Analysis and Synthesis of Low-Gain Integral Controllers for Nonlinear Systems with Application to Feedback-Based Optimization

John W. Simpson-Porco, Member, IEEE

Abstract—Relaxed conditions are given for stability of a feedback system consisting of an exponentially stable multi-input multi-output nonlinear plant and an integral controller. Roughly speaking, it is shown that if the composition of the plant equilibrium input-output map and the integral feedback gain is infinitesimally contracting, then the closed-loop system is exponentially stable if the integral gain is sufficiently low. The main result is applied to analyze stability of an optimal frequency regulation scheme for AC power systems. We demonstrate how the key condition can be checked computationally via semidefinite programming, and how gain matrices can be synthesized for nonlinear systems to achieve guaranteed performance bounds. Finally, we apply the results to show closed-loop stability of recently developed feedback-based schemes which optimize the steady-state behaviour of a dynamic system.

Index Terms—Nonlinear control, integral control, output regulation, linear matrix inequalities (LMIs), slow integrators, singular perturbation

I. INTRODUCTION

Tracking and disturbance rejection in the presence of model uncertainty is one of the fundamental purposes of automatic control. A case which commonly occurs in engineering practice is that the system one wishes to control is complex, and no accurate dynamic model is available, but it is known through experience that the system reasonably well-behaved (i.e., stable and responsive to control inputs). The modest design goal is to improve reference tracking and disturbance rejection via a supplementary integral controller. An instance of this problem is that of secondary frequency regulation for large-scale power systems, where the Automatic Generation Control (AGC) system asymptotically rebalances load and generation by applying integral control to the measured system frequency deviation. In real AGC systems, the integral gain is always kept very low, to ensure the underlying uncertain power system dynamics are not destabilized by the integral feedback loop.

While such a supplemental integral loop is trivial to tune for SISO LTI systems, the general MIMO LTI case is slightly more subtle. Consider the continuous-time LTI state-space model

\[
\dot{x} = Ax + Bu + B_w w, \\
\epsilon = Cx + Du + D_w w
\]

with state \(x \in \mathbb{R}^n\), control input \(u \in \mathbb{R}^m\), constant disturbance/reference signal \(w \in \mathbb{R}^{n_w}\), and error output \(\epsilon \in \mathbb{R}^p\).

Assume that \(A\) is Hurwitz. One interconnects the system (1) with the integral controller

\[
\dot{\eta} = -\epsilon e, \quad u = K\eta,
\]

where \(K \in \mathbb{R}^{n \times p}\) is a gain matrix and \(\epsilon > 0\). Let

\[
G(0) = -CA^{-1}B + D, \quad G_w(0) = -CA^{-1}B_w + D_w
\]

denote the DC gain matrices from \(u\) and \(w\) to \(e\). Implicit in the proof of [1, Lemma 3] is the following: if there exists \(K\) such that \(-G(0)K\) is Hurwitz, then the feedback system (1)–(2) is exponentially stable for small \(\epsilon > 0\). The condition \(-G(0)K\) Hurwitz is equivalent to \(G(0)\) having full row rank, and indeed a suitable gain design is \(K = G(0)^1\), as used in [1, Lemma 3]. This result was stated succinctly by Morari in [2, Theorem 3]; see also [3, Lemma 1, A.2, A.3] for the details. From a singular perturbation point of view [4], the low integral gain \(\epsilon\) induces a time-scale separation in the system (1)–(2). The reduced slow time-scale dynamics are given by

\[
\frac{1}{\epsilon}\dot{\eta} = -G(0)K\eta - G_w(0)w,
\]

and thus \(-G(0)K\) determines stability on the slow time-scale.

Extensions of this LTI result to Lur’e-type systems [5], [6] and to distributed-parameter systems [7] have been pursued. In the full nonlinear setting, the most well-known result is due to Desoer and Lin [8], who proved that if the equilibrium input-to-error map of the plant is strongly monotone, then a similar low-gain stability result holds; a related condition was recently also used in [9, Equation (21)]. When specialized for LTI systems, the Desoer–Lin condition states that \(G(0) + G(0)^T\) should be positive definite; it therefore does not properly generalize the Davison/Morari result. It appears the only attempt to close this gap was reported in [10], where Tseng proposed a design based on inverting the Jacobian of the plant equilibrium input-to-error map. This recovers Davison’s special design \(K = G(0)^{-1}\) in the (square) LTI case, but in general yields a very complicated nonlinear feedback. In sum, the available results in the literature do not reduce as expected in the LTI case, and the literature lacks systematic procedures for constructing low-gain integral controllers for nonlinear and uncertain systems. One goal of this paper is to close this gap.

While the design of traditional low-gain tracking control is important in and of itself, another source of recent interest in such low-gain methods in a nonlinear context has arisen from the study of feedback-based optimizing controllers for dynamic systems; see [11]–[15] for various recent works. In this line
of work, the controller does not attempt to track an explicit reference, but instead attempts to drive the system towards an optimal point of operation in the presence of an unmeasured exogenous disturbance. The results we develop will also be applicable to this class of “tracking-adjacent” problems.

Contributions: The broad goals of this paper are 1) to understand when low-gain MIMO integral feedback can be applied to a MIMO nonlinear system, and 2) to leverage modern robust control tools for the analysis and design of such control loops. These goals are largely inspired by practical problems in power system control, and by the foundational paper [1], which provided definitive constructive answers in the (certain) LTI case. As a result of these goals, this work is somewhat disjoint from the modern literature on nonlinear output regulation (see [9], [16]–[18] for recent contributions) where the focus is on quite different issues, such as nonlinear stabilization, practical vs. asymptotic regulation, and the construction of internal models. This work is therefore best understood as a continuation of the line of research in [1], [8], [10].

There are three specific contributions. The first (Section III) is a generalization the main result of [8], providing relaxed conditions on the plant’s equilibrium input-to-error map which ensure closed-loop stability under low-gain integral control. The main idea is to impose that the reduced time-scale dynamics be infinitesimally contracting, which ensures the existence of a unique and exponentially stable equilibrium point for all asymptotically constant exogenous disturbances. Unlike the conditions reported in [8], [10], this condition recovers the Davison/Morari result that $-G(0)K$ should be Hurwitz when restricted to the LTI case, and allows for additional flexibility over [8], [10]. We apply the results to show stability of an optimal frequency regulation scheme for AC power systems.

Second, in Section IV we outline how semidefinite programming can be used for certification of stability under low-gain integral control, as well as direct convex synthesis of integral gain matrices which achieve robust performance. These results apply equally in the nonlinear or in the uncertain linear contexts, and are illustrated via academic examples.

Third and finally, in Section V we apply the main result to establish closed-loop stability certificates for recently developed techniques for feedback-based optimization of dynamic systems. Our approach yields a parameterization of the feedback gain, and generalizes the design $K = G(0)^\dagger$ of [1]. The results are illustrated on a constrained reference tracking problem.

A. Notation

For column vectors $x_1, \ldots, x_n$, $\text{col}(x_1, \ldots, x_n)$ denotes the associated concatenated column vector. The identity matrix of size $n$ is $I_n$, and $1_n$ denotes the $n$-vector of all unit elements. The notation $P > 0$ (resp. $P \geq 0$) means $P$ is symmetric and positive (semi)definite. In any expression of the form $(\ast)^T X Y$ or $[\begin{bmatrix} X & Y \\ \ast & \ast \end{bmatrix}]$, $(\ast)$ is simply an abbreviation for $Y$. For matrices $X_1, \ldots, X_n$, $\text{diag}(X_1, \ldots, X_n)$ denotes the associated block diagonal matrix. Given two $2 \times 2$ block matrices $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ and $Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$, we define the diagonal augmented matrix

$$\text{daug}(X, Y) = \begin{bmatrix} X_{11} & 0 & X_{12} & 0 \\ 0 & Y_{11} & 0 & Y_{12} \\ X_{21} & 0 & X_{22} & 0 \\ 0 & Y_{21} & 0 & Y_{22} \end{bmatrix}.$$  

Given a function $V(x, y)$, $\nabla_x V(x, y)$ denotes its gradient with respect to $x$. The space $\mathcal{L}_2^p([0, \infty))$ denotes the set of measurable maps $f : \mathbb{R} \to \mathbb{R}^p$ which are zero for $t < 0$ with $\|f(t)\|_2^p$ being integrable over $[0, \infty)$, and $\mathcal{L}_2^p([0, \infty))$ denoting the associated extended signal space; see, e.g., [19] for details.

II. Problem Setup and Assumptions

We consider a physical plant which is described by a finite-dimensional nonlinear time-invariant state-space model

$$\dot{x}(t) = f(x(t), u(t), w), \quad x(0) = x_0$$

$$e(t) = h(x(t), u(t), w)$$

where $x(t) \in \mathbb{R}^n$ is the state with initial condition $x_0$, $u(t) \in \mathbb{R}^m$ is the control input, $e(t) \in \mathbb{R}^p$ is the error to be regulated to zero, and $w \in \mathbb{R}^{n_w}$ is a vector of constant reference signals, disturbances, and unknown parameters. Without loss of generality, we assume that $f(0, 0, 0) = 0$ and $h(0, 0, 0) = 0$, so that the origin is a zero-error equilibrium when both inputs are zero. The map $f$ is locally Lipschitz in $(x, u)$ and continuous in $w$ on a domain of interest $D_x \times D_u \times D_w \subseteq \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n_w}$ containing the origin, with $h$ continuously differentiable in $(x, u)$ and continuous in $w$ on the same domain. For fixed $w$, the possible equilibrium state-input-error triplets $(\bar{x}, \bar{u}, \bar{e})$ are determined by the algebraic equations

$$0 = f(\bar{x}, \bar{u}, w), \quad \bar{e} = h(\bar{x}, \bar{u}, w).$$

We make the following assumptions to capture both the steady-state and dynamic behaviour of the plant.

**Assumption 2.1 (Plant Assumptions):** For (5) there exist (A1) sets $\mathcal{X} \subseteq D_x$ and $\mathcal{I} \subseteq D_u \times D_w$ containing the origin and a map $\pi_x : \mathcal{I} \to \mathcal{X}$ which is class $C^1$ in $u$ and satisfies $0_n = f(\pi_x(u, w), u, w)$, for all $(u, w) \in \mathcal{I}$.

(A2) constants $c_1, c_2, L_{V_i}, \rho_t > 0$ and a function $V_i : \mathcal{X} \times \mathcal{I} \to \mathbb{R}_{\geq 0}$, $(x, (u, w)) \mapsto V_i(x, u, w)$ which is class $C^1$ in $(x, u)$ and satisfies $c_1 \|x - \pi_x(u, w)\|_2^2 \leq V_i(x, u, w) \leq c_2 \|x - \pi_x(u, w)\|_2^2$ \n\n$$\|\nabla_x V_i(x, u, w)\|_2 \leq L_{V_i} \|x - \pi_x(u, w)\|_2$$

for all $x \in \mathcal{X}$ and $(u, w) \in \mathcal{I}$.

(A3) constants $L_{\pi_x}, L_{h_x}, L_h > 0$ such that $\|x - \pi_x(u, w) - x' - \pi_x(u', w')\|_2 \leq L_{\pi_x} \|u - u'\|_2$, $\|h(x, u, w) - h(x', u', w')\|_2 \leq L_h \|x - x'\|_2$ $\|h(x, u, w) - h(x', u', w')\|_2 \leq L_{h_x} \|x' - x\|_2$ for all $x, x' \in \mathcal{X}$ and all $(u, w), (u', w') \in \mathcal{I}$.

1Asymptotically constant references/disturbances are treated similarly [20].
Assumption (A1) states that each constant input-disturbance pair \((\bar{u}, w) \in \mathcal{I}\) yields a unique equilibrium state \(\bar{x} = \pi_x(\bar{u}, w)\) in the set \(\mathcal{X}\). We call the map \(\pi : \mathcal{I} \rightarrow \mathbb{R}^p\) given by
\[
\pi(\bar{u}, w) \triangleq h(\pi_x(\bar{u}, w), \bar{u}, w) \quad (6)
\]
the equilibrium input-to-error map. Note that \(\pi(0, 0) = 0\), and that \(\pi\) is class \(C^1\) in \(\bar{u}\). The conditions (A2) are the existence of a Lyapunov function which establishes exponential stability of \(\bar{x} = \pi_x(\bar{u}, w)\), and (A3) enumerates Lipschitz constants for the mappings \(\pi_x\) and \(h\) on the appropriate sets. When restricted to the LTI case in (1), (A1)–(A3) simply reduce to the matrix \(A\) being Hurwitz, and we may select \(\mathcal{X} = \mathbb{R}^n\) and \(\mathcal{I} = \mathbb{R}^m \times \mathbb{R}^{n_w}\). For later notational use, we let
\[
\mathcal{W} = \{w \in \mathbb{R}^{n_w} \mid \text{there exists } u \in \mathbb{R}^m \text{ s.t. } (u, w) \in \mathcal{I}\}
\]
and for a given \(w \in \mathcal{W}\) we let
\[
\mathcal{U}_w = \{u \in \mathbb{R}^m \mid (u, w) \in \mathcal{I}\} \neq \emptyset
\]
denote the set of constant controls for which the equilibrium map \(u \mapsto \pi_x(u, w)\) is defined.

Example 2.2 (Illustration of Assumption 2.1): Consider the scalar dynamic system
\[
\dot{x} = -\beta \sin(x) + u - w, \quad e = x,
\]
where \(\beta > 0\). Clearly the origin \((x, u, w) = (0, 0, 0)\) is an equilibrium. For fixed \(\gamma \in [0, \frac{\pi}{2}]\), if we define
\[
\mathcal{I}(\gamma) = \{(u, w) \in \mathbb{R}^2 \mid -\beta \sin(\gamma) \leq u - w \leq \beta \sin(\gamma)\}
\]
\[
\mathcal{X}(\gamma) = \{\bar{x} \in \mathbb{R} \mid -\gamma \leq x \leq \gamma\}
\]
then \(\pi(u, w) = \arcsin((u + w)/\beta)\) is a continuously differentiable map from \(\mathcal{I}(\gamma)\) into \(\mathcal{X}(\gamma)\). Note however that the set \(\mathcal{I}(\gamma)\) cannot be nicely expressed as a Cartesian product of a control set and a disturbance set. To fulfill (A2) we may select \(V_1(x, u, w) = (x - \pi(x, u, w))^2\), which is indeed class \(C^1\) in \((x, u)\) and satisfies the required bounds with \(c_1 = c_2 = \frac{1}{2}\), \(\rho_1 = \beta \cos(\gamma), L_{V_1} = 1/(\beta \cos(\gamma))\). We may take \(W = \mathbb{R}\) and \(\mathcal{U}_w(\gamma) = \{u \in \mathbb{R} \mid -\beta \sin(\gamma) + w \leq u \leq \beta \sin(\gamma) + w\}\).

We are interested in the application of a pure integral feedback control scheme to (5) which acts on the error \(e\) as
\[
\begin{align*}
\dot{\eta} &= -\varepsilon e, \quad \eta(0) \in \mathbb{R}^p \\
u &= k(\eta)
\end{align*}
\]
(7)
where \(k : \mathbb{R}^p \rightarrow \mathbb{R}^m\) is a feedback and \(\varepsilon > 0\) is to be determined. We assume that
\[
k(\eta) \text{ is class } C^1 \text{ and } L_k\text{-Lipschitz continuous on } \mathbb{R}^p.
\]

The closed-loop system is the interconnection of the plant (5) and the controller (7). Our goal is to give conditions under which (5),(7) has an exponentially stable equilibrium point for sufficiently small \(\varepsilon > 0\).

III. A Generalization of the Desoer-Lin Result on Low-Gain Integral Control

The main result of this section provides a generalization of the result of [8], where the monotonicity requirement on the equilibrium input-to-error map that was weakened to contraction [21]–[24] of the vector field \(\eta \mapsto -\pi(k(\eta), w)\).

While there are several approaches to contraction analysis, with varying sophistication, we will make use of the formulation based on the matrix measure; this has proved sufficient for our applications of interest. Let \(\|\cdot\|\) denote any vector norm on \(\mathbb{R}^n\), with \(\|\cdot\|\) also denoting the associated induced matrix norm. The matrix measure associated with \(\|\cdot\|\) is the mapping \(\mu : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}\) defined by (e.g., [25, Chap. 2.2.2])
\[
\mu(A) \triangleq \lim_{h \rightarrow 0^+} \frac{1}{h}(\|I_n - hA\| - 1).
\]

Matrix measures associated with standard vector norms \(\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty\) (and their weighted variants) are all explicitly computable and have found substantial use in applications; a summary is beyond our scope, but see [25, Chap. 2.2.2].

Infinitesimal contraction of a vector field is characterized via the matrix measure of its Jacobian matrix [22]. Let \(P\) be a non-empty parameter set, and consider the dynamics
\[
\dot{x}(t) = F(x(t), \alpha), \quad x(0) = x_0, \quad \alpha \in P, \quad (8)
\]
where \(F : \mathbb{R}^n \times P \rightarrow \mathbb{R}^n\) is class \(C^1\) in its first argument. Let \(\|\cdot\|\) be any vector norm. For a given \(\alpha \in P\), the system (8) is infinitesimally contracting with respect to \(\|\cdot\|\) on a set \(\mathcal{C}_\alpha \subseteq \mathbb{R}^n\) if there exists \(\rho_\alpha > 0\) such that
\[
\mu(\frac{\partial F}{\partial x}(x, \alpha)) \leq -\rho_\alpha, \quad \text{for all } x \in \mathcal{C}_\alpha. \quad (9)
\]

If \(\mathcal{C}_\alpha\) is a convex and forward-invariant set for (8), then (9) guarantees that (8) possesses a unique equilibrium point \(x^* \in \mathcal{C}_\alpha\) towards which all trajectories with initial conditions \(x_0 \in \mathcal{C}_\alpha\) will converge exponentially at rate \(e^{-\rho_\alpha t}\) [22, Thm. 1/2].

As the parameter \(\alpha\) varies over \(P\), it is generally the case that the set \(\mathcal{C}_\alpha\) and the contraction rate \(\rho_\alpha\) will need to vary. To ensure a uniform rate, we will say that (8) is uniformly infinitesimally contracting with respect to \(\|\cdot\|\) on a family of sets \(\{\mathcal{C}_\alpha\}_{\alpha \in P}\) if there exists \(\rho > 0\) such that for all \(\alpha \in P\)
\[
\mu(\frac{\partial F}{\partial x}(x, \alpha)) \leq -\rho, \quad \text{for all } x \in \mathcal{C}_\alpha. \quad (10)
\]

We are now ready to state the main result.

Theorem 3.1 (Relaxed Conditions for Exponential Stability under Low-Gain Integral Control): Consider the plant (5) interconnected with the integral controller (7) under assumptions (A1)–(A4). Define the reduced dynamics
\[
\begin{align*}
\dot{\eta} &= F_s(\eta, w), \quad \eta(0) = \eta_0, \quad \eta(0) = \eta_0, \quad w \in \mathcal{W},
\end{align*}
\]
and assume that
\[
\begin{align*}
(\text{A5}) \quad & \text{for each } w \in \mathcal{W} \text{ there exists a convex forward-invariant set } \mathcal{C}_w \text{ for (11) such that } k(\mathcal{C}_w) \subseteq \mathcal{U}_w. \\
(\text{A6}) \quad & \text{the system (11) is uniformly infinitesimally contracting with respect to some norm } \|\cdot\|_s \text{ on } \{\mathcal{C}_w\}_{w \in \mathcal{W}}.
\end{align*}
\]

Then there exists \(\varepsilon^* > 0\) such that for all \(\varepsilon \in (0, \varepsilon^*)\) and all \(w \in \mathcal{W}\), the closed-loop system possesses a unique exponentially stable equilibrium point \((\bar{x}, \bar{\eta})\) satisfying \(\bar{x} = \bar{\eta} = \ldots\).
where $h(\bar{x}, k(\bar{\eta}), w) = 0$. If (A1)--(A6) hold globally, then the equilibrium is globally exponentially stable.

Before proving the result, we examine how Theorem 3.1 reduces to known conditions in special cases. When restricted to weighted Euclidean vector norms $||x||_P = (x^TPx)^{1/2}$ where $P > 0$, (A6) holds if and only if [26]

$$\frac{\partial P}{\partial \eta}(\eta, w)^T P + P \frac{\partial P}{\partial \eta}(\eta, w) \preceq -2\rho_s P, \quad \eta \in C_w,$$  

for some $\rho_s > 0$. If we restrict $k(\tilde{\eta}) = \eta$ and $P = I_P$ and $C_w = \mathbb{R}^p$, (12) reduces to the mapping $u \mapsto \pi(u, w)$ being strongly monotone on $P^R$, which is the Desoer--Lin condition [8].

In the LTI case (1)--(2), as previously mentioned, (A1)--(A3) reduce to $A$ being Hurwitz. The joint input/disturbance set $\mathcal{I}$ may be chosen as $\mathcal{I} = \mathbb{R}^m \times \mathbb{R}^n_w$, and the equilibrium input-to-error mapping $\pi(u, w)$ is given explicitly by $\pi(u, w) = G(0)u + G_w(0)w$, where $G(0)$ and $G_w(0)$ are defined previously in (3). A linear integral feedback gain $u = Kx$ obviously satisfies (A4), and the dynamics (11) reduce to

$$\dot{\eta} = F_\eta(x, \eta, w) = -G(0)K\eta - G_w(0)w.$$  

For Euclidean norms, from (12) we see that (A6) reduces to the existence of $P > 0$ and $\rho_s > 0$ such that

$$-(G(0)K)^T P - P (G(0)K) \preceq -2\rho_s P,$$

which by standard Lyapunov results holds if and only if $-G(0)K$ is Hurwitz. In this case, we may always select $C_w = \mathbb{R}^p$ to satisfy (A5), and we therefore properly recover the classical Davison/Morari result for LTI systems.

Proof of Theorem 3.1: Let $w \in W$. Defining $\tau_c := \epsilon t$ the system (5),(7) may be written in singularly perturbed form as

$$\Sigma_w: \begin{align*}  
\epsilon \frac{dx}{dt} &= f(x, k(\eta), w), \\
\frac{d\eta}{dt} &= -h(x, k(\eta), w). 
\end{align*}$$

Closed-loop equilibria $(\bar{x}, \bar{\eta})$ are characterized by the equations

$$0 = f(\bar{x}, \bar{u}, w), \quad 0 = h(\bar{x}, \bar{u}, w), \quad \bar{u} = k(\bar{\eta}).$$

(14)

Given any $\bar{u} \in U_w$, by (A1) the first equation in (14) can be solved for $\bar{x} = \pi_\bar{x}(\bar{u}, w)$; together, (A1)/(A2) imply that $\bar{x}$ is isolated. Eliminating $\bar{x}$ and $\bar{u}$ from (14), we obtain the error-zeroing equation $0 = \pi(k(\bar{\eta}), w)$. From (A5)--(A6), the dynamics (11) are infinitesimally contracting on a forward-invariant convex set $C_w$; it follows from the main contraction stability theorem (see, e.g., [22]) that (11) possesses a unique equilibrium point $\bar{\eta} \in C_w$, and hence $0 = \pi(k(\bar{\eta}), w)$ is uniquely solvable on $C_w$. By (A5), $\bar{\eta}$ further satisfies $k(\bar{\eta}) \in U_w$, which justifies the initial application of (A1). Thus, there exists a unique closed-loop equilibrium $(\bar{x}, \bar{\eta}) \in X \times C_w$.

Let $V_\tau(x, \eta) = \frac{1}{2}||\eta - \bar{\eta}||^2_2$, and for later use note that by equivalence of norms, there exist constants $0 < c_{2x} \leq c'_{2x}$ such that $c_{2x}||z||_2 \leq ||z||_s \leq c'_{2x}||z||_2$ for any $z \in \mathbb{R}^p$. Following [27, Sec. 11.5], define the composite Lyapunov candidate

$$V(x, \eta) = dV_\eta(\eta) + V_\tau(x, k(\eta), w)$$

where $d > 0$ is to be determined. Since (i) $k$ is continuous by (A4), (ii) $V_\tau$ is continuous in all arguments by (A2), and (iii) $k(\bar{\eta}) \in U_w$, it must hold that $V_\tau(x, k(\eta), w) > 0$ for all $(x, \eta) \in X \times C_w$ such that $(x, \eta) \neq (\bar{x}, \bar{\eta})$. Since $V_\tau$ is positive definite with respect to $\bar{\eta}$, it now follows that $V_\tau$ is positive definite on $X \times C_w$ with respect to $(\bar{x}, \bar{\eta})$. We compute the Lie derivative $L_{\Sigma} V_\tau$ of $V_\tau$ along $\Sigma$ as

$$L_{\Sigma} V_\tau(x, \eta) = \frac{1}{2} \nabla_x V_\tau(x, k(\eta), w)^T f(x, k(\eta), w) - \nabla_\eta V_\tau(x, k(\eta), w)^T \frac{\partial}{\partial \eta}(\eta) h(x, k(\eta), w).$$

It follows from (A4) that $||\frac{\partial}{\partial \eta}(\eta)||_2 \leq L_k$ for all $\eta \in \mathbb{R}^p$.

Using (A2) to bound the first term and (A2)/(A4) to bound the second, we have for $(x, \eta) \in X \subseteq C_w$ that

$$L_{\Sigma} V_\tau(x, \eta) \leq -\mu \left(||x - \pi_x(x, k(\eta), w)||^2 + L_{V_\tau} L_k ||x - \pi_x(x, k(\eta), w)|| ||h(x, k(\eta), w)||\right).$$

(15)

To further bound the last term, we write

$$h(x, k(\eta), w) = h(x, k(\eta), w) - h(\pi_x(x, k(\eta), k(\eta), w) + h(\pi_x(x, k(\eta), k(\eta), w) - h(\pi_x(x, k(\eta), k(\eta), w) - h(\pi_x(\bar{x}, k(\bar{\eta}), k(\bar{\eta}), w)$$

where the last term on the right-hand side is simply $\bar{e} = 0$. Using (A3) it follows that

$$||h(x, k(\eta), w)||_2 \leq L_h \|x - \pi_x(x, k(\eta), w)||_2 + L_h \left(||\pi_x(k(\eta), w) - \pi_x(k(\bar{\eta}))|| \|k(\bar{\eta}) - k(\bar{\eta})\|\right).$$

(16)

The second term can be further bounded using (A3) and (A4) to obtain the inequality

$$||h(x, k(\eta), w)||_2 \leq L_{h, \delta} ||x - \pi_x(x, k(\eta), w)||_2$$

where

$$\Sigma(||\eta - \bar{\eta}||_s \leq \limsup_{\alpha \to 0^+} ||\eta - \bar{\eta} - \alpha h(x, k(\eta), w)||_s - ||\eta - \bar{\eta}||_s \right.$$

Since by (6) it holds that

$$F_\eta(\eta, w) = -h(\pi_x(k(\eta), w), k(\eta), w),$$

we may write

$$\eta - \bar{\eta} - \alpha h(x, k(\eta), w) = \eta - \bar{\eta} + \alpha F_\eta(\eta, w) + \alpha \Delta h(x, \eta),$$

where $\Delta h(x, \eta) = h(\pi_x(k(\eta), w), k(\eta), w) - h(x, k(\eta), w)$. Let $J(\eta, w) = \frac{\partial}{\partial \eta}(\eta, w)$ denote the Jacobian matrix of $F_\eta$. Since $C_w$ is convex, $\pi_x \in C_w$, and $F_\eta(\bar{\eta}, w) = 0$, it follows from the mean value theorem that there exists $\xi \in C_w$ such that $F_\eta(\eta, w) = J(\xi, \eta)(\eta - \bar{\eta})$. As induced norms are submultiplicative, we therefore find that

$$\Sigma(||\eta - \bar{\eta}||_s \leq \limsup_{\alpha \to 0^+} \frac{||J(\xi, \eta)||_s - 1}{\alpha} ||\eta - \bar{\eta}||^2_s + \|\Delta h(x, \eta)||_s ||\eta - \bar{\eta}||_s,$$

$$= -\mu_s(1, \|\eta - \bar{\eta}||^2_s + \|\Delta h(x, \eta)||_s ||\eta - \bar{\eta}||_s,$$
where \( \mu_c \) is the matrix measure associated with \( \| \cdot \|_s \). By (A6) there exists \( \rho_s > 0 \) independent of \( \xi, w \) such that 
\[
\mu_w(J(\xi, w)) \leq -\rho_s \quad \text{for all } \xi \in C_w.
\]
Moreover, by (A3) we have that 
\[
\| \Delta h(x, \eta) \| \leq c'_{2s} \| h \| \| x - x^e(\mu(\eta), w) \|_2. 
\]
Combining these, we obtain the bound 
\[
\mathcal{L}_\Sigma V_s(\eta) \leq -\rho_s \| \eta - \bar{\eta} \|_s + \beta \| x - x^e(\mu(\eta), w) \|_2 \| \eta - \bar{\eta} \|_s, 
\]
where \( \beta = c''_{2s} \rho_s \). Putting together the dissipation inequalities (17) and (16), we find for all \( (x, \eta) \in \mathcal{X} \times C_w \) that 
\[
\mathcal{L}_\Sigma V(x, \eta) = -(\ast)^T Q \begin{bmatrix} \| x - x^e(\mu(\eta), w) \|_2 \\ \| \eta - \bar{\eta} \|_s \end{bmatrix} 
\]
where 
\[
Q = \begin{bmatrix} \frac{\rho_l}{\rho_l} - L_h, L_V L_k \left( -c_{2s} \right) \\ -(\beta + \delta)/2 d\rho_s \end{bmatrix}.
\]
Analysis of \( Q \) as a function of \( d \) and \( \varepsilon \) shows that the choice of \( d \) yielding the largest range for \( \varepsilon = d/\beta \); with this choice \( Q > 0 \) if and only if \( \varepsilon \in (0, \varepsilon^*) \), where 
\[
\varepsilon^* = \frac{\rho_l}{L_V L_k - c_{2s}} \frac{\rho_l}{\rho_l} + \frac{1}{c_s} \rho_s L_k L_W \sqrt{L^2 + 1} > 0. \tag{19}
\]
Standard arguments using (A2) can now be applied to conclude that there exists a constant \( \gamma > 0 \) such that \( \mathcal{L}_\Sigma V(x, \eta) \leq -\gamma V(x, \eta) \) for all \( (x, \eta) \in \mathcal{X} \times C_w \), and it quickly follows that the equilibrium \( (\bar{x}, \bar{\eta}) \) is exponentially stable. \( \square \)

A. Application to Power System Frequency Control

We illustrate the main result with a simple example arising in power system control. Our treatment is terse; we refer to [29, Sec. 11.1.6] for engineering background and to [30, Sec. IV] for some recent control-centric references.

The dynamics of AC power systems are inherently stable and produce the phenomena of frequency synchronization, wherein frequencies of AC signals measured at different buses throughout the system become equal in steady state. The secondary frequency regulation problem is to design an integral control loop which maintains the system frequency at its nominal value. Consider an AC power system consisting of a single interconnected area, and suppose there are \( m \) controllable resources available, with \( u \in \mathbb{R}^m \) denoting the vector of power set-points for those resources. Assume that the power dynamic system model satisfies (A1)–(A3), and that the steady-state system frequency deviation \( \Delta \omega_{ss} \) due to a constant net load disturbance \( w \) can be expressed in terms of the steady-state control set-points \( \bar{u} \) as
\[
\Delta \omega_{ss} = \frac{1}{\beta} \sum_{k=1}^m \Delta \omega_{k} - \bar{w}, \tag{22}
\]
where \( \beta > 0 \) is the stiffness constant of the system; see [29, Sec. 11.1.6] for a derivation of (22). For frequency regulation, the power system is over-actuated, and the operator has flexibility in allocating control actions across many actuators. Optimal set-points can be determined via the minimization problem
\[
\text{minimize } \sum_{i=1}^m J_i(\bar{u}, \bar{w}) \quad \text{s.t. } \Delta \omega_{ss} = 0, \tag{23a}
\]
where \( J_i : \mathbb{R} \rightarrow \mathbb{R} \geq 0 \) models the disutility of the \( i \)-th secondary power provider, and may include penalty functions for enforcing inequality constraints. We assume \( J_i \) is class \( C^1 \) and \( \mu_i \)-strongly convex, with its gradient \( \nabla J_i \) being \( L_i \)-Lipschitz continuous.\(^3\)

Several centralized and distributed feedback controllers have been recently proposed to solve (23) online. As one example, we consider the centralized integral controller [30], [32]
\[
\tau \tilde{\eta} = -\sum_{i=1}^m c_i \Delta \omega_{i}, \quad \bar{u}_i = \nabla J_i^{\ast}(\eta), \tag{24}
\]
with scalar state \( \eta \in \mathbb{R} \), where \( \Delta \omega_i \) is the measured frequency deviation at resource \( i \in \{1, \ldots, m\} \), \( \tau > 0 \) is a time constant, and \( c_i \geq 0 \) are convex combination coefficients satisfying \( \sum_{i=1}^m c_i = 1 \). With \( \varepsilon = \tau / \bar{r} \), the controller (24) is a special case of (7) with error signal \( e = \sum_{i=1}^m c_i \Delta \omega_{i} \) and feedback \( k_i(\eta) = \nabla J_i^{\ast}(\eta) \). Combining (22), (24), and the steady-state synchronization condition \( \Delta \omega_1 = \cdots = \Delta \omega_m = \Delta \omega_{ss} \), a simple calculation shows that
\[
F_s(\eta, w) = -\nabla \| k(\eta), \|_p \leq -\frac{1}{\beta} \sum_{i=1}^m \nabla J_i^{\ast}(\eta) + \frac{1}{\beta} w. \tag{25}
\]
\(^3\)These assumptions imply that \( \nabla J_i \) is invertible with inverse given by the gradient \( \nabla J_i^{\ast} \) of its convex conjugate \( J_i^{\ast} \). Standard results give that \( \nabla J_i^{\ast} \) is \( L_i^{-1} \)-strongly monotone and \( \mu_i^{-1} \)-Lipschitz continuous [31].
For $\eta, \eta' \in \mathbb{R}$ with $\pi = \pi(k(\eta), w)$ and $\pi' = \pi(k(\eta'), w)$, using strong convexity of $J^*_i$ we have
\[
(\eta - \eta')(\pi - \pi') = \frac{1}{\beta} \sum (\nabla J^*_i(\eta) - \nabla J^*_i(\eta'))(\eta - \eta')^2 \geq \left( \frac{1}{\beta} \sum_i \frac{1}{J^*_i} \right) (\eta - \eta')^2
\]
so (20) holds with $P = 1$ and $\rho_k = \beta^{-1} \sum_i \frac{1}{J^*_i}$. It follows that the slow dynamics are uniformly infinitesimally contracting on $\mathbb{R}$; (A5)–(A6) therefore both hold, and we conclude that the power system with controller (24) achieves exponentially stable (optimal) frequency regulation for sufficiently large $\tau > 0$. This extends the result of [32] to general power system models, and allows for heterogeneity in the cost functions.

IV. PERFORMANCE ANALYSIS AND SYNTHESIS OF LOW-GAIN INTEGRAL CONTROLLERS FOR NONLINEAR AND UNCERTAIN SYSTEMS

To complement the analytical results in Section III, we now pursue a computational framework for certifying performance of low-gain integral control schemes and synthesizing controller gains. To motivate our general approach, we return to the simple case of LTI systems in Section IV-A before proceeding to nonlinear/uncertain system analysis and synthesis in Sections IV-B and IV-C. Throughout we restrict our attention to linear feedbacks $k(\eta) = K\eta$.

A. Linear Time-Invariant Systems

We begin by considering a computational approach to the design of an integral feedback matrix $K$ in the LTI case; this will motivate our approach in subsequent sections. Recall from Section III that the low integral feedback gain $\varepsilon$ induces a time-scale separation in the dynamics. In the LTI case, the slow dynamics are given by (4) with associated error output
\[
e(t) = G(0)K\eta(t) + G_w(0)w(t),
\]
and the sensitivity transfer matrix from $w$ to $e$ is
\[
S_{\text{slow}}(s) = s(sI_p + \varepsilon G(0)K)^{-1}G_w(0).
\]
Davison’s design recommendation [1, Lemma 3] $K_{\text{Dav}} = G(0)^\dagger$ leads to the simple sensitivity function $S_{\text{slow}}(s) = \frac{s}{s+\varepsilon}G_w(0)$, and achieves the minimum possible value
\[
\sup_{w \in L^2(0,\infty), w \not\equiv 0} \frac{\|e\|_{L_2}}{\|w\|_{L_2}} = \|S_{\text{slow}}(s)\|_{\mathcal{H}_\infty} = \|G_w(0)\|_2.
\]
for the induced $L_2$-gain of the sensitivity function. This design however does not easily extend to the nonlinear case, may perform poorly in the presence of uncertainty (Figure 2), and has the disadvantage for distributed linear control applications that $G(0)^\dagger$ is usually a dense matrix.

These disadvantages can be overcome by moving to a computational robust control framework. Due to the simple structure of the slow dynamics (4),(26), the design of $K$ can be formulated as an $\mathcal{H}_\infty$ state-feedback problem [34, Chap. 7]: for $\rho > 0$ find $Y > 0$ and $Z \in \mathbb{R}^{m \times p}$ such that
\[
\begin{bmatrix}
G(0)Z + (G(0)Z)^T - 2\rho Y & * & * \\
G_w(0)^T & \gamma I_p & * \\
-G(0)Z & -G_w(0) & \gamma I_p
\end{bmatrix} > 0,
\]
and then minimize over $\gamma > 0$. The resulting integral gain — which is recovered as $K_{\infty} = ZY^{-1}$ — will by construction achieve the same peak sensitivity as Davison’s design, but the computational framework offers significant extensions. For instance, decentralization constraints $K \in K$ where $K \subseteq \mathbb{R}^{m \times p}$ is a subspace can be enforced by appending the additional constraints to (27) that $Y$ be diagonal and that $Z \in K$.

We illustrate these ideas via reference tracking on a randomly generated stable LTI system with 30 states, 7 inputs, and 5 outputs. We wish to design an feedback gain of the form
\[
K = \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix}, \quad K_{11} \in \mathbb{R}^{3 \times 3}, \quad K_{22} \in \mathbb{R}^{4 \times 2}, \tag{28}
\]
for use in a low-gain integral control scheme. The SDP (27) above was solved using SDPT3 with the YALMIP [35] interface in MATLAB. Figures 1a,1b show the response of the resulting full-order closed-loop system to sequential step reference changes for the 5 output channels for the designs $K_{\text{Dav}}$ and $K_{\infty}$, with associated maximum singular values of $S_{\text{slow}}(\omega)$ plotted in Figure 1c. The value of $\varepsilon$ was selected for the second design to match the bandwidth of the first design. The LMI-based design has no significant peaking in the sensitivity function and achieves the desired block-decentralization of the control actions.

B. Nonlinear and Uncertain Systems

Analogous to the LTI case (4),(26), the slow time-scale reduced dynamics of the plant (5) and the controller (7) are
\[
\dot{\eta}(t) = -\pi(K\eta(t), w(t)), \quad \eta(0) = \eta_0 \tag{29a}
\]
\[
e(t) = \pi(K\eta(t), w(t)). \tag{29b}
\]
These dynamics are significantly simpler in structure than the full nonlinear dynamics described by (5),(7). Motivated by this simplicity and by the observations in Section IV-A, our goal is to apply computational techniques from robust control for robust performance analysis and gain synthesis; both these problems will admit convex formulations. To precisely formulate performance criteria for (29), we let
\[
\Sigma_{\eta_0} : \mathcal{L}^p_{2e_w}[0, \infty) \to \mathcal{L}^p_{2e_w}[0, \infty), \quad e = \Sigma_{\eta_0}(w) \tag{30}
\]
denote the input-output signal-space operator defined by (29).

In generalizing the ideas in Section IV-A to nonlinear systems, we wish to establish performance guarantees which are independent of the operating point, which motivates the use of an incremental $L_2$-gain criteria. The system (30) is said to have incremental $L_2$-gain less than or equal to $\gamma \geq 0$ if there exists $\beta \geq 0$ such that
\[
\int_0^T \|e(t) - e'(t)\|_2^2 \, dt \leq \beta\|\eta_0 - \eta_0'\|^2_2 + \gamma^2 \int_0^T \|w(t) - w'(t)\|_2^2 \, dt
\]
where $L$ is coarsely described using point-wise incremental as the set of all maps one obtains. The set of functions is invertible on $\mathbb{R}$ well-posed representation (LFR) $T > \eta$ for all gain integral controllers.

To work towards establishing this property, we define a new $F,G,H,J,E$ ∈ $\mathbb{R}^5$ and $q = \eta$, all initial conditions $q$, in the sense that the mapping $\Delta(q) = q - J\Delta(q)$ is invertible on $\mathbb{R}^5$, and hence $\tilde{\Delta}$ is indeed a function; this holds trivially if $J = 0$. We now allow $\Delta$ to range over a set $\Delta$ of functions and define

$$\Pi \triangleq \{ \tilde{\Delta} \mid \Delta \in \Delta \}$$

as the set of all maps one obtains. The set of functions $\Delta$ is coarsely described using point-wise incremental quadratic constraints: assume we have available a convex cone of symmetric matrices $\Theta \subset \mathbb{S}^{(n_q + n_w) \times (n_q + n_w)}$ such that

$$\left[ \Delta(q) - \Delta(q') \right] \theta \left[ \Delta(q) - \Delta(q') \right]^T \geq 0$$

(33)

for all $q,q' \in \mathbb{R}^n$, all $\Theta \in \Theta$, and all $\Delta \in \Delta$; the constraint $\Theta \in \Theta$ must admit an LMI description. As a simple example, the cone of matrices

$$\Theta = \left\{ \theta \begin{bmatrix} -I_{n_w} & 0 \\ 0 & -\beta^2 I_{n_w} \end{bmatrix} \mid \theta \geq 0 \right\}$$

can be used to describe nonlinearities $\Delta$ which are globally Lipschitz continuous with parameter $L$. While further explanation of this modelling framework is beyond our scope, we refer the reader to [36]–[39] for details. We can now state the main analysis result, the proof of which is available in the online version. For notational convenience, with $\gamma \geq 0$ we set

$$\Theta_\gamma = \left[ -\gamma^2 I_{n_w} \ 0 \right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $\gamma \geq 0$ we set

$$\Theta_\gamma = \left[ -\gamma^2 I_{n_w} \ 0 \right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

then the following statements hold:

(i) the system (29a) is uniformly infinitesimally contracting with respect to norm $\|x\| = (x^TPx)^{1/2}$ on $\mathbb{R}^p$;

(ii) the incremental $L_2$-gain of the reduced dynamics (29) is less than or equal to $\gamma$.

For a fixed $K$ and $\rho$, the matrix inequality of Theorem 4.1 is affine in $(P,\Theta,\gamma^2)$ and the best upper bound on $\gamma$ can be computed via semidefinite programming. We note that the result can just as well be applied to uncertain linear systems as it can to nonlinear systems.

The main user effort in applying Theorem 4.1 is to appropriately select $F,G,H,J,E$ and $\Delta$ such that $\pi \in \Pi$. As a simple illustration of the ideas, we continue our example from Section III-A. For the mapping $\pi \circ k$ in (25) we may select $FK = 0$, $G = \beta^{-1}$, $E_1 = -\beta^{-1}$, $HK = 1$, $J = 0$, and $E_2 = 0$, with $\rho = \Delta(q) = 1 + Jq^T$. The nonlinear mapping $\Delta$ satisfies (33) with

$$\Theta = \left\{ \theta \left[ -2 \mu + L \right] \mid \theta \geq 0 \right\}$$

where $\tilde{\mu} = \sum_{i=1}^{m} \frac{1}{\mu_i}$ and $\tilde{L} = \sum_{i=1}^{m} \frac{1}{\mu_i}$. With $\rho = 0$, the performance LMI (34) reduces to the $3 \times 3$ LMI

$$\begin{bmatrix} 2\tilde{\mu} L \theta \\ P\beta^{-1} - \theta(\tilde{\mu} + \tilde{L}) \end{bmatrix} \begin{bmatrix} * \\ 2\theta - \beta^2 \end{bmatrix} \begin{bmatrix} * \\ \gamma^2 - \beta^{-2} \end{bmatrix} \geq 0.$$
A detailed analysis of this LMI shows that it is feasible in $P, \theta > 0$ if and only if
\[
\gamma \geq \gamma^* = \frac{1}{\beta} \sqrt{\frac{(\kappa + 1)^2}{4\kappa}}, \quad \kappa = \frac{L}{\mu} = \frac{\sum_{i=1}^{m} \frac{\mu_i}{\mu}}{\sum_{i=1}^{m} \frac{l_i}{l}}.
\]

It follows that the incremental $\mathcal{L}_2$-gain of the slow time-scale dynamics is upper bounded by $\gamma^*$. This suggests that control performance will degrade quite gracefully as a function of the aggregate condition ratio $\kappa$ of the objective functions, and provides some support for the use of penalty functions in (23).

C. Synthesizing Feedback Gains for Robust Performance

Theorem 4.1 can be exploited for direct convex synthesis of controller gains for uncertain and nonlinear systems. The methodology here is inspired by procedures for robust state-feedback synthesis, and requires the additional restrictions that the matrix $\Theta$ in (33) be nonsingular and that
\[
\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12} & \Theta_{22} \end{bmatrix}, \quad \Theta \triangleq \Theta^{-1} = \begin{bmatrix} \tilde{\Theta}_{11} & \tilde{\Theta}_{12} \\ \tilde{\Theta}_{12} & \tilde{\Theta}_{22} \end{bmatrix}
\]

To illustrate these analysis and synthesis concepts, we return to the reference tracking example of Section IV-A, and augment the previously generated system with three additional I/O channels $p, q \in \mathbb{R}^3$ with randomly selected coefficients in $G, H, J, E$. These channels are subject to the interconnection
\[
p = \Delta(q) = \text{col}((\text{sat}(q_1), \delta q_2, \delta q_3)
\]

where $\delta \in [-1, 1]$ is an uncertain real parameter and sat denotes the standard saturation function. The associated equilibrium input-to-error map $\pi$ is now both nonlinear and uncertain. For $\Delta$ given in (38), the constraint (33) holds with
\[
\Theta = \text{diag} \left( \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, -Q, S, Q \right)
\]

where $\theta_\delta > 0$, $S = \begin{bmatrix} \delta & 0 \\ 0 & s \end{bmatrix}$, $s \in \mathbb{R}$, and $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} > 0$.

As a nominal controller design for this example, we use the same controller $K = G(0)^T$ from Section IV-A, and attempt to certify robust performance via the SDP
\[
\begin{aligned}
&\text{minimize} & & \gamma^2 \quad \text{subject to} \quad (34) \\
&\text{minimize} & & \gamma, P > 0, \Theta \in \Theta
\end{aligned}
\]

Using SDPT3/YALMIP we can certify a performance bound of $\gamma = 16.9$ for the associated slow dynamics. While this is only an upper bound, it suggests that this nominal controller may perform poorly on instances of the nonlinear/uncertain system. Indeed, Figure 2a shows the step response of the resulting full-order nonlinear closed-loop system to sequential step reference changes for the 5 output channels, for the case $\delta = -0.5$. The nominal design performs poorly in the presence of uncertainty and nonlinearity. To improve the design, we note that the matrix $\Theta$ in (39) satisfies the additional restrictions mentioned in (35) if one restricts $s = 0$, in which case
\[
\tilde{\Theta} = \text{diag} \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -D & 0 \\ 0 & D \end{bmatrix} \right)
\]

with $\tilde{\theta}_\delta > 0$, and $D = \begin{bmatrix} d_{11} & d_{12} & d_{12} \end{bmatrix}$, $d_{11} = \begin{bmatrix} q_{11} & q_{12} \end{bmatrix}^{-1} > 0$. To synthesize a robust feedback gain, we solve the SDP
\[
\begin{aligned}
&\text{minimize} & & \gamma^2 \quad \text{subject to} \quad (37) \\
&\text{minimize} & & \gamma, Z > 0, \Theta \in \Theta
\end{aligned}
\]

For our example, a performance bound of $\gamma = 11.6$ is obtained, with step response shown in Figure 2c. The response is noticeably degraded over the centralized robust optimal design, but still improves on the nominal design, is robustly stable, and achieves decentralization of the integral action.

As a final comment, robust analysis and synthesis procedures based on (weighted) contraction norms $\| \cdot \|_1$ and $\| \cdot \|_\infty$ can also likely be developed, and may be valuable and efficient when the reduced dynamics is a monotone dynamical system; see [40] for related ideas.
where the additional problem structure here will allow us to arrive at simpler designs
$z \in \mathbb{R}^m$ for this class of problems. The functions $f_0, g_0$ are class $C^1$ and convex, with globally
Lipschitz continuous gradients, and may include penalty functions for inequality constraints. The constraint (41b) is the steady-state constraint imposed by the dynamic system (40); we make no assumptions about the row or column rank of $G(0)$. Finally, (41c) represents $n_c$ additional engineering constraints imposed by the designer. We assume (i) that (41) is feasible and has an optimal solution for all $w$, (ii) that the signal $H_z z(t) + H_u u(t) - L w(t)$ is measurable, and (iii) that
\[
\text{rank} \begin{bmatrix} I_r & -G(0) \\ H_z & H_u \end{bmatrix} = r + n_c,
\] meaning the constraints (41c) are independent and are never redundant with the steady-state constraints (41b). This final assumption is merely a matter of the designer appropriately specifying the desired additional constraints (41c).

In [12] a number of designs are presented for this problem, which are all based on constructing a suitable error output to be driven to zero; we analyze a variation. First note that the constraints (41b)–(41c) can be rewritten as
\[
0 = \begin{bmatrix} I_r & -G(0) \\ H_z & H_u \end{bmatrix} T_u, \quad T_u \in \mathbb{R}^{(r+m) \times p_1}
\] such that
\[
\begin{bmatrix} \tilde{z} \\ \tilde{u} \end{bmatrix} = \begin{bmatrix} T_z & T_u \end{bmatrix} \xi + \begin{bmatrix} z_0(w) \\ u_0(w) \end{bmatrix},
\]
where $(z_0(w), u_0(w))$ is any feasible point and $\xi \in \mathbb{R}^{p_1}$ is arbitrary. The rank properties of the sub-matrices $T_z$ and $T_u$ will be crucial going forward.

**Lemma 5.1 (Properties of $T$):** Let $T = [T_z \ 0] \in \mathbb{R}^{(r+m) \times p_1}$ be any matrix of full column rank such that (43) holds. The following statements hold:
(i) $T_u \in \mathbb{R}^{m \times p_1}$ has full column rank;
(ii) $T_z \in \mathbb{R}^{r \times p_1}$ has full column rank if and only if
\[
\text{range}(T_u) \subseteq \text{range}(G(0)^T) = \text{range}(G(0)^\dagger);
\] (iii) $G(0)$ full column rank $\implies$ $T_z$ full column rank;
(iv) If $G(0)$ has full row rank, then $T_z$ may be chosen to have full column rank and only if there exists a full column rank matrix $X$ such that $(H_z + H_u G(0)^\dagger) X = 0$, in which case one choice is $T_z = X$ and $T_u = G(0)^\dagger X$.

Using (44), the problem (41) is now equivalent to the unconstrained minimization problem
\[
\begin{array}{ll}
\text{minimize} & f_0(T_z \xi + z_0) + g_0(T_u \xi + u_0) \\
\text{subject to} & \xi \in \mathbb{R}^{p_1}
\end{array}
\] with $\xi^*$ being an optimal solution if and only if
\[
0 = T_z^T \nabla f_0(T_z \xi^* + z_0) + T_u^T \nabla g_0(T_u \xi^* + u_0).
\]
An error signal $e = (e_1, e_2) \in \mathbb{R}^{n_c + p}$ may therefore be constructed by using $z(t)$ and control $u(t)$ as
\[
e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = \begin{bmatrix} H_z z(t) + H_u u(t) - L w \\ T_z^T \nabla f_0(z(t)) + T_u^T \nabla g_0(u(t)) \end{bmatrix},
\]
with the other primal feasibility condition (41b) being enforced by the plant itself in closed-loop. If the closed-loop system is internally stable, then driving $e$ to zero will drive $(x, u)$ towards an optimal steady-state $(\hat{z}(w), \hat{u}(w))$. To this end, consider the integral feedback design
\[
\begin{align*}
\dot{\hat{z}}_1 &= -e_1 e_1, \\
\dot{\hat{z}}_2 &= -e_2 e_2, \\
\dot{\hat{u}} &= K_1 \hat{z}_1 + K_2 \hat{z}_2
\end{align*}
\] (49)
where $e_1, e_2 > 0$. The closed-loop system consists of the plant (40) with nonlinear error output (48), and the controller (49).

**Theorem 5.2 (Closed-Loop Stability of Optimal Steady-State Control):** Consider the steady-state control specification (41) under the stated assumptions, along with the closed-loop system (40), (48), (49). Select the controller gain $K_1$ such that $- (H_z G(0) + H_u) K_1$ is Hurwitz, define the projection matrix
\[\Pi_c \triangleq I_m - K_1 [(H_z G(0) + H_u) K_1]^{-1} (H_z G(0) + H_u),\] (50)
and subsequently select the controller gain $K_2$ such that $\Pi_c K_2 = T_u P_1$ where $P_2 > 0$ is arbitrary. Assume additionally that either
(i) $g_0$ is $\mu_\theta$-strongly convex, or
(ii) $f_0$ is $\mu_f$-strongly convex and $T_2$ has full column rank.
Then for each $w \in \mathbb{R}^n$, the closed-loop system possesses a unique equilibrium point $(\bar{z}, \bar{e})$ at which $(\bar{z}, \bar{u})$ is an optimizer of (41), and there exist sufficiently small choices $e_1 \gg e_2 > 0$ such that equilibrium point is globally exponentially stable.

**Proof of Theorem 5.2:** The proof is based on two sequential applications of Theorem 3.1: the constructions for the controller gains $K_1, K_2$ will be shown to be well-posed along the way.

**Step 1:** Consider first the plant (40) with error output $e_1$ as defined in (48). We show that the assumptions of Theorem 3.1 are met. Since (40) is LTI with $A$ being Hurwitz and since $e_1$ is an affine function of $z$ and $u$, (A1)-(A3) hold globally. The equilibrium input-to-error map of the plant from $u$ to $e_1$ is
\[\pi_1(u, w) = (H_z G(0) + H_u) u + (H_z G(w)(0) - L)w.\] (51)
We consider the feedback controller
\[\bar{\eta}_1 = -e_1 e_1, \quad \bar{u} = K_1 \bar{\eta}_1 + u_2\] (52)
where $u_2$ is a constant auxiliary input. Assumption (A4) holds since the feedback is affine in $\eta_1$. We need only show that (A6) holds on $\mathbb{R}^m_v$. By the full row rank assumption (42), it follows by elementary row operations that
\[\text{rank} \begin{bmatrix} I_r & -G(0) \\ 0 & H_z G(0) + H_u \end{bmatrix} = r + n_c,\]
and therefore $H_z G(0) + H_u$ has full row rank. Thus, there always exists $K_1$ such that $-(H_z G(0) + H_u) K_1$ is Hurwitz, and for any such $K_1$ there exists $P_1 > 0$ such that
\[(H_z G(0) + H_u) K_1 P_1 + P_1 (H_z G(0) + H_u) K_1 > 0.\]
It follows that (20) holds, which verifies (A6) globally. We conclude that we may select $e_1 > 0$ sufficiently small such that the (LTI) closed-loop system (40) with error output $e_1$ and controller (52) is internally stable and $e_1(t) \to 0$ exponentially for all asymptotically constant signals $w$ and $u_2$.

**Step 2:** We now consider the LTI plant defined by (40) and (52) with state $(x, \eta_1)$, input $u_2$ and nonlinear output $e_2$ defined in (48). By the constructions in Step 1, this secondary plant is LTI and internally stable, and hence (A1)-(A2) once again both hold globally. Since $\nabla f_0$ and $\nabla g_0$ are assumed to be globally Lipschitz, the output $e_2$ defined in (48) satisfies (A3) globally. To determine the equilibrium input-to-error mapping of this secondary plant, note that for constant inputs $w$ and $\bar{u}_2$, the steady-state value of $\eta_1$ is determined by the condition that $\pi_1 = 0$, which yields
\[0 = (H_z G(0) + H_u) \bar{u} + (H_z G(w)(0) - L)w\]
\[\bar{u} = K_1 \bar{\eta}_1 + \bar{u}_2.\] (53)
We quickly find from here that
\[\bar{\eta}_1 = - [(H_z G(0) + H_u) K_1]^{-1} (H_z G(0) + H_u) \bar{u}_2 + Dw\]
where $D = - [(H_z G(0) + H_u) K_1]^{-1} (H_z G(w)(0) - L)$. Substituting $\bar{\eta}_1$ back into (53) we conclude that
\[\bar{u} = \Pi_c \bar{u}_2 + K_1 Dw,\]
where $\Pi_c$ is as in (50). A straightforward calculation shows that $\Pi_c^2 = \Pi_c$, and hence $\Pi_c$ is a projection matrix. Moreover, note that $\text{null}(\Pi_c) = \text{null}(H_z G(0) + H_u)$, and therefore $\text{range}(\Pi_c) = \text{null}(H_z G(0) + H_u)$. Thus, $\Pi_c$ is an oblique projection from $\mathbb{R}^m_v$ onto the subspace $\text{null}(H_z G(0) + H_u)$. It now follows quickly that
\[\pi_2(u_2, w) = T_z^T \nabla f_0 (G(0) \Pi_c u_2 + d_1) + T_z^T \nabla g_0 (\Pi_c u_2 + d_2)\] (54)
where $M_1 = G(0) K_1 D + G(w)(0), M_2 = K_1 D$, and $d_k = M_k w$. We now consider the feedback controller
\[\eta_2 = -e_2 e_2, \quad u_2 = K_2 \eta_2\] (55)
where $K_2$ is selected such that $\Pi_c K_2 = T_u P_2$ for arbitrary $P_2 > 0$. From Lemma 5.1 (i) we know that $T_u$ has full column rank, and hence so does $T_u P_2$. Moreover, by construction from (43), we have that $(H_z G(0) + H_u) T_u = 0$, meaning that $\text{range}(T_u) \subseteq \text{null}(H_z G(0) + H_u)$. It follows that the right-hand side of $\Pi_c K_2 = T_u P_2$ always lies in the range of the $\Pi_c$, and hence the equation is always solvable for a matrix $K_2$ of full column rank.

Assumption (A4) holds since the feedback (55) is linear in $\eta_2$: we will verify that (A6) holds globally. From (54), we note that $j(\eta_2) = \pi_2(K_2 \eta_2, w)$ is given by
\[j(\eta_2) = T_z^T \nabla f_0 (G(0) \Pi_c K_2 \eta_2 + d_1) + T_z^T \nabla g_0 (\Pi_c K_2 \eta_2 + d_2).\]
Substituting for $\Pi_c K_2$ and using from (43) that $T_z = G(0) T_u$, we can simplify this to obtain
\[j(\eta_2) = T_z^T \nabla f_0 (T_z P_2 \eta_2 + d_1) + T_z^T \nabla g_0 (T_z P_2 \eta_2 + d_2).\]
Following (20), we compute that
\[\eta_2 - \eta_2^* P_2 (j(\eta_2) - j(\eta_2^*)) = F(\eta_2, \eta_2^*) + G(\eta_2, \eta_2^*)\]
where
\[F(\xi, \xi') = (\xi - \xi') P_2 T_z^T [\nabla f_0 (T_z P_2 \xi + d_1) - \nabla f_0 (T_z P_2 \xi' + d_1)]\]
\[G(\xi, \xi') = (\xi - \xi') P_2 T_z^T [\nabla g_0 (T_z P_2 \xi + d_2) - \nabla g_0 (T_z P_2 \xi' + d_2)].\]
If (i) holds, then since $T_u$ has full column rank, the mapping $\xi \mapsto P_2T_u^T \nabla g(T_u P_2 \xi)$ is strongly monotone, and we conclude that (20) holds for some $\rho_0 > 0$. Similarly, if (ii) holds, then $\xi \mapsto P_2T_u^T \nabla f(T_u P_2 \xi)$ is strongly monotone, and the same conclusion follows. In either case, we conclude that the condition (20) holds globally, and hence that (A5)–(A6) hold. All conditions of Theorem 3.1 are once again established, which completes the proof. □

We illustrate the ideas in this section on an instance of (41) which describes a disturbance rejection problem

\[
\begin{align}
\text{minimize} & \quad \frac{1}{2} \| \tilde{z} \|^2 + c \text{Penalty}(\bar{u}) \\
\text{subject to} & \quad \tilde{z} = G(0) \bar{u} + G_w(0) w \\
& \quad 0 = \sum_{k=1}^{m} \bar{u}_k - \sum_{j=1}^{n_w} w_k
\end{align}
\]

where $c > 0$ is a design parameter. The objective is to keep $z$ near zero, with the penalty function

\[\text{Penalty}(\bar{u}) = \sum_{k=1}^{m} \max(0, u_{\text{min}} - \bar{u}_k, \bar{u}_k - u_{\text{max}})^2\]

attempts to restrict all steady-state control signals to the range $[u_{\text{min}}, u_{\text{max}}]$. It is additionally required — as a hard constraint — that the total steady-state control balances the disturbance in the sense of (56c). In terms of the standard form (41), we have $f_0(z) = \frac{1}{2} \| z \|^2$ being strongly convex, $g_0(z) = c \text{Penalty}(\bar{u})$ being merely convex, $H_z = 0$ and $H_u = I_m$. The error outputs for integration will be constructed as

\[
\begin{bmatrix}
\epsilon_1(t) \\
\epsilon_2(t)
\end{bmatrix} = \begin{bmatrix}
\frac{1}{T_u} \bar{u} - \frac{1}{T_w} w \\
T_u^T \tilde{z} + c T_u^T \nabla \text{Penalty}(\bar{u})
\end{bmatrix}.
\]

The linear system (40) is a randomly generated stable system with 10 states, 5 inputs, 3 outputs, and 3 disturbances, and is such that $G(0)$ has full row rank. We therefore can follow Lemma 5.1 (iv) to construct $T_u$ and $T_z$, and we find ourselves in case (ii) of Theorem 5.2. The response of the closed-loop system to sequential step disturbances in the three disturbance channels is shown in Figure 3.

**Remark 5.3 (Recovering Davison’s Controller):** The classical low-gain tracking controller $K = G(0)^\dagger$ from [1] can be recovered in two different ways within our framework, by either (i) enforcing perfect asymptotic tracking as an equality constraint, or (ii) by penalizing the asymptotic tracking error in the objective function. For the first approach, assume $G(0)$ has full row rank and specialize (41) to the form

\[
\begin{align}
\text{minimize} & \quad 0 \\
\text{subject to} & \quad \tilde{z} = G(0) \bar{u} + G_w(0) w \\
& \quad 0 = \tilde{z}
\end{align}
\]

where $z$ is to be interpreted as the tracking error. In terms of (41), we have $H_z = I_r$ and $H_u = 0$. There are $n_c = r$ constraints, and the rank assumption (42) obviously holds. From Theorem 5.2, the selection requirement for $K_1$ reduces to $-G(0) K_1$ being Hurwitz, and we may take $K_1 = G(0)^\dagger$. The construction of $K_2$ can be skipped and the state $\eta_2$ omitted, since $c_2 = 0$. The controller (48)–(49) therefore reduces to

\[
\dot{\eta}_1 = -\epsilon_1 z, \quad u = G(0)^\dagger \eta_1.
\]

which is precisely Davison’s low-gain integral design. For the second approach, again assume $G(0)$ has full row rank and consider instead the formulation

\[
\begin{align}
\text{minimize} & \quad \frac{1}{2} \| \tilde{z} \|^2 \\
\text{subject to} & \quad \tilde{z} = G(0) \bar{u} + G_w(0) w.
\end{align}
\]

We may satisfy the criteria of Lemma 5.1 (iv) by selecting $X = I_r$, in which case $T_z = I_r$ and $T_u = G(0)^\dagger$. Since there are no equality constraints, we may omit the state $\eta_1$ and select $\Pi_c = I_m$, and with $P_2 = I_m$, the selection criteria for $K_2$ tells us that $K_2 = T_u = G(0)^\dagger$. We find ourselves in case (ii) from the theorem statement, and the controller reduces to

\[
\dot{\eta}_2 = -\epsilon_2 z, \quad u = G(0)^\dagger \eta_2.
\]

which is again Davison’s design. □

**VI. Conclusions**

Relaxed conditions have been given for stability of a non-linear system under low-gain integral control, generalizing those available in the literature; the key idea is to impose an incremental-stability-type condition on the plant equilibrium input-to-error map. Furthermore, when this map can be represented or over-approximated via a linear fractional representation, robust control techniques can be applied to certify the key condition in the theorem and to synthesize appropriate integral controller gains. The results are easily applicable to the study of feedback-based optimization schemes for dynamic systems.

Future work will focus on the application of these results to control problems in the energy systems domain and to further instances of feedback-based optimization. An open theoretical direction is to generalize the analysis and design approach presented here for tracking of signals generated by an arbitrary linear exosystem, which would yield a full generalization of [1].
This generalization may require that incremental-type stability conditions be imposed on the plant, as considered in [41]. Another open direction is to extend the low-gain results here to anti-windup designs (e.g., [42]), which should require only a modified analysis of the slow time-scale dynamics (11).

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John W. Simpson-Porco (S’11–M’16) received the B.Sc. degree in engineering physics from Queen’s University, Kingston, ON, Canada in 2010, and the Ph.D. degree in mechanical engineering from the University of California at Santa Barbara, Santa Barbara, CA, USA in 2015. He is currently an Assistant Professor of Electrical and Computer Engineering at the University of Waterloo, Waterloo, ON, Canada. He was previously a visiting scientist with the Automatic Control Laboratory at ETH Zurich, Zurich, Switzerland. His research focuses on feedback control theory and applications of control in modernized power grids.

Prof. Simpson-Porco is a recipient of the 2012–2014 IFAC Automatica Prize and the Center for Control, Dynamical Systems and Computation Best Thesis Award and Outstanding Scholar Fellowship. He currently serves as an Associate Editor for the IEEE Transactions on Smart Grid.
APPENDIX A
SUPPORTING LEMMAS AND PROOFS

Proof of Theorem 4.1: Fix any $\Delta \in \Delta$ and let $\tilde{\pi}_\Delta$ be defined by (31). Let $\xi, \xi' \in \mathbb{R}^p$ and $w, w' \in \mathbb{R}^m$ be arbitrary, set $u \triangleq K\xi$ and $u' \triangleq K\xi'$, and correspondingly define $(\tilde{e}, q, p)$ and $(\tilde{e}', q', p')$ via (31). From (31) we find that

$$
\begin{aligned}
\dot{\epsilon} - \tilde{e}' &= FK(\xi - \xi') + G(p - p') + E_1(w - w') \\
q - q' &= HK(\xi - \xi') + J(p - p') + E_2(w - w') \\
p - p' &= \Delta(q) - \Delta(q')
\end{aligned}
$$

Left and right multiplying (34) by $\text{col}(\xi - \xi', p - p', w - w')$ and using (59), we obtain

$$
\begin{aligned}
\left[\begin{array}{c}
\tilde{e} - \tilde{e}' \\
\xi - \xi'
\end{array}\right] &= 
\left[\begin{array}{cc}
2\rho P & -P \\
-P & 0
\end{array}\right]
\left[\begin{array}{c}
\tilde{e} - \tilde{e}' \\
\xi - \xi'
\end{array}\right] + 
\left[\begin{array}{c}
p - p' \\
q - q'
\end{array}\right] \\
+ \|\tilde{e} - \tilde{e}'\|^2 - \gamma^2\|w - w'\|^2_2 &\leq 0
\end{aligned}
$$

(60)

To show statement (i), select $w' = w$, and note that the second term in the above inequality is non-negative. Since $\tilde{e} = \tilde{\pi}_\Delta(K\xi, w)$, it follows that

$$(\pi_\Delta(K\xi, w) - \pi_\Delta(K\xi', w))'P(\xi - \xi') \geq 2\rho\|\xi - \xi'\|^2_2,$$

which establishes (20). Since $\Delta$ was arbitrary and $\pi \in \Pi$, statement (i) holds. To show (ii) consider the extended input-output dynamics

$$
\dot{\xi} = -\tilde{e} \hspace{1cm} \dot{\xi}' = -\tilde{e}' \\
\tilde{e} = \tilde{\pi}_\Delta(K\xi, w) \hspace{1cm} \tilde{e}' = \tilde{\pi}_\Delta(K\xi', w') \\
(61)
$$

and define $V(\xi, \xi') = \|\xi - \xi'\|^2_2$. Differentiating along trajectories of (61) and inserting (60), we find

$$
\dot{V}(\xi, \xi') = -2(\tilde{e} - \tilde{e}')'P(\xi - \xi') \\
\leq -2\rho V(\xi, \xi') - \|e - e'\|^2_2 + \gamma^2\|w - w'\|^2_2
$$

Integrating from 0 to time $T > 0$ we have

$$
V(\xi(T), \xi'(T)) - V(\xi(0), \xi'(0)) \\
\leq - \int_0^T \|e - e'\|^2_2 + \gamma^2\|w - w'\|^2_2 dt.
$$

and the performance result quickly follows with $\beta = \lambda_{\text{max}}(P)$, since $V(\xi, \xi') \geq 0$. Since $\Delta$ was arbitrary and $\pi \in \Pi$, statement (ii) holds.

Proof of Lemma 5.1: (i): Suppose by contradiction that $T_u \xi = 0$ for some $\xi$. Multiplying $T_z = G(0)T_u$ by $\xi$, we find that $T_z \xi = 0$, so $\xi \in \text{null}(T_z) \cap \text{null}(T_u) = \text{null}(T)$, which contradicts the fact that $T$ has full column rank.

(ii): Since $T_u$ has full column rank, $T_z$ has full column rank if and only if $\text{range}(T_u) \subseteq \text{null}(G(0)) = \text{range}(G(0)^\top)$. The final statement follows from the fact that $\text{range}(G(0)^\top) = \text{range}(G(0)^\top)$.

(iii): If $G$ has full column rank then $\text{range}(G(0)^\top) = \mathbb{R}^m$, so (45) holds the result follows from (ii).

(iv): First note that if $G(0)$ has full row rank, then $G(0)^\dagger$ is given explicitly as $G(0)^\dagger = G(0)^\top(G(0)G(0)^\top)^{-1}$ and has full column rank. From (ii), we can then say that $T_z$ has full column rank if and only if $T_u = G(0)^\dagger X$ for some full column rank matrix $X \in \mathbb{R}^{r \times k}$, in which case we simply have $T_z = X$. Substituting into the second block equation, $X$ must satisfy

$$
0 = (H_x + H_u G(0)^\dagger)X
$$

which shows the desired result.

$\square$