BORDERED FLOER HOMOLOGY FOR SUTURED MANIFOLDS

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Abstract. We define a sutured cobordism category of surfaces with boundary and 3-manifolds with corners. In this category a sutured 3-manifold is regarded as a morphism from the empty surface to itself. In the process we define a new class of geometric objects, called bordered sutured manifolds, that generalize both sutured 3-manifolds and bordered 3-manifolds. We extend the definition of bordered Floer homology to these objects, giving a functor from a decorated version of the sutured category to $A_{\infty}$-algebras, and $A_{\infty}$-bimodules.

As an application we give a way to recover the sutured homology $SFH(Y, \Gamma)$ of a sutured manifold from either of the bordered invariants $\hat{CF}A(Y)$ and $\hat{CFD}(Y)$ of its underlying manifold $Y$. A further application is a new proof of the surface decomposition formula of Juhász.

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1. Introduction

There are currently two existing Heegaard Floer theories for 3–manifolds with boundary, both of which require extra structure on the 3–manifold.

The first such theory was developed by Juhász in [5]. It is defined for balanced sutured manifolds, which are 3–manifolds with certain decorations on the boundary. While very fruitful, the theory has the disadvantage that the homology $SFH(Y, \Gamma)$ of a sutured manifold $Y$, with sutures $\Gamma$ tells us nothing about the homology $SFH(Y, \Gamma')$ of the same manifold with a different set of sutures $\Gamma'$. Moreover, if a closed 3–manifold $Y$ is obtained by gluing two manifolds with boundary $Y_1$ and $Y_2$, then the sutured Floer invariants $SFH(Y_1, \Gamma_1)$ and $SFH(Y_2, \Gamma_2)$ do not determine the closed Heegaard Floer invariant $\widehat{HF}(Y)$.

The second and more recent theory was developed by Lipshitz, Ozsváth, and Thurston in [8]. It takes a more algebraically complicated form than sutured Floer homology. To a parametrized connected surface $F$ it assigns a differential graded algebra $A(F)$, and to a 3–manifold $Y$ whose boundary is identified with $F$ it assigns two different modules over $A(F)$—an $A_\infty$–module $\widehat{CFA}(Y)$, and a differential graded module $\widehat{CFD}(Y)$. These modules are well-defined up to homotopy equivalence, and have the property that $\widehat{HF}(Y_1 \cup Y_2)$ is the homology of $\widehat{CFA}(Y_1) \otimes \widehat{CFA}(Y_2)$. While the bordered invariants of a manifold $Y$ still depend on extra structure—in this case the parametrization of $\partial Y$—the invariant of one parametrization can be obtained from that of any other.

In the present work we relate these two invariants by a common generalization. We define a new set of topological objects—bordered sutured manifolds, sutured surfaces, and a sutured cobordism category—and define several invariants for them. We show that these invariants interpolate between sutured Floer homology and bordered Floer homology, containing each of them as special cases. An advantage of the new invariants is that they are well-defined and nontrivial even for unbalanced manifolds, unlike $SFH$.

The aim of the present work is to define this common generalization. As a byproduct, we show how to obtain the sutured invariant $SFH(Y, \Gamma)$ of a manifold with connected boundary from either of its bordered invariants $\widehat{CFD}(Y)$ and $\widehat{CFA}(Y)$. Finally, we give a new proof of the sutured decomposition formula of Juhász.

1.1. The sutured and decorated sutured categories. A simplified description of a sutured manifold is given below. A more precise version is given in section 3.

Definition 1.1. A sutured 3–manifold $(Y, \Gamma)$ is a 3–manifold $Y$, with a multi-curve $\Gamma$ on its boundary, dividing the boundary into a positive and negative region, denoted $R_+$ and $R_-$, respectively. We usually impose the
conditions that $Y$ has no closed components, and that $\Gamma$ intersects every component of $\partial Y$.

We can introduce analogous notions one dimension lower.

**Definition 1.2.** A **sutured surface** $(F, \Lambda)$ is a surface $F$, with a 0–manifold $\Lambda \subset \partial F$, diving the boundary $\partial F$ into a positive and negative region, denoted $S_+$ and $S_-$, respectively. Again, we impose the condition that $F$ has no closed components, and that $\Lambda$ intersects every component of $\partial F$.

**Definition 1.3.** A **sutured cobordism** $(Y, \Gamma)$ between two sutured surfaces $(F_1, \Lambda_1)$ and $(F_2, \Lambda_2)$ is a cobordism $Y$ between $F_1$ and $F_2$, together with a collection of properly embedded arcs and circles $\Gamma \subset \partial Y \setminus (F_1 \cup F_2)$, dividing $\partial Y \setminus (F_1 \cup F_2)$ into a positive and negative region, denoted $R_+$ and $R_-$, respectively, such that $R_\pm \cap F_i = S_\pm(F_i)$, for $i = 1, 2$. Again, we require that $Y$ has no closed components, and that $\Gamma$ intersects every component of $\partial Y \setminus (F_1 \cup F_2)$.

There is a **sutured category** $\mathcal{S}$ whose objects are sutured surfaces, and whose morphisms are sutured cobordisms. The identity morphisms are cobordisms of the form $(F \times [0, 1], \Lambda \times [0, 1])$, where $(F, \Lambda)$ is a sutured surface. As a special case, sutured manifolds are the morphisms from the empty surface $(\emptyset, \emptyset)$ to itself.

Unfortunately, we cannot directly define invariants for the sutured category, and we need impose a little extra structure.

**Definition 1.4.** An **arc diagram** is a relative handle diagram for a 2–manifold with corners, where the bottom and top boundaries are both 1–manifolds with no closed components.

**Definition 1.5.** A **parametrized** or **decorated sutured surface** is a sutured surface $(F, \Lambda)$ with a handle decomposition given by an arc diagram $Z$, expressing $F$ as a cobordism from $S_+$ to $S_-$. A **parametrized** or **decorated sutured cobordism** is a sutured cobordism $(Y, \Gamma)$ from $(F_1, \Lambda_1)$ to $(F_2, \Lambda_2)$, such that $(F_i, \Lambda_i)$ is parametrized by an arc diagram $Z_i$, for $i = 1, 2$.

Examples of a sutured surface and its decorated version are given in Fig. 1. A sutured cobordism and its decorated version are given in Fig. 2. We visualize the handle decomposition coming from an arc diagram by drawing the cores of the 1–handles. The **decorated sutured category** $\mathcal{SD}$ is a category whose objects are decorated sutured surfaces—or alternatively their arc diagrams—and whose morphisms are decorated sutured cobordisms. Note that all decorations on the sutured identity $(F \times [0, 1], \Lambda \times [0, 1])$ are isomorphisms, while the ones where the two parametrizations on $F \times \{0\}$ and $F \times \{1\}$ agree are the identity morphisms in $\mathcal{SD}$. In particular, any two parametrizations of the same
sutured surface are isomorphic, and the forgetful functor $Z \mapsto F(Z)$ is an equivalence of categories.

Sutured cobordisms have another, slightly different topological interpretation. For a sutured cobordism $(Y, \Gamma)$ from $(F_1, \Lambda_1)$ to $(F_2, \Lambda_2)$, we can smooth its corners, and set $\Gamma' = \Gamma \cup S_+(F_1) \cup S_+(F_2)$. This turns $(Y, \Gamma')$ into a regular sutured manifold. Therefore, we can think of a sutured cobordism as a sutured manifold, with two distinguished subsets $F_1$ and $F_2$ of its boundary.
Applying the same procedure to the decorated versions of sutured cobordisms, we come up with the notion of \textit{bordered sutured manifolds}, defined more precisely in section 3.

\textbf{Definition 1.6.} A \textit{bordered sutured manifold} \((Y, \Gamma, Z)\) is a sutured manifold \((Y, \Gamma)\), with a distinguished subset \(F \subset \partial Y\), such that \((F, \partial F \cap \Gamma)\) is a sutured surface, parametrized by the arc diagram \(Z\).

Any bordered sutured manifold \((Y, \Gamma, Z_1 \cup Z_2)\), where \(Z_i\) parametrizes \((F_i, \partial F_i \cap \Gamma)\) gives a decorated sutured cobordism \((Y, \Gamma \setminus (F_1 \cup F_2))\) from \(-F_1\) to \(F_2\), and vice versa.

1.2. \textbf{Bordered sutured invariants and TQFT.} To any arc diagram \(Z\)—or alternatively decorated sutured surface parametrized by \(Z\)—we associate a differential graded algebra \(\mathcal{A}(Z)\), which is a subalgebra of some strand algebra, as defined in [8].

These algebras behave nicely under disjoint union. If \(Z_1\) and \(Z_2\) are arc diagrams, then \(\mathcal{A}(Z_1 \cup Z_2) \cong \mathcal{A}(Z_1) \otimes \mathcal{A}(Z_2)\).

To a bordered sutured manifold \((Y, \Gamma, Z)\) we associate a right \(\mathcal{A}_\infty\)-module \(\widehat{\mathcal{BSA}}(Y, \Gamma)\) over \(\mathcal{A}(Z)\), and a left differential graded module \(\widehat{\mathcal{BSD}}(Y, \Gamma)\) over \(\mathcal{A}(Z)\).

Generalizing this construction, let \((F_1, \Lambda_1)\) and \((F_2, \Lambda_2)\) be two sutured surfaces, parametrized by the arc diagrams \(Z_1\) and \(Z_2\), respectively. To any sutured cobordism \((Y, \Gamma)\) between them we associate (a homotopy equivalence class of) an \(\mathcal{A}_\infty\)-algebra \(\mathcal{A}(Z_1) \otimes \mathcal{A}(Z_2)\)-bimodule, denoted \(\widehat{\mathcal{BSDA}}(Y, \Gamma)\). This specializes to \(\widehat{\mathcal{BSA}}(Y, \Gamma)\), respectively \(\widehat{\mathcal{BSD}}(Y, \Gamma)\), when \(F_1\), respectively \(F_2\), is empty, or to the sutured chain complex \(\mathcal{SFC}(Y, \Gamma)\), when both are empty.

\textbf{Definition 1.7.} Let \(\mathcal{D}\) be the category whose objects are differential graded algebras, and whose morphisms are the graded homotopy equivalence classes of \(\mathcal{A}_\infty\)-bimodules of any two such algebras. Composition is given by the derived tensor product \(\widehat{\otimes}\). The identity is the homotopy equivalence class of the algebra considered as a bimodule over itself.

\textbf{Theorem 1.} The invariant \(\widehat{\mathcal{BSDA}}\) respects compositions of decorated sutured cobordisms. Explicitly, let \((Y_1, \Gamma_1, -Z_1 \cup Z_2)\) and \((Y_2, \Gamma_2, -Z_2 \cup Z_3)\) be two bordered sutured manifolds, representing decorated sutured cobordisms from \(Z_1\) to \(Z_2\), and from \(Z_2\) to \(Z_3\), respectively. Then there are graded homotopy equivalences

\[(1) \quad \widehat{\mathcal{BSDA}}(Y_1, \Gamma_1) \widehat{\otimes}_{\mathcal{A}(Z_2)} \widehat{\mathcal{BSDA}}(Y_2, \Gamma_2) \cong \widehat{\mathcal{BSDA}}(Y_1 \cup Y_2, \Gamma_1 \cup \Gamma_2).\]

Specializing to \(Z_1 = Z_3 = \emptyset\), we get

\[(2) \quad \widehat{\mathcal{BSA}}(Y_1, \Gamma_1) \widehat{\otimes}_{\mathcal{A}(Z_2)} \widehat{\mathcal{BSD}}(Y_2, \Gamma_2) \cong \mathcal{SFC}(Y_1 \cup Y_2, \Gamma_1 \cup \Gamma_2).\]

\textbf{Theorem 2.} The invariant \(\widehat{\mathcal{BSDA}}\) respects the identity. In other words, if \((Y, \Gamma, -Z \cup Z)\) is the identity cobordism from \(Z\) to itself, then \(\widehat{\mathcal{BSDA}}(Y, \Gamma)\) is graded homotopy equivalent to \(\mathcal{A}(Z)\) as an \(\mathcal{A}_\infty\)-bimodule over itself.
Together, theorems 1 and 2 imply that $\mathcal{A}$ and $\widehat{BSDA}_M$ form a functor.

**Corollary 3.** The invariants $\mathcal{A}$ and $\widehat{BSDA}_M$ give a functor from $SD$ to $D$, inducing a (non-unique) functor from the equivalent category $S$ to $D$. In particular, if $Z_1$ and $Z_2$ parametrize the same sutured surface, then $\mathcal{A}(Z_1)$ and $\mathcal{A}(Z_2)$ are isomorphic in $D$. In other words, there is an $\mathcal{A}(Z_1), \mathcal{A}(Z_2)$ $\mathcal{A}_\infty$–bimodule providing an equivalence of the derived categories of $\mathcal{A}_\infty$–modules over $\mathcal{A}(Z_1)$ and $\mathcal{A}(Z_2)$.

**1.3. Applications.** The first application, which motivated the development of the new invariants, is to show that $SFH(Y, \Gamma)$ can be computed from $\widehat{CFA}(Y)$ or $\widehat{CFD}(Y)$.

**Theorem 4.** Suppose $Y$ is a connected 3–manifold with connected boundary. With any set of sutures $\Gamma$ on $\partial Y$ we can associate modules $\widehat{CFA}(\Gamma)$ and $\widehat{CFD}(\Gamma)$ over $\mathcal{A}(\pm \partial Y)$, of the appropriate form, such that the following formula holds.

$$SFH(Y, \Gamma) \cong H_*(\widehat{CFA}(Y) \otimes \widehat{CFD}(\Gamma)) \cong H_*(\widehat{CFA}(\Gamma) \otimes \widehat{CFD}(Y)).$$

We prove a somewhat stronger version of theorem 4 in section 10.3.

Another application of the invariants is a new proof of the surface decomposition formula for $SFH$.

**Theorem 5.** Suppose $(Y, \Gamma)$ is a balanced sutured manifold, and $S$ is a good decomposing surface, i.e. $S$ has no closed components, and any component of $\partial S$ intersects $\Gamma$. If $S$ decomposes $(Y, \Gamma)$ to $(Y', \Gamma')$, then for a certain subset $O$ of relative Spin$^c$ structures on $(Y, \partial Y)$, called outer for $S$, the following equality holds.

$$SFH(Y', \Gamma') \cong \bigoplus_{s \in O} SFH(Y, \Gamma, s).$$

The proofs of both statements rely on the fact that by splitting a sutured manifold into bordered sutured pieces we can localize the calculations.

For theorem 4, we localize the suture information, by considering a (punctured) collar neighborhood of $\partial Y$, which knows about the sutures, but not about the 3–manifold, and the complement, which knows about the 3–manifold, but not about the sutures.

For theorem 5, we can localize near the decomposing surface $S$. An easy calculation shows that an equivalent formula holds for a neighborhood of $S$. The local formula then implies the global one.

**1.4. Further questions.** There is currently work in progress [12] to provide a general gluing formula for sutured manifolds, generalizing theorem 5. There is strong evidence that the theory is closely related to contact topology, and to the gluing maps defined by Honda, Kazez and Matić in [3], and the contact category and TQFT defined in [4]. There are also interesting parallels to the work on chord diagrams by Mathews in [9].
Another area of interest is a generalization, to allow a wider class of arc diagrams and Heegaard diagrams. Currently we consider diagrams coming from splitting a sutured manifold along a surface containing index–1 critical points, but no index–2 critical points. This corresponds to cutting a sutured Heegaard diagram along arcs which intersect some $\alpha$ circles, but no $\beta$ circles. It would be of interest to consider what happens if allow index–2 critical points, or equivalently cutting a diagram along an arc that intersects some $\beta$ circles.

Extending the bordered sutured theory to this setting would require arc diagrams that contain two types of arcs, $+$ and $-$, such that $S_+$ is the result of surgery on the $+$ arcs in $Z$, while $-$ is the result of surgery on $-$ arcs. There seem to be two distinct levels of generalization. If we allow some components of $Z$ to have only $+$ arcs, and the rest of the components to have only $-$ arcs, the theory should be mostly unchanged. If however we allow the same components to have a mix of $+$ and $-$ arcs, a qualitatively different approach is required.

**Organization.** The first few sections are devoted to the topological constructions. First, in section 2 we define arc diagrams, and how they parametrize sutured surfaces, as well as the $A_\infty$–algebra associated to an arc diagram. In section 3 we define bordered sutured manifolds, and in section 4 we define the Heegaard diagrams associated to them.

The next few sections define the invariants and give their properties. In section 5 we talk about the moduli spaces of curves necessary for the definitions of the invariants. In section 7 we give the definitions of the bordered sutured invariants $\widehat{BSD}_M$ and $\widehat{BSA}$, and prove Eq. (2) from theorem 1. In section 8 we extend the definitions and properties to the bimodules $\widehat{BSDA}_M$, and sketch the proof of the rest of theorem 1, as well as theorem 2. The gradings are defined together for all three invariants on the diagram level in section 6.

A lot of the material in these sections is a reiteration of analogous constructions and definitions from [8], with the differences emphasized. The reader who is encountering bordered Floer homology for the first time can skip most of that discussion on the first reading, and use theorems 7.14, 7.15 and 8.8 as definitions.

Section 9 gives some examples of bordered sutured manifolds and computations of their invariants. The reader is encouraged to read this section first, or immediately after section 4. The examples can be more enlightening than the definitions, which are rather involved.

Finally, section 10 gives several applications of the new invariants, in particular proving theorems 4 and 5.

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2. The algebra associated to a parametrized surface

The invariants defined by Lipshitz, Ozsváth and Thurston in [8] work only for connected manifolds with one closed boundary component. There is work in progress [7] to generalize this to manifolds with two or more closed boundary components.

In our construction we parametrize surfaces with boundary, and possibly many connected components. This class of surfaces and of their allowed parametrizations is much wider, so we need to expand the algebraic constructions describing them. We discuss below the generalized definitions and discuss the differences from the purely bordered setting.

2.1. Arc diagrams and sutured surfaces. We start by generalizing the definition of a pointed matched circle in [8].

Definition 2.1. An arc diagram $Z = (Z, a, M)$ is a triple consisting of a collection $Z = \{Z_1, \ldots, Z_l\}$ of oriented line segments, a collection $a = \{a_1, \ldots, a_{2k}\}$ of distinct points in $Z$, and a matching of $a$, i.e. a 2–to–1 function $M: a \rightarrow \{1, \ldots, k\}$. Write $|Z_i|$ for $\#(Z_i \cap a)$. We will assume $a$ is ordered by the order on $Z$ and the orientations of the individual segments. We allow $l$ or $k$ to be 0.

We impose the following condition, called non-degeneracy. After performing oriented surgery on the 1–manifold $Z$ at each 0–sphere $M^{-1}(i)$, the resulting 1–manifold should have no closed components.

Definition 2.2. We can sometimes consider degenerate arc diagrams which do not satisfy the non-degeneracy condition. However, we will tacitly assume all arc diagrams are non-degenerate, unless we specifically say otherwise.

Remark. The pointed matched circles of Lipshitz, Ozsváth and Thurston correspond to arc diagrams where $Z$ has only one component. The arc diagram is obtained by cutting the matched circle at the basepoint.

We can interpret $Z$ as an upside-down handlebody diagram for a sutured surface $F(Z)$, or just $F$. It will often be convenient to think of $F$ as a surface with corners, and will use these descriptions interchangeably.

To construct $F$ we start with a collection of rectangles $Z_i \times [0, 1]$ for $i = 1, \ldots, l$. Then attach 1–handles at $M^{-1}(i) \times \{0\}$ for $i = 1, \ldots, k$. Thus $\chi(F) = l - k$, and $F$ has no closed components. Set $\Lambda = \partial Z \times \{1/2\}$, and $S_+ = Z \times \{1\} \cup \partial Z \times [1/2, 1]$. Such a description uniquely specifies $F$ up to isotopy fixing the boundary.

Remark. The non-degeneracy condition on $Z$, is equivalent to the condition that any component of $\partial F$ intersects $\Lambda$. Indeed, the effect on the boundary of adding the 1–handles is surgery on $Z \times \{0\}$. If $Z$ is non-degenerate,
Figure 3. Arc diagram for an annulus, and three different views of parametrization.

this surgery produces no new closed components, and $F$ is indeed a sutured surface.

Alternatively, instead of a handle decomposition we can consider a Morse function on $F$. Whenever we talk about Morse functions, a (fixed) choice of Riemannian metric is implicit.

Definition 2.3. A $\mathcal{Z}$–compatible Morse function on $F$ is a self-indexing Morse function $f: F \to [-1, 4]$, such that the following conditions hold. There are no index–0 or index–2 critical points. There are exactly $k$ index–1 critical points and they are all interior. The gradient of $f$ is tangent to $\partial F \setminus f^{-1}([-1, 1])$. The preimage $f^{-1}([-1, -1/2])$ is isotopic to a collection of rectangles $[0, 1] \times [-1, -1/2]$ such that $f$ is projection on the second factor. Similarly, $f^{-1}([3/2, 4])$ is isotopic to a collection of rectangles $[0, 1] \times [3/2, 4]$ such that $f$ is projection on the second factor.

Furthermore, we can identify $f^{-1}([3/2])$ with $\mathcal{Z}$ such that the unstable manifolds of the $i$–th index–1 critical point intersect $\mathcal{Z}$ at $M^{-1}(i)$. We require that the orientation of $\mathcal{Z}$ and $\nabla f$ form a positive basis everywhere.

Clearly, a compatible Morse function and a handle decomposition as above are equivalent. Examples of an arc diagram, and the different ways we can interpret its parametrization of a sutured surface, are given in Fig. 3. A slightly more complicated example of an arc diagram, corresponding to the parametrization in Fig. 1b, is given in Fig. 4.

There is one more way to describe the above parametrization. Recall that a ribbon graph is a graph with a cyclic ordering of the edges incident to any
Figure 4. An arc diagram $Z$ for a twice punctured torus, and its graph $G(Z)$.

vertex. An embedding of a ribbon graph into a surface will be considered orientation preserving if the ordering of the edges agrees with the positive direction on the unit tangent circle of the vertex in the surface.

**Definition 2.4.** Let $F$ be a sutured surface obtained from an arc diagram $Z$ as above. The *ribbon graph associated to* $Z$ is the ribbon graph $G(Z)$ with vertices $\partial Z \cup \mathbf{a}$, and edges the components of $Z \setminus \mathbf{a}$ and the cores of the 1–handles, which we denote $e_i$ for $i = 1, \ldots, k$. The cyclic ordering is induced from the orientation of $F$.

In these terms, $F$ is parametrized by $Z$ if we specify an orientation preserving proper embedding $G(Z) \hookrightarrow F$, such that $F$ deformation retracts onto the image.

**Remark.** When we draw an arc diagram $Z$ we are in fact drawing its graph $G(Z)$. An example, with all elements of the graph denoted, is given in Fig. 4.

### 2.2. The algebra associated to an arc diagram.

Recall the definition of the strands algebra from [8].

**Definition 2.5.** The *strands algebra* $\mathcal{A}(n, k)$ is a free $\mathbb{Z}/2$–module with generators of the form $\mu = (T, S, \phi)$, where $S$ and $T$ are $k$–element subsets of $\{1, \ldots, n\}$, and $\phi: S \to T$ is a non–decreasing bijection. (We think of $\phi$ as a collection of strands from $S$ to $T$.) Denote by $\text{inv}(\mu) = \text{inv}(\phi)$ the number of inversions of $\phi$, i.e. the elements of $\text{Inv}(\mu) = \{(i, j) : i, j \in S, i < j, \phi(i) > \phi(j)\}$.

Multiplication is given by

$$
(S, T, \phi) \cdot (U, V, \psi) = \begin{cases} 
(S, V, \psi \circ \phi) & \text{if } T = U, \ \text{inv}(\phi) + \text{inv}(\psi) = \text{inv}(\psi \circ \phi), \\
0 & \text{otherwise}.
\end{cases}
$$

The differential on $(S, T, \phi)$ is given by the sum of all possible ways to “resolve” an inversion, i.e. switch $\phi(i)$ and $\phi(j)$ for some inversion $(i, j) \in \text{Inv}(\mu)$. 

Next, we consider the larger extended strands algebra
\[ A(n_1, \ldots, n_l; k) = \bigoplus_{k_1 + \cdots + k_l = k} A(n_1, k_1) \otimes \cdots \otimes A(n_l, k_l). \]

We will slightly abuse notation and think of elements of \( A(n_i, k_i) \) as functions acting on subsets of \( \{(n_1 + \cdots + n_{i-1}) + 1, \ldots, (n_1 + \cdots + n_{i-1}) + n_i\} \) instead of \( \{1, \ldots, n_i\} \). This allows us to identify \( A(n_1, \ldots, n_l; k) \) with a subalgebra of \( A(n_1 + \cdots + n_l, k) \).

We will sometimes talk about the sums \( A(n) = A(n, 0) \oplus \cdots \oplus A(n, n) \), and \( A(n_1, \ldots, n_l) = A(n_1, \ldots, n_l; 0) \oplus \cdots \oplus A(n_1, \ldots, n_l; n_1 + \cdots + n_l) \).

The definition of \( \mathcal{A}(Z, i) \) as a subalgebra of \( \mathcal{A}([Z_1, \ldots, Z_l]; i) \) below is a straightforward generalization of the definition of the algebra associated to a pointed matched circle in [8]. There is, however, a difference in notation. In [8] \( \mathcal{A}(Z, 0) \) denotes the middle summand and negative summand indices are allowed. Here, \( \mathcal{A}(Z, 0) \) is the bottom summand, and we only allow non-negative indices.

For any \( i \)-element subset \( S \subset \{1, \ldots, 2k\} \), there is an idempotent \( I(S) = (S, S, \text{id}_S) \in \mathcal{A}([Z_1, \ldots, Z_l], i) \). For an \( i \)-element subset \( s \subset \{1, \ldots, k\} \), a section \( S \) of \( s \) is an \( i \)-element set \( S \subset M^{-1}(s) \), such that \( M|s \) is injective. To each \( S \) there is an associated idempotent
\[ I(S) = \sum_{S \text{ is a section of } s} I(S). \]

Consider triples of the form \((S, T, \psi)\), where \( S, T \subset \{1, \ldots, 2k\}, \psi \colon S \to T \) is a strictly increasing bijection. Consider all possible sets \( U \subset \{1, \ldots, 2k\} \) disjoint from \( S \) and \( T \), and such that \( S \cup U \) has \( i \) elements. Let
\[ a_i(S, T, \psi) = \sum_{U \text{ as above}} (S \cup U, T \cup U, \psi_U) \in \mathcal{A}([Z_1, \ldots, Z_l]; i), \]
where \( \psi_U|_T = \psi \), and \( \psi_U|_U = \text{id}_U \). In the language of strands, this means “to a set of moving strands add all possible consistent collections of stationary (or horizontal) strands”.

Let \( \mathcal{I}(Z, i) \) be the subalgebra generated by \( I(S) \) for all \( i \)-element sets \( s \), and let \( I = \sum_i I(S) \) be their sum. Let \( \mathcal{A}(Z, i) \) be the subalgebra generated by \( \mathcal{I}(Z, i) \) and all elements of the form \( I \cdot a_i(S, T, \psi) \cdot I \).

All elements \((S, T, \phi)\) considered have the property that \( M|S \) and \( M|T \) are injective.

**Definition 2.6.** The algebra associated with the arc diagram \( Z \) is
\[ \mathcal{A}(Z) = \bigoplus_{i=0}^{k} \mathcal{A}(Z, i), \]
which is a module over
\[ \mathcal{I}(Z) = \bigoplus_{i=0}^{k} \mathcal{I}(Z, i). \]
To any element of $\mu = (S, T, \phi) \in \mathcal{A}(|Z_1|, \ldots, |Z_l|)$ we can associate its homology class $[\mu] \in H_1(Z, a)$, by setting

$$[\mu] = \sum_{i \in S} [l_i],$$

where $l_i$ is the positively oriented segment $[a_i, a_{\phi(i)}] \subset Z$. It is additive under multiplication and preserved by the differential on $\mathcal{A}(|Z_1|, \ldots, |Z_l|)$. Since it only depends on the moving strands of $(S, T, \phi)$, any element of $\mathcal{A}(Z)$ is homogeneous with respect to this homology class, and therefore we can talk about the homology class of an element in $\mathcal{A}(Z)$.

**Remark.** With a collection $Z_1, \ldots, Z_p$ of arc diagrams we can associate their union $Z = Z_1 \cup \cdots \cup Z_p$, where $Z = Z_1 \sqcup \cdots \sqcup Z_p$, preserving the matching on each piece.

There are natural identifications, of algebras

$$\mathcal{A}(Z) = \bigotimes_{i=1}^p \mathcal{A}(Z_i),$$

and of surfaces

$$F(Z) = \bigsqcup_{i=1}^p F(Z_i).$$

**2.3. Reeb chord description.** We give an alternative interpretation of the strands algebra $\mathcal{A}(Z)$.

Given an arc diagram $Z$ with $k$ arcs, there is a unique positively oriented contact structure on the 1–manifold $Z$, while the 0–manifold $a \subset Z$ is Legendrian. There is a family of Reeb chords in $Z$, starting and ending at $a$ and positively oriented. For a Reeb chord $\rho$ we will denote its starting and ending point by $\rho^-\rho^+$, respectively. Moreover, for a collection $\rho = \{\rho_1, \ldots, \rho_n\}$ of Reeb chords as above, we will write $\rho^- = \{\rho_1^-, \ldots, \rho_n^-\}$, and $\rho^+ = \{\rho_1^+, \ldots, \rho_n^+\}$.

**Definition 2.7.** A collection $\rho = \{\rho_1, \ldots, \rho_n\}$ of Reeb chords is $p$–completable if the following conditions hold:

1. $\rho_i^- \neq \rho_i^+$ for all $i = 1, \ldots, n$.
2. $M(\rho_1^-), \ldots, M(\rho_n^-)$ are all distinct.
3. $M(\rho_1^+), \ldots, M(\rho_n^+)$ are all distinct.
4. $\#(M(\rho^-) \cup M(\rho^+)) \leq k - (p - n)$.

Condition (4) guarantees that there is at least one choice of a $(p - n)$–element set $s \subset \{1, \ldots, k\}$, disjoint from $M(\rho^-)$ and $M(\rho^+)$. Such a set is called a $p$–completion or just completion of $\rho$. Every completion of $\rho$ defines an element of $\mathcal{A}(Z, p)$.
Definition 2.8. For a $p$–completable collection $\rho$ and a completion $s$, their associated element in $\mathcal{A}(\mathbb{Z}, p)$ is
\[
a(\rho, s) = \sum_{S \text{ is a section of } s} (\rho^- \cup S, \rho^+ \cup S, \phi_S),
\]
where $\phi_S(\rho_i^-) = \rho_i^+$, for $i = 1, \ldots, n$, and $\phi|_S = \text{id}_S$.

Definition 2.9. The associated element of $\rho$ in $\mathcal{A}(\mathbb{Z}, p)$ is the sum over all $p$–completions:
\[
a_p(\rho) = \sum_{s \text{ is a } p\text{-completion of } \rho} a(\rho, s).
\]
If $\rho$ is not $p$–completable, we will just set $a_p(\rho) = 0$. We will also sometimes use the complete sum
\[
a(\rho) = \sum_{p=0}^k a_p(\rho).
\]

The algebra $\mathcal{A}(\mathbb{Z}, p)$ is generated over $I(\mathbb{Z}, p)$ by the elements $a_p(\rho)$ for all possible $p$–completable $\rho$. Algebra multiplication of such associated elements corresponds to certain concatenations of Reeb chords.

We can define the homology class $[\rho] \in H_1(\mathbb{Z},p)$ in the obvious way, and extend to a set of Reeb chords $\rho = \{\rho_1, \ldots, \rho_n\}$, by taking the sum $[\rho] = [\rho_1] + \cdots + [\rho_n]$. It is easy to see that $[a(\rho, s)] = [\rho]$, and in particular it doesn’t depend on the completion $s$.

2.4. Grading. There are two ways to grade the algebra $\mathcal{A}(\mathbb{Z})$. The simpler is to grade it by a nonabelian group $\text{Gr}(\mathbb{Z})$, which is a $\frac{1}{2}\mathbb{Z}$–extension of $H_1(\mathbb{Z}, a)$. This group turns out to be too big, and does not allow for a graded version of the pairing theorems. For this a subgroup $\text{Gr}(\mathbb{Z})$ of $\text{Gr}(\mathbb{Z})$ is necessary, that can be identified with a $\frac{1}{2}\mathbb{Z}$–extension of $H_1(F(\mathbb{Z}))$. Unfortunately, there is no canonical way to get a $\text{Gr}(\mathbb{Z})$–grading on $\mathcal{A}(\mathbb{Z})$.

Remark. Our notation differs from that in [8]. In particular, our grading group $\text{Gr}(\mathbb{Z})$ is analogous to the group $G'(\mathbb{Z})$ used by Lipshitz, Ozsváth and Thurston, while $\text{Gr}(\mathbb{Z})$ corresponds to their $G(\mathbb{Z})$. Moreover, our grading function $\text{gr}$ corresponds to their $\text{gr}'$, while $\text{gr}$ corresponds to $\text{gr}$.

We start with the $\text{Gr}(\mathbb{Z})$–grading. Suppose $\mathbb{Z} = \{Z_1, \ldots, Z_l\}$. We will define a grading on the bigger algebra $\mathcal{A}(|Z_1|, \ldots, |Z_l|)$ that descends to a grading on $\mathcal{A}(\mathbb{Z})$.

First, we define some auxiliary maps.

Definition 2.10. Let $m: H_0(a) \times H_1(\mathbb{Z}, a) \rightarrow \frac{1}{2}\mathbb{Z}$ be the map defined by counting local multiplicities. More precisely, given the positively oriented line segment $l = [a_i, a_{i+1}] \subset \mathbb{Z}_p$, set
\[
m([a_j], [l]) = \begin{cases} 
\frac{1}{2} & \text{if } j = i, i+1, \\
0 & \text{otherwise,}
\end{cases}
\]
and extend linearly to all of $H_0(a) \times H_1(Z, a)$.

**Definition 2.11.** Let $L : H_1(Z, a) \times H_1(Z, a) \to \frac{1}{2}Z$, be

$$L(\alpha_1, \alpha_2) = m(\partial(\alpha_1), \alpha_2),$$

where $\partial$ is the connecting homomorphism in homology.

The group $\text{Gr}(Z)$ is defined as a central extension of $H_1(Z, a)$ by $\frac{1}{2}Z$ in the following way.

**Definition 2.12.** Let $\text{Gr}(Z)$ be the set $\frac{1}{2}Z \times H_1(Z, a)$, with multiplication

$$(a_1, \alpha_1) \cdot (a_2, \alpha_2) = (a_1 + a_2 + L(\alpha_1, \alpha_2), \alpha_1 + \alpha_2).$$

For an element $g = (a, \alpha) \in \text{Gr}(Z)$ we call $a$ the homological component, and $\alpha$ the homological component of $g$.

Note that if $Z$ has just one component $Z_1$ and $|Z_1| = n$, then this grading group is the same as the group $G'(n)$ defined in [8, Section 3]. In general, if $Z = \{Z_1, \ldots, Z_l\}$, as a set

$$G'(|Z_1|) \times \cdots \times G'(|Z_l|) \cong \left(\frac{1}{2}Z\right)^l \times H_1(Z, a),$$

since $H_1(Z_1, a \cap Z_1) \oplus \cdots \oplus H_1(Z_l, a \cap Z_l) \cong H_1(Z, a)$. Adding the Maslov components together induces a surjective homomorphism

$$\sigma : G'(|Z_1|) \times \cdots \times G'(|Z_l|) \to \text{Gr}(Z).$$

We can now define the grading $\text{gr} : A(|Z_1|, \ldots, |Z_l|) \to \text{Gr}(Z)$.

**Definition 2.13.** For an element $a = (S, T, \phi)$ of $A(|Z_1|, \ldots, |Z_l|)$, set

$$\iota(a) = \text{inv}(\phi) - m(S, [a]),$$

$$\text{gr}(a) = (\iota(a), [a]).$$

Breaking up $a$ into its components $a = (a_1, \ldots, a_l) \in A(|Z_1|) \oplus \cdots \oplus A(|Z_l|)$, we see that $\text{gr}(a) = \sigma(\text{gr}'(a_1), \ldots, \text{gr}'(a_l))$.

Therefore, we can apply the results about $G'$ and $\text{gr}'$ from [8] to deduce the following proposition.

**Proposition 2.14.** The function $\text{gr}$ is indeed a grading on $A(|Z_1|, \ldots, |Z_l|)$, with the same properties as $G'$ on $A(n)$. Namely, the following statements hold.

1. Under $\text{gr}$, $A(|Z_1|, \ldots, |Z_l|)$ is a differential graded algebra, where the differential drops the grading by the central element $\lambda = (1, 0)$.
2. For any completable collection of Reeb chords $\rho$, the element $a(\rho)$ is homogeneous.
3. The grading $\text{gr}$ descends to $A(Z)$.
4. For any completable collection $\rho$, the grading of $a(\rho, s)$ does not depend on the completion $s$. 
Proof. The proof of (1) follows from the corresponding statement for $gr'$, after noticing that the differential on $A([Z_1],\ldots,[Z_l])$ is defined via the Leibniz rule, and the differentials on the individual components drop one Maslov component by 1, while keeping all the rest fixed.

The rest of the statements then follow analogously to those for $gr'$ in [8]. \hfill \Box

2.5. Reduced grading. We can now define the refined grading group $Gr(Z)$. Recall that the surface $F(Z)$ retracts to the graph $G(Z)$, consisting of the segments $Z$, and the arcs $E = \{e_1,\ldots,e_k\}$, such that $Z \cap E = a$. From the long exact sequence for the pair $(G,E)$ we know that the following piece is exact.

$$0 \rightarrow H_1(G) \rightarrow H_1(G,E) \rightarrow H_0(E)$$

The differential $\partial: H_1(G,E) \rightarrow H_0(E)$ can be identified with the composition $M_s \circ \partial: H_1(Z,a) \rightarrow H_0(E)$, and $H_1(F) = H_1(G)$ can be identified with $\ker \partial \subset H_1(Z,a)$. The identification can also be seen by adding the arcs $e_i$ to cycles in $(Z,a)$ to obtain cycles in $G = Z \cup E$. This induces a map $\partial': Gr(Z) \rightarrow H_0(E)$, and the kernel $Gr(Z) \sim \ker \partial'$ is just the subgroup of $Gr(Z)$, consisting of elements with homological component in $\ker \partial \cong H_1(F)$.

Proposition 2.15. Under the identification $\ker \partial = H_1(F)$, the group $Gr(Z)$ can be explicitly described as a central extension of $H_1(F)$ by $\mathbb{Z}/2\mathbb{Z}$, with multiplication law

$$(a_1,[a_1]) \cdot (a_2,[a_2]) = (a_1 + a_2 + \#(a_1 \cap a_2),[a_1] + [a_2]),$$

where $a_1,a_2 \in \mathbb{Z}/2\mathbb{Z}$, and $a_1$ and $a_2$ are curves in $F$, and $\#(a_1 \cap a_2)$ is the signed intersection number, according to the orientation of $F$.

Proof. First, notice that the intersection pairing is well-defined, as it is, via Poincaré duality, just the pairing $\langle \cdot \cup \cdot,[F,\partial F]\rangle$ on $H^1(F,\partial F)$. The remaining step is to show that under the identification $\ker \partial = H_1(F)$, this agrees with the pairing $L$ on $H_1(Z,a)$. This can be seen by starting with line segments on $Z$ and arcs in $E$, pushing the arcs on $E$ away from each other in the $2\#E$ possible ways. One can then count that $\pm 1$ contributions to $L$ always give rise to an intersection point, while $\pm 1/2$ contributions create an intersection point exactly half of the time. \hfill \Box

In fact, for any generator $a \in A(Z)$ with starting and ending idempotents $I_s$ and $I_e$, respectively, $\partial'(gr(a)) = I_e - I_s$, if we think of the idempotents as linear combinations of the $e_i$. Therefore, for any $a$ with $I_e = I_s$, $gr(a)$ is already in $Gr$, and in general it is “almost” in there. At this point we would like to find a retraction $Gr \rightarrow Gr$ and use this to define the refined grading. However this fails even in simple cases. For instance, when $Z$ is an arc diagram for a disc with several sutures, $Gr(Z) = \mathbb{Z}/2\mathbb{Z}$ is abelian, as $H_1(F)$ vanishes, while the commutator of $Gr(Z)$ is $\mathbb{Z} \subset Gr(Z)$, and there can be no retraction, even if we pass to $\mathbb{Q}$-coefficients.
The solution is to assign a grading to $A(Z)$ with values in $\text{Gr}(Z)$, depending on the starting and ending idempotents. First, note that the generating idempotents come in sets of connected components, where $I$ is connected to $J$ if and only if $I - J$ is in the image of $\partial'$, or equivalently in the kernel of $H_0(E) \to H_0(F)$. These connected components correspond to the possible choices of how many arcs are occupied in each connected component of $F(Z)$.

**Definition 2.16.** A grading reduction $r$ for $Z$ is a choice of a base idempotent $I_0$ in each connected component, and a choice $r(I) \in \partial' - 1(I - I_0)$ for any $I \in [I_0]$.

**Definition 2.17.** Given a grading reduction $r$, define the reduced grading $\text{gr}_r(a) = r(I_s) \cdot \text{gr}(a) \cdot r(I_e)^{-1} \in \text{Gr}(Z)$, for any generator $a \in A(Z)$ with starting and ending idempotents $I_s$ and $I_e$, respectively. When unambiguous, we write simply $\text{gr}(a)$.

For any elements $a$ and $b$, such that $a \cdot b$, or even $a \otimes b$ is nonzero, the $r$–terms in $\text{gr}$ cancel, and $\text{gr}(a \otimes b) = \text{gr}(a \cdot b) = \text{gr}(a) \cdot \text{gr}(b)$. Since $\langle \frac{1}{2}Z, 0 \rangle$ is in the center, there is still a well-defined $Z$–action by $\lambda = \langle 1, 0 \rangle$, and $\text{gr}(\lambda a) = \lambda^{-1} \text{gr}(a)$. Therefore, $\text{gr}$ is indeed a grading.

Notice that for any $a$ with $I_s = I_e$, $\text{gr}(a)$ is the conjugate of $\text{gr}(a) \in \text{Gr}$ by $r(I_s)$. In particular, the homological part of the grading is unchanged, and whenever it vanishes, the Maslov component is also unchanged.

**Remark.** Given a set of Reeb chords $\rho$, the element $a(\rho) \in A(Z)$ is no longer homogeneous under $\text{gr}$. Indeed, $\text{gr}(a(\rho, s))$ depends on the completion $s$.

2.6. **Orientation reversals.** It is sometimes useful to compare the arc diagrams $Z$ and $-Z$ and the corresponding gradings. Recall that $-Z$ and $Z$ differ only by the orientation of $Z$. Consequently, the homology components $H_1(\pm Z, a)$ in $\text{Gr}(\pm Z)$ can be identified, while their canonical bases are opposite in order and sign. In particular, the pairings $L_{\pm Z}$ are opposite from each other. Therefore $\text{Gr}(Z)$ and $\text{Gr}(-Z)$ are anti-isomorphic, via the map fixing both the Maslov and homological components.

Similarly, $F(\pm Z)$ differ only in orientation, the homological components $H_1(F)$ can be naturally identified while the intersection pairings are opposite from each other. Thus $\text{Gr}(Z)$ and $\text{Gr}(-Z)$ are also anti-homomorphic, via the map that fixes both components, which agrees with the restriction of the corresponding map on $\text{Gr}(Z)$.

Thus, left actions by $\text{Gr}(Z)$ or $\text{Gr}(-Z)$ naturally correspond to right actions by $\text{Gr}(-Z)$ or $\text{Gr}(-Z)$, respectively, and vice versa.

3. **Bordered sutured 3–manifolds**

In this section—and for most of the rest of the paper—we will be working from the point of view of bordered sutured manifolds, as sutured manifolds
with extra structure. We will largely avoid the alternative description of decorated sutured cobordisms.

3.1. Sutured manifolds.

**Definition 3.1.** A divided surface \((S, \Gamma)\) is a closed surface \(F\), together with a collection \(\Gamma = \{\gamma_1, \ldots, \gamma_n\}\) of pairwise disjoint oriented simple closed curves on \(F\), called sutures, satisfying the following conditions.

Every component \(B\) of \(F\setminus \Gamma\) has nonempty boundary (which is the union of sutures). Moreover, the boundary orientation and the suture orientation of \(\partial B\) either agree on all components, in which case we call \(B\) a positive region, or they disagree on all components, in which case we call \(B\) a negative region. We denote by \(R^+(\Gamma)\) or \(R^+_\Gamma\) (respectively \(R^-(\Gamma)\) or \(R^-\Gamma\)) the closure of the union of all positive (negative) regions.

Notice that the definition doesn’t require \(F\) to be connected, but it requires that each component contain a suture.

**Definition 3.2.** A divided surface \((F, \Gamma)\) is called balanced if \(\chi(R^+) = \chi(R^-)\).

It is called \(k\)-unbalanced if \(\chi(R^+) = \chi(R^-) + 2k\), where \(k\) could be positive, negative or 0. In particular 0-unbalanced is the same as balanced.

Notice that since \(F\) is closed, and \(\chi(S) = \chi(R^+) + \chi(R^-)\), it follows that \(\chi(R^+) - \chi(R^-)\) is always even.

Now we can express the balanced sutured manifolds of [5] in terms of divided surfaces.

**Definition 3.3.** A balanced sutured manifold \((Y, \Gamma)\) is a 3–manifold \(Y\) with no closed components, such that \((\partial Y, \Gamma)\) is a balanced divided surface.

We can extend this definition to the following.

**Definition 3.4.** A \(k\)-unbalanced sutured manifold \((Y, \Gamma)\) is a 3–manifold \(Y\) with no closed components, such that \((\partial Y, \Gamma)\) is a \(k\)-unbalanced divided surface.

Although our unbalanced sutured manifolds are more general than the balanced ones of Juhász, they are still strictly a subclass of Gabai’s general definition in [2]. For example, he allows toric sutures, while we do not.

3.2. Bordered sutured manifolds. In this section we describe how to obtain a bordered sutured manifold from a sutured manifold, by parametrizing part of its boundary.

**Definition 3.5.** A bordered sutured manifold \((Y, \Gamma, Z, \phi)\) consists of the following.

1. A sutured manifold \((Y, \Gamma)\).
2. An arc diagram \(Z\).
(3) An orientation preserving embedding \( \phi: G(Z) \hookrightarrow \partial Y \), such that \( \phi|_Z \) is an orientation preserving embedding into \( \Gamma \), and \( \phi(G(Z) \setminus Z) \cap \Gamma = \emptyset \). It follows that each arc \( e_i \) embeds in \( R_- \).

Note that a closed neighborhood \( \nu(G(Z)) \subset \partial Y \) can be identified with the parametrized surface \( F(Z) \). We will make this identification from now on.

An equivalent way to give a bordered sutured manifold would be to specify an embedding \( F(Z) \hookrightarrow \partial Y \), such that the following conditions hold. Each 0–handle of \( F \) intersects \( \Gamma \) in a single arc, while each 1–handle is embedded in \( \text{Int}(R_-(\Gamma)) \).

Proposition 3.6. Any bordered sutured manifold \((Y, \Gamma, Z, \phi)\) satisfies the following condition, called homological linear independence.

\[
\pi_0(\Gamma \setminus \phi(Z)) \to \pi_0(\partial Y \setminus F(Z)) \text{ is surjective.}
\]

Proof. Indeed, Eq. (5) is equivalent to \( \Gamma \) intersecting any component of \( \partial Y \setminus F \). But \( \Gamma \) already intersects any component of \( \partial Y \). Any component of \( \partial Y \setminus F \) is either a component of \( \partial Y \), or has common boundary with \( F \). The non-degeneracy condition on \( Z \) guarantees that any component of \( \partial F \) hits \( \Gamma \). \( \square \)

Remark. If we want to work with degenerate arc diagrams (which give rise to degenerate sutured surfaces) we can still get well-defined invariants, as long as we impose homological linear independence on the manifolds. However, in that case there is no category, since the identity cobordism from a degenerate sutured surface to itself does not satisfy homological linear independence.

3.3. Gluing. We can glue two bordered sutured manifolds to obtain a sutured manifold in the following way.

Let \((Y_1, \Gamma_1, Z, \phi_1)\) and \((Y_2, \Gamma_2, -Z, \phi_2)\) be two bordered sutured manifolds. Since \( \phi_1 \) and \( \phi_2 \) are embeddings, and \( G(-Z) \) is naturally isomorphic to \( G(Z) \) with its orientation reversed, there is a diffeomorphism \( \phi_1(G(Z)) \to \phi_2(G(-Z)) \) that can be extended to an orientation reversing diffeomorphism \( \psi: F(Z) \to F(-Z) \) of their neighborhoods. Moreover, \( \psi|_{\Gamma_1}: \Gamma_1 \cap F(Z) \to \Gamma_2 \cap F(-Z) \) is orientation reversing.

Set \( Y = Y_1 \cup \psi Y_2 \), and \( \Gamma = (\Gamma_1 \setminus F(Z)) \cup (\Gamma_2 \setminus F(-Z)) \). By homological linear independence on \( Y_1 \) and \( Y_2 \), the sutures \( \Gamma \) on \( Y \) intersect all components of \( \partial Y \), and \((Y, \Gamma)\) is a sutured manifold.

More generally, we can do partial gluing. Suppose \((Y_1, \Gamma_1, Z_0 \cup Z_1, \phi_1)\) and \((Y_2, \Gamma_2, -Z_0 \cup Z_2, \phi_2)\) are bordered sutured. Then

\[
(Y_1 \cup_{F(Z_0)} Y_2, (\Gamma_1 \setminus F(Z_0)) \cup (\Gamma_2 \setminus F(-Z_0)), Z_1 \cup Z_2, \phi_1|_{G(Z_1)} \cup \phi_2|_{G(Z_2)})
\]

is also bordered sutured.

4. Heegaard diagrams

4.1. Diagrams and compatibility with manifolds.
Definition 4.1. A bordered sutured Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, Z, \psi)$ consists of the following data:

1. A surface with with boundary $\Sigma$.
2. An arc diagram $Z$.
3. An orientation reversing embedding $\psi: G(Z) \hookrightarrow \Sigma$, such that $\psi|_Z$ is an orientation preserving embedding into $\partial \Sigma$, while $\psi|_{G(Z) \setminus Z}$ is an embedding into $\text{Int}(\Sigma)$.
4. The collection $\alpha^a = \{\alpha^a_1, \ldots, \alpha^a_k\}$ of arcs $\alpha^a_i = \psi(e_i)$.
5. A collection of simple closed curves $\alpha^c = \{\alpha^c_1, \ldots, \alpha^c_n\}$ in $\text{Int}(\Sigma)$, which are disjoint from each other and from $\alpha^a$.
6. A collection of simple closed curves $\beta = \{\beta_1, \ldots, \beta_m\}$ in $\text{Int}(\Sigma)$, which are pairwise disjoint and transverse to $\alpha = \alpha^a \cup \alpha^c$.

We also require that $\pi_0(\partial \Sigma \setminus Z) \to \pi_0(\Sigma \setminus \alpha)$ and $\pi_0(\partial \Sigma \setminus Z) \to \pi_0(\Sigma \setminus \beta)$ be surjective. We call this condition homological linear independence since it is equivalent to each of $\alpha$ and $\beta$ being linearly independent in $H_1(\Sigma, Z)$.

Homological linear independence on diagrams is the key condition required for admissibility and avoiding boundary degenerations.

Definition 4.2. A boundary compatible Morse function on a bordered sutured manifold $(Y, \Gamma, Z, \phi)$ is a self-indexing Morse function $f: Y \to [-1,4]$ (with an implicit choice of Riemannian metric $g$) with the following properties.

1. The parametrized surface $F(Z) = \nu(G(Z))$ is totally geodesic, $\nabla f$ is parallel to $F$, and $f|_F$ is a $Z$–compatible Morse function.
2. A closed neighborhood $N = \nu(\Gamma \setminus Z)$ is isotopic to $(\Gamma \setminus Z) \times [-1,4]$, such that $f$ is projection on the second factor (and $f(\Gamma) = 3/2$).
3. $f^{-1}(-1) = R_-(\Gamma) \setminus (N \cup F)$, and $f^{-1}(4) = R_+(\Gamma) \setminus (N \cup F)$.
4. $f$ has no index–0 or index–3 critical points.
5. The are no critical points in $\partial Y \setminus F$, and the index–1 critical points for $F$ are also index–1 critical points for $Y$.

See Fig. 6a for a schematic illustration.

From a boundary compatible Morse function $f$ we can get a bordered sutured Heegaard diagram by setting $\Sigma = f^{-1}(3/2)$, and letting $\alpha$ be the intersection of the stable manifolds of the index–1 critical points with $\Sigma$, and $\beta$ be the intersection of the unstable manifolds of the index–2 critical points with $\Sigma$. Note that the internal critical points give $\alpha^c$ and $\beta$, while the ones in $F \subset \partial Y$ give $\alpha^a$. We notice that $Z \subset F \cap \Sigma$ and $\alpha^a$ form an embedding $\psi: G(Z) \to \Sigma$. Homological linear independence for the diagram follows from that of manifold.

Definition 4.3. A diagram as above is called a compatible bordered sutured Heegaard diagram to $f$.

Proposition 4.4. Compatible diagrams and boundary compatible Morse functions are in a one-to-one correspondence.
Figure 5. Half of a 2–handle attached along an arc. Its critical point and two incoming gradient flow lines are in the boundary.

Proof. We need to give an inverse construction. Start with a bordered sutured diagram \( \mathcal{H} \), and construct a bordered sutured manifold in the following way. To \( \Sigma \times [1, 2] \) attach 2–handles at \( \alpha^c \times \{1\} \), and at \( \beta_i \times \{2\} \). Finally, at \( \alpha^a \times \{1\} \) attach “halves of 2–handles”. These are thickened discs \( D^2 \times [0, 1] \) attached along an arc \( a \times \{1/2\} \subset \partial D^2 \times \{1/2\} \). (See Fig. 5.) Then \( \Gamma \) is \( \Sigma \times \{3/2\} \), and \( F(Z) \) is \( Z \times [1, 2] \), together with the “middles” of the partial handles, i.e. \( (\partial D^2 \setminus a) \times [0, 1] \). To such a handle decomposition on the new manifold \( Y \) corresponds a canonical boundary compatible Morse function \( f \). Note that attaching the half-handles has no effect topologically, but adds boundary critical points.

Proposition 4.5. Any sutured bordered manifold has a compatible diagram in the above sense. Moreover, any two compatible diagrams can be connected by a sequence of moves of the following types:

1. Isotopy of the circles in \( \alpha^c \) and \( \beta \), and isotopy, relative to the endpoints, of the arcs in \( \alpha^a \).
2. Handleslide of a circle in \( \beta \) over another circle in \( \beta \).
3. Handleslide of any curve in \( \alpha \) over a circle in \( \alpha^c \).
4. Stabilization.

Proof. For the proof of this proposition we will modify our definition of a compatible Morse function, to temporarily “forget” about \( F \).

A pseudo boundary compatible Morse function \( f \) for a bordered sutured manifold \( (Y, \Gamma, Z, \phi) \) is a boundary compatible Morse function for the manifold \( (Y, \Gamma, \emptyset, \emptyset) \) (which is just a standard Morse function for the sutured manifold \( (Y, \Gamma) \), in the sense of [5]), with some additional conditions. Namely, we require that \( f^{-1}([-1, 3/2]) \cap \phi(e_i) \) consist of two arcs (at the endpoints of \( \phi(e_i) \)), tangent to \( \nabla f \). We also require that \( \phi(G(Z)) \) be disjoint from the unstable manifolds of index–1 critical points.
(a) A true boundary compatible Morse function. There is one boundary critical point giving rise to $\alpha^a$.

(b) A pseudo boundary compatible Morse function. There is one arc in $f^{-1}(-1)$ giving rise to $\alpha^a$.

Figure 6. Comparison of a boundary compatible and pseudo boundary compatible Morse functions. Several internal critical points are given in each, with gradient flowlines, giving rise to $\alpha^c$ and $\beta$.

Such Morse functions are in 1–to–1 correspondence with compatible diagrams by the following construction. As usual, $\Sigma = f^{-1}(3/2)$, while $\alpha^c$ and $\beta$ are the intersections of $\Sigma$ with stable, respectively unstable, manifolds for index–1 and index–2 critical points. On the other hand, $\alpha^a_i$ is the intersection of $\Sigma$ with the gradient flow from $e_i$. Since the flow avoids index–1 critical points, $\alpha^a$ is disjoint from $\alpha^c$. See Fig. 6 for a comparison between the two types of Morse functions.

The backwards construction is the same as for true boundary-compatible Morse functions, except we do not attach the half 2–handles’ at $\alpha^a \times \{1\}$, and instead just set $e_i = \alpha^a_i \times \{1\} \cup \partial \alpha^a_i \times [1, 3/2]$. 


This alternative construction allows us to use standard results about sutured manifolds. In particular, [5, Propositions 2.13—2.15] imply that \((Y, \Gamma)\) has a compatible Morse function, and hence Heegaard diagram, and any two compatible diagrams are connected by Heegaard moves. Namely, there is a family \(f_t\) of Morse functions, which for generic \(t\) corresponds to an isotopy, and for a finite number of critical points corresponds to a index–1, index–2 critical point creation, (i.e. stabilization of the diagram), or a flowline between critical points of the same index (handleslides between circles in \(\alpha^c\) or between circles in \(\beta\)).

Since the stable manifold of any index–1 critical point intersects \(R^-\) at a pair of points, we can always perturb \(f\) to get a pseudo-compatible diagram for \((Y, \Gamma, Z, \phi)\). Any two such diagrams are connected by a sequence of sutured Heegaard moves (ignoring \(\alpha^a\)). For generic \(t\), a sutured compatible \(f_t\) is also pseudo bordered sutured compatible. At non-generic \(t\), there is a flow from some point on \(e_i\) to an index–1 critical point. This corresponds to sliding \(\alpha_i\) over the corresponding circle in \(\alpha^c\), so we must add those to the list of allowed Heegaard moves. □

4.2. Generators.

**Definition 4.6.** A *generator* for a bordered sutured diagram \(\mathcal{H} = (\Sigma, \alpha, \beta)\) is a collection \(x = (x_1, \ldots, x_g)\) of intersection points in \(\alpha \cap \beta\), such that there is exactly one point on each \(\alpha^c\) circle, exactly one point on each \(\beta\) circle, and at most one point on each \(\alpha^a\) arc.

The set of all generators for \(\mathcal{H}\) is denoted \(\mathcal{G}(\mathcal{H})\) or \(\mathcal{G}\).

As a degenerate case, when \(#\beta = #\alpha^c = 0\), we will let \(\mathcal{G}\) contain a single element, which is the empty collection \(x = ()\).

Notice that if \(\mathcal{G}\) is nonempty, then necessarily \(g = #\beta \geq #\alpha^c\). We call \(g\) the *genus* of \(\mathcal{H}\). Moreover, exactly \(p = g - #\alpha^c\) many of the \(\alpha^a\) arcs are occupied by each generator. Let \(o(x) \subseteq \{1, \ldots, k\}\) denote the set of occupied \(\alpha^a\) arcs, and \(\overline{o}(x) = \{1, \ldots, k\} \setminus o(x)\) denote the set of unoccupied arcs.

**Remark.** If \(\mathcal{H} = (\Sigma, \alpha, \beta)\) is a bordered sutured diagram compatible with a \(p\)–unbalanced bordered sutured manifold, then exactly \(p\) many \(\alpha^a\) arcs are occupied by each generator for \(\mathcal{H}\).

Indeed, let \(g = #\beta\), and \(h = #\alpha\). By the construction of a compatible manifold, \(R_-(\Gamma)\) is diffeomorphic to \(\Sigma\) after surgery at each \(\alpha^c\) circle, while \(R_+(\Gamma)\) is diffeomorphic to \(\Sigma\) after surgery at each \(\beta\) circle. But surgery on a surface at a closed curve increases its Euler characteristic by 2. Therefore, the manifold is \((g - h)\)–unbalanced.

4.3. Homology classes. Later we will look at pseudoholomorphic curves that go “between” two generators. We can classify such curves into homology classes as follows.
Definition 4.7. For given generators $x$ and $y$, the homology classes from $x$ to $y$, denoted by $\pi_2(x, y)$, be the elements of

$$H_2(\Sigma \times [0, 1] \times [0, 1], (\alpha \times \{0\} \times [0, 1]) \cup (\beta \times \{0\} \times [0, 1]),$$

$$\cup (Z \times [0, 1] \times [0, 1]) \cup (x \times [0, 1] \times \{0\}) \cup (y \times [0, 1] \times \{1\})),$$

which map to the relative fundamental class of $x \times [0, 1] \cup y \times [0, 1]$ under the boundary homomorphism, and collapsing the rest of the boundary.

There is a product map $*: \pi_2(x, y) \times \pi_2(y, z) \to \pi_2(x, z)$ given by concatenation at $y \times [0, 1]$. This product turns $\pi_2(x, x)$ into a group, called the group of periodic classes at $x$.

Definition 4.8. The domain of a homology class $B \in \pi_2(x, y)$ is the image

$$[B] = \pi_{\Sigma*}(B) \in H_2(\Sigma, Z \cup \alpha \cup \beta).$$

We interpret it as a linear combination of regions in $\Sigma \setminus (\alpha \cup \beta)$. We call the coefficient of such a region in a domain $D$ its multiplicity.

The domain of a periodic class is a periodic domain.

We can split the boundary $\partial [B]$ into pieces $\partial^0 B \subset Z, \partial^0 B \subset \alpha$, and $\partial^0 B \subset \beta$. We can interpret $\partial^0 B$ as an element of $H_1(Z, a)$.

Definition 4.9. The set of provincial homology classes from $x$ to $y$ is the kernel $\pi_2^0(x, y)$ of $\partial^0: \pi_2(x, y) \to H_1(Z, a)$.

The periodic classes in $\pi_2^0(x, x)$ are provincial periodic class and their domains are provincial periodic domains.

The groups of periodic classes reduce to the much simpler forms

$$\pi_2(x, x) \cong H_2(\Sigma \times [0, 1], Z \times [0, 1] \cup \alpha \times \{0\} \cup \beta \times \{1\}),$$

$$\pi_2^0(x, x) \cong H_2(\Sigma \times [0, 1], \alpha^c \times \{0\} \cup \beta \times \{1\}).$$

Since 2–handles and half-handles are contractible, these groups are isomorphic to $H_2(Y, F)$ and $H_2(Y)$, respectively, by attaching the cores of the handles.

4.4. Admissibility. As usual in Heegaard Floer homology, in order to get well defined invariants, we need to impose certain admissibility conditions on the Heegaard diagrams. Like in [8], there are two different notions of admissibility.

Definition 4.10. A bordered sutured Heegaard diagram is called admissible if every nonzero periodic domain has both positive and negative multiplicities.

A diagram is called provincially admissible if every nonzero provincial periodic domain has both positive and negative multiplicities.

Proposition 4.11. Any bordered sutured Heegaard diagram can be made admissible by performing isotopy on $\beta$. 
Corollary 4.12. Any bordered sutured 3–manifold has an admissible diagram, and any two admissible diagrams are connected, using Heegaard moves, through admissible diagrams.

The analogous statement holds for provincially admissible diagrams.

Since provincially admissible diagrams are a subset of admissible diagrams, the second part of the argument trivially follows from the first. The first part, on the other hand, follows from proposition 4.11, by taking any sequence of diagrams connected by Heegaard moves, and making all of them admissible, through a consistent set of isotopies.

Proof of proposition 4.11. The proof is analogous to those for bordered manifolds and sutured manifolds. We use the isomorphism from the previous section between periodic domains and $H_2(Y,F)$.

Notice that $H_1(\Sigma, \partial \Sigma \setminus Z)$ maps onto $H_1(Y, \partial Y \setminus F)$, and therefore pairs with $H_2(Y,F)$ and periodic domains. Find a basis for $H_2(Y,F)$, represented by pairwise disjoint properly embedded arcs $a_1, \ldots, a_m$. We can always do that since every component of $\Sigma$ hits $\partial \Sigma \setminus Z$. Cutting $\Sigma$ along such arcs will give a collection of discs, each of which contains exactly one component of $Z$ in its boundary.

We can do finger moves of $\beta$ along each $a_i$, and along a push off $b_i$ of $a_i$, in the opposite direction. This ensures that there are regions, for which the multiplicities of any periodic domain $D$ are equal to its intersection numbers with $a_i$ and $b_i$, which have opposite signs. Suppose $D$ has a nonzero region, and pick a point $p$ in such a region. By homological linear independence $p$ can be connected to $\partial \Sigma \setminus Z$ in the complement of $\alpha \cup Z$, as well as in the complement of $\beta$. Connecting these paths gives a cycle in $H_1(\Sigma, \partial \Sigma \setminus Z)$, which pairs non trivially with $D$. Since the $a_i$ span this group, at least one of them pairs non trivially with $D$, which means $D$ has negative multiplicity in some region.

\[ \square \]

4.5. Spin\textsuperscript{c}–structures. Recall that a Spin\textsuperscript{c}–structure on an $n$–manifold is a lift of its principal $SO(n)$–bundle to a Spin\textsuperscript{c}$(n)$–bundle. For 3–manifolds there is a useful reformulation due to Turaev (see [11]). In this setting, a Spin\textsuperscript{c}–structure $s$ on the 3–manifold $Y$ is a choice of a non vanishing vector field $v$ on $Y$, up to homology. We say that two vector fields are homologous if they are homotopic outside of a finite collection of disjoint open balls.

Given a trivialization of $TY$, a vector field $v$ on $Y$ can be thought of as a map $v: Y \to S^2$. This gives an identification of the set $\text{Spin}^c(Y)$ of all Spin\textsuperscript{c}–structures with $H^2(Y)$ via $s(v) \mapsto v^*([S^2])$. The identification depends on the trivialization of $TY$ by an overall shift by a 2–torsion element. This means that $\text{Spin}^c(Y)$ is naturally an affine space over $H^2(Y)$.

Given a fixed vector field $v_0$ on a subspace $X \subset Y$, we can define the space of relative Spin\textsuperscript{c}–structures $\text{Spin}^c(Y,X,v_0)$, or just $\text{Spin}^c(Y,X)$ in the following way. A relative Spin\textsuperscript{c}–structure is a vector field $v$ on $Y$, such
that \( v|_X = v_0 \), considered up to homology in \( Y \setminus X \). If \( \text{Spin}^c(Y, X, v_0) \) is nonempty, it is an affine space over \( H^2(Y, X) \).

To a \( \text{Spin}^c \)-structure \( s \) in \( \text{Spin}^c(X) \) or \( \text{Spin}^c(Y, X, v_0) \), represented by a vector field \( v \), we can associate its Chern class \( c_1(s) \), which is just the first Chern class \( c_1(v^+) \) of the orthogonal complement subbundle \( v^+ \subset TY \).

With a generator in a Heegaard diagram we will associate two types of \( \text{Spin}^c \)-structures. Let \( x \in \mathcal{G}(\mathcal{H}) \) be a generator. Fix a boundary-compatible Morse function \( f \) (and appropriate metric). The vector field \( \nabla f \) vanishes only at the critical points of \( f \). Each intersection point in \( x \) lies on a gradient trajectory connecting an index–1 to an index–2 critical point. If we cut out a neighborhood of that trajectory, we can modify the vector field inside to one that is non vanishing (the two critical points have opposite parity). For any unoccupied \( \alpha^a \) arc, the corresponding critical point is in \( F \subset \partial Y \). We can therefore modify the vector field in its neighborhood to be non vanishing. Call the resulting vector field \( v(x) \).

Notice that \( v_0 = v(x)|_{\partial Y \setminus F} \) = \( \nabla f|_{\partial Y \setminus F} \) does not depend on \( x \), while \( v(x)|_{\partial Y} = v(o(x)) \) only depends on \( o(x) \). Moreover, under a change of the Morse function or metric (even for different diagrams), \( v_0 \) and \( v_0(x) \) can only vary inside a contractible set. Therefore the corresponding sets \( \text{Spin}^c(Y, \partial Y \setminus F, v_0) \) and \( \text{Spin}^c(Y, \partial Y, v_0(x)) \), respectively, are canonically identified. Thus we can talk about \( \text{Spin}^c(Y, \partial Y \setminus F) \) and \( \text{Spin}^c(Y, \partial Y, o) \), where \( o \subset \{ 1, \ldots, k \} \), as invariants of the underlying bordered sutured manifold. This justifies the following definition.

**Definition 4.13.** Let \( s(x) \) and \( s^{rel}(x) \) be the relative \( \text{Spin}^c \)-structures induced by \( v(x) \) in \( \text{Spin}^c(Y, \partial Y \setminus F) \) and \( \text{Spin}^c(Y, \partial Y, o(x)) \), respectively.

We can separate the generators into \( \text{Spin}^c \) classes. Let

\[
\mathcal{G}(\mathcal{H}, s) = \{ x \in \mathcal{G}(\mathcal{H}) : s(x) = s \},
\]

\[
\mathcal{G}(\mathcal{H}, o, s^{rel}) = \{ x \in \mathcal{G}(\mathcal{H}) : o(x) = o, s^{rel}(x) = s^{rel} \}.
\]

The fact that the invariants split by \( \text{Spin}^c \) structures is due to the following proposition.

**Proposition 4.14.** The set \( \pi_2(x, y) \) is nonempty if and only if \( s(x) = s(y) \). The set \( \pi_2^{rel}(x, y) \) is nonempty if and only if \( o(x) = o(y) \) and \( s^{rel}(x) = s^{rel}(y) \).

**Proof.** This proof is, again, analogous to those for bordered and for sutured manifolds.

To each pair of generators \( x, y \in \mathcal{G}(\mathcal{H}) \), we associate a homology class \( \epsilon(x, y) \in H_1(Y, F) \). We do that by picking 1–chains \( a \subset \alpha \), and \( b \subset \beta \), such that \( \partial a = y - x + z \), where \( z \) is a 0–chain in \( Z \), and \( \partial b = y - x \), and setting \( \epsilon(x, y) = [a - b] \). We can interpret \( a - b \) as a set of properly embedded arcs and circles in \( (Y, F) \) containing all critical points.

The vector fields \( v(x) \) and \( v(y) \) differ only in a neighborhood of \( a - b \). One can see that in fact \( s(y) - s(x) = \text{PD}(a - b) = \text{PD}(\epsilon(x, y)) \). On the
other hand, we can interpret $\epsilon(x, y)$ as an element of

$$H_1(\Sigma \times [0, 1], \alpha \times \{0\} \cup \beta \times \{1\} \cup \mathbb{Z} \times [0, 1]) \cong H_1(Y, F).$$

In particular, $\pi_2(x, y)$ is nonempty, if and only if there is a 2–chain in $\Sigma \times [0, 1]$ with boundary which is a representative for $\epsilon(x, y)$ in the relative group above. This is equivalent to $\epsilon(x, y) = 0 \in H_1(Y, F)$. This proves the first part of the proposition.

The second one follows analogously, noticing that we can pick a path $a - b$, such that $a \subset \alpha$, if and only if $o(x) = o(y)$, and in that case $\pi_2^0(x, y)$ is nonempty if and only if $\epsilon^{rel}(x, y) = [a - b] = 0 \in H_1(Y)$, while $s^{rel}(y) - s^{rel}(x) = PD([a - b]) \in H^2(Y, \partial Y)$. \hfill \QED

4.6. Gluing. We can glue bordered sutured diagrams, similar to the way we glue bordered sutured manifolds.

Let $\mathcal{H}_1 = (\Sigma_1, \alpha_1, \beta_1)$ and $\mathcal{H}_2 = (\Sigma_2, \alpha_2, \beta_2)$ be bordered sutured diagrams for $(Y_1, \Gamma_1, Z, \phi_1)$ and $(Y_2, \Gamma_2, -Z, \phi_2)$, respectively. We can identify $Z$ with its embeddings in $\partial \Sigma_1$ and $\partial \Sigma_2$ (one is orientation preserving, the other is orientation reversing).

Let $\Sigma = \Sigma_1 \cup_Z \Sigma_2$. Each $\alpha^a$ arc in $\mathcal{H}_1$ matches up with the corresponding one in $\mathcal{H}_2$ to form a closed curve in $\Sigma$. Let $\alpha$ denote the union of all $\alpha^c$ circles in $\mathcal{H}_1$ and $\mathcal{H}_2$, together with the newly formed circles from all $\alpha^a$ arcs. Finally, let $\beta = \beta_1 \cup \beta_2$.

**Proposition 4.15.** The diagram $\mathcal{H} = (\Sigma, \alpha, \beta)$ is compatible with the sutured manifold $Y_1 \cup_{F(Z)} Y_2$, as defined in section 3.3.

**Proof.** The manifolds $Y_1$ and $Y_2$ are obtained from $\Sigma_1 \times [1, 2]$ and $\Sigma_2 \times [1, 2]$, respectively, by attaching 2–handles (corresponding to $\alpha^c$ and $\beta$ circles), and halves of 2–handles (corresponding to $\alpha^a$ arcs). The surface of gluing $F$ can be identified with the union of $Z \times [1, 2]$ with the middles of the half-handles. Thus, we get a base of $(\Sigma_1 \cup_Z \Sigma_2) \times [1, 2]$, with the combined 2–handles from each side. In addition the half-handles glue in pairs to form actual 2–handles, each of which is glued along matching $\alpha^a$ arcs. \hfill \QED

Similarly, we can do partial gluing. If we have manifolds $(Y_1, \Gamma_1, Z_0 \cup Z_1, \phi_1)$ and $(Y_2, \Gamma_2, -Z_0 \cup Z_2, \phi_2)$ with diagrams $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, $\mathcal{H}_1 \cup_{Z_0} \mathcal{H}_2$ is a diagram compatible with the bordered sutured manifold $Y_1 \cup_{F(Z_0)} Y_2$.

4.7. Nice diagrams. As with the other types of Heegaard Floer invariants, the invariants become a lot easier to compute (at least conceptually) if we work in the category of nice diagrams, developed originally by Sarkar and Wang in [10].

**Definition 4.16.** A bordered sutured diagram $\mathcal{H} = (\Sigma, \alpha, \beta, Z, \psi)$ is nice if every region of $\Sigma \setminus (\alpha \cup \beta)$ is either adjacent to $\partial \Sigma \setminus Z$—in which case we call it a boundary region—or is a topological disc with at most 4 corners.
**Proposition 4.17.** Any bordered sutured diagram can be made nice by isotopies of $\beta$, handleslides among the circles in $\beta$, and stabilizations.

**Proof.** The proof is a combination of those for bordered and sutured manifolds, in [8] and [6], respectively.

First, we make some stabilizations until every component of $\Sigma$ contains both $\alpha$ and $\beta$ curves. Next we do finger moves of $\beta$ curves until any curve in $\alpha$ intersects $\beta$, and vice versa. Then, we ensure all non boundary regions are simply connected. We do that inductively, decreasing the rank of $H_1$ relative boundary for each region.

Then, following [8], we do finger moves of some $\beta$ curves along curves parallel to each component of $Z$ to ensure that all regions adjacent to some Reeb chord in $Z$ are rectangles (where one side is in $Z$, two are in $\alpha$, and one is in $\beta$).

Finally, we label all regions by their distance, i.e. number of $\beta$ arcs in $\Sigma \setminus \alpha$ one needs to cross, to get to a boundary region, and by their badness (how many extra corners they have). We do finger moves of a $\beta$ arc in a bad region through $\alpha$ arcs, until we hit a boundary, a bigon, or another (or the same) bad region. There are several cases depending on what kind of region we hit, but the overall badness of the diagram decreases, so the algorithm eventually terminates. The setup is such that we can never hit a region adjacent to a Reeb chord, so the algorithm for sutured manifolds goes through for bordered sutured manifolds. □

5. **Moduli spaces of holomorphic curves**

In this section we describe the moduli spaces of holomorphic curves involved in the definitions of the bordered invariants and prove the necessary properties. The definitions and arguments are mostly a straightforward generalization of those in [8, Chapter 5].

5.1. **Differences with bordered Floer homology.** For the reader familiar with border Floer homology we highlight the similarities and the differences with our definitions.

In the bordered setting of Lipshitz, Ozsváth, and Thurston, there is one boundary component and one basepoint on the boundary. One counts pseudoholomorphic discs in $\Sigma \times [0,1] \times \mathbb{R}$, but in practice one thinks of their domains in $\Sigma$. Loosely speaking, the curves that do not hit $\partial \Sigma$ correspond to differentials, the ones that do hit the boundary correspond to algebra actions, while the ones that hit the basepoint are not counted at all.

In the bordered sutured setting, the boundary $\partial \Sigma$ has several components, while some subset $Z$ of $\partial \Sigma$ is distinguished. We again count pseudoholomorphic curves in $\Sigma \times [0,1] \times \mathbb{R}$, and again, those curves that do not hit the boundary correspond to differentials. The novel idea is the interpretation of the boundary. Here the algebra action comes from curves that hit any component of $Z \subset \partial \Sigma$, while the curves that hit any component of $\partial \Sigma \setminus Z$ are not counted. In a sense, the set $\partial \Sigma \setminus Z$ plays the role of the basepoint.
With this in mind, most of the constructions in [8] carry over. Below we describe the necessary analytic constructions.

5.2. Holomorphic curves and conditions. We will consider several variations of the Heegaard surface \( \Sigma \), namely the compact surface with boundary \( \Sigma = \Sigma \), the open surface \( \text{Int}(\Sigma) \), which can be thought of as a surface with several punctures \( p = \{p_1, \ldots, p_n\} \), and the closed surface \( \Sigma_\pi \), obtained by filling in those punctures. Alternatively, it is obtained from \( \Sigma \) by collapsing all boundary components to points.

We will also be interested in the surface \( D = [0, 1] \times \mathbb{R} \), with coordinates \( s \in [0, 1] \) and \( t \in \mathbb{R} \).

Let \( \omega_\Sigma \) be a symplectic form on \( \text{Int}(\Sigma) \), such that \( \partial \Sigma \) is a cylindrical end, and let \( j_\Sigma \) be a compatible almost complex structure. We can assume that \( \alpha^p \) is cylindrical near the punctures in the following sense. There is a neighborhood \( U_p \) of the punctures, symplectomorphic to \( \partial \Sigma \times (0, \infty) \subset T^* (\partial \Sigma) \), such that \( j_\Sigma \) and \( \alpha^p \cap U_p \) are invariant with respect to the \( \mathbb{R} \)–action on \( \partial \Sigma \times (0, \infty) \). Let \( \omega_D \) and \( j_D \) be the standard symplectic form and almost complex structure on \( D \subset \mathbb{C} \).

Consider the projections

\[
\pi_{\Sigma}: \text{Int}(\Sigma) \times D \rightarrow \text{Int}(\Sigma), \\
\pi_D: \text{Int}(\Sigma) \times D \rightarrow D, \\
s: \text{Int}(\Sigma) \times D \rightarrow [0, 1], \\
t: \text{Int}(\Sigma) \times D \rightarrow \mathbb{R}.
\]

Definition 5.1. An almost complex structure \( J \) on \( \text{Int}(\Sigma) \times D \) is called admissible if the following conditions hold:

- \( \pi_D \) is \( J \)–holomorphic.
- \( J(\partial_s) = \partial_t \) for the vector fields \( \partial_s \) and \( \partial_t \) in the fibers of \( \pi_\Sigma \).
- The \( \mathbb{R} \)–translation action in the \( t \)–coordinate is \( J \)–holomorphic.
- \( J = j_\Sigma \times j_D \) near \( p \times D \).

Definition 5.2. A decorated source \( S^p \) consists of the following data:

- A topological type of a smooth surface \( S \) with boundary, and a finite number of boundary punctures.
- A labeling of each puncture by one of “+”, “−”, or “e”.
- A labeling of each e puncture by a Reeb chord \( \rho \) in \( \mathbb{Z} \).

Given \( S^p \) as above, denote by \( S_\pi \) the surface obtained from \( S \) by filling in all the e punctures.

We consider maps

\[
u: (S, \partial S) \rightarrow (\text{Int}(\Sigma) \times D, (\alpha \times \{1\} \times \mathbb{R}) \cup (\beta \times \{0\} \times \mathbb{R}))
\]
satisfying the following conditions:

1. \( u \) is \( (j, J) \)–holomorphic for some almost complex structure \( j \) on \( S \).
2. \( u: S \rightarrow \text{Int}(\Sigma) \times D \) is proper.
(3) $u$ extends to a proper map $u_{\Sigma} : S_{\Sigma} \to \Sigma \times \mathbb{D}$.
(4) $u_{\Sigma}$ has finite energy in the sense of Bourgeois, Eliashberg, Hofer, Wysocki and Zehnder [1].
(5) $\pi_2 \circ u : S \to \mathbb{D}$ is a $g$-fold branched cover. (Recall that $g$ is the cardinality of $\beta$, not the genus of $\Sigma$).
(6) At each $+$ puncture $q$ of $S$, $\lim_{z \to q} u(z) = +\infty$.
(7) At each $-$ puncture $q$ of $S$, $\lim_{z \to q} u(z) = -\infty$.
(8) $\pi_\Sigma \circ u : S \to \text{Int} (\Sigma)$ does not cover any of the regions of $\Sigma \setminus (\alpha \cup \beta)$ adjacent to $\partial \Sigma \setminus \mathbb{Z}$.
(9) \textbf{Strong boundary monotonicity}. For each $t \in \mathbb{R}$, and each $\beta_i \in \beta$:
$u^{-1}(\beta_i \times \{0\} \times \{t\})$ consists of exactly one point. For each $\alpha_i \in \alpha^c$,
$u^{-1}(\alpha_i \times \{t\})$ consists of exactly one point.
(11) $u$ is embedded.

Under conditions (1)–(9), at each $+$ or $-$ puncture, $u$ is asymptotic to an arc $z \times [0, \infty)$, where $z$ is some intersection point in $\alpha \cap \beta$. If in addition we require condition (10), then the intersection points $x_1, \ldots, x_q$ corresponding to $-$ punctures form a generator $x$, while the ones $y_1, \ldots, y_q$ corresponding to $+$ punctures form a generator $y$. We call $x$ the incoming generator, and $y$ the outgoing generator for $u$.

If we compactify the $\mathbb{R}$ component of $\mathbb{D}$ to include $\{\pm \infty\}$, we get a compact rectangle $\widetilde{\mathbb{D}} = [0, \infty) \times [-\infty, +\infty]$. Let $u$ be a map satisfying conditions (1)–(10), and with incoming and outgoing generators $x$ and $y$. Let $\tilde{S}$ be $S$ with all punctures filled in by arcs. Then $u$ extends to a map

$$\widetilde{u} : (\tilde{S}, \partial \tilde{S}) \to (\Sigma \times \widetilde{\mathbb{D}}, (\alpha \times \{1\} \times [-\infty, +\infty]) \cup (\beta \times \{0\} \times [-\infty, +\infty])$$

$$\cup (\mathbb{Z} \times \widetilde{\mathbb{D}}) \cup (x \times [0, 1] \times \{-\infty\}) \cup (y \times [0, 1] \times \{+\infty\}).$$

Notice that the pair of spaces on the right is the same as the one in definition 4.7. It is clear that a map $u$ satisfying conditions (1)–(10) has an associated homology class $B = [u] = [\widetilde{u}] \in \pi_2(x, y)$.

We will also impose an extra condition on the height of the $e$ punctures of $S$.

\textbf{Definition 5.3.} For a map $u$ from a decorated source $S^\circ$, and an $e$ puncture $q$ on $\partial S$, the \textit{height} of $q$ is the evaluation $ev(q) = t \circ u_{\Sigma}(q) \in \mathbb{R}$.

\textbf{Definition 5.4.} Let $E(S^\circ)$ be the set of all $e$ punctures in $S$. Let $\overrightarrow{P} = (P_1, \ldots, P_m)$ be an ordered partition of $E(S^\circ)$ into nonempty subsets. We say $u$ is $\overrightarrow{P}$–compatible if for $i = 1, \ldots, m$ all the punctures in $P_i$ have the same height $ev(P_i)$, and moreover $ev(P_i) < ev(P_j)$ for $i < j$. 


To a partition $\overrightarrow{P} = (P_1, \ldots, P_m)$ we can associate a sequence $\overrightarrow{\rho} = (\rho_1, \ldots, \rho_m)$ of sets of Reeb chords, by setting

$$\rho_i = \{ \rho : \rho \text{ labels } q, q \in P_i \}.$$ 

Moreover, to any such sequence $\overrightarrow{\rho}$ we can associate a homology class $[\overrightarrow{\rho}] = [\rho_1] + \cdots + [\rho_m] \in H_1(\mathbb{Z}, a)$, and an algebra element $a(\overrightarrow{\rho}) = a(\rho_1) \cdots a(\rho_m)$.

It is easy to see that $[a(\overrightarrow{\rho})] = [\overrightarrow{\rho}]$. It is also easy to see that for a curve $u$ satisfying conditions (1)–(10) with homology class $[u] = B$, and for any partition $\overrightarrow{P}$ we have $[\overrightarrow{\rho}(\overrightarrow{P})] = \partial^3 B$.

5.3. Moduli spaces. We are now ready to define the moduli spaces that we will consider.

**Definition 5.5.** Let $x, y \in G(\mathcal{H})$ be generators, let $B \in \pi_2(x, y)$ be a homology class, and let $S^\circ$ be a decorated source. We will write

$$\widetilde{M}^B(x, y, S^\circ)$$

for the space of curves $u$ with source $S^\circ$ satisfying conditions (1)–(10), asymptotic to $x$ at $-\infty$ and to $y$ at $+\infty$, and with homology class $[u] = B$.

This moduli space is stratified by the possible partitions of $E(S^\circ)$. More precisely, given a partition $\overrightarrow{P}$ of $E(S^\circ)$, we write

$$\widetilde{M}^B(x, y, S^\circ, \overrightarrow{P})$$

for the space of $\overrightarrow{P}$–compatible maps in $\widetilde{M}^B(x, y, S^\circ)$, and

$$\widetilde{M}^B_{\text{emb}}(x, y, S^\circ, \overrightarrow{P})$$

for the space of maps in $\widetilde{M}^B(x, y, S^\circ, \overrightarrow{P})$ that also satisfy (11).

**Remark.** The definitions in the current section are analogous to those in [8], and a lot of the results in that paper carry over without change. We will cite several of them here without proof.

**Proposition 5.6.** (Compare to [8, Proposition 5.5].) There is a dense set of admissible $J$ with the property that for all generators $x,$ $y$, all homology classes $B \in \pi_2(x, y)$ and all partitions $\overrightarrow{P}$, the spaces $\widetilde{M}^B(x, y, S^\circ, \overrightarrow{P})$ are transversely cut out by the $\overrightarrow{J}$–equations.

**Proposition 5.7.** (Compare to [8, Proposition 5.6].) The expected dimension $\text{ind}(B, S^\circ, \overrightarrow{P})$ of $\widetilde{M}^B(x, y, S^\circ, \overrightarrow{P})$ is

$$\text{ind}(B, S^\circ, \overrightarrow{P}) = g - \chi(S) + 2e(B) + \# \overrightarrow{P},$$

where $e(B)$ is the Euler measure of the domain of $B$. 
It turns out that whether the curve $u \in \mathcal{M}^B(x, y, S^\circ, \overline{P})$ is embedded depends entirely on the topological data consisting of $B$, $S^\circ$, and $\overline{P}$. That is, there are entire components of embedded and of non embedded curves. Moreover, for such curves there is another index formula that does not depend on $S^\circ$. To give it we need some more definitions.

For a homology class $B \in \pi_2(x, y)$, and a point $z \in \alpha \cap \beta$, let $n_z(B)$ be the average multiplicity of $[B]$ at the four regions adjacent to $z$. Let $n_x = \sum_{x \in \Sigma} n_x(B)$, and $n_y = \sum_{y \in \Sigma} n_y(B)$.

For a sequence $\overline{\rho} = (\rho_1, \ldots, \rho_m)$ of Reeb chords, define the embedded Euler characteristic and embedded index
\[
\chi_{\text{emb}}(B, \overline{\rho}) = g + e(B) - n_x(B) - n_y(B) - \iota(\overline{\rho}),
\]
\[
\text{ind}(B, \overline{\rho}) = e(B) + n_x(B) + n_y(B) + \# \overline{\rho} + \iota(\overline{\rho}).
\]

**Definition 5.8.** For a homology class $B \in \pi_2(x, y)$ and a sequence $\overline{\rho} = (\rho_1, \ldots, \rho_m)$ of Reeb chords, define the embedded Euler characteristic and embedded index
\[
\chi_{\text{emb}}(B, \overline{\rho}) = g + e(B) - n_x(B) - n_y(B) - \iota(\overline{\rho}),
\]
\[
\text{ind}(B, \overline{\rho}) = e(B) + n_x(B) + n_y(B) + \# \overline{\rho} + \iota(\overline{\rho}).
\]

**Proposition 5.9.** Suppose $u \in \mathcal{M}^B(x, y, S^\circ, \overline{P})$. Exactly one of the following two statements holds.

1. $u$ is embedded and the following equalities hold.
   \[
   \chi(S^\circ) = \chi_{\text{emb}}(B, \overline{\rho}(\overline{P})),
   \]
   \[
   \text{ind}(B, S^\circ, \overline{P}) = \text{ind}(B, \overline{\rho}(\overline{P})),
   \]
   \[
   \mathcal{M}^B_{\text{emb}}(x, y, S^\circ, \overline{P}) = \mathcal{M}^B(x, y, S^\circ, \overline{P}).
   \]

2. $u$ is not embedded and the following inequalities hold.
   \[
   \chi(S^\circ) > \chi_{\text{emb}}(B, \overline{\rho}(\overline{P})),
   \]
   \[
   \text{ind}(B, S^\circ, \overline{P}) < \text{ind}(B, \overline{\rho}(\overline{P})),
   \]
   \[
   \mathcal{M}^B_{\text{emb}}(x, y, S^\circ, \overline{P}) = \emptyset.
   \]

**Proof.** This is essentially a restatement of [8, Proposition 5.47] \qed

Each of these moduli spaces has an $\mathbb{R}$–action that is translation in the $t$ factor. It is free on each $\mathcal{M}^B(x, y, S^\circ, \overline{P})$, except when the moduli space consists of a single curve $u$, where $\pi_D \circ u$ is a trivial $g$–fold cover of $D$, and $\pi_S \circ u$ is constant (so $B = 0$). We say that $u$ is stable if it is not this trivial case.

For moduli spaces of stable curves we mod out by this $\mathbb{R}$–action:

**Definition 5.10.** For given $x$, $y$, $S^\circ$, and $\overline{P}$, set
\[
\mathcal{M}^B(x, y, S^\circ, \overline{P}) = \mathcal{M}^B(x, y, S^\circ, \overline{P})/\mathbb{R},
\]
\[
\mathcal{M}^B_{\text{emb}}(x, y, S^\circ, \overline{P}) = \mathcal{M}^B_{\text{emb}}(x, y, S^\circ, \overline{P})/\mathbb{R}.
\]
5.4. Degenerations. The properties of the moduli spaces which are necessary to prove that the invariants are well defined and have the expected properties, are essentially the same as in [8]. Their proofs also carry over with minimal change. We sketch below the most important results.

To study degenerations we first pass to the space of holomorphic combs which are trees of holomorphic curves in $\Sigma \times \mathbb{D}$, and ones that live at East infinity, i.e. in $\mathbb{Z} \times \mathbb{R} \times \mathbb{D}$. This is the proper ambient space to work in, to ensure compactness.

The possible degenerations that can occur at the boundary of 1-dimensional moduli spaces of embedded curves are of two types. One is a two story holomorphic building, as usual in Floer theory. The second type consists of a single curve $u$ in $\Sigma \times \mathbb{D}$, with another curve degenerating at East infinity, at the $e$ punctures of $u$. Those curves that can degenerate at East infinity are of several types, join curves, split curves, and shuffle curves, that correspond to certain operations on the algebra $A(\mathcal{L})$. In fact, the types of curves that can appear dictate how the algebra should behave.

There are also corresponding gluing results, that tell us that in the cases we care about, a rigid holomorphic comb is indeed the boundary of a 1-dimensional space of curves. Unfortunately, in some cases the compactified moduli spaces are not compact 1-manifolds. However, we can still recover the necessary result that certain counts of 0-dimensional moduli spaces are even, and thus become 0, when reduced to $\mathbb{Z}/2$.

The only place where significant changes need to be made to the arguments, are in ruling out bubbling and boundary degenerations. The reason for the changes are the different homological assumptions we have made for $\Sigma$, $\mathbb{Z}$, $\alpha$, and $\beta$ in the definition of bordered sutured Heegaard diagrams. We give below the precise statement, and the modified proof. The rest of the arguments are essentially local in nature, and do not depend on these homological assumptions.

**Proposition 5.11.** Suppose $\mathcal{M} = \mathcal{M}^B(x, y, S^\partial, P)$ is 1-dimensional. Then the following types of degenerations cannot occur as the limit $u$ of a sequence $u_j$ of curves in $\mathcal{M}$.

1. $u$ bubbles off a closed curve.
2. $u$ has a boundary degeneration, i.e. $u$ is a nodal curve that collapses one or more properly embedded arcs in $(S, \partial S)$.

**Proof.** For (1) notice that if a closed curve bubbles off, it has to map to $\text{Int}(\Sigma) \times \mathbb{D} \cong \text{Int}(\Sigma)$ which has no closed components. In particular, $H_2(\text{Int}(\Sigma) \times \mathbb{D}) = 0$, and the bubble will have zero energy.

For (2), assume there is such a degeneration $u$ with source $S^\partial$. Repeating the argument in [8, Lemma 5.37], if an arc $a \in S^\partial$ collapses in $u$, then by strong boundary monotonicity its endpoints $\partial a$ lie on the same arc in $\partial \Sigma$. If $b$ is the arc in $\partial S^\partial$ connecting them, then $t \circ u$ is constant on $b$. Therefore, $\pi_\mathbb{D} \circ u$ is constant on the entire component $T$ of $S^\partial$ containing $b$. 

There is a compactification $\overline{T}$ of $T$, filling in the punctures by arcs, and an induced map $\overline{\pi}: \overline{T} \to \Sigma \times \mathbb{D}$. Under $\pi$, the boundary $\partial T$ can only map to either $\alpha \cup Z \times \{1\} \times \mathbb{R}$ or to $\beta \times \{0\} \times \mathbb{R}$, which have different $t$ components. Therefore, the entire boundary maps entirely to one of them, or entirely to the other. In particular, we have either a map

$$\pi_\Sigma \circ \overline{\pi}: (T, \partial T) \to (\Sigma, \alpha \cup Z),$$

or a map

$$\pi_\Sigma \circ \overline{\pi}: (T, \partial T) \to (\Sigma, \beta).$$

By homological linear independence both of the groups $H_2(\Sigma, \alpha \cup Z)$ and $H_2(\Sigma, \beta)$ are 0, and $u|_T$ must have zero energy. □

The equivalent statement in the bordered setting is necessary for [8, Proposition 5.32].

6. Diagram gradings

In this section we define gradings on the set of generators $G(H)$ for a given bordered sutured diagram $H$. More precisely, if $H$ represents the bordered sutured manifold $(Y, \Gamma, Z)$, for each Spin$^c$–structure $s \in \text{Spin}^c(Y, \partial Y \setminus F(Z))$ we define grading sets $\text{Gr}(H, s)$ and $\text{Gr}(H, s)$ which have left actions by $\text{Gr}(Z)$ and $\text{Gr}(Z)$, respectively, and right actions by $\text{Gr}(Z)$ and $\text{Gr}(Z)$, respectively. Then we define maps $G(H, s) \to \text{Gr}(H, s)$ and $G(H, s) \to \text{Gr}(H, s)$, which are well-defined up to a shift to be made precise below.

In the next couple of sections we use these maps to define relative gradings on the bordered sutured invariants.

6.1. Domain gradings. We start by defining a grading on all homology classes in $\pi_2(x, y)$ for $x$ and $y$ generators in $G(H)$. We will abuse notation and will not distinguish between a given homology class and its associated domain in $H_2(\Sigma, Z \cup \alpha \cup \beta)$.

**Definition 6.1.** Given a domain $B \in \pi_2(x, y)$ define

$$\text{gr}(B) = (-e(B) - n_x(B) - n_y(B), \partial^\beta B) \in \text{Gr}(Z).$$

Given a grading reduction $r$ from $\text{Gr}(Z)$ to $\text{Gr}(Z)$, define

$$\underline{\text{gr}}(B) = r(I(o(x))) \cdot \text{gr}(B) \cdot r(I(o(y)))^{-1} \in \text{Gr}(Z).$$

The basic properties of these gradings, and in fact the reason they are called gradings is that they are compatible with composition of domains. They are also compatible with the indices of moduli spaces.

**Proposition 6.2.** Given a domain $B \in \pi_2(x, y)$, for any compatible sequence $\overline{\rho}$ of sets of Reeb chords, we have $\text{gr}(B) = \lambda^{-\text{ind}(B, \overline{\rho}) + \# \overline{\rho} \cdot \text{gr}(\overline{\rho})}$.

For any two domains $B_1 \in \pi_2(x, y)$ and $B_2 \in \pi_2(y, z)$, their concatenation has grading $\text{gr}(B_1 \ast B_2) = \text{gr}(B_1) \cdot \text{gr}(B_2)$.

Similar statements hold for $\underline{\text{gr}}(B)$. 
Proof. For the first statement, recall that \( \text{ind}(B, \overrightarrow{\rho}) = e(B) + n_x(B) + n_y(B) + \nu(\overrightarrow{\rho}) + \# \overrightarrow{\rho} \), and the homological components of \( \text{gr}(\overrightarrow{\rho}) \) and \( \text{gr}(B) \) are both \( \partial^B B \) for a compatible pair. The second statement follows from the first, using the fact that the index is additive for domains, and \( \lambda \) is central.

For the equivalent statement for \( \text{gr} \), we just have to use \( \text{gr}(I_{o(x)} \cdot a(\overrightarrow{\rho}) \cdot I_{o(y)}) \), instead of \( \text{gr}(\overrightarrow{\rho}) \) which is not defined, and notice that the reduction terms match up. \( \square \)

6.2. Generator gradings. We will give a relative grading for the generators in each \( \text{Spin}^c \)-structure. Here a relative grading in a \( G \)-set means a map \( g: \mathcal{G}(\mathcal{H}, s) \rightarrow A \), where \( G \) acts on \( A \), say on the right. Two such gradings \( g \) and \( g' \) with values in \( A \) and \( A' \) are equivalent, if there is a bijection \( \phi: A \rightarrow A' \), such that \( \phi \) commutes with both the \( G \)-actions and with \( g \) and \( g' \). The traditional case of a relative \( \mathbb{Z} \) or \( \mathbb{Z}/n \)-valued grading corresponds to \( \mathbb{Z} \) acting on its quotient, with the grading map defined up to an overall shift by a constant.

**Definition 6.3.** For a Heegaard diagram \( \mathcal{H} \) and generator \( x \in \mathcal{G}(\mathcal{H}) \) define the stabilizer subgroup \( \mathcal{P}(x) = \text{gr}(\pi_2(x, x)) \subset \text{Gr}(\mathcal{Z}) \). For any \( \text{Spin}^c \)-structure \( s \) pick a generator \( x \in \mathcal{G}(\mathcal{H}, s) \) and let \( \text{Gr}(\mathcal{H}, s) \) be the set of right cosets \( \mathcal{P}(x_0) \backslash \text{Gr}(\mathcal{Z}) \) with the usual right \( \text{Gr}(\mathcal{Z}) \)-action. Define the grading \( \text{gr}: \mathcal{G}(\mathcal{H}, s) \rightarrow \text{Gr}(\mathcal{H}, s) \) by \( \text{gr}(x) = \mathcal{P} \cdot \text{gr}(B) \) for any \( B \in \pi_2(x_0, x) \).

**Proposition 6.4.** Assuming \( \mathcal{G}(\mathcal{H}, s) \) is nonempty, this is a well-defined relative grading, independent of the choice of \( x_0 \), and has the property \( \text{gr}(x) \cdot \text{gr}(B) = \text{gr}(y) \) for any \( B \in \pi_2(x, y) \).

Proof. These follow quickly from the fact that concatenation of domains respects the grading. For example, for any two domains \( B_1, B_2 \) from \( x_0 \) to \( x \), the cosets \( \mathcal{P}(x_0) \cdot \text{gr}(B_1) \) are the same. Independence from the choice of \( x_0 \) follows from the fact that \( \mathcal{P}(x) \) is a conjugate of \( \mathcal{P}(x_0) \). \( \square \)

Fixing a grading reduction \( r \), and setting \( \mathcal{P}(x) = \text{gr}(\pi_2(x, x)) = r(I_{o(x)}) \cdot \mathcal{P}(x) \cdot r(I_{o(x)}))^{-1} \), we get a reduced grading set \( \text{Gr}(\mathcal{H}, s) \) with a right \( \text{Gr}(\mathcal{Z}) \)-action, and reduced grading \( \text{gr} \) on \( \text{Gr}(\mathcal{H}, s) \) with the same properties as \( \text{gr} \).

In light of the discussion in section 2.6, the sets \( \text{Gr}(\mathcal{H}, s) \) and \( \text{Gr}(\mathcal{H}, s) \) have left actions by \( \text{Gr}(-\mathcal{Z}) \) and \( \text{Gr}(-\mathcal{Z}) \), respectively. Keep in mind that for the reduced grading, the reduction term used for acting on \( \text{gr}(x) \) is \( r(I_{o(x)}) \), corresponding to the complementary idempotent of \( x \).

To define the grading on the bimodules \( B \mathcal{S} \mathcal{D} A \), we will need to take a mixed approach. Given a bordered sutured manifold \((Y, \Gamma, \mathcal{Z}_1 \cup \mathcal{Z}_2)\), thought of as a cobordism from \( -\mathcal{Z}_1 \) to \( \mathcal{Z}_2 \), we will use the left action of \( \text{Gr}(-\mathcal{Z}_1) \subset \text{Gr}(-\mathcal{Z}_1 \cup \mathcal{Z}_2)) \) and the right action of \( \text{Gr}(\mathcal{Z}_2) \subset \text{Gr}(\mathcal{Z}_1 \cup \mathcal{Z}_2) \). The two actions commute since the correction term \( L \) vanishes on mixed pairs. Moreover, the Maslov components act the same on both sides.
6.3. A simpler description. In the special case when \( \mathcal{Z} = \emptyset \) and the manifold is just sutured, the grading takes a simpler form that is the same as the usual relative grading on \( SFH \). Recall that the divisibility of a Spin\(^c\)-structure \( s \) is the integer \( \mathrm{div}(s) = \gcd_{a \in H_2(Y)} (c_1(s), \alpha) \), and that sutured Floer homology groups \( SFH(Y, s) \) are relatively-graded by the cyclic group \( \mathbb{Z}/\mathrm{div}(s) \). (See [5].)

**Theorem 6.5.** Let \( \mathcal{H} \) be a Heegaard diagram for a sutured manifold \((Y, \Gamma)\), which can also be interpreted as a diagram for the bordered sutured manifold \((Y, \Gamma, \emptyset)\). For any Spin\(^c\)-structure \( s \), the grading sets are \( \overline{\mathrm{Gr}}(\mathcal{H}, s) = \mathrm{Gr}(\mathcal{H}, s) = \frac{1}{2}\mathbb{Z}/\mathrm{div}(s) \), with the usual action by \( \overline{\mathrm{Gr}}(\emptyset) = \mathrm{Gr}(\emptyset) = \frac{1}{2}\mathbb{Z} \). Moreover, the relative gradings \( \mathrm{gr} = \mathrm{gr}^\ast \) on \( \overline{\mathrm{Gr}}(\mathcal{H}, s) \) coincide with relative grading on \( SFH \). In particular, only the integer gradings are occupied. □

**Proof.** The grading on \( SFH \) is defined in essentially the same way, on a diagram level. There a domain \( B \in \pi_2(x, y) \) is graded as \( -\mathrm{ind}(B) = -e(B) - n_x - n_y = \mathrm{gr}(B) \in \mathrm{Gr}(\emptyset) \). The rest of definition is exactly the same, with the result that the gradings coincide, except that in the bordered sutured case we start with the bigger group \( \frac{1}{2}\mathbb{Z} \), while \( \mathrm{gr}(B) = -\mathrm{ind}(B) \) still takes only integer values.

In general, the grading sets \( \mathrm{Gr}(\mathcal{H}, s) \) and \( \mathrm{Gr}^\ast(\mathcal{H}, s) \) can look very complicated, but if we forget some of the structure we can give a reasonably nice description similar to the purely sutured case.

**Proposition 6.6.** There is a projection map \( \pi: \overline{\mathrm{Gr}}(\mathcal{H}, s) \to \mathrm{im}(i_*: H_1(F) \to H_1(Y)) \) with the following properties. Each fiber looks like \( \frac{1}{2}\mathbb{Z}/\mathrm{div}(s) \), with the usual translation action by the central subgroup \( \frac{1}{2}\mathbb{Z}, 0 \). Any element of the form \((*, \alpha)\) permutes the fibers of \( \pi \), sending \( \pi^{-1}(\beta) \) to \( \pi^{-1}(\beta + i_*(\alpha)) \), while preserving the \( \frac{1}{2}\mathbb{Z} \)-action.

**Proof.** Recall that \( \pi_2(x, x) \) is isomorphic to \( H_2(Y, F) \) by attaching the cores of 2–handles and half-handles. Inside, the subgroup \( \pi_2^\partial(x, x) \) of provincial periodic domains is isomorphic to \( H_2(Y) \subset H_2(Y, F) \). Similar to the purely sutured case, for provincial periodic domains \( e(B) + 2n_x(B) = \langle c_1(s), [B] \rangle \).

The subgroup
\[
\mathcal{P}^\partial(x) = \mathrm{gr}(\pi_2^\partial(x, x)) = (\langle c_1(s), H_2(Y) \rangle, 0) = (\mathrm{div}(s)\mathbb{Z}, 0)
\]
is central, and therefore \( \mathcal{P}^\partial(x) = \mathrm{gr}(\pi_2^\partial(x, x)) \subset \mathcal{P}(x) \) is also \( (\mathrm{div}(s)\mathbb{Z}, 0) \) and central in \( \overline{\mathrm{Gr}}(\mathcal{Z}) \).

In particular, taking the quotient \( \overline{\mathrm{Gr}}(\mathcal{Z})/\mathcal{P}^\partial \) has the effect of reducing the Maslov component modulo \( \mathrm{div}(s) \). On the other hand, since any two classes \( B_{1,2} \) with the same \( \partial^\partial \) differ by a provincial domain, \( \mathcal{P}/\mathcal{P}^\partial \) is isomorphic to \( \mathrm{im}(\partial: H_2(Y, F) \to H_1(F)) = \ker(i_*: H_1(F) \to H_1(Y)) \). Ignoring the Maslov component, passing to \( (\mathcal{P}/\mathcal{P}^\partial) \setminus (\overline{\mathrm{Gr}}(\mathcal{Z})/\mathcal{P}^\partial) = \mathcal{P}/\overline{\mathrm{Gr}}(\mathcal{Z}) \) reduces the homological component \( H_1(F) \) modulo \( \ker i_* \). Therefore, the new homological component is valued in \( H_1(F)/\ker i_* \cong \mathrm{im}(\partial) \). □
6.4. Grading and gluing. The most important property of the reduced grading is that it behaves nicely under gluing of diagrams. This will later allow us to show that the pairing on $\text{BSDA}$ respects the grading. First, we define a grading for a pair of diagrams which can be glued together, and then show it coincides with the grading on the gluing.

Suppose $H_1$ and $H_2$ are diagrams for $(Y_1, \Gamma_1, -Z_1 \cup Z_2)$ and $(Y_2, \Gamma_2, -Z_2 \cup Z_3)$, respectively, and fix reductions for $\text{Gr}(Z_1)$, $\text{Gr}(Z_2)$, and $\text{Gr}(Z_3)$. Recall that $\text{Gr}(H_1, s_1)$ has left and right actions by $\text{Gr}(Z_1)$ and $\text{Gr}(Z_2)$, respectively, while $\text{Gr}(H_2, s_2)$ has left and right actions by $\text{Gr}(Z_2)$ and $\text{Gr}(Z_3)$, respectively.

It is easy to see that generators in $H_1 \cup Z_2, H_2$ correspond to pairs of generators with complementary idempotents at $Z_2$, and there are restriction maps on Spin$^c$-structures, such that $s(x_1, x_2)\big|_{Y_i} = s(x_i)$. Let $F_i = F(Z_i)$. From the long exact homology sequence for the triple $(Y_1 \cup Y_2, F_1 \cup F_2 \cup F_3, F_1 \cup F_3)$ and Poincaré duality, we can see that $\{s : s_{Y_i} = s_i\}$ is either empty or an affine set over $\text{im}(i_* : H_1(F_2) \to H_1(Y_1 \cup Y_2, F_1 \cup F_3))$.

**Definition 6.7.** Let $\text{Gr}(H_1, H_2, s_1, s_2)$ be the product

$$\text{Gr}(H_1, s_1) \times_{\text{Gr}(Z_2)} \text{Gr}(H_2, s_2),$$

i.e. the usual product of the two sets, modulo the relation $(a \cdot g, b) \sim (a, g \cdot b)$ for any $g \in \text{Gr}(Z_2)$. It inherits a left action by $\text{Gr}(Z_1)$ and a right action by $\text{Gr}(Z_3)$, which commute and where the Maslov components act in the same way.

Define a grading on $\bigcup_{s_{Y_i} = s_i} \mathcal{G}(H_1 \cup H_2, s)$ by

$$\text{gr}'(x_1, x_2) = ([\text{gr}(x_1), \text{gr}(x_2)] \in \text{Gr}(H_1, H_2, s_1, s_2).$$

**Theorem 6.8.** Assume $s_1$ and $s_2$ are compatible, i.e. there is at least one $s$ restricting to each of them. There is a projection on the mixed grading set

$$\pi : \text{Gr}(H_1, H_2, s_1, s_2) \to \text{im}(i_* : H_1(F_2) \to H_1(Y_1 \cup Y_2, F_1 \cup F_3)),
$$

defined up to a shift in the image, with the following properties.

1. For any two generators $x$ and $y$ with $s(x)|_{Y_i} = s(y)|_{Y_i} = s_i$, we have

$$\text{PD}(s(y) - s(x)) = \pi(\text{gr}'(x)) - \pi(\text{gr}'(y)),$$

i.e. $\pi$ distinguishes Spin$^c$-structures. Moreover, the $\text{Gr}(Z_1)$ and $\text{Gr}(Z_3)$-actions preserve the fibers of $\pi$.

2. For each $s$, such that $s_{Y_i} = s_i$, there is a unique fiber $\text{Gr}_s$ of $\pi$, such that the grading $\text{gr}'|_{\text{Gr}(H_1 \cup H_2, s)}$ is valued in $\text{Gr}_s$, and is equivalent to $\text{gr}$ valued in $\text{Gr}(H_1 \cup H_2, s)$. 

**Proof.** It is useful to pass to only right actions, as the grading sets were originally defined. Recall that $\text{Gr}(H_i, s_i)$ are defined as the cosets $\mathcal{P}(x_i) \backslash \text{Gr}(-Z_i \cup Z_{i+1})$ for some $x_i \in \mathcal{G}(H_i, s_i)$, for $i = 1, 2$. We define the mixed grading as the quotient of the product by the action of the subgroup $H' = \{((a, \alpha), (-a, -\alpha))\} \cong \text{Gr}(Z_2)$, for all $a \in \frac{1}{2}Z$, $\alpha \in H_1(F_2)$ on the right.
Since $H'$ commutes with $\text{Gr}(-Z_1) \times \text{Gr}(Z_3) \subset \text{Gr}(-Z_1 \cup Z_2) \times \text{Gr}(-Z_2 \cup Z_4)$, we can think of $\text{Gr}((H_1, H_2, s_1, s_2))$ as the cosets
$$p' \setminus \text{Gr}(-Z_1 \cup Z_2) \times \text{Gr}(-Z_2 \cup Z_3)/H',$$
with a right action by $\text{Gr}(-Z_1) \times \text{Gr}(Z_3)$. Here $P' = P(x_1) \times P(x_2)$. Since the Maslov components behave nicely and commute with everything, we can take a quotient by the Maslov component $\{(a, 0), (-a, 0)\}$ in $H'$. We get
$$\text{Gr}((H_1, H_2, s_1, s_2)) = P' \setminus \text{Gr}(-Z_1 \cup Z_2 \cup -Z_2 \cup Z_3)/H,$$
with a right action of $\text{Gr}(-Z_1 \cup Z_3)$. Here $H = \{(0, \alpha, -\alpha) : \alpha \in H_1(F_2)\} \subset \text{Gr}(-Z_2 \cup -Z_3)$, while the abelian group $P$ is generated by the two subgroups $P(x_i)$. In other words, the mixed grading set has elements of the form $[a] = P \cdot a \cdot H$, with action $[a] \cdot g = [a \cdot g]$.

Let $\Pi$ be addition of the $H_1(F_2)$-homological component terms together, and ignoring the rest. $H$ is in the kernel of $\Pi$, while $\Pi|_P$ is the restriction of $\Pi$ to $\text{gr}'$ to
$$\pi_2(x_1, x_1) \times \pi_2(x_2, x_2) \cong H_2(Y_1, F_1 \cup F_2) \times H_2(Y_2, F_2 \cup F_3) \cong H_2(Y_1 \cup Y_2, F_1 \cup F_2 \cup F_3),$$
and coincides with the boundary map
$$\partial : H_2(Y_1 \cup Y_2, F_1 \cup F_2 \cup F_3) \rightarrow H_1(F_1 \cup F_2 \cup F_3, F_1 \cup F_3) \cong H_1(F_2)$$
from the long exact sequence of the triple. Therefore $\Pi(P) = \ker(i_* : H_1(F_2) \rightarrow H_1(Y_1 \cup Y_2, F_1 \cup F_3))$, and $\Pi$ descends on the cosets to a projection $\pi$ with values in $H_1(F)/\ker i_* \cong \im i_*$. A different choice of $x_i$ only shifts the homological components, and so the image of $\pi$.

To prove (1), we need to check that for any compatible $y_i \in G((H_i, s_i)$, the difference $s(y_1, y_2) - s(x_1, x_2)$ is the same as $-\pi(\text{gr}((y_1, y_2)) - \pi(\text{gr}((x_1, x_2)))$. Suppose $B_i \in \pi_2(x_i, y_i)$. Then the latter difference is $-\pi(|\text{gr}(B_1), \text{gr}(B_2)|) + \pi([0, 0]) = -i_*(h_1 + h_2)$, where $h_1$ is the $H_1(F_2)$ part of the homological component of $\text{gr}(B_1)$. Since the $Z_2$-idempotents of $x_1$ and $x_2$ are complementary, as well as those of $y_1$ and $y_2$, the reduction terms $r$ cancel, and we can look at $\text{gr}(B_1)$ and $\text{gr}(B_2)$, instead. Therefore $h_1 + h_2 = \partial^B B_1 + \partial^B B_2$, interpreted as an element of $H_1(F_2) \subset H_1(Z_2, a_2)$. Here $\partial^B$ denotes the $Z_2$ part of $\partial^B$. It is indeed in that subgroup, again because of the complementary idempotents.

By the proof of proposition 4.14, we have $s(y) - s(x) = PD([a - b])$, where $a$ and $b$ are any two 1-chains in $\alpha$ and $\beta$, with $\partial a = y - x + z$ and $\partial b = y - x$, where $z$ is a 0-chain in $Z_1 \cup Z_3$. The boundaries of $B_1$ and $B_2$ almost give us such chains. Let $a_i = \partial^B B_1$, $b_i = \partial^B B_2$, and $c_i = \partial^B B_2$, as chains. Then $[c_1 + c_2] = h_1 + h_2$, in $H_1(Z_2, a_2)$, and $[a_1 + a_2 + b_1 + b_2 + c_1 + c_2] = -$\$[\partial^B B_1 + \partial^B B_2] = 0 \in H_1(Y_1 \cup Y_2, F_1 \cup F_3)$. Notice that we can represent $h_1 + h_2 \in H_1(F_2)$ by the 1-chain $c_1 + c_2 + d$, where $d$ is a sum of some $\alpha$-arcs in $H_1$. Let $a = a_1 + a_2 - d$, and $b = b_1 + b_2$. They have the desired properties, and $[a - b] = [a_1 + a_2 + b_1 + b_2 + c_1 + c_2] - [c_1 + c_2 + d] =$
0 – i_*(h_1 + h_2). Thus π does indeed distinguish Spin^c–structures. It is clear that the π|_{Gr(−Z_1∪Z_3)} = 0 and the action preserves the fibers.

For (2), we know the restriction gr'(⊂(H_1∪H_2,s)) takes values in a unique fiber Gr. To see that the grading is equivalent to gr, we need three results. First, we need to show that the action of Gr(−Z_1∪Z_3) is transitive on Gr. Second, we need to show that the stabilizers of gr'(y_0) and gr(y_0) are the same for some y_0 ∈ G(H_1 ∪ H_2, s). These two steps show that Gr and Gr(∪H_1 ∪ H_2, s) are equivalent as grading sets. Finally, we need to show that for any other y ∈ G(H_1 ∪ H_2, s), there is at least one g ∈ Gr(−Z_1∪Z_3), such that gr(y) = gr(y_0) · g and gr'(y) = gr'(y_0) · g.

For the first part, notice that Gr(−Z_1∪Z_3) × H is exactly the kernel of Π, while the reduction to π was exactly by the image of P. Therefore, if π([a_1]) = π([a_2]), then Π(a_1) = Π(p · a_2) for some p ∈ P, so a_1 = p · a_2 · g · h for some g ∈ Gr(−Z_1∪Z_3) and h ∈ H. In other words, [a_1] = [a_2] · g.

For the second part, we can assume (x_1, x_2) are in s, and use that as y_0. In this case the stabilizer for gr' is (P · H) ∩ Gr(−Z_1∪Z_3). We may also assume the base idempotents for r are I_{π_i(x_1)} for Z_1, I_{π_2(x_1)} = I_{π_3(x_2)} for Z_2, and I_{π_2(x_2)} for Z_3. This ensures gr = gr for periodic domains at (x_1, x_2).

This corresponds to the gradings of pairs of periodic classes B_i ∈ π_2(x_i) with ∂_B_1 + ∂_B_2 = 0, canceling those terms. But such pairs are in 1–to–1 correspondence with periodic class B ∈ π_2((x_1, x_2), (x_1, x_2)). The gradings for such pairs are additive, so the stabilizer of gr' is the same as gr.

Finally, to show the relative gradings are the same, pick any B ∈ π_2(y_0, y). It decomposes into two classes B_i connecting the H_i components of y_0 and y with canceling ∂_B_i. Similar to the above discussion, the regular gradings satisfy gr(B) = gr(B_1) gr(B_2). The reduction terms match up, so the same holds for gr. Thus gr(B) is the grading difference between y and y_0 for both gradings.

7. One-sided invariants

7.1. Overview. To a bordered sutured manifold (Y, Γ, Z, φ), we will associate the following invariants. Each of them is defined up to homotopy equivalence, in the appropriate sense.

   (1) A right A_∞–module over A(Z), denoted BS^A(Y, Γ, Z, φ).
   (2) A left type D structure over A(−Z), denoted BSD(Y, Γ, Z, φ).
   (3) A left differential graded module over A(−Z), which we denote BSD_M(Y, Γ, Z, φ).

7.2. Type D structures. Although we can express all of the invariants and their properties in terms of differential graded modules and A_∞–modules, from a practical standpoint it is more convenient to use the language of type D structures introduced in [8]. We recall here the definitions and basic properties. To simplify the discussion we will restrict to the case where the
algebra is differential graded, and not a general $A_\infty$–algebra. This is all that is necessary for the present applications.

Remark. Any algebra or module has an implicit action by a base ring, and any usual tensor product $\otimes$ is taken over such a base ring. In the case of the algebra $A(Z)$ associated with an arc algebra, and any modules over it, the base ring is the idempotent ring $I(Z)$. We will omit the base ring from the notation to avoid clutter.

Let $A$ be a differential graded algebra with differential $\mu_1$ and multiplication $\mu_2$.

**Definition 7.1.** A (left) type $D$ structure over $A$ is a graded module $N$ over the base ring, with a homogeneous operation

$$\delta: N \to (A \otimes N)[1],$$

satisfying the compatibility condition

$$\delta \circ (\mu_1 \otimes \text{id}_N) + (\mu_2 \otimes \text{id}_N) \circ (\text{id}_A \otimes \delta) \circ \delta = 0.$$

We can define induced maps

$$\delta_k: N \to (A^\otimes k \otimes N)[k],$$

by setting

$$\delta_k = \begin{cases} 
\text{id}_N & \text{for } k = 0, \\
(\text{id}_A \otimes \delta_{k-1}) \circ \delta & \text{for } k \geq 1.
\end{cases}$$

**Definition 7.2.** We say a type $D$ structure $N$ is bounded if for any $n \in N$, $\delta_k(n) = 0$ for sufficiently large $k$.

Given two left type $D$ structures $N$, $N'$ over $A$, the homomorphism space $\text{Hom}(N, A \otimes N')$, graded by grading shifts, becomes a graded chain complex with differential

$$Df = (\mu_1 \otimes \text{id}_{N'}) \circ f + (\mu_2 \otimes \text{id}_{N'}) \circ (\text{id}_A \otimes \delta') \circ f + (\mu_2 \otimes \text{id}_{N'}) \circ (\text{id}_A \otimes f) \circ \delta.$$

**Definition 7.3.** A type $D$ structure map from $N$ to $N'$ is a cycle in the above chain complex.

Two such maps $f$ and $g$ are homotopic if $f - g = Dh$ for some $h \in \text{Hom}(N, A \otimes N')$, called a homotopy from $f$ to $g$.

If $f: N \to A \otimes N'$, and $g: N' \to A \otimes N''$ are type $D$ structure maps, their composition $f \circ g: N \to A \otimes N''$ is defined to be

$$f \circ g = (\mu_2 \otimes \text{id}_{N''}) \circ (\text{id}_A \otimes g) \circ f.$$

With the above definitions, type $D$ structures over $A$ form a differential graded category. This allows us, among other things, to talk about homotopy equivalences. (In general, for an $A_\infty$–algebra $A$, this is an $A_\infty$–category.)
Let $M$ be a right $\mathcal{A}_\infty$–module over $A$, with higher $\mathcal{A}_\infty$ actions
\[ m_k: M \otimes A^{\otimes(k-1)} \to M[2-k], \text{ for } k \geq 1. \]

Let $N$ be a left type $D$ structure. We can define a special tensor product between them.

**Definition 7.4.** Assuming at least one of $M$ and $N$ is bounded, let $M \boxtimes_A N$ be the graded vector space $M \otimes N$, with differential
\[ \partial: M \otimes N \to (M \otimes N)[1], \]
defined by
\[ \partial = \sum_{k=1}^{\infty} (m_k \otimes \text{id}_N) \circ (\text{id}_M \otimes \delta_{k-1}). \]

The condition that $M$ or $N$ is bounded guarantees that the sum is always finite. In that case $\partial^2 = 0$ (using $\mathbb{Z}/2$ coefficients), and $M \boxtimes N$ is a graded chain complex.

The most important property of $\boxtimes$, as shown in [7] is that it is functorial up to homotopy and induces a bifunctor on the level of derived categories.

The chain complex $A \boxtimes N$ is in fact a graded differential module over $A$, with differential
\[ \partial = \mu_1 \otimes \text{id}_N + (\mu_2 \otimes \text{id}_N) \circ \delta, \]
and algebra action
\[ a \cdot (b, n) = (\mu_2(a, b), n). \]

In a certain sense working with type $D$ structures is equivalent to working with their associated left modules. In particular, $A \boxtimes -$ is a functor, and $M \boxtimes (A \boxtimes N)$ and $M \boxtimes N$ are homotopy equivalent as graded chain complexes.

### 7.3. $\widehat{\text{BSD}}$ and $\widehat{\text{BSD}}_M$

Let $\mathcal{H} = (\Sigma, \alpha, \beta, \mathcal{Z}, \psi)$ be a provincially admissible bordered sutured Heegaard diagram, and let $J$ be an admissible almost complex structure.

We will define $\widehat{\text{BSD}}$ as a type $D$ structure over $\mathcal{A}(-\mathcal{Z})$.

**Definition 7.5.** Fix a relative Spin$^c$–structure $s \in \text{Spin}^c(Y, \partial Y \setminus F)$. Let $\widehat{\text{BSD}}(\mathcal{H}, J, s)$ be the $\mathbb{Z}/2$ vector space generated by the set of all generators $\mathcal{G}(\mathcal{H}, s)$. Give it the structure of an $\mathcal{I}(-\mathcal{Z})$ module as follows. For any $x \in \mathcal{G}(\mathcal{H}, s)$ set
\[ I(s) \cdot x = \begin{cases} x & \text{if } s = \overline{s}(x), \\ 0 & \text{otherwise.} \end{cases} \]

We consider only discrete partitions $\overline{P} = (\{q_1\}, \ldots, \{q_m\})$. 
Definition 7.6. For \( x, y \in G(\mathcal{H}) \) define
\[
a_{x, y} = \sum_{\text{ind}(B, \overrightarrow{\rho}) = 1, \overrightarrow{P} \text{ discrete}} \# M^B_{\text{emb}}(x, y, S^\omega, \overrightarrow{P}) \cdot a(-P_1) \cdots a(-P_m).
\]

We compute \( a(-P_1) \), since the Reeb chord \( \rho_i \) labeling the puncture \( q_i \) is oriented opposite from \(-Z\).

Definition 7.7. Define \( \delta: \widehat{BSD}(\mathcal{H}, J, s) \to A(-Z) \otimes \widehat{BSD}(\mathcal{H}, J, s) \) as follows.
\[
\delta(x) = \sum_{y \in G(\mathcal{H})} a_{x, y} \otimes y.
\]

Note that, \( \pi_2(x, y) \) is nonempty if and only if \( s(x) = s(y) \), so the range of \( \delta \) is indeed correct.

Theorem 7.8. The following statements are true.

1. \( \widehat{BSD}(\mathcal{H}, J, s) \) equipped with \( \delta \), and the grading \( \text{Gr}(\mathcal{H}, s) \)-valued grading \( \text{gr} \) is a type D structure over \( A(-Z) \). In particular,
\[
\lambda^{-1} \cdot \text{gr}(x) = \text{gr}(a) \cdot \text{gr}(y),
\]
whenever \( \delta(x) \) contains the term \( a \otimes y \).

2. If \( \mathcal{H} \) is totally admissible, \( \widehat{BSD}(\mathcal{H}, J, s) \) is bounded.

3. For any two provincially admissible diagrams \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), equipped with admissible almost complex structures \( J_1 \) and \( J_2 \), there is a graded homotopy equivalence
\[
\widehat{BSD}(\mathcal{H}_1, J_1, s) \simeq \widehat{BSD}(\mathcal{H}_2, J_2, s).
\]

Therefore we can talk about \( \widehat{BSD}(Y, \Gamma, Z, \psi, s) \) or just \( \widehat{BSD}(Y, \Gamma, s) \), relatively graded by \( \text{Gr}(Y, s) \).

Proof. In light of the discussion in section 5.4, the proofs carry over from those for \( \widehat{CFD} \) in the bordered case. We sketch the main steps below.

For (1), first we use provincial admissibility to guarantee the sums in the definitions are finite. Indeed, only finitely many provincial domains \( B \in \pi_2^0(x, y) \) are positive and can contribute. The number of non provincial domains ends up irrelevant, since only finitely many sequences of elements of \( A(-Z) \) have nonzero product.

To show that Eq. (6) is satisfied, we count possible degenerations of 1-dimensional moduli spaces, which are always an even number. Two story buildings correspond to the \((\mu_2 \otimes \text{id}_N) \circ (\text{id}_A \otimes \delta) \circ \delta \) term. The degenerations with a curve at East infinity correspond to the \((\mu_1 \otimes \text{id}_N) \circ \delta \) term.

To show that the grading condition is satisfied, recall that \( a \otimes y \) can be a term in \( \delta(x) \) only if there is a domain \( B \in \pi_2(x, y) \), and a compatible sequence \( \overrightarrow{\rho} = \{(\rho_1), \ldots, (\rho_p)\} \) of Reeb chords, such that \( \text{ind}(B, \overrightarrow{\rho}) = 1 \), and \( a = I_{\pi(x)} \cdot a(-\rho_1) \cdots a(-\rho_p) \cdot I_{\pi(y)} \). We will prove the statement for \( \text{gr} \),
which allows us to ignore the idempotents at the end. The $\text{gr}$-version then follows from using the same reduction terms.

Notice that $\text{gr}_{-Z}(-\rho_i) = (-1/2, [\rho_i])$, and so

$$\text{gr}(a) = (-p/2 + \sum_{i<j} L_{-Z}([-\rho_i], [-\rho_j]), -[\rho]),$$

which we can also interpret as a $\text{Gr}(Z)$-grading acting on the right. On the other hand, $\text{gr}_{Z}(\rho_i) = (-1/2, [\rho_i])$, and

$$\text{gr}(-\rho) = (-p/2 + \sum_{i<j} L_{Z}([\rho_i], [\rho_j]), [\rho]).$$

Recall that $L_{Z}$ and $L_{-Z}$ have opposite signs. Therefore, we have the relation $\text{gr}(-\rho)^{-1} = \lambda^{-p} \text{gr}(a)$. Thus, we have

$$\text{gr}(a \otimes y) = \text{gr}_{-Z}(a) \cdot \text{gr}(y) = \text{gr}(y) \cdot \text{gr}_{Z}(a)$$

$$= \text{gr}(x) \cdot \text{gr}(B) \text{gr}(a) = \text{gr}(x) \cdot \lambda^{-\text{ind}(B, \rho)} + \#(\rho) \text{gr}(\rho) \text{gr}(a)$$

$$= \text{gr}(x) \cdot \lambda^{-1+p} \lambda^{-p} \text{gr}(a)^{-1} \text{gr}(a) = \text{gr}(x) \cdot \lambda^{-1}.$$

For (2), we use the fact that with full admissibility, only finitely many domains $B \in \pi_2(x, y)$ are positive, and could contribute to $\delta_k$, for any $k$. Therefore, only finitely many of the terms of $\delta_k(x)$ are nonzero.

For (3) we use the fact that provincially admissible diagrams can be connected by Heegaard moves. To isotopacies and changes of almost complex structure, we associate moduli spaces, depending on a path $(\mathcal{H}_t, J_t)$ of isotopic diagrams and almost complex structures. Counting 0-dimensional spaces gives a type $D$ map $\hat{BSD}(\mathcal{H}_0, J_0) \to \hat{BSD}(\mathcal{H}_1, J_1)$. Analogous results to those in section 5 and counting the ends of 1-dimensional moduli spaces show that the map is well defined and is in fact a homotopy equivalence. To handleslides, we associate maps coming from counting holomorphic triangles, which also behave as necessary in this special case.

For invariance of the grading, we show that both in time-dependent moduli spaces, and when counting triangles we can grade domains compatibly. In particular, the stabilizers are still conjugate, and the grading set is preserved. In both cases we count domains with index 0, so the relative gradings of individual elements are also preserved. □

If we ignore $\text{Spin}^c$ structures we can talk about the total invariant

$$\hat{BSD}(Y, \Gamma) = \bigoplus_{s \in \text{Spin}^c(Y, \Gamma)} \hat{BSD}(Y, \Gamma, s).$$

We define $\hat{BSD}_M$ in terms of $\hat{BSD}$. The two are essentially different algebraic representations of the same object.
Definition 7.9. Given a bordered sutured manifold \((Y, \Gamma, \mathcal{Z}, \phi)\), let
\[
\widehat{\text{BSD}}_M(Y, \Gamma, s) = A(-\mathcal{Z}) \boxtimes \text{BSD}(Y, \Gamma, s),
\]
\[
\widehat{\text{BSD}}_M(Y, \Gamma) = A(-\mathcal{Z}) \boxtimes \text{BSD}(Y, \Gamma).
\]

Remark. Recall that if \((Y, \Gamma)\) is \(p\)-unbalanced, then any generator has \(p\) many occupied arcs. However, for \(\widehat{\text{BSD}}\) the algebra action depends on unoccupied arcs. Therefore, if \(\mathcal{Z}\) has \(k\) many arcs, then \(\widehat{\text{BSD}}(Y, \Gamma)\) is in fact a type \(D\) structure over \(A(-\mathcal{Z}, k - p)\) only.

7.4. \(\widehat{\text{BSA}}\). The definition of \(\widehat{\text{BSA}}\) is similar to that of \(\widehat{\text{BSD}}\), but differs in some important aspects. In particular, we count a wider class of curves and they are recorded differently.

Let \(\mathcal{H} = (\Sigma, \alpha, \beta, \mathcal{Z}, \psi)\) be a provincially admissible bordered sutured Heegaard diagram, and let \(J\) be an admissible almost complex structure.

We define \(\widehat{\text{BSA}}\) as an \(A_{\infty}\)-module over \(A(\mathcal{Z})\).

Definition 7.10. Fix a relative \(\text{Spin}^c\)-structure \(s \in \text{Spin}^c(Y, \partial Y \setminus F)\). Let \(\widehat{\text{BSA}}(\mathcal{H}, J, s)\) be the \(\mathbb{Z}/2\) vector space generated by the set of all generators \(\mathcal{G}(\mathcal{H}, s)\). Give it the structure of an \(I(\mathcal{Z})\) module by setting
\[
x \cdot I(s) = \begin{cases} x & \text{if } s = o(x), \\
0 & \text{otherwise}. \end{cases}
\]

For generators \(x, y \in \mathcal{G}(\mathcal{H})\), a homology class \(B \in \pi_2(x, y)\), and a source \(S^\circ\) we consider all partitions \(\overline{P} = (P_1, \ldots, P_m)\), not necessarily discrete. We also associate to a sequence of Reeb chords a sequence of algebra elements, instead of a product. Let
\[
\overline{a}(x, y, \overline{P}) = I(o(x)) \cdot (a(\rho_1) \otimes \cdots \otimes a(\rho_m)) \cdot I(o(y)).
\]

Definition 7.11. For \(x, y \in \mathcal{G}(\mathcal{H})\), \(\overline{P} = (\rho_1, \ldots, \rho_m)\) define
\[
c_{x,y,\overline{P}} = \sum_{\overline{P}(\overline{P})=\overline{P}} \#M_{\text{emb}}^B(x, y, S^\circ, \overline{P}).
\]

Definition 7.12. Define \(m_k: \widehat{\text{BSA}}(\mathcal{H}, J, s) \otimes A(\mathcal{Z})^{\otimes (k-1)} \to \widehat{\text{BSA}}(\mathcal{H}, J, s)\) as follows.
\[
m_k(x, a_1, \ldots, a_{k-1}) = \sum_{y \in \mathcal{G}(\mathcal{H})} \overline{a}(x, y, \overline{P}) = a_1 \otimes \cdots \otimes a_{k-1}.
\]

Theorem 7.13. The following statements are true.

1. \(\widehat{\text{BSA}}(\mathcal{H}, J, s)\) equipped with the actions \(m_k\) for \(k \geq 1\), and the \(\text{Gr}(\mathcal{H}, s)\)-valued grading \(\text{gr}\) is an \(A_{\infty}\)-module over \(A(\mathcal{Z})\). In particular,
\[
\text{gr}(m_k(x, a_1, \ldots, a_{k-1})) = \text{gr}(x) \cdot \text{gr}(a_1) \cdots \text{gr}(a_{k-1}) \lambda^{k-2}.
\]
If $H$ is totally admissible, $\widehat{BSA}(H,J,s)$ is bounded.

(3) For any two provincially admissible diagrams $H_1$ and $H_2$, equipped with admissible almost complex structures $J_1$ and $J_2$, there is a graded homotopy equivalence

$$\widehat{BSA}(H_1,J_1,s) \simeq \widehat{BSA}(H_2,J_2,s).$$

Therefore we can talk about $\widehat{BSA}(Y,\Gamma,Z,\psi,s)$ or just $\widehat{BSA}(Y,\Gamma,s)$, relatively graded by $Gr(Y,s)$.

Proof. The proofs are analogous to those for $\widehat{BSD}$, with some differences. The biggest difference is that we count more domains, so we need to use more results about degenerations.

The other major difference is the grading. Again, we prove the statement for $gr$, and the one for $gr$ follows immediately. Suppose $y$ is a term in $m_k(x,a_1,\ldots,a_{k-1})$. Then there is a domain $B \in \pi_2(x,y)$ and a compatible sequence $\vec{\rho} = (\rho_1,\ldots,\rho_{k-1})$ of sets of Reeb chords, such that $\text{ind}(B,\vec{\rho}) = 1$, and $a_i = a(\rho_i,s_i)$, for some appropriate completion $s_i$. In particular, $\text{gr}(a_1) \cdots \text{gr}(a_{k-1}) = \text{gr}(\vec{\rho})$. On the other hand,

$$\text{gr}(y) = \text{gr}(x) \cdot \text{gr}(B) = \text{gr}(x) \cdot \lambda^{-\text{ind}(B,\vec{\rho})+\# \text{gr}(\vec{\rho})} = \text{gr}(x) \cdot \lambda^{-1+(k-1)} \text{gr}(a_1) \cdots \text{gr}(a_{k-1}).$$

As with $\widehat{BSD}$, if we ignore Spin$^c$–structures we can talk about the total invariant

$$\widehat{BSA}(Y,\Gamma) = \bigoplus_{s \in \text{Spin}^c(Y,\Gamma)} \widehat{BSA}(Y,\Gamma,s).$$

Remark. As with $\widehat{BSD}$ the only nontrivial algebra action is by a single component of $A(Z)$. In this case the action depends on occupied arcs. Therefore if $(Y,\Gamma)$ is $p$–unbalanced, then $\widehat{BSA}(Y,\Gamma)$ is an $A_\infty$–module over $A(Z,p)$ only.

### 7.5. Invariants from nice diagrams.

For a nice diagram $H$, the invariants can be computed completely combinatorially, avoiding all discussion of moduli spaces.

**Theorem 7.14.** Let $H$ be a nice diagram. Then for any admissible almost complex structure $J$, the type $D$ structure $\widehat{BSD}(H,J)$ can be computed as follows. The map $\delta(x)$ counts the following types of curves.

(1) A source $S^e$ from $x$ to $y$, consisting of $g$ bigons with no $e$ punctures, where all but one of the bigons are constant on $\Sigma$, while the remaining one embeds as a convex bigon. The interior of the image contains none of the fixed points $x \cap y$. Such curves contribute $I(\vec{\sigma}(x)) \otimes y$ to $\delta(x)$.

(2) A source $S^e$ from $x$ to $y$, consisting of $g-2$ bigons, each of which has no $e$ punctures and is constant on $\Sigma$, and a single quadrilateral with
no e punctures, which embeds as a convex rectangle. The interior of the image contains none of the fixed points $x \cap y$. Such curves contribute $I(\mathcal{P}(x)) \otimes y$ to $\delta(x)$.

(3) A source $S^o$ from $x$ to $y$, consisting of $g - 1$ bigons, each of which has no e punctures and is constant on $\Sigma$, and a single bigon with one e punctures, which embeds as a convex rectangle, one of whose sides is the Reeb chord $-\rho \in \mathbb{Z}$ labeling the puncture. The interior of the image contains none of the fixed points $x \cap y$. Such curves contribute $I(\mathcal{P}(x))a(\rho)I(\mathcal{P}(y)) \otimes y$ to $\delta(x)$.

**Theorem 7.15.** Let $\mathcal{H}$ be a nice diagram. Then for any admissible almost complex structure $J$, the $\mathcal{BSA}(\mathcal{H}, J)$ can be computed as follows.

The differential $m_1(x)$ counts the following types of regions. (These are the same as cases (1) and (2) in theorem 7.14.)

1. A source $S^o$ from $x$ to $y$, consisting of $g$ bigons with no e punctures, where all but one of the bigons are constant on $\Sigma$, while the remaining one embeds as a convex bigon. The interior of the image contains none of the fixed points $x \cap y$. Such curves contribute $y$ to $m_1(x)$.

2. A source $S^o$ from $x$ to $y$, consisting of $g - 2$ bigons, each of which has no e punctures and is constant on $\Sigma$, and a single quadrilateral with no e punctures, which embeds as a convex rectangle. The interior of the image contains none of the fixed points $x \cap y$.

The algebra action $m_2(x, \cdot)$ counts regions of the type below.

1. A source $S^o$ from $x$ to $y$, consisting of $g - p$ bigons, each of which has no e punctures and is constant on $\Sigma$, and a collection of $p$ bigons, each of which has one e puncture and which embeds as a convex rectangle, one of whose sides is the Reeb chord $\rho_i \in \mathbb{Z}$. The height of all e punctures is the same, the interior of any image rectangle contains none of the fixed points $x \cap y$ and no other rectangles. Such curves contribute $y$ to the action $m_2(x, I(o(x))\{a(\rho_1, \ldots, \rho_p)\}I(o(y))$.

In addition, all actions $m_k$ for $k \geq 3$ are zero.

**Proof of theorems 7.14 and 7.15.** The proofs follow the same steps as the ones for nice diagrams in bordered manifolds. By looking at the index formula, and the restricted class of regions, one can show that the only $B, S^o, \widehat{P}$ that have index $\text{ind}(B, S^o, \widehat{P}) = 1$ are of the following two types.

1. $S$ has no e punctures, and consists of $g - 1$ trivial components, and one non-trivial bigon component, or $g - 2$ trivial component and one non-trivial rectangle component.

2. $S$ has several trivial components, and several bigons with a single e puncture each. Moreover, the partition $\widehat{P}$ consists of only one set.

The extra condition that the embedded index is also 1 (so the moduli space consists of embedded curves), is equivalent to having no fixed points in the interior of a region, and no region contained completely inside another.
For such curves, \( \mathcal{M}_{\text{emb}}^{B}(x, y, S^{c}, F) \) has exactly one element, independent of the almost complex structure \( J \), using for example the Riemann mapping theorem.

7.6. **Pairing theorem.** In this section we describe the relationship between the sutured homology of the gluing of two bordered sutured manifolds, and their bordered sutured invariants, proving the second part of theorem 1.

Recall that bordered sutured invariants are homotopy types of chain complexes, while sutured Floer homology is usually regarded as an isomorphism type of homology groups. However, one can also regard the underlying chain complex as an invariant up to homotopy equivalence. To be precise, we will use \( SFH \) to denote sutured Floer homology, and \( SFC \) to denote a representative chain complex defining that homology.

**Theorem 7.16.** Suppose \((Y_{1}, \Gamma_{1}, Z, \phi_{1})\) and \((Y_{2}, \Gamma_{2}, -Z, \phi_{2})\) are two bordered sutured manifolds that glue along \( F = F(Z) \) to form the sutured manifold \((Y, \Gamma)\). Let \( s_{i} \in \text{Spin}^{c}(Y_{i}, \partial Y_{i} \setminus F) \) be relative \( \text{Spin}^{c} \)-structures for \( i = 1, 2 \). Then there is a graded chain homotopy equivalence

\[
\bigoplus_{s_{i}|Y_{i} = s_{i}} \text{SFC}(Y, \Gamma, s_{i}) \simeq \widehat{BSA}(Y_{1}, \Gamma_{1}, s_{1}) \boxtimes_{A(Z)} \widehat{BSD}(Y_{2}, \Gamma_{2}, s_{2}),
\]

provided that at least one of the modules on the right hand-side comes from an admissible diagram.

To identify the gradings, we use the fact that the combined grading set \( \text{Gr}(Y_{1}s_{1}) \times_{\text{Gr}(Z)} \text{Gr}(Y_{2}, s_{2}) \) distinguishes the individual \( \text{Spin}^{c} \)-structures \( s \in \text{Spin}^{c}(Y, \partial Y) \) by its homological component, while the Maslov component agrees with the SFH grading on each \( \text{SFC}(Y, s) \).

**Corollary 7.17.** In terms of modules and derived tensor products, the pairing theorem can be expressed as

\[
\bigoplus_{s_{i}|Y_{i} = s_{i}} \text{SFC}(Y, \Gamma, s_{i}) \simeq \widehat{BSA}(Y_{1}, \Gamma_{1}, s_{1}) \boxtimes \widehat{BSD}(Y_{2}, \Gamma_{2}, s_{2}).
\]

Corollary 7.17 is a restatement of theorem 7.16 in purely \( A_{\infty} \)-module language. This allows us to dispose of type \( D \) structures entirely. However, in practice, the definition of the derived tensor product \( \boxtimes \) involves an infinitely generated chain complex, while that of \( \boxtimes \) only a finitely generated chain complex (assuming both sides are finitely generated).

**Proof of theorem 7.16.** We can prove the theorem using nice diagrams, similar to [8, Chapter 8].

Suppose \( \mathcal{H}_{1} \) and \( \mathcal{H}_{2} \) are nice diagrams for \( Y_{1} \) and \( Y_{2} \), respectively. If we glue them to get a diagram \( \mathcal{H} = \mathcal{H}_{1} \cup_{Z} \mathcal{H}_{2} \) for \( Y = Y_{1} \cup_{F} Y_{2} \), then \( \mathcal{H} \) is also a nice diagram. Indeed, the only regions that change are boundary regions, which are irrelevant, and regions adjacent to a Reeb chord. In the latter case, two rectangular regions in \( \mathcal{H}_{1} \) and \( \mathcal{H}_{2} \), that border the same Reeb chord, glue to a single rectangular region in \( \mathcal{H} \).
Generators in $G(H)$ correspond to pairs of generators in $G(H_1)$ and $G(H_2)$ that occupy complementary sets of arcs. Provincial bigons and rectangles in $H_i$ are also bigons and rectangles in $H$. The only other regions in $H$ that contribute to the differential $\partial$ on $SFC$ are rectangles that are split into two rectangles in $H_1$ and $H_2$, each of which is adjacent to the same Reeb chord $\rho$ in $Z$. Such rectangles contribute terms of the form $(m_2 \otimes \id_{\widetilde{BSD}(H_2)}) \circ (\id_{\widetilde{BSA}(H_1)} \otimes \delta)$. Overall, terms in $\partial : SFC(H) \to SFC(H)$ are in a one-to-one correspondence with terms in $\partial : \widetilde{BSA}(H_1) \otimes \widetilde{BSD}(H_2) \to \widetilde{BSA}(H_1) \otimes \widetilde{BSD}(H_2)$.

This shows that there is an isomorphism of chain complexes

$$SFC(H) \cong \widetilde{BSA}(H_1) \otimes \widetilde{BSD}(H_2).$$

The splitting into $\text{Spin}^c$–structure and the grading equivalence follow from theorems 6.5 and 6.8, where the latter is applied to $(Y_1, \Gamma_1, -\emptyset \cup Z)$ and $(Y_2, \Gamma_2, -Z \cup \emptyset)$. □

8. Bimodule invariants

As promised in the introduction, we will associate to a decorated sutured cobordism, a special type of $A_{\infty}$–bimodule. We will sketch the construction, which closely parallels the discussion of bimodules in [7]. The reader is encouraged to look there, especially for a careful discussion of the algebra involved.

8.1. Algebraic preliminaries. The invariants we will define have the form of type DA structures, which is a combination of a type $D$ structure and an $A_{\infty}$–module.

**Definition 8.1.** Let $A$ and $B$ be differential graded algebras with differential and multiplication denoted $\partial_A$, $\partial_B$, $\mu_A$, and $\mu_B$, respectively. A type DA structure over $A$ and $B$ is a graded vector space $M$, together with a collection of homogeneous operations $m_k: M \otimes B^{\otimes (k-1)} \to A \otimes M[2-k]$, satisfying the compatibility condition

$$\sum_{p=1}^{k} (\mu_A \otimes \id_M) \circ (\id_A \otimes m_{k-p+1}) \circ (m_p \otimes \id_{B^{\otimes (k-p)}}) + (\partial_A \otimes \id_M) \circ m_k$$

$$+ \sum_{p=0}^{k-2} m_k \circ (\id_M \otimes \id_{B^{\otimes p}} \otimes \partial_B \otimes \id_{B^{\otimes (k-p-2)}})$$

$$+ \sum_{p=0}^{k-3} m_k \circ (\id_M \otimes \id_{B^{\otimes p}} \otimes \mu_B \otimes \id_{B^{\otimes (k-p-3)}}) = 0,$$

for all $k \geq 0.$
We can also define \( m^j_k : M \otimes B^{\otimes(k-1)} \to A^{\otimes i} \otimes M[1 + i - k] \), such that \( m^0_1 = \text{id}_M \), \( m^0_k = 0 \) for \( k > 1 \), \( m^1_k = m_k \), and \( m^i_k \) is obtained by iterating \( m^*_k \):

\[
m^i_k = \sum_{j=0}^{k-1} (\text{id}_{A^{\otimes(i-1)}} \otimes m_{j+1}) \circ (m_{k-j}^{i-1} \otimes \text{id}_{B^{\otimes j}}).
\]

In the special case where \( A \) is the trivial algebra \( \{1\} \), this is exactly the definition of a right \( \mathcal{A}_\infty \)-module over \( B \). In the case when \( B \) is trivial, or we ignore \( m^i_k \) for \( k \geq 2 \), this is exactly the definition of a left type \( D \) structure over \( A \). In that case \( m^1_k \) corresponds to \( \delta_i \).

We will use some notation from [7] and denote a type \( DA \) structure over \( A \) and \( B \) by \( ^A M_B \). In the same vein, a type \( D \) structure over \( A \) is \( ^A M \), and a right \( \mathcal{A}_\infty \)-module over \( B \) is \( M_B \). We can extend the tensor \( \boxtimes \) to type \( DA \) structures as follows.

**Definition 8.2.** Let \( ^A M_B \) and \( ^B N_C \) be two type \( DA \) structures, with operations \( m^i_j \), and \( n^i_j \), respectively. Let \( ^A M_B \boxtimes_B ^B N_C \) denote the type \( DA \) structure \( ^A(M \otimes N)_C \), with operations

\[
(m \boxtimes n)^i_j = \sum_{j \geq 1} (m^i_j \otimes \text{id}_N) \circ (\text{id}_M \otimes n^{j-1}_k).
\]

In the case when \( A \) and \( C \) are both trivial, this coincides with the standard operation \( M_B \boxtimes_B N \).

The constructions generalize to mixed multi-modules of type \( ^{A_1, \ldots, A_l} M_{B_1, \ldots, B_l} ^{C_1, \ldots, C_k} \). Such a module is left, respectively right, type \( D \) with respect to \( A_p \), respectively \( C_p \), and left, respectively right \( \mathcal{A}_\infty \)-module with respect to \( B_p \), respectively \( D_p \). The category of such modules is denoted \( ^{A_1, \ldots, A_l} \text{Mod}_{B_1, \ldots, B_l} ^{C_1, \ldots, C_k} \).

We can apply the tensor \( \boxtimes \) to any pair of such modules, as long as one of them has \( X \) as an upper (lower) right index, and the other has \( X \) as a lower (upper) left index.

We will only use a few special cases of this construction. The most important one is to associate to \( ^A M_B \) a canonical \( A, B, \mathcal{A}_\infty \)-bimodule \( _A(^A(M \boxtimes M))_B = A_A \boxtimes_A ^A M_B \). This allows us to bypass type \( D \) and type \( DA \) structures. In particular,

\[
_A(A \boxtimes M)_B \boxtimes_B (_B(B \boxtimes N)_C) \simeq _A(A \boxtimes (M \boxtimes_B N))_C.
\]

### 8.2. \( \text{BSD} \) and \( \text{BSA} \) revisited

Recall that the definition of \( \text{BSD} \) counted a subset of the moduli spaces used to define \( \text{BSA} \), and interpreted them differently. This operation can in fact be described completely algebraically. For any arc diagram \( Z \), there is a bimodule (or type \( DD \) structure) \( ^{\mathcal{A}(-Z)} \mathcal{A}(Z)_1 \), such that

\[
^{\mathcal{A}(-Z)} \text{BSD}(\mathcal{H}, J) = ^{\mathcal{A}(Z)} \text{BSA}(\mathcal{H}, J)_1 \boxtimes_1 ^{\mathcal{A}(Z)} \mathcal{A}(Z)_1.
\]
In fact, we could use this as the definition of \( \hat{BSD} \), and use the naturality of \( \boxtimes \) to prove that it is well-defined for \( \mathcal{H} \) and \( J \), and its homotopy type is an invariant of the underlying bordered sutured manifold.

### 8.3. Bimodule categories

For two differential graded algebras \( A \) and \( B \), the notion of a left-left \( A, B \)-module is exactly the same as that of a left \( A \otimes B \)-module. Similarly, a left type \( D \) structure over \( A \) and \( B \) is exactly the same as a left type \( D \) structure over \( A \otimes B \). In other words, we can interpret a module \( ^A_B M \) as \( A \otimes B M \), and vice versa, and the categories \( ^A_B \text{Mod} \) and \( A \otimes B \text{Mod} \) are canonically identified.

The situation is not as simple for \( A \infty \)-modules. The categories \( ^A_B \text{Mod} \) and \( A \otimes B \text{Mod} \) are not the same, or even equivalent. Fortunately, there is a canonical functor \( F : ^A_B \text{Mod} \to ^A_B \text{Mod} \) which induces an equivalence of the derived categories. For this result, and the precise definition of \( F \) see [7].

### 8.4. \( \hat{BSDA} \) and \( \hat{BSDA}_M \)

We will give two definitions of the bimodules. One is purely algebraic, and allows us to easily deduce that the bimodules are well-defined and invariant, while the other is more analytic, but is more useful in practice.

**Definition 8.3.** Suppose \((Y, \Gamma, Z_1 \cup Z_2, \phi)\) is a bordered sutured manifold—or equivalently, a decorated sutured cobordism from \(F(−Z_1)\) to \(F(Z_2)\). Note that \( A(Z_1 \cup Z_2) = A(Z_1) \otimes A(Z_2) \). Define

\[
\hat{BSDA}(Y, \Gamma, s)_{A(Z_1)} = F(\hat{BSA}(Y, \Gamma, s))_{A(Z_1), A(Z_2)} \boxtimes_{A(Z_1), A(Z_2)} A(Z_1, A(Z_2)).
\]

The invariance follows easily from the corresponding results for \( \hat{BSA} \) and naturality.

As promised, below we give a more practical construction. Fix a provincially admissible diagram \( \mathcal{H} = (\Sigma, \alpha, \beta, Z_1 \cup Z_2, \psi) \), and an admissible almost complex structure \( J \).

Recall that to define both \( \hat{BSD} \) and \( \hat{BSA} \), we looked at moduli spaces \( \mathcal{M}^{B \text{emb}}_e(x, y, S^e, P) \), where \( P \) is a partition of the \( e \) punctures on the source \( S^e \). In our case, we can distinguish two sets of \( e \) punctures—those labeled by Reeb chords in \( Z_1 \), and those labeled by Reeb chords in \( Z_2 \). We denote the two sets by \( E_1 \) and \( E_2 \), respectively. Any partition \( P \) restricts to partitions \( P_i = P|_{E_i} \) on the two sets.

**Definition 8.4.** Define the moduli space

\[
\mathcal{M}^{B \text{emb}}_e(x, y, S^e, P_1, P_2) = \bigcup_{P|_{E_i} = P_i} \mathcal{M}^{B \text{emb}}_e(x, y, S^e, P).
\]
with index
\[
\text{ind}(B, \overrightarrow{\rho}_1, \overrightarrow{\rho}_2) = c(B) + n_x(B) + n_y(B) \\
+ \# \overrightarrow{\rho}_1 + \# \overrightarrow{\rho}_2 + \iota(\overrightarrow{\rho}_1) + \iota(\overrightarrow{\rho}_2),
\]
where \( \overrightarrow{\rho}_i = \overrightarrow{\rho}(\overrightarrow{\rho})_i \) is a sequence of sets of Reeb chords in \( Z_i \), for \( i = 1, 2 \).

This has the effect of forgetting about the relative height of punctures in \( E_1 \) to those in \( E_2 \). Its algebraic analogue is applying the functor \( \mathcal{F} \), which combines the algebra actions \( m_3(x, a \otimes 1, 1 \otimes b) \) and \( m_3(x, 1 \otimes b, a \otimes 1) \) into \( m_{1,1,1}(x, a, b) \).

The general idea is to treat the \( Z_1 \) part of the arc diagram as in \( \overrightarrow{\text{BSD}} \), and the \( Z_2 \) part as in \( \overrightarrow{\text{BSA}} \). First, to a generator \( x \in G(\mathcal{H}) \) we associate idempotents \( I_1(\overrightarrow{\sigma}(x)) \in \mathcal{I}(-Z_1) \) and \( I_2(\iota(x)) \in \mathcal{I}(Z_2) \), corresponding to unoccupied arcs on the \( Z_1 \) side, and occupied arcs on the \( Z_2 \) side, respectively. Next, we will look at discrete partitions \( \overrightarrow{P}_1 = (\{q_1\}, \ldots, \{q_i\}) \) on the \( Z_1 \) side, while allowing arbitrary partitions \( \overrightarrow{P}_2 \) on the \( Z_2 \) side.

If the punctures in \( \overrightarrow{P}_1 \) are labeled by the Reeb chords \( (\rho_1, \ldots, \rho_i) \), set
\[
a_1(x, y, \overrightarrow{P}_1) = I_1(\overrightarrow{\sigma}(x)) \cdot a(-\rho_1) \cdots a(-\rho_i) \cdot I_1(\overrightarrow{\sigma}(y)) \in \mathcal{A}(-Z_1).
\]

If the sets of punctures in \( \overrightarrow{P}_2 \) are labeled by a sequence of sets of Reeb chords \( (\rho_1, \ldots, \rho_i) \), set
\[
a_2(x, y, \overrightarrow{P}_2) = I_2(x) \cdot a(\rho_1) \otimes \cdots \otimes a(\rho_j) \cdot I_2(y) \in \mathcal{A}(Z_2)^{\otimes j}.
\]

**Definition 8.5.** Fix \( \mathcal{H}, J, \) and \( s \). Let \( \overrightarrow{\text{BSDA}}(\mathcal{H}, J, s) \) be freely generated over \( \mathbb{Z}/2 \) by \( G(\mathcal{H}, s) \), with \( \mathcal{I}(-Z_1) \) and \( \mathcal{I}(Z_2) \) actions
\[
I(s_1) \cdot x \cdot I(s_2) = \begin{cases} 
\ x & \text{if } s_1 = \overrightarrow{\sigma}(x) \text{ and } s_2 = o(x), \\
\ 0 & \text{otherwise.}
\end{cases}
\]
It has type \( DA \) operations
\[
m_k(x, b_1, \ldots, b_{k-1}) = \sum_{\text{ind}(B, \overrightarrow{\rho}(\overrightarrow{P}_1), \overrightarrow{\rho}(\overrightarrow{P}_2)) = 1} ^{\# M_{\text{emb}}^B(x, y, S^3, \overrightarrow{P}_1, \overrightarrow{P}_2) \cdot a_1(x, y, \overrightarrow{P}_1) \otimes y.}
\]
It is easy to check that definitions 8.5 and 8.3 are equivalent. Considering pairs of partitions corresponds to the functor \( \mathcal{F} \), while restricting to discrete partitions on the \( Z_1 \) side and multiplying the corresponding Reeb chords corresponds to the functor \( \overrightarrow{\mathcal{X}}_{A(Z_1)} \mathcal{A}(-Z_1) \mathcal{A}(-Z_1) \).

We can use either definition to define
\[
\mathcal{A}(-Z_1) \overrightarrow{\text{BSDA}}_M(Y, \Gamma, s)_{A(Z_2)} = \mathcal{A}(-Z_1) \mathcal{A}(-Z_1) \mathcal{X}_{A(Z_1)} \mathcal{A}(-Z_1) \overrightarrow{\text{BSDA}}(Y, \Gamma, s)_{A(Z_2)}.
\]
As with the one-sided modules, there is a well-defined grading.
Theorem 8.6. The grading $\text{gr}$ on $\hat{\text{BSDA}}(Y, \Gamma, s)$ is well-defined with values in $\text{Gr}(Y, s)$, and makes it a graded DA–structure. In particular, whenever $b \otimes y$ is a term in $m_k(x, a_1, \ldots, a_{k-1})$, we have

$$\text{gr}(b) \cdot \text{gr}(y) = \lambda^{k-2} \cdot \text{gr}(x) \cdot \text{gr}(a_1) \cdots \text{gr}(a_{k-1}).$$

Proof. This is a straightforward combination of the arguments for the gradings on $\hat{\text{BSD}}$ and $\hat{\text{BSA}}$. □

8.5. Nice diagrams and pairing. The key results for bimodules allowing us to talk about a functor from the decorated sutured cobordism category $\text{SD}$ to the category $\text{D}$ of differential graded algebras and $A_\infty$–bimodules are the full version of theorem 1, and theorem 2. Below we give a more precise version of theorem 1, in the vein of theorem 7.16.

Theorem 8.7. Given two bordered sutured manifolds $(Y_1, \Gamma_1, -Z_1 \cup Z_2)$ and $(Y_2, \Gamma_2, -Z_2 \cup Z_3)$, representing cobordisms from $Z_1$ to $Z_2$ and from $Z_2$ to $Z_3$, respectively, there are graded homotopy equivalences of bimodules

$$\bigoplus_{s|Y_i=s_i} \hat{\text{BSDA}}(Y_1 \cup Y_2, s) \simeq \hat{\text{BSDA}}(Y_1, s_1) \boxtimes_{A(Z_2)} \hat{\text{BSDA}}(Y_2, s_2).$$

$$\bigoplus_{s|Y_i=s_i} \hat{\text{BSDA}_M}(Y_1 \cup Y_2, s) \simeq \hat{\text{BSDA}_M}(Y_1, s_1) \hat{\boxtimes}_{A(Z_2)} \hat{\text{BSDA}_M}(Y_2, s_2).$$

The gradings are identified in the sense of theorem 6.8.

The proof is completely analogous to that of theorem 7.16. It relies on the combinatorial form of $\hat{\text{BSDA}}$ from a nice diagram, and the fact that gluing two such diagrams also gives a nice diagram with direct correspondence of domains. The actual result for nice diagrams is given below.

Theorem 8.8. For any nice diagram $H = (\Sigma, \alpha, \beta, Z_1 \cup Z_2, \psi)$ and any admissible almost complex structure $J$ the domains that contribute to $m_k$ are of the following types.

1. Provincial bigons and rectangles, which contribute terms of the form $I \otimes y$ to $m_1(x)$.
2. Rectangles hitting a Reeb chord at $Z_1$, which contribute terms of the form $a \otimes y$ to $m_1(x)$.
3. Collections of rectangles hitting Reeb chords at $Z_2$, at the same height, which contribute terms of the form $I \otimes y$ to $m_2(x, \ldots)$.

Proof. The proof is the same as those for $\hat{\text{BSD}}$ and $\hat{\text{BSA}}$. The only new step is showing that there are no mixed terms, i.e. combinations of (2) and (3). In other words, the actions of $A(-Z_1)$ and $A(Z_2)$ commute for a nice diagram. The reason is that such a combined domain that hits both $Z_1$ and $Z_2$ decomposes into two domains that hit only one side each. There is no constraint of the relative heights, so such a domain will have index at least 2, and would not be counted. □
8.6. **Bimodule of the identity.** In this subsection we sketch the proof of theorem 2. We prove a version for $\overline{BSDA}$, which implies the original statement.

**Definition 8.9.** Given an arc diagram $Z$, define the bimodule $A(Z) \ll A(Z)$, which as an $I(Z)$–bimodule is isomorphic to $I(Z)$ itself, and whose nontrivial operations are

\[(7) \quad m_2(I_i, a) = a \otimes I_f,\]

for all algebra elements $a \in A(Z)$ with initial and final idempotents $I_i$ and $I_f$, respectively. It is absolutely graded by $Gr(Z)$, as a subset of $A(Z)$, i.e. all elements are graded 0.

It is easy to see that $A(Z) \ll A(Z)$ maps to $A(Z)\cong A(Z)$ canonically, and that $A(Z) \ll A(Z) \cong A(Z)$.

**Theorem 8.10.** The identity decorated sutured cobordism $\text{id}_Z = (F(Z) \times [0, 1], \Lambda \times [0, 1])$ from $Z$ to $Z$ has a graded bimodule invariant $A(Z) \overline{BSDA}(\text{id}_Z) A(Z) \cong A(Z) \ll A(Z)$.

**Proof (sketch).** The proof is essentially the same as that of the corresponding statement for pure bordered identity cobordisms in [7]. First we look at an appropriate Heegaard diagram $H$ for $\text{id}_Z$. For any $Z$ there is a canonical diagram of the form in Fig. 10a, only here we interpret the left side as the $-Z$, or type $D$, portion of the boundary, while the right side is the $+Z$, or $A_\infty$–type, portion. Indeed, choosing which of the right arcs are occupied in a generator determines it uniquely, and $G(H)$ is a one-to-one correspondence with elementary idempotents. Thus the underlying space for $\overline{BSDA}(\text{id}_Z)$ is $I(Z)$. For any Reeb chord $\rho$ of length one there is a convex octagonal domain in $H$ that makes Eq. (7) hold for $a = a(\rho, s)$, for any such $\rho$, and any completion $s$.

The rest of the proof is algebraic. Any bimodule with underlying module $I(Z)$ corresponds to some $A_\infty$–algebra morphism $\phi: A(Z) \to A(Z)$. We compute the homology of $A(Z)$ and show it is Massey generated by length one Reeb chords as above. Since Eq. (7) holds for such elements, $\phi$ is a quasiisomorphism. By theorem 8.7, we know $\overline{BSDA}(\text{id}_Z)$ squares to itself, and so does $\phi$, i.e. $\phi \circ \phi \simeq \phi$. Since it is a quasi isomorphism, it is homotopic to the identity morphism, and Eq. (7) holds for all $a$, up to homotopy equivalence.

Finally, for the grading, $Gr(-Z \cup Z)$ has two copies of $H_1(F(Z))$, with opposite pairings. For all Spin$^c$–structures, there are obvious periodic domains, such that $\pi_2(x,x) = H_1(F)$. Taking the quotient by the stabilizer subgroup identifies the subgroups $Gr(-Z)$ and $Gr(Z)$ by the canonical antiisomorphism. All domains have vanishing Maslov grading and canceling homological gradings, so in each Spin$^c$–structure all generators have the same relative grading. Thus, we can identify it with an absolute grading where all gradings are 0. \qed
9. Examples

To help the reader understand the definitions we give some simple examples of bordered sutured manifolds and compute their invariants.

9.1. Sutured surfaces and arc diagrams. First we discuss some simple arc diagrams and their algebras, that parametrize the same sutured surfaces.

Example 9.1. One of the simplest classes of examples is the following. Let $F_n$ be the sutured surface $(D^2, \Lambda_n)$, where $\Lambda_n$ consists of $2n$ distinct points. That is, $F_n$ is a disc, whose boundary circle is divided into $n$ positive and $n$ negative arcs.

There are many different arc diagrams for $F_n$, especially for large $n$, but there are two special cases which we will consider in detail.

Example 9.2. Let $Z = \{Z_1, \ldots, Z_n\}$ be a collection of oriented arcs, and $a = \{a_1, \ldots, a_{2n-2}\}$ be a collection of points, such that $a_1, \ldots, a_{n-1} \in Z_1$ are in this order, and $a_{n+i-1} \in Z_{i+1}$ for $i = 1, \ldots, n-1$. Let $M$ be the matching $M(a_i) = M(a_{2n-i-1}) = i$ for $i = 1, \ldots, n-1$. The arc diagram $W_n = (Z, a, M)$ parametrizes $F_n$, as in Fig. 7a.

Proposition 9.1. For the arc diagram $W_n$ from example 9.2, the algebra $A(W_n)$ satisfies $A(W_n, k) \cong A(n-1, k)$ for all $k = 0, \ldots, n-1$.

Proof. The algebra $A(W_n)$ is a subalgebra of $A(n-1, 1, \ldots, 1) \cong A(n-1) \otimes A(1)^{\otimes(n-1)}$. But $A(1) = A(1, 0) \oplus A(1, 1)$, where both summands are trivial. The projection $\pi$ to $A(n-1) \otimes A(1, 0)^{\otimes(n-1)} \cong A(n-1)$ respects the algebra structure. For each $\rho$ and completion $s$, the projection $\pi$ kills all summands in $a(\rho, s)$, except the one corresponding to the unique section $S$ of $s$, where $S \subset \{1, \ldots, n-1\}$. Therefore $\pi|_{A(W_n)}$ is an isomorphism. □

Example 9.3. Let $Z = \{Z_1, \ldots, Z_n\}$ and $a = \{a_1, \ldots, a_{2n-2}\}$, again but set $a_1 \in Z_1$, $a_{2n-2} \in Z_n$, while $a_{2i}, a_{2i+1} \in Z_{i+1}$ for $i = 1, \ldots, n-2$. Set the matching $M$ to be $M(a_{2i-1}) = M(a_{2i}) = i$ for $i = 1, \ldots, n-1$. The arc diagram $\mathcal{V}_n = (Z, a, M)$ also parametrizes $F_n$, as in Fig. 7b.

Proposition 9.2. For the arc diagram $\mathcal{V}_n$ from example 9.3, its associated algebra $A(\mathcal{V}_n)$ has no differential.

Proof. By definition $A(\mathcal{V}_n)$ is a subalgebra of $A(1) \otimes A(2)^{\otimes(n-2)} \otimes A(1)$. It is trivial to check that neither $A(1)$, nor $A(2)$ have differentials. The differential on their product is defined by the Leibniz rule, so it also vanishes. □

It will be useful for next section to compute the two algebras $A(W_1)$ and $A(\mathcal{V}_1)$ explicitly. Recall definition 2.8, which assigns to a collection $\rho$ of Reeb chords, corresponding to moving strands, and a completion $s$, corresponding to stationary strands, an algebra element $a(\rho, s)$. Abusing notation, we will denote the idempotent $I(\{1, 2, 4\})$ by $I_{124}$, etc.
Example what happens when we fill in a sutured surface with a chord diagram.

The 1–summand is \( \rho_1 \rho_2 = \rho_1 \rho_2 \rho_1 \rho_2 \). The 2–summand \( \mathcal{A}(W_4, 2) = \langle I_{12}, I_{13}, I_{23}, \rho_1'' \rho_2'' \rho_1'' \rho_2'' \rangle \) is the most interesting. Here \( \rho_1'' = a(\{\rho_1\}, \{3\}) \), \( \rho_2'' = a(\{\rho_2\}, \{1\}) \), \( \rho_{12}'' = a(\{\rho_{12}\}, \{2\}) \), and \( \rho_2'' \cdot \rho_1'' = a(\{\rho_1, \rho_2\}, \emptyset) \). There is a nontrivial differential \( \partial \rho_{12}'' = \rho_2'' \cdot \rho_1'' \), and one nontrivial product, which is clear from our notation.

In \( W_4 \) there are three Reeb chords—\( \sigma_1 \) from \( a_2 \) to \( a_3 \), and \( \sigma_2 \) from \( a_1 \) to \( a_5 \). Again, the summands \( \mathcal{A}(W_4, 0) = \langle I_\emptyset \rangle \) and \( \mathcal{A}(W_4, 3) = \langle I_{123} \rangle \) are trivial. The 1–summand is \( \mathcal{A}(W_4, 1) = \langle I_1, I_2, I_3, \sigma_1', \sigma_2' \rangle \), where \( \sigma_1' = a(\{\sigma_1\}, \emptyset) \). It has no nontrivial differentials or products. The 2–summand is \( \mathcal{A}(W_4, 2) = \langle I_{12}, I_{13}, I_{23}, \sigma_1'', \sigma_2'', \sigma_1', \sigma_2' \rangle \), where \( \sigma_1'' = a(\{\sigma_1\}, \{3\}) \), \( \sigma_2'' = a(\{\sigma_2\}, \{1\}) \), and \( \sigma_2'' \cdot \sigma_1'' = a(\{\sigma_1, \sigma_2\}, \emptyset) \). There are no differentials and there is one nontrivial product.

### 9.2. Bordered sutured manifolds

We give three examples of bordered sutured manifolds. Topologically they are all very simple—in fact they are all \( D^2 \times [0, 1] \). They are, nonetheless, interesting and have nontrivial invariants. Bordered sutured manifolds of this type are essential for the study of what happens when we fill in a sutured surface with a chord diagram.

**Example 9.4.** The first example is \( M_1 = (D^2 \times [0, 1], \Gamma_1, -W_4) \), where \( D^2 \times \{0\} \) is parametrized by \(-W_4\), and the rest of the boundary is divided into two positive and three negative regions (see Fig. 8a). An admissible—and in fact nice—Heegaard diagram for \( M_1 \) is given in Fig. 8a. We will compute \( \mathcal{A}(W_4) \text{BSD}(M_1) \).

First, notice that the relative Spin\(^c\)–structures are in one-to-one correspondence with \( H_1(D^2 \times [0, 1], D^2 \times \{0\}) = 0 \), so there is a unique Spin\(^c\)–structure. There are two generators \((x)\) and \((y)\), with idempotents \( I_{13} \cdot (x) = \)
Figure 8. Three examples of bordered sutured manifolds (top row), and their diagrams (bottom row). Capital roman letters denote 1-hand les, lower case roman letters denote intersection points, and Greek letter s denote Reeb chords (always oriented upward). All non-boundary elementary regions have been shaded.

$(x), and I_{12} \cdot (y) = (y)$ (both in $I(\mathcal{W}_4, 2)$). There is a single region contributing to $\delta$. It corresponds to a source $S^c$ which is a bigon from $(y)$ to $(x)$, with one $c$ puncture labeled $-\rho_2$. It contributes $a_2(\rho_2) \otimes (x) = \rho_2'' \otimes (x)$ to $\delta(y)$. Therefore, the only nontrivial term in $\delta$ is

$$\delta((y)) = \rho_2'' \otimes (x).$$

If we want to compute $\widehat{BSA}(M_1)_{\mu_1(\mathcal{W}_4)}$, the same generators have idempotents $(x) \cdot I_2 = (x)$ and $(y) \cdot I_3 = (y)$, and the same region contributes $(x)$.
to $m_2((y), a(-\rho_2))$, instead. The only nontrivial term is

$$m_2((y), -\rho_2^2) = (x).$$

**Example 9.5.** The second example is $M_2 = (D^2 \times [0, 1], \Gamma_1, -\mathcal{V}_4)$, which is the same as $M_1$, except for the different parametrization of $D^2 \times \{0\}$ (see Fig. 8b). An admissible diagram for $M_2$ is given in Fig. 8e.

First, we compute $\mathcal{A}(\mathcal{V}_4)\overline{BSD}(M_2)$. It has two generators, with idempotents $I_{12} \cdot (u) = (u)$ and $I_{23} \cdot (v) = (v)$. There is one region which is a bigon with two $e$ punctures labeled $-\sigma_2$ and $-\sigma_1$, at different heights. It contributes $a_2(\sigma_2)\alpha_2(\sigma_1) \otimes (v) = \sigma_2'' \cdot \sigma_1'' \otimes (v)$ to $\delta((u))$. Therefore the differential is

$$\delta((u)) = \sigma_2'' \cdot \sigma_1'' \otimes (v).$$

For $\overline{BSD}(M_2)_{\mathcal{A}(\mathcal{V}_4)}$, the idempotents are $(u) \cdot I_3 = (u)$ and $(v) \cdot I_1 = (v)$. The region contributes to $m_3$, yielding

$$m_3((u), -\sigma_2'', -\sigma_1^1) = (v).$$

**Example 9.6.** Our last—and richest—example is $M_3 = (D^2 \times [0, 1], \Gamma_2, -\mathcal{V}_4 \cup \mathcal{W}_4)$, where $-\mathcal{V}_4$ parametrizes $D^2 \times \{0\}$, and $\mathcal{W}_4$ parametrizes $D^2 \times \{1\}$ (see Fig. 8c). This is a decorated sutured cobordism from $\mathcal{V}_4$ to $\mathcal{W}_4$, which is an isomorphism in the decorated category $\mathcal{SD}$. An admissible diagram for $M_3$ is given in Fig. 8f.

We will compute (part of) $\mathcal{A}(\mathcal{V}_4)\overline{BSD}(M_3)_{\mathcal{A}(\mathcal{V}_4)}$. In this case, since $H_1(D^2 \times [0, 1], D^2 \times \{0, 1\}) = \mathbb{Z}$, there are multiple Spin$^c$–structures. As in the proof of theorem 10.5, the Spin$^c$–structures correspond to how many $\alpha^a$ arcs are occupied on the $\mathcal{W}_4$ side of $-\mathcal{V}_4 \cup \mathcal{W}_4$. Let $s_k$ be the Spin$^c$–structure with $k$ arcs occupied. There are $3 - k$ arcs occupied on the $-\mathcal{V}_4$ side for each such generator, and therefore $\overline{BSD}(M_3, s_k)$ is a bimodule over $\mathcal{A}(\mathcal{V}_4, k)$ and $\mathcal{A}(\mathcal{W}_4, k)$. Moreover, only $k = 0, 1, 2, 3$ give nonzero invariants.

It is easy to check that $\overline{BSD}(M_3, s_0)$ and $\overline{BSD}(M_3, s_3)$ have unique generators, $(ace)$ and $(fgh)$, respectively, with no nontrivial actions $m_k$. We will leave $\overline{BSD}(M_3, s_1)$ as an exercise and compute $\overline{BSD}(M_3, s_2)$. There are five generators, with idempotents as follows.

- $I_{12} \cdot (agh) \cdot I_{23} = (agh)$
- $I_{12} \cdot (fbh) \cdot I_{13} = (fbh)$
- $I_{13} \cdot (fch) \cdot I_{13} = (fch)$
- $I_{13} \cdot (fgd) \cdot I_{12} = (fgd)$
- $I_{23} \cdot (fge) \cdot I_{12} = (fge)$

There are four elementary domains, each of which contributes one term to $m_1$ or $m_2$. Some of them also contribute to $m_1$ or $m_2$ for $\overline{BSD}(M_3, s_1)$, and there is a composite domain that also contributes in that case. The nontrivial operations for $\overline{BSD}(M_3, s_2)$ are listed below.

- $m_1((fgd)) = \sigma_2'' \otimes (fge)$
- $m_2((fgd), \rho_2'') = I_{13} \otimes (fch)$
- $m_1((fbh)) = \sigma_2'' \otimes (fch)$
- $m_2((fbh), \rho_2'') = I_{12} \otimes (agh)$
9.3. **Gluing.** Our final example is of gluing of bordered sutured manifolds and the corresponding operation on their invariants.

**Example 9.7.** We will use the manifolds from examples 9.4–9.6. If we glue $M_1$ and $M_2$ along $F(W_4)$ we obtain exactly $M_2$. Treating $A(W_4) \hat{B}SD(M_1)_{A(\emptyset)}$, we can compute the product

$$
\hat{B}SDA(M_3) \boxtimes_{A(W_4)} \hat{B}SD(M_1),
$$

which is a type D structure over $A(V_4)$. Since the only relative Spin$^c$–structure on $M_3$ which extends over $M_1$ is $s_2$, the product is equal to $\hat{B}SDA(M_3, s_2) \boxtimes \hat{B}SD(M_1)$. Another way to see this is to notice that if we decompose the product over $\boxtimes_{A(W_4,k)}$, only the $k = 2$ term is nonzero.

After taking the tensor product $\otimes_{I(W_4,2)}$ of the underlying modules, the generators and idempotents are:

$$
I_{13} \cdot (fch) \boxtimes (x) = (fch) \boxtimes (x) \quad I_{12} \cdot (fbh) \boxtimes (x) = (fbh) \boxtimes (x) \\
I_{23} \cdot (fgd) \boxtimes (y) = (fgd) \boxtimes (y) \quad I_{13} \cdot (fgd) \boxtimes (y) = (fgd) \boxtimes (y)
$$

The induced operations are:

$$
\delta((fgd) \boxtimes (y)) = \sigma'' \otimes ((fgd) \boxtimes (y)) + I_{13} \otimes ((fch) \boxtimes (x)) \\
\delta((fbh) \boxtimes (x)) = \sigma'' \otimes ((fbh) \boxtimes (x))
$$

There is one pure differential, from $(fgd) \boxtimes (y)$ to $(fch) \boxtimes (x)$. We can cancel it, and see that the complex is homotopy equivalent to $\hat{B}SD(M_2)$, as expected from the pairing theorem.

10. **Applications**

In this section we describe some applications of the new invariants. First, as a warm-up we describe how both sutured Floer homology and the regular bordered Floer homology appear as special cases of bordered sutured homology. Then we describe how we can recover the sutured Floer homology of a manifold with boundary from its bordered invariants.

Another application is a new proof for the surface decomposition formula [6, Theorem 1.3] of Juhász.

10.1. **Sutured Floer homology as a special case.** We have already seen that the for a bordered sutured manifold $(Y, \Gamma, \emptyset)$, the bordered sutured invariants coincide with the sutured ones. However, there are many more cases when this happens. In fact, for any balanced bordered sutured manifold, the $\hat{B}SD$ and $\hat{B}SA$ invariants still reduce to $SFH$, no matter what the arc diagram is.

**Theorem 10.1.** Let $(Y, \Gamma)$ be a balanced sutured manifold, and $\phi: G(\mathcal{Z}) \to \partial Y$ be a parametrization of any part of $(Y, \Gamma)$ by an arc diagram $\mathcal{Z}$ with $k$ matched pairs. Let $(SFC, \emptyset)$ be the sutured chain complex for $(Y, \Gamma)$.

The following statements hold.
(1) \( \widehat{BSA}(Y, \Gamma, Z, \phi, m_1) \cong (SFC(Y, \Gamma), \partial) \), where \( A(\emptyset) = \{ I(\emptyset) \} \) acts by identity on \( \widehat{BSA} \) and \( A(\emptyset, k) \) kills it for any \( k > 0 \).

(2) \( \widehat{BSD}(Y, \Gamma, Z, \phi) \cong SFC(Y, \Gamma) \) as a set, with
\[ \delta(x) = I \otimes \partial(x), \]
where \( I = I(\emptyset) \) is the unique idempotent in \( A(-Z, k) \).

(3) \( \widehat{BSD}_M(Y, \Gamma, Z, \phi) \cong A(-Z, k) \otimes SFC(Y, \Gamma) \) as a product of chain complexes, with the standard action of \( A(-Z) \) on \( \widehat{BSD}_M \).

Proof. Let \( \mathcal{H} \) be a provincially admissible Heegaard diagram for \( (Y, \Gamma, Z, \phi) \).
If we erase \( Z \) and \( \alpha^a \) from the diagram, we obtain an admissible sutured diagram \( \mathcal{H}' \) for \( (Y, \Gamma) \). (Indeed, any periodic domain for \( \mathcal{H}' \) is a provincial periodic domain for \( \mathcal{H} \).)

Remember that for a balanced, i.e. 0–unbalanced manifold, each generator occupies 0 arcs in \( \alpha^a \). In particular \( \mathcal{G}(\mathcal{H}) = \mathcal{G}(\mathcal{H}') \).

Let \( u \in M^B(x, y, S^0, P) \) be a strongly boundary monotonic curve. Let \( o_t(u) \) denote the set of \( \alpha \in \alpha \), for which \( u^{-1}(\alpha \times \{1\} \times \{t\}) \) is nonempty. Since \( x \) occupies only \( \alpha \) circles, \( o_t(u) \subset \alpha^c \) for small \( t \). The only changes in \( o_t(u) \) can occur at the heights of \( e \) punctures. But at an \( e \) puncture, the boundary goes over a Reeb chord, so \( o_t(u) \) can only change by replacing some arc in \( \alpha^a \) with another. Therefore, \( o_t(u) \subset \alpha^c \) for all \( t \in \mathbb{R} \), and \( S^0 \) has no \( e \) punctures. Thus, \( u \) is a curve with no \( e \) punctures and doesn’t involve \( \alpha^a \). But these are exactly the curves from \( \mathcal{H}' \) counted in the definition of \( SFH \).

Therefore, the curves counted for the definitions of \( \widehat{BSD} \) and \( \widehat{BSA} \) from \( \mathcal{H} \) are in a one-to-one correspondence with curves counted for the definition of \( SFH \) from \( \mathcal{H}' \). Moreover, in \( BSD \) and \( BSA \) these curves are all provincial.

Algebraically, in \( BSD \) a provincial curve from \( x \) to \( y \) contributes \( 1 \otimes y \) to \( \delta(x) \). In \( BSA \) it contributes \( y \) to \( m_1(x) \). Finally, in \( SFH \) it contributes \( y \) to \( \partial(x) \). The first two statements follow. The last is a trivial consequence of the definition of \( BSD_M \). \( \square \)

In particular, the interesting behavior of the bordered sutured invariants occurs when the underlying sutured manifold is unbalanced. In that case sutured Floer homology is not defined, or is trivially set to 0.

10.2. Bordered Floer homology as a special case. The situation in this section is the opposite of that in the previous one. Here we show that if we look at manifolds that are, in a sense, maximally unbalanced, the bordered sutured invariants reduce to purely bordered invariants.

First we recall a basic result from [5].

Proposition 10.2. Let \( \mathcal{C} \) denote the class of all closed connected 3–manifolds, and \( \mathcal{C}' \) denote the class of all sutured 3–manifolds with one boundary
component homeomorphic to $S^2$, and a single suture on it. The following statements hold.

1. $C$ and $C'$ are in a one-to-one correspondence given by the map 
   \[ \xi : C \to C' , \]
   where $\xi(Y)$ is obtained by removing an open 3–ball from $Y$, and putting a single suture on the boundary.

2. There is a natural homotopy equivalence 
   \[ \hat{CF}(Y) \simeq SFC(\xi(Y)). \]

The correspondence is most evident on the level of Heegaard diagrams, where a diagram for $\xi(Y)$ is obtained from a diagram for $Y$ by cutting out a small disc around the basepoint.

There is a natural extension of this result to the bordered category.

**Theorem 10.3.** Let $B$ denote the class of bordered manifolds with one boundary component, and let $B'$ denote the class of bordered sutured manifolds of the following form. 

$$(Y, \Gamma, Z, \phi) \in B'$$

if and only if $D = \partial Y \setminus F(Z)$ is a single disc $D$ and $\Gamma \cap D$ is a single arc. The following statements hold.

1. $B$ and $B'$ are in a one-to-one correspondence given by the map 
   \[ \zeta : B \to B' , \]
   which to a bordered manifold $Y$ parametrized by $Z = (Z, a, M, z)$ associates a bordered sutured manifold $\zeta(Y) = (Y, Z, Z', \phi)$, parametrized by $Z' = (Z \setminus D, a, M)$, where $D$ is a small neighborhood of $z$.

2. For any $Y \in B$, we have 
   \[ \hat{BSD}(\zeta(Y)) \simeq \hat{CFD}(Y) , \]
   \[ \hat{BSA}(\zeta(Y)) \simeq \hat{CFA}(Y) . \]

3. If $Y_1$ and $Y_2$ are bordered manifolds that glue to form a closed manifold $Y$, then $\zeta(Y_1)$ and $\zeta(Y_2)$ glue to form $\xi(Y)$.

**Proof.** In the bordered setting the parametrization of $F(Z) = \partial Y$ means that there is a self-indexing Morse function $f$ on $F$ with one index–0 critical point $p$, one index–2 critical point $q$, and $2k$ many index–1 critical points $r_1, \ldots, r_{2k}$. The circle $Z$ is the level set $f^{-1}(3/2)$, the basepoint $z$ is the intersection of the gradient flow from $p$ to $q$ with $Z$, and the matched points $M^{-1}(i) \in a$ are the intersections of the flowlines from $r_i$ with $Z$.

Note that $F' = F \setminus D$ is a surface with boundary, parametrized by the arc diagram $Z' = (Z', a, M)$, where $Z' = Z \setminus D$. Indeed, if we take $D$ to be a neighborhood of the flowline from $p$ to $q$, then $f|_{F'}$ is a self indexing Morse function for $F'$ with only index–1 critical points, and their stable manifolds intersect the level set $Z'$ at the matched points $a$.

Moreover, the circle $Z$ separates $F$ into two regions—a disc $R_+$ around the index–2 critical point $q$, and a genus $k$ surface $R_-$ with one boundary
component. Thus, \((Y, Z)\) is indeed sutured, and the arc \(Z'\) embeds into the suture \(Z\). Since \(D \cap Z\) is an arc, the manifold we get is indeed in \(B'\).

To see that the construction is reversible we need to check that for any \((Y, \Gamma, Z, \phi) \in B'\) there is only one suture in \(\Gamma\), \(Z\) has only one component, and \(R_+\) is a disc. Indeed, \(Z \cap \Gamma\) consists only of properly embedded arcs in \(F(Z)\). But \(\Gamma \cap \partial F = \Gamma \cap \partial D\) consists of two points, and therefore there is only one arc. Now \(\Gamma = (\Gamma \cap F) \cup (\Gamma \cap D)\) is a circle, and \(R_+ \cap F\) is half a disc, so \(R_+\) is a disc. This proves (1).

To see (2), we will investigate the correspondence on Heegaard diagrams. Consider a boundary compatible Morse function \(f\) on a bordered 3–manifold \(Y\). On the boundary it behaves as described in the first part of the proof. In the interior, there are only index–1 and index–2 critical points. Let \(B\) be a neighborhood in \(Y\) of the flowline from the index–0 to the index–3 critical point, which are the index–0 and index–2 critical points on the surface. Then \(D\) is precisely \(B \cap \partial Y\). Let \(Y' = Y \setminus B\). Topologically, passing from \(Y\) to \(Y'\) has no effect, except for canceling the two critical points. Now \(f|_{Y'}\) is a boundary compatible Morse function for the bordered sutured manifold \(Y' = \xi(Y)\). One can verify this is the same construction as above, except we have pushed \(D\) slightly into the manifold.

Looking at the Heegaard diagrams \(\mathcal{H} = (\Sigma, \alpha, \beta)\) and \(\mathcal{H}' = (\Sigma', \alpha, \beta)\), compatible with \(f\) and \(f|_{Y'}\), respectively, one can see that the effect of removing \(B\) on \(\mathcal{H}\) is that of removing a neighborhood of the basepoint \(z \in \partial \Sigma\). Now \(Z = \partial \Sigma \setminus \nu(z)\), the Reeb chords correspond, and \(\partial \Sigma' \setminus Z\) is a small arc in the region where \(z\) used to be.

Recall that the definitions of \(\widehat{CFD}\) and \(\widehat{CFA}\) on one side, and \(\widehat{BSD}\) and \(\widehat{BSA}\) on the other, are the same, except that \(\partial \Sigma' \setminus Z\) in the latter plays the role of \(z\) in the former. Therefore the corresponding moduli spaces \(\mathcal{M}\) exactly coincide, and for these particular diagrams there is actual equality of the invariants, proving (2).

For (3), it is enough to notice that \(Y = Y_1 \cup_F Y_2\), while \(\zeta(Y_1) \cup_{F \setminus D} \zeta(Y_2) = Y_1 \cup_{F \setminus D} Y_2\), which is \(Y\) minus a ball. \(\Box\)

### 10.3. From bordered to sutured homology.

In the current section we prove theorem 4, which was the original motivation for developing the theory of bordered sutured manifolds and their invariants. Recall that it states that for any set of sutures on a bordered manifold, the sutured homology can be obtained from the bordered homology in a functorial way. A refined version is given below.

**Theorem 10.4.** Let \(F\) be a closed connected surface parametrized by some pointed matched circle \(Z\). Let \(\Gamma\) be any set of sutures on \(F\), i.e. an oriented multi curve in \(F\) that divides it into positive and negative regions \(R_+\) and \(R_-\).

There is a (non unique) left type \(D\) structure \(\widehat{CFD}(\Gamma)\) over \(A(Z)\), with the following property. If \(Y\) is any 3–manifold, such that \(\partial Y\) is identified
with \( F \), making \((Y, \Gamma)\) a sutured manifold, then
\[
SFC(Y, \Gamma) \simeq \widehat{CFA}(Y) \boxtimes \widehat{CFD}(\Gamma).
\]

Similarly, there is a (non-unique) right \( A_\infty \)-module \( \widehat{CFA}(\Gamma) \) over \( A(-Z) \), such that
\[
SFC(Y, \Gamma) \simeq \widehat{CFA}(\Gamma) \boxtimes \widehat{CFD}(Y).
\]

Before we begin the proof, we will note that although \( \widehat{CFD}(\Gamma) \) and \( \widehat{CFA}(\Gamma) \) are not unique (not even up to homotopy equivalence), they can be easily made so by fixing some extra data. The exact details will become clear below.

**Proof.** Fix the surface \( F \), pointed matched circle \( Z = (Z, a, M) \), and the sutures \( \Gamma \). Repeating the discussion in the proof of theorem 10.3, the parametrization of \( F \) means that there is a self-indexing Morse function \( f \) on \( F \) with exactly one index–0 critical point, and exactly one index–2 critical point, where the circle \( Z \) is the level set \( f^{-1}(3/2) \).

The choice that breaks uniqueness is the following. Isotope \( \Gamma \) along \( F \) until one of the sutures \( \gamma \) becomes tangent to \( Z \) at the basepoint \( z \), and so that the orientations of \( Z \) and \( \gamma \) agree. Let \( D \) be a disc neighborhood of \( z \) in \( F \). We can further isotope \( \gamma \) until \( \gamma \cap D = Z \cap D \). We will refer to this operation as *picking a basepoint, with direction*, on \( \Gamma \).

Let \( F' = F \setminus D \), and let \( P \) be the 3–manifold \( F' \times [0, 1] \). Let \( \Delta \) be a set of sutures on \( P \), such that
\[
(F' \times \{1\}) \cap \Delta = (F' \cap \Gamma) \times \{1\},
\]
\[
(F' \times \{0\}) \cap \Delta = (F' \cap Z) \times \{0\},
\]
\[
(\partial D \times [0, 1]) \cap \Delta = (\Gamma \cap \partial D) \times [0, 1].
\]

We orient \( \Delta \) so that on the “top” surface \( F' \times \{1\} \) its orientation agrees with \( \Gamma \), its orientation on the “bottom” is opposite from \( Z \), and on \( \partial D \times [0, 1] \) the two segments are oriented opposite from each other.

As in theorem 10.3, \( F' \) is parametrized by the arc diagram \( Z' = (Z \setminus D, a, M) \). Therefore the “bottom” of \( P \), i.e. \( F' \times \{0\} \) is parametrized by \( -Z' \). (Indeed \( -(Z \setminus D) \) is part of \( \Delta \).) This makes \((P, \Delta)\) into a bordered sutured manifold, parametrized by \( -Z' \).

Isotopies of \( \Gamma \) outside of \( D \) have no effect on \( P \), except for an isotopy of \( \Delta \) in the non parametrized part of \( \partial P \). Therefore the bordered sutured manifold \( P \) is an invariant of \( F, \Gamma \), and the choice of basepoint on \( \Gamma \).

Define
\[
\widehat{CFD}(\Gamma) = \widehat{BSD}(P, \Delta),
\]
\[
\widehat{CFA}(\Gamma) = \widehat{BSA}(P, \Delta).
\]

It is clear that their homotopy types are invariants of \( \Gamma \) and the choice of basepoint (with direction). Since \( \mathcal{A}(Z') = \mathcal{A}(Z) \), they are indeed modules over \( \mathcal{A}(Z) \) and \( \mathcal{A}(-Z) \), respectively.
To prove the rest of the theorem, consider any manifold \( Y \) with boundary \( \partial Y = F \). By the construction in theorem 10.3, \( \zeta(Y) \) is the sutured manifold \( (Y, Z) \), where \( F' \) is parametrized by \( Z' \).

If we glue \( \zeta(Y) \) and \( P \) along \( F' \), we get the sutured manifold

\[
(Y \cup F' \times [0, 1], (Z \setminus F') \cup (\Delta \setminus F' \times \{0\})).
\]

The sutures consist of \( Z \setminus F' = Z \cap D = \Gamma \cap D, \Delta \cap (\partial D \times [0, 1]) = (\Gamma \cap \partial D) \times [0, 1], \) and \( \Delta \cap (F' \times \{1\}) = (\Gamma \setminus D) \times \{1\} \). Up to homeomorphism, \( Y \cup F' \times [0, 1] = Y \), and under that homeomorphism the sutures get collapsed to \( \Gamma \subset F \). Therefore, \( \zeta(Y) \cup F' P \) is precisely \( (Y, \Gamma) \).

Using theorem 10.3, \( BSD(\zeta(Y)) \simeq \widehat{CFD}(Y) \), and \( \widehat{BSA}(\zeta(Y)) \simeq \widehat{CFA}(Y) \).

By theorem 7.16,

\[
SFC(Y, \Gamma) \simeq \widehat{BSA}(\zeta(Y)) \boxtimes BSD(P) \simeq \widehat{CFA}(Y) \boxtimes \widehat{CFD}(\Gamma),
\]

\[
SFC(Y, \Gamma) \simeq \widehat{BSA}(P) \boxtimes BSD(\zeta(Y)) \simeq \widehat{CFA}(\Gamma) \boxtimes \widehat{CFD}(Y).
\]

\[ \square \]

10.4. Surface decompositions. The final application we will show is a new proof of the surface decomposition theorem of Juhász proved in [6].

More precisely we prove the following statement.

**Theorem 10.5.** Let \( (Y, \Gamma) \) be a balanced sutured manifold. Let \( S \) be a properly embedded surface in \( Y \) with the following properties. \( S \) has no closed components, and each component of \( \partial S \) intersects both \( R_- \) and \( R_+ \).

(Juhász calls such a surface a good decomposing surface.)

A \( \text{Spin}^c \) structure \( s \in \text{Spin}^c(Y, \Gamma) \) is outer with respect to \( S \) if it is represented by a vector field \( v \) which is nowhere tangent to a normal vector to \( -S \) (with respect to some metric).

Let \( (Y', \Gamma') \) be the sutured manifold, obtained by decomposing \( Y \) along \( S \). More precisely, \( Y' \) is \( Y \) cut along \( S \), such that \( \partial Y' = \partial Y \cup +S \cup -S \), and the sutures \( \Gamma' \) are chosen so that \( R_-'(\Gamma') = R_-'(\Gamma) \cup -S \), and \( R_+'(\Gamma') = R_+'(\Gamma) \cup +S \). Here \( +S \) (respectively \( -S \)) is the copy of \( S \) on \( \partial Y' \), whose orientation agrees (respectively disagrees) with \( S \).

Then the following statement holds.

\[
\text{SFH}(Y', \Gamma') \cong \bigoplus_{s \text{ outward to } S} \text{SFH}(Y, \Gamma, s).
\]

**Proof.** We will consider three bordered sutured manifolds. Let \( T = S \times [-2, 2] \subset Y \) be a neighborhood of \( S \) in \( Y \) (so the positive normal of \( S \) is in the + direction). Let \( W = Y \setminus T \), and let \( P = S \times ([-2, -1] \cup [1, 2]) \subset T \). We can assume that \( \Gamma \cap \partial T \) consists of arcs parallel to the \([ -2, 2 ] \) factor.

Put sutures on \( T, W \) and \( P \) in the following way. First, notice that \( R_+(\Gamma) \cap \partial S \) consists of several arcs \( a = \{ a_1, \ldots, a_n \} \). Let \( A_+ \subset S \) be a collection of disjoint discs, such that \( A_+ \cap \partial S = a \).
On $T$ put sutures $\Gamma_T$, such that
\[ R_+(\Gamma_T) \cap \partial Y = R_+(\Gamma) \cap \partial T, \]
\[ R_+(\Gamma_T) \cap (S \times \{\pm 2\}) = A_+ \times \{\pm 2\}. \]

On $W$ put sutures $\Gamma_W$, such that
\[ R_+(\Gamma_W) \cap \partial Y = R_+(\Gamma) \cap \partial W, \]
\[ R_+(\Gamma_W) \cap (S \times \{\pm 2\}) = A_+ \times \{\pm 2\}. \]

On $P$ put sutures $\Gamma_P$, such that
\[ R_+(\Gamma_P) \cap \partial Y = R_+(\Gamma) \cap \partial P, \]
\[ R_+(\Gamma_P) \cap (S \times \{\pm 2\}) = A_+ \times \{\pm 2\}, \]
\[ R_+(\Gamma_P) \cap (S \times \{-1\}) = S \times \{-1\}, \]
\[ R_+(\Gamma_P) \cap (S \times \{1\}) = \emptyset. \]

Fix a parametrization of $S$ by a balanced arc diagram $\mathcal{Z}_S$ with $k$ many arcs, such that the positive region of $S$ is $A_+$. This is possible, since $S$ has no closed components, and the arcs all hit every boundary component.

Parametrize the surfaces $S \times \{\pm 2\}$ in each of $T$, $W$, and $P$ by $\pm \mathcal{Z}_S$, depending on orientation. If we set $U = S \times \{\pm 2\} \subset W$, with the boundary orientation from $W$, then $U$ is parametrized by $Z = \mathcal{Z}_1 \cup \mathcal{Z}_2$, where $\mathcal{Z}_1 \cong \mathcal{Z}_S$ parametrizes $S \times \{-2\}$, and $\mathcal{Z}_2 \cong -\mathcal{Z}_S$ parametrizes $S \times \{2\}$. Thus, $W$ is a bordered sutured manifold parametrized by $Z$, while $T$ and $P$ are parametrized by $-Z$ (see Fig. 9). Moreover, gluing along the parametrization,
\[ W \cup_U T = (Y, \Gamma), \]
\[ W \cup_U P = (Y', \Gamma'). \]

We will look at the relationship between $\widehat{BSD}(T)$ and $\widehat{BSD}(P)$. For simplicity we will assume $S$ is connected. The argument easily generalizes to multiple connected components. Alternatively, it follows by induction. The Heegaard diagrams $\mathcal{H}_T$ and $\mathcal{H}_P$ for $T$ and $P$ are shown in figure 10. Since all regions $D$ have nonzero $\partial^2 D$, the diagrams are automatically provincially admissible.

The algebra $\mathcal{A}(Z)$ splits as $\mathcal{A}(Z_1) \otimes \mathcal{Z}(Z_2)$, and each idempotent $I \in \mathcal{I}(Z)$ splits as the product $I = I_1 \otimes I_2$, where $I_1 \in \mathcal{I}(Z_1)$ and $I_2 \in \mathcal{I}(Z_2)$. Moreover, $I_1$ is in a summand $\mathcal{I}(Z,l)$ for some $l = 0, \ldots, k$. Denote this number by $l(I)$. Intuitively, $l(I)$ means “how many arcs on the $Z_1$ portion of $Z$ does $I$ occupy”. Similarly, for a generator $x$ we can define $l(x) = l(I(\overline{x})).$

Notice that $\mathcal{H}_P$ has a unique generator $x_P$, such that $l(x_P) = k$. Moreover, there are only two regions in the diagram, and both of them are boundary regions. Therefore, no curves contribute to $\delta$. Thus, $BSD(P)$ has a unique generator $x_P$, with $\delta(x_P) = 0$.

Now, consider $\mathcal{H}_T$. Every $\alpha^a$ arc intersects a unique $\beta$ curve, and any $\beta$ curve intersects a unique pair of $\alpha^a$ arcs, that correspond in $-Z_1$ and
Figure 9. Bordered sutured decomposition of \((Y, \Gamma)\) and \((Y', \Gamma')\).

\(-\mathbb{Z}_2 \cong \mathbb{Z}_1\). Therefore for any \(s \subset \{1, \ldots, k\}\) there is a unique generator \(x_s \in \mathcal{G}(\mathcal{H}_i)\), such that \(I(\overline{v}(x)) = I_1(s) \otimes I_2(\overline{\pi})\), and \(l(x_s) = \# s\). These are all the elements of \(\mathcal{G}(\mathcal{H}_P)\).

Consider all the Spin\(^c\)–structures in Spin\(^c\)(\(T, \partial T \setminus S \times \{\pm 2\}\)). By Poincaré duality they are an affine space over \(H_1(T, S \times \{\pm 2\}) = H_1(S \times [-2, 2], S \times \{\pm 2\}) \cong \mathbb{Z}\), generated by an arc \(\mu = \{p\} \times [-2, 2]\) for any \(p \in S\). It is easy to see that \(\epsilon(x, y) = (l(x) - l(y)) \cdot [\mu]\). Thus, for any \(x \in \mathcal{G}(\mathcal{H}_T)\), its Spin\(^c\)–structure \(s(x)\) depends only on \(l(x)\). In particular, there is a unique generator \(x_T\), in the Spin\(^c\)–structure \(s_k = s(x_T)\) which corresponds to \(l = k\).

Since \(l(x_T) = k\), any class \(B \in \pi_2(x, x)\) that contributes to \(\delta\) could not hit any Reeb chords on the \(\mathbb{Z}_2\) side, and \(\partial^\partial B \cap \mathbb{Z}_2\) should be empty. But any elementary region in the diagram hits Reeb chords on both sides. Therefore any such \(B\) should be 0, and \(\delta(x_T) = 0\).
Notice that $\mathcal{G}(\mathcal{H}_P) = \{x_P\} \cong \{x_T\} = \mathcal{G}(\mathcal{H}_T, s_k)$, $I(\overline{x}_P) = I(\overline{x}_T) = I_1(\{1, \ldots, k\}) \otimes I_2(\emptyset)$, and $\delta(x_P) = \delta(x_T) = 0$. Therefore $\overline{BSD}(\mathcal{H}_P) \cong \overline{BSD}(\mathcal{H}_T, s_k)$, and $\overline{BSD}(P) \simeq \overline{BSD}(T, s_k)$, as type D structures over $\mathcal{A}(Z)$.

By the pairing theorem,

$$SFC(Y', \Gamma') \simeq \overline{BSA}(W) \boxtimes \overline{BSD}(P) \simeq \overline{BSA}(W) \boxtimes \overline{BSD}(T, s_k) \simeq \bigoplus_{s|T = s_k} SFC(Y, \Gamma, s).$$

To finish the proof, we need to check that $s \in \text{Spin}^c(Y, \partial Y)$ is outward to $S$ if and only if $s|T = s_k$. This follows from the fact that being outward to $S$ is a local condition. In $T = S \times [-2, 2]$ the existence of an outward vector field representing $s_l$ is equivalent to $l = k$. \hfill \Box

In fact, using bimodules the proof carries through even when $W$ has another bordered component $Z'$. Thus we get a somewhat stronger version of the formula.

**Theorem 10.6.** If $(Y, \Gamma, Z, \phi)$ is a bordered sutured manifold, and $S$ is a nice decomposing surface, where $\partial S \subset \partial Y \setminus F(Z)$, and $(Y', \Gamma', Z, \phi)$ is obtained by decomposing along $S$, then the following formulas hold.

$$\overline{BSD}(Y', \Gamma') \simeq \bigoplus_{s \text{ outward to } S} \overline{BSD}(Y, \Gamma, s),$$

$$\overline{BSA}(Y', \Gamma') \simeq \bigoplus_{s \text{ outward to } S} \overline{BSA}(Y, \Gamma, s).$$
Proof. The first statement follows as in theorem 10.5, using $\hat{\text{BSDA}}(W)$. The second follows analogously, replacing the argument for $\hat{\text{BSD}}(T)$ and $\hat{\text{BSD}}(P)$ with one for $\hat{\text{BSA}}(T)$ and $\hat{\text{BSA}}(P)$. □

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