Generalised neo-pseudo projective recurrent Finsler space

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Abstract

Objectives: The purpose of this paper is to obtain several results in the field of generalised neo-pseudo projective recurrent Finsler space. Methods: A generalization technique is employed to solve the resulting problem. We provide its application in the study of space-time. Findings: In section 1, we have defined and studied some of the basic and useful results for later work. Section 2 deals for the neo-pseudo projective recurrent curvature tensor. The notion of neo-pseudo projective recurrent space of second order has been delineated in the section 3. In the section 4 we have studied the generalised neo-pseudo projective recurrent space and established several new results.

Novelty/Conclusion: In this paper we have studied some recurrent properties of neo-pseudo projective curvature tensor in a Finsler space. We have obtained several new results which are as follows:

- If the space Fⁿ admits a neo-pseudo projective curvature tensor Q_{βγδ}^{α}, then Q_{βγδ}^{α} is skew-symmetric with regard to last two indices.
- If the neo-pseudo projective deviation tensor Q_{β}^{α} and pseudo deviation tensor field T_{β}^{α} coincides to each other for q = 1 then space is W-flat.
- If Fⁿ admits the projectively flat Q-recurrent space then the relation \nabla_β Q_γ^ρ + \nabla_γ Q_β^ρ = 0 holds good.
- If a Finsler space Fⁿ admits projectively flat Q-birecurrent space then the relation K_{βγδ} Q_γ^α + K_{βρδ} Q_ρ^α + K_{ργδ} Q_γ^ρ = 0 holds good.
- If the space is Q-birecurrent then the generalised Q-recurrent space is Q-symmetric.
- For the projective flat generalised Q-recurrent space the relation \nabla_β \nabla_γ Q_ρ^α + \nabla_γ \nabla_δ Q_ρ^α + \nabla_ρ \nabla_δ Q_γ^α = 0 holds good.

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1 Introduction

Let $F^n$ be an $n$-dimensional Finsler space with a positive definite metric $g_{\alpha \beta}$, which admit a projective deviation tensor field $W^\alpha_{\beta}$ and pseudo deviation tensor field $T^\alpha_{\beta}$ satisfying

$$Q^\alpha_{\beta} = p W^\alpha_{\beta} + q T^\alpha_{\beta} \quad (1.1)$$

where in $p$ and $q$ are scalars which are positively homogenous of degree zero in $\dot{x}^\alpha$.

Prof. U.P. Singh and Prof. A.K. Singh while developing the theory of neo-pseudo projective curvature tensor, obtain two kinds of curvature tensor $Q^\alpha_{\beta \gamma}$ and $Q^\alpha_{\beta \gamma \delta}$ \(^{(1)}\). With a view to defining the projective deviation tensor field and pseudo deviation tensor field, he constructed the quantities $Q^\alpha_{\beta}(x, \dot{x})$ which behave like neo-pseudo projective deviation tensor.

With the help of tensor $Q^\alpha_{\beta}(x, \dot{x})$ the absolute differential of concerning vector referred to the scalar function $Q(x, \dot{x})$ is defined as follows \(^{(1–3)}\):

$$Q^\alpha_{\beta \gamma} = \frac{1}{3} (\dot{x}^\beta Q^\alpha_{\gamma} - \dot{x}^\gamma Q^\alpha_{\beta}) \quad (1.2)$$

and

$$Q^\alpha_{\beta \gamma \delta} = \dot{x}^\beta Q^\alpha_{\gamma \delta} \quad (1.3)$$

It is easy to verify that the neo-pseudo projective curvature tensor satisfies the following relations \(^{(1,4)}\):

$$Q^\alpha_{\beta \gamma} + Q^\alpha_{\gamma \delta \beta} + Q^\alpha_{\delta \beta \gamma} = 0 \quad (1.4)$$

$$Q^\alpha_{\beta \gamma} \dot{x}^\beta = Q^\alpha_{\gamma} \quad (1.5)$$

and

$$Q^\alpha_{\beta \gamma \delta} \dot{x}^\beta = Q^\alpha_{\gamma \delta} \quad (1.6)$$

Moreover, these curvature tensor also satisfy the following identities

$$Q^\alpha_{\beta \gamma \delta} \dot{x}^\beta \dot{x}^\gamma = Q^\alpha_{\delta} \quad (1.7)$$

and

$$Q^\alpha_{\beta} \dot{x}^\beta = 0 \quad (1.8)$$

As it is well known, in the Finsler space a scalar function $Q(x, \dot{x})$ is given by

$$Q^\alpha_{\alpha} = (n - 1) q T \quad (1.9)$$

Let us consider a curvature tensor $W^\alpha_{\beta}$ in Finsler space, is termed as projective curvature tensor in the Finsler space and is defined as follows \(^{(2,5,6)}\):

$$W^\alpha_{\beta} = H^\alpha_{\beta} + T^\alpha_{\beta} \quad (1.10)$$

Wherein $H^\alpha_{\beta}$ is positively homogeneous of degree one in $\dot{x}^\alpha$.

In analogy with the relation (1.1) the projective curvature tensors $W^\alpha_{\beta \gamma}$ and $W^\alpha_{\beta \gamma \delta}$ in the Finsler space with the condition $p = q = 1$ may be defined as follows \(^{(2,7)}\):

$$W^\alpha_{\beta \gamma} = H^\alpha_{\beta \gamma} + Q^\alpha_{\beta \gamma} \quad (1.11)$$

$$W^\alpha_{\beta \gamma \delta} = H^\alpha_{\beta \gamma \delta} + Q^\alpha_{\beta \gamma \delta} \quad (1.12)$$
and

\[ W_{\beta\gamma\delta}^\alpha = H_{\beta\gamma\delta}^\alpha + Q_{\beta\gamma\delta}^\alpha \]  

(1.13)

In view of above discussions, we have the following theorems:

**Theorem 1.1:**
For the neo-pseudo projective curvature tensor the relation

\[ Q_{\beta\gamma}^\alpha = -Q_{\gamma\beta}^\alpha \]  

(1.14)

holds good.

**Proof:**
Interchanging \( b \) and \( g \) in equation (1.2) and adding this with new equation, we get the desired result.

**Theorem 1.2:**
If the space \( F^n \) admits a neo-pseudo projective curvature tensor \( Q_{\alpha\beta\gamma}^\delta \) then \( Q_{\alpha\beta\gamma}^\delta \) is skew-symmetric with regard to last two indices.

**Proof:**
Interchanging \( g \) and \( d \) in equation (1.3) and adding the new equation to the equation (1.3), we obtain

\[ Q_{\alpha\beta\gamma}^\delta + Q_{\alpha\beta\delta\gamma} = \delta_\beta Q_{\gamma\delta}^\alpha + \delta_\delta Q_{\gamma\beta}^\alpha \]  

(1.15)

From equations (1.14) and (1.15), we get

\[ Q_{\alpha\beta\gamma}^\delta = -Q_{\alpha\beta\delta\gamma} \]  

(1.16)

**Theorem 1.3:**
If the neo-pseudo projective deviation tensor \( Q_{\alpha\beta}^\gamma \) coincides with geodesic deviation tensor field \( H_{\alpha\beta}^\gamma \) in the Finsler space \( F^n \) then projective deviation tensor field \( W_{\alpha\beta}^\gamma \) and the neo-pseudo projective deviation tensor \( Q_{\alpha\beta}^\gamma \) are identically equal to each other.

**Proof:**
If the neo-pseudo projective deviation tensor \( Q_{\alpha\beta}^\gamma \) coincides with geodesic deviation tensor field \( H_{\alpha\beta}^\gamma \). Consequently, from equation (1.11) follows

\[ W_{\alpha\beta}^\gamma = Q_{\alpha\beta}^\gamma \]  

(1.17)

Hence projective deviation tensor field \( W_{\alpha\beta}^\gamma \) and the neo-pseudo projective deviation tensor \( Q_{\alpha\beta}^\gamma \) are identically equal to each other.

**Theorem 1.4:**
If the neo-pseudo projective deviation tensor \( Q_{\alpha\beta}^\gamma \) and pseudo deviation tensor field \( T_{\alpha\beta}^\gamma \) coincides to each other for \( q = 1 \) then Finsler space admits the condition \( W_{\alpha\beta}^\gamma = 0 \) i.e. \( W \)-flat.

**Proof:**
Insitng \( q = 1 \) in equation (1.1), we obtain

\[ Q_{\alpha\beta}^\gamma = p \ W_{\alpha\beta}^\gamma + T_{\alpha\beta}^\gamma \]  

(1.18)

If the neo-pseudo projective deviation tensor \( Q_{\alpha\beta}^\gamma \) and pseudo deviation tensor field \( T_{\alpha\beta}^\gamma \) coincides to each other then from equation (1.18) we observe that

\[ W_{\alpha\beta}^\gamma = 0 \]  

(1.19)

This manifests that the space is \( W \)-flat.

**Theorem 1.5:**
If the neo-pseudo projective deviation tensor \( Q_{\alpha\beta}^\gamma \) and projective deviation tensor field \( W_{\alpha\beta}^\gamma \) coincides to each other then the geodesic deviation tensor field \( H_{\alpha\beta}^\gamma \) vanish identically i.e. \( H \)-flat.

**Proof:**
If the neo-pseudo projective deviation tensor $Q_{\beta}^{\alpha}$ and projective deviation tensor field $W_{\beta}^{\alpha}$ coincides to each other then from equation (1.11) follows the result

$$H_{\beta}^{\alpha} = 0$$

(1.20)

Consequently, the space is H-flat.

**Theorem 1.6:**

If the projective deviation tensor field $W_{\beta}^{\alpha}$ and geodesic deviation tensor field $H_{\beta}^{\alpha}$ coincides to each other then the neo-pseudo projective deviation tensor $Q_{\beta}^{\alpha}$ vanish identically i.e. Q-flat.

**Proof:**

If the projective deviation tensor field $W_{\beta}^{\alpha}$ and geodesic deviation tensor field $H_{\beta}^{\alpha}$ coincides to each other. Consequently, from equation (1.11) follows

$$Q_{\beta}^{\alpha} = 0$$

(1.21)

Therefore the space is Q-flat.

## 2 Recurrent neo-pseudo projective curvature tensor in Finsler space

In view of the investigation of Prof. U.P. Singh and Prof. A.K. Singh\(^{(1)}\) we observe that if the neo-pseudo projective deviation tensor $Q_{\beta}^{\alpha}$ is necessarily recurrent then projective deviation tensor and pseudo deviation tensor are proportional to each other.

As a consequence of this follows the result

$$W_{\beta}^{\alpha} = t T_{\beta}^{\alpha}$$

(2.1)

wherein $t$ is a scalar.

As a consequence of equations (1.1) and (2.1), we obtain

$$Q_{\beta}^{\alpha} = s T_{\beta}^{\alpha}$$

(2.2)

wherein $s = pt + q$ is any scalar and positively homogeneous of degree zero in $x^{\alpha}$.

**Definition 2.1:**

A Finsler space whose curvature tensor is recurrent is called Q-recurrent Finsler space.

In view of the definition it follows that for a recurrent space, we have

$$\nabla_{\epsilon} Q_{\beta \gamma \delta}^{\alpha} = R_{\epsilon} Q_{\beta \gamma \delta}^{\alpha}$$

(2.3)

wherein $R_{\epsilon}$ is a non-zero vector termed as the recurrent vector field.

**Definition 2.2:**

An n-dimensional Finsler space $F^n$ is called Q-symmetric when the covariant derivative of curvature tensor is everywhere zero i.e.

$$\nabla_{\epsilon} Q_{\beta \gamma \delta}^{\alpha} = 0$$

(2.4)

**Definition 2.3:**

A Finsler space $F^n$ is said to be Q-flat when its curvature tensor vanishes identically.

As a consequence of this definition follows the result:

$$Q_{\beta \gamma \delta}^{\alpha} = 0$$

(2.5)

Contracting (2.3) with $x^{\beta}$ and use of equation (1.6), we obtain

$$\nabla_{\epsilon} Q_{\gamma \delta}^{\alpha} = R_{\epsilon} Q_{\gamma \delta}^{\alpha}$$

(2.6)

Again contracting (2.6) with $x^{\gamma}$ and making use of equation (1.5), we get

$$\nabla_{\epsilon} Q_{\delta}^{\alpha} = R_{\epsilon} Q_{\delta}^{\alpha}$$

(2.7)
Thus, we have now the following theorem:

**Theorem 2.1:**
For the recurrence vector space $R_\varepsilon$ in the Finsler space $F^n$ there exists the relation

$$\nabla_\rho \nabla_\varepsilon = (\nabla_\rho R_\varepsilon - \nabla_\varepsilon R_\rho).$$

**Proof:**
Differentiating (2.3) covariantly with regard to $x^\rho$, we get

$$\nabla_\rho \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} = (\nabla_\rho R_\varepsilon + R_\varepsilon R_\rho) Q^\alpha_{\beta\gamma\delta}$$

(2.8)

Interchanging $\varepsilon$ and $\delta$ in equation (2.8) and subtracting the new equation from equation (2.8), we obtain

$$\nabla_\rho \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} - \nabla_\varepsilon \nabla_\rho Q^\alpha_{\beta\gamma\delta} = (\nabla_\rho R_\varepsilon - \nabla_\varepsilon R_\rho) Q^\alpha_{\beta\gamma\delta}$$

(2.9)

Contracting (2.9) with $x^\beta x^\gamma$ and use of equation (1.7), we get

$$\nabla_\rho \nabla_\varepsilon Q^\alpha_{\beta\gamma} = (\nabla_\rho R_\varepsilon) Q^\alpha_{\beta\gamma}$$

(2.10)

Contracting $\alpha$ and $\delta$ in equation (2.10) and use of equation (1.9), we obtain

$$\nabla_\rho \nabla_\varepsilon - \nabla_\varepsilon \nabla_\rho (\log T) = (\nabla_\rho R_\varepsilon - \nabla_\varepsilon R_\rho)$$

(2.11)

**Definition 2.4:**
If the neo-pseudo projective curvature tensor $Q^\alpha_{\beta\gamma\delta}$ in the Finsler space $F^n$ satisfies the relation

$$\nabla_\rho \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} = K_{\rho\varepsilon} Q^\alpha_{\beta\gamma\delta}$$

(3.1)

then $F^n$ is termed as $Q$-recurrent with recurrence vector field $R_\varepsilon$.

Consequently, we have a theorem:

**Theorem 2.2:**
If $F^n$ admits the projectively flat $Q$-recurrent space then the relation

$$\nabla_\rho \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} + \nabla_\rho Q^\alpha_{\beta\gamma\delta} + \nabla_\gamma Q^\alpha_{\varepsilon\beta\delta} = 0$$

holds good.

**Proof:**
If the space is projectively flat then from equation (1.12), we have

$$H^\alpha_{\beta\gamma} + Q^\alpha_{\beta\gamma} = 0$$

(2.13)

Differentiating (2.13) covariantly with respect to $x^\varepsilon$, we get

$$\nabla_\varepsilon H^\alpha_{\beta\gamma} + \nabla_\varepsilon Q^\alpha_{\beta\gamma} = 0$$

(2.14)

Taking the cyclic permutation in $\beta$, $\gamma$, $\varepsilon$ and adding, we have

$$\left( \nabla_\varepsilon H^\alpha_{\beta\gamma} + \nabla_\beta H^\alpha_{\varepsilon\gamma} + \nabla_\gamma H^\alpha_{\varepsilon\beta} \right) + \left( \nabla_\varepsilon Q^\alpha_{\beta\gamma} + \nabla_\beta Q^\alpha_{\varepsilon\gamma} + \nabla_\gamma Q^\alpha_{\varepsilon\beta} \right) = 0$$

(2.15)

The first part of equation (2.15) vanishes due to commutation formula (6), equation (6.13), p.128), hence we obtain

$$\nabla_\varepsilon Q^\alpha_{\beta\gamma} + \nabla_\beta Q^\alpha_{\varepsilon\gamma} + \nabla_\gamma Q^\alpha_{\varepsilon\beta} = 0$$

(2.16)

### 3 Neo-Pseudo Projective Recurrent Space:

**Definition 3.1:**
Neo-pseudo projective curvature tensor $Q^\alpha_{\beta\gamma\delta}$ of a Finsler space satisfies the relation

$$\nabla_\rho \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} = K_{\rho\varepsilon} Q^\alpha_{\beta\gamma\delta}$$

(3.1)

wherein $K_{\rho\varepsilon}$ is non-zero recurrent tensor, then it is called neo-pseudo projective recurrent space of second order or briefly a $Q$-birecurrent space.\(^{(3,4)}\)
**Definition 3.2:**
If the covariant derivative of neo-pseudo projective curvature tensor \( Q^a_{\beta\gamma\delta} \) vanishes identically then the space is termed as Q-bisymmetric.

As a consequence of above definition follows the result
\[ \nabla_\rho \nabla_\epsilon Q^a_{\beta\gamma\delta} = 0 \]  (3.2)

In this regard we shall now establish the following theorem:

**Theorem 3.1:**
The necessary and sufficient condition for Finsler space to be Q-bisymmetric space is that the neo-pseudo projective tensor vanishes identically.

**Proof:**
Since neo-pseudo projective tensor vanishes i.e. \( Q^a_{\beta\gamma\delta} = 0 \). Consequently from equation (3.1) it follows that \( \nabla_\rho \nabla_\epsilon Q^a_{\beta\gamma\delta} = 0 \).

This manifests that the space to be Q-bisymmetric.

Conversely, if the space to be Q-bisymmetric then the converse of theorem is immediately proof.

**Remark 3.1:**
It is noteworthy that every Q-recurrent is necessarily Q-birecurrent.

**Theorem 3.2:**
In a Finsler space \( F^n \), the recurrent tensor field \( K_{\epsilon\rho} \) is not symmetric in general.

**Proof:**
Contracting \( \alpha \) and \( \delta \) in equation (3.1) yields
\[ \nabla_\rho \nabla_\epsilon Q^a_{\beta\gamma} = K_{\epsilon\rho} Q_{\beta\gamma} \]  (3.3)

Interchanging \( \epsilon \) and \( \rho \) in equation (3.3) and subtracting the new equation from equation (3.3), we obtain
\[ \nabla_\rho \nabla_\epsilon Q_{\beta\gamma} - \nabla_\epsilon \nabla_\rho Q_{\beta\gamma} = (K_{\epsilon\rho} - K_{\rho\epsilon}) Q_{\beta\gamma} \]  (3.4)

Contracting (3.4) with \( \dot{x}^\beta \ x^\gamma \) and use of equation (1.7), we get
\[ \nabla_\rho \nabla_\epsilon Q - \nabla_\epsilon \nabla_\rho Q = (K_{\epsilon\rho} - K_{\rho\epsilon}) Q \]  (3.5)

Using commutation formula (5), equation (6.10), p.126 and equation (2.2), we obtain
\[ (K_{\epsilon\rho} - K_{\rho\epsilon}) Q = (\partial_\sigma Q) H^\sigma_{\rho\epsilon} \]  (3.6)

From equations (1.9) and (3.6), consequently, follows
\[ K_{\epsilon\rho} - K_{\rho\epsilon} = \partial_\sigma (\log T) H^\sigma_{\rho\epsilon} \]  (3.7)

Yields the result
\[ K_{\epsilon\rho} \neq K_{\rho\epsilon} \]  (3.8)

**Theorem 3.3:**
If a Finsler space \( F^n \) admits projectively flat Q-birecurrent space then the relation \( K_{\epsilon\rho} Q^a_{\beta\gamma} + K_{\beta\rho} Q^a_{\gamma\epsilon} + K_{\gamma\epsilon} Q^a_{\epsilon\beta} = 0 \) holds good.

**Proof:**
Differentiating equation (2.16) covariantly with respect to \( x^\rho \), we get
\[ \nabla_\rho \nabla_\epsilon Q^a_{\beta\gamma} + \nabla_\epsilon \nabla_\rho Q^a_{\beta\gamma} + \nabla_\rho \nabla_\gamma Q^a_{\gamma\beta} = 0 \]  (3.9)

Since the space is Q-birecurrent then equation (3.2) assumes the form
\[ K_{\epsilon\rho} Q^a_{\beta\gamma} + K_{\beta\rho} Q^a_{\gamma\epsilon} + K_{\gamma\epsilon} Q^a_{\epsilon\beta} = 0 \]  (3.10)
4 Generalised Neo-Pseudo Projective Recurrent Space:

Let us consider the relation

\[ \nabla_\rho \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} = R_\rho \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} + K_{\varepsilon\rho} Q^\alpha_{\beta\gamma\delta} \]  \hspace{1cm} (4.1)

wherein \( R_\rho \) and \( K_{\varepsilon\rho} \) are recurrence vector and recurrence tensor fields respectively.

**Definition 4.1:**

The neo-pseudo projective curvature tensor \( Q^\alpha_{\beta\gamma\delta} \) of Finsler space \( F^n \) satisfying the condition (4.1) is called generalised neo-pseudo projective recurrent curvature tensor\(^{(3,4)}\).

**Definition 4.2:**

Finsler space \( F^n \) equipped with the generalised neo-pseudo projective recurrent curvature tensor \( Q^\alpha_{\beta\gamma\delta} \) is called generalised neo-pseudo projective recurrent Finsler space\(^{(3,9)}\).

In this regard, we have the following theorems:

**Theorem 4.1:**

The necessary and sufficient condition for Finsler space \( F^n \) to be Q-symmetric is that the space has to be Q-birecurrent.

**Proof:**

If the space be Q-symmetric i.e. \( \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} = 0 \), then

\[ \nabla_\rho \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} = K_{\varepsilon\rho} Q^\alpha_{\beta\gamma\delta} \] \hspace{1cm} (4.2)

which is the condition of Q-birecurrent.

Conversely, let us assume that the space be Q-birecurrent, follows the condition (4.2). Inserting equation (4.2) into equation (4.1), we obtain \( \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} = 0 \). Hence the space is Q-symmetric.

**Theorem 4.2:**

If the space \( F^n \) is Q-symmetric and Q-flat then its generalised neo-pseudo projective recurrent space vanishes identically.

**Proof:**

If the space be Q-symmetric i.e. \( \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} = 0 \) and Q-flat i.e. \( Q^\alpha_{\beta\gamma\delta} = 0 \). Consequently, from equation (4.1) follows

\[ \nabla_\rho \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} = 0 \]. This establishes the validity of the theorem.

**Remark 4.1:**

It is noteworthy that if \( F^n \) to be Q-symmetric and Q-flat follows that the generalised neo-pseudo projective recurrent space necessarily vanishes. Consequently, the space is simply generalised Q-symmetric one.

**Theorem 4.3:**

If space \( F^n \) admits Q-symmetric and Q-flat then the space is a generalised Q-symmetric one.

**Proof:**

It follows immediately from theorem 4.2.

**Theorem 4.4:**

In Finsler space \( F^n \), if the space is Q-birecurrent then the generalised Q-recurrent space is Q-symmetric.

**Proof:**

It is obvious from equations (2.4), (3.1) and (4.1).

**Theorem 4.5:**

For the recurrence vector \( R_\rho \) the relation

\[ R_\rho \nabla_\varepsilon T - R_\varepsilon \nabla_\rho T = 0 \] \hspace{1cm} (4.3)

holds good.

**Proof:**

Contracting (4.1) by \( \alpha \) and \( \delta \), we obtain

\[ \nabla_\rho \nabla_\varepsilon Q^\beta_\gamma = R_\rho \nabla_\varepsilon Q^\beta_\gamma + K_{\varepsilon\rho} Q^\beta_\gamma \] \hspace{1cm} (4.4)

Interchanging \( \varepsilon \) and \( \rho \) in equation (4.4) and subtracting the new equation from equation (4.4), we get

\[ \nabla_\rho \nabla_\varepsilon Q^\beta_\gamma - \nabla_\varepsilon \nabla_\rho Q^\beta_\gamma = R_\rho \nabla_\varepsilon Q^\beta_\gamma - R_\varepsilon \nabla_\rho Q^\beta_\gamma + (K_{\varepsilon\rho} - K_{\rho\varepsilon}) Q^\beta_\gamma \] \hspace{1cm} (4.5)
Contracting (4.5) with $\dot{x^\beta}$ $x^\gamma$ and use of equation (1.7), we get

$$\nabla_\rho \nabla_\epsilon Q - \nabla_\epsilon \nabla_\rho Q = R_\rho \nabla_\epsilon Q - R_\epsilon \nabla_\rho Q + (K_{\epsilon \rho} - K_{\rho \epsilon}) Q$$

(4.6)

By virtue of equations (3.5) and (4.6), we get

$$R_\rho \nabla_\epsilon Q - R_\epsilon \nabla_\rho Q = 0$$

(4.7)

Inserting equation (1.9) in equation (4.7), we get the desired result.

Theorem 4.6:
For the projective flat generalised $Q$-recurrent space the relation

$$\nabla_\rho \nabla_\epsilon Q^\alpha _{\gamma \delta} + \nabla_\rho \nabla_\gamma Q^\alpha _{\delta \epsilon} + \nabla_\rho \nabla_\delta Q^\alpha _{\epsilon \gamma} = 0$$

holds good.

Proof:
Contracting (4.1) with $\dot{x^\beta}$, we obtain

$$\nabla_\rho \nabla_\epsilon Q^\alpha _{\gamma \delta} = R_\rho \nabla_\epsilon Q^\alpha _{\gamma \delta} + K_{\epsilon \rho} Q^\alpha _{\gamma \delta}$$

(4.8)

Taking the cyclic permutation in $\epsilon, \gamma, \delta$ and on making use of equations (2.16) and (3.10), we observe that

$$\nabla_\rho \nabla_\epsilon Q^\alpha _{\gamma \delta} + \nabla_\rho \nabla_\gamma Q^\alpha _{\delta \epsilon} + \nabla_\rho \nabla_\delta Q^\alpha _{\epsilon \gamma} = 0$$

(4.9)

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