PATH AND QUASIHOMOTOPY FOR SOBOLEV MAPS BETWEEN MANIFOLDS

ELEFTERIOS SOULTANIS

Abstract. We study the relationship between quasihomotopy and path homotopy for Sobolev maps between manifolds. We employ singular integrals on manifolds to show that, in the critical exponent case, path homotopy implies quasihomotopy – and observe the rather surprising fact that \( n \)-quasihomotopic maps need not be path homotopic. We also study the case where the target is an aspherical manifold, e.g. a manifold with nonpositive sectional curvature, and the contrasting case of the target being a sphere.

1. Introduction

Let \( M \) and \( N \) be compact Riemannian manifolds with \( n = \dim M \geq 2 \). The study of harmonic and \( p \)-harmonic maps between \( M \) and \( N \) naturally leads to questions about homotopies between finite energy Sobolev maps \([10, 9, 8, 36, 29]\).

However classical homotopy is incompatible with Sobolev maps: on one hand Sobolev maps need not be continuous, and on the other classical homotopy classes are not stable under convergence in the Sobolev norm. Indeed, an easy example by B. White \([37]\) showed that the identity map \( S^3 \to S^3 \) is homotopic to maps of arbitrarily small energy, whilst not being homotopic to a constant map.

F. Burstall, in \([5]\), studied energy minimization within classes of maps with prescribed 1-homotopy class, and White \([37]\) introduced the notion of \( d \)-homotopy for an integer \( d \leq n = \dim M \).

Two maps \( u,v \in W^{1,p}(M; N) \) are \( d \)-homotopic, \( d < p \), if the restrictions of \( u \) and \( v \) to a \( d \)-skeleton of a generic triangulation of \( M \) (which are continuous by the Sobolev embedding theorem) are classically homotopic.

White proved \([37, 38]\) that Sobolev maps \( u \in W^{1,p}(M; N) \) \((p \leq n)\) have a well defined \( ([p] - 1) \)-homotopy type (i.e. the homotopy class of the restriction of \( u \) does not depend on the generic \( ([p] - 1) \)-dimensional skeleton) that is stable under weak convergence in \( W^{1,p}(M; N) \), and therefore well suited for variational minimization problems.

Connections of \( d \)-homotopy with the topology of the Sobolev space \( W^{1,p}(M; N) \) are already visible in \([37]\). The notion of path homotopy, introduced by H. Brezis and Y. Li in \([5]\) utilizes this idea.

Two maps \( u,v \in W^{1,p}(M; N) \) \((1 < p < \infty)\) are path homotopic if there exists a continuous path \( h \in C([0,1]; W^{1,p}(M; N)) \) joining \( u \) and \( v \).

Date: November 16, 2018.

Key words and phrases. Function spaces, Sobolev mappings, Riemannian manifolds, Homotopy.

This research was conducted in the University of Jyväskylä and at IMPAN, Warsaw.
They proved [3, Theorem 0.2] that $W^{1,p}(M; N)$ is always path connected when $1 < p < 2$, while a deep result of Hang and Lin [15] states that, for $1 < p < n$, two maps $u, v \in W^{1,p}(M; N)$ are path homotopic if and only if they are $(\lfloor p \rfloor - 1)$-homotopic.

When $p = n$ this equivalence does not remain valid. Instead, Sobolev maps $u, v \in W^{1,n}(M; N)$ have well-defined homotopy classes (due to the density of Sobolev maps [30] and a result of White [37], see Theorem 2.1.) When $p > n$ the Sobolev embedding implies that Sobolev maps are continuous and indeed by results in Appendix A in [3] path homotopy is equivalent to classical homotopy.

With the emergence of analysis on metric spaces (see [14, 18, 17, 33] and the monographs [1, 19]) the study of energy minimization problems between more general spaces has become viable. The first steps in this direction were taken by N. Korevaar and R. Schoen [26] – who studied the existence of minimizers of 2-energy in homotopy classes of maps from a manifold to a nonpositively curved metric space (see [4]) – and J. Jost [20, 21, 22, 23] who studied the related problem of minimizing 2-energy in equivariance classes of maps from $(1, 2)$-Poincaré space spaces to nonpositively curved metric spaces.

In the more general setting both $d$-homotopy and path homotopy become problematic. The lack of triangulations in metric spaces on the one hand, and the fact that the topology of Newton Sobolev spaces $N^{1,p}(X; Y)$ depends on the embedding of $Y$ into a Banach space (see [13]) on the other, make both notions of homotopy difficult to work with.

In [35], for the purpose of studying minimizers of $p$-energy in homotopy classes of maps from a $(1, p)$-Poincaré space to a nonpositively curved metric space a third notion, called $p$-quasihomotopy, was introduced. Here we state the definition for manifolds. It is based on the known fact that Sobolev maps $u \in W^{1,p}(M; N)$ have $p$-quasicontinuous representatives, i.e. for every $\varepsilon > 0$ there is an open set $E \subset M$ with $\text{Cap}_p(E) < \varepsilon$ so that $u|_{M \setminus E}$ is continuous. Quasicontinuity may be seen as a refinement of the almost continuity of measurable maps.

Two quasicontinuous representatives $u, v \in W^{1,p}(M; N)$ ($1 < p < \infty$) are $p$-quasihomotopic if there is a map $H : M \times [0, 1] \to N$ with the following property: for any $\varepsilon > 0$ there is an open set $E \subset M$ with $\text{Cap}_p(E) < \varepsilon$ so that $H|_{M \setminus E \times [0, 1]}$ is a (continuous) homotopy between $u|_{M \setminus E}$ and $v|_{M \setminus E}$.

Capacity is a much finer measure of smallness than the Lebesgue measure; a set $E \subset M$ of zero $p$-capacity has Hausdorff dimension at most $n - p$, and sets of small $p$-capacity have small Hausdorff content,

$$\text{Cap}_p(E) \leq c(n, p, q)H^n_\infty(E)$$ for any $1 < q < p$. (Theorem 5.3 in [27].) Thus, while quasihomotopy allows for discontinuities, it does so in a sense a minimal amount, preserving some amount of topology. For example, a set of zero $p$-capacity, $p > 1$, does not separate a space, whereas a set of measure zero may. There is also a $p$-quasicontinuous counterpart to the fact that if the preimage of a point of a continuous function (from a connected space) is nonempty and open, then the function must be constant (see Lemma 5.3 in [35]).

As such, $p$-quasihomotopy is a natural relaxation of classical homotopy to encompass Sobolev maps. Indeed, under the additional assumption that the target
space has hyperbolic universal cover there always exists minimizers of $p$-energy in quasihomotopy classes in the metric setting, see Theorem 1.1. in [34].

When $p > n$ the fact any nonempty set has $p$-capacity $\geq \varepsilon_0$ for some small number $\varepsilon_0$ implies that $p$-quasihomotopy coincides with classical homotopy, and thus with path homotopy.

However when $1 < p < n$ the notion of $p$-quasihomotopy turns out to differ from the other two. Theorem 1.4 in [35] states that when $1 < p < n$, if $u, v \in W^{1,p}(M; N)$ are $p$-quasihomotopic then they are path homotopic. The proof in fact yields more: if $1 < p \leq n$ and $u, v \in W^{1,p}(M; N)$ are $p$-quasihomotopic then $u$ and $v$ are $d$-homotopic, where $d = \lceil p \rceil - 1$ is the largest integer $< p$. Since $\lceil p \rceil - 1 < \lfloor p \rfloor - 1$ unless $p$ is an integer it is expected that path homotopic maps need not be quasihomotopic. Indeed the constant map and

$$x \mapsto \frac{x}{|x|} \in W^{1,p}(B^2; S^1), \quad 1 < p < 2$$

are path homotopic but not $p$-quasihomotopic (see Section 4.2 in [35]).

The first main theorem in this paper considers the remaining case $p = n$.

**Theorem 1.1.** Let $M$ and $N$ be smooth compact Riemannian manifolds, with $n = \dim M$. If two maps $f, g \in W^{1,n}(M; N)$ are path homotopic then they are $n$-quasihomotopic.

The relationships between path-, quasi-, and $d$-homotopy are summarized in the table below.

| $W^{1,p}(M; N)$ | $1 < p < n$ | $p$-quasihomotopy $\Rightarrow$ $(\lfloor p \rfloor - 1)$-homotopy $\iff$ path homotopy |
|----------------|-------------|----------------------------------------------------------------------------------|
| $p = n$        | path homotopy $\Rightarrow$ $p$-quasihomotopy $\Rightarrow$ $(n - 1)$-homotopy |
| $p > n$        | $p$-quasihomotopy $\iff$ homotopy $\iff$ path homotopy |

Surprisingly, the converse of Theorem 1.1 fails. Namely it can happen that two maps $f, g \in W^{1,n}(M; N)$ are $n$-homotopic but not path homotopic. An example to this effect is given in Corollary 4.2. It is noteworthy that in the example the target has the rational homology type of a sphere (in this case it is in fact a sphere) in light of the discussion in [11] (see in particular Theorems 1.4 and 1.5 there). An $n$-manifold $M$ is a rational homology sphere if

$$H^k_{\text{dR}}(M) = \begin{cases} 0, & k \neq 0, n \\ \mathbb{Z}, & k = 0, k = n, \end{cases}$$

where $H^k_{\text{dR}}(M)$ denotes the de Rham cohomology of $M$.

For generic manifolds $M, N$, particularly rational homology sphere targets, the implications between path- and quasihomotopy depend on $p$.

In contrast, for aspherical target manifolds the situation is simpler. An $m$-manifold $N$ is aspherical if the homotopy groups $\pi_k(N)$ vanish for all $k \geq 2$. Using Whiteheads theorem (Theorem 4.5 in [16]) aspherical manifolds may be characterized as those with contractible universal cover. Aspherical manifolds include, as an important subclass, manifolds of nonpositive sectional curvature.

For general $p \in (1, \infty)$ we have the following theorem.
Theorem 1.2. Suppose $M$ and $N$ are compact smooth Riemannian manifolds, $N$ aspherical and $1 < p < \infty$. If two maps $u, v \in W^{1,p}(M; N)$ are $p$-quasihomotopic then they are path homotopic.

When $p \geq 2$ we can say more.

Theorem 1.3. Let $2 \leq p < \infty$, $M, N$ be smooth compact Riemannian manifolds, $N$ being aspherical. Then two maps $f, g \in W^{1,p}(M; N)$ are path homotopic if and only if they are $p$-quasihomotopic.

The restriction $p \geq 2$ is essential. Indeed by Theorem 0.2 in [3] the space $W^{1,p}(M; N)$ is always path connected when $1 < p < 2$, while there may exists distinct $p$-quasihomotopy classes (see the example above).

Outline. The proof of Theorem 1.1 is based on approximating a given Sobolev map with suitable mollified maps and showing the convergence is quasiuniform (Theorem 2.13). The second section is devoted to mollification and the use of singular integrals to accomplish this.

Section 3 deals with the aspherical case. For nonpositively curved targets Proposition 1.2 follows directly from Theorem 1.1 and Proposition 1.5 in [35] but the more general case of aspherical targets requires somewhat different arguments and the use of Theorem 2.13. Theorem 1.3 is an immediate consequence of Theorem 3.3 presented in this Section.

The last Section is devoted to proving that $W^{1,p}(M; S^k)$ is $p$-quasiconnected, i.e. any two maps in $W^{1,p}(M; S^k)$ are $p$-quasihomotopic, when $p \leq k$ (Proposition 4.1). Some of the auxiliary results (e.g. Proposition 4.3) may be interesting in themselves. Proposition 4.1 serves as an example showing that sometimes – though not in general – path homotopy - and $p$-quasihomotopy classes coincide.

The paper is closed by remarking that $W^{1,p}(B^{k+1}; S^k)$, while path connected when $p < k + 1$, is not $p$-quasiconnected for $k < p < k + 1$.

2. Critical exponent case

The proof strategy of Theorem 1.1 utilizes Brian White’s result.

Theorem 2.1 ([37], Theorem 0 and [2], Theorem 2). Two Lipschitz maps in $W^{1,n}(M; N)$ are path homotopic if and only if they are homotopic. Moreover for each $u \in W^{1,n}(M; N)$ there is a number $\varepsilon > 0$ so that if $\|u - v\|_{1,n} < \varepsilon$ then $u$ and $v$ are path homotopic.

Coupled with the fact, due to Schoen-Uhlenbeck [30], that $Lip(M; N)$ is dense in $W^{1,n}(M; N)$ the question, whether path homotopy implies $n$-quasihomotopy, is reduced to the following statement. For every $u \in W^{1,n}(M; N)$ and $\varepsilon > 0$ there is a Lipschitz map $u_\varepsilon$ with $\|u - u_\varepsilon\|_{1,n} < \varepsilon$ such that $u_\varepsilon$ is $n$-quasihomotopic to $u$.

We will construct such functions by means of mollifying the original function.

2.1. Mollifiers. Suppose $\psi : [0, \infty) \to [0, 1]$ is a Lipschitz cut-off function with $\text{spt} \psi \subset [0, 1)$. Given $r > 0$ define $\psi_r : M \to \mathbb{R}$ by

$$
\psi_r(p) = \int_M \psi \left( \frac{|p - z|}{r} \right) \, dz.
$$
Definition 2.2. Given $u \in L^p(M; \mathbb{R}^\nu)$ and $r > 0$ set

$$
\psi_r * u(p) = \frac{1}{\psi_r(p)} \int_M \psi \left( \frac{|z-p|}{r} \right) u(z) dz, \quad p \in M
$$

Lemma 2.3. For each $u \in L^1_{loc}(M; \mathbb{R}^\nu)$ and $r > 0$ the map $\psi_r * u : M \to \mathbb{R}^\nu$ is Lipschitz continuous. Moreover

$$
\psi_r * u(x) \leq CrM|u|(x)
$$

for almost every $x \in M$, with $C$, depending only on $\psi$, $M$ and $\nu$.

Proof. For $g \in L^1_{loc}(M)$ and arbitrary $x, y \in M$ we have

$$
\left| \int_M \psi \left( \frac{|x-z|}{r} \right) g(z) dz - \int_M \psi \left( \frac{|y-z|}{r} \right) g(z) dz \right| \\
\leq \text{Lip}(\psi) \frac{1}{B(x, r + d(x,y))} \int_{B(x, r + d(x,y))} \frac{|x-z| - |y-z|}{r} |g(z)| dz \\
\leq \text{Lip}(\psi) \frac{d(x, y)}{r} \int_{B(x, r + d(x,y))} |g| dz. 
$$

(2.1)

The lipschitz continuity of $\psi_r * u$ follows from this by expressing the difference $\psi_r * u(x) - \psi_r * u(y)$, where $d(x, y) < r$, as

$$
\frac{\psi_r(y) - \psi_r(x)}{\psi_r(x) \psi_r(y)} \int_M \psi \left( \frac{|x-z|}{r} \right) u(z) dz \\
+ \frac{1}{\psi_r(y)} \left( \int_M \psi \left( \frac{|x-z|}{r} \right) u(z) dz - \int_M \psi \left( \frac{|y-z|}{r} \right) u(z) dz \right)
$$

and applying (2.1) and the doubling property of the measure.

The estimate in the claim follows by a standard decomposition of the integral into annular regions, see [19, 17].

Lemma 2.4. (Schoen-Uhlenbeck) Let $u \in W^{1,p}(M; N)$. For $r > 0$ we have

$$
\text{dist}(N, \varphi_r * u(p)) \lesssim \left( \int_{B_r(p)} |Du|^n dz \right)^{1/n}
$$

for all $p \in M$. Consequently for each $u \in W^{1,n}(M; N)$ there is $r_0 > 0$ so that

$$
\sup_{p \in M} \text{dist}(N, \varphi_r * u(p)) < \varepsilon_0
$$

whenever $r < r_0$.

Proof. Let $p \in M$. For a.e. $z \in B_r(p)$

$$
\text{dist}(N, \varphi_r * u(p)) \leq \|u(z) - \varphi_r * u(p)\|.
$$

Taking an average integral over $B_r(p)$ we obtain

$$
\text{dist}(N, \varphi_r * u(p)) \leq \int_{B_r(p)} \|u(z) - \varphi_r * u(p)\| dz.
$$
By the $(1, n)$-Poincare inequality (which every manifold of dimension $n$ supports)

\[
\int_{B_r(p)} \|u(z) - \varphi_r * u(p)\|dz \leq \frac{1}{\varphi_r(p)} \int_{B_r(p)} \int_{B_r(p)} \varphi\left(\frac{|p-w|}{r}\right) \|u(z) - u(w)\|dzdw
\]

\[
\lesssim \int_{B_r(p)} \int_{B_r(p)} \|u(z) - u(w)\|dzdw
\]

\[
\lesssim r \left(\int_{B_r(p)} |Du|^n dz\right)^{1/n} \simeq \left(\int_{B_r(p)} |Du|^n dz\right)^{1/n}.
\]

The implied constants in the estimates depend only on the data of $M$ and on $N$. The second assertion follows directly from the absolute continuity of the measure $|Du|dz$. \hfill \Box

2.2. Singular integrals. Let us set some notation. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function and define the kernel $k_r : (0, \infty) \to \mathbb{R}$,

\[
k_r(t) = \frac{\varphi(t/r)}{t^{n-1}}.
\]

We abuse notation by writing

\[
k_r(p, q) = k_r(|p-q|), \ p, q \in M
\]

and finally, given $g \in L^p(M) \ (1 < p < \infty)$, we define the convolution

\[
k_r * g(x) = \int_M k_r(x, z)g(z)dz, \ x \in M.
\]

By the compactness of $M$ there exists $r_1$ so that

\[
\exp_x : B^n(r_1) \to B(x, r_1)
\]

is a 2-bilipschitz diffeomorphism for all $x \in M$. Thus, when $r < r_1$ we may use a change of variables given by the exponential map and write the integral above

\[
k_r * g(x) = \int_{B^n(r)} k_r(|\xi|)g(\exp_x \xi)J \exp_x \xi d\xi.
\]

Lemma 2.5. Let $1 < p < \infty$. Given $g \in L^p(M)$ the function $k_r * g$ has distributional gradient

\[
\nabla_x (k_r * g)v = -PV \int_M k'_r(|x-z|) (\nabla_x d_z, v)g(z)dz, \ v \in T_x M.
\]

Proof. We refer to [32, 6] for the existence and basic properties of singular integrals on manifolds (see in particular Chapter IV in [25] and the example in [32, D]).

The distributional derivative is determined by the condition

\[
\int_M \langle \nabla (k_r * g), V \rangle dx = -\int_M (k_r * g) \text{div} V dx
\]
for all smooth vector fields $V$ on $M$. We may write

$$\int_M PV \int_M k_r'(|x-z|)\langle \nabla_x d_z, V_z \rangle g(z) dz \, dx$$

$$= \lim_{\delta \to 0} \int_M \int_{M \setminus B_\delta(z)} k_r'|x-z|\langle \nabla_x d_z, V_z \rangle g(z) dz \, dx$$

(2.2) $$= \lim_{\delta \to 0} \int_M g(z) \int_{M \setminus B_\delta(z)} k_r'|x-z|\langle \nabla_x d_z, V_z \rangle dz \, dx .$$

Note that when $x \neq z$ the vector $\nabla_x d_z$ is the unit vector normal to $\partial B_\delta(z)$ at $x$. Thus $- \nabla_x d_z$ is the unit normal to $\partial (M \setminus B_\delta(z))$ at $x$. The divergence theorem gives

$$\int_{M \setminus B_\delta(z)} k_r'|x-z|\langle \nabla_x d_z, V_x \rangle dx$$

$$= - \int_{M \setminus B_\delta(z)} k_r(|x-z|) \text{div}_V x dx + \int_{\partial B_\delta(z)} k_r(|y-z|)\langle \nabla_y d_z, V_y \rangle d\sigma(y)$$

(2.3) $$= - \int_{M \setminus B_\delta(z)} k_r(|x-z|) \text{div}_V x dx + O(\delta).$$

The second term is $O(\delta)$ since it may be estimated using again the divergence theorem:

$$\left| \int_{\partial B_\delta(z)} k_r(|y-z|)\langle \nabla_y d_z, V_y \rangle d\sigma(y) \right| = \left| k_r(\delta) \int_{B_\delta(z)} \text{div}_V y dx \right| \lesssim \delta^{1-n} \delta^n .$$

Plugging (2.3) in (2.2) we obtain

$$\int_M PV \int_M k_r'|x-z|\langle \nabla_x d_z, V_z \rangle g(z) dz \, dx$$

$$= - \lim_{\delta \to 0} \int_M g(z) \int_{M \setminus B_\delta(z)} k_r(|x-z|) \text{div}_V x dx + \lim_{\delta \to 0} \int_M O(\delta) dz$$

(2.4) $$= - \lim_{\delta \to 0} \int_M \int_{M \setminus B_\delta(z)} k_r(|x-z|) g(z) \text{div}_V x dz dx = - \int_M (k_r * g) \text{div}_V x dx .$$

Thus we are done.

□

**Lemma 2.6.** The operators $g \mapsto k_r * g$ $(r > 0)$ are uniformly bounded

$$L^p(M) \to W^{1,p}(M),$$

i.e.

(2.5) $$\int_M |k_r * g(x)|^p dx + \int_M |\nabla_x (k_r * g)|^p dx \leq C \int_M |g|^p dx ,$$

g $\in L^p(M)$, for all $0 < r < r_1$.

**Proof.** For a.e. $x \in M$ we have, $v \in T_x M$ and $r > 0$

$$|\nabla_x (k_r * g)v| = \left| PV \int_M k_r'|x-z|\langle \nabla_x d_z, v \rangle g(z) dz \right|$$

$$\leq |v| \int_M \left| \frac{\phi'(|x-z|/r)}{r|x-z|^{n-1}} |g(z)| dz \right| + \left| PV \int_M \frac{\phi(|x-z|/r)}{|x-z|^n} \langle \nabla_x d_z, v \rangle g(z) dz \right| .$$
Using this and the estimate in Lemma 2.3 we obtain the estimate
\[ |k_r * g(x)|^p + |\nabla_x (k_r * g)|^p \leq C(r^p + 1)M^g(x)^p \]
\[ + \sup_{|v| = 1} \left| \text{PV} \int_M \frac{\varphi(|x - z|/r)}{|x - z|^n} (\nabla_x dz, v)g(z)dz \right|^p \]
(2.6)
In light of (2.6) it suffices to demonstrate the (uniform) boundedness of
\[ \text{Lemma 2.3} \]
By Definition 4 in [32, B] it is sufficient to prove the boundedness, uniformly in
\[ |x| = \text{dim} \]
which is bilipschitz diffeomorphic to \( B^n(r) \) through the exponential map \( \exp_x : B^n(r) \to B(x, r) \), the operator \( T_j \) may be written
\[ T_j g(x) = \text{PV} \int_{B^n(r)} \varphi(|\xi|/r) \frac{\xi_j}{|\xi|^{n+1}} g(\exp_x \xi) J \exp_x(\xi) d\xi. \]
By Definition 4 in [32, B] it is sufficient to prove the boundedness, uniformly in \( r \), for the Euclidean operator
\[ \tilde{T}_j : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \]
given by the same kernel:
\[ \tilde{T}_j h(x) = \text{PV} \int_{\mathbb{R}^n} \varphi(|\xi|/r) \frac{\xi_j}{|\xi|^{n+1}} h(x - \xi) d\xi. \]
By Theorem 5.4.1 in [12] (cf. Chapter 5, Theorem 5.1 in [7]) this is implied by the following two conditions. Denote
\[ K_r(y) = \varphi(|y|/r) \frac{y_j}{|y|^{n+1}}. \]
1. \( \| \tilde{K}_r \|_\infty \leq A \), and
2. \( |\nabla K_r(y)| \leq \frac{B}{|y|^{n+1}}. \)
A change of variables implies \( \tilde{K}_r(\xi) = \tilde{K}_1(r \xi) \) so that
\[ \| \tilde{K}_r \|_\infty \leq \| \tilde{K}_1 \|_\infty := A. \]
We may estimate
\[ |\nabla K_r(y)| \leq \chi_{B^n(r)}(y) \left[ \frac{\| \varphi \|_\infty}{r^{n+1} |y|^n} + \| \varphi \|_\infty |\nabla (y_j/|y|^{n+1})| \right] \leq C(n, \varphi) \frac{r^n |y|^{n+1}}{|y|^{n+1}}. \]
Consequently both (1) and (2) are satisfied with constants independent of \( r \). This completes the proof of the sublemma.
\[ \square \]
Having a bound \( \| T_j \|_{L^p(M) \to L^p(M)} \leq C \) where \( C \) is independent of \( r \) we obtain the estimate (2.5) with constant \( C \) independent of \( r \). This proves Lemma 2.6. \( \square \)
2.3. The proof of Theorem 1.1. Using Lemma 2.4 we define a net of approximating maps with values in $N$.

**Definition 2.8.** Let $u \in W^{1,n}(M; N)$, and let $r_0$ be the constant in Lemma 2.4. For $0 < r \leq r_0$ set

$$u_r(p) = \pi(\varphi_r * u(p)), \quad p \in M.$$

Additionally, we set

$$u_0 = u.$$

For each $r > 0$ the maps $u_r : M \to N$ are clearly Lipschitz. The resulting map $M \times [0, r_0] \ni (p, t) \mapsto u_t(p)$ is a key component in the proof of Theorem 1.1.

**Lemma 2.9.** Let $r > 0$. Then $u_s \to u_r$ uniformly as $s \to r$.

**Proposition 2.10.** The maps $u_r$ converge $n$-quasiconvexly to $u$, i.e. for each $\varepsilon > 0$ there exists an open set $U$ with $\text{Cap}_n(U) < \varepsilon$ such that $(u_r)|_{M \setminus U} \to u|_{M \setminus U}$ uniformly as $r \to 0$.

**Proof of Lemma 2.9.** We will estimate the difference $\|u_r(p) - u_s(p)\|$ by splitting it into two parts. Let $b$ be any vector in $\mathbb{R}^r$. We will later choose it appropriately.

$$\begin{align*}
\|u_r(p) - u_s(p)\| &\leq \|\varphi_r \ast u(p) - \varphi_s \ast u(p)\| = \|\varphi_r \ast [u - b](p) - \varphi_s \ast [u - b](p)\| \\
&\leq \left| \frac{1}{\varphi_r(p)} - \frac{1}{\varphi_s(p)} \right| \int_{B_r(p)} \varphi \left( \frac{|p - z|}{r} \right) \|u(z) - b\|dz \\
&+ \frac{1}{\varphi_s(p)} \int_{B_{s/r}(p)} \varphi \left( \frac{|p - z|}{r} \right) - \varphi \left( \frac{|p - z|}{s} \right) \|u(z) - b\|dz \tag{2.7}
\end{align*}$$

Let us estimate the two terms $\begin{align*}
\frac{1}{\varphi_s(p)} \int_{B_{s/r}(p)} \varphi \left( \frac{|p - z|}{r} \right) - \varphi \left( \frac{|p - z|}{s} \right) \|u(z) - b\|dz 
\end{align*}$ separately, starting with the latter. Throughout we assume that $|r - s| < r$, which implies that $\varphi_{r/s}(p) \lesssim \varphi_s(p)$ with constant depending only on $M$.

$$\begin{align*}
\frac{1}{\varphi_s(p)} \int_{B_{s/r}(p)} \varphi \left( \frac{|p - z|}{r} \right) - \varphi \left( \frac{|p - z|}{s} \right) \|u(z) - b\|dz 
&\lesssim |r/s - 1| \vee |s/r - 1| \int_{B_{s/r}(p)} \|u - b\|dz.
\end{align*} \tag{2.8}$$

A similar computation yields the same bound for $\begin{align*}
\frac{1}{\varphi_r(p)} \int_{B_{s/r}(p)} \varphi \left( \frac{|p - z|}{s} \right) \|u(z) - b\|dz
\end{align*}$. Thus we arrive at

$$\begin{align*}
\|u_r(p) - u_s(p)\| &\lesssim |r/s - 1| \vee |s/r - 1| \int_{B_{s/r}(p)} \|u - b\|dz.
\end{align*} \tag{2.9}$$

Now we choose $b = u_{B_{s/r}(p)}$ and use the $(1, n)$-Poincare inequality to estimate

$$\begin{align*}
\int_{B_{s/r}(p)} \|u - b\|dz &\lesssim \left( \int_{B_{s/r}(p)} |Du|^n dz \right)^{1/n} \lesssim \|Du\|_{L^n(M)}.
\end{align*}$$

Combining these we arrive at

$$\begin{align*}
\|u_r(p) - u_s(p)\| &\lesssim (|r/s - 1| \vee |s/r - 1|) \|Du\|_{L^n(M)}
\end{align*}$$

for all $p$. Thus $u_s \to u_r$ uniformly as $s \to r$, as long as $r \neq 0$. \hfill \square

Proposition 2.10 requires more work. We begin by estimating the difference of $u$ and $u_r$ by an expression which we study in more detail.
Lemma 2.11. Let \( u \in W^{1,p}(M;N) \). For \( p \)-q.e. \( x \in M \) we have
\[
\| u(x) - u_r(x) \| \leq \int_M \frac{\varphi(|z-x|/r)}{|z-x|^{n-1}} |Du|(z)dz.
\]

Proof. The proof is similar to [17, p. 28, (4.5)]. \( \square \)

Lemma 2.12. Let \( 1 < p < \infty \) and let \( (f_k) \subset N^{1,p}(M) \) be a bounded sequence with \( 0 \leq f_{k+1} \leq f_k \) pointwise and \( \|f_k\|_{L^p} \to 0 \) as \( k \to \infty \). Then \( f_k \to 0 \) \( p \)-quasiumiformly.

Proof. Since \( N^{1,p}(M) \) is reflexive we may pass to a subsequence converging weakly to 0, and by the Mazur lemma a sequence of convex combinations converges to 0 in norm. Passing to another subsequence if needed, we may assume that the sequence of convex combinations,
\[
h_m = \lambda_1^{k_1} f_{k_1} + \cdots + \lambda_m^{k_m} f_{k_m}, \quad (k_1 < \cdots < k_m),
\]
converges to zero \( p \)-quasiumiformly. The monotonicity now implies
\[
0 \leq f_{k_m} \leq h_m
\]
so that a subsequence of \( (f_k) \) converges \( p \)-quasiumiformly to zero. Since the sequence is pointwise nonincreasing the whole sequence converges to zero \( p \)-quasiumiformly. \( \square \)

These auxiliary results yield Proposition 2.10.

Proof of Proposition 2.10. By Lemma 2.11 we have
\[
\| u(x) - u_r(x) \| \lesssim k_r * |Du|(x)
\]
for \( p \)-quasievery \( x \in M \). Choosing \( \varphi \) nonincreasing we get that
\[
k_r * |Du| \leq k_s * |Du|
\]
pointwise whenever \( r < s \), and further,
\[
k_r * |Du| \overset{L^n}{\to} 0
\]
as \( r \to 0 \). By lemma 2.6 the functions \( k_r * |Du| \) have uniformly bounded \( W^{1,n} \) norms (in \( r \)) so by Lemma 2.12 we have that \( k_r * |Du| \to 0 \) \( n \)-quasiumiformly. Consequently \( u_r \to u \) \( n \)-quasiumiformly. \( \square \)

Theorem 2.13. Let \( u \in W^{1,n}(M;N) \). The map \( M \times [0,r_0] \to N \) given by
\[
(p,r) \mapsto u_r(p)
\]
in 2.8 defines an \( n \)-quasihomotopy \( u \simeq u_{r_0} \).

Proof. Denote \( H(p,r) = u_r(p) \) and suppose \( \varepsilon > 0 \) is given. Let \( U \) be the open set satisfying the claim of Proposition 2.10 We claim that \( H|_{M \setminus U \times [0,r_0]} \) is continuous. For this it suffices to show that \( (u_s)|_{M \setminus U} \to (u_r)|_{M \setminus U} \) uniformly as \( s \to r \). This, however, follows immediately from 2.8 and 2.10 \( \square \)

We close this Section with the proof of Theorem 1.1.
Proof of Theorem 1.1. Suppose \( u, v \in W^{1,n}(M; N) \) are path homotopic. For small enough \( \varepsilon \) we have, by Theorems 2.13 and 2.1 that \( u_\varepsilon \) is both \( n \)-quasihomotopic and path homotopic to \( u \). The same holds for \( v \) and \( v_\varepsilon \).

It follows that \( u_\varepsilon \) and \( v_\varepsilon \) are path homotopic and since they are Lipschitz, homotopic (Theorem 2.4).

Thus \( u_\varepsilon \) and \( v_\varepsilon \) are \( n \)-quasihomotopic. Consequently \( u \) and \( v \) are \( n \)-quasihomotopic. \( \square \)

3. Aspherical targets

A topological space \( X \) is called aspherical if \( \pi_i(X) = 0 \) for every \( i > 1 \). It is well known that for smooth Riemannian manifolds the vanishing of higher homotopy groups is equivalent to having contractible universal cover. In particular manifolds with nonpositive sectional curvature are aspherical. The equivalence stated in Theorem 1.3 can be seen as a Sobolev version of Whitehead's theorem [16].

Before turning our attention to Theorem 1.3 let us present a proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose \( N \) is aspherical and let \( f, g \in W^{1,p}(M; N) \) be \( p \)-quasihomotopic. We divide the proof into three cases:

1. \( p < n \): By Theorem 1.4 in [35] \( f \) and \( g \) are path homotopic.
2. \( p > n \): In this case path homotopy and \( p \)-quasihomotopy coincide, see the discussion in the introduction.
3. \( p = n \): This is the only case that requires some work. By Theorem 2.13 \( f, g \) are \( n \)-quasihomotopic to Lipschitz maps \( f_0, g_0 \) so we may assume that \( f \) and \( g \) are themselves Lipschitz. Since \( N \) is aspherical it is path representable [34] Proposition 3.4 and thus by [34] Theorem 1.2 \( (f, g) \in N^{1,n}(M; N) \cap \text{Lip}(M; N) \) has a lift \( h \in N^{1,n}(M; \tilde{N}_{\text{diag}}) \) where \( \tilde{N}_{\text{diag}} \) is the diagonal cover of \( N \) (see [34] Subsection 2.4). Since \( gh = g(f, g) \leq \text{LIP}(f) + \text{LIP}(g) \) almost everywhere (Lemma 4.3 in [34]) it follows that \( h \) is in fact Lipschitz. Thus the continuous map \( (f, g) : M \to N \times N \) admits a (continuous) lift \( h : M \to \tilde{N}_{\text{diag}} \). By Proposition 3.2 in [34] \( f \) and \( g \) are homotopic, hence path homotopic in \( W^{1,n}(M; N) \). \( \square \)

When \( p \geq 2 \), a Sobolev map \( f \in W^{1,p}(M; N) \) induces a homomorphism \( u_* : \pi(M, x_0) \to \pi(N, f(x_0)) \) [31] (see also [38, 28]). For almost every \( x_0 \in M \) an induced homomorphism satisfies, for all \( [\gamma] \in \pi(M, x_0) \):

\begin{itemize}
  \item \( u_*[\gamma] = [u \circ \gamma] \) if \( \gamma \) is such that \( u \circ \gamma \) is continuous
  \item \( u_*[\gamma] = [u \circ \gamma'] \) for some \( \gamma' \sim \gamma \).
\end{itemize}

It is known that no such induced homomorphism need exist for a Sobolev map \( f \in W^{1,p}(M; N) \) when \( 1 < p < 2 \).

To connect induced homomorphisms to \( p \)-quasihomotopies we recall the notion of a fundamental system of loops from [34].

Given a \( p \)-quasicontinuous representative \( u \in W^{1,p}(M; N) \), an upper gradient \( g \in L^p(M) \) and an exceptional path family \( \Gamma_0 \) of curves in \( M \), such that \( g \) is an upper
gradient of $u$ along any curve $\gamma \notin \Gamma_0$, and a basepoint $x_0 \in M$ with $Mg^\beta(x_0) < \infty$, the collection of loops
\[ \mathcal{F}_{x_0}(g, \Gamma_0) = \{ \alpha \beta^{-1} : \Gamma_{x_0} \setminus \Gamma_0, Mg^\beta(x) < \infty \} \]
is called the fundamental system of loops.

Let
\[ \text{spt}_p \Gamma_0 = \bigcap \{ \mathcal{M}\rho^p = \infty \} \]
where the intersection is taken over all admissible metrics $\rho \in L^p(M)$ for which
\[ \int_\gamma \rho = \infty \text{ for all } \gamma \in \Gamma_0. \]

**Lemma 3.1.** There is a constant $C$ with the following property. If $\Gamma_0$ is a path family and $g \in L^p(M)$ a nonnegative Borel function with
\[ \int_\gamma g = \infty, \, \gamma \in \Gamma_0, \]
then for any $x, y \notin \{ Mg^p = \infty \}$ there exists a curve $\gamma \notin \Gamma_0$ joining $x$ and $y$ with
\[ \ell(\gamma) \leq Cd(x, y). \]

**Proof.** By Lemma 4.5 in [34] and Theorem 2 (4) in [24] we have
\[ d(x, y)^{1-p} \leq C \text{Mod}_p(\Gamma_{xy} \setminus \Gamma_g; \mu_{xy}), \]
where
\[ \mu_{xy}(A) = \int_A \left[ \frac{d(x, z)}{\mu(B(x, d(x, z)))} + \frac{d(y, z)}{\mu(B(y, d(y, z)))} \right] d\mu(z), \, A \subset X. \]

In particular $\Gamma_{xy} \setminus \Gamma_g$ is nonempty. Note that $\Gamma_0 \subset \Gamma_g$.

If $\ell(\gamma) \geq Cd(x, y)^p$ for all $\gamma \in \Gamma_{xy} \setminus \Gamma_g \subset \Gamma_{xy} \setminus \Gamma_0$ then $\rho = 1/(Dd(x, y))$ is admissible for $\Gamma_{xy} \setminus \Gamma_g$ and thus
\[ \text{Mod}_p(\Gamma_{xy} \setminus \Gamma_g; \mu_{xy}) \leq CD^{-p}d(x, y)^{1-p}. \]

Combining the two inequalities yields the required bound on $D$.

**Lemma 3.2.** Let $p \geq 2$, and $u \in W^{1,p}(M; N)$ be a quasicontinuous representative. Given an upper gradient $g$ of $u$, a path family $\Gamma_0$ of zero $p$-modulus, and a point $x_0 \notin \text{spt}_p \Gamma_0$ with $Mg^\beta(x_0) < \infty$, we have
\[ u_* \pi(M, x_0) = u_2 \mathcal{F}_{x_0}(g, \Gamma_0). \]

**Proof.** Let $u \in W^{1,p}(M; N)$ and let $g, \Gamma_0$ be as in the claim. Set
\[ E = \{ x_0 : Mg^\beta(x_0) = \infty \} \cup \text{spt} \Gamma_0 \]
and choose and arbitrary point $x_0 \notin E$. For any $\gamma \in \mathcal{F}_{x_0}(g, \Gamma_0)$ clearly $[u \circ \gamma] \in u_* \pi(M, x_0)$. Thus we only need to prove the other inclusion.

To this end, fix a loop $\gamma$ based on $x_0$. Take a tubular neighbourhood $T$ of $\gamma$ so that any loop in $T$ is homotopic with $\gamma$. Take a finite chain of open balls $x_0 \in B_0, B_1, \ldots, B_k$ of radii $r > 0$ such that $2C\overline{B_j} \subset T$, and $B_j \cap B_{j+1} \neq \emptyset$, where $C$ is the constant in Lemma 3.1. Since $|E| = 0$ there exists, for each $j$, points $y_j \in (B_j \cap B_{j+1}) \setminus E$ (with the convention that $y_0 = x_0$ and $y_k \in (B_0 \cap B_k) \setminus E.$)
By Lemma 3.1 there exists a curve $\gamma_j \not\in \Gamma_0$ joining $y_j$ and $y_{j+1}$ with $\ell(\gamma_j) \leq Cd(y_j, y_{j+1})$ (here $y_{k+1} = x_0$). Hence $|\gamma_j| \subset T$. The loop $\gamma' = \gamma_0 \cdots \gamma_{k+1}$ belongs to $\mathcal{F}_{\pi_1}(g, \Gamma_0)$ and is contained in $T$, and therefore homotopic with $\gamma$.

It follows that $|u \circ \gamma'| = u_*[\gamma'] = u_*(\gamma)$ and since $\gamma$ was arbitrary we obtain $u_*\pi(M, x_0) \leq u_2 \mathcal{F}_{\pi_0}(g, \Gamma_0)$. The proof is complete. \qed

**Lemma 3.3.** Let $p \geq 2$. Two maps, $u, v \in W^{1,p}(M; N)$, are $p$-quasihomotopic if and only if $u_2 \pi(M)$ and $v_2 \pi(M)$ are conjugated subgroups of $\pi(N)$.

**Proof.** By [31, Theorem 1.2 and 1.3] the maps $u, v$ are $p$-quasihomotopic if and only if

$$ (u, v)_* \mathcal{F}_{\pi_0}(g, \Gamma_0) \leq p_* \pi(\tilde{\mathcal{N}}_{\text{diag}}, [\alpha]) $$

for some $[\alpha] \in p^{-1}(u(x_0), v(x_0))$, and some $x_0 \in M$. Here $(p, \tilde{\mathcal{N}}_{\text{diag}})$ is the diagonal cover of $N$ which consists of homotopy classes of all paths in $N$ (see [31] for the precise construction). A modification of the proof of [31, Lemma 2.18] yields

$$ p_* \pi(\tilde{\mathcal{N}}_{\text{diag}}, [\alpha]) = \{(\gamma), [\alpha^{-1} v \alpha] \in [\gamma] \in \pi(N, u(x_0)) \} \leq \pi(N, u(x_0)) \times \pi(N, v(x_0)). $$

On the other hand by Lemma 3.2

$$ (u, v)_* \mathcal{F}_{\pi_0}(g, \Gamma_0) = (u, v)_* \pi(M, x_0) = \{(u_*[\gamma], v_*[\gamma]) : [\gamma] \in \pi(M, x_0)\}. $$

By these two identities [31] is equivalent to

$$ u_*[\gamma] = [\alpha^{-1} v_*[\gamma][\alpha] $$

for all $[\gamma] \in \pi(M, x_0)$. Hence we are done. \qed

**Lemma 3.4.** If $u, v \in W^{1,p}(M; N)$ are path homotopic ($p \geq 2$) then for almost every $x_0 \in M$ $u_* \pi(M, x_0)$ and $v_* \pi(M, x_0)$ are conjugated.

**Proof.** Suppose first that $p < n$. Then by [11] Theorem 1.1 $u$ and $v$ are $[p - 1]$-homotopic and, since $p \geq 2$, in particular 1-homotopic. Fix a 1-skeleton $K$ of $M$ containing a point $x_0 \in \{M(|Du|^p + |Dv|^p) < \infty\}$, and such that $|u|^K$ and $|v|^K$ are (continuous and) homotopic by a homotopy $h : K \times [0, 1] \to N$.

To prove that the image subgroups of the homomorphisms are conjugated, take a loop $\gamma$ with basepoint $x_0$. By [11, Section 4.1, Theorem 4.8] $\gamma$ is homotopic to a loop $\gamma'$ which lies in $K$. Thus the image loops $u \circ \gamma'$ and $v \circ \gamma'$ are conjugated by

$$ H(s, t) = h(\gamma(s), t), \quad t, s \in [0, 1]^2. $$

Denoting by $\alpha$ the path $t \mapsto h(x_0, t)$ we thus have

$$ [u \circ \gamma'] = [\alpha^{-1}(v \circ \gamma')\alpha]. $$

Consequently

$$ u_*([\gamma]) = u_*([\gamma']) = (v_*([\gamma']))^{[\alpha]} = (v_*([\gamma]))^{[\alpha]}, \quad [\gamma] \in \pi(M, x_0). $$

This proves the claim in the case $p < n$.

In case $p \geq n$ it follows from Theorem 1.1 and Theorem ?? that $u$ and $v$ are $p$-quasihomotopic. The claim now follows from Lemma 3.3 above. \qed

Combining Proposition 1.2 and Lemmata 3.3 and 3.4 we obtain the following theorem, which directly implies Theorem 1.3.

**Theorem 3.5.** Let $p \geq 2$, and $N$ aspherical. Then two maps $u, v \in W^{1,p}(M; N)$ are path homotopic if and only if the subgroups $u_* \pi(M)$ and $v_* \pi(M)$ are conjugated.
Proof. Suppose \( u, v \) are path homotopic. Then Lemma 3.4 implies the claim. If, conversely, \( u_\ast \pi(M) \) and \( v_\ast \pi(M) \) are conjugated, Lemma 3.3 implies that \( u \) and \( v \) are \( p \)-quasihomotopic. By Proposition 1.2 \( u \) and \( v \) are path homotopic. \( \square \)

4. QUASICONNECTEDNESS OF \( W^{1,p}(M; S^k) \)

In this section the following result is proven.

**Proposition 4.1.** Suppose \( M \) is a smooth compact riemannian manifold, possibly with boundary, and \( 1 < p \leq k \). Then \( u \in W^{1,p}(M; S^k) \) is \( p \)-quasiconnected, i.e. every map is \( p \)-quasihomotopic to a constant.

We single out the following corollary.

**Corollary 4.2.** Suppose \( 2 \leq k \) and \( 1 < p \leq k \). Then any two maps in \( W^{1,p}(S^k; S^k) \) are \( p \)-quasihomotopic.

The proof of Theorem 4.1 is based on the example given in [2] after Theorem 3. We begin by observing that in a suitable range of \( p \)'s points have small preimages under Sobolev maps.

**Lemma 4.3.** Let \( f \in W^{1,p}(M; N) \) be a \( p \)-quasicontinuous representative, \( 1 < p \leq \dim N \). Then for almost every \( y \in N \) we have
\[
\text{Cap}_p(f^{-1}(y)) = 0.
\]

**Proof.** For \( y \in N \), consider the function \( u_k \in W^{1,p}(M) \) given by
\[
u_k(x) = w_k \circ f,
\]
where \( w_k : N \to \mathbb{R} \) is defined by
\[
\begin{cases}
1 & , z \in B(y, 1/k^2) \\
 \frac{1}{\log k} \log \left( \frac{1}{|z-y|} \right) & , z \in A(y, 1/k^2, 1/k) \\
0 & , z \notin B(y, 1/k)
\end{cases}
\]

Then \( u_k|f^{-1}(y) \equiv 1 \) \( p \)-quasieverywhere and therefore
\[
\text{Cap}_p(f^{-1}(y)) \leq \liminf_{k \to \infty} \|u_k\|_{1,p}^p.
\]

We have the pointwise estimates
\[
0 \leq u_k(x) \leq \chi_{B(y, 1/k)}(f(x)),
\]
\[
\begin{align*}
|\nabla u_k|(x) & \leq |\nabla w_k|(f(x))|\nabla f|(x) \leq (\log k)^{-1} \frac{\chi_{A(y, 1/k^2, 1/k)}(f(x))}{|f(x) - y|} |\nabla f|(x)
\end{align*}
\]
almost everywhere. Thus
\[
\begin{align*}
\text{Cap}_p(f^{-1}(y)) & \leq \liminf_{k \to \infty} \left[ \int_M \chi_{B(y, 1/k)} \circ f \, dx \\
& \quad + (\log k)^{-p} \int_M \frac{\chi_{A(y, 1/k^2, 1/k)}(f(x))}{|f(x) - y|^p} |\nabla f|^p \, dx \right].
\end{align*}
\]
Integrating over $y \in N$ and using Fatou and Fubini we obtain
\begin{equation}
\int_N \text{Cap}_p(f^{-1}(y))dy
\leq \liminf_{k \to \infty} \int_M \int_N \left[ \chi_{B(y,1/k)}(f(x)) + (\log k)^{-p} |\nabla f|^p(x) \frac{\chi_{A(y,1/k^2,1/k)}(f(x))}{|f(x) - y|^p} \right] dydx.
\end{equation}
Since
\[ \int_M \int_N \chi_{B(y,1/k)}(f(x))dydx = \int_M \left( \int_N \chi_{B(f(x),1/k)}(y)dy \right) dx \leq C/k^{\dim N} \]
inequality (4.1) becomes
\[ \int_N \text{Cap}_p(f^{-1}(y))dy \leq \liminf_{k \to \infty} \int_M \int_N (\log k)^{-p} |\nabla f|^p(x) \frac{\chi_{A(y,1/k^2,1/k)}(f(x))}{|f(x) - y|^p} dydx. \]
The righthand integral in turn may be written as
\[ (\log k)^{-p} \int_M |\nabla f|^p(x) \left( \int_N \frac{\chi_{A(f(x),1/k^2,1/k)}(y)}{|f(x) - y|^p} dy \right) dx. \]
For sufficiently large $k \geq 1$ one may estimate
\[ \int_N \frac{\chi_{A(f(x),1/k^2,1/k)}(y)}{|f(x) - y|^p} dy \lesssim C \int_{\mathbb{R}^{d\dim N}} \chi_{A(0,1/k^2,1/k)}(y) dy \lesssim \int_{1/k^2}^{1/k} t^{d\dim N - 1 - p} dt. \]
Since $p \leq \dim N$ we obtain
\[ \int_{1/k^2}^{1/k} t^{d\dim N - 1 - p} dt \leq \int_{1/k^2}^{1/k} t^{-1} dt = \log k. \]
Plugging all these inequalities into (4.2) we obtain
\[ \int_N \text{Cap}_p(f^{-1}(y))dy \leq C \liminf_{k \to \infty} \int_M (\log k)^{-p} |\nabla f|^p(x)dx = 0, \]
thus completing the proof. \hfill $\square$

**Corollary 4.4.** Let $2 \leq k$ and $1 < p \leq k$. For a $p$-quasiconstant representative $f \in W^{1,p}(M; S^k)$ the following holds for almost every $y \in S^k$.
\[ \lim_{r \to 0} \text{Cap}_p(f^{-1}B(y, r)) = 0. \]

**Proof.** Let $\varepsilon > 0$ be arbitrary and let $U \subset M$ be open with $\text{Cap}_p(U) < \varepsilon$ and $f|_{M\setminus U}$ continuous. We may estimate
\[ \text{Cap}_p(f^{-1}B(y, r)) \leq \text{Cap}_p((f|_{M\setminus U})^{-1}(B(y, r))) + \text{Cap}_p(U). \]
The sets $(f|_{M\setminus U})^{-1}(B(y, r))$ are compact and decrease to $(f|_{M\setminus U})^{-1}(y)$ as $r > 0$ decreases. By the monotonicity of capacity for compact sets therefore
\[ \limsup_{r \to 0} \text{Cap}_p((f|_{M\setminus U})^{-1}(B(y, r))) = \text{Cap}_p((f|_{M\setminus U})^{-1}(y)). \]
The latter quantity is zero for almost every $y \in S^k$ by Lemma 4.3 above. Thus we obtain
\[ \text{Cap}_p(f^{-1}B(y, r)) \leq 0 + \text{Cap}_p(U) < \varepsilon. \]
Postcomposition with $g$ is inverse to $G$ which preserves $p$-quasicontinuity and $g$ is $p$-quasihomotopic to $id_{S^k}$ by noting that the map $f$ is $p$-quasiconnected. (This easily seen by noting that the map $f$ is $p$-quasiconnected.)

In closing we remark that $W^1, p(B^{k+1}; S^k)$, $k < p < k+1$ provides another example where path and $p$-quasihomotopy differ.

Consider the map $g : (0, 1] \times S^k \to B^{k+1}$ given by $g(t, y) = ty$.

This is a $p$-quasihomotopy equivalence ($p < k+1$) since the map $h(x) = (|x|, x/|x|)$ is $p$-quasiconnected and $g \circ h = id_{B^{k+1}}, h \circ g = id_{(0, 1] \times S^k}$ $p$-quasieverywhere. Thus, postcomposition with $g$ defines a continuous map $G : W^1, p(B^{k+1}; S^k) \to W^1, p((0, 1] \times S^k; S^k)$, $Gf = f \circ g$,

which preserves $p$-quasihomotopy classes and is bijective (the map $f \mapsto f \circ h$ is an inverse to $G$).

It is known ([3], Proposition 0.2) that $W^1, p((0, 1] \times S^k; S^k)$ is path connected when $p < k+1$. However, when $k < p < k+1$, the Sobolev space $W^1, p((0, 1] \times S^k; S^k)$ and consequently $W^1, p(B^{k+1}; S^k)$ is not $p$-quasiconnected. (This easily seen by noting that the map $f(t, y) = y$, $(t, y) \in (0, 1] \times S^k$, is not $p$-quasihomotopic to a constant map.)

**Acknowledgements.** I would like to thank Pekka Pankka for reading the manuscript and making many valuable comments. I also thank Pawel Goldstein for useful discussions.
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ul. Śniadeckich 8, 00-656 Warszawa

*E-mail address*: elefterios.soultanis@gmail.com