Pair production of fundamental unstable particles in modified perturbation theory in NNLO

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Abstract

We consider pair production and decay of fundamental unstable particles in the framework of a modified perturbation theory (MPT) treating resonant contributions in the sense of distributions. The cross-section of the process is calculated within the NNLO of the MPT in a model that admits exact solution. Universal massless-particles contributions are taken into consideration. A comparison of the outcomes with the exact solution demonstrates excellent convergence of the MPT series at the energies near and above the maximum of the cross-section.

1 Introduction

A description of the processes of productions and decays of fundamental unstable particles to satisfy the up-to-date requirements must provide, on the one hand, gauge cancellations and unitarity and, on the other hand, enough high accuracy of calculation of resonant contributions of unstable particles. Unfortunately, in the framework of conventional perturbation theory (PT) a simultaneous fulfilling of these requirements is obstructed by divergences caused by resonant contributions. For this reason in the propagators of unstable particles the Dyson resummation is usually applied, which shifts the resonant singularities out of the region of physical momenta. However, a resummation mixes the PT orders, which generally leads to violation of the gauge cancellations. So simultaneously with using the Dyson resummation an application of additional tricks is required.

Among various approaches that include such tricks, the most known one is based on the Laurent expansion of the amplitude around the complex poles of the resonant propagators. Each term of this expansion is considered expanded in the framework of the conventional PT, as well, but a certain portion of the self-energy is not involved in the latter expansion as having been absorbed by the shift of the point of singularity (the remnant of the Dyson resummation). The gauge cancellations are completely maintained in this approach. However, the precision of the description vastly falls at the increasing of a distance from the resonant region, and an uncertainty arises at calculating the residues in the complex-poles. Nevertheless, in the vicinity of the
resonant region the pole expansion in many cases is suitable for applications. In particular, at LEP2 the loop corrections to the $W$-pair production were calculated in the double pole approximation (DPA) [1, 2], the leading approximation in the pole expansion. Unfortunately, at international linear collider (ILC) [3] the accuracy of DPA is no longer sufficient [4], and the higher-order corrections in the pole expansion unlikely can save the situation. Therefore the pinch-technique method and the method based on the background-field formalism move forward to foreground, which, in principle, can provide the necessary precision (see [5] and [6], and the references therein). However, the consecutive application of the mentioned methods implies a calculation of a huge volume of additional contributions that formally appear outside the limits of required precision, which is impractical [7]. So at present hopes are pinning on the approach of “complex-mass scheme” (CMS), which avoids mentioned difficulties [4,8]. Nevertheless, in the CMS another problem related to the unitarity arises. The point is that the CMS uses the complex-valued renormalized masses for unstable particles and this requires an introduction of the complex-valued counterterms, which violates unitarity. For this reason the CMS cannot be considered as a rigorous procedure [8]. The problem becomes especially topical at calculating the contributions in the next-to-next-to-leading order (NNLO). Thus to make the calculations up to the NNLO alternative approaches are required.

A promising candidate for this role is a modified perturbation theory (MPT), first proposed in [9] and then elaborated in [10] and [11]. For determining the resonant contributions the distribution theory is applied in this approach instead of the Dyson resummation in whatever form. In essence, the MPT implies a systematic expansion in powers of the coupling constant directly of the probability instead of the amplitude. This mode allows one to impart the sense of distributions to the propagators squared of unstable particles, and on this basis to asymptotically expand the propagators squared without the appearance of the divergences in the cross-section. Since the object to be expanded (the cross-section) is gauge invariant and the expansion is made in powers of the coupling constant, the result of the expansion must automatically be gauge-invariant. This implies that the gauge cancellations in the MPT must be automatically maintained. Of course, this should be so if the MPT exists, i.e. if it is a well-determined method. In the case of pair production of unstable particles this property was proved and an algorithm of the calculation of each order of the MPT expansion was elaborated [11].

The aim of the given paper is to perform numerical analysis of the convergence properties of the MPT series in the case of pair production of unstable particles. At once we should notice that in the qualitative sense the outcomes should weakly depend on the model under consideration because the choice of a model implies mainly a definition of the test function in the presence of which the relevant distributions (the propagators squared) are MPT-expanded. So it is reasonable to carry out the examination in the framework of a model possessing an exact solution. As such a model, we consider the improved Born approximation for the process $e^+ e^- \rightarrow \gamma, Z \rightarrow t\bar{t} \rightarrow W^+b W^-\bar{b}$. For simplicity we consider $W$ bosons and $b$ quarks to be stable particles, with $W$ being massive and $b$ being massless. At the same time we consider realistic corrections to the width of the top quark. This should allow us to get an
information about the rapidity of convergence in the realistic case. A similar model has actually been considered in [12] by examining the MPT within the next-to-leading order (NLO). However, the contribution from the soft massless particles to the process of production of unstable particles have been omitted in that work. This was a serious omission because the mentioned contributions include Coulomb singularities [13]-[15] appreciably affecting the cross-section. In this paper we improve the calculations of [12] (in particular eliminate some bug in the calculations), and carry out numerical calculations further up to the NNLO with taking into consideration universal Coulomb singular contributions.

In the next section, we present the basic information about the MPT and detail the model in the framework of which we carry out computations. In Sect. 3 we present outcomes. In Sect. 4 we discuss the results.

2 MPT and a model for its examination

The observable cross-section of production and decay of unstable particles, for example in $e^+e^-$ annihilation, has the form of a convolution of the hard-scattering cross-section with the flux function [1].

$$\sigma(s) = \int_{s_{\text{min}}}^{s} \frac{ds'}{s'} \phi(s'/s) \hat{\sigma}(s').$$ (1)

Here $s$ is the energy squared in the center-of-mass system, $\hat{\sigma}$ is the hard-scattering cross-sections, $\phi$ is the flux function describing contributions of nonregistered photons emitted in the initial state. The $s'/s$ characterizes a fraction of the energy expended on the production of unstable particles. For our purposes it is sufficient to take $\phi$ in the leading-log approximation. So we put

$$\phi(z; s) = \beta_e (1 - z)(\beta_e - 1) - \frac{1}{2} \beta_e (1 + z), \quad \beta_e = \frac{2\alpha}{\pi} \left( \ln \frac{s}{m_e^2} - 1 \right).$$ (2)

In the case of pair production of unstable particles the double-resonant contributions are most crucial. Bearing this in mind we write down the hard-scattering cross-section in the form

$$\hat{\sigma}(s) = \int_{s_{1\text{min}}}^{\infty} \int_{s_{2\text{min}}}^{\infty} ds_1 ds_2 \hat{\sigma}(s; s_1, s_2) (1 + \delta_c).$$ (3)

In this formula $\hat{\sigma}(s; s_1, s_2)$ is an exclusive cross-section, $\delta_c$ stands for soft massless-particles contributions, $s_1$ and $s_2$ are virtualities of unstable particles. In the case of the process $e^+e^- \to \gamma, Z \to t\bar{t} \to W^+b W^-\bar{b}$ with massive $W$ and massless $b$, we have $s_{1\text{min}} = s_{2\text{min}} = M_W^2$, and $s_{\text{min}} = 4M_W^2$. In $\hat{\sigma}(s; s_1, s_2)$ we extract kinematic and Breit-Wigner (BW) factors,

$$\hat{\sigma}(s; s_1, s_2) = \frac{1}{s^2} \theta(\sqrt{s} - \sqrt{s_1} - \sqrt{s_2}) \sqrt{\lambda(s, s_1, s_2)} \Phi(s; s_1, s_2) \rho(s_1) \rho(s_2).$$ (4)
Here λ(s, s₁, s₂) = [s−(√s₁+√s₂)²][s−(√s₁−√s₂)²] is the kinematic function, ρ(s₁) and ρ(s₂) are BW factors. Function Φ(s; s₁, s₂) is the rest of the amplitude squared. Below we consider Φ in the Born approximation, and thus only the BW factors are subject to the MPT expansion. In the general case, we define the BW factors as

\[ ρ(s) = \frac{MΓ₀}{π} \times |Δ(s)|². \]  

(5)

Here M is the renormalized mass of the top quark, Γ₀ is its Born width, Δ(s) is a scalar part of the Dyson-resummed propagator (the spin factor is referred to Φ),

\[ Δ^{-1}(s) = s - M² + ReΣ(s) + i ImΣ(s), \]  

(6)

ReΣ(s) and ImΣ(s) are the real and imaginary parts of the renormalized self-energy.

In the case of a smooth weight, an isolated BW factor may be represented in the form of an asymptotic expansion in the sense of distributions in powers of the coupling constant (generating thus the MPT expansion of isolated BW factor). Up to and including the NNLO this expansion looks as follows [2]:

\[ ρ(s) = δ(s-M²) + \frac{MΓ₀}{π} PV \left\{ \frac{1}{(s-M²)²} - \frac{2α ReΣ₁(s)}{(s-M²)³} \right\} + \sum_{n=0}^{2} c_n(α) \frac{(-s)^n}{n!} δ^{(n)}(s-M²) + O(α³). \]  

(7)

Here α is the coupling constant, δ(⋯) is the δ-function, δ⁽ⁿ⁾ is its nth derivative, PV means the principal-value prescription. The leading term in (7) defines the narrow-width approximation. The contributions in the curly brackets appear as a result of the naive expansion of the propagator squared; PV makes the poles in this expansion integrable. The contributions under the sum-sign correct the contributions of the PV poles (in the singular point) so that the expansion becomes asymptotic. The coefficients cⁿ(α) are polynomials in α, determined by the self-energy contributions of the unstable particle. In an arbitrary UV-renormalization scheme the completely explicit expressions for cⁿ(α) may be found in [10]. In the case of the on-mass-shell (OMS) type scheme, they are found in [11]. In the latter case the coefficients cⁿ within the NNLO are determined by \( I₁, I₂, I₃, I₁', I₁'' \), where \( I_n = ImΣ_n(M²), \)

\( I'_n = ImΣ'_n(M²), \)

\( I''_n = ImΣ''_n(M²), \)

and by \( R₂, R'_₂ \), where \( R_n = ReΣ_n(M²), \)

\( R'_n = ReΣ'_n(M²). \)

Here \( Σ_n \) is the n-loop self-energy defined in accordance with relation \( Σ = αΣ₁ + α²Σ₂ + ⋯ \).

Unfortunately, the weight in our case is not smooth because of the kinematic factor in formula (4). A solution to this problem is found on the basis of analytic regularization via the substitution \( [λ(s, s₁, s₂)]^{½} \rightarrow [λ(s, s₁, s₂)]'' \). Furthermore, the weight \( Φ (1 + δ_ϕ) \) may be expanded in powers of \( s₁ \) and \( s₂ \) around \( s₁ = M² \) and \( s₂ = M² \). Then, it becomes possible to analytically calculate singular integrals irrespective of details of the definition of the weight. After calculating singular integrals and removing the regularization the outcomes remain finite and the expansion remains asymptotic [11]. In principle, this salvages the applicability of the approach, and the problem is reduced to numerical calculations only.
Now we turn to the definition of the model in the framework of which we will carry out calculations. At first we notice that within the NNLO the MPT expansion of the general BW factor based on propagator \( \Delta^{-1}_{\text{NNLO}} \) coincides with that based on the following “minimal” propagator:

\[
\Delta^{-1}_{\text{NNLO}}(s) = s - M^2 + \alpha \Re \Sigma_1(s) + i \alpha \left[ I_1 + (s-M^2) I'_1 + \frac{1}{2} (s-M^2)^2 I''_1 \right] + \alpha^2 \left[ R_2 + i I_2 + (s-M^2) R'_2 \right] + i \alpha^3 I_3. \tag{8}
\]

In fact, after the MPT expansion any contribution not included in (8) appears outside the NNLO. However, under the consideration in the conventional-function sense, the \( R_3 \) and \( I'_2 \) in the region \( s - M^2 \sim O(\alpha) \) may be assigned to the NNLO, as well. So we start from the following modeling propagator:

\[
\Delta^{-1}_{\text{NNLO}}(s) = s - M^2 + \alpha \Re \Sigma_1(s) + i \alpha \Im \Sigma_1(s) + \alpha^2 \left[ R_2 + i I_2 + (s-M^2) (R'_2 + i I'_2) \right] + \alpha^3 (R_3 + i I_3). \tag{9}
\]

Propagator (9) ensures the NNLO precision from the point of view of both the MPT and conventional functions. For uniformity we consider \( \Im \Sigma_1 \) off-shell as well as in the case of \( \Re \Sigma_1 \).

Now let us direct our attention to the definition of the two- and three-loop contributions to propagator (9). Actually they may be determined by using the only fact that they are the on-shell contributions. Specifically, the real parts may be determined by basing on the UV-renormalization conditions. But one should remember that in the unstable-particles case the OMS scheme may be determined in different fashions. In particular, the conventional OMS scheme is determined by the conditions \( R_n = 0 \) and \( R'_n = 0 \) [16]. However, it is inconvenient for the calculations in the higher-orders, because the renormalized mass \( M \) in this scheme beginning with the two-loops is different from the observable mass, and beginning with the two-loops generally is gauge-dependent [17]. The problem is eliminated at considering the first renormalization condition in the form \( M^2 = \Re s_p \), where \( s_p \) is the pole of the propagator, \( \Delta^{-1}(s_p) = 0 \), which means the equating of the renormalized mass to the observable mass. The second renormalization condition may be determined by equating the imaginary part of the on-shell self-energy to the imaginary part of \( s_p \). As a result the equality \( s_p = M^2 - i I \) is established by means of the UV-renormalization conditions, where \( I = \Im \Sigma(M^2) \). So both the renormalized mass and the imaginary part of the on-shell self-energy become gauge-independent. This scheme of the UV renormalization was called the OMS scheme in [18] and the “pole scheme” in [19]. In this scheme the \( R_2, R'_2 \) and \( R_3 \) are determined as

\[
R_2 = -I_1 I'_1, \quad R'_2 = -I_1 I''_1/2, \quad R_3 = -I_2 I'_1 - I_1 I'_2 + I_1 R''_1/2. \tag{10}
\]

The imaginary contributions to the on-shell self-energy are determined, in effect, by the unitarity condition. For \( I_1 \) and \( I_2 \) the appropriate relations are

\[
\alpha I_1 = M \Gamma_0, \quad \alpha^2 I_2 = M \alpha \Gamma_1. \tag{11}
\]
Here $\Gamma_0$ and $\alpha \Gamma_1$ are the Born and the one-loop contributions to the width. The $I_3$, in the general case, is nontrivially related with the two-loop contribution $\alpha^2 \Gamma_2$. In the OMS scheme this relation is

$$\alpha^3 I_3 = M \alpha^2 \Gamma_2 + \Gamma_0^2 / (8M). \quad (12)$$

Unfortunately, the derivatives of $\text{Im} \Sigma$ cannot be determined by similar means. However the $I'_2$, which we need, is a facultative quantity from the point of view of the MPT expansion (see above). So we may determine $I'_2$ by using the approximate relationships

$$\alpha^2 \text{Im} \Sigma_2(s) = \sqrt{s} \alpha \Gamma_1(s), \; \Gamma_1(s) = \Gamma_1 \times \Gamma_0(s)/\Gamma_0, \; \Gamma_0(s) = \alpha \text{Im} \Sigma_1(s)/\sqrt{s}. \quad (13)$$

This yields

$$\alpha^2 I'_2 = \frac{\alpha \Gamma_1}{\Gamma_0} \alpha I'_1. \quad (13)$$

Now we determine the one-loop self-energy $\Sigma_1(s)$. In the framework of the model, we determine it with contributions of the $W$ boson and $b$ quark only. In this way we avoid the IR divergences generally arising at determining $\text{Re} \Sigma_1(s)$. Standard calculation in t’Hooft-Feynman gauge gives

$$\alpha \Sigma_1(s) = A(s) - \text{Re} A(M^2) - (s - M^2) \text{Re} A'(M^2), \quad (14)$$

$$A(s) = -\frac{G_F M_W^2}{4\sqrt{2} \pi^2} s \left[ 2 + \frac{M^2}{M_W^2} \right] B_1(s; 0, M_W) + 1. \quad (15)$$

Here $B_1(s; m_1, m_2)$ is the Passarino-Veltman function [20].

Thus, we have determined all contributions to the propagator (9) and thereby the BW factors in formula (4). Further, by virtue of (7) we can determine the MPT expansion of the BW factors. The coefficients $c_n$ on account of (10)–(12), (14), (15) and [11] are as follows:

$$c_0 = -\alpha \frac{\Gamma_1}{\Gamma_0} + \alpha^2 \left[ \frac{\Gamma_0^2}{\Gamma_0^2} - \frac{\Gamma_2}{\Gamma_0^2} \right] - \frac{\Gamma_0^2}{8M^2} - \left( \frac{\Gamma_0}{M^2 - M_W^2} \right)^2,$$

$$c_1 = 0, \quad c_2 = -M^2 \Gamma_0^2. \quad (16)$$

Recall that each $\Gamma_n$ includes an additional factor $\alpha$, which is conditioned by the vertex origin of the width.

To complete definition of the model, we must determine also the factor $(1 + \delta_c)$ in formula (3). Let us remember that we have ignored in the self-energy all massless-particles contributions that lead to IR divergences. For this reason we have to ignore all other soft-massless-particles contributions whose IR-divergent contributions are to be cancelled in the cross-section. So, there should remain only the Coulomb singular contributions that are not cancelled. Recall that they have the meaning of universal corrections arising due to exchanges by soft massless particles (photons, gluons) between outgoing massive particles in the limit of small relative velocities. In the case

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1The gauge independence should be restored after including the higher-order corrections in $\Phi$, and after including the single- and non-resonant contributions in the cross-section [11].
of strong-interacting top quarks, it is reasonable to ignore the exchanges by photons and to take into account only gluon exchanges. We also restrict our consideration to the one-gluon approximation. Then, with taking into account the off-shell and finite-width effects, we have \(14, 15\)

\[
\delta_c = \frac{\kappa \alpha_s \pi}{2\beta} \left[ 1 - \frac{2}{\pi} \arctan \left( \frac{\beta_M}{\beta \text{Im} \beta_M} \right) \right].
\]  

(17)

Here \(\kappa = 4/3\) is the group factor, \(\alpha_s\) is the strong coupling constant, \(\beta = \frac{1}{s} \sqrt{\lambda(s, s_1, s_2)}\) is the velocity of the unstable particles in the c.m.f., \(\beta_M = \sqrt{1 - \frac{4(M^2 - iM\Gamma)}{s}}\). Further we put \(\Gamma = \Gamma_0\) in the latter formula. The energy-scale dependence in \(\alpha_s\), we take into consideration as described in [21].

So, now the model is completely determined. The cross-section in the model may be straightforwardly calculated. We call the result, the “exact” solution. Simultaneously we can calculate the MPT expansion of the cross-section and compare the outcome with the “exact” result. Ultimately the expansion should have the form

\[
\sigma(s) = \sigma_0(s) + \alpha \sigma_1(s) + \alpha^2 \sigma_2(s) + \cdots.
\]  

(18)

Here \(\sigma_0\) means the cross-section in the LO approximation, \(\alpha \sigma_1\) and \(\alpha^2 \sigma_2\) mean the NLO and NNLO corrections, respectively. So, the \(\sigma_{01} = \sigma_0 + \alpha \sigma_1\) and \(\sigma_{012} = \sigma_0 + \alpha \sigma_1 + \alpha^2 \sigma_2\) determine the NLO and NNLO approximations. Similarly we denote the contributions to the hard-scattering cross-section \(\hat{\sigma}(s)\).

3 Results of numerical calculations

Parameters of the model we determine as follows: \(M = 175\) GeV, \(M_W = 80.4\) GeV, and we use the following previously calculated input-data for the width [23]:

\[
\begin{align*}
\Gamma_0 &= 1.56\ \text{GeV}, \\
\Gamma_0 + \alpha \Gamma_1 &= 1.45\ \text{GeV}, \\
\Gamma_0 + \alpha \Gamma_1 + \alpha^2 \Gamma_2 &= 1.42\ \text{GeV}.
\end{align*}
\]  

(19)

The \(\Gamma = \Gamma_0 + \alpha \Gamma_1 + \alpha^2 \Gamma_2\) we consider as the total width of the top quark. From [19], we get \(\alpha \Gamma_1 = -0.11\) GeV, and \(\alpha^2 \Gamma_2 = -0.03\) GeV. The \(\alpha\) in \(\Phi(s; s_1, s_2)\) and \(\phi(z; s)\), we set equal \(1/137\). All calculations are carried out on the basis of rather general FORTRAN code with double precision written in accordance with the formulas and instructions described in [11].

In Fig. [11(a)] we present the results of the calculation of the total cross-section \(\sigma(s)\) above the threshold. The results in percentages with respect to the exact solution are shown in Fig. [11(b)]. In the latter figure we place also a result in DPA, where \(\sigma_{DPA}(s)\) is determined by the same formulas as in the case of \(\sigma(s)\) but by substituting \(\Phi(s; M^2, M^2)(1 + \delta_c(s; M^2, M^2))\) for \(\Phi(s; s_1, s_2)(1 + \delta_c(s; s_1, s_2))\) and \(s - M^2 + i\Gamma\) for \(\Delta^{-1}(s)\). The similar results for the hard-scattering cross-section \(\hat{\sigma}(s)\) are presented by Fig. [12(a,b)]. Let us remember that \(\hat{\sigma}(s)\) is responsible for the distribution over the
invariant mass of the $t\bar{t}$ system and therefore is of interest, as well [24]. In Fig. 3 we show the results separately for the NLO and NNLO corrections to $\sigma$. In Table 1 the results are represented in the numerical form at the characteristic energies accessible at the planned $e^+e^-$ colliders. In the last column the numbers in parenthesis represent the uncertainties in the last digits. (See discussion of their determination in [25].) In the other columns the uncertainties are omitted as they appear in the digits that are not shown. In the lower positions in the Table the results are presented in percentages with respect to the exact result in the model.

The above outcomes exhibit very stable behavior of the NLO and NNLO approximations in the energy region beginning with approximately 400 GeV. (In this region simultaneously the right hierarchy of the corrections is established, $\sigma_0 < \sigma_1 < \sigma_2$.) The accuracy of the NNLO approximation is established greatly high in this region. In particular, at $400 \text{ GeV} < \sqrt{s} < 600 \text{ GeV}$ it is within $\pm 0.5\%$. At increasing energy the accuracy in relative units is slightly decreasing, but the cross-section is decreasing, too, so that the effective precision of the description remains approximately the same (because the ratio of the discrepancy to the quantity $\sqrt{\sigma}$, which characterizes statistical error, is approximately constant). In contrast to the above picture, the DPA exhibits greatly unstable behavior; its discrepancy varies from $+3.0\%$ to $-7.4\%$ in the energy region $400 \text{ GeV} < \sqrt{s} < 1500 \text{ GeV}$. On the whole, such a behavior

Figure 1: Total cross-section $\sigma(s)$. The exact result in the model, we show by thick curve. Dotted, short-dashed, and continuous thin curves mean the LO, NLO, and NNLO approximations in the MPT, respectively. The results are presented in pb (a) and in percents to the exact result (b). In panel (b), we show by the long-dashed curve the result in the DPA.

Figure 2: Hard-scattering cross-section $\hat{\sigma}(s)$. The notation is the same that in Fig. 1.
Figure 3: Corrections $\sigma_1$ and $\sigma_2$ (dashed and continuous curves, respectively).

Table 1: The results of the calculation of the total cross-section in pb.

| $\sqrt{s}$ (TeV) | $\sigma_0$ | $\sigma_01$ | $\sigma_012$ |
|------------------|-------------|-------------|--------------|
| 0.5              | 0.6724      | 0.6344      | 0.6698(7)    |
|                  | 100%        | 94.3%       | 99.6(1)%     |
| 1                | 0.2255      | 0.2124      | 0.2240(2)    |
|                  | 100%        | 94.2%       | 99.3(1)%     |
| 3                | 0.03697     | 0.03377     | 0.03653(3)   |
|                  | 100%        | 91.4%       | 98.8(1)%     |
| 5                | 0.02032     | 0.01705     | 0.01991(2)   |
|                  | 100%        | 83.9%       | 98.0(1)%     |

coincides with the expected one for DPA, including the order of magnitude of the discrepancy in the Born approximation for $\Phi$ [2].

To conclude this section three important remarks are in order. First we note that the results expressed in relative units are almost insensitive to the choice of the test function $\Phi$. In particular, the turning-on/off of the Coulomb factor has very small effect. For instance, at $\sqrt{s} = 500$ GeV this leads to 0.7%-modification of the ratio $\sigma_{012}/\sigma$ and at $\sqrt{s} = 1500$ GeV does less than 0.1%. (Although, the variation of the absolute value of the cross-section is considerable in both cases: about 35% and 20%, respectively.) The second remark concerns a large value of the correction $\sigma_2$ in comparison with the discrepancy $\sigma - \sigma_{012}$. For example, at $\sqrt{s} = 500$ GeV they constitute 5.3% and 0.4% of $\sigma$, respectively. However we think that this is an incidental unbalance as $\sigma_2$ gains its value mainly due to the correction $\alpha^2 \Gamma_2$ to the width, which in our case exhausts the corrections, and simultaneously $\alpha^2 \Gamma_2$ is quite large (approximately 2% of $\Gamma$). If we put everywhere $\alpha^2 \Gamma_2 = 0$, then at $\sqrt{s} = 500$ GeV the $\sigma_2$ decreases to 2.1% with the discrepancy remaining within the 0.5%-interval. On the other hand, if we put $\alpha^2 \Gamma_2 = 0$ only at calculating the coefficients $c_n$ without the change of the model itself, then the $\sigma_2$ becomes almost the same as in the latter case,
but the discrepancy increases to 3.6%. The third remark concerns the ill-convergent property of the MPT in the near-threshold region. With the energy approaching the threshold the accuracy and stability of MPT rapidly become worse. This manifests itself in the violation of the hierarchy of the corrections and then in the blowing up of the corrections. Actually, this behavior was predicted in [11]. To prevent this difficulty another mode of the MPT near threshold must be applied [11] that implies Taylor expansion of $\sigma(s)$ both in powers of $\alpha$ and in powers of $s - 4M^2$, where $4M^2$ is the threshold. An alternative method implies a secondary Dyson resummation in the framework of the MPT approach [10, 22].

4 Discussion

Although above calculations have been carried out in the framework of a model, the obtained outcomes expressed in relative units to a large extent are model-independent. By this we mean that the outcomes are weakly sensitive to the choice of the test function determined by the model under consideration. We verified this property by carrying out calculations with various test functions and found that the influence of the test function manifests itself mainly in a factor common for different contributions to the cross-section. In particular, even very large variation in the test function that appear at the turning-on/off the Coulomb factor, in relative units leads to small modifications of the outcomes.

On this basis we suppose that the loop corrections to the test function will lead in relative units to small modifications of the outcomes, too. In particular, our result about the 0.5%-accuracy of the NNLO approximation near the maximum of the cross-section, should remain in force at turning-on the loop corrections. Moreover, one can further improve the results if applying the MPT on the background of the loop corrections only, and considering the Born contribution in the old fashion with the Dyson-resummation in the unstable-particles propagators — on analogy of actual practice of application of DPA [11, 2]. In this case the discrepancy in the MPT description will be diminished by a factor $O(\alpha)$.

Another aspect of the problem of model-dependence of our results concerns the corrections to the width of unstable particles. Recall that these corrections determine coefficients $c_n$, which are crucial for the definition of MPT expansion. We have considered the case with rather large corrections to the width (7% and 2% in the NLO and NNLO, respectively). At diminishing these corrections, the MPT corrections to the cross-section should diminish, too. At least, we have observed this property in the framework of the model under consideration. On this ground we can expect the improving of the precision of description at transiting from the top quarks to EW-only interacting particles, for instance to the $W$-bosons, because the corrections to the width are lesser in the latter case.

As regards the application of our results to the description of realistic processes with the top-quark pair production, we should stress that our calculations simulate the main contribution to the cross-section as they cover the double-resonant contributions. So on the basis of our results we can judge about the precision that must be achieved in
realistic calculations. Fortunately, the accuracy of the NNLO approximation detected in our analysis, is satisfactory from the point of view of the ILC requirements. Really, assuming that at the ILC several hundred thousands of the $t\bar{t}$ events is expected, we conclude that the calculation of the cross-section is needed with a few per mille accuracy. As we have seen above, this, in general, is ensured by the NNLO in the MPT.

In summary, we have shown that the MPT stably works at the energies near the maximum of the cross-section and above at the description of the total cross-section for the pair production and decay of fundamental unstable particles. We have found also that in the mentioned energy region the MPT provides very good precision within the NNLO. In particular, at the ILC energies in the case of the top-quark pair production the NNLO approximation provides 0.5%-precision of the description. The further increase of the precision is possible at the proceeding to the NNNLO, possible on the basis of the results of [11], or at the proceeding to the compound use of the MPT, when the loop corrections are treated completely in the framework of the MPT while the Born contribution to the cross-section is taken into consideration in the old fashion with the Dyson resummation in the unstable-particles propagators. On the whole, the MPT method is a real candidate for carrying out high-precision calculations needed for ILC.

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