THICK POINTS FOR A GAUSSIAN FREE FIELD IN 4 DIMENSIONS

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ABSTRACT. This article is concerned with the study of fractal properties of thick points for 4-dimensional Gaussian Free Field. We adopt the definition of Gaussian Free Field on \( \mathbb{R}^4 \) introduced by Chen and Jakobson (2012) viewed as an abstract Wiener space with underlying Hilbert space \( H^2(\mathbb{R}^4) \). We can prove that for \( 0 \leq a \leq 4 \), the Hausdorff dimension of the set of \( a \)-high points is \( 4 - a \). We also show that the thick points give full mass to the Liouville Quantum Gravity measure on \( \mathbb{R}^4 \).

1. Introduction

Random measures defined by means of log-correlated Gaussian fields \( X \) and that can be formally written as “\( m(d\omega) = e^{iX(\omega)}d\omega \)” arise in conformal field theory and in probability. When \( X \) is an instance of the Gaussian Free Field (GFF) then such measures are referred to as Liouville measures. The interest around such objects comes from physics and in particular from the understanding and proving the KPZ relation, formulated by Knizhnik, Polyakov and Zamolodchikov (Knizhnik et al. (1988)), which gives the relation between volume exponents derived using the quantum metric induced by \( m(d\omega) \) and the Euclidean metric. Several interesting papers have been written to show this relation, and we refer to Duplantier and Sheffield (2011), Rhodes and Vargas (2013), Robert and Vargas (2010) for details about these results. To construct such measures one has to rely on an approximation (cut-off) of the field and there are various methods to construct this approximation. While on the one hand a more geometric approach (which explicitly relies on the structure of the field) is present in the work Duplantier and Sheffield (2011), the perspective of Rhodes and Vargas (2013), Robert and Vargas (2008), Robert and Vargas (2010) dates back to the definition of Kahane (1985), Mandelbrot (1972) of multiplicative chaos, which deals with properties of the covariance kernel. These works extended the concept of multiplicative chaos of Kahane to a more general class of covariance kernels.

In this paper we focus our attention on the multifractal formalism of the underpinned Gaussian field, or with an equivalent terminology on its so-called thick points. To our knowledge the first rigorous study in this direction was made by Mandelbrot (in the collection Mandelbrot et al. (2004)) in the context of one-dimensional log-correlated Gaussian fields. In an interesting work, Hu et al. (2010) showed that the Hausdorff dimension of the set of \( a \)-thick points is \( 2 - a \) for \( 0 \leq a \leq 2 \) for the planar GFF, and in Rhodes and Vargas (2013), Kahane (1985) such a result is shown for “nice” covariance kernels leading to multiplicative chaos. The

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The set of thick points is relevant in understanding the support of two-dimensional Liouville quantum gravity (LQG). It was shown in fact in Duplantier and Sheffield (2011) that the LQG measure is almost surely supported on the thick points, in analogy to Kahane’s similar results (Kahane (1985)) on 1D Gaussian multiplicative chaos and to (Rhodes and Vargas, 2013, Theorem 4.1) in higher dimensions. Our work being motivated by the definition of sphere average introduced by Chen and Jakobson (2012) in dimension 4, we prefer to stick to the more geometrical construction of the Gaussian free field rather than handling it as an instance of multiplicative chaos, although both approaches prove to be fruitful to investigate high points. Other than Chen and Jakobson’s recent article and the development of multiplicative chaos, the main motivation for considering such model comes from its discrete analogue which turns out to be related to the membrane model (cf. Kurt (2007)) defined on \( \mathbb{Z}^4 \). It is known that in dimension 4 the model undergoes a phase transition in terms of the behavior of the infinite Gibbs volume measure, as was proved in Kurt (2009). Recently, some work on the fractal dimension of the thick points in this discrete setting has been carried through by Daviaud (2006) for the 2D discrete Gaussian Free Field and Cipriani (2013) on the discrete 4D membrane model.

In this article we adhere to the sphere average process of Chen and Jakobson (2012) and prove in Theorem 2.1 that the set of thick points gives full mass to the LQG measure. In particular, we show in Theorem 2.2 that the set of \( a \)-thick points has Hausdorff dimension \( 4 - a \) when \( 0 \leq a \leq 4 \). When \( a > 4 \), the set of thick points is almost surely empty. The outline of the article is as follows. In Section 2 we recall the model introduced by Chen and Jakobson (2012) and state our main result more precisely. In Section 3 we list some basic properties of the sphere average process and also provide a proof of Theorem 2.1 using a so-called rooted or Peyrière measure. The proof of Theorem 2.2 is given in Sections 4 and 5 and relies on proving two different bounds. For the upper bound we use the version of the Kolmogorov-Centsov theorem derived by Hu et al. (2010). For the lower bound we use a standard finite-energy method and the Markov property of the GFF.

## 2. GFF Model and Statement of the Main Results

To keep the paper self contained we review in this section some definitions of the GFF on \( \mathbb{R}^4 \) from Chen and Jakobson (2012) and state some properties of the sphere average process which will be useful in deriving our main result. In order to do so we begin with the definition of abstract Wiener space.

**Definition 2.1** (Abstract Wiener space, Stroock (2010)). An abstract Wiener space is a triple \((\Theta, H, W)\), where

- \( \Theta \) is a separable Banach space,
- \( H \) is a Hilbert space which is continuously embedded as a dense subspace of \( \Theta \), equipped with the scalar product \( \langle \cdot, \cdot \rangle_H \),
- \( W \) is a Gaussian probability measure on \( \Theta \) defined as follows.

Let \( \Theta^* \) be the dual space of \( \Theta \). Given any \( x^* \in \Theta^* \) there exists a unique \( h_{x^*} \in H \) such that for all \( h \in H \), \( \langle h, x^* \rangle = \langle h, h_{x^*} \rangle_H \) where \( \langle \cdot, \cdot \rangle \) denotes the action of \( x^* \) on \( \Theta \). The sigma algebra \( B(\Theta) \) on \( \Theta \) is such that all the maps \( \theta \mapsto (\theta, x^*) \) are measurable. \( W \) is a probability measure such that for all \( x^* \in \Theta^* \),
(2.1) \[ \mathbb{E}_W [\exp (i \langle \cdot, x^* \rangle)] = \exp \left( - \frac{\| h_x \|^2_H}{2} \right). \]

Although the introduction of the set \( \Theta \) is evidently important for the definition of the GFF, its choice is not unique as explained in Stroock (2010), Corollary 8.3.2 and afterwards. Moreover \( W(H) = 0 \) as \( H \) is dense in \( \Theta \). In our setting, we consider the underlying Hilbert space to be \( H := H^2(\mathbb{R}^4) \) which is the completion of the Schwartz space \( \mathcal{S}(\mathbb{R}^4) \) equipped with the inner product

\[ (f_1, f_2)_H = \int_{\mathbb{R}^4} (I - \Delta)^2 f_1(x) f_2(x) \, dx \quad \text{for all} \quad f_1, f_2 \in \mathcal{S}(\mathbb{R}^4). \]

\( H^{-2}(\mathbb{R}^4) \) is the Hilbert space consisting of tempered distributions \( \mu \) such that

\[ \| \mu \|_{H^{-2}} = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} (1 + |\xi|^2)^{-2} |\hat{\mu}(\xi)|^2 \, d\xi < \infty. \]

where \( \hat{\mu} \) is the Fourier transform. It is possible to identify \( H \) with \( H^{-2} \) through the linear isometry \( (I - \Delta)^{-2} : H^{-2} \to H \). By abuse of notation we will call \( h_v \) the image of \( v \in H^{-2} \) under \( (I - \Delta)^{-2} \), that is, \( h_v \) is the unique element in \( H \) such that \( \langle h, v \rangle = \langle h_v, h \rangle_H \) for all \( h \in H \). At this point we have to introduce another fundamental object for our work, the \textit{Paley-Wiener integral} \( \mathcal{I}(h_v) \). \( \mathcal{I} \) is viewed as a mapping

\[ \mathcal{I} : x^* \in \Theta^* \quad \mapsto \quad \mathcal{I}(h_{x^*}) \in L^2(\mathcal{W}) \]

\[ \theta \in \Theta \mapsto [I(h_x)](\theta) := \langle \theta, x^* \rangle. \]

By (2.1), we have \( \{ \mathcal{I}(h_v) : v \in H^{-2} \} \) is also a Gaussian family whose covariance is given by

\[ \mathbb{E}_W [\mathcal{I}(h_{v_1}) \mathcal{I}(h_{v_2})] = \langle h_{v_1}, h_{v_2} \rangle_H = \langle v_1, v_2 \rangle_{H^{-2}}. \]

Therefore \( \mathcal{I} \) is an isometry from \( \{ h_{x^*} : x^* \in \Theta^* \} \to L^2(\mathcal{W}) \), and since the former set is dense in \( H \), it admits a unique extension to the whole of \( H \). For every \( x \in \mathbb{R}^4 \) and \( \epsilon > 0 \) denote as \( x^\epsilon \in H^{-2} \) the tempered distribution given by

\[ \langle f, \sigma_{x^\epsilon} \rangle = \frac{1}{2\pi^2\epsilon^3} \int_{D(x,\epsilon)} f(y) \, d\sigma(y), \quad \text{for all} \quad f \in \mathcal{S}(\mathbb{R}^4), \]

where \( d\sigma \) is the surface area measure on \( D(x,\epsilon) \), the sphere of radius \( \epsilon \) around \( x \). Interestingly, Chen and Jakobson (2012) noted that \( \{ \mathcal{I}(h_{x^\epsilon}) : \epsilon > 0 \} \) fails to possess the Markov property and considered the following Gaussian family:

\[ \{ \mathcal{I}(h_{x^\epsilon}), \mathcal{I}(h_{d\sigma_{x^\epsilon}}) : x \in \mathbb{R}^4, \epsilon > 0 \}, \]

where \( d\sigma_{x^\epsilon} \) the tempered distribution given by \( \langle f, d\sigma_{x^\epsilon} \rangle := \frac{1}{\epsilon^3} \langle f, \sigma_{x^\epsilon} \rangle \) for all \( f \in \mathcal{S}(\mathbb{R}^4) \). It is important to point out at this juncture that such a collection is reminiscent of the double boundary conditions needed for the membrane model in the discrete case (Kurt (2008)). Let \( \xi := (1,1)^T \) and

\[ B(r) := \left( \begin{array}{ccc} I_1(r) / r & I_1'(r) / r & I_1''(r) / r \\ I_2(r) / r & I_2'(r) / r & I_2''(r) / r \end{array} \right), \]
where $I_k$ are the modified Bessel functions of order $k \in \mathbb{N}$. Define

$$
\mu^x_\epsilon := \zeta^T \mathbf{B}^{-1}(\epsilon) \left( \frac{c^x_\epsilon}{d\sigma^x_\epsilon} \right).
$$

It was shown in Chen and Jakobson (2012) that $\mu^x_\epsilon \in H^{-2}(\mathbb{R}^4)$ and $\{I(h^x_\epsilon) : x \in \mathbb{R}^4, \epsilon > 0\}$ forms a Gaussian family with the correct Markovian properties and is the suitable candidate for the sphere average process.

**Definition 2.2** (Thick points of the sphere average). For the sphere average process the set of $a$-thick points is defined as

$$
T(a) = \left\{ x \in \mathbb{R}^4 : \lim_{\epsilon \to 0} \frac{I(h^x_\epsilon)}{\sqrt{2\pi^2 G(\epsilon)}} = \sqrt{2a} \right\}.
$$

Here $G(\epsilon) = \text{Var}_W(I(h^x_\epsilon))$ and an explicit expression using Bessel functions is given in (3.1).

We would also need a definition of another set quite similar to the above:

$$
T_{\geq}(a) = \left\{ x \in \mathbb{R}^4 : \limsup_{\epsilon \to 0} \frac{I(h^x_\epsilon)}{\sqrt{2\pi^2 G(\epsilon)}} \geq \sqrt{2a} \right\}.
$$

It is easy to see that $T(a) \subset T_{\geq}(a)$.

One of the main results of Chen and Jakobson (2012) (Theorem 5) was to show the existence of the Liouville quantum gravity measure and the validity of the KPZ relation in $\mathbb{R}^4$. Define a random measure on $\mathbb{R}^4$ by

$$
m^0_\epsilon(dx) = E^0_\epsilon(x)dx,
$$

where

$$
E^0_\epsilon = \exp \left( \gamma I(h^x_\epsilon) - \frac{\gamma^2}{2} G(\epsilon) \right).
$$

If $\epsilon_n = \epsilon^n_0$ with $\epsilon_0 \in (0,1)$ and $0 < \gamma^2 < 2\pi^2$, then there exists a non-negative measure $m^\theta$ on $\mathbb{R}^4$ such that the following convergence holds for every $f \in C_c(\mathbb{R}^4)$:

$$
\int_{\mathbb{R}^4} f(x)m^\theta_n(dx) \to \int_{\mathbb{R}^4} f(x)m^\theta(dx) \text{ as } n \to \infty
$$

$\mathcal{W}$-almost surely and also in $L^2(\mathcal{W})$. It is also known that this measure is almost surely positive.

In the following Theorem we show that the set of thick points gives full measure to the LQG measure in $\mathbb{R}^4$.

**Theorem 2.1.** Let $0 < \gamma^2 < 2\pi^2$, then for $a = \gamma^2/4\pi^2$ we have

$$
m^\theta(T(a)^c) = 0 \ \mathcal{W} \text{ - a.s.}
$$

That is, the set $T(a)$ gives full mass to the measure $m^\theta(\cdot)$.
For the proof of Theorem 2.1 we construct the rooted measure or Peyri`ere measure. For the use of rooted measures see Duplantier and Sheffield (2011), Rhodes and Vargas (2013).

Before we state our main result on fractal properties of thick points, we recall the definition of Hausdorff dimension and Hausdorff measure.

**Definition 2.3** (Hausdorff dimension). Let $X$ be a metric space and $S \subseteq X$. For every $d \geq 0$ and $\delta > 0$ define the Hausdorff-$d$-measure in the following way:

$$C^d_\delta(S) := \inf \left\{ \sum_i \text{diam}(E_i)^d : E_1, E_2, E_3, \ldots, \text{cover } S, \text{diam}(E_i) \leq \delta \right\},$$

i.e. we are considering coverings of $S$ by sets of diameter no more than $\delta$. Then

$$C^d_\delta(S) = \sup_{\delta > 0} C^d_\delta(S) = \lim_{\delta \downarrow 0} C^d_\delta(S)$$

is the Hausdorff-$d$-measure of the set $S$. The *Hausdorff dimension* of $S$ is defined by

$$\dim_H(S) := \inf\{d \geq 0 : C^d_\delta(S) = 0\}.$$

**Theorem 2.2.** For $0 \leq a \leq 4$, the Hausdorff dimension of $T(a)$ is $4 - a$. For $a > 4$, we have that $T(a)$ is empty.

**Remark 2.1.** The above result shows similarity with the membrane model. In Cipriani (2013) it was shown that discrete fractal dimension of the $a$-high points is $4 - 4a^2$.

To prove Theorem 2.2 we apply some of the techniques implemented in Dembo et al. (2000, 2001) to show similar results for occupation measures of planar or spatial Brownian motion.

3. **GFF model and some estimates**

This section is devoted to providing some details about the behavior of the sphere average process, such as the covariance structure. We then use them to derive a proof of Theorem 2.1.

3.1. **Some more properties of the sphere average process: covariance structure.** Let us denote as $D(0, R)$ the sphere centered at 0 with radius $R > 0$. Let $I_r, K_r$ be the modified Bessel functions of order $r \in \mathbb{N} \cup \{0\}$. Define the positive function $G : (0, \infty) \mapsto (0, \infty)$ by

$$G(r) := \left(-\frac{1}{4\pi^2}\right)^{\frac{2I_1(r)K_1(r)+2I_2(r)K_0(r)-1}{I_1(r)-K_0(r)I_2(r)}}.$$

It can be shown that $G$ is strictly decreasing and smooth, with $\lim_{r \to 0} G(r) = +\infty$ and $\lim_{r \to +\infty} G(r) = 0$. It also follows from the properties of the Bessel functions that as $r$ decreases to 0, $G(r)$ asymptotically behaves like $\frac{1}{2\pi} \log r$. Then, we have that

(1) given $x \in \mathbb{R}^d$ and $\epsilon_1 \geq \epsilon_2 > 0$,

$$E_W \left[ I \left( h_{\rho_1^2} \right) I \left( h_{\rho_2^2} \right) \right] = E_W \left[ I^2 \left( h_{\rho_1^2} \right) \right] = G(\epsilon_1).$$
(2) Given $x, y \in \mathbb{R}^4$, $x \neq y$, and $\epsilon_1, \epsilon_2 > 0$ with $D(x, \epsilon_1) \cap D(y, \epsilon_2) = \emptyset$,
(3.3) \[ E_W \left[ I \left( h_{\mu_1} \right) I \left( h_{\mu_2} \right) \right] = \frac{1}{2\pi^2} K_0(|x - y|), \]
where $K_0$ is the modified Bessel function of order 0.
(3) Given $x, y \in \mathbb{R}^4$, $x \neq y$, and $\epsilon_1, \epsilon_2 > 0$ with $D(y, \epsilon_2) \subseteq D(x, \epsilon_1)$,
(3.4) \[ E_W \left[ I \left( h_{\mu_1} \right) I \left( h_{\mu_2} \right) \right] = I_0(|x - y|) G(\epsilon_1) - \frac{1}{4\pi^2} \frac{I_2(|x - y|)}{I_1(\epsilon_1) - I_0(\epsilon_1) I_2(\epsilon_1)}. \]

The next lemma states one of the most useful and important properties of the spherical average process and is analogous to the properties of the two dimensional circular average process studied in Duplantier and Sheffield (2011), Hu et al. (2010). It shows that for fixed $x \in \mathbb{R}^4$, the spherical average after a time change is a Brownian motion and in disjoint annuli two such motions are independent. We briefly sketch the proof of the following lemma as it is an easy consequence after one compares the covariance structure.

**Lemma 3.1.**
(a) Let $G(\cdot)$ be as in (3.1) and for $x \in \mathbb{R}^4$, let $B(x, t) = I \left( h_{\mu_x} \right)_{G^{-1}(t)}$.

Then $B(x, t) - B(x, t_1)$ has the same distribution as a standard Brownian motion for $t \geq t_1$.

(b) Given $x, y \in \mathbb{R}^4$ and $t_1 \leq t \leq t_2$ and $s_1 \leq s \leq s_2$ be such that $D(x, G^{-1}(s_1)) \setminus D(x, G^{-1}(s_2))$ and $D(y, G^{-1}(t_1)) \setminus D(y, G^{-1}(t_2))$ are disjoint, then $\{B(x, s) - B(x, s_1)\}_{s_1 \leq s \leq s_2}$ is independent of $\{B(y, t) - B(y, t_1)\}_{t_1 \leq t \leq t_2}$.

**Proof.** (a) It follows from (3.2) that for $t_1 \leq s \leq t$ one has
\[ \text{Cov}_W(B(x, t) - B(x, t_1), B(x, s) - B(x, t_1)) = \]
\[ = G(G^{-1}(s)) - G(G^{-1}(t_1)) + G(G^{-1}(t_1)) - G(G^{-1}(t)) = s - t_1 \]
Here we have used the fact that $G(\cdot)$ and $G^{-1}(\cdot)$ are decreasing functions and hence, as $t_1 \leq s \leq t$ we have $G^{-1}(t_1) \geq G^{-1}(s) \geq G^{-1}(t)$.
(b) As the annuli are disjoint it follows that $|x - y| > G^{-1}(t_1) + G^{-1}(s_1) \geq G^{-1}(t) + G^{-1}(s) \geq G^{-1}(t_1) + G^{-1}(s_1)$ and hence again using (3.3) we obtain
\[ \text{Cov}_W(B(y, t) - B(y, t_1), B(x, s) - B(x, s_1)) = 0. \]

\[ \square \]

3.2. **Proof of Theorem 2.1.** Let $\Gamma$ be a compact subset of $\mathbb{R}^4$. Let $B(\Gamma)$ be the Borel sigma algebra of subsets of $\Gamma$. We define a rooted measure on $B(\Theta) \otimes B(\Gamma)$ as
\[ \mathcal{M}(dxd\theta) = \frac{m^\theta(dx)d\mathcal{W}(d\theta)}{|\Gamma|}. \]
Here $|\Gamma|$ denotes the volume of the set $\Gamma$ with respect to the Lebesgue measure. Note that $\mathcal{M}(\Theta \times \Gamma) = E_W \left[ m^\theta(\Gamma) \right] |\Gamma|^{-1} = 1$ and as such $\mathcal{M}$ is a probability measure on the space $\Gamma \times \Theta$.

Let $r(t) := G^{-1}(t + G(R))$, $R > 0$ fixed and define
\[ \bar{B}(x, t)(\theta) := I \left( h_{\mu_1} \right)_{r(t)}(\theta) - I \left( h_{\mu_1} \right)(\theta). \]

The following lemma allows us to view the random measure $m^\theta$ in a different way. We show that the joint distribution of $(x, \bar{B}(x, t))$ under $\mathcal{M}(dxd\theta)$ is nothing but
the distribution of \((x, \tilde{B}(x, t) + \gamma t)\) under \(\mathcal{W}(d\theta)dx\) and in the latter case the marginal on \(\Theta\) does not depend on \(x\).

**Lemma 3.2.** Let \(0 < \gamma^2 < 2\pi^2\). For any compact set \(\Gamma\) and any \(F \in C_c(\mathbb{R}^4 \times \mathbb{R})\) we have

\[
(3.5) \int_{\Theta} \int_{\Gamma} F(x, \tilde{B}(x, t)(\theta)) \mathcal{M}(dx d\theta) = \frac{1}{|\Gamma|} \int_{\Theta} \int_{\Gamma} F(x, \tilde{B}(x, t)(\theta) + \gamma t) \mathcal{W}(d\theta)dx.
\]

**Proof.** Note that for almost every \(\theta\), the map \(x \in \Gamma \mapsto F(x, \tilde{B}(x, t)(\theta))\) is continuous by Corollary 3 of Chen and Jakobson (2012). So from the weak convergence in (2.5) we have that

\[
\lim_{n \to \infty} \int_{\Theta} \int_{\Gamma} F(x, \tilde{B}(x, t)) m_{\epsilon_n}^\theta(dx) = \int_{\Gamma} F(x, \tilde{B}(x, t)) m^\theta(dx).
\]

Since the function in the integral is bounded we have for some constant \(C\) and for all \(n\)

\[
\int_{\Theta} \int_{\Gamma} F(x, \tilde{B}(x, t)) m_{\epsilon_n}^\theta(dx) \mathcal{W}(d\theta) \leq C|\Gamma|.
\]

So by dominated convergence

\[
(3.6) \lim_{n \to \infty} \frac{1}{|\Gamma|} \int_{\Theta} \int_{\Gamma} F(x, \tilde{B}(x, t)) m_{\epsilon_n}^\theta(dx) \mathcal{W}(d\theta) = \int_{\Theta} \int_{\Gamma} F(x, \tilde{B}(x, t)) \mathcal{M}(dx d\theta).
\]

Note that for small enough \(\epsilon > 0\)

\[
\text{Cov}(\tilde{B}(x, t), h_{\epsilon^2}) = G(r(t)) - G(R) = t
\]

holds, so for \(n\) large enough we have by Cameron-Martin theorem

\[
\int_{\Theta} \int_{\Gamma} F(x, \tilde{B}(x, t)) m_{\epsilon_n}^\theta(dx) \mathcal{W}(d\theta) = \int_{\Theta} \int_{\Gamma} F(x, \tilde{B}(x, t)) E_{\epsilon_n}^\theta(x) dx \mathcal{W}(d\theta)
= \int_{\Gamma} \int_{\Theta} F(x, \tilde{B}(x, t)(\theta) + \gamma t) \mathcal{W}(d\theta)dx.
\]

We have the required statement in the lemma using (3.6).

**Proof of Theorem 2.1.** Using the fact \(E_{\mathcal{W}}[m^\theta(A)] = |A|\) for any bounded set \(A\) it follows that the marginal of \(\mathcal{M}\) on \(\Gamma\) is nothing but the normalized Lebesgue measure on \(\Gamma\). Hence by Theorem 9.2.2. of Stroock (2010) there exists a Borel measurable map

\[
x \in \Gamma \rightarrow \mathcal{L}_x(\cdot) \in M_1(\Theta),
\]

where \(M_1(\Theta)\) is the set of probability measures on \(\Theta\) and the following holds

\[
\mathcal{M}(dx d\theta) = \mathcal{L}_x(d\theta) \frac{dx}{|\Gamma|}.
\]

Note that \(\mathcal{L}_x(d\theta)\) is nothing but the regular conditional probability. Now using the above decomposition we have that

\[
\int_{\Theta} \int_{\Gamma} F(x, \tilde{B}(x, t)) \mathcal{M}(dx d\theta) = \frac{1}{|\Gamma|} \int_{\Gamma} \int_{\Theta} F(x, \tilde{B}(x, t)) \mathcal{L}_x(d\theta)dx.
\]

So from (3.2) we have for any compact set \(\Gamma\) and \(F \in C_c(\mathbb{R}^4 \times \mathbb{R})\)

\[
(3.7) \frac{1}{|\Gamma|} \int_{\Gamma} \int_{\Theta} F(x, \tilde{B}(x, t)) \mathcal{L}_x(d\theta)dx = \frac{1}{|\Gamma|} \int_{\Gamma} \int_{\Theta} F(x, \tilde{B}(x, t) + \gamma t) \mathcal{W}(d\theta)dx.
\]
If we denote \( \mu_x \) to be law of \( \tilde{B}(x,t) \) under \( \mathcal{L}_x(d\theta) \) and \( \nu \) be the law \( \tilde{B}(x,t) + \gamma t \) under \( \mathcal{W}(d\theta) \) on \( \mathbb{R} \) it is possible to see that \( \nu \) is the law of a standard Brownian motion with a drift. Since (3.7) holds for any compact set \( \Gamma \), it is easy to show that for almost every \( x \in \mathbb{R}^d \), \( \mu_x = \nu \). If we take \( a = \gamma^2 / 4\pi^2 \) and use the fact that the sphere average process is a time inversion of a Brownian motion (see Lemma 3.1), then the set of thick points can also be written as

\[
T(a) = \left\{ x \in \mathbb{R}^d : \lim_{\epsilon \to 0} \frac{\tilde{B}(x,t)}{t} = \gamma \right\}.
\]

Now from the discussion above we have that

\[
\mathcal{M}(T(a)^c) = \frac{1}{|\Gamma|} \int_{\Gamma} \mathcal{L}_x(T(a)^c) dx.
\]

Since the law of \( \tilde{B}(x,t) \) under \( \mathcal{L}_x \) is the same as the law of Brownian motion with a drift, the condition for the thick points gets satisfied with probability 1. So we have \( \mathcal{M}(T(a)^c) = 0 \), which together with the fact that \( m^0(\cdot) \) is a positive measure with probability 1 proves the result. \( \square \)

4. Upper bound of Theorem 2.2

In this section we prove the upper bound. By the countable stability property, viz.

\[
\dim_H \left( \bigcup_{i=1}^\infty E_i \right) = \sup_{1 \leq i \leq \infty} \dim_H (E_i)
\]

it is enough to show that for \( R \geq 1 \)

\[
\dim_H T_{\geq}(a, R) = \dim_H \left\{ x \in D(0, R) : \limsup_{\epsilon \to 0} \frac{I(h_{\mu_x}^a)}{\sqrt{2\pi^2 G(\epsilon)}} \geq \sqrt{2a} \right\} \leq 4 - a
\]

almost surely. Hence if we cover \( \mathbb{R}^d \) with a countable union of balls of radius \( R = 1, 2, \ldots \), this will prove the upper bound. The next proposition gives the local Hölder continuity of the process and through it we can determine a modification of the process which has some uniform estimates on the increments. It is similar to Proposition 2.1 of Hu et al. (2010) and uses Lemma C.1 of Hu et al. (2010). The proof also uses some finer estimates on the covariance functions and some bounds on Bessel functions which are provided in the Appendix.

**Proposition 4.1.** There exists a modification \( \tilde{X} \) of the process \( \{ I(h_{\mu_x}^a) : z \in D(0, R), t \in (0, 1) \} \) such that for every \( 0 < \gamma < \frac{1}{2} \) and \( \epsilon, \zeta > 0 \) there exists \( M > 0 \) such that the following holds:

\[
|\tilde{X}(z,r) - \tilde{X}(w,s)| \leq M \left( \log \frac{1}{r} \right)^\beta |(z,r) - (w,s)|^\gamma,
\]

for all \( z, w \in D(0, R) \) and \( r, s \in (0, 1) \) with \( 1/2 \leq r/s \leq 2 \).

**Proof.** Consider now \( x, y \in D(0, R), \epsilon_1, \epsilon_2 \in (0, 1) \) and we abbreviate

\[
H_{\epsilon_1, \epsilon_2}(x, y) := \text{Cov}_W \left( I(h_{\mu_x}^{\epsilon_1}), I(h_{\mu_y}^{\epsilon_2}) \right).
\]

We distinguish between three cases:
Case 1: Let $x = y$. By Lemma 6.1, we have
\[
|H_{e_1, e_1}(x, x) - H_{e_2, e_1}(x, x)| \leq |H_{e_1, e_1}(x, x) - H_{e_1, e_2}(x, x)| + |H_{e_2, e_1}(x, x) - H_{e_1, e_2}(x, x)| \\
\leq |G(e_1) - G(e_1 \lor e_2)| + |G(e_2) - G(e_1 \lor e_2)| \\
\leq C|e_1 - e_2| \overline{e_1 \lor e_2}.
\]

Here we have used that $|\log(x/y)| \leq \frac{|x-y|}{x+y}$.

Case 2: Let $D(x, e_1) \cap D(y, e_2) = \emptyset$. In this case $|x-y| > e_1 + e_2 > e_1$.

Then
\[
|H_{e_1, e_1}(x, x) - H_{e_1, e_2}(x, y)| = |G(e_1) - \frac{1}{2\pi} K_0(|x-y|)| \\
\leq -C(\log e_1 + \log(|x-y|)) \leq \frac{|x-y|}{e_1}.
\]

Similarly one can show that $|H_{e_2, e_2}(y, y) - H_{e_1, e_2}(x, y)| \leq \frac{|x-y|}{e_1}$.

Case 3: Let $D(y, e_2) \subseteq D(x, e_1)$.

\[
|H_{e_1, e_1}(x, x) - H_{e_1, e_2}(x, y)| \leq |G(e_1)(1 - I_0(|x-y|))| + C \frac{I_2(|x-y|)}{I_1^2(e_1) - I_0^2(e_1) I_2(e_1)} \\
\leq -C \log e_1 |x-y|^2 + \frac{|x-y|^2}{e_1} \leq C \frac{|x-y|}{e_1}.
\]

Combining these three cases we obtain that
\[
\text{Var}_W \left( \mathcal{I} \left( h_{\mu_1^y} \right) - \mathcal{I} \left( h_{\mu_2^y} \right) \right) \leq C \frac{|x-y| + |e_1 - e_2|}{e_1 \lor e_2}.
\]

Since $\mathcal{I} \left( h_{\mu_1^x} \right) - \mathcal{I} \left( h_{\mu_2^x} \right)$ is Gaussian,

\[
\mathbb{E}_W \left[ \left| \mathcal{I} \left( h_{\mu_1^y} \right) - \mathcal{I} \left( h_{\mu_2^y} \right) \right|^2 \right] \leq C \left( \frac{|x-y| + |e_1 - e_2|}{e_1 \lor e_2} \right)^{\alpha/2}.
\]

We can find $\alpha$ and $\beta$ large enough such that $|\frac{\beta}{\alpha} - \frac{1}{2}| < \delta$, and consequently by (Hu et al., 2010, Lemma C.1) there exists a modification $\widetilde{X}(x, \epsilon) = \mathcal{I} \left( h_{\mu_2^y} \right)$ a.s. on $L^2(W)$ satisfying (4.2).

In this section for the proof of the upper bound we work with this modification which we also denote by $\mathcal{I} \left( h_{\mu_1^x} \right)$. Recall that $B(x, t) = \mathcal{I} \left( h_{\mu_{G^{-1}(t)}} \right)$.

Proof of the upper bound. Let $\epsilon > 0$ and $\gamma \in (0, 1/2)$, $\zeta \in (0, 1)$ and denote $\gamma := (1+\epsilon)\gamma$. Also let $K := e^{-1}$, $r_n := n^{-K}$.

Define the set
\[
U_R := \left\{ x \in D(0, R) : \limsup_{n \to +\infty} \frac{\mathcal{I} \left( h_{\mu_{G^{-1}(t)}} \right)}{\sqrt{2\pi} G(r_n)} \geq \gamma \right\}.
\]

We first show that
\[
T_{\geq} (a, R) \subset U_R.
\]
For \( x \in T_\geq (a, R) \) and for \( t \in (G(r_n), G(r_{n+1})) \) we write \( B(x, G(r_n)) = B(x, G(r_n)) - B(x, t) + B(x, t) \). By Proposition 4.1 we have

\[
|B(x, t) - B(x, G(r_n))| \leq M \left( \log \left( \frac{1}{G^{-1}(t)} \right) \right) \delta \left( \frac{G^{-1}(t) - r_n}{G^{-1}(t) + t} \right) \leq M(\log(n+1))^{\delta} \left( \frac{r_{n+1} - r_n}{r_{n+1}} \right) = O \left( \left( \log n \right)^{\delta} \right).
\]

(4.5)

Hence using the fact that \( G(r_n) \sim C \log n \) for \( n \to +\infty \) and \( \delta < 1 \) we have

\[
\frac{|B(x, G(r_n)) - B(x, t)|}{\sqrt{2\pi^2 G(r_n)}} = O \left( \left( \log n \right)^{\delta} \right) = o(1).
\]

Now (4.4) follows as we have

\[
\limsup_{n \to +\infty} \frac{B(x, G(r_n))}{\sqrt{2\pi^2 G(r_n)}} \geq \limsup_{t \to +\infty} \frac{B(x, t)}{\sqrt{2\pi^2 t}} \geq \sqrt{2}a.
\]

The next step is to determine a cover for the set \( U_R \). In view of that, let \( (x_{nj})_{j=1}^{\tilde{m}} \) be a maximal collection of points in \( D(0, R) \) such that \( \inf_{j \neq j'} |x_{nj} - x_{n\ell}| \geq r_n^{1+\epsilon} \). Denote

\[
A_n := \left\{ j : \frac{|B(x_{nj}, G(r_n))|}{\sqrt{2\pi^2 G(r_n)}} \geq \sqrt{2}a - \delta(n) \right\}
\]

with \( \delta(n) = C(\log n)^{\delta - 1} \) (the constant \( C \) will be tuned later according to (4.6)). For any \( x \in D(0, R) \), there exists \( j \in \{1, \ldots, \tilde{m} \} \) such that \( x \in D(x_{nj}, r_n^{1+\epsilon}) \). By (4.5) we have,

\[
\frac{|B(x_{nj}, G(r_n)) - B(x, G(r_n))|}{\sqrt{2\pi^2 G(r_n)}} \leq C(\log n)^{\delta} \frac{|x - x_n|^\gamma}{G(r_n)^{1+\gamma}}
\]

(4.6)

\[
= \delta(n) \frac{\log n}{G(r_n)} \leq C\delta(n)
\]

which implies, renaming possibly \( \delta(n) \),

\[
\frac{B(x_{nj}, G(r_n))}{G(r_n)} \geq \sqrt{2}a - \delta(n).
\]

Hence we have \( j \in A_n \). Therefore for all \( N \geq 1, \bigcup_{n \geq N} D(x_{nj}, r_n^{1+\epsilon}) \) covers \( U_R \) with sets having maximal diameter \( 2r_n^{1+\epsilon} \). Next we claim that

\[
E_{W} [||A_n||] \leq C(\log n) r_n^{-4(1+\epsilon)} + o(1).
\]

(4.7)

Assume (4.7) for the moment. If we choose \( \alpha := 4 - a + \epsilon 4 + \epsilon \) we have

\[
E_{W} \left[ \sum_{n \geq N} \sum_{j \in A_n} \text{diam} (D(x_{nj}, r_n^{1+\epsilon}))^a \right] \leq \sum_{n \geq N} (\log n) r_n^{(1+\epsilon)\alpha - 4(1+\epsilon) + o(1)}
\]

\[
\leq \sum_{n \geq N} (\log n) r_n^{a + o(1)} = C \sum_{n \geq N} (\log n) n^{-a + o(1)} < +\infty.
\]
Therefore \( \sum_{n \geq N} \sum_{j < A_n} \text{diam}(D(x_{nj}, r_{n+1}^a))^2 < +\infty \) a.s. and this implies \( \dim_H(T_{\geq}(a, r)) \leq 4 - a \) a.s. by letting \( \varepsilon \downarrow 0 \). This completes the proof of the upper bound provided we show (4.7). We first estimate \( W(j \in A_n) \) as follows:

\[
W(j \in A_n) = W\left( \frac{|B(x_{nj}, G(r_n))|}{G(r_n)} \geq (\sqrt{2} - \delta(n)) \frac{2 \pi^2}{\sqrt{G(r_n)}} \right)
\]

\[
\leq C(a + o(1)) G(r_n) \exp\left(-a (1 - o(1))^2 \frac{2 \pi^2}{G(r_n)}\right) \leq C(1 + o(1)) r_n^{a+o(1)},
\]

since \( G(r_n) \sim -\frac{\log r_n}{2 \pi^2} \) as \( n \to +\infty \). Furthermore

\[
E_W[|A_n|] \leq C(\log n) r_n^{a+o(1)} \leq (\log n) r_n^{a+o(1)} - 4(1 + \varepsilon).
\]

This proves (4.7) and hence the upper bound.

Now we show that for every \( R > 1 \), \( T_{\geq}(a, R) \) is empty for \( a > 4 \) using the above estimates. Note that

\[
\sum_{n \geq 1} W(|A_n| > 1) \leq \sum_{n \geq 1} E_W[|A_n|] \leq \sum_{n \geq 1} r_n^{a-4(1+\varepsilon)} = \sum_{n \geq 1} r_n^4 < +\infty
\]

and hence by the Borel-Cantelli lemma we can conclude that, if \( \varepsilon \) becomes arbitrarily small, \( |A_n| = 0 \) eventually and so \( T_{\geq}(a, R) \) is empty for \( a > 4 \) with probability one.

\[
\square
\]

5. Lower bound of Theorem 2.2

To derive the lower bound we use the energy method. For detailed use of this method see Section 4.3 of M"orters et al. (2010). The \( \alpha \)-th energy of a measure \( \mu \) is given by

\[
I_{\alpha}(\mu) = \iint \frac{d\mu(x) d\mu(y)}{|x - y|^\alpha}.
\]

Given a set \( A \), if we can find a measure \( \rho \) such that \( I_{\alpha}(\rho) < \infty \) then \( \dim_H(A) > \alpha \). For this, partition the hypercube \( J := [0, 1]^4 \) into \( s_n^{-4} \) smaller hypercubes of radius \( s_n = \frac{1}{n^4} \). Let \( x_{ni} \) be the centers of these hypercubes and \( C_n \) be the set of these centers. Define \( t_m := G(s_m) \) for all \( m \leq n \). Note that since \( G \) is decreasing we have that \( t_m \) is increasing and also using the asymptotic expansion of \( G \) we have, \( t_m = -\frac{\log s_m}{2 \pi^2} (1 + o(1)) \). Let \( A_m(x), B_m(x) \) be the events

\[
A_m(x) := \left\{ \sup_{t_m < t \leq t_{m+1}} |B(x, t) - B(x, t_m)| - \sqrt{4a \pi^2(t - t_m)} \leq \sqrt{t_{m+1} - t_m} \right\},
\]

\[
B_m(x) := \left\{ \sup_{t \geq t_m} |B(x, t) - B(x, t_m)| - t \leq 1 - t_m \right\}.
\]

We say that \( x \) is an \( n \)-perfect \( a \)-thick point if \( E^n(x) := \bigcap_{m \leq n} A_m(x) \cap B_{n+1}(x) \) occurs. Note that \( B_{n+1}(x) \) is independent of the other events. We introduce a random variable \( Y_{ni} \) for \( i = 1, \cdots, |C_n| \) such that

\[
Y_{ni} = \begin{cases} 1 & \text{if } x_{ni} \text{ is an } n \text{-perfect } a \text{-thick point,} \\ 0 & \text{otherwise.} \end{cases}
\]

Fix \( t_m < t \leq t_{m+1} \) and on the event \( E^n(x) \) we have, as \( m \to \infty \),

\[
|B(x, t) - B(x, t_1) - \sqrt{4a \pi^2(t - t_1)}| = o(m \log m) = o(t). \tag{5.1}
\]
Define now the set of perfect $a$-thick points as
\[ P(a) := \bigcap_{k \geq 1} \bigcup_{n \geq k} S(z, sn), \]
where $C_n(a)$ is the set of centers of which $x_{ni}$ is a $n$-perfect thick point and $S(z, r)$ is a hypercube of radius $r$ centered around $z$. Let
\[ T(a, J) := \left\{ x \in J : \lim_{t \to \infty} \frac{I(h^{G_{n-1}(t)}}{\sqrt{2\pi t}} = a \right\} \subset T(a). \]

**Lemma 5.1.**
(5.2)
\[ P(a) \subseteq T(a, J). \]

**Proof.** If $z \in P(a)$ there exists a sequence $(z_{nk})_{k \in \mathbb{N}}$ of points s. t. $z_{nk} \in C_n(a)$ for all $k$ and $|z - z_{nk}| \leq s_n$. For $m$ s. t. $t_m < t \leq t_{m+1}$
\[ |B(z_{nk}, t) - B(z_{nk}, t_1) - \sqrt{4\alpha \pi^2}(t - t_1)| = o(t) \]
follows as in (5.1). Since the Brownian motion is a.s. continuous taking the limit for $k \to +\infty$
\[ |B(z, t) - B(z, t_1) - \sqrt{4\alpha \pi^2}(t - t_1)| = o(t) \]
and dividing by $\sqrt{2\pi t}$
\[ \left| I \left( \frac{h^{G_{n-1}(t)}}{\sqrt{2\pi t}} \right) - \sqrt{2\alpha} \right| = o(1) \]
which is an equivalent formulation of the set of thick points. \hfill \Box

Next we make preparations to define a measure $\mu$ supported on $P(a)$ with positive probability. For this purpose define a sequence of measures $\mu_n$ on $J$ supported on $n$-perfect thick points.

\[ (5.3) \quad \mu_n(\cdot) = \sum_{i=1}^{[C_n]} \frac{1}{W(E^{a}(x_{ni}))} 1_{\{Y_{ni}=1\}} \lambda(\cdot \cap S(x_{ni}, s_n)), \]
where $\lambda(\cdot)$ is the Lebesgue measure.

In the following lemma we list down some important properties of this measure.

**Lemma 5.2.** Let $\mu_n(\cdot)$ be as above. Then the following hold:
(a) $E_W[\mu_n(J)] = 1$;
(b) $\sup_n E_W[\mu_n(J)^2] < \infty$;
(c) $\sup_n E_W[I_\alpha(\mu_n)] < \infty$;
(d) there exist $a, b \in (0, \infty)$ such that for all $n$ we have
\[ W \left( b \leq \mu_n(J) < b^{-1}, I_\alpha(\mu_n) < a \right) > 0 \]
for any $\alpha \leq 4 - a$.

The proof of Lemma 5.2 requires a correlation inequality and a lower bound depends on the following lemma. Its proof is similar to the proof of Lemma 3.3 of Hu et al. (2010) and hence we skip it.
Lemma 5.3. Let $A_m(x), B_m(x)$ be as above with $s_m = \frac{1}{m!}$. Let

$$E^n(x) = \bigcap_{m \leq n} A_m(x) \cap B_{n+1}(x).$$

Then for every $y \in S(x, s_l) \setminus S(x, s_{l+1})$, $l > 2$, we have

$$W(E^n(x) \cap E^n(y)) \leq \mathcal{C}_l W(E^n(x)) W(E^n(y)),$$

where $\mathcal{C}_l$ is defined by

$$\mathcal{C}_l := C \prod_{j \leq l+1} \frac{1}{c_j},$$

and $c_j = \exp \left( \frac{1}{2} \sqrt{4 \pi \alpha} \sqrt{t_{j+1} - t_j} - 4 \pi \alpha (t_{j+1} - t_j) \right)$.

Using the above Lemma the proof of Lemma 5.2 is almost immediate.

Proof of Lemma 5.2. Note the series $\sum_{d=1}^{\infty} s_d^4 \mathcal{C}_d$ converges (absolutely) by the ratio test. By means of the same criterion one shows also that $\sum_{d=1}^{\infty} s_d^4 \mathcal{C}_d^d < +\infty$ under the assumption $\alpha \leq 4$. Keeping these facts in mind we proceed to the proof.

(a) As $S(x_{ni}, s_n)$ forms a cover of $\mathcal{I}$ it is easy to show that $E_W[\mu_n(\mathcal{I})] = 1$. In particular,

$$E_W[\mu_n(\mathcal{I})] = \sum_{i=1}^{\infty} \lambda(\mathcal{I}) \lambda(\mathcal{I}) = 1.$$

(b) Using Lemma 5.3 we have,

$$E_W[\mu_n(\mathcal{I})^2] = \sum_{i,j=1}^{\infty} \lambda(\mathcal{I}) \lambda(\mathcal{I}) \lambda(\mathcal{I}) \lambda(\mathcal{I}) < \infty.$$

Above we have used the fact that the number of hypercubes with center at $x_{ni}$ and radius $s_i$ is proportional to $s_i^d / s_n^d$.

(c) For the expected energy we follow the same procedure as above. Note that $|x_{ni} - x_{nj}| > s_{l+1}$ then if we take $x \in S(x_{ni}, s_n)$ and $y \in S(x_{nj}, s_n)$ then $|x - y| > s_{l+1}$. Then

$$E_W[\mu_\alpha(\mathcal{I})] = \sum_{i,j=1}^{\infty} \lambda(\mathcal{I}) \lambda(\mathcal{I}) \lambda(\mathcal{I}) \lambda(\mathcal{I}) < \infty.$$
(d) By Paley-Zygmund inequality and the fact that \( \sup_{n \geq 2} \mathbb{E}_W [\mu_n(J)^2] < \infty \), there exists \( \nu > 0 \)
\[
\mathcal{W} (\mu_n(J) \geq b) \geq (1-b)^2 \frac{1}{\mathbb{E}_W [\mu_n(J)]} \geq \frac{(1-b)^2}{\sup_{n \geq 2} \mathbb{E}_W [\mu_n(J)^2]} \geq \nu > 0.
\]
Also using Markov’s inequality we have that
\[
\mathcal{W} \left( \mu_n(J) \geq b^{-1} \right) \leq b \mathbb{E}_W [\mu_n(J)] = b.
\]
Hence choosing \( b > 0 \) and \( \nu > 0 \) appropriately we have,
\[
\mathcal{W} \left( b \leq \mu_n(J) \leq b^{-1} \right) = \mathcal{W} (\mu_n(J) \geq b) - \mathcal{W} \left( \mu_n(J) \geq b^{-1} \right) \geq 2\nu > 0.
\]
Also note that since \( \mathbb{E}_W [I_a(\mu_n)] \) is uniformly bounded in \( n \), using Markov’s inequality we have
\[
\mathcal{W} (I_a(\mu_n) > a) \leq \nu.
\]
Hence (d) follows from the above observations and the fact that,
\[
\mathcal{W} \left( b \leq \mu_n(J) \leq b^{-1}, I_a(\mu_n) \leq a \right) \geq \mathcal{W} \left( b \leq \mu_n(J) \leq b^{-1} \right) - \mathcal{W} (I_a(\mu_n) > a) \geq 2\nu - \nu = \nu > 0.
\]
\[
\square
\]

**Proof of the lower bound.** Now using Lemma 5.2 we continue with the proof of lower bound. If we define
\[
G := \limsup_{n \to +\infty} \left\{ b \leq \mu_n(J) < b^{-1}, I_a(\mu_n) < a \right\},
\]
then by Lemma 5.2 (d), \( \mathcal{W} (G) \) is bounded away from zero. \( I_a \) being a lower semicontinuous function, the set of measures \( \mu \) for which \( b \leq \mu(J) < b^{-1} \) and \( I_a(\mu) < a \) is compact in the topology of weak convergence. Therefore the sequence \( (\mu_n)_{n \in \mathbb{N}} \) admits surely along a subsequence \( (\mu_n)_{k \in \mathbb{N}} \) a weak limit \( \mu \), which is a finite measure supported on \( P(a) \) and whose \( a \)-energy is finite. Hence, we have
\[
(5.5) \quad \mathcal{W} \left( C_0^4(P(a)) > 0 \right) > 0.
\]
Now by the monotonicity of the Hausdorff-\( a \)-measure, if we can show that
\[
\mathcal{W} \left( C_0^4(T(a,J)) > 0 \right) \in \{0,1\}
\]
then by (5.5), the set \( \left\{ C_0^4(T(a,J)) > 0 \right\} \) will have probability one and hence the proof will be complete.

Now from the construction of \( \mu^\varepsilon_a \), it holds from Equation 7.9 of Chen and Jakobson (2012) that
\[
\mathcal{T} (h_{\mu^\varepsilon_a}) = f_1(e)\mathcal{T}(h_{\varepsilon^2}) + f_2(e)\mathcal{T}(h_{\varepsilon^4}),
\]
where
\[
f_1(e) = \frac{\varepsilon I_1(e) - 2I_2(e)}{I_1(e) - I_0(e)} I_2(e), \quad f_2(e) = \frac{-eI_2(e)}{I_1(e) - I_0(e)} I_2(e).
\]
Since \( \lim_{e \to 0} f_1(e) = 2 \) and \( \lim_{e \to 0} f_2(e) = 0 \), \( \mu^\varepsilon_a \to 2\delta_x \) as \( e \to 0 \) in the sense of distributions. In fact, since \( d\sigma^\varepsilon_a(\xi) = -\frac{\varepsilon^2}{2} f_2(e) |\xi|^2 \exp(i(\xi, x)) x \to 0 \) for all \( \xi \), \( d\sigma^\varepsilon_a \to 0 \) in the sense of distributions. Thus
\[
\limsup_{\varepsilon \to 0} \frac{\mathcal{T}(h_{\mu^\varepsilon_a})}{\sqrt{2\pi G(e)}} = \limsup_{\varepsilon \to 0} \frac{f_1(e)\mathcal{T}(h_{\varepsilon^2})}{\sqrt{2\pi G(e)}}.
\]
By Stroock (2008) (Section 2), if \( \{h_m\}_{m \in \mathbb{N}} \) is an orthonormal basis of \( H \),

\[
[I(h_\varepsilon^T)](\theta) = \langle \theta, \sigma_\varepsilon^x \rangle \text{ w.p.a.s.} = \left\langle \sum_{m \geq 1} [I(h_m)](\theta) h_m, \sigma_\varepsilon^x \right\rangle.
\]

The series will depend then only on its tail, as \( \langle h_m, \sigma_\varepsilon^x \rangle \rightarrow h_m(x) \) and \( G(\varepsilon) \rightarrow +\infty \) as \( \varepsilon \rightarrow 0 \). Using the fact that \( [I(h_m)]_{m \geq 1} \) are i.i.d. we can apply Kolmogorov’s 0-1 law to conclude.

6. Appendix

Here we will collect some of the bounds on the Bessel functions. These bounds are easy to derive but for completeness we provide a short proof for them.

**Lemma 6.1.** (a) For some constant \( C > 0 \) and \( x > 0 \)

\[
|I_1^2(x) - I_0(x)I_2(x)| \geq Cx^2.
\]

(b) Let \( G(\cdot) \) be as in (3.1), then \( G(x) \leq -C \log x \) for all \( x \in [0,1] \), with \( C > 0 \) uniform in \( x \).

**Proof.** (a) Following Joshi and Bissu (1991) we have,

\[
I_1^2(x) - I_0(x)I_2(x) = \frac{I_1^2(x)}{x} \left( \frac{J_1(x)}{I_1(x)} \right)' = \frac{I_1^2(x)}{x} \sum_{n \geq 1} \frac{4xj_{1,n}}{(x^2 + j_{1,n}^2)^2},
\]

where we used the equality \( \left( \frac{J_1(x)}{I_1(x)} \right)' = \sum_{n \geq 1} \frac{4xj_{1,n}}{(x^2 + j_{1,n}^2)^2} \), \( j_{1,n} \) being the \( n \)-th zero of \( J_1(x)/x \) (Watson (1944)). Now using the identity \( I_1(x) = (x/C) \prod_{n \geq 1} \left( 1 + \frac{x^2}{j_{1,n}^2} \right) \) (Watson, 1944, Page 498) we derive

\[
I_1^2(x) - I_0(x)I_2(x) = \frac{I_1^2(x)}{x} \left( \frac{J_1(x)}{I_1(x)} \right)' = \frac{I_1^2(x)}{x} \frac{4xj_{1,1}}{(x^2 + j_{1,1}^2)^2} + \frac{I_1^2(x)}{x} \sum_{n \geq 2} \frac{4xj_{1,n}}{(x^2 + j_{1,n}^2)^2} > \frac{4I_1^2(x)j_{1,1}}{(x^2 + j_{1,1}^2)^2} > C'x^2.
\]

(b) By part (a) and the series expansion of Bessel functions (Abramowitz and Stegun (1965)) one can find a bound for \( G(\cdot) \) as follows (\( \gamma \) is the Euler-Mascheroni constant):

\[
G(x) \leq \frac{C}{x^2} (2h_1(x)K_1(x) + 2I_2(x)K_0(x) - 1)
\]

\[
= \frac{C}{x^2} \left( 2 \left( \frac{x}{2} + \frac{x^3}{16} + O(x^4) \right) \left( \frac{1}{x} + \frac{x}{4} (-1 + 2\gamma - 2 \log 2 + 2 \log x) + O(x^3 \log x) \right) + 2 \left( \frac{x^2}{8} + O(x^4) \right) (-(\gamma + \log 2 - \log x) + O(x^2 \log x)) - 1 \right)
\]

\[
= \frac{C}{x^2} \left( 1 + \frac{x^2}{8} + O(x^3) + \frac{-1 + 2\gamma - 2 \log 2 + 2 \log x}{4} + \frac{-1 + 2\gamma - 2 \log 2}{32} x^4 + \right.
\]

\[
\left. + \frac{1}{4} \left( \frac{-1 + 2\gamma - 2 \log 2}{4} x^2 + \frac{-1 + 2\gamma - 2 \log 2}{32} x^4 + \right) \right).
\]
\[ + O \left( x^2 \log x \right) + \frac{x^2}{4} C + O \left( x^4 \right) - \frac{x^2 \log x}{4} - 1 \right) = -C \log x + C'. \]

Here \( C, C' \) denote positive constants that may vary from line to line. \( \square \)

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