More about wormholes in generalized Galileon theories

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Abstract

We consider a class of generalized Galileon theories within General Relativity in space-times of more than two spatial dimensions. We show that these theories do not admit stable, static, spherically symmetric, asymptotically flat and traversable Lorentzian wormholes.

1 Introduction and summary

Galileons and their generalizations [1, 2, 3, 4, 5, 6, 7, 8] may violate the Null Energy Condition (NEC) without obvious pathologies [6, 7, 9, 10]. It is therefore tempting to make use of these theories for constructing examples of stable, static, asymptotically flat, traversable Lorentzian wormholes [11, 12, 13, 14, 15] in classical General Relativity, whose putative existence necessarily requires NEC-violation [13, 16, 17].

The most commonly studied class of generalized Galileons is described by the Lagrangians of the following form [6, 7] (metric signature (+, −,..., −))

\[ L = -\frac{1}{2\kappa}R + F(\pi, X) + K(\pi, X)\Box\pi, \]

where the first term is the Einstein–Hilbert Lagrangian, \( \kappa = 8\pi G \), \( \pi \) is the Galileon field, \( F \) and \( K \) are arbitrary Lagrangian functions, and \( X = \nabla_\mu \pi \nabla^\mu \pi \), \( \Box\pi = \nabla_\mu \nabla^\mu \pi \). A static,
A spherically symmetric and traversable Lorentzian wormhole in \((d+2)\)-dimensional space-time is described by the metric

\[
ds^2 = a^2(r)dt^2 - dr^2 - c^2(r)\gamma_{\alpha\beta}dx^\alpha dx^\beta ,
\]

where \(x^\alpha\) and \(\gamma_{\alpha\beta}\) are coordinates and metric on unit \(d\)-dimensional sphere. The coordinate \(r\) runs from \(-\infty\) to \(+\infty\), and the metric coefficients are strictly positive and bounded from below:

\[
a(r) \geq a_{\text{min}} > 0 , \quad c(r) \geq R_{\text{min}} > 0 ,
\]

where \(R_{\text{min}}\) is the radius of the wormhole throat. The wormhole geometry is assumed to be asymptotically flat. For \(d \geq 2\) (four or more space-time dimensions) this implies the asymptotic behavior

\[
a \to a_\pm , \quad c(r) \to \pm r , \quad \text{as} \quad r \to \pm \infty ,
\]

where \(a_\pm\) are positive constants. For consistency, the Galileon field supposedly supporting a wormhole is also static and spherically symmetric, \(\pi = \pi(r)\).

It has been observed in Ref. [18] that there is tension between the properties of the Galileon energy-momentum tensor that can support a wormhole, on the one hand, and the requirement of stability (the absence of ghosts and gradient instabilities in the Galileon perturbations about the putative solution \(\pi(r)\)), on the other. In 3-dimensional space-time \((d = 1)\), the analysis of Ref. [18] was sufficient to rule out stable wormholes under very mild assumption on the asymptotic behavior of \(\pi'\) at spatial infinity, cf. Ref. [19]. However, the arguments of Ref. [18] regarding wormholes in higher dimensional space-times were not completely conclusive.

In this paper we consider space-times of more than 3 dimensions \((d \geq 2)\). Our purpose is to complete the analysis and show that stable wormholes with properties \((2), (3), (4)\) do not exist in theories with the Lagrangians of the form \((1)\). The argument we present is quite general; in particular, no assumption on the behavior of \(\pi(r)\) at \(r \to \pm \infty\) is made.

The paper is organized as follows. We begin with generalities of spherically symmetric Galileon backgrounds and perturbations about them in Section 2. We present our argument in Section 3 and conclude in Section 4.

## 2 Galileon and its perturbations

The Galileon energy-momentum tensor reads

\[
T_{\mu \nu} = 2F_X \partial_\mu \pi \partial_\nu \pi + 2K_X \Box \pi \cdot \partial_\mu \pi \partial_\nu \pi - \partial_\mu K \partial_\nu \pi - \partial_\nu K \partial_\mu \pi - g_{\mu \nu} F + g_{\mu \nu} g^{\lambda \rho} \partial_\lambda K \partial_\rho \pi ,
\]
where $F_\pi = \partial F / \partial \pi$, $F_X = \partial F / \partial X$, etc., and $\partial_\mu K = K_\pi \partial_\mu \pi + 2 K_X \nabla^\lambda \pi \nabla_\mu \nabla_\lambda \pi$. Its components in the static case, $\pi = \pi(r)$, are

\begin{align}
T_0^0 &= -F - K_\pi \pi'^2 + 2 \pi''^2 K_X, \\
T_r^r &= -2 \pi'^2 F_X - F + K_\pi \pi'^2 + 2 K_X \pi'^3 \left( \frac{d'}{a} + \frac{d''}{c} \right), \\
T_\alpha^\beta &= \delta_\alpha^\beta T_0^0.
\end{align}

To derive the quadratic Lagrangian for the Galileon perturbations of high momenta and frequencies, one writes the full Galileon field equation

\begin{align}
(-2F_X + 2K_\pi - 2K_X \nabla_\mu \nabla_\mu \pi - 2K_X \Box \pi) \Box \pi + (-4F_{XX} + 4K_{XX}) \nabla^\mu \pi \nabla_\mu \nabla^\nu \pi \\
-4K_{XX} \nabla^\mu \pi \nabla_\nu \nabla_\mu \nabla_\nu \pi + 4K_{XX} \nabla^\nu \pi \nabla^\lambda \pi \nabla_\mu \nabla_\nu \nabla_\mu \nabla_\nu \pi + 2K_X \nabla^\nu \pi \nabla_\mu \nabla_\nu \pi \\
+ 2K_X R_{\mu\nu} \nabla^\mu \pi \nabla^\nu \pi + \ldots = 0;
\end{align}

hereafter dots denote terms without second derivatives. The last line here is a subtlety of the Galileon theories: the Galileon field equation involves the second derivatives of metric, and the Einstein equations involve the second derivatives of the Galileon [6] (see also ref. [7]), and so do the linearized equations for perturbations. However, the linearized theory can be reduced to the theory of purely Galileon perturbations. The trick is to integrate the metric perturbations out of the Galileon field equation by making use of the Einstein equations [6].

The linearized Galileon equation can be written in the following form

\begin{align}
-2[F_X + K_X \Box \pi - K_\pi + \nabla_\nu (K_X \nabla^\nu \pi)] \nabla_\mu \nabla^\mu \chi \\
-2[2(F_{XX} + K_{XX} \Box \pi) \nabla^\mu \pi \nabla^\nu \pi - 2(\nabla^\mu K_X) \nabla^\nu \pi - 2 K_X \nabla^\mu \nabla^\nu \pi] \nabla_\mu \nabla_\nu \chi \\
+ 2K_X R_{\mu\nu}^{(1)} \nabla^\mu \pi \nabla^\nu \pi + \ldots = 0,
\end{align}

where $\pi$ is the background, $\chi$ is the Galileon perturbation about this background, and $R_{\mu\nu}^{(1)}$ is linear in metric perturbations. We now make use of the Einstein equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R = \kappa T_{\mu\nu}$, or

\[ R_{\mu\nu} = \kappa \left( T_{\mu\nu} - \frac{1}{d-1} g_{\mu\nu} T^\lambda_\lambda \right), \]

linearize the energy-momentum tensor and obtain for the last term in eq. (6)

\[ 2K_X R_{\mu\nu}^{(1)} \nabla^\mu \pi \nabla^\nu \pi = -2\kappa K_X^2 \left[ -\frac{2(d-2)}{d-1} X^2 \Box \chi + 4X \nabla^\mu \pi \nabla^\nu \pi \nabla_\mu \nabla_\nu \chi \right] + \ldots. \]

The resulting linearized Galileon field equation is obtained from the following quadratic
Lagrangian:

\[
L^{(2)} = [F_X + K_X \Box \pi - K_\pi + \nabla_\nu (K_X \nabla^\nu \pi)] \nabla_\mu \pi \nabla^\mu \pi \\
- 2(F_X X + K_X X \Box \pi - 2(\nabla^\mu K_X) \nabla_\nu \pi - 2K_X \nabla^\mu \nabla^\nu \pi] \nabla_\mu \pi \nabla_\nu \pi \\
- \frac{2(d-2)}{d-1} \kappa K_X^2 X^2 \nabla_\mu \pi \nabla^\mu \pi + 4 \kappa K_X^2 X \nabla^\mu \pi \nabla_\mu \chi \nabla_\pi \chi \\
- 2(d-2) \kappa \kappa K_X^2 X^2 \nabla_\mu \pi \nabla^\mu \pi \\
\n+ 2K_X \nabla^\mu \nabla_\nu \pi \nabla_\mu \pi \nabla_\nu \pi \\
+ \frac{4}{d-1} \kappa \kappa K_X^2 X^2 \nabla_\mu \pi \nabla^\mu \pi \\
- 2K_X \nabla_\nu \pi \nabla_\mu \chi \nabla^\mu \pi \nabla_\nu \pi \\
\nSpecifying to static, spherically symmetric background, one finds

\[
L^{(2)} = a^{-2} G^{00} \dot{\chi}^2 - G^{rr} (\chi')^2 - c^{-2} G^{\Omega \alpha \beta} \partial_\alpha \chi \partial_\beta \chi + \ldots ,
\]

where the omitted terms do not contain derivatives of \( \chi \), and the effective metric is

\[
G^{00} = F_X - K_\pi - K_X' \pi' - 2K_X \pi'' - 2dK_X \frac{c'}{c} \pi' - \frac{2(d-1)}{d} \kappa K_X^2 \pi'^4 ,
\]

\[
G^{\Omega} = F_X - K_\pi - K_X' \pi' - 2K_X \pi'' - 2(d-1)K_X \frac{c'}{c} \pi' - 2K_X \frac{a'}{a} \pi' - \frac{2(d-1)}{d} \kappa K_X^2 \pi'^4 ,
\]

\[
G^{rr} = F_X - 2F_X X \pi'^2 - K_\pi + K_X' \pi' - 2K_X \pi' \left( \frac{a'}{a} + \frac{c'}{c} \right) \\
+ 2K_X \pi'^2 \pi'' + 2K_X \pi'^3 \left( \frac{a'}{a} + \frac{c'}{c} \right) + \frac{2(d+1)}{d} \kappa K_X^2 \pi'^4 .
\]

The background is stable, provided that \( G^{00} > 0, G^{rr} \geq 0, G^{\Omega} \geq 0 \) for every \( r \), otherwise there are either ghosts or gradient instabilities. We now show that the property \( G^{00} > 0 \) cannot hold for a non-singular wormhole solution.

### 3 The argument

A combination of the Einstein equations gives

\[
T_0^0 - T_r^r = - \frac{d a}{\kappa c} \left( \frac{c'}{a} \right)' .
\]

We combine eqs. (5a), (5b) and (7) to obtain the relation

\[
2 \frac{c}{a} \pi'^2 G^{00} = -\frac{d}{d \tau} \left( 2 \frac{c}{a} K_X \pi'^3 + \frac{d c'}{\kappa a} \right) - \frac{2(d-1)}{d} \kappa K_X \pi'^3 \left( 2 \frac{c}{a} K_X \pi'^3 + \frac{d c'}{\kappa a} \right)
\]

It is now natural to introduce a variable

\[
Q = \frac{1}{cd-1} \left( 2 \frac{c}{a} K_X \pi'^3 + \frac{d c'}{\kappa a} \right)
\]
and cast the relation (8) into

$$\frac{2}{ac^{d-2}} \pi' G^{00} = -Q' - \frac{d-1}{d} \kappa ac^{d-2} Q^2$$

(9)

The absence of ghosts and gradient instabilities, $G^{00} > 0$, is possible only if

$$\frac{Q'}{Q^2} < -\frac{d-1}{d} \kappa ac^{d-2}.$$

(10)

We now recall that we consider the case $d \geq 2$ and that $a(r)$ and $c(r)$ are bounded from below by positive numbers, see eq. (3). Hence, the relation (10) implies that

$$\frac{Q'}{Q^2} < -\mathcal{C},$$

where $\mathcal{C}$ is a positive constant.

We integrate the latter relation from $r$ to $r' > r$ and obtain

$$Q^{-1}(r) - Q^{-1}(r') < -\mathcal{C}(r' - r) \quad \text{for all } r' > r.$$

Suppose now that $Q(r') > 0$ at some value of $r'$. Then

$$Q^{-1}(r) < Q^{-1}(r') - \mathcal{C}(r' - r).$$

As $r$ decreases from $r'$ to $-\infty$, $Q^{-1}(r)$ starts positive, stays bounded from above and eventually becomes bounded by a negative number. Hence, $Q^{-1}(r)$ crosses zero, which means that $Q$ is infinite and the configuration is singular.

$Q(r)$ cannot be negative everywhere either. If $Q(r)$ is negative at some $r$, then

$$Q^{-1}(r') > Q^{-1}(r) + \mathcal{C}(r' - r).$$

As $r'$ increases from $r$ to $+\infty$, the right hand side increases, and eventually becomes positive. $Q^{-1}(r')$ crosses zero, and the configuration is again singular. This completes the argument.

4 Discussion

Let us end up with two remarks. First, adding conventional matter obeying the NEC would not help: the relation (9) would be modified as follows:

$$\frac{2}{ac^{d-2}} \pi' G^{00} = -Q' - \frac{d-1}{d} \kappa ac^{d-2} Q^2 - \frac{1}{ac^{d-2}} (T_0^0 - T^r_r) M,$$

where the last term is due to the conventional matter, and the NEC implies $(T_0^0 - T^r_r) M > 0$. Hence, the inequality (10) would still hold, and our argument would remain valid.
Second, our argument is quite technical. It is reassuring, however, that the generalized Galileon theories of the type (1) refuse to support wormholes, and hence time machines [13]. In this regard, it would be interesting to study whether or not more general theories with second-order Lagrangians and second order field equations, and also \( p \)-form theories [5] have the same property.

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