TIME-CHANGES OF HEISENBERG NILFLOWS

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ABSTRACT. We consider the three dimensional Heisenberg nilflows. Under a full measure set Diophantine condition on the generator of the flow we construct Bufetov functionals which are asymptotic to ergodic integrals for sufficiently smooth functions, have a modular property and scale exactly under the renormalization dynamics. By the asymptotic property we derive results on limit distributions, which generalize earlier work of Griffin and Marklof [GM] and Cellarosi and Marklof [CM]. We then prove analyticity of the functionals in the transverse directions to the flow. As a consequence of this analyticity property we derive that there exists a full measure set of nilflows such that generic (non-trivial) time-changes are mixing and moreover have a “stretched polynomial” decay of correlations for sufficiently smooth functions (this strengthens a result of Avila, Forni, and Ulcigrai [AFU]). Moreover we also prove that there exists a full Hausdorff dimension set of nilflows such that generic non-trivial time-changes have polynomial decay of correlations.

1. INTRODUCTION

This paper concerns the smooth ergodic theory of parabolic flows, that is, flows characterized by polynomial (sub-exponential) divergence of nearby orbits. In particular we prove results on limit distributions of Heisenberg nilflows and on the decay of correlations of their non-trivial reparametrizations (time-changes). Our approach is based on the construction of finitely additive Hölder measures and Hölder cocycles for Heisenberg nilflows, asymptotic to ergodic integrals, following the work of A. Bufetov [Bu] on translation flows and of Bufetov and G. Forni [BF], [Fo1] on horocycle flows. Hölder cocycles for translation flows are closely related to “limit shapes” of ergodic sums for Interval Exchange Transformations, studied in the work of S. Marmi, P. Moussa and J.-C. Yoccoz [MMY] on wandering intervals for affine Interval Exchange Transformations. In fact, roughly speaking, “limit shapes” are related to graphs of Hölder cocycles as functions of time.

We recall that the mixing property for generic, non-trivial time-changes of Heisenberg nilflows was proved by A. Avila, G. Forni and C. Ulcigrai [AFU]. The main result of that paper was that for uniquely ergodic Heisenberg nilflow all non-trivial
time-changes, within a dense subspace of time-changes, are mixing. Under a Diophantine condition the set of trivial time-changes has countable codimension and can be explicitly described in terms of invariant distributions for the nilflow.

Results on limit theorems for skew-translations, which appear as return maps (with constant return time) of Heisenberg nilflows, limited however to a single character function, have more recently been proved by J. Griffin and J. Marklof [GM] and refined by F. Cellarosi and Marklof [CM] by an approach based on theta functions. Their results raised the question of possible relations between theta functions and Bufetov’s Hölder cocycles, developed for other analogous dynamical systems in [Bu] (translation flows), [BF] (horocycle flows), [BS] (tilings), as a formalism to derive asymptotic theorem for ergodic averages and prove limit theorems.

In this paper we generalize the results of Griffin and Marklof [GM] on limit distributions, proving in particular that almost all limits of ergodic averages of arbitrary sufficiently smooth functions are distributions of Hölder continuous functions on the Heisenberg nilmanifold, hence in particular they have compact support. Our main results, however, are on the decay of correlations of smooth functions for time-changes: we prove that it has polynomial (power law) speed for all non-trivial smooth time-changes of Heisenberg nilflows of bounded type, within a generic subspace of time-changes. As mentioned above, the study of limit distributions for parabolic flows has been developed only in recent years after Bufetov’s work [Bu] on translations flows (and Interval Exchange Transformations). A comprehensive study of spatial and temporal limit theorems for dynamical systems of different type has been more recently carried out in the work of D. Dolgopyat and O. Sarig [DS].

The study of mixing properties of elliptic parabolic flows and their time-changes has a longer history. For instance, mixing properties of suspension flows over rotations and Interval Exchange Transformations have been investigated in depth (see for instance [Ko1], [Ko2], [KS], [Sch], [Ul1], [Ul2] and reference therein), mixing for reparametrizations of linear toral flows were investigated by B. Fayad (see for instance [Fa2]), finally mixing for time-changes of classical horocycle flows was proved in a classical paper of B. Marcus [Ma] after a partial result of Kushnirenko [Ku]. As mentioned above mixing for time-changes of Heisenberg nilflows was investigated in [AFU]. Work in progress of Avila, Forni, Ravotti and Ulcigrai indicates that the methods developed there extend to proof of mixing for a dense set of non-trivial time-change for any uniquely ergodic nilflows. Ravotti’s paper [Rav2] is a step in that direction. It should be remarked that there is an important difference between time-changes of linear toral flows and parabolic flows. In the parabolic case there are often countably many obstructions to triviality of time-changes for Diophantine flows, while in the elliptic case of linear toral flows non-trivial time-changes can exist only in the Liouvillean case.

Estimates on the decay of correlations of smooth functions for non-homogenous elliptic or parabolic flows are harder to come by and there are much fewer results in the literature. A classical paper of M. Ratner established the decay rate for classical horocycle flows (as well as geodesic flows) on surfaces of constant negative
curvature. This result was generalized to sufficiently smooth time-changes of horocycle flows by Forni and Ulcigrai [FU], who also proved that the spectrum remains Lebesgue. Fayad [Fa1] proved polynomial decay for a class of Kochergin-type flows on the 2-torus and only recently, in [FFK], it was shown that there exists a class of Kochergin flows on the 2-torus with Lebesgue maximal spectral type. For locally Hamiltonian flows with a saddle loop on surfaces (or, more generally, for suspension flows over Interval Exchange Transformations with asymmetric logarithmic singularities of the roof function), Ravotti [Rav1] was able to prove (logarithmic) estimates on decay of correlations. For these flows mixing was proved by Khanin and Sinai [KS] in the toral case, and by C. Ulcigrai [UI] for suspension flows over Interval Exchange Transformations in the significant special case of roof functions with a single asymmetric logarithmic singularity.

We expect non-trivial time-changes of nilflows to have polynomial decay of correlations. However, we are able to prove this result only for Heisenberg nilflows of bounded type. Our methods do not generalize to higher step nilflows, since they are based on the renormalization dynamics introduced by L. Flaminio and G. Forni in [FlaFo], which has no known generalization to the higher step case. We are also unable to decide whether the spectral measures of time-changes of Heisenberg nilflows are absolutely continuous with respect to Lebesgue. Indeed, the approach of [FU], considerably refined in [FFK], fails since the “stretching of Birkhoff sums” is at best borderline square integrable (it grows at most as the square root of the time, up to logarithmic terms). In fact, our bounds on the decay of correlations are significantly worse than that, and we have no control on the size of the exponent. This follows from the general principle that proving “lower bounds” on Birkhoff sums or ergodic integrals is much harder than proving “upper bounds”. In our case we are able to prove polynomial (power-law) lower bounds outside appropriate sublevel sets of Bufetov’s Hölder cocycles, which are asymptotic to ergodic integrals up to a well-controlled error. Polynomial estimates on the measure of such sublevel sets (for small parameter values) are derived from general results (see [Bru], [BruGa]) on the measure of the sublevel sets of analytic functions. In fact, at the core of our argument we establish the real analyticity of the Bufetov cocycles along the leaves of a foliation transverse to the flow.

This outline is different from the proof of mixing in [AFU]. In that paper the stretching of Birkhoff sums for Heisenberg nilflows was derived from a more general result on the growth of Birkhoff sums of functions which are not coboundaries with measurable transfer function, essentially based on a measurable Gottschalk-Hedlund theorem, and on the parabolic divergence of orbits. However, it is completely unclear whether it is possible to prove an effective version of this argument. For this reason we have followed here a different approach.

Outline of the paper. In Section 2 we give basic definitions on Heisenberg nilflows, the Heisenberg moduli space, renormalization flow and Sobolev spaces. Finally we state two main theorems. In Section 3 we recall some basic results in representation theory of Heisenberg group. In Section 4 we compute the stretching of arcs (in the central direction) under the reparametrized flow. Sections 5 and 6 are crucial
since Bufetov functionals are constructed and their main properties are studied. In particular we prove the expected asymptotic formula according to which Bufetov functionals control orbital integrals. In Section 7 we derive from the asymptotic formula results on limit distributions of ergodic integrals for Diophantine Heisenberg nilflows, following the method developed in [Bu], [BF]. We also give an alternative proof, based on representation theory, of a substantial part of the work of Griffin and Marklof [GM] on limit theorems for skew-shifts of the 2-torus, and generalize most of their conclusions to arbitrary smooth functions. Our approach also naturally gives results on the regularity of limit distributions, in particular their Hölder property (with exponent $1/2-$) first derived for quadratic Weyl sums in the work of Cellarosi and Marklof [CM].

In Section 8 we prove sharp square mean lower bounds for Bufetov functionals along the leaves of a one-dimensional foliation transverse to the flow. Our aim is to prove measure estimates for the sub-level sets of Bufetov functionals, a key result in establishing the stretching of ergodic integrals outside sets of small measure. For that we prove in Section 9 that Bufetov functionals are real analytic on the leaves of a 2-dimensional foliation (the weak-stable foliation of the renormalization dynamics on the Heisenberg nilmanifold). We then recall in Section 10 a result of A. Brudnyi [Bru] on the measure of the sub-level sets of real analytic functions. These estimates depend on the so-called Chebyshev degree and valency of the function. We prove that under certain conditions the valency is uniformly bounded on every normal family of analytic functions. In Sections 11 and 12 we apply results of the previous section to finally prove measure estimates on the sets where Bufetov functionals are small (Lemmas 11.2 and 12.1). We conclude in Section 13 with an analysis of correlations and derive from results of Sections 11 and 12 (Corollary 11.3 and 12.2) the proof of our main Theorems 2.4 and 2.5.

2. Definitions

In this section we will recall definitions of Heisenberg nilflows, moduli space of Heisenberg frames $\mathcal{M}$, the renormalization flow $g_R$ on $\mathcal{M}$ and the renormalization cocycle $p_R$ on the Hilbert bundle of Sobolev distributions. For more details see [ FlaFo] or [ Fo2]. We also introduce an extended renormalization flow $\hat{g}_R$ on an extended moduli space $\hat{\mathcal{M}}$, which is a tautological bundle over $\mathcal{M}$ with fibers isomorphic to the Heisenberg nilmanifold.

2.1. Nilflows. The (three-dimensional) Heisenberg group $\text{Heis}$ is given by

$$\text{Heis} := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}. $$

Let $\Gamma$ be a lattice in $\text{Heis}$. The Heisenberg manifold $M$ is the quotient $\Gamma \backslash \text{Heis}$. It is known that up to an automorphism of $\text{Heis}$

$$\Gamma = \Gamma_K = \left\{ \begin{pmatrix} 1 & m & p \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} : m, n, p \in \mathbb{Z} \right\},$$

where $K = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. 

where $K$ is a positive integer. Notice that $M$ has a probability measure $\text{vol}$ locally given by Haar measure on Heis.

2.1.1. Heisenberg nilflows. Let $W$ be any element of the Lie algebra $\eta$ of Heis. The Heisenberg nilflow for $W$ is given by
\[ \phi^W_t(x) = x \exp(tW) \quad \text{for} \quad x \in M. \]
Notice that $\phi^W_t$ on $M$ preserves $\text{vol}$.

2.2. Renormalization. A Heisenberg frame is any triple $(X, Y, Z)$ of elements generating $\eta$ such that $Z$ is a fixed generator of the center of the Lie algebra $Z(\eta)$ and $[X, Y] = Z$ (of course we have $[X, Z] = [Y, Z] = 0$). One can for instance take
\[ Z = Z_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]
The set of all Heisenberg frames can be identified with the subgroup $A$ of all automorphisms of Heis which are identity on the center. Notice that up to identification, $A$ is equal to the group $\text{SL}(2, \mathbb{R}) \rtimes \mathbb{R}^2$. Let $A_\Gamma$ be the subgroup of $A$ which stabilizes $\Gamma$, i.e. for $a \in A_\Gamma$, $a(\Gamma) = \Gamma$. We have the following definition

**Definition 2.1.** [FlaFo] The moduli space of the Heisenberg manifold $M$ is the quotient space $M = A_\Gamma \backslash A$.

It follows that $A_\Gamma$ is isomorphic to $A_\Gamma \ltimes (K^{-1} \mathbb{Z}^*)^2$ where $A_\Gamma$ is a finite index subgroup of $\text{SL}(2, \mathbb{Z})$. Therefore the space $M$ is a finite volume orbifold which fibers over the homogeneous space $A_\Gamma \backslash \text{SL}(2, \mathbb{R})$ with fiber $T^2$ (see Proposition 3.4. in [FlaFo]).

2.2.1. The renormalization flow. Following the notation from [FlaFo], for an element $a \in A$ we denote $\bar{a} := A_\Gamma a \in M$. Let $(X_0, Y_0, Z_0)$ be a fixed Heisenberg triple. Let $(a_t)$ be the following one-parameter subgroup of $A$:
\[ a_t(X_0, Y_0, Z_0) = (e^t X_0, e^{-t} Y_0, Z_0). \]

**Definition 2.2.** The renormalization flow $g_\mathbb{R}$ on $M$ is defined by
\[ g_t(\bar{a}) = \bar{a} a_t = A_\Gamma a a_t. \]

In what follows we will also consider the extended renormalization flow $\hat{g}_\mathbb{R}$ on extended moduli space $\hat{M}$, defined as follows. The extended moduli space is the quotient
\[ \hat{M} := A_\Gamma \backslash (A \times M), \]
with respect to the action of $A_\Gamma$ on $A \times M$ by multiplication on the left on $A$ and by the embedding $A_\Gamma < \text{Diff}(M)$ on $M$. The extended renormalization flow is the projection to the extended moduli space of the flow
\[ (t, a, x) \rightarrow (a a_t, x), \quad \text{for all} \quad (t, a, x) \in \mathbb{R} \times A \times M. \]
Note that $\hat{M}$ is a fiber bundle over $M$ with fiber diffeomorphic to $M$ and the extended renormalization flow $\hat{g}_\mathbb{R}$ projects onto the renormalization flow $g_\mathbb{R}$. 
2.3. Sobolev spaces. For \((a,x) \in A \times M\) denote \(X_a = a_*(X_0)\), \(Y_a = a_*(Y_0)\). Let \(\Delta_a := -X_a^2 - Y_a^2 - Z_0^2\) be the Laplace operator. For every \(s \in \mathbb{R}\) and any \(C^\infty\) function \(f \in L^2(M)\) we define
\[
\|f\|_{a,s} = (f, (1 + \Delta_a)^s f)^{1/2}.
\]
Let \(W^s_a(M)\) be the completion of \(C^\infty(M)\) with the above norm. Let \(W^{-s}_a(M)\) denote the dual space. Following \([\text{FlaFo}]\) again, we can define
\[
W^s := A_\Gamma \setminus (A \times W^s(M)) \quad \text{and} \quad W^{-s} := A_\Gamma \setminus (A \times W^{-s}(M)),
\]
where \(A \times W^s(M)\) and \(A \times W^{-s}(M)\) denotes the Hilbert bundle over \(A\), where
\[
\|(a,f)\|_s = \|f\|_{a,s} \quad \text{and} \quad \|(a,D)\|_{-s} = \|D\|_{a,-s}.
\]
We denote elements of \(W^s\) (respectively \(W^{-s}\)) by \((a,f)\) (respectively \((a,D)\)).

We now define the renormalization cocycle.

**Definition 2.3.** \([\text{FlaFo}]\) The renormalization cocycle \(\rho_R\) is a flow on \(W^s\) and \(W^{-s}\) given by
\[
\rho^t(a,f) = (ad^t, f) \quad \text{and} \quad \rho^t(a,D) = (ad^t, D).
\]

**Main results.**

Let \(X\) be the generator of \((\phi^X_t)\) and \(\omega_X\) denote the measure preserved by \(X\).

**Theorem 2.4.** There exists a set \(\mathcal{F} \subset \mathcal{M}\) of full Hausdorff dimension and, for all \(s > 7/2\), a generic set \(\Omega \subset W^s(M)\) such that, for \(a = (X,Y,Z) \in \mathcal{F}\) and \(\alpha \in W^s_0(M)\) with \(Z(1/\alpha) \in \Omega\) the following holds. Either \(1/\alpha\) is an \(X\)-coboundary, or there exist constants \(C_{a,\alpha} > 0\) and \(\bar{\delta}_{a,\alpha} > 0\) such that, for every \(h \in W^s(M)\), \(g \in L^2(M)\) such that \(Zg \in L^2(M)\) and for all \(t \in \mathbb{R}\), we have
\[
| < h \circ \phi^X_t \alpha, g >_{L^2(M, \omega_{a\alpha})} | < \frac{C_{a,\alpha}}{(1 + |t|)^{\bar{\delta}_{a,\alpha}}} \|h\|_s (\|g\|_0 + \|Zg\|_0).
\]

**Theorem 2.5.** There exists a set \(\mathcal{F}' \subset \mathcal{M}\) of full measure and, for all \(s > 7/2\), a generic set \(\Omega \subset W^s(M)\) such that for \(a = (X,Y,Z) \in \mathcal{F}'\) and \(\alpha \in W^s_0(M)\) with \(Z(1/\alpha) \in \Omega\) the following holds. Either \(1/\alpha\) is an \(X\)-coboundary, or for every \(\delta > 1/2\) there exists a constant \(C_{a,\alpha,\delta} > 0\) such that, for every \(h \in W^s(M)\), \(g \in L^2(M)\) such that \(Zg \in L^2(M)\) and for all \(t \in \mathbb{R}\), we have
\[
| < h \circ \phi^X_t \alpha, g >_{L^2(M, \omega_{a\alpha})} | < C_{a,\alpha,\delta}(1 + |t|)^{-\frac{1}{1 + \log^2(1 + |t|)}} \|h\|_s (\|g\|_0 + \|Zg\|_0).
\]

3. Representation theory

Recall that the right quasi regular representation \(\mathcal{U}\) of the Heisenberg group \(\text{Heis}\) on \(L^2(M, \mu)\) is given by
\[
\mathcal{U}(g)F = F(R(g)),
\]
here \(R(g)(x) = xg\). Notice that
\[
L^2(M, \mu) = \bigoplus_{n \in \mathbb{Z}} H_n,
\]
where \( H_n = \{ f \in L^2(M, \mu) : \exp(iZ)f = \exp(2\pi inKt)f \} \) are closed and \( \mathcal{U} \)-invariant. Moreover each \( H_n \) further splits into irreducible subrepresentations spaces. A complete classification of irreducible representations (with non-zero central parameter) is given by the Stone-Von Neumann theorem.

**Theorem 3.1 (Stone-Von Neumann).** For any a Heisenberg triple \( a = (X, Y, Z) \) and for any irreducible unitary representation \( \pi \) of Heis of non-zero central parameter \( n \in \mathbb{Z} \setminus \{0\} \) on the Hilbert space \( H \subset H_n \) there exists a unique unitary operator \( U := U^H_a : H \to L^2(\mathbb{R}, \lambda) \) such that

\[
(U \circ D\pi(X) \circ U^{-1})(f)(u) = f'(u),
\]

\[
(U \circ D\pi(Y) \circ U^{-1})(f)(u) = 2\pi nKf(u),
\]

\[
(U \circ D\pi(Z) \circ U^{-1})(f)(u) = 2\pi nKf(u).
\]

Moreover, by Proposition 4.4. in [FlaFo] it follows that for every irreducible representation \( H \subset H_n \) with non-zero central parameter the space of \( X \)-invariant distribution has dimension 1.

**Definition 3.2.** For all \( a = (X, Y, Z) \in A \), let \( D^H_a \) be the unique distribution such that \( D^H_a \) corresponds by the unitary equivalence \( U^H_a \) (given by the Stone-Von Neumann theorem) to the Lebesgue measure on \( \mathbb{R} \).

**Lemma 3.3.** If \( H \) is any irreducible representation of non-zero central parameter, we have

\[
D^H_{g_t(a)} = e^{-t/2}D^H_a.
\]

**Proof.** Let \( H \) be any irreducible unitary representation of central parameter \( n \neq 0 \). The unitary operator \( U_t : L^2(\mathbb{R}, \lambda) \to L^2(\mathbb{R}, \lambda) \) given by

\[
U_t(f) = e^{t/2}f(e^t u)
\]

intertwines \( (X, Y, Z) \) and \( g_t(X, Y, Z) = (e^t X, e^{-t} Y, Z) \), in the sense that

\[
U_t(U \circ D\pi(e^t X) \circ U^{-1})U_t^{-1} = U \circ D\pi(X) \circ U^{-1},
\]

\[
U_t(U \circ D\pi(e^{-t} Y) \circ U^{-1})U_t^{-1} = U \circ D\pi(Y) \circ U^{-1},
\]

\[
U_t(U \circ D\pi(Z) \circ U^{-1})U_t^{-1} = U \circ D\pi(Z) \circ U^{-1}.
\]

It follows by the above definitions and by the uniqueness part of the Stone-Von Neumann theorem that \( U^H_{g_t(X, Y, Z)} = U_t \circ U^H_{X, Y, Z} \), hence by the definition of \( D^H_{X, Y, Z} \) it follows that

\[
D^H_{g_t(a)} = \text{Leb}_\mathbb{R} \circ U^H_{g_t(a)} = (\text{Leb}_\mathbb{R} \circ U_t) \circ U^H_a
\]

\[
= e^{-t/2}(\text{Leb}_\mathbb{R} \circ U^H_t) = e^{-t/2}D^H_a.
\]

This finishes the proof. \( \square \)
4. Stretching of curves

Fix a Heisenberg triple \((X, Y, Z)\). Let \(\alpha > 0\) denote a smooth time-change function (of the flow \(\phi^V_t\) generated by \(X\)) and \(V = \alpha X\). We have the commutations
\[
[V, Y] = [\alpha X, Y] = -\langle Y, \alpha \rangle X + \alpha Z = -\frac{Y\alpha}{\alpha} V + \alpha Z, \\
[V, Z] = [\alpha X, Z] = -\frac{Z\alpha}{\alpha} V.
\]

Let \((\phi^V_t)\) denote the flow generated by the vector field \(V\) on the nilmanifold \(M\). We will compute the tangent vector of the push forwards of curves under the flow \(\phi^V_t\).

Let \(W\) be any vector in the Lie algebra. We write
\[
(\phi^V_t)^* (W) = a_t V + b_t Y + c_t Z.
\]

By differentiation we derive
\[
\frac{da_t}{dt} V + \frac{db_t}{dt} Y + \frac{dc_t}{dt} Z = -Va_t V - Vb_t Y - b_t [V, Y] - Vc_t Z - c_t [V, Z] \\
= -(Va_t - b_t \frac{Y\alpha}{\alpha} - c_t \frac{Z\alpha}{\alpha}) V - Vb_t Y - (b_t \alpha + Vc_t) Z.
\]
or in other terms
\[
\frac{da_t}{dt} = -Va_t + b_t \frac{Y\alpha}{\alpha} + c_t \frac{Z\alpha}{\alpha}, \\
\frac{db_t}{dt} = -Vb_t, \\
\frac{dc_t}{dt} = -Vc_t - b_t \alpha.
\]

It follows that
\[
\frac{d}{dt} (a_t \circ \phi^V_t) = (b_t \circ \phi^V_t) \frac{Y\alpha}{\alpha} \circ \phi^V_t + (c_t \circ \phi^V_t) \frac{Z\alpha}{\alpha} \circ \phi^V_t, \\
\frac{d}{dt} (b_t \circ \phi^V_t) = 0, \\
\frac{d}{dt} (c_t \circ \phi^V_t) = -(b_t \circ \phi^V_t) (\alpha \circ \phi^V_t).
\]

At this point analogously to [AFU] we will look at the case \(W = Z\) (curves tangent to the central direction), hence \((a_0, b_0, c_0) = (0, 0, 1)\). We have
\[
a_t \circ \phi^V_t = \int_0^t \frac{Z\alpha}{\alpha} \circ \phi^V_t \, d\tau, \\
b_t \circ \phi^V_t = 0, \\
c_t \circ \phi^V_t = 1.
\]

In other terms
\[
D\phi^V_t (Z) = \left( \int_0^t \frac{Z\alpha}{\alpha} \circ \phi^V_t \, d\tau \right) V + Z.
\]
To understand the above orbital integrals we write them in terms of the nilflow \( \phi^X_t \). We have relations
\[
\tau_V(x,t) = \int_0^t \alpha^{-1} \circ \phi^X_t(x) dr \quad \text{and} \quad \tau_X(x,t) = \int_0^t \alpha \circ \phi^V_t(x) dr
\]
By these formulas and by change of variables, we have
\[
(3) \quad \int_0^t f \circ \phi^V_t(x) d\tau = \int_0^{\tau(x,t)} \left( \frac{f}{\alpha} \right) \circ \phi^X_t(x) dr
\]
We will therefore investigate time averages
\[
(4) \quad \int_0^t f \circ \phi^X_t(x) dr
\]
for functions \( f \) of zero average with respect to the Haar volume on \( M \).

5. Construction of the Functionals

Let \( \gamma \) be any rectifiable curve. The curve \( \gamma \) defines a current, that is, a continuous functional on 1-forms. We recall that the renormalization cocycle \( \rho_t \) acts on currents (see Definition 2.3).

Fix an irreducible representation \( H \subset L^2_0(M) \) contained in the eigenspace of eigenvalue \( 2\pi i Kn \in 2\pi i K\mathbb{Z} \) of the action of the center of the Heisenberg group on \( M \) and fix a Heisenberg triple \( a = (X,Y,Z) \). There exists a unique basic current \( B^H_a \) (of degree 2 and dimension 1) associated to \( D^H_a \).

\[ B^H_a = D^H_a \eta_X. \]

The above formula means that for every 1-form \( \alpha \) we have
\[ B^H_a(\alpha) = D^H_a (\frac{\eta_X \wedge \alpha}{\omega}). \]

The current \( B^H_a \) is basic in the sense that
\[ t_X B^H_a = L_X B^H_a = 0. \]

The basic current \( B^H_a \) belongs to a dual Sobolev space of currents, defined as follows. We can write any smooth 1-form \( \alpha \) as follows:
\[ \alpha = \alpha_X \hat{X} + \alpha_Y \hat{Y} + \alpha_Z \hat{Z}. \]

It follows that the space of smooth 1-forms is identified to the product \( (C^\infty(M))^3 \) by the isomorphism
\[ \alpha \to (\alpha_X, \alpha_Y, \alpha_Z). \]

By the above isomorphism, it is also possible to define Sobolev spaces of currents
\[ \Omega^j_a(M) \equiv W^j_a(M)^3 \quad \text{for } j \geq 0, \]
and their dual spaces \( \Omega^{-j}_a(M) := (\Omega^j_a(M))^* \) of currents.

By the Sobolev embedding theorem, for every rectifiable arc \( \gamma \), the current \( \gamma \in \Omega^{-j}_a(M) \) for all \( s > 3/2 \). Since \( D^H_a \in W^{-s}_a(M) \) for all \( s > 1/2 \), all basic currents \( B^H_a \in \Omega^{-s}_a(M) \) for all \( s > 1/2 \). Notice that the Hilbert structure of \( \Omega^j_a(M) \) and \( \Omega^{-j}_a(M) \) depends on \( a = (X,Y,Z) \).
Let $\Pi_{H}^{\gamma} : \Omega_{a,t}^{-\gamma}(M) \to \Omega_{a,t}^{-\gamma}(H)$ denote the orthogonal projection on a single irreducible component (of central parameter $n \in \mathbb{Z} \setminus \{0\}$).

Let $\mathcal{B}_{H,a}^{\gamma} : \Omega^{-\gamma}(M) \to \mathbb{C}$ denote the orthogonal component map in the direction of the $1$-dimensional space of basic currents, supported on a single irreducible unitary representation.

We introduce below a crucial Diophantine condition. Let $\delta_{M} : M \to \mathbb{R}^{+} \cup \{0\}$ be the distance function (which projects to the hyperbolic metric of curvature $-1$ on) from the base point $\overline{id} \in M$.

For any $L > 0$, let $DC(L)$ denote the set of $\overline{\pi} \in M$ such that

$$\int_{0}^{+\infty} \exp\left[\frac{1}{4} \delta_{M}(g_{-1}(\overline{\pi})) - \frac{t}{2}\right] dt \leq L. \tag{5}$$

Let $DC$ denote the union of the sets $DC(L)$ over all $L > 0$. By Kinchine’s theorem, or the logarithmic law of geodesics, it follows that, for almost all $\overline{\pi} \in M$, we have

$$\limsup_{t \to +\infty} \frac{\delta_{M}(g_{-1}(\overline{\pi}))}{\log t} = 1.$$ 

It follows immediately that the set $DC \subset M$ has full Haar volume.

The Bufetov functionals are defined for all Diophantine $\overline{\pi} \in DC$ as follows:

**Lemma 5.1.** Let $\overline{\pi} \in DC(L)$. For $s > 7/2$ and every rectifiable arc $\gamma$ on $M$, the limit

$$\hat{\beta}_{H}(a, \gamma) = \lim_{t \to +\infty} e^{-\frac{t}{2}} \mathcal{B}_{H,a,t}(\gamma).$$

exists, is finite and defines a finitely-additive measure on the space of rectifiable arcs. There exists a constant $C^{\prime} > 0$ such that the following estimate holds:

$$|\Pi_{H}^{-\gamma}(\gamma) - \hat{\beta}_{H}(a, \gamma)B_{a,t}|_{a,-s} \leq C^{\prime}(1 + L)(1 + \int_{\gamma} \gamma + \int_{\gamma} \gamma).$$

For every $L > 0$, the function $\hat{\beta}_{H}(\cdot, \gamma)$ is continuous on $DC(L) \subset M$.

**Proof.** For simplicity of notation (since $H$ is fixed) we suppress the dependence on $H \subset L^{2}(M)$. We will use subscript $a,t$ to denote any dependence on $g_{-1}(a) = (X_{t}, Y_{t}, Z)$, for example

$$\Pi_{H}^{-\gamma}(\gamma) := \Pi_{H,a,t}^{-\gamma}(\gamma), \quad B_{H,a,t}^{-\gamma} := \mathcal{B}_{H,a,t}^{-\gamma}(\gamma), \quad B_{a,t} := B_{a,t}^{H}(\gamma).$$

For every $t \in \mathbb{R}$ we have the following splitting:

$$\Pi_{H}^{-\gamma}(\gamma) = B_{a,t}^{-\gamma}(\gamma)B_{a,t} + R_{a,t}.$$ 

Moreover this splitting is orthogonal in $\Omega_{g_{-1}(a)}^{-\gamma}(M)$. By construction, for any $h \in \mathbb{R}$ we have

$$\mathcal{B}_{a,t+h}^{-\gamma}(\gamma)B_{a,t+h} + R_{a,t+h} = \mathcal{B}_{a,t}^{-\gamma}(\gamma)B_{a,t} + R_{a,t}.$$ 

Since by Lemma 3.3 we have $B_{t+h} = e^{-h/2}B_{t}$ it follows that

$$\mathcal{B}_{a,t+h}^{-\gamma}(\gamma) = e^{h/2}\mathcal{B}_{a,t}^{-\gamma}(\gamma) + \mathcal{B}_{a,t+h}^{-\gamma}(R_{a,t}).$$
By differentiating the expression at \( h = 0 \), we get
\[
\frac{d}{dt} B_{a,t}^{-s}(\gamma) = \frac{1}{2} B_{a,t}^{-s}(\gamma) + \left[ \frac{d}{dh} B_{a,t+h}^{-s}(R_{a,t}) \right]_{h=0}.
\]
The derivative on the right hand side of the above equation can be computed in representation. Let \( < \cdot, \cdot >_{g} \) denote the inner product in the Hilbert space \( \Omega_{g}^{-s}(a) \).

From the intertwining formulas (2) it follows that
\[
B_{a,t+h}^{-s}(R_{a,t}) = < R_{a,t}, B_{a,t+h}^{-s} >_{a,t+h}
= < R_{a,t} \circ U_{-h}, \frac{B_{a,t+h}^{-s} \circ U_{-h}}{B_{a,t+h}^{-s}} >_{a,t}
= < R_{a,t} \circ U_{-h}, \frac{B_{a,t}}{B_{a,t+h}^{-s}} >_{a,t} = B_{a,t}^{-s}(R_{a,t} \circ U_{-h}).
\]

Now by the definition of the intertwining operators \( U_{h} \) in formula (1) it follows that, in the sense of distributions,
\[
\frac{d}{dh} (R_{a,t} \circ U_{-h}) = -R_{a,t} \circ (X_{t} + \frac{1}{2}) \circ U_{-h} = [(X_{t} - \frac{1}{2})R_{a,t}] \circ U_{-h}.
\]

We conclude that
\[
\left[ \frac{d}{dh} B_{a,t+h}^{-s}(R_{a,t}) \right]_{h=0} = -B_{a,t}^{-s}((X_{t} - \frac{1}{2})R_{a,t}).
\]

We finally claim that the following estimate holds: for all rectifiable curve \( \gamma \) on \( M \) and all \( t \in \mathbb{R} \), we have
\[
|B_{a,t}^{-s}((X_{t} - \frac{1}{2})R_{a,t}(\gamma))| \leq |R_{a,t}(\gamma)|_{g_{-t}(a),-s+1}
\leq C_{s} \exp\left[ \frac{1}{4} \delta_{M}(g_{-t}(a)) \right] (1 + \int_{\gamma} |\hat{Y}| + \int_{\gamma} |\hat{Z}|).
\]

The above remainder estimate will be proved in the lemma below. We get therefore a scalar differential equation
\[
\frac{d}{dt} B_{a,t}^{-s}(\gamma) = \frac{1}{2} B_{a,t}^{-s}(\gamma) + \mathcal{R}_{a,t}(\gamma)
\]
with a bounded non-negative function \( \mathcal{R}_{a,t}(\gamma) \) satisfying the estimate
\[
\mathcal{R}_{a,t}(\gamma) \leq C_{s} \exp\left[ \frac{1}{4} \delta_{M}(g_{-t}(a)) \right] (1 + \int_{\gamma} |\hat{Y}| + \int_{\gamma} |\hat{Z}|).
\]

The solution of the above differential equation is
\[
B_{a,t}^{-s}(\gamma) = e^{\frac{t}{2}} [B_{a,0}^{-s}(\gamma) + \int_{0}^{t} e^{-\frac{\tau}{2}} \mathcal{R}_{a,\tau}(\gamma) d\tau].
\]

It follows that, under the Diophantine assumption that \( \tilde{a} \in DC(L) \),
\[
\lim_{t \to +\infty} e^{-\frac{t}{2}} B_{a,0}^{-s}(\gamma) = \hat{\beta}_{H}(a, \gamma)
\]
exists. Since by definition the distributions $B_{a,t}^{-1}(\gamma)$ and $R_{a,t}(\gamma)$ depend continuously on $(a,t) \in A \times \mathbb{R}$, by the Diophantine bound (6), which implies the convergence of the integral
\[ \int_0^{+\infty} e^{-\frac{s}{r}} R_{a,t}(\gamma) \, d\tau, \]
it follows that the complex number
\[ \hat{\beta}_H(a, \gamma) = B_{a,0}^{-1}(\gamma) + \int_0^{+\infty} e^{-\frac{s}{r}} R_{a,t}(\gamma) \, d\tau \]
depends continuously on $a \in DC(L)$. Moreover, we have
\[ \Pi_{H,t}^{-1}(\gamma) - \hat{\beta}_H(a, \gamma) B_{a}^{H} = R_0 - \left( \int_0^{+\infty} e^{-\frac{s}{r}} R_{a,t} \, d\tau \right) B_{a}^{H} \]
and by the above bound on the remainder terms $R_{a,t}$ and by the Diophantine condition on $\gamma \in A$, it follows that
\[ |\Pi_{H,a}^{-1}(\gamma) - \hat{\beta}_H(a, \gamma) B_{a}^{H}|_{a,-s} \leq C_s^0 (1 + L) (1 + \int_{\gamma} |\hat{Y}| + \int_{\gamma} |\hat{Z}|). \]
The argument is thus concluded, up to the above claim on the remainder bounds.
\[ \square \]

We then prove the claim on the remainder bounds.

**Lemma 5.2.** There exists $C_s > 0$ such that, for all $t \geq 0$ and all rectifiable arcs $\gamma$ we have
\[ |R_t(\gamma)|_{a,-s} \leq C_s \exp \left( \frac{1}{4} \delta M(g_t(a)) \right) (1 + \int_{\gamma} |\hat{Y}| + \int_{\gamma} |\hat{Z}|). \]

**Proof.** Let $\alpha$ be any 1-form. For simplicity, for all $t \in \mathbb{R}$, we let $g_t(a) = (X_t, Y_t, Z_t)$. We can write
\[ \alpha = \alpha_X \hat{X}_t + \alpha_Y \hat{Y}_t + \alpha_Z \hat{Z}_t. \]
Let us assume now that $\alpha$ is supported on a single irreducible component $H$. Since
\[ \omega = \hat{X}_t \wedge \hat{Y}_t \wedge \hat{Z}_t, \quad \text{hence } \eta_{X_t} = \hat{Y}_t \wedge \hat{Z}_t, \]
we have the identity
\[ B_{t}^{H}(\alpha) = D_{t}^{H}(\alpha_X \eta_{X_t} \wedge \hat{X}_t + \alpha_Y \eta_{X_t} \wedge \hat{Y}_t + \alpha_Z \eta_{X_t} \wedge \hat{Z}_t) = D_{t}^{H}(\alpha_X) \cdot \]
Let us then assume that
\[ B_{t}^{H}(\alpha) = D_{t}^{H}(\alpha_X) = 0. \]
It follows that $\alpha_X$ is a coboundary for the cohomological equation, that is, there exists a smooth function $u$ on $M$ (with a loss of Sobolev regularity of $1+$) such that
\[ \alpha_X = X_t u. \]
By the Sobolev embedding theorem, for any $s > r + 1 > 7/2$, there exists a constant $B_{r}(g_t(a))$ we have
\[ |u|_{C^s(M)} + |Y_t u|_{C^s(M)} + |Z u|_{C^s(M)} \leq B_{r}(g_t(a)) |u|_{g_t(a), s} \leq B_{r}(g_t(a)) |\alpha_X|_{g_t(a), s}. \]
By [FlaPo], Corollary 3.11, there exists a universal constant $C_r > 0$ such that the best Sobolev constant $B_r(a)$ is bounded above as follows:

$$B_r(a) \leq C_r \exp\left[\frac{1}{4} \delta_M(a)\right].$$

We remark that we can write

$$du = (X_t u) \hat{X}_t + (Y_t u) \hat{Y}_t + (Z u) \hat{Z},$$

hence, by the Sobolev embedding theorem and the fact that $\hat{Y}_t = e^{-t} \hat{Y}$, for all $s > 7/2$, we have

$$|\int_\gamma \alpha| = |\int_\gamma du + (\alpha_{\gamma_1} - Y_t u) \hat{Y}_t + (\alpha_{\gamma_2} - Z u) \hat{Z}|$$

$$\leq C_s |\alpha|_{(g_t(a), s)} \exp\left[\frac{1}{4} \delta_M(g_t(a))\right](1 + \int_\gamma |\hat{Y}| + \int_\gamma |\hat{Z}|).$$

Let us now consider an arbitrary smooth 1-form $\alpha$ on $M$ supported on a single irreducible component. There exists a orthogonal decomposition

$$\alpha = \alpha_0 + \alpha_0^\perp \in \Omega_{g_t(a)}^s(M)$$

such that $\alpha_0 \in \text{Ker}(B_t^H)$. Since $R_t(\gamma) \in \{B_t^H\}^\perp \in \Omega_{g_t(a)}^{-s}(M)$, and

$$\alpha_0^\perp \in \text{Ker}(B_t^H)^\perp = \text{Ker}(\{B_t^H\}^\perp) \in \Omega_{g_t(a)}^s(M),$$

it follows that

$$R_t(\gamma)(\alpha) = R_t(\gamma)(\alpha_0) = \int_\gamma \alpha_0,$$

hence the above estimate leads to the bound

$$|R_t(\gamma)(\alpha)| \leq C_s |\alpha_0|_{(g_t(a), s)} \exp\left[\frac{1}{4} \delta_M(g_t(a))\right](1 + \int_\gamma |\hat{Y}| + \int_\gamma |\hat{Z}|).$$

The conclusion immediately follows by the orthogonality of the decomposition.

□

6. Main Properties of the Functionals

By definition, the Bufetov functional has the additive property, that is, for all rectifiable arcs $\gamma_1$ and $\gamma_2$ on $M$, by linearity of projections and limits, we have

$$\hat{\beta}_H(a, \gamma_1 + \gamma_2) = \hat{\beta}_H(a, \gamma_1) + \hat{\beta}_H(a, \gamma_2);$$

it has the scaling property, that is, for every rectifiable arc $\gamma$ and $t > 0$, we have

$$\hat{\beta}_H(g_t(a), \gamma) = e^{-t/2} \hat{\beta}_H(a, \gamma),$$

The Bufetov functional also has the following invariance property: for all rectifiable arc $\gamma$ and for all $\tau > 0$,

$$\hat{\beta}_H(a, (\phi^\gamma_\tau)(\gamma)) = \hat{\beta}_H(a, \gamma).$$
The above invariance property follows from the fact that by the Sobolev embedding theorem we have
\[ |\phi_t^Y(\gamma) - \gamma|_{\mathcal{GB}(f^{-1}(\tau))} \leq C \tau \delta_{\mathcal{M}}(g_{-t}(a)) \leq C \tau \exp\left[\frac{\delta_{\mathcal{M}}(g_{-t}(a))}{4}\right]. \]

In fact, the current \( \phi_t^Y(\gamma) - \gamma \) is equal, up to two bounded orbit arcs of the flow \( \phi_{\mathbb{R}}^Y \), to the boundary of a 2-dimensional current \( \Delta \), which has uniformly bounded 2-dimensional area with respect to the frame \( g_{-t}(a) := (X, Y, Z) \). This follows from the fact that \( \Delta \) can be taken to be surface tangent to the flow \( \phi_{\mathbb{R}}^Y \), hence, not only \( \hat{X} \cap \hat{Y} = \hat{X} \cap \hat{Y} \), but also
\[
\int_\Delta |\hat{Y} \wedge \hat{Z}| = e^{-t} \int_0^T \left[ \int_{\phi_t^Y(\gamma)} |Z| \right] d\sigma \quad \text{and} \quad \int_\Delta |\hat{X} \wedge \hat{Z}| = 0.
\]

The above invariance property then follows immediately from the Diophantine condition and from the existence of the Bufetov functional.

Finally, the Bufetov functional has the following vanishing property: for every rectifiable arc \( \gamma \) tangent to the central-stable foliation of the extended renormalization flow \( \hat{g}_{\mathbb{R}} \) on the fibers of the extended moduli space \( \hat{\mathcal{M}} \), that is, the foliation generated by the integrable distribution \( \{Y, Z\} \) above each point \( a = (X, Y, Z) \), we have
\[
(11) \quad \hat{\beta}_H(a, \gamma) = 0.
\]
The vanishing property is a direct consequence of the definition, as the length of any arc \( \gamma \) tangent to the central-stable foliation is uniformly bounded along the backward orbit of the renormalization flow:
\[
\int_Y |\hat{X}| = 0, \quad \int_Y |\hat{Y}| = e^{-t} \int_Y |\hat{Y}|, \quad \int_Y |\hat{Z}| = \int_Y |\hat{Z}|.
\]

Let now \( \gamma_t^X(x) \) denote the arc of orbit of the flow \( \phi_{\mathbb{R}}^X \), that is,
\[ \gamma_t^X(x) = \{ \phi_t^X(x) | t \in [0, T] \}, \]
and, for all \( (x, T) \in M \times \mathbb{R}^+ \), let
\[ \beta_H(a, x, T) := \hat{\beta}_H\left(a, \gamma_t^X(x)\right). \]
From the additive property we derive the following cocycle property: for all \( (x, T_1, T_2) \in M \times \mathbb{R} \times \mathbb{R} \) we have
\[
\beta_H(a, x, T_1 + T_2) = \beta_H(a, x, T_1) + \beta_H(a, \phi_t^X(x), T_2).
\]
Moreover, for \( \alpha \in A_\Gamma \), we have
\[
(12) \quad \beta_H(\alpha a, \alpha(x), T) = \beta_H(a, x, T),
\]
which means that the function \( \beta_H(\cdot, \cdot, T) \) is a well defined function on the extended moduli space \( \hat{\mathcal{M}} \). By Lemma 5, for any smooth function \( f \) which belongs to a single irreducible component \( H \), we have
\[
(13) \quad |\int_0^T f \circ \phi_t^X(x) dt - \beta_H(a, x, T)D_H^a(f)| = |< \gamma_t^X(x), f \hat{X} > - \hat{\beta}_H(a, \gamma_t^X(x))B_a(f \hat{X})| \leq C_\pi''(1 + L)|f|_{1, a}.
\]
We have therefore derived the following asymptotic formula for ergodic averages.

\[
\beta_H(a, x, T t) = T^{1/2} \beta_H(g_{\log T}(a), x, t) = T^{1/2} \beta_H(g_{\log T}([a, x]_{A_T}), t).
\]

We have therefore derived the following asymptotic formula for ergodic averages. For every \( x \in M \) and \( t, T > 0 \) we have

\[
\left| \int_0^T f \circ \phi_t^X(x) d\tau - T^{1/2} \beta_H(g_{\log T}[a, x]_{A_T}, t) D^H_a(f) \right| \leq C_s'' (1 + L |f|_{a,s}).
\]

As an immediate consequence of the above asymptotic property we can derive the following orthogonality property: for all \( a \in DC \) and for all \( t \in \mathbb{R}^+ \), for any smooth function \( f \in H \) we have

\[
\beta_H(a, \cdot, t) \in H \subset L^2(M).
\]

It follows that \( \beta_H(a, \cdot, t) \in H \) as a pointwise uniform limit of (normalized) ergodic integrals functions of any given function \( f \in H \).

It can also be proved (as in the work of Bufetov [Bu], or [BF]) that the Bufetov functionals are Hölder for exponent \( 1/2 - \) along the orbits of the flow \( \phi^X_{\beta} \).

In fact, the Hölder property for the Bufetov functionals on rectifiable arcs takes the following form: there exists a constant \( C > 0 \) such that, for every (admissible) rectifiable arc \( \gamma \) on \( M \) we have

\[
|\hat{\beta}_H(a, \gamma)| \leq C \left( 1 + \int_{\gamma} |\hat{X}| + \int_{\gamma} |\hat{Y}| + \int_{\gamma} |\hat{Z}| \right) \left( \int_{\gamma} |\hat{X}| \right)^{1/2}.
\]

The Hölder property is an immediate consequence of the scaling property and of uniform bounds for the Bufetov functionals on arcs of bounded length. From the above property we can easily derive the Hölder property for the Hölder cocycles \( \beta(a, x, T) \) with respect to \( x \in M \) along the orbits of the flow \( \phi^X_{\beta} \) or with respect to the time \( T \in \mathbb{R} \). We conclude this section by constructing Bufetov functionals of smooth functions. By the theory of unitary representations we can write

\[
L^2(M) = \bigoplus_{n \in \mathbb{Z}} H_n := \bigoplus_{n \in \mathbb{Z}} \bigoplus_{i = 1}^{\mu(n)} H_{i,n},
\]

where \( H_n = \bigoplus_{i = 1}^{\mu(n)} H_{i,n} \) is the space with central parameter \( n \in \mathbb{Z} \setminus \{0\} \), and \( H_{i,n} \) are irreducible representation spaces, for \( i = 1, \ldots, \mu(n) \). It follows from the Howe-Richardson multiplicity formula (or by a direct calculation of irreducible representations) that

\[
\mu(n) = |n|, \quad \text{for all } n \neq 0.
\]
For every \( n \neq 0 \) and every \( i \in \{1, \ldots, \mu(n)\} \), let \( D_{i,n}^{j,n} \) denote the unique normalized \( X_n \)-invariant distribution supported on \( W^{-s}(H_{i,n}) \) and \( \beta^{i,n} = \beta_{H_{i,n}} \), the associated Bufetov functional. Since any function \( f \in W^s(M) \) has a decomposition

\[
f = \sum_{n \in \mathbb{Z}} \sum_{i = 1}^{\mu(n)} f_{i,n}
\]

where each component \( f_{i,n} \in W^s(H_{i,n}) \), we can define the Bufetov cocycle associated to the function \( f \in W^s(M) \) as the sum

\[
\beta^f(a,x,T) := \sum_{n \in \mathbb{Z}} \sum_{i = 1}^{\mu(n)} D_{i,n}^{j,n}(f) \beta^{i,n}(a,x,T).
\]

The following result is a version for Bufetov functionals of bound on ergodic integrals proved in [FG2], Lemma 1.4.9. For every \( (a,T) \in A \times \mathbb{R}^+ \), we introduce the excursion function

\[
E_M(a,T) := \int_{0}^{\log T} \exp \left( \frac{\delta_M(g_{\log T}^{-1}(\tilde{a}))}{4} - \frac{t}{2} \right) dt
\]

\[
= T^{-1/2} \int_{0}^{\log T} \exp \left( \frac{\delta_M(g_{\tilde{a}})}{4} + \frac{t}{2} \right) dt.
\]

**Lemma 6.1.** *For all Diophantine \( a \in DC(L) \) and for all function \( f \in W^s(M) \) for \( s > 2 \), the Bufetov functional \( \beta^f \) is defined by a uniformly convergent series, hence the function \( \beta^f_a \) is a Hölder function on \( M \times \mathbb{R} \). In addition there exists a constant \( C_s > 0 \) such that whenever \( a \in DC(L) \) we have, for all \( (x,t,T) \in M \times (\mathbb{R}^+)^2 \),

\[
|\beta^f(a,x,tT)| \leq C_s \left( L + T^{1/2} (1 + t + E_M(a,T)) \right) |f|_{a,s}.
\]

**Proof:** It follows from Lemma 5.1 that there exists a constant \( C > 0 \) such that whenever \( a \in DC(L) \) then

\[
|\beta^{i,n}(a,x,t)| \leq C(1 + L + t), \quad \text{for all } (x,t) \in M \times \mathbb{R}^+.
\]

By the exact scaling property in formula (14), we have

\[
\beta^{i,n}(a,x,tT) = T^{1/2} \beta^{i,n}(g_{\log T}(a),x,t) = T^{1/2} \beta^{i,n}(g_{\log T}([a,x]_{A_T}),t).
\]

By the definition of the set \( DC(L) \) in formula (5) we have that whenever \( a \in DC(L) \) then \( g_{\log T}(a) \in DC(L_T) \) with

\[
L_T \leq LT^{-1/2} + E_M(a,T),
\]

hence by the bound in formula (20) we have that

\[
|\beta^{i,n}(g_{\log T}(a),x,t)| \leq C(1 + L_T + t), \quad \text{for all } (x,t,T) \in M \times (\mathbb{R}^+)^2.
\]
It follows that for all $r > 1/2$ we have

$$|\beta^f(a,x,T)| \leq C_r T^{1/2}(1 + L_T + t) \sum_{n \in \mathbb{Z}} \sum_{i=1}^{\mu(n)} |f_{i,n}|_{a,r}$$

$$\leq C_r T^{1/2}(1 + L_T + t) \left( \sum_{n \in \mathbb{Z}} (1 + n^2)^{-r'} \right)^{-1/2} \left( \sum_{n \in \mathbb{Z}} \sum_{i=1}^{\mu(n)} |(1 - Z^2)^{r'/2} f_{i,n}|_{a,r} \right)^{1/2}.$$ 

We therefore conclude that for all $r' > 3/2$ there exists a constant $C_{r,r'} > 0$ such that

$$|\beta^f(a,x,tT)| \leq C_{r,r'} T^{1/2}(1 + L_T + t)|f|_{a,r+r'},$$

hence, in view of formula (21), the statement is proved. □

It follows from the convergence result given in Lemma [6.1] that all properties of the Bufetov functionals $\beta_H$, each associated to a single irreducible component, extend to the Bufetov functionals $\beta^f$ for any $f \in W^s(M)$ ($s > 2$).

In particular, for every Diophantine $a = (X, Y, Z) \in DC$ the function $\beta^f_H$ on $M \times \mathbb{R}$ is a Hölder cocycle for the flow $\phi^X$, which satisfies the scaling property (14), that is, for all $(x, t, T) \in M \times (\mathbb{R}^+)^2$, we have

$$(22) \quad \beta^f(a,x,Tt) = T^{1/2} \beta^f(g_{\log T}(a), x, t) = T^{1/2} \beta^f(g_{\log T}(\Lambda a), x, t).$$

Finally from the asymptotic formula (15) on each irreducible component we derive the following asymptotic result:

**Theorem 6.2.** For all $s > 7/2$ there exists a constant $C_s > 0$ such that for all $a = (X, Y, Z) \in DC(L)$, for all $f \in W^s(M)$ and for all $(x, T) \in M \times \mathbb{R}^+$, we have

$$(23) \quad \left| \int_0^T f \circ \phi^X(x) dt - \beta^f(a,x,T) \right| \leq C_s (1 + L)|f|_{a,s}.$$ 

All the results of this paper, about limit distributions and about decay of correlations for time-changes, are derived from the above asymptotic result.

7. LIMIT DISTRIBUTIONS

In this section we derive some corollaries on limit distributions of ergodic integrals, which generalize results of J. Griffin and J. Marklof [CM] to arbitrary smooth functions and recover the Hölder property of limit distributions, proved by F. Cellarosi and J. Marklof [CM].

**Lemma 7.1.** There exists a continuous modular function $\theta_H : A \to H \subset L^2(M)$ such that for all $f \in W^s_0(H)$ with $s > 1/2$, we have

$$\lim_{T \to +\infty} \frac{1}{T^{1/2}} \int_0^T f \circ \phi^X(\cdot) dt - \theta_H \left( g_{\log T}(a) \right) D^H_t(f) \|L^2(M) = 0.$$ 

The family $\{\theta_H(a) | a \in A\}$ has (positive) constant norm in $L^2(M)$: there exists a constant $C > 0$ such that for all $a \in A$ we have

$$\|\theta_H(a)\|_{L^2(M)} = C.$$
Proof: We refer the reader to [FalFo] or [Fo2] for background on the application of this theory to the cohomological equation of Heisenberg nilflows. By the Stone-Von Neumann theorem, the space $H := H_z$ is unitarily equivalent to the space $L^2(\mathbb{R}, du)$ and such a unitary equivalence can be chosen so that the group $\phi^X_{\mathbb{R}}$ is represented as the group of translations on the real line and the group $\phi_0^X$ is a group of the form $\{e^{it}Id\}$. In other terms we have the infinitesimal representation

$$X \rightarrow \frac{d}{du}, \quad Y \rightarrow izu.$$  

The space of smooth vectors (for a irreducible unitary representation of central parameter $z \neq 0$) is the Schwartz space $S(\mathbb{R})$ and the space of translation invariant tempered distribution on the real line is given by all scalar multiples of the Lebesgue measure. It follows that invariant distributions supported on $H$ for the flow $\phi^X_{\mathbb{R}}$ are represented as scalar multiples of the Lebesgue measure. Finally the Sobolev space $W^s_a(H)$ is represented as the space $S^s(\mathbb{R})$ of functions $f \in L^2(\mathbb{R}, du)$ such that

$$\int_\mathbb{R} |(1 + \frac{d^2}{du^2} + z^2u^2)^{s/2} \hat{f}(u)|^2 du < +\infty.$$  

The statement is equivalent to the claim that there exists $\theta(a) \in L^2(\mathbb{R}, du)$ such that for all $f \in S^s(\mathbb{R})$ we have

$$\lim_{T \rightarrow +\infty} \| \frac{1}{T^{1/2}} \int_0^T f(u + t) dt - \theta (g_{\log T}(a)) \operatorname{Leb}(f) \|_{L^2(\mathbb{R}, du)} = 0.$$  

An equivalent formulation, by the standard Fourier transform on $\mathbb{R}$:

$$\lim_{T \rightarrow +\infty} \| \frac{1}{T^{1/2}} \int_0^T e^{i\hat{u}t} \hat{f}(\hat{u}) dt - \hat{\theta} (g_{\log T}(a)) \hat{f}(0) \|_{L^2(\mathbb{R}, d\hat{u})} = 0.$$  

Let $\chi \in L^2(\mathbb{R}, d\hat{u})$ denote the function defined as

$$\chi(\hat{u}) = \frac{e^{i\hat{u}} - 1}{i\hat{u}}, \quad \text{for all } \hat{u} \in \mathbb{R}.$$  

Let $\hat{\theta}(a)(\hat{u}) := \chi(\hat{u})$, for all $\hat{u} \in \mathbb{R}$. Let us compute $\theta (g_{\log T}(a))$. By definition $g_{\log T}(a) = (TX, T^{-1}Y, Z)$, hence the induced representation

$$TX \rightarrow iT\hat{u}, \quad T^{-1}Y \rightarrow T^{-1} z \frac{d}{d\hat{u}}$$  

is intertwined to the normalized representation by the unitary equivalence $U_T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ defined as

$$U_T(f)(\hat{u}) = T^{1/2} f(T\hat{u}), \quad \text{for all } u \in \mathbb{R}.$$  

In fact, we have

$$iT\hat{u} = U_T^{-1} \circ (iT\hat{u}) \circ U_T, \quad \frac{d}{d\hat{u}} = U_T^{-1} \circ (T^{-1} z \frac{d}{d\hat{u}}) \circ U_T.$$  

It follows that, for all $a \in A$ and all $T > 0$,

$$\hat{\theta} (g_{\log T}(a))(\hat{u}) = U_T(\chi)(\hat{u}) = T^{1/2} \chi(T\hat{u}).$$
The function \( \theta_H(a) \in H \) is uniquely defined in representation as the unique Fourier anti-transform \( \theta(a) \in L^2(\mathbb{R}) \) of the function \( \hat{\theta}(a) \in L^2(\mathbb{R}) \). By its definition the function \( \theta_H \) is modular, that is, it is invariant under the action of the lattice \( A_\Gamma \) on \( A \). As a consequence, it induces a well-defined function on the moduli space \( \mathcal{M} = A_\Gamma \backslash A \). By unitary equivalence

\[
\|\theta_H(a)\|_H = \|\theta(a)\|_{L^2(\mathbb{R})} = \|\hat{\theta}(a)\|_{L^2(\mathbb{R})} = \left\|\frac{\epsilon_i^\alpha - 1}{i\hat{\alpha}}\right\|_{L^2(\mathbb{R},d\hat{\alpha})} := C > 0.
\]

By integration we have

\[
\int_0^T e^{i\hat{\alpha} t} \hat{f}(\hat{u}) dt = T \chi(T \hat{u}) \hat{f}(\hat{u}) = T \chi(T \hat{u}) (\hat{f}(\hat{u}) - \hat{f}(0)) + T^{1/2} \hat{\theta} (g_{\log T}(a)) (\hat{u}) \hat{f}(0).
\]

The claim is therefore reduced to the following statement

\[
\lim_{T \to +\infty} \left\|T^{1/2} \chi(T \hat{u}) (\hat{f}(\hat{u}) - \hat{f}(0))\right\|_{L^2(\mathbb{R},d\hat{u})} = 0.
\]

Since by hypothesis \( f \in S^s(\mathbb{R}) \) with \( s > 1/2 \), the function \( \hat{f} \in C^0(\mathbb{R}) \) and it is bounded, hence

\[
\left\|T^{1/2} \chi(T \hat{u}) (\hat{f}(\hat{u}) - \hat{f}(0))\right\|_{L^2(\mathbb{R},d\hat{u})} = \left\|\chi(v) \left(\hat{f} \left(\frac{v}{T}\right) - \hat{f}(0)\right)\right\|_{L^2(\mathbb{R},dv)} \to 0,
\]

by the dominated convergence theorem.

The continuous dependence of the function \( \theta_H(a) \in H \) on \( a \in A \) follows from the continuous dependence of the intertwining operator \( U_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \), which conjugates a representation of the form

\[
X_a \to \alpha \frac{d}{du} + i\gamma u + \nu, \quad Y_a \to \beta \frac{d}{du} + i\delta u + w
\]

to the standard representation, on the parameters

\[
\left(\begin{array}{cc}
\alpha & \beta \\
\gamma & \delta
\end{array}\right) \in SL(2,\mathbb{R}) \quad \text{and} \quad (\nu, w) \in \mathbb{R}^2,
\]

of the automorphism \( a \in A \). The intertwining operator \( U_a \) can be computed explicitly. The details are left to the reader.

\[\square\]

**Corollary 7.2.** There exists a constant \( C > 0 \) such that, for any \( s > 1/2 \), for any \( a = (X, Y, Z) \in A \) and for any \( f \in W^s_a(\mathcal{M}) \), we have

\[
\lim_{T \to +\infty} \frac{1}{T^{1/2}} \left\|\int_0^T f \circ \phi_t^X dt\right\|_{L^2(\mathcal{M})} = C|D_a^H(f)|.
\]

The above statement strengthens Lemma 15 of \( [\text{AFU}] \). In fact, there the authors proved a slightly weaker statement for linear skew-shifts of the torus \( \mathbb{T}^2 \), that is for maps of the form

\[
T_{\rho,\sigma}(y,z) = (y + \rho, z + y + \sigma), \quad \text{for all } (y,z) \in \mathbb{T}^2.
\]
As explained in [AFU], the minimal flow $\phi^X_\beta$ has constant return time to a transverse torus with return map a linear skew-shift, that is, a map of the form $T_{\rho,\sigma}$ for constants $\rho \in \mathbb{R} \setminus \mathbb{Q}$ and $\sigma \in \mathbb{R}$.

From Corollary 7.2 we derive the following limit result for the $L^2$ norm of Bufetov functionals.

**Corollary 7.3.** There exists a constant $C > 0$ such that for all irreducible components $H \subset L^2(M)$ and $a \in DC$, we have

$$\lim_{T \to +\infty} \frac{1}{T^{1/2}} \| \beta_H(a,\cdot,T) \|_{L^2(M)} = C.$$ 

**Proof.** By the normalization of the invariant distributions in the Sobolev space $W^s(M)$ for any given $s > 1/2$ the following holds. For all irreducible components $H$ and all $a \in A$, there exists a (non-unique) function $f_a^H \in W^s(H)$ such that

$$D_a^H(f_a^H) = \| f_a^H \|_s = 1.$$ 

For all $a \in DC(L)$, and for $s > 7/2$, we derive from the asymptotic formula in Theorem 6.2 that

$$\left| \int_0^T f_a^H \circ \phi^X(x) dt - \beta_H(a,x,T) \right| \leq C_s(1 + L).$$

The $L^2$ estimates in the statement then follow from Corollary 7.2. 

A relation between the Bufetov functionals and the above theta functionals is established below.

**Corollary 7.4.** For any irreducible component $H \subset L^2(M)$ the following holds. For any $L > 0$ and for any $g_\mathbb{R}$-invariant probability measure $\mu$ supported on $DC(L) \subset \mathcal{M}$, we have

$$\beta_H(a,\cdot,1) = \theta_H(a) \in L^2(M), \quad \text{for } \mu \text{-almost all } \bar{a} \in \mathcal{M}.$$ 

**Proof.** By Theorem 6.2 and Lemma 7.1 we immediately derive that there exists a constant $C_\mu > 0$ such that for all $a \in \text{supp}(\mu) \subset DC(L)$, for all $T > 0$ we have

$$\| \beta_H(g_{\log T}(a),\cdot,1) - \theta_H(g_{\log T}(a)) \|_{L^2(M)} \leq \frac{C_\mu}{T^{1/2}}.$$ 

By Luzin’s theorem, for any $\delta > 0$ there exists a compact subset $\mathcal{E}(\delta) \subset \mathcal{M}$ such that we have the measure bound $\mu(\mathcal{M} \setminus \mathcal{E}(\delta)) < \delta$ and such that the function $\beta_H(a,\cdot,1) \in L^2(M)$ depends continuously on $\bar{a} \in \mathcal{E}(\delta)$. By Poincaré recurrence there is a full measure subset $\mathcal{E}'(\delta) \subset \mathcal{E}(\delta)$ of $g_\mathbb{R}$-recurrent points. In particular, for every $\bar{a}_0 \in \mathcal{E}'(\delta)$ there is a diverging sequence $(t_n)$ such that $\{g_\mathbb{R}(\bar{a}_0)\} \subset \mathcal{E}(\delta)$ with

$$\lim_{n \to \infty} g_\mathbb{R}(\bar{a}_0) = \bar{a}_0.$$ 

By the continuity of the function $\theta_H : \mathcal{M} \to L^2(M)$, by the continuity at $\bar{a}_0$ of the function $\beta_H(\bar{a},\cdot,1) \in L^2(M)$ on the set $\mathcal{E}(\delta)$, and by the above $L^2$ estimate, we have

$$\| \beta_H(\bar{a}_0,\cdot,1) - \theta_H(\bar{a}_0) \|_{L^2(M)} = \lim_{n \to \infty} \| \beta_H(g_\mathbb{R}(\bar{a}_0),\cdot,1) - \theta_H(g_\mathbb{R}(\bar{a}_0)) \|_{L^2(M)} = 0.$$
We have thus proved that \( \beta_H(\bar{a}, \cdot, 1) = \theta_H(a) \in L^2(M) \) for all \( a \in \mathcal{E}(\delta) \). It follows that the set where the latter identity fails has \( \mu \)-measure less than any \( \delta > 0 \), hence the identity holds for \( \mu \)-almost all \( \bar{a} \in A \), as stated.

\[ \square \]

**Corollary 7.5.** There exists a constant \( C > 0 \) such that for all irreducible components \( H \subset L^2(M) \) the following holds. For any \( L > 0 \) and for any \( g_{\mathbb{R}} \)-invariant probability measure \( \mu \) supported on \( DC(L) \subset \mathcal{M} \), and for all \( T > 0 \) we have

\[
\| \beta_H(a, \cdot, T) \|_{L^2(M)} = CT^{1/2} \quad \text{for } \mu \text{-almost all } \bar{a} \in \mathcal{M}.
\]

**Remark 7.6.** Since by Lemma 7.1 the function \( \theta_H : A \to L^2(M) \) is continuous and approximates ergodic integrals, it is possible to write it (at least for the first irreducible component) in terms of the theta function \( \Theta_\chi \) introduced in [GM], as both functions are continuous, modular, and provide a similar asymptotic formula for ergodic averages (sums) (see formula (13) in [GM]). It follows that Bufetov functional \( \beta_H \) (for the first irreducible component) essentially coincide with the function \( \Theta_\chi \) almost everywhere on the moduli space. The main advantage of the approach of Cellarosi and Marklof [CM] is that it provides an explicit Diophantine condition which describes the set where the function \( \Theta_\chi \) is absolutely convergent and \( 1/2 \)-Hölder (see [CM], Theorem 3.10).

For general smooth functions we proceed as in the previous section. Since any function \( f \in W^s(M) \) has a decomposition

\[
f = \sum_{n \in \mathbb{Z}} \sum_{i=1}^{\mu(n)} f_{i,n}
\]

where each component \( f_{i,n} \in W^s(H_{i,n}) \), we can define the functional \( \theta^f : A \to L^2(M) \) associated to the function \( f \in W^s(M) \) as the weighted sum over all functionals \( \theta^{i,n} := \theta_{H_{i,n}} \) associated to the irreducible representations \( H_{i,n} \):

\[
\theta^f(a) := \sum_{n \in \mathbb{Z}} \sum_{i=1}^{\mu(n)} D_{i,n}^a(f) \theta^{i,n}(a).
\]

**Lemma 7.7.** For all \( a \in A \) and for all function \( f \in W^s(M) \) for \( s > 2 \), the functional \( \theta^f \) is defined by a convergent series, hence the function \( \theta^f(a) \) is an \( L^2 \) function on \( M \) and \( \theta^f : A \to L^2(M) \) is a continuous function.

From Lemma 7.1 we derive a general asymptotic theorem:

**Theorem 7.8.** For all \( a \in A \) and for all \( f \in W^s(M) \) with \( s > 2 \) we have

\[
\lim_{T \to +\infty} \frac{1}{T^{1/2}} \int_0^T f \circ \phi_t \, dt - \theta^f \left( g_{\log T}(a) \right) \|_{L^2(M)} = 0.
\]

We leave to the reader the derivation of Lemma 7.7 and Theorem 7.8.

From Theorem 7.8 we can derive most of a result of Griffin and Marklof [GM] on limit distribution of theta sums in the related context of Heisenberg nilflows. We also prove the result established later by Cellarosi and Marklof [CM] (see in
particular [CM], §3) that limit distributions are the distributions of Hölder function of exponent equal to $1/2−$. Our results have the advantage of holding for all sufficiently smooth functions, while the work of Griffin and Marklof [GM], and Cellarosi and Marklof [CM] holds only for a single (toral) character. However, they are much less explicit and less detailed, especially as far as the the behavior at infinity in the moduli space is concerned, and in particular we have no results on limit distributions for time sequences corresponding to escape in the cusp of the moduli space.

The following result summarizes our results on limit distributions of ergodic averages of sufficiently smooth functions for Heisenberg nilflows:

**Theorem 7.9.** Let $a = (X,Y,Z) \in A$ and let $(T_n)$ be any sequence such that

$$\lim_{n \to +\infty} g \log T_n(\bar{a}) = a_\infty \in \mathcal{M}.$$  

For every zero average function $f \in W^s(M)$ with $s > 7/2$ which is not a coboundary, the limit distribution of the family of random variables

$$E_{T_n}(f) := \frac{1}{T_n^{1/2}} \int_0^{T_n} f \circ \phi^{X_t} dt$$

exists and is equal to the distribution of the function $\theta^f(a_\infty) \in L^2(M)$. In particular, if $a_\infty \in DC$ belongs to the $\omega$-limit set of any $g_\mathbb{R}$-orbit on the set DC of Diophantine points and is a continuity point of the Bufetov functional $\beta^f$ on $DC \subset A$, then $\theta^f(a_\infty)$ is almost everywhere equal to a bounded $\frac{1}{2}$-Hölder function on $M$, hence in particular the limit distribution has compact support.

**Proof.** Since $a_\infty \in \mathcal{M}$, the existence and characterization of the limits follows from Lemma 7.7 and Theorem 7.8. The regularity statement follows from Corollary 7.4. 

With the exception of the Hölder regularity statement, equivalent results were proved in [GM] for linear toral skew-shift and for function cohomologous to the principal toral character. For such functions, the authors also investigated the case when the limit point $a_\infty = +\infty$ and proved that in that case the limit distribution is the Dirac delta $\delta_0$ at $0 \in \mathbb{R}$. The Hölder regularity property was proved in [CM].

**8. Square Mean Lower Bounds**

In this section we prove transverse square mean lower bound for ergodic integrals.

Let $T^2_\Gamma$ denote the 2-dimensional torus transverse to flow, defined as follows:

$$T^2_\Gamma := \{ \Gamma \exp(yY_0 + zZ) | (y,z) \in \mathbb{R}^2 \}.$$  

The torus $T^2_\Gamma$ is transverse to the nilflow $\phi^{X_0}_\mathbb{R}$ on $M$, hence transverse to all nilflows $\phi^X_\mathbb{R}$ such that $<X,X_0> \neq 0$. For all $a = (X,Y,Z)$, let

$$t_a := \frac{1}{|<X,X_0>|}$$
denote the return time of the flow $\phi^X_t$ to the transverse tori $\mathbb{T}_1^2$.

We will prove bounds for the square mean of ergodic integrals along the leaves of the foliation of the torus $\mathbb{T}_1^2$ into circles transverse to the central direction:

$$\{ \xi \exp(yb) | y \in \mathbb{T} \}_{\xi \in \mathbb{T}_1^2}.$$ 

**Lemma 8.1.** There exists a constant $C > 0$ such that for all $a = (X, Y, Z)$ and for every irreducible component $H$ of central parameter $n \neq 0$, there exist a function $f_H \in C^\infty(H)$ such that

$$|f_H|_{L^\infty(M)} \leq Ct_a^{-1}|D^H_a(f_H)|,$$

$$|f_H|_{a,s} \leq Ct_a^{-1}|D^H_a(f_H)|(1 + t_a^{-1}\|Y\|)^s(1 + n^2)^{s/2},$$

and such that, for all $x \in \mathbb{T}_1^2$ and $T \in \mathbb{Z}t_a$, we have

$$|\int_0^T f_H(\phi^X_t(x))dt| \leq 1,$$

$$\|\int_0^T f_H(\phi^X_t(x))dt\|_{L^2(T,dy)} = |D^H_{t_a}(f_H)|(\frac{T}{t_a})^{1/2}.$$ 

In addition, whenever $H \perp H' \subset L^2(M)$ the functions

$$\int_0^T f_H(\phi^X_t(x))dt \quad \text{and} \quad \int_0^T f_{H'}(\phi^X_t(x))dt$$

are orthogonal in $L^2(T,dy)$.

**Proof.** We recall that whenever $<X, X_0> \neq 0$ the return map of the flow $\phi^X_t$ to the transverse torus $\mathbb{T}_1^2$ is a linear skew-shift, that is, a map of the form

$$T_{\rho, \sigma}(y, z) = (y + \rho, z + y + \sigma) \quad \text{on} \quad \mathbb{R}/\mathbb{Z} \times \mathbb{R}/K^{-1}\mathbb{Z},$$

for constants $\rho, \sigma \in \mathbb{R}$. The operator $I_a : L^2(M) \to L^2(\mathbb{T}_1^2)$ defined as

$$f \to I_a(f) := \int_0^{t_a} f \circ \phi^X_t(\cdot)dt,$$

for all $f \in L^2(M)$

is a surjective linear map of $L^2(M)$ onto $L^2(\mathbb{T}_1^2)$ with right inverse $R^X_a$ defined as follows: let $\chi \in C^\infty(0, 1)$ denote any function of integral equal to one and, for any $F \in L^2(\mathbb{T}_1^2)$, let $R^X_a(F) \in L^2(M)$ be the function uniquely defined by the identity

$$R^X_a(F)(\phi^X_t(x)) = t_a^{-1}\chi(t/t_a)F(x), \quad \text{for all} \ (x,t) \in \mathbb{T}_1^2 \times [0, t_a].$$

It is immediate from the definition that there exists a constant $C_\chi > 0$ such that

$$|R^X_a(F)|_{a,s} \leq C_\chi t_a^{-1}(1 + t_a^{-1}\|Y\|)^s\|F\|_{W^{s,1}(\mathbb{T}_1^2)}.$$ 

As explained in [AFU], §5, the space $L^2(\mathbb{T}_1^2)$ can be decomposed as a direct sum of irreducible subspaces invariant under the action of the map $T_{\rho, \sigma}$ on $L^2(\mathbb{T}_1^2)$ by composition. The subspace of functions with non-zero central character can be split as a direct sum of components $H_{(m,n)}$ with $(m,n) \in \mathbb{Z}K[n] \times \mathbb{Z} \setminus \{0\}$. These
As described in [AFU], §5, the functions $F$ are generators of the space of invariant distributions. The distribution $T\,\tilde{\chi}$ of central parameter $n \neq 0$ is such that $\hat{\tilde{\chi}}(\hat{\sigma}T_0)=\hat{\tilde{\chi}}(\hat{\sigma}T_0)\in C^\omega(\mathcal{S})$ has the property that such that $\hat{\tilde{\chi}}(\hat{\sigma}T_0)=\hat{\tilde{\chi}}(\hat{\sigma}T_0)$ is characterized by a Fourier basis given by the functions $f^{\gamma_0}(y)\,\tilde{\chi}(T_0\,\tilde{\chi})(y)$, $\forall \gamma_0 \in \mathbb{Z}^2$.

The argument is therefore complete. $\square$

In addition the following estimates hold:

$$\|c_H\|_{L^\infty} \leq C_\chi t^{-1}_a \quad \text{and} \quad \|c_H\|_{L^\infty} \leq C_\chi t^{-1}_a \left(1 + t^{-1}_a \|Y\|\right)^4 \left(1 + K^2 n^2\right)^{1/2}.$$ 

Since for every $n \in \mathbb{Z} \setminus \{0\}$ and $m \in \mathbb{Z}K[n]$, the system

$$\{\exp[2\pi i (m \cdot j \rho + Kn(z + j(\sigma + y) + \rho \left(\frac{j}{2}\right))]\}_{j \in \mathbb{Z}} \subset L^2(\mathbb{T}, dy)$$

is orthonormal, we have the identity

$$\|\sum_{j=0}^{J-1} \exp[2\pi i (m \cdot j \rho + Kn(z + j(\sigma + y) + \rho \left(\frac{j}{2}\right))]\|_{L^2(\mathbb{T}, dy)}^2 = J.$$ 

In addition, by an immediate computation

$$\int_{\mathbb{T}} \left(\sum_{j=0}^{J-1} \exp[2\pi i (m \cdot j \rho + Kn(z + j(\sigma + y) + \rho \left(\frac{j}{2}\right))]\right) dy \in \{0, 1\}.$$ 

By the above formula it follows that, whenever $T/t_a \in \mathbb{Z}$ we have

$$\left|\int_{\mathbb{T}} \left(\int_0^T f_H \circ \phi^X_n(y)\,\tilde{\chi}(x)\,dy\right) dt\right| = \left|\int_{\mathbb{T}} \left(\sum_{j=0}^{J-1} e_{m,n} \circ T^j_{\rho,\sigma}(y,z)\right) dy\right| \leq 1,$$

$$\|\int_0^T f_H \circ \phi^X_n(y)\,\tilde{\chi}(x)\,dy\|_{L^2(\mathbb{T}, dy)} = \left\|\sum_{j=0}^{J-1} e_{m,n} \circ T^j_{\rho,\sigma}\right\|_{L^2(\mathbb{T}, dy)} = \left(\frac{T}{t_a}\right)^{1/2},$$

The argument is therefore complete.
For any infinite dimensional vector \( c := (c_{i,n}) \in \ell^2 \), let \( \beta_c \) denote the Bufetov functional defined as follows

\[
\beta_c = \sum_{n \in \mathbb{Z} \neq 0} \sum_{i=1}^{\mu(n)} c_{i,n} \beta_i^{n}.
\]

It follows from the orthogonality property and from Corollary 7.3 that the function \( \beta_c(a, \cdot, T) \in L^2(M) \) for all \( (a, T) \in A \times \mathbb{R}^+ \). In fact,

\[
\|\beta_c(a, \cdot, T)\|_{L^2(M)} = \sum_{n \in \mathbb{Z} \neq 0} \sum_{i=1}^{\mu(n)} |c_{i,n}|^2 \|\beta_i^{n}(a, \cdot, T)\|_{L^2(M)}^2 \leq C^2 |c|^2 |c| T.
\]

For any vector \( c := (c_{i,n}) \in \ell^2 \), let \( |c|_s \) denote the norm defined as

\[
|c|^2_s = \sum_{n \in \mathbb{Z} \neq 0} \sum_{i=1}^{\mu(n)} (1 + K^2 n^2)^s |c_{i,n}|^2.
\]

For any \( a = (X, Y, Z) \in A \) such that \( <X, X_0> \neq 0 \) or, equivalently, such that the return time \( t_a > 0 \) is finite, and for any \( x \in M \) let \( S_{a,x} \) denote the transverse cylinder defined as follows:

\[
S_{a,x} = \{ x \exp(yY + zZ) | (y', z') \in [0, t_a^{-1}] \times T \} \subset M.
\]

Let \( \Phi_{a,x} : T^2_\mathbb{F} \to S_{a,x} \) denote the maps defined as follows. For any \( \xi \in T^2_\mathbb{F} \), let \( \xi' \in S_{a,x} \) denote the first intersection of the orbit \( \{ \phi^X_t(\xi) | t \geq 0 \} \) with the transverse cylinder \( S_{a,x} \). By definition there exists \( t(\xi) \geq 0 \) such that

\[
\xi' = \Phi_{a,x}(\xi) = \phi^X_{t(\xi)}(\xi), \quad \text{for all } \xi \in T^2_\mathbb{F}.
\]

Let \( (y, z) \) and \( (y', z') \) denote the coordinates, respectively on \( T^2_\mathbb{F} \) and \( S_{a,x} \), given by the exponential map, as follows

\[
(y, z) \to \xi_{y,z} := \Gamma \exp(yY_0 + zZ) \in T^2_\mathbb{F} \quad \text{and} \quad (y', z') \to \xi'_{y', z'} := x \exp(y'Y + z'Z).
\]

Let \( X = \alpha X_0 + \beta Y_0 + vZ \) and \( Y = \gamma Y_0 + \delta Y_0 + wZ \) with \( \alpha \neq 0 \) and \( \alpha \delta - \beta \gamma = 1 \). Let \( x = \Gamma \exp(yY_0 + zZ) \exp(t_x X_0) \) with \( (y_x, z_x) \in \mathbb{T} \times \mathbb{R}/KZ \) and \( t_x \in (0, 1) \). By the Baker-Campbell-Hausdorff formula, we derive that

\[
|t(\xi)| = |\delta t_x + \gamma(y - y_x)| \leq \|Y\|.
\]

and that the map \( \Phi_{a,x} : T^2_\mathbb{F} \to S_{a,x} \) is given by formulas of the following form: there exists a polynomial \( P(a, x, y) \) of total degree 4, quadratic with respect to each of the variables \( (a, x, y) \in \mathbb{M} \times M \times \mathbb{R} \), such that

\[
\Phi_{a,x}(y, z) =: \begin{cases} 
y' = \alpha(y - y_x) + \beta t_x, 
\end{cases}
\]

In particular, the map \( \Phi_{a,x} \) is invertible and such that

\[
\Phi_{a,x}^\ast(dy' \land dz') = \alpha dy \land dz = t_a^{-1} dy \land dz.
\]

The cylinders \( S_{a,x} \) are foliated by images of the circles \( \{ \xi \exp(yY_0) | y \in \mathbb{T} \} \subset T^2_\mathbb{F} \) under the map \( \Phi_{a,x} \).
Lemma 8.2. For any \( s > 7/2 \) there exists a constant \( C_s > 0 \) such that, for all Diophantine \( a \in DC(L) \), for all \( c \in \ell^2 \), for all \( z \in \mathbb{T} \) and all \( T > 0 \), we have

\[
|\int_{\mathbb{T}} \beta_c(a, \Phi_{a,x}(\xi_{y,z}), T) dy| \leq C_s(t_a + t_a^{-1})(1 + L)(1 + t_a^{-1})|\|Y\|s|c|, \\
\|\beta_c(a, \Phi_{a,x}(\xi_{y,z}), T)\|_{L^2(\mathbb{T},dy)} - \left(\frac{T}{t_a}\right)^{1/2} |c|_0 \leq C_s(t_a + t_a^{-1})(1 + L)(1 + t_a^{-1})|\|Y\|s|c|. 
\]

Proof: By Lemma [8.1] for every \( n \neq 0 \) and every \( i \in \{1, \ldots, \mu(n)\} \), there exists a function \( f_{i,n} \in C^\infty(H_{i,n}) \) with \( D^{i,n}(f_{i,n}) = 1 \) such that, for all \( T \in \mathbb{Z}_{t_a} \) we have, for all \( \xi \in \mathbb{T}_T^2 \) the identities

\[
|\int_{\mathbb{T}} \left( \int_0^T f_{i,n}(\phi^X_s(\phi^Y_{z}(\xi))) ds \right) dy| \in \{0, 1\}, \\
\|\int_0^T f_{i,n}(\phi^X_s(\phi^Y_{z}(\xi))) ds\|_{L^2(\mathbb{T},dy)} = \left(\frac{T}{t_a}\right)^{1/2}. 
\]

In addition, the integrals

\[
\int_0^T f_{i,n}(\phi^X_s(\phi^Y_{z}(\xi))) ds
\]

form an orthogonal system of functions in \( L^2(\mathbb{T},dy) \). Let then

(30)

\[
f_c := \sum_{n \in \mathbb{Z} \neq 0} \sum_{i=1}^{\mu(n)} c_{i,n} f^{i,n},
\]

By construction we have \( \beta_c = \beta^{c} \). It is immediate that

\[
|\int_{\mathbb{T}} \left( \int_0^T f_c(\phi^X_s(\phi^Y_{z}(\xi))) ds \right) dy| \leq |c|_{\ell^1}. 
\]

By orthogonality we also have

\[
\|\int_0^T f_c(\phi^X_s(\phi^Y_{z}(\xi))) ds\|_{L^2(\mathbb{T},dy)} = \left(\frac{T}{t_a}\right)^{1/2}|c|_0. 
\]

From the estimates on the functions \( f_{i,n} \) stated in Lemma [8.1] we derive the bounds

\[
|f_c|_{L^\infty(M)} \leq C|c|_{\ell^1} \quad \text{and} \quad |f_c|_{a,s} \leq C t_a^{-1}(1 + t_a^{-1})|\|Y\|s|c|_s. 
\]

From this estimate it follows that, for every \( z \in \mathbb{T} \) and for all \( T > 0 \), we have

\[
\|\int_0^T f_c(\phi^X_s \circ \Phi_{a,x}(\xi_{y,z})) ds - \int_0^T f_c(\phi^X_s(\xi_{y,z})) ds\|_{L^2(\mathbb{T},dy)} \leq 2 |f_c|_{L^\infty(M)}|\|Y\|. 
\]

Finally let \( T_a := t_a([T/t_a]+1) \in \mathbb{Z}_{t_a} \). We have

\[
\|\int_0^T f_c(\phi^X_s(\xi_{y,z})) ds - \int_0^{T_a} f_c(\phi^X_s(\xi_{y,z})) ds\|_{L^2(\mathbb{T},dy)} \leq t_a |f_c|_{L^\infty(M)}. 
\]
We have therefore derived that, for some constant $C' > 0$ and for all $T > 0$, the following bounds hold:

$$\left| \int_0^T \left( \int_0^T f_c(\phi_{a,s}(x)) ds \right) dy \right| \leq C't_a(1 + t_a^{-1}||Y||)|c|_{\beta},$$

$$\left\| \int_0^T f_c(\phi_{a,s}(x)) ds \right\|_{L^2(T,dy)} - \left( \frac{T}{t_a} \right)^{1/2} |c|_0 \leq C't_a(1 + t_a^{-1}||Y||)|c|_{\beta}.$$

By the asymptotic property of Theorem 6.2, for all $s > 7/2$ there exists a constant $C_s > 0$ such that we have the uniform estimate

$$\left| \int_0^T f \circ \phi_{a}(x) dt - \beta^{\prime}(a,x,T) \right| \leq C_s(1 + L)|f|_{a,s}.$$

Since, by the above bounds on the function $f_c$, there exists constant $C'_s > 0$ such that

$$C't_a(1 + t_a^{-1}||Y||)|c|_{\beta} + C_st_a^{-1}(1 + L)|f_c|_{a,s} \leq C'_s(t_a + t_a^{-1})(1 + L)(1 + t_a^{-1}||Y||)^{\gamma}|c|_s,$$

we arrive at the estimates claimed in the statement. \qed

9. Analyticity of the Functionals

In this section we will prove that, for all $a = (X,Y,Z) \in DC$ and for all $T \in \mathbb{R}$, the Bufetov functionals $\beta_H(a,\cdot,T)$ are real analytic along the foliation tangent to the integrable distribution $\{Y,Z\}$. This result is crucial in deriving measure estimates for the level sets of the Bufetov functionals and for our results on decay of correlations of time changes.

By the orthogonality property, for every $T > 0$, the Bufetov cocycle belongs to a single irreducible component $H$ (with central parameter $n \in \mathbb{Z} \setminus \{0\}$), hence in particular (or from its definition), for all $(x,T) \in M \times \mathbb{R}$ and for all $t \in \mathbb{R}$,

$$\beta_H(a,\phi^T_n(x),T) = e^{2\pi innT} \beta_H(a,x,T).$$

Let $\gamma : [0,T] \to M$ a $C^1$ (or piece-wise $C^1$ parametrized path). For every $t \in \mathbb{R}$ we define

$$\gamma^T_n(s) = \phi^T_n(\gamma(s)) \quad \text{for all } s \in [0,T].$$

Lemma 9.1. The following formula holds:

$$\dot{\beta}_H(a,\gamma^T_n) = e^{2\pi innKT} \dot{\beta}_H(a,\gamma) - 2\pi inKT \int_0^T e^{2\pi innKs} \dot{\beta}_H(a,\gamma|_{[0,s]} ds.$$

Proof. Let $a = (X,Y,Z)$ and let $\alpha$ be a 1-form supported on a single irreducible component $H$. As above we have the decomposition

$$\alpha = \alpha_X \hat{X} + \alpha_Y \hat{Y} + \alpha_Z \hat{Z}.$$

Let us compute the pairing of the current $\gamma^T_n$ with the 1-form $\alpha$ on $M$. By definition the tangent vector of the path $\gamma^T_n$ is given by the formula

$$\frac{d\gamma^T_n}{ds} = D\phi^T_n \left( \frac{d\gamma}{ds} \right) + tZ \circ \gamma^T_n.$$

$$\dot{\beta}_H(a,\gamma^T_n) = e^{2\pi innKT} \dot{\beta}_H(a,\gamma) - 2\pi inKT \int_0^T e^{2\pi innKs} \dot{\beta}_H(a,\gamma|_{[0,s]} ds.$$
It follows that the pairing is given by the formula

\[
< \gamma^Z_t, \alpha > = \int_0^T [\alpha(D\phi^Z_t(s)\frac{d\gamma}{ds}(s)) + \iota_Z \alpha \circ \gamma^Z_t(s)] \, ds.
\]

Since \( Z \) belongs to the center of the Lie algebra and the coefficients \( \alpha_X, \alpha_Y \) and \( \alpha_Z \) of \( \alpha \) are eigenfunctions of the subgroup generated by \( Z \) of eigenvalue \( 2\pi nK \in 2\pi iKZ \), it follows that

\[
< \gamma^Z_t, \alpha > = \int_0^T [e^{2\pi i nKts} \alpha(\frac{d\gamma}{ds}(s)) + \iota_Z \alpha \circ \gamma^Z_t(s)] \, ds.
\]

Integration by parts gives

\[
\int_0^T e^{2\pi i nKts} \alpha(\frac{d\gamma}{ds}(s)) \, ds = e^{2\pi i nKt} \int_0^T \alpha(\frac{d\gamma}{ds}(s)) \, ds - 2\pi i nK \int_0^T e^{2\pi i nKts} \int_0^s \alpha(\frac{d\gamma}{dr}(r)) \, dr \, ds,
\]

hence we have the formula

\[
< \gamma^Z_t, \alpha > = e^{2\pi i nKt} < \gamma, \alpha > - 2\pi i nK \int_0^T e^{2\pi i nKts} < \gamma|_{[0,s]}, \alpha > \, ds + \int_0^T (\iota_Z \alpha \circ \gamma^Z_t(s)) \, ds.
\]

Since the flow \( g_{\mathbb{R}} \) is identity on the center \( Z \), it follows that

\[
\lim_{t \to +\infty} e^{-\frac{t}{2}} \int_0^T (\iota_Z (\rho_{-t})^* \alpha \circ \gamma^Z_t(s)) \, ds = 0,
\]

hence the stated formula follows by the definition of the Bufetov functional and by the linearity of the projection operators.

**Lemma 9.2.** The following formula holds:

\[
\beta_H(a, \phi^Y_t(x), T) = e^{-2\pi i nKT} \beta_H(a, x, T) + 2\pi i nK \int_0^T e^{-2\pi i nKs} \beta_H(a, x, s) \, ds.
\]

**Proof.** We have the following commutation identities:

\[
x \exp(sX) \exp(tY) = x \exp(tY) \exp(sX) \exp(tsZ).
\]

Let then \( \gamma^X_t(s) := \phi^X_t(x) \) for all \( s \in [0, T] \). By definition the symbol \( [\gamma^X_t]_T \) denotes the path given by the formula

\[
[\gamma^X_t]_T(s) := \phi^Z_t(\gamma^X_t(s)), \quad \text{for all } s \in [0, T].
\]

It then follows by the definitions that

\[
\phi^Y_t(\gamma^Y_t(x)) = [\gamma^Y_t(\phi^Y_t(x))]_T^Z.
\]
By the invariance property of the Bufetov functional and by Lemma 6.1, we have
\[ \beta_H(a, x, T) = \hat{\beta}_H(a, \phi_x^T Y (x)) = e^{2\pi inKT} \hat{\beta}_H(a, Y (\phi_x^T (x))) \]
\[ - 2\pi nKt \int_0^T e^{2\pi nKs} \hat{\beta}_H(a, Y (\phi_x^T (x)))|_{[0, s]} ds \]
\[ = e^{2\pi nKT} \beta_H(a, \phi_x^T (x), T) - 2\pi nKt \int_0^T e^{2\pi nKs} \beta_H(a, \phi_x^T (x), s) ds \]
The statement is an immediate consequence of the above formula. □

It follows from Lemma 9.2 and formula (31) that the Bufetov functional is real analytic (real and complex part are real analytic) on every leaf of the foliation tangent to the integrable distribution \{ Y, Z \}.

For any \( R > 0 \) let us introduce the analytic norm defined for all \( c \in \ell^2 \) as
\[ ||c||_{\omega, R} := \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} e^{\rho R} |c_{i,n}| \]
Let \( \Omega_R \) denote the subspace of \( c \in \ell^2 \) such that \( ||c||_{\omega, R} \) is finite.

**Lemma 9.3.** For any \( c \in \Omega_R \), any \( a \in \text{DC}(L) \) and \( T > 0 \), the functions defined as
\[ \beta_c(a, \phi_x^T \phi_z^T (x), T) , \quad \text{for all } (y, z) \in \mathbb{R} \times T , \]
extends to a holomorphic function in the domain
\[ (32) \quad D_{R,T} := \{ (y, z) \in \mathbb{C} \times \mathbb{C} | \text{Im}(y)|T + |\text{Im}(z)| < \frac{R}{2\pi K} \} . \]
The following bounds hold: for any \( R' < R \) there exists a constant \( C_{R,R'} > 0 \) such that, for all \( (y, z) \in D_{R',T} \) we have
\[ |\beta_c(a, \phi_x^T \phi_z^T (x), T)| \leq C_{R,R'} ||c||_{\omega, R} \left( L + T^{1/2} (1 + E_M(a, T)) \right) (1 + K |\text{Im}(y)|T) . \]

**Proof.** By Lemma 9.2 and formula (31), for all \( x \in M \) we have
\[ \beta_H(a, \phi_x^T \phi_z^T (x), T) = e^{2\pi i (z-y)T} nK \beta_H(a, x, T) \]
\[ + 2\pi nK e^{2\pi i nK} \int_0^T e^{-2\pi i nK s} \beta_H(a, x, s) ds . \]
As a consequence, by Lemma 6.1, for \( (y, z) \in \mathbb{C} \times \mathbb{C}/\mathbb{Z} \) we have
\[ |\beta_c(a, \phi_x^T \phi_z^T (x), T)| \leq C \left( L + T^{1/2} (1 + E_M(a, T)) \right) \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} |c_{i,n}| e^{2\pi |\text{Im}(z-y)|T} nK \]
\[ + 2\pi C \left( L + T^{1/2} (1 + E_M(a, T)) \right) K |\text{Im}(y)|T \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} n |c_{i,n}| e^{2\pi |\text{Im}(y)|T + |\text{Im}(z)|} nK . \]
It follows that the function \( \beta_H(a, \phi_x^T \phi_z^T (x), T) \) is given by a series of holomorphic functions on \( \mathbb{C} \times \mathbb{C}/\mathbb{Z} \), which converges uniformly on compact subsets of the domain \( D_{R,T} \), hence it is holomorphic there. The stated uniform bound follows immediately from the proof. □
Theorem 10.1. Let \( \eta \) be a constant \( C > 0 \) (the proof of Lemma 9.3, by Lemma 6.1 for \( c \)).

Proof. Since \( \eta \) is such that for any convex set \( D \)\
\( \beta \) function \( C \) extends to a holomorphic function in the domain \( \mathbb{C} \times \mathbb{C}/\mathbb{Z} \). In addition, there exists a constant \( C_\eta > 0 \) such that, for all \( T > 0 \) and for all \( (y, z) \in \mathbb{C} \times \mathbb{C}/\mathbb{Z} \), we have
\[
|\beta_c(a, \phi^y \phi^Z(x), T)| \leq C_\eta \left( L + T^{1/2} (1 + E_M(a, T)) \right) \times (1 + 2\pi K |\text{Im}(y)|T) \exp(\|\text{Im}(y)|T + |\text{Im}(z)|)^{2-\eta}].
\]

Lemma 9.4. For any \( c \in \Omega_\infty^{(\eta)} \), any \( a \in DC(L) \) and \( T > 0 \), the functions defined as
\[
\beta_c(a, \phi^y \phi^Z(x), T), \quad \text{for all } (y, z) \in \mathbb{R} \times \mathbb{T},
\]
extends to a holomorphic function in the domain \( \mathbb{C} \times \mathbb{C}/\mathbb{Z} \). In addition, there exists a constant \( C_\eta > 0 \) such that, for all \( T > 0 \) and for all \( (y, z) \in \mathbb{C} \times \mathbb{C}/\mathbb{Z} \), we have
\[
|\beta_c(a, \phi^y \phi^Z(x), T)| \leq C_\eta \left( L + T^{1/2} (1 + E_M(a, T)) \right) \times (1 + 2\pi K |\text{Im}(y)|T) \exp(\|\text{Im}(y)|T + |\text{Im}(z)|)^{2-\eta}].
\]

Proof. Since \( \Omega_\infty^{(\eta)} \subset \Omega_R \), for all \( R > 0 \), it follows already from Lemma 9.3 that the function \( \beta_c(a, \phi^y \phi^Z(x), T) \) extends to a holomorphic function on \( \mathbb{C} \times \mathbb{C}/\mathbb{Z} \). As in the proof of Lemma 9.3 by Lemma 6.1 for \( (y, z) \in \mathbb{C} \times \mathbb{C}/\mathbb{Z} \) we have
\[
|\beta_c(a, \phi^y \phi^Z(x), T)| \leq C' \left( L + T^{1/2} (1 + E_M(a, T)) \right) \times (1 + 2\pi K |\text{Im}(y)|T) \sum_{\ell \geq 1} n c_{\ell, n} \left( e^{2\pi K |\text{Im}(y)|T + |\text{Im}(z)|} n \right).
\]

Since by assumption \( c \in \Omega_\infty^{(\eta)} \), the stated estimates is proved.

In the Sections 11 and 12 we will use Lemmas 9.3 and 9.4 to get uniform measure estimates on sets where the Bufetov functional is small. This is possible thanks to results on the measure of small sets for analytic functions (see [BruGa], [Bru]).

10. BOUNDS ON THE VALENCE

For convenience of the reader we recall a result of A. Brudnyi on the measure of level sets of analytic functions.

For any \( r > 1 \), let \( \mathcal{O}_r \) denote the space of holomorphic functions on the ball \( B_\mathbb{C}(0, r) \subset \mathbb{C}^n \). Let \( B_{\mathbb{R}}(0, 1) := B_\mathbb{C}(0, 1) \cap \mathbb{R}^n \) denote the real euclidean unit ball.

Theorem 10.1. ([Bru], Thm. 1.9) For any \( f \in \mathcal{O}_r \), there is a constant \( d := d(f, r) > 0 \) such that for any convex set \( D \subset B_{\mathbb{R}}(0, 1) \), for any measurable subset \( \omega \subset D \)
\[
\sup_D |f| \leq \left( \frac{4n \text{Leb}(D)}{\text{Leb}(\omega)} \right)^d \sup_{\omega} |f|.
\]
The best constant $d$ in the above theorem is called the Chebyshev degree, denoted by $d_f(r)$, of the function $f \in \mathcal{O}_r$ in $B_C(0, 1)$. The Chebyshev degree can be estimated by the valency of the function. We recall the definition of the valency.

A holomorphic function $f$ defined in a disk is called $p$-valent if it assumes no value more than $p$-times there (counting multiplicities). We also say that $f$ is 0-valent if it is a constant. For any $t \in [1, r]$, let $L_t$ denote the set of one-dimensional complex affine lines such that $L_t \cap B_C(0, t) \neq \emptyset$.

**Definition 10.2.** ([Bru], Def. 1.6) Let $f \in \mathcal{O}_r$. The number
\[
 v_f(t) := \sup_{L \in L_t} \{ \text{valency of } f|L \cap B_C(0, t) \}
\]
is called the valency of $f$ in $B_C(0, t)$.

By [Bru] Prop. 1.7, for any $f \in \mathcal{O}_r$ and any $t \in [1, r)$ the valency $v_f(t)$ is finite and there is a constant $c := c(r) > 0$ such that
\[(34) \quad d_f(r) \leq cv_f\left(\frac{1+r}{2}\right).
\]

In this section we prove the following result.

**Lemma 10.3.** Let $R > r > 1$. For any normal family $\mathcal{F} \subset \mathcal{O}_R$ such that no functions in $\mathcal{F} = \emptyset$ is constant along a one-dimensional complex line, we have
\[
 \sup_{f \in \mathcal{F}} v_f(r) < +\infty.
\]

**Proof.** We argue by contradiction. If the statement does not hold, then there exists a sequence of functions $(f_k) \subset \mathcal{F}$, a sequence of affine one-dimensional subspaces $(L_k)$ and a bounded sequence of complex numbers such that
\[
 \# f_k^{-1}\{w_k\} \cap L_k \cap B_C(0, r) \to +\infty.
\]

By compactness, up to passing to subsequence we can assume that $f_k \to f$ uniformly on all compact subset of the ball $B_C(0, R)$, that $L_k \to L$, a one-dimensional affine complex line such that $L \cap B_C(0, r) \neq \emptyset$, in the Hausdorff topology, and that $w_k \to w \in \mathbb{C}$. By hypothesis $f|L$ is non-constant, hence we can assume that $f_k|L_k$ is also non-constant for all $k \in \mathbb{N}$. Since for any $r' > r$ the valency $v_f(r')$ of the function $f$ on $B_C(0, r')$ is finite we have that
\[
 \# f^{-1}\{w\} \cap L \cap B_C(0, r') < +\infty.
\]

Let $f^{-1}\{w\} = \{p_1, \ldots, p_v\} \subset L \cap B_C(0, r')$. Let $\varepsilon > 0$ be chosen so that $B_C(p_i, \varepsilon) \cap B_C(p_j, \varepsilon) = \emptyset$ and $f|\partial B_C(p_i, \varepsilon) \neq 0$ for all $i, j \in \{1, \ldots, v\}$. Since $L_k \to L$ there exists a sequence of affine holomorphic maps $A_k : \mathbb{C}^n \to \mathbb{C}^n$ such that $A_k \to \text{Id}$ uniformly on compact sets and $A_k(L) = L_k$ for all $k \in \mathbb{N}$. By uniform convergence we have that for $n \in \mathbb{N}$ sufficiently large all the solutions $z \in L \cap B_C(0, r')$ of the equation $f_k \circ A_k(z) - w_k = 0$ are contained in the union of the balls $B_C(p_i, \varepsilon) \cap L$. For all $k \in \mathbb{N}$, let $\phi_k := (f_k \circ A_k)|L$, and let $\phi = f|L$. The sequence of holomorphic
We conclude that for $k$ sufficiently large, we have that
\[
\#(f_k \circ A_k)^{-1}(w_k) \cap B_{C}(p_i, \varepsilon) \cap L = \frac{1}{2\pi i} \int_{\partial B_{C}(p_i, \varepsilon)} \frac{\phi'(z)_k}{\phi_k(z) - w_k} \, dz.
\]
By uniform convergence on compact sets it follows that
\[
\frac{1}{2\pi i} \int_{\partial B_{C}(p_i, \varepsilon)} \frac{\phi'(z)}{\phi_k(z) - w} \, dz \to \frac{1}{2\pi i} \int_{\partial B_{C}(p_i, \varepsilon)} \frac{\phi'(z)}{\phi(z) - w} \, dz,
\]
hence we have
\[
\#(f_k \circ A_k)^{-1}(w_k) \cap B_{C}(p_i, \varepsilon) \cap L \to #f^{-1}(w) \cap B_{C}(p_i, \varepsilon) \cap L.
\]
We conclude that for $k$ sufficiently large
\[
\#f_k^{-1}(w_k) \cap L \cap B_{C}(0, r) \leq #f^{-1}(w) \cap L \cap B_{C}(0, r') < +\infty.
\]
Since by assumption the LHS in the above inequality diverges, we have reached a contradiction. The argument is concluded.

In one complex dimension we prove a quantitative bound on the valency.

**Lemma 10.4.** For any $R > r > 3t > 3$, there exists a constant $C_{rt} > 0$ such that the following holds. For any non-constant holomorphic function of one complex variable $f \in O_R$, let $M_f(r)$ denote the maximum modulus of $f$ on the closed ball $B_C(0, r) \subset \mathbb{C}$ and let $O_f(t)$ its oscillation in the ball $B_C(0, t)$. The valency $v_f(t)$ of the function $f$ in the ball $B_C(0, t)$ satisfies the following estimate
\[
v_f(t) \leq C_{rt} \log \left( \frac{4M_f(r)}{O_f(t)} \right).
\]

**Proof.** Since there exists a single complex one-dimensional affine space $L \subset \mathbb{C}$, it suffices to estimate the valency of the function $f$ on $B_C(0, t)$, that is, the number of solutions $z \in B_C(0, t)$ of equations of the form $f(z) = w$.

By definition, the above equation has solutions only if $|w| \leq M_f(r)$. Let $f_w \in O_R$ denote the holomorphic function $f_w(z) = f(z) - w$. By definition, the maximum modulus of $f_w$ on the closed ball $B_C(0, r) \subset \mathbb{C}$ is at most $2M_f(r)$. Let $w \in f(B_C(0, t)$ and let $z_w \in B_C(0, t)$ be any point such that
\[
|f_w(z_w)| = |f_w(z) - w| \geq O_f(t)/2.
\]
Let $\{z_1, \ldots, z_v\} \subset B_C(z_w, 2t) \setminus \{z_w\}$ denote the (non-empty) set of zeros of the function $f_w$ in $B_C(z_w, 2t)$ listed with their multiplicities. Since $B_C(0, t) \subset B_C(z_w, 2t)$ it follows that the number of solution of the equation $f_w(z) = 0$ in $B_C(0, t)$ is at most $v \in \mathbb{N}$. Let us define
\[
g_w(z) = f_w(z_w + z)^{\prod_{k=1}^{v} (1 - \frac{z_w + z}{z_k})^{-1}}, \quad z \in B_C(0, R - t).
\]
By definition the function \( g_w \) in holomorphic in \( B_\mathcal{C}(0, R - t) \). By the maximum modulus principle
\[
\frac{1}{2} O_f(t) \leq |f_w(z_w)| = |g_w(0)| \leq \max \limits_{|z|=r-t} |g_w(z)| \leq 2M_f(r)\left(\frac{r-t}{3t} - 1\right)^{-\nu},
\]
which immediately implies, by taking logarithms,
\[
\nu \leq \log\left(\frac{r-t}{3t} - 1\right) \log \left(\frac{4M_f(r)}{O_f(t)}\right).
\]
The statement is therefore proved. \( \square \)

11. Measure estimates: The bounded-type case

Finally, we derive a bound on the valency, hence on the Chebyshev degree of the holomorphic extensions of Diophantine Bufetov functionals, uniform over compact invariant subset of the moduli space.

Lemma 11.1. Let \( L > 0 \) and let \( \mathcal{B} \subset DC(L) \) be a bounded subset. Given \( R > 0 \), for all \( c \in \Omega_R \) and \( T > 0 \), let \( \mathcal{F}(c, T) \) denote the family of real analytic functions of the variable \( y \in [0, 1) \) defined as follows:
\[
\mathcal{F}(c, T) := \{\beta_c(a, \Phi_{a,x}(\xi_{y,z}), T) | (a, x, z) \in \mathcal{B} \times \mathcal{M} \times \mathbb{T}\}.
\]
There exist \( T_B > 0 \) and \( \rho_B > 0 \), such that for every \((R, T)\) such that \( R/T \geq \rho_B \) and \( T \geq T_B \), and for all \( c \in \Omega_R \setminus \{0\} \), we have
\[
\sup \limits_{f \in \mathcal{F}(c, T)} \nu_f < +\infty.
\]

Proof. Since \( \mathcal{B} \subset \mathcal{M} \) is bounded, we get
\[
0 < t_B^{\min} = \inf \limits_{a \in \mathcal{B}} t_a \leq \sup \limits_{a \in \mathcal{B}} t_a = t_B^{\max} < +\infty.
\]
For any \( a \in \mathcal{B} \) and \( x \in \mathcal{M} \), the map \( \Phi_{a,x} : [0, 1) \times \mathbb{T} \to [0, t_a^{-1}) \times \mathbb{T} \) introduced in formula (29) extends to a complex analytic diffeomorphism \( \hat{\Phi}_{a,x} : \mathcal{C} \times \mathbb{C}/\mathbb{Z} \to \mathcal{C} \times \mathbb{C}/\mathbb{Z} \). By Lemma 9.3 it follows that the real analytic function
\[
\beta_c(a, \Phi_{a,x}(\xi_{y,z}), T), \quad (y, z) \in [0, 1) \times \mathbb{T},
\]
extends to a holomorphic function on the domain \( \hat{\Phi}_{a,x}^{-1}(D_{R,T}) \subset \mathcal{C} \times \mathbb{C}/\mathbb{Z} \). In particular, for every \( z \in \mathbb{T} \) the function
\[
\beta_c(a, \Phi_{a,x}(\xi_{y,z}), T), \quad y \in [0, 1)
\]
extends to a holomorphic function defined on a strip
\[
H_{a,x,R,T} := \{y \in \mathbb{C} | \text{Im}(y) < h_{a,x,R,T}\}.
\]
Moreover by the boundedness of the set \( \mathcal{B} \subset \mathcal{M} \) it follows that
\[
\inf \limits_{(a, x) \in \mathcal{B} \times \mathcal{M}} h_{a,x,R,T} := h_{R,T} > 0.
\]
In fact, the function \( h_{a,x,R,T} \) and its lower bound \( h_{R,T} \) can be computed explicitly from the formula (29) for the polynomial map \( \Phi_{a,x} \) and from definition of the domain \( D_{R,T} \) in formula (32). In particular, it follows that for every \( r > 1 \)
there exists \( \rho_B \gg 1 \) such that, for every \( R, T \) such that \( R/T \geq \rho_B \), then for every \((a, x, z) \in B \times M \times \mathbb{T} \) we have that, as a function of \( y \in [0, 1] \),

\[
\beta_c(a, \Phi_{a,x}(\xi_{y,z}), T) \in O_r.
\]

It then follows from Lemma\textsuperscript{9.3} that the family \( \mathcal{F}(\epsilon, T) \) is uniformly bounded and hence normal. Moreover, from Lemma \textsuperscript{8.2} it follows that for sufficiently large \( T > 0 \) no sequence from \( \mathcal{F}(\epsilon, T) \) can converge to a constant function. The statement finally follows from Lemma\textsuperscript{10.3}.

We can finally derive crucial measure estimates on Bufetov functionals.

**Lemma 11.2.** Let \( a \in DC \) be such that the forward orbit \( \{g_t(\tilde{a})\}_{t \in \mathbb{R}^+} \) is contained in a compact subset of \( \mathcal{M} \). There exist \( R > 0 \), \( T_0 > 0 \) and \( C > 0 \), \( \delta > 0 \) such that, for every \( c \in \Omega_R \setminus \{0\} \), \( T \geq T_0 \) and for every \( \epsilon > 0 \), we have

\[
\text{vol}(\{x \in M | |\beta_c(a, x, T) - \epsilon T^{1/2}| \leq \epsilon T^{1/2}\}) \leq C\epsilon^\delta.
\]

**Proof.** Let \( R > 0 \) and \( T_0 > 0 \) be chosen so that the conclusions of Lemma\textsuperscript{11.1} hold and let \( c \in \Omega_R \). By the scaling property of Bufetov functionals

\[
\beta_c(a, x, T) = (T/T_0)^{1/2} \beta_c(g_{\log(T/T_0)}(a), x, T_0).
\]

Since \( a \in DC \) and the \( g_{\mathbb{R}} \)-forward orbit \( \{g_t(\tilde{a})\}_{t \in \mathbb{R}^+} \) is contained in a compact set, there exists \( L > 0 \) such that \( g_t(a) \in DC(L) \) for all \( t > 0 \). Since the volume on \( M \) is invariant under the action \( A_T \), it is enough to estimate (uniformly over \((a, x) \in B \times M \) for any given bounded subset \( B \subseteq DC(L) \) such that \( \{g_t(a)\}_{t \in \mathbb{R}^+} \subseteq A_T \setminus B \), for any \( \epsilon > 0 \), the volume \( \text{vol}(\{x \in M | |\beta_c(a, x, T_0) - \epsilon \|T\| | \leq \epsilon \}) \). By Fubini’s theorem it is enough to estimate, uniformly over \((a, x, z) \in B \times M \times \mathbb{T} \),

\[
\text{Leb}(\{y \in [0, 1] | |\beta_c(a, \Phi_{a,x}(\xi_{y,z}), T_0) - \epsilon | \leq \epsilon \}).
\]

Let \( \delta^{-1} := c(r) \sup_{f \in \mathcal{F}(c, T_0)} V_f(1+\frac{r}{2}) < +\infty \). Since by Lemma\textsuperscript{8.2} we have

\[
\inf_{(a, x, z) \in B \times M \times \mathbb{T}} \sup_{y \in [0, 1]} |\beta_c(a, \Phi_{a,x}(\xi_{y,z}), T_0)| > 0.
\]

it follows from Theorem\textsuperscript{10.1} for \( D = B_{\mathbb{R}}(0, 1) \) and

\[
\omega := \{y \in [0, 1] | |\beta_c(a, \Phi_{a,x}(\xi_{y,z}), T_0) - \epsilon | \leq \epsilon \},
\]

and by the bound in formula \textsuperscript{34} for the Chebychev degree, that the following estimate holds: there exists a constant \( C > 0 \) such that, for all \((a, x, z) \in B \times M \times \mathbb{T} \) and for all \( \epsilon > 0 \), we have

\[
\text{Leb}(\{y \in [0, 1] | |\beta_c(a, \Phi_{a,x}(\xi_{y,z}), T_0) - \epsilon | \leq \epsilon \}) \leq C\epsilon^\delta.
\]

The statement then follows by the Fubini theorem. \( \square \)

**Corollary 11.3.** Let \( a = (X, Y, Z_0) \) be as in Lemma\textsuperscript{11.2} There exist \( R > 0 \), \( T_0 > 0 \) and \( C > 0 \), \( \delta > 0 \) such that, for every \( c \in \Omega_R \setminus \{0\} \), \( T \geq T_0 \) and for every \( \epsilon > 0 \), we have

\[
\text{vol}(\{x \in M | \int_0^T f_c(\Phi^x_t) dt | \leq \epsilon T^{1/2}\}) \leq C\epsilon^\delta.
\]
12. Measure estimates: the general case

Bufetov functionals were constructed for $a = (X, Y, Z) \in A$ under a (full measure) Diophantine condition (DC) on $\tilde{a} \in \mathcal{M}$, which depends on the backward orbit under the renormalization flow $g_\tilde{a}$ in the moduli space $\mathcal{M}$. Throughout this section we assume that $a \in DC$ satisfies an additional (full measure) Diophantine condition $DC_{log}$ (which depends on the forward orbit): $a \in DC_{log}$ if $\tilde{a} \in \mathcal{M}$ satisfies the logarithmic law of geodesics, that is, if

$$\limsup_{t \to +\infty} \frac{\delta_M(g_t(\tilde{a}))}{\log t} = 1.$$  

Lemma 12.1. Let $a \in DC \cap DC_{log}$. Let $\eta \in (0, 1)$ and let $c \in \Omega_\omega(\eta)$. For every $\delta \in (1 - \eta/2, 1)$ and for every $\zeta > 0$, there exist constants $C_{\delta, \zeta} > 0$ and $C_\zeta > 0$ such that, for every $\varepsilon > 0$, we have

$$\text{vol}\{x \in M| |\beta_c(a, x, T)| \leq \varepsilon \frac{T^{1/2}}{C_\zeta \log^{1/2 + \varepsilon} T}\} \leq 4e^{\delta_\zeta \text{log}^{\delta_\zeta}}.$$  

Proof: Let us assume $a \in DC_{log} \cap DC(L)$. By the Diophantine condition $DC_{log}$, there exists a bounded set $B \subset \mathcal{M}$ such that the following holds. For any $\zeta > 0$ there exist a constant $C_\zeta > 0$ and a sequence $(t_n)$ with $g_{t_n}(\tilde{a}) \in B$ such that

$$e^{\varepsilon_1 - t_n} \leq C_\zeta t_n^{1 + \zeta}.$$  

Let $T \gg 1$ and for all $n \in \mathbb{N}$ let $T_n = e^{-t_n}T$. By the Diophantine condition $DC_{log}$, for any $\xi > 0$ there exists a constant $C_\xi > 0$ such that $g_{t_n}(a) \in DC(L_n)$ with

$$L_n = e^{-t_n/2}L + \varepsilon_M(a, e^{t_n}) \leq e^{-t_n/2}L + C_\xi t_n^{1 + \xi}$$

$$= (T/T_n)^{-1/2}L + C_\xi \log^{1/2 + \xi} (T/T_n).$$

By the scaling property of the Bufetov functionals

$$\beta_c(a, x, T) = (T/T_n)^{1/2} \beta_c(g_{t_n}(a), x, T_n).$$

Since $B$ is a bounded, hence relatively compact set, we have

$$0 < t_{B, \min} = \inf_{a \in B} t_a \leq \sup_{a \in B} t_a = t_{B, \max} < +\infty.$$  

By Lemma 9.4 the real analytic function

$$\beta_c(g_{t_n}(a), \phi_y^T \phi_z^T(x), T_n), \quad \text{for all } (y, z) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$$

extends to a complex analytic function on the strip $D_R^{(n)} \subset \mathbb{C}^2$ defined as

$$D_R^{(n)} = \{(y, z) \in \mathbb{C} \times \mathbb{C}/\mathbb{Z}| |\text{Im}(y)| T_n + |\text{Im}(z)| \leq \frac{RT_n}{2\pi K}\},$$

and, for any $c \in \Omega_\omega(\eta)$ there exists $C_\eta > 0$ such that for any $(y, z) \in D_R^{(n)}$ we have the uniform upper bound

$$|\beta_c(g_{t_n}(a), \phi_y^T \phi_z^T(x), T_n)| \leq C_\eta \left(L_n + T_n^{1/2} (1 + E_M(g_{t_n}(a), T_n))\right) (1 + RT_n) e^{(RT_n)^{2-\eta}}.$$
In particular it follows that, for any \( z \in \mathbb{T} \), the function \( \beta_c(g_n(a), \Phi_{\gamma_n(a),z}(\xi_{y,z}), T_n) \), defined for \( y \in [0, 1] \) extends to a holomorphic function on the strip

\[
H_R = \{ y \in \mathbb{C} | |\text{Im}(y)| \leq \frac{R_{\text{min}}}{4 \pi} \},
\]

and there exists a constant \( C'_{\eta} > 0 \) such that the following uniform upper bound holds: for any \( y \in H_R \) and any \( z \in \mathbb{T} \), we have

\[
|\beta_c(g_n(a), \Phi_{\gamma_n(a),z}(\xi_{y,z}), T_n)| \leq C'_{\eta} \left( L_n + T_n^{1/2} (1 + E_M(g_n(a), T_n)) \right) (1 + RT_n) e^{(RT_n)^{2-\eta}}.
\]

By a calculation, for all \( t_n \geq 0 \) and for \( T_n = e^{-t_n} T \in [0, T] \) we have that

\[
E_M(g_n(a), T_n) \leq E_M(a, T) \leq C_{\zeta} \log^{1/4 + \frac{\zeta}{2}} T.
\]

By Lemma 8.2 it follows that, for any \( s > 7/2 \), whenever we have

\[
\left( \frac{T_n}{t_{\text{max}}} \right)^{1/2} |e|_0 \geq 10C_s (1 + L_n)|e|_s \sup_{a \in B} \{ t_n^{-1} (1 + t_n^{-1} ||Y||)^{s+1} \},
\]

then there exists a constant \( C_B > 0 \) such that

\[
\| \beta_c(g_n(a), \Phi_{\gamma_n(a),z}(\xi_{y,z}), T_n) \|_{L^2(T, dy)} \geq C_B |e|_0 T_n^{1/2};
\]

\[
| \int_T \beta_c(g_n(a), \Phi_{\gamma_n(a),z}(\xi_{y,z}), T_n) dy | \leq \frac{C_B}{4} |e|_0 T_n^{1/2}.
\]

In particular, we derive a uniform lower bound for the oscillation \( O_n(e, T) \) of the function \( \beta_c(g_n(a), \Phi_{\gamma_n(a),z}(\xi_{y,z}), T_n) \) for \( y \in [0, 1] \):

\[
O_n(e, T) \geq \frac{C_B}{2} |e|_0 T_n^{1/2}.
\]

It remains to optimize the choice of \( t_n > 0 \), hence of \( T_n \in [0, T] \), given \( T > 0 \). It follows from formulas (39) and (42) that for any \( \zeta > 0 \), there exists a constants \( L_{\zeta} > 0 \) such that we want to choose \( T_n \) to be the smallest solution of the inequality

\[
T_n \geq L_{\zeta}^2 \left( 1 + \log^{1+\frac{\zeta}{2}} (T/T_n) \right)^2.
\]

By this definition and by the condition in formula (38) we then have

\[
e^{t_n} \leq L_{\zeta} e^{t_n (1 + t_n^{1+\frac{\zeta}{2}}) / 2} \leq T \leq L_{\zeta} e^{t_{n+1} (1 + t_{n+1}^{1+\frac{\zeta}{2}}) / 2} \leq L_{\zeta} C_{\zeta} e^{t_n (1 + t_n^{1+\frac{\zeta}{2}}) / 2},
\]

which in turn implies

\[
\frac{T}{T_n} = e^{t_n} \geq (L_{\zeta} C_{\zeta})^{-1} \frac{T}{(1 + \log^{1+\frac{\zeta}{2}} T)^2}.
\]

It follows in particular that

\[
T_n \leq (L C_{\zeta}) (1 + \log^{1+\frac{\zeta}{2}} T)^2, \quad L_n \leq L + C_{\zeta} L_{\zeta}^{-1} T_n^{1/2}.
\]
With this choice there exists a constant \( C_{\eta, \zeta} > 0 \) such that
\[
|\beta_c(g_{a, a}(\xi, \zeta), T)| \leq C_{\eta, \zeta} \left( 1 + T_n^{1/2} \log^{1/4 + \zeta} T \right) \\
\times (1 + R \log^{1/2 + 2\zeta} T) \exp\left[ (LC_{\zeta}^0 \rho(1 + \log^{1/4 + \zeta} T))^{4 - 2\eta} \right].
\]
(46)

For any \( R > r > 10\pi K(\mu^{\text{min}})^{-1} \), by formulas (44) and (46), from Lemma \( \ref{lem:main} \) we derive that there exists a constant \( C > 0 \) such that, for all \( t \in (1, 3/2) \), for all fixed \( \zeta \in \mathbb{T} \), and for all sufficiently large \( T > 0 \), the valency \( \nu_\beta(t) \in \mathbb{N} \) of the function of one-complex variable \( \beta_c(g_{a, a}(\xi, \zeta), T) \) is bounded above as follows:
\[
\nu_\beta(t) \leq C \log \left( C_{\eta, \zeta}^0 \rho(1 + \log^{3/4 + 3\zeta} T) \exp\left[ (LC_{\zeta}^0 \rho(1 + \log^{1/4 + \zeta} T))^{2 - \eta} \right] \right) \\
\leq C_{\eta, \zeta}^0 \left( \log(C_{\eta, \zeta}^0 \rho) + \log \log T + \log \log \log \log T \right).
\]

By Theorem \( \ref{thm:main} \), it follows that, for all \( \delta \in (1 - \eta/2, 1) \), there exists a constant \( C_{\delta, \zeta} > 0 \) such that, for all \( x, z, n \in M \times \mathbb{Z} \times \mathbb{N} \), and for all \( \epsilon > 0 \) we have
\[
\text{Leb} \left( \{ y \in [0, 1] | |\beta_c(g_{a, a}(\xi, \zeta), T)| < \epsilon \} \right) < 4\epsilon \delta^{-\log^\delta T}.
\]
(47)
Finally, from the scaling identity (40), and from the measure estimates (47) and Fubini’s theorem, it follows that for all \( \zeta > 0 \) and for all \( \delta \in (1 - \eta/2, 1) \) there exists a constant \( C_{\zeta} > 0 \) such that
\[
\text{vol}(\{ x \in M | |\beta_c(a, x, T)| \leq \epsilon \frac{T^{1/2}}{1 + C_{\zeta} \log^{1/4 + \zeta} T} \}) \leq 4\epsilon \delta^{-\log^\delta T},
\]
as claimed in the statement.

\[ \square \]

**Corollary 12.2.** Let \( a = (X, Y, Z_0) \) be as in Lemma \( \ref{lem:main} \). Let \( \eta \in (0, 1) \) and let \( c \in \Omega_\infty^{\eta} \). For every \( \delta \in (1 - \eta/2, 1) \) and for every \( \zeta > 0 \), there exist constants \( C_{\delta, \zeta} > 0 \) and \( C_{\zeta} > 0 \) such that, for every \( \epsilon > 0 \), we have
\[
\text{vol}(\{ x \in M | |\beta_c(a, x, T)| \leq \epsilon \frac{T^{1/2}}{C_{\zeta} \log^{1/4 + \zeta} T} \}) \leq 4\epsilon \delta^{-\log^\delta T}.
\]
(48)

**13. Correlations. Proof of Theorems \[2.4\] and \[2.5\]**

We will first analyze correlations and then derive Theorems \( \ref{thm:main} \) and \( \ref{thm:main2} \) from respectively Corollaries \( \ref{cor:main} \) and \( \ref{cor:main2} \). We analyze correlations as follows. Let \( \omega_\nu \) the \( V \)-invariant volume form. It follows from the definition of \( V = \alpha X \) that \( \omega_\nu = \alpha^{-1} d\nu \). We have
\[
\langle h \circ \phi^V_\xi, g \rangle_{L^2(M, \omega_\nu)} = \langle h \circ \phi^V_\xi, \frac{g}{\alpha} \rangle_{L^2(M, d\nu)}.
\]

hence it is equivalent to analyze correlations with respect to Haar volume form \( d\nu \). As the Haar volume form is \( \mathbb{Z} \)-invariant we have
\[
\langle h \circ \phi^V_\xi, g \rangle_{L^2(M, d\nu)} = \frac{1}{S} \int_0^S \langle h \circ \phi^Z_{\xi, t}, g \circ \phi^Z_{\xi, t} \rangle_{L^2(M, d\nu)} dt\]
Integrating by parts we finally derive the formula (see the paper [FU]):

\[
\frac{1}{S} \int_0^S < h \circ \phi_t^V \circ \phi_s^Z, g \circ \phi_s^Z >_{L^2(M, d\text{vol})} ds \]

\[
= \frac{1}{S} \int_0^S h \circ \phi_t^V \circ \phi_s^Z ds, g \circ \phi_s^Z >_{L^2(M, d\text{vol})} \]

\[- \frac{1}{S} \int_0^S < h \circ \phi_t^V \circ \phi_s^Z dr, Z g \circ \phi_s^Z >_{L^2(M, d\text{vol})} ds. \]

We have thus written correlations in terms of integrals

\[
\int_0^S h \circ \phi_t^V \circ \phi_s^Z ds. \]

Let \( D_t \) denote the function on \( M \) defined as

\[(49)\]

\[D_t(x) := \int_0^t \frac{Z \alpha}{\alpha} \circ \phi_t^V d\tau.\]

We then have the formula

\[
\int_0^S h \circ \phi_t^V \circ \phi_s^Z ds = \int_0^S \frac{1 + D_t^2 \circ \phi_s^Z}{1 + D_t^2 \circ \phi_s^Z} h \circ \phi_t^V \circ \phi_s^Z ds \]

\[- \int_0^S \frac{1 + D_t^2 \circ \phi_s^Z}{1 + D_t^2 \circ \phi_s^Z} h \circ \phi_t^V \circ \phi_s^Z ds. \]

We also have

\[
\int_0^S \frac{D_t \circ \phi_s^Z}{1 + D_t^2 \circ \phi_s^Z} h \circ \phi_t^V \circ \phi_s^Z ds = \int_0^S \frac{D_t \circ \phi_s^Z}{1 + D_t^2 \circ \phi_s^Z} [(h \circ \phi_t^V \circ \phi_s^Z)(D_t \circ \phi_s^Z)] ds \]

\[- \int_0^S \frac{D_t \circ \phi_s^Z}{1 + D_t^2 \circ \phi_s^Z} [(h \circ \phi_t^V \circ \phi_s^Z)(D_t \circ \phi_s^Z)] ds \]

For every \( \sigma > 0 \), let \( \gamma_{s,t}^\sigma \) be the path defined as

\[
\gamma_{s,t}^\sigma(s) = (\phi_t^V \circ \phi_s^Z)(x), \quad \text{for all } s \in [0, \sigma]. \]

We have computed above that

\[
\frac{d \gamma_{s,t}^\sigma}{ds}(s) = [D_t \circ \phi_s^Z(x)]V(\gamma_{s,t}^\sigma(s)) + Z(\gamma_{s,t}^\sigma(s)). \]

It follows that

\[
\int_{K_t} h \hat{V} = \int_0^\sigma (h \circ \phi_t^V \circ \phi_s^Z)(x) (D_t \circ \phi_s^Z)(x) ds. \]
In other terms, we have the following identity:

\[
\int_0^S \frac{D_t^2 \circ \phi_s^Z(x)}{1 + D_t^2 \circ \phi_s^Z(x)} h \circ \phi_s^V \circ \phi_s^Z(x) ds = \frac{D_t \circ \phi_s^Z(x)}{1 + D_t^2 \circ \phi_s^Z(x)} \int_{\mathcal{E}_t^s} h\hat{V} ds,
\]

\[
- \int_0^S \frac{d}{ds} \left[ \frac{D_t \circ \phi_s^Z(x)}{1 + D_t^2 \circ \phi_s^Z(x)} \right] \int_{\mathcal{E}_t^s} h\hat{V} ds.
\]

It remains to estimate the term

\[
\frac{d}{ds} \left[ \frac{D_t \circ \phi_s^Z(x)}{1 + D_t^2 \circ \phi_s^Z(x)} \right] = \left[ \frac{d}{ds} D_t \circ \phi_s^Z(x) \right] \left[ 1 - \frac{D_t^2 \circ \phi_s^Z(x)}{(1 + D_t^2 \circ \phi_s^Z(x))^2} \right].
\]

Our estimate is thus reduced to a bound on the term

\[
\frac{d}{ds} D_t \circ \phi_s^Z(x) = \int_0^t \left\{ D_t [V \left( \frac{Z\alpha}{\alpha} \right) \circ \phi_s^V] + Z \left( \frac{Z\alpha}{\alpha} \right) \circ \phi_s^V \right\} \circ \phi_s^Z(x) d\tau
\]

\[
= [D_t \left( \frac{Z\alpha}{\alpha} \right) \circ \phi_s^V] \circ \phi_s^Z(x) + \int_0^t \left[ Z \left( \frac{Z\alpha}{\alpha} \right) - \left( \frac{Z\alpha}{\alpha} \right) \right] \circ \phi_s^V \circ \phi_s^Z(x) d\tau
\]

\[
= [D_t \left( \frac{Z\alpha}{\alpha} \right) \circ \phi_s^V] \circ \phi_s^Z(x) + \int_0^t \left( \frac{Z\alpha}{\alpha^2} \right) \circ \phi_s^V \circ \phi_s^Z(x) d\tau
\]

In particular we conclude that there exists a constant \( C_\alpha > 0 \) such that

\[
\left| \frac{d}{ds} D_t \circ \phi_s^Z(x) \right| \leq C_\alpha (1 + t^{1/2}).
\]

Since the arc \( \gamma_{\epsilon,t}^\alpha \) is smooth and contained in the weak stable leaf, it follows from Lemma \[5.1\] and by the Hölder property (or from the scaling property) that, for all \( s > 7/2 \) there exists a constant \( C_s > 0 \) such that, for all \( t > 0 \), we have

\[
\left| \int_{\mathcal{E}_t^s} h\hat{V} \right| \leq C_s |h|_s \left( \int_{\mathcal{E}_t^s} |\hat{V}| \right)^{1/2} \leq C_s |h|_s (1 + \sigma^{1/2} t^{1/4}),
\]

For a given \( t > 1 \) and \( \epsilon > 0 \), let now consider the set

\[
M_\alpha(t, \epsilon) := \{ x \in M : |D_t(x)| \geq \epsilon t^{1/2} \}.
\]

There exists a constant \( C_\alpha > 0 \) such that

\[
\phi_s^Z(M_\alpha(t, 2\epsilon)) \subset M_\alpha(t, \epsilon), \quad \text{for all } s \in \left( -\epsilon \frac{\epsilon}{C_\alpha}, \epsilon \frac{\epsilon}{C_\alpha} \right).
\]

**Proof of Theorems \[2.4\] and \[2.5\]** By Corollary \[11.3\] for \( f = \frac{Z\alpha}{\alpha} \in \Omega \) there exist constants \( C_{a,\alpha} > 0 \) and \( \delta_{a,\alpha} > 0 \) such that we have

\[
\text{vol} (M \setminus M_\alpha(t, \epsilon)) \leq C_{a,\alpha} t^{1/4} \epsilon^\delta_{a,\alpha}.
\]

It follows that correlations on the set \( M_\alpha(t, 2\epsilon) \) can be estimated by the expression

\[
(1 + t^{3/4}) \parallel \frac{1}{1 + D_t^2} \parallel_{L^1(M_\alpha(t, \epsilon))} \leq \frac{(1 + t^{3/4})}{1 + \epsilon^2 t},
\]
while the correlation on the complementary set $M \setminus M_{\alpha}(t, 2\varepsilon)$ can be estimated simply as follows: there exists a constant $C'_{\alpha, \varepsilon} > 0$ such that
\[
\langle h \circ \phi^t, g \rangle_{L^2(M \setminus M_{\alpha}(t, 2\varepsilon))} \leq C'_{\alpha, \varepsilon} e^{\delta_h \alpha} \|h\|_{\|g\|_0 + \|Zg\|_0}.
\]
By optimizing the above estimates we derive a bound for the decay of correlations of the following form: for every $h, g \in W^s(M) \cap L^2_0(M)$ and for all $t > 0$, we have
\[
\langle h \circ \phi^t, g \rangle_{L^2(M)} \leq C_{\alpha, \varepsilon}(1 + t)^{-h \alpha \frac{\log(1 + t)}{1 + \log(1 + t)}} \|h\|_s (\|g\|_0 + \|Zg\|_0).
\]
This finishes the proof of Theorem 2.4. For Theorem 2.5, by an analogous reasoning based on Corollary 12.2 we get that for a generic set of time-changes we have the following bound on correlations. For every $\delta > 1/2$, there exists a constant $C_{\alpha, \varepsilon, \delta} > 0$ such that, for all $h, g \in W^s(M) \cap L^2_0(M)$ and for all $t \in \mathbb{R}$, we have
\[
\langle h \circ \phi^t, g \rangle_{L^2(M)} \leq C_{\alpha, \varepsilon, \delta}(1 + |t|)^{-h \alpha \frac{\log(1 + |t|)}{1 + \log(1 + |t|)}} \|h\|_s (\|g\|_0 + \|Zg\|_0).
\]
This finishes the proof of Theorem 2.5. □

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