Speed limits in Liouville space for open quantum systems

RAAM UZDIN\(^{(a)}\) and RONNIE KOSLOFF

Fritz Haber Research Center for Molecular Dynamics, The Hebrew University of Jerusalem
Jerusalem 9190401, Israel

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Abstract – One of the defining properties of an open quantum system is the variation of its purity in time. We derive speed limits on the rate of purity change for systems coupled to a Markovian environment. Our speed limits are based on Liouville space where density matrices are represented as vectors. This approach leads to speed limits that are always tighter compared to their parallel speed limits in Hilbert space. These bounds depend solely on the generators of the nonunitary dynamics and are independent of the particular state of the systems. Thus, they are perfectly suited to investigate dephasing, thermalization, and decorrelation processes of arbitrary states. We show that our speed limits can be attained and are therefore tight. As an application of our results we study dephasing of interacting spins, and the speed of classical and quantum correlation erasure in multi-particle system.

Determining the maximal rate of evolution of an open system is of prime importance in quantum physics. Any quantum system is unavoidably coupled to external degrees of freedom (the environment) that lead to loss of phase coherence and/or to thermalization [1]. In most applications, the main challenge is to minimize and slow down the effect of the environment. In quantum computing [2] and coherent control [3], it is vital to achieve low dephasing rates in order to obtain the target transformation. On the other hand, in cooling and pure state preparation it is desirable to speed up the influence of the surroundings. In addition, in quantum thermodynamics, a bound on the rate of thermalization of a heat engine with its reservoirs will limit its cycle time and therefore impose a restriction on the maximal power output [4]. An interaction with the environment is associated with changes of the system’s purity \(\mathcal{P} = \text{tr}(\rho^2)\) [1], where \(\rho\) is the density operator of the system. Despite the ability of the purity to quantify the impact of the environment on the system, to the best of our knowledge no general result on the maximal rate of purity variation has been obtained so far within the basic Markovian formalism of open quantum systems.

Bounds on the rate of quantum evolution are useful to assess if a process can be completed in a given time, without having to explicitly solve the equations of motion [5–10]. Quantum speed limits, have been studied in [11–24]. Their evaluation is of importance in essentially all areas of quantum physics where the determination of the minimal time of a process is of interest [25–29]. The traditional speed limits are associated with measures of distinguishability between an initial state and its corresponding final state (e.g., the angle in the projective Hilbert space for pure states [7], or the Bures angle for mixed states [19]). However speed limits can be studied for any quantity of interest that changes in time [30,31]. In this paper we focus on purity for the reason mentioned earlier. When speaking about speed limits of some quantity \(G\) (e.g. entropy [30,31], purity [31], or angle between states) it is important to distinguish between a bound on the instantaneous rate of change \(\frac{dG}{dt}\) and a bound on the cumulative change \(|G(t_f) - G(t_i)|\) carried out over a time interval \([t_i, t_f]\). The two are related via

\[
|G(t_f) - G(t_i)| = \left| \int_{t_i}^{t_f} \frac{dG}{dt} \, dt \right| \leq \int_{t_i}^{t_f} \left| \frac{dG}{dt} \right| \, dt. \quad (1)
\]

There is, however, an important difference between instantaneous and cumulative speed limits. Instantaneous speed limits often use the state of the system \(\rho\) to evaluate \(\frac{dG}{dt}\). In quantum metrology where the optimal state for phase estimation is needed [17,18], the state dependence is very useful. Other well-known speed limits such as the Mandelstam-Tamm bound [11], and the Margolus-Levitin bound [12] are state dependent. For cumulative bounds,
this dependence becomes a problem when \( H = H(t) \). For open-systems state-dependent speed limits see [30,31]. However, for the purpose of bounding cumulative changes, state-dependent speed limits cannot be used. If the integrand on the right-hand side of (1) depends on the state, it means that \( \rho(t) \) must be known in order to calculate the bound. However, if \( \rho(t) \) is available, it is possible to calculate \( G(t_f) - G(t_i) \) exactly and directly. Thus, for the speed limit to be useful, we impose the restriction that the bound on the rate of change must be state-independent (see [22,32,33] for state-independent bounds) and must be expressed solely in terms of the generators of motion (e.g., Hamiltonian and Lindblad operators [1]).

The goal of this paper is to provide cumulative state-independent speed limits for the purity change. Although this can accomplished within the standard density matrix formalism (Hilbert Space, hereafter) we find that the Liouville space approach produces speed limits that are always better than the Hilbert space speed limits. Next, we further improve our results by introducing the purity deviation. Finally, we employ our formalism to derive speed limits for multi-particle dephasing channels and for decorrelating channels. Although our purity deviation speed limits is state-independent and cumulative, it can be attained and is therefore tight.

**Purity speed limit in Hilbert space.** — We begin by deriving a state-independent purity speed limit using the standard density matrix formulation. We consider a possibly time-dependent \( N \)-level quantum system with Hamiltonian \( H = H(t) \). The effect of the environment on the dynamics of the system in the weak-coupling limit is described by a Markovian master equation of the Lindblad type for the density matrix of the system [1],

\[
d_t \rho = i[H, \rho] + L_i(\rho) + e^{i\phi(t)} \sum_k \left( A_k \rho A_k^\dagger - \frac{1}{2} A_k^\dagger A_k \rho \right),
\]

where the operators \( A_k \in \mathbb{C}^{N \times N} \) describe the interaction with the environment. For Markovian systems \( \phi(t) = 0 \), however (2) also describes some non-Markovian systems [34-36].

From (2) it follows that \( d_t \ln \text{tr}(\rho^2) = 2 \text{tr}(L_i(\rho))/\text{tr}(\rho^2) \). Integrating over time and using the triangle inequality, we have

\[
\ln \left( \frac{\mathcal{P}(t_f)}{\mathcal{P}(t_i)} \right) \leq \int_{t_i}^{t_f} \frac{2 |\text{tr}(L_i(\rho))|}{\text{tr}(\rho^2)} dt.
\]

Next, we exploit the fact that \( \mathcal{P}(t) = \text{tr}(\rho^2) = \|\rho\|^2 \), where \( \|B\| = \sqrt{\text{tr}(B^\dagger B)} \) denotes the standard Hilbert-Schmidt norm. An upper bound to (3) can be derived with the help of elementary matrix algebra [37,38]. Combining the Cauchy-Schwarz inequality, \( |\text{tr}(\rho L_i(\rho))| \leq \|\rho\|_2 \|L_i(\rho)\|_2 \), the triangle inequality, the submultiplicativity property of the norm, and the master equation, we find \( \|\rho\|_2 \|L_i(\rho)\|_2 \leq 2 \sum_k \|A_k^\dagger\rho A_k\|_2^2 \). Inserting this expression into (3), we obtain a “\( \text{norm integral} \)” inequality of type (1) for the logarithm of the purity

\[
- \ln \mathcal{P}(t_f) + \ln \mathcal{P}(t_i) \leq \int_{t_i}^{t_f} \sum_k \|A_k\|^2 dt,
\]

By using \( |\text{tr}(\rho L_i(\rho))| \leq \|\rho\|_2 \|L_i(\rho)\|_2 \leq \|L_i(\rho)\|_2 \) one can also get a speed limit for \( \mathcal{P} \) rather than for \(-\ln \mathcal{P} \) but it will be less tight than (4). Equation (4) provides a speed limit to the variation of the second-order Rényi entropy \( -\ln \mathcal{P} \) in terms of the Hilbert-Schmidt norm of the Lindblad operators. Its practical usefulness stems from the fact that the operators \( A_k \) can be experimentally determined in large variety of quantum systems [40-45]. As we shall see this simple bound is not very tight and scales badly with the number of levels. Fortunately, this can be remedied by using the Liouville space formalism.

**Purity speed limit in Liouville space.** — Quantum dynamics is traditionally described in Hilbert space. However, for open quantum systems, it is sometimes advantageous to use an alternative space where density operators are represented by vectors, and time evolution is generated by superoperators that operate on vectors just from the left (as in Schrödinger equation). This space is referred to as Liouville space [46]. We denote the “density vector” by \( |\rho\rangle \in \mathbb{C}^{1 \times N^2} \). It is obtained by reshaping the density matrix \( \rho \) into a larger single vector with index \( \alpha \in \{1,2,\ldots,N^2\} \). The vector \( |\rho\rangle \) is not normalized to unity in general. The norm squared is equal to the purity, \( \mathcal{P} = \text{tr}(\rho^2) = \langle \rho | \rho \rangle \), where \( \langle \rho | = | \rho \rangle^\dagger \), as usual. From the identity \( i\partial_t \rho \alpha = \sum_k [i\partial_t (A_k \rho \alpha)] \partial \beta \rho \beta = \sum_k \text{H}_\alpha \rho \beta \) it follows that in Liouville space the equation of motion (2) takes a Schrödinger-like form,

\[
i \partial_t |\rho\rangle = \mathcal{H} |\rho\rangle,
\]

when using the common matrix to vector index mapping \( \alpha = (\text{row} - 1)N + \text{column} \), the Hamiltonian superoperator \( \mathcal{H} \in \mathbb{C}^{N^2 \times N^2} \) is given by [38,47]

\[
\mathcal{H} = -i(H \otimes I - I \otimes H^t) + i \sum_k A_k \otimes A_k^\dagger - \frac{1}{2} I \otimes (A_k^\dagger A_k)^t - \frac{1}{2} A_k^\dagger A_k \otimes I.
\]

The superoperator \( \mathcal{H} \) is non-Hermitian for open quantum systems. The skew Hermitian part \( \mathcal{H} - \mathcal{H}^\dagger \)/2 is responsible for purity changes and stems uniquely from the Lindblad operators \( A_k \) of the master equation (2). To derive a purity speed limit in Liouville space we use (5), and obtain the equality, \( \partial_t \ln |\rho\rangle = -i\langle \rho | \mathcal{H} - \mathcal{H}^\dagger \rangle / |\rho\rangle \). Integrating this expression over time and using the triangle inequality, we get

\[
\ln \left( \frac{\mathcal{P}(t_f)}{\mathcal{P}(t_i)} \right) \leq \int_{t_i}^{t_f} \frac{|\langle \rho | \mathcal{H} - \mathcal{H}^\dagger \rangle |}{|\rho\rangle} dt.
\]
The integrand may be further bounded by the spectral (or operator) norm
\[
\left| \frac{\langle \rho | H - H^\dagger | \rho \rangle}{\langle \rho | \rho \rangle} \right| \leq \| H - H^\dagger \|_{sp}. \tag{8}
\]
For skew Hermitian operators like \( H - H^\dagger \) with eigenvalues \( \lambda_i \), the spectral norm is equal to \( \max |\lambda_i| \) [37]. Combining (7) and (8), we eventually obtain a cumulative speed limit for the second-order Rényi entropy of the form (1):
\[
-\ln \mathcal{P}(t_f) + \ln \mathcal{P}(t_i) \leq \int_{t_i}^{t_f} \| H - H^\dagger \|_{sp} dt. \tag{9}
\]
Here as well, instead of a speed limit for \(-\ln \mathcal{P}\) it is possible to get a less tighter speed limit for \(\mathcal{P}\). We point out that the norm \( \| H - H^\dagger \|_{sp} \) can be directly determined from the measurable Lindblad operators \( A_k \) using (6).

**Liouville space vs. Hilbert space speed limits.**

Next, we show that the Liouville space speed limit is always tighter compared to the Hilbert space bound (4). From (6) and the triangle inequality
\[
\| H - H^\dagger \|_{sp} \leq \sum_k \| A_k \otimes A_k^\dagger \|_{sp} + \| A_k^\dagger \otimes A_k \|_{sp}
\]
\[
+ \| I \otimes [A_k^\dagger A_k] \|_{sp} + \| A_k^\dagger A_k \otimes I \|_{sp},
\]
we get
\[
\| H - H^\dagger \|_{sp} \leq \sum_k 4 \| A_k \|_{sp}^2 + \sum_k 4 \| A_k^\dagger \|_{sp}^2.
\]
Using \( A \otimes B = \| A \|_{sp} \| B \|_{sp} \), and submutiplicativity of the spectral norm we get
\[
\| H - H^\dagger \|_{sp} \leq \sum_k 4 \| A_k \|_{sp}^2 \leq \sum_k 4 \| A_k \|_{sp}^2,
\]
where the last inequality is a general relation between the two norms [38]. Equation (10) gives two important results. First, it proves our claim that the Liouville space speed limit is always tighter compared to the Hilbert space speed limit (4). The second result is that by replacing \( 4 \| A_k \|_{sp}^2 \) with \( 4 \| A_k \|_{sp}^2 \) in (8), a new tighter bound in Hilbert space is obtained. On the one hand, it uses the original Lindblad operator \( A_k \), and, on the other hand, it is always tighter than (4). In addition, it does not suffer from the scaling problems we discuss next.

**Scaling behavior.** Consider \( M \) \( N \)-level non-interacting particles. The system is in a product state at all times. Only the first particle is coupled to an environment described by a Lindblad operator \( A_1 \). The change of the purity of the system is dictated by the first particle (the others evolve unitarily). Thus, we expect the purity speed limit to be independent of \( M \). Indeed, the Liouville space bound (9) is \( M \) independent, \( |\ln \mathcal{P}_M/\mathcal{dt}| = |\| H_1 - H_1^\dagger \|_{sp}| \), where \( H_1 \) is the single-particle super-Hamiltonian that arises from \( A_1 \). In contrast, the Hilbert space bound (4) is \( |\ln \mathcal{P}_M/\mathcal{dt}| \leq N^{M-1} |A_1|/2 \). The \( M \)-dependence exponentially overestimates the speed. Another scaling problem appears in the dephasing of a single \( N \)-level particle. Let the eigenvalues of the dephasing operator be \( \lambda_j(A) = \exp(i\varphi_j) \). The Liouville space speed limit is \( \max |\lambda_i - \lambda_j|^2 \leq 4 \), while the Hilbert space speed limit is equal to \( 4N \). Hence the Hilbert space bound (4) overestimates the purity value for \( N \geq 2 \). Note that the scaling problems are resolved if the Hilbert-Schmidt norm is replaced by the spectral norm (the Liouville formalism and (10) show why this is legitimate).

**Purity deviation.** In what follows we introduce the purity deviation. It has two main advantages over the regular purity. The first is that it enables to get a speed limit which is even tighter than (9). The second advantage is that it provides information that is sometimes more useful than the purity. Essentially, it quantifies the distance between two different solutions of the system as a function of time. Such a measure is very useful in studying relaxation to steady-state dynamics. Let \( \rho_0 \) be a specific solution of the quantum evolution \( i\partial_t |\rho\rangle = H|\rho\rangle \), and the corresponding purity deviation as \( \mathcal{P}_D = |\rho_D|/|\rho| \). The purity deviation has a simple geometrical meaning as the square of the Euclidean distance, \( \text{tr}[(\rho - \rho_0)|\rho|] \), between the states \( \rho \) and \( \rho_0 \) (the regular purity is the distance to the origin \( \rho_0 = 0 \)).

By taking the time derivative of \( \mathcal{P}_D \) and repeating the previous derivation, we find
\[
|\ln \mathcal{P}_D(t_f)/\mathcal{P}_D(t_i)| \leq \int_{t_i}^{t_f} \| H - H^\dagger \|_{sp} dt, \tag{11}
\]
where the purity \( \mathcal{P} \) has now been replaced by the purity deviation \( \mathcal{P}_D \). While (11) is valid for all vectors \( |\rho_\alpha\rangle \) that solve (5), it becomes particularly useful when \( |\rho_\alpha\rangle \) is given by the steady state, \( i\partial_t |\rho_\alpha\rangle = 0 \). The benefit of the replacement \( \mathcal{P} \rightarrow \mathcal{P}_D \) is that only the part of the purity that changes in time is taken into account. The purity deviation bound (11) has the remarkable property that it may be tight at all times for purely dephasing qubit channels and for the erasure of classical correlations (see below).

**Applications.**

a) **Dephasing channels and other unital maps.** To illustrate the strength of the purity deviation bound even for the simplest scenarios, we first apply it to a pure dephasing of a qubit. The system is described by \( H = \sigma_z \) and \( A = \sigma_x \). In this case, the Hilbert space bound (4) is exactly two times larger than the Liouville space bound (9). For an initial density matrix of the form \( \rho(t_i) = \{a, b\}, \{b^*, 1 - a\} \), we find \( |\ln \mathcal{P}(t_f)/\mathcal{P}(t_i)| \leq 2(t_f - t_i) \) in Hilbert space and \( |\ln \mathcal{P}(t_f)/\mathcal{P}(t_i)| \leq t_f - t_i \) in Liouville space. Remarkably, the purity deviation bound (11) is tight at all times in this case. To see this we choose \( \rho_0 \) to be the steady state given by the fully mixed state \( \rho_0 = \{a, 0\}, \{0, 1 - a\} \) (there may be several steady states, and one may choose one of them, depending on the initial state), and obtain the equality \( |\ln \mathcal{P}_D(t_f)/\mathcal{P}_D(t_i)| = |\ln[2(\lambda_0^2 - e^{-t_f})/2(\lambda_0^2 - e^{-t_i})]| = t_f - t_i \), which is exactly equal to the right-hand side of (11).
for erasing quantum correlation while leaving the classical information intact. As shown next, this happens naturally in the presence of dephasing. Let $\{0\},\{1\}\otimes M$ be the basis of interest (e.g. the energy basis) in which the erasure takes place. Given some initial density matrix of the whole $N$-particle system in this basis $\rho_i$, the corresponding state with no quantum correlation is $\rho_f = \text{Diagonal}(\rho_i)$. To quantify the speed at which we approach the quantum correlation free state $\rho_f$, we use the standard Hilbert-Schmidt norm

$$R(t) = \text{tr}[(\rho(t) - \rho_f)^2].$$

This quantity has the structure of purity deviation since both $\rho(t)$ and $\rho_f$ are valid solution of the equation of motion. In order to remove quantum correlation a dephasing operation is needed. This dephasing can be achieved in two different ways. Either by local dephasing on each particle, or by global dephasing operators. We start from local dephasing by applying $M$ Lindblad terms of the form $A_1 = \sqrt{\gamma_{\text{local}}I \otimes I \otimes \ldots \otimes I \ldots A_2 = \sqrt{\gamma_{\text{local}}I \otimes \sigma_2 \otimes I \otimes \ldots \text{and so on}}$. From (6) one can verify that $\|H - H\|^2 = 2M [\gamma_{\text{global}}]$ and that the maximal rate is achieved for GHZ [48] pure states: $\frac{1}{\sqrt{M}}[(a,b,c,...) + |1 - a,1 - b,1 - c,\ldots|]$, where $a,b,c \in \{0,1\}$. Using (11), the speed limit for local erasure of quantum correlation is

$$R(t) \geq R(0)e^{-\gamma_{\text{local}}t},$$

where the speed limit is tight for GHZ states. Next we consider a global dephasing operator of the form $A_{m,n}^{(k)} = \sqrt{\gamma_{\text{global}}I \otimes \ldots I}$, $M$. Using (6) one can verify that $H - H\|$ has $M$ singular values that are equal to zero and the rest are equal to $[\gamma_{\text{global}}]$ and, therefore,

$$R(t) = R(0)e^{-\gamma_{\text{global}}t}.$$  

Remarkably, the bound yields an exact tight result for any initial state (since all the modes (coherences) decay at the same rate). These two different erasure processes not only show the difference between local and global erasure but also demonstrate that the speed limits we derived based on Liouville space and the purity deviation, can be tight for arbitrarily large and entangled system.

c) Interacting particles. To demonstrate that our bound is also applicable for systems with strongly interacting particles, we consider the case of local dephasing for each particle in a chain of $M$ interacting particles. This is exactly the scenario in recent studies of heat transport in ion chains [49,50]. Since the fully mixed state is a solution for any type of interaction we obtain that (14) holds where this time $R(t) = \text{tr}(\rho(t) - \rho_f)$, where $\rho_f$ is the density matrix of the fully mixed $M$-particle state. In fig. 2 we plot $R(t)/R(0)$ for five interacting spins with time-dependent nearest-neighbor interaction of the form $V_{i,i+1} = V_0\cos(t)\sigma_i^x \otimes \sigma_{i+1}^x$. The Hamiltonian of particle “i” is $H_i = \sigma_z$ and the local dephasing operator of each particle is $A_i = \sigma_x$. The red line is our bound on $R(t)/R(0)$, the black dashed lines are numerical results for
random initial condition with \(V_0 = 0.1\), and the solid blue lines are numerical results with strong interaction \(V_0 = 10\).

d) Erasing classical and quantum correlations by resetting a subsystem. The creation of quantum correlations between different systems [2] is a key ingredient in quantum information. Recently, however, there is also a growing interest in the converse problem of correlation erasure. That is, the removal of correlations between two (or more) systems, while leaving the local information (the reduced density operators) intact [51–56]. The problem of quantum state decorrelation (also called quantum decoupling or disentanglement) plays an important role in quantum state merging [57], the computing of channel capacities [58], quantum cryptography [59], as well as in the study of thermalization [60]. Experimental realizations of decorrelation have been reported in [61,62]. We shall use our approach to determine the maximal rate of correlation erasure. This example serves two purposes. The first is to show an application of our theory to quantum information. The second purpose is to show that even in cases where the purity deviation cannot be explicitly related to the standard purity, it can still carry valuable information that is absent in the standard purity.

Consider two systems \(A\) and \(B\) that are initially correlated (correlations may be classical and/or quantum). The joint density matrix is denoted by \(\rho_{AB}\) and the respective reduced density matrices are \(\rho_A = \text{tr}_B(\rho_{AB})\) and \(\rho_B = \text{tr}_A(\rho_{AB})\). We wish to decorrelate \(A\) from \(B\) by using a Markovian reservoir. We assume that there is no interaction between \(A\) and \(B\) during the decorrelation process. The corresponding quantum decorrelator generates the transformation

\[
\rho_{AB} \rightarrow \rho_A \otimes \rho_0,
\]

where \(\rho_0\) is some predetermined state that is independent of \(\rho_{AB}\). Ideally, one would wish the final state to be \(\rho_A \otimes \rho_B\) for any \(\rho_{AB}\). However, this operation was shown to be nonlinear in general [51]. In cases in which \(\rho_B\) is known, one may choose \(\rho_0 = \rho_B\). However, this would lead to a decorrelator that is tuned to a specific \(\rho_B\).

We use the standard \(L_2\) norm to define the distance between the initial correlated state \(\rho_{AB}\) and the final decorrelated state \(\rho_A \otimes \rho_0\),

\[
R_{\text{dec}} = \text{tr}[(\rho_{AB} - \rho_A \otimes \rho_0)^2].
\]

If \(\rho_0 = \rho_B\) this measure is very similar to the geometric discord introduced in [63]; However, \(R_{\text{dec}}\) describes the distance to a state where system \(A\) has neither quantum nor classical correlation to system \(B\). Demanding that \(\rho_A \otimes \rho_0\) is a solution of the equation of motion, eq. (17) takes the form of a purity deviation. Next we show that the one-partite Lindblad operator \(L_{\text{dec}} = 1_A \otimes L_B\), where \(L_B\) has a single stationary state \(\rho_0\), achieves the decorrelating transformation given in (16). After showing that, we shall apply our purity deviation speed limit to understand how fast \(R_{\text{dec}}\) decreases and \(A\) becomes decorrelated from \(B\).

Consider the Lindblad equation of motion that describes an interaction with a bath in the Markovian limit

\[
\partial_t \rho_{AB} = L[\rho_{AB}],
\]

where the Lindblad form of \(L\) is given in (2). To leave system \(A\) intact we choose the form

\[
L_{\text{dec}} = 1_A \otimes L_B,
\]

where \(\rho_0\) is the only zero state of \(L_B\): \(L_B[\rho] = 0 \iff \rho = \rho_0\). Since the Lindblad operator has a tensor product form we can write the evolution as

\[
\rho_{AB}(t) = e^{L_{\text{dec}} t}[\rho_{AB}(0)] = I_{N \times N} \otimes e^{L_B t}[\rho_{AB}(0)].
\]

The only steady state of \(K_B = e^{L_B}\) is \(\rho_0\) and therefore we can write

\[
\lim_{t \to \infty} K_B[\sigma] = \text{tr}(\sigma)\rho_0,
\]

where we considered the slightly more general case where the initial trace is different from one (the Lindblad map conserves the trace). To show how \(L_{\text{dec}}\) operates on a general density matrix we decompose the initial density matrix in the following way:

\[
\rho_{AB} = \frac{1}{N^2} \left[ I_{N^2 \times N^2} + \sum_{i=1}^{N_Z} r_{A,i} Z_i \otimes I_{N \times N} \right. \\
+ \sum_{i=1}^{N_Z} r_{B,i} I_{N \times N} \otimes Z_i + \sum_{i,j=1}^{N_Z} t_{ij} Z_i \otimes Z_j \right],
\]

where \(N_Z = N^2 - 1\), and \(Z_i\) are traceless orthonormal basis operators for \(N \times N\) Hermitian traceless matrices. \(r_A\) and \(r_B\) determine the reduced density matrices:

\[
\text{tr}_B \rho_{AB} = \frac{1}{N} \left[ I_{N \times N} + \sum r_{A,i} Z_i \right] = \rho_A.
\]
Next we use (21), and apply \( \lim_{t \to \infty} K \) to the initial state (23) and get
\[
\rho_{AB}(t \to \infty) = \frac{1}{N} \left[ I_{N \times N} \otimes \rho_0 + \sum r_{A,i} Z_i \otimes \rho_0 \right] = \rho_A \otimes \rho_0,
\]
which shows that \( L_{\text{dec}} \) realizes the decorrelating transformation \( \rho_{AB} \to \rho_A \otimes \rho_0 \). An immediate conclusion from this result follows: even when removing the correlation is not the objective, it occurs naturally when the dynamics is generated by a one-party (local) single-steady-state Markovian map.

Now that we have established \( \rho_{AB} \to \rho_A \otimes \rho_0 \) readily apply (11), and get that the norm integral needed to change the decorrelation purity from \( R_{\text{dec},i} \) to \( R_{\text{dec},f} \) in time \( T \) is
\[
\left| \ln \frac{R_{\text{dec},f}}{R_{\text{dec},i}} \right| \leq \int_0^T \left\| \mathcal{H}_B - \mathcal{H}_B^i \right\| dt,
\]
where \( \mathcal{H}_B \) is the Liouville space representation of \( L_B \).

We illustrate the usefulness of (26) by considering the initial state \( \rho_{AB} = (\lambda/4)I_{4 \times 4} + (1-\lambda)|\Psi^+\rangle \langle \Psi^+| \), where \( I_{4 \times 4} \) is the unit operator, \( |\Psi^+\rangle \) is the usual Bell state, and \( 0 \leq \lambda \leq 1 \). The reduced density operators are \( \rho_A = \rho_B = \frac{1}{2} I_{2 \times 2} \). The quantum discord of the joint state is monotonically increasing from zero to one as \( \lambda \) changes from zero to one [64]. We take the quantum decorrelator \( L_{\text{dec}} \) as the sum of the Lindblad operators \( I_{2 \times 2} \otimes \sigma_{\pm} \), where \( \sigma_{\pm} \) are the raising and lowering operators in spin space. This is equivalent to a thermal bath whose temperature is much larger than the energy gap of the system. Figure 3 shows the relative decorrelation distance, \( R_{\text{dec}}(t)/R_{\text{dec}}(0) \), as a function of time for different values of the parameter \( \lambda \). Interestingly, in this case, classical correlations (blue) are erased faster than quantum correlations (red). In fact the classical case constitutes an example where the decorrelation speed bound (26) is tight.

Finally we point that the decorrelation process described here is quite different from simply replacing system \( B \) with a new system that is in a state \( \rho_0 \) (that is not correlated to system \( A \)). In this case nothing is accomplished since system \( A \) is still fully correlated to the old system \( B \). However, in the weak-coupling limit (Lindblad dynamics) the correlation is spread over the vast number of degrees of freedom in the bath. Consequently it is effectively lost and cannot be retrieved. The slightest noise will make it impossible to get the correlations back by time reversal.

**Conclusions.**—The Liouville space approach was used to derive a cumulative state-independent quantum speed limit for purity changes that is much tighter than the speed limit obtained by using the Hilbert space approach. The tools used in this paper can be exploited to derive cumulative state-independent speed limits for the change of observable quantities. By introducing the purity deviation with respect to the steady-state of the system, an even more accurate speed limit was obtained. Remarkably, the purity deviation bound can be attained, and therefore constitutes a tight speed limit. To demonstrate the utility of our results we derived speed limits for dephasing processes of interacting particles and for correlation erasure. We expect the purity speed limit to become a useful tool in the investigation of the dynamics of open quantum systems, from coherent control and unitary gates implementation to quantum thermodynamics.

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