ON DESCRIPTIONS OF PRODUCTS OF SIMPLICES

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Abstract. We give several new criteria to judge whether a simple convex polytope in a Euclidean space is combinatorially equivalent to a product of simplices. These criteria are mixtures of combinatorial, geometrical and topological conditions that are inspired by the ideas from toric topology.

1. Background

A convex polytope $P$ is the convex hull of a finite set of points in a Euclidean space $\mathbb{R}^d$. The dimension of $P$ is the dimension of the affine hull of these points. Any codimension-one face of $P$ is called a facet of $P$. We call an $n$-dimensional convex polytope $P$ simple if each vertex of $P$ is the intersection of exactly $n$ different facets of $P$. Two convex polytopes are combinatorially equivalent if their face lattices are isomorphic. Topologically, combinatorial equivalence corresponds to the existence of a (piecewise linear) homeomorphism between the two polytopes that restricts to homeomorphisms between their facets, and hence all their faces (see [19, Chapter 2.2]).

In this paper, we will give several new criteria to judge whether a simple convex polytope is combinatorially equivalent to product of simplices (Theorem 2.4 and Theorem 2.7). Some of these criteria are purely combinatorial, while others are phrased in geometrical or topological terms. These criteria are mainly inspired by the ideas from toric topology. So in the following we first explain some basic constructions and facts in toric topology that are relevant to our discussion.

An abstract simplicial complex on a set $[m] = \{v_1, \ldots, v_m\}$ is a collection $K$ of subsets $\sigma \subseteq [m]$ such that if $\sigma \in K$, then any subset of $\sigma$ also belongs to $K$. We always assume that the empty set belongs to $K$ and refer to $\sigma \in K$ as a simplex of $K$. In particular, one-element simplices are called vertices of $K$. If $K$ contains all one-element subsets of $[m]$, then we say that $K$ is a simplicial complex on the vertex set $[m]$. To avoid ambiguity in our argument, we also use $V(K)$ and $V(\sigma)$ to refer to the vertex sets of $K$ and any simplex $\sigma$ in $K$.

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Any abstract simplicial complex $K$ admits a geometric realization in some Euclidean space. But sometimes we also use $K$ to denote its geometric realization when the meaning is clear in the context.

Given a finite abstract simplicial complex $K$ on a set $[m]$ and a pair of spaces $(X, A)$ with $A \subset X$, we can construct a topological space $(X, A)^K$ by:

$$(X, A)^K = \bigcup_{\sigma \in K} (X, A)^\sigma, \text{ where } (X, A)^\sigma = \prod_{v_j \in \sigma} X \times \prod_{v_j \notin \sigma} A. \quad (1)$$

Here $\prod$ means Cartesian product. So $(X, A)^K$ is a subspace of the Cartesian product of $m$ copies of $X$. It is called the polyhedral product or the generalized moment-angle complex of $K$ and $(X, A)$. In particular, $\mathcal{Z}_K = (D^2, S^1)^K$ and $\mathbb{R}\mathcal{Z}_K = (D^1, S^0)^K$ are called the moment-angle complex and the real moment-angle complex of $K$, respectively (see [11 Section 4.1]). The natural actions of $(\mathbb{Z}_2)^m$ on $(D^1)^m$ and $(S^1)^m$ on $(D^2)^m$ induce canonical actions of $(\mathbb{Z}_2)^m$ on $\mathbb{R}\mathcal{Z}_K$ and $(S^1)^m$ on $\mathcal{Z}_K$, respectively.

When $K$ is the boundary of the dual of a simple convex polytope $P$, the $\mathcal{Z}_K$ and $\mathbb{R}\mathcal{Z}_K$ are closed manifolds, also denoted by $\mathcal{Z}_P$ and $\mathbb{R}\mathcal{Z}_P$ respectively. In this case, $\mathcal{Z}_P$ and $\mathbb{R}\mathcal{Z}_P$ are called the moment-angle manifold and the real moment-angle manifold of $P$, respectively (see [3 Section 6.1]). These manifolds can be constructed in another way as described below (see [10 Construction 4.1]).

Let $P^n$ be an $n$-dimensional simple convex polytope. Let $\mathcal{F}(P^n) = \{F_1, \ldots, F_m\}$ be the set of facets of $P^n$. Let $\{e_1, \ldots, e_m\}$ be a basis of $(\mathbb{Z}_2)^m$ and define a map $\lambda : \mathcal{F}(P^n) \to (\mathbb{Z}_2)^m$ by $\lambda(F_i) = e_i$. Then we can construct a space

$$M(P^n, \lambda) := P^n \times (\mathbb{Z}_2)^m / \sim \quad (2)$$

where $(p, g) \sim (p', g')$ if and only if $p = p'$ and $g^{-1}g' \in G_p^m$ where $G_p^m$ is the subgroup of $(\mathbb{Z}_2)^m$ generated by the set $\{\lambda(F_i) | p \in F_i\}$. Let $\pi_\lambda : M(P^n, \lambda) \to P^n$ be the quotient map. One can show that $\mathbb{R}\mathcal{Z}_{P^n}$ is homeomorphic to $M(P^n, \lambda)$ and the canonical action of $(\mathbb{Z}_2)^m$ on $\mathbb{R}\mathcal{Z}_{P^n}$ can be written on $M(P^n, \lambda)$ as:

$$g' \cdot [(p, g)] = [(p, g' + g)], \quad p \in P^n, \quad g, g' \in (\mathbb{Z}_2)^m. \quad (3)$$

The moment-angle manifold $\mathcal{Z}_{P^n}$ can be similarly constructed from $P^n$ and a map $\Lambda : \mathcal{F}(P^n) \to \mathbb{Z}^m$ where $\{\Lambda(F_1), \ldots, \Lambda(F_m)\}$ is a unimodular basis of $\mathbb{Z}^m$. Indeed, if we indentify the torus $(S^1)^m = \mathbb{R}^m/\mathbb{Z}^m$, then we have

$$\mathcal{Z}_{P^n} \cong P^n \times (S^1)^m / \sim \quad (4)$$

where $(p, g) \sim (p', g')$ if and only if $p = p'$ and $g^{-1}g' \in T_p^\Lambda$ where $T_p^\Lambda$ is the subtorus of $(S^1)^m$ determined by the linear subspace of $\mathbb{R}^m$ spanned by the set $\{\Lambda(F_i) | p \in F_i\}$.

In addition, $\mathbb{R}\mathcal{Z}_{P^n}$ and $\mathcal{Z}_{P^n}$ are smooth manifolds. In fact, there exists an equivariant smooth structure on $\mathbb{R}\mathcal{Z}_{P^n}$ (or $\mathcal{Z}_{P^n}$) with respect to the canonical
\((\mathbb{Z}_2)^m\)-action (or \((S^1)^m\)-action). The reader is referred to [3, Ch.6] or [4, Ch.6] for the discussion of smooth structures on (real) moment-angle manifolds. Moreover, for any proper face \(f\) of \(P^n\), \(\pi^{-1}_\lambda(f)\) is an embedded closed smooth submanifold of \(\mathbb{R}Z_{P^n}\) which is the fixed point set of the subgroup of \((\mathbb{Z}_2)^m\) generated by \(\{\lambda(F_i) \mid f \in F_i\}\) under the canonical \((\mathbb{Z}_2)^m\)-action.

2. Descriptions of products of simplices

For any \(k \in \mathbb{N}\), let \(\Delta^k\) denote the standard \(k\)-dimensional simplex, which is
\[
\Delta^k = \{(x_1, \ldots, x_k, x_{k+1}) \in \mathbb{R}^{k+1} \mid x_1 + \cdots + x_{k+1} = 1, x_1, \ldots, x_{k+1} \geq 0\}.
\]
For any \(n_1, \ldots, n_q \in \mathbb{N}\), consider \(\Delta^{n_1} \times \cdots \times \Delta^{n_q}\) as a product of \(\Delta^{n_1}, \ldots, \Delta^{n_q}\) in the Cartesian product \(\mathbb{R}^{n_1+1} \times \cdots \times \mathbb{R}^{n_q+1}\).

Next, we first list some descriptions of products of simplices that appeared in Wiemeler’s paper [18].

Theorem 2.1 (Wiemeler [18]). Let \(P^n\) be an \(n\)-dimensional simple convex polytope with \(m\) facets, \(n \geq 3\). Then the following statements are equivalent.

(a) \(P^n\) is combinatorially equivalent to a product of simplices.

(b) Any 2-dimensional face of \(P^n\) is either a 3-gon or a 4-gon.

(c) There exists a quasitoric manifold \(M^{2n}\) over \(P^n\) which admits a nonnegatively curved Riemannian metric that is invariant under the canonical \((S^1)^n\)-action on \(M^{2n}\).

A quasitoric manifold \(M^{2n}\) over \(P^n\) is the quotient space of \(Z_{P^n}\) under a free action of a rank \(m - n\) toral subgroup of \((S^1)^m\) (see [10]). There is a canonical \((S^1)^n\)-action on \(M^{2n}\) induced from the canonical action of \((S^1)^m\) on \(Z_{P^n}\), which makes \(M^n\) a torus manifold (see [11]).

Theorem 2.1(b) is a corollary of [18, Proposition 4.5] and Theorem 2.1(c) is a corollary of [18, Lemma 4.2]. Note that Theorem 2.1(b) is a particularly useful description of products of simplices. Indeed, the proofs of many other descriptions of products of simplices in this paper boil down to this one first. But the proof of [18, Proposition 4.5] is a little long and not particularly easy to follow. We will give a shorter proof of Theorem 2.1(b) in the appendix to make our paper more self-contained.

Next, we give more descriptions of products of simplices from combinatorial and topological viewpoints. For convenience let us introduce some notations first.

- For any topological space \(X\) and any field \(k\), let
  \[
  hrk(X; k) = \sum_{i=0}^{\infty} \dim_k H^i(X; k).
  \]
For any vertex $v$ in a simplicial complex $K$, we denote by $\text{link}_K v$ the link of $v$ in $K$. We denote a simplex spanned by vertices $v_0, v_1, \ldots, v_p$ in $K$ by $[v_0, v_1, \ldots, v_p]$ and its boundary complex by $\partial[v_0, v_1, \ldots, v_p]$.

In addition, for a simplicial complex $K$ on the vertex set $[m] = \{v_1, \ldots, v_m\}$, we can define a new simplicial complex $L(K)$ from $K$, called the double of $K$, where $L(K)$ is a simplicial complex on the vertex set $[2m] = \{v_1, v'_1, \ldots, v_m, v'_m\}$ determined by the following condition: $\omega \subset [2m]$ is a minimal (by inclusion) missing simplex of $L(K)$ if and only if $\omega$ is of the form $\{v_{i_1}, v'_{i_1}, \ldots, v_{i_k}, v'_{i_k}\}$ where $\{v_{i_1}, \ldots, v_{i_k}\}$ is a minimal missing simplex of $K$. Note that any minimal missing simplex in $L(K)$ must have even number of vertices. The double of $K$ is a special case of iterated simplicial wedge construction (also called simplicial wedge $J$-construction). Indeed, by the notation introduced in [1], $L(K) = K(2, \ldots, 2)$.

The following are some basic facts about $L(K)$ (see Ustinovsky [15, 16]).

- $\dim(L(K)) = m + \dim(K)$ (Lemma 1.2).
- $L(K_1 \ast K_2) = L(K_1) \ast L(K_2)$ (here $\ast$ is the join of two simplicial complexes).
- If $K = \partial P^*$ where $P^*$ is the simplicial polytope dual to a simple convex polytope $P$, then $L(K) = \partial L(P)^*$ where $L(P)$ is a simple convex polytope called the double of $P$ (see [15] for the construction of $L(P)$).
- $L(\partial \Delta^k) = \partial \Delta^{2k+1}$.

The following are some easy or well known facts on products of simplices. We want to list them and give a simple proof for reference.

**Proposition 2.2.** Let $P$ be an $n$-dimensional simple polytope with $m$ facets and let $K$ be the boundary of the simplicial polytope dual to $P$. Then the following statements are all equivalent.

(a) $P$ is combinatorially equivalent to a product of simplices.

(b) $K$ is simplicially isomorphic to $\partial \Delta^{n_1} \ast \cdots \ast \partial \Delta^{n_q}$ for some $n_1, \ldots, n_q \in \mathbb{N}$.

(c) The vertex sets of all the minimal missing faces of $K$ form a partition of $V(K)$.

(d) $L(K)$ is simplicially isomorphic to $\partial \Delta^{l_1} \ast \cdots \ast \partial \Delta^{l_q}$ for some $l_1 \cdots, l_q \in \mathbb{N}$.

(e) There exists some field $\mathbf{k}$ so that $\text{hrk}(\mathbb{R}Z_K; \mathbf{k}) = 2^{m-\dim(K)-1}$, or equivalently $\text{hrk}(\mathbb{R}Z_P; \mathbf{k}) = 2^{m-n}$.

(f) There exists some field $\mathbf{k}$ so that $\text{hrk}(Z_K; \mathbf{k}) = 2^{m-\dim(K)-1}$, or equivalently $\text{hrk}(Z_P; \mathbf{k}) = 2^{m-n}$.

**Proof.** The equivalences of (a) $\Leftrightarrow$ (b) and (b) $\Leftrightarrow$ (c) are easy to see.

(b) $\Rightarrow$ (d). If $K = \partial \Delta^{n_1} \ast \cdots \ast \partial \Delta^{n_q}$, then $L(K) = L(\partial \Delta^{n_1} \ast \cdots \ast \partial \Delta^{n_q}) = L(\partial \Delta^{n_1}) \ast \cdots \ast L(\partial \Delta^{n_q}) = \partial \Delta^{2n_1+1} \ast \cdots \ast \partial \Delta^{2n_q+1}$. 

(d) ⇒ (c). Suppose \( L(K) = \partial \Delta^{t_1} \ast \cdots \ast \partial \Delta^{t_q} \). Notice that for each \( 1 \leq j \leq q \), \( \Delta^{t_j} \) is a minimal missing simplex of \( L(K) \). So \( \Delta^{t_j} \) must have even number of vertices, which implies that \( l_j \) is an odd integer. Then by (b) ⇔ (c), the vertex sets of all the minimal missing faces of \( L(K) \) form a partition of \( V(L(K)) \). This forces the vertex sets of all the minimal missing faces of \( K \) to form a partition of \( V(K) \) as well, which is (c).

(a) ⇒ (e) and (f). If \( P = \Delta^{n_1} \times \cdots \times \Delta^{n_q}, n_1 + \cdots + n_q = n \), then
\[
Z_P = S^{2n_1+1} \times \cdots \times S^{2n_q+1}, \quad \mathbb{R} Z_P = S^{n_1} \times \cdots \times S^{n_q}.
\]
The number of facets of \( P \) is \( m = n + q \). It is clear that for any field \( k \),
\[
\text{hrk}(Z_P; k) = \text{hrk}(\mathbb{R} Z_P; k) = 2^q = 2^{m-n}.
\]

(e) ⇒ (a). For any vertex \( v \) of \( K \), let \( m_v \) be the number of vertices in \( \text{link}_K v \). According to the proof of [16, Theorem 3.2] (note that the argument there works for any coefficient), there is a subspace \( X \) of \( \mathbb{R} Z_K \) so that
\[
\text{hrk}(\mathbb{R} Z_P; k) = \text{hrk}(\mathbb{R} Z_K; k) \geq \text{hrk}(X; k),
\]
where \( X \) is the disjoint union of \( 2^{m-v-1} \) copies of \( \mathbb{R} Z_{\text{link}_K v} \). So we have
\[
2^{m-n} = \text{hrk}(\mathbb{R} Z_K; k) \geq 2^{m-v-1} \text{hrk}(\mathbb{R} Z_{\text{link}_K v}; k).
\]
Then \( \text{hrk}(\mathbb{R} Z_{\text{link}_K v}; k) \leq 2^{m-v+1} \). On the other hand, [16, Theorem 3.2] tells us that \( \text{hrk}(\mathbb{R} Z_{\text{link}_K v}; k) \geq 2^{m-v-1} \) (since \( \dim(\text{link}_K v) = n - 2 \)). So we obtain
\[
\text{hrk}(\mathbb{R} Z_{\text{link}_K v}; k) = 2^{m-v-1}.
\]

Note if \( v \) is the vertex corresponding to a facet \( F \) of \( P \), then \( \mathbb{R} Z_{\text{link}_K v} = \mathbb{R} Z_F \). So we have shown that if the condition (e) holds for \( P \), it should hold for any facet of \( P \) as well.

By iterating the above argument, we deduce that the condition (e) holds for all the two dimensional faces of \( P \). It is not hard to see that the real moment-angle manifold of a \( k \)-gon is a closed connected orientable surface with genus \( 1 + (k-4)2^{k-3} \) (see [4, Proposition 4.1.8]). So any 2-dimensional face of \( P \) is either a 3-gon or a 4-gon. Then by Theorem 2.1(b), the polytope \( P \) is combinatorially equivalent to a product of simplices.

(f) ⇒ (a). First of all, [16, Lemma 2.2] says that there is a homeomorphism \( Z_K \cong \mathbb{R} Z_{L(K)} \). Since \( K \) has \( m \) vertices, \( \dim(L(K)) = m + \dim(K) = m + n - 1 \). So if \( \text{hrk}(Z_K; k) = 2^{m-n} \), we have
\[
\text{hrk}(\mathbb{R} Z_{L(K)}; k) = 2^{m-n} = 2^{2m-(m+n-1)-1} = 2^{2m-\dim(L(K))-1}.
\]
So (e) holds for \( L(K) \). Since we have already shown (e) ⇒ (a) ⇔ (b), \( L(K) \) is simplicially isomorphic to \( \partial \Delta^{n_1} \ast \cdots \ast \partial \Delta^{n_q} \) for some \( n_1, \ldots, n_q \in \mathbb{N} \). Then we finish the proof by the equivalence of (d) and (a). \( \square \)
Remark 2.3. The equivalences of (b), (e) and (f) in Proposition 2.2 are stated in [4] Section 4.8] as an exercise.

Moreover, we can judge whether a simple polytope $P$ is combinatorially equivalent to a specific product of simplices via some combinatorial invariants called bigraded Betti numbers, which are derived from the Stanley-Reisner ring of $P$ (see [4] Sec 3.2 for the definition). Indeed, it is shown in [7] that a simple polytope $P$ is combinatorially equivalent to $\Delta^{n_1} \times \cdots \times \Delta^{n_q}$ if and only if $P$ has the same bigraded Betti numbers as $\Delta^{n_1} \times \cdots \times \Delta^{n_q}$. Simple polytopes with this kind of property are called combinatorially rigid (see [8, 9]).

Next, we give a new way to judge whether a simple polytope is combinatorially equivalent to a product of simplices.

Theorem 2.4. Let $K$ be the boundary of the simplicial polytope dual to a simple polytope $P$. Then $P$ is combinatorially equivalent to a product of simplices if and only if the following conditions hold for $K$: for any maximal simplex $\sigma$ in $K$ and any vertex $v$ of $\sigma$, the full subcomplex of $K$ by restricting to $V(K) - V(\sigma)$ is a simplex of $K$, denoted by $\xi_\sigma$, and moreover the intersection of $\xi_\sigma$ and link$_K v$ is a simplex (could be empty) as well.

Proof. Suppose $P$ is a product of simplices. Then $K = \partial\Delta^{n_1} \times \cdots \times \partial\Delta^{n_q}$ for some $n_1, \ldots, n_q \in \mathbb{N}$. Denote the vertices of $\partial\Delta^{n_k}$ by $v_0^k, v_1^k, \ldots, v_{n_k}^k$ for each $k = 1, \ldots, q$. Then for a maximal simplex $\sigma$ in $K$, there exists $0 \leq l_k \leq n_k$, $k = 1, \ldots, q$, so that

$$\sigma = [v_0^1, \ldots, \hat{v}_{l_1}^1, \ldots, v_{n_1}^1] \ast [v_0^2, \ldots, \hat{v}_{l_2}^2, \ldots, v_{n_2}^2] \ast \cdots \ast [v_0^q, \ldots, \hat{v}_{l_q}^q, \ldots, v_{n_q}^q]$$

where $[v_0^k, \ldots, \hat{v}_{l_k}^k, \ldots, v_{n_k}^k]$ is the simplex spanned by all the vertices of $\partial\Delta^{n_k}$ except $v_{l_k}^k$ for each $1 \leq k \leq q$. It is easy to see that the full subcomplex of $K$ by restricting to $V(K) - V(\sigma)$ is just the simplex $[v_0^1, v_1^2, \ldots, v_{l_q}^q] = v_{l_1}^1 \ast v_{l_2}^2 \ast \cdots \ast v_{l_q}^q$. All the vertices of $\sigma$ are $\{v_{i_k}^k; 0 \leq i_k \neq l_k \leq n_k, 1 \leq k \leq q\}$. And we have

$$\text{link}_K v_{i_k}^k = \partial\Delta^{n_1} \ast \cdots \ast \partial[v_0^1, \ldots, \hat{v}_{i_k}^k, \ldots, v_{n_k}^k] \ast \cdots \ast \partial\Delta^{n_q}.$$ 

Note that when $n_k = 1$, $\partial[v_0^1, \ldots, \hat{v}_{i_k}^k, \ldots, v_{n_k}^k]$ is empty. Then the intersection of $[v_{l_1}^1, v_{l_2}^2, \ldots, v_{l_q}^q]$ and $\text{link}_K v_{i_k}^k$ is exactly the simplex $[v_{l_1}^1, v_{l_2}^2, \ldots, v_{l_q}^q]$ if $n_k > 1$, and is $[v_{l_1}^1, \ldots, \hat{v}_{i_k}^k, \ldots, v_{l_q}^q]$ if $n_k = 1$. The necessity of these conditions is proved.

For the sufficiency, we first show that if these conditions holds for $K$, then they also hold for the link of any vertex of $K$. When $\dim(K) \leq 1$, the theorem is obviously true. So we assume $\dim(K) \geq 2$ below. Let $u$ be an arbitrary vertex of $K$. Let $\sigma$ be a maximal simplex of $K$ containing $u$ and let $v$ be an arbitrary vertex of $\sigma$ different from $u$. By our assumption, the intersection $\xi_\sigma \cap \text{link}_K u$ and
$\xi_{\sigma} \cap \text{link}_K v$ are both simplices. Let $\tau$ be the simplex with $V(\tau) = V(\sigma) - \{u\}$. Then $\tau$ is a maximal simplex in $\text{link}_K u$. Since $V(\xi_{\sigma}) = V(K) - V(\sigma)$, we have

$$V(\text{link}_K u) - V(\tau) = V(\xi_{\sigma}) \cap V(\text{link}_K u).$$

Then since $\xi_{\sigma} \cap \text{link}_K u$ is a simplex, the full subcomplex of $\text{link}_K u$ by restricting to $V(\text{link}_K u) - V(\tau)$ must agree with $\xi_{\sigma} \cap \text{link}_K u$. Moreover, since $\tau$ could be any vertex of $\tau$, we need to show that the intersection of the simplex $\xi_{\sigma} \cap \text{link}_K u$ with $\text{link}_{\text{link}_K u} v$ is also a simplex. Observe that $\text{link}_{\text{link}_K u} v = \text{link}_K u \cap \text{link}_K v$.

So we have

$$(\xi_{\sigma} \cap \text{link}_K u) \cap \text{link}_{\text{link}_K u} v = (\xi_{\sigma} \cap \text{link}_K u) \cap (\text{link}_K u \cap \text{link}_K v) = (\xi_{\sigma} \cap \text{link}_K u) \cap (\xi_{\sigma} \cap \text{link}_K v)$$

The intersection of the two simplices $\xi_{\sigma} \cap \text{link}_K u$ and $\xi_{\sigma} \cap \text{link}_K v$ has to be a simplex (could be empty) by the definition of simplicial complex. Moreover when $\sigma$ ranges over all the maximal simplices of $K$ containing $u$, the vertex $v$ will range over all the vertices in $\text{link}_K u$. So our argument shows that these conditions hold for $\text{link}_K u$.

By iterating the above argument, we can prove that for any codimension-two simplex $\eta$ of $K$, the link of $\eta$ is a simplicial circle which satisfies the conditions. This forces the link of $\eta$ is either $\partial \Delta^2$ or $\partial \Delta^1 \ast \partial \Delta^1$. Dually it means that any 2-dimensional face of $P$ is either a 3-gon or a 4-gon. Then by Theorem 2.1(b), the polytope $P$ is combinatorially equivalent to a product of simplices.

Next, we give some descriptions of products of simplices in terms of geometric conditions on real moment-angle manifolds of simple convex polytopes. We first recall a concept in metric geometry (see [5, Definition 3.1.12]).

**Definition 2.5 (Quotient Metric Space).** Let $(X, d)$ be a metric space and let $\mathcal{R}$ be an equivalence relation on $X$. The quotient semi-metric $d_{\mathcal{R}}$ is defined as

$$d_{\mathcal{R}}(x, y) = \inf \left\{ \sum_{i=1}^{k} d(p_i, q_i) : p_1 = x, q_k = y, k \in \mathbb{N} \right\},$$

where the infimum is taken over all choices of $\{p_i\}$ and $\{q_i\}$ such that the point $q_i$ is $\mathcal{R}$-equivalent to $p_{i+1}$ for all $i = 1, \cdots, k-1$. Moreover, by identifying points with zero $d_{\mathcal{R}}$-distance, we obtain a metric space $(X/\mathcal{R}, d)$ called the quotient metric space of $(X, d)$.

Suppose $P$ is a simple convex polytope in a Euclidean space $\mathbb{R}^d$. Consider $P$ to be equipped with the intrinsic metric. More precisely, the intrinsic metric on $P$ defines the distance between any two points $x$ and $y$ in $P$ to be the infimum
of lengths of piecewise smooth paths in $P$ that connect $x$ and $y$. Note that the
intrinsic metric on $P$ coincides with the subspace metric on $P$ since $P$ is convex.

By the construction in (4), $\mathbb{R}Z_P = M(P, \lambda)$ is a closed manifold obtained by
gluing $2^m$ copies of $P$ along their facets. We can assume that the $2^m$ copies of $P$
are congruent convex polytopes inside the same Euclidean space and the gluings
of their facets are all isometries. Then by Definition 2.5 we obtain a quotient
metric on $\mathbb{R}Z_P$, denoted by $d_P$. It is clear that the metric $d_P$ is invariant with
respect to the canonical action of $(\mathbb{Z}_2)^m$ on $\mathbb{R}Z_P$ (see (3)).

**Remark 2.6.** We can also call $(\mathbb{R}Z_P, d_P)$ a Euclidean polyhedral space,
which just means that it is built from Euclidean polyhedra (see [5, Definition 3.2.4]).

Note that if $P'$ is another simple convex polytope combinatorially equivalent
to $P$ but not congruent to $P$, the two metric spaces $(\mathbb{R}Z_P, d_P)$ and $(\mathbb{R}Z_P, d_P')$
are not isometric in general (though $\mathbb{R}Z_P$ is homeomorphic to $\mathbb{R}Z_P$).

**Theorem 2.7.** Let $P$ be an $n$-dimensional simple convex polytope, $n \geq 2$, with
$m$ facets. Then the following statements are all equivalent.

(a) $P$ is combinatorially equivalent to a product of simplices.

(b) There exists a non-negatively curved Riemannian metric on $\mathbb{R}Z_P$ that is
invariant under the canonical $(\mathbb{Z}_2)^m$-action on $\mathbb{R}Z_P$.

(c) There exists a simple convex polytope $P'$ combinatorially equivalent to $P$
so that the metric space $(\mathbb{R}Z_P', d_{P'})$ is non-negatively curved.

(d) There exists a simple convex polytope $P'$ combinatorially equivalent to $P'$
so that all the dihedral angles of $P'$ are non-obtuse.

Note that a Riemannian metric on a manifold is non-negatively curved means
that its sectional curvature is everywhere non-negative, while a metric space being
non-negatively curved is defined via comparison of triangles (see [5, Section 4]).

**Proof.** (a) $\Rightarrow$ (b) The real moment-angle manifold of a product of simplices
$\Delta^{n_1} \times \cdots \times \Delta^{n_q}$ is diffeomorphic to a product of standard spheres
$S^{n_1} \times \cdots \times S^{n_q}$ where $S^k = \{(x_1, \cdots, x_{k+1}) \in \mathbb{R}^{k+1} | x_1^2 + \cdots + x_{k+1}^2 = 1\}$ for any $k \in \mathbb{N}$. Let $S^k$
be equipped with the induced Riemannian metric from $\mathbb{R}^{k+1}$. Then it is easy to
check that $S^{n_1} \times \cdots \times S^{n_q}$ is a nonnegatively curved Riemannian manifold with
respect to the product of the Riemannian metrics on $S^{n_1}, \cdots, S^{n_q}$.

(b) $\Rightarrow$ (a) Recall the definition of $\pi_\lambda : M(P, \lambda) = \mathbb{R}Z_P \rightarrow P$ in (2). For any
proper face $f$ of $P$, let $M_f = \pi_\lambda^{-1}(f)$. It is easy to see the following.

- $M_f$ is an embedded closed submanifold of $\mathbb{R}Z_P$ which has $2^{m+\dim(f)-n-m_f}$
  connected components, where $m_f$ is the number of facets of $f$.
- Each connected component of $M_f$ is diffeomorphic to $\mathbb{R}Z_f$. 
Note that $M_f$ is the fixed point set of a rank $n - \dim(f)$ subgroup of $(\mathbb{Z}_2)^m$ under the canonical action of $(\mathbb{Z}_2)^m$ on $\mathbb{R}Z_P$. Then since the Riemannian metric is $(\mathbb{Z}_2)^m$-invariant, each component of $M_f$ is a totally geodesic submanifold of $\mathbb{R}Z_P$ (see [13, Theorem 5.1]), and so is non-negatively curved with respect to the induced Riemannian metric from $\mathbb{R}Z_P$. This implies that the condition (b) holds for $\mathbb{R}Z_f$ as well.

In particular when $\dim(f) = 2$, the $\mathbb{R}Z_f$ is a closed connected surface with non-negatively curved Riemannian metric. Then by Gauss-Bonnet Theorem, the Euler characteristic $\chi(\mathbb{R}Z_f) \geq 0$, which implies that $f$ has to be a 3-gon or a 4-gon. Then by Theorem [2,1b), the polytope $P$ is combinatorially equivalent to a product of simplices.

(a) ⇒ (c) Suppose $P$ is combinatorially equivalent to $\Delta^{n_1} \times \cdots \times \Delta^{n_q}$ where $n_1 + \cdots + n_q = n$. Consider the standard simplex $\Delta^k$ as a metric subspace of $\mathbb{R}^{k+1}$ with the intrinsic metric. Let $P' = \Delta^{n_1} \times \cdots \times \Delta^{n_q}$ be the product of the $q$ metric spaces $\Delta^{n_1}, \ldots, \Delta^{n_q}$. For each $1 \leq i \leq q$, let $\{v_{0i}, \ldots, v_{ni}\}$ be the set of vertices of $\Delta^{n_i}$. Then all the facets of $P'$ are (see [6])

$$\{F'_{ki} = \Delta^{n_1} \times \cdots \times \Delta^{n_{i-1}} \times f^i_{ki} \times \Delta^{n_{i+1}} \times \cdots \times \Delta^{n_q} \mid 0 \leq k_i \leq n_i, \ i = 1, \ldots, q\}$$

where $f^i_{ki}$ is the codimension-one face of the simplex $\Delta^{n_i}$ which is opposite to the vertex $v^i_{ki}$. The total number of facets of $P'$ is $m = n + q$.

Claim: As a metric space $(\mathbb{R}Z_{P'}, d_{P'})$ is isometric to the product of the $q$ metric spaces $(\mathbb{R}Z_{\Delta^{n_1}}, d_{\Delta^{n_1}}), \ldots, (\mathbb{R}Z_{\Delta^{n_q}}, d_{\Delta^{n_q}})$.

Indeed if we glue two copies of $P'$ along the facet $F^i_{ki}$, we obtain

$$\Delta^{n_1} \times \cdots \times \Delta^{n_{i-1}} \times \left(\Delta^{n_i} \cup_{f^i_{ki}} \Delta^{n_i}\right) \times \Delta^{n_{i+1}} \times \cdots \times \Delta^{n_q}.$$ 

We can decompose the gluing procedure in the construction (2) for $\mathbb{R}Z_{P'}$ into $q$ steps. The $i$-th step only glues those facets of the form $\{F^i_{ki}, 0 \leq k_i \leq n_i\}$ in the $2^{n_i}$ copies of $P'$, which gives us the factor $(\mathbb{R}Z_{\Delta^{n_i}}, d_{\Delta^{n_i}})$, while fixing all other factors in the product. So after the first step we obtain $2^{m-n_1-1}$ copies of $\mathbb{R}Z_{\Delta^{n_1}} \times \Delta^{n_2} \times \cdots \times \Delta^{n_q}$. After the second step we obtain $2^{m-n_1-n_2-2}$ copies of $\mathbb{R}Z_{\Delta^{n_1}} \times \mathbb{R}Z_{\Delta^{n_2}} \times \Delta^{n_3} \times \cdots \times \Delta^{n_q}$ and so on. Then our claim follows.

Moreover, observe that for any $k \in \mathbb{N}$, $(\mathbb{R}Z_{\Delta^k}, d_{\Delta^k})$ is isometric to the boundary of the $(k+1)$-dimensional cross-polytope $Q^{k+1}$ whose vertices are

$$\{(0, \cdots, 0, \underbrace{1, 0, \cdots, 0}_{i}, 0, \cdots, 0) ; (0, \cdots, 0, \underbrace{1, 0, \cdots, 0}_{i}, 0, \cdots, 0) ; i = 1, \cdots, k+1\}.$$ 

Recall that the $n$-dimensional cross-polytope is the simplicial polytope dual to the $n$-dimensional cube (see Figure 1 for the case $n = 2, 3$).

It is well known that the intrinsic metric on any convex hypersurface (i.e. the boundary of a compact convex set with nonempty interior) in a Euclidean space
$\mathbb{R}^n$ ($n \geq 3$) is non-negatively curved (see [5, p.359]). Then since $Q^{k+1}$ is a convex polytope in $\mathbb{R}^{k+1}$, $(\mathbb{R}Z_{\Delta^k}, d_{\Delta^k})$ is non-negatively curved for any $k \geq 2$. When $k = 1$, the boundary of $Q^2$ is a piecewise smooth simple curve in $\mathbb{R}^2$. But by definition (see [5, Definition 4.1.9]), the intrinsic metric on any piecewise smooth simple curve is non-negatively curved because any geodesic triangle on the curve is degenerate. So we can conclude that $(\mathbb{R}Z_{P'}, d_{P'})$ is non-negatively curved because the product of non-negatively curved Alexandrov spaces is still non-negatively curved (see [5, Chapter 10]).

(c) $\Rightarrow$ (d) If the metric $d_{P'}$ on $\mathbb{R}Z_{P'}$ is non-negatively curved, we want to show that the dihedral angle between any two adjacent facets $F_1$ and $F_2$ of $P'$ is non-obtuse. Otherwise, assume that the dihedral angle $\theta$ between $F_1$ and $F_2$ is obtuse. Choose a point $O$ in the relative interior of $F_1 \cap F_2$, a point $A \in F_1$ and $B \in F_2$ so that the line segments $OA$ and $OB$ are perpendicular to $F_1 \cap F_2$. Then $\angle AOB = \theta$. Suppose the lengths of the line segments $OA$, $OB$ and $AB$ are $|OA| = |OB| = a$, $|AB| = b$.

In the gluing construction for $\mathbb{R}Z_{P'}$, consider two copies of $P'$ glued along the facet $F_1$. We then have an isosceles triangle $\triangle AB_1B_2$ in $\mathbb{R}Z_{P'}$ (see Figure 2). When $a$ is small enough, the distance between $B_1$ and $B_2$ in $(\mathbb{R}Z_{P'}, d_{P'})$ is $2a$ by the definition of the quotient metric because $\bar{B}_1O \cup \bar{O}B_2$ is the shortest path between $B_1$ and $B_2$ in $(\mathbb{R}Z_{P'}, d_{P'})$. Moreover, let $\triangle \bar{A}B_1B_2$ be a triangle in the Euclidean plane $\mathbb{R}^2$ which have the same lengths of sides as $\triangle AB_1B_2$. Then since $\theta$ is obtuse, it is clear that $\triangle AB_1B_2$ is strictly thinner than $\triangle \bar{A}B_1B_2$, i.e.

$$\angle AB_1B_2 < \triangle \bar{A}B_1B_2, \quad \angle AB_2B_1 < \triangle \bar{A}B_2B_1, \quad \angle B_1AB_2 < \triangle \bar{B}_1\bar{A}\bar{B}_2.$$ But this contradicts our assumption that the metric $d_{P'}$ on $\mathbb{R}Z_{P'}$ is non-negatively curved (see [5, Section 4.1.5]). Therefore, $\theta$ has to be non-obtuse.

(d) $\Rightarrow$ (a) Suppose $F_1, F_2$ and $F_3$ are three facets of $P'$ with $F_1 \cap F_2 \cap F_3 \neq \emptyset$. Then $F_1 \cap F_2$ and $F_1 \cap F_3$ are codimension-one faces of $F_1$. By our assumption, the dihedral angles of $(F_1, F_2)$, $(F_1, F_3)$ and $(F_2, F_3)$ are all non-obtuse. We claim that the dihedral angle between $F_1 \cap F_2$ and $F_1 \cap F_3$ in $F_1$ is non-obtuse as well.
Indeed, we can assume that $P'$ sits inside $\mathbb{R}^n$ and let $\eta_i \in \mathbb{R}^n$ ($i = 1, 2, 3$) be a normal vector of $F_i$ pointing to the interior of $P$ (see Figure 3). By choosing a proper coordinate system of $\mathbb{R}^n$, we can assume that $\eta_1 = (0, \cdots, 0, 1) \in \mathbb{R}^n$ and $F_1$ lies in the coordinate hyperplane \{x_n = 0\} $\subset \mathbb{R}^n$. Let $\eta_2 = (a_1, \cdots, a_{n-1}, a_n)$, $\eta_3 = (b_1, \cdots, b_{n-1}, b_n)$. Since the dihedral angles of $(F_1, F_2)$, $(F_1, F_3)$ and $(F_2, F_3)$ are all non-obtuse, the inner products of $\eta_1$, $\eta_2$, $\eta_3$ satisfy

$$\eta_1 \cdot \eta_2 = a_n \leq 0, \quad \eta_1 \cdot \eta_3 = b_n \leq 0, \quad (\eta_2, \eta_3) = a_1 b_1 + \cdots + a_{n-1} b_{n-1} + a_n b_n \leq 0.$$  

(5)

Note that $(a_1, \cdots, a_{n-1}, 0)$ and $(b_1, \cdots, b_{n-1}, 0)$ are normal vectors of $F_1 \cap F_2$ and $F_1 \cap F_3$ inside $F_1$ respectively. So (5) implies that the dihedral angle between $F_1 \cap F_2$ and $F_1 \cap F_3$ in $F_1$ is non-obtuse. Our claim is proved.

By iterating the above arguments, we can show that for any 2-dimensional face $f$ of $P'$, any interior angle of $f$ is non-obtuse. Since $f$ is a Euclidean polygon, it must be either a 3-gon or a 4-gon. So since $P$ is combinatorially equivalent to $P'$, any 2-face of $P$ is either a 3-gon or a 4-gon, too. Then by Theorem 2.1(b), the polytope $P$ is combinatorially equivalent to a product of simplices. \hfill $\square$
Remark 2.8. In the statement of Theorem 2.7(b), if we do not require the Riemannian metric on $\mathbb{R}Z_P$ to be $(\mathbb{Z}_2)^m$-invariant, it is still likely that $P$ has to be combinatorially equivalent to a product of simplices (see [14] Section 5.2). But we do not know how to prove this so far.

3. Appendix

Here we give another proof of Theorem 2.1(b). For brevity, we say that a simplicial complex is a sphere join if it is isomorphic to $\partial\Delta^{n_1} \ast \cdots \ast \partial\Delta^{n_q}$ for some $n_1, \ldots, n_q \in \mathbb{N}$. One dimensional sphere join is either $\partial\Delta^2$ (boundary of a triangle) or $\partial\Delta^1 \ast \partial\Delta^1$ (boundary of a square). Let us first prove the following theorem.

**Theorem 3.1.** Let $K$ be a simplicial complex of dimension $n$. Suppose that $K$ satisfies the following two conditions:

(a) $K$ is a pseudomanifold,
(b) the link of any vertex of $K$ is a sphere join of dimension $n - 1$.

Then $K$ is a sphere join.

Recall that $K$ is an $n$-dimensional pseudomanifold if the following conditions hold:

(i) Every $(n - 1)$-simplex of $K$ is a face of exactly two $n$-simplices for $n > 1$.
(ii) For every pair of $n$-simplices $\sigma$ and $\sigma'$ in $K$, there exists a sequence of $n$-simplices $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \sigma'$ such that the intersection $\sigma_i \cap \sigma_{i+1}$ is an $(n - 1)$-simplex for all $i$.

The Condition (ii) means that $K$ is a strongly connected simplicial complex.

**Proof.** First of all, assumption (b) implies that the link of any $k$-simplex in $K$ is a sphere join of dimension $n - k - 1$. Let $w$ be a vertex of $K$. By assumption (b), the link $\text{link}_K w$ is of the form $\text{link}_K w = \partial\Delta^{n_1} \ast \cdots \ast \partial\Delta^{n_q}$ where $n_1 + \cdots + n_q = n$. Denote the vertices of $\partial\Delta^{n_k}$ by $v_0^k, v_1^k, \ldots, v_{n_k}^k$ for $k = 1, \ldots, q$, so that

$$\text{link}_K w = \partial[v_0^1, v_1^1, \ldots, v_{n_1}^1] \ast \cdots \ast \partial[v_0^q, v_1^q, \ldots, v_{n_q}^q].$$

(6)

Let $I$ be the set of vertices $v_1^1, \ldots, v_{n_1}^1, \ldots, v_1^q, \ldots, v_{n_q}^q$. Then $[I]$ is a maximal simplex in $\text{link}_K w$ and the simplex $[I, w]$ spanned by $I$ and $w$ is of dimension $n$. Since $K$ is a pseudomanifold by assumption (a), there is a unique vertex $v$ in $K$ such that $[I, v] \cap [I, w] = [I]$. We have two cases below.

Case 1. The case where $v \notin \text{link}_K w$. In this case we claim $K = \partial[v, w] \ast \text{link}_K w$. The proof is as follows. Choose an element from $I$ arbitrarily, say $v_j^i$ ($1 \leq i \leq q$, $1 \leq j \leq n_i$). Set $\tilde{I} = (I \setminus \{v_j^i\}) \cup \{v_j^0\}$. Then $[\tilde{I}]$ is an $(n - 1)$-simplex of $\text{link}_K w$ by (6), so there is a unique vertex $\tilde{v}$ of $K$ such $[I, \tilde{v}] \cap [I, w] = [I]$ as before since
$K$ is a pseudomanifold. Now we shall observe the link of an $(n-2)$-simplex $[I \cap \bar{I} = \{v_j^i\}]$ in $K$. By our construction, the following are four $n$-simplices in $K$ containing $[I \cap \bar{I}]$:

$$[I \cap \bar{I}, v_j^i, w], \ [I \cap \bar{I}, v_0^i, w], \ [I \cap \bar{I}, v_j^i, v], \ [I \cap \bar{I}, v_0^i, \bar{v}].$$

Therefore the vertices $v_j^i, w, v_0^i, v, \bar{v}$ are in the link of the $(n-2)$-simplex $[I \cap \bar{I}]$. But by assumption (b), this link is a sphere join of dimension one which can have at most four vertices. Note that $v_j^i, w, v_0^i$ are mutually distinct and $v, \bar{v}$ are different from $v_j^i, w, v_0^i$. So we must have $\bar{v} = v$. Now let $v_j^i$ run over all elements of $I$, then $\bar{I}$ runs over all $(n-1)$-simplices in link$_K w$ that share a $(n-2)$-simplex with $I$. Moreover by (6), link$_K w$ is a strongly connected simplicial complex. So applying our argument to $[I]$ and all other $(n-1)$-simplices in $K$, we can show that $\partial[v, w] \ast $link$_K w$ is a subcomplex of $K$. However, $\partial[v, w] \ast $link$_K w$ and $K$ are both pseudomanifolds and have the same dimension, so they must agree. This proves the claim.

Case 2. The case where $v \in $link$_K w$, so $v$ is one of $v_0^1, v_0^2, \ldots, v_0^q$. We may assume $v = v_0^1$ without loss of generality. Then

$$[v, I] = [v_0^1, v_1^1, \ldots, v_{n_1}^1, v_1^2, \ldots, v_{n_2}^2, \ldots, v_1^q, \ldots, v_{n_q}^q]$$

is an $n$-simplex in $K$. (7)

We look at link$_K v$. Since $v = v_0^1$, it follows from (6) that link$_K v$ contains

$$\partial[v_1^1, \ldots, v_1^1] \ast \partial[v_0^2, v_1^2, \ldots, v_{n_2}^2] \ast \ldots \ast \partial[v_0^q, v_1^q, \ldots, v_{n_q}^q]$$

as a subcomplex. This together with assumption (b) implies that there is a vertex $w'$ different from vertices in (8) such that link$_K v$ is one of the following:

$$\partial[w', v_1^1, \ldots, v_{n_1}^1] \ast \partial[v_0^2, v_1^2, \ldots, v_{n_2}^2] \ast \ldots \ast \partial[v_0^q, v_1^q, \ldots, v_{n_q}^q],$$

$$\partial[v_1^1, \ldots, v_{n_1}^1] \ast \partial[w', v_0^2, v_1^2, \ldots, v_{n_2}^2] \ast \ldots \ast \partial[v_0^q, v_1^q, \ldots, v_{n_q}^q],$$

$$\vdots$$

$$\partial[v_1^1, \ldots, v_{n_1}^1] \ast \partial[v_0^2, v_1^2, \ldots, v_{n_2}^2] \ast \ldots \ast \partial[w', v_0^q, v_1^q, \ldots, v_{n_q}^q].$$

However, the fact (7) implies that none of the above occurs except the first one. So we have

$$\text{link}_K v = \partial[w', v_1^1, \ldots, v_{n_1}^1] \ast \partial[v_0^2, v_1^2, \ldots, v_{n_2}^2] \ast \ldots \ast \partial[v_0^q, v_1^q, \ldots, v_{n_q}^q].$$

(9)

The simplex $[I]$ is in link$_K v$ by (7) and the $n$-simplices $[I, v]$ and $[I, w]$ share $[I]$. Note that $w \in $link$_K v$ in this case but $w \neq v_j^i$ for all $i$ and $j$. So from (9) we can conclude $w = w'$. Then

$$\text{link}_K v = \partial[w, v_1^1, \ldots, v_{n_1}^1] \ast \partial[v_0^2, v_1^2, \ldots, v_{n_2}^2] \ast \ldots \ast \partial[v_0^q, v_1^q, \ldots, v_{n_q}^q].$$

(10)
Remember that \( v = v_0^1 \). We claim that \( K \) contains
\[
\partial[w, v_0^1, v_1^1, \ldots, v_{n_1}^1] \ast \partial[v_0^2, v_1^2, \ldots, v_{n_2}^2] \ast \cdots \ast \partial[v_0^q, v_1^q, \ldots, v_{n_q}^q]
\] (11)
as a subcomplex. Indeed, any \( n \)-simplex in (11) is spanned by \( n + 1 \) vertices which consist of \( n_1 + 1 \) vertices from \( \partial[w, v_0^1, v_1^1, \ldots, v_{n_1}^1] \) and \( n_i \) vertices from \( \partial[v_0^i, v_1^i, \ldots, v_{n_i}^i] \) for \( i = 2, 3, \ldots, q \). Since \( v_0^1 = v \), either \( w \) or \( v \) is in the \( n_1 + 1 \) vertices from \( \partial[w, v_0^1, v_1^1, \ldots, v_{n_1}^1] \). If \( w \) (resp. \( v \)) is in the \( n_1 + 1 \) vertices from \( \partial[w, v_0^1, v_1^1, \ldots, v_{n_1}^1] \), then any \( n \)-simplex formed this way is in \( K \) by (10) (resp. (6)). This proves the claim.

Finally, since \( K \) and the subcomplex (11) are both pseudomanifolds and have the same dimension, they must agree. So we finish the proof of the theorem.

\textbf{Proof of Theorem 2.1(b):} Suppose any 2-dimensional face of \( P \) is either a 3-gon or a 4-gon. We want to show that \( P \) is combinatorially equivalent to a product of simplices, or equivalently \( \partial P^* \) is a sphere join. Let us do induction on the dimension of \( P \). When \( \dim P = 2 \), the proof is trivial. If \( \dim P \geq 3 \), we will show that \( \partial P^* \) satisfies the two conditions in Theorem 3.1. Condition (a) is obvious. By induction assumption, all facets of \( P \) are product of simplices which means that \( \partial P^* \) satisfies condition (b). So we finish the induction by Theorem 3.1.

\textbf{Remark 3.2.} Theorem 2.1(b) is equivalent to saying that if all the facets of \( P^n \) are products of simplices, then so is \( P^n \). In fact this statement has already appeared in an old paper [12, Lemma 2.7] where a product of simplices is called a “simplicial prism”. But the proof of this statement in [12] is a bit vague in the final step. In addition, Theorem 2.7(d) is also stated in [12, Lemma 2.8].

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