Analytic Torsion for Surfaces with Cusps I: Compact Perturbation Theorem and Anomaly Formula

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Abstract: We define the analytic torsion associated with a Riemann surface endowed with a metric having Poincaré-type singularities in the neighborhood of a finite number of points and a Hermitian vector bundle with at most logarithmic singularities at those points, coming from the metric on the negative power of the canonical line bundle twisted by the divisor of the points. Then we provide a relation between this analytic torsion and the Ray–Singer analytic torsion of the compactified surface. From this relation we then establish the anomaly formula, which describes how the analytic torsion changes under the change of the metric on the surface and on the vector bundle.

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1. Introduction

The goal of this article is to define and study the analytic torsion associated with a Riemann surface with hyperbolic cusps and a holomorphic Hermitian vector bundle with at most logarithmic singularities around the cusps.

To define the analytic torsion, we use the regularization of the heat trace, obtained by subtracting a universal contribution coming from the model case of CP[1] \ {0, 1, ∞}. We provide a relation between this analytic torsion and the Ray–Singer analytic torsion of compactified surface. Then we prove the anomaly formula, which describes how this analytic torsion changes with the change of metric and Hermitian structure on the vector bundle.

In our setting we do not require the metric to be of constant scalar curvature everywhere and we do not put any restriction neither on the holomorphic vector bundle, nor on the Hermitian metric over it. In particular, we do not suppose that it comes from some representation of the associated Fuchsian group.

More precisely, let \( M \) be a compact Riemann surface, \( D_M = \{ P^M_1, \ldots, P^M_m \} \) be a finite set of distinct points in \( M \). Let \( g^{TM} \) be a Kähler metric on the punctured Riemann surface \( M := M \setminus D_M \).

For \( \epsilon \in ]0, 1[ \), \( i = 1, \ldots, m \), let \( z^M_i : \overline{M} \supset V^M_i(\epsilon) \to D(\epsilon) := \{ z \in \mathbb{C} : |z| \leq \epsilon \} \) be a local holomorphic coordinate around \( P^M_i \). We denote \( V^M_i(\epsilon) := \{ x \in M : |z^M_i(x)| < \epsilon \} \).

We say that \( g^{TM} \) is Poincaré-compatible with coordinates \( z^M_1, \ldots, z^M_m \) if for any \( i = 1, \ldots, m \), there is \( \eta > 0 \) such that \( g^{TM}|_{V^M_i(\eta)} \) is induced by the Hermitian form

\[
\sqrt{-1} dz^M_i d\bar{z}^M_i \left| z^M_i \right|^2. \tag{1.2}
\]

We say that \( g^{TM} \) is a metric with cusps if it is Poincaré-compatible with some holomorphic coordinates near \( D_M \). A triple \( (\overline{M}, D_M, g^{TM}) \) of a Riemann surface \( \overline{M} \), a set of punctures \( D_M \) and a metric with cusps \( g^{TM} \) is called a surface with cusps (cf. [40]).

For example, if a pointed surface \((M, D_M)\) is stable, i.e. the genus \( g(M) \) of \( M \) satisfies

\[
2g(M) - 2 + m > 0, \tag{1.3}
\]

then, by the uniformization theorem (cf. [21, Chapter IV], [5, Lemma 6.2]), there is the canonical hyperbolic metric \( g^{TM}_{hyp} \) of constant scalar curvature \(-1\) on \( M \). Once again, by the uniformization theorem, there are local holomorphic coordinates \( z^M_i \) of \( P^M_i \), \( i = 1, \ldots, m \), such that \( g^{TM}_{hyp} \) is induced by (1.2) in the neighbourhood of \( D_M \). Thus, \( (\overline{M}, D_M, g^{TM}_{hyp}) \) is a surface with cusps.

Let \( \xi \) be a holomorphic vector bundle over a complex manifold \( X \) with a Hermitian metric \( h^\xi \) over \( X \). A pair \((\xi, h^\xi)\) is called a Hermitian vector bundle over \( X \).

From now on, we fix a surface with cusps \((\overline{M}, D_M, g^{TM})\) and a Hermitian vector bundle \((\xi, h^\xi)\) over it. We denote by \( \omega_{\overline{M}} := T^*(1,0)\overline{M} \) the canonical line bundle over \( \overline{M} \). Let \( \mathcal{O}_{\overline{M}}(D_M) \) be the line bundle associated to the divisor \( D_M \). The twisted canonical line bundle on \( \overline{M} \) is defined as

\[
\omega_M(D) := \omega_{\overline{M}} \otimes \mathcal{O}_{\overline{M}}(D_M). \tag{1.4}
\]
The metric \( g^{TM} \) endows the line bundle \( \omega_M \) (resp. \( \omega_M(D) \)) with the induced Hermitian metric \( ||\cdot||^\omega_M \) (resp. with \( ||\cdot||_M \) via the canonical isomorphism \( \omega_M(D) \simeq \omega_M \)) over \( M \). In other worlds, there is \( \epsilon > 0 \), such that for the canonical section \( s_{DM} \) of \( \mathcal{O}_M(D_M) \), over \( \mathcal{V}_i(\epsilon) \), we have
\[
\|dz_i^M\|_M = \ln|z_i^M|, \quad \|dz_i^M \otimes s_{DM}/z_i^M\|_M = \ln|z_i^M|.
\] (1.5)

We denote by \( \Box_{\xi_0 \otimes \omega_M(D)^n} \) the Kodaira Laplacian associated with \( (M, g^{TM}) \) and \( (\xi_0 \otimes \omega_M(D)^n, h^{\xi_0} \otimes ||\cdot||^{2n}_M) \).

In this article, apart from the discussion of the \( L^2 \)-norm, we only consider the restriction of \( \Box_{\xi_0 \otimes \omega_M(D)^n} \) on the sections of degree 0.

Assume first \( m = 0 \), i.e. the surfaces are compact, then Ray–Singer in [43, Definition 1.2] defined the analytic torsion as the regularized determinant of \( \Box_{\xi_0 \otimes \omega_M(D)^n} \). More precisely, let \( \lambda_i, i \in \mathbb{N} \) be the non-zero eigenvalues of \( \Box_{\xi_0 \otimes \omega_M(D)^n} \). By Weyl’s law, for \( \text{Re}(s) > 1 \), the zeta-function
\[
\zeta_M(s) := \sum_{i} 1/\lambda_i^s,
\] (1.6)
is well-defined and it is holomorphic in that region. Moreover, as it can be seen by the small-time expansion of the heat kernel and the classical properties of the Mellin transform, it extends meromorphically to \( \mathbb{C} \). This extension is holomorphic at 0, and the analytic torsion is defined by
\[
T(g^{TM}, h^{\xi_0} \otimes ||\cdot||^{2n}_M) := \exp(-\zeta_M'(0)).
\] (1.7)

By (1.6) and (1.7), we may interpret the analytic torsion as
\[
T(g^{TM}, h^{\xi_0} \otimes ||\cdot||^{2n}_M) := \prod_{i=0}^{\infty} \lambda_i.
\] (1.8)

Now, assume \( m > 0 \). Then \( M \) is non-compact, and the heat operator associated to \( \Box_{\xi_0 \otimes \omega_M(D)^n} \) is no longer of trace class. Also the spectrum of \( \Box_{\xi_0 \otimes \omega_M(D)^n} \) is not discrete in general. Thus, neither the definition (1.7), nor the interpretation (1.8) are applicable, and another approach should be used.

Suppose for the moment that \((\overline{M}, D_M)\) satisfies (1.3). Let \( g^{TM}_{\text{hyp}} \) be the canonical hyperbolic metric of constant scalar curvature \(-1\). We denote by \( Z_{(\overline{M}, D_M)}(s), s \in \mathbb{C} \) the Selberg zeta-function, which is given for \( \text{Re}(s) > 1 \) by the absolutely converging product:
\[
Z_{(\overline{M}, D_M)}(s) = \prod_{\gamma} \prod_{k=0}^{\infty} (1 - e^{-(s+k)l(\gamma)})^2,
\] (1.9)
where \( \gamma \) runs over the set of all simple closed non-oriented geodesics on \( M \) with respect to \( g^{TM}_{\text{hyp}} \), and \( l(\gamma) \) is the length of \( \gamma \). The function \( Z_{(\overline{M}, D_M)}(s) \) admits a meromorphic extension to the whole complex \( s \)-plane with a simple zero at \( s = 1 \) (see for example [19, (5.3)]). We denote by \( ||\cdot||^{\text{hyp}}_M \) the norm induced by \( g^{TM}_{\text{hyp}} \) on \( \omega_M(D) \) over \( M \).
In this situation, for \( l \in \mathbb{Z}, l < 0 \), Takhtajan–Zograf in [45, (6)] proposed the analogue of the analytic torsion defined via the Selberg zeta function as

\[
T_{TZ}(g_{\text{hyp}}^M, (||\cdot||_M)^{2n}) := \begin{cases} 
\exp \left( \frac{\log(2)}{2} - \chi(M) \frac{c}{2} \right) \cdot Z'_{(M, D_M)}(1), & \text{for } n = 0, \\
\exp(-c \cdot \chi(M)/2) \cdot Z'_{(M, D_M)}(-n + 1), & \text{for } n < 0,
\end{cases}
\]

where \( \chi(M) = 2 - 2g(M) - m \) is the Euler characteristics of \( M \), and \( c_k, k \in \mathbb{N}^* \), are given by

\[
c_0 = 4\zeta'(-1) - \frac{1}{2} + \ln(2\pi), \\
c_k = \sum_{l=0}^{k-1} (2k - 2l - 1)(\ln(2k + 2kl - l^2 - l) - \ln(2)) + \left(\frac{1}{3} + k + k^2\right) \ln(2) + (2k + 1) \ln(2\pi) + 4\zeta'(-1) - 2(\frac{1}{2} + k + k^2)^2 - 4 \sum_{l=1}^{k-1} \ln(l!) - 2 \ln(k!).
\]

Remark 1.1. To explain the values \( c_k, k \in \mathbb{N} \), it was shown by D’Hoker-Phong [19, (7.30)], [20, (3.6)] (see also [44], [12, (50)] and [41, (9)]), that the definition (1.10) coincides with (1.7). In other words, the two definitions are compatible for \( M \) stable, \( m = 0, gTM = g_{\text{hyp}}^M \) and \( n \leq 0 \).

The advantage of the definition (1.10) is an explicit formula in terms of “simple” geometric objects and, thus, suitability for the variational-type arguments (see [23,45]). However, it only works for the complete hyperbolic metric \( g_{\text{hyp}}^M \) of constant scalar curvature \(-1\) on \( M \) and trivial Hermitian vector bundle \((\xi, h^\xi)\).

Our first goal of this article is to give a definition (see Definition 2.16) of the analytic torsion \( T(g_{TM}, h^\xi \otimes ||\cdot||_M^{2n}) \) for \( n \leq 0, \) which generalizes both (1.7) and (1.10). Our definition is done using formula (1.7), where in place of a trace we use a regularized version of it, obtained by subtracting a universal “spectral contribution” of \( \mathbb{C} P^1 \setminus \{0, 1, \infty\} \). Later in [25] we show that our definition actually coincides with (1.10) for hyperbolic surfaces of constant scalar curvature and \((\xi, h^\xi)\) trivial (thus, extending the results of D’Hoker-Phong [19, (7.30)], [20, (3.6)]).

In this article, after giving a formal definition of the determinant of the Laplacian, we provide two results for computing it. The first one, Theorem A, which we also call the relative compact perturbation theorem, expresses the quotient of two Quillen norms associated with surfaces with the same number of cusps through a quotient of two Quillen norms associated with surfaces without cusps. The second one, Theorem B, which we also call the anomaly formula, explains how the Quillen norm changes under the change of \( g_{TM}, h^\xi \). It shows that although the Quillen norm is a global invariant, the variation of it, induced by the change of the metric and the Hermitian structure, can be expressed as an integral of a local quantity. We see that this local quantity has an explicit contribution localized near the cusps. This contribution does not have analogues for compact surfaces and it describes the variation of the Poincaré-compatible coordinates induced by the variation of the Kähler metric. The study of the heat kernel associated to \( h^\xi \otimes ||\cdot||_M^{2n} \) on a surface with cusps \((\overline{M}, D_M, gTM)\) plays the foremost role in our approach.

---

1. It’s easy to see that \( T(g_{\text{hyp}}^M, (||\cdot||_M^{2n})^n) \) corresponds to \( \det'((\frac{1}{2} \Delta_M^-)^n) \) in the notation of [20, (1.1)], [12, (3)]. Since for \( c > 0 \), by [20, §3], we have \( \det'((c \Delta_M^-) = \det'((c \Delta_M^+)^n) \), coefficients (1.11) for \( k \in \mathbb{N}^* \) can be read off from [12, (50)] for \( c = 1/2 \) (cf. [28, Definition 4.2]) and for \( k = 0 \) from [19, (7.23), (7.30)], [44, Corollary 1] (cf. [27, (6.3)]).

2. By Serre duality, if one prefers to work with positive line bundles, we can interpret it as the analytic torsion of the vector bundle \( \xi^* \otimes \omega_M^{n+1} (D_M)^{-n} \) associated to \( (gTM, (h^\xi)^* \otimes ||\cdot||_M^{2n} \otimes (||\cdot||_M^{2n})^2) \), for \( n \leq 0 \).
Now let’s describe our results more precisely. For \( n \leq 0 \), we may endow the complex line
\[
\left( \det H^0(\overline{M}, \xi \otimes \omega_M(D)^n) \right)^{-1} := \left( \Lambda^\text{max} H^0(\overline{M}, \xi \otimes \omega_M(D)^n) \right)^{-1} \otimes \Lambda^\text{max} H^1(\overline{M}, \xi \otimes \omega_M(D)^n),
\]
with the \( L^2 \)-norm \( \| \cdot \|_{L^2} \). In the compact case it coincides with the \( L^2 \)-norm induced by the harmonic forms associated with \( g^{TM} \) and \( h^\xi \otimes \| \cdot \|_{M}^{2n} \), in the non-compact, we define it by identifying cohomologies as subspaces of the kernel of the Laplacian, see Sect. 2.1. Then we define the Quillen norm on the complex line (1.12) by
\[
\| \cdot \|_{\text{Q}}(g^{TM}, h^\xi \otimes \| \cdot \|_{M}^{2n}) = T(g^{TM}, h^\xi \otimes \| \cdot \|_{M}^{2n})^{1/2} \cdot \| \cdot \|_{L^2}(g^{TM}, h^\xi \otimes \| \cdot \|_{M}^{2n}).
\]
To motivate, when \( m = 0 \), i.e. the surface \( M \) is compact, this coincides with the usual definition of the Quillen norm from [9, 1.64] and [10, Definition 1.5].

Now let’s give some definitions, which are essential for our first theorem.

**Definition 1.2. (Flattening of a metric)** Let \((\overline{M}, D_M, g^{TM})\) be a surface with cusps. We say that a (smooth) metric \( g^{TM}_f \) over \( M \) is a flattening (Fig. 1) of \( g^{TM} \) if there is \( \nu > 0 \) such that \( g^{TM} \) is induced by (1.2) over \( V_M(\nu) \), and
\[
\left. g^{TM}_f \right|_{M \setminus \bigcup_i V_M(\nu)} = \left. g^{TM} \right|_{M \setminus \bigcup_i V_M(\nu)}.
\]
(1.14)

The supremum of all \( \nu > 0 \), satisfying (1.14) is called the tightness of the flattening.

Let \((\overline{N}, D_N, g^{TN})\) be another surface with cusps and let \( g^{TN}_f \) be a flattening of \( g^{TN} \). We say that the flattenings \( g^{TM}_f \) and \( g^{TN}_f \) are compatible, if for any \( i = 1, \ldots, m \), we have
\[
((z^N_i)^{-1} \circ z^M_i)^* \left( \left. g^{TM}_f \right|_{V_M(\nu)} \right) = \left. g^{TN}_f \right|_{V^N(\nu)},
\]
for some \( \nu > 0 \), satisfying (1.14) and
\[
\left. g^{TN}_f \right|_{N \setminus \bigcup_i V^N(\nu)} = \left. g^{TN} \right|_{N \setminus \bigcup_i V^N(\nu)}.
\]
(1.16)

Similarly, we define the notion of flattenings \( \| \cdot \|_M^f, \| \cdot \|_N^f \) for Hermitian norms \( \| \cdot \|_M, \| \cdot \|_N \).

We say that the flattenings \( \| \cdot \|_M^f, \| \cdot \|_N^f \) are compatible (Fig. 2) if they satisfy similar conditions to (1.14), (1.16), and for any \( i = 1, \ldots, m \), we have
\[
((z^N_i)^{-1} \circ z^M_i)^* \left( \| \cdot \|_M/\| \cdot \|_M^f \right) \left|_{V^{M}(\nu)} = \left( \| \cdot \|_N/\| \cdot \|_N^f \right) \right|_{V^{N}(\nu)}.
\]
(1.17)

**Remark 1.3.** The definitions of flattenings \( g^{TM}_f \) of \( g^{TM} \) and \( \| \cdot \|_M^f \) of \( \| \cdot \|_M \) are independent, and there is no relation between them as in (1.5).
Theorem A. (Relative compact perturbation) Let \((\overline{M}, D_M, g^{TM}), (\overline{N}, D_N, g^{TN})\) be two surfaces with the same number of cusps. Let \((\xi, h^\xi)\) be a Hermitian vector bundle over \(\overline{M}\) of rank \(\text{rk}(\xi)\). We denote by \(||\cdot||_M, ||\cdot||_N\) the norms induced by \(g^{TM}, g^{TN}\) as in (1.5) on \(\omega_M(D)\) and \(\omega_N(D)\) over \(M\) and \(N\) respectively. Let \(g^{TM}_f, g^{TN}_f, ||\cdot||^f_M, ||\cdot||^f_N\) be compatible flattenings of \(g^{TM}, g^{TN}, ||\cdot||_M, ||\cdot||_N\) respectively. Then for any \(n \in \mathbb{Z}, n \leq 0\), we have

\[
2 \ln \left( \frac{||\cdot||_Q(g^{TM}, h^\xi \otimes ||\cdot||^f_M^{2n})}{||\cdot||_Q(g^{TM}_f, h^\xi \otimes (||\cdot||^f_M)^{2n})} \right)
- \text{rk}(\xi) \ln \left( \frac{||\cdot||_Q(g^{TN}, ||\cdot||^f_N^{2n})}{||\cdot||_Q(g^{TN}_f, (||\cdot||^f_N)^{2n})} \right)
= \int_M c_1(\xi, h^\xi) \left( 2n \ln \left( \frac{||\cdot||^f_M}{||\cdot||_M} \right) + \ln (g^{TM}_f/g^{TM}) \right).
\]

(1.18)

In other words, the relative Quillen norm can be computed through a compact perturbation.

Remark 1.4. (a) It is possible to restate Theorem A in the way, which doesn’t use the language of compatible flattenings. It says that the quantity

\[
2\ln \left( \frac{||\cdot||_Q(g^{TM}, h^\xi \otimes ||\cdot||^f_M^{2n})}{||\cdot||_Q(g^{TM}_f, h^\xi \otimes (||\cdot||^f_M)^{2n})} \right)
- \text{rk}(\xi) \ln \left( \frac{||\cdot||_Q(g^{TN}, ||\cdot||^f_N^{2n})}{||\cdot||_Q(g^{TN}_f, (||\cdot||^f_N)^{2n})} \right)
= \int_M c_1(\xi, h^\xi) \left( 2n \ln \left( \frac{||\cdot||^f_M}{||\cdot||_M} \right) + \ln (g^{TM}_f/g^{TM}) \right)
\]

(1.19)

depends only on the integer \(n \in \mathbb{Z}, n \leq 0\), and the functions \((g^{TM}_f/g^{TM})|_{V_i^{M(1)}} \circ (c^M_i)^{-1}: \mathbb{D}^n \to \mathbb{R}, (||\cdot||^f_M/||\cdot||_M)|_{V_i^{M(1)}} \circ (c^M_i)^{-1}: \mathbb{D}^n \to \mathbb{R}, \) for \(i = 1, \ldots, m\). This reformulation is particularly useful when one studies the variation of the Quillen norm in a family setting.

(b) For \(n = 0\) and \((\xi, h^\xi)\) trivial, Theorem A was proved by Jorgenson-Lundelius in [33, Theorem 7.3] and Albin-Aldana-Rochon in [2, Theorem 5.2]. The fact that the geometry near the cusps of \((M, g^{TM})\) and \((N, g^{TN})\) coincides is used extensively in their proofs. This doesn’t hold in our case due to the presence of \((\xi, h^\xi)\), and the techniques we use are different even in the case when \((\xi, h^\xi)\) is trivial. We note that in [2, Definition 2.2], authors also consider funnel singularities.

The main feature of our techniques is that they are implicit, and unlike [33], we avoid studying the precise contribution of the continuous spectrum to the heat kernel.

Our next result explains how the Quillen norm changes under the conformal change of the metric with cusps. Let’s recall that by [9, Theorem 1.27], the Bott-Chern classes of a vector bundle \(\xi\) with Hermitian metrics \(h^\xi_1, h^\xi_2\) are natural differential forms (strictly
speaking, those are classes of differential forms, see Remark 1.7b)) defined so that they satisfy
\[
\frac{\partial}{\partial \bar{\partial}} \frac{1}{2\pi \sqrt{-1}} \tilde{\omega}(\xi, h^\xi_1, h^\xi_2) = \omega(\xi, h^\xi_1) - \omega(\xi, h^\xi_2),
\]
(1.20)
where \(\partial\), ch are Todd and Chern forms. By [9, Theorem 1.27], we have the following identities
\[
\tilde{\omega}(\xi, h^\xi_1, h^\xi_2)[0] = 2\tilde{\omega}(\xi, h^\xi_1, h^\xi_2)[0] = \ln \left( \det (h^\xi_1 / h^\xi_2) \right).
\]
(1.21)
If, moreover, \(\xi := L\) is a line bundle, we have
\[
\tilde{\omega}(L, h^\xi_1, h^\xi_2)[2] = 6\tilde{\omega}(L, h^\xi_1, h^\xi_2)[2] = \ln(h^\xi_1 / h^\xi_2) \left( c_1(L, h^\xi_1) + c_1(L, h^\xi_2) \right)/2,
\]
(1.22)
where \(c_1\) is the first Chern form.

**Definition 1.5.** For a surface with cusps \((\overline{M}, D_M, g^{TM})\), the Wolpert norms \(\|\cdot\|_i^W\) on the complex lines \(\omega|_{p_i}^M, i = 1, \ldots, m\), are defined by \(\|dz_i^{M}\|_i^W = 1\). It induces the Wolpert norm \(\|\cdot\|_W\) on the complex line \(\otimes_{i=1}^m \omega|_{p_i}^M\).

**Remark 1.6.** Poincaré-compatible coordinates are uniquely defined up to a multiplication by a unimodular complex number, so the norms \(\|\cdot\|_W\) are well-defined. They were originally introduced by Wolpert in [48, Definition 1] for surfaces of constant scalar curvature \(-1\), and he used it in his geometric interpretation of Takhtajan–Zograf forms over the moduli space of pointed curves.

**Theorem B.** (Anomaly formula for metrics with cusps) Let \(g^{TM}, g_0^{TM}\) be two metrics on \(M\) such that both triples \((\overline{M}, D_M, g^{TM}), (\overline{M}, D_M, g_0^{TM})\) are surfaces with cusps. We denote by \(\|\cdot\|_M, \|\cdot\|_0^M\) the norms induced by \(g^{TM}, g_0^{TM}\) on \(\omega(M,D)\), and by \(\|\cdot\|_W, \|\cdot\|_0^W\) the associated Wolpert norms. Let \(h^\xi, h_0^\xi\) be two Hermitian metrics on \(\xi\) over \(M\). Then the right-hand side of the following equation is finite, and
\[
2 \ln \left( \|\cdot\|_Q (g_0^{TM}, h_0^\xi \otimes (\|\cdot\|_M^0)^{2n}) / \|\cdot\|_Q (g^{TM}, h^\xi \otimes (\|\cdot\|_M^0)^{2n}) \right)
\]
\[
= \int_M \left[ \tilde{\omega}(\omega(M)^{-1}, (\|\cdot\|_M^0)^{-2}) \tilde{\omega}(\xi, h^\xi) \tilde{\omega}(\omega(M)^n, (\|\cdot\|_M^0)^{2n}) \right. \\
\quad + \tilde{\omega}(\omega(M)^{-1}, (\|\cdot\|_M^0)^{-2}) \tilde{\omega}(\xi, h^\xi) \tilde{\omega}(\omega(M)^n, (\|\cdot\|_M^0)^{2n}) \right]
\]
\[
+ \frac{\det(h^\xi / h_0^\xi)}{2} \sum \ln \left( \det(h^\xi / h_0^\xi) \right)_{p_i^M}. \]
(1.23)

**Remark 1.7.** a) The anomaly formula was firstly proved by Polyakov in [42] for \(m = 0, n = 0\) and \((\xi, h^\xi)\) trivial, who used it to compute some integrals over moduli spaces of embedded surfaces which arise in mathematical physics. It was generalized by Bismut-Gillet-Soulé [10, Theorem 1.23] to any dimension (for compact manifolds, which correspond to \(m = 0\) in our case). For \(m = 0\), in [22], Fay gave another proof of (1.23). Our proof relies on the anomaly formula for \(m = 0\).
b) Strictly speaking, the integral in (1.23) is not well-defined, since $\tilde{c}, \tilde{d}$ are only well-defined as classes up to an element of the form $\partial \alpha + \partial \beta$. Since a priori nothing is known about the growth of $\alpha, \beta$ near $D_M$, the integrals of $\partial \alpha$ and $\partial \beta$ over $M$ might not converge (let alone being equal to 0 by “Stokes theorem”). For the purposes of this article, however, it is enough to think of $\tilde{c}, \tilde{d}$ as forms, defined by (1.21) and (1.22).

An alternative way to interpret those classes is through the Bott-Chern theory for pre-log-log Hermitian vector bundles, introduced by Burgos Gil-Kramer-Kühn in [14] (cf. [24, §2.4]).

(c) Experts will notice the difference between the terms under the integral in the right-hand side of (1.23) and the terms, which appear in the right-hand side of the anomaly formula of Bismut-Gillet-Soulé [10, Theorem 1.23] (see (3.3)), where in the arguments of Todd class and secondary Todd class we have $\omega_M$ in place of $\omega_M(D)$. However, this difference is not a real issue, since for the current of integration $\delta_{D_M}$ along $D_M$, we have the following identities over $M$:

$$
\tilde{d}(\omega_M^{-1}, (||\omega_M||^0_M)^{-2}, (||\omega_M^0||^0_M)^{-2}) = \tilde{d}(\omega_M(D)^{-1}, ||\omega_M||^2_M, (||\omega_M^0||^0_M)^{-2})
$$

$$
[\tilde{d}(\omega_M^{-1}, (||\omega_M^0||^0_M)^{-2})]^{[2]} = [\tilde{d}(\omega_M(D)^{-1}, (||\omega_M^0||^0_M)^{-2})]^{[2]} + \frac{1}{2} \delta_{D_M},
$$

(1.24)

$$
[\tilde{c}(\omega_M(D)^n, ||\omega_M^n||^0_M, (||\omega_M^0||^0_M)^{2n})]^{[0]}|_{D_M} = 0,
$$

where [0], [2] stand for the part of degree 0 and 2, and in the second identity we used Poincaré-Lelong equation. Nevertheless, we prefer to state Theorem B in the given form, since in the sequel we will use that the Hermitian line bundles $(\omega_M(D), ||\omega_M||^0_M), (\omega_M(D), ||\omega_M^0||^0_M)$ are pre-log-log with singularities along $D_M$ in the terminology of Burgos Gil-Kramer-Kühn [14], and the Hermitian line bundles $(\omega_M, ||\omega_M||^0_M), (\omega_M, ||\omega_M^0||^0_M)$ do not satisfy those properties.

d) Let $\phi : M \to \mathbb{R}$ be a smooth function such that

$$
\mathcal{g}_0^{TM} = e^{2\phi} \mathcal{g}^{TM}.
$$

(1.25)

In the case when $\phi$ has compact support in $M$, Theorem B follows directly from the anomaly formula of Bismut-Gillet-Soulé (see Theorem 3.1), Theorem A and (1.24).

The difference between Theorem B and Theorem 3.1 is in the last two terms of (1.23):

$$
\frac{-\text{rk}(\xi)}{6} \ln \left( ||\omega_M^0||^0_M / ||\omega_M^0||^0_W \right) + \frac{1}{2} \sum \ln \left( \det(h^\xi / h^\xi_0) \right)_{p_i M}.
$$

(1.26)

For $n = 0$ and $(\xi, h^\xi)$ trivial, Albin-Aldana-Rochon in [1, Theorem 2.9] got a version of Theorem B. Here authors do not require $\phi$ to be of compact support but have some extra decay at cusps (see [1, (2.11)]). The conformal transformations for $\phi$ with this type of decay assumptions do not alter the Wolpert norm, and, thus, the terms (1.26) do not appear in their theory. We note that in [1], authors also consider funnel singularities. The anomaly formula for surfaces with only funnel singularities was proved before by Borthwick-Judge-Perry, [13].

In our applications [24, Theorems C, D], [25, Theorem 1.2], it is crucial that by applying anomaly formula, we can trivialize the Poincaré-compatible coordinates horizontally in the family of Riemann surfaces with hyperbolic cusps. Thus, the appearance of the terms (1.26) is of fundamental importance in what follows.

e) A similar theorem appeared in the paper of Lundelius [36, Theorem 1.1] for $n = 0$ and $(\xi, h^\xi)$ trivial. However, we disagree with his result, as it differs from ours in the
last two terms of (1.23). From [36, p. 226, line 4], his proof should only work for \( \phi \) of compact support in \( M \).

To motivate this paper, we discuss several applications of Theorems A, B, which are proved in the sequel [24,25]. All those results are done in a family setting, i.e. we fix a holomorphic, proper map \( \pi : X \to S \) of complex manifolds such that for every \( t \in S \), the space \( X_t := \pi^{-1}(t) \) is a complex curve with at most double point singularities. We also fix disjoint sections \( \sigma_1, \ldots, \sigma_m : S \to X \), which avoid singular points of the fibers, and we denote by \( D_{X/S} \) the divisor, given by \( \text{Im}(\sigma_1) + \ldots + \text{Im}(\sigma_m) \). Those sections will model the position of hyperbolic cusps at the fibers in our family.

1. Regularity and asymptotics of the Quillen norm in a degenerating family of surfaces, [24, Theorem C]. We consider the determinant line bundle \( \lambda(j^*(\xi \otimes \omega_{X/S}(D)^n)) := (\det R^*\pi_*(\xi \otimes \omega_{X/S}(D)^n))^{-1}, n \leq 0 \), where \( \xi \) is a holomorphic vector bundle over \( X \) and \( \omega_{X/S}(D) := \omega_{X/S} \otimes \Omega^1_X(D_{X/S}) \) is the twisted relative canonical line bundle. We endow the vector bundles \( \xi, \omega_{X/S}(D) \) with Hermitian metrics \( h^\xi, ||\cdot||_{X/S} \) satisfying some mild hypotheses. Let \( |\Delta| \) be the locus of singular curves of \( \pi \). We define the Quillen norm \( ||\cdot||_Q \) on \( \lambda(j^*(\xi \otimes \omega_{X/S}(D)^n)) \) over \( S \setminus |\Delta| \), as a family version of (1.13). Then we study the regularity and singularities of \( ||\cdot||_Q \) near \( |\Delta| \). We also explicit some conditions under which the renormalized Quillen norm extends continuously at the singular locus.

The hypotheses, which we put on \( ||\cdot||_{X/S} \) are mild enough to include the case of degenerating hyperbolic surfaces. In this particular case, the asymptotics of the associated analytic torsion was studied before by Wolpert [47], Lundelius [36], Jorgenson-Lundelius [34], and many others. Our approach is very different from those references and it is based on the paper of Bismut-Bost [8], who prove the curvature theorem for singular families of curves endowed with smooth metric.

2. Curvature theorem for surfaces with cusps, [24, Theorem D]. We show that the metric \( ||\cdot||_Q \) from the previous paragraph is good enough, so that one can define its Chern form as a current. Then we give an explicit formula for this current, which refines the Riemann-Roch-Grothendieck theorem on the level of currents.

In particular, if we consider the family of hyperbolic surfaces, this extends the curvature theorem of Takhtajan–Zograf [45, Theorem 1] over the moduli space of curves to its Deligne-Mumford compactification. If we consider the case when there is no cusps, we get a generalization of Bismut-Bost [8, Théorème 2.1] to the case of degenerating metrics. We discuss a relation between this result and the arithmetic Riemann-Roch theorem for pointed stable curves due to Gillet-Soulé [30,31], Deligne [17] and Freixas [27,28].

3. Restriction and compatibility theorems, [25]. We relate the restriction of the renormalized Quillen norm \( ||\cdot||_Q \) at the locus of singular fibers \( |\Delta| \) with the Quillen norm of the normalization of singular fibers. By a combination of this result with the analogous statement for Takhtajan–Zograf analytic torsion, see (1.10), we deduce the compatibility between our definition of the analytic torsion and the one of Takhtajan–Zograf. This generalizes the result of Jorgenson-Lundelius [34, (4.7)], where authors did it for hyperbolic surfaces, \((\xi, h^\xi)\) trivial and \( n = 0 \).

Let’s describe how the present article is related to mathematical physics. Indeed, in [35], Kleptsov–Ma–Marinescu–Wiegmann related the asymptotics of the generating functional for the integer quantum Hall effect when the flux of the magnetic field through a Riemann surface tends to infinity, and the asymptotics of the analytic torsion associated to an increasing power of a positive line bundle. As the anomaly formula for Riemann surfaces played an essential role in their study (see [35, Theorem 2]), the present article lays a foundation to extend their result to the case of surfaces with hyperbolic cusps.
Finally, due to recent interest in orbifold setting (see [29,46]), let’s discuss how the theory developed here can be adapted to the orbifold Riemann surfaces. By combining the definition of the analytic torsion here and of the orbifold analytic torsion due to Ma [37], for an orbisurface \((M, g^{TM})\) with cusps \(D_M \subseteq \overline{M}\), we may define the analytic torsion \(T(g^{TM}, h^{\xi} \otimes \cdot |\cdot|_M^n)\), where \(n \leq 0\) and \(|\cdot|_M^n\) is the induced norm on the orbifold twisted line bundle \(\omega_M(D)\). Similarly to the manifolds case, this definition should generalize the definitions of the analytic torsion for stable hyperbolic orbisurfaces and \((\xi, h^{\xi})\) trivial due to Takhtajan–Zograf [46], Freixas–von Pippich [29].

Since our methods in the proof of Theorem A are purely local, the analogue of Theorem A would still hold for orbisurfaces. Since we got Theorem B by combining the calculations of the norm for the Mumford isomorphism in the orbifold setting due to Freixas–von Pippich [29] and the anomaly formula, it should be possible to get the orbifold analogue of Mumford isometry for any orbisurface with metric with cusps and a Hermitian vector bundle over it. We hope to return to this question very soon.

We note that our definition of the analytic torsion is related to the definition of the relative analytic torsion due to Lundelius and Jorgenson-Lundelius, which was given for \((\xi, h^{\xi})\) trivial and \(n = 0\) in [32,33,36] (see Remarks 1.7e), 2.17c)), and the definition of Albin-Rochon (see Remark 2.17d)), which was given for \((\xi, h^{\xi})\) trivial and \(n = 0\) in [3, §7.1].

The \(b\)-trace of Melrose [39], used in the definition of Albin-Rochon, should also give the definition of the analytic torsion in our case, however we decided to work in a relative setting, and \(b\)-trace does not appear here explicitly. This gives us more flexibility to establish some estimates on the heat kernel which are used in the proof of Theorem A.

Now, let’s describe the structure of this paper. In Sect. 2, we develop spectral theory for surfaces with cusps. We introduce the notion of the analytic torsion and Quillen norm, which are used throughout the article. In Sect. 3, we prove Theorem A. For this, we study the families of metrics which “converge” to the metric with cusps in a special way. In Sect. 4, we prove Theorem B. The main idea is to use Theorem A and to obtain Theorem B by the anomaly formula of Bismut-Gillet-Soulé [9, Theorem 1.23].

**Notation.** For \(\epsilon > 0\) and \((\overline{M}, D_M), (\overline{N}, D_N), \xi\) as in the statement of Theorem A, we denote

\[
\begin{align*}
D(\epsilon) &= \{u \in \mathbb{C} : |u| < \epsilon\}, & D^*(\epsilon) &= \{u \in \mathbb{C} : 0 < |u| < \epsilon\}, \\
\mathbb{D} &= D(1), & \mathbb{D}^* &= D^*(1), \\
\omega_M(D) &= \omega_M^{-1} \otimes \omega_M^*(D_M), \\
E^\xi_M &:= \xi \otimes \omega_M(D)^n, & E^\xi_N &:= \omega_N(D)^n.
\end{align*}
\]

By \(g^{TM}\) we denote the metric on \(\mathbb{D}^*\), induced by (1.2). By Spec(A) we denote the spectrum of a self-adjoint operator \(A\), acting on some Hilbert space. We denote by \(B^M(x, r)\) the geodesic ball of radius \(r > 0\) around \(x \in M\) in a Riemannian surface \(M\) with Riemannian metric \(g^{TM}\), by \(d_M(x, y), x, y \in M\) the induced distance function, and by \(dv_M\) the associated Riemannian volume form. We will sometimes drop \(M\) from the notation when this doesn’t cause any conflict.

We denote by \(L_X \boxtimes L_Y\) the holomorphic line bundle over \(X \times Y\), which is given by \(\pi_X^* L_X \otimes \pi_Y^* L_Y\) for some line bundles \(L_X, L_Y\) over the complex manifolds \(X\) and \(Y\) respectively and natural projections \(\pi_X : X \times Y \to X\) and \(\pi_Y : X \times Y \to Y\).
2. Spectral Theory of Surfaces with Cusps

In this section we study spectral properties of surfaces with cusps and define the analytic torsion.

More precisely, in Sect. 2.1 we set up the notation and state the spectral gap theorem. In Sect. 2.2 we state several estimates of the heat kernel associated with a hyperbolic surface, we define the regularized heat trace and the analytic torsion. Section 2.3 is the most technical one. Here we prove the estimates on the heat kernel of punctured disc endowed with Poincaré metric. Finally, in Sect. 2.4 we prove the statements from Sects. 2.1 and 2.2.

2.1. The setting of the problem and the spectral gap theorem. Let \((\overline{M}, D_M, g^{TM})\) be a Riemann surface with cusps and let \((\xi, h^\xi)\) be a Hermitian vector bundle over \(\overline{M}\). We denote by \(|\cdot|_M\) the Hermitian norm induced by \(g^{TM}\) on \(\omega_M(D)\) (see (1.4)) over \(M\).

By \(h^\xi TM\) we note the Hermitian metric on \(T(1,0)M\) induced by \(g^{TM}\) by the natural identification \(TM \ni Y \mapsto \frac{1}{2}(Y + \sqrt{-1}JY) \in T(0,1)M\), where \(J\) is the complex structure of \(M\). Let \(\alpha, \alpha' \in \mathcal{C}_c^\infty(M, E^\xi_M)\) or \(\alpha, \alpha' \in \mathcal{C}_c^\infty(M, T^*(0,1)\overline{M} \otimes E^\xi_M)\). The \(L^2\)-scalar product is defined by

\[
|\alpha, \alpha'|_{L^2} := \int_M \langle \alpha(x), \alpha'(x) \rangle_h dv_M(x),
\]

where \(dv_M\) is the Riemannian volume form on \((M, g^{TM})\), and \(\langle \cdot, \cdot \rangle_h\) is the pointwise Hermitian product induced by \(h^\xi\), \(h^{TM}\), \(|\cdot|_M\). To make the right-hand side of (2.1) finite for any \(\alpha, \alpha' \in \mathcal{C}_c^\infty(\overline{M}, E^\xi_M)\) or \(\alpha, \alpha' \in \mathcal{C}_c^\infty(\overline{M}, T^*(0,1)\overline{M} \otimes E^\xi_M)\), by (1.2), we need to suppose \(n \leq 0\), which we do from now on.

We define the Hilbert space \((L^2(E^\xi_M), |\cdot|_{L^2})\), as the \(L^2\)-completion of the space \(\mathcal{C}_c^\infty(M, E^\xi_M)\) with respect to \(\langle \cdot, \cdot \rangle_{L^2}\). Sometimes when we want to insist on the choice of \(g^{TM}, h^\xi\) and \(|\cdot|_M\), we denote this space by \(L^2(g^{TM}, h^\xi \otimes (|\cdot|_M)^{2n})\).

We denote by \(\Box^{E^\xi_M}\) the Kodaira Laplacian on \(\mathcal{C}_c^\infty(M, E^\xi_M)\), given by

\[
\Box^{E^\xi_M} := (\partial^{E^\xi_M})^* \partial^{E^\xi_M},
\]

where \((\partial^{E^\xi_M})^*\) is the formal adjoint of \(\partial^{E^\xi_M}\) with respect to \(\langle \cdot, \cdot \rangle_{L^2}\). Since \((M, g^{TM})\) is complete, the operator \(\Box^{E^\xi_M}\) is essentially self-adjoint on \(L^2(E^\xi_M)\) (cf. [38, Corollary 3.3.4]). We denote its closure by the same symbol.

In this article we are mostly interested in the heat operator \(\exp(-t\Box^{E^\xi_M})\), \(t > 0\). We denote

\[
\exp(\perp(-t\Box^{E^\xi_M})) := \exp(-t\Box^{E^\xi_M}) - P_M,
\]

where \(P_M\) is the orthogonal projection onto \(\ker(\Box^{E^\xi_M})\). We denote by

\[
\exp(-t\Box^{E^\xi_M})(x, y), \exp(\perp(-t\Box^{E^\xi_M})(x, y) \in (E^\xi_M)_x \otimes (E^\xi_M)_y^*, \quad \text{for } x, y \in M,
\]

the smooth kernels of \(\exp(-t\Box^{E^\xi_M})\), \(\exp(\perp(-t\Box^{E^\xi_M})\) with respect to \(dv_M\). Then

\[
\exp(-t\Box^{E^\xi_M})(x, x), \exp(\perp(-t\Box^{E^\xi_M})(x, x) \in \End(\xi)_x, \quad \text{for } x \in M.
\]
In Sect. 2, we fix $g^TM$, $h^ξ$, $||\cdot||_M$ and remove them from some notation: by $|\cdot|_{h×h}$ we mean the pointwise norm on $(ω^k_M ⊗ E^ξ,n_M)^* ⊗ (ω^l_M ⊗ E^ξ,n_M)$, $k, l ∈ ℤ$ induced by $h^ξ$, $||\cdot||_M$, $g^TM$; by $|\cdot|$ we mean either the modulus of a complex number, or the pointwise norm on the vector bundle $\text{End}(ξ)$ induced by $h^ξ$. We defer the proof of the next theorem until Sect. 2.4.

**Theorem 2.1.** For $n ≤ 0$, the operator $□^{E^ξ,n_M}$ has a spectral gap near 0. More precisely, we have

$$\ker(□^{E^ξ,n_M}) = H^0(\overline{M}, E^{ξ,n}_M), \quad (2.6)$$

and there is $μ > 0$ such that

$$\text{Spec}(□^{E^ξ,n_M}) \cap [0, μ] = \emptyset. \quad (2.7)$$

**Remark 2.2.** As it would follow from our proof, there are $c_1, c_1 > 0$ such that the set

$$\text{Spec}(□^{E^ξ,n_M}) \cap [0, \sqrt{-n} + c_2] \quad (2.8)$$

is discrete for any $(M, g^TM)$, $(ξ, h^ξ)$, $||\cdot||_M$ and $n ≤ 0$. For $n = 0$, $(ξ, h^ξ)$ trivial, and $c_2 = 1/4$, this was proved by Müller in [40, §6].

For $n = 0$, our proof of Theorem 2.1 relies on the result of Müller [40, §6, Proposition 6.9], who proves Theorem 2.1 for $n = 0$ and $(ξ, h^ξ)$ trivial. In case of $n < 0$, we obtain Theorem 2.1 by gluing the estimates in the neighbourhood of cusp, coming from Nakano’s inequality (cf. [38, Theorem 1.4.14]), and the estimates away from the cusps coming from the spectral gap for the Laplacian on a surface with boundary with Dirichlet condition on the boundary.

Finally, let’s discuss the construction of the $L^2$-norm $||\cdot||_{L^2}^2(g^TM, h^ξ ⊗ ||\cdot||_M^n_M)$ on the line bundle (1.12). By the isomorphism (2.6), we may endow $H^0(\overline{M}, E^{ξ,n}_M)$ with the $L^2$-scalar product induced by (2.1). Similarly to the analysis in the proof of (2.6), we have a natural isomorphism

$$\ker(□_1^{E^ξ,n_M}) = \begin{cases} H^1(\overline{M}, E^{ξ,n}_M), & \text{for } n = 0, \\ H^1(\overline{M}, E^{ξ,n}_M ⊗ \mathcal{O}_M(D_M)), & \text{for } n ≤ -1, \end{cases} \quad (2.9)$$

where $□_1^{E^ξ,n_M} = \overline{\partial}^{E^ξ,n_M} (\overline{\partial}^{E^ξ,n_M})^*$ is the Kodaira Laplacian associated with (0, 1)-forms with values in $E^{ξ,n}_M$. For $n = 0$, the isomorphism (2.9) induces the scalar product on $H^1(\overline{M}, E^{ξ,n}_M)$ by (2.1). For $n ≤ -1$, we induce the scalar product on $H^1(\overline{M}, E^{ξ,n}_M)$ by (2.1), (2.9) and the inclusion

$$H^1(\overline{M}, E^{ξ,n}_M) \hookrightarrow H^1(\overline{M}, E^{ξ,n}_M ⊗ \mathcal{O}_M(D_M)), \quad \alpha \mapsto \alpha ⊗ s_{DM}, \quad (2.10)$$

where $s_{DM}$ is the canonical holomorphic section of $\mathcal{O}_M(D_M)$. Those scalar products induce the natural $L^2$-norm $||\cdot||_{L^2}^2(g^TM, h^ξ ⊗ ||\cdot||_M^n_M)$ on the line bundle (1.12).
2.2. Relative spectral theory for surfaces with cusps. The main goal of this section is to define the analytic torsion for any surface with cusps \((\overline{M}, D_M, \overline{g}^{TM})\), any Hermitian vector bundle \((\xi, h^\xi)\) over \(\overline{M}\) twisted by a power \(n \leq 0\) of the twisted canonical line bundle \(\left(\omega_M(D), ||\cdot||_M\right)\). This definition extends the relative definition due to Jorgenson-Lundelius [33, Definition 1.9], which they gave in the case \(n = 0\) and \((\xi, h^\xi)\) trivial.

Our main idea is to define the analytic torsion by the formula analogous to (1.7), where the zeta function is defined not by (1.6) but using an integral involving a regularized heat trace (see Definition 2.15). More formally, we regularize the definition of the heat trace by taking out the divergent asymptotics in the integral of the heat kernel over a truncated surface converging to the surface with cusps (see Definition 2.9). Once the necessary technicalities are sorted out, we prove that the zeta function defined as the Mellin transform of this function is meromorphic and 0 is a holomorphic point of it. Then we define the analytic torsion in Definition 2.16. In the compact case it coincides with the definition of Ray–Singer (cf. (1.7)), and it has a renormalization constant introduced for non-compact surfaces, which makes our definition compatible with the definition of Takhtajan–Zograf (see Remark 2.17a and (1.10)).

The challenge here is that unlike in [33], the precise contribution of the continuous spectrum to the heat kernel is unknown, moreover the local geometry near the cusp of (2.12) is different.

To define the analytic torsion

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The challenge here is that unlike in [33], the precise contribution of the continuous spectrum to the heat kernel is unknown, moreover the local geometry near the cusp depends on \((\xi, h^\xi)\). We circumvent this difficulty by the analytic localization techniques of Bismut-Lebeau [11, §11], by the parametrix construction for the heat kernel (cf. [7, §2.4, 2.5]) and by careful study of the heat kernel on the punctured disc endowed with Poincaré metric. The parametrix construction is particularly useful when we would estimate the effect of non-triviality of \((\xi, h^\xi)\) (see Proposition 2.6 and (2.19)).

We fix \(n \in \mathbb{Z}\). Define the function \(\rho_M : M \to [1, +\infty[\) by (see (1.1) for a definition of \(V_i^M(\cdot)\))

\[
\rho_M(x) = \begin{cases} 
1 & \text{for } x \in M \setminus (\cup_i V_i^M(1/2)), \\
\sqrt{|\ln |z_i(x)||} & \text{for } x \in V_i^M(1/2), \quad i = 1, \ldots, m.
\end{cases}
\]

(2.11)

**Remark 2.3.** The function \((\rho_M(x))^{-2}\) is proportional to the injectivity radius at point \(x\) of \((M, g^{TM})\).

We denote by \(d(\cdot, \cdot)\) the distance function on \((M, g^{TM})\). Now we can state the main theorems of this section. Their proofs are delayed until Sect. 2.4.

**Proposition 2.4.** For any \(l, l' \in \mathbb{N}\), there are \(c, c', C > 0\) such that for any \(t > 0\), \(x, x' \in M\), we have

\[
|\langle \nabla_x \cdot \nabla_{x'} \rangle_{h} \exp(-t\Box_{\xi}^{E_x, n}(x, x'))|_{h \times h} \leq C\rho_M(x)\rho_M(x')t^{-1-(l+l')/2} \cdot \\
\cdot \exp(ct - c' \cdot d(x, x')^2/t),
\]

(2.12)

where \(\nabla\) is induced by the Levi-Civita connection and the Chern connections of \((\xi, h^\xi)\) and \((\omega_M(D), ||\cdot||_M)\). Also, if \(n \leq 0\), then there are \(c, C > 0\) such that for any \(t > 0\), we have

\[
|\langle \nabla_x \cdot \nabla_{x'} \rangle_{h} \exp(-t\Box_{\xi}^{E_x, n}(x, x'))|_{h \times h} \leq C\rho_M(x)\rho_M(x')t^{-4-l-l'} \exp(-ct).
\]

(2.13)

**Remark 2.5.** By Remark 2.3, we see that for \(n = 0\), \((\xi, h^\xi)\) trivial and \(k, l = 0\), (2.12) is exactly [15, Theorem VIII.8], applied for a surface with hyperbolic cusps. Our proof of (2.12) is different.
Now, let $M$, $N$ and all related notions be as in the statement of Theorem A. Let $z^M_i$ be as in (1.2) and let $z^N_i$ be defined analogously for $N$.

**Proposition 2.6.** For any $k \in \mathbb{N}$, there are $\epsilon, c, c', C > 0$ such that for any $t > 0$, $u \in \mathbb{C}$, $|u| \leq \epsilon$:

$$\left| \exp(-t \Box^{E^M_i})((z^M_i)^{-1}(u), (z^M_i)^{-1}(u)) - \Id_{\xi} \cdot \exp(-t \Box^{E^N_i})((z^N_i)^{-1}(u), (z^N_i)^{-1}(u)) \right| \leq C \ln |u|\exp(ct) \cdot \min \left\{ |\ln |u||^{-k} + \exp(-c'(\ln |\ln |u||)^2/t) \right\};$$  

(2.14)

$$|u|^{1/3} + \exp(-c'/t) \right\}. \quad (2.15)$$

Moreover, if $n \leq 0$, then there are $\zeta < 1$ and $C > 0$ such that

$$\left| \exp(-t \Box^{E^M_i})((z^M_i)^{-1}(u), (z^M_i)^{-1}(u)) - \Id_{\xi} \cdot \exp(-t \Box^{E^N_i})((z^N_i)^{-1}(u), (z^N_i)^{-1}(u)) \right| \leq C \ln |u|^\zeta \exp(-ct). \quad (2.16)$$

**Remark 2.7.** As we explain in the course of the proof of Proposition 2.6, if $(\xi, h^\xi)$ is trivial around the cusps, then the estimates (2.14), (2.15) could be easily improved to

$$\left| \exp(-t \Box^{E^M_i})((z^M_i)^{-1}(u), (z^M_i)^{-1}(u)) - \Id_{\xi} \cdot \exp(-t \Box^{E^N_i})((z^N_i)^{-1}(u), (z^N_i)^{-1}(u)) \right| \leq C \ln |u|\exp(-c'(\ln |\ln |u||)^2/t). \quad (2.17)$$

To prove (2.14), (2.15) in full generality, we use Duhamel’s formula and (2.12).

**Proposition 2.8.** There are smooth bounded functions $a^{M,n}_{\xi,j}: M \to \text{End}(\xi)$, $j \geq -1$ such that for any $x \in M$, $t_0 > 0$, $k \in \mathbb{N}$, there is $C > 0$ such that for any $t \in [0, t_0]$, we have

$$\left| \exp(-t \Box^{E^M_i})((z^M_i)^{-1}(u), (z^M_i)^{-1}(u)) - \sum_{j=-1}^{k} a^{M,n}_{\xi,j}(x)t^j \right| \leq Ct^k. \quad (2.18)$$

Moreover, if $x \in M \setminus (\bigcup_i V_i^M (e^{-t^{-1/3}}))$, then $C$ can be chosen independently of $t \in [0, t_0]$ and $x$.

Also, there is $\epsilon > 0$, such that for any $l \in \mathbb{N}$, $j \geq -1$, there is $C > 0$ such that for any $u \in \mathbb{C}$, $0 < |u| \leq \epsilon$, $i = 1, \ldots, m$, we have

$$\left| (\nabla_u)^iy \left( a^{M,n}_{\xi,j}((z^M_i)^{-1}(u)) - \Id_{\xi}a^{N,n}_{j}(z^N_i)^{-1}(u)) \right) \right|_{\mathfrak{h}} \leq C|u|^{1/3}, \quad (2.19)$$

where $\nabla$ is induced by the Levi-Civita connection and Chern connections associated with $(\xi, h^\xi)$ and $(\omega_{\mathfrak{D}}(0), ||||_{\mathfrak{D}})$.

From now on till the end of this section, we denote by

$$P := \mathbb{C}P^1 \setminus \{0, 1, \infty\}, \quad (2.20)$$

and by $g^{TP}$ the unique hyperbolic metric of constant scalar curvature $-1$ over $P$ with cusps at $D_P = \{0, 1, \infty\}$. We use the notations $||\cdot||_P$, $V^P_i(\epsilon)$, $E^P_{\epsilon}$, $\ldots$ and denote by $z^P$ the Poincaré-compatible coordinate of $0 \in \mathbb{C}P^1$ of $(P, g^{TP})$. 
Definition 2.9. We define the regularized heat trace by

\[
\text{Tr}^r[\exp^{-t\Box^E_M^n}] := \int_{M \setminus (\cup_i V_i^M(\eta))} \text{Tr}[\exp^{-t\Box^E_M^n}(x, x)] d\nu_M(x)
- \frac{m \cdot \operatorname{rk}(\xi)}{3} \int_{P \setminus (\cup_i V_i^P(\eta))} \text{Tr}[\exp^{-t\Box^E_P^n}(x, x)] d\nu_P(x)
+ \sum_i \int_{D^+(\eta)} \left( \text{Tr}[\exp^{-t\Box^E_M^n}(z_{M}^{1}(u), z_{M}^{1}(u))] - \operatorname{rk}(\xi) \text{Tr}[\exp^{-t\Box^E_P^n}(z_{P}^{1}(u), z_{P}^{1}(u))] \right) d\nu_{D^+}(u),
\]

where \( \eta > 0 \) is such that Proposition 2.6 holds for \( \epsilon := \eta \) and (1.2) holds.

Remark 2.10. (a) From the fact that there is a holomorphic automorphism of \( \mathbb{C}P^1 \) permuting \( D_P \) and inducing the isometry on \( (P, g^P) \), the coordinate \( z_P \) in (2.21) can be changed to a Poincaré-compatible coordinate associated with 1 or \( \infty \), and this would result in the same definition.

(b) Essentially, in our definition of the regularized heat trace, we take out the diverging part of the usual heat trace. This idea is very similar to the famous \( b \)-trace, defined by Melrose in [39, Lemma 4.62], which was used in the context of Riemann surfaces with cusps by Albin and Rochon [3].

Proposition 2.11. The integrals on the right-hand side of (2.21) converge and the right-hand sided is independent of \( \eta > 0 \). We also have

\[
\text{Tr}^r[\exp^{-t\Box^E_M^n}] := \lim_{r \to 0} \left( \int_{M \setminus (\cup_i V_i^M(r))} \text{Tr}[\exp^{-t\Box^E_M^n}(x, x)] d\nu_M(x)
- \operatorname{rk}(\xi) \int_{P \setminus (\cup_i V_i^P(r))} \text{Tr}[\exp^{-t\Box^E_P^n}(x, x)] d\nu_P(x) \right).
\]

Proof. The first two integrals in the right-hand side of (2.21) are bounded by (2.13). The last one is bounded by (2.16) and the fact that for any \( \varsigma < 1 \), we have

\[
\int_{D(\epsilon)} \frac{\sqrt{-1}dud\overline{u}}{|u|^2|\ln|u||^{2-\varsigma}} < +\infty.
\]

The independence on \( \eta > 0 \) is trivial. The formula (2.22) follows from (2.16). \( \square \)

A similar quantity \( \text{Tr}^r[\exp(-t\Box^E_M^n)] \) (see also [33, Definition 1.1] for the relative version) is defined analogically to (2.21), where we put \( \exp \) in place of \( \exp^{-} \). It is well-defined by (2.14), (2.23) and the fact that for any \( \epsilon > 0 \) small enough, there is \( C > 0 \) such that for any \( t > 0 \):

\[
\int_{D(\epsilon)} \exp(-c' (\ln |\ln|u||)^2/t) \frac{\sqrt{-1}dud\overline{u}}{|u|^2|\ln|u||^{2}} \leq Ct^{1/2} \exp(-c'/2(\ln |\ln\epsilon|)^2/t).
\]

By (2.6), the relation between Definition 2.9 and \( \text{Tr}[\exp(-t\Box^E_M^n)] \) is given by

\[
\text{Tr}^r[\exp^{-t\Box^E_M^n}] = \text{Tr}^r[\exp(-t\Box^E_M^n)] - \dim H^0(M, E^n_M) + \frac{\operatorname{rk}(\xi)}{3} \dim H^0(M, E^n_P).
\]
Remark 2.12. In [33, §3], Jorgenson-Lundelius defined the relative heat trace
\[ \text{Tr}^{\text{rel}}[\exp(-t\Box^{E_{M}^{n}}); \exp(-t\Box^{E_{N}^{p}})] \quad (2.26) \]
for \((\xi, h^{\xi})\) trivial and \(n = 0\). Directly from the definition, in this case we have
\[
\begin{align*}
\text{Tr}^{\text{rel}}[\exp(-t\Box^{E_{M}^{n}}); \exp(-t\Box^{E_{N}^{p}})] &= \text{Tr}^{F}[\exp(-t\Box^{E_{M}^{n}})] - \text{rk}(\xi)\text{Tr}^{F}[\exp(-t\Box^{E_{N}^{p}})], \\
\text{Tr}^{F}[\exp(-t\Box^{E_{M}^{n}})] &= \frac{1}{3}\text{Tr}^{\text{rel}}[3\exp(-t\Box^{E_{M}^{n}}); m\exp(-t\Box^{E_{p}^{n}})],
\end{align*}
\]
where \(3\exp(-t\Box^{E_{M}^{n}})\) (resp. \(m\exp(-t\Box^{E_{p}^{n}})\)) means the heat operator on \(M \sqcup M \sqcup M\) (resp. on \(P \sqcup \cdots \sqcup P\)) with the induced geometry.

By Proposition 2.8, the functions \(\text{Tr}[a_{*, j}^{M, n}(x)], a^{P, n}_{*, j}(x)\) are integrable over \(M\) and \(P\) respectively. For \(j \geq -1\), we denote
\[
\begin{align*}
A_{\xi, j, 0}^{M, n} &:= \int_{M} \text{Tr}[a_{\xi, j}^{M, n}(x)]d\nu_{M}(x) - \frac{\text{rk}(\xi)}{3}\int_{P} a^{P, n}_{\xi, j}(x)d\nu_{P}(x), \\
A_{\xi, j}^{M, n} &= A_{\xi, j, 0}^{M, n} - \dim H^{0}(\overline{M}, E_{M}^{\xi}) + \frac{\text{rk}(\xi)}{3}\dim H^{0}(\overline{P}, E_{p}^{n}).
\end{align*}
\]

Proposition 2.13. For any \(t_{0} > 0, k \in \mathbb{N}\), there is \(C > 0\) such that for any \(t \in [0, t_{0}]\), we have
\[
\left| \text{Tr}^{F}[\exp(-t\Box^{E_{M}^{n}})] - \sum_{j=-1}^{k} A_{\xi, j}^{M, n}t^{j} \right| \leq Ct^{k}. \quad (2.29)
\]

Proof. First of all, by (2.25), it is enough to prove that for any \(t_{0} > 0, k \in \mathbb{N}\), there is \(C > 0\) such that for any \(t \in [0, t_{0}]\), we have
\[
\left| \text{Tr}^{F}[\exp(-t\Box^{E_{M}^{n}})] - \sum_{j=-1}^{k} A_{\xi, j, 0}^{M, n}t^{j} \right| \leq Ct^{k}. \quad (2.30)
\]

By Proposition 2.8, for any \(t_{0} > 0, k \in \mathbb{N}\), there is \(C > 0\) such that for any \(t \in [0, t_{0}]\), we have
\[
\begin{align*}
\left| \int_{M \setminus (\cup_{i} V^{M}_{i}(e^{-t-1/3}))} \left[ \text{Tr}[\exp(-t\Box^{E_{M}^{n}})(x, x)] - \sum_{j=-1}^{k} \text{Tr}[a_{\xi, j}^{M, n}(x)]t^{j} \right]d\nu_{M}(x) \right| &\leq Ct^{k}, \\
\left| \int_{P \setminus (\cup_{i} V^{P}_{i}(e^{-t-1/3}))} \left[ \exp(-t\Box^{E_{p}^{n}})(x, x) - \sum_{j=-1}^{k} a^{P, n}_{\xi, j}(x)t^{j} \right]d\nu_{P}(x) \right| &\leq Ct^{k}.
\end{align*}
\]
\[
(2.31)
\]
Since for \(u \in \mathbb{C}, 0 < |u| \leq e^{-t-1/3}\), we have \(t^{-1/3} \leq |\ln|u||, by (2.14), (2.23) and (2.24), for any \(k \in \mathbb{N}\) there are \(c, C > 0\) such that for any \(t \in [0, t_{0}], i = 1, \ldots, m\), we have
\[
\int_{D(e^{-t-1/3})} \left| \text{Tr} \left[ \exp(-t \Box g_{E_{M}}^{E,n}(z)M^{-1}(u), (z_i M)^{-1}(u)) \right] - \text{rk}(\xi) \exp(-t \Box g_{E_{P}}^{P}(z)P^{-1}(u), (z_P)^{-1}(u)) \right| dv_{\mathbb{D}^*}(u) \leq C t^k + C \exp(-ct^{-1/2}).
\]

(2.32)

Also, by (2.19), for any \( j \in \mathbb{N}, i = 1, \ldots, m \) there are \( c, C > 0 \), such that we have

\[
\int_{D(e^{-t-1/3})} \left| \text{Tr} \left[ a_{\xi,j}^{M,n}(z^M_i)^{1/2}(u) \right] - \text{rk}(\xi) a_{j}^{P,n}(z_P)^{-1}(u) \right| dv_{\mathbb{D}^*}(u) \leq C \exp(-ct^{-1/3}).
\]

(2.33)

We see that (2.30) holds by (2.31), (2.32) and (2.33).

\[\Box \]

Proposition 2.14. For any \( t_0 > 0 \), there are \( c, C > 0 \) such that for any \( t \geq t_0 \), we have

\[
\left| \int_{M \backslash (\bigcup_i V_i M(\eta))} \text{Tr} \left[ \exp(-t \Box g_{E_{M}}^{E,n}(x, x)) \right] dv_{M}(x) \right| \leq C \exp(-ct),
\]

(2.35)

By (1.2), (2.16) and (2.23), we deduce that there are \( c, C > 0 \) such that for any \( t \geq t_0 \), we have

\[
\left| \int_{D(t)} \left( \text{Tr} \left[ \exp(-t \Box g_{E_{M}}^{E,n}(z)M^{-1}(u), (z_i M)^{-1}(u)) \right] - \text{rk}(\xi) \text{Tr} \left[ \exp(-t \Box g_{E_{P}}^{P}(z)P^{-1}(u), (z_P)^{-1}(u)) \right] \right) dv_{\mathbb{D}^*}(u) \right| \leq C \exp(-ct).
\]

(2.36)

We conclude by (2.35) and (2.36).

\[\Box \]

Definition 2.15. We define the \textit{regularized zeta function} \( \xi_M(s) \) for \( s \in \mathbb{C} \), \( \text{Re}(s) > 1 \) by

\[
\xi_M(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \text{Tr}^{\mathbb{F}} \left[ \exp(-t \Box g_{E_{M}}^{E,n}(0)) \right] t^s \frac{dt}{t}.
\]

(2.37)

By Propositions 2.13 and 2.14, the function \( \xi_M(s) \) is holomorphic for \( \text{Re}(s) > 1 \) and has a meromorphic extension to the entire \( s \)-plane. From the classical properties of the Mellin transform, this extension, which we also denote by \( \xi_M(s) \), is holomorphic at \( s = 0 \).

Definition 2.16. We define the \textit{analytic torsion} by

\[
T(g^T, h^g \otimes ||-||^{2n}_{M} ) := \exp(-\xi_M'(0)) \cdot T_{TZ}(g^T, ||-||^{2n}_{P})^{\text{rk}(\xi)/3}.
\]

(2.38)
Remark 2.17. a) In the forthcoming paper we show that under the conditions (1.3), we have
\[ T(g_{TM}^{TM}, (||\cdot||_{M}^{TM})^{2n}) = T_{TZ}(g_{hyp}^{TM}, (||\cdot||_{M}^{TM})^{2n}). \]  
(2.39)

For the moment, we content ourselves by noting that (2.39) holds for \( M = P \) by the choice of the last multiplicand in (2.38).

b) Explicitly, we have the following identity (see Proposition 2.13 for the definition of \( a_{-1}^{M} \)):
\[ \zeta'_{M}(0) = \int_{0}^{1} \left( \text{Tr}^{\perp}[\exp(-t\Box^{E_{n}^{\xi,n}})] - \frac{A_{\xi,-1}^{M,n}}{t} - A_{\xi,0}^{M,n} \right) \frac{dt}{t} \\
+ \int_{1}^{+\infty} \text{Tr}^{\perp}[\exp(-t\Box^{E_{n}^{\xi,n}})] \frac{dt}{t} + A_{\xi,-1}^{M,n} - \Gamma'(-1)A_{\xi,0}^{M,n}. \]  
(2.40)

c) By (2.37), the relation between the relative analytic torsion, defined by Jorgensen-Lundelius [33] for \((\xi, h^{\xi})\) trivial and \( n = 0 \), and our definition is
\[ T_{rel}(g^{TM}, 1; g^{TN}, 1) = T(g^{TM}, 1) T(g^{TN}, 1). \]  
(2.41)

d) In [3], Albin-Rochon, for \( n = 0 \), gave an alternative definition of the analytic torsion \( T_{AR}(g^{TM}) \). By [4, (1.24)], [3, §7] and (2.14), the relation between their definition and ours is
\[ \frac{T_{AR}(g^{TM})}{T_{AR}(g^{TN})} = \frac{T(g^{TM}, 1)}{T(g^{TN}, 1)}, \]  
(2.42)

for \( M, N \) as in the statement of Theorem A. Their definition is based on \( b \)-trace of Melrose [39], see Remark 2.10b).

e) In his thesis [26, Corollary 8.2.2], Freixas explicitly evaluated (see (1.9))
\[ \log Z'_{(P, D_{p})}(1) = 4\zeta'(-1) + \log 2 + \frac{10}{9} \log 2. \]  
(2.43)

By combining (1.10), (2.43), we may give an explicit formula for \( T_{TZ}(g^{TP}, 1) \) in (2.38). Evaluating \( T_{TZ}(g^{TP}, (||\cdot||_{P}^{2n}) \) for any \( n \leq 0 \) is an interesting problem, which is related to the arithmetic Riemann-Roch theorem for families of Riemann surfaces with cusps (see the thesis of Freixas [26, §6, 8] and his articles [27, 28]).

2.3. Heat kernel on the punctured hyperbolic disc and elliptic estimates. In this section we recall the well-known construction [7, §2.4, 2.5] of the parametrix, applied for the heat kernel on the punctured disc, endowed with Poincaré metric and a Hermitian vector bundle. We also prove the elliptic estimates for Kodaira Laplacian on this punctured disc.

Let’s explain the setting in this section. Let \((\xi, h^{\xi})\) be Hermitian vector bundle over \( \mathbb{D} \). Let
\[ \omega_{\mathbb{D}}(0) := \omega_{\mathbb{D}} \otimes \mathcal{O}_{\mathbb{D}}(0) \]  
(2.44)
be the twisted canonical line bundle as in (1.4), and let \( ||\cdot||_{\mathbb{D}} \) be the norm on \( \omega_{\mathbb{D}}(0) \) over \( \mathbb{D}^{*} \), induced by \( g^{TM} \) as in (1.5). We denote the restriction of \( h^{\xi} \) to \( \mathbb{D}^{*} \) by the same symbol. By Cartan’s Theorem A, we fix a holomorphic trivialization \( e_{1}, \ldots, e_{rk(\xi)} \) of
Let \( \xi \) over \( \mathbb{D} \). We chose it in such a way that it becomes a normal trivialization (cf. [18, Proposition V.12.10]), i.e. we have
\[
h^\xi(e_i, e_j)(u) = \delta_{ij} + O(|u|^2). \tag{2.45}
\]

Let \( \Box^\xi \otimes \omega_\mathbb{D}(0)^n \), \( n \in \mathbb{Z} \) be the Kodaira Laplacian associated with \( h^\xi \otimes \| \cdot \|_{\mathbb{D}}^{2n} \) on \( (\mathbb{D}, g^T \mathbb{D}^\ast) \). Let
\[
\exp(-t \Box^\xi \otimes \omega_\mathbb{D}(0)^n)(z_1, z_2) \in (\xi \otimes \omega_\mathbb{D}(0)^n)^{z_1} \otimes (\xi \otimes \omega_\mathbb{D}(0)^n)^{z_2}, \quad \text{for } z_1, z_2 \in \mathbb{D}^\ast, \tag{2.46}
\]
be the smooth kernel of the heat operator \( \exp(-t \Box^\xi \otimes \omega_\mathbb{D}(0)^n) \) with respect to the volume form \( dv_\mathbb{D}^\ast \).

We consider the covering
\[
\rho : \mathbb{H} \to \mathbb{D}^\ast, \quad z \mapsto e^{\sqrt{-1}z}. \tag{2.47}
\]

Easily, the metric \( g^T \mathbb{H} := \rho^\ast(g^T \mathbb{D}^\ast) \) is equal to the standard hyperbolic metric on the upper half-plane. The Deck transformations of \( \rho \) are generated by the isometry
\[
U : \mathbb{H} \to \mathbb{H}, \quad z \mapsto z + 2\pi. \tag{2.48}
\]

Let \( \| \cdot \|_{\mathbb{H}} \) be the norm on \( \omega_\mathbb{H} \), given by \( \rho^\ast(\| \cdot \|_{\mathbb{D}}) \). For \( z = (x, y) := x + \sqrt{-1}y \), we have
\[
g^T \mathbb{H}_z = \frac{dx^2 + dy^2}{y^2}, \quad \|dz\|_{\mathbb{H}}(z) = y. \tag{2.49}
\]

Let \( \Box^\xi \otimes \omega_\mathbb{H}^n \) be the Kodaira Laplacian associated with \( g^T \mathbb{H}, \rho^\ast(h^\xi) \otimes \| \cdot \|_{\mathbb{H}}^{2n} \) on \( (\mathbb{H}, g^T \mathbb{H}) \), and let
\[
\exp(-t \Box^\xi \otimes \omega_\mathbb{H}^n)(z_1, z_2) \in (\rho^\ast(\xi) \otimes \omega_\mathbb{H}^n)^{z_1} \otimes (\rho^\ast(\xi) \otimes \omega_\mathbb{H}^n)^{z_2}, \quad \text{for } z_1, z_2 \in \mathbb{H}. \tag{2.50}
\]
be the smooth kernel of the heat operator \( \exp(-t \Box^\xi \otimes \omega_\mathbb{H}^n) \) with respect to the Riemannian volume form \( dv_\mathbb{H} \) on \( \mathbb{H} \), induced by \( g^T \mathbb{H} \). For \( z_1, z_2 \in \mathbb{D} \), the relation between (2.46) and (2.50) is given by
\[
\exp(-t \Box^\xi \otimes \omega_\mathbb{D}(0)^n)(z_1, z_2) = \sum_{i \in \mathbb{Z}} \exp(-t \Box^\xi \otimes \omega_\mathbb{H}^n)(\tilde{z}_i, U^i \tilde{z}_2), \tag{2.51}
\]
where \( \tilde{z}_i \in \mathbb{H}, \rho(\tilde{z}_i) = z_i \) for \( i = 1, 2 \).

Since \( (\mathbb{H}, g^T \mathbb{H}) \) is a complete manifold, we may use the framework of [7, §2.4, 2.5] to construct the parametrix of \( \exp(-t \Box^\xi \otimes \omega_\mathbb{H}^n) \). Let us briefly recall the main steps of this construction. By doing so, we also provide some uniform estimates on the heat kernels.

We denote by \( d_\mathbb{H}(z_1, z_2), z_1, z_2 \in \mathbb{H} \) the Riemannian distance associated with \( g^T \mathbb{H} \), we have
\[
d_\mathbb{H}((x_1, y_1), (x_2, y_2)) = 2 \ln \left( \frac{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + \sqrt{(x_1 - x_2)^2 + (y_1 + y_2)^2}}{2\sqrt{y_1 y_2}} \right). \tag{2.52}
\]

Let \( \psi : \mathbb{R} \to [0, 1] \) be a smooth even function such that
\[
\psi(u) = \begin{cases} 
1 & \text{for } |u| < 1/2, \\
0 & \text{for } |u| > 1.
\end{cases} \tag{2.53}
\]
For $k \in \mathbb{N}$, $z_1, z_2 \in \mathbb{H}$, $t > 0$, let $k_{t,k}^\xi \otimes \omega_1^n \in \mathcal{C}^\infty(\mathbb{H} \times \mathbb{H}, (\rho^*(\xi) \otimes \omega_1^n) \boxtimes (\rho^*(\xi) \otimes \omega_1^n)*)$ be given by (cf. [7, (2.7)])

$$k_{t,k}^\xi \otimes \omega_1^n(z_1, z_2) := \frac{\psi(z_1, z_2)^2}{t} \exp \left( - \frac{d_{\mathbb{H}}(z_1, z_2)^2}{4t} \right) \left( \sum_{i=0}^k i^i \Phi_{i,n}^\xi(z_1, z_2) \right), \quad (2.54)$$

where $\Phi_{i,n}^\xi \in \mathcal{C}^\infty(\mathbb{H} \times \mathbb{H}, (\rho^*(\xi) \otimes \omega_1^n) \boxtimes (\rho^*(\xi) \otimes \omega_1^n)*)$, $i \geq 0$ are symmetric (i.e. $\Phi_{i,n}^\xi(z_1, z_2) = (\Phi_{i,n}^\xi(z_2, z_1))^*$) and given by the procedure, described in [7, Theorem 2.26]. We denote by $\Phi_{i,n}$, $i \geq 0$ those sections associated to $(\xi, h^k)$ trivial. Now let’s state the main result of this section.

**Proposition 2.18.** The sections $\Phi_{i,n}$ are uniformly $\mathcal{C}^\infty$-bounded in the following sense: for any $l, l' \in \mathbb{N}$, there is $C > 0$ such that for any $z_1, z_2 \in \mathbb{H}$, we have

$$|\nabla_{z_1}^l(\nabla_{z_2})^l \Phi_{i,n}^\xi(z_1, z_2)|_{h \times h} \leq C, \quad (2.55)$$

where $\nabla$ is induced by the Levi-Civita connection and Chern connections associated with $(\xi, h^k)$, $(\omega_1^n(0), ||\cdot||_{\mathbb{H}_2})$, and $||\cdot||_{h \times h}$ is the associated pointwise norm.

Moreover, for any $l, l' \in \mathbb{N}$, there is $C > 0$ such that for any $z_1, z_2 \in \mathbb{H}$, we have

$$|\nabla_{z_1}^l(\nabla_{z_2})^l (\Phi_{i,n}^\xi - \text{Id}_\xi \cdot \Phi_{i,n}(z_1, z_2))|_{h \times h} \leq C \exp(- (\text{Im}(z_1 + \text{Im}(z_2))/6). \quad (2.56)$$

**Proof.** Let’s fix $z_0 \in \mathbb{H}, z_0 = (x_0, y_0)$. For $z \in \mathbb{H}, r > 0$ we denote by $B^n(z, r) \subset \mathbb{H}$ the hyperbolic disc of radius $r$ around $z$. We consider the isometry

$$g_{z_0} : (\mathbb{H}, g^{\mathbb{T}_\mathbb{H}}) \rightarrow (\mathbb{H}, g^{\mathbb{T}_\mathbb{H}}), \quad (x, y) \mapsto ((x - x_0)/y_0, y/y_0). \quad (2.57)$$

As $g_{z_0}(z_0) = (0, 1) := \sqrt{-1}$, we have $g_{z_0}(B^n(z_0, 1)) = B^n(\sqrt{-1}, 1)$. We recall that by the procedure, described in [7, Theorem 2.26], the sections $\Phi_{i,n}^\xi(z, \cdot)$ are defined locally, i.e. they depend only on the restriction of $(\mathbb{H}, g^{\mathbb{T}_\mathbb{H}}), (\xi, h^k)$ over $B^n(z, 1)$, and if $d_{\mathbb{H}}(z, z_2) > 1$, then $\Phi_{i,n}^\xi(z, z_2) = 0$. Moreover, if one changes “smoothly” the parameters $g^{\mathbb{T}_\mathbb{H}}, h^k$, then the sections $\Phi_{i,n}^\xi(z, \cdot)$ change smoothly “at the same rate”. Let’s make the last point precise and adapt it for our situation.

Let $h^k_z, h^{k,0}_z, z \in \mathbb{H}$ be two families of Hermitian metrics on $(g_z^{-1})^* \xi$ over $B^n(\sqrt{-1}, 1)$, and let $\Phi_{i,n}^\xi(\sqrt{-1}, \cdot), \Phi_{i,n}^{k,0}(\sqrt{-1}, \cdot)$ be the corresponding sections from (2.54). Suppose that there is $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for any $l \in \mathbb{N}$, there is $C > 0$ such that for any $z_2 \in B^n(\sqrt{-1}, 1)$, we have

$$|\nabla^l(h^k_z)(z_2)|_h \leq C, \quad |\nabla^l(h^k_z - h^{k,0}_z)(z_2)|_h \leq Cf(\text{Im} z). \quad (2.58)$$

From the procedure, described in [7, Theorem 2.26], the sections $\Phi_{i,n,z}^\xi(\sqrt{-1}, \cdot), \Phi_{i,n,z}^{k,0}(\sqrt{-1}, \cdot)$ are obtained iteratively by applying the Laplacian associated with $h^k_z$ and $h^{k,0}_z$ to $\Phi_{i-1,n,z}^\xi(\sqrt{-1}, \cdot)$ and $\Phi_{i-1,n,z}^{k,0}(\sqrt{-1}, \cdot)$ respectively and integrating over the geodesics.
of length \(\leq 1\), emanating from \(\sqrt{-1}\). Thus, for any \(l \in \mathbb{N}\) there is \(C > 0\) such that for any \(z_2 \in B^\mathbb{H}(\sqrt{-1}, 1)\), we have

\[
\left| (\nabla_{z_2})^j \Phi_{i,n,z}^E(\sqrt{-1}, z_2) \right|_h \leq C,
\]

\[
\left| (\nabla_{z_2})^j (\Phi_{i,n,z}^E - \Phi_{i,n,z}^E,0)(\sqrt{-1}, z_2) \right|_h \leq Cf(\text{Im } z),
\]

(2.59)

or, as we stated before, the sections \(\Phi_{i,n,z}^E(\sqrt{-1}, \cdot)\), \(\Phi_{i,n,z}^E(\sqrt{-1}, \cdot)\), \(i \geq 0\) change “at the same rate”.

Now, let \(h_z^E, z \in \mathbb{H}\) be defined by

\[
h_z^E := ((g_z^1 \rho)^* h_z^E)_{B^\mathbb{H}(\sqrt{-1}, 1)}.
\]

(2.60)

Let the frame \(e_1, \ldots, e_{kE}\) be as in (2.45). Then for \(z \in \mathbb{H}, z_2 = (x, y) \in B^\mathbb{H}(\sqrt{-1}, 1)\), we have

\[
h_z^E((g_z^{-1} \rho)^* e_i, (g_z^{-1} \rho)^* e_j)(z_2) = h^E(e_i, e_j)(e^{-y_0+\sqrt{-1} (x_0+y_0)}).
\]

(2.61)

Let \(h_z^{E,0}, z \in \mathbb{H}\) be defined by

\[
h_z^{E,0}((g_z^{-1} \rho)^* e_i, (g_z^{-1} \rho)^* e_j)(z_2) = \delta_{ij},
\]

(2.62)

where \(\delta_{ij}\) is the Kronecker delta symbol. Let \(\Phi_{i,n,z}^E(\sqrt{-1}, \cdot)\), \(\Phi_{i,n,z}^{E,0}(\sqrt{-1}, \cdot)\) be the sections from (2.54), associated with \(\|\cdot\|_{\mathbb{H}}\) \(B^\mathbb{H}(\sqrt{-1}, 1), \mathcal{L}^{TH}_{B^\mathbb{H}(\sqrt{-1}, 1)}\) and \(h_z^E, h_z^{E,0}\) respectively. Then by the locality of \(\Phi_{i,n,z}^E(\sqrt{-1}, \cdot)\), \(\Phi_{i,n,z}^{E,0}(\sqrt{-1}, \cdot)\), for any \(z_2 \in B^\mathbb{H}(\sqrt{-1}, 1)\), we have

\[
\Phi_{i,n,z}^E(\sqrt{-1}, z_2) = \Phi_{i,n}^E(z, g_z^{-1}(z_2)), \quad \Phi_{i,n,z}^{E,0}(\sqrt{-1}, z_2) = \text{Id}_\mathbb{E} \cdot \Phi_{i,n}(z, g_z^{-1}(z_2)).
\]

(2.63)

By the symmetry of \(\Phi_{i,n,z}^E(\sqrt{-1}, \cdot)\) and (2.63), to complete the proof of Proposition 2.18, it is enough to prove the analogue of (2.58) for \(f(x) = \exp(-x/3)\).

Now, by the formula (2.52), we have

\[
\min \{ \text{Im } z : z \in B^\mathbb{H}(\sqrt{-1}, 1) \} \geq 1/6.
\]

(2.64)

By (2.45), (2.61) and (2.64), we have (2.58) for \(f(x) = \exp(-x/3)\), which finishes the proof. \(\Box\)

To compare \(k_{i,k}^{E \otimes \omega_{\mathbb{H}}^E}(x, y)\) with the heat kernel, we recall the definition of the “defect”:

\[
r_{i,k}^{E \otimes \omega_{\mathbb{H}}^E}(z_1, z_2) := (\partial_t + \Box_{D,x}^{E \otimes \omega_{\mathbb{H}}^E})k_{i,k}^{E \otimes \omega_{\mathbb{H}}^E}(z_1, z_2).
\]

(2.65)

The following theorem says, in particular, that as one increases \(k \in \mathbb{N}\), the kernel \(k_{i,k}^{E \otimes \omega_{\mathbb{H}}^E}(z_1, z_2)\) more and more accurately “satisfies” the properties defined by the heat kernel.
Proposition 2.19. For any \( t_0 > 0 \), the family of kernels \( k_{t,k}^{\xi \otimes \omega^n_H}(z_1, z_2), t \in [0, t_0], z_1, z_2 \in \mathbb{H} \) defines a uniformly bounded family of operators \( K_{t,k}^{\xi \otimes \omega^n_H} \) on \( C_c^\infty(\mathbb{H}), \rho^*(\xi) \otimes \omega^n_H \) such that for any \( s \in C_c^\infty(\mathbb{H}), \rho^*(\xi) \otimes \omega^n_H \), the sections \( K_{t,k}^{\xi \otimes \omega^n_H}(s) \) converge, as \( t \to 0 \), to \( s \) over any compact subset of \( \mathbb{H} \) with all its derivatives.

Moreover, for any \( l, l', l'' \in \mathbb{N} \), there are \( c', C > 0 \) such that for any \( t \in [0, t_0] \), \( z_1, z_2 \in \mathbb{H} \):

\[
\left| (\nabla_{z_1})^l (\nabla_{z_2})^{l'} (\partial_t)^{l''} k_{t,k}^{\xi \otimes \omega^n_H}(z_1, z_2) \right|_{h \times h} \leq C t^{-1 - (l + l')/2 - l''} \cdot \psi(d_H(z_1, z_2)^2/2) \cdot \exp(-c' \cdot d_H(z_1, z_2)^2/t).
\]

(2.66)

\[
\left| (\nabla_{z_1})^l (\nabla_{z_2})^{l'} (\partial_t)^{l''} r_{t,k}^{\xi \otimes \omega^n_H}(z_1, z_2) \right|_{h \times h} \leq C t^{-1 - (l + l')/2 - l''} \cdot \psi(d_H(z_1, z_2)^2/2) \cdot \exp(-c' \cdot d_H(z_1, z_2)^2/t).
\]

(2.67)

**Proof.** The first statement is established in the same way as in [7, Theorem 2.29]. The estimate (2.66) follows directly from (2.54) and (2.65). The proof of (2.67) uses (2.55), but otherwise it is done in the same way as [7, Theorem 2.29]. \( \Box \)

This theorem means that \( k_{t,k}^{\xi \otimes \omega^n_H}(z_1, z_2) \) is the parametrix of the heat equation in the sense of [7, p.77]. Thus, we may construct the heat kernel as follows. For \( k, k' \in \mathbb{N} \), \( z, z' \in \mathbb{H} \), we denote

\[
q_{t,k,k'}^{\xi \otimes \omega^n_H}(z, z') := \int_{t \Delta_{k'}}^t \int_{\mathbb{H}^{k'}} k_{t-t',k}^{\xi \otimes \omega^n_H}(z, z') r_{t',k-1,k}^{\xi \otimes \omega^n_H}(z'_{k'}, z_{k'-1}) \cdots \cdot r_{t,k}^{\xi \otimes \omega^n_H}(z_1, z'_1) d\nu_H(z_{k'}) \otimes \cdots \otimes d\nu_H(z_1) d\nu_{t \Delta_{k'}}(t_1, \ldots, t_{k'}),
\]

(2.68)

where \( \Delta_{k'} \) is the standard \( k' \)-simplex, and \( d\nu_{t \Delta_{k'}}(t_1, \ldots, t_{k'}) \) is the standard volume form over \( t \Delta_{k'} \). Now let’s explain why (2.68) is well-defined. The integration over \( \mathbb{H}^{k'} \) in (2.68) is well-defined since by (2.53), (2.54) and (2.65), the functions under the integral vanish if the arguments are too distant, so all the integrations are done in a compact subset. The integration over \( t \Delta_{k'} \) is well-defined for \( k \geq 1 \) by (2.54) and (2.67). By the same reasons, it is easy to see that if \( k \geq (l + l')/2 + l'' + 1 \), then the partial derivatives \( (\partial_{z_j})^l (\partial_{z_j})^{l'} (\partial_t)^{l''} q_{t,k,k'}^{\xi \otimes \omega^n_H}(z_1, z_2) \) exists.

**Proposition 2.20.** For any \( t_0 > 0 \), \( k \in \mathbb{N}^* \) and \( l, l', l'' \in \mathbb{N} \), there are \( c', C > 0 \) such that for any \( t \in [0, t_0], z_1, z_2 \in \mathbb{H} \), and \( k' \in \mathbb{N} \) satisfying \( k \geq (l + l')/2 + l'' + 1 \), we have

\[
\left| (\nabla_{z_1})^l (\nabla_{z_2})^{l'} (\partial_t)^{l''} q_{t,k,k'}^{\xi \otimes \omega^n_H}(z_1, z_2) \right|_{h \times h} \leq \frac{C k' k^{kk'-l-l'-2} e^{-2l''}}{(k'-1)!} \exp(-c' \cdot d_H(z_1, z_2)^2/t).
\]

(2.69)

Moreover, for any \( t \in [0, t_0], z_1, z_2 \in \mathbb{H}, k \in \mathbb{N}^* \), the series

\[
\sum_{k'=0}^{\infty} (-1)^k q_{t,k,k'}^{\xi \otimes \omega^n_H}(z_1, z_2),
\]

(2.70)

converges to \( \exp(-t \sum_{k=0}^{\infty} q_{t,k,k'}^{\xi \otimes \omega^n_H}(z_1, z_2)) \) in \( C^{2k-2}(H \times H) \), and for any \( l, l', l'' \in \mathbb{N} \), satisfying \( k \geq (l + l')/2 + l'' + 1 \), there is \( C > 0 \) such for any \( t \in [0, t_0], z_1, z_2 \in \mathbb{H} \), we
have
\[|\nabla_{z_1}^l(\nabla_{z_2}^m(\partial_t)^n \exp(-t\square_{\gamma}(0)^n) - k^l_{t,k})|_{h\times h} \leq Ct^{-l'-2l''} \exp(-(c' \cdot d_{\mathbb{H}}(z_1, z_2)^2)/t). \] (2.71)

Proof. First of all, we note that by the weighted mean inequality and the triangle inequality, for \( k' \in \mathbb{N}, t > t_k' > \ldots > t_1 > 0, \) and \( z, z', z_1, \ldots, z_{k'} \in \mathbb{H}, \) we have
\[ \exp\left(-\frac{c' \cdot d_{\mathbb{H}}(z, z_k')^2}{t - t_k'}\right) \exp\left(-\frac{c' \cdot d_{\mathbb{H}}(z_k', z_{k'-1})^2}{t_k' - t_{k'-1}}\right) \ldots \exp\left(-\frac{c' \cdot d_{\mathbb{H}}(z_1, z')^2}{t_1}\right) \leq \exp\left(-\frac{c' \cdot d_{\mathbb{H}}(z, z')^2}{t}\right). \] (2.72)

We also note that the integration over each variable \( z_1, \ldots, z_{k'} \) is done over a hyperbolic ball of radius 1, which has a constant volume, independently of the choice of its center. From now on, the proof remains verbatim with [7, Lemma 2.22, Theorem 2.23], where one has to replace the appropriate estimates by (2.66), (2.67) and use (2.72) to bound the exponentials. \( \square \)

Now let’s apply all this theory to the study of the heat kernel on the punctured hyperbolic disc. We summarize all the important results, which will be used in Sect. 2.4, in the following theorem, which is a local analogue of (2.12) and Proposition 2.8.

**Proposition 2.21.** For any \( l, l', l'' \in \mathbb{N}, \) there are \( t_0 > 0, c, c', C > 0 \) such that for any \( t \in ]0, t_0[, u, v \in \mathbb{D}^*, \) we have
\[ |\nabla_u^l(\nabla_v^m(\partial_t)^n \exp(-t\square_{\gamma}(0)^n))(u, v)|_{h\times h} \leq Ct^{-1-(l+l')/2-l''} \cdot (1 + |\ln |u||)^{1/2} \cdot (1 + |\ln |v||)^{1/2} \exp\left(-\frac{c' \cdot d_{\mathbb{H}}(u, v)^2}{t}\right). \] (2.73)

Moreover, there are bounded sections \( a_{\xi,j}^{D^*,n} \in \mathcal{C}^\infty(\mathbb{D}^*, End(\xi)), j \geq -1 \) such that there are \( c', C > 0 \) such that for any \( u \in \mathbb{D}^*, k \in \mathbb{N} \) and \( t \in ]0, t_0[, \) we have
\[ \left|\exp(-t\square_{\gamma}(0)^n) - \sum_{j=-1}^k a_{\xi,j}^{D^*,n} t^j\right| \leq \left(1 + |\ln |u||\right) \left(C t + \frac{C}{t} \exp\left(-\frac{c' \cdot |\ln |u||^2}{t}\right)\right). \] (2.74)

Moreover, for any \( j \geq -1, \) there is \( C > 0 \) such that for any \( u \in \mathbb{D}^*, \) we have
\[ |(\nabla_u^l)(a_{\xi,j}^{D^*,n} - 1_{tk(\xi)} a_j^{D^*,n})|_{h\times h} \leq C |u|^{1/3}, \] (2.75)

where we trivialized \( \xi \) as in the beginning of this section.

Before proving this theorem, let’s prove the following technical

**Lemma 2.22.** There is \( t_0 > 0 \) such that for any \( z_1, z_2 \in \mathbb{H}, t \in ]0, t_0[, \) satisfying \( d_{\mathbb{H}}(z_1, U^t z_2) \leq d_{\mathbb{H}}(z_1, z_2) \) for any \( i \in \mathbb{Z}, \) we have
\[ \sum \exp\left(-d_{\mathbb{H}}(z_1, U^i z_2)^2/t\right) \leq C \left((\text{Im}(z_1) + 1)(\text{Im}(z_2) + 1)^{1/2} \exp\left(-d_{\mathbb{H}}(z_1, z_2)^2/(2t)\right)\right). \] (2.76)
Proof. We decompose the sum in (2.76) into two parts: for \( i^2 \leq 4 \text{Im}(z_1) \text{Im}(z_2) \) and the complementary. Trivially, by the assumption, the first part is bounded by

\[
4\left((\text{Im}(z_1) + 1)(\text{Im}(z_2) + 1)\right)^{1/2} \exp\left(-\frac{d_{\mathbb{H}}(z_1, z_2)^2}{2t}\right).
\]

(2.77)

Now, by choosing \( t_0 \) small enough, we see that

\[
\exp\left(-\left(\ln\frac{i^2}{\text{Im}(z_1) \text{Im}(z_2)}\right)^2/t\right) \leq \frac{\text{Im}(z_1) \text{Im}(z_2)}{i^2}.
\]

(2.78)

By (2.81) and (2.78), we see that

\[
\sum_{i^2 > 4 \text{Im}(z_1) \text{Im}(z_2)} \exp(-d_{\mathbb{H}}(z_1, U^i z_2)^2/t) \leq \left(\text{Im}(z_1) \text{Im}(z_2)\right) \cdot \exp\left(-d_{\mathbb{H}}(z_1, z_2)^2/(2t)\right) \sum_{i^2 > 4 \text{Im}(z_1) \text{Im}(z_2)} i^{-2} \leq \left(\text{Im}(z_1) \text{Im}(z_2)\right)^{1/2} \cdot \exp\left(-d_{\mathbb{H}}(z_1, z_2)^2/(2t)\right).
\]

(2.79)

We conclude by (2.77) and (2.79). \( \square \)

Proof of Proposition 2.21. Let \( u, v \in \mathbb{D}^* \), and let \( \tilde{u}, \tilde{v} \in \mathbb{H} \) be such that \( \rho(\tilde{u}) = u \), \( \rho(\tilde{v}) = v \). Then \( \text{Im}(\tilde{u}) = |\log |u||, \text{Im}(\tilde{v}) = |\log |v||. \) By (2.51), (2.71) and Lemma 2.22, we have

\[
\left| (\nabla u)^I(\nabla v)^J (\partial_t)^I'' \left( \exp(-t\xi \otimes \omega_{\mathbb{D}^*})(u, v) - \sum_{i \in \mathbb{Z}} k_{i,k}^* \otimes \omega_{\mathbb{H}}(\tilde{u}, U^i \tilde{v}) \right) \right|_{h \times h} \leq C i^{k-(l+l'/2)-2-\ln(1 + |\log |u||)^{1/2}} \exp\left(-c' \cdot d_{\mathbb{D}^*}(u, v)^2/t\right).
\]

(2.80)

Now, by (2.52), for any \( i \neq 0 \) and \( z_1, z_2 \in \mathbb{H} \) as in Lemma 2.22, we have

\[
d_{\mathbb{H}}(z_1, U^i z_2) \geq |\log \left(i^2/(\text{Im}(z_1) \text{Im}(z_2))\right)|.
\]

(2.81)

From (2.81), there is \( C > 0 \) such that

\[
\#\left\{ i \in \mathbb{Z} : d_{\mathbb{H}}(z_1, U^i z_2) < 2 \right\} \leq C((\text{Im}(z_1) + 1)(\text{Im}(z_2) + 1))^{1/2}.
\]

(2.82)

Thus, the number of non-zero terms in the sum under the module in (2.80) is bounded by the right-hand side of (2.82). So, by (2.66), (2.80) and (2.82), we get (2.73).

Now, by (2.52), there is \( C > 0 \) such that for any \( z \in \mathbb{D}^*, \tilde{z} \in \mathbb{H}, \rho(\tilde{z}) = z, \) and \( i \in \mathbb{Z}^* \), we have

\[
d_{\mathbb{H}}(\tilde{z}, U^i \tilde{z}) \geq \frac{C}{|\log |z||}.
\]

(2.83)

Thus, from Lemma 2.22, (2.54), (2.80) and (2.83), for \( j \geq -1 \), we get (2.74) by setting

\[
ed_{\xi,j}^{\mathbb{D}^*,n}(z) := \Phi_{j+1}^{\xi \otimes \omega_{\mathbb{D}^*}}(\tilde{z}, \tilde{z}).
\]

(2.84)

Now, (2.75) follows from (2.56) and (2.84). \( \square \)
Finally, as an application of the ideas from the proof of Proposition 2.18, let’s establish the following elliptic estimates.

**Lemma 2.23.** For any $\alpha > 0$, $k \in \mathbb{N}$, there is $C > 0$, such that for any $n \in \mathbb{Z}$, $\sigma \in \mathcal{C}^\infty(\mathbb{D}^*, \xi \otimes \omega_D(0)^n)$, $x \in \mathbb{D}^*$, we have

$$\left| \nabla^k \sigma(x) \right|_h \leq C \left| \log |x| \right|^{1/2} \sum_{i=0}^{2k} \left( n^{4+2k-i} + 1 \right) \left\| (\square_z^{\mathbb{H}} \sigma)^i \right\|_{L^2(B^D(x, |x|))}. \quad (2.85)$$

**Remark 2.24.** Similar results have appeared in a recent article of Auvray–Ma–Marinescu [5, §4], [6, §4]. Our methods of proof are, however, fundamentally different.

**Proof.** We conserve the notations from the proof of Proposition 2.18.

We denote by $\square_z^{\mathbb{H}} \sigma$ the Kodaira Laplacian on $B^\mathbb{H}(\sqrt{-1}, 1)$ associated to $g^\mathbb{H}$, $h_z^\mathbb{H} \otimes ||\cdot||^n_\mathbb{H}$. Let $\nabla_z$ be the connection on $B^\mathbb{H}(\sqrt{-1}, 1)$ induced by the Chern connection associated to $h_z^\mathbb{H}$, $||\cdot||_\mathbb{H}$ and the Levi-Civita connection on $(\mathbb{H}, g^\mathbb{H})$.

The family of metrics $h_z^\mathbb{H}$ over $B^\mathbb{H}(\sqrt{-1}, 1)$ has bounded geometry by (2.58). From this, the fact that $g_z \in \text{Aut}(\mathbb{H})$ preserves $g^\mathbb{H}$ and [38, Lemma 1.6.2], we deduce that for any $0 < \alpha < 1$, $k \in \mathbb{N}$, there is $C > 0$, such that for any $z \in \mathbb{H}$, $n \in \mathbb{Z}$, $\sigma_1 \in \mathcal{C}^\infty(B^\mathbb{H}(\sqrt{-1}, 1), \rho^*(\xi) \otimes \omega^n_\mathbb{H})$:

$$\left| (\nabla_z^{\mathbb{H}} \sigma_1)(\sqrt{-1}) \right|_h \leq C \sum_{i=0}^{2k} \left( n^{4+2k-i} + 1 \right) \left\| (\square_z^{\mathbb{H}} \sigma_1)^i \right\|_{L^2(B^\mathbb{H}((\sqrt{-1}, \alpha)))}. \quad (2.86)$$

Now, for $\tilde{\sigma} \in \mathcal{C}^\infty(\mathbb{H}, \rho^*(\xi) \otimes \omega^n_\mathbb{H})$, we denote $\sigma_1 := ((g_z)^{-1} \ast \tilde{\sigma})$. Then by the fact that $g_z \in \text{Aut}(\mathbb{H})$ preserves $g^\mathbb{H}$, we trivially have

$$\left| (\nabla^{\mathbb{H}} \sigma_1)(\sqrt{-1}) \right|_h = \left| (\nabla^\mathbb{H} \tilde{\sigma})(z) \right|_h,$$

$$\left\| (\square_z^{\mathbb{H}} \sigma_1)^i \right\|_{L^2(B^\mathbb{H}(\sqrt{-1}, \alpha))} = \left\| (\square_z^{\mathbb{H}} \tilde{\sigma})^i \right\|_{L^2(B^\mathbb{H}(\tilde{\sigma}_, \alpha))}. \quad (2.87)$$

From (2.86) and (2.87), we deduce the following elliptic estimate on $\mathbb{H}$:

$$\left| (\nabla^{\mathbb{H}} \tilde{\sigma})(\tilde{x}) \right|_h \leq C \sum_{i=0}^{2k} \left( n^{4+2k-i} + 1 \right) \left\| (\square_z^{\mathbb{H}} \tilde{\sigma})^i \right\|_{L^2(B^\mathbb{H}(\tilde{x}, \alpha))}. \quad (2.88)$$

By (2.81), we deduce that for any $\alpha$, there is $C' > 0$ such that for any $x \in D^*(1/2)$, $\tilde{x} \in \mathbb{H}$, such that $\rho(\tilde{x}) = x$, we have

$$\# \left\{ \tilde{y} \in B^\mathbb{H}(\tilde{x}, \alpha) : \rho(\tilde{y}) = y \right\} \leq C' \cdot |\log |x||. \quad (2.89)$$

Thus, by (2.89) and the fact that the restriction $\rho|_{B^\mathbb{H}(\tilde{x}, \alpha)} : B^\mathbb{H}(\tilde{x}, \alpha) \to B^D(x, |x|)$ is a surjection, we deduce that for any $\sigma \in \mathcal{C}^\infty(\mathbb{D}^*, \xi \otimes \omega_D(0)^n)$, $x \in \mathbb{D}^*$ and $\tilde{x} \in \mathbb{H}$ such that $\rho(\tilde{x}) = x$, we have

$$\left\| (\square_z^{\mathbb{H}} \sigma)^i (\sigma \circ \rho) \right\|_{L^2(B^\mathbb{H}(\tilde{x}, \alpha))} \leq (C')^{1/2} |\log |x||^{1/2}. \left\| (\square_z^{\mathbb{H}} \sigma)^i \right\|_{L^2(B^D(x, |x|))} \quad (2.90)$$

By (2.88) and (2.90) applied for $\tilde{\sigma} := \rho^* \sigma$, we deduce (2.85) for $C := C(C')^{1/2}$. □

**Lemma 2.25.** For any $\beta > 1$, $k \in \mathbb{N}$, $n \in \mathbb{Z}$, there is $C > 0$, such that for any $\sigma \in \mathcal{C}^\infty(\mathbb{D}^*, \xi \otimes \omega_D(0)^n)$, $x \in D(x/(2\beta)) \setminus \{0\}$, we have

$$\left| \nabla^k \sigma(x) \right|_h \leq C |\log |x||^{3+k} \sum_{i=0}^{2k} \left( n^{4+2k-i} + 1 \right) \left\| (\square_z^{\mathbb{H}} \sigma)^i \right\|_{L^2(D(\beta|x|) \setminus D(|x|/\beta))}. \quad (2.91)$$
Proof. We use the notation from the proof of Lemma 2.23.

First, let’s prove that for any $\gamma > 1$, $k \in \mathbb{N}$, $n \in \mathbb{Z}$, there is $C > 0$, such that for any $z \in \mathbb{H}$, $\tilde{\sigma} \in \mathcal{C}^{\infty}(\mathbb{H}, \rho^*(\xi) \otimes \omega^n_{\mathbb{H}})$, we have the following elliptic estimate on $\mathbb{H}$:

$$
|\nabla^k \tilde{\sigma}(z)|_h \leq C |\Im z|^{2+k} \sum_{i=0}^{2+k} \left\| (\Box_z^{\xi} \otimes \omega^n_{\mathbb{H}})^i \tilde{\sigma} \right\|_{L^2(B^\mathbb{H}(z, \gamma/\Im z))}.
$$

(2.92)

Similarly to (2.87), we see that in the notations (2.86), to prove (2.92), it is enough to prove that for any $1 > \delta > 0$, there exists $C > 0$ such that the following estimate holds

$$
|\nabla^k (\sigma_1)(\sqrt{-1})|_h \leq C \delta^{-(2+k)} \sum_{i=0}^{2+k} \left\| (\Box_z^{\xi} \otimes \omega^n_{\mathbb{H}})^i \sigma_1 \right\|_{L^2(B^\mathbb{H}(\sqrt{-1}, \delta))}.
$$

(2.93)

However, as the family of metrics $h^\xi_z$, $z \in \mathbb{H}$ has bounded geometry and $g^\mathbb{H}$ differs from the standard Euclidean metric over $B^\mathbb{H}(\sqrt{-1}, 1) \subset \mathbb{C}$ by a smooth function, we deduce that it is enough to prove that for a standard Kodaira Laplacian $\Box$ on $\mathbb{C}$, for any $1 > \delta > 0$, $k \in \mathbb{N}$, $n \in \mathbb{Z}$, $\sigma' \in \mathcal{C}^{\infty}(D(\delta))$, we have

$$
|\nabla^k (\sigma')(0)|_h \leq C \delta^{-(2+k)} \sum_{i=0}^{2+k} \left\| \Box^i \sigma' \right\|_{L^2(D(\delta))}.
$$

(2.94)

But (2.94) follows from a standard elliptic estimate on a disc (cf. [38, Lemma 1.6.2]) by using the transformation $\sigma''(x) := \sigma'(\delta x)$.

Directly from (2.90), there is $C' > 0$ such that for any $\sigma \in \mathcal{C}^{\infty}(\mathbb{D}, \xi \otimes \omega_\mathbb{D}(0)^n)$, $x \in D(1/2) \setminus \{0\}$ and $\tilde{x} \in \mathbb{H}$ such that $\rho(\tilde{x}) = x$, we have

$$
\left\| (\Box^{\xi, \omega^n_{\mathbb{D}}} \otimes \omega^n_{\mathbb{D}})^i (\sigma \circ \rho) \right\|_{L^2(B^\mathbb{D}(\tilde{x}, \gamma/\Im(\tilde{x})))} \leq C' \|\log |x||^{1/2} \left\| (\Box^{\xi, \omega^n_{\mathbb{D}}} \otimes \omega^n_{\mathbb{D}})^i \sigma \right\|_{L^2(B^\mathbb{D}(x, \gamma/|\log |x||))}
$$

(2.95)

From (2.95) and (2.92) applied for $\tilde{\sigma} = \sigma \circ \rho$, we get

$$
|\nabla^k \sigma(x)|_h \leq C \|\log |x||^{k/2} \sum_{i=0}^{k/2} \left\| (\Box^{\xi, \omega^n_{\mathbb{D}}} \otimes \omega^n_{\mathbb{D}})^i \sigma \right\|_{L^2(B^\mathbb{D}(x, \gamma/|\log |x||))}.
$$

(2.96)

However, by (2.52), we see that for any $\beta > 1$, there is $\gamma > 0$ such that for any $z \in D(1/(2\beta))$, we have $B^\mathbb{D}(z, \gamma/|\log |z||) \subset D(\beta z) \setminus D(z/\beta)$. From this and (2.96), we deduce (2.91). □

Remark 2.26. Let’s choose a family of Hermitian metrics $h^\xi_{\eta}$, $\eta \in [0, 1]$ instead of $h^\xi$, for

$$
h^\xi_{\eta}(e_i, e_j)(u) := (1 - \psi(|u|^2/\eta)) h^\xi(e_i, e_j)(u) + \psi(|u|^2/\eta) \delta_{ij},
$$

(2.97)

where $\psi$ is defined in (2.53), $e_i, i = 1, \ldots, \text{rk}(\xi)$ is as in (2.45). Then all the estimates of this chapter would continue to hold uniformly over $\eta \in [0, 1]$.

Let’s briefly explain this point. First of all, as all the results of this section rely on Proposition 2.18, it is enough to explain why the uniform analogue of (2.58) holds, as it is the main step in the proof of Proposition 2.18. But this is due to the fact that for the Hermitian metrics $h^\xi_{z, \eta}$, $z = (x_0, y_0) \in \mathbb{H}$, defined in the notation of Proposition 2.18 by (compare with (2.60))

$$
h^\xi_{z, \eta} := (g^{-1}_z \rho)^* h^\xi_{\eta}|_{B^\mathbb{H}(\sqrt{-1}, 1)},
$$

(2.98)
we have (compare with (2.61))

\[ h^\xi_{z,\eta}(\xi g_{z}^{-1})^* e_i, (g_{z}^{-1})^* e_j)(z_2) = h^\xi_{\eta}(e_i, e_j)(e^{-y_0}+\sqrt{-1}(x_0+y_0)). \]  

(2.99)

Now, for any \( c > 0 \) the function \((1 - \psi(e^{-y_0}/\eta)) \cdot e^{-2y_0}, y \in [1, c]\) has bounded derivatives uniformly on \( y_0 > 0 \) and \( \eta \in [0, 1] \). From this observation, (2.99) implies that for \( h^\xi_{z,0} \) defined as in (2.62), the following uniform analogue of (2.58) holds:

\[
\begin{align*}
|\langle (\partial x)^{(i)}(\partial y)^{(j)} (h^\xi_{z,\eta})(z_2) \rangle_h | & \leq C, \\
|\langle (\partial x)^{(i)}(\partial y)^{(j)} (h^\xi_{z,\eta} - h^\xi_{z,0})(z_2) \rangle_h | & \leq C e^{-\text{Im} z/3}.
\end{align*}
\]  

(2.100)

Finally, let’s mention one consequence of Remark 2.26. We add a subscript \( \eta \) to all the objects which depend on \( h^\xi_{\eta} \) instead of \( h^\xi \).

**Lemma 2.27.** For any \( \alpha > 0, k \in \mathbb{N}, \) there is \( C > 0, \) such that for any \( n \in \mathbb{Z}, \sigma \in \mathcal{C}^\infty(\mathbb{D}, \xi \otimes \omega_{\mathbb{D}}(0)^n), x \in \mathbb{D}, \) we have

\[
|\nabla^\xi_{\eta}\sigma(x)|_{h,\eta} \leq C |\log |x||^{1/2} \sum_{i=0}^{2k} (n^{4(2k+i-1)} + 1) \left\| (\square^\xi_{\eta} \otimes \omega_{\mathbb{D}}(0))^i \sigma \right\|_{L^2_n(B^0(x,\alpha))}.
\]  

(2.101)

**Proof.** Same as the proof of Lemma 2.23, as by (2.100), the family \( h^\xi_{z,\eta} \) is bounded. \( \square \)

### 2.4. Proofs of Theorem 2.1 and Propositions 2.4, 2.6, 2.8.

In this section we finally present the proofs of Theorem 2.1 and Propositions 2.4, 2.6, 2.8.

**Proof of Theorem 2.1.** First of all, for \( n \leq 0 \), there is \( C > 0 \) such that for any \( z \in D^*(1/2): \)

\[
C \leq |z|^2 (\ln |z|)^{2-2n}.
\]  

(2.102)

Let \( g_{sm}^{TM} \) be some smooth metric over \( \overline{M} \), and let \( |||\cdot|||_{M}^{sm} \) be some smooth Hermitian norm on \( \omega_{M}(D) \) over \( \overline{M} \). By (2.102), there is \( C > 0 \) such that \( g_{sm}^{TM} \otimes (|||\cdot|||_{M}^{sm})^{2n} \leq C g^{TM} \otimes (|||\cdot|||_{M}^{2n}). \) Thus, we have

\[
\ker(\square^{E_{M}}_{M}) \subset L^2(g^{TM} \otimes (|||\cdot|||_{M}^{2n}), \mathcal{H}^{\xi,n}_{\mathcal{E}_{M}}\otimes (|||\cdot|||_{M}^{2n})).
\]  

(2.103)

Let \( s \in \ker(\square^{E_{M}}_{M}). \) By (2.103) and the classical \( L^2 \)-extension theorem (cf. [38, Lemma 2.3.22]), \( s \) extends holomorphically to \( V^{M}_{i}(\epsilon). \) In other words

\[
\ker(\square^{E_{M}}_{M}) \subset H^{0}(\overline{M}, E^{E_{M}}_{\mathcal{E}_{M}}).
\]  

(2.104)

On the other hand, by the finiteness of the volume of \( (M, g^{TM}) \), see (2.23), we see that that each holomorphic section lies in \( L^2(g^{TM} \otimes (|||\cdot|||_{M}^{2n}), \mathcal{H}^{\xi,n}_{\mathcal{E}_{M}}), \) i.e.

\[
H^{0}(\overline{M}, E^{E_{M}}_{\mathcal{E}_{M}}) \subset \ker(\square^{E_{M}}_{M}).
\]  

(2.105)

We deduce (2.6) by (2.104) and (2.105).

For \( n = 0 \), our proof of Theorem 2.1 relies on the result of Müller [40, §6, Proposition 6.9], who proved Theorem 2.1 for \( n = 0 \) and \( (\xi, h^\xi) \) trivial. In case of \( n < 0, \) we
obtain Theorem 2.1 by gluing the estimates in the neighbourhood of cusp, coming from Nakano’s inequality (cf. [38, Theorem 1.4.14]), and the estimates away from the cusps coming from the spectral gap for the Dirichlet Laplacian of a surface with boundary.

Let’s show that (2.7) holds for $n = 0$ and any $(\xi, h_\xi)$. In [40, §6], Müller proved (2.8) for $(\xi, h_\xi)$ trivial, $n = 0$ and $c_2 = 1/4$, see Remark 2.2. This implies, in particular, that (2.7) holds for $(\xi, h_\xi)$ trivial and $n = 0$ (see [40, Proposition 6.9]). He proved (2.8) in this case by studying explicitly the spectrum of Kodaira Laplacian of the Neumann problem in the cusp and using the scattering matrix to relate the continuous spectrum of the cusp and the initial manifold. If the Hermitian vector bundle $(\xi, h_\xi)$ is trivial around the cusps (i.e. we can choose a holomorphic frame trivializing the Hermitian structure), then the Hermitian structure around $D_M$ is exactly as in $(\mathbb{D}^s, g^TM^s)$ and $(\mathbb{C}^{\text{rk}(\xi)}, h_{st})$ for a standard Hermitian product $h_{st}$ on $\mathbb{C}^{\text{rk}(\xi)}$. Thus, the result of Müller extends line by line to the case $n = 0$ and $(\xi, h_\xi)$ trivial around the cusps, i.e.

$$\text{Spec}(\Box_{E^\xi,\eta} M) \cap [0, 1/4] \text{ is discrete.} \quad (2.106)$$

Now, let $h_\xi$ be any Hermitian metric on $\xi$. We will prove that there is $k \in \mathbb{N}$ and $F \subset L^2(E^\xi,\eta^n, M)$, codim$F = k$, such that we have

$$\inf_{s \in F} \left\{ \left\langle \Box_{E^\xi,\eta} M, \Box_{E^\xi,\eta} M \right\rangle_{\mathcal{L}^2} / \langle s, s \rangle_{\mathcal{L}^2} \right\} > 0. \quad (2.107)$$

Then, by the min-max theorem (cf. [38, (C.3.3)]), (2.6) and (2.107), we get (2.7).

We choose $\eta \in ]0, 1/2]$ small enough, so that (1.2) is satisfied for any $i = 1, \ldots, m$. For each $i = 1, \ldots, m$, we fix a normal trivialization of $\xi$ over $V_i^M(\eta)$, i.e. a local holomorphic frame $e_1, \ldots, e_{\text{rk}(\xi)}$ of $\xi$ over $V_i^M(\eta)$ as in (2.45). Let $h_\eta^\xi$ be a Hermitian metric on $\xi$ such that it coincides with $h_\xi$ over $M \setminus (\cup_i V_i^M(\eta))$ and over $V_i^M(\eta)$ it is given by (compare with (2.97))

$$h_\eta^\xi((z_i^M)^{-1}(u))(e_i, e_j) = (1 - \psi(|u|^2/\eta))h_\xi((z_i^M)^{-1}(u))(e_i, e_j) + \psi(|u|^2/\eta)\delta_{ij},$$

(2.108)

where $\psi$ is defined in (2.53), $e_i, i = 1, \ldots, \text{rk}(\xi)$ is as in (2.45), and $\delta_{ij}$ is the Kronecker delta symbol. Then $(\xi, h_\eta^\xi)$ is trivial around the cusps, and there is $C > 0$ such that for any $\eta \in ]0, 1/2]$, we have

$$(h_\eta^\xi)^{-1} \frac{\partial h_\eta^\xi}{\partial z_i^M}(z_i^M)^{-1}(u) \leq C|u|. \quad (2.109)$$

We denote by $\Box_{\eta M}^\xi$ the Kodaira Laplacian on $(M, g^TM)$, associated with $h_\eta^\xi$. Then over $V_i^M(\eta)$, we have

$$\left(\Box_{\eta M}^\xi \right)^* = \left(\|d z_i^M\|_{\omega_M} \right)^2 \left( \frac{\partial}{\partial z_i^M} + (h_\eta^\xi)^{-1} \frac{\partial h_\eta^\xi}{\partial z_i^M} \right) \cdot \iota_{\partial/\partial z_i^M}, \quad (2.110)$$

where $\iota$ is the contraction and $\left(\Box_{\eta M}^\xi \right)^*$ is the adjoint of $\Box_{\eta M}^\xi$ with respect to the $L^2$-scalar product induced by $h_\eta^\xi$. By (2.2) and (2.110), we deduce

$$\Box_{\eta M}^\xi - \Box_{\eta M} = \sum_i |z_i^M|^2 (\ln |z_i^M|)^2 \left( (h_\eta^\xi)^{-1} \frac{\partial h_\eta^\xi}{\partial z_i^M} - (h_\eta^\xi)^{-1} \frac{\partial h_\eta^\xi}{\partial z_i^M} \right) \frac{\partial}{\partial z_i^M}. \quad (2.111)$$
We denote by $(\cdot, \cdot)_{L^2_0}$ the $L^2$-scalar induced by $g^{TM}$, $h^\xi$. We fix $\eta > 0$ small enough so that $2h^\xi > h^\xi_\eta > h^\xi/2$. Then we have $2(\cdot, \cdot)_{L^2_0} > (\cdot, \cdot)_{L^2_0} > (\cdot, \cdot)_{L^2/2}$. Now, by (2.111) and the Cauchy inequality, for $s \in \mathcal{C}^\infty_c(M, E^{\xi,n}_M)$, as the support of (2.111) lies in $\cup V^\ell_i M(\eta^{1/2}/2)$, by (2.109):

\[
(\Box^{E^{\xi,n}_M} s, s)_{L^2_0} \geq \frac{1}{2} (\Box^{E^{\xi,n}_M} s, s)_{L^2_0} - 2Cm|\eta|^2 \ln |\eta| \langle (s, s)_{L^2_0} \cdot (\Box^{E^{\xi,n}_M} s, s)_{L^2_0} \rangle^{1/2}. \tag{2.112}
\]

We fix $\eta > 0$ small enough so that $4Cm|\eta|^2 \ln |\eta| \leq 1/16$, and put

\[
F := \left\{ s \in \text{dom}(\Box^{E^{\xi,n}_M}) : \Box^{E^{\xi,n}_M} s = \lambda s, \quad \lambda < 1/4 \right\}, \tag{2.113}
\]

where the orthogonal complement is taken with respect to $(\cdot, \cdot)_{L^2_0}$. Since $(\xi, h^\xi)$ is trivial around the cusps, by (2.106), the space $F$ is of finite codimension. By (2.112) and (2.113), for $s \in F$:

\[
\frac{(\Box^{E^{\xi,n}_M} s, s)_{L^2_0}}{(s, s)_{L^2_0}} \geq 1 + \frac{1}{4} \left( \frac{(\Box^{E^{\xi,n}_M} s, s)_{L^2_0}}{(s, s)_{L^2_0}} \right)^{1/2} \left( \frac{\langle (\Box^{E^{\xi,n}_M} s, s)_{L^2_0} \rangle^{1/2}}{(s, s)_{L^2_0}} - \frac{1}{4} \right) \geq \frac{1}{32}. \tag{2.114}
\]

Also, by the Cauchy inequality, we have

\[
\left( \frac{(\Box^{E^{\xi,n}_M} s, \Box^{E^{\xi,n}_M} s)_{L^2_0}}{(s, s)_{L^2}} \right)^{1/2} \geq \frac{(\Box^{E^{\xi,n}_M} s, s)_{L^2_0}}{(s, s)_{L^2_0}}. \tag{2.115}
\]

Then (2.114) and (2.115) imply (2.107), and thus (2.7) holds for $n = 0$ and any $(\xi, h^\xi)$.

We remark that similarly to (2.112), we have

\[
(\Box^{E^{\xi,n}_M} s, s)_{L^2_0} \geq \frac{1}{2} (\Box^{E^{\xi,n}_M} s, s)_{L^2} - 2m|\eta|^2 \ln |\eta| \langle (s, s)_{L^2} \cdot (\Box^{E^{\xi,n}_M} s, s)_{L^2} \rangle^{1/2}. \tag{2.116}
\]

From (2.107) and (2.116), in a similar fashion as we got (2.107), we deduce that there exists $\mu > 0$ such that for any $\eta$ small enough, we have

\[
\text{Spec} \left( \Box^{E^{\xi,n}_M} \cap [0, \mu] \right) = \emptyset. \tag{2.117}
\]

Now let’s show that (2.7) holds for $n < 0$ and any $(\xi, h^\xi)$. Similarly, we prove that there are $k \in \mathbb{N}$, $F \subset L^2(E^{\xi,n}_M)$, $\text{codim} F = k$ satisfying (2.107). Then, as before, we would get (2.7).

Let $\eta_0 > 0$ be chosen such that $g^{TM}$ is induced by (1.2) over $\cup_i V_i^\ell M(\eta_0)$, and

\[
|\sqrt{-1} R^\xi, \Lambda^{TM}| \leq 1/4, \quad \text{over } \cup_i V_i^\ell M(\epsilon_0), \tag{2.118}
\]

where $R^\xi$ is the curvature of the Chern connection on $(\xi, h^\xi)$, and $\Lambda^{TM}$ is the contraction with the Hermitian norm induced by $g^{TM}$. Such $\epsilon_0$ exists since $(\xi, h^\xi)$ is a Hermitian vector bundle over $\bar{M}$ and $\Lambda^{TM} = O(|z^i_M \ln |z^i_M|^2|)dz^i_M d\bar{z}^i_M$ can be made arbitrarily small by replacing $\epsilon_0$ by a smaller number.
Let $\rho : \overline{M} \to [0, 1]$ be a smooth cut-off function satisfying

$$
\rho(x) = \begin{cases} 
1 & \text{for } x \in \bigcup_i V_i^M(\epsilon_0/2), \\
0 & \text{for } x \in M \setminus (\bigcup_i V_i^M(\epsilon_0)). 
\end{cases}
$$

(2.119)

For $s \in \mathcal{C}^\infty_c(M, E_M^{\xi,n})$, we have

$$
\langle \Box^{E_M^n} s, s \rangle_{L^2} = \langle \Box^{E_M^n} (\rho s), \rho s \rangle_{L^2} \\
+ \langle \Box^{E_M^n} ((1 - \rho)s), (1 - \rho)s \rangle_{L^2} + 2 \langle \Box^{E_M^n} (\rho s), (1 - \rho)s \rangle_{L^2}.
$$

(2.120)

Trivially, we have

$$
||\langle \Box^{E_M^n} (\rho s), (1 - \rho)s \rangle_{L^2}|| \leq ||\rho(\Box^{E_M^n} s), (1 - \rho)s \rangle_{L^2}|| + ||\Box^{E_M^n} (\rho s), (1 - \rho)s \rangle_{L^2}||.
$$

(2.121)

Since $[\Box^{E_M^n}, \rho]$ is a differential operator of order 1 with support in a compact subspace of $M$, there is $C > 0$ such that for any $s \in \mathcal{C}^\infty_c(M, E_M^{\xi,n})$, we have

$$
||\Box^{E_M^n} (\rho s) s \rangle_{L^2} ||^2 \leq C \left( ||\Box^{E_M^n} s \rangle_{L^2} ||^2 + ||s||_{L^2}^2 \right).
$$

(2.122)

By (2.22) and Cauchy inequality, there is $c_2 > 0$ such that for any $\epsilon > 0$, we have

$$
||\Box^{E_M^n} (\rho s) s \rangle_{L^2} || \leq \epsilon \left( ||\Box^{E_M^n} s \rangle_{L^2} ||^2 + ||s||_{L^2}^2 \right) + \left( c_2/\epsilon \right) ||(1 - \rho)s \rangle_{L^2}^2,
$$

(2.123)

Thus, by (2.120), (2.121) and (2.123), we see that

$$
\langle \Box^{E_M^n} s, s \rangle_{L^2} + (2 + 2\epsilon) \langle \Box^{E_M^n} s \rangle_{L^2}^2 \geq \langle \Box^{E_M^n} (\rho s), \rho s \rangle_{L^2} \\
+ \langle \Box^{E_M^n} ((1 - \rho)s), (1 - \rho)s \rangle_{L^2} - 2\epsilon ||s||_{L^2}^2 - (2 + 2c_2/\epsilon) ||(1 - \rho)s \rangle_{L^2}^2.
$$

(2.124)

Recall that by Nakano’s inequality (cf. [38, Theorem 1.4.14]), we have

$$
\langle \Box^{E_M^n} (\rho s), \rho s \rangle_{L^2} \geq \langle [\sqrt{-1} R^{E_M^n}, \Lambda^{TM}](\rho s), \rho s \rangle_{L^2},
$$

(2.125)

where $R^{E_M^n}$ is the curvature of the Chern connection on $E_M^{\xi,n}$. We decompose

$$
R^{E_M^n} = R^n + n\text{Id} \cdot R^{oM(D)},
$$

(2.126)

where $R^{oM(D)}$ is the curvature of the Chern connection on $(oM(D), ||-||M)$. Now, by (1.2), over $V_i^M(\eta_0)$, we have

$$
[\sqrt{-1} R^{oM(D)}, \Lambda^{TM}] = -1/2.
$$

(2.127)

We conclude by (2.118), (2.125), (2.126) and (2.127) that for $d := -n/2 - 1/4 > 0$, we have

$$
\langle \Box^{E_M^n} (\rho s), \rho s \rangle_{L^2} \geq d ||\rho s||_{L^2}^2.
$$

(2.128)
As the closure of \( M \setminus (\cup_i V_i^M(\epsilon)) \) is a compact manifold with boundary, the Dirichlet problem for \( \square^{E^\epsilon_n}_M \) on \( M \setminus (\cup_i V_i^M(\epsilon)) \) has a discrete set of eigenvalues. Let \( \phi_1, \phi_2, \ldots \) be the eigenvectors corresponding to the eigenvalues in the increasing order. There exists \( k \in \mathbb{N} \) such that for any \( s \), satisfying \( s \perp (1 - \rho)\phi_i, i = 1, \ldots, k \), we have

\[
\langle \square^{E^\epsilon_n}_M ((1 - \rho)s), (1 - \rho)s \rangle_{L^2} \geq (2 + d + 2c_2/\epsilon)|| (1 - \rho)s ||^2_{L^2}.
\] (2.129)

Thus, we conclude from (2.124), (2.128) and (2.129) that for any \( \epsilon > 0 \) there are \( c_1, c_2 > 0, k \in \mathbb{N} \) such that for any \( s \) satisfying \( s \perp (1 - \rho)\phi_i, i = 1, \ldots, k \), we have

\[
\langle \square^{E^\epsilon_n}_M s, s \rangle_{L^2} + (2 + 2c_1/\epsilon)\| \square^{E^\epsilon_n}_M s \|_{L^2}^2 \geq (d/2 - 4\epsilon)||s||^2_{L^2}.
\] (2.130)

We take \( \epsilon = d/16 \) and set \( F = ((1 - \rho)\phi_1, \ldots, (1 - \rho)\phi_k)^\perp \), where the orthogonal complement is taken with respect to the \( L^2 \)-scalar product. Then we deduce (2.107) from (2.115) and (2.130). \( \square \)

We recall that the function \( \rho_M : M \to [1, +\infty[ \) was defined in (2.11). To prove Propositions 2.4, 2.6, we need the following technical

**Lemma 2.28.** For any \( \alpha > 0 \), \( k \in \mathbb{N} \), there is \( C > 0 \), such that for any \( n \in \mathbb{Z}, \sigma \in C^\infty(M, E^\epsilon_n)_M, \) \( x \in M \), we have

\[
\| \nabla^k \sigma(x) \|_h \leq C \rho_M(x) \sum_{i=0}^{2k} (n^{4(2k-i)} + 1) \| (\square^{E^\epsilon_n}_M)^i \sigma \|_{L^2(B_M(x, \alpha))}.
\] (2.131)

**Proof.** Let \( \epsilon > 0 \). For \( x \in M \setminus (\cup_i V_i^M(\epsilon)) \), the estimate (2.131) follows from [38, Lemma 1.6.2]. For \( x \in V_i^M(\epsilon) \), the estimate (2.131) follows from Lemma 2.23. \( \square \)

To prove Proposition 2.4, we need the following

**Lemma 2.29.** Let \( f(t), t > 0 \) be a semigroup of operators acting on \( L^2(E^\epsilon_n)_M \) with smooth kernels \( f(t, x, y), x, y \in M \) associated with \( dV_M(y) \). Suppose that for any \( l, l', l'' \in \mathbb{N} \), there are some \( t_0 > 0, c', C_1 > 0 \), such that for any \( t \in [0, t_0] \), \( x, y \in M \), we have

\[
\left| (\nabla_x)^l (\nabla_y)^{l'} (\partial_y)^{l''} f(t, x, y) \right|_{h \times h} \leq C t^{1-(l+l')/2-l''} \rho_M(x)\rho_M(y) \exp(-c'd(x, y)^2/t).
\] (2.132)

Then there are \( c, C > 0 \) such that for any \( t > 0, x, y \in M \), we have

\[
\left| (\nabla_x)^l (\nabla_y)^{l'} (\partial_y)^{l''} f(t, x, y) \right|_{h \times h} \leq C t^{1-(l+l')/2-l''} \rho_M(x)\rho_M(y) \exp(ct - c'd(x, y)^2/t).
\] (2.133)

**Proof.** There are essentially three different cases to consider \( x, y \in M \setminus (\cup_i V_i^M(1/2)) \), \( x \in V_i^M(1/2), y \in V_j^M(1/2) \) for some \( i \neq j \) and \( x, y \in V_i^M(1/2) \) for some \( i = 1, \ldots, m \). We only treat the last one, which is the most difficult, and we leave the rest to the reader.

We denote \( u = z_i^M(x), v = z_i^M(y) \). Let’s prove by induction that there exists \( c, C > 0 \) such that for any \( k \in \mathbb{N}, t < 2^k t_0 \), we have
\[
\left| (\nabla_u) (\nabla_v)^{l''} (\partial_t)^{l'''} f(t, u, v) \right|_{h \times h} \leq C t^{\frac{1}{2} - (l' + l'' + l''') / 2} (1 + |\ln|u||) / 2 (1 + |\ln|v||) / 2 \\
\cdot \exp \left( (c / 2 - k) - \frac{c'}{t} \cdot d(u, v)^2 \right).
\]

(2.134)

Now, for \( k = 0 \), (2.134) is simply (2.132). Once the induction step is done, (2.134) would imply (2.133). For simplicity, we treat the case \( l = l' = l'' = 0 \), as the generalization is straightforward.

Let \( k \in \mathbb{N} \) and \( 2^{k-1} t_0 \leq t < 2^k t_0 \), then by the semigroup property, we have

\[
\left| f(2t, u, v) \right|_{h \times h} \leq \int_M \left| f(t, u, z) \right|_{h \times h} |(f(t, z, v))_{h \times h} | d v_M(z).
\]

(2.135)

Without losing the generality, suppose \(|u| \leq |v|\). We decompose the integration over \( M \) into four parts: over \( V_i^M(|u|) \), over \( V_i^M(|v|) \setminus V_i^M(|u|) \), over \( V_i^M(1/2) \setminus V_i^M(|v|) \) for \( i = 1, \ldots, m \), and over \( M \setminus (\cup V_i^M(1/2)) \). We will suppose that \(|u|\) is small enough, as if it is not, then the treatment of all those cases reduces to the last one, which is the easiest one. Before treating those cases, let’s recall some facts about the geometry of \( (\mathbb{D}^*, \mathbb{D}^T \mathbb{D}^*) \) and the induced \( S^1 \)-action by rotations. First of all, by (1.2), for any \( u_0 \in V_i^M(1/2) \), the length of the \( S^1 \)-orbit of \( u_0 \) is given by \( 2\pi / |\ln|u_0||\). Thus, by triangle inequality and \( S^1 \)-symmetry, for any \( u \in V_i^M(1/2) \), we have

\[
d_{\mathbb{D}^*}(|u_0|, |u_1|) \leq d_{\mathbb{D}^*}(u_0, u_1) \leq d_{\mathbb{D}^*}(|u_0|, |u_1|) + \min \left\{ \frac{2\pi}{|\ln|u_0||}, \frac{2\pi}{|\ln|u_1||} \right\}.
\]

(2.136)

In (2.136) and after, for \( u \in \mathbb{D}^* \), we interpret \(|u| \in \mathbb{R}^* \) as an element in \( \mathbb{D}^* \), given by the standard inclusion \( \mathbb{R}^* \subset \mathbb{C}^* \). Also, by a trivial calculation, we have

\[
d_{\mathbb{D}^*}(|u_0|, |u_1|) = |\ln |u_0|| - |\ln |u_1||.
\]

(2.137)

Let \( z \in V_i^M(1/2) \), by abuse of notation, we denote \( z := z_i^M(z) \).

Let’s treat the integration over \(|z| < |u|\). By (2.136), we have

\[
d_{\mathbb{D}^*}(z, v) \geq d_{\mathbb{D}^*}(|u|, |v|) \geq d_{\mathbb{D}^*}(u, v) - \frac{2\pi}{|\ln|u||}.
\]

(2.138)

By (2.137) and (2.138), since \( u \) is small enough, we deduce

\[
d_{\mathbb{D}^*}(z, v)^2 \geq d_{\mathbb{D}^*}(|u|, |v|)^2 \geq d_{\mathbb{D}^*}(u, v)^2 - 4\pi.
\]

(2.139)

From the induction hypothesis (2.134), (2.138) and (2.139), we deduce

\[
\int_{|z| < |u|} \left| f(t, u, z) \right|_{h \times h} |(f(t, z, v))_{h \times h} | d v_M(z) \leq 2C^2 t^{-2} (1 + |\ln|u||)^{1/2} \\
\cdot (1 + |\ln|v||)^{1/2} \exp \left( (c / 2 - 2(k - 1) + \frac{4\pi c'}{t} - \frac{c'}{t} d_{\mathbb{D}^*}(u, v)^2) \right) \\
\cdot \int_{|z| < |u|} \exp \left( - \frac{c'}{t} d_{\mathbb{D}^*}(|u|, |z||)^2 \frac{1}{|z|^2} \ln |z|| \right) d z d z.
\]

(2.140)

Now, by (2.137), there exists \( C_2 > 0 \) such that for any \( t > 0 \), we have
\[
\int_{|z|<|u|} \exp \left( - \frac{c'}{t} d_{\mathbb{D}^*}(|u|, |z|)^2 \right) \frac{\sqrt{-1}dzd\overline{z}}{|z|^2 \ln |z|} = 4\pi \int_0^\infty \exp \left( - \frac{c'}{t} r^2 \right) dr \leq C_2 \sqrt{t}.
\]
(2.141)

From (2.140) and (2.141), we deduce
\[
\int_{|z|<|u|} \left| f(t, u, z) \right|_{h \times h} \cdot \left| f(t, z, v) \right|_{h \times h} dv_M(z)
\leq 2C^2 C_2 \exp(4\pi^2 c'/t) t^{-3/2} \left( 1 + |\ln |u|| \right)^{1/2} \left( 1 + |\ln |v|| \right)^{1/2} \left( 2 \cdot 2^{k-1} - 2(k-1) \right) \leq C_2 \sqrt{t}.
\]
(2.142)

Thus, by choosing $c$, $C$ appropriately, by using the bounds on $t$, we bound the contribution from the integral over $|u| < |z| < |v|$ by the right-hand side of (2.134).

Now let’s treat the integral over $|u| < |z| < |v|$. From (2.136), (2.137) and the boundness of the Gaussian integral, for some $C > 0$, we deduce
\[
\int_{|u|<|z|<|v|} \exp \left( - \frac{c'}{t} \left( d_{\mathbb{D}^*}(|u|, z)^2 + d_{\mathbb{D}^*}(|v|, z)^2 \right) \right) \frac{\sqrt{-1}dzd\overline{z}}{|z|^2 \ln |z|} \leq 2\pi \int_{\ln |\ln |u||}^{\ln |\ln |v||} \exp \left( - \frac{c'}{t} \left( (y - \ln |\ln |u||)^2 + (\ln |\ln |v|| - y)^2 \right) \right) dy
\leq 2\pi \int_0^{d_{\mathbb{D}^*}(|u|, |v|)/2} \exp \left( - \frac{c'}{t} \left( d_{\mathbb{D}^*}(|u|, |v|) - r \right)^2 \right) dr
\leq C \sqrt{t} \exp \left( - \frac{c'}{2t} d_{\mathbb{D}^*}(|u|, |v|)^2 \right). \quad (2.143)
\]

From the induction hypothesis (2.134), (2.139) and the bounds on $t$, we bound the contribution of the integration over $|u| < |z| < |v|$ by the right-hand side of the induction step (2.134).

The integral over $|v| < |z| < 1/2$ is treated similarly to the integral over $|z| < |u|$. The integral over $z \in M \setminus \bigcup_i V_i M (1/2)$ is the easiest one and it follows from (2.72).

\[ \square \]

**Proof of Proposition 2.4.** Let’s prove (2.13) first. From Lemma 2.28, there is $C > 0$, such that for any $x$, $x' \in M$, we have
\[
\left| (\nabla_x)^I (\nabla_x)^{I'} \exp \left( -t \Box^{E^{e,n}_M} \right)(x, x') \right|_{h \times h} \leq C \rho_M(x) \rho_M(x') \cdot \sum_{i=0}^{2+l} \sum_{j=0}^{2+l'} \left\| (\Box^{E^{e,n}_M})^i \exp \left( -t \Box^{E^{e,n}_M} \right)(\Box^{E^{e,n}_M})^j \right\|^{0.0}, \quad (2.144)
\]
where $\left\| \cdot \right\|_{0.0}$ is the operator norm between the corresponding $L^2$ spaces. For any $l \in \mathbb{N}$, $c > 0$, there is $C > 0$ such that for any $t > 0$, we have
\[
\sup_{u \geq c} u^l \exp(-tu) \leq C t^{-l} \exp(-ct/2). \quad (2.145)
\]
By Theorem 2.1, for any $i$, $j \in \mathbb{N}$, there are $c$, $C > 0$ such that for any $t > 0$, we have
From (2.144) and (2.146), we get (2.13).

Let’s proceed with a proof of (2.12). By Lemma 2.29, it’s enough to prove it for \( t < t_0 \) for some \( t_0 > 0 \). We fix \( \varepsilon > 0 \) small enough, and consider several cases.

Case 1: \( x, x' \in M \setminus (\cup_i V_i^M(\varepsilon)) \). The estimate (2.12) for small \( t \) is classical and it is proved by using finite propagation speed of solutions of hyperbolic equations (cf. [38, Theorems D.2.1, 4.2.8]) and the parametrix estimates of the heat kernel similar to [7, §2.4, 2.5].

Case 2: \( x \in V_i^M(\varepsilon), x' \notin V_i^M(2\varepsilon) \), for some \( i = 1, \ldots, m \). In this case, we prove the estimate (2.12) for \( t < t_0 \) by using finite propagation speed of solutions of hyperbolic equations.

More precisely, for \( r > 0 \), we introduce smooth even functions (cf. [38, (4.2.11)])

\[
K_{t,r}(a) = \int_{-\infty}^{+\infty} \exp(\sqrt{-1}v\sqrt{2ta}) \exp(-\frac{v^2}{2})(1 - \psi\left(\frac{\sqrt{2}tv}{r}\right)) \frac{dv}{\sqrt{2\pi}},
\]

\[
G_{t,r}(a) = \int_{-\infty}^{+\infty} \exp(\sqrt{-1}v\sqrt{2ta}) \exp(-\frac{v^2}{2})\psi\left(\frac{\sqrt{2}tv}{r}\right) \frac{dv}{\sqrt{2\pi}},
\]

where \( \psi : \mathbb{R} \to [0,1] \) was defined in (2.53). Let \( \widetilde{K}_{t,r}, \widetilde{G}_{t,r} : \mathbb{R}_+ \to \mathbb{R} \) be the smooth functions given by \( \widetilde{K}_{t,r}(a^2) = K_{t,r}(a), \widetilde{G}_{t,r}(a^2) = G_{t,r}(a) \). Then the following identities hold

\[
\exp(-t\Box^{E^n}_M) = \widetilde{G}_{t,r}(\Box^{E^n}_M) + \widetilde{K}_{t,r}(\Box^{E^n}_M).
\]

By the finite propagation speed of solutions of hyperbolic equations (cf. [38, Theorems D.2.1, 4.2.8]), the section \( \widetilde{G}_{t,r}(\Box^{E^n}_M)(y, \cdot), y \in M \), depends only on the restriction of \( \Box^{E^n}_M \) onto \( B^M(y, r) \). Moreover, we have

\[
\text{supp} \ \widetilde{G}_{t,r}(\Box^{E^n}_M)(y, \cdot) \subset B^M(y, r).
\]

From (2.148) and (2.149), we get

\[
\exp(-t\Box^{E^n}_M)(y, z) = \widetilde{K}_{t,r}(\Box^{E^n}_M)(y, z) \quad \text{if} \quad d(y, z) > r.
\]

From (2.147), for any \( r_0 > 0 \) fixed, there exists \( c' > 0 \) such that for any \( m \in \mathbb{N} \), there is \( C > 0 \) such that for any \( t \in [0, 1], r > r_0, a \in \mathbb{R} \), the following inequality holds (cf. [38, (4.2.12)])

\[
|a|^m |K_{t,r}(a)| \leq C \exp(-c' r^2 / t).
\]

Thus, by (2.151), for \( t \in [0, 1], r > r_0, a \in \mathbb{R}_+ \), we have

\[
|a|^m |\widetilde{K}_{t,r}(a)| \leq C \exp(-c' r^2 / t).
\]

Now, by (2.152), there exists \( c' > 0 \) such that for any \( k, k' \in \mathbb{N} \), there is \( C > 0 \) such that for any \( t \in [0, 1] \) and \( r > r_0 \), we have

\[
|||((\Box^{E^n}_M)^k \widetilde{K}_{t,r}(\Box^{E^n}_M)(\Box^{E^n}_M)^{k'})^{0,0} \leq C \exp(-c' r^2 / t),
\]

where \( ||| \cdot |||^{0,0} \) is the operator norm between the corresponding \( L^2 \)-spaces. Thus, by Lemma 2.28, for any \( l, l' \in \mathbb{N} \), there are \( c', C > 0 \) such that for any \( x, x' \in M, r > r_0 \), we have
We get (2.12) from (2.150) and (2.154) by taking \( r_0 = \frac{1}{4} d(V_i^M(\epsilon), M \setminus V_i^M(2\epsilon)) \) and \( r = \frac{1}{2} d(x, y) \).

**Case 3:** \( x, x' \in V_i^M(2\epsilon) \) for some \( i = 1, \ldots, m \). In this case, we prove the estimate (2.12) for \( t < t_0 \) by (2.73) and by finite propagation speed of solutions of hyperbolic equations.

We choose a holomorphic trivialization \( e_1, \ldots, e_{\text{rk} (\xi)} \) of \( \tilde{V}_i^M(\epsilon) \). From the fact that the restriction of  is trivial away from a compact set, and by abuse of notation, we denote the resulting Hermitian metric by \( h_0^\epsilon \).

We choose a holomorphic trivialization \( e_1, \ldots, e_{\text{rk} (\xi)} \) of \( \tilde{V}_i^M(\epsilon) \). From the fact that the restriction of  is trivial away from a compact set, and by abuse of notation, we denote the resulting Hermitian metric by \( h_0^\epsilon \).

We denote \( u := z_i^M(x), u' := z_i^M(x'), r := d_{\mathbb{D}^n}(u, 2\epsilon) \). Without losing the generality, we suppose \( r < d_{\mathbb{D}^n}(u, 2\epsilon) \).

By (2.136) and (2.137), for some \( c > 0 \), we have

\[
d_{\mathbb{D}^n}(u, 2\epsilon) \geq \ln |\ln |u|| - c. \tag{2.155}
\]

From the fact that the restriction of \( \Box_{V_i^M}^{E_0^\epsilon, n} \) onto \( B^M(x, r) \) coincides with the restriction of \( \Box_{\mathbb{D}^n}^{E_0^\epsilon, n} \) onto \( B(0, r) \), by finite propagation speed of solutions of hyperbolic equations, we have

\[
\tilde{G}_{t, r}(\Box_{V_i^M}^{E_0^\epsilon, n})(x, x') = \tilde{G}_{t, r}(\Box_{\mathbb{D}^n}^{E_0^\epsilon, n})(u, u'), \tag{2.156}
\]

for \( E_0^\epsilon, n := \tilde{V}_i^M(\epsilon) \otimes \omega_{\mathbb{D}^n}(D)^n \). Now, from (2.148) and (2.156), we get

\[
\exp(-t\Box_{V_i^M}^{E_0^\epsilon, n})(x, x') = \exp(-t\Box_{\mathbb{D}^n}^{E_0^\epsilon, n})(u, u') = \tilde{K}_{t, r}(\Box_{V_i^M}^{E_0^\epsilon, n})(x, x') - \tilde{K}_{t, r}(\Box_{\mathbb{D}^n}^{E_0^\epsilon, n})(u, u'). \tag{2.157}
\]

Now, we conclude by (2.73), (2.154), (2.155) and (2.157). \( \square \)

**Proof of Proposition 2.6.** First of all, in the case when \( (\xi, h^\epsilon) \) is trivial around the cusps, by choosing \( \epsilon \) small enough in Case 3 of the proof of Proposition 2.4, we see that the Hermitian vector bundle \( (\xi_0, h_0^\epsilon) \) becomes trivial. Thus, (2.17) follows from (2.154), (2.157).

**Now let’s prove the estimates** (2.14), (2.15). Consider a family of Hermitian metrics \( h_\epsilon^\epsilon, \epsilon \in [0, 1] \) on \( \xi \) such that they coincide with \( h_0^\epsilon \) over \( M \setminus (\cup_i V_i^M(1/2)) \) and over \( V_i^M(1/2), \) we have

\[
h_\epsilon^\epsilon((z_i^M)^{-1}(u))(e_i, e_j) := (1 - \epsilon \psi(4|u|^2))h_0^\epsilon((z_i^M)^{-1}(u))(e_i, e_j) + \epsilon \psi(4|u|^2)\delta_{ij}, \tag{2.158}
\]

where \( \psi \) is defined in (2.53), \( e_i, i = 1, \ldots, \text{rk}(\xi) \) is as in (2.45), and \( \delta_{ij} \) is the Kronecker delta symbol. We denote by \( \Box_{V_i^M}^{E_0^\epsilon, n} \) the Kodaira Laplacian on \( (M, g^T M) \), associated with \( h_\epsilon^\epsilon \otimes |\cdot|_M^n \). Then we have (2.111) for \( \eta = \epsilon \). Moreover, (2.109) still holds uniformly on \( \eta := \epsilon \in [0, 1] \). By Duhamel’s formula (cf. [7, Theorem 2.48]), there exists \( \epsilon_0 > 0 \) such for any \( u \in V_i^M(\epsilon_0) \), we have
\[
\partial_\epsilon \exp(-t \Box_{\epsilon}^{E_{n}}) (u, u) = -\int_0^t \int_{v \in M} \exp(-(t - s) \Box_{\epsilon}^{E_{n}}) (u, v) \cdot \left( \partial_\epsilon \left( \Box_{\epsilon}^{E_{n}} \right) v \exp(-s \Box_{\epsilon}^{E_{n}}) (v, u) \right) dv_M (v) ds.
\]

(2.159)

Now, the operator (2.111) has support over \( V_i^M (1/2) \), thus, the integration in (2.159) is done only over \( V_i^M (1/2) \). Using the coordinate function \( \epsilon_i^M \), we identify \( V_i^M (1/2) \) with \( D^*(1/2) \subset \mathbb{D} \). Now, since the family of Hermitian metrics (2.158) is smooth, the estimate (2.12) holds uniformly in \( \epsilon \), and by (2.12), (2.109), (2.111), there is \( C > 0 \) such that

\[
\left| \partial_\epsilon \exp(-t \Box_{\epsilon}^{E_{n}}) (u, u) \right| \leq C \left( 1 + |\ln |u|| \right) \exp(ct) \int_0^t \int_{v \in D^*(\frac{1}{2})} |v|(1 + |\ln |v||) \frac{1}{t - s} \cdot \frac{1}{s^{3/2}} \cdot \exp \left( - \frac{d(u, v)^2}{4} (s^{-1} + (t - s)^{-1}) \right) dv_{D^*} (v) ds.
\]

(2.160)

For \( r \in \mathbb{R}_+ \), we decompose

\[
\int_{v \in D^*(\frac{1}{2})} |v|(1 + |\ln |v||) \exp \left( - \frac{d_{D^*}(u, v)^2}{4} (s^{-1} + (t - s)^{-1}) \right) dv_{D^*} (v)
= \int_{v \in B_{D^*}(u, r) \cap D^*(\frac{1}{2})} + \int_{v \in D^*(\frac{1}{2}) \setminus B_{D^*}(u, r)}.
\]

(2.161)

Since for \( \tilde{u} \in \mathbb{H} \), \( \rho(\tilde{u}) = u \), the restriction \( \rho_{B_{\mathbb{H}}(\tilde{u}, r)} : B_{\mathbb{H}}(\tilde{u}, r) \rightarrow B_{\mathbb{D}}(u, r) \) of the covering \( \rho \) from Sect. 2.3 is a surjection, which reduces the distances, we have

\[
\int_{v \in B_{\mathbb{D}}(u, r)} \exp \left( - \frac{d_{D^*}(u, v)^2}{4} (s^{-1} + (t - s)^{-1}) \right) dv_{D^*} (v)
\leq \int_{\tilde{v} \in B_{\mathbb{H}}(\tilde{u}, r)} \exp \left( - \frac{d_{\mathbb{H}}(\tilde{u}, \tilde{v})^2}{4} (s^{-1} + (t - s)^{-1}) \right) dv_{\mathbb{H}} (\tilde{v}).
\]

(2.162)

However, since \( (\mathbb{H}, g^{\mathbb{H}}) \) is isometrically transitive, the right-hand side of (2.162) doesn’t depend on \( \tilde{u} \), i.e. it is a function of \( r > 0 \). Thus, in further estimation of the right-hand side of (2.162), we may suppose that \( \tilde{u} = \sqrt{-1} \).

Now let’s take \( r = 1 \). Over \( B_{\mathbb{H}}(\sqrt{-1}, 1) \), the metric \( g^{\mathbb{H}} \) is equivalent to the standard Euclidean metric. Thus, by the Gaussian integral on \( \mathbb{C} \), for some \( C > 0 \), we have

\[
\int_{\tilde{v} \in B_{\mathbb{H}}(\sqrt{-1}, 1)} \exp \left( - \frac{d_{\mathbb{H}}(\sqrt{-1}, \tilde{v})^2}{4} (s^{-1} + (t - s)^{-1}) \right) dv_{\mathbb{H}} (\tilde{v}) \leq \frac{C}{s^{-1} + (t - s)^{-1}}.
\]

(2.163)

Now, there is \( C > 0 \) such that

\[
\int_{v \in D^*(\frac{1}{2}) \setminus B_{D^*}(u, 1)} \exp \left( - \frac{d_{D^*}(u, v)^2}{4} (s^{-1} + (t - s)^{-1}) \right) dv_{D^*} (v)
\leq \int_{v \in D^*(\frac{1}{2})} \exp \left( - (s^{-1} + (t - s)^{-1})/4 \right) dv_{D^*} (v)
\leq C \exp \left( - (s^{-1} + (t - s)^{-1})/4 \right),
\]

(2.164)
where in the last line we used the fact that the volume of $D^*(1/2)$ is finite. By (2.161), (2.162), (2.163), (2.164), and by the fact that from (1.2) and (2.137), for $v \in B^{D^*}(u, 1)$, we have $|v| \leq |u|^{1/\epsilon}$, we deduce that there are $c, C > 0$ such that

$$\int_{v \in D^*(\frac{1}{2})} |v|(1 + |\ln |v||) \exp \left( - \frac{d_{D^*}(u, v)^2}{4} (s^{-1} + (t - s)^{-1}) \right) dv_{D^*}(v)$$

$$\leq C |u|^{1/\epsilon} |\ln |u|| \frac{s^{-1} + (t - s)^{-1}}{} + C \exp \left( - c(s^{-1} + (t - s)^{-1}) \right).$$

(2.165)

From (2.160) and (2.165), we get (2.15).

**Now let’s prove (2.14).** Now let’s fix $k \in \mathbb{N}$ and take $r = d_{D^*}(|u|, |\ln |u||)^{-k}$. By (2.137):

$$r = - \int_{|u|} |\ln |u||^{-k} \frac{dr}{r |\ln r|} \approx \ln |\ln |u||.$$  

(2.166)

Then by (2.163) and (2.164), as $r \geq 1$, for some $c, C > 0$, we have

$$\int_{v \in B^{D^*}(u, 1) \cap D^*(\frac{1}{2})} \exp \left( - \frac{d_{D^*}(u, v)^2}{4} (s^{-1} + (t - s)^{-1}) \right) dv_{D^*}(v)$$

$$= \int_{v \in B^{D^*}(u, 1) \cap D^*(\frac{1}{2})} + \int_{v \in (B^{D^*}(u, 1) \cap D^*(\frac{1}{2})) \setminus B^{D^*}(u, 1)} \leq C \frac{s^{-1} + (t - s)^{-1}}{s^{-1} + (t - s)^{-1}}$$

$$+ C \exp \left( - c(s^{-1} + (t - s)^{-1}) \right).$$

(2.167)

Also, by (2.166) and the fact that the volume of $D^*(\frac{1}{2})$ is finite, there are $c, C > 0$, such that

$$\int_{v \in D^*(\frac{1}{2}) \setminus B^{D^*}(u, 1)} \exp \left( - \frac{d_{D^*}(u, v)^2}{4} (s^{-1} + (t - s)^{-1}) \right) dv_{D^*}(v)$$

$$\leq C \exp \left( - c(|\ln |\ln |u||)^2(s^{-1} + (t - s)^{-1}) \right).$$

(2.168)

By (2.161), (2.167) and (2.168), we have

$$\int_{v \in D^*(\frac{1}{2})} |v|(1 + |\ln |v||) \exp \left( - \frac{d_{D^*}(u, v)^2}{4} (s^{-1} + (t - s)^{-1}) \right) dv_{D^*}(v)$$

$$\leq C \left( \left( \frac{1}{s^{-1} + (t - s)^{-1}} + \exp \left( - c(s^{-1} + (t - s)^{-1}) \right) \right)(1 + |\ln |\ln |u||)^k \right)$$

$$\cdot |\ln |u||^{-k} + C \exp \left( - c(|\ln |\ln |u||)^2(s^{-1} + (t - s)^{-1}) \right).$$

(2.169)

By (2.160) and (2.169), we get (2.14).

**Now let’s prove the estimate (2.16).** We have the identity

$$\exp(-t \square_{M, E_{M}^{\xi, n}})(x, x') = \exp(-t \square_{M, E_{M}^{\xi, n}})(x, x') + \sum s_i(x)s_i(x')^*,$$

(2.170)

where $s_i$ is an orthonormal basis of $H^0(M, E_{M}^{\xi, n})$ with respect to $\langle \cdot, \cdot \rangle_{L^2}$, see (2.1). From (2.6), (2.14) and (2.170), we conclude that there are $c', C > 0$, such that for any $t > 0$, $u \in D^*(1/2)$:

$$\left| \exp(t \square E_{M}^{\xi, n})(z_i^M)^{-1}(u), (z_i^M)^{-1}(u)) - \exp(-t \square E_{M}^{\xi, n})(z_i^N)^{-1}(u), (z_i^N)^{-1}(u)) \right|$$

$$\leq C \frac{s^{-1} + (t - s)^{-1}}{s^{-1} + (t - s)^{-1}} + C \exp \left( - c(s^{-1} + (t - s)^{-1}) \right).$$
\[ \leq C \exp(ct)\left( |\ln |u|| \exp(-c'(\ln |u|)^2/t) + 1 \right). \] (2.171)

Also, from (2.13), there are \(c', C > 0\), such that for any \(t > 0, u \in D^*(1/2)\), we have
\[ \left| \exp\left(-t\Box_{M}^{\xi,n}(z_i^M)^{-1}(u), (z_i^M)^{-1}(u)\right) - \mathrm{Id}_\xi \cdot \exp(-t\Box_{\xi,n}((z_i^M)^{-1}(u), (z_i^M)^{-1}(u)) \leq C|\ln |u||t^{-4} \exp(-ct). \right. \] (2.172)

By Cauchy inequality, we have
\[ \exp(-ct - c'(|\ln |u||)^2/t) \leq |\ln |u||^{-2\sqrt{cc'}}. \] (2.173)

We get (2.16) by multiplying appropriate powers of (2.171) with (2.172) and using (2.173).

\(\square\)

**Proof of Proposition 2.8.** By finite propagation speed of solutions of hyperbolic equations and small-time asymptotics of the heat kernel in a compact manifold, we get (2.18). Moreover, the constant \(C\) from (2.18) could be chosen independently of \(x \in M \setminus (\bigcup_i V_i^M(\epsilon))\), for some \(\epsilon > 0\).

Now let’s suppose \(x \in V_i^M(\epsilon)\), for some \(i = 1, \ldots, m\). We note \(u = z_i^M(x)\), and we use (2.157) for \(h = d_{\xi,n}(u, 2\epsilon)\). Then by (2.74), (2.154) and (2.157), we see that there are smooth sections \(a_i^{M,n} : M \to \text{End}(\xi)\), as described, and there is \(C > 0\) such that for any \(x \in M, t \in [0, t_0]\):
\[ \left| \exp(-t\Box_{M}^{\xi,n})(x, x) - \sum_{j=-1}^{k} a_i^{M,n}(x) t^j \right| \leq C\rho_M(x)^2 \left( t^k + \frac{1}{t} \exp\left( -\frac{c'}{t |\ln |z_i^M(x)||^2} \right) + \exp\left( -c(|\ln |z_i^M(x)||)^2/t \right) \right). \] (2.174)

and for \(a_i^{\xi,n},\) defined as in Proposition 2.21, we have
\[ a_i^{M,n}(x) = a_i^{\xi,n}(z_i^M(x)). \] (2.175)

From (2.174), we conclude that if \(x \in M \setminus (\bigcup_i V_i^M(\epsilon^{-1/3}))\), then \(C\) in (2.74) can be chosen independently of \(t \in [0, t_0]\) and \(x\).

The statement (2.19) and the boundness of \(a_i^{M,n}(x)\) follows from (2.75) and (2.175).

\(\square\)

### 3. Compact Perturbation of the Cusp: A Proof of Theorem A

In this section we prove Theorem A. The proof consists of two steps. In the first step, Sect. 3.2, we prove that by successive “flattenings” of the Hermitian metric \(h^\xi\), the associated Quillen norm converges to the Quillen norm associated with \(h^\xi\). For this, essentially, we use the estimates developed in Sect. 2.3 along with analytic localization techniques of Bismut–Lebeau [11, §11]. In the second step, Sect. 3.3, we restrict ourselves to the case when \((\xi, h^\xi)\) is trivial near the cusps, and we construct a family of flattenings which “approach” the cusp metric in such a way that the associated analytic
torsion converges. In this step we use the analytic localization techniques of Bismut-Lebeau [11, §11] along with the maximal principle. Finally, as we explain in Sect. 3.1, those two results are enough to give a complete proof of Theorem A. Moreover, as we will see along the way, we actually prove Theorem B for $g_0^T = g^T$, i.e. for the variation of $h^\xi$.

3.1. General strategy of a proof of Theorem A. In this section we describe the main idea of the proof more precisely. Let’s recall the setting of the problem. We fix surfaces with cusps $(\overline{M}, D_M, g^T_M)$, $(\overline{N}, D_N, g^T_N)$, a Hermitian vector bundle $(\xi, h^\xi)$ over $\overline{M}$ and $n \in \mathbb{Z}$ as in the statement of Theorem A. We consider the family of Hermitian metrics $h^\xi_\eta$, $\eta \in ]0, 1/2]$ on $\xi$ constructed in (2.108). The main goal of Sect. 3.2 is to prove the following formula

$$\lim_{\eta \to 0} ||\cdot|| Q(g^T_M, h^\xi_0 \otimes ||\cdot||_M^{2n}) = ||\cdot|| Q(g^T_M, h^\xi \otimes ||\cdot||_M^{2n}).$$

As $h^\xi_0|_{D_M} = h^\xi|_{D_M}$, we see that (3.1) is compatible with Theorem B.

In Sect. 3.3 we construct a family of flattenings $g^{TM}_{1, \theta}$, $||\cdot||_M^{f, \theta}$ such that the corresponding $\nu$ from (1.14) tends to 0, as $\theta \to 0$. We consider the flattenings $g^{TN}_{1, \theta}$, $||\cdot||_N^{f, \theta}$, which are compatible to $g^{TM}_{1, \theta}$, $||\cdot||_M^{f, \theta}$, see (1.15), (1.17). Then we prove that for any Hermitian metric $h^\xi_2$ on $\xi$ over $\overline{M}$, for which $(\xi, h^\xi_2)$ is trivial around the cusps, we have

$$\lim_{\theta \to 0} \frac{||\cdot|| Q(g^{TM}_{1, \theta}, h^\xi_2 \otimes (||\cdot||_M^{f, \theta})^{2n})}{||\cdot|| Q(g^{TN}_{1, \theta}, (||\cdot||_N^{f, \theta})^{2n} \chi(\xi))} = \frac{||\cdot|| Q(g^{TM}, h^\xi \otimes ||\cdot||_M^{2n})}{||\cdot|| Q(g^{TN}, ||\cdot||_N^{2n} \chi(\xi))}.$$  

This is the most technical and challenging part of this section.

Now let’s explain how (3.1) and (3.2) imply Theorem A. Recall that $\tilde{d}$ and $\tilde{\chi}$ are given by (1.21) and (1.22). Let’s recall the following theorem of Bismut-Gillet-Soulé [10, Theorem 1.23]:

**Theorem 3.1.** (Anomaly formula) Let $\overline{M}$ be endowed with two (smooth) metrics $g^{TM}_1$, $g^{TM}_2$ over $\overline{M}$. We denote by $||\cdot||_1^o$, $||\cdot||_2^o$ the Hermitian norms on $\omega_M$ induced by $g^{TM}_1$, $g^{TM}_2$ over $\overline{M}$. Let $\xi$ be a holomorphic vector bundle with Hermitian metrics $h^\xi_1$, $h^\xi_2$ over $\overline{M}$. We have the following identity

$$2 \ln \left( ||\cdot||_Q(g^{TM}_2, h^\xi_2) / ||\cdot||_Q(g^{TM}_1, h^\xi_1) \right) = \int_{\overline{M}} \left[ \tilde{d}(\omega^{-1}_M, (||\cdot||_1^o)^{-2}, (||\cdot||_2^o)^{-2}) \chi(\xi, h^\xi_1) + \tilde{d}(\omega^{-1}_M, (||\cdot||_2^o)^{-2}) \tilde{\chi}(\xi, h^\xi_1, h^\xi_2) \right].$$

Now, by Theorem 3.1, the fact that the flattenings $g^{TM}_{1, \theta}$, $||\cdot||_M^{f, \theta}$ and $g^{TN}_{1, \theta}$, $||\cdot||_N^{f, \theta}$ are compatible, and the fact that $(\xi, h^\xi)$ is trivial around the cusps, we see that the term inside the limit on the left-hand side of (3.2) doesn’t depend on the choice of the flattenings for $\theta$ small enough. Thus, for any $\theta > 0$ such that $(\xi, h^\xi_\eta)$ is trivial over $\bigcup_i V^M_i(\theta)$ (for example, for $\theta^2 < \eta$), by (3.2), we have
\[
\frac{||\cdot||_Q(g^{TM}_{f,\theta}, h_\eta^\xi \otimes (||\cdot||^f_M)^{2n})}{||\cdot||_Q(g^{TN}_{f,\theta}, (||\cdot||^f_M)^{2n})^{rk(\xi)}} = \frac{||\cdot||_Q(g^{TM}, h_\eta^\xi \otimes ||\cdot||^f_M^{2n})}{||\cdot||_Q(g^{TN}, ||\cdot||^f_M^{2n})^{rk(\xi)}}. \tag{3.4}
\]

Now, by Theorem 3.1, for any \(\theta \in [0, 1]\) and any Hermitian metric \(h^\xi\) over \(\xi\), we have

\[
2 \ln \left(\frac{||\cdot||_Q(g^{TM}_{f,\theta}, h_\eta^\xi \otimes (||\cdot||^f_M)^{2n})}{||\cdot||_Q(g^{TN}_{f,\theta}, h_2^\xi \otimes (||\cdot||^f_M)^{2n})}\right) = \int_M \text{td}(\omega_M^{-1}, g^{TM}_{f,\theta}) \tilde{\text{c}}h\left(\xi, h_2^\xi, h_\eta^\xi\right) \text{ch}(\omega_M(D)^n, (||\cdot||^f_M)^{2n}). \tag{3.5}
\]

From (3.4) and (3.5), for any \(\theta^2 < \eta\), we have

\[
2 \ln \left(\frac{||\cdot||_Q(g^{TM}, h_\eta^\xi \otimes ||\cdot||^f_M^{2n})}{||\cdot||_Q(g^{TN}, ||\cdot||^f_M^{2n})^{rk(\xi)}}\right) = \int_M \text{td}(\omega_M^{-1}, g^{TM}_{f,\theta}) \tilde{\text{c}}h\left(\xi, h_2^\xi, h_\eta^\xi\right) \text{ch}(\omega_M(D)^n, (||\cdot||^f_M)^{2n}). \tag{3.6}
\]

Trivially, the following identity holds

\[
\int_M \text{td}(\omega_M^{-1}, g^{TM}_{f,\theta}) \tilde{\text{c}}h\left(\xi, h_2^\xi, h_\eta^\xi\right) \text{ch}(\omega_M(D)^n, (||\cdot||^f_M)^{2n}) = \int_M \text{td}(\omega_M(D)^{-1}, ||\cdot||^{-2}_M) \tilde{\text{c}}h\left(\xi, h_2^\xi, h_\eta^\xi\right). \tag{3.7}
\]

Now, by (1.20), (1.21) and (1.24), we have

\[
\int_M \text{td}(\omega_M^{-1}, g^{TM}) \tilde{\text{c}}h\left(\xi, h_2^\xi, h_\eta^\xi\right) = \int_M \text{td}(\omega_M(D)^{-1}, ||\cdot||^{-2}_M) \tilde{\text{c}}h\left(\xi, h_2^\xi, h_\eta^\xi\right). \tag{3.8}
\]

Now, by (1.20) and Green identities, we have

\[
\int_M \left(\text{td}(\omega_M^{-1}, g^{TM}_{f,\theta}) - \text{td}(\omega_M^{-1}, g^{TM})\right) \tilde{\text{c}}h\left(\xi, h_2^\xi, h_\eta^\xi\right) = \int_M \tilde{\text{c}}d\left(\omega_M^{-1}, g^{TM}_{f,\theta}, g^{TM}\right)\left(c_1(\xi, h_2^\xi) - c_1(\xi, h_\eta^\xi)\right) + \frac{1}{2} \sum \ln \left(\det(h_2^\xi / h_\eta^\xi)|_P^M\right). \tag{3.9}
\]

Similarly, by (1.20), we have

\[
\int_M \tilde{\text{c}}h\left(\xi, h_2^\xi, h_\eta^\xi\right) \left(\text{ch}(\omega_M(D)^n, (||\cdot||^f_M)^{2n}) - \text{ch}(\omega_M(D)^n, ||\cdot||^{2n}_M)\right) = \int_M \left(c_1(\xi, h_2^\xi) - c_1(\xi, h_\eta^\xi)\right) \tilde{\text{c}}h(\omega_M(D)^n, (||\cdot||^f_M)^{2n}, ||\cdot||^{2n}_M). \tag{3.10}
\]
By (3.6)-(3.10), we get

\[
2 \ln \left( \frac{||\cdot||_Q(g^{TM}, h^\xi_\eta \otimes ||\cdot||^2_{M})}{||\cdot||_Q(g^{TM}, h^\xi_\theta \otimes (||\cdot||^2_{M})^2)} \right) - 2r_k(\xi) \ln \left( \frac{||\cdot||_Q(g^{TN}, ||\cdot||^2_N)}{||\cdot||_Q(g^{TN}, (||\cdot||^2_{N})^2)} \right) = \int_M \left( \widetilde{t}d(\omega^{-1}_M, g^{TM}, g^{TM}) + \widetilde{c}(\omega(D)^n, (||\cdot||^2_{M})^2, ||\cdot||^2_{M}) \right) (c_1(\xi, h^n_\xi) - c_1(\xi, h^n_\theta)) \\
+ \int_M \frac{\partial(\omega(D)^{-1}, ||\cdot||^2_M)}{2} \widetilde{c}(\xi, h^n_\xi, h^n_\eta) + \int_M \widetilde{c}(\xi, h^n_\xi, h^n_\eta) ch(\omega(D)^n, ||\cdot||^2_{M}) \\
- \int_M \frac{\partial(\xi, h^n_\xi, h^n_\eta)}{2} + \frac{1}{2} \sum \ln \left( \det(h^n_\xi / h^n_\eta) \right) \right). 
\]

(3.11)

We make \( \theta \to 0 \) in (3.11). By (3.2), the uniform bounds on \( g^{TM}_\theta \) and \( ||\cdot||^2_{M} \) from (3.43), Lebesgue dominated convergence theorem and the fact that the Bott-Chern representatives of Chern and Todd classes appear only in degree 0 in the first term of the right hand side of (3.11), we deduce

\[
2 \ln \left( \frac{||\cdot||_Q(g^{TM}, h^\xi_\eta \otimes ||\cdot||^2_{M})}{||\cdot||_Q(g^{TM}, h^\xi_\theta \otimes (||\cdot||^2_{M})^2)} \right) \\
= \int_M \frac{\partial(\omega(D)^{-1}, ||\cdot||^2_M)}{2} \widetilde{c}(\xi, h^n_\xi, h^n_\eta) + \int_M \widetilde{c}(\xi, h^n_\xi, h^n_\eta) ch(\omega(D)^n, ||\cdot||^2_{M}) \\
- \int_M \frac{\partial(\xi, h^n_\xi, h^n_\eta)}{2} + \frac{1}{2} \sum \ln \left( \det(h^n_\xi / h^n_\eta) \right) \right). 
\]

(3.12)

Now we let \( \eta \to 0 \). Then by (3.1), (3.7), the fact that the first Chern forms of \( (\xi, h^n_\eta) \), \( \eta \in [0, 1] \) are uniformly bounded and by Lebesgue dominated convergence theorem, we get Theorem B for \( g^{TM}_0 = g^{TM} \) and \( h^\xi_0 := h^\xi_2 \), i.e. trivial around the cusps. By applying this result twice for \( h^\xi := h^n_\xi, h^n_0 := h^n_\xi, h^n_0 := h^n_\xi, h^n_0 := h^n_\xi \), and by taking the difference, we get Theorem B for \( g^{TM}_0 = g^{TM} \) and any \( h^n_0 \). By this, Theorem 3.1, (3.4), (3.9) and (3.10) we deduce Theorem A.

3.2. Flattening the Hermitian metric: a proof of (3.1). In this section, we reduce Theorem A to the case where \( (\xi, h^\xi_\eta) \) is trivial near the cusps. For this, we prove (3.1). As we explained in Sect. 3, we consider the family of Hermitian metrics \( h^\xi_\eta, \eta \in [0, 1/2] \) on \( \xi \) constructed in (2.108). We denote by \( \square^{E^\xi_\eta}_M \) the Kodaira Laplacian on \( (M, g^{TM}) \), associated with \( (\xi \otimes \omega(M)^n, h^\xi_\eta \otimes ||\cdot||^2_{M}) \). Similarly, for all the geometric objects we considered before, the subscript \( \eta \) means that instead of \( h^\xi \), we use \( h^\xi_\eta \).

**Proposition 3.2.** For \( n \leq 0 \), there is \( \eta_0 > 0 \) such that the operators \( \square^{E^\xi_\eta}_M, \eta \in [0, \eta_0] \) have a uniform spectral gap near 0, i.e. there is \( \mu > 0 \) such that for any \( \eta \in [0, \eta_0] \), we have

\[
\ker(\square^{E^\xi_\eta}_M) = H^0(M, E^{E^\xi_\eta}_M), \\
\text{Spec}(\square^{E^\xi_\eta}_M) \cap [0, \mu] = \emptyset.
\]

(3.13) (3.14)
Proof. For \( n = 0 \), the statement of Proposition 3.2 is exactly (2.117). For \( n < 0 \), the proof of Theorem 2.1 remains unchanged, since the first Chern form of \((\xi, h^\xi_\eta)\) is bounded, and thus the inequality (2.118) continues to hold. □

In this section, we denote by \( \nabla \) the connection, induced by the Levi-Civita connection and the Chern connections associated with \((\xi, h^\xi_\eta) \) and \((\omega_M(D), ||\cdot||_M)\). We denote by \( d(\cdot, \cdot) \) the distance function on \((M, g^TM)\).

Lemma 3.3. For any \( l, l' \in \mathbb{N}, n \in \mathbb{Z} \), there are \( \eta_0, C > 0 \), such that for any \( \sigma \in \mathcal{C}^\infty(M \times M, (E^E_M) \otimes (E^E_M)^*) \), \( x, x' \in M \) and any \( \eta \in ]0, \eta_0[ \), we have

\[
\left| (\nabla_x)^l(\nabla_{x'})^{l'} \sigma(x, x') \right|_{h \times h} \leq C \rho_M(x) \rho_M(x') \sum_{i=0}^{2+l} \sum_{i=0}^{2+l'} \left| (\Box_{E^E_M}^{E^E_M})^i \sigma(z, z') \right|_{L^2,\eta}.
\] (3.15)

Proof. Let \( \epsilon > 0 \). For \( x \in M \setminus (\bigcup_i V_i^M(\epsilon)) \), the estimate (3.15) follows from [38, Lemma 1.6.2]. From \( x \in V_i^M(\epsilon) \), the estimate (3.15) follows from Lemma 2.27. □

Proposition 3.4. For any \( l, l' \in \mathbb{N} \), there are \( \eta_0, c, c', C > 0 \) such that for any \( t > 0 \), \( x, x' \in M \), \( \eta \in ]0, \eta_0[ \), we have

\[
\left| (\nabla_x)^l(\nabla_{x'})^{l'} \exp(-t\Box_{E^E_M}^{E^E_M})(x, x') \right|_{h \times h} \leq C \rho_M(x) \rho_M(x') t^{-1-\frac{l+l'}{2}} \cdot \exp\left( ct - c' \cdot d(x, x')^2/t \right).
\] (3.16)

Also, if \( n \leq 0 \), then there are \( c, C > 0 \) such that for any \( t > 0 \), \( \eta \in ]0, \eta_0[ \), we have

\[
\left| (\nabla_x)^l(\nabla_{x'})^{l'} \exp(+t\Box_{E^E_M}^{E^E_M})(x, x') \right|_{h \times h} \leq C \rho_M(x) \rho_M(x') t^{-4-l-l'} \exp(-ct).
\] (3.17)

Proof. By Remark 2.26, the proof of (2.12) works uniformly on \( \eta \), thus, we get (3.16). Now, (3.17) follows from Proposition 3.2 and Lemma 3.3. □

Proposition 3.5. For any \( k \in \mathbb{N} \), there are \( \eta_0, \epsilon_1, c, c', C > 0 \) such that for any \( t > 0 \), \( u \in \mathbb{C}, |u| \leq \epsilon_1 \), \( \eta \in ]0, \eta_0[ \), \( i = 1, \ldots, m \), we have

\[
\left| \left( \exp(-t\Box_{E^E_M}^{E^E_M}) - \exp(-t\Box_{E^E_M}^{E^E_M}) \right) (z_i^M)^{-1}(u), (z_i^M)^{-1}(u) \right| \leq C |\ln |u|| |\exp(ct)| \left( |\ln |u||^{-k} + \exp(-c'(\ln |\ln |u||)^2/t) \right).
\] (3.18)

Moreover, if \( n \leq 0 \), then there are \( \zeta < 1 \) and \( c, C > 0 \) such that

\[
\left| \left( \exp(-t\Box_{E^E_M}^{E^E_M}) - \exp(-t\Box_{E^E_M}^{E^E_M}) \right) (z_i^M)^{-1}(u), (z_i^M)^{-1}(u) \right| \leq C |\ln |u||^{\zeta} t^{-4} \exp(-ct).
\] (3.19)

Proof. This statement is a uniform version of Proposition 2.6. As the proof of Proposition 2.6 is based on (2.12), which holds uniformly on \( \eta \in ]0, 1/2[ \) by Remark 2.26, the analogue of (2.14) also holds uniformly on \( \eta \). This implies (3.18). The proof of (3.19) remains identical to the proof of (2.16), one only has to use (3.17) instead of (2.13). □
Theorem 3.7. There are \( \eta_0, c', C > 0 \) such that for any \( t > 0, \eta \in [0, \eta_0] \) and \( x \in M \setminus (\bigcup_i V_i^M(\rho \eta^{-1} \ln \eta^{-1})) \), we have

\[
\left| \exp(-t\square_{\eta}^{E,n}) - \exp(-t\square_{\eta}^{E,n}) \right|(x, x) \leq C \rho_M(x)^2 \exp(-c'(\ln |\ln \eta|)^2/t). \tag{23.23}
\]

Proof. We put \( r = d(V_i^M(\rho \eta^{-1} / 2), M \setminus V_i^M(\rho \eta^{-1} \ln \eta^{-1})) \). Then by (2.173), we have \( r \approx \ln |\ln \eta| \). Similarly to (2.156), using (2.173) and the fact that \( h_{\eta}^\xi \) coincides with \( h^\xi \) over \( M \setminus (\bigcup_i V_i^M(\rho \eta^{-1} \ln \eta^{-1})) \), by the finite propagation speed of solutions of hyperbolic equations, there is \( c > 0 \) such that for any \( x \in M \setminus (\bigcup_i V_i^M(\rho \eta^{-1} \ln \eta^{-1})) \), we have

\[
\tilde{G}_{t,r}(\square_{\eta}^{E,n})(x, \cdot) \asymp \tilde{G}_{t,r}(\square_{\eta}^{E,n})(x, \cdot).
\tag{23.24}
\]

Then, similarly to (2.177), we have (see (2.147))

\[
\left( \exp(-t\square_{\eta}^{E,n}) - \exp(-t\square_{\eta}^{E,n}) \right)(x, x) = \left( \tilde{K}_{t,r}(\square_{\eta}^{E,n}) - \tilde{K}_{t,r}(\square_{\eta}^{E,n}) \right)(x, x). \tag{23.25}
\]

Now, similarly to (2.153), for any \( k, k' \in \mathbb{N} \), there are \( c', C > 0 \) such that for any \( t > 0 \), we have

\[
\left\| \left( \square_{\eta}^{E,n} \right)^k \left( \tilde{K}_{t,r}(\square_{\eta}^{E,n}) \right) \left( \square_{\eta}^{E,n} \right)^{k'} \right\|(0,0) \leq C \exp(-c'r^2/t). \tag{23.26}
\]

From (3.15) and (3.26), similarly to (2.154), for some \( c', C > 0 \) and for any \( x \in M \), we get

\[
\left| \tilde{K}_{t,r}(\square_{\eta}^{E,n})(x, x) \right| \leq C \rho_M(x)^2 \exp(-c'r^2/t). \tag{23.27}
\]

Now, from (2.154), (23.25) and (23.27), we get (23.23). \( \Box \)
Now we can relate the regularized heat traces associated with $h^x_{\eta}$ and $h^z$.

**Theorem 3.8.** There are $c, C > 0$, $\xi > 0$, $t_0 > 0$, such that for any $t > t_0$, $\eta \in ]0, e^{-3}]$, we have

$$|\text{Tr}^r[\exp^+(-t\square_{\eta}^{E^\xi,n})] - \text{Tr}^r[\exp^+(-t\square_{\eta}^{E^z,n})]| \leq C(\ln |\ln \eta|)^{-\xi} \exp(-ct). \quad (3.28)$$

**Proof.** First of all, by (3.17) and (3.19), in the same way as in Proposition 2.14, by replacing the use of (2.16) by (3.19), we get

$$|\text{Tr}^r[\exp^+(-t\square_{\eta}^{E^\xi,n})] - \text{Tr}^r[\exp^+(-t\square_{\eta}^{E^z,n})]| \leq C \exp(-ct). \quad (3.29)$$

Now, by (2.25), we have

$$\text{Tr}^r[\exp^+(-t\square_{\eta}^{E^\xi,n})] - \text{Tr}^r[\exp^+(-t\square_{\eta}^{E^z,n})] = \text{Tr}^r[\exp(-t\square_{\eta}^{E^\xi,n})] - \text{Tr}^r[\exp(-t\square_{\eta}^{E^z,n})]. \quad (3.30)$$

Trivially, there is $C > 0$ such that for any $\eta \in ]0, e^{-3}]$, we have

$$\int_{D(1/2) \setminus D(|\ln \eta|^{-1})} \sqrt{\det g} d^2z \leq C \ln |\ln \eta|. \quad (3.31)$$

We decompose the integration in the definition of $\text{Tr}^r[\exp(-t\square_{\eta}^{E^\xi,n})]$, analogous to Definition 2.9, into two parts: over $\cup_j V_j M (|\ln \eta|^{-1})$ and over $M \setminus (\cup_j V_j M (|\ln \eta|^{-1}))$. By bounding the first part of the integral corresponding to the right-hand side of (3.30) by (2.24), (3.18) and second part by (3.23) and (3.31), we see that there are $c, c', C > 0$ such that for any $t > 0$, $\eta \in ]0, e^{-3}]$, we have

$$\text{Tr}^r[\exp(-t\square_{\eta}^{E^\xi,n})] - \text{Tr}^r[\exp(-t\square_{\eta}^{E^z,n})] \leq \frac{C \exp(ct)}{\ln |\ln \eta|} + C(1 + t) \ln |\ln \eta| \exp \left( ct - \frac{c'}{t} (\ln |\ln \eta|)^2 \right). \quad (3.32)$$

By multiplying (3.29) and (3.32) with suitable powers, and using (2.173), (3.30), we get (3.28). \hfill \Box

Now, for $j \geq -1$, we denote (compare with (2.28))

$$A_{M,n}^{M,n} = \int_M \text{Tr}[a_{M,n}^{M,n}(x)] d\nu_M(x) - \frac{\text{rk}(\xi)}{3} \int_P a_{M,n}^{P,n}(x) d\nu_N(x) - \dim H^0(\overline{M}, E^\xi_{M,n}) + \frac{\text{rk}(\xi)}{3} \dim H^0(\overline{P}, E^n_{P}). \quad (3.33)$$

The integrals in (3.33) converge by Proposition 2.8.

**Proposition 3.9.** For any $t_0 > 0$, $k \in \mathbb{N}$, there is $C > 0$ such that for any $t \in ]0, t_0]$, we have

$$|\text{Tr}^r[\exp^+(-t\square_{\eta}^{E^\xi,n})] - \sum_{j=-1}^k A_{M,n}^{M,n} t^j| \leq Ct^k. \quad (3.34)$$
Proof. It is proved in the same way as Proposition 2.13 with one modification: instead of using Propositions 2.6, 2.8 we use Propositions 3.5, 3.6. □

Proposition 3.10. For any \( t_0 > 0 \), there is \( C > 0 \), such that for any \( t \in [0, t_0] \), \( \eta \in [0, e^{-3}] \):

\[
\left| \frac{1}{t} \left( \sum_{j=1}^{0} A_{\xi, j}^{M, n} t^j \right) \right| \leq C (\ln \ln |\ln \eta|)^{-1/3}. \quad (3.35)
\]

Proof. First of all, by Propositions 2.13, 3.9, we get

\[
\left| \left( \sum_{j=1}^{0} A_{\xi, j}^{M, n} t^j \right) \right| \leq C t. \quad (3.36)
\]

Now, by (3.21) and (3.22), there are \( \eta_0, C > 0 \) such that for any \( \eta \in [0, \eta_0] \), \( j = -1, 0 \), we have

\[
|A_{\xi, j}^{M, n} - A_{\xi, j}^{M, n}| \leq C \eta^{1/6}. \quad (3.37)
\]

Also, by Theorem 3.7 and (3.31), there are \( \eta_0, C' > 0 \) such that for any \( t \in [0, t_0] \), \( \eta \in [0, \eta_0] \):

\[
\int_{M \setminus \cup_{i} V_{i}^{M}(|\ln \eta|^{-1})} \left| \left( \exp(-t \square_{\eta}^{E_{i}^{n}} M) - \exp(-t \square_{\eta}^{E_{i}^{n}} M) \right) (x, x) \right| d\nu_{M}(x) \leq C \ln \ln |\ln \eta| \exp \left( -\frac{C'}{t} (\ln \ln \eta)^2 \right). \quad (3.38)
\]

Also, by (2.24) and (3.18), there are \( \eta_0, C' > 0 \) such that for any \( t \in [0, t_0] \), \( \eta \in [0, \eta_0] \), we have

\[
\int_{V_{i}^{M}(|\ln \eta|^{-1})} \left| \left( \exp(-t \square_{\eta}^{E_{i}^{n}} M) - \exp(-t \square_{\eta}^{E_{i}^{n}} M) \right) (x, x) \right| d\nu_{M}(x) \leq \frac{C}{\ln |\ln \eta|} + C \exp \left( -\frac{C'}{t} (\ln \ln \eta)^2 \right). \quad (3.39)
\]

Thus, by (3.30), (3.37), (3.38) and (3.39), there are \( C', C > 0 \) such that for any \( t \in [0, t_0] \), we have

\[
\left| \frac{1}{t} \left( \sum_{j=1}^{0} A_{\xi, j}^{M, n} t^j \right) \right| \leq C \eta^{1/6} \frac{C}{\ln |\ln \eta|} + C \ln \ln |\ln \eta| \exp \left( -\frac{C'}{t} (\ln \ln \eta)^2 \right). \quad (3.40)
\]

Now, by multiplying (3.36) and (3.40) with appropriate powers, and integrating on \( t \) from 0 to 1, we deduce Proposition 3.10. □
**Proof of (3.1).** By Theorem 3.8, Proposition 3.10, (2.40), (3.37) and Lebesgue dominated convergence theorem, we have

$$\lim_{n \to 0} T(g^M, h^\xi \otimes ||\cdot||_M^{2n}) = T(g^M, h^\xi \otimes ||\cdot||_M^{2n}).$$

(3.41)

However, trivially from (2.1), we have

$$\lim_{n \to 0} ||\cdot||_{L^2}(g^M, h^\xi \otimes ||\cdot||_M^{2n}) = ||\cdot||_{L^2}(g^M, h^\xi \otimes ||\cdot||_M^{2n}).$$

(3.42)

By (3.41) and (3.42), we get (3.1). □

### 3.3. Flattening the Riemannian metric: a proof of (3.2).

In this section we introduce the notion of a right family of flattenings, which is a family of metrics “approaching” the cusped metric. We study how the relative heat trace behaves as this family of flattenings “converges” to the cusped metric, and from this study we deduce (3.2).

From now and till the end of Sect. 3, we suppose that \( (\xi, h^\xi) \) is trivial near the cusps.

**Definition 3.11.** We say that the flattenings \( s^T_{i,\theta} \), \( ||\cdot||_{i,\theta} \in \{0, 1\} \) (cf. Definition 1.2) of \( g^M \), \( ||\cdot||_M \) are \( n \)-tight, \( n \in \mathbb{Z} \) if they satisfy the following requirements:

1. We have \( s^T_{i,\theta} |_{M \setminus (\cup_i V_i^M(\theta))} = g^T |_{M \setminus (\cup_i V_i^M(\theta))} \) and similarly for \( ||\cdot||_{i,\theta} \).
2. For all \( i = 1, \ldots, m \), the following identity holds over \( V_i^M(\theta^2) \):

$$||d\xi_i^M \otimes s_{D_i^M}/\xi_i^M||_{i,\theta} = |\ln \theta|, \text{ where } s_{D_i^M} \text{ is the canonical section of } \theta_i^M(D_i^M).$$

3. There are flattenings \( s^{T_{i,\theta}}_{m} \), \( ||\cdot||_{m,\theta} \) of \( g^T \), \( ||\cdot||_M \) and \( \tau > 0 \) such that for any \( \theta \in [0, e^{-3}] \):

$$s^{T_{i,\theta}}_{m} \otimes (||\cdot||_{m,\theta})^{2n} \leq s^T_{i,\theta} \otimes (||\cdot||_{i,\theta})^{2n} \leq \tau \cdot g^T \otimes ||\cdot||_M^{2n},$$

(3.43)

$$||\cdot||_{m,\theta} \leq \tau ||\cdot||_{i,\theta} \leq ||\cdot||_M.$$

4. We have the following analogue of Lemma 2.28: for any \( m \in \{-n, n\} \), there is \( C > 0 \) such that for any \( \sigma \in \mathcal{C}_c(M, E_{M}^{\xi, m}(\theta) \), \( \theta \in [0, e^{-3}] \), and for any \( x \in M \), we have

$$|\sigma(x)|_{h,\theta} \leq C \rho_{M,\theta}(x) \sum_{i=0}^2 \left\| \left( D_{i,\theta}^{E_{M}^{\xi, m}} \right)^i \sigma \right\|_{L^2,\theta},$$

(3.44)

where \( D_{i,\theta}^{E_{M}^{\xi, m}} \) is the Laplacian associated with \( s^T_{i,\theta} \) and \( h^\xi \otimes (||\cdot||_M^{2m}) \); \( |\cdot|_{h,\theta} \) is the pointwise norm induced by \( h^\xi \) and \( ||\cdot||_{i,\theta} \); \( ||\cdot||_{L^2,\theta} \) is the \( L^2 \) norm induced by \( s^T_{i,\theta}, h^\xi \), \( ||\cdot||_{i,\theta} \); and the function \( \rho_{M,\theta} : M \to [1, \infty[ \) is given by

$$\rho_{M,\theta}(x) = \begin{cases} 1 & \text{for } x \in M \setminus (\cup_i V_i^M(1/2)), \\ \frac{1}{|\ln |z_i^M(x)||} & \text{for } x \in V_i^M(1/2) \setminus V_i^M(\theta^3), \\ (\ln \theta)^6 & \text{for } x \in V_i^M(\theta^3). \end{cases}$$

(3.45)
In Sect. 3.5, we show that for any \( n \in \mathbb{Z}, n < 0, \) \( n \)-tight families of flattenings exist. We fix \( n \in \mathbb{Z}, n \leq 0 \) and \( n \)-tight families of flattenings \( g_{t, \theta}^{TM}, ||\cdot||_{M}, \theta \in ]0, 1[. \) From (3.43):

\[
g_{t, \theta}^{TM} \leq \tau \cdot g^{TM}.
\]  

(3.46)

Recall that \( \Box_{E_{1, \theta}}^{E_{1, \theta}} \) is the Kodaira Laplacian associated to \( g^{TM}, h_{1, \theta} \) and \( ||\cdot||_{M}. \) We set \( \mu > 0 \) as in (2.7), and let \( \tau \) be as in (3.43). We defer the proof of the following theorem until Sect. 3.4.

**Proposition 3.12.** The operator \( \Box_{E_{1, \theta}}^{E_{1, \theta}} \) has a uniform spectral gap near 0, \( \mu \) for any \( \theta \in ]0, e^{-3}[, \)

\[
\ker(\Box_{E_{1, \theta}}^{E_{1, \theta}}) \simeq H^{0}(M, E_{1, \theta}),
\]  

(3.47)

\[
\text{Spec}(\Box_{E_{1, \theta}}^{E_{1, \theta}}) \cap ]0, \mu/\tau[, \theta = \emptyset.
\]  

(3.48)

In what follows, we denote the smooth kernels of \( \exp(-t\Box_{E_{1, \theta}}^{E_{1, \theta}}), \exp(-t\Box_{E_{1, \theta}}^{E_{1, \theta}}) \) with respect to the Riemannian volume form \( dv_{M, \theta} \) induced by \( g_{t, \theta}^{TM} \) by

\[
\exp(-t\Box_{E_{1, \theta}}^{E_{1, \theta}})(x, y), \exp(-t\Box_{E_{1, \theta}}^{E_{1, \theta}})(x, y) \in (E_{1, \theta})^{*} \otimes (E_{1, \theta})_{y}, \text{ for } x, y \in M.
\]  

(3.49)

**Proposition 3.13.** There are \( c, C > 0 \) such that for any \( t > 0, x \in M, \theta \in ]0, e^{-3}[, \)

\[
\left| \exp(-t\Box_{E_{1, \theta}}^{E_{1, \theta}})(x, x) \right| \leq C\rho_{M, \theta}(x)^{2}t^{-3}e^{-ct}.
\]  

(3.50)

**Proof.** The proof is the same as the proof of Proposition 2.4. One only has to change the use of Lemma 2.28 by (3.44) and of Theorem 2.1 by Proposition 3.12. \( \square \)

**Proposition 3.14.** There are \( c', C > 0 \) such that for any \( t > 0, \theta \in ]0, e^{-3}[, \) \( x \in M \setminus (\cup_{i} V_{i}^{M}(||\ln \theta||^{-1})), \) we have

\[
\left| \left( \exp(-t\Box_{E_{1, \theta}}^{E_{1, \theta}}) - \exp(-t\Box_{M}^{E_{1, \theta}}) \right)(x, x) \right| \leq C\rho_{M, \theta}(x)^{2}e^{-ct}e^{-(c'(||\ln \theta||)^{2}/t)}.
\]  

(3.51)

**Proof.** The proof is the same as the proof of Theorem 3.7. \( \square \)

We construct the flattenings \( g_{t, \theta}^{TN}, ||\cdot||_{N}, \theta \in ]0, 1[ \) of \( g^{TN}, ||\cdot||_{N} \), which are compatible with \( g_{t, \theta}^{TM}, ||\cdot||_{M}, \theta \in ]0, 1[ \). Trivially, the flattenings \( g_{t, \theta}^{TN}, ||\cdot||_{N}, \theta \in ]0, 1[ \) are \( n \)-tight. The following theorem is an analogue of Proposition 2.6, and it forms the core of this section. Its proof is deferred to Sect. 3.4.

**Proposition 3.15.** There are \( c, c', C > 0, \zeta < 1 \) such that for any \( t > 0, \theta \in ]0, e^{-3}[, \) \( i = 1, \ldots, m, \) \( u \in C, ||u|| \leq ||\ln \theta||^{-1}, \) we have

\[
\left| \exp(-t\Box_{E_{1, \theta}}^{E_{1, \theta}})((z_{i})^{M})^{-1}(u), (z_{i})^{M}(u)) - \text{Id}_{E_{1, \theta}}\exp(-t\Box_{E_{1, \theta}}^{E_{1, \theta}})((z_{i})^{-N}(u), (z_{i})^{-N}(u)) \right|
\]

\[
\leq C||\ln \max(\theta, |u|)|| \cdot e^{-ct}e^{-(c'(||\ln \max(\theta, |u|)||)^{2}/t)}.
\]  

(3.52)

\[
\left| \exp(-t\Box_{E_{1, \theta}}^{E_{1, \theta}})((z_{i})^{M})^{-1}(u), (z_{i})^{M}(u)) - \text{Id}_{E_{1, \theta}}\exp(-t\Box_{E_{1, \theta}}^{E_{1, \theta}})((z_{i})^{-N}(u), (z_{i})^{-N}(u)) \right|
\]

\[
\leq C||\ln \max(\theta, |u|)||^{5} \cdot t^{-4}e^{-ct}.
\]  

(3.53)
Now, for brevity, we denote

\[ X_\theta(t) := \text{Tr}[\exp^\perp(-t\square_{E_M}^{E_n})] - \text{rk}(\xi)\text{Tr}[\exp^\perp(-t\square_{E_N}^{E_n})] \]

\[ -\text{Tr}^r[\exp^\perp(-t\square_{E_M}^{E_n})] + \text{rk}(\xi)\text{Tr}^r[\exp^\perp(-t\square_{E_N}^{E_n})]. \]  

(3.54)

Let’s use Propositions 3.12–3.15 to study the convergence of heat traces. The main theorem here is

**Theorem 3.16.** There are \( c, c', C > 0 \) such that for any \( t > 0, \theta \in [0, e^{-3}] \), we have

\[ |X_\theta(t)| \leq C \exp(-ct - c'(\ln \ln |\ln \theta|)^2/t). \]  

(3.55)

**Proof.** Let’s denote

\[ A_M^\perp(t) = \int_{M \setminus (\cup_i V_i^M(\ln \theta)^{-1}))} \left( \text{Tr}[\exp^\perp(-t\square_{E_M}^{E_n})(x, x)] - \text{Tr}[\exp^\perp(-t\square_{E_M}^{E_n})(x, x)]\right) dv_{M, \theta}(x), \]

\[ A_N^\perp(t) = \int_{N \setminus (\cup_i V_i^N(\ln \theta)^{-1})} \left( \text{Tr}[\exp^\perp(-t\square_{E_N}^{E_n})(x, x)] - \text{Tr}[\exp^\perp(-t\square_{E_N}^{E_n})(x, x)]\right) dv_{N, \theta}(x), \]

\[ B_\theta^\perp(t) = \sum_i \int_{D(\ln \theta)^{-1})} \left( \text{Tr}[\exp^\perp(-t\square_{E_M}^{E_n})((z_i^M)^{-1}(u), (z_i^M)^{-1}(u))] - \text{rk}(\xi)\text{Tr}[\exp^\perp(-t\square_{E_N}^{E_n})((z_i^N)^{-1}(u), (z_i^N)^{-1}(u))\right) dv_\theta(u), \]

\[ B^\perp(t) = \sum_i \int_{D(\ln \theta)^{-1})} \left( \text{Tr}[\exp^\perp(-t\square_{E_M}^{E_n})((z_i^M)^{-1}(u), (z_i^M)^{-1}(u))] - \text{rk}(\xi)\text{Tr}[\exp^\perp(-t\square_{E_N}^{E_n})((z_i^N)^{-1}(u), (z_i^N)^{-1}(u))\right) dv_\theta(u), \]  

(3.56)

and \( dv_{N, \theta}, dv_\theta \) are the Riemannian volume forms induced by \( g_{E_M}^{TM} \) and \((z_i^M)^{-1}g_{E_N}^{TM}\) correspondingly. Then we have

\[ X_\theta(t) = A_M^\perp(t) + A_N^\perp(t) + B_\theta^\perp(t) + B^\perp(t). \]

(3.57)

By (1.2), (3.31), (3.45) and (3.46), there is \( C > 0 \) such that for any \( \theta \in [0, e^{-3}] \), we have

\[ \int_{M \setminus (\cup_i V_i^M(\ln \theta)^{-1})} \rho_{M, \theta}(x)^2 dv_{M, \theta}(x) \leq C(\ln \ln |\ln \theta|). \]  

(3.58)

By Proposition 3.13, (2.13) and (3.58), there are \( c, C > 0 \) such that

\[ |A_M^\perp(t)|, |A_N^\perp(t)| \leq C(\ln \ln |\ln \theta|)t^{-4} \exp(-ct). \]

(3.59)

By (1.2), (2.23) and (3.46), for any \( \zeta < 1 \), there is \( C > 0 \) such that for any \( \theta \in [0, e^{-3}] \), we have

\[ \int_{V_i^M(\ln \theta)^{-1})} \left| \ln \max(\theta, |z_i^M(x)|) \right|^\zeta dv_{M, \theta}(x) \leq C. \]  

(3.60)
By (3.53) and (3.60), there are $c, C > 0$ such that

$$|B^\perp_\theta (t)| \leq C t^{-4} \exp(-ct). \quad (3.61)$$

By (2.16) and (2.23), there are $c, C > 0$ such that

$$|B^\perp(t)| \leq C t^{-4} \exp(-ct). \quad (3.62)$$

By (3.59), (3.61) and (3.62), for some $c, C > 0$, and for any $t > 0, \theta \in [0, e^{-3}]$, we have

$$|X_\theta(t)| \leq C (1 + t^{-4}) (\ln \ln |\ln \theta|) \exp(-ct). \quad (3.63)$$

Now, alternatively, we may also write

$$X_\theta(t) = A_M(t) + A_N(t) + B_\theta(t) + B(t), \quad (3.64)$$

where $A_M(t), A_N(t), B_\theta(t), B(t)$ are as in (3.56), but we put $\exp$ in place of $\exp^\perp$.

By Proposition 3.14 and (3.58), there are $c', C > 0$ such that for any $\theta \in [0, e^{-3}]$, we have

$$|A_M(t)|, |A_N(t)| \leq C (\ln \ln |\ln \theta|) \exp(-c' (\ln \ln |\ln \theta|)^2 / t). \quad (3.65)$$

By (3.66), we have

$$\int_{V_t^M(\ln |\ln \theta|)^{-1}} |\ln \max(\theta, |u|)| dV_{t,\theta}(x) \leq C \ln |\ln \theta|. \quad (3.66)$$

By (3.52) and (3.66), for some $c', C > 0$, we have

$$|B_\theta(t)| \leq C \ln |\ln \theta| \exp(-c' (\ln \ln |\ln \theta|)^2 / t). \quad (3.67)$$

By (2.17) and (2.24), for some $c', C > 0$, we have

$$|B(t)| \leq C (1 + t) \exp(-c' (\ln \ln |\ln \theta|)^2 / t). \quad (3.68)$$

By (3.65), (3.67) and (3.68), we conclude that

$$|X_\theta(t)| \leq C (1 + t) (\ln |\ln \theta|) \exp(-c' (\ln \ln |\ln \theta|)^2 / t). \quad (3.69)$$

By multiplying (3.63) with power $1 - \mu \in [1/2, 1]$ and (3.69) with power $\mu$, for some $c, c', C > 0$

$$|X_\theta(t)| \leq C (1 + t) (1 + t^{-4}) (\ln |\ln \theta|)^{\mu} (\ln \ln |\ln \theta|) \exp(-ct - c' \mu (\ln \ln |\ln \theta|)^2 / t). \quad (3.70)$$

By (2.173) and (3.70), we deduce (3.55) by taking $\mu$ small enough. \qed

For $s \in \mathbb{C}, \Re(s) > 1$, let’s denote the approximated regularized zeta-function by

$$\zeta_M^\theta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \text{Tr} \left[ \exp^{\perp}(-t E_{t,\theta}^\alpha) \right] t^{s-1} dt. \quad (3.71)$$

As usually, $\zeta_M^\theta(s)$ has a meromorphic extension to the entire $s$-plane, and this extension is holomorphic at 0. We recall that the zeta-function $\zeta_M$ was defined in Definition 2.15.
Proposition 3.17. For any $\theta \in [0, e^{-3}]$, the difference $\zeta_M^\theta(s) - \text{rk}(\xi)\zeta_N^\theta(s) - \zeta_M(s) + \text{rk}(\xi)\zeta_N(s)$ is a holomorphic function on $\mathbb{C}$. Moreover, as $\theta \to 0$, we have

$$\zeta_M^\theta(s) - \text{rk}(\xi)\zeta_N^\theta(s) - \zeta_M(s) + \text{rk}(\xi)\zeta_N(s) \to 0,$$  

uniformly for $s$ varying in a compact subset of $\mathbb{C}$. In particular, as $\theta \to 0$, we have

$$\frac{T(g_{T,\theta}^M, h^\xi \otimes (\cdot || \cdot |_{M}^{f,\theta})^{2n})}{T(g_{T,\theta}^N, (\cdot || \cdot |_{N}^{f,\theta})^{2n})} \to \frac{T(g_{T,\theta}^M, h^\xi \otimes (\cdot || \cdot |_{M}^{2n})}{T(g_{T,\theta}^N, (\cdot || \cdot |_{N}^{2n}){\text{rk}(\xi)}}).$$  

(3.73)

Proof. First of all, by Definition 2.15, (3.54) and (3.71), we have

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} X_\theta(t) t^s dt = \zeta_M^\theta(s) - \text{rk}(\xi)\zeta_N^\theta(s) - \zeta_M(s) + \text{rk}(\xi)\zeta_N(s).$$  

(3.74)

Now, by Theorem 3.16, the function $X_\theta(t)$ has subexponential growth near 0 and $\infty$, thus, by (3.74), the left-hand side of (3.72) is a holomorphic function over $\mathbb{C}$ for any $\theta \in [0, e^{-3}]$.

Also, by Theorem 3.16, there are $c, c', C > 0$, $\zeta > 0$ such that for any $t > 0$, $\theta \in [0, e^{-3}]$:

$$X_\theta(t) \leq C |\ln \theta|^{-\zeta} \exp(-ct - c'/t).$$  

(3.75)

In particular, by (3.75), as $\theta \to 0$, we have

$$X_\theta(t) \to 0.$$  

(3.76)

By (3.74), (3.75), (3.76) and Lebesgue dominated convergence theorem, we deduce (3.72). Now, (3.73) follows from Definitions 2.15, 2.16, (3.71) and (3.72). □

We denote by $||\cdot||_{L^2}(g_{T,\theta}^M, h^\xi \otimes (||\cdot||_{M}^{f,\theta})^{2n})$ the $L^2$-norm over the line bundle (1.12) induced by $g_{T,\theta}^M, h^\xi, ||\cdot||_{M}^{f,\theta}$. By properties 1,3 of tight families and Lebesgue dominated convergence, we have

$$\lim_{\theta \to 0} ||\cdot||_{L^2}(g_{T,\theta}^M, h^\xi \otimes (||\cdot||_{M}^{f,\theta})^{2n}) = ||\cdot||_{L^2}(g_{T}^M, h^\xi \otimes ||\cdot||_{M}^{2n}),$$  

(3.77)

$$\lim_{\theta \to 0} ||\cdot||_{L^2}(g_{T,\theta}^N, (||\cdot||_{N}^{f,\theta})^{2n}) = ||\cdot||_{L^2}(g_{T}^N, ||\cdot||_{N}^{2n}).$$  

From (3.73) and (3.77), as $\theta \to 0$, we get (3.2) for $h_2^\xi := h^\xi$.

3.4. Proofs of Propositions 3.12, 3.15. In this section we prove Propositions 3.12, 3.15, which were announced in Sect. 3.3. In the proof of Proposition 3.12 we use the homogeneity of the Laplacian. In the proof of Proposition 3.15, we use the analytic localization techniques, the maximal principle and sup-characterization of the Bergman kernel. We recall that we suppose that $(\xi, \rho^\xi)$ is trivial around the cusps.

Proof of Proposition 3.12. First of all, (3.47) is a consequence of Hodge theory for compact manifolds. To prove (3.48), by (3.43), it is enough to prove the following: let $\tau > 0$, $g_{0}^T$ be a Kähler metric on $\overline{M}$ and let $||\cdot||_{M}^{0}$ be a Hermitian norm on $\omega_M(D)$ over $M$ such that over $\overline{M}$, we have

$$g_{0}^T \otimes (||\cdot||_{M}^{0})^{2n} \leq \tau \cdot g_{T,\theta}^M \otimes ||\cdot||_{M}^{2n},$$  

(3.78)
\[ ||\cdot||_M^0 \leq ||\cdot||_M. \quad (3.79) \]

Let \((\cdot, \cdot)_{L_2^0}\) be the \(L^2\)-scalar product associated with \(\mathfrak{g}^{TM}_0, h^k, ||\cdot||_M^0\), and let \(\square^{E,n}_0\) be the associated Kodaira Laplacian. Then for \(n \leq 0\), we have

\[ \text{inf} \left\{ \text{Spec} \left( \square^{E,n}_M \right) \setminus \{0\} \right\} \leq \tau \cdot \text{inf} \left\{ \text{Spec} \left( \square^{E,n}_0 \right) \setminus \{0\} \right\}. \quad (3.80) \]

Let’s prove this statement. By (2.23), we deduce

\[ \mathcal{C}^\infty(M, E^k_M) \subset \text{Dom}(\square^{E,n}_M). \quad (3.81) \]

In the following series of transformations, we use (3.79) and \(n \leq 0\) to get the inequality.

For \(s \in \mathcal{C}^\infty(M, E^k_M)\) we have

\[ \left( \square^{E,n}_0 s, s \right)_{L^2_0} = \left( \overline{\delta}^{E,n}_M s, \overline{\delta}^{E,n}_M s \right)_{L^2_0} \geq \left( \delta^{E,n}_M s, \delta^{E,n}_M s \right)_{L^2} = \left( \square^{E,n}_M s, s \right)_{L^2}. \quad (3.82) \]

Also, from (3.78), we have

\[ \left( s, s \right)_{L^2_0} \leq \tau \cdot \left( s, s \right)_{L^2}. \quad (3.83) \]

From (3.82) and (3.83), we deduce

\[ \frac{\left( \square^{E,n}_0 s, s \right)_{L^2_0}}{\left( s, s \right)_{L^2_0}} \geq \frac{\left( \square^{E,n}_M s, s \right)_{L^2}}{\tau \cdot \left( s, s \right)_{L^2}}. \quad (3.84) \]

We denote \(k = \dim H^0(M, E^k_M)\). By the min-max theorem (cf. [38, (C.3.3)]) and (3.81), we have

\[ \text{inf} \left\{ \text{Spec} \left( \square^{E,n}_M \right) \setminus \{0\} \right\} = \text{inf}_{F \subset \mathcal{C}^\infty(M, E^k_M)} \left\{ \sup_{s \in F} \left\{ \left( \square^{E,n}_M s, s \right)_{L^2} \right\} : \dim F = k + 1 \right\}. \quad (3.85) \]

Then (3.80) follows from (3.84) and (3.85). \(\square\)

**Proof of Proposition 3.15.** This proof uses all the properties of tight families. The presence of the line bundle \(\omega_M(D)\) makes analysis more difficult, and we have to consider 2 cases: \(\theta^3 < |u| < |\log \theta|^{-1}\) and \(|u| \leq \theta^3\). The main feature exploited in the first case is that we have elliptic estimate with the needed power of logarithm (3.44), (3.45). The main feature exploited in the second case is the property 2 of tight families along with the maximal principle (cf. [15, p. 180]).

**Let’s prove (3.52)** for \(\theta^3 \leq |u| \leq |\log \theta|^{-1}\). We put \(r = d(u, 1/2)\), then by (2.137), \(r \approx \ln |\ln |u||.\) In this case, similarly to (2.156), by the fact that our flattenings are compatible, \((\xi, h^k)\) is trivial near the cusps, and by the finite propagation speed of solutions of hyperbolic equations, we have

\[ \tilde{G}_{t,r}(\square^{E,n}_{f,\theta})(z_i^M)^{-1}(u), \cdot) = \text{Id}_\xi \cdot \tilde{G}_{t,r}(\square^{N}_{f,\theta})(z_i^N)^{-1}(u), \cdot), \quad (3.86) \]

where \(\tilde{G}_{t,r}\) is as in (2.147). Then, similarly to (2.157), by (3.86), we have

\[ \exp(-t\square^{E,n}_{f,\theta})(z_i^M)^{-1}(u), \cdot) - \text{Id}_\xi \cdot \exp(-t\square^{N}_{f,\theta})(z_i^N)^{-1}(u), \cdot) \]
\[ = \tilde{K}_{t,r}(\Box_{t,\theta}^{E_{\xi,n}^M})(z_i^M)^{-1}(u), \cdot) - \text{Id}_\xi \cdot \tilde{K}_{t,r}(\Box_{t,\theta}^{E_{\xi,n}^n})(z_i^N)^{-1}(u), \cdot). \quad (3.87) \]

Now, similarly to (3.27), from (2.152), (3.44) and (3.87), for any \( \theta^3 \leq |u|, |v| \leq |\log \theta|^{-1} \), we get

\[
\left| \exp(-t\Box_{t,\theta}^{E_{\xi,n}^M})(z_i^M)^{-1}(u), (z_i^M)^{-1}(v)) - \text{Id}_\xi \exp(-t\Box_{t,\theta}^{E_{\xi,n}^n})(z_i^N)^{-1}(u), (z_i^N)^{-1}(v)) \right|_{h \times h, \theta} \\
\leq C|u||v|\exp(-c'(\ln|\ln|u||)^2/t). \quad (3.88)
\]

In particular, (3.88) implies (3.52) for \( \theta^3 \leq |u| \leq |\log \theta|^{-1} \).

**Let’s prove (3.52)** for \( |u| \leq \theta^3 \). We trivialize \((\omega_M(D), || \cdot ||_M), (\omega_N(D), || \cdot ||_N)\) as in property 2 of tight families. Then, since \((\xi, h^\xi)\) is trivialized around the cusps, for \( v, w \in D(\theta^3) \), we may look at \( \exp(-t\Box_{t,\theta}^{E_{\xi,n}^M})(z_i^M)^{-1}(v), (z_i^M)^{-1}(w)) \) and \( \text{Id}_\xi \exp(-t\Box_{t,\theta}^{E_{\xi,n}^n})(z_i^N)^{-1}(v), (z_i^N)^{-1}(w)) \) as the functions over \( D(\theta^3) \times D(\theta^3) \) with values in \( \text{End}(\xi|_{P_M}) \).

For \( v, w \in D(\theta^3) \), we denote

\[
F(v, w, t) := \exp(-t\Box_{t,\theta}^{E_{\xi,n}^M})(z_i^M)^{-1}(v), (z_i^M)^{-1}(w)) \\
- \text{Id}_\xi \exp(-t\Box_{t,\theta}^{E_{\xi,n}^n})(z_i^N)^{-1}(v), (z_i^N)^{-1}(w)). \quad (3.89)
\]

We write \( F(v, w, t) = (F_{kl}(v, w, t))_{l=1}^{\dim \xi} \) for the components of the matrix from \( \text{End}(\xi|_{P_M}) \). We notice that the functions \( F_{kl}(v, w, t) \) satisfy the heat equation with zero initial data in \( D(\theta^3) \times D(\theta^3) \times ]0, +\infty[, \) i.e. for any \( k, l = 1, \ldots, \dim \xi \), we have

\[
\left( \frac{\partial}{\partial t} + \Box_{t,\theta} \right) F_{kl}(u, v, t) = 0 \quad \text{and} \quad \lim_{t \to 0} F_{kl}(u, v, t) = 0, \quad (3.90)
\]

where \( \Box_{t,\theta} \) is the Laplace-Beltrami operator induced by \((z_i^M)^{-1} \otimes \theta^TM_{\xi,\theta} \) on \( D(\theta^3) \). Thus, by the maximal principle (cf. [15, p. 180]), for \( |u| \leq \theta^3 \), we get

\[
|F_{kl}(u, u, t)| \leq \sup_{\tau' \in [0, t]} \sup_{|w| = \theta^3} |F_{kl}(u, w, \tau')|. \quad (3.91)
\]

By applying the maximal principle again, we get

\[
|F_{kl}(u, w, \tau')| \leq \sup_{\tau \in [0, \tau']} \sup_{|u| = \theta^3} |F_{kl}(v, w, \tau)|. \quad (3.92)
\]

By (3.88), there are \( c', C > 0 \) such that for any \( \theta \in ]0, e^{-3}[ \), and \( |v|, |w| = \theta^3 \), we have

\[
|F_{kl}(v, w, \tau)| \leq |\ln \theta| \exp(-c'(\ln|\ln|\theta||)^2/t). \quad (3.93)
\]

By (3.91), (3.92) and (3.93), we get (3.52) for \(|u| \leq \theta^3 \). Thus, (3.52) is completely proved.

**Now let’s prove (3.53).** By Proposition 3.13, there are \( c, C > 0 \) such that for any \(|u| \leq |\ln \theta|^{-1}, \theta \in ]0, e^{-3}[ \), \( t > 0 \), we have

\[
\left| \exp(-t\Box_{t,\theta}^{E_{\xi,n}^M})(z_i^M)^{-1}(u), (z_i^M)^{-1}(u)) - \text{Id}_\xi \exp(-t\Box_{t,\theta}^{E_{\xi,n}^n})(z_i^N)^{-1}(u), (z_i^N)^{-1}(u)) \right|
\]
Now, for any \( x, x' \in M \), we have

\[
\exp(-t \Box_{\ell, \theta}^{E_{\ell, n}^g})(x, x') = \exp^{1/2}\left(-t \Box_{\ell, \theta}^{E_{\ell, n}^g}\right)(x, x') + B_{M}^{E_{\ell, n}^g}(x, x'),
\]

where \( B_{M}^{E_{\ell, n}^g}(x, x') \) is the Bergman kernel, defined by

\[
B_{M}^{E_{\ell, n}^g}(x, x') = \sum s_i(x)(s_i(x'))^{*}_{\theta},
\]

for an orthonormal base \( \{s_i\} \) of \( H^0(M, E_{M}^{\ell, n}) \) with respect to the \( L^2 \)-scalar product induced by \( g_{T_M}, \|\cdot\|_{M}^{\ell, n} \), and \( (\cdot)^{*}_{\theta} \) is the dual with respect to \( |\cdot|_{h, \theta} \). By [16, Lemma 3.1], we have

\[
B_{M}^{E_{\ell, n}^g}(x, x) = \max \left\{ \left| s(x) \right|_{h, \theta}^2 : s \in H^0(M, E_{M}^{\ell, n}) \setminus \{0\} \right\},
\]

By (3.43) and the fact that \( n \leq 0 \), we see that for any \( s \in C^\infty(M, E_{M}^{\ell, n}) \), we have

\[
|s(x)|_{h, sm} \leq |s(x)|_{h, sm}, \quad ||s||_{L^2, \theta} \geq ||s||_{L^2, sm},
\]

where \( |\cdot|_{h, sm} \) is the pointwise norm induced by \( h^g, ||\cdot||_{sm} \), and \( ||\cdot||_{L^2, sm} \) is the \( L^2 \)-norm induced by \( h^g, ||\cdot||_{sm}, g_{T_M} \). From (3.97) and (3.98), we deduce

\[
B_{M}^{E_{\ell, n}^g}(x, x) \leq B_{sm}^{E_{\ell, n}^g}(x, x),
\]

where \( B_{sm}^{E_{\ell, n}^g}(x, x') \) is the Bergman kernel associated with \( h^g, ||\cdot||_{sm}, g_{T_M} \). Thus, from (3.52), (3.95) and (3.99), there is \( C > 0 \) such that for any \( \theta \in [0, 1/2], |u| < \theta^3 \), we have

\[
\left| \exp^{1/2}\left(-t \Box_{\ell, \theta}^{E_{\ell, n}^g}\right)((z_i^M)^{-1}(u), (z_i^M)^{-1}(u)) - \Id_{g} \exp^{1/2}\left(-t \Box_{\ell, \theta}^{E_{\ell, n}^g}\right)((z_i^N)^{-1}(u), (z_i^N)^{-1}(u)) \right| \\
\leq C \left( 1 + |\ln \max(\theta, |u|)\exp(-c'(|\ln |\ln \max(\theta, |u|)|)^2/4t) \right),
\]

By multiplying (3.94) with power \( \mu \in [0, 1/2] \) and (3.100) with power \( 1 - \mu \), we have

\[
\left| \exp^{1/2}\left(-t \Box_{\ell, \theta}^{E_{\ell, n}^g}\right)((z_i^M)^{-1}(u), (z_i^M)^{-1}(u)) - \Id_{g} \exp^{1/2}\left(-t \Box_{\ell, \theta}^{E_{\ell, n}^g}\right)((z_i^N)^{-1}(u), (z_i^N)^{-1}(u)) \right| \\
\leq C |\ln \max(\theta, |u|)|^{1+11\mu}t^{-4} \exp(-c\mu t - c'(|\ln |\ln \max(\theta, |u|)|)^2/(2t)) + C |\ln \max(\theta, |u|)|^{12\mu}t^{-4} \exp(-c\mu t),
\]

By (2.173) and (3.101), we finally get (3.53) by taking \( \mu \) small enough. \( \square \)
3.5. Existence of tight families of flattenings. Here we prove by an explicit construction that for any \( n \in \mathbb{Z}, n \leq 0 \), there are \( n \)-tight families of flattenings \( g_{f,\theta}^{TM}, ||\cdot||^{f,\theta}_{M} \) (see Definition 3.11). To simplify the notation, for \( 0 < a < b, i = 1, \ldots, m \), we denote (see (1.1))
\[
C_{i}^{M}(a, b) := V_{i}^{M}(b) \setminus V_{i}^{M}(a).
\]

(3.102)

As in Sect. 3.3, we suppose that \((\xi, h^{\xi})\) is trivial near the cusps.

Before giving the details, let’s describe in words our construction. The metrics \( g_{TM}^{M}(\theta) \), \( ||\cdot||^{f,\theta}_{M} \) are equal to \( g^{TM} \), \( ||\cdot||_{M} \) over \( M \setminus (\cup_{i} V_{i}^{M}(\theta)) \). The metric \( ||\cdot||^{f,\theta}_{M} \) gets “flattened” over the set \( C_{i}^{M}(\theta^{2}, \theta) \) so that it differs from \( ||\cdot||_{M} \) by a multiplication by a function, which is bounded by a constant independent of \( \theta \). Over \( V_{i}^{M}(\theta^{2}) \), \( ||\cdot||^{f,\theta}_{M} \) is flat with a normalization as in property 2 of tight families. The metric \( g_{f,\theta}^{TM} \) coincide with \( g^{TM} \) over \( M \setminus (\cup_{i} V_{i}^{M}(\theta^{4})) \). It gets “flattened” over the set \( C_{i}^{M}(\theta^{4}/4, \theta^{4}) \), so that it differs from \( g^{TM} \) by a bounded function. Finally, over \( V_{i}^{M}(\theta^{4}/4) \) it is flat such that the Riemannian manifolds \((V_{i}^{M}(\theta^{4}/4), g_{f,\theta}^{TM})\) and \((D(2), (dx^{2} + dy^{2})/(\ln \theta)^{2})\) are isometric up to a multiplication by a constant independent of \( \theta \).

Now let’s make this description more precise by giving explicit formulas.

Let \( \phi : [0, +\infty[ \rightarrow [0, 1] \) be some smooth decreasing function satisfying
\[
\phi(u) = \begin{cases} 
1 & \text{for } u \in [0, 1], \\
0 & \text{for } u \in [2, +\infty[. 
\end{cases}
\]

(3.103)

Fix \( \theta \in ]0, 1/2[ \). We denote by \( ||\cdot||^{f,\theta}_{M} \) the Hermitian norm on \( \omega_{M}(D) \) such that \( ||\cdot||^{f,\theta}_{M} \) coincides with \( ||\cdot||_{M} \) over \( M \setminus (\cup_{i} V_{i}^{M}(\theta)) \), and over \( V_{i}^{M}(\theta) \), it satisfies
\[
||dz_{i}^{M} \otimes s_{M}/z_{i}^{M}||^{f,\theta}_{M}(x) = |\ln \theta| \cdot \frac{\ln |z_{i}^{M}(x)|}{\ln \theta} \cdot \frac{\phi(|z_{i}^{M}(x)/|\ln \theta|)}{\phi(|z_{i}^{M}(x)/|\ln \theta|)}.
\]

(3.104)

Let the metric \( g_{f,\theta}^{TM} \) coincide with \( g^{TM} \) over \( M \setminus (\cup_{i} V_{i}^{M}(\theta^{4})) \), and over \( V_{i}^{M}(\theta^{4}) \) be induced by
\[
\left( |z_{i}^{M} \ln |z_{i}^{M}|^{2}/|\theta^{4}|^{2} \right)^{2} = \sqrt{-1}dz_{i}^{M}d\bar{z}_{i}^{M}/|z_{i}^{M}\ln |z_{i}^{M}|^{2}.
\]

(3.105)

Then we see that the metrics \( g_{f,\theta}^{TM}, ||\cdot||^{f,\theta}_{M} \) verify the description given in the beginning of the section. Let \( \square_{L^{2},\theta}^{E_{n}^{\xi}} \), \( ||\cdot||_{L^{2},\theta} \) be the Kodaira Laplacian and the \( L^{2} \)-norm induced by \( g_{f,\theta}^{TM}, h^{\xi} \otimes (||\cdot||^{f,\theta}_{M})^{2n} \).

Theorem 3.18. The flattenings \( g_{f,\theta}^{TM}, ||\cdot||^{f,\theta}_{M}, \theta \in ]0, 1/2[ \) are \( n \)-tight.

Proof. We see directly from (3.104) and (3.105) that all the requirements for tightness are trivially satisfied with only one exception - the estimate (3.44). As \( g_{f,\theta}^{TM} \) and \( ||\cdot||^{f,\theta}_{M} \) coincide with \( g^{TM} \) and \( ||\cdot||_{M} \) over \( M \setminus (\cup_{i} V_{i}^{M}(\theta)) \), by Lemma 2.28, (2.137), it is enough to prove that for any \( n \in \mathbb{Z} \), there is \( C > 0 \) such that for any \( \sigma \in \mathcal{C}^{\infty}(\overline{M}, E_{M}^{n}) \), \( x \in V_{i}^{M}(\theta^{1/2}) \), the estimate (3.44) holds.
Let’s prove (3.44) for $x \in C_i^M(\theta^3, \theta^{1/2})$. Let’s denote by $\||\cdot||_{f,\theta}^{i}$ the Hermitian norm on $\omega_\emptyset(0)$ (see (2.44)), given by the formula (compare with (3.104))

$$||dz \otimes s_0/z||_{f,\theta}^{i}(z) = \frac{|ln|z||}{ln \theta} \phi(|ln|z||/ln \theta).$$  \hspace{1cm} (3.106)

We denote by $\Box_{f,\theta}^{i}$ the Kodaira Laplacian on $(\mathbb{D}^*, \gamma^{TM})$ associated to $(\omega_\emptyset(0)^n, (\||\cdot||_{f,\theta}^{i})^{2n})$.

By (2.137), for $x \in M \setminus (\cup_i V_i^M(\theta^3))$, we have $B(x, ln(4/3)) \subset M \setminus (\cup_i V_i^M(\theta^4))$. By this, the fact that $\gamma^{TM}$ coincides with $\gamma^{i,\theta}$ over $M \setminus (\cup_i V_i^M(\theta^4))$ and the fact that $(\xi, h^\xi)$ is trivial around the cusps, we can prove that for (3.44) for $x \in C_i^M(\theta^3, \theta^{1/2})$, it is enough to prove that for any $n \in \mathbb{Z}$, there is $C > 0$ such that for any $\sigma' \in \mathcal{C}^\infty(\mathbb{D}, \omega_\emptyset(0)^n)$, $z \in D(\theta^{1/2}) \setminus D(\theta^3)$, we have

$$|\sigma'(z)|_{h,\theta} \leq C \log |z|^{1/2} \sum_{i=0}^{2} \left\| \Box_{f,\theta}^{i}\right\|_{L^2(B^{\emptyset}(\emptyset,ln(4/3)),\theta')}. \hspace{1cm} (3.107)$$

Let’s prove (3.107). Recall that for $z_0 \in \mathbb{H}$, in (2.57), we have defined $g_{z_0} \in \text{Aut}(\mathbb{H})$, and in (2.47), we have defined the covering map $\rho : \mathbb{H} \to \mathbb{D}^*$. By proceeding in the same way as in the proof of Lemma 2.23, we see that it is enough to prove that for any $z_0 \in \mathbb{H}$, the family of metrics

$$\||\cdot||_{z_0,\mathbb{H}}^{i} := \left(\left(\gamma_{z_0}^{-1}\rho\right)^{*}||\cdot||_{\mathbb{D}}^{i}\right)_{B^{\emptyset}(\sqrt{-1},1)}$$  \hspace{1cm} (3.108)

is uniformly $\mathcal{C}^\infty$-bounded for $\theta \in [0, 1/2]$ and $z_0 \in \mathbb{H}$ such that $\rho(z_0) \in D(\theta^{1/2}) \setminus D(\theta^3)$. However, similarly to (2.61), for any $z_0 = (x_0, y_0) \in \mathbb{H}$, $z_2 = (x, y) \in B^{\emptyset}(\sqrt{-1}, 1)$, by (3.104), we have

$$\left\| (g_{z_0}^{-1}\rho)^{*}(dz \otimes s_0/z) \right\|_{z_0,\mathbb{H}}^{i} = \left\| dz \otimes s_0/z \right\|_{\mathbb{D}}^{i}(e^{-y_0+\sqrt{-1}(x_0y_0+x_0)}) = (y_0/\log \theta)^{\phi(y_0/|\log \theta|)}, \hspace{1cm} (3.109)$$

which is uniformly bounded for $(x, y) \in B^{\emptyset}(\sqrt{-1}, 1)$ and $|\log \theta|/2 \leq y_0 \leq 3|\log \theta|$. But since $\rho(z_0) \in D(\theta^{1/2}) \setminus D(\theta^3)$ if and only if $|\log \theta|/2 \leq y_0 \leq 3|\log \theta|$, we conclude that (3.107) holds for $z \in D(\theta^{1/2}) \setminus D(\theta^3)$. Thus, (3.44) holds for $x \in V_i^M(\theta^{1/2}) \setminus V_i^M(\theta^3)$.

Let’s prove (3.44) for $x \in C_i^M(\theta^4/4, \theta^3)$. Since the Hermitian line bundle $(\omega_M(D), \||\cdot||_M^{i})$ is trivial over $V_i^M(\theta^2)$, without loss of generality we may and we will suppose $n = 0$. Since $C_i^M(\lfloor x \rfloor/2, 2\lfloor x \rfloor) \subset C_i^M(\theta^4/8, 2\theta^3)$, by Lemma 2.25, we have

$$|\sigma(x)| \leq C(ln \theta)^3 \sum_{j=0}^{2} \left\| \Box_{M}^{j} \right\|_{L^2(C_i^M(\theta^4/2,2\theta^3))}. \hspace{1cm} (3.110)$$

By a trivial calculation, we have

$$\left(\frac{|z \ln |z||^2}{\theta^8 |ln \theta^4|^2}\right)^{-\psi(|z|^2/\theta^8)} \partial \left(\frac{|z \ln |z||^2}{\theta^8 |ln \theta^4|^2}\right)^\psi(|z|^2/\theta^8)$$
\[
\frac{\psi(|z|^2/\theta^8)\ln|z| + 1/2}{z\ln|z|} + \ln\left(\frac{|z\ln|z||^2}{\theta^8|\ln\theta|^2}\psi'(|z|^2/\theta^2)z\theta^{-8}\right).
\]

(3.111)

By (3.105) and (3.111), there is \(C > 0\) such that for any \(\theta \in [0, 1/2]\), we have

\[
|\Box(g^{TM}/g_{1,\theta}^{TM})| < C\ln\theta^2, \quad |\partial(g^{TM}/g_{1,\theta}^{TM})|_{h,\theta} < C|\ln\theta|,
\]

(3.112)

over \(C_i^M(\theta^4/2, \theta^3)\). As \((\omega_1(D), ||\cdot||^{f,\theta}_{L^2})\) is trivial over \(V_i^M(2\theta^3)\), the following identity holds

\[
\Box_{E_{\xi,n}^{TM}/g_{1,\theta}^{TM}} = (g^{TM}/g_{1,\theta}^{TM}) \cdot \Box_{E_{\xi,n}^{TM}}.
\]

(3.113)

By (3.110), (3.112) and (3.113), we get (3.44) for \(x \in C_i^M(\theta^4, \theta^3)\).

Let’s prove (3.44) for \(x \in V_i^M(\theta^4)\). First of all, we recall that by Sobolev inequality and standard elliptic estimates, we have for some \(C > 0\) and any \(h \in \mathcal{C}\infty(D(2))\), \(x \in D(1)\):

\[
|h(x)| \leq C \sum_{i=0}^2 \|\Box_i h\|_{L^2_{g_{st}}},
\]

(3.114)

where \(\|\cdot\|_{L^2_{g_{st}}}\) is the \(L^2\)-norm induced by the standard Euclidean metric \(g_{st}\) over \(D(2)\), and \(\Box\) is the Kodaira Laplacian induced by \(g_{st}\). We denote by \(g_{st,\theta}\) the rescaled Euclidean metric given by

\[
g_{st,\theta} := \frac{dx^2 + dy^2}{(\ln\theta)^2}.
\]

(3.115)

Let \(\|\cdot\|_{L^2_{g_{st,\theta}}}\) be the \(L^2\)-norm induced by \(g_{st,\theta}\), let \(\Box_{\theta}\) be the Kodaira Laplacian induced by \(g_{st,\theta}\). Analogously to (3.113), the estimation (3.114) implies that

\[
|h(x)| \leq C(\ln\theta)^4 \sum_{i=0}^2 \|\Box_i h\|_{L^2_{g_{st,\theta}}}.
\]

(3.116)

By (3.105), the spaces \((D(2), g_{st,\theta})\) and \((V_i^M(\theta^4), g_{1,\theta}^{TM})\) are isometric up to a constant independent of \(\theta\). Thus, by (3.116), we deduce (3.44) for \(x \in V_i^M(\theta^4)\).

Now, all the cases have been considered, thus, the proof of Theorem 3.18 is finished. \(\square\)

4. The Anomaly Formula: A Proof of Theorem B

In this section we prove Theorem B. First of all, we recall that in Sect. 3 we proved Theorem B for \(g_{0}^{TM} = g_{TM}\), i.e. when we have only the variation of \(h^\xi\). Thus, it’s left to prove Theorem B for \(h^\xi = h^\xi\) and under the supposition that \((\xi, h^\xi)\) is trivial around the cusps. Let’s describe the idea of the proof. We construct a family of flattenings which “approach” the cusp metric and we use Theorem A to relate the corresponding relative Quillen norms. Then we apply the anomaly formula of Bismut-Gillet-Soulé [10, Theorem 1.23] (see Theorem 3.1) and calculate the limit of the right-hand side of (3.3), as the family of flattenings “approach” the cusp metric.

Before giving a proof of Theorem B, let’s fix some notation. By the assumptions of Theorem B, for \(\epsilon > 0\), there are holomorphic functions \(h_\phi^\epsilon : D(\epsilon) \to D(1)\),
\[ i = 1, \ldots, m, \text{ such that } g_0^{TM} \text{ is Poincaré-compatible with coordinates } h_i^\phi(z_i^M) \text{ around } P_i^M \in D_M. \]

We note
\[ z_i^{0,M} := h_i^\phi(z_i^M). \] (4.1)

By Definition 1.5 of the Wolpert norm, we have the following identity
\[ \ln \left( \frac{||\cdot||^W}{||\cdot||^0_W} \right) = \sum \ln \left| (h_i^\phi)'(0) \right|. \] (4.2)

**First of all, let’s describe why the right-hand side of (1.23) is finite.** For \( \epsilon > 0 \), in \( V_i^M(\epsilon) \):

\[ c_1(\omega_M(D), (||\cdot||^M)^2)|_M = \frac{\partial^2}{\partial \bar{\theta} \partial \theta} \ln (||s||^2_M) = O(|z_i^M \ln |z_i^M||^{-2}), \] (4.3)

where \( s \) is a local holomorphic frame of \( \omega(D_M) \). Similar estimation holds for the norm \( ||\cdot||^0_M \). The identity (1.25) says

\[ e^{2\phi} dz_i^M dz_i^M = \frac{dz_i^{0,M} dz_i^{0,M}}{||z_i^M \ln |z_i^M||^2}. \] (4.4)

By (1.5) and (4.4), we see that over \( V_i^M(\epsilon) \), we have

\[ \ln (||\cdot||^0_M/||\cdot||_M) = O \left( |\ln |z_i^M||^{-1} \right). \] (4.5)

By (1.21), (1.22), (2.23), (4.3) and (4.5), we conclude that the right-hand side of (1.23) is finite.

**Now let’s describe the precise family of flattenings we choose.** Recall that the function \( \psi : \mathbb{R} \to [0, 1] \) was defined in (2.53). Let \( g_{f,\theta}^{TM} \) be a metric over \( \overline{M} \) such that it coincides with \( g^{TM} \) away from \( \cup_i V_i^M(\theta) \), and over \( V_i^M(\theta) \) it is induced by

\[ \frac{dz_i^M \ln |z_i^M|}{z_i^M \ln |z_i^M|} \psi(\ln |z_i^M|/\ln \theta), \] (4.6)

for all \( i = 1, \ldots, m \). Similarly, let \( ||\cdot||_{f,\theta}^M \) be the smooth metric on \( \omega_M(D_M) \) over \( \overline{M} \) such that it coincides with \( ||\cdot||_M \) away from \( \cup_i V_i^M(\theta) \), and over \( V_i^M(\theta) \), \( i = 1, \ldots, m \), we have

\[ ||dz_i^M \otimes s_{DM}/z_i^M||_{f,\theta}^M = \frac{\psi(\ln |z_i^M|/\ln \theta)}{\ln |z_i^M|}, \] (4.7)

where \( s_{DM} \) is the canonical section of \( \mathcal{O}_{\overline{M}}(D_M), \) \( \text{div}(s_{DM}) = D_M. \)

For \( \epsilon > 0, i = 1, \ldots, m \), we denote

\[ V_i^{0,M}(\epsilon) := \{ x \in M : |z_i^{0,M}(x)| \leq \epsilon \}. \] (4.8)

Let \( g^{TM}_{0,f,\theta}, ||\cdot||_{0,M}^{f,\theta} \) be the flattenings of \( g_0^{TM}, ||\cdot||_M^{0} \), compatible with the flattenings \( g_{0,f,\theta}^{TM}, ||\cdot||_{0,M}^{f,\theta} \) (cf. (1.15), (1.16)). More precisely, the metrics \( g^{TM}_{0,f,\theta}, ||\cdot||_{0,M}^{f,\theta} \) coincide with \( g_0^{TM}, ||\cdot||_M^{0} \) away from \( \cup_i V_i^{0,M}(\theta) \), and over \( V_i^{0,M}(\theta) \) the metric \( g^{TM}_{0,f,\theta} \) is induced by

\[ \frac{dz_i^{0,M} \ln |z_i^{0,M}|}{z_i^{0,M} \ln |z_i^{0,M}|} \psi(\ln |z_i^{0,M}|/\ln \theta), \] (4.9)
Also, for \( s_{DM} \) as in (4.7), we have
\[
\left\| dz_i^0, M / s_{DM} / z_i^0, M \right\|_{0, M}^{f, \theta} = \ln |z_i^0, M| \psi(\ln |z_i^0, M| / \ln \theta). \tag{4.10}
\]

Let’s denote by \( ||\cdot||_{f, \theta, M}^0, ||\cdot||_{f, \theta, M}^0 \) the norms on \( \omega_M \) over \( M \) induced by \( g_{0, f, \theta}^M \) and \( g_{1, f, \theta}^M \) respectively.

**Proof of (1.23).** By Theorem A, for any \( \theta \in [0, 1] \), we have
\[
2 \ln \left( \frac{\||\cdot\||_Q(g_{0, f, \theta}^M, h^\xi \otimes ||\cdot||_{0, M}^2)}{\||\cdot\||_Q(g_{1, f, \theta}^M, h^\xi \otimes ||\cdot||_{0, M}^2)} \right) = 2 \ln \left( \frac{\||\cdot\||_Q(g_{0, f, \theta}^M, h^\xi \otimes ||\cdot||_{0, M}^2)}{\||\cdot\||_Q(g_{1, f, \theta}^M, h^\xi \otimes ||\cdot||_{0, M}^2)} \right). \tag{4.11}
\]

Now we will show that the limit of the right-hand side of (4.11), as \( \theta \to 0 \) is exactly the right-hand side of (1.23). Then Theorem B will follow from (4.11).

We denote by \( ||\cdot||_{f, \theta, M}^0, ||\cdot||_{f, \theta, M}^0 \) the norms on \( \omega_M \) induced by \( g_{0, f, \theta}^M \) and \( g_{1, f, \theta}^M \). Set
\[
\Phi(\theta) := \left[ \frac{\tilde{t}(\omega_{M}^{-1}, (||\cdot||_{f, \theta, M}^0)^{-2}, (||\cdot||_{f, \theta, M}^0)^{-2}) \text{ch}(\xi, h^\xi) \text{ch}(\omega_M(D), (||\cdot||_{f, \theta, M}^0)^2n)}{\frac{\text{ch}(\omega_M(D), (||\cdot||_{f, \theta, M}^0)^2n)}{2}} \right]. \tag{4.12}
\]

Then, by Theorem 3.1, we have
\[
2 \ln \left( \frac{\||\cdot\||_Q(g_{0, f, \theta}^M, h^\xi \otimes ||\cdot||_{0, M}^2)}{\||\cdot\||_Q(g_{1, f, \theta}^M, h^\xi \otimes ||\cdot||_{0, M}^2)} \right) = \int_M \Phi(\theta). \tag{4.13}
\]

where \( \tilde{t} \) and \( \text{ch} \) are given by (1.21) and (1.22). We decompose the right-hand side of (4.13) into integrals over \( M \setminus (\cup_i(V_i^M(\theta) \cup V_i^{0, M}(\theta))) \) and over \( V_i^M(\theta) \cup V_i^{0, M}(\theta) \), \( i = 1, \ldots, m \). Since the flattenings \( g_{0, f, \theta}^M, g_{1, f, \theta}^M \) and \( ||\cdot||_{f, \theta, M}^0, ||\cdot||_{f, \theta, M}^0 \) coincide with \( g_{0, f, \theta}^M, g_{1, f, \theta}^M \) and \( ||\cdot||_{0, M}^0, ||\cdot||_{0, M}^0 \) over \( M \setminus (\cup_i(V_i^M(\theta) \cup V_i^{0, M}(\theta))) \), and the quantities under the integration in the anomaly formula are local, we see by Lebesgue dominated convergence theorem, by the finiteness of the right-hand side of (1.23) and by (1.24), that the integral of \( \Phi(\theta) \) over \( M \setminus (\cup_i(V_i^M(\theta) \cup V_i^{0, M}(\theta))) \) converges to the integral part in the right-hand side of (1.23), as \( \theta \to 0 \).

Now let’s study the contribution over \( \cup_i(V_i^M(\theta) \cup V_i^{0, M}(\theta)) \) of the integral in (4.13). We note that in the case when \( \phi \) from (1.25) has compact support in \( M \), this integral is actually zero for \( \theta \) sufficiently small (which is consistent with the statement of Theorem B).

From the discussion above, (4.2), (4.13), and the fact that we restrict ourselves to the case \( (\xi, h^\xi) \) trivial around the cusps, Theorem B will follow from the following

**Lemma 4.1.** As \( \theta \to 0 \), we have
\[
\int_{V_i^M(\theta) \cup V_i^{0, M}(\theta)} \Phi(\theta) \to -\frac{\text{rk}(\xi)}{6} \ln |(h_i^{\phi})'(0)|. \tag{4.14}
\]
Proof. All the subsequent formulas should be regarded as being valid over $V_i^M(\theta) \cup V_i^{0,M}(\theta)$. By (4.6) and (4.7), we have

$$
c_1(\omega_M, (||\cdot||_{f,\theta,M}^0)^2) = -\frac{1}{2\pi} \left( \psi(\ln |z_i^M|/\ln \theta) \cdot (2 \ln |z_i^M| + 2 \ln |\ln |z_i^M||) \right)
$$

$$
= \left[ \ln |z_i^M|/\psi'(\ln |z_i^M|/\ln \theta) \right] + \frac{\psi'(\ln |z_i^M|/\ln \theta)}{2z_i^M \ln \theta} + O\left( \frac{\ln |\ln |z_i^M||}{|z_i^M|^{1/2} \ln z_i^M} \right)
$$

$$
+ O\left( \frac{\ln |\ln |z_i^M||}{|z_i^M|^{1/2} \ln z_i^M} \right)^2 \frac{z_i^M}{2\pi}.
$$

(4.15)

$$
c_1(\omega_M(D), (||\cdot||_{f,\theta,M}^0)^2) = -\frac{1}{2\pi} \left( \psi(\ln |z_i^M|/\ln \theta) \cdot (2 \ln |\ln |z_i^M||) \right)
$$

$$
= O\left( \frac{\ln |\ln |z_i^M||}{|z_i^M|^{1/2} \ln z_i^M} \right) d\bar{z}_i^M.
$$

(4.16)

By $d\bar{z}_i^M = (N_i^\phi)'(z_i^M) \cdot d\bar{z}_i^M$ and $\ln |\ln |z_i^M|| = \ln |\ln |z_i^M|| + O(1/\ln |\ln |z_i^M||)$, we deduce

$$
\ln (||\cdot||_{f,\theta,M}^0/||\cdot||_{f,\theta,M}^0) = \psi(\ln |z_i^M|/\ln \theta) \left( \ln |z_i^M| + \ln |\ln |z_i^M|| \right)
$$

$$
- \left( \psi(\ln |z_i^M|/\ln \theta) \left( \ln |z_i^M| + \ln |\ln |z_i^M|| \right) - \ln |(h_i^\phi)'(z_i^M)| \right)
$$

$$
= \ln |(h_i^\phi)'(0)| \left( 1 + \psi(\ln |z_i^M|/\ln \theta) + \psi'(\ln |z_i^M|/\ln \theta) \ln |z_i^M|/\ln \theta \right)
$$

$$
+ O\left( \frac{\ln |\ln |z_i^M||}{|\ln |z_i^M||} \right),
$$

(4.17)

$$
\ln (||\cdot||_{0,M}/||\cdot||_{0,M}) = \psi(\ln |z_i^M|/\ln \theta) \ln |\ln |z_i^M||
$$

$$
- \left( \psi(\ln |z_i^M|/\ln \theta) \ln |\ln |z_i^M|| \right) = O\left( \frac{\ln |\ln |z_i^M||}{|\ln |z_i^M||} \right).
$$

(4.18)

Finally, from (4.15) and the analogous statement for $||\cdot||_{f,\theta,M}^0$, we easily get

$$
\partial \bar{\theta} \ln (||\cdot||_{f,\theta,M}^0/||\cdot||_{f,\theta,M}^0) = O\left( \frac{\ln |\ln |z_i^M||}{|z_i^M| \ln |z_i^M||^2} \right) d\bar{z}_i^M.
$$

(4.19)

From Theorem 3.1, (1.21), (1.22) and (4.15)–(4.19), we get

$$
\int_{V_i^M(\theta) \cup V_i^{0,M}(\theta)} \Phi(\theta) = \frac{-rk(\xi)}{3} \int_{V_i^M(\theta) \cup V_i^{0,M}(\theta)} c_1(\omega_M, (||\cdot||_{f,\theta,M}^0)^2) \ln \left( \frac{||\cdot||_{f,\theta,M}^0}{||\cdot||_{f,\theta,M}^0} \right)
$$

$$
+ O\left( \frac{\ln |\ln |z_i^M||}{|z_i^M| \ln |z_i^M||^2} \right) d\bar{z}_i^M.
$$

(4.20)

From (4.15), (4.17) and (4.20), we get

$$
\lim_{\theta \to 0} \int_{V_i^M(\theta) \cup V_i^{0,M}(\theta)} \Phi(\theta) = \frac{2 \ln |(h_i^\phi)'(0)|}{3} \cdot rk(\xi)
$$
\[
\lim_{\theta \to 0} \int_{\theta^{1/2}}^{\theta} \frac{1}{r} \left( \frac{\psi''(\ln r)}{\ln \theta} \right) \frac{\ln r}{2(\ln \theta)^2} + \psi'(\ln r) \left( \frac{1}{\ln \theta} + \psi''(\ln r) \ln r \right) dr
\]
\[
= -\frac{2 \ln |h^\theta_0(0)|}{3} \cdot \text{rk}(\xi) \cdot \int_{1/2}^1 \left( -\psi'(u) + \psi'(u) \psi(u) + u \psi'(u)^2 \right.
\]
\[
- u \psi''(u)/2 + u \psi''(u) \psi(u)/2 + u^2 \psi'(u) \psi'(u)/2 \Big) du,
\]
where in the last identity we used the change of variables \( u := \ln r / \ln \theta \). By the integration by parts and (2.53), we have
\[
\int_{1/2}^1 \psi'(u) du = -1, \quad \int_{1/2}^1 u \psi''(u) \psi(u) du = \frac{1}{2} - \int_{1/2}^1 u \psi'(u)^2 du,
\]
\[
\int_{1/2}^1 u \psi''(u) du = 1, \quad \int_{1/2}^1 u^2 \psi'(u) \psi'(u) du = -\int_{1/2}^1 u \psi'(u)^2 du,
\]
\[
\int_{1/2}^1 \psi(u) \psi(u) du = -\frac{1}{2}.
\]
We get (4.14) from (4.21), (4.22).

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