LIFTING COMMUTATION RELATIONS IN CUNTZ ALGEBRAS

BRUCE BLACKADAR

Abstract. We examine splitting of the quotient map from the full free product $A \ast B$, or the unital free product $A \ast C B$, to the (maximal) tensor product $A \otimes B$, for unital C*-algebras $A$ and $B$. Such a splitting is very rare, but we show there is one if $A$ and $B$ are both the Cuntz algebra $O_2$ or $O_\infty$, and in a few other cases. The splitting is not explicit (and in principle probably cannot be). We also describe severe $K$-theoretic obstructions to a splitting.

1. Introduction

Lifting commutation relations from quotients of C*-algebras is a difficult and often unsolvable problem. We consider the essentially generic case where $A$ and $B$ are C*-algebras (to avoid unnecessary technicalities, we will only consider the case of unital $A$ and $B$), and we have the quotient map from the full free product or full unital free product to the tensor product.

Recall that the full free product $A \ast B$ is the universal C*-algebra generated by copies of $A$ and $B$ with no relations (note that this free product is nonunital even if $A$ and $B$ are unital), the unital free product $A \ast C B$ is the universal C*-algebra generated by copies of $A$ and $B$ with a common unit, and the tensor product $A \otimes B$ is the universal C*-algebra generated by commuting copies of $A$ and $B$ with a common unit (all tensor products in this paper are maximal; our examples are nuclear, so this is not much of an issue). There is a natural quotient map $\pi$ from the full free product $A \ast B$, or the unital free product $A \ast C B$, to the tensor product $A \otimes B$.

We consider the question of whether there is a splitting (cross section) for this quotient map, i.e. a *-homomorphism $\sigma : A \otimes B \to A \ast B$ (or to $A \ast C B$, not necessarily unital) with $\pi \circ \sigma$ the identity on $A \otimes B$. The short answer is “rarely.”

Existence of a splitting for the quotient map from $A \ast B$ to $A \otimes B$ is equivalent to having a functorial procedure for beginning with two (not necessarily unital) *-homomorphisms $\phi : A \to D$ and $\psi : B \to D$ of $A$ and $B$ into a C*-algebra $D$ and manufacturing new *-homomorphisms $\tilde{\phi} : A \to D$ and $\tilde{\psi} : B \to D$ such that $\tilde{\phi}(A)$ and $\tilde{\psi}(B)$ commute and have a common unit. “Functorial” means:

(i) If $f : D \to D'$ is a *-homomorphism, then $\tilde{f} \circ \tilde{\phi} = f \circ \phi$ and $\tilde{f} \circ \tilde{\psi} = f \circ \psi$.

(ii) If $\phi(A)$ and $\psi(B)$ already commute and have a common unit, then $\tilde{\phi} = \phi$ and $\tilde{\psi} = \psi$.

Date: March 24, 2015.
[Such \( \phi \) and \( \psi \) define a *-homomorphism \( \phi \circ \psi \) from \( A \ast B \) to \( D \); let \( \tilde{\phi} \) and \( \tilde{\psi} \) be defined by \( (\phi \circ \psi) \circ \sigma \) for a splitting \( \sigma \)]. Such a functorial procedure is automatically point-norm continuous in the sense that if \( \phi_n : A \to D \) and \( \psi_n : B \to D \) converge point-norm to \( \phi \) and \( \psi \) respectively, then \( \tilde{\phi}_n \to \tilde{\phi} \) and \( \tilde{\psi}_n \to \tilde{\psi} \) point-norm respectively [it suffices to note that \( \phi_n \ast \psi_n \to \phi \ast \psi \) point-norm].

Existence of a splitting is also equivalent to being able to always solve the following lifting problem: given a C*-algebra \( D \), a (closed) ideal \( J \) of \( D \), and *-homomorphisms \( \phi : A \to D/J \) and \( \psi : B \to D/J \) such that \( \phi(A) \) and \( \psi(B) \) commute and have a common unit, and \( \phi \) and \( \psi \) separately lift to *-homomorphisms \( \phi : A \to D \) and \( \psi : B \to D \), find lifts \( \tilde{\phi} : A \to D \) and \( \tilde{\psi} : B \to D \) such that \( \tilde{\phi}(A) \) and \( \tilde{\psi}(B) \) commute and have a common unit.

We now give examples suggesting that splittings are “rare.”

**Example 1.1.** Consider the simplest nontrivial C*-algebra \( C^2 \). There is an explicit description of \( C^2 \ast C^2 \), which is the universal unital C*-algebra generated by two projections, as the continuous functions from \([0,1] \) to \( M_2 \) which are diagonal at the endpoints (cf. [Bla06], IV.1.4.2). From this description it is easy to see that there is no splitting for the quotient map from \( C^2 \ast C^2 \) to \( C^2 \otimes C^2 \) (which is just evaluation at the endpoints of \([0,1] \)). There is a fortiori no splitting for the quotient map from \( C^2 \ast C^2 \) to \( C^2 \otimes C^2 \), since this quotient map factors through \( C^2 \ast C^2 \).

There is also no splitting for the quotient map from \( C^2 \ast C^2 \) to \( C^2 \ast C^2 \) \([EE02]\).

**Example 1.2.** Here is an easier example. Let \( n > 1 \). Then the quotient map from \( M_n \ast C M_n \) to \( M_n \otimes M_n \) cannot split (unitally). For \( M_n \ast C M_n \) has \( M_n \) as a quotient, but there is no nonzero homomorphism from \( M_n \otimes M_n \cong M_n \otimes M_n \) to \( M_n \). (Actually it cannot split nonunitally either; cf. Theorem 3.3 \$4 \).

Similarly, if \( m \) and \( n \) (\( > 1 \)) are not relatively prime, there is no splitting for the quotient map from \( M_m \ast C M_n \) to \( M_m \otimes M_n \), since \( M_m \ast C M_n \) has \( M_p \) as a quotient, where \( p \) is the least common multiple of \( m \) and \( n \), and \( p < mn \). If \( m \) and \( n \) are relatively prime, the question is more delicate, but it seems unlikely that \( M_m \ast C M_n \) contains a unital copy of \( M_{mn} \) (there is no such copy if either \( m \) or \( n \) is prime \([RV98]\)).

These obstructions to a splitting are K-theoretic; see section 3 for a discussion.

There is somewhat more hope in general of getting a splitting if the free product is replaced by a “soft tensor product” \( A \hat{\otimes} e B \) where the copies of \( A \) and \( B \) are required to approximately commute (there are various ways to make this precise; cf. [211]). In the case of \( C^2 \otimes C^2 \), we want to restrict to the case where the commutator \( pq - qp \) of the two generating projections has small norm, say \( \leq \epsilon \) for a specified \( \epsilon > 0 \). In fact, in this case, as soon as \( \epsilon < 1/2 \) there is a splitting, as is easily seen. A similar result holds for \( M_m \hat{\otimes} e M_n \), where the matrix units of the two matrix algebras \( e \)-commute, and more generally for \( A \hat{\otimes} e B \) for \( A \) and \( B \) finite-dimensional, for small enough \( \epsilon \).

But in only slightly more general settings there is no splitting even in the soft tensor product case:

**Example 1.3.** If we regard \( C(T) \otimes C(T) \) as the universal C*-algebra generated by two commuting unitaries, and we let \( C(T) \hat{\otimes} e C(T) \) be the “soft torus,” the universal C*-algebra generated by two unitaries \( u \) and \( v \) with \( \|uv - vu\| \leq \epsilon \) \([Exe93], [EEL91], [EE02]\), then there is no splitting for the natural quotient map from \( C(T) \hat{\otimes} e C(T) \)
to $C(T) \otimes C(T)$ for any $\epsilon > 0$ (the Voiculescu matrices ([Voi83], [EL89]) can be used to show this is impossible.) Thus a fortiori there can be no splitting for the quotient map from $C(T) \ast_C C(T)$ to $C(T) \otimes C(T)$, since this quotient map factors through $C(T) \otimes_C C(T)$. This example can be essentially summarized by saying that $C(T^2) \cong C(T) \otimes C(T)$ is not semiprojective ([2.3]).

**Example 1.4.** Not all obstructions to a splitting are $K$-theoretic. For example, there is no splitting for the quotient map from $C([0, 1]) \circledast_{\epsilon} C([0, 1])$ to $C([0, 1]) \otimes C([0, 1]) \cong C([0, 1]^2)$ for any $\epsilon > 0$ since $C([0, 1]^2)$ is not semiprojective ([2.3]), and hence no splitting for $C([0, 1]) \ast_C C([0, 1]) \to C([0, 1]) \otimes C([0, 1])$ even though there is a splitting on the $K$-theory level. (There is, however, a splitting for the quotient map from $C([0, 1]) * C([0, 1])$ to $C([0, 1]) *_C C([0, 1])$; in fact, there is a simple explicit cross section for the map from $A * C([0, 1])$ to $A *_C C([0, 1])$ for any unital $A$.)

One can ask whether the quotient map from $A \ast_C B$ to $A \otimes B$ ever splits (if neither $A$ nor $B$ is $C$). Perhaps surprisingly, the answer is yes: it splits if $A$ and $B$ are certain Kirchberg algebras (but far from all pairs of Kirchberg algebras). For example, in one of our main results we show (Corollary [4.0]) it splits if $A = B = O_2$ or if $A$ is any semiprojective Kirchberg algebra and $B = O_\infty$. In fact, even the quotient map from $A \ast B$ to $A \otimes B$ splits in these cases. It has been known that in each of these cases, the quotient map from $A \ast B$ to $A \otimes B$ splits if $\epsilon > 0$ is sufficiently small, where $A \ast B$ denotes the universal $C^*$-algebra in which the standard generators and their adjoints in the two algebras $\epsilon$-commute (this is easily proved from the definition of semiprojectivity, cf. [2.3]): but it came as a surprise to the author that there is a splitting even when no commutation condition (or any other relation) between the algebras is assumed.

See Section [2] for definitions and a brief discussion of semiprojectivity and Kirchberg algebras, and [Bla06] for general information about $C^*$-algebras. [Lor97] contains an extensive discussion of lifting problems and semiprojectivity for $C^*$-algebras.

### 2. Preliminaries

**2.1.** If $A$ and $B$ are separable and unital, and $G$ and $H$ are sets of generators for $A$ and $B$ respectively which are finite or sequences converging to 0, and $\epsilon > 0$, we define $A \circledast_{\epsilon} B$ to be the universal unital $C^*$-algebra generated by unital copies of $A$ and $B$ such that $\|[a, b]\| \leq \epsilon$ for all $a \in G$, $b \in H$. The notation does not reflect the dependence on $G$ and $H$, which is not important qualitatively; if we instead write $A \circledast_{\epsilon, G, H} B$, and $G'$ and $H'$ are different sequences of generators converging to 0, then for any $\epsilon > 0$ there is an $\epsilon' > 0$ such that the natural quotient map from $A \circledast_{\epsilon, G, H} B$ to $A \otimes B$ factors through $A \circledast_{\epsilon', G', H'} B$. We will thus fix $G$ and $H$ and just write $A \circledast_{\epsilon} B$.

The natural quotient map from $A \circledast_{\epsilon} B$ to $A \otimes B$ factors through $A \circledast_{\epsilon'} B$ for any $\epsilon' < \epsilon$.

**2.2.** Recall the definition of semiprojectivity ([Bla85], [Bla06] II.8.3.7]): A separable $C^*$-algebra $A$ is semiprojective if, whenever $D$ is a $C^*$-algebra, $(J_n)$ an increasing sequence of closed (two-sided) ideals of $D$, and $J = \bigcup J_n$, then any homomorphism $\phi: A \to D/J$ can be partially lifted to a homomorphism $\psi: A \to D/J_n$ for some sufficiently large $n$.
If $A$ is unital, then in checking semiprojectivity from the definition $D$ and $\phi$ may be chosen unital, and then the partial lift $\psi$ can be required to be unital (in fact, $\psi$ will automatically become unital if $n$ is sufficiently increased).

There are many known examples of semiprojective $C^*$-algebras, but semiprojectivity is quite restrictive. For example, if $X$ is a compact metrizable space, $C(X)$ is semiprojective if and only if $X$ is an ANR of dimension $\leq 1$ [ST12]; the dimension restriction is essentially exactly the fact that commutation relations cannot be partially lifted in general. See also [2,5]

**Proposition 2.3.** Let $A$ and $B$ be unital semiprojective $C^*$-algebras. Then the quotient map from $A \otimes B$ to $A \otimes B$ splits (unitally) for some $\epsilon > 0$ (hence for all sufficiently small $\epsilon > 0$) if and only if $A \otimes B$ is semiprojective.

**Proof.** Suppose $A \otimes B$ is semiprojective. Set $D = A \ast C B$ and $J_n$ the kernel of the natural quotient map from $D$ onto $A \otimes /n B$. Then $(J_n)$ is an increasing sequence of closed ideals of $D$, and if $J = [\cup_n J_n]^{-1}$, then $D/J \cong A \otimes B$. By semiprojectivity there is a (unital) partial lift of the identity map on $A \otimes B$ for some $n$.

Conversely, suppose there is a splitting $\sigma$ for the quotient map from $A \otimes B$ to $A \otimes B$ for some $\epsilon > 0$. Let $D = (J_n) \ast C B$ be unital semiprojective $C^*$-algebra. Then $D/J_n$ is a lift of $\phi \circ \pi$, where $\pi$ is the quotient map from $A \ast C B$ to $A \otimes B$ (this is essentially the argument that shows that $A \ast C B$ is semiprojective). Since the partial lifts of the generators of $A$ and $B$ asymptotically commute as $n \to \infty$, if $n$ is sufficiently increased, the map $\psi$ factors through $A \otimes B$. Then $\psi \circ \sigma$ is a partial lift of $\phi$, so $A \otimes B$ is semiprojective. \qed

**2.4.** A Kirchberg algebra is a separable nuclear purely infinite (simple unital) $C^*$-algebra. It is not known whether such a $C^*$-algebra is automatically in the bootstrap class for the Universal Coefficient Theorem [Bla98, 22.3.4]; a Kirchberg algebra in this bootstrap class is called a UCT Kirchberg algebra.

The first Kirchberg algebras to be studied were the Cuntz algebras $O_n$, $2 \leq n \leq \infty$. If $2 \leq n < \infty$, $O_n$ is the universal (unital) $C^*$-algebra generated by $n$ isometries with mutually orthogonal range projections adding up to the identity. $O_n$ is the universal (unital) $C^*$-algebra generated by a sequence of isometries with mutually orthogonal range projections. The next class was the (simple) Cuntz-Krieger algebras $O_A$. These are all UCT Kirchberg algebras.

Kirchberg (in part done also independently by Phillips) showed that the UCT Kirchberg algebras are classified by their $K$-theory. If $A$ is a unital $C^*$-algebra, write $L(A)$ for the triple $(K_0(A), K_1(A), [1_A])$; $L(A)$ is part of the Elliott invariant of $A$, and is the entire Elliott invariant of $A$ if $A$ is a Kirchberg algebra. A unital $*$-homomorphism $\phi$ from $A$ to $B$ induces a morphism from $L(A)$ to $L(B)$, i.e. a homomorphism $\phi_{*0} : K_0(A) \to K_0(B)$ with $\phi_{*0}([1_A]) = [1_B]$ and a homomorphism $\phi_{*1} : K_1(A) \to K_1(B)$. The following facts are known (see [Ror02] for a good exposition):

(i) If $A$ and $B$ are UCT Kirchberg algebras and $\alpha : L(A) \to L(B)$ is a morphism, then there is a unital $*$-homomorphism $\phi : A \to B$ with $\phi_{*} = \alpha$. If $\alpha$ is an isomorphism, $\phi$ can be chosen to be an isomorphism. In particular, $L(A)$ is a complete isomorphism invariant of $A$ among UCT Kirchberg algebras.
(ii) If \( G_0 \) and \( G_1 \) are any countable abelian groups (recall that the \( K \)-groups of any separable C*-algebra are countable), and \( u \) is any element of \( G_0 \), then there is a UCT Kirchberg algebra \( A \) (unique up to isomorphism) with \( L(A) \cong (G_0, G_1, u) \).

Cuntz showed that \( L(O_n) = (\mathbb{Z}_{n-1}, 0, 1) \) if \( n < \infty \) and \( L(O_\infty) = (\mathbb{Z}, 0, 1) \). As technical results toward the above classification, Kirchberg ([KP00]; cf. [Rør02]) showed the following facts about tensor products:

(iii) For any Kirchberg algebra \( A \) (UCT or not), \( O_2 \otimes A \cong O_2 \). In particular, \( O_2 \otimes O_2 \cong O_2 \).

(iv) For any Kirchberg algebra \( A \) (UCT or not), \( O_\infty \otimes A \cong A \). In particular, \( O_\infty \otimes O_\infty \cong O_\infty \).

(v) The infinite tensor products \( O_2^\infty \) and \( O_\infty^\infty \) are isomorphic to \( O_2 \) and \( O_\infty \) respectively.

The isomorphism \( O_2 \otimes O_2 \cong O_2 \) (which was first shown by Elliott; cf. [Rør94]) and the others are quite deep results. They are highly nonconstructive; in fact, it is in principle essentially impossible to give an explicit isomorphism of \( O_2 \otimes O_2 \) and \( O_\infty \) by the results of [AC13].

2.5. It has recently been shown [End] that if \( A \) is a UCT Kirchberg algebra, then \( A \) is semiprojective if and only if \( K_*(A) \) is finitely generated (in fact, this problem was the original motivation for the work of the present paper). Special cases were previously known: it is easy to show that \( O_n \) \((n < \infty)\) and, more generally, \( O_A \) for any \( A \) is semiprojective [Bla85]. \( O_\infty \) was shown to be semiprojective in [Bla04], and more general results were obtained in [Szy02] and [Spi09].

3. \( K \)-Theoretic Obstructions, Unital Free Product Case

It turns out that there are rather severe \( K \)-theoretic obstructions to a splitting for the quotient map from \( A \ast C B \) to \( A \otimes B \). In this section, we will restrict attention to separable nuclear C*-algebras in the UCT class [Bla88 22.3.4], so that the Künneth Theorem for Tensor Products [Bla98 23.1.3] and the results of [Ger97] hold, although some of what we do here works in greater generality. The analysis is largely an elementary (but moderately complicated) exercise in group theory, so some details are omitted.

Let \( A \) and \( B \) be unital C*-algebras of the above form. We recall the results of the Künneth theorem and of [Ger97]:

\[
\begin{align*}
K_0(A \ast B) & \cong K_0(A) \oplus K_0(B) \\
K_1(A \ast B) & \cong K_1(A) \oplus K_1(B) \\
K_0(A \ast_C B) & \cong [K_0(A) \oplus K_0(B)]/([1_A], -[1_B]) \\
K_1(A \ast_C B) & \cong K_1(A) \oplus K_1(B)
\end{align*}
\]

\[
\begin{align*}
K_0(A \otimes B) & \cong [K_0(A) \otimes ZK_0(B)] \oplus [K_1(A) \otimes ZK_1(B)] \oplus \text{Tor}_1^Z(K_0(A), K_1(B)) \oplus \text{Tor}_1^Z(K_1(A), K_0(B)) \\
K_1(A \otimes B) & \cong [K_0(A) \otimes ZK_1(B)] \oplus [K_1(A) \otimes ZK_0(B)] \oplus \text{Tor}_1^Z(K_0(A), K_0(B)) \oplus \text{Tor}_1^Z(K_1(A), K_1(B))
\end{align*}
\]

where \( \text{Tor}_1^Z \) denotes the Tor-functor of homological algebra.
The quotient map from $A \ast B$ to $A \ast_C B$ induces the obvious maps on the $K$-groups. The quotient map $\pi$ from $A \ast_C B$ to $A \otimes B$ induces the following maps: on $K_0$, $\pi_0(x, y) = x \otimes [1_B] + [1_A] \otimes y$ in the $K_0(A) \otimes_Z K_0(B)$ summand; on $K_1$, $\pi_1(x, y) = ([1_A] \otimes y) \oplus (x \otimes [1_B])$ in the $[K_0(A) \otimes_Z K_1(B)] \oplus [K_1(A) \otimes_Z K_0(B)]$ summands.

If there is a splitting, there must be cross sections for these maps. Thus there is a $K$-theoretic obstruction almost any time both $A$ and $B$ have nontrivial $K_1$, or if the rank of $K_0$ of both $A$ and $B$ is at least 2. Specifically, we have:

**Proposition 3.1.** A splitting for the quotient map from $A \ast_C B$ to $A \otimes B$ is impossible under any of the following conditions:

1. $(\text{rank}(K_0(A)) \oplus \text{rank}(K_0(B))) > \text{rank}(K_0(A)) + \text{rank}(K_0(B))$, or $\text{rank}(K_0(A)) + \text{rank}(K_0(B)) - 1$ if $[1_A]$ or $[1_B]$ has infinite order.

2. $\text{Tor}_1^G(K_*(A), K_*(B)) \neq 0$.

This is only the beginning of the $K$-theoretic obstructions. We will analyze the case where $K_*(A)$ and $K_*(B)$ are finitely generated. Then we have

$$K_0(A) = \mathbb{Z}^a \oplus G_0, \quad K_1(A) = \mathbb{Z}^n \oplus G_1$$

$$K_0(B) = \mathbb{Z}^b \oplus H_0, \quad K_1(B) = \mathbb{Z}^m \oplus H_1$$

for nonnegative integers $a, b, n, m$ and finite abelian groups $G_0, G_1, H_0, H_1$.

In order to have a splitting, from Proposition 3.1(iii) we need the orders of $G_0$ and $G_1$ to be relatively prime to the orders of $H_0$ and $H_1$, so we will assume this from now on. We also have that $K_1(A) \otimes_Z K_1(B) = 0$ is necessary for a splitting by Proposition 3.1(i), which implies that $n$ and $m$ cannot both be nonzero; without loss of generality we assume $m = 0$. We must have $\mathbb{Z}^n \otimes_Z H_1 \cong H_1^n = 0$, so either $n = 0$ or $H_1 = 0$. We then obtain:

1. $$K_0(A \ast_C B) = (\mathbb{Z}^{a+b} \oplus G_0 \oplus H_0)/([1_A], -[1_B])$$

2. $$K_0(A \otimes B) = \mathbb{Z}^{ab} \oplus G_0^b \oplus H_0^a$$

3. $$K_1(A \ast_C B) = \mathbb{Z}^n \oplus G_1 \oplus H_1$$

4. $$K_1(A \otimes B) = \mathbb{Z}^{ab} \oplus G_1^b \oplus H_1^a \oplus H_0^b$$

since $G_i \otimes_Z H_j = 0$ for all $i, j$.

We need for the induced maps from $K_i(A \ast_C B)$ to $K_i(A \otimes B)$ to be surjective and have a cross section. For this, we need $H_0^n = 0$, so $n = 0$ or $H_0 = 0$.

**3.2.** There are two trivial cases: $K_*(A)$ is either $(0, 0)$, or $(\mathbb{Z}, 0)$ and $[1_A] = 1$, in which case there is no restriction on $K_*(B)$. (The situation is, of course, symmetric in $A$ and $B$.)

**3.3.** Next we can easily dispose of the case where $n > 0$. In this case we observed earlier that $H_0 = H_1 = 0$, and from (4) we see that $b \leq 1$. Thus $K_*(B)$ is either $(0, 0)$ or $(\mathbb{Z}, 0)$. Suppose $K_0(B) = \mathbb{Z}$. The map $\pi_1$ from $K_1(A \ast_C B) \cong \mathbb{Z}^n \oplus G_1$ to $K_1(A \otimes B) \cong \mathbb{Z} \oplus G_1$ is multiplication by $[1_B]$: for this to be surjective, we need for $[1_B]$ to be a generator of $K_0(B)$. Thus we are in one of the trivial cases.
3.4. From now on we assume $n = 0$, i.e. $K_1(A)$ and $K_1(B)$ are finite groups. The situation is now symmetric in $A$ and $B$. The map from $K_0(A \ast \mathcal{C} B)$ to $K_0(A \otimes B)$ must send $\mathbb{Z}^a \oplus \langle [1_A] \rangle \oplus \langle [1_B] \rangle$ onto $\mathbb{Z}^{ab}$, and thus we must have $ab \leq a + b$, and if either the torsion-free part of $[1_A]$ or $[1_B]$ is nonzero, we must have $ab \leq a + b - 1$. Thus we have that either $a$ or $b$ is $\leq 1$, or $a = b = 2$ and the torsion-free part of both $[1_A]$ and $[1_B]$ is $0$. But in the case $a = b = 2$, the maps $\pi_{*,0}$ on the torsion-free part of $K_0$ are zero and cannot be surjective. Thus there is no splitting in this case.

3.5. Now assume without loss of generality that $a = 0$ or $1$. We first dispose of the cases where $[1_A]$ or $[1_B]$ has finite order (which includes the case $a = 0$). Suppose
\[
(K_0(A), K_1(A), [1_A]) = (\mathbb{Z} \oplus G_0, G_1, (u, r))
\]
\[
(K_0(B), K_1(B), [1_B]) = (\mathbb{Z}^b \oplus H_0, H_1, (v, s))
\]
Then
\[
K_*(A \ast \mathcal{C} B) = ((\mathbb{Z} \oplus G_0 \oplus \mathbb{Z}^b \oplus H_0)/(u, r, v, s)), G_1 \oplus H_1)
\]
\[
K_*(A \otimes B) = (\mathbb{Z}^b \oplus G_0^b \oplus H_0, G_1^b \oplus H_1)
\]
If $u = 0$, then $\pi_{*,0}$ maps $\mathbb{Z}^b \oplus H_0$ to $0$, so for the map to be surjective we must have $b \leq 1$ and $H_0 = 0$; if $b = 1$, for $\pi_{*,0}$ to map $\mathbb{Z} \oplus G_0$ onto $K_0(A \otimes B)$ we must have $v = 1$. Similarly, $\pi_{*,1}$ maps the $H_1$ to $0$, so $H_1 = 0$. Thus we are in a trivial case (for $B$).

If $v = 0$, we argue similarly that $G_0 = G_1 = 0$ and $u = 1$, so we are again in a trivial case (for $A$).

Now consider the case
\[
(K_0(A), K_1(A), [1_A]) = (G_0, G_1, (r))
\]
\[
(K_0(B), K_1(B), [1_B]) = (\mathbb{Z}^b \oplus H_0, H_1, (v, s))
\]
where $b \geq 1$ and $v = 0$. Then as before $G_0 = G_1 = 0$ and we are in a trivial case (for $A$).

The last case is where both $K_*(A)$ and $K_*(B)$ are torsion (finite) groups. Then
\[
K_*(A \otimes B) = (0, 0)
\]
and there is no $K$-theoretic restriction.

3.6. The remaining case is where $a = 1$, $b \geq 1$, and $[1_A]$ and $[1_B]$ have infinite order. This is the most delicate case. Write $u$ and $v$ for the components of $[1_A]$ and $[1_B]$ in the torsion-free parts of $K_0(A)$ and $K_0(B)$ respectively; we may assume without loss of generality that $u > 0$ and that $v = (w, 0, \ldots, 0)$ with $w > 0$.

First consider the case $b = 1$. We have
\[
K_0(A \ast \mathcal{C} B) \cong (\mathbb{Z} \oplus G_0 \oplus \mathbb{Z} \oplus H_0)/(u, r, -w, -s) \cong \mathbb{Z}^{2} / \langle (u, -w) \rangle \oplus G_0 \oplus H_0
\]
and the map $\pi_{*,0}$ to $K_0(A \otimes B) = \mathbb{Z} \oplus G_0^b \oplus H_0$ must send $\mathbb{Z}^2 / \langle (u, -w) \rangle$ onto the first coordinate. This map is multiplication by $w$ on the first coordinate plus multiplication by $u$ on the second; thus it can only be surjective if $u$ and $w$ are relatively prime. In this case, it is both injective and surjective on this piece.

The map $\pi_{*,0}$ maps the $G_0$ to the $G_0$ and is multiplication by $w$. For this to be surjective (hence bijective), we must have $wG_0 = G_0$, i.e. $w$ must be relatively prime to the order of $G_0$. Similarly, $u$ must be relatively prime to the order of $H_0$.

The same argument for $\pi_{*,1}$ shows that $u$ and $w$ are relatively prime to the orders of $H_1$ and $G_1$ respectively.
We cannot rule out the case where \(u\) and \(v\) are greater than 1 and relatively prime. For example, consider the case \(u = 2, v = 3\). For simplicity suppose the \(K\)-groups are torsion-free, i.e. \(K_*(A) = (Z, 0, 2)\) and \(K_*(B) = (Z, 0, 3)\). Then \(K_0(A \ast C B) = Z^2/\langle(2, -3)\rangle\) with order unit \([2, 0] = [0, 3]\). \(K_0(A \otimes B) = Z\) with order unit 6, and the map \(\pi_{*0}\) sends \([x, y]\) to \(3x + 2y\), and is an isomorphism; the inverse map sends \(n\) to \([n, -n]\). It is possible that this inverse map is induced by a homomorphism on the algebra level in some cases (e.g. possibly in the case \(A = M_2(O_\infty)\) and \(B = M_3(O_\infty)\)), although it is not in the case \(A = M_2, B = M_3\).

Thus the possibilities are

\[
L(A) = (Z \oplus G_0, G_1, (u, r)), \quad L(B) = (Z \oplus H_0, H_1, (w, s))
\]

where \(u\) is relatively prime to \(w\) and to the orders of \(H_0\) and \(H_1\), and \(w\) is relatively prime to the orders of \(G_0\) and \(G_1\).

**3.7.** Now consider the case \(b > 1\). We have

\[
K_0(A \ast C B) \cong (Z \oplus G_0 \oplus Z^b \oplus H_0)/\langle(u, r, -w, 0, \ldots, 0, -s)\rangle
\]

\[
\cong Z^2/\langle(u, -w)\rangle \oplus Z^{b-1} \oplus G_0 \oplus H_0
\]

and the map \(\pi_{*0}\) to \(K_0(A \otimes B) = Z^b \oplus G_0^b \oplus H_0\) must send \(Z^2/\langle(u, -w)\rangle\) onto the first coordinate. This map is multiplication by \(w\) on the first coordinate plus multiplication by \(u\) on the second; thus it can only be surjective if \(u\) and \(w\) are relatively prime. In this case, it is both injective and surjective on this piece.

The map \(\pi_{*0}\) sends \(G_0\) into \(G_0^b\) and sends \(x\) to \((wx, 0, \ldots, 0)\). This can only be surjective if \(G_0 = 0\). Similarly, \(\pi_{*1}\) sends \(G_1\) into \(G_1^b\) by the same formula, so \(G_1 = 0\). Thus \(K_*(A) = (Z, 0)\).

However, we do not need to have \(u = 1\). But there is a restriction: the map \(\pi_{*0}\) sends \(H_0\) into \(H_0\) by multiplication by \(u\); this needs to be surjective, i.e. \(u\) is relatively prime to the order of \(H_0\). Similarly, \(uH_1 = H_1\), so \(u\) is relatively prime to the order of \(H_1\). Thus, for any \(b > 1\), we have the possibilities

\[
L(A) = (Z, 0, u), \quad L(B) = (Z^b \oplus H_0, H_1, (w, 0, \ldots, 0, s))
\]

where \(u\) is relatively prime to \(w\) and to the orders of \(H_0\) and \(H_1\).

We summarize:

**Theorem 3.8.** Let \(A\) and \(B\) be separable nuclear unital \(C^*\)-algebras in the UCT class with finitely generated \(K\)-theory. Then there can be a splitting for the quotient map from \(A \ast C B\) to \(A \otimes B\) only in the following situations:

(i) \(L(A) = (0, 0, 0)\) or \((Z, 0, 1)\), \(L(B)\) arbitrary (or vice versa). In the first case \(L(A \otimes B) = (0, 0, 0)\), and in the second \(L(A \otimes B) \cong L(B)\).

(ii) \(L(A) = (G_0, G_1, r)\), \(L(B) = (H_0, H_1, s)\). In this case \(L(A \otimes B) = (0, 0, 0)\).

(iii) \(L(A) = (Z \oplus G_0, G_1, (u, r))\), \(L(B) = (Z \oplus G_0, H_1, (w, s))\), where \(u\) is relatively prime to \(w\) and to the orders of \(H_0\) and \(H_1\), and \(w\) is relatively prime to the orders of \(G_0\) and \(G_1\).

(iv) \(L(A) = (Z, 0, u)\), \(L(B) = (Z^b \oplus H_0, H_1, (w, 0, \ldots, 0, s))\), where \(b > 1\), and \(u\) is relatively prime to \(w\) and to the orders of \(H_0\) and \(H_1\).

In (ii)–(iv), \(G_0, G_1, H_0, H_1\) are any finite abelian groups with the orders of \(G_0\) and \(G_1\) relatively prime to the orders of \(H_0\) and \(H_1\).
In all cases except (ii), $K_*(A \otimes B)$ is the same as $K_*(A \otimes B)$ and the maps $\pi_{*0}$ and $\pi_{*1}$ are isomorphisms. In case (ii), we have

$$L(A \otimes B) = ((G_0/rG_0) \oplus (H_0/sH_0), G_1 \oplus H_1, 0)$$

i.e. the identity always has class 0.

Theorem 3.8 does not guarantee a splitting in these cases; there often is not. For one thing, there may be more obstructions coming from ordered $K$-theory:

**Example 3.9.** (cf. Example 1.2) Suppose $K_*(A) = K_*(B) = (\mathbb{Z}, 0)$ but $[1_A] = m$, $[1_B] = n$ with $m,n > 1$. Then $K_*(A \otimes B) \cong (\mathbb{Z}, 0)$ and $K_*(A \otimes B) \cong (\mathbb{Z}^2/((m,-n)), 0)$, so there is potentially a splitting on the group level. But depending on the actual map $\pi_*$ there may not be one at the scaled ordered group level, or even at the group level.

Suppose $n = m$. Then $K_0(A \otimes B) \cong \mathbb{Z} \oplus \mathbb{Z}_n$, and the order unit in $K_0(A \otimes B)$ is $(n,0)$. The order unit in $K_0(A \otimes B)$ is $n^2$. The map $\pi_* : K_0(A \otimes B) \to K_0(A \otimes B)$ is multiplication by $n$, which is not surjective, so there can be no cross section. More generally, in $n$ and $m$ are not relatively prime, and $d$ is their greatest common divisor and $p$ is their least common multiple, then $K_0(A \otimes B) \cong \mathbb{Z} \oplus \mathbb{Z}_d$ and the order unit in $K_0(A \otimes B)$ is $(p,0)$. The order unit in $K_0(A \otimes B)$ is $mn$; $\pi_*$ is multiplication by $\frac{mn}{p} = d$ and is not surjective.

In these cases there is not a splitting at the group level. But suppose $m$ and $n$ are relatively prime. Then there is a splitting at the group level ($\pi_*$ is an isomorphism).

But the positive cone in $K_0(A \otimes B)$ can be larger than the positive cone in the unital free product $K_0(A \otimes B)$, for example if $A = \mathbb{M}_m$, $B = \mathbb{M}_n$; in this case the positive cone in $K_0(A \otimes B) \cong \mathbb{Z}$ is the usual one, but it is shown in [RV98] that (at least if $m$ or $n$ is prime) the positive cone in $K_0(A \otimes B)$ (or $K_0(A \otimes B)$ is the subsemigroup of $\mathbb{Z}$ generated by $m$ and $n$.

**Example 3.10.** The situation can be nicer with Kirchberg algebras: the unital free product $M_2(\mathbb{O}_\infty) \ast \mathbb{C} M_3(\mathbb{O}_\infty)$ contains a unital copy of $\mathbb{M}_6$. Let $\{ij : 1 \leq i, j \leq 3\}$ be the standard matrix units in the $M_3(\mathbb{O}_\infty)$. There are mutually orthogonal subprojections $f_1, f_2, f_3, g_1, g_2$ of $e_{11}$ in $M_3(\mathbb{O}_\infty)$ adding up to $e_{11}$ such that $f_1, f_2, f_3$ are equivalent to $e_{11}$ and $g_1$ and $g_2$ are equivalent (take $[f_k] = [e_{11}]$ and $[g_k] = [-e_{11}]$ for all $k$ in $K_0(\mathbb{O}_\infty) \cong \mathbb{Z}$). Then

$$f_1 + f_2 + f_3 \sim 1_{M_3(\mathbb{O}_\infty)}.$$  

We have that $1_{M_3(\mathbb{O}_\infty)} = 1_{M_3(\mathbb{O}_\infty)}$ since the free product is unital, and $1_{M_3(\mathbb{O}_\infty)}$ is the sum of two equivalent projections in the $M_2(\mathbb{O}_\infty)$, so $f_1 + f_2 + f_3$ is the sum of two equivalent projections $p_1, p_2$. Set $p_{11} = r_1 + g_1$ and $p_{12} = r_2 + g_2$; then $e_{11} = p_{11} + p_{12}$, and $p_{11} \sim p_{12}$. For $j = 2, 3$ set $p_{j1} = e_{j1}p_{11}e_{j1}$ and $p_{j2} = e_{j1}p_{12}e_{j1}$; then

$$\{p_{jk} : 1 \leq j \leq 3, 1 \leq k \leq 2\}$$

are six mutually orthogonal equivalent projections in $M_2(\mathbb{O}_\infty) \ast \mathbb{C} M_3(\mathbb{O}_\infty)$ adding up to the identity.

This argument generalizes to show that if $m$ and $n$ are relatively prime, then $M_m(\mathbb{O}_\infty) \ast \mathbb{C} M_n(\mathbb{O}_\infty)$ contains a unital copy of $\mathbb{M}_{mn}$. Thus there is no $K$-theoretic obstruction to a splitting in this case.

**Example 3.11.** If $A$ or $B$ does not have finitely generated $K$-theory, the situation can be much more complicated. For example, if $A$ is the CAR algebra, there does
not appear to be any \( K \)-theoretic obstruction to a cross section for the quotient map from \( A \ast_c A \) to \( A \otimes A \), but it is questionable that there is a splitting in this case.

One unresolved question is: if there is a splitting for the quotient map from \( A \ast_c B \) to \( A \otimes B \), is there necessarily a unital splitting (and is a splitting even necessarily unital)? It follows from Theorem 3.8 and the comment afterward that in the UCT case with finitely generated \( K \)-theory, any splitting must necessarily at least send the identity of \( A \otimes B \) to a projection in \( A \ast_c B \) whose \( K_0 \)-class is the same as the \( K_0 \)-class of the identity.

4. \( K \)-Theoretic Obstructions, General Free Product Case

We can do an essentially identical analysis of obstructions to splitting of the quotient map from \( A \ast B \) to \( A \otimes B \). But this case can also be handled more easily: if there is a splitting for this map, there is also a splitting for the quotient map from \( A \ast_c B \) to \( A \otimes B \), so the restrictions in this case are more severe. However, it is easily seen that in all the cases in Theorem 3.8 where a splitting is possible, there is actually a splitting (at the \( K \)-theory level) of the quotient map from \( A \ast B \) to \( A \ast_c B \). So the \( K \)-theoretic restrictions are identical in the two cases. (Restrictions from ordered \( K \)-theory may be more severe in the full free product case; cf. Example 1.2.)

The quotient map from \( A \ast B \) to \( A \ast_c B \) does not split in general. There are \( K \)-theoretic obstructions in many cases; for example, a quotient map between finite abelian groups generally does not split.

Ordered \( K \)-theory also implies nonsplitting of the quotient map from \( A \ast B \) to \( A \ast_c B \) in some cases even when there is a splitting at the \( K \)-theory level:

Example 4.1. The positive cone of \( K_0(\mathbb{C}^2 \ast \mathbb{C}^2) \cong \mathbb{Z}^2 \oplus \mathbb{Z}^2 = \mathbb{Z}^4 \) is the set of 4-tuples with nonnegative entries, and the scale consists of points with each entry 0 or 1 (not all such points; in fact, it can be shown that the scale only consists of \((1,1,0,0),(0,0,1,1)\) and the 4-tuples with at most one 1). The positive cone and scale of \( K_0(\mathbb{C}^2 \ast_c \mathbb{C}^2) \cong \mathbb{Z}^4/\langle(1,1,-1,-1)\rangle \) are the images of the ones in \( \mathbb{Z}^4 \). If there is a cross section \( \sigma : \mathbb{C}^2 \ast_c \mathbb{C}^2 : \mathbb{C}^2 \ast \mathbb{C}^2 \) for the quotient map, \( \sigma_* \) must send \([(1,0,0,0)]\) to an element of the scale in \( \mathbb{Z}^4 \) which is in the equivalence class; the only such element is \((1,0,0,0)\) (in fact, this is the only element of the equivalence class in the positive cone). Similarly, we must have \( \sigma_*([[(0,1,0,0)])] = (0,1,0,0), \) etc. But there is no such homomorphism since then \( 0 = \sigma_*([[(1,1,-1,-1)]]) = (1,1,-1,-1) \neq 0 \). Thus there can be no cross section.

5. Lifting Commutation Relations

This section contains the main results of this paper. In this section, let \( P \) be a separable unital \( \text{C}^* \)-algebra with the following properties:

(i) \( P \) is semiprojective.

(ii) \( P \) is isomorphic to \( P^n = \bigotimes_{k=1}^n P \) for any \( n \) and to \( P^\infty = \bigotimes_{k=1}^\infty P \).

For example, \( P = O_2 \) or \( P = O_\infty \) [2,4]. (In fact, \( O_2 \) and \( O_\infty \) may be the only \( \text{C}^* \)-algebras besides \( \mathbb{C} \) satisfying the two conditions; they are the only Kirchberg algebras.)
We will also let $A$ be a unital $P$-absorbing C*-algebra, i.e. $A \otimes P \cong A$ (for example, any Kirchberg algebra is $O_\infty$-absorbing). Then $A \otimes P^n \cong A \otimes P^\infty \cong A$. Note that $A = P$ is an allowed case.

**Theorem 5.1.** Let $P$ be as above, and $A$ a unital $P$-absorbing semiprojective C*-algebra. Let $Q$ be the full free product of $A$ and a sequence of copies of $P$, i.e. $Q$ is the universal C*-algebra generated by a copy of $A$ and a sequence of copies of $P$ with no relations. The canonical quotient map $\pi : Q \to A \otimes P^\infty$ splits, i.e. there is a *-homomorphism $\sigma : A \otimes P^\infty \to Q$ with $\pi \circ \sigma = id$.

**Proof.** For each $n$, let

$$Q_n = (A \otimes P^n) \ast P \ast P \ast \cdots$$

which is the universal C*-algebra generated by a copy of $A$ and a sequence of copies of $P$ such that the copy of $A$ and the first $n$ copies of $P$ commute and have a common unit. There is an obvious canonical quotient map $\pi_n$ from $Q_n$ onto $Q$, for all $n$ and a quotient map $\pi_{n,m} : Q_n \to Q_m$ for $n < m$ satisfying $\pi_{n,p} = \pi_{m,p} \circ \pi_{n,m}$ for $n < m < p$. Thus we have an inductive system $(Q_n, \pi_{n,m})$ with surjective connecting maps, and

$$\lim_{\rightarrow}(Q_n, \pi_{n,m}) \cong A \otimes P^\infty$$

where the infinite tensor product is regarded as the universal C*-algebra generated by a copy of $A$ and a sequence of copies of $P$ which commute and have a common unit, and the isomorphism with the inductive limit is the canonical one.

By assumption, $A \otimes P^\infty$ is isomorphic to $A$ and is thus semiprojective. By definition of semiprojectivity, there is a lifting $\sigma : A \otimes P^\infty \to Q_n$ of the identity map on $A \otimes P^\infty$, for some $n$. Thus we have a cross section

$$(A \otimes P^n) \ast P \ast P \ast \cdots \xrightarrow{\sigma} (A \otimes P^n) \otimes P \otimes P \otimes \cdots$$

and the result follows by fixing an isomorphism of $A$ with $A \otimes P^n$. \hfill \Box

**Corollary 5.2.** Let $P$ and $A$ be as in the theorem. For any $n \geq 1$, the quotient map from $A \ast P \ast \cdots \ast P$ (n copies of $P$) to $A \otimes P^n$ splits.

**Proof.** The quotient map $\pi$ from $Q$ to $A \otimes P^\infty$ factors through

$$\rho : Q \to A \ast P \ast \cdots \ast P \ast P^\infty$$

($n-1$ copies of $P$), the universal C*-algebra generated by a copy of $A$ and a sequence of copies of $P$ where the copies of $P$ for $k \geq n$ commute and have a common unit. If $\sigma : A \otimes P^\infty \cong A \otimes P^{n-1} \otimes P^\infty \to Q$ is the cross section from the theorem, then $\rho \circ \sigma$ is the desired cross section once $P^\infty$ is identified with $P$. \hfill \Box

Since the quotient map from $Q$ to $A \otimes P^\infty$ factors through the infinite unital free product of copies of $A$ and $P$, the theorem and corollary remain true if “free product” is replaced by “unital free product.” However, this does not directly give a unital splitting. But we can simply repeat the argument replacing $Q$ by the unital infinite free product to obtain unital versions:

**Theorem 5.3.** Let $P$ be as above, and $A$ a unital $P$-absorbing semiprojective C*-algebra. Let $Q$ be the full unital free product of $A$ and a sequence of copies of $P$, i.e. $Q$ is the universal unital C*-algebra generated by a copy of $A$ and a sequence of copies of $P$ all with a common unit but with no other relations. The canonical
quotient map \( \pi : Q \to A \otimes P^\infty \) splits unitally, i.e. there is a unital \(*\)-homomorphism \( \sigma : A \otimes P^\infty \to Q \) with \( \pi \circ \sigma = \text{id} \).

**Corollary 5.4.** Let \( P \) and \( A \) be as in the theorem. For any \( n \geq 1 \), the quotient map from \( A \ast P \ast_C \cdots \ast_C P \) (\( n \) copies of \( P \)) to \( A \otimes P^n \) splits unitally.

**Corollary 5.5.** If \( A \) and \( P \) are as in the theorem, \( B \) is a unital \( C^* \)-algebra, \( J \) a closed two-sided ideal of \( B \), and \( A_0 \) and \( P_0 \) are commuting unital copies of \( A \) and \( P \) in \( B/J \), and \( A_0 \) and \( P_0 \) lift to commuting unital copies of \( A \) and \( P \) in \( B \), then \( A_0 \) and \( P_0 \) lift to commuting unital copies of \( A \) and \( P \) in \( B \).

**Corollary 5.6.** (i) The canonical quotient map from \( O_2 \ast O_2 \) to \( O_2 \otimes O_2 \) splits.

(ii) Let \( A \) be a semiprojective Kirchberg algebra. Then the canonical quotient map from \( A \ast O_\infty \) to \( A \otimes O_\infty \) splits.

For (ii), note that any Kirchberg algebra absorbs \( O_\infty \) (\[EE02\] iv)).

6. How Explicit and General?

The quotient maps from \( O_2 \ast O_2 \) to \( O_2 \otimes O_2 \cong O_2 \) and from \( O_n \ast O_\infty \) to \( O_n \otimes O_\infty \cong O_n \) for any \( n \), \( 2 \leq n \leq \infty \), split. It would be very useful to have an explicit formula or description of a cross section in these cases (note that no uniqueness for the splitting should be expected); but there may not be any such explicit description, just as there is no explicit isomorphism between \( O_2 \otimes O_2 \) and \( O_2 \).

There are still unresolved cases where both \( A \) and \( B \) are semiprojective Kirchberg algebras, most obviously the cases \( O_m \otimes O_n \) where \( m - 1 \) and \( n - 1 \) are relatively prime (e.g. if \( m = 2 \) and \( n > 2 \)). Note that if \( 2 < n < \infty \), then there is a splitting for \( O_2 \otimes O_2 \) but not for \( O_n \otimes O_n \) (ruled out by Theorem 3.8); how about for \( O_2 \otimes O_n \)?

The case where \( A \) and \( B \) are Kirchberg algebras which are not semiprojective is also open. The question of existence of a splitting would not seem to depend on any essential dependence on semiprojectivity (even though the present proof techniques use semiprojectivity). For example, one can write \( A \) and \( B \) as inductive limits of semiprojective Kirchberg algebras \( A_n \) and \( B_n \) (cf. \[Rør02\]) and try to make splittings for \( A_n \ast B_n \) approximately compatible using approximate unitary equivalence theorems for embeddings of Kirchberg algebras.

**References**

[AC13] Pere Ara and Guillermo Cortiñas. Tensor products of Leavitt path algebras. *Proc. Amer. Math. Soc.*, 141(8):2629–2639, 2013.

[Bla85] Bruce Blackadar. Shape theory for \( C^* \)-algebras. *Math. Scand.*, 56(2):249–275, 1985.

[Bla04] Bruce Blackadar. *K-theory for operator algebras*, volume 5 of *Mathematical Sciences Research Institute Publications*. Cambridge University Press, Cambridge, second edition, 1998.

[Bla06] Bruce Blackadar. *Operator algebras*, volume 122 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2006. Theory of \( C^* \)-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[EE02] Søren Eilers and Ruy Exel. Finite-dimensional representations of the soft torus. *Proc. Amer. Math. Soc.*, 130(3):727–731 (electronic), 2002.

[EEL91] George A. Elliott, Ruy Exel, and Terry A. Loring. The soft torus. III. The flip. *J. Operator Theory*, 26(2):333–344, 1991.
[EL89] Ruy Exel and Terry Loring. Almost commuting unitary matrices. Proc. Amer. Math. Soc., 106(4):913–915, 1989.

[End] Dominic Enders. Semiprojectivity for Kirchberg algebras. preprint.

[Exe93] Ruy Exel. The soft torus and applications to almost commuting matrices. Pacific J. Math., 160(2):207–217, 1993.

[Ger97] Emmanuel Germain. KK-theory of the full free product of unital C*-algebras. J. Reine Angew. Math., 485:1–10, 1997.

[KP00] Eberhard Kirchberg and N. Christopher Phillips. Embedding of exact C*-algebras in the Cuntz algebra O_2. J. Reine Angew. Math., 525:17–53, 2000.

[Lor97] Terry A. Loring. Lifting solutions to perturbing problems in C*-algebras, volume 8 of Fields Institute Monographs. American Mathematical Society, Providence, RI, 1997.

[Ror94] Mikael Rørdam. A short proof of Elliott’s theorem: $O_2 \otimes O_2 \cong O_2$. C. R. Math. Rep. Acad. Sci. Canada, 16(1):31–36, 1994.

[Ror02] M. Rørdam. Classification of nuclear, simple C*-algebras. In Classification of nuclear C*-algebras. Entropy in operator algebras, volume 126 of Encyclopaedia Math. Sci., pages 1–145. Springer, Berlin, 2002.

[RV98] Mikael Rørdam and Jesper Villadsen. On the ordered $K_0$-group of universal, free product C*-algebras. K-Theory, 15(4):397–322, 1998.

[Spi09] Jack Spielberg. Semiprojectivity for certain purely infinite C*-algebras. Trans. Amer. Math. Soc., 361(6):2805–2830, 2009.

[ST12] Adam P. W. Sørensen and Hannes Thiel. A characterization of semiprojectivity for commutative C*-algebras. Proc. Lond. Math. Soc. (3), 105(5):1021–1046, 2012.

[Szy02] Wojciech Szmyański. On semiprojectivity of C*-algebras of directed graphs. Proc. Amer. Math. Soc., 130(5):1391–1399 (electronic), 2002.

[Voi83] Dan Voiculescu. Asymptotically commuting finite rank unitary operators without commuting approximants. Acta Sci. Math. (Szeged), 45(1-4):429–431, 1983.

Department of Mathematics/0084, University of Nevada, Reno, Reno, NV 89557, USA
E-mail address: bruceb@unr.edu