Ising nematic fluid phase of hard-core dimers on the square lattice

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We present a model of classical hard-core dimers on the square lattice that contains an Ising nematic phase in its phase diagram. We consider a model with an attractive interaction for parallel dimers on a given plaquette of the square lattice and an attractive interaction for neighboring parallel dimers on the same row (viz column) of the lattice. By extensive Monte Carlo simulations we find that with a finite density of holes the phase diagram has, with rising temperatures, a columnar crystalline phase, an Ising nematic liquid phase and a disordered fluid phase, separated by Ising continuous phase transitions. We present strong evidence for the Ising universality class of both transitions. The Ising nematic phase can be interpreted as either an intermediate classical thermodynamic phase (possibly of a strongly correlated antiferromagnet) or as a phase of a 2D quantum dimer model using the Rokhsar-Kivelson construction of exactly solvable quantum Hamiltonians.

I. INTRODUCTION

Hard-core dimer models have been useful in identifying and handling basic aspects of the configurational frustration in quantum spin systems with non-magnetic ground states such as RVB wavefunctions and valence bond singlet crystals. Hard-core dimer models have been used extensively in the context of quantum condensed matter model constructions, such as the quantum dimer models by Rokhsar and Kivelson to characterize topological fluid quantum phases strongly correlated superconducting phases as well as crystalline phases and stripe phases (see, e.g Ref.[5]). Such modeling led to useful insights and basic understanding of aspects of quantum frustration. Similar models can be used to understand the high-temperature behavior of such strongly correlated quantum antiferromagnets, assuming that the valence bond spin gap has an onset at much higher energies.

Ising nematic liquid phases and more generally, electronic liquid crystalline phases have been observed in several physical systems, notably in the two-dimensional electron gas (2DEG) in large magnetic fields, in the bilayer ruthenate Sr3Ru2O7 in magnetic fields, in the iron-based superconductors and most interestingly, and in the cuprate superconductors. In particular, nematic order has been reported in YBa2Cu3O6+δ in much of the pseudogap regime and in very underdoped YBa2Cu3O6+δ over a large temperature range as well as in Bi2Sr2CaCu2O8+δ and in the iron-based superconductors LaFeAsO and Ba(Fe1−xCox)2As2. In this paper, we study the emergence of an Ising nematic dimer phase in a classical dimer model, using extensive numerical classical Monte Carlo simulations and analytical arguments. We consider dimer models with two types of interactions: a) an attractive interaction with strength u for parallel dimers on the same plaquette of the square lattice (which favors columnar ordered states), and b) an attractive interaction of strength v for dimers on next-nearest-neighbor bonds on the same row (viz column) of the square lattice (which favors all tilted columnar states). We investigated the phase diagram of this model, shown schematically in Fig. 2 A, B and C, at full dimer coverage and at a finite density δ of holes as a function of temperature T and for varying dimer interactions u and v.

In the fully packed regime, 2D dimer models can describe ordered phases, columnar phases with varying degree of translation symmetry breaking as well as quantum critical systems and topological phases. The nematic phase of this system is characterized by spatially disordered dimer configurations that globally break rotational invariance, as shown in Fig. 4 i.e. long range orientational order that breaks spontaneously the C4 point group symmetry of the square lat-

FIG. 1. A typical configuration of an Ising nematic phase for a hard-core dimer model with a finite density of holes (see text).
Two neighboring spins means that it is occupied by a valence bond singlet of quantum antiferromagnets, a bond occupied by a dimer classical dimer model at various coverings. In the context of corresponding (classical) 2D Ising phase transitions.

Finite temperature range (see Fig. 2). Evolution of phase diagram for different hole densities \( \delta \), and for different attractive dimer plaquette interactions \( u \) and attractive dimer-aligning interactions \( v \) as a function of temperature \( T \). A) When only the plaquette interactions are present there is a single phase transition is present between a featureless dimer-hole liquid and the columnar solid. At zero hole density this transition is in the KT universality class. At finite hole density there is a line of varying critical exponents lead to a rather unconventional multicritical point at \( \delta_m = 0.11, T_m = 0.4 \) (endpoint of solid line). At lower temperatures \( T < T_m \) the transition is discontinuous, and when in a fixed-density ensemble there is phase coexistence. The dotted lines represent the range of observed coexistence, derived from simulations. B) As soon as the dimer-aligning interaction \( v \) is finite, a new phase emerges in-between the dimer-hole liquid and the columnar solid, the Ising nematic dimer liquid. The phase transition from the columnar solid to the nematic liquid appears to resemble quantitatively (almost overlapping) the columnar solid to dimer-hole liquid transition line at zero nematicity. The transition line is in the Ising universality class. At higher hole densities there is a subsequent Ising transition line separating the Ising nematic phase from the isotropic dimer-hole liquid. C) When only the dimer-aligning interaction \( v \) is present, the columnar solid phase is absent. Only a single Ising transition line separates the nematic liquid from the dimer-hole liquid. In the close-packed limit \( \delta = 0 \) and for \( u = 0 \), the nematic liquid is infinitesimally unstable to a columnar solid, possibly at finite tilt due to a sub-extensive entropy of aligned dimer lines. The solid line for \( T > 1 \) signifies the presence of the finite tilt columnar solid. Due to the presence of this sub-extensive entropy, the nature of the transition, expected to be in the KT universality class, is unclear from the simulations.

These phases can be interpreted as occurring in a classical dimer model at various coverings. In the context of quantum antiferromagnets, a bond occupied by a dimer means that it is occupied by a valence bond singlet of two neighboring spins or as a label of an unsatisfied bond in frustrated quantum magnets. In doped antiferromagnets with a spin gap, as well as in valence bonds one has to include the charge degrees of freedom in the form of holes. A simple model of such systems is the doped quantum dimer model. Dimer models of the type we discuss here can also serve to represent, qualitatively, forms of orbital order of the type used in models of the iron superconductors and in the analysis of the STM data in Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ as well as for the analysis on valley order in certain 2DEGs in magnetic fields. Although the dimer models do not represent many aspects of the microscopic physics of these systems faithfully, its symmetries are the same as the point groups of the more microscopic models and, hence, should be useful to describe many aspects of these systems, particularly their phase transitions.

Here we show that a classical dimer model with a finite density of holes can lead, with suitable interactions, to a nematic phase in dimer models arising from the melting of a columnar state. However, it has been unknown how to stabilize such a nematic liquid in classical or/and quantum dimer models by using isotropic Hamiltonians with local interactions. By using the RK-construction it is straightforward to find an exactly solvable quantum dimer models with a stable quantum nematic phase whose ground-state wave function \( |\Psi\rangle \) has local weights of the same form as the Gibbs weights of the classical dimer model we study here, given the configurations are assumed to be orthonormal. Namely, 

\[
|\Psi\rangle = \frac{1}{\sqrt{Z}} \exp(-\beta E_C/2)|C\rangle \tag{1.1}
\]

where

\[
Z = \sum_{\langle C\rangle} \exp(-\beta E_C) \tag{1.2}
\]

is the norm of the wave function and also the partition function of a classical system with Gibbs weight \( \exp(-\beta E_C)/Z \) in the same dimension.
We fully characterize the behavior of the corresponding classical model through analytic arguments and extensive simulations. First, we show that the key ingredient for stabilizing nematic dimer liquids is a strong same-row or same-column dimer-aligning, spatially isotropic, interaction. In a fully packed lattice this interaction leads to crystalline phases. Here we will show that at finite hole density it stabilizes an Ising nematic fluid phase. Further, we demonstrate numerically that all relevant transitions at finite hole density are of Ising nature and that the nematic phase, even though it breaks explicitly rotational symmetry, by having most dimers point to the same direction, also has disordered liquid-like correlations along the orientation direction. However, in the close-packed case, and when only the dimer-aligning interaction is present, it is unclear whether the transition is in the Ising or the KT universality class. Theoretical arguments predict the KT transition but the simulations give, as we will discuss, inconsistent results possibly due to finite-size effects.

This work is organized as follows. In Sec. II we present the generalized dimer model and discuss its symmetry properties. In Sec. III we present and analyze the results of our Monte Carlo simulations which lead to the phase diagrams of Fig. 2. In Sec. IV we present results form a mean-field theory based on a formulation of the generalized dimer model in a Grassmann variables formulation. Our conclusions are presented in Sec. V.

II. THE MODEL

The classical dimer model is on a square lattice of linear dimension $L$ (total number of sites $N = L^2$) covered by $N_d$ hard-core dimers and $N_h = L^2 - 2N_d$ holes with the hole density being $\delta = N_h/L^2$. The configuration space consists of all the possible dimer coverings satisfying the hard-core constraint with a given number of holes. The energy of a dimer configuration $C$ is

$$E_C = \sum_{\mathbf{r}, \alpha = x, y} \left[ v n_\alpha(\mathbf{r}) n_\alpha(\mathbf{r} + 2\mathbf{e}_\alpha) + u n_\alpha(\mathbf{r}) n_\alpha(\mathbf{r} + \mathbf{e}_{\beta \neq \alpha}) \right] \tag{2.1}$$

where $n_\alpha(\mathbf{r})$ is the dimer occupation number of the link of the square lattice with endpoints at the two nearest neighbor sites $\mathbf{r}$ and $\mathbf{r} + \mathbf{e}_\alpha$ (with $\alpha = x, y$). The first term of the total energy $E_C$ of Eq. (2.1) is proportional to the number of parallel dimers on the same row (or column) on next-nearest-neighbor sites, while the second term is proportional to the number of plaquettes with two parallel dimers, $\beta = 1/T$ is the inverse temperature of the classical model and we assume that the interaction of parallel dimers on a plaquette is attractive, $u > 0$. Below, we will refer to the first term of Eq. (2.1) the dimer-aligning interaction (with strength $v$) and to the second term the plaquette interaction (with strength $u$).

The model with configurational energy $E_C$ of Eq. (2.1) is explicitly invariant under translations (by one lattice spacing) in both directions of the square lattice and under $C_4$ point group symmetries. However, in the special case $u = 0$, the configurational energy is also invariant under the independent rigid shifts of the dimer configurations on the links of each separate rows and columns. This discrete “sliding” symmetry formally relates dimer configurations with different tilts. As it stands, this symmetry formally leads to a sub-extensive (on the square lattice) entropy even at $T = 0$ which hinders the equilibration of this system for large lattice sizes. A discrete sliding symmetry of this type also arises in the 2D classical Ising model with only four spin interactions. However this symmetry can be broken by fixed boundary conditions to select a particular tilted state. This symmetry is also explicitly broken by the plaquette interactions even for arbitrarily small values of $u$. For this reason, it is important to focus on the case $u > 0$.

The partition function for a dimer model with no aligning interaction $v = 0$ was studied in detail both in the fully packed case and with a finite density of holes. In the fully packed regime it was found that, for $u \geq 0$, the system is characterized by a line of fixed points parametrized by $u = 1/T$. At temperatures below a Kosterlitz-Thouless (KT) transition at $u_c \approx 1.5$, the fully-packed dimer model enters a four-fold degenerate columnar phase which has a spontaneously broken translation and rotational symmetry. For any density of holes, the line of fixed points is replaced by a disordered phase, but the columnar phase survives, unless the density of holes is large enough.

The phase diagram in the presence of the aligning interaction $v$ as well as the plaquette interaction $u$ at various hole densities is complex. Apart from the already described disordered liquid and columnar solid phases, it is expected that a new phase emerges, which only breaks the rotational symmetry but not the translational one. The existence of this Ising nematic phase has been proven for the case $u = 0$ on dilute systems by Heilmann and Lieb. Here, we will explore the emergence of this phase primarily through numerical simulations, supplemented by theoretical arguments.

Intuition for this model can come from the fact that the close-packed square lattice dimer model has an effective field theoretic description in terms of a continuum Gaussian (free boson) field. The mapping of the square lattice classical dimer model to a height model proceeds by assigning a height variable on each plaquette. On the even sublattice, going clockwise, the height changes by $+3$ if a dimer is on a link, and by $-1$ when there isn’t one. On the odd sublattice, the changes reverse. Moreover, the dimer hard-core constraint implies that there are only 4 local dimers configurations on each site, guaranteed in the dual height model by the compactification $h \rightarrow h + 4$. In terms of the rescaled field $\phi = \pi h/2$ the
actual action is
\[ S = \int d^2 x \left( \frac{K}{2} (\nabla \phi)^2 + g \cos(4\phi) \right) \]  \hspace{1cm} (2.2)
with \( K = 1/4\pi \) at the non-interacting dimer point and with a charge 4 perturbation, \( \cos(4\phi) = \cos(2\pi h) \), biasing the coarse-grained height field to take integer values.

The different observables (and hence perturbations) of this theory consist of the \( O_{n,m} \) composite operators with \( n \) units of charge and \( m \) units of vorticity. Their scaling dimensions are
\[ \Delta(n, m)(K) = \frac{n^2}{4\pi K} + \pi Km^2 \]  \hspace{1cm} (2.3)
As shown in Ref.[4], the columnar order parameter corresponds to the elementary charge operator \( O_{\pm 1, 0} \), which is relevant with \( \Delta_{1,0} = 1 \), while the nematic order parameter corresponds to \( O_{\pm 2, 0} \) which is irrelevant at the free dimer point. The operator \( O_{\pm 4, 0} \), which naturally appears due to discreteness of the height values is a strongly irrelevant operator, but it is the one that drives the KT transition, when the plaquette attractive interaction is added. Moreover, the hole operator, corresponding to a hole density, is represented by the fundamental vortex operator \( O_0 \), a typically strongly relevant operator which drives the system to a high temperature liquid. Finally, by using OPE based on the density operator definitions, it is easy to show that the leading effect of the dimer-aligning interaction on the same row or the same column at the non-interacting dimer point is just a renormalization of the free field stiffness. This conclusion is verified in the simulations, since in the close-packed limit (and \( u \) infinitesimal) the columnar crystalline phase appears. On the dilute lattice, such a picture breaks down and the emergence of the phase diagram is not understood analytically.

### III. RESULTS FROM MONTE CARLO SIMULATIONS

We simulate the partition function of this dimer model with configurational energy \( E_C \) of Eq.(2.1) by means of a Monte Carlo worm-algorithm with a local heat-bath detailed balance condition. We have set the energy scale to be \( v = 1 > 0 \). The study is made in the canonical ensemble, similar to previous analogous studies.\[4,23,35,36\]

The observables, apart from the energy related specific heat, are related to the possible symmetry breaking, the columnar and nematic order. The local columnar order parameter \( c(\mathbf{r}) \) is defined with respect to the dimer occupation number at each site \( n_\alpha(\mathbf{r}) \),
\[ c(\mathbf{r}) = \sum_{\alpha=x,y} (-1)^\alpha n_\alpha(\mathbf{r}) \hat{e}_\alpha \]  \hspace{1cm} (3.1)
with the global order parameter being
\[ C = \frac{2}{L^2} \sum_\mathbf{r} c(\mathbf{r}) \]  \hspace{1cm} (3.2)
and is normalized such that the four columnar states correspond to
\[ C = \{ \pm 1, 0 \}, \{ 0, \pm 1 \}. \]  \hspace{1cm} (3.3)
A columnar state breaks the rotation and the translation symmetries of the lattice.

The rotation symmetry breaking (nematic) order parameter is defined to be
\[ R = \frac{2}{L^2} |N^h - N^v| \]  \hspace{1cm} (3.4)
where \( N^h \) and \( N^v \) are, respectively, the number dimers on horizontal and vertical bonds in configuration \( C \). One also considers the corresponding susceptibilities \( \chi \). In particular, for a second order transition
\[ \frac{\chi}{N} = \frac{\langle R^2 \rangle - \langle |R| \rangle^2}{L^3} \propto L^{2-\eta} \]  \hspace{1cm} (3.5)
while for a first order transition
\[ \frac{\chi}{N} \propto L^2 \]  \hspace{1cm} (3.6)
Finally, the Binder cumulant
\[ B = \frac{\langle R^4 \rangle}{\langle R^2 \rangle^2} \]  \hspace{1cm} (3.7)
is a scale-invariant quantity in the case of a continuous transition, and should thus exhibit a crossing point at criticality as a function of the system size. In addition, the finite size-scaling (FSS) of its derivative with respect to the temperature,
\[ \frac{dB}{dT} \propto L^{1/\nu} \]  \hspace{1cm} (3.8)
provides a direct way to determine the critical exponent \( \nu \) of the correlation length for nematic fluctuations.

#### A. Case \( u = 0, \delta = 0 \): attractive dimer-aligning interactions (but no plaquette interactions) on a close-packed dimer lattice

Firstly, we investigate the simplest case, being the case where only the dimer-aligning attractive interaction is present, \( u = 0 \) and \( v = 1 \), on a close-packed dimer lattice. In this limit the system has a discrete “sliding” symmetry which, for periodic boundary conditions leads to a sub-extensive ground state entropy. This feature indicates that the ground state may be any of the possible tilted phases, such as those discussed in Ref.[4]. This feature
FIG. 3. Columnar order parameter as a function of temperature for multiple system sizes in a close-packed dimer model without a plaquette interaction, $u = 0$ and zero hole density, $\delta = 0$. The columnar order vanishes quickly in the thermodynamic limit, with no signs of an instability.

poses serious numerical difficulties. Nevertheless, as we will see in Section III B these difficulties are suppressed once $u > 0$, which favors columnar order, no matter how small it is. Since in the case of $v = 0$ and $u > 0$ the transition from the four-old degenerate columnar order state is in the KT universality class, here we might expect the same scaling. However the existence of many tilted ground states (each with a four-fold global degeneracy) makes the analysis more difficult and convergence problematic.

Close-packed dimer configurations on the square lattice present critical correlations at $T = \infty$ as aforementioned. While from the field theory (and renormalization group arguments) we expect, as described, a renormalization of the KT-transition temperature toward a solid phase, which should resemble a columnar solid under appropriate boundary conditions on an infinite lattice, our simulations present a complex picture shown in Figs. 3, 4, and 5. From this data, it is reasonable to conclude that this is an Ising transition, given that the anomalous exponent of correlation function of the rotational order parameter (having a parent Ising symmetry) appears to be $\eta = 0.25$, consistent with both KT and with Ising universality, and that the columnar order is absent. On the other hand, the specific heat does seem to diverge is consistent with either a cusp or with a weak a logarithmic divergence, both of which have an exponent $\alpha = 0$. This may be consistent with an Ising transition, which has its prototypical logarithmic divergence. In a KT transition, the specific heat displays a peak clearly above the critical temperature (for example detected through the peak of the rotational susceptibility) and does not diverge; in our case, the specific heat is clearly non-divergent but the peak appears unconventionally close ($\pm 0.006$) to the
and we present strong evidence that the Ising signatures mask the true nature of the transition. The simulations were performed on double-periodic boundary conditions for reasons of numerical efficiency.

However, in this situation the sub-extensive entropy discussed above is not suppressed. Hence our simulations cannot distinguish a columnar state from any state with broken translational invariance but at finite tilt. Since the entropy is sub-extensive the associated discrete “sliding” symmetry must be spontaneously broken. However, performing fixed boundary condition simulations appears numerically intractable. For all these reasons we believe that our simulations with periodic boundary conditions have not converged at low temperatures for $u = 0$.

B. Case $u > 0, v > 0, \delta = 0$: Competition between attractive dimer-aligning and plaquette interactions on a close-packed dimer lattice

The next case study is for $u$ and $v$ present in the system for a close-packed lattice, but with $u = 0.1v$: since we aim to focus on the non-perturbative effects of the dimer-aligning interaction. It is clear that in the opposite limit $v \ll u$ the dimer-aligning perturbation would only shift the location of the KT transition with no change in the universality class. However, in the limit we probe $u \ll v$, the system’s behavior is rather complex, exactly due to the non-trivial crossover that we find in the case where $u = 0$. While the maximum system size we studied is many times larger than the sizes studied in Ref. 8, it appears still that the system has not converged to a well defined transition. The expectation is that the system should display a KT transition since the sub-extensive entropic degeneracy is lifted for any finite value of $u \neq 0$. In Figs. 4, 5, 10 and 11 we present strong evidence that this transition has not converged. The rotational order parameter displays a clear transition with its susceptibility displaying a large peak with scaling similar to Ising transitions ($\eta = 0.25$), and its Binder cumulant...
FIG. 10. Specific heat for different system sizes displaying a divergence with an exponent $\alpha/\nu \approx 0.5$ for a close-packed dimer model with dimer-aligning, and plaquette interactions at $v = 0$.

FIG. 11. Binder cumulants for the columnar (top) and nematic (bottom) order parameters in a close-packed dimer model with dimer-aligning interactions and plaquette interactions, at zero hole density. Here, $u = 0.1$ and $v = 1.0$.

C. Case $u = 0$, $\delta > 0$: Only dimer-aligning interaction on a dilute dimer lattice and the emergence of an Ising phase transition towards a nematic fluid

Table I. Critical behavior at the nematic-disordered phase transition line in the absence of a plaquette interaction, $u = 0$. From Left to Right the columns denote: hole density, critical temperature, specific heat exponent ratio $\alpha/\nu$, rotational order parameter anomalous exponent $\eta$, correlation length exponent $\nu$ and finally, the value of the Binder cumulant at the crossing point. The Ising universality class prescribes the value 1.162 for this quantity.
FIG. 12. Nematic (top) and columnar (bottom) order parameters in a dimer model with only dimer-aligning interactions. The plaquette interaction is $u = 0$ and the dimer-aligning interactions $v = 1$. The hole density is $\delta = 0.03125 = 1/32$. Top: the nematic order parameter as a function of temperature for increasing system size, displaying a strong transition. Bottom: the columnar order parameter, clearly vanishing in the thermodynamic limit.

FIG. 14. Temperature dependence of the specific heat of a dimer model with only dimer-aligning interactions, with $v = 1$. The plaquette interaction is absent, $u = 0$. The hole density is $\delta = 0.03125 = 1/32$. The temperature dependence of the specific heat displays a clear phase transition with corresponding exponent from the peak scaling $\alpha/\nu \simeq 0.1 \pm 0.1$, consistent with the expected logarithmic singularity ($\alpha = 0$) at a 2D Ising transition.

FIG. 15. Binder ratio crossings for the nematic order parameter of a dimer model with only dimer-aligning interactions, with $v = 1$, at finite hole density. The plaquette interaction is not included, $u = 0$. The hole density is $\delta = 0.03125 = 1/32$. The observed crossing point is consistent with the expected 2D Ising transition and its derivative at the crossing gives clearly $\nu \simeq 1$.

At finite small dilution but no plaquette interaction $u = 0, v = 1$, we find a clear transition line that separates the isotropic hole-dimer liquid phase from the Ising nematic phase. This transition is clearly in the Ising universality class, as verified by the values of the anomalous exponents as well as the Binder cumulant crossing values, being in consistency with the Ising universality. The evidence is presented in Figs. 12, 13, 14 and Table I. While in the figures we show a typical case at dilution $\delta = 1/32$ and the consistency shown by the order parameters, their susceptibilities, the specific heat and the Binder cumulants, at the table we show our estimations for particular critical exponents and universal cumulant values for a number of hole densities we studied, displaying a strong consistency. For $\delta = 1/32$, a clear phase
we show how the nematic correlations evolve with system size and hole density at a low temperature $T = 0.5$. As can be seen from Fig. 13, at this temperature and all but the close-packed case, the system is an Ising nematic liquid which is stable to perturbations (such as the plaquette interaction). Through the scaling with the system size, it is clear that at long distances dimers have liquid correlations, albeit double than at infinite temperature, due to the fact that deep in the nematic phase they are primarily oriented in the same direction. The correlations have clearly converged for $L \geq 64$. As a function of hole density, we observe the dramatic effect of having non-decaying correlations at $\delta = 0$, signifying the absence of liquid correlations, but they immediately become liquid-like as the system gets dilute. Starting from the perfect close-packed crystal (at zero dilution, cf. Fig. 2C), assuming fixed boundary conditions, then to first order, if some dimers $n$ (introducing $2n$ holes) are removed from the lattice, then there are $\binom{N}{n}$ ways to remove them, therefore contributing an extensive entropy without though destroying the columnar order. Further, to second order, at the expense of energy $1/T$, each pair of holes has a sub-extensive contribution by moving freely on the line it was created, again winning over the crystalline confining tendency. Since a pair can be formed on each line, it leads to an extensive entropy contribution as well, that can dominate over the energy cost. Therefore, the nematic liquid is also deconfining along the oriented direction.

We should note that the Ising transition line, should end at a tricritical point and coexistence at a higher $\delta$ from the ones we investigated. The investigation of this discontinuous transition is beyond the scope of this work, but should be in the tricritical Ising universality class.

Finally, in Figs. 16, 17, we show how the nematic correlations evolve with system size and hole density at a low temperature $T = 0.5$. As can be seen from Fig. 13, at this temperature and all but the close-packed case, the system is an Ising nematic liquid which is stable to perturbations (such as the plaquette interaction). Through the scaling with the system size, it is clear that at long distances dimers have liquid correlations, albeit double than at infinite temperature, due to the fact that deep in the nematic phase they are primarily oriented in the same direction. The correlations have clearly converged for $L \geq 64$. As a function of hole density, we observe the dramatic effect of having non-decaying correlations at $\delta = 0$, signifying the absence of liquid correlations, but they immediately become liquid-like as the system gets dilute. Starting from the perfect close-packed crystal (at zero dilution, cf. Fig. 2C), assuming fixed boundary conditions, then to first order, if some dimers $n$ (introducing $2n$ holes) are removed from the lattice, then there are $\binom{N}{n}$ ways to remove them, therefore contributing an extensive entropy without though destroying the columnar order. Further, to second order, at the expense of energy $1/T$, each pair of holes has a sub-extensive contribution by moving freely on the line it was created, again winning over the crystalline confining tendency. Since a pair can be formed on each line, it leads to an extensive entropy contribution as well, that can dominate over the energy cost. Therefore, the nematic liquid is also deconfining along the oriented direction.

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D. Case $\alpha > 0, \nu > 0, \delta > 0$: Competition between dimer-aligning and plaquette interactions in a dimer model at finite hole density displaying two Ising phase transitions

Finally, we investigate the case of a finite hole density, where the system contains both the dimer-aligning and plaquette interactions. In this case, we observe a more complex phase diagram, where two Ising transition lines emerge from the close-packed limit (where the transition is in the KT universality class possibly, as we argued) (shown in Fig. 2B), separating the Ising nematic dimer liquid from the disordered isotropic dimer-hole liquid at high temperatures and the columnar solid phase at low temperatures. The evidence for our conclusion that both transitions are in the Ising universality class are displayed in Figs. [18] [19] [20] [21] [22] [23] for a particular hole density.
FIG. 18. Evolution of the columnar and the nematic order parameters as a function of temperature in dimer model at finite hole density with both dimer-aligning, and plaquette interactions. Here $u = 0.1$ and $v = 1$. The hole density is $\delta = 0.0625 = 1/16$. Two Ising phase transitions are observed as temperature decreases. The columnar order parameter shows a distinctly different transition temperature from the nematic order. The isotropic-nematic transition takes place at a higher temperature due to the presence of the attractive columnar interaction.

FIG. 19. Specific heat for a dimer model with both dimer-aligning, and plaquette interactions at finite hole density with both dimer-aligning, and plaquette interactions. The plaquette coupling is $u = 0.1$ and $v = 1$. The hole density is $\delta = 0.0625 = 1/16$. The specific heat shows a signature for both transitions, but the lower temperature transition’s peak is much weaker, between the columnar solid and the Ising nematic liquid, converging much slower as the system size increases.

FIG. 20. Columnar order susceptibility of the dimer model with both dimer-aligning and plaquette interactions at finite hole density. The plaquette coupling is $u = 0.1$ and the two (and column) coupling are $v = 1$. The hole density is $\delta = 0.0625 = 1/16$. The susceptibility of the columnar order parameter displays a strong divergent peak with Ising exponents at the lower transition temperature.

FIG. 21. Susceptibility for the nematic order parameter in a dimer model with both dimer-aligning and plaquette interactions at finite hole density. The plaquette coupling is $u = 0.1$ and $v = 1$. The hole density is $\delta = 0.0625 = 1/16$. The susceptibility of the nematic order parameter displays a strong divergent peak with the Ising exponents at the higher transition temperature at $T = T_N \approx 0.8$. The weakness of the low temperature transition can be further observed in the behavior of the specific heat, which displays a strong peak (with a very weak divergence, as expected for Ising transitions) at $T_N$ but a very weak peak at $T_C$. The columnar order susceptibility displays a strong peak at $T_C$ with a peak which shows an anomalous exponent $\eta \sim 0.2$, lower than the Ising-expected $1/4$, but expected given the weak convergence we observe throughout for this transition with the system size. However, the
nematic-disordered transition, where there is a strong crossing at $1.13$, which given our definitions, is the universal value expected for the Ising universality class. Further, the cumulant’s derivative scales in a consistent way with the Ising expectations. However, the columnar order’s cumulant displays a complex behavior with two crossings at both transitions, since the columnar order parameter displays fluctuations sensitive at both transitions. The lower transition’s Binder ratio displays a crossing around $\sim 2.1$ that appears to drift toward lower value and the Binder ratio derivative displays Ising-like scaling (with $\nu \sim 1$). We believe that the Binder ratio has not yet converged to the Ising value for finite-size and finite-anisotropy reasons. Namely, the columnar order parameter is defined in a $2 \times 1$ unit cell while the simulation box has always been square $L \times L$. In studies of the Binder ratio for the square-lattice Ising model on rectangular boxes, it was shown that the critical Binder cumulant value had a strong dependence on the system’s aspect ratio. We expect a strong dependence also in our case, and simulations in thin strips are expected to show a much faster convergence for this transition.

IV. MEAN-FIELD THEORY

The understanding of this Ising nematic phase transition can be elucidated also through a mean-field study, similar to the approach followed in Ref.\[4\]. Using a Grassmann representation for the dilute dimer model and performing two Hubbard-Stratonovich transformations, with the exact same field definitions, we find the following effective thermodynamic potential,

$$\Gamma(z, V) = \frac{V}{4} \sum_{ijkl} m_{ij} \tilde{M}_{ijkl} m_{kl}$$

$$+ \sum_{ij} n_i \left( \frac{V}{2} \sum_{kl} \tilde{M}_{ijkl} m_{kl} + \frac{z}{2} M_{ij} \right) n_j$$

$$- \sum_i \ln \left[ 2 \sum_j \left( \frac{V}{2} \sum_{kl} \tilde{M}_{ijkl} m_{kl} + \frac{z}{2} M_{ij} \right) n_j + 1 \right]$$

(4.1)

where $m_{ij}, n_i$ are order parameters coupled to neighboring same-axis aligned dimers and hole density respectively, while $z = e^{\beta \mu}$ and $V = z^2(e^{\beta} - 1)$ where $\beta = 1/T$ and $\mu$ the hole chemical potential. Further, $\tilde{M}_{ijkl}$ is non-zero and unity when $i, j, k, l$ are neighboring sites on the same lattice line, i.e. $M_{ij}$ is non-zero and unity when $ij$ are nearest neighboring sites. The only difference with the study of Ref.\[4\] is that the attractive neighboring interaction is in a different direction (aligning instead of plaquette).

We use the ansatz $n_i = n$ and $m_{ij} = (-1)^i m$ and we
look for solutions of the extremal equation
\[ \frac{\partial \Gamma}{\partial m} = 0, \quad \text{with} \quad \beta z \frac{\partial \Gamma}{\partial z} = n \] (4.2)

The equations can be either solved numerically or analytically, with the latter being in an expansion in \( z \) (small dimer density) and then, \( \beta \) (high temperature), giving finally the leading term for the critical hole density at high temperatures to be
\[ \delta_c(T) = 2\left(1 + \beta - \sqrt{1 + 2\beta}\right)/\beta = \beta + O(\beta^2) \] (4.3)

leading to the dashed line shown in Fig. 24.

![Graph showing phase transition](#)

**FIG. 24.** Mean-Field Approach to the Phase diagram in the \( u = 0, v = 1 \) case. The thick dashed line describes a mean-field estimate from an associated mean-field theory discussed in the text. The points signify the transition line, as estimated from numerical simulations.

### V. CONCLUSIONS

In this paper we examined the phase diagram of a 2D dimer model on the square lattice at zero and finite hole densities. The important difference between this work and previous studies is the consideration of the interplay of attractive interactions on a plaquette (which was considered previously) with attractive (aligning) dimer interactions on the same row (and column) for dimers on next-nearest neighboring bonds of the lattice. By means of large-scale Monte Carlo simulations, combined with a scaling analysis and a mean-field theory we find that the phase diagram of this simple model, shown in Fig. 2 in addition to a line of fixed points at zero hole density, generally contains three phases: a dimer-hole liquid, a columnar phase (valence bond solid) and a novel Ising nematic phase. We presented detailed results for the scaling behavior of the columnar and nematic (orientational) order parameters, specific heat and Binder cumulants at the various transitions. In the general case our results are consistent with two Ising transitions as the interaction and hole density are varied.

These results are of interest to both the understanding of classical dimer models and to quantum dimer models which arise in the context of frustrated quantum magnets. In the latter context the theory we presented here describes the behavior of RVB-type wave functions at finite hole density. All phases described here are non-magnetic as the dimers represent valence bond singlets.

Experimentally relevant models of nematic phases should have a columnar phase dominating at low temperatures, consistent with strong experimental evidence in cuprate and iron pnictides materials. In this work, we considered a framework that includes geometric frustration originating in the hard-core dimer exclusion effect on the square lattice. We find that exactly because of the geometric frustration, a nematic phase at low temperatures is always unstable to columnar fluctuations, due to entropic reasons, implying that negligible amounts of the plaquette interaction \( u \) yields a columnar phase at low temperatures. It would be very interesting to investigate similar phase diagrams on different lattices such as the kagome, where a \( Z_2 \) spin-liquid has been identified as the ground-state of the Heisenberg antiferromagnet\(^{41}\) pointing toward useful dimer descriptions for the ground-state wavefunction\(^{42,43}\).

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