Sunklodas’ Approach to Normal Approximation for Time-Dependent Dynamical Systems

Juho Leppänen1 · Mikko Stenlund2

Received: 9 February 2020 / Accepted: 12 September 2020 / Published online: 25 September 2020
© Springer Science+Business Media, LLC, part of Springer Nature 2020

Abstract
We consider time-dependent dynamical systems arising as sequential compositions of self-maps of a probability space. We establish conditions under which the Birkhoff sums for multivariate observations, given a centering and a general normalizing sequence \( b(N) \) of invertible square matrices, are approximated by a normal distribution with respect to a metric of regular test functions. Depending on the metric and the normalizing sequence \( b(N) \), the conditions imply that the error in the approximation decays either at the rate \( O(N^{-1/2}) \) or the rate \( O(N^{-1/2} \log N) \), under the additional assumption that \( \|b(N)^{-1}\| \lesssim N^{-1/2} \). The error comes with a multiplicative constant whose exact value can be computed directly from the conditions. The proof is based on an observation due to Sunklodas regarding Stein’s method of normal approximation. We give applications to one-dimensional random piecewise expanding maps and to sequential, random, and quasistatic intermittent systems.

Keywords Stein’s method · Multivariate normal approximation · Time-dependent dynamical system · Intermittency

Mathematics Subject Classification 60F05 · 37A05 · 37A50 · 37C60

1 Introduction

In this note we revisit the topic of statistical limit laws by Stein’s method for dynamical systems, studied previously in [10,19,20,24,26,29–31,35,46]. We consider discrete time-dependent dynamical systems described by sequential compositions \( T_n = T_n \circ \cdots \circ T_1 \),...
where each $T_i : X \to X$ is a transformation of a probability space $(X, \mathcal{B}, \mu)$. The measure $\mu$ is not assumed to be invariant under any of the maps $T_i$. Given a bounded observable $f : X \to \mathbb{R}^d$ and a sequence $b = b(N) \in \mathbb{R}^{d \times d}$ of invertible matrices, we are interested in approximating the law of the sums

$$W = W(N) = \sum_{n=1}^{N-1} b^{-1}(f \circ T_n - \mu(f \circ T_n))$$

by a multivariate normal distribution. More precisely, we want to identify conditions that cover a wide range of chaotic time-dependent systems and imply a good upper bound on

$$\sup_{h \in \mathcal{H}} |\mu[h(W) - \Phi_\Sigma(h)]|,$$

where $\mathcal{H}$ is a class of regular test functions $h : \mathbb{R}^d \to \mathbb{R}$, and $\Phi_\Sigma(h)$ denotes the expectation of $h$ with respect to the multivariate normal distribution $\mathcal{N}(0, \Sigma)$ with covariance matrix $\Sigma = \text{Cov}_\mu(W) = \mu(W \otimes W)$.

Since its introduction in [50], Stein’s method has seen extensive development in the literature of probability theory. In the present context of dynamical systems, the simple basic idea of the method can be described as follows. If for each test function $h \in \mathcal{H}$ the solution $A : \mathbb{R}^d \to \mathbb{R}$ to the differential equation (called a Stein equation)

$$\text{tr } \Sigma D^2 A(w) - w^T \nabla A(w) = h(w) - \Phi_\Sigma(h) \quad (w \in \mathbb{R}^d)$$

lies in another class of functions $\mathcal{A}$, then it follows that

$$(1) \leq \sup_{A \in \mathcal{A}} |\mu[\text{tr } \Sigma D^2 A(W) - W^T \nabla A(W)]|.$$  \hspace{1cm} (3)

In this way the original problem of approximating the law of $W$ by $\mathcal{N}(0, \Sigma)$ is reduced to bounding the right hand side of (3), which interestingly only depends on the law of $W$ and the class $\mathcal{A}$. It was observed in [30,31] that, when $b(N) = \sqrt{N} I_d \otimes \cdots \otimes I_d$, Taylor expanding $\nabla A(W)$ about the punctured sums

$$W^n.K = \sum_{0 \leq i \leq N-1 : |i-n| > K} b^{-1}(f \circ T_i - \mu(f \circ T_i))$$

with a suitably chosen $K = K(N) \gg 1$ leads to certain correlation-decay conditions for an upper bound on $|\mu[\text{tr } \Sigma D^2 A(W) - W^T \nabla A(W)]|$. Such an approach calls for bounds on partial derivatives of $A$, which are known to follow from bounds on partial derivatives of $h$. In [30,31], $\mathcal{H}$ was taken to be the class of three times differentiable functions with bounded derivatives in the case of a general $d > 1$, and the class of Lipschitz continuous functions in the case $d = 1$.

The approach described above was applied in [30] to stationary Sinai billiards and in [31] to time-dependent smooth uniformly expanding circle maps. Both systems are (the latter in a certain sense) exponentially mixing, which is essentially the reason why replacing $W$ with $W^n.K$ in the application of Stein’s method causes only a small error. Indeed, upper bounds of order $O(N^{-1/2} \log N)$ on (1) for sufficiently regular observables could be obtained this way. While such a “fixed gap” approach works also for polynomially mixing systems, it yields a larger error depending on the rate of mixing. This can be seen from the results of [29], where time-dependent systems in the spirit of [1,43] described by sequential compositions $T_{\alpha_n} \circ \cdots \circ T_{\alpha_1}$ of polynomially mixing intermittent maps $T_{\alpha_n} : [0, 1] \to [0, 1]$ with parameters $0 \leq \alpha_n \leq \beta_n < 1/3$ were considered. Under the condition that $\Sigma = \text{Cov}_\mu(W)$ is positive definite,
an upper bound of order $O(N^{\beta_*-1/2}(\log N)^{1/\beta_*})$ was obtained for Lipschitz continuous observables. The result was used to establish central limit theorems for quasistatic and random compositions of intermittent maps.

The purpose of the present note is to describe an adaptation of Stein’s method that is more suitable than those of [30,31] for normal approximation of polynomially mixing systems, and investigate some of its implications. The starting point is a decomposition of $\mu[\text{tr} \Sigma D^2 A(W) - W^T \nabla A(W)]$ due to Sunklodas [56], which allows to identify correlation-decay conditions that imply a rate of decay for (1) depending on the “growth of $b(N)$”. In the case of a general $b(N)$ such that $\|b(N)^{-1}\| \lesssim N^{-1/2}$, the conditions yield the rate $O(N^{-1/2})$ for a class of smooth test functions $\mathcal{H}$, and in the special self-norming case $b(N) = [\text{Cov}_\mu(\sum_{n<N} (f \circ T_n - \mu(f \circ T_n)))]^{1/2}$ the rate $O(N^{-1/2} \log N)$ for Lipschitz continuous test functions. A key ingredient in the proof of the latter estimate is a recent result due to Gallouët–Mijoule–Swan [16] concerning the regularity of solutions to Stein equation. As applications we establish rates of convergence in the central limit theorem for the random piecewise expanding model studied by Dragičević et al. [12] and for sequential, random, and quasistatic intermittent systems. The results for intermittent systems notably improve those of [29].

Statistical properties of time-dependent dynamical systems have been studied in several previous works including [2,15,33,44,53,54,57]. Central limit theorems were shown by Bakhtin [3,4], Conze–Raugi [8], and more recently by Nándori–Szász–Varjú [41] and Nicol–Török–Vaieni [43]. Heinrich [27] showed a Berry–Esseen type upper bound for sequences of uniformly expanding interval maps admitting a Markov partition. Haydn–Nicol–Török–Vaieni [25] established almost sure invariance principles (ASIP) for piecewise-expanding and other related models, also in higher dimension. ASIPs were obtained also by Castro–Rodrigues–Varandas [6] for convergent sequences of Anosov diffeomorphisms and expanding maps on compact Riemannian manifolds. Recently Su [55] proved a vector valued ASIP for a general class of polynomially mixing time-dependent systems. Among its many implications is a self-norming CLT for the sequential intermittent system with $\beta_* < 1/2$, under a (polynomial) variance growth condition. Finally, Hafouta [23] showed several limit theorems, including a Berry–Esseen theorem and a local limit theorem, for sequential compositions of maps belonging to a certain class of distance expanding maps of a compact metric space.

**Notation**

For a function $A: \mathbb{R}^d \to \mathbb{R}$, we write $D^k A$ for the $k$th derivative of $A$, and also denote $\nabla A = D^1 A$. We define

$$\|D^k A\|_\infty = \max\{\|\partial_1^{t_1} \cdots \partial_d^{t_d} A_\alpha\|_\infty : t_1 + \cdots + t_d = k, \ 1 \leq \alpha \leq d'\}.$$ 

The spectral norm of a matrix $A \in \mathbb{R}^{d \times d}$ is denoted by

$$\|A\|_s = \sup\{\|Ax\| : \|x\| = 1\},$$

where $\| \cdot \|$ is the Euclidean norm of $\mathbb{R}^d$. We use $B_d(x, r)$ to denote the open ball in $\mathbb{R}^d$ with center $x$ and radius $r > 0$.

Given a measure space $(X, \mathcal{B}, \mu)$ and a $\mu$-integrable function $f: X \to \mathbb{R}^d$ we set $\mu(f) = \int f \, d\mu$. The components of $f$ are denoted by $f_\alpha$, where $\alpha \in \{1, \ldots, d\}$. The Lebesgue measure is denoted by $m$.

For two vectors $v, w \in \mathbb{R}^d$ we set $v \otimes w = [v_\alpha w_\beta]_{\alpha,\beta}$. 

\textcopyright Springer
We denote by $C$ a generic positive constant whose value might change from one line to the next. We use $C(a_1, \ldots, a_n)$ to denote a positive constant that depends only on the parameters $a_1, \ldots, a_n$.

**Structure of the Paper**

In Sect. 2 we present our main results concerning normal approximation of abstract discrete time-dependent dynamical systems. Sections 3 and 4 contain applications to one-dimensional dynamics. The model of Sect. 3 is a random dynamical system of piecewise smooth uniformly expanding maps, while in Sect. 4 we consider sequential, quasistatic, and random intermittent systems. Finally, in Sect. 5 we prove the main results.

**2 Main Results**

Consider a sequence $(T_n)_{n \geq 1}$ of measurable maps $T_n : X \to X$ of a probability space $(X, B, \mu)$. For each $i \geq 0$ let $g^i : X \to \mathbb{R}^d$ be a bounded measurable function and define

$$f^i = g^i \circ T_i \circ \cdots \circ T_1 \quad \text{and} \quad \bar{f}^i = f^i - \mu(f^i).$$

Given $N \geq 1$ and an invertible matrix $b = b(N) \in \mathbb{R}^{d \times d}$, we write

$$W = W(N) = \sum_{i=0}^{N-1} b^{-1} \bar{f}^i.$$

Given also $n, k \geq 0$, we write

$$\bar{f}^{n,k} = \sum_{0 \leq i < N; |i-n| = k} \bar{f}^i.$$

The covariance matrix of $W$ is denoted by

$$\Sigma = \text{Cov}_\mu(W) = \mu(W \otimes W).$$

**2.1 General Normalization**

First we consider a general invertible $b = b(N)$ and give conditions that imply an upper bound on the distance between the law of $W$ and the normal distribution $\mathcal{N}(0, \Sigma)$ with respect to a smooth metric.

Suppose that $\|g^i\|_\infty = \sup_{x \in X} \|g^i(x)\| \leq M$ for all $0 \leq i \leq N - 1$. Then, given a smooth test function $h : \mathbb{R}^d \to \mathbb{R}$ and $(s, t, z) \in [0, 1]^2 \times \mathbb{R}^d$ we define the matrix-valued function $G_h = G_h^{(s,t,z)} : \mathbb{R}^d \times B_d(0, 4M + 1) \to \mathbb{R}^{d \times d}$ by

$$G_h(x, y) = b^{-1} \left[ D^2 h(sb^{-1}(x + ty) + z) - D^2 h(sb^{-1}x + z) \right] b^{-1},$$

where $D^2 h(x) = \left[ \partial_\alpha \partial_\beta h(x) \right]_{\alpha, \beta}$. For a differentiable function $F : \mathbb{R}^d \times B_d(0, 4M + 1) \to \mathbb{R}^{d \times d}$ we set

$$\|F\|_\infty = \sup \{ \|F(x, y)\|_s : (x, y) \in \mathbb{R}^d \times B_d(0, 4M + 1) \}$$

and

$$\|\nabla F\|_\infty = \max_{1 \leq i \leq 2d} \sup \{ \|\partial_i F(x, y)\|_s : (x, y) \in \mathbb{R}^d \times B_d(0, 4M + 1) \}.$$
where \( \partial_t F(x, y) = [\partial_t F_{\alpha, \beta}(x, y)]_{\alpha, \beta} \)

Here is the first main result:

**Theorem 2.1** Fix \( N \geq 1 \) and let \( h : \mathbb{R}^d \to \mathbb{R} \) be three times differentiable with \( \|D^p h\|_\infty < \infty \) for \( 1 \leq p \leq 3 \). Suppose \( M = \max_{i, \leq N} \|g_i\|_\infty < \infty \), and that there exist a function \( \rho : \mathbb{N} \to \mathbb{R}_+ \) with \( \lim_{n \to \infty} \rho(n) = 0 \) and constants \( C_i > 0 \), \( 1 \leq i \leq 3 \), such that the following conditions hold for all \( 0 \leq n, m \leq N - 1 \):

(A1) For all \( \alpha, \beta \in \{1, \ldots, d\} \),

\[ |\mu(\tilde{f}^n_\alpha \tilde{f}^m_\beta)| \leq C_1 \rho(|n - m|). \]

(A2) Whenever \( (s, t, z) \in [0, 1]^2 \times \mathbb{R}^d \) and \( m \leq k \leq N - 1 \),

\[ \left| \mu \left( (\tilde{f}^n)^T G_{\tilde{h}} \left( \sum_{0 \leq i \leq N - 1: |i - n| > k} \tilde{f}^i, \tilde{f}^n,k \right) \tilde{f}^n,m \right) \right| \leq C_2 (\|G_{\tilde{h}}\|_\infty + \|\nabla G_{\tilde{h}}\|_\infty) \rho(m). \]

(A3) Whenever \( (s, t, z) \in [0, 1]^2 \times \mathbb{R}^d \) and \( 2m \leq k \leq N - 1 \),

\[ \left| \mu \left( (\tilde{f}^n)^T G_{\tilde{h}} \left( \sum_{0 \leq i \leq N - 1: |i - n| > k} \tilde{f}^i, \tilde{f}^n,k \right) \tilde{f}^n,m \right) \right| \leq C_3 (\|G_{\tilde{h}}\|_\infty + \|\nabla G_{\tilde{h}}\|_\infty) \rho(k - m). \]

(A4) The matrix \( \Sigma = \mu(W \otimes W) \) is positive definite.

Then

\[ |\mu[h(W)] - \Phi_\Sigma(h)| \leq C_* \|D^3 h\|_\infty N \|b^{-1}\|_3 \sum_{m=1}^{N-1} m \rho(m), \]

where

\[ C_* = M^3 d^4 10(C_1 + C_2 + C_3 + 4). \]

Here \( \Phi_\Sigma(h) \) denotes the expectation of \( h \) with respect to \( \mathcal{N}(0, \Sigma) \).

**Remark 2.2** Theorem 2.1, as well as Theorems 2.3 and 2.6 given below, continue to hold if \( f^i \) are replaced with general random vectors.

We postpone proving Theorem 2.1 and other results in this section until Sect. 5. Due to the smooth metric, the constant \( C_* \) in the upper bound (4) is independent of the covariance matrix \( \Sigma \). Note that under the additional assumptions \( \sum_{m=1}^{\infty} m \rho(m) < \infty \) and \( \|b^{-1}\|_3 \lesssim N^{-1/2} \) we obtain \( |\mu[h(W)] - \Phi_\Sigma(h)| = O(N^{-1/2}) \) as \( N \to \infty \), which is the optimal rate in this generality. Conditions (A1)-(A3) are designed for time-dependent systems with sufficiently good (polynomial) mixing properties. Condition (A1) requires the decay of non-stationary correlations at the rate \( \rho \). Condition (A2) requires that, for large \( m \), the random vectors

\[ \tilde{f}^n \quad \text{and} \quad G_{\tilde{h}} \left( \sum_{0 \leq i \leq N - 1: |i - n| > k} \tilde{f}^i, \tilde{f}^n,k \right) \tilde{f}^n,m \]

are componentwise nearly uncorrelated. This is reasonable because the function on the right depends on \( \tilde{f}^i \) with \( |i - n| \geq m \) only. The function \( G_{\tilde{h}} \) is differentiable and its \( C^1 \) norm.
appears as a factor in the upper bound. Condition (A3) is similar in spirit to condition (A2), for it requires
\[ G_h \left( \sum_{0 \leq i \leq N-1: |i-n|>k} \tilde{f}^i, \tilde{f}^{n,k} \right) \text{ and } \tilde{f}^n \otimes \tilde{f}^{n,m} \]
to be nearly componentwise uncorrelated, which is again reasonable when \( k \gg m \).

Recall that the Wasserstein distance between two random vectors \( Y_1 \) and \( Y_2 \) is defined by
\[ d_W(Y_1, Y_2) = \sup_{h \in \mathcal{W}} |\mu(h(Y_1)) - \mu(h(Y_2))|, \]
where
\[ \mathcal{W} = \{ h : \mathbb{R}^d \to \mathbb{R} : |h(x) - h(y)| \leq \|x - y\| \} \]
is the class of all 1-Lipschitz functions. When \( d = 1 \) we obtain a result similar to Theorem 2.1 for the Wasserstein distance. The relaxed smoothness of \( h \) comes with the expense that conditions (A2) and (A3) have to be verified for a whole class of regular functions.

For a function \( G : \mathbb{R}^d \to \mathbb{R} \) we denote
\[ \text{Lip}(G) = \sup_{x \neq y} \frac{|G(x) - G(y)|}{\|x - y\|}. \]

**Theorem 2.3** Let \( d = 1 \) and fix \( N \geq 1 \). Take \( b = \text{Var}_\mu(\sum_{i<N} \tilde{f}^i)^{1/2} \). Suppose that \( M = \max_{i<N} \|g^i\|_\infty < \infty \), that \( b > 0 \), and that there exist constants \( C_i > 0 \), \( 1 \leq i \leq 3 \), and a function \( \rho : \mathbb{N} \to \mathbb{R}_+ \) with \( \lim_{n \to \infty} \rho(n) = 0 \) such that the following conditions hold for all \( 0 \leq n, m \leq N - 1 \):

1. **(B1)** \( |\mu(\tilde{f}^n \tilde{f}^m)| \leq C_1 \rho(|n - m|) \).
2. **(B2)** Whenever \( m \leq k \leq N - 1 \) and \( G : \mathbb{R} \times B_1(0, 4M + 1) \to \mathbb{R} \) is a bounded Lipschitz continuous function,
   \[ |\mu \left[ \tilde{f}^n G \left( \sum_{0 \leq i \leq N-1: |i-n|>k} \tilde{f}^i, \tilde{f}^{n,k} \right) \tilde{f}^{n,m} \right] \leq C_2(\|G\|_\infty + \text{Lip}(G)) \rho(m). \]
3. **(B3)** Whenever \( 2m \leq k \leq N - 1 \) and \( G : \mathbb{R} \times B_1(0, 4M + 1) \to \mathbb{R} \) is a bounded Lipschitz continuous function,
   \[ |\mu \left[ \tilde{f}^n \tilde{f}^{n,m} G \left( \sum_{0 \leq i \leq N-1: |i-n|>k} \tilde{f}^i, \tilde{f}^{n,k} \right) \right] \leq C_3(\|G\|_\infty + \text{Lip}(G)) \rho(k - m). \]

Then
\[ d_W(W, Z) \leq C_\ast Nb^{-3} \sum_{m=1}^{N-1} m \rho(m), \]
where
\[ C_\ast = 96M^3 \left( C_1 + C_2 + C_3 + 1 \right), \]
and \( Z \sim \mathcal{N}(0, 1) \) is a random variable with standard normal distribution.
The following easy observation allows for normalizing constants other than

\[ b = \left[ \text{Var}_\mu \left( \sum_{i=0}^{N-1} \tilde{f}^i \right) \right]^{1/2}. \]

\[ \text{Lemma 2.4 Suppose (5) of Theorem 2.3 holds. Then, for any } c > 0, \]

\[ d_W \left( c^{-1} \sum_{i=0}^{N-1} \tilde{f}^i, c^{-1} b Z \right) \leq C_N c^{-1} b^{-2} \sum_{m=1}^{N-1} m \rho(m). \]  

\[ \text{Proof} \quad \text{For any random variables } X, Y \text{ and any } a > 0, \text{ the Wasserstein metric satisfies} \]

\[ d_W (aX, aY) = a d_W (X, Y). \]

\[ \square \]

\[ \text{Remark 2.5 There is a notable difference between the upper bounds (4) and (6): unlike (4), (6) always depends on } \text{Var}_\mu (\sum_{i=0}^{N-1} \tilde{f}^i)^{1/2} \text{ in addition to the normalizing constant } c. \text{ This difference is due to the choice of metric.} \]

\[ \text{2.2 Self-normalization} \]

\[ \text{We now assume that } \text{Cov}_\mu (\sum_{i=0}^{N-1} \tilde{f}^i) \text{ is positive definite and set } b = \text{Cov}_\mu (\sum_{i=0}^{N-1} \tilde{f}^i)^{1/2} \text{ so that } \Sigma = \mu (W \otimes W) = I_d \otimes d. \text{ In this case we establish an upper bound on the distance between the law of } W \text{ and a standard normal random vector } Z \sim \mathcal{N}(0, I_d \otimes d) \text{ with respect to the Wasserstein metric. Unlike Theorem 2.3, the result applies for a general } d \geq 1. \text{ We denote by } \lambda_{\min} \text{ the least eigenvalue of } \text{Cov}_\mu (\sum_{i=0}^{N-1} \tilde{f}^i). \]

\[ \text{Theorem 2.6 Let } N \geq 1. \text{ Suppose that max}_{0 \leq i < N} \| g^i \|_\infty \leq M \text{ where } M \geq 1, \text{ that } \lambda_{\min} > 1, \text{ and that there exist a non-increasing function } \rho : \mathbb{N} \rightarrow \mathbb{R}_+ \text{ with } \lim_{n \rightarrow \infty} \rho(n) = 0 \text{ and constants } C_i > 0, 1 \leq i \leq 3, \text{ such that the following conditions hold for all } 0 \leq n, m \leq N - 1:} \]

\[ \text{(C1) For all } \alpha, \beta \in \{1, \ldots, d\}, \]

\[ \| \mu (\tilde{f}^n_\alpha \tilde{f}^m_\beta) \| \leq C_1 \rho (|n - m|). \]

\[ \text{(C2) Whenever } m \leq k \leq N - 1 \text{ and } G : \mathbb{R}^d \times B_d (0, 4M + 1) \rightarrow \mathbb{R}^{d \times d} \text{ is a bounded } C^1 \text{-function with bounded gradient,} \]

\[ \| \mu \left( (\tilde{f}^n)^T G \left( \sum_{0 \leq i \leq N-1: |i-n|>k} \tilde{f}^i, \tilde{f}^n.k \right) \tilde{f}^n,m \right) \| \leq C_2 (\| G \|_\infty + \| \nabla G \|_\infty) \rho (m). \]

\[ \text{(C3) Whenever } 2m \leq k \leq N - 1 \text{ and } G : \mathbb{R}^d \times B_d (0, 4M + 1) \rightarrow \mathbb{R}^{d \times d} \text{ is a bounded } C^1 \text{-function with bounded gradient,} \]

\[ \| \mu \left( (\tilde{f}^n)^T G \left( \sum_{0 \leq i \leq N-1: |i-n|>k} \tilde{f}^i, \tilde{f}^n,k \right) \tilde{f}^n,m \right) \| \leq C_3 (\| G \|_\infty + \| \nabla G \|_\infty) \rho (k - m). \]
Then
\[ d_w(W, Z) \leq C_* N(1 + \log N) \lambda_{\min}^{-3} \sum_{m=1}^{N-1} (1 + \log(\rho(m)^{-1})) m \rho(m), \]
where
\[ C_* = \left( \frac{8 + 16d}{d} \frac{\Gamma(\frac{1+d}{2})}{\Gamma(\frac{d}{2})} + \sqrt{d} M + 2 \right) 2019 d^2 4^d M^4 \]
\times \left[ (C_2 + C_3)(1 + \rho(0)^{-1})(2\rho(0) + 1) + C_1 + 1 \right].

**Remark 2.7** If in addition \( \sum_{m=1}^{\infty} (1 + \log(\rho(m)^{-1})) m \rho(m) < \infty \) and \( \lambda_{\min} \gtrsim N \) hold, then we obtain the rate \( d_w(W, Z) = O(N^{-1/2} \log N) \), as \( N \to \infty \).

### 2.3 Pène’s CLT for Stationary Dynamics

The theorems given above apply in the stationary case where \( T_n = T \) preserves the measure \( \mu \) for all \( n \geq 1 \). In this case the problem of normal approximation has been studied in several important articles including \([9,13,14,22,32,44]\), using different methods, conditions, and metrics. In the multidimensional case \( d > 1 \), Pène \([45]\) formulated a correlation-decay condition for stationary processes, based on the inductive proof of Rio \([48]\). Let \( S_n = \sum_{i=0}^{n-1} f^i \), where \( f^i = f \circ T^i, f : X \to \mathbb{R}^d \) is bounded and \( \mu(f) = 0 \). In this context of measure preserving transformations, Pène’s condition can be stated as follows:

\[ \text{(D)} \quad \text{There exist } r \in \mathbb{Z}_+, C > 0, M \geq \max\{1, \|f\|_\infty\}, \text{and a sequence of real numbers } (\varphi_{p,l}) \text{ with } |\varphi_{p,l}| \leq 1 \text{ and } \sum_{p=1}^{\infty} p \max_{0 \leq |l| \leq |p/(r+1)|} \varphi_{p,l} < \infty, \text{ such that for any integers } a, b, c \geq 0 \text{ satisfying } 1 \leq a + b + c \leq 3, \text{ for any integers } i, j, k, p, q, l \text{ with } 0 \leq i \leq j \leq k \leq k + p \leq k + p + q \leq k + p + l, \text{ for any } a, \beta, \gamma \in \{1, \ldots, d\}, \text{ and for any bounded differentiable function } F : \mathbb{R}^d \times ([-M, M]^d)^3 \to \mathbb{R} \text{ with bounded gradient,} \]
\[ |\text{Cov}_\mu[F(S_i, f^i, f^j, f^k), (f^{k+p})^a, (f^{k+p+q})^\beta, (f^{k+p+l})^\gamma]| \leq C (\|F\|_\infty + \|\nabla F\|_\infty) \varphi_{p,l}. \]

Condition (D) is satisfied by chaotic dynamical systems such as Sinai billiards. It was shown in \([45]\) that condition (D) implies the existence of the limit \( \Sigma_0 := \lim_{n \to \infty} n^{-1} \mu(S_n \otimes S_n) \) and, whenever \( \Sigma_0 \) is nonnull, the existence of a constant \( B > 0 \) such that
\[ d_w(N^{-\frac{1}{2}} S_N, U) \leq B N^{-\frac{1}{2}} \quad \forall N \geq 1, \quad (7) \]
where \( U \) is a Gaussian random variable with expectation 0 and covariance matrix \( \Sigma_0 \).

Compared to Theorem 2.1, (7) gives an upper bound of the same order \( O(N^{-1/2}) \) for stationary systems whose correlations decay at a rate which has a finite first moment, for test functions that are only assumed to be Lipschitz continuous. On the other hand, Theorem 2.1 is more general in that it applies for rather arbitrary matrix-valued normalizing sequences \( b(N) \). Furthermore, the constant \( C_* \) in (4) is more explicit than the one in (7) in terms of its dependence on \( d, f \) and the underlying dynamical system. The same can be said about the constant \( C_* \) in Theorem 2.6, which gives an upper bound for the same metric as (7) but with a slightly weaker rate of convergence due to the logarithmic factor. Note that, similarly to conditions (B2)–(B3) and (C2)–(C3), condition (D) has to be verified for a whole class of regular functions \( F \).
3 Application I: Random 1D Piecewise Expanding Maps

In this section we apply Theorem 2.3 to estimate the rate of convergence in the quenched CLT for a class of piecewise expanding random dynamical systems. Namely we consider the setup studied by Dragičević et al. [12]. Below we recall some definitions and results from [12] as they are necessary for understanding the application given in this section.

Set \((X, \mathcal{B}) = ([0, 1], \text{Borel}(0, 1])\) and for a function \(g : X \to \mathbb{R}\) define its total variation by

\[
V(g) = \inf_{h = g_m \text{-a.e.}} \sup_{x_0 < \cdots < x_n = 1} \sum_{k=1}^{n} |h(x_k) - h(x_{k-1})|.
\]

Moreover, define

\[
\| g \|_{BV} = V(g) + \| g \|_{L^1(m)}.
\]

The Banach space \(BV\) consists of all functions \(g\) with \(V(g) < \infty\) and is equipped with the norm \(\| \cdot \|_{BV}\).

Let us denote by \(\mathcal{E}\) the collection of all maps \(T : X \to X\) for which there exists a finite partition \(\mathcal{A}(T)\) of \(X\) into subintervals such that for every \(I \in \mathcal{A}(T)\):

1. \(T \rvert I\) extends to a \(C^2\) map in a neighborhood of \(I\);
2. \(\delta(T) := \inf |T'| > 1\).

The map \(T\) is monotonous on each element \(I \in \mathcal{A}(T)\). From now on we take \(\mathcal{A}(T)\) to be the minimal such partition and set \(N(T) = |\mathcal{A}(T)|\).

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \(\tau : \Omega \to \Omega\) be an invertible \(\mathbb{P}\)-preserving transformation. We consider a map \(\omega \mapsto T_\omega\) from \(\Omega\) into \(\mathcal{E}\). Random compositions of maps are denoted by

\[
T^n_\omega = T_{\tau^{-1}_\omega} \circ \cdots \circ T_\omega
\]

and

\[
\mathcal{L}^n_\omega = \mathcal{L}_{\tau^{-1}_\omega} \cdots \mathcal{L}_\omega,
\]

where \(\mathcal{L}_\omega : L^1(m) \to L^1(m)\) is the transfer operator associated to \(T_\omega\):

\[
\mathcal{L}_\omega f(x) = \sum_{T_\omega(y) = x} \frac{f(y)}{|(T_\omega)'y|}.
\]

Conditions (H):

1. \(\tau : \Omega \to \Omega\) is invertible, \(\mathbb{P}\)-preserving, and ergodic.
2. The map \((\omega, x) \mapsto (\mathcal{L}_\omega H(\omega, \cdot))(x)\) is measurable for every measurable function \(H : \Omega \times X \to \mathbb{R}\) such that \(H(\omega, \cdot) \in L^1(X, m)\).
3. \(N := \sup_{\omega \in \Omega} N(T_\omega) < \infty; \delta := \inf_{\omega \in \Omega} \delta(T_\omega) > 1; D := \sup_{\omega \in \Omega |T_\omega''| < \infty}\)
4. There is \(r \geq 1\) such that \(\delta^r > 2\) and \(\text{ess inf}_{\omega \in \Omega} \min_{J \in \mathcal{A}(T_\omega)} m(J) > 0\).
5. For every subinterval \(J \subset X\) there is \(k = k(J) \geq 1\) such that \(T_\omega^k(J) = X\) holds for almost every \(\omega \in \Omega\).

Remark 3.1 It was shown in [12] that conditions (H) imply several nice properties for the transfer operators \(\mathcal{L}_\omega\), including a Lasota–Yorke inequality and exponential decay in the BV-norm. The authors used such properties to establish an almost sure invariance principle.
Lemma 3.2 (See Proposition 1 in [12]) Assume conditions (H). Then there exists a unique measurable and non-negative function \( h : \Omega \times X \to \mathbb{R} \) such that \( h_\omega := h(\omega, \cdot) \in BV \), \( m(h_\omega) = 1 \) and \( \mathcal{L}_\omega(h_\omega) = h_{\tau(\omega)} \) for almost every \( \omega \in \Omega \). Moreover, \( \text{ess sup}_{\omega \in \Omega} \| h_\omega \|_{BV} < \infty \).

3.1 Statement of Result

Let \( f : X \to \mathbb{R} \) be a bounded measurable function and set
\[
\tilde{f}(\omega, x) = f(x) - \mu_\omega(f),
\]
where \( d \mu_\omega = h_\omega \, dm \) and \( h \) is the function from Lemma 3.2. Set
\[
W(\omega) = b^{-1} \sum_{n=0}^{N-1} \tilde{f}_{\tau^n(\omega)} \circ T^n_\omega = b^{-1} \sum_{n=0}^{N-1} (f \circ T^n_\omega - \mu_\omega(f \circ T^n_\omega)),
\]
where \( b \) is the square root of \( \mu_\omega\left( \sum_{n=0}^{N-1} \tilde{f}_{\tau^n(\omega)} \circ T^n_\omega \right)^2 \). We denote by \( \varphi : \Omega \times X \to \Omega \times X \) the skew product \( \varphi(\omega, x) = (\tau(\omega), T_\omega(x)) \), which preserves the measure \( \mu \) on \( \Omega \times X \) defined by
\[
\mu(A \times B) = \int_A \int_B h \, d(\mathbb{P} \times m), \quad A \in \mathcal{F}, \ B \in \mathcal{B}.
\]

Theorem 3.3 Consider a family of piecewise expanding maps \((T_\omega)_\omega \in \Omega\) such that conditions (H) hold. Fix \( N \geq 1 \) and suppose \( f \) is Lipschitz continuous such that \( \tilde{f} \) can not be written as \( g - g \circ \varphi \) for any \( g \in L^2(\Omega \times X, \mu) \). Then there is \( C_* > 0 \) independent of \( N \) such that
\[
d_W(W, Z) \leq C_* N^{-\frac{1}{2}}
\]
holds for almost every \( \omega \in \Omega \). Here \( Z \sim \mathcal{N}(0, 1) \) is a random variable with standard normal distribution.

Remark 3.4 The proof of Theorem 3.3 is based on Theorem 2.3. Theorem 2.1 or 2.6 could be used instead to obtain similar central limit theorems for multivariate observables \( f : X \to \mathbb{R}^d \).

3.2 A Functional Correlation Bound

Conditions (B2) and (B3) of Theorem 2.3 will be verified by applying the auxiliary result given below, which facilitates bounding integrals of the form \( \int F \circ (T^m_\omega)_{0 \leq m < k} \, d\mu \), where \( F : [0, 1]^k \to \mathbb{R} \) is not necessarily a product of one-dimensional observables. Such functional correlation bounds were established for stationary Sinai billiards in [39] and for time-dependent intermittent maps in [38].

For a function \( F : [0, 1]^k \to \mathbb{R} \), \( \theta \in (0, 1) \), and \( 1 \leq \beta \leq k \) we denote
\[
[F]_{\theta, \beta} = \sup_{x \in [0,1]^k} \sup_{a \neq a'} \frac{|F(x(a/\beta)) - F(x(a'/\beta))|}{|a-a'|^\theta},
\]
where \( x(a/\beta) \in [0,1]^k \) is obtained from \( x \) by replacing the \( \beta \)th coordinate with \( a \in [0,1] \). We say that \( F \) is \( \theta \)-Hölder continuous in the coordinate \( \beta \) if \( [F]_{\theta, \beta} < \infty \).
Proposition 3.5 Let \( k \geq 2 \). Consider integers \( 0 \leq n_1 \leq \cdots \leq n_k \) blocked according to a set of indices \( 0 = \ell_0 < \ell_1 < \cdots < \ell_p < \ell_{p+1} = k \), where we assume that \( n_{\ell_i+1} < \cdots < n_{\ell_{i+1}} \) hold for all \( 0 \leq i \leq p \). Suppose \( (T_\omega)_{\omega \in \Omega} \subset \mathcal{E} \) is a family of maps such that conditions (H) hold, and that \( F_\omega : [0, 1]^k \to \mathbb{R} \) is a function with \( \text{ess sup}_{\omega \in \Omega} \| F_\omega \|_\infty < 0 \) and

\[
L = \text{ess sup}_{\omega \in \Omega} \sup_{1 \leq \beta \leq \ell_p} [F_\omega]_{\theta, \beta} < \infty.
\]

Denote by \( H_\omega(x_1, \ldots, x_{p+1}) \) the function

\[
F_\omega(T_\omega^{n_1}(x_1), \ldots, T_\omega^{n_{\ell_1}}(x_1), T_\omega^{n_{\ell_1+1}}(x_2), \ldots, T_\omega^{n_{\ell_2}}(x_2), \ldots, T_\omega^{n_{\ell_p+1}}(x_{p+1}), \ldots, T_\omega^{n_k}(x_{p+1})).
\]

Then, for any probability measures \( \mu_1, \ldots, \mu_{p+1} \) whose densities belong to \( BV \), and for almost every \( \omega \in \Omega \),

\[
\left| \int H_\omega(x, \ldots, x) d\mu_1(x) - \int \cdots \int H_\omega(x_1, \ldots, x_{p+1}) d\mu_1(x_1)d\mu_2(x_2) \cdots d\mu_{p+1}(x_{p+1}) \right|
\]

\[
\leq C( \text{ess sup}_{\omega \in \Omega} \| F_\omega \|_\infty + L) \left( \sum_{i=1}^{p+1} \| h_i \|_{BV} \right) \sum_{i=1}^{p} \gamma^{n_{\ell_i+1} - n_{\ell_i}},
\]

where \( 0 < \gamma < 1 \), and \( C = C(p, (T_\omega)_{\omega \in \Omega}, \theta) > 0 \).

**Remark 3.6** The upper bound (9) is independent of \( k \).

The proof for Proposition 3.5 is based on two auxiliary results. The first result is an immediate consequence of Corollary 8 in [2] due to Aimino and Rousseau, who considered sequential (non-random) compositions of piecewise-expanding maps. The second result is Lemma 2 in the paper [12] by Dragičević et al.

**Lemma 3.7** Suppose conditions (H) hold. There is \( C > 0 \) such that for almost every \( \omega \in \Omega \),

\[
\sum_{I \in \mathcal{A}(T_\omega^m)} V_I \left( \frac{1}{|T_\omega^m|} \right) \leq C,
\]

where \( V_I(f) \) denotes the total variation of \( f \) over the subinterval \( I \subset [0, 1] \).

**Proof** As is explained on p. 2252 of [12], condition (iv) implies that there exists \( \alpha' \in (0, 1) \) and \( K' > 0 \) such that, for almost every \( \omega \in \Omega \),

\[
V(\mathcal{L}_\omega^{\ell'} \phi) \leq (\alpha')^{\ell} V(\phi) + \frac{K'}{1 - \alpha'} \| \phi \|_{L^1(m)}
\]

holds for all \( \phi \in BV \) and \( \ell \geq 1 \). It suffices to fix \( \Omega^* \subset \Omega \) with \( \mathbb{P}(\Omega^*) = 1 \) such that (11) holds for all \( \omega \in \Omega^* \). Then the proof of Corollary 8 in [2] shows that (10) holds for all \( \omega \in \Omega^* \).

**Lemma 3.8** (See Lemma 2 in [12]) Assume conditions (H). There is \( K > 0 \) and \( \eta \in (0, 1) \) such that, for almost every \( \omega \in \Omega \),

\[
\| \mathcal{L}_\omega^n \phi \|_{BV} \leq K \eta^n \| \phi \|_{BV}
\]

holds for all \( n \geq 0 \) and \( \phi \in BV \) with \( m(\phi) = 0 \).
Proof for Proposition 3.5. The proof proceeds by induction on \( p \). First let \( p = 1 \) and denote \( \ell_1 = \ell \). Then the function \( H_\omega(x, y) \) in Proposition 3.5 becomes

\[
H_\omega(x, y) = F_\omega(T_{\omega_1}^{n_1}(x), \ldots, T_{\omega_\ell}^{n_\ell}(x), T_{\omega_{\ell+1}}^{n_{\ell+1}}(y), \ldots, T_{\omega_k}^{n_k}(y)),
\]

where \( n_1 < \ldots < n_\ell \leq n_{\ell+1} < \ldots < n_k \). Set \( n_* = n_\ell + \lfloor (n_{\ell+1} - n_\ell) / 2 \rfloor \). Then,

\[
\int H_\omega(x, x) \, d\mu_1(x) = \sum_{J \in \mathcal{A}(T_{\omega_1}^{n_*)})} \left( \int J H_\omega(x, y) d\mu_1(x) d\mu_2(y) \right) d\mu_1(x).
\]

Claim. If \( a, b \in J \in \mathcal{A}(T_{\omega_1}^{n_*)}) \), then almost surely

\[
\left| H_\omega(a, x) - \int H_\omega(a, y) d\mu_2(y) - \left( H_\omega(b, x) - \int H_\omega(b, y) d\mu_2(y) \right) \right| \leq L_1 \kappa^{n_{\ell+1} - n_\ell},
\]

where \( L_1 = \text{ess sup}_{\omega \in \Omega} \max_{1 \leq \alpha \leq \ell} F_\omega[\theta, \alpha, \kappa] = (\delta^{\theta})^{-1/2} \in (0, 1) \), and \( C = C(\kappa) > 0 \). We recall that by definition \( \delta = \inf_{\omega \in \Omega} \delta(T_\omega) > 1 \).

Proof for Claim. Since \( F_\omega \) is \( \theta \)-Hölder continuous for a.e. \( \omega \in \Omega \) in the first \( \ell \) coordinates,

\[
|F_\omega(a_1, \ldots, a_\ell, x, \ldots, x) - F_\omega(b_1, \ldots, b_\ell, x, \ldots, x)| \leq L_1 \sum_{\alpha=1}^\ell |a_\alpha - b_\alpha|^\theta
\]

holds for a.e. \( \omega \in \Omega \). Consequently,

\[
|H_\omega(a, x) - H_\omega(b, x)| \leq L_1 \sum_{\alpha=1}^\ell |T_{\omega_\alpha}^{n_\alpha}(a) - T_{\omega_\alpha}^{n_\alpha}(b)|^\theta \leq L_1 \sum_{\alpha=1}^\ell m(T_{\omega_\alpha}^{n_\alpha}(J))^\theta.
\]

For each \( 1 \leq \alpha \leq \ell \), \( T_{\omega_\alpha}^{n_\alpha} \) maps \( T_{\omega_\alpha}^{n_\alpha}(J) \) diffeomorphically onto \( T_{\omega_\alpha}^{n_\alpha}(J) \), which implies the upper bound \( m(T_{\omega_\alpha}^{n_\alpha}(J)) \leq (\delta^{n_\alpha - n_{\alpha}})^{-1} m(T_{\omega_\alpha}^{n_\alpha}(J)) \leq \delta^{-n_\alpha + n_\alpha} \). That is, for a.e. \( \omega \in \Omega \),

\[
|H_\omega(a, x) - H_\omega(b, x)| \leq L_1 \sum_{\alpha=1}^\ell (\delta^{\theta})^{-n_\alpha + n_\alpha} \leq \sum_{k=n_* - n_\ell}^{\infty} \kappa^k \leq C(\kappa) \kappa^{n_{\ell+1} - n_\ell}.
\]

This proves the claim. \( \square \)

We fix a point \( c_J \in J \) for each \( J \in \mathcal{A}(T_{\omega_1}^{n_*)}) \). Then (12) implies for a.e. \( \omega \in \Omega \) the upper bound

\[
\left| \sum_{J \in \mathcal{A}(T_{\omega_1}^{n_*)})} \left( \int J \left( H_\omega(x, x) - \int H_\omega(x, y) d\mu_2(y) \right) d\mu_1(x) \right) \right| \leq L_1 \kappa^{n_{\ell+1} - n_\ell}.
\]

1 We denote by \( \lfloor x \rfloor \) the greatest non-negative integer \( n \) with \( n \leq x \).
Let \( h_1 \in BV \) denote the density of \( \mu_1 \), and let \( h_2 \in BV \) denote the density of \( \mu_2 \). Moreover, let \( H_\omega(c_J, x) \) be the function that satisfies \( H_\omega(c_J, T_\omega^{n+1}(x)) = H_\omega(c_J, x) \). Fix \( J \in A(T_\omega^{n+1}) \). Then, for a.e. \( \omega \in \Omega \),

\[
\left| \int_J \left( H_\omega(c_J, x) - \int H_\omega(c_J, y) \, d\mu_1(y) \right) \, d\mu_2(x) \right| \\
\leq \left| \int_0^1 \left( H_\omega(c_J, x) - \int H_\omega(c_J, y) \, d\mu_1(y) \right) \left( 1_J(x)h_2(x) - \mu_2(J)h_1(x) \right) \, dm(x) \right| \\
\leq 2 \sup_{\omega \in \Omega} \| F_\omega \|_\infty \| L_\omega^{n+1}(1_Jh_2 - \mu_2(J)h_1) \|_{L^1(m)}.
\]

Let \( x \in [0, 1] \). Since \( J \in A(T_\omega^{n+1}) \), either \( x \in T_\omega^{n+1} \) (J) and

\[
L_\omega^{n+1}(1_Jh_2) = \frac{h_2((T_\omega^{n+1}\restriction J)^{-1}x)}{|(T_\omega^{n+1}\restriction J)^{-1}x|},
\]

or \( L_\omega^{n+1}(1_Jh_2)x = 0 \). It follows easily from this and the strict monotonicity of \( T_\omega^{n+1} \restriction J \) that

\[
V(L_\omega^{n+1}(1_Jh_2)) \leq 2\|h_2\|_\infty \sup_{y \in J} \frac{1}{|(T_\omega^{n+1})'y|} + \|h\|_\infty V_J \left( \frac{1}{|(T_\omega^{n+1})'y|} \right) + V_f(h_2) \sup_{y \in J} \frac{1}{|(T_\omega^{n+1})'y|},
\]

where

\[
\sup_{y \in J} \frac{1}{|(T_\omega^{n+1})'y|} \leq V_J \left( \frac{1}{|(T_\omega^{n+1})'y|} \right) + \int_0^1 \frac{1}{|(T_\omega^{n+1})'(T_\omega^{n+1} \restriction J)^{-1}y|} \, dy \leq V_J \left( \frac{1}{|(T_\omega^{n+1})'y|} \right) + m(J).
\]

We conclude that

\[
V(L_\omega^{n+1}(1_Jh_2)) \leq 6\|h_2\|_BV \left( V_J \left( \frac{1}{|(T_\omega^{n+1})'y|} \right) + m(J) \right). \tag{14}
\]

On the other hand there is \( C > 0 \) such that, for any \( \phi \in BV \), \( \sup_{k \geq 0} V(L_\omega^{k}(\phi)) \leq C\|\phi\|_BV \) holds for almost every \( \omega \in \Omega \). This follows from (11) together with the fact that \( \|L_\omega(\phi)\|_BV \leq C\|\phi\|_BV \) for almost every \( \omega \in \Omega \); see p. 2257 of [12]. In particular,

\[
V(L_\omega^{n+1}(\mu_2(J)h_1)) \leq \mu_2(J)C\|h_1\|_BV \quad \text{a.e.} \omega \in \Omega. \tag{15}
\]

Next we combine Lemma 3.8, (14) and (15) to obtain

\[
\|L_\omega^{n+1}(1_Jh_2 - \mu_2(J)h_1)\|_{L^1(m)} \\
\leq K\eta^{n+1-n}\|L_\omega^{n+1}(1_Jh_2 - \mu_2(J)h_1)\|_BV \\
\leq C\eta^{n+1-n}(\|h_1\|_BV + \|h_2\|_BV) \left( V_J \left( \frac{1}{|(T_\omega^{n+1})'y|} \right) + m(J) + \mu_2(J) \right),
\]

for a.e. \( \omega \in \Omega \), where \( \eta_1 \in (0, 1) \). Then, by Lemma 3.7,

\[
(13) \leq \sum_{J \in A(T_\omega^{n+1})} 2 \text{ess} \sup_{\omega \in \Omega} \| F_\omega \|_\infty \| L_\omega^{n+1}(1_Jh_2 - \mu_2(J)h_1) \|_{L^1(m)} \\
\leq \sum_{J \in A(T_\omega^{n+1})} \text{ess} \sup_{\omega \in \Omega} \| F_\omega \|_\infty C\eta^{n+1-n}(\|h_1\|_BV + \|h_2\|_BV).
\]
\[ \times \left( V_f \left( \frac{1}{|T_{n+}^*|} \right) + m(J) + \mu_2(J) \right) \leq C \text{ess sup}_{\omega \in \Omega} \| F_\omega \|_{\infty} C_{\gamma_1}^{^{n+1-\eta_1}} (\| h_1 \|_{BV} + \| h_2 \|_{BV}), \]

for a.e. \( \omega \in \Omega \). Taking \( \gamma = \max\{\eta_1, \kappa\} \) completes the proof for the case \( p = 1 \).

Suppose that we have shown (9) for \( p - 1 \), and fix integers \( 0 = \ell_0 < \ell_1 < \ldots < \ell_p < \ell_{p+1} = k \) as in the proposition. Recall that \( H_\omega(x_1, \ldots, x_{p+1}) \) denotes the function

\[ F_\omega(T_{\omega}^{n_1}(x_1), \ldots, T_{\omega}^{n_1}(x_1), T_{\omega}^{n_1+1}(x_2), \ldots, T_{\omega}^{n_2}(x_2), \ldots, T_{\omega}^{n_{p+1}}(x_{p+1}), \ldots, T_{\omega}^{n_k}(x_{p+1})). \]

From the case \( p = 1 \) we know that, for a.e. \( \omega \in \Omega \),

\[ \left| \int H_\omega(x, \ldots, x) d\mu_1(x) - \int H_\omega(x, \ldots, x, x_{p+1}) d\mu_1(x) d\mu_{p+1}(x_{p+1}) \right| \leq C (\text{ess sup}_{\omega \in \Omega} \| F_\omega \|_{\infty} + L) (\| h_1 \|_{BV} + \| h_{p+1} \|_{BV}) \gamma^{n_{p+1} - n_{\ell_1}} (16) \]

where \( h_1 \) is the density of \( \mu_1 \).

Next for each \( x_{p+1} \in [0, 1] \), we apply the induction hypothesis to the function

\[ (y_1, \ldots, y_k) \mapsto F_\omega(y_1, \ldots, y_{p}, T_{\omega}^{n_{p+1}}(x_{p+1}), \ldots, T_{\omega}^{n_k}(x_{p+1})). \]

This implies for a.e. \( \omega \in \Omega \) the upper bound

\[ \left| \int H_\omega(x, \ldots, x, x_{p+1}) d\mu_1(x) - \int \cdots \int H_\omega(x_1, \ldots, x_p, x_{p+1}) d\mu_1(x_1) \ldots d\mu_p(x_p) \right| \leq C (\text{ess sup}_{\omega \in \Omega} \| F_\omega \|_{\infty} + L) \left( \sum_{i=1}^{p} \| h_i \|_{BV} \right) \sum_{i=1}^{p-1} \gamma^{n_{p+1} - n_{\ell_1}}. (17) \]

for all \( x_{p+1} \in [0, 1] \). Now, to complete the proof for Proposition 3.5, it suffices to combine (16) and (17).

\[ \square \]

### 3.3 Proof for Theorem 3.3

It was shown in [12] that there exists a non-random \( \sigma^2 \geq 0 \) such that

\[ \sigma^2 = \lim_{n \to \infty} \mu_\omega \left[ \left( \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \tilde{f}_{\tau k} \circ T_{\omega}^k \right)^2 \right] \]

for almost every \( \omega \in \Omega \). Moreover, \( \sigma^2 = 0 \) if and only if there exists \( g \in L^2(\Omega \times X, \mu) \) such that \( \tilde{f} = g \circ \varphi \). Hence, under our assumption there exists \( C > 0 \) and \( n_0 \geq 1 \) such that, for a.e. \( \omega \in \Omega \),

\[ b^2 = \mu_\omega \left[ \left( \sum_{k=0}^{n-1} \tilde{f}_{\tau k} \circ T_{\omega}^k \right)^2 \right] \geq Cn \]

holds for all \( n \geq n_0 \).

Next we show that, with \( \mu_\omega \) as the initial measure, conditions (B1)–(B3) hold with \( \rho(m) = \gamma^m \) for a.e. \( \omega \in \Omega \), where \( \gamma \in (0, 1) \) is the same as in Proposition 3.5. To this end recall that, by Lemma 3.2, the density \( h_\omega \) of \( \mu_\omega \) lies in \( BV \) for a.e. \( \omega \in \Omega \).
Sunklodas' Approach to Normal Approximation for Time-Dependent...

(B1): For brevity, we introduce the notation $\tilde{f}_n^\omega = f \circ T_n^\omega - \mu_\omega(f \circ T_n^\omega)$. Taking $k = 1$, $p = 1$, $F_\omega(x, y) = f(x)f(y)$, and $\mu_1 = \mu_\omega = \mu_2$ in Proposition 3.5 yields the upper bound

$$|\mu_\omega(\tilde{f}_n^\omega, \tilde{f}_m^\omega)| \leq C \|f\|_{\text{Lip}} \sup_{\omega \in \Omega}\|h_\omega\|_{BV}\gamma^{n-m},$$

(18)

for a.e. $\omega \in \Omega$.

(B2): Let $m \leq k \leq N - 1$ and let $G : \mathbb{R} \times B_1(0, 4\|f\|_\infty + 1) \to \mathbb{R}$ be a bounded Lipschitz continuous function. We define $F_\omega(x_0, \ldots, x_{n-k}, x_{n-m}, x_n, x_{n+m}, x_{n+k}, \ldots, x_{N-1})$ by the formula

$$\psi_{n, \omega}(x_n)G\left(\sum_{|i-n|>k} \psi_{1, \omega}(x_i), \sum_{|i-n|=k} \psi_{1, \omega}(x_i)\right) \sum_{|i-n|=m} \psi_{1, \omega}(x_i),$$

where $\psi_{1, \omega}(x) = f(x) - \mu_\omega(f \circ T_\omega^i)$ and the summations are over $i$. Then

$$\mu_\omega\left[F_\omega(T_0^0, \ldots, T_\omega^n, T_\omega^0, T_\omega^{n-m}, T_\omega^{n+m}, T_\omega^{n+2}, \ldots, T_\omega^{N-1})\right]$$

$$= \mu_\omega\left[\tilde{f}_n^\omega G\left(\sum_{|i-n|>k} \tilde{f}_\omega^i, \sum_{|i-n|=k} \tilde{f}_\omega^i\right) \sum_{|i-n|=m} \tilde{f}_\omega^i\right],$$

which is the integral we need to control. It is easy to verify that $F_\omega$ is Lipschitz continuous with

$$\sup_{\omega \in \Omega}\|F_\omega\|_\infty \leq \|G\|_\infty 4\|f\|_\infty^2$$

and

$$\sup_{\omega \in \Omega}\sup_{\beta \in \mathcal{I}}[F_\omega]_{1, \beta} \leq 8\|f\|_{\text{Lip}}^3(\|G\|_\infty + \text{Lip}(G)),$$

where $\mathcal{I} = \{0 \leq i \leq N - 1 : |i - n| \geq k\} \cup \{0 \leq i \leq N - 1 : |i - n| = m\} \cup \{n\}$ is an indexing for the arguments of $F$. Observe that, since $\mu_\omega(\tilde{f}_\omega^m) = 0$,

$$\int \int \int F_\omega(T_\omega^{n-k}x, T_\omega^{n-m}x, T_\omega^ny, T_\omega^{n+m}z, T_\omega^{n+k}z) d\mu_\omega(x) d\mu_\omega(y) d\mu_\omega(z) = 0,$$

where $T_\omega^{n-k}x = (T_\omega^0x, \ldots, T_\omega^{n-k}x)$ and $T_\omega^{n+k}z = (T_\omega^{n+k}z, \ldots, T_\omega^{N-1}z)$. It follows by Proposition 3.5 applied with $F_\omega$ and $p = 2$ that, for a.e. $\omega \in \Omega$,

$$\left|\mu_\omega\left[\tilde{f}_\omega^m G\left(\sum_{|i-n|>k} \tilde{f}_\omega^i, \sum_{|i-n|=k} \tilde{f}_\omega^i\right) \sum_{|i-n|=m} \tilde{f}_\omega^i\right]\right|$$

$$\leq C \left(\sup_{\omega \in \Omega}\|F_\omega\|_\infty + \sup_{\omega \in \Omega}\sup_{\beta \in \mathcal{I}}[F_\omega]_{1, \beta}\right) \sup_{\omega \in \Omega}\|h_\omega\|_{BV}\gamma^m$$

$$\leq C \left(\|G\|_\infty 4\|f\|_\infty^2 + 8\|f\|_{\text{Lip}}^3(\|G\|_\infty + \text{Lip}(G))\right) \sup_{\omega \in \Omega}\|h_\omega\|_{BV}\gamma^m$$

$$\leq C (\|G\|_\infty + \text{Lip}(G))(\|f\|_\infty^2 + \|f\|_{\text{Lip}}^3) \sup_{\omega \in \Omega}\|h_\omega\|_{BV}\gamma^m.$$
(B3): This is obtained in the same way as condition (B2). Namely, whenever \(2m \leq k \leq N - 1\), applying Proposition 3.5 with \(p = 2\) and the function 

\[
\psi_{n,\omega}(x_n)G_*(\sum_{|i-n|>k} \psi_{i,\omega}(x_i), \sum_{|i-n|=k} \psi_{i,\omega}(x_i)) \sum_{|i-n|=m} \psi_{i,\omega}(x_i),
\]

where 

\[
G_*(x, y) = G(x, y) - \mu_\omega \left[ \sum_{|i-n|=m} \tilde{f}_n^i \tilde{f}_{n,k} \right],
\]

implies for a.e. \(\omega \in \Omega\) the upper bound 

\[
\mu_\omega \left[ \sum_{|i-n|=m} \tilde{f}_n^i \tilde{f}_{n,k} \right] \leq C(\|G\|_\infty + \text{Lip}(G))(\|f\|_{L^2}^2 + \|f\|_{L^1} \ess \sup_{\omega \in \Omega} \|h_\omega\|_{BV})^m.
\]

Since \(\sum_{m=1}^\infty m \gamma_m < \infty\), Theorem 3.3 now follows by Theorem 2.3.

**Remark 3.9** Another example of a random dynamical system that satisfies the conditions of Theorem 2.3 is the Sinai Billiard of [51], in which a scatterer configuration on the torus is randomly updated between consecutive collisions. The key technical lemmas necessary for obtaining an analog of Proposition 3.5 were proven in [51,53], including a statistical memory loss starting from an initial measure supported on a single homogeneous local unstable manifold (Lemma 12 of [51]), and a tail estimate on the prevalence of short local unstable manifolds (Lemma 13 of [51]). The application would imply a rate of convergence in the annealed CLT but we will not treat it here.

## 4 Application II: Intermittent Maps

Following [40] we define for each \(\alpha \in (0, 1)\) the map \(T_\alpha : [0, 1] \rightarrow [0, 1]\) by 

\[
T_\alpha(x) = \begin{cases} 
  x(1 + 2^\alpha x^\alpha) & \forall x \in [0, 1/2), \\
  2x - 1 & \forall x \in [1/2, 1].
\end{cases}
\]

Associated to each map \(T_\alpha\) is its transfer operator \(\mathcal{L}_\alpha : L^1(m) \rightarrow L^1(m)\) defined by 

\[
\mathcal{L}_\alpha h(x) = \sum_{y \in T_\alpha^{-1}\{x\}} \frac{h(y)}{T_\alpha'(y)}.
\]

We denote by \(d\hat{\mu}_\alpha = \hat{h}_\alpha dm\) the invariant absolutely continuous probability measure associated to \(T_\alpha\). It follows from [40] that the density \(\hat{h}_\alpha\) belongs to the convex cone of functions 

\[
\mathcal{C}_\alpha(\alpha) = \{f \in C([0, 1]) \cap L^1 : f \geq 0, f \text{ decreasing,} \\
x^{\alpha+1} f \text{ increasing,} f(x) \leq 2^\alpha(2 + \alpha)x^{-\alpha}m(f) \}.
\]

We recall from [1,40] that 

\[
0 < \alpha \leq \beta \implies \mathcal{C}_\alpha(\alpha) \subset \mathcal{C}_\alpha(\beta).
\]
and that
\[ 0 < \alpha \leq \beta \Rightarrow \mathcal{L}_\alpha C_*(\beta) \subset C_*(\beta). \]

### 4.1 Sequential Compositions

First we consider sequential compositions
\[ \tilde{T}_n = T_{\alpha_n} \circ \cdots \circ T_{\alpha_1} \]
of intermittent maps with parameters \( 0 < \alpha_n \leq \beta_n < 1 \). The notation below is adapted from Sect. 2.2: \( \mu \) is a Borel probability measure on \([0, 1]\); \( g^n : [0, 1] \to \mathbb{R}^d \) is a bounded observable for all \( n \geq 1 \);

\[ W = \sum_{i=0}^{N-1} b^{-1} \tilde{f}^i; \quad b = \left[ \text{Cov}_\mu \left( \sum_{i=0}^{N-1} \tilde{f}^i \right) \right]^{1/2}; \quad \tilde{f}^n = g^n \circ \tilde{T}_n - \mu(g^n \circ \tilde{T}_n); \]

\( \lambda_{\text{min}} \) is the least eigenvalue of \( \text{Cov}_\mu \left( \sum_{i=0}^{N-1} \tilde{f}^i \right) \).

For a Lipschitz continuous function \( g : [0, 1] \to \mathbb{R}^d \) we set \( \|g\|_{\text{Lip}} = \|g\|_{\infty} + \text{Lip}(g) \), where
\[ \|g\|_{\infty} = \sup_{x \in [0, 1]} \|g(x)\| \]
and
\[ \text{Lip}(g) = \sup_{x \neq y} \frac{\|g(x) - g(y)\|}{|x - y|}. \]

**Theorem 4.1** Let \( N \geq 1 \) and let \( \mu \) be a measure whose density lies in the cone \( C_*(\beta_n) \).

Suppose that \( g^n : [0, 1] \to \mathbb{R}^d \) are Lipschitz continuous with \( \sup_{n \leq N} \|g^n\|_{\text{Lip}} + 1 \leq L \) and that \( \lambda_{\text{min}} > 1 \). Denote by \( Z \sim N(0, I_{d \times d}) \) a standard normal random vector.

1. If \( \beta_n < 1/3 \), then there is \( C_* = C_*(L, d, \beta_n) > 0 \) such that
   \[ d_{\mathbb{W}}(W, Z) \leq C_* N(1 + \log N) \lambda_{\text{min}}^{-\frac{3}{2}}. \]
   In particular, if \( \lambda_{\text{min}} \gg N^{2/3}(\log N)^{2/3} \), then \( W \xrightarrow{d} Z \) as \( N \to \infty \).

2. If \( 1/3 \leq \beta_n < 2/5 \), then for any \( \delta > 0 \) there is \( C_* = C_*(L, d, \beta_n, \delta) > 0 \) such that
   \[ d_{\mathbb{W}}(W, Z) \leq C_* N^{4 - \frac{1}{\beta_n} + \delta} \lambda_{\text{min}}^{-\frac{3}{2}}. \]
   In particular, if \( \lambda_{\text{min}} \gg N^{8/3 + 2\delta/3 - 2/3\beta_n} \), then \( W \xrightarrow{d} Z \) as \( N \to \infty \).

**Remark 4.2** A couple of remarks are in order:

1. The proof is based on Theorem 2.6. In the special case \( d = 1 \) let us denote \( S = \sum_{i=0}^{N-1} \tilde{f}^i \) and \( \sigma^2 = \mu(S^2) \). Assuming \( \beta_n < 1/3 \), the sharper upper bound
   \[ d_{\mathbb{W}}(\sigma^{-1} S, Z) \leq C_* N \sigma^{-3} \]
is obtained by applying Theorem 2.3 instead of Theorem 2.6, provided that \( \sigma^2 > 0 \). Consequently, by Lemma 2.4, for any \( c > 0 \),
\[
d_w(c^{-1}S, c^{-1}\sigma Z) \leq C_\ast Nc^{-1}\sigma^{-2}.
\]
(20)

Without any assumption on \( \sigma^2 \) we still obtain the weaker bound
\[
d_w(N^{-\frac{1}{2}}S, N^{-\frac{1}{2}}\sigma Z) \leq C_\ast N^{-1/6}.
\]

This follows easily by combining (20) with the fact that, for any random variables \( X \) and \( Y \) with finite variances \( \sigma_X^2 \) and \( \sigma_Y^2 \), respectively, the Wasserstein metric satisfies
\[
d_w(X, Y) \leq \sigma_X + \sigma_Y \quad \text{(see e.g. [31] for the last statement)}.
\]

(ii) In the stationary case of a single intermittent map \( T \) show that conditions (C1)–(C3) of Theorem 2.6 hold with
\[
k = C_\ast \psi \quad \text{where} \quad C_\ast = C(\beta_\ast).
\]

Proof for Theorem 4.1

Set \( \rho(n) = n^{1-1/\beta_\ast} (\log n)^{1/\beta_\ast} \) for \( n \geq 2 \) and \( \rho(0) = \rho(1) = 1 \). We show that conditions (C1)–(C3) of Theorem 2.6 hold with \( \rho \) using Theorem 1.1 in [38].

(C1): Let \( \alpha, \beta \in \{1, \ldots, d\} \) and \( 0 \leq n, \ m \leq N - 1 \). Applying Theorem 1.1 in [38] with \( k = 2, \ p = 1, \ F(x, y, z) = g^n_\alpha(y)g_m^\beta(z) \), and \( \mu_1 = \mu \) yields the upper bound
\[
|\mu(T^n_\alpha \mathcal{F}_\beta^m)| \leq C L^2 \rho(|n - m|),
\]
where \( C = C(\beta_\ast) > 0 \).

(C2): Let \( 0 \leq n, \ m \leq N - 1, \ m \leq k \leq N - 1 \), and \( G : \mathbb{R}^d \times B_d(0, 4L + 1) \to \mathbb{R}^{d \times d} \) be a bounded \( C^1 \)-function with bounded gradient. We define
\[
F(x_0, \ldots, x_{n-k}, x_{n-m}, x_n, x_{n+m}, x_{n+k}, \ldots, x_{N-1})
\]
by the formula
\[
\psi^n(x_n)^T G \left( \sum_{|i-n|>k} \psi^i(x_i), \sum_{|i-n|=k} \psi^i(x_i), \sum_{|i-n|=m} \psi^i(x_i) \right)
\]
where \( \psi^i(x) = g^i(x) - \mu(g^i \circ \tilde{T}_i) \) and the summations are over \( i \). Then,
\[
\mu \left[ F(\tilde{T}_0, \ldots, \tilde{T}_{n-k}, \tilde{T}_{n-m}, \tilde{T}_n, \tilde{T}_{n+m}, \tilde{T}_{n+k}, \ldots, \tilde{T}_{N-1}) \right]
\]
\[
= \mu \left[ (f^n)^T G \left( \sum_{|i-n|>k} \tilde{f}^i, \sum_{|i-n|=k} \tilde{f}^i, \sum_{|i-n|=m} \tilde{f}^i \right) \right],
\]
which is the integral we need to control. It is easy to verify that
\[
\| F \|_\infty \leq 8L\| G \|_\infty
\]
(21)
and
\[ 
\sup_{\beta \in \mathcal{I}} [F]_{1, \beta} \leq 8L^3 d^2 (\|G\|_\infty + \|\nabla G\|_\infty). 
\] (22)

Here
\[ 
\|G\|_\infty = \sup_{x \in \mathbb{R}^d, \|y\| < 4L+1} \|G(x, y)\|_s \quad \text{and} \quad \|\nabla G\|_\infty = \max_{1 \leq i \leq 2d} \sup_{x \in \mathbb{R}^d, \|y\| < 4L+1} \|\partial_i G(x, y)\|_s.
\]

\([F]_{1, \beta}\) is defined by (8), and \(\mathcal{I} = \{0 \leq i \leq N - 1 : |i - n| \geq k\} \cup \{0 \leq i \leq N - 1 : |i - n| = m\} \cup \{n\}\) is an indexing for the arguments of \(F\). Theorem 1.1 in [38] together with (21) and (22) implies the upper bound
\[
\left| \mu \left( (\tilde{f}^n)^T G \left( \sum_{|i-n|>k} \tilde{f}^i, \sum_{|i-n|=m} \tilde{f}^i \right) \right) \right| \leq C(\beta_s) 8L^3 d^2 (\|G\|_\infty + \|\nabla G\|_\infty) \rho(m).
\]

(C3): This is shown in the same way as condition (C2). Namely Theorem 1.1 in [38] is applied with the function
\[
\psi^n(x_n)^T G_1 \left( \sum_{|i-n|>k} \psi^i(x_i), \sum_{|i-n|=m} \psi^j(x_i) \right) \sum_{|i-n|=m} \psi^i(x_i),
\]
where
\[
G_1(x, y) = G(x, y) - \mu \left[ G \left( \sum_{|i-n|>k} \tilde{f}^i, \sum_{|i-n|=k} \tilde{f}^i \right) \right].
\]

We leave the details to the reader.

If \(\beta_s < 1/3\), it follows by the foregoing that conditions (C1)–(C3) hold also with \(\rho(n) = n^{-\kappa}\) for some \(\kappa > 2\). In particular \(\sum_{n=1}^\infty (1 + \log(\rho(m)^{-1}))m \rho(m) < \infty\), so that item (1) of Theorem 4.1 follows by Theorem 2.6. If instead \(1/3 \leq \beta_s < 2/5\) we obtain conditions (C1)–(C3) with \(\rho(n) = n^{1-1/\beta_s+\delta}\) for any \(\delta > 0\). Then \(\sum_{m=1}^{N-1} (1 + \log(\rho(m)^{-1}))m \rho(m) \leq C(\beta_s, \delta \beta') N^{3-1/\beta_s+\delta+\delta'}\) holds for arbitrarily small \(\delta' > 0\) and item (2) of Theorem 4.1 follows again by Theorem 2.6.

In the remainder of this section we look at situations where we have control on the limiting behavior of \(\text{Cov}_\mu(\sum_{i=0}^{N-1} \tilde{f}^i)\).

### 4.2 Quasistatic Dynamics

We apply Theorem 4.1 to a model described by time-dependent (non-random) compositions of slowly transforming intermittent maps. More precisely we consider the following subclass of quasistatic dynamical systems (QDS); for background and earlier results on quasistatic systems we refer the reader to [11,29,31,36,37,52].

**Definition 4.3** (Intermittent QDS) Let \(\mathbf{T} = \{T_{a_n,k} : 0 \leq k \leq n, n \geq 1\}\) be a triangular array of intermittent maps with parameters \(a_{n,k} \in [0, 1]\). If there is a piecewise continuous curve \(\gamma : [0, 1] \to [0, 1]\) satisfying
\[
\lim_{n \to \infty} a_{n,\lfloor nt \rfloor} = \gamma_t
\]
for all \(t\), we say that \((\mathbf{T}, \gamma)\) is an intermittent QDS.
Given an intermittent QDS \((T, \gamma)\), we define the functions \(S_n : [0, 1] \times [0, 1] \to \mathbb{R}\) by
\[
S_n(x, t) = \int_0^{nt} f_{n,[s]}(x) \, ds, \quad n \geq 1,
\]
where
\[
f_{n,k} = f \circ T_{n,k} \circ \cdots \circ T_{n,1}, \quad 0 \leq k \leq n,
\]
\[
f_{n,0} = f, \quad \text{and} \quad f : [0, 1] \to \mathbb{R}^d \text{ is a bounded function.}
\]
Set \(W_n(x, t) = b^{-1} \bar{S}_n(x, t)\) by \(N(0, I_{d \times d})\), where \(\bar{S}_n(x, t) = S_n(x, t) - \mu(S_n(x, t))\) and \(b = b(n, t) = \text{Cov}_\mu(\bar{S}_n(\cdot, t))^{1/2}\).

**Theorem 4.4** Let \(f : [0, 1] \to \mathbb{R}^d\) be a Lipschitz continuous function and \(\mu\) be such that its density lies in \(C_2(\beta_a)\). Suppose that the limiting curve \(\gamma\) is Hölder-continuous, that for some \(\eta \in (0, 1)\) we have
\[
\sup_{n \geq 1} \sup_{t \in [0,1]} |\alpha_{\eta,[nt]} - \gamma_t| < \infty,
\]
and that there exists \(t_0 \in (0, 1)\) such that \(f\) is not a co-boundary for \(T_{t_0}\) in any direction\(^2\).

1. If \(\gamma([0, 1]) \subset [0, \beta_a]\) and \(\beta_a < 1/3\), then there exists \(C_* = C_*(t_0, d, f, \gamma)\) such that for all \(t \geq t_0\) and \(n \geq 2\),
\[
d_W(W_n(t), Z) \leq C_* n^{-\frac{1}{2}} \log n.
\]
2. If \(\gamma([0, 1]) \subset [0, \beta_a]\) and \(1/3 \leq \beta_a < 2/5\), then for any \(\delta > 0\) there exists \(C_* = C_*(t_0, d, f, \delta, \gamma)\) such that for all \(t \geq t_0\) and \(n \geq 1\),
\[
d_W(W_n(t), Z) \leq C_* n^{\frac{2}{3} - \frac{1}{p_\alpha} + \frac{1}{\delta}}.
\]

**Proof** Set \(\xi_n(x, t) = n^{-\frac{1}{2}} b W_n(x, t)\). By Lemma 4.4 in [29], uniformly in \(t \in [0, 1]\),
\[
\lim_{n \to \infty} [\text{Cov}_\mu(\xi_n(t))]_{\alpha, \beta} = \int_0^t [\hat{\Sigma}_s(f)]_{\alpha, \beta} \, ds \quad \forall \alpha, \beta \in \{1, \ldots, d\},
\]
where
\[
\hat{\Sigma}_s(f) = \hat{\mu}_{\gamma_s}[\hat{f}_t \otimes \hat{f}_t] + \sum_{k=1}^{\infty} (\hat{\mu}_{\gamma_s}[\hat{f}_t \otimes \hat{f}_t \circ T_{\gamma_s}^k] + \hat{\mu}_{\gamma_s}[\hat{f}_t \circ T_{\gamma_s}^k \otimes \hat{f}_t]),
\]
and \(\hat{f}_t = f - \hat{\mu}_{\gamma_t}(f)\). By theorem 2.11 in the same article the limit covariance \(\Sigma_t(f) := \int_0^t \hat{\Sigma}_s(f) \, ds\) is positive definite for all \(t \geq t_0\) (this is where the co-boundary condition on \(f\) is needed). In particular, \(\lambda_{\min}(\Sigma_t(f)) > 0\), where \(\lambda_{\min}(A)\) denotes the least eigenvalue of the matrix \(A \in \mathbb{R}^{d \times d}\). It follows by the same argument as in p. 20 of [29] that there exists \(n_0\) and \(C > 0\) such that \(\lambda_{\min}(\text{Cov}_\mu(\xi_n(t))) \geq C\) holds for all \(t \geq t_0\) and all \(n \geq n_0\). In other words,
\[
\lambda_{\min}(\text{Cov}_\mu(\bar{S}_n(t))) \geq C n \quad \forall t \geq t_0 \forall n \geq n_0.
\]

\(^2\) i.e. there does not exist a unit vector \(v \in \mathbb{R}^d\), a constant \(c \in \mathbb{R}^d\), and a function \(\psi \in L^2(\hat{\mu}_{\gamma_0})\) such that \(v^T f = c + \psi - \psi \circ T_{\gamma_0}\).
Next we show the wanted upper bound on $d_W (W_n(t), Z)$ by controlling separately the following three terms:

\begin{align*}
&d_W \left( b \left( \frac{\lfloor nt \rfloor}{n} \right)^{-1} \bar{S}_n \left( \frac{\lfloor nt \rfloor}{n} \right), Z \right), \quad (24) \\
&d_W \left( b \left( \frac{\lfloor nt \rfloor}{n} \right)^{-1} \bar{S}_n \left( \frac{\lfloor nt \rfloor}{n} \right), b(t)^{-1} \bar{S}_n \left( \frac{\lfloor nt \rfloor}{n} \right) \right), \quad (25) \\
&d_W \left( b(t)^{-1} \bar{S}_n \left( \frac{\lfloor nt \rfloor}{n} \right), b(t)^{-1} \bar{S}_n (t) \right), \quad (26)
\end{align*}

where $b(s) = \text{Cov}_\mu (\bar{S}_n (s))^{1/2}$.

Note that

$$\bar{S}_n \left( \frac{\lfloor nt \rfloor}{n} \right) = \sum_{k=0}^{\lfloor nt \rfloor - 1} \bar{f}_{n,k}.$$

It follows immediately by (23) and Theorem 4.1 that for all $n \geq n_0$ and $t \geq t_0$,

\begin{equation}
(24) \leq \begin{cases} 
C (\| f \|_{\text{Lip}}, d, \beta_*, t_0) n^{-\frac{1}{2}} \log n, & 0 \leq \beta_* < 1/3 \\
C (\| f \|_{\text{Lip}}, d, \beta_*, t_0, \delta) n^{\frac{5}{2} - \frac{1}{2\delta} + \delta}, & 1/3 \leq \beta_* < 2/5,
\end{cases}
\end{equation}

where $\delta > 0$ can be made arbitrarily small.

In the remainder of this proof we assume that $\beta_* < 1/2$ and $\gamma ([0, 1]) \subset [0, \beta_*]$. Whenever $t \geq t_0$ and $n \geq n_0$,

\begin{equation}
(25) \leq \| b \left( \frac{\lfloor nt \rfloor}{n} \right)^{-1} - b(t)^{-1} \|_{s, \mu} \left( \| \bar{S}_n \left( \frac{\lfloor nt \rfloor}{n} \right) \| \right) \leq \| b \left( \frac{\lfloor nt \rfloor}{n} \right)^{-1} \|_{s} \| b \left( \frac{\lfloor nt \rfloor}{n} \right) - b(t) \|_{s} \| b(t)^{-1} \|_{s, \mu} \left( \| \bar{S}_n \left( \frac{\lfloor nt \rfloor}{n} \right) \|^2 \right)^{1/2}.
\end{equation}

Since the density of $\mu$ belongs to $C_\gamma (\beta_*)$, it follows by Lemma 3.3 in [36] that

$$\mu \left( \| \bar{S}_n \left( \frac{\lfloor nt \rfloor}{n} \right) \|^2 \right) \leq C n.$$

where $C = C (d, \| f \|_{\text{Lip}}, \gamma, \beta_*) > 0$ is a constant independent of $t$. Moreover (see Lemma 5.4),

$$\| b \left( \frac{\lfloor nt \rfloor}{n} \right)^{-1} \|_{s} = \lambda_{\text{min}} \left( \text{Cov}_\mu \left( \bar{S}_n \left( \frac{\lfloor nt \rfloor}{n} \right) \right) \right)^{-1/2} \leq C n^{-1/2}$$

and

$$\| b(t)^{-1} \|_{s} = \lambda_{\text{min}} \left( \text{Cov}_\mu \left( \bar{S}_n (t) \right) \right)^{-1/2} \leq C n^{-1/2}.$$

That is,

\begin{equation}
(25) \leq C n^{-1/2} \| b \left( \frac{\lfloor nt \rfloor}{n} \right) - b(t) \|_{s}.
\end{equation}

For brevity denote $\Sigma_{n,s} = \text{Cov}_\mu (\bar{S}_n (s))$. Then we have for $t \geq t_0$ and $n \geq n_0$ the upper bound (see [49] for the first inequality)

$$\| b \left( \frac{\lfloor nt \rfloor}{n} \right) - b(t) \|_{s} \leq \frac{\| \Sigma_{n,[\lfloor nt \rfloor]/n} - \Sigma_{n,t} \|_{s}}{\lambda_{\text{min}} (\Sigma_{n,[\lfloor nt \rfloor]/n})^{1/2} + \lambda_{\text{min}} (\Sigma_{n,t})^{1/2}} \leq C n^{-1/2} \| \Sigma_{n,[\lfloor nt \rfloor]/n} - \Sigma_{n,t} \|_{s}.$$

To bound the remaining spectral norm we fix $\alpha, \beta \in \{1, \ldots, d\}$, denote $\varphi = f_\alpha$ and $\psi = f_\beta$.

For a a real-valued function $g : [0, 1] \to \mathbb{R}$ and integers $0 \leq k \leq n$ we denote $\bar{g}_{n,k} = \varphi$.
Whenever \( n \geq 2 / t_0 \), we can use Theorem 1.1 in [36] to find \( \kappa > 1 \) such that

\[
\left| \Sigma_{n,\lfloor nt \rfloor} - \Sigma_{n,\lfloor nt \rfloor} \right| \leq C (2^{n-t} \| f \|_{\text{Lip}})^{\kappa} \frac{1}{n^{1-\delta}}.
\]

Hence, the upper bound \( \| \Sigma_{n,\lfloor nt \rfloor} - \Sigma_{n,\lfloor nt \rfloor} \|_S \leq d C \| f \|_{\text{Lip}}^2 \) follows by Lemma 5.4. We have shown that \( (25) \leq C n^{-1/2} \) whenever \( t \geq t_0 \) and \( n \geq n_0 \).

Finally, by (23) and Lemma 5.4,

\[
(26) \leq \mu \left( \left\| (b(t)^{-1} (\bar{S}_n \lfloor \lfloor nt \rfloor n) - \bar{S}_n (t)) \right\| \right) \leq \| b(t)^{-1} \|_S \mu (\| \bar{f}_n \|) dr ds
\]

whenever \( t \geq t_0 \) and \( n \geq n_0 \). Now to finish the proof for Theorem 4.4 it suffices to combine the foregoing upper bounds on (24), (25), and (26).

\[
\square
\]

### 4.3 Rate in the Quenched CLT

We consider a sequence \((T_{\omega})_{i \geq 1}\) of intermittent maps with parameters \((\omega_i)_{i \geq 1}\) drawn randomly from the probability space \((\Omega, \mathcal{F}, \mathbb{P}) = ([0, \beta_n]^{\mathbb{Z}^+}, \mathcal{E}^{\mathbb{Z}^+}, \mathbb{P})\), where \(\mathcal{E}\) is the Borel \(\sigma\)-algebra of \([0, \beta_n]\) and \(\mathbb{Z}^+ = \{1, 2, \ldots\}\). Let \(\tau : \Omega \to \Omega\) denote the shift \((\tau(\omega))_i = \omega_i+1\).

**Conditions (RDS):**

(i) The shift \(\tau : \Omega \to \Omega : (\tau(\omega))_i = \omega_{i+1}\) preserves \(\mathbb{P}\).

(ii) There is \(C > 0\) and \(\gamma > 0\) such that, for all \(n \geq 1\),

\[
\sup_{i \geq 1} \sup_{A \in \mathcal{F}_i, B \in \mathcal{F}_i^{\infty}} \left| \mathbb{P}(A \cap B) - \mathbb{P}(A) \mathbb{P}(B) \right| \leq C n^{-\gamma},
\]

where \(\mathcal{F}_i^1\) is the sigma-algebra generated by the projections \(\pi_1, \ldots, \pi_i, \pi_k(\omega) = \omega_k\), and \(\mathcal{F}_i^{\infty}\) is generated by \(\pi_{i+n}, \pi_{i+n+1}, \ldots\).

We set

\[
W = W(\omega) = \sum_{n=1}^{N-1} N^{-\frac{1}{2}} \bar{f}^n, \quad \bar{f}^n = f \circ \varphi(n, \omega) - \mu (f \circ \varphi(n, \omega));
\]

where \(f : [0, 1] \to \mathbb{R}^d\) is a bounded observable with \(d \geq 1\), and \(\varphi(n, \omega) = T_{\omega_0} \circ \cdots \circ T_{\omega_1}\).

That is, we take

\[
b = \sqrt{N} I_{d \times d}
\]
as the normalizing matrix.

**Theorem 4.5** Suppose that \( \beta_* < 1/3 \), that \( f : [0, 1] \to \mathbb{R}^d \) is Lipschitz continuous, and that \( \mu \) is a measure whose density belongs to \( C_\ast(\beta_\ast) \). Assume conditions (RDS). Then,

\[
\Sigma = \sum_{k=0}^{\infty} (2 - \delta_{k0}) \lim_{i \to \infty} \mathbb{E}[\mu(f_i(f_{i+k})^T) - \mu(f_i)\mu((f_{i+k})^T)]
\]

is well-defined and positive semi-definite. Moreover, \( \Sigma \) is positive definite if and only if

\[
\sup_{N \geq 1} \mathbb{E} \mu \left( \sum_{i < N} v^T f \circ \varphi(i, \omega) \right)^2 = \infty \tag{27}
\]

holds for all \( v \neq 0 \). Fix an arbitrarily small \( \delta > 0 \). If \( \Sigma \) is positive definite, then there is \( \Omega^* \subset \Omega \) with \( \mathbb{P}(\Omega^*) = 1 \) such that for any three times differentiable function \( h : \mathbb{R}^d \to \mathbb{R} \) with \( \max_{1 \leq k \leq 1} \|D^k h\|_\infty < \infty \), any \( \omega \in \Omega^* \), and any \( N \geq 2 \),

\[
|\mu[h(W)] - \Phi_{\Sigma}(h)| \leq C_\ast(\|D^3 h\|_\infty + \|h\|_{Lip}) \theta(N),
\]

where \( C_\ast = C_\ast((T_\omega)_{\omega \in \Omega}, \ell, f) > 0 \) and

\[
\theta(N) = \begin{cases} O(N^{-\frac{1}{2}}(\log N)^{\frac{1}{2} + \delta}), & \gamma > 1, \\ O(N^{-\frac{1}{2} + \delta}), & \gamma = 1, \\ O(N^{-\frac{1}{2}}(\log N)^{\frac{1}{2} + \delta}), & 0 < \gamma < 1. \end{cases}
\]

**Remark 4.6** Nicol–Török–Vaienti [43], Su [55], and Nicol–Pereira–Török [42] proved CLTs without rates of convergence for random dynamical systems of intermittent maps with parameters \( \omega_i \leq \beta_* < 1/2 \). Theorem 4.5 gives a better rate of convergence than the following upper bound established in [29] for univariate \( f : [0, 1] \to \mathbb{R} \):

\[
d_{\mathbb{W}}(W, \sigma Z) = \begin{cases} O(N^{-\frac{1}{2}}(\log N)^{\frac{1}{2} + \delta}), & \gamma \geq 1, \\ O(N^{-\frac{1}{2} + \delta}(\log N)^{\frac{1}{2} + \delta}) + O(N^{-\frac{1}{2}}(\log N)^{\frac{3}{2} + \delta}), & 0 < \gamma < 1. \end{cases}
\]

Here \( Z \sim \mathcal{N}(0, 1) \) and \( \sigma^2 = \Sigma \). The proof below can be modified to obtain the upper bound \( d_{\mathbb{W}}(W, \sigma Z) \leq C_\ast \theta(N) \) for univariate \( f : [0, 1] \to \mathbb{R} \).

**Proof** Given any vector \( v \in \mathbb{R}^d \) denote

\[
\ell_n(v) = v^T \Sigma_N v,
\]

where \( \Sigma_N = \text{Cov}_\mu(W \otimes W) \). In other words, \( \ell_n(v) \) is the variance of

\[
W(v^T f) = \sum_{n=1}^{N-1} N^{-\frac{1}{2}} v^T \tilde{f}^n.
\]

Let \( v \in \mathbb{R}^d \). By Theorem 2.6 in [29], \( \lim_{n \to \infty} \ell_n(v) = v^T \Sigma v \) exists and \( v^T \Sigma v > 0 \) if and only if

\[
\sup_{N \geq 1} \mathbb{E} \mu \left( \sum_{i < N} v^T f \circ \varphi(i, \omega) \right)^2 = \infty.
\]
Hence (27) is equivalent to the positive definiteness of $\Sigma$. The proof of Theorem 2.6 in [29] also shows that, for almost every $\omega \in \Omega$,

$$\left| \ell_N(v) - v^T \Sigma v \right| = \begin{cases} O(N^{-\frac{1}{2}} (\log N)^{\frac{3}{2}+\delta}), & \gamma > 1, \\ O(N^{-\frac{1}{2}+\delta}), & \gamma = 1, \\ O(N^{-\frac{\gamma}{2}} (\log N)^{\frac{3}{2}+\delta}), & 0 < \gamma < 1, \end{cases}$$  

(28)

Hence, by Lemma 4.4 in [28], for almost every $\omega \in \Omega$,

$$\| \Sigma_N - \Sigma \|_s = \begin{cases} O(N^{-\frac{1}{2}} (\log N)^{\frac{3}{2}+\delta}), & \gamma > 1, \\ O(N^{-\frac{1}{2}+\delta}), & \gamma = 1, \\ O(N^{-\frac{\gamma}{2}} (\log N)^{\frac{3}{2}+\delta}), & 0 < \gamma < 1. \end{cases}$$  

(29)

From now on we assume that $\Sigma$ is positive definite. We split $|\mu[h(W)] - \Phi_\Sigma(h)|$ into two terms:

$$|\mu[h(W)] - \Phi_\Sigma(h)|$$  

(30)

and

$$|\Phi_\Sigma(h) - \Phi_\Sigma(h)|.$$  

(31)

It follows by (29) that there is $N_0$ such that $\Sigma_N$ is positive definite for $N \geq N_0$ and a.e. $\omega \in \Omega$. Then, for all $\omega \in \Omega$ and $N \geq N_0$, the upper bound

$$\| \Sigma_N - \Sigma \|_s \leq C \| D^3 h \|_\infty N^{-\frac{1}{2}}$$  

(32)

holds for some $C = C(\beta_*, d, \| f \|_{\text{lip}}) > 0$. The proof for (32) is almost verbatim the same as the proof for Theorem 4.1: Theorem 2.1 is applied with $b = \sqrt{N} I_{d \times d}$ after verifying conditions (A1)–(A3) using Theorem 1.1 in [36]. We will not repeat the argument here.

Finally, it is easy to show that, for some absolute constant $C > 0$,  

$$\| \Sigma_N - \Sigma \|_s \leq C \lambda_{\min}(\Sigma_N)^{-\frac{1}{2}} + \lambda_{\min}(\Sigma)^{-\frac{1}{2}} \| \Sigma_N - \Sigma \|_s.$$  

(33)

Hence, for $N \geq N_0$ and a.e. $\omega \in \Omega$ (see [49] for the first inequality),

$$\| \Sigma_N - \Sigma \|_s \leq C \lambda_{\min}(\Sigma_N)^{-\frac{1}{2}} + \lambda_{\min}(\Sigma)^{-\frac{1}{2}} \| \Sigma_N - \Sigma \|_s.$$  

The obtained upper bound combined with (29) finishes the proof for Theorem 4.5.  

\hfill $\square$

5 Proofs for Main Results

5.1 On the Regularity of Solutions to Stein Equation

Let the matrix $\Sigma \in \mathbb{R}^{d \times d}$ be symmetric and positive definite. Denote respectively by $\phi_\Sigma$ and $\Phi_\Sigma$ the density and expected value of the $d$-dimensional normal distribution with mean 0 and covariance matrix $\Sigma$. Given a test function $h : \mathbb{R}^d \rightarrow \mathbb{R}$, define

$$A(w) = -\int_0^\infty \left\{ \int_{\mathbb{R}^d} h(e^{-s}w + \sqrt{1 - e^{-2s}} z) \phi_\Sigma(z) \, dz - \Phi_\Sigma(h) \right\} \, ds.$$  

(33)

Then, we have the following result for smooth test functions $h$; see [5,17,18,21]:

\begin{footnote}{Springer}
Lemma 5.1 Let $h : \mathbb{R}^d \to \mathbb{R}$ be three times differentiable with $\|D^k h\|_\infty < \infty$ for $1 \leq k \leq 3$. Then, $A \in C^3(\mathbb{R}^d, \mathbb{R})$, and $A$ solves the Stein equation (2). Moreover, the partial derivatives of $A$ satisfy the bounds

$$\|\partial_{t_1} \cdots \partial_{t_d} A\|_\infty \leq k^{-1} \|\partial_{t_1} \cdots \partial_{t_d} h\|_\infty$$

whenever $t_1 + \cdots + t_d = k$, $1 \leq k \leq 3$.

Note that the bounds on the partial derivatives of $A$ are independent of the covariance matrix $\Sigma$.

Recently Gallouët–Mijoule–Swan [16] obtained notable improvements on the regularity of solutions to Stein’s equation in the case $\Sigma = I_d \times \mathbb{R}$, for test functions $h$ that are assumed to be Hölder continuous:

Lemma 5.2 (See Proposition 2.2 in [16]) Set $\Sigma = I_d \times \mathbb{R}$ and let $h : \mathbb{R}^d \to \mathbb{R}$ be $\eta$-Hölder continuous with some $\eta \in (0, 1]$. Then the function $A : \mathbb{R}^d \to \mathbb{R}$ defined by (33) solves the Stein equation (2). Moreover, $A \in C^2(\mathbb{R}^d, \mathbb{R})$ and its second derivative satisfies the following bound:

$$\|D^2 A(w) - D^2 A(w')\|_s \leq \|w - w'\|[h]_\eta(C_\# + \log \|w - w'\|), \quad \forall w, w' \in \mathbb{R}^d,$$  

where

$$[h]_\eta = \sup_{x \neq y} \frac{|h(x) - h(y)|}{\|x - y\|^\eta}$$

and

$$C_\# = 2^{\frac{\eta}{2}+1} \eta + 2d \frac{\Gamma(\frac{\eta+1}{2})}{\Gamma(\frac{d}{2})}.$$

We will apply the result with $\eta = 1$. In this case the result is known to be optimal in terms of regularity of $D^2 A$. More precisely it was shown in [16] that, when $d = 2$ and $h(x, y) = \max\{0, \min\{x, y\}\}$ (an example considered first by Raič in [47]),

$$\partial_i \partial_j A(u, u) - \partial_i \partial_j A(0, 0) \sim_{u \to 0+} \frac{1}{\sqrt{2\pi} u \log u}.$$

5.2 Sunklodas’ Decomposition

Set $Y^i = b^{-1} \tilde{f}^i$ so that

$$W = \sum_{i=0}^{N-1} Y^i.$$

Next, we define punctured modifications of the sum $W$, namely

$$W^{n, m} = W - \sum_{i \in [n]_m} Y^i,$$

where

$$[n]_m = \{0 \leq i < N : |i - n| \leq m\}.$$
Moreover, set
\[ Y^{n,m} = b^{-1} \hat{f}^{n,m} = \sum_{|i-n|=m} b^{-1} f^i. \]

Note that
\[ W = \sum_{k=0}^{N-1} Y^{n,k} = W^{n,-1} \quad \text{and} \quad W^{n,N-1} = 0 \quad (35) \]
as well as
\[ W^{n,m-1} = W^{n,m} + Y^{n,m} \quad (36) \]
for any \( n \) and \( m \).

The proofs for the main results are based on the following decomposition, which is a multivariate version of Proposition 4 in [56] due to Sunklodas.

**Proposition 5.3** Suppose \( A \in C^2(\mathbb{R}^d, \mathbb{R}) \). Denote
\[ \delta^{n,m}(u) = D^2 A(W^{n,m} + u Y^{n,m}) - D^2 A(W^{n,m}) \]
and
\[ \delta^{n,m} = \delta^{n,m}(1) = D^2 A(W^{n,m-1}) - D^2 A(W^{n,m}). \]

Then
\[ \mu[\text{tr} \Sigma D^2 A(W) - W^T \nabla A(W)] = E_1 + \cdots + E_7, \]
where
\begin{align*}
E_1 &= -\sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \int_0^1 \mu[(Y^n)^T \delta^{n,m}(u) Y^{n,m}] du, \\
E_2 &= -\sum_{n=0}^{N-1} \int_0^1 \mu[(Y^n)^T \delta^{n,0}(u) Y^n] du, \\
E_3 &= -\sum_{n=0}^{N-1} \sum_{m=1}^{2m} \sum_{k=m+1}^{N-1} \mu[(Y^n)^T \delta^{n,k} Y^{n,m}], \\
E_4 &= -\sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \sum_{k=2m+1}^{N-1} \mu[(Y^n)^T \delta^{n,k} Y^{n,m}], \\
E_5 &= -\sum_{n=0}^{N-1} \sum_{k=1}^{N-1} \mu[(Y^n)^T \delta^{n,k} Y^n], \\
E_6 &= \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \mu \left[ (Y^n)^T \sum_{k=0}^{m} \mu(\delta^{n,k}) Y^{n,m} \right], \\
E_7 &= \sum_{n=0}^{N-1} \mu[(Y^n)^T \mu(\delta^{n,0}) Y^n].
\end{align*}
**Proof** For any \( n \), by (35),

\[
\nabla A(W) - \nabla A(0) = \nabla A(W^{n,-1}) - \nabla A(W^{n,N-1}) = \sum_{m=0}^{N-1} [\nabla A(W^{n,m-1}) - \nabla A(W^{n,m})].
\]

By (36),

\[
\nabla A(W^{n,m-1}) - \nabla A(W^{n,m}) = \left[ D^2 A(W^{n,m}) + \int_0^1 D^2 A(W^{n,m} + u Y^{n,m}) - D^2 A(W^{n,m}) \, du \right] Y^{n,m}
\]

\[
= \left[ D^2 A(W^{n,m}) + \int_0^1 \delta^{n,m}(u) \, du \right] Y^{n,m}.
\]

Since \( \mu[(Y^n)^T \nabla A(0)] = 0 \), it follows by the above identities that

\[
\mu[W^T \nabla A(W)] = \sum_{n=0}^{N-1} \mu[(Y^n)^T (\nabla A(W) - \nabla A(0))] = -E_1 - E_2 + \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \mu[(Y^n)^T D^2 A(W^{n,m})] Y^{n,m}.
\]

Note that

\[
\mu[tr \Sigma D^2 A(W)] = \mu[W^T \mu(D^2 A(W)) W] = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \mu[(Y^n)^T \mu(D^2 A(W))] Y^{n,m},
\]

so what remains of \( \mu[tr \Sigma D^2 A(W) - W^T \nabla A(W)] \) after subtracting \( E_1 \) and \( E_2 \) is

\[
\sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \mu[(Y^n)^T \mu(D^2 A(W))] Y^{n,m} - (Y^n)^T D^2 A(W^{n,m}) Y^{n,m}
\]

\[
= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \left( \mu[(Y^n)^T \mu(D^2 A(W) - D^2 A(W^{n,m})) Y^{n,m}] - \mu[(Y^n)^T D^2 A(W^{n,m}) Y^{n,m}] \right),
\]

where

\[
\overline{D^2 A(W^{n,m})} = D^2 A(W^{n,m}) - \mu[D^2 A(W^{n,m})].
\]

Next note that

\[
D^2 A(W^{n,m}) - D^2 A(0) = D^2 A(W^{n,m}) - D^2 A(W^{n,N-1}) = \sum_{k=m+1}^{N-1} \delta^{n,k}.
\]
Since $\mu[(Y^n)^T D^2 A(0) Y^n] = 0$, this yields

$$
\sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \mu[(Y^n)^T D^2 A(W^{n,m}) Y^n] = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \sum_{k=m+1}^{N-1} \mu[(Y^n)^T \delta_{n,k} Y^n]
$$

$$
= \sum_{n=0}^{N-1} \sum_{m=0}^{2m} \mu[(Y^n)^T \delta_{n,k} Y^n] + \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \sum_{k=2m+1}^{N-1} \mu[(Y^n)^T \delta_{n,k} Y^n]
$$

$$
+ \sum_{n=0}^{N-1} \sum_{k=1}^{N-1} \mu[(Y^n)^T \delta_{n,k} Y^n] = -E_3 - E_4 - E_5.
$$

Finally, since

$$
D^2 A(W) - D^2 A(W^{n,m}) = D^2 A(W^{n+1}) - D^2 A(W^{n,m}) = \sum_{k=0}^{m} \delta_{n,k},
$$

we have

$$
\sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \mu[(Y^n)^T \mu(D^2 A(W) - D^2 A(W^{n,m})) Y^n]
$$

$$
= \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \sum_{k=0}^{m} \mu(\delta_{n,k}) \mu(Y^n) + \sum_{n=0}^{N-1} \mu[(Y^n)^T \mu(\delta_{n,0}) Y^n] = E_6 + E_7.
$$

This completes the proof for Proposition 5.3. \hfill \Box

5.3 Proof for Theorem 2.1

We gather in the following lemma some useful basic inequalities involving the spectral norm.

**Lemma 5.4** For all $A, B \in \mathbb{R}^{d \times d}$, $x \in \mathbb{R}^d$, and $\alpha, \beta \in \{1, \ldots, d\}$:

(i) $\|Ax\| \leq \|A\|_s \|x\|$;

(ii) $\|AB\|_s \leq \|A\|_s \|B\|_s$;

(iii) $|A_{\alpha\beta}| \leq \|A\|_s \leq \left(\max_{1 \leq j \leq d} \sum_{i=1}^{d} |A_{ij}| \right)^{1/2} \left(\max_{1 \leq i \leq d} \sum_{j=1}^{d} |A_{ij}| \right)^{1/2}$;

(iv) $|trA| \leq d \|A\|_s$;

(v) $\|A\|_s = \sqrt{\lambda_{\max}(A^T A)} \leq \sqrt{tr A^T A}$, where $\lambda_{\max}(A^T A)$ denotes the largest eigenvalue of the positive-semidefinite matrix $A^T A$.

**Lemma 5.5** Let $h : \mathbb{R}^d \to \mathbb{R}$ be three times differentiable with $\|D^k h\|_\infty < \infty$ for $1 \leq k \leq 3$ and let $A$ be the function (33) that solves Stein’s equation. Define $\delta^{n,k}(u)$ as in Proposition 5.3. Then, conditions (A2) and (A3) imply that, for all $0 \leq n, m \leq N - 1$, the following two conditions hold:

(A2′) Whenever $u \in [0, 1]$ and $m \leq k \leq N - 1$,

$$
|\mu[(Y^n)^T \delta_{n,k}(u) Y^n]| \leq C_2 5M d^2 \|D^3 h\|_\infty \|b^{-1}\|_s^3 \rho(m).
$$

(A3′) Whenever $2m \leq k \leq N - 1$,

$$
|\mu[(Y^n)^T \delta_{n,k} Y^n]| \leq C_3 5M d^2 \|D^3 h\|_\infty \|b^{-1}\|_s^3 \rho(k - m).
$$

Springer
Proof We denote
\[ G_h(x, y) = G_h(x, y; s, t, z) = b^{-1} \left[ D^2 h( sb^{-1}(x + ty) + z) - D^2 h( sb^{-1}x + z) \right] b^{-1}, \]
where \((s, t, z) \in [0, 1]^2 \times \mathbb{R}^d\). Then,
\[
| (bG_h)_{\alpha, \beta}(x, y) | \leq \sup_{\xi} \| \nabla \partial_{\alpha, \beta} h(\xi) \| \| b^{-1} \| y \| \leq \sqrt{d} \| D^3 h \|_{\infty} \| b^{-1} \|_s \| y \|,
\]
which together with Lemma 5.4 implies
\[
\| (bG_h)_{\alpha, \beta}(x, y) \|_s \leq d^2 \| D^3 h \|_{\infty} \| b^{-1} \|_s \| y \|.
\] (37)
Hence,
\[
\| G_h(x, y) \|_s \leq d^2 \| D^3 h \|_{\infty} \| b^{-1} \|_s^3 \| y \|.
\] (38)
Similarly we see that, for all \(1 \leq i \leq 2d\),
\[
\| \partial_i G_h(x, y) \|_s \leq 2d^2 \| D^3 h \|_{\infty} \| b^{-1} \|_s^3.
\] (39)

For (A1’) Suppose that \(m \leq k \leq N - 1\). Recall from Lemma 5.1 that
\[
A(w) = - \int_0^\infty \left\{ \int_{\mathbb{R}^d} h(e^{-w} + \sqrt{1 - e^{-2w}} z) \phi(\zeta) \frac{dz}{\Phi(\zeta)} \right\} d\zeta
\]
solves the Stein equation (2). Since \(h\) is three times differentiable with \(\| D^k h \|_{\infty} < \infty\) for \(1 \leq k \leq 3\), we can use dominated convergence to compute
\[
D^2 A(w) = - \int_0^\infty e^{-2w} \int_{\mathbb{R}^d} D^2 h(e^{-w} + \sqrt{1 - e^{-2w}} z) \phi(\zeta) \frac{dz}{\Phi(\zeta)} d\zeta.
\]
Recall that, for a function \(F : \mathbb{R}^d \times B_d(0, 4M + 1) \rightarrow \mathbb{R}^{d \times d}\), we denote
\[
\| F \|_{\infty} = \sup \{ \| F(x, y) \|_s : (x, y) \in \mathbb{R}^d \times B_d(0, 4M + 1) \}
\]
and
\[
\| \nabla F \|_{\infty} = \max_{1 \leq i \leq 2d} \sup \{ \| \partial_i F(x, y) \|_s : (x, y) \in \mathbb{R}^d \times B_d(0, 4M + 1) \}.
\]
By Fubini’s theorem,
\[
\mu((Y^n)^T \delta^{n,k}(u) Y^{n,m})
\]
\[
= - \int_0^\infty e^{-2w} \int_{\mathbb{R}^d} \mu((Y^n)^T (D^2 h(e^{-w}(W^{n,k} + u Y^{n,m}) + \sqrt{1 - e^{-2w}} z)) Y^{n,m}) \phi(\zeta) \frac{dz}{\Phi(\zeta)} d\zeta
\]
\[
= - \int_0^\infty e^{-2w} \int_{\mathbb{R}^d} \mu \left( \left( \sum_{|i-n|>m} \bar{f}^i, \bar{f}^{n,m} ; e^{-w}, u, \sqrt{1 - e^{-2w}} z \right) \right) \phi(\zeta) \frac{dz}{\Phi(\zeta)} d\zeta
\]
so that an application of condition (A2) combined with (38) and (39) yields
\[
| \mu((Y^n)^T \delta^{n,k}(u) Y^{n,m}) |
\]
\[
\leq C_2 (\| G_h \|_{\infty} + \| \nabla G_h \|_{\infty}) \rho(m) \int_0^\infty e^{-2w} \int_{\mathbb{R}^d} \phi(\zeta) \frac{dz}{\Phi(\zeta)} d\zeta
\]
\[
= \frac{C_2}{2} (\|G_h\|_{\infty} + \|\nabla G_h\|_{\infty}) \rho(m)
\leq 2C_25Md^2\|D^3h\|_{\infty}\|b^{-1}\|_s^3 \rho(m),
\]
which proves condition (A2'). The proof for condition (A3') is essentially the same which is why we omit it.
\[\square\]

We now proceed to show Theorem 2.1. Combining Lemma 5.1 with Proposition 5.3 yields
\[
|\mu[h(W)] - \Phi_\Sigma(h)| = |\mu[\text{tr} D^2 A(W) - W^T \nabla A(W)]| \leq \sum_{i=1}^7 |E_i|,
\]
where \(A\) is given by (33) and \(E_i\) are as in Proposition 5.3. We bound each term \(E_i\) separately, using conditions (A1), (A2') and (A3').

By condition (A2'),
\[
|E_1| = \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \int_0^1 |\mu[(Y^n)^T \delta^{n,m}(u) Y^{n,m}]| \, du 
\leq \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} C_25Md^2\|D^3h\|_{\infty}\|b^{-1}\|_s^3 \rho(m).
\]

Moreover,
\[
|E_2| = \sum_{n=0}^{N-1} \int_0^1 |\mu[(Y^n)^T \delta^{n,0}(u) Y^n]\| d\mu = \sum_{n=0}^{N-1} \int_0^1 |\mu[(Y^n\| \|\delta^{n,0}(u)\|_s Y^n]\| |\, du 
\leq N\|b^{-1}\|_s^2 4M^2 \cdot d^2\|D^3h\|_{\infty}\|b^{-1}\|_s(4M + 1) 
\leq 20M^3 d^2\|D^3h\|_{\infty}N\|b^{-1}\|_s^3,
\]
where (37) was used in the third inequality.

For \(E_3\) first note that
\[
\mu[(Y^n)^T \mu(\delta^{n,k}) Y^{n,m}] = \text{tr} \mu(Y^n \otimes Y^{n,m}) \mu(\delta^{n,k}),
\]
so that Lemma 5.4 and condition (A1) can be used to obtain
\[
|\mu[(Y^n)^T \mu(\delta^{n,k}) Y^{n,m}]| \leq d\|\mu(Y^n \otimes Y^{n,m})\|_s \|\mu(\delta^{n,k})\|_s 
\leq d\|b^{-1}\|_s^2 \|\mu(\hat{f}^n \otimes \hat{f}^{n,m})\|_s \|\mu(\delta^{n,k})\|_s 
\leq d^2\|b^{-1}\|_s^2 C_1 \rho(m) \cdot d^2\|D^3h\|_{\infty}\|b^{-1}\|_s(4M + 1) 
\leq C_15Md^4\|D^3h\|_{\infty}\|b^{-1}\|_s^3 \rho(m).
\]

Combining (40) with an application of condition (A2') yields
\[
|E_3| = \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \sum_{k=m+1}^{2m} \mu[(Y^n)^T \delta^{n,k} Y^{n,m}] 
\leq \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \sum_{k=m+1}^{2m} ((\mu[(Y^n)^T \delta^{n,k} Y^{n,m}] + \mu[(Y^n)^T \mu(\delta^{n,k}) Y^{n,m}])).
\[
\leq (C_2d^2 + C_1d^4) 5M \|D^3h\|_{\infty} \|b^{-1}\|^3 N \sum_{m=1}^{N-1} m \rho(m).
\]

Condition (A3') is used to bound \(E_4\) and \(E_5\):

\[
|E_4| = \left| \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \sum_{k=2m+1}^{N-1} |\mu(\left( Y^n \right)^T \delta^{n,k} Y^n) m| \right|
\leq C_35Md^2 \|D^3h\|_{\infty} N \|b^{-1}\|^3 \sum_{m=1}^{N-1} m \rho(m),
\]

and

\[
|E_5| = \left| \sum_{n=0}^{N-1} \sum_{k=1}^{N-1} |\mu(\left( Y^n \right)^T \delta^{n,k} Y^n) m| \right|
\leq C_35Md^2 \|D^3h\|_{\infty} N \|b^{-1}\|^3 \sum_{m=1}^{N-1} m \rho(m).
\]

Again by (40),

\[
|E_6| = \left| \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \sum_{k=0}^{m} |\mu(\left( Y^n \right)^T \delta^{n,k} Y^n) m| \right|
\leq C_15Md^4 \|D^3h\|_{\infty} N \|b^{-1}\|^3 \sum_{m=1}^{N-1} m \rho(m).
\]

Finally,

\[
|E_7| = \left| \sum_{n=0}^{N-1} \mu(\left( Y^n \right)^T \delta^{n,0} Y^n) m \right|
\leq \sum_{n=0}^{N-1} \mu(\|Y^n\|^2) \|\mu(\delta^{n,0})\|_s
\leq 20M^3d^2 \|D^3h\|_{\infty} N \|b^{-1}\|^3 \sum_{m=1}^{N-1} m \rho(m).
\]

Gathering the foregoing upper bounds we obtain

\[
|\mu[h(W)] - \Phi \Sigma (h)|
\leq N \|b^{-1}\|^3 \sum_{m=1}^{N-1} m \rho(m) \|D^3h\|_{\infty} \left( C_25Md^2 + 20M^3d^2 + (C_2d^2 + C_1d^4) 5M
\right.
\]
\[
+ 2C_35Md^2 + C_15Md^4 + 20M^3d^2)
\]
\[
\leq N \|b^{-1}\|^3 \sum_{m=1}^{N-1} m \rho(m) \|D^3h\|_{\infty} M^3d^4 10(C_1 + C_2 + C_3 + 4).
\]

The proof for Theorem 2.1 is complete.

### 5.4 Proof for Theorem 2.3

Since the proof for Theorem 2.3 is very similar to the proof for Theorem 2.1, we omit most of the details and only give an outline, emphasizing differences between the two proofs.

Now \(b^2 = \text{Var}_\mu (\sum_{i < N} f_i) > 0\) so that \(\text{Var}_\mu (W) = \mu(W^2) = 1\). Then the univariate Stein equation is defined by

\[
A'(w) - w A(w) = h(w) - \Phi_1(h),
\]
where \( w \in \mathbb{R} \). Note that the order of (41) is one smaller than the order of the multivariate Stein equation (2). We have the following result regarding the regularity of \( A \):

**Lemma 5.6** (See [7])

Whenever \( h : \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous with \( \text{Lip}(h) \leq 1 \) the solution \( A : \mathbb{R} \to \mathbb{R} \) to (41) belongs to the class \( \mathcal{F}_1 \) consisting of all differentiable functions with an absolutely continuous derivative, satisfying the bounds

\[
\|A\|_{\infty} \leq 2, \quad \|A'\|_{\infty} \leq \sqrt{2/\pi}, \quad \text{and} \quad \|A''\|_{\infty} \leq 2.
\]

The lemma implies that

\[
d_{\mathcal{F}}(W, Z) \leq \sup_{A \in \mathcal{F}_1} |\mu[A'(W) - WA(W)]|,
\]

where \( Z \sim \mathcal{N}(0, 1) \). Next \( \mu[A'(W) - WA(W)] \) is decomposed precisely as in Proposition 4 of [56]. The decomposition is the same as that given in Proposition 5.3 except that \( \delta^{n,m}(u) \) there is replaced with

\[
\delta^{n,m}(u) = A'(W^{n,m} + u Y^{n,m}) - A'(W^{n,m}).
\]

Then

\[
\delta_{n,m}(u) = G_u \left( \sum_{|i-n|>m} \bar{f}^i, \bar{f}^{n,m} \right),
\]

where

\[
G_u(x, y) = A' \left( b^{-1}x + b^{-1}uy \right) - A' \left( b^{-1}x \right).
\]

By Lemma 5.6

\[
|G_u(x, y)| \leq \text{Lip}(A')|b^{-1}uy| \leq 2b^{-1}|y|
\]

and

\[
\text{Lip}(G_u) \leq 4b^{-1}.
\]

Hence, conditions (B2) and (B3) can be applied with \( G_u \upharpoonright (\mathbb{R} \times B_1(0, 4M + 1)) \) as in the proof for Theorem 2.1. Using also condition (B1) we obtain bounds to each of the terms \( E_i \) appearing in the univariate version of Proposition 5.3, which then lead to the upper bound (5).

### 5.5 Proof for Theorem 2.6

From now on we assume that \( \text{Cov}_\mu \left( \sum_{i=0}^{N-1} \bar{f}^i \right) \) is positive definite and take

\[
b = \left[ \text{Cov}_\mu \left( \sum_{i=0}^{N-1} \bar{f}^i \right) \right]^{1/2},
\]

in which case \( \Sigma = \mu(W \otimes W) = I_{d \times d} \). By Lemma 5.4,

\[
\|b^{-1}\|_2^2 = \lambda_{\text{max}} \left( \left[ \text{Cov}_\mu \left( \sum_{i=0}^{N-1} \bar{f}^i \right) \right]^{-1} \right) = \lambda_{\text{min}}^{-1},
\]
where we recall that $\lambda_{\min}$ is the least eigenvalue of $\text{Cov}_\mu(\sum_{i=0}^{N-1} \tilde{f}^i)$.

By Lemma 5.2,

$$d_{\mathcal{W}}(W, Z) \leq \sup_{A \in A} |\mu[\text{tr} D^2 A(W) - W^T \nabla A(W)]|,$$

where $Z \sim \mathcal{N}(0, I_{d \times d})$ and $A$ denotes the class of all $C^2$ functions satisfying (34). The proof then proceeds as follows. First we decompose $\mu[\text{tr} D^2 A(W) - W^T \nabla A(W)] = \sum_{i=1}^{s} E_i$ using Proposition 5.3, which reduces the proof to bounding each term $E_i$ for functions $A \in \mathcal{A}$. For example, to obtain an upper bound on $E_1$ we have to control the integral

$$\int_{0}^{1} \mu[(Y^n)^T \delta^{n,m}(u)Y^{n,m}] du,$$

where we recall that $\delta^{n,m}(u) = D^2 A(W^{n,m} + u Y^{n,m}) - D^2 A(W^{n,m})$. To this end we will describe a class $\mathcal{G}$ of regular functions $G : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ such that

$$\int_{0}^{1} \mu[(Y^n)^T \delta^{n,m}(u)Y^{n,m}] du = \mu \left[ (\tilde{f}^n)^T G \left( \sum_{|i-n| > k} \tilde{f}^i, \tilde{f}^n,k \right) \tilde{f}^{n,m} \right]. \quad (42)$$

The integral on the right is bounded by condition (C2), provided that $G$ is a $C^1$-function. This might not be the case, since functions in $\mathcal{G}$ will have the same regularity as the second derivatives of functions in $\mathcal{A}$, which according to Lemma 5.2 is Lipschitz up to a logarithmic factor. But we can approximate such functions by $C^\infty$-functions, which in combination with condition (C2) then leads to an upper bound on (42) and consequently on $E_1$. The other terms $E_i$ will be treated similarly. We now proceed to detail the foregoing argument.

We denote by $\mathcal{G}$ the collection of all functions $G : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ that satisfy the following upper bounds:

$$\sup_x \|G(x, y)\|_x \leq K \lambda_{\min}^{-\frac{3}{2}}(1 + \log N)(\|y\|^2 + 1) \quad \forall y \in \mathbb{R}^d;$$

$$\sup_y \|G(a, y) - G(a', y)\|_y$$

$$\leq K \lambda_{\min}^{-\frac{3}{2}}(1 + \log N)\|a - a'\|(1 + \|a - a'\|) \quad \forall a, a' \in \mathbb{R}^d;$$

$$\sup_x \|G(x, a) - G(x, a')\|_x$$

$$\leq K \lambda_{\min}^{-\frac{3}{2}}(1 + \log N)\|a - a'\|(1 + \|a - a'\|) \quad \forall a, a' \in \mathbb{R}^d.$$

where $K = 2C_\eta + \sqrt{d}4M + 2$ and $C_\eta$ is the constant from Lemma 5.2 with $\eta = 1$.

Lemma 5.7 Assume $\lambda_{\min} > 1$. Then, given any $A \in \mathcal{A}$ and $0 \leq n, m \leq N - 1$, there is a function $G_a : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ satisfying

$$G_a \left( \sum_{|i-n| > m} \tilde{f}^i, \tilde{f}^{n,m} \right) = b^{-1} \delta^{n,m}(u)b^{-1}, \quad (43)$$

where $\delta^{n,m}(u)$ is defined as in Proposition 5.3, such that

$$G_1 \in \mathcal{G} \quad \text{and} \quad G = \int_{0}^{1} G_{\tilde{f}} du \in \mathcal{G}.$$
Proof It is easy to see that (43) holds with $G_u(x, y)$ defined as
\[
b^{-1} \left[ D^2 A \left( b^{-1} x + b^{-1} uy \right) - D^2 A \left( b^{-1} x \right) \right] b^{-1},
\]
We show that $G \in \mathcal{G}$ and leave the similar verification of $G_1 \in \mathcal{G}$ to the reader. Observe that, by Lemma 5.4 and (34),
\[
\|G_u(x, y)\|_s \leq \|b^{-1}\|_s^2 \|ub^{-1}y\| \left( C_\# + | \log \|ub^{-1}y\| \right) \tag{44}
\]
holds for all $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d \setminus \{0\}$, and $u \in (0, 1]$. Then assume (as we may) that $y \neq 0$. We use (44) and $\|b^{-1}\|_s = \lambda_{\min}^{-1/2} < 1$ to obtain
\[
\|G(x, y)\|_s \leq \int_0^1 \|G_u(x, y)\|_s \, du \leq \|b^{-1}\|_s^2 \int_0^1 \|ub^{-1}y\| \left( C_\# + | \log \|ub^{-1}y\| \right) \, du \\
\leq \|b^{-1}\|_s^3 \|y\| \left( C_\# + 1 + | \log \|b^{-1}y\| \right) \\
\leq \lambda_{\min}^{-3/2} (1 + \|y\|^2) \left( C_\# + 2 + \log \|b^{-1}\|_s^{-1} \right).
\]
Since $1 \leq \|b\|_s \|b^{-1}\|_s$,
\[
\|b^{-1}\|_s^{-1} \leq \|b\|_s \leq \left( \text{tr Cov}_\mu \left( \sum_{i=0}^{N-1} \bar{f}^i \right) \right)^{1/2} \leq \sqrt{d}2MN, \tag{45}
\]
where we used Lemma 5.4. Hence,
\[
\|G(x, y)\|_s \leq \lambda_{\min}^{-3/2} (1 + \|y\|^2)(\log N + 1)(C_\# + 2 + \sqrt{d}2M).
\]
Next let $a, a', y \in \mathbb{R}^d$. Then,
\[
\|G(a, y) - G(a', y)\|_s \leq \int_0^1 \|G_u(a, y) - G_u(a', y)\|_s \, du \\
\leq 2\|b^{-1}\|_s^3 \left( \|b^{-1}(a - a')\| \left( C_\# + \log \|b^{-1}(a - a')\| \right) \right) \\
\leq 2\|b^{-1}\|_s^3 \left( \|a - a'\| \left( C_\# + \log (\sqrt{d}2MN) \right) + \|a - a'\| \log \|a - a'\| \right) \\
\leq 2\lambda_{\min}^{-3/2} \|a - a'\| \left( 1 + \log \|a - a'\| \right) \left( C_\# + \sqrt{d}2M + \log N + 1 \right) \\
\leq \lambda_{\min}^{-3/2} \|a - a'\| \left( 1 + \log \|a - a'\| \right) \left( \log N + 1 \right) (2C_\# + \sqrt{d}4M + 2),
\]
where (44) was used in the second inequality, and (45) in the third inequality.
Finally, for all $a, a', x \in \mathbb{R}^d$,
\[
\|G(x, a) - G(x, a')\|_s \leq \int_0^1 \|G_u(x, a) - G_u(x, a')\|_s \, du \\
\leq \|b^{-1}\|_s^2 \int_0^1 \|ub^{-1}(a - a')\| \left( C_\# + \log \|ub^{-1}(a - a')\| \right) \, du \\
\leq \|b^{-1}\|_s^3 \|a - a'\| \left( C_\# + 1 + \log \|b^{-1}\|^{-1} \right) \left( 1 + \log \|a - a'\| \right) \\
\leq \|b^{-1}\|_s^3 \|a - a'\| \left( 1 + \log \|a - a'\| \right) \left( \log N + 1 \right) (2C_\# + \sqrt{d}2M + 1),
\]
where (45) was used in the second last inequality. This completes the proof for $G \in \mathcal{G}$. \qed
The following lemma is established by a standard approximation argument. See Appendix 1 for the proof.

**Lemma 5.8** Conditions (C2) and (C3) imply that, for all $0 \leq n, m \leq N - 1$, the following two conditions hold:

(C2') Whenever $m \leq k \leq N - 1$ and $G \in \mathcal{G}$,

$$\left| \mu \left[ (\hat{f}^n)^T G \left( \sum_{|i-n|>k} \hat{f}^i, \hat{f}^n, \hat{f}_m \right) \right] \right| \leq C_2' \lambda_{\min}^{-\frac{3}{2}} (1 + \log N) (1 + \log(\rho(m)^{-1})) \rho(m),$$

where

$$C_2' = C_2 d^4 K 673 M^2 \left( 1 + \frac{1}{\rho(0)} \right) (2 \rho(0) + 1).$$

(C3') Whenever $2m \leq k \leq N - 1$ and $G \in \mathcal{G}$,

$$\left| \mu \left[ (\hat{f}^n)^T G \left( \sum_{|i-n|>k} \hat{f}^i, \hat{f}^n, \hat{f}_m \right) \right] \right| \leq C_3' \lambda_{\min}^{-\frac{3}{2}} (1 + \log N) (1 + \log(\rho(k-m)^{-1})) \rho(k-m),$$

where

$$C_3' = C_3 d^4 K 673 M^2 \left( 1 + \frac{1}{\rho(0)} \right) (2 \rho(0) + 1).$$

We proceed to bound the terms $E_i$ in Proposition 5.3 using conditions (C1), (C2') and (C3'). Let $G_u$ be a function as in Lemma 5.7 and set $G = \int_0^1 G_u \, du$. Then for $E_1$ we have by condition (C2') the upper bound

$$|E_1| = \left| \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \mu \left[ (\hat{f}^n)^T G \left( \sum_{|i-n|>m} \hat{f}^i, \hat{f}^n, \hat{f}_m \right) \hat{f}_m \right] \right| \leq \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} C_2' \lambda_{\min}^{-\frac{3}{2}} (1 + \log N) (1 + \log(\rho(m)^{-1})) \rho(m) \leq N (1 + \log N) \lambda_{\min}^{-\frac{3}{2}} C_2' \sum_{m=1}^{N-1} (1 + \log(\rho(m)^{-1})) \rho(m).$$

Since $G \in \mathcal{G}$,

$$|E_2| = \left| \sum_{n=0}^{N-1} \mu \left[ (\hat{f}^n)^T G \left( \sum_{|i-n|>0} \hat{f}^i, \hat{f}^n \right) \hat{f}^n \right] \right| \leq N 4 M^2 K \lambda_{\min}^{-\frac{3}{2}} (1 + \log N)((2M)^2 + 1) \leq N (1 + \log N) \lambda_{\min}^{-\frac{3}{2}} K 20 M^4.$$
For $E_3$ we note that

$$
\mu[(Y^n)^T \mu(\delta^{n,k}) Y^{n,m}] = \text{tr} \mu(\tilde{f}^n \otimes \tilde{f}^{n,m}) \mu \left[ G_1 \left( \sum_{|i-n|>k} \tilde{f}^i, \tilde{f}^{n,k} \right) \right].
$$

Hence, by Lemma 5.4 and condition (C1),

$$
|\mu[(Y^n)^T \mu(\delta^{n,k}) Y^{n,m}]| \
\leq d \|\mu(\tilde{f}^n \otimes \tilde{f}^{n,m})\|_s \|G_1\|_\infty \
\leq d^2 C_1 \rho(m) K \lambda_{\min}^{-\frac{3}{2}} (1 + \log N)((4M + 1)^2 + 1).
$$

Combining (46) with condition (C2') implies the upper bound

$$
|E_3| \leq \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \left| \mu[(Y^n)^T \delta^{n,k} Y^{n,m}] \right| \
\leq \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \sum_{k=m+1}^{2m} \left| \mu[(Y^n)^T \delta^{n,k} Y^{n,m}] \right| \
\leq \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \sum_{k=m+1}^{2m} \left| \mu[(Y^n)^T \delta^{n,k} Y^{n,m}] \right| \
+ \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \sum_{k=m+1}^{2m} d^2 C_1 \rho(m) K \lambda_{\min}^{-\frac{3}{2}} (1 + \log N)((4M + 1)^2 + 1) \
\leq N (1 + \log N) \lambda_{\min}^{-\frac{3}{2}} \left( \sum_{m=1}^{N-1} C_2' (1 + \log(\rho(m)^{-1})) m \rho(m) \right) \
+ \sum_{m=1}^{N-1} d^2 C_1 m \rho(m) K (\|4M + 1\|^2 + 1) \
\leq N (1 + \log N) \lambda_{\min}^{-\frac{3}{2}} \left( 2C_2' + d^2 C_1 K 52M^2 \right) \sum_{m=1}^{N-1} (1 + \log(\rho(m)^{-1})) m \rho(m).
$$

Next condition (C3') is used to bound $E_4$ and $E_5$:

$$
|E_4| = \left| \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \sum_{k=2m+1}^{N-1} \mu[(\tilde{f}^n)^T G_1 \left( \sum_{|i-n|>k} \tilde{f}^i, \tilde{f}^{n,k} \right) \tilde{f}^{n,m}] \right| \
\leq N \sum_{m=1}^{N-1} \sum_{k=2m+1}^{N-1} C_3' \lambda_{\min}^{-\frac{3}{2}} (1 + \log N)(1 + \log(\rho(k-m)^{-1})) \rho(k-m) \
\leq N (1 + \log N) \lambda_{\min}^{-\frac{3}{2}} C_3' \sum_{m=1}^{N-1} (1 + \log(\rho(m)^{-1})) m \rho(m).
$$
and

\[
|E_5| = \left| \sum_{n=0}^{N-1} \sum_{k=1}^{N-1} \mu \left( (\bar{f}_n)^T G_1 \left( \sum_{|i-n|>k} \bar{f}_i, \bar{f}_n \right) \right) \bar{f}_n \right| \\
\leq N (1 + \log N) \lambda_{\min}^{-3} C'_3 \sum_{m=1}^{N-1} (1 + \log(\rho(m)^{-1})) \rho(m).
\]

Again by (46),

\[
|E_6| = \left| \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \sum_{k=0}^{m} \mu \left( (Y^n)^T \mu(\delta^{n,k}) Y^n, m \right) \right| \\
\leq \sum_{n=0}^{N-1} \sum_{m=1}^{N-1} \sum_{k=0}^{m} d^2 C_1 \rho(m) K \lambda_{\min}^{-3} (1 + \log N)((4M + 1)^2 + 1).
\]

\[
= N (1 + \log N) \lambda_{\min}^{-3} d^2 C_1 K 26 M^2 \sum_{k=1}^{N-1} m \rho(m).
\]

Finally,

\[
|E_7| = \left| \sum_{n=0}^{N-1} \mu \left( (\bar{f}_n)^T \mu \left( G_1 \left( \sum_{|i-n|>0} \bar{f}_i, \bar{f}_n \right) \right) \right) \bar{f}_n \right| \\
\leq N 4 M^2 \|G_1\|_\infty \leq N (1 + \log N) \lambda_{\min}^{-3} 104 M^4 K.
\]

Recall that, by Lemma 5.2,

\[d_W(W, Z) \leq \sup_{A \in A} |\mu[tr D^2 A(W) - W^T \nabla A(W)]|.
\]

Hence, Proposition 5.3 together with the above bounds implies

\[
d_W(W, Z) \leq N (1 + \log N) \lambda_{\min}^{-3} \left[ 3 C'_2 + 2 C'_3 + d^2 C_1 K 52 M^2 \right] \sum_{m=1}^{N-1} (1 + \log(\rho(m)^{-1})) m \rho(m)
\]

\[
+ K 20 M^4 + d^2 C_1 K 26 M^2 \sum_{k=1}^{N-1} m \rho(m) + 104 M^4 K
\]

\[
\leq N (1 + \log N) \lambda_{\min}^{-3} \sum_{m=1}^{N-1} (1 + \log(\rho(m)^{-1})) m \rho(m) \left[ 3 C'_2 + 2 C'_3 
\right.
\]

\[
+ d^2 C_1 K 78 M^2 + 124 M^4 K.
\]

The proof for Theorem 2.6 is complete.

**Acknowledgements**  
JL was supported by DOMAST (University of Helsinki) and by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 787304). He would like to thank Viviane Baladi for helpful discussions. MS was supported by Emil Aaltosen Säätiö, and the Jane and Aatos Erkko Foundation.
Appendix A: Proof for Lemma 5.8

Let us define the mollifier $\eta : \mathbb{R}^d \to \mathbb{R}$ by $\eta(x) = c\varphi(1 - \|x\|^2)$ where

$$\varphi(t) = \begin{cases} e^{-1/t^2} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases}$$

and $c > 0$ is such that $\int_{\mathbb{R}^d} \eta(x) \, dx = 1$. Then

$$c^{-1} = \int_{\mathbb{R}^d} \varphi(1 - \|x\|^2) \, dx \geq \int_{B_d(0, 1/2)} \varphi(1 - \|x\|^2) \, dx \geq \varphi(\frac{3}{4}) m(B_d(0, \frac{1}{2}))$$

$$\geq e^{-2}(\frac{1}{2})^d m(B_d(0, 1)).$$

(47)

Let $G \in \mathcal{G}$. We approximate the components of $G$ by convolutions $G_{\alpha,\beta}^\varepsilon : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$,

$$G_{\alpha,\beta}^\varepsilon(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} G_{\alpha,\beta}(y) j_\varepsilon(x - y) \, dy = \int_{\mathbb{R}^d \times \mathbb{R}^d} G_{\alpha,\beta}(x - \varepsilon y) j(y) \, dy,$$

where $x = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$, $j_{\varepsilon}(x) = \varepsilon^{-2d} j(x/\varepsilon)$, and $j(x) = \eta(x_1) \eta(x_2)$.

For all $\alpha, \beta \in \{1, \ldots, d\}$, $x = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$, and $\varepsilon \in (0, 1)$:

$$|G_{\alpha,\beta}^\varepsilon(x) - G_{\alpha,\beta}(x)| \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |G_{\alpha,\beta}(x - \varepsilon y) - G_{\alpha,\beta}(x)| j(y) \, dy$$

$$\leq \int_{B_d(0, 1) \times B_d(0, 1)} \|G(x - \varepsilon y) - G(x)\|_s j(y) \, dy$$

$$\leq K\lambda_{\min}^{-\frac{3}{2}} (1 + \log N) \int_{B_d(0, 1) \times B_d(0, 1)} \left[\varepsilon \|y_1\| (1 + \log \varepsilon \|y_1\|^{-1}) + \|y_2\| (1 + \log \varepsilon \|y_2\|^{-1})\right] j(y) \, dy$$

$$\leq K\lambda_{\min}^{-\frac{3}{2}} (1 + \log N) 6\varepsilon \log \varepsilon^{-1}.$$

Lemma 5.4 was used in the second inequality and $G \in \mathcal{G}$ in the third inequality. It follows by Lemma 5.4 that

$$\|G^\varepsilon(x) - G(x)\|_s \leq dK\lambda_{\min}^{-\frac{3}{2}} (1 + \log N) 6\varepsilon \log \varepsilon^{-1}. \quad (48)$$

Since $G \in \mathcal{G}$,

$$|G_{\alpha,\beta}^\varepsilon(x)| \leq \sup_{y_1 : \|y_1\| \leq \|x_1\| + \varepsilon} |G_{\alpha,\beta}(y_1, y_2)| \leq \sup_{y_1 : \|y_1\| \leq \|x_1\| + \varepsilon} \|G(y_1, y_2)\|_s$$

$$\leq K\lambda_{\min}^{-\frac{3}{2}} (1 + \log N) ((\|x_2\| + \varepsilon)^2 + 1),$$

so that Lemma 5.4 implies

$$\|G^\varepsilon(x)\| \leq dK\lambda_{\min}^{-\frac{3}{2}} (1 + \log N) ((\|x_2\| + \varepsilon)^2 + 1). \quad (49)$$

Since

$$G_{\alpha,\beta}^\varepsilon(x) - G_{\alpha,\beta}(y) = \int_{\mathbb{R}^d \times \mathbb{R}^d} G_{\alpha,\beta}(z) (j_\varepsilon(x - z) - j_\varepsilon(y - z)) \, dz$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d} (G_{\alpha,\beta}(z) - G_{\alpha,\beta}(x))(j_\varepsilon(x - z) - j_\varepsilon(y - z)) \, dz,$$
we have
\[
\partial_i G^e(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} (G_{\alpha,\beta}(z) - G_{\alpha,\beta}(x))(\partial_i j(x))(x - z) \, dz
\]
\[
= \int_{B_d(x,\varepsilon) \times B_d(x,\varepsilon)} \|G(z) - G(x)\| \varepsilon^{-2d-1} \partial_i j(x) \, dz.
\]
An easy computation shows that \(|\partial_i j(x)| \leq 12c^2\). Using this, \(G \in G\), and (47) we obtain for all \(\varepsilon \in (0, 1)\) the upper bound
\[
|\partial_i G^e(x)| \leq \int_{B_d(x,\varepsilon) \times B_d(x,\varepsilon)} \|G(z) - G(x)\| \varepsilon^{-2d-1} |\partial_i j(x)| \, dz
\]
\[
\leq \varepsilon^{-2d-1} \left(2K\lambda_{\min}^{-\frac{3}{2}}(1 + \log N)\varepsilon(1 + \log \varepsilon^{-1}) \cdot 12c^2 \cdot m(B_d(x, \varepsilon))^2\right)
\]
\[
= 2K\lambda_{\min}^{-\frac{3}{2}}(1 + \log N)(1 + \log \varepsilon^{-1}) \cdot 12c^2 \cdot m(B_d(0, 1))^2
\]
\[
\leq \log(\varepsilon^{-1})48K\lambda_{\min}^{-\frac{3}{2}} \cdot 12 \cdot 4^d = \log(\varepsilon^{-1})576K\lambda_{\min}^{-\frac{3}{2}} \cdot 4^d.
\]
Hence, by Lemma 5.4,
\[
\|\partial_i G^e(x)\| \leq \log(\varepsilon^{-1})576K\lambda_{\min}^{-\frac{3}{2}} d4^d, \quad 1 \leq i \leq 2d, \quad (50)
\]
where \(\partial_i G^e(x) = [\partial_i G^e_{\alpha,\beta}(x)]_{\alpha,\beta}\).

We combine (48)–(50) with condition (C2) to obtain

\[
\left| \mu \left[ (\tilde{f}^n)^T G \left( \sum_{|i-n|>k} \tilde{f}^i, \tilde{f}^n, k \right) \tilde{f}^{n,m} \right] \right|
\]
\[
\leq \mu \left[ (\tilde{f}^n)^T G \left( \sum_{|i-n|>k} \tilde{f}^i, \tilde{f}^n, k \right) \tilde{f}^{n,m} \right] + 8M^2 \|G^e - G\|_\infty
\]
\[
\leq C_2 \left( \|G^e\|_\infty + \|\nabla G^e\|_\infty \right) \rho(m)
\]
\[
+ 8M^2 d K\lambda_{\min}^{-\frac{3}{2}}(1 + \log N)6\varepsilon \log \varepsilon^{-1}
\]
\[
\leq C_2 d K\lambda_{\min}^{-\frac{3}{2}}\left[ (1 + \log N)(4M + 1 + \varepsilon)^2 + 1 \right] + \log(\varepsilon^{-1})576 \cdot 4^d \rho(m)
\]
\[
+ 8M^2(1 + \log N)6\varepsilon \log \varepsilon^{-1}
\]
\[
\leq C_2 d K\lambda_{\min}^{-\frac{3}{2}}(1 + \log N)(97M^2 + 576 \cdot 4^d) \log(\varepsilon^{-1})(\rho(m) + \varepsilon).
\]
Choosing \(\varepsilon = \frac{1}{2} \rho(m) < 1\) implies condition (C2'). The proof for condition (C3') is omitted as it is almost verbatim the same.

**References**

1. Aimino, R., Rousseau, J.: Concentration inequalities for sequential dynamical systems of the unit interval. Ergodic Theory Dyn. Syst. 36(8), 2384–2407 (2016). https://doi.org/10.1017/etds.2015.19
2. Aimino, R., Huyi, H., Nicol, M., Török, A., Vaienti, S.: Polynomial loss of memory for maps of the interval with a neutral fixed point. Discrete Contin. Dyn. Syst. 35(3), 793–806 (2015). https://doi.org/10.3934/dcds.2015.35.793

3. Bakhtin, V.I.: Random processes generated by a hyperbolic sequence of mappings. I. Izv. Ross. Akad. Nauk. Ser. Mat. 58(2), 40–72 (1994). https://doi.org/10.1070/IM1995v044n02ABEH001596

4. Bakhtin, V.I.: Random processes generated by a hyperbolic sequence of mappings II. Izv. Ross. Akad. Nauk Ser. Mat. 58(3), 184–195 (1994). https://doi.org/10.1070/IM1995v044n03ABEH001616

5. Barbour, A.: Stein’s method for diffusion approximations. Probab. Theory Related Fields 84(3), 297–322 (1990). https://doi.org/10.1007/BF01197887

6. Castro, A., Rodrigues, F.B., Varandas, P.: Stability and limit theorems for sequences of uniformly hyperbolic dynamics. (2017). Preprint. arXiv:1709.01652

7. Chen, L.H.Y., Goldstein, L., Shao, Q.-M.: Normal approximation by Stein’s method. Probability and its Applications (New York). Springer, Heidelberg (2011). https://doi.org/10.1007/978-3-642-15007-4

8. Conze, J.-P., Raugi, A.: Limit theorems for sequential expanding dynamical systems on [0, 1]. In: Ergodic theory and related fields, vol. 430. Contemp. Math., pp. 89–121. American Mathematical Society, Providence (2007)

9. Dedecker, J., Rio, E.: On mean central limit theorems for stationary sequences. Ann. Inst. Henri Poincaré Probab. Stat. 44(4), 693–726 (2008). https://doi.org/10.1214/07-AIHP117

10. Denker, M., Gordin, M., Sharova, A.: A Poisson limit theorem for toral automorphisms. Ill. J. Math. 48(1), 1–20 (2004). http://projecteuclid.org/euclid.ijm/1258136170

11. Dobbs, N., Stenlund, M.: Quasistatic dynamical systems. Ergodic Theory Dyn. Syst. 37(8), 2556–2596 (2017). https://doi.org/10.1017/etds.2016.9

12. Dragičević, D., Froyland, G., González-Tokman, C., Vaienti, S.: Almost sure invariance principle for random piecewise expanding maps. Nonlinearity 31(5), 2252–2280 (2018). https://doi.org/10.1088/1361-6544/aaaf4b

13. Dubois, L.: An explicit Berry-Esseen bound for uniformly expanding maps on the interval. Israel J. Math. 186, 221–250 (2011). https://doi.org/10.1007/s11856-011-0137-y

14. Fernando, K., Liverani, C.: Edgeworth expansions for weakly dependent random variables. (2018). Preprint. arXiv:1803.07667

15. Freitas, A.C.M., Freitas, J.M., Vaienti, S.: Extreme value laws for sequences of intermittent maps. Proc. Am. Math. Soc. 146(5), 2103–2116 (2018). https://doi.org/10.1090/proc/13892

16. Gallouët, T., Mijoule, G., Swan, Y.: Regularity of solutions of the stein equation and rates in the multivariate central limit theorem. (2018). Preprint. arXiv:1805.01720

17. Gaunt, R.E.: Rates of convergence in normal approximation under moment conditions via new bounds on solutions of the Stein equation. J. Theoret. Probab. 29(1), 231–247 (2016). https://doi.org/10.1007/s10959-014-0562-z

18. Goldstein, L., Rinott, Y.: Multivariate normal approximations by Stein’s method and size bias couplings. J. Appl. Probab. 33(1), 1–17 (1996). https://doi.org/10.2307/23325259

19. Gordin, M.: A homoclinic version of the central limit theorem. M. J. Math. Sci. 68(4), 451–458 (1994). https://doi.org/10.1007/BF01254269

20. Gordin, M., Denker, M.: The Poisson limit for automorphisms of two-dimensional tori driven by continued fractions. M. J. Math. Sci. 199(2), 139–149 (2014). https://doi.org/10.1007/s10958-014-1841-z

21. Götze, F.: On the rate of convergence in the multivariate CLT. Ann. Probab. 19(2), 724–739 (1991). https://doi.org/10.1214/aop/1176990448

22. Gouëzel, S.: Berry-Esseen theorem and local limit theorem for non uniformly expanding maps. Ann. Inst. H. Poincaré Probab. Statist. 41(6), 997–1024 (2005). https://doi.org/10.1016/j.anihpbb.2004.09.002

23. Hafouta, Y.: A sequential rpf theorem and its applications to limit theorems for time dependent dynamical systems and inhomogeneous Markov chains. (2019). Preprint. arXiv:1903.04018

24. Haydn, N.: Entry and return times distribution. Dyn. Syst. 28(3), 333–353 (2013). https://doi.org/10.1080/14689367.2013.822459

25. Haydn, N., Yang, F.: Entry times distribution for mixing systems. J. Stat. Phys. 163(2), 374–392 (2016). https://doi.org/10.1007/s10955-016-1487-y

26. Haydn, N., Nicol, M., Török, A., Vaienti, S.: Almost sure invariance principle for sequential and non-stationary dynamical systems. Trans. Am. Math. Soc. 369(8), 5293–5316 (2017). https://doi.org/10.1090/tran/6812

27. Heinrich, L.: Mixing properties and central limit theorem for a class of non-identical piecewise monotonic $C^2$-transformations. Math. Nachr. 181, 185–214 (1996). https://doi.org/10.1002/mana.3211810107

28. Hella, O.: Central limit theorems with a rate of convergence for sequences of transformations. (2018). Preprint. arXiv:1811.06062
29. Hella, O., Leppänen, J.: Central limit theorems with a rate of convergence for time-dependent intermittent maps. Stoch. Dyn. (2019). https://doi.org/10.1142/S0219493720500252
30. Hella, O., Stenlund, M.: Quenched normal approximation for random sequences of transformations. J. Stat. Phys. (2019). https://doi.org/10.1007/s10955-019-02390-5
31. Hella, O., Leppänen, J., Stenlund, M.: Stein’s method of normal approximation for dynamical systems. Stoch. Dyn. (2019). https://doi.org/10.1142/S0219493720500215
32. Jan, C.: Vitesse de convergence dans le TCL pour des chaînes de Markov et certains processus associés à des systèmes dynamiques. C. R. Acad. Sci. Paris Sér. I Math. 331(5), 395–398 (2000)
33. Kawan, C., Latushkin, Y.: Some results on the entropy of non-autonomous dynamical systems. Dyn. Syst. 31(3), 251–279 (2016). https://doi.org/10.1080/14689366.2015.1111299
34. Kawan, C., Leppänen, J., Stenlund, M.: A note on the finite-dimensional distributions of dispersing billiard processes. J. Stat. Phys. 168(1), 128–145 (2017). https://doi.org/10.1007/s10955-017-1790-2
35. King, J.L.: On M. Gordin’s homoclinic question. Intl. Math. Res. Notices 5, 203–212 (1997). https://doi.org/10.1155/S1073792897000147
36. Leppänen, J.: Functional correlation decay and multivariate normal approximation for non-uniformly expanding maps. Nonlinearity 30(11), 4239 (2017). http://stacks.iop.org/0951-7715/30/i=11/a=4239
37. Leppänen, J.: Continuous dependence of the normal approximation on the distribution of the random environment. Nonlinearity 30(5), 1671–1685 (2017). https://doi.org/10.1088/1361-6544/30/5/1671
38. Leppänen, J., Stenlund, M.: Quasistatic dynamics with intermittency. Math. Phys. Anal. Geom. 19(2), Art. 8, 23 (2016)
39. Leppänen, J., Stenlund, M.: A note on the finite-dimensional processes of dispersing billiard processes. J. Stat. Phys. 168(1), 128–145 (2017). https://doi.org/10.1007/s10955-017-1790-2
40. Liverani, C., Saussol, B., Varjú, T.: A probabilistic approach to intermittency. Ergodic Theory Dyn. Syst. 19(3), 671–685 (1999). https://doi.org/10.1017/S0143385799133856
41. Nándori, P., Szász, D., Varjú, T.: A central limit theorem for time-dependent dynamical systems. J. Stat. Phys. 146(6), 1213–1220 (2012). https://doi.org/10.1007/s10955-012-0451-8
42. Nicol, M., Pereira, F. P., Török, A.: Large deviations and central limit theorems for sequential and random systems of intermittent maps. (2019). To appear in Ergodic Theory and Dynamical Systems. arXiv:1909.07435
43. Nicol, M., Török, A., Vaienti, S.: Central limit theorems for sequential and random intermittent dynamical systems. Ergodic Theory Dyn. Syst. 38(3), 1127–1153 (2018). https://doi.org/10.1017/etds.2016.69
44. Pène, F.: Rates of convergence in the CLT for two-dimensional dispersive billiards. Commun. Math. Phys. 225(1), 91–119 (2002). https://doi.org/10.1007/s0022001000573
45. Pène, F.: Rate of convergence in the multidimensional central limit theorem for stationary processes. Application to the Knudsen gas and to the Sinai billiard. Ann. Appl. Probab. 15(4), 2331–2392 (2005). https://doi.org/10.1214/105051605000000476
46. Psiloyenis, Y.: Mixing conditions and return times on Markov Towers. ProQuest LLC, Ann Arbor (2008). Ph.D. thesis, University of Southern California. http://search.proquest.com/docview/304461750/
47. Raič, M.: A multivariate CLT for decomposable random vectors with finite second moments. J. Theoret. Probab. 17(3), 573–603 (2004). https://doi.org/10.1023/B:JOTP.0000040290.44087.68
48. Rio, E.: Sur le théorème de Berry-Esseen pour les suites faiblement dépendantes. Probab. Theory Related Fields 104(2), 255–282 (1996). https://doi.org/10.1007/BF01247840
49. Schmitt, B.A.: Perturbation bounds for matrix square roots and Pythagorean sums. Linear Algebra Appl. 174, 215–227 (1992). https://doi.org/10.1016/0024-3795(92)90052-C
50. Stein, C.: A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In: Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory, pages 583–602. Univ. California Press, Berkeley (1972). http://projecteuclid.org/euclid.bsmsp/1200514239
51. Stenlund, M.: An almost sure ergodic theorem for quasistatic dynamical systems. Math. Phys. Anal. Geom. 19(3), Art. 14, 18 (2016)
52. Stenlund, M.: A vector-valued almost sure invariance principle for Sinai billiards with random scatterers. Commun. Math. Phys. 325(3), 879–916 (2014). https://doi.org/10.1007/s00220-013-1870-3
53. Stenlund, M., Young, L.-S., Zhang, H.: Dispersing billiards with moving scatterers. Commun. Math. Phys. 322(3), 909–955 (2013)
54. Su, Y.: Vector-valued almost sure invariance principle for non-stationary dynamical systems. (2019). Preprint. arXiv:1903.09763
55. Sunklodas, J.: On normal approximation for strongly mixing random variables. Acta Appl. Math. 97(1–3), 251–260 (2007). https://doi.org/10.1007/s10440-007-9122-1
56. Tanzi, M., Pereira, T., van Strien, S.: Robustness of ergodic properties of nonautonomous piecewise expanding maps. (2016). Preprint. arXiv:1611.04016
57. Yaofeng, S.: Almost surely invariance principle for non-stationary and random intermittent dynamical systems. Discrete Contin. Dyn. Syst. 39(11), 6585–6597 (2019). https://doi.org/10.3934/dcds.2019286

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.