OPTIMAL GRADIENT ESTIMATES FOR THE INSULATED
CONDUCTIVITY PROBLEM WITH DIMENSIONS
MORE THAN TWO

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Abstract. In high-contrast composite materials, the electric (or stress) field may blow up in the narrow region between inclusions. The gradient of solutions depend on \( \varepsilon \), the distance between the inclusions, where \( \varepsilon \) approaches to 0. By using the maximum principle techniques, we give another proof of the Dong-Li-Yang estimates [17] for any convex inclusions of arbitrary shape with \( n \geq 3 \). This result solves the problem raised by [31], where the spherical inclusions with \( n \geq 4 \) is considered. Moreover, we also generalize the above results with flatter boundaries near touching points.

1. Introduction and main results

1.1. Background. Let \( D \) be a bounded open set in \( \mathbb{R}^n, n \geq 3 \), containing two subdomains \( D_1 \) and \( D_2 \), with \( \varepsilon \)-apart, for a small positive constant \( \varepsilon \). For a given appropriate function \( g \), we consider the following conductivity problem with Dirichlet boundary data

\[
\begin{cases}
-\nabla(a_k(x)\nabla u_{k,\varepsilon}) = 0, & \text{in } D, \\
u_{k,\varepsilon} = g, & \text{on } \partial D,
\end{cases}
\]

where

\[
a_k(x) = \begin{cases}
k \in [0,1) \cup (1,\infty], & \text{in } D_1 \cup D_2, \\
1, & \text{in } D_0 := D \setminus (D_1 \cup D_2).
\end{cases}
\]

In the context of electric conduction, the elliptic coefficients \( a_k \) refer to conductivity, and the solution \( u_{k,\varepsilon} \) represents voltage potential. From an engineering point of view, the most important quality is \( \nabla u_{k,\varepsilon} \), representing the electric field. The above model arises from the study of composite material [7], where Babuška etc. analyzed numerically that the high concentration of extreme electric field will occur in the narrow region between the adjacent inclusions or between inclusions and boundaries. Bonnetier and Vogelius [13] proved that \( \nabla u_{k,\varepsilon} \) is bounded for a fixed \( k \), which is far away from 0 and \( \infty \), and circular inclusions in two dimension. Later, Li and Vogelius [26] proved the boundedness of \( \nabla u_{k,\varepsilon} \) for general second order elliptic equations of divergence form with piecewise Hölder coefficients and general shape of inclusions in any dimensions. In [25], Li and Nirenberg extended the results in [26] to general second order elliptic systems of divergence form.

When \( k \) equals to \( \infty \) (perfect conductor) or \( 0 \) (insulator), the gradient of solutions is much different. It was shown in [14, 20, 30] that the gradient general become unbounded, as \( \varepsilon \rightarrow 0 \). Ammari et.al. in [4, 5] considered the perfect and insulate conductivity problem for the disk inclusions in dimension two, and gave the blow

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up rate $\varepsilon^{-1/2}$ in both cases. They also showed that the blow up rate is optimal. For the perfect conductivity problem in high dimensions, Yun extended the results to any bounded strictly convex smooth domains \cite{32,34}. Bao, Li and Yin in \cite{8,9} considered the higher dimensions and gave the optimal blow up rate, $\varepsilon^{-1/2}$ for $n = 2$, $\varepsilon^{-1/2} \ln \varepsilon$ for $n = 3$, $\varepsilon^{-1}$ for $n \geq 4$. For further works, see e.g. \cite{1,3,6,8,10,12,15,16,18,19,21–24,29,32,33} and their references therein.

When $k$ goes to $0$, $u_{k,\varepsilon}$ converges to the solution of the following insulated conductive problem:

$$
\begin{align*}
\Delta u_\varepsilon &= 0 \quad \text{in } D_0, \\
\frac{\partial u_\varepsilon}{\partial \nu} &= 0 \quad \text{on } \partial D_1 \cup \partial D_2, \\

u_\varepsilon &= g \quad \text{on } \partial D.
\end{align*}
(1.2)
$$

where $\nu$ is the outward unit normal vector. For the insulated conductivity problem, it was proved in \cite{9} that the optimal blow up rate is $\varepsilon^{-1/2}$ in $\mathbb{R}^2$. Yun in \cite{34} considered two circle balls and gave the optimal blow rate $\varepsilon^{n/2-2}$. Li and Yang in \cite{27,28} improved the upper bound in dimension $n \geq 3$ to be of order $\varepsilon^{-1/2+\beta}$ for some $\beta > 0$. Later, Weinkove in \cite{31} gave the blow-up rate $\gamma^*$ as the positive solution of the quadratic equation:

$$
(n - 2)(\gamma^*)^2 + (n^2 - 4n + 5)\gamma^* - (n^2 - 5n + 5) = 0
$$

(1.3)

for $n \geq 4$, which improved the result in \cite{27}. Several months later, Dong, Li and Yang considered the optimal gradient estimate in \cite{17} and gave the optimal blow up rate for $n \geq 3$:

$$
-\left(\frac{n - 1}{2}\right) + \sqrt{\left(\frac{n - 1}{2}\right)^2 + 4(n - 2)}.
$$

(1.4)
Until now, for the insulated conductivity problem (1.2) with any dimensions, the blow-up rate has been determined. But there are still some interesting questions to consider. As we know, Weinkove in [31] used the maximum principle techniques to deal with the problem, which is completely different from [17]. Whether this techniques can be used to give another proof of Dong-Li-Yang optimal estimates (1.4) is an interesting question raised by Weinkove himself in [31].

In this paper, we try to consider this open problem. As we know, Weinkove in [31] didn’t give the blow-up rate for $n = 3$ and only deal with the case for spherical inclusions rather than any convex inclusions of arbitrary shape. These two points are the main difficulties to overcome in this paper.

1.2. Our domain. Before stating our main result, we firstly fix our domain. We use $x = (x', x_n)$ to denote a point in $\mathbb{R}^n$, $x' = (x_1, x_2, \ldots, x_{n-1})$, $n \geq 3$. Let $D$ be a bounded open set in $\mathbb{R}^n$ that contains a pair of subdomain $D_1$ and $D_2$ with $2\varepsilon$ distance.

**Figure 2. The narrow region $\Omega_\varepsilon$.**

Fix a constant $R_0 < 1$, independent of $\varepsilon$, such that the portions of $\partial D_j$ near the origin (which denoted by $\Gamma_\pm$) can be parameterized by $(x', h_1(x') + \varepsilon)$ and $(x', h_2(x') - \varepsilon)$, respectively. That is,

$$
\Gamma_+ = \{ x_n = h_1(x') + \varepsilon, \ |x'| < R_0 \},
\Gamma_- = \{ x_n = h_2(x') - \varepsilon, \ |x'| < R_0 \},
$$

where $h_1$ and $h_2$ satisfy the following assumptions:

$$
h_1(x') > h_2(x') \quad \text{for} \quad |x'| < R_0. \tag{1.5}
$$

Moreover, by the convexity assumptions on $\partial D_1$ and $\partial D_2$, after a rotation of the coordinates, if necessary, we assume that

$$
h_1(x') = \lambda_1 |x'|^2 + O(|x'|^{2+\alpha}), \quad h_2(x') = -\lambda_2 |x'|^2 + O(|x'|^{2+\alpha}) \quad \text{for} \quad |x'| < R_0, \tag{1.6}
$$

where $\alpha \in (0, 1)$, $\lambda_1$ and $\lambda_2$ are some positive constants depending on the curvature of $\partial D_1$ and $\partial D_2$, and

$$
\|h_1\|_{C^{2,\alpha}(B_{2R_0})} + \|h_2\|_{C^{2,\alpha}(B_{2R_0})} \leq \mu, \tag{1.7}
$$

for some constant $\mu$. Here and throughout the paper, we use the notation $O(A)$ to denote a quantity that can be bounded by $CA$, where $C$ is some positive constant.
independent of $\varepsilon$. For $0 < r \leq R_0$, define
\[
\Omega_r := \left\{ (x', x_n) \in \mathbb{R}^n \left| h_2(x') - \varepsilon < x_n < h_1(x') + \varepsilon, \ |x'| < r \right. \right\}.
\]
By standard elliptic estimates, the solution $u \in H^1(D_0)$ of (1.2) satisfies
\[
\|u\|_{C^1(\Omega_{R_0/2})} \leq C.
\]
We will focus on the following problem near the origin:
\[
\begin{aligned}
\Delta u &= 0, \quad \text{in } D_0, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma_+ \cup \Gamma_-, \\
\|u\|_{L^\infty} &\leq 1.
\end{aligned}
\tag{1.8}
\]

The idea of this paper comes from [31], which using the maximum principle to deal with a specific quantity in the narrow region between the insulators:
\[
\left( (|x'|^2 + \sigma)^{1-\gamma} + \varepsilon^{1-\gamma}(1-\delta) - A(bx_n^2 + |x'|^4 + \sigma)^{\gamma-1/2} \right) |\nabla u|^2. \tag{1.9}
\]
\(|x'|^2 + \sigma)^{1-\gamma} + \varepsilon^{1-\gamma}(1-\delta) is the main term, and $A(bx_n^2 + |x'|^4 + \sigma)^{\gamma-1/2}$ is the lower order term which is used to adjust the quantity in boundary and interior. $\sigma$ and $\delta$ are small positive constants. We revise (1.9) by
\[
\begin{aligned}
\left( (|x'|^2 + \varepsilon)^{1-\gamma} - \frac{b}{2} \varepsilon^{1-\gamma} - A(|x'|^2 + \varepsilon)^{2-\gamma} \cos \frac{x_n}{|x'|^2 + \varepsilon} \\
+ B(|x'|^2 + \varepsilon)^{1-\gamma} \left( \frac{|x'|^2}{2} + \varepsilon \right) \cos \left( \frac{\pi x_n - \frac{\lambda_1-\lambda_2}{\pi} |x'|^2}{|x'|^2 + \varepsilon} \right) \right) |\nabla u|^2,
\end{aligned}
\tag{1.10}
\]
Here, $(|x'|^2 + \varepsilon)^{1-\gamma} - \frac{b}{2} \varepsilon^{1-\gamma} \geq C(|x'|^2 + \varepsilon)^{1-\gamma}$. Compared to the main term in (1.9) and (1.10), they are equivalent. The lower order term $-A(|x'|^2 + \varepsilon)^{2-\gamma} \cos \frac{x_n}{|x'|^2 + \varepsilon} + B(|x'|^2 + \varepsilon)^{1-\gamma} \left( \frac{|x'|^2}{2} + \varepsilon \right) \cos \left( \frac{\pi x_n - \frac{\lambda_1-\lambda_2}{\pi} |x'|^2}{|x'|^2 + \varepsilon} \right)$ plays a very important role in this quantity: $B(|x'|^2 + \varepsilon)^{1-\gamma} \left( \frac{|x'|^2}{2} + \varepsilon \right) \cos \left( \frac{\pi x_n - \frac{\lambda_1-\lambda_2}{\pi} |x'|^2}{|x'|^2 + \varepsilon} \right)$ is used to keep the normal derivative along the boundaries with good sign and $-A(|x'|^2 + \varepsilon)^{2-\gamma} \cos \frac{x_n}{|x'|^2 + \varepsilon}$ is used to make the second order derivative have a positive term, which is very useful to keep this quantity from blowing up in the interior of the narrow region. The constants $A, B, b$ are chosen to optimal the estimates.

Next, we give our main results.

1.3. Main results.

**Theorem 1.1.** Let $D, D_1, D_0$ be defined as above and satisfy (1.5)-(1.7). $g \in C^{1,\alpha}(\partial D)$. Assume that $u \in H^1(D) \cap C^1(D_0)$ is a solution of system (1.2), then for $n \geq 3$, we have
\[
\|\nabla u\|_{L^\infty(\Omega_{R/2})} \leq \frac{C\|g\|_{C^{1,\alpha}(\partial D)}}{(\varepsilon + |x'|^2)^{(1-\gamma)/2}}, \quad 0 < \gamma \leq \gamma^*, \tag{1.11}
\]
where
\[
\gamma^* = \gamma^*(n) := \frac{-(n-1) + \sqrt{(n-1)^2 + 4(n-2)}}{2} \in (0, 1). \tag{1.12}
\]
Remark 1.2. From (1.12), when \( n = 3 \), \( \gamma^* = \sqrt{2} - 1 \), the blow up rate is \( \sqrt{2 - 2} \), which is consistent with [17] and [34]. In table 1, we give the exact and approximate numerical values of blow up rate \( -\frac{1-\gamma^*}{2} \) for \( n = 3, 4, 5, 6, \infty \).

| \( n \) | \( \gamma^* \) | \( -\frac{1-\gamma^*}{2} \) | approx. |
|-------|---------|----------------|--------|
| 3     | \( \sqrt{2} - 1 \) | \( -\frac{2-\sqrt{2}}{2} \) | -0.2929 |
| 4     | \( \sqrt{17} - 3 \) | \( -\frac{5-\sqrt{17}}{2} \) | -0.2192 |
| 5     | \( \sqrt{7} - 2 \) | \( -\frac{3-\sqrt{7}}{2} \) | -0.1771 |
| 6     | \( \sqrt{41} - 5 \) | \( -\frac{7-\sqrt{41}}{4} \) | -0.1492 |
| \( \infty \) | 1 | 0 | 0 |

Obviously, the blow up rate is monotonically increasing about \( n \), that means the electric field concentration phenomenon will disappear as \( n \to \infty \).

The above procedure can be applied to deal with the following generalized \( m \)-convex inclusion cases. For simplicity, we assume that for \( x \in \Omega_r \),

\[
h_1(x') = \lambda_1 |x'|^m + O(|x'|^{m+\alpha}), \quad h_2(x') = -\lambda_2 |x'|^m + O(|x'|^{m+\alpha}), \quad m > 2.
\]

(1.13)

**Theorem 1.3.** Let \( D, D_1, D_0 \) be defined as above and satisfy (1.5), (1.7) and (1.13), \( g \in C^{1,\alpha}(\partial D) \). Assume that \( u \in H^1(D) \cap C^1(D_0) \) is a solution of system (1.2), then for \( n \geq 3 \), we have

\[
\|\nabla u\|_{L^\infty(\Omega_R/2)} \leq \frac{C\|g\|_{C^{1,\alpha}(\partial D)}}{(\varepsilon + |x'|^m)^{(1-\gamma)/2}}, \quad 0 < \gamma \leq \gamma^*,
\]

where

\[
\gamma^* = \gamma^*(n) := \frac{-(n-1) + \sqrt{(n-1)^2 + 4(n-2)}}{2} \in (0, 1).
\]

(1.14)

The proofs of Theorem 1.1 and Theorem 1.3 are given in section 2.

2. Proof of Theorem 1.1 and Theorem 1.3

Firstly, we have the following lemma, which is similar to Lemma 2.1 in [31].

**Lemma 2.1.** Under the assumption of (1.5)-(1.7), at any point of \( \Gamma_+ \) and \( \Gamma_- \), we have

\[
\frac{\partial}{\partial \nu}(|\nabla u|^2) \leq \frac{4\lambda_k}{\sqrt{1 + 4\lambda_k^2 |x'|^2}} |\nabla u|^2,
\]

where \( k = 1 \) on \( \Gamma_+ \), \( k = 2 \) on \( \Gamma_- \).
Proof. We only give the proof of the point on \( \Gamma_+ \), the proof on \( \Gamma_- \) is similar. From (2.3) in [31], one has

\[
\frac{\partial}{\partial \nu} (|\nabla u|^2) = 2 \sum_{i,j=1}^{n-1} \left( \frac{h_{1,x_i,x_j} - h_{1,x_i}}{1 + \sum_{k=1}^{n-1} h_{1,x_k}} \right) u_{x_i} u_{x_j} + 2 u_{x_n}^2
\]

on \( \Gamma_+ \),

where we use the boundary data \( \frac{\partial u}{\partial \nu} = 0 \), which is equivalent to

\[
- \sum_{i=1}^{n-1} h_{1,x_i} u_{x_i} + u_{x_n} = 0.
\]

Then by the assumptions, (2.1) holds. \( \square \)

Proof of Theorem 1.1. Without loss of general, we assume that \( \lambda_1 + \lambda_2 = 1 \). We consider the quantity

\[ Q = F |\nabla u|^2 \text{ in } \Omega_r, \]

where

\[
F := (|x'|^2 + \varepsilon)^{1-\gamma} - \frac{b}{2} \varepsilon^{1-\gamma} - A(|x'|^2 + \varepsilon)^{2-\gamma} \cos \frac{x_n}{|x'|^2 + \varepsilon} \cos \left( \frac{\pi x_n - \lambda_1 \lambda_2 |x'|^2}{2 |x'|^2 + \varepsilon} \right),
\]

\( A, B, b \) are uniform positive constants satisfying

\[
A > \frac{4\pi \gamma \max\{\lambda_1, \lambda_2\} + 54n - 70 - \left( 42 + 54n - \frac{40}{n-1} \right) \gamma}{\cos 1 - \pi \sin 1}, \quad (2.4)
\]

\[
\frac{2}{\pi} \left( 4\pi \max\{\lambda_1, \lambda_2\} + 4\sin 1 \right) < B < \frac{2}{\pi^2} \left[ A \cos 1 + \left( 42 + 54n - \frac{40}{n-1} \right) \gamma - 54n + 70 \right], \quad (2.5)
\]

\[
2\gamma (1 - \gamma) < b \leq \frac{4}{5}. \quad (2.6)
\]

By inequality \( a^p + b^p \geq (a+b)^p \geq 2^{p-1} (a^p + b^p) \) for \( a, b > 0, 0 < p < 1 \), we know that

\[
F \geq \frac{1}{2\gamma} |x'|^{2(1-\gamma)} + \frac{1-b}{2\gamma} \varepsilon^{1-\gamma} \geq \frac{1-b}{2\gamma} (|x'|^2 + \varepsilon)^{1-\gamma}, \quad (2.7)
\]

where we use that

\[
-A(|x'|^2 + \varepsilon)^{2-\gamma} \cos \frac{x_n}{|x'|^2 + \varepsilon} \cos \left( \frac{\pi x_n - \lambda_1 \lambda_2 |x'|^2}{2 |x'|^2 + \varepsilon} \right) \ll (|x'|^2 + \varepsilon)^{1-\gamma}. \quad (2.8)
\]
We assume that the quantity \( Q \) achieves a maximum at \( p \) in \( \mathcal{P}_r \). If \( p \) is in \( \partial \Omega_r \setminus (\Gamma_+ \cup \Gamma_-) \), by (2.7), we have

\[
(|x'|^2 + \varepsilon)^{1-\gamma} |\nabla u|^2 \leq C\|g\|_{C^{1,\alpha}\left(\partial D\right)}^2,
\]

(2.9)

thus (1.11) holds.

In the following, we will prove that the quantity \( Q \) can only achieve its maximum on \( \partial \Omega_r \setminus (\Gamma_+ \cup \Gamma_-) \).

Firstly, we assume that \( Q \) achieves its maximum at a point \( p \in \Gamma_+ \), then by Lemma 2.1, we have

\[
0 \leq \frac{\partial Q}{\partial \nu} = \frac{\partial F}{\partial \nu} |\nabla u|^2 + F \frac{\partial}{\partial \nu} (|\nabla u|^2) \leq \left( \frac{\partial F}{\partial \nu} + \frac{4\lambda_1}{\sqrt{1 + 4\lambda_1^2|x'|^2}} F \right) |\nabla u|^2 \quad \text{on } \Gamma_+.
\]

(2.10)

Since \( x_n = \lambda_1|x'|^2 + \varepsilon + O(|x'|^{2+\alpha}) \) on \( \Gamma_+ \) and the fact that

\[
\frac{x_n - \frac{\lambda_1 - \lambda_2}{2} |x'|^2}{|x'|^2 + \varepsilon} = 1 \quad \text{on } \Gamma_+,
\]

we have

\[
\frac{\partial F}{\partial \nu} \bigg|_{\Gamma_+} = -\frac{1}{\sqrt{1 + 4\lambda_1^2|x'|^2}} \left( \sum_{i=1}^{n-1} 2\lambda_1 x_i F_{x_i} - F_{x_n} \right)
\]

\[
= \frac{1}{\sqrt{1 + 4\lambda_1^2|x'|^2}} \left\{ -4\lambda_1 (1 - \gamma) (|x'|^2 + \varepsilon)^{-\gamma} |x'|^2
+ 4A\lambda_1 (2 - \gamma) (|x'|^2 + \varepsilon)^{1-\gamma} |x'|^2 \cos \frac{x_n}{|x'|^2 + \varepsilon}
+ 4A\lambda_1 (|x'|^2 + \varepsilon)^{-\gamma} |x'|^2 \sin \frac{x_n}{|x'|^2 + \varepsilon}
- B\pi\lambda_1 (\lambda_1 - \lambda_2 + 1) (|x'|^2 + \varepsilon)^{1-\gamma} |x'|^2
+ A(|x'|^2 + \varepsilon)^{1-\gamma} \sin \frac{x_n}{|x'|^2 + \varepsilon} - \frac{\pi}{2} B(|x'|^2 + \varepsilon)^{1-\gamma} + O(|x'|^{1-\gamma+\alpha}) \right\}
\]

\[
\leq \frac{1}{\sqrt{1 + 4\lambda_1^2|x'|^2}} \left\{ \left( -4\lambda_1 (1 - \gamma) + A \sin 1 - \frac{\pi}{2} B \right) (|x'|^2 + \varepsilon)^{1-\gamma}
+ 4\lambda_1 (1 - \gamma) \varepsilon (|x'|^2 + \varepsilon)^{-\gamma} + 4A\lambda_1 (2 - \gamma) (|x'|^2 + \varepsilon)^{1-\gamma} |x'|^2
+ 4A\lambda_1 (|x'|^2 + \varepsilon)^{-\gamma} |x'|^2 \sin 1 - B\pi\lambda_1 (\lambda_1 - \lambda_2 + 1) (|x'|^2 + \varepsilon)^{1-\gamma} |x'|^2
+ O(|x'|^{1-\gamma+\alpha}) \right\}.
\]
Then
\[
\frac{\partial F}{\partial \nu} + \frac{4\lambda_1}{\sqrt{1 + 4\lambda_1^2|x'|^2}} F
\leq \frac{1}{\sqrt{1 + 4\lambda_1^2|x'|^2}} \left\{ \left(4\lambda_1 \gamma + A \sin 1 - \frac{\pi B}{2}\right)(|x'|^2 + \varepsilon)^{1-\gamma}
\right.
\]
\[
+ [4(1 - \gamma) - 2^{2-\gamma}b] \lambda_1 \varepsilon^{1-\gamma} - 4A\lambda_1 (|x'|^2 + \varepsilon)^{2-\gamma} \cos 1
\]
\[
+ 4A\lambda_1 (2 - \gamma)(|x'|^2 + \varepsilon)^{1-\gamma}|x'|^2 + 4A\lambda_1 (|x'|^2 + \varepsilon)^{-\gamma}|x'|^2 x_n \sin 1
\]
\[
- B\pi \lambda_1 (\lambda_1 - \lambda_2 + 1) (|x'|^2 + \varepsilon)^{1-\gamma}|x'|^2 + O(|x'|^{1-\gamma+a})\}\}
\]
\begin{equation}
< 0 \text{ on } \Gamma_+.
\end{equation}
(2.11)

where for the last line, we used the inequalities \((2.5), (2.6)\) and the fact that

\[-4A\lambda_1(|x'|^2 + \varepsilon)^{2-\gamma} \cos 1 + 4A\lambda_1 (2 - \gamma)(|x'|^2 + \varepsilon)^{1-\gamma}|x'|^2 + 4A\lambda_1 (|x'|^2 + \varepsilon)^{-\gamma}|x'|^2 x_n \sin 1 - B\pi \lambda_1 (\lambda_1 - \lambda_2 + 1) (|x'|^2 + \varepsilon)^{1-\gamma}|x'|^2 \]
\[< (|x'|^2 + \varepsilon)^{1-\gamma}.
\]

Combining \((2.10)\) and \((2.11)\),
\[
0 \leq \frac{\partial Q}{\partial \nu} < 0 \text{ on } \Gamma_+,
\]
which is a contraction.

Similarly, we can also prove that the maximum cannot attained on \(\Gamma_-.\)

Next, we assume that the quantity \(Q\) achieves a maximum at a point \(p \in \Omega_r\), then we have
\[
0 \geq \Delta Q = \Delta F |\nabla u|^2 + 2\nabla F \cdot \nabla (|\nabla u|^2) + 2F |\nabla \nabla u|^2.
\]
(2.12)

In the following, we prove that
\[
\Delta Q > 0.
\]
(2.13)

**Step 1. Estimates of** \(2\nabla F \cdot \nabla (|\nabla u|^2)\).
For \(2\partial_{x_n} F \cdot \partial_{x_n} (|\nabla u|^2)\), by Cauchy inequality, immediately we have
\[
2\partial_{x_n} F \cdot \partial_{x_n} (|\nabla u|^2) = 4F_{x_n} \sum_{j=1}^n u_{x_j} u_{x_j x_n}
\]
\[
\geq - 2\eta |\nabla u|^2 - \frac{2}{\eta} |F_{x_n}|^2 |\nabla \nabla u|^2,
\]
(2.14)

where \(\eta\) is some small positive constant which may differ from line to line, and which can be shrunk at the expense of shrinking \(\varepsilon\) or \(r\).

Since at maximum of \(Q\), it holds that
\[
0 = Q_{x_i} = F_{x_i} |\nabla u|^2 + F \partial_{x_i} (|\nabla u|^2),
\]
for \(i = 1, \ldots, n - 1\), we have that
\[
\partial_{x_i} (|\nabla u|^2) = - \frac{F_{x_i} |\nabla u|^2}{F},
\]
which leads
\[
2\partial_{x_i} F \cdot \partial_{x_i} (|\nabla u|^2) = - \frac{2}{F} F_{x_i}^2 |\nabla u|^2.
\]
Sum above for \(i\) from 1 to \(n - 1\), one has
\[
\sum_{i=1}^{n-1} 2 \partial_{x_i} F \cdot \partial_{x_i} (|\nabla u|^2) = - \frac{\sum_{i=1}^{n-1} 2 F_{x_i}^2}{F} |\nabla u|^2. \tag{2.15}
\]
Combining (2.15) and (2.14) together, we get
\[
F_1 := 2 \nabla F \cdot \nabla (|\nabla u|^2) \geq - \left[ \sum_{i=1}^{n-1} 2 F_{x_i}^2 \right] |\nabla u|^2 - \frac{2}{\eta} |F_{x_n}|^2 |\nabla u|^2. \tag{2.16}
\]
On the other hand, we may make a change of coordinates so that \(x_2 = \cdots = x_{n-1} = 0\) and \(x_1 \geq 0\) and hence
\[
\nabla |x'|^2 = (2|x'|, 0, \ldots, 0).
\]
Next, we use the fact that at the maximum of \(Q\) we have
\[
0 = Q_{x_1} = F_{x_1} |\nabla u|^2 + F \partial_{x_1} (|\nabla u|^2),
\]
that is
\[
F_{x_1} = \frac{-F \partial_{x_1} (|\nabla u|^2)}{|\nabla u|^2}.
\]
Then by Cauchy-Schwarz inequality,
\[
2 \partial_{x_1} F \cdot \partial_{x_1} (|\nabla u|^2) = -2F \left[ \frac{\partial_{x_1} (|\nabla u|^2)}{|\nabla u|^2} \right]^2 = -8F \frac{\left( \sum_{j=1}^{n} u_{x_j} u_{x_1} \right)^2}{|\nabla u|^2}
\geq -8F \sum_{j=1}^{n} u_{x_j x_1}.
\]
Using the fact that \(u\) is harmonic, one has
\[
\sum_{j=1}^{n} u_{x_j x_1}^2 = \frac{n-1}{n} u_{x_1 x_1}^2 + \frac{1}{n} u_{x_1 x_1}^2 + \sum_{j=2}^{n} u_{x_j x_1}^2
\leq \frac{n-1}{n} u_{x_1 x_1}^2 + \frac{1}{n} \left( \sum_{k=2}^{n} u_{x_k x_k} \right)^2 + \frac{1}{2} \sum_{i,j=1}^{n} u_{x_i x_j}^2
\leq \frac{n-1}{n} u_{x_1 x_1}^2 + \frac{n-1}{n} \sum_{k=2}^{n} u_{x_k x_k}^2 + \frac{n-1}{n} \sum_{i,j=1}^{n} u_{x_i x_j}^2
\leq \frac{n-1}{n} \sum_{i,j=1}^{n} u_{x_ix_j}^2 = \frac{n-1}{n} |\nabla \nabla u|^2,
\]
we have
\[
2 \partial_{x_1} F \cdot \partial_{x_1} (|\nabla u|^2) \geq - \frac{8F(n-1)}{n} |\nabla \nabla u|^2. \tag{2.17}
\]
From (2.17) and (2.14), one has
\[
F_2 := 2 \nabla F \cdot \nabla (|\nabla u|^2) = 2 \partial_{x_1} F \cdot \partial_{x_1} (|\nabla u|^2) + 2 \partial_{x_n} F \cdot \partial_{x_n} (|\nabla u|^2)
\geq - 2\eta |\nabla u|^2 - \left[ \frac{8F(n-1)}{n} + \frac{2}{\eta} |F_{x_n}|^2 \right] |\nabla \nabla u|^2. \tag{2.18}
\]
Combining (2.16) and (2.18), for $0 < \xi < 1$, we can write

$$2\nabla F \cdot \nabla (|\nabla u|^2) = \xi F_1 + (1 - \xi) F_2$$

$$\geq - \left[ \sum_{i=1}^{n-1} \frac{2F_i^2}{F} \xi + 2\eta \right] |\nabla u|^2$$

$$- \left[ \frac{8F(n-1)}{n} (1 - \xi) + \frac{2}{\eta} |F_{x_n}|^2 \right] |\nabla \nabla u|^2. \quad (2.19)$$

Substituting (2.19) into (2.12), we have

$$\Delta Q \geq \left[ \Delta F - \sum_{i=1}^{n-1} \frac{2F_i^2}{F} \xi - 2\eta \right] |\nabla u|^2$$

$$+ \left[ 2F - \frac{8F(n-1)}{n} (1 - \xi) - \frac{2}{\eta} |F_{x_n}|^2 \right] |\nabla \nabla u|^2. \quad (2.20)$$

**Step 2: Estimates of** $\left[ \Delta F - \sum_{i=1}^{n-1} \frac{2F_i^2}{F} \xi - 2\eta \right] |\nabla u|^2$.

By simple computation,

$$\Delta F = 2(n-1)(1-\gamma)(|x'|^2 + \varepsilon)^{-\gamma} - 4\gamma(1-\gamma)(|x'|^2 + \varepsilon)^{-\gamma-1}|x'|^2$$

$$+ 2A(|x'|^2 + \varepsilon)^{-\gamma} \cos \frac{x_n}{|x'|^2 + \varepsilon} \left\{ 2(|x'|^2 + \varepsilon)^{-2}|x'|^2 x_n^2 \right\}$$

$$- (n-1)(2-\gamma)(|x'|^2 + \varepsilon) - 2(2-\gamma)(1-\gamma)|x'|^2 \right\}$$

$$+ 2A(|x'|^2 + \varepsilon)^{-\gamma} \sin \frac{x_n}{|x'|^2 + \varepsilon} \left\{ 6(|x'|^2 + \varepsilon)^{-1}|x'|^2 x_n - (n-1)x_n \right\}$$

$$+ B(|x'|^2 + \varepsilon)^{-\gamma} \sin \left( \frac{\pi x_n - \frac{\lambda_1 - \lambda_2}{2} |x'|^2}{\frac{|x'|^2}{2} + \varepsilon} \right) \left( \lambda_1 - \lambda_2 + \frac{x_n - \frac{\lambda_1 - \lambda_2}{2} |x'|^2}{\frac{|x'|^2}{2} + \varepsilon} \right)$$

$$\cdot \left\{ \frac{\pi}{2} (n-1)(|x'|^2 + \varepsilon) + 2\pi(1-\gamma)|x'|^2 \right\}$$

$$+ B(|x'|^2 + \varepsilon)^{-\gamma} \cos \left( \frac{\pi x_n - \frac{\lambda_1 - \lambda_2}{2} |x'|^2}{\frac{|x'|^2}{2} + \varepsilon} \right) \left\{ [(2-\gamma)|x'|^2 + (3-2\gamma)\varepsilon] \right.$$

$$\cdot (n-1 - 2\gamma|x'|^2(|x'|^2 + \varepsilon)^{-1}) + 2(2-\gamma)|x'|^2$$

$$- \frac{\pi^2}{4} (|x'|^2 + \varepsilon)|x'|^2 \left( \frac{|x'|^2}{2} + \varepsilon \right)^{-1} \left( \lambda_1 - \lambda_2 + \frac{x_n - \frac{\lambda_1 - \lambda_2}{2} |x'|^2}{\frac{|x'|^2}{2} + \varepsilon} \right)^2 \right\}$$

$$+ A(|x'|^2 + \varepsilon)^{-\gamma} \cos \frac{x_n}{|x'|^2 + \varepsilon}$$

$$- \frac{\pi^2}{4} B(|x'|^2 + \varepsilon)^{-1-\gamma} \left( \frac{|x'|^2}{2} + \varepsilon \right)^{-1} \cos \left( \frac{\pi x_n - \frac{\lambda_1 - \lambda_2}{2} |x'|^2}{\frac{|x'|^2}{2} + \varepsilon} \right). \quad (2.21)$$
Since

\[2A(|x'|^2 + \varepsilon)^{-\gamma} \cos \frac{x_n}{|x'|^2 + \varepsilon} \left\{ 2(|x'|^2 + \varepsilon)^{-2}|x'|^2 x_n^2 \right. \]
\[- (n-1)(2-\gamma)(|x'|^2 + \varepsilon) - 2(2-\gamma)(1-\gamma)|x'|^2 \}
\[+ 2A(|x'|^2 + \varepsilon)^{-\gamma} \sin \frac{x_n}{|x'|^2 + \varepsilon} \left\{ 6(|x'|^2 + \varepsilon)^{-1}|x'|^2 x_n - (n-1)x_n \right\}
\[+ B(|x'|^2 + \varepsilon)^{-\gamma} \sin \left( \frac{\pi x_n - \frac{\lambda_1-\lambda_2}{2} |x'|^2}{2} \right) \lambda_1 - \lambda_2 + \frac{x_n - \frac{\lambda_1-\lambda_2}{2} |x'|^2}{|x'|^2 + \varepsilon} \}
\[\cdot \left\{ \frac{\pi}{2} (n-1)(|x'|^2 + \varepsilon) + 2\pi(1-\gamma)|x'|^2 \right\}
\[+ B(|x'|^2 + \varepsilon)^{-\gamma} \cos \left( \frac{\pi x_n - \frac{\lambda_1-\lambda_2}{2} |x'|^2}{2} \right) \left\{ [(2-\gamma)|x'|^2 + (3-2\gamma)\varepsilon] \right. \]
\[\cdot (n-1-2\gamma|x'|^2(|x'|^2 + \varepsilon)^{-1}) + 2(2-\gamma)|x'|^2 \]
\[\left. - \frac{\pi^2}{4}(|x'|^2 + \varepsilon)|x'|^2 \left( \frac{|x'|^2}{2} + \varepsilon \right)^{-1} \left( \lambda_1 - \lambda_2 + \frac{x_n - \frac{\lambda_1-\lambda_2}{2} |x'|^2}{|x'|^2 + \varepsilon} \right)^2 \right\} \ll (|x'|^2 + \varepsilon)^{-\gamma}, \]

one has

\[\Delta F \geq \left[ 2(n-1)(1-\gamma) - 4\gamma(1-\gamma) + A \cos 1 - \frac{\pi^2}{2} B - \eta \right] (|x'|^2 + \varepsilon)^{-\gamma}. \quad (2.22)\]

For \(F_{x_i}^2\), we can write

\[F_{x_i}^2 = \left\{ 2(1-\gamma)(|x'|^2 + \varepsilon)^{-\gamma} x_i - 2A(2-\gamma)(|x'|^2 + \varepsilon)^{1-\gamma} x_i \cos \frac{x_n}{|x'|^2 + \varepsilon} \right. \]
\[- 2A(|x'|^2 + \varepsilon)^{-\gamma} x_i x_n \sin \frac{x_n}{|x'|^2 + \varepsilon} \]
\[+ B(|x'|^2 + \varepsilon)^{-\gamma} x_i \left( \frac{(2-\gamma)|x'|^2 + (3-2\gamma)\varepsilon}{2} \right) \cos \left( \frac{\pi x_n - \frac{\lambda_1-\lambda_2}{2} |x'|^2}{2} \right) \]
\[\left. + \frac{B\pi}{2} (|x'|^2 + \varepsilon)^{1-\gamma} x_i \left( \lambda_1 - \lambda_2 + \frac{x_n - \frac{\lambda_1-\lambda_2}{2} |x'|^2}{|x'|^2 + \varepsilon} \right) \sin \left( \frac{\pi x_n - \frac{\lambda_1-\lambda_2}{2} |x'|^2}{2} \right) \right\}^2 \]
\[\leq 4(1+\eta)(1-\gamma)^2(|x'|^2 + \varepsilon)^{-2\gamma} x_i^2, \quad (2.23)\]
where we use that
\[ -2A(2 - \gamma)(|x'|^2 + \varepsilon)^{1-\gamma} x_i \cos \frac{x_n}{|x'|^2 + \varepsilon} \]
\[ -2A(|x'|^2 + \varepsilon)^{-\gamma} x_i x_n \sin \frac{x_n}{|x'|^2 + \varepsilon} \]
\[ + B(|x'|^2 + \varepsilon)^{-\gamma} x_i \left((2 - \gamma)|x'|^2 + (3 - 2\gamma)\varepsilon\right) \cos \left(\frac{\pi}{2} \frac{x_n - \frac{\lambda_1 - \lambda_2}{2} |x'|^2}{|x'|^2 + \varepsilon}\right) \]
\[ + \frac{B\pi}{2} (|x'|^2 + \varepsilon)^{1-\gamma} x_i \left(\lambda_1 - \lambda_2 + \frac{x_n - \frac{\lambda_1 - \lambda_2}{2} |x'|^2}{|x'|^2 + \varepsilon}\right) \sin \left(\frac{\pi}{2} \frac{x_n - \frac{\lambda_1 - \lambda_2}{2} |x'|^2}{|x'|^2 + \varepsilon}\right) \]
\[ \leq 2\eta(1 - \gamma)(|x'|^2 + \varepsilon)^{-\gamma}|x_i|. \]
Hence, from (2.23) and (2.6), we can write
\[ -\sum_{i=1}^{n-1} \frac{2F_i^2}{F} \geq \frac{8(1 + \eta)(1 - \gamma)^2(|x'|^2 + \varepsilon)^{-2\gamma} |x'|^2}{\frac{1}{2\gamma} (|x'|^2 + \varepsilon)^{1-\gamma}} \geq -\frac{80(1 + \eta)(1 - \gamma)^2(|x'|^2 + \varepsilon)^{-2\gamma} |x'|^2}{(|x'|^2 + \varepsilon)^{1-\gamma}}. \] (2.24)
Combining (2.22) and (2.24), one has
\[ \Delta F - \sum_{i=1}^{n-1} \frac{2F_i^2}{F} \zeta - 2\eta \]
\[ \geq \left[2(n - 1)(1 - \gamma) - 4\gamma(1 - \gamma) + A \cos 1 - \frac{\pi^2}{2} B - 80\xi_0(1 - \gamma)^2 - 3\eta\right] (|x'|^2 + \varepsilon)^{-\gamma} \]
\[ := M(n, \gamma)(|x'|^2 + \varepsilon)^{-\gamma}, \] (2.25)
where \(\xi_0 = \xi(1 + \eta)\) and
\[ M(n, \gamma) := 2(n - 1)(1 - \gamma) - 4\gamma(1 - \gamma) + A \cos 1 - \frac{\pi^2}{2} B - 80\xi_0(1 - \gamma)^2 - 3\eta. \]
Define
\[ \rho := - \left[\gamma^2 + (n - 1)\gamma - (n - 2)\right] \geq 0, \] (2.26)
and choose
\[ \xi_0 = 1 - \frac{n}{4(n - 1)} + \eta, \] (2.27)
then \(M\) can be written by
\[ M(n, \gamma) = \left(56 - \frac{20}{n - 1}\right) \rho + \left(42 + 54n - \frac{40}{n - 1}\right) \gamma - 54n + 70 + A \cos 1 \]
\[ - \frac{\pi^2}{2} B - 3\eta \]
\[ \geq \left(42 + 54n - \frac{40}{n - 1}\right) \gamma - 54n + 70 + A \cos 1 - \frac{\pi^2}{2} B - 3\eta > 0. \]
where in the last inequality we use (2.5). Thus,
\[ \left[\Delta F - \sum_{i=1}^{n-1} \frac{2F_i^2}{F} \zeta - 2\eta \right] |\nabla u|^2 > 0. \] (2.28)
Step 3: Estimates of $\left[ 2F - \frac{8F(n-1)}{n} (1 - \xi) - \frac{2}{\eta} |F_{x_n}|^2 \right] |\nabla \nabla u|^2$.

Since

$$|F_{x_n}|^2 = \left[ A(|x'|^2 + \epsilon)^{1-\gamma} \sin \frac{x_n}{|x'|^2 + \epsilon} - \frac{\pi}{2} B(|x'|^2 + \epsilon)^{1-\gamma} \sin \left( \frac{\pi x_n - \frac{A-\lambda_2}{2} |x'|^2}{2 |x'|^2 + \epsilon} \right) \right]^2 \leq \eta^2 (|x'|^2 + \epsilon)^{1-\gamma},$$

we have

$$\frac{2}{\eta} |F_{x_n}|^2 \leq \eta (|x'|^2 + \epsilon)^{1-\gamma}.$$

From (2.27), we know

$$1 - \xi = \frac{n}{4(n-1)(1+\eta)},$$

which leads

$$2F - \frac{8F(n-1)}{n} (1 - \xi) - \frac{2}{\eta} |F_{x_n}|^2 \geq 2\eta F - \eta (|x'|^2 + \epsilon)^{1-\gamma} > 0.$$ Thus

$$\left[ 2F - \frac{8F(n-1)}{n} (1 - \xi) - \frac{2}{\eta} |F_{x_n}|^2 \right] |\nabla \nabla u|^2 > 0. \quad (2.29)$$

Combining (2.28), (2.29) and (2.20), we have (2.13), which is contradictory with (2.12).

Hence, we have ruled out the possibility that $Q$ obtains its maximum point at the boundary $\partial \Omega_{r} \backslash (\Gamma_+ \cup \Gamma_-)$ Thus, increasing $r$ if necessary, we have

$$Q \leq C \|g\|_{C^{1,\alpha}(\partial D)},$$

(1.11) follows.

Proof of Theorem 1.3. Under the assumption of (1.5), (1.7) and (1.13), from Lemma 2.1 we know that at any point of $\Gamma_+$ and $\Gamma_-$, we have

$$\frac{\partial}{\partial \nu} (|\nabla u|^2) \leq \frac{2m(n-1)\lambda_k |x'|^{m-2}}{\sqrt{1 + m^2\lambda_k^2 |x'|^m}} |\nabla u|^2, \quad (2.30)$$

where $k = 1$ on $\Gamma_+, k = 2$ on $\Gamma_-.$

We consider the quantity

$$Q = F |\nabla u|^2 \quad \text{in} \quad \Omega_{r},$$

where

$$F = (|x'|^m + \epsilon)^{1-\gamma} - A(|x'|^m + \epsilon)^{2-\gamma} |x'|^{m-2} \cos \frac{x_n}{|x'|^m + \epsilon}$$

$$+ B(|x'|^m + \epsilon)^{1-\gamma} |x'|^{m-2} \left( \frac{|x'|^m}{2} + \epsilon \right) \cos \left( \frac{\pi x_n - \frac{\lambda_1-\lambda_2}{2} |x'|^m}{2 |x'|^m + \epsilon} \right) \quad (2.31)$$

$A, B$ are uniform constants satisfying

$$A > \frac{(m - 2 + m\gamma)m\pi \max\{\lambda_1, \lambda_2\} - m \left( \frac{m}{2} - \frac{m}{n-1} - n + 3 \right) \gamma - m (m + n - 3 - \frac{m}{2})}{\cos 1 - \pi \sin 1},$$

and
\[
\frac{2}{\pi} \left\{ (m - 2 + m \gamma) m \max \{ \lambda_1, \lambda_2 \} + A \sin 1 \right\} \\
< B < \frac{2}{\pi^2} \left\{ m \left( \frac{mn}{2} - \frac{m}{n - 1} - n + 3 \right) \gamma + m \left( m + n - 3 - \frac{mn}{2} \right) + A \cos 1 \right\},
\]

Similar to the proof of Theorem 1.1 we have the result. \(\square\)

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