Quantized Quasi-Two Dimensional Bose-Einstein Condensates with Spatially Modulated Nonlinearity

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We investigate the localized nonlinear matter waves of the quasi-two dimensional Bose-Einstein condensates with spatially modulated nonlinearity in harmonic potential. It is shown that the whole BEC can be quantized which is unexpected before. Their quantum and topological properties can be simply described by two quantum numbers. We also formulate an experimental procedure for the realization of these novel phenomena in \(^7\)Li condensate. This opens the door to the investigation of new matter waves in the high dimensional BEC with spatially modulated nonlinearities.

**Introduction.**—Since the remarkable experimental realization of Bose-Einstein condensations (BEC), there has been an explosion of the experimental and theoretical activity devoted to the physics of dilute ultracold bosonic gases. It is known that the properties of BEC including their shape, collective nonlinear excitations are determined by the sign and magnitude of the s-wave scattering length. A prominent way to adjust scattering length is to tune an external magnetic field in the vicinity of a Feshbach resonance. Alternatively, one can use a Feshbach resonance induced by optical or electric field. Since all quantities of interest in the BEC crucially depend on scattering length, a tunable interaction suggests very interesting studies of the many-body behavior of condensate systems.

In the past years, techniques for adjusting the scattering length globally have been crucial to many experimental achievements. More recently, condensates with a spatially modulated nonlinearity by manipulating scattering length locally have been proposed. This is experimentally feasible due to the flexible and precise control of the scattering length with tunable interactions. The spatial dependence of scattering length can be implemented by a spatially inhomogeneous external magnetic field in the vicinity of a Feshbach resonance.

However, so far, the studies of BEC with spatially modulated nonlinearity are limited in the quasi-one dimensional cases. Moreover, in the study of nonlinear problems no one discusses their quantum properties which are common in linear systems such as the linear harmonic oscillator. In this Letter, we extend the similarity transformation to the quasi-two dimensional (quasi-2D) BEC with spatially modulated nonlinearity in harmonic potential, and find a family of stable localized nonlinear matter wave solutions. Similar to the linear harmonic oscillator, we discover that the whole BEC can be quantized which is unexpected before. Their quantum and topological properties can be simply described by two quantum numbers. We also formulate an experimental procedure for the realization of these novel phenomena in \(^7\)Li condensate. This opens the door to the investigation of new matter waves in the high dimensional BEC with spatially modulated nonlinearities.

**Model and exact localized solutions.**—The system considered here is a BEC confined in a harmonic trap \(V(\mathbf{r}) = m(\omega_x^2 r_x^2 + \omega_y^2 r_y^2)/2\), where \(m\) is atomic mass, \(\omega_x^2 = x^2 + y^2\), and \(\omega_x, \omega_y\) are the confinement frequencies in the radial and axial directions, respectively. In the mean-field theory, the BEC system at low temperature is described by the Gross-Pitaevskii (GP) equation in three dimensions. If the trap is pancake-shaped, i.e. \(\omega_z \gg \omega_x\), it is reasonable to reduce the GP equation for the condensate wave function to a quasi-2D equation.

\[
i\psi_t = -\frac{1}{2} (\psi_{xx} + \psi_{yy}) + \frac{1}{2} \omega^2 (x^2 + y^2) \psi + g(x, y)^2 \psi, \tag{1}
\]

where \(\omega = \omega_x/\omega_z\), the length, time and wave function \(\psi\) are measured in units of \(a_h = h/\sqrt{m \omega_z}, a_h^{-1}\) and \(g(x, y) = 4\pi a_s(x, y)\) represents the strength of interatomic interaction characterized by the s-wave scattering length \(a_s(x, y)\), which can be spatially inhomogeneous by magnetically tuning the Feshbach resonances.

Now we consider the spatially localized stationary solution \(\psi(x, y, t) = \phi(x, y) e^{-i \mu t}\) of Eq. 1 with \(\phi(x, y)\) being a real function for \(\lim_{|x|, |y| \to \infty} \phi(x, y) = 0\). This maps Eq. 1 onto a stationary nonlinear Schrödinger equation \(\frac{1}{2} \phi_{xx} + \frac{1}{2} \phi_{yy} - \frac{i}{2} \omega^2 (x^2 + y^2) \phi - g(x, y) \phi^3 + i \mu \phi = 0\) [17]. Here \(\mu\) is the real chemical potential. Solving this stationary equation by similarity transformation [12], we
is a positive real constant, $K$ two secondary quantum numbers: $(a)$ when $n = 0$ and develops singularity when $l$ gets large.

obtain a families of exact localized nonlinear wave solutions for Eq. (1) as

$$\psi_n = \frac{(n+1)K(k)\eta}{\sqrt{\nu}} \mathrm{cn}(\theta, k)e^{-i\mu t}, \quad n = 0, 2, 4, \cdots$$ (2)

$$\psi_n = \frac{(n+1)K(k)\eta}{\sqrt{2\nu}} \mathrm{sn}(\theta, k)e^{-i\mu t}, \quad n = 1, 3, 5, \cdots$$ (3)

where $k = \sqrt{2}/2$ is the modulus of elliptic function, $\nu$ is a positive real constant, $K(k) = \int_0^\pi \frac{1 - k^2 \sin^2 \xi}{\sqrt{1 - k^2 \sin^2 \xi}} d\xi$ is elliptic integral of the first kind, $\mathrm{sd} = \mathrm{sn}/dn$ with $\mathrm{sn}, \mathrm{cn}$ and $dn$ being Jacobi elliptic functions, $\theta, \eta$ and $g$ are determined by

$$\theta = (n + 1)K(k) \frac{\eta}{2\nu} \mathrm{erf}\left[\sqrt{2\nu}(x + y)/2\right],$$

$$\eta = e^{\nu xy} \mathrm{KummerU}\left[\frac{\mu}{2\nu}, 1/2, \nu (x - y)^2 / 2\right],$$

$$g(x, y) = -2\nu \eta/(\pi \nu^2) e^{-\nu (x+y)^2},$$

where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\tau^2} d\tau$ is error function, and $\mathrm{KummerU}(a, c, s)$ is Kummer function of the second kind which is a solution of ordinary differential equation $s^{''}A(s) + (c - s)A(s) - aA(s) = 0$. It is easy to see that when $|x|, |y| \to \infty$ we have $\psi_n \to 0$ for solutions $\psi_n$ in Eqs. (2)-(3) with Eq. (4), thus they are localized bound state solutions.

In the above construction, it is observed that the number of zero points of function $\eta$ in Eq. (4) is equal to that of function $\mathrm{KummerU}\left[\frac{\mu}{2\nu}, 1/2, \nu (x - y)^2 / 2\right]$, which strongly depends on $\omega$ and the ratio $\mu/\omega$. We assume the number of zero points in $\eta$ along line $y = -x$ is $l$. In the following, we will see that integer $n$ is associated with the energy levels of the atoms and integers $n, l$ determine the topological properties of atom packets, so $n$ and $l$ are named the principal quantum number and secondary quantum number in quantum mechanics. In addition, the three free parameters $\omega, \mu$ and $\nu$ are positive, so the dimensionless interaction function $g(x, y)$ is negative, which indicates an attractive interaction between atoms. There are known atomic gases with attractive interactions realized by modulating magnetic technique, for examples, the $^{85}$Rb and $^{7}$Li atoms.

To translate our results into units relevant to the experiments $^{85}$Rb $^{\text{[14]}}$, we take the $^{7}$Li condensate containing $10^3 \sim 10^5$ atoms in a pancake-shaped trap with radial frequency $\nu = 2\pi \times 100 \text{ Hz}$, axial frequency $\omega = 2\pi \times 500 \text{ Hz}$ $^{21}$. In this case, the ratio of trap frequency $\omega$ in Eq. (1) is 0.02 which is determined by $\nu/\omega_z$. The unit of length is $1.69 \mu m$, the unit of time is $0.32 \text{ ms}$ and the unit of chemical potential is $nK$. The spatially inhomogeneous interaction parameter $g(x, y)$ is independent of principal quantum number $n$ but is strongly related to the secondary quantum number $l$. In the Fig. 1, we show that for $\omega = 0.02, \nu = 0.1$, function $g(x, y)$ is smooth in space when $l = 0$ and develops singularity when the $l$ gets large.

Quantized quasi-2D BEC.—In order to investigate the quantum and topological properties of the localized non-

FIG. 1: (color online). The interaction parameter $g(x, y)$ for two secondary quantum numbers: (a) $l=0$ and (b) $l=1$ with $\omega = 0.02, \nu = 0.1$. It is seen that $g(x, y)$ is a smooth function when $l=0$ and develops singularity when $l$ gets large.

FIG. 2: (color online). The density profiles of the even parity wave function (2) for $n=1, 2, 4$ and (3) for $n=3$ and $5$, respectively.
FIG. 3: (color online). The density distributions of the quasi-2D BEC in harmonic potential for different secondary quantum number $l$. Figs. 3(a)-3(d) show the density profiles of the even parity wave function (2) for each quantum state is equal to $(n + 1) \times (l + 1)$, and Figs. 3(e)-3(h) show the density profiles of the odd parity wave function (3) with Eq. (4) for $n = 0, 1, 2, 3$. We see that the number of density packets increases pair by pair when $l$ increases. The number of density packets for each quantum state is equal to $(n + 1) \times (l + 1)$, and all the density packets are symmetrical with respect to lines $y = \pm x$, as shown in Figs. 2-3.

**Normalization energy vs chemical potential.** Next we calculate the normalization energy of each quantum states numerically. The total energy of the quasi-2D BEC is $E(\psi) = \int \int dx dy |\nabla \psi|^2 + \frac{1}{2}\omega^2(x^2 + y^2)|\psi|^2 + \frac{1}{2}g(x,y)|\psi|^4$. So the normalized energy is given by $E(\psi)/N = \mu - \frac{1}{2N} \int \int dx dy g(x,y)|\psi|^4$ with $N = \int \int dx dy |\psi|^2$. Fig. 4 shows the normalized energy for the even parity wave function (2) increases when the principal quantum number $n$ increases. So does the odd parity wave function (3), as shown in Fig. 4(b). It is shown that the energy levels of the atoms are only associated with the principal quantum number $n$. These are similar to energy level distribution of the energy eigenvalue problem for the linear harmonic oscillator described by linear Schrödinger equation.

**Stability analysis.**—Stability of exact solutions with respect to perturbation is very important, because only stable localized nonlinear matter waves are promising for experimental observations and physical applications. To study the stability of our exact solutions (2)-(3) with Eq. (4), we consider a perturbed solution $\psi(x,y,t) = [\phi_n(x,y) + \Psi(x,y,t)]e^{-i\mu t}$ of Eq. (1). Here $\phi_n(x,y)$ are the exact solutions of the stationary nonlinear Schrödinger equation $\frac{\partial \phi}{\partial x} + \frac{1}{2}\frac{\partial \phi}{\partial y} - \frac{1}{2}\mu^2(x^2 + y^2)\phi - g(x,y)\phi^5 + \mu \phi = 0$. $\Psi(x,y,t) \ll 1$ is a small perturbation to the exact solutions and $\Psi(x,y,t) = \ldots$
[\{R(x, y) + I(x, y)\}e^{i\Delta t}] is decomposed into its real and imaginary parts \(^{[23]}\). Substituting this perturbed solution to the quasi-2D GP equation (1) and neglecting the higher-order terms in \((R, I)\), we obtain a standard eigenvalue problem \(L_+ R = \lambda I\), \(L_- I = \lambda R\), where \(\lambda\) is eigenvalue, \(R, I\) are eigenfunctions with \(L_+ = -\frac{1}{2}(\partial_x^2 + \partial_y^2) + 3g(x, y)\phi_n(x, y)^2 + \frac{1}{2}\omega^2(x^2 + y^2) - \mu\) and \(L_- = -\frac{1}{2}(\partial_x^2 + \partial_y^2) + g(x, y)\phi_n(x, y)^2 + \frac{1}{2}\omega^2(x^2 + y^2) - \mu\). Numerical experiments show that when \(\omega = 0.02\) and \(\mu, \nu\) are arbitrary non-negative constants, only for principle quantum number \(n = 0, 1, 2, 3, 4, 5\) are the eigenvalues \(\lambda\) of this eigenvalue problem real. This suggests that for \(\omega = 0.02\) the exact localized nonlinear matter wave solution (2) is linear stability only for \(n = 0, 2\) and solution (3) is linear stability only for \(n = 1, 3, 5\), see Fig. 5. It is seen that when the frequencies of pancake-shaped trap is fixed, the stability of the exact solutions (2)-(3) with Eq. (4) rests only on the principle quantum number \(n\).

Experimental protocol.—We now provide an experimental protocol for creating the quasi-2D localized nonlinear matter waves. To do so, we take the attractive \(^{7}\)Li condensate \(^{[2]}\) \(^{[14]}\), containing about \(10^3 \sim 10^5\) atoms, confined in a pancake-shaped trap with radial frequency \(\omega_1 = 2\pi \times 10\) Hz and axial frequency \(\omega_2 = 2\pi \times 500\) Hz \(^{[21]}\). This trap can be determined by combination of spectroscopic observations, direct magnetic field measurement, and the observed spatial cylindrical symmetry of the trapped atom cloud \(^{[21]}\). The next step is to realize the spatial variation of the scattering length. Near the Feshbach resonance \(^{[3]}\) \(^{[13]}\) \(^{[16]}\) \(^{[24]}\), the scattering length \(a_s(B)\) varies dispersively as a function of magnetic field \(B\), i.e. \(a_s(B) = a[1 + \Delta/(B_0 - B)]\), with \(a\) being the asymptotic value of the scattering length far from the resonance, \(B_0\) being the resonant value of the magnetic field, and \(\Delta\) being the width of the resonance. For the magnetic field in \(z\) direction with gradient \(\alpha\) along \(x-y\) direction, we have \(\vec{B} = [B_0 + \alpha B_1(x, y)]e_z\). In this case, the scattering length is dependent on \(x\) and \(y\). In real experiments, the spatially dependent magnetic field may be generated by a microfabricated ferromagnetic structure integrated on an atom chip \(^{[23]}\) \(^{[24]}\), such that interaction in Fig. 1 can be realized. In order to observe the density distributions in Figs. 2-3 clearly in experiment, the \(^{7}\)Li atoms should be evaporatively cooled to low temperatures, say in the range of \(10\) to \(100\) \(nK\). After the interaction parameter in Fig. 1(a) is realized by modulating magnetic field properly, the density distributions in Fig. 2 can be observed for different numbers of atoms by evaporative cooling, for example, the numbers of atoms in Fig. 2(a)-2(c) are \(3.76 \times 10^3, 6.84 \times 10^3, 2.633 \times 10^3\), respectively. The density distributions in Fig. 3 can also be observed by changing the scattering lengths through magnetic field for various atom numbers.

Conclusion.—In summary, we have discovered a new family of stable exact localized nonlinear matter wave solutions of the quasi-2D BEC with spatially modulated nonlinearities in harmonic potential. Similar to the linear harmonic oscillator, we introduce two classes of quantum numbers: the principle quantum number \(n\) and secondary quantum number \(l\). The matter wave functions have even parity for the even principle quantum number and odd parity for the odd one, the energy levels of the atoms are only associated with the principle quantum number, and the number of density packets for each quantum state is equal to \((n + 1) \times (l + 1)\). We also provide an experimental scheme to observe these novel phenomena in future.
experiments. Our results are of particular significance to matter wave management in high dimensional BEC.

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