THE SPECTRUM OF CHARACTERS OF ULTRAFILTERS ON $\omega$

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Abstract. We show the consistency of statement: “the set of regular cardinals which are the character of some ultrafilter on $\omega$ is not convex”. We also deal with the set of $\pi$-characters of ultrafilters on $\omega$.

§0 Introduction

Some cardinal invariants of the continuum are actually the minimum of a natural set of cardinals $\leq 2^{\aleph_0}$ which can be called the spectrum of the invariant. Such a case is $\text{Sp}_\chi$, the set of characters $\chi(D)$ of non-principal ultrafilters on $\omega$ (the minimal number of generators). On the history see [BnSh 642]; there this spectrum and others were investigated and it was asked if $\text{Sp}_\chi$ can be non-convex (formally 0.2(2) below).

The main result is 1.1, it solves the problem (starting with a measurable). This was presented in a conference in honor of Juhasz, quite fitting as he had started

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the investigation of consistency on $\chi(D)$. In §2 we note what we can say on the strict $\pi$-character of ultrafilters.

The investigation is continued in [Sh:915] trying to get more “disorderly” behaviours in smaller cardinals and in particular answering negatively the original question, 0.2(2).

Recall

0.1 Definition. 1) $\text{Sp}_\chi = \text{Sp}(\chi)$ is the set of cardinals $\theta$ such that: $\theta = \chi(D)$ for some non-principal ultrafilter $D$ on $\omega$ where
2) For $D$ an ultrafilter on $\omega$ let $\theta = \chi(D)$ be the minimal cardinality $\theta$ such that $D$ is generated by some family of $\theta$ members, i.e. $\text{Min}\{|A| : A \subseteq D \text{ and } (\forall B \in D)(\exists A \in A)[A \subseteq^* B]|\}$, it does not matter if we use “$A \subseteq B$”.

Now, Brendle and Shelah [BnSh 642, Problem 5], asked the question formulated in 0.2(2) below, but it seems to me now that the question is really 0.2(1)+(3).

0.2 Problem 1) Can $\text{Sp}(\chi) \cap \text{Reg}$ have gaps, i.e., can it be that $\theta < \mu < \lambda$ are regular, $\theta \in \text{Sp}(\chi), \mu \notin \text{Sp}(\chi), \lambda \in \text{Sp}(\chi)$?
2) In particular does $\aleph_1, \aleph_3 \in \text{Sp}(\chi)$ imply $\aleph_2 \in \text{Sp}(\chi)$?
3) Are there any restrictions on $\text{Sp}(\chi) \cap \text{Reg}$?

We thank the referee for helpful comments and in particular 2.5(1).

Discussion: This rely on [Sh 700, §4], there is no point to repeat it but we try to give a description.

Let $S = \{\alpha < \lambda : \text{cf}(\alpha) \neq \kappa\}$ or any unbounded subset of it. We define ([Sh 700, 4.3]) the class $\mathcal{R} = \mathcal{R}_{\lambda,S}$ of objects $t$ approximating our final forcing. Each $t \in K$ consists mainly of a finite support iteration $\langle P^t_i, Q^t_i : i < \mu \rangle$ of c.c.c. forcing of cardinality $\leq \lambda$ with limit $P^t = P^t_\mu$, but also $\tau^t_i(i < \mu)$ of $Q^t_i$ satisfying a strong version of the c.c.c. and for $i \in S$, also $D^t_i$, a $P^t_i$-name of a non-principal ultrafilter on $\omega$ from which $Q^t_i$ is nicely defined and $A^t_i$, a $Q^t_i$-name (so $P^t_{i+1}$-name) of a pseudo-intersection (and $Q^t_i, i \in S$, nicely defined) of $D^t_i$ such that $i < j \in S \Rightarrow A^t_i \in D^t_j$.

So $\{A_i : i \in S\}$ witness $u \leq \mu$ in $V^{P^t_\mu}$; not necessarily we have to use non-nicely defined $Q^t_i$, though for $i \in S$ we do.

The order $\leq_{\mathcal{R}}$ is natural order, we prove the existence of the so-called canonical limit.

Now a major point of [Sh 700] is: for $s \in \mathcal{R}$ letting $\mathcal{D}$ be a uniform $\kappa$-complete ultrafilter on $\kappa$, (or just $\kappa_1$-complete $\mathcal{R}_0 < \theta < \kappa$), we can consider $t = s^\kappa/\mathcal{D}$; by Los theorem, more exactly by Hanf’s Ph.D. Thesis, (the parallel of) Los theorem
for $\mathbb{L}_{\kappa_1,\kappa_2}$ apply, it gives that $t \in \mathcal{R}$, well if $\lambda = \lambda^\kappa / \mathcal{D}$; and moreover $s \leq \mathcal{R} t$ under the canonical embedding.

The effect is that, e.g. being “a linear order having cofinality $\theta \neq \kappa$” is preserved, even by the same witness whereas having cardinality $\theta < \lambda$ is not and sets of cardinality $\geq \kappa$ are increased. As $\mathfrak{d}$ is the cofinality (not of a linear order but) of a partial order there are complications, anyhow as $\mathfrak{d}$ is defined by cofinality whereas $\mathfrak{a}$ by cardinality of sets this helps in [Sh 700], noting that as we deal with c.c.c. forcing, reals are represented by $\omega$-sequences of conditions, the relevant thing are preserved. So we use a $\leq_{\mathcal{R}}$-increasing sequence $\langle t_\alpha : \alpha \leq \lambda \rangle$ such that for unboundedly many $\alpha < \lambda$, $t_{\alpha+1}$ is essentially $\langle t_\alpha \rangle^\kappa / \mathcal{D}$.

What does “nice” $Q = Q(D)$, for $D$ a non-principal ultrafilter over $\omega$ mean? We need that

(α) $Q$ satisfies a strong version of the c.c.c.

(β) the definition commute with the ultra-power used

(γ) if $P$ is a forcing notion then we can extend $D$ to an ultrafilter $D^+$ for every (or at least some) $P$-name of an ultrafilter $D$ extending $D$ we have $Q(D) \leq P \ast Q(D^+)$ (used for the existence of canonical limit).

Such a forcing is combining Laver forcing and Mathias forcing for an ultrafilter $D$ on $\omega$, that is: if $p \in D$ iff $p$ is a subtree of $\omega$ with trunk $\text{tr}(p) \in p$ such that for $\eta \in p$ we have $\ell g(\eta) < \ell g(\text{tr}(p)) \to (\exists! n)(\eta^\ast \langle n \rangle \in p)$ and $\ell g(\eta) \geq \ell g(\text{tr}(p)) \Rightarrow \{ n : \eta^\ast \langle n \rangle \in p \} \in D$.

§1 Using measurables and FS iterations with non-transitive memory

We use [Sh 700] in 1.1 heavily. We use measurables (we could have used extenders to get more). The question on $\aleph_1, \aleph_2, \aleph_3$, i.e. Problem 0.2(2) remains open.

1.1 Theorem. There is a c.c.c. forcing notion $P$ of cardinality $\lambda$ such that in $V^P$ we have $a = \lambda$, $b = \mathfrak{d} = \mu$, $u = \mu$, $\{ \mu, \lambda \} \subseteq \text{Sp}_\chi$ but $\kappa_2 \notin \text{Sp}(\chi)$ if

$\otimes_1 \kappa_1, \kappa_2$ are measurable and $\kappa_1 < \mu = \text{cf}(\mu) < \kappa_2 < \lambda = \lambda^\mu = \lambda^{\kappa_2} = \text{cf}(\lambda)$.

Proof. Let $\mathcal{D}_\ell$ be a normal ultrafilter on $\kappa_\ell$ for $\ell = 1, 2$. Repeat [Sh 700, §4] with $(\kappa_1, \mu, \lambda)$ here standing for $(\kappa, \mu, \lambda)$ there, getting $t_\alpha \in \mathcal{R}$ for $\alpha \leq \lambda$ which is $\leq_{\mathcal{R}}$-increasing and letting $P_\alpha^\mathfrak{a} = P_\mathfrak{a}^\lambda$ we have $\mathcal{Q}^\alpha = \langle P^\alpha_\varepsilon : \varepsilon < \mu \rangle$ is a $\ll$-increasing continuous sequence of c.c.c. forcing notions, $P_\mu^\mathfrak{a} = P^\mathfrak{a} = P^\mathfrak{a}_\alpha := \lim(\mathcal{Q}^\alpha) = \cup\{ P^\alpha_\varepsilon : \varepsilon < \mu \}$ but add the demand that for unboundedly many $\alpha < \lambda$
$\mathbb{P}^{\alpha+1}_\alpha$ is isomorphic to the ultrapower $(\mathbb{P}^{\alpha})^{\kappa_2}/\mathcal{D}_2$, by an isomorphism extending the canonical embedding.

More explicitly we choose $t_\alpha$ by induction on $\alpha \leq \lambda$ such that

1. $t_\alpha \in \mathfrak{A}$, see Definition [Sh 700, 4.3] so the forcing notion $\mathbb{P}^{t_\alpha}_i$ for $i \leq \mu$ is well defined and is $\prec$-increasing with $i$

2. $\langle t_\beta : \beta \leq \alpha \rangle$ is $\leq_{\mathcal{A}}$-increasing continuous which means that:
   a. $\gamma \leq \beta \leq \alpha \Rightarrow t_\gamma \leq_{\mathcal{A}} t_\beta$, see Definition [Sh 700, 4.6](1) so $\mathbb{P}^{t_\gamma}_i \prec \mathbb{P}^{t_\beta}_i$ for $i \leq \mu$
   b. if $\alpha$ is a limit ordinal then $t_\alpha$ is a canonical $\leq_{\mathcal{A}}$-u.b. of $\langle t_\beta : \beta < \alpha \rangle$, see Definition [Sh 700, 4.6](2)
   c. if $\alpha = \beta + 1$ and $\text{cf}(\beta) \neq \kappa_2$ then $t_\alpha$ is essentially $t^{\kappa_1}_\beta/\mathcal{D}_1$
      i.e. we have to identify $\mathbb{P}^{t_\beta}_\xi$ with its image under the canonical embedding of it into $(\mathbb{P}^{t_\beta}_\xi)^{\kappa_1}/\mathcal{D}_1$, in particular this holds for $\varepsilon = \mu$, see Subclaim [Sh 700, 4.9])
   d. if $\alpha = \beta + 1$ and $\text{cf}(\beta) = \kappa_2$ then $t_\alpha$ is essentially $t^{\kappa_2}_\beta/\mathcal{D}_2$.

So we need

2. Subclaim [Sh 700, 4.9] applies also to the ultrapower $t^{\kappa_2}_\beta/\mathcal{D}$.

[Why? The same proof applies as $\mu^{\kappa_2}/\mathcal{D}_2 = \mu$, i.e., the canonical embedding of $\mu$ into $\mu^{\kappa_2}/\mathcal{D}_2$ is one-to-one and onto (and $\lambda^{\kappa_1}/\mathcal{D}_1 = \lambda^{\kappa_2}/\mathcal{D}_2 = \lambda$, of course).]

Let $\mathbb{P}^{\alpha}_\varepsilon = \mathbb{P}^{t_\alpha}_\varepsilon$ for $\varepsilon \leq \mu$ so $\mathbb{P}^{\alpha}_\varepsilon = \bigcup\{\mathbb{P}^{\alpha}_\zeta : \varepsilon < \mu\}$ and $\mathbb{P} = \mathbb{P}^{\lambda}$. It is proved in [Sh 700, 4.10] that in $\mathbb{V}^{\mathbb{P}}$, by the construction, $\mu \in \text{Sp}(\chi)$, $\alpha \leq \lambda$ and $\mu = 2^{\aleph_0} = \lambda$. By [Sh 700, 4.11] we have $\alpha \geq \lambda$ hence $\alpha = \lambda$, and always $2^{\aleph_0} \in \text{Sp}(\chi)$ hence $\lambda = 2^{\aleph_0} \in \text{Sp}(\chi)$. So what is left to be proved is $\kappa_2 \notin \text{Sp}(\chi)$. Assume toward contradiction that $p^* \Vdash \text{"}D\text{" is a non-principal ultrafilter on }\omega \text{ and } \chi(D) = \kappa_2$ and let it be exemplified by $\langle A_\varepsilon : \varepsilon < \kappa_2 \rangle$.

Without loss of generality $p^* \Vdash \text{"}A_\varepsilon \notin D\text{" does not belong to the filter on }\omega \text{ generated by } \{A_\zeta : \zeta < \varepsilon\} \cup \{\omega \setminus n : n < \omega\}$, for each $\varepsilon < \kappa_2$ and trivially also $\omega \setminus A_\varepsilon$ does not belong to this filter”.

As $\lambda$ is regular $> \kappa_2$ and the forcing notion $\mathbb{P}^{\lambda}$ satisfies the c.c.c., clearly for some $\alpha < \lambda$ we have $p^* \in \mathbb{P}^{\alpha}$ and $\varepsilon < \kappa_2 \Rightarrow A_\varepsilon$ is a $\mathbb{P}^{\alpha}$-name.

So for every $\beta \in [\alpha, \lambda)$ we have
\[ \mathfrak{S}_\beta \] \( p^* \models_{\mathbb{P}_\beta} \) “for each \( i < \kappa_2 \) the set \( A_i \in [\omega]^{\aleph_0} \) is not in the filter on \( \omega \) which \( \{ A_j : j < i \} \cup \{ \omega \setminus n : n < \omega \} \) generates, and also the complement of \( A_i \) is not in this filter (as \( D \) exemplifies this).”

But for some such \( \beta \), the statement \( \mathfrak{S}_1 \) holds, i.e. \( \mathfrak{S}_1(d) \) apply, so in \( \mathbb{P}^{\beta+1} \) which essentially is a \( (\mathbb{P}^{\beta})^{\kappa_2}/\mathcal{D}_2 \) we get a contradiction. That is, let \( j_\beta \) be an isomorphism from \( \mathbb{P}^{\beta+1} \) onto \( (\mathbb{P}^{\beta})^{\kappa_2}/\mathcal{D}_2 \) which extends the canonical embedding of \( \mathbb{P}^{\beta} \) into \( (\mathbb{P}^{\beta})^{\kappa_2}/\mathcal{D}_2 \). Now \( j_\beta \) induces a map \( \hat{j}_\beta \) from the set of \( \mathbb{P}^{\beta+1} \)-names of subsets of \( \omega \) into the set of \( (\mathbb{P}^{\beta})^{\kappa_2}/\mathcal{D}_2 \)-names of subsets of \( \omega \), and let \( A^* = \hat{j}_\beta^{-1}(\langle A_i : i < \kappa_2 \rangle/\mathcal{D}_2) \) so \( p^* \models_{\mathbb{P}^{\beta+1}} \) “\( A^* \in [\omega]^{\aleph_0} \) and the sets \( A^*, \omega \setminus A^* \) do not include any finite intersection of \( \{ A_\varepsilon : \varepsilon < \kappa_2 \} \cup \{ \omega \setminus n : n < \omega \} \).” So \( p^* \models_{\mathbb{P}^{\beta+1}} \) “\( \{ A_\varepsilon : \varepsilon < \kappa_2 \} \) does not generate an ultrafilter on \( \omega \)” but \( \mathbb{P}^{\beta+1} \triangleleft \mathbb{P} \), contradiction. \( \square_{1.1} \)

1.2 Remark. 1) As the referee pointed out we can in 1.1, if we waive “\( u < a \)” we can forget \( \kappa_1 \) (and \( \mathcal{D}_1 \)) so not taking ultra-powers by \( \mathcal{D}_1 \), so \( \mu = \aleph_0 \) is allowed, but we have to start with \( t_0 \) such that \( \mathbb{P}_0^{t_0} \) is adding \( \kappa_2 \)-Cohen.

2) Moreover, in this case we can demand that \( \mathbb{Q}_\alpha^t = \mathbb{Q}(D_\alpha^t) \) and so we do not need the \( t_\alpha^t \). Still this way was taken in [Sh:915, §1]. But this gain in simplicity has a price in lack of flexibility in choosing the \( t \). We use this mildly in §2; mildly as only for \( \mathbb{P}_1 \). See more in [Sh:915, §2,§3].

§2 Remarks on \( \pi \)-bases

2.1 Definition. 1) \( \mathcal{A} \) is a \( \pi \)-base if:

(a) \( \mathcal{A} \subseteq [\omega]^{\aleph_0} \)

(b) for some ultrafilter \( D \) on \( \omega \), \( \mathcal{A} \) is a \( \pi \)-base of \( D \), see below, note that \( D \) is necessarily non-principal

1A) We say \( \mathcal{A} \) is a \( \pi \)-base of \( D \) if \( (\forall B \in D)(\exists A \in \mathcal{A})(A \subseteq^* B) \).

1B) \( \pi \chi(D) = \text{Min}\{ |\mathcal{A}| : \mathcal{A} \text{ is a } \pi \text{-base of } D \} \).

2) \( \mathcal{A} \) is a strict \( \pi \)-base if:

(a) \( \mathcal{A} \) is a \( \pi \)-base of some \( D \)

(b) no subset of \( \mathcal{A} \) of cardinality \( < |\mathcal{A}| \) is a \( \pi \)-base.
3) $D$ has a strict $\pi$-base when $D$ has a $\pi$-base $\mathcal{A}$ which is a strict $\pi$-base.

4) $\text{Sp}^*_\pi = \{|\mathcal{A}|: \text{there is a non-principal ultrafilter} D \text{ on } \omega \text{ such that } \mathcal{A} \text{ is a strict } \pi\text{-base of } D\}$.

2.2 Definition. For $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ let $\text{Id}_\mathcal{A} = \{B \subseteq \omega: \text{for some } n < \omega \text{ and partition } \langle B_\ell : \ell < n \rangle \text{ of } B \text{ for no } A \in \mathcal{A} \text{ and } \ell < n \text{ do we have } A \subseteq^* B_\ell\}$.

2.3 Observation. For $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ we have:

(a) $\text{Id}_\mathcal{A}$ is an ideal on $\mathcal{P}(\omega)$ including the finite sets, though may be equal to $\mathcal{P}(\omega)$

(b) if $B \subseteq \omega$ then: $B \in [\omega]^{\aleph_0} \setminus \text{Id}_\mathcal{A}$ iff there is a (non-principal) ultrafilter $D$ on $\omega$ to which $B$ belongs and $\mathcal{A}$ is a $\pi$-base of $D$

(c) $\mathcal{A}$ is a $\pi$-base iff $\omega \notin \text{Id}_\mathcal{A}$.

Proof.

Clause (a): Obvious.

Clause (b):

The “if" direction: Let $D$ be a non-principal ultrafilter on $\omega$ such that $B \in D$ and $\mathcal{A}$ is a $\pi$-base of $D$. Now for any $n < \omega$ and partition $\langle B_\ell : \ell < n \rangle$ of $B$ as $B \in D$ and $D$ is an ultrafilter clearly there is $\ell < n$ such that $B_\ell \in D$ hence by Definition 2.1(1A) there is $A \in \mathcal{A}$ such that $A \subseteq^* B_\ell$. By the definition of $\text{Id}_\mathcal{A}$ it follows that $B \notin \text{Id}_\mathcal{A}$ but $[\omega]^{<\aleph_0} \subseteq \text{Id}_\mathcal{A}$ so we are done.

The “only if" direction: So we are assuming $B \notin \text{Id}_\mathcal{A}$ so as $\text{Id}_\mathcal{A}$ is an ideal of $\mathcal{P}(\omega)$ there is an ultrafilter $D$ on $\omega$ disjoint to $\text{Id}_\mathcal{A}$ such that $B \in D$. So if $B' \in D$ then $B' \subseteq \omega \land B' \notin \text{Id}_\mathcal{A}$. $\text{Id}_\mathcal{A}$ hence by the definition of $\text{Id}_\mathcal{A}$ it follows that $(\exists A \in \mathcal{A})(A \subseteq^* B')$. By Definition 2.1(1A) this means that $\mathcal{A}$ is a $\pi$-base of $D$.

Clause (c): Follows from clause (b). $\square_{2.4}$

2.4 Observation. 1) If $D$ is an ultrafilter on $\omega$ then $D$ has a $\pi$-base of cardinality $\pi_\chi(D)$.

2) $\mathcal{A}$ is a $\pi$-base iff for every $n \in [1, \omega)$ and partition $\langle B_\ell : \ell < n \rangle$ of $\omega$ to finitely many sets, for some $A \in \mathcal{A}$ and $\ell < n$ we have $A \subseteq^* B_\ell$.

3) $\text{Min}\{\pi_\chi(D) : D \text{ a non-principal ultrafilter on } \omega\} = \text{Min}\{|\mathcal{A}| : \mathcal{A} \text{ is a } \pi\text{-base}\} = \text{Min}\{|\mathcal{A}| : \mathcal{A} \text{ is a strict } \pi\text{-base}\}.$
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Proof. 1) By the definition.
2) For the “only if” direction, assume $\mathcal{A}$ is a $\pi$-base of $D$ then $\text{Id}_{\mathcal{A}} \subseteq \mathcal{P}(\omega) \setminus D$ (see the proof of 2.2) so $\omega \notin \text{Id}_{\mathcal{A}}$ and we are done.
   For the “if” direction, use 2.2.
3) Easy. \qed_{2.4}

2.5 Theorem. In $V^\mathbb{P}$ as in 1.1, we have $\{\mu, \lambda\} \subseteq \text{Sp}^*_{\pi\chi}$ and $\kappa_2 \notin \text{Sp}^*_{\pi\chi}$.

Proof. Similar to the proof of 1.1 but with some additions. The main change is in the proof of $\Vdash_{\mathbb{P}} “\lambda \in \text{Sp}_{\chi}”$. The main addition is that choosing $t_\alpha$ by induction on $\alpha$ we also define $\mathcal{A}_\alpha$ such that

$\otimes_1'$ \(a\), \(b\) as in $\otimes_1$
\(c\) as in $\otimes_2(c)$ but only if $\alpha \neq 2 \text{ mod } \omega$ (and $\alpha = \beta + 1$)
\(d\) $A_\alpha$ is a $\mathbb{P}_0^\delta$-name of an infinite subset of $\omega$
\(e\) if $\alpha \neq 2 \text{ mod } \omega$ then $\Vdash_{\mathbb{P}_t} A_\alpha = \omega$ (or do not define $A_\alpha$)
\(f\) if $\alpha < \beta$ are $= 2 \text{ mod } \omega$ then $\Vdash_{\mathbb{P}_t} A_\beta \subseteq^* A_\alpha$
\(g\) if $\beta = \alpha + 1$ and $\beta = 2 \text{ mod } \omega$ and $B$ is a $\mathbb{P}_t^\delta$-name of an
   infinite subset of $\omega$ then $\Vdash_{\mathbb{P}_t} B \notin^* A_\alpha$.

This addition requires that we also prove

$\otimes_3$ if $s \in \mathcal{A}$ and $D$ is a $\mathbb{P}_0^\delta$-name of a filter on $\omega$ including all co-finite subsets of $\omega$ (such that $\emptyset \notin D$) then for some $(t, A)$ we have

\(a\) $s \leq_{\mathcal{A}} t$
\(b\) $\Vdash_{\mathbb{P}_t} “A$ is an infinite subset of $\omega$
\(c\) if $B$ is a $\mathbb{P}^\delta$-name of an infinite subset of $\omega$ then $\Vdash_{\mathbb{P}_t} “B \notin^* A$”.

[Why $\otimes_3$ holds? Without loss of generality $\Vdash_{\mathbb{P}_0} “D$ is an ultrafilter on $\omega”$. We can find a pair $(\mathbb{P}', A')$

\(\alpha\) $\mathbb{P}'$ is a c.c.c. forcing notion
\(\beta\) $\mathbb{P}_0^\delta \ll \mathbb{P}'$ moreover $\mathbb{P}' = \mathbb{P}_0^\delta \ast Q(D)$
\(\gamma\) $|\mathbb{P}'| \leq \lambda$
(δ) $\vdash \Pi \left< A \right>$ is an almost intersection of $D$ (i.e. $A \in [\omega]^{\aleph_0}$ and $(\forall B \in D)(A \subseteq^* B)$)

(e) $A'$ $\in \omega$ is the generic of $\mathbb{Q}[D]$ and $A' = \text{Rang}(\eta)$ so both are $\mathbb{P}'$-names.

Now we define $t'$: for $i \leq \mu$ we choose $\mathbb{P}^i_t = \mathbb{P}^i \ast_{\nu^i_0} \mathbb{P}'$ and we choose $A^i_t$ naturally.

Let $\langle n_\rho : \rho \in \omega > 2 \rangle$ be a $\mathbb{P}^i_t$-name listing the members of $A$.

Now we choose $t$ such that $t' \leq \mathring{t}$ and for some $\mathbb{P}_0$-name $\rho$ of a member of $\omega > 2$ we have $\vdash_{\mathbb{P}_i} \rho \neq \nu$ for any $\mathbb{P}_\nu$-name (clearly exists, e.g. when $(t, t')$ is like $(t, s)$ above). Now $A := \{ n_\rho | k : k < \omega \}$ is forced to be an infinite subset of $A'$, and if it includes a member of $\mathcal{P}(\omega)^{\mathbb{P}_i}$ or even $\mathcal{P}(\omega)^{\mathbb{P}_i} \mathcal{V}^{\mathbb{P}_i}$ we get that $\rho$ is from $\langle \omega > 2 \rangle^{\mathbb{P}_i}$, contradiction.

$(\ast)_1 \mu \in \text{Sp}_{\pi X}^* \in \mathcal{V}^\mathbb{P}$, of course.

[Why? As there is a $\subseteq^*$-decreasing sequence $\langle B_\alpha : \alpha < \mu \rangle$ of sets which generates a (non-principle ultrafilter). We can use $B_\alpha$ is the generic of $\mathbb{P}^{\lambda\lambda}_{\alpha + 1}/\mathbb{P}^{\lambda\lambda}_{\alpha}$]  

$(\ast)_2 \kappa_2 \notin \text{Sp}_{\pi X}^*$.  

[Why? Toward contradiction assume $p^* \in \mathbb{P}$ and $p^* \vdash \Pi \left< D \right>$ is a non-principal ultrafilter on $\omega$ and $\{ \mathcal{U}_\varepsilon : \varepsilon < \kappa_2 \}$ is a sequence of infinite subsets of $\omega$ which is a strict $\pi$-base of $D^\varepsilon$; so $p^* \vdash \Pi \left< \mathcal{U}_\varepsilon : \varepsilon < \zeta \right>$ is not a $\pi$-base of any ultrafilter on $\omega$ for every $\zeta < \kappa_2$, hence for some $\langle B_\zeta, \ell : \ell < n_\zeta \rangle$ we have $p^* \vdash \left< n_\ell \mathring{<} \omega \right.$ and $\langle B_\zeta, \ell : \ell < n_\ell \rangle$ is a partition of $\omega$ and $\varepsilon < \zeta \wedge \ell < n_\zeta \Rightarrow \mathcal{U}_\varepsilon \notin \mathcal{P}^\ast B_\zeta, \ell^\ast$. We now as in the proof of 1.1, choose suitable $\beta < \lambda$ and consider $\langle B_\ell' : \ell < n \rangle = 3^{-1}\left( \langle B_\zeta, \ell : \ell < n_\zeta \rangle : \zeta < \kappa_2 \right) / \mathcal{D}_2$ so $p^* \vdash_{\mathbb{P}^{\beta + 1}} \left< B_\ell' : \ell < n \right>$ is a partition of $\omega$ to finitely many sets and $\varepsilon < \kappa_2 \wedge \ell < n \Rightarrow \mathcal{U}_\varepsilon \notin \mathcal{P}^\ast B_\ell'^\ast$. But this contradicts $p^* \vdash \Pi \left< \mathcal{U}_\varepsilon : \varepsilon < \kappa_2 \right>$ is a $\pi$-base.]

$(\ast)_3 \lambda \in \text{Sp}_{\pi}^*$.  

[Why? Clearly it is forced (i.e. $\vdash_{\mathbb{P}_\lambda}$) that $\langle A_{\alpha + 2} : \alpha < \lambda \rangle$ is a $\subseteq^*$-decreasing sequence of infinite subsets of $\omega$, hence there is an ultrafilter of $D$ on $\omega$ including it. Now $A_{\omega + 2}$ witness that $\mathcal{P}(\omega)^{\mathbb{P}_{\omega + 2}}$ is not a $\pi$-base of $D$ (recalling clause (h) of $\text{\dag}_1$). As $\lambda$ is regular we are done.]
REFERENCES.

[BnSh 642] Jörg Brendle and Saharon Shelah. Ultrafilters on \( \omega \) — their ideals and their cardinal characteristics. *Transactions of the American Mathematical Society*, **351**:2643–2674, 1999. math.LO/9710217.

[Sh:915] Saharon Shelah. 

[Sh 700] Saharon Shelah. Two cardinal invariants of the continuum (d < a) and FS linearly ordered iterated forcing. *Acta Mathematica*, **192**:187–223, 2004. Also known under the title ”Are a and d your cup of tea?”. math.LO/0012170.