GENERALIZATIONS OF THE DURAND-KERNER METHOD

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Dedicated to the memory of my school teacher Alexander L. Smirnov.

ABSTRACT. We propose an approach to constructing iterative methods for finding polynomial roots simultaneously. One feature of this approach is using the fundamental theorem of symmetric polynomials. The new results presented in this paper are simultaneous Householder’s method and some generalizations of the Durand-Kerner method.

1. Introduction

Let \( f(z) \) be a polynomial of degree \( n \) with coefficients in \( \mathbb{C} \) and let its factorization over the complex numbers be

\[
f(z) = \prod_{j=1}^{n} (z - \lambda_j),
\]

where \( \lambda_j \) (\( j = 1, 2, ..., n \)) are roots (zeros) of \( f(z) \).

Let us consider some known methods for simultaneous approximation of roots. The classical (Weierstrass) Durand-Kerner method \([1, 2, 3, 4]\) is related to

\[
z_{i}^{(k+1)} = z_{i}^{(k)} - \frac{f(z_{i}^{(k)})}{\prod_{j=1, j \neq i}^{n} (z_{i}^{(k)} - z_{j}^{(k)})} \quad (i = 1, \ldots, n),
\]

here \( k \) is the iteration number. Further in similar formulas we will use \( z_i \) and \( \hat{z}_i \) instead \( z_i^{(k)} \) and \( z_i^{(k+1)} \), respectively. If the roots \( \lambda_i \) (\( i = 1, 2, ..., n \)) are distinct and the initial approximations \( z_i^{(0)} \) (\( i = 1, 2, ..., n \)) are close to them, then the method is of quadratic convergence proven by Dochev \([3]\).

The Maehly-Ehrlich-Alberth method \([5, 6, 7]\) with cubic convergence deals with

\[
\hat{z}_i = z_i - \left[ \frac{f'(z_i)}{f(z_i)} - \sum_{j \neq i} \frac{1}{z_i - z_j} \right]^{-1} \quad (i = 1, \ldots, n).
\]

In practice, it is convenient to use a formula which does not contain division by a near-zero value \( f(z_i) \), since it may lead to loss of significance. So the following formula is used:

\[
\hat{z}_i = z_i - f(z_i) \left[ f'(z_i) - f(z_i) \sum_{j \neq i} \frac{1}{z_i - z_j} \right]^{-1}.
\]

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\( ^1 \)Here and further we imply only the case of simple roots and good initial approximations.
Further, when division by \( f(z_i) \) occurs, we mean that before using the formula, we transform it. There are modifications that significantly improve the iterative schemes above (see Petcovic and Milovanovic [13, 14, 15] and references therein).

The Ostrowski-Gargantini method [8, 9] having the fourth order of convergence is based on the following iterative formula:

\[
\hat{z}_i = z_i - \left[ \left( \frac{f'(z_i)}{f(z_i)} \right)^2 - \frac{f''(z_i)}{f(z_i)} - \sum_{j \neq i} \frac{1}{(z_i - z_j)^2} \right]^{-1/2} (i = 1, \ldots, n).
\]

The symbol * denotes that one of the values of the square root (more appropriate) is chosen. In using such notation we follow [10, 11]. A criterion for the choice of an appropriate value of the square root is given in [9]; we need to choose such a value of the square root so that the following is minimal:

\[
\left| \frac{f'(z_i)}{f(z_i)} - \left[ \left( \frac{f'(z_i)}{f(z_i)} \right)^2 - \frac{f''(z_i)}{f(z_i)} - \sum_{j \neq i} \frac{1}{(z_i - z_j)^2} \right] \right|^{1/2}.
\]

Since (1.4) contains only the terms which must be calculated in the current iteration step, the direct way of choosing a value of the square root, that is the way consisting in the minimization of \(|f(\hat{z}_i)|\), requires more calculations in a general case.

The generalization of (1.2), (1.3) was presented in [10, 12]. This result is as follows:

\[
\hat{z}_i = z_i - \left[ F_m(z_i) - \sum_{j \neq i} \frac{1}{(z_i - z_j)^m} \right]^{-1/m} (i = 1, \ldots, n),
\]

where

\[
F_m(z) = \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left( \frac{f'(z)}{f(z)} \right) (m \in \mathbb{Z}^+).
\]

To choose an appropriate value of the \( m \)th root we can use the minimization of

\[
\left| \frac{f'(z_i)}{f(z_i)} - \left[ F_m(z_i) - \sum_{j \neq i} \frac{1}{(z_i - z_j)^m} \right] \right|^{1/m}.
\]

The generalized iterative formula (1.5) is locally of \((m + 2)\)th order of convergence. For more information about simultaneous root-finding methods see [16, 17, 18, 19, 20, 21].

2. Constructing iterative formulas

The elementary symmetric polynomials are defined as follows

\[
e_0(x_1, \ldots, x_n) = 1, \quad e_k(x_1, \ldots, x_n) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} x_{j_1} \cdots x_{j_k} \quad (1 \leq k \leq n).
\]

It is known that any symmetric polynomial in \( x_1, \ldots, x_n \) can be expressed as a polynomial in \( e_k(x_1, \ldots, x_n) (1 \leq k \leq n) \), moreover, such a representation is unique.
For example, we consider the $m$th power sum of $n$ variables, i.e., $p_m(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j^m$. There is the following recursive procedure:

$$
\begin{align*}
p_1 &= e_1, \\
p_2 &= e_1p_1 - 2e_2, \\
p_3 &= e_1p_2 - e_2p_1 + 3e_3, \\
p_4 &= e_1p_3 - e_2p_2 + e_3p_1 - 4e_4, & \text{and so on.}
\end{align*}
$$

The recurrence relation is

$$
p_m = \sum_{j=1}^{m-1} (-1)^{m-1+j} e_{m-j} p_j + (-1)^{m-1} me_m, \quad m \geq 1.
$$

Therefore, we can obtain the representation of $p_m$ via $e_k$ ($1 \leq k \leq m$). Also, there are explicit formulas which express power sums in terms of elementary symmetric polynomials, see [22].

**Lemma 2.1.** For an integer $0 \leq k \leq n$ the following holds:

$$
\frac{1}{k!} f^{(k)}(z) = e_k \left( \frac{1}{z - \lambda_1}, \ldots, \frac{1}{z - \lambda_n} \right).
$$

This lemma is used to construct iterative formulas. The main idea is as follows: suppose we take some symmetric polynomial in the variables $1/(z - \lambda_j)$ ($1 \leq j \leq n$) and express it via elementary symmetric polynomials, then using (2.1), we obtain a formula which, after simple transformations, gives us a simultaneous root-finding method.

**Example 2.2.** Let us consider the polynomial $p_3\left((z - \lambda_1)^{-1}, \ldots, (z - \lambda_n)^{-1}\right)$. There is the representation $p_3 = e_1^3 - 3e_2e_1 + 3e_3$. Using (2.1), we obtain

$$
\sum_{j=1}^{n} \frac{1}{(z - \lambda_j)^3} = \left( \frac{f'(z)}{f(z)} \right)^3 - \frac{3f'(z)f''(z)}{2f(z)^2} + \frac{f'''(z)}{2f(z)}.
$$

Making simple transformations, we get

$$
\lambda_i = z - \left[ \left( \frac{f'(z)}{f(z)} \right)^3 - \frac{3f'(z)f''(z)}{2f(z)^2} + \frac{f'''(z)}{2f(z)} - \sum_{j \neq i} \frac{1}{(z - \lambda_j)^3} \right]^{-1/3}.
$$

Finally, we have the following iterative method

$$
\tilde{z}_i = z_i - \left[ \left( \frac{f'(z_i)}{f(z_i)} \right)^3 - \frac{3f'(z_i)f''(z_i)}{2f(z_i)^2} + \frac{f'''(z_i)}{2f(z_i)} - \sum_{j \neq i} \frac{1}{(z_i - z_j)^3} \right]^{-1/3}.
$$

This is exactly (1.5) when $m = 3$. Usually, the method (2.2) is not used in practice.

**Remark 2.3.** If we consider $p_m\left((z - \lambda_1)^{-1}, \ldots, (z - \lambda_n)^{-1}\right)$, then we obtain (1.6) and derive the following relation

$$
F_m(z) = u_m \left( \frac{f'(z)}{f(z)}, \frac{1}{2!} \frac{f''(z)}{f(z)}, \ldots, \frac{1}{n!} \frac{f^{(n)}(z)}{f(z)} \right),
$$

where the polynomial $u_m$ is defined by

$$
p_m(x_1, \ldots, x_n) = u_m(e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n)).
$$
Simultaneous Householder’s method. Let $\alpha$ and $\beta$ be nonzero elements in $\mathbb{C}$. We consider the symmetric polynomial $\alpha p_2 + \beta p_1^2$ in variables $1/(z-\lambda_j)$ ($1 \leq j \leq n$). Let us introduce the notation:

\begin{equation}
q = \frac{1}{z - \lambda_i}, \quad S_r = \sum_{j \neq i} \frac{1}{z - \lambda_j} \quad (r \in \mathbb{Z}^+).
\end{equation}

Then $p_r = q^r + S_r$. Using this, we have the following:

\begin{align*}
\alpha p_2 + \beta p_1^2 &= \alpha(q^2 + S_2) + \beta(q^2 + 2qS_1 + S_1^2) \\
&= \alpha(q^2 + S_2) + \beta(q^2 + 2q(p_1 - q) + S_1^2) \\
&= (\alpha - \beta)q^2 + 2\beta p_1 q + \alpha S_2 + \beta S_1^2.
\end{align*}

We see that it is convenient to put $\alpha = \beta = 1$. Then $q = (p_2 - S_2 + p_1^2 - S_1^2)/(2p_1)$. Since $p_1 = e_1$, $p_2 = e_1^2 - 2e_2$, with the help of $\alpha$ we get

\begin{equation}
\lambda_i = z - \frac{2f(z)f'(z)}{2[f'(z)]^2 - f(z)f''(z) - [f(z)]^2(S_2 + S_1^2)}.
\end{equation}

Finally, this formula leads to the simultaneous root-finding method (2.7)

\begin{equation}
\hat{z}_i = z_i - \frac{2f(z_i)f'(z_i)}{2[f'(z_i)]^2 - f(z_i)f''(z_i) - [f(z_i)]^2(S_2 + S_1^2)}.
\end{equation}

In the last section we will show that this method is locally of the fourth order of convergence. The formula (2.6) is related to Halley’s method for solving a nonlinear equation, so we call (2.7) simultaneous Halley’s method.

Now we consider $\alpha p_3 + \beta p_1 p_2 + \gamma p_1^3$ in variables $1/(z - \lambda_j)$ ($1 \leq j \leq n$). Then

\begin{align*}
\alpha p_3 + \beta p_1 p_2 + \gamma p_1^3 &= \alpha(q^3 + S_3) + \beta(q + S_1)(q^2 + S_2) + \gamma(q + S_1)^3 \\
&= (\alpha + \beta + \gamma)q^3 + (\beta + 3\gamma)S_1 q^2 + (\beta S_2 + 3\gamma S_1^2)q + \alpha S_3 + \beta S_1 S_2 + \gamma S_1^3 \\
&= (\alpha - \beta + \gamma)q^3 + (\beta - 3\gamma)p_1 q^2 + (\beta p_2 + 3\gamma p_1^2)q + \alpha S_3 + \beta S_1 S_2 + \gamma S_1^3.
\end{align*}

We put $\alpha = 2$, $\beta = 3$, $\gamma = 1$, then

\begin{equation}
q = \frac{2(p_3 - S_3) + 3(p_1 p_2 - S_1 S_2) + p_1^3 - S_1^3}{3(p_2 + p_1^2)}.
\end{equation}

Therefore, we have

\begin{equation}
\lambda_i = z - \frac{6f f'^2 - 3f^2 f''}{6f^3 - 6ff'f'' + f^2f''' - f^3(2S_3 + 3S_1 S_2 + S_1^3)}.
\end{equation}

Using this, we can get the corresponding simultaneous root-finding method, which is connected to so-called Householder’s method for solving a nonlinear equation $g(x) = 0$, where $g$ is a function in one real variable. Indeed, the iterative formula of $d$th-order Householder’s method is

\begin{equation}
\hat{x} = x + d \frac{(1/g)^{(d-1)}(x)}{(1/g)^{(d)}(x)} \quad (d \in \mathbb{Z}^+),
\end{equation}

\footnote{The rate of convergence of the method has order $d + 1$.}
then for \( d = 3 \) we obtain

\[
\hat{x} = x - \frac{6gg'^2 - 3g^2g''}{6g'^3 - 6gg'' + g^2g'''}.
\]

Also, we note that when \( d = 2 \) we obtain Halley’s method mentioned above.

Let us consider \( \alpha p_4 + \beta p_1 p_3 + \gamma p_2^2 + \delta p_1^2 p_2 + \epsilon p_1^4 \); the number of summands is equal to the integer partition of 4. By analogy with the previous we get

\[
\begin{align*}
\alpha(p_4 - S_4) + \beta(p_1p_3 - S_1S_3) + \gamma(p_2^2 - S_2^2) + \delta(p_1^2 p_2 - S_1^2 S_2) + \epsilon(p_1^4 - S_1^4) \\
= (\alpha - \beta - \gamma + \delta - \epsilon)q^4 + (\beta - 2\delta + 4\epsilon)p_1 q^3 + ((2\gamma - \delta)p_2 + (\delta - 6\epsilon)p_1^2)q^2 \\
+ (\beta p_3 + 2\delta p_1 p_2 + 4\epsilon p_1^3)q.
\end{align*}
\]

We put \( \epsilon = 1 \), then in order to obtain a linear equation with respect to the variable \( q \) we need to solve the following system:

\[
\begin{align*}
\alpha - \beta - \gamma + \delta - 1 &= 0, \\
2\gamma - \delta &= 0, \\
\delta - 6 &= 0.
\end{align*}
\]

The solution is \( \alpha = 6, \beta = 8, \gamma = 3, \delta = 6 \). Finally, we have

\[
\lambda_i = z - \frac{4f(6f'^3 - 6ff'f'' + f^2f^{(3)})}{24f'^4 - 36ff'^2f'' + 6f^2f'^2 + 8f^2f'f^{(3)} - f^3f^{(4)} - fT}, \tag{2.10}
\]

where \( T = 6S_4 + 8S_1 S_3 + 3S_2^2 + 6S_1^2 S_2 + S_1^4 \). Since \( (2.10) \) is also related to \( (2.9) \), we can represent \( (2.6), (2.9), (2.10) \) in the following form

\[
\lambda_i = z + d \frac{(1/f)^{(d-1)}}{(1/f)^{(d)}}(z) + (-1)^{d-1} H_d/f(z), \tag{2.11}
\]

where \( d = 2, 3, 4 \), respectively and

\[
\begin{align*}
H_2 &= S_2 + S_1^2, \\
H_3 &= 3S_3 + 3S_1 S_2 + S_1^3, \\
H_4 &= 6S_4 + 8S_1 S_3 + 3S_2^2 + 6S_1^2 S_2 + S_1^4.
\end{align*}
\]

Using the considered approach, we can get formulas similar to \( (2.11) \) for another positive integer \( d \). It is easy to show that when \( d = 1 \), we have \( H_1 = S_1 \). This case corresponds to the Maehly-Ehrlich-Alberth method.

The explicit formula for \( H_d \). The homogeneous symmetric polynomial of degree \( k \) in \( x_1, \ldots, x_n \) is

\[
h_k(x_1, \ldots, x_n) = \sum_{1 \leq j_1 \leq \cdots \leq j_k \leq n} x_{j_1} \cdots x_{j_k}.
\]

As is known, \( h_k \) can be expressed in terms of power sums; the formula is as follows:

\[
h_k = \sum_{r_1 + 2r_2 + \cdots + kr_k = k} \prod_{j=1}^k \frac{p_j^{r_j}}{r_j! j_0^{r_j}}. \tag{2.12}
\]
Using this, we get
\[ h_2 = (p_2 + p_1^2)/2, \]
\[ h_3 = (2p_3 + 3p_1p_2 + p_1^3)/6, \]
\[ h_4 = (6p_4 + 8p_1p_3 + 3p_2^2 + 6p_1^2p_2 + p_1^4)/24. \]

Therefore, we see that
\[ (3.4) \]
\[ n \]
The cases \( d \) Here, as above \( q \)
\[ (3.2) \]
for \( k \)
\[ (3.1) \]
We begin by introducing the following notation:

By (2.13), (2.13) is of \((d + 2)\)th order of convergence.

**Remark 2.4.** In fact, we have proved that (2.11), (2.13) hold for \( d = 1, 2, 3, 4, 5, 6 \).
The cases \( d = 5, 6 \) are not included in the paper because they are cumbersome.

### 3. Generalizations of the Durand-Kerner method

Let \( f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \) and, as before, \( f(z) = \prod_{j=1}^{n} (z - \lambda_j) \).

We begin by introducing the following notation:

\[ (3.1) \]
\[ e_{k;i} = e_k \left( \frac{1}{z - \lambda_1}, \ldots, \frac{1}{z - \lambda_{i-1}}, \frac{1}{z - \lambda_{i+1}}, \ldots, \frac{1}{z - \lambda_n} \right) \quad (0 \leq k \leq n - 1). \]

Also, for convenience, we assume that if \( k \geq n \), then \( e_{k;i} = 0 \). It is easy to see that for \( k \geq 1 \) the following identity holds:

\[ (3.2) \]
\[ e_k = q e_{k-1;i} + e_{k;i}. \]

Here, as above \( q = 1/(z - \lambda_1) \); also \( e_k \) is the elementary symmetric polynomial in variables \( 1/(z - \lambda_j) \) \((1 \leq j \leq n)\). If we put \( k = n \) in (3.2), then

\[ 1/q = \frac{e_{n-1;i}}{e_n}. \]

By (2.11) and (3.1) we have

\[ e_n = \frac{1}{n!} \frac{f^{(n)}(z)}{f(z)} = \frac{1}{f(z)} \]
and \( e_{n-1;i} = \prod_{j \neq i} \frac{1}{z - \lambda_j} \).

Finally, we get

\[ (3.3) \]
\[ \lambda_i = z - f(z)/\prod_{j \neq i} (z - \lambda_j). \]

As is seen, this is the main formula for the Durand-Kerner method (11). Although the derivation of (3.3) from the full factorization of \( f(z) \) is simpler, we have shown the technique that will be used below.

We put \( k = n \) in (3.2), then \( e_{n-1} = q e_{n-2;i} + e_{n-1;i} \). Since \( f(z) e_{n-1} = nz + a_{n-1} \)
and \( f(z) e_{n-1;i} = 1/q \), we have the following:

\[ (3.4) \]
\[ nz + a_{n-1} = f(z) q e_{n-2;i} + 1/q. \]
Dividing this formula by \( q \) and taking into account that \( 1/q = z - \lambda_i \), we obtain

\[
(z - \lambda_i)^2 - (nz + a_{n-1})(z - \lambda_i) + f(z)e_{n-2;i} = 0.
\]

This implicit formula can be used to obtain Weierstrass-like methods. We have two possible ways: the first is to solve the equation (3.5) in the variable \( \lambda_i \), the second is to use (3.3) so that the equation becomes linear, which is to be solved in \( \lambda_i \). In addition, we use the following formula, which can be proved by simple transformations,

\[
e_{n-2;i} = (nz - z + \sum_{j \neq i} \lambda_j)/\prod_{j \neq i}(z - \lambda_j).
\]

Then, following the second way, we have

\[
\lambda_i = z - \frac{1}{nz + a_{n-1} \prod_{j \neq i}(z - \lambda_j)} \left[ (n - 1)z + \sum_{j \neq i} \lambda_j + \frac{f(z)}{\prod_{j \neq i}(z - \lambda_j)} \right].
\]

In this paper we will not study the corresponding iterative method in detail, we will only make a few remarks. If initial approximations are good and all the roots of \( f(z) \) are distinct, then the method has quadratic convergence. If this polynomial has a root of multiplicity \( n \), then this significantly affects the convergence (up to its absence). This is due to the fact that in this case \( -a_n/n \) is the root of \( f(z) \).

Now we generalize the method obtained. We have

\[
e_{n-m} = qe_{n-m-1;i} + e_{n-m;i}.
\]

The following holds:

\[
e_{n-m} = \frac{v_m(z)}{f(z)}, \text{ where } v_m(z) = \sum_{i=0}^{m} a_{n-m+i} \left( \frac{n-m+i}{n-m} \right) z^i
\]

and

\[
e_{k;i} = \frac{c_{n-k-1;i}}{\prod_{j \neq i}(z - \lambda_j)},
\]

where \( c_{m;i} = e_m(z - \lambda_1, \ldots, z - \lambda_{i-1}, z - \lambda_{i+1}, \ldots, z - \lambda_n) \) (0 \( \leq m \leq n - 1 \) ). From these formulas and (3.8) it follows that

\[
(z - \lambda_i)^2 e_{m-1;i} - (z - \lambda_i)v_m(z) + \frac{f(z)}{\prod_{j \neq i}(z - \lambda_j)} c_{m;i} = 0.
\]

Using (3.8), we obtain a linear equation in \( \lambda_i \), solving which we find

\[
\lambda_i = z - \frac{1}{v_m(z)} \frac{f(z)}{\prod_{j \neq i}(z - \lambda_j)} \left[ c_{m;i} + \frac{f(z)}{\prod_{j \neq i}(z - \lambda_j)} c_{m-1;i} \right].
\]

The first values of \( c_{m;i} \) are given below:

\[
c_{0;i} = 1,
\]

\[
c_{1;i} = (n - 1)z + b_1,
\]

\[
c_{2;i} = (n - 1)(n - 2)z^2/2 - (n - 2)b_1z + (b_1^2 - b_2)/2,
\]
where \( b_k = \sum_{j \neq i} x_j^k \) \((k \in \mathbb{Z}^+)\). The general formula is

\[
c_m = \sum_{l=0}^{m} \left( \begin{array}{c} n - 1 - m + l \\ l \end{array} \right) \left( \sum_{r_1 + 2r_2 + \ldots + (m-1)r_{m-l}}^{m-l} \prod_{j=1}^{m-l} \frac{(-b_j)^{r_j}}{r_j!} \right) z^l.
\]

In this formula, we assume that the sum over \( r_1, \ldots, r_{m-l} \) is equal to 1 if \( m - l = 0 \).

**Remark 3.1.** We note that (3.11) can be linearized by (3.12). But this should be done after studying the method based on (3.12).

### 4. Convergence analysis

In this section, we show that simultaneous Halley’s method is of the fourth order of convergence. We consider only the case when all the roots of \( f(z) \) are distinct, in other words, we assume that there exists a positive real number \( M \) such that \(|\lambda_i - \lambda_k| > M\) for any \( l \neq k \). Let us denote the right side of the formula (2.7) by \( \varphi_i(z_1, \ldots, z_n) \). To study the convergence of the method we put \( z_k = \lambda_k + \alpha_k \varepsilon \) \((1 \leq k \leq n)\), where \( \varepsilon \) is real and \( \alpha_k, \ldots, \alpha_n \) are arbitrary complex numbers. Then we consider the expression \( \varphi_i(\lambda_1 + \alpha_1 \varepsilon, \ldots, \lambda_n + \alpha_n \varepsilon) \) as a function of the variable \( \varepsilon \). We note that (2.7) is obtained from the exact formula (2.6). So if \( z_i \) is arbitrary and the remaining \( z_j \) are equal to \( \lambda_j \), then \( \varphi_i(z_1, \ldots, z_n) = \lambda_i \). In this case only one iterative step is necessary to obtain \( \lambda_i \). Thus, a computational error in some iteration step is caused by errors related to the sums in \( \varphi_i \). Therefore, it is convenient to get the following:

\[
f(z_i)^2 \left( \sum_{j \neq i} (z_i - z_j)^{-2} + \left[ \sum_{j \neq i} (z_i - z_j)^{-1} \right]^2 \right) = f(z_i)^2 \left( \sum_{j \neq i} (z_i - \lambda_j)^{-2} + \left[ \sum_{j \neq i} (z_i - \lambda_j)^{-1} \right]^2 \right) + O(\varepsilon^3) \text{ as } \varepsilon \to 0.
\]

Here, we use that \( f(z_i) = f(\lambda_i + \alpha_i \varepsilon) = O(\varepsilon) \) and (since the roots are distinct)

\[
\sum_{j \neq i} (z_i - z_j)^{-2} + \left[ \sum_{j \neq i} (z_i - z_j)^{-1} \right]^2 = \sum_{j \neq i} (z_i - \lambda_j)^{-2} + \left[ \sum_{j \neq i} (z_i - \lambda_j)^{-1} \right]^2 + O(\varepsilon).
\]

Also, since the roots are distinct, it follows that \( f'(\lambda_i) \neq 0 \). Then using this, we obtain

\[
2f(z_i) f'(z_i) 2 [f'(z_i)]^2 - f(z_i) f''(z_i) - [f(z_i)]^2 \left( \sum_{j \neq i} (z_i - \lambda_j)^{-2} + \left[ \sum_{j \neq i} (z_i - \lambda_j)^{-1} \right]^2 \right) + O(\varepsilon^3)
\]

\[
= \frac{2f(z_i) f'(z_i)}{2 [f'(z_i)]^2 - f(z_i) f''(z_i) - [f(z_i)]^2 \left( \sum_{j \neq i} (z_i - \lambda_j)^{-2} + \left[ \sum_{j \neq i} (z_i - \lambda_j)^{-1} \right]^2 \right) + O(\varepsilon^3)}.
\]

Finally, we have \( \varphi_i(\lambda_1 + \alpha_1 \varepsilon, \ldots, \lambda_n + \alpha_n \varepsilon) = \varphi_i(\lambda_1, \ldots, \lambda_i + \alpha_i \varepsilon, \ldots, \lambda_n) + O(\varepsilon^4) \) as \( \varepsilon \to 0 \). As discussed above, \( \varphi_i(\lambda_1, \ldots, \lambda_i + \alpha_i \varepsilon, \ldots, \lambda_n) = \lambda_i \). So we conclude that the method has the fourth order of convergence.
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