High-fidelity universal quantum gates through quantum interference

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Abstract. - Twisted rapid passage is a type of non-adiabatic rapid passage that gives rise to controllable quantum interference effects that were first observed experimentally in 2003. We show that twisted rapid passage sweeps can be used to implement a universal set of quantum gates that operate with high-fidelity. For each gate in the universal set, sweep parameter values are provided which simulations indicate will yield a quantum gate with error probability $P_e < 10^{-4}$. Note that all gates in this universal set are driven by a single type of control field (twisted rapid passage), and the error probability for each gate falls below the rough-and-ready estimate for the accuracy threshold $P_a \sim 10^{-4}$. The simulations suggest that the universal gate set produced by twisted rapid passage shows promise for use in a fault-tolerant scheme for quantum computing.

Introduction. – The physical context for our discussion is the accuracy threshold theorem [1–8] which established that a quantum computation of arbitrary duration could be done, with arbitrarily small error probability, in the presence of noise, and using imperfect quantum gates, under the following conditions. (1) The computational data is protected by a sufficiently layered concatenated quantum error correcting code. (2) Fault-tolerant protocols for quantum computation, error correction, and measurement are used. (3) A universal set of unencoded quantum gates is available with the property that each gate in the set has an error probability $P_e$ that falls below a value $P_a$ known as the accuracy threshold. The value of the threshold is model-dependent, though for many, $P_a \sim 10^{-4}$ has become a rough-and-ready estimate. Thus gates are anticipated to be approaching the accuracies needed for fault-tolerant quantum computing when $P_e < 10^{-4}$. One of the principal challenges facing the field of quantum computing is finding a way to implement a universal set of unencoded quantum gates for which all gate error probabilities satisfy $P_e < 10^{-4}$.

In this Letter numerical simulation results are presented which suggest that a class of non-adiabatic rapid passage sweeps, first realized experimentally in 1991 [9], and known as twisted rapid passage (TRP), should be capable of implementing a universal set of unencoded quantum gates $G_u$ that operate non-adiabatically, and with gate error probabilities satisfying $P_e < 10^{-4}$. $G_u$ consists of the one-qubit Hadamard and NOT gates, together with variants of the one-qubit $\pi/8$ and phase gates, and the two-qubit controlled-phase gate. The universality of $G_u$ was demonstrated in Ref. [10]. This level of gate accuracy is due to controllable quantum interference effects that arise during a TRP sweep [11, 12], and which were observed using NMR in 2003 [13]. To find sweep parameter values that yield such high-performance gates, it proved necessary: (i) to combine the simulations with an optimization procedure that searches for minima of $P_e$ [10,12]; and (ii) for the modified controlled-phase gate, to also apply the symmetrized evolution of Ref. [14].

The outline of this Letter is as follows. We begin with a summary of the essential properties of TRP. This is followed by a discussion of how the simulation and optimization are done, and how symmetrized evolution is incorporated into the two-qubit dynamics. We then present our simulation results for the different gates in $G_u$. We close with a discussion of our results and of future work.

Preliminaries. – To introduce TRP [11,12], we consider a single qubit interacting with an external control
field $\mathbf{F}(t)$ via the Zeeman interaction $H_z(t) = -\mathbf{σ} \cdot \mathbf{F}(t)$, where the $\mathbf{σ}$ are the Pauli matrices ($i = x, y, z$). TRP is a generalization of adiabatic rapid passage (ARP). In ARP, the field $\mathbf{F}(t)$ is slowly inverted over a time $T_0$ such that $\mathbf{F}(t) = a\mathbf{z} + b\mathbf{x}$. In TRP, the control field is allowed to twist in the $x$-$y$ plane with time-varying azimuthal angle $\phi(t)$, while simultaneously undergoing inversion along the $z$-axis:

$$\mathbf{F}(t) = a\mathbf{z} + b \cos(\phi(t))\mathbf{x} + b \sin(\phi(t))\mathbf{y}.$$  \hfill (1)

Here $-T_0/2 \leq t \leq T_0/2$, and the TRP inversion can be non-adiabatic.

**Controllable Quantum Interference.** As shown in Ref. [12], the qubit undergoes resonance when $at = -h\phi(t)/2 = 0$. For polynomial twist with $\phi_n(t) = (2/n)Bt^n$, this condition has $n - 1$ roots, though only real-valued roots correspond to resonance. Ref. [11] showed that for $n \geq 3$, the qubit undergoes resonance multiple times during a single TRP sweep: (i) for all $n \geq 3$, when $B > 0$; and (ii) for odd $n \geq 3$, when $B < 0$. For the remainder of this Letter we restrict ourselves to $B > 0$, and to quartic twist for which $n = 4$ in $\phi_n(t)$. For quartic twist, the qubit passes through resonance at time $t = 0, \pm \sqrt{a/hB}$ [11]. Thus the time separating the qubit resonances can be controlled through variation of the sweep parameters $B$ and $a$. Ref. [11] showed that these multiple resonances have a strong influence on the qubit transition probability, allowing transitions to be strongly enhanced or suppressed through a small variation of the sweep parameters. Ref. [15] calculated the qubit transition amplitude to all orders in the non-adiabatic coupling. The result found there can be re-expressed as the following diagrammatic series:

$$T_-(t) = \ldots + \ldots + \ldots.$$  \hfill (2)

Lower (upper) lines correspond to propagation in the negative (positive) energy level, and the vertical lines correspond to transitions between the two energy levels. The calculation sums the probability amplitudes for all interfering alternatives that allow the qubit to end up in the positive energy level at time $t$ given that it was initially in the negative energy level. As we have seen, varying the TRP sweep parameters varies the time separating the resonances. This in turn changes the value of each diagram in eq. (2), and thus alters the interference between alternatives in the quantum superposition. It is the sensitivity of the individual alternatives/diagrams to the time separation of the resonances that allows TRP to manipulate this quantum interference. Zwanziger et al. [13] observed these interference effects in the transition probability using NMR and found quantitative agreement between theory and experiment. It is the link between the TRP sweep parameters and this quantum interference that we believe makes it possible for TRP to drive highly accurate non-adiabatic one- and two-qubit gates.

**Simulation and Optimization.** A detailed presentation of our simulation and optimization protocols appears in Refs. [10, 12]. We can only give a brief sketch of that presentation here. As is well-known, the Schrodinger dynamics applies a unitary transformation $U$ to an initial quantum state $|\psi\rangle$ which is driven by the system Hamiltonian $H(t)$. The Hamiltonian (see below) is assumed to contain terms that Zeeman-couple each qubit to the TRP control field $\mathbf{F}(t)$. Assigning values to the TRP sweep parameters $(a, b, B, T_0)$ determines $H(t)$, which then determines the actual unitary transformation $U_a$ applied. The task is to find sweep parameter values that produce a $U_a$ that approximates a target gate $U_t$ sufficiently closely that its error probability (defined below) satisfies $P_e < 10^{-4}$.

In the following, the target gate $U_t$ will be one of the gates in the universal set $G_a$. Since $G_a$ contains only one- and two-qubit gates, our simulations will only involve one- and two-qubit systems. For the one-qubit simulations, the Hamiltonian $H_1(t)$ is the Zeeman Hamiltonian introduced earlier. Ref. [10] showed that it can be written in the following dimensionless form:

$$\mathcal{H}_1(\tau) = (1/\lambda) \{-\tau\sigma_z - \cos(4\tau)\sigma_x - \sin(4\tau)\sigma_y\}.$$  \hfill (3)

Here: $\tau = (a/b)t$; $\lambda = h\gamma_2/b^2$; and for quartic twist, $\phi_4(\tau) = (\eta_4/2\lambda)\tau^4$ with $\eta_4 = hbb^2/a^3$. For the two-qubit simulations, the Hamiltonian $H_2(t)$ contains terms that Zeeman-couple each qubit to the TRP control field, and an Ising interaction that couples the two qubits. Note that alternative two-qubit interactions can easily be considered, though we focus on the Ising interaction here. The energy-levels for the resulting Hamiltonian contain a resonance-frequency degeneracy that was found to spoil gate performance. Specifically, the resonance-frequency for transitions between the ground- and first-excited states ($E_1 \leftrightarrow E_2$) is the same as that for transitions between the second- and third-excited states ($E_3 \leftrightarrow E_4$). To remove this degeneracy a term $c_4|E_4(\tau)/E_4(\tau)|$ was added to $H_2(t)$. Combining all these remarks, one arrives at the following dimensionless two-qubit Hamiltonian [10]:

$$\mathcal{H}_2(\tau) =
\begin{align*}
&\left[-(d_1 + d_2)/2 + \tau/\lambda\sigma_z^1 - (d_3/\lambda)\cos(4\tau)\sigma_x^1 + \sin(4\tau)\sigma_y^1\right] \\
&+ \left[ -(d_2 - d_3)/2 + \tau/\lambda\sigma_z^2 - (1/\lambda)\cos(4\tau)\sigma_x^2 + \sin(4\tau)\sigma_y^2\right] \\
&+ (\pi d_4/2)\sigma_z^1\sigma_z^2 + c_4|E_4(\tau)/E_4(\tau)|.
\end{align*}$$  \hfill (4)

Here: (1) $b_i = h\gamma_i B_i/t$, $\omega_i = \gamma_i B_0$, and $i = 1, 2$; (2) $\tau = (a/b)t$, $\lambda = h\gamma_2/b^2$, and $\eta_4 = hbb^2/a^3$; and (3) $d_1 = (\omega_1 - \omega_2)b_2/a$, $d_2 = (\Delta/a)b_2$, $d_3 = b_1/b_2$, and $d_4 = (J/a)b_2$, where $\Delta$ is a detuning parameter [10].

The numerical simulation assigns values to the TRP sweep parameters and then integrates the Schrodinger equation to obtain the unitary transformation $U_a$ produced by the sweep. To assess how closely $U_a$ approximates the target gate $U_t$, it proves useful to introduce the positive operator $P = (U_a^\dagger - U_t^\dagger)(U_a - U_t)$. Given $U_a$, $U_t$, and an initial state $|\psi\rangle$, one can work out the error
probability $P_e(\psi)$ for the TRP final state $|\psi_n\rangle = U_n|\psi\rangle$, relative to the target final state $|\psi_t\rangle = U_t|\psi\rangle$. The gate error probability $P_e$ is defined to be the worst-case value of $P_e(\psi): P_e \equiv \max_{\psi} P_e(\psi)$. Ref. [12] showed that $P_e$ satisfies the bound $P_e \leq TrP$, where the RHS is the trace of the positive operator $P$ introduced above. Once $U_n$ is known, $TrP$ is easily evaluated, and so it makes a convenient proxy for $P_e$, which is harder to calculate. To find TRP sweep parameter values that yield highly accurate non-adiabatic quantum gates, it proved necessary to combine the numerical simulations with function minimization algorithms that search for sweep parameter values that minimize the $TrP$ upper bound [16]. The multidimensional downhill simplex method was used for the one-qubit gates, while simulated annealing was used for the two-qubit modified controlled-phase gate. This produced the one-qubit gate results that will be presented below. However, for the modified controlled-phase gate, simulated annealing was only able to find parameter values that gave $P_e \leq 1.27 \times 10^{-3}$ [10]. To further improve the performance of this two-qubit gate, it proved necessary to incorporate the symmetrized evolution of Ref. [14] to obtain a modified controlled-phase gate with $P_e < 10^{-4}$. We now briefly describe how symmetrized evolution is incorporated into our simulations.

**Symmetrized Evolution and TRP.** Ref. [14] introduced a unitary group-symmetrization procedure that yields an effective dynamics that is invariant under the action of a finite group $G$. We incorporate this group-symmetrization into a TRP sweep by identifying the group $G$ with a finite symmetry group of the target gate $U_t$, and then applying the procedure of Ref. [14] to filter out the $G$-noninvariant part of the TRP dynamics. As the $G$-noninvariant dynamics is manifestly bad dynamics relative to $U_t$, group-symmetrized TRP yields a better approximation to $U_t$. We briefly describe the group-symmetrization procedure, and then show how it can be incorporated into a TRP sweep.

Consider a quantum system $Q$ with time-independent Hamiltonian $H$ and Hilbert space $\mathcal{H}$. The problem is to provide $Q$ with an effective dynamics that is invariant under a finite group $G$, even when $H$ itself is not $G$-invariant. This symmetrized dynamics manifests as a $G$-invariant effective propagator $\tilde{U}$ that evolves the system state over a time $t$. Let $\{\rho_i = \rho(\eta_i)\}$ be a unitary representation of $G$ on $\mathcal{H}$, and let $[G]$ denote the order of $G$. The procedure begins by partitioning the time-interval $(0,t)$ into $N$ subintervals of duration $\Delta t_N = t/N$, and then further partitioning each subinterval into $[G]$ smaller intervals of duration $\delta t_N = \Delta t_N/[G]$. Let $\delta U_N = \exp[-(i/\hbar)\delta t_N H]$ denote the $H$-generated propagator for a time-interval $\delta t_N$, and assume that the time to apply each $\rho_i \in G$ is negligible compared to $\delta t_N$ (bang-bang limit [17]). In each subinterval, the following sequence of transformations is applied: $U(\Delta t_N) = \prod_{i=1}^{[G]} \rho_i^t \delta U_N \rho_t$. Ref. [14] showed that: (i) $U(\Delta t_N) \rightarrow \exp[-(i/\hbar)\Delta t_N \tilde{H}]$ as $N \rightarrow \infty$, where

$$\tilde{H} = (1/|G|) \sum_{i=1}^{[G]} \rho_i^{-1} H \rho_i;$$

(ii) $\tilde{H}$ is $G$-invariant ($[\tilde{H}, \rho_i] = 0$ for all $\rho_i \in G$); and (iii) the propagator $\tilde{U}$ over $(0,t)$ is $\tilde{U} = \exp[-(i/\hbar)t \tilde{H}]$, which is $G$-invariant due to the $G$-invariance of $\tilde{H}$. The end result is an effective propagator $\tilde{U}$ that is $G$-invariant as desired.

This procedure can be generalized to allow for a time-dependent Hamiltonian $H(t)$. To do this, the time interval $(0,t)$ must be divided into sufficiently small subintervals that $H(t)$ is effectively constant in each. Within each subinterval, the above time-independent argument is applied, yielding a $G$-symmetrized propagator for that subinterval. Combining the effective propagators for each of the subintervals then gives the full propagator $\tilde{U} = \tilde{U}[\exp[-(i/\hbar)\int_0^t \tilde{H}(\tau)d\tau]]$, where $\tilde{U}$ denotes a time-ordered exponential, and $\tilde{H}(t) = (1/|G|) \sum_{i=1}^{[G]} \rho_i t H(t) \rho_i$.

For our two-qubit simulations, the target gate is the modified controlled-phase gate $V_{cp} = (1/2)[(1 + \sigma_z^2)/2 - (1 - \sigma_z^2)\sigma_z^2]$ which is invariant under the group $G = \{1, \sigma_z^2\}$. Thus $|G| = 4$, and we set $\rho_1 = 1, \rho_2 = \bar{\sigma}_x, \rho_4 = \sigma_z^2$. Switching over to dimensionless time, we partition the sweep time-interval $(-\tau_0/2, \tau_0/2)$ into sufficiently small subintervals that our two-qubit Hamiltonian $H_2(\tau)$ is effectively constant within each. We then apply the time-independent symmetrization procedure to each subinterval with the $V_{cp}$ symmetry group acting as $G$. Combining the effective propagators for each of the subintervals as above gives the $G$-symmetrized propagator for the full TRP sweep $\tilde{U} = \tilde{U}[\exp[-(i/\hbar)\int_{-\tau_0/2}^{\tau_0/2} \tilde{H}(\tau)d\tau]]$, with $\tilde{H}(\tau) = (1/4) \sum_{i=1}^{[G]} \rho_i^t H_2(\tau) \rho_i$. We shall see that $G$-symmetrized TRP yields an approximation to $V_{cp}$ with $P_e < 10^{-4}$.

**Simulation Results.**

**One-qubit gates.** Operator expressions for the target gates are: (i) Hadamard—$U_0 = (1/\sqrt{2})(\sigma_x + \sigma_z)$; (ii) NOT—$U_{not} = \sigma_x$; (iii) modified phase—$V_{cp} = (1/\sqrt{2})(\sigma_x - \sigma_y)$. The gate fidelity is calculated using $F_n = (1/2^n)\text{Re}[Tr(U_n^\dagger U)]$, where $n$ is the number of qubits acted on by the gate. This fidelity is especially convenient as it is related to our $TrP$ upper bound: $F_n = 1 - (1/2^{n+1})TrP$ [10]. Finally, the connection between the TRP experimental and theoretical parameters is given in Refs. [10], [11], and [12] for superconducting, NMR, and atom-based qubits, respectively.

A study of the TRP-implementation of these one-qubit gates was first reported in Ref. [12]. It proves useful to reparameterize the TRP sweep parameters $\lambda \rightarrow \lambda^*$ and $\eta_i \rightarrow \eta_i$, where $\lambda = \lambda^* \exp[-(\lambda^* - \lambda^0)/\lambda^0]$ and $\eta_i = \eta_i^0 \exp[-(\eta_i^0 - \eta_i^0)/\lambda^0]$. For each one-qubit gate in $G_u$, the fixed-point of the reparameterization $(\lambda^0, \eta_i^0)$ is given by the optimum sweep parameter values found in Ref. [12]. Table 1 presents the values for the optimum sweep parameters $(\lambda^*, \eta_i^0)$ that produced our best results for $TrP$ for each of the gates in $G_u$. The fixed-point $(\lambda^0, \eta_i^0)$ for each gate is also listed. In all one-qubit simulations, the dimen-
Table 1: Simulation results for the one-qubit gates in $G_a$. The gate error probability $P_e$ satisfies $P_e \leq \text{Tr} P$.

| Gate | $\lambda^*$ | $\eta^*_h$ | $\eta^*_U$ | $\eta^*_P$ |
|------|------------|-----------|-----------|-----------|
| $U_h$ | 5.85      | 2.93 x 10^{-4} | 9.30 x 10^{-6} | 5.8511 |
| $U_{not}$ | 7.32      | 2.93 x 10^{-4} | 1.12 x 10^{-5} | 7.3205 |
| $U_{\pi/8}$ | 6.02      | 8.15 x 10^{-4} | 3.55 x 10^{-5} | 6.0150 |
| $U_p$ | 5.98      | 3.81 x 10^{-4} | 8.70 x 10^{-5} | 5.9750 |

Table 2: Variation of $\text{Tr} P$ for the Hadamard gate when the TRP sweep parameters are altered slightly from their optimum values. Variation of $\text{Tr} P$ for the other one-qubit gates in $G_a$ is comparable to that of the Hadamard gate and so corresponding Tables for these other gates are not shown.

| $\lambda^*$ | $\eta^*_h$ | $\eta^*_U$ | $\eta^*_P$ |
|------------|-----------|-----------|-----------|
| 5.85      | 2.92 x 10^{-4} | 2.24 x 10^{-6} | 2.93 x 10^{-4} |
| 2.93 x 10^{-4} | 9.30 x 10^{-6} | 5.85      | 9.30 x 10^{-6} |
| 2.94 x 10^{-4} | 6.06 x 10^{-5} | 5.86      | 1.12 x 10^{-5} |

The reparametrization fixed-point to be $c_4^0 = 2.173$, $d_4^0 = 0.8347$. The (optimized) parameter values $\lambda = 5.04$, $\eta_4 = 3.0 \times 10^{-4}$, $\tau_0 = 120$, $d_1 = 99.3$, $d_2 = 0.0$, $d_3 = -0.41$, $d_4 = 0.835$, and $c_4^* = 2.17$ produced a gate $U_{\lambda}$ for which $\text{Tr} P = 8.87 \times 10^{-5}$, gate fidelity $F_{cp} = 0.999989$, and gate error probability satisfying $P_e \leq 8.87 \times 10^{-5}$. We see that simulating symmetrization evolution to a TRP sweep we obtain an approximation to $V_{cp}$ with $P_e \leq 10^{-4}$. In Table 3 we show how $\text{Tr} P$ varies when the parameters $c_4$ and $d_4$ vary slightly from their optimum values. Sensitivities of gate performance to the remaining parameters is comparable to that of $c_4^*$ and $d_4^*$ and so corresponding tables are not shown. We see that gate performance is a slowly-varying function of the parameters $(\lambda^*, \eta^*).

 Modified controlled-phase gate. We complete the universal set $G_a$ by presenting our simulation results for the $G$-symmetrized TRP implementation of the modified controlled-phase gate $V_{\text{cp}}$. In the two-qubit computational basis (eigenstates of $\sigma_z^1\sigma_z^2$), $V_{\text{cp}} = \text{diag}(1, 1, -1, 1)$. TRP implementation of $V_{\text{cp}}$ without symmetrized evolution was reported in Ref. [10]. The results presented there are superceded by the $G$-symmetrized TRP results presented below. For purposes of later discussion, note that the parameters appearing in $H_2(\tau)$ fall into two sets. The first consists of the TRP sweep parameters $(\lambda, \eta_4, \tau_0)$, while the second set $(c_4, d_1, \ldots, d_4)$ consists of parameters for degeneracy-breaking, detuning, and coupling. We partitioned the TRP sweep into $N_{\text{seq}} = 2500$ pulse sequences, with each sequence based on the four element symmetry group for $V_{\text{cp}}$ introduced earlier.

For the modified controlled-phase gate $V_{\text{cp}}$, gate performance was not found to be very sensitive to small variations of the TRP sweep parameters. Instead, for $V_{\text{cp}}$ without symmetrized evolution [10], gate performance was most sensitive to $c_4$, $d_1$, and $d_4$. However, when symmetrized evolution was incorporated into the TRP sweep, $d_1$ ceased to be a critical parameter. Thus it only proved necessary to reparameterize $c_4 \rightarrow c_4^*$ and $d_4 \rightarrow d_4^*$, where $c_4 = c_4^* \exp[-(c_4^* - c_4^0)/c_4^0]$, and $d_4 = d_4^* \exp[-(d_4^* - d_4^0)/d_4^0]$. Simulations incorporating symmetrized evolution determined the reparameterization fixed-point to be $c_4^0 = 2.173$, $d_4^0 = 0.8347$. The (optimized) parameter values $\lambda = 5.04$, $\eta_4 = 3.0 \times 10^{-4}$, $\tau_0 = 120$, $d_1 = 99.3$, $d_2 = 0.0$, $d_3 = -0.41$, $d_4 = 0.835$, and $c_4^* = 2.17$ produced a gate $U_{\lambda}$ for which $\text{Tr} P = 8.87 \times 10^{-5}$, gate fidelity $F_{cp} = 0.999989$, and gate error probability satisfying $P_e \leq 8.87 \times 10^{-5}$. We see that by adding symmetrized evolution to a TRP sweep we obtain an approximation to $V_{cp}$ with $P_e \leq 10^{-4}$. In Table 3 we show how $\text{Tr} P$ varies when the parameters $c_4$ and $d_4$ vary slightly from their optimum values. Sensitivities of gate performance to the remaining parameters is comparable to that of $c_4^*$ and $d_4^*$ and so corresponding tables are not shown. We see that gate performance is a slowly-varying function of the parameters $c_4^*$ and $d_4^*$, as well as of $(\lambda, \eta_4, \tau_0)$ and $(d_1, d_2, d_3)$.

Discussion. We have presented simulation results which suggest that TRP sweeps should be capable of implementing a universal set of quantum gates $G_a$ that operate non-adiabatically and with gate error probability satisfying $P_e \leq 10^{-4}$. We note that all gates in the universal set $G_a$ are driven by a single type of control field (TRP), and that the gate error probability for all gates in $G_a$ falls below the rough-and-ready estimate for the accuracy threshold $P_a \sim 10^{-4}$. The simulation results presented in this Letter suggest that the universal quantum gate set $G_a$ produced by TRP shows promise for use in a fault-tolerant scheme for quantum computing. Refs. [10–12] have shown how TRP sweeps can be applied to NMR, atomic, and superconducting qubits. It should also be possible to apply them to spin-based qubits in quantum dots using a magnetic field since the same Zeeman-coupling acts as with NMR qubits. Although we have studied a number of forms of polynomial twist, as well as periodic twist [18], we have found that quartic twist provides best all-around performance when it comes to making the gates in $G_a$. At present we do not have arguments that explain why quartic twist works better than other forms of twist. We are currently working to develop a theory of the optimum...
Table 3: Variation of $Tr P$ for the modified controlled-phase gate when $c_4^*$ and $d_4^*$ are altered slightly from their optimum values.

| $c_4^*$ | $d_4^*$ | $Tr P$ | $c_4^*$ | $d_4^*$ | $Tr P$ |
|---------|---------|--------|---------|---------|--------|
| 2.17    | 0.834   | $8.77 \times 10^{-3}$ | 0.835   | 2.16    | $8.15 \times 10^{-3}$ |
| 0.835   | 8.77 $\times 10^{-5}$ | 2.17    | 8.77 $\times 10^{-5}$ | 0.836   | 8.44 $\times 10^{-5}$ |

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