DOUBLY COMMUTING SUBMODULES OF THE HARDY MODULE OVER POLYDISCS

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Abstract. In this note we establish a vector-valued version of Beurling’s Theorem (the Lax-Halmos Theorem) for the polydisc. As an application of the main result, we provide necessary and sufficient conditions for the “weak” completion problem in $H^\infty(D^n)$.

1. Introduction and Statement of Main Results

In [B], Beurling described all the invariant subspaces for the operator $M_z$ of “multiplication by $z$” on the Hilbert space $H^2(D)$ of the disc. In [L], Peter Lax extended Beurling’s result to the (finite-dimensional) vector-valued case (while also considering the Hardy space of the half plane). Lax’s vectorial case proof was further extended to infinite-dimensional vector spaces by Halmos, see [NF]. The characterization of $M_z$-invariant subspaces obtained is the following famous result.

Theorem 1.1 (Beurling-Lax-Halmos). Let $S$ be a closed nonzero subspace of $H^2_{E^*}(D)$. Then $S$ is invariant under multiplication by $z$ if and only if there exists a Hilbert space $E$ and an inner function $\Theta \in H^\infty_{E \to E^*}(D)$ such that $S = \Theta H^2_{E^*}(D)$.

For $n \in \mathbb{N}$ and $E_*$ a Hilbert space, $H^2_{E_*}(D^n)$ is the set of all $E_*$-valued holomorphic functions in the polydisc $D^n$, where $D := \{z \in \mathbb{C} : |z| < 1\}$ (with boundary $T$) such that

$$\|f\|_{H^2_{E_*}(D^n)} := \sup_{0 < r < 1} \left( \int_{T^n} \|f(rz)\|^2_{E_*} \, dz \right)^{1/2} < +\infty.$$ 

On the other hand, if $\mathcal{L}(E, E_*)$ denotes the set of all continuous linear transformations from $E$ to $E_*$, then $H^\infty_{E \to E_*}(D^n)$ denotes the set of all $\mathcal{L}(E, E_*)$-valued holomorphic functions with $\|f\|_{H^\infty_{E \to E_*}(D^n)} := \sup_{z \in D^n} \|f(z)\|_{\mathcal{L}(E, E_*)} < \infty$.

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An operator-valued $\Theta \in H^\infty_{E \to E_*}(\mathbb{D}^n)$ is inner if the pointwise a.e. boundary values are isometries:

$$(\Theta(\zeta))^* \Theta(\zeta) = I_E \text{ for almost all } \zeta \in \mathbb{T}^n.$$ 

A natural question is then to ask what happens in the case of several variables, for example when one considers the Hardy space $H^2_{E_*}(\mathbb{D}^n)$ of the polydisc $\mathbb{D}^n$. It is known that in general, a Beurling-Lax-Halmos type characterization of subspaces of the Hardy Hilbert space is not possible \cite{R}. It is however, easy to see that the Hardy space on the polydisc $H^2_{E_*}(\mathbb{D}^n)$, when $n > 1$, satisfies the **doubly commuting** property, that is, for all $1 \leq i < j \leq n$

$$M^*_z M_j = M_j M^*_z.$$ 

We impose this additional assumption to the submodules of $H^2_{E_*}(\mathbb{D}^n)$ and call that class of submodules as doubly commuting submodules. More precisely:

**Definition 1.2.** A commuting family of bounded linear operators $\{T_1, \ldots, T_n\}$ on some Hilbert space $\mathcal{H}$ is said to be **doubly commuting** if 

$$T_i T_j^* = T_j^* T_i,$$

for all $1 \leq i, j \leq n$ and $i \neq j$.

A closed subspace $\mathcal{S}$ of $H^2_{E}(\mathbb{D}^n)$ which is invariant under $M_{z_1}, \ldots, M_{z_n}$ is said to be a **doubly commuting submodule** if $\mathcal{S}$ is a submodule, that is, $M_{z_i} \mathcal{S} \subseteq \mathcal{S}$ for all $i$ and the family of module multiplication operators \{$R_{z_1}, \ldots, R_{z_n}$\} where

$$R_{z_i} := M_{z_i}|_{\mathcal{S}},$$

for all $1 \leq i \leq n$, is doubly commuting, that is,

$$R_{z_i} R_{z_j}^* = R_{z_j}^* R_{z_i},$$

for all $i \neq j$ in $\{1, \ldots, n\}$.

In this note we completely characterize the doubly commuting submodules of the vector-valued Hardy module $H^2_{E_*}(\mathbb{D}^n)$ over the polydisc, and this is the content of our main theorem. This result is an analogue of the classical Beurling-Lax-Halmos Theorem on the Hardy space over the unit disc.

**Theorem 1.3.** Let $\mathcal{S}$ be a closed nonzero subspace of $H^2_{E_*}(\mathbb{D}^n)$. Then $\mathcal{S}$ is a doubly commuting submodule if and only if there exists a Hilbert space $E$ with $E \subseteq E_*$, where the inclusion is up to unitary equivalence, and an inner function $\Theta \in H^\infty_{E \to E_*}(\mathbb{D}^n)$ such that

$$\mathcal{S} = M_\Theta H^2_{E}(\mathbb{D}^n).$$
In the special scalar case $E_\ast = \mathbb{C}$ and when $n = 2$ (the bidisc), this characterization was obtained by Mandrekar in [M], and the proof given there relies on the Wold decomposition for two variables [S]. Our proof is based on the more natural language of Hilbert modules and a generalization of Wold decomposition for doubly commuting isometries [Sa].

As an application of this theorem, we can establish a version of the “Weak” Completion Property for the algebra $H^\infty(\mathbb{D}^n)$. Suppose that $E \subset E_c$. Recall that the Completion Problem for $H^\infty(\mathbb{D}^n)$ is the problem of characterizing the functions $f \in H^\infty_{E \rightarrow E_c}(\mathbb{D}^n)$ such that there exists an invertible function $F \in H^\infty_{E_c \rightarrow E}(\mathbb{D}^n)$ with $F|_E = f$.

In the case of $H^\infty(\mathbb{D})$, the Completion Problem was settled by Tolokonnikov in [To]. In that paper, it was pointed out that there is a close connection between the Completion Problem and the characterization of invariant subspaces of $H^2(\mathbb{D})$. Using Theorem 1.3 we then have the following analogue of the results in [To].

**Theorem 1.4** (Tolokonnikov’s Lemma for the Polydisc). Let $f \in H^\infty_{E \rightarrow E_c}(\mathbb{D}^n)$ with $E \subset E_c$ and $\dim E, \dim E_c < \infty$. Then the following statements are equivalent:

(i) There exists a function $g \in H^\infty_{E_c \rightarrow E}(\mathbb{D}^n)$ such that $gf \equiv I$ in $\mathbb{D}^n$ and the operators $M_{z_1}, \ldots, M_{z_n}$ doubly commute on the subspace $\ker M_g$.

(ii) There exists a function $F \in H^\infty_{E_c \rightarrow E}(\mathbb{D}^n)$ such that $F|_E = f$, $F|_{E_c \oplus E}$ is inner, and $F^{-1} \in H^\infty_{E_c \rightarrow E}(\mathbb{D}^n)$.

**Remark 1.5.** Theorem 1.4 for the polydisc is different from Tolokonnikov’s lemma in the disc in which one does not demand that the completion $F$ has the property that $F|_{E_c \oplus E}$ is inner. But, from the proof of Tolokonnikov’s lemma in the case of the disc (see [N]), one can see that the following statements are equivalent for $f \in H^\infty_{E \rightarrow E_c}(\mathbb{D})$ with $E \subset E_c$ and $\dim E < \infty$:

(i) There exists a function $g \in H^\infty_{E_c \rightarrow E}(\mathbb{D})$ such that $gf \equiv I$ in $\mathbb{D}$.

(ii) There exists a function $F \in H^\infty_{E_c \rightarrow E}(\mathbb{D})$ such that $F|_E = f$, and $F^{-1} \in H^\infty_{E_c \rightarrow E}(\mathbb{D})$.

(ii’) There exists a function $F \in H^\infty_{E_c \rightarrow E}(\mathbb{D})$ such that $F|_E = f$, $F|_{E_c \oplus E}$ is inner, and $F^{-1} \in H^\infty_{E_c \rightarrow E}(\mathbb{D})$.

In the polydisc case it is unclear how the conditions

(II) There exists a function $F \in H^\infty_{E_c \rightarrow E}(\mathbb{D}^n)$ such that $F|_E = f$, and $F^{-1} \in H^\infty_{E_c \rightarrow E}(\mathbb{D}^n)$.

(II’) There exists a function $F \in H^\infty_{E_c \rightarrow E}(\mathbb{D}^n)$ such that $F|_E = f$, $F|_{E_c \oplus E}$ is inner, and $F^{-1} \in H^\infty_{E_c \rightarrow E}(\mathbb{D}^n)$.
are related. We refer to the Completion Problem in (II) as the Strong Completion Problem, while the one in (II) as the Weak Completion Problem. Whether the two are equivalent is an open problem.

We also remark that in the disc case, Tolokonnikov’s Lemma was proved by Sergei Treil \cite{T} without any assumptions about the finite dimensionality of $E, E_c$. However, our proof of Theorem 1.4 relies on Lemma 3.1, whose validity we do not know without the assumption on the finite dimensionality of $E$ and $E_c$. 

**Example 1.6.** As a simple illustration of Theorem 1.4, take $n = 3$, $\dim E = 1$, $\dim E_c = 3$ and

$$f := \begin{bmatrix} e^{z_1} \\ e^{z_2} \\ e^{z_3} \end{bmatrix} \in (H^\infty(\mathbb{D}^3))^{3 \times 1}.$$ 

With $g := \begin{bmatrix} e^{-z_1} & 0 & 0 \end{bmatrix} \in (H^\infty(\mathbb{D}^2))^{1 \times 3}$, we see that $gf = 1$. We have

$$\ker M_g = \left\{ \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} \in (H^2(\mathbb{D}^3))^{3 \times 1} : e^{-z_1} \varphi_1 = 0 \right\}$$

$$= \left\{ \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} \in (H^2(\mathbb{D}^3))^{3 \times 1} : \varphi_1 = 0 \right\} = \Theta(H^2(\mathbb{D}^2))^{2 \times 1},$$

where $\Theta$ is the inner function

$$\Theta := \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \in (H^\infty(\mathbb{D}^3))^{3 \times 2}.$$ 

As $\Theta$ is inner, it follows from Theorem 1.3 that $M_{z_1}, M_{z_2}, M_{z_3}$ doubly commute on the submodule $\Theta(H^2(\mathbb{D}^3))^{2 \times 1} = \ker M_g$. Hence $f$ can be completed to an invertible matrix. In fact, with

$$F := \begin{bmatrix} f & \Theta \end{bmatrix} = \begin{bmatrix} e^{z_1} & 0 & 0 \\ e^{z_2} & 1 & 0 \\ e^{z_3} & 0 & 1 \end{bmatrix},$$

one can easily see that $F$ is invertible as an element of $(H^\infty(\mathbb{D}^3))^{3 \times 3}$.

In Section 2 we give a proof of Theorem 1.3 and subsequently, in Section 3 we use this theorem to study the Weak Completion Problem for $H^\infty(\mathbb{D}^n)$, providing a proof of Theorem 1.4.

2. Beurling-Lax-Halmos Theorem for the Polydisc

In this section we present a complete characterization of “reducing submodules” and a proof of the Beurling-Lax-Halmos theorem for doubly commuting submodules of $H^2_E(\mathbb{D}^n)$. 
Recall that a closed subspace $S \subseteq H^2_E(D^n)$ is said to be a reducing submodule of $H^2_E(D^n)$ if $M_{z_i}S, M_{z_i}^*S \subseteq S$ for all $i = 1, \ldots, n$.

We start by reviewing some definitions and some well-known facts about the vector-valued Hardy space over polydisc. For more details about reproducing kernel Hilbert spaces over domains in $\mathbb{C}^n$, we refer the reader to [DMS]. Let

$$S(z, w) = \prod_{j=1}^{n}(1 - \overline{w}_j z_j)^{-1}. \quad ((z, w) \in D^n \times D^n)$$

be the Cauchy kernel on the polydisc $D^n$. Then for some Hilbert space $E$, the kernel function $S_E$ of $H^2_E(D^n)$ is given by

$$S_E(z, w) = S(z, w)I_E. \quad ((z, w) \in D^n \times D^n)$$

In particular, $\{S(\cdot, w)\eta : w \in D^n, \eta \in E\}$ is a total subset for $H^2_E(D^n)$, that is,

$$\text{span}\{S(\cdot, w)\eta : w \in D^n, \eta \in E\} = H^2_E(D^n),$$

where $S(\cdot, w) \in H^2(D^n)$ and

$$(S(\cdot, w))(z) = S(z, w),$$

for all $z, w \in D^n$. Moreover, for all $f \in H^2_E(D^n)$, $w \in D^n$ and $\eta \in E$ we have

$$\langle f, S(\cdot, w)\eta \rangle_{H^2_E(D^n)} = \langle f(w), \eta \rangle_E.$$  

Note also that for the multiplication operator $M_{z_i}$ on $H^2_E(D^n)$

$$M_{z_i}^*(S(\cdot, w)\eta) = \bar{w}_i(S(\cdot, w)\eta),$$

where $w \in D^n$, $\eta \in E$ and $1 \leq i \leq n$.

We also have

$$S^{-1}(z, w) = \sum_{0 \leq i_1 < \ldots < i_l \leq n} (-1)^l z_{i_1} \cdots z_{i_l} \bar{w}_{i_1} \cdots \bar{w}_{i_l},$$

for all $z, w \in D^n$.

For $H^2(D^n)$ we set

$$S^{-1}_E(M_z, M_{z_i}) := \sum_{0 \leq i_1 < \ldots < i_l \leq n} (-1)^l M_{z_{i_1}} \cdots M_{z_{i_l}} M_{z_{i_1}}^* \cdots M_{z_{i_l}}^*.$$  

The following Lemma is well-known in the study of reproducing kernel Hilbert spaces.

**Lemma 2.1.** Let $E$ be a Hilbert space. Then

$$S^{-1}_E(M_z, M_z) = P_E,$$

where $P_E$ is the orthogonal projection of $H^2(D^n)$ onto the space of all constant functions.
Proof. for all $z, w \in \mathbb{D}^n$ and $\eta, \zeta \in E$ we have

$$\langle S^{-1}_E(M_z, M_z) (S(\cdot, z)\eta), (S(\cdot, w)\zeta) \rangle_{H^2_E(\mathbb{D}^n)}$$

$$= \left\langle \sum_{0 \leq i_1 < \ldots < i_n} (-1)^i M_{z_{i_1}} \cdots M_{z_{i_n}} M^\ast_{z_{i_n}} \cdots M^\ast_{z_{i_1}} (S(\cdot, z)\eta), (S(\cdot, w)\zeta) \right\rangle_{H^2_E(\mathbb{D}^n)}$$

$$= \sum_{0 \leq i_1 < \ldots < i_n} (-1)^i \left\langle M^\ast_{z_{i_n}} \cdots M^\ast_{z_{i_1}} (S(\cdot, z)\eta), M_{z_{i_n}} \cdots M_{z_{i_1}} (S(\cdot, w)\zeta) \right\rangle_{H^2_E(\mathbb{D}^n)}$$

$$= \sum_{0 \leq i_1 < \ldots < i_n} (-1)^i \langle \hat{z}_{i_n} \cdots \hat{z}_{i_1} w_{i_1} \ldots w_i (S(\cdot, z), S(\cdot, w)) \rangle_{H^2_E(\mathbb{D}^n)} \langle \eta, \zeta \rangle_E$$

$$= \langle S^{-1}(\mathbf{w}, \mathbf{z})\mathbf{S}(\mathbf{w}, \mathbf{z}), S(\cdot, z)\eta, (S(\cdot, w)\zeta) \rangle_{H^2_E(\mathbb{D}^n)}$$

$$= \langle P_E S(\cdot, z)\eta, S(\cdot, w)\zeta \rangle_{H^2_E(\mathbb{D}^n)}$$

Since $\{S(\cdot, z)\eta : z \in \mathbb{D}^n, \eta \in E\}$ is a total subset of $H^2_E(\mathbb{D}^n)$, we have that

$$S^{-1}_E(M_z, M_z) = P_E.$$

This completes the proof. \qed

In the following proposition we characterize the reducing submodules of $H^2_E(\mathbb{D}^n)$.

**Proposition 2.2.** Let $S$ be a closed subspace of $H^2_E(\mathbb{D}^n)$. Then $S$ is a reducing submodule of $H^2_E(\mathbb{D}^n)$ if and only if

$$S = H^2_E(E, \mathbb{D}^n),$$

for some closed subspace $E_\ast$ of $E$.

**Proof.** Let $S$ be a reducing submodule of $H^2_E(\mathbb{D}^n)$, that is, for all $1 \leq i \leq n$ we have

$$M_{z_i} P_S = P_S M_{z_i}.$$

By Lemma 2.1

$$P_E P_S = S^{-1}_E(M_z, M_z) P_S = P_S S^{-1}_E(M_z, M_z) = P_S P_E.$$

In particular, that $P_S P_E$ is an orthogonal projection and

$$P_S P_E = P_E P_S = P_{E_\ast},$$

where $E_\ast := E \cap S$. Hence, for any

$$f = \sum_{k \in \mathbb{N}^n} a_k \mathbf{z}^k \in S,$$

where $a_k \in E$ for all $k \in \mathbb{N}^n$, we have

$$f = P_S f = P_S \left( \sum_{k \in \mathbb{N}^n} M_z^k a_k \right) = \sum_{k \in \mathbb{N}^n} M_z^k P_S a_k.$$
But $P_S a_k = P_S P_E a_k \in E_*$. Consequently, $M_k^k P_S a_k \in H_E^2(\mathbb{D}^n)$ for all $k \in \mathbb{N}^n$ and hence $f \in H_E^2(\mathbb{D}^n)$. That is, $\mathcal{S} \subseteq H_E^2(\mathbb{D}^n)$. For the reverse inclusion, it is enough to observe that $E_* \subseteq \mathcal{S}$ and that $\mathcal{S}$ is a reducing submodule. The converse part is immediate. Hence the lemma follows.

Let $\mathcal{S}$ be a doubly commuting submodule of $H_E^2(\mathbb{D}^n)$. Then

$$R_{z_i} R_{z_i}^* = M_{z_i} P_S M_{z_i}^* P_S = M_{z_i} P_S M_{z_i}^*, \quad z_i \in \mathbb{D}^n,$$

implies that $R_{z_i} R_{z_i}^*$ is an orthogonal projection of $\mathcal{S}$ onto $z_i \mathcal{S}$ and hence $I_{\mathcal{S}} - R_{z_i} R_{z_i}^*$ is an orthogonal projection of $\mathcal{S}$ onto $\mathcal{S} \ominus z_i \mathcal{S}$, that is,

$$I_{\mathcal{S}} - R_{z_i} R_{z_i}^* = P_{\mathcal{S} \ominus z_i \mathcal{S}},$$

for all $i = 1, \ldots, n$. Define

$$W_i = \text{ran}(I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) = \mathcal{S} \ominus z_i \mathcal{S},$$

for all $i = 1, \ldots, n$, and

$$W = \bigcap_{i=1}^n W_i.$$

Now let $\mathcal{S}$ be a doubly commuting submodule of $H_E^2(\mathbb{D}^n)$. By doubly commutativity of $\mathcal{S}$ it follows that (also see [Sa])

$$(I_{\mathcal{S}} - R_{z_i} R_{z_i}^*)(I_{\mathcal{S}} - R_{z_j} R_{z_j}^*) = (I_{\mathcal{S}} - R_{z_j} R_{z_j}^*)(I_{\mathcal{S}} - R_{z_i} R_{z_i}^*),$$

for all $i \neq j$. Therefore $\{(I_{\mathcal{S}} - R_{z_i} R_{z_i}^*)\}_{i=1}^n$ is a family of commuting orthogonal projections and hence

$$(2.1) \quad W = \bigcap_{i=1}^n W_i = \bigcap_{i=1}^n (\mathcal{S} \ominus z_i \mathcal{S}) = \bigcap_{i=1}^n \text{ran}(I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) = \text{ran}\left(\bigcap_{i=1}^n (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*)\right).$$

Now we present a wandering subspace theorem concerning doubly commuting submodules of $H_E^2(\mathbb{D}^n)$. The result is a consequence of a several variables analogue of the classical Wold decomposition theorem as obtained by Gaspar and Suciu [GS]. We provide a direct proof (also see Corollary 3.2 in [Sa]).

**Theorem 2.3.** Let $\mathcal{S}$ be a doubly commuting submodule of $H_E^2(\mathbb{D}^n)$. Then

$$\mathcal{S} = \bigoplus_{k \in \mathbb{N}^n} \mathcal{S} \ominus z^k \mathcal{W}.$$

**Proof.** First, note that if $\mathcal{M}$ is a submodule of $H_E^2(\mathbb{D}^n)$ then

$$\bigcap_{k \in \mathbb{N}} R_{z_i}^k \mathcal{M} \subseteq \bigcap_{k \in \mathbb{N}} M_{z_i}^k H_E^2(\mathbb{D}^n) = \{0\},$$

implies that $\mathcal{M}$ is a submodule of $H_E^2(\mathbb{D}^n)$.
for each \( i = 1, \ldots, n \). Therefore, \( R_{z_i} \) is a shift, that is, the unitary part \( \bigcap_{k \in \mathbb{N}} R_{z_i}^k \mathcal{M} \) in the Wold decomposition (cf. [NF], [Sa]) of \( R_{z_i} \) on \( \mathcal{M} \) is trivial for all \( i = 1, \ldots, n \). Moreover, if \( \mathcal{S} \) is doubly commuting then

\[
R_{z_i} (I_{\mathcal{S}} - R_{z_j} R_{z_j}^*) = (I_{\mathcal{S}} - R_{z_j} R_{z_j}^*) R_{z_i},
\]

for all \( i \neq j \). Therefore \( \mathcal{W}_j \) is a \( R_{z_i} \)-reducing subspace for all \( i \neq j \). Note also that for all \( 1 \leq m < n \),

\[
\bigcap_{i=1}^{m+1} \mathcal{W}_i = \text{ran} \left( \prod_{i=1}^{m} (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) \right)
= \text{ran} \left( \prod_{i=1}^{m} (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) - R_{z_{m+1}} R_{z_{m+1}}^* \prod_{i=1}^{m} (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) \right)
= \text{ran} \left( \prod_{i=1}^{m} (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) - R_{z_{m+1}} \prod_{i=1}^{m} (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) R_{z_{m+1}}^* \right)
= (\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m) \ominus z_{m+1} (\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m),
\]

and hence

\[
(\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m) \ominus z_{m+1} (\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m) = \bigcap_{i=1}^{m+1} \mathcal{W}_i.
\]

We use mathematical induction to prove that for all \( 2 \leq m \leq n \), we have

\[
\mathcal{S} = \sum_{k \in \mathbb{N}^m} \oplus z^k (\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m).
\]

First, by Wold decomposition theorem for the shift \( R_{z_1} \) on \( \mathcal{S} \) we have

\[
\mathcal{S} = \sum_{k_1 \in \mathbb{N}} \oplus R_{z_1}^{k_1} \mathcal{W}_1 = \sum_{k_1 \in \mathbb{N}} \oplus z_1^{k_1} \mathcal{W}_1.
\]

Again by applying Wold decomposition for \( R_{z_2} |_{\mathcal{W}_1} \in \mathcal{L}(\mathcal{W}_1) \) we have

\[
\mathcal{W}_1 = \sum_{k_2 \in \mathbb{N}} \oplus R_{z_2}^{k_2} (\mathcal{W}_1 \cap z_2 \mathcal{W}_1) = \sum_{k_2 \in \mathbb{N}} \oplus z_2^{k_2} (\mathcal{W}_1 \cap \mathcal{W}_2),
\]

and hence

\[
\mathcal{S} = \sum_{k_1 \in \mathbb{N}} \oplus z_1^{k_1} \left( \sum_{k_2 \in \mathbb{N}} \oplus z_2^{k_2} (\mathcal{W}_1 \cap \mathcal{W}_2) \right) = \sum_{k_1, k_2 \in \mathbb{N}} \oplus z_1^{k_1} z_2^{k_2} (\mathcal{W}_1 \cap \mathcal{W}_2).
\]

Finally, let

\[
\mathcal{S} = \sum_{k \in \mathbb{N}^m} \oplus z^k (\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m),
\]

for some \( m < n \). Then we again apply the Wold decomposition on the isometry

\[
R_{z_{m+1}} |_{\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m} \in \mathcal{L}(\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m)
\]
to obtain
\[ \mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m = \sum_{k_{m+1} \in \mathbb{N}} \oplus z_{k_{m+1}}^{k_{m+1}} ( (\mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m) \oplus z_{m+1} \mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m ) \]
\[ = \sum_{k_{m+1} \in \mathbb{N}} \oplus z_{k_{m+1}}^{k_{m+1}} ( \mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m \cap \mathcal{W}_{m+1} ), \]
which yields
\[ S = \sum_{k \in \mathbb{N}_{m+1}} \oplus z^k ( \mathcal{W}_1 \cap \ldots \cap \mathcal{W}_{m+1} ). \]
This completes the proof.

We now turn to the proof of Theorem 1.3.

**Proof of Theorem 1.3.** By Theorem 2.3 we have
\[ (2.2) \quad S = \sum_{k \in \mathbb{N}^n} \oplus z^k ( \bigcap_{i=1}^n \mathcal{W}_i ). \]

Now define the Hilbert space \( E \) by
\[ E = \bigcap_{i=1}^n \mathcal{W}_i, \]
and the linear operator \( V : H_{E}^2 (\mathbb{D}^n) \to H_{E^*}^2 (\mathbb{D}^n) \) by
\[ V \left( \sum_{k \in \mathbb{N}^n} a_k z^k \right) = \sum_{k \in \mathbb{N}^n} M_z^k a_k, \]
where
\[ \sum_{k \in \mathbb{N}^n} a_k z^k \in H_{E}^2 (\mathbb{D}^n) \]
and \( a_k \in E \) for all \( k \in \mathbb{N}^n \). Observe that
\[ \| \sum_{k \in \mathbb{N}^n} M_z^k a_k \|_{H_{E}^2 (\mathbb{D}^n)}^2 = \| \sum_{k \in \mathbb{N}^n} z^k a_k \|_{H_{E}^2 (\mathbb{D}^n)}^2 = \sum_{k \in \mathbb{N}^n} \| z^k a_k \|_{H_{E}^2 (\mathbb{D}^n)}^2, \]
where the last equality follows from the orthogonal decomposition of \( S \) in (2.2).

Therefore,
\[ \| \sum_{k \in \mathbb{N}^n} M_z^k a_k \|_{H_{E}^2 (\mathbb{D}^n)}^2 = \sum_{k \in \mathbb{N}^n} \| z^k a_k \|_{H_{E}^2 (\mathbb{D}^n)}^2 = \sum_{k \in \mathbb{N}^n} \| a_k \|_{H_{E^*}^2 (\mathbb{D}^n)}^2 = \sum_{k \in \mathbb{N}^n} \| z^k a_k \|_{H_{E^*}^2 (\mathbb{D}^n)}^2, \]
and hence \( V \) is an isometry. Moreover, for all \( k \in \mathbb{N}^n \) and \( \eta \in E \) we have
\[ VM_z (z^k \eta) = V (z^{k+e_i} \eta) = M_z^{k+e_i} \eta = M_z (M_z^k \eta) = M_z V (z^k \eta), \]
that is, \( VM_z = M_z V \) for all \( i = 1, \ldots, n \). Hence \( V \) is a module map. Therefore,
\[ V = M_\Theta, \]
for some bounded holomorphic function \( \Theta \in H^\infty_{E \rightarrow E_n}(\mathbb{D}^n) \) (cf. page 655 in [BLTT]). Moreover, since \( V \) is an isometry, we have

\[
M^*_\Theta M_\Theta = I_{H^2_E(\mathbb{D}^n)},
\]

that is, that \( \Theta \) is an inner function. Also since \( M_{z_i}E \subseteq S \) for all \( i = 1, \ldots, n \) we have that

\[
\text{ran} V \subseteq S.
\]

Also by (2.2) that \( S \subseteq \text{ran} V \). Hence it follows that

\[
\text{ran} V = \text{ran} M_\Theta = S,
\]

that is,

\[
S = \Theta H^2_E(\mathbb{D}^n).
\]

Finally, for all \( i = 1, \ldots, n \), we have

\[
S \ominus z_iS = \Theta H^2_E(\mathbb{D}^n) \ominus z_i\Theta H^2_E(\mathbb{D}^n) = \{ \Theta f : f \in H^2_E(\mathbb{D}^n), M^*_z \Theta f = 0 \},
\]

and hence by (2.1)

\[
E = \bigcap_{i=1}^n W_i = \bigcap_{i=1}^n (S \ominus z_iS) = \{ \Theta f : M^*_z \Theta f = 0, f \in H^2_E(\mathbb{D}^n), \forall i = 1, \ldots, n \}
\]

\[
\subseteq \{ g \in H^2_{E_{z_i}}(\mathbb{D}^n) : M^*_z g = 0, \forall i = 1, \ldots, n \} = E_*,
\]

that is,

\[
E \subseteq E_*.
\]

To prove the converse part, let \( S = M_\Theta H^2_E(\mathbb{D}^n) \) be a submodule of \( H^2_{E_{z_i}}(\mathbb{D}^n) \) for some inner function \( \Theta \in H^\infty_{E \rightarrow E_{z_i}}(\mathbb{D}^n) \). Then

\[
P_S = M_\Theta M^*_\Theta,
\]

and hence for all \( i \neq j \),

\[
M_{z_i}P_SM^*_z = M_{z_i}M_\Theta M^*_\Theta M^*_z = M_\Theta M_{z_i} M^*_z M^*_\Theta = M_{z_i} M^*_j M^*_z M^*_\Theta
\]

\[
= M_{z_i} M^*_j M^*_\Theta M_{z_i} M^*_\Theta = M_{z_i} M^*_j M^*_z M_{z_i} M^*_\Theta = P_SM^*_z M_{z_i} P_S.
\]

This implies

\[
R^*_{z_j}R_{z_i} = P_SM^*_z P_SM^*_z |_S = P_SM^*_z M_{z_i} |_S = M_{z_i}P_SM^*_z = R_{z_i} R^*_{z_j},
\]

that is, \( S \) is a doubly commuting submodule. This completes the proof. \( \blacksquare \)
3. Tolokonnikov's Lemma for the Polydisc

We will need the following lemma, which is a polydisc version of a similar result proved in the case of the disc in Nikolski’s book \[N]\* p.44-45. Here we use the notation \(M_g\) for the multiplication operator on \(H^2_E\) induced by \(g \in H^\infty_{E \to E} \).

**Lemma 3.1 (Lemma on Local Rank).** Let \(E, E_c\) be Hilbert spaces, with \(\dim E, \dim E_c < \infty\). Let \(g \in H^\infty_{E_c \to E}(\mathbb{D}^n)\) be such that

\[
\ker M_g = \{ h \in H^2_{E_c}(\mathbb{D}^n) : g(z)h(z) \equiv 0 \} = \Theta H^2_{E_a}(\mathbb{D}^n),
\]

where \(E_a\) is a Hilbert space and \(\Theta\) is a \(\mathcal{L}(E_a, E_c)\)-valued inner function. Then

\[
\dim E_c = \dim E_a + \text{rank } g,
\]

where \(\text{rank } g := \max_{\zeta \in \mathbb{D}^n} \text{rank } g(\zeta)\).

**Proof.** We have \(\ker M_g = \{ h \in H^2_{E_c}(\mathbb{D}^n) : gh \equiv 0 \}\). If \(\zeta \in \mathbb{D}^n\), then let

\[
[\ker M_g](\zeta) := \{ h(\zeta) : h \in \ker M_g \}.
\]

We claim that \([\ker M_g](\zeta) = \Theta(\zeta)E_a\). Indeed, let \(v \in [\ker M_g](\zeta)\). Then \(v = h(\zeta)\) for some element \(h \in \ker M_g \supseteq \Theta H^2_{E_a}(\mathbb{D}^n)\). So \(h = \Theta \varphi\), for some \(\varphi \in H^2_{E_a}(\mathbb{D}^n)\). In particular, \(v = h(\zeta) = \Theta(\zeta)\varphi(\zeta)\), where \(\varphi(\zeta) \in E_a\). So

\[
(3.1) \quad [\ker M_g](\zeta) \subset \Theta(\zeta)E_a.
\]

On the other hand, if \(w \in \Theta(\zeta)E_a\), then \(w = \Theta(\zeta)x\), where \(x \in E_a\). Consider the constant function \(x\) mapping \(\mathbb{D} \ni \zeta \mapsto x \in E_a\). Clearly \(x \in H^2_{E_a}(\mathbb{D}^n)\). So \(h := \Theta x \in \Theta H^2_{E_a}(\mathbb{D}^n) = \ker M_g\). Hence \(w = \Theta(\zeta)x = (\Theta x)(\zeta) = h(\zeta)\), and so \(w \in [\ker M_g](\zeta)\). So we also have that

\[
(3.2) \quad \Theta(\zeta)E_a \subset [\ker M_g](\zeta).
\]

Our claim that \([\ker M_g](\zeta) = \Theta(\zeta)E_a\) follows from (3.1) and (3.2).

Suppose that for a \(\zeta \in \mathbb{D}^n\), \(v \in [\ker M_g](\zeta)\). Then \(v = h(\zeta)\) for some \(h \in \ker M_g\). Thus \(gh \equiv 0\) in \(\mathbb{D}^n\), and in particular, \(g(\zeta)v = g(\zeta)h(\zeta) = 0\). Thus \(v \in \ker g(\zeta)\). So we have that \([\ker M_g](\zeta) \subset \ker g(\zeta)\). Hence \(\dim [\ker M_g](\zeta) \leq \dim \ker g(\zeta)\). Consequently

\[
\dim \Theta(\zeta)E_a = \dim [\ker M_g](\zeta) \leq \dim \ker g(\zeta) = \dim E_c - \text{rank } g(\zeta),
\]

where the last equality follows from the Rank-Nullity Theorem. Since \(\Theta\) is inner, we have that the boundary values of \(\Theta\) satisfy \(\Theta(\zeta)^*\Theta(\zeta) = I_{E_a}\) for almost all \(\zeta \in \mathbb{T}^n\). So there is an open set \(U \subset \mathbb{D}^n\) such that for all \(\zeta \in U\)

\[
\dim E_a = \dim \Theta(\zeta)E_a.
\]
But from the definition of the rank of \( g \), we know that there is a \( \zeta_* \in \mathbb{D}^n \) such that we have \( k := \text{rank } g = \text{rank } g(\zeta_*) \). So there is a \( k \times k \) submatrix of \( g(\zeta_*) \) which is invertible. Now look at the determinant of this \( k \times k \) submatrix of \( g \). This is a holomorphic function, and so it cannot be identically zero in the open set \( U \). So there must exist a point \( \zeta_1 \in U \subset \mathbb{D}^n \) such that rank \( g = \text{rank } g(\zeta_1) \) and \( \dim E_a = \dim \Theta(\zeta_1)E_a \). Hence \( \dim E_a \leq \dim E_c - \text{rank } g \).

For the proof of the opposite inequality, let us consider a principal minor \( g_1(\zeta_1) \) of the matrix of the operator \( g(\zeta_1) \) (with respect to two arbitrary fixed bases in \( E_c \) and \( E \) respectively). Then \( \det g_1 \in H^\infty \), \( \det g_1 \neq 0 \). Let \( E_c = E_{c,1} \oplus E_{c,2} \), \( E = E_1 \oplus E_2 \) (\( \dim E_{c,1} = \dim E_1 = \text{rank } g(\zeta_1) \)) be the decompositions of the spaces \( E_c \) and \( E \) corresponding to this minor, and let

\[
 g(\zeta) = \begin{bmatrix} g_1(\zeta) & g_2(\zeta) \\ \gamma_1(\zeta) & \gamma_2(\zeta) \end{bmatrix}, \quad \zeta \in \mathbb{D}^n,
\]

be the matrix representation of \( g(\zeta) \) with respect to this decomposition. Owing to our assumption on the rank, it follows that there is a matrix function \( \zeta \mapsto W(\zeta) \) such that

\[
 \begin{bmatrix} \gamma_1(\zeta) & \gamma_2(\zeta) \end{bmatrix} = W(\zeta) \begin{bmatrix} g_1(\zeta) & g_2(\zeta) \end{bmatrix}.
\]

So \( \gamma_2(\zeta) = W(\zeta)g_2(\zeta) = (\gamma_1(\zeta)(g_1(\zeta))^{-1})g_2(\zeta) \). Thus with \( g_1^{co} := (\det g_1)g_1^{-1} \), we have

\[
 \gamma_2 \det g_1 = \gamma_1 g_1^{co} g_2,
\]

and using this we get the inclusion \( M_\Omega H^2_{E_{c,2}}(\mathbb{D}^n) \subset \ker M_g \), where \( \Omega \in H^\infty_{E_{c,2} \rightarrow E_c}(\mathbb{D}^n) \) is given by

\[
 \Omega = \begin{bmatrix} g_1^{co} g_2 \\ - \det g_1 \end{bmatrix}.
\]

We have rank \( \Omega = \dim E_{c,2} = \dim E_c - \text{rank } g = \dim \ker(g(\zeta_1)) \). Consequently, we obtain

\[
 \dim [\ker M_g](\zeta_1) \geq \dim \ker(g(\zeta_1)).
\]

We now turn to the extension of Tolokonnikov’s Lemma to the polydisc.

**Proof of Theorem 1.4.** (ii) \( \Rightarrow \) (i): If \( g := P_c F^{-1} \), then \( gf = I \). It only remains to show that the operators \( M_{z_1}, \ldots, M_{z_n} \) are doubly commuting on the space \( \ker M_g \). Let \( \Theta, \Gamma \) be such that:

\[
 F = \begin{bmatrix} f & \Theta \\ \Gamma \end{bmatrix} \quad \text{and} \quad F^{-1} = \begin{bmatrix} g \\ \Gamma \end{bmatrix}.
\]

Since \( FF^{-1} = I_{E_c} \), it follows that \( fg + \Theta \Gamma = I_{E_c} \). Thus if \( h \in H^2_{E_c}(\mathbb{D}^n) \) is such that \( gh = 0 \), then \( \Theta(\Gamma h) = h \), and so \( h \in \Theta H^2_{E_{c,2} \oplus E}(\mathbb{D}^n) \). Hence \( \ker M_g \subset \text{ran } M_\Theta \). Also, since \( F^{-1} F = I \), it follows that \( g \Theta = 0 \), and so \( \text{ran } M_\Theta \subset \ker M_g \). So \( \ker M_g = \text{ran } M_\Theta = \Theta H^2_{E_{c,2} \oplus E}(\mathbb{D}^2) \). By Theorem 1.3, the operators \( M_{z_1}, \ldots, M_{z_n} \) must doubly commute on the subspace \( \ker M_g \).
(i) ⇒ (ii): Let
\[ S := \{ h \in H^2_{E_c}(\mathbb{D}^n) : g(z)h(z) \equiv 0 \} = \ker g. \]

\( S \) is a closed non-zero invariant subspace of \( H^2_{E_c}(\mathbb{D}^n) \). Also, by assumption, \( M_{z_1}, \ldots, M_{z_n} \) are doubly commuting operators on \( S \). Then by the above Theorem 1.3, there exists an auxiliary Hilbert space \( E_a \) and an inner function \( \tilde{\Theta} \) with values in \( \mathcal{L}(E_a, E_c) \) with \( \dim E_a \leq \dim E_c \) such that
\[ S = \tilde{\Theta}H^2_{E_a}(\mathbb{D}^n). \]

By the Lemma on Local Rank, \( \dim E_a = \dim E_c - \text{rank } g = \dim E_c - \dim E = \dim (E_c \ominus E) \). Let \( U \) be a (constant) unitary operator from \( E_c \ominus E \) to \( E_a \) and define \( \Theta := \tilde{\Theta}U \). Then \( \Theta \) is inner, and we have that \( \ker g = \Theta H^2_{E_c \ominus E}(\mathbb{D}^n) \). To get \( F \in H^\infty_{E_c \rightarrow E_a}(\mathbb{D}^n) \) define the function \( F \) for \( z \in \mathbb{D}^n \) by
\[ F(z)e := \begin{cases} f(z)e & \text{if } e \in E \\ \Theta(z)e & \text{if } e \in E_c \ominus E. \end{cases} \]

We note that \( F \in H^\infty(\mathbb{D}^n) \) and \( F|_E = f \). We now show that \( F \) is invertible. With this in mind, we first observe that
\[ (I - fg)H^2_{E_c}(\mathbb{D}^n) \subset \Theta H^2_{E_c \ominus E}(\mathbb{D}^n) = \ker M_g. \]

This follows since \( g(I - fg)h = gh - gh = 0 \) for all \( h \in H^2_{E_c}(\mathbb{D}^n) \). Thus we have that \( \Theta^*(I - fg) \in H^\infty_{E_c \rightarrow E_c \ominus E}(\mathbb{D}^n) \). Now, define \( \Omega = g \oplus \Theta^*(I - fg) \). Clearly \( \Omega \in H^\infty_{E_c \rightarrow E_c}(\mathbb{D}^n) \). Next, note that
\[ F\Omega = fg + \Theta \Theta^*(I - fg) = I. \]

Similarly,
\[ \Omega F = g f \mathbb{P}_E + \Theta^*(I - fg)(f \mathbb{P}_E + \Theta \mathbb{P}_{E_c \ominus E}) = \mathbb{P}_E + \Theta^*(f \mathbb{P}_E - fgf \mathbb{P}_E + \Theta \mathbb{P}_{E_c \ominus E}) = \mathbb{P}_E + \Theta^* \Theta \mathbb{P}_{E_c \ominus E} = I. \]

Thus we have that \( F^{-1} \in H^\infty(\mathbb{D}^n; E_c \rightarrow E_c) \).

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