Boosting the Kerr-geometry
into
an arbitrary direction

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Abstract

We generalize previous work [1] on the ultrarelativistic limit of the Kerr-geometry by lifting the restriction on boosting along the axis of symmetry.

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1) **Introduction**

Although the Kerr-geometry, when considered classically, represents a vacuum solution of the Einstein-equations, there have been various attempts \([2, 3, 4, 5]\) to find a singular matter distribution as its source. The natural candidate for locating such an energy-momentum distribution is the singular region of the spacetime.

Using distributional techniques it is indeed possible to find a tensor-distribution \([6]\), which is supported in the singular region and which from the distributional viewpoint represents the right-hand side of the Einstein equations. A central ingredient in this endeavour is the Kerr-Schild decomposition of the metric which is the main reason for the applicability of distributional techniques to the non linear Einstein equations. Moreover, the flat (background) part of the decomposition provides us with a natural notion of boosts as being its associated isometries. However, as already noted in the Schwarzschild-case \([7]\), boosting the metric itself does not produce a sensible result. Therefore the strategy is to shift from the metric to the energy-momentum tensor which has a well-defined limit \([8]\). The resulting limit is a pp-wave \([9]\). Solving the Einstein-equations with this inhomogeneity produces the so-called Aichelburg–Sexl (AS) geometry describing the gravitational field of a massless point-particle. The result is independent of the direction of the boost which is due to the spherical symmetry of the original geometry.

This is, however, no longer the case for Kerr, which is only axis-symmetric. In a first step \([1]\) the authors considered therefore a boost along the preferred direction, namely the axis of symmetry. The aim of the present work is to lift this restriction and to investigate the form of the limit by boosting along an
arbitrary direction. Finally we will discuss the extremal case where the boost
direction becomes perpendicular to the axis of symmetry in some detail.

2) The general boost

The main ingredient of our approach is the Kerr-Schild decomposition of the Kerr
geometry

\[ g_{ab} = \eta_{ab} + f k^a k_b, \] (1)

where \( \eta_{ab} \) denotes the flat background part, with respect to which boosts
find their natural home. \( k^a \) denotes a geodetic null vector field and \( f \) a scalar function.

With respect to Kerr-Schild coordinates \([10]\) \( \eta_{ab} \) becomes manifestly flat and \( k^a \) and \( f \) are given by \((\rho^2 = x^2 + y^2)\)

\[ k^a = (1, k^i), \quad k^i = \frac{\rho r}{r^2 + a^2}e^i_\rho - \frac{a \rho}{r^2 + a^2}e^i_\phi, \]

\[ f = \frac{2mr}{\Sigma}, \quad \Sigma = \frac{r^4 + a^2 z^2}{r^2}, \]

where \( r \) is subject to \( r^4 - r^2(\rho^2 + z^2 - a^2) - a^2 z^2 = 0 \). The Ricci-tensor for
geometries in the Kerr-Schild class takes the form

\[ R^a_b = \frac{1}{2}(\partial^a \partial_c (f k^c k_b) + \partial_b \partial_c (f k^c k^a) - \partial^2(f k^a k_b)), \] (2)

which in the Kerr case gives rise to the distributional energy-momentum tensor
\[
T^a_{\, b} = \frac{m\delta(z)}{8\pi} \left\{ \frac{2}{a} \left( \frac{a^2 \vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} - \frac{\rho \vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} - \frac{\delta(\rho - a)}{a} \right) (dt)^a (\partial_t)_b \right. \\
+ \left. ((\partial_\rho)^a (e_\varphi)_b - (e_\varphi)^a (dt)_b) \left( 2 \left[ \frac{\rho \vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} \right] - \frac{\pi}{a} \delta(\rho - a) \right) \\
+ \frac{2}{a} \left( - \left[ \frac{\rho^2 \vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} \right] - \frac{\rho \vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} + 2\delta(\rho - a) \right) (e_\varphi)^a (e_\varphi)_b \\
- \frac{2}{a} \frac{\vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} (e_\rho)^a (e_\rho)_b \right\} 
\] (3)

The square-bracket terms in (3) represent distributional extensions of the corresponding non-locally integrable functions to the whole of test function space. Their definition may be exemplified by

\[
\left( \frac{\vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}}, \varphi \right) := \int_{\rho \leq a} d^2 x \frac{1}{\sqrt{a^2 - \rho^2}} (\varphi(x) - \varphi(\rho e_\rho)),
\]

where \( e_\rho \) denotes the radial unit-vector with respect to polar coordinates. For a more detailed discussion on the origin of these terms the reader is referred to [6].

Interpreting Kerr-Schild coordinates as being asymptotically at rest we may rewrite \( T^a_{\, b} \) with respect to an arbitrary Lorentz-frame. The boost-plane is spanned by the timelike vector \( P^a = m(\partial_t)^a \) and its orthogonal spacelike counterpart \( Q^a = m e^a \). Without loss of generality we may take \( e^a \) to lie in the \( x-z \) plane

\[
m(e_z)^a = Q^a \cos \alpha - m \bar{e}^a \sin \alpha \\
m(e_x)^a = Q^a \sin \alpha + m \bar{e}^a \cos \alpha
\] (4)

where \( \alpha \) denotes the angle between the axis of symmetry and the direction of the boost and \( \bar{e}^a \) the spacelike direction, which spans together with \( (e_y)^a \) the two-plane orthogonal to the boost. With respect to (4) the \( \delta(z) \) factor in (3) fixes
\( (Qx) = m \tilde{x} \tan \alpha, \ \tilde{x} = (\tilde{e}x) \), which in turn implies that
\[ \rho^2 = x^2 + y^2 = \frac{1}{m^2}((Qx) \sin \alpha + m \tilde{x} \cos \alpha)^2 + y^2 = \frac{\tilde{x}^2}{\cos^2 \alpha} + y^2 = : \tilde{\rho}^2. \] (5)

So one ends up with the following expression for the energy-momentum tensor
\[
T^a_b &= \frac{\delta(Qx - m \tilde{x} \tan \alpha)}{8 \pi \cos \alpha} \left\{ -\frac{2}{a} \left( \frac{a^2 \vartheta(a - \tilde{\rho})}{\sqrt{a^2 - \tilde{\rho}^2}^3} - \vartheta(a - \tilde{\rho}) \sqrt{a^2 - \tilde{\rho}^2} - \delta(\tilde{\rho} - a) \right) P^a_P^b \\
+ \left( \frac{\tilde{\rho}^2 \vartheta(a - \tilde{\rho})}{\sqrt{a^2 - \tilde{\rho}^2}^3} - \frac{\pi}{a} \delta(\tilde{\rho} - a) \right) \{ P^a m(e_\phi)_b + m(e_\phi)^a P^b \} \\
+ \frac{2}{a} \left( \frac{\tilde{\rho}^2 \vartheta(a - \tilde{\rho})}{\sqrt{a^2 - \tilde{\rho}^2}^3} - \vartheta(a - \tilde{\rho}) \sqrt{a^2 - \tilde{\rho}^2} + 2\delta(\tilde{\rho} - a) \right) m(e_\phi)^a m(e_\phi)_b \\
- \frac{2}{a} \frac{\vartheta(a - \tilde{\rho})}{\sqrt{a^2 - \tilde{\rho}^2}^3} m(e_\rho)^a m(e_\rho)_b \right\} \quad (6)
\]

where due to the \( \delta \)-factor in (6)
\[
m(e_\phi)^a &= \frac{1}{\rho} (x m(e_y)^a - y m(e_x)^a) = \frac{1}{\tilde{\rho}} \left( -y \sin \alpha Q^a - \frac{m}{\cos \alpha} (y \cos^2 \alpha (\tilde{e})^a - \tilde{x} (e_y)^a) \right) \\
m(e_\rho)^a &= \frac{1}{\rho} (x m(e_x)^a + y m(e_y)^a) = \frac{1}{\tilde{\rho}} (\tilde{x} \tan \alpha Q^a + m(\tilde{x} (\tilde{e})^a + y (e_y)^a)) \quad (7)
\]

Although this expression may look rather unwieldy in comparison with (3) it allows a simple ultrarelativistic limit by letting \( m \to 0 \) and replacing \( P^a \) and \( Q^a \) by their null limit \( p^a \).
\[
T^a_b = \frac{\delta(px)}{8 \pi \cos \alpha} \left\{ -\frac{2}{a} \left( \frac{a^2 \vartheta(a - \tilde{\rho})}{\sqrt{a^2 - \tilde{\rho}^2}^3} - \vartheta(a - \tilde{\rho}) \sqrt{a^2 - \tilde{\rho}^2} - \delta(\tilde{\rho} - a) \right) \\
+ \left( \frac{\tilde{\rho}^2 \vartheta(a - \tilde{\rho})}{\sqrt{a^2 - \tilde{\rho}^2}^3} - \frac{\pi}{a} \delta(\tilde{\rho} - a) \right) \frac{2y \sin \alpha}{\tilde{\rho}} \\
+ \left( - \frac{\tilde{\rho}^2 \vartheta(a - \tilde{\rho})}{\sqrt{a^2 - \tilde{\rho}^2}^3} + 2\delta(\tilde{\rho} - a) \right) \frac{2y^2 \sin^2 \alpha}{a \tilde{\rho}^2} - \frac{\vartheta(a - \tilde{\rho})}{\sqrt{a^2 - \tilde{\rho}^2}^3} \right\} p^a p_b =: -\frac{1}{16 \pi} g(\tilde{x}, y) \delta(px) p^a p_b \quad (8)
\]
As expected the resulting energy-momentum tensor is that of a pp-(shock)wave. For $\alpha \to 0$ only the first term in the curly bracket survives and (8) coincides with the result obtained in [1].

3) Perturbative evaluation

In order to find the metric corresponding to the distributional energy-momentum tensor (8), one has to solve the Einstein equations that in this setting take the form of the Poisson equation

$$(\partial_x^2 + \partial_y^2) f(\bar{x}, y) = g(\bar{x}, y)$$

for the profile-function $f(x) \delta(px)$ of the pp-wave. It can be solved straightforwardly in a perturbative way except for the particular case of the orthogonal boost $\alpha \to \pi/2$ which needs special care and will therefore be dealt with in the next chapter. Rescaling $\bar{x}$ by $\cos \alpha$ and denoting the new variable by $x$ (9) becomes

$$\left(\Delta + \tan^2 \alpha \partial_x^2 \right) \sum_{n=0}^{\infty} f_n \sin^n \alpha = \frac{1}{\cos \alpha} (g_0 + \sin \alpha g_1 + \sin^2 \alpha g_2)$$

where the $g_i$ may be read off from (8). Expanding $\cos \alpha$ and $\tan \alpha$ into power series with respect to $\sin \alpha$ and grouping corresponding powers together yields

$$\Delta f_0 = g_0, \quad \Delta f_1 = g_1,$$

$$\Delta f_{2n} + \sum_{k=0}^{n-1} \partial_x^2 f_{2k} = \frac{(1/2)_n}{n!} g_0 + \frac{(1/2)_{n-1}}{(n-1)!} g_2, \quad n \geq 1,$$

$$\Delta f_{2n+1} + \sum_{k=0}^{n-1} \partial_x^2 f_{2k+1} = \frac{(1/2)_n}{n!} g_1, \quad n \geq 1.$$
Let us explicitly derive the first order perturbation $f_1$ which is determined by

$$\Delta f_1(x) = -8 \frac{y}{\rho} \left( \frac{\rho \vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} \right) - \frac{\pi}{2a} \delta(\rho - a).$$  \hspace{1cm} (12)

In the region $0 < \rho < a$ the classical analogue of (12) may be separated employing polar coordinates. Decomposing $f_1(x)$ into $\tilde{f}(\rho) \sin \phi$ we obtain the radial equation

$$\frac{1}{\rho} \partial_\rho (\rho \partial_\rho \tilde{f}_1) - \frac{1}{\rho^2} \tilde{f}_1 = -\frac{8\rho}{\sqrt{a^2 - \rho^2}}$$

which may be simplified by replacing $\rho$ by $ae^u$

$$(\partial_u^2 - 1) \tilde{f}_1 = -\frac{8e^{3u}}{\sqrt{1 - e^{2u^2}}}.$$  

This equation is easily solved by using the Green-function

$$\vartheta(u) \sinh u = \frac{1}{2} \vartheta(\rho - a) \left( \frac{\rho}{a} - \frac{a}{\rho} \right)$$

that gives rise to the particular solution

$$f_1(x) = \frac{8}{\rho} \sqrt{a^2 - \rho^2} \sin \phi.$$  

Taking into account the distributional identities

$$\Delta \left( \frac{\vartheta(a - \rho)}{\rho} \sqrt{a^2 - \rho^2} \sin \phi \right) = -\left[ \frac{\rho \vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} \right] \sin \phi + \frac{\pi}{2a} \delta(\rho - a) \sin \phi + 2\pi a \partial_\rho \delta^{(2)}(x)$$

$$\Delta \left( \frac{1}{\rho} \sin \phi \right) = 2\pi \partial_\rho \delta^{(2)}(x),$$  \hspace{1cm} (13)

we find

$$f_1(x) = \vartheta(a - \rho) \left( \frac{8}{\rho} (\sqrt{a^2 - \rho^2} - a) \right) \sin \phi \sin \phi - 8 \vartheta(\rho - a) \frac{a}{\rho} \sin \phi.$$  \hspace{1cm} (14)
The choice of the solution is dictated by the distributional nature of the inhomogeneity and natural boundary conditions for $\rho \to \infty$ which ensure that $f_1$ vanishes asymptotically, which is equivalent to the fact that the whole solution tends to the AS-geometry at large distances. The last property may also be derived from the fact that $f_1$ tends to zero in the limit $a \to 0$. Comparing (14) with the result of the boost along the axis of symmetry [1] shows that the rotational contributions being inverse powers die off only in the limit and not at some finite value. This behaviour is easily understood by interpreting (11) as a 2-dimensional electrostatic problem. In the spherically symmetric case the corresponding charge distribution produces only a monopole momentum and may therefore be replaced by a pointlike distribution of total charge outside its support.

4) Transversal Limit

In order to calculate the limit $\alpha \to \pi/2$ of (8) one has to evaluate the expression on an arbitrary test function $\varphi$. Only two different types of expressions occur in the calculation, namely those which arise from the square-bracket terms and the concentrated (delta) contributions respectively. In the following we will exemplify the respective limits on typical representatives from each class.

\[ \lim_{\alpha \to \pi/2} \frac{1}{\cos \alpha} \left( \frac{\partial (a - \tilde{\rho})}{\sqrt{a^2 - \rho^2}} \right) \cdot \varphi = \lim_{\alpha \to \pi/2} \left( \frac{\partial (a - \rho)}{\sqrt{a^2 - \rho^2}} \right) \cdot \tilde{\varphi} = \]

\[ \lim_{\alpha \to \pi/2} \int_{\rho \leq a} d^2x \frac{1}{\sqrt{a^2 - \rho^2}} (\varphi(\cos \alpha x, y) - \varphi(\cos \alpha a \cos \phi, a \sin \phi)) = \]

\[ \int_{\rho \leq a} d^2x \frac{1}{\sqrt{a^2 - \rho^2}} (\varphi(0, y) - \varphi(0, a \sin \phi)), \quad (15) \]
where $\tilde{\varphi}(x, y) := \varphi(\cos \alpha x, y)$. Unfortunately the above result is not in a very useful form. Further simplification of (15) may be achieved by integrating out $x$ in the first term, and $\rho$ in the second. However, in order to perform these integrations we have to restrict the domain of integration to a disk of radius $\bar{a} < a$ and do the limit $\bar{a} \to a$ in the end. More explicitly we find

$$\int_{\rho \leq \bar{a}} d^2x \frac{1}{\sqrt{a^2 - \rho^2}} \varphi(0, y) = \int_{-\bar{a}}^{\bar{a}} dy \varphi(0, y) \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dx \frac{1}{\sqrt{a^2 - \rho^2}} =$$

$$\frac{2}{\sqrt{a^2 - \bar{a}^2}} \int_{-\bar{a}}^{\bar{a}} dy \varphi(0, y) \frac{\sqrt{\bar{a}^2 - y^2}}{a^2 - y^2}$$

and

$$\int_{\rho \leq \bar{a}} d^2x \frac{1}{\sqrt{a^2 - \rho^2}} \varphi(0, a \sin \phi) = \int_0^{\bar{a}} \rho d\rho \int_{-\sqrt{a^2-\rho^2}}^{\sqrt{a^2-\rho^2}} d\phi \varphi(0, a \sin \phi) =$$

$$\left( \frac{1}{\sqrt{a^2 - \bar{a}^2}} - \frac{1}{a} \right) 2 \int_{-a}^{a} dy \frac{y}{\sqrt{a^2 - y^2}} \varphi(0, y).$$

Using l’Hospital’s rule (15) becomes

$$\frac{2}{a} \int_{-a}^{a} \frac{dy}{\sqrt{a^2 - y^2}} \varphi(0, y) - \frac{2 \sin a}{a} \int_{-\bar{a}}^{\bar{a}} dy \frac{y}{\sqrt{a^2 - y^2}} \varphi(0, y) =$$

$$\frac{2}{a} \int_{-a}^{a} \frac{dy}{\sqrt{a^2 - y^2}} \varphi(0, y) - \frac{\pi}{a} (\varphi(0, a) + \varphi(0, -a)) =$$

$$\left( \frac{2 \vartheta(a - |y|)}{a \sqrt{a^2 - y^2}} \delta(x), \varphi \right) - \frac{\pi}{a} (\delta(a - |y|) \delta(x), \varphi).$$

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The limit of the simplest concentrated contribution gives

\[
\lim_{\alpha \to \pi/2} \frac{1}{\cos \alpha} (\delta(\rho - a), \varphi) = \lim_{\alpha \to \pi/2} (\delta(\rho - a), \tilde{\varphi}) = \lim_{\alpha \to \pi/2} a \frac{2\pi}{\pi} \int_0^{2\pi} d\phi \varphi(\cos \alpha a \cos \phi, a \sin \phi) = a \int_0^{2\pi} d\phi \varphi(0, a \sin \phi) = 2a \int_{-a}^{a} dy \frac{\varphi(0, y)}{\sqrt{a^2 - y^2}} \delta(\bar{x}), \varphi \]

Dealing with the remaining terms of (8) in the same way we finally end up with the “transversal” energy-momentum tensor

\[
T^\alpha_b = \delta(px) \delta(\bar{x}) \left\{ \frac{1}{2} \delta(a - |y|) - \frac{1}{2} \left( \frac{y}{a} \right) \delta(a - |y|) \right\} p^a p_b = \delta(px) \delta(\bar{x}) \delta(y + a) p^a p_b. \tag{17}
\]

It is interesting to note that all contributions localized on the line segment \(|y| \leq a\) compensate each other and the energy-momentum tensor turns out to be concentrated on a pointlike region only. The corresponding profile function is obtained by solving the Poisson equation with (17) as inhomogeneity, which gives

\[
f(\bar{x}) = -8 \log \left( \frac{\sqrt{\bar{x}^2 + (y + a)^2}}{\rho_0} \right), \tag{18}\]

where \(\rho_0\) denotes the length-scale of the AS-geometry \([\text{I}].\) The limit \(a \to 0\) in fact reproduces the AS-profile function.
5) Conclusion

In the present paper we showed how to calculate the ultrarelativistic limit of the Kerr-geometry without putting any restriction on the direction of the boost. Our method is based upon the energy-momentum tensor of the original Kerr-geometry, which in contrast to metric admits a well-defined limit, thus avoiding possible ambiguities arising from the removal of the infinities of the metric-limit. The resulting energy-momentum tensor has the form of a pp-(shock)wave, depending parametrically on the angle $\alpha$ between the boost direction and the axis of rotation. This dependence turns the support of the energy-momentum tensor into an elliptical region in the two-dimensional subspace of the $px = 0$-plane. Therefore only the limiting cases $\alpha = 0$ and $\alpha = \pi/2$, where the support becomes a circle and a line-segment respectively, admit a solution in closed form. Nevertheless the general case allows a perturbative treatment if one suitably rescales the coordinates and expands the resulting expression with respect to $\sin \alpha$. An explicit calculation shows that the ”screening” behaviour of the longitudinal ($\alpha = 0$) case, where the solution turned into that of AS outside the disk with radius $a$, gets modified by contributions such that the whole solution displays only asymptotic AS-behaviour. The perturbative expansion breaks down in the limiting case $\alpha \to \pi/2$ where the direction of the boost becomes perpendicular to the axis of symmetry. Taking into account the distributional nature of the limit it is nevertheless possible to calculate the profile function of the limiting case in closed form.

It would be interesting to investigate the dependence of particle scattering on the angle $\alpha$ in comparison to the $\alpha = 0$ case, since the latter displays exactly the AS-behaviour outside the disk with radius $a$. Work in this direction is currently
under progress.
References

[1] Balasin H and Nachbagauer H, Class. Quantum Grav. 12, 707 (1995).

[2] Israel W, Phys. Rev. D2, 641, (1970).

[3] Israel W, Phys. Rev. D15, 935, (1977).

[4] Burinskii Ya, String-like Structures in Complex Kerr-geometry, gr-qc 9303003, and references therein.

[5] Lopez, Nouvo Cimento 66B, 17, (1981).

[6] Balasin H and Nachbagauer H, Class. Quantum Grav. 11, 1453 (1994).

[7] Aichelburg P and Sexl R, J. Gen. Rel. Grav. 2 (1971) 303.

[8] Balasin H and Nachbagauer H, Class. Quantum Grav. 10, 2271 (1993).

[9] Ehlers J and Kundt W, in Gravitation, An Introduction to Current Research ed. L. Witten, Wiley NY, (1962) 85.

[10] Hawking S and Ellis G, The Large Scale Structure of Space-time, UCP 1978.

[11] Parker P E, J. Math. Phys., 20, 1423 (1979).