INSTABILITY OF FLAG VARIETIES

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Abstract. Let $E$ be an ample vector bundle on a smooth projective variety $X$ over the complex numbers and let $\text{Fl}(r, E)$ be the Flag variety of codimension $r = (r_1, \ldots, r_t)$ planes in $E$. We compute the Donaldson-Futaki invariant of test configurations induced by subbundles of the vector bundle $E$ and derive instability results for certain polarisations of $\text{Fl}(r, E)$ where $E$ is an unstable vector bundle.

1. Introduction

$K$-stability is a notion which has become important in the recent work on Fano varieties by Tian, Donaldson and others. On the other hand, slope stability is a notion defined for coherent sheaves on a polarised variety. The work of many authors has shown that these two notions of stability are closely related when varieties are constructed from sheaves. Morrison [11] proved that given a vector bundle of rank 2 on a curve, bundle stability of the vector bundle and Chow stability of its projectivisation are equivalent for certain naturally defined polarisations. Over an arbitrary base, Ross and Thomas showed that unstable sheaves have $K$-unstable polarisations [12]. If we assume that the base $(X, L)$ has a constant scalar curvature Kähler metric in $c_1(L)$, the $K$-stability of the projectivisation of a stable vector bundle follows from results of Hong [9] and Stoppa [15]. Further related results have been more recently been proven by Seyyedali [14] and Keller and Ross [10].

The natural generalisation of the results mentioned above are to flag varieties associated to vector bundles. We show how a destabilising subbundle of a vector bundle $E$ determines a destabilising test configurations for certain flag varieties of $E$. Our result depends on the knowledge of the polynomials appearing in the Chern character of Schur bundles for which we give a conjectural formula. We verify this conjecture in a number of cases.

For the remainder of the paper, $X$ is a smooth complex projective variety of dimension $n$ with a very ample polarisation $L$ and $E$ is an algebraic vector bundle of rank $e$. Given a partition $r$, we denote the flag variety of dimension $r$ quotients in $E$ by $\text{Fl}(r, E)$, or simply by $F$ when $r$ and $E$ are clear from context. Let $\pi$ be the projection map $\pi : F \rightarrow X$ and $L$ any $\pi$-ample line bundle on $\text{Fl}(r, E)$. $\mathbb{P}(E)$ denotes the projective bundle of lines.

Theorem 1. Let $(X, L)$ be a smooth polarised curve and $E$ an ample vector bundle of rank $e$ on $X$. Let $r$ be one of the partitions $(2)$, $(3)$, $(3, 1)$, $(3, 2)$ or $(3, 2, 1)$. If $E$ is slope unstable, then the flag variety $\text{Fl}(r, E)$ of $r$-flags of quotients in $E$ with the polarisation $L$ is $K$-unstable.

For polarisations that make the fibre of $\pi$ small, a similar result is true without imposing a restriction on the dimension of the base. We denote $L_m = \mathcal{L} \otimes L^m$.

Theorem 2. Assume that $(X, L)$ is a smooth polarised variety and $E$ a vector bundle of rank $e$ on $X$ with a destabilising subbundle $F$. Let $r$ be one of the partitions listed
Then there exists an $m_0$ such that the flag variety $\text{Fl}(r, E)$ of $r$-flags of quotients in $E$ with the polarisation $\mathcal{L}_m$ is $K$-unstable for $m > m_0$. 

Remark 3. We expect both of these theorems to hold for all partitions $r$. This would follow from a combinatorial conjecture that we present in Section 2.3.

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2. Preliminaries

2.1. Stability definitions. First we briefly review the required parts of the stability definitions that we use. For the complete definition of $K$-stability and its relations to Chow stability see [13].

**Definition 4.** Let $X$ be a projective variety with an ample polarisation $L$. A test configuration for the polarised variety $(X, L)$ is determined by the following data. Let $f : X \to \mathbb{C}$ be a flat morphism, $V$ an ample line bundle on $X$ and $\rho : \mathbb{C}^* \times X \to X$ a $V$-linearised action on $X$ that covers the usual action on $\mathbb{C}$. Furthermore, we require that the general fibre of $f$ be projectively isomorphic to $(X, L)$. We refer to a given test configuration as the triple $(X, V, \rho)$.

The central fibre of $0 \in \mathbb{C}$ is fixed under the action of $\rho$. We define the integers $w(k)$ to be the total weights of the $\mathbb{C}^*$ action induced on the vector spaces

$$\Lambda^\text{top} \det H^0(X_0, V^k).$$

It is a well known fact that $w(k)$ coincides with a polynomial of degree $n+1$, for $k > k_0$, where $k_0$ is sufficiently large. Assuming $k > k_0$ we write

$$w(k) = w_0 k^{n+1} + w_1 k^n + O(k^{n-1})$$

and similarly let

$$h^0(X, L_k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}),$$

again for $k$ sufficiently large. Given the above data, we define the Donaldson-Futaki invariant

$$F(X, V, \rho) = \frac{w_0 a_1 - w_1 a_0}{a_0^2}.$$ 

**Definition 5.** A test configuration $(X, V, \rho)$ destabilises $(X, L)$ if its Futaki-invariant is negative, and we say that $(X, L)$ is $K$-unstable.

We also recall the definition of slope stability of a sheaf. Given a torsion free sheaf $E$ on $(X, L)$, we write

$$\deg E = \int_X c_1(E).c_1(L)^{n-1}.$$ 

Define the slope of $E$ to be

$$\mu(E) = \deg E / \text{rank } E.$$
Definition 6. A vector bundle $E$ is slope stable if every torsion free subsheaf $F$ satisfies $\mu(F) < \mu(E)$. If the non-strict version of the inequality holds, $E$ is slope semi-stable. A torsion free subsheaf (subbundle) $F$ with $\mu(F) > \mu(E)$ is called a destabilising subsheaf.

2.2. Schur powers and the pushforward formula. In this section we discuss the correspondence between Schur powers of a vector bundle and line bundles of flag varieties, and state a pushforward formula which will be used later. A partition $\lambda = (\lambda_1, \ldots, \lambda_t)$ of a finite nonincreasing sequence of natural numbers. The largest index $l$ is called the length of $\lambda$. Also define the size $|\lambda| := \sum_{i=1}^{l} \lambda_i$ of $\lambda$.

The Schur power of a free module $M$ associated to $\lambda$ is a free module that we denote by $M^\lambda$. The definition and basic properties of Schur functors can be found in the references (e.g. [4, p. 106-111]). The Schur power construction is functorial, so it is well-defined on the category of vector bundles over schemes [16, Section II.2]. Given a vector bundle $E$ on a projective variety $X$ and a partition $\lambda$, we denote the Schur power by $E^\lambda$.

On the flag variety $\mathrm{Fl}(r, E)$, there is a sequence of tautological vector bundles

$$0 = \mathcal{R}_0 \subseteq \mathcal{R}_1 \subseteq \ldots \subseteq \mathcal{R}_t \subseteq \pi^* E^*,$$

where $\mathrm{rank} \mathcal{R}_i = r_{t-i}$ and $\pi$ is the projection $\mathrm{Fl}(r, E) \rightarrow X$. The pullbacks of the $(s_1, \ldots, s_t)$ line bundles under $\mathrm{Fl}(r, E) \hookrightarrow \mathbb{P}(\Lambda^{r_1}E^*) \times \ldots \times \mathbb{P}(\Lambda^{r_t}E^*)$ can also be written as the line bundles

$$(\det \mathcal{R}_1)^{s_1} \otimes \cdots \otimes (\det \mathcal{R}_t)^{s_t}.$$ We will use partitions to index these line bundles. First, we make the following definition.

Definition 8. Let $\nu + \lambda$ denote the componentwise addition of two partitions $\nu$ and $\lambda$ of the same length $l$. Similarly define the componentwise product of two partitions $\nu \lambda$, and the product of a positive integer $k$ and a partition $\lambda$ to be $k \lambda = (k\lambda_1, \ldots, k\lambda_l)$. Let $\alpha_i$ for $i = 1, \ldots, t$ be partitions and define

$$Q_i = (\mathcal{R}_i/\mathcal{R}_{i-1})^*$$

The higher direct images of any vector bundle of the form

$$Q_i^\alpha_1 \otimes \cdots \otimes (Q_i^\alpha_t)$$

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Recall that the restrictions of ample line bundles on $\mathrm{Fl}(r, V)$ onto fibres must have a simple form.

Proposition 7 (Ample cone of the ordinary flag variety [1]). Let $V$ be a vector space and $r$ a strictly decreasing partition. The flag variety $\mathrm{Fl}(r, V)$ can be naturally regarded as a projective subvariety $\iota : \mathrm{Fl}(r, V) \hookrightarrow \mathbb{P} := \mathbb{P}(\Lambda^{r_1}V^*) \times \ldots \times \mathbb{P}(\Lambda^{r_t}V^*)$. Ample line bundles on $\mathrm{Fl}(r, V)$ are pullbacks of the form $\iota^*(s_1, \ldots, s_t)$, where $s_j > 0$ for all $j$.

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The higher direct images of any vector bundle of the form

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are described by the Borel-Bott-Weil theorem \cite{16} (4.1.4 Theorem). For relatively ample line bundles, the theorem takes the following form.

**Proposition 9.** Let $\nu$ and $r$ be partitions of the same length $t$, and let

$$\hat{\nu} = (\nu_1^\delta_1, \nu_2^\delta_2, \ldots, \nu_t^\delta_t) - 1,$$

where the powers mean repeated entries and $\delta_i = r_i - r_{i-1}$. Define the line bundle

$$\mathcal{L}(\nu) = \prod_{i=1}^{t} (\det Q_i)^{\nu_i + t - i}$$

Then by the Borel-Bott-Weil theorem we have

$$\pi_* \mathcal{L}(\nu) = E^{\hat{\nu}} + r',$$

and

$$R^i \pi_* \mathcal{L}(\nu) = 0,$$

for $i > 0$.

**Remark 10.** Conversely, for any Schur power $E^\lambda$ of a vector bundle one can find a unique partitions $\nu$ and $r$ such that the line bundle constructed using Equation (1) pushes forward to $E^\lambda$.

**Lemma 11.** There exists an $m_0$ such that

$$\mathcal{L}(\nu)_m := \mathcal{L}(\nu) \otimes \pi^* L^m,$$

is ample for $m > m_0$.

**Proof.** There is an $m > 0$ such that

$$(\text{Fl}(r, E), \mathcal{L}(\nu)_m) \simeq (\text{Fl}(r, E \otimes L^k), \mathcal{L}(\nu)),$$

where $k$ depends on $m$ and $E \otimes L^k$ is ample. The vector bundles $\Lambda^i(E \otimes L^k)$ are ample for all $i = 1, \ldots, e$ by \cite{7} Corollary 5.3, so by definition the (1) line bundles $\mathbb{P}(\Lambda^i(E \otimes L^k))$ are ample for $i = 1, \ldots, e$. We can regard $\text{Fl}(r, E)$ as a subvariety in the product

$$\mathbb{P}(\Lambda^{r_1}(E \otimes L^k)^*) \times \cdots \times \mathbb{P}(\Lambda^{r_t}(E \otimes L^k)^*),$$

where the line bundle $\mathcal{L}(\nu)$ is the restriction of

$$(\nu_1 - \nu_2 + 1, \nu_2 - \nu_3 + 1, \ldots, \nu_{t-1} - \nu_t + 1, \nu_t),$$

which is ample. \hfill \Box

### 2.3. Chern characters of Schur powers.

In this section we compute Chern classes that will be needed later. This is done in standard fashion using the splitting principle \cite{5} Remark 3.2.3, which reduces the problem to the case of decomposable vector bundles, and the following lemma.

**Lemma 12** (Determinantal identity, \cite{2}). Let $\lambda$ be a partition of length $l$. The Chern character of a Schur bundle is

$$\text{ch}E^\lambda = \det \left( \text{ch} (\text{Sym}^{\lambda} + j - i E) \right)_{i,j}$$  (3)
Proof. By the splitting principle we may assume that \( E = L_1 \oplus \ldots \oplus L_e \). Let \( p \) be a polynomial function on the set of factors \( L_1, \ldots, L_e \) with integral coefficients \( a_I \) for \( I = (i_1, \ldots, i_e) \). We denote

\[
p(E) = \bigoplus_I (L_1^{i_1} \otimes \cdots \otimes L_e^{i_e})^{\otimes a_I},
\]

In particular, the Schur powers of decomposable vector bundles can be expressed in as

\[
E^\lambda = s_\lambda(E),
\]

where \( s_\lambda \) is the Schur polynomial of \( \lambda \) [4, p. 72]. We can write the Schur polynomial \( s_\lambda \) in terms of complete symmetric polynomials \( h_k \) as

\[
s_\lambda(E) = \det \left( h_{\lambda_i - j} \right)_{i,j},
\]

using the Giambelli formula [6, Appendix A]. Taking Chern characters on both sides completes the proof of the Lemma. \( \square \)

Lemma 13. Let \( E \) be a vector bundle of rank \( e \). The Chern character of the bundle \( \text{Sym}^k E \) is

\[
\left( \begin{array}{c} k + e - 1 \\ e - 1 \end{array} \right) \left( 1 + \frac{c_1(E)}{e} + A_1(E)k^2 + A_2(E)k + Z \right),
\]

where \( A_1(E), A_2(E) \in \mathbb{Q}[c_1(E), c_2(E)] \) are given by

\[
A_1(E) = \frac{c_1(E)^2 - c_2(E)}{e(e + 1)},
\]

\[
A_2(E) = \frac{c_1(E)^2}{2e(e + 1)} - \frac{c_2(E)}{e + 1}
\]

and \( Z \) is a sum of terms of Chow degree 3 and higher.

Proof. Given a partition \( \mu \), let \( m_\mu \) denote the monomial symmetric function of shape \( \mu \) and \( h_k \) the complete symmetric function of degree \( k \). We have

\[
\text{ch}(\text{Sym}^k E) = \text{ch} h_k(E)
\]

\[
= \text{ch} \sum_\mu m_\mu(E)
\]

\[
= \sum_\mu (1 + \mu_1 x_1 + \mu_1^2 x_1^2/2 + \ldots) \cdots (1 + \mu_e x_e + \mu_e^2 x_e^2/2 + \ldots)
\]

where the sum is over all \( e \)-tuples that sum to \( k \). The rest of the computation is an elementary enumeration problem. The Chow-degree one part of \( \text{ch}(\text{Sym}^k E) \) is

\[
\text{rank}(\text{Sym}^k E)_1 = \text{rank}(\text{Sym}^k E) \frac{c_1(E)}{e},
\]

where

\[
\text{rank}(\text{Sym}^k E) = \left( \begin{array}{c} k + e - 1 \\ e - 1 \end{array} \right).
\]

The degree two term can be written as

\[
\sum_{i=1}^{k} \sum_{j=1}^{k-i} ij \left( \begin{array}{c} n - 3 + k_i - j \end{array} \right) \sum_{l=0}^{e} x_l x_m \sum_{i=1}^{k} i^2 \left( \begin{array}{c} e - 2 + k_i - i \end{array} \right) \sum_{i=1}^{e} x_i^2/2,
\]

1E.g. \( m_{(2,1)}(x, y) = x^2 y + xy^2 \).
which using the combinatorial identities proved in the appendix simplifies to
\[
\frac{(k + e - 1)!}{(k - 2)!(n + 1)!} \sum_{m < l}^{e} x_m x_l + \frac{(n + 2k - 1)(k + n - 1)!}{(k - 1)!(n + 1)!} \sum_{m}^{e} x_m^2/2.
\]
Picking out rank \(S^k E\) as a common factor yields
\[
\text{ch}_2(S^k E) = \text{rank}(S^k E) \left( \frac{k(k - 1)}{e(e + 1)} \sum_{m < l}^{e} (x_m x_l) + (2k^2 + k(e - 1)) \sum_{m}^{e} x_m^2/2 \right)
\]
Recall that the Chern classes of \(E\), when written in terms of the \(x_i\), are
\[
c_1(E)^2 = \sum_{m=1}^{e} x_m^2 + 2 \sum_{m < l}^{e} x_m x_l
\]
and
\[
c_2(E) = \sum_{m < l}^{e} x_m x_l.
\]
Thus we have
\[
\text{ch}(S^k E) = \text{rank}(\text{Sym}^k E) \left( 1 + \frac{c_1(E)}{e} k + A_1(E) k^2 + A_2(E) k + Z \right),
\]
where
\[
A_1(E) = \frac{c_1(E)^2 - c_2(E)}{e(e + 1)}
\]
\[
A_2(E) = \frac{(e - 1)c_1(E)^2}{2e(e + 1)} - \frac{c_2(E)}{e + 1}
\]
and \(Z\) is a sum of terms of Chow degree 3 and higher.

**Remark 14.** A similar calculation for higher degree terms shows that there exists a polynomial \(T(k) \in \mathbb{Q}[c_j(E)]\) such that
\[
\text{ch}(\text{Sym}^k E) = \text{rank}(\text{Sym}^k E) T(k)
\]
such that the coefficient of any Chow degree \(j\) term of \(T(k)\) is a polynomial of degree \(j\) in \(k\).

**Conjecture 1.** Let \(\lambda\) be an arbitrary partition. For a fixed \(e = \text{rank} E\) there exist \(G_i(E, \lambda) \in \mathbb{Q}[c_j(E), j = 1, 2, \ldots, n]\)
\[
\text{ch} E^\lambda = \text{rank} E^\lambda (1 + G_1(E, \lambda) + G_2(E, \lambda) + \ldots + G_n(E, \lambda))
\]
where \(G_i(E, \lambda)\) is of degree \(i\) both as elements of the Chow ring of \(X\) and as polynomials in \(\lambda\). Let \(\Lambda_i\) be the \(i\)th elementary symmetric function on \(\lambda\). The polynomials \(G_1(E, \lambda)\) and \(G_2(E, \lambda)\) are given by
\[
G_1(E, \lambda) = \frac{\Lambda_1}{c_1(E)}
\]
and
\[
G_2(E, \lambda) = H_1(\lambda)c_1(E)^2 + H_2(\lambda)c_2(E) + H_3(\lambda)A_2(E),
\]
where
\[
H_1(\lambda) = \frac{\Lambda_2}{e(e + 1)}
\]
\[
H_2(\lambda) = \frac{2e\Lambda_2 - (e - 1)\Lambda_1^2}{(e - 1)e(e + 1)}
\]
\[ H_3(\lambda) = \frac{e\Lambda_1 - \sum_i (2i-1)\lambda_i}{e-1}. \]

**Theorem 15.** Conjecture [1] holds for partitions up to length 3.

**Proof.** This is an elementary calculation using Lemma 12 and Lemma 13. \(\square\)

**Remark 16.** If \(r\) is a strictly decreasing partition and \(\lambda\) is its dual, the length of \(\hat{\nu} + \lambda\) (see Equation (2)) is the largest flag parameter \(r_1\).

3. **Test configurations induced from extensions**

3.1. **Definition of the test configuration.** In this section we construct a test configurations for \((\text{Fl}(r, E), \mathcal{L}_{m})\) which is destabilising if and only if a corresponding subbundle \(F\) of \(E\) destabilising. The test configuration is the flag variety of a vector bundle \(\mathcal{E}\) that carries a natural \(\mathbb{C}^*\)-action. Recall the following construction.

**Definition 17** (Turning off an extension). Let \(F\) be a subbundle of \(E\) with cokernel \(G\). We construct a vector bundle \(\mathcal{E}\) over \(\mathbb{C} \times X\) with general fibre \(E\) and special fibre \(F \oplus G\) as follows. Let \(\mathcal{U}\) be the universal family over \(\text{Ext}^1(E/G, F)\) and let \(\iota : \mathbb{C} \times X \to \text{Ext}^1(E/G, F) \times X\) be the inclusion of the line through the extension

\[ 0 \to F \to E \to G \to 0 \]

Set \(\mathcal{E} = \iota^* \mathcal{U}\). We have an exact sequence

\[ 0 \to F \to \mathcal{E} \to G \to 0 \]

of vector bundles on \(\mathbb{C} \times X\), where \(F \to \mathcal{E}\) is isomorphic to the pullback of \(F\). The usual \(\mathbb{C}^*\)-action on \(\mathcal{E}\) lifts to a \(\mathbb{C}^*\)-action on \(E\) by letting \(t.v = tv\) for \(v \in F\) and extending to \(\mathcal{E}\) by zero. This action restricts to the central fibre \(F \oplus G\) by scaling \(F\) with weight +1 and acting trivially on \(G\).

Let \(\alpha > \beta > 0\) be integers. By rescaling the trivial action on \(\mathcal{E}\) and \(F\), we may construct an action on \(\mathcal{E}\) that acts on the central fibre by scaling \(F\) and \(G\) with weights \(\alpha\) and \(\beta\), respectively. This action lifts to any Schur bundles of \(\mathcal{E}\).

Let \(\mathcal{L}_\mathcal{E} = \mathcal{L}_\mathcal{E}(\nu)\) be the line bundle on \(\text{Fl}(r, \mathcal{E})\) defined in Section 2.2. The \(\mathbb{C}^*\)-action on \(E\) induces a linearised action on \(\text{Fl}(r, \mathcal{E})\). We extend this action trivially to the line bundles \(\mathcal{L}_{\mathcal{E}, m}\) for all \(m > 0\). These \(\mathbb{C}^*\)-actions will all be denoted by \(\rho\) when \(\alpha\) and \(\beta\) and the polarisation are clear from context.

**Claim 18.** There is an \(m_0\) such that \((\mathcal{X}, \mathcal{L}_{\mathcal{E}, m}, \rho_{\alpha, \beta})\) is a test configuration for all \(m > m_0\).

**Proof.** The family is flat since the projections are Zariski locally trivial. By applying Lemma 11 to the general fibre and the fibre over 0, we find an \(m_0\) such that the line bundle \(\mathcal{L}_{\mathcal{E}, m}\) is ample relative to the projection to \(\mathbb{C}\) for \(m > m_0\). \(\square\)

**Claim 19.** If \(X\) is a curve, \(E\) is ample and \(F\) is has maximal slope, then \((\mathcal{X}, \mathcal{L}_{\mathcal{E}, m}, \rho_{\alpha, \beta})\) is a test configuration.

**Proof.** By the proof of Claim 18 we just need to show that \(F \oplus G\) is ample. A stable bundle on a curve is ample if and only if its degree is positive [8, Section 2]. \(F\) is stable since its slope is maximal, and all of its subbundles have positive degree since \(E\) is ample. Therefore \(F\) is ample. \(G\) is ample since it is a quotient of an ample bundle [7, Proposition 2.2]. Finally, direct sums of ample bundles are ample so \(F \oplus G\) is ample. \(\square\)
3.2. Computation of the Futaki invariant. We compute the integers $w(k)$ for the action defined in Section 3 using an idea due to Donaldson [3] of relating the total weight with the Hilbert polynomial of a fibre bundle over $\mathbb{P}^1$. Let $\pi_1 : X \times \mathbb{P}^1 \to X$ be the first projection.

Lemma 20. Let $E = F \oplus G$ and assume that $E \otimes L^\frac{m}{n}$ is ample. Let $\rho$ be a $\mathbb{C}^*$-action on $E$ that scales the subbundles $F$ and $G$ by weights $\alpha > 0$ and $\beta > 0$, respectively. Finally, fix a partition $\lambda$. Then there exists a vector bundle $E_{\alpha, \beta}$ such that the total weight $w(\lambda)$ of the induced action on the vector space

$$H^0 \left( X, \Lambda^\text{top} \left( E^\lambda \otimes L^m \right) \right)$$

and the dimension of the vector space $H^0(\mathbb{P}^1 \times \mathbb{P}^1, E_{\alpha, \beta}^\lambda \otimes \pi^* L^m)$ are related by

$$h^0(\mathbb{P}^1 \times \mathbb{P}^1, E_{\alpha, \beta}^\lambda \otimes \pi^* L^m) = w(\lambda) + h^0(X, E^\lambda \otimes L^m).$$

Proof. Use the natural isomorphism

$$(\text{Fl}(r, E), L_{k(\lambda)}) \cong (\text{Fl}(r, E \otimes L^k), L),$$

and choose the parameter $m$ such that $L^\frac{m}{n}$ is ample. The line bundle $L^m$ plays no other role in this calculation since $(E \otimes L^k)^\lambda = L^{k(\lambda)} \otimes E^\lambda$. Therefore we may as well assume that $m = 0$ and that $E$ is ample. Define the vector bundle

$$E_{\alpha, \beta} = F(\alpha) \oplus G(\beta) = \pi_1^* F \otimes \pi_2^* (\alpha) \oplus \pi_1^* G \otimes \pi_2^* (\beta)$$
on $X \times \mathbb{P}^1$. By the Littlewood-Richardson rule (see [16, Section 2.3]) we can write

$$E^\lambda = \bigoplus_{\nu, \mu, \lambda} (F(\alpha)^\nu \otimes G(\beta)^\mu)^{N_{\nu, \mu, \lambda}}$$

where the sum is over all partitions $\nu$ and $\mu$ whose sizes sum up to the size of $\lambda$ and the coefficient $N_{\nu, \mu, \lambda}$ is the Littlewood-Richardson coefficient. Using the projection formula, Riemann-Roch and additivity of the Euler characteristic we see that

$$\chi(X \times \mathbb{P}^1, E_{\alpha, \beta}^\lambda) = \sum_{\nu, \mu, \lambda} N_{\nu, \mu, \lambda} \chi(X \times \mathbb{P}^1, F^\nu \otimes G^\mu \otimes \mathbb{P}^1(\nu|\alpha + |\mu|\beta))$$

$$= \sum_{\nu, \mu, \lambda} (\nu|\alpha + |\mu|\beta + 1)N_{\nu, \mu, \lambda} \chi(X, F^\nu \otimes G^\mu)$$

$$= \chi(X, E^\lambda) + \sum_{\nu, \mu, \lambda} (\nu|\alpha + |\mu|\beta) \chi \left( X, (F^\nu \otimes G^\mu)^{\otimes N_{\nu, \mu, \lambda}} \right).$$

Higher cohomology in the Euler characteristics above vanishes as $F, G$ and $E$ are ample and Schur powers of ample bundles are ample [21, Theorem 5.2]. The vector space

$$H^0(X, (F^\nu \otimes G^\mu)^{\otimes N_{\nu, \mu, \lambda}})$$

is precisely the space vectors of weight $|\nu|\alpha + |\mu|\beta$. This completes the proof. □

Let $\nu$ be a partition of length $t$ and $L(\nu)$ the corresponding line bundle and denote $\lambda = (\nu + r^t)$. The coefficients of the Hilbert polynomials

$$h^0(X, E^{k, \lambda}) = a_0 k^n + a_1 k^{N-1} + O(k^{N-2}) \quad (9)$$

and

$$h^0(\mathbb{P}^1, E_{\alpha, \beta}^{k, (\nu + r^t)}) = b_0 k^{N+1} + b_1 k^N + O(k^{N-1}). \quad (10)$$
The Donaldson-Futaki invariant can then be computed from
\[
F(X, L, E, m, \rho_{\alpha, \beta}) = \frac{b_0 a_1}{a_0^2} - \frac{b_1}{a_0} + 1. \tag{12}
\]

Remark 21. Notice that any common factor in the Hilbert polynomials appearing in equations (9) and (10) cancels out in the Donaldson-Futaki invariant.

3.3. Flag variety over a curve. We will carry out the computation of the Donaldson-Futaki invariant for \(X\) a smooth curve of genus \(g\). Let \(L\) be an ample line bundle on \(X\) and \(E\) an ample unstable vector bundle of rank \(e\) on \(X\). Assume that \(r = (r_1, \ldots, r_t)\) is one of the partitions listed in Theorem 1. Fix a destabilising subsheaf \(F\) of \(E\) with maximal slope. The saturation, which by definition has a torsion free quotient, also destabilises. Torsion free coherent sheaves on a curve are locally free, so we may assume that \(F\) is a subbundle.

Let \(\mathcal{L} = \mathcal{L}(\nu)\) be a relatively ample line bundle on \(\mathbb{F} = \text{Fl}(r, E)\) that corresponds to a partition \(\nu\), and \(\lambda = (\hat{\nu} + \nu')\), where \(\hat{\nu}\) is defined in Proposition 9. We will show that the induced test configuration \((X, L, E, m, \rho_{\alpha, \beta})\) constructed in Section 3.1 has the Futaki invariant
\[
C(\mu(E) - \mu(F)),
\]
where \(C\) is a positive number depending on \(X, L, E, F\) and \(\lambda\). It follows that \(\text{Fl}(r, E)\) is destabilised by the induced test configuration.

By Riemann-Roch we have
\[
h^0(\mathbb{F}, \mathcal{L}^k) = \text{rank } E^\lambda (a_1 k + a_0),
\]
where
\[
a_0 = \frac{A_1}{e} \deg E,
\]
\[
a_1 = 1 - g.
\]
Using equations (7) and (8) and Riemann-Roch on \(X \times \mathbb{P}^1\) we have
\[
h^0(X \times \mathbb{P}^1, E^{k, \lambda}_{\alpha, \beta}) = \text{rank } E^{k, \lambda}_{\alpha, \beta} (b_0 k^2 + b_1 k + b_2),
\]
where
\[
b_0 = H_1(\lambda)c_1(E_{\alpha+1, \alpha})^2 + H_2(\lambda)c_2(E_{\alpha+1, \alpha}) - \frac{A_1}{2e} c_1(E_{\alpha+1, \alpha}).K_{X \times \mathbb{P}^1}
\]
and \(H_1, H_2\) and \(H_3\) are from Conjecture 1.

Plugging the intersection classes
\[
c_1(E_{\alpha, \beta})^2 = 2 \deg E(\alpha e + f)
\]
\[
c_2(E_{\alpha, \beta}) = \deg E + (c_1(E).f)(\alpha(e - 1) + f)
\]
into Equation (5) we see that
\[
A_2(E_{\alpha+1, \alpha}) = \frac{(e - 1)c_1(E_{\alpha+1, \alpha})^2}{2e(e + 1)} - \frac{c_2(E_{\alpha+1, \alpha})}{e + 1}
\]
\[
= -\frac{f(\mu(E) - \mu(F))}{e + 1}.
\]
Notice that we have
\[ a_1 b_0 = -(g - 1)(e + 1)H_2(\lambda)A_2(E_{\alpha+1,\alpha}) - \frac{(g - 1)\Lambda_1^2}{2e^2} c_1(E_{\alpha+1,\alpha})^2 \] (13)
and
\[
\begin{align*}
a_0 b_1 &= \frac{\Lambda_1}{e} \deg E \cdot H_3(\lambda)A_2(E_{\alpha+1,\alpha}) - \frac{\Lambda_1^2 \deg E}{2e^2} c_1(E_{\alpha,\beta})K_{X \times \mathbb{P}^1} \\
&= \frac{\Lambda_1}{e} \deg E \cdot H_3(\lambda)A_2(E_{\alpha+1,\alpha}) + a_0^2 - \frac{(g - 1)\Lambda_1^2}{2e^2} c_1^2(E_{\alpha+1,\alpha}),
\end{align*}
\] (14)
which follows from
\[
c_1(E_{\alpha+1,\alpha}).K_{X \times \mathbb{P}^1}/2 = ((e \alpha + f)\mu E \cdot g)(-f + (g - 1)\mu g) = -\deg E + \frac{(g - 1)c_1(E_{\alpha+1,\alpha})^2}{2 \deg E}
\]
where \( f \cong X \) and \( g \cong \mathbb{P}^1 \) are the fibres of the two projections.
Substituting the expressions from equations (13) and (14) into Equation (12) leads to
\[
a_0^2 \cdot F(X, L_e, \rho_{\alpha+1,\alpha}) = a_1 b_0 - a_0 b_1 + a_0^2 = ((e + 1)(g - 1)H_2(\lambda) - \Lambda_1 H_3(\lambda) \deg E/e) A_2(E_{\alpha+1,\alpha}) = C(\mu(E) - \mu(F)),
\]
where
\[
C = \frac{f}{(e - 1)e(e + 1)} \left( (g - 1) ((e - 1)\Lambda_1^2 - 2e\Lambda_2) + \Lambda_1 (e\Lambda_1 - \sum_i (2i - 1)\lambda_i) \deg E \right).
\]

Using \( e - 1 \geq r_1 \) and recalling that \( r_1 \) is the length of \( \lambda \), we can write
\[
(e - 1)\Lambda_1^2 - 2e\Lambda_2 = (e - 1) \sum_{i=1}^{r_1} \lambda_i^2 - 2 \sum_{1 \leq i < j \leq r_1} \lambda_i \lambda_j \geq \sum_{1 \leq i < j \leq r_1} (\lambda_i - \lambda_j)^2 + \sum_{i=1}^{r_1} \lambda_i^2 > 0
\]
To see that the second term of \( C \) is positive, notice that
\[
\sum_i (2i - 1)\lambda_i = \sum_{j=1}^{s} (\lambda'_j)^2,
\]
where \( \lambda' \) denotes the conjugate partition of \( \lambda \). Letting \( s \) denote the length of \( \lambda' \), we have
\[
e\Lambda_1 - \sum_i (2i - 1)\lambda_i = \sum_{j=1}^{s} \lambda'_j (e - \lambda'_i) > 0,
\]
which is positive since \( e > r_1 \geq \lambda'_i \) for all \( i \). This completes the proof of Theorem [1].

3.4. The twisted polarisation \( L_m \). Let \( X \) be a smooth projective variety of dimension \( n \) with an ample polarisation \( L \). Let \( E \) be a vector bundle of rank \( e \) with a destabilising subbundle \( F \). Let \( F = Fl(r, E) \) be the flag variety of quotients in \( E \) with \( r \) as in Theorem 2. Fix a \( \pi \)-ample line bundle \( L = L(\nu) \) and denote \( \lambda = \nu + \nu' \) (see Proposition 7). We will show that the leading term in the Futaki invariant of
\[
(X, L_{e,m}, \rho_{\alpha+1,\alpha})
\]
is

\[ D(\mu(E) - \mu(F)), \]

where \( D \) is a positive number depending on \( X, L, E, F \) and \( \lambda \).

The Hilbert polynomial of \((E \otimes L^m)^\lambda\) will be computed from

\[
\chi(\mathcal{F}, L^k) = \chi(X, E^\lambda \otimes L^{mk}) \\
= \int_X e^{mk\omega} \text{ch}(E^\lambda) \text{Td}(X) \\
= \frac{(mk)^n}{n!}\omega^n \cdot \text{rank}(E^\lambda) \\
+ \frac{(mk)^{n-1}}{(n-1)!}\omega^{n-1} \left( \text{rank}(E^\lambda) \frac{c_1(X)}{2} + \frac{Rc_1(E^\lambda)}{e} \right) \\
+ \frac{(mk)^{n-2}}{(n-2)!}\omega^{n-2} \left( \text{rank}(E^\lambda) \text{Td}_2(X) + \frac{Rc_1(E^\lambda) c_1(X)}{2e} + \text{ch}_2(E^\lambda) \right) \\
+ (k^{n-3}),
\]

which follows from Riemann-Roch and the pushforward formula of Proposition 9. Here \( \text{Td}_2(X) \) is the second Todd class of \( X \). Using Riemann-Roch on \( X \times \mathbb{P}^1 \), we similarly compute the Hilbert polynomial of \( E_{a+1, a}^\lambda \otimes \pi^*_1 L^m \), where \( \pi_1 \) is the first projection.

To apply Lemma 20, choose \( m_0 \) so that the bundle \( E \otimes L^{m_0} \) is ample and assume from now on that \( m > m_0 \).

As in Section 3.3 we write

\[
h^0(X, E^\lambda \otimes L^{mk}) = \text{rank}(E^\lambda)(a_0 k^n + a_1 k^{n-1} + O(k^{n-2})), \\
h^0(X \times \mathbb{P}^1, E_{a+1, a}^\lambda \otimes L^{mk}) = \text{rank}(E^\lambda)(b_0 k^n + b_1 k^{n-1} + O(k^{n-2})).
\]

Next, we expand the \( a_i \) and the \( b_i \) in powers of \( m \) as

\[
b_0 = b_{0,0} m^{n+1} + b_{0,1} m^n + (m^{n-1}) \\
b_1 = b_{1,0} m^{n+1} + b_{1,1} m^n + (m^{n-1}) \\
a_0 = a_{0,0} m^n + a_{0,1} m^{n-1} + (m^{n-2}) \\
a_1 = a_{1,0} m^n + a_{1,1} m^{n-1} + (m^{n-2}).
\]
Let $\omega = c_1(L)$ and $\eta = \pi^*_1 \omega$. From equation (15), we see that

\[
\begin{align*}
    b_{0,0} &= \frac{R}{e \cdot n!} \eta^n \cdot c_1(E_{\alpha+1,\alpha}) \\
    b_{0,1} &= \frac{1}{(n-1)!} \eta^{n-1} \cdot (H_1(\lambda)c_1(E_{\alpha+1,\alpha})^2 + H_2(\lambda)c_2(E_{\alpha+1,\alpha})) \\
    b_{1,0} &= -\frac{\eta^n \cdot K_{X \times \mathbb{P}^1}}{2 \cdot n!} \\
    b_{1,1} &= \frac{1}{(n-1)!} \left( \eta^{n-1} \cdot H_3(\lambda) A_2(E_{\alpha+1,\alpha}) - R \eta^{n-1} \frac{K_{X \times \mathbb{P}^1} \cdot c_1(E_{\alpha+1,\alpha})}{2e} \right) \\
    a_{0,0} &= \frac{\omega^n}{n!} = \deg X \\
    a_{0,1} &= \frac{\omega^{n-1} \cdot c_1(E)}{e(n-1)!} \\
    a_{1,0} &= 0 \\
    a_{1,1} &= -\frac{\omega^{n-1} \cdot K_X}{2(n-1)!} = -\frac{\deg K_X}{2(n-1)!}.
\end{align*}
\]

The proof of the following lemma is a straightforward calculation.

**Lemma 22.** The intersection numbers appearing above are

\[
\begin{align*}
    \omega^n &= \deg X \\
    \omega^{n-1} \cdot c_1(E) &= \deg E \\
    \omega^{n-1} \cdot K_X &= \deg K_X \\
    \eta^n \cdot c_1(E_{\alpha+1,\alpha}) &= \deg X(e\alpha + f) \\
    \eta^{n-1} \cdot c_1(E_{\alpha+1,\alpha})^2 &= 2 \deg(e\alpha + f) \\
    \eta^{n-1} \cdot c_2(E_{\alpha+1,\alpha}) &= \deg F + \deg E(\alpha(e-1) + f)\alpha \\
    \eta^{n-1} \cdot K_{X \times \mathbb{P}^1} \cdot c_1(E_{\alpha+1,\alpha}) &= \deg K_X(e\alpha + f) - 2 \deg E \\
    \eta^n \cdot K_{X \times \mathbb{P}^1} &= -2 \deg X \\
    \eta^n \cdot A_2(E_{\alpha+1,\alpha}) &= \frac{f(\mu(E) - \mu(F))}{e + 1}.
\end{align*}
\]

We expand the Futaki-invariant in powers of $m$

\[
a_0^2 F(X, \mathcal{L}_m, \rho_{\alpha+1,\alpha}) = F_0 m^{2n+1} + F_1 m^{2n} + (m^{2n-1}),
\]

where

\[
F_0 = a_{1,0} b_{0,0} - a_{0,0} b_{1,0} + a_{0,0}^2 = -\left( \frac{\deg X}{n!} \right)^2 + \left( \frac{\deg X}{n!} \right)^2 = 0
\]

and

\[
F_1 = a_{1,0} b_{0,1} + a_{1,1} b_{0,1} - a_{0,1} b_{1,0} - b_{1,1} a_{0,0} + 2 a_{0,0} a_{0,n-1}.
\]

An elementary calculation similar to the one we did in Section 3.3 shows terms without the factor $A_2(E_{\alpha+1,\alpha})$ cancel and we get

\[
F(X, \mathcal{L}_m) = D(\mu(E) - \mu(F)) + O(m^{-1})
\]
where
\[ D = \frac{\int u(e) \lambda_i - \sum_i (2i - 1) \lambda_i}{(e - 1)(e + 1) \deg X} \]
is a positive constant by the same argument as in Section 3.3. This completes the proof of Theorem 2.

APPENDIX A. COMBIPROOFS

We include the proofs of the combinatorial formulae for completeness.

Lemma 23. Let \( k \) and \( n \) be integers and let \( p(k) = \binom{k + n - 1}{n - 1} \). Then
\[ \sum_i i^2 \binom{n - 2 + k - i}{n - 2} = \frac{(n + 2k - 1)(k + n - 1)!}{(k - 1)!(n + 1)!} = (2k^2 + k(n - 1))p(k) \]
and
\[ \sum_{i,j} ij \binom{n - 3 + k - i - j}{n - 3} = \frac{(k + n - 1)!}{(k - 2)!(n + 1)!} = k(k + 1)p(k). \]

Proof. We prove the first identity by induction on \( n \) and \( k \). Let
\[ f(k, n) = \sum_i i^2 \binom{n - 2 + k - i}{n - 2} \]
Using the identity
\[ \binom{n}{k} = \binom{n - 1}{k - 1} + \binom{n - 1}{k} \]
which holds for all \( 0 \leq k \leq n - 1 \) we see that
\[ f(k, n) = \sum_{i=1}^{k} i^2 \binom{n - 2 + k - i}{n - 2} = k^2 + \sum_{i=1}^{k-1} i^2 \left( \binom{n - 3 + k - i}{n - 3} + \binom{n - 2 + (k - 1) - i}{n - 2} \right) \]
\[ = k^2 + f(k, n - 1) - k^2 + f(k - 1, n) = f(k - 1, n) + f(k, n - 1). \]
Finally we verify that
\[ \frac{(n + 2k - 3)(k + n - 2)!}{(k - 2)!(n + 1)!} + \frac{(n + 2k - 2)(k + n - 2)!}{(k - 1)!(n + 1)!} = \frac{(n + 2k - 1)(k + n - 1)!}{(k - 1)!(n + 1)!}. \]
This completes the induction step. The base case follows from verifying the cases \( f(k, 2) \) and \( f(1, n) \).

The proof of the second identity is almost identical. Let
\[ g(k, n) = \sum_{i=1}^{k} \sum_{j=1}^{k-i} ij \binom{n - 3 + k - i - j}{n - 3}. \]
Again we have
\[
g(k, n) = \sum_{i=1}^{k} i(k - i) + \sum_{i=1}^{k-1} \sum_{j=1}^{k-i-1} ij \left( \frac{n - 4 + k - i - j}{n - 4} + \frac{n - 3 + k - 1 - i - j}{n - 3} \right)
\]
\[
= \sum_{i=1}^{k} i(k - i) + g(k, n - 1) - \sum_{i=1}^{k} i(k - i) + g(k - 1, n)
\]
\[
= g(k - 1, n) + g(k, n - 1).
\]
Verify the right hand side as above by computing
\[
\frac{(k + n - 2)!}{(k - 3)!(n + 1)!} + \frac{(k + n - 2)!}{(k - 2)!n!} = \frac{(k + n - 1)!}{(k - 2)!(n + 1)!}.
\]
The base case follows from verifying the cases \(g(k, 2)\) and \(g(1, n)\). □

Remark 24. Higher Chern classes of symmetric bundles can be computed from a general formula for \(f(k, n, J)\), where
\[
f(k, n, J) = \sum_{i_1=1}^{k-1} \sum_{i_2=1}^{k-(i_1+\ldots+i_{q-1})} \cdots \sum_{i_q=1}^{k-(i_1+\ldots+i_{q-1})} i_1^{j_1} \cdots i_q^{j_q} \left( \frac{n + k - \sum_i (J_i) - q - 1}{n - q - 1} \right),
\]
where we denote \(|J| = q\).

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