Abstract

We consider the Constrained-degree percolation model in random environment (CDPRE) on the square lattice. In this model, each vertex \( v \) has an independent random constraint \( \kappa_v \) which takes the value \( j \in \{0, 1, 2, 3\} \) with probability \( \rho_j \). The dynamics is as follows: each edge \( e \) attempts to open at a random time \( U_e \sim U(0, 1] \), independently of all other edges. It succeeds if at time \( U_e \) both its end-vertices have degrees strictly smaller than their respective constraints. We obtain almost exponential decay of the radius of the open cluster of the origin at all times when its expected size is finite. Since CDPRE is dominated by Bernoulli percolation, such result is meaningful only if the supremum of all values of \( t \) for which the expected size of the open cluster of the origin is finite is larger than 1/2. We prove this last fact by showing a sharp phase transition for an intermediate model.

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1 Introduction

Let \( \mathbb{L}^2 = (\mathbb{Z}^2, \mathcal{E}) \) be the usual square lattice. To each site \( v \in \mathbb{Z}^2 \) we associate independently a random variable \( \kappa_v \) which takes the value \( j \in \{0, 1, 2, 3\} \) with probability \( \rho_j \). Denote by \( \mathbb{P}_\rho \) the corresponding product measure on \( \{0, 1, 2, 3\}^{\mathbb{Z}^2} \). Consider the following dependent continuous time percolation process: let \( \{U_e\}_{e \in \mathcal{E}} \) be a collection of i.i.d. random variables with uniform distribution on the interval \((0, 1]\). At time \( t = 0 \) all edges are closed; each edge \( e = \langle u, v \rangle \) opens at time \( U_e \) if \( |\{z \in \mathbb{Z}^2 \setminus \{u\} : \langle z, u \rangle \text{ is open at time } U_e \}| < \kappa_u \) and \( |\{z \in \mathbb{Z}^2 \setminus \{v\} : \langle z, v \rangle \text{ is open at time } U_e \}| < \kappa_v \).

In words, at the random time \( U_e \) the edge \( e \) attempts to open. It succeeds if both its endpoints have degrees smaller than their respective attached constraints. Once an edge is open, it remains open.

The model described above was inspired by its deterministic constraint version introduced in [2]. In that model, constraints are equal to some fixed deterministic \( \kappa \) for every vertex. The authors of [2] prove a non-trivial phase transition for the model on \( \mathbb{L}^2 \) when \( \kappa = 3 \). In contrast, they show there is no percolation when \( \kappa = 2 \), even at time \( t = 1 \). In a recent paper, see [7], the authors extend some of the results of [2], proving a phase transition for the model on \( \mathbb{L}^d \), \( d \geq 2 \), for several values of a constant deterministic \( \kappa \). See [4, 5, 6, 8] for other models with some type of constraint.

The random constraint version we approach in this work was introduced in [10]. In that work, the authors show a non-trivial phase transition for the model on \( \mathbb{L}^2 \) when \( \rho_3 \) is sufficiently large, thus extending the main result of [2].
A formal definition of the CDPRE model reads as follows. To each edge \( e \in \mathcal{E} \) we assign independently a random variable \( U_e \sim \text{U}(0,1) \), independent of \( \{\kappa_v\}_{v \in \mathbb{Z}^2} \). The corresponding product measure is denoted by \( \mathbb{P} \). We think of \( U_e \) as the time when edge \( e \) attempts to open and usually refer to \( \{U_e\}_{e \in \mathcal{E}} \) as a configuration of \textit{clocks}. Given a collection of constraints \( \kappa = \{\kappa_v\}_{v \in \mathbb{Z}^2} \) and a clock configuration \( U = \{U_e\}_{e \in \mathcal{E}} \), let

\[ \omega_t : \{0,1,2,3\}^{\mathbb{Z}^2} \times [0,1]^{\mathcal{E}} \rightarrow \{0,1\}^{\mathcal{E}} \]

be the function that associates the pair \((\kappa,U)\) to a configuration \( \omega_t(\kappa,U) \) of open and closed edges at time \( t \). From now on, we use the short notation \( \omega_t \) and denoted by \( \omega_t(e) \) the status of the edge \( e \) in the configuration \( \omega_t \). We say an edge \( e \) is \( t \)-open (\( t \)-closed) if \( \omega_t(e) = 1 \) (\( \omega_t(e) = 0 \)). Formally, writing \( \mathbb{1}_A \) for the indicator function of the event \( A \) and \( \text{deg}(v,t) \) for the degree of vertex \( v \) in \( \omega_t \), the configuration at edge \( e = \langle u,v \rangle \) is written as

\[ \omega_t(e) = \mathbb{1}_{\{U_e \leq t\}} \times \mathbb{1}_{\{\text{deg}(u,U_e) < \kappa_u\}} \times \mathbb{1}_{\{\text{deg}(v,U_e) < \kappa_v\}}. \]

Denote by \( \mathbb{P}_{\rho,t} \) the pushforward product law governing \( \omega_t \), that is, for any measurable set \( A \subset \{0,1\}^{\mathcal{E}} \),

\[ \mathbb{P}_{\rho,t}(A) = (\mathbb{P} \times \mathbb{P})(\omega_t^{-1}(A)). \]

What makes this model interesting is that, at any fixed time \( t > 0 \), the configurations exhibit infinite range dependencies. However, as we will show later, the dependence of the states of any two edges decays faster than exponential when the distance between the two edges increases (see Proposition 1 in Section 2.1), a fact that will be important in this work. We also stress that the model does not possess the FKG property, making the analysis harder.

### 1.1 Results and discussion

Before we state our results, let us introduce some notation. A \textit{path} of \( \mathbb{L}^2 \) is an alternating sequence \( v_0, e_0, v_1, e_1, \ldots, e_{n-1}, v_n \) of distinct vertices \( v_j \) and edges \( e_j = \langle v_j, v_{j+1} \rangle \). Such path has length \( n \) and is said to connect \( v_0 \) to \( v_n \). A path is said to be open if all of its edges are open. Write \( C_v \) for the open cluster of \( v \in \mathbb{Z}^2 \), i.e., the set of vertices connected to \( v \) by an open path. By translation invariance of the probability measure, we take this vertex to be the origin and define the percolation and susceptibility critical thresholds

\[ t_c(\rho) = \sup\{t \in [0,1] : \mathbb{P}_{\rho,t}(|C| = \infty) = 0\}, \]

\[ \bar{t}_c(\rho) = \sup\{t \in [0,1] : \mathbb{E}_{\rho,t}(|C|) < \infty\}, \]

respectively. Here \( \mathbb{E}_{\rho,t} \) denotes expectation with respect to \( \mathbb{P}_{\rho,t} \). Clearly, \( \bar{t}_c(\rho) \leq t_c(\rho) \).

Let \( u,v \in \mathbb{Z}^2 \) and write \( d(u,v) \) for the graph distance from \( u \) to \( v \). Set \( B(n) = [-n,n]^2 \) for the box of side-length \( 2n \) centered at the origin. For \( x \in \mathbb{Z}^2 \), we define \( B(x,n) = x + B(n) \). Given \( \Gamma \subset \mathbb{Z}^2 \), we write \( \mathcal{E}(\Gamma) \) to denote the set of edges inherit from \( \mathbb{L}^2 \) with both endpoints in \( \Gamma \). We use \( \partial \Gamma \) to denote the vertex boundary of \( \Gamma \), being the set of vertices in \( \Gamma \) which are adjacent to some vertex not in \( \Gamma \).
We also write \( \partial^e \Gamma \) for the external edge boundary of \( \Gamma \), i.e., the set of edges \( e = \langle u, v \rangle \), with \( u \in \Gamma \) and \( v \notin \Gamma \).

It is clear that the CDPRE model is stochastically dominated by independent Bernoulli percolation for all \( t \in [0, 1] \), and hence we have exponential tail decay of the radius of the open cluster when \( t < 1/2 \). Theorem 1 below, whose proof is deferred to Section 2.2, gives that \( \bar{t}_c(\rho) \) (and consequently \( t_c(\rho) \)) is strictly larger than \( 1/2 \) when \( \rho_0 = 0 \) (we strongly believe this is true for all \( \rho \), although we could not prove it). It is therefore natural to ask: do we have exponential decay for all \( t \) smaller than \( t_c(\rho) \)? We prove almost exponential decay of the radius of the open cluster for all \( t < \bar{t}_c(\rho) \), giving a partial answer to this question. A nice open problem consists in proving that the model exhibits a sharp phase transition, i.e., that the radius of the open cluster decays exponentially fast for all \( t < t_c(\rho) \). In particular, this would give \( t_c(\rho) = \bar{t}_c(\rho) \).

**Theorem 1.** Consider the CDPRE model with \( \rho_0 = 0 \). It holds that \( \bar{t}_c(\rho) > \frac{1}{2} \).

Let \( \theta_n(t) \) denote the probability that the origin is connected to \( \partial B(n) \) at time \( t \). We omit \( \rho \) from the notation to keep it clean. We will prove the following theorem.

**Theorem 2.** Let \( \rho \) and \( t < \bar{t}_c(\rho) \) be given. Then, for all \( \epsilon > 0 \), there exists a constant \( \alpha = \alpha(\epsilon, \rho, t) \) such that

\[
\theta_n(t) \leq \exp \left( -\alpha n^{1-\epsilon} \right), \quad \forall n \geq 1.
\]

In the next section we will prove Theorem 1 and Theorem 2. The proof of Theorem 1 is obtained by showing a sharp phase transition for an intermediate model. The proof of Theorem 2 consists of an application of a Simon-Lieb type inequality and a bootstrap argument.

## 2 Proofs

### 2.1 Proof of Theorem 2

In the proof of Theorem 2 we will apply a Simon-Lieb type inequality to boxes of several length scales. Let \( L_n^k = \lfloor n^{\frac{k}{k+1}} \rfloor \), \( k = 1, 2, \ldots \), and observe that if the origin is connected by an open path to \( \partial B(n) \), then the origin must be connected by an open path to \( \partial B(\lfloor \frac{1}{2} L_n^k \rfloor) \) and there must exist a vertex \( w \in \partial B(L_n^k) \) such that \( w \) is connected to \( \partial B(n) \) by an open path using edges on the complement of \( B(L_n^k) \) only. The main difficulty here is to control the scales \( L_n^k \) in order to balance the decay of connectivity and the decay of correlations between events whose occurrence depends only on the state of edges inside \( B(\lfloor \frac{1}{2} L_n^k \rfloor) \) and those depending on the state of edges outside \( B(L_n^k) \). A bootstrap argument will be useful to attain such control of scales. We observe that the decay of correlations obtained in Theorem 2 of [10] is no longer enough here, and we derive a new decay rate which is improved by a \( \log n \) factor.

In what follows, the notation \( \{ w \leftarrow A \rightarrow B \} \) means that the vertex \( w \) is connected to some vertex in \( B \) using only edges with both endpoints in \( A \). All constants \( c_1, \ldots, c_6 \) appearing in this section are universal and do not depend on \( n \) or \( t \).
Proof of Theorem 2. Fix $\rho > m$. By translation invariance we have

$$\mathbb{P}_{\rho,t} \left( 0 \leftrightarrow B(m), w \xrightarrow{B(2m)^c} \partial B(n) \right) \leq \mathbb{P}_{\rho,t} \left( 0 \leftrightarrow B(m) \right) \mathbb{P}_{\rho,t} \left( w \leftrightarrow \partial B(n) \right) + c_1 m \exp \left( -\frac{1}{2} m \log m \right).$$

Proof. We follow the argument in Section 2.1 of [1]. Fix $t > 0$ and let $M_t(x)$ be the set of vertices $y$ such that there is a path $x, e_1, x, e_2, x, \ldots, e_k, y$ with $t > U_{e_1} > U_{e_2} > \cdots > U_{e_k}$. This gives, with the aid of Stirling’s formula,

$$\mathbb{P}_{\rho,t}(M_t(x) \cap \{ x + \partial B(m) \} \neq \emptyset) \leq \frac{4.3^{m-1}}{m!} \leq \frac{1}{2} \exp \left( -m \log \left( \frac{m}{3e} \right) \right). \quad (1)$$

Write $M_t(B(n)) = \bigcup_{x \in \partial B(n)} M_t(x)$ and let $w \in \partial B(2m)$. Union bound and (1) yields

$$\mathbb{P}_{\rho,t}(M_t(B(m)) \cap M_t(w) \neq \emptyset) \leq 2 \mathbb{P}_{\rho,t}(M_t(B(m)) \cap \partial S([3m/2]) \neq \emptyset) \leq c_2 m \exp \left( -\frac{1}{2} m \log m \right),$$

for $m > 9e^2$. Note that on the event $\{ M_t(B(m)) \cap M_t(w) = \emptyset \}$, the events $\{ 0 \leftrightarrow \partial B(m) \}$ and $\{ w \xrightarrow{B(2m)^c} \partial B(n) \}$ are determined by disjoint sets of edges, and therefore we obtain

$$\text{Cov} \left( \mathbb{1}\{ 0 \leftrightarrow \partial B(m) \}, \mathbb{1}\{ w \xrightarrow{B(2m)^c} \partial B(n) \} \right) \leq c_1 m \exp \left( -\frac{1}{2} m \log m \right).$$

The proof follows by observing that

$$\mathbb{P}_{\rho,t} \left( 0 \leftrightarrow \partial B(m), w \xrightarrow{B(2m)^c} \partial B(n) \right) \leq \mathbb{P}_{\rho,t} \left( 0 \leftrightarrow \partial B(m) \right) \mathbb{P}_{\rho,t} \left( w \leftrightarrow \partial B(n) \right) + \text{Cov} \left( \mathbb{1}\{ 0 \leftrightarrow \partial B(m) \}, \mathbb{1}\{ w \xrightarrow{B(2m)^c} \partial B(n) \} \right).$$

Proof of Theorem 2. Fix $\rho, t < \ell_c(\rho)$, and write $\theta_n(t) \equiv \theta_n$. Following the discussion at the beginning of this section, let us consider boxes of side length $\sqrt{n}$. Proposition 1 gives

$$\theta_n \leq \theta_{\left[ \frac{1}{2} \sqrt{n} \right]} \left( \sum_{w \in \partial B(\lfloor \sqrt{n} \rfloor)} \mathbb{P}_{\rho,t}(w \leftrightarrow \partial B(n)) \right) + c_3 \lfloor \sqrt{n} \rfloor \exp \left( -\frac{1}{2} \lfloor \sqrt{n} \rfloor \log \lfloor \sqrt{n} \rfloor \right). \quad (2)$$

By translation invariance we have

$$\mathbb{P}_{\rho,t}(w \leftrightarrow \partial B(n)) \leq \theta_{n-\lfloor \sqrt{n} \rfloor}.$$
for any $w \in \partial B(\lfloor \sqrt{n} \rfloor)$. Hence

$$\theta_n \leq 8 \lfloor \sqrt{n} \rfloor \theta_{\lfloor \sqrt{n} \rfloor} \theta_{n-\lfloor \sqrt{n} \rfloor} + c_3 \lfloor \sqrt{n} \rfloor \exp \left( -\frac{1}{2} \lfloor \sqrt{n} \rfloor \log \lfloor \sqrt{n} \rfloor \right).$$

Iterating the above $\lfloor \sqrt{n} \rfloor$ times and using the same argument for $\theta_{n-j\lfloor \sqrt{n} \rfloor}$, $j \in \{1, 2, \ldots, \lfloor \sqrt{n} \rfloor\}$, we obtain

$$\theta_n \leq \left( 8 \lfloor \sqrt{n} \rfloor \theta_{\lfloor \sqrt{n} \rfloor} \right)^{\lfloor \sqrt{n} \rfloor} + c_3 \lfloor \sqrt{n} \rfloor \exp \left( -\frac{1}{2} \lfloor \sqrt{n} \rfloor \log \lfloor \sqrt{n} \rfloor \right) \sum_{i=0}^{\lfloor \sqrt{n} \rfloor-1} \left( 8 \lfloor \sqrt{n} \rfloor \theta_{\lfloor \sqrt{n} \rfloor} \right)^i. \quad (3)$$

It is easy to see that, if $\mathbb{E}_{\rho,t}(|C|) < \infty$, then $\sum_{n \geq 1} \theta_n(t) < \infty$. Since $\{\theta_n(t)\}_n$ is decreasing, an exercise in analysis gives $\lim_{n \to \infty} n\theta_n(t) = 0$. Hence we can find some $n_0 \in \mathbb{N}$ such that

$$8 \lfloor \sqrt{n} \rfloor \theta_{\lfloor \sqrt{n} \rfloor} < e^{-2},$$

for all $n \geq n_0$. This gives

$$\theta_n \leq \exp \left( -\sqrt{n} \right) + c_4 \lfloor \sqrt{n} \rfloor \exp \left( -\frac{1}{2} \lfloor \sqrt{n} \rfloor \log \lfloor \sqrt{n} \rfloor \right),$$

for all $n \geq n_0$. Note that

$$c_4 \lfloor \sqrt{n} \rfloor \exp \left( -\frac{1}{2} \lfloor \sqrt{n} \rfloor \log \lfloor \sqrt{n} \rfloor \right) = \exp \left\{ - \left\lfloor \frac{1}{2} \log \lfloor \sqrt{n} \rfloor \right\rfloor + \frac{(\log \lfloor \sqrt{n} \rfloor - 1) \lfloor \sqrt{n} \rfloor}{2} - \log \left( c_4 \lfloor \sqrt{n} \rfloor \right) \right\} \leq \exp \left( -\frac{1}{4} \sqrt{n} \right),$$

for all $n$ large enough, and hence

$$\theta_n \leq 2 \exp \left( -\frac{1}{4} \sqrt{n} \right). \quad (4)$$

Now we implement a bootstrap argument to obtain better bounds. We will successively apply the construction in [2] with boxes of radius $\lfloor \frac{1}{2} \log n \rfloor$ and $\lfloor n \rfloor$, and obtain a sequence of natural numbers $\{n_k\}_{k \geq 1}$ such that, for fixed $k \in \mathbb{N},$

$$\theta_n \leq 2 \exp \left[ -\frac{1}{10} n \log n \right], \quad \forall n \geq n_k. \quad (5)$$

Plugging the bound $(4)$ in $(3)$ we obtain

$$\theta_n \leq \left[ 16 \lfloor \sqrt{n} \rfloor \exp \left( -\frac{1}{2} \lfloor \sqrt{n} \rfloor \right) \right]^{\lfloor \sqrt{n} \rfloor} + c_5 \lfloor \sqrt{n} \rfloor \exp \left( -\frac{1}{2} \lfloor \sqrt{n} \rfloor \log \lfloor \sqrt{n} \rfloor \right)$$

$$\leq \exp \left\{ - \left( \frac{1}{4} n^{\frac{1}{2}} - n^{\frac{1}{2}} \log (16 \lfloor n^{\frac{1}{2}} \rfloor) \right) \right\} + \exp \left\{ - \left( \frac{1}{2} \lfloor \sqrt{n} \rfloor \log \lfloor \sqrt{n} \rfloor - \log \left( c_5 \lfloor \sqrt{n} \rfloor \right) \right) \right\}$$

$$\leq 2 \exp \left\{ - \left( \frac{1}{2} \lfloor \sqrt{n} \rfloor + \frac{1}{2} \left( \log \lfloor \sqrt{n} \rfloor - 1 \right) \right) \right\} \leq 2 \exp \left( -\frac{1}{10} n \log n \right),$$

for all $n$ sufficiently large, say, for all $n \geq n_1$. 

5
Assume we have found a sequence \( n_1, n_2, \ldots, n_{\ell-1} \) such that \([5]\) is true. The same construction as in \([2]\) with boxes of radius \([16 n^{\frac{3}{r+1}}]\) and \([6 n^{\frac{1}{5}}]\) yields

\[
\theta_n \leq \left[16 |n^{\frac{3}{r+1}}| \exp\left(-\frac{1}{4} |n^{\frac{1}{5}}| \log |n^{\frac{1}{5}}|\right)\right]|n^{\frac{1}{5}}|
\]

\[
+ c_6 |n^{\frac{1}{5}}| \exp\left(-\frac{1}{2} |n^{\frac{1}{5}}| \log |n^{\frac{1}{5}}|\right)
\]

\[
\leq \exp\left\{-\left(\frac{1}{2} |n^{\frac{1}{5}}| \log |n^{\frac{1}{5}}| - |n^{\frac{1}{5}}| \log(16 |n^{\frac{3}{r+1}}|)\right)\right\}
\]

\[
+ \exp\left\{-\left(\frac{1}{2} |n^{\frac{1}{5}}| \log |n^{\frac{1}{5}}| - |n^{\frac{1}{5}}| \log (c_6 |n^{\frac{1}{5}}|)\right)\right\}
\]

\[
\leq 2 \exp\left\{-\left(\frac{1}{2} |n^{\frac{1}{5}}| + \left(\log(16 |n^{\frac{3}{r+1}}|) - 1\right)\frac{1}{2} |n^{\frac{1}{5}}| - \log (c_6 |n^{\frac{1}{5}}|)\right)\right\}.
\]

It is clear that one can choose some \( n_\ell \) large enough such that

\[
\theta_n \leq 2 \exp\left[-\frac{1}{16} n^{\frac{2}{r+1}} \log n^{\frac{1}{5}}\right], \quad \forall n \geq n_\ell.
\]

(6)

Given \( \epsilon > 0 \), choose \( k^* \) such that \( \frac{k^*}{r+1} > 1 - \epsilon \). By (6), there is \( n_{k^*} \in \mathbb{N} \) such that

\[
\theta_n \leq 2 \exp\left(-\frac{1}{16} n^{\frac{k^*}{r+1}} \log n^{\frac{1}{5}}\right) \leq \exp\left(-\frac{1}{16} n^{1-\epsilon}\right), \quad \forall n \geq n_{k^*}.
\]

\[
\square
\]

### 2.2 Proof of Theorem 1

In this section we prove Theorem 1. The idea of the proof is to construct an intermediate model that dominates the CDPRE process when \( \rho_0 = 0 \) and is dominated by independent Bernoulli percolation. We will show that the intermediate model phase transition is sharp, which will give us the desired result.

Let \( \Lambda = \{(x_1, x_2) \in \mathbb{Z}^2 : x_1 = 0, 1, 2, 3, 4, 5 \text{ and } x_2 = 0, 1, 2, 3, 4, 5\} \) and \( \overline{\Lambda} = \{(x_1, x_2) \in \mathbb{Z}^2 : x_1 = 1, 2, 3, 4, 5 \text{ and } x_2 = 1, 2, 3\} \). For each \((r, s) \in \mathbb{Z}^2\), define \( \Lambda_{r,s} = \Lambda + (6r, 5s) \) and \( \overline{\Lambda}_{r,s} = \overline{\Lambda} + (6r, 5s) \).

Consider the following sets of edges in \( \mathcal{E}(\Lambda_{r,s}) \):

\[
g_{r,s} = (\{6r + 2, 5s + 2\}, \{6r + 3, 5s + 2\}),
\]

\[
A_{r,s} = \{e \in \mathcal{E}(\overline{\Lambda}_{r,s}) : |e \cap \partial \overline{\Lambda}_{r,s}| = 1\}.
\]

The intermediate model is constructed as follows: let \( \{U_e\}_{e \in \mathcal{E}} \) be an independent collection of uniform random variables on \([0,1]\) with corresponding product measure \( \mathbb{P} \) and define the event

\[
C_{r,s} = \left\{ U \in [0,1]^\mathcal{E} : \max_{e \in A_{r,s}} U_e < \min_{e \in \mathcal{E}(\Lambda_{r,s}) \setminus A_{r,s}} U_e \right\}.
\]

See Figure 1 for a sketch of the boxes and edges involved in the construction. A configuration of the
Figure 1: \( \Lambda_{r,s} \) (larger box), \( \overline{\Lambda}_{r,s} \) (in gray) and the edge \( g_{r,s} \). \( A_{r,s} \) consists of the dashed edges.

The intermediate model is a function

\[
\tilde{\omega}_t : [0,1]^E \rightarrow \{0,1\}^E
\]

such that

\[
\tilde{\omega}_t(e) = \begin{cases} 
1_{\{U_e \leq t\}} & \text{if } e \notin \cup_{r,s} g_{r,s}, \\
1_{\{U_e \leq t\}}1_{\mathcal{C}_{r,s}} & \text{if } e = g_{r,s}.
\end{cases}
\]

Write \( \check{t}_c \) and \( \tilde{t}_c \) for the susceptibility critical thresholds (the supremum of \( t \in [0,1] \) such that the mean size of the open cluster is finite a.s.) of Bernoulli percolation and the intermediate model, respectively. Note that \( \tilde{\omega}_t(e) \) can be obtained through a standard coupling (using the same variables \( U_e \)) with the CDPRE model. In particular, we obtain \( \tilde{\omega}_t(e) \geq \omega_t(e) \) for all \( t \in [0,1] \) and for all \( e \in \mathcal{E} \), whenever \( \rho_0 = 0 \). Also, if we denote the independent Bernoulli configuration of parameter \( t \) by \( \hat{\omega}_t \), then \( \tilde{\omega}_t \) is an essential diminishment of the configuration \( \hat{\omega}_t \) (see Section 2.1.1 in [2]). Suppose that the intermediate model phase transition is sharp. Then, we have the inequality

\[
\frac{1}{2} = \check{t}_c < \tilde{t}_c.
\]

Since the CDPRE model is dominated by the intermediate model, this gives us

\[
\frac{1}{2} < \tilde{t}_c \leq \bar{t}_c(\rho),
\]

for all \( \rho \) with \( \rho_0 = 0 \). We will prove sharpness of the phase transition for the intermediate model with an application of the OSSS inequality for boolean functions and a suitable randomized algorithm.

**Remark 1.** The fact that the intermediate model phase transition is sharp also gives us the existence of some constant \( \eta > \frac{1}{2} \) such that the radius of the open cluster of the origin decays exponentially fast for all \( t \in (0, \eta) \).

Let us introduce further notation. Assume \( I \) is a countable set, and let \( (\Omega^I, \pi^\otimes I) \) be a product probability space with elements denoted by \( \omega = (\omega_i)_{i \in I} \). Consider a boolean function \( f : \Omega^I \rightarrow \{0,1\} \).
An algorithm \( T \) determining \( f \) takes a configuration \( \omega \) as an input, and reveals the value of \( \omega \) in different coordinates, one by one. At each step, the next coordinate to be revealed depends on the values of \( \omega \) revealed so far. This process keeps going until the value of \( f \) is the same no matter the values of \( \omega \) on the remaining coordinates. For a formal description of a randomized algorithm we refer the reader to \([9]\).

Denote by \( \delta_i(T) \) and \( \text{Inf}_i(f) \) the revealment and the influence of the \( i \)-th coordinate, respectively, which are defined by

\[
\delta_i(T) := \pi \otimes I(T \text{ reveals the value of } \omega_i),
\]

\[
\text{Inf}_i(f) := \pi \otimes I(f(\omega) \neq f(\tilde{\omega})),
\]

where \( \tilde{\omega} \) is equal to \( \omega \) in every coordinate, except the \( i \)-th coordinate which is resampled independently.

The OSSS inequality, introduced in \([9]\) by O’Donnel, Saks, Schramm and Servedio, gives a bound on the variance of \( f \) in terms of the influence and the computational complexity of an algorithm for this function. It says that, for any function \( f : \Omega \rightarrow \{0, 1\} \) and any algorithm \( T \) determining \( f \),

\[
\text{Var}(f) \leq \sum_{i \in I} \delta_i(T) \text{Inf}_i(f).
\] (7)

The intermediate model is a 5-dependent percolation process and the OSSS inequality can not be directly applied. To overcome this difficulty, we introduce a suitable product space to encode the measure of the intermediate model. We take \( \Omega = [0, 1], I = \mathcal{E} \) and \( \pi \otimes I = P \). Writing \( \mathcal{B}_n = \{0 \leftrightarrow \partial B(n)\} \), we are interested in bounding the variance of the boolean function \( 1_{\tilde{\omega}_t^{-1}(\mathcal{B}_n)} \) considered as a function from \([0, 1]^\mathcal{E}\) onto \( \{0, 1\} \).

### 2.2.1 Bound on the revealment

Denote by \( \tilde{P}_t \) the law of the intermediate model, that is,

\[
\tilde{P}_t(A) = P(U : \tilde{\omega}_t(U) \in A),
\]

for all \( A \subset \{0, 1\}^\mathcal{E} \), and write \( \tilde{\theta}_n(t) = \tilde{P}_t(\mathcal{B}_n) \) and \( S_n(t) = \sum_{k=1}^n \tilde{\theta}_k(t) \). The next lemma shows the existence of an algorithm determining the boolean function \( 1_{\tilde{\omega}_t^{-1}(\mathcal{B}_n)} \) and gives an upper bound on its revealment.

**Lemma 1.** For each \( 1 \leq k \leq n \), there exists an algorithm \( T_k \) determining \( 1_{\tilde{\omega}_t^{-1}(\mathcal{B}_n)} \) such that, for each \( e \in \mathcal{E} \),

\[
\sum_{k=1}^n \delta_e(T_k) \leq 4\beta S_n(t),
\] (8)

for some universal constant \( \beta > 0 \).

To introduce the algorithm \( T_k \), let \( I_n \) be the set of edges \( g_{r,s} \) such that \( \Lambda_{r,s} \cap B(n) \neq \emptyset \). For each \( g = g_{r,s} \in I_n \), define an auxiliary algorithm \( \text{Determine}(g) \), which reveals the random variables \( U_e \) for all \( e \in \mathcal{E}(\Lambda_{r,s}) \cup \partial^c \Lambda_{r,s} \).
Definition 1 (Algorithm $T_k$). Let $\Lambda_1, \Lambda_2, \ldots$ be a fixed ordering of the edges $g_{r,s} \in I_n$. The algorithm $T_k$ is defined as follows. Set $D_0 = \emptyset$ and $Z_0 = \partial B(k)$. Assume $D_m \subset \bigcup_{g_{r,s} \in I_n} g_{r,s}$ and $Z_m \subset \mathbb{Z}^2$ are given. If there is no $g_{r,s} \in I_n \setminus D_m$ such that $\Lambda_{r,s} \cap Z_m \neq \emptyset$, the algorithm stops. If there is such $g_{r,s}$, the algorithm chooses the earliest one according to the fixed ordering and proceeds as follows:

- run $\text{Determine}(g_{r,s}),$
- $D_{m+1} = D_m \cup g_{r,s},$
- $Z_{m+1} = Z_m \cup \{\text{all vertices incident to } \partial \Lambda_{r,s} \text{ and connected to } Z_m \text{ by a path of } t\text{-open edges in } \mathcal{E}(\Lambda_{r,s}) \cup \partial \mathcal{E}(\Lambda_{r,s})\}.$

We are ready to prove Lemma 1.

Proof of Lemma 1: First, note that the algorithm $T_k$ discovers the union of all open components of $\partial B(k)$ at time $t$, in particular it determines the function $\mathbb{1}_{\omega_k^{-1}(\emptyset_n)}$. Let us bound the revealment of $T_k$. Write $\Lambda_{r,s}(e)$ for the element of $\{\Lambda_{r,s} : (r, s) \in I_n\}$ with the property that $e \in \mathcal{E}(\Lambda_{r,s})$ (if $e \in \mathcal{E}(\Lambda_{r,s})$ for no $(r, s) \in I_n$, then we abuse notation and set $\Lambda_{r,s}(e)$ to be the union of the two boxes which contain exactly one end-vertex of $e$). Observe that if $e \in B(n)$ is revealed, then there is a path of $t\text{-open edges connecting } \partial B(k) \text{ to } \partial \Lambda_{r,s}(e)$. Hence

$$\sum_{k=1}^{n} \delta_t(T_k) \leq 2 \sum_{k=1}^{n} \bar{P}_t(\partial \Lambda_{r,s}(e) \leftrightarrow \partial B(k)) \leq 2 \sum_{k=1}^{n} \sum_{v \in \partial \Lambda_{r,s}(e)} \bar{P}_t(v \leftrightarrow \partial B(k))$$

$$\leq 2 \sum_{k=1}^{n} \sum_{v \in \partial \Lambda_{r,s}(e)} \bar{P}_t(v \leftrightarrow \partial B(v, d(v, \partial B(k)))) \leq 4\beta \sum_{k=1}^{n} \bar{\theta}_k(t).$$

Last inequality follows by translation invariance, and the proof is complete.

2.2.2 A Russo type formula

Let $F$ be an event that depends only on the state of edges with both endpoints in a finite set $\Gamma \subset \mathbb{Z}^2$. Such event is called a $\Gamma$-local event. We have the following Russo’s type formula.

Lemma 2. Let $F$ be an increasing $\Gamma$-local event and $0 < \alpha_1 < \alpha_2 < 1$. There exists a constant $q > 0$ such that, for all $t \in [\alpha_1, \alpha_2],$

$$\frac{d}{dt} \bar{P}_t(F) \geq q \sum_{e \in \mathcal{E}^{+}(\Gamma)} \bar{P}_t(e \text{ is pivotal for } F).$$

Proof. Let $\delta > 0$. For an increasing event $F$,

$$\bar{P}_{t+\delta}(F) - \bar{P}_t(F) = \mathbb{P}(\bar{\omega}_{t+\delta} \in F, \bar{\omega}_t \notin F)$$

$$= \mathbb{P}(\bar{\omega}_{t+\delta} \in F, \bar{\omega}_t \notin F, \exists e \in \mathcal{E}^{+}(\Gamma) \text{ s.t. } t < U_e \leq t + \delta). \quad (9)$$
Let $W_{t,\delta}$ be the random set of edges $f$ such that $t < U_f \leq t + \delta$. Clearly,

$$P(|W_{t,\delta}| \geq 2) = o(\delta). \quad \text{(10)}$$

From (9) and (10) we obtain

$$\bar{P}_{t+\delta}(F) - \bar{P}_t(F) = P(\bar{\omega}_{t+\delta} \in F, \bar{\omega}_t \notin F, |W_{t,\delta}| = 1) + o(\delta) = \sum_{e \in \mathcal{E}(t)} P(\bar{\omega}_{t+\delta} \in F, \bar{\omega}_t \notin F, W_{t,\delta} = \{e\}) + o(\delta).$$

We now consider three cases. Remember that $\mathcal{E}(\Lambda_{r,s}) = \{g_{r,s}\} \cup A_{r,s} \cup B_{r,s}$. First, let $e \in \mathcal{E}(\Gamma) - \bigcup_{r,s} \mathcal{E}(\Lambda_{r,s})$. Then,

$$P(\bar{\omega}_{t+\delta} \in F, \bar{\omega}_t \notin F, W_{t,\delta} = \{e\}) = P(\text{e is pivotal for } F \text{ em } \bar{\omega}_t, W_{t,\delta} = \{e\}) = \delta \times P(\text{e is pivotal for } F \text{ em } \bar{\omega}_t) = \delta \times \bar{P}_t(\text{e is pivotal for } F).$$

Now let $e = g_{r,s} = \langle a, b \rangle$ for some pair $(r, s)$. Write $u_1, u_2 \in \mathbb{R}^2$ for the canonical basis of $\mathbb{R}^2$ and consider the event $X = \{U_{a-u_1, b-u_1} > U_{a+u_2, b+u_2}\} \subset C_{r,s}$. This gives the inclusion

$$\{\bar{\omega}_{t+\delta} \in F, \bar{\omega}_t \notin F, W_{t,\delta} = \{e\}\} \supset \{X, e \text{ is pivotal for } F \text{ in } \bar{\omega}_t, W_{t,\delta} = \{e\}\}.$$

Note that the event $X \cap \{e \text{ is pivotal for } F \text{ in } \bar{\omega}_t\}$ depends only on the variables $U_f$ with $f \neq g_{r,s}$. Hence

$$P(X, e \text{ is pivotal for } F \text{ in } \bar{\omega}_t, W_{t,\delta} = \{e\}) = P(X, e \text{ is pivotal for } F \text{ in } \bar{\omega}_t) P(W_{t,\delta} = \{e\}).$$

Since $P(X | e \text{ is pivotal for } F \text{ in } \bar{\omega}_t) > 0$ for all $t \in [\alpha_1, \alpha_2]$, and $P$ is continuous for local events, Weierstrass Theorem implies the existence of a constant $M_1 > 0$ such that

$$P(X, e \text{ is pivotal for } F \text{ in } \bar{\omega}_t, W_{t,\delta} = \{e\}) \geq M_1 \delta \times \bar{P}_t(\text{e is pivotal for } F).$$

Finally, let $e \in A_{r,s} \cup B_{r,s}$ and denote $Y = \{U_{g_{r,s}} > t\}$. Note that

$$\{\bar{\omega}_{t+\delta} \in F, \bar{\omega}_t \notin F, W_{t,\delta} = \{e\}\} = \{e \text{ is pivotal for } F \text{ in } \bar{\omega}_t, W_{t,\delta} = \{e\}\} \supset \{Y, e \text{ is pivotal for } F \text{ in } \bar{\omega}_t, W_{t,\delta} = \{e\}\}.$$

Note that the event $Y \cap \{e \text{ is pivotal for } F \text{ in } \bar{\omega}_t\}$ depends only on the variables $U_f$ with $f \neq e$. 

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Therefore, as in the previous case, there exists a constant $M_2 > 0$ such that

$$\mathbb{P}(Y, e \text{ is pivotal for } F \text{ in } \tilde{\omega}_t, W_{t, \delta} = \{e\}) = \mathbb{P}(Y, e \text{ is pivotal for } F \text{ in } \tilde{\omega}_t)\mathbb{P}(W_{t, \delta} = \{e\}) \geq M_2 \delta \times \tilde{P}_t(e \text{ is pivotal for } F).$$

Taking $q = \min\{M_1, M_2\}$ we obtain

$$\tilde{P}_{t+\delta}(F) - \tilde{P}_t(F) \geq \delta q \sum_{e \in \mathcal{E}(\Gamma)} \tilde{P}_t(e \text{ is pivotal for } F) + o(\delta).$$

The result follows by dividing both sides by $\delta$ and taking the limit when $\delta$ goes to zero.

2.2.3 A bound on the influences

We now seek for a bound on the influence of an edge $e \in \mathcal{E}(B(n))$ on $\mathbb{1}_{\mathcal{B}_n}$, that is, we seek for a bound on

$$\text{Inf}_e(\mathbb{1}_{\mathcal{B}_n}) := \mathbb{P}(U: \mathbb{1}_{\mathcal{B}_n}(\tilde{\omega}_t(U)) \neq \mathbb{1}_{\mathcal{B}_n}(\tilde{\omega}_t(U^*)))),$$

where $U$ is equal to $U^*$ in every edge, except edge $e$ which is resampled independently. We do this in two steps. First, assume $e \in \mathcal{E}(\Gamma) - \bigcup_{r,s} \mathcal{E}(\Lambda_{r,s})$ or $e = g_{r,s}$ for some pair $(r,s)$. In this case we have

$$\text{Inf}_e(\mathbb{1}_{\mathcal{B}_n}) = \tilde{P}_t(e \text{ is pivotal for } \mathcal{B}_n).$$

Now let $e \in \mathcal{E}(\Lambda_{r,s}) \setminus g_{r,s}$. In this case,

$$\text{Inf}_e(\mathbb{1}_{\mathcal{B}_n}) = \mathbb{P}(U: \mathbb{1}_{\mathcal{B}_n}(\tilde{\omega}_t(U)) \neq \mathbb{1}_{\mathcal{B}_n}(\tilde{\omega}_t(U^*))), U_{g_{r,s}} > t) + \mathbb{P}(U: \mathbb{1}_{\mathcal{B}_n}(\tilde{\omega}_t(U)) \neq \mathbb{1}_{\mathcal{B}_n}(\tilde{\omega}_t(U^*))), U_{g_{r,s}} \leq t).$$

If $U_{g_{r,s}} > t$ and the indicator of $\mathcal{B}_n$ is changed, then $e$ must be pivotal for $\mathcal{B}_n$. If $U_{g_{r,s}} \leq t$ and the indicator of $\mathcal{B}_n$ is changed, then either $e$ or $g_{r,s}$ must be pivotal for $\mathcal{B}_n$. Putting all together, we obtain

$$\sum_{e \in B(n)} \text{Inf}_e(\mathbb{1}_{\mathcal{B}_n}) \leq \gamma \sum_{e \in B(n)} \tilde{P}_t(e \text{ is pivotal for } \mathcal{B}_n),$$

for some constant $\gamma > 0$.

Let $t^*_c$ denote the percolation critical threshold for the intermediate model. By stochastic dominance and the results of [10] we know that $1/2 < t^*_c < 1$. We now prove that the intermediate model undergoes a sharp phase transition, a fact from which Theorem 1 is a corollary.

**Theorem 3.** Consider the intermediate model on $\mathbb{Z}^2$.

1. For $t < t^*_c$, there exists $c_t > 0$ such that for all $n \geq 1$, $\bar{\theta}_n(t) \leq \exp(-c_t n)$.

2. There exists $c > 0$ such that for $t > t^*_c$, $\tilde{P}_t(0 \longleftrightarrow \infty) \geq c(t - t^*_c)$. 

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Proof. Applying the OSSS inequality \([7]\) for each \(k\) and then summing on \(k\), Lemma \([1]\) gives

\[
\widetilde{\theta}_n(t)(1 - \widetilde{\theta}_n(t)) \leq \frac{4\beta S_n(t)}{n} \sum_{e \in B(n)} \text{Inf}_e(1_{\mathcal{G}_n}).
\]

Equation \([11]\) and Lemma \([2]\) give

\[
\sum_{e \in \partial(B(n))} \text{Inf}_e(1_{\mathcal{G}_n}) \leq \gamma q^{-1} \frac{d}{dt} \widetilde{\theta}_n(t).
\]

Hence, there is a constant \(\nu > 0\) such that

\[
\frac{d}{dt} \widetilde{\theta}_n(t) \geq \frac{\nu n}{S_t(n)} \widetilde{\theta}_n(t)(1 - \widetilde{\theta}_n(t)).
\]

Fix \(t_0 \in (t^*_c, \alpha_2)\). Since \(\widetilde{\theta}_n(t)\) is increasing in \(t\), we have \(1 - \theta_n(t) \geq 1 - \theta_1(t_0)\) for all \(t \leq t_0\). The result follows with an application of Lemma 3 in \([3]\) to the function \(f_n = \frac{\nu \theta_n(t)}{(1 - \theta_1(t_0)))}. \)

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