Non-linear description of hardening zone of steel

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Abstract. The most well-known classical descriptions of the steel’s hardening area are the non-linear descriptions by Nadai and Ludwik. However, all these approachings have a large relative error (up to 35–45 %) with respect to the experimental hardening curves obtained on the modern universal tensile machines. In addition, the significant drawback of the Nadai’s and Ludwik’s approachings is the infinite derivative (the tangent of inclination angle) of the stress curve at the beginning of the stretching of the rod (ε = 0). It contradicts the classical Hooke’s law for the small elastic deformations and is not observed in none of the metals in practice. Below we propose other method of the direct and inverse non-linear descriptions using the finite or infinite power series with the displacement of the relative deformation. It is shown that this method is much more accurate than the classical description methods.

1. Introduction

The plastic deformation of steels in the hardening area is widely used in the production of metal products from the steel rods, sheets and pipes [1–23].

Let the long rod with the initial cross-section $A_0$ and the initial length $l_0$ be stretched by the longitudinal force $F$. Let $l$ and $\Delta l$ be the length and absolute lengthening of the rod under tension.

The dependence of the stress on the relative elongation of the rod in the elastic area obeys the classical linear Hooke’s law

$$
\sigma(\varepsilon) = E\varepsilon, \quad 0 \leq \varepsilon \leq \varepsilon_y = \frac{\sigma_y}{E};
$$

where $\sigma = F/A_0$ is a normal stress in the cross section of the steel rod, $\varepsilon = \Delta l/l_0$ is the relative elongation of the rod under tension, $E$ is the young’s modulus (the modulus of elasticity) of steel, $\sigma_y$ is the yield strength of steel.

The classical Prandtl’s approaching for the hardening area has the form

$$
\sigma(\varepsilon) = \begin{cases} 
E\varepsilon, & 0 \leq \varepsilon \leq \varepsilon_y = \frac{\sigma_y}{E}; \\
\sigma_y, & \varepsilon \geq \varepsilon_y;
\end{cases}
$$

where $\varepsilon_y$ is the elongation of the rod corresponding to the yield strength $\sigma_y$.

The linear approaching [3–7, 12, 13] for the hardening area has the form
\[ \sigma(\varepsilon) = \begin{cases} E\varepsilon, & 0 \leq \varepsilon \leq \varepsilon_y = \frac{\sigma_y}{E}; \\ \sigma_y + P_y(\varepsilon - \varepsilon_y), & \varepsilon \geq \varepsilon_y; \end{cases} \]

where \(P_y\) is the hardening module of steel at \(\varepsilon = \varepsilon_y\).

Note that the Prandtl’s approaching is a special case of the linear approaching (under \(P_y = 0\)).

The linear approaching well describes the stress curve at the beginning of the hardening area.
However, the yield point is often to replace with some close value from those considerations so that
the areas under the experimental non-linear curve and under the linear analytical hardening curve was
the same on the studied interval of the relative elongation of the rod.

Note that in metallurgy in the production of the large-diameter pipes made from the high-strength
steels, the deformation of the metal at the end of the hardening area is often not advisable, since this
can lead to the defects inside the steel sheet and increased “fragility” of steel, when the ratio of the
yield strength to the ultimate strength becomes close to one.

The classical Nadai’s approaching \([1]\) for the hardening area has the form

\[ \sigma(\varepsilon) = P \cdot \ln^n(1 + \varepsilon), \quad \varepsilon \geq 0, \quad P = \text{const}, \quad 0 < n < 1, \]

where \(n\) is some real number.

The disadvantage of the Nadai’s approaching is the infinite derivative (the tangent of inclination
angle) of the stress curve at the beginning of the stretching of the rod (\(\varepsilon = 0\)). It contradicts the
classical Hooke’s law for the small elastic deformations (\(\sigma(\varepsilon) = E\varepsilon, \quad d\sigma(\varepsilon)/d\varepsilon = E \neq \infty\)) and is not
observed in none of the metals in practice.

In addition, when the stress reaches the ultimate strength \(\sigma_u\), the inclination angle of the hardening
curve is not equal to zero (the Nadai’s hardening curve has no maximum at the moment of neck’s
formation in the tested specimen). In deriving his formula, Nadai used the assumption of
incompressibility of the material, which is not true for steels.

The classical Ludwik’s approaching \([2]\) for the hardening area has the form

\[ \sigma(\varepsilon) = \frac{\sigma_y + P \cdot \ln^n(1 + \varepsilon)}{1 + \varepsilon}, \quad \varepsilon \geq 0, \quad P = \text{const} > 0, \quad 0 < m < 1, \]

where \(m\) is some real number.

It is believed that the Ludwik’s approaching describes the hardening area more accurately than the
Nadai’s approaching. However, the disadvantages of the Ludwik’s approaching are the same as that of
the Nadai’s approaching.

2. The direct approaching of the hardening area of steel

Let us consider in detail the peculiarities of the static stretching diagram of a steel specimen
\([12, 13, 18, 19]\).

The results of mechanical tests (the stretching diagram) of two flat specimens from the pipe steel of
the strength class 485 are shown in figure 1. The length and width of the specimens, respectively, is
180 mm and 39.8 mm, the test temperature is \(+20\) °C, the test procedure is ISO 6892.

The test results of specimens: the conventional yield strength \(\sigma_{0.5}\) (0.5 \%) for specimens,
respectively, equals 523 MPa and 525 MPa, the ultimate strength equals 612 MPa and 610 MPa, the
relative elongation of specimens at the time of rupture equals 27 \% and 22 \%, the ratio of the yield
strength to the ultimate strength equals 0.858 and 0.857.

In figure 1 it is clearly seen that the hardening curve in the stretching diagram is clearly nonlinear.
The derivative (the tangent of an inclination angle) of the stress curve at the point, corresponding to
the yield strength of the steel, has a bend and some finite (not infinite) value. In the hardening area,
the stress curve is a monotonically increasing convex curve, having a maximum (the zero derivative) at a
point corresponding to the ultimate strength \(\sigma_u\) (at the time of the neck formation).
Figure 1. Experimental stress-strain stretching diagram of two specimens from the pipe steel of strength class 485.

Further, for the approaching of the stress curve we will use the classic stretching diagram $\sigma - \varepsilon$ (the stress – relative elongation diagram) (figure 2) [6, 7, 18, 19].

Sometimes at the beginning of the hardening area, a small yield area can be observed, when the elongation of the rod occurs at an almost constant value of the stretching force $F$. The length of the yield area is incomparably smaller than the size of the hardening area, so the yield area can usually be neglected in the practical calculations. The yield area is not visible in figure 1.

The direct non-linear Shinkin’s approaching for the hardening area has the form

$$
\sigma(\varepsilon) = \begin{cases} 
E\varepsilon, & 0 \leq \varepsilon \leq \varepsilon_y = \frac{\sigma_y}{E}; \\
\sigma_y + \sum_n P_n \left( \varepsilon - \frac{\sigma_y + \varepsilon_y}{E} \right)^n - \varepsilon_y^n, & \varepsilon \geq \varepsilon_y, \varepsilon \varepsilon 
\end{cases}
$$

$$
\sigma(\varepsilon_y) = \sigma_y, \quad \frac{d\sigma(\varepsilon_y)}{d\varepsilon} = \sum_n nP_n\varepsilon_y^{n-1} = P_y;
$$

where $P_y$ is the hardening module of steel at $\varepsilon = \varepsilon_y$, $P_n$ are the coefficients of the series (may have both positive and negative values), $\varepsilon_n$ are the displacements of the relative elongation $\varepsilon$ relatively $\varepsilon_y$, $n$ are the real numbers (can have both positive and negative values).

In the steel’s hardening area, the derivative $d\sigma(\varepsilon)/d\varepsilon$ is a monotonically decreasing function of $\varepsilon$.

Under $P_n = 0$ we obtain the Prandtl’s approaching. Under one term of the series, $\varepsilon_n = 0$ and $n = 1$ we obtain the linear approaching.
Figure 2. The idealized diagram of steel’s stretching.

The peculiarities of the Shinkin’s approaching are the introduction of the displacements $\varepsilon_n$ (which allow us to have a finite value of the derivative of the hardening curve at the point corresponding to the yield strength) and the approaching of the hardening curve in the form of a finite or infinite series (which allows us to approximate the hardening curve from the yield strength to the ultimate strength (the moment of the formation of the neck) with any precision).

**Remark 1.** Under the approaching of the whole hardening area of steel (from the yield strength to the ultimate strength) by three members of the series and under the boundary conditions (the four boundary conditions)

$$
\sigma(e_y) = \sigma_y, \quad \sigma(e_u) = \sigma_u, \quad \frac{d\sigma(e_y)}{de} = P_y, \quad \frac{d\sigma(e_u)}{de} = 0;
$$

the direct non-linear Shinkin’s approaching is a cubic multinomial (polynomial) by $e$ and has the form

$$
\sigma(e) = \begin{cases}
Ee, & 0 \leq e \leq e_y = \frac{\sigma_y}{E}; \\
\sigma_y + P_y (e - e_y) - \frac{2P_y (\sigma_u - \sigma_y)}{(e_u - e_y)^3} (e - e_y) - \frac{P_y (\sigma_u - \sigma_y)}{(e_u - e_y)^3} (e - e_y)^3, & e_u \geq e \geq e_y;
\end{cases}
$$

$$
\sigma(e_y) = \sigma_y, \quad \sigma(e_u) = \sigma_u, \quad \frac{d\sigma(e_y)}{de} = P_y, \quad \frac{d\sigma(e_u)}{de} = 0,
$$

where $\sigma_u$ is the ultimate strength of steel, $e_u$ is the relative elongation of the rod corresponding to the ultimate strength.

3. The inverse approaching of the hardening area of steel

If the relative elongation $e_u$, at which the steel’s hardening curve reaches the maximum value (the ultimate strength $\sigma_u$), is determined stably (has a small deviation from the mean value under the experimental stretching of steel specimens), then it is convenient to apply the inverse approaching of the hardening area, that is, to approximate the hardening curve not from the yield strength $\sigma_y$, but from the ultimate strength $\sigma_u$. 
The inverse non-linear Shinkin’s approaching for the hardening area has the form

\[
\sigma(\varepsilon) = \begin{cases} 
E\varepsilon, & 0 \leq \varepsilon \leq \varepsilon_y = \frac{\sigma_y}{E}; \\
\sigma_u - \sum_n P_n (\varepsilon_u - \varepsilon)^n, & \varepsilon_u \geq \varepsilon \geq \varepsilon_y, \quad P_n > 0, \quad n > 1; 
\end{cases}
\]

\[
\sigma(\varepsilon_y) = \sigma_u - \sum_n P_n (\varepsilon_u - \varepsilon_y)^n = \sigma_y, \quad \sigma(\varepsilon_u) = \sigma_u,
\]

\[
\frac{d\sigma(\varepsilon_y)}{d\varepsilon} = \sum_n nP_n (\varepsilon_u - \varepsilon_y)^{n-1} = P_y, \quad \frac{d\sigma(\varepsilon_u)}{d\varepsilon} = 0,
\]

where \(P_y\) is the steel’s hardening module, \(P_n\) are the positive coefficients of the series, \(n\) are the positive real numbers.

In the hardening area, the curve \(\sigma(\varepsilon)\) is a convex monotonically increasing curve having the maximum at the point \(\varepsilon = \varepsilon_u\) corresponding to the ultimate strength \(\sigma_u\).

**Remark 2.** Under the approaching of the whole steel’s hardening area (from the yield strength to the ultimate strength), with the help of one member of the series and at the boundary conditions (the four boundary conditions)

\[
\sigma(\varepsilon_y) = \sigma_y, \quad \sigma(\varepsilon_u) = \sigma_u, \quad \frac{d\sigma(\varepsilon_y)}{d\varepsilon} = P_y, \quad \frac{d\sigma(\varepsilon_u)}{d\varepsilon} = 0,
\]

the inverse non-linear Shinkin’s approaching has the form

\[
\sigma(\varepsilon) = \begin{cases} 
E\varepsilon, & 0 \leq \varepsilon \leq \varepsilon_y = \frac{\sigma_y}{E}; \\
\sigma_u - \frac{P_y (\varepsilon_u - \varepsilon)}{\sigma_u - \sigma_y}, & \varepsilon_u \geq \varepsilon \geq \varepsilon_y;
\end{cases}
\]

\[
\sigma(\varepsilon_y) = \sigma_y, \quad \sigma(\varepsilon_u) = \sigma_u, \quad \frac{d\sigma(\varepsilon_y)}{d\varepsilon} = P_y, \quad \frac{d\sigma(\varepsilon_u)}{d\varepsilon} = 0.
\]

4. **Numerical calculations**

Consider the approaching of the steel’s hardening area of the two steel rods, the stretching diagram of which is shown in figure 1. In this case, the hardening modulus at \(\varepsilon = \varepsilon_y\) is equal to \(P_y = 1441\) MPa, the average yield strength is equal to \(\sigma_y = (523 + 525)/2 = 524\) MPa, the average ultimate strength is equal to \(\sigma_u = (612 + 610)/2 = 611\) MPa, the average value of the relative elongation, corresponding to the ultimate strength \(\varepsilon_u\), is equal to \(\varepsilon_u = 0.12\).

Next, we approximate the steel’s hardening area completely (from the yield strength to the ultimate strength) under the boundary conditions (or a part of these boundary conditions)

\[
\sigma(\varepsilon_y) = \sigma_y, \quad \sigma(\varepsilon_u) = \sigma_u, \quad \frac{d\sigma(\varepsilon_y)}{d\varepsilon} = P_y, \quad \frac{d\sigma(\varepsilon_u)}{d\varepsilon} = 0.
\]

In this case, the Nadai’s approaching (two boundary conditions) has the form
\[
\sigma(\varepsilon) = \frac{P \cdot \ln^n(1 + \varepsilon)}{1 + \varepsilon}, \quad n = \frac{\ln \left( \frac{\sigma_u(1 + \varepsilon_u)}{\sigma_y(1 + \varepsilon_y)} \right)}{\ln(1 + \varepsilon_y)}, \quad P = \frac{\sigma_y(1 + \varepsilon_y)}{\ln^n(1 + \varepsilon_y)},
\]

\[
\sigma(\varepsilon) = \sigma_y, \quad \sigma(\varepsilon_u) = \sigma_u; \quad \frac{d\sigma(\varepsilon)}{d\varepsilon} \neq P_m, \quad \frac{d\sigma(\varepsilon_u)}{d\varepsilon} \neq 0;
\]

and the Ludwik’s approaching (two boundary conditions) has the form

\[
\sigma(\varepsilon) = \frac{\sigma_y + P \cdot \ln^n(1 + \varepsilon)}{1 + \varepsilon}, \quad m = \frac{\sigma_u(1 + \varepsilon_u) \ln(1 + \varepsilon_u)}{\sigma_u(1 + \varepsilon_u) - \sigma_y}, \quad P = \frac{\sigma_u(1 + \varepsilon_u) - \sigma_y}{\ln^n(1 + \varepsilon_u)},
\]

\[
\sigma(\varepsilon_u) = \sigma_u, \quad \frac{d\sigma(\varepsilon_u)}{d\varepsilon} = 0; \quad \sigma(\varepsilon_y) \neq \sigma_y, \quad \frac{d\sigma(\varepsilon_y)}{d\varepsilon} \neq P_y.
\]

**Figure 3.** The stress curves in the hardening area of steel: the experimental (dotted) curve (1), the curve by Nadai (2), the curve by Ludwik (3), the curves of the direct (4) and inverse (5) approachings by Shinkin; \( \varepsilon_y \leq \varepsilon \leq \varepsilon_u \).

For the steel's hardening area, the experimental stress curve (figure 1), the curves of the Nadai’s and Ludwik’s approachings, the curves of the forward and backward Shinkin’s approachings are shown in figure 3.

In figure 3 we see that the curves of the forward and reverse Shinkin’s approachings practically coincide with the experimental hardening curve (figure 1), and the curves of the Nadai’s and Ludwik’s approachings have a significant relative error.
\[
\Delta(\varepsilon) = \frac{\sigma(\varepsilon) - \sigma_{\text{exp}}(\varepsilon)}{\sigma_u - \sigma_\gamma},
\]

where \(\sigma_{\text{exp}}(\varepsilon)\) is the experimental dependence of the stress (figure 1) from the relative elongation \(\varepsilon\).

So, in the steel’s hardening area, the maximum relative error \(\Delta(\varepsilon)\) for the Nadai’s approaching is equal to 47.3 %, and for the Ludwik’s approaching is equal to 35.5 %.

5. Conclusion

The new non-linear (direct and inverse) approachings of the steel’s hardening area under the stretching of rod in the form of a power series with the displacement of the argument (relative elongation) of the power functions of the series are proposed. It is shown that the proposed approachings for the steel’s hardening area are much more accurate than the classical Prandtl’s, Nadai’s, and Ludwik’s approachings.

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