Weyl matrix functions and inverse problems for discrete Dirac type self-adjoint system: explicit and general solutions

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Abstract

Discrete Dirac type self-adjoint system is equivalent to the block Szegö recurrence. Representation of the fundamental solution is obtained, inverse problems on the interval and semiaxis are solved. A Borg-Marchenko type result is obtained too. Connections with the block Toeplitz matrices are treated.

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1 Introduction

Continuous self-adjoint Dirac type system

\[
\frac{dY}{dx}(x,z) = i(zj + jV(x))Y(x,z), \quad j = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}
\] (1.1)

is a classical object of analysis with various applications (in mathematical physics and nonlinear integrable equations, in particular). Here \(I_p\) is the \(p \times p\) identity matrix and \(v\) is a \(p \times p\) matrix function. In this paper, we treat a discrete self-adjoint Dirac type system:

\[
W_{k+1}(\lambda) - W_k(\lambda) = -\frac{i}{\lambda}jC_kW_k(\lambda) \quad (k \geq 0), \quad (1.2)
\]
where \( C_k \) are \( m \times m \) matrices, \( m = 2p \),

\[
C_k = C_k^*, \quad C_kjC_k = j, \tag{1.3}
\]

To see that (1.2) is a discrete analog of system (1.1), notice that (1.1) is equivalent to a subclass of canonical systems \( W_x = izjH(x)W \) (see [33, 37] and references therein). One can follow also the arguments from [24], where the skew self-adjoint discrete Dirac type system have been studied and explicit solutions of the isotropic Heisenberg magnet model have been obtained. As suggested in [24] introduce matrix functions \( U \) and \( W \) by the relations

\[
W(x, z) = U(x)Y(x, z), \quad \frac{dU}{dx}(x) = -iU(x)jV(x), \quad U(0) = I_m. \tag{1.4}
\]

Since \( V \) is self-adjoint, we get from (1.4) that \( U \) is \( j \)-unitary, i.e., \( UjU^* \equiv j \). Now (1.1) and the first relation in (1.4) yield

\[
\frac{dW}{dx} = \frac{dU}{dx}U^{-1}W + iU(zj + jV)U^{-1}W = izjHW, \tag{1.5}
\]

where \( H = jUU^*j = H^*, \quad HjH \equiv j \). Compare system (1.2), where matrices \( C_k \) satisfy (1.3), and system (1.5) to see that (1.2) is an immediate discrete analog of (1.1).

When \( p = 1 \) and \( C_k > 0 \), system (1.2) is equivalent to the well-known self-adjoint Szegő recurrence, which plays an important role in the orthogonal polynomials theory and is also an auxiliary system for the Ablowitz-Ladik hierarchy (see, for instance, [19, 20, 38] and various references therein). The equivalence of system (1.2), where \( C_k > 0 \) and \( C_kjC_k = j \), to the block (matrix-valued) Szegő recurrence is given in Proposition 2.1.

We consider representation of the fundamental solution of system (1.2) and solve direct and inverse problems directly in terms of the Weyl functions. Both explicit and general solutions are obtained. First, we obtain explicit solutions of the direct and inverse problems for system (1.2) for the case of the so called pseudo-exponential potentials \( C_k \) (the case of the rational Weyl functions). Our case includes as a subcase the rapidly decaying strictly pseudo-exponential potentials. Recall that discrete and continuous systems with the potentials, which belong to the subclass of the strictly pseudo-exponential
potentials, have been actively studied in [1]-[6], [8]-[10]. In particular, direct and inverse problems for Szegö recurrence on the semiaxis with the scalar \((p = 1)\) strictly pseudo-exponential potentials have been treated in [5, 6]. Direct and inverse problems for the pseudo-exponential potentials (continuous case) have been studied in a series of Gohberg-Kaashoek-Sakhnovich papers [21, 22] (see references therein and see also [17] for the case of the generalized pseudo-exponential potentials). The case of the discrete skew-self-adjoint Dirac system have been studied in [24]. Notice that similar to [21, 31] (see also [5, 6, 8, 21, 24]) we start our explicit constructions with the explicit formula for the fundamental solution.

For a more general (non-rational) situation of the Weyl functions \(\phi(z) = \sum_{k=0}^{\infty} \phi_k z^k\) such that \(\sum_{k=0}^{\infty} \|\phi_k\| < \infty\) the direct problem for a block (matrix-valued) Szegö recurrence on the semiaxis (including non-self-adjoint case and under some additional conditions) is treated in a recent important Alpay-Gohberg paper [7]. In Sections 5 and 6 we solve direct and inverse problems for the general type potentials \(C_k > 0\) (and thus for the general type self-adjoint block Szegö recurrence) on the interval and semiaxis. Borg-Marchenko type uniqueness result for system (1.2) is obtained too. Connections with the well known Toeplitz matrices appear. For the interesting discussions on the connections between Toeplitz matrices, Szegö recurrences and orthogonal polynomials see also [3, 13] and references therein. Interesting spectral theoretical results on the discrete canonical systems, where \(C_k j C_k = 0\), one can find in [27, 36]. A complete Weyl theory for the Jacobi matrices and various useful references are contained in [39].

2 Preliminaries

An important discrete analog of Dirac type system takes the form

\[
X_{k+1}(z) = \theta_k R_k \begin{bmatrix} zI_p & 0 \\ 0 & I_p \end{bmatrix} X_k(z), \quad R_k = R_k^*, \quad R_k j R_k^* = j \quad (2.1)
\]
System (2.1) can be presented in the form (1.2) after transformation

$$W_k(\lambda) = \frac{(i - \lambda^{-1})^k}{\prod_{r=0}^{k-1} \theta_r} U_k \begin{bmatrix} zI_p & 0 \\ 0 & I_p \end{bmatrix} X_k(z), \quad C_k = (U_k^*)^{-1} R_k^2 U_k^{-1},$$

(2.2)

where $U_0 := I_m$, $U_k := (ijR_0)(ijR_1) \times \ldots \times (ijR_{k-1})$ $(k > 0)$, $z = \frac{1 + i\lambda}{1 - i\lambda}$.

A particular scalar case ($p = 1$) of system (2.1) is a well known Szegö recurrence, where

$$R_k = \frac{1}{\sqrt{1 - |\rho_k|^2}} \begin{bmatrix} -\rho_k & 1 \\ \rho_k & 1 \end{bmatrix} > 0, \quad |\rho_k| < 1, \quad \theta_k = \sqrt{1 - |\rho_k|^2}.$$  

(2.4)

Further we assume that some expression for $\theta_k$ via $R_k$ is fixed for $p > 1$ too. For $p = 1$ representations (2.4) of $R_k$ follow from the relations $R_k = R_k^* > 0$ and $R_k j R_k = j$. Coefficients $\rho_k$ are called Schur (or sometimes Verblunsky) coefficients (see, for instance, [12, 38] and various references therein). Notice that matrices $C_k$ given by the second relation in (2.2) are positive definite. Vice versa, if matrices $C_k$ are positive definite, Szegö recurrence is uniquely recovered from system (1.2). The same is true for the block Szegö recurrences (2.1), where $R_k > 0$ and $\theta_k = \theta(R_k)$ for some function $\theta$.

**Proposition 2.1** There is a one to one correspondence between the subclass of system (1.2), where the matrices $C_k > 0$ satisfy (1.3), and block Szegö recurrences (2.1), where $R_k > 0$ and $\theta_k = \theta(R_k)$. This correspondence is given by (2.2), (2.3) to map block Szegö recurrences into Dirac type systems and is given by the equality

$$U_0 = I_m, \quad R_k = \left(U_k^* C_k U_k\right)^{\frac{1}{2}} > 0, \quad U_k = U_{k-1}(ijR_{k-1})$$  

(2.5)

to map Dirac type systems into block Szegö recurrences.

**Proof.** The first part of the proposition is already proved. Moreover, according to (2.2) and (2.3) matrices $R_k$ ($k \geq 0$) and $U_k$ ($k > 0$) are uniquely
defined by the relations (2.5). Clearly we have $R_k = R_k^* > 0$, and it remains to prove $R_k j R_k = j$. We shall use for this purpose a unitary equivalence of $U_k^* C_k U_k$ to diagonal matrix $D_k > 0$:

$$U_k^* C_k U_k = \hat{U}_k^* D_k \hat{U}_k, \quad \hat{U}_k^* \hat{U}_k = \hat{U}_k \hat{U}_k^* = I_m.$$  \hspace{1cm} (2.6)

When $k > 0$ we assume that $R_r j R_r = j$ is already proved for $r < k$, and so $U_k j U_k^* = j$. Then formula (2.6) implies that $\hat{U}_k^* D_k \hat{U}_k j \hat{U}_k^* D_k \hat{U}_k = j$, i.e.,

$$D_k^{-1} = J_k D_k J_k, \quad J_k := \hat{U}_k j \hat{U}_k^*, \quad J_k = J_k^* = J_k^{-1}.$$  \hspace{1cm} (2.7)

By (2.7), without loss of generality we can assume that the diagonal matrix $D_k$ has the following form:

$$D_k = \text{diag}\{d_0 I_{l_0}, d_1 I_{l_1}, d_1^{-1} I_{l_1}, \ldots, d_r I_{l_r}, d_r^{-1} I_{l_r}\},$$  \hspace{1cm} (2.8)

where $d_0 = 1$ and $d_l \neq d_s, d_l \neq d_s^{-1}$ for $l \neq s$. Formulae (2.7) and (2.8) imply, in their turn, that the matrix $J_k$ has a block diagonal structure

$$J_k = \text{diag}\{u_0, j_1, \ldots, j_r\}, \quad j_s = \begin{bmatrix} 0 & u_s \\ u_s^* & 0 \end{bmatrix},$$  \hspace{1cm} (2.9)

where $u_s$ are $l_s \times l_s$ unitary matrices, and $u_0 = u_0^*$. According to (2.8) and (2.9) we have

$$J_k D_k^{\frac{1}{2}} = D_k^{-\frac{1}{2}} J_k.$$  \hspace{1cm} (2.10)

By (2.6) we have $R_k = \hat{U}_k^* D_k^{\frac{1}{2}} \hat{U}_k$, and, taking into account (2.10), we get

$$R_k j R_k = \hat{U}_k^* D_k^{\frac{1}{2}} J_k D_k^{\frac{1}{2}} \hat{U}_k = \hat{U}_k^* J_k \hat{U}_k = \hat{U}_k^* \hat{U}_k j \hat{U}_k^* \hat{U}_k = j.$$

\[\blacksquare\]

The spectral theory of the discrete and continuous systems is strongly related to the construction of the fundamental solutions (see, for instance, [6]-[10], [21, 22, 24, 26], [31]-[37] and references therein). The $j$-properties of the fundamental solutions play an important role [10, 12, 14, 15, 16, 23, 26, 36, 37].

For the case of the explicit construction the version of the Backlund-Darboux transformation (BDT) introduced in [29, 30, 31] proves very fruitful.
Choose \( n > 0 \), two \( n \times n \) parameter matrices \( A \) (det \( A \neq 0 \)) and \( S_0 = S_0^* \), and \( n \times m \) parameter matrix \( \Pi_0 \) such that
\[
AS_0 - S_0 A^* = i \Pi_0 j \Pi_0^*.
\]
(2.11)

Define sequences \( \{ \Pi_k \} \) and \( \{ S_k \} \) \( (k > 0) \) by the relations
\[
\Pi_{k+1} = \Pi_k + i A^{-1} \Pi_k j,
\]
(2.12)
\[
S_{k+1} = S_k + A^{-1} S_k (A^*)^{-1} + A^{-1} \Pi_k \Pi_k^* (A^*)^{-1}.
\]
(2.13)

It follows that the matrix identity
\[
AS_{k+1} - S_{k+1} A^* = i \Pi_{k+1} j \Pi_{k+1}^* \quad (k \geq 0)
\]
(2.14)
is true. Following the lines of the discrete BDT version for the skew self-adjoint discrete Dirac type system presented in [24], we get the theorem.

**Theorem 2.2** Suppose det \( S_r \neq 0 \) \( (0 \leq r \leq N) \). Then the fundamental solution \( W_{k+1} \) of system (1.2), where
\[
C_k := I_m + \Pi_k^* S_k^{-1} \Pi_k - \Pi_{k+1}^* S_{k+1}^{-1} \Pi_{k+1},
\]
(2.15)

admits representation
\[
W_{k+1}(\lambda) = w_A(k + 1, \lambda)(I_m - i \lambda j)^{k+1} w_A(0, \lambda)^{-1} \quad (0 \leq k < N).
\]
(2.16)

Here \( W_{k+1} \) is normalized by the condition \( W_0(\lambda) = I_m \), and
\[
w_A(k, \lambda) := I_m - ij \Pi_k^* S_k^{-1} (A - \lambda I_n)^{-1} \Pi_k,
\]
(2.17)

The right hand side of (2.17) with fixed \( k \) is a so called transfer matrix function in Lev Sakhnovich form [35]-[37].

We say that system (1.2), where matrices \( C_k \) are given by (2.15), is determined by the parameter matrices \( A, S_0 \) and \( \Pi_0 \).

**Proof of theorem.** Formula (2.16) easily follows from the basic for this proof equality
\[
w_A(k + 1, \lambda)(I_m - \frac{i}{\lambda} j) = (I_m - \frac{i}{\lambda} j C_k) w_A(k, \lambda),
\]
(2.18)
that we shall derive now. Taking into account (2.17) one can see that (2.18) is equivalent to the equality

\[-\frac{i}{\lambda} (I_m - C_k) = - (I_m - \frac{i}{\lambda} jC_k) i j \Pi_k^* S_k^{-1} (A - \lambda I_n)^{-1} \Pi_k
\]

\[+ i j \Pi_{k+1}^* S_{k+1}^{-1} (A - \lambda I_n)^{-1} \Pi_{k+1} (I_m - \frac{i}{\lambda} j), \quad (2.19)\]
i.e., the Taylor coefficients at infinity of the matrix functions in both sides of (2.19) coincide. Hence, by the series expansion 

\[(A - \lambda I_n)^{-1} = -\lambda^{-1} \sum_{r=0}^{\infty} (\lambda^{-1} A)^r \]
and formula (2.12), formula (2.19) is equivalent to a family of equalities:

\[I_m - C_k = -\Pi_k^* S_k^{-1} \Pi_k + \Pi_{k+1}^* S_{k+1}^{-1} \Pi_{k+1} \quad (2.20)\]
and

\[K_k A^{r-2} \Pi_k = 0 \quad (r > 0), \quad (2.21)\]

where

\[K_k := \Pi_{k+1}^* S_{k+1}^{-1} (A^2 + I_n) - \Pi_k^* S_k^{-1} A^2 + i C_k j \Pi_k^* S_k^{-1} A. \quad (2.22)\]

Notice that (2.20) is immediate from (2.15). If we prove also \(K_k = 0\), then (2.21) will follow, and so we will get (2.19) or equivalently (2.18), which implies (2.16). It remains to show that \(K_k = 0\). For this purpose we shall rewrite (2.22) using (2.12) and (2.15):

\[K_k = \Pi_{k+1}^* S_{k+1}^{-1} (A^2 + I_n) - \Pi_k^* S_k^{-1} A^2 + i j \Pi_k^* S_k^{-1} A + i j \Pi_k^* S_k^{-1} \Pi_k j \Pi_k^* S_k^{-1} A
\]

\[= -i \Pi_{k+1}^* S_{k+1}^{-1} (\Pi_k + i A^{-1} \Pi_k j) j \Pi_k^* S_k^{-1} A. \quad (2.23)\]

According to (2.14) we have \(i \Pi_k j \Pi_k^* S_k^{-1} = A - S_k A^* S_k^{-1}\). Therefore, from (2.23) we derive

\[K_k = \Pi_{k+1}^* S_{k+1}^{-1} (I_n + S_k A^* S_k^{-1} A + A^{-1} \Pi_k \Pi_k^* S_k^{-1} A) - \Pi_k^* A^* S_k^{-1} A + i j \Pi_k^* S_k^{-1} A.\]

In view of (2.13) we simplify our last formula:

\[K_k = \Pi_{k+1}^* A^* S_k^{-1} A - \Pi_k^* A^* S_k^{-1} A + i j \Pi_k^* S_k^{-1} A. \quad (2.24)\]

Finally, by (2.12) and (2.24) we have \(K_k = 0\). ■
Proposition 2.3 Suppose \( \det S_r \neq 0 \) \((0 \leq r \leq N)\). Then the matrices \( C_k \) \((0 \leq k < N)\) given by (2.15) satisfy conditions (1.3).

Proof. The first equality in (1.3) is immediate. To prove the second equality notice that by the standard in the S-node theory [35]-[37] calculations (see also, for instance, formula (2.10) in [17] it follows from (2.11) and (2.13) that

\[
w_A(r, \lambda)^* j w_A(r, \lambda) = j + i(\overline{\lambda} - \lambda) \Pi_r^*(A^* - \overline{\lambda}I_n)^{-1} S_r^{-1}(A - \lambda I_n)^{-1} \Pi_r. \tag{2.25}
\]

In particular, we have

\[
w_A(r, \lambda)^* j w_A(r, \lambda) = j. \quad r \geq 0. \tag{2.26}
\]

It is easily checked also that

\[
(I_m + \frac{i}{\lambda} j) j (I_m - \frac{i}{\lambda} j) = \left(1 + \frac{1}{\lambda^2}\right) j. \tag{2.27}
\]

According to (2.18) formulas (2.26) and (2.27) yield the equality

\[
(I_m + \frac{i}{\lambda} C_k j) j (I_m - \frac{i}{\lambda} j C_k) = \left(1 + \frac{1}{\lambda^2}\right) j. \tag{2.28}
\]

Therefore the second equality in (1.3) holds.

3 Auxiliary propositions

Recall that the invertibility of matrices \( S_k \) is essential for our constructions. On the other hand the important subcase of Szegö recursion corresponds to system (1.2), where \( C_k > 0 \). A natural condition, when all \( S_k > 0 \) and \( C_k > 0 \) is given in our next proposition.

Proposition 3.1 Let the parameter matrix \( S_0 \) be positive definite, i.e., \( S_0 > 0 \). Then we have

\[
S_k > 0 \quad (k \geq 0), \quad C_k > 0 \quad (k \geq 0). \tag{3.1}
\]
Proof. The inequalities for $S_k$ in (3.1) follow from (2.13) by induction. To derive the relations $C_k > 0$, introduce first two block matrices:

$$G = \begin{bmatrix} S_k & \Pi_k \\ \Pi_k^* & cI_m \end{bmatrix}, \quad F = \begin{bmatrix} A^{-1} & aA^{-1}\Pi_k \\ 0 & -ibj \end{bmatrix},$$

(3.2)

where

$$a(2 + ac) = 1, \quad b(1 + ac) = 1,$$

(3.3)

and, moreover, $c$ is sufficiently large so that $G > 0$. We shall discuss the choice of $a$ and $b$ satisfying (3.3) later on. According to (2.12), (2.13), (3.2) and (3.3), direct calculations show that

$$G + FGF^* = \begin{bmatrix} S_{k+1} & \Pi_{k+1} \\ \Pi_{k+1}^* & c(1 + b^2)I_m \end{bmatrix}.$$  

(3.4)

As $G + FGF^* > G > 0$ we have $G^{-1} > (G + FGF^*)^{-1}$, and, therefore, the inequality holds also for the $m \times m$ right lower blocks of these matrices: $(G^{-1})_{22} > ((G + FGF^*)^{-1})_{22}$. Finally, we obtain

$$\left((G^{-1})_{22}\right)^{-1} < \left((G + FGF^*)^{-1}\right)_{22}^{-1}.$$  

(3.5)

Taking into account (3.2), we can rewrite (3.5) in the form

$$cI_m - \Pi_k^*S_k^{-1}\Pi_k < c(1 + b^2)I_m - \Pi_{k+1}^*S_{k+1}^{-1}\Pi_{k+1}.$$  

(3.6)

Let us fix $c$ and choose a root $a$ ($0 < a < 1/2$), of the equation

$$a^2 + \frac{2}{c}a - \frac{1}{c} = 0,$$

which is always possible. Putting also $b = a(1 - a)^{-1}$, we see that relations (3.3) hold. Moreover, the first relation in (3.3) means that $a^2c = 1 - 2a$, and so

$$cb^2 = ca^2(1 - a)^2 = (1 - 2a)(1 - a)^2 < 1.$$  

(3.7)

From (3.6) and (3.7) it follows that

$$I_m + \Pi_k^*S_k^{-1}\Pi_k - \Pi_{k+1}^*S_{k+1}^{-1}\Pi_{k+1} > 0.$$  

(3.8)

Recall the definition (2.15) to see that inequality (3.8) implies $C_k > 0$. ■

In this section we shall need as well another property of $C_k$.
Proposition 3.2 Let relations (1.3) hold, and assume that $C_k > 0$. Then we have $C_k \pm j \geq 0$.

Proof. It follows from (1.3) that
\[
(C_k + \varepsilon j)(C_k + \varepsilon j) = 2\varepsilon (C_k + \frac{1 + \varepsilon^2}{2\varepsilon} j).
\]
(3.9)
If $(C_k + \varepsilon j)f = 0$, then by (3.9) we have also $(C_k + \frac{1 + \varepsilon^2}{2\varepsilon} j)f = 0$, and so $(1 - \varepsilon^2)jf = 0$. Therefore, we have $\det(C_k + \varepsilon j) \neq 0$, when $|\varepsilon| < 1$. Thus, the inequality $C_k > 0$ yields $(C_k + \varepsilon j) \geq 0$ for $|\varepsilon| \leq 1$. ■

4 Weyl functions, direct and inverse problem: the case of the pseudoexponential potentials

Following definitions of the Weyl functions for Sturm-Liouville, Dirac type and canonical systems on the semiaxis (see, for instance, [25, 37] and references therein), we can define also Weyl functions for system (1.2). Namely, let matrices $C_k > 0$ satisfy (1.3). Then, a $p \times p$ matrix function $\varphi$ holomorphic in the lower halfplane $C_-$ is said to be a Weyl function for system (1.2) on the semiaxis $k \geq 0$, if the inequality
\[
\sum_{k=0}^{\infty} [\varphi(\lambda)^* \ I_p] q(\lambda)^k W_k(\lambda)^* C_k W_k(\lambda) \begin{bmatrix} \varphi(\lambda) \\ I_p \end{bmatrix} < \infty
\]
holds, where $q(\lambda) = |\lambda^2|(|\lambda^2| + 1)^{-1}$.

Remark 4.1 Similar to the continuous case we have a summation formula:
\[
\sum_{k=0}^{r} q(\lambda)^k W_k(\lambda)^* C_k W_k(\lambda) = \frac{|\lambda^2| + 1}{i(\lambda - \bar{\lambda})} \left(q(\lambda)^r W_{r+1}(\lambda)^* j W_{r+1}(\lambda) - j\right).
\]
(4.2)
Indeed, according to (1.2) and (1.3) we have
\[
W_{k+1}(\lambda)^* j W_{k+1}(\lambda) = W_k(\lambda)^* \left(I_m + \frac{i}{\lambda} C_k j\right) j \left(I_m - \frac{i}{\lambda} j C_k\right) W_k(\lambda)
\]
\[ q(\lambda)^{-1}W_k(\lambda)^*jW_k(\lambda) + \frac{i(\lambda - \overline{\lambda})}{|\lambda^2|}W_k(\lambda)^*C_kW_k(\lambda), \]
i.e.,
\[
\frac{|\lambda^2| + 1}{i(\lambda - \overline{\lambda})} \left( q(\lambda)^{k+1}W_{k+1}(\lambda)^*jW_{k+1}(\lambda) - q(\lambda)^kW_k(\lambda)^*jW_k(\lambda) \right)
\]
\[ = q(\lambda)^kW_k(\lambda)^*C_kW_k(\lambda). \tag{4.3} \]

Formula (4.3) yields (4.2).

To construct the Weyl function, partition first matrix \( \Pi_0 \) and matrix-function \( w_A(0, \lambda) \) into blocks:
\[ \Pi_0 = [\Phi \quad \Psi], \quad w_A(0, \lambda) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix}. \tag{4.4} \]

Similar to the considerations in [21] it follows from (2.17) that
\[ b(\lambda)d(\lambda)^{-1} = -i\Phi^*S_0^{-1}(A^\times - \lambda I_n)^{-1}\Psi, \quad A^\times = A + i\Psi^*\Psi S_0^{-1}. \tag{4.5} \]

To calculate \( d(\lambda)^{-1} \) here, we use the fact from the system theory:
\[ \left( I_p + C(\lambda I_n - A)^{-1}B \right)^{-1} = I_p - C(\lambda I_n - (A - BC))^{-1}B. \tag{4.6} \]

**Theorem 4.2** Let parameter matrices be fixed, assume \( S_0 > 0 \), and define \( C_k \) by (2.15). Then system (1.2) is well-defined on the semiaxis and its unique Weyl function, which satisfies (4.1), takes the form
\[ \varphi(\lambda) = -i\Phi^*S_0^{-1}(A^\times - \lambda I_n)^{-1}\Psi, \quad A^\times = A + i\Psi^*\Psi S_0^{-1}. \tag{4.7} \]

**Proof.** By Proposition 3.1 system (1.2) is well-defined. Now, relations (4.5) imply \( \varphi = bd^{-1} \) for the matrix function \( \varphi \) given by (4.7). According to (2.25) we have
\[ w_A(0, \lambda)^*jw_A(0, \lambda) \leq j \quad (\lambda \in \mathbb{C}_-), \]
and it follows, in particular, that \( d(\lambda)^*d(\lambda) \geq I_p + b(\lambda)^*b(\lambda) \). Therefore, we get
\[ \varphi(\lambda)^*\varphi(\lambda) < I_p \quad (\lambda \in \mathbb{C}_-), \tag{4.8} \]
and so $\varphi$ is holomorphic in $\mathbb{C}_-$. Notice that the equality $\varphi = bd^{-1}$ is equivalent to the formula

$$
\begin{bmatrix}
\varphi(\lambda) \\
I_p
\end{bmatrix} = w_A(0, \lambda) \begin{bmatrix}
0 \\
I_p
\end{bmatrix} d(\lambda)^{-1}.
$$

(4.9)

Taking into account (4.9) and $w_A(r + 1, \lambda)^* j w_A(r + 1, \lambda) \leq j$, we derive from representation (2.16) of $W_{r+1}(\lambda)$ that

$$
[\varphi(\lambda)^* I_p] W_{r+1}(\lambda)^* j W_{r+1}(\lambda) \begin{bmatrix}
\varphi(\lambda) \\
I_p
\end{bmatrix} = |\lambda + i|^{2r+2} |\lambda|^{-2r-2} (d(\lambda)^*)^{-1}
$$

$$
\times [0 I_p] w_A(r + 1, \lambda)^* j w_A(r + 1, \lambda) \begin{bmatrix}
0 \\
I_p
\end{bmatrix} d(\lambda)^{-1} < 0.
$$

(4.10)

By (4.2) and (4.10) the inequality

$$
[\varphi(\lambda)^* I_p] \sum_{k=0}^{r} q(\lambda)^k W_k(\lambda)^* C_k W_k(\lambda) \begin{bmatrix}
\varphi(\lambda) \\
I_p
\end{bmatrix} < \frac{|\lambda|^2 + 1}{i(\lambda - \lambda)} I_p
$$

(4.11)

is true. From (4.11) inequality (4.1) is immediate, i.e., $\varphi$ defined by (4.7) is a Weyl function.

Let us show that $\varphi$ is a unique Weyl function. First notice that by Proposition 3.2 we have inequality $W_s^* C_s W_s \geq W_s^* j W_s$. Now, use relation (4.3) to derive inequality $q^s W_s^* j W_s \geq q^{s-1} W_{s-1}^* j W_{s-1}$. From the inequalities above we get

$$
q(\lambda)^k W_k(\lambda)^* C_k W_k(\lambda) \geq j.
$$

(4.12)

Therefore, the following equality is immediate for any $f \in \mathbb{C}^p$:

$$
\sum_{k=0}^{\infty} f^*[I_p] 0] q(\lambda)^k W_k(\lambda)^* C_k W_k(\lambda) \begin{bmatrix}
I_p \\
0
\end{bmatrix} f = \infty.
$$

(4.13)

According to (4.1) and (4.13), the dimension of the subspace $L \in \mathbb{C}^m$, such that for all $h \in L$ we have

$$
\sum_{k=0}^{\infty} h^* q(\lambda)^k W_k(\lambda)^* C_k W_k(\lambda) h < \infty,
$$

(4.14)
equals \( p \). Now, suppose that there is a Weyl function \( \tilde{\varphi} \neq \varphi \), where \( \varphi \) is given by (4.7). Then the columns of \( \begin{bmatrix} \varphi(\lambda) \\ I_p \end{bmatrix} \) and the columns of \( \begin{bmatrix} \tilde{\varphi}(\lambda) \\ I_p \end{bmatrix} \) belong to \( L \). Therefore, \( \dim L > p \) for those \( \lambda \), where \( \tilde{\varphi}(\lambda) \neq \varphi(\lambda) \), and we come to a contradiction. ■

**Remark 4.3** If \( S_0 > 0 \), then by (2.12)-(2.15) we can substitute parameter matrices \( A, S_0 \) and \( \Pi_0 \) by the parameter matrices \( S_0^{-\frac{1}{2}}AS_0^{\frac{1}{2}}, I_n \) and \( S_0^{-\frac{1}{2}}\Pi_0 \), which determine the same system. For \( S_0 = I_p \) formula (4.7) takes the form

\[
\varphi(\lambda) = -i\Phi^*(A^x - \lambda I_n)^{-1}\Psi, \quad A^x = A + i\Psi\Phi^*,
\]

and we have also \( A^x - (A^x)^* = i(\Phi\Phi^* + \Psi\Psi^*) \), \( \det(A^x - i\Psi\Psi^*) \neq 0 \).

**Example 4.4** Consider the simplest example: \( p = 1, n = 1, A = a \in \mathbb{R} (a \neq 0), S_0 = 1 \). From (2.11) and (2.12) it follows that \( |\Phi| = |\Psi| \) and

\[
\Pi_k = \left[ \left( \frac{a + i}{a} \right)^k \Phi \left( \frac{a - i}{a} \right)^k \Psi \right], \quad \Pi_k\Pi_k^* = 2|\Phi|^2\left( \frac{a^2 + 1}{a^2} \right)^k.
\]

Now, in view of \( S_0 = 1 \), (2.13) and the second relation in (4.16) one can check that

\[
S_k = (k\zeta + 1)\left( \frac{a^2 + 1}{a^2} \right)^k, \quad \zeta = \frac{2|\Phi|^2}{a^2 + 1}.
\]

Finally, using (2.15), (4.16) and (4.17) we get the entries \((C_k)_{ij}\) of \( C_k \):

\[
(C_k)_{11} = (C_k)_{22} = 1 + \zeta|\Phi|^2(k\zeta + 1)^{-1}((k + 1)\zeta + 1)^{-1},
\]

\[
(C_k)_{21} = (C_k)_{12} = \Phi\Psi\left((k\zeta + 1)^{-1}\left( \frac{a + i}{a - i} \right)^k - ((k + 1)\zeta + 1)^{-1}\left( \frac{a + i}{a - i} \right)^{k+1} \right).
\]

The Weyl function of system (1.2), where the matrices \( C_k \) are given by (4.18) and (4.19), is easily calculated using (4.15):

\[
\varphi(\lambda) = i\Phi\Psi(\lambda - a - i|\Psi|^2)^{-1}.
\]

Notice that our matrices \( C_k \) belong to the class of the so called pseudoexponential potentials. An important subclass of the strictly pseudoexponential potentials, that is, a subclass with an additional requirement \( \sigma(A) \subset \mathbb{C}_- (\sigma - \)
spectrum), have been treated for $p = 1$ in [5, 6]. In particular, for the strictly pseudoexponential subcase the inequality $|\varphi(\lambda)| < 1$ for $\lambda \in \mathbb{C}_-$ is true. On the other hand, in the simple example above we have $\sigma(A) = a \in \mathbb{R}$ and $|\varphi| = 1$ for $\lambda = a$.

According to (4.7) and (4.8) Weyl function $\varphi$ is a rational, strictly proper and contractive in $\mathbb{C}_-$ matrix function. By the proof of Theorem 9.4 [22] such matrix functions admit representation (realisation):

$$\varphi(\lambda) = -i\Phi^*(\theta - \lambda I_n)^{-1}\Psi,$$

(4.21)

where

$$\theta - \theta^* = i(\Phi\Phi^* + \Psi\Psi^*).$$

(4.22)

Direct calculation shows also that formulas (4.21) and (4.22) yield $I_p - \varphi^*\varphi \geq 0$ for $\lambda \in \mathbb{C}_-$. So, realization (4.21), (4.22) is equivalent to function being rational, strictly proper and contractive in $\mathbb{C}_-$.

**Theorem 4.5** Matrix function $\varphi$ is the Weyl function of some system (1.2) determined by the parameter matrices with $S_0 > 0$ if and only if it admits representation (4.21), (4.22) such that $\det(\theta - i\Phi\Phi^*) \neq 0$. Then $\varphi$ is the Weyl function of some system (1.2), where $C_k > 0$. To recover such system put

$$S_0 = I_n, \quad A = \theta - i\Phi\Phi^*, \quad \Phi = \Phi, \quad \Psi = \Psi, \quad \Pi_0 = [\Phi \quad \Psi],$$

(4.23)

and define matrices $C_k$ by formula (2.15), where matrices $\Pi_k$ and $S_k$ ($k > 0$) are given by formulas (2.12) and (2.13).

**Proof.** The necessity of theorem’s conditions follows from Remark 4.3. Now, suppose these conditions are fulfilled. Then, from (4.22) and (4.23) it follows that the identity (2.11) holds for the parameter matrices. Therefore, system (1.2) is defined. So, by Theorem 4.2 $\varphi$ is the Weyl function of this system. ■

**Remark 4.6** The Weyl functions in the upper halfplane can be treated in a quite similar way. That is, we define Weyl functions in $\mathbb{C}_+$ by the inequality

$$\sum_{k=0}^{\infty} [I_p \varphi(\lambda)^* q(\lambda)^k W_k(\lambda)^* C_k W_k(\lambda) \left[ I_p \varphi(\lambda) \right] < \infty.$$

(4.24)
Then the Weyl function of system (1.2), where matrices $C_k$ are given by (2.15) and $S_0 > 0$, takes the form

$$\varphi(\lambda) = c(\lambda)a(\lambda)^{-1} = i\Psi^*S_0^{-1}(A^* - \lambda I_n)^{-1}\Phi, \quad A^* = A - i\Phi\Phi^*S_0^{-1}. \quad (4.25)$$

A definition of a Weyl function in $\mathbb{C}_-$ can be also given in a more general form.

**Definition 4.7** Let matrices $C_k > 0$ satisfy (1.3). Then, a $p \times p$ matrix function $\varphi$ holomorphic in $\mathbb{C}_-$ is said to be a Weyl function for system (1.2) on the semiaxis $k \geq 0$, if the following inequality holds:

$$\sum_{k=0}^{\infty} [i\varphi(\lambda)^* I_p q(\lambda)^k K W_k(\lambda)^* C_k W_k(\lambda) K^* \begin{bmatrix} -i\varphi(\lambda) \\ I_p \end{bmatrix}] < \infty. \quad (4.26)$$

Here $K^* = K^{-1}$ and $q(\lambda) = |\lambda|^2(|\lambda|^2 + 1)^{-1}$.

When $K$ in the inequality (4.26) equals $I_n$, this inequality coincides with the inequality (4.1). In general, the choice of the matrix $K$ is related to the choice of the domain of the operator corresponding to the Dirac system, and usually $K$ is chosen so that the Weyl functions are Herglotz functions. Further we assume that

$$K = \frac{1}{\sqrt{2}} \begin{bmatrix} I_p & -I_p \\ I_p & I_p \end{bmatrix}. \quad (4.27)$$

Simple transformations show that the Weyl function $\varphi_I$ defined via (4.1) and the Weyl function $\varphi_K$ defined via (4.26) and (4.27) are connected by the relation

$$\varphi_K = -i(I_p - \varphi_I)(I_p + \varphi_I)^{-1}. \quad (4.28)$$

From (4.28) it follows that

$$\varphi_K(\lambda) - \varphi_K(\lambda)^* = -2i(I_p + \varphi_I(\lambda)^*)^{-1}(I_p - \varphi_I(\lambda)^*\varphi_I(\lambda))(I_p + \varphi_I(\lambda))^{-1}, \quad (4.29)$$

$\lambda \in \mathbb{C}_-$. Thus, according to (4.8) and (4.29) $\varphi_K$ is a Herglotz function with a non-positive imaginary part in $\mathbb{C}_-$.
5 Weyl functions, direct and inverse problem on the interval: general case

In this section we shall consider the self-adjoint matrix discrete Dirac type system (1.2) on the interval \(0 \leq k \leq N\). We assume that (1.3) holds and \(C_k > 0\). It was shown in Proposition 2.1 that \(C_k > 0\) yields \(C_k = (U_k^*)^{-1}R_k^2U_k^{-1}\), where \(R_k = R_k^*\) and \(R_kjR_k = j\). Hence, we get

\[
C_k = (U_k^*)^{-1}(R_k^2 + R_kjR_k)U_k^{-1} - j = (U_k^*)^{-1}R_k(I_m + j)R_kU_k^{-1} - j
\]

where

\[
\hat{\beta}(k) = [I_p \ 0]R_kU_k^{-1}, \quad \hat{\beta}(k)j\hat{\beta}(k)^* = I_p.
\]

Further we shall use these relations:

\[
C_k = 2\hat{\beta}(k)^*\hat{\beta}(k) - j, \quad \hat{\beta}(k)j\hat{\beta}(k)^* = I_p, \quad 0 \leq k \leq N.
\]

(5.2)

**Remark 5.1** Relations (5.2) are equivalent to the relations \(C_k > 0\) and (1.3). Indeed, we have just derived (5.2) from \(C_k > 0\) and \(C_kjC_k = j\), and vice versa: direct calculation shows that (5.2) yields (1.3). To derive also from (5.2) the inequality \(C_k > 0\), choose a matrix \(\tilde{\beta}(k)\) such that

\[
\tilde{\beta}(k)j\tilde{\beta}(k)^* = 0, \quad \tilde{\beta}(k)j\tilde{\beta}(k)^* = -I_p.
\]

(5.3)

Notice, that in view of the second relations in (5.2) the maximal subspace, which is \(j\)-orthogonal to the rows of \(\tilde{\beta}_k\), proves to be \(p\)-dimensional and \(j\)-negative, i.e., \(\tilde{\beta}(k)\) always exists. According to (5.2) and (5.3) we have

\[
\begin{bmatrix}
\hat{\beta}(k) \\
\tilde{\beta}(k)
\end{bmatrix}
\begin{bmatrix}
j \\
j
\end{bmatrix}
= j
\begin{bmatrix}
\hat{\beta}(k) \\
\tilde{\beta}(k)
\end{bmatrix}
\begin{bmatrix}
j \\
j
\end{bmatrix}
\begin{bmatrix}
\hat{\beta}(k) \\
\tilde{\beta}(k)
\end{bmatrix}.
\]

(5.4)

Finally, by the first relations in (5.2) and by (5.4) we obtain

\[
C_k = \hat{\beta}(k)^*\hat{\beta}(k) + \tilde{\beta}(k)^*\tilde{\beta}(k) = \begin{bmatrix}
\hat{\beta}(k) \\
\tilde{\beta}(k)
\end{bmatrix}^*\begin{bmatrix}
\hat{\beta}(k) \\
\tilde{\beta}(k)
\end{bmatrix} > 0.
\]

(5.5)
From the second relation in (5.2) for \( k \geq 0 \) it follows also that
\[
\det \left( \hat{\beta}(k)j\hat{\beta}(k+1)^* \right) \neq 0, \quad 0 \leq k \leq N - 1.
\]
(5.6)

Indeed, if (5.6) does not hold, we have \( \hat{\beta}(k)j\hat{\beta}(k+1)^*f = 0 \) for some \( f \neq 0 \). Then, in view of the second relations in (5.2) for \( k \geq 0 \), we see that the linear span of the rows of \( \beta_k \) and of \( f^*\beta_{k+1} \) forms a \( p + 1 \)-dimensional \( j \)-positive subspace of \( \mathbb{C}_m \), which is impossible.

Notice also that according to (2.1) and (2.3) we have
\[
U^{-1}_k = -ijR_{k-1}U_{k-1}^{-1}, \quad U^*_k = -iR_{k-1}U_{k-1}^{-1}j \quad (k > 0).
\]
(5.7)

Compare (5.1) and (5.2) to see that a matrix \( \hat{\beta}(k-1) \), which satisfies the equalities
\[
C_{k-1} = 2\hat{\beta}(k-1)^*\hat{\beta}(k-1) - j, \quad \hat{\beta}(k-1)j\hat{\beta}(k-1)^* = I_p,
\]
coincides with the upper block row of \( -iR_{k-1}U_{k-1}^{-1} \) up to a \( p \times p \) unitary factor \( \hat{u}(k-1) \). Moreover, \( \hat{\beta}(k-1) \) given by (5.3) defines the lower block row of \( -iR_{k-1}U_{k-1}^{-1} \) up to some \( p \times p \) unitary factor \( \tilde{u}(k-1) \). Taking into account the second equalities in (2.5) and (5.7), we get
\[
R_k = \left( D(k-1) \begin{bmatrix} \hat{\beta}(k-1) \\ \hat{\beta}(k-1) \end{bmatrix} jC_{k,j} \begin{bmatrix} \hat{\beta}(k-1)^* & \tilde{\beta}(k-1)^* \end{bmatrix} D(k-1)^* \right)^{1/2},
\]
(5.8)

where
\[
D(k-1) = \begin{bmatrix} \hat{u}(k-1) & 0 \\ 0 & \tilde{u}(k-1) \end{bmatrix}.
\]
(5.9)

**Remark 5.2** We see that the two matrices \( C_k \) and \( C_{k-1} \) determine \( R_k \) by (5.8) up to a unitary block diagonal matrix \( D(k-1) \).

Similar to the continuous case, Weyl functions of the discrete system on the interval are defined via Möbius (linear-fractional) transformation
\[
\varphi(\lambda) = i\left( W_{21}(\lambda)R(\lambda) + W_{22}(\lambda)Q(\lambda) \right) \left( W_{11}(\lambda)R(\lambda) + W_{12}(\lambda)Q(\lambda) \right)^{-1},
\]
(5.10)
where $R$ and $Q$ are $p \times p$ analytical functions in the neighbourhood of $\lambda = -i$, and

$$\mathcal{W}(\lambda) = \{W_{ij}(\lambda)\}_{i,j=1}^{2} = KW_{N+1}(\overline{\lambda})^*, \quad (5.11)$$

Here, coefficients $W_{ij}$ of the Möbius transformation are the $p \times p$ blocks of $\mathcal{W}$, the matrix $K$ is given by (4.27) and $K^* = K^{-1}, \quad K_jK^* = J, \quad J = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}. \quad (5.12)$

It would be convenient to put $\beta(k) := \hat{\beta}(k)K^*$ and rewrite (5.2) as

$$C_k = 2K^*\beta(k)^*\beta(k)K - j, \quad \beta(k)J\beta(k)^* = I_p, \quad 0 \leq k \leq N. \quad (5.13)$$

We shall need the following analog (for the self-adjoint case) of Theorem 3.4 [34].

**Theorem 5.3** Suppose $W$ ($W_0(\lambda) = I_m$) is the fundamental solution of system (1.2), which satisfies conditions (5.13). Suppose also that a $p \times p$ matrix function $\varphi$ is given by formulas (5.10) and (5.11), where

$$\det(W_{11}(-i)R(-i) + W_{12}(-i)Q(-i)) \neq 0. \quad (5.14)$$

Then system (1.2) satisfies (1.3), $C_k > 0$ ($k \geq 0$), and the inequalities

$$\det\left(\beta(k)J\beta(k+1)^*\right) \neq 0, \quad 0 \leq k \leq N - 1 \quad (5.15)$$

hold. Moreover, system (1.2) is uniquely recovered from the first $N+1$ Taylor coefficients $\{\alpha_k\}_{k=0}^{N}$ of $i\varphi\left(i\left(\frac{z+1}{z-1}\right)\right)$ at $z = 0$ by the following procedure.

First, introduce $(N+1)p \times p$ matrices $\Phi_1, \Phi_2$:

$$\Phi_1 = \begin{bmatrix} I_p \\ I_p \\ \vdots \\ I_p \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} \alpha_0 \\ \alpha_0 + \alpha_1 \\ \vdots \\ \alpha_0 + \alpha_1 + \ldots + \alpha_N \end{bmatrix}. \quad (5.16)$$
Then, introduce an \((N + 1)p \times 2p\) matrix \(\Pi\) and an \((N + 1)p \times (N + 1)p\) block lower triangular matrix \(A\) by the blocks:

\[
\Pi = [\Phi_1 \Phi_2],
\]

\[
A := A(N) = \{a_{j-k}\}_{k,j=0}^{N} = \begin{cases} 
0 & \text{for } r > 0 \\
\frac{i}{2} I_p & \text{for } r = 0 \\
i I_p & \text{for } r < 0
\end{cases}.
\]  

(5.17)

Next, we recover \((N + 1)p \times (N + 1)p\) matrix \(S\) as a unique solution of the matrix identity

\[
AS - SA^* = i\Pi J \Pi^*.
\]  

(5.18)

This solution is invertible and positive, i.e., \(S > 0\). Finally, matrices \(\beta(k)^* \beta(k)\) are easily recovered from the formula

\[
\Pi^* S^{-1} \Pi = B^* B, \quad B := B(N) = \begin{bmatrix} \beta(0) \\ \beta(1) \\ \vdots \\ \beta(N) \end{bmatrix}.
\]  

(5.19)

Now, matrices \(C_k\) and system (1.2) are defined via the first equality in (5.13).

Proof. Step 1. According to Remark 5.1 relations \(C_k > 0\) and (1.3) follow from (5.13). Relations (5.15) follow from (5.6). Now, let

\[
K(r) = \begin{bmatrix} K_0(r) \\ K_1(r) \\ \vdots \\ K_r(r) \end{bmatrix},
\]  

(5.20)

where \(K_l(r)\) are \(p \times (r + 1)p\) matrices of the form

\[
K_l(r) = i\beta(l)J[\beta(0)^* \ldots \beta(l-1)^* \beta(l)^*/2 \ldots 0].
\]  

(5.21)

From (5.19)-(5.21) it follows that

\[
K(r) - K(r)^* = iB(r)JB(r)^*.
\]  

(5.22)

By induction we shall show in the next step that \(K\) is similar to \(A\):

\[
K(r) = V_-(r) A(r) V_-(r)^{-1} \quad (0 \leq r \leq N),
\]  

(5.23)
where $V(r)^\pm$ are block lower triangular matrices. Taking into account (5.23) and multiplying both sides of (5.22) by $V(r)^{-1}$ from the left and by $(V(r)^*)^{-1}$ from the right, we get

$$A(r)S(r) - S(r)A(r)^* = i\Pi(r)J\Pi(r)^*,$$

$$S(r) := V(r)^{-1}(V(r)^*)^{-1}, \quad \Pi(r) := V(r)^{-1}B(r).$$

Moreover, Step 3 will show that matrix $V(N)$ can be chosen so that the equality

$$\Pi = [\Phi_1 \Phi_2] = V(N)^{-1}B(N)$$

holds, i.e., $\Pi = \Pi(N)$. (Here $\Phi_1$ and $\Phi_2$ are given by (5.16).)

Identities (5.24) have unique solutions $S(r)$ as the spectra of $A(r)$ and $A(r)^*$ do not intersect. (The statement follows from the rewriting of (5.24) in the form

$$S(r)(A(r)^* - \lambda I)^{-1} - (A(r) - \lambda I)^{-1}S(r) = i(A(r) - \lambda I)^{-1}\Pi(r)J\Pi(r)^*(A(r)^* - \lambda I)^{-1},$$

and from the following integration of both sides of the obtained identity along a contour, such that the spectra of $A$ is inside and the spectra of $A^*$ outside it. In particular, by (5.18) and (5.24) one can see that $S := S(N)$. Hence, we derive from (5.25) and (5.26) that $S > 0$ and the first equality in (5.19) holds. It remains only to prove (5.23) and (5.26).

Step 2. Now, we shall consider block lower triangular matrices $V_(k)$ ($0 \leq k \leq N$):

$$V_(0) = v_(0) = \beta_1(0), \quad V_(k) = \begin{bmatrix} V_-(k-1) & 0 \\ X(k) & v_-(k) \end{bmatrix} \quad (k > 0),$$

where $v_-(k)$ are $p \times p$ matrices, where $\beta_1(k)$ and $\beta_2(k)$ are $p \times p$ blocks of $\beta(k) = [\beta_1(k) \beta_2(k)]$, and where $X(k) = [X_0(k) \tilde{X}(k)]$ are $p \times kp$ matrices. Here $X_0(k)$ are arbitrary $p \times p$ blocks, and the matrices $\tilde{X}(k), v_-(k)$ are given by the formulas

$$\tilde{X}(k) = i(\beta(k)J[\beta(0)^* \ldots \beta(k-1)^*]V_-(k-1) \begin{bmatrix} I_{(k-1)p} \\ 0 \end{bmatrix} - v_-(k)[I_p \ldots I_p]),$$

20
\[
\times \left( A(k-2) + \frac{i}{2} I_{(k-1)p} \right)^{-1}, \quad v_-(k) = \beta(k) J \beta(k-1)^* v_-(k-1). \quad (5.28)
\]

According to (5.17) we have \( A(0) = (i/2) I_p \). From the second relation in (5.13) and definitions (5.20) and (5.21) it is immediate that \( K(0) = (i/2) I_p \), and so (5.23) is valid for \( r = 0 \). Assume that (5.23) is true for \( r = k - 1 \), and let us show that (5.23) is true for \( r = k \) too. It is easy to see that
\[
V_-(k) - 1 = \left[ \begin{array}{cc}
V_-(k-1)^{-1} & 0 \\
-v_-(k)^{-1} X(k) V_-(k-1)^{-1} & v_-(k)^{-1}
\end{array} \right]. \quad (5.29)
\]

Then, in view of definitions (5.17) and (5.27), our assumption implies
\[
V_-(k) A(k) V_-(k)^{-1} = \left[ \begin{array}{cc}
K(k-1) & 0 \\
Y(k) & i/2 I_p
\end{array} \right], \quad (5.30)
\]

where \( Y(k) = \left[ (X(k) A(k-1) + i v_-(k) [I_p \ldots I_p]) i/2 v_-(k) \right] \)
\[
\times \left[ \begin{array}{cc}
V_-(k-1)^{-1} & 0 \\
-v_-(k)^{-1} X(k) V_-(k-1)^{-1}
\end{array} \right].
\]

Rewrite the product on the right-hand side of the last formula as
\[
Y(k) = \left( X(k) (A(k-1) - \frac{i}{2} I_{kp}) + i v_-(k) [I_p \ldots I_p] \right) V_-(k-1)^{-1}. \quad (5.31)
\]

From (5.17) and (5.31) it follows that
\[
Y(k) = \left[ \left( \tilde{X}(k) (A(k-2) + \frac{i}{2} I_{(k-1)p}) + i v_-(k) [I_p \ldots I_p] \right) i v_-(k) \right] V_-(k-1)^{-1}. \quad (5.32)
\]

Notice that the sequence \([I_p \ldots I_p]\) of identity matrices in (5.32) is one block smaller than in (5.31). By (5.28) and (5.32) we have
\[
Y(k) = i \beta(k) J \left[ [\beta(0)^* \ldots \beta(k-1)^*] V_-(k-1) \left[ \begin{array}{c}
I_{(k-1)p}^{}
\end{array} \right] \beta(k-1)^* v_-(k-1) \right]
\times V_-(k-1)^{-1}. \quad (5.33)
\]

Finally, formulas (5.27) and (5.33) imply
\[
Y(k) = i \beta(k) J [\beta(0)^* \ldots \beta(k-1)^*] (k > 0). \quad (5.34)
\]
According to the second relation in (5.13) and formulas (5.21) and (5.34) we get

\[
Y(k) \frac{i}{2} I_p = K_k(k).
\] (5.35)

Now, using (5.20) and (5.35), one can see that the right-hand side of (5.30) equals \( K(k) \). Thus, (5.23) is true for \( r = k \) and, therefore, it is true for all \( 0 \leq r \leq n \).

Step 3. To derive (5.26) we shall first prove that the matrices \( V_-(r) \) given by (5.27) and (5.28) can be chosen so that

\[
V_-(r)^{-1} B_1(r) = I_p, \quad B_1(r) := B(r) \left[ \begin{array}{c} I_p \\ 0 \end{array} \right] = \left[ \begin{array}{c} \beta_1(0) \\ \cdots \\ \beta_1(r) \end{array} \right].
\] (5.36)

In other words, the blocks \( X_0(r) \), arbitrary till now, can be chosen so. Indeed, by the definition in (5.19) and the first equality in (5.27) formula (5.36) is true for \( r = 0 \). Assume that (5.36) is true for \( r = k - 1 \). Then, from (5.29) it follows that (5.36) is true for \( r = k \), if only

\[
-v_-(k)^{-1} X(k) \left[ \begin{array}{c} I_p \\ \cdots \\ I_p \end{array} \right] + v_-(k)^{-1} \beta_1(k) = I_p.
\] (5.37)

It implies that we get equality (5.36) for \( r = k \) by letting

\[
X_0(k) = \beta_1(k) - v_-(k) - \tilde{X}(k) \left[ \begin{array}{c} I_p \\ \cdots \\ I_p \end{array} \right].
\] (5.38)

Hence, by a proper choice of the matrices \( X_0(r) \) we obtain (5.36) for all \( r \leq N \).

It remains to prove that

\[
V_-(N)^{-1} B_2(N) = \Phi_2, \quad B_2(N) := \left[ \begin{array}{c} \beta_2(0) \\ \cdots \\ \beta_2(N) \end{array} \right].
\] (5.39)

For that purpose we shall consider the matrix function \( W_{N+1}(\lambda) \), which is used in (5.11) to define the coefficients of the Möbius transformation
Namely, we shall prove the transfer matrix function representation of $W_{N+1}(\lambda)$:

$$W_{N+1}(\lambda) = \left(\frac{\lambda + i}{\lambda}\right)^{N+1} K^* w_A \left(N, -\frac{\lambda}{2}\right) K; \quad (5.40)$$

where

$$w_A(r, \lambda) = I_{2p} - i J \Pi(r)^* S(r)^{-1} \left(A(r) - \lambda I_{(r+1)p}\right)^{-1} \Pi(r). \quad (5.41)$$

Identity (5.24) implies a similar to (2.25) equality

$$w_A(r, \mu)^* J w_A(r, \lambda) = J + i(\overline{\mu} - \lambda) \Pi(r)^* (A(r)^* - \overline{\mu} I_{(r+1)p})^{-1} S(r)^{-1} (A(r) - \lambda I_{(r+1)p})^{-1} \Pi(r). \quad (5.42)$$

Moreover, according to factorization theorem 4 from [35] (see also [37], p. 188) we have

$$w_A(r, \lambda) = \left(I_{2p} - i J \Pi(r)^* S(r)^{-1} P^* (PA(r) P^* - \lambda I_p)^{-1} (PS(r)^{-1} P^*)^{-1}
\times PS(r)^{-1} \Pi(r)\right) w_A(r-1, \lambda), \quad P = [0 \ldots 0 \ I_p]. \quad (5.43)$$

Taking into account (5.17), (5.25), and (5.27) we obtain

$$\left(P A(r) P^* - \lambda I_p\right)^{-1} = \left(\frac{i}{2} - \lambda\right)^{-1} I_p, \quad PS(r)^{-1} P^* = v_-(r)^* v_-(r), \quad (5.44)$$

$$PS(r)^{-1} \Pi(r) = v_-(r)^* P B(r) = v_-(r)^* \beta(r). \quad (5.45)$$

Substitute (5.44) and (5.45) into (5.43) to get

$$w_A(r, \frac{\lambda}{2}) = \left(I_{2p} - \frac{2i}{i - \lambda} J \beta(r)^* \beta(r)\right) w_A(r-1, \frac{\lambda}{2}). \quad (5.46)$$

From the definitions (5.17), (5.25), and (5.41) we also easily derive

$$w_A(0, \frac{\lambda}{2}) = I_{2p} - \frac{2i}{i - \lambda} J B(0)^* B(0) = I_{2p} - \frac{2i}{i - \lambda} J \beta(0)^* \beta(0). \quad (5.47)$$

On the other hand system (1.2) with additional conditions (5.13) can be rewritten as

$$W(r + 1, \lambda) = \frac{\lambda + i}{\lambda} \left(I_{2p} - \frac{2i}{i + \lambda} j K^* \beta(r)^* \beta(r) K\right) W(r, \lambda). \quad (5.48)$$
In view of the normalization $W(0) = I_{2p}$, formulas (5.46)-(5.48) imply (5.40). From (5.40) and (5.42) it follows that

$$W(N + 1, \lambda)jW(N + 1, \lambda^*) = \left(\frac{\lambda + i}{\lambda}\right)^{N+1} \left(\frac{\lambda - i}{\lambda}\right)^{N+1} j.$$  \hspace{1cm} (5.49)

Let us include functions $\varphi$ into consideration. Introduce

$$\mathcal{A}(\lambda) := \left|\frac{\lambda}{\lambda + i}\right|^{2N+2} \left[i\varphi(\lambda)^* I_p\right]KW(N+1, \lambda)^* jW(N+1, \lambda)K^* \left[-i\varphi(\lambda) I_p\right].$$  \hspace{1cm} (5.50)

According to (5.10), (5.11), and (5.49) we have

$$\mathcal{A}(\lambda) = \left|\frac{\lambda - i}{\lambda}\right|^{2N+2} (\mathcal{W}_{11}(\lambda)R(\lambda) + \mathcal{W}_{12}(\lambda)Q(\lambda))^* \mathcal{A}(\lambda)^* \left[-i\varphi(\lambda) I_p\right].$$  \hspace{1cm} (5.51)

By (5.14) and (5.51) $\mathcal{A}$ is bounded in the neighborhood of $\lambda = -i$:

$$\|\mathcal{A}(\lambda)\| = O(1) \text{ for } \lambda \to -i.$$  \hspace{1cm} (5.52)

Now, substitute (5.40) and (5.42) into (5.50) to obtain

$$\mathcal{A}(\lambda) = [i\varphi(\lambda)^* I_p] \left(J + \frac{i}{2}(\lambda - \overline{\lambda})\Pi(N)^* A(N)^* + \frac{\lambda}{2}I_{(N+1)p}\right)^{-1} S(N)^{-1}$$

$$\times (A(N) + \frac{\lambda}{2}I_{(N+1)p})^{-1}\Pi(N) \left[-i\varphi(\lambda) I_p\right].$$  \hspace{1cm} (5.53)

Notice that $S(N) > 0$. Hence, formulas (5.52) and (5.53) imply that

$$\left\|\left(A(N) + \frac{\lambda}{2}I_{(N+1)p}\right)^{-1}\Pi(N) \left[-i\varphi(\lambda) I_p\right]\right\| = O(1) \text{ for } \lambda \to -i.$$  \hspace{1cm} (5.54)

Recall that $\Pi(N) = V_-(N)^{-1} B(N)$ and that $A(N)$ is denoted by $A$. Now, represent $\Pi(N)$ in the block form

$$\Pi(N) = [\Phi_1(N) \quad \Phi_2(N)], \quad \Phi_k(N) = V_-(N)^{-1} B_k(N) \quad (k = 1, 2).$$  \hspace{1cm} (5.55)
According to (5.16) and (5.36) we have $\Phi_1(N) = \Phi_1$. Hence, multiplying the matrix function on the left-hand side of (5.54) by $i(\Phi_1^*(A + \frac{\lambda}{2} I_{(N+1)p})^{-1} \Phi_1)$ we derive

$$
\phi(\lambda) + i \left( \Phi_1^*(A + \frac{\lambda}{2} I_{(N+1)p})^{-1} \Phi_1 \right)^{-1} \Phi_1^*(A + \frac{\lambda}{2} I_{(N+1)p})^{-1} \Phi_1 = O \left( \left\| \Phi_1^*(A + \frac{\lambda}{2} I_{(N+1)p})^{-1} \Phi_1 \right\| \right)
$$

for $\lambda \to -i$. (5.56)

The matrix $A + \frac{\lambda}{2} I_{(N+1)p}$ is easily inverted explicitly (see, for instance, formula (1.10) in [32]). As a result one obtains

$$
\Phi_1^*(A + \frac{\lambda}{2} I_{(N+1)p})^{-1} = \frac{2}{i + \lambda} [\hat{q}^N \hat{q}^{N-1} \ldots \hat{q} I_p], \quad \hat{q} := \frac{\lambda - i}{\lambda + i} I_p.
$$

Moreover, we get

$$
\Phi_1^*(A + \frac{\lambda}{2} I_{(N+1)p})^{-1} \Phi_1 = \frac{2}{i + \lambda} \left( \hat{q}^{N+1} - I_p \right) \left( \hat{q} - I_p \right)^{-1}.
$$

(5.57)

Let $\lambda = i \left( \frac{z+1}{z-1} \right)$, i.e., $z = \left( \frac{\lambda + i}{\lambda - i} \right)$. Then, we derive from (5.58) that

$$
\left( \Phi_1^*(A + \frac{\lambda}{2} I_{(N+1)p})^{-1} \Phi_1 \right)^{-1} = (-iz^{N+1} + O(z^{2N+2})) I_p \quad (z \to 0).
$$

(5.59)

Taking into account (5.57) and (5.59), we rewrite (5.56) as

$$
\phi \left( i \left( \frac{z+1}{z-1} \right) \right) + i(1 - z)[I_p \quad zI_p \quad z^2 I_p \quad \ldots \Phi_2(N)] = O(z^{N+1})
$$

for $z \to 0$. From (5.16) and (5.60) it follows that $\Phi_2(N) = \Phi_2$, i.e., (5.39) is true. As $\Phi_1(N) = \Phi_1$ and $\Phi_2(N) = \Phi_2$, so $\Pi(N) = \Pi$ and formula (5.26) is finally proved. ■

**Definition 5.4** Let matrices $C_k$ satisfy (5.13). Then, a $p \times p$ matrix function $\varphi$ holomorphic in $\mathbb{C}_-$ is said to be a Weyl function for system (1.2) on the interval $0 \leq k \leq N$, if $\varphi$ admits representation (5.10), where a pair $R, Q$ is meromorphic in $\mathbb{C}_-$, well-defined at $\lambda = -i$, and nonsingular with $j$-property, i.e.,

$$
R(\lambda)^* R(\lambda) + Q(\lambda)^* Q(\lambda) > 0, \quad R(\lambda)^* R(\lambda) \leq Q(\lambda)^* Q(\lambda).
$$

(5.61)

The set of Weyl functions is denoted by $\mathcal{N}(N)$.
Using notation (5.11), we deduce from (4.2) the inequality
\[ q(\lambda)^{N+1}W(\lambda)jW(\lambda)^* \leq J, \quad \lambda \in \mathbb{C}_-. \] \hspace{1cm} (5.62)

According to [26] we can change the order of factors in (5.62):
\[ q(\lambda)^{N+1}W(\lambda)^* JW(\lambda) \leq j, \quad \lambda \in \mathbb{C}_-. \] \hspace{1cm} (5.63)

Moreover, after excluding \( \lambda = -i \) the inequality is strict
\[ q(\lambda)^{N+1}W(\lambda)^* JW(\lambda) < j, \quad \lambda \in \mathbb{C}_- \setminus -i. \] \hspace{1cm} (5.64)

In view of (1.2), (5.11), (5.4), and (5.5) at \( \lambda = -i \) we get
\[ W(-i) = KW_{N+1}(i)^* = (-2)^{N+1}K \prod_{k=0}^{N} (\tilde{\beta}_k^j \tilde{\beta}_k^* j). \] \hspace{1cm} (5.65)

From the second relations in (5.3) and from (5.61) we, analogously to the proof of (5.6), derive:
\[ \det [I_p \, I_p] j \tilde{\beta}_0^* \neq 0, \quad \det \tilde{\beta}_k j \tilde{\beta}_{k+1}^* \neq 0, \quad \det \tilde{\beta}_N j \begin{bmatrix} R(-i) \\ Q(-i) \end{bmatrix} \neq 0. \] \hspace{1cm} (5.66)

By (5.65) and (5.66) the next proposition is valid.

**Proposition 5.5** Let the pair \( R, Q \) satisfy (5.61) Then inequality (5.14) is fulfilled.

By Proposition 5.5 and the proof of Theorem 5.3 we get a corollary.

**Corollary 5.6** Weyl functions of system (1.2), which satisfies conditions (5.13), are Herglotz functions and admit the Taylor representation
\[ \varphi\left( i \left( \frac{z+1}{z-1} \right) \right) = -i \left( \psi_0 + (\psi_1 - \psi_0)z + \ldots + (\psi_N - \psi_{N-1})z^N \right) + O(z^{N+1}), \] \hspace{1cm} (5.67)

where \( z \to 0 \) and the \( p \times p \) matrices \( \psi_k \) are the blocks of
\[ \Phi_2 = \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_N \end{bmatrix} = V_-(N)^{-1}B(N). \] \hspace{1cm} (5.68)
Proof. From (5.10), (5.61) and (5.63) it follows that

\[
[I_p \quad i\varphi^*]J \begin{bmatrix} I_p \\ -i\varphi \end{bmatrix} \leq 0,
\]
i.e., \(\Im \varphi(\lambda) \leq 0\) for \(\lambda \in \mathbb{C}_-\), and so \(\varphi\) is a Herglotz function.

By Proposition 5.5 the Weyl functions satisfy conditions of Theorem 5.3. Then, by the second relation in (5.16) we have representation (5.67) of \(\varphi\) via the blocks of \(\Phi_2\). By the proof of Theorem 5.3 we get also \(\Phi_2 = \Phi_2(N)\), i.e., (5.68) holds. Here \(V_-(N)\) and \(B(N)\) are recovered from the matrices \(\beta(k)\) and do not depend on \(\varphi\). ■

Remark 5.7 As Weyl functions \(\varphi\) satisfy conditions of Theorem 5.3, so the procedure given in Theorem 5.3 provides a recovery of system (1.2) from a Weyl function (i.e., provides a solution of the inverse problem).

The following proposition is also true

**Proposition 5.8** The set \(\mathcal{N}(N)\) \((N > M)\) is imbedded in \(\mathcal{N}(M)\), i.e., \(\mathcal{N}(N) \subset \mathcal{N}(M)\).

**Proof.** By (4.2) we have

\[
q(\lambda)^{N+1}W_{N+1}(\lambda)^*jW_{N+1}(\lambda) \leq q(\lambda)^{M+1}W_{M+1}(\lambda)^*jW_{M+1}(\lambda), \quad \lambda \in \mathbb{C}_+.
\]

(5.69)

Insert the length \(N\) of the interval into the notation \(W\):

\[
W(N, \lambda) = W(\lambda) = KW_{N+1}(\lambda)^*.
\]

(5.70)

From (5.69) and (5.70) it follows that

\[
q(\lambda)^{N-M} \left(W(M, \lambda)^{-1}W(N, \lambda)\right)^*jW(M, \lambda)^{-1}W(N, \lambda) \leq j.
\]

(5.71)

Moreover, in view of (1.2) and (5.70) we have

\[
W(M, \lambda)^{-1}W(N, \lambda) = \prod_{k=M+1}^{N} (I_m + \frac{i}{\lambda}C_{kj}),
\]

(5.72)
and the expression on the left-hand side of (5.72) is analytic at \( \lambda = -i \).

Suppose now that \( \varphi \in \mathcal{N}(N) \) is a Weyl function generated by some pair \( R, Q \), which satisfies (5.61). Then, according to (5.61), (5.71) and (5.72) the pair
\[
\left[ \begin{array}{c} \tilde{R}(\lambda) \\ \tilde{Q}(\lambda) \end{array} \right] = W(M,\lambda)^{-1} W(N,\lambda) \left[ \begin{array}{c} R(\lambda) \\ Q(\lambda) \end{array} \right]
\]

satisfies conditions of Definition 5.4 too. Moreover, it is easy to see that
\[
i(W_{21}(M,\lambda)\tilde{R}(\lambda) + W_{22}(M,\lambda)\tilde{Q}(\lambda))(W_{11}(M,\lambda)\tilde{R}(\lambda) + W_{12}(M,\lambda)\tilde{Q}(\lambda))^{-1},
\]
\[
= i(W_{21}(N,\lambda)R(\lambda) + W_{22}(N,\lambda)Q(\lambda))(W_{11}(N,\lambda)R(\lambda) + W_{12}(N,\lambda)Q(\lambda))^{-1}
\]
\[
= \varphi(\lambda),
\]
which completes the proof.  

Theorem 5.3 and Proposition 5.8 imply a Borg-Marchenko type result:

**Theorem 5.9** Let \( \tilde{\varphi} \) and \( \hat{\varphi} \) be Weyl functions of the two discrete Dirac type systems (1.2), which satisfy conditions (5.13). Denote by \( \tilde{C}_k \) \((0 \leq k \leq \tilde{N})\) the potentials \( C_k \) of the first system and by \( \hat{C}_k \) \((0 \leq k \leq \hat{N})\) the potentials of the second system. Denote Taylor coefficients of \( i\tilde{\varphi}\left(i\left(\frac{z+1}{z-1}\right)\right) \) and \( i\hat{\varphi}\left(i\left(\frac{z+1}{z-1}\right)\right) \) at \( z = 0 \) by \( \{\tilde{\alpha}_k\} \) and \( \{\hat{\alpha}_k\} \), respectively, and assume that \( \tilde{\alpha}_k = \hat{\alpha}_k \) for all \( k \leq N \leq \min\{\tilde{N},\hat{N}\} \). Then we have \( \tilde{C}_k = \hat{C}_k \) for \( k \leq N \).

**Proof.** According to Proposition 5.8, \( \tilde{\varphi} \) and \( \hat{\varphi} \) are Weyl functions of the first and second systems, respectively, on the interval \( 0 \leq k \leq N \). By Theorem 5.3 these systems on the interval \( 0 \leq k \leq N \) are uniquely recovered by the first \( N + 1 \) Taylor coefficients of the Weyl functions.  

An interesting Borg-Marchenko type result for supersymmetric Dirac difference operators have been obtained earlier in [11].
6 Toeplitz matrices and Dirac system on the semiaxis

By [28], p. 116 it is easy to recover a block Toeplitz matrix $S$ which satisfies (5.18), where the blocks $\Phi_1$ and $\Phi_2$ of $\Pi$ are given by (5.16). Namely, we have

$$S = \{s_{j-k}\}_{k,j=0}^N, \quad s_{-k} = \alpha_k = s_k^* \quad (k > 0), \quad s_0 = s_0^* = \alpha_0 + \alpha_0^*. \quad (6.1)$$

Moreover, this $S$ is a unique solution of (5.18). A description of all extensions of $S$ preserving the number of negative eigenvalues, which uses transfer matrix function $w_A$, is given in [28] (see also Theorem 4.1 in [32]) in terms of the linear fractional transformation

$$\hat{\varphi}(\lambda) = \left( \hat{R}(\lambda)w_{11}(\lambda) + \hat{Q}(\lambda)w_{21}(\lambda) \right)^{-1} \left( \hat{R}(\lambda)w_{12}(\lambda) + \hat{Q}(\lambda)w_{22}(\lambda) \right), \quad (6.2)$$

where $\{w_{kj}(\lambda)\}_{k,j=1}^2 = w_A(N,\lambda)$, and the meromorphic pairs $\hat{R}, \hat{Q}$ have $J$-property, i.e.,

$$\hat{R}(\lambda)\hat{R}(\lambda)^* + \hat{Q}(\lambda)\hat{Q}(\lambda)^* > 0, \quad \hat{R}(\lambda)\hat{Q}(\lambda)^* + \hat{Q}(\lambda)\hat{R}(\lambda)^* \geq 0, \quad \lambda \in \mathbb{C}_+. \quad (6.3)$$

In particular, for the case $S > 0$, which is treated here, the matrix functions $\hat{\varphi}\left( -i\frac{(z+1)}{2(z-1)} \right)$ are always analytic at $z = 0$ and admit the Taylor representation

$$\hat{\varphi}\left( -i\frac{(z+1)}{2(z-1)} \right) = \hat{s}_0 + \hat{s}_{-1}z + \hat{s}_{-2}z^2 + \ldots \quad (6.4)$$

Our next statement follows from Theorem 4.1 [32].

**Theorem 6.1** Assume that $S = \{s_{j-k}\}_{k,j=0}^N > 0$, and fix $\alpha_0$ such that $\alpha_0 + \alpha_0^* = s_0$. Using (5.16), (5.41), and (6.1) introduce $\Pi = [\Phi_1 \Phi_2]$ and $\{w_{kj}(\lambda)\}_{k,j=1}^2 = w_A(N,\lambda)$. Now, let matrix functions $\hat{\varphi}$ be given by (6.2), where the pairs $\hat{R}, \hat{Q}$ satisfy (6.3) and are well defined at $\lambda = \frac{i}{2}$. Then the Taylor coefficients $\hat{s}_{-k}$ at $z = 0$ of the matrix functions $\hat{\varphi}\left( -i\frac{(z+1)}{2(z-1)} \right)$ satisfy relations

$$\hat{s}_{-k} = s_{-k} \quad (0 < k \leq N), \quad \hat{s}_0 = \alpha_0. \quad (6.5)$$
Moreover, putting $s_{-k} = s^*_{-k} = \tilde{s}_{-k}$ for $k > N$, we have $\{s_{j-k}\}_{k,j=0}^M \geq 0$ for all $M > N$. In other words, the Taylor coefficients of the matrix functions $\hat{\varphi}\left(-\frac{i(z + 1)}{2(z - 1)}\right)$ generate nonnegative extensions of $S$. All the nonnegative extensions of $S$ are generated in this way.

Taking into account that $w_A(N, \lambda)Jw_A(N, \overline{\lambda})^* = J$, we derive equality $\hat{\varphi} = \tilde{\varphi}$ for the matrix function

$$\tilde{\varphi}(\lambda) = -\left(w_{12}(\overline{\lambda})^*\tilde{R}(\lambda) + w_{22}(\overline{\lambda})^*\tilde{Q}(\lambda)\right)\left(w_{11}(\overline{\lambda})^*\tilde{R}(\lambda) + w_{21}(\overline{\lambda})^*\tilde{Q}(\lambda)\right)^{-1},$$

(6.6)

where

$$\tilde{R}(\lambda)^*\tilde{R}(\lambda) + \tilde{Q}(\lambda)^*\tilde{Q}(\lambda) > 0, \quad \tilde{R}(\lambda)\tilde{Q}(\lambda) + \tilde{Q}(\lambda)\tilde{R}(\lambda) = 0, \quad \lambda \in \mathbb{C}_+.$$  

(6.7)

Notice also that relations (6.3) and (6.7) yield

$$\tilde{R}(\lambda)^*\tilde{Q}(\lambda) + \tilde{Q}(\lambda)^*\tilde{R}(\lambda) \leq 0,$$

(6.8)

and vice versa relations (6.7) and (6.8) yield the second relation in (6.3). Hence, Theorem 6.1 can be reformulated in terms of the linear fractional transformations (6.6), where $\tilde{R}, \tilde{Q}$ have $J$-property (6.8). Finally, use (5.11), (5.12) and (5.40) to rewrite (5.10) in the form

$$i\varphi(\lambda) = -\left(w_{12}(-\overline{\lambda}/2)^*\tilde{R}(-\lambda/2) + w_{22}(-\overline{\lambda}/2)^*\tilde{Q}(-\lambda/2)\right)$$

$$\times \left(w_{11}(-\overline{\lambda}/2)^*\tilde{R}(-\lambda/2) + w_{21}(-\overline{\lambda}/2)^*\tilde{Q}(-\lambda/2)\right)^{-1},$$

(6.9)

where we put

$$\begin{bmatrix}
\tilde{R}(-\lambda/2) \\
\tilde{Q}(-\lambda/2)
\end{bmatrix} = K
\begin{bmatrix}
R(\lambda) \\
Q(\lambda)
\end{bmatrix}. $$

(6.10)

Here, formula (6.10) is a one to one mapping of the pairs satisfying (5.61) into pairs satisfying the first relation in (6.7) and relation (6.8). By (6.6) and (6.9) we have $\hat{\varphi}\left(-\frac{i(z + 1)}{2(z - 1)}\right) = \tilde{\varphi}\left(-\frac{i(z + 1)}{2(z - 1)}\right) = i\varphi\left(\frac{z + 1}{z - 1}\right)$ Therefore Theorem 6.1 can be rewritten.
Theorem 6.2 Assume that \( S = \{s_{j-k}\}_{k,j=0}^{N} > 0 \), fix \( \alpha_0 \) such that \( \alpha_0 + \alpha_0^* = s_0 \), and introduce \( \mathcal{W} \) via (5.11) and (5.40). Let matrix functions \( \varphi \) be given by (5.10), where the pairs \( R, Q \) satisfy (5.61) and are well defined at \( \lambda = -i \).

Then \( i\varphi(-i) = \alpha_0 \), and the following Taylor coefficients \( \alpha_k \) at \( z = 0 \) of the matrix functions \( i\varphi\left(\frac{i(z+1)}{(z-1)}\right) \) satisfy relations

\[
\alpha_k = s_{-k} \quad (0 < k \leq N). \tag{6.11}
\]

Moreover, putting \( s_{-k} = s_k^* = \alpha_k \) for \( k > N \), we have \( \{s_{j-k}\}_{k,j=0}^{M} \geq 0 \) for all \( M > N \). In other words, the Taylor coefficients of the matrix functions \( i\varphi\left(\frac{i(z+1)}{(z-1)}\right) \) generate nonnegative extensions of \( S \). All the nonnegative extensions of \( S \) are generated in this way.

Remark 6.3 By Definition 5.4 and Theorem 6.2 the Weyl functions from the Weyl disk \( N(N) \) generate all the nonnegative extensions of \( S \). It provides, in particular, another proof of Proposition 5.8.

Consider now system (1.2), which satisfies (5.13) on the semiaxis \( k \geq 0 \).

Theorem 6.4 Let system (1.2) be given on the semiaxis \( k \geq 0 \) and let matrices \( C_k \) satisfy (5.13). Then, there is a unique function \( \varphi_\infty \), which belongs to all the Weyl discs \( \mathcal{N}(N) \):

\[
\bigcap_{N=0}^{\infty} \mathcal{N}(N) = \varphi_\infty. \tag{6.13}
\]

Proof. According to Corollary 5.6 matrices \( \{C_k\}_{k=0}^{N} \) (\( N < \infty \)) or equivalently matrices \( \{\beta(k)\}_{k=0}^{N} \) uniquely define blocks \( \{s_{-k}\}_{k=0}^{N} \), where \( s_{-k} = \alpha_k = \psi_k - \psi_{k-1} \) for \( k > 0 \) and \( s_0 = \alpha_0 + \alpha_0^* \) (\( \alpha_0 = \psi_0 \)). Moreover, by Proposition 5.8 these \( s_{-k} \) do not depend on \( N \geq k \), and so system (1.2) on the semiaxis determines an infinite sequence \( \{s_{-k}\}_{k=0}^{\infty} \). By Theorem 5.3 we have \( \{s_{j-k}\}_{k,j=0}^{N} > 0 \) for all \( N \geq 0 \). Apply now Theorem 6.2 to see that

\[
i\varphi\left(\frac{i(z+1)}{(z-1)}\right) = \alpha_0 + \sum_{k=1}^{\infty} s_{-k}z^k = i\varphi_\infty\left(\frac{i(z+1)}{(z-1)}\right), \tag{6.14}
\]
i.e., this $\varphi$ belongs to $\bigcap_{N=0}^{\infty} \mathcal{V}(N)$. Moreover, as the sequence $\{s_{-k}\}_{k=0}^{\infty}$ is unique, so by Theorem 6.2 the function $\varphi \in \bigcap_{N=0}^{\infty} \mathcal{V}(N)$ is unique. ■

Recall that a Weyl function on the semiaxis is defined by Definition 4.7, where $K$ is given by formula (4.27). Theorem 6.4 yields our next result.

**Theorem 6.5** Let system (1.2) be given on the semiaxis $k \geq 0$ and let matrices $C_k$ satisfy (5.13). Then, the matrix function $\varphi_\infty$ given by (6.13) is the unique Weyl function of system (1.2) on the semiaxis.

**Proof.** By (5.10) and (6.13), we have

$$
\begin{bmatrix}
-i\varphi_\infty(\lambda) \\
I_p
\end{bmatrix} = JW(r + 1, \lambda) \begin{bmatrix} R(\lambda) \\
Q(\lambda) \end{bmatrix}
$$

(6.15)

for all $r \geq 0$ and for some depending on $r$ pairs $R, Q$ which satisfy (5.61).

In view of (5.11), (5.12), (5.49), and (6.15) we obtain

$$
[i\varphi_\infty(\lambda)^* I_p] (q(\lambda)^{r+1}KW_{r+1}(\lambda)^*jW_{r+1}(\lambda)K^* - J) \begin{bmatrix} -i\varphi_\infty(\lambda) \\
I_p
\end{bmatrix}
$$

$$
= i \left( \varphi_\infty(\lambda) - \varphi_\infty(\lambda)^* \right) + \left( \frac{\lambda^2 + 1}{\lambda^2(\lambda^2 + 1)} \right)^{r+1} [R(\lambda)^* Q(\lambda)^*] j \begin{bmatrix} R(\lambda) \\
Q(\lambda) \end{bmatrix}.
$$

(6.16)

Now, formulas (5.61) and (6.16) imply

$$
[i\varphi_\infty(\lambda)^* I_p] (q(\lambda)^{r+1}KW_{r+1}(\lambda)^*jW_{r+1}(\lambda)K^* - J) \begin{bmatrix} -i\varphi_\infty(\lambda) \\
I_p
\end{bmatrix}
$$

$$
\leq i \left( \varphi_\infty(\lambda) - \varphi_\infty(\lambda)^* \right).
$$

(6.17)

It follows from (4.2) and (6.17) that

$$
\sum_{k=0}^{r} [i\varphi_\infty(\lambda)^* I_p] q(\lambda)^kKW_k(\lambda)^*C_kW_k(\lambda)K^* \begin{bmatrix} -i\varphi_\infty(\lambda) \\
I_p
\end{bmatrix}
$$

$$
\leq \frac{\lambda^2 + 1}{\lambda - \lambda} \left( \varphi_\infty(\lambda) - \varphi_\infty(\lambda)^* \right).
$$

(6.18)
Finally, by (6.18) the inequality (4.26) is immediate, and \( \varphi_\infty \) given by (6.13) is a Weyl function.

To prove the uniqueness of the Weyl function notice that by Proposition 3.2 and by relation (4.3) we have

\[
q^k W_k^* C_k W_k \geq q^k W_k^* j W_k \geq q^{k-1} W_{k-1}^* j W_{k-1} \geq \ldots \geq W_0^* j W_0 = j \quad (\lambda \in \mathbb{C}_-).
\]

Hence, in view of (6.19) we obtain

\[
\sum_{k=0}^r [I_p I_p] q(\lambda)^k K W_k(\lambda)^* C_k W_k(\lambda) K^* \left[ \begin{array}{c} I_p \\ I_p \end{array} \right] \geq 2(r + 1) I_p,
\]

and it follows that

\[
\sum_{k=0}^\infty [I_p I_p] q(\lambda)^k K W_k(\lambda)^* C_k W_k(\lambda) K^* \left[ \begin{array}{c} I_p \\ I_p \end{array} \right] = \infty. \tag{6.20}
\]

Taking into account Definition 4.7 and inequality (6.20), we can show the uniqueness of the Weyl function similar to the proof of the uniqueness in Theorem 4.2. ■

Now, we formulate a solution of the inverse problem.

**Theorem 6.6** The set of the Weyl functions \( \varphi(\lambda) \) of systems (1.2), given on the semiaxis \( k \geq 0 \) and such that matrices \( C_k \) satisfy (5.13), coincides with the set of functions \( \varphi \) such that

\[
\tilde{i} \varphi \left( \frac{(z + 1)}{(z - 1)} \right) = \alpha_0 + \sum_{k=1}^\infty s_{-k} z^k
\]

are Caratheodory matrix functions in the unit disk and \( \{s_{j-k}\}_{k,j=0}^N > 0 \) for all \( 0 \leq N < \infty \) \( (s_0 := \alpha_0 + \alpha_0^*) \). These systems (1.2) are uniquely recovered from their Weyl functions via the procedure given in Theorem 5.3.

**Proof.** According to Theorem 6.5 the Weyl function on the semiaxis is also a Weyl function on the intervals. Hence, the procedure to construct a solution of the inverse problem follows from Theorem 5.3. It follows from Theorem
5.3 also, that the matrices \( \{s_{j-k}\}_{k,j=0}^N \) generated by the Weyl functions are positive definite.

Hence, it remains to show that all the functions such that (6.21) holds and \( \{s_{j-k}\}_{k,j=0}^r > 0 \) \((r \geq 0)\) are Weyl functions. Indeed, fixing such a matrix function \( \varphi \), we get a sequence of matrices \( S(r) = \{s_{j-k}\}_{k,j=0}^r > 0 \). Therefore we get a sequence of the transfer matrix functions \( w_A(r, \lambda) \) of the form (5.41), where

\[
\Pi(r) = \begin{bmatrix}
I_p & \alpha_0 \\
I_p & \alpha_0 + s_{-1} \\
& \ldots & \ldots \\
I_p & \alpha_0 + s_{-1} + \ldots + s_{-r}
\end{bmatrix},
\]

and (5.24) holds. Taking into account formulas (5.41), (5.47) and (5.43), (5.46), we obtain matrix functions

\[
\beta(r)^* \beta(r) := \Pi(r)^* S(r)^{-1} P^* \left( PS(r)^{-1} P^* \right)^{-1} PS(r)^{-1} \Pi(r) \quad (r \geq 0).
\]

In view of the matrix identity (5.24) we have

\[
PS(r)^{-1} \Pi(r) J \Pi(r)^* S(r)^{-1} P^* = -iP \left( S(r)^{-1} A(r) - A(r)^* S(r)^{-1} \right) P^* = PS(r)^{-1} P^*.
\]

In other words \( \beta(r) \) satisfies the second relation in (5.13). Therefore formulas \( C_r = 2K \beta(r)^* \beta(r) K - j \) define a system of our class on the semiaxis. In fact, this construction coincides with (5.19) and the function \( \varphi \) is the Weyl function of our system. Indeed, similar to the proof of Theorem 5.3 we derive from (6.23) the equality (5.40). Compare now Definition 5.4 and Theorem 6.2 to see that \( \varphi \in \mathcal{N}(N) \) for any \( N \). According to Theorems 6.4 and 6.5 it means that \( \varphi \) is the Weyl function. ■

Finally, consider the upper halfplane and define holomorphic Weyl functions in \( \mathbb{C}_+ \) via relations (4.26) and (4.27) too. Put

\[
\overline{\mathcal{N}}(N) := \{ \varphi(\overline{\lambda})^* : \varphi \in \mathcal{N}(N) \}.
\]
Remark 6.7 Similar to the proof that $\hat{\varphi} = \tilde{\varphi}$, where $\hat{\varphi}$ and $\tilde{\varphi}$ are given by (6.2) and (6.6), respectively, one can show that the set $\mathcal{N}(N)$ consists of linear fractional transformations (5.10), where the pairs $R, Q$ are meromorphic in $\mathbb{C}_+$, are well defined at $\lambda = i$, and have the property

$$R(\lambda)^* R(\lambda) + Q(\lambda)^* Q(\lambda) > 0, \quad R(\lambda)^* R(\lambda) \geq Q(\lambda)^* Q(\lambda), \quad \lambda \in \mathbb{C}_+. \quad (6.26)$$

In view of Remark 6.7, we obtain in $\mathbb{C}_+$ the analog of Theorem 6.5, and the proof is similar.

Theorem 6.8 Let system (1.2) be given on the semiaxis $k \geq 0$ and let matrices $C_k$ satisfy (5.13). Then, the matrix function $\varphi_\infty(\lambda)^* = \bigcap_{N=0}^\infty \mathcal{N}(N)$ is the unique Weyl function in $\mathbb{C}_+$ of system (1.2) on the semiaxis.

Proof. Substitute $\varphi_\infty(\lambda)^*$ instead of $\varphi_\infty(\lambda)$ into (6.16) and take into account (6.26) to derive

$$[i \varphi_\infty(\lambda) I_p] \begin{pmatrix} J - q(\lambda)^{r+1} K W_{r+1}(\lambda)^* j W_{r+1}(\lambda) K^* \end{pmatrix} \begin{pmatrix} -i \varphi_\infty(\lambda)^* \cr I_p \end{pmatrix} \leq i \left( \varphi_\infty(\lambda) - \varphi_\infty(\lambda)^* \right). \quad (6.27)$$

Now, inequalities (6.18) and (4.26) are straightforward, i.e., $\varphi_\infty(\lambda)^*$ is a Weyl function.

Instead of (6.19) we use the inequality

$$q(\lambda)^k W_k(\lambda)^* C_k W_k(\lambda) \geq -q(\lambda)^k W_k(\lambda)^* j W_k(\lambda) \geq -j \quad (\lambda \in \mathbb{C}_+), \quad (6.28)$$

which yields inequality

$$\sum_{k=0}^\infty [I_p - I_p] q(\lambda)^k K W_k(\lambda)^* C_k W_k(\lambda) K^* \begin{pmatrix} I_p \cr -I_p \end{pmatrix} = \infty. \quad (6.29)$$

The uniqueness of the Weyl function follows from (6.29) \qed

Theorem 6.8 for the scalar case $p = 1$ has been proved earlier in [18, 23] (see also Theorem 3.2.11 [38]).

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