VOLUME ENTROPY FOR MINIMAL PRESENTATIONS OF SURFACE GROUPS IN ALL RANKS

LLUÍS ALSEDA, DAVID JUHER, JÉRÔME LOS AND FRANCESC MAÑOSAS

Abstract. We study the volume entropy of a class of presentations (including the classical ones) for all surface groups, called minimal geometric presentations. We rediscover a formula first obtained by Cannon and Wagreich [6] with the computation in a non published manuscript by Cannon [5]. The result is surprising: an explicit polynomial of degree $n$, the rank of the group, encodes the volume entropy of all classical presentations of surface groups. The approach we use is completely different. It is based on a dynamical system construction following an idea due to Bowen and Series [3] and extended to all geometric presentations in [15]. The result is an explicit formula for the volume entropy of minimal presentations for all surface groups, showing a polynomial dependence in the rank $n > 2$. We prove that for a surface group $G_n$ of rank $n$ with a classical presentation $P_n$ the volume entropy is $\log(\lambda_n)$, where $\lambda_n$ is the unique real root larger than one of the polynomial

$$x^n - 2(n-1) \sum_{j=1}^{n-1} x^j + 1.$$

1. Introduction

In the beginning of the 80’s several breakthroughs occurred in group theory. The main one was the development of large scale geometry for groups, largely due to M. Gromov with, for instance, the classification of groups with polynomial growth function [14] or the introduction of the now standard notion of hyperbolic groups [13]. At about the same time R. Grigorchuck [12] found a class of groups with intermediate growth function. In all these classes of groups the growth function plays a central role. The growth function depends on the generating set $X$ or on the presentation $P = \langle X/R \rangle$ of the group $G$. It is defined as the map $\mathbb{N} \to \mathbb{N}$ such that

$$n \mapsto f_{G,P}(n) = \text{Card}\{g \in G : \text{length}_X(g) \leq n\}.$$

From the growth function $f_{G,P}$ several asymptotic functions are defined such as the volume entropy or the growth series also called the Poincaré series.
The computational issues appeared also at about the same period. An idea due to J. Cannon [7] allows an inductive way to describe geodesics in the Cayley graph $\text{Cay}^1(G, P)$ via the notion of cone types. This notion has been intensively used later on by Epstein, Cannon, Levy, Holt, Patterson, Thurston [9] with the introduction of a very large class of groups, called automatic, that contains the hyperbolic groups of Gromov. The computation of the growth function or the growth series becomes possible in principle from a geodesic automatic structure, when it exists. This is the case for hyperbolic groups. This computation, as it was noticed in [6], can also be obtained using the Floyd-Plotnick method [10].

In practice, finding an explicit geodesic automatic structure from the presentation is not so simple. For free groups with the free presentation all the computations are easy and, for instance, the volume entropy is simply $\log(2n - 1)$, for the free group of rank $n$ (see for instance [8]). The next simple case is the class of surface groups. For the classical presentations of surface groups, the growth series appeared in a paper by Cannon and Wagreich [6] without the explicit computation, leading to those series that were earlier obtained in a non published manuscript of Cannon [5]. For hyperbolic groups, the existence of a geodesic automatic structure for each presentation implies that the growth series is a rational function (see [9, 7]). In this case the volume entropy (sometimes called the critical exponent) is related to the largest pole of the growth series, i.e. the largest root of the denominator of the growth series (see for instance [4]). The result of Cannon and Wagreich for the classical presentations of surface groups shows that the denominator $Q_n$ of the growth series is an explicit polynomial depending on the rank $n \geq 3$ of the surface group:

\begin{equation}
Q_n(x) := x^n - 2(n - 1) \sum_{j=1}^{n-1} x^j + 1.
\end{equation}

The fact that a single, explicit polynomial could encode the volume entropy for all surface groups is mysterious a priori, specially since the original computations of Cannon did not appear in published form.

In this paper we rediscover the polynomial $Q_n(x)$ from a completely different point of view and we hope that a part of the mystery will disappear. In our approach we compute the volume entropy of the group presentations from a dynamical system argument based on an idea due to R. Bowen and C. Series [3] and generalized in [15].

The original idea of Bowen and Series was to associate a specific map $\Phi_{B-S} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ to a particular action of the group $G = \pi_1(S)$ on the hyperbolic plane $\mathbb{H}^2$, where $\mathbb{S}^1$ is seen as the space at infinity of $\mathbb{H}^2$. In [15], $\mathbb{S}^1$ is considered as the Gromov boundary of the group $\partial G$ and a map $\Phi_P : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is constructed for each presentation $P$ in a class, called geometric, characterized by the fact that the two dimensional Cayley complex $\text{Cay}^2(G, P)$ is planar. The maps $\Phi_P$ are called Bowen-Series-Like and they satisfy several interesting properties, in particular the volume entropy of the presentation $P$ equals the topological entropy of the map $\Phi_P$. In addition the map $\Phi_P$ admits a finite Markov partition and the computation of the topological entropy for such maps is standard.
For any surface $S$, the classical presentation of the corresponding surface group $\Gamma = \pi_1(S)$ is geometric. These classical presentations are given by the minimal number of generators $n$ and one relation of length $2n$. For orientable surfaces, $n$ is even and equals $2g$, where $g$ is the genus of the surface. In this case, the classical relation is a product of $g$ commutators. In the non-orientable case, there is no restriction on the parity of $n$ and the relation is given by the product of the squares of all generators (see for instance \cite{18}). A presentation with the minimal number of generators is called minimal. The rank 2 cases (torus and Klein bottle) are, as usual, special: they are not hyperbolic, the growth function is quadratic and thus the volume entropy is 0. For $n > 2$ all minimal geometric presentations are proved to have the minimal volume entropy, among geometric presentations. It is conjectured that this minimum should be an absolute minimum in \cite{15}.

We rediscover here the surprising explicit polynomial $Q_n(x)$, $r \geq 3$. We will see that $Q_n(x)$ has a unique real root larger than one denoted $\lambda_n$. More precisely we prove:

**Theorem 1.1.** For $n > 2$, let $\Gamma$ be a surface group of rank $n$ with a minimal geometric presentation $P$. Then, the volume entropy of $\Gamma$ with respect to the presentation $P$ is $\log(\lambda_n)$. Moreover, for $n \geq 4$, $\lambda_n$ satisfies:

$$2n - 1 - \frac{1}{(2n - 1)^{n-2}} < \lambda_n < 2n - 1.$$

The above inequalities show that the difference between the volume entropy for the surface group and for the free group of the same rank is explicitly very small.

The genesis of this rediscovery is interesting. The dynamical system approach discussed above allows to compute the volume entropy of any geometric presentation $P$ from an explicit Bowen-Series-Like map: $\Phi_P: S^1 \to S^1$. We developed an algorithm to compute the entropy of such maps, for the classical presentations of orientable surfaces, via the well known kneading invariant technique of Milnor and Thurston \cite{16}. The polynomial $Q_n(x)$ appears that way in the computation for all orientable surfaces of genus $g \leq 43$. To obtain the theorem we needed to compute the determinant of a matrix whose size grows either linearly in $n$ with polynomial entries using the Milnor-Thurston method, or quadratically in $n$ with integer entries using the Markov matrix method. The computation leading to the proof of the theorem became possible by a succession of two surprises. First, by a particular choice of a minimal presentation $P_n^+$ the corresponding BSL map $\Phi_{P_n^+}$ admits an explicit symmetry of order $2n$. By a quotient process, the Markov matrix is reduced to an integer matrix whose size grows linearly in $n$. Then a method, developed in \cite{2} under the nice name of the “Rome technique”, was directly applicable to our case and reduced the computation to a $2 \times 2$ matrix with polynomial entries and no computer was necessary.

The paper is organized as follows. In Section 2 we recall the necessary ingredients for the construction of the map $\Phi_P$ in some particular geometric presentations. The map is then given explicitly for the particular minimal geometric presentations with a symmetry property, together with its Markov partition. In Section 3 we obtain a first formula for the volume entropy in...
the orientable case, in terms of the Markov matrix of the Bowen-Series-Like map. We exploit the symmetry of the presentation to obtain a formula for the volume entropy in terms of the spectral radius of a simpler matrix called the compacted matrix of rank $n$. In Section 4 we extend these results to the non-orientable case by showing that the volume entropy in this case is also the logarithm of the spectral radius of the compacted matrix of rank $n$. The computation of the spectral radius of this new matrix is still somewhat difficult. In Section 5 we obtain a simpler matrix with the same spectral radius. Finally, in Section 6, the “Rome method” is explained and applied to compute this spectral radius, and proving Theorem 1.1.

2. Bowen-Series-Like maps for geometric presentations

In this section we review the necessary ingredients for the construction of the Bowen-Series-Like maps defined in [15].

2.1. Geometric presentations.

Let $P = \langle X/R \rangle = \langle x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, R_1, \ldots, R_k \rangle$ be a presentation of a group $\Gamma$. Recall that the Cayley graph $\text{Cay}^1(\Gamma, P)$ is a metric space and let $B_m$ be the ball of radius $m$ centred at the identity. We denote the cardinality of any finite set $A$ by $|A|$. The volume entropy of $\Gamma$ with respect to the presentation $P$ is denoted by $h_{\text{vol}}(\Gamma, P)$ and defined as:

$$\lim_{m \to \infty} \frac{1}{m} \log |B_m|.$$ 

A presentation $P$ of a surface group $\Gamma = \pi_1(S)$ is called geometric if the Cayley 2-complex $\text{Cay}^2(\Gamma, P)$ is a plane. In particular the Cayley graph $\text{Cay}^1(\Gamma, P)$ is a planar graph. A geometric presentation $P$ is called minimal if the number of generators is minimal. For a group of an orientable surface of genus $g$ it is well known that the minimal number of generators is $2g$ (see [18] for instance) and, in this case, there is a presentation with a single relation of length $4g$. The standard classical presentation in this case is the following:

$$\langle x_1^{\pm 1}, y_1^{\pm 1}, x_2^{\pm 1}, y_2^{\pm 1}, \ldots, x_g^{\pm 1}, y_g^{\pm 1}, \prod_{i=1}^{g} [x_i, y_i] \rangle,$$

where $[x_i, y_i] = x_i \cdot y_i \cdot x_i^{-1} \cdot y_i^{-1}$ is a commutator.

For a rank $n$ group of a non-orientable surface there is also a classical presentation with a single relation of length $2n$:

$$\langle x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, \prod_{i=1}^{n} x_i^2 \rangle.$$

It is easy to check that such classical presentations are geometric (see below).

Geometric presentations satisfy very simple combinatorial properties:

**Lemma 2.1** (Floyd and Plotnick [10]). If $P = \langle x_1^{\pm 1}, \ldots, x_n^{\pm 1}/R_1, \ldots, R_k \rangle$ is a geometric presentation of a surface group $\Gamma$ then $P$ satisfies the following properties:
The set \( \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\} \) admits a cyclic ordering that is preserved by the \( \Gamma \)-action.

(b) Each generator appears exactly twice (with plus or minus exponent) in the set \( R = \{R_1, \ldots, R_k\} \) of relations.

(c) Each pair of adjacent generators, according to the cyclic ordering (a), appears exactly once in \( R \) and defines uniquely a relation \( R_i \in R \).

The following statement is the main ingredient to compute the volume entropy of a geometric presentation. The statement also contains the main result about minimal geometric presentations. In what follows \( S^1 \) will denote a (topological) circle. Recall that any surface group \( \Gamma \) is Gromov-hyperbolic \cite{13} and its boundary is: \( \partial \Gamma \cong S^1 \).

Let us introduce the notion of a Markov partition. Let \( W \) be a finite set of \( S^1 \). An interval of \( S^1 \) will be called \( W \)-basic if it is the closure of a connected component of \( S^1 \setminus W \). Observe that two different \( W \)-basic intervals have pairwise disjoint interiors. Let \( \phi: S^1 \to S^1 \) and let \( W \subset S^1 \) be finite. We say that \( W \) is a Markov partition of \( \phi \) if \( W \) is \( \phi \)-invariant (i.e., \( \phi(W) \subset W \)) and the image by \( \phi \) of every basic interval is a union of basic intervals.

**Theorem 2.2** (Los \cite{15}). Let \( P \) be a geometric presentation of a surface group \( \Gamma \). Then there exists a map \( \Phi_P: \partial \Gamma = S^1 \to \partial \Gamma = S^1 \) with the following properties:

(a) The map \( \Phi_P \) is Markov, i.e., it admits a finite Markov partition.

(b) The topological entropy of \( \Phi_P \), \( h_{top}(\Phi_P) \), is equal to the volume entropy \( h_{vol}(\Gamma, P) \).

In addition, the volume entropy is minimal, among geometric presentations, for all minimal geometric presentations.

The map \( \Phi_P \) satisfies more properties that are not needed here. Property (a) is specially interesting for computations since, for Markov maps, it is classical that the topological entropy is nothing but the logarithm of the spectral radius of a finite integer matrix, the Markov transition matrix (see \cite{17} or \cite{1} for instance). The goal of the next sections is to make such a Markov partition explicit in the particular cases of minimal geometric presentations.

### 2.2. Construction of the Bowen-Series-Like map.

In this subsection we review the definition and the necessary properties of the BSL maps, in the particular case of minimal geometric presentations.

#### 2.2.1. Bigons.

As we have seen, a presentation \( P = \langle X/R \rangle \) defines the Cayley graph \( \text{Cay}^1(\Gamma, P) \) and the Cayley 2-complex \( \text{Cay}^2(\Gamma, P) \). A bigon in \( \text{Cay}^1(\Gamma, P) \) is a pair of distinct geodesics \( \{\gamma_1, \gamma_2\} \) connecting two vertices \( \{v, v'\} \in \text{Cay}^1(\Gamma, P) \). We denote by \( B_v(x,y) \) the set of bigons \( \{\gamma_1, \gamma_2\} \) whose initial vertex is \( v \) and so that the geodesic \( \gamma_1 \) starts at \( v \) by the edge labelled \( x \) and \( \gamma_2 \) starts at \( v \) by the edge labelled \( y \), with \( x \neq y \). By the \( \Gamma \)-action we can fix the initial vertex \( v \) to be the identity and we denote \( B_{id}(x,y) \) by \( B(x,y) \).

For geometric presentations of surface groups the set of bigons is particularly simple.
Lemma 2.3. If $P = \langle X/R \rangle$ is a geometric presentation of a surface group $\Gamma$ then the set of bigons $B(x, y)$ is non empty if and only if $(x, y)$ is an adjacent pair of generators, according to the cyclic ordering of Lemma 2.1(a). In addition, if $(x, y)$ is an adjacent pair of generators there is a unique bigon $\beta(x, y) \in B(x, y)$ of finite minimal length, called minimal bigon.

This lemma is proved in [13] Lemmas 2.6 and 2.12. Observe that each minimal bigon is particularly simple for a geometric presentation where all relations have even length. Indeed, each pair of adjacent generators $(x, y)$ defines a unique relation by Lemma 2.1(c). The relation can be written, up to cyclic permutation and inversion $(R_i \rightarrow (R_i)^{-1})$, as: $R_i = \gamma_1 \cdot (\gamma_2)^{-1}$, where $\gamma_1$ is a word (or a path) starting by the letter $x$, $\gamma_2$ starts by the letter $y$ and $l(\gamma_1) = l(\gamma_2)$. The observation is that for geometric presentations the two paths $\gamma_1$ and $\gamma_2$ start at the identity and end at the same vertex (since $R_i$ is a relation) and are geodesics. In other words the pair $\{\gamma_1, \gamma_2\}$ is a bigon. It is minimal and unique by Lemma 2.1(c).

2.2.2. Bigon-Rays. We describe a canonical way to define a point on the boundary $\partial \Gamma$ associated to an adjacent pair of generators $(x, y)$. Recall that a surface group is hyperbolic in the sense of Gromov [13] and its boundary $\partial \Gamma$ is the circle $S^1$. By definition of $\partial \Gamma$, a point $\xi \in \partial \Gamma$ is the limit of geodesic rays, for instance starting at the identity, modulo the equivalence relation among rays that two rays are equivalent if they stay at a uniform bounded distance from each others (c.f. [13]). If $\xi \in \partial \Gamma$ is a point on the boundary we denote by $\{\xi\}$ a geodesic ray starting at identity and converging to $\xi$.

In what follows, given two integers $k$ and $l$ we will denote $k \pmod l$ by $[k]_l$. Also, we choose $1, 2, \ldots, l$ as the representatives of the classes modulo $l$; that is, $[0]_l = [l]_l = l$. However, unless necessary we omit the modulo part in the notations.

Notation 2.4. In what follows we denote the $n$ generators (and their inverses) by $y_1, y_2, \ldots, y_n$, in such a way that $y_{[i+1]_2n}$ are the elements adjacent to $y_i$ with respect to the cyclic ordering from Lemma 2.1(a). We denote an adjacent pair by $(y_i, y_{[i+1]_2n})$ where, by convention, the edges denoted $y_i$ and $y_{[i+1]_2n}$ are adjacent and oriented from the vertex. We also adopt the convention that $y_i$ is on the left of $y_{[i+1]_2n}$ (see Figure 1). This convention defines an orientation of the plane $\text{Cay}^2(\Gamma, P)$.

The parity of the number of adjacent pairs at each vertex implies that $(y_i, y_{[i+1]_2n})$ defines an opposite pair, with respect to the cyclic ordering of Lemma 2.1(a), defined by:

$$(y_i, y_{[i+1]_2n})^{\text{opp}} := (y_{[i+n]_2n}, y_{[i+n+1]_2n})$$

(see Figure 2).

We construct a unique infinite sequence of adjacent pairs, bigons and vertices from any given pair $(y_i, y_{[i+1]_2n})$ by the following process:

Step 1. Each adjacent pair, at the identity, defines a unique minimal bigon $\beta(y_i, y_{i+1})$ by Lemma 2.3. The bigon $\beta(y_i, y_{i+1})$ is a pair of geodesics $\{\gamma_l, \gamma_r\}$, where the indices $l, r$ stand for left and right, with respect to an orientation of the plane $\text{Cay}^2(\Gamma, P)$. The geodesics $\{\gamma_l, \gamma_r\}$ connect the identity to a vertex $v_1 = v_1[\beta(y_i, y_{i+1})]$. 

Step 2. The two geodesics \( \{ \gamma_l, \gamma_r \} \) end at \( v_1 \) by two generators that are adjacent by Lemma 2.3. Therefore the bigon \( \beta(y_i, y_{i+1}) \) defines a unique adjacent pair at \( v_1 \), called a top pair of \( \beta(y_i, y_{i+1}) \), which is denoted: \( \text{topp}[\beta(y_i, y_{i+1})] \), based at \( v_1 = v_1[\beta(y_i, y_{i+1})] \) and is uniquely defined by \( (y_i, y_{i+1}) \).

Step 3. The pair \( \text{topp}[\beta(y_i, y_{i+1})] \) defines an opposite pair at \( v_1 \), denoted by:

\[
(\text{topp}[\beta(y_i, y_{i+1})])^{\text{opp}} := (y_i, y_{i+1})^{(1)}.
\]

Step 4. We consider then the unique minimal bigon, at \( v_1 \), defined by the pair \( (y_i, y_{i+1})^{(1)} \) by Lemma 2.3:

\[
\beta^{(1)}(y_i, y_{i+1}) := \beta_{v_1}[(y_i, y_{i+1})^{(1)}].
\]

Step 5. The bigon \( \beta^{(1)}(y_i, y_{i+1}) \) defines a new top pair \( \text{topp}[\beta^{(1)}(y_i, y_{i+1})] \), at the vertex \( v_2 \).

The Steps 1–5 define, by induction, a unique infinite sequence of vertices and bigons (see Figure 2):

\[
\text{id}, v_1, v_2, \ldots,
\]

Each bigon in the infinite sequence \( \{ \beta^{(k)}(y_i, y_{i+1}) : k \in \mathbb{N} \} \) is a pair of geodesics \( \{ \gamma_l^{(k)} \}, \gamma_r^{(k)} \} \) with \( k \in \mathbb{N} \) connecting the vertices \( v_k \) and \( v_{k+1} \).

By definition, the terminal vertex \( v_{k+1} \) of \( \beta^{(k)} \) is the initial vertex of the next bigon \( \beta^{(k+1)} \) in the sequence. Therefore a finite concatenation of bigons

\[
\beta^{(0)}(y_i, y_{i+1}) \beta^{(1)}(y_i, y_{i+1}) \cdots \beta^{(k)}(y_i, y_{i+1})
\]

makes sense. It is defined by the finite collection of paths:

\[
\{ \gamma_l^{(0)}, \gamma_r^{(0)}, \gamma_l^{(1)}, \gamma_r^{(1)}, \ldots, \gamma_l^{(k)}, \gamma_r^{(k)} : \epsilon(j) \in \{ l, r \}, j \in \{ 0, 1, \ldots, k \} \}.
\]

We denote the infinite concatenation of all these paths as:

\[
\beta^{\infty}(y_i, y_{i+1}) := \lim_{k \to \infty} \beta^{(0)}(y_i, y_{i+1}) \beta^{(1)}(y_i, y_{i+1}) \cdots \beta^{(k)}(y_i, y_{i+1}).
\]

Lemma 2.5 (Los [15, Lemma 3.1]). With the above notation the following statements hold.
(a) Each path in the collection: \( \beta^{(0)}(y_i, y_{i+1}) \beta^{(1)}(y_i, y_{i+1}) \cdots \beta^{(k)}(y_i, y_{i+1}) \) is a geodesic segment, for all \( k \in \mathbb{N} \).

(b) Two geodesic segments in (a) stay at a uniform distance from each other for any \( k \in \mathbb{N} \).

In consequence, the infinite concatenation \( \beta^\infty(y_i, y_{i+1}) \) defines infinitely many geodesic rays with a unique limit point in \( \partial \Gamma \). It will be denoted by \( (y_i, y_{i+1})^\infty \).

2.2.3. Cylinders, definition of the BSL map. We define the cylinder of length one as the subset of the boundary:

\[
C_x := \{ \xi \in \partial \Gamma : \text{there is a geodesic ray } \{ \xi \} \text{ starting at id by } x \in X \}.
\]

Lemma 2.6. Let \( P = \langle X/R \rangle \) be a geometric presentation of \( \Gamma \). The boundary \( \partial \Gamma = S^1 \) is covered by the cylinder sets \( C_x, \ x \in X \) and:

(a) Two cylinders have non-empty intersection: \( C_x \cap C_y \neq \emptyset \) if and only if \((x, y)\) is an adjacent pair of generators.

(b) Each cylinder \( C_x, \ x \in X \) is a non trivial connected interval of \( \partial \Gamma \).

This lemma is proved in [15, Lemmas 2.13 and 2.14]. Observe that the point \( (y_i, y_{i+1})^\infty \) of Lemma 2.5 belongs, by definition, to the intersection \( C_{y_i} \cap C_{y_{i+1}} \).

In what follows we consider the points in the circle ordered *clockwise*. That is, if \( r, s, t \) are pairwise different points of \( S^1 \) we will write \( r < s < t \) if \( s \) belongs to the clockwise arc starting at \( r \) and ending at \( t \). The notation \( r \leq s \leq t \) will also be used in the natural way. Then the interval \([r, t]\) is defined as the set \( \{ s \in S^1 : r \leq s \leq t \} \). Also, if \( I, J, K \) are closed connected subsets of \( S^1 \) with pairwise disjoint interiors we will write \( I < J < K \) whenever \( r \leq s \leq t \) for every \( r \in I, \ s \in J \) and \( t \in K \).

Definition 2.7. If \( P = \langle X/R \rangle \) is a geometric presentation of a hyperbolic surface group \( \Gamma \), then we denote by \( I_{y_i} \) the interval \([ (y_{i-1}, y_i)^\infty, (y_i, y_{i+1})^\infty]\). Clearly \( I_{y_i} \) is a subset of \( C_{y_i} \) for every \( y_i \in X \).
We define the Bowen-Series-Like map $\Phi_P : \partial \Gamma \to \partial \Gamma$ by

$$\Phi_P(\xi) = x^{-1}(\xi) \text{ if } \xi \in I_x,$$

where $x^{-1}(\xi)$ is the action, by homeomorphism, on $\partial \Gamma$ by the group element $x^{-1}$.

The map $\Phi_P$ satisfies the following elementary properties:

(i) It depends explicitly on the presentation $P$ (the exact dependence will be explained below).

(ii) Since $I_x \subset C_x$, each $\xi \in I_x$ has a writing, as a limit of a ray, as $\{\xi\} = x \cdot \omega$. The image under $\Phi_P$ is given by:

$$\{\Phi_P(\xi)\} = \{x^{-1}(x \cdot \omega)\} = \{\omega\}.$$

In other words, the map $\Phi_P$ is a shift map, on this particular writing as a ray.

2.3. Markov partition for minimal geometric presentations.

Theorem 2.2 states that the map $\Phi_P$ admits a Markov partition. In this subsection we will define a particular presentation, which will be called symmetric, and we will make the Markov partition explicit for this presentation.

The first step is to define subdivision points in each interval $I_x$, $x \in X$. Let us recall that the extreme points $(y, x)^\infty$ and $(x, z)^\infty$ of the intervals $I_x$ are limit points of bigon rays $\beta^\infty(y, x)$ and $\beta^\infty(x, z)$. Let us focus on $(y, x)^\infty$. Let $\beta^\infty(y, x)$ be the bigon ray starting at the vertex $v \in \text{Cay}^1(\Gamma, P)$. Observe that with this definition we can write:

$$\beta^\infty(y, x) = \beta(y, x) \cdot \beta^\infty_{v_1}[(y, x)^{(1)}],$$

with the notations of Subsection 2.2.2.

The particular property of a minimal geometric presentation that is useful here is that there is only one relation $R$ of even length $2n$, when $\Gamma$ is a surface group of rank $n$. In this case, any bigon $\beta(y, x)$ has the form $\{\gamma_l, \gamma_r\}$ with $\gamma_l \cdot (\gamma_r)^{-1}$ being one of the words representing the relation $R$, up to cyclic permutation and inversion. This word starts with the letter $y$ and terminates with the letter $x^{-1}$.

Since the relation $R$ has length $2n$, let us write the two paths $\{\gamma_l, \gamma_r\}$ as:

$$\{y \cdot x_{i_1} \cdot \cdots \cdot x_{i_n}, x \cdot x_{i_2} \cdot \cdots \cdot x_{i_n}\}.$$

We focus on the “$x$” side of Equations (3),(4), i.e. on the infinite collection of rays:

$$x \cdot x_{i_2} \cdot \cdots \cdot x_{i_n} \cdot \beta^\infty_{v}[(y, x)^{(1)}],$$

where $v$ is the group element written: $v = x \cdot x_{i_2} \cdots x_{i_n}$. The vertices $v^{1} = x$ and $v^{j} = x \cdot x_{i_2} \cdots x_{i_j}$, for $j = 2, 3, \ldots, n-1$ of $\text{Cay}^1(\Gamma, P)$ belong to $\gamma_r$ and are ordered along $\gamma_r$ (this notation is consistent with $v = v_n$).

The following pairs of consecutive edges:

$$\{(x, x_{i_2}), (x_{i_2}, x_{i_3}), \ldots, (x_{i_{n-1}}, x_{i_n})\}$$

at the vertices $\{v^{1}, \ldots, v^{n-1}\}$, are crossed by the path $\gamma_r$, where the notation $x_{i_j}$ means the edge $x_{i_j}$ with the opposite orientation. Observe that each pair of consecutive letters along the paths $\gamma_r$ are adjacent generators.
Lemma 2.8. If the relation defining $\beta(y,x)$ has even length $2n$ then the collection:

$$R^x_L := \left\{ x \cdot x_{i_2} \cdots x_{i_j} \cdot \beta^{(*)}(\infty)(x_{i_j}, x_{i_{j+1}})^{\text{opp}} : j = 1, \ldots, n - 1 \right\},$$

(see Figure 3) is called the left (with respect to $x$) subdivision rays. They satisfy the following properties:

(a) Each path in the infinite collection $R^x_L$ is a ray starting at the identity.

(b) For a given $j \in \{1, 2, \ldots, n - 1\}$, all the rays in $R^{(x,j)}_L = x \cdot x_{i_2} \cdots x_{i_j} \cdot \beta^{(*)}(\infty)(x_{i_j}, x_{i_{j+1}})^{\text{opp}}$ converge to the same point $\lambda^j_x \in \partial \Gamma$.

(c) For any $j \neq p$, the rays in $R^{(x,j)}_L$ and in $R^{(x,p)}_L$ have a common beginning: $x \cdot x_{i_2} \cdots x_{i_\nu}$ where $\nu := \min\{j, p\}$ and are otherwise disjoint.

(d) Each $\lambda^j_x$, $j \in \{1, 2, \ldots, n - 1\}$ belongs to the interior of the interval $I_x$ of Definition 2.7.

(e) The limit points $\lambda^j_x$ are inversely ordered with respect to the index $j \in \{1, 2, \ldots, n - 1\}$ along $\partial \Gamma$ (that is, $\lambda^{n-1}_x < \lambda^{n-2}_x < \cdots < \lambda^2_x < \lambda^1_x$).

This lemma is proved in [15, Lemma 4.1].

![Figure 3. Subdivision rays.](image)

We denote $L_x = \{\lambda^1_x, \ldots, \lambda^{n-1}_x\}$ this set of left (with respect to $x$) limit points. By the same analysis the adjacent pair $(x, y)$ defines the set of right (with respect to $x$) limit points $R_x = \{\rho^1_x, \ldots, \rho^{n-1}_x\}$, which are ordered with respect to the superindex. Observe that we use here the fact that a minimal geometric presentation has only one relation (of length $2n$). Consider now the set of all such points:

$$S = \bigcup\limits_{x \in X} (R_x \cup L_x \cup \partial I_x),$$

called the subdivision points.
Lemma 2.9. If $P$ is a geometric presentation of a hyperbolic surface group $\Gamma$ so that all relations have even length, then the set of subdivision points $\mathcal{S}$ is invariant under the map $\Phi_P$ of Definition 2.7 and defines a finite Markov partition of $\partial \Gamma$.

This statement is a particular case of [15] Theorem 4.3. For a minimal geometric presentation there is only one relation of length $2n$ for a surface group of rank $n$. In this case the partition of each interval $I_{\mathcal{S}}$ above is given by the points $\mathcal{R}_{x} \cup \mathcal{L}_{x} \cup \partial I_{x}$ which are ordered in the following way:

$$\lambda_{x}^{2} := (y, x)^{\infty} < \lambda_{x}^{1-1} < \cdots < \lambda_{x}^{2} < \rho_{x}^{1} < \lambda_{x}^{2} < \cdots < \rho_{x}^{n} := (x, z)^{\infty}.$$  

We also observe here that the intervals $I_{x}$ are ordered, along $S^1$, by the (cyclic) ordering of the generators at the identity. Then, we can define a partition of each of the intervals $I_{x}$ consisting on the following subintervals:

\begin{align}
L_{x}^{i} &= [\lambda_{x}^{i}, \lambda_{x}^{i-1}] \quad \text{and} \quad R_{x}^{i} = [\rho_{x}^{i-1}, \rho_{x}^{i}] \quad \text{for } i \in \{3, 4, \ldots, n\},
C_{x}^{l} &= [\lambda_{x}^{1}, \rho_{x}^{1}] \quad \text{and} \quad C_{x}^{r} = [\lambda_{x}^{2}, \rho_{x}^{2}], \quad \text{and }
C_{x} &= [\rho_{x}^{1}, \lambda_{x}^{2}].
\end{align}

Recall that a subdivision point (left or right) has the following writing:

$$\{\lambda_{x}^{2}\} = x \cdot x_{i2} \cdot \cdots \cdot x_{ij} \cdot \beta_{x_{i2} \cdots x_{ij}}^{\infty}[(\pi_{ij}, x_{ij+1})]_{\text{opp}}, \quad \text{for } j \in \{1, 2, \ldots, n\}.$$  

Since the map $\Phi_P$ acts, on each interval $I_{x}$, on the ray writing as a shift map we obtain:

\begin{align}
\{\Phi_{P}(\lambda_{x}^{2})\} &= \beta^{\infty}[(\pi, x_{i2})]_{\text{opp}}, \quad \text{and} \\
\{\Phi_{P}(\lambda_{x}^{1})\} &= x_{i2} \cdots x_{ij} \cdot \beta_{x_{i2} \cdots x_{ij}}^{\infty}[(\pi_{ij}, x_{ij+1})]_{\text{opp}} \quad \text{for } j \in \{2, 3, \ldots, n\}
\end{align}

and there is a similar writing for the points $\rho_{x}^{j}$.  

Lemma 2.10. If $P$ is a geometric presentation of a surface group with all relations of even length then the image of the central interval $C_{x} = [\rho_{x}^{1}, \lambda_{x}^{2}]$ under $\Phi_P$ is a single interval $I_{u}$, $u \in X$, where $u$ is the generator that is opposite to $x^{-1}$ for the cyclic ordering of Lemma 2.1(a) at the vertex $x$.

Proof. By [10] we observe that $\{\Phi_{P}(\lambda_{x}^{2})\} = \beta^{\infty}[(\pi, x_{i2})]_{\text{opp}}$ and similarly $\{\Phi_{P}(\rho_{x}^{1})\} = \beta^{\infty}[(\pi', x_{i2})]_{\text{opp}}$. Since the two adjacent pairs $(\pi, x_{i2})$ and $(\pi', x_{i2})$ are adjacent at the vertex $v_1 = x$ then the two opposite pairs $(\pi, x_{i2})_{\text{opp}}$ and $(\pi', x_{i2})_{\text{opp}}$ are also adjacent. That means that they share one edge $u$. This edge is just the one that is opposite to $x^{-1}$ at the vertex $x$ (see Figure 4). □

Next we define a particular presentation, which we call symmetric, for the rank $n$ group $\pi_1(S)$, where $S$ is an orientable surface. Recall that $n = 2g$, where $g$ is the genus of $S$.

Definition 2.11. Given a surface group $\pi_1(S)$ of rank $n = 2g$, where $S_g$ is orientable of genus $g$, the presentation

$$\langle x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1} / x_1 x_2 \cdots x_n x_1^{-1} x_2^{-1} \cdots x_n^{-1} \rangle$$

will be called symmetric and denoted by $P^n_+$.  

Proposition 2.12. The symmetric presentation $P^n_+$ is minimal and geometric.
Proof. Consider the polygon $\Delta_n$ with $2n$ sides, labelled by the elements of \{${x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}}$\} in the ordering:

$$x_1 \cdot x_2 \cdots x_n \cdot x_1^{-1} \cdot x_2^{-1} \cdots x_n^{-1}.$$ 

The identification of the side labelled $x_i$ with the one labelled $x_i^{-1}$ defines an orientable surface of genus $g$. The identification is an equivalence relation $\sim$ and $\Delta_n/\sim$ is the surface of genus $g$. The presentation $P_n^+$ is minimal since it has $n$ generators and it is geometric because the universal cover of the surface $\Delta_n/\sim$ is nothing but the Cayley 2-complex $\text{Cay}^2(\Gamma, P_n^+)$ that is a plane.

Lemma 2.1 says that for geometric presentations the generators have a cyclic ordering at each vertex. For the presentation $P_n^+$ the cyclic ordering is

$$x_1 < x_2^{-1} < x_3 < x_4^{-1} < \cdots < x_{n-1} < x_n^{-1} < x_1^{-1} < x_2 < \cdots < x_{n-1} < x_n.$$

3. The topological entropy of the map $\Phi_{P_n^+}$

The aim of this section is to start the computation of the topological entropy of the Bowen-Series-Like map $\Phi_{P_n^+}$ for the symmetric presentation $P_n^+ = \langle X/R \rangle$ of the orientable surface group of rank $n$.

Since the surface is orientable and the presentation is geometric and minimal, then all generators $x \in X$ act on $\partial \Gamma$ as an orientation preserving homeomorphism. By Definition 2.7(ii), $\Phi_P|_{\partial \Gamma}$ is an orientation preserving homeomorphism for every $x \in X$ and from Lemma 2.9, the set $S$ defines a Markov partition of $\Phi_P$. Since $\partial \Gamma_{x_i} \subset S$ we also have that $\Phi_P$ is a homeomorphism on every $S$–basic interval.

In this situation the topological entropy can be easily computed as the logarithm of the spectral radius of the associated Markov matrix. Let us recall such result.

Let $W$ be a Markov partition of a map $\phi: \mathbb{S}^1 \to \mathbb{S}^1$, and let $U_1, U_2, \ldots, U_{|W|}$ be a labelling of the $W$–basic intervals. The Markov matrix of $W$ is
defined as the $|W| \times |W|$ $(0,1)$--matrix $M = (m_{ij})_{i,j=1}^{W}$ such that $m_{ij} = 1$ if and only if $\phi(U_i) > U_j$.

For any square matrix $M$, we will denote its *spectral radius* by $\rho(M)$.

It is well known (see for instance [2] or [1, Theorem 4.4.5]), that if $\phi$ is monotone on each basic interval then

$$h_{\text{top}}(\phi) = \log \max\{\rho(M), 1\}. \quad (11)$$

We will use (11) to compute $h_{\text{top}}(\Phi_{P^+})$. To this end we first have to compute the Markov matrix of $S$ that, in what follows, will be denoted by $M_{n}^+$. As we will see, a direct computation of $\rho(M_{n}^+)$ is infeasible at a practical level because the size of the matrix grows quadratically with $n$. So, the computation of $\rho(M_{n}^+)$ will be done in two steps by using spectral radius preserving transformations of the matrix $M_{n}^+$. In this section we will compute the Markov matrix $M_{n}^+$ for a symmetric presentation in the orientable case.

To do this, we need to specify completely the map $\Phi_{P^+}$ and then compute its Markov matrix. Recall that, for the symmetric presentation $P_{n}^+$ (see comments after Proposition 2.12), the cyclic ordering of Lemma 2.1 at any vertex is given by:

$$x_1 < x_2^{-1} < \cdots < x_{n-1} < x_n^{-1} < x_1^{-1} < x_2 < \cdots < x_{n-1}^{-1} < x_n < x_1$$

(see Figure 5). The main property of this cyclic ordering which makes the symmetric presentation very special and useful is that the edge that is opposite to $x$ at any vertex is simply the edge $x^{-1}$.

The above cyclic ordering of the generators induces the following ordering of the intervals $I_x$ along the boundary $\partial \Gamma = S^1$:

$$I_{x_1} < I_{x_2}^{-1} < \cdots < I_{x_n}^{-1} < I_{x_1}^{-1} < I_{x_2} < \cdots < I_{x_n} < I_{x_1}.$$ 

The fact that the symmetric presentation has associated the above cyclic ordering gives the following immediate corollary of Lemma 2.10:

**Corollary 3.1.** Let $P_{n}^+$ be the symmetric presentation of an orientable surface group of rank $n$. Then, $\Phi_{P^+}(C_x) = I_x$ for each generator $x$.

For notational reasons we denote the ordered generators as

$$y_1 < y_2 < \cdots < y_{2n} < y_1$$
and the corresponding intervals as

\[ I_{y_1} < I_{y_2} < \cdots < I_{y_{2n}} < I_{y_1}, \]

where \( y_i = x_{i-n}^{(-1)^i+1} \) for \( 1 \leq i \leq n \), and \( y_i = x_{i-n}^{(-1)^i} \) for \( n+1 \leq i \leq 2n \). Also, the fact that the edge that is opposite to \( x \) at any vertex is the edge \( x^{-1} \) now gives

\[ y_i^{-1} = y_{i+n}^{(2-n)} . \]

Observe from (9) that each of the \( 2n \) intervals \( I_{y_i} \) is divided into \( 2n-1 \) intervals

\[ I_{y_1}^{(n)} < \cdots < I_{y_3}^{(n)} < C_{y_2}^{(L)} < C_{y_3}^{(R)} < R_{y_1}^{(3)} < \cdots < R_{y_2}^{(n)} , \]

Hence, \(|S| = 2n(2n-1)\) and thus, the matrix \( M_n^+ \) is \( 2n(2n-1) \times 2n(2n-1) \).

Equations (10), (12) and Corollary 3.1 give the following images of the partition intervals defined in (9) (see Figure 6):

\[
\begin{align*}
\Phi_{P_n^+} (L_{y_i}^j) &= L_{y_{i+n}+1}^{j-1} \quad \text{for } j \in \{4, 5, \ldots, n\}, \\
\Phi_{P_n^+} (L_{y_i}^3) &= C_{y_{i+n+1}}^{(L)} \cup C_{y_{i+n+1}}^{(R)} , \\
\Phi_{P_n^+} (C_{y_i}^L) &= C_{y_{i+n+1}}^{(L)} \cup \left( \bigcup_{j=2}^{n} R_{y_{i+n+1}}^{j} \right) \cup \left( \bigcup_{k=[i+n+2]}^{[i+1]} I_{y_k} \right) , \\
\Phi_{P_n^+} (C_{y_i}^R) &= C_{y_{i+n+1}}^{(L)} \cup \left( \bigcup_{j=2}^{n} L_{y_{i+n+1}}^{j} \right) \cup \left( \bigcup_{k=[i+1]}^{[i+n-2]} I_{y_k} \right) , \\
\Phi_{P_n^+} (R_{y_i}^3) &= C_{y_{i+n+1}}^{(L)} \cup C_{y_{i+n+1}}^{(R)} , \\
\Phi_{P_n^+} (R_{y_i}^1) &= R_{y_{i+n+1}}^{j-1} \quad \text{for } j \in \{4, 5, \ldots, n\} .
\end{align*}
\]

From the formulae (14) it follows that the Markov matrix \( M_n^+ \) has a structure in blocks, all of size \( (2n-1) \times (2n-1) \). So, it is convenient to write the matrix \( M_n^+ \) as

\[
\begin{pmatrix}
M_{11} & M_{12} & \cdots & M_{1,2n} \\
M_{21} & M_{22} & \cdots & M_{2,2n} \\
\vdots & \vdots & \ddots & \vdots \\
M_{n1} & M_{n2} & \cdots & M_{n,2n} \\
M_{2n,1} & M_{2n,2} & \cdots & M_{2n,2n}
\end{pmatrix}
\]

(15)

where each of the matrices \( M_{ij} = (m_{ij}^{(L)})_{i,j=1}^{2n-1} \) is of size \( (2n-1) \times (2n-1) \).

Accordingly, we will label the basic intervals contained in \( I_{y_i} \) as \( U_j^i \) in such a way that they preserve the ordering given in (13). So, for \( i = 1, 2, \ldots, 2n \)
and $j = 1, 2, \ldots, 2n - 1$ it follows that

$$U^i_j = \begin{cases} L^{(n+1)-j}_{y_n} & \text{for } j = 1, 2, \ldots, n - 1, \\ C_{y_n} & \text{for } j = n - 1, \\ C_{y_n} & \text{for } j = n, \\ C_{y_n} & \text{for } j = n + 1, \\ R^{(n-1)}_{y_n} & \text{for } j = n + 2, n + 3, \ldots, 2n - 1. \end{cases}$$

With this labelling we define the matrix $M^+_n$ so that $m^+_n = 1$ if and only if $\phi(U^i_j) \supset U^i_j$.

The next theorem is a first reduction in the effective computation of $h_{\text{top}}(\Phi P^+_n)$. 

**Figure 6.** The intervals $I_{y_i}$ in the circle together with the interior intervals. The outer curve is the image $\Phi_{P_n^+} |_{I_{y_i}}$ (which is order preserving). The intervals $L^i_{y_i}$, $R^i_{y_i}$ and their images are drawn with a continuous black line, $L^3_{y_i}$, $R^3_{y_i}$ and their images are drawn with a dotted line, $C^L_{y_i}$, $C^R_{y_i}$ and their images are drawn with a continuous thick black line and finally, $C^r_{y_i}$ and its image are drawn with a continuous thick grey line.
Theorem 3.2.

\[ h_{\text{top}}(\Phi_{P_n}^+) = \log \max \left\{ \rho(M_n^+), 1 \right\} = \log \max \left\{ \rho \left( \sum_{k=1}^{2n} M_{1k} \right), 1 \right\}. \]

An \((r, s)\)-block circulant matrix is a matrix of the form

\[
\begin{pmatrix}
A_1 & A_2 & A_3 & \ldots & A_r \\
A_r & A_1 & A_2 & \ldots & A_{r-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_2 & A_3 & A_4 & \ldots & A_1
\end{pmatrix}
\]

where each \(A_i\) is an \(s \times s\) matrix. Notice that a circulant matrix is completely determined by its first block row \((A_1 A_2 A_3 \ldots A_r)\).

The next lemma will be crucial in effectively computing the spectral radius of \(M_n^+\) (see Figure 7 for an example in the case of rank 4).

Lemma 3.3. The Markov matrix \(M_n^+\) is a \((2n, 2n-1)\)-block circulant matrix.

Proof. From the formulae (14) it follows that \(\Phi_{P_n}^+(U_i^l) \supset U_j^t\) if and only if \(\Phi_{P_n}^+(U_i^{[l+1]2n}) \supset U_j^{[t+1]2n}\). In terms of the Markov matrix this amounts to \(m_{ij}^l = m_{ij}^{[l+1]2n,[t+1]2n}\) for every \(l, t \in \{1, 2, \ldots, 2n\}\) and \(i, j \in \{1, 2, \ldots, 2n - 1\}\). This implies that \(M_{lt} = M_{[l+1]2n,[t+1]2n}\). \(\square\)

The next technical lemma provides a nice and useful result about the spectral radius of block circulant matrices.
Lemma 3.4. Let
\[
A = \begin{pmatrix}
A_1 & A_2 & A_3 & \ldots & A_r \\
A_r & A_1 & A_2 & \ldots & A_{r-1} \\
A_{r-1} & A_r & A_1 & \ldots & A_{r-2} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
A_2 & A_3 & A_4 & \ldots & A_1
\end{pmatrix}
\]
be a non-negative block circulant matrix. Then
\[
\rho(A) = \rho \left( \sum_{i=1}^r A_i \right).
\]

Proof. Since \(A\) is a block matrix, for every \(m \geq 1\), \(A^m\) is a block matrix
\[
\begin{pmatrix}
A_{11}^{(m)} & A_{12}^{(m)} & \ldots & A_{1r}^{(m)} \\
A_{21}^{(m)} & A_{22}^{(m)} & \ldots & A_{2r}^{(m)} \\
\ldots & \ldots & \ldots & \ldots \\
A_{r1}^{(m)} & A_{r2}^{(m)} & \ldots & A_{rr}^{(m)}
\end{pmatrix}
\]
where each block has size \(s \times s\) and is a sum of \(r^{m-1}\) non-commutative products of \(m\) matrices among the blocks \(A_1, A_2, A_3, \ldots, A_r\). That is, each \(A_{ij}^{(m)}\) is the sum of \(r^{m-1}\) matricial products of the form \(A_{r1} A_{r2} \cdots A_{rm}\) with \(r_1, r_2, \ldots, r_m \in \{1, 2, \ldots, r\}\).

Moreover, since every block \(A_i\) appears exactly once in every block row and every block column it can be proved by induction that every product of the form \(A_{r1} A_{r2} \cdots A_{rm}\) appears exactly once in every block row and every block column of \(A^m\). Therefore, for every \(q \in \{1, 2, \ldots, r\}\), \(\sum_{i=1}^r A_{qi}^{(m)}\) and \(\sum_{i=1}^r A_{iq}^{(m)}\) are the sum of \(r^m\) matricial products, and every product \(A_{r1} A_{r2} \cdots A_{rm}\) appears exactly once in each of these expressions. Hence,
\[
\sum_{i=1}^r A_{qi}^{(m)} = \sum_{i=1}^r A_{iq}^{(m)} = \left( \sum_{i=1}^r A_i \right)^m.
\]

Consequently, the classical matrix norms satisfy:
\[
\|A^m\|_\infty = \| (\sum_{i=1}^r A_i)^m \|_\infty \quad \text{and} \quad \|A^m\|_1 = \| (\sum_{i=1}^r A_i)^m \|_1.
\]

Since the spectral radius is defined as: \(\rho(A) = \lim_{m \to \infty} \|A^m\|_1^{1/m}\) for any matrix norm, it follows that \(\rho(A) = \rho \left( \sum_{i=1}^r A_i \right)\).

Remark 3.5. In fact the above lemma holds for every matrix for which a given block appears exactly once in every block row and every block column.

Proof of Theorem 3.2. It follows from (11) and Lemmas 3.3 and 3.4.

Next, to complete the first reduction in the computation of \(h_{\top}(\Phi_{F_n^+})\), we will give an explicit formula for the matrix \(\sum_{k=1}^{2n} M_k\).
Proof of Lemma 3.6.

From formulae (14) and taking into account the labelled matrix

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[(17) \quad c_{ij} = \begin{cases} 
1 & j = i + 1 \text{ and } i \in \{1, 2, \ldots, n - 3\}, \\
1 & j \in \{n - 1, n\} \text{ and } i = n - 2, \\
n - 2 & j \in \{1, 2, \ldots, n\} \text{ and } i = n - 1, \\
n - 1 & j \in \{n + 1, n + 2, \ldots, 2n - 1\} \text{ and } i = n - 1, \\
1 & j = i - 1 \text{ and } i \in \{n + 3, n + 4, \ldots, 2n - 1\}, \text{ and otherwise.} 
\end{cases}
\]

In matrix form, \( C_n \) is

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Lemma 3.6.

\[
\sum_{k=1}^{2n} M_{1k} = C_n.
\]

To prove this lemma we need to explicitly describe the matrix \( M_n^+ \). To this end we introduce the following notation. The zero matrix of size \( k \times k \) will be denoted by \( 0_k \), and \( J_k \) will denote the \( k \times k \) \((0, 1)\)-matrix with ones in the anti-diagonal. Also, if \( i \in \{1, 2, \ldots, k\} \), \( U_i^k \) will denote the \( k \times k \) matrix such that all entries in the \( i\)-th row are 1 and all other entries are 0. Finally, for \( k \geq 5 \) odd, \( T_k = (t_{ij}) \) is the \( k \times k \) \((0, 1)\)-matrix such that \( t_{ij} = 1 \) if and only if (see Figure 8 for examples):

- \( j = i + 1 \) and \( i \in \{1, 2, \ldots, k - 3\} \), or
- \( j \in \{k - 1, k\} \) and \( i = k - 2 \) or
- \( k + 1 \leq j \leq k \) and \( i = k - 1 \),

where \( \tilde{k} = \frac{k+1}{2} \). Observe that (see again Figure 8) \( J_k T_k J_k \) is the matrix obtained from \( T_k \) by a symmetry with respect to the central coordinate \( l_{k, k}^i \).

Proof of Lemma 3.6. From formulae (14) and taking into account the labelling of the basic intervals (16) it follows that \( m_{ij}^k = 1 \) if and only if (see
Figure 7 for an example in the case of rank 4:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix}
\]

Figure 8. Examples of the matrices \(\mathbf{U}_k, \mathbf{T}_k, \mathbf{J}_k\) and \(\mathbf{J}_k \mathbf{T}_k \mathbf{J}_k\) with \(k = 7\).

In matrix block form, the above formulae become (see again Figure 7):

\[
\begin{align*}
M_{t, [l + n + 1]_{2n}} &= \mathbf{T}_{2n-1} \\
M_{lt} &= \mathbf{U}^{n-1}_{2n-1} \quad \text{for } [t + n + 2]_{2n} \leq t \leq [l - 1]_{2n} \\
M_{ll} &= \mathbf{U}^{n-1}_{2n-1} \\
M_{lt} &= \mathbf{U}^{n+1}_{2n-1} \quad \text{for } [l + 1]_{2n} \leq t \leq [l + n - 2]_{2n} \\
M_{l, [l + n + 1]_{2n}} &= \mathbf{J}_{2n-1} \mathbf{T}_{2n-1} \mathbf{J}_{2n-1} \\
M_{l, [l + n]_{2n}} &= \mathbf{0}_{2n-1}
\end{align*}
\]

for \(l \in \{1, 2, \ldots, 2n\}\). Consequently,

\[
\sum_{t=1}^{n} M_{lt} = \mathbf{T}_{2n-1} + \mathbf{J}_{2n-1} \mathbf{T}_{2n-1} \mathbf{J}_{2n-1} + \mathbf{U}^{n}_{2n-1} + (n - 2) \left( \mathbf{U}^{n-1}_{2n-1} + \mathbf{U}^{n+1}_{2n-1} \right) = C_n.
\]

\(\square\)
The next corollary gives an explicit formula for the entropy in the orientable case in terms of the spectral radius of a \((2n - 1) \times (2n - 1)\) matrix which is a “compacted” version of the Markov matrix \(M_n^+\).

**Corollary 3.7.**

\[
h_{\text{top}}(\Phi_{P_n^+}) = \log \max\{\rho(C_n), 1\}.
\]

**Proof.** It follows from Theorem 3.2 and Lemma 3.6. \qed

**Remark 3.8.** Note that the map \(\Phi_{P_n^+}\) commutes with a rigid rotation \(R\) of period 2\(n\). The quotient space obtained by identifying each orbit of \(R\) to a point is a circle. The map induced by \(\Phi_{P_n^+}\) on this quotient space is also a Markov map. The matrix \(C_n\) is nothing but the Markov matrix of this induced map (see [14] and Figure 6).

### 4. The non-orientable case

We start this section by extending the definition of symmetric presentation (Definition 2.11) to non orientable surface groups.

**Definition 4.1.** Given a surface group \(\Gamma = \pi_1(S)\) of rank \(n\), where \(S\) is a non-orientable surface, the following presentation of \(\Gamma\) will be called symmetric and denoted by \(P_n^-\). Its definition depends on the parity of \(n\) as follows. For \(n\) odd, we define \(P_n^-\) as

\[
\langle x_1^\pm 1, x_2^\pm 1, \ldots, x_n^\pm 1 / x_1x_2\cdots x_n x_{n-1}x_{n-2}\cdots x_1x_n \rangle
\]

while, for \(n\) even, \(P_n^-\) is defined as

\[
\langle x_1^\pm 1, x_2^\pm 1, \ldots, x_n^\pm 1 / x_1x_2\cdots x_n x_{n-1}x_{n-2}\cdots x_1x_n^{-1} \rangle.
\]

Similar arguments to the ones used in the proof of Proposition 2.12 yield that the symmetric presentation \(P_n^-\) is minimal and geometric.

As in the orientable case, the nomenclature symmetric for the presentation \(P_n^-\) accounts for the fact that, at each vertex, the cyclic ordering of the generators (Lemma 2.1) exhibits the useful property that the edge opposite to \(x\) at any vertex is simply the edge \(x^{-1}\). Indeed, one can check that the ordering of the generators at any vertex is

\begin{align*}
x_1 < x_2^{-1} < x_3 < \cdots < x_{n-1} < x_n^{-1} < x_1^{-1} < x_2 < x_3^{-1} < \cdots < x_{n-1}^{-1} < x_n
\end{align*}

when \(n\) is even, and

\begin{align*}
x_1 < x_2^{-1} < x_3 < \cdots < x_{n-1} < x_n < x_1^{-1} < x_2 < x_3^{-1} < \cdots < x_{n-1}^{-1} < x_n^{-1}
\end{align*}

when \(n\) is odd.

The fact that the symmetric presentation has associated the above cyclic ordering implies that Corollary 3.1 also holds for the non-orientable case:

**Corollary 4.2.** Let \(P_n^-\) be the symmetric presentation of a non-orientable surface group of rank \(n\). Then, \(\Phi_{P_n^-}(C_x) = I_x\) for each generator \(x\).

Notice that map \(\Phi_{P_n^+}\) and the Markov matrix \(M_n^+\) are only defined for \(n\) even since the group corresponds to an orientable surface. However, all associated formulae extend to the case \(n\) odd. In this sense below we will
compare the maps \( \Phi_{P^+} \) and \( \Phi_{P^-} \) and the associated Markov matrices \( M^+_n \) and \( M^-_n \), independently on the parity of \( n \).

Using the notations introduced in the previous sections one can check that the Markov map \( \Phi_{P^-} \) behaves essentially as \( \Phi_{P^+} \) in all intervals \( I_y \), except when \( i \in \{n, 2n\} \). In these two intervals the map reverses orientation. So, when \( i \notin \{n, 2n\} \) the equation (14) holds with \( \Phi_{P^-} \) instead of \( \Phi_{P^+} \). When \( i = n \),

\[
\begin{align*}
\Phi_{P^-}(L^j_{yn}) & = R^{j-1}_{yn} \quad \text{for } j \in \{4, 5, \ldots, n\}, \\
\Phi_{P^-}(L^3_{yn}) & = C_{yn-2} \cup C^{R}_{yn-1}, \\
\Phi_{P^-}(C^L_{yn}) & = C^L_{yn-2} \cup \left( \bigcup_{j=2}^{n} L^j_{yn-1} \right) \cup \left( \bigcup_{k=n+1}^{2n-2} I_{y_k} \right), \\
\Phi_{P^-}(C^R_{yn}) & = I_{yn}, \\
\Phi_{P^-}(C^R_{yn}) & = C^R_{yn} \cup \left( \bigcup_{j=2}^{n} R^j_{yn} \right) \cup \left( \bigcup_{k=2}^{n-1} I_{y_k} \right), \\
\Phi_{P^-}(R^3_{yn}) & = C^L_{yn} \cup C_{yn}, \\
\Phi_{P^-}(R^j_{yn}) & = L^{-1}_{yn} \quad \text{for } j \in \{4, 5, \ldots, n\},
\end{align*}
\]

and analogous formulae hold for the interval \( I_{yn} \). Hence, in a similar way to the previous section it follows that the Markov matrix \( M^-_n \) of \( \Phi_{P^-} \) is of the form (15) with \( M_{ij} \) replaced by \( J_{2n-1} \) for \( i \in \{n, 2n\} \) and \( j = 1, 2, \ldots, 2n \) (see Figure 9).

Next we want to prove Lemma 4.3 which is an analogue of Lemma 3.4 for this case. This will allow us to simplify the computation of the spectral radius of \( M^-_n \). As expected, a further consequence of Lemmas 4.3 and 3.4 will be that the orientation-reversing character of \( \Phi_{P^-} \) in the intervals \( I_{yn} \) and \( I_{yn} \) has no effects in the entropy. To do this it is convenient to introduce the

Figure 9. The first three (of the total of six) block rows of the Markov matrix \( M_{P^-} \) corresponding to the symmetric presentation of a non-orientable surface group of rank 3.
notion of disoriented block circulant matrix as follows. An \((r,s)\)–disoriented block circulant matrix is a matrix of the form

\[
A = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1r} \\
A_{21} & A_{22} & \cdots & A_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
A_{r1} & A_{r2} & \cdots & A_{rr}
\end{pmatrix}
\]

where each \(A_{ij}\) is an \(s \times s\) matrix for which there exists an \((r,s)\)–block circulant matrix

\[
\tilde{A} = \begin{pmatrix}
\tilde{A}_{11} & \tilde{A}_{12} & \cdots & \tilde{A}_{1r} \\
\tilde{A}_{21} & \tilde{A}_{22} & \cdots & \tilde{A}_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{A}_{r1} & \tilde{A}_{r2} & \cdots & \tilde{A}_{rr}
\end{pmatrix}
\]

such that given \(i \in \{2, \ldots, r\}\), either \(A_{ij} = \tilde{A}_{ij}\) for every \(j = 1, 2, \ldots, r\) or \(A_{ij} = J_s \tilde{A}_{ij}\) for every \(j = 1, 2, \ldots, r\). That is, every block row of \(A\) coincides with the corresponding block row of \(\tilde{A}\) or is obtained from the corresponding block row of \(\tilde{A}\) by pre-multiplying each block by \(J_s\). Observe that this last operation permutes the individual rows of the block row symmetrically with respect to the central horizontal axis. The matrix \(\tilde{A}\) will be called the parallelization of \(A\). Observe that the assumption that the first block row of \(A\) and \(\tilde{A}\) coincide implies that the parallelization of \(A\) is unique.

**Lemma 4.3.** Let

\[
A = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1r} \\
A_{21} & A_{22} & \cdots & A_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
A_{r1} & A_{r2} & \cdots & A_{rr}
\end{pmatrix}
\]

be a non-negative disoriented \((r,s)\)–block circulant matrix such that

\[
\left( \sum_{j=1}^{r} A_{1j} \right) J_s = J_s \left( \sum_{j=1}^{r} \tilde{A}_{1j} \right).
\]

Then

\[
\rho(A) = \rho \left( \sum_{j=1}^{r} A_{1j} \right).
\]

**Proof.** Let

\[
\tilde{A} = \begin{pmatrix}
\tilde{A}_{11} & \tilde{A}_{12} & \cdots & \tilde{A}_{1r} \\
\tilde{A}_{21} & \tilde{A}_{22} & \cdots & \tilde{A}_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{A}_{r1} & \tilde{A}_{r2} & \cdots & \tilde{A}_{rr}
\end{pmatrix}
\]

be the unique parallelization of \(A\). We will prove that \(\|A^m\|_\infty = \|\tilde{A}^m\|_\infty\) for every \(m \geq 0\). Then,

\[
\rho(A) = \lim_{m \to \infty} \|A^m\|_\infty^{1/m} = \lim_{m \to \infty} \|\tilde{A}^m\|_\infty^{1/m} = \rho(\tilde{A}),
\]

and the result follows from Lemma 3.3.
For every $m \in \mathbb{N}$ we will write $A^m$ and $\tilde{A}^m$ as
\[
\begin{pmatrix}
A_{11}^{(m)} & A_{12}^{(m)} & \cdots & A_{1r}^{(m)} \\
A_{21}^{(m)} & A_{22}^{(m)} & \cdots & A_{2r}^{(m)} \\
\vdots & \vdots & \ddots & \vdots \\
A_{r1}^{(m)} & A_{r2}^{(m)} & \cdots & A_{rr}^{(m)}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\tilde{A}_{11}^{(m)} & \tilde{A}_{12}^{(m)} & \cdots & \tilde{A}_{1r}^{(m)} \\
\tilde{A}_{21}^{(m)} & \tilde{A}_{22}^{(m)} & \cdots & \tilde{A}_{2r}^{(m)} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{A}_{r1}^{(m)} & \tilde{A}_{r2}^{(m)} & \cdots & \tilde{A}_{rr}^{(m)}
\end{pmatrix},
\]
respectively. Then,
\[
\|A^m\|_\infty = \max_{i=1,2,\ldots,r} \left\| \sum_{j=1}^{r} A_{ij}^{(m)} \right\|_\infty = \max_{i=1,2,\ldots,r} \left( \sum_{j=1}^{r} A_{ij}^{(m)} \right) u_s
\]
and, similarly,
\[
\|\tilde{A}^m\|_\infty = \max_{i=1,2,\ldots,r} \left( \sum_{j=1}^{r} \tilde{A}_{ij}^{(m)} \right) u_s,
\]
where $u_s$ denotes the (column) vector of size $s$ with all the entries equal to 1. So, to prove the lemma it is enough to show that
\[
\left( \sum_{j=1}^{r} A_{ij}^{(m)} \right) u_s = \left( \sum_{j=1}^{r} \tilde{A}_{ij}^{(m)} \right) u_s
\]
for every $i = 1, 2, \ldots, r$ and $m \in \mathbb{N}$.

Before proving this claim we will prove two necessary technical results on the block rows of the matrix $A$ that are stronger versions of assumption (19). The first one is the following:
\[
\left( \sum_{j=1}^{r} A_{ij} \right) J_s = J_s \left( \sum_{j=1}^{r} A_{ij} \right) \quad \text{and} \quad
\left( \sum_{j=1}^{r} A_{ij} \right) w_s = \left( \sum_{j=1}^{r} A_{rj} \right) w_s
\]
for every non-negative vector $w_s$ of size $s$ such that $J_s w_s = w_s$ and every $i, r \in \{1, 2, \ldots, r\}$.

From the definitions of disoriented block circulant matrix, parallelization and block circulant matrix we get:
\[
\sum_{j=1}^{r} A_{ij} = \sum_{j=1}^{r} \tilde{A}_{ij} = \sum_{j=1}^{r} \tilde{A}_{ij}
\]
and $\sum_{j=1}^{r} A_{ij}$ is either $\sum_{j=1}^{r} \tilde{A}_{ij}$ or $J_s \left( \sum_{j=1}^{r} \tilde{A}_{ij} \right)$. Thus, from (22) and (19) we get, respectively,
\[
\sum_{j=1}^{r} A_{ij} = \left\{ \begin{array}{l}
\sum_{j=1}^{r} \tilde{A}_{ij} = \sum_{j=1}^{r} A_{ij} \\
J_s \left( \sum_{j=1}^{r} \tilde{A}_{ij} \right) = J_s \left( \sum_{j=1}^{r} A_{ij} \right) = \left( \sum_{j=1}^{r} A_{ij} \right) J_s
\end{array} \right.
\]
Consequently, if $w_s$ is a non-negative vector of size $s$ such that $J_s w_s = w_s$, we have $\left( \sum_{j=1}^{r} A_{ij} \right) w_s = \left( \sum_{j=1}^{r} A_{rj} \right) w_s$ for every $i \in \{1, 2, \ldots, r\}$. This proves the second equality of (21). To prove the first one we use the above expression for $\sum_{j=1}^{r} A_{ij}$ and (19). In the first case we have
\[
\left( \sum_{j=1}^{r} A_{ij} \right) J_s = \left( \sum_{j=1}^{r} A_{ij} \right) J_s = J_s \left( \sum_{j=1}^{r} A_{ij} \right) = J_s \left( \sum_{j=1}^{r} A_{ij} \right),
\]
and in the second one,
\[
\left(\sum_{j=1}^{r} A_{ij}\right) J_s = J_s \left(\sum_{j=1}^{r} A_{1j}\right) J_s = J_s \left(\sum_{j=1}^{r} A_{ij}\right) = J_s \left(\sum_{j=1}^{r} A_{ij}\right).
\]
This ends the proof of (21).

The second technical result that we need is a weaker version of (21) but that holds for all powers of \(A\). More precisely, for every \(i, r \in \{1, 2, \ldots, r\}\) and \(m \in \mathbb{N}\),
\[
(23) \quad J_s \left(\sum_{j=1}^{r} A_{ij}^{(m)}\right) u_s = \left(\sum_{j=1}^{r} A_{ij}^{(m)}\right) u_s.
\]
In fact this implies that \( \left(\sum_{j=1}^{r} A_{ij}^{(m)}\right) u_s \) is a vector independent on \(\ell\), call it \(w_{s,\ell}^{m}\) such that \(J_s w_{s,\ell}^{m} = w_{s,\ell}^{m}\).

For \(m = 1\), (23) follows directly from (21) and the fact that \(J_s u_s = u_s\).

Now we assume that (23) holds for \(m \geq 1\) and prove it for \(m + 1\). Clearly,
\[
(24) \quad \sum_{j=1}^{r} A_{ij}^{(m+1)} = \sum_{j=1}^{r} \left(\sum_{\ell=1}^{r} A_{ij} A_{\ell j}^{(m)}\right) = \sum_{\ell=1}^{r} A_{ij} \left(\sum_{j=1}^{r} A_{ij}^{(m)}\right) = \left(\sum_{\ell=1}^{r} A_{ij}^{(m)}\right) u_s.
\]
for every \(i \in \{1, 2, \ldots, r\}\). Thus, from the induction hypothesis and (21),
\[
J_s \left(\sum_{j=1}^{r} A_{ij}^{(m+1)}\right) u_s = J_s \sum_{\ell=1}^{r} A_{ij} \left(\sum_{j=1}^{r} A_{ij}^{(m)}\right) u_s = J_s \left(\sum_{\ell=1}^{r} A_{ij}^{(m)}\right) u_s
\]
\[
= \left(\sum_{\ell=1}^{r} A_{ij}^{(m)}\right) w_{s,\ell}^{m} = \left(\sum_{\ell=1}^{r} A_{ij}^{(m)}\right) u_s
\]
\[
= \left(\sum_{j=1}^{r} A_{ij}^{(m+1)}\right) u_s.
\]
This completes the induction step and, thus, (23) is proved.

Now we will prove formula (20) by induction on \(m\) for a fixed but arbitrary \(i \in \{1, 2, \ldots, r\}\). Assume that \(m = 1\). If \(A_{ij} = \tilde{A}_{ij}\) for \(j = 1, 2, \ldots, r\) then the equality is trivially true. Otherwise, \(A_{ij} = J_s \tilde{A}_{ij}\) for \(j = 1, 2, \ldots, r\). Hence, from (21),
\[
\left(\sum_{j=1}^{r} A_{ij}\right) u_s = \left(\sum_{j=1}^{r} A_{ij}\right) (J_s u_s) = J_s \left(\sum_{j=1}^{r} A_{ij}\right) u_s
\]
\[
= J_s \left(\sum_{j=1}^{r} \tilde{A}_{ij}\right) u_s = \left(\sum_{j=1}^{r} \tilde{A}_{ij}\right) u_s,
\]
because \(J_s\) is an involution (i.e. \(J_s^{2}\) is the identity of size \(s\)).

Assume that (20) holds for \(m \geq 1\). As above, we will consider two cases. If \(A_{ij} = \tilde{A}_{ij}\) for \(\ell = 1, 2, \ldots, r\), then from (24), the analogous formula for \(\tilde{A}\) and the induction assumption we get
\[
\left(\sum_{j=1}^{r} \tilde{A}_{ij}^{(m+1)}\right) u_s = \sum_{\ell=1}^{r} \tilde{A}_{ij} \left(\sum_{j=1}^{r} \tilde{A}_{ij}^{(m)}\right) u_s
\]
\[
= \sum_{\ell=1}^{r} \tilde{A}_{ij} \left(\sum_{j=1}^{r} \tilde{A}_{ij}^{(m)}\right) u_s = \left(\sum_{j=1}^{r} \tilde{A}_{ij}^{(m+1)}\right) u_s.
\]
So, we are left with the case \( A_{i\ell} = J_s \tilde{A}_{i\ell} \) for \( \ell = 1, \ldots, r \). In a similar way to the previous case we get
\[
\left( \sum_{j=1}^r A_{ij}^{(m+1)} \right) u_s = J_s \sum_{\ell=1}^r \tilde{A}_{i\ell} \left( \sum_{j=1}^r \tilde{A}_{ij}^{(m)} \right) u_s
= J_s \left( \sum_{j=1}^r \tilde{A}_{ij}^{(m+1)} \right) u_s.
\]
Consequently, from (23) we get:
\[
\left( \sum_{j=1}^r A_{ij}^{(m+1)} \right) u_s = J_s \left( \sum_{j=1}^r \tilde{A}_{ij}^{(m+1)} \right) u_s = \left( \sum_{j=1}^r \tilde{A}_{ij}^{(m+1)} \right) u_s.
\]
This ends the proof of the lemma. \( \square \)

With the help of Lemma 4.3 as in the orientable case we obtain:

**Corollary 4.4.**
\[
h_{top}(\Phi_{P_n^-}) = \log \max \{ \rho(C_n), 1 \}.
\]

5. **Computation of the topological entropy — second reduction: Super compacting the matrix \( C_n \)**

The *super compacted matrix of rank* \( n \) is the \( n \times n \) matrix \( SC_n = (s_{ij}) \) defined as follows:
\[
s_{ij} = \begin{cases} 
1 & \text{if } i \leq n - 2 \text{ and } j = i + 1 \text{ or } i = n, \\
2 & \text{if } i = n - 2 \text{ and } j = n, \\
2n - 3 & \text{if } i = n - 1 \text{ and } j < n, \\
2n - 4 & \text{if } i = n - 1 \text{ and } j = n, \text{ and} \\
0 & \text{otherwise.}
\end{cases}
\]

In matrix form we have:
\[
SC_n = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \\
2n - 3 & 2n - 3 & 2n - 3 & 2n - 3 & \cdots & 2n - 3 & 2n - 3 & 2n - 4 \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1
\end{pmatrix}
\]

In this section we prove

**Proposition 5.1.** *For every* \( n \geq 3 \),
\[
\max \{ \rho(C_n), 1 \} = \max \{ \rho(SC_n), 1 \}.
\]

To prove the above result we need another intermediate matrix which we obtain from \( C_n \). We introduce the *divided compacted matrix of rank* \( n \) of size
2n × 2n, denoted by $DC_n = (d_{ij})$, which we define as follows:

$$d_{ij} = \begin{cases} c_{ij} & \text{if } i < n \text{ and } j \leq n, \\ c_{i,j-1} & \text{if } i < n \text{ and } j \geq n + 1, \\ c_{i-1,j} & \text{if } i > n + 1 \text{ and } j \leq n, \\ c_{i-1,j-1} & \text{if } i > n + 1 \text{ and } j \geq n + 1, \\ 1 & \text{if } i = n \text{ and } j \leq n \text{ or } i = n + 1 \text{ and } j \geq n + 1, \\ 0 & \text{if } i = n \text{ and } j \geq n + 1 \text{ or } i = n \text{ and } j \leq n, \end{cases}$$

where $C_n = (c_{ij})$. In matrix form, $DC_n$ is (compare with the definition of the matrix $C_n$ in Page 18):

$$DC_n = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & \vdots & \vdots & \vdots \\
n - 2 & n - 2 & n - 2 & n - 2 & n - 2 & n - 2 & n - 2 & n - 2 & n - 2 & n - 2 & n - 2 & n - 2 & n - 2 & n - 2 \\
1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 & 0 & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
\end{pmatrix}$$

Notice that the matrix $DC_n$ is indeed the Markov matrix of a topological model obtained by subdividing the central interval of the topological model from Remark 3.8 at a fixed point (that exists because the central interval covers itself).

Proof of Proposition 5.1. First we will prove that

$$\text{Spec}(DC_n) = \text{Spec}(C_n) \cup \{1\},$$

where Spec(·) denotes the set of eigenvalues of a matrix. To do this observe that 1 is an eigenvalue of $DC_n$ with eigenvector $(0, 0, \ldots, 0, 1, -1, 0, \ldots, 0, 0)$, where the two non-zero elements of this vector are at the $n$ and $n + 1$ entries. Also, if $\mu$ is an eigenvalue of $C_n$ with eigenvector $(v_1, v_2, \ldots, v_{2n-1})$, then, it follows directly from the definition of the matrix $DC_n$ that $\mu$ is also an eigenvalue of $DC_n$ with eigenvector

$$(v_1, v_2, \ldots, v_{n-1}, \frac{v_n}{2}, v_{n+1}, \ldots, v_{2n-1}).$$

Conversely, if $\mu$ is an eigenvalue of $DC_n$ with eigenvector $(v_1, v_2, \ldots, v_{2n})$, then, again from the definition of the matrix $DC_n$, it follows that $\mu$ is also an eigenvalue of $C_n$ with eigenvector

$$(v_1, v_2, \ldots, v_{n-1}, v_n + v_{n+1}, v_{n+2}, \ldots, v_{2n}).$$

This proves the statement.

The second step of the proof will be to show that $\rho(DC_n) = \rho(SC_n)$. To do this it is convenient to write the matrix $DC_n$ in block form, with blocks of size $n \times n$:

$$DC_n = \begin{pmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22} \\
\end{pmatrix}.$$
Observe that $DC_n$ is symmetric with respect to the central point. So, $D_{21} = J_nD_{12}J_n$ and $D_{22} = J_nD_{11}J_n$, which amounts to:

$$DC_n = \begin{pmatrix} D_{11} & D_{12} \\ J_nD_{12}J_n & J_nD_{11}J_n \end{pmatrix}.$$ 

Now let us consider the block matrix of size $2n \times 2n$ defined by

$$Z := \begin{pmatrix} I_n & 0_n \\ 0_n & J_n \end{pmatrix},$$

where $I_n$ denotes the identity matrix of size $n \times n$. Clearly $Z$ is non-singular and $Z^{-1} = Z$. Hence,

$$ZDC_nZ^{-1} = \begin{pmatrix} D_{11} & D_{12} \\ D_{12}J_n & D_{11} \end{pmatrix}$$

because $J_nJ_n = I_n$. Observe that $ZDC_nZ^{-1}$ is a non-negative block circulant matrix. Thus,

$$\rho(DC_n) = \rho(ZDC_nZ^{-1}) = \rho(D_{11} + D_{12}J_n)$$

by Lemma 3.4. Moreover, by direct inspection it follows that $D_{11} + D_{12}J_n = SC_n$.

Summarizing, we have proved

$$\max \{ \rho(C_n), 1 \} = \max \{ \rho(DC_n), 1 \} = \max \{ \rho(SC_n), 1 \}.$$ 

\[\Box\]

**Remark 5.2.** The topological model in Remark 3.8 has a fixed point and commutes with the symmetry of degree -1 with respect to this fixed point. The quotient space obtained by identifying each orbit of the symmetry to a point is a closed interval, and the induced map on this quotient space is also a Markov map. The matrix $SC_n$ is in fact the Markov matrix of this quotient map.

6. THE SPECTRAL RADIUS OF $SC_n$ AND PROOF OF THEOREM 1.1

To compute the spectral radius of $SC_n$ we will use the rome method proposed in [2]. To this end we have to introduce some notation.

Let $M = (m_{ij})$ be a $k \times k$ matrix. Given a sequence $p = (p_j)_{j=0}^{\ell}$ of elements of $\{1, 2, \ldots, k\}$ we define the width of $p$, denoted by $w(p)$, as the number $\prod_{j=1}^{\ell} m_{p_{j-1}p_j}$. If $w(p) \neq 0$ then $p$ is called a path of length $\ell$. The length of a path $p$ will be denoted by $\ell(p)$. A loop is a path such that $p_\ell = p_0$ i.e. that begins and ends at the same index.

A subset $R$ of $\{1, 2, \ldots, k\}$ is called a rome if there is no loop outside $R$, i.e. there is no path $(p_j)_{j=0}^{\ell}$ such that $p_\ell = p_0$ and $(p_j : 0 \leq j \leq \ell)$ is disjoint from $R$. For a rome $R$ a path $(p_j)_{j=0}^{\ell}$ is called simple if $p_i \in R$ for $i = 0, \ell$ and $p_{i} \notin R$ for $i = 1, 2, \ldots, \ell - 1$.

If $R = \{r_1, r_2, \ldots, r_k\}$ is a rome of a matrix $M$ then we define an $\ell \times \ell$ matrix-valued real function $M_R(x)$ by setting $M_R(x) = (a_{ij}(x))$, where $a_{ij}(x) = \sum_{p} w(p) \cdot x^{-\ell(p)}$, where the summation is over all simple paths originating at $r_i$ and terminating at $r_j$. 


Theorem 6.1 (Theorem 1.7 of [2]). If \( R \) is a rome of cardinality \( \ell \) of a \( k \times k \) matrix \( M \) then the characteristic polynomial of \( M \) is equal to

\[
(-1)^{k-\ell} x^k \det(M_R(x) - I_\ell).
\]

To use Theorem 6.1 it is helpful to represent the matrix \( M \) in form of a combinatorial graph which amounts to draw all paths of length 1 associated to \( M \). To do this we introduce the following notation. A path \((i,j)\) of length 1 will be written as \( i \xrightarrow{w} j \), where \( w \) denotes the width of the path. For the matrix \( M \) the width \( w \) of the path \( i \xrightarrow{w} j \), is just the entry \( (M)_{i,j} \neq 0 \).

Observe that, with this notation, a path \( p = (p_j)_{j=0}^{k} \) is written as

\[
p_0 \xrightarrow{w_0} p_1 \xrightarrow{w_1} \cdots p_{k-1} \xrightarrow{w_{k-1}} p_k
\]

and \( w(p) = \prod_{i=0}^{k-1} w_i \).

We will compute the spectral radius of \( SC_n \) in Proposition 6.3 by using Theorem 6.1. In Figure 10 we show the combinatorial graph associated to \( SC_n \).

Remark 6.2. The combinatorial graph associated to \( SC_n \) shown in Figure 10 is in fact the generalized Markov graph of the topological model obtained in Remark 5.2.

Proposition 6.3. The spectral radius of \( SC_n \) is the largest root of the polynomial \( Q_n(x) \).

Proof. By direct inspection of the graph of Figure 10 it follows that \( SC_n \) is an irreducible non-negative integer matrix. Hence, by the Perron-Frobenius Theorem (see [11]), we get that the spectral radius of \( SC_n \) is the largest eigenvalue of the characteristic polynomial of \( SC_n \). It is larger than 1 and simple. So, to prove the theorem, we have to show that the characteristic polynomial of \( SC_n \) is \( Q_n(x) \).
Clearly $R = \{n - 1, n\}$ is a root of $SC_n$ (see Figure 1). Hence,

$$M_R(x) = \left( \frac{\beta(x^{-1} + z(x))}{x^{-1} + z(x)} \right) \left( \frac{(\beta - 1)x^{-1} + 2\beta z(x)}{x^{-1} + 2z(x)} \right)$$

where $\beta = 2n - 3$, $z(x) := \sum_{\ell=2}^{n-1} x^{-\ell}$.

By Theorem 6.1 the characteristic polynomial of $SC_n$ is

$$(-1)^{n-2} x^n \left| \begin{array}{cc} \beta(x^{-1} + z(x)) & 2\beta(x^{-1} + z(x)) - (\beta + 1)x^{-1} \\ x^{-1} + z(x) & 2(x^{-1} + z(x)) - x^{-1} - 1 \end{array} \right|$$

$$= x^n \left( (x^{-1} - \beta - 2)(x^{-1} + z(x)) + x^{-1} + 1 \right)$$

$$= x^n(\beta - 2(2n - 1)) \left( \sum_{\ell=1}^{n-1} x^{-\ell} \right) + x^{n-1} + x^n$$

$$= x^n - (2n - 2) \sum_{j=1}^{n-1} x^j + 1.$$

This ends the proof of the proposition. \(\Box\)

Next we prove a technical lemma that studies the polynomial (1) and gives the bounds for $\lambda_n$.

**Lemma 6.4.** For every $n \geq 3$, $Q_n(x)$ has a unique real root $\lambda_n$ larger than one. Moreover, for $n \geq 4$,

$$2n - 1 - \frac{1}{(2n - 1)^n - 2} < \lambda_n < 2n - 1.$$

**Proof.** Observe that $Q_n(0) = 1$, $Q_n(1) = -2n(n - 2) < 0$ and

$$Q'_n(x) = n \left( x^{n-1} - \frac{2(n-1)}{n} \sum_{j=1}^{n-1} j x^{-j} \right) \leq n \left( x^{n-1} - \frac{2(n-1)}{n} \right).$$

Since $x^{n-1} - \frac{2(n-1)}{n}$ is negative for every $n \geq 3$ and $x \in [0, 1]$, it follows that $Q_n$ has a unique root in $(0, 1)$.

On the other hand, it is easy to see that $Q_n(x) = x^n Q_n(x^{-1})$ (that is, $Q_n$ is a reciprocal polynomial) and, hence, $z$ is a root of $Q_n$ if and only if $z^{-1}$ is a root of $Q_n$. Consequently, $Q_n(x)$ has a unique real root larger than one.

Also,

$$Q_n(2n - 1) = (2n - 1)^n - 2(n - 1) \frac{(2n - 1)^n - (2n - 1)}{2(n - 1)} + 1 = 2n.$$

So, $\lambda_n < 2n - 1$.

To end the proof of the lemma it is enough to show that

$$Q_n \left( 2n - 1 - \frac{1}{(2n-1)^{n-2}} \right) < 0$$

for $n \geq 4$. We have

$$Q_n(z) = z^n - 2(n-1)s \frac{z^n - z}{2(n-1)s - 1} + 1$$

$$= \frac{z^n - 4n(n-1)s + 2n - 1}{2(n-1)s - 1}.$$
where \( z := 2n - 1 - \frac{1}{s} \) and \( s := (2n - 1)^{n-2} \).

Since \( 2(n - 1)s - 1 > 2n - 1 > 0 \), \( z^n - 4n(n - 1)s > 0 \) implies that \( Q_n(z) < 0 \).

An exercise shows that \( z^n > (2n - 1)^n - n(2n - 1) \). Hence,

\[
z^n - 4n(n - 1)s > (2n - 1)^{n-2} \left( 1 - \frac{n}{(2n - 1)^{n-3}} \right)
\]

and \( z^n - 4n(n - 1)s > 0 \) for \( n \geq 4 \). So, \( Q_n(z) < 0 \) and the lemma is proved. \( \square \)

\textbf{Proof of Theorem 1.1.} It follows from Theorem 2.2, Corollaries 3.7 and 4.3, Propositions 5.1 and 6.3, and Lemma 6.4. \( \square \)

\textbf{References}

[1] Lluís Alsedà, Jaume Llibre, and Michał Misiurewicz. \textit{Combinatorial dynamics and entropy in dimension one}, volume 5 of \textit{Advanced Series in Nonlinear Dynamics}. World Scientific Publishing Co. Inc., River Edge, NJ, second edition, 2000.

[2] Louis Block, John Guckenheimer, Michał Misiurewicz, and Lai Sang Young. Periodic points and topological entropy of one-dimensional maps. In \textit{Global theory of dynamical systems (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1979)}, volume 819 of \textit{Lecture Notes in Math.}, pages 18–34. Springer, Berlin, 1980.

[3] Rufus Bowen and Caroline Series. Markov maps associated with Fuchsian groups. \textit{Inst. Hautes Études Sci. Publ. Math.}, (50):153–170, 1979.

[4] D. Calegari. The ergodic theory of hyperbolic groups. \textit{Contemp. Math.}, 597:15–52, 2013.

[5] J. W. Cannon. The growth of the closed surface groups and the compact hyperbolic Coxeter groups. 1980.

[6] J. W. Cannon and Ph. Wagreich. Growth functions of surface groups. \textit{Math. Ann.}, 293(2):239–257, 1992.

[7] James W. Cannon. The combinatorial structure of cocompact discrete hyperbolic groups. \textit{Geom. Dedicata}, 16(2):123–148, 1984.

[8] Pierre de la Harpe. \textit{Topics in geometric group theory}. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.

[9] David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston. \textit{Word processing in groups}. Jones and Bartlett Publishers, Boston, MA, 1992.

[10] William J. Floyd and Steven P. Plotnick. Growth functions on Fuchsian groups and the Euler characteristic. \textit{Invent. Math.}, 88(1):1–29, 1987.

[11] F. R. Gantmacher. \textit{The theory of matrices. Vols. 1, 2}. Translated by K. A. Hirsch. Chelsea Publishing Co., New York, 1959.

[12] R. I. Grigorchuk. On the Milnor problem of group growth. \textit{Dokl. Akad. Nauk SSSR}, 271(1):30–33, 1983.

[13] M. Gromov. Hyperbolic groups. In \textit{Essays in group theory}, volume 8 of \textit{Math. Sci. Res. Inst. Publ.}, pages 75–263. Springer, New York, 1987.

[14] Mikhail Gromov. Groups of polynomial growth and expanding maps. \textit{Inst. Hautes Études Sci. Publ. Math.}, (53):53–73, 1981.

[15] Jérôme Los. Volume entropy for surface groups via Bowen-Series-like maps. \textit{J. Topol.}, 7(1):120–154, 2014.

[16] John Milnor and William Thurston. On iterated maps of the interval. In \textit{Dynamical systems (College Park, MD, 1986–87)}, volume 1342 of \textit{Lecture Notes in Math.}, pages 465–563. Springer, Berlin, 1988.

[17] Michael Shub. \textit{Stabilité globale des systèmes dynamiques}, volume 56 of \textit{Astérisque}. Société Mathématique de France, Paris, 1978. With an English preface and summary.
[18] John Stillwell. *Classical topology and combinatorial group theory*, volume 72 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.

Departament de Matemàtiques, Edifici Cc, Universitat Autònoma de Barcelona, 08913 Cerdanyola del Vallès, Barcelona, Spain
E-mail address: alseda@mat.uab.cat

Departament d'Informàtica i Matemàtica Aplicada, Universitat de Girona, Lluís Santaló s/n, 17071 Girona, Spain
E-mail address: juher@ima.udg.edu

Aix-Marseille Université, Institut Mathématiques de Marseille UMR 7373, 39 Rue F. Joliot Curie, 13013 Marseille, France
E-mail address: los@cmi.univ-mrs.fr

Departament de Matemàtiques, Edifici Cc, Universitat Autònoma de Barcelona, 08913 Cerdanyola del Vallès, Barcelona, Spain
E-mail address: manyosas@mat.uab.cat