Special Functions:
Integral properties of Jack polynomials, hypergeometric functions and invariant polynomials

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Abstract
Some integral properties of Jack polynomials, hypergeometric functions and invariant polynomials are studied for real normed division algebras.

1 Introduction
During the 1960s, real and complex zonal polynomials were studied exhaustively by James (1961, 1964), Constantine (1963) and Khatri (1966), among many others. Excellent reference books include those by Muirhead (1982), Takemura (1984), Farrell (1985) and Mathai (1997), which summarise many of the results published to date.

Hypergeometric functions with a matrix argument were first studied by Herz (1955) and defined in terms of zonal polynomials by Constantine (1963). Hypergeometric functions of one or two matrix arguments have been applied in many areas of science and technology, including multivariate statistical analysis (Muirhead 1982 and Mathai 1997), random matrix theory (Metha 1991 and Forrester 2009), wireless communications (Ratnarajah et al. 2005a,b), shape theory (Goodall and Mardia 1993 and Caro-Lopera et al. 2009).

Later Davis (1979), Davis (1980), Chikuse (1980) and Chikuse and Davis (1980) introduced a class of homogeneous invariant polynomials with two or more matrix arguments, which generalise the zonal polynomials; many of their basic and integral properties are studied in real cases.

In the context of multivariate statistics, zonal polynomials were initially used to express many noncentral matrix variate distributions. However, there were other distributional problems that could not be solved using zonal polynomials. In these latter cases, invariant polynomials were used to obtain explicit expressions of doubly noncentral matrix variate distributions, matrix variate distribution functions and the joint density of eigenvalues of matrix variate beta type I and II distributions, etc. see James (1964) and Davis (1980).

During the 1990s, zonal polynomials regained prominence but from a more general point of view, in which it was observed that zonal polynomial for real and complex cases are particular cases of Jack polynomials, see Sawyer (1997) and Goulden and Jackson (1996), among many others. In terms of Jack polynomials, it is possible to give a general definition

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for hypergeometric functions, see Gross and Richards (1987), Gross and Richards (1989), among many others. In both cases, Jack polynomials and generalised hypergeometric functions are written in term of a parameter, denoted by $\alpha$ or $\beta$ and with which, for example for $\beta = 1, 2$ or $4$, the zonal polynomials and hypergeometric functions are obtained for real, complex and quaternion cases, respectively, see Gross and Richards (1987), Dimitriu et al. (2005) and Koev and Edelman (2006).

The properties for Jack polynomials and hypergeometric functions with a matrix argument have been studied by Herz (1955), James (1964), Constantine (1963), Khatri (1966) and Muirhead (1982) in the real case (zonal polynomials); by James (1964), Takemura (1984), Farrell (1985) and Ratnarajah et al. (2005a,b) in the complex case (Schur functions); by Li and Xue (2009) in the quaternion case and by Gross and Richards (1987) and Caro-Lopera et al. (2007) in the general case (real, complex and quaternion cases), among many others.

A serious obstacle encountered when Jack polynomials, hypergeometric functions and invariant polynomials are to be used is the question of their calculation. Fortunately, with the excellent algorithm proposed by Koev and Edelman (2006) and Koev (2004), it is now possible to use these techniques in many applications, see Ratnarajah et al. (2005b) and Caro-Lopera et al. (2009). Unfortunately, this obstacle remains for the general case of invariant polynomials.

Let us take into account that there are exactly four normed division algebras: the real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$), and octonions ($\mathbb{O}$); moreover, these are the only alternative division algebras, and all division algebras have a real dimension of 1, 2, 4 or 8, see Baez (2002, Theorems 1, 2 and 3). Furthermore, according to Baez (2002), there is still no proof that octonions are useful for understanding the real world.

In this paper, we generalise diverse integral properties of Jack polynomials, hypergeometric functions and invariant polynomials for normed division algebras. Note that we can only conjecture the results for the octonion case, because many of its related matrix problems are still under study. However, for example in Forrester (2009, Section 1.4.5, pp. 22-24) it is proved that the bi-dimensional density function of the eigenvalue, for a $2 \times 2$ octonionic matrix with symmetric normal distribution, is obtained from the general joint density function of the eigenvalues for the symmetric normal distribution, assuming $m = 2$ and $\beta = 8$. The material in the present paper is organised as follows: Section 2 provides some notation and preliminary results about Jacobians, gamma and beta multivariate functions and invariant measures. The definition and many integral properties of Jack polynomials are obtained in Section 3. Many extensions of the integral properties of hypergeometric functions with one and two arguments are studied in Section 4. For invariant polynomials with two matrix arguments, in Section 5 we derive diverse integral properties, such as the inverse Laplace transform, gamma and beta integrals, etc. Finally, in Section 6 we show diverse applications of some results derived previously, such as the distribution function of a central Wishart distribution for normed division algebras, its joint eigenvalue density and the distribution function of the largest and smallest eigenvalues. We emphasise the conditions that must be met by the parameters that take part in many integral properties in the cases discussed, because, even in the original references, these conditions were omitted or established inexactly.

## 2 Preliminary

Let $L_{m,n}^{\beta}$ be the linear space of all $n \times m$ matrices of rank $m \leq n$ over $\mathfrak{F}$ with $m$ distinct positive singular values, where $\mathfrak{F}$ denotes a real finite-dimensional normed division algebra. In particular, let $GL(m, \mathfrak{F})$ be the space of all invertible $m \times m$ matrices over $\mathfrak{F}$. Let $\mathfrak{F}^{n \times m}$
be the set of all \( n \times m \) matrices over \( \mathfrak{F} \). The dimension of \( \mathfrak{F}^{n \times m} \) over \( \mathbb{R} \) is \( \beta mn \). And let us recall that the parameter \( \beta \) has traditionally been used to count the real dimension of the underlying normed division algebra. In other branches of mathematics, the parameter \( \alpha = 2/\beta \) is used, see Baez (2002).

Table 1: Values of the \( \beta = 2/\alpha \) parameter.

| \( \beta \) | \( \alpha \) | Normed division algebra |
|---|---|---|
| 1 | 2 | real (\( \mathbb{R} \)) |
| 2 | 1 | complex (\( \mathbb{C} \)) |
| 4 | 1/2 | quaternion (\( \mathbb{H} \)) |
| 8 | 1/4 | octonion (\( \mathbb{O} \)) |

Let \( A \in \mathfrak{F}^{n \times m} \), then \( A^* = \overline{A}^T \) denotes the usual conjugate transpose. The set of matrices \( H_1 \in \mathfrak{F}^{n \times m} \) such that \( H_1^2 = I_m \) is a manifold denoted \( \mathcal{V}^\beta_{m,n} \), termed the Stiefel manifold (\( H_1 \) are also known as semi-orthogonal (\( \beta = 1 \)), semi-unitary (\( \beta = 2 \)), semi-symplectic (\( \beta = 4 \)) and semi-exceptional type (\( \beta = 8 \)) matrices, see Dray and Manogue (1999). The dimension of \( \mathcal{V}^\beta_{m,n} \) over \( \mathbb{R} \) is \( [\beta mn - m(m - 1)/2] \). In particular, \( \mathcal{V}^\beta_{m,m} \) with dimension over \( \mathbb{R} \), \( [m(m + 1)/2] \), is the maximal compact subgroup \( \mathcal{U}^\beta(m) \) of \( \mathcal{L}^\beta_{m,m} \) and consists of all matrices \( H \in \mathfrak{F}^{m \times m} \) such that \( H^T H = I_m \). Therefore, \( \mathcal{U}^\beta(m) \) is the real orthogonal group \( \mathcal{O}(m) \) (\( \beta = 1 \)), the unitary group \( \mathcal{U}(m) \) (\( \beta = 2 \)), the compact symplectic group \( \mathcal{Sp}(m) \) (\( \beta = 4 \)) or exceptional type matrices \( \mathcal{O}(m) \) (\( \beta = 8 \)) for \( \mathfrak{F} = \mathbb{R}, \mathbb{C}, \mathbb{H} \) or \( \mathbb{O} \), respectively. Denote by \( \mathfrak{S}^\beta_m \) the real vector space of all \( S \in \mathfrak{F}^{m \times m} \) such that \( S = S^* \). Let \( \mathfrak{P}^\beta_m \) be the cone of positive definite matrices \( S \in \mathfrak{F}^{m \times m} \); then \( \mathfrak{P}^\beta_m \) is an open subset of \( \mathfrak{S}^\beta_m \). Over \( \mathbb{R} \), \( \mathfrak{S}^\beta_m \) consist of symmetric matrices; over \( \mathbb{C} \), Hermitian matrices; over \( \mathbb{H} \), quaternionic Hermitian matrices (also termed self-dual matrices) and over \( \mathbb{O} \), octonionic Hermitian matrices. Generically, the elements of \( \mathfrak{S}^\beta_m \) are termed as **Hermitian matrices**, irrespective of the nature of \( \mathfrak{F} \). The dimension of \( \mathfrak{S}^\beta_m \) over \( \mathbb{R} \) is \( [m(m - 1)/2] \). Let \( \mathfrak{D}^\beta_m \) be the diagonal subgroup of \( \mathcal{L}^\beta_{m,m} \) consisting of all \( D \in \mathfrak{F}^{m \times m} \), \( D = \text{diag}(d_1, \ldots, d_m) \). Let \( \mathfrak{T}^\beta_U(m) \) be the subgroup of all upper triangular matrices \( T \in \mathfrak{F}^{m \times m} \) such that \( t_{ij} = 0 \) for \( 1 < i < j \leq m \); and let \( \mathfrak{T}^\beta_L(m) \) be the opposed lower triangular subgroup \( \mathfrak{T}^\beta_L(m) = \left( \mathfrak{T}^\beta_U(m) \right)^T \). For any matrix \( X \in \mathfrak{F}^{n \times m} \), \( dX \) denotes the matrix of differentials \( (dx_{ij}) \). Finally, we define the measure or volume element \((dX)\) when \( X \in \mathfrak{F}^{n \times m}, \mathfrak{S}^\beta_m, \mathfrak{D}^\beta_m \) or \( \mathcal{V}^\beta_{m,n} \), see Dimitriu (2002).

If \( X \in \mathfrak{F}^{n \times m} \) then \((dX)\) (the Lebesgue measure in \( \mathfrak{F}^{n \times m} \)) denotes the exterior product of the \( \beta mn \) functionally independent variables

\[
(dX) = \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m} \bigwedge_{k=1}^{\beta} dx_{ij}^{(k)}.
\]

**Remark 2.1.** Note that for \( x_{ij} \in \mathfrak{F} \)

\[
dx_{ij} = \bigwedge_{k=1}^{\beta} dx_{ij}^{(k)}.
\]

In particular for \( \mathfrak{F} = \mathbb{R}, \mathbb{C}, \mathbb{H} \) or \( \mathbb{O} \) we have
\begin{itemize}
  \item $x_{ij} \in \mathbb{R}$ then
    \[ dx_{ij} = \bigwedge_{k=1}^{1} dx_{ij}^{(k)} = dx_{ij}. \]
  \item $x_{ij} = (x_{ij}^{(1)} + i x_{ij}^{(2)}) \in \mathbb{C}$, then
    \[ dx_{ij} = dx_{ij}^{(1)} \wedge dx_{ij}^{(2)} = \bigwedge_{k=1}^{2} dx_{ij}^{(k)}. \]
  \item $x_{ij} = (x_{ij}^{(1)} + i x_{ij}^{(2)}) + (j x_{ij}^{(3)} + k x_{ij}^{(4)}) \in \mathbb{H}$, then
    \[ dx_{ij} = dx_{ij}^{(1)} \wedge dx_{ij}^{(2)} \wedge dx_{ij}^{(3)} \wedge dx_{ij}^{(4)} = \bigwedge_{k=1}^{4} dx_{ij}^{(k)}. \]
  \item $x_{ij} = (x_{ij}^{(1)} + e_{1} x_{ij}^{(2)} + e_{2} x_{ij}^{(3)} + e_{3} x_{ij}^{(4)} + e_{4} x_{ij}^{(5)} + e_{5} x_{ij}^{(6)} + e_{6} x_{ij}^{(7)} + e_{7} x_{ij}^{(8)}) \in \mathbb{O}$, then
    \[ dx_{ij} = dx_{ij}^{(1)} \wedge dx_{ij}^{(2)} \wedge dx_{ij}^{(3)} \wedge dx_{ij}^{(4)} \wedge dx_{ij}^{(5)} \wedge dx_{ij}^{(6)} \wedge dx_{ij}^{(7)} \wedge dx_{ij}^{(8)} = \bigwedge_{k=1}^{8} dx_{ij}^{(k)}. \]
\end{itemize}

If $S \in \mathfrak{S}^\beta_m$ (or $S \in \mathfrak{S}^\beta_L(m)$) then $(dS)$ (the Lebesgue measure in $\mathfrak{S}^\beta_m$ or in $\mathfrak{S}^\beta_L(m)$) denotes the exterior product of the $m(m+1)/2$ functionally independent variables (or denotes the exterior product of the $m(m-1)/2 + n$ functionally independent variables, if $s_{ii} \in \mathbb{R}$ for all $i = 1, \ldots, m$)

\[
(dS) = \left\{ \begin{array}{l}
\bigwedge_{i \leq j \leq k=1}^{m} ds_{ij}^{(k)}, \\
\bigwedge_{i=1}^{m} ds_{ii} \bigwedge_{i<j}^{m} ds_{ij}^{(k)}, \text{ if } s_{ii} \in \mathbb{R}.
\end{array} \right.
\]

\textbf{Remark 2.2.} Since generally the context establishes the conditions on the elements of $S$, that is, if $s_{ij} \in \mathbb{R}, \in \mathbb{C}, \in \mathbb{H}$ or $\in \mathbb{O}$. It shall be considered

\[
(dS) = \bigwedge_{i \leq j \leq k=1}^{m} ds_{ij}^{(k)} \equiv \bigwedge_{i=1}^{m} ds_{ii} \bigwedge_{i<j}^{m} ds_{ij}^{(k)}. \]

Observe, too, that for the Lebesgue measure $(dS)$ defined thus, it is required that $S \in \mathfrak{S}^\beta_m$, that is, $S$ must be a non singular Hermitian matrix (Hermitian definite positive matrix). In the real case, when $S$ is a positive semidefinite matrix, its corresponding measure is studied in \cite{Uhlig1994}, \cite{Diaz-Garcia1997}, \cite{Diaz-Garcia2005a} and \cite{Diaz-Garcia2005b} under different coordinate systems.

If $\Lambda \in \mathfrak{D}^\beta_m$ then $(d\Lambda)$ (the Legesgue measure in $\mathfrak{D}^\beta_m$) denotes the exterior product of the $\beta m$ functionally independent variables

\[
(d\Lambda) = \bigwedge_{i=1}^{n} \bigwedge_{k=1}^{\beta} d\lambda_{i}^{(k)}. \]

If $H_1 \in \mathfrak{V}^\beta_{m,n}$ then

\[
(H_1^* dH_1) = \bigwedge_{i=1}^{n} \bigwedge_{j=i+1}^{m} h_{ij}^* dh_{ij}. \]
where \( H = (H_1|H_2) = (h_1, \ldots, h_m|h_{m+1}, \ldots, h_n) ∈ \mathcal{U}^\beta(m) \). It can be proved that this differential form does not depend on the choice of the matrix \( H_2 \) and that it is invariant under the transformations

\[
H_1 → QHP_1, \quad Q ∈ \mathcal{U}^\beta(n) \text{ and } P ∈ \mathcal{U}^\beta(m).
\]

When \( m = 1; \mathcal{V}^\beta_{1,n} \) defines the unit sphere in \( \mathbb{S}^n \). This is, of course, an \( (n-1)\beta \)-dimensional surface in \( \mathbb{S}^n \). When \( m = n \) and denoting \( H_1 \) by \( H \), \((H^*dH)\) is termed the Haar measure on \( \mathcal{U}^\beta(m) \) and defines an invariant differential form of a unique measure \( ν \) on \( \mathcal{U}^\beta(m) \) given by

\[
ν(\mathcal{M}) = \int_{\mathcal{M}} (H^*dH).
\]

It is unique in the sense that any other invariant measure on \( \mathcal{U}^\beta(m) \) is a finite multiple of \( ν \) and invariant because is invariant under left and right translations, that is

\[
ν(Q\mathcal{M}) = ν(\mathcal{M}) = ν(\mathcal{M}), \quad ∀Q ∈ \mathcal{U}^\beta(m).
\]

The surface area or volume of the Stiefel manifold \( \mathcal{V}^\beta_{m,n} \) is

\[
\text{Vol}(\mathcal{V}^\beta_{m,n}) = \int_{H_1 ∈ \mathcal{V}^\beta_{m,n}} (H^*_1dH_1) = \frac{2\pi^mn^{\beta/2}}{\Gamma_m[n\beta/2]}, \quad (2.2)
\]

and therefore

\[
(dH_1) = \frac{1}{\text{Vol}(\mathcal{V}^\beta_{m,n})} (H^*_1dH_1) = \frac{\Gamma^\beta_m[n\beta/2]}{2\pi^mn^{\beta/2}} (H^*_1dH_1).
\]

is the normalised invariant measure on \( \mathcal{V}^\beta_{m,n} \) and \((dH)\), i.e. with \((m = n)\), it defines the normalised Haar measure on \( \mathcal{U}^\beta(m) \). In \((2.2)\), \( \Gamma^\beta_m[a] \) denotes the multivariate Gamma function for the space \( \mathbb{S}^\beta_m \), and is defined by

\[
\Gamma^\beta_m[a] = \int_{A ∈ \mathbb{S}^\beta_m} \text{etr}\{-A\}|A|^{a-(m-1)\beta/2-1} (dA)
\]

\[
= \pi^m(m-1)\beta/4 \prod_{i=1}^m \Gamma[a-(i-1)\beta/2]
\]

\[
= \pi^m(m-1)\beta/4 \prod_{i=1}^m \Gamma[a-(m-i)\beta/2], \quad (2.3)
\]

where \( \text{etr}\{\cdot\} = \exp\{\text{tr}(\cdot)\} \) denotes the determinant and \( \text{Re}(a) > (m-1)\beta/2 \), see Gross and Richards \( (1987) \). This can be obtained as a particular case of the generalised gamma function of weight \( \kappa \) for the space \( \mathbb{S}^\beta_m \) with \( \kappa = (k_1, k_2, \ldots, k_m) \), \( k_1 ≥ k_2 ≥ \cdots ≥ k_m ≥ 0 \), taking \( \kappa = (0, 0, \ldots, 0) \) and which for \( \text{Re}(a) ≥ (m-1)\beta/2 - k_m \) is defined by, see Gross and Richards \( (1987) \),

\[
\Gamma^\beta_m[a, \kappa] = \int_{A ∈ \mathbb{S}^\beta_m} \text{etr}\{-A\}|A|^{a-(m-1)\beta/2-1} q_\kappa(A)(dA)
\]

\[
= \pi^m(m-1)\beta/4 \prod_{i=1}^m \Gamma[a+k_i-(i-1)\beta/2]
\]

\[
= \pi^m(m-1)\beta/4 \prod_{i=1}^m \Gamma[a+k_i-(m-i)\beta/2]
\]

\[
= [a]^{\beta} \Gamma^\beta_m[a], \quad (2.4)
\]
where for \( A \in \mathfrak{S}_m \)
\[
q_\kappa(A) = |A_m|^{k_m} \prod_{i=1}^{m-1} |A_i|^{k_i - k_{i+1}}.
\] (2.5)

with \( A_p = (a_{rs}) \), \( r, s = 1, 2, \ldots, p \), \( p = 1, 2, \ldots, m \) is termed the highest weight vector, see Gross and Richards [1987].

Remark 2.3. Let \( \mathcal{P}(\mathfrak{S}_m^\beta) \) denote the algebra of all polynomial functions on \( \mathfrak{S}_m^\beta \), and \( \mathcal{P}_k(\mathfrak{S}_m^\beta) \) the subspace of homogeneous polynomials of degree \( k \) and let \( \mathcal{P}_\kappa(\mathfrak{S}_m^\beta) \) be an irreducible subspace of \( \mathcal{P}(\mathfrak{S}_m^\beta) \) such that
\[
\mathcal{P}_k(\mathfrak{S}_m^\beta) = \bigoplus_\kappa \mathcal{P}_\kappa(\mathfrak{S}_m^\beta).
\]

Note that \( q_\kappa \) is a homogeneous polynomial of degree \( k \), moreover \( q_\kappa \in \mathcal{P}_\kappa(\mathfrak{S}_m^\beta) \), see Gross and Richards [1987].

In (2.4), \( [a]_\kappa^\beta \) denotes the generalised Pochhammer symbol of weight \( \kappa \), defined as
\[
[a]_\kappa^\beta = \prod_{i=1}^{m} (a - (i - 1)\beta/2)_{k_i} = \frac{\pi^{m(m-1)\beta/4} \prod_{i=1}^{m} \Gamma[a + k_i - (i - 1)\beta/2]}{\Gamma_m[a]^{\kappa}} = \frac{\Gamma_m^\beta[a, \kappa]}{\Gamma_m^\beta[a]},
\]
where \( \text{Re}(a) > (m - 1)\beta/2 - k_m \) and
\[
(a)_i = a(a + 1) \cdots (a + i - 1),
\]
is the standard Pochhammer symbol.

A variant of the generalised gamma function of weight \( \kappa \) is obtained from Khatri [1966] and is defined as
\[
\Gamma_m^\beta[a, -\kappa] = \int_{A \in \mathfrak{P}_m^\beta} \text{etr}(-A)|A|^{a - (m-1)\beta/2 - 1} q_\kappa(A^{-1})(dA)
\]
\[
= \pi^{m(m-1)\beta/4} \prod_{i=1}^{m} \Gamma[a - k_i - (m - i)\beta/2]
\]
\[
= \pi^{m(m-1)\beta/4} \prod_{i=1}^{m} \Gamma[a - k_i - (i - 1)\beta/2]
\]
\[
= \frac{(-1)^k \Gamma_m^\beta[a]}{[-a + (m - 1)\beta/2 + 1]_\kappa^{\beta}},
\] (2.6)

where \( \text{Re}(a) > (m - 1)\beta/2 + k_1 \).

The two expressions of \( \Gamma_m^\beta[a, \kappa] \), \( \Gamma_m^\beta[a, \kappa] \) and \( \Gamma_m^\beta[a, -\kappa] \) as the product of ordinary gamma functions are obtained using the proofs corresponding to \( A = TT^* \) and \( A = T^*T \) with the corresponding Jacobian given in Lemma 2.5. Alternatively, note that for any function \( g(y) \)
\[
\prod_{i=1}^{q} g(x + i - 1) = \prod_{i=1}^{q} g(x + q - i),
\] (2.7)

and
\[
\prod_{i=1}^{q} g(x - i + 1) = \prod_{i=1}^{q} g(x - q + i).
\] (2.8)
Similarly, from [Herz (1955, p. 480)] the multivariate beta function for the space $\mathcal{G}_m^\beta$, can be defined as

$$B_m^\beta[b, a] = \int_{0 < S < I_m} |S|^{a-(m-1)/2} (dS)$$

$$= \int_{R \in \mathcal{G}_m^\beta} |R|^{a-(m-1)/2} (dR)$$

$$= \frac{\Gamma_m^\beta[a] \Gamma_m^\beta[b]}{\Gamma_m^\beta[a+b]}$$  \hspace{1cm} (2.9)$$

where $R = (I - S)^{-1} - I$, $\text{Re}(a) > (m - 1)\beta/2$ and $\text{Re}(b) > (m - 1)\beta/2$.

Some Jacobians in the quaternionic case are obtained in Li and Xue (2009). We now cite some Jacobians in terms of the parameter $\beta$, based on the work of Dimitriu (2002). We also include a parameter count (or number of functionally independent variables, $\#fiv$), that is, if $A$ is factorised as $A = BC$, then the parameter count is written as $\#fiv$ in $A = [\#fiv$ in $B] + [\#fiv$ in $C]$, see Dimitriu (2002).

**Lemma 2.4.** Let $X$ and $Y \in \mathcal{G}_m^\beta$, and let $Y = AXA^* + C$, where $A$ and $C \in L_U^\beta(m)$ are constant matrices. Then

$$(dY) = |A^*A|^\beta(m-1)/2+1 (dX).$$  \hspace{1cm} (2.10)$$

**Lemma 2.5** (Cholesky’s decomposition). Let $S \in \mathcal{G}_m^\beta$ and $T \in \mathcal{G}_U^\beta(m)$ with $t_{ii} > 0$, $i = 1, 2, \ldots, m$. Then

- parameter count: $\beta m (m-1)/2 + m = \beta m (m-1)/2 + m$

$$(dS) = \begin{cases} 2^m \prod_{i=1}^{m} t_{ii}^{\beta(m-i)+1} (dT) & \text{if } S = T^*T; \\ 2^m \prod_{i=1}^{m} t_{ii}^{\beta(i-i)+1} (dT) & \text{if } S = TT^*. \end{cases}$$  \hspace{1cm} (2.11)$$

**Lemma 2.6** (Spectral decomposition). Let $S \in \mathcal{G}_m^\beta$. Then, the spectral decomposition can be written as $S = W\Lambda W^*$, where $W \in \mathcal{U}^\beta(m)$ and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m) \in \mathcal{D}_m^\beta$, with $\lambda_1 > \cdots > \lambda_m > 0$. Then

- parameter count: $\beta m (m-1)/2 + m = [\beta m (m+1)/2 - m - (\beta - 1)m] + [m]$ and

$$(dS) = 2^{-m} \pi^{\beta} \prod_{i<j}^{m} (\lambda_i - \lambda_j)^{\beta} (d\Lambda)(W^*dW),$$  \hspace{1cm} (2.12)$$

where

$$\theta = \begin{cases} 0, & \beta = 1; \\ -m, & \beta = 2; \\ -2m, & \beta = 4; \\ -4m, & \beta = 8. \end{cases}$$

### 3 Integral properties of Jack polynomials

In this section we review and study diverse integral properties of Jack polynomials for normed division algebras. However, let us first consider the following remarks and definitions.

**Remark 3.1.** Note that Jack polynomials and hypergeometric functions with one or two matrix arguments are valid for $\beta > 0$ (Koev and Edelman (2006)), but in our case $\beta$ denotes the real dimension of $\mathfrak{g}$. Also, we use the parameter $\beta$ instead of $\alpha$ in the definition of the Jack polynomials and hypergeometric functions, with the equivalence shown in [Table I](#).
Then:

Let us characterise the Jack symmetric function $J^{(\beta)}_\kappa(\lambda_1, \ldots, \lambda_m)$ of parameter $\beta$, see Sawyer (1997). A decreasing sequence of nonnegative integers $\kappa = (k_1, k_2, \ldots)$ with only finitely many nonzero terms is said to be a partition of $k = \sum k_i$. Let $\kappa$ and $\tau = (t_1, t_2, \ldots)$ be two partitions of $k$. We write $\tau \leq \kappa$ if $\sum_{i=1}^t t_i \leq \sum_{i=1}^r k_i$ for each $t$. The conjugate of $\kappa$ is $\kappa' = (k'_1, k'_2, \ldots)$ where $k'_i = \text{card}\{j : k_j \geq i\}$. The length of $\kappa$ is $l(\kappa) = \max\{i : k_i \neq 0\} = k'_1$. If $l(\kappa) \leq m$, it is often written that $\kappa = (k_1, k_2, \ldots, k_m)$.

The monomial symmetric function $M_\kappa(\cdot)$ indexed by a partition $\kappa$ can be regarded as a function of an arbitrary number of variables such that all but a finite number are equal to 0: if $\lambda_i = 0$ for $i > m \geq l(\kappa)$ then $M_\kappa(\lambda_1, \ldots, \lambda_m) = \sum \lambda_1^{k_1} \cdots \lambda_m^{k_m}$, where the sum is over all distinct permutations $\{\delta_1, \ldots, \delta_m\}$ of $\{k_1, \ldots, k_m\}$, and if $l(\kappa) > m$ then $M_\kappa(\lambda_1, \ldots, \lambda_m) = 0$. A symmetric function $f$ is a linear combination of monomial symmetric functions. If $f$ is a symmetric function then $f(\lambda_1, \ldots, \lambda_m, 0) = f(\lambda_1, \ldots, \lambda_m)$. For each $m \geq 1$, $f(\lambda_1, \ldots, \lambda_m)$ is a symmetric polynomial in $m$ variables.

Then the Jack symmetric function $J^{(\beta)}_\kappa(\lambda_1, \ldots, \lambda_m)$ with a parameter $\beta$, satisfies the following conditions:

$$J^{(\beta)}_\kappa(\lambda_1, \ldots, \lambda_m) = \sum_{\tau \leq \kappa} \nu_{\kappa, \tau}(\beta) M_\tau(\lambda_1, \ldots, \lambda_m), \quad (3.1)$$

$$J^{(\beta)}_\kappa(1, \ldots, 1) = \left(\frac{2}{\beta}\right)^k \prod_{i=1}^m ((m - i + 1)\beta/2)^{k_i}, \quad (3.2)$$

$$D^{(\beta)}_2 J^{(\beta)}_\kappa(\lambda_1, \ldots, \lambda_m) = \sum_{i=1}^m k_i(k_i - 1 + \beta(m - i)) J^{(\beta)}_\kappa(\lambda_1, \ldots, \lambda_m). \quad (3.3)$$

where

$$D^2 = \sum_{i=1}^m \lambda_i^2 \frac{\partial^2}{\partial \lambda_i^2} \beta \sum_{i=1}^m \lambda_i^2 \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \frac{\partial}{\partial \lambda_i}. $$

Here, the constants $\nu_{\kappa, \tau}(\beta)$ do not depend on $\lambda_i$’s but on $\kappa$ and $\tau$. Note that if $m < l(\kappa)$ then $J^{(\beta)}_\kappa(\lambda_1, \ldots, \lambda_m) = 0$. The conditions include the case $\beta = 0$ and then $J^{(0)}(\lambda_1, \ldots, \lambda_m) = e_{e_\kappa} \prod_{i=1}^m (m - i + 1)^{k_i}$, where $e_{e_\kappa}(\lambda_1, \ldots, \lambda_m) = \prod_{i=1}^{l(\kappa)} e_{k_i}(\lambda_1, \ldots, \lambda_m)$ are the elementary symmetric functions indexed by partitions $\kappa$, if $m \geq l(\kappa)$ then $e_{e_\kappa}(\lambda_1, \ldots, \lambda_m) = \sum_{i_1 < i_2 < \cdots < i_r} \lambda_{i_1} \ldots \lambda_{i_r}$, and if $m < l(\kappa)$ then $e_{e_\kappa}(\lambda_1, \ldots, \lambda_m) = 0$, see Sawyer (1997).

Now, from Koev and Edelman (2006), the Jack functions

$$J^{(\beta)}(\lambda_1, \ldots, \lambda_m) = J^{(\beta)}_\kappa(\lambda_1, \ldots, \lambda_m),$$

wher $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of the matrix $X \in \mathfrak{S}_m^\beta$, can be normalised in such a way that

$$\sum_\kappa C^{(\beta)}_\kappa(X) = (\text{tr}(X))^k, \quad (3.4)$$

or equivalently, such that

$$\sum_{k=1}^\infty \sum_\kappa \frac{C^{(\beta)}_\kappa(X)}{k!} = \text{etr}\{X\}, \quad (3.5)$$

where $C^{(\beta)}_\kappa(X)$ denotes the Jack polynomials (for simplicity, we have replaced $(\beta)$ by $\beta$ as the superindex for the Jack polynomials). These are related to the Jack functions by

$$C^{(\beta)}_\kappa(X) = \frac{2^k k!}{\beta^k \nu_{\kappa}} J^{(\beta)}_\kappa(X), \quad (3.6)$$

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where

\[ \nu_\kappa = \prod_{(i,j) \in \kappa} h^*_\kappa(i,j) h^*_\kappa(i,j), \]

and \( h^*_\kappa(i,j) = k_j - i + 2(k_j - j + 1)/\beta \) and \( h^*_\kappa(i,j) = k_j - i + 2(k_i - j)/\beta \) are the upper and lower hook lengths at \( (i,j) \in \kappa \), respectively. Also, observe that for \( X = S^*S \) and \( Y = W^*W \) we have

\[ C^\beta_\kappa(WXW^*) = C^\beta_\kappa(YS^*Y). \]  

(3.7)

In particular for \( A^{1/2} \) such that \((A^{1/2})^2 = A\)

\[ C^\beta_\kappa(Y^{1/2}XY^{1/2}) = C^\beta_\kappa(X^{1/2}YX^{1/2}). \]

(3.8)

Therefore, given that \( XY, YX, Y^{1/2}XY^{1/2} \) and \( X^{1/2}YX^{1/2} \) all have the same eigenvalues, we opt for convenience of notation rather than strict adherence to rigor, and write \( C^\beta_\kappa(XY) \) or \( C^\beta_\kappa(YX) \) rather than \( C^\beta_\kappa(Y^{1/2}XY^{1/2}) \), even though \( XY \) or \( YX \) need not lie in \( S_\beta \). Note that

\[ C^\beta_\kappa(Z^{1/2}XZ^{1/2}Y) = C^\beta_\kappa(XZ^{1/2}YZ^{1/2}), \]  

(3.9)

for all \( X, Y \in S_\beta \) and \( Z \in RZ_\beta \). From Gross and Richards \([1987] \) Equation 4.8(2) and Definition 5.3) we have

\[ C^\beta_\kappa(X) = C^\beta_\kappa(I) \int_{H \in U^{\beta}(m)} q_\kappa(H^*XH)(dH) \]  

(3.10)

for all \( X \in S_\beta \); where \((dH)\) is the normalised Haar measure on \( U^{\beta}(m) \). Finally, for the \( c \) constant we have that \( C^\beta_\kappa(cX) = c^\beta C^\beta_\kappa(X) \).

Some basic integral properties are cited below. For this purpose, we utilise the complexification \( S^{\beta,\epsilon}_m = S_\beta^\epsilon + iS_\beta^\epsilon \) of \( S_\beta \). That is, \( S^{\beta,\epsilon}_m \) consist of all matrices \( X \in (\mathbb{C})^{m \times m} \) of the form \( Z = X + iY \), with \( X, Y \in S_\beta^\epsilon \). We refer to \( X = \text{Re}(Z) \) and \( Y = \text{Im}(Z) \) as the real and imaginary parts of \( Z \), respectively. The generalised right half-plane \( \Phi = \mathbb{R}^{\beta,\epsilon}_m + i\mathbb{R}^{\beta,\epsilon}_m \) in \( S^{\beta,\epsilon}_m \) consists of all \( Z \in S^{\beta,\epsilon}_m \) such that \( \text{Re}(Z) \in \Phi \), see Gross and Richards \([1987] \) p. 801.

For any \( X, Y \in S^{\beta,\epsilon}_m \),

\[ \int_{H \in U^{\beta}(m)} C^\beta_\kappa(XH^*Y)(dH) = \frac{C^\beta_\kappa(X)C^\beta_\kappa(Y)}{C^\beta_\kappa(I)}. \]

(3.11)

For all \( R \in S^{\beta,\epsilon}_m, Z \in \Phi \) and \( \text{Re}(a) > (m - 1)\beta/2 - k_m \),

\[ \int_{X \in \Phi} \text{etr}\{-XZ\}|X|^{a-(m-1)\beta/2-1}C^\beta_\kappa(XR)(dX) \]

\[ = \Gamma^\beta_m[a,\kappa]|Z|^{-a}C^\beta_\kappa(RZ^{-1}) \]

\[ = [a]^\beta_m\Gamma^\beta_m[a] |Z|^{-a}C^\beta_\kappa(RZ^{-1}). \]  

(3.12)

Remark 3.2. In general, the result \((3.12)\) has been established under the condition \( \text{Re}(a) > (m-1)\beta/2 \), see \( \text{Constantine} [1963], \text{Muirhead} [1982], \text{Ratnarahaj et al.} [2005b] \) and \( \text{Li and Xue} [2009] \), but in reality the correct condition is \( \text{Re}(a) > (m - 1)\beta/2 - k_m \). This fact is immediate, observing that \( [a]^\beta_m\Gamma^\beta_m[a] = \Gamma^\beta_m[a,\kappa] \) and the different expressions for \( \Gamma^\beta_m[a,\kappa] \) in \((2.4)\).
Let \( \operatorname{Re}(a) > (m-1)\beta/2 \) and \( \operatorname{Re}(b) > (m-1)\beta/2 \). Then

\[
\int_{0 \leq x < 1} |X|^{a-(m-1)\beta/2-1} (I - X)(b-(m-1)\beta/2-1) C_\kappa^\beta (XR)(dX)
= \frac{\Gamma_m^\beta [a, \kappa] \Gamma_m^\beta [b]}{\Gamma_m^\beta [a+b, \kappa]} C_\kappa^\beta (R)
= \frac{[a]^\beta [b]^\beta}{[a+b]^\beta} C_\kappa^\beta (R),
\]
for all \( R \in \mathbb{S}_m^\beta \), see Gross and Richards (1987), Theorems 5.5 and 5.9 and Corollary 5.10 for real, complex and quaternion cases.

**Remark 3.3.** Observe that result (3.13) was established under the conditions \( \operatorname{Re}(a) > (m-1)\beta/2 \) and \( \operatorname{Re}(b) > (m-1)\beta/2 \), see Constantine (1963), Muirhead (1982), Ratnarajah et al. (2005b) and Li and Xue (2005), but the correct conditions are in fact \( \operatorname{Re}(a) > (m-1)\beta/2 - k_m \) and \( \operatorname{Re}(b) > (m-1)\beta/2 \). This fact is verified by observing that \([a]^\beta [m] = \Gamma_m^\beta [a, \kappa]\) and the different expressions for \( \Gamma_m^\beta [a, \kappa] \) in (3.12).

We now extend several integral properties of zonal polynomials in the real and complex cases to normed division algebras. Our first result is a generalisation of one studied by Teng et al. (1989) for real case, see also Caro-Lopera et al. (2009). From this result, we can obtain diverse particular integral properties of Jack polynomials.

**Theorem 3.4.** Let \( Z \in \Phi \) and \( U \in \mathbb{S}_m^\beta \). Assume \( \gamma = \int_0^\infty f(z)z^{am-k-1}dz < \infty \). Then

\[
\int_{X \in \mathbb{S}_m^\beta} f(tr(XZ)||X|^{a-(m-1)\beta/2-1} C_\kappa^\beta (X^{-1}U)(dX)
= \frac{\Gamma_m^\beta [a, \kappa]}{\Gamma_m^\beta [am-k]} |Z|^{-a} C_\kappa^\beta (UZ^{-1}) \cdot \gamma,
\]
for \( \operatorname{Re}(a) > (m-1)\beta/2 + k_1 \), and

\[
\int_{X \in \mathbb{S}_m^\beta} f(tr(XZ)||X|^{a-(m-1)\beta/2-1} C_\kappa^\beta (XU)(dX)
= \frac{\Gamma_m^\beta [a, \kappa]}{\Gamma_m^\beta [am+k]} |Z|^{-a} C_\kappa^\beta (UZ^{-1}) \cdot \vartheta,
\]
where \( \vartheta = \int_0^\infty f(z)z^{am+k-1}dz < \infty \), \( \operatorname{Re}(a) > (m-1)\beta/2 - k_m \) and \( \kappa = (k_1, \ldots, k_m) \) and \( k = k_1 + \cdots + k_m \).

**Proof.** Denote the left side of (3.14) by \( I(U, Z) \). By (3.10) and interchange of order on integration

\[
I(I, I) = \int_{X \in \mathbb{S}_m^\beta} f(tr(X)||X|^{a-(m-1)\beta/2-1} C_\kappa^\beta (X^{-1})(dX)
= C_\kappa^\beta (I) \int_{X \in \mathbb{S}_m^\beta} f(tr(X)||X|^{a-(m-1)\beta/2-1}
\times \left( \int_{H \in \mathbb{W}_m^\beta} q_\kappa (H^* X^{-1} H)(dH) \right) (dX)
= C_\kappa^\beta (I) \int_{X \in \mathbb{S}_m^\beta} f(tr(X)||X|^{a-(m-1)\beta/2-1} q_\kappa (X^{-1})(dX).
\]
Let $X = TT^*$, from Lemma 2.5

$$(dX) = 2^m \prod_{i=1}^{m} t^{\beta(i-i)+1}(dT).$$

Then

$$I(I, I) = 2^m C^\beta_I(\I) \cdot \int \cdot \int f \left( \sum_{i=1}^{m} t^2_{ij} \right) \prod_{i=1}^{m} (t_{ij})^{2(m-k_i-(m-i)\beta/2)-1} (dT).$$

Applying the Fang and Zhang (1990, Lemma 2.4.3, p. 51) we obtain

$$\text{Corollary 3.5.}$$

Let $X \in P_m^\beta$ and if $a_i < 0$, then

$$\int_{\Phi \in S_m^\beta} f(\text{tr} \ XZ) |X|^{a-(m-1)\beta/2-1} (dX) = \frac{\Gamma_m[a]}{\Gamma[a m]} |Z|^{-a} \cdot \gamma, \quad (3.16)$$

for $\text{Re}(a) > (m-1)\beta/2$.

If we take $f(y) = \exp\{-y\}$ in Corollary 3.5, we obtain

$$\int_{X \in P_m^\beta} \text{etr}\{-XZ\} |X|^{a-(m-1)\beta/2-1} (dX) = \Gamma_m[a] |Z|^{-a}, \quad (3.17)$$

and if $Z = I$ we obtain the multivariate gamma function for the space $S_m^\beta$.

Other particular results of Theorem 3.4 are summarised below.
Corollary 3.6. Let $Z \in \Phi$ and $U \in \mathcal{G}_m^\beta$.

$$
\int_{X \in \mathcal{G}_m^\beta} \text{etr}\{-XZ\}|X|^{a-(m-1)\beta/2-1}C^\beta_\kappa (X^{-1}U) (dX) = \frac{\Gamma_m^\beta [a, \kappa]}{\Gamma\lceil ma - k \rceil} |Z|^{-a}C^\beta_\kappa (UZ),
$$

(3.18)

for $\text{Re}(a) > (m-1)\beta/2 + k_1$.

Proof. This is obtained by taking $f(y) = \exp\{-y\}$ in (3.14).

Corollary 3.7. Let $Z \in \Phi$ and $U \in \mathcal{G}_m^\beta$ and $j \in \mathbb{R}$, such that $\text{Re}(ma + j - k) > 0$, then

$$
\int_{X \in \mathcal{G}_m^\beta} \text{etr}\{-XZ\} \text{tr}X^j |X|^{a-(m-1)\beta/2-1}C^\beta_\kappa (X^{-1}U) (dX) = \frac{\Gamma_m^\beta [a, -\kappa] \Gamma [ma + j - k]}{\Gamma [ma - k]} |Z|^{-a}C^\beta_\kappa (UZ),
$$

(3.19)

for $\text{Re}(a) > (m-1)\beta/2 + k_1$. And if $j$ is such that $\text{Re}(ma + j + k) > 0$, then

$$
\int_{X \in \mathcal{G}_m^\beta} \text{etr}\{-XZ\} \text{tr}X^j |X|^{a-(m-1)\beta/2-1}C^\beta_\kappa (XU) (dX) = \frac{\Gamma_m^\beta [a, \kappa] \Gamma [ma + j + k]}{\Gamma [ma + k]} |Z|^{-a}C^\beta_\kappa (UZ^{-1}),
$$

(3.20)

for $\text{Re}(a) > (m-1)\beta/2 - k_m$.

Proof. The desired result is obtained by taking $f(y) = \exp\{-y\}y^j$ in Theorem 3.4.

Corollary 3.8. Let $Z \in \Phi$ and $U \in \mathcal{G}_m^\beta$ and $\eta > 0$ then

$$
\int_{X \in \mathcal{G}_m^\beta} (1 + 2\eta^{-1} \text{tr}XZ)^{-\beta (am+\eta)} |X|^{a-(m-1)\beta/2-1}C^\beta_\kappa (X^{-1}U) (dX) = \frac{\Gamma_m^\beta [a, -\kappa] \Gamma [(\beta - 1)am + \beta\eta + k]}{(2\eta^{-1})^{am+k} \Gamma [\beta (ma + \eta)]} |Z|^{-a}C^\beta_\kappa (UZ),
$$

(3.21)

for $\text{Re}(a) > (m-1)\beta/2 + k_1$. And

$$
\int_{X \in \mathcal{G}_m^\beta} (1 + 2\eta^{-1} \text{tr}XZ)^{-\beta (am+\eta)} |X|^{a-(m-1)\beta/2-1}C^\beta_\kappa (XU) (dX) = \frac{\Gamma_m^\beta [a, \kappa] \Gamma [(\beta - 1)am + \beta\eta - k]}{(2\eta^{-1})^{am+k} \Gamma [\beta (ma + \eta)]} |Z|^{-a}C^\beta_\kappa (UZ^{-1}),
$$

(3.22)

for $\text{Re}(a) > (m-1)\beta/2 - k_m$.

Proof. The desired result is obtained by taking $f(y) = (1 + 2\eta^{-1}y)^{-\beta (am+\eta)}$ in Theorem 3.4.

Many other interesting particular cases of Theorem 3.4 can be found, for example by defining $f(\text{tr}XZ)$ as the kernel of matrix variate generalised Wishart distributions, see Fang and Zhang (1990) and Gupta and Varga (1993).

Important analogues of the beta function integral are given in the following theorems. Theorem 3.9 is discussed by Khatri (1966) in the real case.
Theorem 3.9. If $R \in \mathcal{S}_{m}^{\beta, \varepsilon}$, then

\[
\int_{X \in \mathcal{P}_{m}^{\beta}} |X|^{a-(m-1)\beta/2-1} |I + X|^{-(a+b)} C_{\kappa}^{\beta}(RX^{-1})(dX) = \frac{\Gamma_{m}^{\beta}[a, -\kappa]\Gamma_{m}^{\beta}[b, \kappa]}{\Gamma_{m}^{\beta}[a + b]} C_{\kappa}^{\beta}(R), \tag{3.23}
\]

for $\text{Re}(a) > (m-1)\beta/2 + k_1$ and $\text{Re}(b) > (m-1)\beta/2 - k_m$. And

\[
\int_{X \in \mathcal{P}_{m}^{\beta}} |X|^{a-(m-1)\beta/2-1} |I + X|^{-(a+b)} C_{\kappa}^{\beta}(RX)(dX) = \frac{\Gamma_{m}^{\beta}[a, \kappa]\Gamma_{m}^{\beta}[b, -\kappa]}{\Gamma_{m}^{\beta}[a + b]} C_{\kappa}^{\beta}(R), \tag{3.24}
\]

for $\text{Re}(a) > (m-1)\beta/2 - k_m$ and $\text{Re}(b) > (m-1)\beta/2 + k_1$.

Proof. By Corollary 3.6 we have for any $Z \in \Phi$

\[
\int_{X \in \mathcal{P}_{m}^{\beta}} \text{etr}\{-XZ\} |X|^{a-(m-1)\beta/2-1} C_{\kappa}^{\beta}(X^{-1}R) |Z|^{a}(dX) = \Gamma_{m}^{\beta}[a, -\kappa] C_{\kappa}^{\beta}(RZ), \tag{3.25}
\]

Multiplying both sides of (3.25) by $\text{etr}\{-Z\} |Z|^{b-(m-1)\beta/2-1}$ and integrating with respect to $Z$ we have

\[
\int_{X \in \mathcal{P}_{m}^{\beta}} \left( \int_{Z \in \mathcal{P}_{m}^{\beta}} \text{etr}\{-I + XZ\} |Z|^{a+b-(m-1)\beta/2-1} (dZ) \right) |X|^{a-(m-1)\beta/2-1} C_{\kappa}^{\beta}(X^{-1}R) (dX)
\]

\[
= \Gamma_{m}^{\beta}[a, -\kappa] \int_{Z \in \mathcal{P}_{m}^{\beta}} \text{etr}\{-Z\} |Z|^{b-(m-1)\beta/2-1} C_{\kappa}^{\beta}(RZ) (dZ). \tag{3.26}
\]

The desired result in (3.23) is obtained by using Corollary 3.6 and 3.17 in the left and right sides of (3.26), respectively. The result in (3.24) is obtained similarly. 

Corollary 3.10. If $R \in \mathcal{S}_{m}^{\beta, \varepsilon}$, then

\[
\int_{0 < X < 1} |X|^{a-(m-1)\beta/2-1} |I - X|^{b-(m-1)\beta/2-1} C_{\kappa}^{\beta}(RX^{-1})(dX) = \frac{\Gamma_{m}^{\beta}[a, -\kappa]\Gamma_{m}^{\beta}[b]}{\Gamma_{m}^{\beta}[a + b, -\kappa]} C_{\kappa}^{\beta}(R), \tag{3.27}
\]

for $\text{Re}(a) > (m-1)\beta/2 + k_1$ and $\text{Re}(b) > (m-1)\beta/2$.

Proof. It is obtained in a similar way to that given for (3.13), see Gross and Richards (1987). 

Now, taking $b = (m-1)\beta/2 + 1 > (m-1)\beta/2$, from (3.13) and Corollary 3.10 we have the following result.
Corollary 3.11. If $R \in \mathcal{G}_m^\beta$, then
\[
\int_{0 < X < 1} |X|^{a-(m-1)\beta/2-1}C_\kappa^\beta(RX^{-1})(dX) = \frac{\Gamma_m^\beta[a,-\kappa] \Gamma_m^\beta[(m-1)\beta/2+1]}{\Gamma_m^\beta[a+(m-1)\beta/2+1,-\kappa]} C_\kappa^\beta(R),
\tag{3.28}
\]
for $\text{Re}(a) > (m-1)\beta/2 + k_1$. And
\[
\int_{0 < X < 1} |X|^{-a-(m-1)\beta/2-1}C_\kappa^\beta(XR)(dX) = \frac{\Gamma_m^\beta[a,\kappa] \Gamma_m^\beta[(m-1)\beta/2+1]}{\Gamma_m^\beta[a+(m-1)\beta/2+1,\kappa]} C_\kappa^\beta(R),
\tag{3.29}
\]
for $\text{Re}(a) > (m-1)\beta/2 - k_m$.

Similarly, taking $a = (m-1)\beta/2 + 1 > (m-1)\beta/2 - k_m$ in (3.13), we have the following result.

Corollary 3.12. If $R \in \mathcal{G}_m^\beta$, then
\[
\int_{0 < X < 1} |I - X|^{-b-(m-1)\beta/2-1}C_\kappa^\beta(XR)(dX) = \frac{\Gamma_m^\beta[(m-1)\beta/2+1,\kappa] \Gamma_m^\beta[b]}{\Gamma_m^\beta[(m-1)\beta/2+1+b,\kappa]} C_\kappa^\beta(R),
\tag{3.30}
\]
for $\text{Re}(b) > (m-1)\beta/2$.

4 Hypergeometric functions

In this section, we study diverse integral properties of hypergeometric functions for normed division algebras. First, let us consider the following definition.

Fix complex numbers $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$, and for all $1 \leq i \leq q$ and $1 \leq j \leq m$ do not allow $-b_i + (j-1)\beta/2$ to be a nonnegative integer. Then the hypergeometric function with one matrix argument $pF_q^\beta$ is defined to be the real-analytic function on $\mathcal{G}_m^\beta$ given by the series
\[
pFq{\beta}{a_1, \ldots, a_p}{b_1, \ldots, b_q}{X} = \sum_{k=0}^{\infty} \frac{[a_1]_\kappa^\beta \cdots [a_p]_\kappa^\beta [b_1]_\kappa^\beta \cdots [b_q]_\kappa^\beta}{k!} C_\kappa^\beta(X).
\tag{4.1}
\]

Some known properties are, see Gross and Richards (1987, Section 6, pp. 803-810):

Convergence hypergeometric functions.

1. If $p \leq q$ then the hypergeometric series (4.1) converges absolutely for all $X \in \mathcal{G}_m^\beta$.

2. If $p = q + 1$ then the series (4.1) converges absolutely for $||X|| = \max\{|\lambda_i| : i = 1, \ldots, m\} < 1$, and diverges for $||X|| > 1$, where $\lambda_1, \ldots, \lambda_m$ are the $i$-th eigenvalues of $X \in \mathcal{G}_m^\beta$.

3. If $p > q$ then the series (4.1) diverges unless it terminates.

For all $X \in \mathcal{G}_m^\beta$, indeed, for all $X \in \mathcal{G}_m^\beta$. This is characteristic of the general situation when $p \leq q$.
\[
nFq{\beta}{a_1, \ldots, a_q}{X} = \sum_{k=0}^{\infty} \frac{C_\kappa^\beta(X)}{k!} = \sum_{k=0}^{\infty} \frac{(\text{tr } X)^k}{k!} = \text{etr}(X),
\tag{4.2}
\]
If \( \text{Re}(a) > (m - 1)\beta/2 \), and \( ||X|| < 1 \),
\[
1F_0^\beta(a; X) = \frac{1}{\Gamma_m[a]} \int_{Y \in \mathcal{S}^\beta_m} \text{etr}\{- (I - X) Y\}|Y|^{a-(m-1)\beta/2-1}(dY) \quad (4.3)
\]
gives the full analytic continuation of \( 1F_0^\beta(a; \cdot) \) to any simply-connected domain in \( \mathcal{S}^\beta_m \).
The right side is determined by the principal branch of the argument. The fact that the hypergeometric series \( 1F_0^\beta \) has \( \{X \in \mathcal{S}^\beta_m : ||X|| < 1\} \) as its domain of convergence is characteristic of \( p+1F_p^\beta \) for all \( p \geq 0 \).

Let \( \text{Re}(c) > \text{Re}(a) + (m - 1)\beta/2 > (m - 1)\beta \) and \( ||X|| < 1 \). Then
\[
p+1F_p^\beta(a_1, \ldots, a_p; b_1, \ldots, b_q; X) = \frac{1}{\mathcal{E}_m[a, c - a]} \int_{0 < Y < I_m} pF_p^\beta(a_1 \cdots a_p; b_1 \cdots b_q; XY) \times |Y|^{a-(m-1)\beta/2-1} |I - Y|^{c-a-(m-1)\beta/2-1}(dY), \quad (4.4)
\]
for \( p = q + 1 \). In particular, for \( p = 2 \), we have the Euler formula
\[
2F_1^\beta(a_1, a; c; X) = \frac{1}{\mathcal{E}_m[a, c - a]} \times \int_{0 < Y < I_m} 1F_0^\beta(a_1; XY)|Y|^{a-(m-1)\beta/2-1} \times |I - Y|^{c-a-(m-1)\beta/2-1}(dY). \quad (4.5)
\]
for arbitrary \( a_1 \), \( \text{Re}(c) > \text{Re}(a) + (m - 1)\beta/2 > (m - 1)\beta \) and \( ||X|| < 1 \).

Remark 4.1. Observe that (4.1) (and of course (4.5), too) is a consequence of (3.13). And then the condition over \( a \) and \( c \) must be \( \text{Re}(c) > \text{Re}(a) + (m - 1)\beta/2 > (m - 1)\beta - k_m \).

Laplace transform of hypergeometric functions. Assume \( p \leq q \), \( \text{Re}(a) > (m - 1)\beta/2 \) and \( U \in \mathcal{S}_m^\beta \). Then
\[
\int_{X \in \mathcal{S}^\beta_m} \text{etr}\{- XZ\} pF_q^\beta(a_1 \cdots a_p; b_1 \cdots b_q; XU)|X|^{a-(m-1)\beta/2-1}(dX) = |Z|^{-a} \Gamma_m^\beta[a] \quad p+1F_p^\beta(a_1 \cdots a_p; b_1 \cdots b_q; UZ^{-1}). \quad (4.6)
\]
When \( p < q \), the integral in (4.6) converges absolutely for all \( Z \in \Phi \). When \( p = q \), the integral converges absolutely for all \( Z \in \mathcal{S}^{\beta,\xi}_m \), such that \( ||(\text{Re}(Z))^{-1}|| < 1 \).

Remark 4.2. Herz (1955, p. 485) wrote...”The convergence of the integral (4.6) requires at least \( \text{Re}(a) > (m - 1)\beta/2 \)...” However, Gross and Richards (1987) based their proof of (4.6) in their Theorem 5.9 which is valid for \( \text{Re}(a) > (m - 1)\beta/2 - k_m \). The same remark should be considered in the versions of Constantine (1963), Muirhead (1982), Ratnarajah et al (2005) and Li and Xue (2009).

Similarly, the hypergeometric function of two matrix arguments \( pF_q^{(m),\beta} \) is defined to be the real-analytic function on \( \mathcal{S}^{\beta}_m \) given by the series
\[
pF_q^{(m),\beta}(a_1, \ldots, a_p; b_1, \ldots, b_q; X, Y) = \sum_{k=0}^\infty \sum_{\mathbf{r}} \frac{[a_1]^{\beta} \cdots [a_p]^{\beta}}{[b_1]^{\beta} \cdots [b_q]^{\beta}} \frac{C_k^\beta(X)C_k^\beta(Y)}{k!} \quad (4.7)
\]
Some basic properties of (1.7) are shown below, see Gross and Richards (1987).

Convergence hypergeometric functions with two matrix arguments.
1. If \( p \leq q \) then the hypergeometric series (4.7) converges absolutely for all \( X \) and \( Y \) in \( \mathbb{G}_m^\beta \).

2. If \( p = q + 1 \) then the series (4.7) converges absolutely for \( ||X|| \cdot ||Y|| < 1 \), and diverges for \( ||X|| \cdot ||Y|| > 1 \).

Also

\[
\int_{H \in \mathbb{H}^{\beta}(m)} p F_q^\beta(a_1, \ldots, a_p; b_1, \ldots, b_q; XHY^*) (dH) = p F_q^\beta(a_1, \ldots, a_p; b_1, \ldots, b_q; X, Y) \tag{4.8}
\]

In particular

\[
\int_{H \in \mathbb{H}^{\beta}(m)} 0 F_0^\beta(XHY^*) (dH) = \int_{H \in \mathbb{H}^{\beta}(m)} \text{etr}(XHY^*) (dH) \tag{4.10}
\]

\[
= 0 F_0^\beta(X, Y) \tag{4.11}
\]

**Laplace transform of hypergeometric functions with two matrix arguments.**

Assume \( p \leq q \), \( \text{Re}(a) > (m - 1)\beta/2 \) and \( U \in \mathbb{G}_m^\beta \). Then

\[
\int_{X \in \mathbb{G}_m^\beta} \text{etr}(-XZ) p F_q^\beta(a_1 \cdots a_p; b_1 \cdots b_q; XU, Y)|X|^{a-(m-1)\beta/2-1} (dX)
= |Z|^{-a} \Gamma_m^\beta[a] \int_{\Phi_\alpha^\beta(a, \kappa)} \text{etr}(XZ) |Z|^{-a} C_\kappa^\beta(UZ^{-1}, Y) (dZ) \tag{4.12}
\]

When \( p < q \), the integral in (4.12) converges absolutely for all \( Z \in \Phi \) and \( Y \in \mathbb{G}_m^\beta \). When \( p = q \), the integral converges absolutely for all \( Z \) and \( Y \in \mathbb{G}_m^\beta \), such that \( \|\text{Re}(Z)\| - \|Y\| < 1 \). Remark similar to (4.2) should be considered for (4.12).

We now propose further integral properties of hypergeometric functions for normed division algebras. The first result is the inverse Laplace transformation. In the real case, this result was obtained by Herz (1955), Constantine (1963), James (1964) and Muirhead (1982, p. 261), and in the complex case by Mathai (1997, p. 370). Let us first consider the following extension of similar results discussed in Constantine (1963), see also Muirhead (1982, p. 253).

**Lemma 4.3.** Assume that \( Z \) and \( X \in \mathbb{G}_m^\beta, \kappa \), \( U \in \mathbb{G}_m^\beta \) and \( \text{Re}(a) > a_0 \). Then

\[
\frac{2(m-1)\beta/2}{(2\pi i)^{m-1})\beta/2+m} \int_{X-Z_0 \in \Phi} \text{etr}(XZ) |Z|^{-a} C_\kappa^\beta(UZ^{-1}, Y) (dZ) = \frac{1}{\Gamma_m^\beta[a, \kappa]} |X|^{a-(m-1)\beta/2-1} C_\kappa^\beta(XU), \tag{4.13}
\]

where \( Z_0 \in \Phi_\alpha^\beta \).

**Theorem 4.4.** Assume that \( Z \) and \( X \in \mathbb{G}_m^\beta, \kappa \), \( U \in \mathbb{G}_m^\beta \) and \( \text{Re}(b) > b_0 \). Then

\[
\frac{\Gamma_m^\beta[b] 2^{m-1)\beta/2}}{(2\pi i)^{m-1})\beta/2+m} \int_{X-Z_0 \in \Phi} \text{etr}(XZ) |Z|^{-b} p F_q^\beta(a_1, \ldots, a_p; b_1, \ldots, b_q; UZ^{-1}, Y) (dZ) = |X|^{b-(m-1)\beta/2-1} p F_q^\beta(a_1, \ldots, a_p; b_1, \ldots, b_q, b; XU), \tag{4.14}
\]

and if \( Y \in \mathbb{G}_m^\beta \)

\[
\frac{\Gamma_m^\beta[b] 2^{m-1)\beta/2}}{(2\pi i)^{m-1})\beta/2+m} \int_{X-Z_0 \in \Phi} \text{etr}(XZ) |Z|^{-b} \times p F_q^\beta(a_1, \ldots, a_p; b_1, \ldots, b_q; UZ^{-1}, Y)(dZ)
\]
\[ \mathbf{Z}_0 \in \mathcal{W}_m^\beta. \]

**Proof.** Proof of both (4.14) and (4.15) follows by expanding the \( pF_q^\beta \) and \( pF_q^{(m)} \) functions in the integrands and integrating term by term using (4.13). \( \square \)

**Theorem 4.5.** The \( 1F_1^\beta \) function has the integral representation

\[
1F_1^\beta(a,c;\mathbf{X}) = \frac{1}{\mathcal{B}_m[a,c-a]} \int_{0<|\mathbf{Y}|<1} \text{etr}\{\mathbf{X}\mathbf{Y}\}|\mathbf{Y}|^{a-(m-1)\beta/2-1} \\
\times |\mathbf{I} - \mathbf{Y}|^{-a-(m-1)\beta/2-1} (d\mathbf{Y}),
\]

valid for \( \text{Re}(c) > \text{Re}(a) + (m-1)\beta/2 > (m-1)\beta - k_m \) and all \( \mathbf{X} \in \mathfrak{S}_m^\beta. \)

**Proof.** The desired result is obtained by expanding \( \text{etr}\{\mathbf{X}\mathbf{Y}\} \) using (3.30) and integrating term by term using (3.13). \( \square \)

The generalised Kummer and Euler relations are given in the following result.

**Theorem 4.6.**

\[
1F_1^\beta(a,c;\mathbf{X}) = \text{etr}\{\mathbf{X}\}1F_1^\beta(c-a,c; -\mathbf{X}),
\]

for \( \mathbf{X} \in \mathfrak{S}_m^\beta. \) And

\[
2F_1^\beta(a,b;c;\mathbf{X}) = |\mathbf{I} - \mathbf{X}|^b 2F_1^\beta(c-a,b;c; -\mathbf{X}(\mathbf{I} - \mathbf{X})^{-1})
\]

\[= |\mathbf{I} - \mathbf{X}|^{c-a-b} 2F_1^\beta(c-a,c-b;c;\mathbf{X})\]  

for \( ||\mathbf{X}|| < 1.\)

**Remark 4.7.** Observe that, for \( 1F_1^\beta(a;\mathbf{X}) \) the condition \( \text{Re}(a) > (m-1)\beta/2 \) over \( a \) is determined by its integral representation (4.30). However \( 1F_0^\beta(a;\mathbf{X}) \) is easily seen to be analytic for all \( a \) and \( ||\mathbf{X}|| < 1, \) see [Herz (1955), p. 486]. Similarly, the conditions \( \text{Re}(c) > \text{Re}(a) + (m-1)\beta/2 > (m-1)\beta - k_m \) over \( a \) and \( c \) given in Theorem 4.5, valid for Theorem 4.6, (4.17) too, are determined by the existence of the integral (4.10). However, these conditions can be extended to other possible values if we use the inverse Laplace transformation to define \( 1F_1^\beta(a,c;\mathbf{X}) \). In this case \( 1F_1^\beta(a,c;\mathbf{X}) \) is valid for the arbitrary complex \( a, \text{Re}(c) > (m-1)\beta/2 \) and \( \mathbf{X} \in \mathfrak{S}_m^\beta, \) see [Herz (1955), p. 487]. Also, the conditions \( \text{Re}(c) > \text{Re}(a) + (m-1)\beta/2 > (m-1)\beta - k_m \) over \( a \) and \( c \) for (4.5) and Theorem 4.6, (4.18) and (4.19) are determined by the absolutely convergence of the integral (4.5). Again, these conditions about \( a \) and \( c \) can be extended to other possible values using the inverse Laplace transformation and the results for \( 1F_1^\beta(a,c;\mathbf{X}) \) obtained as described before, see [Herz (1955), p. 489]. Finally, let us take into account that, for any analysis if the integral representation of \( 1F_1^\beta(a,c;\mathbf{X}) \) or \( 2F_1^\beta(a,b;c;\mathbf{X}) \) is not used explicitly, then the extended conditions for \( a \) and \( c \) could be considered.

**Theorem 4.8.** Let \( \mathbf{X} \in \mathcal{L}_m^\beta, \) and \( \mathbf{H} = (\mathbf{H}_1|\mathbf{H}_2) \in \mathcal{W}_m^\beta(m) \), \( \mathbf{H}_1 \in \mathcal{V}_m^\beta \). Then

\[
0F_1^\beta(\beta n/2; \beta^2 \mathbf{X}^* / 4) = \int_{\mathbf{H} \in \mathcal{W}_m^\beta(m)} \text{etr}(\beta \mathbf{X}\mathbf{H}_1)(d\mathbf{H})
\]

\[
= \int_{\mathbf{H}_1 \in \mathcal{V}_m^\beta} \text{etr}(\beta \mathbf{X}\mathbf{H}_1)(d\mathbf{H}_1)
\]
Proof. The proof is analogous to that given in the real case for Muirhead (1982, Theorem 7.4.1) and in the quaternion case by [Li and Xue (2009)]. Alternative proofs can be established in an analogous form to those given by James (1961) and Herz (1955, p. 494-495). For (4.21) it might be necessary to consider Lemma 9.5.3, p. 397 in Muirhead (1982).

On the basis of Theorem 3.3 we now discuss diverse integral properties of generalised hypergeometric functions, which contain as particular cases many of the results established above.

**Theorem 4.9.** Assume \( p \leq q \) and \( \text{Re}(a) > (m - 1)\beta/2 - k_m \) and \( U \in \mathbb{S}_m^\beta \). Then for \( \theta = \int_0^\infty f(z)z^{am+k-1}dz < \infty \),

\[
\int_{x \in \Psi_m^\beta} f(tr \{ XZ \} p F_q^{(m)} (a_1 \cdots a_p; b_1 \cdots b_q; XU)) |X|^{a - (m - 1)\beta/2 - 1} (dX) = |Z|^{-a} \Gamma^\beta_m[a] \sum_{k=0}^{\infty} \sum_{\kappa} \frac{|a_1|_\kappa^\beta \cdots |a_p|_\kappa^\beta |a_q|_\kappa^\beta}{|b_1|_\kappa^\beta \cdots |b_q|_\kappa^\beta} \frac{C^\beta_k(\text{UZ})^{\gamma}}{\Gamma[am + k]k!} \cdot \theta. \tag{4.22}
\]

When \( p < q \), the integral in (4.22) converges absolutely for all \( Z \in \Phi \). When \( p = q \), the integral converges absolutely for all \( Z \in \mathbb{S}_m^\beta \), such that \( ||\text{Re}(Z)||^{-1} \leq 1 \).

Similarly, let \( p \leq q \), \( \text{Re}(a) > (m - 1)\beta/2 - k_m \) and \( U \in \mathbb{S}_m^\beta \). Then

\[
\int_{x \in \Psi_m^\beta} f(tr \{ XZ \} p F_q^{(m)} (a_1 \cdots a_p; b_1 \cdots b_q; XU, Y)) |X|^{a - (m - 1)\beta/2 - 1} (dX) = |Z|^{-a} \Gamma^\beta_m[a] \sum_{k=0}^{\infty} \sum_{\kappa} \frac{|a_1|_\kappa^\beta \cdots |a_p|_\kappa^\beta |a_q|_\kappa^\beta}{|b_1|_\kappa^\beta \cdots |b_q|_\kappa^\beta} \frac{C^\beta_k(\text{UZ}) C^\beta_k(Y)^\gamma}{\Gamma[am + k]k! C_k^\beta(I)} \cdot \theta. \tag{4.23}
\]

When \( p < q \), the integral in (4.23) converges absolutely for all \( Z \in \Phi \) and \( Y \in \mathbb{S}_m^\beta \). When \( p = q \), the integral converges absolutely for all \( Z \) and \( Y \in \mathbb{S}_m^\beta \), such that \( ||\text{Re}(Z)||^{-1} \cdot ||Y|| < 1 \).

Observe that if \( f(tr \{ XZ \} = \text{etr}\{-XZ\} \) in Theorem 4.9 then we obtain (4.6) and (4.12).

**Theorem 4.10.** Assume \( p \leq q \) and \( \text{Re}(a) > (m - 1)\beta/2 - k_m \) and \( U \in \mathbb{S}_m^\beta \). Then for \( \gamma = \int_0^\infty f(z)z^{am+k-1}dz < \infty \),

\[
\int_{x \in \Psi_m^\beta} f(tr \{ XZ \} p F_q^{(m)} (a_1 \cdots a_p; b_1 \cdots b_q; X^{-1}U)) |X|^{a - (m - 1)\beta/2 - 1} (dX) = |Z|^{-a} \Gamma^\beta_m[a] \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-1)^k |a_1|_\kappa^\beta \cdots |a_p|_\kappa^\beta |a_q|_\kappa^\beta}{|b_1|_\kappa^\beta \cdots |b_q|_\kappa^\beta} \frac{C^\beta_k(\text{UZ})^{\gamma}}{\Gamma[am + k]k!} \cdot \gamma. \tag{4.24}
\]

When \( p < q \), the integral in (4.24) converges absolutely for all \( Z \in \Phi \). When \( p = q \), the integral converges absolutely for all \( Z \in \mathbb{S}_m^\beta \), such that \( ||\text{Re}(Z)||^{-1} \cdot ||y|| < 1 \).

Similarly, let \( p \leq q \), \( \text{Re}(a) > (m - 1)\beta/2 - k_m \) and \( U \in \mathbb{S}_m^\beta \). Then

\[
\int_{x \in \Psi_m^\beta} f(tr \{ XZ \} p F_q^{(m)} (a_1 \cdots a_p; b_1 \cdots b_q; X^{-1}U, Y)) |X|^{a - (m - 1)\beta/2 - 1} (dX) = |Z|^{-a} \Gamma^\beta_m[a] \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-1)^k |a_1|_\kappa^\beta \cdots |a_p|_\kappa^\beta |a_q|_\kappa^\beta}{|b_1|_\kappa^\beta \cdots |b_q|_\kappa^\beta} \frac{C^\beta_k(\text{UZ}) C^\beta_k(Y)^\gamma}{\Gamma[am + k]k! C_k^\beta(I)} \cdot \gamma. \tag{4.25}
\]
When $p < q$, the integral in (4.25) converges absolutely for all $Z \in \Phi$ and $Y \in \mathcal{S}_m^p$. When $p = q$, the integral converges absolutely for all $Z$ and $Y \in \mathcal{S}_m^p$, such that $\|(\text{Re}(Z))^{-1}\| \cdot ||Y|| < 1$.

Now, let us define $f(\text{tr} XZ) = \text{etr}(-XZ)$ in Theorem 4.10 then we obtain:

$$\int_{X \in \mathcal{P}_m^a} \text{etr}(-XZ) p F_m^{\beta}(a_1 \cdots a_p) \cdot b_1 \cdots b_q; X^{-1} U) |X|^{a -(m-1)\beta/2 -1} (dX)$$

$$= |Z|^{-a} \Gamma_m^0[a] \cdot p F_{q+1}^\beta(a_1 \cdots a_p; b_1 \cdots b_q, -a + (m - 1)\beta/2 + 1; -UZ).$$

and

$$\int_{X \in \mathcal{P}_m^a} \text{etr}(-XZ) p F_{(m)}^{\beta}(a_1 \cdots a_p; b_1 \cdots b_q; X^{-1} U, Y) |X|^{a -(m-1)\beta/2 -1} (dX)$$

$$= |Z|^{-a} \Gamma_m^0[a] \cdot p F_{q+1}^{\beta}(a_1 \cdots a_p; b_1 \cdots b_q, -a + (m - 1)\beta/2 + 1; -UZ, Y),$$

where for both Re$(a) > (m - 1)\beta/2 + k_1$.

Similar results to (4.6) and (4.12) or (4.26) and (4.27) can be obtained from Corollaries 3.7 and 3.8.

Now, we propose the incomplete gamma and beta functions for normed division algebras.

**Theorem 4.11.** Let $A \in \mathcal{S}_m^\beta$ and $\Omega \in \Phi$. Then

$$\int_{0 < \zeta < \Omega} \text{etr}(-AX) |X|^{a -(m-1)\beta/2 -1} (dX)$$

$$= B_m^\beta[a, (m-1)\beta/2 + 1] |\Omega|^{a_1} F_m^\beta(a; a + (m-1)\beta/2 + 1; -\Omega A),$$

for Re$(a) > (m - 1)\beta/2 - k_m$. And, let $0 < \Xi < I$, then

$$\int_{0 < Y < \Xi} |Y|^{a -(m-1)\beta/2 -1} |I - Y|^{b -(m-1)\beta/2 -1} (dY) = B_m^\beta[a, (m-1)\beta/2 + 1]$$

$$\times \cdot |\Xi|^{a_1} F_m^\beta(a, -b + (m - 1)\beta/2 + 1; a + (m - 1)\beta/2 + 1; \Xi),$$

for Re$(a) > (m - 1)\beta/2 - k_m$ and Re$(b) > (m - 1)\beta/2$.

**Proof.** For (4.28), let us make the transformation $X = \Omega^{1/2} R \Omega^{1/2}$ and by applying Lemma 2.4 we have, $(dX) = |\Omega|^{(m+1)\beta/2 + 1} (dR)$, with $0 < R < I$. Then, expanding $\text{etr}(-AX)$ as a series of Jack polynomials and integrating term by term using Corollary 3.4 the desired result is obtained. Similarly, (4.29) is proved by making the transformation $Y = \Xi^{1/2} R \Xi^{1/2}$ from where, applying the Lemma 2.4 we obtain that $(dX) = |\Xi|^{(m+1)\beta/2 + 1} (dR)$, with $0 < R < I$, expanding $|I - X \Xi|^{b -(m-1)\beta/2 -1} = 1 F_m^\beta(-b + (m - 1)\beta/2 + 1; X \Xi)$ and integrating term by term using Corollary 3.11.

**Theorem 4.12.** Let $A \in \mathcal{S}_m^\beta$ and $\Omega \in \Phi$. If $r = a - (m-1)\beta/2 - 1$ is a positive integer, then

$$\int_{X > \Omega} \text{etr}(-AX) |X|^{a -(m-1)\beta/2 -1} (dX)$$

$$= \Gamma_m^0[a] |\Omega|^{-a} \text{etr}(-\Omega A) \sum_{k=0}^r \sum_{\kappa} \frac{C_m^\kappa(\Omega A)}{k!},$$

for Re$(a) > (m - 1)\beta/2 + k_1$ and $\sum_\kappa$ denotes summation over those partitions $\kappa = (k_1, \ldots, k_m)$ of $k$ with $k_1 \leq r$. 

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Theorem 4.13. Let $X \in \mathcal{P}_m^\beta$ be a random matrix with density function $f(X)$. Then the joint density function of the eigenvalues $\lambda_1, \ldots, \lambda_m$ of $X$ is

$$\frac{\pi^{m^2/2+\theta}}{\Gamma^\beta_m(m\beta/2)} \prod_{i<j} (\lambda_i - \lambda_j)^\beta \int_{H \in \mathfrak{U}^\beta(m)} f(\Omega \Lambda H^\dagger)(dH) \quad (4.31)$$

where $L = \text{diag}(\lambda_1, \ldots, \lambda_m)$, $\lambda_1 > \cdots > \lambda_m > 0$, $\theta$ is defined in Lemma 2.6 and $(dH)$ is the normalised Haar measure.

Proof. The proof follows immediately from Lemma 2.6.

5 Invariant polynomials

In this section, we extend many of the properties of a class of homogeneous polynomials for normed division algebras of degrees $k$ and $t$ in the elements of matrices $X$ and $Y \in \mathfrak{E}_m^\beta$, respectively, see Davis (1979), Davis (1980), Chikuse (1980) and Chikuse and Davis (1986); these are denoted as $c_{\phi}^{[\beta]_{k,\tau}}(X, Y)$. These homogeneous polynomials are invariant under the simultaneous transformations

$$X \to U^\dagger XU, \quad Y \to U^\dagger YU, \quad H \in \mathfrak{U}^\beta(m).$$

The most important relationship of these polynomials is

$$\int_{H \in \mathfrak{U}^\beta(m)} C^\beta_\lambda(A H^\dagger X H) C^\beta_\tau(B H^\dagger Y H)(dH) = \sum_{\phi \in \kappa, \tau} C_{\phi, \lambda}^{[\beta]_{k,\tau}}(A, B) C_{\phi, \tau}^{[\beta]_{k,\tau}}(X, Y) C^\beta_\phi(1), \quad (5.1)$$

where $(dH)$ is the normalised Haar measure and $C^\beta_\lambda$, $C^\beta_\tau$ and $C^\beta_\phi$ are Jack polynomials indexed by ordered partitions $\kappa$, $\tau$ and $\phi$ of nonnegative integers $k$, $t$ and $f = k + t$, respectively, into not more than $m$ parts. $\phi \in \kappa, \tau$ denotes the irreducible representation of $\text{GL}(m, \mathbb{F})$ indexed by $2\phi$ that occurs in the decomposition of the Kronecker product $2\kappa \otimes 2\tau$ of the irreducible representations indexed by $2k$ and $2\tau$, see Davis (1979) and Davis (1980).

In a similar way to the case of Jack polynomials, let $A = A^\dagger A$ and $B = B^\dagger B$. We opt for convenience of notation rather than strict adherence to rigour, and write $C^{[\beta]_{k,\tau}}_\phi(A X A, Y B)$,
or $C^{[\beta]\kappa,\tau}_\phi(AX, BY)$ rather than $C^{[\beta]\kappa,\tau}_\phi(AXA^*, BYB^*)$, even though $XA$, $YB$, $AX$, or $BY$ need not lie in $\mathcal{S}_m^\beta$.

Some of the elementary properties and results on invariant polynomials are extended below:

**Elementary properties of $C^{[\beta]\kappa,\tau}_\phi$.**

Let $X$ and $Y \in \mathcal{S}_m^\beta$, then

$$C^{[\beta]\kappa,\tau}_\phi(X, X) = \theta^{[\beta]\kappa,\tau}_\phi C^{[\beta]}_\phi(X), \quad \text{where} \quad \theta^{[\beta]\kappa,\tau}_\phi = \frac{\beta C^{[\kappa,\tau]}_\phi(I, I)}{C^{[\beta]}_\phi(I)}.$$  \hfill (5.2)

$$C^{[\beta]\kappa,\tau}_\phi(X, Y) = \begin{cases} \frac{\theta^{[\beta]\kappa,\tau}_\phi C^{[\beta]}_\phi(I)}{\theta^{[\beta,\kappa,\tau]}_\phi C^{[\beta]}_\phi(I)} C^{[\beta]}_\kappa(X), & \text{for } Y = I; \\
\frac{\theta^{[\beta]\kappa,\tau}_\phi C^{[\beta]}_\phi(I)}{C^{[\beta]}_\tau(I)} C^{[\beta]}_\tau(Y), & \text{for } X = I. \end{cases}$$ \hfill (5.3)

$$C^{[\beta]\kappa,0}_\phi(X, Y) = C^{[\beta]}_\kappa(X), \quad \text{and} \quad C^{[\beta,0,\tau]}_\phi(X, Y) = C^{[\beta]}_\tau(Y).$$ \hfill (5.4)

$$C^{[\beta]}_\kappa(X) C^{[\beta]}_\tau(Y) = \sum_{\phi \in \kappa, \tau} \theta^{[\beta]\kappa,\tau}_\phi C^{[\beta]\kappa,\tau}_\phi(X, Y),$$ \hfill (5.5)

therefore,

$$(\text{tr } X)^k (\text{tr } Y)^t = \sum_{\kappa, \tau, \phi \in \kappa, \tau} \theta^{[\beta]\kappa,\tau}_\phi C^{[\beta]\kappa,\tau}_\phi(X, Y).$$ \hfill (5.6)

From (5.2) and (5.5)

$$C^{[\beta]}_\kappa(X) C^{[\beta]}_\tau(X) = \sum_{\phi \in \kappa, \tau} \left( \theta^{[\beta]\kappa,\tau}_\phi \right)^2 C^{[\beta]}_\phi(X).$$ \hfill (5.7)

For constant $a$ and $b$

$$C^{[\beta]\kappa,\tau}_\phi(aX, bY) = a^k b^t C^{[\beta]\kappa,\tau}_\phi(X, Y).$$ \hfill (5.8)

The next expansion can be used to derive several useful results of invariant polynomials. From (5.1), (5.5) and (4.2) we obtain

$$\int_{H \in \mathcal{H}^\beta(m)} \text{etr}\{AH^*XH + BH^*YH\} (dH) = \sum_{\kappa, \tau, \phi} \sum_{k=0}^{\infty} \sum_{\tau} \sum_{\phi \in \kappa, \tau} \frac{C^{[\beta]\kappa,\tau}_\phi(A, B) C^{[\beta]\kappa,\tau}_\phi(X, Y)}{k!} C^{[\beta]}_\phi(I).$$ \hfill (5.9)

where

$$\sum_{\kappa, \tau, \phi} \sum_{k=0}^{\infty} \sum_{\tau} \sum_{\phi \in \kappa, \tau} \sum_{\kappa, \tau, \phi}.$$

From (5.9) we obtain, see Díaz-García (2008),

$$\int_{H \in \mathcal{H}^\beta(m)} C^{[\beta]\kappa,\tau}_\phi(AH^*XH, BH^*YH) (dH) = \frac{C^{[\beta]\kappa,\tau}_\phi(A, B) C^{[\beta]\kappa,\tau}_\phi(X, Y)}{\theta^{[\beta]\kappa,\tau}_\phi C^{[\beta]}_\phi(I)},$$ \hfill (5.10)

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in particular using (5.3)

\[ \int_{H \in \Omega^\beta(m)} C_{\phi}^{[\beta] \kappa, \tau}(A^* H^* X H A, B)(dH) = \frac{C_{\phi}^{[\beta] \kappa, \tau}(A^* A, B)C_{\phi}^{\beta}(X)}{C_{\phi}^{\beta}(I)}, \quad (5.11) \]

analogously

\[ \int_{H \in \Omega^\beta(m)} C_{\phi}^{[\beta] \kappa, \tau}(A, B^* H^* Y H B)(dH) = \frac{C_{\phi}^{[\beta] \kappa, \tau}(A, B^* B)C_{\phi}^{\beta}(Y)}{C_{\phi}^{\beta}(I)}, \quad (5.12) \]

Laplace transform.

For all \( A \) and \( B \in \mathcal{S}_m^\beta, Z \in \Phi \)

\[ \int_{X \in \Psi^\alpha_m} \text{etr}\{-XZ\} |X|^{a-(m-1)\beta/2-1} C_{\phi}^{[\beta] \kappa, \tau}(A X, B X)(dX) = \Gamma_m^\beta[a, \phi]|Z|^{-a} C_{\phi}^{[\beta] \kappa, \tau}(A Z^{-1} B Z^{-1}), \quad (5.13) \]

valid for \( \text{Re}(a) > (m - 1)\beta/2 + (k + t)_1 \). In particular

\[ \int_{X \in \Psi^\alpha_m} \text{etr}\{-XZ\} |X|^{a-(m-1)\beta/2-1} C_{\phi}^{[\beta] \kappa, \tau}(A X A^*, B)(dX) = \Gamma_m^\beta[a, \kappa]|Z|^{-a} C_{\phi}^{[\beta] \kappa, \tau}(A Z^{-1} A^*, B), \quad (5.14) \]

where \( \text{Re}(a) > (m - 1)\beta/2 + k_1 \). And

\[ \int_{X \in \Psi^\alpha_m} \text{etr}\{-XZ\} |X|^{a-(m-1)\beta/2-1} C_{\phi}^{[\beta] \kappa, \tau}(A, B X B^*)(dX) = \Gamma_m^\beta[a, \tau]|Z|^{-a} C_{\phi}^{[\beta] \kappa, \tau}(A, B Z^{-1} B^*), \quad (5.15) \]

with \( \text{Re}(a) > (m - 1)\beta/2 + t_1 \). Similarly, for all \( A \) and \( B \in \mathcal{S}_m^\beta, Z \in \Phi \),

\[ \int_{X \in \Psi^\alpha_m} \text{etr}\{-XZ\} |X|^{a-(m-1)\beta/2-1} C_{\phi}^{[\beta] \kappa, \tau}(A X^{-1}, B X^{-1})(dX) = \Gamma_m^\beta[a, -\phi]|Z|^{-a} C_{\phi}^{[\beta] \kappa, \tau}(A Z, B Z), \quad (5.16) \]

where \( \text{Re}(a) > (m - 1)\beta/2 - (k + t)_m \). In particular

\[ \int_{X \in \Psi^\alpha_m} \text{etr}\{-XZ\} |X|^{a-(m-1)\beta/2-1} C_{\phi}^{[\beta] \kappa, \tau}(A X^{-1} A^*, B)(dX) = \Gamma_m^\beta[a, -\kappa]|Z|^{-a} C_{\phi}^{[\beta] \kappa, \tau}(A Z A^*, B), \quad (5.17) \]

with \( \text{Re}(a) > (m - 1)\beta/2 - k_m \). And

\[ \int_{X \in \Psi^\alpha_m} \text{etr}\{-XZ\} |X|^{a-(m-1)\beta/2-1} C_{\phi}^{[\beta] \kappa, \tau}(A, B X^{-1} B^*)(dX) = \Gamma_m^\beta[a, -\tau]|Z|^{-a} C_{\phi}^{[\beta] \kappa, \tau}(A, B Z B^*), \quad (5.18) \]

valid for \( \text{Re}(a) > (m - 1)\beta/2 - t_m \).
Inverse Laplace transform.

Assume that $Z$ and $X \in \mathcal{G}^{\bar{\beta},\bar{\epsilon}}_m$, $A$ and $B \in \mathcal{G}^{\bar{\beta}}_m$ and $\text{Re}(b) > b_0$. Then

$$
\frac{\Gamma_m^\beta [b, \phi] 2^{m(m-1)/\beta}}{(2\pi)\Gamma_m^{m(m-1)/\beta/2+m}} \int_{Z - Z_0 \in \Phi} \text{etr}(XZ) |Z|^{-b} C^{[\beta]_{\kappa,\tau}}_{\phi}(AZ^{-1}, BZ^{-1})(dZ) = |X|^{b-(m-1)/\beta/2-1} C^{[\beta]_{\kappa,\tau}}_{\phi}(AX, BX), \quad (5.19)
$$

and

$$
\frac{\Gamma_m^\beta [b, -\phi] 2^{m(m-1)/\beta}}{(2\pi)\Gamma_m^{m(m-1)/\beta/2+m}} \int_{Z - Z_0 \in \Phi} \text{etr}(XZ) |Z|^{-b} C^{[\beta]_{\kappa,\tau}}_{\phi}(AZ, BZ)(dZ) = |X|^{b-(m-1)/\beta/2-1} C^{[\beta]_{\kappa,\tau}}_{\phi}(AX^{-1}, BX^{-1}). \quad (5.20)
$$

Similar expressions are obtained for $C^{[\beta]_{\kappa,\tau}}_{\phi}(AZ^{-1}, B)$ and $C^{[\beta]_{\kappa,\tau}}_{\phi}(A, BZ^{-1})$ from (5.19); and for $C^{[\beta]_{\kappa,\tau}}_{\phi}(AZ, B)$ and $C^{[\beta]_{\kappa,\tau}}_{\phi}(A, BZ)$ from (5.20).

Beta type I integrals.

For all $A$ and $B \in \mathcal{G}^{\bar{\beta},\bar{\epsilon}}_m$ and $\text{Re}(b) > (m-1)/\beta/2$,

$$
\int_{0 < X < 1} |X|^{a-(m-1)/\beta/2-1} |I - X|^{b-(m-1)/\beta/2-1} C^{[\beta]_{\kappa,\tau}}_{\phi}(AX, BX)(dX) = \frac{\Gamma_m^\beta [a, \phi] \Gamma_m^\beta [b]}{\Gamma_m^{3}[a + b, \phi]} C^{[\beta]_{\kappa,\tau}}_{\phi}(A, B), \quad (5.21)
$$

valid for $\text{Re}(a) > (m-1)/\beta/2 - (k + t)_m$. In particular

$$
\int_{0 < X < 1} |X|^{a-(m-1)/\beta/2-1} |I - X|^{b-(m-1)/\beta/2-1} C^{[\beta]_{\kappa,\tau}}_{\phi}(AXA^*, B)(dX) = \frac{\Gamma_m^\beta [a, \kappa] \Gamma_m^\beta [b]}{\Gamma_m^{3}[a + b, \kappa]} C^{[\beta]_{\kappa,\tau}}_{\phi}(A A^*, B), \quad (5.22)
$$

with $\text{Re}(a) > (m-1)/\beta/2 - k_m$. And

$$
\int_{0 < X < 1} |X|^{a-(m-1)/\beta/2-1} |I - X|^{b-(m-1)/\beta/2-1} C^{[\beta]_{\kappa,\tau}}_{\phi}(A, BXB^*)(dX) = \frac{\Gamma_m^\beta [a, \tau] \Gamma_m^\beta [b]}{\Gamma_m^{3}[a + b, \tau]} C^{[\beta]_{\kappa,\tau}}_{\phi}(A, B B^*). \quad (5.23)
$$

where $\text{Re}(a) > (m-1)/\beta/2 - t_m$. Another particular integral given in the real case by Davis (1979) is

$$
\int_{0 < X < 1} |X|^{a-(m-1)/\beta/2-1} |I - X|^{b-(m-1)/\beta/2-1} C^{[\beta]_{\kappa,\tau}}_{\phi}(X, I - X^*)(dX) = \frac{\Gamma_m^\beta [a, \kappa] \Gamma_m^\beta [b, \tau]}{\Gamma_m^{3}[a + b, \phi]} \delta^{[\beta]_{\kappa,\tau}}_{\phi} C^{[\beta]_{\kappa,\tau}}_{\phi}(I), \quad (5.24)
$$

valid for $\text{Re}(a) > (m-1)/\beta/2 - k_m$ and $\text{Re}(b) > (m-1)/\beta/2 - k_m$. 

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Analogously, for all $A$ and $B \in \mathcal{S}_m^{\beta, \gamma}$ and $\text{Re}(b) > (m - 1)\beta/2$,
\[
\int_{0 < x < 1} |X|^{a-(m-1)\beta/2-1}|I - X|^{b-(m-1)\beta/2-1}C_{\phi}^{\beta, \kappa, \tau}(AX^{-1}, BX^{-1})(dX) = \frac{\Gamma_{m}^{\beta}[a, -\phi]\Gamma_{m}^{\beta}[b]}{\Gamma_{m}[a + b, -\phi]}C_{\phi}^{\beta, \kappa, \tau}(A, B), \tag{5.25}
\]
where $\text{Re}(a) > (m - 1)\beta/2 + (k + t)_1$. In particular
\[
\int_{0 < x < 1} |X|^{a-(m-1)\beta/2-1}|I - X|^{b-(m-1)\beta/2-1}C_{\phi}^{\beta, \kappa, \tau}(AX^{-1}A^*, B)(dX) = \frac{\Gamma_{m}^{\beta}[a, -\kappa]\Gamma_{m}^{\beta}[b]}{\Gamma_{m}[a + b, -\kappa]}C_{\phi}^{\beta, \kappa, \tau}(A^*, B), \tag{5.26}
\]
valid for $\text{Re}(a) > (m - 1)\beta/2 + k_1$. And
\[
\int_{0 < x < 1} |X|^{a-(m-1)\beta/2-1}|I - X|^{b-(m-1)\beta/2-1}C_{\phi}^{\beta, \kappa, \tau}(A, BX^{-1}B^*)(dX) = \frac{\Gamma_{m}^{\beta}[a, -\tau]\Gamma_{m}^{\beta}[b]}{\Gamma_{m}[a + b, -\tau]}C_{\phi}^{\beta, \kappa, \tau}(A, B B^*), \tag{5.27}
\]
with $\text{Re}(a) > (m - 1)\beta/2 + t_1$.

Now, taking $b = (m - 1)\beta/2 + 1 > (m - 1)\beta/2$ in (5.21) and (5.25) we have the following results.

For all $A$ and $B \in \mathcal{S}_m^{\beta, \gamma}$,
\[
\int_{0 < x < 1} |X|^{a-(m-1)\beta/2-1}C_{\phi}^{\beta, \kappa, \tau}(AX, BX)(dX) = \frac{\Gamma_{m}^{\beta}[a, \phi]\Gamma_{m}^{\beta}[(m - 1)\beta/2 + 1]}{\Gamma_{m}[a + (m - 1)\beta/2 + 1, \phi]}C_{\phi}^{\beta, \kappa, \tau}(A, B), \tag{5.28}
\]
valid for $\text{Re}(a) > (m - 1)\beta/2 - (k + t)_m$. And,
\[
\int_{0 < x < 1} |X|^{a-(m-1)\beta/2-1}C_{\phi}^{\beta, \kappa, \tau}(AX^{-1}, BX^{-1})(dX) = \frac{\Gamma_{m}^{\beta}[a, -\phi]\Gamma_{m}^{\beta}[(m - 1)\beta/2 + 1]}{\Gamma_{m}[a + (m - 1)\beta/2 + 1, -\phi]}C_{\phi}^{\beta, \kappa, \tau}(A, B), \tag{5.29}
\]
where $\text{Re}(a) > (m - 1)\beta/2 + (k + t)_1$.

**Beta type II integrals.**

For all $A$ and $B \in \mathcal{S}_m^{\beta, \gamma}$,
\[
\int_{X \in \mathcal{P}_m^\beta} |X|^{a-(m-1)\beta/2-1}|I + X|^{-(a+b)}C_{\phi}^{\beta, \kappa, \tau}(AX, BX) = \frac{\Gamma_{m}^{\beta}[a, \phi]\Gamma_{m}^{\beta}[b, -\phi]}{\Gamma_{m}[a + b]}C_{\phi}^{\beta, \kappa, \tau}(A, B), \tag{5.30}
\]
with $\text{Re}(a) > (m - 1)\beta/2 - (k + t)_m$ and $\text{Re}(b) > (m - 1)\beta/2 + (k + t)_1$. In particular
\[
\int_{X \in \mathcal{P}_m^\beta} |X|^{a-(m-1)\beta/2-1}|I + X|^{-(a+b)}C_{\phi}^{\beta, \kappa, \tau}(AXA^*, B) = \frac{\Gamma_{m}^{\beta}[a, \kappa]\Gamma_{m}^{\beta}[b, -\kappa]}{\Gamma_{m}[a + b]}C_{\phi}^{\beta, \kappa, \tau}(A^*, B), \tag{5.31}
\]
\[24\]
such that $\text{Re}(a) > (m - 1)\beta/2 - k_m$ and $\text{Re}(b) > (m - 1)\beta/2 + k_1$. And

$$
\int_{X \in \Psi_m^\alpha} |X|^{a - (m-1)\beta/2 - 1} |I + X|^{-(a+b)} C_{\phi}^{[\beta] \kappa, \tau}(A, BX^*) \\
= \frac{\Gamma_m^\beta [a, \tau] \Gamma_m^\beta [b, -\tau]}{\Gamma_m^\beta [a + b]} C_{\phi}^{[\beta] \kappa, \tau}(A, BB^*),
$$

(5.32)
valid for $\text{Re}(a) > (m - 1)\beta/2 - t_m$ and $\text{Re}(b) > (m - 1)\beta/2 + t_1$.

In a similar way, for all $A$ and $B \in \mathcal{S}_m^\beta$, $\mathcal{E}$,

$$
\int_{X \in \Psi_m^\alpha} |X|^{a - (m-1)\beta/2 - 1} |I + X|^{-(a+b)} C_{\phi}^{[\beta] \kappa, \tau}(AX^{-1}B, BX^{-1}) \\
= \frac{\Gamma_m^\beta [a, -\phi] \Gamma_m^\beta [b, \phi]}{\Gamma_m^\beta [a + b]} C_{\phi}^{[\beta] \kappa, \tau}(A, B),
$$

(5.33)
where $\text{Re}(a) > (m - 1)\beta/2 + (k + t)_1$ and $\text{Re}(b) > (m - 1)\beta/2 - (k + t)_m$. In particular

$$
\int_{X \in \Psi_m^\alpha} |X|^{a - (m-1)\beta/2 - 1} |I + X|^{-(a+b)} C_{\phi}^{[\beta] \kappa, \tau}(AX^{-1}A^*, B) \\
= \frac{\Gamma_m^\beta [a, -\kappa] \Gamma_m^\beta [b, \kappa]}{\Gamma_m^\beta [a + b]} C_{\phi}^{[\beta] \kappa, \tau}(AA^*, B),
$$

(5.34)
with $\text{Re}(a) > (m - 1)\beta/2 + k_1$ and $\text{Re}(b) > (m - 1)\beta/2 - k_m$. And

$$
\int_{X \in \Psi_m^\alpha} |X|^{a - (m-1)\beta/2 - 1} |I + X|^{-(a+b)} C_{\phi}^{[\beta] \kappa, \tau}(A, BX^{-1}B^*) \\
= \frac{\Gamma_m^\beta [a, -\tau] \Gamma_m^\beta [b, \tau]}{\Gamma_m^\beta [a + b]} C_{\phi}^{[\beta] \kappa, \tau}(A, BB^*),
$$

(5.35)
such that $\text{Re}(a) > (m - 1)\beta/2 + t_1$ and $\text{Re}(b) > (m - 1)\beta/2 - t_m$.

**Incomplete gamma and beta functions.**

First consider the following results

For all $A$ and $B \in \mathcal{S}_m^\beta$ and $0 < E < I$,

$$
\int_{0 < X < \Xi} |X|^{a - (m-1)\beta/2 - 1} C_{\phi}^{[\beta] \kappa, \tau}(AX, BX)(dX) \\
= \frac{\Gamma_m^\beta [a, \phi] \Gamma_m^\beta [(m - 1)\beta/2 + 1, \phi]}{\Gamma_m^\beta [a + (m - 1)\beta/2 + 1, \phi]} |\Omega|^\beta C_{\phi}^{[\beta] \kappa, \tau}(A\Xi, B\Xi).
$$

(5.36)
valid for $\text{Re}(a) > (m - 1)\beta/2 - (k + t)_m$. And

$$
\int_{0 < X < \Xi} |X|^{a - (m-1)\beta/2 - 1} C_{\phi}^{[\beta] \kappa, \tau}(AXA^*, B)(dX) \\
= \frac{\Gamma_m^\beta [a, \kappa] \Gamma_m^\beta [(m - 1)\beta/2 + 1, \kappa]}{\Gamma_m^\beta [a + (m - 1)\beta/2 + 1, \kappa]} |\Omega|^\beta C_{\phi}^{[\beta] \kappa, \tau}(A\Xi A^*, B).
$$

(5.37)
valid for $\text{Re}(a) > (m - 1)\beta/2 - k_m$. 

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The next result is obtained immediately, expanding $\text{etr}\{-XA\}$ in terms of Jack polynomials, making use of the property (5.5) and integrating term by term using (5.36). Thus, for all $A$ and $B \in \mathbb{S}_m$ and $\Omega \in \Phi$,}

$$
\int_{0 < X < \Omega} \text{etr}\{-XA\} |X|^{a-(m-1)\beta/2-1} C_\beta(AX) (dX) = \frac{\Gamma_\beta^m[a] \Gamma_\beta^m[(m-1)\beta/2 + 1]}{\Gamma_\beta^m[a + (m-1)\beta/2 + 1]} \sum_{k=0}^{\infty} \sum_{\kappa,\phi \in \kappa,\tau} \frac{[a]_\kappa^\beta [\beta]_\kappa^\tau C_\beta(\kappa,\tau)(-A\Omega, B\Omega)}{k! [a + (m-1)\beta/2 + 1]_\phi^\beta}.
$$

valid for $\text{Re}(a) > (m-1)\beta/2 + (k+t)$.

Similarly, we expand $|I - X|^{b-(m-1)\beta/2-1}$ in terms of Jack polynomials, make use of the property (5.5) and integrate term by term using (5.36). For all $A \in \mathbb{S}_m^d$ and $0 < \Xi < I$,

$$
\int_{0 < X \leq \Xi} |X|^{a-(m-1)\beta/2-1} |I - X|^{b-(m-1)\beta/2-1} C_\beta(AX) (dX) = \frac{\Gamma_\beta^m[a] \Gamma_\beta^m[(m-1)\beta/2 + 1]}{\Gamma_\beta^m[a + (m-1)\beta/2 + 1]} \sum_{k=0}^{\infty} \sum_{\kappa,\phi \in \kappa,\tau} \frac{[-b + (m-1)\beta/2 + 1]_\kappa^\beta [\beta]_\kappa^\tau C_\beta(\kappa,\tau)(\Omega, A\Omega)}{k! [a + (m-1)\beta/2 + 1]_\phi^\beta}.
$$

valid for $\text{Re}(a) > (m-1)\beta/2 - (k+t)m$.

6 Application

As an application, in this section we found the joint density eigenvalue of the central Wishart distribution for normed division algebras, and also derived the largest and smallest eigenvalue distributions. First, from [Díaz-García and Gutiérrez (2009)], let us consider the following definitions.

**Definition 6.1.** Let $X \in \mathcal{L}_{m,n}^\beta$ be a random matrix. Then $X$ is said to have a matrix variate normal distribution $X \sim \mathcal{N}_{n,m}^\beta(\mu, \Sigma, \Theta)$, of mean $\mu$ and $\text{Cov}(\text{vec}X^T) = \Theta \otimes \Sigma$, if its density function is given by

$$
\frac{1}{(2\pi)^{mn/2} |\Sigma|^{n/2} |\Theta|^{m/2} \text{etr}\{-\frac{\beta}{2} \Sigma^{-1}(X - \mu)^\top \Theta^{-1}(X - \mu)\}}.
$$

Also

**Definition 6.2.** Let $X \in \mathcal{L}_{m,n}^\beta$ with distribution $X \sim \mathcal{N}_{n,m}^\beta(0, \Sigma, I_n)$ and define $S = X^\top X$, then $S$ is said to have a central Wishart distribution $S \sim \mathcal{W}_{m,n}^\beta$ with $n$ degrees of freedom and parameter $\Sigma$. Moreover, its density function is given by

$$
\frac{1}{(2\pi)^{mn/2} \Gamma_m^\beta [\beta n/2] |\Sigma|^{\beta n/2} \text{etr}\{-\beta \Sigma^{-1}S/2\}}
$$

with $n \geq (m-1)\beta$.

Therefore, from (4.31) and (4.11) the joint density of the eigenvalues, $\lambda_1 > \cdots > \lambda_m > 0$, of $S$ is

$$
\frac{\pi^{m^2 \beta/2 + g}}{(2\pi)^{mn/2} \Gamma_m^\beta [\beta n/2] \prod_{i=1}^{m} \lambda_i^{\beta(n-m+1)/2-1}} \prod_{i=1}^{m} \lambda_i^{\beta(n-m+1)/2-1} \times \prod_{i<j}^{m} (\lambda_i - \lambda_j)^{\beta} F_0(-\beta \Sigma^{-1}/2, L)
$$
where \( L = \text{diag}(\lambda_1, \ldots, \lambda_m) \).

In addition, as an immediate consequence of Theorems \[ \text{Theorem 6.3. Let } S \sim W_m^\beta(n, \Sigma) \text{ and } \Omega \in \Phi, \text{ then}
\]

\[
P(S < \Omega) = \frac{\Gamma_m^\beta[(m-1)\beta/2 + 1]}{(2\beta - 1)^{3mn/2} \Gamma_m^\beta[(n + m - 1)\beta/2 + 1]} \frac{\Omega^{3n/2}}{\Sigma^{3n/2}} \times 1 F_1^\beta(\beta n/2; (n + m - 1)\beta/2 + 1; -\beta \Omega \Sigma^{-1}/2),
\]

valid for \( \text{Re}(n) > (m-1)\beta - 2k_m \). And if \( r = (n - m + 1)\beta/2 - 1 \) is a positive integer, then

\[
P(\Omega > S) = \text{etr}(-\beta \Omega \Sigma^{-1}/2) \sum_{k=0}^{mr} \sum_{\kappa} \frac{C^\beta_\kappa(\beta \Omega \Sigma^{-1}/2)}{k!},
\]

for \( \text{Re}(n) > (m-1)\beta + 2k_1 \) and \( \sum_\kappa \) denotes summation over those partitions \( \kappa = (k_1, \ldots, k_m) \) of \( k \) with \( k_1 \leq r \).

Observing that if \( \lambda_{\max} \) and \( \lambda_{\min} \) are the largest and smallest eigenvalues of \( S \), respectively, then the inequalities \( \lambda_{\max} < x \) and \( \lambda_{\min} > y \) are equivalent to \( S < xI \) and \( S > yI \), respectively and the following result is obtained.

**Corollary 6.4. Assume that \( S \sim W_m^\beta(n, \Sigma) \) and \( x > 0 \). Then**

\[
P(\lambda_{\max} < x) = \frac{\Gamma_m^\beta[(m-1)\beta/2 + 1]}{(2\beta - 1)^{3mn/2} \Gamma_m^\beta[(n + m - 1)\beta/2 + 1]} \frac{x^{3n/2}}{\Sigma^{3n/2}} \times 1 F_1^\beta(\beta n/2; (n + m - 1)\beta/2 + 1; -\beta x \Sigma^{-1}/2),
\]

valid for \( \text{Re}(n) > (m-1)\beta - 2k_m \). And if \( r = (n - m + 1)\beta/2 - 1 \) is a positive integer and \( y > 0 \), then

\[
P(\lambda_{\min} < y) = 1 - \text{etr}(-\beta y \Sigma^{-1}/2) \sum_{k=0}^{mr} \sum_{\kappa} \frac{C^\beta_\kappa(\beta y \Sigma^{-1}/2)}{k!},
\]

for \( \text{Re}(n) > (m-1)\beta + 2k_1 \) and \( \sum_\kappa \) denotes summation over those partitions \( \kappa = (k_1, \ldots, k_m) \) of \( k \) with \( k_1 \leq r \).

As a numerical example we plot the distribution function of \( \lambda_{\max} \) on Figure 1 and the distribution function of \( \lambda_{\min} \) on Figure 2. First note that applying the generalised Kummer relation \[ \text{(4.17 in (6.3)) we obtain}
\]

\[
P(\lambda_{\max} < x) = \frac{\Gamma_m^\beta[(m-1)\beta/2 + 1] \text{etr}(-\beta x \Sigma^{-1}/2)}{(2\beta - 1)^{3mn/2} \Gamma_m^\beta[(n + m - 1)\beta/2 + 1]} \frac{x^{3n/2}}{\Sigma^{3n/2}} \times 1 F_1^\beta((m-1)\beta/2 + 1; (n + m - 1)\beta/2 + 1; \beta x \Sigma^{-1}/2).
\]
The largest eigenvalue distribution of real central Wishart distribution

The largest eigenvalue distribution of complex central Wishart distribution

The largest eigenvalue distribution of quaternion central Wishart distribution

The largest eigenvalue distribution of octonion central Wishart distribution

Figure 1: Distribution functions of \( \lambda_{\text{max}} \) of \( W_{\beta}^2(4, \text{diag}(1, 2)) \), \( \beta = 1, 2, 4 \) and 8.
The smallest eigenvalue distribution of real central Wishart distribution

\[ F(y) = P(Y < y) \]

The smallest eigenvalue distribution of complex central Wishart distribution

\[ F(y) = P(Y < y) \]

The smallest eigenvalue distribution of quaternion central Wishart distribution

\[ F(y) = P(Y < y) \]

The smallest eigenvalue distribution of octonion central Wishart distribution

\[ F(y) = P(Y < y) \]

Figure 2: Distribution functions of \( \lambda_{\text{min}} \) of \( W_\beta^2(7, \text{diag}(1,2)) \), \( \beta = 1, 2, 4 \) and 8.

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