Abstract

Modeling spatial overdispersion requires point processes models with finite dimensional distributions that are overdispersive relative to the Poisson. Fitting such models usually heavily relies on the properties of stationarity, ergodicity, and orderliness. And, though processes based on negative binomial finite dimensional distributions have been widely considered, they typically fail to simultaneously satisfy the three required properties for fitting. Indeed, it has been conjectured by Diggle & Milne that no negative binomial model can satisfy all three properties. In light of this, we change perspective, and construct a new process based on a different overdispersive count model, the Generalized Waring Distribution. While comparably tractable and flexible to negative binomial processes, the Generalized Waring process is shown to possess all required properties, and additionally span the negative binomial and Poisson processes as limiting cases. In this sense, the GW process provides an approximate resolution to the conundrum highlighted by Diggle & Milne.

Keywords and Phrases: additivity; stationarity; ergodicity; orderliness; overdispersion; Poisson process; negative binomial process; generalized Waring process; complete separable metric space

Running Head: Spatial Overdispersion and the GWP

1 Introduction

The definition of an appropriate probability model for spatial count data typically requires the determination of an additive point process on the domain in question. Additivity is a minimal requirement, requiring that when the region of observation changes, or when non-overlapping regions are aggregated in a systematic manner, the corresponding count distribution remains in the
same family. Additional assumptions that are often made for convenience include stationarity/ergodicity (allowing estimation of the model based on a single realization) and orderliness (to avoid the apparition of coincident events).

In cases where the spatial counts to be modelled are prone to exhibit overdispersion, however, it may be challenging to specify a point process model that simultaneously features additivity, stationarity/ergodicity, and orderliness. A popular modeling strategy is to construct a point process with finite-dimensional laws of the negative binomial form. Such processes are known as negative binomial processes, and have been defined and studied on general state spaces (Gregoire (1983)). Owing to their combination of flexibility and mathematical tractability, they have been employed in many practical situations (see for example Bates (1955), Boswell & Patil (1977), Cliff & Ord (1973), Ramakrishnan (1951) etc.). However, they have been shown to fail in simultaneously accommodating the three properties listed above. As a matter of fact, it has been conjectured by Diggle & Milne (1983), that additive/stationary/orderly spatial point processes processes with negative binomial finite-dimensional distributions may not even exist. In their words, it would seem that one is "unable to exhibit a negative binomial point process that is statistically interesting according to the criteria we laid down" [these criteria being additivity/stationarity/orderliness].

To elaborate, the construction of a negative binomial process $N$ usually hinges on one of two schemes. The first scheme is based on compounding Poisson processes by means of the logarithmic distribution (see Feller (1968)). One defines $N(B) = \sum_{k=1}^{M(B)} X_k$ to denote the count corresponding to where $M$ is a stationary Poisson process with mean (intensity) measure $E(N(B)) = \lambda \cdot \mu(B)$, where $\mu(B)$ denotes the area (Lebesgue measure) of $B$. Given $M$, the random variables $X_i$ are taken to be independently and identically distributed (i.i.d) according to the logarithmic series distribution with parameters $(\delta, z)$, having probability generating function (p.g.f.) $\frac{-\ln(1 - \frac{\delta z}{1 + \delta})}{\ln(1 + \delta)}$, $\delta > 0$. The resulting process can be seen to be of Negative binomial form with p.g.f. $E(z^{N(B)}) = \left\{ 1 + \delta(1 - z) \right\} \frac{\ln(1 + \delta)}{\ln(1 + \delta)}$. This is a Poisson cluster scheme (cf. Daley & Vere-Jones, 1972, Example 2.4.B and Cox & Isham, 1980, Fisher, 1972, Example 5.6, Burnett & Wasan, 1980), and, as remarked by Diggle & Milne,1983, is always stationary/ergodic (any stationary Poisson cluster process is known to be mixing; Westcott, 1971, p. 300 ), but clearly non-orderly. A second scheme
is based on mixing Poisson processes, generating so-called Polya processes (see Matern (1971), cf. Daley & Vere-Jones, (1972), Example 2.1.C; Fisher, (1972), p. 500). Here, one samples a gamma random variable $\Lambda$ with parameters $\alpha$ and $\beta > 0$, and conditionally specifies $N(B)$ to be Poisson given $\Lambda$, with intensity $\Lambda \cdot \mu(B)$. The resulting process is again of the negative binomial type, with p.g.f. $E\{z^{N(B)}\} = (1 + \beta(1 - z))^{-\alpha \mu(B)}$. Polya processes on the real line are well-established in the literature on accident proneness cf. Cane (1972). As mentioned again by Diggle & Milne (1983), they are stationary by construction. However, the only stationary mixed Poisson processes which are ergodic are those for which the mixing distribution is concentrated at a single point, thus giving an (ordinary) Poisson process (cf. Westcott, 1972, p. 464). It follows that non-trivial processes of this type can be orderly but never ergodic.

In summary, the first approach yields ergodic but non-orderly processes, whereas the second approach yields orderly but non-ergodic processes. In this paper, therefore, rather than make a new attempt at finding a point process with precisely negative binomial one-dimensional distributions (which may not even be possible), we change strategy, and consider a different choice of over-disperse one-dimensional distributions. An established competitor to the negative binomial distribution is the Generalized Waring Distribution (GWD; see, e.g. Irwin (1975), Xekalaki (1983b, 1984)). This has long been used to fit overdispersed count data, particularly in the field of accident studies, providing a more plausible model for the interpretation of the data generating mechanism; and, it can approximate the negative binomial and the Poisson distribution as limiting cases. A corresponding (temporal) stochastic process has been defined and studied by Xekalaki & Zografi (2008), in the context of temporally evolving data featuring clustering or contagion.

Using the GWD as a building block, we construct an additive, stationary, ergodic, and orderly spatial point process, and study its basic properties. We develop our results on a general separable metric state space, before focussing on the practically relevant case of $\mathbb{R}^d$. The process is seen to satisfy several useful closure properties (under projection, marginalization, and superposition) and to be easy to simulate. We further show that, in the limit as certain parameters of the process diverge, this Generalized Waring Point Process approximates a negative binomial process. In doing so, we give an approximate positive solution to the task set out by Diggle & Milne: while a stationary, ergodic and orderly point process with one-dimensional negative binomial distributions may not
exist, there exists a point process that is stationary, ergodic and orderly point process and has one dimensional distributions that are approximately negative binomial (depending on parameter choice).

The paper is organised as follows. In Section 2, we provide some necessary background notions related to the generalized Waring distribution, its moments, and properties that will be used in subsequent sections. Specifically, it is shown that the generalized Waring distribution possesses the property of countable additivity, which is fundamental to our later construction. The definition and existence of the generalized Waring process in a complete separable metric space is given in Section 3. In particular, the process is shown to be orderly, and to be characterised by the property that \( N(A) \) follows a Univariate Generalized Waring Distribution (UGWD) with parameters \((a, k\mu(A), \rho)\) for all bounded sets \(A\) in a dissecting ring \(A\) of the complete separable metric space. The same section includes the determination of the corresponding intensity measure, factorial moment measures and the \(n^{th}\) - order moment measures. The generalized Waring process in \(\mathbb{R}^d\) with Lebesgue measure as parameter measure \(\mu(\cdot)\) is then defined in paragraph 4. It is shown to be orderly, ergodic and \(n^{th}\)-order stationary. The existence of the \(n^{th}\)-order reduced moments of a generalized Waring process in \(\mathbb{R}^d\), if \(\rho > n\), useful for applications, is obtained as a corollary. Finally, multivariate extensions are considered in Section 5, where we define the multivariate GWP as a special case of the GWP on the product space \(S \times \{1, 2, \ldots, m\}\), and is shown to satisfy several appealing closure properties with respect to marginalization.

2 The Generalized Waring Distribution and Additivity

In this section we provide some background on the generalized Waring distribution and discuss some of its structural properties that will be essential in what follows. In particular, we extend the previously established finite additivity property to countable additivity, as a first important step in the construction of the GW point process.

A random variable \(X\) is said to have the generalized Waring distribution
with parameters $a, k$ and $\rho$, denoted by $GW \ D (a, k; \rho)$, if

$$P \{X = n\} = \pi_n (a, k; \rho) = \frac{\rho^k}{(\rho + a)_k (\rho + a + k)} \frac{1}{n!} \quad n=0,1,... \tag{1}$$

where $a(\beta) = \frac{\Gamma (\alpha + \beta)}{\Gamma (\alpha)}$, $P \{X = x\} = 0$, $x \in \{0, 1, 2, ...\}$ (see e.g. Irwin (1975), Xekalaki (1981), Xekalaki (1983b)). Here $a > 0$, $k > 0$, $\rho > 0$ and $k$ need not be integers. The distribution is symmetric in $a$ and $k$.

The probability generating function of the generalized Waring distribution is given by

$$E (z^X) = \sum_{n=0}^{\infty} z^n \pi_n (a, k; \rho) = \frac{\rho^k}{(\rho + a)_k} 2F_1 (a, k; \rho + a + k; z) \tag{2}$$

where

$$2F_1 (a, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{a\beta\gamma}{\gamma(n)} \frac{z^n}{n!}.$$ 

The $r$th factorial moments are

$$\mu_1 = \frac{a[xr]}{(\rho - 1) (\rho - 2) ... (\rho - r)} \tag{3}$$

where $x_{[r]} = x(x + 1)...(x + r)$.

From (3) it follows immediately that all $r$th moments, ordinary moments about any origin, central moments as well as factorial moments are infinite if $\rho \leq r$. Moments about any origin, including central moments, can be obtained from (3) by the usual transformation formula (see Irwin (1975), PartI ). In particular the mean is given by

$$E (X) = \frac{ak}{\rho - 1}, \quad \rho > 1 \tag{4}$$

while the variance is

$$\sigma^2 = \mu_2 = \frac{ka(\rho + a - 1)(\rho + k - 1)}{(\rho - 1)^2 (\rho - 2)}, \quad \rho > 2 \tag{5}$$

The multivariate generalized Waring distribution with parameter vector $(\alpha, k_1, ..., k_s; \rho)$, denoted by $MGW \ D (a; k; \rho)$, is the probability distribution of a random vector $(X_i, i = 1, 2, ..., s)$ of nonnegative integer-valued components,
with probability function given by

\[ P_{x_1, \ldots, x_s} = P(X_i = x_i, i = 1, 2, \ldots, s) = \frac{\rho \left( \sum_{i=1}^{s} k_i \right) a \left( \sum_{i=1}^{s} x_i \right) \prod_{i=1}^{s} k_i(x_i)}{(\rho + a) \left( \sum_{i=1}^{s} k_i + \sum_{i=1}^{s} x_i \right) x_i!} \quad (6) \]

(see Xekalaki (1986)). The special case for \( s = 2 \) is known in the literature as the bivariate generalized Waring distribution, denoted by \( BGWD(a; k_1, k_2; \rho) \).

The probability generating function of the multivariate Generalized Waring distribution can be expressed in terms of Lauricella’s hypergeometric function of type D as

\[ G(z) = \frac{\rho \left( \sum_{i=1}^{s} k_i \right)}{(\rho + a) \sum_{i=1}^{s} k_i} F_D(a, k_1, k_2, \ldots, k_s; \rho + a + \sum_{i=1}^{s} k_i; z) \]

where

\[ F_D(a, \beta_1, \beta_2, \ldots, \beta_s; \gamma; z) = \sum_{r_1, r_2, \ldots, r_s} \frac{a^{(\sum_{i=1}^{s} r_i)} \prod_{i=1}^{s} (\beta_i(r_i)) \left( z_i \right)^{r_i}}{r_1!}, \quad z = (z_1, z_2, \ldots, z_s) \]

The factorial moments of the \( MGWD(a; k; \rho) \) (see Xekalaki (1985a), Xekalaki (1986)) are then given by

\[ \mu_{(r_1, r_2, \ldots, r_s)} = E \left[ (X_1)_{[r_1]} (X_2)_{[r_2]} \ldots (X_s)_{[r_s]} \right] \]

\[ = \frac{a^{(\sum_{i=1}^{s} r_i)} \prod_{i=1}^{s} (k_i(r_i))}{(\rho - 1)(\rho - 2) \ldots (\rho - \sum_{i=1}^{s} r_i)}, \quad r_i = 0, 1, \ldots; i = 1, 2, \ldots, s \quad (8) \]

and are finite for \( \rho > \sum r_i \), the latter being a necessary condition for the series \( F_D(a, k_1 + r_1, k_2 + r_2, \ldots, k_s + r_s; \rho + a + \sum_{i=1}^{s} (k_i + r_i); 1) \) to converge. Moments of order \( n \) can be derived from these factorial moments.

The marginal means and marginal variances are respectively given by

\[ \mu_{X_i} = E(X_i) = \frac{ak_i}{\rho - 1}, \quad \rho > 1 \quad (9) \]
\[ \sigma^2_{X_i} = \frac{k_i a (\rho + a - 1) (\rho + k_i - 1)}{(\rho - 1)^2 (\rho - 2)}, \rho > 2 \]  

(10)

\(i = 1, 2, \ldots, s\) (see Xekalaki (1986)

The second moment and the pairwise covariances are

\[ \mu_{X_iX_j} = E(X_iX_j) = \frac{a (a + 1) k_i k_j}{(\rho - 1)(\rho + a - 1)}, i, j = 1, 2, \ldots, s; \rho > 2 \]  

(11)

\[ \sigma_{X_iX_j} = \frac{a (\rho + a - 1) k_i k_j}{(\rho - 1)^2 (\rho - 2)}, i, j = 1, 2, \ldots, s; \rho > 2 \]  

(12)

One of the most important features of the GWD is \textbf{additivity}. Specifically, if \(X\) and \(Y\) are random variables with marginal distributions \(UGWD (a, k_1; \rho)\) and \(UGWD (a, k_2; \rho)\), respectively, and with joint distribution \(BGWD (a, k_1, k_2; \rho)\), then \(X + Y\) is a \(UGWD (a, k_1 + k_2; \rho)\) random variable. More generally, letting \(X_j\) be \(UGWD (a, k_j; \rho)\) or each \(j, j = 1, 2, \ldots, n\) and jointly distributed as \(MGWD (a, k_1, k_2, \ldots, k_n; \rho)\), then, if we denote \(m = \sum_{j=1}^{n} k_j\), we have that \(S = \sum_{j=1}^{n} X_j\) also has a \(UGWD (a, m; \rho)\) distribution.

These last two properties hint at the possibility of using the GWD as a basis for the construction of overdisperse point processes. This requires extending additivity to countable additivity, which we do in the form of the next theorem:

**Theorem 1** Let \(X_j\) be \(UGWD (a, k_j; \rho)\) variables for each \(j, j = 1, 2, \ldots\) and for each \(n \geq 3\) let their joint distribution be the \(MGWD (a, k_1, k_2, \ldots, k_n; \rho)\). If \(m = \sum_{j=1}^{\infty} k_j\) converges, then \(S = \sum_{j=1}^{\infty} X_j\) converges with probability 1, and \(S\) has a \(UGWD (a, m; \rho)\) distribution. If on the other hand, \(\sum_{j=1}^{\infty} k_j\) diverges, then \(S\) diverges with probability 1.

\[ P \{S_n \leq r\} = \sum_{i=0}^{r} \pi_i (a, m_n; \rho) \]  

(13)
The sequence \( \{S_n \leq r\} \) is a decreasing sequence of events for fixed \( r \), and their intersection is \( \{S \leq r\} \). Thus, using continuity from above,

\[
P\{S \leq r\} = \lim_{n \to \infty} P\{S_n \leq r\} = \lim_{n \to \infty} \sum_{i=0}^{r} \pi_i(a, m_n; \rho) .
\]

If \( m_n \) converges to a finite limit \( m \), the continuity of \( \pi_j \) implies that

\[
P\{S \leq r\} = \sum_{i=0}^{r} \pi_i(a, m; \rho)
\]

leading to

\[
P\{S = r\} = \pi_r(a, m; \rho) .
\]

This in turn implies that \( S \) is finite and distributed as generalized Waring with parameters \( a, m; \rho \) \((UGWD(a, m; \rho))\).

On the other hand, if \( m_n \to \infty \),

\[
\sum_{i=0}^{r} \pi_i(a, m_n; \rho) = \sum_{i=0}^{r} \frac{\rho(m_n)}{(\rho + a)(m_n)} \frac{a(i)m_n(i)}{(\rho + a + m_n)(i)} \frac{1}{i!} = \]

\[
\left[ \frac{\rho(a)}{(\rho + m_n)(a)} \frac{m_n}{(\rho + a + m_n)} \frac{(m_n + 1)}{(\rho + a + m_n + 1)} ... \frac{(m_n + i - 1)}{(\rho + a + m_n + i - 1)} \right] \to 0
\]

so that \( P\{S > r\} = 1 \). Since this holds for all \( r \), \( S \) diverges with probability 1.

### 3 The Generalized Waring Process

We now proceed to the definition of the generalized Waring process on a complete separable metric space and the investigation of some of its basic properties. The construction starts from postulating the existence of a point process with finite dimensional distributions of the generalized Waring form (Subsection 3.1), and then demonstrating the existence and uniqueness of such a process (Subsection 3.2). Basic features of the process such as a conditional property useful for simulation, as well as its intensity measure, factorial moment measures and \( n \)th - order moment measures are then derived in Subsection 3.3.
3.1 Definition and Basic Properties

Let $S$ be a complete separable metric space, $A$ a semiring of bounded Borel sets generating the Borel $\sigma$-algebra $B_S$ of subsets of $S$ (Appendix 2. Lemma A2.I.III, Daley and Vere-Jones (1988)) and $\mu(\cdot)$ a boundedly finite Borel measure. The distribution of a random measure is completely determined by its finite dimensional (fidi) distributions, i.e. the joint distribution of arbitrary finite families $\{A_i, i = 1, ..., s\}$ of disjoint sets from $A$ (Proposition 6.2.III, Daley & Vere-Jones (1988)). Now consider the space of all boundedly finite, integer-valued measures $(\mathcal{N}_S, B(\mathcal{N}_S))$ and let $(\Omega, \mathcal{F}, P)$ be some probability space.

**Definition 1** Let

$$N : (\Omega, \mathcal{F}, P) \rightarrow (\mathcal{N}_S, B(\mathcal{N}_S))$$

be a point process for whose finite dimensional distributions over disjoint bounded Borel sets $\{A_i, i = 1, ..., l\}$ are given by

$$P\{N(A_i) = n_i; i = 1, ..., l\} = \frac{\rho (k \sum_{i=1}^{l} \mu(A_i))^a (\sum_{i=1}^{l} n_i)^l}{(\rho + a) \left( k \sum_{i=1}^{l} \mu(A_i) + \sum_{i=1}^{l} n_i \right) n_1!}.$$ (16)

Then $N$ is called a generalized Waring process with parameters $a, \rho, k > 0$ and parameter measure $\mu(\cdot)$.

In other words, for every finite family of disjoint bounded Borel sets $\{A_i, i = 1, ..., l\}$ the joint distribution of $\{N(A_i) = n_i, i = 1, ..., l\}$ is the MGWD $(a, k\mu(A_1), k\mu(A_2), ..., k\mu(A_l); \rho)$. As usual, the process $\{N(A) : A \in B_S\}$ is to be thought of as a random measure. In particular, for any $A \in B_S$, $N(A)$ is a $\mathbb{Z}^l$-valued random variable, while for any $\omega \in \Omega$, $N(\omega, \cdot)$ is a discrete Radon measure.

We remark that, if such a process exists, it will necessarily be countably additive. To see this, let $\{A_i, i = 1, 2, ...\}$ be disjoint and have union $A$. Using Theorem 1, and the fact that $\mu(A) = \sum_{i=1}^{\infty} \mu(A_i)$ converges, we immediately obtain that $N(A) = \sum_{i=1}^{\infty} N(A_i)$ is distributed as UGWD $(a, k\mu(A); \rho)$. Furthermore, such a process will be orderly provided the parameter measure is diffuse:
Theorem 2  A process as in Definition 1 is an orderly point process if and only if its parameter measure has no fixed atoms.

Proof  A point process is orderly when given any bounded $A \in \mathcal{B}_S$, there is a dissecting system $\mathcal{T} = \{T_n\} = \{ \{ A_{ni} : i = 1, \ldots, k_n \} \}$ such that $\inf \sum_{i=1}^{k_n} P\{ N(A_{ni}) > 2 \} = 0$. (see Daley & Vere-Jones (1988)). Hence it is sufficient to examine when the ratio $P\{ N(A_{\varepsilon,x}) > 1 \} / P\{ N(A_{\varepsilon,x}) > 0 \}$ tends to 0, where $A_{\varepsilon,x}$ is the open sphere of radius $\varepsilon$ and center $x \in A$. In the case of a GW process, $N(A_{\varepsilon,x})$ has a generalized Waring distribution with parameters $a > 0$, $\rho > 0$ and $\mu(A_{\varepsilon,x}) = \mu_\varepsilon$, so that

$$P\{ N(A_{\varepsilon,x}) > 0 \} = 1 - P\{ N(A_{\varepsilon,x}) = 0 \} = 1 - \frac{\rho(\mu_\varepsilon)}{(\rho + a)(\mu_\varepsilon)}.$$  

$$P\{ N(A_{\varepsilon,x}) > 1 \} = 1 - \frac{\rho(\mu_\varepsilon)}{(\rho + a)(\mu_\varepsilon)} - \frac{\rho(\mu_\varepsilon)}{(\rho + a)(\mu_\varepsilon)} \frac{a \cdot k\mu_\varepsilon}{(\rho + a + k\mu_\varepsilon)}.$$  

If $x$ is a fixed atom of $\mu$, then $\mu_\varepsilon \to \mu_0 = \mu \{ x \} > 0$ as $\varepsilon \to 0$, while if $x$ is not a fixed atom, then $\mu(A_{\varepsilon,x}) \to 0$.

In the first case, the ratio $P\{ N(A_{\varepsilon,x}) > 1 \} / P\{ N(A_{\varepsilon,x}) > 0 \}$ tends to the constant $1 - \frac{\rho(\mu_0) \cdot a \cdot k\mu_0}{(\rho + a)(\mu_0 + 1)} - \rho(\mu_0)$, while in the second case it tends to 0, and the proof is complete.

From now and on we will consider only orderly generalized Waring processes. Indeed, any orderly point process with finite dimensional distributions of the generalized Waring type is necessarily a GWP with a non-atomic parameter measure.

Theorem 3  Let $N(\cdot)$ be an orderly point process. For $N(\cdot)$ to be a generalized Waring process with parameters $a > 0$, $\rho > 0$, $k > 0$ and parameter measure $\mu(\cdot)$, it is necessary and sufficient that there exist a boundedly finite nonatomic measure $\mu$ on the Borel sets $\mathcal{B}_S$ such that $N(A)$ has generalized Waring distribution with parameters $a, k\mu(A), \rho$ for each bounded set $A$ of a dissecting ring $\mathcal{A}$ of the complete separable metric space $S$.

Proof  We begin with necessity. Let $N(\cdot)$ be a generalized Waring Process and $A$ a bounded set of a dissecting ring $\mathcal{A}$ (A is also a Borel set). Then, by definition, there exists a boundedly finite Borel measure $\mu(\cdot)$ such that for every finite family of disjoint bounded Borel sets $\{A_i, i = 1, \ldots, s\}$, $P\{ N(A_i) = n_i, \}$...
\( i = 1, \ldots, s \) is given by \( 16 \). From this, it follows that the distribution of \( N(A) \) is the \( GWD(a, k\mu(A); \rho) \).

To prove sufficiency, suppose that there exists a boundedly finite nonatomic measure \( \mu \) on the Borel sets \( B_s \) such that \( N(A) \) has generalized Waring distribution with parameter \( a, k\mu(A), \rho \) for each bounded set \( A \) of a dissecting ring. According to Theorem 7.3.II of Daley & Vere-Jones (1988), the values of the avoidance function \( P_0(A) = P \{ N(A) = 0 \} = \frac{\rho(k\mu(A))}{(\rho + a)(k\mu(A))} \) on the bounded sets of a dissecting ring for the complete separable metric space, determine the distribution of a simple point process \( N(\cdot) \) on this space.

### 3.2 Existence and Uniqueness

To prove that the point process stipulated in the previous section does indeed exist, it is sufficient to establish that the fidi distributions given by (16) fulfill Kolmogorov’s consistency conditions, combined with the measure requirements given by the basic existence theorem for point processes (Theorem 7.1.XI Daley & Vere-Jones (1988)).

**Theorem 4 (Kolmogorov’s Consistency Conditions)** A collection of finite dimensional distributions as defined via Definition 2 satisfies Kolmogorov’s consistency conditions. That is, for every finite family of disjoint bounded Borel sets \( \{ A_i, i = 1, \ldots, l \} \),

(I) for any permutation \( i_1, \ldots, i_l \) of the indexes \( 1, \ldots, l \),

\[
P_l(A_{i_1}, \ldots, A_{i_l}; n_{i_1}, \ldots, n_{i_l}) = P_l(A_1, \ldots, A_l; n_1, \ldots, n_l) \quad (17)
\]

(II) \( \sum_{r=0}^{\infty} P_l(A_1, \ldots, A_1, n_1, \ldots, n_{l-1}, r) = P_{l-1}(A_1, \ldots, A_{l-1}, n_1, \ldots, n_{l-1}) \)

**Proof** To show (I), we notice that one can write \( \sum_{j=1}^{l} \mu = \sum_{j=1}^{l} (A_j), \sum_{j=1}^{l} n_{i_j} = \sum_{j=1}^{l} n_j, \sum_{j=1}^{l} \left[ k\mu(A_j) \right]_{(n_j)} = \prod_{j=1}^{l} \frac{[k\mu(A_j)]_{(n_j)}}{n_j!} \), which proves (17).
To show (II), we write
\[
\sum_{r=0}^{\infty} P_l (A_1, \ldots, A_l; n_1, \ldots, n_l; r) = P_l (A_1, \ldots, A_l; n_1, \ldots, n_l),
\]
\[
= P_l (A_1, \ldots, A_{l-1}; n_1, \ldots, n_{l-1})
\]

**Theorem 5 (Measure Requirements) Suppose that**

(I) \(N\) is bounded finite a.s. and has no fixed atoms.

(II) \(N\) satisfies Definition 2.

Then, there exists a boundedly finite nonatomic Borel measure \(\mu (\cdot)\) such that
\[
P_0 (A) = \Pr \{ N (A) = 0 \} = \rho_{(k\mu (A))} \frac{\rho (k\mu (A))}{(\rho + a) (k\mu (A))}
\]
for all bounded borel sets \(A\) and \(\forall i, i = 1, \ldots, s \mu (A_i) = \mu_i\).

**Proof** Let \(A \in B_s\) and let \(\mu (A) > 0\) be the root of the equation
\[
P_0 (A) = \Pr \{ N (A) = 0 \} = \rho_{(k\mu (A))} \frac{\rho (k\mu (A))}{(\rho + a) (k\mu (A))}
\]
which does exist (see Appendix, Lemma 10).

a) We first prove that \(\mu (\cdot)\) is a measure. To show finite additivity, we observe that
\[
P_0 (A) = \Pr \{ N (A) = 0 \} = \frac{\rho (k\mu (A))}{(\rho + a) (k\mu (A))}.
\]

Hence for each family of bounded, disjoint, Borel sets \(\{ A_i, i = 1, \ldots, s \}\), the joint distribution of \(\{ N (A_i) = n_i, i = 1, \ldots, s \}\) is the \(MGWD(\{ A_i, k\mu (A_1), k\mu (A_2), \ldots, k\mu (A_s); \rho \})\), and if \(A = \sum_{i=1}^{s} A_i\) then \(N (A) = \sum_{i=1}^{s} N (A_i)\) has distribution \(GWD(\{ A_i; \rho \})\). So \(\mu (A) = \sum_{i=1}^{s} \mu (A_i)\) which establishes finite additivity of \(\mu (\cdot)\). To extend this to countable additivity, it suffices to prove that \(\mu (A_i) \to 0\) a.s. for any decreasing sequence \(\{ A_i \}\) of bounded Borel sets for which \(\mu (A_i) < \infty\) and \(A_i \downarrow O\). For \(A_i \downarrow O\) \(N (A_i) \to 0\) a.s. and thus \(P_0 (A_i) = \Pr \{ N (A_i) = 0 \} \to 1\) a.s. hence \(\mu (A_i) = \frac{\rho (1 - P_0 (A_i))}{kP_0 (A_i)}\) \(\to 0\) a.s.

b) To show that \(\mu (\cdot)\) is non-atomic, we can consider by (I) that for every \(x\) that \(\Pr \{ N (\{ x \}) > 0 \} = (1 - P_0 (\{ x \})) = 0\) so \(\mu (\{ x \}) = \frac{\rho (1 - P_0 (\{ x \}))}{kP_0 (\{ x \})} = 0\).
c) To show that μ(·) is boundedly finite it is enough to prove that $P_0(A) > 0$ for every bounded borel set $A$. By supposing the contrary that for some set $A$, $P_0(A) = 0$, one, following Daley & Vere-Jones (1988), Lemma 2.4.VI, can find that in this case there exists a fixed atom of the process, contradicting (I) which proves that $P_0(A) > 0$ for every bounded borel set $A$.

### 3.3 Conditional Property and Moment Measures

A useful property of the GWP is the conditional property, which provides a straightforward way of simulating the process:

**Theorem 6** (Conditional Property). Consider a Generalized Waring point process in $\Omega$ with parameters $a > 0$, $\rho > 0$, $k > 0$. Let $W \subset \Omega$ be any region with $0 < \mu(W) < +\infty$. Given that $N(W) = n$, the conditional distribution of $N(B)$ for $B \subset W$ is the beta-binomial distribution with parameters $\mu(B)$, $\mu(W) - \mu(B)$ and $n$:

$$p(N(B) = k \mid N(W) = n) = \binom{n}{k} \frac{(\mu(B))^k (\mu(W) - \mu(B))^{n-k}}{(\mu(W))^n}$$

**Proof**

$$p(N(B) = k \mid N(W) = n) = \frac{p(N(B) = k, N(W - B) = n - k)}{p(N(W) = n)}$$

$$= \frac{\rho_{(a)} \binom{n}{k} (\mu(B))^k (\mu(W) - \mu(B))^{n-k}}{(\rho + \mu(W))^n (\rho + \mu(W) + a)^n} \cdot \frac{1}{k! (n-k)!}$$

$$= \frac{n!}{k! (n-k)!} \frac{(\mu(W))^n}{(\mu(B))^k (\mu(W) - \mu(B))^{n-k}}$$

Using the conditional property, we can generate a realization of a Generalized Waring process with parameters $a > 0$, $\rho > 0$, $k > 0$ in $W$, through the following steps:

1. Generate a random variable $M$ with a Generalized Waring distribution with parameters $a, \rho, k \cdot \mu(W)$. 

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2. Given $M = m$, generate $m$ points $Z_1, Z_2, \ldots, Z_m$ in $W$ where $Z_i \sim \text{Bernoulli}(\pi_i)$ having first simulated a draw from the beta process i.e. a countably infinite collection of weighted atoms in $W$, with weights that lie in the interval $[0; 1]$ (Hjort (1990)).

We now turn our attention to determining the $n$th-order moment measures of the process, needed to establish $n$th-order stationary, as discussed in the next section.

Let $N$ be a generalized Waring process with parameters $(a, k; \rho)$ and parameter measure $\mu(\cdot)$. For $A$ a Borel set, the distribution of $N(A)$ is the $GW D_{a,k\mu(A);\rho}$. Therefore, its first moment measure is

$$\lambda(A) = E(N(A)) = \frac{ak\mu(A)}{\rho-1}, \quad \rho > 1$$

and its intensity rate is the Radon-Nikodym derivative

$$\eta(A) = \frac{d\lambda}{d\mu} = \frac{ak}{\rho-1}, \quad \rho > 1.$$  

For $A, B$ two Borel sets the joint distribution of $(N(A), N(B))$ is the $BG WD(a, k\mu(A), k\mu(B);\rho)$, hence the second-order moment measure of the process is

$$M_2(A \times B) = E(N(A)N(B)) = \frac{a(a+1)k^2\mu(A)\mu(B)}{(\rho-1)(\rho-2)}, \quad \rho > 2$$  \hspace{1cm} (18)

Given a finite family of disjoint bounded Borel sets $\{A_i, i = 1, \ldots, s\}$ the joint distribution of $\{N(A_i) = n_i, i = 1, \ldots, s\}$ is the $MG WD(a, k\mu(A_1), k\mu(A_2), \ldots, k\mu(A_s);\rho)$, hence the factorial moment measure, $E [N(A_1)_{[r_1]} N(A_2)_{[r_2]} \ldots N(A_s)_{[r_s]}]$, of the process is

$$\mu(r_1,r_2,\ldots,r_s)(A_1 \times A_2 \times \ldots \times A_s) = \frac{a(\sum r_i)^k \prod_{i=1}^{s} (\mu(A_i))_{(r_i)}}{(\rho-1)(\rho-2)\ldots(\rho-\sum r_i)}, \quad r_i = 0,1,\ldots; \quad i = 1,2,\ldots,s.$$  \hspace{1cm} (19)
obtained from (19)

\[
M_n (A_1 \times A_2 \times \ldots \times A_n) = \mathbb{E} \left[ \left( N(A_1) \right)_{[1]} \left( N(A_2) \right)_{[1]} \ldots \left( N(A_s) \right)_{[1]} \right] \quad (20)
\]

\[
= a(n)k^n \prod_{i=1}^{n} (\mu(A_i))_{(r_i)} \frac{1}{(\rho - 1)(\rho - 2) \ldots (\rho - n)} \quad \text{for } \rho > n.
\]

for \( \rho > \sum r_i \).

4 The Generalized Waring Process in \( \mathbb{R}^d \)

We now focus on the generalized Waring process on the state-space \( \mathbb{R}^d \), with Lebesgue measure as its parameter measure \( \mu(\cdot) \). We show that this constitutes an orderly, stationary, ergodic and \( n \)th-order stationary point process.

4.1 The Generalized Waring Process as a Simple Point Process

Let \( S = \mathbb{R}^d \) and let \( \mu(\cdot) \) be the Lebesgue measure on \( \mathbb{R}^d \). The Borel algebra \( \mathcal{B}_{\mathbb{R}^d} \) in \( \mathbb{R}^d \) is the smallest \( \sigma \)-algebra on \( \mathbb{R}^d \) which contains all the open rectangles of \( d \)-dimensions. The generalized Waring process \( \{ N(A); A \in \mathcal{B}_{\mathbb{R}^d} \} \) can be defined by assuming that for every finite family of disjoint bounded Borel sets \( \{ A_i; i = 1, \ldots, s \} \) the joint distribution of \( \{ N(A_i) = n_i; i = 1, \ldots, s \} \) is the \( \text{MGWD} (a, k\mu(A_1), k\mu(A_2), \ldots, k\mu(A_s); \rho) \), \( a > 0 \), \( \rho > 0 \), \( k > 0 \).

The Lebesgue measure in \( \mathbb{R}^d \) has no atoms. Thus, the process is orderly.

**Theorem 7** The generalized Waring Process is a simple point process

This follows directly from the Proposition 7.2.V, Daley & Vere-Jones (1988), since the generalized Waring Process in \( \mathbb{R}^d \) is orderly.

4.2 Stationarity, \( n \)th-order Stationarity and Ergodicity

The Lebesgue measure in \( \mathbb{R}^d \) is also invariant under translations, hence the following can be proved:

**Theorem 8** Let \( N(\cdot) \) be a generalized Waring process in \( \mathbb{R}^d \) with parameters \( a > 0 \), \( \rho > 0 \), \( k \in N \). Then \( N(\cdot) \) is stationary.
Proof. We need to prove that for each \( u \in \mathbb{R}^d \) and all bounded Borel sets \( A \in \mathcal{B}_{\mathbb{R}^d} \), the avoidance function \( P_0 (\cdot) \) of the generalized Waring process defined above satisfies \( P_0 (A) = P_0 (A + u) \) (see Daley & Vere-Jones (1988), Theorem 10.1.III).

From the invariance of the Lebesgue measure on \( \mathbb{R}^d \) one can write
\[
P_0 (A) = \frac{\rho (k\mu (A))}{\rho + a} = \frac{\rho (k\mu (A + u))}{\rho + a} = P_0 (A + u)
\]
which proves the theorem.

A stationary Point process for which the \( n \)th-order moment measure exists is \( n \)th-order stationary (see Daley & Vere-Jones (1988)). Hence, using Diggle & Milne the following theorem and its corollary are trivial.

**Theorem 9** The generalized Waring process in \( \mathbb{R}^d \) with parameters \( a > 0, \rho > n, k \in \mathbb{N} \) is \( n \)th-order stationary.

**Theorem 10** The generalized Waring process in \( \mathbb{R}^d \) is ergodic

Proof. A necessary and sufficient criteria for a stationary process to be ergodic is to be metrically transitive. From Lemma 15, Appendix B, there exists one and only one root \( x > 0 \) of the equation \( \Gamma (\rho + x + a) \Gamma (\rho + k\mu (A)) = b > 0 \). Let us consider \( A \) a set in \( \mathbb{R}^d \) and let \( S_x \) be the shift operator. If \( A \) is such that \( P (S_x A \cap A) = P (A) \) then \( \frac{\rho (k\mu (S_x A \cap A))}{\rho + a} = \frac{\rho (k\mu (A))}{\rho + a} \). Hence we obtain that \( \frac{\Gamma (\rho + k\mu (S_x A \cap A) + a)}{\Gamma (\rho + k\mu (A))} = \frac{\Gamma (\rho + k\mu (A) + a)}{\Gamma (\rho + k\mu (A))} \) and from Lemma 15, Appendix B, follows that \( \mu (S_x A \cap A) = \mu (A) \). The last relation stands if \( A = \Phi \) or \( A = \mathbb{R}^d \) which does mean that \( P (A) = 0 \) or 1. This proves the theorem.

5 Special Cases of the Generalized Waring Process

In this section, we consider three instances of Generalized Waring Processes that may arise by multivariate extension, marginalization, projection, and limiting arguments. Specifically, define the multivariate generalized Waring process as a special case of the generalized Waring process on the product space \( S \times \{1, 2, \ldots, m\} \) and show that marginals of a multivariate GWP, as well as their
sums, are all GWP as well. We then show that GWP are closed under projection, and finally demonstrate how negative binomial and Poisson processes can be seen as special (limiting) cases of the GWP as some parameters are allowed to suitably diverge.

5.1 The Multivariate Generalized Waring Process

Consider the product space $S \times \{1, 2, \ldots, m\}$ and let $B_{S \times \{1, 2, \ldots, m\}}$ be the associated product Borel $\sigma$-algebra. Define the function $\nu : B_{S \times \{1, 2, \ldots, m\}} \to R^+$ such that for each $B = \sum_{i=1}^{\infty} A_i \times C_i \in B_{S \times \{1, 2, \ldots, m\}}(A_i \in B_S, C_i \in \mathcal{P} \{\{1, 2, \ldots, m\}\}$, $\nu(B) = \sum_{i=1}^{\infty} \mu(A_i)$ where $B_S$ is the Borel $\sigma$-algebra and $\mu(\cdot)$ some boundedly finite Borel measure. It is clear that $\nu(\cdot)$ is a boundedly finite Borel measure on $S \times \{1, 2, \ldots, m\}$. This allows us to define:

Definition 2 The generalized Waring process with parameters $a$, $k$, $\rho$ and parameter measure $\nu(\cdot)$ on $S \times \{1, 2, \ldots, m\}$ is called the multivariate generalized Waring process with parameters $a$, $k$, $\rho$ and parameter measure $\mu(\cdot)$ on $S$.

The multivariate GWP satisfies a number of convenient closure properties:

Theorem 11 Let $N(\cdot)$ be a multivariate generalized Waring process with parameters $a$, $k$, $\rho$ and parameter measure $\mu(\cdot)$ on $S$. Then the following hold:

1. For every $i \in \{1, 2, \ldots, m\}$, the marginal process $N_i(\cdot) = N(\cdot \times \{i\})$ is a GW process with parameters $a$, $k$, $\rho$ and parameter measure $\mu(\cdot)$.

2. $\sum_{i=1}^{l} N_i(\cdot)$ is a generalized Waring process with parameters $a$, $k$, $\rho$ and parameter measure $k\mu(\cdot)$.

3. For every finite collection of distinct indices $i_1, i_2, \ldots, i_l \in \{1, 2, \ldots, m\}$, $\{N_{i_1}(\cdot), N_{i_2}(\cdot), \ldots, N_{i_l}(\cdot)\}$ is a multivariate generalized Waring process with parameters $a$, $\rho$, $k$ and parameter measure $\mu(\cdot)$.

4. $\left\{N_i(\cdot), \sum_{j \neq i} N_j(\cdot)\right\}$ is a bivariate generalized Waring process with parameters $a$, $\rho$, $k$ and parameter measure $k\mu(\cdot), (m-1)k\mu(\cdot)$.

Proof For each bounded Borel set $A \in B_s$, the joint distribution of $\{N_1(\cdot), N_2(\cdot), \ldots, N_m(\cdot)\}$ is the MGWD $\{a; k\mu(A_1), k\mu(A_2), \ldots, k\mu(A_m) ; \rho\}$. From the structural properties of the multivariate generalized Waring distribution (see Xekalaki (1986)), one has:

1. The distribution of $\{N_i(\cdot) = x_i\}$, for $i$ a given value on $\{1, 2, \ldots, m\}$ is the generalized Waring distribution with parameters $a$, $k\mu(A), \rho$. By Theorem 3
this is a sufficient condition for the process $N_i(\cdot)$ to be a generalized Waring process.

2. The distribution of \( \left\{ \sum_{j=1}^{l} N_{ij}(A) = x_{ij} \right\} \), is the generalized Waring distribution with parameters $a, kl\mu(A), \rho$. By Theorem 3 this is a sufficient condition for the process $\sum_{j=1}^{l} N_{ij}(\cdot)$ to be a generalized Waring process.

3. For every $\{A_{i1}, A_{i2}, \ldots, A_{il} \in B_S\}$, let us consider $\{B_1, B_2, \ldots, B_m \in B_S\}$ where $B_i = B$ for $i \notin i_1, i_2, \ldots, i_l$ and $B_i = A_{ij}$ for $i = i_j$. The joint distribution of $\{N_1(B_1), N_2(B_2), \ldots, N_m(B_m)\}$ is the MGWD $(a; k\mu(B_1), k\mu(B_2), \ldots, k\mu(B_m); \rho)$. From the structural properties of the multivariate generalized Waring distribution (see Xekalaki (1986)), it follows that the joint distribution of $\{N_{i1}(A_{i1}), N_{i2}(A_{i2}), \ldots, N_{il}(A_{il})\}$ is the MGWD $(a; k\mu(A_{i1}), k\mu(A_{i2}), \ldots, k\mu(A_{il}); \rho)$ which proves part 3.

4. For every $A, B \in B_S$ let us consider $\{B_1, B_2, \ldots, B_m \in B_S\}$ where $B_i = A$ and $B_j = B$ for $j \neq i$. The joint distribution of $\{N_1(B_1), N_2(B_2), \ldots, N_m(B_m)\}$ is the

$$MGWD(a; k_1\mu(B_1), k_2\mu(B_2), \ldots, k_m\mu(B_m); \rho).$$

From the structural properties of the multivariate generalized Waring distribution (see Xekalaki (1986)), it follows that the joint distribution of $\{N_i(A) \sum_{j \neq i} N_j(B)\}$ is the BGWD $(a; k\mu(A), (m - 1)k\mu(B); \rho)$ which proves part 4.

### 5.2 Projections of Generalized Waring Processes

Assume one has a product measurable space $(S_1 \times S_2, \mathcal{B}_{S_1} \otimes \mathcal{B}_{S_2}, \mu_1 \times \mu_2)$ and let $N(\cdot)$ be a GWP on that space, with parameters $a, k, \rho$. Define $N_{S_1}(\cdot)$ and $N_{S_2}(\cdot)$ to be the projections of $N(\cdot)$ onto $(S_1, \mathcal{B}_{S_1}, \mu_1)$ and $(S_2, \mathcal{B}_{S_2}, \mu_2)$, respectively, defined by $N_{S_1}(A) = N(A \times S_1)$ and $N_{S_2}(B) = N(S_2 \times B)$. These projections will also be generalized Waring processes:

**Theorem 12** The projections $N_{S_1}(\cdot)$ and $N_{S_2}(\cdot)$ of a GW process $N(\cdot)$ with parameters $a, k, \rho$, onto the product measurable space $(S_1 \times S_2, \mathcal{B}_{S_1} \otimes \mathcal{B}_{S_2}, \mu_1 \times \mu_2)$ are also GW processes with parameters $a, b, \rho$ respectively onto $(S_1, \mathcal{B}_{S_1}, \mu_1)$ and $(S_2, \mathcal{B}_{S_2}, \mu_2)$. 

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Proof Let \( \{ A_i \in \mathcal{B}_{S_1}, i = 1, 2, ..., l \} \) be finite family of disjoint bounded Borel sets. The family \( \{ A_i \times S_1 \in \mathcal{B}_{S_1} \otimes \mathcal{B}_{S_2}, i = 1, 2, ..., l \} \) is also a finite family of disjoint bounded sets. Hence, the joint distribution of \( \{ N_{S_1}(A_i) = n_i, i = 1, ..., l \} \) is the MGWD \( (a, k\mu(A_1), k\mu(A_2), ..., k\mu(A_l); \rho) \) which proves that the \( N_{S_1} (\cdot) \) is a GWP with parameters \( a, k, \rho \). The same argument yields the result for \( N_{S_2} (\cdot) \).

5.3 The Poisson and the NB Processes as Limiting Cases of the Generalized Waring Process

We finally turn to demonstrate how negative binomial processes can be obtained as limiting cases of the GWP. Doing so establishes that, even though negative binomial processes cannot be orderly and stationary/ergodic simultaneously, they can be approximated by a process with these properties.

**Theorem 13** Let \( N (\cdot) \) be a generalized Waring process with parameters \( a > 0, \rho > 0, \) and \( k \in \mathbb{N} \), and with parameter measure \( \mu (\cdot) \). Letting \( k \to \infty \) and setting \( \rho = c \cdot k \) for \( c > 0 \) a constant, the generalized Waring process converges weakly to a Negative Binomial process with parameters \( a \) and \( c \).

**Proof** Denote \( N_k (\cdot), k > 0 \) the generalized Waring process indexed by the parameter \( k \) and \( N (\cdot) \) the Negative Binomial process with parameters \( a \) and \( c \). In order to prove that \( N_k (\cdot) \to N (\cdot) \) weakly, it is sufficient to prove (see e.g. Daley & Vere-Jones (1988), Kallenberg (2002)):

(i). \( P (N_k (A) = 0) \to \frac{\rho (k\mu(A))}{(\rho + \alpha)(k\mu(A))} \) for all bounded \( A \) of a dissecting ring \( T \) of \( S \).

(ii) That the generalized Waring process is uniformly tight.

In order to prove (i) we consider \( P (N_k (A) = 0) = \frac{\rho (k\mu(A))}{(\rho + \alpha)(k\mu(A))} \).

We calculate:

\[
\frac{\rho (k\mu(A))}{(\rho + \alpha)(k\mu(A))} = \frac{\rho (a)}{(\rho + k\mu(A))} = \frac{ck(a)}{ck(k+1) \cdots (ck+a-1)}
\]

\[
\frac{ck+k\mu(A)}{ck (ck+1) \cdots (ck+a-1)}
\]

\[
= \frac{k^a c (c + \frac{1}{k}) \cdots (c + \mu(A) + \frac{a-1}{k})}{(c + \mu(A)) (c + \mu(A) + \frac{1}{k}) \cdots (c + \mu(A) + \frac{a-1}{k})}
\]

\[
= \frac{c^a}{(c + \mu(A))^a} = P (N (A) = 0)
\]

To establish uniform tightness as required in (ii), we use two results concerning regular and tight measures in a complete separable metric space \( S \). A
Borel measure is tight if and only if it is compact regular (see e.g Lema A2.2.IV Daley & Vere-Jones (1988)). In turn, a finite, finitely additive, and nonnegative set function defined on the Borel sets of a complete separable metric space \( S \) is compact regular if and only if it is countably additive (see e.g Corollary A2.2.VII Daley & Vere-Jones (1988)). Therefore (ii) follows from the countable additivity theorem (Theorem 1), proven in earlier.

In turn, a Poisson process can be approximated by a negative binomial process, so that it can also be approximated by a GWP:

**Theorem 14** Let \( N(\cdot) \) be the limit process \( N(\cdot) \) of the previous Theorem, i.e. a Negative Binomial process with parameters \( a \) and \( c \). If \( c \to \infty \) and \( a = \lambda \cdot c \) where \( \lambda > 0 \) is a constant, \( N(\cdot) \) converges weakly to a Poisson process with parameter \( \lambda \).

**Proof** Write \( \{N_c(\cdot)\}, c > 0 \), to highlight that the negative binomial process in question is indexed by \( c \). We need to show that there exists a Poisson process \( M(\cdot) \) such that: \( N_c(\cdot) \to M(\cdot) \) weakly. Following Daley & Vere-Jones (1988), Lemma 9.I.IV, weak convergence of the process and convergence of finite dimensional (fidi) distributions are equivalent. So, in order to prove that \( N_c(\cdot) \to M(\cdot) \) weakly, it is sufficient to prove that the fidi distributions of \( N_c(\cdot) \) converge weakly to those of \( M(\cdot) \).

For every \( \{A_1, A_2, ..., A_n \in B_S\} \) we consider the probability generating function \( G_c(A_1, A_2, ..., A_n; z_1, ..., z_n) \) of \( N_c(\cdot) \) and obtain

\[
G_n(A_1, A_2, ..., A_n; z_1, ..., z_n) = \frac{1}{e^c \left( c + \sum_{i=1}^{n} (1 - z_i) \mu(A_i) \right)^{-\lambda \cdot c}}
\]

But

\[
\frac{1}{e^c \left( c + \sum_{i=1}^{n} (1 - z_i) \mu(A_i) \right)^{-\lambda \cdot c}} \to \exp \left( -\lambda \sum_{i=1}^{n} (1 - z_i) \mu(A_i) \right),
\]

which is the probability generating function \( G(A_1, A_2, ..., A_n; z_1, ..., z_n) \) of the Poisson process with parameter \( \lambda \).

### 6 Discussion

We have been able to define a new spatial point process for phenomena characterised by over-dispersion, in great generality. The Generalized Waring Process (GWP) has been shown to be able to simultaneously satisfy the properties that negative binomial processes fail to (orderliness, stationary, and ergodicity). Moreover, we have demonstrated that the new process features appealing closure properties, in the sense that projection, marginalization, and superposition
all yield processes of the same GWD type, with easily determinable parameters. By means of a conditional property, we have also illustrated that the process is straightforward to simulate. These properties offer substantial advantages relative to existing competitors of the negative binomial type, both from the theoretical and the practical viewpoints, especially in terms of fitting the process on the basis of a single realization. Indeed, we have shown that Generalized Waring Point Process can even approximate negative binomial processes, giving a positive resolution to the quandary posed in the conclusion of the paper by Diggle and Milne: "Any view we adopt seems to fall in a situation from which progress looks difficult, and we conjecture that no stationary, ergodic, orderly negative binomial processes exist." Though negative binomial processes may fail to simultaneously verify orderliness/stationarity/ergodicity, they can be well approximated by flexible and tractable processes of the GWP class that do verify these properties.

Potential further advantages of the generalized Waring Process relative to negative binomial processes may arise in the context of compounding (or clustering) and mixing (or heterogeneity). In particular, Cane (1974,1977) has demonstrated that one cannot distinguish between compounding and heterogeneity under a negative binomial distribution: given a total of \( n \) events, the distribution of event times is the same, whether one the model arose out of mixing or compounding. In contrast, Xekalaki (1983b) demonstrated that discriminating between clustering and mixing may well be possible in the context of the Generalized Waring Distribution, by showing that the conditional distribution of the times of events given their total is different under compounding and under mixing (see also Xekalaki (2006, 2014, 2015). This property can then be used in order to distinguish clustering, which may otherwise be confounded with compounding.

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8 Appendix
8.1 Existence Lemma

The following Lemma proves that the equation $P_0(A) = \frac{\rho(k_{\mu}(A))}{(\rho + a)(k_{\mu}(A))}$ always has a unique solution. This result is used in the proof of existence of GWP.

**Lemma 15** For each $a > 0$, $\rho > 0$, $0 \leq P_0 \leq 1$, there exists one and only one root $x > 0$ of the equation

$$\frac{\Gamma(\rho+x+a)}{\Gamma(\rho+x)} = \frac{\Gamma(\rho+a)}{\Gamma(\rho)}$$

Proof

It has been proved (see Bai-ni, Ying-jie and Feng [?] Theorem 3) that

$$\frac{\Gamma(y)}{\Gamma(x)} > \frac{y^{y-\gamma} e^{-y}}{x^{x-\gamma} e^{-x}}$$

for all $y > x \geq 1$, where $\gamma$ stands for the Euler-Mascheroni constant.

Hence we can obtain

$$\frac{\Gamma(\rho+x+a)}{\Gamma(\rho+x)} > \frac{1}{e^a} \frac{(\rho+x+a)^{\rho+x+a-\gamma}}{(\rho+x)^{\rho+x-\gamma}}$$

Now consider the function

$$f(x) = \frac{(\rho+x+a)^{\rho+x+a-\gamma}}{(\rho+x)^{\rho+x-\gamma}} e^{-a}$$

with derivative

$$f'(x) = e^{-a} \left[ \ln(\rho+x+a) + 1 - \frac{\gamma}{\rho+x+a} \right] (\rho+x)^{\rho+x+a-\gamma}$$

$$\frac{1}{(\rho+x)^{2(\rho+x-\gamma)}}$$

$$e^{-a} \left[ \ln(\rho+x) + 1 - \frac{\gamma}{\rho+x} \right] (\rho+x)^{\rho+x-\gamma}$$

$$\frac{1}{(\rho+x)^{2(\rho+x-\gamma)}}$$

The functions $\varphi(x) = (1 + \ln x - \frac{\gamma}{x})$ and $\omega(x) = x^x$ are increasing for $x > 1$, since $\varphi'(x) = \left( \frac{1}{x} + \frac{\gamma}{x^2} \right) > 0$ if $x > -\gamma$, $\omega'(x) = (1 + \ln x) x^x > 0$ if $x > 1$. Hence the function $g(x) = \varphi(x) \omega(x) = \left( 1 + \ln x - \frac{\gamma}{x} \right) x^x$ is increasing for $x > 1$. 

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Therefore, \[
\left[ \ln \left( \frac{\rho}{\rho + x + a} \right) + 1 - \frac{\gamma}{\rho + x + a} \right] (\rho + x + a)^{\rho + x + a - \gamma} - \ln (\rho + x) + 1 - \frac{\gamma}{\rho + x} \right] (\rho + x)^{\rho + x - \gamma} > 0 \]
for \( x > 1 \) so that \( f'(x) > 0 \), proving that \( f(x) \) is increasing for \( x > 1 \).

In summary, we can state that \( \forall b \in \mathbb{R}, \exists x > 1, \) such that \( f(x) > b \).

Let us now consider \( b = \frac{\Gamma (\rho + a)}{\Gamma (\rho)} \). For that value \( \exists x > 0 \), such that

\( f(x) > b \). Clearly for \( x = 0 \), \( \frac{\Gamma (\rho + x + a)}{\Gamma (\rho + x)} = \frac{\Gamma (\rho + a)}{\Gamma (\rho)} < b \). The function \( \frac{\Gamma (\rho + x + a)}{\Gamma (\rho + x)} \) is continuous for \( x > 0 \) as a ratio of two continuous functions. So, applying Bolzano’s Theorem to \( \frac{\Gamma (\rho + x + a)}{\Gamma (\rho + x)} - b \), we obtain that \( \exists x > 0 \) such that \( \frac{\Gamma (\rho + x + a)}{\Gamma (\rho + x)} - b = 0 \).

On the other hand

\[
\frac{d}{dx} \left[ \frac{\Gamma (\rho + x + a)}{\Gamma (\rho + x)} \right] = \frac{\Gamma (\rho + x + a)}{\Gamma (\rho + x)} \frac{d}{dx} \left[ \exp \left( \ln \frac{\Gamma (\rho + x + a)}{\Gamma (\rho + x)} \right) \right]
\]

\[
= \frac{\Gamma (\rho + x + a)}{\Gamma (\rho + x)} \frac{d}{dx} \left[ \ln (\rho + x + a) - \ln (\rho + x) \right]
\]

\[
= \frac{\Gamma (\rho + x + a)}{\Gamma (\rho + x)} [\Psi (\rho + x + a) - \Psi (\rho + x)]
\]

and using the relation \( \Psi (t) = -\gamma + \sum_{i=0}^{\infty} \left( \frac{1}{i + 1} - \frac{1}{i + t} \right) \) where \( \gamma \) is the Euler-Mascheroni constant, we obtain

\[
\sum_{i=0}^{\infty} \left( \frac{1}{i + \rho + x} - \frac{1}{i + \rho + x + a} \right) > 0
\]

which proves the Lemma.