Modeling the Lower Bound for a Marine Acoustic Waveguide with Elastic Bottom

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Abstract. At the present time, theory of sound propagation in a sea is based on the theory of non-ideal waveguides. In particular, plane-layered waveguide models are used as the basis for computer simulation of sound fields in inhomogeneous waveguides, where the unconsolidated sedimentary layer is of greatest interest. A characteristic feature of this layer is the presence of inhomogeneity in the physical properties and geometry of bottom, which cause local acoustic field perturbations. In the case when parameters of the waveguide are weakly dependent on distance, the use of plane-layered models proved to be fruitful. Such models are taking into account the geoacoustic properties of the bottom and the velocity profile of sound. The paper is devoted to the modelling lower bound of a waveguide with elastic bottom. It should be noted that solution for classical Pekeris waveguide contains branch line integrals, which to not allow us the method of particular domains to modelling waveguides with more complex structure of bottom. Thus we will consider a waveguide on a rigid basement, where sufficiently large elastic bottom layer with attenuation will adequately describe bottom properties. The solution for such waveguide does not presented in literature and may be a basis for modelling sound propagation in the waveguides with stepped elastic bottom.

1. Introduction
Underwater acoustic has always been an area of intense research activity. Today, the theory of layered waveguides is the main area of acoustics dealing with sound propagation in fluids. Among the first mathematical models of waveguides to an underwater acoustic propagation problem was the plane-layered Pekeris waveguide on a fluid [1] or an elastic halfspace [2]. These solutions have a great importance in a large number of geophysical and engineering problems, because media are actually layered or do not greatly differ from them.

For waveguides with a more complex bottom geometry new solutions have recently been obtained by [3, 4] by applying the method of particular domains. It was ascertained that for waveguides with stepped bottom the boundary value problems could be reduced to the infinite systems of linear equations. It should be noted that this approach is impossible for the Pekeris waveguide on an elastic half-space due to arising the branch line integrals (continuum contribution). To calculate the sound field in a waveguide with elastic stepped bottom according to the method of particular domains, it is important to obtain solution without branch line integrals. Quite obviously, that to avoid mentioned contour integrals we have to consider a waveguide on a rigid base, where sufficiently large elastic bottom layer with attenuation will adequately describe bottom properties.
2. Formulation of the problem

We consider as a basis a plane-layered model of a marine acoustic waveguide of constant depth $H$ on a rigid base. The bottom is assumed to be an isospeed elastic layer of constant density $\rho_1$, compressional wave speed $c_1$ and shear wave speed $c_2$. The water layer of constant depth $h$ is characterized by a sound speed $c(z)$ and constant density $\rho_0$. Loss in the elastic bottom is described by using complex wave speeds $c_j = c_j / (1 + i \alpha_j)$, where $\alpha_1$ and $\alpha_2$ are the compressional and shear wave attenuations. The waveguide has cylindrical symmetry, thus the $Oz$ axis is oriented toward the bottom (Fig. 1). A point harmonic sound source with coordinates $S(0, z_0)$ emitting a wave of angular frequency $\omega = 2\pi f$.

Pressure fields in the water layer $P$ and the displacements $u_r$, $u_z$ and stresses $\sigma_{zz}$, $\sigma_{rz}$ in the bottom are determined [5] with help of the compressional displacement potentials $\Phi(r, z)$, $\phi(r, z)$ and share displacement potential $\theta(r, z)$ by

$$P = \rho_0 \omega^2 \Phi$$

$$u_r = \frac{\partial \Phi}{\partial r} + \frac{\partial^2 \theta}{\partial r \partial z}, \quad u_z = \frac{\partial \phi}{\partial z} + \frac{\partial^2 \theta}{\partial z^2} + k_1^2 \theta$$

$$\sigma_{zz} = 2G \frac{\partial}{\partial z} \left( k_2^2 \theta + \frac{\partial^2 \theta}{\partial z^2} \right) + 2G \frac{\partial^2 \phi}{\partial z^2} - \lambda k_2^2 \phi, \quad \sigma_{rz} = 2G \frac{\partial}{\partial r} \left( \frac{\partial \phi}{\partial z} + \frac{\partial^2 \theta}{\partial z^2} + \frac{k_1^2}{2} \theta \right)$$

where $\lambda$ and $G$ are elastic constants, $k_j = \omega / C_j$ ($j=1,2$).

The potentials are solutions to the Helmholtz equations

$$\Delta \Phi + \frac{\omega^2}{c^2(z)} \Phi = -\frac{\delta(r) \delta(z-z_0)}{r}$$

$$\Delta \phi + k_2^2 \phi = 0, \quad \Delta \theta + k_1^2 \theta = 0$$

where $\Delta$ is the Laplace operator and $\delta$ is the Dirac delta function.

At the waveguide boundaries, we use the conditions for sound absorption by the water surface of the waveguide and the condition of total reflection from its rigid base

$$\Phi = 0 \text{ at } z = 0 \text{ and } u_r = u_z = 0 \text{ at } z = H$$

The potential must also satisfy continuity of vertical displacements $u_z$, normal stress $\sigma_{zz}$ and tangential stress $\sigma_{rz}$ at the bottom interface $z = h$

$$\frac{\partial \Phi}{\partial z} = u_z, \quad P = -\sigma_{zz} \text{ and } \sigma_{rz} = 0$$
3. Constructing the solution of the problem

Boundary conditions (6) and (7) can be satisfied exactly with help of normal mode approach. Actually, the normal mode technique assumes that solution has the following form

\[
\begin{pmatrix} \Phi \\ \phi \\ \theta \end{pmatrix} = \sum_{n=1}^{\infty} A_n(z_n) \begin{pmatrix} \Phi_n^*(z, \xi_n) \\ \phi_n(z, \xi_n) \\ \theta_n(z, \xi_n) \end{pmatrix} H_{0}^{(1)}(\xi_n r)
\]

where \( H_{0}^{(1)}(\cdot) \) is an 0-order Hankel function of the first kind.

Substituting (8) into (4), (5) and boundary conditions (6), (7) we can see that the vertical eigenfunctions must satisfy the following boundary value problem

\[
\begin{align*}
\frac{d^2 \Phi^*}{dz^2} + \left( \frac{\omega^2}{c^2(z)} - \xi^2 \right) \Phi^* &= 0, \\
\frac{d^2 \phi^*}{dz^2} + (k_z^2 - \xi^2) \phi^* &= 0, \\
\frac{d^2 \theta^*}{dz^2} + (k_z^2 - \xi^2) \theta^* &= 0
\end{align*}
\]

\( \frac{d\phi^*}{dz} = 0 \) at \( z = 0 \)

\( \frac{d\phi^*}{dz} = 0 \) at \( z = h \)

\( \frac{d\phi^*}{dz} = -\xi^2 \phi^* \) at \( z = H \)

Obviously, the differential equations for elastic potentials lead to

\[
\Phi^* = C_4 \cos \mu_1 z + C_5 \sin \mu_1 z
\]

and \( \theta^* = C_4 \cos \mu_2 z + C_6 \sin \mu_2 z \)

where \( \mu_j = (k_j^2 - \xi_j^2)^{1/2} \).

For the case of constant sound speed in the water layer \( c(z) = c_0 \), we obtain the similar formula

\[
\Phi^* = C_2 \sin \mu_2 z
\]

where \( \mu = (k^2 - \xi^2)^{1/2} \), \( k = \omega / c_0 \).

It should be noted that the form (14) allows us to satisfy the first boundary condition (10). Substituting (10), (11) in boundary and interface conditions (11), (12) the following homogeneous system of algebraic equations are obtained

\[
\begin{align*}
\mu \cos \mu h & \quad 0 & 0 & 0 & -0.5k_1^2 \cos \mu_1 h & -0.5k_1^2 \sin \mu_1 h & C_1 \\
0 & 0.5k_2^2 \sin \mu_1 h & -0.5k_2^2 \sin \mu_2 h & -\xi_1 \mu_1 \sin \mu_1 h & -\xi_2 \mu_1 \sin \mu_1 h & C_2 \\
0 & \mu_1 \sin \mu_1 h & -\mu_1 \cos \mu_1 h & 0.5k_1^2 - \xi_2 \cos \mu_1 h & 0.5k_2^2 - \xi_2 \sin \mu_1 h & C_3 \\
0 & -\mu_1 \sin \mu_2 H & \mu_1 \cos \mu_2 H & -\xi_2 \mu_2 \cos \mu_2 H & -\xi_2 \mu_2 \sin \mu_2 H & C_4 \\
0 & \cos \mu_1 H & \sin \mu_1 H & -\mu_2 \sin \mu_2 H & \mu_2 \cos \mu_2 H & C_5
\end{align*}
\]

Quite obviously, system (15) will have a non-trivial solution for \( C_j \) if the determinant formed by the coefficients of system is set to zero. This leads to the characteristic equation in \( \xi_j \) and its complex roots \( \xi_n \) are found numerically. Then choosing one of values \( C_j \) to be an arbitrary constant, we obtain non-trivial solution of system (15) and the vertical wavefunctions \( \Phi_n^*(z) \), \( \varphi_n(z) \) and \( \theta_n(z) \) of the problem.

The point source singularity is treated using the approach of [6], which allows to construct normal mode solution in the water layer for any plane-layered waveguide as follows

\[
\Phi = \frac{i}{4} \sum_{n=1}^{\infty} \frac{\Phi_n^*(z_n) \Phi_n^*(z) H_{0}^{(1)}(\xi_n r)}{\gamma_n}
\]

where \( \gamma_n = \int_0^1 (\Phi_n^*(z))^2 dz - \frac{g'(\xi_n) \Phi_n^*(h)}{2\xi_n g'(\xi_n)} \).
The function \( g(\zeta) \) describes the properties of waveguide bottom [6] as follows

\[
\Phi^* + g(\zeta) \frac{d\Phi^*}{dz} = 0 \text{ at } z = h
\]  

(17)

i.e. this function determines the acoustic impedance for each normal mode [7].

Making some algebraic manipulation with boundary conditions (11), the following relationship can be obtained

\[
\Phi^* + \frac{2(2\zeta^2 - k_z^2)p^* + 4\xi^2 \frac{d\Phi^*}{dz}}{k^2_0 \theta^*} = 0
\]  

(18)

Taking into account that \( \theta^* = \frac{2}{(k_z^2 - 2\zeta^2)} \frac{d\phi^*}{dz} \), we can derive from (18)

\[
\frac{k^2_0}{4} g(\zeta) = -(2\zeta^2 - k_z^2)^2 \theta^*\left(\frac{d\phi^*}{dz}\right)^{-1} + \xi^2 \frac{d\theta^*}{dz}
\]  

(19)

Next, using explicit representations for the eigenfunctions of boundary value problems (13) we can find

\[
\phi^* \left(\frac{d\phi^*}{dz}\right)^{-1} = \frac{\cos \mu_i h + (C_4 / C_3) \sin \mu_i h}{\mu_i (-\sin \mu_i h + (C_4 / C_3) \cos \mu_i h)} \text{ and } \frac{1}{\theta^*} \frac{d\theta^*}{dz} = \frac{\mu_i (-\sin \mu_i h + (C_6 / C_5) \cos \mu_i h)}{\cos \mu_i h + (C_6 / C_5) \sin \mu_i h}.
\]

Thus, to finish constructing of function \( g(\zeta) \) we must estimate the relations of constants \( C_4 / C_3 \) and \( C_6 / C_5 \). Using the equations of linear system (15) these relations can be found as

\[
\phi^* \left(\frac{d\phi^*}{dz}\right)^{-1} = 2\mu_2 \left(k_z^2 - 2\zeta^2\right) \cos \mu_i (H - h) + 4\mu_2 \left(\xi^2 + \zeta^2\right) \cos \mu_i (H - h) - k_z^2 \tan h \mu_2 \sin \mu_i (H - h)
\]

\[
\left(\frac{d\theta^*}{dz}\right)^{-1} = \left(\frac{k_z^2 - 2\zeta^2}{2}\right) + \left(\xi^2 - \mu_2\right) \sin \mu_i (H - h) + \left(\xi^2 + \mu_2\right) \cos \mu_i (H - h)
\]

\[
\left(\frac{d\theta^*}{dz}\right)^{-1} = \left(\frac{\xi^2 - \mu_2}{\xi^2 + \mu_2}\right) \sin \mu_i (H - h) + \left(\xi^2 + \mu_2\right) \cos \mu_i (H - h)
\]

Substituting these values in (19), we obtain the explicit expression for function \( g(\zeta) \), which makes it possible to write the solution of non-homogeneous Helmholtz equations (4) in the water layer according to formula (16).

4. Numerical results and conclusions

The proposed approach was numerically realized with help of the Mathematica package, while the complex roots of characteristic equation were founded using the Newton procedure. It should be noted that for the case of attenuation coefficients \( \alpha_j = 0 \) this equation has finite set of real roots which corresponds to propagating normal waves, and infinite set of complex roots. For the case of non-zero values of \( \alpha_j \), all values of horizontal wavenumbers a little moves. This fact allows us to use horizontal wavenumbers \( \xi_n \) for the waveguide without attenuation in the bottom as a first approximation for waveguide with attenuation. Fig. 2 shows the horizontal wavenumbers \( \{\xi_n\} \) for considered model of waveguide with elastic bottom which situated on a absolutely rigid basement with the following parameters: \( c_0 = 1450 \text{ m/s, } c_1 = 1800 \text{ m/s, } c_2 = 1000 \text{ m/s, } \rho_0 = 1.0 \text{ kg/m}^3, \rho_1 = 1.6 \text{ kg/m}^3, f = 16 \text{ Hz, } h=100 \text{ m, } H = 200 \text{ m, } \alpha_j = 0 \). In comparison, the horizontal wavenumbers \( \{\xi_n\} \) for waveguide with liquid bottom which also situated on an absolutely rigid basement are presented on fig. 3 for the same parameters.
It is well known, that a change in the amplitude of the field in the waveguide is characterized by the magnitude of transmission losses (intensity):

$$TL = -20 \log_{10} \left( \frac{\Phi}{\Phi_0} \right),$$

where $\Phi = Qe^{ikr}/4\pi r$ is the amplitude of the potential created by a point sound source at a distance $r=1$ m in an unlimited medium. Fig. 4 shows the change in $TL$ $(r, z)$ for different values $z$ on distance $r$ from 10 to 1000 m (parameters of waveguide are the same and $z_0 = 10$ m, normal line is corresponded to the case of $z = 10$ m, dashed line is to 50 m and dot-dashed line to the case of $z=90$ m). As expected, the $TL$ values decrease with distance, sound pressure increases to elastic bottom.

**Figure 2.** Horizontal wavenumbers for waveguide on an elastic bottom.

**Figure 3.** Horizontal wavenumbers for waveguide on a liquid bottom.

**Figure 4.** The magnitude of transmission losses for different values of depth $z$. 
Thus to describe the sound wave propagation process in a layered medium, one can use both the model of a bottom situated on a homogeneous elastic half-space and a bottom representing a sufficiently deep layer on a rigid basement. Nevertheless, the solution presented here has some advantage, as we noted earlier, it allows us to avoid contour integrals in the expressions of potentials and to use this approach for more complex bottom relief.

5. References

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