Interval colorings of complete balanced multipartite graphs

Petros A. Petrosyan\textsuperscript{a,b,*}

\textsuperscript{a}Institute for Informatics and Automation Problems, National Academy of Sciences, 0014, Armenia

\textsuperscript{b}Department of Informatics and Applied Mathematics, Yerevan State University, 0025, Armenia

A graph $G$ is called a complete $k$-partite ($k \geq 2$) graph if its vertices can be partitioned into $k$ independent sets $V_1, \ldots, V_k$ such that each vertex in $V_i$ is adjacent to all the other vertices in $V_j$ for $1 \leq i < j \leq k$. A complete $k$-partite graph $G$ is a complete balanced $k$-partite graph if $|V_1| = |V_2| = \cdots = |V_k|$. An edge-coloring of a graph $G$ with colors $1, \ldots, t$ is an interval $t$-coloring if all colors are used, and the colors of edges incident to each vertex of $G$ are distinct and form an interval of integers. A graph $G$ is interval colorable if $G$ has an interval $t$-coloring for some positive integer $t$. In this paper we show that a complete balanced $k$-partite graph $G$ with $n$ vertices in each part is interval colorable if and only if $nk$ is even. We also prove that if $nk$ is even and $(k-1)n \leq t \leq (\frac{3}{2}k - 1)n - 1$, then a complete balanced $k$-partite graph $G$ admits an interval $t$-coloring. Moreover, if $k = p2^q$, where $p$ is odd and $q \in \mathbb{N}$, then a complete balanced $k$-partite graph $G$ has an interval $t$-coloring for each positive integer $t$ satisfying $(k-1)n \leq t \leq (2k - p - q)n - 1$.

Keywords: edge-coloring, interval coloring, complete multipartite graph, complete graph, complete bipartite graph

1. Introduction

Throughout this paper all graphs are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of $G$, respectively. For $F \subseteq E(G)$, the subgraph obtained by deleting the edges of $F$ from $G$ is denoted by $G - F$. The maximum degree of $G$ is denoted by $\Delta(G)$. The terms and concepts that we do not define can be found in \cite{19}.

An edge-coloring of a graph $G$ is a mapping $\alpha : E(G) \to \mathbb{N}$. A proper edge-coloring of a graph $G$ is an edge-coloring $\alpha$ of $G$ such that $\alpha(e) \neq \alpha(e')$ for any pair of adjacent edges $e, e' \in E(G)$. The edge-chromatic number $\chi'(G)$ of $G$ is the least number of colors needed for a proper edge-coloring of $G$. Clearly, $\chi'(G) \geq \Delta(G)$ for every graph $G$. On the other hand, the well-known theorem of Vizing \cite{16} states that the edge-chromatic number of any graph $G$ is either $\Delta(G)$ or $\Delta(G) + 1$. One of the most important, interesting and

*email: pet_petros@{ipia.sci.am, ysu.am, yahoo.com}
long-standing problem in this field is the problem of determining the exact value of the edge-chromatic number of graphs. There are many results in this direction, in particular, the exact value of the edge-chromatic number is known for bipartite graphs [13], complete graphs [4, 18], complete multipartite graphs [9, 14], split graphs with odd maximum degree [5], outerplanar graphs [7], planar graphs $G$ with $\Delta(G) \geq 7$ [17, 20].

A graph $G$ is a complete $k$-partite ($k \geq 2$) graph if its vertices can be partitioned into $k$ independent sets $V_1, \ldots, V_k$ such that each vertex in $V_i$ is adjacent to all the other vertices in $V_j$ for $1 \leq i < j \leq k$. A complete $k$-partite graph $G$ is a complete balanced $k$-partite graph if $|V_1| = |V_2| = \cdots = |V_k|$. Clearly, if $G$ is a complete balanced $k$-partite graph with $n$ vertices in each part, then $\Delta(G) = (k-1)n$. Note that the complete graph $K_n$ and the complete balanced bipartite graph $K_{n,n}$ are special cases of the complete balanced $k$-partite graph. In [14], Laskar and Hare proved the following:

**Theorem 1** If $G$ is a complete balanced $k$-partite graph with $n$ vertices in each part, then

$$\chi'(G) = \begin{cases} (k-1)n, & \text{if } nk \text{ is even,} \\ (k-1)n + 1, & \text{if } nk \text{ is odd.} \end{cases}$$

A more general result was obtained by Hoffman and Rodger [9]. Before we formulate this result we need a definition of the overfull graph. A graph $G$ is overfull if $|E(G)| > \left\lfloor \frac{|V(G)|}{2} \right\rfloor \Delta(G)$. Clearly, if $G$ is overfull, then $\chi'(G) = \Delta(G) + 1$.

**Theorem 2** If $G$ is a complete $k$-partite graph, then

$$\chi'(G) = \begin{cases} \Delta(G), & \text{if } G \text{ is not overfull,} \\ \Delta(G) + 1, & \text{if } G \text{ is overfull.} \end{cases}$$

An edge-coloring of a graph $G$ with colors $1, \ldots, t$ is an interval $t$-coloring if all colors are used, and the colors of edges incident to each vertex of $G$ are distinct and form an interval of integers. A graph $G$ is interval colorable if $G$ has an interval $t$-coloring for some positive integer $t$. For an interval colorable graph $G$, the least and the greatest values of $t$ for which $G$ has an interval $t$-coloring are denoted by $w(G)$ and $W(G)$, respectively. The concept of interval edge-coloring was introduced by Asratian and Kamalian [1]. In [1, 2], they proved the following:

**Theorem 3** If $G$ is a regular graph, then

1. $G$ is interval colorable if and only if $\chi'(G) = \Delta(G)$.
2. If $G$ is interval colorable and $w(G) \leq t \leq W(G)$, then $G$ has an interval $t$-coloring.

In [10], Kamalian investigated interval colorings of complete bipartite graphs and trees. In particular, he proved the following:

**Theorem 4** For any $r, s \in \mathbb{N}$, the complete bipartite graph $K_{r,s}$ is interval colorable, and

1. $w(K_{r,s}) = r + s - \gcd(r, s)$,
(2) \( W(K_{r,s}) = r + s - 1 \),

(3) if \( w(K_{r,s}) \leq t \leq W(K_{r,s}) \), then \( K_{r,s} \) has an interval \( t \)-coloring.

Later, Kamalian [11] obtained an upper bound on \( W(G) \) for an interval colorable graph \( G \) depending on the number of vertices of \( G \).

**Theorem 5** If \( G \) is a connected interval colorable graph, then \( W(G) \leq 2|V(G)| - 3 \).

Clearly, this bound is sharp for the complete graph \( K_2 \), but if \( G \neq K_2 \), then this upper bound can be improved to \( 2|V(G)| - 4 \) [8]. For an \( r \)-regular graph \( G \), Kamalian and Petrosyan [12] showed that if \( G \) with at least \( 2r + 2 \) vertices admits an interval \( t \)-coloring, then \( t \leq 2|V(G)| - 5 \). For a planar graph \( G \), Axenovich [3] showed that if \( G \) has an interval \( t \)-coloring, then \( t \leq \frac{1}{6}|V(G)| \). In [15], Petrosyan investigated interval colorings of complete graphs and \( n \)-dimensional cubes. First note that \( K_{2n+1} \) is not interval colorable, but \( K_{2n} \) is interval colorable and \( w(K_{2n}) = 2n - 1 \) for any \( n \in \mathbb{N} \). For \( W(K_{2n}) \), Petrosyan [15] proved the following:

**Theorem 6** If \( n = p2^q \), where \( p \) is odd and \( q \) is nonnegative, then \( W(K_{2n}) \geq 4n - 2 - p - q \).

In this paper we investigate interval colorings of complete balanced \( k \)-partite graphs. In particular, we generalize Theorem 5 for complete balanced \( k \)-partite graphs. Also, we discuss some other corollaries of our result.

2. Results

If \( \alpha \) is a proper edge-coloring of \( G \) and \( v \in V(G) \), then \( S(v, \alpha) \) denotes the set of colors appearing on edges incident to \( v \).

Let \( [t] \) denote the set of the first \( t \) natural numbers. Let \( [a] \) denote the largest integer less than or equal to \( a \). For two positive integers \( a \) and \( b \) with \( a \leq b \), the set \( \{a, \ldots , b\} \) is denoted by \( [a,b] \) and called an interval. For an interval \( [a,b] \) and a nonnegative number \( p \), the notation \( [a,b] \oplus p \) means: \( [a+p, b+p] \).

We also need the following lemma.

**Lemma 7** If \( K_{n,n} \) is a complete balanced bipartite graph with a bipartition \( (U,V) \), where \( U = \{u_1, \ldots , u_n\} \) and \( V = \{v_1, \ldots , v_n\} \), then \( K_{n,n} \) has an interval \((2n-1)\)-coloring \( \alpha \) such that \( S(u_i, \alpha) = S(v_i, \alpha) = [i, i+n-1] \) for \( 1 \leq i \leq n \).

**Proof.** Let \( (U,V) \) be a bipartition of \( K_{n,n} \), where \( U = \{u_1, \ldots , u_n\} \) and \( V = \{v_1, \ldots , v_n\} \). Define a coloring \( \alpha \) of the edges of \( K_{n,n} \) as follows: for each edge \( u_iv_j \in E(K_{n,n}) \), let \( \alpha(u_iv_j) = i+j-1 \), where \( 1 \leq i \leq n, 1 \leq j \leq n \). Clearly \( \alpha \) is an interval \((2n-1)\)-coloring of \( K_{n,n} \) and \( S(u_i, \alpha) = S(v_i, \alpha) = [i, i+n-1] \) for \( 1 \leq i \leq n \). \( \square \)
Let \( G \) be a complete balanced \( k \)-partite graph with \( n \) vertices in each part. By Theorems 1 and 3 we have that \( G \) is interval colorable if and only if \( nk \) is even. Moreover, if \( nk \) is even, then \( w(G) = \Delta(G) = (k - 1)n \). On the other hand, by Theorem 5 we obtain \( W(G) \leq 2nk - 3 \) whenever \( nk \) is even. Now we derive a lower bound for \( W(G) \).

**Theorem 8** If \( G \) is a complete balanced \( k \)-partite graph with \( n \) vertices in each part and \( nk \) is even, then \( W(G) \geq \left( \frac{3}{2}k-1 \right) n-1 \).

**Proof.** We distinguish our proof into two cases.

Case 1: \( k \) is even.

Let \( V(G) = \{ v_j^{(i)} : 1 \leq i \leq k, 1 \leq j \leq n \} \) and

\[ E(G) = \{ v_p^{(i)} v_q^{(j)} : 1 \leq i < j \leq k, 1 \leq p \leq n, 1 \leq q \leq n \} \]

Define an edge-coloring \( \alpha \) of the graph \( G \).

For each edge \( v_p^{(i)} v_q^{(j)} \in E(G) \) with \( 1 \leq i < j \leq k \) and \( p = 1, \ldots, n, q = 1, \ldots, n \), define a color \( \alpha \left( v_p^{(i)} v_q^{(j)} \right) \) as follows:

for \( i = 1, \ldots, \left\lfloor \frac{k}{4} \right\rfloor, j = 2, \ldots, \frac{k}{2}, i + j \leq \frac{k}{2} + 1, \) let

\[ \alpha \left( v_p^{(i)} v_q^{(j)} \right) = (i + j - 3) n + p + q - 1; \]

for \( i = 2, \ldots, \frac{k}{2} - 1, j = \left\lfloor \frac{k}{4} \right\rfloor + 2, \ldots, \frac{k}{2}, i + j \geq \frac{k}{2} + 2, \) let

\[ \alpha \left( v_p^{(i)} v_q^{(j)} \right) = (i + j + \frac{k}{2} - 4) n + p + q - 1; \]

for \( i = 3, \ldots, \frac{k}{2}, j = \frac{k}{2} + 1, \ldots, k - 2, j - i \leq \frac{k}{2} - 2, \) let

\[ \alpha \left( v_p^{(i)} v_q^{(j)} \right) = \left( \frac{k}{2} + j - i - 1 \right) n + p + q - 1; \]

for \( i = 1, \ldots, \frac{k}{2}, j = \frac{k}{2} + 1, \ldots, k, j - i \geq \frac{k}{2}, \) let

\[ \alpha \left( v_p^{(i)} v_q^{(j)} \right) = \left( j - i - 1 \right) n + p + q - 1; \]

for \( i = 2, \ldots, 1 + \left\lfloor \frac{k-2}{4} \right\rfloor, j = \frac{k}{2} + 1, \ldots, \frac{k}{2} + \left\lfloor \frac{k-2}{4} \right\rfloor, j - i = \frac{k}{2} - 1, \) let

\[ \alpha \left( v_p^{(i)} v_q^{(j)} \right) = (2i - 3) n + p + q - 1; \]

for \( i = \left\lfloor \frac{k-2}{4} \right\rfloor + 2, \ldots, \frac{k}{2}, j = \frac{k}{2} + 1 + \left\lfloor \frac{k-2}{4} \right\rfloor, \ldots, k - 1, j - i = \frac{k}{2} - 1, \) let

\[ \alpha \left( v_p^{(i)} v_q^{(j)} \right) = (i + j - 3) n + p + q - 1; \]

for \( i = \frac{k}{2} + 1, \ldots, \frac{k}{2} + \left\lfloor \frac{k}{4} \right\rfloor - 1, j = \frac{k}{2} + 2, \ldots, k - 2, i + j \leq \frac{3}{2}k - 1, \) let
Interval colorings of complete balanced multipartite graphs

\[
\alpha \left( v_p^{(i)} v_q^{(j)} \right) = (i + j - k - 1) n + p + q - 1;
\]

for \( i = \frac{k}{2} + 1, \ldots, k - 1, j = \frac{k}{2} + \left\lceil \frac{k}{4} \right\rceil + 1, \ldots, k, i + j \geq \frac{3}{2} k, \) let

\[
\alpha \left( v_p^{(i)} v_q^{(j)} \right) = (i + j - \frac{k}{2} - 2) n + p + q - 1.
\]

Let us prove that \( \alpha \) is an interval \( \left( (\frac{3}{2} k - 1) n - 1 \right) \)-coloring of the graph \( G \).

First we show that for each \( t \in \left( \frac{3}{2} k - 1 \right) n - 1 \), there is an edge \( e \in E(G) \) with \( \alpha(e) = t \).

Consider the vertices \( v_1^{(1)}, \ldots, v_n^{(1)}, v_1^{(k)}, \ldots, v_n^{(k)} \). Now, by Lemma 7 and the definition of \( \alpha \), for \( 1 \leq j \leq n \),

\[
S \left( v_j^{(1)}, \alpha \right) = \bigcup_{i=1}^{k-1} \left( \left[ j, j + n - 1 \right] \oplus (l - 1)n \right) \text{ and } S \left( v_j^{(k)}, \alpha \right) = \bigcup_{i=\frac{k}{2}}^{2k-2} \left( \left[ j, j + n - 1 \right] \oplus (l - 1)n \right).
\]

Let \( \overline{C} \) and \( \overline{\overline{C}} \) be the subsets of colors appear on edges incident to the vertices \( v_1^{(1)}, \ldots, v_n^{(1)} \) and \( v_1^{(k)}, \ldots, v_n^{(k)} \) in the coloring \( \alpha \), respectively, that is:

\[
\overline{C} = \bigcup_{j=1}^{n} S \left( v_j^{(1)}, \alpha \right) \text{ and } \overline{\overline{C}} = \bigcup_{j=1}^{n} S \left( v_j^{(k)}, \alpha \right).
\]

It is straightforward to check that \( \overline{C} \cap \overline{\overline{C}} = \left( \left( \frac{3}{2} k - 1 \right) n - 1 \right), \) so for each \( t \in \left( \frac{3}{2} k - 1 \right) n - 1 \), there is an edge \( e \in E(G) \) with \( \alpha(e) = t \).

Next we show that the edges incident to any vertex of \( G \) are colored by \( (k - 1)n \) consecutive colors.

Let \( v_j^{(i)} \in V(G) \), where \( 1 \leq i \leq k, 1 \leq j \leq n \).

Subcase 1.1. \( 1 \leq i \leq 2, 1 \leq j \leq n \).

By Lemma 7 and the definition of \( \alpha \), we have

\[
S \left( v_j^{(1)}, \alpha \right) = S \left( v_j^{(2)}, \alpha \right) = \bigcup_{i=1}^{k-1} \left( \left[ j, j + n - 1 \right] \oplus (l - 1)n \right) = \left[ j, j + (k - 1)n - 1 \right].
\]

Subcase 1.2. \( 3 \leq i \leq \frac{k}{2}, 1 \leq j \leq n \).

By Lemma 7 and the definition of \( \alpha \), we have

\[
S \left( v_j^{(i)}, \alpha \right) = \bigcup_{i=\frac{k-3+i}{2}}^{k-3+i} \left( \left[ j, j + n - 1 \right] \oplus (l - 1)n \right) = \left[ j + (i - 2)n, j + (k - 3 + i)n - 1 \right].
\]

Subcase 1.3. \( \frac{k}{2} + 1 \leq i \leq k - 2, 1 \leq j \leq n \).

By Lemma 7 and the definition of \( \alpha \), we have

\[
S \left( v_j^{(i)}, \alpha \right) = \bigcup_{i=\frac{k}{2}-i}^{k-3+i} \left( \left[ j, j + n - 1 \right] \oplus (l - 1)n \right) = \left[ j + (i - \frac{k}{2})n, j + (\frac{k}{2} - 1 + i)n - 1 \right].
\]

Subcase 1.4. \( k - 1 \leq i \leq k, 1 \leq j \leq n \).

By Lemma 7 and the definition of \( \alpha \), we have
\( S \left( v_j^{(k-1)}, \alpha \right) = S \left( v_j^{(k)}, \alpha \right) = \bigcup_{l=\frac{3}{2}k-2}^{2k-2} \left( [j, j+n-1] \oplus (l-1)n \right) = \left[ j + \left( \frac{k}{2} - 1 \right), j + \left( \frac{3k}{2} - 2 \right) \right] n - 1 \).

This shows that \( \alpha \) is an interval \( (\left( \frac{3}{2}k - 1 \right)n - 1) \)-coloring of \( G \); thus \( W(G) \geq \left( \frac{3}{2}k - 1 \right)n - 1 \) for even \( k \geq 2 \).

Case 2: \( n \) is even.

Let \( n = 2m \), \( m \in \mathbb{N} \). Let \( U_i = \{ v_1^{(i)}, \ldots, v_m^{(i)}, v_1^{(k+i)}, \ldots, v_m^{(k+i)} \} \) \((1 \leq i \leq k)\) be the \( k \) independent sets of vertices of \( G \). For \( i = 1, \ldots, 2k \), define the set \( V_i \) as follows: \( V_i = \{ v_1^{(i)}, \ldots, v_m^{(i)} \} \). Clearly, \( V(G) = \bigcup_{i=1}^{2k} V_i \). For \( 1 \leq i < j \leq 2k \), define \( \langle V_i, V_j \rangle \) as the set of all edges between \( V_i \) and \( V_j \). It is easy to see that for \( 1 \leq i < j \leq 2k \), \(|\langle V_i, V_j \rangle| = m^2 \) except for \(|\langle V_i, V_{k+i} \rangle| = 0 \) whenever \( i = 1, \ldots, k \). If we consider the sets \( V_i \) as the vertices and the sets \( \langle V_i, V_j \rangle \) as the edges, then we obtain that \( G \) is isomorphic to the graph \( K_{2k} - F \), where \( F \) is a perfect matching. Now we define an edge-coloring \( \beta \) of the graph \( G \).

For each edge \( v_p^{(i)}v_q^{(j)} \in E(G) \) with \( 1 \leq i < j \leq 2k \) and \( p = 1, \ldots, m, q = 1, \ldots, m, \) define a color \( \beta \left( v_p^{(i)}v_q^{(j)} \right) \) as follows:

for \( i = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor \), \( j = 2, \ldots, k \), \( i + j \leq k + 1 \), let
\[
\beta \left( v_p^{(i)}v_q^{(j)} \right) = (i + j - 3) m + p + q - 1;
\]

for \( i = 2, \ldots, k - 1 \), \( j = \left\lfloor \frac{k}{2} \right\rfloor + 2, \ldots, k \), \( i + j \geq k + 2 \), let
\[
\beta \left( v_p^{(i)}v_q^{(j)} \right) = (i + j + k - 5) m + p + q - 1;
\]

for \( i = 3, \ldots, k \), \( j = k + 1, \ldots, 2k - 2 \), \( j - i \leq k - 2 \), let
\[
\beta \left( v_p^{(i)}v_q^{(j)} \right) = (k + j - i - 2) m + p + q - 1;
\]

for \( i = 1, \ldots, k - 1 \), \( j = k + 2, \ldots, 2k \), \( j - i \geq k + 1 \), let
\[
\beta \left( v_p^{(i)}v_q^{(j)} \right) = (j - i - 2) m + p + q - 1;
\]

for \( i = 2, \ldots, 1 + \left\lfloor \frac{k-1}{2} \right\rfloor \), \( j = k + 1, \ldots, k + \left\lfloor \frac{k-1}{2} \right\rfloor \), \( j - i = k - 1 \), let
\[
\beta \left( v_p^{(i)}v_q^{(j)} \right) = (2i - 3) m + p + q - 1;
\]

for \( i = \left\lfloor \frac{k-1}{2} \right\rfloor + 2, \ldots, k \), \( j = k + 1 + \left\lfloor \frac{k-1}{2} \right\rfloor, \ldots, 2k - 1 \), \( j - i = k - 1 \), let
\[
\beta \left( v_p^{(i)}v_q^{(j)} \right) = (i + j - 4) m + p + q - 1;
\]

for \( i = k + 1, \ldots, k + \left\lfloor \frac{k}{2} \right\rfloor - 1 \), \( j = k + 2, \ldots, 2k - 2 \), \( i + j \leq 3k - 1 \), let
Interval colorings of complete balanced multipartite graphs

\[
\beta \left( v_p^{(i)} v_q^{(j)} \right) = (i + j - 2k - 1) m + p + q - 1;
\]

for \( i = k + 1, \ldots, 2k - 1, \ j = k + \lfloor \frac{k}{2} \rfloor + 1, \ldots, 2k, \ i + j \geq 3k, \) let

\[
\beta \left( v_p^{(i)} v_q^{(j)} \right) = (i + j - k - 3) m + p + q - 1.
\]

Let us prove that \( \beta \) is an interval \((\frac{3}{2}k - 1) n - 1\)-coloring of the graph \( G \).

First we show that for each \( t \in \left[ \frac{3}{2}k - 1, n - 1 \right], \) there is an edge \( e \in E(G) \) with \( \beta(e) = t. \)

Consider the vertices \( v_1^{(1)}, \ldots, v_m^{(1)}, v_1^{(2k)}, \ldots, v_m^{(2k)}. \) Now, by Lemma \( \square \) and the definition of \( \beta, \) for \( 1 \leq j \leq m, \)

\[
S \left( v_j^{(1)}, \beta \right) = \bigcup_{l=1}^{2k-2} ([j, j + m - 1] \oplus (l-1)m) \quad \text{and} \quad S \left( v_j^{(2k)}, \beta \right) = \bigcup_{l=k}^{3k-3} ([j, j + m - 1] \oplus (l-1)m).
\]

Let \( \tilde{C} \) and \( \tilde{C} \) be the subsets of colors appear on edges incident to the vertices \( v_1^{(1)}, \ldots, v_m^{(1)} \) and \( v_1^{(2k)}, \ldots, v_m^{(2k)} \) in the coloring \( \beta, \) respectively, that is:

\[
\tilde{C} = \bigcup_{j=1}^{m} S \left( v_j^{(1)}, \beta \right) \quad \text{and} \quad \tilde{C} = \bigcup_{j=1}^{m} S \left( v_j^{(2k)}, \beta \right).
\]

It is straightforward to check that \( \tilde{C} \cup \tilde{C} = \left[ \frac{3}{2}k - 1, n - 1 \right], \) so for each \( t \in \left[ \frac{3}{2}k - 1, n - 1 \right], \) there is an edge \( e \in E(G) \) with \( \beta(e) = t. \)

Next we show that the edges incident to any vertex of \( G \) are colored by \((k - 1)n\) consecutive colors.

Let \( v_j^{(i)} \in V(G), \) where \( 1 \leq i \leq 2k, \ 1 \leq j \leq m. \)

Subcase 2.1. \( 1 \leq i \leq 2, \ 1 \leq j \leq m. \)

By Lemma \( \square \) and the definition of \( \beta, \) we have

\[
S \left( v_j^{(1)}, \beta \right) = S \left( v_j^{(2)}, \beta \right) = \bigcup_{l=1}^{2k-2} ([j, j + m - 1] \oplus (l-1)m) = [j, j + (2k - 2)m - 1].
\]

Subcase 2.2. \( 3 \leq i \leq k, \ 1 \leq j \leq m. \)

By Lemma \( \square \) and the definition of \( \beta, \) we have

\[
S \left( v_j^{(i)}, \beta \right) = \bigcup_{l=i-4}^{2k+i} ([j, j + m - 1] \oplus (l-1)m) = [j + (i - 2)m, j + (2k - 4 + i)m - 1].
\]

Subcase 2.3. \( k + 1 \leq i \leq 2k - 2, \ 1 \leq j \leq m. \)

By Lemma \( \square \) and the definition of \( \beta, \) we have

\[
S \left( v_j^{(i)}, \beta \right) = \bigcup_{l=i-k}^{k+i} ([j, j + m - 1] \oplus (l-1)m) = [j + (i - k)m, j + (k - 2 + i)m - 1].
\]

Subcase 2.4. \( 2k - 1 \leq i \leq 2k, \ 1 \leq j \leq m. \)

By Lemma \( \square \) and the definition of \( \beta, \) we have
Corollary 10

Let \( nk \) be an even number with \( n \) vertices in each part. If \( G \) is a complete balanced \( k \)-partite graph with \( nk \) vertices, then we have:

\[
\begin{align*}
S \left( v_j^{(2k-1)}, \beta \right) &= S \left( v_j^{(2k)}, \beta \right) = \bigcup_{t=k}^{3k-3} ([j, j + m - 1] \oplus (l - 1)m) = [j + (k - 1)m, j + (3k - 3)m - 1].
\end{align*}
\]

This shows that \( \beta \) is an interval \( (\frac{3}{2}k - 1) \cdot n - 1 \)-coloring of \( G \); thus \( W(G) \geq (\frac{3}{2}k - 1) \cdot n - 1 \) for even \( n \geq 2. \Box \)

From Theorems 3(1) and 8, and taking into account that a complete balanced \( k \)-partite graph with \( n \) vertices in each part is overfull when \( nk \) is odd, we have:

Corollary 9

If \( G \) is a complete balanced \( k \)-partite graph with \( n \) vertices in each part, then \( \chi'(G) = (k - 1)n \) if and only if \( nk \) is even.

From Theorems 3(2) and 8 we have:

Corollary 10

Let \( G \) be a complete balanced \( k \)-partite graph with \( n \) vertices in each part and \( nk \) is even. If \( (k - 1)n \leq i \leq (\frac{3}{2}k - 1) \cdot n - 1 \), then \( G \) has an interval \( t \)-coloring.

Also, note that the proof of the case 2 implies that if a graph \( G \) with \( n \) vertices is \((n - 2)\)-regular, then \( \chi'(G) = n - 2. \)

The next theorem improves the lower bound in Theorem 8 for complete balanced \( k \)-partite graphs with even \( k \).

Theorem 11

Let \( G \) be a complete balanced \( k \)-partite graph with \( n \) vertices in each part. If \( k = p2^q \), where \( p \) is odd and \( q \in \mathbb{N} \), then \( W(G) \geq (2k - p - q)n - 1 \).

Proof. Let \( V(G) = \{v_j^{(i)}: 1 \leq i \leq k, 1 \leq j \leq n\} \) and \( V(K_k) = \{u_1, \ldots, u_k\} \). Also, let \( E(G) = \{v_r^{(i)}v_s^{(j)}: 1 \leq i < j \leq k, 1 \leq r \leq n, 1 \leq s \leq n\} \) and \( E(K_k) = \{u_iu_j: 1 \leq i < j \leq k\} \).

Since \( k = p2^q \), where \( p \) is odd and \( q \in \mathbb{N} \), by Theorem 6, there exists an interval \((2k - 1 - p - q)\)-coloring \( \alpha \) of \( K_k \). Now we define an edge-coloring \( \beta \) of the graph \( G \).

For each edge \( v_r^{(i)}v_s^{(j)} \in E(G) \) with \( 1 \leq i < j \leq k \) and \( r = 1, \ldots, n, s = 1, \ldots, n \), define a color \( \beta \left( v_r^{(i)}v_s^{(j)} \right) \) as follows:

\[
\beta \left( v_r^{(i)}v_s^{(j)} \right) = (\alpha(u_iu_j) - 1)n + r + s - 1.
\]

By Lemma 7 and the definition of \( \beta \), and taking into account that \( \max S(u_i, \alpha) - \min S(u_i, \alpha) = k - 2 \) for \( i = 1, \ldots, k \), we have

\[
S \left( v_j^{(i)}, \beta \right) = \bigcup_{t=\min S(u_i, \alpha)}^{\max S(u_i, \alpha)} ([j, j + n - 1] \oplus (l - 1)n) = [j + (\min S(u_i, \alpha) - 1)n, j + \max S(u_i, \alpha)n - 1]
\]
for $i = 1, \ldots, k$ and $j = 1, \ldots, n$, and

$$\bigcup_{i=1}^{k} \bigcup_{j=1}^{n} S \left( v_j^{(i)}, \beta \right) = [(2k - p - q)n - 1].$$

This shows that $\beta$ is an interval $((2k - p - q)n - 1)$-coloring of the graph $G$; thus $W(G) \geq (2k - p - q)n - 1$. □

From Theorems 3(2) and 11 we have:

**Corollary 12** Let $G$ be a complete balanced $k$-partite graph with $n$ vertices in each part and $k = p2^q$, where $p$ is odd and $q \in \mathbb{N}$. If $(k - 1)n \leq t \leq (2k - p - q)n - 1$, then $G$ has an interval $t$-coloring.

### 3. Problems

In the previous section we obtained some results on interval colorings of complete balanced multipartite graphs, but very small is known about interval colorings of complete unbalanced multipartite graphs. In fact, there are only two results on interval colorings of complete unbalanced multipartite graphs. Let $n_1 \leq \cdots \leq n_k$ be positive integers. The complete multipartite graph $K_{n_1, \ldots, n_k}$ is a complete $k$-partite graph for which $|V_i| = n_i$, $i = 1, \ldots, k$. The first result is Theorem 4 which gives all possible values of the number of colors in interval colorings of $K_{n_1,n_2}$. The second result was obtained by Feng and Huang [6]. In [6], they proved that the complete 3-partite graph $K_{1,1,n}$ is interval colorable if and only if $n$ is even. Now we would like to formulate some problems on interval colorings of complete multipartite graphs:

**Problem 1** Characterize all interval colorable complete multipartite graphs.

**Problem 2** Find the exact values of $w(G)$ and $W(G)$ for interval colorable complete multipartite graphs $G$.

**Problem 3** Find the exact value of $W(K_{n_{\ldots,n}})$ for interval colorable complete balanced $k$-partite graphs $K_{n_{\ldots,n}}$.

Note that even a special case of Problem 3 is open: the problem of determining the exact value of $W(K_{2n})$ for complete graph $K_{2n}$.

### REFERENCES

1. A.S. Asratian and R.R. Kamalian, Interval colorings of edges of a multigraph, Appl. Math. 5 (1987), 25-34 (in Russian).
2. A.S. Asratian and R.R. Kamalian, Investigation on interval edge-colorings of graphs, J. Combin. Theory Ser. B 62 (1994), 34-43.
3. M.A. Axenovich, On interval colorings of planar graphs, Congr. Numer. 159 (2002), 77-94.
4. M. Behzad, G. Chartrand and J. K. Cooper, Jr., The colour numbers of complete graphs, J. London Math. Soc. 42 (1967), 226-228.
5. B.-L. Chen, H.-L. Fu and M. T. Ko, Total chromatic number and chromatic index of split graphs, J. Combin. Math. Combin. Comput. 17 (1995), 137-146.
6. Y. Feng and Q. Huang, Consecutive edge-coloring of the generalized $\theta$-graph, Discrete Appl. Math. 155 (2007), 2321-2327.
7. S. Fiorini, On the chromatic index of outerplanar graphs, J. Combin. Theory Ser. B 18 (1975), 35-38.
8. K. Giaro, M. Kubale and M. Malafiejski, Consecutive colorings of the edges of general graphs, Discrete Math. 236 (2001), 131-143.
9. D.G. Hoffman and C.A. Rodger, The chromatic index of complete multipartite graphs, J. Graph Theory 16 (1992), 159-163.
10. R.R. Kamalian, Interval colorings of complete bipartite graphs and trees, preprint, Comp. Cen. of Acad. Sci. of Armenian SSR, Erevan, 1989 (in Russian).
11. R.R. Kamalian, Interval edge colorings of graphs, Doctoral Thesis, Novosibirsk, 1990.
12. R.R. Kamalian and P.A. Petrosyan, A note on interval edge-colorings of graphs, Math. Probl. of Comp. Sci. 36 (2012), 13-16.
13. D. König, Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, Math. Ann. 77 (1916), 453-465.
14. R. Laskar and W. Hare, Chromatic numbers for certain graphs, J. London Math. Soc. (2), 4 (1972), 489-492.
15. P.A. Petrosyan, Interval edge-colorings of complete graphs and $n$-dimensional cubes, Discrete Math. 310 (2010), 1580-1587.
16. V.G. Vizing, On an estimate of the chromatic class of a $p$-graph, Diskret. Analiz 3 (1964), 25-30 (in Russian).
17. V.G. Vizing, Critical graphs with a given chromatic class, Diskret. Analiz 5 (1965), 9-17 (in Russian).
18. V.G. Vizing, The chromatic class of a multigraph, Kibernetika 3 (1965), 29-39 (in Russian).
19. D.B. West, Introduction to Graph Theory, Prentice-Hall, New Jersey, 2001.
20. L. Zhang, Every planar graph with maximum degree 7 is of class 1, Graphs and Combin. 16 (2000), 467-495.