THE FIRST TWO BETTI NUMBERS OF THE MODULI SPACES OF VECTOR BUNDLES ON SURFACES

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0. Introduction

This paper is a continuation of our effort in understanding the geometry of the moduli space of stable vector bundles. For any polarized smooth projective surface $(X, H)$ and for any choice of $(I, d) \in \text{Pic}(X) \times H^4(X, \mathbb{Z})$, there is a coarse moduli space $\mathcal{M}(I, d)^0$ of rank two $\mu$-stable (with respect to $H$) locally free sheaves $\mathcal{E}$ of $\wedge^2 \mathcal{E} \cong I$ and $c_2(\mathcal{E}) = d$. This moduli space has been studied extensively recently. One important discovery is that the moduli space $\mathcal{M}(I, d)^0$ exhibits remarkable properties at stable range. To cite a few, for arbitrary surface the moduli space $\mathcal{M}(I, d)^0$ has the expected dimension, is smooth at general points and is irreducible, and for a large class of surfaces of general type $\mathcal{M}(I, d)^0$ are of general type, all true for $d$ sufficiently large [Fr, GL, Li2, Do, Zh]. In this paper, we will investigate another aspect of this moduli space. Namely, the Betti numbers of $\mathcal{M}(I, d)^0$. So far, there have been a lot of progress along this direction based on two different approaches: Algebro-geometric approach and gauge theoretic approach. The algebraic geometry approach is relatively new. In [ES,Ki,Yo], they studied in detail the Betti numbers of the moduli space of stable sheaves over $\mathbb{P}^2$ (for the rank two and higher rank cases). Beauville [Be] has a nice observation concerning some rational surfaces and Göttsche and Huybrechts [GH] have worked out the case for K3 surfaces. The gauge theory approach has been around for quite a while. To begin with, let $(M, g)$ be a compact oriented Riemannian four-manifold and let $P_d$ be a smooth $\text{SO}(3)$ (or $\text{SU}(2)$)-bundle over $M$ associated to a rank two vector bundle of $c_1 = I$ and $c_2 = d$. Consider the pair

\begin{equation}
\mathcal{N}(P_d) \subset B(P_d)^*,
\end{equation}

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where $\mathcal{B}(P_d)^*$ is the space of gauge equivalent classes of irreducible connections on $P_d$ and $\mathcal{N}(P_d)$ is the subspace of Anti-Self-Dual connections. By a celebrated theorem of Donaldson, when $M = X$ is an algebraic surface with a Kahler metric associated to the ample divisor $H$, $\mathfrak{M}(I,d)^0$ is canonically isomorphic to $\mathcal{N}(P_d)$. The advantage of looking at the pair (0.1) is that $H_*(\mathcal{B}(P_d)^*)$ is calculable, at least modulo torsions, in terms of the homotopy type of $X$ and so does $H_*(\mathfrak{M}(I,d)^0)$ if we know the induced homomorphism

\[
\ell(d)_i : H_i(\mathfrak{M}(I,d)^0; \mathbb{Z}) \rightarrow H_i(\mathcal{B}(P_d)^*; \mathbb{Z}).
\]

In [AJ], Atiyah and Jones conjectured that for $M = S^4$ and SU(2)-bundle $P_d$, there is a sequence of (explicit) integers $\{q_k\}$ such that for $d \geq q_k$, (0.2) is an isomorphism for $i \leq k$. Later, Taubes’ work [Ta] suggests that similar conjecture should hold for arbitrary 4-manifold with possibly different sequence $\{q_k\}$. This conjecture has been confirmed for $S^4$, $\mathbb{CP}^2$ and $K3$ surfaces, see [HB, HBM, ES, GH, Ki, Ti1, Ti2, Yo].

In this paper, we will study $H_*(\mathfrak{M}(I,d)^0)$ for arbitrary algebraic surface. Due to technical difficulties, we are unable to prove the generalized Atiyah-Jones conjecture for all Betti numbers. Instead, we will prove the following theorems that will determine the first two Betti numbers of the moduli space.

**Theorem 0.1.** For any smooth projective surface $(X,H)$ and any $I \in \text{Pic}(X)$, there is an integer $N$ depending on $(X,I,H)$ so that whenever $d \geq N$, then

\[
\ell(d)_i : H_i(\mathfrak{M}(I,d)^0; \mathbb{Q}) \rightarrow H_i(\mathcal{B}(P_d)^*; \mathbb{Q})
\]

is an isomorphism for $i \leq 2$, where $P_d$ is the SO(3) (or SU(2))-bundle associated to the rank two complex vector bundle $E$ with $\wedge^2 E \cong I$ and $c_2(E) = d$.

By calculating the first and second Betti numbers of $\mathcal{B}(P_d)^*$, we get

**Theorem 0.2.** With the notation as in theorem 0.1, then there is an integer $N$ depending on $(X,I,H)$ so that for $d \geq N$, $\dim H_1(\mathfrak{M}(I,d)^0)$ and $\dim H_2(\mathfrak{M}(I,d)^0)$ are $b_1$ and $b_2 + \frac{1}{2}b_1(b_1 - 1)$ respectively, where $b_i = \dim H_i(X)$.

In algebraic geometry, there is a moduli space $\mathfrak{M}(I,d)$ of $H$-stable rank two sheaves $\mathcal{E}$ with $\det \mathcal{E} \cong I$ and $c_2(\mathcal{E}) = d$. $\mathfrak{M}(I,d)$ is quasi-projective and contains $\mathfrak{M}(I,d)^0$ as a Zariski open subset. We calculate the first two Betti numbers of $\mathfrak{M}(I,d)$ as well.

**Theorem 0.3.** With the notation as in theorem 0.1, then there is an integer $N$ depending on $(X,I,H)$ so that for $d \geq N$, $\dim H_1(\mathfrak{M}(I,d))$ and $\dim H_2(\mathfrak{M}(I,d))$ are $b_1$ and $b_2 + \frac{1}{2}b_1(b_1 - 1) + 1$ respectively.
There is a general principle [Mu] that explains why the Betti numbers of $M(I, d)$ take the form in theorem 0.3. For simplicity, let us assume $M(I, d)$ is projective and admits a universal family, say $E$ over $X \times M(I, d)$. Then $E$ is expected to contain all information of $M(I, d)$. For instance, the cohomology ring $H^*(M(I, d))$ (with rational coefficient) should be generated by the Kunneth components of $c_i(E)$. Put it differently, each $c_i(E)$ defines homomorphisms

$$\mu_i^{[*]}: H_*(X) \rightarrow H^{2i-*}(M(I, d))$$

via slant product. Then the Mumford principle states that $H^*(M(I, d))$ is generated by the image of $\{\mu_i\}_{i \geq 2}$ and within certain degree, say $r(d)$ ($r(d) \rightarrow \infty$ when $d \rightarrow \infty$), their images obey no restraint other than the obvious commutativity law of the cohomology ring $H^*$. In particular, if we look at $H^1(M(I, d))$, then it should be generated (freely) by the images of $\mu_2^{[3]}: H_3(X) \rightarrow H^1(M(I, d))$ which has dimension $b_1$ by Poincare duality. For $H^2(M(I, d))$, it should be generated freely by the wedge product of $H^1(M(I, d))$, the image of $\mu_2^{[3]}: H_2(X) \rightarrow H^2(M(I, d))$ and the image of $\mu_3^{[3]}: H_4(X) \rightarrow H^2(M(I, d))$, which span a linear space of total dimension $b_2 + \frac{1}{2}b_1(b_1 - 1) + 1$.

One motivation of the current work is to determine the Picard group of the moduli space $M(I, d)^0$ and $M(I, d)$. As is known, Pic($M(I, d)$) is largely determined by $\dim H^1(M(I, d))$ and $\dim H^2(M(I, d))$. In [Li3], we have determine their Picard groups based on the information gained here.

Now we explain the strategy in establishing these Theorems. In the following, we let $i = 1$ or $2$. According to Taubes [Ta], for large $d$ there are canonical homomorphisms $\tau(d)_i$ and $\tilde{\tau}(d)_i$ making the following diagram commutative:

$$
\begin{array}{ccc}
H_i(M(I, d)^0) & \xrightarrow{\ell(d)_i} & H_i(B(P_d)^*) \\
\downarrow{\tau(d)_i} & & \downarrow{\tilde{\tau}(d)_i} \\
H_i(M(I, d+1)^0) & \xrightarrow{\ell(d+1)_i} & H_i(B(P_{d+1})^*)
\end{array}
$$

(0.3)

(Here and in the remainder of this paper, all homologies are with rational coefficients unless otherwise is stated.) Further, $\tilde{\tau}(d)_i$ is an isomorphism, $\ell(d)_i$ is surjective for sufficiently large $d$ and the composition of $\tau(\cdot)_i$’s

$$
\tau(d, d + k)_i: H_i(M(I, d)^0) \rightarrow H_i(M(I, d + k)^0)
$$

has the property that

$$
\ker{\ell(d)_i} \subset \ker{\tau(d, d + k)_i}
$$

(0.4)

$^1$Taubes constructed a diagram using the space of based connections. His diagram is identical to ours because SO(3) and SU(2) are rational 3-sphere.
for some \( k(d) \). Thus, if for large \( d \) the homomorphism \( \tau(d)_i \) is surjective, \( \tau(d)_i \) must be an isomorphism for sufficiently large \( d \). Therefore by (0.4), \( \ell(d)_i \) will be an isomorphism for sufficiently large \( d \) as well, thus establishing theorem (0.1).

The homomorphism

\[ (0.5) \quad \tau(d)_i : H_i(\mathcal{M}(I, d)^0) \rightarrow H_i(\mathcal{M}(I, d+1)^0), \quad i \leq 2 \]

can be defined easily in our context. We fix an \( x \in X \) and let \( S^x_\mathcal{M}(I, d+1) \subset \mathcal{M}(I, d+1) \) be those \( E \) such that \( E^{\vee\vee}/E \cong \mathbb{C}_x \). \( S^x_\mathcal{M}(I, d+1) \) is a \( \mathbb{P}^1 \)-bundle over \( \mathcal{M}(I, d) \) by sending \( E \) to \( E^{\vee\vee} \). Let \( V_0 \subset \mathcal{M}(I, d+1) \) and the bundle \( S^x_\mathcal{M}(I, d+1) \rightarrow \mathcal{M}(I, d)^0 \) induce a commutative diagram

\[ (0.6) \quad \begin{array}{c}
0 \rightarrow H_i(V_0) \rightarrow H_i(S^x_\mathcal{M}(I, d+1)) \rightarrow H_i(\mathcal{M}(I, d)^0) \rightarrow 0 \\
\| \quad \tau(d)_i \\
0 \rightarrow H_i(V_0) \rightarrow H_i(\mathcal{M}(I, d+1)) \leftarrow H_i(\mathcal{M}(I, d+1)^0).
\end{array} \]

When \( d \) is sufficiently large, \( H_i(\mathcal{M}(I, d+1)) \) is a direct sum of the images of \( H_i(V_0) \) and \( H_i(\mathcal{M}(I, d+1)^0) \). Therefore, (0.6) induces a homomorphism \( H_i(\mathcal{M}(I, d)^0) \rightarrow H_i(\mathcal{M}(I, d+1)^0) \) that is the mentioned homomorphism \( \tau(d)_i \). (See lemma 4.3 for details.)

From (0.6), \( \tau(d)_i \) is surjective if \( r(d)_i \) is surjective. In this paper, we will prove the surjectivity of \( r(d)_i \) by establishing the following two theorems:

**Theorem 0.4.** With the notation as in theorem 0.1, then there is an integer \( N \) so that whenever \( d \geq N \), then for \( i = 1, 2 \),

\[ H_i(\mathcal{M}(I, d), S\mathcal{M}(I, d)) = 0, \]

where \( S\mathcal{M}(I, d) = \mathcal{M}(I, d) - \mathcal{M}(I, d)^0 \).

**Theorem 0.5.** With the notation as in theorem 0.1, then there is an integer \( N \) so that whenever \( d \geq N \), then for \( i = 1, 2 \) and closed \( x \in X \), the image of

\[ H_i(S\mathcal{M}(I, d)) \rightarrow H_i(\mathcal{M}(I, d)) \]

is contained in the image of

\[ H_i(S^x_\mathcal{M}(I, d)) \rightarrow H_i(\mathcal{M}(I, d)). \]
The strategy to establish theorem 0.4 is to apply the Lefschetz hyperplane theorem to the moduli space. The classical Lefschetz hyperplane theorem states that for any smooth, projective variety $Z$ of complex dimension $n$ and any smooth very ample divisor $Z_1 \subset Z$, the pair $(Z, Z_1)$ has vanishing homology group up to degree $n - 1$. Concerning our situation, the ideal pair to look at is $(\mathcal{M}(I, d), SM(I, d))$. But Lefschetz hyperplane theorem does not apply directly to this pair because $SM(I, d)$ is definitely not ample. Instead, we will first find an ample subvariety $W$ of $\mathcal{M}(I, d)$ and apply the generalized Lefschetz hyperplane theorem to the pair $(\mathcal{M}(I, d)^0, W)$ to establish the surjectivity of

$$(0.7) \quad H_i(W, SM(I, d) \cap W) \longrightarrow H_i(\mathcal{M}(I, d), SM(I, d)), \quad i \leq 2.$$ 

The set $W$ has an explicit geometric description: Let $C \in |nH|$ be a fixed smooth divisor for some $n > 0$. Then $W \subset \mathcal{M}(I, d)$ consists of those $E$ such that $E|_C$ is not semistable. By work of [Li1], there is a morphism

$$(0.8) \quad \Psi_C : \mathcal{M}(I, d) \longrightarrow \mathbb{P}^M$$

and a codimension $3g(C) - 2$ linear subspace $V \subset \mathbb{P}^N$ such that $\Phi_C^{-1}(V) = W$. Hence we obtain the surjectivity of (0.7) by applying the stratified Morse theory developed in [GM] to $\Psi_C$ and $V \subset \mathbb{P}^N$.

The next step is to show that

$$(0.9) \quad H_i(W, SM(I, d) \cap W), \quad i \leq 2$$

is trivial. The tactic is to construct explicitly a homology between any class in (0.9) with the null class by exploiting the fact that restriction to $C$ of sheaves in $W$ are not semistable. Here is an outline: For any locally free sheaf $E \in W$, let $\mathcal{L}$ be the destabilizing quotient sheaf of $E|_C$ and let $\mathcal{F}$ be the elementary transformation of $E$ defined by the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow \mathcal{L} \longrightarrow 0.$$ 

Then $E$ can be reconstructed from $\mathcal{F}$ via

$$0 \longrightarrow E \longrightarrow \mathcal{F}(C) \stackrel{\alpha}{\longrightarrow} I \otimes \mathcal{L}^{-1} \longrightarrow 0.$$ 

If we vary $\alpha$ and $\mathcal{L}$, we get deformations of $E$ within $W$. In certain cases, we can deform $E$ to non-locally free sheaves this way. This method was used by O'Grady in showing that $H_0(W, SM(I, d) \cap W) = 0$ [OG1]. In this paper, we will work out this construction in the relative case to prove the vanishing of (0.9).
A large portion of the current work is devoted to study the singularities of various sets. This is necessary because generalized Lefschetz hyperplane theorem only apply to varieties with “mild” singularities. In principle, the current approach should work for all homology groups $H_i$ through a range that depends on the (local) topology of the singularities of $W$. For the moment, the author can only show that $W$ is locally irreducible but has no idea whether it enjoys any higher local connectivity. Nevertheless, the local irreducibility of $W$ is sufficient to show the vanishing of (0.9) and thus establishing theorem 0.4. Theorem 0.5 is proved by carefully studying the inclusion $\mathcal{SM}(I, d) \subset \mathcal{M}(I, d)$.

The layout of the paper is as follows: In §1, we will gather all relevant properties of the moduli space $\mathcal{M}(I, d)$ of which we will need. These include some discussion of singularities of algebraic sets. In §2, by studying deformation of sheaves over curves, we will show that the set $W \subset \mathcal{M}(I, d)$ is locally irreducible. §3 is a refinement of [La, OG] in which we will demonstrate how one can deform a family of locally free sheaves to non-locally free sheaves and thus deriving the vanishing of (0.9). The theorem 0.4 and 0.5 will be proved in §5. Most of the materials concerning singularity are drawn from the excellent book of Goresky and MacPherson [GM].

1. Preliminaries

In the first part of this section, we will gather results of $\mathcal{M}(I, d)$ that are important to our study. Some of them have already appeared or known to the experts and others are improvements of the earlier results. We will give the reference to each result and provide proof if necessary. In the second part, we will review some materials concerning topology of singular sets.

First, let us recall the convention that will be used throughout this paper. In this paper, all schemes considered are of finite type and are over complex number field. All points of schemes are closed points. We will use Zariski topology throughout the paper unless otherwise mentioned. Thus a closed subset is a union of finite closed subvarieties. We will use algebraic subset to mean finite union of locally closed subsets. By dimension of an algebraic set we mean complex dimension. We will only consider coherent sheaves in this paper and will not distinguish a vector bundle from the sheaf of its sections.

Throughout the rest of this paper, we fix a smooth algebraic surface $X$ and a line bundle $I \in \text{Pic}(X)$. Let $H$ be an ample divisor on $X$. We say a rank two sheaf $\mathcal{E}$ is $H$-stable (resp. $H$-semistable) if for any proper quotient sheaf $\mathcal{E} \to \mathcal{F}$, (i.e. it has non-trivial kernel) we have

$$\frac{1}{\text{rank } \mathcal{E}} \chi_\mathcal{E}(n) < \frac{1}{\text{rank } \mathcal{F}} \chi_\mathcal{F}(n) \quad \text{(resp. } \leq \text{)}$$
for sufficiently large \( n \), where \( \chi_E(n) = \chi(E \otimes H^\otimes n) \) is the value of the Hilbert polynomial of \( E \). Note that \( H \)-semistable sheaves are necessarily torsion free. Similarly, we say \( E \) is \( H \)-\( \mu \)-stable (resp. \( H \)-\( \mu \)-semistable) if for any rank one torsion free quotient sheaf \( E \to F \), we have \( \mu(E) < \mu(F) \) (resp. \( \leq \)), where \( \mu(E) = \frac{1}{\text{rank}E} c_1(E) \cdot H \). We define stable and \( \mu \)-stable sheaves on curves similarly. We also need the concept of \( e \)-stable.

For any constant \( e \), a rank two sheaf \( E \) is said to be \( e \)-stable if for any rank one torsion free quotient sheaf \( E \to F \), we have \( \mu(E) < \mu(F) + e \). One notices that for torsion free sheaves, \( H \)-\( \mu \)-stable implies \( H \)-stable and \( H \)-semistable implies \( H \)-\( \mu \)-semistable. This is not the case for sheaves with torsion. For instance, it is easy to construct a sheaf with torsion on curve that is \( \mu \)-stable but not stable. In case the choice of \( H \) is apparent from the context, we will simply call them stable or \( \mu \)-stable. We agree that by unstable we mean not semistable. According to [Gi], for any \( d \in \mathbb{Z} \), there is a moduli scheme \( \overline{\mathcal{M}}_H(I,d) \) of rank two \( H \)-semistable sheaves \( E \) with \( \det E = I \) and \( c_2(E) = d \) (modulo equivalence relation). \( \overline{\mathcal{M}}_H(I,d) \) is projective. In the following, we will freely refer a semistable sheaf \( E \) as a point of \( \overline{\mathcal{M}}_H(I,d) \).

There are several open subsets of \( \overline{\mathcal{M}}_H(I,d) \) that are relevant to our study. The first is the open subset \( \mathcal{M}_H(I,d) \subseteq \overline{\mathcal{M}}_H(I,d) \) of all \( H \)-stable sheaves (called the moduli of stable sheaves) and \( \mathcal{M}_H(I,d)^0 \subseteq \mathcal{M}_H(I,d) \) of all \( \mu \)-stable locally free sheaves. In most cases, \( \mathcal{M}_H(I,d)^0 \) is a Zariski dense open subset of \( \overline{\mathcal{M}}_H(I,d) \). We let \( S\mathcal{M}_H(I,d) = \overline{\mathcal{M}}_H(I,d) - \mathcal{M}_H(I,d)^0 \). \( S\mathcal{M}_H(I,d) \) contains non-locally free sheaves as well as locally free but non \( \mu \)-stable sheaves. For integer \( l \geq 1 \), we let \( S_l\mathcal{M}_H(I,d) \subseteq S\mathcal{M}_H(I,d) \) be the set of non-locally free sheaves \( E \) such that the length \( l(E^{\vee \vee}/E) = l \), where \( E^{\vee \vee} \) is the double dual of \( E \), and let \( S_l^0\mathcal{M}_H(I,d) \subseteq S_l\mathcal{M}_H(I,d) \) be the subset of those \( E \) such that \( E^{\vee \vee}/E \cong \bigoplus_{i=1}^l \mathcal{O}_{x_i} \) with \( x_i \) distinct. Note that \( S_l^0\mathcal{M}_H(I,d) \subseteq S_l\mathcal{M}_H(I,d) \) is open.

Usually, the algebraic subset \( S\mathcal{M}_H(I,d) \subseteq \overline{\mathcal{M}}_H(I,d) \) is not Cartier (Cartier means that set-theoretically it is locally definable by one equation) which makes the study of the topology difficult. However, there is one situation that it does. Namely, when \( H \) is \( (I,d) \)-generic.

**Definition 1.1.** 1. An ample divisor \( H \) is called \( (I,d) \)-generic if for any strictly \( H \)-semistable sheaf \( E \) with \( \det E = I \) and \( c_2(E) \leq d + 10 \), \( E \) is \( S \)-equivalent to a direct sum (of rank one sheaves) \( L_1 \oplus L_2 \) such that \( c_1(L_1) = c_1(L_2) \in H^2(X, \mathbb{R}) \).

2. Let \( H_0 \) be any ample divisor. An ample divisor \( H \) is called \((H_0, I,d)\)-suitable if not only \( H \) is \( (d,I) \)-generic but also has the property that any \( H \)-semistable sheaves \( E \) with \( \det E = I \) and \( c_2(E) \leq d \) are necessarily \( H \)-\( \mu \)-semistable.

Here we face a dilemma: In proving the main theorems, we need to work on \( \mathcal{M}_H(I,d) \) inductively on \( d \). Although for fixed \((I,d)\) there are plenty of \((I,d)\)-generic ample divisors, for fixed \( H \), it will cease to be \((I,d)\)-generic for large enough \( d \), assuming
dim $H^{1,1}(X) > 1$. Thus we need to adjust $H$ constantly as $d$ becoming large. To get by this, we will work with a set of polarizations simultaneously.

To this end, some discussion on the selection of polarizations is in order. First, because of [Li1,p458], any two ample divisors $H_1$ and $H_2$ will give rise to (canonically) isomorphic moduli spaces $\mathcal{M}_{H_1}(I, d)$ and $\mathcal{M}_{H_2}(I, d)$ if $c_1(H_1)$ and $c_1(H_2)$ lie on the same (real) line in $H^2(X; \mathbb{R})$. Now let

$$\text{NS}_\mathbb{R} = (H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Q})) \otimes_\mathbb{Q} \mathbb{R},$$

let $\text{NS}_\mathbb{Q}^+$ be the ample cone and let $\text{NS}_\mathbb{Q}$ and $\text{NS}_\mathbb{Q}^+$ be the intersection with $H^2(X, \mathbb{Q})$ of the corresponding spaces. For any $\xi \in \text{NS}_\mathbb{Q}^+$, we define the moduli space $\mathcal{M}_\mathbb{Q}(I, d)$ to be $\mathcal{M}_H(I, d)$ for some ample $H$ such that $c_1(H) = n\xi$ for some $n$. By abuse of notation, in the following we will use $H \in \text{NS}_\mathbb{Q}^+$ to mean $H$ a $\mathbb{Q}$-divisor with $c_1(H) \in \text{NS}_\mathbb{Q}^+$. Next, let $H_0$ be any ample divisor and let $C_\varepsilon \subset \text{NS}_\mathbb{Q}$ be an $\varepsilon$-ball in $\text{NS}_\mathbb{Q}$ centered at $H_0 \in \text{NS}_\mathbb{Q}^+$, after fixing an Euclidean metric on $\text{NS}_\mathbb{Q}$. For sufficiently small $\varepsilon > 0$, the closure $\text{cl}(C_\varepsilon)$ of $C_\varepsilon$ in $\text{NS}_\mathbb{R}$ is still contained in $\text{NS}_\mathbb{Q}^+$. We call such $C_\varepsilon$ precompact neighborhood of $H_0 \in \text{NS}_\mathbb{Q}^+$ and denoted by $C_\varepsilon \subset \text{NS}_\mathbb{Q}^+$.

**Lemma 1.2.** Let $H_0$ be an ample line bundle and let $C \subset \text{NS}_\mathbb{Q}^+$ be a precompact neighborhood of $H_0 \in \text{NS}_\mathbb{Q}^+$. Then for any choice of $(I, d)$, we can find an $(H_0, I, d)$-suitable $\mathbb{Q}$-ample divisor $H$ in $C$.

**Proof.** It follows from theorem 1 on page 398 of [Qi] and the Hodge index theorem. □

From now on, we fix an $H_0 \in \text{NS}_\mathbb{Q}^+$ and a precompact neighborhood $C \subset \text{NS}_\mathbb{Q}^+$ of $H_0 \in \text{NS}_\mathbb{Q}^+$. We will study moduli space $\mathcal{M}_H(I, d)$ with $H$ an arbitrary $\mathbb{Q}$-divisor in $C$ and derive estimate that depend on the set $C \subset \text{NS}_\mathbb{Q}^+$ rather than individual $H \in C$. We choose a smooth $C \in [2n_0H_0]$ for some $n_0$ such that

$$C \cdot C - |K_X \cdot C| - I \cdot C \geq 10.$$  

(1.1)

We will fix such a $C$ and denote by $g$ its genus in the remainder of this paper.

Let $H \in C$. Since usually the moduli space $\mathcal{M}_H(I, d)$ is singular, it is convenient to work with a smooth subset of it:

$$\mathcal{M}_d = \{ E \in \mathcal{M}_H(I, d) \mid E \text{ is } \mu \text{-stable and } \text{Ext}^2(E^\vee, E^\vee)^0 = 0 \}.$$  

(1.2)

(For any sheaf $E$, we let $\text{End}^0(E)$ be the sheaf of traceless endomorphisms of $E$ and let $\text{Ext}^1(E, E)^0$ be the trace-less part of $\text{Ext}^1(E, E)$. $\mathcal{M}_d$ is smooth. We let $\mathcal{M}_d^0$, $S\mathcal{M}_d$ and $S_I\mathcal{M}_d$ etc. be $\mathcal{M}_d \cap \mathcal{M}_H(I, d)^0$, $\mathcal{M}_d \cap S\mathcal{M}_H(I, d)$ and $\mathcal{M}_d \cap S_I\mathcal{M}_H(I, d)$ etc. respectively. For any constant $e < 0$, we let $\mathcal{M}_H(I, d)_e \subset \mathcal{M}_H(I, d)$ be the set of all $e$-stable sheaves. We summarize some properties of these sets in the following lemma.
Lemma 1.3. There is an $N$ depending on $(X, I, C)$ so that whenever $d \geq N$, then for any $H \in C$,

1. $\mathcal{M}_H(I, d)$ is normal, irreducible and has pure dimension $4d - I^2 - 3\chi(O_X)$;
2. $\mathcal{M}_H(I, d)$ is a local complete intersection scheme;
3. Both $\mathcal{M}_d$ and $S_1\mathcal{M}_d$ are smooth;
4. $\mathcal{M}_H(I, d)^0 \subset \mathcal{M}_H(I, d)$, $\mathcal{M}_H(I, d) \subset \mathcal{M}_H(I, d)$ and $\mathcal{M}_d \subset \mathcal{M}_H(I, d)$ are dense, $S_0^0\mathcal{M}_H(I, d)$ has dimension $\dim \mathcal{M}_H(I, d) - l$ and is dense in $S_l\mathcal{M}_H(I, d)$ for $l \leq 4$;
5. The codimension of the sets $\mathcal{M}_H(I, d) - \mathcal{M}_H(I, d)_e$ and $\mathcal{M}_H(I, d) - \mathcal{M}_d$ in $\mathcal{M}_H(I, d)$ are at least $10g$, where $e = -2C^2$.

Proof. (1) and (2) were proved in [GL, Li2]. (3) is obvious and (4) and (5) can be found in [Do, Fr, Li1, Qi, Zu]. □

We now introduce some subsets of $\mathcal{M}_H(I, d)$ associated with $C$. Let $2C \subset X$ be the obvious non-reduced subscheme supported on $C$. We define

(1.3) $\Lambda_1^C = \{ E \in \mathcal{M}_H(I, d) \mid \text{Ext}_X^1(E^\vee, (E^\vee(-2C))^0 \neq 0) \}$.

(1.4) $\Lambda_2^C = \{ E \in \mathcal{M}_H(I, d) \mid \text{Ext}_X^0(E^\vee, (E^\vee(-2C))^0) > g + 1 \}$.

(1.5) $\Lambda_3^C = \{ F \in \mathcal{M}_H(I, d) \mid \mathcal{E}_{|2C} \cong E_{|2C} \}$, where $E$ is a sheaf locally free along $C$.

It is clear that all these sets are locally closed in $\mathcal{M}_H(I, d)$. Following [GL lemma 6.6], $\Lambda_3^C$ is a subscheme of $\mathcal{M}_H(I, d)$. Let $F \in \Lambda_3^C$. The Zariski tangent space of $\Lambda_3^C$ at $F$ is isomorphic to the kernel of the restriction homomorphism $\text{Ext}_X^1(F, F)^0 \rightarrow \text{Ext}_X^1(F_{|2C}, F_{|2C})^0$.

Lemma 1.4. Let $E$ be any locally free sheaf on $2C$. Then $\Lambda_3^C$ is smooth and meets $S_1\mathcal{M}_H(I, d)$ transversally away from $\Lambda_1^C$.

Proof. This is a standard deformation calculation. We shall leave the details to the readers. □

Next we estimate the codimension of the sets $\Lambda_1^C$ and $\Lambda_2^C$.

Lemma 1.5. There is an integer $N$ depending on $(X, I, C)$ such that whenever $d \geq N$ and $H \in C$, then $\text{codim}(\Lambda_1^C, \mathcal{M}_H(I, d)) \geq 10g$ and $\text{codim}(\Lambda_2^C, \mathcal{M}_H(I, d)) \geq 3g$.

Proof. The proof given by [Do, Fr, Zu] can be adopted to cover the estimate $\text{codim}(\Lambda_1^C, \mathcal{M}_H(I, d)) \geq 10g$. Now we show that by choosing $N$ large, $\text{codim}(\Lambda_2^C, \mathcal{M}_H(I, d)) \geq 3g$. 

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First of all, for any $E \in \Lambda^2_C$, $\mathcal{E}_C$ is a direct sum of line bundles, say $L_1 \oplus L_2$ with $\deg L_1 > \deg L_2$ and $h^0(L_1 \otimes L_2^{-1}) \geq g + 3$ by Riemann-Roch. Because $L_1 \otimes L_2 \cong I_C$, the set $\{ \mathcal{E}_C \mid E \in \Lambda^2_C \}$ is isomorphic to an open subset of $\text{Pic}(C)$. Next, for stable $E \in \Lambda^2_C - \Lambda^0_C$, the tangent space of the set $\{ E' \in \Lambda^2_C \mid \mathcal{E}_C' \cong \mathcal{E}_C \}$ at $E$ is the kernel of $\text{Ext}_X^1(\mathcal{E}, \mathcal{E}^0) \to \text{Ext}_C^1(\mathcal{E}_C, \mathcal{E}_C)^0$ which has dimension no more than

$$\dim \text{Ext}_X^1(\mathcal{E}, \mathcal{E}^0) - (3g - 3 + h^0(\text{End}(\mathcal{E}_C))) \leq \dim \mathcal{M}_H(I, d) - 4g.$$

Finally, since $\text{codim}(\Lambda^2_C, \mathcal{M}_H(I, d)) \geq 3g$,

$$\text{codim}(\Lambda^2_C, \mathcal{M}_H(I, d)) \geq \min\{4g - \dim \text{Pic}(C), 3g\} = 3g.$$

This is exactly what we want. □

In studying $\mathcal{M}_H(I, d)$, we often need to use the local universal family. Let $w \in \mathcal{M}_H(I, d)$ be any closed point. A local universal family of $\mathcal{M}_H(I, d)$ at $w$ is an analytic (or étale) neighborhood $g: U \to \mathcal{M}_H(I, d)$ of $w$ and a flat family of sheaves $\mathcal{E}_U$ on $X \times U$ flat over $U$ such that for any $u \in U$, the restriction sheaf $\mathcal{E}_{U|X \times \{u\}}$ is represented by the $g(u) \in \mathcal{M}_H(I, d)$. By expressing $\mathcal{M}_H(I, d)$ as G.I.T. quotient of the Grothendieck’s quotient scheme and applying the étale slicing theorem, we have

**Lemma 1.6.** Any point $w \in \mathcal{M}_H(I, d)$ admits a local universal family.

Now we discuss how to construct the morphism (0.8). We fix an $N$ given by lemma 1.3. For any $d \geq N$, let $H \in \mathcal{C}$ be $(H_0, I, d)$-suitable. In [Li1], the author constructed a line bundle $L_C$ over $\mathcal{M}_H(I, d)$ and the associated morphism $\Psi_C: \mathcal{M}_H(I, d) \to \mathbb{P}^R$. We summarize the property of this morphism as follows:

**Lemma 1.7.** [Li1,§2] Let $N$ be given in lemma 1.3 and let $d \geq N$, $H \in \mathcal{C}$ be $(H_0, I, d)$-suitable. Then there is a line bundle $L_C$ over $\mathcal{M}_H(I, d)$ of which the following holds:

1. For some large $m > 0$, $H^0(\mathcal{M}_H(I, d), L_C \otimes m)$ is base point free and induces a morphism $\Psi_C: \mathcal{M}_H(I, d) \to \mathbb{P}^R$;
2. For any $E \in \mathcal{M}_H(I, d)$, $\Psi_C^{-1}(\mathcal{E})$ consists of all $F \in \mathcal{M}_H(I, d)$ such that $F^\vee \cong \mathcal{E}^\vee$ and $\ell((F^\vee/F) \otimes \mathcal{O}_x) \cong \ell((\mathcal{E}^\vee/\mathcal{E}) \otimes \mathcal{O}_x)$ for all $x \in X$;
3. There is a codimension $3g - 2$ linear subspace $V \subset \mathbb{P}^R$ such that $\Psi_C^{-1}(V)$ is exactly the set

$$\mathcal{W} = \{ E \in \mathcal{M}_H(I, d) \mid \mathcal{E}_C \text{ is not semistable} \}.$$
Proof. (1) and (2) follows directly from [Li1]. Now we prove (3). Let $\mathcal{M}(C)$ be the moduli space of semistable vector bundles over $C$ and let $\rho: \overline{\mathcal{M}}_{H}(I, d) \rightarrow \mathcal{M}(C)$ be the rational map sending $E$ to $E|_C$ when it is semistable. Then there is an ample line bundle $L_C$ on $\mathcal{M}(C)$ such that $\rho^*(L_C) \cong L_C$ over where $\rho$ is defined. Further, for any $s \in H^0(\mathcal{M}(C), L_C^{\otimes m})$, $\rho^*(s)$ extends on $\overline{\mathcal{M}}_{H}(I, d)$ with vanishing locus

$$\{E \in \overline{\mathcal{M}}_{H}(I, d) \mid E|_C \text{ is not semistable or } E|_C \in s^{-1}(0)\}.$$  

(See [Li] for details.) Thus, if we choose $3g-2$ sections of $L_C^{\otimes m}$ with no common vanishing locus, then the extensions of their pull backs will define a codimension $3g-2$ linear subspace $V \subset \mathbb{P}^2$ that has the desired property.  

Note that in the proof, we only used the fact that all sheaves in $\overline{\mathcal{M}}_{H}(I, d)$ are $H_0$-$\mu$-semistable. The extra requirement that $H$ be $(I, d)$-generic will be useful later because of the following lemma.

**Lemma 1.8.** Let $N$ be as before and let $d \geq N$, $H \in \mathcal{C}$ be $(I, d)$-generic. Then the subset $S\overline{\mathcal{M}}_{H}(I, d) \subset \overline{\mathcal{M}}_{H}(I, d)$ is Cartier.

Proof. The proof will appear in [Li3]. (See [DN] for a special case.)  

In the remainder of this section, we will look more closely the local geometry of various subsets of $\overline{\mathcal{M}}_{H}(I, d)$. This is necessary because later we are unable to apply Lefschetz hyperplane theorem directly to the space we want but rather a Zariski open subset of it.

We still fix $H_0 \in \mathcal{C} \equiv \text{NS}_Q^+$, the $N$ given by lemma 1.3 and 1.5. For any $d \geq N$, we choose an $(H_0, I, d)$-suitable $H \in \mathcal{C}$. We let $S_1\mathcal{M}_{H}(I, d) \subset \mathcal{M}_{H}(I, d)$ and $\mathcal{W} \subset \mathcal{M}_{H}(I, d)$ be subsets defined preceding definition 1.1 and in lemma 1.7 respectively. Note that $S\overline{\mathcal{M}}_{H}(I, d) \subset \overline{\mathcal{M}}_{H}(I, d)$ is Cartier, $\overline{\mathcal{M}}_{H}(I, d)$ normal, $\mathcal{M}_{H}(I, d)$ locally is a complete intersection and $\mathcal{W} \subset \mathcal{M}_{H}(I, d)$ is defined by $3g-2$ equations. We fix an embedding $\overline{\mathcal{M}}_{H}(I, d) \subset \mathbb{P}^R$ and an analytic Riemannian metric on $\mathbb{P}^R$. For any closed subset $A \subset \overline{\mathcal{M}}_{H}(I, d)$ and $\delta > 0$, we define the $\delta$-neighborhood of $A \subset \overline{\mathcal{M}}_{H}(I, d)$, denoted by $A^\delta$ or $B_\delta(A)$, to be the set $\{z \in \overline{\mathcal{M}}_{H}(I, d) \mid \text{dist}(z, A) < \delta\}$.

We first look at the subset $S_1\mathcal{M}_{d} \subset \mathcal{M}_{d}$ (cf. (1.2)). $S_1\mathcal{M}_{d}$ is a $\mathbb{P}^1$-bundle over $X \times \mathcal{M}_{0,d-1}$ via the projection

$$\mu_{XM}: \mathcal{E} \in S_1\mathcal{M}_{d} \mapsto (\text{supp}(\mathcal{E}^{\vee\vee}/\mathcal{E}), \mathcal{E}^{\vee\vee}).$$

Let $V_0$ be a general fiber of $\mu_{XM}$ and let $\mathcal{N}$ be a normal slice of $S_1\mathcal{M}_{d} \subset \mathcal{M}_{d}$ along $V_0$.

(i.e. $\mathcal{N}$ is an analytic neighborhood of $V_0 \subset \mathcal{N}$, where $\mathcal{N} \subset \mathcal{M}_{d}$ is a smooth analytic subvariety containing $V_0$ that meets $\mathcal{S}\mathcal{M}_{d}$ transversally along $V_0$. For more details, see [GM].)
Lemma 1.9. For $0 < \delta \ll 1$, $\mathcal{N} \cap B_\delta(V_0)$ is a smooth analytic surface that contains $V_0$ as a (-2)-curve. In particular, $\mathcal{N} \cap B_\delta(V_0) - V_0$ is homeomorphic to $(\mathbb{R}^4 - \{0\})/\mathbb{Z}_2$.

Proof. Since $V_0$ is a general fiber of $\mu_{XM}$, $\mathcal{M}_H(I, d)$ is smooth along $V_0$ and for $0 < \delta \ll 1$, $\mathcal{N} \cap B_\delta(V_0)$ is a smooth surface. The fact that $V_0 \subset \mathcal{N}$ is a (-2)-curve can be checked directly by using the fact that sheaves in $V_0$ are kernels of $E_0 \to C_x$, where $V_0$ lies over $(x, E_0) \in X \times \mathcal{M}_H(I, d - 1)$ and the tangent bundle of $\mathcal{M}_H(I, d)$ at $z \in V_0$ is $\operatorname{Ext}^1(E_z, E_z)$. We shall omit the details here. □

One type of technical results that we need in the future says that for closed subset $\Lambda \subset \mathcal{M}_H(I, d)$ of large codimension,

$$H_i(\mathcal{M}_H(I, d) - \Lambda, \mathcal{S}\mathcal{M}_H(I, d) - \Lambda) \longrightarrow H_i(\mathcal{M}_H(I, d), \mathcal{S}\mathcal{M}_H(I, d))$$

is an isomorphism. This type of results are certainly known to the experts. Due to the lack of reference, we now provide proofs of them.

Lemma 1.10. Let $Z \subset \mathbb{P}^{n+r}$ be any irreducible quasi-projective variety of pure dimension $n$ and $\Lambda, V \subset Z$ two closed algebraic subsets. Assume $V$ is Cartier, $\Lambda$ has codimension at least $k$ and $Z - V$ is locally defined by at most $r + l$ equations, then

$$H_i(Z - \Lambda, V - \Lambda) \longrightarrow H_i(Z, V)$$

is an isomorphism for $i < (k - l) - 1$ and is surjective for $i = (k - l) - 1$.

Proof. Let $\mathcal{S}$ be a Whitney stratification of $Z$ by algebraic subsets so that $V$ and $\Lambda$ are union of strata. Let $S_1, \cdots, S_k$ be strata of $\Lambda$ with non-decreasing dimensions and let $\Lambda_i = \cup_{j > i} S_j$. Then the lemma follows if the homomorphism

$$H_i(Z - \Lambda_j, V - \Lambda_j) \longrightarrow H_i(Z - \Lambda_{j-1}, V - \Lambda_{j-1})$$

has the stated property for all $j$. Thus we only need to prove the lemma under the assumption that $\Lambda$ is already a stratum in $\mathcal{S}$. Since $\Lambda \subset Z$ is a stratum and $\Lambda \subset Z$ is closed, there is a compact $\Lambda_0 \subset \Lambda$ such that $((Z - \Lambda) \cup \Lambda_0, (V - \Lambda) \cup \Lambda_0)$ has the same homology group as $(Z, V)$. Hence it suffices to show that

$$H_i(Z - \Lambda, V - \Lambda) \longrightarrow H_i((Z - \Lambda) \cup \Lambda_0, (V - \Lambda) \cup \Lambda_0)$$

has the desired property. Let $p_0 \in \Lambda_0$ and let $N_{p_0}$ be the normal slice of $\Lambda \subset Z$ at $p_0$. We claim that the pair

$$(\partial B_\varepsilon(p_0) \cap N_{p_0}, \partial B_\varepsilon(p_0) \cap N_{p_0} \cap V)$$
Lemma 1.12. Let \( i \) be a real dimension at most \( \#(\Sigma' \cap \Lambda) \) for 0 \( \ll 1 \). Indeed, the case when \( p_0 \in V \) follows directly from theorem 2 and the remark preceding it on page 156 of [GM]. When \( p_0 \notin V \), we let \( (p_0, \mathcal{N}_{p_0}) \subset (0, \mathbb{C}^R) \) be an embedding and let \( h_1, \ldots, h_t \), \( t = R - \dim \mathcal{N}_{p_0} + r \), be the defining equations of \( \mathcal{N}_{p_0} \). Then the claim follows from applying the same theorem to map \( \mathcal{N} : \mathbb{C}^R \to \mathbb{C}^{R+t}, \mathcal{N}(z) = (z, h(z)) \), and the linear subspace \( \mathbb{C}^R \times \{0\} \subset \mathbb{C}^{R+t} \).

Next by Thom’s first isotopy lemma [GM], we can find a set \( U \subset (Z - \Lambda) \cup \Lambda_0 \) containing \( \Lambda_0 \) with a projection \( U \to \Lambda_0 \) restricting to identity on \( \Lambda_0 \) such that \( U \to \Lambda_0 \) is an \( \mathcal{N}_{p_0} \) bundle over \( \Lambda_0 \). (i.e. \( U \) is a union of normal slices of \( \Lambda \subset Z \) at \( p \in \Lambda_0 \).) Because (1.7) is \( (k - l - 2) \)-connected, \((U - \Lambda_0, (U - \Lambda_0) \cap V) \) is \( (k - l - 2) \)-connected as well. Finally, we apply the Mayer-Vietoris sequence to pairs \((Z - \Lambda, V - \Lambda) \) and \((U, U \cap V) \). Because \((U, U \cap V) \) has trivial homology groups and \((U - \Lambda_0, (U - \Lambda_0) \cap V) \) is \( (k - l - 2) \)-connected, (1.6) is an isomorphism for \( i < k - l - 1 \) and surjective for \( i = k - l - 1 \). \( \square \)

Let \( S \) be a Whitney stratification of \( Z \) as before and let \( S \in \mathcal{S} \) be a stratum. The following two lemmas concern the intersection with \( S \) of representatives \( \Sigma \subset Z \) of elements in \( H_i(Z) \). We still denote by \( N_p \), a normal slice of \( S \in Z \) at \( p \in S \).

**Lemma 1.11.** Let \( S \in \mathcal{S} \) be any stratum that is closed in \( Z \). Then if \( \partial B_\varepsilon(p) \cap N_p \), \( 0 < \varepsilon \ll 1 \), is \( l \)-connected \((-1 \)-connected if it is non-empty and \(-\infty \)-connected if it is empty), then any \( v \in H_i(Z) \) can be represented by a cycle \( \Sigma \subset Z \) such that \( \Sigma \cap S \) has real dimension at most \( i - l - 2 \).

**Lemma 1.12.** Let \( S \in \mathcal{S} \) be any stratum and let \( q \in \overline{S} \subset Z \) be any point. Then if \( \Sigma \subset Z \) is a closed cycle contained in \( \cup_{i \in A} S_i \), where \( A \) is a subset of \( \mathcal{S} \), and \( \Sigma \cap S \) is discrete, then we can find a new representative \( \Sigma' \subset Z \) of \( [\Sigma] \in H_i(Z) \) such that \( \Sigma' \subset (\cup_{j \in A} S_j - S) \cup \{q\} \).

**Proof.** The proof of lemma 1.11 is obvious and we shall omit it. We now prove lemma 1.12. Let \( U \) be a cone neighborhood of \( q \in Z \) that respects the stratification \( \mathcal{S} \). Let \( p \in \Sigma \cap S \). We choose a differentiable path \( \rho : [0, 1] \to S \) connecting \( p \) with \( p_1 \in U \) and let \( N_{[0,1]} \) be a continuous family of normal slices of \( S \subset Z \) along \( \rho([0,1]) \). By shrinking \( N_{[0,1]} \) if necessary, we can assume \( N_{[0,1]} \) is homeomorphic to \( N_p \times [0,1] \), say by \( \Psi : N_p \times [0,1] \to N_{[0,1]} \). Next, by perturb \( \Sigma \) near \( p \), we can assume \( \Sigma \cap B_\varepsilon(p) \subset N_p \) for \( 0 < \varepsilon \ll 1 \). We fix a sufficiently small \( \varepsilon > 0 \) so that \( \Psi((B_\varepsilon(p) \cap N_p) \times \{1\}) \subset U \). Now let \( A_1 = \Sigma - B_\varepsilon(p), A_2 = \Psi(\Gamma \times [0,1]), \) where \( \Gamma = \partial(\Sigma \cap B_\varepsilon(p)) \) and \( A_3 \) is the cone over \( \Psi(\Gamma \times \{1\}) \) in \( U \). Then \( \Sigma' = A_1 \cup A_2 \cup A_3 \) is a representative of \( [\Sigma] \) with \( \#(\Sigma' \cap S) < \#(S) \). By performing the above perturbation at each \( p \in \Sigma \cap S \), we will get a desired representative of \( [\Sigma] \in H_*(Z) \). \( \square \)
2. Unstable sheaves over curves

In this section, we will show that for any \( \mu \)-unstable sheaves \( \mathcal{E} \) over \( C \), the germ of the space of \( \mu \)-unstable sheaves \( \mathcal{E} \) that are deformations of \( \mathcal{E} \) is irreducible. To make it precise, we first recall the notion of algebraic versal deformation space of \( \mathcal{E} \).

**Definition 2.1.** Let \( Z \) be a projective scheme and let \( \mathcal{E} \) be a rank two sheaf over \( Z \) with \( \det \mathcal{E} \cong M \). An algebraic versal deformation space of \( \mathcal{E} \) (of fixed determinant) is a tuple \((A, \mathcal{F}_A; 0, \mathcal{E})\), where \( 0 \in A \) is a quasi-projective scheme and \( \mathcal{F}_A \) is a flat algebraic family of sheaves on \( Z \times A \) with \( \det \mathcal{F}_A \cong p^*ZM \), such that

1. the restriction of \( \mathcal{F}_A \) to \( Z \times \{0\} \), say \( \mathcal{F}_0 \), is isomorphic to \( \mathcal{E} \), and further the Kodaira-Spencer map \( T_0A \to \text{Ext}^1(\mathcal{F}_0, \mathcal{F}_0)^0 \) induced by the family \( \mathcal{F}_A \) is an isomorphism;
2. For any marked variety \( s_0 \in S \) coupled with a flat family of sheaves \( \mathcal{E}_S \) on \( Z \times S \) with \( \det \mathcal{E}_S \cong p^*_ZM \) and \( \mathcal{E}_{s_0} \cong \mathcal{E} \), there is an analytic neighborhood \( U \) of \( s_0 \in S \) and an analytic map \( \eta: (U, s_0) \to (A, 0) \) so that the restriction of \( \mathcal{E}_S \) to \( Z \times U \) is isomorphic to \( (1_Z \times \eta)^*\mathcal{F}_A \), extending the given isomorphism \( \mathcal{F}_0 \cong \mathcal{E} \) and \( \mathcal{E}_{s_0} \cong \mathcal{E} \).

**Lemma 2.2.** Assume \( Z \) is a projective curve and \( H^0(Z, \mathcal{O}_Z) = \mathbb{C} \). Then for any rank 2 sheaf \( \mathcal{E} \) on \( Z \) of \( \det \mathcal{E} \cong M \), there is an algebraic versal deformation space (of fixed determinant) of \( \mathcal{E} \).

*Proof.* The existence of this space in our setting is known for long time (see [At]). Here we outline the proof since we need some properties of this deformation space that can not be found in reference.

By choosing a sufficiently ample line bundle \( H \) on \( Z \), we can express \( \mathcal{E} \) as a quotient sheaf of \( \mathcal{R} = \oplus^NH^{-1} \) with \( N = h^0(\mathcal{E} \otimes H) \). Let \( \mathcal{Q} \) be the Grothendieck’s Quot-scheme parameterizing all quotient sheaves \( \mathcal{F} \) of \( \mathcal{R} \) with \( \det \mathcal{F} \cong M \). We fix a point \( 0 \in \mathcal{Q} \) so that \( \mathcal{F}_0 \cong \mathcal{E} \) and the associated quotient sheaf \( \mathcal{R} \overset{\sigma}{\to} \mathcal{F}_0 \) induces isomorphism \( \mathbb{C}^N \cong H^0(\mathcal{F}_0 \otimes H) \). Because \( \dim Z = 1 \) and \( H \) is sufficiently ample, \( \mathcal{Q} \) is smooth at 0 \([Ma, p594]\) and the tangent space of \( \mathcal{Q} \) at 0 belongs to the exact sequence

\[
0 \to \text{Hom}(\mathcal{F}_0, \mathcal{F}_0) \to \text{Hom}(\mathcal{R}, \mathcal{F}_0) \overset{j}{\to} T_0\mathcal{Q} \to \text{Ext}^1(\mathcal{F}_0, \mathcal{F}_0)^0 \to 0.
\]

Now we let \( A \subset \mathcal{Q} \) be an affine smooth subvariety containing 0 so that the induced homomorphism

\[
T_0A \to T_0\mathcal{Q} \to \text{Ext}^1(\mathcal{F}_0, \mathcal{F}_0)^0
\]

is an isomorphism. Let \( \mathcal{F}_A \) be the restriction to \( Z \times A \) of the universal quotient family. Then the data \( (A, \mathcal{F}_A; 0, \mathcal{F}_0) \) satisfies 1) of the definition 2.1.
Now, we show that they also satisfy 2) of the definition 2.1. Let \( s_0 \in S \) and \( \mathcal{E}_S \) be given in (2). Because \( H \) is sufficiently ample, by shrinking \( S \) is necessary, there is a morphism \( g : S \to \Omega \) with \( g(s_0) = 0 \) so that \( \mathcal{E}_S \) is isomorphic to the pullback of the universal quotient family. It remains to find a neighborhood \( U \) of 0 in \( \Omega \) and an analytic map \( \pi_+ : U \to A \) so that \( \pi_+ \circ g \) is the desired map \( \eta \). We argue as follows: \( \Omega \) is a \( G \)-scheme with \( G = GL(N, \mathbb{C}) = \text{Hom}(\mathcal{R}, \mathcal{R}) \) (because \( H^0(Z, \mathcal{O}_Z) = \mathbb{C} \)). Let \( G_0 \) be the stabilizer of 0 in \( \Omega \) and let \( N \) be a normal slice of \( G_0 \subset G \) at 1 in \( G_0 \). Then the orbit \( N \cdot 0 \) is smooth at 0 and its tangent space at 0 is the image \( j(\text{Hom}(\mathcal{R}, \mathcal{F}_0)) \subset T_0 \Omega \). Thus \( N \cdot 0 \) and \( A \) meets transversally at 0. In particular, there is an analytic neighborhood \( V_\epsilon \) of 1 in \( N \) and an analytic neighborhood \( V_\epsilon \cap V_\epsilon \) of 0 in \( A \) so that \( V_\epsilon \cap V_\epsilon \to \Omega \), \((u_-, u_+) \mapsto u_- \cdot u_+\), is one-to-one. Let \( U = \text{image of } V_\epsilon \times V_\epsilon \) and let \( (2.2) \)

\[
\pi_\pm : U \to V_\pm
\]

be the induced projections. Then for any analytic variety \( B \) and any analytic map \( \xi : B \to V_\epsilon \times V_\epsilon \subset \Omega \), \( \xi(z) = f(z) \cdot (\pi_+ \circ \xi(z)) \), where \( f(z) = \pi_\cdot \circ \xi(z) \in N \). Hence the sheaf \( \mathcal{E}_B \) in the pullback quotient family \( h_B : p_2^* \mathcal{R} \to \mathcal{E}_B \) via \( 1 \times \xi \) and the sheaf \( \mathcal{E}_B' \) in the pullback quotient family \( h_B' : p_2^* \mathcal{R} \to \mathcal{E}_B' \) via \( 1 \times (\pi_+ \circ \xi) \) are isomorphic (analytically). Back to \( g : S \to \Omega \), we let \( U = g^{-1}(U) \) and \( \eta = \pi_+ \circ g : U \to A \). The previous reasoning shows that \((\eta, U)\) is the desired map. \( \square \)

**Remark 2.3.** For \( w \in V_\epsilon \subset A \), because \( T_w \Omega \to \text{Ext}^1(\mathcal{F}_w, \mathcal{F}_w)^0 \) is surjective and \( T_w(V_\cdot \cdot w) \to \text{Ext}^1(\mathcal{F}_w, \mathcal{F}_w)^0 \) is trivial, \( T_w A \to \text{Ext}^1(\mathcal{F}_w, \mathcal{F}_w)^0 \) must be surjective.

From now on, we fix a smooth connected curve \( C \) of genus \( \geq 4 \), a rank two \( \mu \)-unstable sheaf \( \mathcal{E} \) over \( C \) of even degree and a versal deformation space \( (A, \mathcal{F}_A; 0, \mathcal{E}) \) of \( \mathcal{E} \). By tensoring \( \mathcal{E} \) with \( M^{-1} \), where \( M = \text{det } \mathcal{E} \), we can assume without loss of generality that \( \text{det } \mathcal{E} \cong \mathcal{O}_C \). We first stratify \( A \) as follows: For any \( l > 0 \), we let \( S_l \) be the set of all rank two \( \mu \)-unstable sheaves with \( \text{det } \mathcal{E} \cong \mathcal{O}_C \) whose destabilizing quotient sheaves have degree \(-l \) and let \( S_l^0 \) be those in \( S_l \) that have \( H^0(\text{End}^0(\mathcal{E})) = 0 \). In light of our discussion in §1, we are interested in the set of points in \( A \) whose associated sheaves are \( \mu \)-unstable. Namely,

\[
(2.3) \quad A_0 = \{ w \in A | \mathcal{F}_w \text{ is } \mu\text{-unstable} \}.
\]

\( A_0 \) is a locally closed algebraic set. Since \( \mathcal{E} \) is \( \mu \)-unstable, \( 0 \in A_0 \). We need a technical lemma that follows from the proof of lemma 2.2. In the following, for any point \( w \) in an algebraic set \( W \), we will use germ(\( W, w \)) to denote the \( \varepsilon \)-ball \( B_\varepsilon(w) \) for \( 0 < \varepsilon \ll 1 \) of \( w \in W \), under some Riemannian metric.

**Lemma 2.4.** Let \( \mathcal{E}_S \), where \( s_0 \in S \) is a smooth curve, be a family of \( \mu \)-unstable sheaves satisfying \( \mathcal{E}_{s_0} \cong \mathcal{E} \). Then if \( s_0 \in U \subset S \) and \( \eta_1 \) and \( \eta_2 \) are two analytic maps
from \((U, s_0)\) to \((A, 0)\) given by (2) of the definition 2.1 based on the family \(E_S\), then the images \(\eta_1(\text{germ}(U, s_0))\) and \(\eta_2(\text{germ}(S, s_0))\) are contained in the same irreducible component of \(\text{germ}(A_0, 0)\).

Proof. We continue to use the notation developed in lemma 2.2. Let \(E_U = E_S|_{C \times U}\). Since \(A \subset \Omega\), the map \(\eta_i: U \rightarrow A\) is given by quotient sheaf homomorphism \(f_i: p_U^*(R) \rightarrow E_U\). Since \(f_i\) is uniquely determined by the induced isomorphism \(\tilde{f}_i: \oplus^N \mathcal{O}_{Z \times U} \rightarrow p_U^*(E_U) \otimes p_U^*H\) (at least near \(0 \in \Omega\)), \(\tilde{g} = f_2^{-1} \circ f_1 \in \text{Hom}(\mathcal{O}_{Z \times U}, \mathcal{O}_{Z \times U})\) induces a homomorphism \(g: p_Z^*R \rightarrow p_Z^*R\) that makes \(f_1 = f_2 \circ g\). Since \(\eta_1(s_0) = \eta_2(s_0)\) as quotient sheaves, \(\tilde{g}(s_0) = c \cdot \text{id}\). Next, let \(T\) be a connected analytic neighborhood of \(1 \in GL(N)\) and let \(\Psi: T \times U \rightarrow \Omega\) be the map defined by \(\Psi(h, s) = h \cdot f_2(s)\). By shrinking \(T\) and \(U\) if necessary, we can assume the composition of \(\Psi\) with the local projection \(\pi_+\) of (2.2) is well-defined and such that

\[
\eta_i(\text{germ}(U, s_0)) \subset \pi_+ \circ \Psi(\text{germ}(T \times U, (1, s_0))) \subset \text{germ}(A_0, 0) \quad \text{for } i = 1, 2.
\]

Because \(T\) is irreducible, \(\eta_1(\text{germ}(U, s_0))\) and \(\eta_2(\text{germ}(U, s_0))\) must be contained in the same irreducible component of \(\text{germ}(A_0, 0)\). This completes the proof of the lemma.

\[\square\]

In the remainder of this section, we shall prove two technical lemmas concerning \(A_0\).

**Proposition 2.5.** Assume \(\bar{E}\) is locally free, then the subvariety \(A_0 \subset A\) has pure complex codimension \(g + 1\) at \(0 \in A_0\).

When \(E\) is a sheaf with torsion, we denote by \(E^t\) the torsion subsheaf of \(E\).

**Proposition 2.6.** Assume \(\ell(\bar{E}^t) \leq 1\), then the subvariety \(A_0\) is locally irreducible at \(0 \in A_0\).

Here we say a variety \(Z\) is locally irreducible at \(z \in Z\) if for \(0 < \varepsilon \ll 1\), the smooth locus of the \(\varepsilon\)-neighborhood \(B_\varepsilon(z) \cap Z\) is connected. It is easy to see that if \(w \in A_0\) corresponds to a locally free sheaf in \(S_0\), then \(w\) is a smooth point of \(A_0\). Also, an easy tangent space calculation shows that \(\text{codim}(S_0^0 \cap A, A) = g + 1\). Therefore, Proposition 2.5 will be established if we can show that \(A_0 \cap S_0^0\) is dense in \(A_0\). To achieve this, we need to study the deformation of sheaves in \(S_l\).

We first study the locally free case. Let \(E \in S_l, l \geq 2\), be locally free and let \(L \subset E\) be the destabilizing subsheaf of \(E\). We choose an effective divisor \(D \subset C\) of degree \(l - 1\) and choose a homomorphism \(E \rightarrow \mathcal{O}_D\) so that the composition \(L \rightarrow E \rightarrow \mathcal{O}_D\) is
surjective. Let $F$ be the kernel of $E \to O_D$. Then $F$ belongs to the exact sequence

$$0 \to L(-D) \to F \to L^{-1} \to 0$$

and $E$ can be recovered as the kernel of $F(D) \xrightarrow{\sigma_0} O_D$. Next we fix a homomorphism $\eta : F(D) \to O_D$ so that $\text{ker}\{\eta \oplus \sigma_0 : F(D) \to O_D^\oplus\}$ is isomorphic to $F$. Then for any $s \in H^0(O_D)$, $\sigma_s = \sigma_0 + sn : F(D) \to O$ is always surjective and consequently, $E_s = \text{ker}\{\sigma_s\}$ is locally free. \{E_s \mid s \in H^0(O_D)\} provides us an algebraic family of locally free sheaves parameterized by the total space $S$ of $H^0(O_D)$. We denote this family by $E_S$. In case $s \in S$ corresponds to $0 \in H^0(O_D)$, then $E_s$ is isomorphic to $E$. Further, for any $s \in S$, $E_s$ is $\mu$-unstable because $L(-D) \subset E_s$ and $\text{deg } L(-D) = 1$. It is straightforward to check that for general $s \in S$, $L(-D)$ is the destabilizing subsheaf of $E_s$. Therefore, we have proved

\textbf{Lemma 2.7.} Let $D \subset C$ be any effective divisor of degree $l-1$ and let $E_S$ be the family of locally free sheaves constructed with $S$ the total space of $H^0(O_D)$. Then $S \cap S_1$ is dense in $S$.

Our next task is to analyze the set $S \cap S_1$. Let $F$ be given by

$$(2.4) \quad 0 \to L \to F \to L^{-1} \to 0$$

with $L$ locally free and has degree $> 0$. We claim that when (2.4) does not split, then $h^0(\text{End}^0(F)) = h^0(L^\otimes)$. Indeed, since $\text{deg } L > 0$, there is a surjective $\text{Hom}(F,F) \to \text{Hom}(L^{-1}, L^{-1}) = \mathbb{C}$. Let $\Lambda$ be its kernel. Since (2.4) is non-split, all elements in $\Lambda$ lift to $\text{Hom}(F,L)$, which lift to $\text{Hom}(L^{-1}, L)$ for the same reason. Therefore, $h^0(\text{End}^0(F)) = h^0(L^\otimes) + 1$. This proves the claim. Now let $s \in S \cap S_1$ be any point, where $S$ is as in lemma 2.7, and let $E_s$ be the locally free sheaf. Then $E_s$ is non-split and then $h^0(\text{End}^0(E_s)) = 0$ if and only if $h^0(L^\otimes(-2D)) = 0$. To this end, we need

\textbf{Lemma 2.8.} For any $L \in \text{Pic}(C)$ of degree $l \geq 2$, there is an effective divisor $D \subset C$ of degree $l-1$ such that $h^0(L^\otimes(-2D)) = 0$.

\textit{Proof.} Recall that for any subspace $V \subset H^0(L^\otimes)$, there is an $x \in C$ so that $\dim(V \cap H^0(L^\otimes(-2x))) \leq \dim V - 2$. Hence, the lemma follows if $h^0(L^\otimes) \leq 2 \deg L - 2$. But this follows from the Clifford’s theorem and Riemann-Roch theorem. \qed

Combining lemma 2.7, 2.8 and the fact that $E_s$ is non-split for general $s \in S$, we have proved
**Proposition 2.9.** Let $\mathcal{E} \in S_1$, $l \geq 2$ be any locally free sheaf (with $\det \mathcal{E} \cong \mathcal{O}_C$). Then for general effective divisor $D \subset C$ of degree $l - 1$ and for the family of locally free sheaves $\mathcal{E}_S$ parameterized by $S$ (the total space of $H^0(\mathcal{O}_D)$) constructed in lemma 2.7, we have $S \cap S_1^0$ is dense in $S$.

**Remark 2.10.** Let $\mathcal{E} \in S_1$, $l \geq 2$ be locally free, let $Z$ be the set of degree $l - 1$ effective divisors (which is the symmetric product $S^{l-1}C$), let $D_Z \subset C \times Z$ be the universal divisor and let $\bar{Z}$ be the total space of the underlining vector bundle of the locally free sheaf $\pi_{Z*}(\mathcal{O}_{D_Z})$. Then all $\mathcal{E}_z$’s constructed in lemma 2.7 are parameterized by $\bar{Z}$. Further, if we let $Y \subset \bar{Z}$ be the subset corresponding to the zero section of $\pi_{Z*}(\mathcal{O}_{D_Z})$, then sheaves associated to $w \in Y$ are isomorphic to $\mathcal{E}$. Now let $\bar{Z}_0 \subset \bar{Z}$ be the set of points whose associated sheaves belongs to $S_1^0$. Then lemma 2.9 says that $\bar{Z}_0$ is dense in $\bar{Z}$.

Now we study the deformation of locally free sheaves in $S_1$. Let $\mathcal{E} \in S_1$ and let $\mathcal{L}$ be the destabilizing subsheaf of $\mathcal{E}$. We choose an affine neighborhood $U$ of $\{\mathcal{L}\} \in \text{Pic}(C)$ and let $\mathcal{L}_U$ be the restriction of the Poincare line bundle to $C \times U$. We denote by $s_0 \in U$ the point $\mathcal{L}$. Now we construct a set that parameterize sheaves having $\mathcal{L}_u$, $u \in U$, as their destabilizing subsheaves. This can be done as follows: Consider the extension group $\text{Ext}^1_{C \times U}(\mathcal{L}_U^{-1}, \mathcal{L}_U)$. Since $U$ is affine, this group is an $\mathcal{O}_U$-module and because $\text{Ext}^2(\mathcal{L}_U^{-1}, \mathcal{L}_U) = 0$, by cohomology and base change theorem, the restriction homomorphism

$$\text{Ext}^1_{C \times U}(\mathcal{L}_U^{-1}, \mathcal{L}_U) \rightarrow \text{Ext}^1_{C}(\mathcal{L}_u^{-1}, \mathcal{L}_u)$$

is surjective for each $u \in U$. Let $Z$ be the total space of $\text{Ext}^1_{C \times U}(\mathcal{L}_U^{-1}, \mathcal{L}_U)$. There is an extension sheaf over $C \times Z$

$$0 \rightarrow \mathcal{E}_Z \rightarrow \mathcal{E}_{\bar{Z}} \rightarrow \mathcal{L}_U^{-1} \rightarrow 0$$

such that whose restriction to each $C \times \{z\}$, $z \in Z$ over $u \in U$, is the extension sheaf defined by $\mathcal{E}_{\bar{Z}_0} \cong \text{Ext}^1_{C}(\mathcal{L}_u^{-1}, \mathcal{L}_u)$ that is the image of $z$ under (2.5). Here $p_{C \times U}$ is the projection $C \times Z \rightarrow C \times U$. Because (2.5) is surjective, any extension sheaf of $\mathcal{L}_u^{-1}$ by $\mathcal{L}_u$ appears in this family. Therefore, $\mathcal{E}_Z$ is a deformation of $\mathcal{E}$ whose destabilizing subsheaves are members of $\mathcal{L}_U$. Of course, for general $z \in Z$, the sheaf $\mathcal{E}_z$ is non-split. Hence, $h^0(\mathcal{E}_z) = h^0(\mathcal{L}_U) = 0$ for general $z \in Z$ because $g(C) \geq 4$.

With the material prepared, now it is easy to prove the following proposition.

**Proposition 2.11.** Let $\tilde{E}$ be any $\mu$-unstable rank two locally free sheaf of $\det \mathcal{E} \cong \mathcal{O}_C$ and let $A$ be the versal deformation space of $\tilde{E}$ with $A_0 \subset A$ the locus of $\mu$-unstable sheaves. Then $A_0 \cap S_1^0$ is dense in $A_0$.
Proof. It suffices to show that any $\mu$-unstable locally free sheaf $\mathcal{E}$ admits a deformation whose general member belongs to $\mathcal{S}_l^0$. Such deformation was constructed in proposition 2.8 and (2.6). □

Proof of proposition 2.5. Since $A_0 \cap \mathcal{S}_l^0$ is dense in $A_0$, it suffices to show that for $w \in A_0 \cap \mathcal{S}_l^0$ close to 0, $\text{codim}(A_0 \cap \mathcal{S}_l^0$ at $w) = g + 1$. Let $\mathcal{F}_A$ be the family associated to the versal deformation space $A$ of $\mathcal{E}$, let $w \in A_0 \cap \mathcal{S}_l^0$ and let $(A', \mathcal{F}_{A'}; 0', \mathcal{F}_w)$ be the versal deformation space of $\mathcal{F}_w$. Then by lemma 2.2, there is an analytic neighborhood $U$ of $w \in A$ and an analytic map $f : (U, w) \to (A', 0')$ such that $(1_C \times f)^* \mathcal{F}_{A'}$ is (analytically) isomorphic to $\mathcal{F}_A$ restricting to $C \times U$. Note, $U \cap \mathcal{S}_l^0 = f^{-1}(A' \cap \mathcal{S}_l^0)$. Therefore proposition 2.5 follows from (a) $\text{codim}(A' \cap \mathcal{S}_l^0, A') = g + 1$ and (b) $f$ is a submersion at $w$. We first prove (b). Since both $A$ and $A'$ are smooth, it suffices to show that the differential $df : T_wA \to T_{0'}A'$ is surjective. But this follows from the remark 2.3 since $w \in A_0$ is close to 0. (a) is a straight forward dimension counting argument based on the exact sequence (2.4). This completes the proof of proposition 2.5. □

We now turn our attention to the proof of proposition 2.6 for locally free $\mathcal{E}$.

Proof of Proposition 2.6. We first consider the case where $\mathcal{E} \in \mathcal{S}_l$ is locally free and $l \geq 2$. Following the remark 2.10, there is a pair of irreducible varieties $Y \subset \tilde{Z}$ and a family of locally free sheaves $\mathcal{E}_Z$ on $C \times \tilde{Z}$ so that any deformation $\mathcal{E}_w$ of $\mathcal{E}$ constructed in lemma 2.7 is part of this family and for $w \in Y \subset \tilde{Z}$, $\mathcal{E}_w \cong \mathcal{E}$. By lemma 2.2, there is an analytic map $f_w : \text{germ}(\tilde{Z}, w) \to (A, 0)$ associated to $w \in Y$ that is induced by the family $\mathcal{E}_Z$. Since $\tilde{Z}$ is irreducible, $f_w$ defines a unique irreducible component of $\text{germ}(A_0, 0)$ and by lemma 2.4, this irreducible component is independent of the choice of $f_w$ and $w \in Y$. We denote this component by $B_0$. In the following, we will show that $B_0$ is the only irreducible component of $\text{germ}(A_0, 0)$.

Let $B$ be any irreducible component of $\text{germ}(A_0, 0)$. Since $B \cap \mathcal{S}_l^0$ is dense in $B$, there is a smooth analytic curve $s_0 \in S$, an analytic map $\varphi : (S, s_0) \to (B, 0)$ so that $\varphi(S - s_0) \subset B \cap \mathcal{S}_l^0$. Let $\mathcal{E}_S$ be the pull-back of $\mathcal{F}_A$. Because $\varphi(S - s_0) \subset B \cap \mathcal{S}_l^0$ and because destabilizing subsheaves of unstable (locally free) sheaf is unique, there is an invertible sheaf $\mathcal{L}_S$ on $C \times S$ so that for $s \in S - s_0$, $\mathcal{L}_S$ is the destabilizing subsheaf of $\mathcal{E}_S$. Hence there is a homomorphism $\mathcal{L}_S \to \mathcal{E}_S$ that induces an exact sequence

\begin{equation}
0 \to \mathcal{L}_S \to \mathcal{E}_S \xrightarrow{\beta} \mathcal{L}_S^{-1} \otimes \mathcal{I}_\Sigma \to 0,
\end{equation}

where $\mathcal{I}_\Sigma$ is the ideal sheaf of a zero-scheme $\Sigma \subset C \times S$ supported on $C \times \{s_0\}$. We claim that there is an effective divisor $D_S \subset C \times S$ containing $\Sigma$ flat over $S$ such that $D_S$ has degree $l - 1$ along fibers of $C \times S \to S$. Indeed, for any $z \in \text{supp}(\Sigma)$, since $\mathcal{E}_S$...
is locally free, \(I_{\Sigma,z}\) is generated by, say \(f_1^z, f_2^z \in \mathcal{O}_{C \times S, z}\). Without loss of generality, we can assume

\[(2.8) f_1^z|_{C \times \{s_0\}} \text{ generates the locally free part of } I_{\Sigma}|_{C \times \{s_0\} \text{ at } z}.\]

Then the union of \(\{f_1^z = 0\} \subset C \times S\) for \(z \in \text{supp}(\Sigma)\) form a divisor \(D_S\) flat over \(S\) near the fiber \(C \times \{s_0\}\). By shrinking \(S\) if necessary, we can assume \(D_S\) is flat over \(S\). We now check that it has degree \(l - 1\) along the fiber \(C \times \{s_0\}\). Because of (2.8), by restricting (2.7) to \(C \times \{s_0\}\), we get

\[
0 \rightarrow \mathcal{L}_{s_0} \rightarrow \mathcal{E}_{s_0} \xrightarrow{\beta_{s_0}} \mathcal{L}_{s_0}^{-1}(-D_{s_0}) \oplus \tau \rightarrow 0,
\]

where \(D_{s_0} = D_S \cap (C \times \{s_0\})\). From this we see \(\deg D_{s_0} = l - 1\) since \(\mathcal{E}_{s_0} = \mathcal{E}\) has destabilizing subsheaf of degree \(l\) and \(\deg \mathcal{L}_{s_0} = 1\).

Now by taking the preimage of \(\mathcal{L}_S^{-1}(-D_S) \subset \mathcal{L}_S^{-1} \otimes I_\Sigma\) under \(\beta\) in (2.7), we get a locally free sheaf \(\mathcal{F}_S\) that belongs to the exact sequence

\[(2.9) 0 \rightarrow \mathcal{L}_S \rightarrow \mathcal{F}_S \rightarrow \mathcal{L}_S^{-1}(-D_S) \rightarrow 0\]

and the exact sequence

\[
0 \rightarrow \mathcal{F}_S \rightarrow \mathcal{E}_S \rightarrow \mathcal{T} \rightarrow 0,
\]

where \(\mathcal{T} = \mathcal{L}_S^{-1} \otimes I_\Sigma/\mathcal{L}_S^{-1}(-D_S)\). Since \(I_\Sigma\) is generated by \(f_1^z\) and \(f_2^z\) and \(\mathcal{L}_S^{-1}(-D_S)\) is generated by \(f_1^z\) at \(z \in \text{supp}(\Sigma)\), \(\mathcal{T}\) is a rank one locally free \(\mathcal{O}_{D_S}\)-modules. Therefore, imitating the argument proceeding lemma 2.7, we can recover \(\mathcal{E}_S\) by

\[
0 \rightarrow \mathcal{E}_S \rightarrow \mathcal{F}_S(D_S) \xrightarrow{\sigma_S} \mathcal{O}_{D_S} \rightarrow 0.
\]

Because of (2.9), \(\mathcal{F}_s\) has degree 1 destabilizing quotient sheaf for any \(s \in S\).

Now we show that \(\varphi(\text{germ}(S, s_0)) \subset B_0\). Let \(0 \in T\) be a smooth analytic curve and let

\[
\tilde{\sigma}_{S \times T} : \mathcal{F}_S(D_S) \otimes_{\mathcal{O}_{C \times S}} \mathcal{O}_{C \times S \times T} \rightarrow \mathcal{O}_{D_S \times T}
\]

be a surjective homomorphism whose restriction to \(C \times S \times 0\) is \(\sigma_S\). Let \(\mathcal{E}_{S \times T}\) be the kernel of \(\tilde{\sigma}_{S \times T}\). By definition 2.1, this family induces an analytic map \(\eta : \text{germ}(S \times T, (s_0, 0)) \rightarrow (A, 0)\). Thanks to lemma 2.4, we know that \(\eta(\text{germ}(S, s_0) \times \{0\}) \subset B\) and because the restriction to \(C \times \{s_0\} \times T\) of \(\mathcal{E}_{S \times T}\) belongs to the family constructed in the Remarks 2.10, \(\eta(\{s_0\} \times \text{germ}(T, 0)) \subset B_0\). Further, by choosing \(\tilde{\sigma}_{S \times T}\) generic, we can assume for \(t \in T - 0\), \(\mathcal{E}_{(s_0, t)} \in \mathcal{S}_0^t\). Therefore, the general points of \(\eta(\text{germ}(\{s_0\} \times T, (s_0, 0))\) are smooth points of \(A_0\). On the other hand, general points of \(\eta(\text{germ}(S, s_0) \times \{0\})\) are smooth points of \(A_0\) as well. Hence \(B_0 = B\) because \(S \times T\) is irreducible. This completes the proof of Proposition 2.6 for \(l \geq 2\).
It remains to show that germ($A_0, 0$) is irreducible when $\bar{E} \in S_1$. Let $\mathcal{L}$ be the destabilizing subsheaf of $\bar{E}$ and let $\mathcal{E}_Z$ be the family of sheaves on $C \times Z$ constructed in (2.6). Because the set $Z_0 = \{z \in Z | \mathcal{E}_z \cong \bar{E}\}$ is connected and $Z$ is irreducible, the data $Z_0 \subset Z$ defines a unique irreducible component $B \subset \text{germ}(A_0, 0)$. On the other hand, because all deformations of $\bar{E}$ can be realized as a sub-family in $Z$, by lemma 2.4, germ($A_0, 0$) ⊂ $B$. Therefore, $A_0$ is locally irreducible at 0. □

In the remainder of this section, we will study the deformation of $\mu$-unstable non-locally free sheaves. First, we collect some information of non-locally free sheaves in $S_1$.

**Lemma 2.12.** Let $\mathcal{E} \in S_1$ be any non-locally free sheaf, then there is a deformation of $\mathcal{E}$ so that whose generic member are locally free in $S_1$. Further, if $l \geq 2$, then we can find deformation of $\mathcal{E}$ so that whose generic member are non-locally free in $S_1$.

**Proof.** Let $\mathcal{E}^l \subset \mathcal{E}$ be the torsion subsheaf and let $\mathcal{E}^l = \mathcal{E}/\mathcal{E}^l$. Then $\mathcal{E} = \mathcal{E}^l \oplus \mathcal{E}^l$. Let $S$ be the total space of $\text{Ext}^1(\mathcal{E}^l, \mathcal{E}^l)$. Then $S$ defines a family of sheaves that are extension of $\mathcal{E}^l$ by $\mathcal{E}^l$. The desired families can be chosen as subfamilies of $S$ easily. □

**Proposition 2.13.** Let $\mathcal{E} \in S_1$ be any sheaf with length one torsion subsheaf and let $(A, 0, F_A, \bar{E})$ be the versal deformation space given by lemma 2.2. Then $A_0$ is locally irreducible at 0, where $A_0$ is the locus of $\mu$-unstable sheaves.

**Proof.** The proof is similar to that of proposition 2.6. Let $B$ be an irreducible component of germ($A_0, 0$). Then by the previous lemma, the general sheaves in $B$ are locally free sheaves in $S_1$. Hence we can find a smooth curve $s_0 \in S$ and a morphism $f: (S, s_0) \to (A, 0)$ so that $f(S - s_0) \subset B \cap S_0$ and are locally free. Let $F_s$ on $C \times S$ be the pull back of $F_A$ and let $L_s$ be the family of (invertible) destabilizing quotient sheaves of $\mathcal{E}_s$. Then there is a zero scheme $Z \subset C \times S$ such that $\mathcal{E}_s$ is the extension of $L_s$ by $L_s^{-1} \otimes \mathcal{I}_{C \times S}$. Because $\ell((\mathcal{E}_{s_0})^l) = 1$, $Z \cap (C \times \{s_0\}) = \{z\}$ (reduced) for some $z \in C$. Therefore,

$$Z = \{t + h(t, s) = 0, \quad s^k = 0\}, \quad h(s, t) \in (s, t^2)\mathcal{O}_{C \times S, (z, s_0)}$$

for some $k \geq 1$, where $t$ and $s$ are analytic coordinate of $z \in C$ and $s_0 \in S$ respectively. Since $Z$ and $\{t = 0, s^k = 0\}$ are contained in the family

$$Z_u = \{t + u \cdot h(t, s) = 0, \quad s^k = 0\}, \quad u \in \mathbb{C},$$

any family of sheaves $\mathcal{E}_Z$ that restricts to $\mathcal{E}_{s_0}$ on $C \times \{s_0\}$ and is an extension of $L_s$ by $L_s^{-1} \otimes \mathcal{I}_{Z_u \subset C \times S}$, $u \in \mathbb{C}$, will be in $B$ thanks the lemma 2.4 (at least near $s_0$). Therefore,
depends only on the choice of $k$ and the family $\mathcal{L}_S$. But because $\text{Pic}(C)$ is smooth, $A$ does not depend on $\mathcal{L}_S$ either. Hence for families $\mathcal{E}_S$ whose corresponding zero schemes $Z$ have identical degree $k$ (considered as cycles) after restricting to fiber $C \times \{s_0\}$, they identify the same irreducible component of germ($A_0, 0$). We denote this component by $A[k]$. However, if we take a base change of $S$ branched at $s_0$ of ramification index $m$, the new family certainly is still contained in $A[k]$. Hence $A[k] = A[mk] = A[m]$. This proves that germ($A_0, 0$) is irreducible. □

3. Deformation of locally free sheaves

In this section, we will construct the homology mentioned in the introduction that will lead us to the proof of the vanishing of (0.9).

We fix $(H_0, I)$, a $C \in |2n_0H_0|$ as in (1.1), a precompact neighborhood $C \subseteq \text{NS}_Q^+$ of $H_0 \in \text{NS}_Q^+$ and the integer $N$ given in lemma 1.3 and 1.5. In this section, we will prove

**Proposition 3.1.** Let $d \geq N$ and let $H \in \mathcal{C}$ be $(H_0, I, d)$-suitable. Then the subset $W \subset \mathfrak{M}_H(I, d)$ defined in (3) of lemma 1.7 (i.e. $W$ consists of sheaves in $\mathfrak{M}_H(I, d)$ restricting to unstable sheaves on $C$) satisfies

\begin{equation}
H_i(W, W \cap \mathfrak{M}_H(I, d)) = 0, \quad i \leq 2.
\end{equation}

To prove the theorem, we need to show that every homology cycle $(\Sigma^i, \partial \Sigma^i) \rightarrow (W, W \cap \mathfrak{M}_H(I, d))$ is homologous to zero. We will construct such homology directly by constructing deformation of sheaves.

We first study deformation of sheaves following [La, OG1]. Let $W$ be any reduced (quasi-projective) scheme and let $\mathcal{E}_W$ be a family of rank two torsion free sheaves on $X \times W$ flat over $W$ so that $\wedge^2 \mathcal{E}_W \cong \mathcal{O}_{X \times W}$. We assume that $\mathcal{E}_W$ is locally free along $C \times W$ and for any $w \in W$, the restriction of $\mathcal{E}_w$ to $C$ is unstable and whose destabilizing quotient sheaf has degree $-1$. Because the destabilizing quotient sheaf is unique, there is an invertible quotient sheaf $\mathcal{E}_{w|C \times W} \rightarrow \mathcal{L}_w$ on $C \times W$ whose restriction to $C \times \{w\}$, $w \in W$, realizes $\mathcal{L}_w$ as the destabilizing quotient sheaf of $\mathcal{E}|C\times W$. Let $\iota:C \times W \rightarrow X \times W$ be the obvious immersion and let $\mathcal{E}_W \rightarrow \iota_* \mathcal{L}_w$ be the induced (surjective) homomorphism. We denote its kernel by $\mathcal{F}_W$. Then $\mathcal{F}_W, \mathcal{E}_W$ and $\mathcal{L}_w$ are related by the following exact sequence

\begin{equation}
0 \rightarrow \mathcal{E}_W(-C \times W) \rightarrow \mathcal{F}_W \xrightarrow{\sigma_w} \iota_* \mathcal{L}_w^{-1} \rightarrow 0.
\end{equation}

Of course, $\mathcal{E}_W$ is determined by the section

\begin{equation}
\sigma_W \in H^0(C \times W, \mathcal{H}om(\mathcal{F}_W, \mathcal{L}_w^{-1})).
\end{equation}
Next we will enlarge the family $E_W$ by varying the sheaf $L_W$ and the homomorphism $\sigma_W$. Let $J = \text{Pic}^0(C)$ and let $P$ be the normalized Poincare line bundle so that $P|_{p_0 \times J} \cong O_J$, where $p_0 \in C$ is fixed. (For $\lambda \in J$, we will use $P_\lambda$ to denote the bundle $P|_{C \times \{\lambda\}}$.) For the moment we assume for some large $n$, the sheaf $\mathcal{H}(F_W, L_w^{-1})$ belongs to the exact sequence

$$0 \to \mathcal{H}(F_W, L_w^{-1}) \to N_W \to T \to 0,$$

where $N_W = O_C^{\oplus 2}(np_0 \times W)$ and $T$ is a family of torsion sheaves on $C \times W$ flat over $W$. Let $p_{C \times W}$, $p_{C \times J}$ and $p_{W \times J}$ be projections from $C \times W \times J$ to $C \times W$, $C \times J$ and $W \times J$ respectively. We consider the following direct image sheaves on $W \times J$:

$$A_{W \times J} = p_{W \times J*}(p_{C \times W}^*N_W \otimes p_{C \times J}^*P);$$

$$B_{W \times J} = p_{W \times J*}(p_{C \times W}^*T \otimes p_{C \times J}^*P).$$

Both $A_{W \times J}$ and $B_{W \times J}$ are locally free. We let $A$ be the vector bundle associated to $A_{W \times J}$ and let $O_P(-1)$ be the tautological line bundle of the projective bundle $P(A)$ of $A$. We use the convention that $O_P(-1)$ is a subsheaf of $\pi^*A_{W \times J}$, where $\pi: P(A) \to W \times J$ is the projection. Composed with the induced homomorphism $\pi^*A_{W \times J} \to \pi^*B_{W \times J}$, we get

$$f : O_P(-1) \to \pi^*B_{W \times J}.$$

Since both $A$ and $B$ are locally free, for any closed $(w, \lambda) \in W \times J$ and

$$v \in \text{ the total space of } O_P(-1) \text{ over } (w, \lambda),$$

$v$ corresponds to a section (unique up to scalars) $\phi \in H^0(O_C^{\oplus 2}(np_0) \otimes P_\lambda)$. Further, $f(v) = 0$ if and only if the image of $\phi$ in $H^0(T|_{C \times \{w\}} \otimes P_\lambda)$ is trivial, and by the exactness of (3.4) it occurs exactly when $\phi$ can be lifted to a homomorphism $\tilde{\phi} \in H^0(F_w, L_w^{-1} \otimes P_\lambda)$. Hence, the set of all non-trivial homomorphisms $\text{Hom}_C(F_w, L_w^{-1} \otimes P_\lambda)$ modulo scalars is parameterized by the scheme $f^{-1}(0) \subset P(A)$. We let $\tilde{Z}$ be $f^{-1}(0)$ endowed with reduced scheme structure.

**Lemma 3.2.** Let $C_{W \times J} = p_{W \times J*}(p_{C \times W}^*(F_W^\vee \otimes L_W^\vee) \otimes p_{C \times J}^*P)$. Then there is a homomorphism

$$\alpha_Z : O_P(-1)|_{\tilde{Z}} \to \pi^*C_{W \times J}$$

so that for any closed $z \in \tilde{Z}$ over $(w, \lambda) \in W \times J$, the image

$$\text{Im}(\alpha_z) \subset \text{Hom}_C(F_w|_{C}, L_w^{-1} \otimes P_\lambda)$$
induced by the restriction homomorphism

\[ \pi^*C_{W \times J} \otimes k(z) \longrightarrow \text{Hom}_C(F_w|_C, L^{-1}_w \otimes P_\lambda) \]

is non-trivial. Further, there is a section \( \rho: W \to \tilde{Z} \) of the projection \( \pi_W: \tilde{Z} \to W \) so that for any closed \( w \in W \) and \( z = \rho(w) \), \( \text{Im}(\alpha_z) \) coincides with \( \sigma_w \otimes k(w) \) of (3.3) up to scalars.

**Proof.** \( \pi^*A_{W \times J}, \pi^*B_{W \times J} \) and \( \pi^*C_{W \times J} \) certainly belong to the exact sequence

\[ \pi^*C_{W \times J} \longrightarrow \pi^*A_{W \times J} \frac{\beta}{\gamma} \pi^*B_{W \times J}. \]

Because the composition of

\[ \mathcal{O}_P(-1)|\tilde{Z} \longrightarrow \pi^*(A_{W \times J}) \]

with \( \beta \) is trivial, it lifts to a unique homomorphism \( \alpha_{\tilde{Z}} \) in (3.6). The non-triviality of \( \text{Im}(\alpha_z) \) for \( z \in \tilde{Z} \) is obvious. It remains to prove that there is a section \( \rho: W \to \tilde{Z} \) with the desired property. Indeed, if we let \( \lambda_0 \in J \) be the trivial line bundle and let \( i: W \to W \times \{\lambda_0\} \subset W \times J \) be the immersion, then the section \( \sigma_W \) in (3.3) provides a section \( \sigma_W \in H^0(C \times W, N_W) \) that induces a section \( \rho: W \to \mathbb{P}(A) \) whose image is contained in \( \tilde{Z} \). \( \rho: W \to \tilde{Z} \) is the desired section. \( \square \)

In the following, we will use the section \( \alpha_{\tilde{Z}} \) to construct a family of torsion free sheaves on \( X \times \tilde{Z} \). Let \( \pi_{C \times W}, \pi_{C \times J} \) and \( \pi_{\tilde{Z}} \) be projections from \( C \times \tilde{Z} \) to \( C \times W \), \( C \times J \) and \( \tilde{Z} \) respectively. Then with \( p_{X \times W}: X \times \tilde{Z} \to X \times W \), the section \( \alpha_{\tilde{Z}} \) provides us a homomorphism (on \( X \times \tilde{Z} \))

\[ (3.7) \quad \alpha_{\tilde{Z}}: p_{X \times W}^*F_W \longrightarrow \iota_*(-\pi_{\tilde{Z}}^*(\mathcal{O}_{\mathbb{P}(1)}|\tilde{Z}) \otimes \pi_{C \times W}^*L^{-1}_W \otimes \pi_{C \times J}^*P). \]

\( \iota: C \times \tilde{Z} \to X \times \tilde{Z} \). We denote the right hand side of (3.7) by \( \iota_*\mathcal{L}_Z \), where \( \mathcal{L}_Z \) is an invertible sheaf on \( C \times \tilde{Z} \). For technical reason, we let \( Z \subset \tilde{Z} \) be the union of those irreducible components \( A \subset \tilde{Z} \) such that at general \( z \in A \), the restriction to \( X \times \{z\} \) of \( \alpha_{\tilde{Z}} \) is surjective. Note that \( \rho(W) \) is contained in \( Z \). We let \( \alpha_Z \) and \( \mathcal{L}_Z \) be restriction of \( \alpha_{\tilde{Z}} \) and \( \mathcal{L}_Z \) to \( X \times Z \) and \( C \times Z \) respectively. We let \( \Sigma \subset C \times Z \) be the subsheaf so that \( \text{Im}(\alpha_Z) = \iota_*\mathcal{L}_Z \otimes \mathcal{I}_\Sigma \) (\( \mathcal{I}_\Sigma \) is the ideal sheaf of \( \Sigma \subset C \times Z \)) and let \( \mathcal{E}_Z(-C \times Z) \) be the kernel of \( \alpha_Z \). Then we have the following exact sequence (on \( X \times Z \))

\[ (3.8) \quad 0 \longrightarrow \mathcal{E}_Z(-C \times Z) \longrightarrow p_{X \times W}^*F_W \longrightarrow \alpha_Z \iota_*\mathcal{L}_Z \otimes \mathcal{I}_\Sigma \longrightarrow 0. \]

We need a technical lemma which says that \( \mathcal{E}_Z \) is a family of torsion free sheaves flat over \( Z \). Once we know that it is flat over \( Z \), then \( \mathcal{E}_Z \) is a family that contains \( \mathcal{E}_Z \) as a subfamily.
Lemma 3.3. Let $\tilde{E}_Z$ be the sheaf in (3.8). Then $\tilde{E}_Z$ is a family of torsion free sheaves on $X \times Z$ flat over $Z$. Further, if we let $Z_0 = \{ z \in Z \mid \alpha_z \text{ is surjective} \}$, then for closed $z \in Z$, $\tilde{E}_z$ is locally free along $C$ (resp. $X - C$) if and only if $z \in Z_0$ (resp. $E_w$ is locally free, where $w$ lies over $w \in W$).

Proof. When $z \in Z_0$, then $\alpha_Z$ is surjective near $C \times \{ z \}$ and the kernel $E_Z$ is locally free and flat there. Now we assume $z \in Z - Z_0$. Because $p^*_{X \times W}F_W$ is locally free at $C \times \{ z \}$, $\Sigma$ is locally defined by at most two equations. On the other hand, $\Sigma$ contains no fibers of $C \times Z \to Z$ and $Z_0$ is dense in $Z$ by our selection of $Z \subset \tilde{Z}$. Therefore, $\text{codim}(\Sigma) \geq 2$. Thus $\Sigma$ is a local complete intersection scheme of codimension 2. Next, because (3.8) is exact and because $\iota_*L_Z \otimes \mathcal{I}_\Sigma$ is flat over $Z$, $\tilde{E}_Z$ is flat over $Z$. As to the torsion freeness, it suffices to show that

\[(3.9) \quad \text{Tor}(\mathcal{I}_\Sigma, \mathcal{O}_{C \times \{ z \}}) = 0.\]

Now we prove (3.9). Let $p \in \Sigma$ be any point over $z$. Since $\mathcal{I}_\Sigma,p$ is generated by two sections, say $f_1, f_2$, $\mathcal{I}_{\Sigma,p}$ belongs to the exact sequence

\[0 \to \mathcal{I}_{D_1 \subset C \times Z,p} \to \mathcal{I}_{\Sigma,p} \to \mathcal{I}_{D_2 \subset D_1,p} \to 0,\]

where $D_1$ is the divisor $\{ f_1 = 0 \}$ in $C \times Z$ and $D_2$ is the divisor $\{ f_2 = 0 \}$ in $D_1$, and $\mathcal{I}_{D_1 \subset C \times Z}$ and $\mathcal{I}_{D_2 \subset D_1}$ are ideal sheaves of $D_1 \subset C \times Z$ and $D_2 \subset D_1$ respectively. Without loss of generality, we can assume $D_1$ is a divisor flat over $Z$ (because $\text{supp}(\Sigma)$ contains no fibers of $C \times Z \to Z$). Therefore, $\mathcal{I}_{D_1 \subset C \times Z,p}$ and $\mathcal{I}_{D_2 \subset D_1,p}$ are locally free $\mathcal{O}_{C \times Z,p}$-modules and $\mathcal{O}_{D_1,p}$-module respectively, and hence have trivial $\text{Tor}(\cdot, \mathcal{O}_{C \times \{ z \}})$. Thus, (3.9) holds and $\tilde{E}_Z$ is a flat family of torsion free sheaves.

Finally, for any $z \in Z$, the restriction of (3.8) to $X \times \{ z \}$ is still exact:

\[(3.10) \quad 0 \to \mathcal{E}_z(-C) \to F_w \to (\iota_*L_z \otimes \mathcal{I_\Sigma})|_{X \times \{ z \}} \to 0.\]

In particular, when $p \in C$, $\mathcal{E}_z,p$ is locally free if and only if $(\mathcal{I}_\Sigma)|_{C \times \{ z \}}$ is a locally free $\mathcal{O}_{C \times \{ z \}}$-module. Therefore, $\mathcal{E}_z$ is locally free at $p \in C$ if and only if $(p, z) \notin \Sigma$ and locally free at $p \notin C$ if and only if $F_w$ is locally free at $p$. This completes the proof of the lemma. \qed

We have the following lemma which states that the previous construction is independent of the choices made.

Lemma 3.4. The scheme $\tilde{Z} = f^{-1}(0)$, $Z \subset \tilde{Z}$ and the family $\tilde{E}_Z$ are independent of the choice of the inclusion $\mathcal{H}om(F_W, L_W^{-1}) \to N_W$. 


Proof. We only need to check the following: Assume $\mathcal{F}_W^' \otimes \mathcal{L}_W^{-1} \to \mathcal{O}_{C \times W}^{\oplus 2}(m_{p_0} \times W)$ is another inclusion with cokernel $\mathcal{T}'$ making the diagram

$$
\begin{array}{c}
0 \longrightarrow \mathcal{Hom}(\mathcal{F}, \mathcal{L}_W^{-1}) \longrightarrow \mathcal{O}_{C \times W}^{\oplus 2}(m_{p_0} \times W) \longrightarrow \mathcal{T} \longrightarrow 0 \\
\downarrow \quad \downarrow \\
0 \longrightarrow \mathcal{Hom}(\mathcal{F}, \mathcal{L}_W^{-1}) \longrightarrow \mathcal{O}_{C \times W}^{\oplus 2}(m_{p_0} \times W) \longrightarrow \mathcal{T}' \longrightarrow 0
\end{array}
$$

commutative and assume the vertical arrows are injective, then if we let $Z \subset \tilde{Z} \subset \mathbf{P}(A)$ and $Z' \subset \tilde{Z}' \subset \mathbf{P}(A')$ be the corresponding subschemes, there is an isomorphism between $Z \subset \tilde{Z}$ and $Z' \subset \tilde{Z}'$ (canonically) and isomorphism between $\tilde{\mathcal{E}}_Z$ and $\tilde{\mathcal{E}}_{Z'}$ (non-canonically). This is obvious because $A$ is a subbundle of $A'$ and under the inclusion $\mathbf{P}(A) \subset \mathbf{P}(A')$, $\tilde{Z}' \subset \mathbf{P}(A)$. We shall omit the details here. □

Remark. Indeed, more is true: Let $W_i, i = 1, 2$, be two varieties with family $\mathcal{E}_i$ on $X \times W_i$ and let $Z_i$ be the corresponding schemes constructed. Assume there is a morphism $\varphi: W_1 \to W_2$ that induces an isomorphism $((1_X \times \varphi)^* \mathcal{E}_2)_{|_{2C \times W_1}} \cong \mathcal{E}_1_{|_{2C \times W_1}}$, then there is a canonical isomorphism $Z_1 \cong W_1 \times_f Z_2$ that respects $\rho_i: W_i \to Z_i$. The proof of it is similar to that of lemma 3.4. The key observation is that the construction of $Z$ involves twice elementary transformations of sheaves $\mathcal{E}_i$ along $C$ which depends only on their restrictions to $2C$.

We now study a subset of $Z$ of non-locally free sheaves: We keep $p_0 \in C$ and define

$$Z^{p_0} = \{ z \in Z \mid \tilde{\mathcal{E}}_z \text{ is not locally free at } p_0 \}. $$

Clearly, $Z^{p_0} \subset Z$ is the the vanishing locus of the restriction to $\{p_0\} \times Z$ of (3.7)

$$(3.11) \quad \alpha_{Z, p_0}: \pi^* \mathcal{F}_0 \to \mathcal{O}_{\mathbf{P}(1)} |_Z \otimes (\pi_{W \times C}^* \mathcal{L}_W) |_{\{p_0\} \times Z},$$

where $\pi: Z \to W$ is the projection and $\mathcal{F}_0 \cong \mathcal{F}_W |_{\{p_0\} \times W}$ is viewed as a sheaf on $W$. Because the restriction of $(\pi_{W \times C}^* \mathcal{L}_W) |_{\{p_0\} \times Z}$ to any fiber of $\pi$ is trivial and the line bundle $\mathcal{O}_{\mathbf{P}(1)}$ is $\pi$-relatively ample [La,p46], $Z^{p_0} \subset Z$ is $\pi$-relatively ample and has codimension at most 2. In the following, we will apply the generalized Lefschetz hyperplane theorem to the pair $(Z^{p_0}, Z)$. Let $Z_{\operatorname{reg}}$ be the largest open subset such that the restriction of $\pi$ to $Z_{\operatorname{reg}}$ is smooth [Ha, p271].
Lemma 3.5. Assume \( \pi \) is smooth at general points of \( \rho(W) \subset Z \). Then there is a maximal dense open subset \( W_0 \subset W \) of which the following holds: Let \( Z_0 = \pi^{-1}(W_0) \). Then \( \rho(W_0) \subset Z_{reg} \) and the induced pairs \( Z_0 \to W_0, Z_{reg} \cap Z_0 \to W_0 \) and \( Z^{p_0} \cap Z_0 \to W_0 \) are topological fiber bundles. Further, there is a Riemannian metric on \( Z \) so that for any \( \delta \)-neighborhood \( Z^{p_0,\delta} \) of \( Z^{p_0} \subset Z, \ 0 < \delta \ll 1 \), and any \( w \in W_0 \),

\[
(3.12) \quad H_i(Z_{reg} \cap \pi^{-1}(w), Z_{reg} \cap Z^{p_0,\delta} \cap \pi^{-1}(w)) = 0
\]

for \( i \leq 3 \).

Proof. First we prove the existence of such an open subset \( W_0 \subset W \). Since \( \pi : Z \to W \) is projective, there are Whitney stratification \( S_Z \) of \( Z \) and \( S_W \) of \( W \) by algebraic subvarieties such that \( \pi \) is a stratified map (see [GM] for definition). Without loss of generality, we can assume \( Z_{reg}, Z^{p_0} \) and \( \rho(W) \subset Z \) are union of strata of \( S_Z \). Let \( W_0 \) be union of top-dimensional strata in \( S_W \) and let \( Z_0 = \pi^{-1}(W_0) \). \( W_0 \subset W \) is open and dense. Because \( \pi \) is smooth at general points of \( \rho(W) \subset Z \), \( \rho(W_0) \subset Z_{reg} \). Further, by Thom’s first isotopy lemma, \( Z_0 \to W_0, Z_{reg} \cap Z_0 \to W_0 \) and \( Z^{p_0} \cap W_0 \) are topological fiber bundles. As to the vanishing of (3.12), we notice that by (1.1), \( \dim \pi^{-1}(w) \geq 10 \) for any \( w \in W \) [La, p45]. On the other hand, \( Z^{p_0} \cap \pi^{-1}(w) \) is the vanishing locus of two sections of the ample line bundle \( O_P(1)|_{Z_w} \). By the theorem on page 150 and remarks on page 152 of [GM], we can find a Riemannian metric on \( Z \) that provides us the vanishing of (3.12). Here, we have used the fact that for \( w \in W_0, Z_{reg} \cap \pi^{-1}(w) \) is smooth.

Finally, by using the fact that if \( W_0, W'_0 \subset W \) are two open subsets satisfying the conclusion of the lemma, then \( W_0 \cup W'_0 \) also satisfies the conclusion of the lemma. Thus, there is a maximal open subset \( W_0 \subset W \) of which the lemma holds. \( \square \)

Now we are ready to construct the homology between any class in \( H_i(W) \) with class in \( H_i(W \cap \mathcal{M}_{I,H}(I,d)) \). Let \( e = -2C \cdot C \). In light of the previous construction, we will take \( W \) to be the set

\[
(3.13) \quad W = \left\{ \mathcal{E} \in W \mid \mathcal{E} \text{ is e-stable, } \mathcal{E}|_C \text{ is locally free and the destabilizing quotient sheaf of } \mathcal{E}|_C \text{ has degree } \frac{1}{2}I \cdot C - 1. \right\}
\]

To construct the corresponding \( \pi : Z \to W \) and sheaves \( \tilde{E}_Z \) on \( X \times Z \), we argue as follows: We first take an (analytic or étale) open covering \( \{W_{\alpha}\} \) of \( W \) so that over each \( W_\alpha \), there is a universal family \( \mathcal{E}_\alpha \) on \( X \times W_\alpha \) that belongs to the exact sequence (3.4) for some integer \( n_\alpha \). Then we apply lemma 3.2 and 3.4 to pairs \( (W_\alpha, \mathcal{E}_\alpha) \) to get \( \pi_\alpha : \tilde{Z}_\alpha \to W_\alpha \) and sheaves \( \tilde{E}_\alpha \) on \( X \times \tilde{Z}_\alpha \). By lemma 3.4, when \( W_{\alpha \beta} = W_\alpha \cap W_\beta \neq \emptyset \),
\( \pi^{-1}(W_{\alpha\beta}) \) and \( \pi^{-1}(W_{\alpha\beta}) \) are canonically isomorphic and \( \tilde{E}_\alpha \) and \( \tilde{E}_\beta \) are isomorphic (non-canonically) on the overlap as well. Hence we can patch \{ \tilde{Z}_\alpha \} (resp. \( Z_{\alpha} \subset \tilde{Z}_\alpha \); \( \rho_\alpha : W_{\alpha} \rightarrow Z_{\alpha} \); resp. \( Z_{p0} \subset Z \)) together to get a scheme \( \pi : \tilde{Z} \rightarrow W \) (resp. \( Z \subset \tilde{Z} \); \( \rho : W \rightarrow Z \); resp. \( Z_{p0} \rightarrow Z \)). Also, the collection \{ \tilde{E}_\alpha \} provides us a rational map \( \eta : Z -\rightarrow \mathcal{M}_H(I, d) \) by sending \( z \in Z \) to \( \tilde{E}_z \) when it is \( H \)-stable. Finally, because sheaves in \( W \) are \( e \)-stable, sheaves in \( Z \) are necessarily \( H \)-stable [OG1,p597]. Hence \( \eta \rightarrow \mathcal{M}_H(I, d) \) is well-defined everywhere. Note that \( \eta \circ \rho \) is the identity map, \( \eta(Z_{p0}) \subset \mathcal{S}\mathcal{M}_H(I, d) \) and for any \( w \in W \cap \mathcal{S}\mathcal{M}_H(I, d) \), \( \eta(\pi^{-1}(w)) \subset \mathcal{S}\mathcal{M}_H(I, d) \).

**Proposition 3.6.** Let \( H_0, H_0 \in \mathcal{C} \subset NS^C_0, C \subset X \) and \( N \) be as before. For any \( d \geq N \) and \((H_0, I, d)\)-suitable \( H \in \mathcal{C} \), let \( W \) be the set \((3.13) \) and let \( \pi : Z \rightarrow W, Z_{p0} \subset Z \) and \( \rho : W \rightarrow Z \) be the sets constructed. Then \( W \) is dense in \( W \) and has pure codimension \( g + 1 \) in \( \mathcal{M}_H(I, d) \) \( (g = g(C)) \). Further there is a maximal open dense subset \( W_0 \subset W \) such that

1. Let \( Z_0 = \pi^{-1}(W_0), Z_{00} = Z_0 \cap Z_{p0} \) and \( Z_{0,\text{reg}} = Z_0 \cap Z_{\text{reg}} \), then \( \rho(W_0) \subset Z_{0,\text{reg}} \) and \( \pi : Z_0 \rightarrow W_0, Z_{00} \rightarrow W_0 \) and \( Z_{0,\text{reg}} \rightarrow W_0 \) are topological fiber bundles;
2. There is a Riemannian metric on \( Z \) such that for any \( \delta \)-neighborhood \( Z_{p0,\delta} \) of \( Z_{p0} \subset Z, 0 < \delta \ll 1 \), and any \( w \in W_0 \),

\[
H_i(Z_{\text{reg}} \cap \pi^{-1}(w), Z_{p0,\delta} \cap Z_{\text{reg}} \cap \pi^{-1}(w)) = 0
\]

for \( i \leq 3 \).

**Proof.** We first prove that \( W \) has pure codimension \( g + 1 \). Let \( p \in W \) be any point corresponding to \( \mathcal{E} \) and let \( U \subset \mathcal{M}_H(I, d) \) be an analytic neighborhood of \( p \) so that there is a well-defined map

\[
\varphi : (U, p) \rightarrow (A, 0)
\]

provided by lemma 2.2, where \((A, 0)\) is the versal deformation space of \( \mathcal{E}|_C \). Since \( \varphi^{-1}(A_0) = U \cap W \) and \( \text{codim}(A_0, A) = g + 1 \), the codimension of each component of \( W \) is at most \( g + 1 \). Further, when \( p \not\in \Lambda^C_f \) (cf. (1.3)), \( \varphi \) is a submersion at \( p \) and the codimension of \( W \) at \( p \) is exact \( g + 1 \). On the other hand, by choosing \( N \) large, we can assume \( \dim \Lambda^C_f \leq \dim \mathcal{M}_H(I, d) - 10g \) (lemma 1.3). Thus \( \mathcal{W} - \Lambda^C_f \) is dense in \( \mathcal{W} \) and therefore \( W \) has pure codimension \( g + 1 \) as claimed. As to the inclusion \( W \subset \mathcal{W} \), it is clear that \( \mathcal{W} = W \cup R_1 \cup R_2 \), where \( R_1 \) consists of sheaves in \( W \) that are not \( e \)-stable and \( R_2 \) consists of sheaves in \( \mathcal{W} \) that are not locally free along \( C \). By lemma 1.1, there is an integer \( N \) such that for \( d \geq N, \dim R_1 \leq \text{dim} \mathcal{W} - 4 \). Since \( \dim \Lambda^C_f < \dim \mathcal{W} \), \( W \) will be dense in \( \mathcal{W} \) if we can show that any sheaf \( \mathcal{E} \in R_2 - \Lambda^C_f \) admits a deformation whose generic member is in \( W \) (i.e. whose restriction to \( C \) is locally free and unstable). But this follows from lemma 2.12 and [GL,p84-87].
We now prove that $\pi : Z \to W$ is smooth at general points of $\rho(W)$. Let $w \in W$ be any point corresponding to $E_w$ and let $F_w$ be the sheaf in (3.2). Namely, $F_w$ is the kernel of $E_w \to L_w$, where $L_w$ is the destabilizing quotient sheaf of $E_w|C$. If we assume $w \in W$ is general, then $E_w$ will be $2e$-stable. Hence $F_w$ will be $e$-stable. In this way, we can define a morphism $f : W \to Y$ that sends $E_w$ to $F_w$, where $Y$ is a moduli scheme of stable sheaves with appropriate Chern classes. We let the image scheme be $Y$. From the construction of $Z$, we know that a neighborhood of $\rho(w) \in \pi^{-1}(w)$ is isomorphic to the set of surjective homomorphisms $\alpha : F_w \to L'$ up to scalars, where $L$ is an invertible sheaf on $C$ with $\deg L = \deg L_w$. Let $E_{\alpha}$ be the kernel of $\alpha$. We claim that if $\beta : F_w \to L''$ is another homomorphism and $E_{\beta}$ its kernel, then $E_{\alpha} \cong E_{\beta}$ if and only if $L' \cong L''$ and $\alpha = \lambda \beta$ for some $\lambda \in \mathbb{C}$. Indeed, the isomorphism $E_{\alpha} \cong E_{\beta}$ will induce homomorphism $h : F_w(-C) \to F_w$ whose determinant $\det h$ vanishes along $C$. Because $F_w$ is $e$-stable, $h$ must be a multiple of identity homomorphism at each $x \in X$. Thus $h$ must vanishes along $C$ and thus we get the following commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & E_{\alpha} \\
\cong & \downarrow & \downarrow h(-C) \\
0 & \longrightarrow & E_{\beta} \\
\end{array}
\begin{array}{ccc}
\longrightarrow & F_w & \alpha \longrightarrow L' & \longrightarrow & 0 \\
\beta \longrightarrow & F_w & \longrightarrow & L'' & \longrightarrow & 0.
\end{array}
$$

Hence $h(-C) = \lambda \cdot \text{id}$, $L' \cong L''$ and $\alpha = \lambda \beta$ as claimed.

Now let $z \in Z$ be a point over $w \in W$, corresponding to sheaf $E_z$ and $E_w$ respectively. Then by sending $z$ to $(E_z, E_w) \in W \times_Y W$ we obtain a morphism

$$
\mathcal{H} : Z \longrightarrow W \times_Y W.
$$

By previous argument, when $w \in W$ is a general point and $z \in Z$ is near $\rho(w)$, $\mathcal{H}$ is one-to-one. On the other hand, since all schemes involved are with reduced scheme structures, $\mathcal{H}$ must be an isomorphism at $\rho(w)$ for general $w \in W$. Now let $\text{pr}_2 : W \times_Y W \to W$ be the projection onto the second copy. It is straightforward to check that at general $w \in W$, both $\text{pr}_2$ and $W \times_Y W$ are smooth at $(w, w)$. Finally, because the $\pi : Z \to W$ is exactly $\text{pr}_2 \circ \mathcal{H}$ (at least near $(w, w)$), $\pi$ will be smooth at $\rho(w) \in Z$ as well. This proves that $\pi$ is smooth at general points of $\rho(W)$.

The last statement follows from lemma 3.5

In the following, we will show how to use the pair $(W_0, Z_0)$ to get the desired vanishing theorem. We begin with a simple version of theorem 3.1.

**Proposition 3.7.** Let the notation be as before, then for $i \leq 2$,

$$
(3.15) \quad H_i(W_0, W_0 \cap \mathfrak{M}_H(I, d)) \longrightarrow H_i(W, W \cap \mathfrak{M}_H(I, d))
$$
is trivial.

Proof. Let \( \xi \in H_i(W_0, W_0 \cap \mathcal{M}_H(I, d)) \) be any element represented by \( \Sigma \subset W_0 \) with \( \partial \Sigma \subset W_0 \cap \mathcal{M}_H(I, d) \). Let \( \rho : W_0 \to Z_0 \) be the section and let \( \Sigma' = \rho(\Sigma) \subset Z_{0,\text{reg}} \).

Because for each \( w \in W_0 \), the pair

\[
(Z_{0,\text{reg}} \cap Z_0^{p_0, \delta} \cap \pi^{-1}(w), Z_{0,\text{reg}} \cap \pi^{-1}(w))
\]

is 2-connected for \( 0 < \delta \ll 1 \), we can find an \((i + 1)\)-chain \( T' \subset Z_{0,\text{reg}} \) whose boundary \( \partial T' \) has a decomposition \( \partial T' = \Sigma' \cap A'_1 \cap A'_2 \) such that \( A'_1 \subset Z_0^{p_0, \delta} \) and

\[
A'_2 \subset \bigcup \{ \pi^{-1}(w) \mid w \in \partial \Sigma' \}. 
\]

Because \( Z_0 \to W_0 \) is proper, we can find \( T \subset Z_0 \) so that \( \partial T = \Sigma' \cup A_1 \cup A_2 \) with \( A_1 \subset Z_0^{p_0} \) and \( A_2 \) satisfies (3.16) with \( A'_2 \) replaced by \( A_2 \). Thus \( \partial(\eta(T)) = \Sigma \) modulo \( \mathcal{M}_H(I, d) \). In particular, the image of \( \xi \) in \( H_i(W, W \cap \mathcal{M}_H(I, d)) \) is trivial. \( \square \)

It remains to show that (3.15) is surjective. We first let \( S_1 = A_1^\prime \). By proposition 3.6, \( W \subset \mathcal{M}_H(I, d) \) has pure codimension \( g + 1 \) and \( S_1 \subset W \) has codimension at least \( 6g \). Note that \( \mathcal{M}_H(I, d) \) is a local complete intersection and \( W \subset \mathcal{M}_H(I, d) \) is defined by \( 3g - 2 \) equations. Hence by lemma 1.10,

\[
H_i(W_1, W_1 \cap \mathcal{M}_H(I, d)) \to H_i(W, W \cap \mathcal{M}_H(I, d))
\]

is surjective for \( i \leq 2 \), where \( W_1 = W - S_1 \).

Lemma 3.8. With the notation as before and let \( S_2 = (W_1 - W_0) \cap \mathcal{M}_H(I, d) \), then for \( i \leq 2 \) the induced homomorphism

\[
H_i(W_1 - S_2, (W_1 - S_2) \cap \mathcal{M}_H(I, d)) \to H_i(W_1, W_1 \cap \mathcal{M}_H(I, d))
\]

is surjective.

Proof. We first let \( S_3 \) be those \( E \in W_1 \) so that \( E \in \mathcal{M}_H(I, d) - S_1 \mathcal{M}_H(I, d) \) and \( E_{|C} \) is not locally free. \( S_3 \subset W_1 \) is closed and has codimension 3. By the proof of lemma 1.10, (3.17) is surjective with \( S_2 \) replaced by \( S_3 \) because \( W_1 - \mathcal{M}_H(I, d) \) is locally irreducible. Let \( W_2 = W_1 - S_3 \) and let \( S_4 = (W_2 - W_0) \cap \mathcal{M}_H(I, d) \). We claim that \( W_2 \) is locally irreducible and transversal to \( \mathcal{M}_H(I, d) \). Let \( p \in W_2 \) be a point associated to \( E \in S_1 \mathcal{M}_H(I, d) \) and let \( 0 \in A \) be a versal deformation space of \( E_{|C} \) given in lemma 2.2. Then the map \( \varphi \) in (3.14) is a submersion at \( p \) and hence \( W_1 \) is locally irreducible at \( p \) since \( A_0 \) is locally irreducible (proposition 2.6). Also \( \mathcal{M}_H(I, d) \) and \( W_1 \) are transversal at \( p \) as stratified sets. This is because the tangent space of fiber of
\(\varphi\) is the image \(\text{Ext}^1(\mathcal{E}, \mathcal{E}(-C))^0 \to \text{Ext}^1(\mathcal{E}, \mathcal{E})^0\) and the tangent space of \(S_1\mathcal{M}_H(I, d)\) is the image of \(H^1(\mathcal{E}^\vee \otimes \mathcal{E})\), and they together span \(\text{Ext}^1(\mathcal{E}, \mathcal{E})^0\). This proves that \(S_1\mathcal{M}_H(I, d) \cap \mathcal{W}_1\) is locally irreducible.

Next, we claim that \((\mathcal{W}_2 - S_4) \cap \mathcal{S}\mathcal{M}_H(I, d)\) is dense in \(\mathcal{W}_2 \cap \mathcal{S}\mathcal{M}_H(I, d)\). Indeed, because of the vanishing of \(\text{Ext}^2(\mathcal{E}, \mathcal{E}(-2C))^0\), a general sheaf \(\mathcal{E} \in \mathcal{W}_2 \cap \mathcal{S}\mathcal{M}_H(I, d)\) is locally free along \(C \subset X\) whose restriction to \(2C\) is isomorphic to \(\mathcal{F}|_{2C}\) for a general \(\mathcal{F} \in \mathcal{W}\). Then by the remark after lemma 3.4 and the maximal of \(W_0 \subset W\), \(\mathcal{E}\) is contained in \((\mathcal{W}_2 - S_4) \cap \mathcal{S}\mathcal{M}_H(I, d)\). This establishes the claim. Because \(\mathcal{W}_2\) is locally irreducible, is transversal to \(\mathcal{S}\mathcal{M}_H(I, d)\) and because \((\mathcal{W}_2 - S_4) \cap \mathcal{S}\mathcal{M}_H(I, d)\) is dense in \(\mathcal{W}_2 \cap \mathcal{S}\mathcal{M}_H(I, d)\), the following homomorphism

\[
H_i(\mathcal{W}_2 - S_4, (\mathcal{W}_2 - S_4) \cap \mathcal{S}\mathcal{M}_H(I, d)) \longrightarrow H_i(\mathcal{W}_2, \mathcal{W}_2 \cap \mathcal{S}\mathcal{M}_H(I, d))
\]

must be surjective for \(i \leq 2\). This proves the lemma because \(\mathcal{W}_2 - S_4 = \mathcal{W}_1 - S_2\). \(\square\)

Now we prove theorem 3.1. We only need to show that (3.17) is surjective. Let \(\mathcal{W}_3 = \mathcal{W} - \Lambda^C_1 - (\mathcal{W} - \mathcal{W}_0) \cap \mathcal{S}\mathcal{M}_H(I, d)\). We already know that

\[
H_i(\mathcal{W}_3, \mathcal{W}_3 \cap \mathcal{S}\mathcal{M}_H(I, d)) \to H_i(\mathcal{W}, \mathcal{W} \cap \mathcal{S}\mathcal{M}_H(I, d))
\]

is surjective. Because \(\mathcal{W}_3 \cap \mathcal{S}\mathcal{M}_H(I, d) = \mathcal{W}_0 \cap \mathcal{S}\mathcal{M}_H(I, d)\) and because \(\mathcal{W}_3\) is locally irreducible, the case for \(H_1\) follows from lemma 1.11. Now let \(\xi \in H_2(\mathcal{W}_3, \mathcal{W}_3 \cap \mathcal{S}\mathcal{M}_H(I, d))\). By lemma 1.11 again, we can find a representative \(\Sigma \subset \mathcal{W}_0\) with \(\partial \Sigma \subset \mathcal{W}_3 \cap \mathcal{S}\mathcal{M}_H(I, d)\) such that \(\Sigma \cap (\mathcal{W}_3 - \mathcal{W}_0)\) is discrete. Since the codimension of \(\Lambda^C_2\) in \(\mathcal{M}_H(I, d)\) is at least \(3g\) (lemma 1.3), by perturb \(\Sigma\) as we did in lemma 1.10, we can assume \(\Sigma \cap (\mathcal{W}_3 - \mathcal{W}_0) \cap \Lambda^C_2 = \emptyset\). In the following, we will construct a new representative of \(\xi\) that is contained in \(\mathcal{W}_0\).

We still keep the representative \(\Sigma\). Let \(p \in \Sigma \cap (\mathcal{W}_3 - \mathcal{W}_0)\) be associated to \(\mathcal{E}\). Since \(p \not\in \Lambda^C_2\), the set \(\Lambda^C_2 \subset \mathcal{M}_H(I, d)\) (see (1.5)) has pure codimension at most \(3(-\chi(\mathcal{O}_{2C})) + (g + 1)\). Let \(R\) be an irreducible component of \(\Lambda^C_2\) containing \(p\). Then by a recent work of [OG2], we can choose \(N\) (depending only on \(C\)) such that whenever \(d \geq N\), then \(\overline{R} \cap \mathcal{S}_1\mathcal{M}_H(I, d) \neq \emptyset\). On the other hand, by [GL,p80] we indeed have \(R \cap \mathcal{S}_1\mathcal{M}_H(I, d) = \emptyset\). Note that by dimension count, \(R \cap \mathcal{S}_1\mathcal{M}_H(I, d) - \Lambda^C_2 \neq \emptyset\). Now we pick a differentiable path \(\rho:\{0, 1\} \to R\) connecting \(p\) and \(\bar{p} \in R \cap \mathcal{S}_1\mathcal{M}_H(I, d) - \Lambda^C_2\) and let \(U\) be a (classical) neighborhood of \(\rho([0, 1])\) in \(\mathcal{M}_H(I, d)\). Without loss of generality, we can assume there is a universal family \(\mathcal{E}_U\) on \(X \times U\). Now let \(0 \in A\) be the versal deformation space of \(\mathcal{E}|_{2C}\). Based on the proof of lemma 2.2, we can find an analytic map \(\varphi:(U, \rho([0, 1])) \to (A, 0)\) induced by the family \(\mathcal{E}_U\), after shrinking \(U\) if necessary. By further shrink \(U\) if necessary, we can assume \(\varphi\) is a submersion at \(\rho([0, 1])\) and thus realize \(U\) as a product \((-\varepsilon, 1 + \varepsilon) \times U_0\) with \(\varepsilon\) factor through \(\varphi_0:U_0 \to A\) ((\(t\)) \times U_0 is a normal slice of \(\rho([0, 1])\) at \(\rho(t)\)). Thus by the remark after lemma 3.4 and the maximal
of $W_0 \subset W$, we can assume without loss of generality that $W_0 \cap U$ is a (topological) fiber bundle over $\varphi([0,1]) \subset A$. Finally, because $S_1 \mathcal{M}_H(I,d)$ is transversal to $R$ at $\tilde{p}$, we can apply the technique in the proof of lemma 1.11 to find a 2-chain $T$ contained in $W_0 \cap U$ with $\partial T = \Gamma_1 \cup \Gamma_2$ such that $\Gamma_2 \subset W_0 \cap S_1 \mathcal{M}_H(I,d)$ and $\Gamma_1 \subset \Sigma$ is the boundary of $B_{\varepsilon}(p) \cap \Sigma$. (Indeed, $T$ can be made to be homeomorphic to $\Gamma_1 \times [0,1]$ and is the result of pulling $\Gamma_1$ to $U \cap S_1 \mathcal{M}_H(I,d)$ along $W_0 \cap U$.) Hence $\Sigma' = (\Sigma - B_{\varepsilon}(p)) \cup T$ represents the homological cycle $[\Sigma]$ while $\Sigma' \cap (W_3 - W_0)$ is one point less than $\Sigma \cap (W_3 - W_0)$. Therefore, by iterating this process for each point in $\Sigma \cap (W_3 - W_0)$, we eventually get a $\tilde{\Sigma} \subset W_0$ that represents $[\Sigma] \in H_2^2(W, W \cap S_1 \mathcal{M}_H(I,d))$. This shows that (3.15) is surjective. This and proposition 3.7 together prove the proposition 3.1. □

4. Proof of the main theorems

In this section, we will first prove theorem 0.4 by using Lefschetz hyperplane theorem. After that, we will study the pair $S \mathcal{M}_d(H) \subset \mathcal{M}_d(H)$ in detail to establish both theorem 0.5 and 0.1.

We will keep the notation developed in the previous sections. We fix $H_0, H_0 \in C \subset NS^+_Q$ and the $N$ given in lemma 1.3 and 1.5. For $d \geq N$, we pick a $(H_0, I, d)$-suitable $H \in C$. $\mathcal{M}_{H_0}(I,d')$ is birational to $\mathcal{M}_H(I,d')$, $d' = d$ or $d + 1$, which induces the following commutative square

$$
\begin{array}{ccc}
H_i(\mathcal{M}_{H_0}(I,d) \cap U) & \cong & H_i(\mathcal{M}_H(I,d) \cap U) \\
\tau(d) & & \tilde{\tau}(d) \\
H_i(\mathcal{M}_{H_0}(I,d + 1) \cap U) & \cong & H_i(\mathcal{M}_H(I,d + 1) \cap U)
\end{array}
$$

for $i \leq 2$. Following the discussion in the introduction, theorem 0.1 follows from the surjectivity of $\tilde{\tau}(d)_i$, which is equivalent to the following theorem.

**Theorem 4.1.** Let $d \geq N$ and $H \in C$ be $(H_0, I, d)$-suitable. Then the homomorphism $\tilde{\tau}(d)_i : H_i(\mathcal{M}_H(I,d) \cap U) \rightarrow H_i(\mathcal{M}_H(I,d + 1) \cap U)$ is surjective.

As explained in the introduction, theorem 4.1 will be proved in two steps: The first is to use the Lefschetz hyperplane theorem to prove the following:

**Theorem 4.2.** With the choice of $d$ and $H$ made in theorem 4.1, then for $i \leq 2$,

$$H_i(\mathcal{M}_H(I,d), S \mathcal{M}_H(I,d)) = 0.$$
Proof. Let $\Lambda = \overline{\mathcal{M}}_H(I, d) - \mathcal{M}_H(I, d)$. Since $H$ is $(I, d)$-generic, $\Lambda \subset S\overline{\mathcal{M}}_H(I, d)$. Also, since $\text{codim}(\Lambda) \geq 10g$ (lemma 1.7), and $\mathcal{M}_H(I, d)$ is a local complete intersection, by lemma 1.10,

$$H_i(\mathcal{M}_H(I, d), S\mathcal{M}_H(I, d)) \rightarrow H_i(\overline{\mathcal{M}}_H(I, d), S\overline{\mathcal{M}}_H(I, d))$$

is an isomorphism for $i \leq 2$. For the same reason, the subset $W \subset \mathcal{M}_H(I, d)$ (defined in (1.1)) and its closure $\overline{W}$ in $\overline{\mathcal{M}}_H(I, d)$ induces a surjective homomorphism

$$H_i(W, W \cap S\mathcal{M}_H(I, d)) \rightarrow H_i(\overline{W}, \overline{W} \cap S\overline{\mathcal{M}}_H(I, d))$$

for $i \leq 2$. Since $H_i(W, W \cap S\mathcal{M}_H(I, d)) = 0$ (proposition 3.1), theorem 4.2 follows from the surjectivity of the homomorphism

$$H_i(\overline{W}, \overline{W} \cap S\overline{\mathcal{M}}_H(I, d)) \rightarrow H_i(\overline{\mathcal{M}}_H(I, d), S\overline{\mathcal{M}}_H(I, d))$$

for $i \leq 2$.

For simplicity, in the following we will denote by $\mathcal{M}_d$ the set of all $\mu$-stable sheaves $E$ with $\text{Ext}^2(E^{\vee}, E^{\vee}) = 0$ and by $\overline{\mathcal{M}}_d$ the space $\overline{\mathcal{M}}_H(I, d)$. Let

$$\Psi : \overline{\mathcal{M}}_d \rightarrow \mathbf{P}^R$$

be the morphism constructed in lemma 1.5 and let $V \subset \mathbf{P}^R$ be the codimension $3g - 2$ linear subspace such that $\Psi^{-1}(V) \cap \mathcal{M}_d = W \cap \mathcal{M}_d$. For $\delta > 0$, we let $V^{\delta} \subset \mathbf{P}^R$ be the $\delta$-neighborhood of $V \subset \mathbf{P}^R$ under the Fubini-Study metric and let $\overline{V}^{\delta} = \Psi^{-1}(V^{\delta})$ and $\mathcal{W}^{\delta} = \Psi^{-1}(V^{\delta}) \cap \mathcal{M}_d$. We first consider the triple $(\mathcal{M}_d, S\mathcal{M}_d, S\mathcal{M}_d \cap \mathcal{W}^{\delta})$ and its induced long exact sequence

$$H_i(\mathcal{M}_d, S\mathcal{M}_d \cap \mathcal{W}^{\delta}) \rightarrow H_i(\mathcal{M}_d, S\mathcal{M}_d) \rightarrow H_{i-1}(S\mathcal{M}_d, S\mathcal{M}_d \cap \mathcal{W}^{\delta}).$$

Let $U \subset \mathcal{M}_d$ be the open subset consisting of sheaves $E \in \mathcal{M}_d$ such that $\ell(E^{\vee} \cap E) \leq 4$. $U$ is smooth, the compliment of $U$ in $\overline{\mathcal{M}}_d$ has codimension 5 and for any $u \in U$, $\dim \Psi^{-1}(\Psi(u)) \leq 12$. Then by lemma 1.10,

$$H_i(U, U \cap S\mathcal{M}_d \cap \mathcal{W}^{\delta}) \rightarrow H_i(\mathcal{M}_d, S\mathcal{M}_d \cap \mathcal{W}^{\delta})$$

is surjective for $i \leq 2$ because $S\mathcal{M}_d \subset \mathcal{M}_d$ is Cartier. Next, because the fibers of $\Psi|_U : U \rightarrow \mathbf{P}^R$ have dimension at most 12 and $U$ has pure dimension much bigger than $3g + 20$, we can apply the stratified Morse theory technique (exactly the same as in the proof of theorem 4.1 on page 195 of [GM]) to the map $\Psi|_U$ to conclude that

$$H_i(U \cap \mathcal{W}^{\delta}, U \cap S\mathcal{M}_d \cap \mathcal{W}^{\delta}) \rightarrow H_i(U, U \cap S\mathcal{M}_d \cap \mathcal{W}^{\delta})$$
is surjective for \( i \leq 2 \). Then by (4.3) and (4.4), for \( i \leq 2 \)

\[
(4.5) \quad H_i(U \cap W^\delta, U \cap SM_d \cap W^\delta) \to H_i(M_d, SM_d)
\]

will be surjective if \( H_{i-1}(SM_d, SM_d \cap W^\delta) = 0 \).

We claim that for \( j \leq 1 \), \( H_j(SM_d, S_1M_d) = 0 \). \( H_0 = 0 \) because \( S_1M_d \) is dense in \( SM_d \) (lemma 1.1). For \( H_1 \), let \( f : ([0, 1], \partial[0, 1]) \to (SM_d, S_1M_d) \) be a continuous map. Since \( S_1M_d \) is dense in \( SM_d \) (by lemma 1.11), we can assume without loss of generality that \( f^{-1}(SM_d - S_1M_d) \) is a finite set, say \( \{p_1, \cdots, p_k\} \). Because \( SM_d \) is a local complete intersection and the compliment of \( S_1M_d \cup S_2^1M \subset SM_d \) has codimension 2 by lemma 1.10, we can choose \( f \) so that the points \( p_i \) are all contained in \( S_2M_d \). Let \( R_1 = S_2^1M \) and let \( R_2 = S_2M_d - S_2^1M \). Since \( R_1 \) is irreducible and dense in \( S_2M_d \), by lemma 1.12 we can choose \( f \) so that all \( p_i \) belong to \( R_2 \). On the other hand, because \( SM_d \) is locally irreducible at \( R_2 \) (this can be checked directly), we can perturb \( f \) within \( \bigcup(B_\delta(f(p_i)) \cap SM_d) \) to obtain a representative \( f' \) of \( [f] \) whose image is contained in \( S_1M_d \). Therefore, \( [f] = 0 \) and hence \( H_1(SM_d, S_1M_d) = 0 \).

Now we consider the triple \((SM_d, S_1M_d, S_1M_d \cap W^\delta)\) and its induced long exact sequence \((j \leq 1)\)

\[
\to H_j(S_1M_d, S_1M_d \cap W^\delta) \to H_j(SM_d, SM_d \cap W^\delta) \to H_j(SM_d, S_1M_d) = 0.
\]

Because \( H_j(SM_d, S_1M_d \cap W^\delta) \to H_j(SM_d, SM_d \cap W^\delta) \) is surjective \((S_1M_d \subset SM_d \) is dense), by the above exact sequence to show \( H_j(SM_d, SM_d \cap W^\delta) = 0 \) it suffices to show that \( H_j(S_1M_d, S_1M_d \cap W^\delta) = 0 \). For this, we will look at the restriction to \( S_1M_d \) of \( \Psi \) in (4.2),

\[
\Psi' : S_1M_d \to \mathbb{P}^R.
\]

Because fibers of \( \Psi' \) has dimension 1, we can apply theorem on page 153 of [GM] to conclude that \( H_j(S_1M_d, S_1M_d \cap W^\delta) = 0 \) for \( j \leq 1 \) and then \( H_j(SM_d, SM_d \cap W^\delta) = 0 \). Therefore, we have proved the surjectivity of (4.5) for \( i \leq 2 \) which combined with lemma 1.10 yields the surjectivity of

\[
(4.6) \quad H_i(W^\delta, \overline{SM_d} \cap \overline{W}^\delta) \to H_i(M_d, \overline{SM_d}), \quad i \leq 2.
\]

Finally, since both \( \overline{M}_d \) and \( \overline{SM}_d \) are complete and (4.6) holds for all \( 0 < \delta \ll 1 \), we obtain the surjectivity of (4.1) by applying proposition 4.A.1 on page 206 of [GM]. This completes the proof of the theorem. \( \square \)

In the remainder of this section, we will use theorem 4.2 to establish theorem 4.1. But first, we will fill in the details of the definition of the homomorphism (0.5). For any \( x \in X \), we let \( S_1^xM_d \subset S_1M_d \) be the set of \( E \in S_1M_d \) that are non-locally free at \( x \).
Note that $S_1^d M_d$ is a $\mathbb{P}^1$-bundle over $\mathcal{M}^0_{d-1}$. Let $V_0$ be a general fiber of this bundle. Then the inclusion $V_0 \subset \mathcal{M}_d$ and the bundle $S_1^d M_d \to \mathcal{M}^0_{d-1}$ induce the commutative diagram

$$
\begin{array}{ccc}
0 & \to & H_i(V_0) \\
\downarrow & & \downarrow r(d)_i \\
0 & \to & H_i(M_d)
\end{array}
$$

(4.7)

where $i \leq 2$.

**Lemma 4.3.** Let $d \geq N$. For $i \leq 2$, the induced homomorphism $H_i(\mathcal{M}_d^0) \to H_i(\mathcal{M}_d)$ is injective and whose image $H_i(\mathcal{M}_d^0)^\sim$ is a (linear) compliment of the image $H_i(\mathcal{M}_d^0)^\sim$ of $H_i(V_0) \to H_i(\mathcal{M}_d)$. In particular, since the top row of (4.7) is exact, we get a unique homomorphism $\rho(d)_i : H_i(\mathcal{M}_{d-1}^0) \to H_i(\mathcal{M}_d^0)$ that coincides with Taubes’ homomorphism $\tilde{\tau}(d)_i$.

**Proof.** We need to show that $H_i(\mathcal{M}_d^0)^\sim$ is a compliment of $H_i(\mathcal{M}_d^0)^\sim$. Let $i = 1, 2$. We first show that $H_i(\mathcal{M}_d) \to H_i(\mathcal{M}_d, \mathcal{M}_d^0)$ is surjective. Since $\mathcal{M}_d$ is smooth, $H_i(\mathcal{M}_d) \cong H_i(\mathcal{M}_d^0 \cup S_1 \mathcal{M}_d)$. Thus it suffices to show

$$
H_i(\mathcal{M}_d^0 \cup S_1 \mathcal{M}_d) \longrightarrow H_i(\mathcal{M}_d^0 \cup S_1 \mathcal{M}_d, \mathcal{M}_d^0)
$$

(4.8)

is surjective. Let $\mathcal{U}$ be a tubular neighborhood of $S_1 \mathcal{M}_d \subset \mathcal{M}_d^0 \cup S_1 \mathcal{M}_d$ such that $\mathcal{U} - S_1 \mathcal{M}_d$ is an $(\mathbb{R}^4 - 0)/\mathbb{Z}_2$-bundle over $\mathcal{M}_{d-1}^0$. Let $\xi \in H_i(\mathcal{M}_d^0 \cup S_1 \mathcal{M}_d, \mathcal{M}_d^0)$ be represented by $\Sigma \subset (\mathcal{M}_d^0 \cup S_1 \mathcal{M}_d) \cap \mathcal{U}$. Then $\partial \Sigma$ represents a trivial cycle in $H_{i-1}(\mathcal{U}) = H_{i-1}(\mathcal{U} - S_1 \mathcal{M}_d)$. Thus we can find another chain $\Sigma' \subset \mathcal{U} - S_1 \mathcal{M}_d$ with $\partial \Sigma' = -\partial \Sigma$. Hence $\Sigma' \cup \Sigma$ represents a cycle $\tilde{\xi} \in H_i(\mathcal{M}_d^0 \cup S_1 \mathcal{M}_d)$ whose image in $H_i(\mathcal{M}_d^0 \cup S_1 \mathcal{M}_d, S_1 \mathcal{M}_d)$ is $\xi$. Thus (4.8) is surjective and hence $H_i(\mathcal{M}_d^0) \to H_i(\mathcal{M}_d)$ is injective.

On the other hand, the composition $H_i(V_0) \to H_i(\mathcal{M}_d) \to H_i(\mathcal{M}_d, \mathcal{M}_d^0)$ is an isomorphism because the intersection $(V_0, SM_d)$ in $\mathcal{M}_d$ is -2 (this is well defined because $V_0$ is proper) and $SM_d$ is irreducible. Therefore, $H_i(\mathcal{M}_d^0)^\sim$ is a compliment of $H_i(V_0)^\sim$ and the homomorphism $H_i(\mathcal{M}_{d-1}^0) \to H_i(\mathcal{M}_d^0)$ is well-defined. It is straight forward to check that the homomorphism $\rho(d)_i$ coincides with that of Taubes’.

**Corollary 4.4.** Assume $d \geq N$, then the homomorphism $\tilde{\tau}(d)_i$ is surjective if and only if the homomorphism

$$
r(d)_i : H_i(S_1^d M_d) \longrightarrow H_i(\mathcal{M}_d)
$$

(4.9)
is surjective.

Before we prove the surjectivity of (4.9), let us state a technical lemma whose proof will be postponed until the end of this section. For convenience, in the following whenever a space $Z$ admits an obvious map $Z \to \mathcal{M}_d$, then we will denote by $H_i(Z)^\sim$ the image of $H_i(Z) \to H_i(\mathcal{M}_d)$.

**Lemma 4.5.** Let $\mathcal{F} \in S_j \mathcal{M}_d$, $j \leq 3$, be any sheaf and let $V(\mathcal{F}) \subset S_j \mathcal{M}_d$ be the set of those $\mathcal{E}$ such that $\mathcal{F}^\vee \cong \mathcal{E}^\vee$ and $\ell((\mathcal{E}^\vee/\mathcal{E}) \otimes \mathcal{O}_x) = \ell((\mathcal{F}^\vee/\mathcal{F}) \otimes \mathcal{O}_x)$ for all $x \in X$. Then the image $H_i(V(\mathcal{F}))^\sim$ is contained in $H_i(V_0)^\sim$.

Now we prove the following proposition that is equivalent to the surjectivity of (4.9).

**Proposition 4.6.** Let $H_0, H_0 \in \mathcal{C} \in NS_f^+ \mathbb{Q}$ be fixed. Then there is an $N$ such that for any $d \geq N$ and $(H_0, I, d)$-suitable $H \in \mathcal{C}$, the homomorphism

$$r(d)_i : H_i(S_1^d \mathcal{M}_d) \to H_i(\mathcal{M}_d)$$

is an isomorphism for $i \leq 2$.

**Proof.** The statement for $i = 0$ follows from [GL,OG1]. The proof that $r(d)_1$ is an isomorphism is similar and easier than that of $r(d)_2$, which we will prove now.

Let $N$ be chosen so that all requirement of $N$ in the previous results have been met and the proposition is true for $r(d)_1$. The surjectivity of $r(d)_2$ will be proved by first establishing $H_2(S_1 \mathcal{M}_d)^\sim \subset H_2(S_1^d \mathcal{M}_d)^\sim$ and then $H_2(S_1 \mathcal{M}_d)^\sim \subset H_2(S_1^d \mathcal{M}_d)^\sim$. The difficult in showing the first inclusion lies in the fact that $S_1 \mathcal{M}_d$ is not locally irreducible along $S_2^0 \mathcal{M}_d$. Thus $H_2(S_1 \mathcal{M}_d) \to H_2(S \mathcal{M}_d)$ is not surjective. However, there images in $H_2(\mathcal{M}_d)$ actually coincide.

Before we go any further, we need to introduce some subsets of $S \mathcal{M}_d$ that will help us understand the geometry of the compliment of $S_1 \mathcal{M}_d \subset S \mathcal{M}_d$. The first space is $Z_1$ that is an algebraic space whose closed points are pairs

$$\{\mathcal{E} \subset \mathcal{F}\} : \mathcal{F} \in \mathcal{M}_{d-1}, \mathcal{E} \in S \mathcal{M}_d \text{ and } \mathcal{F}/\mathcal{E} \cong \mathbb{C}_p \text{ for some } p \in X.$$

$Z_1$ admits projections $\pi^1_M$ and $\pi^1_X$ onto $S \mathcal{M}_d$ and $X$ respectively by sending $\{\mathcal{E} \subset \mathcal{F}\}$ to $\{\mathcal{E}\} \in S \mathcal{M}_d$ and $\text{supp}(\mathcal{E}/\mathcal{F})$ respectively. Note that $Z_1$ is irreducible and $\pi^1_M$ is one-to-one restricting to $(\pi^1_M)^{-1}(S_1 \mathcal{M}_d)$. The second space is $Z_2$ consisting of tuples

$$\{\mathcal{E} \subset \mathcal{F}_1 \subset \mathcal{F}_2\} : \mathcal{F}_2 \in \mathcal{M}_{d-2}, \mathcal{F}_1/\mathcal{E} \cong \mathbb{C}_{p_1} \text{ and } \mathcal{F}_2/\mathcal{F}_1 \cong \mathbb{C}_{p_2} \text{ for some } p_1, p_2 \in X.$$
We let \( \pi_1^2 : \mathcal{Z}_2 \rightarrow \mathcal{Z}_1 \) be the map sending \( \{ E \subset F \} \) to \( \{ E \subset F \} \). Let \( \pi_M^2 (= \pi_M^1 \circ \pi_1^2) \), \( \pi_X^2 \), and \( \pi_X^1 \) be the projections from \( \mathcal{Z}_2 \) to \( \mathcal{S}\mathcal{M}_d \) and \( X \) respectively by sending \( \{ E \subset F \} \) to \( E \), \( p_1 \) and \( p_2 \) respectively.

Now let \( W_1 \subset S_2 \mathcal{M}_d - S_0^0 \mathcal{M}_d \) and let \( W_2 \subset S_3 \mathcal{M}_d \) be subsets consisting of points \( w \)'s so that \( (\pi_M^2)^{-1}(w) \) are single point sets. We claim that \( \dim S_2 \mathcal{M}_d - (W_1 \cup S_0^0 \mathcal{M}_d) \leq \dim \mathcal{M}_d - 4 \) and \( W_2 \not= \emptyset \). Indeed, let \( E \subset \mathcal{S}\mathcal{M}_d \) be such that \( \ell(\mathcal{E}^{\vee}/\mathcal{E}) = k \geq 4 \). Then \( (\pi_M^2)^{-1}(E) \) is a point if and only if there is a unique filtration \( T_0 \subset T_2 \subset T_2 = \mathcal{E}^{\vee}/\mathcal{E} \) such that \( \ell(T_j/T_{j-1}) = 1 \), \( j = 1, 2 \). When \( \ell(\mathcal{E}^{\vee}/\mathcal{E}) = 2 \) and \( E \not\subset S_0^0 \mathcal{M}_d \), then the uniqueness of the above filtration is equivalent to \( (\mathcal{E}^{\vee}/\mathcal{E}) \otimes \mathbb{C}_p = \mathbb{C}_p \) for some \( p \in X \). From this description, it is easy to see that \( W_1 \) is dense in \( S_2 \mathcal{M}_d - S_0^0 \mathcal{M}_d \). Therefore, \( \dim S_2 \mathcal{M}_d - (S_0^0 \mathcal{M}_d \cup W_1) \leq \dim \mathcal{M}_d - 4 \). To show \( W_2 \not= \emptyset \), one notices that any sheaf that has the form \( \mathcal{O}_p \oplus (z_1, z_2^2) \mathcal{O}_p \) at \( p \in X \), where \( (z_1, z_2) \) is a local coordinate of \( p \in X \), and locally free elsewhere belongs to \( W_2 \). Such sheaves do exist in \( S_3 \mathcal{M}_d \). Finally, we note that \( S_3 \mathcal{M}_d \) is irreducible because its generic points are kernel of \( F \rightarrow \oplus_{i=1}^3 \mathbb{C}_p \), with \( F \) locally free and \( p_1, p_2, p_3 \in X \) distinct.

Now let \( \xi \in H_2(\mathcal{S}\mathcal{M}_d) \) be any element. Since \( \mathcal{M}_d \) is smooth and \( \mathcal{S}\mathcal{M}_d \) is Cartier, by lemma 1.11 and 1.12, we can find a representative \( f : \Sigma \rightarrow \mathcal{S}\mathcal{M}_d \) of a multiple of \( \xi \) such that \( \Sigma \) is a Riemann surface and

\[
(4.10) : f(\Sigma) \subset S_1 \mathcal{M}_d \cup S_0^0 \mathcal{M}_d \cup W_1 \cup W_2;
\]

\[
(4.11) : f^{-1}(S_0^0 \mathcal{M}_d \cup W_1 \cup W_2) \text{ is an (at most real 1-dimensional) stratified set.}
\]

Now let \( C \) be the closure of \( f^{-1}(S_0^0 \mathcal{M}_d) \subset \Sigma \) and let \( \eta : S \rightarrow \Sigma \) be the Riemann surface with boundary obtained by cutting \( \Sigma \) along \( C \). Because \( \pi_M^1 : \mathcal{Z}_1 \rightarrow \mathcal{S}\mathcal{M}_d \) is one-to-one over \( f(\Sigma) - S_0^0 \mathcal{M}_d \), \( f \) lifts to a unique \( g' : S - \partial S \rightarrow \mathcal{Z}_1 \) and further, because \( \pi_M^1 \) is finite over \( f(\Sigma) \), \( g' \) extends to \( g : S \rightarrow \mathcal{Z}_1 \). By the choice of \( W_1 \) and \( W_2 \), the preimage of any \( z \in g(\partial S) \) of the map \( \pi_1^2 : \mathcal{Z}_2 \rightarrow \mathcal{Z}_1 \) is a one point set. Therefore, if we let \( \Gamma = \partial S \), the restriction of \( g \) to \( \Gamma \) lifts to \( \tilde{g} : \Gamma \rightarrow \mathcal{Z}_2 \), which gives rise to the following commutative diagram of maps:

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{c} & S \\
\downarrow \tilde{g} & & \downarrow g \\
\mathcal{Z}_2 & \xrightarrow{\pi_1^2} & \mathcal{Z}_1 \xrightarrow{\pi_M^1} \mathcal{S}\mathcal{M}_d.
\end{array}
\]

**Lemma 4.7.** \( \tilde{g} : \Gamma \rightarrow \mathcal{Z}_2 \) represents the trivial class in \( H_1(\mathcal{Z}_2) \).

**Proof.** As chains,

\[
\partial(\pi_X^1 \circ g(S)) = \pi_X^1 \circ g(\partial S) = \pi_X^1 \circ g(\Gamma).
\]
Thus $\pi^1_X \circ g(\Gamma)$ represents the trivial class in $H_1(X)$. Because $\pi^2_X \circ \tilde{g} = \pi^1_X \circ g$ as maps from $\Gamma$ to $X$. $[\pi^2_X \circ \tilde{g}(\Gamma)] = 0$ in $H_1(X)$ as well. We claim $\pi^2_X \circ \tilde{g}(\Gamma)$ also represents the trivial class in $H_1(X)$. Indeed, let $I_1, \ldots, I_k$ (each is homeomorphic to $[0,1]$) be segments of $C \subset S$ with fixed orientations. Then $\partial S$ can be divided into segments $I_1^\pm, \ldots, I_k^\pm$ such that $\eta(I_i^\pm) = \pm I_i$ as (oriented) chains and $\partial S = \sum_{i=1}^k (I_i^+ + I_i^-)$. Next, we let $\Lambda_1$ be those $I_i$ such that $\text{Im} \, g(I_i^+)$ $\subset \text{Im} \, g(I_i^-)$ (which means that we do not need to cut $\Sigma$ along $I_i$ in order to get a lift $g$) and let $\Lambda_2$ be the remainder $I_i$’s. Note that for $i \in \Lambda_1$, $g(I_i^+) = -g(I_i^-)$. Then we have the following identities of chains:

$$\pi^2_X \circ \tilde{g}(I_i^+) + \pi^2_X \circ \tilde{g}(I_i^-) = \pi^2_X \circ \tilde{g}(I_i^+) + \pi^2_X \circ \tilde{g}(I_i^-) = 0, \quad I_i \in \Lambda_1;$$

$$\pi^2_X \circ g(I_i^\pm) = -\pi^2_X \circ \tilde{g}(I_i^\mp), \quad I_i \in \Lambda_2.$$

Therefore, we have

$$(4.12) \quad [\pi^2_X \circ \tilde{g}(\Gamma)] = -[\pi^2_X \circ \tilde{g}(\Gamma)] = 0 \in H_1(X).$$

Now we consider the class $[\tilde{g}(\Gamma)] \in H_1(Z_2)$. Let $p : Z_2 \to M_{d-2}$ be the map sending $\{E \in F_1 \subset F_2\}$ to $F_2$. Because $M_{d-2}$ is smooth and $Z_2$ is locally irreducible, $\tilde{g}(\Gamma)$ is homotopic to a $\Gamma' \subset Z_2$ such that $p(\Gamma') \subset M^0_{d-2}$. Let $X' = p^{-1}(M^0_{d-2})$. It is easy to check that fibers of $X'$ over $M^0_{d-2} \times X \times X$ (via $p \times \pi^2_X \times \pi^2_X$) are connected and have trivial first homology groups. By Leray spectral sequence, $H_1(X')$ is isomorphic to $H_1(M^0_{d-2} \times X \times X)$. Because of (4.12), for any pair of distinct points $(x_1, x_2) \in X \times X$, there is a 1-cycle $\Gamma''$ contained in $S_2^x \times x_2 M_d = (\pi^2_X \times \pi^2_X)^{-1}(x_1, x_2)$ such that $\Gamma''$ is homologous to $\tilde{g}(\Gamma)$ in $Z_2$. Let the homology be given by $\Sigma'$. Namely, $\Sigma' \subset Z_2$ such that $\partial \Sigma' = \Gamma'' - \tilde{g}(\Gamma)$. By lemma 1.11 and 1.12, we can choose $\Sigma'$ such that $\Sigma' \subset (\pi^2_M)^{-1}(S_2^x M_d \cup W_2)$. Finally, because $\pi^2_M(\Sigma') \subset S_2^x \times x_2 M_d$ is the boundary of $\pi^2_M(\Sigma') \subset S_2^x M_d$, $[\pi^2_M(\Gamma'')] \in H_1(S_2^x \times x_2 M_d)$ is contained in the kernel of $H_1(S_2^x \times x_2 M_d) \to H_1(M_d)$. By the induction hypothesis, $[\pi^2_M(\Gamma'')] = 0 \in H_1(S_2^x \times x_2 M_d)$ and then $[\tilde{g}(\Gamma')] = 0 \in H_1(Z_2)$. This completes the proof of the lemma. □

Continuation of the proof of proposition 4.6. From the proof, we see that we can find a 2-chain $\Sigma' \subset Z_2$ so that $\partial \Sigma' = \tilde{g}(\Gamma)$ and $\pi^2_M(\Sigma') - S_2 M_d$ is a discrete point set contained in $W_2$. In summary, for any $\xi \in H_2(S_2 M_d)$, we can find a closed stratifiable set $S \subset Z_1$ that $\pi^2_M(S)$ is a representative of a multiple of $\xi \in H_2(M_d)$ and the boundary $\Gamma = \partial S \subset Z_1$ can be lifted to $Z_2$, say $\tilde{\Gamma} \to Z_2$, with $[\tilde{\Gamma}] = 0 \in H_1(Z_2)$. Hence we can find a 2-chain $S' \subset Z_2$ such that

$$(4.13) \quad \partial S' = -\tilde{\Gamma}, \quad \pi^2_M(S') \subset S_2 M_d \cup W_2$$

and $\pi^2_M(S') - S_2 M_d \subset W_2$ is discrete.

Let $T_1 = S \cup \pi^2_M(S') \subset Z_1$ and $T_2 = \pi^2_M(S') \subset S_2 M_d - S_1 M_d$. Then $u_1 = [T_1] \in H_1(Z_1)$ and $u_2 = [T_2] \in H_2(S_2 M_d - S_1 M_d)$ has the property that their images $\bar{u}_1$ and $\bar{u}_2$ in
$H_2(\mathcal{M}_d)$ provide a decomposition $\xi = \bar{u}_1 + \bar{u}_2$. Finally, by the construction, $T_1$ is contained in $(\pi_1^d)^{-1}(S_1\mathcal{M}_d \cup S_2\mathcal{M}_d \cup W_2)$ and $T_2$ is contained in $S_2\mathcal{M}_d \cup W_2$.

Our next step is to prove that $\bar{u}_1 \in H_2(S_1\mathcal{M}_d)^\sim$ and $\bar{u}_2 \in H_2(S_2\mathcal{M}_d)^\sim$. We first prove $\bar{u}_1 \in H_2(S_1\mathcal{M}_d)^\sim$. We take the representative $T_1 \subset Z_1$. By perturb $T_1$ if necessary, we can assume $T_1 - (\pi_1^d)^{-1}(S_1\mathcal{M}_d)$ is discrete whose image in $S_1\mathcal{M}_d$ lies in $S_2\mathcal{M}_d \cup W_2$. Let $\{p_1, \ldots, p_k\} \subset T_1$ be these points and let $\mathcal{F}_i$ be the sheaf in $\mathcal{M}_d - 1$ that corresponds to the image of $p_i$ under the projection $p_M : Z_1 \to \mathcal{M}_d - 1$ by sending $\{E \subset F\}$ to $F$. We let $V(\mathcal{F}) \subset \mathcal{M}_d - 1$ be the closed set defined in lemma 4.5. Then there is an analytic deformation retract neighborhood $U_i$ of $V(\mathcal{F})$ such that $H_1(U_i - S\mathcal{M}_d - 1) = 0$. Because fibers of $Z_1 \to \mathcal{M}_d - 1$ are $\mathbb{P}^1$ over $\mathcal{M}_d - 1$, the homomorphism

$$H_2((p_M)^{-1}(\mathcal{M}_{d-1}^0)) \oplus \bigoplus_{i=1}^k H_2(V(\mathcal{F}_i)) \to H_2\left((p_M)^{-1}(\mathcal{M}_{d-1}^0 \cup \bigcup_{i=1}^k V(\mathcal{F}_i))\right)$$

is surjective by Mayer-Vietoris sequence. However, we know that $H_2(V(\mathcal{F}_i)) \subset H_2(V_0)^\sim \subset H_2(S_1\mathcal{M}_d)^\sim$ (lemma 4.5). Therefore, $\bar{u}_1 \in H_2(S_1\mathcal{M}_d)^\sim$.

Next, we prove $\bar{u}_2 \in H_2(S_2\mathcal{M}_d)^\sim$. Because $\bar{u}_2$ is the image of $[\pi_2^d(S')]$ with $S'$ satisfies (4.13) and because $S_2\mathcal{M}_d$ is locally irreducible, similar to the previous reasoning, $\bar{u}_2$ is contained in the linear span of $H_2(S_2^d\mathcal{M}_d)^\sim$ and $H_2(V(\mathcal{F}))^\sim$ for all possible $\mathcal{F}$ in $W_2$ and $S_2\mathcal{M}_d - S_2^d\mathcal{M}_d$. But by lemma 4.5, all these $H_2(V(\mathcal{F}))^\sim$ are contained in $H_2(S_2^d\mathcal{M}_d)^\sim$. Hence, $\bar{u}_2 \in H_2(S_2^d\mathcal{M}_d)^\sim$.

It remains to show that

(4.14) $H_2(S_1\mathcal{M}_d)^\sim \subset H_2(S_1^d\mathcal{M}_d)^\sim$ and $H_2(S_2\mathcal{M}_d)^\sim \subset H_2(S_2^d\mathcal{M}_d)^\sim$.

We will prove the first inclusion and leave the proof of the second inclusion to the readers. Since $S_1\mathcal{M}_d$ is a $\mathbb{P}^1$-bundle over $X \times \mathcal{M}_{d-1}^0$ with projection $\pi$, by Kunneth decomposition

$$H_2(S_1\mathcal{M}_d) = \pi_*^{-1}(H_2(X \times \mathcal{M}_{d-1}^0))$$

$$= \pi_*^{-1}(H_2(M_{d-1}^0)) + \pi_*^{-1}(H_1(M_{d-1}^0) \otimes H_1(X)) + \pi_*^{-1}(H_2(X)),$n

where $\pi_* : H_2(S_1\mathcal{M}_d) \to H_2(X \times \mathcal{M}_{d-1}^0)$. Let $\pi_*^{-1}(\cdot)^\sim$ be the image in $H_2(\mathcal{M}_d)$ of the respective space. We claim that all of them are contained in $H_2(S_1^d\mathcal{M}_d)^\sim$. First, $\pi_*^{-1}(H_2(M_{d-1}^0)^\sim) = H_2(S_1^d\mathcal{M}_d)^\sim$ by definition. Next, we fix a ball $B \subset X$ containing $x$ and a compact $S \subset \mathcal{M}_{d-1}$ such that

(4.15) $\pi_*^{-1}(H_1(M_{d-1}^0) \otimes H_1(X)) \subset H_2(W_S)$,

where $W_S$ is the set of $E \in S_1\mathcal{M}_d$ such that for some $\mathcal{F} \in S, E \subset \mathcal{F}$ and $\text{supp}(\mathcal{F}/E) \cap B = \emptyset$. Because $H_1(S_1^d\mathcal{M}_{d-1}) \to H_1(\mathcal{M}_{d-1})$ is an isomorphism, by induction hypothesis we can replace $S$ by a set $S' \subset S_1^d\mathcal{M}_{d-1}$ and still have (4.15) with $S$ replaced by
Now we are ready to prove the main theorems. Let $H_0$ be any ample divisor, let $H_0 \in C \subseteq \text{NS}_Q^+$ be a precompact neighborhood of $H_0 \in \text{NS}_Q^+$ and let $N$ be the constant given by proposition 4.6 and the preceding propositions. Then for any $H \in C$ and $d \geq N$, $H_i(\mathfrak{M}_H(I, d)^0)$ is isomorphic to $H_i(\mathfrak{M}_{H_0}(I, d)^0)$ ($i \leq 2$ here and in the later discussion). However, for $d \geq N$ and $(H_0, I, d)$-suitable $H \in C$, by theorem 4.1

$$H_i(\mathfrak{M}_{H_0}(I, d)^0, \mathbb{Q}) \rightarrow H_i(\mathfrak{M}_{H_0}(I, d + 1)^0, \mathbb{Q})$$

is surjective. Therefore,

$$(4.16) \quad H_i(\mathfrak{M}_{H_0}(I, d)^0, \mathbb{Q}) \rightarrow H_i(\mathfrak{M}_{H_0}(I, d + 1)^0, \mathbb{Q})$$

is surjective for all $d \geq N$. Since $H_i(\mathfrak{M}_{H_0}(I, d)^0)$ are linear spaces, the system (4.16) has to stabilize at some stage. Namely, for some $N_1 \geq N$, (4.16) is an isomorphism for all $d \geq N_1$. Further, combined with the work of [Tà] (see (0.3) and (0.4)), we can find an $N_2$ so that for $d \geq N_2$,

$$H_i(\mathfrak{M}_{H_0}(I, d)^0, \mathbb{Q}) \cong H_i(\mathcal{B}(P_d)^*, \mathbb{Q}).$$

This proves theorem 0.1. For theorem 0.2, we simply apply the above isomorphism to the fact

$$\dim H_1(\mathcal{B}(P_d)^*) = b_1 \quad \text{and} \quad \dim H_2(\mathcal{B}(P_d)^*) = b_2 + \frac{1}{2} b_1 (b_1 - 1),$$

where $b_i = \dim H_i(X)$ (see page 181-182 of [DK]). Finally, by the proof of lemma 4.3, $h_1(\mathfrak{M}_{H_0}(I, d)) = h_1(\mathfrak{M}_{H_0}(I, d)^0)$, and $h_2(\mathfrak{M}_{H_0}(I, d)) = h_2(\mathfrak{M}_{H_0}(I, d)^0) + 1$ because $\mathfrak{M}_{H_0}(I, d) \subset \mathfrak{M}_{H_0}(I, d)$ is an irreducible Cartier divisor. This proves theorem 0.3.

In the remainder of this section, we shall prove lemma 4.5 promised earlier. First, we will introduce set similar to $\mathcal{Z}_2$ that is a desingularization of $V(F)$ as topological space. Let $\mathcal{E}_0$ be any sheaf and let $x \in X$ be fixed. We define $R_i$ be the set of all filtrations

$$(4.17) \quad \{\mathcal{E}_i \subset \mathcal{E}_{i-1} \subset \cdots \subset \mathcal{E}_0\}$$

such that $\mathcal{E}_j/\mathcal{E}_{j-1} \cong \mathbb{C}_x$. Let $\pi_{ij}: R_i \rightarrow R_j$, $i \geq j$, be the map sending (4.17) to the subfiltration $\{\mathcal{E}_j \subset \cdots \subset \mathcal{E}_0\}$ and let $p_i$ be the map sending (4.17) to $\mathcal{E}_i$. Obviously,
$R_1 \cong \mathbb{P}^1$ and for any $z \in R_1$, $\pi_2^{-1}(z) \cong \mathbb{P}^2$. Thus $R_2$ is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$. Similarly, because for each $\{E_2 \subset E_1 \subset E_0\} \in R_3$ the tensor product $E_2 \otimes \mathbb{C}_x = \mathbb{C}^\oplus 4$, the fibers of $\pi_{32} : R_3 \to R_2$ are isomorphic to $\mathbb{P}^3$. Hence $R_3$ is a $\mathbb{P}^3$-bundle over $R_2$ as topological spaces. Therefore, $R_2$ and $R_3$ are simply connected, $H_2(R_2) = \mathbb{Q}^\oplus 2$ and $H_2(R_3) = \mathbb{Q}^\oplus 3$.

Now we prove lemma 4.5. We first study the case where $F \in S_2 \mathcal{M}_d$ and the support of $F^\vee \vee / F$ is $x$. Let $\mathcal{E}_0 = F^\vee \vee$ and let $R_2$ be the set introduced based on $\mathcal{E}_0$ and $x$. Then $p_2$ maps $R_2$ onto $V(F)$. Let $w \in V(F)$ be the sheaf that is the kernel of $\mathcal{E}_0 : \mathbb{C}^\oplus 2$. Then $p_2$ is one-to-one away from $Z = p_2^{-1}(w)$ and $Z \cong \mathbb{P}^1$. Hence $H_1(R_2) \to H_1(V(F))$ is an isomorphism and $p_{2*} : H_2(R_2) \to H_2(V(F))$ is surjective whose kernel contains $[Z]$. Thus $\dim H_2(V(F)) \leq 1$. On the other hand, $H_2(V_0)^-$ is one dimensional and is obviously contained in $H_2(V(F))^\sim$. Therefore, $H_2(V(F))^\sim = H_2(V_0)^\sim$. The case $F \in S_2 \mathcal{M}_d$ can be checked similarly. This completes the proof of lemma 4.5 for $j = 2$.

Now we study the case where $F \in S_3 \mathcal{M}_d$. We will deal with the case where $\text{supp}(F^\vee \vee / F) = \{x\}$ and leave the other cases to the readers. Let $\mathcal{E}_0 = F^\vee \vee$ and let $R_3$ be as before. Then $p_3_1 : R_3 \to V(F)$ is surjective induces surjective homomorphisms $H_1(R_3) \to H_1(V(F))$ and $p_{3*} : H_2(R_3) \to H_2(V(F))$. Since $H_2(R_3) = \mathbb{Q}^\oplus 3$, as before to prove the lemma it suffices to show that $\ker(p_{3*})$ is two dimensional. We will prove this by construct two generators of $\ker(p_{3*})$ explicitly. First, let $\mathcal{E}_2'$ be the kernel of $\mathcal{E}_0 \to \mathbb{C}^\oplus 2$ and let $\mathcal{E}_3' \subset \mathcal{E}_2'$ be any sheaf with $\mathcal{E}_3' / \mathcal{E}_3 \cong \mathbb{C}_x$. We define $Z_1$ be the subset of $R_3$ consists of filtrations of type

$$\{\mathcal{E}_3' \subset \mathcal{E}_2' \subset \mathcal{E}_1 \subset \mathcal{E}_0\}.$$

$Z_1$ is isomorphic to $\mathbb{P}^1$ and $[Z_1] \neq 0 \in H_2(R_3)$ belongs to ker($p_{3*}$). Secondly, we let $w \in V(F)$ be a sheaf that is isomorphic to $O_x \oplus (u_1^2, u_1 u_2, u_2^2)O_x$ at $x$, where $(u_1, u_2)$ is an analytic coordinate of $x \in X$. Then $Z_2 = p_3^{-1}(w)$ is again a projective line that generates a nontrivial kernel of $p_{2*}$. Since $\pi_{31}(Z_2)$ is a point and $\pi_{31}(Z_1) = R_1$, $[Z_1]$ and $[Z_2]$ generates two dimensional subspace in $H_2(R_3)$. Therefore $\dim H_2(V(F)) \leq 1$. Finally, because $Q = H_2(V_0)^\sim \subset H_2(V(F))^\sim$, they must be identical. This proves the lemma 4.5.

References

[Ar] Artin, M., Algebraic approximation of structures over complete local rings, Publ. Math. IHES. 36 (1969), 23-58.

[AJ] Atiyah, M.F. and Jones, J.D., Topological aspects of Yong-Mills theory, Comm. Math. Phys. 61 (1978), 97-118.

[Be] Beauville, A., preprint (1993).

[BHM²] Boyer, C.P., Hurtubise, J.C., Mann, B.M. and Milgram, R.J., The topology of instanton moduli spaces. I: The Atiyah-Jones conjecture, Ann of Math 137 (1993), 561-609.
Donaldson, S.K., *Polynomial invariants for smooth four-manifolds*, Topology 29 No. 3 (1986), 257-315.

Donaldson, S.K. and Kronheimer, P.B., *The Geometry of Four-Manifolds*, Oxford Mathematical Monographs, Oxford Science Publications, 1990.

Drezet, J.M. and Narasimhan, S., *Group de Picard des varietes de modules de fibres semi-stables sur les coubes algebriques*, Invent. Math. 97 (1989), 53-94.

Ellingsrud, G. and Stromme, S.A., *preprint* (1993).

Friedman, R., *Vector bundles over surfaces*, to be published.

Gieseker, D. and Li, J., *Irreducibility of moduli of rank two vector bundles*, J. Diff. Geom. 40 (1994), 23-104.

Goresky, M. and MacPherson, R., *Stratified Morse Theory*, vol. 3 Band 14, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, 1988.

Hamm, H.A., *Lefschetz theorems for singular varieties*, Proceeding of Symposia in Pure Mathematics 40 (1983), 547-557.

Hurtubise, J.C. and Milgram, R.J., *The Atiyah-Jones conjecture for ruled surfaces*, preprint (1994).

Kirwan, F., *Geometric invariant theory and Atiyah-Jones conjecture*, preprint (1993).

Lazarsfeld, R., *Some applications of the theory of positive vector bundles*, Lecture Notes in Mathematics 1092 (1984), 29-61.

Li, J., *Algebraic geometric interpretation of Donaldson’s polynomial invariants*, J. of Diff. Geometry 37 (1993), 417-466.

Li, J., *Kodaira dimension of moduli space of vector bundles on surfaces*, Invent. Math. 115 (1994), 1-40.

Li, J., *The Picard group of moduli space of stable sheaves over algebraic surfaces*, to appear in Taniguchi Symposium on Mathematics.

Maruyama, M., *Moduli of stable sheaves*, II, J. Math. Kyoto Univ. 18-3 (1978), 557-614.

Mumford, D., *Towards an enumerative geometry of the moduli space of curves*, Progress in Mathematics (1983), Birkhauser Boston, Inc..

O’Grady, K., *The irreducible components of moduli spaces of vector bundles on surfaces*, Invent. Math. 112 (1993), 586-613.

O’Grady, K., *Moduli of vector bundles on projective surfaces: some basic results*, preprint.

Qin, Z., *Birational properties of moduli spaces of stable locally free rank-2 sheaves on algebraic surfaces*, Manuscripta Math., 72 (1991), 163-180.

Taubes, C., *The stable topology of self-dual moduli spaces*, J. Diff. Geom. 19 (1984), 337-392.

Tian, Y-L., *The based SU(n)-instanton moduli spaces*, Math. Ann. 298 (1994), 117-139.

Tian, Y-L., *The Atiyah-Jones conjecture for classical groups*, preprint (1994).

Yoshioka, K., *The Betti numbers of the moduli space of stable sheaves of rank 2 on $\mathbb{P}^2$*, J. reine angew. Math. 453 (1994), 193-220.

Zhu, K., *Generic smoothness of the moduli of rank two stable bundles over an algebraic surface*, Math. Z. 207 No. 4 (1991), 629-643.