Stabilizing Error Correction Codes
for Controlling LTI Systems over Erasure Channels

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Abstract—We propose \((k, k')\) stabilizing codes, which is a type of delayless error correction codes that are useful for control over networks with erasures. For each input symbol, \(k\) output symbols are generated by the stabilizing code. Receiving any \(k'\) of these outputs guarantees stability. Thus, the system to be stabilized is taken into account in the design of the erasure codes. Our focus is on LTI systems, and we construct codes based on independent encodings and multiple descriptions. The theoretical efficiency and performance of the codes are assessed, and their practical performances are demonstrated in a simulation study. There is a significant gain over other delayless codes such as repetition codes.

I. INTRODUCTION

There has been a vast amount of literature on networked control systems over erasure channels, cf. [1]–[22]. In [2], it was shown that for a given unstable linear time invariant (LTI) system, there exists a critical limit on the packet dropout rate beyond which the system cannot be stabilized in the usual mean-square sense. To go beyond this critical limit, several techniques have been proposed ranging from error correction codes [5], [8] and multiple descriptions [18] to packetized predictive control [7] to name a few.

Assume the output of the plant is to be encoded and transmitted over a digital erasure channel, where packets are either completely lost or received without errors. To recover from erasures, error correction codes can be utilized [8], [23]. Error-correction codes are often designed with a certain loss rate of the channel in mind, and do not necessarily rely on the plant (exceptions include the work in [8] which tracks the plant state). For example, \((n, k)\) erasure channel codes, take \(k\) source packets and outputs \(n\) channel packets. If any \(k\) of the channel packets are received, the original \(k\) source packets can be completely recovered. If more than \(k\) packets are received, the additional received data packets are not useful since they do not contain any further information about the plant state than what is already known. Finally, if less than \(k\) packets are received, the source packets can generally not be recovered at all and all the transmitted information is in this case wasted.

An alternative to error correction codes are multiple descriptions [25], which combines source and channel coding. With multiple descriptions, the source is encoded into a number of descriptions, which are individually transmitted over the channel. There is no priority on the descriptions, and any subset of the descriptions can be jointly decoded to achieve a desired performance. Multiple descriptions were, for example, used for state-estimation in [3] and combined with packetized predictive control in [18]. One of the problems with multiple descriptions is that it is generally very hard to design good multiple-description codes. Another problem is that the descriptions generally contains redundant information except in the limit of vanishing data rates or when used in the extreme asymmetric situation, where the descriptions are prioritized and a successive refinement scheme is obtained. If one is able to construct a successive refinement source coder, then it was shown in [26] and [27], that the layers in the successive refinement code can be combined with traditional error correction codes in order to obtain a (sub-optimal) multiple-description code. It was recently shown that a combination of successive refinement and multiple descriptions with feedback becomes rate-distortion optimal under certain asymptotical conditions [28].

We will in this paper focus on discrete-time LTI plants, stationary Gaussian disturbances, Gaussian initial state, scalar-valued control inputs and sensor outputs. Thus, the plant state can have an arbitrary dimensionality but the control signal as well as the output of the plant are both scalar valued. For such a system, the minimal information rate required to guarantee stability and a desired performance (measured in terms of the variance of the plant output) was completely characterized in [24] for the case of communications over error-free digital channels. An illustration of the system is shown in Fig. 1.

We show that simple stabilizing erasure codes can be obtained from properly designed independent encodings [28] or multiple descriptions [25]. Specifically, for a given LTI plant we design a \((k, k')\) stabilizing code such that when combining any \(k'\) descriptions of the code, the resulting SNR is above a critical limit, which guarantees that the decoded control signal contains sufficient information to stabilize the plant. We show that simple codes based on independent encodings are asymptotically efficient for nearly stable plants. In general, for unstable plants, it is advantageous to use a
design based on multiple descriptions. In a simulation study, we demonstrate that for the same sum-rate and delay, it is possible to achieve a significant gain in performance over that which is possible with repetition coding.

II. BACKGROUND

Let us begin by considering the networked control system presented in [24], and which is shown in Fig. 1. Here $P$ is an LTI plant that is open-loop unstable, $u$ is the scalar control input, and $y$ is the scalar sensor output of the plant. The external disturbance is denoted by $d$ and $e$ is an error signal that is related to the output performance. The plant output $y$ is to be encoded by the causal encoder $Enc$, transmitted over the ideal noise-less digital channel, and then decoded by the causal decoder $Dec$. The encoder-decoder pair $(Enc, Dec)$ also contains the controller. Thus, the output of the decoder is the control signal to the plant. For a fixed data rate of the coder, the performance will be measured by the variance $\sigma^2_e$ of the output $e$. We have the following linear input-output relationship through the plant $P$:

$$\begin{bmatrix} e \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} d \\ u \end{bmatrix}. \tag{1}$$

It was shown in [24], that if the initial state $x_0$ and the external disturbances are arbitrarily colored but jointly Gaussian, then the optimal encoder-decoder pair constitute a linear system + noise. This implies that the system in Fig. 1 can be modelled by the linear system shown in Fig. 2. In this system, $F$ and $L$ are LTI systems, and $q$ is an additive white Gaussian noise, which simulates the coding noise due to source coding. In this equivalent form, we have the following relationship [24]:

$$u = Fw, \quad w = v + q, \quad v = L \begin{bmatrix} z^{-1}w \\ y \end{bmatrix}, \quad L = [L_w, L_y]; \tag{2}$$

where $z^{-1}$ indicates a one-step delay operator. The signal-to-noise ratio $\gamma$ of the system is defined as:

$$\gamma \triangleq \frac{\sigma^2_v}{\sigma^2_q}. \tag{3}$$

It was shown in [24], that for any proper LTI filters $F$ and $L$ that makes the system in Fig. 2 internally stable and well-posed, we have the following explicit expressions:

$$\gamma = \|S - 1\|^2 + \frac{1}{\sigma^2_q} \| P_{21}S \|^2 \tag{4}$$

$$\sigma^2_v = \| P_{11} + P_{12}K(1 - P_{22}K)^{-1} P_{21} \|^2 + \| P_{21}FS \|^2 \sigma^2_q \tag{5}$$

$$S = (1 - L_wz^{-1} - P_{22}FL_y)^{-1} \tag{6}$$

$$K = FL_y(1 - L_wz^{-1})^{-1}. \tag{7}$$

To find the optimal filters $(F, L)$ that minimizes $\sigma^2_v$ subject to a constraint on $\gamma$, one needs to solve a convex optimization problem [24]. A lower bound on the minimal coding rate $R$ achievable when using optimal filters $(F, L)$ is given by:

$$R \geq \frac{1}{2} \log_2 (1 + \gamma). \tag{8}$$

It is clear from (4) that asymptotically as $\sigma^2_q \to \infty$, $\gamma \to \|S - 1\|^2$, which shows that the minimum SNR required for stability is $\|S - 1\|^2$, and the minimum rate required for stability is given by $\frac{1}{2} \log_2 (1 + \|S - 1\|^2)$.

III. CAUSAL CODERS

The encoder $E_i : \mathbb{R}^i \times S^i \rightarrow Y^k$ at time $i$ is a (possibly) time-varying causal one-to-many map, which at each time instance produces $k$ outputs, that is:

$$(y_{i}^{(1)}(i), \ldots, y_{i}^{(k)}(i)) = E_i(y^i, s^i), \tag{9}$$

where $y_{i}^{(j)}(i) \in Y_i$ denotes the $j$th output of the encoder at time $i$, and $y^i = y_1, \ldots, y_i$ indicates that the encoder is only using the sequence of current and past plant outputs. The sequence $s^i$ denotes side information. Thus, the encoder can be randomized via the side information, which for example allows one to obtain a stochastic encoder. The outputs of the encoder are discrete. However, by use of subtractive dithering techniques, the resulting reconstructed values at the decoder are continuous. With this, the quantizer can be modelled as an additive white noise source [29].

Let $I(i) \subseteq \{1, \ldots, k\}$ denote the set of indices of the received descriptions at time $i$. At each time instance, $k$ descriptions are produced and transmitted over the digital erasure channels. The set of causal decoders at time $i$ is $D_i^T : Y_{i}^{I(i)} \times S_i^{I(i)} \rightarrow \mathbb{R}$, $\forall I(i) \subseteq \{1, \ldots, k\}$. For a particular choice of decoder, say $D_i^{T^i}$, the reconstructed signal $u(i) \in \mathbb{R}$ at time $i$ is given by:

$$u(i) = D_i^{T^i}(y_{i}^{I(i)}, s_i^{T(i)}). \tag{10}$$

Section IV considers lower bounds on the coding rates based on Gaussian coding schemes. The operational data rates obtained when using a practical coding scheme is generally greater than these lower bounds. These operational issues regarding the stabilizing codes are treated in the longer version of the paper [30]. In particular, since we are here focusing on the situation with a scalar output, we need to use scalar quantizers. It is well known that scalar quantizers suffers from at rate-loss compared to vector quantizers except at very low bit rates. In addition, we need to entropy encode the output of the quantizer to further reduce the bitrate. Since
the entropy coder is operating on one sample at a time, it will generally not be possible to reach the entropy of the output.

IV. STABILIZING ERROR CORRECTION CODES

We will first introduce some definitions, which we will be needing in the sequel.

**Definition 1:** We will denote by \((F, L, P, \gamma)\) a linear system on the form shown in Fig. 2 which has coding rate \(R = 0.5 \log_2(1 + \gamma)\) and performance \(D = \sigma_k^2\), where \(\sigma_k^2\) is given by \((5)\).

**Definition 2:** A \((k, k')\) stabilizing code for the system \((F, L, P, \gamma)\) produces \(k\) descriptions such that using any \(k'\) of them is sufficient to stabilize the system.

To quantify the efficiency of a \((k, k')\) stabilizing code when used on a particular system \((F, L, P, \gamma)\), we will compare the sum-rate \(R_S\) of the \(k\) descriptions to the rate \(R\) required for a single-description code to achieve the same performance as that obtained when using all \(k\) descriptions (without erasures). In the linear Gaussian case, the efficiency can be assessed by simple means as shown in the definition below.

**Definition 3:** The efficiency \(\eta\) of a \((k, k')\) stabilizing code for the system \((F, L, P, \gamma)\) is defined as:

\[
\eta \triangleq \frac{\log_2(1 + \hat{\gamma})}{k \log_2(1 + \tilde{\gamma})}, \quad 0 \leq \eta \leq 1,
\]

where \(\hat{\gamma}\) is the SNR when using any single description out of the \(k\) descriptions, and \(\tilde{\gamma}\) is the SNR when combining all \(k\) descriptions.

When measuring efficiency in \((11)\), we need to make sure that we compare the coding rates of systems having similar performance (in terms of \(\sigma_k^2\)). The best performance of a \((k, k')\) stabilizing code is obtained when using all \(k\) descriptions, which results in an SNR of \(\hat{\gamma}\). The rate of each description is \(0.5 \log_2(1 + \hat{\gamma})\) and the sum-rate is \(0.5k \log_2(1 + \hat{\gamma})\). On the other hand, when not using a stabilizing code we need a coding rate of \(0.5 \log_2(1 + \tilde{\gamma})\) to achieve an SNR of \(\tilde{\gamma}\).

For a classical \((n,k)\) error correction code that produces \(n\) outputs for each \(k\) input sample (or block of samples), the efficiency is \(k/n\), and the delay is \(k-1\) samples (blocks). A repetition code that duplicates the same source block \(k\) times has efficiency \(1/k\) and zero delay. The stabilizing codes that we propose are also delayless and are able to improve upon the efficiency of repetition codes due to the property that descriptions can synergistically improve upon each other.

A. Stabilizing codes based on independent encodings

**Definition 4:** Let \(w_i = v + q_i, i = 1, \ldots, k\). If \(w_j\) and \(w_i, i \neq j\), are conditionally independent given \(v\), then we refer to \(w_1, \ldots, w_k\) as independent encodings \((28)\).

**Lemma 1:** \((k, k')\) Stabilizing Code Based on Independent Encodings. Consider the system \((F, L, P, \gamma)\), which is illustrated in Fig. 2. Let \(v\) be Gaussian and let \(w_i = v + q_i, i = 1, \ldots, k\), be \(k\) independent encodings of \(v\), where \(q_i, i = 1, \ldots, k\), are mutually independent, zero-mean Gaussian distributed, and having a common variance \(\sigma^2\). If for some \(1 \leq k' \leq k\), the common variance satisfies

\[
\sigma^2 \leq \frac{\gamma k' ||L_y P_{21}S||^2}{||S - 1||^2 (\gamma - ||S - 1||^2)},
\]

where \(S\) is given in \((6)\), then \(w_i, i = 1, \ldots, k\), form a \((k, k')\) stabilizing code for the system \((F, L, P, \gamma)\).

**Proof:** The variance of \(\frac{1}{k} (w_{i_1} + \cdots + w_{i_k})\) is \((k')^{-1} \sigma^2\) for any subset of \(k'\) encodings. The resulting SNR \(\gamma' = k' \sigma^2 \sigma^{-2}\), when combining \(k'\) descriptions, needs to satisfy:

\[
\gamma' = k' \sigma^2 \sigma^{-2} > ||S - 1||^2,
\]

since \(||S - 1||^2\) is the minimal SNR required to guarantee stability. We now use that \(\gamma' \sigma^2 = \sigma_k^2\), and from \((4)\) we get:

\[
\sigma_v^2 = \gamma (||S - 1||^2)^{-1} ||L_y P_{21}S||^2.
\]

Inserting into \((13)\) and re-arranging terms leads to:

\[
\sigma^2 < \gamma k' ||S - 1||^{-2} (\gamma - ||S - 1||^2)^{-1} ||L_y P_{21}S||^2,
\]

which leads to \((12)\).

The following lemma provides a lower bound on the sum-rate required for a \((k, k')\) stabilizing code based on independent encodings. We note that if one is not interested in the performance when receiving less than \(k'\) descriptions, then the sum-rate can generally be further reduced by use of distributed source coding techniques such as Slepian-Wolf coding \((31)\). However, at low coding rates, the bound becomes asymptotically optimal as is shown by Lemma \(3\).

**Lemma 2:** The minimum sum-rate \(R_S\) of a \((k, k')\) stabilizing code based on independent encodings for the system \((F, L, P, \gamma)\) is:

\[
R_S \geq \frac{k}{2} \log_2 \left(1 + \frac{||S + 1||^2}{k'}\right).
\]

**Proof:** Let \(\sigma^2\) be the variance of the coding noise for a single description of the \((k, k')\) stabilizing code. Then, the resulting variance when linearly combining \(k'\) descriptions is \(\sigma^2/k'\). Thus, \(\text{SNR} = k \sigma^2/k' \geq ||S - 1||^2\), where the inequality follows since \(||S - 1||^2\) is the minimum SNR that guarantees stability. Isolating \(\sigma^2\) leads to:

\[
\sigma^2 \leq ||S - 1||^{-2} k' \sigma_v^2.
\]

We can now express the sum-rate in terms of \(\sigma^2\), that is:

\[
R_S = \frac{k}{2} \log_2 \left(1 + \frac{\sigma^2}{\sigma_v^2}\right) \geq \frac{k}{2} \log_2 \left(1 + (k')^{-1} ||S - 1||^2\right).
\]

**Lemma 3:** Consider the system \((F, L, P, \gamma)\). The efficiency of a minimum sum-rate \((k, k')\) stabilizing code based on independent encodings is given by:

\[
\eta = \frac{\log_2 \left(1 + (k')^{-1} ||S - 1||^2\right)}{k \log_2 \left(1 + (k')^{-1} ||S - 1||^2\right)},
\]

and the code is asymptotically efficient in the sense of:

\[
\lim_{||S - 1||^2 \to 0} \eta = 1.
\]
Proof: The first part follows immediately from (17), since the SNR for a single description is \( \sigma_i^2 / \sigma^2 \) and for \( k \) descriptions it is \( k \sigma_i^2 / \sigma^2 \). The second part follows since the logarithm of the number 1 + \( kc \) is approximately linear in \( k \) when \( c \ll 1 \), i.e., \( \log(1 + kc) \approx k \log(1 + c) \) for small \( c \).

For a \((F, L, P, \gamma)\) system, if \( \|S - 1\| = 0 \) it means that the system is stable. Thus, the second part of Lemma 5 considers the situation where the plant is either stable or nearly stable, i.e., the unstable poles are near the unit circle. In this case, the coding rates are arbitrary small, and the \( k \) descriptions of the \((k, k')\) stabilizing code becomes mutually independent.

Thus, there is no redundancy by using \( k \) descriptions each of rate \( R/k \) over a single description of rate \( R \) [28].

**Lemma 4:** Consider the system \((F, L, P, \gamma)\). The performance (in terms of \( \sigma_i^2 \)) for this system when using \( \ell \geq k' \) descriptions of a minimum sum-rate \((k, k')\) stabilizing code based on independent encodings is:

\[
\sigma_e^2 = \|P_{11} + P_{12}K(1 - P_{22}K)^{-1}P_{21}\|^2 + \frac{k' \gamma}{\ell} \frac{1}{\|S - 1\|^2} \gamma \|F_2 FS\|^2 \|L_y P_{21} S\|^2, \quad \ell = k', \ldots, k. \tag{20}
\]

**Proof:** Follows from (5) by inserting (15) and the fact that the noise variance satisfies \( \sigma_e^2 \) for \( \ell = 1, \ldots, k \).

**B. Example 1**

Consider a plant \( P \) that provides the following input-output relationship between \( (u, d) \) and \( y \):

\[
y = \frac{0.165}{(z - 2)(z - 0.5789)}(u + d), \quad c = y, \tag{21}
\]

where the external disturbance \( d \) has a standard normal distribution. Notice that the plant has an unstable pole at \( z = 2 \), which implies that the minimum SNR required for stability is \( \|S - 1\|^2 = 3 \), and equivalently the minimum coding rate is \( 0.5 \log_2(1 + 3) = 1 \) bit. For this plant, we can choose a particular \( \gamma \) and find the optimal filters \( L, F \) and associated \( S, K \) by using the method described in [24]. From these we can find the performance \( \sigma_e^2 \) using (5) and coding rate \( R = 0.5 \log_2(1 + \gamma) \). Changing \( \gamma \) leads to another set of \( L, F, K, S \) variables and different performances and coding rates.

Let us now design a \((4, 2)\) stabilizing code, so that receiving any 2 descriptions implies that the minimum SNR requirement is fulfilled. We choose \( \sigma_e^2 \) so that the resulting SNR is \( \frac{4}{\gamma} \), when linearly combining \( \ell = 1, \ldots, 4 \) descriptions. For \( \gamma = 7.2 \), we obtain: SNR = 1.8, 3.6, 5.4, and 7.2 for \( \ell = 1, \ldots, 4 \), respectively. For \( \ell = 2 \) it is clear that the resulting SNR is greater than the minimum of 3, and we therefore have a \((4, 2)\) stabilizing code.

The coding rate per description is \( 0.5 \log_2(1 + 1.8) = 0.74 \) bits, and the sum-rate is \( R_S = 2.96 \) bits. The coding rate required for a single-description system to achieve SNR = 7.14 is \( R = 0.5 \log_2(1 + 7.14) = 1.51 \) bits. Thus, the efficiency is \( \eta = 1.51/2.96 = 0.51 \). For comparison, a repetition code with 4 descriptions would have an efficiency of \( \eta = 0.25 \).

In Fig. 3, we have illustrated the resulting SNR when combining the \( \ell = 1, \ldots, 4 \) descriptions as a function of \( \gamma \).

**C. Stabilizing codes based on multiple descriptions**

It is possible to introduce correlation between the quantization noises \( q_i, i = 1, \ldots, k \), of the encodings in Definition 4 which makes it possible to exploit the benefits of multiple descriptions. Of course, zero correlation is a special case of multiple descriptions, which is usually referred to as the no excess marginal rate case [32]. When introducing correlation, the sum-rate is no longer simply just given by the sum of the optimal marginal (description) rates. The sum-rate also becomes a function of the amount of correlation introduced; the greater (negative) correlation, the greater sum-rate [33].

**Lemma 5:** \((k, k')\) Stabilizing Code Based on Multiple Descriptions. Consider the system \((F, L, P, \gamma)\), which is illustrated in Fig. 2. Let \( v \) be Gaussian and let \( w_i = v + q_i, i = 1, \ldots, k \), where \( q_i, i = 1, \ldots, k \), are zero-mean Gaussian distributed with variance \( \sigma_i^2 \), and pairwise correlated with correlations coefficient -1 < \( \rho \) < 0. If
for some $k'$ and $\rho$, the common variance $\sigma^2$ satisfies
\[
\sigma^2 \leq \frac{\gamma k' ||L_y P_{21}S||^2}{||S - 1||^2(\gamma - ||S - 1||^2)(1 + (k' - 1)\rho)},
\]
where $S$ is given in (8), then $w_i, i = 1, \ldots, k$, form a $(k, k')$ stabilizing code for the system $(F, L, P, \gamma)$.

Proof: We need to ensure that $\text{SNR} = ||S - 1||^2$, when receiving at least $k'$ descriptions. The noise variance when combining any $k'$ descriptions is given by:
\[
\text{var}\left(\frac{1}{k'} \sum_{i=1}^{k'} q_i\right) = \frac{\sigma^2}{k'}(1 + (k' - 1)\rho).
\]
Using (23), the SNR is given by:
\[
\frac{\sigma^2}{k'}(1 + (k' - 1)\rho) \geq ||S - 1||^2.
\]
Isolating $\sigma^2$ and inserting (44) leads to:
\[
\begin{align*}
\sigma^2 &\leq k' ||S - 1||^{-2}(1 + (k' - 1)\rho)^{-1}\sigma^2_v^{-2} \\
&= k' ||S - 1||^{-2}(1 + (k' - 1)\rho)^{-1}\gamma(\gamma - ||S - 1||^2)^{-1} \\
&\times ||L_y P_{21}S||^2.
\end{align*}
\]

Let $\rho \in (\frac{-1}{2}, 0]$ be the common correlation coefficient between all noise pairs $q_i, q_j, \forall i \neq j$, and let $\sigma^2$ be their common variance. If we are only interested in the performance when receiving $k'$ descriptions or all $k$ descriptions, then the sum-rate $R_S$ can be explicitly expressed [34]:
\[
R_S = \frac{1}{2k'} \log_2 \left( k' + \frac{\sigma^2(1 + (k' - 1)\rho)}{\sigma^2(1 - \rho)} \right) + \frac{1}{2k} \log_2 \left( \frac{1 - \rho}{1 + (k' - 1)\rho} \right),
\]
where it is assumed the source is standard normal.

V. Simulation Study

We consider the same system as that of Example 1, and we will assume i.i.d. packet losses. The encoder is informed about the packet loss probability but does not know when an erasure occurs. Knowledge of the packet loss probability makes it possible to design an efficient entropy coder (lossless coder).

We will be using a subtractively dithered scalar quantizer, which is a stochastic quantizer that provides different outputs, when encoding the same source multiple times [29]. We will use this to form the $k$ independent encodings.

We encode the output $v$ of Fig. 2 using a subtractively dithered scalar quantizer $Q_{\Delta}$ with step-size $\Delta$ to obtain:
\[
w_i = Q_{\Delta}(v + \xi_i) - \xi_i, i = 1, \ldots, k.
\]
where $\xi$ denotes the dither signal. We choose the step-size $\Delta$ so that the resulting SNR when using only a single description $w_i$ is 1.76, which is below that required for stability. Combining any two descriptions yields SNR = 3.57 and combining all three yields SNR = 5.29. Thus, using at least two descriptions is sufficient to stabilize the system. Based on this we design a $(3, 2)$ stabilizing code, which for each input sample produces three outputs using the quantizer three times. The theoretical efficiency of this scheme is $\log 2(1 + 5.29)/(3\log_2(1 + 1.76)) = 0.6$. In practice, we suffer from a rate loss due to using a scalar quantizer. The theoretical rate is $1/2\log_2(1 + 1.76) = 0.73$. However, transmitting less than one bit per sample is only possible when encoding vectors. The measured entropy of the quantized output is 1.57 bits per description.

We have plotted the performance of the $(3, 2)$ stabilizing code in Fig. 5 as a function of the packet-loss probability. We assume i.i.d. packet losses, and simply average the received descriptions to form the reconstruction. Also shown is the performance when transmitting one of the descriptions three times. This corresponds to a $(3, 1)$ repetition code having similar sum-rate as the $(3, 2)$ stabilizing code. For each packet-loss probability, the performance and rates are averages over a realization having $10^6$ samples. It can be observed that using a stabilizing code is up till 3 dB better than a repetition code at low packet-loss probabilities.

We also show in Fig. 5 the performance of a $(2, 1)$ stabilizing code, which is compared to a $(2, 1)$ repetition code. The SNR is 3.45 and 6.91, when using 1 or 2 descriptions, respectively, of the $(2, 1)$ stabilizing code. The measured output entropy after scalar quantization is in this case 1.76 bits per description, and the sum-rate is 3.52 bits.

Finally, we design a $(3, 2)$ stabilizing code based on multiple descriptions. It is not straightforward to obtain correlated noises between the descriptions, and we use here the approach described in [35], which is based on nested lattices and index assignments. The source is first quantized using a fine-grained quantizer referred to as the central quantizer. Then, a one-to-many map is applied, which maps the quantized value to $k$ points in a nested (coarser) lattice. If all $k$ coarser points are received, the map is invertible and the point of the central quantizer is used for reconstruction. If less than $k$ descriptions are received, the reconstruction is given by the average of the received points in the coarser lattice [35]. We are using a nesting factor of 5, and the result-
ing pairwise correlation coefficient between the descriptions is $\rho = -0.41$. The SNR of a single description is 1.68 and that of two descriptions is 5.6, which is above the critical value for stability. The SNR when all descriptions are used is 12.0. The step-size of the fine lattice is chosen such that the resulting bitrate is similar to that of the $(3, 2)$ stabilizing code based on independent encodings.

It can be seen in Fig. 5 that stabilizing codes outperform repetition coding. Moreover, using MD coding when constructing the stabilizing codes is better than using independent encodings, except at very low bitrates or very high packet-loss rates.

VI. CONCLUSIONS

A new construction of error correction codes were proposed, which takes the stability of the plant into account. For linear systems with scalar input and output, explicit designs were provided, and it was shown that there is a significant gain over using traditional repetition codes. Similar to repetition coding, the proposed codes do not add additional delays but operate on each sample at a time.

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