ON STABLE MODULES THAT ARE NOT GORENSTEIN PROJECTIVE

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Abstract. In [AB], Auslander and Bridger introduced Gorenstein projective modules and only about 40 years after their introduction a finite dimensional algebra $A$ was found in [JS] where the subcategory of Gorenstein projective modules did not coincide with $\perp A$, the category of stable modules. The example in [JS] is a commutative local algebra. We explain why it is of interest to find such algebras that are non-local with regard to the homological conjectures. We then give a first systematic construction of algebras where the subcategory of Gorenstein projective modules does not coincide with $\perp A$ using the theory of gendo-symmetric algebras. We use Liu-Schulz algebras to show that our construction works to give examples of such non-local algebras with an arbitrary number of simple modules.

Introduction

Let $A$ always be a finite dimensional connected non-semisimple algebra over a field $K$ and modules are finite dimensional right modules if nothing is stated otherwise. For a module $M$ we have the left module $D(M) := \text{Hom}_K(M,K)$. Recall that the dominant dimension of a module $M$ is defined as $\text{domdim}(M) := \sup \{ \text{pd}(M) | \text{pd}(M) < \infty \}$. The dominant dimension of a module $M$ with minimal injective coresolution $(I_i)$ is defined as $\text{domdim}(M) = 0$ if the injective envelope of $M$ is not projective and $\text{domdim}(M) = \sup \{ n \geq 0 | I_i$ is projective for $i = 0,1,...,n \} + 1$ in case $I_0$ is projective. The codominant dimension of a module $M$ is defined as the dominant dimension of $D(M)$. The dominant dimension of an algebra is defined as the dominant dimension of the regular module. Recall the following famous homological conjectures, which are all open:

1. (Finitistic dimension conjecture) The finitistic dimension of an algebra is finite.
2. (Strong Nakayama conjecture) For any non-zero module $M$ we have $\text{Ext}^i(M,A) \neq 0$ for some $i \geq 0$.
3. (Generalised Nakayama conjecture) For any simple module $M$ we have $\text{Ext}^i(M,A) \neq 0$ for some $i \geq 0$.
4. (Nakayama conjecture) Any non-selfinjective algebra has finite dominant dimension.
5. (First Tachikawa conjecture) For any non-selfinjective algebra, there is $i \geq 1$ with $\text{Ext}^i(D(A),A) \neq 0$.
6. (Second Tachikawa conjecture) For any selfinjective algebra and non-projective module $M$, there is $i \geq 1$ with $\text{Ext}^i(M,M) \neq 0$.

It is well known that one has the implications $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 4)$ and that $4)$ is equivalent to $5)$ and $6)$, see for example [Yam].

Recall that a module $M$ is called Gorenstein projective in case $\text{Ext}^i(M,A) = 0$ and $\text{Ext}^i(D(A),\tau(M)) = 0$ (here $\tau$ denotes the Auslander-Reiten translate) for all $i \geq 1$ and it is called Gorenstein injective in case $D(M)$ is Gorenstein projective. A module $M$ is called a stable module in case $\text{Ext}^i(M,A) = 0$ for all $i \geq 1$ and it is called a costable module in case $D(M)$ is stable. We write $\perp N := \{ M | \text{Ext}^i(M,N) = 0 \}$ and $N^\perp := \{ M | \text{Ext}^i(N,M) = 0 \}$ for all $i \geq 1$. Following [Mar2], we call a finite dimensional algebra nearly Gorenstein in case the Gorenstein projective modules coincide with the stable modules and dually the Gorenstein injective modules coincide with the costable modules. For example all Gorenstein algebras are nearly Gorenstein. In [Mar2] it was shown that every nearly Gorenstein algebra satisfies the strong Nakayama conjecture and thus also the generalized Nakayama conjecture. Thus in order to really challenge those conjecture, one first has to find a systematic way to
construct non-nearly Gorenstein algebras. This means that one has to construct stable modules that are not Gorenstein projective. Only around 40 years after the definition of Gorenstein projective modules by Auslander and Bridger in [AB], a finite dimensional algebra was found that is not nearly Gorenstein in [JS]. However, the algebra was a local algebra and thus not of interested for the homological conjectures, since the homological conjectures are trivially true for local algebras (local algebras have finitistic dimension equal to zero). Thus it is of interest to find a construction of non-local and non-nearly Gorenstein algebras to challenge the homological conjectures and study non-Gorenstein projective stable modules.

In our first section we introduce two new homological conjectures between the finitistic dimension conjecture and the first Tachikawa conjecture and we look at the finitistic dominant dimension of an algebra, that is defined as the supremum of all dominant dimensions of modules having finite dominant dimension. The finitistic codominant dimension is defined dually. This is an analog of the finitistic conjecture and the first Tachikawa conjecture and we look at the finitistic dominant dimension of an algebras to challenge the homological conjectures and study non-Gorenstein projective stable modules.

The second section gives our main theorem, see 2.3

**Theorem.** Let $A$ be a symmetric algebra and $X$ a direct sum of indecomposable non-projective modules. Let $M$ be an indecomposable module such that $\text{Ext}^l(X, M) \neq 0$ for some $l \geq 1$ and $\text{Ext}^l(X, M) = 0$ for all $i \geq l + 1$. Then the gendo-symmetric algebra $B := \text{End}_A(A \oplus X)$ is not nearly Gorenstein and has infinite finitistic codominant dimension.

We remark that in the situation of the previous theorem, our proof of the theorem will show that the $B$-module $\text{Hom}_A(A \oplus X, \Omega^{-l}(M))$ will be a costable module but not Gorenstein injective. Thus applying the duality one also has a stable module over the opposite algebra of $B$ that is not Gorenstein projective. After proving the previous theorem, the rest of the second section is devoted to give a concrete construction of non-local algebras that are not nearly Gorenstein with an arbitrary number of simple modules.

Liu-Schulz example algebras (we call them Liu-Schulz algebras in the following) are defined as $K < x, y, z > / (x^2, y^2, z^2, xy + qx, xz + qxy, yz + qyx)$ with $q$ being a non-zero field element. They are local Frobenius algebras and were first studied in [LS]. For more information on those algebras we refer for example to [Rin] and [Sm]. They appear also as the trivial extension algebras of the algebras $K < x, y > / (x^2, y^2, xy + axy)$ for $a$ being a non-zero field element. We call the algebras of the form $K < x, y > / (x^2, y^2, xy + agx)$ quantum 2-exterior algebras in the following.

Liu-Schulz algebras and quantum 2-exterior algebras have appeared often in the representation theory of algebras as construction of counterexamples to conjectures or examples of exotic behavior of algebras. We mention three such examples:

- (1) The first non-periodic modules with syzygies of bounded dimension were found over quantum 2-exterior algebras by Schulz in [Sch].
- (2) A conjecture on the shape of Auslander-Reiten components was proven to be wrong using Liu-Schulz algebras in [LS].
- (3) A question of Happel about finite Hochschild cohomology and global dimension was shown to have a negative answer using quantum 2-exterior algebras in [BCMS].

We will look at certain endomorphism rings of generators over Liu-Schulz algebras. Those algebras are gendo-symmetric algebras in the sense of Fang and Koenig (see [FanKoe]) and we will use the theory of gendo-symmetric algebras and dominant dimension for the construction of non-nearly Gorenstein algebras and non-Gorenstein injective costable modules. We also assume knowledge of basic representation theory and Auslander-Reiten theory of finite dimensional algebras. See for example [SkoYam] and [TaN]. The author is thankful to Osamu Iyama for a useful suggestion.

1. Finitistic dominant dimension and two new conjectures

The finitistic dominant dimension of $A$ is defined as $\text{findomdim}(A) := \text{sup} \{ \text{domdim}(M) | \text{domdim}(M) < \infty \}$, this was first introduced in [Mar]. Dually, one can define the finitistic codominant dimension as the supremum of all codominant dimensions of modules having finite codominant dimension. We will show in forthcoming work that the finitistic dominant and codominant dimension of large classes of algebras, including monomial algebras, is always finite. We have the following easy lemma:
Lemma 1.1. \(1\) Let \(A\) be an algebra of finite finitistic dimension, then every non-projective module \(M\) with infinite dominant dimension has infinite projective dimension.

\(2\) Let \(A\) be an algebra of finite finitistic codominant dimension, then every non-projective module \(M\) with infinite dominant dimension has infinite codominant dimension.

Proof. \(1\) Assume there is a non-projective module \(M\) with finite projective dimension and infinite dominant dimension. Then the modules \(\Omega^{-1}(M)\) have arbitrary large finite projective dimension and thus the finitistic dimension is infinite.

\(2\) Assume there is a non-projective module \(M\) with finite codominant dimension and infinite dominant dimension. Then the modules \(\Omega^{-1}(M)\) have arbitrary large finite codominant dimension and thus the finitistic codominant dimension is infinite.

Note that in case one algebra \(A\) has infinite finitistic dominant dimension then the opposite algebra has infinite finitistic codominant dimension.

The previous lemma motivates to study modules of infinite dominant dimension and look at their minimal projective resolution in order to test algebras for finite finitistic dimension and finite finitistic codominant dimension. As an analog to the finitistic dimension conjecture, one can ask whether the finitistic dominant dimension is always finite. We will give a negative answer in the next section using the codominant dimension. As an analog to the finitistic dimension conjecture, one can ask whether the minimal projective resolution in order to test algebras for finite finitistic dimension and finite finitistic codominant dimension is always finite.

We define the \(\text{add}(M)\)-resolution dimension of \(N\) as the minimal \(n\) such that there is an exact sequence of the form

\[0 \to M_n \xrightarrow{f_n} M_{n-1} \cdots M_0 \xrightarrow{f_0} N \to 0,\]

where the maps \(f_0\) and \(f_n : M_n \to K_n\) are all minimal right approximations when \(K_n\) denotes the kernel of \(f_n-1\).

We also need the following well-known theorem of Mueller, see [Mu]:

Theorem 1.2. Let \(A\) be an algebra with generator-cogenerator \(M\) and \(B := \text{End}_A(M)\). Let \(N\) be an \(A\)-module. Then the \(B\)-module \(\text{Hom}_A(M, N)\) has dominant dimension equal to \(\inf\{i \geq 1 | \text{Ext}^i(M, N) \neq 0\}\) + 1. Especially: The dominant dimension of \(B\) equals \(\inf\{i \geq 1 | \text{Ext}^i(M, M) \neq 0\}\) + 1.

We will also use that in the situation of the previous theorem, the functor \(\text{Hom}_A(M, -)\) is an equivalence between \(\mod - A\) and the full subcategory \(\text{Dom}_2\) of \(\mod - B\) consisting of modules of dominant dimension at least 2, see for example [APT] section 3.

The next definition is taken from [CIM], where such algebras are used to give a new characterisation of representation-finite hereditary algebras.

Definition 1.3. Let \(A\) be an algebra, then define the SGC-extension algebra of \(A\) to be the algebra \(\text{End}_A(D(A) \oplus A)\), that is obtained from \(A\) by taking the endomorphism ring of the smallest generator-cogenerator.

Proposition 1.4. Look at the following four statements:

\(1\) (Finitistic dimension conjecture) Any algebra has finite finitistic dimension.

\(2\) Any non-projective module with infinite dominant dimension has infinite projective dimension.

\(3\) Any non-injective costable module has infinite \(\text{add}(A \oplus D(A))\)-resolution dimension.

\(4\) (First Tachikawa conjecture) For any non-selfinjective algebra, there is an \(i \geq 1\) with \(\text{Ext}^i(D(A), A) \neq 0\).

We have \((1) \implies (2) \implies (3) \implies (4)\).

Proof. We saw \((1) \implies (2)\) already in [1.1]. Assume now \((2)\) and let \(M\) be a costable module with finite \(\text{add}(A \oplus D(A))\)-resolution dimension. The defining condition \(\text{Ext}^i(D(A), M) = 0\) for all \(i \geq 1\) gives that the module \(\text{Hom}_A(D(A) \oplus A, M)\) has infinite dominant dimension over the SGC-extension algebra \(B = \text{End}_A(D(A) \oplus A)\), by [1.2]. Applying the functor \(\text{Hom}_A(A \oplus D(A), -)\) to a finite \(\text{add}(A \oplus D(A))\)-resolution of \(M\), one obtains that the \(B\)-module \(\text{Hom}_A(D(A) \oplus A, M)\) has finite projective dimension, contradicting \((2)\). Now assume \((3)\) and \(\text{Ext}^i(D(A), A) = 0\) for all \(i \geq 1\) in a non-selfinjective algebra. Then
the module $M = A$ is a costable module with finite $add(A \oplus D(A))$-resolution dimension, contradicting (3).

The previous proposition motivates us to state (2) and (3) as new homological conjectures:

**Conjecture.**

1. Any non-projective module with infinite dominant dimension has infinite projective dimension.
2. Any non-injective costable module has infinite $add(A \oplus D(A))$-resolution dimension.

Those conjectures motivate us to study modules of infinite dominant dimension and costable modules.

We prove the conjecture (2) for Gorenstein algebras:

**Proposition 1.5.** Let $A$ be a Gorenstein algebra, then any non-injective costable module has infinite $add(A \oplus D(A))$-resolution dimension.

**Proof.** First assume $A$ has Gorenstein dimension equal to zero, which is equivalent to $A$ being selfinjective. In this case $add(A \oplus D(A))$-resolutions coincide with projective resolutions and the result follows since any non-projective module over a selfinjective algebra has infinite projective dimension. Assume $A$ is a non-selfinjective Gorenstein algebra for the rest of the proof. We use three known facts about Gorenstein algebras, for proofs see for example [Che]:

1. An $A$-module $N$ has finite injective dimension iff it has finite projective dimension.
2. An $A$-module $N$ is Gorenstein projective iff it is a costable module and an $A$-module $N$ is Gorenstein injective iff it is a costable module.
3. A non-projective Gorenstein projective module has infinite projective dimension and a non-injective Gorenstein injective module has infinite injective dimension.

Assume $N$ is a non-injective costable module with finite $add(A \oplus D(A))$-resolution dimension. Then there is an $add(A \oplus D(A))$-resolution of the form: $0 \to M_n \xrightarrow{f_n} \cdots M_0 \xrightarrow{f_0} N \to 0$ with $M_n \in add(A \oplus D(A))$. Let $K_r$ denote the kernel of $f_{r-1}$. Note that all $M_i$ have finite projective dimension, since $A$ is Gorenstein and thus $D(A)$ has finite projective dimension. Then there is a short exact sequence $0 \to M_n \to M_{n-1} \to K_{n-1} \to 0$, which shows that also $K_{n-1}$ has finite projective dimension. Using induction, we see that also $N$ has finite projective dimension. But since $A$ is Gorenstein, the costable modules coincide with the Gorenstein injective modules, which always have infinite injective dimension when they are non-injective. But for Gorenstein algebras, a module has infinite injective dimension iff it has infinite projective dimension and thus $N$ also has infinite projective dimension. This is a contradiction and thus $N$ has infinite $add(A \oplus D(A))$-resolution dimension.

2. **The Liu-Schulz algebra and non-nearly Gorenstein algebras**

We first need the following result in order to calculate the codominant dimension of algebras:

**Proposition 2.1.** Let $A$ be a selfinjective algebra and $M = A \oplus X$ a generator with $X$ having no projective direct summands. Let $B := \text{End}_A(M)$ and $Y := \text{Hom}_A(M, N)$ for an $A$-module $N$.

1. The codominant dimension of the $B$-module $Y$ equals $\inf\{i \geq 1 | Ext^1(M, \Omega^i(N)) \neq 0\} - 1$.
2. $Y$ has infinite dominant and infinite codominant dimension iff $Ext^1(M, \Omega^i(N)) = 0$ for all $i \in \mathbb{Z}$.

**Proof.** Note that minimal $add(M)$-approximations of an $A$-module $N$ correspond to the projective cover of the $B$-module $\text{Hom}_A(M, N)$ by applying the functor $\text{Hom}_A(M, -)$. Let $R$ be the direct sum of all projective-injective indecomposable $B$-modules. Then one has $add(R) = add(\text{Hom}_A(M, A))$. Thus the codominant dimension of the module $Y$ is at least one iff its projective cover is in $add(R)$ iff the minimal $add(M)$-approximation of $N$ is equal to the projective cover of $N$ in $mod - A$. Let $f : P \to N$ be the projective cover of $N$ in $mod - A$. Then this is a minimal $add(M)$-approximation iff $\text{Hom}_A(M, f)$ is surjective. Applying to the short exact sequence $0 \to \Omega^1(N) \to P \to N \to 0$ the functor $\text{Hom}_A(M, -)$ we get the exact sequence: $0 \to \text{Hom}_A(M, \Omega^1(N)) \to \text{Hom}_A(M, P) \to \text{Hom}_A(M, N) \to Ext^1(M, \Omega^1(N)) \to 0$. This gives us that the projective cover is the minimal $add(M)$-approximation iff $Ext^1(M, \Omega^1(N)) = 0$. We obtain (1) by applying the same argument to $\Omega^l(N)$ for $l \geq 1$. (2) is a consequence of (1) in combination with Mueller’s theorem [12].
Recall that an algebra $B$ is gendo-symmetric in case $B \cong \text{End}_A(M)$, when $A$ is a symmetric algebra and $M$ a generator-cogenerator. See [FanKoe] for more on such algebras. We need the following results in this section:

**Proposition 2.2.** Let $A$ be an algebra of dominant dimension $n \geq 1$.

(1) Then every Gorenstein projective module has dominant dimension at least $n$ and every Gorenstein injective module has codominant dimension at least $n$.

(2) Let $A$ be a gendo-symmetric algebra and $M$ an $A$-module. Then $M$ has dominant dimension at least two iff $M \cong \text{Hom}_A(D(A),M)$ and in this case the dominant dimension of $M$ is equal to $\inf\{i \geq 1|\text{Ext}^i(D(A),M) \neq 0\} + 1$.

**Proof.**

(1) See [Mar2], lemma 3.9.

(2) See [FanKoe], proposition 3.3.

We can now prove the main theorem of this article.

**Theorem 2.3.** Let $A$ be a symmetric algebra and $X$ a direct sum of indecomposable non-projective modules. Let $M$ be an indecomposable module such that $\text{Ext}^{i}(X, M) \neq 0$ for some $i \geq 1$ and $\text{Ext}^{i}(X, M) = 0$ for all $i \geq l + 1$. Then the gendo-symmetric algebra $B := \text{End}_A(A \oplus X)$ is not nearly Gorenstein and has infinite finitistic codominant dimension.

**Proof.** Let $N := A \oplus X$ and $B := \text{End}_A(N)$. Then $B$ is by definition gendo-symmetric. We now look at the $B$-module $R := \text{Hom}_A(N, \Omega^{-i}(M))$. Since we have $\text{Ext}^{i}(X, \Omega^{-i}(M)) = \text{Ext}^{i+1}(X, M) = 0$ for all $i \geq 1$, by [1] the module $R$ has infinite dominant dimension. Now using [2] $R$ has codominant dimension zero since $\text{Ext}^{i}(X, \Omega^{i}(\Omega^{-i}(M))) = \text{Ext}^{i}(X, \Omega^{-(i-1)}(M)) = \text{Ext}^{i}(X, M) = 0$. Now one has by [2] (2) that $R$ is a costable module, since modules $W$ with infinite dominant dimension over gendo-symmetric algebras have $\text{Ext}^{i}(D(A),W) = 0$ for all $i \geq 1$. By (1) of [2] it can not be Gorenstein injective since it has codominant dimension zero, while all Gorenstein injective modules have codominant dimension at least two. Thus $B$ is not nearly Gorenstein, as it contains a module that is costable but not Gorenstein injective. Now look at the $B$-modules $\Omega^{-i}(R)$ for $i \geq 1$. Since $R$ has infinite dominant dimension and codominant dimension zero, those modules have codominant dimension equal to $i$. Thus the algebra $B$ has infinite finitistic codominant dimension.

We remark that in the proof of the previous theorem we constructed a costable module $R$ over an algebra $B$ that is not Gorenstein injective. To obtain stable modules that are not Gorenstein projective, one can just look at $D(R)$ over the opposite algebra of $B$. We now construct non-local algebras with costable modules that are not Gorenstein injective with an arbitrary large number of simple modules. Let $A$ be the Liu-Schulz algebra, see for example Ringel [Rin] for an article about this algebra. That is $A = A_r := K < x, y, z >/(x^2, y^2, z^2, yx + rxy, zy + ryz, xz + rxz)$ for some non-zero field element $r$. We often write short $A$ instead of $A_r$. For non-zero field elements $c$, we define the right $A$-modules $M_c := A/(c + xy)A$. In the following $c, d, e, f$ will denote field elements. We assume that $r^2 \neq 1$ and $r^3 \neq 1$ for simplicity in the following.

We collect some results for this algebra and those modules, where we refer to Ringel for proofs:

**Proposition 2.4.** Let $A$ be the Liu-Schulz algebra, then

(1) $A$ is a local 8-dimensional symmetric algebra.

(2) $M_c$ is $4$-dimensional and $M_c$ is isomorphic to $M_d$ iff $c = d$.

Note that $\Omega^{i}(M_c) = (x + cy)A \cong M_{cr}$.

**Lemma 2.5.** Let $A$ be the Liu-Schulz algebra, then

(1) $\text{dim}(\text{Hom}_A(M_c, M_e)) = 2 + \delta_{c,e} + \delta_{cr^2,e}$, where $\delta_{s,t}$ is the Kronecker delta.

(2) $\text{Ext}^{i}(M_c, M_d) = 0$ iff $d \neq c, cr \neq d, cr^2 \neq d$ and $cr^3 \neq d$.

(3) $\text{Ext}^{i}(M_c, M_d) = 0$ for all $i \geq 1$ iff $d \neq r^2c$ for all $l \geq 0$.

**Proof.**

(1) We calculate $\text{dim}(\text{Hom}_A(A/(x + cy)A,(x + dy)A)))$, which gives the result by setting $e := dr$ since $(x + dy)A \cong A/(x + dxy)A$. The elements in $\text{Hom}_A(A/(x + cy)A,(x + dy)A)$ are of the form $t_z$, which is the homomorphism by right multiplication by an element $z \in (x + dy)A$
with \( z(x + cy)A = 0 \), which is equivalent to \( z(x + cy) = 0 \). Now a general \( z \in (x + dy)A \) can be written as \( z = a_1(x + dy) + a_2xy + a_3xyz + a_4(x + dyz) \) and the condition \( z(x + cy) = 0 \) leads to \( a_1(x + dy)(x + cy) + a_4(x + dyz)(x + cy) \). This is simplified to \( a_1(cxy - rdxz) + a_4(-rcyzz + dyzx) = 0 \). Now this gives that we can choose \( a_2 \) and \( a_3 \) arbitrary and \( a_1 = 0 \) if \( e \neq c \) and \( a_4 = 0 \) if \( e = r^2c \). This gives the result.

(2) Look at the short exact sequence:

\[
0 \to A/(x + cy)A \to A \to A/(x + cy)A \to 0
\]

and apply the functor \( \text{Hom}_A(-, A/(x + cy)A) \) to obtain the exact sequence:

\[
0 \to \text{Hom}_A(A/(x + cy)A, A/(x + dy)A) \to \text{Hom}_A(A/(x + cy)A, A/(x + dy)A) \to \text{Ext}_1^A(A/(x + cy)A, A/(x + dy)A) \to 0.
\]

Thus, \( \dim(\text{Ext}^1(A/(x + cy)A, A/(x + dy)A)) = \dim(\text{Hom}_A(A/(x + cy)A, A/(x + dy)A)) + \dim(\text{Hom}_A(A/(x + cy)A, A/(x + dy)A)) - \dim(\text{Hom}_A(A/(x + cy)A, A/(x + dy)A)) = 2 + \delta_{c,d} + \delta_{cr^2,d} + 2 + \delta_{cr^2,d} + \delta_{cr^2,d} - 4 \) by (1). This gives \( \dim(\text{Ext}^1(A/(x + cy)A, A/(x + dy)A)) = 0 \) iff \( c \neq d, cr \neq d, cr^2 \neq d \) and \( cr^3 \neq d \).

(3) This follows directly by (2) using that \( \text{Ext}^i(M_d, M_d) = \text{Ext}^i(\Omega^{-1}(M_c), M_d) = \text{Ext}^i(M_{d-i}, M_d) \).

\[\square\]

**Theorem 2.6.** Let \( K \) be a field element with an element \( r \) such that \( r^l \neq 1 \) for all \( l \geq 1 \) and let \( c_1, ..., c_n \) be pairwise distinct field elements. Let \( N := A_r \oplus M_{c_1} \oplus ... \oplus M_{c_n} \).

(1) \( B \) is a gendo-symmetric algebra of dominant dimension equal to two with \( n + 1 \) simple modules.

(2) The algebra \( B := \text{End}_A(N) \) has infinite finitistic codominant dimension and is not nearly Gorenstein.

**Proof.**

(1) \( B \) is by definition gendo-symmetric, since \( A_r \) is symmetric. By \( (2.5) \), \( \text{Ext}^1(M_{c_i}, M_{c_j}) \neq 0 \) and thus \( B \) has dominant dimension equal to two by Mueller’s theorem. Since the module \( N \) is basic, the number of simple modules of \( B \) equals the number of indecomposable summands of \( N \).

(2) We want to use \( (2.3) \) to prove the result and show that the assumptions of \( (2.3) \) are satisfied. Choose \( Z := M_{c_i} \) with \( i \) chosen so that for \( d := \frac{1}{c_i} \), \( d \neq r^i c_j \) for any \( l \geq 1 \) and any \( j \). Note that \( M_d \oplus \Omega^{-1}(M_c) \). Then we have \( \text{Ext}^1(N, Z) \neq 0 \) since \( \text{Ext}^1(Z, Z) \neq 0 \) by \( (2.5) \). But for \( l \geq 2 \) we have \( \text{Ext}^l(N, Z) = \text{Ext}^{l-1}(N, \Omega^{-1}(Z)) = \text{Ext}^{l-1}(N, M_d) = 0 \) by \( (2.5) \), since \( \text{Ext}^1(M_{c_i}, M_d) = 0 \) for any \( i \geq 1 \) and any \( j \) because \( d \neq r^i c_j \) for any \( i \geq 0 \) and any \( j \).

\[\square\]

3. Open Questions

We note that Liu-Schulz algebras are a special case of quantum complete intersection algebras that are those of the form \( K < x_1, x_2, ..., x_n > / (x_i^{a_i}, x_ix_j + q_{ij}x_jx_i) \) for \( i > j \), \( a_i > 2 \) natural numbers and \( q_{ij} \) non-zero field elements. Those are local Frobenius algebras that are not always symmetric, see for example \( [Op] \). One can construct non-nearly Gorenstein algebra from such algebras that are not necessarily Liu-Schulz algebras by finding modules \( M \) with \( \text{Ext}^1(M, M) \neq 0 \) and \( \text{Ext}^1(M, M) = 0 \) for \( i \geq 2 \) as we did in \( (2.6) \). We thus formulate some questions more generally for quantum complete intersection algebras. We call an indecomposable module \( M \) \( l \)-special for some \( l \geq 1 \) in case \( \text{Ext}^1(M, M) \neq 0 \) and \( \text{Ext}^1(M, M) = 0 \) for all \( i \geq 1 \).

(1) Can one construct \( l \)-special modules over symmetric algebras over an arbitrary field? Can one construct non-nearly Gorenstein algebras over an arbitrary field? Note that in our construction we had to assume that there are field elements that are not roots of unity so our construction did not work for example for finite fields.

(2) Can one classify all \( l \)-special modules over quantum complete intersection algebras?

(3) Are there non-projective modules \( M \) with \( \text{Ext}^1(M, M) = 0 \) over quantum complete intersection algebras? We remark that it seems to be an open problem in general to find modules over local Frobenius algebras with \( \text{Ext}^1(M, M) = 0 \). We refer to \( [Mar3] \) for more on this.

(4) Do algebras of the form \( B = \text{End}_A(A \oplus X) \) have finite finitistic dimension in case the assumption of \( (2.3) \) are fulfilled? We note that we do not know the answer even for the algebras in \( (2.6) \).
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References

[AB] Auslander, M.; Bridger, M.: Stable Module Theory. Memoirs of the American Mathematical Society, No. 94 American Mathematical Society, Providence, R.I. (1969) 146 pp.

[APT] Auslander, M.; Platzeck, M.; Todorov, G.: Homological theory of idempotent ideals. Transactions of the American Mathematical Society, Volume 332, Number 2, August 1992.

[BGMS] Buchweitz, R.; Green, R.; Madsen, D.; Solberg, O.: Finite Hochschild cohomology without finite global dimension. Math. Res. Lett. 12 (2005), no. 5-6, 805-816.

[CIM] Chan, A.; Iyama, O.; Marczinzik, R.: Auslander-Gorenstein algebras from Serre-formal algebras via replication. https://arxiv.org/abs/1707.03996.

[Che] Chen, X.: Gorenstein Homological Algebra of Artin Algebras. http://home.ustc.edu.cn/~xwchen/Personal2Papers/postdoc-Xiao-Wu%20Chen%202010.pdf retrieved 18.06.2017.

[FanKoe] Fang, M.; Koenig, S.: Independence of the total reflexivity conditions for modules. Algebras and Representation Theory 9 (2006), 217-226.

[LSR] Liu, S.; Schulz, R.: The existence of bounded infinite DTr-orbits. Proc. Amer. Math. Soc. 122 (1994) 1003-1005.

[JS] Jorgensen, D.; Sega, L.: Endomorphism algebras of generators over symmetric algebras. Journal of Algebra 332 (2011) 428-433.

[JS] Jorgensen, D.; Sega, L.: Independence of the total reflexivity conditions for modules. Algebras and Representation Theory 9 (2006), 217-226.

[Mar] Marczinzik, R.: Upper bounds for the dominant dimension of Nakayama and related algebras. http://arxiv.org/abs/1605.09634.

[Mar2] Marczinzik, R.: Gendo-symmetric algebras, dominant dimensions and Gorenstein homological algebra. https://arxiv.org/abs/1608.04212.

[Mar3] Marczinzik, R.: Upper bounds for dominant dimensions of gendo-symmetric algebras. Archiv der Mathematik September 2017, Volume 109, Issue 3, pp 231-243.

[Mue] Mueller, B.: The classification of algebras by dominant dimension. Canadian Journal of Mathematics, Volume 20, pages 398-409, 1968.

[Op] Oppermann, S.: Hochschild cohomology and homology of quantum complete intersections. Algebra Number Theory 4(7)(2010)821-838.

[Rin] Ringel, C.: The Liu-Schulz example. Representation theory of algebras. Seventh International Conference on Representations of Algebras August 22-26, 1994 Cocoyoc, Mexico. CMS conference proceedings Volume 18.

[Sch] Schulz, R.: A non-projective module without self-extensions. Archiv der Mathematik June 1994, Volume 62, Issue 6, pp 497-500.

[Sm] Smith, S.P.: Some finite dimensional algebras related to elliptic curves. Seventh International Conference on Representations of Algebras August 22-26, 1994 Cocoyoc, Mexico. CMS conference proceedings Volume 18.

[SkoYam] Skowronski, A.; Yamagata, K.: Frobenius Algebras I: Basic Representation Theory. EMS Textbooks in Mathematics, (2011).

[Ta] Tachikawa, H.: Quasi-Frobenius Rings and Generalizations: QF-3 and QF-1 Rings (Lecture Notes in Mathematics 351) Springer; (1973).

[Yam] Yamagata, K.: Frobenius Algebras Hazewinkel, M. (editor): Handbook of Algebra, North-Holland, Amsterdam, Volume I, 841-887, (1996).

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