Fermionic counting of RSOS-states and Virasoro character formulas for the unitary minimal series $M(\nu, \nu + 1)$. Exact results.

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Dedicated to the memory of Mikhail Nirenberg

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Abstract

The Hilbert space of an $RSOS$-model, introduced by Andrews, Baxter, and Forrester, can be viewed as a space of sequences (paths) $\{a_0, a_1, \ldots, a_L\}$, with $a_j$-integers restricted by $1 \leq a_j \leq \nu$, $|a_j - a_{j+1}| = 1$, $a_0 \equiv s$, $a_L \equiv r$. In this paper we introduce different basis which, as shown here, has the same dimension as that of an $RSOS$-model. This basis appears naturally in the Bethe ansatz calculations of the spin $\frac{\nu - 1}{2}$ $XXZ$-model. Following McCoy et al, we call this basis – fermionic (FB).

Our first theorem $\text{Dim}(FB) = \text{Dim}(RSOS - basis)$ can be succinctly expressed in terms of some identities for binomial coefficients. Remarkably, these binomial identities can be $q$-deformed. Here, we give a simple proof of these $q$-binomial identities in the spirit of Schur’s proof of the Rogers-Ramanujan identities. Notably, the proof involves only the elementary recurrences for the $q$-binomial coefficients and a few creative observations.

Finally, taking the limit $L \to \infty$ in these $q$-identities, we derive an expression for the character formulas of the unitary minimal series $M(\nu, \nu + 1)$ ”Bosonic Sum $\equiv$ Fermionic Sum”. Here, Bosonic Sum denotes Rocha-Caridi representation $(\chi_{r,s=1}^{\nu,\nu+1}(q))$ and Fermionic Sum stands for the companion representation recently conjectured by the McCoy group [3].
1. Introduction

The last decade has witnessed a remarkable convergence of ideas in such diverse areas of mathematical physics as theory of knots and links, classical and quantum exactly integrable systems, two-dimensional gravity, string and conformal field theories and others. The emerging mathematical structure, however, makes one turn around and examine number theoretical questions which often date back to the time of ancient Greece.

In a very interesting development [1]-[6], celebrated identities of the Rogers-Ramanujan type were revisited and many new \(q\)-series for the characters of irreducible highest weight representations of conformal field theories (CFT) were proposed. As a result, numerous new links between Bethe ansatz approach [7], CFT, modular forms, Rogers dilogarithmic functions and geometry of flag manifold were established [1]-[6], [8]-[12].

In particular, Rogers-Ramanujan identities [13]

\[
\sum_{m=0}^{\infty} \frac{q^{m(m+r)}}{(q, q)_m} = \frac{1}{(q, q)_{\infty}} \sum_{j=-\infty}^{\infty} \{ q^{j(10j+1+2r)} - q^{(2j+1)(5j+2-r)} \}; \quad r = 0, 1
\]

(1.1)

can be expressed in the modern language as "Fermionic Sum \(\equiv\) Bosonic Sum" representation of characters of the non-unitary minimal model \(M(2, 5)(c = -\frac{22}{5})\), with \(r = 1(0)\) being a label for a primary field of conformal dimension \(0(-\frac{1}{5})\). We also mention that the character itself can be thought as a "\(q\)-dimension" of the Bethe ansatz basis.

In this paper we consider the following generalization of (1.1) related to \(M(\nu, \nu + 1)\) conformal series \((c = 1 - \frac{6}{\nu(\nu+1)})\)

\[
\sum_{\substack{m_1, m_2, \ldots, m_{\nu-2} = 0 \\text{mod} 2 \\ m_i \equiv V_{C,B}^r \text{ (mod 2)}}}^{\infty} \frac{q^{m_i C m_i}}{(q, q)_{m_{\nu-2}}} \prod_{i=1}^{\nu-3} \left[ \frac{1}{2}(K_{\nu-2} \cdot \bar{m}_i) + \frac{1}{2} \delta_{i, b^+} \right]_{q} =
\]

\[
\frac{q^r}{(q, q)_{\infty}} \sum_{j=-\infty}^{\infty} (q^{j\nu(\nu+1)+r(\nu+1)-\nu} - q^{(j\nu+r)(j(\nu+1)+1)}); \quad r = 1, 2, \ldots, \nu - 1;
\]

(1.2)

where for \(i = 1, 2, \ldots, \nu - 2\) and \(m_i \in \mathbb{Z} \geq 0\)

\[
(\bar{m})_i = m_i; \quad b^+ = \nu - r - 1; \quad b^- = r - 1
\]

(1.3)

and \((\nu - 2) \times (\nu - 2)\)-dimensional Cartan matrix \(C\) is related to the incidence matrix \(K_{\nu-2}\)

\[
K_{\nu-2} = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
& \cdots & \cdots & \cdots \\
0 & \cdots & 1 & 0 \\
0 & \cdots & 0 & 1
\end{pmatrix}
\]

(1.4)
in a standard fashion
\[ C_{\nu-2} = 2 - K_{\nu-2}. \] (1.5)

The rest of notations in (1.2) will be defined later (2.20, 2.21, 3.4). The r.h.s. in (1.2) can be easily recognized as Rocha-Caridi formula \[ \chi_{\nu,\nu+1}(q) \sim Tr_{r,s=1}(q^{L_0}) \] for the \( M(\nu, \nu + 1) \) character of a primary field with conformal dimension
\[ \Delta_{r,s} = \frac{[r(\nu + 1) - s \cdot \nu] - 1}{4\nu(\nu + 1)}; \quad r = 1, 2, \ldots, \nu - 1; \quad s = 1, 2, \ldots, \nu \] (1.6)
and the l.h.s. in (1.2) stands for the companion fermionic representation, proposed by the Stony Brook group [3]. Note that in the simplest case \( \nu = 3 \) (Ising model) identity (1.2) was known to be true for quite some time [15]. The first important step towards general proof was taken in [6] where polynomial generalization of (1.2) was proposed (see (3.54)) and \( \nu = 4 \) (and 3) case was proven. The object of the present paper is to provide proof for the remaining cases: \( 4 < \nu < \infty \).

Let us now introduce some important background. Among many techniques available to prove identities of the Rogers-Ramanujan type ([16] and references there), proof by Schur [17] still stands out as, perhaps, a monument to the last days of the Kaiser Reich. The main idea of Schur’s approach (see also [18]) was to convert \( q \)-series in (1.1) into polynomials by introducing a finitization parameter \( L \), which roughly measures the degree of polynomials. Letting \( L \) tend to infinity, one recovers original identities. An advantage of having \( L \) is that it can be employed to prove polynomial identities by means of recursion relations. Remarkably, this finitization parameter has a direct physical interpretation. Depending on a situation, one can think of \( L \) either as a size of a system, number of particles or as an ultraviolet cut-off for the truncated conformal basis ([13], and below).

Another hero of our story is a \( \nu \)-state RSOS-model introduced by Andrews, Baxter, and Forrester (ABF) [19] about ten years ago. Unlike some papers which go through a long latent period before (if ever) they are in the limelight, the ABF paper received immediate attention due to Huse’s identification [20] of the critical points of RSOS-models (in a first critical regime) with those described by the unitary minimal series \( M(\nu, \nu + 1) \). Huse’s observation was later confirmed in [21] and [22], where central charge \( c \) for a second critical regime of a \( \nu \)-state RSOS-model was found to be equal to that of the Fateev-Zamolodchikov parafermion model [23]
\[ c = 2 - \frac{6}{\nu + 1}. \]

The states of an RSOS-model can be thought as paths, labeled by sequences of integers \( \{a_0, a_1, \ldots, a_L\} \) with \( 1 \leq a_i \leq \nu \) (\( i = 0, 1, \ldots, L \)), \( a_0 \equiv s, a_L \equiv r \). In what follows, we will

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¹Since Schur discovered and proved identities (1.1) independently, it would be appropriate to correct the historical injustice and call (1.1) Rogers-Ramanujan-Schur identities, as attempted by some.
refer to the fixed values \( r \) and \( s \) as boundary conditions \((r, s)\).

Boltzmann weights of a \( \nu \)-state RSOS-model (their explicit form will not be needed here)

\[
W_{\nu} \begin{bmatrix}
  a'_{i+1} \\
  a_i \\
  a_{i+2} \\
  a_{i+1}
\end{bmatrix}
\]

vanish, unless integers \( a_i \)'s satisfy the constraint

\[
| a_i - a_{i+1} | = 1. \tag{1.7}
\]

Dimension of the path space (subject to restriction (1.7) above and boundary conditions \((r, s)\)) can be concisely expressed in terms of the incidence matrix \( K_{\nu} \) as

\[
\text{Dim}_{r,s}(\text{path space}) = ((K_{\nu})^{L})_{s,r}. \tag{1.8}
\]

In the limit \( L \to \infty \), it is possible to think of the path states with boundary conditions \((s, r)\) as \( M(\nu, \nu + 1) \) Virasoro states in the module of a primary field with conformal dimension \( \Delta_{r,s} \) (1.6).

The corner transfer matrix calculations of ABF [19] suggest that the action of a Virasoro (energy) operator \( L_0 \) in the path space \((r, s)\) can be described as

\[
\frac{1}{4} \sum_{j=1}^{\infty} j | a_j - a_{j+2} |. 
\]

Not much is known about the action of other Virasoro generators in the path space. The interested reader may consult [24]-[29] for the recent developments. Path-Virasoro isomorphism, once rigorously established, would imply the natural interpretation of finite \( L \) as a kind of an ultraviolet cut-off for the truncated conformal basis.

A connection between critical RSOS-model and XXZ-model with rational anisotropy \( \gamma = \frac{\pi}{\nu+1} \) was noticed in [22], [30] and [31], where it was shown that the spectrum of an RSOS-model can be obtained from that of XXZ-model, provided one implements certain projection mechanism. Moreover, Reshetikhin proposed in [32] that the RSOS-Boltzmann weights appear as factors in the \( S \)-matrix, describing scattering among the elementary excitations of spin \((\frac{\nu}{2}) \) XXZ-model (in a weak anisotropic regime). According to [32], two-particle \( S \)-matrix has the following structure

\[
S = W_{\nu} \otimes S_{\text{SG}}, \tag{1.9}
\]

where the second factor in the r.h.s. of (1.9) denotes a two-particle amplitude for the Sine-Gordon model.

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2Throughout this paper, subindex of the incidence matrix refers to its dimension.
We remind the reader that a fundamental particle of spin \((\frac{\nu-1}{2})\) \(XXZ\)-model is associated with a hole in the Dirac sea of negative energy Bethe strings of length \((\nu-1)\). Other Bethe strings have zero energy. These strings can be used to label internal quantum numbers of a fundamental particle. In deriving (1.5), Reshetikhin identified short strings (of length \(1 \leq l_i \leq \nu - 2\)) with \(RSOS\) degrees of freedom and long ones \((l_i \geq \nu)\) with vertex (spin \(\frac{1}{2}\)) degrees of freedom of a particle.

In [33] we used the Bethe ansatz technique to evaluate the dimension of an "\(RSOS\)-string" space directly. Our calculations led us to the following system of equations for the non-negative Bethe integers \(n_i, m_i(i = 1, 2, \ldots, \nu - 2)\)

\[
\begin{align*}
    n_1 + m_1 &= \frac{1}{2}m_2 \\
    &\vdots \\
    n_i + m_i &= \frac{1}{2}(m_{i+1} + m_{i-1}) \\
    &\vdots \\
    n_{\nu-2} + m_{\nu-2} &= \frac{1}{2}(L + m_{\nu-3})
\end{align*}
\]

(1.10)

with \(L\), now being a number of physical excitations. Depending on \(\nu\)-parity (integer or half-integer spin), \(L\) can be either

\[L = 0, 1, 2, \ldots ; \quad \nu \equiv \text{even}\]

or

\[L = 0, 2, 4, \ldots ; \quad \nu \equiv \text{odd}.
\]

System (1.10) describes the \((\nu-2)\) Fermi bands. Each band consists of the \((m_i + n_i)\) consecutive integers with only \(n_i\) distinct integers being "occupied". Because of this exclusion rule, we would refer to (1.10) as a fermionic system.

It is a simple matter to convert (1.10) into a partition problem for \(\frac{L}{2}\)

\[
\sum_{i=1}^{\nu-2} n_i(\nu - 1 - i) + m_1 \frac{\nu - 1}{2} = \frac{L}{2}.
\]

(1.11)

To any set \(\{m_1, n_1, n_2, \ldots, n_{\nu-2}\}\) satisfying (1.11) one adds a companion set \(\{m_2, m_3, \ldots, m_{\nu-2}\}\)

\[m_i = 2 \sum_{j=1}^{i} (i - j) \cdot n_j + i \cdot m_1 ; \quad i = 2, \ldots, \nu - 2\]

to obtain all solutions to system (1.10).

Equation (1.11) clearly shows that it is impossible to remove any "occupied" integer from a given Fermi band without affecting the rest of them in a profound way. Thus, the fermionic system (1.10) provides an interesting example of a kinematic interaction.
At a first glance, path space dimension (1.8) and equations (1.10) have nothing in common. However, motivated by the extensive numerical checks, we conjectured in [33] that

\[ ((K_{\nu}^L)_{1,1} = \sum_{m_1 \equiv \text{even}} \prod_{i=1}^{\nu-2} \left[ \begin{array}{c} n_i + m_i \\ n_i \end{array} \right]; \quad L \equiv \text{even} \quad (1.12) \]

\[ ((K_{\nu}^L)_{1,\nu} = \sum_{m_1 \equiv \text{odd}} \prod_{i=1}^{\nu-2} \left[ \begin{array}{c} n_i + m_i \\ n_i \end{array} \right]; \quad L + \nu \equiv \text{odd} \quad (1.13) \]

with sum taken over the subset of all solutions to (1.11) which obey extra parity restrictions.

The original motivation for this paper was to provide proof for the conjecture above. While searching for proof, I became acquainted with results of the Stony Brook group and got convinced that fermionic description of the CFT characters is related to a $q$-deformation of the counting formulas (1.12, 1.13). The remainder of this paper is organized as follows.

In Section 2, we generalize the fermionic system (1.10) in order to treat a more general class of boundary conditions. Expressing generating functions for a number of paths and fermionic states in terms of Chebyshev polynomials of the second kind, we establish the equivalences of two counting procedures. In Section 3, we first, propose and prove a $q$-generalization of certain binomial identities derived in Section 2 and then, obtain the proof of fermionic sum representation for $M(\nu, \nu + 1)$-characters. In Section 4, we discuss the physical significance of our results and speculate about a possible generalization. I conclude on a personal note.

2. Equivalence of fermionic and path counting of RSOS-states

In this section we find a closed form expression for generating functions of number of path and fermionic states and, as a result, establish the equivalence of different counting procedures. For the sake of clarity, some technical details are relegated to Appendix A.

The incidence matrix $K_{\nu}$ introduced in (1.4) has a well-known spectral decomposition

\[ K_{\nu}v^{(n)} = (2 \cos \frac{\pi n}{\nu + 1})v^{(n)}; \quad n = 1, 2, \ldots, \nu \]

\[ v_i^{(n)} = \sqrt{\frac{2}{\nu + 1}} \cdot \sin \frac{\pi ni}{\nu + 1}; \quad i = 1, 2, \ldots, \nu. \quad (2.1) \]

We comment that it became a habit for some to refer to the parameters $n$ and $(\nu + 1)$ in formula (2.1) as Coxeter exponents and Coxeter number, respectively. Keeping in mind that

\[ \binom{N}{m} \text{ denotes the usual binomial coefficient.} \]
\[ L + r \equiv \text{odd} \] positive integer we introduce a generating function

\[ G_r^{(1)}(x) = \sum_{L=0}^{\infty} x^{L/2} ((K_\nu)^L)_{1,r} = \left( \frac{1 + p_+(r)(\sqrt{xK_\nu} - 1)}{1 - xK_\nu^2} \right)_{1,r}, \]  

where \( p_\pm(r) = \frac{1 \pm (-1)^r}{2} \). Making use of a spectral decomposition of the matrix \( K_\nu \) (2.1), we find

\[ G_r^{(1)}(x) = \frac{2}{\nu + 1} \sum_{n=1}^{\nu} \frac{1 + p_+(r)(2\sqrt{x} \cos \frac{\pi n}{\nu + 1} - 1)}{1 - 4x \cos^2 \frac{\pi n}{\nu + 1}} \sin \frac{\pi n}{\nu + 1} \sin \frac{\pi n}{\nu + 1}. \]  

Introducing \( z \) by

\[ \frac{1}{\sqrt{x}} = z + \frac{1}{z} \]

we can present \( G_r^{(1)} \) as

\[ G_r^{(1)}(z) = \frac{2}{\nu + 1} (z^2 + 1)^{p_-(r)+1} z^{p_+(r)} \sum_{n=1}^{\nu} \frac{\sin \left( \frac{\pi n}{\nu + 1} \right) \cdot \sin \left( \frac{\pi n}{\nu + 1} (p_+(r) + 1) \right)}{\left[ z^2 - \exp \left( \frac{2\pi n}{\nu + 1} \right) \right] \left[ z^2 - \exp \left( - \frac{2\pi n}{\nu + 1} \right) \right]} . \]  

To proceed further we need the formula below

\[ (z^2 + 1)^{p_-(r)} (z^{2(\nu+1)} - 1) \sum_{n=1}^{\nu} \frac{\sin \left( \frac{\pi n}{\nu + 1} \right) \cdot \sin \left( \frac{\pi n}{\nu + 1} (p_+(r) + 1) \right)}{\left[ z^2 - \exp \left( \frac{2\pi n}{\nu + 1} \right) \right] \left[ z^2 - \exp \left( - \frac{2\pi n}{\nu + 1} \right) \right]} = \frac{\nu + 1}{2} (z^{2(\nu+1-r)} - 1)z^{r-p_+(r)-1} . \]  

The simplest way to prove (2.6) is as follows. First, we check that the l.h.s. of (2.6), considered as function of \( z^2 \), has no poles. This being so, we infer that both sides in (2.6) are polynomials of degree not higher than \( \nu \). To complete the proof we evaluate both sides at

\[ z^2 = \exp \left( i \frac{2\pi}{\nu + 1} j \right) ; \quad j = 0, 1, 2, \ldots , \nu . \]  

After elementary (if tedious) calculations we see that identity (2.6) holds for \( z^2 \), given by (2.7).

According to the fundamental theorem of algebra, two polynomials which appear in (2.6) must be equal. Thus, formula (2.6) is proven.

With help of (2.6) we can process \( G_r^{(1)} \) further

\[ G_r^{(1)} = (z + \frac{1}{z}) \times z^{\nu+1-r} \frac{1}{z^{\nu+1} - \frac{1}{z^{\nu+1}}} . \]  

Introducing \( \theta \)

\[ \frac{1}{\sqrt{x}} = z + \frac{1}{z} = 2 \cos \theta , \]

one can express \( G_r^{(1)} \) in terms of Chebyshev polynomials of the second kind \( U_m(\theta) \)

\[ G_r^{(1)}(\theta) = U_1(\theta) \cdot \frac{U_{\nu-r}(\theta)}{U_\nu(\theta)} ; \quad U_m(\theta) = \frac{\sin(m + 1)\theta}{\sin \theta} . \]  

\[ \text{(2.10)} \]
It is well known that orthogonal polynomials $U_m(\theta)$ satisfy recurrences
\[
2 \cos \theta \cdot U_m(\theta) = U_{m-1}(\theta) + U_{m+1}(\theta) \tag{2.11}
\]
along with initial conditions
\[
U_0(\theta) = 1 ; \quad U_1(\theta) = 2 \cos \theta = \frac{1}{\sqrt{x}}. \tag{2.12}
\]
There is, however, another way to specify these polynomials uniquely
\[
U_{m-1}(\theta) \cdot U_{m+1}(\theta) = U_m^2(\theta) - 1
\]
\[
U_0(\theta) = 1 ; \quad U_1(\theta) = 2 \cos \theta. \tag{2.13}
\]
For a reason which will become apparent later, we’d like to call (2.13) fermionic recurrences.

As immediate consequences of results (2.8, 2.10, 2.11) we have for $L + r \equiv \text{odd}
\[
\left( (K_\nu)^L \right)_{1,r} = \frac{1}{2\pi i} \int dx \frac{G_r^{(1)}}{x} \frac{x^{L/2}}{x^{L/2}} = \frac{1}{2\pi i} \int \frac{dz}{z} \left( z + \frac{1}{z} \right)^L z^{1-r} \frac{1-z^{2(\nu+1-r)}}{1-z^{-2(\nu+1)}} = \sum_{j=-\infty}^{\infty} \left\{ \left[ \frac{L+1-r}{2} - j(\nu+1) \right] - \left[ \frac{L-r}{2} - j(\nu+1) \right] \right\} \tag{2.14}
\]
and
\[
\left( (K_\nu)^L \right)_{1,r+1} + \left( (K_\nu)^L \right)_{1,r-1} = \frac{1}{2\pi i} \int dx \frac{1}{x} \frac{U_1}{U_\nu} (U_{\nu-r-1} + U_{\nu-r+1}) = \frac{1}{2\pi i} \int dx \frac{1}{x} \frac{U_1}{U_\nu} U_{\nu-r} = \left( (K_\nu)^{L+1} \right)_{1,r}. \tag{2.15}
\]
In what follows, we would frequently refer to the r.h.s. of (2.14) as bosonic counting of the path space. The last equation clearly exhibits a remarkable connection between recurrences for path counting function $\left( (K_\nu)^L \right)_{1,r}$ and that for the orthogonal polynomials (2.11). We intend to pursue this observation further elsewhere.

We now move on to introduce a different set of definitions, related to what we call fermionic counting of the RSOS states. First, we generalize system (1.10) by adding to it the inhomogeneous term, parametrized by $r = 1, 2, \ldots, \nu - p_+(L + \nu)$. For $i = 1, 2, \ldots, \nu - 2$ let it be written
\[
n_i + m_i = \frac{1}{2} \left\{ \theta(i > 1) \cdot m_{i-1} + \theta(\nu - 2 > i) \cdot m_{i+1} + L \cdot \delta_{i,\nu-2} \right\}
+ \frac{1}{2} \left\{ p_-(L+r) \cdot \delta_{i,r-1} + p_+(L+r) \cdot \delta_{i,\nu-1-r} \right\}
\]
\[
m_1 = 2n_0 + p_-(L+r) \cdot \theta(r > 1) - p_+(L+r) \cdot \delta_{r,\nu-1} \tag{2.16}
\]
where \( n_0, n_1, \ldots, n_{\nu-2} \) are non-negative integers and \( \delta_{i,j}, \theta(i > j) \) denote Kronecker delta and a step function, respectively. Again, it is trivial to show that system (2.16) is equivalent to the following partition problem
\[
\sum_{i=0}^{\nu-2} n_i (\nu - 1 - i) = \frac{L + r \cdot p_+ (L + r) - (r - 1) \cdot p_- (L + r)}{2}; \quad n_i \geq 0 \quad \text{for} \quad i = 0, 1, \ldots, \nu - 2.
\]
(2.17)

Once a set of non-negative integers \( \{n_i\} \) which satisfies constraint (2.17) is found, set \( \{m_i\} \) can be determined for \( i = 1, 2, \ldots, \nu - 2 \) as
\[
m_i = \tilde{n}_i - V_{i,r} p_+ (L + r) + \tilde{V}_{i,r} p_- (L + r)
\]
(2.18)

where
\[
\tilde{n}_i = 2 \sum_{l=1}^{i} \ln_{i-l}
\]
(2.19)
\[
V_{i,r} \equiv V_{i,r}^+ = [i - (\nu - 1 - r)] \cdot \theta(i > \nu - 1 - r)
\]
(2.20)
\[
\tilde{V}_{i,r} \equiv V_{i,r}^- = (r - 1) + [i - (r - 1)] \cdot \theta(r - 1 > i)
\]
(2.21)

Next we define fermionic counting functions \( I(L, r) \) and \( J(L, r) \)
\[
I(L, r) = \sum \prod_{i=1}^{\nu-2} \left[ \frac{n_i + m_i}{n_i} \right] = \sum \prod_{i=1}^{\nu-2} \left[ \frac{n_i + \tilde{n}_i + \tilde{V}_{i,r}}{n_i} \right]; \quad L + r \equiv \text{odd}
\]
(2.22)
\[
J(L, r) = \sum \prod_{i=1}^{\nu-2} \left[ \frac{n_i + m_i}{n_i} \right] = \sum \prod_{i=1}^{\nu-2} \left[ \frac{n_i + \tilde{n}_i - V_{i,r}}{n_i} \right]; \quad 1 \leq r \leq \nu - 1
\]
(2.23)

where symbol \( \sum \) above stands for the sum over all solutions to constraint (2.17). We’d like to point out that even though we did not impose any constraints on \( \{m_i\} \), except (2.18), it is clear from definition (2.22, 2.23) that effectively \( m_i \geq 0 \), since for \( m_i < 0 \) corresponding binomial coefficient \( \left[ \frac{n_i + m_i}{m_i} \right] \) vanishes. By the same argument, one may disregard requirement \( n_i \geq 0 \).

We are now in a position to formulate the main assertion of this section
\[
((K_\nu)_1, r) = I(L, r) = J(L, r - 1); \quad L + r \equiv \text{odd}.
\]
(2.24)

For the proof, let’s construct two fermionic generating functions
\[
G_r^{(2)}(x) = \sum_{L=0}^{\infty} \left. x^\frac{L}{2} I(L, r) \right|_{L \equiv p_+ (r) \ (mod \ 2)}; \quad L + r \equiv \text{odd}
\]
(2.25)
\[
G_r^{(3)}(x) = \sum_{L=0}^{\infty} \left. x^\frac{L}{2} J(L, r) \right|_{L \equiv p_- (r) \ (mod \ 2)}; \quad L + r \equiv \text{even}.
\]
(2.26)
Taking into account constraint (2.17), one finds
\[
G^{(2)}_r(x) = \sum_{n_0, n_1, \ldots, n_{\nu-2}=0}^{\infty} x^{r-1 + \sum_{i=0}^{\nu-2} n_i (\nu-1-i)} \times \prod_{i=1}^{\nu-2} \left[ n_i + \tilde{n}_i + \tilde{V}_{i,r} \right].
\] (2.27)

To proceed further we shall recall the binomial theorem
\[
\sum_{n_i=0}^{\infty} \left[ \frac{n_i + \tilde{n}_i + \tilde{V}_{i,r}}{n_i} \right] \tilde{x}^{n_i} = \frac{1}{(1 - \tilde{x})^{\tilde{n}_i + 1 + \tilde{V}_{i,r}}},
\] (2.28)

Our strategy for obtaining closed form expression for \( G^{(2)}_r \) is to apply (2.28) in repetitive fashion to sum out all variables \( n_i \)'s. Let’s demonstrate how it works. Having summed out \( n_{\nu-2} \)-variable in (2.27) we obtain with help of \( \tilde{n}_{\nu-2} = \sum_{l=0}^{\nu-3} (\nu-2-l)n_l \)
\[
G^{(2)}_r = \sum_{n_0, n_1, \ldots, n_{\nu-3}=0}^{\infty} x^{r-1} \times \frac{1}{(1 - T_{1,0})^{1+\tilde{V}_{\nu-2,r}}} \times \prod_{i=0}^{\nu-3} \left[ (T_{\nu-1-i,1})^{n_i} \times \prod_{i=1}^{\nu-3} \left[ n_i + \tilde{n}_i + \tilde{V}_{i,r} \right] \right],
\] (2.29)

where
\[
T_{i,1} = \frac{T_{i,0}}{(1 - T_{1,0})^{2(i-1)}}
\] (2.30)

and
\[
T_{i,0} = x^i; \quad i = 1, 2, \ldots.
\] (2.31)

Obviously, one can use the binomial theorem (2.28) again to eliminate \( n_{\nu-3} \)-variable.
\[
G^{(2)}_r = \sum_{n_0, n_1, \ldots, n_{\nu-4}=0}^{\infty} x^{r-1} \times \frac{1}{(1 - T_{1,0})^{1+\tilde{V}_{\nu-2,r}}} \times \frac{1}{(1 - T_{2,1})^{1+\tilde{V}_{\nu-3,r}}} \times \prod_{i=0}^{\nu-4} \left[ (T_{\nu-1-i,2})^{n_i} \times \prod_{i=1}^{\nu-4} \left[ n_i + \tilde{n}_i + \tilde{V}_{i,r} \right] \right],
\] (2.32)

where
\[
T_{i,2} = \frac{T_{i,1}}{(1 - T_{2,1})^{2(i-2)}}.
\] (2.33)

Proceeding as above, we derive
\[
G^{(2)}_r = x^{r-1} \prod_{i=1}^{\nu-1} \frac{1}{(1 - T_{i,i-1})^{1+\tilde{V}_{\nu-1-i,r}}}
\] (2.34)

with quantities \( T_{i,m} \) being determined recursively as
\[
\begin{align*}
T_{i,m} = \frac{T_{i,m-1}}{(1 - T_{m,m-1})^{2(i-m)}}; & \quad m = 0, 1, 2, \ldots \\
T_{i,0} = x^i; & \quad i = 1, 2, \ldots.
\end{align*}
\] (2.35)
The reader is encouraged to prove (2.35) by carrying out the inductive step $m \rightarrow m + 1$. As shown in Appendix A, recurrences (2.35) imply interesting connection between $T_{j,m}$’s and Chebyshev polynomials $U_m$’s. In particular,

$$T_{m,m-1} = U_m^{-2}(\theta).$$

(2.36)

Now we substitute (2.21) and (2.36) into (2.34) to get

$$G_r(2) = x^{r-1} \cdot \prod_{m=1}^{r} \left( \frac{U_m^2}{U_m^2 - 1} \right)^r \times \prod_{m=\nu-r+1}^{\nu-1} \left( \frac{U_m^2}{U_m^2 - 1} \right)^{\nu-m}.$$  

(2.37)

Using fermionic recurrences (2.13) for $U_m$, two products above telescope to yield a final result

$$G_r(2) = x^{r-1} \cdot \left( \frac{U_{\nu-r} U_1}{U_{\nu-r+1}} \right)^r \cdot \left[ \left( \frac{U_{\nu-r+1}}{U_{\nu-r}} \right)^r \cdot \frac{U_{\nu-r}}{U_{\nu}} \right] = U_1 \cdot \frac{U_{\nu-r}}{U_{\nu}}.$$  

(2.38)

Comparing (2.10) and (2.38) it is plain

$$G_r^{(1)} = G_r^{(2)}$$

(2.39)

and therefore,

$$((K_\nu)^L)_{1,r} = I(L, r).$$

(2.40)

Treating $G_r^{(3)}$ in a similar fashion we first find

$$G_r^{(3)} = x^{-\frac{r}{2}} \cdot \prod_{m=1}^{r-1} (1 - U_m^{-2})^{r-1-m} \cdot \prod_{m'=r}^{\nu-1} \frac{1}{(1 - U_m^{-2})}.$$  

(2.41)

and then, use fermionic recurrences (2.13) again to arrive at a simple form

$$G_r^{(3)} = \frac{U_1 \cdot U_r \cdot U_{\nu-1}}{U_{\nu}} = G_{r+1}^{(2)} + U_1 \cdot U_{r-1}.$$  

(2.42)

Since

$$\int \frac{dx}{x} \cdot \frac{U_1 \cdot U_{r-1}}{x^{\frac{r}{2}}} = 0; \quad L = 0, 1, 2, \ldots$$

(2.43)

we immediately have

$$((K_\nu)^L)_{1,r} = I(L, r) = \frac{1}{2\pi i} \int \frac{dx}{x} \cdot \frac{G_r^{(2)}}{x^{\frac{r}{2}}} = \frac{1}{2\pi i} \int \frac{dx}{x} \cdot \frac{G_r^{(3)}}{x^{\frac{r}{2}}} = J(L, r - 1).$$

(2.44)

That concludes our proof of (2.24).

At this point it is natural to combine $I(L, r)$ and $J(L, r)$ into a single fermionic object $F(L, r)$

$$F(L, r) = \begin{cases} I(L, r); & L + r \equiv odd \\ J(L, r); & L + r \equiv even. \end{cases}$$

(2.45)
We shall now use identities (2.14) and (2.24) to express the main result of this section in a following form

\[
F(L, r) = \sum \prod_{i=1}^{\nu-2} \left[ \frac{n_i + m_i}{n_i} \right] = \\
\sum_{j=-\infty}^{\infty} \left\{ \left[ \frac{L}{\left(\frac{L+1-r}{2}\right) - j(\nu+1)} \right] - \left[ \frac{L}{\left(\frac{L-1-r}{2}\right) - j(\nu+1)} \right] \right\},
\]

(2.46)

where \((x)\) denotes the integer part of \(x\). Introducing compact vector notations

\[
(\vec{m})_i = m_i
\]

(2.47)

\[
(\vec{V}_r(L))_i = -p_+(L + r) \cdot \vec{V}_{i,r} + p_-(L + r) \cdot \vec{V}_{i,r}
\]

(2.48)

\[
(\vec{u}_r(L))_i = p_+(L + r) \cdot \delta_{i,\nu-r-1} + p_-(L + r) \cdot \delta_{i,r-1}
\]

(2.49)

and remembering the system of equations (2.16), we can rewrite the fermionic sum in (2.46) entirely in terms of \(\vec{m}\)-variables.

\[
\sum_{\vec{V}_r(L)} \cdot \prod_{i=1}^{\nu-2} \left[ \frac{1}{2} (K_{\nu-2} \cdot \vec{m} + \vec{u}_r(L))_i + \frac{1}{2} L \delta_{i,\nu-2} \right] = \\
\sum_{\vec{V}_r(L)} \cdot \prod_{i=1}^{\nu-2} \left[ \frac{1}{2} \left( \frac{L}{\left(\frac{L+1-r}{2}\right) - j(\nu+1)} \right) - \left( \frac{L}{\left(\frac{L-1-r}{2}\right) - j(\nu+1)} \right) \right]
\]

(2.50)

where symbol \(\sum\vec{V}_r(L)\) stands for the sum over

\[
m_i \equiv (\vec{V}_r(L))_i (mod 2) ; \quad m_i \in \mathbb{Z} ; \quad i = 1, 2, \ldots, \nu - 2.
\]

(2.51)

The advantage of representation (2.50) is that the summation variables \(\vec{m}\) are almost free, subject only to constraint (2.51). Also, as we shall see in the next section, a \(q\)-analogue of representation (2.50) is particularly well - suited to study \(L \to \infty\) limit. It should be added that identities similar to (2.50) for \(\nu = 5\) case (3-state Potts model) were proven in [34].

It follows from elementary recurrences

\[
\left[ \begin{array}{c} N \\ m \end{array} \right] = \left[ \begin{array}{c} N - 1 \\ m - 1 \end{array} \right] + \left[ \begin{array}{c} N - 1 \\ m \end{array} \right]
\]

(2.52)

and (2.46) that for \(1 < r < \nu\)

\[
F(L, r) = F(L - 1, r) + \left\{ \begin{array}{ll} F(L - 1, r - 1) ; & L + r \equiv odd \\ F(L - 1, r + 1) ; & L + r \equiv even \end{array} \right.
\]

(2.53)
or, equivalently
\[
\begin{align*}
J(L, r) &= I(L - 1, r) + J(L - 1, r + 1) \\
I(L, r) &= J(L - 1, r) + I(L - 1, r - 1).
\end{align*}
\] (2.54)

Surely, since \(I(L, r) = J(L, r - 1)\), one may be tempted to rewrite the above in a simpler form
\[
\begin{align*}
J(L, r) &= J(L - 1, r + 1) + J(L - 1, r - 1) \\
I(L, r) &= I(L - 1, r + 1) + I(L - 1, r - 1).
\end{align*}
\] (2.55)

I believe that recurrences (2.54) are more natural than (2.55), if one insists on interpreting \(J(L, r)\) and \(I(L, r)\) as components of a single object \(F(L, r)\). As further evidence for this point of view, we mention (somewhat anticipating things to come) that it is (2.54) rather than (2.55) which admits straightforward \(q\)-generalization and that for \(q \neq 1\), identity
\[
I(L, r) = J(L, r - 1)
\] (2.56)

no longer holds true.

Nevertheless, if one insists on casting (2.55) in a form which mimics \(q\)-case, then (2.55) should be rewritten as
\[
\begin{align*}
\{J(L, r) - J(L - 1, r + 1)\} &= J(L - 2, r) + \{J(L - 1, r - 1) - J(L - 2, r)\} \\
\{I(L, r) - I(L - 1, r - 1)\} &= I(L - 2, r) + \{I(L - 1, r + 1) - I(L - 2, r)\}.
\end{align*}
\] (2.57)

In this section we’ve demonstrated the equivalence of fermionic and bosonic counting of the \(RSOS\)-states by means of a generating function approach. As an alternative way of establishing this equivalence, one may try to prove directly that \(F(L, r)\) satisfies recurrences (2.54). This alternative becomes especially valuable since the author did not (yet) succeed in finding a \(q\)-analogue of a generating function method.

We now venture into the realm of \(q\)-identities.

3. Proof of the \(q\)-identities

We start this section by reminding the reader a few \(q\)-definitions. The \(q\)-generalization of number \(X\), due to Heine [35], can be written as
\[
X_q = \frac{1 - q^X}{1 - q}.
\] (3.1)

As \(q\) tends to one
\[
\lim_{q \to 1} X_q = X.
\] (3.2)

Next, we introduce \(q\)-shifted factorial \((a, q)_n\)
\[
(a, q)_n = \begin{cases} 
1 & ; \quad n = 0 \\
\prod_{j=0}^{n-1} (1 - aq^j) & ; \quad n \geq 1,
\end{cases}
\] (3.3)
where \((n \in \mathbb{Z})\) and \((N, m \in \mathbb{Z})\) \(q\)-binomial coefficient

\[
\begin{bmatrix} N \\ m \end{bmatrix}_q = \begin{cases} \frac{(q,q)_N}{(q,q)_m(q,q)_{N-m}} ; & 0 \leq m \leq N \\ 0 ; & \text{otherwise.} \end{cases}
\] (3.4)

It is quite amusing that \(q\)-numbers were rediscovered recently in the context of quantum groups. There, symbol \(q\) naturally refers to the quantum deformation parameter. But Heine and others used the same \(q\)-symbol almost a century before the advent of Quantum mechanics. What is it? Mere coincidence or, maybe, Heine somehow knew about Quantum mechanics? If so, then this is just another hint that the plane of knowledge exists outside of time and space, in some ideal Plato’s world of ideas [36]. Returning to our mundane business, we note elementary recursion relations for the \(q\)-binomial coefficients

\[
\begin{bmatrix} N \\ m \end{bmatrix}_q = \begin{bmatrix} N-1 \\ m-1 \end{bmatrix}_q + q^m \begin{bmatrix} N-1 \\ m \end{bmatrix}_q
\] (3.5)

\[
\begin{bmatrix} N \\ m \end{bmatrix}_q = \begin{bmatrix} N-1 \\ m \end{bmatrix}_q + q^{N-m} \begin{bmatrix} N-1 \\ m-1 \end{bmatrix}_q
\] (3.6)

which are similar to those for the usual binomial coefficients (2.52). On the other hand, there are certain limiting procedures which are not well defined for \(q = 1\), but may be perfectly well defined for \(|q| < 1\). For instance, letting \(N\) tend to infinity, one finds

\[
\lim_{N \to \infty} \begin{bmatrix} N \\ m \end{bmatrix}_q = \frac{1}{(q,q)_m} ; \quad |q| < 1
\] (3.7)

and

\[
\lim_{N \to \infty} \lim_{m \to \infty} \begin{bmatrix} N \\ m \end{bmatrix}_q = \frac{1}{(q,q)_\infty} ; \quad |q| < 1, \quad N > m.
\] (3.8)

We are now well equipped to propose a \(q\)-analogue of the fermionic counting functions (2.22, 2.23). For \(r = 1, 2, \ldots, \nu\) let us introduce

\[
I_q(L, r) = \sum q^{Y_r(L)} \prod_{i=1}^{\nu-2} \begin{bmatrix} n_i + m_i \\ n_i \end{bmatrix}_q = \sum q^{Y_r(L)} \prod_{i=1}^{\nu-2} \begin{bmatrix} n_i + \tilde{n}_i + \tilde{V}_{i,r} \\ n_i \end{bmatrix}_q ; \quad L + r = \text{odd}
\] (3.9)

and for \(r = 1, 2, \ldots, \nu - 1\)

\[
J_q(L, r) = \sum q^{X_r(L)} \prod_{i=1}^{\nu-2} \begin{bmatrix} n_i + m_i \\ n_i \end{bmatrix}_q = \sum q^{X_r(L)} \prod_{i=1}^{\nu-2} \begin{bmatrix} n_i + \tilde{n}_i - V_{i,r} \\ n_i \end{bmatrix}_q ; \quad L + r = \text{even}
\] (3.10)
\[ X_r(L) = -\frac{1}{2} \sum_{i=1}^{\nu-2} \left( n_i - \frac{L}{2} \delta_{i,\nu-2} - \frac{1}{2} \delta_{i,\nu-r-1} \right) \cdot (\bar{n}_i - V_{i,r}) \]  \hspace{1cm} (3.11)

and

\[ Y_r(L) = -\frac{1}{2} \sum_{i=1}^{\nu-2} \left( n_i - \frac{L}{2} \delta_{i,\nu-2} - \frac{1}{2} \delta_{i,r-1} \right) \cdot (\bar{n}_i + \tilde{V}_{i,r}) \]  \hspace{1cm} (3.12)

with the rest of notations the same as in Section 2.

From the definition above, it is obvious

\[ I_q(L, \nu) = I_q(L-1, \nu-1) = J_q(L, \nu-1) \]  \hspace{1cm} (3.13)

and

\[ J_q(L-1, 1) = I_q(L, 1). \]  \hspace{1cm} (3.14)

What, perhaps, is not so obvious that \( I_q \)'s and \( J_q \)'s satisfy the following recursion relations

\[ \{ I_q(L, r) - \theta(r > 1) \cdot q^{\frac{L-1}{2}} I_q(L-1, r-1) \} = I_q(L-2, r) + \theta(\nu - 1 > r) \cdot q^{\frac{L-1}{2}} \{ I_q(L-1, r+1) - q^{\frac{L-1}{2}} \cdot I_q(L-2, r) \} \]  \hspace{1cm} (3.15)

and

\[ \{ J_q(L, r) - \theta(\nu - 1 > r) \cdot q^{\frac{L-1}{2}} \cdot J_q(L-1, r+1) \} = J_q(L-2, r) + \theta(r > 1) \cdot q^{\frac{L-1}{2}} \{ J_q(L-1, r-1) - q^{\frac{L-1}{2}} \cdot J_q(L-2, r) \}, \]  \hspace{1cm} (3.16)

with \( \theta(i > j) \) being a step function

\[ \theta(i > j) \equiv \begin{cases} 1, & i > j \\ 0, & i \leq j. \end{cases} \]

We’d like to comment here that from the technical point of view the choice of phase factors (3.11, 3.12) is custom-tailored to support a proof of recurrences (3.15, 3.16), which in turn were motivated by \( q = 1 \) case. Yet, from the physical point of view, the choice (3.11, 3.12) is intimately related to the assumption of linear dispersion law for quasi-particles as we shall later see.

To set a stage for the technically involved proof of (3.13) and (3.16), in this section, we consider only (3.16) with \( r = 1, 2 \) and relegate our general discussion to Appendices B and C. Before moving on, a few remarks are in order. In what follows, the expression "telescopic expansion" will be frequently used. Since it is hardly a standard terminology we have to comment on its meaning. For \( a = 0, 1 \) and \( i = 0, 1, \ldots, n \) let \( A_i^{(a)} \) be some product of the \( q \)-binomial coefficients. We call the sum \( A_0^{(1)} + \sum_{i=1}^{n} A_i^{(0)} \) telescopic expansion for \( A_n^{(1)} \) if

\[ A_i^{(1)} + A_{i+1}^{(0)} = A_{i+1}^{(1)} \]  \hspace{1cm} (3.17)
holds. Properties (3.17), which are somewhat similar to that of the sliding tubes of a jointed telescope, clearly imply

\[ A^{(1)}_n = A^{(1)}_0 + \sum_{i=1}^{n} A^{(0)}_i. \] (3.18)

We note that recurrences (3.5, 3.6) provide a simple example of a telescopic expansion \(^4\).

Our second remark concerns with interesting features of the solutions to system (2.16). To make notations more manageable we introduce symbol \{\vec{n}, \vec{\tilde{n}}\}_{L,r} which denotes some particular set of integers \((\vec{n})_i \equiv n_i\) and \((\vec{\tilde{n}})_i \equiv \tilde{n}_i\), defined by (2.17) and (2.19). The following properties of \{\vec{n}, \vec{\tilde{n}}\}_{L,r} sets are indispensable for our treatment

\[
\{\vec{n}, \vec{\tilde{n}}\}_{L-2,r} = \{\vec{n}, \vec{\tilde{n}}\}_{L,r} - \{\vec{e}_{\nu-2,0}\}
\] (3.20)

with unit vector \(\vec{e}_a\), defined by its components as

\[(\vec{e}_a)_i = \delta_{i,a}; \quad a = 1, 2, \ldots, \nu - 2.\] (3.21)

Confirmation of (3.19) and (3.20) is a simple matter and we leave it as an exercise for the reader.

Let us now turn to the proof of (3.16) for \(r = 1\), which is an easiest possible case. Using \(q\)-binomial identities (3.5), we expand \(J_q(L, 1)\) as

\[
J_q(L, 1) = \sum q^{X_1(L)} \prod_{i=1}^{\nu-2} \left[ \begin{array}{c} n_i - \delta_{i,\nu-2} + \tilde{n}_i \\ n_i - \delta_{i,\nu-2} \end{array} \right]_q + \\
+ \sum q^{X_1(L)+n_{\nu-2}} \prod_{i=1}^{\nu-2} \left[ \begin{array}{c} n_i + \tilde{n}_i - \delta_{i,\nu-2} \\ n_i \end{array} \right]_q. \] (3.22)

Keeping in mind (3.19, 3.20), it is trivial to verify

\[
X_1(L) = X_1(L - 2)
\]

\[
X_1(L) + n_{\nu-2} = \frac{L}{2} + X_2(L - 1). \] (3.23)

Results (3.22) and (3.23), together with

\[
\{\vec{n}, \vec{\tilde{n}}\}_{L-1,1} = \{\vec{n}, \vec{\tilde{n}}\}_{L,1} - \{\vec{e}_{\nu-2,0}\}
\] (3.24)

\(^4\)It is also possible to give a ”particle” interpretation to properties (3.17, 3.18), however, I prefer ”telescopic” analogy because it is more visual
\[ \{\vec{n}, \vec{n}\}_{L-1,2} = \{\vec{n}, \vec{n}\}_{L,1} \]  
\[ V_{i,2} = \delta_{i,\nu-2} \] (3.25)

allow us to recognize expansion (3.22) for \( J_q(L, 1) \) as

\[ J_q(L, 1) = J_q(L - 2, 1) + q^{L} \cdot J_q(L - 1, 2). \] (3.27)

This proves recurrences (3.16) for \( r = 1 \).

Treatment of \( r = 2 \) case is more interesting, since it involves a new element which plays an important role in a general case. Once again, we construct a telescopic expansion for \( J_q(L, 2) \)

\[
J_q(L, 2) = \sum q^{X_2(L)} \prod_{j=1}^{\nu-2} \left[ \frac{n_i + \tilde{n}_i - 2\delta_{i,\nu-2} - \delta_{i,\nu-3}}{n_i - \delta_{i,\nu-2}} \right]
\]

which can be easily verified with an aid of the \( q \)-binomial recurrences (3.5). Then, using

\[ \{\vec{n}, \vec{n}\}_{L,2} = \{\vec{n}, \vec{n}\}_{L-1,3} \] (3.29)

and

\[ \{\vec{n}, \vec{n}\}_{L-2,2} = \{\vec{n}, \vec{n}\}_{L,2} - \{\vec{e}_{\nu-2}, 0\}, \] (3.30)

one finds that

\[ X_2(L) + n_{\nu-2} + n_{\nu-3} = \frac{L}{2} + X_3(L - 1) \] (3.31)

and

\[ X_2(L) = X_2(L - 2). \] (3.32)

Now it is obvious that the first (third) term in (3.28) is \( q^L J_q(L - 1, 3)(J_q(L - 2, 2)) \). Hence, equation (3.28) becomes

\[
J_q(L, 2) - q^{L} J_q(L - 1, 3) - J_q(L - 2, 2) = \sum q^{X_2(L)} \prod_{j=1}^{\nu-2} \left[ \frac{n_i + \tilde{n}_i - 2\delta_{i,\nu-2} - \delta_{i,\nu-3}}{n_i - \delta_{i,\nu-2}} \right].
\] (3.33)

To proceed further, we shall make the change of summation variables

\[
\vec{n} \rightarrow \vec{n} + \vec{e}_{\nu-3} - 2\vec{e}_{\nu-2}
\]

\[
\vec{n} \rightarrow \vec{n} + 2\vec{e}_{\nu-2}.
\] (3.34)
This transforms expression (3.33) to
\[ J_q(L, 2) - q^{\frac{L}{2}} \cdot J_q(L - 1, 3) - J_q(L - 2, 2) = \]
\[ \sum q^{\frac{L}{2} + x_i(L-3)} \prod_{i=1}^{\nu-2} \left[ \begin{array}{c} n_i + \tilde{n}_i - 2\delta_{i,\nu-2} \\ 1 \\
_i - 2\delta_{i,\nu-2} \end{array} \right] . \] (3.35)

Making use of
\[ \{\tilde{n}, \tilde{n}\}_L, 2 - \{2\tilde{e}_{\nu-2}, 0\} = \{\tilde{n}, \tilde{n}\}_{L-3,1} \] (3.36)

implied by (3.19) and (3.20), we arrive at a simple form
\[ J_q(L, 2) - q^{\frac{L}{2}} \cdot J_q(L - 1, 3) - J_q(L - 2, 2) = q^{\frac{L}{2} - 1} \cdot J_q(L - 3, 1). \] (3.37)

Recalling (3.27), we finally have
\[ J_q(L, 2) - q^{\frac{L}{2}} \cdot J_q(L - 1, 3) = J_q(L - 2, 2) + q^{\frac{L}{2} - 1}) \cdot \{J_q(L - 1, 1) - q^{\frac{L}{2} - 1} \cdot J_q(L - 2, 2)\}. \] (3.38)

This completes our proof of (3.16) for \( r = 2 \).

Let us now highlight the main points of a general proof of recurrences (3.15) and (3.16). We start as before, by expanding \( J_q(L, r)(I_q(L, r)) \) in a telescopic fashion. We then, recognize the first term in this expansion as \( q^{\frac{L}{2}} J_q(L - 1, r + 1)(q^{\frac{L}{2}} I_q(L - 1, r - 1)) \) and the last one as \( J_q(L - 2, r)(I_q(L - 2, r)) \). Subtracting these two terms, we are left with the sum of \( (r - 1)((\nu - r - 1)) \) terms, each one being the sum of \( q \)-binomial products. We now relabel the summation variables in a systematic, but individual way (i.e. re-labeling procedure is different for each term!). At this point it is natural to introduce interpolating functions \( Z_t \) and \( \tilde{Z}_t \) which have the following remarkable properties

\[ \begin{align*}
Z_0 &= J_q(L, r) - \theta(\nu - 1 > r) \cdot q^{\frac{L}{2}} \cdot J_q(L - 1, r + 1) - J_q(L - 2, r) \\
Z_{r-2} &= q^{\frac{L}{2} - 1} \cdot \{J_q(L - 1, r - 1) - q^{\frac{L}{2} - 1} \cdot J_q(L - 2, r)\} \\
Z_t &= Z_{t+1} ; \quad t = 0, 1, 2, \ldots, r - 2
\end{align*} \] (3.39)

and

\[ \begin{align*}
\tilde{Z}_0 &= \theta(\nu - 1 > r) \{I_q(L, r) - \theta(r + 1) \cdot q^{\frac{L}{2}} \cdot I_q(L - 1, r - 1) - I_q(L - 2, r)\} \\
\tilde{Z}_{\nu-2-r} &= \theta(\nu - 1 > r) \cdot q^{\frac{L}{2} - 1} \{I_q(L - 1, r + 1) - q^{\frac{L}{2} - 1} \cdot I_q(L - 2, r)\} \\
\tilde{Z}_t &= \tilde{Z}_{t+1} , \quad t = 0, 1, 2, \ldots, \nu - 2
\end{align*} \] (3.40)

The equation (3.39) and (3.40) clearly imply recurrences (3.15) and (3.16). We now refer the motivated reader to the Appendices B and C for a detailed discussion.

Reflecting on formulas (3.15) and (3.16), one can’t help but notice that recurrences for \( J_q(L, r) \) and \( I_q(L, 2) \) are quite similar. To strengthen this similarity we introduce
\[ \bar{I}_q(L, r) \equiv J_q(L + 1, r) - q^{\frac{L}{2} - 1} \cdot J_q(L, r + 1) \cdot \theta(\nu - 1 > r) \] (3.41)
and rewrite equation (3.16) as
\[ \tilde{I}_q(L, r) = J_q(L - 1, r) + \theta(r > 1) \cdot q^{\frac{r}{2}} \tilde{I}_q(L, r - 1). \] (3.42)

Using (3.41) and (3.42) above, we derive recursion relations for \( \tilde{I}_q(L, r) \)
\[ \{ \tilde{I}_q(L, r) - \theta(r > 1) \cdot q^{\frac{r}{2}} \tilde{I}_q(L - 1, r - 1) \} - \theta(\nu - 1 > r) \times q^{\frac{r}{2}} \cdot \{ \tilde{I}_q(L - 1, r + 1) - q^{\frac{r}{2}} \tilde{I}_q(L - 2, r) \} = \tilde{I}_q(L - 2, r). \] (3.43)

Comparing (3.43) and (3.15) it is plain that \( I_q(L, r) \) and \( \tilde{I}_q(L, r) \) satisfy identical recurrences. This fact together with initial conditions
\[ \tilde{I}_q(L = 1, r) = I_q(L = 1, r) \quad \tilde{I}_q(L = 1, r) = I_q(L = 0, r) \] (3.44)
implies that
\[ \tilde{I}_q(L, r) = I_q(L, r). \] (3.45)

Hence, we have a \( q \)-analogue of (2.54)
\[ \begin{cases} J_q(L, r) = I_q(L - 1, r) + q^{\frac{r}{2}} \cdot J_q(L - 1, r + 1) \cdot \theta(\nu - 1 > r) \\ I_q(L, r) = J_q(L - 1, r) + q^{\frac{r}{2}} \cdot I_q(L - 1, r - 1) \cdot \theta(r > 1) \end{cases} \] (3.46)

\( J_q(L, r) \) and \( I_q(L, r) \) can be fused into a single object
\[ F_q(L, r) \equiv \begin{cases} I_q(L, r) ; & L + r \equiv odd \\ J_q(L, r) ; & L + r \equiv even \end{cases} \] (3.47)
which satisfies the following recurrences
\[ F_q(L, r) = F_q(L - 1, r) + q^{\frac{r}{2}} \begin{cases} F_q(L - 1, r - 1) \cdot \theta(r > 1) ; & L + r \equiv odd \\ F_q(L - 1, r + 1) \cdot \theta(\nu - 1 > r) ; & L + r \equiv even \end{cases} \] (3.48)
and initial conditions
\[ F_q(L = 0, r) = \delta_{1,r} \quad F_q(L = 1, r) = \delta_{1,r} + q^{\frac{r}{2}} \cdot \delta_{2,r}. \] (3.49)

Remarkably, recurrences (3.48) derived above, are practically the same as those studied by Andrews, Baxter, and Forrester [19]. Following along the lines of Schur’s polynomial proof of the Rogers-Ramanujan identities [17], Andrews, Baxter, and Forrester introduced another useful representation for the solution of (3.48)
\[ B_q(L, r) = \sum_{j=-\infty}^{\infty} \left\{ q^{\tilde{r}(j)} \left[ \begin{array}{c} L \\ \left( \frac{L+1-\nu}{2} \right) - j(\nu + 1) \end{array} \right]_q - q^{\tilde{r}(j)} \left[ \begin{array}{c} L \\ \left( \frac{L-1-\nu}{2} \right) - j(\nu + 1) \end{array} \right]_q \right\} \] (3.50)

\(^5\)To make reading easier, we took the liberty to present results of [19] in the notations of this paper.
where
\[ \varphi_r(j) = \frac{r(r-1)}{4} + j\nu(\nu+1) + r(\nu+1) - \nu \] (3.51)
and
\[ \tilde{\varphi}_r(j) = \frac{r(r-1)}{4} + (j\nu + r) \cdot [j(\nu+1) + 1]. \] (3.52)
Making use of \( q \)-binomial identities (3.5, 3.6), it is a simple matter to verify that \( B_q(L, r) \) satisfy (3.48) and (3.49). Since equations (3.48) and (3.49) specify polynomials uniquely, we can state the main result of this section
\[ F_q(L, r) = B_q(L, r). \] (3.53)
Result (3.53) can be entirely expressed in terms of \( \vec{m} \)-variables to obtain a \( q \)-analogue of (2.50)
\[ \sum_{\vec{V}_r(L)} q^{\vec{m}^tC\vec{m}/4} \prod_{i=1}^{\nu-2} \left[ \frac{1}{2}(K_{\nu-2} \cdot \vec{m} + \vec{u}_r(L))_i + \frac{L}{2}\delta_{i,\nu-2} \right] = \sum_{j=-\infty}^{\infty} \left\{ q^{\varphi_r(j)} \left[ \left( L - \frac{L_1}{2} \right) - j(\nu+1) \right]_q - q^{\tilde{\varphi}_r(j)} \left[ \left( L - \frac{L_1}{2} \right) - j(\nu+1) \right]_q \right\} \] (3.54)
with Cartan matrix \( C \) defined by (1.5). Once again, the advantage of representation (3.54) is that \( \vec{m} \)-variables are practically free, subject only to constraint (2.51). This fact makes it easy to perform \( L \to \infty \) limit. We note that the \( q \)-identities in a form (3.54) (modulo minor notational differences) were conjectured (proven for \( \nu = 3, 4 \)) by Melzer in [6]. In my opinion variables \( \vec{n}, \vec{\tilde{n}} \) used in this paper are more convenient when it comes to the proof.
Finally, letting \( L \) tend to infinity in (3.54), and using formulas (3.7, 3.8), we complete the proof of (1.2)
\[ \sum_{m_1, m_2, \ldots, m_{\nu-2}=0}^{\infty} \prod_{i=1}^{\nu-3} \left[ \frac{1}{2}(K_{\nu-2} \cdot \vec{m})_i + \frac{1}{2}\delta_{i,b^\pm} \right]_q = \frac{1}{(q, q)_{\infty}} \sum_{j=-\infty}^{\infty} \{ q^{\varphi_r(j)} - q^{\tilde{\varphi}_r(j)} \} \] (3.55)
with \( b^\pm \), defined by (1.3).
To the best of my knowledge, the proof of (3.55) is given here for the first time.
Both sums appearing in (3.55) have their own technical merits. In particular, the bosonic sum is best when it comes to the modular properties; on the other hand, it is the fermionic representation which enables one to study \( q \to 1^- \) limiting behavior in terms of Rogers dilogarithms [3] (see also [3], [10], [37]). By the way, Ramanujan himself was quite aware of the dilogarithms and knew how to evaluate them at special points.
In the limit \( \nu \) tends to infinity, identities (3.54, 3.55) imply the new character formula for affine algebra \( A_1^{(1)} \) [8]. This formula (which can be interpreted as equivalence of two-particle and infinite-particle description of XXX-model excitations) plays an important role in the "Yangian" representation of CFT (39 and references there).
To conclude this section, we’d like to point out that one can generalize polynomial identities (3.54) further [6] by adding to the fermionic system (2.16) one more inhomogeneous term. This would enable us to treat a most general class of boundary conditions 1 ≤ r ≤ ν − 1; 1 ≤ s ≤ ν which correspond to the primary fields with conformal dimensions Δ_{r,s} (1.6). We anticipate that it will be straightforward to extend our technique to deal with most general boundary conditions. The details will be given elsewhere [40].

4. Discussion

In this section we’d like to comment on some ”physical” matters which were left out of the discussion, because of the rather rigid format of the main body of this paper. To shed even light on the origin of a fermionic (bosonic) sum terminology, we shall recall two important results from the theory of partitions [41]. The first one is

$$\frac{1}{(q, q)_\infty} = 1 + \sum_{n=1}^{\infty} p_b(n) \cdot q^n$$

(4.1)

where $p_b(n)$ is a number of additive partitions on $n$ into unrestricted number of positive integers which may or may not be different (order of integers in partition is irrelevant). Since no exclusion rule is imposed on the parts, it is natural to refer to $(q, q)^{-1}_\infty$ as a bosonic character. The second one deals with a number of additive partitions $p_f(n, m, N)$ of $n ≥ 0$ into $m$ unequal, non-negative parts which do not exceed $N − 1$. This result can be written as

$$\left[ \begin{array}{c} N \\ m \end{array} \right]_q = q^{m(m-1)/2} \cdot \sum_{n=0}^{\infty} q^n \cdot p_f(n, m, N).$$

(4.2)

It is obvious that the exclusion rule for $p_f(n, m, N)$ is of a fermionic nature.

We can now give a physical interpretation of phase factors appearing in (3.9, 3.10). Think of $q$ as a Boltzmann factor

$$q = e^{-2\pi v_s M/T},$$

where $T, M, v_s$ are temperature, length of the system and speed of sound for the excitations, respectively. Then (4.2) implies that a typical term in (3.9, 3.10)

$$q^{(n_i-\delta_{i,k,\pm} - \delta_{i,\nu-2}) m_i} \left[ \begin{array}{c} n_i + m_i \\ m_i \end{array} \right]_q$$

(4.3)

is a character (re-scaled partition function) for a massless system of the $m_i$ fermions with energy $\varepsilon_k$ and momentum $p_k$, subject to the linear dispersion law

$$\begin{cases} \varepsilon_k^\pm = v_s \cdot p_k^\pm ; & k = 0, 1, \ldots, n_i + m_i - 1 \\ p_k^\pm = \frac{2\pi}{M}(k - \frac{n_i+m_i-1}{2} + \frac{L\delta_{i,\nu-2}+\delta_{i,\nu\pm}}{4}) \end{cases}$$

(4.4)
and restrictions \( p_{k'} \neq p_k \) for \( k' \neq k \).

Equation (4.4) shows that the action of \( L_0 \) (energy) operator takes a particularly trivial form in fermionic basis. It is natural to pose a question whether the action of some other operators on fermionic states can be described in a straightforward fashion. One may hope that the answer to this question is positive. In fact, it was demonstrated in the early paper by this author [42] (where \( c = 1 \)-model was analyzed) that the matrix elements of a \( U(1) \)-current assume a miraculously simple form in the Bethe (fermionic) basis. That observation made it possible to establish a map between Virasoro and Bethe states and to obtain exact results for correlation functions [12]. [13]. Provided one can generalize this approach, it may be possible to study the finite - volume spectrum of integrable massive perturbations of a rational CFT analytically, without resorting to the aid of a computer [44].

Let us now turn to the most intriguing feature of fermionic sums. They are not unique. A number of interesting examples was given in [3], [4], [5]. In particular, for Ising model one has either \( (\nu = 3) \) fermionic sum related to a coset construction \( (A_1(1))_1 \otimes (A_1(1))_1 \otimes (A_1(1))_2 \) or a sum related to construction \( (E_8(1))_1 \otimes (E_8(1))_1 \otimes (E_8(1))_2 \) with first (second) sum involving one (eight) type(s) of quasi-particles. Note that both group structures appear naturally in the analysis of two known integrable perturbations of the Ising model [45]. Another important example, suggesting a possible ”infinite degeneracy” of a fermionic sum description of CFT characters, can be found in [46]. There it was shown that a simplest bosonic character \( (q, q)^{-1}_\infty \) can be written in a fermionic language as

\[
\frac{1}{(q, q)_\infty} = \sum_{m_1, m_2, \ldots, m_p \geq 0} q^{\sum_{i=1}^{p} N_i^2} (q, q)_{m_p} \cdot \prod_{i=1}^{p} (q, q)_{m_i}
\]

where \( N_i = \sum_{l=1}^{p} m_l \). A graph interpretation of a phase factor \( \sum_{i=1}^{p} N_i^2 \) is

\[
\sum_{i=1}^{p} N_i^2 = \tilde{m}'(2 - \tilde{K})^{-1} \tilde{m}
\]

with \( \tilde{K} \) being the incidence matrix of a tadpole graph \( A_{2p}/Z_2 \) [9]. Most importantly \( p \), which can be thought as a number of different particle types, is an arbitrary positive integer.

Now, if one assumes that CFT ”knows” about integrable off-critical extensions, it would be natural to interpret the above mentioned non-uniqueness of fermionic sums as an indication
that there exist many (perhaps infinitely many) different perturbations of CFT which preserve integrability [2]-[3]. If it is indeed the case, one should face up to the possibility that several relevant operators (with finely tuned ratios of coupling constants) may be added to CFT in such a way that a resulting theory is still integrable. It is well established that the information about a mass spectrum and spins of higher conserved quantities is encoded into the Cartan matrix. As for the information related to the ratios of relevant operator coupling constants, translating the phenomenological observations made by Di Francesco et al [4] into the language of this paper might provide the necessary insight.

Finally I’d like to point out that it is possible to obtain many new $q$-identities of the Rogers-Ramanujan type through studies of the higher spin XXZ-models in the regime of strong anisotropy. Hopefully all these challenging questions will be addressed in my future publications.

**Concluding Remark**

I’d like to finish this paper on a personal note. The author firmly believes that ascertaining the role of a fermionic-bosonic sum equivalence in the theory of form factors [18] and in parallel development related to the $q$-affine algebra [19] would deepen the synthesis between mathematics and physics, and would lead to further breakthroughs.

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**Note added**

After this work was submitted for publication, three papers dealing with related issues appeared [50]. In the last reference of [10] it was suggested that identity (3.54) proven here implies (via Bailey-Andrews construction) the validity of similar identities for characters

$\chi^{(\nu, k\nu+\nu-1)}_{1, r(k+1)}(q)$, $\chi^{(\nu, k\nu+\nu-1)}_{1, r(k+1)+k}(q)$ and $\chi^{(\nu, k\nu+1)}_{1,kr}(q)$, $\chi^{(\nu, k\nu+1)}_{1,k(r+1)+1}(q)$ for $\nu \geq 4$, $1 \leq r \leq \nu - 2$ and $k \geq 1$. 
Finally, I’d like to mention the latest work by A. Kirillov \[51\] which, in my opinion, is a "must read" paper for anyone interested in $q$-identities.

**Appendix A**

Here, we will establish useful connection between Chebyshev polynomials of the second kind $U_m$ (2.11) and quantities $T_{j,m}$, defined recursively as

$$T_{j,m} = \frac{T_{j,m-1}}{(1 - T_{m,m-1})^{2(j-m)}}$$  \hspace{1cm} (A.1)

$$T_{j,0} = x^j$$  \hspace{1cm} (A.2)

for $j, m + 1 = 1, 2, 3, \ldots$. Substituting

$$T_{1,0} = x = U_1^{-2}$$  \hspace{1cm} (A.3)

into (A.1), we get

$$T_{2,1} = \frac{1}{(x-1)^2} = U_2^{-2}.$$  \hspace{1cm} (A.4)

Let us now consider the ratio $T_{j,m} / T_{j+1,m}$. Using (A.1, A.2) we find

$$\frac{T_{j,m}}{T_{j+1,m}} = (1 - T_{m,m-1})^2 \frac{T_{j,m-1}}{T_{j+1,m-1}} = \ldots$$

$$= \left( \prod_{l=1}^{m} (1 - T_{l,l-1}) \right) \cdot \frac{T_{j,0}}{T_{j+1,0}} = \frac{1}{x} \cdot \prod_{l=1}^{m} (1 - T_{l,l-1}).$$  \hspace{1cm} (A.5)

Note that identity (A.5) implies that the ratio $T_{j,m} / T_{j+1,m}$ does not depend on $j$! This observation, along with two trivial consequences of (A.1)

$$T_{m,m-1} = T_{m,m}$$

$$T_{m+2,m} = T_{m+2,m+1} \cdot (1 - T_{m+1,m})^2,$$  \hspace{1cm} (A.6)

leads to the following chain

$$\frac{T_{m,m-1}}{T_{m+1,m}} = \frac{T_{m,m}}{T_{m+2,m}} = \frac{T_{m+1,m}}{T_{m+2,m+1}} = \frac{T_{m+1,m}}{T_{m+2,m+1}(1 - T_{m+1,m})^2}.$$  \hspace{1cm} (A.7)

Denoting $T_{m,m-1}^{-1}$ as

$$\tilde{U}_m^2 = \frac{1}{T_{m,m-1}}$$  \hspace{1cm} (A.8)

we obtain from (A.1)

$$\tilde{U}_{m+2}^2 \cdot \tilde{U}_m^2 = (\tilde{U}_{m+1}^2 - 1)^2.$$  \hspace{1cm} (A.9)
Extracting the square root and replacing $m$ by $m - 1$, expression (A.9) becomes
\[
\tilde{U}_{m+1} \cdot \tilde{U}_{m-1} = (\tilde{U}_m^2 - 1). \tag{A.10}
\]

Comparing (A.10) and (2.13) we see that the recurrences for $\tilde{U}_m$ are identical to those for the Chebyshev polynomials $U_m$. This fact together with initial conditions (A.3, A.4), implies that
\[
\tilde{U}_m = U_m \tag{A.11}
\]
and, therefore,
\[
\frac{1}{T_{m,m-1}} = U_m^2. \tag{A.12}
\]

Now using (A.5) and (A.12) it is a simple matter to express all $T_{j,m}$’s in terms of Chebyshev polynomials
\[
T_{j,m} = U_m^{2(m-j)} \cdot U_m^{2(j-m-1)}. \tag{A.13}
\]

**Appendix B**

In this appendix we will prove the following claim for $r = 1, 2, \ldots, \nu - 1$
\[
\{ J_q(L, r) - q \frac{1}{2} \cdot J_q(L - 1, r + 1) \cdot \theta(\nu - 1 > r) \} = J_q(L - 2, r) + \theta(r > 1) \cdot q \frac{1}{2} \{ J_q(L - 1, r - 1) - q \frac{1}{2} \cdot J_q(L - 2, r) \} \tag{B.1}
\]
where for $L + r \equiv \text{even}$ integer
\[
J_q(L, r) = q^{X_r(L)} \prod_{i=1}^{\nu-2} \left[ \begin{array}{c} n_i + \tilde{n}_i - V_{i,r} \\ n_i \end{array} \right]_q \tag{B.2}
\]
and $V_{i,r}$, $X_r(L)$ and $\{\tilde{n}, \tilde{n}\}_{L,r}$ were defined by (2.20), (3.11), (2.17, 2.19) respectively. For notational simplicity we’ve suppressed the summation symbol in (B.2). Nevertheless, it should be remembered that the sum over all solutions to constraint (2.17) is always assumed in (B.2).

We start by expanding $J_q(L, r)$ in a telescopic fashion
\[
J_q(L, r) = q^{X_r(L)} \left\{ \theta(\nu - 1 > r) \cdot q^{\sum_{\nu-r-1}^{\nu-2} n_i} \prod_{i=1}^{\nu-2} \left[ \begin{array}{c} n_i + \tilde{n}_i - V_{i,r+1} \\ n_i \end{array} \right]_q + \right.
\]
\[
+ \sum_{l=0}^{r-1} q^{\theta(r-1 > l)} \cdot q^{\sum_{\nu-r+l}^{\nu-2} n_i} \prod_{i=1}^{\nu-2} \left[ \begin{array}{c} n_i + \tilde{n}_i - V_{i,r} - \theta(i > \nu - 2 - r + l) \\ n_i - \delta_{i,\nu-1-r+l} \end{array} \right]_q \left\} \tag{B.3}
\]
To prove formula (B.3) above, one should simply check a telescopic properties (3.17, 3.18) using binomial recurrences (3.5).

With the aid of easily verifiable identities

\[
\{ \tilde{n}, \tilde{n} \}_{L,r} = \{ \tilde{n}, \tilde{n} \}_{L-1,r+1}
\]

\[
X_r(L) + \sum_{\nu=r-1}^{\nu-2} n_i = \frac{L}{2} + X_{r+1}(L-1)
\]

(B.4)

implied by (3.19), we can immediately identify the first term in the expansion (B.3) as

\[
q^{\frac{L}{2}} \cdot J_q(L - 1, r + 1) \cdot \theta(\nu - 1 > r)
\]

and hence,

\[
J_q(L, r) - q^{\frac{L}{2}} \cdot J_q(L - 1, r + 1) \cdot \theta(\nu - 1 > r) = q^{X_r(L)} \left\{ \sum_{i=0}^{r-2} q^{\nu-r-1} n_i \times \right.
\]

\[
\left. \prod_{i=1}^{\nu-2} \left[ n_i + \tilde{n}_i - V_{i,r} - \theta(i > \nu - 2 - r + l) \right] + \prod_{i=1}^{\nu-2} \left[ (n_i - \delta_{i,\nu-2}) + \tilde{n}_i - V_{i,r} \right] \right\}. \tag{B.5}
\]

Next, using (B.3) along with two simple equations below

\[
V_{i,r} - \theta(l > 1) \sum_{m=0}^{l-2} \delta_{i,\nu-r+m} = V_{i,r-1} + \theta(i > \nu - r - 2 + l)
\]

\[
\{ \tilde{n}, \tilde{n} \}_{L-1,r-1} = \{ \tilde{n}, \tilde{n} \}_{L,r} - \{ \bar{e}^{\nu-2}, 0 \} \tag{B.6}
\]

one derives expansion for \{J_q(L - 1, r - 1) - q^{\frac{L}{2}} \cdot J_q(L - 2, r)\}

\[
J_q(L - 1, r - 1) - q^{\frac{L}{2}} \cdot J_q(L - 2, r) = q^{X_{r-1}(L-1)} \left\{ \sum_{i=1}^{r-2} q^{\nu-r+i} n'_i \times \right.
\]

\[
\left. \prod_{i=1}^{\nu-2} \left[ n'_i + \tilde{n}_i - V_{i,r} + \theta(l > 1) \cdot \sum_{m=0}^{l-2} \delta_{i,\nu-r+m} \right] + \right. \nonumber
\]

\[
+ \prod_{i=1}^{\nu-2} \left[ n'_i + \tilde{n}_i - V_{i,r} + \sum_{m=0}^{r-3} \delta_{i,\nu-r+m} \right] \right\}. \tag{B.7}
\]

with \(n'_i = n_i - \delta_{i,\nu-2}\). This expansion will come in handy later.

Let us now return to expansion (B.3). Since

\[
\{ \tilde{n}, \tilde{n} \}_{L-2,r} = \{ \tilde{n}, \tilde{n} \}_{L,r} - \{ \bar{e}^{\nu-2}, 0 \} \tag{B.8}
\]

and

\[
X_r(L - 2) = X_r(L) \tag{B.9}
\]
we recognize the last term in the r.h.s. of (B.5) as \( J_q(L - 2, r) \). Thus,

\[
J_q(L, r) - q^{L} \cdot J_q(L - 1, r + 1) \cdot \theta(\nu - 1 > r) - J_q(L - 2, r) = \\
\sum_{l=0}^{r-2} q^{L+1+X_{r-1}(L-1)} \cdot \theta(t > 0) \prod_{i=1}^{\nu-2} \left[ n_i + \tilde{n}_i - V_{i,r} - \theta(i > \nu - 2 - r + l) \right]_q,
\]

(B.10)

At this point it is expedient to perform the change of summation variables in (B.10). For the \( l \)-th term, this change takes the form

\[
\tilde{n} \rightarrow \tilde{n} + \tilde{e}_{\nu-r+l-1} - \tilde{e}_{\nu-r+1} - \tilde{e}_{\nu-2} \\
\tilde{n} \rightarrow \tilde{n} + 2 \sum_{i=\nu-r+l}^{\nu-2} \tilde{e}_i.
\]

(B.11)

The equation (B.10) becomes

\[
J_q(L, r) - q^{L} \cdot J_q(L - 1, r + 1) \cdot \theta(\nu - 1 > r) - J_q(L - 2, r) = \\
q^{\frac{L-1}{2}+X_{r-1}(L-1)} \cdot \sum_{l=0}^{r-2} q^{\theta(t > 0) \sum_{\nu-r}^{\nu-1} n_i} \prod_{i=1}^{\nu-2} \left[ n_i' + \tilde{n}_i - V_{i,r} + \theta(i > \nu - r + l) \right]_q
\]

where, once again, \( n_i' = n_i - \delta_{i,\nu-2} \). We want to stress that the change of variables (B.11) explicitly depends on \( l \), i.e. it is different for each term in the sum appearing in (B.10). Next we define for \( t = 0, 1, \ldots, r - 2 \)

\[
Z_t = q^{\frac{L-1}{2}+X_{r-1}(L-1)} \left\{ \theta(t > 0) \sum_{l'=1}^{t} q^{\sum_{\nu-r}^{\nu-1} n_i'} \prod_{i=1}^{\nu-2} \left[ n_i' + \tilde{n}_i - V_{i,r} + \theta(l' > 1) \sum_{m=0}^{t-2} \delta_{i,\nu-r+m} \right]_q \\
+ \sum_{l=t}^{r-2} q^{\theta(t > l) \sum_{\nu-r+l}^{\nu-1} n_i} \prod_{i=1}^{\nu-2} \left[ n_i' + \tilde{n}_i - V_{i,r} + \theta(i > \nu - r + l) + \theta(t > 0) \sum_{m=0}^{t-1} \delta_{i,\nu-r+m} \right]_q \right\}.
\]

(B.13)

Reflecting upon (B.7), (B.12) and (B.13), one notices

\[
Z_0 = J_q(L, r) - q^{L} \cdot J_q(L - 1, r + 1) \cdot \theta(\nu - 1 > r) - J_q(L - 2, r) \\
Z_{r-2} = q^{\frac{L-1}{2} \{ J_q(L - 1, r - 1) - q^{\frac{L-1}{2}} \cdot J_q(L - 2, r) \}}.
\]

(B.14)

Identities (B.14) may be regarded as a motivation for introducing new objects \( Z_t \). To complete the proof of (B.1) we show for \( t = 0, 1, \ldots, r - 3 \)

\[
Z_t = Z_{t+1}.
\]

(B.15)
To this end we expand "l = t"-term in the r.h.s. of (B.13) as
\[
\sum_{l=1}^{\nu-2} q^l \prod_{i=1}^{\nu-2} \left[ n_i' + \tilde{n}_i - V_{i,r} + \theta(i > \nu - r + t) + \theta(t > 0) \sum_{m=0}^{t-1} \delta_{i,v-r+m} n_i' - \delta_{i,v-r+t} \right] = \sum_{l=t+1}^{r-1} q^{\theta(l>t)} \sum_{l'=1}^{\nu-2} \prod_{i=1}^{\nu-2} \left[ n_i' + \tilde{n}_i - V_{i,r} + \theta(i > \nu - r + l') + \theta(t > 0) \sum_{m=0}^{l'-1} \delta_{i,v-r+m} \right] = \sum_{l=t+1}^{r-1} q^{\theta(l>t)} \sum_{l'=1}^{\nu-2} \prod_{i=1}^{\nu-2} \left[ n_i' + \tilde{n}_i - V_{i,r} + \theta(i > \nu - r + l) + \theta(t > 0) \sum_{m=0}^{t-1} \delta_{i,v-r+m} \right] = \sum_{l=t+1}^{r-1} q^{\theta(l>t)} \sum_{l'=1}^{\nu-2} \prod_{i=1}^{\nu-2} \left[ n_i' + \tilde{n}_i - V_{i,r} + \theta(i > \nu - r + l) + \theta(t > 0) \sum_{m=0}^{t-1} \delta_{i,v-r+m} \right]
\]

The proof of a telescopic expansion (B.16) is, by now, a standard operating procedure, so we leave it as an exercise for the reader. Substituting (B.16) into (B.13) and, using the binomial identity (B.5), we find
\[
Z_t = q^{\frac{l-1}{2} + X_{r-1}(L-1)} \left\{ \theta(t + 1 > 0) \sum_{l'=1}^{t+1} q^{\nu-2} \prod_{i=1}^{\nu-2} \left[ n_i' + \tilde{n}_i - V_{i,r} + \theta(l' > 1) \sum_{m=0}^{l'-1} \delta_{i,v-r+m} \right] + \right. \\
+ \sum_{l=t+1}^{r-2} q^{\theta(l>t)} \sum_{l'=1}^{\nu-2} \prod_{i=1}^{\nu-2} \left[ n_i' + \tilde{n}_i - V_{i,r} + \theta(i > \nu - r + l) + \theta(t > 0) \sum_{m=0}^{t-1} \delta_{i,v-r+m} \right] \times \\
\times \left( q^{\nu-2} \prod_{i=1}^{\nu-2} \left[ n_i' + \tilde{n}_i - V_{i,r} + \theta(i > \nu - r + t) + \theta(t > 0) \sum_{m=0}^{t-1} \delta_{i,v-r+m} \right] \right) \right\} = Z_{t+1} \quad (B.17)
\]

which proves (B.15). Now (B.14) and (B.15) imply that
\[
J_q(L, r) - q^\frac{L-1}{2} \cdot J_q(L, r + 1) \cdot \theta(\nu - 1 > r) - J_q(L, r) = Z_0 = Z_1 = \ldots = Z_{r-2} = q^\frac{L-1}{2} \cdot \{ J_q(L, r + 1) - q^\frac{L-1}{2} \cdot J_q(L, r) \}. \quad (B.18)
\]

This completes the proof of our claim (B.1).

**Appendix C**

Here we will prove the following claim for \( r = 1, 2, \ldots, \nu - 1 \)
\[
\{ I_q(L, r) - q^\frac{L-1}{2} \cdot I_q(L, r + 1) \cdot \theta(r > 1) \} = I_q(L - 2, r) + \theta(\nu - 1 > r) \cdot q^\frac{L-1}{2} \cdot \{ I_q(L - 1, r + 1) - q^\frac{L-1}{2} \cdot I_q(L - 2, r) \} \quad (C.1)
\]

where for \( L + r \equiv \text{odd integer} \)
\[
I_q(L, r) = q^{V(L)} \prod_{i=1}^{\nu-2} \left[ n_i + \tilde{n}_i + \tilde{V}_{i,r} \right] \quad (C.2)
\]
and \( \bar{V}_{i,r} \), \( Y_r(L) \) and \( \{ \vec{n}, \vec{n} \}_{L,r} \) were defined by (2.21), (3.12), (2.17), 2.19), respectively. Once again, the sum over all solutions to constraint (2.17) is assumed in (C.1). Since the proof of (C.1) is very similar to that of (B.1), we will leave the details out and will refer the reader to the Appendix B for clarification when the need arises.

We start by expanding \( I_q(L, r) \) in a telescopic fashion (3.18):

\[
I_q(L, r) = q^{Y_r(L)} \left\{ \theta(r > 1) \cdot q^{\sum_{r-1} n_i} \prod_{i=1}^{\nu-2} \left[ n_i + \bar{n}_i + \bar{V}_{i,r-1} \right] + \sum_{l=r-1}^{\nu-2} q^{\theta(\nu-2>l)} \prod_{i=1}^{\nu-2} \left[ n_i + \bar{n}_i + \bar{V}_{i,r} - \theta(i > l - 1) \right] \right\}. \tag{C.3}
\]

Observing

\[
\begin{align*}
\{ \vec{n}, \vec{n} \}_{L,r} &= \{ \vec{n}, \vec{n} \}_{L-1,r-1} \\
Y_r(L) + \sum_{r-1} n_i &= \frac{L}{2} + Y_{r-1}(L - 1)
\end{align*}
\]

we identify the first term inside of the figure brackets (C.3) as \( \theta(r > 1) \cdot q^{\frac{L}{2}} \cdot I_q(L - 1, r - 1) \) and \\
\( l = \nu - 2 \) -term as \( I_q(L - 2, r) \). Hence, we have

\[
\Delta \equiv I_q(L, r) - \theta(r > 1) \cdot q^{\frac{L}{2}} \cdot I_q(L - 1, r - 1) - I_q(L - 2, r) = \]

\[
\theta(\nu - 1 > r) \cdot \sum_{l=r-1}^{\nu-3} q^{Y_r(L) + \sum_{i+1}^{\nu-2} n_i} \prod_{i=1}^{\nu-2} \left[ n_i + \bar{n}_i + \bar{V}_{i,r} - \theta(i > l - 1) \right] \tag{C.6}
\]

and

\[
I_q(L, r) - \theta(r > 1) \cdot q^{\frac{L}{2}} \cdot I_q(L - 1, r - 1) =
\]

\[
\theta(\nu > r) \cdot \sum_{l=r-1}^{\nu-2} q^{Y_r(L) + \theta(\nu-2>l)} \prod_{i=1}^{\nu-2} \left[ n_i + \bar{n}_i + \bar{V}_{i,r} - \theta(i > l - 1) \right] \tag{C.7}
\]

Replacing \( r \) by \( r + 1 \), \( L \) by \( L - 1 \) in (C.7) and making use of

\[
\begin{align*}
\{ \vec{n}, \vec{n} \}_{L-1,r+1} &= \{ \vec{n}, \vec{n} \}_{L,r} - \{ \vec{e}_{\nu-2}, 0 \} \\
\bar{V}_{i,r+1} - \theta(i > l - 1) &= \bar{V}_{i,r} + \theta(l > r) \cdot \sum_{m=r}^{l-1} \delta_{i,m}
\end{align*}
\]

we have

\[
\Delta = \theta(\nu > r) \cdot \sum_{l=r}^{\nu-2} q^{Y_r(L) + \theta(\nu-2>l)} \prod_{i=1}^{\nu-2} \left[ n_i + \bar{n}_i + \bar{V}_{i,r} - \theta(i > l - 1) \right] \tag{C.8}
\]

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we find

\[ \theta(\nu - 1 > r)\{I_q(L - 1, r + 1) - q^{\frac{L-1}{r}} \cdot I_q(L - 2, r)\} = \]

\[ = q^{Y_{+1}(L-1)} \left\{ \theta(\nu - 2 > r) \sum_{l=r}^{\nu-3} q^{\theta((\nu - 2)l)} \sum_{i=1}^{\nu-2} n'_i \prod_{i=1}^{\nu-2} \left[ n'_i + \tilde{n}_i + \tilde{V}_{i,r} + \theta(l > r) \sum_{m=r}^{l-1} \delta_{i,m} \right] \right\} \]

\[ + \theta(\nu - 1 > r) \cdot \sum_{i=1}^{\nu-2} \left[ n'_i + \tilde{n}_i + \tilde{V}_{i,r} + \sum_{m=r}^{\nu-3} \delta_{i,m} \right] \] (C.9)

with \( n'_i = n_i - \delta_{i,\nu-2} \). Returning to \( \Delta \) (C.6), we perform the change of summation variables, which for \( l \)-th term takes the form

\[ \tilde{n} \rightarrow \tilde{n} + \tilde{e}_l - \tilde{e}_{l+1} - \tilde{e}_{\nu-2} \]

\[ \tilde{n} \rightarrow \tilde{n} + 2 \sum_{i=l+1}^{\nu-2} \tilde{e}_i, \] (C.10)

and obtain

\[ \Delta = \theta(\nu - 1 > r) \cdot q^{\frac{L-1}{r} + Y_{+1}(L-1)} \sum_{i=1}^{\nu-3} q^{\theta((\nu - 2)l)} \sum_{i=1}^{\nu-2} n'_i \prod_{i=1}^{\nu-2} \left[ n'_i + \tilde{n}_i + \tilde{V}_{i,r} + \theta(l > r) \sum_{m=r}^{l-1} \delta_{i,m} \right] \] (C.11)

Next, we introduce for \( t = 0, 1, \ldots, \nu - 2 - r \) interpolating function \( \tilde{Z}_t \), defined as

\[ \tilde{Z}_t = \theta(\nu - 1 > r) \cdot q^{\frac{L-1}{r} + Y_{+1}(L-1)} \times \]

\[ \times \left\{ \theta(t > 0) \sum_{l=t}^{r+t-1} q^{\theta((\nu - 2)l')} \sum_{i=1}^{\nu-2} n'_i \prod_{i=1}^{\nu-2} \left[ n'_i + \tilde{n}_i + \tilde{V}_{i,r} + \theta(l' > r) \sum_{m=r}^{l'-1} \delta_{i,m} \right] \right\} + \]

\[ + \sum_{l=t+r}^{\nu-3} q^{\theta((\nu - 2)l')} \sum_{i=1}^{\nu-2} n'_i \prod_{i=1}^{\nu-2} \left[ n'_i + \tilde{n}_i + \tilde{V}_{i,r} + \theta(l > l' + 1) + \theta(t > 0) \sum_{m=r}^{l-1} \delta_{i,m} \right] \] (C.12)

Inspecting (C.9), (C.11) and (C.12), we notice

\[ \tilde{Z}_0 = \Delta \]

\[ \tilde{Z}_{\nu - 2 - r} = \theta(\nu - 1 > r) \cdot q^{\frac{L-1}{r}} \{I_q(L - 1, r + 1) - q^{\frac{L-1}{r}} \cdot I_q(L - 2, r)\}. \] (C.13)

Following along the lines of Appendix B, we can prove for \( t = 0, 1, \ldots, \nu - 3 - r \n
\[ \tilde{Z}_t = \tilde{Z}_{t+1}. \] (C.14)

Finally, equation (C.13) together with (C.14) implies

\[ \Delta = \tilde{Z}_0 = \tilde{Z}_1 = \ldots = \tilde{Z}_{\nu - 2 - r} = \theta(\nu - 1 > r) \cdot q^{\frac{L-1}{r}} \{I_q(L - 1, r + 1) - q^{\frac{L-1}{r}} \cdot I_q(L - 2, r)\} \] (C.15)

which proves the claim (C.1).
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