Abstract

We review some facts about various T-dualities and sigma models on group manifolds, with particular emphasis on supersymmetry. We point out some of the problems in reconciling Poisson-Lie duality with the bi-hermitean geometry of N=2 supersymmetric sigma models. A couple of examples of supersymmetric models admitting Poisson-Lie duality are included.
1 Introduction

Supersymmetric sigma models are of interest, e.g., as gauge-fixed string actions, as representing exact string vacua (WZW-models), for their intimate connection to complex geometry (of the target manifold) and for their role as effective low-energy actions for supergravity scalars.

The various (generalized) T-dualities for sigma models are important in the context of strings, where, e.g., the usual T-duality relates different geometries describing one and the same physical configuration.

In this article, we review some facts about various T-dualities with emphasis on sigma models on group manifolds. Particular emphasis is put on the requirements for $N = (2, 2)$
supersymmetry, (bi-hermitean target space). We comment on the situation for general models not describable as WZW models. The latter half of the paper consists of a discussion of Poisson-Lie duality for $N = 1$ supersymmetric nonlinear sigma models and examples based on $SU(2) \otimes U(1)$.

This paper grew out of an effort to understand how the stringent requirements on the target space geometry of $N = (2, 2)$ supersymmetric sigma models might be made to agree with Poisson-Lie duality in a more general case than abelian or non-abelian T-duality. While we have not resolved the original problem, we believe that the discussion contained in this paper will serve as a necessary background and starting point. Along the way we have collected a number of observations and comments which may find their use in other contexts as well.

## 2 Nonlinear Sigma Models

In this section we collect the necessary background on (supersymmetric) two-dimensional non-linear sigma models.

The action for a general non-linear sigma model is

$$S = \int d^2 \xi \partial X^\mu E_{\mu \nu}(X) \delta X^\nu$$

where the metric $G_{\mu \nu} = \frac{1}{2} E_{(\mu \nu)}$ and the torsion potential $B_{\mu \nu} = \frac{1}{2} E_{[\mu \nu]}$.

In $N = 1$ superspace this becomes

$$S = i \int d^2 \xi d^2 \theta D_+ X^\mu E_{\mu \nu} D_- X^\nu$$

where again $E_{\mu \nu} = G_{\mu \nu} + B_{\mu \nu}$ and where $G_{\mu \nu}(X)$ and $B_{\mu \nu}(X)$ and $X^\mu(\xi, \theta)$ are superfields whose lowest components enters in (2.1) above, (we use the same notation for superfields as for their lowest components).

As first described in [1], the action (2.2) has $N = (2, 2)$ supersymmetry $^2$ i.e., an additional non-manifest supersymmetry of the form

$$\delta X^\mu = \varepsilon^+(D_+ X^\nu) \mathcal{J}_\nu^{(+)\mu} + \varepsilon^-(D_- X^\nu) \mathcal{J}_\nu^{(-)\mu},$$

provided that $\mathcal{J}^{(\pm)}$ are complex structures: they square to minus one,

$$\mathcal{J}^{2(\pm)} = -\mathbb{I},$$

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$^1$We use (anti-) symmetrization without a combinatorial factor

$^2$The target-space geometry for models with less supersymmetry, e.g, (2, 1), is also very interesting, but will not be discussed here. See [2].
and have vanishing Nijenhuis tensors\(^3\),
\[
N_{\mu\nu}^{(\pm)\kappa} \equiv \mathcal{J}_\mu^{(\pm)\gamma} \partial_\gamma \mathcal{J}_\nu^{(\pm)\kappa} - (\mu \leftrightarrow \nu) = 0 .
\] (2.5)

In addition, the metric has to be bi-hermitean, i.e., hermitean with respect to both complex structures
\[
\mathcal{J}_\mu^{(\pm)\gamma} G_{\gamma\rho} \mathcal{J}_\nu^{(\pm)\rho} = G_{\mu\nu} ,
\] (2.6)
and the complex structures should be covariantly constant with respect to certain connections \(\Gamma^{(\pm)}\), respectively
\[
\nabla_\mu^{(\pm)} \mathcal{J}_\nu^{(\pm)\gamma} = 0 .
\] (2.7)

These connections are
\[
\Gamma_\mu^{(\pm)\gamma} = \Gamma^{(0)}_\mu^{\gamma} \pm T^{\gamma}_\mu ,
\] (2.8)
with \(\Gamma^{(0)}\) the Christoffel connection for the metric \(G\), and the torsion given by
\[
T^{\gamma}_\mu = H_{\mu\rho\gamma} G^{\rho\gamma} .
\] (2.9)

This relates the complex structures to the field-strength for the \(B\)-field,
\[
H_{\mu\nu\rho} = \partial^\mu \partial_\nu B_\rho ,
\] (2.10)
which implies
\[
H_{\mu\nu\rho} = \mathcal{J}_\mu^{(+)\gamma} \mathcal{J}_\nu^{(+)\kappa} \mathcal{J}_\rho^{(+)\lambda} d\mathcal{J}_\gamma^{(+)} - \mathcal{J}_\mu^{(-)\gamma} \mathcal{J}_\nu^{(-)\kappa} \mathcal{J}_\rho^{(-)\lambda} d\mathcal{J}_\gamma^{(-)} ,
\] (2.11)
where \(d\mathcal{J}^{(\pm)}\) is the exterior derivative of the two forms with components \(\mathcal{J}_\mu^{(\pm)\gamma} = \mathcal{J}_\mu^{(\pm)\gamma} G_{\gamma\nu}\), (antisymmetrical because of (2.6))\(^4\).

In fact, the set of conditions (2.5)-(2.10), derived from requiring that the action (2.2) is invariant under the variation (2.3) along with closure of the algebra, is a minimal set of requirements and may be modified as seen from the following.

Condition (2.11) may be equivalently expressed as a relation which states that the complex structures preserve the torsion, i.e., the field strength of the \(B\)-field. We may strengthen this condition to preservation of the \(B\)-field itself by replacing condition (2.6) by
\[
\mathcal{J}_\mu^{(\pm)\gamma} E_{\gamma\rho} \mathcal{J}_\nu^{(\pm)\rho} = E_{\mu\nu} ,
\] (2.12)
since the antisymmetric part of this reads
\[
\mathcal{J}_\mu^{(\pm)\gamma} B_{\gamma\rho} \mathcal{J}_\nu^{(\pm)\rho} = B_{\mu\nu} .
\] (2.13)

\(^3\)More general models with non-vanishing \(\mathcal{N}\) have also been considered [3]

\(^4\)For a recent discussion of the relevant geometry, see [4]
(The symmetric part is (2.6).) Imposing (2.12) in the variation of (2.2), we find that (2.7) is weakened to

\[ E_{\mu \nu} \partial_{\rho} J^{(\pm)}_{\alpha} - \partial_{[\mu} E_{\nu \rho]} J^{(\pm)\nu}_{\alpha} + \partial_{\nu} E_{[\mu \rho]} J^{(\pm)}_{\alpha} = 0 \]  
(2.14)

\[ E_{\nu [\mu} \partial_{\rho} J^{(-)}_{\alpha]} - \partial_{[\mu} E_{\nu \rho]} J^{(-)\nu}_{\alpha} + \partial_{\nu} E_{[\mu \rho]} J^{(-)}_{\alpha} = 0 . \]  
(2.15)

We recover the previous conditions as follows: Combining (2.15) with the derivative of (2.12) returns (2.7). Antisymmetrizing (2.15) in all three indices and multiplying with \( J^{(\pm)\gamma}_\nu J^{(\pm)\kappa}_\rho J^{(\pm)\lambda}_\mu \) yields (2.11). Hence the new conditions represent a special case of the general structure.

Since we have a condition (2.12) which is stronger than necessary for \( N = 2 \), we may ask if it is compatible with other conditions on the theory. We first note that when the two complex structures commute, \( [J^{(+)}, J^{(-)}] = 0 \), their product gives an almost product structure, i.e., \( \Pi^2 = \Pi \) where \( \Pi = J^{(+)}, J^{(-)} \) [1]. While the individual integrability of \( J^{(+)\nu} \) and \( J^{(-)\nu} \) is not sufficient to guarantee integrability of \( \Pi \), in conjunction with (2.7) it is [5].

We may then choose coordinates where

\[ \Pi = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} . \]  
(2.16)

It is shown in [1] that the subspaces projected out by \( \hat{P}_\pm \equiv \frac{1}{2}(\Pi \pm \Pi) \) are Kähler, i.e., these sectors contain no \( B \)-field. Since (2.12) implies \( \hat{P}_+ E \hat{P}_T = 0 \) it also follows that \( E \) has no “mixed” components in these coordinates. Hence, we conclude\(^5\) that (2.12) is compatible with \( B \neq 0 \) only if \( [J^{(+)\nu}, J^{(-)\nu}] \neq 0 \). This excludes formulations in terms of chiral and twisted chiral superfields [1], but may allow (anti-)semichiral superfields [6] as coordinates, as discussed in [7]. (For \( N = 4 \) the geometric structure is even more restricted [1], and there are additional superfield coordinates available [8].)

Another property one might want to study is the gauge transformation of the \( B \)-field. In the absence of boundaries, at least, this field only enters the field equations through its field strength (2.10). Under what conditions is that compatible with (2.13)? To answer this we define the projection operators

\[ P^{(\pm)}_{\nu \rho} \equiv \frac{1}{2} (\mathbb{I} \pm iJ^{(\pm)}) . \]  
(2.17)

Using these, we may restate (2.12), or equivalently (2.6) and (2.13), as expressing

\[ P_{\nu \rho} E_{\nu \rho} = 0 \Rightarrow P_{\nu \rho} \tilde{P}_{\nu \rho} G_{\nu \rho} = P_{\nu \rho} \tilde{P}_{\nu \rho} B_{\nu \rho} = 0 . \]  
(2.18)

\(^5\)Using the explicit forms in [1] one verifies that \( J^{(-)\nu} \) indeed does not preserve \( B \).
The $(\pm)$-labels distinguishing the two complex structures are suppressed for easier readability. These relations mean that in the canonical coordinates of the complex structures the tensors have only mixed components, $G_{ij}$, $G_{\bar{i}j}$, $B_{ij}$ and $B_{\bar{i}j}$. A gauge transformation of $B$ infinitesimally reads $\delta B_{\mu\nu} = \partial_{[\mu} \Lambda_{\nu]}$. We deduce that we must have

$$P_{\pm\mu} P_{\pm\nu} \partial_{[\nu} \Lambda_{\rho]} = 0 ,$$

or $\partial_{[i} \Lambda_{j]} = \partial_{[i} \Lambda_{j]} = 0$. A strong version of this condition is that the transformation parameters have to have (anti-) holomorphic components $\Lambda_i(\bar{z})$, $(\Lambda_{\bar{i}}(z))$ in both sets of canonical coordinates. (Of course $\Lambda_i = \partial_i \Lambda$ and c.c. also solve the constraints.)

This shows that the requirement of $N = 2$ supersymmetry leads to restrictions on the gauge symmetry for the $B_{\mu\nu}$-field. We shall see in Sec. 5 below, that this symmetry is also in conflict with the Poisson-Lie condition.

3 Sigma Models on Group Manifolds

We shall be particularly interested in non-linear sigma models on group manifolds. For open models, interesting relations between the geometry and boundary conditions were discovered in $[11, 12, 13]$ and, generally, they are the appropriate setting for Poisson-Lie duality, which we discuss in Sec. 5. In a group $G$, we parametrize the group elements $g \in G$ using coordinates $X^\mu$, and define the left and right frames by

$$g^{-1} \delta g = L_\mu \delta X^\mu; \quad \delta gg^{-1} = R_\mu \delta X^\mu ,$$

where $L_\mu \equiv L^A_\mu T_A$, $R_\mu \equiv R^A_\mu T_A$, with $T_A$ the generators of the corresponding Lie algebra $[T_A, T_B] = f^C_{AB} T_C$. In these coordinates, a general sigma model on the group space may be written

$$S = i \int d^2 \xi d^2 \theta D_+ X^\mu E_{\mu\nu} D_- X^\nu = i \int d^2 \xi d^2 \theta Tr(g^{-1} D_+ g) A E_{AB}(g^{-1} D_- g)^B$$

where $E_{\mu\nu} = L^A_\mu E_{AB} L^B_\nu$. In the special case of a Wess-Zumino-Witten model we also have (in the bosonic sector)

$$S = \int d^2 \xi T r(g^{-1} \partial g)(g^{-1} \bar{\partial} g) + \frac{1}{3} \int Y T r(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg) ,$$

which means that the symmetric part of $E_{AB}$ is the Killing form and that the torsion is $H_{\mu\nu\rho} = L^A_\mu L^B_\nu L^C_\rho f_{ABC}$, where $f_{ABC}$ are the completely antisymmetric structure constants.

$^6$Based on previous results for general supersymmetric sigma models in $[9, 10]$
Further, the two-dimensional space $\partial Y$ has been extended to $Y$ with an additional auxiliary coordinate. For this case one can show that the $N = 2$ conditions in Sec 1. above require

$$
\nabla_{\rho}(-) L_\mu^A = 0, \quad \nabla_{\rho}(+) R_\mu^A = 0 .
$$

(3.23)

In the more general case we want to consider here, we derive the following relation for the covariant derivatives of the left frames:

$$
\nabla^\rho L_\mu^A = -\frac{1}{2} \left( f_A^{(\rho\mu)} + f_A^{\rho\mu} + 2T_A^{(\rho\mu)} + L_B^\rho L_C^\mu \nabla_A G^{BC} + L^B[\rho \nabla^\mu] G_{AB} \right)
$$

(3.24)

$$
\nabla^\rho R_\mu^A = -\frac{1}{2} \left( -f_A^{(\rho\mu)} - f_A^{\rho\mu} + 2T_A^{(\rho\mu)} + R_B^\rho R_C^\mu \nabla_A G^{BC} + R^B[\rho \nabla^\mu] G_{AB} \right),
$$

(3.25)

where Lie-algebra indices are raised and lowered with $G_{AB} = \frac{1}{2} E(AB)$, transformed into target space indices with $L^\mu_A$ or $R^\mu_A$, and we have allowed for a torsion $T$ and assumed that $\nabla_\mu G_{\nu\rho} = 0$. Target space indices are raised and lowered using $G^{\mu\nu}$. Clearly, when $G_{AB}$ is constant and the structure constants $f_A^{(\rho\mu)} + f_A^{\rho\mu}$ equal to ($\mp 2$ times) the torsion, we recover (3.23). (In the WZW case, in addition, $f_{ABC} \equiv f_{AB}^D G_{DC}$ are completely antisymmetric.)

4 Isometry-based T-Dualities

The idea of dual formulations describing the same physical situation had been around a long time in the context of sigma models when it found its application in string theory. In fact the geometry changing aspects make it particularly interesting for two-dimensional models, but there are many features that are fascinating in general. See, e.g., [14]-[25] for reviews and general aspects of sigma model duality.

When the sigma model (2.1) or (2.2) has (generalized) isometries with Killing vector fields $k_A = k_A^\mu \partial_\mu$ with algebra $[k_A, k_B] = f_A^{CD} k_C$

$$
\delta X^\mu = \varepsilon^A k^\mu_A = \mathcal{L}_{\varepsilon k} X^\mu , \quad \mathcal{L}_{\varepsilon k} E_{\mu\nu} = 0 ,
$$

(4.26)

there exists a “parent action” from which the sigma model and its T-dual can be derived. In the bosonic sector it reads

$$
\mathcal{S} = \int d^2 \xi \left( DX^\mu E_{\mu\nu}(X) DX^\nu + tr \Lambda F \right),
$$

(4.27)

7The conditions stated here are stronger than necessary. Typically we need only require $\mathcal{L}_k B = d\omega$. For a thorough discussion of conditions on isometries and their gauging in the context of susy WZW models, see [26].
where the covariant derivatives and the field strength are

\[ DX^\mu = \partial X^\mu + A^A k_A^\mu , \quad \bar{D} X^\mu = \bar{\partial} X^\mu + \bar{A}^A k_A^\mu , \quad F = [D, \bar{D}] = \partial \bar{A} - \bar{\partial} A + [A, \bar{A}] \].  \tag{4.28} \]

Varying \( \Lambda \) gives that \( F \) is pure gauge. Plugging this back into (4.27) we recover the original action (2.1), whereas the \( A \)-field equations instead yield the dual action in terms of \( \Lambda \). For abelian isometries this prescription is unproblematic. For one isometry and in coordinates adapted to this isometry it yields the famous Buscher rules [27][28] relating the original background \( G, B \) to the dual background \( \tilde{G}, \tilde{B} \):

\[
\begin{align*}
\tilde{G}_{00} &= G_{00}^{-1} , \\
\tilde{G}_{0i} &= G_{00}^{-1} B_{0i} , \\
\tilde{G}_{ik} &= G_{ik} - G_{00}^{-1} (G_{i0} G_{0k} + B_{i0} B_{0k}) , \\
\tilde{B}_{ik} &= B_{ik} + G_{00}^{-1} (G_{i0} B_{0k} + B_{i0} G_{0k}) , \\
\tilde{B}_{0i} &= G_{00}^{-1} G_{0i} ,
\end{align*}
\]

(4.29)

where 0 is the isometry direction and \( i \) denotes the rest of the coordinates (the spectators). With the appropriate superfield interpretation, these rules apply also to \( N = 1 \) supersymmetric models.

The relations (4.29) are expressed in adapted coordinates where \( G \) and \( B \) are independent of the isometry direction \( X^0 \), (although one may formulate the rules in a covariant fashion using the Killing vectors). It is interesting that in the dualization for \( N = 2,4 \) models in superspace, which is achieved via a gauging of holomorphic isometries [29], the dual model is described directly in canonical complex coordinates [30]. This is related to the fact that there duality relates the K"ahler potentials rather than the metric.

The above rules also generalize to the case of several commuting isometries.

Several items may be mentioned at this stage. Firstly, as is obvious from the factors of \( G_{00}^{-1} \), the case of a lightlike isometry has to be treated separately. Secondly, although T-duality is always compatible with supersymmetry, it is sometimes necessary to take non-local world-sheet effects into account [31],[32]. Thirdly, typically the complex geometries of the \( N = 2,4 \) target spaces will only be preserved if the isometries active in the duality commute with the supersymmetries.

Non-abelian duality generalizes the above relations for the case of a non-abelian isometries [33]. It is somewhat more problematic, partly due to the fact that the dual of the dual model does not return the original, i.e., unlike the abelian case the non-abelian duality is not idempotent. It is perhaps best studied within the framework of Poisson-Lie duality, which we now describe.
5 Poisson-Lie Duality

A very interesting generalization of T-duality to the situation when the group action is not an isometry of the sigma model was constructed in [34], and has since been discussed extensively, e.g., in [35],[36],[37],[38],[39], [23],[40]. Supersymmetric versions are treated in, e.g., [41], [42],[43], [44].

5.1 Definitions

In Poisson-Lie duality the isometries in (4.26) are generalized to the following relation

\[ \mathcal{L}_A E_{\mu\nu} \equiv \mathcal{L}_R E_{\mu\nu} = -E_{\rho\sigma} R^\rho_B \tilde{f}^{BC} R^\sigma_C E_{\sigma
u}. \]  

(5.30)

Here \( \tilde{f}^{AB} \) are structure constants in a dual Lie algebra. For the sake of greater clarity we will not consider spectators, i.e., we will only keep the target space coordinates affected by the transformations (5.30). We are thus effectively studying a \( \sigma \)-model on the corresponding group space.

Klimčik and Ševera [34], show that the condition (5.30) can be solved and the dual model found provided that the Lie algebra \( \mathcal{G} \) and its dual \( \tilde{\mathcal{G}} \) form what is called a Drinfel’d double [45, 46, 47].

Let \( \mathcal{G} \) and \( \tilde{\mathcal{G}} \) be groups obeying (5.30) on the original \( \sigma \)-model and its dual, respectively, with \( \text{dim} \mathcal{G} = \text{dim} \tilde{\mathcal{G}} \). The corresponding Lie algebras are \( \mathcal{G} \) and \( \tilde{\mathcal{G}} \). Then the Drinfel’d double \( \mathcal{D}^2 \equiv \mathcal{G} \otimes \tilde{\mathcal{G}} \) and comes equipped with an invariant inner product \( \langle , \rangle \). The corresponding algebra \( \mathfrak{d} \) consists of the two subalgebras \( \mathcal{G} \) and \( \tilde{\mathcal{G}} \) that are null-spaces w.r.t. this product. We choose two sets of generators \( \{T_A\} \) and \( \{\tilde{T}^A\} \) so that \( \{T_A\} \) span \( \mathcal{G} \) and \( \{\tilde{T}^A\} \) span \( \tilde{\mathcal{G}} \). The set \( T_A \in \{T_A, \tilde{T}^B\} \) then span \( \mathfrak{d} \). The Lie algebra of the Drinfel’d double generated by \( T_A \) and \( \tilde{T}^A \) (\( A = 1, \ldots, \text{dim} \mathcal{G} \)), is

\[
[T_A, T_B] = f_{AB}^C T_C, \\
[\tilde{T}^A, \tilde{T}^B] = \tilde{f}_{AB}^{CD} \tilde{T}^C, \\
[T_A, \tilde{T}^B] = \tilde{f}_{ABC}^{R} T_C - f_{AC}^R \tilde{T}^C, 
\]

(5.31)

where \( f_{AB}^C \) and \( \tilde{f}_{AB}^{CD} \) are the structure constants of \( \mathcal{G} \) and \( \tilde{\mathcal{G}} \), respectively, and satisfy the Lie bi-algebra \( (\mathcal{G}, \tilde{\mathcal{G}}) \) consistency condition

\[
f_{DC}^A \tilde{f}_{RA}^S = \tilde{f}_{DA}^C f_{RA}^S + \tilde{f}_{DA}^C f_{RA}^S - \tilde{f}_{DA}^S f_{RA}^C - \tilde{f}_{DA}^S f_{RA}^C. 
\]

(5.32)
This condition arises in the PL duality context as the condition \([\mathcal{L}_a, \mathcal{L}_b] = f_{ab}^c \mathcal{L}_c\) applied to (5.30). The invariant inner product between the generators has the following properties

\[
\langle T_A, T_B \rangle = \langle \tilde{T}^A, \tilde{T}^B \rangle = 0, \quad \langle T_A, \tilde{T}^B \rangle = \delta_A^B
\]

and obeys the invariance condition

\[
\langle \mathcal{X} T_A \mathcal{X}^{-1}, T_B \rangle = \langle T_A, \mathcal{X}^{-1} T_B \mathcal{X} \rangle,
\]

where \(\mathcal{X}\) is any element of the Drinfel’d double or one of its subgroups.

We define,

\[
\begin{align*}
\mu^{AB}(g) &= \langle g \tilde{T}^A g^{-1}, \tilde{T}^B \rangle; \quad \nu^A_B(g) = \langle g \tilde{T}^A g^{-1}, T_B \rangle; \\
\alpha_B^A(\tilde{g}) &= \langle \tilde{g} T_B \tilde{g}^{-1}, \tilde{T}^A \rangle; \quad \beta_{AB}(\tilde{g}) = \langle \tilde{g} T_A \tilde{g}^{-1}, T_B \rangle
\end{align*}
\]

which obey \(\mu(g^{-1}) = \mu^t(g)\), \(\nu(g^{-1}) = \nu^{-1}(g)\), \(\alpha(\tilde{g}^{-1}) = \alpha^{-1}(\tilde{g})\) and \(\beta(\tilde{g}^{-1}) = \beta^t(\tilde{g})\) where \(t\) stands for transpose.

We return to the solution of (5.30) given by Klimčík and Ševera. With \(E_{\mu\nu} = L^A_{\mu} E_{AB} L^B_{\nu}\) as in (3.21) the solution is

\[
E_{AB} = ((E^0)^{-1} + \Pi)^{-1}_{AB}; \quad \Pi^{AB} = \mu^{AC} \nu^C_B,
\]

where \(E^0\) is a constant matrix. Similarly, in the dual theory one has relations corresponding to (5.30) and (3.21) and

\[
\tilde{E}^{AB} = [(E^0 + \tilde{\Pi})^{-1}]^{AB}; \quad \tilde{\Pi}_{AB} = \beta_{AC} \alpha^C_B.
\]

The abelian and non-abelian dualities described previously are special cases of the more general PL duality. In the non-abelian case we have \(\mu^{AB} = 0\), \(\alpha_B^A = \delta^A_B\) and \(\beta_{AB} = f_{AB}^C \tilde{x}_C\), where \(\tilde{x}_C\) is the dual non-inert coordinates, so that \(E_{AB} = E^0_{AB}\) and \(\tilde{E}^{AB} = [(E^0 + f^C_{AB} \tilde{x}_C)^{-1}]^{AB}\).

We now include spectators and give the generalized Buscher rules (in the notation of [44])

\[
\begin{align*}
\tilde{E}^{-1} - \beta \alpha &= (E^{-1} + \mu \nu)^{-1} = E^0(x^\alpha); \\
\tilde{E}^{-1} \tilde{F}^R &= E^0 E^{-1} F^R = F^R(x^\alpha); \\
-\tilde{F}^L \tilde{E}^{-1} &= F^L E^{-1} E^0 = F^L(x^\alpha); \\
\tilde{F} - \tilde{F}^L \tilde{E}^{-1} \tilde{F}^R &= F + F^L (E^{-1} E^0 E^{-1} - E^{-1} F^R = \tilde{F}(x^\alpha).
\end{align*}
\]

Here the indices are split according to \(\mu \rightarrow (i, \alpha)\), with \(\alpha\) representing the spectators. To be able to use a condensed notation, we have replace \(E\) by \(\tilde{F}\) when it carries curved indices. Further, \(\tilde{F}^L_{\alpha B} \equiv \tilde{F}_{\alpha j} \tilde{L}^j_B, \tilde{F}^R_{\alpha B} \equiv \tilde{L}^A_{\alpha} \tilde{F}^A_{i B}\), \(\tilde{F}^{LB}_{\alpha} \equiv \tilde{F}_{\alpha} \tilde{L}^A_B\) and \(\tilde{F}^{RA}_{\beta} \equiv \tilde{L}^A_{\alpha} \tilde{F}^A_{\beta}\).
These relations apply verbatim also to $N = 1$ models \[44\]. For $N = 2$, the general rules that take into consideration the bi-hermitean geometry have not been worked out. In fact, the whole Poisson-Lie structure is easily applied to $N = 1$ models, whereas for $N = 2$ only certain superconformal models have lent themselves to a Poisson-Lie description \[42\].

5.2 The Poisson-Lie Condition Rewritten

The Poisson-Lie condition (5.30) can be rewritten in a form from which its solution may be found via integration. In a particular case this may turn out to be just as efficient as calculating the objects that enter the general solution (5.36) and (5.37) above. Using the definition of the Lie derivative of the frame fields and the fact that the left and right fields commute, $[R_A, L_B]^{\mu} = 0$, we find

$$L_A E_{\mu \nu} = R^\rho_A L^{\rho B}_{\mu} (\partial_\rho E_{BC}) L^C_{\nu} \tag{5.39}$$

It follows that the Poisson-Lie condition can be rewritten as$^8$

$$R^\rho_A \partial_\rho E_{BC} = -E_{BD} L^{D}_{\rho} R^\rho_{EF} f^{EF}_{A} R^\lambda_A L^C_{\lambda} E_{GC} \tag{5.40}$$

or in terms of the inverse matrix elements

$$R^\rho_A \partial_\rho E^{DE} = L^{D}_{\rho} \tilde{R}^{\rho BC}_{A} R^\lambda_A L^E_{\lambda} \tag{5.41}$$

The dual relation is

$$\tilde{R}_A^\rho \partial^\rho \tilde{E}_{DE} = \tilde{L}^\rho_D \tilde{R}^B_{\rho} f^{A}_{BC} \tilde{R}^C_{A} L^E_{\lambda} \tag{5.42}$$

Here $\tilde{E}_{BC}$ is the components of the inverse matrix of $\tilde{E}$. In the case of non-abelian duality, the dual vector fields are trivial (i.e. $\tilde{R}_A^\rho \sim \delta^A_\rho$ and $\tilde{L}^\rho_D \sim \delta^D_\rho$ etc). Then

$$\partial^A \tilde{E}_{BC} = f^{A}_{BC} \tag{5.43}$$

For this case, the solution is

$$\tilde{E}_{BC} = E^0_{BC} + f^{A}_{BC} x_A \tag{5.44}$$

where we included spectator fields. ($A$ does not run through spectator degrees of freedom and $E^0_{BC}$ depends only on these spectators.)

$^8$Note that $LR$ is the group element $g$ in the adjoint representation.
5.3 The B-field Gauge Symmetry

In this section we briefly touch on the gauge symmetry for the B field $\delta B_{\mu \nu} = \partial_{[\mu} \Lambda_{\nu]}$ previously mentioned in Sec. 2. The argument is applicable to $N = 0, 1, 2$.

In abelian T-duality this symmetry may be treated as an enlargement of the duality group, at least in certain cases [48]. It would be interesting to understand if a similar interpretation is also possible for Poisson-Lie duality. We thus ask if this gauge symmetry is compatible with the condition (5.30). For the $B$-field this condition reads (in form language)

$$L_A B = i_A H + d(i_A B) = \frac{1}{2} \tilde{f}^{BC}_A (i_B B \wedge i_C B + R_B \wedge R_C) ,$$

(5.45)

where $R_A \equiv R^\mu_A G_{\mu \nu} dX^\nu$, and $i_A$ represents the contraction with $R^\mu_A$. Since the field strength $H$ is invariant, the variation $\delta B = d\Lambda$ gives

$$d(i_A d\Lambda) + \tilde{f}^{BC}_A i_C B \wedge i_B d\Lambda = 0 .$$

(5.46)

We first consider the possibility that the relation (5.46) is in fact an identity. This can be shown to be the case if $2\tilde{f}^{BC}_A i_C B = -\frac{1}{2} f^{B}_{AC} R^C$, an equation which may be solved to express $B_{AB}$ as a function of the structure constants $f^{B}_{AC}$ and $\tilde{f}^{BC}$. Clearly this represents a very special configuration. Otherwise, when (5.46) is not an identity, it may be interpreted as a structure equation. Viewing it like this, it is immediately clear that it generically gives $\omega^B_A = \tilde{f}^{BC}_A i_C B$ as a function of the gauge parameter. This cannot be the case and we conclude that there is an incompatibility. Perhaps it is possible to amend the Poisson-Lie condition with terms that take care of this, but we will not pursue this topic further here.

6 Supersymmetric Examples

In this section we present two examples which illustrates some of the previous discussion. Generally, there are several different ways to decompose a Drinfeld double into bi-algebras and an organizing principle is needed [49],[50]. Typically, in an application the choice will be dictated by additional requirements, e.g., tracelessness of the structure constants, imposed to preserve the conformal invariance of string theory [39]. Further, while there is a full classification of all six-dimensional Drinfeld doubles [49], a similar classification for the eight dimensional doubles is lacking. Since these are the smallest doubles of interest for $N = 2$, looking for such examples will be somewhat hampered by this lack of classification.

We take our starting point in the well known example of the $N = 4$ supersymmetrical WZW model on $SU(2) \times U(1)$, [51]. We want to find a $N = 1$ supersymmetric model instead,
based on a Drinfeld double with this group as part of the double. We find the double via a slight generalization of the $SU(2), E_3$ double of Sfetsos. ($E_3$ is described in more detail below). It would also be interesting to extend this to $N = 2$. We shall see that although we will find an almost complex structure, it fails to satisfy the conditions needed for $N = 2$.

A group element is

$$g = \frac{e^{i\theta}}{\sqrt{\phi \bar{\phi} + \lambda \bar{\lambda}}} \begin{pmatrix} \lambda & \bar{\phi} \\ -\phi & \bar{\lambda} \end{pmatrix}$$

(6.47)

where $\theta \equiv -\frac{1}{2} \ln(\phi \bar{\phi} + \lambda \bar{\lambda})$. The right and left invariant frames are found from (3.20) with generators $t_A = (\sigma_a, \frac{\sigma_a}{2})$; $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $Tr(t_A t_B) = \frac{1}{2} \delta_{AB}$; $\mu = \{ \phi, \bar{\phi}, \lambda, \bar{\lambda} \}$; $A = \{0, a\}$; $a = \{1, 2, 3\}$. These vector fields generate the $su(2) \oplus u(1)$ algebra

$$[L_0, L_b] = 0; \quad [L_a, L_b] = i\epsilon_{abc} L_c$$

$$[R_0, R_b] = 0; \quad [R_a, R_b] = -i\epsilon_{abc} R_c$$

and their explicit form is given in the appendix.

The algebra of the other component in the Drinfeld double $D^2$ is that of $e_3 \oplus u(1)$:

$$[t_i, t_j] = [t_0, t_i] = 0, \quad [t_3, t_j] = t_i, \quad i = 1, 2.$$ (6.49)

The structure constants $f_{AB}^C$ and $\tilde{f}_{AB}^C$ may be read off from (6.48) (left frames) and (6.49), respectively. Defining the generators of the $D^2$ algebra according to

$$T_a = \left( \frac{\sigma_a}{2}, \frac{\sigma_a}{2} \right); \quad T_0 = \left( \frac{1}{2}, \frac{1}{2} \right)$$

$$\tilde{T}^1 = (\sigma_+,-\sigma_-); \quad \tilde{T}^2 = -i(\sigma_+,-\sigma_-)$$

$$\tilde{T}^3 = \left( \frac{\sigma_3}{2}, -\frac{\sigma_3}{2} \right); \quad \tilde{T}^0 = \left( \frac{1}{2}, -\frac{1}{2} \right),$$

where $\sigma_a, a = 1, 2, 3$ and $\sigma_\pm$ refer to the usual Pauli matrices and their $\pm$ combinations. With the definitions in (6.50), the generators satisfy conditions (5.31). The invariant product $\langle \mid \rangle$ needed on the double is defined as

$$\langle (A, B)\mid (C, D) \rangle \equiv \langle A\mid B \rangle - \langle C\mid D \rangle,$$ (6.51)

where $(A, B) \in D^2$ and $\langle A\mid B \rangle = 2tr AB$. With (6.51) the generators in (6.50) satisfy the condition (5.33).

Having found a double and the left and right frames on one of the components, we plug the frames into (5.41) and solve it. The solution is given by

$$E^{AB} = E_0^{AB} + \Pi^{AB} = q^{AB} + c^{AB} + \Pi^{AB},$$ (6.52)
where $\eta^{AB}$ is a constant symmetric matrix, $c^{AB}$ is a constant antisymmetric matrix and $\Pi$ an antisymmetric coordinate-dependent matrix which reads

$$
\Pi = \frac{1}{D^2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 2i\lambda\bar{\lambda} & -\bar{\lambda}\phi + \lambda\phi \\
0 & 2i\lambda\bar{\lambda} & 0 & -i(\bar{\lambda}\phi + \lambda\phi) \\
0 & \bar{\lambda}\phi - \lambda\phi & i(\bar{\lambda}\phi + \lambda\phi) & 0
\end{pmatrix}, \quad (6.53)
$$

where $D^2 \equiv \lambda\bar{\lambda} + \phi\bar{\phi}$. The solution (6.52) is the inverse of the one given for the general case in (5.36). To write down the sigma model, we need the inverse $E^{AB}$ of the solution in (6.52), $E^{AB}E^{BC} = E^{CB}E^{BA} = \delta^C_A$. Introducing $\theta^{AB} \equiv c^{AB} + \Pi^{AB}$ and using the notation for $E^{AB} = G^{AB} + B^{AB}$ introduced in Sec.1, we find the following relations useful in looking for the inverse

$$
\eta^{AB}B_{BC} + \theta^{AB}G_{BC} = 0
$$

$$
\eta^{AB}G_{BC} + \theta^{AB}B_{BC} = \delta^A_C
$$

This implies that

$$
B_{AB} = -\eta_{AC}\theta^{CD}G_{DB}, \quad (6.55)
$$

where $\eta_{AB}$ is the (constant) inverse of $\eta^{AB}$, and that the inverse of $G$ is

$$
G^{AB} = \eta^{AB} - \theta^{AC}\eta_{CD}\theta^{DB}.
$$

Equivalently

$$
G_{AB} = (\eta - \eta\theta)^{-1} = (E^{-1}\eta(E^{-1})^T)_{AB}. \quad (6.57)
$$

Inserting (6.52) into these relations we calculate $E^{AB}$ and hence find a $N = 1$ sigma model and its dual on the double by inserting the result into (3.21). The various $N = 1$ supersymmetric models possible are determined by the choices of $\eta^{AB}$ and $c^{AB}$ in (6.52). We present the result for two different choices.

To find the explicit form of the double is straightforward. When we know $E^{AB}$ in (6.52), we compute $\alpha$ and $\beta$ in (5.35) using (6.50) and the invariant product (6.51). In doing this, we also need to coordinatize the dual group elements $\tilde{g}$. Finally (5.37) yields the dual metric and $B$-field.

The supersymmetric actions result from inserting $E$ or $\tilde{E}$ into (2.2).

---

9Since $B$ is antisymmetric, the symmetric part of the RHS of (6.55) has to vanish, which one can check that it does, writing it in terms of $E$ and $E^{-1}$. 

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14
6.1 Example I

If we choose $\eta^{AB} = \delta^{AB}$ and $c^{A0} = 0$, we find

$$G_{AB} = \Delta^{-1} \begin{pmatrix} \Delta & 0 & 0 & 0 \\ 0 & D^4 - s_+^2 & -is_+s_- & -2ls_+ \\ 0 & -is_+s_- & D^4 + s_-^2 & 2ils_- \\ 0 & -2ls_+ & 2ils_- & D^4 - 4l^2 \end{pmatrix},$$

(6.58)

where we use the condensed notation

$$l \equiv -\lambda \bar{\lambda}$$

$$s_\pm \equiv \bar{\lambda}\phi \pm \lambda\phi$$

$$(D^4 - 4l^2 - s_-^2 + s_+^2)D^{-4} \equiv \Delta.$$

(6.59)

In the same notation, the antisymmetric tensor reads

$$B_{AB} \equiv \Delta^{-1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -il & s_- \\ 0 & il & 0 & is_+ \\ 0 & -s_- & -is_+ & 0 \end{pmatrix},$$

(6.60)

6.2 Example II

In this example, with an eye towards $N = 2$, we attempt to find a complex structure that preserves the new metric.

The hermiticity condition (2.6) will be satisfied for a metric $G_{\mu\nu}$ if the corresponding relation is satisfied for the Lie-algebra components $J^B_{AC}J^C_{BD} = G_{AD}$, and this is equivalent to preservation of the inverse (6.56). From (6.52) it may be shown that it is sufficient to require preservation of $E^{-1}$, (or equivalently of $E$), a relation that we discussed in the paragraphs surrounding (2.12). In fact, choosing\textsuperscript{10}

$$J = \begin{pmatrix} -i & -2iq - p(\phi/\bar{\lambda} - \bar{\phi}/\lambda) & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & \phi/\bar{\lambda} - \bar{\phi}/\lambda \\ 0 & 0 & 0 & i \end{pmatrix},$$

(6.61)

gives a $J$ which preserves $E^{-1}$ with antisymmetric part $\theta$, provided that $c^{02} = c^{03} = 0$, and

\textsuperscript{10} One of several possibilities
symmetric part
\[
\eta^{-1} = \begin{pmatrix}
0 & 0 & 0 & n \\
0 & 0 & p & q \\
0 & p & 0 & 0 \\
n & q & 0 & 0
\end{pmatrix},
\]
where we may take \( n = 1 \) without loss of generality. We thus have an almost complex structure associated with the sigma model given by this choice of parameters. Unfortunately it does not pass the next test for \( N = 2 \); it does not satisfy (2.15). In fact, a further check shows that it is not integrable, its Nijenhuis tensor (2.5) is non-zero. (In calculating these relations we need to go to the coordinate expressions.) We finally record the expression for \( E \) in this case (with \( q = 0 = c^{01}, p = 1 \):
\[
E = \begin{pmatrix}
2i((\bar{\lambda}\phi)^2 - (\lambda\phi)^2)/A_+A_- & -i(\bar{\lambda}\phi + \lambda\phi)/A_+ & (\bar{\lambda}\phi - \lambda\phi)/A_- & 1 \\
-i(\bar{\lambda}\phi + \lambda\phi)/A_- & 0 & -D^2/A_- & 0 \\
(\bar{\lambda}\phi - \lambda\phi)/A_+ & D^2/A_+ & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\]
where
\[
A_\pm \equiv -2i\lambda\bar{\lambda} \pm D^2.
\]

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### 7 Appendix

#### 7.1 The right- and left-invariant frames

The components of the right-invariant forms are
\[
R^0_{\phi} = -\frac{i\bar{\phi}}{D^2}; \quad R^0_{\phi} = -\frac{i\phi}{D^2}; \quad R^0_{\lambda} = -\frac{i\bar{\lambda}}{D^2}; \quad R^0_{\lambda} = -\frac{i\lambda}{D^2}
\]
\[ R^1_\phi = -\frac{\lambda}{D^2}; \quad R^1_\lambda = \frac{\phi}{D^2}; \quad R^1_\phi = -\frac{\bar{\phi}}{D^2}; \quad R^1_\lambda = \frac{\bar{\lambda}}{D^2} \]

\[ R^2_\phi = \frac{i\lambda}{D^2}; \quad R^2_\lambda = \frac{i\phi}{D^2}; \quad R^2_\phi = -\frac{i\bar{\phi}}{D^2}; \quad R^2_\lambda = -\frac{i\bar{\lambda}}{D^2} \]

\[ R^3_\phi = -\frac{\bar{\phi}}{D^2}; \quad R^3_\lambda = \frac{\phi}{D^2}; \quad R^3_\phi = \frac{\bar{\lambda}}{D^2}; \quad R^3_\lambda = -\frac{\lambda}{D^2} \] (7.65)

and the components of the right-invariant vectors are

\[ R^0_\phi = \frac{i\phi}{2}; \quad R^1_\phi = -\frac{\lambda}{2}; \quad R^2_\phi = -\frac{i\lambda}{2}; \quad R^3_\phi = -\frac{\phi}{2} \]

\[ R^0_\lambda = \frac{i\bar{\phi}}{2}; \quad R^1_\lambda = \frac{\bar{\lambda}}{2}; \quad R^2_\lambda = -\frac{i\bar{\lambda}}{2}; \quad R^3_\lambda = -\frac{\bar{\phi}}{2} \]

\[ R^0_\phi = \frac{i\lambda}{2}; \quad R^1_\phi = \frac{\bar{\phi}}{2}; \quad R^2_\phi = \frac{\phi}{2}; \quad R^3_\phi = \frac{\bar{\lambda}}{2} \]

\[ R^0_\lambda = \frac{i\bar{\lambda}}{2}; \quad R^1_\lambda = \frac{\phi}{2}; \quad R^2_\lambda = \frac{i\bar{\phi}}{2}; \quad R^3_\lambda = \frac{\bar{\phi}}{2} \] (7.66)

The components of the left-invariant forms are

\[ L^0_\phi = -\frac{i\bar{\phi}}{D^2}; \quad L^0_\phi = -\frac{i\bar{\phi}}{D^2}; \quad L^0_\lambda = -\frac{i\bar{\lambda}}{D^2}; \quad L^0_\lambda = -\frac{i\bar{\lambda}}{D^2} \]

\[ L^1_\phi = -\frac{\bar{\lambda}}{D^2}; \quad L^1_\phi = -\frac{\bar{\lambda}}{D^2}; \quad L^1_\lambda = \frac{\phi}{D^2}; \quad L^1_\lambda = \frac{\phi}{D^2} \]

\[ L^2_\phi = \frac{i\bar{\lambda}}{D^2}; \quad L^2_\phi = \frac{i\bar{\lambda}}{D^2}; \quad L^2_\lambda = -\frac{i\bar{\phi}}{D^2}; \quad L^2_\lambda = -\frac{i\bar{\phi}}{D^2} \]

\[ L^3_\phi = \frac{\bar{\phi}}{D^2}; \quad L^3_\phi = -\frac{\phi}{D^2}; \quad L^3_\lambda = \frac{\bar{\phi}}{D^2}; \quad L^3_\lambda = -\frac{\phi}{D^2} \] (7.67)

and the components of the left invariant vectors are

\[ L^0_\phi = \frac{i\phi}{2}; \quad L^1_\phi = \frac{\bar{\phi}}{2}; \quad L^2_\phi = \frac{\phi}{2}; \quad L^3_\phi = \frac{\bar{\lambda}}{2} \]

\[ L^0_\lambda = \frac{i\bar{\phi}}{2}; \quad L^1_\lambda = \frac{i\bar{\phi}}{2}; \quad L^2_\lambda = -\frac{i\bar{\phi}}{2}; \quad L^3_\lambda = \frac{\bar{\phi}}{2} \]

\[ L^0_\phi = \frac{i\lambda}{2}; \quad L^1_\phi = \frac{i\lambda}{2}; \quad L^2_\phi = \frac{i\lambda}{2}; \quad L^3_\phi = \frac{i\lambda}{2} \]

\[ L^0_\lambda = \frac{i\bar{\lambda}}{2}; \quad L^1_\lambda = \frac{i\bar{\lambda}}{2}; \quad L^2_\lambda = \frac{i\bar{\lambda}}{2}; \quad L^3_\lambda = \frac{i\bar{\lambda}}{2} \] (7.68)

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