Electronic transport and the related anomalous effects in silicene-like hexagonal lattice

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We investigate the anomalous effects due to the Berry correction and the considerable perturbations in the silicene-like hexagonal lattice system. The Berry curvature in periodic Bloch band system which related to the electromagnetic field is explored, the induced transverse anomalous velocity gives rise to the intrinsic Hall conductivity (without the vertex correction) especially in the quantum anomalous Hall phase. The quantum anomalous Hall effect which related to the anomalous velocity term is detected, including the band avoided crossing effect and the generated special band gap. The topological spin transport is affected by the Berry curvature and the spin-current-induced Skyrmion spin texture motion is contrasted between the quantum spin Hall effect and quantum anomalous Hall effect. Since silicene involving the orbital degree of freedom, the orbital magnetic moment and orbital magnetization contributes significantly to the electronic transport properties of silicene as explored in this article. We also investigate the electronic tunneling properties of silicene in Josephson junction with the electric-field-induced Rashba-coupling, the anomalous effect due to the Berry phase is mentioned. Our results is meaningful to the application of the spintronics and valleytronics base on the silicene-like topological insulators.

1 Introduction

In this paper, we investigate the anomalous effects induced by the Berry correction (semiclassical correction) to the electronic transport (tunneling) of the non-relativistic electrons (or quasiparticles) in silicene. The non-relativistic case origin from the small Fermi velocity compared to the speed of light, which is $v_F = 5.5 \times 10^5 \approx c/500$ for silicene and smaller than that of graphene which is $v_F \approx c/300$. There is also a distinction between the relativistic and non-relativistic case: The main parameter during the semiclassical motion of particles is momentum (quasimomentum) for non-relativistic case, while it is wavelength for relativistic case. Due to the semiclassical correction to the equation of motion of particles, the trajectory of particle is distincted from the traditional one, especially when it’s under an electromagnetic field or a circular light field, which are related to the Berry curvature (Berry gauge field). Moreover, artificial gauge field has been successfully created base on the non-Abelian Berry phase in the nonadiabatic case[1]. The unconventional semiclassical motion with nonzero Berry curvature also leads to the topological nontrivial spin transport as well as the intriguing spin/valley Hall conductivity in the topological insulators like silicene or MoS$_2$ or germanene. The topological spin transport as well as the momentum of center-of-mass of the wave package are affected by the applied electric field, magnetic field, and the off-resonance circularly polarized light. The anomalous velocity due to the Berry curvature (including a Lorentz-like term) shifts the

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electrons in the direction transverse to the electric field and magnetic field (which is also the direction of the spin accumulation by the spin Hall effect\cite{2}), and gives rise to the spin or valley (transverse) Hall conductivity, while the circularly polarized light results in the opposite-spin configuration of the neighbor valleys\cite{3} which will affects the spin polarization and charge current, and the motion of electrons is along the applied fields which can be obtained through the expression of $\hbar \partial_t \mathbf{k}$ as we state in this article. The electron and hole are be endowed with the velocities in opposite direction and thus give rise to the Hall conductivity in the presence of longitudinal charge current. The spin accumulation caused by spin current in quantum spin Hall phase is orthogonal to the electric field and the charge current, note that the spin current here is unlike the one caused by the conduction electron flow which is along the direction of applied electric field, thus it’s recently observed\cite{2} experimentally for the anomalous motion of skyrmion carried by the spin current in quantum spin Hall phase. Base on the optical spin-valley-coupled selection rule, this spin accumulation can be facilitated by the applied circularly polarized light (the so-called photo-current-induced spin accumulation\cite{4}) or the magneto-optic Kerr effect.

The largest difference between the effective Dirac Hamiltonian in relativistic case and non-relativistic case is the emergence of the Zeeman-like exchange field term and the (intrinsic and external) spin-orbit coupling (SOC) term, and the classical mass is replaced by the Dirac-mass term (or the mass about the interlayer hopping and intralayer hopping for the bilayer silicene\cite{5}). It has also been proved experimentally\cite{6} early that the SOC-induced Berry geometric phase affects deeply the quantum transport (like the spin or valley) just like the Aharonov-Anandan geometric phase induced by magnetic field. The intrinsic spin or valley Hall conductivity (or polarization)\cite{7} is also related to the Berry curvature which induce the anomalous electron motion under the electromagnetic field. Intriguingly, the intrinsic quantum spin Hall phase in silicene can be changed to the trivial insulator with obvious variance of conductivity under high external gate voltage\cite{8}. The anomalous velocity term also related to the quantum anomalous Hall effect\cite{9}, which with a special nonzero Chern number and an anticrossing band-induced gap as we explored in this paper. We also found that the anomalous Hall effect can be generated by the exchange field and the electric field. The Berry phase is just like the Aharonov-Bohm phase induced by the magnetic field\cite{10, 11}, while for the case that only have the electric field, it cause a topological spin transport which induce a spin current\cite{11} like the quantum spin Hall effect.

Distinct to the Dirac fermions (nonrelativitic when $m_D^{\sigma_z \tau_z} \neq 0$) in the low-energy tight-binding midel as shown in Appendix.A, the Bulk effective Hamiltonian of the weyl semimetal or three-dimension topological insulator requires the nonzero longitudinal momentum $k_z$ and quadratic dispersion $k^2$ which for silicene or graphene (hexagonal quantum spin Hall materials) can emerges only for bilayer or multilayer form. Here the linear $k_z$-dependence rised with the increase of overlap between the $p_z$ orbit of atoms, in the case of $k_x = k_y = 0$. In the mean time, nonzero Berry curvature (like the case of quantum anomalous Hall effect) also leads to the anomalous velocity\cite{12, 13} (in semiclassical correction of solid), and it’s related to the Anomalous Rabi oscillation in the Dirac-Weyl Fermionic systems\cite{14} as well as the dispersion of the Andreev bound state\cite{15, 16, 17}. The anomalous Rabi oscillation frequency is unlike the conventional Rabi frequency, its related to the Chern number: When the Chern number of silicene (or for other topological insulator system) is zero, then the induced (by anomalous Rabi oscillation frequency) mass is nonzero for this trivial system and thus with the gapped edge states; while when the Chern number is nonzero, the induced mass is zero and corresponds to the non-trivial system which with the gapless edge states. While for the Andreev bound states, It becomes gapless for the topological state (or with an infinitesimal gap like the topological band with nonzero Chern number). Consider the amplitude of the envelope function of the two sublattices, the quasienergy of silicene can also be described by the Weyl function as
\( \varepsilon = \gamma (\sigma \cdot \mathbf{k}) \), with the band parameter \( \gamma = 2k_F/(\pi \Pi(\mathbf{q},0)\hbar)(\sim v_F \) for monolayer silicene with large carriers density, and \( \Pi(\mathbf{q},0) \) is the dynamical polarization (which is inversely proportional to the static polarization function).

2 Low-energy tight-binding model

The time-dependent Dirac Hamiltonian in tight-binding model of the monolayer silicene under time-dependent vector potential reads\[17 \ 15\]

\[
H(t) = \hbar v_F (\eta \tau_x P_x(t) + \tau_y P_y(t)) + \eta \lambda_{\text{SOC}} \tau_z \sigma_z + a \lambda_{R_z} \eta \tau_z (P_y \sigma_x - P_x \sigma_y)
- \frac{\Delta}{2} E_\perp \tau_z + \frac{\lambda_{R_1}}{2} (\eta \sigma_y \tau_x - \sigma_x \tau_y) + M_e s_z + M_c - \eta \tau_z \hbar v_F^2 A \Omega + \mu,
\]

where \( P_x(t) = k_x - eA_x(t) = k_x - eA \sin \Omega t \). \( E_\perp \) is the perpendicularly applied electric field, \( a = 3.86 \) is the lattice constant, \( \mu \) is the chemical potential, \( \Delta \) is the buckled distance between the upper superlattice and lower superlattice, \( \sigma_x \) and \( \tau_z \) are the spin and pseudospin (sublattice) degrees of freedom, respectively. \( \eta = \pm 1 \) for K and K’ valley, respectively. \( M_e \) is the spin-dependent exchange field and \( M_c \) is the charge-dependent exchange field. \( \lambda_{\text{SOC}} = 3.9 \) meV is the strength of intrinsic spin-orbit coupling (SOC) and \( \lambda_{R_z} = 0.7 \) meV is the intrinsic Rashba coupling which is a next-nearest-neighbor (NNN) hopping term and breaks the lattice inversion symmetry. \( \lambda_{R_1} \) is the electric field-induced nearest-neighbor (NN) Rashba coupling which has been found that linear with the applied electric field in our previous works\[19 \ 17 \ 18 \ 20 \ 5\], which as \( \lambda_{R_1} = 0.012E_\perp \). Note that we ignore the effects of the high-energy bands on the low-energy bands.

Due to the perpendicular electric field \( E_\perp \) and the off-resonance circularly polarized light which with frequency \( \Omega \gg 1000 \) THz, the Dirac-mass and the corresponding quasienergy spectrum (obtained through the diagonalization procedure)

\[
\varepsilon = \frac{m_D^{\sigma_z \tau_z} = |\eta| \sqrt{\lambda_{\text{SOC}}^2 + a^2 \lambda_{R_z}^2 k^2 s_z \tau_z - \frac{\Delta}{2} E_\perp \tau_z + M_e s_z - \eta \hbar v_F^2 A \Omega|}}{s \left( a^2 \lambda_{R_z}^2 k^2 + \left( \hbar v_F^2 k^2 + (\eta \lambda_{\text{SOC}} s_z \tau_z - \frac{\Delta}{2} E_\perp \tau_z - \eta \hbar v_F^2 A \Omega)^2 + M_e s_z + s\mu \right)^2 \right)^{1/2}},
\]

respectively, where the dimensionless intensity \( A = e A a / \hbar \) is in a form similar to the Bloch frequency, and \( s = \pm 1 \) is the electron/hole index, and the subscript e and h denotes the electron and hole, respectively. Note that here this Dirac-mass is correct for exchange field \( |M_s| \leq \lambda_{\text{SOC}} (1 + a^2 \lambda_{R_z}^2 / \hbar^2 v_F^4) \), and the resonance will be presented in the following. The off-resonance circularly polarized light results in the asymmetry band gap in two valleys (see Ref.\[5\]) and breaks the time-reversal symmetry in the mean time, and thus provides two pairs of the different incident electrons that may leads to the Josephson current reversal due to the valley-polarization. The effects of the exchange-field-term \( M_e s_z \) and \( M_c \) are presented in the Fig.1(a)-(c), where we can see that the \( M_e \) which is spin-dependent generally close the gap and shift the band with up-spin and down-spin upward and downward, respectively, while the charge-dependent \( M_c \) just move the whole band structure upward but does not breaks the spin degenerate which shows it may related to the valley degree of freedom \( \eta \). It’s obviously that the avoided crossing effect emerges and it is enhanced with the increase of electric field (or magnetic field), that also contributes to the Chern number by the Skyrmion spin texture as we discuss below. Here we note that the motion of the Skyrmion spin texture here carried by the spin current due to the angular momentum conservation\[2\] is rather weak than the one in the bulk sample, and the
Skyrmion is also much more stable in the quantum anomalous Hall phase which with chiral edge than that in the quantum spin Hall phase which with helical edge. The avoided crossing effect also observable in the Floquet system under the electromagnetic field\cite{21}. For the band structure that both the intrinsic SOC and the NNN electric field-induced Rashba-coupling $\lambda_{R_1}$ are taken into account, the symmetry between the conduction band and valence band is broken, as shown in Fig.1(d), and the band splitting between the two conduction bands is decreased while that of the two valence bands is increased. Due to the spin mixing by the extrinsic Rashba-coupling $\lambda_{R_1}$, the band splitting here is no more the spin-splitting but related to the index $(\alpha, \beta)$.

$$\varepsilon = a^2 \lambda_{R_2} k^2 + (\alpha \lambda_{R_1} + \beta \sqrt{\frac{\hbar^2 \nu_F}{2} k^2 + (\eta \lambda_{R_1} - \alpha \lambda_{soc}) s_z \tau_z - \frac{\Delta}{2} E_{\perp} \tau_y - \eta \hbar \nu_F A^2} + M s_z + s \mu)^2, \quad (3)$$

Through Fig.1(d), we can see that the energy splitting of band structure is controlled by the index $(\alpha, \beta)$, and the configuration of spin helical is symmetry between the conduction band and valence band. Such phenomenon also emerges for the case of off-resonance light to the MoS$_2$ \cite{23}, which has a much larger intrinsic SOC than graphene or silicene, and in the mean time, due to the $\eta$-dependent optical term in Eq.1, the valley asymmetry is rised (see \cite{17, 5}) and results in the possible 100% valley polarization with the almost pure valley transport like a valley filter. The reduction of the Dirac-mass is also accompanied by the rise of the longitudinal conductivity\cite{2}. In addition, the off-resonance light also enhance the difference of the orbital magnetic moment between two inequivalent valleys, as we discuss in following text.

In the presence of vertical electric field as well as the first-order and second-order Rashba-coupling, the system of silicene can be described by Hamiltonian $H = \Psi^\dagger H_{eff}^\pm \Psi / 2$ in the low-energy Dirac theory; with the two-component spinor-valued field operators

$$\Psi = [(\psi_+^A, \psi_+^B, \psi_-^A, \psi_-^B), ((\psi_+^{A\dagger}, \psi_+^{B\dagger}, \psi_-^{A\dagger}, \psi_-^{B\dagger})]^T. \quad (4)$$

The BCS-like effective Hamiltonians $H_{eff}$ in the basis of $\{\tau \otimes \sigma\} \text{ read}$

$$H_{eff}^K = \begin{pmatrix}
    m_{D}^{++} & \hbar \nu_F (k_x - i k_y) & ia \lambda_{R_2} (k_x - i k_y) & 0 \\
    -i a \lambda_{R_2} (k_x + i k_y) & m_{D}^{+-} & -i \lambda_{R_1} & -i a \lambda_{R_2} (k_x - i k_y) \\
    0 & ia \lambda_{R_2} (k_x + i k_y) & m_{D}^{-+} & \hbar \nu_F (k_x + i k_y) \\
    0 & 0 & \hbar \nu_F (k_x + i k_y) & m_{D}^{--}
\end{pmatrix}, \quad (5)$$

$$H_{eff}^{K'} = \begin{pmatrix}
    m_{D}^{++} & \hbar \nu_F (k_x - i k_y) & ia \lambda_{R_2} (k_x - i k_y) & 0 \\
    -i a \lambda_{R_2} (k_x + i k_y) & m_{D}^{-+} & 0 & -i \lambda_{R_1} \\
    -i a \lambda_{R_2} (k_x - i k_y) & -i a \lambda_{R_2} (k_x + i k_y) & m_{D}^{--} & \hbar \nu_F (k_x + i k_y) \\
    0 & 0 & \hbar \nu_F (k_x + i k_y) & m_{D}^{--}
\end{pmatrix}. \quad (6)$$

While for the bilayer silicene with the NN interlayer hopping $t' = 2$ eV and we ignore the NNN interlayer hopping. The interlayer SOC is estimated as 0.5 meV here\cite{23} and since the trigonal warping term between two layers has a non-negligible impact when apply the off-resonance light in terahertz range\cite{24}, we set the trigonal warping hopping parameter as $t_w = 0.16$ eV here. Then the low-energy Dirac effective model can be written in a matrix form\cite{5}:

$$H_{eff}^{bi} = \eta \begin{pmatrix}
    m_{D}^{++} - \eta \hbar \nu_F A_4 \frac{A\Omega}{\nu_F} & \hbar \nu_F (k_x + i k_y) & 0 & \hbar \nu_F (k_x - i k_y) \\
    \hbar \nu_F (k_x - i k_y) & m_{D}^{+\text{\prime}+} + \eta \hbar \nu_F A_4 \frac{A\Omega}{\nu_F} & \hbar \nu_F (k_x + i k_y) & 0 \\
    0 & \hbar \nu_F (k_x - i k_y) & m_{D}^{\text{\prime}++} & \eta t' \\
    \hbar \nu_F (k_x + i k_y) & 0 & \eta t' & m_{D}^{\text{\prime}++} 
\end{pmatrix}, \quad (7)$$
where \( v_w = \sqrt{3at_w/2\hbar} \) is the velocity associates with the trigonal warping\[5\]. The valley symmetry is broken due to the trigonal warping term here, which may leads to the single-Dirac-cone state, that implies that the light in a finite intensity has the same effect with the out-of-plane antiferromagnetic exchange field. Since the time-reversal-invariance is broken by the off-resonance light, the valley symmetry is broken and the up-spin and down-spin flow with different velocities in each edge, and that is the fundament of the realization of single-Dirac cone state\[5\].

Through the procedure of block diagonalization as mentioned in Ref.\[25\], the above matrices can be simplified as\[5\]

\[
H_{\text{eff}} = \begin{pmatrix}
\nu F(k_x + ik_y) & \hbar v_F(k_x - ik_y)e^{-i\theta} \\
\hbar v_F(k_x + ik_y)e^{i\theta} & -\nu F(k_x + ik_y)
\end{pmatrix},
\]

with

\[
\hbar v_F k = \hbar v_F \begin{pmatrix} 0 & \kappa_x - ik_y \\ \kappa_x + ik_y & 0 \end{pmatrix} = \pm t|1 + e^{i\kappa_x} + e^{-i\kappa_y}|,
\]

where \( \theta \) is the angle between \( \mathbf{k} \) and \( \kappa_x \). \( \kappa_x + ik_y = \hat{s}_{AB} \cdot \mathbf{k} \)\[26\][27\]. Here we comment that the exchange field for the out-of-plane polarization is much larger than the in-plane one. Note that we don’t consider the \( \lambda_{R_1} \) and \( \lambda_{R_2} \) term in the above Dirac-mass. While for the bilayer silicene, the simplified matrix becomes

\[
H_{\text{eff}}^b = \begin{pmatrix}
\nu F(k_x + ik_y)^2 e^{2i\theta} & \frac{\hbar^2}{2m^*}(k_x - ik_y)^2 e^{-2i\theta} \\
\frac{\hbar^2}{2m^*}(k_x - ik_y)^2 e^{2i\theta} & -\nu F(k_x + ik_y)
\end{pmatrix},
\]

with

\[
\frac{\hbar^2 k^2}{2m^*} = \frac{k^2t^2}{t'} = \frac{\hbar^2}{2m^*} \begin{pmatrix} 0 & (k_x - ik_y)^2 \\ (k_x + ik_y)^2 & 0 \end{pmatrix} = \pm t' \pm \sqrt{\left(\frac{t'}{2}\right)^2 + t'^2|1 + e^{i\kappa_x} + e^{-i\kappa_y}|^2},
\]

where \( t' \) is the interlayer hopping, and that can be easily deduced by \( m^* = \hbar^2t'/(2t^2) \sim 1/v_F^2 \) and \( \hbar v_F = \sqrt{\frac{3}{2}} at \). Note that the \( m^* \) here is much smaller than the free electron mass, and it’s related to the interlayer and intralayer hopping and the velocity of these hopping is much slower than the speed of light thus give rises the non-relativity effect as we introduced at the beginning. Due to the possible quadratic dispersion in the energy band bottom, the momentum-independent effective mass reads \( m^*_{\text{bottom}} = \frac{4m^*_{\nu F} v_F^2}{\hbar^2 t^2} \). The above expression results in the four band structure in the spin degenerate case for the silicene bilayer.

The momentum above can be replaced by the canonical (covariant) momentum through the minimal substitution as \( k_x \rightarrow P_x - \frac{e}{c}A_x \) where \( A_x \) is the \( x \)-component of the vector potential \( \mathbf{A} \). For relativitic particle, the canonical momentum satisfies \( P = -i\hbar \partial_c \mathcal{E} \) since \( \partial_c = -\frac{i}{\hbar} \partial [11] \), and it’s useful in the block diagonalization of Dirac equation as well as the use of BMT equation. Since we applying the magnetic field perpendicular to the silicene, i.e., \( \mathbf{B} = \nabla \times \mathbf{A} = (0, 0, B_z) \), with the Landau gauge \( \mathbf{A} = (-Bz, y, 0, 0) \), and since the momentum can be replaced by the covariant one in Peierls phase, the ladder operators satisfy \( [\mathbf{P}, \mathbf{P}^\dagger] = 1 \), with \( \mathbf{P} = \hbar(y\ell_B - \ell_B k_x + \partial_y) \) and \( \mathbf{P}^\dagger = \hbar(x\ell_B - \ell_B k_x - \partial_y) \).

### 3 Berry curvature with external electromagnetic field

In the presence of scattering by the charged impurity (spin-orbit scattering) with a scattering potential larger than the lattice constant\[25\] and the strong SOC (compared to the graphene or black phosphorus), the elastic back-scattering which with the conserving spin is suppressed since the spin rotates as the wave vector changes direction during the scattering (in fact, the
spin is always in the direction of the wave vector due to the helicity of the dirac equation in the case of time-reversal invariant). In the presence of time-reversal-invariance, the spin-momentum locking can be observed. That also implies that, for the quadratic edge state dispersion, the back-scattering is only possible in low-energy region where the difference between spin directions is $\ll 180^\circ$. As the Bloch wave vector undergoes a rotation around the whole pseudospin space, the Berry phase of monolayer silicene which is gauge invariant (in contrast to the anomalous case with the perturbations) can be obtained by the integral of the time within the period of a full rotation (a loop winding around the K-point)

$$\Gamma = -i \lim_{\phi \to 0} \int_0^{1+(\phi)} \frac{1}{t} dt \frac{2\pi}{\phi} = \pi,$$

with the position-dependent phase $\phi = \arctan \frac{k_{i+1} - k_i}{k}$. For bilayer silicene, the Berry phase can be obtained as $2\pi$ through the similar procedure. In fact, for the adiabatic case without the perturbations, the trivial Berry phase is constant and the $e^{i\Gamma}$ is gauge-invariant, which is equivalent to the symmetry case. Note that here we don’t consider the SOC, and thus the gauge invariance is obtained through the winding number of pseudospin in momentum space. While the SOC will slightly reduce the Berry phase in monolayer silicene and enlarge the Berry phase in bilayer silicene.

For the wave vector rotates as a function of time in an adiabatic way, $\mathbf{k}(t = 0) = \mathbf{k}(t = T)$, the berry curvature, which well describe the local property of the band structure for adiabatic transport, can be obtained as a triple integral with the second-rank tensor field, $\Omega_i(k)$, which well describe the local property of the band structure for adiabatic transport, can be obtained as a triple integral with the second-rank tensor field, $\Omega_i(k) = -\text{Im}[\varepsilon_{i\mu
u}\partial_{k\mu}\langle\psi|\partial_{k\nu}\rangle]$ where $\psi$ is the Bloch band state whose period is $L$, $\varepsilon_{i\mu
u}$ is the Levi-Civita tensor. Using $\mathbf{I} = \frac{i}{2\pi} \sum_{\psi'} \langle \psi' | \psi' \rangle$ (here we assume that the silicene is subjected to an one-dimensional periodic potential along the zigzag direction with the period $L = 2\pi$ this normalization term can be omitted), $\langle \psi' | \partial_{k\mu} \psi \rangle (\varepsilon_{\psi} - \varepsilon_{\psi'}) = \langle \psi' | \partial_{k\mu} H_k | \psi \rangle$, then the well known Berry curvature formula can be obtained as

$$\Omega_i(k) = -\text{Im} \left[ \sum_{\psi' \neq \psi} \frac{\langle \psi' | \partial_{k\mu} H_k | \psi \rangle \times \langle \psi' | \partial_{k\mu} H_k | \psi \rangle}{(\varepsilon_{\psi} - \varepsilon_{\psi'})^2} \right].$$

Here the Bloch wave function is related to the eigenfunction of the system Hamiltonian by $\psi(r) = e^{-i\mathbf{k} \cdot \mathbf{r}}\Psi(r)$. The sum of Berry curvature around the curl is constant thus it’s divergence-free and obeys $\partial_t \langle \psi | \psi \rangle = 0$, thus $\partial_t \Omega(k) = 0$ for the Bloch band which does not degenerate with other bands in momentum space. By using the Berry connection in low energy level, $\mathbf{A}(k) = i\langle \psi | \partial_{k\mu} \psi \rangle = -\text{Im} \langle \psi | \partial_{k\psi} \rangle$, the Berry curvature can be rewritten as $\Omega_i(k) = \varepsilon_{i\mu\nu}\partial_{k\mu} A_{\nu}(k) \mathbf{s} = \frac{k}{k^2} \mathbf{s} = \frac{1}{2\pi} \mathbf{s} \cdot \mathbf{e}_k$ where $\mathbf{e}_k$ labels the radiation direction of the wave vector. where $\mathbf{s}$ denotes the spin operator as well as its helicity and it’s an important indicator for the system under magnetic field or light field, and it can also be replaced by a pseudospin operator. Here we note that, distinguished from the case of free pseudospin-1 Maxwell particle, for the massless two-dimension Dirac Fermions near the Dirac cone where the Berry curvature has obvious peak, the coupling between the magnetic moment and the magnetic field will leads to a magnetic field-induced energy shift which cancel the effect of the Berry curvature-induced energy-shift in the case of only the magnetic field exist. While for the internal magnetic moment for a magnetism matter, it can be directly coupled to the supercurrent for a Josephson device.

The momentum-pseudospin space Berry curvature in unit of $e^2/h$ is in a similar distribution with the orbital magnetic moment which couples to the magnetic field in z-direction and reads

$$m(k) = \frac{e}{h} e(k) \Omega(k)$$

$$= \frac{e}{2h} \frac{2\eta h^2 v_F^2 m_D^{(i+s)^2}}{2(4(m_D^{(i+s)^2})^2 + h^2 v_F^2 k^2)},$$

[30]
We can see that the orbital magnetic moment is opposite in sign for two valleys, which is required by the time-reversal invariant and the accompanied spin-momentum locking. Note that the relation $\hbar v_F = \frac{2}{\sqrt{7}} at$ is used here. The orbital magnetic moment can be probed by X-ray circular dichroism or other electron probes\[31\]. The Berry curvature thus can be expressed as

$$\Omega(k) = \frac{1}{2} \frac{2 \hbar^2 v_F^2 m_D^{ns \tau_z}}{2 \varepsilon (4 (m_D^{ns \tau_z})^2 + \hbar^2 v_F^2 k^2)}$$

(13)

The orbital magnetic moment is shown in Fig.2 for two different Dirac-mass. The orbital magnetic moment decays faster for smaller Dirac-mass, and through the above expression we can obtain that the Berry curvature has the same law. It is worth noting that the nonzero Berry curvature and orbital magnetic moment require the broken symmetry\[32\]. Here we list some possible symmetry-broken case in our system: (I) the Rashba-coupling and perpendicular electric field breaks the inversion symmetry; and the spatial inversion symmetry can be broken by the buckled structure like in the GaAs; (II) the off-resonance light breaks the time-reversal-invariance (or due to the competition between Zeeman coupling and Rashba-coupling\[33\]); (III) the Rashba-coupling also breaks the chiral symmetry as well as the symmetry between conduction band and valence band (when in the absence of exchange field); (IV) the magnetic-field-induced shift (according to Löwdin perturbation theory) in quasimomenta space breaks the reflection symmetry. Except that, the distortions origin from, e.g., strain, breaks the inversion and nonsymmorphic symmetries.

As mentioned above, the coupling between magnetic moment and the magnetic field induce the energy shift (drift in the motion) $\delta E_B = -\mu \cdot B$, where $\mu = \hbar v_F \cdot e \mathbf{k}/(2\varepsilon) \times \hat{s}$ is the magnetic effective moment. However, for the electromagnetic field, the induced-energy-shift is

$$\delta E = \left[ \delta E_B - \frac{e \hbar^2 v_F^2}{2 \varepsilon (\varepsilon + \hbar v_F k)} \mathbf{E} \times \mathbf{k} \right] \cdot \hat{s},$$

(14)

which is similar to the BMT equation about the spin precession in the semiclassical limit\[31\], and here the non-relativity correction factor is $1/(\sqrt{1 - (v_F/c)^2}) \approx 1$. The above equation describes the motion of particle in the U(1) electromagnetic gauge field $(\mathbf{B} + \frac{k^2 v_F^2}{\varepsilon + \hbar v_F k} \mathbf{E} \times \mathbf{k})$. We can see that the shift (or spin precession) $\delta E \sim 1/k^2(1/\varepsilon^2)$. That similar to the expression of the precession of the transported spin, which can be reads

$$\partial_t \mathbf{s} = \delta E \times \mathbf{s}.$$  

(15)

The shift of momentum in the armchair direction of Brillouin zone can also be induced simply by the in-plane magnetic field\[35\] which breaks the reflection symmetry (reflection operator in armchair direction is $R = \tau_x s_y$) and the time-reversal-invariance (the time-reversal operator which is antiunitary is $\Theta = is_y K$ where $K$ denotes the complex conjugation while $is_y$ is the mirror operator). That implies that the effect of the in-plane magnetic field is similar to the in-plane exchange field (ferromagnetic or antiferromagnetic), which is effective in breaking the topologically protected gapless edge model and thus rise the nonzero orbital magnetic moment (also the orbital effect) and the Berry curvature. Both the gap under the in-plane exchange field $M$ and the perpendicular magnetic field $B_z$ exhibit linear relations: for in-plane exchange field, gap $\Delta \approx 2eV/(M)eV$, while for $B_z$ it’s $\Delta \approx 0.2meV/T$\[35\]. Thus the effect of gap opening for magnetic field is much lower than the in-plane exchange field.

The single-component Landau gauge cause a conventional circular orbits in the presence of Lorentz-like force. However, if there is a harmonic trap potential in the presence of symmetry gauge (with two components like $A = (B_2 y, -B_2 x, 0)$ which lose the translation invariance in both the $x$- and $y$-direction), the cyclotron orbits may becomes more complex with variant curvatures especially for the electron gas which has larger diffusion coefficient. For the bilayer
silicene (or bilayer MoS\(_2\)), the nonzero effective mass \(m^*\) (related to the interlayer hopping) is also shifted by the magnetic field, as \(m^* \rightarrow m^* + eB\frac{\kappa}{m^*}\) \([9]\), where \(\kappa\) is an exotic parameter in Lie algebra\([36]\). Here we note that, the \(\kappa\) is inversely proportional to the frequency of external harmonic trap potential if it exist, i.e., \(\kappa \propto 1/\omega\), thus the manipulation of both the harmonic trap and the magnetic field as well as the initial momentum can be used to control the effective mass. Note that here the effective mass is related to the initial momentum as long as the \(m^* \rightarrow \infty\) \([9]\). Note that the trap here only affects the particles with mass, but invalid for the massless particle no matter what the frequency it is. In fact, even without the harmonic trap, the magnetic field itself also has a coupling effect by suppressing the diffusion of the wave package. A singular configuration emerges at \(B = \frac{(m^*)^2}{\epsilon \sigma}\) (for gyromagnetic ratio \(g \approx 2\)), where the effective mass is zero and thus give rise the Hall motion \(\frac{\sqrt{2}}{2}\partial_{ki}\varepsilon = \varepsilon_{\mu\nu}E_{\nu}/B\) in the presence of disorder come from the electric field and magnetic field (we only consider the part which affects the band structure). Since for zero effective mass under such critical magnetic field, the electric field and exchange field where we set \(\partial_{ki}\varepsilon = \varepsilon_{\mu\nu}E_{\nu}/B\) is approximation result\([21]\). Here we note that, the \(\kappa\) curvature turns to a \(s\)-function at Dirac-cone in the zero-Dirac-mass-limit as we mentioned, the \(\kappa\) radius \(R_{\perp} = 0\) (thus \(\eta\tau_{z} = -1\)), while the radius expanded by the wave vector \(k_{R}\) in momentum space reads\([23]\)

\[
k_{R} = \sqrt{\frac{\hbar^{2}v_{F}^{2}(M_{s}^{2} - \lambda_{SOC}^{2}) - a^{2}\lambda_{R_{2}}^{2}\lambda_{SOC}^{2}(2\hbar^{2}v_{F}^{2} + a^{2}\lambda_{R_{2}}^{2})}{\hbar^{2}v_{F}^{2} + a^{2}\lambda_{R_{2}}^{2}}}.
\]

(17)

as labeled in the last panel of Fig.1(c) for the case of \(M_{s} = M_{c} = 7.8\) meV where can easily see the quantum anomalous Hall phase. The orbital magnetic moment of the electron moves along the loop with radius \(k_{R}\) at the lowest energy can be analytically obtained as \(m(k) = k_{R}e\nu F/2\). The Fig.1(d) shows the evolution of the band gap at K valley with different \(M_{s}\) and \(M_{c}\). For the case of \(\lambda_{R_{2}} = 0\), the gap vanish and the above radius reduce to the conventional form

\[
k_{R}^{*} = \sqrt{\frac{M_{s}^{2} - \lambda_{SOC}^{2}}{\hbar F}},
\]

(18)

and it’s similar to the form of Ref.\([37]\) which is for the zero-trap and zero-exchange-field case with nonzero particle mass. In Fig.3, we show the gap \(\Delta\) and the radius \(k_{R}\) as a function of the electric field and exchange field where we set \(M_{s} = M_{c}\). From Fig.3(c), when \(M_{s} = M_{c} =
spin-independent perturbation potential. Through the above Hall motion, the group velocity can be rewritten as

\[ v_A(k) = i(\langle \partial_t \partial_k \psi \vert \partial_k \psi \rangle - \langle \partial_k \partial_t \psi \vert \partial_t \psi \rangle) = -2i \text{Im}[\partial_t (\psi \vert \partial_k \psi \rangle], \]

where \( k \) is the quasimomentum of the center-of-mass, while for the two-dimensional system or the higher one, the anomalous velocity owns the Lorentz force term as

\[ v_A(k) = -\partial_t k \times \Omega(k) \]

and it’s related to the effective force \( F \) by

\[ \partial_t k = \frac{F}{\hbar} = \partial_r V(r) - \frac{e}{\hbar} E(k) - \frac{e}{\hbar} v_g \times B(k) \]

\[ = \partial_r V(r) + \frac{e}{\hbar} \partial_r \Phi(k)e^\mu - \frac{e}{\hbar} v_g \times B(k), \]

where \( \Phi(k) \) is the scalar potential of the electric field and \( B(k) \) is the vector potential. Both the scalar and vector potential are incorporated in the Bloch wave function \( \psi \). Note that Bloch oscillation emerges when the effective force is constant, and it’s largely affected by the anomalous effect in the presence of perturbations. Here we note that the effective Hamiltonian here contains the position-dependent period-potential term, \( H = \varepsilon(k) + V(r) \), where \( \varepsilon = k^2/2\hbar^2 |_{k \in BZ} \) is independent of the position.

Here the Levi-Civita tensor is missing which distinguished from the multidimensional case with Einstein summation. \( v_g = \partial H/\partial k + v_A \) is group velocity (after Berry correction) of the center of the wave package, where \( H = \varepsilon(k) + V(r) \) is the summation of the quasieenergy and the spin-independent perturbation potential. Through the above Hall motion, the group velocity can be rewritten as

\[ v_g = \frac{\hbar}{eB} \partial_t k + \frac{E}{B}, \]

thus Berry curvature here vanish for \( B = 0 \). Through the form of expression of the \( \hbar \partial_t k \), we can see that the motion of electrons is along the applied fields, i.e., along the \( k_z \)-direction. While in the case of without the perturbations (without electric field or magnetic field), the charge density \( \rho \) at the band crossing point obeys the continuity equation

\[ \partial_t \rho + \nabla_k \cdot (\rho \hbar \partial_t k) = 0 \]

in quasimomentum space, and that’s also valid for the multiband touching models including the bilayer silicene or Weyl semimetal\[38, 39\]. The above continuity equation is also valid in phase space (including the coordinate and quasimomentum). But that needs the distribution function to satisfies \( \partial_t f = 0 \), or in the scattering form of Boltzmann-Vlasov equation\[40, 41\]:

\[ \partial_t f + v_g f - \frac{e}{\hbar} \nabla_k = (\partial_t f)_{\text{collision}} = 0, \]

which is satisfied when the scattering broadening is smaller than the bandwidth. Through the above corrected group velocity, the interband transition can be described by the momentum operator

\[ \mathcal{P}^\eta = m_0 \left[ \langle \psi \vert \partial H/\partial k \vert \psi' \rangle + \hbar \partial_t k \cdot \Omega(k) \right] \]

\[ = m_0 \left[ v_F(1 + \frac{e}{\hbar} \partial_r \Phi(k)e^\mu - \frac{e}{\hbar} v_g \times B(k)) \right] \left[ \frac{2\eta v_F^2 m_D^\eta_{\tau_z}}{22 \varepsilon (4m_D^{\eta_{\tau_z}})^2} \right], \]

\[ \eta, \tau_z = 0012, \text{ and } 0012, \text{ and } k_z. \]
where \( m_0 \) is the free electron mass. The plot of \( \mathcal{P}^\eta \) for \( \eta = 1 \) (K valley) is presented in Fig.4. During the interband transition, the kinetic term won’t induce the spin flip while the Rashba-coupling \( \lambda_{Rz} \) will induce the spin flip in a topological insulator\(^4\), which makes the spin \( s_z \) no more a good quantum number.

It’s obvious that the symmetry broken (or parity broken) together with the time-dependence (of band) give rise the anomalous velocity and the non-adiabatic correction. While the adiabatic approximation requires the large band gap \( m_{s}^{\eta_{\tau_z}} > L\!F/2 \) (to prevent the excited particles through the gap) in the absence of the scattering and perturbations. The anomalous velocity is vanishes for the moment-free spin(or pseudospin)-1 massless particle which can easily be found in the Maxwell metal. For such a particle, the above effective Hamiltonian matrices (4)-(6) are still valid just by replacing \( \frac{\hbar v_F}{\sqrt{2}} (k_x \pm i k_y) \) with \( \frac{1}{\sqrt{2}} \hbar v_F (k_x \pm i k_y) \). The quasienergy here is with a period of \( 2\pi/L \) in momentum space which is estimated as 1 here, thus in first Brillouin zone it has \( \varepsilon(k) = \varepsilon(k + 2\pi/L) \).

The intrinsic quantum spin Hall conductivity due to the anomalous velocity and electron trajectories reads\(^7\)

\[
\sigma_{xy}^s = \frac{e^2}{h} \int_{BZ} \frac{d^2k}{4\pi^2} [f_{s_z=1} - f_{s_z=-1}] \Omega(k).
\] (25)

For the case Fermi-level within the band gap, the spin Hall conductivity reads

\[
\sigma_{xy}^s = \frac{e^2}{4\hbar} \sum_{\eta \tau_z} (\text{sgn}[m_{D}^{\eta_{\tau_z},s_z=1}] - \text{sgn}[m_{D}^{\eta_{\tau_z},s_z=-1}]).
\] (26)

According to the Dirac-mass mentioned above, the requirements for the nonzero spin Hall conductivity can be obtained as

\[
\eta \lambda_{SOC} \tau_z + M > \frac{\Delta}{2} E_{\perp} \tau_z + \eta \hbar v_F^2 A^2/\Omega,
\] (27)

and \( \frac{\Delta}{2} E_{\perp} \tau_z + \eta \hbar v_F^2 A^2/\Omega < 0 \),

or

\[
\eta \lambda_{SOC} \tau_z + M < \frac{\Delta}{2} E_{\perp} \tau_z + \eta \hbar v_F^2 A^2/\Omega,
\] (28)

and \( \frac{\Delta}{2} E_{\perp} \tau_z + \eta \hbar v_F^2 A^2/\Omega > 0 \).

For the case Fermi level lies within the conduction band, the spin Hall conductivity can be obtained as

\[
\sigma_{xy}^s = \frac{e^2}{4\hbar} \sum_{\eta \tau_z} \frac{[m_{D}^{\eta_{\tau_z},s_z=1}] - [m_{D}^{\eta_{\tau_z},s_z=-1}]}{\varepsilon_{s_z=1} - \varepsilon_{s_z=-1}},
\] (29)

Our above analytical calculations are also agree with the results of Ref.\(^4\). While for the intrinsic quantum valley Hall conductivity, it can be obtained through the similar procedure but focus on the difference between the cases of \( \eta = 1 \) and \( \eta = -1 \). Related to the Boltzman equation in finite temperature but in the absence of vertex correction between different tiles with different self-energy, the above Hall conductivity in quantum anomalous Hall phase can
be rewritten as
\[
\sigma_{xy} = \frac{e^2}{2h} \int_{BZ} \frac{d^2k}{4\pi^2} \sum_{\psi\psi'} (\varepsilon(\psi) - \varepsilon(\psi'))^{-1} \partial_{\varepsilon} f(\varepsilon(\psi)) \times \Omega(k),
\]
(30)
where \( f(\varepsilon(\psi)) \) is the Dirac-Fermi distribution function.

4 Perturbation from the impurity scattering potential and periodic off-resonance light

The impurity is important to the intervalley scattering, especially in the bulk part where the valley mixing is missing\[3\]. In the presence of nonzero impurity scattering angle with a single nonmagnetic (without the coupling with spin) impurity, i.e., the position-dependent Gaussian scattering potential in the \( T \)-matrix approximation\[45\], the density of state (DOS) reads
\[
D(k, \omega) = \int_{BZ} \frac{d^2k}{(2\pi)^2} (f_{\varepsilon_m} - f_{\varepsilon_n}) \text{Im}[G_k(\varepsilon_m - \varepsilon_n + \Delta k) - G_k(\varepsilon_n - \varepsilon_m)T(\omega)G_k(\varepsilon_n - \varepsilon_m + \Delta k)],
\]
(31)
Here the effect of \( T(\omega) \) is similar to the vertex function except that the vertex function is a connection between different frequencies but with the same momentum while the \( T(\omega) \) here is a connection between different momentums which is related to the scattered wave vector but with the same frequency\[19\]. The direction of \( T(\omega) \) is perpendicular to the boundary between the two distinguish momentum tiles, and weighted by the (momentum-resolved) DOS or the spectral function, which is also related to the low-energy behaviours of optical conductivity and the Hall conductivity. For the nonmagnetic impurity (or the weak magnetic ordering impurities like W- or Mo-silicene), the \( T(\omega) \) has
\[
T^{-1}(\omega) = \frac{1}{V_s\sigma_z - \frac{1}{\hbar^2} \int_{BZ} \frac{d^2k}{4\pi^2} G_k(\varepsilon_m - \varepsilon_n)},
\]
(32)
where \( V_s \) is the single scalar impurity scattering potential with the \( z \)-direction spin-polarization (here we note that the conductance in the presence of scalar impurity is larger than that in the presence of magnetic impurity). The impurity scattering potential here is assumed to be momentum-independent but energy (frequency)-dependent, and thus it is in a Gaussian scattering form with a \( \delta \)-function:
\[
\delta(r - r_{\text{imp}}).
\]
The Keldysh Green’s function here \( G_k(E_m - E_n) \) independent of the scattering, and it reads\[45\]
\[
G_k(\varepsilon_m - \varepsilon_n) = (\varepsilon - H - n_{\text{imp}} T(\omega))^{-1},
\]
(33)
where \( n_{\text{imp}} T(\omega) \) is the scattering-independent self-energy term. The perturbed Hamiltonian under the impurity scattering potential is
\[
H_V = H_0 + V \frac{1}{\varepsilon - H_0} V + V \frac{1}{\varepsilon - H_0} V \frac{1}{\varepsilon - H_0} V + O((\varepsilon - H_0)^{-3}).
\]
(34)
Note that here the nonmagnetic impurity scattering potential is assumed independent of momentum and thus has a \( \delta \)-function singularity.

According to the second-order effect (which is unique to the off-resonance light) of the effective photocoupling process, where the quasienergy is shifted by absorbing (emitting) a certain quantity photons, as \( \varepsilon \pm \hbar\Omega/2 \), i.e., it’s shifted by a half-integer multiples of \( \hbar\Omega/2 \). The transport properties during such process can be detected by the Green’s function with
complex variable (quasienergy), and the decimation method\cite{46}. For the perturbation from the time-dependent periodic off-resonance light $V(T + t) = V(t)$, the monochromatic-perturbed Hamiltonian can be obtained by the simple Schrödinger equation $H_V \phi = E\phi$ using the Floquet technique, with Floquet Hamiltonian in a tridiagonal form as

$$H_V = \begin{pmatrix}
\ddots & V_{-N} & H_{-N} & V_{+N} & 0 & 0 & \cdots \\
\cdots & V_{-N} & H_0 & V_{+N} & 0 & \cdots & \\
\cdots & 0 & V_{-N} & H_0 & V_{+N} & 0 & \cdots \\
\cdots & 0 & 0 & V_{-N} & H_N & V_{+N} & \cdots \\
& & & & & & \ddots
\end{pmatrix}, \tag{35}$$

and

$$\phi = (\cdots, \phi_{-N}, \phi_0, \phi_{+N}, \cdots)^T \tag{36}$$

thus the $H_V$ can be obtained as

$$H_V = H_0 + V_{-N} \frac{1}{\varepsilon - H_{-N}} V_{+N} + V_{+N} \frac{1}{\varepsilon - H_{+N}} V_{-N} + O(V^2) \tag{37}$$

$$\approx H_0 + \frac{[H_{-1}, H_{+1}]}{\hbar \Omega} + O(\frac{A^4}{\Omega^2}),$$

with the interaction term $V_N = \frac{1}{T} \int_0^T H(t) e^{-iN \hbar \omega t}$ where $H(t)$ is the one appear in Appendix A. $H_0$ can be evaluated as the one-period mean value of $H(t)$, i.e., $H_0 = \frac{1}{T} \int_0^T H(t) dt$. The above perturbed effective Hamiltonian can be related to the evolution operator with the effect of Berry curvature by\cite{47}

$$H_V = i\hbar \frac{T}{T} \log U, \tag{38}$$

with

$$U = e^{-iTH_0} \mathcal{P} e^{i\int_C A(k) dk}, \tag{39}$$

where $\mathcal{P}$ is the path operator of the electron contour $C$.

While for the five-diagonal one, we can using the following equation with real space renormalization group

$$\begin{pmatrix}
H_{-N} & V_{+N} & V_{+N+1} \\
V_{-N} & H_0 & V_{+N} \\
V_{-N-1} & V_{-N} & H_N
\end{pmatrix}
\begin{pmatrix}
\phi_{-N} \\
\phi_0 \\
\phi_{+N}
\end{pmatrix}
= \varepsilon
\begin{pmatrix}
\phi_{-N} \\
\phi_0 \\
\phi_{+N}
\end{pmatrix}, \tag{40}$$

with the Fourier index $N$ and the Bloch wave function in zeroth Fourier model is

$$\phi_0 = \frac{V_{-N}}{\varepsilon - H_0} \phi_{-N} + \frac{V_{+N}}{\varepsilon - H_0} \phi_{+N}. \tag{41}$$

The circular polarized light-induced periodically-driven nonequilibrium system results in a dc-driven charge current, which disobey the current continuity $\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$ and the probability current conserved $-i\frac{\hbar}{2m}(\phi^* \nabla \phi - \phi \nabla \phi^*)$ and the Gaussian distribution (wigner-Dyson type), which can be represented by a variant reservoir response form in a determined Landau level.
with the frequency $\omega$ before the optical coupling:

$$J_{\text{res}} = \sum_{n,b} \int \frac{d\omega}{2\pi} \epsilon_n \rho_a(\omega + n\hbar \omega_l) t_{n,b}^2 \rho_b(\epsilon_n) G_{ij}(n, \omega)(f_b(\omega) - f_a(\omega + n\hbar \omega_l)),$$

$$G_{ij}(n, \omega) = \int dt e^{i n \hbar \omega t} \int dt' \rho_{ij}^{\text{ret}}(t, t') e^{i (\omega + \mu_b)(t-t')} ,$$

$$\rho_a(\omega + n\hbar \omega_l) = \frac{1}{N} \sum_k \delta(\omega + n\hbar \omega_l - t | \epsilon_k |) ,$$

$$\rho_b(\omega) = \frac{1}{N} \sum_k \delta(\omega - t | \epsilon_k |) , \quad t = \frac{2\sqrt{3} \hbar v_F}{3a} \approx 1.6 eV , \quad v_F \approx 5.5 \times 10^5 \text{ m/s}$$

$$f_b(\omega) = \frac{1}{1 + \exp[\beta_b(\omega - \mu_b)]} ,$$

$$f_a(\omega + n\hbar \omega_l) = \frac{1}{1 + \exp[\beta_a(\omega + n\hbar \omega_l - \mu_a)]} ,$$

where the $a$ corresponds the channel which coupling with the photons (absorbs or emits) $n$ photons, while the $b$ is the one which not couple with the photons (i.e., describe the transport between two leads $a$ and $b$). $f_{a/b}$ is the Fermi-Dirac distribution function, $\omega_l$ denotes the frequency of the light, $\rho_{a/b}$ is DOS per unit cell of each channel, and $G^{\text{ret}}_{ij}(t, t')$ is the retarded Green’s function $G^{\text{ret}}_{ij}(t, t') = -i \theta(t - t') \langle \{ c_i(t), c_j^\dagger(t) \} \rangle$ ($\theta(t - t')$ is the Heaviside step function), and the term $G^{\text{ret}}_{ij}(t, t') e^{i(\omega + \mu_b)t-t'}$ can be replaced by the advanced Green’s function

$$G^{\text{adv}}_{ij}(t, t') e^{i(\omega + \mu_b)(t-t')},$$

with $G^{\text{adv}}_{ij}(t, t') = i \theta(t' - t) \langle \{ c_i(t), c_j^\dagger(t) \} \rangle$ in the above equation since the relation $G^{\text{ret}}_{ij}(t, t') = (G^{\text{adv}}_{ij}(t, t'))^*$. While for the Matsubara frequency $\omega_M$, we just need to replace the real time by the imaginary time $\tau$. The reservoir variables can be well described by the master equation in the Liouville space with the unperturbed density operator $\mathcal{J}$

$$\partial_\tau \mathcal{J} = -i [ \mathcal{H}, \mathcal{J} ] + \kappa \sum_i (O_i \mathcal{J} O_i^\dagger - \frac{1}{2} (O_i^\dagger O_i \mathcal{J} + \mathcal{J} O_i^\dagger O_i )) \equiv \mathcal{L} \mathcal{J} ,$$

where $\mathcal{J}$ corresponds to the pure state or mixed state and $O_i$ is the Lindblad operator describing the bath coupling.

The diagonal Floquet Green’s function

$$G_{\text{dia}}^N = \frac{1}{\varepsilon - (H_{-1} + V_{+1} \frac{1}{\varepsilon - H_0} V_{-1}^\dagger) - (V_{+N+1} + V_{N+1} \frac{1}{\varepsilon - H_0} V_{+N}^\dagger) \frac{1}{\varepsilon - (H_{N+N} + V_{-N} \frac{1}{\varepsilon - H_0} V_{-N})} (V_{-N-1} + V_{-N} \frac{1}{\varepsilon - H_0} V_{-N})} ,$$

The retarded or advanced diagonal Floquet Green’s functions obeys

$$G_{\text{dia}}^{N+1} = [\varepsilon \mathcal{I} - (H_{N+N+1} + V_{-N} G_{\text{dia}}^{N} V_{-N})]^{-1} .$$

5 Anomalous Andreev reflection in Josephson device under electric field and off-resonance light

The Josephson effect\cite{48} is the earliest explanation about the tunneling between two superconductors separated by a oxide layer or quantum dot\cite{49, 50}. The tunneling strength is $S = 2\pi |V|^2 \delta(\omega - \mu_n - \varepsilon_n (m_{Dn}^{\mu} \tau))$, where

$$V = \sum_{q, s_z} T_{k,(k+q)\tau} c_{k, s_z}^\dagger d_{(k+q), s_z} + H.c. - m_{Dn}^{\mu \sigma \tau} (e^{i \phi c_{k, s_z}^\dagger d_{(k+q), s_z} + H.c.)}$$

(46)
is the tunneling matrix element, where $c^\dagger d^\dagger$ are the creation operators of single-particle state in the left and right leads, respectively. $\omega$ is the excitation energy of single-particle state, $\mu$ is the chemical potential in the middle region. $m_D^{\eta_s, \tau_z}$ is the Dirac-mass (excitation gap) in the normal region (see Appendix), thus $\varepsilon(m_D^{\eta_s, \tau_z} = 0)$ is the quasienergy (single particle energy) in the middle region. The many-body wave function which decrease exponentially in the normal region can be represented as $\Psi(x) \propto e^{-x/\xi_s} e^{i\phi/2} e^{-i\phi/2}$, where $\xi$ is the superconducting coherence length, $x$ is the mean free path in the bulk region (shorter than the middle normal region), $\ell = 0.47$ Å is the characteristic length scale of silicene in the normal region (without the proximity-induced superconductivity or the magnetism), $d$ is the vertical distance to the nonmagnetic impurity within the substrate, $\phi$ is the phase difference between the left and right superconducting leads which can be ignored if the middle region is replaced by an ordinary superconductor[51]. Here note that we imaging the ideal interface where the Fermi wavelength in superconducting leads is much shorter than the superconducting coherence length.

For excitation gap in middle region smaller than the superconducting gap, $m_D^{\eta_s, \tau_z} < \Delta_s$, the Josephson effect emerges through the formation and disruption of Cooper pairs with the process of Andreev reflection and the mixing between the conduction band and valence band in the interface state with superconductor. $\Delta_s$ is the superconducting gap (complex pair potential) which obeys the BCS relation and can be estimated as $\Delta_s = \Delta_0 \tanh(1.74 \sqrt{T_c/T - 1}) e^{i\phi/2}$ (here we only consider the right superconducting lead) with $\phi$ the macroscopic phase-difference between the left and right superconducting leads, $\Delta_0$ the zero-temperature energy gap which estimated as 0.001 eV here and $T_c \approx 5.66 \times 10^{-4}$ eV the superconducting critical temperature.

The perturbations including the electric field and the off-resonance light can be taken into account within the computation of Andreev bound state levels, it’s found that, for $\lambda_{R_1} = \mu = m_D^{\eta_s, \tau_z} \ll \varepsilon$, the anomalous Andreev reflection is dominating[52] during the electronic tunneling in the Josephon junction. In the retro-reflection regime with subgap energies $\varepsilon < \Delta_s$, in contrast to the conventional Andreev reflection whose backscattered hole pass through the valence band[52] and with a spin-flip process (due to the $\lambda_{R_2}$), the anomalous Andreev reflection happen with an another electron with opposite spin come from the valence band and scatter a hole lies still in the conduction band. Consider the $\lambda_{R_2}$, the anomalous Andreev reflection is still possible due to the band splitting as presented in Fig.1(d). Another anomalous effect in the presence of $\lambda_{R_2}$ is due to the reversing of Berry phase[53]. Distinct from the Berry phase obtained above, the additional phase factor induced by $\lambda_{R_1}$ will cause the Berry phase of monolayer silicene changes to $2\pi$ at valley K from the previous value $\pi$, while the Berry phase of bilayer silicene is changes to $\pi$ at valley K from the previous value $2\pi$. Note that here the result is valid for $\lambda_{R_1} \gg \lambda_{SOC}$ by the modulation of electric field. The Andreev reflection is enhanced in the monolayer silicene but reduced in bilayer silicene due to the transition of Berry phase, due to the constructive interference by the $\pi$ Berry phase and destructive interference by the $2\pi$ Berry phase[54 55]. this is similar to the effect of bias voltage in finite doped system; the low bias voltage gives rise the retro-reflection while high bias voltage gives rise the specular one.

For the superconductor-ferromagnet-superconductor (SFS) junction, the Cooper pairs can be described by the above many-body wave function as $\Psi(x) \propto e^{-x/\xi_s} e^{i\phi/2} e^{-i\phi/2} \cos(x/\xi_f)$, where $\xi_f \propto \sqrt{D_f/M}$[55] is the characteristic length for ferromagnetic silicene with the diffusion coefficient $D_f \propto \nu_F$ and the exchange field $M$. The quasiparticle mean free time $\tau \rightarrow \infty$ for ballistic transport. In fact both the exchange field and the magnetic field can suppress the diffusion by coupling the motions in each direction as we mentioned above, and that can be described by the covariant derivative as shown in Ref.56. In the regime of excitation quasienergy $\varepsilon \leq \Delta_0$ and $\mu \approx m_D^{\eta_s, \tau_z}$, it’s dominated by the conventional normal reflection while the reflected particles with minority spin is much less[52]. In Fig.5, we show the Andreev bound state level in SNS-junction versus the phase different under different conditions. The
6 Conclusions

We discuss the effect of time-dependent scalar or vector potential and nonzero Berry curvature on the electronic transport in the silicene, where the degrees of freedom (spin, pseudospin, and valley) related to the center-of-mass play an important role. The anomalous effects due to the Berry correction raised by the time-dependent scalar or vector potential as well as the time-dependent band structure and Bloch band states, like the anomalous velocity term, anomalous Bloch oscillation in the presence of an effective force, and the anomalous Andreev reflection. The topological spin/valley transport as well as the momentum of center-of-mass of the wave package are affected by the applied electric field, magnetic field, and the off-resonance circularly polarized light. The anomalous velocity due to the Berry curvature (including a Lorentz-like term) shifts the electrons in the direction transverse to the electric field and magnetic field, and gives rise to the spin (transverse) Hall conductivity as we investigate in this article. The model detected here is the silicene-like two-dimension hexagonal system, where the motion of the Skyrmion spin texture (in quantum anomalous Hall phase) carried by the spin current is rather weak (compared to the one in quantum spin Hall phase).

The momentum operator about the interband transition involving the Berry correction is obtained, where the corrected velocity term is used including the effects of external periodic potential and the effective force in semi classical dynamics. For perturbation related to the valley index only, like the right(+) or left(-)-handed off-resonance light, the momentum operator satisfies: \( P^K_+ = P^K_- \), i.e., the momentum operator in one valley is just the time-reversal of the another valley, and that also valid for the selection excitation. We also found that, in the presence of orbital degree of freedom and the off-resonance circularly polarized light, the valley mixing (through the edge states), valley polarization and the orbital magnetic moment (or orbital magnetization) affected largely by the light field with the selection rule. The symmetry broken due to the perturbations together with the time-dependence of band give rise the anomalous velocity and the non-adiabatic correction, while the adiabatic approximation requires the large band gap \( m_D^{\eta_s,\tau_z} > LF/2 \) (to prevent the excited particles through the gap) in the absence of the scattering and perturbations, thus our result won’t be valid for the single band case which with very small effective force \( F \) and thus can be treated as the adiabatic case. For the Andreev reflection in the presence of electric-field-induced Rashba-coupling, where we imaging a ideal interface between the conductor and the superconductor leads (note that here the Andreev reflection can’t be happen in the middle regime if the conductor is replaced by a insulator barrier), the spin-flip becomes possible even during the process of backscattering, and the anomalous equal-spin Andreev reflection\[52\] can happen in regime \( \varepsilon < \mu + m_D^{\eta_s,\tau_z} \) in the absence of \( \lambda_{R_t} \), where the reflected hole lies in the conduction band through the backscattering. The investigation of the anomalous effects induced by the Berry curvature is helpful to understanding the semiclassical dynamics, quantum anomalous Hall effect\[57, 58\], Bloch electron system, and even the condensate matter system\[59\] or magneto-electronic devices\[60\], we mainly focus on the silicene-like hexagonal lattice system in low-energy Dirac tight-binding model in this article. Our results can also be used to the silicene-like topological insulators, like the germanene, tinene, MoS\(_2\), black phosphorus.
7 Appendix: Computation of the Andreev bound level

We focus on the computation of the dispersion of Andreev bound level in this section. The Andreev level reads \[15\]

\[\varepsilon_A = \pm \frac{\Delta_s}{\sqrt{2}} \sqrt{1 - \frac{A(C - \cos \phi) + s_x \sqrt{B^2 A^2 + B^2 - (C - \cos \phi)^2}}{A^2 + B^2}}, \tag{47}\]

where we have used the definitions

\[A = C_1 C_2 + \frac{(S_1 S_2 (\frac{f_1}{f_3} + 1)(\frac{f_2}{f_3} - 1))}{4 \sqrt{\hbar^2 v_F^2 k_y^2 f_2/f_4 + 1 \sqrt{-f_1/f_3}} \sqrt{-\hbar^2 v_F^2 k_y^2 f_1/f_3 + 1 \sqrt{f_2/f_1}}}, \]

\[B = \frac{S_1 C_2 (\frac{f_1}{f_2} + \frac{1}{2})}{\sqrt{-\hbar^2 v_F^2 k_y^2 f_2/f_3 + 1 \sqrt{f_2/f_1}} - \sqrt{(\hbar^2 v_F^2 k_y^2 f_2/f_4 + 1 \sqrt{-f_4/f_3})^2}}, \]

\[C = \frac{\hbar^2 v_F^2 k_y^2 S_1 S_2}{\sqrt{\hbar^2 v_F^2 k_y^2 f_2/f_4 + 1 \sqrt{-f_4/f_3}} \sqrt{-(\hbar^2 v_F^2 k_y^2 f_1/f_3 + 1 \sqrt{f_2/f_1})^2}} \]

\[- \frac{[1 \cdot \Theta(\varepsilon_n - \mu_n - m_D^{ns_{\tau_z}}) + (-1) \cdot \Theta(-\varepsilon_n + \mu_n + m_D^{ns_{\tau_z}}] \times \frac{(S_1 S_2 (f_2/f_1 - 1)(f_4/f_3 + 1))}{4 \sqrt{\hbar^2 v_F^2 k_y^2 f_2/f_4 + 1 \sqrt{-f_4/f_3}} \sqrt{-(\hbar^2 v_F^2 k_y^2 f_1/f_3 + 1 \sqrt{f_2/f_1})^2}}, \tag{48}\]

with the Heaviside step function \(\Theta\) which distinguish the two kinds of AR: retroreflection and specular AR, and thus makes this expression valid for both of these two case. The wave vectors \(f_1 \sim f_4\) and parameters \(C_1,\ C_2,\ S_1,\ S_2\) are defined as

\[f_1 = m_D^{ns_{\tau_z}} + \varepsilon + \mu_s, \quad f_2 = m_D^{ns_{\tau_z}} + \varepsilon - \mu_s, \]

\[f_3 = \varepsilon - m_D^{ns_{\tau_z}} + \mu_s, \quad f_4 = m_D^{ns_{\tau_z}} - \varepsilon + \mu_s, \]

\[C_1 = \cos(L \sqrt{f_1 f_3/\hbar^2 v_F^2 - k_y^2}), \]

\[C_2 = \cos(L \sqrt{-f_2 f_4/\hbar^2 v_F^2 - k_y^2}), \]

\[S_1 = \sin(L \sqrt{f_1 f_3/\hbar^2 v_F^2 - k_y^2}), \]

\[S_2 = \sin(L \sqrt{-f_2 f_4/\hbar^2 v_F^2 - k_y^2}), \tag{49}\]

where \(\mu_s\) is the chemical potential of the highly doped superconducting regime. The \(x\)-component of the wave vector for the electron channel and hole channel, \(k_{xe}\) and \(k_{xh}\), are incorporated in the above wave vectors, specially, the electron and hole wave vectors here are both complex, which implies the inclusion of the subgap solutions with the evanescent scattering waves, and it has \(\frac{\Delta_s}{2e^2} 2\cos \beta = 1\) for \(|\varepsilon_n| < \Delta_s\). The \(k_y \sim 2\) meV is conserved during our computation, while the \(k_x\) is unconserved during the scattering,

\[k_x^e = \sqrt{\frac{f_1 f_2}{\hbar^2 v_F^2} - k_y^2}, \]

\[k_x^h = [1 \cdot \Theta(\varepsilon_n - \mu_n - m_D^{ns_{\tau_z}}) + (-1) \cdot \Theta(-\varepsilon_n + \mu_n + m_D^{ns_{\tau_z}}] \sqrt{\frac{f_4(-f_2)}{\hbar^2 v_F^2} - k_y^2}, \tag{50}\]
That’s similar to the result of Ref.\[52\] which is for graphene and thus with valley degenerates:

\[
k_x^{(h)} = \frac{1}{\hbar v_x} (\mu + (-)\varepsilon) \sqrt{1 + \frac{2\eta \lambda_{SOC}}{\mu + (-)\varepsilon} \sqrt{\frac{\hbar v_y^2 k_y^2}{\hbar v_x^2 k_x^2 (h^2)} + 1}} \tag{51}
\]

in the presence of Rashba-coupling.

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Figure 1: Band gap evolution in valley K under the strength of of $M_s$ (a), $M_c$ (b), and both of them (c). Both the external fields and the on-site interaction $U$ are setted as zero here. (d) is the band gap evolution in valley K with the intrinsic SOC and NNN Rashba-coupling $\lambda_{R_2}$ [18]. The avoided crossing effect is obvious in the last panel of (c). In (d), the index $(\alpha, \beta)$ are explained in the text, the index $\gamma$ denotes the spin helical; $\gamma = -$ for anticlockwise spin helical and $\gamma = +$ for clockwise spin helical.
Figure 2: (Color online) Orbital magnetic moment $m(k)$ in the two dimension momentum space for Dirac mass $m_D^{\sigma_z \tau_z} = 0.005$ eV (a) and $m_D^{\sigma_z \tau_z} = 0.32$ eV (b). The vertical axis is in unit of $e/\hbar$. 
Figure 3: (Color online) (a) is the band gap $\Delta$ (in unit of eV) in the case of $|\lambda_{SOC} - \frac{\Delta}{2} E_{\perp} + M_s| \geq 0$. (b) is the band gap (in unit of eV) created by the anticrossing of bands, such band gap enlarge with the increase of $M_s$, and decrease with the increase of $E_{\perp}$, it’s corresponding radius $k_R$ is presented in the (c). Here we set $\sigma_z = -1$, $\tau_z = -1$ for (a) and $\sigma_z = 1$, $\tau_z = -1$ for (b) and (c). (d) shows the enlarged view at $M_s = 2\lambda_{SOC} = 0.0078$ eV (i.e., the gray dash-line in (c)), which is precisely the case of the last panel of Fig.1(c). The red arrow indicates the radius of quantum anomalous Hall phase $(M_s = 2\lambda_{SOC}, E_{\perp} = 0)$ is about $k_R = 0.00126$ (i.e., the one labeled in the Fig.2(c)), and the radius $k_R$ for $M_s = 2\lambda_{SOC}$ vanish at $E_{\perp} = \frac{4\lambda_{SOC}}{N} \approx 0.0339$ eV.
Figure 4: Momentum operator $\mathcal{P}$ about the interband process in $E_{\perp} - \mathbf{k}$ space with different chemical potential and magnetic field. Note that here we set $c = m_0 = \hbar = v_F = 1$ for simplicity, and to obtain the result efficiently, the Berry curvature is also treated as 1 to avoid the large peak in the small $\mathbf{k}$ regime (as shown in the Fig. 2).
Figure 5: Andreev bound state level versus the phase difference for $E_\perp = 0.034$ eV, $\mu = 0.2$ eV (a) and $E_\perp = 1.54$ eV, $\mu = 0.2$ eV (b) and $E_\perp = 1.54$ eV, $\mu = 0.02$ eV (c). The range of $\phi$-axis is setted as one period, $4\pi$. The intensity of off-resonance light is setted as 0.3.