The fundamental group of a toroidal compactification of a Hermitian locally symmetric space

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Abstract

The present work obtains the fundamental group of a toroidal compactification \((D/\Gamma)_{\Sigma}^{'}\) of a non-compact quotient \(D/\Gamma\) of a Hermitian symmetric space \(D\) of non-compact type by a lattice \(\Gamma\) in the isometry group \(G\) of \(D\). As a consequence it derives the equality \(\text{rk}_{\mathbb{Z}}H_1((D/\Gamma)_{\Sigma}^{'},\mathbb{Z}) = \text{rk}_{\mathbb{Z}}H_1(D/\Gamma,\mathbb{Z})\) of the ranks of the first homology groups of \((D/\Gamma)_{\Sigma}^{'},\mathbb{Z}\) and \(D/\Gamma,\mathbb{Z}\) with integral coefficients. The work provides also a sufficient condition on a torsion free non-uniform lattice \(\Gamma\), under which the fundamental group of \((D/\Gamma)_{\Sigma}^{'}\) is residually finite. Articles of Hummel-Schroeder, Hummel and Di Cerbo imply that the toroidal compactifications \((\mathbb{B}^n/\Gamma)_{\Sigma}^{'}\) of generic non-compact torsion free \(\mathbb{B}^n/\Gamma\) satisfy this sufficient condition.

1 Introduction

The Hermitian symmetric spaces of noncompact type are quotients \(D = G/K\) of non-compact semisimple Lie groups \(G\) by connected maximal compact subgroups \(K\) of \(G\). A non-uniform lattice \(\Gamma\) of \(G\) is a discrete subgroup whose quotient \(D/\Gamma\) is noncompact and has finite invariant volume. A parabolic subgroup \(Q\) of \(G\) is called \(\Gamma\)-rational if its unipotent radical \(N_Q\) intersects \(\Gamma\) in a lattice \(\Gamma_Q := \Gamma \cap N_P\) of \(N_Q\). Let us denote by \(\Gamma P(G)\) the set of the \(\Gamma\)-rational parabolic subgroups of \(G\). In [2] Ash, Mumford, Rapoport and Tai construct complex analytic compactifications \((D/\Gamma)_{\Sigma}^{'}\) of the quotients \(D/\Gamma\) by arithmetic non-uniform lattices \(\Gamma < G\), which depend on \(\Gamma\)-admissible families \(\Sigma = \{\Sigma(Q)\}_{Q \in \Gamma P(G)}\) of polyhedral cone decompositions \(\Sigma(Q)\) of the centers \(U_Q \simeq (\mathbb{R}^m,+)\) of \(N_Q\). These compactifications are obtained from \(D/\Gamma\) by adjoining subsets of toric varieties, associated with \(Q \in \Gamma P(G)\) and are called toroidal compactifications. An extensive reference on toroidal and other types of compactifications of local Hermitian symmetric spaces \(D/\Gamma\) is the monograph [4] of Borel and Ji.

According to Margulis (cf. [25]), all irreducible lattices \(\Gamma\) of semisimple Lie groups \(G\) of real rank \(\text{rk}_\mathbb{R}G \geq 2\) are arithmetic. The only Hermitian symmetric spaces \(D = G/K\) of non-compact type with \(\text{rk}_\mathbb{R}G = 1\) are the complex balls \(\mathbb{B}^n = SU_{n,1}/S(U_n \times \mathbb{C})\).

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and their toroidal compactifications $(\mathbb{B}^n/\Gamma)'$ do not depend on $\Sigma$. Non-arithmetic lattices of $SU_{n,1}$ are constructed by Mostow in [29], Deligne-Mostow in [6], Deraux, Parker and Paupert in [9] and others. In [21] Hummel and Schroeder apply a differential geometric technique, called cusp closure, towards the construction of the toroidal compactification $(\mathbb{B}^n/\Gamma)'$ for a not necessarily arithmetic lattice $\Gamma < SU_{n,1}$. Mok’s [26], Hummel’s [22] and Di Cerbo’s [7] study polarization of type $(1,p)$ and level $p^2$. Hullek and Sankaran’s [20] extends Knöller’s result by proving the simply connectedness of any smooth model of the moduli space $A_{1,p}$ of the abelian surfaces with a polarization of type $(1,p)$ and a level structure.

The fundamental group of a toroidal compactification of an arithmetic noncompact quotient $D/\Gamma$ of an arbitrary Hermitian symmetric space $D = G/K$ of non-compact type is studied by Sankaran in [35]. A lattice $\Gamma < G$ is neat if the eigenvalues of any $\gamma \in \Gamma$ generate a torsion free subgroup of $\mathbb{C}^*$. For an arbitrary lattice $\Gamma < G$ let us denote by $\Gamma^U$ the subgroup of $\Gamma$, generated by the intersections $\Gamma^U_Q := \Gamma \cap U_Q$ with the centers $U_Q$ of the unipotent radicals $N_Q$ of the $\Gamma$-rational parabolic subgroups $Q$ of $G$. According to $\gamma U_Q \gamma^{-1} = U_{\gamma Q \gamma^{-1}}$ for $\forall \gamma \in \Gamma$, $\Gamma^U$ is a normal subgroup of $\Gamma$. For any arithmetic non-uniform lattice $\Gamma < G$ Sankaran shows in [35] that the fundamental group $\pi_1(D/\Gamma)_{\Sigma}'$ of a toroidal compactifications $(D/\Gamma)'_{\Sigma}$ is a quotient group of $\Gamma/\Gamma^U$.

In the case of a neat arithmetic non-uniform lattice $\Gamma < G$, he establishes the equality $\pi_1(D/\Gamma)'_{\Sigma} = \Gamma/\Gamma^U$. This is done by construction of a simply connected complex analytic space $(D/\Gamma)'_{\Sigma}$, which is an etale $\Gamma/\Gamma^U$-Galois cover of $(D/\Gamma)'_{\Sigma}$ in the case of a neat $\Gamma$. In the same article, Sankaran shows that for any finite group $\Phi$ there is an integer $g \geq 2$ and a subgroup $\Gamma \subseteq Sp(2g, \mathbb{Z})$ of finite index, such that $\pi_1(H_g/\Gamma)'_{\Sigma} = \Phi$ for the Siegel upper half-space $H_g = Sp(2g, \mathbb{R})/U_g$.

The present note proves that for an arbitrary (not necessarily arithmetic) non-uniform lattice $\Gamma < G$, the fundamental group of a toroidal compactification $(D/\Gamma)'_{\Sigma}$ is $\pi_1(D/\Gamma)'_{\Sigma} = \Gamma/\Gamma^{\text{Fix}}\Gamma^U$, where $\Gamma^{\text{Fix}}$ is the subgroup of $\Gamma$, generated by $\gamma \in \Gamma$ with at least one fixed point on $D$. Due to $\text{Stab}_D(\gamma p) = \gamma \text{Stab}_D(p) \gamma^{-1}$ for $\forall p \in D$, $\forall \gamma \in \Gamma$, the generating set of $\Gamma^{\text{Fix}}$ is invariant under conjugations by $\gamma \in \Gamma$ and $\Gamma^{\text{Fix}}$ is a normal subgroup of $\Gamma$. As a consequence of the calculation of $\pi_1(D/\Gamma)'_{\Sigma}$, the article shows the equality $\text{rk}_2 H_1((D/\Gamma)'_{\Sigma}, \mathbb{Z}) = \text{rk}_2 H_1(D/\Gamma, \mathbb{Z})$ of the ranks of the first integral homology groups. It derives a sufficient condition for the residual finiteness of $\pi_1(D/\Gamma)'_{\Sigma}$ in the case of a torsion free non-uniform $\Gamma < G$. We believe that the toroidal compactifications $(D/\Gamma)'_{\Sigma}$ of generic torsion free $D/\Gamma$ satisfy this sufficient condition.

Here is a synopsis of the paper. Section 2 introduces some notations and terminology, which are used throughout. The first two subsections recall the refined Langlands decomposition of a parabolic subgroup $Q < G$ and the refined horospherical decomposition of $D$, associated with $Q$. The third subsection discuss the Siegel domain presentations $D = (U_Q + iC_Q) \times V_Q \times D_{Q,h}$ of $D$, associated with $Q \in \Gamma P(G)$. These are products of the vector groups $U_Q \simeq (\mathbb{R}^m, +)$, $V_Q := N_Q/U_Q \simeq (\mathbb{C}^n, +)$.
with strongly convex polyhedral cones \( C_Q \subset U_Q \) and the \( \Gamma \)-rational analytic boundary components \( D_{Q,h} \), normalized by \( Q \). The product \( iC_Q \) of \( C_Q \) with the imaginary unit \( i \) are viewed as open subset of the pure imaginary part of the complexifications \( U_Q \otimes \mathbb{R} \subset U_Q \). The exposition follows the monograph [4] of Borel and Ji. The last subsection 2.4 collects some data, needed for the definition of a \( \Gamma \)-admissible family \( \Sigma = \{ \Sigma(Q) \}_{Q \in \Gamma P(G)} \) of polyhedral cone decompositions \( \Sigma(Q) \) of \( U_Q \).

Section 3 is devoted to the toroidal compactifications \( (D/\Gamma)_{\Sigma} \) and their coverings \( (D/\Gamma_\alpha)_{\Sigma} \) for normal subgroups \( \Gamma_\alpha < \Gamma \), containing \( \Gamma^U \). For an arbitrary \( \Gamma \)-rational parabolic subgroup \( Q < G \) the intersection \( \Gamma^U_Q := \Gamma \cap U_Q \cong (\mathbb{Z}^m,+) \) is a uniform lattice of \( U_Q \cong (\mathbb{R}^m,+) \). The quotient \( T(Q) := U_Q \otimes \mathbb{C}/\Gamma^U_Q \cong (\mathbb{C}^*)^m \) is an algebraic torus over \( \mathbb{C} \). The toric variety \( X_{\Sigma(Q)} \), associated with a \( \Gamma \)-admissible cone decomposition \( \Sigma(Q) \) of \( U_Q \) contains \( T(Q) \) as an open dense subset. The action of \( T(Q) \) on itself by multiplication extends to a \( T(Q) \)-action on \( X_{\Sigma(Q)} \). Let \( Y_{\Sigma(Q)} \) be the interior of the closure of \( U_Q + iC_Q/\Gamma^U_Q \) in \( X_{\Sigma(Q)} \). The considerations from subsection 2.4 imply the existence of \( \Sigma = \{ \Sigma(Q) \}_{Q \in \Gamma P(G)} \) with smooth \( X_{\Sigma} \) and \( Y_{\Sigma} \). Subsection 3.1 introduces the complex analytic spaces \( Z_{\Sigma(Q)} = (D/\Gamma^U_Q)_{\Sigma(Q)} = Y_{\Sigma(Q)} \times V_Q \times D_{Q,h} \) for \( Q \in \Gamma P(G) \). The next subsection 3.2 constructs the gluing maps \( \mu_Q^P : Z_{\Sigma(Q)} \to Z_{\Sigma(P)} \) for \( P, Q \in \Gamma P(G), P \subseteq Q \), which are needed for the definition of \( (D/\Gamma_\alpha)_{\Sigma} \). Subsection 3.3 defines the complex analytic space \( (D/\Gamma_\alpha)_{\Sigma} = \bigoplus_{Q \in \Gamma P(G)} Z_{\Sigma(Q)} / \sim_{\Gamma_\alpha} \) as the quotient of the disjoint union of \( Z_{\Sigma(Q)} \) with respect to the equivalence relation \( \sim_{\Gamma_\alpha} \), induced by the \( \Gamma_\alpha \)-action on \( D \). From now on, let \( \Gamma \text{Min} P(G) \) be the set of the \( \Gamma \)-rational minimal parabolic subgroups \( P \) of \( G \). Proposition 1 from 3.3 checks that the spaces \( Z_{\Sigma(P)} \), associated with \( P \in \Gamma \text{Min} P(G) \) generate \( (D/\Gamma_\alpha)_{\Sigma} = \bigoplus_{P \in \Gamma \text{Min} P(G)} Z_{\Sigma(P)} / \sim_{\Gamma_\alpha} \).

This is in the spirit of Sankaran’s observation from [35] that \( \Gamma^U \) is generated by \( \Gamma^U_P := \Gamma \cap U_P \) for \( P \in \Gamma \text{Min} P(G) \). The construction of \( (D/\Gamma_\alpha)_{\Sigma} \) by the spaces \( Z_{\Sigma(P)} \), \( P \in \Gamma \text{Min} P(G) \) enables to view \( (D/\Gamma_\alpha)_{\Sigma} \) as a union of discrete quotients of \( Z_{\Sigma(P)} = Y_{\Sigma(P)} \times V_Q \cong \Delta^r \times \mathbb{C}^n \), where \( \Delta = \{ t \in \mathbb{C} \mid |t| < 1 \} \) is the unit disc in \( \mathbb{C} \), \( r = \text{rk}_G \mathbb{R}, n = \text{dim}_\mathbb{C} D - r \). Subsection 3.3 illustrates also the dependence of the smooth simply connected coverings \( (D/\Gamma^U)^{\cdot}_{\Sigma} \) of \( (D/\Gamma)^{\cdot}_{\Sigma} \) on \( \Gamma \). More precisely, for any normal subgroup \( \Upsilon \triangleleft \Gamma \), the simply connected complex manifold \( (D/\Upsilon^U)^{\cdot}_{\Sigma} \) is a ramified covering of \( (D/\Gamma^U)^{\cdot}_{\Sigma} \), regardless of the lack or presence of torsions in \( \Gamma \). Lemma 2 from 3.4 shows that for any element \( \gamma \Gamma^U \in \Gamma/\Gamma^U \) with a fixed point on \( (D/\Gamma^U)^{\cdot}_{\Sigma} \) there is a representative \( \gamma \in \Gamma \) with a fixed point on \( D \).

Section 4 comprises the main results of the article. After proving that \( \pi_1(D/\Gamma)^{\cdot}_{\Sigma} = \Gamma/\Gamma^{\text{Fix}\Gamma^U} \) in 4.1, subsection 4.2 establishes the existence of a finite subgroup \( F \) of the first homology group \( H_1(D/\Gamma, \mathbb{Z}) \), such that \( H_1(D/\Gamma, \mathbb{Z})/F \approx H_1((D/\Gamma)^{\cdot}_{\Sigma}, \mathbb{Z}) \). In particular, \( \text{rk}_\mathbb{Z} H_1(D/\Gamma, \mathbb{Z}) = \text{rk}_\mathbb{Z} H_1((D/\Gamma)^{\cdot}_{\Sigma}, \mathbb{Z}) \). Subsection 4.3 provides a sufficient condition \( SC \) for residual finiteness of \( \pi_1(D/\Gamma)^{\cdot}_{\Sigma} = \Gamma/\Gamma^U \) in the case of a torsion free non-uniform lattice \( \Gamma < G \). If \( (D/\Gamma)^{\cdot}_{\Sigma} \) satisfies \( SC \) then the rank of the first homology group \( H_1((D/\Gamma)^{\cdot}_{\Sigma}, \mathbb{Z}) \) is shown to be bounded above by \( 2(\text{dim}_\mathbb{C} D - \text{rk}_G \mathbb{R}) \). According to Hummel-Schroeder’s [21] and Hummel’s [22], any torsion free non-uniform lattice \( \Gamma_\alpha < SU_{n,1} \) has a normal subgroup \( \Gamma \triangleleft \Gamma_\alpha \) of finite index \( [\Gamma_\alpha : \Gamma] < \infty \), such that the ball quotient compactification \( (\mathbb{B}^n/\Gamma)^{\prime} \) satisfies \( SC \), whenever the \( \Gamma \)-action on the \( \Gamma \)-rational boundary points \( \partial_{\Gamma} \mathbb{B}^n \) of \( \mathbb{B}^n \) is not transitive. We believe
that a generic \((D/\Gamma)^o\) is subject to SC. Subsection 4.4 is devoted to the Kobayashi non-hyperbolicity of \((D/\Gamma)^o\). This property implies that any locally trivial holomorphic fibration \((D/\Gamma)^o\) over a complex analytic space \(M\) of \(\dim \mathbb{C} M \geq \text{rk}_\mathbb{C} G\) has Kobayashi non-hyperbolic base. The final, fifth subsection of section 4 formulates two open questions. More precisely, in [35] Sankaran observes that the spaces \(Z_{\Sigma}(Q)\) for \(Q \in \Gamma P(G)\) are mapped biholomorphically onto their images in \((D/\Gamma^U)^o\).

Combining with Proposition 1, one concludes that the simply connected complex manifold \((D/\Gamma^U)^o\) is a (generically infinite) union of subspaces, which are biholomorphic to \(\Delta^* \times \mathbb{C}^n\) and have fairly complicated intersections. One questions whether \((D/\Gamma^U)^o\) is a Stein space. It is also interesting to compute that higher homotopy groups \(\pi_i(D/\Gamma^U)^o = \pi_i(D/\Gamma)^o, i \geq 2\) in the case of a torsion-free non-uniform lattice \(\Gamma < G\).

The final section discusses some properties of the toroidal compactifications \(X' = (\mathbb{B}^2/\Gamma')\) of the quotients \(\mathbb{B}^2/\Gamma\) of the complex 2-ball \(\mathbb{B}^2 = SU_{2,1}/S(U_2 \times U_1)\) by torsion-free non-uniform lattices \(\Gamma < SU_{2,1}\). Let \(X\) be the minimal model of \(X'\). Proposition 15 from subsection 5.1 observes that \(\kappa(X) \geq 0\) and derives the inequality 
\[
e^2 c^2(X) = K^2_X \leq e(X) = c_2(X)
\]
for the Chern numbers \(c_2(X), c_2(X)\) of \(X\) in the case of \(\kappa(X) = 2\). If \(\kappa(X) = 1\) and \(f : X \to C\) is an elliptic fibration then by Proposition 16 the base \(C\) is of genus \(g(C) = 0\) or 1 with irregularity \(q(X) = g(C) = 1\) in the case of an elliptic base \(C\). Dürr’s Theorem 2.9 from [10] specifies the fundamental group of \(X\) for \(\kappa(X) = 1\). According to Parker’s [32], the minimal volume of a quotient \(\mathbb{B}^2/\Gamma\) by a torsion-free non-uniform lattice \(\Gamma_o < SU_{2,1}\) with respect to the invariant metric with holomorphic sectional curvature \(-1\) is \(\frac{8\pi^2}{3}\). Subsection 5.2 studies the non-compact torsion-free \(\mathbb{B}^2/\Gamma\) with \(\text{vol}(\mathbb{B}^2/\Gamma_o) = \frac{8\pi^2}{3}\). More precisely, Corollary 18 establishes that the minimal model \(X_o\) of the toroidal compactification \(X'_o = (\mathbb{B}^2/\Gamma'_o)^o\) of such \(\mathbb{B}^2/\Gamma'_o\) is either an abelian surface or an elliptic fibration \(X_o \to C\) of Kodaira dimension \(\kappa(X_o) = 1\) and Euler number \(e(X_o) = 0\), whose base \(C\) is of genus \(g(C_o) = 0\) or 1. By Stover’s [36], the minimal volume of a quotient \(\mathbb{B}^2/\Gamma\) by a non-uniform (eventually torsion) arithmetic lattice \(\Gamma < SU_{2,1}\) is \(\frac{8\pi^2}{3}\) and this lower bound is attained by exactly two conjugacy classes of Picard modular lattices \(\Gamma'_1, \Gamma'_2\) over \(\mathbb{Q}(\sqrt{-3})\). Any arithmetic torsion-free non-uniform \(\Gamma_o < SU_{2,1}\) with minimal volume \(\text{vol}(\mathbb{B}^2/\Gamma_o) = \frac{8\pi^2}{3}\) is shown to be contained in \(\Gamma'_1\) or in \(\Gamma'_2\). The appendix of [36] provides complete list of the arithmetic torsion-free non-uniform lattices \(\Gamma_j\) with \(\text{vol}(\mathbb{B}^2/\Gamma_j) = \frac{8\pi^2}{3}\), which are contained in \(\Gamma'_1 \cap \Gamma'_2\). Namely, there are eight such lattices \(\Gamma_1, \ldots, \Gamma_8\), which are characterized by their generators, abelianizations and the number of the cusps of \(\mathbb{B}^2/\Gamma\). (The cusps of \(\mathbb{B}^2/\Gamma\) are the \(\Gamma\)-orbits of the \(\Gamma\)-rational boundary points of \(\mathbb{B}^2\).) Our Corollary 18 determines the Kodaira-Enriques classification types of the minimal models \(X_j\) of Stover’s examples \(X'_j = (\mathbb{B}^2/\Gamma_j)^o\), \(1 \leq j \leq 8\). On the other hand, these examples justify that any possible Kodaira-Enriques classification type is realized by an arithmetic torsion-free, non-compact ball quotient \(\mathbb{B}^2/\Gamma\) with minimal volume \(\text{vol}(\mathbb{B}^2/\Gamma_o) = \frac{8\pi^2}{3}\).

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2 Notations and terminology

2.1 Refined Langlands decomposition of a parabolic subgroup of $G$

The present subsection recalls some facts for the parabolic subgroups $Q$ of $G$, which can be found in I.1. and III.1 from the book [4] of Borel and Ji.

For an arbitrary parabolic subgroup $Q$ of $G$ there is Langlands decomposition

$$Q = N_Q A_Q M_Q$$

where $N_Q$ is the unipotent radical of $Q$, $R_Q = N_Q \rtimes A_Q$ is the solvable radical of $Q$ and $M_Q$ is a semisimple complement of $R_Q$. The unipotent radical $N_Q$ of $Q$ is a 2-step nilpotent group, i.e. $[[N_Q, N_Q], N_Q] = 0$. The commutant $[N_Q, N_Q]$ of $N_Q$ coincides with the center $U_Q$ of $N_Q$. The abelian group $U_Q$ is isomorphic to its Lie algebra and will be identified with $(\mathbb{R}^m, +)$ for $m = \dim \mathbb{R} U_Q$. The quotient $V_Q = N_Q/U_Q$ is also an abelian group, isomorphic to its Lie algebra. The group $V_Q$ is isomorphic to $(\mathbb{C}^n, +)$ and $N_Q = U_Q \rtimes V_Q$ is a semi-direct product of $U_Q$ and $V_Q$. For a Hermitian symmetric space $D = G/K$ of non-compact type, the semi-simple complement $M_Q$ of the solvable radical $R_Q$ of $Q$ is a product $M_Q = G'_{Q,l} \ltimes G_{Q,h}$ of semisimple groups $G'_{Q,l}$, $G_{Q,h}$ of noncompact type. Altogether, there is a refined Langlands decomposition

$$Q = [(U_Q \rtimes V_Q) \rtimes A_Q] \rtimes (G'_{Q,l} \ltimes G_{Q,h})$$

of an arbitrary parabolic subgroup $Q$ of $G$.

In particular, if $P$ is a minimal parabolic subgroup of $G$ then $M_P = Z(A_P) \cap K$ for the centralizer $Z(A_P)$ of $A_P$ in $G$ and the refined Langlands decomposition of $P$ reads as

$$P = [(U_P \rtimes V_P) \rtimes A_P] \rtimes (Z(A_P) \cap K).$$

The group $A_P \simeq (\mathbb{R}^r_0)^r$ is a maximal flat $\mathbb{R}$-split torus of $G$ and its dimension $r = \dim \mathbb{R} A_P = \text{rk}_\mathbb{R} G$ equals the real rank of $G$. Moreover, $A_P$ is a maximal flat totally geodesic subspace of $D = G/K$.

Any pair of parabolic subgroups $P \subset Q$ of $G$ is associated with relative Langlands decompositions of $P$ and $Q$ (cf. [4], I.1.11). More precisely, there is a parabolic subgroup $P_1$ of $M_Q = P_1(K \cap Q)$, such that

$$N_P = N_Q N_{P_1}, \quad A_P = A_Q A_{P_1}, \quad M_P = M_{P_1}.$$ (3)

As a result, one has

$$P = [(N_Q N_{P_1}) \rtimes (A_Q A_{P_1})] \rtimes M_{P_1} = (N_Q \rtimes A_Q) P_1,$$

$$Q = [(N_Q \rtimes A_Q) \rtimes [(N_{P_1} \rtimes A_{P_1}) \rtimes M_{P_1}]](K \cap Q) = P(K \cap Q).$$ (5)

Bearing in mind that the center $U_Q$ of the unipotent radical $N_Q$ of a parabolic subgroup $Q$ of $G$ coincides with the commutant $[N_Q, N_Q]$ of $N_Q$, one concludes that

$$U_Q \subset U_P$$

for any parabolic subgroups $P \subset Q$ of $G$.
2.2 Refined horospherical decomposition of $D = G/K$, associated with a parabolic subgroup of $G$

For an arbitrary parabolic subgroup $Q$ of $G$ and an arbitrary maximal compact subgroup $K$ of $G$ there holds $G = QK$. Therefore $Q$ acts transitively on $D = G/K$. Moreover, $Q \cap K = M_Q \cap K$ for the semisimple complement $M_Q$ of the solvable radical $R_Q$ of $Q$. As a result, the refined Langlands decomposition (1) of $Q$ induces the refined horospherical decomposition

$$D = U_Q \times V_Q \times A_Q \times D_{Q,l} \times D_{Q,h}$$

of $D$ with $D_{Q,l} = G'_{Q,l}/G'_{Q,l} \cap K$, $D_{Q,h} = G_{Q,h}/G_{Q,h} \cap K$. The above equality is to be interpreted as a real analytic diffeomorphism. The factor $D_{Q,h}$ is a Hermitian symmetric space of noncompact type, while $D_{Q,l}$ is a Riemannian symmetric space of noncompact type. The parabolic subgroups $Q$ of $G$ are in a bijective correspondence with the complex analytic boundary components $D_{Q,h}$ of $D$. Namely,

$$Q = \{ g \in G \mid g(D_{Q,h}) \subseteq D_{Q,h} \}$$

is the normalizer of the analytic boundary component $D_{Q,h} \subset \partial D$ in $G$. We say that $D_{Q,h}$ is $\Gamma$-rational if $Q$ is $\Gamma$-rational. Let us denote by $\partial \Gamma D$ the set of the $\Gamma$-rational analytic boundary components of $D$.

The parabolic group $Q = (N_Q \times A_Q) \rtimes (G'_{Q,l} \times G_{Q,h})$ acts on the Hermitian symmetric space $D = N_Q \times A_Q \times D'_{Q,l} \times D_{Q,h}$ by the rule

$$(n_0, a_0, g_0)(n, a, z') = ((a_0, g_0, g_0)^{-1}n_0(a_0, g_0, g_0)n, \ a_0a, \ g_0z', \ g_0z)$$

for $\forall (n_0, a_0, g_0) \in (N_Q \times A_Q) \rtimes (G'_{Q,l} \times G_{Q,h})$, $\forall (n, a, z') \in N_Q \times A_Q \times D'_{Q,l} \times D_{Q,h}$.

If $P$ is a minimal parabolic subgroup of $G$ then $P \cap K = Z(A_P) \cap K$ and the refined Langlands decomposition (2) implies the refined horospherical decomposition

$$D = U_P \times V_P \times A_P.$$  

In particular, the analytic boundary components $D_{P,h} \subset \partial D$, associated with minimal parabolic subgroups $P$ of $G$ are analytically isolated points, i.e., any holomorphic curve in $\partial D$, starting from the point $D_{P,h}$ is constant. Let us denote by $\partial^\Gamma_P$ the set of the isolated $\Gamma$-rational boundary points and note that $\partial^\Gamma_P D$ is $\Gamma$-invariant.

2.3 Siegel domain presentation of $D = G/K$, associated with a parabolic subgroup of $G$

Pyatetskii-Shapiro’s [33] provides realizations of the Hermitian symmetric spaces $D = G/K$ of noncompact type as Siegel domains of genus 3. These are families of open cones, varying over products of complex Euclidean spaces and Hermitian symmetric spaces of noncompact type. Our Siegel domain realizations are products of cones, real Euclidean spaces and Hermitian symmetric spaces of noncompact type. The term “Siegel domain presentation” is used to emphasize the presence of a cone factor in the considered decomposition.
Let us recall that a cone $C$ in $\mathbb{R}^m$ with vertex at the origin $\hat{0} = (0, \ldots, 0) \in \mathbb{R}^m$ is a subset $C$ of $\mathbb{R}^m$, which together with any point $p \in C$ contains the entire segment from $\hat{0}$ to $p$. The subsets of $\mathbb{R}^m$ of the form

$$\sigma = \langle u_1, \ldots, u_s \rangle = \left\{ \sum_{i=1}^s x_i u_i \mid x_i \in \mathbb{R}^{>0}, \ 1 \leq i \leq s \right\}$$

for some $u_1, \ldots, u_s \in \mathbb{R}^m$ are called open polyhedral cones. It is clear that any polyhedral cone is a cone. A polyhedral cone $\sigma = \langle u_1, \ldots, u_s \rangle$ is strongly convex if it does not contain an entire real line through the origin.

Any parabolic subgroup $Q$ of $G$ is associated with a refine horospherical decomposition (7). The $A_Q$-orbit $C_Q = A_Q D_{Q,l}'$ of the Riemannian symmetric space $D_{Q,l}'$ of non-compact type is an open, strongly convex polyhedral cone in $C_Q$ (cf. [4], Definition III.7.16). Note that the reductive group $G_{Q,l}' = A_Q \times G_{Q,l}'$ acts transitively on the polyhedral cone $C_Q = G_{Q,l}' / G \cap K = G_{Q,l}' / G_{Q,l}' \cap K$. The abelian group $V_Q$, isomorphic to its Lie algebra has even real dimension $\dim \mathbb{R} V_Q = 2n$ and inherits a natural complex structure form the bounded symmetric domain $D = G / K$. From now on, we identify $V_Q$ with $(\mathbb{C}^n, +)$. Combining with (7), one obtains a real analytic diffeomorphism of $D$ onto the product

$$(U_Q + iC_Q) \times V_Q \times D_{Q,h},$$

which will be called a Siegel domain realization of $D$, associated with $Q$.

If $P$ is a minimal parabolic subgroup of $G$ then the Siegel domain realization of $D$, associated with $P$ is of the form

$$D = (U_P + iC_P) \times V_P.$$  

(11)

Bearing in mind that $G_{Q,l}' = Z(A_P) \cap K$ coincides with the intersection of the centralizer $Z(A_P)$ of $A_P$ in $G$ with the maximal compact subgroup $K$ of $G$, one concludes the the open polyhedral cone $C_P = A_P \simeq (\mathbb{R}^{>0})^r$ is of dimension, equal to the real rank $r = \text{rk}_{\mathbb{R}}G$ of $G$.

In the next subsection we introduce $\Gamma$-admissible polyhedral cone decompositions $\Sigma(Q)$ of $U_Q$ for $Q \in \Gamma P(G)$, which are not only $\Gamma = \Gamma \cap G_{Q,l}'$-invariant but consist of cones $\sigma = \langle u_1, \ldots, u_s \rangle$ with vertices $u_i \in \Gamma_{Q,l}' := \Gamma \cap G_{Q,l}'$. Let $\varphi : D = G / K \to D_g = \text{Sp}(g, \mathbb{R}) / U_g$ be a holomorphic horizontal embedding of the bounded symmetric domain $D$ in a period domain $D_g$ of weight 1 Hodge structures with Hodge numbers $h^{1,0} = h^{0,1} = g$. Then any point of $D$ is associated with a polarized Hodge structure $H^1 = H^{1,0} + H^{0,1}$ of weight 1. In the case of the Siegel upper half-space $D = D_g = \text{Sp}(g, \mathbb{R}) / U_g$, Yau and Zhang show in [37] that the analytic boundary components $D_{Q,h}$ of $D = \text{Sp}(g, \mathbb{R}) / U_g$ parameterize mixed Hodge structures of weight 1, whose weight filtrations are induced by elements $u \in \text{Lie}(U_Q)$. More precisely, the Lie algebra $\text{Lie}(N_Q)$ of the unipotent radical $N_Q$ of a parabolic subgroup $Q$ of $\text{Sp}(g, \mathbb{R})$ is two-step nilpotent with center $\text{Lie}(U_Q) = [\text{Lie}(N_Q), \text{Lie}(N_Q)]$. Representing $\text{Lie}(N_Q)$ by strictly upper triangular matrices, one observes that $u^2 = 0$ for all $u \in \text{Lie}(U_Q)$. The underlying real vector spaces $H_{1,h}^1$ of the Hodge structures
$H^1_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = H^{1,0} + H^{0,1}$, classified by $D = Sp(g, \mathbb{R})/U_g$ is acted by $Lie Sp(g, \mathbb{R})$. In particular, the action of $u \in Lie(U_Q)$ on $H^1_{\mathbb{R}}$ defines a weight filtration

$$\text{im}[u : H^1_{\mathbb{R}} \rightarrow H^1_{\mathbb{R}}] \subseteq \ker[u : H^1_{\mathbb{R}} \rightarrow H^1_{\mathbb{R}}] \subset H^1_{\mathbb{R}}$$

due to $u^2 = 0$.

The generalized balls $B(p, q) = SU(p, q)/SU(p) \times SU(q)$ are known to admit a holomorphic horizontal $SU(p, q)$-equivariant embedding in the period domain $D_1 = Sp(p+q, \mathbb{R})/U_{p+q}$ of the weight 1 Hodge structures with Hodge numbers $h^{1,0} = h^{0,1} = p + q$. We expect that some of the analytic boundary components of $B(p, q)$ could be interpreted as classifying spaces of mixed Hodge structures of weight 1 with specific degeneration behavior.

The open quadrics $Quad(p, 2) = SO(p, 2)/SO(p) \times SO(2)$ are period domains of weight 2 Hodge structures $H^2$ with Hodge numbers $h^{2,0} = h^{0,2} = 1$, $h^{1,1} = p$. It is interesting to study the degenerations of $H^2$ into mixed Hodge structures over the analytic boundary components of $Quad(p, 2)$. The corresponding weight filtration, arising from an element $u \in Lie(U_Q)$ with $u^2 = 0$ is again of length 3.

### 2.4 $\Gamma$-admissible family $\Sigma = \{\Sigma(Q)\}_{Q \in \Gamma P(G)}$ of polyhedral cone decompositions $\Sigma(Q)$ of $U_Q$

If $Q$ is a $\Gamma$-rational parabolic subgroup of $G$ and $\Gamma^N_Q := \Gamma \cap N_Q$ is a lattice in $N_Q$ then $\Gamma^U_Q := \Gamma \cap U_Q$ is a lattice of the center $U_Q$ of $N_Q$. A polyhedral cone $\sigma = \langle u_1, \ldots, u_s \rangle \subset U_Q$ is $\Gamma^U_Q$-rational if its vertices $u_1, \ldots, u_s \in \Gamma^U_Q$ belong to $\Gamma^U_Q$.

If $\sigma = \langle u_1, \ldots, u_s \rangle$ is a polyhedral cone with vertices $u_1, \ldots, u_s$ then for any subset $\{u_{j_1}, \ldots, u_{j_t}\} \subset \{u_1, \ldots, u_s\}$, $1 \leq j_1 < \ldots < j_t \leq s$, the polyhedral cone $\langle u_{j_1}, \ldots, u_{j_t} \rangle$ is called a face of $\sigma$. A polyhedral cone decomposition $\Sigma(Q)$ of $U_Q$ or a fan of $U_Q \simeq \mathbb{R}^m$ is a family of strongly convex polyhedral cones $\sigma \subset U_Q$, such that:

(i) every face $\tau$ of a cone $\sigma \in \Sigma(Q)$ is a cone $\tau \in \Sigma(Q)$ and

(ii) for any $\sigma, \tau \in \Sigma(Q)$ the intersection $\sigma \cap \tau$ is a face of $\sigma$ and $\tau$.

A polyhedral cone decomposition $\Sigma(Q)$ of $U_Q$ is $\Gamma^U_Q$-rational if all polyhedral cones $\sigma \in \Sigma(Q)$ are $\Gamma^U_Q$-rational. A $\Gamma^U_Q$-rational polyhedral cone decomposition $\Sigma(Q)$ of $U_Q$ is $\Gamma$-admissible if $\Gamma^U_Q \subset \Gamma \cap G_Q$ acts on $\Sigma(Q)$ with finite number of orbits. In other words, $\gamma \sigma \in \Sigma(Q)$ for $\forall \sigma \in \Sigma(Q)$, $\forall \gamma \in \Gamma^U_Q$ and there exist finitely many cones $\sigma_1, \ldots, \sigma_k \in \Sigma(Q)$ such that $\Sigma(Q) = \cup_{i=1}^k \cup_{\gamma \in \Gamma^U_Q} \gamma \sigma$ is depleted by the union of their $\Gamma^U_Q$-orbits.

Recall that the lattice $\Gamma$ acts on the set $\Gamma P(G)$ of the $\Gamma$-rational parabolic subgroups of $G$ by conjugation. For any $\Gamma$-invariant subset $\mathcal{P} \subseteq \Gamma P(G)$, the family $\Sigma = \{\Sigma(Q)\}_{Q \in \mathcal{P}}$ of the $\Gamma$-admissible cone decompositions $\Sigma(Q)$ of $U_Q$ for $Q \in \mathcal{P}$ is $\Gamma$-admissible if:

(i) $\gamma \Sigma(Q) = \Sigma(\gamma Q \gamma^{-1})$ for $\forall \gamma \in \Gamma$, $\forall Q \in \mathcal{P}$

(ii) $\Sigma(Q) = \Sigma(P) \cap U_Q$ for $\forall P \subset Q$, $Q, P \in \mathcal{P}$.

In order to explain the last condition, recall that $U_Q \simeq (\mathbb{R}^m, +)$ is an $\mathbb{R}$-linear subspace of $U_P \simeq (\mathbb{R}^r, +)$. The intersection $\Sigma(P) \cap U_Q$ is defined as the set of the faces $\sigma \cap U_Q$ of the cones $\sigma \in \Sigma(P)$.
A polyhedral cone decomposition $\Sigma'(Q)$ of $U_Q$ is a refinement of a polyhedral cone decomposition $\Sigma(Q)$ of $U_Q$ is every polyhedral cone $\sigma \in \Sigma(Q)$ belongs to $\Sigma'(Q)$. A $\Gamma_Q^U$-rational polyhedral cone decomposition $\Sigma(Q)$ of $U_Q$ is regular if any cone $\sigma = (u_1, \ldots, u_s) \in \Sigma(Q)$ has vertices $u_i = (u_i^{(1)}, \ldots, u_i^{(m)}) \in \Gamma_Q^U \simeq (\mathbb{Z}^m, +)$ with co-prime coordinates $u_i^{(1)}, \ldots, u_i^{(m)} \in \mathbb{Z}$. A $\Gamma$-admissible family $\Sigma = \{\Sigma(Q)\}_{Q \in \mathcal{P}}$ of polyhedral cone decompositions is regular if consists of regular $\Sigma(Q)$. According to Theorem 6.3 from [5], any $\Gamma_Q^U$-rational polyhedral cone decomposition $\Sigma(Q)$ of $U_Q$ has a regular refinement $\Sigma'(Q)$. One can choose $\Sigma'(Q)$ to be $\Gamma_{Q,T}$-invariant with finitely many $\Gamma_{Q,T}$-orbits. Thus, any $\Gamma$-admissible polyhedral cone decomposition $\Sigma(Q)$ of $U_Q$ has a $\Gamma$-admissible regular refinement $\Sigma'(Q)$ and any $\Gamma$-admissible family $\Sigma = \{\Sigma(Q)\}_{Q \in \mathcal{P}}$ has a $\Gamma$-admissible regular refinement $\Sigma' = \{\Sigma'(Q)\}_{Q \in \mathcal{P}}$.

3 Preliminaries on toroidal compactifications

3.1 The spaces $Z_{\Sigma(Q)} = (D/\Gamma_Q^U)_{\Sigma(Q)} = Y_{\Sigma(Q)} \times V_Q \times D_{Q,h}$

For any $\Gamma$-rational parabolic subgroup $Q$ of $G$ the quotient

$$\Gamma(Q) := (U_Q \otimes_{\mathbb{R}} \mathbb{C})/\Gamma_Q^U \simeq (\mathbb{C}^*)^m$$

of the complexification $\mathbb{C}_Q^U := U_Q \otimes_{\mathbb{R}} \mathbb{C}$ is an algebraic torus over $\mathbb{C}$. If the algebraic variety $X$ contains a torus $(\mathbb{C}^*)^m$ as an open dense subset and the action of $(\mathbb{C}^*)^m$ on itself extends to $X$, we say that $X$ is a toric variety.

Any polyhedral cone decomposition $\Sigma(Q)$ of $U_Q$ is associated to a toric variety $X_{\Sigma(Q)}$, compactifying the torus $\Gamma(Q) \simeq (\mathbb{C}^*)^m$. In [24] Konter constructs $X_{\Sigma(Q)}$ as a union of $\Gamma(Q)$ with the quotients $\Gamma(Q)/\text{Span}_C(\sigma)$ by the complex spans $\text{Span}_C(\sigma)$ of $\sigma \in \Sigma(Q)$. More precisely, the subspace $\text{Span}_C(\sigma)$ of $U_Q^C := U_Q \otimes_{\mathbb{R}} \mathbb{C}$ acts on the universal covering $\widetilde{\Gamma(Q)} = U_Q^C$ of $\Gamma(Q)$ by translations. The quotient $U_Q^C/\text{Span}_C(\sigma)$ is a complex vector space, generated by the elements of $\Gamma_Q^U/\Gamma_Q^U \cap \text{Span}_C(\sigma)$. We define $\Gamma(Q)/\text{Span}_C(\sigma)$ as the quotient of the complex vector space $U_Q^C/\text{Span}_C(\sigma)$ by its uniform lattice $\Gamma_Q^U/\Gamma_Q^U \cap \text{Span}_C(\sigma)$. By their very definition, $\Gamma(Q)/\text{Span}_C(\sigma)$ are complex algebraic tori $(\mathbb{C}^*)^l$ of dimension $l < m$. By Theorem 3.2 (1) from Brasselett's [5] the toric variety $X_{\Sigma(Q)}$, associated with a polyhedral cone decomposition of the entire real vector space $U_Q \simeq \mathbb{R}^m$ is compact. All toric varieties $X_{\Sigma(Q)}$ are rational by Fulton’s book [13].

Note that $U_Q + iC_Q/\Gamma_Q^U$ is an open subset of the algebraic torus $\Gamma(Q) = U_Q^C/\Gamma_Q^U \simeq (\mathbb{C}^*)^m$. We choose $Y_{\Sigma(Q)}$ to be the interior of the closure of $U_Q + iC_Q/\Gamma_Q^U$ in $X_{\Sigma(Q)}$.

Theorem 6.3 (2) from [5] asserts that the toric variety $X_{\Sigma(Q)}$ is smooth exactly when the polyhedral cone decomposition $\Sigma(Q)$ of $U_Q$ is regular. Moreover, by Theorem 6.3 [5], every regular $\Gamma$-admissible refinement $\Sigma_0(Q)$ of $\Sigma(Q)$ induces a resolution $X_{\Sigma_0(Q)} \rightarrow X_{\Sigma(Q)}$ of the singularities of $X_{\Sigma(Q)}$. From now on, we assume that the $\Gamma$-admissible family $\Sigma = \{\Sigma(Q)\}_{Q \in \mathcal{P}}$ of polyhedral cone decompositions $\Sigma(Q)$ of $U_Q$ is regular, so that $X_{\Sigma(Q)}$ and $Y_{\Sigma(Q)}$ are smooth. Yau and Zhang prove in [37] that for a regular $\Gamma$-admissible family $\Sigma = \{\Sigma(Q)\}_{Q \in \mathcal{P}}$, the smooth irreducible components of the complement $Y_{\Sigma(Q)} \setminus (U_Q + iC_Q)/\Gamma_Q^U$ intersect transversely with each other.
In the special case of a $\Gamma$-rational minimal parabolic subgroup $P$ of $G$, the Siegel domain presentation (11) of $D$, associated with $P$ implies that

$$Z_{\Sigma(P)} = Y_{\Sigma(P)} \times V_P$$

with $V_P \simeq (\mathbb{C}^n, +)$ for $n = \frac{1}{2} \dim_{\mathbb{R}} V_P$. According to subsection 2.3, the open polyhedral cone $C_P = A_P \simeq (\mathbb{R}^{>0})^r$ for $r = \mathrm{rk} G = \dim_{\mathbb{R}} U_P$. Let us fix an isomorphism $\Gamma_P^U \simeq (\mathbb{Z}, +)$ and identify $U_P + iC_P = \mathbb{H}^r$ with $r$-fold product of upper half-planes $\mathbb{H} = \{ t \in \mathbb{C} \mid \Im(t) > 0 \}$. Then note that the map

$$\varepsilon_P : U_P^\mathbb{R} := U_P \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}^r \to (\mathbb{C}^*)^r,$$

$$\varepsilon_P(u_1, \ldots, u_r) = (e^{2\pi i u_1}, \ldots, e^{2\pi i u_r})$$

is $\Gamma_P^U \simeq (\mathbb{Z}, +)$-Galois covering, which transforms $U_P + iC_P \simeq \mathbb{H}^r$ into the $r$-fold product

$$(U_P + iC_P)/\Gamma_P^U = \varepsilon_P(U_P + iC_P) = (\Delta^r)^r$$

of punctured discs $\Delta^r = \{ t \in \mathbb{C} \mid 0 < |t| < 1 \}$. The closure of $(U_P + iC_P)/\Gamma_P^U \simeq (\Delta^*)^r$ in the rational, smooth, compact complex toric variety $X_{\Sigma(P)}$ is isomorphic to $\overline{\Delta}$, where $\overline{\Delta} = \{ t \in \mathbb{C} \mid |t| \leq 1 \}$ stands for the closed unit disc. The interior of that closure is $Y_{\Sigma(P)} \simeq \Delta^r$ with $\Delta = \{ t \in \mathbb{C} \mid |t| < 1 \}$. Thus, for any $\Gamma$-rational minimal parabolic subgroup $P$ of $G$ the space

$$Z_{\Sigma(P)} = Y_{\Sigma(P)} \times V_P \simeq \Delta^r \times \mathbb{C}^n.$$

If $\Gamma_P^N := \Gamma \cap N_P$ then the subset $L_P := 0^r \times V_P \subseteq (Y_{\Sigma(P)} \times V_P) \setminus (U_P + iC_P/\Gamma_P^U)$ is acted by the quotient group $\Lambda(P) = \Gamma_P^N/\Gamma_P^U = \Gamma_P^N/\Gamma_P^N \cap U_P \simeq \Gamma_P^N U_P/U_P$, which can be considered as a subgroup of $N_P/U_P = V_P \simeq (\mathbb{C}^n, +)$. For a $\Gamma$-rational minimal parabolic subgroup $P < G$, the discrete subgroup $\Lambda(P) \simeq (\mathbb{Z}^n, +)$ of $(\mathbb{C}^n, +)$ is a uniform lattice and the quotient $T_P = (V_P, +)/\Lambda(P)$ is a compact complex $n$-dimensional torus. From now on, we call $T_P$ boundary compact complex torus, associated with $P \in \Gamma \text{Min}(P, G)$.

In the case of $\text{rk} G = 1$, note that $D = \mathbb{B}^n$ and $Z_{\Sigma(P)} \setminus (\mathbb{B}^n/\Gamma_P^U) = L_P$. Therefore, the toroidal compactification divisor $T := (\mathbb{B}^n/\Gamma)/ \setminus (\mathbb{B}^n/\Gamma)$ consists of discrete quotients of boundary compact complex tori $T_P$ of $\dim_{\mathbb{C}} T_P = n - 1$, associated with $P \in \Gamma \text{Min}(PSU_{n,1}) = \Gamma P(SU_{n,1})$.

### 3.2 The gluing maps $\mu_P^Q$ of the families of partial completions, associated with $\Gamma$-rational parabolic subgroups $Q \supset P$ of $G$

For an arbitrary $\Gamma$-rational parabolic subgroup $Q \subset G$, the quotient $D/\Gamma_Q^U$ is diffeomorphic to the product

$$(U_Q + iC_Q)/\Gamma_Q^U \times V_Q \times D_{Q,h}$$

according to the Siegel domain presentation (10). One extends $D/\Gamma_Q^U$ to

$$Z_{\Sigma(Q)} = (D/\Gamma_Q^U)_{\Sigma(Q)} := Y_{\Sigma(Q)} \times V_Q \times D_{Q,h}.$$
For arbitrary $\Gamma$-rational parabolic subgroups $P \subset Q$ of $G$ we are going to construct a holomorphic map

$$\mu_P^Q : Z_{\Sigma(Q)} \to Z_{\Sigma(P)}.$$  

According to (6), $U_Q$ is an $\mathbb{R}$-linear subspace of $U_P$. Therefore $(\Gamma_Q^U, +) := (\Gamma \cap U_Q, +)$ is a subgroup of $(\Gamma_P^U, +) := (\Gamma \cap U_P, +)$ and the identity inclusion

$$\text{Id}_D : D = (U_Q + iC_Q) \times V_Q \times D_{Q,h} \to D = (U_P + iC_P) \times V_P \times D_{P,h}$$

induces a holomorphic covering

$$\mu_P^Q : D/\Gamma_Q^U = (U_Q + iC_Q)/\Gamma_Q^U \times V_Q \times D_{Q,h} \to D/\Gamma_P^U = (U_P + iC_P)/\Gamma_P^U \times V_P \times D_{P,h},$$

$$\mu_P^Q(\Gamma_Q^U x) = \Gamma_P^U x \quad \forall x \in D,$$

where $\Gamma_Q^U, \Gamma_P^U x$ stand for the corresponding orbits of $x$.

On one hand, the inclusion $U_Q^C \subset U_P^C$ of the corresponding complexifications induces an inclusion $\varepsilon_P^Q : T(Q) = U_Q^C/\Gamma_Q^U \to U_P^C/\Gamma_P^U$ of their $\Gamma_Q^U$-quotients. On the other hand, due to $\Gamma_Q^U < \Gamma_P^U$, there is a covering map $k_P^Q : U_P^C/\Gamma_P^U \to U_Q^C/\Gamma_Q^U = T(P)$. The composition

$$\mu_{P,1}^Q := k_P^Q \circ \varepsilon_P^Q : T(Q) = (\mathbb{C}^*)^m \to T(P) = (\mathbb{C}^*)^n$$

is a homomorphism of algebraic tori, as far as the liftings of $\varepsilon_P^Q$ and $k_P^Q$ to the corresponding universal coverings are homomorphisms of additive groups. In order to extend the holomorphic map $\mu_{P,1}^Q$ to $X_{\Sigma(Q)}$, recall that $\Sigma(Q) = \{ \sigma \cap U_Q \mid \sigma \in \Sigma(P) \}$ by the $\Gamma$-admissibility of $\Sigma = \{ \Sigma(Q) \}_{Q \subset P}$. The composition of the embedding $U_Q^C/\text{Span}_C(\sigma \cap U_Q) \to U_P^C/\text{Span}_C(\sigma \cap U_Q)$ of $\mathbb{C}$-vector spaces with the projection $U_P^C/\text{Span}_C(\sigma \cap U_Q) \to U_P^C/\text{Span}_C(\sigma)$ of $\mathbb{C}$-vector spaces is equivariant under the action of $\Gamma_Q^U/\Gamma_P^U \cap \text{Span}_C(\sigma \cap U_Q)$, respectively, $\Gamma_P^U/\Gamma_P^U \cap \text{Span}_C(\sigma)$. Its quotient map $T(Q)/\text{Span}_C(\sigma \cap U_Q) \to T(P)/\text{Span}_C(\sigma)$ is a holomorphic homomorphism of algebraic tori. Altogether, one obtains a holomorphic map

$$\mu_{P,1}^Q : X_{\Sigma(Q)} \to X_{\Sigma(P)}$$

of the smooth compact toric varieties, associated with the $\Gamma$-admissible polyhedral cone decompositions $\Sigma(Q), \Sigma(P)$. The closure $(U_Q + iC_Q)/\Gamma_Q^U$ of $(U_Q + iC_Q)/\Gamma_Q^U$ in $X_{\Sigma(Q)}$ is mapped into the closure $(U_P + iC_P)/\Gamma_P^U$ of $(U_P + iC_P)/\Gamma_P^U$, as far as $\mu_{P,1}^Q((U_Q + iC_Q)/\Gamma_Q^U) = (U_P + iC_P)/\Gamma_P^U$ and the extension of the map $\mu_{P,1}^Q$ over the closure $(U_Q + iC_Q)/\Gamma_Q^U$ is continuous. The interior $Y_{\Sigma(Q)}$ of $(U_Q + iC_Q)/\Gamma_Q^U$ is mapped into the interior $Y_{\Sigma(P)}$ of $(U_P + iC_P)/\Gamma_P^U$. More precisely, $\mu_{P,1}^Q$ is a composition of an embedding and a covering map. The embedding transforms an open subset in a subset of an open subset and the covering is an open map. Thus, for any open subset $W_Q \subset (U_Q + iC_Q)/\Gamma_Q^U$ there is an open subset $W_P \subset (U_P + iC_P)/\Gamma_P^U$, containing $\pi_{P,1}^Q(W_Q)$ and $\mu_{P,1}^Q(\Sigma(Q)) \subseteq Y_{\Sigma(P)}$. In such a way, one obtains a holomorphic gluing map

$$\mu_P^Q : Z_{\Sigma(Q)} \times V_Q \times D_{Q,h} \to Y_{\Sigma(P)} \times V_P \times D_{P,h} = Z_{\Sigma(P)},$$
\[ \mu^Q_{\Gamma} \lim_{n \to \infty} (t_n, v, z) = \lim_{n \to \infty} \mu^Q_{\Gamma}(t_n, v, z) = \lim_{n \to \infty} (t_n + \Gamma^U_P/\Gamma^U_Q, v, z) \]

for any sequence of points \( \{t_n\}_{n=1}^{\infty} \subseteq (U_Q + iC_Q)/\Gamma^U_Q \), converging to some point \( \lim_{n \to \infty} t_n \in Y_{\Sigma(Q)} \cap (U_Q + iC_Q)/\Gamma^U_Q \). By the very definition, \( \mu^Q_{\Gamma} \) coincides with the identity of \( Z_{\Sigma(Q)} = (D/\Gamma^U_Q)_{\Sigma(Q)} \).

### 3.3 The toroidal compactifications \((D/\Gamma)'_\Sigma\) and their coverings

The present subsection recalls the construction of a toroidal compactification \((D/\Gamma)'_\Sigma\) of a local Hermitian symmetric space \(D/\Gamma\), associated with a \(\Gamma\)-admissible family \(\Sigma = \{\Sigma(Q)\}_{Q \in \Gamma P(G)}\) of polyhedral cone decompositions \(\Sigma(Q)\) of \(U_Q\). In the notations from subsection 3.1, let us consider the disjoint union \(\coprod_{Q \in \Gamma P(G)} Z_{\Sigma(Q)}\). Denote by \(\Gamma^U\) the subgroup of \(\Gamma\), generated by \(\Gamma^U_Q\) for all \(Q \in \Gamma P(G)\). For an arbitrary normal subgroup \(\Gamma_o\) of \(\Gamma\), containing \(\Gamma^U\), the \(\Gamma_o\)-action on \(D\) induces an equivalence relation \(\sim_{\Gamma_o}\) on \(\coprod_{Q \in \Gamma P(G)} Z_{\Sigma(Q)}\). According to [2], p. 255 or [35]. Proof of Theorem 2.1, \(z_1 \sim_{\Gamma_o} z_2\) for \(z_j \in Z_{\Sigma(Q)_j}\), \(1 \leq j \leq 2\) if there exist \(\gamma \in \Gamma_o, Q \in \Gamma P(G)\) and \(z \in Z_{\Sigma(Q)}\), such that \(Q_1 \subseteq Q, \gamma Q_2 \gamma^{-1} \subseteq Q\), \(\mu^Q_{\Gamma}\gamma_1(\gamma) = z_1\) and \(\mu^Q_{\Gamma}\gamma_2(\gamma^{-1}) = z_2\). Let us put

\[ (D/\Gamma)'_\Sigma := \left( \coprod_{Q \in \Gamma P(G)} Z_{\Sigma(Q)} \right) / \sim_{\Gamma_o}. \]

For \(\Gamma_o = \Gamma\) one obtains the toroidal compactification \((D/\Gamma)'_\Sigma\) of \(D/\Gamma\), associated with \(\Sigma = \{\Sigma(Q)\}_{Q \in \Gamma P(G)}\). In [35] Sankaran constructs the \((\Gamma/\Gamma^U')\)-Galois covering \((D/\Gamma)'_\Sigma\) of \((D/\Gamma)'_\Sigma\) and shows that \((D/\Gamma)'_\Sigma\) is a simply connected complex analytic space.

The next proposition establishes that the spaces \(Z_{\Sigma(P)}\), associated with the \(\Gamma\)-rational minimal parabolic subgroups \(P\) generate \((D/\Gamma)'_\Sigma\) for any \(\Gamma^U \subseteq \Gamma_o \subseteq \Gamma\). This result is analogous to Sankaran’s observation that \(\Gamma^U\) is generated by \(\Gamma^U_P\) for all \(\Gamma\)-rational minimal parabolic subgroups \(P\) of \(G\). In order to mention a result of Yau and Zhang of the same kind, let us note that the Siegel upper half-space \(D = H_g = Sp(2g, \mathbb{R})/U_{g} \) is an open strongly convex polyhedral cone. Yau and Zhang prove in [37] that any \(\Gamma\)-admissible polyhedral cone decomposition of \(H_g\) induces a \(\Gamma\)-admissible family \(\Sigma = \{\Sigma(Q)\}_{Q \in \Gamma P(G)}\) of polyhedral cone decompositions of \(U_Q\).

**Proposition 1.** For an arbitrary normal subgroup \(\Gamma_o\) of \(\Gamma\), containing \(\Gamma^U\) the complex analytic space

\[ (D/\Gamma)'_\Sigma := \coprod_{Q \in \Gamma P(G)} Z_{\Sigma(Q)} / \sim_{\Gamma_o} \]

coincides with

\[ (D/\Gamma)'_{\Sigma}^{\text{Min}} := \coprod_{P \in \Gamma \text{Min} P(G)} Z_{\Sigma(P)} / \sim_{\Gamma_o} \]

where \(\Gamma \text{Min} P(G)\) is the set of the \(\Gamma\)-rational minimal parabolic subgroups of \(G\).
Proof. The identical inclusion
\[
\prod_{P \in \Gamma \text{Min} P(G)} Z_{\Sigma(P)} \hookrightarrow \prod_{Q \in \Gamma P(G)} Z_{\Sigma(Q)}
\]
induces a holomorphic embedding
\[
\Phi_0 : \left(\frac{D}{\Gamma_0}\right)_\Sigma^{\text{Min}} = \prod_{P \in \Gamma \text{Min} P(G)} Z_{\Sigma(P)}/\sim_{\Gamma_0} \hookrightarrow \prod_{Q \in \Gamma P(G)} Z_{\Sigma(Q)}/\sim_{\Gamma_0} = \left(\frac{D}{\Gamma_0}\right)_\Sigma^1,
\]
where \(z_1 \sim_{\Gamma_0} z_2\) for \(z_j \in Z_{\Sigma(P)} = Y_{\Sigma(P)} \times V_P \times D_{Q,h}\) if there exist \(\gamma \in \Gamma_0, Q \in \Gamma P(G)\) and \(z \in Z_{\Sigma(Q)}\) with \(P_1 \subset Q, \gamma P_2 \gamma^{-1} \subset Q\), \(\mu^Q_{P_1}(z) = z_1, \mu^Q_{\gamma P_2 \gamma^{-1}}(z) = z_2\). It suffices to show that any \(z_Q \in Z_{\Sigma(Q)}\) with \(Q \in \Gamma P(G)\) is \(\sim_{\Gamma_0}\)-equivalent to some \(z_P \in Z_{\Sigma(P)}\) with \(P \in \Gamma \text{Min} P(G)\), in order to conclude that \(\Phi_0\) is surjective, whereas a biholomorphism of \(\left(\frac{D}{\Gamma_0}\right)_\Sigma^{\text{Min}}\) with \(\left(\frac{D}{\Gamma_0}\right)_\Sigma^1\). Indeed, for any \(Q \in \Gamma P(G)\) the \(\Gamma\)-rational analytic boundary component \(D_{Q,h} \subset \partial \Delta\) has an isolated \(\Gamma\)-rational point. The stabilizer of that point if a \(\Gamma\)-rational minimal parabolic subgroup \(P\) of \(G\), contained in \(Q\). Now \(\mu^Q_{\gamma}(z_Q) = z_P \in Z_{\Sigma(P)}\) is \(\sim_{\Gamma_0}\)-equivalent to \(z_Q\).

\bigskip

In the proof of Theorem 1.5 from [35] Sankaran shows that the complex analytic spaces \(Z_{\Sigma(Q)} := Y_{\Sigma(Q)} \times V_Q \times D_{Q,h}\) are simply connected for \(\forall Q \in \Gamma P(G)\). Further, the proof of Theorem 2.1 from [35] establishes that the natural coverings \(D/\Gamma_Q^U \to D/\Gamma^U\) extend to holomorphic maps \(\pi^U_{\Sigma(Q)} : Z_{\Sigma(Q)} \to (D/\Gamma^U)_\Sigma\), which are biholomorphic onto their images. Therefore \((D/\Gamma^U)_\Sigma = \bigcup_{Q \in \Gamma P(G)} \pi^U_{\Sigma(Q)}(Z_{\Sigma(Q)})\) is a simply connected complex analytic space, as a union of simply connected complex analytic subspaces \(\pi^U_{\Sigma(Q)}(Z_{\Sigma(Q)})\), regardless of the intersections \(\pi^U_{\Sigma(Q)}(Z_{\Sigma(Q)}) \cap \pi^U_{\Sigma(P)}(Z_{\Sigma(P)})\) for \(Q, P \in \Gamma P(G)\). By Proposition 1, one represents
\[
(D/\Gamma^U)_\Sigma = \bigcup_{P \in \Gamma \text{Min} P(G)} \pi^U_{\Sigma(P)}(Z_{\Sigma(P)})
\]
as a union of \(\pi^U_{\Sigma(P)}(Z_{\Sigma(P)}) \simeq Z_{\Sigma(P)} = Y_{\Sigma(P)} \times V_P \simeq \Delta^r \times \mathbb{C}^n\).

We claim that for a normal subgroup \(\Upsilon \trianglelefteq \Gamma\) of finite index \([\Gamma : \Upsilon]\) \(\ll \infty\) the simply connected complex analytic cover \((D/\Upsilon^U)_\Sigma\) of \((D/\Gamma)_\Sigma\) is a ramified cover of \((D/\Gamma)^U_\Sigma\) even if \(\Gamma < G\) is a torsion free lattice. More precisely, for a fixed \(\Gamma\)-rational minimal parabolic subgroup \(P\) of \(G\) one can view \(\Upsilon^U_P := \Upsilon \cap U_P \simeq (\mathbb{Z}^r, +)\) for \(r = \text{rk}_G\). The intersection \(\Upsilon^U_P \subset \Upsilon \cap U_P\) is of finite index in \(\Gamma^U_P\), as far as
\[
\Gamma^U_P/\Upsilon^U_P = \Gamma^U_P/\Gamma^U \cap \Upsilon \simeq \Gamma^U_P/\Upsilon \leq \Gamma/\Upsilon.
\]
Let \(\Upsilon^U_P \simeq (m_1 \mathbb{Z} \times \ldots \times m_r \mathbb{Z}, +)\) for some \(m_i \in \mathbb{N}\). Any \(\Upsilon\)-admissible family \(\Sigma'' = \{\Sigma''(Q)\}_{Q \in \Gamma P(G)}\) of polyhedral cone decompositions \(\Sigma''(Q)\) of \(U_Q\) has a \(\Gamma\)-admissible refinement \(\Sigma = \{\Sigma(Q)\}_{Q \in \Gamma P(G)}\), which is obtained from \(\Sigma''\) by adjoining the \(\Gamma\)-images of \(\sigma \in \Sigma''\). Thus, the \(\Gamma^U_P/\Upsilon^U_P\)-Galois coverings
\[
\Upsilon^U_P(P) = U^U_P/\Upsilon^U_P \longrightarrow U^U_P/\Gamma^U_P = \Upsilon(P)
\]
of $r$-dimensional algebraic tori extend to $\Gamma_P^U/Y_P^U$-Galois coverings $X^\Sigma(P) \to X_\Sigma(P)$ of the corresponding smooth compact complex toric varieties. These coverings induce $\Gamma_P^U/Y_P^U$-Galois coverings $Y^\Sigma(P) \to Y_\Sigma(P)$ and

$$Z^\Sigma(P) = Y^\Sigma(P) \times V_P \longrightarrow Z_\Sigma(P) = Y_\Sigma(P) \times V_P.$$  

In such a way, one obtains a $\Gamma_P^U/Y_P^U$-Galois covering

$$\varphi^U_\Sigma : \left( D/\Gamma_P^U \right)_\Sigma \longrightarrow \left( D/\Gamma_P^U \right)_\Sigma,$$

of simply connected compact complex analytic varieties. Note that $\varphi^U_\Sigma$ is ramified over $(D/\Gamma_P^U)_\Sigma \setminus (D/\Gamma_P^U)_\Sigma$. For instance, $\varphi^U_\Sigma$ is totally ramified over $0^r \times V_P \subset Z_\Sigma(P)$ for any $P \in \Gamma \text{Min}(G)$. More precisely,

$$\zeta^U_P : U^\Sigma_P = U_P \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow (\mathbb{C}^*)^r,$$

$$\zeta^U_P(t_1, \ldots, t_r) = \left( e^{2\pi i \frac{m_1 t_1}{m_1}}, \ldots, e^{2\pi i \frac{m_r t_r}{m_r}} \right)$$

is $Y_P^U \simeq (m_1 \mathbb{Z} \times \ldots \times m_r \mathbb{Z})$-Galois covering and

$$\zeta^\Gamma_P : U^\Sigma_P \longrightarrow (\mathbb{C}^*)^r,$$

$$\zeta^\Gamma_P(t_1, \ldots, t_r) = \left( e^{2\pi i t_1}, \ldots, e^{2\pi i t_r} \right)$$

is $\Gamma_P^U \simeq \mathbb{Z}^r$-Galois covering, closing the commutative diagram

$$\begin{array}{ccc}
U^\Sigma_P & \xrightarrow{\zeta^\Gamma} & \Sigma^\Gamma(U^\Sigma_P) \\
\zeta^\Delta & \downarrow & \downarrow \zeta^\Sigma \\
\zeta^U_P(U^\Sigma_P) & \xrightarrow{\psi^U_\Sigma} & \zeta^\Gamma(U^\Sigma_P)
\end{array}$$

with $\psi^U_\Sigma(s_1, \ldots, s_r) = (s_1^{m_1}, \ldots, s_r^{m_r})$. It is clear that $\psi^U_\Sigma(0^r) = 0^r$ and

$$\varphi^U_\Sigma = \psi^U_\Sigma \times \text{Id}_{V_P} : Z^\Sigma(P) \longrightarrow Z_\Sigma(P)$$

is totally ramified over $0^r \times V_P$.

In the special case of a complex ball $D = \mathbb{B}^n = SU_{n,1}/SU_n \times U_1$ (i.e., for $\text{rk}_G = 1$), note that $\Gamma P(SU_{n,1}) = \Gamma \text{Min}(P(SU_{n,1})$ and the toroidal compactifications of $\mathbb{B}^n/\Gamma$ are independent of $\Sigma = \{ \Sigma(P) \}_{P \in \Gamma P(SU_{n,1})}$ of polyhedral cone decompositions $\Sigma(P)$ of $U_P \simeq \mathbb{R}$. The reason for this is that the $\sim_\Gamma$-equivalence classes are exactly the $\Gamma$-orbits on $\coprod_{P \in \Gamma \text{P}(G)} Z_\Sigma(P)$. 

14
3.4 The $\Gamma/\Gamma^U$-action on $(D/\Gamma^U)^\prime_\Sigma$

Let us denote by $\xi^U_\Sigma: D \to D/\Gamma^U$, respectively, $\zeta_\Gamma: D \to D/\Gamma$ the Galois coverings, associated with the biholomorphism groups $\Gamma^U$, respectively, $\Gamma$ of $D$. In order to describe the action of $\Gamma/\Gamma^U$ on $(D/\Gamma^U)^\prime_\Sigma$, note that the $\Gamma$-action on $\Gamma P(G)$ by conjugation extends to holomorphic maps

$$
\gamma: Y_{\Sigma(Q)} \times V_Q \times D_{Q,h} \to Y_{\Sigma(\gamma Q^\gamma^{-1})} \times V_{\gamma Q^\gamma} \times D_{\gamma Q^\gamma^{-1}}
$$

for $\forall \gamma \in \Gamma$ and $\forall Q \in \Gamma P(G)$. Any $\gamma \in \Gamma$ transforms the $\sim_{\Gamma^U}$-equivalence class of $x \in Y_{\Sigma(Q)} \times V_Q \times D_{Q,h}$ into the $\sim_{\Gamma^U}$-equivalence class of $\gamma x$ and reduces to a biholomorphic map

$$
\gamma: (D/\Gamma^U)^\prime_\Sigma \to (D/\Gamma^U)^\prime_\Sigma.
$$

By the very definition of $\sim_{\Gamma^U}$, all $\gamma \in \Gamma^U$ act identically on $(D/\Gamma^U)^\prime_\Sigma$ and the $\Gamma$-action on $(D/\Gamma^U)^\prime_\Sigma$ reduces to a $(\Gamma/\Gamma^U)$-action

$$(\Gamma/\Gamma^U) \times (D/\Gamma^U)^\prime_\Sigma \to (D/\Gamma^U)^\prime_\Sigma,$$

$$(\gamma \Gamma^U) \pi^{U}_{\Sigma(Q)}(x) = \pi^{U}_{\Sigma(\gamma Q^\gamma^{-1})}(\gamma x) \quad \text{for} \quad \forall \gamma \Gamma^U \in \Gamma/\Gamma^U, \ \forall z \in Z_{\Sigma(Q)}.
$$

The next lemma establishes that the $\Gamma/\Gamma^U$-fixed points on $(D/\Gamma^U)^\prime_\Sigma$ are associated with $\Gamma$-fixed points on $D$.

**Lemma 2.** If $\gamma_0 \Gamma^U \in \Gamma/\Gamma^U \setminus \{\Gamma^U\}$ has a fixed point on $(D/\Gamma^U)^\prime_\Sigma$ then there exists a representative $\gamma \in \Gamma$ of $\gamma \Gamma^U = \gamma_0 \Gamma^U$, which has a fixed point $p \in Fix_D(\gamma)$ on $D$.

**Proof.** Let us suppose that $\pi^{U}_{\Sigma(P)}(x)$ with $x \in Z_{\Sigma(P)} = Y_{\Sigma(P)} \times V_P$ is a fixed point of $\gamma_0 \Gamma^U \in \Gamma/\Gamma^U$. Then $x \sim_{\Gamma^U} \gamma_0 x$, i.e., there exist $\gamma_1 \in \Gamma^U$, $Q \in \Gamma P(G)$ and $y \in Z_{\Sigma(Q)}$, such that $P \subseteq Q$, $(\gamma_1 \gamma_0) P(\gamma_1 \gamma_0)^{-1} \subseteq Q$, $\mu^Q_P(y) = x$, $\mu^Q_{(\gamma_1 \gamma_0) P(\gamma_1 \gamma_0)^{-1}}(y) = \gamma_1 \gamma_0 x$. If $y \in D/\Gamma^U$ then $\mu^Q_P(y) = x \in D/\Gamma^U$ and the $\Gamma^U$-equivalence of $x$ and $\gamma_0 x$ amounts to the existence of $\gamma_2 \in \Gamma^U$ with $\gamma_0 q = \gamma_2 q$ for some preimage $q \in D$ of $x$ under the $\Gamma^U_P$-Galois covering $D \to D/\Gamma^U_P$. Now, $\gamma := \gamma_2^{-1} \gamma_0 \in Stab_p(q)$ and $\gamma = \gamma_2^{-2} \gamma_0 \in \Gamma^U \gamma_0 = \gamma_0 \Gamma^U$, because $\Gamma^U$ is a normal subgroup of $\Gamma$. As a result, $\gamma \Gamma^U = \gamma_0 \Gamma^U$ and $Fix_D(\gamma) \ni q$ is non-empty.

In the case of $y \in Z_{\Sigma(Q)} \setminus (D/\Gamma^U_Q)$, it suffices to produce $y' \in D/\Gamma^U_Q$ with $\mu^Q_\gamma(y') \sim_{\Gamma^U} \gamma_0 \mu^Q_\gamma(y')$ and to apply the previous considerations. To this end, note that the commutative diagram

$$
\begin{array}{ccc}
D/\Gamma^U_Q & \overset{\mu^Q_{(\gamma_1 \gamma_0) P(\gamma_1 \gamma_0)^{-1}}}{\longrightarrow} & D/\Gamma^U_P \\
\mu^Q_{\gamma_0} \downarrow & & \downarrow \gamma_1 \gamma_0
\end{array}
$$
extends to a commutative diagram

\[
\begin{array}{ccc}
Z_{\Sigma(Q)} & \xrightarrow{\mu^Q} & Z_{\Sigma(P)} \\
\downarrow{\mu^Q_{(\gamma_1\gamma_0)P(\gamma_1\gamma_0)^{-1}}} & & \downarrow{\gamma_1\gamma_0} \\
Z_{\Sigma((\gamma_1\gamma_0)P(\gamma_1\gamma_0)^{-1})} & & 
\end{array}
\]

Then for an arbitrary \( y' \in D/\Gamma_Q \) one has \( \mu^Q_{(\gamma_1\gamma_0)P(\gamma_1\gamma_0)^{-1}}(y') = \gamma_1\gamma_0\mu^Q_P(y') \), which amounts to \( \mu^Q_P(y') \sim_{\Gamma^U} \gamma_0\mu^Q_P(y') \).

4 The fundamental group and the first homology group of a toroidal compactification

4.1 The fundamental group of \((D/\Gamma)'_{\Sigma}\)

In order to formulate Sankaran’s results on \( \pi_1(D/\Gamma)'_{\Sigma} \), let us recall that a lattice \( \Gamma < G \) is neat if for any \( \gamma \in \Gamma \) the eigenvalues of \( \gamma \) generate a torsion free subgroup of \( \mathbb{C}^* \). Any neat lattice \( \Gamma < G \) is torsion free.

**Theorem 3.** (Sankaran [35] Corollary 1.6, Theorem 2.1) Let \( \Gamma \) be a non-uniform arithmetic lattice in \( G \) and \( \Gamma^U \) be the subgroup of \( \Gamma \), generated by \( \Gamma^U_Q = \Gamma \cap U_Q \) for \( \forall Q \in \Gamma P(G) \). Then the fundamental group \( \pi_1(D/\Gamma)'_{\Sigma} \) of a toroidal compactification \((D/\Gamma)'_{\Sigma}\) of \( D/\Gamma \) is a quotient group of \( \Gamma/\Gamma^U \).

In particular, if \( \Gamma \) is a neat arithmetic non-uniform lattice then \( \pi_1(D/\Gamma)'_{\Sigma} = \Gamma/\Gamma^U \).

The present subsection generalizes Sankaran’s result to arbitrary (not necessarily arithmetic, not necessarily neat) lattices \( \Gamma < G \).

**Theorem 4.** For an arbitrary non-uniform lattice \( \Gamma < G \) and an arbitrary \( \Gamma \)-admissible family \( \Sigma = \{\Sigma(Q)\}_{Q \in \Gamma P(G)} \) of polyhedral cone decompositions of \( U_Q \), the fundamental group

\[
\pi_1(D/\Gamma)'_{\Sigma} = \Gamma/\Gamma^{\text{Fix}}\Gamma^U,
\]

where \( \Gamma^U \) is the subgroup of \( \Gamma \), generated by the intersections \( \Gamma^U_Q := \Gamma \cap U_Q \) of \( \Gamma \) with the centers \( U_Q \) of the unipotent radicals \( N_Q \) of the \( \Gamma \)-rational parabolic subgroups \( Q < G \) and \( \Gamma^{\text{Fix}} \) is the subgroup of \( \Gamma \), generated by \( \gamma \in \Gamma \) with a fixed point on \( D \).

**Proof.** Recall that \( (D/\Gamma^U)'_{\Sigma} \) is a path connected simply connected locally compact topological space and the quotient group \( \Gamma/\Gamma^U \) acts properly discontinuously on \( D/\Gamma^U)'_{\Sigma} \) by homeomorphisms. The quotient \( (D/\Gamma^U)'_{\Sigma}/(\Gamma/\Gamma^U) = (D/\Gamma)'_{\Sigma} \) is the toroidal compactification of \( D/\Gamma \), associated with \( \Sigma \). According to a theorem of Armstrong from [1], the fundamental group

\[
\pi_1(D/\Gamma)'_{\Sigma} = (\Gamma/\Gamma^U)/(\Gamma/\Gamma^U)^{\text{Fix}},
\]
where \((\Gamma/\Gamma^U)^{\text{Fix}}\) is the subgroup of \(\Gamma/\Gamma^U\), generated by \(\gamma\Gamma^U \in \Gamma/\Gamma^U\) with a fixed point on \((D/\Gamma^U)_\Sigma\). There remains to be shown that

\[
(\Gamma/\Gamma^U)^{\text{Fix}} = \Gamma^{\text{Fix}}\Gamma^U/\Gamma^U.
\]

Towards the inclusion \(\Gamma^{\text{Fix}}\Gamma^U/\Gamma^U \subseteq (\Gamma/\Gamma^U)^{\text{Fix}}\), let us note that \(\gamma p = p\) for some \(p \in D\) implies that \((\gamma\Gamma^U)(\zeta^U(p)) = \zeta^U(\gamma p) = \zeta^U(p)\). Thus, for any generator \(\gamma\) of \(\Gamma^{\text{Fix}}\) the coset \(\gamma\Gamma^U \in \Gamma/\Gamma^U\) is a generator of \((\Gamma/\Gamma^U)^{\text{Fix}}\). For the opposite inclusion \((\Gamma/\Gamma^U)^{\text{Fix}} \subseteq \Gamma^{\text{Fix}}\Gamma^U\), let us assume that \(\gamma\Gamma^U \in \Gamma/\Gamma^U\) has a fixed point on \((D/\Gamma^U)_\Sigma\).

By Lemma 2, one can assume that \(\gamma\) has a fixed point on \(D\). Therefore any generator of \((\Gamma/\Gamma^U)^{\text{Fix}}\) has a representative, which generates \(\Gamma^{\text{Fix}}\) and \((\Gamma/\Gamma^U)^{\text{Fix}} = \Gamma^{\text{Fix}}\Gamma^U/\Gamma^U\).

\[
\square
\]

4.2 The first homology group of \((D/\Gamma)'_\Sigma\)

Corollary 5. For an arbitrary non-uniform lattice \(\Gamma\) of the isometry group \(G\) of a Hermitian symmetric space \(D = G/K\) of non-compact type the first homology group of a toroidal compactification \((D/\Gamma)'_\Sigma\) is isomorphic to

\[
H_1((D/\Gamma)'_\Sigma, \mathbb{Z}) = H_1(D/\Gamma, \mathbb{Z})/F
\]

for a finite abelian group \(F\).

Proof. The abelianization of a group \(\mathfrak{G}\) is its quotient \(ab(\mathfrak{G}) = \mathfrak{G}/[\mathfrak{G}, \mathfrak{G}]\) by the commutant \([\mathfrak{G}, \mathfrak{G}]\). For an arbitrary complex analytic space \(S\) the first homology group with integral coefficients

\[
H_1(S, \mathbb{Z}) = ab\pi_1(S) = \pi_1(S)/[\pi_1(S), \pi_1(S)]
\]

is isomorphic to the abelianization of the fundamental group \(\pi_1(S)\) of \(S\). Note that the commutant of a quotient \(\mathfrak{G}/\mathfrak{H}\) of a group \(\mathfrak{G}\) by its normal subgroup \(\mathfrak{H}\) is generated by \([g_1, g_2]\mathfrak{H} = (g_1g_2g_1^{-1}g_2^{-1})\mathfrak{H}\) for \(\forall g_1, g_2 \in \mathfrak{G}\). Therefore \([\mathfrak{G}/\mathfrak{H}, \mathfrak{G}/\mathfrak{H}] = [\mathfrak{G}, \mathfrak{G}]\mathfrak{H}/\mathfrak{H}\) and the abelianization of \(\mathfrak{G}/\mathfrak{H}\) is

\[
ab(\mathfrak{G}/\mathfrak{H}) = (\mathfrak{G}/\mathfrak{H})/[\mathfrak{G}/\mathfrak{H}, \mathfrak{G}/\mathfrak{H}] = (\mathfrak{G}/\mathfrak{H})/(\mathfrak{G}/\mathfrak{H}\mathfrak{H}/\mathfrak{H}) \simeq \mathfrak{G}/[\mathfrak{G}, \mathfrak{G}].
\]

Making use of Theorem 4, one concludes that

\[
H_1((D/\Gamma)'_\Sigma, \mathbb{Z}) \simeq ab(\Gamma/\Gamma^{\text{Fix}}\Gamma^U) \simeq \Gamma/[[\Gamma, \Gamma]]^{\text{Fix}}\Gamma^U.
\]

On the other hand, the Hermitian symmetric space \(D\) is path connected, simply connected locally compact space with a properly discontinuous action of \(\Gamma\) by homeomorphisms. By Armstrong’s theorem from [1], the fundamental group of \(D/\Gamma\) is \(\pi_1(D/\Gamma) = \Gamma/\Gamma^{\text{Fix}}\). Therefore \(H_1(D/\Gamma, \mathbb{Z}) \simeq ab(\Gamma/\Gamma^{\text{Fix}}) \simeq \Gamma/[[\Gamma, \Gamma]]^{\text{Fix}}\) and

\[
H_1((D/\Gamma)'_\Sigma, \mathbb{Z}) \simeq \Gamma/[[\Gamma, \Gamma]]^{\text{Fix}}/[[\Gamma, \Gamma]]^{\text{Fix}}\Gamma^U/[[\Gamma, \Gamma]]^{\text{Fix}} \simeq H_1(D/\Gamma, \mathbb{Z})/F
\]

for the group

\[
F := \Gamma^U[[\Gamma, \Gamma]]^{\text{Fix}}/[[\Gamma, \Gamma]]^{\text{Fix}}.
\]
Note that $F$ is a subgroup of $\Gamma/[\Gamma, \Gamma]^F_{\text{Fix}}$, so that the commutant
\[
[F, F] \leq [\Gamma/[\Gamma, \Gamma]^F_{\text{Fix}}, \Gamma/[\Gamma, \Gamma]^F_{\text{Fix}}] = [\Gamma, \Gamma]^F_{\text{Fix}}/[\Gamma, \Gamma]^F_{\text{Fix}} = \{1\}
\]
is trivial and the group $F$ is abelian. In [35] Sankaran observes that $\Gamma^U_P$ is generated by $\Gamma^U_P = \Gamma \cap U_P$ for all $P \in \Gamma \text{Min} P(G)$. This is due to the inclusions $U_Q \subset U_P$ for all $Q < \Gamma P(G)$, containing $P$. If $P_2 = \gamma P_1 \gamma^{-1}$ for some $P_1, P_2 \in \Gamma \text{Min} P(G)$, then $\gamma \in \Gamma$, the unipotent radicals $N_{P_2} = \gamma N_{P_1} \gamma^{-1}$ are $\Gamma$-conjugate, as well as the commutants $[N_{P_1}, N_{P_2}] = U_{P_1}$, i.e., $U_{P_2} = \gamma U_{P_1} \gamma^{-1}$. Therefore $\Gamma^U_{\gamma P_1 \gamma^{-1}} = \gamma \Gamma^U_{P_1} \gamma^{-1}$ and
\[
\Gamma^U_{\gamma P_1 \gamma^{-1}}[\Gamma, \Gamma]^F_{\text{Fix}}/[\Gamma, \Gamma]^F_{\text{Fix}} = \Gamma_{P_1}[\Gamma, \Gamma]^F_{\text{Fix}}/[\Gamma, \Gamma]^F_{\text{Fix}}
\]
coincide as subgroups of the abelian group $F$. There is a finite number of $\Gamma$-conjugacy classes of $\Gamma$-rational minimal parabolic subgroups of $G$. For any $\Gamma$-orbit representative set $\{P_1, \ldots, P_m\}$ of $\Gamma \text{Min} P(G)$, the group $F$ is generated by $\Gamma^U_{P_j}$, $\Gamma^U_{\Gamma \cap U_P}$ with $1 \leq j \leq m$. The lattices $\Gamma^U_{P_j} \simeq (\mathbb{Z}^r, +)$, $r = \text{rk}_R G$ are finitely generated, so that $F$ is a finitely generated abelian group. It suffices to show that any element of $F$ is of finite order, in order to conclude that $F$ is a finite abelian group. To this end, let us consider $\Gamma^N_P := \Gamma \cap N_P$ for an arbitrary $P \in \Gamma \text{Min} P(G)$ and observe that the commutant
\[
[\Gamma^N_P, \Gamma^N_P] \leq \Gamma \cap [N_P, N_P] = \Gamma \cap U_P = \Gamma^U_P.
\]
It suffices to prove that $\text{Span}_R[\Gamma^N_P, \Gamma^N_P] = U_P$. Then by the structure theorem for finitely generated abelian groups and their subgroups, there is a $\mathbb{Z}$-basis $\lambda_1, \ldots, \lambda_r$ of $\Gamma^U_P$ and $m_1, \ldots, m_r \in \mathbb{N}$, such that $m_1 \lambda_1, \ldots, m_r \lambda_r$ is a $\mathbb{Z}$-basis of $[\Gamma^N_P, \Gamma^N_P]$. If $m \in \mathbb{N}$ is the least common multiple of $m_1, \ldots, m_r$, then $(\Gamma^U_P)^m \subseteq [\Gamma^N_P, \Gamma^N_P] \subseteq \Gamma$ and any element of $\Gamma^U_P[\Gamma, \Gamma]^F_{\text{Fix}}/[\Gamma, \Gamma]^F_{\text{Fix}}$ is of order, dividing $m$.

Towards $\text{Span}_R[\Gamma^N_P, \Gamma^N_P] = U_P$, let $\beta_1, \ldots, \beta_{r+2n} \in \text{Lie} N_P$ be such elements of the Lie algebra $\text{Lie} N_P$ of $N_P$ whose exponents $b_j = \text{Exp}(\beta_j) \in N_P$, $1 \leq j \leq r+2n$, generate the lattice $\Gamma^N_P$. The group $N_P$ is 2-step nilpotent, so that its Lie algebra $\text{Lie} N_P$ is 2-step nilpotent and $b_j = \text{Exp}(\beta_j) = I_k + \beta_j + \frac{\beta_j^2}{2}$ for some realization of $\text{Lie} N_P$ and $N_P$ by $(k \times k)$-upper triangular matrices. Then $\text{Lie} N_P = \text{Span}_R(\beta_1, \ldots, \beta_{r+2n})$ and the Lie algebra commutator
\[
\text{Lie} U_P = [\text{Lie} N_P, \text{Lie} N_P] = \text{Span}_R([\beta_i, \beta_j]_{\text{Lie}} \mid 1 \leq i \neq j \leq r + 2n).
\]
The abelian group $U_P \simeq (\mathbb{R}^r, +)$ is isomorphic to its Lie algebra, so that one can identify $[\beta_i, \beta_j]_{\text{Lie}}$ with the group commutators $\text{Exp}[\beta_i, \beta_j]_{\text{Lie}} = [\text{Exp}(\beta_i), \text{Exp}(\beta_j)] = [b_i, b_j]$. As a result, $U_P = \text{Span}_R([b_i, b_j] \mid 1 \leq i \neq j \leq r + 2n)$. The commutant $[\Gamma^N_P, \Gamma^N_P]$ is generated by $[b_i, b_j]$ with $1 \leq i \neq j \leq r + 2n$, as far as $\Gamma^N_P$ is generated by $b_1, \ldots, b_{r+2n}$. Therefore $\text{Span}_R([b_i, b_j] \mid 1 \leq i \neq j \leq r + 2n) = \text{Span}_R[\Gamma^N_P, \Gamma^N_P]$ and $U_P = \text{Span}_R[\Gamma^N_P, \Gamma^N_P]$.

\[
\Box
\]

4.3 A sufficient condition for residual finiteness of $\pi_1(D/\Gamma)'_\Sigma$

The present subsection provides a sufficient condition for a toroidal compactification $(D/\Gamma)'_\Sigma$ of a torsion free non-compact $D/\Gamma$, under which the fundamental group
$\pi_1(D/\Gamma)^{\mathbb{S}_1} = \Gamma/\Gamma^U$ of $(D/\Gamma)^{\mathbb{S}_1}$ is residually finite. Hummel-Schroeder’s [21] and Hummel’s [22] imply that the toroidal compactifications of generic noncompact torsion free ball quotients $\mathbb{B}^n/\Gamma$ satisfy these sufficient condition. More precisely, Hummel and Schroeder show in [21] that any torsion free non-uniform lattice $\Gamma_o < SU_{n,1}$ contains a finite subset $\{\gamma_1, \ldots, \gamma_k\} \subset \Gamma_o$, such that for any normal subgroup $\Gamma \triangleleft \Gamma_o$ with $\Gamma \cap \{\gamma_1, \ldots, \gamma_k\} = \emptyset$ the toroidal compactification $(\mathbb{B}^n/\Gamma)^\prime$ admits a Riemannian metric of non-positive sectional curvature. In [7], Di Cerbo refers to such $(\mathbb{B}^n/\Gamma)^\prime$ as to Hummel-Schroeder toroidal compactifications. Recall that $rk_S SU_{n,1} = 1$ and all the $\Gamma$-rational parabolic subgroups of $SU_{n,1}$ are minimal. According to Hummel’s [22], if $(\mathbb{B}^n/\Gamma)^\prime$ is a Hummel-Schroeder toroidal compactification then all the parabolic isometries $\gamma \in \Gamma$ are unipotent, i.e., $\Gamma \cap P = \Gamma \cap NP$ for any $P \in T(\mathbb{B}^n/\Gamma_{n,1})$ with unipotent radical $NP$. In the case of complex dimension 2, Di Cerbo shows in [7] that the Hummel-Schroeder toroidal compactifications $X^\prime = (\mathbb{B}^2/\Gamma)^\prime$ are minimal surfaces $X^\prime = X$ of general type with ample canonical bundle. By Nadel’s article [31] the universal covering $\tilde{X}$ of such $X = X^\prime$ has discrete biholomorphism group, containing $\pi_1(X) = \Gamma/\Gamma^U$ as a subgroup of finite index.

Here is an immediate consequence of the aforementioned results, illustrating the abundance of Hummel-Schroeder compactifications $(\mathbb{B}^n/\Gamma)^\prime$.

**Lemma 6.** (Hummel-Schroder [21]) Any non-uniform torsion free lattice $\Gamma^\prime < SU_{n,1}$ has a normal subgroup $\Gamma_o \triangleleft \Gamma^\prime$ of finite index $[\Gamma^\prime : \Gamma_o] < \infty$, such that any $\Gamma \triangleleft \Gamma^\prime$ with $\Gamma \subseteq \Gamma_o$ is associated with a Hummel-Schroeder toroidal compactification $(\mathbb{B}^n/\Gamma)^\prime$.

**Proof.** According to [21], there is a finite subset $\{\gamma_1, \ldots, \gamma_k\} \subset \Gamma^\prime$, such that any $\Gamma \triangleleft \Gamma^\prime$ with $\Gamma \cap \{\gamma_1, \ldots, \gamma_k\} = \emptyset$ is associated with a Hummel-Schroeder toroidal compactification $(\mathbb{B}^n/\Gamma)^\prime$. It suffices to observe the existence of $\Gamma_o \triangleleft \Gamma^\prime$ with finite index $[\Gamma^\prime : \Gamma_o] < \infty$ and $\Gamma_o \cap \{\gamma_1, \ldots, \gamma_k\} = \emptyset$. The lattice $\Gamma^\prime$ of the simple Lie group $SU_{n,1}$ is residually finite, so that for any $1 \leq i \leq k$ there is a normal subgroup $\Gamma_i$ of finite index $[\Gamma^\prime : \Gamma_i] < \infty$ with $\gamma_i \notin \Gamma_i$. The quotient group $\Gamma_i/\Gamma_i \cap \Gamma_j \cong \Gamma_i \Gamma_j/\Gamma_j$ is finite as a subgroup of finite group $\Gamma^\prime/\Gamma_j$. Therefore $[\Gamma^\prime : \Gamma_i \cap \Gamma_j] = [\Gamma^\prime : \Gamma_i][\Gamma_i : \Gamma_i \cap \Gamma_j] < \infty$. By an induction on $k$ there follows $[\Gamma^\prime : \Gamma_i \cap \Gamma_j \cap \Gamma_l] < \infty$, so that $\Gamma_o := \Gamma_i \cap \Gamma_j \cap \Gamma_l$ is a normal subgroup of $\Gamma^\prime$ of finite index $[\Gamma^\prime : \Gamma_o] < \infty$ with $\Gamma_o \cap \{\gamma_1, \ldots, \gamma_k\} = \emptyset$. 

An arbitrary non-uniform lattice $\Gamma^\prime$ of the isometry group $G$ of a Hermitian symmetric space $D = G/K$ of noncompact type has a neat normal subgroup $\Gamma_o \triangleleft \Gamma^\prime$ of finite index. By Theorem 2, p. 301 from Ash-Mumford-Rapoport-Tai’s book [2], $\Gamma_o$ has a subgroup $\Gamma < \Gamma_o$ of finite index $[\Gamma_o : \Gamma] < \infty$, such that $D/\Gamma$ is of general type. According to the considerations from subsection 3.3 after Proposition 1, the covering $(D/\Gamma^U)^{\mathbb{S}_1} \to (D/\Gamma_o^U)^{\mathbb{S}_1}$ is ramified over $(D/\Gamma_o^U)^{\mathbb{S}_1} \setminus (D/\Gamma^U)^{\mathbb{S}_1}$. Therefore $\zeta_{\Gamma_o^U} : (D/\Gamma)^{\mathbb{S}_1} \to (D/\Gamma_o)^{\mathbb{S}_1}$ is ramified over $(D/\Gamma_o)^{\mathbb{S}_1} \setminus (D/\Gamma_o)^{\mathbb{S}_1}$. For subgroups $\Gamma \triangleleft \Gamma_o$ of sufficiently large index $[\Gamma_o : \Gamma]$ the canonical divisor of the smooth projective variety $(D/\Gamma)^{\mathbb{S}_1}$ is expected to be ample as a sum of the pull-back of the canonical divisor of $(D/\Gamma_o)^{\mathbb{S}_1}$ with the ramification divisor of $\zeta_{\Gamma_o^U}$. Therefore $(D/\Gamma)^{\mathbb{S}_1}$ has an ample canonical bundle, i.e., the global holomorphic sections of sufficiently large tensor powers of the canonical bundle provide a projective embedding of $(D/\Gamma)^{\mathbb{S}_1}$. The majority of
the smooth complex projective varieties with ample canonical bundle are expected to
admit a Riemannian metric with non-positive sectional curvature.

We say that the $\Gamma$-rational minimal parabolic subgroups $P_1$ and $P_2$ are $\Gamma$-equivalent
if $P_2 = \gamma P_1 \gamma^{-1}$ for some $\gamma \in \Gamma$. Bearing in mind the gluings in the construction of
the toroidal compactification $(D/\Gamma)_\Sigma^{'},$ one gives the following

**Definition 7.** The $\Gamma$-rational minimal parabolic subgroups $P_1$ and $P_2$ of $G$ are weakly
$\Gamma$-equivalent if there exist $\gamma \in \Gamma$ and a $\Gamma$-rational parabolic subgroup $Q < G,$ such that
$\gamma P_1 \gamma^{-1} \subseteq Q$ and $P_2 \subseteq Q.$

Clearly, any $\Gamma$-equivalent $\Gamma$-rational minimal parabolic subgroups $P_1$ and $P_2 = \gamma P_1 \gamma^{-1}$ are weakly $\Gamma$-rational. In the case of $\text{rk}_{\mathbb{R}} G = 1,$ i.e., for $G = SU_{n,1}$ the weak
$\Gamma$-equivalence coincides with the $\Gamma$-equivalence.

**Theorem 8.** Let us suppose that a toroidal compactification $(D/\Gamma)_\Sigma^{'},$ of a noncom-
pact torsion free $D/\Gamma$ admits a Riemannian metric of non-positive sectional curvature,
$\Gamma \cap P = \Gamma \cap N_P$ for all $\Gamma$-rational minimal parabolic subgroups $P < G$ with unipotent
radicals $N_P$ and there are at least two weak $\Gamma$-equivalence classes of $\Gamma$-rational
minimal parabolic subgroups $P < G.$ Then the fundamental group $\pi_1(D/\Gamma)_\Sigma^{'}, = \Gamma/\Gamma^U$ of
$(D/\Gamma)_\Sigma^{'},$ is residually finite and the first homology group $H_1((D/\Gamma)_\Sigma^{'},, \mathbb{Z})$ is of rank
$\text{rk}_{\mathbb{Z}} H_1((D/\Gamma)_\Sigma^{'},, \mathbb{Z}) \leq 2(\dim_{\mathbb{C}} D - \text{rk}_{\mathbb{R}} G).$

**Proof.** The boundary compact complex tori $T_P = (V_P, +)/((\Lambda(P), +),$ associated with
$P \in \Gamma\text{Min}P(G)$ are mapped isomorphically onto their images in $(D/\Gamma)_\Sigma^{'},$ for a torsion
free non-uniform lattice $\Gamma < G$ with $\Gamma \cap P = \Gamma \cap N_P.$ The reason for this is that any $\gamma \in \Gamma \setminus P$ moves the isolated $\Gamma$-rational boundary point $p = Fix_{\partial P} (P) \in \partial P.$ The corresponding coset class transforms isomorphically $[Z_{\Gamma(P)}(\Lambda(P))] \setminus (D/\Gamma_P^N)$ onto
$[Z_{\Gamma(P)}(\Lambda(P))] \setminus (D/\Gamma^N_P)$ without identifying points from the boundary
compact complex torus $T_P \subseteq [Z_{\Gamma(P)}(\Lambda(P))] \setminus (D/\Gamma^N_P).$

After Di Cerbo’s [7], we claim that the natural inclusions $T_P \hookrightarrow (D/\Gamma)_\Sigma^{'},$ induce injective homomorphisms
$$\varphi : (\mathbb{Z}^{2n}, +) \simeq \pi_1(T_P) = \Lambda(P) = \Gamma^N_P/\Gamma^U_P \hookrightarrow \pi_1(D/\Gamma)_\Sigma^{'}, = \Gamma/\Gamma^U$$
of the corresponding fundamental groups. That is due to the existence of a Riemannian
metric on $(D/\Gamma)_\Sigma^{'},$ with non-positive sectional curvature. The homomorphism $\varphi$ is
induced by the composition
$$\varphi_0 : \Gamma^N_P \cap \Gamma = \Gamma \cap N_P = \Gamma \cap P \hookrightarrow \pi_1(D/\Gamma)_\Sigma^{'}, = \Gamma/\Gamma^U$$
of the inclusion $\Gamma \cap P \subseteq \Gamma$ with the natural epimorphism $\Gamma \to \Gamma/\Gamma^U.$ The kernel $\ker \varphi_0 = \Gamma^U \cap P$ contains $\Gamma^U_P = \Gamma \cap U_P$ as a subgroup with quotient group
$$0 = \ker \varphi = \ker \varphi_0/\Gamma^U_P = \Gamma^U \cap P/\Gamma^U_P.$$ Therefore $\Gamma^U \cap P \subseteq \Gamma^U_P = \Gamma \cap U_P$ and
$$\pi_1(T_P) = \Gamma \cap P/\Gamma^U \cap P = (\Gamma \cap P)/(\Gamma \cap P) \cap \Gamma^U \simeq (\Gamma \cap P)\Gamma^U/\Gamma^U.$$
The coset space $\Gamma / (\Gamma \cap P) \simeq \Gamma P / P$ is a discrete and, therefore, finite subset of the compact homogeneous space $G / P = PK / P \simeq K / K \cap P$. The inclusions

$$\Gamma \supset (\Gamma \cap P) \Gamma^U \supset \Gamma \cap P$$

imply that the index $[\Gamma : (\Gamma \cap P) \Gamma^U] < \infty$ is finite. As a result, $\pi_1(T_P)$ is of finite index

$$[\pi_1(D / \Gamma)^U \Sigma : \pi_1(T_P)] = [\Gamma / \Gamma^U : (\Gamma \cap P) \Gamma^U / \Gamma^U] = [\Gamma : (\Gamma \cap P) \Gamma^U] < \infty$$

in $\pi_1(D / \Gamma)^U \Sigma$. Towards the residual finiteness of $\pi_1(D / \Gamma)^U \Sigma = \Gamma / \Gamma^U$, it suffices to show that $\cap_{P \in \Gamma \text{MinP}(G)} \pi_1(T_P) = \{\Gamma^U\}$, in order to assert that all subgroups of $\pi_1(D / \Gamma)^U \Sigma = \Gamma / \Gamma^U$ of finite index intersect in the neutral element $\Gamma^U \in \Gamma / \Gamma^U$ alone. For any $P \in \Gamma \text{MinP}(G)$ denote by $\zeta^P U \Sigma(P) \rightarrow \zeta^P U (Z\Sigma(P))$ the $\Gamma^U / \Gamma^U P$-Galois covering and observe that an arbitrary element $\gamma \in \Gamma^U$ transforms $\zeta^P U (Z\Sigma(P))$ into $(\gamma \Gamma^U) \zeta^P U (Z\Sigma(P)) = \zeta^P \Sigma(\gamma \Gamma^U - 1)$. Let us pick up $\Gamma$-rational minimal parabolic subgroups $P_j$ and $P_2$ of $G$, which are not weakly $\Gamma$-equivalent and assume that there exists $\Gamma^U \neq \gamma_2 \Gamma^U = \gamma_2 \Gamma^U \in \pi_1(T_{P_j}) \cap \pi_1(T_{P_2})$ with $\gamma_j \in \Gamma \cap P_j, 1 \leq j \leq 2$. On the other hand, $(\gamma_1 \Gamma^U) \zeta^P \Sigma(P_3) = \zeta^P \Sigma(\gamma_2 \Gamma^U - 1) \cap (\gamma_1 \Gamma^U - 1) \gamma_2 \Gamma^U \neq P_2$, as far as the normalizer of $P_2$ in $G$ coincides with $P_2$ and $P_1 \cap P_2 = \{1\}$. The contradiction justifies $\pi_1(T_{P_j}) \cap \pi_1(T_{P_2}) = \{\Gamma^U\}$, whereas $\cap_{P \in \Gamma \text{MinP}(G)} \pi_1(T_P) = \{\Gamma^U\}$ and the residual finiteness of the fundamental group $\pi_1(D / \Gamma)^U \Sigma = \Gamma / \Gamma^U$.

In order to derive the upper bound on the rank of $H_1((D / \Gamma)^U \Sigma, Z)$, let us fix a $\Gamma$-rational minimal parabolic subgroup $P < G$ and note that the intersection

$$F_0 = \cap_{\gamma \Gamma^U \in \Gamma / \Gamma^U} (\gamma \Gamma^U \pi_1(T_P)(\gamma \Gamma^U)^{-1}$$

of the $\Gamma / \Gamma^U$-conjugates of the finite index subgroups $\pi_1(T_P)$ of $\Gamma / \Gamma^U$ is a normal subgroup $F_0 < \Gamma / \Gamma^U$ with finite quotient group $F_1 = (\Gamma / \Gamma^U) / F_0$. There arises an exact sequence of groups

$$1 \longrightarrow F_0 \longrightarrow \Gamma / \Gamma^U \longrightarrow F_1 \longrightarrow 1.$$ 

The subgroup $F_0$ of the abelian group $\pi_1(T_P) = \Lambda(P) \simeq (\mathbb{Z}^{2n}, +), n = \dim_C D - \text{rk}_\mathbb{R} G$ is abelian. Bearing in mind that the abelianization is a right exact functor, one obtains the exact sequence

$$F_0 \xrightarrow{\psi} ab(\Gamma / \Gamma^U) \longrightarrow ab(F_1) \longrightarrow 1$$

with a finite abelian group $ab(F_1)$. Therefore, the subgroup $\psi(F_0)$ of $ab(\Gamma / \Gamma^U) = H_1((D / \Gamma)^U \Sigma, Z)$ is of finite index and the rank

$$\text{rk}_\mathbb{Z} H_1((D / \Gamma)^U \Sigma, Z) = \text{rk}_\mathbb{Z} \psi(F_0) \leq \text{rk}_\mathbb{Z} (F_0) \leq \text{rk}_\mathbb{Z} \pi_1(T_P) = 2n.$$ 

\[\square\]
Corollary 9. If \((\mathbb{B}^n/\Gamma)'\) is a Hummel-Schroeder toroidal compactification of a non-compact torsion free ball quotient \(\mathbb{B}^n/\Gamma\) of \(\dim_{\mathbb{C}}\mathbb{B}^n = n \geq 2\) and the \(\Gamma\)-action on the \(\Gamma\)-rational boundary points \(\partial_{\mathbb{R}}\mathbb{B}^n = \partial_{\mathbb{R}}\mathbb{B}^n\) is not transitive then the fundamental group \(\pi_1(\mathbb{B}^n/\Gamma)' = \Gamma/\Gamma^U\) is residually finite and the homology group \(H_1((\mathbb{B}^n/\Gamma)', \mathbb{Z})\) is of rank \(\text{rank} H_1((\mathbb{B}^n/\Gamma)', \mathbb{Z}) \leq 2n - 2\).

As a matter of fact, let us observe that any uniform torsion free lattice \(\Gamma < SU_{n,1}\) is Gromov hyperbolic, so that the fundamental group \(\pi_1(\mathbb{B}^n/\Gamma) = \Gamma\) does not contain a subgroup, isomorphic to \((\mathbb{Z}^2, +)\). As a contrast, by Theorem 8 and Corollary 9, the fundamental group \(\pi_1(\mathbb{B}^n/\Gamma)' = \Gamma/\Gamma^U\) of a generic \((\mathbb{B}^n/\Gamma)'\) with a non-uniform torsion free lattice \(\Gamma < SU_{n,1}\) contains a subgroup, isomorphic to \((\mathbb{Z}^{2n-2}, +)\).

According to by McReynolds’ terminology from [30], the quotient group \(\Gamma/\Gamma^U\) of a torsion free non-uniform lattice \(\Gamma < G\) is residually finite exactly when \(\Gamma^U\) is separable in \(\Gamma\). Namely, a subgroup \(\mathcal{H}\) of \(\mathcal{G}\) is said to be separable in \(\mathcal{G}\) if for any \(g \in \mathcal{G} \setminus \mathcal{H}\) there is a subgroup \(\mathcal{G}_g \leq \mathcal{G}\) of finite index \(|\mathcal{G}_g : \mathcal{G}_g| < \infty\) with \(\mathcal{H} \subseteq \mathcal{G}_g\) and \(g \not\in \mathcal{G}_g\).

The subgroups of \(\Gamma/\Gamma^U\) of finite index are of the form \(\Gamma_\alpha/\Gamma^U\) for subgroups \(\Gamma_\alpha \leq \Gamma\), containing \(\Gamma^U\), and having a finite index \(|\Gamma : \Gamma_\alpha| = |\Gamma/\Gamma^U : \Gamma_\alpha/\Gamma^U| < \infty\). Thus, the quotient group \(\Gamma/\Gamma^U\) is residually finite if for any \(\gamma \Gamma^U \in \Gamma/\Gamma^U\) there is a subgroup \(\Gamma_\alpha\) of \(\Gamma^U\) of finite index with \(\Gamma^U \subseteq \Gamma_\alpha\) and \(\gamma \not\in \Gamma_\alpha\). In other words, \(\Gamma/\Gamma^U\) is residually finite if and only if \(\Gamma^U\) is separable in \(\Gamma\). Corollary 4.2 from McReynolds [30] establishes the separability of the nilpotent subgroups \(N\) of the arithmetic lattices \(\Gamma < SU_{n,1}\) in \(\Gamma\). The groups \(\Gamma^U\) are not necessarily nilpotent but generated by the nilpotent subgroups \(\Gamma^U_Q = \Gamma \cap U_Q\) of \(\Gamma\) for \(\forall Q \in \Gamma P(G)\). That is why we expect that a generic toroidal compactification \((D/\Gamma)'_\Sigma\) of a torsion free \(D/\Gamma\) has residually finite fundamental group \(\Gamma/\Gamma^U\).

4.4 Locally trivial holomorphic fibrations of \((D/\Gamma)'_\Sigma\)

For an arbitrary \(P \in \Gamma MinP(G)\) and an arbitrary \(\Gamma\)-admissible polyhedral cone decomposition \(\Sigma(P)\) of \(U_P\) the covering \(D/\Gamma^U_P = (U_P + iC_P)/\Gamma^U_P \times V_P \to D/\Gamma\) extends to a covering \(\pi_{\Sigma(P)} : Z_{\Sigma(P)} = Y_{\Sigma(P)} \times V_P \to \pi_{\Sigma(P)}(Z_{\Sigma(P)})\) of its image \(\pi_{\Sigma(P)}(Z_{\Sigma(P)}) \subseteq (D/\Gamma)'_\Sigma\). At the end of subsection 3.1 we have introduced the boundary compact complex tori \(T_P = (V_P, +)/(\Lambda(P), +)\) as subsets of \(Z_{\Sigma(P)}/(\Gamma^U_P/\Gamma^U_P)\). The images of \(T_P\) in \((D/\Gamma)'_\Sigma\) admits nonconstant holomorphic maps from \(L_P = 0^r \times V_P \simeq \mathbb{C}^n\). That justifies the following

Corollary 10. The toroidal compactifications \((D/\Gamma)'_\Sigma\) are not Kobayashi hyperbolic.

Note that the discrete quotient \(D/\Gamma\) of the bounded symmetric domain \(D\) is Kobayashi hyperbolic, so that the Kobayashi non-hyperbolicity of \((D/\Gamma)'_\Sigma\) is due to the compactification divisor

\[
(D/\Gamma)'_\Sigma \setminus (D/\Gamma) = \bigcup_{P \in \Gamma MinP(G)} \pi_{\Sigma(P)} \left( \left[ Y_{\Sigma(P)} \setminus (U_P + iC_P)/\Gamma^U_P \right] \times V_P \right)
\]

with \(\left[ Y_{\Sigma(P)} \setminus (U_P + iC_P)/\Gamma^U_P \right] \times V_P \simeq \left[ \Delta^r \setminus (\Delta^s)^r \right] \times \mathbb{C}^n\).

In order to trace out the impact of the Kobayashi non-hyperbolicity of \((D/\Gamma)'_\Sigma\) on its holomorphic fibrations, we fix the notion of a locally trivial holomorphic fibration.
Definition 11. A holomorphic fibration \( f : N \to M \) of complex analytic spaces is locally trivial if for any point \( x \in M \) there are a neighborhood \( U_x \subseteq M \) and a proper complex analytic subspace \( A_x \subset U_x \), such that for any \( y \in U_x \setminus A_x \) the fiber \( f^{-1}(y) \) over \( y \) is biholomorphic to the fiber \( f^{-1}(x) \) over \( x \).

Proposition 12. If a locally trivial holomorphic fibration \( f : (D/\Gamma)^{\prime}_\Sigma \to M \) has a complex analytic base \( M \) of \( \dim_{\mathbb{C}} M \geq \text{rk}_\mathbb{R} G \) then \( M \) is not Kobayashi hyperbolic.

Proof. Let us assume that \( M \) is Kobayashi hyperbolic. Then for any point \( y \in Y_{\Sigma(P)} \setminus (U_P + iC_P)/\Gamma_P^U \) with \( P \in \Gamma_{\text{Min}}P(G) \) the image \( \pi_{\Sigma(P)}(y \times V_P) \) of \( y \times V_P \) is contained in a fiber of \( f \). This is due to the distance non-increasing property of the holomorphic maps with respect to Kobayashi pseudo-distance and the Kobayashi non-hyperbolicity of \( y \times V_P \cong \mathbb{C}^n \).

The generic fiber of \( f : (D/\Gamma)^{\prime}_\Sigma \to M \) is claimed to be Kobayashi hyperbolic. More precisely, the compactification divisor

\[
(D/\Gamma)^{\prime}_\Sigma \setminus (D/\Gamma) = \bigcup_{P \in \Gamma_{\text{Min}}P(G)} \pi_{\Sigma(P)}([\Delta^r \setminus (\Delta^r)^*] \times \mathbb{C}^n)
\]

is a union of complex hypersurfaces \( \pi_{\Sigma(P)}(\Delta^r \setminus (\Delta^r)^* \times \mathbb{C}^n) \) for \( 1 \leq i \leq r \). Therefore \( f \pi_{\Sigma(P)}(\Delta^r \setminus (\Delta^r)^* \times \mathbb{C}^n) = f \pi_{\Sigma(P)}(\Delta^r \setminus (\Delta^r)^* \times \mathbb{C}^n) \) are of complex dimension \( \leq r - 1 \), so that there is a complex analytic subspace \( A \subset M \) with \( f((D/\Gamma)^{\prime}_\Sigma \setminus (D/\Gamma)) \subseteq A \). Now, \( f^{-1}(M \setminus A) \) is contained in \( D/\Gamma \) and, therefore, Kobayashi hyperbolic. The distance non-increasing property of the holomorphic embeddings \( f^{-1}(y) \to f^{-1}(M \setminus A) \) of the fibers \( f^{-1}(y) \) over \( y \in M \setminus A \) implies the Kobayashi hyperbolicity of \( f^{-1}(y) \) for all \( y \in M \setminus A \). By the local triviality of \( f : (D/\Gamma)^{\prime}_\Sigma \to M \) there follows the Kobayashi hyperbolicity of all the fibers of \( f \). Namely, for any \( x \in M \) there exist a neighborhood \( U_x \) of \( x \) on \( M \) and an analytic subspace \( A_x \subset U_x \), such that for any \( y \in U_x \setminus A_x \) the fiber \( f^{-1}(y) \) admits a biholomorphic map \( \varphi^x_y : f^{-1}(x) \to f^{-1}(y) \). The proper complex analytic subspace \( A \subset M \) intersects \( U_x \) in a proper complex analytic subspace \( A \cap U_x \), so that \( U_x \setminus (A_x \cup A) \neq \emptyset \) is non-empty. The presence of a biholomorphic maps \( \varphi^x_y : f^{-1}(x) \to f^{-1}(y) \) for all \( y \in U_x \setminus (A_x \cup A) \) implies the Kobayashi hyperbolicity of \( f^{-1}(x) \) by the distance non-increasing property of \( \varphi^x_y \) and the Kobayashi hyperbolicity of \( f^{-1}(y) \). Thus, the assumptions of the corollary suffice for the Kobayashi hyperbolicity of all the fibers of \( f : (D/\Gamma)^{\prime}_\Sigma \to M \), which contradicts the presence of compact complex analytic Kobayashi non-hyperbolic spaces \( \pi_{\Sigma(P)}(0^r \times V_P) \) of complex analytic subspace \( \pi_{\Sigma(P)}(Z_{\Sigma(P)}) \cong Z_{\Sigma(P)} \times (\Delta^r)^* \times \mathbb{C}^n \).

4.5 Open problems

In this subsection we list two open problems, arising from the representation

\[
(D/\Gamma^U)^{\prime}_\Sigma = \bigcup_{P \in \Gamma_{\text{Min}}P(G)} \pi^U_{\Sigma(P)}(Z_{\Sigma(P)})
\]

of the simply connected covering \( (D/\Gamma^U)^{\prime}_\Sigma \) of the toroidal compactification \( (D/\Gamma)^{\prime}_\Sigma \) as a union of contractible Stein spaces

\[
\pi^U_{\Sigma(P)}(Z_{\Sigma(P)}) \cong Z_{\Sigma(P)} = Y_{\Sigma(P)} \times V_P \cong \Delta^r \times \mathbb{C}^n
\]
with $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$. Bearing in mind that $(D/\Gamma')_2^0$ has the same higher homotopy groups as $(D/\Gamma')_2^1$, one asks

**Problem 13.** What are the higher homotopy groups $\pi_i((D/\Gamma')_2^0) \cong \pi_i((D/\Gamma')_2^1)$ for $i \geq 2$?

**Problem 14.** Is $(D/\Gamma^U)'_2$ a Stein space?

## 5 Toroidal compactifications of torsion free quotients of the complex 2-ball

### 5.1 Toroidal compactifications of positive Kodaira dimension

The present subsection studies some geometric properties of the minimal models $X$ of the toroidal compactifications $X' = (\mathbb{B}^2/\Gamma)'$ of the quotients $\mathbb{B}^2/\Gamma$ of the complex 2-ball $\mathbb{B}^2 = SU_{2,1}/SU_2 \times U_1$ by torsion free non-uniform lattices $\Gamma < SU_{2,1}$. Examples of such $X'$ with an abelian minimal model $X$ are constructed by Holzapfel in [18] and [19]. His considerations from [19] imply the existence of torsion free noncompact $\mathbb{B}^2/\Gamma$ with $X' = (\mathbb{B}^2/\Gamma)'$ of general type. In [28] Momot constructs a series of minimal surfaces $X = X' = (\mathbb{B}^2/\Gamma)'$ with torsion free $\Gamma$ and Kodaira dimension $\kappa(X') = 1$. After an etale base changes, any of Momot’s examples $X$ admits an elliptic fibration $f : X \rightarrow C$ with an elliptic base $C$ and $e(X)/6$ singular fibers of type $I_6$. The toroidal compactification divisor $T = (\mathbb{B}^2/\Gamma)' \setminus (\mathbb{B}^2/\Gamma) = \sum_{i=1}^{36} T_i$ consists of 36 smooth elliptic irreducible components $T_i$ with self-intersection number $T_i^2 = -\frac{e(X)}{12} < 0$. The results of Hummel-Schroeder’s from [21] and Di Cerbo from [7] imply that a generic torsion free non-uniform lattice $\Gamma < SU_{2,1}$ has $X = X' = (\mathbb{B}^2/\Gamma)'$ of general type.

The present subsection observes that the toroidal compactifications $X' = (\mathbb{B}^2/\Gamma)'$ of noncompact torsion free $\mathbb{B}^2/\Gamma$ are of Kodaira dimension $\kappa(X') \geq 0$. In the case of $\kappa(X) = 2$, Proposition 15 establishes the inequality $c_1^2(X) = K_X^2 \leq e(X) = c_2(X)$ of the Chern numbers of $X$. Proposition 16 discusses $X' = (\mathbb{B}^2/\Gamma)'$ of Kodaira dimension $\kappa(X') = 1$ and their minimal models $X$. It establishes that the elliptic fibrations $f : X \rightarrow C$ have base $C$ of genus $g(C) = 0$ or 1. This fact is not an immediate consequence from Proposition 12, as far as the elliptic fibrations are not locally trivial (cf. Definition 11). In the case of $g(C) = 1$ the Albanese variety of $X$ is shown to coincide with $C$ and the irregularity $q(X) = g(C) = 1$. The fundamental group of a minimal surface, which admits an elliptic fibration $f : X \rightarrow C$ is computed by Dürr in [10]. Corollary 17 specifies the application of Dürr’s results to the minimal models $X$ of the toroidal compactifications $X' = (\mathbb{B}^2/\Gamma)'$ of torsion free non-compact $\mathbb{B}^2/\Gamma$.

**Lemma 15.** Let $\Gamma < SU_{2,1}$ be a torsion free non-uniform lattice, $X' = (\mathbb{B}^2/\Gamma)'$ be the toroidal compactification of the ball quotient $\mathbb{B}^2/\Gamma$ by $\Gamma$ and $X$ be the minimal model of $X'$. Then:

- (i) the Kodaira dimension $\kappa(X) \geq 0$;
- (ii) if $\kappa(X) = 2$ then $K_X^2 \leq e(X)$;
(iii) if $X' = (\mathbb{B}^2/\Gamma)'$ is a Hummel-Schroeder toroidal compactification then

$$\max \left( \frac{e(X)}{5} - \frac{36}{5}, 1 \right) \leq K_X^2 \leq e(X).$$

**Proof.** Without mentioning explicitly, we assume that all the ball quotients $\mathbb{B}^2/\Gamma$ from the present proof are by torsion free non-uniform lattices $\Gamma < SU_{2,1}$.

(i) The assertion $\kappa(X') \geq 0$ is equivalent to the non-existence of $X' = (\mathbb{B}^2/\Gamma)'$ with $\kappa(X') = -\infty$. The surfaces $X'$ of $\kappa(X') = -\infty$ are the rational ones and the ruled surfaces $\Sigma_g \to C_g$ with a base $C_g$ of genus $g \geq 1$. The projective varieties with ample anti-canonical bundle are called Fano varieties. By Corollary 2.6 from Di Cerbo shows in [7], there are no Fano varieties $(\mathbb{B}^n/\Gamma)'$. In particular, there are no rational surfaces $X' = (\mathbb{B}^2/\Gamma)'$. Bogomolov has proved in [27] that the toroidal compactifications of Kodaira dimension $\kappa(X') \leq 0$ and irregularity $q(X') = h^{0,1}(X') \geq 1$ have abelian minimal models $X$. Bearing in mind that the ruled surfaces $\Sigma_g$ with a base $C_g$ of genus $g \geq 1$ have irregularity $q(\Sigma_g) = g \geq 1$, one concludes the non-existence of $X' = (\mathbb{B}^2/\Gamma)'$ with $\kappa(X') = -\infty$.

(ii) Bogomolov-Miyaoka-Yau inequality for the minimal surfaces $X$ of $\kappa(X) = 2$ asserts that $K_X^2 \leq 3e(X)$ with equality $K_X^2 = 3e(X)$ exactly when $X = \mathbb{B}^2/\Gamma$ is a compact torsion free quotient of $\mathbb{B}^2$. Thus, $K_X^2 = 2e(X)$ for the minimal model $X'$ of $X' = (\mathbb{B}^2/\Gamma)'$. In the case of $K_X^2 = 2e(X)$ the surface $X = \Delta \times \Delta$ is a compact torsion free quotient of the bi-disc $\Delta \times \Delta$. That implies $K_X^2 \leq e(X)$ for the minimal model $X$ of $X' = (\mathbb{B}^2/\Gamma)'$.

(iii) Di Cerbo shows in [7] that the Hummel-Schroeder toroidal compactifications $X' = (\mathbb{B}^2/\Gamma)'$ are minimal surfaces of general type with ample canonical bundle. By definition, $X$ is of general type if the Kodaira dimension $\kappa(X) = 2$. Noether inequality for the minimal surfaces $X' = X$ of general type asserts that $5K_X^2 - e(X) + 36 \geq 0$, which is equivalent to $K_X^2 \geq \frac{e(X)}{5} - \frac{36}{5}$. On the other hand, the minimal surfaces $X$ of general type have $c_1^2 = K_X^2 \geq 1$.

\[\square\]

**Proposition 16.** Let $\Gamma < SU_{2,1}$ be a torsion free non-uniform lattice, whose quotient $\mathbb{B}^2/\Gamma$ has toroidal compactification $X' = (\mathbb{B}^2/\Gamma)'$ of Kodaira dimension $\kappa(X') = 1$ and $f : X \to C$ be an elliptic fibration of the minimal model $X$ of $X'$ over a smooth compact complex curve $C$. Then:

(i) the genus of $C$ is $g(C) = 0$ or 1

(iv) if $C$ is an elliptic curve then the irregularity $q(X) = g(C) = 1$.

**Proof.** (i) Any minimal surface $X$ of $\kappa(X) = 1$ admits at least one elliptic fibration $f : X \to C$ over a smooth compact complex curve $C$. Let us suppose that an elliptic fibration $f : X \to C$ has a base $C$ of genus $g(C) \geq 2$. Then the image of an irreducible component $T_i$ of $T = (\mathbb{B}^2/\Gamma)' \setminus (\mathbb{B}^2/\Gamma)$ on $X$ is contained in a fiber of $f$. The complement $X \setminus \bigcup_{i=1}^k f^{-1}(c_i)$ of finitely many fibers is Kobayashi hyperbolic and the generic fiber of $f$ is Kobayashi hyperbolic. By Shabat’s thesis [34], the universal covering $\tilde{X}$ of the family $f : X \to C$ of curves of genus $\geq 2$ over a curve $C$ of genus $g(C) \geq 2$ is a bounded domain in $\mathbb{C}^2$. Therefore $X$ is Kobayashi hyperbolic.
However, the universal covering map $C = \tilde{T}_i \to T_i$ of any irreducible component $T_i$ of $T$ provides a non-constant holomorphic map $C \to X' \to X$ and contradicts the Kobayashi hyperbolicity of $X$. The above argument establishes that any fibration $X \to C$ of the minimal model $X$ of the toroidal compactification $X' = (\mathbb{B}^2/\Gamma)'$ of a torsion free $\mathbb{B}^2/\Gamma$ has base $C$ of genus $g(C) = 0$ or $1$.

(ii) Let us assume the opposite and recall that the Albanese variety of a compact Kähler manifold $M$ is the compact complex torus

$$\text{Alb}(M) := H^0(M, \Omega^1_M)^*/H_1(M, \mathbb{Z}),$$

where $\Omega^1_M$ stands for the sheaf of the holomorphic differential $1$-forms on $M$. Note that $\text{Alb}(M)$ is of dimension $\dim_{\mathbb{C}} \text{Alb}(M) = h^{0,1}(M) = q(M)$. For an elliptic curve $C$, the Albanese variety $\text{Alb}(C) = \text{Jac}(C) = C$. The correspondence

$$p \mapsto (\omega \mapsto \int_{\gamma} \omega) \quad \text{for} \quad \forall p \in M, \quad \forall \omega \in H^0(M, \Omega^1_M)$$

induces the Albanese map $\text{alb}_M : M \to \text{Alb}(M)$. By Beauville’s [3], if $f : X \to C$ is a fibration of a smooth minimal surface $X$ with elliptic generic fibre $F$ and elliptic base $C$ then either $\text{Alb}(X) = \text{Jac}(C) = C$ or there is an exact sequence

$$0 \longrightarrow F' \longrightarrow \text{Alb}(X) \longrightarrow C \longrightarrow 0 \quad (12)$$

of abelian varieties for an elliptic curve $F'$, isogeneous to $F$. Moreover, $\text{Alb}(X) = C$ exactly when the irregularity $q(X) = g(C) = 1$, while $q(X) = g(C) + 1$ is equivalent to (12).

Let us assume that there holds (12). By the universal property of the Albanese variety $\text{Alb}(X)$, the surjective morphism $f : X \to C$ onto the compact complex torus $C$ admits a factorization

$$\begin{array}{cccc}
X & \overset{\text{alb}_X}{\longrightarrow} & \text{Alb}(X) & \longrightarrow C & \longrightarrow 0 \\
\downarrow f & & \downarrow f_o & & \\
C & & & & 
\end{array}$$

through the Albanese map $\text{alb}_X$ of $X$ and a surjective morphism $f_o : \text{Alb}(X) \to C$ of compact complex tori. Therefore all the fibres $f_o^{-1}(p)$, $p \in C$ of $f_o$ are compact complex tori of $\dim_{\mathbb{C}} f_o^{-1}(p) = \dim_{\mathbb{C}} \text{Alb}(X) - \dim_{\mathbb{C}} C = 1$, i.e., elliptic curves.

The image $\text{alb}_X(X)$ of the irreducible projective surface $X$ under its Albanese map $\text{alb}_X$ is a closed irreducible subvariety of $\text{Alb}(X)$. If $\dim_{\mathbb{C}} \text{alb}_X(X) \leq 1$ then $\dim_{\mathbb{C}} \text{alb}_X(X) = \dim_{\mathbb{C}} f_o^{-1}(\text{alb}_X(X)) = \dim_{\mathbb{C}} \text{alb}_X(X) - \dim_{\mathbb{C}} f_o^{-1}(p) \leq 0$ for any $p \in C$. The contradiction justifies that $\dim_{\mathbb{C}} \text{alb}_X(X) = 2 = \dim_{\mathbb{C}} \text{Alb}(X)$, whereas $\text{alb}_X(X) = \text{Alb}(X)$. Now, the surjective morphism $\text{alb}_X : X \to \text{Alb}(X)$ is a finite covering of compact complex surfaces.
For an arbitrary \( p \in C \) the elliptic fibre \( f^{-1}_o(p) \) of \( f_o \) is a finite quotient of the fibre \( f^{-1}(p) \) of \( f \), due to the presence of a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_X & \xrightarrow{alb_X} & \mathcal{O}_{X(h)}(1) \\
\downarrow f & & \downarrow f_o \\
C & & C
\end{array}
\]

of surjective morphisms. Therefore the irreducible components of \( f^{-1}(p) \) are of genus \( \geq 1 \) and the elliptic fibration \( f : X \to C \) has no singular fibres. Note that the finite ramified coverings of the elliptic curves \( f^{-1}_{\alpha}(p) \) are of genus \( \geq 2 \). Therefore, the finite coverings \( alb_X : f^{-1}(p) \to alb_X(f^{-1}(p)) \) are unramified for \( \forall p \in C \) and the Albanese map \( alb_X : X \to Alb(X) \) is a finite unramified covering. As a result, \( X \) is of Kodaira dimension \( \kappa(X) = \kappa(Alb(X)) = 0 \), contrary to the assumption \( \kappa(X) = 1 \). The contradiction justifies the non-existence of torsion-free toroidal compactifications \( X' = (\mathbb{B}^2/\Gamma)' \), whose minimal models \( X \) are elliptic fibrations \( f : X \to C \) with elliptic base and \( q(X) = g(C) + 1 = 2 \).

\[ \square \]

Let \( f : X \to C \) be an elliptic fibration of a smooth minimal projective surface \( X \) with multiple fibers \( m_iF_i \), \( 1 \leq i \leq k \) and a base \( C \) of genus \( g \). By Dürer’s Theorem 2.9 [10], if the holomorphic Euler characteristic \( \chi(O_X) > 0 \) then the fundamental group of \( X \) is isomorphic to the orbifold fundamental group of \( C \),

\[
\pi_1^{Orb}(C) = \langle a_1, b_1, \ldots, a_g, b_g, s_1, \ldots, s_k \mid s_1^{m_1}, \prod_{i=1}^{a_i} a_i \prod_{j=1}^{b_j} b_j \rangle.
\]

If \( \chi(X) = 0 \) then the fundamental group \( \pi_1(F) \simeq (\mathbb{Z}^2, +) \) of a generic fiber \( F \) of \( f \) embeds in \( \pi_1(X) \) as a normal subgroup with quotient group \( \pi_1(X)/\pi_1(F) = \pi_1^{Orb}(C) \). Let \( p_q(X) = h^{0,1}(X) = h^{1,0}(X) \) be the geometric genus and \( q(X) = h^{1,1}(X) \) be the irregularity of \( X \). By Noether’s formula for a smooth compact complex surface \( X \), the holomorphic Euler characteristic

\[
\chi(O_X) = p_q(X) - q(X) + 1 = \frac{K_X^2 + e(X)}{12}.
\]

In the case of \( \kappa(X) = 1 \) one has \( K_X^2 = 0 \) and \( e(X) \geq 0 \), so that \( \chi(X) > 0 \) exactly when \( e(X) \in 12\mathbb{N} \) and \( \chi(X) = 0 \) is equivalent to \( e(X) = 0 \). In such a way, we have justified the following

**Corollary 17.** (Dürer, Theorem 2.9, [10]) Let \( \Gamma < SU_{2,1} \) be a torsion free non-uniform lattice, whose quotient \( \mathbb{B}^2/\Gamma \) has toroidal compactification \( X' = (\mathbb{B}^2/\Gamma)' \) of Kodaira dimension \( \kappa(X') = 1 \) and \( f : X \to C \) be an elliptic fibration of the smooth projective minimal model \( X \) of \( X' \) with multiple fibers \( m_iF_i \), \( 1 \leq i \leq k \). Then:

(i) if \( e(X) > 0 \) then \( e(X) \in 12\mathbb{N} \), the fundamental group

\[
\pi_1(X) \simeq \pi_1^{Orb}(C) \simeq \langle s_1, \ldots, s_k \mid s_1^{m_1}, s_1 \ldots s_k \rangle \quad \text{or}
\]

(ii) if \( e(X) = 0 \) then \( \kappa(X) \geq 2 \), the fundamental group

\[
\pi_1(X) \simeq \pi_1^{Orb}(C) \simeq \langle s_1, \ldots, s_k \mid s_1^{m_1}, s_1 \ldots s_k \rangle \quad \text{or}
\]

(iii) if \( e(X) < 0 \) then \( \kappa(X) < 2 \), the fundamental group

\[
\pi_1(X) \simeq \pi_1^{Orb}(C) \simeq \langle s_1, \ldots, s_k \mid s_1^{m_1}, s_1 \ldots s_k \rangle \quad \text{or}
\]

27
\[\langle a_1, b_1, s_1, \ldots, s_k \mid s_i^{m_i}, [a_1, b_1]s_1 \ldots s_k \rangle\]

and the irregularity \(q(X) = g(C) = 0 \text{ or } 1;\)

(ii) if \(e(X) = 0\) then the fundamental group \(\pi_1(F) \simeq \langle \mathbb{Z}^2, + \rangle\) of a generic fiber \(F\) of \(f\) embeds in \(\pi_1(X)\) as a normal subgroup with quotient group

\[\pi_1(X)/\pi_1(F) \simeq \pi_1^{orb}(C) = \langle s_1, \ldots, s_k \mid s_i^{m_i}, s_1 \ldots s_k \rangle\]

or

\[\langle a_1, b_1, s_1, \ldots, s_k \mid s_i^{m_i}, [a_1, b_1]s_1 \ldots s_k \rangle\]

and the irregularity \(q(X) \in \{g(C), g(C) + 1\} \cap \{0, 1\}.

5.2 Toroidal compactifications \((\mathbb{B}^2/\Gamma)'\) of minimal volume

Hirzebruch has shown in [15], [16] that the volume of a complex ball quotient \(\mathbb{B}^n/\Gamma\) by a torsion free lattice \(\Gamma < SU_{n,1}\) is

\[
\text{vol}(\mathbb{B}^n/\Gamma) = \frac{(-\pi)^{n/2}}{n+1} e(\mathbb{B}^n/\Gamma),
\]

where \(e(\mathbb{B}^n/\Gamma)\) stands for the Euler number and the invariant metric is normalized in such a way that the holomorphic sectional curvature equals \(-1\). In particular,

\[
\text{vol}(\mathbb{B}^2/\Gamma) = \frac{8\pi^2}{3} e(\mathbb{B}^2/\Gamma) \tag{13}
\]

with \(e(\mathbb{B}^2/\Gamma) \in \mathbb{N}.

Let \(\mathbb{Q}(\sqrt{-d})\) be an imaginary quadratic number field with integers ring \(\mathcal{O}_{-d}\). Denote by \(PU_{n,1}(\mathcal{O}_{-d})\) the projective unitary group with entries from \(\mathcal{O}_{-d}\). The lattices \(\Gamma\) of \(PU_{n,1}\), which are commensurable with \(PU_{n,1}(\mathcal{O}_{-d})\) are called Picard modular over \(\mathbb{Q}(\sqrt{-d})\). Any arithmetic lattice of \(PU_{n,1}\) is Picard modular over \(\mathbb{Q}(\sqrt{-d})\) for some \(d \in \mathbb{N}.

In [14] Hersonsky and Paulin show that the minimal volume of a compact torsion free \(\mathbb{B}^2/\Gamma_o\) is \(8\pi^2\). For a noncompact torsion free \(\mathbb{B}^2/\Gamma\), Parker establishes in [32] that \(\text{vol}(\mathbb{B}^2/\Gamma) \geq \frac{8\pi^2}{3}\) and constructs two examples, attaining this lower bound. By Stover’s [36], the minimal volume of an arithmetic (eventually torsion) \(\mathbb{B}^2/\Gamma\) is \(\frac{\pi^2}{27}\).

The subsequent work [11] of Emery and Stover generalizes [36] and expresses the minimal volume \(V_{-d,n}\) of a quotient \(PU_{n,1}/\Gamma\) by a non-uniform Picard modular lattice \(\Gamma < PU_{n,1}\) over \(\mathbb{Q}(\sqrt{-d})\) by the L-function \(L_{-d} = \frac{\zeta_{-d}}{\zeta}\), associated with Dedekind zeta function \(\zeta_{-d}\) of \(\mathbb{Q}(\sqrt{-d})\) and the Riemann zeta function \(\zeta\). It provides an estimate the number of the isomorphism classes of the Picard modular lattices \(\Gamma < PU_{n,1}\) over \(\mathbb{Q}(\sqrt{-d})\) with minimal volume \(\text{vol}(PU_{n,1}/\Gamma) = V_{-d,n}\). For all \(n \geq 2\), the non-uniform arithmetic lattices \(\Gamma < PU_{n,1}\) of smallest \(\text{vol}(PU_{n,1}/\Gamma) = V_{-d,n}\) are shown to be Picard modular over \(\mathbb{Q}(\sqrt{-3})\). For an odd \(n = 2k + 1 \neq 7(\text{mod } 8)\) there is a unique isomorphism class of non-uniform lattices of \(PU_{2k+1,1}\) with minimal co-volume \(V_{-3,2k+1}\). For an even \(n = 2k\), there are exactly two isomorphism classes of non-uniform lattices of \(PU_{2k,1}\) with minimal co-volume \(V_{-3,2k}\).
Let $\Gamma_1', \Gamma_2' < SU_{2,1}(\mathcal{O}_{-3})$ be the non-conjugate torsion Picard modular lattices with minimal co-volume

$$\text{vol}(\mathbb{B}^2/\Gamma_1') = \text{vol}(\mathbb{B}^2/\Gamma_2') = \frac{\pi^2}{27}.$$ 

In [36] Stover shows that any arithmetic torsion free lattice $\Gamma < SU_{2,1}$, whose quotient has minimal volume $\text{vol}(\mathbb{B}^2/\Gamma) = \frac{8\pi^2}{3}$ is contained in $\Gamma_1'$ or in $\Gamma_2'$. He provides a complete list $\Gamma_j$, $1 \leq j \leq 8$, of the torsion free $\Gamma_j < SU_{2,1}$ with $\text{vol}(\mathbb{B}^2/\Gamma_j) = \frac{8\pi^2}{3}$, which are contained in $\Gamma_1' \cap \Gamma_2'$. More precisely, the generators of $\Gamma_j$ are expressed by the two generators of $SU_{2,1}(\mathcal{O}_{-3})$, found by Falbel and Parker in [12]. Moreover, [36] specifies the abelianizations $ab\Gamma_j = \Gamma_1'/[\Gamma_j, \Gamma_j]$ of $\Gamma_j$ and the number $h_j$ of the cusps of $\mathbb{B}^2/\Gamma_j$. The next table recalls Stover’s examples from [36] by listing the number $k_j$ of the generators of $\Gamma_j$, $ab\Gamma_j$ and $h_j$.

| $\Gamma_j$ | $k_j$ | $h_j$ | $ab(\Gamma_j) = \Gamma_1'/[\Gamma_j, \Gamma_j]$ |
|------------|-------|-------|---------------------------------------------|
| $\Gamma_1$ | 3     | 4     | $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_9$ |
| $\Gamma_2$ | 5     | 4     | $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ |
| $\Gamma_3$ | 4     | 4     | $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ |
| $\Gamma_4$ | 3     | 2     | $\mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}$ |
| $\Gamma_5$ | 3     | 2     | $\mathbb{Z} \times \mathbb{Z}$ |
| $\Gamma_6$ | 2     | 2     | $\mathbb{Z}_3 \times \mathbb{Z}_9$ |
| $\Gamma_7$ | 3     | 2     | $\mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}$ |
| $\Gamma_8$ | 3     | 2     | $\mathbb{Z} \times \mathbb{Z}$ |

The next Corollary 18 shows that if $\Gamma < SU_{2,1}$ is a torsion free non-uniform lattice with $\text{vol}(\mathbb{B}^2/\Gamma) = \frac{8\pi^2}{3}$ then the minimal model $X$ of $X' = (\mathbb{B}^2/\Gamma)'$ is either an abelian surface or an elliptic fibration $f: X \to C$ of Kodaira dimension $\kappa(X) = 1$ with $e(X) = 0$ and a base $C$ of genus 0 or 1. It determines the Kodaira-Enriques classification types of the minimal models $X_j$ of Stover’s examples $X_j' = (\mathbb{B}^2/\Gamma_j)'$. In such a way, it establishes that all the possibilities for the noncompact torsion free $\mathbb{B}^2/\Gamma$ of $\text{vol}(\mathbb{B}^2/\Gamma) = \frac{8\pi^2}{3}$ do really occur.

**Corollary 18.** Let $\Gamma < SU_{2,1}$ be a torsion free non-uniform lattice, whose quotient has minimal volume $\text{vol}(\mathbb{B}^2/\Gamma) = \frac{8\pi^2}{3}$ with respect to the invariant metric with holomorphic sectional curvature $-1$. Then:

(i) the minimal model $X$ of the toroidal compactification $X' = (\mathbb{B}^2/\Gamma)'$ is either an abelian surface or an elliptic fibration $f: X \to C$ of Kodaira dimension $\kappa(X) = 1$ with Euler number $e(X) = 0$ and a base $C$ of genus $g(C) = 0$ or 1;

(ii) the toroidal compactification $X' = (\mathbb{B}^2/\Gamma)'$ has one rational $(-1)$-curve $L$ and $1 \leq h \leq 4$ irreducible components of the toroidal compactifying divisor $T = X'\setminus(\mathbb{B}^2/\Gamma)$;

(iii) in the case of an abelian minimal model $X$, the toroidal compactifying divisor $T$ has $h = 4$ irreducible components $T_i$ and $L.T_i = 1$ for all $1 \leq i \leq 4$.

The aforementioned possibilities are realized by Stover’s examples form [36]. Namely, the minimal models $X_j$ of the toroidal compactifications $X_j' = (\mathbb{B}^2/\Gamma_j)'$ of $\Gamma_j$, $1 \leq j \leq 8$ from the Appendix of [36] are as follows: For $j \in \{1, 2, 6\}$ there are elliptic
fibrations $f_j : X_j \rightarrow \mathbb{P}^1$ with $\kappa(X_j) = 1$, $e(X_j) = 0$, $q(X_j) = 0$. The surface $X_3$ is abelian. The surfaces $X_j$ with $j \in \{4, 5, 7, 8\}$ are elliptic fibrations $f_j : X_j \rightarrow C_j$ of $\kappa(X_j) = 0$, $e(X_j) = 0$, $q(X_j) = 1$ with an elliptic or a rational base $C_j$.

Proof. According to Hirzebruch’s formula (13) and Parker’s result $\text{vol}(\mathbb{P}^2 / \Gamma) \geq \frac{8\pi^2}{3}$, the problem reduces to the description of the minimal models $X$ of $X' = (\mathbb{P}^2 / \Gamma)'$ with $e(\mathbb{P}^2 / \Gamma) = 1$. Bearing in mind that the toroidal compactifying divisor $T = (\mathbb{P}^2 / \Gamma)' \setminus (\mathbb{P}^2 / \Gamma)$ is a finite disjoint union of elliptic curves, one observes that $e(\mathbb{P}^2 / \Gamma) = e(X')$. If $X$ contains $s \geq 0$ rational $(-1)$-curves then $1 = e(X') = e(X) + s$. Any smooth minimal complex projective surface $X$ has Euler number $e(X) \geq 0$. Thus, either $e(X) = 1$, $s = 0$ or $e(X) = 0$, $s = 1$. By Kodaira-Enriques classification of the smooth minimal projective surfaces and Corollary 17, there is no $X$ with $e(X) = 1$. In the case of $e(X) = 0$, one makes use of $\kappa(X) \geq 0$ by Lemma 15 (i), in order to conclude that $X$ is an abelian, hyperelliptic or of Kodaira dimension $\kappa(X) = 1$. Momot’s [27] establishes that if $\kappa(X) \leq 0$ and $q(X) \geq 1$ then $X$ is an abelian surface. That excludes the case of a hyperelliptic $X$. Proposition 16 specifies that for an elliptic fibration $f : X \rightarrow C$ of Kodaira dimension $\kappa(X) = 1$, the base $C$ is a rational or an elliptic curve.

Let $h$ be the number of the disjoint smooth elliptic irreducible components $T_i$ of the toroidal compactifying divisor $T = (\mathbb{P}^2 / \Gamma)' \setminus (\mathbb{P}^2 / \Gamma)$. Holzapfel’s proportionality from [17] asserts that

$$3e(X) - K_X^2 = \sum_{i=1}^{h} (-T_i^2) - 4s.$$  

For an abelian surface $X$ or a minimal elliptic fibration $f : X \rightarrow C$ of $\kappa(X) = 1$ one has $K_X^2 = 0$. Combining with $s = 1$, $e(X) = 0$, one concludes that

$$\sum_{i=1}^{h} (-T_i^2) = 4.$$  

(14)

An abelian surface $X$ does not support rational curves. Therefore, the blow-down $\xi : X' \rightarrow X$ of the smooth rational $(-1)$-curve $L$ on $X'$ transforms $T_i$ into smooth elliptic curves $D_i = \xi(T_i)$ on $X$ and $L.T_i \leq 1$. By adjunction formula,

$$0 = -e(D_i) = D_i(D_i + K_X) = D_i(D_i + \mathcal{O}_X) = D_i^2.$$  

The contractibility of the toroidal compactifying divisor $T$ of $X'$ requires $T_i^2 \leq -1$. That is why, the blown-up point $\xi(L) \in X$ belongs to $D_i$ for all $1 \leq i \leq h$ and $T_i^2 = D_i^2 - 1 \leq -1$, $L.T_i = 1$. As a result, $h = 4$.

If $f : X \rightarrow C$ is an elliptic fibration of $\kappa(X) = 1$ and $e(X) = 0$ then (14) and $T_i^2 \leq -1$ imply $h \leq \sum_{i=1}^{h} (-T_i^2) = 4$.

In order to classify the minimal models $X_j$ of Stover’s examples $X_j' = (\mathbb{P}^2 / \Gamma_j)'$, note that $\text{rk}_Z H_1(X_j, \mathbb{Z}) = \text{rk}_Z H_1(X_j', \mathbb{Z})$ is a birational invariant. Combining with Corollary 5, one concludes that

$$\text{rk}_Z H_1(X_j, \mathbb{Z}) = \text{rk}_Z H_1(X_j', \mathbb{Z}) = \text{rk}_Z(ab \Gamma_j).$$
Thus, $X_j$ is an abelian surface exactly when $r_k(ab\Gamma_j) = 4$ and an elliptic fibration $f_j : X_j \to C_j$ of $\kappa(X_j) = 1$ and $e(X_j) = 0$ for $r_k(ab\Gamma_j) \in \{2, 0\}$. More precisely, $X_3$ is an abelian surface with $\pi_1(X_3) = H_1(X_3, \mathbb{Z}) \cong (\mathbb{Z}^4, +)$ and all the other $X_j$, \( j \in \{1, \ldots, 8\} \setminus \{3\} \) are elliptic fibrations with $\kappa(X_j) = 1$ and $e(X_j) = 0$. Note that $r_k(ab\Gamma_j) = 0$ for $j \in \{1, 2, 6\}$, so that the corresponding $X_j$ have vanishing irregularity $q(X_j) = 0$. That requires the bases $C_j = \mathbb{P}^1(\mathbb{C})$ of the elliptic fibrations $f_j : X_j \to C_j$ to be rational for $j \in \{1, 2, 6\}$. In the case of $j \in \{4, 5, 7, 8\}$ one has $q(X_j) = 1$ and the base $C_j$ of $f_j : X_j \to C_j$ could be either elliptic or a rational curve.

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