A statistical approach to the QCD phase transition
— A mystery in the critical temperature

Noriyoshi Ishii
Radiation Laboratory,
The Institute of Physical and Chemical Research (RIKEN),
2-1 Hirosawa, Wako, Saitama 351-0198, Japan

Hideo Suganuma
Faculty of Science, Tokyo Institute of Technology,
2-12-1 Ohokayama, Meguro, Tokyo 152-8551, Japan

We study the QCD phase transition based on the statistical treatment with the bag-model picture of hadrons, and derive a phenomenological relation among the low-lying hadron masses, the hadron sizes and the critical temperature of the QCD phase transition. We apply this phenomenological relation to both full QCD and quenched QCD, and compare these results with the corresponding lattice QCD results. Whereas such a statistical approach works well in full QCD, it results in an extremely large estimate of the critical temperature in quenched QCD, which indicates a serious problem in understanding of the QCD phase transition. This large discrepancy traces back to the fact that enough number of glueballs are not yet thermally excited at the critical temperature $T_c \approx 280$ MeV in quenched QCD due to the extremely small statistical factor as $e^{-m_G/T_c} \approx 0.00207$. This fact itself has a quite general nature independent of the particular choice of the effective model framework. We are thus arrive at a mystery, namely, what is really the trigger of the deconfinement phase transition.

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I. INTRODUCTION

The quark-gluon-plasma (QGP) is one of the most interesting targets in the finite-temperature quark-hadron physics. Currently, the QGP creation experiment is being performed in RHIC project at BNL, and much progress in understanding the finite-temperature QCD is desired. Historically, the instability of the hadron phase was first argued by Hagedorn before the discovery of QCD. He pointed out the possibility of a phase transition at finite temperature, based on the string or the flux-tube picture of hadrons. After QCD was established as the fundamental theory of the strong interaction, this transition was recognized as the deconfinement phase transition to the QGP phase, where quarks and gluons are liberated with the restored chiral symmetry. The QCD phase transition has been studied using various infrared effective models of QCD such as the linear $\sigma$ model, the Nambu-Jona-Lasinio model, the dual Ginzburg-Landau theory and so on.

In order to study nonperturbative features of the QCD phase transition, the lattice QCD Monte Carlo calculation serves as a powerful tool directly based on QCD. It has been already extensively used to study the nature of the QCD phase transition. At the quenched level, SU(3) lattice QCD indicates the existence of the deconfinement phase transition of a weak first order at $T_c \approx 260 - 280$ MeV. On the other hand, in the presence of dynamical quarks, it indicates the chiral phase transition at $T_c = 173(3)$ MeV for $N_f = 2$ and $T_c = 154(8)$ MeV for $N_f = 3$ in the chiral limit.

In this paper, we attempt to understand the physical implications of the recent lattice QCD results on the QCD phase transition, and point out an abnormal nature of the quenched QCD phase transition based on the statistical argument. In the actual calculation, we adopt the statistical approach to the QCD phase transition with the bag-model picture of hadrons. Although it is a simply-minded phenomenological effective model, it can reproduce the critical temperature of the full-QCD phase transition “amazingly” well. In spite of this success, this approach terribly overestimates the critical temperature in quenched QCD. We consider the essential cause of this overestimate, and point out a serious problem hidden in the QCD phase transition in a model-independent manner.

The contents are organized as follows. In Sect. I, we give a brief review of the statistical approach to the QCD phase transition with the bag-model picture of hadrons, and derive a phenomenological relation among the hadron masses, the hadron sizes and the critical temperature. We apply this relation to full QCD. In Sect. II, we apply this relation to quenched QCD without dynamical quarks, and point out a serious problem on the critical temperature. We discuss an abnormal nature of the deconfinement phase transition in quenched QCD.
II. A STATISTICAL APPROACH TO THE FULL-QCD PHASE TRANSITION

We investigate analytically the features of the QCD phase transition based on the statistical treatment with the bag-model picture of hadrons. We begin by deriving a phenomenological relation between the critical temperature \( T_c \) and the properties of the hadrons, i.e., the mass and the size. In the bag-model picture, quarks and gluons are assumed to be confined inside a spherical bag. Here, color confinement is simply taken into account through the bag-like intrinsic structure of hadrons. At low temperature, only a small number of such bags are thermally excited, and the thermodynamic properties of the system are described in terms of these spatially isolated bags. With the increasing temperature, the number of the thermally excited bags increases. Gradually, these bags begin to overlap one another, and they finally cover the whole space region. Hence, the QCD phase transition is described in terms of the overlaps of the thermally excited bags. Such a physical interpretation of the QCD phase transition is called as the closed packing picture.

To proceed, we define the spatial occupation ratio \( r_V(T) \) at temperature \( T \) to be the ratio of the total volume of the spatial regions inside the thermally excited bags to the volume \( V \) of the whole space region. In the closed packing picture of the QCD phase transition, \( r_V(T) \) plays the key role, which is estimated as

\[
r_V(T) = \frac{1}{V} \sum_n \frac{4\pi}{3} R_n^3 \cdot \lambda_n N_n(T) \tag{1}
\]

\[
= \sum_n \lambda_n R_n^3 \frac{4\pi}{3} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{e^{\sqrt{m_n^2 + k^2}/T} - 1}
\]

\[
= \sum_n \lambda_n R_n^3 T^3 f(m_n/T),
\]

where \( N_n(T), \lambda_n, m_n \) and \( R_n \) are the number at temperature \( T \), the degeneracy, the mass and the bag radius of the \( n \)-th elementary excitation, respectively. Here, \( f(\bar{m}) \) is defined by

\[
f(\bar{m}) = \frac{4\pi}{3} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{e^{\sqrt{\bar{m}^2 + k^2}} - 1}, \tag{2}
\]

and its functional form is plotted against \( \bar{m} \equiv m/T \) in Fig. 1. Note that \( f(m/T) \) is a characteristic function to describe the thermal contribution of the boson with the mass \( m \) at the temperature \( T \) \cite{10}. For \( m \gg T \), \( f(m/T) \) decreases exponentially with \( m/T \), and the thermal contribution is expected to become negligible.

As mentioned before, in the closed packing picture, the phase transition takes place, when the thermally excited bags almost cover the whole space region. Hence, the critical temperature \( T_c \) is estimated by solving

\[
r_V(T_c) = 1. \tag{3}
\]

Now, we investigate the phase transition in full QCD using the statistical approach. In full QCD, the lightest physical excitation is the pion, and all the other hadrons are rather heavy as \( m \gg m_\pi, T_c \). In fact, the pion is considered to play the key role in describing the thermodynamic properties of full QCD below \( T_c \) from the viewpoint of the statistical physics. Hence, in most cases, only the pionic degrees of freedom are taken into account in the hadron phase in the argument of the full-QCD phase transition. By using the isospin degeneracy \( \lambda_\pi = 3 \), the mass \( m_\pi = 140 \) MeV and the radius \( R_\pi \approx 1 \) fm, we solve Eq. (3) with \( r_V(T) = 3R_\pi^3 T^3 f(m_\pi/T) \) to estimate the critical temperature as \( T_c \approx 183 \) MeV. Considering its closeness to the full lattice QCD result with \( N_f = 2 \), i.e., \( T_c \approx 170 \) MeV, the statistical approach to the full-QCD phase transition seems to be rather good.

We now consider the \( m_\pi \)-dependence of the critical temperature \( T_c \). Note that, in the actual lattice QCD calculations, the pion mass is taken to be still rather heavy as \( m_\pi \gtrsim 400 \) MeV for the technical reasons. From these data, the critical temperature \( T_c \) in the chiral limit is obtained using the chiral extrapolation. In Ref. \cite{8}, the authors parametrized the full lattice QCD result of \( T_c \) as

\[
\left( \frac{T_c}{\sqrt{\sigma}} \right) (m_\pi) = 0.40(1) + 0.039(4) \left( \frac{m_\pi}{\sqrt{\sigma}} \right), \tag{4}
\]

where \( \sigma \) denotes the string tension. Strictly speaking, the phase transition becomes just a cross-over for intermediate values of \( m_\pi \). Hence, in Ref. \cite{8}, the pseudo critical temperature is adopted as \( T_c \), which is determined from the peak positions of the susceptibilities of the Polyakov loop and so on.
For the $N_f = 2$ case, we plot, in Fig. 2, the theoretical estimate of $T_c$ against $m_\pi$, based on Eq. (2) in the statistical approach with the bag-model picture. The dashed curve denotes $T_c$ retaining only the contribution from pions with $R_\pi  \simeq 1$ fm. The solid curve denotes the results including also the contributions from the low-lying vector mesons, such as $\rho$ and $\omega$, which are the next lightest particles in $N_f = 2$ full QCD. Here, we have used $\lambda_\rho = 3 \times 3 = 9$, $m_\rho = 770$ MeV, $R_\rho  \simeq 1$ fm, $\lambda_\omega = 3$, $m_\omega = 783$ MeV, $R_\omega  \simeq 1$ fm as inputs, which are treated as $m_\pi$-independent constants. These vector mesons give an additional contribution to $r_V(T)$ as $\delta r_V(T) = 9 R_3^2 T^3 f(m_\rho/T) + 3 R_3^2 T^3 f(m_\omega/T)$. The triangle and the square in Fig. 2 denote the lattice data taken from Refs. [8, 11]. In spite of its roughness, the resulting behavior of the critical temperature $T_c$ seems to coincide rather well with the lattice QCD result. In this calculation, the low-lying vector mesons, $\rho$ and $\omega$, give a small but non-negligible contribution to the critical temperature.

We next investigate the idealized SU(3)$_f$ symmetric case to analyze the other lattice QCD data. In this case, the pseudo-scalar (PS) octet mesons, such as pions, kaons, and $\eta_8$, are the lightest and possess the same mass $m_{PS}$ in common with the degeneracy as $\lambda_{PS} = 8$. We plot, in Fig. 3, the critical temperature $T_c$ against $m_{PS}$. The dashed curve denotes $T_c$ retaining only the contribution from PS-mesons with $R_{PS}  \simeq 1$ fm, which leads to $r_V(T) = 8 R_3^3 T^3 f(m_{PS}/T)$. The dot-dashed curve in Fig. 3 denotes the results including also the contributions from the octet vector mesons with $\lambda_V = 8 \times 3 = 24$, $m_V \simeq 770$ MeV, $R_V  \simeq 1$ fm. (Inclusion of the flavor-singlet vector meson does not change the result so much.)

These vector mesons give an additional contribution to $r_V(T)$ as $\delta r_V(T) = 24 R_3^3 T^3 f(m_V/T)$. The circle in Fig. 3 denotes the lattice data taken from Ref. [11]. We see that both the dashed and the dot-dashed curves are roughly consistent with the lattice QCD results. We note that the small deviation almost disappears by slightly adjusting the bag size as $R_{PS} = R_V = 0.75$ fm, as is shown in Fig. 3 with the solid curve.

In this way, this simple statistical approach with the bag-model picture works rather well in reproducing the critical temperature $T_c$ of the full-QCD phase transition. Here, the main contribution is given by the thermal pion, and the low-lying vector mesons give a small but non-negligible contribution to $T_c$.

III. A MYSTERY IN THE QUENCHED QCD PHASE TRANSITION

In spite of the absence of the dynamical quarks, quenched QCD provides various important nonperturbative features such as the color confinement and instantons. Hence, to understand some of the nonperturbative natures of QCD, quenched QCD plays the primary role, serving as a simplified version of the complicated real problems. In this section, we apply the statistical approach with the bag-model picture to quenched QCD in a similar manner done in the previous section.

Due to the color confinement, only the color-singlet modes can appear as physical excitations, and such modes are called as glueballs in quenched QCD. The mass spectrum of the glueballs is known through the quenched...
lattice QCD calculations. The lightest physical excitation is the 0++ glueball with $m_G = 1.5 - 1.7$ GeV. Following the similar argument as in the full QCD case, the lowest 0++ glueball is expected to play the key role in describing the thermodynamic properties of quenched QCD in the confinement phase. Hence, we first take into account only the 0++ glueball. As for the size $R$ of the scalar glueball, we note that there is no widely agreed value on it. We adopt a rather small value as $R \sim 0.4$ fm, which has been indicated by recent lattice QCD studies.

We use the degeneracy $\lambda_G(S) = 1$, the mass $m_{G(S)} \simeq 1730$ MeV, and $R_{G(S)} \simeq 0.4$ fm as inputs. Then, the spatial occupation ratio is given by $r_V(T) = R_{G(S)}^2 \sigma T^3 f(m_{G(S)}/T)$, and the critical temperature is estimated as $T_c \simeq 827$ MeV from Eq. (3). This estimate is too much larger than the quenched lattice QCD result as $T_c = 0.629(3) \sqrt{\sigma} \simeq 280$ MeV in Ref. [3] with $\sqrt{\sigma} = 450$ MeV. (In other words, only a tiny fraction of the space region is covered by the thermally excited bags as $r_V(T) = 0.0021$ at $T = 280$ MeV.) This large discrepancy would be a serious problem in the quenched QCD phase transition.

To seek a possible solution, we examine the case with a larger value of the glueball size $R$, since there is no established value on $R$. In Fig. 4, the critical temperature $T_c$ is plotted against the lightest 0++ glueball mass $m_{G(S)}$ in the statistical approach for various glueball sizes as $R_{G(S)} = 0.5, 1.0, 1.5$ fm. The cross (x) indicates the quenched lattice QCD results, $T_c = 280$ MeV and $m_G = 1730$ MeV. In this argument, to reproduce the critical temperature as $T_c \simeq 280$ MeV, the glueball size is to be abnormally large as $R_{G(S)} \simeq 3.1$ fm in the vicinity of $T_c$. However, such a drastic thermal swelling of the lowest scalar glueball was rejected by the recent lattice QCD studies, which states that the thermal glueball size is almost unchanged even near $T_c$. In any case, the glueball size does not seem to provide the solution on this discrepancy.

We may seek for the solution in the drastic pole-mass reduction of the 0++ glueball near the critical temperature as was suggested in Ref. [6] in the context of the dual Ginzburg-Landau theory. In Fig. 4, we see that, for the problem to be settled, the pole-mass reduction must be as significant as $m_{G(S)}(T_c) \lesssim 500$ MeV. However, in the recent lattice QCD calculations, it has been reported that the thermal 0++ glueball persists to hold a rather large pole-mass as $m_{G(S)}(T \simeq T_c) \simeq 1250$ MeV. Hence, we have to seek for another possibility by including the contribution of the excited-state glueballs.

In addition to the lowest 0++ glueball with $m_{G(S)} \simeq 1730$ MeV and $R_{G(S)} \simeq 0.4$ fm, we consider the thermal contribution from the lowest 2++ glueball, which is the next lightest hadron in quenched QCD. We take $\lambda_G(T) = 5$, $m_G(T) \simeq 2400$ MeV [4], $R_G(T) \simeq 1$ fm. (Here, we assume it to have a typical hadron size.) Then, the spatial occupation ratio receives a correction as $\delta r_V(T) = 5R_{G(T)}^2 T^3 f(m_G(T)/T)$, and the correction amounts to $\delta r_V(T) = 0.0223$ at $T = 280$ MeV. The resulting critical temperature is given as $T_c \simeq 432$ MeV, which is still too large. We note that the realistic glueball size would be more compact. However, if so, its contribution becomes more negligible.

Besides these two low-lying glueballs, the following excited states are predicted in Ref. [2] as $0^+(2590)$, $0^{++}(2670)$, $1^{++}(2940)$, $2^{++}(3100)$, $3^{++}(3550)$, $0^{--}(3640)$, $3^{++}(3690)$, $1^{--}(3850)$, $2^{++}(3890)$, $2^{--}(3930)$, $3^{--}(4130)$, $2^{++}(4140)$, $0^{++}(4740)$. We include all the contribution from these excited states, assuming the unknown glueball size as a typical hadron size, i.e., $R_G \simeq 1$ fm. The correction by these excited states amounts to only $\delta r_V(T) = 0.0113$ at $T = 280$ MeV, and the resulting critical temperature is estimated as $T_c = 395$ MeV, which is again too large.

In this way, we observe that the statistical approach does not work at all in quenched QCD. The direct cause of this failure is the extremely small statistical factor $e^{-m_{G(S)}/T_c} \simeq 0.0021$, which strongly suppresses the excitations of the glueballs. As a consequence, only a remarkably tiny fraction of the space region can be covered by the thermally excited bags of glueballs at $T = 280$ MeV, which cannot be the driving force of the QCD phase transition.

IV. SUMMARY AND DISCUSSIONS — WHAT IS THE TRIGGER OR THE DRIVING FORCE OF THE QCD PHASE TRANSITION?

We have analytically studied the QCD phase transition based on the statistical treatment with the bag-model picture. We have derived a phenomenological relation among the critical temperature $T_c$, the mass and the size...
of the low-lying hadrons. First, we have applied this relation to full QCD, and have compared the analytical results with the full lattice QCD results. In full QCD, we have found that this approach works amazingly well. As is expected, the pionic contribution seems to play the main role in determining the critical temperature, and the low-lying vector mesons are found to provide a small but non-negligible contribution.

Unlike full QCD, we have found that the statistical approach terribly overestimates the critical temperature as \( T_c \approx 827 \text{ MeV} \) in quenched QCD, and that only a tiny fraction of the space region is covered by the thermally excited bags of glueballs at \( T \approx 280 \text{ MeV} \). We have considered the possibility of a swelling of the glueball size and reduction of the glueball mass near the critical temperature. However, both of these two have not provided us with the solution on the large discrepancy. In spite of including all the contributions from the 15 low-lying glueballs up to 5 GeV predicted in quenched lattice QCD [12], the discrepancy remains to be still large. In other words, the number of the thermally-excited glueballs is still too small at \( T \approx 280 \text{ MeV} \), even after so many glueball excited states are taken into account. The direct origin of this discrepancy is the strong suppression of the thermal excitation of glueballs due to the extremely small statistical factor as \( e^{-m_{G(S)}/T} = 0.00207 \) at \( T = 280 \text{ MeV} \) even for the lightest glueball, which leads to the insufficient amount of the covered space by the thermally excited bags of glueballs.

Although several arguments given so far have been based on the bag-model picture, this problem itself has a quite general nature, which can go beyond the reliability of the model framework. We finally reformulate this problem in a model-independent general manner by considering the inter-particle distance \( l \) instead of the bag size \( R \). Note that the reliability of the statistical argument becomes improved in the dilute glueball gas limit. Now, taking into account all the low-lying 15 glueball modes up to 5 GeV predicted in quenched lattice QCD [12], we calculate the inter-particle distance \( l \) of the glueballs based on only the statistical argument. At \( T = 280 \text{ MeV} \), the inter-particle distance is estimated as \( l \approx 5 \text{ fm} \). It follows that the deconfinement phase transition takes place at \( T \approx 280 \text{ MeV} \), when the glueball density becomes \( \rho = 1/\left\{ \frac{4\pi}{25}(2.5 \text{ fm})^3 \right\} \approx 1/(4.0 \text{ fm})^3 \). Remember that the theoretical estimates of the glueball size are rather small as \( R_{G(S)} \lesssim 0.4 \text{ fm} \) [13]. Hence, this density seems to be rather dilute. Since the long-range interaction among glueballs is mediated by the virtual one-gluon exchange process in quenched QCD, the interactions among glueballs are exponentially suppressed beyond its Compton length \( 1/m_{G(S)} = 0.112 \text{ fm} \). Therefore, one cannot expect the strong long-range interaction acting among the spatially-separated thermal glueballs.

In this way, it is quite difficult to imagine how such a too rare excitation of thermal glueballs can lead to the phase transition. We are thus arrive at the mystery. What is really the trigger or the driving force of the deconfinement phase transition in quenched QCD? In order to understand the QCD phase transition, this problem should be seriously considered.

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