Bringing a Superposition $\alpha|11\rangle + \beta|00\rangle \ (\alpha > \beta)$ to a Bell State

Through Quantum Zeno-like Measurements

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Abstract

We show rather surprising features of Zeno-like measurements on two pairs of qubits system interacting pairwise. For the superposition state $\alpha|11\rangle + \beta|00\rangle \ (\alpha > \beta)$ and a finite set of measurements we find an entanglement enhancement. For $N \to \infty$ the initial state is preserved. Moreover we show that a single measurement after the “sudden death of entanglement” induces a new entangled state. For specific measurement times the entanglement will be maximum.

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The possibility of freezing the evolution of a system by frequent measurements was firstly pointed out by von Neumann in 1932 [1], and formally presented by B. Misra and E. C. Sudarshan [2]. Although some experimental realizations of the effect have been reported [3, 4, 5, 6] it remained, for several years, the center of fervorous debates [7, 8, 9, 10, 11, 12]. Recently a conclusive experiment on an unstable quantum system was reported [13].

Interestingly enough Quantum Zeno-like measurements have become an important tool for practical applications in the growing field of quantum information theory. Error prevention [14, 15, 16, 17, 18], quantum state engineering [19], state preservation [20] and decoherence control [21, 22, 23] are some examples. Recently Zeno-like measurements have been used to protect an entangled state from its environment [24].

In the present contribution we show that Zeno-like measurements in the context of a double Jaynes-Cummings model [25] may not only preserve the maximum entanglement of a Bell state, but also, surprisingly, enhance the entanglement of the state $\alpha|11\rangle + \beta|00\rangle$ ($\alpha > \beta$), bringing it to a Bell state.

Many important features of entanglement dynamics have been extensively studied in such models, namely: relation between energy and entanglement [26], pairwise concurrence dynamics [27], entanglement invariant for this model [25] and sudden death of entanglement [28].

In this case (sudden death of entanglement) we were able to show, for this system, that a single measurement after the event may yield a new entangle state.

Let us consider four qubits system coupled 2(ab) by 2(AB) with identical coupling coefficient($g$). The interaction is such that it couples a only to A, and b to B as in Jaynes-Cummings models quoted above.

Let us consider the initial state $|\phi+\rangle_{ab}|0,0\rangle_{AB} = (\alpha_0|1,1\rangle_{ab} + \beta_0|0,0\rangle_{ab})|0,0\rangle_{AB}$, whose dynamical evolution is given as

$$|\phi+\rangle_{ab}|0,0\rangle_{AB} \longrightarrow (\alpha_0 a^2(t)|1,1\rangle_{ab} + \beta_0|0,0\rangle_{ab})|0,0\rangle_{AB}$$

$$-\alpha_0 b(t) [ia(t)(|1,0\rangle_{ab}|0,1\rangle_{AB} + |0,1\rangle_{ab}|1,0\rangle_{AB}) + b(t)|0,0\rangle_{ab}|1,1\rangle_{AB}],$$

where $a(t) = \cos(gt)$ and $b(t) = \sin(gt)$. Note that at $t = T = \frac{\pi}{2g}$ there is an entanglement swapping
\[ |\phi_+\rangle_{ab}|0, 0\rangle_{AB} \longrightarrow |0, 0\rangle_{ab}|\phi_-\rangle_{AB}, \]  

where \(|0, 0\rangle_{ab}|\phi_-\rangle_{AB} = |0, 0\rangle_{ab}(\alpha_0|1, 1\rangle_{AB} - \beta_0|0, 0\rangle_{AB}).\]

Let us introduce a probe system composed of \(N\) two level systems all prepared on \(|0\rangle_{AB}\) state, i.e. \(\bigotimes_{k=1}^{N} |0\rangle_{M}^{(k)}\). This probe system interacts one at the time (at times \(t_N = N\tau\)) with subsystem \(AB\). This interaction will discriminate, after each interaction, between the two possibilities: no excitation in \(AB\) and otherwise. We also assume that the probe system can be measured after every interaction with \(AB\).

After a sequence of \(N\) interactions keeping only those with null result, i.e., the probe remains unexcited; the vector state will become

\[ |\Phi_N(T)\rangle = \frac{1}{(|\alpha_0|^2 \cos^{4N}(g\tau) + |\beta_0|^2)^{1/2}} \left( \alpha_0 \cos^{2N}(g\tau) |1, 1\rangle_{ab} + \beta_0 |0, 0\rangle_{ab} \right) |0, 0\rangle_{AB}, \]  

where we fixed the total time of the evolutions as \(T = \frac{\pi}{2g}\) (the entanglement swap time), and \(\tau = \frac{T}{N}\).

It is apparent from this expression that the relative contribution of the state \(|1, 1\rangle_{ab}\) can be manipulated. In what follows we derive the conditions under which the entanglement enhancement of this system is possible.

For this purpose we calculate the concurrence \[29\] of the initial state

\[ C_0 = \frac{2|\alpha_0||\beta_0|}{|\alpha_0|^2 + |\beta_0|^2} = 2|\alpha_0||\beta_0|, \]  

and the one after \(N\) measurements

\[ C_N = \frac{2|\alpha_0||\beta_0| \cos^{2N}(g\tau)}{|\alpha_0|^2 \cos^{4N}(g\tau) + |\beta_0|^2}. \]  

Taking the limit \(N \to \infty\) on (3) and (5), we can see the dynamics inhibition.

\[ \lim_{N \to \infty} |\Phi_N(T)\rangle = |\phi_+\rangle_{ab}|0, 0\rangle_{AB}, \]  

\[ \lim_{N \to \infty} C_N = C_0. \]

The vector state of the system after \(N\) measurements tends to its initial state as does the concurrence to its initial value when \(N \to \infty\).
From equation (4) and the normalization of $|\phi_+\rangle_{ab}|0,0\rangle_{AB}$ ($|\alpha_0|^2 + |\beta_0|^2 = 1$), we may write:

$$
|\alpha_0| = \frac{1}{\sqrt{2} + \sqrt{1 - C_0^2/4}} \geq |\beta_0|
$$

$$
|\alpha_0| = \frac{1}{\sqrt{2} - \sqrt{1 - C_0^2/4}} \leq |\beta_0|
$$

Substituting (8) in (5), and after some algebra we may write the concurrence $C_N$ as function of the initial concurrence and the number of measurements.

$$
C_N^\pm = C_N^\pm; |\alpha_0| \geq |\beta_0|
$$

$$
C_N^\pm; |\alpha_0| \leq |\beta_0|
$$

where

$$
C_N^\pm = \frac{2C_0 \cos^{2N}(g\tau)}{1 + \cos^{4N}(g\tau) \mp \sqrt{1 - C_0^2[1 - \cos^{4N}(g\tau)]}}.
$$

The functions ($C_N^+$ and $C_N^-$) show a similar behavior as $N \to \infty$, both tend to the initial concurrence value. However, their behavior is quite different when $N$ is finite, this is shown on Fig.1.

If only one measurement is performed at $\tau = \frac{\pi}{2g}$, no inhibition can be observed and the entanglement is completely transferred to the subsystem $AB$. Therefore, the concurrence of subsystem $ab$ is null and both curves in Fig. 1 start from zero. As $N$ increases $C_N^+$ grows steadily tending to $C_0$ when $N \to \infty$. We conclude that for initial states with $|\alpha_0| \leq |\beta_0|$, the effect of entanglement freezing is the only one that can be induced.

On the other hand, the function $C_N^+$ shows an interesting effect besides the entanglement freezing. When the initial state respects the relation $|\alpha_0| > |\beta_0|$, the concurrence of the subsystem $ab$ can exceed its initial value, and become very close to the maximum value of 1. This is best visualized in the coefficient $\alpha_0 \cos^{2N}(g\tau)$ (in (3)) which decreases as $N$ increases, and when its value approaches $\beta_0$ the concurrence grows. If $\alpha_0 \cos^{2N}(g\tau) = \beta_0$ the concurrence is maximum $C_N = 1$.

It is also possible to bring the superposition $\alpha_0|11\rangle + \beta_0|00\rangle$ ($\alpha_0 > \beta_0$) to a Bell state with only one measurement. The procedure is simple, one just has to let the system evolve freely and perform a measurement on subsystem $AB$ at the time $t = \frac{1}{g} \arccos(\sqrt{\beta_0/\alpha_0})$ (when
\(\alpha_0 a^2(t) = \beta_0\). Selecting the states with null result, one will have a Bell state prepared in subsystem \(ab\).

The appearance of entanglement sudden death for the initial states \((\alpha|11\rangle + \beta|00\rangle)|0, 0\rangle\) \((\alpha > \beta)\), in the context of the double Jaynes-Cummings model is well known [28]. In order to explicitate this effect we give the expression for the concurrence \((C_f)\) of the state evolving without external interference, as a function of the initial concurrence.

\[
C_f^\pm(t) = \max(0, \Lambda^\pm(t)), \quad (11)
\]

where

\[
\Lambda^\pm(t) = \sqrt{1 \pm \sqrt{1 - C_0^2}} \cos^2(gt) \left( \sqrt{1 \mp \sqrt{1 - C_0^2} - \sqrt{1 \pm \sqrt{1 - C_0^2} \sin^2(gt)}} \right). \quad (12)
\]

Notice that for times

\[
t = \frac{1}{g} \arcsin \left( \frac{\sqrt{\frac{1 - \sqrt{1 - C_0^2}}{1 + \sqrt{1 - C_0^2}}}}{\sqrt{1 - C_0^2}} \right), \quad (13)
\]
which are prior to the swap time, there will be a sudden death of entanglement. Fig.2 illustrate this. However, in Fig.3 we show how a single measurement can induce an early entanglement resurrection.

The graphic on Fig.3 shows the concurrence \( C^+_1 \) after a single measurement on \( AB \) with null result performed at times \( t \), for different values of the initial concurrence of the states \((\alpha|11\rangle + \beta|00\rangle)|0, 0\rangle (\alpha > \beta)\). Notice that (from Fig.2) the concurrence is zero after the sudden death time. In Fig.3, surprisingly enough, the set of vanishing final concurrences are “brought to life” by a single measurement performed not necessarily immediately after the time at which sudden death takes place.

![Graph](image)

**FIG. 2:** Concurrence without measurements \( (C_f) \) as function of \( gt \) and \( C_0 \) for initial states \((\alpha|11\rangle + \beta|00\rangle)|0, 0\rangle (\alpha > \beta)\).

The reason for this is that after the sudden death and before the entanglement swap time, there is no entanglement on \( ab \), but the excitations are not completely transferred to \( AB \). Then if one performs a measurement on \( AB \), and gets a null result, a state with finite entanglement is prepared in \( ab \). It is even possible to resurrect the entanglement to the maximum value \( (C^+_1 = 1) \), exceeding the initial one, if the measurement is performed at \( t = \frac{1}{g} \arccos \left( \sqrt{\frac{\beta}{\alpha}} \alpha_0 \right) \).

To summarize, we have shown that QZE is capable of enhancing the entanglement of the state \((\alpha|11\rangle + \beta|00\rangle)|0, 0\rangle (\alpha > \beta)\). Entanglement freezing is possible for all the others Bell states, however not the enhancement. Also surprisingly is the fact that the sudden death, observed in other calculations [28], can be avoided by a single measurement.
FIG. 3: Concurrence with one measurement at time $t$, as function $gt$ and $C_0$ for initial states
$(\alpha|11\rangle + \beta|00\rangle)|0,0\rangle$ $(\alpha > \beta)$.

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