DIOPHANTINE PROBLEMS FOR $q$-ZETA VALUES¹

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1. Introduction. As usual, quantities depending on a number $q$ and becoming classical objects as $q \to 1$ (at least formally) are regarded as $q$-analogues or $q$-extensions. A possible way to $q$-extend the values of the Riemann zeta function reads as follows (here $q \in \mathbb{C}$, $|q| < 1$):

$$\zeta_q(k) = \sum_{n=0}^{\infty} \sigma_{k-1}(n) q^n = \sum_{\nu=0}^{\infty} \frac{\nu^{k-1} q^\nu}{1-q^\nu} = \sum_{\nu=0}^{\infty} \frac{q^\nu \rho_k(q^\nu)}{(1-q^\nu)^k}, \quad k = 1, 2, \ldots, \quad (1)$$

where $\sigma_{k-1}(n) = \sum d|n d^{k-1}$ is the sum of powers of the divisors and the polynomials $\rho_k(x) \in \mathbb{Z}[x]$ can be determined recursively by the formulae $\rho_1 = 1$ and $\rho_{k+1} = (1 + (k-1)x)\rho_k + x(1-x)\rho_k'$ for $k = 1, 2, \ldots$ (see [1, Part 8, Chapter 1, Section 8, Problem 75] for the case $k = 2$). Then the limit relations

$$\lim_{q \to 1; |q| < 1} (1-q)^k \zeta_q(k) = \rho_k(1) \cdot \zeta(k) = (k-1)! \cdot \zeta(k), \quad k = 2, 3, \ldots, \quad (2)$$

hold; the equality $\rho_k(1) = (k-1)!$ is proved in [2, formula (7)]. The above defined $q$-zeta values (1) present several new interesting problems in the theory of diophantine approximations and transcendental numbers; these problems are extensions of the corresponding problems for ordinary zeta values and we state some of them in Section 3 of this note. Our nearest aim is to demonstrate how some recent contributions to the arithmetic study of the numbers $\zeta(k), k = 2, 3, \ldots$, successfully work for $q$-zeta values. Namely, we mean the hypergeometric construction of linear forms (proposed in the works of E. M. Nikishin [3], L. A. Gutnik [4], Yu. V. Nesterenko [5]) and the arithmetic method (due to G. V. Chudnovsky [6], E. A. Rukhadze [7], M. Hata [8]) accompanied with the group-structure scheme (due to G. Rhin and C. Viola [9], [10]). The next section contains new irrationality measures of the numbers $\zeta_q(1)$ and $\zeta_q(2)$ for $q^{-1} = p \in \mathbb{Z} \setminus \{0, \pm 1\}$, and our starting point is the following table illustrating a connection of some objects and their $q$-extensions (here $\lfloor \cdot \rfloor$ denotes the integral part of a number and the notation ‘l.c.m.’ means the least common multiple). We refer the reader to the book [11] and the works [12]–[14], where a motivation and a ground are presented.

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 ordinary objects & \textbf{\textit{q}}-\textit{extensions, }p=1/q\in\mathbb{Z}\setminus\{0,\pm1\}\\
\hline
\text{numbers }n\in\mathbb{Z} & \text{‘numbers’ }[n]_p = \frac{p^n - 1}{p - 1} \in \mathbb{Z}[p] \\
\text{primes }l\in\{2,3,5,7,\ldots\} \in \mathbb{Z} & \text{irreducible reciprocal polynomials} \\
\text{Euler’s gamma function }\Gamma(t) & \text{Jackson’s }\textit{q}-\text{gamma function} \\
\text{the factorial }n! = \Gamma(n+1) & \text{the }q\text{-factorial }[n]_q! = \Gamma_q(n+1) \\
\text{ord }n! = \left\lfloor \frac{n}{l} \right\rfloor + \left\lfloor \frac{n}{l^2} \right\rfloor + \cdots & \text{ord}_{\Phi_l(p)}[n]_p! = \left\lfloor \frac{n}{l} \right\rfloor, \ l = 2,3,4,\ldots \\
\text{the prime number theorem } & \text{Mertens’ formula} \\
\text{lim}_{n\to\infty} \frac{\log D_n}{n} = 1 & \text{lim}_{n\to\infty} \frac{\log |D_n(p)|}{n^2 \log |p|} = \frac{3}{\pi^2} \\
\hline

If \(\psi(x)\) is the logarithmic derivative of Euler’s gamma function and \(\{x\} = x - \lfloor x\rfloor\) is the fractional part of a number \(x\), then, for each semi-interval \([u,v) \subset (0,1)\), Mertens’ formula yields the limit relation

\[
\lim_{n \to \infty} \frac{1}{n^2 \log |p|} \sum_{l \cdot \{n/l\} \in [u,v)} \log |\Phi_l(p)| = \frac{3}{\pi^2} \left( \psi'(u) - \psi'(v) \right) = \frac{3}{\pi^2} \int_u^v d(-\psi'(x))
\]

(3)

(see [15, Lemma 1]), which can be regarded as a \(\textit{q}\)-extension of the formula

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{\text{primes }l > \sqrt{n} \in [u,v)} \log l = \psi(v) - \psi(u) = \int_u^v \! d\psi(x)
\]

in the arithmetic method [6]–[10].

2. \textbf{Rational approximations to }\textit{q}-\textit{zeta values and basic transformations.} Let \(a_0, a_1, a_2,\) and \(b\) be positive integers satisfying the condition \(a_1 + a_2 \leq b\). Then, Heine’s series

\[
F(a,b) = \frac{\Gamma_q(b-a_2)}{(1-q)\Gamma_q(a_1)} \sum_{t=0}^{a_0} \frac{\Gamma_q(t+a_1) \Gamma_q(t+a_2)}{\Gamma_q(t+1) \Gamma_q(t+b)} q^{a_0 t}
\]

becomes a \(\mathbb{Q}(p)\)-linear form

\[
F(a,b) = A \zeta_q(1) - B
\]

with the property

\[
p^{-M} D_m(p) \cdot F(a,b) \in \mathbb{Z}[p] \zeta_q(1) + \mathbb{Z}[p];
\]

(4)

here \(M = M(a,b)\) is some (explicitly defined) integer and \(m\) is the maximum of the 6-element set

\[
c_{00} = a_0 + a_1 + a_2 - b - 1, \quad c_{01} = a_0 - 1, \quad c_{11} = a_1 - 1, \quad c_{21} = a_2 - 1,
\]

\[
c_{12} = b - a_1 - 1, \quad c_{22} = b - a_2 - 1.
\]
Taking $H(c) = F(a, b)$ and using the stability of the quantity

$$\frac{F(a_0, a_1, a_2, b)}{\Gamma_q(a_0) \Gamma_q(a_2) \Gamma_q(b - a_2)} = \frac{H(c)}{\Pi_q(c)},$$

where $\Pi_q(c) = [c_{01}]_q! [c_{21}]_q! [c_{22}]_q! = p^{-N(c)} \Pi_p(c)$,

under the action of the transformations

$$\tau = (c_{22} c_{21} c_{01} c_{12} c_{00}): (a_0, a_1, a_2, b) \mapsto (a_1, b - a_1, a_0, a_0 + a_2),$$

$$\sigma = (c_{11} c_{21})(c_{12} c_{22}): (a_0, a_1, a_2, b) \mapsto (a_0, a_2, a_1, b)$$

we arrive at the better than (4) inclusions

$$p^{-M} D_m(p) \Omega^{-1}(p) \cdot F(a, b) \in \mathbb{Z}[p] \zeta_q(1) + \mathbb{Z}[p]$$

(5)

with

$$\Omega(p) = \prod_{l=1}^m \Phi(p), \quad \nu_l = \max_{g \in \langle \tau^2, \sigma \rangle} \text{ord}_{g}(p) \frac{\Pi_p(c)}{\Pi_p(\infty)}.$$  

(6)

In addition, trivial estimates for $F(a, b)$ and explicit formulae for the coefficient $A$ imply that

$$|F(a, b)| = |p|^{O(b)}, \quad |A| \leq |p|^{(a_0 + a_1 + a_2)b - (a_1^2 + a_2^2 + b^2)/2 + O(b)}$$

(7)

with some absolute constant in $O(b)$.

Note that the non-trivial transformation $\tau$ of the quantity $H(c)/\Pi_q(c)$ has been obtained (in other notation) by E. Heine still in 1847. The transformation group $\mathfrak{G} = \langle \tau, \sigma \rangle$ of order 12 has no ordinary analogue since corresponding (in limit $q \to 1$) Gauß’s hypergeometric series are divergent. We use the group $\langle \tau^2, \sigma \rangle$ of order 6 instead of the total available group $\mathfrak{G}$ to ensure the required condition $a_1 + a_2 \leq b$.

Now, choosing $a_0 = a_2 = 8n + 1$, $a_1 = 6n + 1$, and $b = 15n + 1$, and taking in mind (5), (7), and (3) we derive the following result.

**Theorem 1.** For each $q = 1/p$, $p \in \mathbb{Z} \setminus \{0, \pm 1\}$, the number $\zeta_q(1)$ is irrational and its irrationality exponent satisfies the estimate

$$\mu(\zeta_q(1)) \leq 2.42343562 \ldots .$$

(8)

A value $\mu = \mu(\alpha)$ is said to be the irrationality exponent of a real irrational number $\alpha$ if $\mu$ is the least possible exponent such that for any $\varepsilon > 0$ the inequality $|\alpha - a/b| \leq b^{-(\mu + \varepsilon)}$ has only finitely many solutions in integers $a$ and $b$. The estimate (8) can be compared with the previous result $\mu(\zeta_q(1)) \leq 2\pi^2/(\pi^2 - 2) = 2.50828476 \ldots$ of P. Bundschuh and K. Väänänen in [12] corresponding to the choice $a_0 = a_1 = a_2 = n + 1$ and $b = 2n + 2$ in the above notation.

Similar arguments with a simpler group $\langle \sigma \rangle$ of order 2 can be put forward to improve W. Van Assche’s estimate $\mu(\log_q(2)) \leq 3.36295386 \ldots$ in [13] for the following $q$-extension of log(2):

$$\log_q(2) = \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1} q^{\nu}}{1 - q^{\nu}} = \sum_{\nu=1}^{\infty} \frac{q^{\nu}}{1 + q^{\nu}}.$$  

Namely, in [14] we obtain the inequality $\mu(\log_q(2)) \leq 3.29727451 \ldots$ for $q^{-1} = p \in \mathbb{Z} \setminus \{0, \pm 1\}$. 
In the case of the numbers $\zeta_q(2)$, consider the positive integers $(a, b) = (a_1, a_2, a_3, b_2, b_3)$ satisfying the conditions $a_j < b_k$, $a_1 + a_2 + a_3 < b_2 + b_3$ and the $q$-basic hypergeometric series

$$\bar{F}(a, b) = \frac{\Gamma_q(b_2 - a_2) \Gamma_q(b_3 - a_3)}{(1 - q)^2 \Gamma_q(a_1)} \sum_{t=0}^{\infty} \frac{\Gamma_q(t + a_1) \Gamma_q(t + a_2) \Gamma_q(t + a_3)}{\Gamma_q(t + 1) \Gamma_q(t + b_2) \Gamma_q(t + b_3)} q^{b_2 + b_3 - a_2 - a_3} t$$

$$= \bar{A} \zeta_q(2) - \bar{B}.$$

Then $p^{-M} D_{m_1}(p) D_{m_2}(p) \cdot \bar{F}(a, b) \in \mathbb{Z}[p] \zeta_q(2) + \mathbb{Z}[p]$, where $m_1 \geq m_2$ are the two successive maxima of the 10-element set

$$c_{00} = (b_2 + b_3) - (a_1 + a_2 + a_3) - 1, \quad c_{jk} = \begin{cases} a_j - 1 & \text{if } k = 1, \\ b_k - a_j - 1 & \text{if } k = 2, 3, \end{cases} \quad j = 1, 2, 3,$$

and, in addition,

$$|\bar{F}(a, b)| = |p|^{O(\max\{b_2, b_3\})}, \quad |\bar{A}| \leq |p|^{b_2 b_3 - (a_1^2 + a_2^2 + a_3^2)/2 + O(\max\{b_2, b_3\})}.$$  

The $c$-permutation group $\mathcal{G} \subset \mathcal{G}_{10}$ generated by all permutations of $a_1, a_2, a_3$, the permutation of $b_2, b_3$, and the permutation $(c_{00} c_{22})(c_{11} c_{33})(c_{13} c_{31})$ has order 120 and is known in connection with the Rhin–Viola proof [9] of the new irrationality measure for $\zeta(2)$ (see also [16, Section 6]). In notation $\bar{H}(c) = \bar{F}(a, b)$, the quantity

$$\bar{H}(c) = [c_{00}]q! [c_{21}]q! [c_{22}]q! [c_{33}]q! [c_{31}]q!$$

is stable under the action of the group $\mathcal{G}$. This $\mathcal{G}$-stability yields the inclusions

$$p^{-M} D_{m_1}(p) D_{m_2}(p) \bar{G}^{-1}(p) \cdot \bar{F}(a, b) \in \mathbb{Z}[p] \zeta_q(2) + \mathbb{Z}[p]$$

with a quantity $\bar{\Omega}(p)$ defined like in (6). Finally, choosing $a_1 = 5n + 1$, $a_2 = 6n + 1$, $a_3 = 7n + 1$, and $b_2 = 14n + 2$, $b_3 = 15n + 2$ we deduce the following result [17].

**Theorem 2.** For each $q = 1/p$, $p \in \mathbb{Z} \setminus \{0, \pm 1\}$, the number $\zeta_q(2)$ is irrational and its irrationality exponent satisfies the estimate

$$\mu(\zeta_q(2)) \leq 4.07869374 \ldots.$$  

(9)

The quantitative estimates of type (9) for $\zeta_q(2)$ have been not known before, although the transcendence of $\zeta_q(2)$ for any algebraic number $q$ with $0 < |q| < 1$ follows from Nesterenko’s theorem [18].

It is nice to mention that the simpler choice of the parameters $a_1 = a_2 = a_3 = n + 1$, $b_2 = b_3 = 2n + 2$ also proves the irrationality of $\zeta_q(2)$ for $q^{-1} \in \mathbb{Z} \setminus \{0, \pm 1\}$, and the limit $q \to 1$ produces Apéry’s original sequence [19] of rational approximations to $\zeta(2)$.

We would like to stress that using, like in [7]–[10], (multiple) $q$-integrals for the both series $F(a, b)$ and $\bar{F}(a, b)$ in study of arithmetic properties of the numbers $\zeta_q(1)$ and $\zeta_q(2)$ is in great difficulties. The reason of this is due to non-existence of a concept of changing the variable of $q$-integration (see [20] and [21, Section 2.2.4]).
3. General problems for \( q \)-zeta values. We start with mentioning that, for an even integer \( k \geq 2 \), the series \( E_k(q) = 1 - 2k \zeta_q(k)/B_k \), where \( B_k \in \mathbb{Q} \) are Bernoulli numbers, is known to be the Eisenstein series. Therefore the modular origin (with respect to the parameter \( \tau = \frac{\log q}{2\pi i} \)) of the functions \( E_4, E_6, E_8, \ldots \) gives the algebraic independence of the functions \( \zeta_q(2), \zeta_q(4), \zeta_q(6) \) over \( \mathbb{Q}[q] \), while all other even \( q \)-zeta values are polynomials in \( \zeta_q(4) \) and \( \zeta_q(6) \). In this sense, the consequence of Nesterenko’s theorem \([18]\) “the numbers \( \zeta_q(2), \zeta_q(4), \zeta_q(6) \) are algebraically independent over \( \mathbb{Q} \) for algebraic \( q, 0 < |q| < 1 \)” reads as a complete \( q \)-extension of the consequence of Lindemann’s theorem \([22]\) “\( \zeta(2) \) is transcendental”. Moreover, the transcendence of values of the function

\[
1 + 4 \sum_{\nu=0}^{\infty} \frac{(-1)^\nu q^{2\nu+1}}{1 - q^{2\nu+1}} = \left( 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \right)^2 \tag{10}
\]

at algebraic points \( q, 0 < |q| < 1 \), also follows from Nesterenko’s theorem (a proof of Jacobi’s identity (10) can be found, e.g., in \([23, \text{Theorem 2}]\)); the series on the left-hand-side of (10) is a \( q \)-analogue of the series

\[
4 \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{2\nu + 1} = \pi.
\]

The best known estimate for the irrationality exponent of (10) in the case \( q^{-1} \in \mathbb{Z} \setminus \{0, \pm1\} \) is obtained in \([24]\).

The limit relations (2) as well as the expected algebraic structure of the ordinary zeta values motivate the following questions (we also regard \( \zeta_q(1) \) to be an odd \( q \)-zeta value, although the corresponding ordinary harmonic series is divergent).

**Problem 1.** Prove that the \( q \)-zeta values \( \zeta_q(1), \zeta_q(2), \zeta_q(3), \ldots \) as functions of \( q \) are linearly independent over \( \mathbb{C}(q) \).

**Problem 2.** Prove that the \( q \)-functional set involving the three even \( q \)-zeta values \( \zeta_q(2), \zeta_q(4), \zeta_q(6) \) and all odd \( q \)-zeta values \( \zeta_q(1), \zeta_q(3), \zeta_q(5), \ldots \) consists of functions that are algebraically independent over \( \mathbb{C}(q) \).

The associated diophantine problems consist in proving the corresponding linear and algebraic independences over the algebraic closure of \( \mathbb{Q} \) for algebraic \( q \) with \( 0 < |q| < 1 \). In this direction, even irrationality and \( \mathbb{Q} \)-linear independence results for \( q \)-zeta values at the point \( q \in \mathbb{Q} \) with \( q^{-1} \in \mathbb{Z} \setminus \{0, \pm1\} \) would be very interesting.

A problem of other type is to construct a model of multiple \( q \)-zeta values involving \( q \)-zeta values (1) and possessing similar properties with the model of multiple zeta values \([25]\).

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