We give explicit solutions to the two-component Hunter-Saxton system on the unit circle. Moreover, we show how global weak solutions can be naturally constructed using the geometric interpretation of this system as a re-expression of the geodesic flow on the semi-direct product of a suitable subgroup of the diffeomorphism group of the circle with the space of smooth functions on the circle. These spatially and temporally periodic solutions turn out to be conservative.

Keywords: The Hunter-Saxton system; semi-direct product; weak geodesic flow; global conservative solutions

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1 Introduction

In this paper, we are concerned with the two-component Hunter-Saxton system with periodic boundary conditions:

\[
\begin{align*}
  u_{txx} + uu_{xxx} + 2u_xu_{xx} &= \rho_x, \quad t > 0, \ x \in \mathbb{S} \simeq \mathbb{R}/\mathbb{Z}, \\
  \rho_t + (u\rho)_x &= 0 \\
  u(0, x) &= \tilde{u}(x), \ \rho(0, x) = \tilde{\rho}(x).
\end{align*}
\]

The Hunter-Saxton system \cite{33, 31, 23, 24} is a two-component generalization of the well-known Hunter-Saxton equation \( u_{txx} + uu_{xxx} + 2u_xu_{xx} = 0 \) modeling the propagation of nonlinear orientation waves in a massive nematic liquid crystal (cf. \cite{15, 1, 2, 19, 20, 21, 17, 30}), to which it reduces if \( \tilde{\rho} \) is chosen to vanish identically.

In mathematical physics, the Hunter-Saxton system \eqref{1.1} arises as a model for the nonlinear dynamics of one-dimensional non-dissipative dark matter (the so-called Gurevich-Zybin system, see \cite{23} and the references therein). Additionally, it is the short wave limit (using the scaling \((t,x) \mapsto (\varepsilon t, \varepsilon x)\), and letting \(\varepsilon \to 0\) in the resulting equations) of the two-component Camassa-Holm system originating in the Green-Naghdi equations which approximate the governing equations for water waves \cite{3, 8, 10, 11}. The Hunter-Saxton system is embedded in a more general family of coupled third-order systems \cite{34}.

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encompassing the axisymmetric Euler flow with swirl [14] and a vorticity model equation [6, 27], among others (cf. [29, 26, 34]).

Geometric aspects of (1.1) have recently been described in [7]: The Hunter-Saxton system can be realized as a geodesic equation on the semi-direct product of a subgroup of the group of circle diffeomorphisms with the space of smooth functions on the circle. This geometrical interpretation is closely linked to the re-expression of the Camassa-Holm and Degasperis-Procesi systems as (non-metric) Euler-Arnold equation (see, e.g., [9, 22]).

In this paper, we adopt the geometric viewpoint of [7] and adapt techniques of [21] developed for the Hunter-Saxton equation to construct global weak solutions to the two-component Hunter-Saxton system (1.1). Weak solutions to the Hunter-Saxton system can, in analogy to the Hunter-Saxton equation [10, 1, 21], be classified as either dissipative or conservative: While the energy $E(t) := \|u_x(t,.)\|_{L^2(S)}^2 + \|\rho(t,.)\|_{L^2(S)}^2$ is a decreasing functional for dissipative solutions, one has

$$E(t) = E(0) \quad \text{for almost every } t > 0$$

for conservative solutions [33]. Here, we give a construction of conservative weak solutions with periodic boundary conditions.

**Remark 1.1.** Recently, the authors of [24] have proved that there are dissipative solutions to the $\mu$-Hunter-Saxton system which exist globally. The main differences to our results are that

- we construct conservative weak solutions;
- we do not exclude zero values of the initial datum $\tilde{\rho}(x)$;
- our solutions are not only periodic in space, but also in time;
- our strategy involves the geometric interpretation of (1.1) (see [7]).

The rest of this article is organized as follows: Section 2 presents novel solution formulae of (1.1) and an explicit expression for the maximal time of existence. In Section 3, we outline the analytic framework in which system (1.1) can be recast as a geodesic equation, following the approach of [7]. Finally, in Section 4, we show that the geodesic flow can be extended indefinitely in a weak sense, and we utilize this result in order to construct global conservative solutions to the Hunter-Saxton system.

**Remark 1.2 (Notation).** The Hilbert spaces of functions (function equivalence classes) $f : S \to \mathbb{R}$ which, together with their derivatives of order $k \geq 0$, are square-integrable, will be denoted by $H^k(S)$, and the corresponding norm by $\|f\|_k$. If $k = 0$, we suppress the subscript and just write $\|f\|$. For the subspace of functions in $H^k(S)$ with vanishing mean, we will use the symbol $H^k(S)/\mathbb{R}$. Lastly, the abbreviation $\{f > 0\}$ for the set $\{x \in S : f(x) > 0\}$ (and other analogous short forms) will be used throughout the text.

\[1\] For the geometry of the Camassa-Holm equation, see, e.g., [18, 25, 4, 5].
2 Explicit solution formulae

In this section, we provide new solution formulae for the Hunter-Saxton system. Recall that the first component equation can be rewritten in terms of the gradient $u_x(t, x)$,

$$u_{tx} + uu_{xx} + \frac{1}{2} u_x^2 = \frac{1}{2} \rho^2 + a(t), \quad (2.2)$$

where the nonlocal term $a(t) = -\frac{1}{2} \int_\mathbb{S} \rho^2 + u_x^2 \, dx$ is determined by periodicity. It turns out (see [32, 31]) that $a'(t) = 0$, so that we may, without loss of generality, set $a(t) = a(0) = -2$.

Let us introduce the Lagrangian flow map $\varphi$ solving

$$\varphi_t(t, x) = u(t, \varphi(t, x)), \quad \varphi(0, x) = x \in \mathbb{S}. \quad (2.3)$$

In terms of $U(t, x) := u_x(t, \varphi)$, $\rho(t, x) = \rho(\varphi)$, we rephrase (1.1) as

$$\begin{cases}
U_t + \frac{1}{2} U^2 = \frac{1}{2} \varrho^2 - 2 \\
\varrho_t + U \varrho = 0 \\
U(0, x) = \tilde{u}_x(x), \quad \varrho(0, x) = \tilde{\rho}(x).
\end{cases} \quad (2.4)$$

Introducing the complex function $z(t, x) = U(t, x) + \sqrt{-1} \, \varrho(t, x)$, one sees after a short calculation that $z$ solves the Riccati-type differential equation

$$z_t(t, x) = -\frac{1}{2} \left( z(t, x)^2 + 4 \right), \quad z(0, x) = \tilde{z}(x) = \tilde{u}_x(x) + \sqrt{-1} \, \tilde{\rho}(x). \quad (2.5)$$

Equation (2.5) is explicitly solvable:

$$z(t, x) = \frac{2 \tilde{z}(x) - 4 \tan t}{\tilde{z}(x) \tan t + 2}$$

Separating real and imaginary parts, we get the following formulae for system (2.4):

$$u_x(t, \varphi(t, x)) = \frac{4 \cos(2t) \, \tilde{u}_x(x) + \sin(2t) \left[ \tilde{u}_x(x)^2 + \tilde{\rho}(x)^2 - 4 \right]}{[2 \cos t + \tilde{u}_x(x) \sin t]^2 + \tilde{\rho}(x)^2 \sin^2 t}, \quad (2.6)$$

$$\rho(t, \varphi(t, x)) = \frac{4 \tilde{\rho}(x)}{[2 \cos t + \tilde{u}_x(x) \sin t]^2 + \tilde{\rho}(x)^2 \sin^2 t}. \quad (2.7)$$

Interesting information can be drawn from the denominator in (2.6), (2.7): both solution components become unbounded at points $x^* \in \mathbb{S}$ which are both zeroes of $\tilde{\rho}$ and for which $\tilde{u}_x(x^*) = -2 \cot t$. The time-dependent and periodically recurring break-down set thus is the intersection

$$B(t) = \{ \tilde{\rho} = 0 \} \cap \{ \tilde{u}_x = -2 \cot t \}.$$

Summing up, we have proven the subsequent theorem.

---

2Observe that the Hunter-Saxton system is invariant under the scalings $u(t, x) \mapsto \alpha u(\alpha t, x)$, $\rho(t, x) = \alpha \rho(\alpha t, x)$, $\alpha \in \mathbb{R}$. 

---
Theorem 2.1. Suppose \((\tilde{u}, \tilde{\rho})^\dagger \in H^k(\mathbb{S}) \times H^{k-1}(\mathbb{S}), k \geq 3\) satisfies \(\|\tilde{u}_x\|^2 + \|\tilde{\rho}\|^2 = 4\). Let
\[
\begin{pmatrix}
    u(t, .) \\
    \rho(t, .)
\end{pmatrix} \in C([0, T*); (H^k(\mathbb{S})/\mathbb{R}) \times H^{k-1}(\mathbb{S})) \cap C^1([0, T*); (H^{k-1}(\mathbb{S})/\mathbb{R}) \times H^{k-2}(\mathbb{S}))
\]
be the solution to the Hunter-Saxton system with initial data \((\tilde{u}, \tilde{\rho})^\dagger\), and let \(\varphi(t, x)\) with \(\varphi(0, x) = 0, t \in [0, T^*], \) solve the Lagrangian flow map equation (2.3).
Then
\[
\varphi(t, x) = \int_0^x \left\{ \left( \cos t + \frac{\tilde{u}_x(y) - \tilde{u}_x(x)}{2} \sin t \right)^2 + \frac{\tilde{\rho}(y)^2}{4} \sin^2 t \right\} dy,
\]
and the first time when the diffeomorphism \(\varphi(t,.) : \mathbb{S} \to \mathbb{S}\) flattens out is given by 
\[T^* = \frac{\pi}{2} + \arctan \left\{ \frac{1}{2} \min_{\tilde{\rho} = 0} \tilde{u}_x(x) \right\}.\]
Lastly, the solution in terms of \(\varphi\) reads
\[
(u(t, .), \rho(t, .))^\dagger = \left( \varphi(t) \circ \varphi(t)^{-1}, \frac{\tilde{\rho}}{\varphi_x(t) \circ \varphi(t)^{-1}} \right)^\dagger.
\]
Remark 2.1. From the expression for the maximal time of existence of solutions, we see that \(T^* < \pi/2\) if there exist points \(x\) where \(\tilde{\rho}\) vanishes; else, one has \(T^* = +\infty\), i.e., global existence in time (see [31]).
The importance of the set \(\{\tilde{\rho} = 0\}\) for the formation of singularities was described in [33], Remark 5.2, for the case of (dissipative) solutions on the real line.

3 Semidirect products over the circle

In this section, we briefly outline the geometric approach of [1] to the two-component Hunter-Saxton system (1.1).

Let DIFF(\(\mathbb{S}\)) be the Lie group of the group of orientation-preserving diffeomorphisms of the circle, and denote by \(C^\infty(\mathbb{S})\) its Lie algebra. Also, define the following Fréchet Lie subgroup of DIFF(\(\mathbb{S}\)) fixing one point:
\[
\text{DIFF}_1(\mathbb{S}) := \{ \varphi \in \text{DIFF}(\mathbb{S}) : \varphi(1) = 1 \}.
\]
Observe that the Lie algebra of DIFF_1(\(\mathbb{S}\)) is the closed subspace
\[
C_0^\infty(\mathbb{S}) := \{ f \in C^\infty(\mathbb{S}) : f(0) = 0 \}.
\]
Now, we introduce the semidirect-product Lie group (cf. [12] [13])
\[
C^\infty G = \text{DIFF}_1(\mathbb{S}) \times C^\infty(\mathbb{S}),
\]
with the group multiplication \((\varphi, f) \cdot (\psi, g) := (\varphi \circ \psi, f + g \circ \psi)\), where \((\varphi, f), (\psi, g)\) are two given elements of \(G\), and \(\circ\) stands for composition. The neutral element of \(G\) is \((id, 0)\), and the inverse of \((\varphi, f)\) is \((\varphi^{-1}, -f \circ \varphi^{-1})\). On the Lie algebra \(T_{id}C^\infty G \simeq C_0^\infty(\mathbb{S}) \times C^\infty(\mathbb{S})\), the Lie bracket is given by
\[
[(u, \rho), (v, \sigma)] = (u_x v - u v_x, u \sigma_x - v \rho_x), \quad (u, \rho), (v, \sigma) \in C^\infty g := C_0^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}).
\]
The semi-direct product \(C^\infty G\) can be equipped with a Riemannian metric via the inertia operator \(\mathbb{A} : C^\infty g \to C^\infty g, (u, \rho)^\dagger \mapsto (-u_{xx}, \rho)^\dagger\) by setting
\[
\langle \begin{pmatrix} u \\ \rho \end{pmatrix}, \begin{pmatrix} v \\ \sigma \end{pmatrix} \rangle_{(\mathbb{A})} := \frac{1}{4} \int_S \left\{ (u, \rho)^\dagger |\mathbb{A}(v, \sigma)^\dagger| \right\} \, dx := \frac{1}{4} \int_S u_x v_x + \rho \sigma \, dx.
\]
for \((u, \rho)^\dagger, (v, \sigma)^\dagger \in C^\infty g\), and by extending it to all of \(C^\infty G\) by right translation \([9]\). The topology induced by this Hilbert structure is weaker than the original Fréchet topology, which is why such a structure on \(C^\infty G\) is called a weak Riemannian metric. Observe also that \(A\) gives rise to the norm
\[
\| (u, \rho)^\dagger \|_A := \frac{1}{2} \sqrt{\| u_x \|^2 + \| \rho \|^2}.
\]
This analytic framework facilitates the following geometric interpretation of the Hunter-Saxton system \((1.1)\):

**Proposition 3.1** ([7], Corollary 3.2). The system \((1.1)\) describes the geodesic flow of the right-invariant \(\tilde{H}^1(S) \times H^0(S)\) metric on the Fréchet Lie group \(\text{Diff}_1(S) \otimes C^\infty(S)\).

**The space \(M^0_{AC}\)**

For the weak formulation of the geodesic flow of the right-invariant \(\tilde{H}^1(S) \times H^0(S)\) metric, we define the space \(M^0_{AC}\) as the semi-direct product \(M^{AC}(S) \otimes H^0(S)\), where \(M^{AC}\) is the set of nondecreasing absolutely continuous functions \(\varphi : [0, 1] \to [0, 1]\) with \(\varphi(0) = 0\) and \(\varphi(1) = 1\) (see [21]). The tangent space at the identity is naturally defined as
\[
T_{(id, 0)} M^0_{AC} := \{ u \in H^1(S) : u(0) = 0 \} \times H^0(S).
\]

**Characterization of tangent spaces.**

In analogy with [21], we have the subsequent lemma.

**Lemma 3.1.** Let \(\varphi \in AC(S)\), the set of absolutely continuous functions \(S \to \mathbb{R}\), and write
\[
N := \{ x \in S : \varphi_x(x) \text{ exists and equals } 0 \}.
\]
Also, let \(\varrho \in H^0(S)\). Then we have the characterization
\[
T_{(\varphi, \varrho)} M^0_{AC} = \left\{ (U, F)^\dagger \in AC(S) \times H^0(S) : U(0) = 0, \right. \\
\left. U_x = 0 \text{ a.e. on } N \text{ and } \int_{S \setminus N} \frac{U_x^2}{\varphi_x} + (F - \varrho)^2 \varphi_x \ dx < +\infty \right\}.
\]

Furthermore, for any elements \((U, F), (V, G) \in T_{(\varphi, \varrho)} M^0_{AC}\), the inner product reads
\[
\langle (U, F), (V, G) \rangle_{(\varphi, \varrho), M^0_{AC}}^A = \frac{1}{4} \int_{S \setminus N} \frac{U_x V_x}{\varphi_x} + (F - \varrho)(G - \varrho) \varphi_x \ dx.
\]

**Proof.** Define the set \(A := \{ x \in S : \varphi_x(x) \text{ exists and } \varphi_x(x) > 0 \}\). The unit circle \(S\) is thus the union of the three sets \(A, N, \) and \(Z\), the latter being of measure zero. Moreover, the sets \(\varphi(A), \varphi(N), \) and \(\varphi(Z)\) are measurable, where \(\varphi(A)\) has full Lebesgue measure \(\lambda\), while for the other two one has \(\lambda(\varphi(N)) = \lambda(\varphi(Z)) = 0\).

Suppose that \((U, F) \in T_{(\varphi, \varrho)} M^0_{AC}\) for some \((\varphi, \varrho) \in M^0_{AC}\). By definition, this means that there exist \(u \in H^1(S)\) with \(u(0) = 0\) such that \(U = u \circ \varphi\), and that there is a \(\rho \in H^0(S)\)
such that $F = g + \rho \circ \varphi$. One can see that $U(0) = 0$ and that $U_x = 0$ a.e. on $\mathbb{S}$ (cf. [21]).

Also,

$$\int_A \frac{u^2}{\varphi_x} + (F - g)^2 \varphi_x \, dx = \int_A [(u_x \circ \varphi)^2 + (\rho \circ \varphi)^2] \varphi_x \, dx$$

$$= \int_S [(u_x \circ \varphi)^2 + (\rho \circ \varphi)^2] \varphi_x \, dx = \int_S u^2_x + \rho^2 \, dy < +\infty.$$  \hfill (3.14)

This proves the inclusion $\subset$ in (3.12).

Conversely, for $U \in AC(\mathbb{S})$ satisfying $U(0) = 0$, $U_x = 0$ a.e. on $N$ and $\int_S \frac{u^2}{\varphi_x} \, dx < \infty$, one can show as in [21] that there exists a $u \in H^0(\mathbb{S})$ with $u(0) = 0$. As for the second component, we define a function $\rho$ a.e. on $\mathbb{S}$ by

$$\rho(y) = (F - g)(\varphi^{-1}(y)), \quad y \in \varphi(A).$$

Now we see that

$$\int_S \rho^2 \, dy = \int_S [(F - g) \circ \varphi^{-1}]^2 \, dy = \int_S [(F - g) \circ \varphi^{-1}]^2 \circ \varphi \varphi_x \, dy,$$

which equals to $\int_A (F - g)^2 \, dx$ and is finite by Young’s inequality, since both $F$ and $g$ are square-integrable. This shows the inclusion $\supset$ in (3.12). We finish the proof by noticing that (3.13) is a consequence of (3.14).

\hfill \Box

4 Main results

Before stating our main results, let us introduce the Christoffel operator $\Gamma_{\varphi,\theta}(\,;\,)$ defined in [9] on the product $H^k(\mathbb{S}) \times H^{k-1}(\mathbb{S})$, $k > 5/2$, as

$$\Gamma_{(id,0)}((u,\rho),(v,\sigma)) = \begin{pmatrix} \Gamma^0_{id}(u,v) - \frac{1}{2} A^{-1} \partial_x (\rho \sigma) \\ -\frac{1}{2} (u_x \sigma + v_x \rho) \end{pmatrix},$$

where $A = -\partial^2_x$. This operator is then extended by right translation to the tangent spaces of the semi-direct product $C^\infty G$ (cf. [7]). Recall that the inverse of $A$ is given by

$$(A^{-1} f)(x) = -\int_0^x \int_0^y f(z) \, dz \, dy + x \int_0^1 \int_0^y f(z) \, dz \, dy$$

for all $f \in H^k(\mathbb{S})/\mathbb{R}$, $k \geq 1$ (see, e.g., [21]).

The following statement asserts the global existence of a weak geodesic flow on $M^0_{AC}$.

**Theorem 4.1.** Let $(\tilde{u},\tilde{\rho}) \in T_{(id,0)}M^0_{AC} = \{u \in H^1(\mathbb{S}) : u(0) = 0\} \times H^0(\mathbb{S})$ with $\|\tilde{u}_x\|^2 + \|\tilde{\rho}\|^2 = 4$. Define

$$\varphi(t,x) := \int_0^t \left(f^2(t,y) + g^2(t,y)\right) \, dy,$$

where

$$f(t,x) := (\cos t + \frac{\tilde{u}}{2} \sin t), \quad \text{and} \quad g(t,x) := \frac{\tilde{u}}{2} \sin t, \quad (t,x) \in [0,\infty) \times \mathbb{S}.$$  \hfill (4.15)

Moreover, let

$$\rho(t,x) := \tilde{\rho}(x) \int_0^t \frac{\chi_{\varphi_x(s,x) > 0}}{\varphi_x(s,x)} \, ds.$$  \hfill (4.16)

Then the following statements are true.
(i) For each time $t \geq 0$, $(\varphi(t, \cdot), \varrho(t, \cdot)) \in \mathcal{M}^0_{\text{AC}}$.

(ii) For each time $t \geq 0$, $(\varphi(t, \cdot), \varrho(t, \cdot)) \in T_{(\varphi, 0)} \mathcal{M}^0_{\text{AC}}$.

(iii) If $\lambda$ denotes Lebesgue measure on $\mathbb{S}$, then, for $t \geq 0$,

$$
\left\langle \frac{\varphi_t}{\varrho_t}, \frac{\varphi_t}{\varrho_t} \right\rangle^\Lambda_{(\text{id,0})} = \begin{cases} 1, & \sin t = 0 \\
1 - \frac{1}{\sin^2 t} \lambda(\{\bar{u}_x = -2\cot t\} \cap \{\bar{\rho} = 0\}) , & \sin t \neq 0. 
\end{cases}
$$

Specifically, the geodesic has constant energy almost everywhere:

$$
\left\langle \frac{\varphi_t}{\varrho_t}, \frac{\varphi_t}{\varrho_t} \right\rangle^\Lambda_{(\text{id,0})} = \left\langle \frac{\varphi_t(0)}{\varrho_t(0)}, \frac{\varphi_t(0)}{\varrho_t(0)} \right\rangle^\Lambda_{(\text{id,0})} \quad \text{for a.e. } t \in [0, \infty).
$$

(iv) The geodesic equation holds in the weak form

$$
\left( \frac{\varphi_u}{\varrho_u} \right) = \Gamma_{(\varphi, \varrho)}((\varphi, \varrho_t), (\varphi, \varrho_t)) \quad \text{for a.e. } t \in [0, \infty).
$$

The weak geodesic formulation (4.19) allows us to study weak solutions to the Hunter-Saxton system (1.1).

**Definition 1** (Definition of a weak solution to (1.1)). The pair $(u, \rho)^\dagger : ([0, \infty) \times \mathbb{S})^2 \to \mathbb{R}$ is a global conservative solution of equation (1.1) with initial data $(\bar{u}, \bar{\rho})^\dagger \in H^1 \times H^0(\mathbb{S})$ if

(a) for each $t \in [0, \infty)$, the map $x \mapsto u(t, x)$ is in $H^1(\mathbb{S})$;

(b) $u \in C([0, \infty) \times \mathbb{S}; \mathbb{R})$ and $u(0, \cdot) = \bar{u}$ point-wise on $\mathbb{S}$; $\rho(0, \cdot) = \bar{\rho}$ a.e. on $\mathbb{S}$;

(c) the maps $t \mapsto u_x(t, \cdot)$ and $t \mapsto \rho(t, \cdot)$ belong to the space $L^\infty([0, \infty); H^0(\mathbb{S}))$;

(d) the map $t \mapsto u(t, \cdot)$ is absolutely continuous from $[0, \infty)$ to $H^0(\mathbb{S})$ and satisfies

$$
u_t + uu_x = \frac{1}{2} \left\{ \int_0^x (u_x^2 + \rho^2) \, dy + x \int_\mathbb{S} (u_x^2 + \rho^2) \, dy \right\} \quad \text{in } H^0(\mathbb{S}) \text{ for a.e. } t \in [0, \infty);
$$

and the map $t \mapsto \rho(t, \cdot)$ satisfies

$$
\rho_t + (up)_x = 0 \quad \text{in } H^0 \text{ for a.e. } t \in [0, \infty).
$$

The next theorem asserts that there are global conservative solutions to the Hunter-Saxton system (1.1).

**Theorem 4.2** (Global weak solutions to the periodic Hunter-Saxton system). For any initial data $(\bar{u}, \bar{\rho})^\dagger \in H^1(\mathbb{S}) \times H^0(\mathbb{S})$ satisfying $\|\bar{u}_x\|^2 + \|\bar{\rho}\|^2 = 4$ and $\bar{u}(0) = 0$. Then the pair

$$
\begin{pmatrix} u(t, \varphi(t, x)) \\ \rho(t, \varphi(t, x)) \end{pmatrix} := \begin{pmatrix} \varphi_t(t, x) \\ \varrho_t(t, x) \end{pmatrix}, \quad (t, x) \in [0, \infty) \times \mathbb{S},
$$

(4.20)

constitutes a global weak solution of the Hunter-Saxton system (1.1) with initial data $(\bar{u}, \bar{\rho})^\dagger$. It furthermore holds that, for $t \geq 0$,

$$
\|u(t, \cdot)\|^2_{\Lambda} + \|\rho(t, \cdot)\|^2_{\Lambda} = \|\bar{u}_x\|^2 + \|\bar{\rho}\|^2 \quad \text{for a.e. } t \in [0, \infty).
$$

(4.21)

This solution is conservative in the sense that

$$
\|u_x(t, \cdot)\|^2 + \|\rho(t, \cdot)\|^2 = \|\bar{u}_x\|^2 + \|\bar{\rho}\|^2 \quad \text{for a.e. } t \in [0, \infty).
$$

7
Proof of the main results

Proof of Theorem 4.1 (i). The result follows in view of the definition of \( \varphi \) (equation (2.8)) and of the condition \( \| \tilde{u}_x \|^2 + \| \tilde{\rho} \|^2 = 4 \).

Proof of Theorem 4.1 (ii). By Lemma 3.1 we have to verify the following five conditions:
1. \( \varphi_t \in AC(S) \), \( \varrho_t \in H^0(S) \);
2. \( \varphi_t(t, 0) = 0 \);
3. \( \varphi_{tx} = 0 \) a.e. on \( N \); and
4. \( \int_{S \cap N} \frac{\varphi_{tx}^2}{\varphi_x} + \varrho_t^2 \varphi_x \, dx < +\infty \).

Condition 4. is a consequence of the computations in the proof of (iii) below. The map \( x \mapsto \varphi_t(t, x) \) is absolutely continuous, \( \varrho_t \in H^0(S) \) by (4.10), and \( \varphi_t(t, 0) = \varphi_t(t, 1) = 0 \). Moreover, we see that \( \varphi_{tx} = (u_x \circ \varphi) \varphi_x = 0 \) whenever \( \varphi_x = 0 \). This proves Conditions 1–3.

Proof of Theorem 4.1 (iii). Elementary but slightly tedious calculations reveal that

\[
\int_{\{ \varphi_x > 0 \}} \left( \frac{\varphi_{tx}^2}{\varphi_x} + \varrho_t^2 \varphi_x \right) \, dx = 4 \int_{\{ \varphi_x > 0 \}} (f_t^2 + g_t^2) \, dx \tag{4.22}
\]

\[
= 4 \int_S (f_t^2 + g_t^2) \, dx - 4 \int_{\{ f = 0 \} \cap \{ g = 0 \}} (f_t^2 + g_t^2) \, dx
\]

\[
= 4 - 4 \int_{\{ f = 0 \} \cap \{ g = 0 \}} (f_t^2 + g_t^2) \, dx, \tag{4.23}
\]

where we used that

\[
4 \int_S (f_t^2 + g_t^2) \, dx = 4 \sin^2 t + \cos^2 t \int_S (\tilde{u}_x^2 + \tilde{\rho}^2) \, dx = 4
\]

to derive equality in (4.23). The set \( \{ f = 0 \} \cap \{ g = 0 \} \) is empty whenever \( \sin t = 0 \), whence

\[
\left\langle \left( \begin{array}{c} \varphi_t \\ \varrho_t \end{array} \right), \left( \begin{array}{c} \varphi_t \\ \varrho_t \end{array} \right) \right\rangle_{\varphi, \varrho} = 1, \quad t \geq 0, \quad \sin t = 0. \tag{4.24}
\]

Whenever \( \sin t \neq 0 \), then

\[
4 - 4 \int_{\{ f = 0 \} \cap \{ g = 0 \}} \left( \sin^2 t - \tilde{u}_x \cos t \sin t + \frac{\tilde{u}_x^2 + \tilde{\rho}^2}{4} \cos^2 t \right) \, dx
\]

\[
= 4 - \frac{4}{\sin^2 t} \lambda \left( \{ \tilde{u}_x = -2 \cot t \} \cap \{ \tilde{\rho} = 0 \} \right).
\]

Summing up, we gain

\[
\int_{\{ \varphi_x > 0 \}} \left( \frac{\varphi_{tx}^2}{\varphi_x} + \varrho_t^2 \varphi_x \right) \, dx = \begin{cases} 4 & t \geq 0, \quad \sin t = 0 \\ 4 - \frac{4}{\sin^2 t} \lambda \left( \{ \tilde{u}_x = -2 \cot t \} \cap \{ \tilde{\rho} = 0 \} \right) & t \geq 0, \quad \sin t \neq 0. \end{cases} \tag{4.25}
\]

This proves the identity (4.17). In order to complete the proof of (iii), it remains to show that (4.17) implies (4.18). This follows once we realize that \( \lambda(\{ f = 0 \} \cap \{ g = 0 \}) = 0 \). But it has already been demonstrated in [21] that \( \{ f = 0 \} \) is a set of measure zero, which, in view of the measurability of the map \( (t, x) \mapsto g(t, x) : [0, \infty) \times S \to \mathbb{R} \), proves (4.18).
Proof of Theorem 4.1 (iv). Define
\[ \phi(t, x) := \int_0^x f^2(t, y) \, dy, \quad \psi(t, x) := \int_0^x g^2(t, y) \, dy, \]
so that \( \varphi(t, x) = \phi(t, x) + \psi(t, x) \) (see equation (2.8)). Then
\[ \varphi_t(t, x) = 2 \int_0^x f f_t + g g_t \, dy, \quad \varphi_{tt} = 2 \int_0^x f_t^2 + g_t^2 + f f_{tt} + g g_{tt} \, dy. \]
Since \( f_{tt} = -f \) and \( g_{tt} = -g \), the second-order derivative of \( \varphi \) can be rewritten as
\[ \varphi_{tt}(t, x) = 2 \int_0^x \left( f_t^2 + g_t^2 \right) \, dy - 2 \int_0^x \left( f^2 + g^2 \right) \, dy. \]
This integral is split into two terms:
\[ 2 \int_0^x \chi_{\{\varphi_x > 0\}} \left( f^2 + g^2 \right) \left( \frac{f_t^2 + g_t^2}{f^2 + g^2} - 1 \right) \, dx + 2 \int_0^x \chi_{\{\varphi_x = 0\}} \left( f_t^2 + g_t^2 \right) \, dx. \tag{4.26} \]
From equation (1.22), we deduce that \( f_t^2 + g_t^2 = \frac{1}{1 + \varphi_x} \) on \( \{\varphi_x > 0\} \), so that, for the first term in (4.26), we may compute
\[ 2 \int_0^x \chi_{\{\varphi_x > 0\}} \left( f^2 + g^2 \right) \left( \frac{f_t^2 + g_t^2}{f^2 + g^2} - 1 \right) \, dy 
\begin{align*}
&= \frac{1}{2} \int_0^x \chi_{\{\varphi_x > 0\}} (u_x \circ \varphi + (\rho \circ \varphi)^2 - 4) \, dy \\
&= \frac{1}{2} \int_0^{\varphi(x)} \chi_{\varphi(t, \varphi_x > 0)} \left( u_x^2 + \rho^2 - 4 \right) \, dy \\
&= \frac{1}{2} \int_0^{\varphi(x)} \chi_{\varphi(t, \varphi_x > 0)} \left( u_x^2 + \rho^2 - 4 \right) \, dy - 2 \varphi(x).
\end{align*} \]
Since the second term in (4.26) vanishes identically on \( S \) for a.e. \( t \in [0, \infty) \) (cf. [21]), we infer (recalling that \( \|u_x(t,.)\|^2 + \|\rho(t,.)\|^2 \equiv 4 \))
\[ \varphi_t = \Gamma_{(\varphi, \theta)}^{(1)} ((\varphi_t, \theta_t), (\varphi_t, \theta_t)) \quad \text{a.e. in time.} \]
As \( \theta_t = \rho \circ \varphi \), one immediately sees that \( \theta_t = -(u_x \rho) \circ \varphi \), and so
\[ \theta_t = \Gamma_{(\varphi, \theta)}^{(2)} ((\varphi_t, \theta_t), (\varphi_t, \theta_t)) \quad \text{a.e. in time.} \]
This finishes the proof of Theorem 4.1. \( \square \)

Proof of Theorem 4.3. Condition (a). The first condition of Definition 4.1 is fulfilled as a consequence of Theorem 4.1(ii) and Lemma 3.1 In view of (4.18) and the identity
\[ \| (u(t, .), \rho(t, .)) \|_{\Lambda}^2 = \left\langle \left( \varphi_t, \theta_t \right), \left( \varphi_t, \theta_t \right) \right\rangle_{(\text{id},0)}, \quad t \geq 0, \]
we moreover see that equation (4.21) holds. \( \square \)

Proof of Theorem 4.3. Condition (b). The identity \( \rho(0, .) = \hat{\rho} \) a.e. on \( S \) follows directly from the definition of \( \rho \) [4.16], and the continuity of the map \( (t, x) \mapsto u(t, x) : [0, \infty) \times S \to \mathbb{R} \) is a result of arguments akin to those of [21]. \( \square \)
Proof of Theorem 4.2, Condition (c). That the map \( t \mapsto u_x(t,.) \) belongs to the space \( L_\infty([0,\infty);H^0(S)) \) can be proved, with the proper adaptations, as in [21]. Concerning the map \( t \mapsto \rho(t,.) \), we first notice that

\[
\int_S \rho(t,y) \theta(y) \, dy = \int_S \rho(t,\varphi(t,x)) \theta(\varphi(t,x)) \varphi_x(t,x) \, dx
\]

which shows that

\[
t \mapsto \int_S \rho(t,y) \theta(y) \, dy
\]

is continuous for each \( \theta \in C_\infty(S) \). For a general \( \theta \in H^0(S) \), we choose an approximating sequence \( \{ \theta_n \}_n \subseteq C_\infty(S) \). Since strong convergence implies weak convergence, it follows, for every fixed \( t \), that

\[
\int_S \rho(t,y) \theta_n(y) \, dy \xrightarrow{n \to \infty} \int_S \rho(t,y) \theta(y) \, dy.
\]

This implies, for any \( \theta \in H^0(S) \), that the map (4.27) is a point-wise limit of measurable maps and thus itself measurable. We finish the argument by invoking Pettis’ theorem. \( \square \)

Proof of Theorem 4.2, Condition (d). We make two claims.

Claim 1. For any test functions \( \eta \in C_\infty_c((0,\infty)) \) and \( \vartheta \in C_\infty(S) \), it holds that

\[
\int_S \left\{ \int_0^\infty \left( \Gamma^{(1)}_{(id,0)}(\vartheta) - uu_x \right) \eta(t) \, dt \right\} \vartheta(y) \, dy = 0.
\]

Claim 2. Moreover, for any test functions \( \eta \in C_\infty_c((0,\infty)) \) and \( \vartheta \in C_\infty(S) \), one has that

\[
\int_S \left\{ \int_0^\infty \left( \Gamma^{(2)}_{(id,0)}(\vartheta) - up_x \right) \eta(t) \, dt \right\} \vartheta(y) \, dy = 0.
\]

For the moment, let us assume that Claim 1 is true. Then

\[
\int_0^\infty u(t,y) \eta(t) \, dt = \int_0^\infty \left( \Gamma^{(1)}_{(id,0)}(\vartheta) - uu_x \right) (t,\eta(t)) \, dt
\]

for any \( \eta \in C_\infty_c((0,\infty)) \), so that we may conclude that the map \( t \mapsto u(t,.) \), \([0,\infty) \to H^0(S)\) is absolutely continuous and that the equality

\[
u(t) = \hat{u} + \int_0^t \left( \Gamma^{(1)}_{(id,0)}((u,\rho),(u,\rho)) - uu_x \right) (s) \, ds
\]

holds in \( H^0(S) \). Since

\[
\Gamma^{(1)}_{(id,0)}((u,\rho),(u,\rho)) = \left( x \mapsto \frac{1}{2} \int_0^x u_x^2 + \rho^2 \, dy - \frac{x}{2} \int_0^x u_x^2 + \rho^2 \, dy \right)
\]

this proves that \( u \) is a weak solution to the first component of the Hunter-Saxton system (1.1). As for the second component, (4.29) yields

\[
\int_0^\infty \rho(t,y) \eta_t \, dt = \int_0^\infty \Gamma^{(2)}_{(id,0)}((u,\rho),(u,\rho)) (t,y) \, \eta(t) \, dt,
\]
for any test function $\eta \in C^\infty_c((0, \infty))$, whence $t \mapsto \rho(t, \cdot), \, [0, \infty) \to H^0(S)$ solves
\[
\rho(t) = \int_0^t \Gamma^{(2)}_{(id,0)}((u, \rho), (u, \rho)) \, ds.
\]

In view of
\[
\Gamma^{(2)}_{(id,0)}((u, \rho), (u, \rho)) = (x \mapsto -(u \rho)_x),
\]
this shows that $\rho$ solves the second component of the Hunter-Saxton system (1.1). The proof is thus complete.

**Proof of Claim 1.** This proof follows, mutatis mutandis, the lines of [21] (p. 655), so we omit it here.

**Proof of Claim 2.** We study the spatial integrand in (4.29) first:
\[
\int_S \rho(t, y) \, \vartheta(y) \, dy = \int_S \rho(t, \varphi(t, x)) \, \vartheta(\varphi(t, x)) \, \varphi_x(t, x) \, dx
\]
\[
= \int_S \varrho_t(t, x) \, \vartheta(\varphi(t, x)) \, \varphi_x(t, x) \, dx.
\]

Integrating by parts in (4.29) now gives
\[
\int_0^\infty \int_S \rho(t, y) \, \vartheta(y) \, dy \, \eta(t) \, dt
\]
\[
= - \int_0^\infty \int_S \frac{d}{dt} \left( \varrho_t(t, x) \, \vartheta(\varphi(t, x)) \, \varphi_x(t, x) \right) \eta(t) \, dt
\]
\[
= - \int_0^\infty \int_S \left\{ \varrho_{tt}(t, x) \, \vartheta(\varphi(t, x)) \varphi_x(t, x) + \varrho_t(t, x) \, \varphi_t(t, x) \, \vartheta_x(\varphi(t, x)) \varphi_x(t, x)
\]
\[
- \varrho_t(t, x) \, \vartheta(\varphi(t, x)) \varphi_{tx}(t, x) \right\} \, dx \eta(t) \, dt.
\]

If we employ the relation $\varrho_{tt} = \Gamma^{(2)}_{(id,0)}((\varphi_t, \varrho_t), (\varphi_t, \varrho_t))$ (cf. (4.19)), the change of variables $y = \varphi(t, x)$ yields
\[
- \int_0^\infty \int_S \left\{ \Gamma^{(2)}_{(id,0)}((u, \rho), (u, \rho))(t, y) \vartheta(y)
\]
\[
+ \rho(t, y) \, u(t, y) \, \vartheta_x(y) + \rho(t, y) \vartheta(y) \varrho_x(t, y) \right\} dy \eta(t) \, dt
\]
\[
= - \int_0^\infty \int_S \left( \Gamma^{(2)}_{(id,0)}((u, \rho), (u, \rho))(t, y) - u(t, y) \rho_x(t, y) \right) \vartheta(y) \, dy \, \eta(t) \, dt.
\]

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