Abstract. Symplectic forms taming complex structures on compact manifolds are strictly related to Hermitian metrics having the fundamental form $\partial \bar{\partial}$-closed, i.e. to strong Kähler with torsion (SKT) metrics. It is still an open problem to exhibit a compact example of a complex manifold having a tamed symplectic structure but non-admitting Kähler structures. We show some negative results for the existence of symplectic forms taming complex structures on compact quotients of Lie groups by discrete subgroups. In particular, we prove that if $M$ is a nilmanifold (not a torus) endowed with an invariant complex structure $J$, then $(M, J)$ does not admit any symplectic form taming $J$. Moreover, we show that if a nilmanifold $M$ endowed with an invariant complex structure $J$ admits an SKT metric, then $M$ is at most 2-step. As a consequence we classify 8-dimensional nilmanifolds endowed with an invariant complex structure admitting an SKT metric.

1. Introduction

Let $(M, \Omega)$ be a compact $2n$-dimensional symplectic manifold. An almost complex structure $J$ on $M$ is said to be tamed by $\Omega$ if

$$\Omega(X, JX) > 0$$

for any non-zero vector field $X$ on $M$. When $J$ is a complex structure (i.e. $J$ is integrable) and $\Omega$ is tamed by $J$, the pair $(\Omega, J)$ has been called a Hermitian-symplectic structure in [40]. Although any symplectic structure always admits tamed almost complex structures, it is still an open problem to find an example of a compact complex manifold admitting a Hermitian-symplectic structure, but no Kähler structures. From [40] there exist no examples in dimension 4. Moreover, the study of tamed symplectic structures in dimension 4 is related to a more general conjecture of Donaldson (see for instance [8] [15] [20]).

A natural class where to search examples of Hermitian-symplectic manifolds is provided by compact quotients of Lie groups by discrete subgroups, since this set typically contains examples of manifolds admitting both complex structures and symplectic structures, but no Kähler structures. Indeed, it is well known that a nilmanifold, i.e. a compact quotient of a nilpotent Lie group by a discrete subgroup, cannot admit any Kähler metric unless it is a torus (see for instance [4] [24]). Moreover, in the case of solvmanifolds, i.e. compact quotients of solvable Lie groups by discrete subgroups, Hasegawa...
showed in [25, 26] that a solvmanifold has a Kähler structure if and only if
it is covered by a finite quotient of a complex torus, which has the structure
of a complex torus bundle over a complex torus.

In this paper we show some negative results for the existence of Hermitian-
symplectic structures on compact quotients of Lie groups by discrete sub-
groups. First of all we show that Hermitian-symplectic structures are strictly
related to a “special” type of Hermitian metric, the SKT one. We recall (see
[18]) that a $J$-Hermitian metric $g$ on a complex manifold $(M, J)$ is called
SKT (Strong Kähler with Torsion) or pluriclosed if the fundamental form
$\omega(\cdot, \cdot) = g(J\cdot, \cdot)$ satisfies
$$\partial\bar{\partial} \omega = 0.$$  
For complex surfaces a Hermitian metric satisfying the SKT condition is
standard in the terminology of Gauduchon [20] and one standard metric
can be found in the conformal class of any given Hermitian metric on a
compact manifold. However, the theory is different in higher dimensions.
SKT metrics have a central role in type II string theory, in 2-dimensional
supersymmetric $\sigma$-models (see [18, 41]) and they have also relations with
generalized Kähler geometry (see for instance [18, 23, 27, 2, 6, 15]).

SKT metrics have been recently studied by many authors. For instance,
new simply-connected compact strong KT examples have been recently con-
structed by Swann in [42] via the twist construction. Moreover, in [16] it
has been shown that the blow-up of an SKT manifold $M$ at a point or along
a compact submanifold admits an SKT metric.

For real Lie groups admitting SKT metrics there are only some classi-
cification results in dimension 4 and 6. More precisely, 6-dimensional SKT
nilpotent Lie groups have been classified in [14] and a classification of SKT
solvable Lie groups of dimension 4 has been recently obtained in [30].

We outline our paper. In Section 2 we show that the existence of a
Hermitian-symplectic structure on a complex manifold $(M, J)$ is equivalent
to the existence of a $J$-compatible SKT metric $g$ whose fundamental form
$\omega$ satisfies $\partial\omega = \partial\bar{\partial}\beta$ for some $\partial$-closed $(2, 0)$-form $\beta$ (see Proposition 2.1).
This result allows us to write down a natural obstruction to the existence
of a Hermitian-symplectic structure on a compact complex manifold (see
Lemma 2.2).

Let $G/\Gamma$ a compact quotient of a simply-connected Lie group $G$ by a
discrete subgroup $\Gamma$. By an invariant complex (resp. symplectic) structure
on $G/\Gamma$ we will mean one induced by a complex (resp. symplectic) structure
on the Lie algebra $g$ of $G$. In Section 3 we start by showing that the existence
of a symplectic form taming an invariant complex structure $J$ on $G/\Gamma$ implies
the existence of an invariant one (see Proposition 3.3). This can be used to
prove the following

**Theorem 1.1.** Let $(M = G/\Gamma, J)$ be a compact quotient of a Lie group by
a discrete subgroup endowed with an invariant complex structure $J$. Assume
that the center $\xi$ of the Lie algebra $\mathfrak{g}$ associated to $G$ satisfies
\begin{equation}
J\xi \cap [\mathfrak{g}, \mathfrak{g}] \neq \{0\},
\end{equation}
then $(M, J)$ does not admit any symplectic form taming $J$.

In the case of a a simply-connected nilpotent Lie group $G$ if the structure equations of its Lie algebra $\mathfrak{g}$ are rational, then there exists a discrete subgroup $\Gamma$ of $G$ for which the quotient $G/\Gamma$ is compact $[32]$. A general classification of real nilpotent Lie algebras exists only for dimension less or equal than 7 and 6 is the highest dimension in which there do not exist continuous families $(31, 21)$. It turns out that in dimension 6 by $[14]$ any 6-dimensional non-abelian nilpotent Lie algebra admitting an SKT metric is 2-step nilpotent. Moreover, by $[12]$ any 6-dimensional SKT nilmanifold is the twist of the Kähler product of the two torus $\mathbb{T}^4$ and $\mathbb{T}^2$ by using two integral 2-forms supported on $\mathbb{T}^4$. In this paper we extend the previous result to any dimension.

**Theorem 1.2.** Let $(M = G/\Gamma, J)$ be a nilmanifold (not a torus) endowed with an invariant complex structure $J$. Assume that there exists a $J$-Hermitian SKT metric $g$ on $M$. Then $g$ is 2-step nilpotent and $M$ is a total space of a principal holomorphic torus bundle over a torus.

Moreover, we show that an invariant SKT metric compatible with an invariant complex structure on a nilmanifold is always standard (Corollary 3.8).

Theorem 1.1 and Theorem 1.2 will be used to prove the following

**Theorem 1.3.** Let $(M = G/\Gamma, J)$ be a nilmanifold (not a torus) with an invariant complex structure. Then $(M, J)$ does not admit any symplectic form taming $J$.

It can be observed that there exist examples of nilpotent Lie algebras not satisfying condition (1.1) (see Example 1). Hence Theorem 1.3 cannot be directly deduced from Theorem 1.1.

In dimension 8 some examples of nilpotent Lie algebras endowed with an SKT metric have been found in $[33, 17]$ and $[37]$, but there is no general classification result. In the last part of the paper we describe the 8-dimensional nilmanifolds endowed with an invariant complex structure admitting an SKT metric. In dimension 6 by $[14]$ the existence of a strong KT structure on a nilpotent Lie algebra $\mathfrak{g}$ depends only on the complex structure of $\mathfrak{g}$. We show that in dimension 8 the previous property is no longer true.

A good quaternionic analog of Kähler geometry is given by *hyper-Kähler with torsion* (shortly HKT) geometry. This geometry was introduced by Howe and Papadopoulos $[28]$ and later studied for instance in $[22]$. Nilmanifolds also provides examples of compact HKT manifolds (see $[22, 10]$). In the last section we investigate which 8-dimensional SKT nilmanifolds admit also HKT structures.
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2. Link with SKT metrics

The study of SKT metrics is strictly related to the study of the geometry of the Bismut connection. Indeed, any Hermitian structure \((J, g)\) admits a unique connection \(\nabla^B\) preserving \(g\) and \(J\) and such that the tensor 
\[
c(X, Y, Z) = g(X, T^B(Y, Z))
\]
is totally skew-symmetric, where \(T^B\) denote the torsion of \(\nabla^B\) (see [19]). This connection was used by Bismut in [5] to prove a local index formula for the Dolbeault operator for non-Kähler manifolds. The torsion 3-form \(c\) is related to the fundamental form \(\omega\) of \(g\) by 
\[
c(X, Y, Z) = -d\omega(JX, JY, JZ)
\]
and it is well known that \(\partial \overline{\partial} \omega = 0\) is equivalent to the condition 
\[
dc = 0.
\]

By [39] it turns out that the Hermitian-symplectic structures are related to static solutions of a new metric flow on complex manifolds called Hermitian curvature flow. Indeed, Streets and Tian constructed a flow using the Ricci tensor associated to the Chern connection instead of the Levi-Civita connection. In this way they obtained an elliptic flow and proved some results on short-time existence of solutions for this flow and on stability of Kähler-Einstein metrics. A modified Hermitian curvature flow was used in [40] to study the evolution of SKT metrics, showing that the existence of some particular type of static SKT metrics implies the existence of a Hermitian-symplectic structure on the complex manifold. Static SKT metrics on Lie groups have been also recently studied in [12].

The aim of this section is to point out a natural link between Hermitian-symplectic structures (also called holomorphic-tamed by de Bartolomeis and Tomassini [43]) and SKT metrics. We have the following

**Proposition 2.1.** Let \((M, J)\) be a complex manifold. Giving a Hermitian-symplectic structure \(\Omega\) on \((M, J)\) is equivalent to assigning an SKT metric \(g\) such that the associated fundamental form \(\omega\) satisfies
\[
\partial \omega = \overline{\partial} \beta
\](2.1)
for some \(\partial\)-closed \((2,0)\)-form \(\beta\).
Proof. Let $\Omega$ be a Hermitian-symplectic structure on $(M, J)$, then:

$$d\Omega = 0, \quad \Omega^{1,1} > 0,$$

where $\Omega^{1,1}$ denotes the $(1, 1)$-component of $\Omega$. We can write the real 2-form $\Omega$ as

$$\Omega = \omega - \beta - \bar{\beta},$$

where $\beta = -\Omega^{2,0}$ is the opposite of the $(2, 0)$-component of $\Omega$ and $\omega = \Omega^{1,1}$ is the $(1, 1)$-component. Then

$$d\Omega = 0 \iff \begin{cases} \partial \omega = \bar{\partial} \beta \\ \partial \beta = 0 \end{cases}$$

and $\omega$ is the fundamental form associated to an SKT metric.

Conversely if $g$ is an SKT metric whose fundamental form $\omega$ satisfies (2.1) for some $\partial$-closed $(2, 0)$-form $\beta$, then $\Omega = \omega - \beta - \bar{\beta}$ defines a Hermitian-symplectic structure on $(M, J)$.

In order to write down a natural obstruction to the existence of a Hermitian-symplectic structure, we recall some basic facts about Hermitian Geometry. Let $(M, J, g)$ be a $2n$-dimensional compact Hermitian manifold and let $\omega$ be the associated fundamental form. The complex structure $J$ is extended to act on $r$-forms by

$$J\alpha(X_1, \ldots, X_r) = (-1)^r \alpha(JX_1, \ldots, JX_r).$$

With respect to this extension $J$ commutes with the duality induced by $g$. On $M$ the Hodge star operator

$$*: \Lambda^{r,s}(M) \to \Lambda^{n-s,n-r}(M)$$

is defined by the relation

$$\alpha \wedge *\beta = g(\alpha, \bar{\beta}) \frac{\omega^n}{n!}.$$

Moreover, $g$ induces the $L^2$ product

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta \frac{\omega^n}{n!}$$

and the differential operators

$$\partial^* : \Lambda^{r,s}(M) \to \Lambda^{r-1,s}(M), \quad \bar{\partial}^* : \Lambda^{r,s}(M) \to \Lambda^{r,s-1}(M)$$

defined as

$$\partial^* = -*\bar{\partial}^*, \quad \bar{\partial}^* = -*\partial^*.$$

It is well known that

$$(\partial^* \alpha, \beta) = (\alpha, \partial \beta), \quad (\bar{\partial}^* \alpha, \beta) = (\alpha, \bar{\partial} \beta).$$
Lemma 2.2. Let \((M, \Omega, J)\) be a compact Hermitian-symplectic manifold and let \(\eta\) be a \((2,1)\)-form. Fix an arbitrary Hermitian metric \(g\) on \((M, J)\) with associated Hodge star operator \(*\). Then 
\[(\partial^* \eta, \Omega^{1,1}) \neq 0 \implies \bar{\partial} \eta \neq 0.\]

Proof. We have 
\[(\partial^* \eta, \Omega^{1,1}) = (\eta, \partial \Omega^{1,1}).\]
By Proposition 2.1
\[\partial \Omega^{1,1} = \partial \beta,\]
for some \(\partial\)-closed \((2,0)\)-form. Therefore
\[(\partial^* \eta, \Omega^{1,1}) = (\eta, \partial \beta) = (\partial^* \eta, \beta)\]
which implies the statement. 

3. Main results

In this section we prove the results stated in the introduction.

We recall that an almost complex structure \(J\) on a real Lie algebra \(g\) is said to be integrable if the Nijenhuis condition
\[[X, Y] - [JX, JY] + J[JX, Y] + J[X, JY] = 0\]
holds for all \(X, Y \in g\). By a complex structure on a Lie algebra we will always mean an integrable one. We can give the following

Definition 3.1. Let \((g, J)\) be a real Lie algebra endowed with a complex structure \(J\). A Hermitian-symplectic structure on \((g, J)\) is a symplectic form \(\Omega\) which tames \(J\), that is a real \(2\)-form \(\Omega\) such that
\[\Omega(X, JX) > 0,\]
for any non-zero vector \(X \in g\).

In order to prove Theorem 1.1 we consider the following

Lemma 3.2. Let \(g\) be a real Lie algebra endowed with a complex structure \(J\) such that \(J\xi \cap [g, g] \neq \{0\}\), where \(\xi\) denotes the center of \(g\). Then \((g, J)\) cannot admit any Hermitian-symplectic structure.

Proof. Suppose that \((g, J)\) admits a Hermitian-symplectic structure \(\Omega\), then by definition \(\Omega\) satisfies the conditions
\[d\Omega(X, Y, Z) = -\Omega([X, Y], Z) - \Omega([Y, Z], X) - \Omega([Z, X], Y) = 0,\]
for every \(X, Y, Z \in g\) and
\[\Omega(X, JX) > 0, \quad \forall X \neq 0.\]
Therefore, in particular one has:
\[d\Omega(X, Y, Z) = -\Omega([Y, Z], X) = 0, \quad \forall X \in \xi, \forall Y, Z \in g,\]
or, equivalently, that
\[\Omega(X, W) = 0,\]
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for any \( X \in \xi \) and \( W \in [\mathfrak{g}, \mathfrak{g}] \cap J\xi \); then \( JW \in \xi \) and one has \( \Omega(W, JW) = 0 \) which is not possible since \( J \) is tamed by \( \Omega \).

\[ \square \]

If \( G \) is a real Lie group with Lie algebra \( \mathfrak{g} \), then assigning a left-invariant almost complex structure on \( G \) is equivalent to choosing an almost complex structure \( J \) on \( \mathfrak{g} \). Such a \( J \) is integrable if and only if it is integrable as an almost complex structure on \( G \). Therefore, a complex structure on \( \mathfrak{g} \) induces a complex structure on \( G \) by the Newlander-Nirenberg theorem and \( G \) becomes a complex manifold. The elements of \( G \) act holomorphically on \( G \) by multiplication on the left but \( G \) is not a complex Lie group in general.

Let now \( M = G/\Gamma \) be a compact quotient of a simply-connected Lie group \( G \) by a uniform discrete subgroup \( \Gamma \). By an invariant complex structure (resp. an invariant Hermitian-symplectic structure) on \( G/\Gamma \) we will mean a complex structure (resp. Hermitian-symplectic structure) induced by one on the Lie algebra \( \mathfrak{g} \).

We can prove the following:

**Lemma 3.3.** If \( (M = G/\Gamma, J) \) with \( J \) an invariant complex structure admits a Hermitian-symplectic structure \( \Omega \), then it admits an invariant Hermitian symplectic structure \( \tilde{\Omega} \).

**Proof.** Suppose that there exists a non-invariant Hermitian-symplectic structure \( \Omega \), then by using the property that \( G \) has a bi-invariant volume form \( d\mu \) (see [34]) and by applying the symmetrization process of [13] we can construct a new symplectic invariant form \( \tilde{\Omega} \), defined by

\[ \tilde{\Omega}(X, Y) = \int_{m \in M} \Omega_m(X_m, Y_m) \, d\mu, \]

for any left-invariant vector field \( X, Y \). The 2-form \( \tilde{\Omega} \) tames the complex structure \( J \).

\[ \square \]

Now are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( (M = G/\Gamma, J) \) be a compact quotient of a Lie group with a left-invariant complex structure. Assume that there exists a \( J \)-compatible Hermitian-symplectic structure on \( M \). Then using Lemma 3.3, we can construct a left-invariant Hermitian-symplectic structure \( \Omega \) on \( (G, J) \). In view of Lemma 3.2, we have \( J\xi \cap [\mathfrak{g}, \mathfrak{g}] = \{0\} \), as required.

Consider a Lie algebra \( (\mathfrak{g}, J, g) \) with a complex structure and a \( J \)-compatible inner product. Let \( \nabla^B \) be the Bismut connection associated to \( (J, g) \). Then the inner product \( g \) is SKT if and only if the torsion 3-form \( c(X, Y, Z) = g(X, T^B(Y, Z)) \) of the Bismut connection \( \nabla^B \) is closed. In view of [10], the Bismut connection \( \nabla^B \) and the torsion 3-form \( c \) can be written
in terms of Lie brackets as
\[ (\nabla^B_X Y, Z) = \frac{1}{2} \{ g([X, Y] - [JX, JY], Z) - g([Y, Z] + [JY, JZ], X) + g([Z, X] - [JZ, JX], Y) \} \]
(3.1)

\[ c(X, Y, Z) = -g([JX, JY], Z) - g([JY, JZ], X) - g([JZ, JX], Y). \]
(3.2)

for any \( X, X, Z \in \mathfrak{g} \). We have the following

**Lemma 3.4.** Let \((\mathfrak{g}, J, g)\) be a Lie algebra with a complex structure and a 
\( J \)-compatible inner product. Let \( \xi \) be the center of \( \mathfrak{g} \) and let \( X \in \xi \). Then
(3.3)

\[ dc(X, Y, JX, JY) = 2 \left( \| [Y, JX] \|^2 - g([JX, Y], JX) - g([Y, JY], [JX, Y], X) \right). \]

**Proof.** Let \( X \in \xi \). Then since \( J \) is integrable and \( X \) belongs to the center, we have
(3.4)

\[ [JX, JY] = J[JX, Y]. \]

Using (3.1) and (3.2) we have
\[
dc(X, Y, JX, JY) =
- c([Y, JX], X, JY) + c([Y, JY], X, JX) - c([JX, JY], X, Y)
\]
\[
= - c([Y, JX], X, JY) + c([Y, JY], X, JX) - c([JX, JY], X, Y)
\]
\[
= g([J[Y, JX], JX], JY) + g([Y, JX], [Y, JX]) - g([JY, J[X, JX]], X)
\]
\[
- g([JY, J[X, JX]], JX) - g([JX, Y], JX, Y)
\]
\[
+ g([JX, JY], [JX, JY]) - g([JY, [JX, Y]], X).
\]

Now, by (3.3) we get 
\[ [JY, JX], JX] = J[[Y, JX], JX], \]
so
\[ g([JY, JX], JX, JY) = g([JX, Y], JX, Y) = -2g([JX, Y], JX, Y). \]

Moreover
\[ g([Y, J[Y, JX]], X) + g([JY, J[X, JX]], JX) + g([JY, [JX, Y]], X) =
= 2g([Y, JY], [JX, Y], X) \]
and therefore
\[ dc(X, Y, JX, JY) =
= 2g([Y, JX], [Y, JX]) - 2g([JX, Y], JX, Y) - 2g([Y, JY], [JX, Y], X), \]
as required.

Now we consider the nilpotent case. We recall that a real Lie algebra \( \mathfrak{g} \) is nilpotent if there exists a positive \( s \) such that for the descending central series \( \{ \mathfrak{g}^k \}_{k \geq 0} \) defined by
\[
\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \quad \cdots, \quad \mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}],
\]
one has $g^s = \{0\}$ and $g^{s-1} \neq \{0\}$. The Lie algebra $g$ is therefore called $s$-step nilpotent and one has that $g^{s-1}$ is contained in the center $\xi$ of $g$.

The next proposition gives an obstruction to the existence of an SKT inner product on a nilpotent Lie algebra with a complex structure.

**Proposition 3.5.** Let $(g, J)$ be a nilpotent Lie algebra endowed with a complex structure $J$. If the center $\xi$ is not $J$-invariant, then $(g, J)$ does not admit any SKT metric.

**Proof.** Let $g$ be an arbitrary $J$-compatible inner product on $g$ and let $\nabla^B$ be the associated Bismut connection. Assume that the center $\xi$ is not $J$-invariant, then there exists $X \in \xi$ such that $JX \notin \xi$, or equivalently there exists $X \in \xi$ such that the map $ad_{JX} : g \to g$ is not identically zero. But certainly the restriction $ad_{JX}|_{g^{s-1}}$ is zero because $g^{s-1} \subseteq \xi$. So there exists an integer $k$ such that $ad_{JX}|_{g^k}$ is not identically zero and $ad_{JX}|_{g^{k+1}}$ is zero. Then we can choose $Y \in g^k$ such that $[JX, Y] \neq 0$ and $[[Y, Z], JX] = 0$ for each $Z \in g$; so, using (3.3) we have

$$dc(X, Y, JX, JY) = 2 \| [Y, JX] \|^2 \neq 0$$

and $g$ cannot be SKT. \qed

Let $J$ be a complex structure on a $2n$-dimensional nilpotent Lie algebra $g$. We introduce the ascending series $\{g^l\}_{l \geq 0}$ defined inductively by

\[
\begin{align*}
    g^0 &= \{0\} \\
    g^l &= \{X \in g \mid [J^kX, g] \subseteq g^{l-1}, k = 1, 2, \ldots, l \geq 1\}.
\end{align*}
\]

We say that $J$ is nilpotent if $g^h = g$ for some positive integer $h$. Equivalently by (7), a complex structure on a $2n$-dimensional nilpotent Lie algebra $g$ is nilpotent if there is a basis $\{\alpha^1, \ldots, \alpha^n\}$ of $(1, 0)$-forms satisfying $d\alpha^1 = 0$ and

$$d\alpha^j \in \Lambda^2(\alpha^1, \ldots, \alpha^{j-1}, \bar{\alpha}^1, \ldots, \bar{\alpha}^{j-1}),$$

for $j = 2, \ldots, n$.

Let $(g, J)$ be a nilpotent Lie algebra with a left-invariant complex structure admitting an SKT metric. Since the center $\xi$ of $g$ is $J$-invariant, $J$ induces a complex structure $\hat{J}$ on the quotient $\hat{g} = g/\xi$ by the relation

$$\hat{J}(X + \xi) = JX + \xi$$

for any $X \in g$. Moreover, if $\hat{J}$ is nilpotent, then $J$ is also nilpotent. Indeed, we have

$$g^1 = \xi, \quad g^2 = \text{span}(\hat{g}^1, \xi)$$

and more in general $g^k = \text{span}(\hat{g}^k, \xi)$, for any $k \geq 1$. Therefore, if $J$ is non-nilpotent, $\hat{J}$ has to be non-nilpotent.

We have the following
Proposition 3.6. Let \((\mathfrak{g}, J)\) be a nilpotent Lie algebra endowed with an SKT inner product \(g\). Then there exists an SKT inner product \(\hat{g}\) on \((\hat{\mathfrak{g}} = \mathfrak{g}/\xi, \hat{J})\), where \(\hat{J}\) is defined by (3.5).

Proof. We have the decomposition
\[
\mathfrak{g} = \xi \oplus \xi^\perp,
\]
where \(\xi^\perp\) denotes the orthogonal complement of \(\xi\) with respect to the SKT metric \(g\). So any \(X \in \mathfrak{g}\) splits in
\[
X = X^\xi + X^\perp.
\]
Then we can identify at the level of vector spaces \(\mathfrak{g}/\xi\) with \(\xi^\perp\) in the following way
\[
\mathfrak{g}/\xi \cong \{X + \xi \mid X \in \xi^\perp\}.
\]
We claim that the inner product \(\hat{g}\) on \(\hat{\mathfrak{g}}\) defined by
\[
\hat{g}(X + \xi, Y + \xi) = g(X^\perp, Y^\perp), \quad X, Y \in \mathfrak{g}
\]
is SKT. We note that \([X, Y]^\perp = [X^\perp, Y^\perp]\), and applying Proposition 3.5 we obtain \((JX)^\perp = JX^\perp\). Then by a direct calculation the torsion 3-form of the Bismut connection of \((\hat{J}, \hat{g})\) satisfy
\[
\hat{c}(X^\xi + X^\perp, Y^\xi + Y^\perp, Z^\xi + Z^\perp) = c(X^\perp, Y^\perp, Z^\perp)
\]
for any \(X, Y, Z \in \mathfrak{g}\). Therefore the closure of \(c\) implies the closure of \(\hat{c}\) since \(dc(X, Y, Z, W) = 0\) for every \(X, Y, Z, W \in \xi^\perp\). \(\Box\)

Now are ready to prove Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2. Let \((M = G/\Gamma, J)\) be a nilmanifold with an invariant complex structure \(J\). Assume that there exists a \(J\)-Hermitian SKT metric \(g\) on \(M\). We can suppose that \(g\) is left-invariant (see [14, 13]). Then \((J, g)\) can be regarded as an SKT structure on the Lie algebra \(\mathfrak{g}\) associated to \(G\). Since \(J\) preserves the center \(\xi\) of \(\mathfrak{g}\) and \((\mathfrak{g}/\xi, \hat{J})\) admits an SKT metric, the center of \(\mathfrak{g}/\xi\)
\[
\hat{\xi} = \{X^\perp + \xi \mid [X^\perp, \mathfrak{g}] \subseteq \xi\},
\]
is \(\hat{J}\)-invariant. Then the ideal of \(\mathfrak{g}\)
\[
\pi^{-1}\hat{\xi} = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subseteq \xi\}
\]
is \(J\)-invariant.

We prove that \(\mathfrak{g}\) is 2-step nilpotent by induction on the dimension of \(\mathfrak{g}\). If \(\dim \mathfrak{g} = 6\), then the claim follows from [14]. Suppose now that the statement is true for \(\dim \mathfrak{g} < 2n\) and assume \(\dim \mathfrak{g} = 2n\). Let \(\xi\) be the center of \(\mathfrak{g}\) and let \(\hat{\mathfrak{g}} = \mathfrak{g}/\xi\). In view of Proposition 3.6 we have that \((\hat{\mathfrak{g}}, \hat{J})\) admits an SKT inner product. Hence, since \(\dim \hat{\mathfrak{g}} < 2n\) by our assumption we get that \(\hat{\mathfrak{g}}\) is at most 2-step nilpotent. This implies that \(\mathfrak{g}\) is at most 3-step nilpotent. If \(\mathfrak{g}\) is 3-step, then there exist \(X, Y \in \mathfrak{g}\) such that
\[
[X, Y] \neq 0.
\]
with \( Y = [X_1, X_2] \in \mathfrak{g}^1 \subseteq \pi^{-1} \xi \). Since \( J(\pi^{-1} \xi) = \pi^{-1} \xi \), we have that \( JY \in \pi^{-1} \xi \). Now
\[
\text{dc}(X, Y, JX, JY) =
\]
\[
= -c([X, Y], JX, JY) + c([X, JX], Y, JY) - c([X, JY], Y, JX)
- c([Y, JX], X, JY) + c([Y, JY], X, JX) - c([JX, JY], X, Y)
\]
\[
= g([X, Y], [X, Y]) - g([J[X, JX], JY], JY) + g([Y, J[X, JX]], JY)
- 2g([Y, JY], [X, JX]) + g([X, JY], [X, JY])
+ g([Y, JX], [Y, JX]) + g([JX, JY], [JX, JY]).
\]
As a consequence of the Jacobi identity we get
\[
[[W_1, W_2], Z] = 0,
\]
for any \( W_1, W_2 \in \mathfrak{g} \) and \( Z \in \pi^{-1} \xi \). Therefore \([Y, JY] \) and \([Y, J[X, JX]] \) vanish. Now we can show also that \([J[X, JX], JY]\) is zero. Indeed, by using the integrability of \( J \) and (3.6) we get
\[
[J[X, JX], JY] = [[X, JX], Y] + J[J[X, JX], Y] + J[[X, JX], JY] = 0,
\]
and so
\[
0 = \text{dc}(X, Y, JX, JY) = \|[X, Y]\|^2 + \|[X, JY]\|^2 + \|[Y, JX]\|^2 + \|[JX, JY]\|^2,
\]
which is a contradiction with the assumption that \( \mathfrak{g} \) is 3-step nilpotent.

Since the center \( Z(G) \) of \( G \) is \( J \)-invariant and \( \Gamma \cap Z(G) \) is a uniform discrete subgroup of \( Z(G) \), we have that the surjective homomorphism \( \pi : \mathfrak{g} \to \mathfrak{g}/\xi \) induces a holomorphic principal torus bundle \( \tilde{\pi} : G/\Gamma \to \mathbb{T}^{2p} \) over a \( 2p \)-dimensional torus \( \mathbb{T}^{2p} \) with \( 2p = \dim \mathfrak{g} - \dim \xi \). \( \square \)

**Remark 3.7.**

1. Since by [36] on a 2-step nilpotent Lie algebra every complex structure is nilpotent, Theorem 1.2 implies that if a nilmanifold endowed with a left-invariant complex structure \( J \) admits a \( J \)-compatible SKT metric, then the complex structure has to be nilpotent.

2. By [42] any 6-dimensional SKT nilmanifold is the twist of the Kähler product of the two torus \( \mathbb{T}^4 \) and \( \mathbb{T}^2 \) by using two integral 2-forms supported on \( \mathbb{T}^4 \). In a similar way as a consequence of Theorem 1.2 we have then that any SKT nilmanifold is the twist of a torus.

**Corollary 3.8.** Let \((M = G/\Gamma, J)\) be a \( 2n \)-dimensional nilmanifold endowed with an invariant complex structure \( J \). If there exists a \( J \)-compatible SKT metric \( g \) on \( M \) and \( n > 2 \), then \( b_1(M) \geq 4 \) and every invariant \( J \)-compatible metric on \( M \) is standard.

**Proof.** By Theorem 1.2 we have the Lie algebra \( \mathfrak{g} \) of \( G \) has to be 2-step nilpotent. Since \( \mathfrak{g}^1 \subseteq \xi \), the annihilator of \( \xi \) in \( \mathfrak{g}^* \) is contained in the
annihilator of $\mathfrak{g}^1$, therefore its elements are all $d$-closed. If $\dim \xi = 2n - 2p$ we can choose a basis of $(1,0)$-forms $\{\alpha^1, \ldots, \alpha^n\}$ such that

\[
\begin{align*}
&\{ da_j = 0, \quad j = 1, \ldots, p, \\
&\{ da_l \in \Lambda^2(\alpha^1, \ldots, \alpha^p, \bar{\alpha}^1, \ldots, \bar{\alpha}^p), \quad l = p + 1, \ldots, n,
\end{align*}
\]

with $\{\alpha^1, \ldots, \alpha^p\}$ in the annihilator of $\xi$ and $\{\alpha^{p+1}, \ldots, \alpha^n\}$ in the complexification of the dual $\xi^*$ of $\xi$. Using the previous basis we can show that there exist at least two $d$-closed $(1,0)$-forms. Indeed, if $p = 1$, then we have

\[
\begin{align*}
\{ da_1 = 0, \\
\{ da_l \in \Lambda^2(\alpha^1, \bar{\alpha}^1), \quad l = 2, \ldots, n,
\end{align*}
\]

and therefore $\mathfrak{g}$ is isomorphic to the direct sum $\mathfrak{h}_3^\mathbb{R} \oplus \mathbb{R}^{2n-3}$ of the 3-dimensional real Heisenberg Lie algebra $\mathfrak{h}_3^\mathbb{R}$ with $\mathbb{R}^{2n-3}$ and $b_1(\mathfrak{g}) = 2n - 1 \geq 4$ since $n > 2$.

By using Nomizu's Theorem [35] we have that the de Rham cohomology of $M$ is isomorphic to the Chevalley-Eilenberg cohomology of the Lie algebra $\mathfrak{g}$. Since there exist at least four real $d$-closed 1-forms we get that $b_1(M) = b_1(\mathfrak{g}) \geq 4$.

We recall that by [20] a $J$-compatible metric $h$ on $(M, J)$ is standard if and only if the Lee form $\theta = Jd^*\omega$ is co-closed, where $\omega$ denotes the fundamental form of $(J, h)$. Since $\mathfrak{g}$ is nilpotent, it is also unimodular, so any $(2n - 1)$-form is $d$-closed and in particular, if $\theta$ is the Lee form of an invariant $J$-compatible metric $h$, then $d(\theta) = 0$.

**Proof of Theorem 1.3.** Let $(M, G/\Gamma, J)$ be a nilmanifold with an invariant complex structure and let $(\mathfrak{g}, J)$ be its associated Lie algebra. Assume that there exists a Hermitian-symplectic structure $\Omega$ on $(M, J)$. Using Lemma 3.3 we may assume that $\Omega$ is left-invariant. Hence $\Omega$ can be regarded as a Hermitian-symplectic structure on $(\mathfrak{g}, J)$. Using Lemma 3.3 we have that $\Omega$ induces an SKT inner product on $(\mathfrak{g}, J)$. Then Theorem 1.2 implies that $\mathfrak{g}$ is 2-step nilpotent and therefore $\mathfrak{g}^1$ is contained in the center $\xi$. But since $\xi$ is $J$-invariant, $J(\mathfrak{g}^1) \subseteq \xi$ and Theorem 1.1 implies the statement.

Theorem 1.3 cannot be directly deduced from Theorem 1.1 since there exist examples of nilpotent Lie algebras not satisfying (1.1). For instance, we can consider the following

**Example 1.** Let $\mathfrak{g}$ be the 10-dimensional nilpotent Lie algebra with structure equations

\[
\begin{align*}
&\{ de^j = 0, \quad j = 1, \ldots, 7, \\
&de^8 = e^1 \wedge e^5 + e^1 \wedge e^6 + e^3 \wedge e^5 + e^3 \wedge e^6, \\
&de^9 = e^2 \wedge e^5 + e^2 \wedge e^6 + e^4 \wedge e^5 + e^4 \wedge e^6, \\
&de^{10} = e^1 \wedge e^8 + e^3 \wedge e^8 + e^2 \wedge e^9 + e^4 \wedge e^9.
\end{align*}
\]

(see [36] Example 3.13) endowed with the non-nilpotent complex structure

\[
Je_1 = e_2, \quad Je_3 = e_4, \quad Je_5 = e_7, \quad Je_8 = e_9, \quad Je_6 = e_{10},
\]
where \( \{e_1, \ldots, e_{10}\} \) denotes the dual basis of \( \{e^1, \ldots, e^{10}\} \). We have

\[
\xi = \text{span}\langle e_7, e_{10}\rangle, \quad g^1 = \text{span}\langle e_8, e_9, e_{10}\rangle
\]

and thus \( J_\xi \cap g^1 = \{0\} \). In this case a basis of \((1,0)\)-forms is then given by

\[
\alpha^1 = e^1 + i e^2, \quad \alpha^2 = e^3 + i e^4, \quad \alpha^3 = e^5 + i e^7, \quad \alpha^4 = e^8 + i e^9, \quad \alpha^5 = e^6 + i e^{10}
\]

with complex structure equations

\[
\begin{aligned}
\frac{d\alpha^j}{d\alpha^j} &= 0, \quad j = 1, 2, 3, \\
\frac{d\alpha^4}{d\alpha^4} &= \frac{1}{2}(\alpha^{12} + \alpha^{13} + \alpha^{23} + \alpha^{25} + \alpha^{15} + \alpha^{25} + \alpha^{27}), \\
\frac{d\alpha^5}{d\alpha^5} &= \frac{1}{2}(\alpha^{14} + \alpha^{17} + \alpha^{54} + \alpha^{57}),
\end{aligned}
\]

where by \( \alpha^j \) we denote \( \alpha^i \wedge \overline{\alpha}^j \).

4. A classification of 8-dimensional SKT nilmanifolds

In this section we classify 8-dimensional nilpotent Lie algebras admitting an SKT structure. We begin describing the following two families of nilpotent Lie algebras.

1. **First family:** Consider the family of 8-dimensional nilpotent Lie algebras with complex structure equations

\[
\begin{aligned}
\frac{d\alpha^j}{d\alpha^j} &= 0, \quad j = 1, 2, \\
\frac{d\alpha^3}{d\alpha^3} &= B_1 \alpha^{12} + B_4 \alpha^{1T} + B_5 \alpha^{2T} + C_3 \alpha^{2T} + C_4 \alpha^{27}, \\
\frac{d\alpha^4}{d\alpha^4} &= F_1 \alpha^{12} + F_4 \alpha^{1T} + F_5 \alpha^{17} + G_3 \alpha^{2T} + G_4 \alpha^{27},
\end{aligned}
\]

where the capitol letters are arbitrary complex numbers and by \( \alpha^j \) we denote \( \alpha^i \wedge \overline{\alpha}^j \). Then the inner product

\[
g = \sum_{k=1}^{4} \alpha^k \otimes \overline{\alpha}^k
\]

is SKT if and only if the complex numbers satisfy the equation

\[
|B_1|^2 + |F_1|^2 + |G_3|^2 + |B_5|^2 + |C_3|^2 + |F_5|^2 = 2 \Re \left( C_4 \overline{B}_4 + F_4 \overline{G}_4 \right).
\]

Notice that in terms of \( g \) and of the derivatives of the \( \alpha^k \)'s, this last equation can be rewritten as

\[
\sum_{j=3}^{4} \left( ||d\alpha^j||^2 + \Re \left[ g(d\alpha^j, \alpha^{1T})g(d\alpha^j, \alpha^{2T}) \right] - \sum_{k=1}^{2} |g(d\alpha^j, \alpha^{kT})|^2 \right) = 0
\]

and by a direct computation we have that the Lee form \( \theta = Jd^* \omega \) is in general not closed.

Note that if both \( |B_4|^2 + |C_4|^2 \) and \( |F_4|^2 + |G_4|^2 \) vanish, then the Lie algebra is abelian. So we can suppose for instance that

\[
\alpha^1 \mapsto \alpha^1, \quad \alpha^2 \mapsto \alpha^2, \quad \alpha^3 \mapsto \alpha^4, \quad \alpha^4 \mapsto \alpha^3
\]
and arguing as in the proof of Lemma 2.14 in [44] we can change the basis in order to have

\[
\begin{align*}
\alpha^j &= 0, \quad j = 1, 2, \\
\alpha^3 &= \rho \alpha^{12} + \alpha^1T + B \alpha^2T + D \alpha^4T, \\
\alpha^4 &= F_1 \alpha^{12} + F_4 \alpha^1T + F_5 \alpha^2T + G_3 \alpha^3T + G_4 \alpha^4T,
\end{align*}
\]

with \(\rho \in \{0, 1\}\), \(B, D \in \mathbb{C}\). With the new change

\[
\alpha^1 \mapsto \alpha^1, \quad \alpha^2 \mapsto \alpha^2, \quad \alpha^3 \mapsto \alpha^3, \quad \alpha^4 \mapsto \alpha^4 - F_4 \alpha^3,
\]

we then get

\[
\begin{align*}
\alpha^j &= 0, \quad j = 1, 2, \\
\alpha^3 &= \rho \alpha^{12} + \alpha^1T + B \alpha^2T + D \alpha^4T, \\
\alpha^4 &= F'_1 \alpha^{12} + F'_4 \alpha^1T + G'_3 \alpha^2T + G'_4 \alpha^4T.
\end{align*}
\]

Therefore, if \((F'_1)^2 + (F'_4)^2 + (G'_3)^2 + (G'_4)^2 = 0\), the Lie algebra is a direct sum of \(\mathbb{R}^2\) with a 6-dimensional SKT nilpotent Lie algebra.

So for instance the direct sum \(\mathfrak{h}^{\mathbb{C}}_3 \oplus \mathbb{R}^2\) of the 3-dimensional complex Heisenberg Lie algebra \(\mathfrak{h}^{\mathbb{C}}_3\) with \(\mathbb{R}^2\) admits an SKT structure. If \((F'_1)^2 + (F'_4)^2 + (G'_3)^2 + (G'_4)^2 \neq 0\), then \(Z_3\) and \(Z_4\) belong to the complexification of the center \(\xi\), so \(\dim \xi \geq 4\). Note that \(\dim \mathfrak{g}^1 = 1\) if and only if the Lie algebra is isomorphic to the direct sum \(\mathfrak{h}^{\mathbb{C}}_3 \oplus \mathbb{R}^3\).

Moreover \(\dim \mathfrak{g}^1 = 3\) if and only one of the following cases occur

a) \((F'_1)^2 + (F'_4)^2 + (G'_3)^2 \neq 0\), \(\rho = B = 0\) and \(D\) real;

b) \(F'_1 = F'_4 = G'_3 = 0\), \(G'_4 \neq 0\) and \(\rho^2 + B^2 \neq 0\).

In case a) we get the Lie algebra

\[
\begin{align*}
\alpha^j &= 0, \quad j = 1, 2, \\
\alpha^3 &= \alpha^1T + D \alpha^4, \\
\alpha^4 &= F'_1 \alpha^{12} + F'_4 \alpha^1T + G'_3 \alpha^3T + G'_4 \alpha^4T.
\end{align*}
\]

with \((F'_1)^2 + (F'_4)^2 + (G'_3)^2 \neq 0\) and \(D\) real. In particular, for \(D = 1\), \(F'_1 = \sqrt{2}\), \(F'_4 = G'_3 = G'_4 = 0\) we get the Lie algebra with structure equations

\[
\begin{align*}
\alpha^j &= 0, \quad j = 1, 2, \\
\alpha^3 &= \alpha^1T + \alpha^2, \\
\alpha^4 &= \sqrt{2} \alpha^{12},
\end{align*}
\]

which is isomorphic to the direct sum \(\mathfrak{h}^Q_7 \oplus \mathbb{R}\), where \(\mathfrak{h}^Q_7\) is the real 7-dimensional Lie algebra of Heineberg type and with 3-dimensional center.

In case b) we get a Lie algebra isomorphic to

\[
\begin{align*}
\alpha^j &= 0, \quad j = 1, 2, \\
\alpha^3 &= \rho \alpha^{12} + \alpha^1T + B' \alpha^2, \\
\alpha^4 &= \alpha^4T,
\end{align*}
\]

with \(\rho^2 + (B')^2 \neq 0\).
2. **Second family:** Consider the family of 8-dimensional nilpotent Lie algebras equipped with a complex structure having structure equations

\[
\begin{align*}
d\alpha^j &= 0, \quad j = 1, 2, 3, \\
d\alpha^4 &= F_1\alpha^{12} + F_2\alpha^{13} + F_4\alpha^{17} + F_5\alpha^{17} + F_6\alpha^{17} + G_1\alpha^{23} \\
&\quad + G_3\alpha^{27} + G_4\alpha^{27} + G_5\alpha^{27} + H_2\alpha^{37} + H_3\alpha^{37} + H_4\alpha^{37}
\end{align*}
\]

(4.3)

where the capitol letters are arbitrary complex numbers and \( H_4 \neq 0 \).

In this case requiring that the inner product \( g = \sum_{k=1}^{4} \alpha^k \otimes \overline{\alpha}^k \) is SKT is equivalent to require that the following equations are satisfied

\[
\begin{align*}
-H_4\overline{F}_4 + H_2\overline{G}_3 + F_3\overline{F}_6 - F_4\overline{G}_5 + F_2\overline{F}_1 &= 0, \\
-H_3\overline{F}_5 + G_4\overline{F}_6 + H_2\overline{G}_4 - G_5\overline{G}_5 + G_1\overline{F}_1 &= 0, \\
-H_4\overline{F}_5 + G_5\overline{F}_6 + H_2\overline{G}_3 - G_3\overline{G}_4 + G_1\overline{F}_2 &= 0, \\
|F_2|^2 + |F_3|^2 + |H_2|^2 &= 2 \text{Re}(H_4\overline{F}_4), \\
|F_1|^2 + |F_5|^2 + |G_3|^2 &= 2 \text{Re}(H_4\overline{F}_4), \\
|G_1|^2 + |G_5|^2 + |H_3|^2 &= 2 \text{Re}(H_4\overline{F}_4).
\end{align*}
\]

(4.4)

Every Lie algebra \( g \) of this family splits in \( g = V_1 \oplus V_2 \), where \( V_1 \) are \( J \)-invariant vector subspaces such that

\[
\dim V_1 = 6, \quad \dim V_2 = 2, \quad [V_1, V_1] \subseteq V_2, \quad V_2 \subseteq \xi
\]

and there exists a real \( J \)-invariant 2-dimensional vector subspace \( V_3 \) (generated by \( Z_3 \) and \( \overline{Z}_3 \)) of \( V_1 \) such that \([V_3, V_3] \neq 0\). In particular any Lie algebra of this family is 2-step nilpotent, has \( \dim g^1 = 2 \) and by a direct computation we have that the Lee form \( \theta = J\alpha^*\omega \) is in general not closed.

Now we are ready to classify 8-dimensional nilmanifolds endowed with an invariant complex structure admitting an SKT metric.

**Theorem 4.1.** Let \( M^8 = G/\Gamma \) be an 8-dimensional nilmanifold (not a torus) with an invariant complex structure \( J \). There exists an SKT metric \( g \) on \( M \) compatible with \( J \) if and only if the Lie algebra \((g, J)\) belongs to one of the two families described above.

**Proof.** Assume that there exists an SKT metric \( g \) on \((M^8, J)\). In view of [14, 13], we may assume that the metric \( g \) is invariant, and so that it is induced by an inner product \( g \) on the Lie algebra \( g \).

In the proof of Theorem 1.2 we have already shown that there exists a basis of \((1, 0)\)-forms \( \{\alpha^1, \alpha^2, \alpha^3, \alpha^4\} \) such that

\[
\begin{align*}
\{\alpha^j\} &= 0, \quad j = 1, \ldots, p, \\
\{\alpha^l\} &= \Lambda^2(\alpha^1, \ldots, \alpha^p, \overline{\alpha}^1, \ldots, \overline{\alpha}^p), \quad l = p + 1, \ldots, 4,
\end{align*}
\]

with \( \{\alpha^1, \ldots, \alpha^p\} \) in the annihilator of \( \xi \) and \( \{\alpha^{p+1}, \ldots, \alpha^4\} \) in the complexification of the dual \( \xi^* \) of the center. Applying the Gram-Schmidt process to
the previous basis \( \{ \alpha^1, \ldots, \alpha^4 \} \) we get a unitary coframe \( \{ \tilde{\alpha}^1, \ldots, \tilde{\alpha}^4 \} \) such that
\[
\begin{align*}
\{ & d\tilde{\alpha}^1 = 0, \ j = 1, \ldots, p, \\
& d\tilde{\alpha}^l \in \Lambda^2(\tilde{\alpha}^1, \ldots, \tilde{\alpha}^p, \tilde{\alpha}^4), \quad l = p + 1, \ldots, 4,
\end{align*}
\]

since \( \text{span}(\tilde{\alpha}^1, \ldots, \tilde{\alpha}^4) = \text{span}(\alpha^1, \ldots, \alpha^4) \), for any \( j = 1, \ldots, 4 \). Then it is not restrictive to assume that \( \{ \alpha^1, \ldots, \alpha^4 \} \) is a \( g \)-unitary coframe, i.e. that the fundamental form \( \omega \) of \( g \) with respect to \( \{ \alpha^1, \ldots, \alpha^4 \} \) takes the standard expression:
\[
\omega = -\frac{i}{2} \sum_{j=1}^{4} \alpha^j \wedge \tilde{\alpha}^j.
\]

We can distinguish three cases according to the dimension \( 2n - 2p \) of the center \( \xi \).

(a) If \( p = 1 \), then \( g \cong \mathfrak{h}_3^5 \oplus \mathbb{R}^5 \) and it belongs to the first family.
(b) For \( p = 2 \) we get the first family \( (4.1) \).
(c) For \( p = 3 \) we obtain the second family \( (4.3) \).

\[\Box\]

**Remark 4.2.** In dimension 8 it is no longer true that the existence of a strong KT structure on a nilpotent Lie algebra \( g \) depends only on the complex structure of \( g \). Indeed, for any of two families of 8-dimensional nilpotent Lie algebras with complex structure equations \((4.1)\) and \((4.3)\) consider a generic \( J \)-Hermitian metric \( g \). The fundamental form \( \omega \) associated to the Hermitian structure \( (J, g) \) can be then expressed as
\[
\begin{align*}
\omega = & a_1 \alpha^{1\uparrow} + a_2 \alpha^{2\uparrow} + a_3 \alpha^{3\uparrow} + a_4 \alpha^{4\uparrow} + a_5 \alpha^{1\uparrow} - a_5 \alpha^{2\uparrow} + a_6 \alpha^{3\uparrow} - a_6 \alpha^{4\uparrow} \\
& + a_7 \alpha^{4\uparrow} - a_7 \alpha^{1\uparrow} + a_8 \alpha^{2\uparrow} - a_8 \alpha^{3\uparrow} + a_9 \alpha^{4\uparrow} - a_9 \alpha^{1\uparrow} + a_{10} \alpha^{2\uparrow} - a_{10} \alpha^{3\uparrow},
\end{align*}
\]

where \( a_l, \ l = 1, \ldots, 10, \) are arbitrary complex numbers (with \( \overline{a_l} = -a_l \)) for any \( l = 1, \ldots, 4 \) such that \( \omega \) is positive definite.

For the first family the SKT equation for a generic \( J \)-Hermitian metric \( g \) is:
\[
\begin{align*}
- a_3 C_4 B_4 - 2a_{10} B_4 C_4 + a_3 B_4 C_4 + a_1 B_1 F_1 + a_3 |B_1|^2 - a_{10} B_1 F_1 + a_4 |F_5|^2 \\
+ a_4 |F_1|^2 - a_{10} G_3 C_3 + a_4 |G_3|^2 + a_3 |B_5|^2 - a_{10} B_5 F_5 - a_4 G_5 F_4 + a_3 |C_3|^2 \\
+ a_{10} C_4 F_4 + a_{10} F_5 B_5 - a_4 F_4 G_4 + a_{10} G_4 B_4 + a_{10} G_3 C_3 - a_{10} C_4 F_4 = 0,
\end{align*}
\]

so it is no longer true that equation \((4.2)\) implies that any \( J \)-Hermitian metric is SKT.

For the second family the SKT equations for the generic \( J \)-Hermitian metric \( g \) are \((4.4)\) and so as in the 6-dimensional case the SKT condition depends only on the complex structure.

Nilmanifolds provide also examples of compact HKT manifolds. We recall that a hyperhermitian manifold \((M, J_1, J_2, J_3, g)\) is called HKT if the fundamental forms \( \omega_l(\cdot, \cdot) = g(J_l \cdot, \cdot), \ l = 1, 2, 3, \) satisfy
\[
J_1 d\omega_1 = J_2 d\omega_2 = J_3 d\omega_3.
\]
This is equivalent to the property that the three Bismut connections associated to the Hermitian structures \((J_l, g)\) coincide and this connection is said to be an HKT connection. The HKT structure is called strong or weak depending on whether the torsion 3-form of the HKT connection is closed or not.

By [3, Theorem 4.2] if a \(4n\)-dimensional nilmanifold \(M^{4n} = G/\Gamma\) is endowed with an HKT structure \((J_1, J_2, J_3, g)\) induced by a left-invariant HKT structure on \(G\), then the hypercomplex structure has to be abelian, i.e

\[
[J_lX, J_lY] = [X, Y], \quad l = 1, 2, 3,
\]

for any \(X, Y\) in the Lie algebra \(\mathfrak{g}\) of \(G\). By [10] every abelian hypercomplex structure on a (non-abelian) nilpotent Lie algebra gives rise to a weak HKT structure.

In [11] a description of hypercomplex 8-dimensional nilpotent Lie algebras was given. More precisely, it was shown that a 8-dimensional hypercomplex nilpotent Lie algebra \(\mathfrak{g}\) has to be 2-step nilpotent and with \(b_1(\mathfrak{g}) \geq 4\).

In particular, by [9] there exist only three (non-abelian) nilpotent Lie algebras of dimension 8 admitting an abelian hypercomplex structure and they are abelian extensions of a Lie algebra of Heinsenberg type, i.e. are isomorphic to one of the following:

\[
\mathfrak{h}_5 \oplus \mathbb{R}^3, \quad \mathfrak{h}_3^C \oplus \mathbb{R}^2, \quad \mathfrak{h}_7^Q \oplus \mathbb{R},
\]

where \(\mathfrak{h}_5\) is the 5-dimensional Lie algebra of Heinsenberg type with 1-dimensional center.

As a consequence of Theorem 4.1 we have the following

**Corollary 4.3.** The nilpotent Lie algebra \(\mathfrak{h}_5 \oplus \mathbb{R}^3\) has weak HKT structures and it does not admit any SKT structure. The Lie algebras \(\mathfrak{h}_3^C \oplus \mathbb{R}^2\) and \(\mathfrak{h}_7^Q \oplus \mathbb{R}\) have SKT structures and weak HKT structures.

**Proof.** The nilpotent Lie algebra \(\mathfrak{h}_5 \oplus \mathbb{R}^3\) has 1-dimensional commutator, but by the description of the two families a 8-dimensional SKT nilpotent Lie algebra \(\mathfrak{g}\) with \(\dim \mathfrak{g}^1 = 1\) is isomorphic to \(\mathfrak{h}_5^5 \oplus \mathbb{R}^5\). In the first family there are Lie algebras isomorphic to \(\mathfrak{h}_3^C \oplus \mathbb{R}^2\) and \(\mathfrak{h}_7^Q \oplus \mathbb{R}\).

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