The Universality of Einstein Equations

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Abstract
It is shown that for a wide class of analytic Lagrangians which depend
only on the scalar curvature of a metric and a connection, the application
of the so–called “Palatini formalism”, i.e., treating the metric and the
connection as independent variables, leads to “universal” equations. If
the dimension $n$ of space–time is greater than two these universal equa-
tions are Einstein equations for a generic Lagrangian and are suitably
replaced by other universal equations at bifurcation points. We show that
bifurcations take place in particular for conformally invariant Lagrangians
$L = R^{n/2} \sqrt{g}$ and prove that their solutions are conformally equivalent to
solutions of Einstein equations. For 2–dimensional space–time we find in-
stead that the universal equation is always the equation of constant scalar
curvature; the connection in this case is a Weyl connection, containing the
Levi–Civita connection of the metric and an additional vectorfield ensuing
from conformal invariance. As an example, we investigate in detail some
polynomial Lagrangians and discuss their bifurcations.

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1 Introduction

In this paper we shall investigate a large class of gravitational Lagrangians depending on a metric and a torsionless connection as basic dynamical variables. We shall show that Einstein equations play always a very central role in the formalism, because of a strong “universality” property.

We first remark that, according to a widespread opinion, one is usually led to believe that any physically meaningful set of field equations having a variational character should be in correspondence with an “essentially unique” Lagrangian, in the sense that for any given set of field variables the Lagrangian generating the equations should be fixed modulo the addition of (dynamically irrelevant) divergencies. However, this is not true, because one can produce examples of non-trivially related families of Lagrangians which are “strongly dynamically equivalent”, in the sense that for a fixed set of field variables they generate the same field equations, although they do not differ by divergencies.

There exist other approaches to “dynamical equivalence” which allow instead to recognize equivalence through suitable changes of variables, e.g. by Legendre transformation or exploiting invariance properties. In gravitational theories their early history begins with Einstein, Eddington and Weyl (see e.g. [1, 2, 3]), passes through Schrödinger [4] and still today is the subject of interest in the literature (see e.g. [5, 6, 7, 8] and ref.s quoted therein). In this paper we shall consider equivalence in the “strong” sense above, i.e., by assuming field variables to be fixed and not changing them.

This raises an important problem, which we believe should be given renewed attention for better understanding both the classical and the quantum properties of physical fields, namely: “which is the most central object in a field theory? Is it the set of field equations or the Lagrangian?”.

As we shall see below, Einstein equations are a strikingly important example for the discussion of this problem. We shall in fact show and analyze their “universality”, by proving that a very large class of profoundly different metric–affine Lagrangians (i.e., depending on a metric and a connection), which one should expect in principle to generate rather different equations, always produce the same set of “universal” field equations. These will be in fact Einstein equations in generic cases. In degenerate situations and in dimension $n = 2$, one gets instead equations which express the constancy of the scalar curvature (which contain Einstein equations as a sector) or, for conformally invariant Lagrangians, equations which are conformally equivalent to Einstein equations.

Since the early days of General Relativity, the so-called “Palatini formalism”, which is based on independent variations of the metric and the connection, has been known to be equivalent to the corresponding metric formulation for the Lagrangian which depends linearly on the scalar curvature constructed out of the metric and the Ricci tensor of the connection. In this case, in fact, when $n \geq 2$ field equations imply that the connection has to be the Levi–Civita connection of the metric, which in turn satisfies Einstein equations (see, e.g., [3]).
This means that there is a full dynamical equivalence with the corresponding metric Lagrangian, which a posteriori coincides with the Hilbert Lagrangian of General Relativity. (For a historical discussion about Palatini formalism see [10]).

In this paper we will show that the method of independent variations of the metric and the connection leads in fact to Einstein equations for a much wider class of Lagrangians which depending on the scalar curvature in a non-linear way. In the process of showing that the relevant field equations reduce to Einstein equations a cosmological constant $\Lambda$ arises, whose only allowed values are fully determined by the Lagrangian itself. One thus finds that Einstein equations are not strictly related with the linear Lagrangian, since any other Lagrangian belonging to the family can be used to derive them. This displays the claimed “universality” of Einstein equations for a rather wide class of Lagrangians.

We emphasize that, incidentally, our result shows also that the metric–affine approach is not equivalent to the purely metric one for non–linear Lagrangians. In fact, for Lagrangians which depend non–linearly on the scalar curvature of a metric, it is known that higher derivatives effects entail the existence of a further scalar field dynamically interacting with the graviton (see [11, 12, 13, 14, 8, 15, 16] and ref.s quoted therein), while here only a cosmological constant appears. The relations between these two formalisms and with conformal invariance are discussed elsewhere [17].

Although for our present purposes we are interested mainly in 4–dimensional space–times, we shall discuss field equations in the arbitrary dimension $n \geq 2$, in view of full generality and applications to 2–dimensional gravity theories. As is well known, in fact, in two dimensions Einstein equations do no longer exist but can be appropriately replaced by other equations involving the curvature (see, e.g., [18]). Our results will in fact provide a new “universal” equation also for $n = 2$, which states that the scalar curvature of the metric and the connection is a constant. This universal equation has a purely geometrical origin and does not require the introduction of extra fields having no geometrical origin. Its important relations with a new topological theory of gravity based on a purely affine Lagrangian are also discussed elsewhere [19].

2 The Universal Field Equations

Let us consider an action of the form

$$S(\Gamma, g) = \int_M L(R) \sqrt{g} \, d^n x$$

on a n–dimensional manifold $M$ endowed with a metric $g_{\mu\nu}$ and a torsionless connection $\Gamma^\sigma_{\mu\nu}$. We use the following standard notation:

$$R^\lambda_{\mu\nu\sigma} = \partial_\nu \Gamma^\lambda_{\mu\sigma} - \partial_\sigma \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\alpha\nu} \Gamma^\alpha_{\mu\sigma} - \Gamma^\lambda_{\alpha\sigma} \Gamma^\alpha_{\mu\nu}$$
The Lagrangian \( L \) is a given function of one real variable, which we assume to be real analytic on its domain of definition.

The Euler–Lagrange equations for the action (1) with respect to independent variations of \( g \) and \( \Gamma \) can be written in the following form

\[
L'(R)R_{(\mu\nu)}(\Gamma) - \frac{1}{2} L(R)g_{\mu\nu} = 0 \tag{2}
\]

\[
\nabla_\alpha (L'(R)\sqrt{g} g^{\mu\nu}) = 0 \tag{3}
\]

where \( \nabla_\alpha \) is the covariant derivative with respect to \( \Gamma \), \( R_{(\mu\nu)} \) denotes the symmetric part of the Ricci tensor of \( \Gamma \) and we assume \( n \geq 2 \). In fact, variation of the action (1) with respect to \( \Gamma \) gives the equation:

\[
\nabla_\alpha (L'(R)\sqrt{g} g^{\mu\nu}) - \nabla_\rho L'(R)\sqrt{g} g^{\rho(\mu} \delta^\nu_{\alpha)} = 0
\]

which, in any dimension \( n \geq 2 \), reduces to (3) by taking a trace.

Quadratic Lagrangians have been investigated by several authors, mainly in the four–dimensional case and also in the presence of matter (see, e.g., \[20, 21, 22, 23, 24, 25\] and ref.s quoted therein); their generalizations in the presence of torsion have also received a lot of attention in the literature (see \[23, 26, 27, 28, 29, 30\] J and ref.s quoted therein). General non–linear Lagrangians of the form (1) in dimension \( n = 4 \) and in the presence of matter have been independently investigated in \[31\] from a different viewpoint, with the aim of discussing conservation laws and the validity of the strong equivalence principle.

The results of \[31\] concerning field equations are in agreement with ours and are in fact a particular case of ours; it should be mentioned that earlier results were previously found for quadratic Lagrangians in \[20, 21, 25\].

In this paper we discuss from a different and apparently new viewpoint the general structure of equations (2) and (3) above, in an arbitrary dimension \( n \geq 2 \) and for an arbitrary analytic function \( L(R) \). We recognize and emphasize a “universal property” of this whole family of Lagrangians which, to our knowledge, has been never clearly formulated in the existing literature. We prove, in fact, that the family generates “universal equations”, independently from the particular Lagrangian chosen, and that in dimension \( n > 2 \) these equations coincide with Einstein equations in the generic case. We further discuss the “universal equations” in dimension \( n = 2 \) and for all ”degenerate” cases, which include the constant scalar curvature equation and conformally invariant equations for conformally invariant Lagrangians.

To solve equations (2)–(3) we proceed as follows. First of all, by taking the trace of eq. (2) one obtains the following equation:
\[ L'(R)R - \frac{n}{2}L(R) = 0 \]  

(4)

We shall then distinguish the following three mutually exclusive cases: (i) eq. (4) has no real solutions; (ii) eq. (4) has real solutions; (iii) eq. (4) is identically satisfied. Their discussion proceeds as follows:

(Case 1) If equation (4) has no real solutions, then also the system (2)–(3) has no real solution.

(Case 2) Let us now suppose that eq. (4) is not identically satisfied and has at least one real solution. In this case, since analytic functions can have at most a countable set of zeroes on the real line, eq. (4) can have no more than a countable set of solutions \( R = c_i (i = 1, 2, \ldots) \), where \( c_i \) are constants. Consider then any solution

\[ R = c_i \]  

(5)

of eq. (4). We have again two possibilities, depending on the value of the first derivative \( L'(c_i) \) at the point \( R = c_i \):

(Subcase 2.1) Let us assume that \( L'(c_i) \neq 0 \). Then eq. (3) takes the form

\[ \nabla_\alpha (\sqrt{g} g^{\mu\nu}) = 0 \]  

(6)

while equation (2) reduces to

\[ R_{(\mu\nu)}(\Gamma) = \Lambda(c_i) g_{\mu\nu} \]  

(7)

where a cosmological constant \( \Lambda = \Lambda(c_i) \) arises according to:

\[ \Lambda = \Lambda(c_i) = L(c_i)/2L'(c_i) = c_i/n \]  

(8)

We shall discuss separately the subcases \( n > 2 \) and \( n = 2 \).

(Subcase 2.1.1) For \( n > 2 \) and any metric \( g_{\mu\nu} \) the general solution of eq. (6) is the Levi–Civita connection of the given metric \( g \), so that the Ricci tensor \( R_{\mu\nu}(\Gamma) \) is automatically symmetric and in fact identical to the Ricci tensor \( R_{\mu\nu}(g) \) of the metric \( g \) itself. Accordingly, the general solution of the system (3)–(4) is given by

\[ \Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\mu\nu}(g) = \frac{1}{2} g^{\sigma\alpha}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}) \]  

(9)

where the metric \( g_{\mu\nu} \) is any solution of the equations

\[ R_{\mu\nu}(g) = \Lambda(c_i) g_{\mu\nu} \]  

(10)

with cosmological constant \( \Lambda \) given by (8). This constant may vanish if and only if \( c_i = 0 \) is one of the solutions of eq. (4).
In the case \( n = 2 \), instead, equation (6) allows a further degree of freedom (related with conformal invariance) and it has in fact the following general solution:

\[
\Gamma^\sigma_{\mu\nu} = W^\sigma_{\mu\nu}(g, B) = \frac{1}{2} g^{\sigma\alpha} (\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}) + \frac{1}{2} (\delta^\mu_\nu B_\sigma + \delta^\nu_\mu B_\sigma - g_{\mu\nu} B^\sigma) \tag{11}
\]

where \( B^\sigma \) is an arbitrary vectorfield. This is due to the fact that only in dimension \( n = 2 \) eq. (6) cannot be reduced to \( \nabla_\alpha g^{\mu\nu} = 0 \) (since only in two dimensions it does not imply that the Riemannian volume element of \( g \) is covariantly constant along \( \Gamma \)). A connection \( W(g, B) \) having the form (11) will be hereafter called a Weyl connection \( \mathcal{W} \) (\[3\]). Using eq. (11), from the definition of \( R(\mu\nu)(\Gamma) \) one has:

\[
R(\mu\nu) \equiv \frac{1}{2} (R(g) - D_\alpha B^\alpha) g_{\mu\nu} \tag{12}
\]

so that eq. (7) reduces to the following scalar equation

\[
R(g) - D_\alpha B^\alpha = 2\Lambda \tag{13}
\]

where \( D_\alpha \) denotes the covariant derivative with respect to the metric \( g_{\mu\nu} \). Equation (13) is the “universal” equation for 2–dimensional space–times; it replaces Einstein equations, which are the “universal equations” in dimension \( n > 2 \). Equation (13) is in fact the equation of constant scalar curvature for the metric \( g \) and the Weyl connection (11), because from (12) one has:

\[
\mathcal{R}(g, B) = g^{\mu\nu} R_{\mu\nu}(W(g, B)) = R(g) - D_\alpha B^\alpha \tag{14}
\]

We remark that eq. (13) has always infinitely many local solutions, but it might have no global analytic solution (depending on the topology of the 2-dimensional manifold \( M \)).

(Subcase 2.2) Suppose now \( L'(c_i) = 0 \). Then eq. (4) implies that also \( L(c_i) = 0 \), i.e., \( c_i \) is a zero of order at least two of \( L(R) \). In this case, eqs (2)–(3) are identically satisfied and the only relation between \( g \) and \( \Gamma \) is contained in the following equation

\[
R(g, \Gamma) = c_i \tag{15}
\]

This equation represents a genuine dynamical relation between the metric and the connection of \( M \), although it is not enough to single out a connection \( \Gamma \) for any given metric \( g \) (as it happened, instead, in subcase (2.a) above, where \( \Gamma \) turns out to be the Levi–Civita connection of \( g \)). In fact, defining a tensorfield \( \Delta^\lambda_{\mu\nu} \) by:

\[
\Delta^\lambda_{\mu\nu} = \Delta^\lambda_{\mu\nu}(g, \Gamma) = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\mu\nu}(g) \tag{16}
\]
equation (14) can be turned into a quasi–linear first–order PDE for the unknown \( \Delta \), having the term \( R(g) - c_i \) as a source. The space of solutions of this last equation, as functions of the metric \( g \) together with its first derivatives and a number of auxiliary fields, has a complicated structure. In any case, it is easy to see that this space contains as a subspace the space of all couples \((g, \Gamma)\) satisfying eq. (7) for \( \Lambda = c_i \).

(Case 3) We consider now the case in which eq. (4) is identically satisfied. Under this hypothesis the Lagrangian is proportional to:

\[
L(R) = |R|^{n/2}
\]  

(17)

This Lagrangian is analytic everywhere on the real line, except at the point \( R = 0 \) (which is singular), unless \( n = 4k \) for some positive integer \( k \). We shall then consider for simplicity the case \( R \geq 0 \) (analogous results will be valid for \( R \leq 0 \) and they may extend across \( R = 0 \) at least if \( n = 4k \)). We stress that for \( n = 4 \) this is exactly the quadratic Lagrangian \( L(R) = R^2 \) considered in [20, 21] so that our results will suitably complete and extend the earlier discussion appearing therein.

In this case equations (2) and (3) read as follows:

\[
R^\alpha \nabla (R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu}) = 0
\]

(18)

\[
\nabla_\alpha (R^{n/2} \sqrt{g} g^{\mu\nu}) = 0
\]

(19)

Notice first of all that, under conformal transformations \( \tilde{g}_{\mu\nu} = e^{\omega} g_{\mu\nu} \), \( \tilde{\Gamma} = \Gamma \)

(20)

setting

\[
R = g^{\mu\nu} R_{\mu\nu}(\Gamma), \tilde{R} = \tilde{g}^{\mu\nu} R_{\mu\nu}(\tilde{\Gamma}) = \tilde{g}^{\mu\nu} R_{\mu\nu}(\Gamma)
\]

(21)

one has

\[
\tilde{R} = e^{-\omega} R , \quad \tilde{R} g_{\mu\nu} = R g_{\mu\nu} , \quad \tilde{R}^{n/2} \sqrt{g} = R^{n/2} \sqrt{g} , \quad \tilde{R}^{n/2} \sqrt{g} g^{\mu\nu} = R^{n/2} \sqrt{g} g^{\mu\nu}
\]

(22.a, 22.b, 22.c, 22.d)

Therefore, the action

\[
S(g, \Gamma) = \int_M |R|^{n/2} \sqrt{g} d^n x
\]

as well as equations (18) and (19) are invariant under the transformation (20), i.e. \( S(\tilde{g}, \tilde{\Gamma}) = S(g, \Gamma) \).
Also in this case we shall consider separately the two cases $n > 2$ and $n = 2$.

(Subcase 3.1)

If $n > 2$ we have two possibilities. If $R = 0$, then we have only the equation

$$R(g, \Gamma) = 0$$

(23)

whose discussion proceeds as for eq. (15) above. When $R > 0$ there is instead an additional conformal degree of freedom, as noticed earlier in [21]J (and later exploited in [3]). In this case, in fact, eq.s (18) and (19) reduce to the following:

$$R_{(\mu\nu)} - \frac{1}{n} R g_{\mu\nu} = 0$$

(24)

$$\nabla_{\alpha}(R_{\mu\nu} \sqrt{g} g^{\mu\nu}) = 0$$

(25)

The following proposition is then true:

**Proposition 1** (i) If $h_{\mu\nu}$ is a solution of the equation

$$R_{\mu\nu}(h) = h_{\mu\nu},$$

(26)

where $R_{\mu\nu}(h)$ is the Ricci tensor of the metric $h_{\mu\nu}$, then the pair $(g, \Gamma)$, where

$$\Gamma = \Gamma_{LC}(h)$$

(27)

$$g_{\mu\nu} J = e^{-\omega} h_{\mu\nu}$$

(28)

is a solution of eq.s (24) and (25) for any function $\omega$ (here, $\Gamma_{LC}$ is the Levi–Civita connection of $h$).

(ii) If $(g, \Gamma)$ is a solution of eq.s (24) and (25), then they have to satisfy the relations

$$R_{\mu\nu}(a) - \frac{1}{n} a_{\mu\nu} = 0$$

(29)

$$\Gamma = \Gamma_{LC}(a)$$

(30)

where

$$a_{\mu\nu} J = R(g, \Gamma) g_{\mu\nu}$$

(31)

Proof. From eq. (27) one has

$$R_{(\mu\nu)}(\Gamma) = R_{\mu\nu}(h)$$

(32)

We shall now use eq.s (20)–(22) for $\tilde{g} = h$ to find:
Using finally eq. (22.b), (26), (32) and (33) one gets:

\[ R_{(\mu\nu)}(\Gamma) - \frac{1}{n} R g_{\mu\nu} = R_{\mu\nu}(h) - \frac{1}{n} \tilde{R} \tilde{g}_{\mu\nu} = h_{\mu\nu} - \frac{1}{n} n h_{\mu\nu} = 0 \]

which shows that (24) holds for the couple \((g, \Gamma)\). To prove that the couple satisfies also eq. (25) we use (22.d) and (33), to obtain:

\[ \nabla_\alpha (R^{\frac{n}{n-2}} \sqrt{g} g^{\mu\nu}) = D_\alpha (R^{\frac{n}{n-2}} \sqrt{\tilde{g} \tilde{g}^{\mu\nu}}) = n^{\frac{n}{n-2}} D_\alpha (\sqrt{h} h^{\mu\nu}) = 0 \]

where \(D_\alpha\) denotes the covariant derivative with respect to the Levi–Civita connection of \(h\). This proves (i). We proceed now to prove (ii). Using formulae (20)–(22) for \(\tilde{g}_{\mu\nu} J = a_{\mu\nu}\), \(\tilde{e}^\omega = R(g, \Gamma)\) one finds \(\tilde{R} = 1\). From (22.d) one gets then:

\[ R^{\frac{n-2}{n}} \sqrt{g} g^{\mu\nu} = \sqrt{a} a^{\mu\nu} \]

so that eq. (25) now reads as follows:

\[ \nabla_\alpha (\sqrt{a} a^{\mu\nu}) = O \]

(34)

which gives immediately eq. (30). From this it follows in turn that \(R_{(\mu\nu)}(\Gamma) = R_{\mu\nu}(a)\). Using then \(\tilde{R} = 1\) together with (22.b) gives \(R g_{\mu\nu} = a_{\mu\nu}\). Therefore, eq. (29) follows from (24). This completes our proof. (Q.E.D)

The above proposition can be re–phrased as follows. If we define a new metric \(h_{\mu\nu}\) by setting:

\[ \sqrt{h} h^{\mu\nu} = R^{\frac{n-2}{n}} \sqrt{g} g^{\mu\nu} \]

(35)

then eq.s (19) and (22) imply that \(\Gamma\) is the Levi–Civita connection of the new metric \(h\) and eq. (18) reduces to

\[ R_{\mu\nu}(h) - \frac{1}{n} h_{\mu\nu} = 0 \]

(36)

This, in turn, leads to a constant scalar curvature for the new metric \(h\) (in fact, it is \(R(h) = 1\)) and \(M\) will be an Einstein manifold with respect to the new metric \(h\). According to the earlier discussion of [21, 3] we can also restate the result as follows: if \((g, \Gamma)\) is a solution of eq.s (18) and (19), then there exists a scalar field \(\psi\) such that the conformally related metric \(\psi g_{\mu\nu}\) satisfies Einstein equations and the connection \(\Gamma\) is the Levi–Civita connection of \(\psi g_{\mu\nu}\); moreover, the scalar curvature of the original metric \(g\) and \(\Gamma\) equals \(\psi\). The origin of this extra scalar field \(\psi\) is discussed elsewhere ([17]) in the framework of Legendre transformation for metric–affine theories.

(Subcase 3.2)
If $n = 2$ then equations (18) and (19) simplify to

$$R(\mu\nu)(\Gamma) - \frac{1}{2} R(g, \Gamma) g_{\mu\nu} = 0 \quad (37)$$

$$\nabla_\alpha (\sqrt{g} g^{\mu\nu}) = 0 \quad (38)$$

Because of (12) and (14), the general solution of equation (38) is again represented by (11), and this in turn implies that equation (37) is identically satisfied. Therefore, the general solution of equations (37)–(38) is given by a pair $(g, W(g, B))$ where $g$ is an arbitrary metric and $W(g, B)$ is a Weyl connection, determined by the same metric and an arbitrary vectorfield $B$. Equation (13) is now replaced by

$$R(g) - D_\alpha B^\alpha = \Lambda(x) \quad (39)$$

where now $\Lambda(x)$ is an arbitrary function.

We can then summarize our results above in the following:

**Theorem 1** Let $L(R)$ be an arbitrary analytic Lagrangian in a $n$–dimensional manifold $M$, which depends on the scalar curvature $R(g, \Gamma)$ of a metric $g$ and a torsionless connection $\Gamma$. The dynamical behaviour of $(g, \Gamma)$ is governed by the equation

$$L'(R)R - \frac{n}{2} L(R) = 0 \quad (*)$$

Then, either one of the following holds:

(1) Equation $(*)$ has no real solutions.

(2) Equation $(*)$ has a discrete set of real solutions $R = c_i, i = 1, 2, \ldots$

(3) Equation $(*)$ is identically satisfied; in this case the Lagrangian is proportional to the power $R^{n/2}$.

Accordingly, either one of the following holds:

(1) If eq. $(*)$ has no real solutions than there are no consistent field equations.

(2.1) If $n > 2$ and $R = c_i$ is a solution of eq. $(*)$ such that $L'(c_i) \neq 0$ then $\Gamma$ is the Levi–Civita connection of $g$ and $g$ satisfies Einstein equations with cosmological constant $\Lambda = c_i/n$.

(2.2) If $n = 2$ and $R = c_i$ is a solution of eq. $(*)$ such that $L'(c_i) \neq 0$ then $g$ is an arbitrary metric and $\Gamma$ is the Weyl connection $W(g, B)$ generated by the Levi–Civita connection of $g$ together with an arbitrary vectorfield $B$. The pair $(g, B)$ satisfies the equation $R(g, B) = R(g, W(g, B)) = c_i$.
If $n \geq 2$ and $R = c_i$ is a solution of eq. (*) such that $L'(c_i) = 0$ then the only dynamical relation between $\Gamma$ and $g$ tells that $R(g, \Gamma) = c_i$.

(3.1) If $n > 2$ then either $R(g, \Gamma) = 0$ is the only dynamical relation between $\Gamma$ and $g$ or (if $R \neq 0$) the following holds: if $(g, \Gamma)$ is a solution then there exists a scalar field $\psi$ such that the conformally related metric $\psi g_{\mu\nu}$ satisfies Einstein equations and the connection $\Gamma$ is the Levi–Civita connection of $\psi g_{\mu\nu}$; moreover, the scalar curvature of the original metric $g$ and $\Gamma$ equals $\psi$.

(3.2) If $n = 2$ then $g$ is an arbitrary metric and $\Gamma$ is the Weyl connection $W(g, B)$ generated by the Levi–Civita connection of $g$ and by an arbitrary vectorfield $B$.

Let us make some further comments on the “exceptional” Lagrangian (17), i.e. $L(R) = a |R|^{\frac{n}{2}}$ (which is degenerate in the appropriate dimension $n$). We first remark that this Lagrangian is in fact invariant under conformal rescalings of the metric $g$, with $\Gamma$ fixed (further comments on this may be found in [17]). This is particularly relevant for 4-dimensional space–times, where the “exceptional case” is just the quadratic Lagrangian $L(R) = R^2$; this case was already considered in [21], where it was argued that it always leads to Einstein equations (for a conformal family of metrics). It turns out that this is fact true only for $R \neq 0$, as a particular case of our general discussion. We remark, however, that the case $R(g, \Gamma) = 0$, which was a priori excluded in [21], has in fact a great relevance as it was discussed above; it does not lead to Einstein equations but to a larger space of solutions, contradicting the conclusions of [21].

As a final remark, we notice that if we add to the action (1) a matter Lagrangian $L_{\text{mat}}(g, \psi, \partial \psi)$ describing the minimal coupling of the metric $g$ with external matterfields $\psi$, the eq. (3) remains unchanged, while instead of equation (2) one finds:

$$L'(R) R_{\mu\nu}(\Gamma) - \frac{n}{2} L(R) g_{\mu\nu} = T_{\mu\nu}$$ (40)

where $T_{\mu\nu} \equiv \delta L_{\text{mat}} / \delta g_{\mu\nu}$ is the energy–momentum tensor of matter. Taking the trace of equation (40) one gets then:

$$L'(R) R - \frac{n}{2} L(R) = T$$ (41)

where $T = g^{\mu\nu} T_{\mu\nu}$. If $T$ is zero or constant the same considerations as above lead to analogous conclusions about the universality of Einstein equations (or of their counterpart (13) for special cases). More general interactions of the connection $\Gamma$ with matter will be considered elsewhere [33].
3 Examples

As an example, we shall consider the space of all Lagrangians of the form

\[ L(R) = aR^2 + bR + c \]  \hspace{1cm} (42)

which is identified to the three–dimensional space \( \mathbb{R}^3 \) with parameters \((a, b, c)\).

If \( n = 4 \), for any point \((a, b, c) \in \mathbb{R}^3\) with \( b \neq 0 \) and \( b^2 - 4ac \neq 0 \) one gets Einstein equations (7) with \( \Lambda = -c/2b \). Therefore, in this case the space \( E \subset \mathbb{R}^3 \) of Lagrangians leading to Einstein equations contains all points of \( \mathbb{R}^3 \) with \( b \neq 0 \) and \( b^2 - 4ac \neq 0 \). On the contrary, on the surface \( \{b^2 - 4ac = 0, b \neq 0\} \), instead, we have not Einstein equations and the only dynamical relation between \( g \) and \( \Gamma \) is given by the equation

\[ R(\Gamma, g) = -2c/b \]  \hspace{1cm} (43)

This surface is therefore a “bifurcation surface” in the space of all Lagrangians; in fact, when coupling constants in (42) let \( L(R) \) tend to the surface, the functional space of solutions \((g, \Gamma)\) enlarges from the space of pairs \((g, \Gamma(g))\) satisfying Einstein equations to the larger space of all pairs \((g, \Gamma)\) satisfying eq. (43). The line \( \{b = c = 0, a \neq 0\} \) corresponds to the exceptional case \( L(R) = aR^2 \). In this case there are solutions \((g, \Gamma)\) of eqs. (2)–(3) which are described by Einstein equations and there are also pairs \((g, \Gamma)\) with the only restriction \( R(g, \Gamma) = 0 \). Finally, we mention that for \( b = 0 \) and \( c \neq 0 \) one gets an inconsistent system of equations.

Another interesting example is

\[ L(R) = R + aR^k \]  \hspace{1cm} (44)

This Lagrangian in \( n \)–dimensional space–time \((n > 2)\) gives Einstein equations for any \( a \) and any \( k = 2, 3, \ldots \) (if \( k \neq n/2) \). In particular, in 4–dimensional space–times if \( k \) is odd and \( a \leq 0 \) the corresponding Einstein equations are \( R_{\mu\nu}(g) = 0 \). Notice here also that the Lagrangian \( L(R) = R + aR^2 \) gives Einstein equations \( R_{\mu\nu}(g) = 0 \) for any \( a \).

4 Conclusions

We have shown that non–linear Lagrangians depending on the scalar curvature \( R(g, \Gamma) \) always lead to “universal” equations as Euler–Lagrange equations, unless it does not lead to any consistent equation at all. Our results were obtained under the explicit hypothesis that the Lagrangian \( L \) is an analytic function. However, since they depend on solutions of eq. (4), our results above trivially extend to all \( C^2 \) Lagrangians such that eq. (4) has a discrete set of solutions in the domain of definition of \( L \). In dimension \( n > 2 \) these universal equations are either Einstein equations with a cosmological constant, in a generic case, or the
constant scalar curvature equations $R(g, \Gamma) = \text{constant}$, in degenerate cases. A notable exception is the case $L(R) = aR^{n/2}$, in which an extra conformal degree of freedom appears. In dimension $n = 2$ conformal invariance entails instead that the connection is a Weyl connection, in which a vectorfield $B$ appear along-with the Levi-Civita connection of a metric; the universal equations still involve $R(g, \Gamma)$.

From a functional viewpoint, we argue that “most” (analytic) Lagrangians depending on the scalar curvature $R(g, \Gamma)$ either lead to Einstein equations or do not give any consistent equations at all, provided we endow the space of all (analytic) Lagrangians with a reasonable topology (i.e., the Lagrangians considered above represent a “generic” case). Out of these generic points we have in fact a bifurcation and Einstein equations are replaced by other universal equations which state that $R(g, \Gamma)$ is a constant. However, from a “practical viewpoint”, the physical parameters in a Lagrangian are known only approximately. This means that we are not dealing with a uniquely given Lagrangian with fixed coupling constants, but rather with a family of Lagrangians, and even small perturbations may destroy the structure of equations at bifurcation points. It would be therefore interesting to investigate in greater detail the geometry of the functional space of all Lagrangians leading to Einstein equations. This should have relevance also to the problem of quantization of the gravitational field by means of non–linear Lagrangians and also when viewing non–linear gravity as a low–energy limit of string theory.

We also point out that in this paper we have proved “universality” for field equations in a large class of non–linear Lagrangians, but we have not addressed the important problem of “universality” of physical observables (like, e.g., conserved quantities). It should be therefore important to investigate the role of our results above in connection with energy and conservation laws (e.g., by means of the Poincaré–Cartan formalism), in order to see to what extent universality holds or not also at this level. In particular, it will be interesting to compare the notion of energy one obtains directly from Einstein equations with the energy one calculates starting from the Lagrangian. We aim to discuss this problem in a future investigation.

It should be stressed that, as we mentioned above, in the purely metric formalism an additional scalar field appears when dealing with non–linear Lagrangians $L(R)$ (see [3, 16]), while here we obtain only the standard Einstein equations and a cosmological constant (unless $L(R) = aR^{n/2}$). Further comments on this difference of behaviour are discussed elsewhere [15] through the method of Legendre transformation and related with conformal invariance under rescalings of the metric.

A difference between the metric and vierbein–connection formalisms was discussed by Witten in [34], where it was shown that the application of the vierbein–connection formalism to 3–dimensional gravity allows to prove its solvability and renormalizability. As it follows from our discussion, by using the metric–affine formalism in any dimension $n \geq 3$ one can add to the Hilbert Lagrangian any
counterterms depending on the scalar curvature without changing the physical content of the theory (i.e., the Einstein equations, if one does not use “fine tuning”, i.e. one assume to be in the generic case). Only the cosmological constant will in fact be renormalized, according to (8).

It would also be interesting to study under which conditions the approach presented in this note can be extended to more general classes of Lagrangians leading again to Einstein equations, including a dependence on invariants more complicated than the scalar curvature (e.g., the square of the Ricci tensor). This is currently under investigation ([33, 32]).

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