a cancellation of the ambiguity will be crucial for the enterprise of a fully semi-classical understanding of the low-energy physics of gauge theory; see Ref. [18] and references therein. A crucial element in such a semi-classical argument is an $S^1$ compactification; the $S^1$ radius $R$ provides a mass scale and if it is sufficiently small compared to the dynamical scale $\Lambda$, $\Lambda R \ll 1$, it would allow a semi-classical (weak coupling) treatment. On the other hand, a non-trivial holonomy along $S^1$, or equivalently, twisted boundary conditions along $S^1$ allows the fractional (anti-)instanton.

For the four-dimensional (4D) $SU(N)$ gauge theory, the above cancellation between the IR renormalon and the bion is still conjecture. In particular, the ambiguity associated with the bion does not exactly correspond to that of the renormalon on $\mathbb{R}^4$, which is typically characterized by $e^{-2S_I/(N\beta_0)}$. Here, $S_I$ is the instanton action and $\beta_0 = 11/3 - 2n_W/3$ is the coefficient of the one-loop beta function for the ’t Hooft coupling; $n_W$ is the number of Weyl fermions in the adjoint representation. It remains as a non-trivial task to investigate if the ambiguity due to the bion coincides with the above form by possible renormalization and de-compactification effects. For the two-dimensional (2D) $CP^{N-1}$ model, which shares many similarities with 4D gauge theory, the beta function is simple $\beta_0 = 1$ and one might further push the above semi-classical interpretation of the renormalon.

In a recent interesting paper [19], the authors carried out a systematic investigation on this issue in the 2D supersymmetric $CP^{N-1}$ model on $\mathbb{R} \times S^1$ with the $\mathbb{Z}_N$ twisted boundary conditions. They computed a one-loop order effective action for quasi-collective coordinates associated with the bion configuration. Then, by employing the Lefschetz thimble method [20, 21], they computed the bion contribution to the vacuum energy as the function of a supersymmetry breaking parameter $\delta \epsilon$. They found that the bion induces the ambiguity in the $O(\delta \epsilon^2)$ term of the vacuum energy with the strength $\sim e^{-2S_I/N\beta_0}$; they then inferred that this ambiguity is canceled by the ambiguity caused by the IR renormalon.

Although the above computation is very explicit, the interpretation of the result may be disputable. It is generally not well understood whether the ambiguity due to the bion truly corresponds to the renormalon ambiguity. In particular, there is a study [22] that sounds incompatible with the above result; this study asserts that in gauge theory on $\mathbb{R}^3 \times S^1$, the $S^1$ compactification works as an IR cutoff and, as a consequence, the IR renormalon disappears in an $S^1$ compactified space. If we assume this mechanism to be general, what cancels the ambiguity caused by the bion? On the other hand, there is also a study where the ambiguity due to the bion cancels the perturbative ambiguity. It was found [23, 24] that the bion and perturbative ambiguities in the ground state energy indeed have the same magnitude with opposite signs in an (approximately) supersymmetric $CP^1$ quantum mechanics; this system
is obtained by a reduction of the 2D $\mathcal{N} = (2, 0)$ supersymmetric $CP^1$ model to lower-lying Kaluza–Klein (KK) momentum modes.

To have some insight in this confused situation and investigate the relation between the bion and the renormalon more directly, it would be useful to consider the large $N$ limit (see Ref. [25] for a classical exposition). The large $N$ limit distinguishes the instanton and the renormalon because their effects differ by the factor $N$. Typically, as the matrix model illustrates [26], the perturbative series tends to become a convergent series in the large $N$ limit, possibly leaving the effect of renormalons. Motivated by this reasoning, in this paper, we consider the large $N$ approximation of the 2D supersymmetric $CP^{N-1}$ model on $\mathbb{R} \times S^1$ with the $\mathbb{Z}_N$ twisted boundary conditions and investigate the IR renormalon via systematic calculations. For the system in the un-compactified space $\mathbb{R}^2$, the large $N$ solution is well-known [27]. We generalize this solution to the compactified space, $\mathbb{R} \times S^1$.

In our large $N$ limit, we assume

$$\Lambda R = \text{const. as } N \to \infty, \quad (1.1)$$

where $\Lambda$ is a dynamical scale (i.e., the $\Lambda$ parameter) and $R$ is the $S^1$ radius. We also assume that $\Lambda R$ is sufficiently small ($\Lambda R \ll 1$) so that the perturbative expansion with respect to the coupling constant at the mass scale $1/R$ is meaningful.

In this paper, as a simple and non-trivial quantity, we study the gluon (or more appropriately, photon) condensate in the above system in the leading order of the large $N$ approximation. We use the following definitions in studying a factorially divergent series. For the perturbative series of a quantity $f(\lambda)$,

$$f(\lambda) \sim \sum_{k=0}^{\infty} f_k \left( \frac{\lambda}{4\pi} \right)^{k+1}, \quad (1.3)$$

where $\lambda$ denotes the 't Hooft coupling constant, we define the Borel transform as

$$B(u) \equiv \sum_{k=0}^{\infty} \frac{f_k}{k!} u^k. \quad (1.4)$$

Then the Borel sum is given by

$$f(\lambda) \equiv \int_0^{\infty} du \, B(u) \, e^{-4\pi u/\lambda}. \quad (1.5)$$

If the perturbative coefficient $f_k$ in Eq. [1.3] grows factorially $f_k \sim b^{-k} k!$ as $k \to \infty$, the Borel transform $B(u)$ [1.4] develops a singularity located at $u = b$. If this singularity is on the positive real $u$ axis, $b > 0$, then the Borel integral [1.5] becomes ill-defined and produces the ambiguity proportional to $\sim e^{-4\pi b/\lambda}$. In this convention, the IR renormalon in the present system in the large $N$ limit is generally expected to produce Borel singularities at $u = 1, 2, \ldots$, at positive integers. Since the minimum action of the bion is $4\pi/\lambda$ (when the constituent

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1 We assume this for technical reasons. An alternative large $N$ limit, in which

$$\Lambda R N = \text{const. as } N \to \infty, \quad (1.2)$$

might be natural for a semi-classical consideration, because the potential height between $N$ degenerate classical vacua of the present system is characterized by the scale $1/(RN)$. We could not find, however, a convincing way to analyze resulting expressions in this large $N$ limit.
fractional instanton and the anti-instanton are infinitely separated), the ambiguity caused by the bion for the vacuum energy corresponds to \( u = 1 \). On the other hand, as we will see below, the gluon condensate in the system in \( \mathbb{R}^2 \) suffers from the IR renormalon at \( u = 2 \); the associated factor in the exponential, \( 8\pi/\lambda \) is twice the minimum bion action.

In this paper, we show that for the gluon condensate in the compactified space \( \mathbb{R} \times S^1 \), the renormalon singularity at \( u = 2 \), which exists for \( \mathbb{R}^2 \), disappears. This sounds consistent with the claim in Ref. \[22\] that there is no IR renormalon in an \( S^1 \) compactified space. However, for \( \mathbb{R} \times S^1 \), we find that an unfamiliar renormalon singularity emerges at \( u = 3/2 \). Our observation thus indicates that an \( S^1 \) compactification significantly affects the renormalon structure. Furthermore, we do not know any semi-classical interpretation of this singularity. It appears that our finding prompts reconsideration on the above semi-classical picture of the IR renormalon.

This paper is organized as follows: In Sect. 2 we define our system starting from the expressions in Ref. \[19\] but using the 't Hooft coupling \( \lambda \equiv g^2 N \) (the 't Hooft coupling is extensively used in this paper). For the large \( N \) approximation, it is highly convenient to employ the homogeneous coordinate of \( \mathbb{C}P^{N-1} \) and its “superpartner”; these are introduced in Sect. 2.2. Then, as a standard procedure in the large \( N \) approximation, we introduce auxiliary fields to impose constraints among original \( N \) fields and to make the action quadratic in the \( N \) fields. In Sect. 3 we obtain the effective action for the auxiliary fields by integrating over the \( N \) fields. We find the saddle point in the large \( N \) limit and then compute the effective action for fluctuations around the saddle point to the quadratic order. In Sect. 4 we compute the gluon condensate in the leading order of the large \( N \) approximation. We then extract the perturbative part from the large \( N \) expression and obtain the corresponding Borel transform. By studying singularities of the Borel transform, we arrive at the above conclusion. We also argue that a perturbative calculation of a physical observable inherits the renormalon ambiguity found in the gluon condensate. Section 5 is devoted to conclusion and discussion. Appendix A contains some useful formulas to translate expressions in the inhomogeneous coordinate to those in the homogeneous coordinate. Appendix B gives some rigorous bounds on the functions appearing in the effective action.

2. \( \mathcal{N} = (2, 2) \) supersymmetric \( \mathbb{C}P^{N-1} \) model on \( \mathbb{R} \times S^1 \)

2.1. Definition of the system

We suppose that the spacetime is \( \mathbb{R} \times S^1 \) and denote the coordinate of \( \mathbb{R} \) by \( x \) and that of \( S^1 \) by \( y \); the radius of \( S^1 \) is \( R \) and thus \( 0 \leq y < 2\pi R \). The Greek indices \( \mu, \nu \), etc. run
over $x$ and $y$. We start with Eq. (4.1) of Ref. [19] in a somewhat different notation:\footnote{Unless stated otherwise, the summation over repeated indices is always understood. This $N = (2, 2)$ action can be obtained by the dimensional reduction of the four-dimensional $N = 1$ Wess–Zumino model [28], by setting $\Phi^a = \varphi^a + \sqrt{2} \psi \bar{\nu} + \theta F$, where $\psi^i = (\psi^a_+ + \psi^a_-)/\sqrt{2}$, $\psi^i_+ = (\psi^a_+ + \psi^a_-)/\sqrt{2}$, and $\psi^i_2 = (-\psi^a_+ + \psi^a_-)/\sqrt{2}$. The Kähler potential is taken as $K = \frac{N}{\lambda} \ln (1 + \sum_{a=1}^{N} \Phi^a \Phi^a)$. Note that we are working in the Euclidean space; the time coordinate is $x^0 = -i x$; the Boltzmann weight in the functional integral is thus $e^{-S}$.}

$$S \equiv \int d^2 x \frac{2N}{\lambda} \left\{ \bar{G}_{ab} \left[ \partial \varphi^a \bar{\partial} \varphi^b + \bar{\partial} \varphi^a \partial \bar{\varphi}^b ight] 
+ \frac{1}{2} \bar{R}_{abcd} \bar{\psi}^a_+ \bar{\psi}^b_+ \bar{\psi}^c_+ \bar{\psi}^d_+ \right\} + S_{\text{top}}. \tag{2.1}$$

In this paper, we use the 't Hooft coupling $\lambda$ that is defined by $\lambda \equiv g^2 N$ from the coupling constant $g$ in Ref. [19]. The lowercase indices $a, b$ etc. run over $1, \ldots, N - 1$, and

$$G_{ab} \equiv \frac{\partial^2}{\partial \varphi^a \partial \bar{\varphi}^b} \ln \left( 1 + \sum_{c=1}^{N-1} |\varphi^c|^2 \right) = \frac{\delta^{ab}}{1 + \sum_c |\varphi^c|^2} - \frac{\bar{\varphi}^a \varphi^b}{(1 + \sum_c |\varphi^c|^2)^2}, \tag{2.2}$$

is the Fubini–Study metric on $\mathbb{C}P^{N-1}$. The connection and the curvature on $\mathbb{C}P^{N-1}$ are given by

$$\Gamma^a_{bc} \equiv G^{ba} \partial_b G_{ac}, \quad \Gamma^a_{bc} \equiv G^{ba} \partial_b G_{ac}, \quad R^a_{bcde} \equiv \partial_c \Gamma^a_{de}. \tag{2.3}$$

In Eq. (2.1), the spacetime derivatives are denoted as

$$\partial \equiv \frac{1}{2} (\partial_x - i \partial_y), \quad \bar{\partial} \equiv \frac{1}{2} (\partial_x + i \partial_y), \tag{2.4}$$

and the topological term $S_{\text{top}}$ is defined by

$$S_{\text{top}} \equiv \int d^2 x \frac{i \theta}{\pi} G_{ab} \left( \partial \varphi^a \bar{\partial} \varphi^b - \bar{\partial} \varphi^a \partial \bar{\varphi}^b \right). \tag{2.5}$$

Now, along $S^1$, we impose the following $\mathbb{Z}_N$ invariant twisted boundary conditions:

$$\varphi^a(x, y + 2\pi R) = e^{2\pi i m_a R} \varphi^a(x, y), \quad \psi^a_\pm(x, y + 2\pi R) = e^{2\pi i m_a R} \psi^a_\pm(x, y),$$

$$\tilde{\psi}^a_\pm(x, y + 2\pi R) = e^{-2\pi i m_a R} \tilde{\psi}^a_\pm(x, y), \quad \tilde{\psi}^a_\mp(x, y + 2\pi R) = e^{2\pi i m_a R} \tilde{\psi}^a_\mp(x, y), \tag{2.6}$$

where the twist angles are proportional to the index $a$:

$$m_a = \frac{a}{N R}, \quad a = 1, \ldots, N - 1. \tag{2.7}$$

These boundary conditions allow the so-called fractional instanton with a particular index $b$ ($b = 1, 2, \ldots, N - 1$):

$$\varphi^{a \neq b} = 0, \quad \varphi^b = C e^{m_b x + iy}, \quad C \in \mathbb{C}, \tag{2.8}$$
which has the classical action
\[ S = 2\pi i \left( \frac{\theta}{2\pi} - i \frac{N}{\lambda} \right) m_b R = \left( \frac{2\pi}{\lambda} + i \frac{\theta}{N} \right) b. \]  
(2.9)

Thus the action of a pair of the \( b \)th fractional instanton and the \( b \)th fractional anti-instanton—called the \( b \)th bion—approaches to
\[ S \sim 4\pi \frac{N}{\lambda} m_b R = \frac{4\pi}{\lambda} b, \]  
(2.10)
as the separation between the fractional instanton and the fractional anti-instanton goes to infinity. Since the \( b \)th bion possesses the action \( S \sim 4\pi b/\lambda \), this would produce a singularity for the Borel transform \( B(u) \) in Eq. (1.5) (of a quantity in the topologically trivial sector) at the value of the classical action \( [13–15] \), i.e., \( u = b \) \( (b = 1, 2, \ldots, N - 1) \).

2.2. Homogeneous coordinates

In the large \( N \) approximation, it is highly convenient to express the above system in terms of the homogeneous coordinate of \( \mathbb{C}P^{N-1} \). That is, introducing new variable \( z^N \in \mathbb{C} \), we set
\[ \varphi^a \equiv \frac{z^a}{z^N}, \quad \bar{z}^A z^A = 1. \]  
(2.11)

Here and in what follows, uppercase indices \( A, B, \) etc. run over \( 1, \ldots, N \). We call fields with the indices \( A, B, \ldots \) by \( N \) fields. Note that this description of the system in terms of \( z^A \) is redundant; i.e., the original variables \( \varphi^a \) are invariant under the \( U(1) \) gauge transformation,
\[ z^A \to g z^A, \quad g \in U(1). \]  
(2.12)

For the fermionic fields, we find that the following variables work quite well (see also Ref. [29]). We introduce new variables \( \chi^N_\pm \) and set
\[ \psi^a_\pm \equiv \frac{1}{z^N} \chi^a_\pm - \frac{z^a}{(z^N)^2} \chi^N_\pm. \]  
(2.13)

Original variables \( \psi^a_\pm \) are invariant under the \( U(1) \) gauge transformation, defined by the combination of Eq. (2.12) and
\[ \chi^A_\pm \to g \chi^A_\pm, \quad g \in U(1). \]  
(2.14)

We then impose the constraint,
\[ \bar{z}^A \chi^A_\pm = 0. \]  
(2.15)

This constraint has the solution, because we can solve this with respect to \( \chi^N_\pm \):
\[ \chi^N_\pm = -(z^N)^2 z^a \psi^a_\pm, \]  
(2.16)
under the condition \( \bar{z}^A z^A = 1 \).

We may assume the following boundary conditions for the homogeneous coordinate variables,
\[ z^A(x, y + 2\pi R) = e^{2\pi i m_A R} z^A(x, y), \]
\[ \chi^A_\pm(x, y + 2\pi R) = e^{2\pi i m_A R} \chi^A_\pm(x, y), \quad \bar{\chi}^A_\pm(x, y + 2\pi R) = e^{-2\pi i m_A R} \bar{\chi}^A_\pm(x, y), \]  
(2.17)

where we have defined
\[ m_N \equiv 0, \]  
(2.18)
as these conditions are consistent with Eq. (2.6).
Then, using various relations given in Appendix A, we find a rather simple expression:

\[
S = \int d^2 x \frac{2N}{\lambda} \left[ \partial z^A \bar{\partial} z^A + \bar{\partial} z^A \partial z^A - 2 j_z j_{\bar{z}} 
+ \chi^A_c (\partial - i j_z) \chi^A_c + \bar{\chi}^A_\tilde{c} (\bar{\partial} - i j_{\bar{z}}) \bar{\chi}^A_\tilde{c} 
- \frac{1}{2} \chi^A_c \bar{\chi}^B_c \chi^B_\tilde{c} - \frac{1}{2} \bar{\chi}^A_\tilde{c} \bar{\chi}^B_\tilde{c} \chi^B_c \right] 
+ S_{\text{top}},
\]

(2.19)

where

\[
j_z = \frac{1}{2i} (z^A \partial z^A - z^A \bar{\partial} z^A), \quad j_{\bar{z}} = \frac{1}{2i} (\bar{z}^A \bar{\partial} z^A - \bar{z}^A \partial \bar{z}^A),
\]

(2.20)

and

\[
S_{\text{top}} = \int d^2 x \frac{\theta}{\pi} (\partial j_{\bar{z}} - \bar{\partial} j_z).
\]

(2.21)

This is basically the action given in Eq. (15) of Ref. [27]. Note that under the \( U(1) \) gauge transformation (2.12), the current (2.20) transforms inhomogeneously,

\[
j_z \rightarrow j_z + \frac{1}{i} g^{-1} \partial g, \quad j_{\bar{z}} \rightarrow j_{\bar{z}} + \frac{1}{i} g^{-1} \bar{\partial} g.
\]

(2.22)

2.3. Auxiliary fields

We now introduce various auxiliary fields. One of their roles is to impose the constraint of Eq. (2.11) and the fermionic constraint (2.15); corresponding Lagrange multiplier fields are \( f \) and \((\eta_\pm, \bar{\eta}_\pm)\), respectively. We also introduce auxiliary fields \( A_{\pm, \tilde{c}} \) and \((\sigma, \bar{\sigma})\) to make the action quadratic in the homogeneous coordinate variables. We thus set

\[
S' \equiv S + \int d^2 x \frac{2N}{\lambda} \left[ \frac{1}{2} f(\bar{z}^A z^A - 1) + \bar{\eta}_- z^A \chi^A_+ + \bar{\eta}_+ \bar{z}^A \chi^A_- + \bar{\chi}^A_+ z^A \eta_- + \bar{\chi}^A_- \bar{z}^A \eta_+ 
+ 2 \left( A_{\bar{z}} + j_{\bar{z}} + \frac{1}{2} i \chi^A_- \chi^A_\tilde{c} \right) \left( A_z + j_z + \frac{1}{2} i \bar{\chi}^B_\tilde{c} \chi^B_+ \right) 
+ \frac{1}{2} \left( \sigma + \bar{\chi}^A_\tilde{c} \chi^A_+ \right) \left( \sigma + \bar{\chi}^B_\tilde{c} \chi^B_+ \right) \right] 
- \frac{\theta}{\pi} \left[ \partial \left( A_{\bar{z}} + j_{\bar{z}} + \frac{1}{2} i \chi^A_- \chi^A_\tilde{c} \right) - \bar{\partial} \left( A_z + j_z + \frac{1}{2} i \bar{\chi}^B_\tilde{c} \chi^B_+ \right) \right].
\]

(2.23)

We impose the periodic boundary conditions for all the auxiliary fields. The action \( S' \) can be cast into the form,

\[
S' = \int d^2 x \frac{N}{\lambda} \left\{ - f - \bar{\sigma} \sigma + z^A \left[ - D_\mu D_\mu + f - 4 \bar{\eta}_+ (\bar{\sigma} P_+ + \sigma P_-) \right] \bar{z}^A 
+ (\bar{\chi}^A_\tilde{c} \chi^A_+) (\bar{\sigma} P_+ + \sigma P_-) \left( \bar{\chi}^A_\tilde{c} \chi^A_+ \right) \right\} 
- \int d^2 x \frac{i \theta}{2\pi} \epsilon_{\mu \nu} \partial_\mu A_\nu,
\]

(2.24)

where, setting \( A_x = A_z + A_{\bar{z}} \) and \( A_y = i (A_z - A_{\bar{z}}) \),

\[
D_\mu z^A \equiv (\partial_\mu + i A_\mu) z^A, \quad \bar{\sigma} \left( \frac{\chi^A_+}{\chi^A_-} \right) \equiv 2 \gamma_\mu (\partial_\mu + i A_\mu) \left( \frac{\chi^A_+}{\chi^A_-} \right),
\]

(2.25)
\( \gamma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad P_\pm \equiv \frac{1 \pm \gamma_5}{2}, \quad \gamma_5 \equiv -i\gamma_x\gamma_y, \) (2.26)

and

\[
\bar{\eta} \equiv (\bar{\eta}_- \bar{\eta}_+), \quad \eta \equiv \begin{pmatrix} \eta_+ \\ \eta_- \end{pmatrix},
\]

(2.27)

and \( \epsilon_{xy} = -\epsilon_{yx} = +1. \) Also, we have defined

\[
(\tilde{\chi}_-^{A} \tilde{\chi}_+^{A}) \equiv (\chi_-^{A} \chi_+^{A}) + 2\bar{\eta}z^{A}(\mathcal{D} + \bar{\sigma}P_+ + \sigma P_-)^{-1},
\]

\[
\begin{pmatrix} \tilde{\chi}_-^{A} \\ \tilde{\chi}_+^{A} \end{pmatrix} \equiv \begin{pmatrix} \chi_-^{A} \\ \chi_+^{A} \end{pmatrix} + 2(\mathcal{D} + \bar{\sigma}P_+ + \sigma P_-)^{-1}z^{A}\eta.
\]

(2.28)

Since the current (2.20) transforms inhomogeneously under the \( U(1) \) gauge transformation as Eq. (2.22), in order for the term added in Eq. (2.23) to be gauge invariant, the auxiliary field \( A_\mu \) receives the \( U(1) \) gauge transformation of the form,

\[
A_\mu \rightarrow A_\mu - \frac{1}{i}g^{-1}\partial_\mu g.
\]

(2.29)

\( A_\mu \) is therefore regarded as a \( U(1) \) gauge potential and the last term of Eq. (2.24) can give rise to a non-trivial topological charge.

3. Leading order large \( N \) approximation

3.1. The saddle point

The large \( N \) approximation consists of the saddle point approximation of the functional integral of auxiliary fields, after the Gaussian integration over the original \( N \) fields [25]. From Eq. (2.24), the integration over \( N \) fields yields the effective action of the auxiliary fields,

\[
S_{\text{eff}} = \int d^2x \sum_A \frac{N}{A}(-f + \bar{\sigma}\sigma)
\]

\[
+ \sum_A \text{Tr} \ln \left[ -D_\mu D_\mu + f - 4\bar{\eta}(\mathcal{D} + \bar{\sigma}P_+ + \sigma P_-)^{-1}\eta \right]
\]

\[
- \sum_A \text{Tr} \ln (\mathcal{D} + \bar{\sigma}P_+ + \sigma P_-).
\]

(3.1)

In this expression, the twisted boundary conditions in Eq. (2.17) that depend on the index \( A \) have to be taken into account.

First, we look for the saddle point of the effective action, by assuming that it is given by

\[
A_{\mu 0} = \text{const.}, \quad f_0 = \text{const.}, \quad \sigma_0 = \text{const.}, \quad \eta_0 = \bar{\eta}_0 = 0.
\]

(3.2)

For such a constant configuration, we can see that

\[
\text{Tr} \ln (\mathcal{D} + \bar{\sigma}P_+ + \sigma P_-) = \text{Tr} \ln (-D_\mu D_\mu + \bar{\sigma}\sigma),
\]

(3.3)
by using the charge conjugation invariance. Then, for the configuration (3.2), going to the momentum space by taking the twisted boundary conditions (2.17) into account, we have

\[
S_{\text{eff}} = \int d^2x \frac{N}{\lambda} (-f_0 + \bar{\sigma}_0 \sigma_0) + \sum_A \int d^2x \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \ln \left[ (p_x + A x_0)^2 + (p_y + m_A + A y_0)^2 + f_0 \right] \\
- \sum_A \int d^2x \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \ln \left[ (p_x + A x_0)^2 + (p_y + m_A + A y_0)^2 + \bar{\sigma}_0 \sigma_0 \right],
\]

(3.4)

where the KK momentum \( p_y \) is discrete:

\[
p_y = \frac{n}{R}, \quad n \in \mathbb{Z}.
\]

(3.5)

Then, we use the identity,

\[
\sum_{n=\infty}^{\infty} e^{ip_y 2\pi R n} = \frac{1}{R} \sum_{n=\infty}^{\infty} \delta(p_y - n/R),
\]

(3.6)

or

\[
\frac{1}{2\pi R} \sum_{n=\infty}^{\infty} F(n/R) = \sum_{n=\infty}^{\infty} \int \frac{dp_y}{2\pi} e^{ip_y 2\pi R n} F(p_y),
\]

(3.7)

in Eq. (3.4) to make the sum \( \sum_{p_y} \) into an integral \( \int dp_y \). This enables us to shift the integration variables as \( p_x \to p_x - A x_0 \) and \( p_x \to p_x - m_A - A x_0 \) to yield

\[
S_{\text{eff}} = \int d^2x \frac{N}{\lambda} (-f_0 + \bar{\sigma}_0 \sigma_0) + \sum_A \int d^2x \int \frac{dp_x}{(2\pi)^2} e^{i(p_x - A x_0 - m_A)2\pi R n} \left[ \ln(p^2 + f_0) - \ln(p^2 + \bar{\sigma}_0 \sigma_0) \right].
\]

(3.8)

We then carry out the sum over \( A \) by noting, from Eqs. (2.17) and (2.18),

\[
\sum_A e^{-im_A 2\pi R n} = \sum_{j=0}^{N-1} \left( e^{-2\pi i j / N} \right)^j = \begin{cases} N & \text{for } n = 0 \text{ mod } N, \\ 0 & \text{for } n \neq 0 \text{ mod } N. \end{cases}
\]

(3.9)

We thus obtain

\[
S_{\text{eff}} = \int d^2x \frac{N}{\lambda} (-f_0 + \bar{\sigma}_0 \sigma_0) + \int d^2x N \int \frac{dp_x}{(2\pi)^2} e^{i(p_x - A x_0)2\pi R nm} \left[ \ln(p^2 + f_0) - \ln(p^2 + \bar{\sigma}_0 \sigma_0) \right].
\]

(3.10)

In this form, the \( m = 0 \) term is ultraviolet (UV) divergent whereas \( m \neq 0 \) terms are the Fourier transforms and UV finite. To the \( m = 0 \) term, we apply dimensional regularization where the dimension of spacetime is set to be \( 2 \to D = 2 - 2\varepsilon \). The result of the momentum
integrations is then
\[
S_{\text{eff}} = \int d^2 x \frac{N}{4\pi} \left[ \frac{4\pi}{\lambda} - \frac{1}{\varepsilon} + \ln \left( \frac{e^{\gamma_E}}{4\pi} \right) \right] (-f_0 + \bar{\sigma}_0 \sigma_0)
+ \int d^2 x \frac{N}{4\pi} \left\{ -f_0 (\ln f_0 - 1) + \bar{\sigma}_0 \sigma_0 [\ln(\bar{\sigma}_0 \sigma_0) - 1] \right\}
+ \int d^2 x \frac{N}{4\pi} (-4) \sum_{m \neq 0} e^{-iA_y \varphi} 2\pi R N m
\times \left[ \frac{\sqrt{f_0}}{2\pi R N |m|} K_1(\sqrt{f_0} 2\pi R N |m|) - \frac{\sqrt{\bar{\sigma}_0 \sigma_0}}{2\pi R N |m|} K_1(\sqrt{\bar{\sigma}_0 \sigma_0} 2\pi R N |m|) \right].
\] (3.11)

Here and in what follows, \( K_\nu(z) \) denotes the modified Bessel function of the second kind. To remove the UV divergence in Eq. (3.11), we introduce the renormalized 't Hooft coupling \( \lambda_R(\mu) \) in the “\( \overline{\text{MS}} \) scheme” by
\[
\lambda = \left( \frac{e^{\gamma_E} \mu^2}{4\pi} \right)^{\varepsilon} \lambda_R(\mu) \left[ 1 + \lambda_R(\mu) \frac{\mu}{4\pi} \right]^{-1}.
\] (3.12)

where \( \mu \) is a renormalization scale. From this result, we obtain the beta function,
\[
\frac{\partial}{\partial \mu} \lambda_R(\mu) \bigg|_\lambda = -2\varepsilon \lambda_R(\mu) - \frac{1}{2\pi} \lambda_R(\mu)^2,
\] (3.13)
and the renormalization-group invariant dynamical scale (the \( \Lambda \) parameter),
\[
\Lambda \equiv \mu e^{-2\pi/\lambda_R(\mu)}.
\] (3.14)

In terms of \( \Lambda \), the effective action (3.11) reads
\[
S_{\text{eff}} = \int d^2 x \frac{N}{4\pi} [V(f_0) - V(\bar{\sigma}_0 \sigma_0)],
\] (3.15)
where the function \( V(z) \) is defined by
\[
V(z) \equiv V_\infty(z) + \tilde{V}(z),
\] (3.16)
with
\[
V_\infty(z) \equiv -z \left[ \ln(z/\Lambda^2) - 1 \right],
\] (3.17)
and
\[
\tilde{V}(z) \equiv -4 \sum_{m \neq 0} e^{-iA_y \varphi} 2\pi R N m \frac{\sqrt{z}}{2\pi R N |m|} K_1(\sqrt{2\pi R N |m|}).
\] (3.18)

The infinite sum in \( \tilde{V}(z) \) is convergent because \( K_\nu(z) \sim e^{-z} \sqrt{\pi/(2z)} e^{-z} \). Moreover, as shown in Appendix B, \( \tilde{V}(z) \rightarrow 0 \) in the large \( N \) limit (1.1). Therefore, as \( N \rightarrow \infty \), \( f_0 \) and \( \bar{\sigma}_0 \sigma_0 \) are given by the solution of
\[
V_\infty'(z) = -\ln(z/\Lambda^2) = 0,
\] (3.19)
that is,
\[
f_0 = \bar{\sigma}_0 \sigma_0 = \Lambda^2.
\] (3.20)

These are identical to the values in the system in \( \mathbb{R}^2 \) [27].

At the above saddle point, from Eq. (3.15), \( S_{\text{eff}} \equiv 0 \) and becomes independent of \( A_y \). Therefore, \( A_y \) is not determined from the saddle point condition in the present supersymmetric system on \( \mathbb{R} \times S^1 \). We should perform the integration over this “vacuum moduli” \( A_y \).
in the functional integral. We note that the original system is invariant under the “center transformation”,
\[ z^A \rightarrow gz^A, \quad \chi^A_{\pm} \rightarrow g\chi^A_{\pm} , \]
where \( g \in U(1) \) obeys the non-trivial boundary condition,
\[ g(x,y + 2\pi R) = e^{2\pi i/N} g(x,y) , \]
and an element \( g = e^{iy/(RN)} \) induces the constant shift on \( A_y^0 \) through Eq. (2.29) as
\[ A_y^0 \rightarrow A_y^0 - \frac{1}{RN} . \]
Hence, the integration over \( A_y^0 \) should be restricted in the “fundamental domain” as
\[ \int_0^1 d(A_y^0RN) . \]

3.2. Effective action for fluctuations

We next compute the effective action for fluctuations of the auxiliary fields around the above large \( N \) saddle point. That is, setting,
\[ A_\mu \equiv A_\mu^0 + \delta A_\mu, \quad f \equiv f_0 + \delta f, \quad \sigma \equiv \sigma_0 + \delta \sigma, \]
we compute \( S_{\text{eff}} \) to the quadratic order in the fluctuations.

For illustration, let us consider
\[ \sum_A \text{Tr} \ln \left[ -D_\mu D_\mu + f - 4\bar{\eta}(\bar{\psi} + \bar{\sigma}P_+ + \sigma P_-)^{-1}\eta \right]_{O(\delta f^2)} . \]

Going to the momentum space and using the relations (3.7) and (3.9), we have
\[ \sum_A \text{Tr} \ln \left[ -D_\mu D_\mu + f - 4\bar{\eta}(\bar{\psi} + \bar{\sigma}P_+ + \sigma P_-)^{-1}\eta \right]_{O(\delta f^2)} = N \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_x} \left( -\frac{1}{2} \right) \tilde{\delta f}(p)\tilde{\delta f}(-p) \sum_{m=-\infty}^{\infty} e^{-iA_y^02\pi RNm} \times \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} e^{ik_y2\pi RNm} \frac{1}{(k^2 - 2xp + f_0 + xp^2)^2} , \]
where we have set
\[ \delta f(x) \equiv \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_x} e^{ipx} \tilde{\delta f}(p) . \]

The momentum integration then yields
\[ \sum_A \text{Tr} \ln \left[ -D_\mu D_\mu + f - 4\bar{\eta}(\bar{\psi} + \bar{\sigma}P_+ + \sigma P_-)^{-1}\eta \right]_{O(\delta f^2)} = \frac{N}{4\pi} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_x} \left( -\frac{1}{2} \right) \tilde{\delta f}(p)\tilde{\delta f}(-p)\mathcal{L}(p) . \]

Here, we have introduced the combination,
\[ \mathcal{L}(p) \equiv \mathcal{L}_\infty(p) + \tilde{\mathcal{L}}(p) , \]
where (using $f_0 = \Lambda^2$ (3.20))

$$L_\infty(p) \equiv \frac{2}{\sqrt{p^2(p^2 + 4\Lambda^2)}} \ln \left( \frac{\sqrt{p^2 + 4\Lambda^2} + \sqrt{p^2}}{\sqrt{p^2 + 4\Lambda^2} - \sqrt{p^2}} \right), \quad (3.31)$$

is the expression common to the uncompactified space $\mathbb{R}^2$, and

$$\hat{L}(p) \equiv \int_0^1 dx \sum_{m \neq 0} e^{-iA_\mu 2\pi R N m} e^{i p y 2\pi R N m} \times \frac{2\pi R |m|}{\sqrt{\Lambda^2 + x(1-x)p^2}} K_1(\sqrt{\Lambda^2 + x(1-x)p^2} 2\pi R N |m|), \quad (3.32)$$

is a part peculiar to the compactified space $\mathbb{R} \times S^1$. Note that $L_\infty(p)$ and $\hat{L}(p)$ are real and

$$L_\infty(-p) = L_\infty(p), \quad \hat{L}(-p) = \hat{L}(p). \quad (3.33)$$

To show the latter property, we note that the change $e^{i p y 2\pi R N m} \to e^{-i p y 2\pi R N m} = e^{i(1-x)p y 2\pi R N m}$ caused by $p \to -p$ can be absorbed by the change of the integration variable $x \to 1 - x$ (recall that $p_y = n/R$ with $n \in \mathbb{Z}$).

Repeating similar calculations by setting

$$\delta A_\mu(x) \equiv \int \frac{dp_x}{2\pi} \sum_{p_y} e^{i p x} \tilde{A}_\mu(p), \quad \delta \sigma(x) \equiv \int \frac{dp_x}{2\pi} \sum_{p_y} e^{i p x} \tilde{\sigma}(p),$$

$$\eta(x) \equiv \int \frac{dp_x}{2\pi} \sum_{p_y} e^{i p x} \tilde{\eta}(p), \quad \bar{\eta}(x) \equiv \int \frac{dp_x}{2\pi} \sum_{p_y} e^{i p x} \bar{\eta}(p), \quad (3.34)$$

and

$$\tilde{\bar{R}}(p) \equiv \frac{1}{2} \left[ \sigma_0 \tilde{\bar{\sigma}}(p) + \sigma_0 \tilde{\sigma}(p) \right], \quad \tilde{I}(p) \equiv \frac{1}{2i} \left[ \sigma_0 \tilde{\bar{\sigma}}(p) - \sigma_0 \tilde{\sigma}(p) \right], \quad (3.35)$$
we obtain

$$ S_{\text{eff} \mid \text{quadratic}} = \frac{N}{4\pi} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \left[ \frac{1}{2} (p^2 \delta_{\mu\nu} - p_{\mu} p_{\nu}) \mathcal{L}(p) \tilde{A}_\mu(p) \tilde{A}_\nu(-p) \right. $$

$$ + \frac{1}{2\Lambda^2} (p^2 + 4\Lambda^2) \mathcal{L}(p) \tilde{R}(p) \tilde{R}(-p) + \frac{1}{2\Lambda^2} p^2 \mathcal{L}(p) \delta I(p) \tilde{I}(-p) $$

$$ - \frac{1}{2} \mathcal{L}(p) \tilde{f}(p) \tilde{f}(-p) $$

$$ - 2\tilde{\eta}(p)(i\tilde{\phi} + 2\sigma_0 P_+ + 2\sigma_0 P_-) \mathcal{L}(p) \tilde{\eta}(-p) $$

$$ + \epsilon_{\mu\nu} p_\mu \mathcal{L}(p) \tilde{A}_\nu(p) \tilde{\eta}(-p) - \epsilon_{\mu\nu} p_\mu \mathcal{L}(p) \tilde{I}(p) \tilde{A}_\nu(-p) $$

$$ + \left( \delta_{\mu y} - \frac{p_\mu p_y}{p^2} \right) K(p) $$

$$ \times \left\{ \tilde{A}_\mu(p) \left[ 2\tilde{R}(p) - \tilde{f}(p) \right] + \left[ 2\tilde{R}(p) - \tilde{f}(p) \right] \tilde{A}_\mu(-p) \right\} $$

$$ - \frac{1}{2\Lambda^2} \epsilon_{\mu y} p_\mu K(p) \left[ \tilde{R}(p) \delta I(-p) - \delta I(p) \tilde{R}(-p) \right] $$

$$ + 4i \left( \delta_{\mu y} - \frac{p_\mu p_y}{p^2} \right) K(p) \tilde{\eta}(p) \gamma_\mu \tilde{\eta}(-p) \right\} $$

$$ + S_{\text{local}}, \quad \text{(3.36)} $$

where we have used the fact that $\sigma_0 \sigma_0 = \Lambda^2$ (recall Eq. (3.20)) and introduced another combination,

$$ K(p) \equiv i \int_0^1 dx \sum_{m \neq 0} e^{-iA_\sigma 2\pi R N m} e^{i x p_\mu 2\pi R N m} 2\pi R N m K_0(\sqrt{\Lambda^2 + x(1-x)p^2} 2\pi R N |m|). $$

Note that $K(p)$ is real and

$$ K(-p) = K(p). \quad \text{(3.38)} $$

The last term of Eq. (3.36) is given by

$$ S_{\text{local}} \equiv \frac{N}{4\pi} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \left\{ \frac{1}{2} \left[ \tilde{\sigma}(p) \tilde{\sigma}(-p) + \tilde{\sigma}(p) \tilde{\sigma}(-p) \right] $$

$$ - \tilde{\sigma}(p) \tilde{\sigma}(-p) \left[ 1 + 2 \sum_{m \neq 0} e^{-iA_\sigma 2\pi R N m} K_0(\Lambda 2\pi R N |m|) \right]\right\}. \quad \text{(3.39)} $$

This term breaks the $U(1)$ chiral symmetry, the invariance of the classical action under

$$ \sigma \to e^{2i\alpha} \sigma, \quad \tilde{\sigma} \to e^{-2i\alpha} \tilde{\sigma}, \quad \eta \to e^{-i\alpha\gamma_5} \eta, \quad \text{and} \quad \tilde{\eta} \to \tilde{\eta} e^{-i\alpha\gamma_5}, $$

and may be regarded as a “quantum

\footnote{To obtain this simplified form, we have to do integration by parts with respect to the Feynman parameter $x$ by using the relations such as $K'_0(z) = -K_1(z)$ and $zK'_1(z) + K_1(z) = -zK_0(z)$. Also, we have defined the $\gamma_5$ in dimensional regularization such that $\gamma_5 = -i\gamma_x \gamma_y$ for any $D$ \cite{30}; thus it commutes with $\gamma_\mu$ when $\mu \neq x$ or $\mu \neq y$.}
anomaly.” However, since this term is local in position space, we may simply remove this by
a local counterterm as an artifact arising from our particular definition of the $\gamma_5$ matrix in
dimensional regularization. In what follows, we assume this and neglect $S_{\text{local}}$.

Equation (3.36) provides the effective action for the fluctuations around the large $N$ saddle
point; this generalizes Eq. (58) of Ref. [27] to the case of the compactified space $\mathbb{R} \times S^1$.

### 3.3 Propagators

To obtain the propagators of the auxiliary fields from Eq. (3.36), we add a gauge fixing term

$$S_{gf} = \frac{N}{4\pi} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \frac{1}{2} p_{\mu} p_{\nu} \mathcal{L}(p) \delta \tilde{A}_\mu(p) \delta \tilde{A}_\nu(-p)$$

(3.40)

to Eq. (3.36). Then, after some calculation, we find the $A_\mu$ propagator,

$$\left\langle \delta \tilde{A}_\mu(p) \delta \tilde{A}_\nu(q) \right\rangle = \frac{4\pi}{N} \mathcal{D}(p) \left( \Lambda^2 + \left(1 - p_y^2/p^2 \right) \mathcal{K}(p)^2 \right) \frac{p_{\mu} p_{\nu}}{(p^2)^2} 2\pi \delta(p_x + q_x) 2\pi R \delta_{p_y + q_y, 0}.$$ (3.41)

For completeness, we list all the propagators:

$$\left\langle \delta \tilde{A}_\mu(p) \tilde{R}(q) \right\rangle = \left\langle \tilde{R}(p) \delta \tilde{A}_\mu(q) \right\rangle = 0,$$

$$\left\langle \delta \tilde{A}_\mu(p) \tilde{I}(q) \right\rangle = -\left\langle \tilde{I}(p) \delta \tilde{A}_\mu(q) \right\rangle = \frac{4\pi}{N} \mathcal{D}(p) \frac{2\Lambda^2 \bar{p}_{\mu}}{p^2} 2\pi \delta(p_x + q_x) 2\pi R \delta_{p_y + q_y, 0},$$

$$\left\langle \delta \tilde{A}_\mu(p) \tilde{f}(q) \right\rangle = \left\langle \tilde{f}(p) \delta \tilde{A}_\mu(q) \right\rangle = \frac{4\pi}{N} \mathcal{D}(p) \frac{2\Lambda^2 \bar{p}_{\mu} \bar{p}_{y}}{p^2} 2\pi \delta(p_x + q_x) 2\pi R \delta_{p_y + q_y, 0},$$

$$\left\langle \delta \tilde{R}(p) \tilde{A}_\mu(q) \right\rangle = \left\langle \tilde{A}_\mu(p) \tilde{R}(q) \right\rangle = \frac{4\pi}{N} \mathcal{D}(p) \Lambda^2 \bar{p}_{\mu} 2\pi \delta(p_x + q_x) 2\pi R \delta_{p_y + q_y, 0},$$

$$\left\langle \delta \tilde{R}(p) \tilde{I}(q) \right\rangle = -\left\langle \tilde{I}(p) \delta \tilde{R}(q) \right\rangle = \frac{4\pi}{N} \mathcal{D}(p) \frac{2\Lambda^2 \bar{p}_{y}}{p^2} 2\pi \delta(p_x + q_x) 2\pi R \delta_{p_y + q_y, 0},$$

$$\left\langle \tilde{R}(p) \tilde{f}(q) \right\rangle = \left\langle \tilde{f}(p) \tilde{R}(q) \right\rangle = \frac{4\pi}{N} \mathcal{D}(p) \frac{2\Lambda^2 \bar{p}_{\mu} \bar{p}_{y}}{p^2} 2\pi \delta(p_x + q_x) 2\pi R \delta_{p_y + q_y, 0},$$

$$\left\langle \delta \tilde{f}(p) \tilde{A}_\mu(q) \right\rangle = \left\langle \tilde{A}_\mu(p) \tilde{f}(q) \right\rangle = \frac{4\pi}{N} \mathcal{D}(p) \left( -1 \right)(p^2 + 4\Lambda^2) 2\pi \delta(p_x + q_x) 2\pi R \delta_{p_y + q_y, 0},$$ (3.42)

and

$$\left\langle \bar{\eta}(p) \tilde{\eta}(q) \right\rangle = \frac{4\pi}{N} \left( i\bar{p} + 2\sigma_0 P_+ + 2\sigma_0 P_- \right) \mathcal{L}(p) + 2i(\gamma_5 - \bar{p} p_y / p^2) \mathcal{K}(p) \left( -\frac{1}{2} \right) 2\pi \delta(p_x + q_x) 2\pi R \delta_{p_y + q_y, 0}. $$ (3.43)

In the above expressions, we have defined

$$\mathcal{D}(p) \equiv (p^2 + 4\Lambda^2) \mathcal{L}(p)^2 + 4(1 - p_y^2/p^2) \mathcal{K}(p)^2, \quad \bar{p}_\mu \equiv \epsilon_{\mu\nu} p_\nu,$$ (3.44)
Fig. 1 The Feynman diagram corresponding to the gluon condensate \(\langle F_{\mu\nu}(x)F_{\mu\nu}(x)\rangle\). The blob is the combination in Eq. (4.1). The \(A_\mu\) propagator (3.41) is given by the chain of the one-loop vacuum polarization diagrams owing to the \(N\) fields.

and, in deriving those expressions, we have noted the relation holding in the two dimensions,

\[
\bar{p}_\mu \bar{p}_\nu = p^2 \delta_{\mu\nu} - p_\mu p_\nu.
\] (3.45)

4. **IR renormalon in the gluon condensate**

In this section, we compute the gluon condensate in the leading order of the large \(N\) approximation and extract the perturbative part from it. We then obtain the corresponding Borel transform \(B(u)\) and study its singularities.

The gluon condensate is given by

\[
\langle F_{\mu\nu}(x)F_{\mu\nu}(x)\rangle = \left(\partial_\mu \delta A_\nu(x) - \partial_\nu \delta A_\mu(x)\right)^2.
\] (4.1)

The contraction of this by the propagator (3.41) (see Fig. 1) gives the leading large \(N\) result as

\[
\langle F_{\mu\nu}(x)F_{\mu\nu}(x)\rangle = \frac{4\pi}{N} \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} e^{ip_\mu 2\pi Rn} \frac{2p^2\mathcal{L}(p)}{(p^2 + 4\Lambda^2)\mathcal{L}(p)^2 + 4(1 - p_y^2/p^2)\mathcal{K}(p)^2}.
\] (4.2)

where in the second equality we have used Eq. (3.7).

First, in the large \(N\) limit (1.1), as shown in Appendix B, we can set \(\hat{\mathcal{L}}(p) \to 0\), \(\mathcal{K}(p) \to 0\). Equation (4.2) thus reduces to (recall Eq. (3.30))

\[
\langle F_{\mu\nu}(x)F_{\mu\nu}(x)\rangle = \frac{4\pi}{N} \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} e^{ip_\mu 2\pi Rn} \frac{2p^2\mathcal{L}(p)}{(p^2 + 4\Lambda^2)\mathcal{L}(p)^2 + 4(1 - p_y^2/p^2)\mathcal{K}(p)^2}.
\] (4.4)

Next, we extract the perturbative part from this expression. If we expand \(\mathcal{L}_\infty(p)\) (3.31) and Eq. (4.4) with respect to \(\Lambda^2/p^2\), the terms in positive powers of \(\Lambda^2/\Lambda^2\)^{k}\ are

\[\text{The integration over } A_{y0} \text{ in Eq. (3.24) is implicitly assumed in this expression. However, since this expression reduces to Eq. (4.4) in the large } N \text{ limit (1.1) that is independent of } A_{y0}, \text{ the integration over } A_{y0} \text{ in Eq. (3.24) is trivial.}\]
regarded as the non-perturbative part, because $\Lambda^2 \sim e^{-4\pi/\lambda R}$. This reasoning tells us that the gluon condensate in perturbation theory (PT) is given by

$$\langle F_{\mu \nu}(x)F_{\mu \nu}(x) \rangle_{\text{PT}} = \frac{4\pi}{N} \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} e^{ip_n 2\pi Rn} \frac{p^2}{\ln(p^2/\Lambda^2)} \left| \text{expansion in } \lambda R \right|,$$

where we explicitly indicate that the integrand should be expanded in $\lambda R$ in the perturbative evaluation. In this expression, we analyze the $n = 0$ term and the $n \neq 0$ terms separately.

### 4.1. The $n = 0$ term

The $n = 0$ term exhibits the quartic UV divergence and we thus introduce the UV cutoff $q$:

$$\langle F_{\mu \nu}(x)F_{\mu \nu}(x) \rangle_{\text{PT}, n = 0} = \frac{4\pi}{N} \int_{|p| \leq q} \frac{d^2p}{(2\pi)^2} \frac{p^2}{\ln(p^2/\Lambda^2)} \left| \text{expansion in } \lambda R \right| .$$

Then, noting

$$\ln(p^2/\Lambda^2) = \ln(p^2/q^2) + \frac{4\pi}{\lambda R(q)},$$

where $\lambda R(q)$ is the renormalized coupling at the cutoff scale $q$ (see Eq. (3.14)), we have the perturbative expansion with respect to $\lambda R(q)$:

$$\langle F_{\mu \nu}(x)F_{\mu \nu}(x) \rangle_{\text{PT}, n = 0} = \frac{4\pi}{Nq^4} \sum_{k=0}^{\infty} \int_{|p| \leq 1} \frac{d^2p}{(2\pi)^2} p^2 (-\ln p^2)^k \left[ \frac{\lambda R(q)}{4\pi} \right]^{k+1} .$$

From this perturbative series, we define the corresponding Borel transform as Eq. (1.4),

$$B_{n=0}(u) = \frac{4\pi}{Nq^4} \int_{|p| \leq 1} \frac{d^2p}{(2\pi)^2} p^2 \sum_{k=0}^{\infty} (-\ln p^2)^k \frac{u^k}{k!}$$

$$= \frac{1}{Nq^4} \frac{-1}{u - 2} .$$

Thus, the Borel transform of the $n = 0$ term of Eq. (4.5) develops a pole singularity at $u = 2$. We had to know this, because the $n = 0$ term in Eq. (4.4) is basically identical to the loop integral appearing in the scalar condensate in the 2D $O(N)$ non-linear sigma model in the un-compactified space $\mathbb{R}^2$, which suffers from the $u = 2$ renormalon ambiguity [31]. See also Ref. [6].

Through the Borel integral (4.5), the Borel singularity at $u = 2$ gives rise to the renormalon ambiguity on the gluon condensate (focusing only on the $n = 0$ term):

$$\langle F_{\mu \nu}(x)F_{\mu \nu}(x) \rangle_{n = 0, \text{IR renormalon at } u = 2} \sim \int_0^{\infty} du \frac{1}{Nq^4} e^{-4\pi u/\lambda R(q)}$$

$$\sim \frac{1}{N} q^4 e^{-8\pi/\lambda R(q)} (\pm \pi i)$$

$$= \frac{1}{N} \Lambda^4 (\pm \pi i) ,$$

where the sign depends on how one avoids the pole singularity at $u = 2$ (+ for a contour in the upper plane, – for a contour in the lower plane). We note that the ambiguity caused

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5 The integral (4.3) is convergent in the IR region and it is real once the UV divergence is regularized by a cutoff as done in the following. Thus, in this sense, the gluon condensate obtained in the large-$N$ expansion is an unambiguous object and the ambiguity found in the following argument stems from the artifact of the perturbative evaluation. This means that the resurgence structure is already assured in this quantity.
by the IR renormalon itself is independent of the UV cutoff \( q \) we introduced; the last line of Eq. (4.10) does not refer to the scale \( q \).

4.2. The \( n \neq 0 \) terms

The \( n \neq 0 \) terms in Eq. (4.5),

\[
\langle F_{\mu \nu}(x)F_{\mu \nu}(x) \rangle_{PT, n \neq 0} = \frac{4\pi}{N} \sum_{n \neq 0} \sum_{k=0}^{\infty} \int \frac{d^2 p}{(2\pi)^2} e^{i p \cdot x} \frac{\lambda_R(p^2)}{\ln(p^2/\Lambda^2)} \exp(-\ln p^2)^k \left[ \frac{\lambda_R(1/R)}{4\pi} \right]^{k+1},
\]

are the Fourier transforms and thus UV convergent. This time, instead of Eq. (4.7), we use

\[
\ln(p^2/\Lambda^2) = \ln(p^2R^2) + \frac{4\pi}{\lambda_R(1/R)},
\]

where \( \lambda_R(1/R) \) is the renormalized coupling at the scale \( 1/R \). Then the perturbative expansion with respect to \( \lambda_R(1/R) \) is given by

\[
\langle F_{\mu \nu}(x)F_{\mu \nu}(x) \rangle_{PT, n \neq 0} = \frac{4\pi}{N} \sum_{n \neq 0} \sum_{k=0}^{\infty} \int \frac{d^2 p}{(2\pi)^2} e^{i p \cdot x} \frac{\lambda_R(p^2)}{\ln(p^2/\Lambda^2)} \exp(-\ln p^2)^k \left[ \frac{\lambda_R(1/R)}{4\pi} \right]^{k+1},
\]

The corresponding Borel transform is thus

\[
B_{n \neq 0}(u) = \frac{4\pi}{N} \frac{1}{R^4} \sum_{n \neq 0} \sum_{k=0}^{\infty} \int \frac{d^2 p}{(2\pi)^2} e^{i p \cdot x} \frac{\lambda_R(p^2)}{\ln(p^2/\Lambda^2)} \exp(-\ln p^2)^k \left[ \frac{\lambda_R(1/R)}{4\pi} \right]^{k+1}
\]

This Borel transform has a pole at \( u = 2 \),

\[
B_{n \neq 0}(u) \sim \frac{1}{N} \frac{1}{R^4} \frac{1}{u - 2},
\]

and thus the corresponding renormalon ambiguity is given by

\[
\langle F_{\mu \nu}(x)F_{\mu \nu}(x) \rangle_{n \neq 0, IR \ renormalon \ at \ u = 2} \sim \int_0^\infty du B_{n \neq 0}(u) e^{-4\pi u/\lambda_R(q)}
\]

\[
\sim \frac{1}{N} \frac{1}{R^4} e^{-8\pi/\lambda_R(1/R)(\mp \pi i)}
\]

\[
= \frac{1}{N} \Lambda^4(\mp \pi i).
\]

This is precisely opposite to the ambiguity in Eq. (4.10) arising from the \( n = 0 \) term. Thus, for the \( u = 2 \) singularity, the contribution from the compactification (i.e., terms survive for a finite \( R \)) cancels the singularity which exists in the un-compactified space.

One can be skeptical about the above cancellation of the Borel singularities, because our argument used different renormalization scales for the coupling constant; \( q \) for the \( n = 0 \) term and \( 1/R \) for the \( n \neq 0 \) terms. However, this is just for simplicity of expressions. If we want, we may use a general mass scale \( \mu \) and make use of

\[
\frac{\lambda_R(q)}{4\pi} = \frac{\lambda_R(\mu)}{4\pi} \left[ 1 - \ln(\mu^2/q^2) \lambda_R(\mu) \right]^{-1},
\]

\[
\frac{\lambda_R(1/R)}{4\pi} = \frac{\lambda_R(\mu)}{4\pi} \left[ 1 - \ln(\mu^2 R^2) \lambda_R(\mu) \right]^{-1},
\]

to obtain the perturbative series in \( \lambda_R(\mu) \). This change of the renormalization scale does not affect the cancellation of the renormalon ambiguity at \( u = 2 \).
4.3. New Borel singularity at $u = 3/2$

We showed that the Borel singularity at $u = 2$, which exists in the un-compactified space, disappears. However, this is not the end of the story. We note that the $\zeta$ function in Eq. (4.13), $\zeta(z)$, possesses a simple pole at $z = 1$. This produces the Borel singularity at $u = 3/2$:

$$B_n\neq 0(u) \sim 3/2 \frac{1}{N} \frac{1}{R^4} \left( -\frac{1}{\pi} \right) \frac{1}{u - 3/2}. \quad (4.18)$$

Since $B_n=0(u)$ [13] has no corresponding singularity, we conclude that the perturbative part [4.5] possesses the renormalon ambiguity at $u = 3/2$ as

$$\langle F_{\mu\nu}(x)F_{\mu\nu}(x) \rangle_{IR \text{ renormalon at } u = 3/2} = \frac{1}{N} \frac{1}{R} \frac{1}{\pi} (\pm i). \quad (4.19)$$

This is the net ambiguity of perturbative evaluation of the gluon condensate in our system. Since this is proportional to $1/R$, it is clear that this renormalon ambiguity is peculiar to the compactified space $\mathbb{R} \times S^1$. Also, the location of the singularity $u = 3/2$ is not dividable by the minimal bion action (corresponding to $u = 2$); we do not know of any semi-classical interpretation of this renormalon ambiguity.

4.4. Renormalon in a physical observable

Since the gluon condensate exhibits the quartic UV divergence, the gluon condensate itself may not be regarded as a physical observable. However, our result implies that there is a physical observable whose perturbative evaluation suffers from the renormalon ambiguity in the same way as the gluon condensate. An explicit example is provided by the gradient flow [32] and its small flow time expansion [33]. For the “$U(1)$ gauge field” $A_\mu(x)$, we introduce the gradient flow for $t \geq 0$ by

$$\partial_t B_\mu(t, x) = \partial_\nu G_{\nu\mu}(t, x) + \alpha_0 \partial_\nu \partial_\rho B_\rho(t, x), \quad B_\mu(t = 0, x) = A_\mu(x), \quad (4.20)$$

where $G_{\mu\nu}(t, x) = \partial_\nu B_\mu(t, x) - \partial_\nu B_\mu(t, x)$ is the field strength of the flowed gauge field and $\alpha_0$ is the “gauge parameter” [32]. Since this equation can be solved as

$$B_\mu(t, x) = A_\mu 0 \delta_{\mu y} + \int d^2 x' \int \frac{dp}{2\pi} \frac{1}{2\pi R} \sum p_\nu e^{ip(x-x')} \left[ \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) e^{-tp^2} + \frac{p_\mu p_\nu}{p^2} e^{-\alpha_0 tp^2} \right] \delta A_\nu(x'),$$

the “gluon condensate” of the flowed gauge field is simply given by putting the Gaussian factor $e^{-2p^2}$ to Eq. (4.2):

$$\langle G_{\mu\nu}(t, x)G_{\nu\rho}(t, x) \rangle = \frac{4\pi}{N} \sum_{n=-\infty}^{\infty} \int \frac{dp}{(2\pi)^2} e^{ip_{\mu\rho}Rn} \frac{2p^2\mathcal{L}(p)e^{-2tp^2}}{(p^2 + 4\Lambda^2)\mathcal{L}(p)^2 + 4(1 - p_\delta^2/p^2)\mathcal{K}(p)^2} \quad (4.21)$$

We then repeat the argument developed so far in this section. This time, the $n = 0$ term is UV convergent thanks to the Gaussian factor and we can take $1/\sqrt{t}$ as the renormalization
scale. Then the Borel sum gives
\[
\langle G_{\mu\nu}(t,x)G_{\mu\nu}(t,x) \rangle_{\text{PT},n=0} = \frac{1}{N} \frac{1}{R} \int_0^\infty du \ 2\pi^{2n-2} \zeta(4-2u) \frac{\Gamma(2-u)}{\Gamma(u-1)} e^{-4\pi u/\Lambda_R (1/\sqrt{t})} = \frac{1}{N} \Lambda^4 \pm \pi i + O(t),
\]
where in the second line, we indicated only the ambiguous part. In the \( n \neq 0 \) terms, on the other hand, we may expand the Gaussian factor as \( e^{-2\pi p^2} = 1 + O(t) \). The first term in this expansion is nothing but Eq. (4.13). Therefore, using Eq. (4.14),
\[
\langle G_{\mu\nu}(t,x)G_{\mu\nu}(t,x) \rangle_{\text{PT},n\neq0} = \frac{1}{N} \frac{1}{R} \int_0^\infty du \ 2\pi^{2n-4} \zeta(4-2u) \frac{\Gamma(2-u)}{\Gamma(u-1)} e^{-4\pi u/\Lambda_R (1/R)} + O(t) \sim \frac{1}{N} \frac{1}{R} \Lambda^3 \frac{1}{\pi} (\pm \pi i) + \frac{1}{N} \Lambda^4 \mp \pi i + O(\Lambda^6 R^2) + O(t). \tag{4.24}
\]
In the sum of Eqs. (4.23) and (4.24), the renormalon ambiguity at \( u = 2 \) cancels out and the leading ambiguity (for \( \Lambda R \ll 1 \)) stems from the \( u = 3/2 \) singularity. This example clearly illustrates that the \( u = 3/2 \) renormalon on \( \mathbb{R} \times S^1 \) indeed appears in the perturbative calculation of a physical observable.

5. Conclusion and discussion

In this paper, in the leading order of the large \( N \) approximation, we studied the renormalon ambiguity in the gluon condensate in the 2D supersymmetric \( \mathbb{C}P^{N-1} \) model on \( \mathbb{R} \times S^1 \) with the \( \mathbb{Z}_N \) twisted boundary conditions. We found that the Borel singularity at \( u = 2 \), which exists in the un-compactified space \( \mathbb{R}^2 \), disappears in the compactified space \( \mathbb{R} \times S^1 \). Instead, we found an unfamiliar singularity at \( u = 3/2 \), which is peculiar to the compactified space and has no obvious semi-classical interpretation. We also showed that this renormalon indeed appears in the perturbative calculation of a physical observable. We emphasize that this result, which may be unexpected, was obtained in a very straightforward and systematic calculation.

The \( u = 3/2 \) singularity peculiar to the compactified space can also be found from the perturbative part \([4.5] \) in the original form with the discrete KK momentum:
\[
\langle F_{\mu\nu}(x)F_{\mu\nu}(x) \rangle_{\text{PT}} = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum \frac{p_y^2}{\ln(p^2/\Lambda^2)} \bigg|_{\text{expansion in } \lambda_R}. \tag{5.1}
\]
If we pick up only the zero KK modes \( p_y = 0 \) from this, we have
\[
\langle F_{\mu\nu}(x)F_{\mu\nu}(x) \rangle_{\text{PT}, p_y = 0} = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \frac{p_y^2}{\ln(p^2/\Lambda^2)} \bigg|_{\text{expansion in } \lambda_R}
= \frac{4\pi}{N} \frac{1}{2\pi R} \int_0^\infty \frac{dp_x}{2\pi} \frac{p_x^2}{\ln(p_x^2/\Lambda^2)} \bigg[ \frac{\lambda_R(q)}{4\pi} \bigg]^{k+1} \tag{5.2}
\]

\footnote{In terms of the small flow time expansion \([3, 32] \), the \( O(t^0) \) terms in Eqs. (4.23) and (4.24) correspond to the gluon condensate \( \langle F_{\mu\nu}(x)F_{\mu\nu}(x) \rangle \).}

\footnote{It might be possible to detect this leading renormalon on \( \mathbb{R} \times S^1 \) by using the stochastic perturbation theory \([34, 37] \).}
The corresponding Borel transform is thus
\[ B(u) = \frac{4\pi}{N} \frac{1}{2\pi R q^3} \int_{-1}^{1} dp_x \frac{p_x^2}{2\pi} \sum_{k=0}^{\infty} \frac{(-\ln p_x^2)^k}{k!} u^k \]
\[ = \frac{1}{N} \frac{q^3}{R} \left( -\frac{1}{\pi} \right) \frac{1}{u - 3/2}, \] (5.3)
and precisely reproduces the singularity in Eq. (4.18). This observation strongly indicates that the Borel singularity at \( u = 3/2 \) is closely related to the divergence of the perturbative series in the supersymmetric \( \mathbb{C}P^{N-1} \) quantum mechanics. This picture seems to be not quite correct, however. The action of the bion in the quantum mechanics is \[ \frac{2}{g_{1D}^2} m, \] (5.4)
where \( g_{1D} \) is the coupling constant in the quantum mechanics. The parameter \( m \) is a remnant of the twist angle (2.7) before the reduction to the quantum mechanics. Thus, if we make the following identifications with the 2D model,
\[ \frac{1}{g_{1D}^2} = \frac{2\pi R}{g^2}, \quad m = \frac{b}{RN}, \] (5.5)
then the bion action in the quantum mechanics becomes \( 4\pi/(g^2 N) b = (4\pi/\lambda) b \). Thus, the bion corresponds to Borel singularities at \( u = 1, 2, \ldots \), but not \( u = 3/2 \).

We finally mention the relation between the present work and preceding analysis based on the bion. In Ref. [19], the vacuum energy is computed as the function of the supersymmetry breaking parameter \( \delta \epsilon \)
\[ \delta S \equiv \int d^2 x \frac{\delta \epsilon}{\pi R} \sum_{a=1}^{N-1} m_a \left( \frac{|\varphi^a|^2}{1 + \sum_{b=1}^{N-1} |\varphi^b|^2} - \frac{1}{N} \right) = \int d^2 x \frac{\delta \epsilon}{\pi R} \sum_{A=1}^{N} m_A \left( \bar{z}^A z^A - \frac{1}{N} \right), \] (5.6)
as
\[ E(\delta \epsilon) = E^{(0)} + E^{(1)} \delta \epsilon + E^{(2)} \delta \epsilon^2 + \cdots . \] (5.7)
The leading ambiguity from the bion calculus was found in the \( E^{(2)} \) term. Since these expansion coefficients can be obtained as the correlation functions,
\[ E^{(1)} = 2 \sum_{A} m_A \left( \bar{z}^A z^A - \frac{1}{N} \right), \]
\[ E^{(2)} = -\frac{1}{\pi R} \int d^2 x \sum_{A} m_A \sum_{B} m_B \langle \bar{z}^A z^A(x) \bar{z}^B z^B(0) \rangle_c, \] (5.8)
in the supersymmetric theory (where \( E^{(0)} = 0 \)), it is interesting to compute these numbers by the large \( N \) technique developed in this paper. We hope to come back this problem in the near future.

Acknowledgements
Discussions with Toshiaki Fujimori, Nobuyuki Ishibashi, Tatsuhiko Misumi, Norisuke Sakai, and, especially, Kazuya Yonekura at the YITP workshop “Strings and Fields 2018” motivated the present work. This work was supported by JSPS Grant-in-Aid for Scientific Research Grant Numbers, JP18J20935 (O.M.), JP16H03982 (H.S.), and JP19K14711 (H.T.).
A. Basic formulas with the homogeneous coordinate

In this Appendix, we summarize some useful formulas to obtain Eq. (2.19). For the bosonic part of the action, see Ref. [25]. In terms of the homogeneous coordinate, the Fubini–Study metric (2.2) is written as

\[ G_{\bar{a}b} = |z|^2 (\delta_{ab} - \bar{z}^a z^b). \]  

(A1)

Then, the fermionic fields satisfy the relation

\[ G_{\bar{a}b} \psi^b_s \psi^a_{s'} = \bar{\chi}_s A \chi_{s'}, \]  

(A2)

where \( s, s' = + \) or \( - \). From the connection on \( \mathbb{C}P^{N-1} \),

\[ \Gamma^a_{bc} = -\frac{\delta^{ab}\varphi^c + \delta^{ac}\varphi^b}{1 + \sum_d |\varphi^d|^2}, \]  

\( \Gamma^{\bar{a}}_{\bar{b} \bar{c}} = -\frac{\delta^{ab}\varphi^c + \delta^{ac}\varphi^b}{1 + \sum_d |\varphi^d|^2}, \]  

(A3)

we have the kinetic term of the homogeneous coordinate variables \( \chi \) and \( \bar{\chi} \) as

\[ G_{\bar{a}b} \psi^b_s \psi^a_{s'} = \bar{\chi}_s A \chi_{s'}, \]  

(A4)

To translate the four-fermion interaction, \( R_{\bar{a}bc} \psi^c + \psi^a \psi^d \), into that of the homogeneous coordinate, we note that the Riemann curvature on \( \mathbb{C}P^{N-1} \) satisfies

\[ R_{\bar{a}bc} \psi^c + \psi^a \psi^d = \chi_A^+ A^+ A^- - \chi_A^+ A^- A^+. \]  

(A5)

This relation and Eq. (A2) immediately indicate

\[ R_{\bar{a}bc} \psi^c + \psi^a \psi^d = \chi_A^+ A^+ A^- - \chi_A^+ A^- A^+. \]  

(A6)

B. Bounds for the functions, \( \hat{V}(z) \), \( \hat{L}(p) \), and \( \mathcal{K}(p) \)

The modified Bessel function of the second kind \( K_\nu(z) \) with \( \nu = 0 \) or \( 1 \) has the upper bound,

\[ K_\nu(z) < \frac{2}{z} e^{-z/2}, \quad \text{for } z > 0. \]  

(B1)

This follows from the integral representation,

\[ K_\nu(z) = \frac{1}{2} \int_0^\infty dx \, x^{-\nu-1} e^{-\frac{x}{2} \left( x + \frac{1}{x} \right)}, \]  

(B2)

as

\[ K_\nu(z) = \frac{1}{2} \left( \int_0^1 + \int_1^\infty \right) dx \, x^{-\nu-1} e^{-\frac{x}{2} \left( x + \frac{1}{x} \right)} \]

\[ = \frac{1}{2} \int_0^1 dx \, x^{-\nu-1} e^{-\frac{1}{2} x \left( x + \frac{1}{x} \right)} + \frac{1}{2} \int_1^\infty dx \, x^{-\nu-1} e^{-\frac{1}{2} x \left( x + \frac{1}{x} \right)} \]

\[ < \int_0^1 dx \, x^{-2} e^{-\frac{1}{2} x} = \frac{2}{z} e^{-z/2}, \]  

(B3)

where, in the second equality, we have changed the integration variable \( x \to 1/x \) in the second integral. The last inequality follows from \( x^{\nu-1} \leq x^{-2} \) for \( 0 \leq x \leq 1 \) and \( \nu = 0, 1 \).
First, for Eq. (3.18), by using Eq. (B3),
\[
|\hat{V}(z)| \leq 4 \sum_{m \neq 0} \frac{\sqrt{z}}{2\pi RN|m|} K_1(\sqrt{z}2\pi RN|m|)
\]
\[
< 4 \sum_{m \neq 0} \frac{2}{(2\pi RN|m|)^2} e^{-\sqrt{z}2\pi RN|m|}
\]
\[
< \frac{16}{(2\pi RN)^2} \frac{e^{-\sqrt{z}2\pi RN}}{1 - e^{-\sqrt{z}2\pi RN}}.
\]
(B4)

Thus, $\hat{V}(z) \to 0$ under the large $N$ limit (1.1).

For Eq. (3.32),
\[
|\hat{L}(p)| \leq \int_0^1 dx \sum_{m \neq 0} \frac{2\pi RN|m|}{\sqrt{\Lambda^2 + x(1-x)p^2}} K_1(\sqrt{\Lambda^2 + x(1-x)p^2}2\pi RN|m|)
\]
\[
< 2 \int_0^1 dx \frac{1}{\Lambda^2 + x(1-x)p^2} \sum_{m \neq 0} e^{-\sqrt{\Lambda^2 + x(1-x)p^2}2\pi RN|m|}
\]
\[
= 4 \int_0^1 dx \frac{1}{\Lambda^2 + x(1-x)p^2} \frac{e^{-\sqrt{\Lambda^2 + x(1-x)p^2}2\pi RN}}{1 - e^{-\sqrt{\Lambda^2 + x(1-x)p^2}2\pi RN}}
\]
\[
< \frac{4}{\Lambda^2} \frac{e^{-\Lambda\pi RN}}{1 - e^{-\Lambda\pi RN}}.
\]
(B5)

For Eq. (3.37), starting from
\[
|\hat{K}(p)| \leq \int_0^1 dx \sum_{m \neq 0} 2\pi RN|m| K_0(\sqrt{\Lambda^2 + x(1-x)p^2}2\pi RN|m|),
\]
(B6)
a calculation parallel to the above leads to
\[
|\hat{K}(p)| < \frac{4}{\Lambda} \frac{e^{-\Lambda\pi RN}}{1 - e^{-\Lambda\pi RN}}.
\]
(B7)

The functions $\hat{L}(p)$ and $\hat{K}(p)$ thus vanish in the large $N$ limit (1.1).

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