Post-Newtonian Expansion of Gravitational Waves from a Particle in Circular Orbit around a Schwarzschild Black Hole

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(Received May 19, 1994)

Based upon the formalism recently developed by one of us (MS), we analytically perform the post-Newtonian expansion of gravitational waves from a test particle in circular orbit of radius $r_0$ around a Schwarzschild black hole of mass $M$. We calculate gravitational wave forms and luminosity up to $\nu^8$ order beyond Newtonian, where $\nu = (M/r_0)^{1/2}$. In particular, we give the exact analytical values of the coefficients of $\ln \nu$ terms at $\nu^6$ and $\nu^8$ orders in the luminosity and confirm the numerical values obtained previously by one of us (HT) and Nakamura. Our result is valid in the small mass limit of one body and gives an important guideline for the gravitational wave physics of coalescing compact binaries.

§ 1. Introduction

Among the possible sources of gravitational waves, coalescing compact binaries are the most promising candidates which can be detected by the near-future laser interferometric gravitational wave detectors such as LIGO$^7$ and VIRGO.$^8$ One reason is that we expect such events to occur 3/yr within 200 Mpc.$^9$ The other reason is that we expect enough amplitude of gravitational waves to be detected by LIGO and VIRGO if such events occur.

Gravitational radiation from coalescing compact binaries contains rich information about physics of neutron stars, cosmological parameters, a test of general relativity and so on. Such information can be extracted out from the gravitational wave form by the matched filtering technique, that is, by cross-correlating the incoming noisy signal with theoretical templates. If the signal and the templates get out of phase with each other by one cycle as the waves sweep through the LIGO/VIRGO band, their cross correlation will be significantly reduced. This means that it is important to construct theoretical templates which are accurate to better than one cycle during entire sweep through the LIGO/VIRGO band.$^4$ Thus, much effort has been recently made to construct accurate theoretical templates.$^{5-9}$

To construct theoretical templates, the post-Newtonian approximations are usually employed to solve the Einstein equations. However, based on numerical calculations of the gravitational radiation from a particle in circular orbit around a non-rotating black hole, Cutler et al.$^{10}$ showed that evaluation of the gravitational wave luminosity to a post-Newtonian order much higher than presently achieved level will be required to construct the templates. Then in order to find out the necessary post-Newtonian order, the same problem was investigated by Tagoshi and Nakamura$^{11}$ with much higher accuracy. They calculated the coefficients of the post-Newtonian expansion of the gravitational wave luminosity to (post)$^4$-Newtonian order (i.e., $O(\nu^8)$ beyond Newtonian) and concluded that the accuracy to at least (post)$^3$-Newtonian order is required for the construction of theoretical templates. In
addition, they found logarithmic terms in the luminosity at (post)$^3$ and (post)$^4$-Newtonian orders.

It is then highly desirable to reproduce these results in a purely analytical way, that is, to derive the exact analytical expression for the coefficients of the post-Newtonian expansion. Poisson$^{13}$ first developed such a method and calculated the luminosity to $O(v^3)$. Then extending Poisson's method, a more systematic method was developed by Sasaki$^{12}$ (hereafter Paper I) and analytical expressions for the ingoing-wave Regge-Wheeler functions $X_n^{lm}$ were derived with the accuracy required to calculate gravitational wave forms and luminosity up to $O(v^8)$ beyond Newtonian. There it was also shown that as long as we are concerned with the radiation going out to infinity, the effect of the presence of the black hole horizon does not appear until we calculate to an extremely higher order, $O(v^{18})$. Hence the expansion of $X_n^{lm}$ can be used in a situation in which there is a non-rotating compact star instead of a black hole.

In this paper, using the result obtained in Paper I, we calculate analytical expressions for the gravitational wave forms and luminosity from a particle in circular orbit around a Schwarzschild black hole to $O(v^8)$ beyond Newtonian. This paper is organized as follows. In § 2, we show the general formulae and conventions used in this paper. In § 3, we first briefly review the result of Paper I and discuss the relation between the Regge-Wheeler and Teukolsky functions with emphasis on the consistency of the post-Newtonian orders in the conversion formula. In § 4, we present the analytical expressions for the gravitational wave forms and luminosity to $O(v^8)$ beyond Newtonian and compare the result with that of Tagoshi and Nakamura.$^{11}$ We find an extremely good agreement between the two. In § 5, applying our result of the luminosity formula, we discuss the effect of the higher order post-Newtonian terms to the accumulated phase of gravitational waves from coalescing compact binaries. Section 6 is devoted to a conclusion. Throughout the paper we use geometrized units, $c = G = 1$.

§ 2. General formulation

We consider the case when a test particle of mass $\mu$ travels a circular orbit around a Schwarzschild black hole of mass $M \gg \mu$. We mostly follow notation used by Poisson,$^{13}$ but for definiteness, we recapitulate necessary formulae and definitions of symbols in this section.

To calculate the gravitational wave forms and luminosity, we consider the inhomogeneous Teukolsky equation,$^{14,15}$

$$\left[ \mathcal{D}^2 \frac{d}{dr} \left( \frac{1}{\mathcal{D}} \frac{d}{dr} \right) - U(r) \right] R_{lm\omega}(r) = T_{lm\omega}(r),$$

where

$$U(r) = \frac{r^2}{\mathcal{D}} \left[ \omega^2 r^2 - 4i\omega(r - 3M) \right] - (l-1)(l+2), \quad \mathcal{D} = r(r - 2M),$$

and $T_{lm\omega}$ is the source term whose explicit form will be given in Eq. (7) below.
We solve Eq. (1) by the Green function method. For this purpose, we need a homogeneous solution $R_{t0}^{in}(r)$ of Eq. (1) which satisfies the following boundary condition,

\[
R_{t0}^{in}(r) = \begin{cases} 
D_{t0} e^{i\omega r^*} & \text{for } r^* \to -\infty, \\
\rho^2 B_{t0} e^{i\omega r^*} + r^{-1} B_{t0} e^{-i\omega r^*} & \text{for } r^* \to +\infty,
\end{cases}
\]

where $r^* = r + 2M\ln(r/2M - 1)$. Then the outgoing-wave solution of Eq. (1) at infinity with the appropriate boundary condition at horizon is given by

\[
R_{t0}(r \to \infty) = \frac{\rho^2 e^{i\omega r^*}}{2i\omega B_{t0}} \int_{2M}^{\infty} dr' R_{t0}^{in}(r') \Delta^{-2} = \rho^2 e^{i\omega r^*} \tilde{Z}_{t0}.
\]

In the case of a circular orbit, the specific energy $\tilde{E}$ and angular momentum $\tilde{L}$ of the particle are given by

\[
\tilde{E} = \frac{(r_0 - 2M)}{\sqrt{r_0(r_0 - 3M)}} \quad \text{(5)}
\]

and

\[
\tilde{L} = \sqrt{Mr_0} / \sqrt{1 - 3M/r_0}, \quad \text{(6)}
\]

where $r_0$ is the orbital radius. The angular frequency is given by $\Omega = (M/r_0)^{1/2}$. Then $T_{t0}(r)$ is given by

\[
\frac{T_{t0}}{\pi} = (-2b_{t0}(r_0 - 2M)^3 \delta(r - r_0)
- b_{t0}2r_0[(r_0 - 2M)^3 \delta'(r - r_0) - (r_0 - 2M)(2 - i\omega r_0) \delta(r - r_0)]
+ 2b_{t0}[r_0^2(r_0 - 2M)^2 \delta''(r - r_0)]
+ (2i\omega r_0^3 - 2r_0^2(3r_0^2 - 8r_0M + 4M^2)) \delta'(r - r_0)
+ (4r_0^2 - 8M^2 - \omega^2 r_0^4 - 6i\omega r_0^2(r_0 - M)) \delta(r - r_0)] \delta(\omega - m\Omega),
\]

where $\delta(r)$ is the Dirac delta function and $' = d/dr$. The coefficients $b_{t0}$ are given by

\[
-b_{t0} = \frac{1}{2} [(l - 1)(l + 1)(l + 2)]^1 \rho_0 Y_{l0} \left( \frac{\pi}{2}, 0 \right) E_0 / (r_0 - 2M),
-1b_{l0} = [(l - 1)(l + 2)]^1 \rho_0 Y_{l0} \left( \frac{\pi}{2}, 0 \right) L / r_0,
-2b_{l0} = \rho_0 Y_{l0} \left( \frac{\pi}{2}, 0 \right) L \Omega,
\]

where $\rho Y_{l0}(\theta, \varphi)$ are the spin-weighted spherical harmonics. From Eqs. (4) and (7), we see that $\tilde{Z}_{t0}$ takes the form,

\[
\tilde{Z}_{t0} = Z_{t0} \delta(\omega - m\Omega),
\]

where
We note the symmetry, \( Z_{l,-m} = (-1)^l Z_{l,m} \), which may be seen from Eqs. (8), (10) and the symmetry of the spin weighted spherical harmonics, \( sYlm(\pi/2,0) = (-1)^{(s+l)s} Ylm(\pi/2,0) \). In terms of the amplitudes \( Z_{l,m} \), the gravitational wave form at infinity is given by

\[
h_+ - i h_\times = -\frac{2\mu}{r^2} \sum_{l=1}^{\infty} \frac{1}{\omega^2} Z_{l,m} - 2 Y_{l,m}(\theta, \varphi) e^{-i\omega(t-r^*)},
\]

and the luminosity is given by

\[
\frac{dE}{dt} = \sum_{l=1}^{\infty} \sum_{m=1}^{l} |Z_{l,m}|^2 / 2\pi \omega^2,
\]

where \( \omega = m\Omega \). Thus the only remaining task is to calculate the ingoing-wave Teukolsky function \( R_{l,m} \).

Instead of directly calculating \( R_{l,m} \) from the homogeneous Teukolsky equation, it is much easier to calculate the corresponding Regge-Wheeler function \( X_{l,m} \) first and then transform it to \( R_{l,m} \). The homogeneous Regge-Wheeler equation takes the form,

\[
\left[ \frac{d^2}{dr*^2} + \omega^2 - V(r) \right] X_{l,m}(r) = 0,
\]

where

\[
V(r) = \left( 1 - \frac{2M}{r} \right) \left( \frac{l(l+1)}{r^2} - \frac{6M}{r^3} \right).
\]

It is known that this equation is obtained by transforming \( R_{l,m} \) as

\[
R_{l,m} = \Delta \left( \frac{d}{dr*} + i\omega \right) \frac{r^2}{\Delta} \left( \frac{d}{dr*} - i\omega \right) rX_{l,m}.
\]

Conversely, we can express \( X_{l,m} \) in terms of \( R_{l,m} \) by the inverse transformation formula as

\[
X_{l,m} = \frac{r^5}{c_0 \Delta} \left( \frac{d}{dr*} - i\omega \right) \frac{r^2}{\Delta} \left( \frac{d}{dr*} + i\omega \right) R_{l,m} \frac{r^2}{\Delta},
\]

where \( c_0 = (l-1)(l+1)(l+2) - 12iM\omega \). Then we obtain the asymptotic forms of \( X_{l,m} \) as

\[
X_{l,m}(r) = \begin{cases} 
C_{l,m} e^{-i\omega r*}, & r* \to -\infty, \\
A_{l,m} e^{i\omega r*} + A_{l,m}^{\text{in}} e^{-i\omega r*}, & r* \to +\infty,
\end{cases}
\]
where \(A_{\omega}^{in}, A_{\omega}^{out}\) and \(C_{\omega}\) are respectively related to \(B_{\omega}^{in}, B_{\omega}^{out}\) and \(D_{\omega}\) defined in Eq. (3) as

\[
B_{\omega}^{in} = -\frac{C_{\omega}}{4 \omega^{2}} A_{\omega}^{in},
\]

\[
B_{\omega}^{out} = -4 \omega^{2} A_{\omega}^{out},
\]

\[
D_{\omega} = \frac{C_{\omega}}{16(1-2iM\omega)(1-4iM\omega)} C_{\omega} .
\]  

(18)

\[\text{§ 3. Post-Newtonian expansion of the Teukolsky function}\]

3.1. Method

In Paper I, the post-Newtonian expansion of the ingoing-wave Regge-Wheeler functions \(X_{\omega}^{in}\) was formulated and they were calculated to \(O(\varepsilon^{2})\) where \(\varepsilon = 2M\omega\). In this subsection we briefly review the method.

We rewrite the homogeneous Regge-Wheeler equation as

\[
\left[ \frac{d^{2}}{dz^{2}} + 1 - \left( 1 - \frac{\varepsilon}{z} \right) \left( \frac{l(l+1)}{z^{2}} - \frac{3 \varepsilon}{z^{2}} \right) \right] X_{\omega}^{in} = 0 ,
\]  

(19)

where \(z = \omega r, \ z^{*} = z + i\varepsilon \ln(z - \varepsilon)\) and we have suppressed the index \(\omega\) since it is trivially absorbed in \(\varepsilon\) and \(z\). Note that for \(\omega = mQ, \varepsilon = 2mv^{2}\) and \(z = mv\) at \(r = r_{0}\). Hence the post-Newtonian expansion corresponds to expanding \(X_{\omega}^{in}\) with respect to \(\varepsilon\) and evaluating \(X_{\omega}^{in}\) at \(z \ll 1\) as well as \(A_{\omega}^{in}\) to required orders in \(\varepsilon\). Now setting

\[
X_{\omega}^{in} = e^{-i\varepsilon \ln(z - \varepsilon)} z^{\ell}(z),
\]  

(20)

we find that Eq. (19) becomes

\[
\left[ \frac{d^{2}}{dz^{2}} + \frac{2}{z} \frac{d}{dz} + \left( 1 - \frac{l(l+1)}{z^{2}} \right) \right] \xi_{\ell} = i e^{-iz} \frac{d}{dz} \left[ \frac{1}{z^{2}} \frac{d}{dz} \left( e^{iz} z^{\ell} \xi_{\ell}(z) \right) \right].
\]  

(21)

Thus expanding \(\xi_{\ell}\) with respect to \(\varepsilon\) as

\[
\xi_{\ell}(z) = \sum_{n=0}^{m} \varepsilon^{n} \xi_{\ell}^{(n)}(z),
\]  

(22)

we obtain the recursive equations,

\[
\left[ \frac{d^{2}}{dz^{2}} + \frac{2}{z} \frac{d}{dz} + \left( 1 - \frac{l(l+1)}{z^{2}} \right) \right] \xi_{\ell}^{(n)}(z) = i e^{-iz} \frac{d}{dz} \left[ \frac{1}{z^{2}} \frac{d}{dz} \left( e^{iz} z^{\ell} \xi_{\ell}^{(n-1)}(z) \right) \right].
\]  

(23)

First for \(n = 0\), we have

\[
\xi_{\ell}^{(0)} = a^{(0)} j_{\ell} + b^{(0)} n_{\ell},
\]  

(24)

where \(j_{\ell}\) and \(n_{\ell}\) are the usual spherical Bessel functions. The boundary condition is that \(\xi_{\ell}^{(0)}\) be regular at \(z = 0\).\(^{13,12}\) Hence \(b^{(0)} = 0\) and for convenience we set \(a_{\ell}^{(0)} = 1\). For \(n \geq 1\), we rewrite Eq. (23) in the indefinite integral form,
\[ \xi^{(n)}_t = n! \int^z dz z^2 e^{-iz} \left[ \frac{1}{z^2} (e^{iz} z^{(n-1)}(z))^2 \right] - jz^2 e^{-iz} n! \left[ \frac{1}{z^2} (e^{iz} z^{(n-1)}(z))^2 \right]. \] (25)

If the above indefinite integrals can be explicitly performed, it is easy to obtain \( \xi^{(n)}_t \) with a desired boundary condition. In Paper I, it is shown that this is indeed the case for \( n=1 \) and 2, and the boundary condition is that \( \xi^{(n)}_t \) be also regular at \( z=0 \) at least for \( n \leq 3 \). We then find for \( n=1 \),

\[ \xi^{(1)}_t = \frac{(l-1)(l+3)}{2(l+1)(2l+1)} j_{l+1} - \left( \frac{l^2-4}{2l(2l+1)} + \frac{2l-1}{l(l-1)} \right) j_{l-1} + z^2 (n j_0 - j_m) j_0 + \sum_{k=1}^{l-1} \left( \frac{1}{k} + \frac{1}{k+1} \right) z^2 (n j_k - j_m) j_k + n l (Ci 2z - \gamma - \ln 2z) - j Si 2z + ij \ln z + a^{(1)}_l j_l, \] (26)

where \( Ci(x) = -\int_x^\infty \frac{\cos t}{t} dt \) and \( Si(x) = \int_x^\infty \frac{\sin t}{t} dt \) are the cosine and sine integral functions, respectively, and \( a^{(1)}_l \) is an arbitrary integration constant which represents the arbitrariness of the normalization of \( X^{(1)}_n \). We set \( a^{(1)}_l = 0 \). As for \( n=2 \), closed analytical expressions of \( \xi^{(2)}_t \) for \( l=2, 3 \) and 4 can be found in Paper I, § 4.2.

To obtain \( X^{(n)}_n \) from \( \xi^{(n)}_t \) is straightforward. Decomposing the real and imaginary parts of \( \xi^{(n)}_t \) as

\[ \xi^{(n)}_t = f^{(n)}_t + i g^{(n)}_t, \] (27)

we find the imaginary parts \( g^{(n)} \) are given by

\[ g^{(0)}_l = 0, \quad g^{(1)}_l = j_i \ln z, \quad g^{(2)}_l = -\frac{1}{z} j_i + f^{(1)}_l \ln z. \] (28)

These relations confirm that \( X^{(n)}_n \) is real at least up to \( O(\epsilon^2) \). In fact, inserting Eq. (27) into Eq. (20) and expanding the result with respect to \( \epsilon \) by assuming \( z \gg \epsilon \), we find

\[ X^{(n)}_n = e^{-i\epsilon \ln(z - \epsilon)} (j_i + \epsilon (f^{(1)}_l + i g^{(1)}_l) + \epsilon^2 (f^{(2)}_l + i g^{(2)}_l) + \cdots) = z (j_i + \epsilon f^{(1)}_l + \epsilon^2 (f^{(2)}_l + g^{(1)}_l) \ln z - \frac{1}{2} j_i (\ln z)^2 + \cdots) = z (j_i + \epsilon f^{(1)}_l + \epsilon^2 (f^{(2)}_l + \frac{1}{2} j_i (\ln z)^2 + \cdots). \] (29)

### 3.2. Calculation

Once we have closed analytical expressions of \( \xi^{(n)}_t \), \( A^{(n)}_n \) can be readily obtained by gathering all the coefficients of leading terms proportional to \( e^{-iz}/z \) from the asymptotic forms of \( \xi^{(n)}_t \) at \( z = \infty \). The explicit forms of \( A^{(n)}_n \) from \( l=2 \) to 4 which are correct up to \( \epsilon^2 \) order are given by

\[ A^{(2)}_n = -\frac{1}{2} i e^{-i\epsilon (\ln 2 + \epsilon)} \exp \left( i \epsilon \frac{5}{3} - i \epsilon \frac{107}{420} \pi \right). \]
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\[
A_3^{in} = \frac{1}{2} i e^{-i (n \pi + \gamma)} \exp \left( i e^{\frac{13}{6}} - i e^{\frac{13}{84} \pi} \right) 
\times \left( 1 - \frac{\pi}{2} e^2 \left[ \frac{25}{18} + \frac{5}{24} \pi^2 + \frac{107}{210} (\gamma + \text{ln}2) \right] + \cdots \right),
\]

\[
A_4^{in} = \frac{1}{2} i e^{-i (n \pi + \gamma)} \exp \left( i e^{\frac{149}{60}} - i e^{\frac{1571}{13860} \pi} \right) 
\times \left( 1 - \frac{\pi}{2} e^2 \left[ \frac{22201}{7200} + \frac{5}{24} \pi^2 + \frac{1571}{6930} (\gamma + \text{ln}2) \right] + \cdots \right).
\]

For \( l \geq 5 \), explicit forms of \( A_l^{in} \) which are valid up to order \( O(\epsilon) \) are given by

\[
A_l^{in} = \frac{1}{2} e^{i l + 1} e^{-i (n \pi + \gamma)} \left[ 1 - \frac{\pi}{2} e^2 \left( \frac{1}{k^2} + \frac{1}{k^3} + \frac{(l-1)(l+3)}{l(l+1)} \right) \right].
\]

As discussed in Paper I, and will be mentioned below, the above formulae are what we all need for \( A_r \) in order to calculate the wave forms and luminosity to \( O(\epsilon^8) \) beyond Newtonian.

In addition to \( A_l^{in} \), we need the series expansion formulae of \( X_l^{in} \) at \( z \ll 1 \). To clarify their required orders of accuracy, we first consider the qualitative behavior of \( X_l^{in} \) as \( z \to 0 \). We find

\[
X_2^{(in)} = z^2 [O(1) + \epsilon O(z) + \epsilon^2 (O(1) + O(1) \text{ln} z) + \epsilon^3 O(z^{-1}) + \cdots],
\]

\[
X_3^{(in)} = z^3 [O(z) + \epsilon O(1) + \epsilon^2 (O(z) + O(z) \text{ln} z) + \cdots],
\]

\[
X_l^{(in)} = z^{l-2} [O(z^{l-2}) + \epsilon O(z^{l-3}) + \epsilon^2 (O(z^{l-4}) + O(z^{l-2}) \text{ln} z) + \cdots]. \quad (l \geq 4)
\]

The above result tells us the accuracy of \( X_l^{in} \) needed to achieve \( O(\epsilon^8) \) beyond Newtonian. For convenience, we set

\[
X_l^{in} = \sum_{n=0}^{\infty} e^n X_l^{(n)}.
\]

First, note that we must calculate \( X_l^{(0)} \) to \( O(\epsilon^1) \), \( X_l^{(1)} \) to \( O(\epsilon^4) \), \( X_l^{(2)} \) to \( O(\epsilon^5) \) and \( X_l^{(3)} \) to \( O(\epsilon^8) \). Then we see from Eq. (36) that we need the series expansions of \( X_l^{in} \) at \( z = 0 \) to the above required order for \( l = 2 \) to \( O(\epsilon^3) \), for \( l = 3, 4, 5 \) and 6 to \( O(\epsilon^2) \), for \( l = 7 \) and 8 to \( O(\epsilon^1) \), and for \( l = 9 \) and 10 to \( O(\epsilon^5) \).

Since we have closed analytical expressions of \( X_l^{(0)} \) and \( X_l^{(1)} \) for all \( l \), and \( X_l^{(2)} \) and \( X_l^{(3)} \) for \( l \leq 4 \), whose series expansion formulae can be trivially obtained, what we additionally need to do is to derive the series expansions of \( X_2^{(3)}, X_3^{(2)} \) and \( X_6^{(3)} \). Taking into account the required accuracy mentioned above, we find that only the leading terms in their series expansions are necessary. First consider \( X_8^{(2)} \) and \( X_6^{(3)} \). With the boundary condition that these be regular at \( z = 0 \), and using Eq. (25) with \( n = 2 \) and Eq. (26), they can be easily calculated.
$X_5^{(2)} = \frac{2}{7425} z^4 + O(z^5), \quad (37)$

$X_6^{(2)} = -\frac{4}{21235} z^5 + O(z^7), \quad (38)$

where possible ambiguities due to the choice of integration constants appear only at $O(z^6)$ and $O(z^7)$, respectively, hence do not affect the coefficients of the leading terms, as it should be so. In the same way, the leading behavior of $X_3^{(3)}$ at $z=0$ is found to be

$X_3^{(3)} = -\frac{319}{6300} z^3 + O(z^4), \quad (39)$

where the aforementioned ambiguity appears at $O(z^4)$, hence again does not affect the result. In Appendix A, we show the series expansions of $X_i^{(n)}$ up to the required orders.

To calculate gravitational waves from the Teukolsky equation, we need to know $R_i^{(n)}(z)$ at $z \ll 1$, which we calculate by using the conversion formula from $X_i^{(n)}$ to $R_i^{(n)}$, Eq. (15). In terms of $X_i^{(n)}$, the result is expressed as

$$\omega R_i^{(n)} = z(z - \epsilon) \left( \frac{d}{dz} + i \right) \left( \frac{z^2}{z(z - \epsilon)} \right) \left( \frac{d}{dz} + i \right) z X_i^{(n)}(z)$$

$$= -z^2(zX_i^{(0)}) + z^2(zX_i^{(0)})'' + 2iz^2(zX_i^{(0)})'$$

$$+ \epsilon(-2z(zX_i^{(0)})'' + z^2(zX_i^{(1)})'' - z^2(zX_i^{(1)}))$$

$$-2iz(zX_i^{(0)})' - i(zX_i^{(0)}) + 2iz^2(zX_i^{(1)})'$$

$$+ \epsilon^2(-2z(zX_i^{(1)})' - 2z(zX_i^{(2)})'' + z^2(zX_i^{(2)})'' - z^2(zX_i^{(2)}))$$

$$- i(zX_i^{(1)}) - 2iz(zX_i^{(1)})' - 2iz^2(zX_i^{(2)})'$$

$$+ \epsilon^3(-2z(zX_i^{(2)})' - 2z(zX_i^{(3)})' + z^2(zX_i^{(3)})'' - z^2(zX_i^{(3)}))$$

$$- i(zX_i^{(2)}) - 2iz(zX_i^{(2)})' - 2iz^2(zX_i^{(3)})'), \quad (40)$$

where $'=d/dz$.

Here a problem might arise, since what we have calculated are the series expansions of $X_i^{(n)}(z_0)$ to $O(\nu^8)$ relative to the leading order, but what we actually need are those of $R_i^{(n)}$. Hence we must examine if there is consistency of the post-Newtonian orders between the series expansions of $X_i^{(n)}$ and $R_i^{(n)}$.

A straightforward way to check this consistency is to insert the truncated series expressions of $X_i^{(n)}$ accurate to $O(\nu^8)$ beyond Newtonian to Eq. (40), insert the resulting expression of $R_i^{(n)}$ to the inverse transformation formula (16), and examine if the final result is the same as the starting expression of $X_i^{(n)}$. We have verified this is indeed the case.

As for $B_i^{(n)}$, it should be now obvious from Eqs. (11), (12) and (18) that the calculated $A_i^{(n)}$ have sufficient accuracy to obtain $B_i^{(n)}$ to the required orders.

All these facts support our confidence that $X_i^{(n)}$ and $R_i^{(n)}$ are equivalent also within the framework of our post-Newtonian expansion.
§ 4. Wave forms and luminosity to $O(v^8)$

Let us now calculate the gravitational wave forms and luminosity up to $O(v^8)$ beyond Newtonian. The task is straightforward but tedious. So we only show the key equations. The calculations have been performed with the help of the computer manipulation software Mathematica. We follow the notation of Poisson\textsuperscript{13} to describe the post-Newtonian expansion of gravitational wave forms and luminosity. Namely, we express them as

\begin{equation}
\frac{dE}{dt} = \frac{32}{5} \left( \frac{\mu}{M} \right)^2 \left( \frac{M}{r_0} \right)^5 \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \eta_{lm},
\end{equation}

\begin{equation}
\mathbf{h}_{+}^{\nu^{\phi}} + \mathbf{h}_{x}^{\nu^{\phi}} = - \left( \frac{\mu}{r} \right) \left( \frac{M}{r_0} \right) \xi_{lm}^{\nu^{\phi}}.
\end{equation}

4.1. Luminosity

The procedure is straightforward. First we obtain $R_{ln}$ from Eq. (40) and $B_{ln}$ from Eq. (18). Then we insert them to Eq. (10) and set $z = mv$ and $e = 2mv^3$ to obtain $Z_{lm}$. Finally, inserting them to Eq. (12) and expanding the results with respect to $v$, we obtain $\eta_{lm}$.

The explicit forms of $\eta_{lm}$ are given by

\begin{align*}
\eta_{2,2} &= 1 - \frac{107 v^2}{21} + 4 \pi v^3 + \frac{4784 v^4}{1323} - \frac{428 \pi v^5}{1323} + \frac{19136 \pi v^7}{1323} + \cdots, \\
\eta_{2,1} &= \frac{17 v^2}{36} + \frac{17 \pi v^5}{504} + \frac{17 \pi v^7}{252} + \cdots, \\
\eta_{3,3} &= \frac{1215 v^2}{896} - \frac{1215 v^4}{112} + \frac{243729 v^6}{9856} - \frac{3645 \pi v^7}{56} + \cdots, \\
\eta_{3,2} &= \frac{5 v^4}{63} - \frac{193 v^6}{567} + \frac{20 \pi v^7}{63} + \frac{68111 v^8}{280665}, \\
\eta_{3,1} &= \frac{v^2}{8064} - \frac{v^4}{1512} + \frac{437 v^6}{4032} - \frac{\pi v^7}{266112} + \cdots.
\end{align*}
\[ \eta_{4,4} = \frac{1280v^4}{567} + \frac{15180v^5}{6237} + \frac{10240\pi v^7}{567} + \frac{560069632v^8}{6243237}, \]
\[ \eta_{4,3} = \frac{729v^6}{4480} - \frac{28431v^8}{24640}, \]
\[ \eta_{4,2} = \frac{5v^4}{3969} + \frac{437v^6}{43659} + \frac{20\pi v^7}{3969} + \frac{7199152v^8}{218513295}, \]
\[ \eta_{4,1} = \frac{v^6}{282240} - \frac{101v^8}{4656960}, \]
\[ \eta_{5,5} = \frac{9765625v^6}{2433024} - \frac{2568359375v^8}{47443968}, \]
\[ \eta_{5,4} = \frac{4096v^8}{13365}, \]
\[ \eta_{5,3} = \frac{2187v^6}{450560} - \frac{150903v^8}{2928640}, \]
\[ \eta_{5,2} = \frac{4v^8}{40095}, \]
\[ \eta_{5,1} = \frac{v^6}{127733760} - \frac{179v^8}{2490808320}, \]
\[ \eta_{5,0} = \frac{26244}{3575}v^8, \]
\[ \eta_{5,4} = \frac{131072}{955975}v^8, \]
\[ \eta_{5,2} = \frac{4}{573585}v^8. \]

From the above results, we obtain \( \frac{dE}{dt} \) which is correct up to \( O(v^8) \) as

\[
\frac{dE}{dt} = \left( \frac{dE}{dt} \right)_n \left( 1 - \frac{1247v^2}{336} + \frac{44711v^4}{9072} + \frac{8191\pi v^5}{672} + \frac{16285\pi v^7}{504} \right) + v^8 \left( \frac{6643739519}{69854400} - \frac{1712\gamma}{105} + \frac{16\pi^2}{3} + \frac{3424\ln2}{105} - \frac{1712\ln v}{105} \right) + v^{10} \left( \frac{323105549467}{3178375200} - \frac{232597\gamma}{4410} - \frac{1369\pi^2}{126} + \frac{39931\ln2}{294} \right) + v^{12} \left( \frac{47385\ln3}{1568} + \frac{232597\ln v}{4410} \right) \right). \tag{43}
\]

4.2. Wave forms

Noting the form of the spherical harmonics, \( s Y_{lm}(\theta, \varphi) = s P_{lm}(\theta) e^{im\varphi} \), and that of \( A_{l}^{*n} \) given by Eqs. (30)~(33), we find the wave form is in the form,

\[ h_{l}^{*m} - ih_{l}^{*m} \propto e^{i\theta(\ln2+\tau)} e^{-\imath\theta D(t-r^*)} e^{\imath m\varphi} \]
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$$= e^{im2\nu^2(\gamma+2\ln2+3\ln\nu)-\Omega(t-r^*)+\varphi} e^{2im\nu^3\ln m}.$$  (44)

This suggests that it is more convenient to introduce a new phase variable $\psi$ as

$$\psi = \Omega(t-r^*) - \varphi - 2\nu^3(\gamma+2\ln2+3\ln\nu).$$  (45)

An advantage of introducing the phase variable $\psi$ is that we directly see the post-Newtonian corrections at $O(\nu^3)$ to the phase of waves. These phase corrections at $O(\nu^3)$ delay the arrival time of the wave to the observer and express the tail effects. From Eqs. (30)~(32), we also see the tail effects at $O(\nu^6)$ on the phase.

In passing, we note that if one expanded $A_{l,m}^{\nu}$ in terms of $\nu$, there would appear terms like $(\ln\nu)^2$ at $O(\nu^6)$. Then one would be surprised to see the cancellation of them in the calculation of $O(\nu^6)$. However in our approach, we see from the beginning that such terms do not exist. Together with the fact that there are no $\ln\nu$ terms at orders smaller than $O(\nu^6)$ and no $(\ln\nu)^2$ terms at all in $X_{l,m}^{\nu}$, we readily foresee that there will be no $(\ln\nu)^2$ terms in $dE/dt$ to $O(\nu^6)$, as we have explicitly seen in the previous subsection.

Using $\psi$ we obtain

$$h_{l,m}^+ + h_{l,m}^x = \frac{\mu}{r} \frac{1}{\nu} (\omega^2 P_{l,m}(\omega) + (-1)^l P_{l,-m}(\omega)) [W_{l,m} e^{-i\psi} + \tilde{W}_{l,m} e^{i\psi}],$$  (46)

$$h_{l,m}^x + h_{l,m}^x = \frac{\mu}{r} \frac{i}{\nu} (\omega^2 P_{l,m}(\omega) - (-1)^l P_{l,-m}(\omega)) [W_{l,m} e^{-i\psi} - \tilde{W}_{l,m} e^{i\psi}],$$  (47)

where $W_{l,m}$ is defined as

$$Z_{l,m} = W_{l,m} e^{i\psi} e^{2\nu^3(\gamma+2\ln2+3\ln\nu)}.$$  (48)

Now from Eq. (42) and using the explicit forms of $\omega^2 P_{l,m}(\omega)$, it is straightforward to obtain $\xi_{l,m}$. The explicit wave form for each $l$ and $m$ are given in Appendix B.

In Figs. 1 and 2, we plot $\xi_{l,m}^+ + \xi_{l,m}^x$ as a function of the orbital phase. We can see from these figures how the higher order post-Newtonian terms alter the shape of the wave forms.

![Fig. 1. Plots of the gravitational wave forms, which are defined as $\xi_{l,m}^+ + \xi_{l,m}^x$, as a function of the orbital phase. The solid line and the dashed line in (a) show $\xi_{l,m}^+$ up to the order $O(\nu^3)$ and $O(\nu^6)$, respectively. The solid line, dashed line and dotted line in (b) show the contribution from $O(\nu^4)$, $O(\nu^5)$ and $O(\nu^6)$ to $\xi_{l,m}$, respectively. In each figure, $\theta = \pi/4$ and $r_0 = 8M$.](image-url)
4.3. Comparison with the numerical result

In this section we compare our analytical result for $dE/dt$ with the numerical result calculated by Tagoshi and Nakamura.\textsuperscript{11)}

The numerical values of the coefficients in the luminosity formula calculated in this paper are given in Table I(a). Comparing them with the coefficients given in their paper, we see that their coefficient of $\nu^8$ differs from the true value about $\sim 20\%$, while all the other coefficients are in very good agreement with each other. This difference is within the error estimated in their paper. However, we find that in fact their numerical data turn out to have much better accuracy. The reason is as follows. In Ref. 11), the coefficients are calculated by least square fitting, but taking account of $\ln \nu$ terms only at $O(\nu^6)$ and $O(\nu^8)$. However, since they calculated the gravitational waves to $l=6$, their data correctly contain contributions from the orders up to $O(\nu^9)$ of the post-Newtonian expansion and some contributions from yet higher orders, provided the data have enough accuracy. Hence we have recalculated the

| $\nu^4$        | $-4.92846119929533$ | $-4.928461199295258$ |
|----------------|---------------------|-----------------------|
| $\nu^5$        | $-38.29283545469344$| $-38.29283545329089$  |
| $\nu^6$        | $+115.7317166756113$| $+115.73172132$       |
| $\nu^6 \ln \nu$| $-16.3047619047619$ | $-16.304761151$       |
| $\nu^7$        | $-101.5095595937416$| $-101.5095959597416$  |
| $\nu^8$        | $-117.504390226773$ | $-117.504390226773$   |
| $\nu^8 \ln \nu$| $+52.74308390022676$| $+52.74308390022676$  |

Table I. (a) The numerical values of the analytically calculated coefficients of the post-Newtonian expansion of $dE/dt$ given in Eq. (43), and (b) those numerically calculated by means of least square fitting, where the same numerical data of $dE/dt$ as in Tagoshi and Nakamura\textsuperscript{11)} are used, but additionally including $\ln \nu$ terms at $O(\nu^9)$ and $O(\nu^{10})$ in the fitting.
coefficients using their numerical data by least square fitting, including \( v^9 \ln v \) and \( v^{10} \ln v \) terms. The results are given in Table I(b). We find that even the coefficient of \( v^8 \) agrees with the analytical value within 1\%. This suggests that their data do actually give the coefficients to \( O(v^9) \) of the post-Newtonian expansion with high accuracy, hence the values of the coefficients of \( v^9 \) and \( v^9 \ln v \) given in Table I(b) are expected to be good approximations to the true values.

§ 5. An estimate of orbital phase of coalescing binaries

In this section, based on our results, we discuss the accuracy of the post-Newtonian expansion to construct the template waveforms from inspiraling compact binaries. Although our results are valid only for the test particle limit, \( \mu/M \ll 1 \), we ignore this fact in the following. Since the effect of non-vanishing \( \mu/M \) would only increase errors in the estimate, our estimate below may be regarded as an optimistic one.

The total cycle \( N \) of gravitational waves from an inspiraling compact binary during sweep through, say, the LIGO band is

\[
N = \int_{t_i}^{t_f} f dt = \int_{r_i}^{r_f} dr \frac{\partial}{\partial t} \frac{dE/dt}{dE/dr},
\]

(49)

where \( f \) is the orbital frequency, \( t_i(t_f) \) and \( r_i(r_f) \) are the initial (final) time and the orbital separation of the binary, respectively, and we have assumed quasi-periodicity of the inspiral orbit. Then expanding both the numerator and denominator with respect to \( v \), \( N \) is expressed as

\[
N = \frac{5}{32\pi} \frac{M}{\mu} \int_{t_i}^{t_f} dt \left( \frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k} \right),
\]

(50)

where \( x = (M/r)^{1/2} \) and the series forms in the denominator and numerator symbolically represent the post-Newtonian corrections to the \( dE/dr \) and \( dE/dt \), respectively, i.e., \( \sum b_k x^k = (dE/dr)/(dE/dr)_{\text{N}} \) and \( \sum a_k x^k = (dE/dt)/(dE/dt)_{\text{N}} \).

To examine the accuracy of the post-Newtonian expansion, we introduce \( N^{(n)} \) which is defined by

\[
N^{(n)} = \frac{5}{32\pi} \frac{M}{\mu} \int_{t_i}^{t_f} dt \left( \frac{\sum_{k=0}^{n} b_k x^k}{\sum_{k=0}^{n} a_k x^k} \right).
\]

(51)

To be specific, we assume the detectable frequency band to be from 10Hz to 1000Hz and the quasi-periodic inspiral stage ends at \( r=6M \). So, \( r_i \) is the radius at which \( f(r_i)=10\text{Hz} \), and \( r_f \) is the one at which \( f(r_f)=1000\text{Hz} \) if \( r_f > 6M \) and \( r_f = 6M \) otherwise.

For \((1.4 \, M_\odot, 1.4 \, M_\odot)\) neutron star binary, we have \( r_i = 175M \) and \( r_f = 8M \). Then we obtain \( N^{(8)} = 16220.13, \, N^{(6)} = 16211.03, \, N^{(7)} = 16211.93 \) and \( N^{(8)} = 16211.91 \). For \((1.4 \, M_\odot, 10 \, M_\odot)\) binary, \( r_i = 68M \) and \( r_f = 6M \), and we obtain \( N^{(8)} = 3484.83, \, N^{(6)} = 3466.56, \, N^{(7)} = 3468.60 \) and \( N^{(8)} = 3468.27 \). For \((10 \, M_\odot, 10 \, M_\odot)\) binary, \( r_i = 47M \) and \( r_f = 6M \), and we obtain \( N^{(5)} = 581.39, \, N^{(6)} = 574.49, \, N^{(7)} = 575.29 \) and \( N^{(8)} = 575.15 \).

From these results we see that for \((1.4 \, M_\odot, 1.4 \, M_\odot)\) binaries, the post-Newtonian
expansion to $O(v^7)$ will be necessary to accurately predict the total cycle. However for $(1.4 \ M_\odot, \ 10 \ M_\odot)$ and $(10 \ M_\odot, \ 10 \ M_\odot)$ binaries, the convergence seems to be slow. The error in the total cycle seems marginal, $\Delta N < 0.5$, even including up to $O(v^8)$. Further, if we regard the value of the coefficient of $v^9$ in Table I(b) to be approximately correct, about 700, the contribution from $O(v^9)$ may be comparable to $O(v^8)$. For these binaries, more detailed analyses are needed.

§ 6. Conclusion

Basing upon formulae developed in Paper I, we have analytically performed the post-Newtonian expansion of the gravitational waves from a test particle in circular orbit around a Schwarzschild black hole.

We have calculated both the gravitational wave forms and luminosity to order $v^8$ beyond Newtonian. In particular, we have given the exact analytical values of the coefficients of $\ln v$ terms at $O(v^6)$ and $O(v^8)$ in the luminosity. The existence of such terms was found previously in a numerical analysis by Tagoshi and Nakamura. We have found that their numerical results are in very good agreement with our analytical ones.

We have also estimated the accuracy of the post-Newtonian expansion to predict the total cycle of coalescing binaries. For $(1.4 \ M_\odot, \ 1.4 \ M_\odot)$ binaries, we have found that the post-Newtonian expansion up to $O(v^7)$ will be sufficient to construct accurate theoretical templates. However for $(1.4 \ M_\odot, \ 10 \ M_\odot)$ and $(10 \ M_\odot, \ 10 \ M_\odot)$ binaries, accuracy to order higher than $v^8$ seems to be necessary.

Although valid only in the test particle limit, our analytical results should be reproduced in the conventional post-Newtonian calculations which are not restricted to the test particle case. Hence our results given a useful guideline for the future researches in the gravitational wave physics of coalescing compact binaries.

Acknowledgements

We thank T. Nakamura, M. Shibata and T. Tanaka for useful discussions. Numerical calculations in this paper are based on the work of HT with T. Nakamura. HT thanks Professor H. Sato for continuous encouragement. This work is supported by the Grant-in-Aid for Scientific Research on Priority Area of the Ministry of Education (04234104).

Appendix A

— The Expansion of $X_i^{\text{in}}$ —

In this appendix, we give the expansion forms of $X_i^{(n)}$ which are required to calculate $X_i^{\text{in}}$ up to order $v^8$.

\[
X_i^{(0)} = \frac{23}{15} - \frac{2}{210} + \frac{2}{7560} - \frac{2}{498960} + \frac{1}{51891840},
\]

\[
X_i^{(1)} = \frac{-13}{630} + \frac{2}{810} - \frac{53}{178200} + \frac{227}{567567000},
\]
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\[ X_2^{(2)} = \frac{1}{110250} \left( \frac{26743}{3150} - \frac{107\ln z}{3150} \right) + \frac{z^5}{926100} \left( -\frac{140953}{926100} + \frac{107\ln z}{44100} \right), \]

\[ X_2^{(3)} = \frac{319z^2}{6300}, \]

\[ X_3^{(0)} = \frac{z^4}{105} - \frac{z^5}{1890} + \frac{z^8}{83160} - \frac{z^{10}}{6486480}, \]

\[ X_3^{(1)} = -\frac{z^3}{126} + \frac{z^5}{630} + \frac{221z^7}{2494800}, \]

\[ X_3^{(2)} = \frac{76369z^2}{1852200} - \frac{2327663z^5}{1100206800} + \frac{13z^3\ln z}{4410} + \frac{13z^3\ln z}{79380}, \]

\[ X_4^{(0)} = \frac{z^5}{945} - \frac{z^7}{20790} + \frac{z^9}{1081080} - \frac{z^{11}}{97297200}, \]

\[ X_4^{(1)} = -\frac{z^4}{630} - \frac{z^6}{9900} + \frac{79z^8}{13899600}, \]

\[ X_4^{(2)} = \frac{z^3}{1764} + \frac{z^5(942578261 - 43548120 \ln z)}{181534122000}, \]

\[ X_5^{(0)} = \frac{z^6}{10395} - \frac{z^8}{270270} + \frac{z^{10}}{16216200}, \]

\[ X_5^{(1)} = -\frac{z^5}{4950} - \frac{41z^7}{8108100}, \]

\[ X_5^{(2)} = \frac{z^4}{7425}, \]

\[ X_6^{(0)} = \frac{z^7}{135135} - \frac{z^9}{4054050} + \frac{z^{11}}{275675400}, \]

\[ X_6^{(1)} = -\frac{8z^6}{405405} - \frac{z^8}{5675670}, \]

\[ X_6^{(2)} = \frac{4z^5}{212355}, \]

\[ X_7^{(0)} = \frac{z^8}{2027025} - \frac{z^{10}}{68918850}, \]

\[ X_7^{(1)} = -\frac{z^7}{630630}, \]

\[ X_7^{(2)} = \frac{z^6}{34459425} - \frac{z^{11}}{1309458150}, \]

\[ X_8^{(1)} = -\frac{z^8}{9189180}, \]

\[ X_9^{(0)} = \frac{z^{10}}{654729075}, \]
Appendix B
—— Wave Forms to $O(v^8)$ ——

In this appendix, we give the gravitational wave forms for all the relevant $l$ and $m$ that contribute up to $O(v^8)$.

\[
\xi_{24}^+ = -(3 + \cos(2\theta)) \left( \cos(2\phi) - \frac{107v^8\cos(2\phi)}{42} \right) \\
+ v^7 \left[ 2\pi\cos(2\phi) + \left( -\frac{17}{3} + 4\ln2 \right)\sin(2\phi) \right] - \frac{2173v^4\cos(2\phi)}{1512} \\
+ v^6 \left( \frac{-107}{21}\pi\cos(2\phi) + \left( \frac{1819}{126} - \frac{214\ln2}{21} \right)\sin(2\phi) \right) \\
+ v^5 \left( \cos(2\phi) \left( \frac{49928027}{1940400} - \frac{856\gamma}{105} + \frac{2\pi^2}{3} + \frac{668\ln2}{105} - \frac{8(\ln2)^2}{105} - \frac{856\ln\nu}{105} \right) \\
+ \left( \frac{-254\pi}{35} + 8\pi\ln2 \right)\sin(2\phi) \right) \\
+ v^4 \left( -\frac{2173\pi\cos(2\phi)}{756} + \left( \frac{36941}{4536} - \frac{2173\ln2}{378} \right)\sin(2\phi) \right) \\
+ v^3 \left( \cos(2\phi) \left( -\frac{326531600453}{12713500800} + \frac{45796\gamma}{2205} - \frac{107\pi^2}{63} \\
- \frac{35738\ln2}{2205} + \frac{428(\ln2)^2}{2205} + \frac{45796\ln\nu}{105} \right) + \left( \frac{13589\pi}{735} - \frac{428\pi\ln2}{21} \right)\sin(2\phi) \right) \right) \\
+ v^2 \left( -\frac{4\sin(\theta)}{3} \left( v\sin(\phi) - \frac{17v^8\sin(\phi)}{28} + v^6 \left( \frac{10\cos(\phi)}{3} + \pi\sin(\phi) \right) \\
- \frac{43v^8\sin(\phi)}{126} + v^6 \left( -\frac{85\cos(\phi)}{42} - \frac{17\pi\sin(\phi)}{28} \right) \right) \\
+ v^5 \left( \frac{81\pi\cos(\phi)}{35} + \left( \frac{14641367}{2910600} - \frac{214\gamma}{105} + \frac{\pi^2}{6} - \frac{214\ln2}{105} - \frac{214\ln\nu}{105} \right)\sin(\phi) \right) \\
+ v^4 \left( -\frac{215\cos(\phi)}{189} - \frac{43\pi\sin(\phi)}{126} \right) \right) \\
+ v^3 \left( 9 \left( 5\sin(\theta) + \sin(3\theta) \right) \left( v\sin(3\phi) - 4v^8\sin(3\phi) \right) \\
+ v^6 \left( \cos(3\phi) \left( \frac{127}{10} - 6\ln3 \right) + 3\pi\sin(3\phi) \right) + \frac{123v^6\sin(3\phi)}{110} \\
+ v^5 \left( \cos(3\phi) \left( \frac{-254}{5} + 24\ln3 \right) - 12\pi\sin(3\phi) \right) \right) \\
+ v^4 \left( \cos(3\phi) \left( \frac{2277\pi}{70} - 18\pi\ln3 \right) + \left( \frac{-185741}{70070} - \frac{78\gamma}{7} \right) \right)
\]
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\[ + \frac{3\pi^2}{2} - \frac{78\ln 2}{7} + \frac{227\ln 3}{35} - 18(\ln 3)^2 - \frac{78\ln 4}{7} \sin(3\phi) \]

\[ + v^5 \left( \cos(3\phi) \left( \frac{15621}{1100} - \frac{369\ln 3}{55} \right) + \frac{369\pi \sin(3\phi)}{110} \right), \]

\[ \zeta_{3,2}^+ = -\frac{4}{3} \cos(2\theta) (v^5 \cos(2\phi) - \frac{193v^4 \cos(2\phi)}{90}) \]

\[ + v^5 \left( 2\pi \cos(2\phi) + \left( -\frac{26}{3} + 4\ln 2 \right) \sin(2\phi) \right) - \frac{1451v^5 \cos(2\phi)}{3960} \]

\[ + v^7 \left( \frac{-193\pi \cos(2\phi)}{45} + \left( \frac{2509}{135} - \frac{386\ln 2}{45} \right) \sin(2\phi) \right) \]

\[ + v^8 \left( \cos(2\phi) \left( -\frac{34098173}{75675600} + \frac{104\gamma}{21} + \frac{2\pi^2}{3} + \frac{520\ln 2}{21} - \frac{8(\ln 2)^2}{21} \right) \sin(2\phi) \right) \]

\[ + \left( \frac{104\ln 4}{21} \right) + \left( \frac{-104\pi}{7} + 8\pi \ln 2 \right) \sin(2\phi) \right), \]

\[ \zeta_{3,1}^+ = \left( -\sin(\theta) + 3\sin(3\theta) \right) \left( v \sin(\phi) - \frac{8v^3 \sin(\phi)}{3} \right) \]

\[ + v^4 \left( \frac{127\cos(\phi)}{30} - \pi \sin(\phi) \right) + \frac{607v^5 \sin(\phi)}{198} + v^6 \left( -\frac{508\cos(\phi)}{45} - \frac{8\pi \sin(\phi)}{3} \right) \]

\[ + v^7 \left( \frac{253\pi \cos(\phi)}{70} + \left( -\frac{1656587}{1891890} + \frac{26\gamma}{21} + \frac{\pi^2}{6} + \frac{26\ln 2}{21} - \frac{26\ln 4}{21} \right) \sin(\phi) \right) \]

\[ + v^8 \left( \frac{77089\cos(\phi)}{5940} + \frac{607\pi \sin(\phi)}{198} \right) \right), \]

\[ \zeta_{4,4}^+ = \frac{1}{3} \left( 5 - 4\cos(2\theta) - \cos(4\theta) \right) \left( v^5 \cos(4\phi) - \frac{593v^4 \cos(4\phi)}{110} \right) \]

\[ + v^6 \left( 4\pi \cos(4\phi) + \left( -\frac{296}{15} + 8\ln 4 \right) \sin(4\phi) \right) + \frac{1068671v^6 \cos(4\phi)}{200200} \]

\[ + v^7 \left( \frac{-1186\pi \cos(4\phi)}{55} + \left( \frac{87764}{825} - \frac{2372\ln 4}{55} \right) \sin(4\phi) \right) \]

\[ + v^8 \cos(4\phi) \left( -\frac{36840955871}{499458960} - \frac{50272\gamma}{3465} + \frac{8\pi^2}{3} + \frac{50272\ln 2}{3465} + \frac{496736\ln 4}{3465} \right) \]

\[ - 32(\ln 4)^2 - \frac{50272\ln 4}{3465} \right) + \left( \frac{-248368\pi}{3465} + 32\pi \ln 4 \right) \sin(4\phi) \right), \]

\[ \zeta_{1,3}^+ = \frac{-27}{80} \left( \sin(\theta) - 3\sin(3\theta) \right) \left( v^5 \sin(3\phi) - \frac{39v^5 \sin(3\phi)}{11} \right) \]

\[ + v^6 \cos(3\phi) \left( \frac{149}{10} - 6\ln 3 \right) + 3\pi \sin(3\phi) \right) + \frac{7206v^6 \sin(3\phi)}{5005} \]

\[ + v^7 \cos(3\phi) \left( -\frac{5811}{110} + \frac{234\ln 3}{11} - \frac{117\pi \sin(3\phi)}{11} \right) \right), \]
\[ \xi_{42} = -\frac{1}{84} \left( 5 + 4\cos(2\theta) + 7\cos(4\theta) \right) \left( \nu^3\cos(2\phi) - \frac{437\nu^4\cos(2\phi)}{110} \right) \\
+ \nu^5 \left( 2\pi\cos(2\phi) + \left( \frac{-148}{15} + 4\ln2 \right) \sin(2\phi) \right) + \frac{1038039\nu^6\cos(2\phi)}{200200} \\
+ \nu^7 \left( -\frac{437\pi\cos(2\phi)}{55} + \left( \frac{32338}{825} + \frac{874\ln2}{55} \right) \sin(2\phi) \right) \\
+ \nu^8 \left( \cos(2\phi) \left( -\frac{54548715967}{2497294800} - \frac{12568\gamma}{3465} + \frac{2\pi^2}{3} + \frac{111616\ln2}{3465} - 8(\ln2)^2 \\
- \frac{12568\ln\nu}{3465} \right) + \left( \frac{-62092\pi}{3465} + 8\pi\ln2 \right) \sin(2\phi) \right) \), \\
\xi_{41} = \frac{1}{560} \left( 3\sin(\theta) + 7\sin(3\theta) \right) \left( \nu^3\sin(\phi) - \frac{101\nu^4\sin(\phi)}{33} + \nu^5 \left( \frac{149\cos(\phi)}{30} + \pi\sin(\phi) \right) \right) \\
+ \frac{42982\nu^6\sin(\phi)}{15015} + \nu^7 \left( \frac{-15049\cos(\phi)}{990} - \frac{101\pi\sin(\phi)}{33} \right) \), \\
\xi_{55} = \frac{625}{3072} \left( -14\sin(\theta) + 3\sin(3\theta) + \sin(5\theta) \right) \left( \nu^3\sin(5\phi) - \frac{263\nu^4\sin(5\phi)}{39} \right) \\
+ \nu^5 \left( \cos(5\phi) \left( \frac{569}{21} - 10\ln5 \right) + 5\pi\sin(5\phi) \right) + \frac{9185\nu^7\sin(5\phi)}{819} \\
+ \nu^8 \left( \cos(5\phi) \left( -\frac{149647}{819} + \frac{2630\ln5}{39} \right) - \frac{1315\pi\sin(5\phi)}{39} \right) \), \\
\xi_{54} = \frac{32}{45} \left( \cos(2\theta) - \cos(4\theta) \right) \left( \nu^3\cos(4\phi) - \frac{4451\nu^4\cos(4\phi)}{910} \right) \\
+ \nu^5 \left( 4\pi\cos(4\phi) + \left( \frac{-326}{15} + 8\ln4 \right) \sin(4\phi) \right) + \frac{10715\nu^6\cos(4\phi)}{2184} \), \\
\xi_{53} = \frac{5120}{27} \left( 14\sin(\theta) + 13\sin(3\theta) + 15\sin(5\theta) \right) \left( \nu^3\sin(3\phi) - \frac{69\nu^4\sin(3\phi)}{13} \right) \\
+ \nu^5 \left( \cos(3\phi) \left( \frac{569}{35} - 6\ln3 \right) + 3\pi\sin(3\phi) \right) + \frac{12463\nu^7\sin(3\phi)}{1365} \\
+ \nu^8 \left( \cos(3\phi) \left( -\frac{39261}{455} + \frac{414\ln3}{13} \right) - \frac{207\pi\sin(3\phi)}{13} \right) \), \\
\xi_{52} = \frac{-2}{135} \left( \cos(2\theta) + 3\cos(4\theta) \right) \left( \nu^3\cos(2\phi) - \frac{3911\nu^4\cos(2\phi)}{910} \right) \\
+ \nu^5 \left( 2\pi\cos(2\phi) + \left( \frac{-163}{15} + 4\ln2 \right) \sin(2\phi) \right) + \frac{63439\nu^6\cos(2\phi)}{10920} \), \\
\xi_{51} = \frac{-2\sin(\theta) - 3\sin(3\theta) + 15\sin(5\theta)}{23040} \left( \nu^3\sin(\phi) - \frac{179\nu^4\sin(\phi)}{39} \right) \)
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\[ + v^8 \left( \frac{569 \cos(\phi) + \pi \sin(\phi)}{105} + \frac{5023 v^8 \sin(\phi)}{585} \right) + v^7 \left( -\frac{101851 \cos(\phi) - 179 \pi \sin(\phi)}{4095} \right), \]

\[ \xi_{6,0} = -\frac{81}{40} (3 + \cos(2\theta)) \sin^2(\theta) \left( v^8 \cos(6\phi) - \frac{113 v^6 \cos(6\phi)}{14} \right) + v^7 \left( 6 \pi \cos(6\phi) + \left(-\frac{487}{14} + 12 \ln 6 \right) \sin(6\phi) \right) + \frac{1372317 v^6 \cos(6\phi)}{73304}, \]

\[ \xi_{6,5} = -\frac{3125}{2016} (1 - 5 \cos(\theta)^2) \sin^3(\theta) \left( v^8 \sin(5\phi) - \frac{149 v^6 \sin(5\phi)}{24} \right) + v^7 \left( \cos(5\phi) \left( \frac{1219}{42} - 10 \ln 5 \right) + 5 \pi \sin(5\phi) \right) + v^6 \left( \cos(5\phi) \left( \frac{1219}{42} - 10 \ln 5 \right) + 5 \pi \sin(5\phi) \right), \]

\[ \xi_{6,4} = -\frac{16 \cos^2(\frac{\theta}{2}) \sin^3(\frac{\theta}{2})}{495} (67 + 92 \cos(2\theta) + 33 \cos(4\theta)) \]

\[ \times \left( v^8 \cos(4\phi) - \frac{93 v^6 \cos(4\phi)}{14} \right) + v^7 \left( 4 \pi \cos(4\phi) + \left(-\frac{487}{21} + 8 \ln 4 \right) \sin(4\phi) \right) + \frac{3261767 v^6 \cos(4\phi)}{219912}, \]

\[ \xi_{6,3} = -\frac{243}{98560} (21 + 52 \cos(2\theta) + 55 \cos(4\theta)) \sin(\theta) \]

\[ \times \left( v^8 \sin(3\phi) - \frac{133 v^6 \sin(3\phi)}{24} + v^6 \left( \cos(3\phi) \left( \frac{1219}{70} - 6 \ln 3 \right) + 3 \pi \sin(3\phi) \right) \right), \]

\[ \xi_{6,2} = -\frac{1}{190080} (210 + 289 \cos(2\theta) + 30 \cos(4\theta) + 495 \cos(6\theta)) \]

\[ \times \left( v^8 \cos(2\phi) - \frac{81 v^6 \cos(2\phi)}{14} + v^7 \left( 2 \pi \cos(2\phi) + \left(-\frac{487}{42} + 4 \ln 2 \right) \sin(2\phi) \right) \right) + \frac{14482483 v^6 \cos(2\phi)}{1099560}, \]

\[ \xi_{6,1} = -\frac{35}{266112} (35 + 60 \cos(2\theta) + 33 \cos(4\theta)) \sin(\theta) \]

\[ \times \left( v^8 \sin(\phi) - \frac{125 v^6 \sin(\phi)}{24} + v^6 \left( \frac{1219 \cos(\phi)}{210} + \pi \sin(\phi) \right) \right), \]

\[ \delta_{7,7} = -\frac{117649}{46080} (3 + \cos(2\theta)) \sin^3(\theta) \left( v^8 \sin(7\phi) - \frac{319 v^6 \sin(7\phi)}{34} \right) + v^7 \left( \cos(7\phi) \left( \frac{7699}{180} - 14 \ln 7 \right) + 7 \pi \sin(7\phi) \right) + v^6 \left( \cos(7\phi) \left( \frac{7699}{180} - 14 \ln 7 \right) + 7 \pi \sin(7\phi) \right), \]
\[ \xi_{ts} = -\frac{243(2 + 3\cos(2\theta))\sin^4(\theta)}{140} \left( v^6\cos(6\psi) - \frac{1787v^8\cos(6\phi)}{238} \right), \]

\[ \xi_{ts} = -\frac{15625(233 + 316\cos(2\theta) + 91\cos(4\theta))\sin^3(\theta)}{3354624} \left( v^5\sin(5\phi) - \frac{271v^7\sin(5\phi)}{34} + v^9\left( \cos(5\phi) \left( \frac{7699}{252} - 10\ln 5 \right) + 5\pi\sin(5\phi) \right) \right), \]

\[ \xi_{ts} = \frac{32(9 + 18\cos(2\theta) + 13\cos(4\theta))\sin^2(\theta)}{1365} \times \left( v^6\cos(4\phi) - \frac{14543v^8\cos(4\phi)}{2142} \right), \]

\[ \xi_{ts} = \frac{729\sin(\theta)}{8200192} \left( 1134 + 1863\cos(2\theta) + 1122\cos(4\theta) + 1001\cos(6\theta) \right) \times \left( v^6\sin(3\phi) - \frac{239v^8\sin(3\phi)}{34} + v^9\left( \cos(3\phi) \left( \frac{7699}{420} - 6\ln 3 \right) + 3\pi\sin(3\phi) \right) \right), \]

\[ \xi_{ts} = -\frac{1}{192192} \left( 25\cos(2\theta) + 88\cos(4\theta) + 143\cos(6\theta) \right) \times \left( v^6\cos(2\phi) - \frac{13619v^8\cos(2\phi)}{2142} \right), \]

\[ \xi_{ts} = \frac{(350 + 775\cos(2\theta) + 946\cos(4\theta) + 1001\cos(6\theta))\sin(\theta)}{147603456} \times \left( v^6\sin(\phi) - \frac{223v^8\sin(\phi)}{34} + v^9\left( \frac{7699\cos(\phi)}{1260} + \pi\sin(\phi) \right) \right), \]

\[ \xi_{ts} = \frac{1024\sin^4(\theta)}{315} \left( v^6\cos(8\phi) - \frac{3653v^8\cos(8\phi)}{342} \right)(3 + \cos(2\theta)), \]

\[ \xi_{ts} = \frac{-823543v^7(5 + 7\cos(2\theta))\sin(7\phi)\sin^5(\theta)}{829440}, \]

\[ \xi_{ts} = \frac{-729\cos^2(\theta)\sin^4(\theta)}{140} \left( v^6\cos(6\phi) - \frac{353v^8\cos(6\phi)}{38} \right), \]

\[ \xi_{ts} = \frac{-78125}{4644864} \left( 33 + 60\cos(2\theta) + 35\cos(4\theta) \right)\sin^3(\theta)v^7\sin(5\phi), \]

\[ \xi_{ts} = \frac{(186 + 309\cos(2\theta) + 182\cos(4\theta) + 91\cos(6\theta))\sin^2(\theta)}{4095} \times \left( v^6\cos(4\phi) - \frac{2837v^8\cos(4\phi)}{342} \right), \]

\[ \xi_{ts} = \frac{-243}{7454720}v^7\sin(3\phi)(1 + 3\cos(2\theta))(15 - 52\cos(2\theta) - 91\cos(4\theta))\sin(\theta), \]

\[ \xi_{ts} = \frac{-23063040}{342} \times \left( v^6\cos(2\phi) - \frac{2633v^8\cos(2\phi)}{342} \right), \]
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\[ \xi_{\pm,1} = \frac{v^7}{379551744} (210 + 385\cos(2\theta) + 286\cos(4\theta) + 143\cos(6\theta))\sin(\phi)\sin(\theta), \]

\[ \xi_{\pm,9} = \frac{-4782969 v^7}{1146880} (3 + \cos(2\theta))\sin(9\phi)\sin^7(\theta), \]

\[ \xi_{\pm,8} = \frac{32768 v^8}{14175} \cos(8\phi)(3 + 4\cos(2\theta))\sin^6(\theta), \]

\[ \xi_{\pm,7} = \frac{5764801 v^7}{902430720} \sin(7\phi)(515 + 676\cos(2\theta) + 153\cos(4\theta))\sin^5(\theta), \]

\[ \xi_{\pm,6} = \frac{-729 v^8\cos(6\phi)(39 + 67\cos(2\theta) + 34\cos(4\theta))\sin^4(\theta),}{23800} \]

\[ \xi_{\pm,5} = \frac{-390625 v^7}{1263403008} \sin(5\phi)(462 + 759\cos(2\theta) + 418\cos(4\theta) + 153\cos(6\theta))\sin^3(\theta), \]

\[ \xi_{\pm,4} = \frac{16 v^8}{34425} \cos(4\phi)(33 + 66\cos(2\theta) + 59\cos(4\theta) + 34\cos(6\theta))\sin^2(\theta), \]

\[ \xi_{\pm,3} = \frac{-243 v^7}{579338240} \sin(3\phi)(693 + 1232\cos(2\theta) + 884\cos(4\theta) \]
\[ + 624\cos(6\theta) + 663\cos(8\theta))\sin(\theta), \]

\[ \xi_{\pm,2} = \frac{-v^8}{57283200} \cos(2\phi)(49\cos(2\theta) + 182\cos(4\theta) + 351\cos(6\theta) + 442\cos(8\theta)), \]

\[ \xi_{\pm,1} = \frac{v^7}{93852794880} \sin(\phi)(735 + 1568\cos(2\theta) + 1820\cos(4\theta) \]
\[ + 2080\cos(6\theta) + 1989\cos(8\theta))\sin(\theta), \]

\[ \xi_{\pm,10} = \frac{-390625 v^8}{72576} \cos(10\phi)(3 + \cos(2\theta))\sin^8(\theta), \]

\[ \xi_{\pm,9} = \frac{4096 v^8}{269325} \cos(8\phi)(349 + 452\cos(2\theta) + 95\cos(4\theta))\sin^6(\theta), \]

\[ \xi_{\pm,8} = \frac{-4374 v^8}{231017675} \cos(6\phi)(19318 + 31299\cos(2\theta) \]
\[ + 16218\cos(4\theta) + 4845\cos(6\theta))\sin^4(\theta), \]

\[ \xi_{\pm,7} = \frac{-v^8}{4578525} \cos(4\phi)(10791 + 19272\cos(2\theta) + 13868\cos(4\theta) \]
\[ + 8568\cos(6\theta) + 4845\cos(8\theta))\sin^2(\theta), \]

\[ \xi_{\pm,6} = \frac{-v^8}{139312742400} \cos(2\phi)(16170 + 28322\cos(2\theta) + 17576\cos(4\theta) \]
\[ + 4693\cos(6\theta) + 1326\cos(8\theta) + 62985\cos(10\theta)), \]

\[ \xi_{\pm,5} = 4\cos(\theta)\left(\sin(2\phi) - \frac{107 v^2 \sin(2\phi)}{42} + v^4\cos(2\phi)^{\left(\frac{17}{3} - 4\log(2)\right)} \right). \]
\[ \begin{align*}
+ 2\pi \sin(2\phi) &- \frac{2173\nu \sin(2\phi)}{1512} + \nu^5 \left( \cos(2\phi) \left( -\frac{1819}{126} + \frac{214 \log(2)}{21} \right) \right) \\
- \frac{107\pi \sin(2\phi)}{21} &+ \nu^8 \left( \cos(2\phi) \left( \frac{254\pi}{35} - 8\pi \log(2) \right) \right) \\
+ \left( \frac{49928027}{1940400} - \frac{856\gamma}{105} + \frac{2\pi^2}{3} + \frac{668\log(2)}{105} - \frac{8\log(2)^2}{105} - \frac{856\log(\nu)}{105} \right) \sin(2\phi) \\
+ \nu^7 \left( \cos(2\phi) \left( -\frac{36941}{4536} + 2173\log(2) \right) - 2173\pi \sin(2\phi) \right) \\
+ \nu^8 \left( \cos(2\phi) \left( \frac{-1358\pi}{735} + \frac{428\pi \log(2)}{21} \right) \right) &+ \left( -\frac{326531600453}{12713500800} \right) \\
+ \left( \frac{45796\gamma}{2205} - \frac{107\pi^2}{63} - \frac{35738\log(2)}{2205} + \frac{428\log(2)^2}{21} - \frac{45796\log(\nu)}{2205} \right) \sin(2\phi) \bigg) \bigg), \\
\xi_2 &= \frac{2\sin(2\theta)}{3} \left( \nu \cos(\phi) - \frac{17\nu^3 \cos(\phi)}{28} + \nu^4 \left( \pi \cos(\phi) - \frac{10\sin(\phi)}{3} \right) \right) \\
- \frac{43\nu^5 \cos(\phi)}{126} &+ \nu^8 \left( \frac{-17\pi \cos(\phi)}{28} + \frac{85\sin(\phi)}{42} \right) \\
+ \nu^7 \left( \cos(\phi) \left( \frac{14641367}{2910600} - \frac{214\gamma}{105} + \frac{\pi^2}{6} - \frac{214\ln(2)}{105} - \frac{214\ln(\nu)}{105} \right) - \frac{81\pi \sin(\phi)}{35} \right) \\
+ \nu^8 \left( \frac{-43\pi \cos(\phi)}{126} + \frac{215\sin(\phi)}{189} \right) \bigg), \\
\xi_3 &= \frac{9\sin(2\theta)}{4} \left( \nu \cos(3\phi) - 4\nu^3 \cos(3\phi) \right) \\
+ \nu^4 \left( 3\pi \cos(3\phi) + \left( -\frac{127}{10} + \frac{6\ln(3)}{2} \right) \sin(3\phi) \right) &+ \frac{123\nu^5 \cos(3\phi)}{110} \\
+ \nu^6 \left( -12\pi \cos(3\phi) + \left( \frac{254}{5} - 24\ln(3) \right) \sin(3\phi) \right) \\
+ \nu^7 \left( \cos(3\phi) \left( -\frac{185741}{70070} - \frac{78\gamma}{7} + \frac{3\pi^2}{2} - \frac{78\ln(2)}{7} + \frac{2277\ln(3)}{35} \right) \right) \\
\left( -18\ln(3)^2 - \frac{78\ln(\nu)}{7} \right) &+ \left( \frac{-2277\pi}{70} + 18\pi \ln(3) \right) \sin(3\phi) \bigg) \\
+ \nu^8 \left( \frac{369\pi \cos(3\phi)}{110} + \frac{15621}{1100} + \frac{369\ln(3)}{55} \right) \sin(3\phi) \bigg), \\
\xi_4 &= \frac{5\cos(\theta) + 3\cos(3\theta)}{6} \left( \nu^2 \sin(2\phi) - \frac{193\nu^4 \sin(2\phi)}{90} \right) \\
+ \nu^6 \left( \cos(2\phi) \left( \frac{26}{3} - \frac{4\ln(2)}{2} \right) + 2\pi \sin(2\phi) \right) - \frac{1451\nu^8 \sin(2\phi)}{3960} \\
+ \nu^7 \left( \cos(2\phi) \left( -\frac{2509}{135} + \frac{386\ln(2)}{45} \right) - \frac{193\pi \sin(2\phi)}{45} \right). \end{align*} \]
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\[ + v^8 \left( \cos(2\phi) \left( \frac{104\pi}{7} - 8\pi\ln 2 \right) + \left( -\frac{340998173}{75675600} \cdot \frac{104\pi}{21} \right) \sin(2\phi) \right) \],

\[ + \frac{2\pi^2}{3} + \frac{520\ln 2}{21} - 8(\ln 2)^2 - \frac{104\pi v}{21} \sin(2\phi) \right) \],

\[ \xi_{\phi,i} = \frac{\sin(2\theta)}{12} \left( v \cos(\phi) - \frac{8v^3 \cos(\phi)}{3} + v^4 \left( \pi \cos(\phi) - \frac{127\sin(\phi)}{30} \right) \right) \]

\[ + \frac{607\pi^5 \cos(\phi)}{198} + v^6 \left( -\frac{8\pi \cos(\phi)}{3} + \frac{508\sin(\phi)}{45} \right) \]

\[ + v^7 \left( \cos(\phi) \left( -\frac{1656587}{1891890} \cdot \frac{26\pi}{21} + \frac{\pi^2}{6} \cdot \frac{26\ln 2}{21} - \frac{26\ln v}{21} - \frac{253\pi \sin(\phi)}{70} \right) \right) \]

\[ + v^8 \left( \frac{607\pi \cos(\phi)}{198} - \frac{77089\sin(\phi)}{5940} \right) \right),

\[ \xi_{\phi,4} = \frac{-4(\cos(\theta) - \cos(3\theta))}{3} \left( v^3 \sin(4\phi) - \frac{593v^4 \sin(4\phi)}{110} \right) \]

\[ + v^5 \left( \cos(4\phi) \left( \frac{296}{15} - 8\ln 4 \right) + 4\pi \sin(4\phi) \right) + \frac{1068671v^6 \sin(4\phi)}{200200} \]

\[ + v^7 \left( \cos(4\phi) \left( -\frac{87764}{825} + \frac{2372\ln 4}{55} \right) - \frac{1186\pi \sin(4\phi)}{55} \right) \]

\[ + v^8 \left( \cos(4\phi) \left( \frac{243836\pi}{3465} - 32\pi \ln 4 \right) + \left( -\frac{36840955871}{499458960} - \frac{50272\pi}{3465} \right) \right) \]

\[ + \frac{8\pi^2}{3} - \frac{50272\ln 2}{3465} + \frac{496736\ln 4}{3465} - \frac{32(\ln 2)^2 - \frac{50272\ln v}{3465}}{3465} \sin(4\phi) \right) \],

\[ \xi_{\phi,4} = \frac{27(2\sin(2\theta) + \sin(4\theta))}{80} \left( v^3 \cos(3\phi) - \frac{39v^4 \cos(3\phi)}{11} \right) \]

\[ + v^5 \left( 3\pi \cos(3\phi) + \left( -\frac{149}{10} + 6\ln 3 \right) \sin(3\phi) \right) + \frac{7206v^7 \cos(3\phi)}{5005} \]

\[ + v^6 \left( -\frac{117\pi \cos(3\phi)}{11} + \left( \frac{5811}{110} - \frac{234\ln 3}{11} \right) \sin(3\phi) \right) \right),

\[ \xi_{\phi,4} = \frac{(\cos(\theta) + 7\cos(3\theta))}{42} \left( v^3 \sin(2\phi) - \frac{437v^4 \sin(2\phi)}{110} \right) \]

\[ + v^5 \left( \cos(2\phi) \left( \frac{148}{15} - 4\ln 2 \right) + 2\pi \sin(2\phi) \right) + \frac{1038039v^6 \sin(2\phi)}{200200} \]

\[ + v^7 \left( \cos(2\phi) \left( -\frac{32338}{825} + \frac{874\ln 2}{55} \right) - \frac{437\pi \sin(2\phi)}{55} \right) \]

\[ + v^8 \left( \cos(2\phi) \left( \frac{62092\pi}{3465} - 8\ln 2 \right) + \left( -\frac{54548715967}{2497294800} - \frac{12568\pi}{3465} \right) \right) \]

\[ + \frac{2\pi^2}{3} + \frac{111616\ln 2}{3465} - 8(\ln 2)^2 - \frac{12568\ln v}{3465} \sin(2\phi) \right) \right) \],
\[ \xi_{s,1} = \left( -2\sin(2\theta) + 7\sin(4\theta) \right) \left( v^s\cos(\phi) - \frac{101v^s\cos(\phi)}{33} + \frac{42982v^s\cos(\phi)}{15015} \right) \]
\[ + v^\delta \left( \pi\cos(\phi) - \frac{149\sin(\phi)}{30} \right) + v^\delta \left( -\frac{101\pi\cos(\phi)}{33} + \frac{15049\sin(\phi)}{990} \right), \]
\[ \xi_{s,2} = \frac{625}{768} \left( -2\sin(2\theta) + 7\sin(4\theta) \right) \left( v^s\cos(5\phi) - \frac{263v^s\cos(5\phi)}{39} \right) \]
\[ + v^\delta \left( 5\pi\cos(5\phi) + \left( -\frac{569}{21} + 10\ln5 \right)\sin(5\phi) \right) + \frac{9185v^s\cos(5\phi)}{819} \]
\[ + v^\delta \left( -\frac{1315\pi\cos(5\phi)}{39} + \left( \frac{1496}{819} - \frac{2630\ln5}{39} \right)\sin(5\phi) \right), \]
\[ \xi_{s,3} = \frac{-2(14\cos(\theta) - 9\cos(3\theta) - 5\cos(5\theta))}{45} \left( v^s\sin(4\phi) - \frac{4451v^s\sin(4\phi)}{910} \right) \]
\[ + v^\delta \left( \cos(4\phi) \left( \frac{326}{15} - 8\ln4 \right) + 4\pi\sin(4\phi) \right) + \frac{10715v^s\sin(4\phi)}{2184} \]
\[ \xi_{s,4} = \frac{-27(2\sin(2\theta) - 9\sin(4\theta))}{1280} \left( v^s\cos(3\phi) - \frac{69v^s\cos(3\phi)}{13} \right) \]
\[ + v^\delta \left( 3\pi\cos(3\phi) + \left( -\frac{569}{35} + 6\ln3 \right)\sin(3\phi) \right) + \frac{12463v^s\cos(3\phi)}{1365} \]
\[ + v^\delta \left( -\frac{207\pi\cos(3\phi)}{13} + \left( \frac{39261}{455} - \frac{414\ln3}{13} \right)\sin(3\phi) \right), \]
\[ \xi_{s,5} = \frac{(14\cos(\theta) + 3\cos(3\theta) + 15\cos(5\theta))}{540} \left( v^s\sin(2\phi) - \frac{3911v^s\sin(2\phi)}{910} \right) \]
\[ + v^\delta \left( \cos(2\phi) \left( \frac{163}{15} - 4\ln2 \right) + 2\pi\sin(2\phi) \right) + \frac{63439v^s\sin(2\phi)}{10920} \]
\[ \xi_{s,6} = \frac{(2\sin(2\theta) + 3\sin(4\theta))}{5760} \left( v^s\cos(\phi) - \frac{179v^s\cos(\phi)}{39} + v^\delta \left( \pi\cos(\phi) - \frac{569\sin(\phi)}{105} \right) \right) \]
\[ + \frac{5023v^s\cos(\phi)}{585} + v^\delta \left( -\frac{179\pi\cos(\phi)}{39} + \frac{101851\sin(\phi)}{4095} \right), \]
\[ \xi_{s,7} = \frac{81\cos(\theta)\sin^4(\theta)}{10} \left( v^s\sin(6\phi) - \frac{113v^s\sin(6\phi)}{14} \right) \]
\[ + v^\delta \left( \cos(6\phi) \left( \frac{487}{14} - 12\ln6 \right) + 6\pi\sin(6\phi) \right) + \frac{1372317v^s\sin(6\phi)}{73304} \]
\[ \xi_{s,8} = -\frac{3125\cos(\theta)(1 + 3\cos^2(\theta))\sin^2(\theta)}{2016} \left( v^s\cos(5\phi) - \frac{149v^s\cos(5\phi)}{24} \right) \]
\[ + v^\delta \left( 5\pi\cos(5\phi) + \left( -\frac{1219}{42} + 10\ln5 \right)\sin(5\phi) \right), \]
\[ \xi_{s,9} = \frac{-64\cos(\theta)(1 + 11\cos(2\theta))\sin^2(\theta)}{495} \left( v^s\sin(4\phi) - \frac{93v^s\sin(4\phi)}{14} \right) \]
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\[ + v^7 \left( \cos(4\phi) \left( \frac{487}{21} - 8\ln 4 \right) + 4\pi \sin(4\phi) \right) + \frac{3261767 v^8 \sin(4\phi)}{219912}, \]

\[ \xi_{6,\theta} = 243 \cos(\theta) \sin(\theta) \left( 53 + 20 \cos(2\theta) + 55 \cos(4\theta) \right) \left( v^5 \cos(3\phi) \right) \]

\[ - \frac{133 v^7 \cos(3\phi)}{24} + v^8 \left( 3\pi \cos(3\phi) + \left( -\frac{1219}{70} + 6\ln 3 \right) \sin(3\phi) \right), \]

\[ \xi_{6,\theta} = \cos(\theta) \left( 47 - 84 \cos(2\theta) + 165 \cos(4\theta) \right) \left( v^4 \sin(2\phi) - \frac{81 v^6 \sin(2\phi)}{14} \right) \]

\[ + v^7 \left( \cos(2\phi) \left( \frac{487}{42} - 4\ln 2 \right) + 2 \pi \sin(2\phi) \right) + \frac{14482483 v^8 \sin(2\phi)}{1099560}, \]

\[ \xi_{6,\theta} = \cos(\theta) \sin(\theta) \left( 41 - 12 \cos(2\theta) + 99 \cos(4\theta) \right) \]

\[ \times \left( v^5 \cos(\phi) - \frac{125 v^7 \cos(\phi)}{24} + v^8 \left( \pi \cos(\phi) - \frac{1219 \sin(\phi)}{210} \right) \right), \]

\[ \xi_{7,\theta} = -117649 \cos(\theta) \sin(\theta) \left( v^5 \cos(7\phi) - \frac{319 v^7 \cos(7\phi)}{34} \right) \]

\[ + v^8 \left( 7 \pi \cos(7\phi) + \left( -\frac{7699}{180} + 14 \ln 7 \right) \sin(7\phi) \right), \]

\[ \xi_{7,\theta} = 243 \cos(\theta) \sin^4(\theta) \left( 13 + 7 \cos(2\theta) \right) \times \left( v^6 \sin(6\phi) - \frac{1787 v^8 \sin(6\phi)}{238} \right), \]

\[ \xi_{7,\theta} = -78125 \cos(\theta) \sin^6(\theta) \left( 3 + 13 \cos(2\theta) \right) \left( v^5 \cos(5\phi) \right) \]

\[ - \frac{271 v^7 \cos(5\phi)}{34} + v^8 \left( 5 \pi \cos(5\phi) + \left( -\frac{7699}{252} + 10 \ln 5 \right) \sin(5\phi) \right), \]

\[ \xi_{7,\theta} = -4 \cos(\theta) \left( 113 + 116 \cos(2\theta) + 91 \cos(4\theta) \right) \sin^2(\theta) \]

\[ \times \left( v^6 \sin(4\phi) - \frac{14543 v^8 \sin(4\phi)}{2142} \right), \]

\[ \xi_{7,\theta} = -729 \cos(\theta) \left( 167 + 44 \cos(2\theta) + 429 \cos(4\theta) \right) \sin(\theta) \]

\[ \times \left( v^6 \cos(3\phi) - \frac{239 v^7 \cos(3\phi)}{34} + v^8 \left( 3 \pi \cos(3\phi) + \left( -\frac{7699}{420} + 6 \ln 3 \right) \sin(3\phi) \right) \right), \]

\[ \xi_{7,\theta} = \cos(\theta) \left( -338 + 1351 \cos(2\theta) - 990 \cos(4\theta) + 1001 \cos(6\theta) \right) \]

\[ \times \left( v^6 \sin(2\phi) - \frac{13619 v^8 \sin(2\phi)}{2142} \right), \]

\[ \xi_{7,\theta} = \cos(\theta) \sin(\theta) \left( 109 + 132 \cos(2\theta) + 143 \cos(4\theta) \right) \]

\[ \times \left( v^6 \sin(4\phi) - \frac{18450432 v^8 \sin(4\phi)}{18450432} \right). \]
\[ \left( v^8 \cos(\phi) - \frac{223}{34} v^6 \cos(\phi) + v^4 \left( \pi \cos(\phi) - \frac{7699}{1260} \sin(\phi) \right) \right), \]

\[ \zeta_{s,8} = \frac{-4096}{315} v^7 \cos(\theta) \sin(\theta) \left( v^8 \sin(8\phi) - \frac{3653}{342} v^8 \sin(8\phi) \right), \]

\[ \zeta_{s,7} = -\frac{823543}{207360} v^7 \cos(7\phi)(2 + \cos(2\theta)) \sin^3(\theta), \]

\[ \zeta_{s,6} = \frac{729}{560} \cos(\theta) \sin(4\phi) \sin^4(\theta) \times \left( v^8 \sin(6\phi) - \frac{353}{38} v^8 \sin(6\phi) \right), \]

\[ \zeta_{s,5} = \frac{-78125}{1161216} v^7 \cos(5\phi) \cos(\theta)(11 + 14\cos(2\theta) + 7\cos(4\theta)) \sin^3(\theta), \]

\[ \zeta_{s,4} = -\frac{4}{4095} \cos(\theta)(49 + 52\cos(2\theta) + 91\cos(4\theta)) \sin^2(\theta), \]

\[ \zeta_{s,3} = \frac{-243}{3727360} v^7 \cos(3\phi) \cos(\theta)(24 + 141\cos(2\theta) + 91\cos(6\theta)) \sin(\theta), \]

\[ \zeta_{s,2} = \frac{\cos(\theta)(-274 + 583\cos(2\theta) - 286\cos(4\theta) + 1001\cos(6\theta))}{11531520} \]

\[ \times \left( v^8 \sin(2\phi) - \frac{2633}{342} v^8 \sin(2\phi) \right), \]

\[ \zeta_{s,1} = \frac{v^7}{94887936} \cos(\phi) \cos(\theta)(-8 + 121\cos(2\theta) + 143\cos(6\theta)) \sin(\theta), \]

\[ \zeta_{s,0} = -\frac{-4782969}{286720} v^7 \cos(\phi) \cos(\theta) \sin^7(\theta), \]

\[ \zeta_{s,9} = -\frac{-8192}{14175} v^8 \sin(8\phi) \cos(\theta)(19 + 9\cos(2\theta)) \sin^6(\theta), \]

\[ \zeta_{s,7} = \frac{40353607}{112803840} v^7 \cos(7\phi) \cos(\theta)(7 + 17\cos(2\theta)) \sin^5(\theta), \]

\[ \zeta_{s,6} = \frac{-390625}{47600} v^8 \sin(6\phi) \cos(\phi)(97 + 132\cos(2\theta) + 51\cos(4\theta)) \sin^4(\theta), \]

\[ \zeta_{s,5} = -\frac{390625}{157925376} v^7 \cos(5\phi) \cos(\theta)(59 + 80\cos(2\theta) + 85\cos(4\theta)) \sin^3(\theta), \]

\[ \zeta_{s,4} = -\frac{-4}{34429} v^8 \sin(4\phi) \cos(\phi)(142 + 343\cos(2\theta) + 130\cos(4\theta) + 153\cos(6\theta)) \sin^2(\theta), \]

\[ \zeta_{s,3} = \frac{243}{72417280} v^7 \cos(3\phi) \cos(\phi)(18 + 195\cos(2\theta) + 78\cos(4\theta) + 221\cos(6\theta)) \sin(\theta), \]

\[ \zeta_{s,2} = \frac{v^8}{114566400} \sin(2\phi) \cos(\phi)(1279 - 1480\cos(2\theta)). \]
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\[ + 2236 \cos(4\theta) - 1976 \cos(6\theta) + 1989 \cos(8\theta) , \]

\[ \xi_{10,1} = \frac{\nu^7}{11731599360} \cos(\phi) \cos(\theta) (166 + 403 \cos(2\theta)) \]

\[ + 234 \cos(4\theta) + 221 \cos(6\theta) \sin(\theta) , \]

\[ \xi_{10,3} = \frac{-131072 \nu^8 \sin(\phi)}{269325} \cos(\theta) (9 + 19 \cos(2\theta)) \sin^4(\theta) , \]

\[ \xi_{10,4} = \frac{2187 \nu^8 \sin(6\phi)}{14470400} \cos(\theta) (2449 + 3604 \cos(2\theta) + 2907 \cos(4\theta)) \sin^4(\theta) , \]

\[ \xi_{10,5} = \frac{-16 \nu^8 \sin(4\phi)}{4578525} \cos(\theta) (446 + 1319 \cos(2\theta)) \]

\[ + 850 \cos(4\theta) + 969 \cos(6\theta) \sin^2(\theta) , \]

\[ \xi_{10,6} = \frac{\nu^8 \sin(2\phi)}{17414092800} \cos(\theta) (3007 - 5720 \cos(2\theta)) \]

\[ + 8268 \cos(4\theta) - 1768 \cos(6\theta) + 12597 \cos(8\theta)) . \]

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