Effects of the equation of state on the bulk properties of rapidly-rotating neutron stars

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Neutron stars are among the densest known objects in the universe and an ideal laboratory for the strange physics of super-condensed matter. While the simultaneously measurement of mass and radius of non-rotating neutron stars may impose strong constraints on the properties of the dense nuclear matter, the observation and study of rapidly-rotating ones, close to the mass-shedding limit, may lead to significantly further constraints. Theoretical predictions allow neutron stars to rotate extremely fast (even more than 2000 Hz). However, until this moment, the fastest observed rotating pulsar has a frequency of 716 Hz, much lower compared to the theoretical predictions. There are many suggestions for the mechanism that leads to this situation. In any case, the theoretical study of uniformly rotating neutron stars, along with the accurate measurements, may offer rich information concerning the constraints on the high density part of the equation of state. In addition, neutron stars through their evolution, may provide us with a criteria to determine the final fate of a rotating compact star. Sensitivity of bulk neutron stars properties on the equation of state at the mass-shedding limit, are the main subject of the present study.

I. INTRODUCTION

Neutron stars are considered as extraordinary astronomical laboratories for the physics of nuclear matter because, from astrophysical point of view, are the objects with the most fascinating constitution of energy and matter in the Universe [3]. To be more specific, the observation of mass, as well as the radius, of slowly-rotating (or non-rotating) neutron stars may provide us with very useful constraints on the equation of state (EoS) of nuclear matter. In addition, neutron stars, due to their compactness, may rotate very fast compared to other astrophysical objects [4]. Henceforth, measurements of specific properties of rapidly-rotating neutron stars (including mainly the mass and radius, frequency, moment of inertia, quadrupole moment etc.) may lead to robust constraints on the EoS as well as on the constitue of nuclear matter at very high densities.

The determination of the maximum neutron star mass is a long-standing issue in astrophysics, due to the identification of the black holes and the unknown behavior of the nuclear matter at very high densities. Until this moment, the most massive neutron stars measurements, include: a) the PSR J1614-2230 (M = 1.97 ± 0.04 M⊙) [5] (or from recent elaboration of the observation M = 1.928 ± 0.017 M⊙ [6]), b) the PSR J0348+0432 (M = 2.01 ± 0.04 M⊙) [7], c) the PSR J0740+6620 (M = 2.17^{+0.11}_{-0.10} M⊙) [8] and d) the PSR J2215+5135 (M = 2.27^{+0.17}_{-0.15} M⊙) [9]. In addition, there is a very detailed study concerning the spin frequency of rotating neutron stars (for a review see Refs. [10]). The fastest rotating pulsar that has been found, is the J1748-244ad with a spin frequency of 716 Hz [11]. However, the issue is still open: why we have not observed pulsars with higher values of frequency, which predicted from the majority of theoretical models? And even more, is it possible to exist an upper limit to the spin frequency of millisecond pulsars and why? [12]. Future measurements of the moment of inertia [13] and Keplerian frequency may be the answer to these questions by improving considerably our knowledge on the properties of rapidly-rotating neutron stars.

The effects of the EoS on the properties of rotating neutron stars (see Ref. [14] for introduction and relevant bibliography), had begun to gain ground almost thirty years ago from Shapiro, Teukolsky and their colleagues [15,20]. A significant contribution on these issues, had been also made from Friedman and his colleagues [21,25], Haensel and co-workers [26,31], as well as Glendenning and his colleagues [32,55]. Rapid rotation and its effects on the EoS, had been studied also in Refs. [36–41] and most recently in Refs. [42–57]. Moreover, in nuclear astrophysics, hot neutron stars in correlation with the rapid rotation, had been studied in Refs. [58–59]. In addition, rapidly-rotating neutron stars in modified gravity theories, have been studied in detail by Kokkotas and his colleagues [60,61].

In this work, we extend the previous fundamental work of Cook, Shapiro and Teukolsky [19], as well as the most recent work of Cipolletta et al. [51]. In particular, we employ a large number of modern realistic EoSs (combined with a few previous ones), which all of them, at least marginally (few of them), predict the upper bound of the maximum neutron star mass of M = 1.928 ± 0.017 M⊙ [6], while also reproducing accurately the bulk properties of finite nuclei (for more details see [62]). Among the number of realistic equations that we used here, we have constructed two EoSs predicted by the Momentum-Dependent Interaction model (MDI). This model reproduces the results of

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microscopic calculations of symmetric nuclear matter and neutron star matter at zero temperature, with the advantage of its extension to finite temperature. To be more specific, an effort was made to systematically study the most of the bulk properties of uniformly rotating neutron stars at the Keplerian sequence, including the mass, polar and equatorial radius, angular velocity, moment of inertia, Kerr parameter, eccentricity, braking index and etc.

Furthermore, we explore the possibility to update the previous empirical universal relations, which connecting the Keplerian angular velocity with the mass and radius at the maximum mass configuration. We systematically study the dependence of the Kerr parameter on the EoS and also provide the evolution of the angular momentum of a neutron star, in order to examine the case where neutron stars considered to be progenitors of black holes. In particular, we examine (according to the terminology of Ref. [19]), two equilibrium sequences of rotating neutron stars, normal and supramassive. While normal evolutionary sequences have a spherical, non-rotating (stable) end point, supramassive ones, which by definition have masses higher than the maximum mass of the non-rotating neutron star, they don’t have a stable end point, and as a consequence the collapse to a black hole is inevitable. However, the construction of normal and mainly supramassive sequences, is a complicated procedure in the framework of General Relativity [19].

In addition, we systematically study the moment of inertia, a quantity which plays important role on the properties of rotating neutron stars, and eccentricity, which can inform us for the deformation of neutron stars. Following the previous work of Lattimer and Prakash [62], we also provide an absolute upper limit of the higher density of matter in the Universe, based on the upper limit imposed by the maximum mass of a neutron star. In fact, we try to improve the bound which was introduced in Ref. [63], by using updated EoSs and including also the case of rapidly-rotating neutron stars. Finally, we study the effects of the EoS on the braking index of pulsars. We mainly focus on values near the Keplerian frequency (70% and more), where the braking index begins to be affected by the rest mass.

The article is organized as follows. In Section II we briefly review the model for the nuclear EoS and the computational hypothesis. In Section III we present the rapidly-rotating configuration for neutron stars. In particular, we introduce the effects of the Keplerian angular velocity on the bulk properties of neutron stars and also, we describe two properties of the EoS, the moment of inertia and eccentricity. In addition, we provide a discussion for the Kerr parameter and the fully described rest mass sequences. The upper bound for density of cold baryonic matter and the effects of the braking index on the EoS, are also obtained. Finally, Section IV contains the summarize and discussion of the present study.

II. THE NUCLEAR EQUATION OF STATE

In the present study we have selected and employed a large number of hadronic EoSs [62] in order to reproduce (at least marginally) the lower upper limit of the maximum neutron star mass ($M = 1.928 \pm 0.017 \, M_\odot$ found in Ref. [63]). In order to study specific properties and evolutionary process of neutron stars, we employ the MDI model. In this model the energy per particle is given by

\begin{equation}
E_b(n, I) = \frac{3}{10} E_F^0 u^{2/3} \left[ (1 + I)^{5/3} + (1 - I)^{5/3} \right] + \frac{1}{3} A \left[ \frac{3}{2} - X_0 I \right]^2 u + \frac{2}{3} B \left[ \frac{3}{2} - X_3 I^2 \right] u^\sigma \left[ \frac{2}{3} \right] \left[ \frac{2}{3} - X_3 I^2 \right] u^{\sigma - 1}
+ \frac{3}{2} \sum_{i=1,2} \left( C_i + \frac{C_i}{5} \right) \left( \frac{\Lambda_i}{k_F^2} \right)^2 \left[ \frac{(1 + I) u^{1/3}}{\Lambda_i} - \tan^{-1} \left( \frac{(1 + I) u^{1/3}}{\Lambda_i} \right) \right] \left( \frac{(1 - I) u^{1/3}}{\Lambda_i} - \tan^{-1} \left( \frac{(1 - I) u^{1/3}}{\Lambda_i} \right) \right)
+ \frac{3}{2} \sum_{i=1,2} \left( C_i - \frac{C_i}{5} \right) \left( \frac{\Lambda_i}{k_F^2} \right)^2 \left[ \frac{((1 + I) u)^{1/3}}{\Lambda_i} - \tan^{-1} \left( \frac{((1 + I) u)^{1/3}}{\Lambda_i} \right) \right] \left( \frac{((1 - I) u)^{1/3}}{\Lambda_i} - \tan^{-1} \left( \frac{((1 - I) u)^{1/3}}{\Lambda_i} \right) \right)
\end{equation}

where $I = (n_n - n_p)/n$, $X_0 = x_0 + 1/2$ and $X_3 = x_3 + 1/2$.

To be more specific, in Eq. 1, the ratio $u$ is defined as $u = n/n_s$, with $n_s$ denoting the equilibrium symmetric nuclear matter density (or saturation density) and equals to 0.16 fm$^{-3}$. The parameters $A$, $B$, $\sigma$, $C_1$, $C_2$ and $B'$, which are called coupling constants and appear in the description of symmetric nuclear matter (SNM), are determined so that the relation $E_b(n_s, 0) = -16$ MeV holds. The finite range parameters are $\Lambda_1 = 1.5 k_F^0$ and $\Lambda_2 = 3 k_F^0$ with $k_F^0$ being the Fermi momentum at the saturation density $n_s$. By suitably choosing the rest of the parameters $x_0$, $x_3$, $Z_1$, and $Z_2$, which appear in the description for the asymmetric nuclear matter (ANM), it is possible to obtain different forms for the density dependence of the symmetry energy as well as on the value of the slope parameter $L$ and the value of the symmetry
energy at the saturation density \(E_{\text{sym}}\). Actually, for each value of \(L\), the density dependence of the symmetry energy is adjusted so that the energy of pure neutron matter is comparable with those of the existing state-of-the-art calculations \[64, 65\].

We have many reasons to support the reliability of the present model. In particular, a) reproduces with high accuracy the properties of symmetric nuclear matter at the saturation density, b) the theoretical prediction of the value and the slope of the symmetry energy at the saturation density are close to the experimental predictions, c) reproduces other properties of SNM (including \(K_0\) and \(Q_0\), for more details see the Appendix A) inside the limiting area of the experimental data, d) reproduces correctly the microscopic calculation of the Chiral model \[66\] for pure neutron matter (for low densities) and the results of the state-of-art calculations of Akmal et al. \[67\] (for high densities), e) has the flexibility that the energy per particle depends not only of the density, but also on the momentum, f) can be easily extended to include temperature dependence (which needed to study core-collapse supernova, proto-neutron stars, neutron stars merger etc.) and g) predicts maximum neutron star mass higher than the observed ones \[6, 9\].

A. Data from microscopic calculations

The parametrization of Eq. \(1\) is performed by using the data originated from previous work of Akmal et al. \[67\]. In particular, we employ the data concerning the energy per particle of symmetric and pure neutron matter (in the area \(0.04 \text{ fm}^{-3} \leq n \leq 0.96 \text{ fm}^{-3}\)) and for the models A18+UIX (hereafter APR-1) and A18+\(\delta v\)+UIX\(\ast\) (hereafter APR-2). In order to achieve the best fitting to Akmal’s data using the Eq. \(1\), we divide our region of study in three sections : a) Low Density Region (LDR) \((0.04 \text{ fm}^{-3} \leq n \leq 0.2 \text{ fm}^{-3})\), b) Medium Density Region (MDR) \((0.2 \text{ fm}^{-3} \leq n \leq 0.56 \text{ fm}^{-3})\) and c) High Density Region (HDR) \((0.56 \text{ fm}^{-3} \leq n \leq 0.96 \text{ fm}^{-3})\). The main properties of the nuclear symmetry energy at the saturation density \(n_s\) are presented in Table I. It is worth pointing out that the parametrization of pure neutron matter leads to results in agreement with the predictions of chiral effective field theory \[66\] for low densities. For high values of densities, the parametrization leads to the prediction of Akmal et al. \[67\]. The main drawback of these two EoSs, APR-1 and APR-2, is related with the violation of causality; the speed of sound becomes greater than the speed of light at high densities. However, the parametrization of the MDI model, has the advantage that prevents the onset from violate the causality.

Table I. Properties of the symmetric nuclear matter. Extended reference on these properties exist in the Appendix A.

| Properties of SNM | APR-1 | APR-2 |
|-------------------|-------|-------|
| \(E_{\text{sym}}\) (MeV) | 482.34 | 568.91 |
| \(K_0\) (MeV)       | -103.7 | -118.78 |
| \(Q_0\) (MeV)       | -88.26 | -99.81 |
| \(E_{\text{sym}}\) (MeV) | 33.61 | 33.59 |
| \(L_{\text{sym}}\)  | 63.31 | 57.4 |
| \(Q_{\text{sym}}\)  | 450.5 | 538.44 |
| \(K_{\text{sym}}\)  | -88.26 | -99.81 |

In addition, the schematic presentations of Eq. \(1\) and the data from Akmal et al. \[67\], are presented in Fig. 1.

![Figure 1](image_url)

**Figure 1.** The symmetric nuclear matter and pure neutron matter fits for the APR-1 and APR-2 EoSs using the Akmal’s \[67\] data and the MDI model, where the data are also presented with circles for the SNM, triangles for the PNM of APR-1 and squares for the PNM of APR-2.

B. The selected Equations of State

The EoSs that we use \[62\], as it was already mentioned, are in consistent with the current observed limits of neutron star mass and also with the observed frequency (716 Hz) limit. In Fig. 2 we present the dependence of the gravitational mass on the corresponding radius for the 23 realistic EoSs at the non-rotating configuration, where the current observed limits are also presented.
III. RAPIDLY-ROTATING NEUTRON STARS

In the framework of General Relativity, rapidly-rotating neutron stars can be described a) by the stationary axisymmetric spacetime metric \[ ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}(dr^2 + r^2d\theta^2) \] (2) where the metric functions \( \nu, \psi, \omega \) and \( \mu \) depend only on the coordinates \( r \) and \( \theta \), and b) the matter inside the neutron star. If we neglect sources of non-isotropic stresses, viscous stresses and heat transport, then the matter inside the neutron star can be fully described by the stress-energy tensor and modeled as a perfect fluid \[ T^{\alpha\beta} = (\varepsilon + P)u^\alpha u^\beta + Pg^{\alpha\beta} \] (3) where \( u^\alpha \) is the fluid’s 4-velocity. The energy density and pressure is denoted as \( \varepsilon \) and \( P \).

A. Keplerian angular velocity

The derivation of the Keplerian frequency, the frequency where a rotating star would shed matter at its equator, is a complicated problem. In Newtonian theory it has its origin on the balance between gravitational and centrifugal forces and takes a very simple form. However, in General Relativity (GR), exhibits a more complicated dependence on the structure of the star through the interior metric. To be more specific, in GR is expressed as a self-consistency condition that must be satisfied by the solution to Einstein’s equations.

It has been shown by Friedman et al. \[23\] that the turning-point method, which is leading to the points of secular instability, can also be used in the case of uniformly rotating neutron stars. With this consideration, in a constant angular momentum sequence, the turning-point of a sequence of configurations with increasing central density, separates the secularly stable from the secularly unstable configuration and consequently, the condition

\[
\frac{\partial M(\varepsilon_c, J)}{\partial \varepsilon_c} \bigg|_{J=\text{ constant}} = 0
\]

(4)

where \( \varepsilon_c \) is the energy density in the center of the neutron star and \( J \) is the angular momentum, defines the possible maximum mass.

The absence of analytical solutions for rapidly-rotating neutron stars leads to numerical estimations for the Keplerian frequency. A significant number of empirical formulas for the Keplerian frequency, using realistic EoSs, have been produced along the years. The expression is given by the formulæ

\[
\Omega_k = \mathcal{F}_{\text{max}} \left( \frac{G M_{\text{max}}}{R_{\text{max}}^3} \right)^{1/2} \text{s}^{-1}
\]

(5)

Although this relation is well established, the unknown parameter \( (\mathcal{F}_{\text{max}}) \) depends highly on the various approximations and of course the EoS. Since, in general, \( f = \Omega/2\pi \), Eq. \( (\mathcal{F}_{\text{max}}) \) can be written in a different form, as follows \[47, 56\]

\[
f_k = C \left( \frac{M_{\text{max}}}{M_\odot} \right)^{1/2} \left( \frac{10 km}{R_{\text{max}}} \right)^{3/2} \text{Hz}
\]

(6)

It is worth pointing out, that while the maximum rotation rate is an increasing function of the softness of the EoS, the maximum mass is a decreasing one (considering a fixed mass). The latter, it has been already noticed by Lattimer et al. \[38\]. These two constraints restrict the EoS in a narrow region. The above statement is one of the main subjects of the present work.

1. The Keplerian frequency, the maximum mass and the corresponding radius of non-rotating neutron stars

In order to study the effects of the Keplerian frequency on the bulk properties of a non-rotating neutron star, we use Eq. \( (\mathcal{F}_{\text{max}}) \), with the following parametrization

\[
f_k = C_{\text{st}} \left( \frac{M_{\text{st}}}{M_\odot} \right)^{1/2} \left( \frac{10 km}{R_{\text{st}}^3} \right)^{3/2} \text{Hz}
\]

(7)

where \( C_{\text{st}} \) corresponds to the unknown parameter for the non-rotating case, \( M_{\text{st}} \) to the maximum mass of a non-rotating neutron star and \( R_{\text{st}} \) to the corresponding
radius. In this case, the value of the parameter $C_{st}$ is 1266.68, updating with this way the work of Haensel et al.\cite{47} ($C_H = 1220$). For reasons of brevity, we define

$$x_{\text{st}}^a = \left(\frac{M_{\text{max}}^a}{M_{\odot}}\right)^{1/2} \left(\frac{10\text{km}}{R_{\text{max}}^{\text{st}}}\right)^{3/2}$$

(8)

where $a$ (st, rot, rm; rot) takes the form of the superscript of the corresponding $C$, respectively.

In Fig. 3a we present the relation (7). This value is in very good agreement with the current EoSs to a linear term (the maximum possible error is less than 5.6%).

2. The Keplerian frequency, the maximum mass and the corresponding radius of rapidly-rotating neutron stars

An interesting relation is also the one between the Keplerian frequency and the macroscopic properties of rapidly-rotating neutron stars (maximum mass and the corresponding radius). Using Eq. (6), but now with the following parametrization

$$f_k = C_{\text{rot}} \left(\frac{M_{\text{max}}^\text{rot}}{M_{\odot}}\right)^{1/2} \left(\frac{10\text{km}}{R_{\text{max}}^\text{rot}}\right)^{3/2} \text{(Hz)}$$

(9)

where $C_{\text{rot}}$ corresponds to the unknown parameter for the rapidly-rotating case, $M_{\text{max}}^\text{rot}$ to the maximum mass of a rapidly-rotating neutron star and $R_{\text{max}}^\text{rot}$ to the corresponding radius. It is remarkable that in this scenario, as Fig. 3b shows, the linear fit between these quantities ($f_k$, $x_{\text{rot}}^\text{rot}$) leads to nearly perfect results (the maximum possible error is less than 1%). In this case, the value of the parameter $C_{\text{rot}}$ is 1781.9.

3. The Keplerian frequency, the maximum rest mass and the corresponding radius of rapidly-rotating neutron stars

In the macroscopic properties of a neutron star, rest mass plays an important role. In order to understand the effects of the rest mass on the Keplerian sequence, we perform the same analysis as in Section III A 2 as Fig. 4 shows.

We study the dependence of the Keplerian frequency on the rest mass using Eq. (6), with the following parametrization

$$f_k = C_{\text{rm,rot}} \left(\frac{M_{\text{max}}^\text{rm,rot}}{M_{\odot}}\right)^{1/2} \left(\frac{10\text{km}}{R_{\text{max}}^\text{rm,rot}}\right)^{3/2} \text{(Hz)}$$

(10)
where $C_{\text{rot} \text{rm}}$ corresponds to the unknown parameter for the rest mass at the rapidly-rotating case, $M_{\text{max} \text{rot}}$ to the rest mass at the maximum mass configuration of a rapidly-rotating neutron star and $R_{\text{max} \text{rot}}$ to the corresponding radius. In Fig. 4 we can see the almost linear relation that holds on between these two quantities ($f_k$, $x_{\text{max} \text{rot}}$) (the maximum possible error is less than 2.2%), enhancing with this way the existence of a relation between the rest mass and the gravitational mass on neutron stars at the Keplerian frequency. In this case, the value of the parameter $C_{\text{rot} \text{rm}}$ is 1644.75.

4. Rest mass and gravitational mass at the maximum mass configuration of rapidly-rotating neutron stars

As a follow-up to the Section III A 3 we study here the dependence of the rest mass on the gravitational mass at the maximum mass configuration for the Keplerian frequency. In Fig. 5, we can see the almost linear relation between these two quantities, as expected from Section III A 3.

![Figure 5](image-url)

Figure 5. The dependence of the rest mass on the gravitational mass of a rapidly-rotating neutron star at the maximum mass configuration. The blue line corresponds to the best linear trend that fits the data. The data from the 23 EoSs are also presented with red circles.

The relation which describes our data, it is given via the form

$$
\left( \frac{M_{\text{max} \text{rot}}}{M_{\odot}} \right) = 1.17 \left( \frac{M_{\text{max} \text{rot}}}{M_{\odot}} \right)
$$

(11)

(the maximum possible error is less than 3.3%) concluding with this way that the percentage difference between these quantities is 17%.

B. Moment of Inertia and Eccentricity

Rotating neutron stars can provide us with more quantities than non-rotating ones that we could study. Among them, there is the moment of inertia and eccentricity. Both these quantities can give us information about the deformation of the mass while its spinning.

The moment of inertia $I_{\text{rot}}$ [51, 69], which have a prominent role in pulsar analysis, can be estimated as

$$
I = \frac{J}{\Omega}
$$

(12)

where $J$ is the angular momentum and $\Omega$ is the angular velocity. This property of neutron stars, quantifies how fast an object can spin with a given angular momentum.

We study here the dependence of the moment of inertia on the gravitational mass for the Keplerian sequence. From Fig. 6a, we can see that all EoSs, present similar behavior. For this reason, inside Fig. 6a, we plot the moment of inertia values corresponding to maximum mass configuration versus the corresponding gravitational mass. A relation, given by the formulae

$$
I_{\text{rot}}^{\text{max}} = -1.568 + 0.883 \exp \left[ 0.7 \left( \frac{M_{\text{max} \text{rot}}}{M_{\odot}} \right) \right] \left( 10^{45} \text{ gr cm}^2 \right)
$$

(13)

describes with high accuracy our data, concluding with this way that moment of inertia, at the maximum mass configuration for the Keplerian frequency, can provide us with a universal relation between the moment of inertia and the corresponding gravitational mass.

We also study here the dependence of the dimensionless moment of inertia on the corresponding compactness parameter $\beta$, which, in general, it is defined as

$$
\beta = \frac{GM}{Rc^2}
$$

(14)

In Fig. 6b, we present a window where the moment of inertia and the compactness parameter can lie (gray region), constraining with this way both these quantities. There is an empirical relation, derived from the data, that can describe this window. The form of this empirical relation, is

$$
I/MR^2 = \alpha_1 + \alpha_2\beta + \alpha_3\beta^2 + \alpha_4\beta^3 + \alpha_5\beta^4
$$

(15)

where the coefficients for the two edges are shown in Table 11. It is clear from Fig. 6a and Eq. (15), that if we have a measurement of moment of inertia, or compactness parameter, we could extract the interval where the other parameter can lie. An average empirical relation had been also created, given by the expression

$$
I/MR^2 = 0.005 + (3.695 \pm 0.315) \beta - (21.12 \pm 3.67) \beta^2 + (68.17 \pm 18.49) \beta^3 - (82.845 \pm 27.485) \beta^4
$$

(16)
As a consequence, by constraining simultaneously these two quantities, we could impose strong constraints on the radius of neutron stars, which still remains an open problem.

Table II. Coefficients of the empirical relation \( \alpha \) for the two edges of the window presented in Fig. 6b.

| Edges   | \( \alpha_1 \) | \( \alpha_2 \) | \( \alpha_3 \) | \( \alpha_4 \) | \( \alpha_5 \) |
|---------|----------------|----------------|----------------|----------------|----------------|
| Upper   | 0.005          | 4.01           | -24.79         | 86.66          | -110.33        |
| Lower   | 0.005          | 3.38           | -17.45         | 49.68          | -55.36         |

From Fig. 6b, we can see that all EoSs, present similar behavior. For this reason, inside Fig. 6b, we plot the dimensionless moment of inertia values corresponding to maximum mass configuration versus the corresponding compactness parameter. A relation, given by the formulae

\[
I_{max}/MR^2 = -0.006 + 1.379 \beta_{max}
\]

(17)

describes with high accuracy our data, concluding with this way that the dimensionless moment of inertia, at the maximum mass configuration for the Keplerian frequency, can provide us with a universal relation between the dimensionless moment of inertia and the corresponding compactness parameter.

Eccentricity, is the main quantity that is related to the deformation of the star. Rapid rotation deforms the models of equilibrium and in order to see how the models change, we calculate the eccentricity, which is given by the form [51]

\[
\epsilon = \sqrt{1 - \left(\frac{r_{pol}}{r_{eq}}\right)^2}
\]

(18)

where the \( r_{pol} \) and \( r_{eq} \) are the polar and equatorial radii of the star, respectively.

Performing the same analysis as for moment of inertia, we study here the dependence of the eccentricity on the gravitational mass for the Keplerian sequence and the eccentricity values corresponding to maximum mass configuration on the corresponding gravitational mass, as Fig. 7 shows. A relation, given by the formulae

\[
\epsilon_{max} = 0.799 + 0.01 \left(\frac{M_{max}}{M_\odot}\right)
\]

(19)

describes with high accuracy our data, concluding with this way that eccentricity, at the maximum mass configuration for the Keplerian frequency, is an EoS-independent property.
C. The Kerr parameter

The Kerr space-time provided from the Einstein’s field equations, give us the so-called Kerr black holes \cite{48, 51}. These rotating black holes can be fully described from the gravitational mass (M) and the angular momentum (J). In order to have a meaningful Kerr black hole, the relation \( J \geq GM^2/c \) (Kerr bound) must hold, or otherwise, we have a naked singularity. A naked singularity is a black hole without a horizon and can be considered as closed timelike curves, where causality would be violated. While there is no rigorous proof from Einstein’s field equations, the cosmic-censorship conjecture implies that a generic gravitational collapse cannot form a naked singularity. This is the reason why the astrophysical black holes should satisfy the Kerr bound \cite{48, 51}.

The gravitational collapse of a massive rotating neutron star, constrained to mass-energy and angular momentum conservation, creates a black hole with almost the same mass and angular momentum as the prior neutron star. In this case, an important quantity to study, directly related with the black holes as well as the neutron stars, is the dimensionless angular momentum \cite{50}, which is defined as

\[
j = \frac{cJ}{GM_0^2} \quad (20)
\]

and it is known as dimensionless spin parameter. As a consequence of this parameter, we can define a new one, starting from the parameter \( \alpha \), which is the angular momentum in units of mass and it is given by the form \cite{15}

\[
\alpha \equiv \frac{J}{M} = \frac{GM_0^2}{c} \frac{1}{M} \quad (21)
\]

As a follow, using Eq. (21), the well-known Kerr parameter, takes the form

\[
K = \frac{\alpha}{MG} = j \left( \frac{M_0}{M} \right)^2 \quad (22)
\]

The dependence of this parameter on the gravitational mass at the Keplerian sequence can be seen in Fig. 9.

While the meaning of this parameter at black-holes physics is so interesting and fundamental (there is a maximum value at 0.998 \cite{71}), that’s not the case for other compact objects such as neutron stars. In order to find a way to constrain the value of the Kerr parameter in neutron stars, we study here the dependence of this parameter on the total gravitational mass for the Keplerian sequence. From Fig. 9, we can see that the maximum value of the Kerr parameter for neutron stars is around 0.75. While there is a number of realistic EoSs that hold on near this value, the maximum value achieved from HLPS-3. This EoS is the stiffest one and produces maximum mass greater than all the others. Strictly speaking, if we consider this EoS as the one that produces the max-
imum possible mass in the maximum mass configuration at the Keplerian sequence, then we could constrain the maximum value of the Kerr parameter in neutron stars.

In Fig. 9 we present also a window (gray region) where the Kerr parameter can lie. There is an empirical relation, derived from the data, that can describe this window. The form of this empirical relation, is

\[ K = d_1 + d_2 \coth \left( d_3 \left( \frac{M_{\text{max}}}{M_\odot} \right) \right) \]  \hspace{1cm} (23)

where the coefficients for the two edges are shown in Table III. It is clear from Fig. 9 and Eq. (23), that if we have a measurement of gravitational mass, or spin parameter, we could extract the interval where the other parameter can lie. An average empirical relation had been also created, given by the expression

\[ K = 0.86 - (1.165 \pm 0.045) \coth \left( 2.105 \pm 0.565 \left( \frac{M_{\text{max}}}{M_\odot} \right) \right) \]  \hspace{1cm} (24)

As a consequence, by constraining simultaneously these two quantities, we could impose strong constraints on the EoS.

Table III. Coefficients of the empirical relation (23) for the two edges of the window presented in Fig. 9

| Edges  | \(d_1\) | \(d_2\) | \(d_3\) |
|--------|--------|--------|--------|
| Upper  | 0.86   | -0.12  | 1.54   |
| Lower  | 0.86   | -0.21  | 2.67   |

In addition, in Fig. 9 we plot the maximum values of the Kerr parameter versus the corresponding gravitational mass. It seems that a linear relation holds between these two quantities, given by the equation

\[ K_{\text{max}} = 0.488 + 0.074 \left( \frac{M_{\text{max}}}{M_\odot} \right) \]  \hspace{1cm} (25)

There are two important reasons for constraining the Kerr parameter at neutron stars: First, the existence of a maximum value at the Kerr parameter, can lead us to possible limits for the compactness on neutron stars; strictly speaking, the maximum value of the Kerr parameter for neutron stars implies a maximum value on the possible maximum mass of rotating neutron stars in the universe and second, can be a criteria for determining the final fate of the collapse of a rotating compact star [48].

D. Constant rest mass sequences

The rest mass sequences, also called as time evolutionary sequences, based on an EoS, are roughly horizontal lines that extend from the Keplerian sequence to the axisymmetric instability limit [18 20]. For a given EoS, the sequences that are below the rest mass value that corresponds to the maximum mass configuration at the non-rotating model, they have a non-rotating member, and as a consequence, are stable and terminate at the non-rotating model sequence. Above this value, none of the sequences have a non-rotating member. Instead, they are unstable and terminate at the axisymmetric instability limit. The onset of instability limit from the Keplerian to the non-rotating sequence, which divides the stable from unstable region, it is called secular line. In particular, models to the right of the secular line are unstable while those that are left, stable. However, the total region that models are unstable is defined via the rest mass that corresponds to the maximum mass configuration of the non-rotating model and the Keplerian sequence, as Fig. 10 shows (gray region). Above this value, the models have masses larger than the maximum mass of the non-rotating model and in that case, are called supramassive models [4].

To be more specific, if a neutron star spin-up by accretion and become supramassive, then it would subsequently spin-down along the constant rest mass sequence until it reaches the axisymmetric instability limit and collapse to a black hole. There is a case, where some relativistic stars could be born as supramassive ones, or even more, become one as a result of a binary merger. In this case, the star would be initially differentially rotating and collapse would be triggered by a combination between spin-down effect and viscosity (the force that driving the star to uniform rotation) [4].

Although, the sequence with rest mass corresponding to the maximum mass configuration of the non-rotating model extends to the right area of the secular orbit, the unstable one, it is the last sequence that has a stable part (half of the sequence terminates at the maximum mass configuration of the non-rotating model). While, below this sequence, all the remaining ones are unconditionally stable against gravitational collapse, above this sequence, all sequences would evolve toward catastrophic collapse to a black hole. In Fig. 10 we can see that if we have a neutron star with rest mass in the white region, it would evolve toward stable configuration at the non-rotating sequence, but if we
massive evolutionary sequences, because their unstable normal evolutionary sequences, neutron stars on supramassive evolutionary sequences. In contradiction to neutron stars on evolutionary sequences, never spin-up as they lose angular momentum on the Kerr parameter for the APR-1 EoS. Non-rotating case is presented with the red curve while the rapidly-rotating, with the blue curve. Constant rest mass sequences are presented as the dependence of the gravitational mass on the central energy density and (b) the gravitational mass on the corresponding radius for the APR-1 EoS.

have a star in the gray region, it would subsequently spin-up and evolve toward catastrophic collapse to a black hole \([72–74]\). The latter is shown clearly in Fig. 11.

Following the concept from Fig. 11 we have constructed the last stable rest mass sequence (LSRMS) for the 23 realistic EoSs, as shown in Fig. 12. This sequence is the one that divides the stable from unstable region, or in other words, the normal from supramassive evolutionary sequences. In Fig. 12 we present a window (gray region) where the last stable rest mass sequence can lie and in fact, because the last stable rest mass sequence is the one that corresponds to the maximum mass configuration at the non-rotating model, this is also the region where the EoS can lie, constraining with this way, simultaneously, the spin parameter and the angular velocity. There is an empirical relation, derived from the data, that can describe this window. The form of this empirical relation, is

\[
\Omega = \left( b_1 K + b_2 K^2 + b_3 K^3 \right) 10^4 \quad (s^{-1}) \quad (26)
\]

where the coefficients for the two edges are shown in Table IV. It is clear from Fig. 12 and Eq. (26), that if we have a measurement of angular velocity, or spin parameter, we could extract the interval where the other parameter can lie. An average empirical relation had been also created, given by the expression

\[
\Omega = \left( (1.645 \pm 0.295) K - (0.094 \pm 0.211) K^2 - (0.755 \pm 0.31) K^3 \right) 10^4 \quad (s^{-1}) \quad (27)
\]
As a consequence, by constraining simultaneously these two quantities, we could significantly narrow the existing area of EoS.

![Graph](image)

Figure 12. Last stable rest mass sequences for the 23 EoSs as the dependence of the angular velocity on the Kerr parameter. Supramassive and normal area are shown to guide the eye. The maximum value of the Kerr parameter is also noted.

| Edges  | $b_1$ | $b_2$ | $b_3$ |
|-------|-------|-------|-------|
| Upper | 1.94  | 0.117 | -1.058|
| Lower | 1.35  | -0.305| -0.449|

Table IV. Coefficients of the empirical relation \(20\) for the two edges of the window presented in Fig. 12

E. Upper bound for density of cold baryonic matter

Although we study realistic EoSs in neutron stars, analytical solutions are far from being insignificant. Useful information can be gained by the comparison between the Einstein’s field equations with numerical solutions for different models of EoSs and the analytical solutions \[63\]. Two classes derive from analytical solutions: (a) normal neutron stars and (b) self-bound neutron stars. In the first case, the energy density vanishes at the surface where the pressure vanishes and in the second one, the energy density is finite at the surface.

In this work, only the first case scenario will be studied. While there are three analytical solutions in this scenario, we take into consideration here only the Tolman VII solution. It has been shown by Lattimer et al. \[63\] that the Tolman VII solution forms an absolute upper limit in density inside any compact star. This is also the case for rotating stars with rotation rates up to the Keplerian (mass-shedding) rate.

At that time, the maximum masses of the existed EoSs were fully included in the region under the Tolman VII solution; the same holds for the rotating models. In recent years, new EoSs have been introduced and old ones that could not describe the maximum observed neutron star mass \[6–9\] have been rejected. In this work, using a total of 23 realistic EoSs that predict the observed maximum mass \[6–9\], we have confirmed that the Tolman VII curve marks the upper limit to the energy density inside a star but without taking into account the rotation. If we add rotation to our models, then this curve is not able to describe anymore the new data as they shift, concerning the plotted area, up and left. For this reason, we propose here, that if there is a curve, like the Tolman VII solution, shifted to the right, that would be a suitable solution to fully describe the maximum energy density inside a star. In other words, the existence of this curve, can help us to form an absolute upper limit in density inside any compact object.

The proposed expression, described by the form

\[
\frac{M}{M_\odot} = 4.25 \sqrt{\frac{10^{15} \text{ gr cm}^{-3}}{\varepsilon_c/c^2}}
\]

(28)

can fully describe both non-rotating and rapidly-rotating configuration.

In Fig. 13 we present the results of the 23 realistic EoSs, for the non-rotating and rapidly-rotating case, Cook’s \[20\] and Salgado’s \[27\] data, Tolman VII analytical solution and the proposed solution \[28\]. The observed
Figure 14. Constant rest mass sequences as the dependence of moment of inertia on the angular velocity for five representative EoSs and with rest mass corresponding to (a) $M_{\text{max}}^{gr} = 1.45M_\odot$, (b) $M_{\text{max}}^{gr} = 2M_\odot$ and (c) $M_{\text{max}}^{gr} = 2.2M_\odot$. The data are presented with the circles and the fits with the dashed lines.

Figure 15. Constant rest mass sequences as the dependence of braking index on the $\Omega/\Omega_k$ for five representative EoSs and and with rest mass corresponding to (a) $M_{\text{max}}^{gr} = 1.45M_\odot$, (b) $M_{\text{max}}^{gr} = 2M_\odot$ and (c) $M_{\text{max}}^{gr} = 2.2M_\odot$.

neutron star mass limits are also presented to guide the eye.

Another interesting effect that presented via the Fig. 13 is the connection that establishes between the gravitational mass at the maximum mass configuration and the corresponding central energy density. Besides the fact that can provide us with the absolute upper limit in density inside any compact star, it also can directly connect the macroscopic properties of neutron star with the microscopic ones.

F. Equation of state effects on the braking index of pulsars

It is well-known that the angular velocity $\Omega$ of an isolated pulsar decreases very slowly with the time. Various energy loss mechanisms are responsible for this effect, including mainly dipole radiation, charged particles ejections and gravitational waves radiation \[2, 75, 81\]. In this case, and in the most simple model, the evolution of the angular velocity is given by the power law

$$\dot{\Omega} = \frac{d\Omega}{dt} = -H\Omega^n$$  \hspace{1cm} (29)

The braking index, $n$, of a pulsar, which describes the dependence of the braking torque on rotation frequency, is a fundamental parameter of pulsar electrodynamics. Simple theoretical arguments, based on the assumption of a constant dipolar magnetic field, predict $n = 3$. It is easy to show that Eq. (29) leads to the fundamental relation

$$n(\Omega) = \frac{\Omega\dot{\Omega}}{\dot{\Omega}^2}$$  \hspace{1cm} (30)

where dot corresponds to the derivative with time. Finally, Eq. (30) leads to

$$n(\Omega) = 3 - \frac{3\Omega' + \Omega^2\Omega''}{2\Omega' + \Omega''}$$  \hspace{1cm} (31)

where $\Omega' = d\Omega/d\Omega$ and $I'' = d^2I/d\Omega^2$. Now, considering the simple power law dependence $I \sim \Omega^\lambda$, the braking index takes the simple and transparent value

$$n(\Omega) = 3 - \lambda$$  \hspace{1cm} (32)

While for $\lambda = 0$ (moment of inertia independent from angular velocity) we recover the well-known result $n = 3$, in general we expect that the inequality $n(\Omega) \leq 3$ must hold. There is a special case where for some reasons when the denominator of Eq. (31) goes to zero, then the braking index exhibits a singularity which leads to increasing of $\Omega$ with time \[2, 75, 82, 83\]. This is an interesting effect (which may be caused due to a phase transition in
the interior of a pulsar) but we are not going to study it further in this work. Instead, we study the effects of the EoS on the braking index as well as on the evolution of the angular velocity of a pulsar, especially for very young, at their birth, with angular velocity being at the mass-shedding limit.

In particular, we study the dependence of the moment of inertia on the angular velocity for five representative EoSs and for three different rest masses. In each case, we produced a fit as shown in Fig. 14, according to the formulae

\[ I = g_1 + g_2 \exp[g_3 \Omega] \]

where \( g_1, g_2 \) are in units of moment of inertia \((10^{45} \text{gr cm}^2)\) and \( g_3 \) in units of time (s). Using Eq. (31) and Eq. (33) we extract, for each case, the relation between the braking index and the angular velocity, as shown in Fig. 15.

From Fig. 15, we can see that from the \( \sim 70\% \) of the Kepler angular velocity until it reaches the Kepler angular velocity, the angular velocity conjugates significant changes to the braking index. This effect is presented for each case with different rest mass. In order to see how the rest mass effects the braking index, we present at Fig. 16 the five representative EoSs for the different rest masses in a single figure. From Fig. 16, it is clear that the rest mass plays an important role on the braking index, i.e. by increasing the rest mass value, the braking index decreases more sharply. This effect will remain valid for all EoSs studied in this paper.

IV. DISCUSSION AND CONCLUSIONS

Different sequences of uniformly rotating neutron stars have been constructed for a large number of EoSs based on various theoretical nuclear models. For the numerical integration of the equilibrium equations, we used the public RNS code \cite{86} by Stergioulas and Friedman \cite{87} (This code is based on the method developed by Komatsu, Eriguchi and Hachisu \cite{88} and modifications introduced by Cook, Shapiro and Teukolsky \cite{17}). In this paper, we have studied the bulk properties of neutron stars in correlation with the mass-shedding limit (Keplerian frequency). To be more specific, we have calculated their gravitational and rest mass, equatorial and polar radii, dimensionless angular momentum, angular velocity, moment of inertia and eccentricity. Relations between the Keplerian frequency and the bulk properties of neutron stars, have been found and shown in the corresponding figures.

The dependence of the moment of inertia, eccentricity and Kerr parameter on the total gravitational mass at the Keplerian sequence, is also obtained. In all cases, the EoSs presented similar behavior, so as a follow up, we have studied the dependence of these parameters on the gravitational mass at the maximum mass configuration. We have concluded with this way, that moment of inertia and Kerr parameter can provide us with universal relations as a function of the gravitational mass at the maximum mass configuration for the Keplerian frequency. It is also interesting the effect of the eccentricity at the maximum mass configuration for the Keplerian frequency on the corresponding gravitational mass, where it seems that eccentricity behaves as an EoS-independent property. Moreover, we find that the Kerr parameter reaches a maximum value at around 0.75 (stiffest EoS) for neutron stars. The importance of this result falls under the fact that the gravitational collapse of a uniformly rotating neutron star, constrained to mass-energy and angular momentum conservation, cannot lead to a naked singularity, or in other words, a maximally rotating Kerr black hole \cite{48}.

Normal and supramassive sequences of constant rest mass for a specific EoS, have been constructed. In the corresponding figures, we present the stability and instability region of a neutron star. This is possible by plotting the evolution of a neutron star along the constant rest mass sequences. The extraordinary effect of supramassive sequences, is that they can inform us for the gravitational collapse to a black hole. The gravitational collapse of a rotating neutron star to a black hole, creates a black hole with almost the same mass and angular momentum as the initial star (small amount of total mass and angular momentum carried away by gravitation radiation \cite{89}), and therefore, the same Kerr parameter. Henceforth, this effect may provide us an observable precursor to gravitational collapse to a black hole. It is important to add here that this effect will remain valid for all the EoSs studied in this paper.

![Figure 16. The dependence of the braking index on the angular velocity for the five representative EoSs with constant rest masses. The lines correspond to the \( M_{\text{max}}^{\text{grav}} = 1.45M_\odot \), the dashed lines to the \( M_{\text{max}}^{\text{grav}} = 2M_\odot \) and the dotted lines to the \( M_{\text{max}}^{\text{grav}} = 2.2M_\odot \).](image-url)
In order to imply possible constraints on the EoS, we have constructed the LSRMS for the variety of the EoSs and the dimensionless moment of inertia. In particular, we have presented them in a figure of the angular velocity as a function of the Kerr parameter and the dimensionless moment of inertia as a function of the compactness parameter, respectively. In both cases, we have extracted a window where these properties can lie. In the first case, concerning the LSRMS, because this sequence is the one that corresponds to the maximum mass configuration at the non-rotating model, this is also the window where the EoS can lie, constraining with this way, simultaneously, the angular velocity and the spin parameter (or Kerr parameter) on neutron stars. In the second case, the window that is formed can help us to constrain the moment of inertia and the compactness parameter. The latter, can impose strong constraints in radius of neutron stars, which is one of the open problems in nuclear astrophysics.

Afterwards, we have updated the work of Lattimer and Prakash [62] by using EoSs which are in consistent with the current observed limits of neutron star mass [6–8]. In this work, we propose the possible existence of an empirical solution, similar to the Tolman VII analytical solution, for neutron stars with rotation rate close to the mass-shedding limit, in order to describe the rotating configuration. The existence of this solution can help us to define the ultimate density of cold baryonic matter by setting an absolute upper limit at the central energy density.

Finally, we have studied the effects of the EoSs on the braking index of pulsars. Braking index, as an intrinsic property of neutron star’s structure, can inform us about the rate of change of angular velocity. Although we know it is very slow, after the 70% of Keplerian angular velocity, braking index is undergoing significant changes through the influence of the rest mass. This specific area, from 70% through the 100% of Keplerian angular velocity, may provide us with useful insights on the constitution of the dense nuclear matter.

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**Appendix A: Properties of symmetric nuclear matter**

We present here the formulas for the properties of symmetric nuclear matter that appear in Table I.

Assuming that the neutron-proton asymmetry is characterized by the parameter [90, 91]

\[ I = \frac{n_n - n_p}{n} = 1 - 2x \]  \hspace{1cm} (A1)

where \( x \) is the proton fraction, the total energy per particle, can be expanded as follow

\[ E(n, I) = E(n, 0) + \sum_{k=2,4} E_{\text{sym}, k}(n) I^k \]  \hspace{1cm} (A2)

where

\[ E_{\text{sym}, k}(n) = \left. \frac{\partial^k E(n, I)}{\partial I^k} \right|_{I=0} \]  \hspace{1cm} (A3)

We studied two cases in this paper, the parabolic (symbolized as \( pa \)) and the full (symbolized as \( f \)) approximation. In the case of the parabolic approximation, we consider that

\[ E_{\text{sym,pa}}(n) = E(n, I = 1) - E(n, I = 0) \]  \hspace{1cm} (A4)

while in the full approximation

\[ E_{\text{sym,f}}(n) = E_{\text{sym,2}}(n) = S_2(n) \]  \hspace{1cm} (A5)

The other relevant properties that appear in Table I are defined as [90, 91]

\[ L = 3n_s \frac{dS_2(n)}{dn} \bigg|_{n_s}, \hspace{1cm} K = 9n_s^2 \frac{d^2 E(n, 0)}{dn^2} \bigg|_{n_s}, \]  \hspace{1cm} (A6)

\[ Q = 27n_s^3 \frac{d^3 E(n, 0)}{dn^3} \bigg|_{n_s}, \hspace{1cm} K_0 = 9n_s^2 \frac{d^2 S_2(n)}{dn^2} \bigg|_{n_s}, \]  \hspace{1cm} (A7)

\[ Q_0 = 27n_s^3 \frac{d^3 S_2(n)}{dn^3} \bigg|_{n_s}, \]  \hspace{1cm} (A8)

where \( L, K, Q \) are related to the first, second and third derivative of the symmetry energy, respectively. \( K_0 \) is the compression modulus and \( Q_0 \) is related to the third derivative of \( E \). The last property is the ratio of the Landau effective mass to mass in vacuum for the MDI model [64, 65, 90, 92], and given by
where $\tau$ corresponds to neutrons or protons.

Appendix B: Observed frequency limit

The 716 Hz limit from the present work is based on work of Lattimer and Prakash [68] and Riahi et al. [57]. Constructing the dependence of the $f_k/f_S$ on the parameter $M/R$ in units of $M_\odot/km$, where $f_S$ is

$$f_S = 1833 \left( \frac{M}{M_\odot} \right)^{1/2} \left( \frac{10km}{R} \right)^{3/2} (\text{Hz})$$

and it is the orbital frequency of a test particle spins around a spherical mass $M$ at a distance $R$, we have extracted a universal relation, which follows the form

$$f_k/f_S = 0.559 + 2.69 \left( \frac{M}{M_\odot} \right) \left( \frac{km}{R} \right) - 20.28 \left[ \left( \frac{M}{M_\odot} \right) \left( \frac{km}{R} \right) \right]^2 + 55.74 \left[ \left( \frac{M}{M_\odot} \right) \left( \frac{km}{R} \right) \right]^3$$

For the observed frequency of the fastest known pulsar, PSR J1748-2446ad, which rotates with a frequency of 716 Hz, we obtained the relation which is shown in Fig. 2.
