The spectral gap for transfer operators of torus extensions over expanding maps

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Abstract

We study the spectral gap for transfer operators of the skew product $F : \mathbb{T}^d \times \mathbb{T}^\ell \to \mathbb{T}^d \times \mathbb{T}^\ell$ given by $F(x, y) = (Tx, y + \tau(x) \pmod{\mathbb{Z}^\ell})$, where $T : \mathbb{T}^d \to \mathbb{T}^d$ is a $C^\infty$ uniformly expanding endomorphism, and the fiber map $\tau : \mathbb{T}^d \to \mathbb{R}^\ell$ is a $C^\infty$ map. We construct a Hilbert space $W^{-s}$ for any $s < 0$, which contains all the Hölder functions of Hölder exponents $|s|$ on $\mathbb{T}^d \times \mathbb{T}^\ell$. Applying the method of semiclassical analysis, we obtain the dichotomy: either the transfer operator has a spectral gap on $W^{-s}$, or $\tau$ is an essential coboundary. In the former case, $F$ mixes exponentially fast for Hölder observables with Hölder exponents $|s|$; and in the latter case, either $F$ is not weak mixing and it is semiconjugate to a circle rotation, or $F$ is unstably mixing, i.e. it can be approximated by non-mixing skew products.

Keywords: spectral gap, transfer operators and Koopman operators, semiclassical analysis, torus extension

Mathematics Subject Classification numbers: 37D20, 37C30, 81Q20

Introduction

In this paper, we study the spectral gap property for transfer operators of torus extensions over expanding maps. The systems $F : \mathbb{T}^d \times \mathbb{T}^\ell \to \mathbb{T}^d \times \mathbb{T}^\ell$ that we consider are of the form $F(x, y) = (Tx, y + \tau(x) \pmod{\mathbb{Z}^\ell})$, which are skew products with expanding $T : \mathbb{T}^d \to \mathbb{T}^d$ on the base and torus rotations on the fibers $\mathbb{T}^\ell$ with rotation vectors $\tau(x) \in \mathbb{R}^\ell$ for any $x \in \mathbb{T}^d$. We obtain the following dichotomy: either the transfer operator of $F$ has a spectral gap on a certain Hilbert space and therefore the system has exponential decay of correlations with
respect to the smooth invariant measure, or the rotation function $\tau(x)$ over $\mathbb{T}^d$ is an essential coboundary. When the base map $T$ is fixed, the former case is open and dense in the $C^0$ topology of the space of skew products. The latter implies that either the system is not weak mixing and is semiconjugate to an expanding endomorphism crossing a circle rotation, or it is unstably mixing and can be approximated by non-mixing skew products.

A Hilbert space, denoted by $\mathcal{W}^{\nu}_{-s}$ for any $s < 0$, is constructed using the Sobolov spaces, which is contained in the distribution space and contains all Hölder functions of Hölder exponents $|s|$ over the phase space $\mathbb{T}^d \times \mathbb{T}^d$ of $F$ (see remark 2.2). It is well known that restricted to $L^2(\mathbb{T}^{d+\ell})$, the transfer operator does not have spectral gap. We define the Hilbert space $\mathcal{W}^{-s}$ such that its norm is stronger along $\mathbb{T}^d$-direction and weaker along $\mathbb{T}^d$-direction.

The method we use to get the spectral gap and thus the exponential mixing property is the semiclassical analysis. Instead of the Ruelle–Perron–Frobenius transfer operator, we use dual, operators hence prove that the Koopman operator has exponential decay of correlations under a so-called partially captive condition. It was recently shown that a simple but intuitive model—a circle extension of a circle expanding map has exponential decay of correlations, or $\tau$ is cohomologous to a piecewise constant. We remark that their analysis did not provide a finer spectral structure of the transfer operators.

In a somewhat different direction, the semiclassical analysis approach is used to study Ruelle–Pollicott resonances for some hyperbolic systems, see [14, 15, 17, 19–23], [27], etc. Applying this approach in the context of partially hyperbolic systems, Faure showed in [18] that a simple but intuitive model—a circle extension of a circle expanding map—has exponential decay of correlations under a so-called partially captive condition. It was recently pointed out in [32] that the partially captive condition is generic but much stronger than the non-coboundary condition of $\tau(x)$ for this two-dimensional model. Using similar techniques, Arnoldi established in [1] the asymptotic spectral gap and the fractal Weyl law for $SU(2)$ extensions of circle expanding maps under the partially captive condition, and later Arnoudi, Faure and Weich obtained a similar result in [2] for circle extensions of certain one-dimensional open expanding maps under a stronger condition called minimal captivity. An improved estimate of the spectral gap was recently obtained by Faure and Weich [24] for the model in [2].

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3 Butterley and Eslami [11] only obtained non-uniform norm estimates for twisted transfer operators over the base map $T$, which is sufficient to establish exponential mixing of $F$ but not enough to prove the spectral gap for the transfer operator associated to $F$. 
Let us mention some similar results in the context of suspension semi-flows over expanding maps. Pollicott [35] showed that a generic suspension semi-flow over an expanding Markov interval map is exponentially mixing. In the case when the base is a linear expanding map, Tsujii [41] constructed an anisotropic Sobolev space on which the transfer operator has spectral gap. Also, Baladi and Vallée [9] proved exponential mixing property for surface semi-flows without finite Markov partitions. We also point out that the construction of anisotropic distributional space for hyperbolic diffeomorphisms, contact Anosov flows and other hyperbolic systems has been developed during the last two decades, see e.g. [4–8], [42], etc.

The main technique we use in this paper is the semiclassical analysis, inspired by Faure [18] and other related works. A key ingredient in our analysis is that we introduce non-standard Sobolev spaces associated with dynamical weights. Although equivalent to the standard ones, these spaces are much more effective in extracting the spectral properties of the Koopman operator \( \hat{F} \) and its decompositions \( \{ \hat{F}_\nu \}_{\nu \in \mathbb{Z}} \). In fact, we convert each \( \hat{F}_\nu \) by unitary conjugation into a new operator \( \tilde{Q}_\nu \) such that \( \tilde{Q}_\nu^* \tilde{Q}_\nu \) is a pseudo-differential operator, whose symbol provides an upper bound for the operator norm of \( \hat{F}_\nu \). Moreover, we prove directly that the rotation vector \( \tau(x) \) is not an essential coboundary if and only if those upper bounds vanish uniformly exponentially fast as \( n \to \infty \) for all high frequencies \( \nu \), from which we conclude that \( \hat{F} \) has spectral gap. We remark that our approach bypasses the captive conditions, and has no dimension restrictions to either the base or the fiber.

This paper is organized as the following. The setting and statements of results are given in section 1. In section 2 we introduce some notions and results from classical and semiclassical analysis, including Fourier transform, Sobolev spaces, Pseudo-differential operators, Fourier Integral Operators, Egorov’s theorem, and \( L^2 \)-continuity theorems. This section is not necessary for the reader who is familiar with the theory. We prove the theorems of the paper in section 3 based on propositions 3.1 and 3.2, which provides the spectral properties of the Koopman operator and its decompositions. These two propositions are proved in section 4, using the classical and semiclassical analysis. A key estimates in the proof, stated in lemma 5.1, is postponed in section 5.

1. Statement of results

Let \( T = \mathbb{R}/\mathbb{Z} \), and let \( T : \mathbb{T}^d \to \mathbb{T}^d \) be a \( C^\infty \) uniformly expanding map such that

\[
\gamma := \inf_{(x,v) \in \mathbb{S}\mathbb{T}^d} |D_x T(v)| > 1,
\]

where \( \mathbb{S}\mathbb{T}^d \) is the unit tangent bundle over \( \mathbb{T}^d \). It is well known that \( T \) has a unique smooth invariant probability measure \( d\mu = h(x)dx \), where the density function \( h \in C^\infty (\mathbb{T}^d, \mathbb{R}^+) \). Further, \( T \) is mixing with respect to \( \mu \). Here and throughout this paper, we fix the expanding map \( T \).

Given a function \( \tau \in C^\infty (\mathbb{T}^d, \mathbb{R}^\ell) \), we define the skew product \( F : \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{T}^d \times \mathbb{T}^d \) by

\[
F \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} Tx \\ y + \tau(x) \pmod{\mathbb{Z}^\ell} \end{array} \right),
\]

which preserves the product measure \( dA = d\mu(x)dy \). We also denote the skew product by \( F_\tau \) when we aim to emphasize the rotation function \( \tau \).

**Definition 1.1.** A real-valued function \( \varphi \in C^\infty (\mathbb{T}^d, \mathbb{R}) \) is called an essential coboundary over \( T \) if there exist \( c \in \mathbb{R} \) and a measurable function \( u : \mathbb{T}^d \to \mathbb{R} \) such that
\[ \varphi(x) = c + u(x) - u(Tx), \quad \mu - \text{a.e.} \, x. \]

Let \( \mathcal{B} \) be the space of real-valued essential coboundaries over \( T \).

A vector-valued function \( \tau = (\tau_1, \tau_2, \ldots, \tau_\ell) \in C^\infty(\mathbb{T}^d, \mathbb{R}^\ell) \) is called an essential coboundary over \( T \) if \( \tau_1, \tau_2, \ldots, \tau_\ell \) are linearly dependent mod \( \mathcal{B} \), that is, there exist \( v \in \mathbb{R}^\ell \setminus \{0\} \), \( c \in \mathbb{R} \) and a measurable function \( u : \mathbb{T}^d \to \mathbb{R} \) such that
\[ v \cdot \tau(x) = c + u(x) - u(Tx), \quad \mu - \text{a.e.} \, x. \quad (1.3) \]

**Remark 1.2.**

(i) By Livsic theory (see e.g. [28]), the measurable function \( u : \mathbb{T}^d \to \mathbb{R} \) in (1.3) is in fact of class \( C^\infty \).

(ii) The functions \( \tau_1, \tau_2, \ldots, \tau_\ell \) are called integrally dependent mod \( \mathcal{B} \) if (1.3) holds for some \( v \in \mathbb{Z}^\ell \setminus \{0\} \). There are functions \( \tau_1, \tau_2, \ldots, \tau_\ell \) that are linearly dependent but not integrally dependent mod \( \mathcal{B} \), unless \( \ell = 1 \).

Let \( \mathcal{D}(\mathbb{T}^{d+\ell}) = C^\infty(\mathbb{T}^{d+\ell}) \). Its dual space \( \mathcal{D}'(\mathbb{T}^{d+\ell}) \) is the space of distributions on \( \mathbb{T}^{d+\ell} \).

The Koopman operator \( \hat{F} : \mathcal{D}(\mathbb{T}^{d+\ell}) \to \mathcal{D}(\mathbb{T}^{d+\ell}) \) is defined by \( \hat{F} \phi = \phi \circ F \). The (Ruelle–Perron–Frobenius) transfer operator \( \hat{F}^\ell : \mathcal{D}'(\mathbb{T}^{d+\ell}) \to \mathcal{D}'(\mathbb{T}^{d+\ell}) \) is the dual operator of \( \hat{F} \), defined by the duality
\[ (\hat{F}^\ell \psi)(\phi) = \psi(\hat{F} \phi) \quad \text{for any } \phi \in \mathcal{D}(\mathbb{T}^{d+\ell}), \quad \psi \in \mathcal{D}'(\mathbb{T}^{d+\ell}). \quad (1.4) \]

**Definition 1.3.** We say that a linear operator \( T : \mathcal{B} \to \mathcal{B} \) of a Banach space \( \mathcal{B} \) has a spectral gap if its spectrum
\[ \text{Spec}(T) = \{1\} \cup \mathcal{K}, \]
where 1 is a simple eigenvalue and \( \mathcal{K} \) is a compact subset of the unit open disk \( D = \{z \in \mathbb{C} : |z| < 1\} \).

Our main result is the following.

**Theorem 1.** Let \( (\mathbb{T}^{d+\ell}, F, dA) \) be the skew product as described above. Then the following dichotomy holds:

1. Either \( \tau(x) \) is an essential coboundary over \( T \);
2. or for any \( s < 0 \), there is an \( \hat{F}^\ell \)-invariant Hilbert space \( \mathcal{W}^{-s} \), which is contained in \( \mathcal{D}'(\mathbb{T}^{d+\ell}) \) and contains \( C^{-s}(\mathbb{T}^{d+\ell}) \), such that \( \hat{F}^\ell | \mathcal{W}^{-s} \) has a spectral gap.

**Remark 1.4.** It is well known that \( \hat{F}^\ell | L^2(\mathbb{T}^{d+\ell}) \) does not have spectral gap. Statement (2) of theorem 1 holds for some Hilbert space \( \mathcal{W}^{-s} \) whose norm is stronger along \( \mathbb{T}^d \)-direction and weaker along \( \mathbb{T}^{d+\ell} \)-direction than that of \( L^2(\mathbb{T}^{d+\ell}) \) (see definition 2.8 and remark 2.2).

Since the spectrum of a linear operator on a Hilbert space coincides with that of its dual operator on the dual space, we will instead prove the spectral gap for the Koopman operator in the latter case of the above dichotomy. That is,
\[ * \]
\[ \text{Here in equation (1.3) } \cdot \cdot \cdot \text{ denotes the standard inner product of two vectors in } \mathbb{R}^\ell. \]
Theorem 2. The following dichotomy holds:

(1) Either \( \tau(x) \) is an essential coboundary over \( T \);
(2) or for any \( s < 0 \), there is an \( \hat{F} \)-invariant Hilbert space \( \mathcal{W}^s \), which is contained in \( \mathcal{D}'(\mathbb{T}^{d+\ell}) \) and contains \( C^{-s}(\mathbb{T}^{d+\ell}) \), such that \( \hat{F}|\mathcal{W}^s \) has a spectral gap.

Theorem 1 is equivalent to theorem 2 if the space \( \mathcal{W}^s \) is defined to be the dual space of \( \mathcal{W}^s \). We will specify the construction of the Hilbert space \( \mathcal{W}^\tau \) in section 2.1.2 (see (2.7)), and prove the spectral gap for \( \hat{F}|\mathcal{W}^s \) in section 3.3.

Remark 1.5. The first case in theorems 1 and 2 is very rare in the sense that the closed subspace that consists of all essential coboundaries has infinite codimension in \( C^\infty(\mathbb{T}^d, \mathbb{R}^\ell) \). It means that the second case is generic in the space of skew products, i.e. there is a subset \( \mathcal{U} \) in \( C^\infty(\mathbb{T}^d, \mathbb{R}^\ell) \) that is open and dense in the \( C^0 \) topology such that for all \( \tau \in \mathcal{U} \), the transfer operator of the corresponding skew product \( F_\tau \) given in (1.2) has a spectral gap.

The infinite codimension of essential coboundaries is a crucial property in showing the stable ergodicity of skew products over general hyperbolic systems. Among tremendous results on this topic, we refer the reader to [10, 25, 26, 33], etc.

The mixing property of the system \((\mathbb{T}^{d+\ell}, F, dA)\) is quantified by the rates of decay of correlations. We say that the skew product \( F \) is exponentially mixing with respect to the smooth measure \( dA \) if there exists \( \rho \in [0, 1) \) such that for any pair of Hölder observables \( \phi, \psi \in C^\alpha(\mathbb{T}^{d+\ell}) \), \( \alpha > 0 \), the correlation function

\[
C_n(\phi, \psi; F, dA) = \left| \int \phi \circ F^n \cdot \psi \, dA - \int \phi \, dA \int \psi \, dA \right|
\]

satisfies \( C_n(\phi, \psi; F, dA) \leq C_{\phi, \psi} \rho^n \) for all \( n \geq 1 \), where \( C_{\phi, \psi} > 0 \) is a constant depending on \( \phi \) and \( \psi \).

Theorem 3. Let \( F = F_\tau : \mathbb{T}^d \times \mathbb{T}^\ell \rightarrow \mathbb{T}^d \times \mathbb{T}^\ell \) be defined as in (1.2). If \( \tau(x) \) is not an essential coboundary over \( T \), then \( F \) is exponentially mixing with respect to \( dA \) for any pair of Hölder observables \( \phi, \psi \in C^{-s}(\mathbb{T}^{d+\ell}) \) for any \( s > 0 \).

Remark 1.6. Theorem 3 follows directly from theorem 2. The conclusions of theorems 2 and 3 immediately imply the following dichotomy:

(1) Either \( F \) is exponentially mixing with respect to \( dA \); 
(2) Or \( \tau(x) \) is an essential coboundary over \( T \).

If \( d = \ell = 1 \), the dichotomy in the remark is proved by Butterley and Eslami [11], in which the circle expansion \( T \) and the rotation \( \tau \) are allowed to have a finite number of discontinuities.

In our context, we say that \( F = F_\tau \) is stably ergodic if \( F_\tau \) is ergodic for any \( \tau' \) that is \( C^0 \)-close to \( \tau \). The stable mixing property and stable exponential mixing property are defined in a similar fashion.

It was shown by Parry and Pollicott [33], and also by Field and Parry [26], that \((\mathbb{T}^{d+\ell}, F, dA)\) is weak mixing (or stably mixing) if and only if the functions \( \tau_1, \tau_2, \ldots, \tau_r \) are integrally independent (or linearly independent) mod \( \mathbb{Z}^5 \). In other words, theorem 3 asserts that if \( F \) is stably mixing, then it is exponentially mixing, and furthermore, it is stably exponentially mixing by

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5 Originally in [26], the independence is modulo \( \mathbb{V} + \mathbb{B} \) for some finite-dimensional subspace \( \mathbb{V} \) of \( C^\infty(\mathbb{T}^d, \mathbb{R}) \). In our setting, \( \mathbb{V} = \{0\} \) because the rotation function \( \tau \) is null-homotopic in \( C^\infty(\mathbb{T}^d, \mathbb{R}) \).
remark 1.5. We can say more about the ergodic properties of the skew product over an expanding map: Dolgopyat [13] proved that $F$ is stably ergodic if and only if it is exponentially mixing; Field and Parry [26] showed that stable ergodicity implies stable mixing property for skew products. Combining all these results and theorem 3, we immediately obtain the following corollary.

**Corollary 1.7.** Let $F$ be the skew product given by (1.2). The following statements are equivalent:

1. $F$ is stably ergodic;
2. $F$ is stably mixing;
3. $F$ is exponentially mixing;
4. $F$ is stably exponentially mixing.

**Remark 1.8.**

(i) In the case when $\ell = 1$, if $F$ is mixing, then $F$ is stably mixing and thus (stably) exponentially mixing. This is simply because that integral independence and linear independence mod $\mathcal{B}$ are the same for $\tau \in C^\infty(T^d, \mathbb{R})$.

(ii) Let $F_0 : T^3 \to T^3$ be given by

$$F_0(x, y_1, y_2) = (2\pi x, y_1 + \tau_0(x), y_2 + \sqrt{3}\tau_0(x)) \pmod{\mathbb{Z}^3},$$

where $\tau_0(x)$ is not a real-valued essential coboundary over the linear expanding map $x \mapsto 2x \pmod{\mathbb{Z}}$ of the circle. It is clear that $\tau_0(x)$ and $\sqrt{3}\tau_0(x)$ are integrally independent but not linearly independent mod $\mathcal{B}$, and hence $F_0$ is unstably mixing and thus unstably ergodic by [26, 33].

(iii) In [43], Zhang considered a circle extension $F$ of a linear circle endomorphism, and showed that if $F$ is stably ergodic in $C(T^2, T^2)$, then the SRB densities vary smoothly for maps in a neighborhood of $F$. The technique therein suggests that we may be able to obtain the dichotomy in theorem 1 for small perturbations of $F$, which need not be skew products.

Next we characterize the dynamical properties of $F = F_{\tau}$ when the rotation vector $\tau = (\tau_1, \tau_2, \ldots, \tau_\ell)$ is an essential coboundary, that is, the functions $\tau_1, \tau_2, \ldots, \tau_\ell$ are linearly dependent mod $\mathcal{B}$. There are two cases:

1. If $\tau_1, \tau_2, \ldots, \tau_\ell$ are integrally dependent mod $\mathcal{B}$, then the behaviors of $F_{\tau}$ in the $T^\ell$ direction become very simple, as we see in Part (iii) of the next theorem. In particular, $F_{\tau}$ is not weak mixing.
2. If $\tau_1, \tau_2, \ldots, \tau_\ell$ are linearly dependent but integrally independent mod $\mathcal{B}$, then $F_{\tau}$ is unstably mixing. We can approximate $F_{\tau}$ by a sequence of non-mixing skew products $F_{\tau(n)}$ as follows. Pick real-valued sequences $\{c_{n,i}\}_{n \in \mathbb{N}}, i = 1, 2, \ldots, \ell$, such that $\lim_{n \to \infty} c_{n,i} = 1$ and $c_{n,1}\tau_1, c_{n,2}\tau_2, \ldots, c_{n,\ell}\tau_\ell$ are integrally dependent mod $\mathcal{B}$. Then set

$$\tau(n) = (c_{n,1}\tau_1, c_{n,2}\tau_2, \ldots, c_{n,\ell}\tau_\ell).$$

A foliation $\mathcal{L}$ of a smooth manifold $M$ is of dimension $m$ if the leaves of $\mathcal{L}$ are $m$ dimensional submanifolds. For a smooth dynamical system $(F, M)$, a foliation $\mathcal{L}$ of $M$ is $F$ invariant if $F$ preserves the leaves, that is, $F(\mathcal{L}(z)) = \mathcal{L}(F(z))$ for any $z \in M$, where $\mathcal{L}(z)$ is the leaf of $\mathcal{L}$ containing $z$.

A smooth dynamical system $(F, M)$ is semiconjugate to a smooth system $(G, N)$ if there is a smooth map $\pi : M \to N$ such that $\pi \circ F = G \circ \pi$.  


Theorem 4. Let $F = F_\tau : \mathbb{T}^d \times \mathbb{T}^\ell \to \mathbb{T}^d \times \mathbb{T}^\ell$ be defined as in (1.2). The following conditions are equivalent.

(i) $\tau_1, \tau_2, \ldots, \tau_\ell$ are integrally dependent mod $\mathcal{B}$;
(ii) There is an $F$ invariant $d + \ell - 1$ dimensional foliation $\mathcal{L}$ of $\mathbb{T}^d \times \mathbb{T}^\ell$ and a vector $v \in \mathbb{Z}^\ell \setminus \{0\}$ such that restricted to each fiber $\{x\} \times \mathbb{T}^\ell$, the leaves of $\mathcal{L}|_{\{x\} \times \mathbb{T}^\ell}$ are $\ell - 1$ dimensional and normal to $v$.
(iii) $F$ is semiconjugate to the map $G = T \times R_c : \mathbb{T}^d \times \mathbb{T} \to \mathbb{T}^d \times \mathbb{T}$ through a continuous map $\pi : \mathbb{T}^d \times \mathbb{T}^\ell \to \mathbb{T}^d \times \mathbb{T}$, where $R_c : \mathbb{T} \to \mathbb{T}$ is a circle rotation with rotation number $c \in \mathbb{R}$. Further, $F$ is semiconjugate to $R_c$.
(iv) $F$ is not weak mixing.

2. Semiclassical analysis: preliminaries

In this section we introduce some notions and basic properties in semiclassical analysis which we are going to use. The distribution spaces and Sobolev spaces will be used in construction of the Hilbert space $W^s$ in theorem 2. The pseudo-differential operators (PDO) and Fourier integral operators (FIO) will be used to prove propositions 3.1 and 3.2, where the Egorov’s theorems and the $L^2$-continuity theorems are also used. For more information and details on the general theory of PDOs and FIOs, one can see in standard references (e.g. [16, 31, 39, 44]).

2.1. Function spaces

In this subsection we introduce the notion of distribution spaces and Sobolev spaces, and then construct the Hilbert space $W^s$ in theorem 2 using Sobolev spaces.

2.1.1. Distribution spaces and Sobolev spaces. Let $\mathcal{D}(\mathbb{T}^{d+\ell}) = C^\infty(\mathbb{T}^{d+\ell})$. Its dual space $\mathcal{D}'(\mathbb{T}^{d+\ell})$ is the space of distributions on $\mathbb{T}^{d+\ell}$. There is a natural injection $i : \mathcal{D}(\mathbb{T}^{d+\ell}) \hookrightarrow \mathcal{D}'(\mathbb{T}^{d+\ell})$ given by

$$\psi(\varphi) := \int_{\mathbb{T}^{d+\ell}} \psi(x,y)\varphi(x,y) \, dx \, dy$$

(2.1)

for any $\varphi, \psi \in \mathcal{D}(\mathbb{T}^{d+\ell})$. Hence we can regard that $\mathcal{D}(\mathbb{T}^{d+\ell}) \subset \mathcal{D}'(\mathbb{T}^{d+\ell})$.

The Fourier transform of $\varphi \in L^2(\mathbb{T}^d)$ is defined by

$$\hat{\varphi}(\xi) = \int_{\mathbb{T}^d} \varphi(x)e^{-i2\pi \xi \cdot x} \, dx, \quad \xi \in \mathbb{Z}^d.$$  

(2.2)

Note that the Fourier transform is an isometry from $L^2(\mathbb{T}^d)$ to $\ell^2(\mathbb{Z}^d)$, and the inverse transform is given by

$$\varphi(x) = \sum_{\xi \in \mathbb{Z}^d} \hat{\varphi}(\xi)e^{i2\pi \xi \cdot x}, \quad x \in \mathbb{T}^d.$$  

(2.3)

It is well known that if $\varphi \in \mathcal{D}(\mathbb{T}^d)$, then $\hat{\varphi}(\xi)$ converges to zero super-polynomially fast as $|\xi| \to \infty$.

Let $\omega$ be the counting measure over the lattice $\mathbb{Z}^d$ on $\mathbb{R}^d$, i.e. $\omega(\xi) = \sum_{\eta \in \mathbb{Z}^d} \delta(\xi - n)$ for $\xi \in \mathbb{R}^d$. Then by the above equations we have
\[ \varphi(x) = \int_{\mathbb{R}^d} \hat{\varphi}(\xi) e^{2\pi y \cdot \xi} \, dy \omega(\xi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2\pi (x-y) \cdot \xi} \varphi(y) \, dy \omega(\xi), \quad x \in \mathbb{T}^d. \]  

(2.4)

Denote \( \langle \xi \rangle = \sqrt{1+|\xi|^2} \), and introduce the standard \( s \)-inner product

\[ \langle \varphi, \psi \rangle_s = \sum_{\xi \in \mathbb{Z}^d} \langle \xi \rangle^{2s} \hat{\varphi}(\xi) \overline{\hat{\psi}(\xi)}, \quad \varphi, \psi \in \mathcal{D}(\mathbb{T}^d) \]  

(2.5)

for any \( s \in \mathbb{R} \). The Sobolev space \( H^s(\mathbb{T}^d) \) is the completion of \( \mathcal{D}(\mathbb{T}^d) \) under \( \langle \cdot, \cdot \rangle_s \).

**Proposition 2.1.** Sobolev spaces have the following properties:

(i) \( \mathcal{D}(\mathbb{T}^d) \subset H^s(\mathbb{T}^d) \subset \mathcal{D}'(\mathbb{T}^d) \) for any \( s \in \mathbb{R} \);

(ii) \( H^s(\mathbb{T}^d) = L^2(\mathbb{T}^d) \) if \( s = 0 \), and \( H^s(\mathbb{T}^d) = \{ \varphi : D^\beta \varphi \in L^2(\mathbb{T}^d) \text{ for any } |\beta| \leq s \} \) if \( s \in \mathbb{N} \), where \( D^\beta \varphi \) are weak derivatives of \( \varphi \);

(iii) \( H^s(\mathbb{T}^d) \subset H^s(\mathbb{T}^d) \) if \( s > s' \);

(iv) \( C^s(\mathbb{T}^d) \subset H^s(\mathbb{T}^d) \) for all \( s \geq 0 \), and if \( s > \frac{d}{2} \), then \( H^s(\mathbb{T}^d) \subset C^{-\frac{d}{2} - s}(\mathbb{T}^d) \) for any small \( \varepsilon > 0 \);

(v) the dual space of \( H^s(\mathbb{T}^d) \), \( s > 0 \), is \( H^{-s}(\mathbb{T}^d) \), and the dual action of \( \phi \in L^2(\mathbb{T}^d) \subset H^{-s}(\mathbb{T}^d) \) on the function \( \psi \in H^s(\mathbb{T}^d) \) is given by (2.1).

For technical treatments, besides the standard \( s \)-inner product given in (2.5), we will also use \( t \)-scaled \( s \)-inner product on \( H^s(\mathbb{T}^d) \) for \( t > 0 \), that is,

\[ \langle \varphi, \psi \rangle_s = t^{2s} \langle \varphi, \psi \rangle_s = \sum_{\xi \in \mathbb{Z}^d} t^{2s} \langle \xi \rangle^{2s} \hat{\varphi}(\xi) \overline{\hat{\psi}(\xi)}. \]  

(2.6)

When equipped with \( \langle \cdot, \cdot \rangle_s \), the space is denoted by \( H^s_\tau(\mathbb{T}^d) \). See section 3.1 for our particular choices of the scaling factor \( t \). We shall also introduce another different but equivalent inner product on \( H^s(\mathbb{T}^d) \) in section 4.1.

2.1.2. **The Hilbert space** \( \mathcal{W}^s \) and \( \mathcal{W}^{-s} \). The Hilbert space \( \mathcal{W}^s \) which we will use in theorem 2 is of the form

\[ \mathcal{W}^s = H^s(\mathbb{T}^d) \otimes H^{-s}(\mathbb{T}^d), \]  

(2.7)

where \( s < 0 \), equipped with the inner product given by

\[ \langle \varphi_1 \otimes \psi_1, \varphi_2 \otimes \psi_2 \rangle_{\mathcal{W}^s} = \langle \varphi_1, \varphi_2 \rangle_{H^s(\mathbb{T}^d)} \langle \psi_1, \psi_2 \rangle_{H^{-s}(\mathbb{T}^d)} \]

and extended by linearity\(^6\). We shall give a more explicit formula of \( \langle \cdot, \cdot \rangle_{\mathcal{W}^s} \) in section 3.1.

Using the duality (2.1), we see that the dual Hilbert space is

\[ \mathcal{W}^{-s} = (\mathcal{W}^s)' = H^{-s}(\mathbb{T}^d) \otimes H^s(\mathbb{T}^d). \]  

(2.8)

**Remark 2.2.** By proposition 2.1(ii) and (iii), we have \( L^2(\mathbb{T}^d) \subset H^s(\mathbb{T}^d) \) and \( C^{-s}(\mathbb{T}^d) \subset H^{-s}(\mathbb{T}^d) \) when \( s < 0 \), thus

\[ \mathcal{W}^s \supset L^2(\mathbb{T}^d) \otimes C^{-s}(\mathbb{T}^d) \supset C^{-s}(\mathbb{T}^{d+s}). \]

Similarly, the space \( \mathcal{W}^{-s} \) contains \( C^{-s}(\mathbb{T}^{d+s}) \) as well. By proposition 2.1(i), it is obvious that \( \mathcal{W}^s \) and \( \mathcal{W}^{-s} \) are both contained in \( \mathcal{D}'(\mathbb{T}^{d+s}) \).

\(^6\) In this paper, the tensor product of two Banach or Hilbert spaces always refers to the metric space completion of their algebraic tensor product.
Recall that the Koopman operator $\hat{T} : \mathcal{D}(\mathbb{T}^{d+\ell}) \to \mathcal{D}(\mathbb{T}^{d+\ell})$ for $F$ is defined by $\hat{T}\phi = \phi \circ F$, and the dual operator $\hat{T}^*: \mathcal{D}'(\mathbb{T}^{d+\ell}) \to \mathcal{D}'(\mathbb{T}^{d+\ell})$ is the RPF (Ruelle–Perron–Frobenius) transfer operator given by (1.4). With the duality given by (2.1), the transfer operator can be explicitly expressed as
\[
\hat{T}\psi(x,y) = \sum_{F(z,w) = (x,y)} \frac{\psi(z,w)}{|\text{Jac}(F)(z,w)|} \quad \text{for any } \psi \in \mathcal{D}(\mathbb{T}^{d+\ell}).
\]
See [36] for more details.

By the duality (2.1) again, we can extend the domain of the Koopman operator to $\mathcal{D}'(\mathbb{T}^{d+\ell})$ such that for any $\psi \in \mathcal{D}'(\mathbb{T}^{d+\ell})$, $\hat{T}\psi$ is the distribution satisfying
\[
(\hat{T}\psi)(\phi) = \psi(\hat{T}\phi) \quad \text{for any } \phi \in \mathcal{D}(\mathbb{T}^{d+\ell}).
\]
It is easy to see that $\hat{T}$ can act on $\mathcal{W}_h$ and $\hat{T}$ can act on $\mathcal{W}^{-\ell}$.

2.2. Semiclassical analysis on the torus

In this subsection we introduce the pseudo-differential operators and Fourier integral operators. The underlying manifold that we analyze is the torus. In standard references the quantization on manifolds is usually defined locally in charts. For the global definitions on tori we recommend chapter 4 in [38].

2.2.1. Symbols. The cotangent bundle over $\mathbb{T}^d$ can be identified as $T^*\mathbb{T}^d \cong \mathbb{T}^d \times \mathbb{R}^d$. Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}_0^d = (\mathbb{N}_0)^d$.

**Definition 2.3.** A complex-valued function $a \in C^\infty(T^*\mathbb{T}^d)$ is called a classical symbol of order $m \in \mathbb{R}$ on $T^*\mathbb{T}^d$ if
\[
\mathcal{N}_{\alpha,\beta,m}(a) := \sup_{(x,\xi) \in T^*\mathbb{T}^d} \frac{|\partial_\alpha^\alpha \partial_\beta^\beta a(x,\xi)|}{\langle \xi \rangle^{m-|\beta|}} < \infty
\]
for any $\alpha, \beta \in \mathbb{N}_0^d$, where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$.

A complex-valued function $a \in C^\infty(T^*\mathbb{T}^d \times (0,1])$ is called a semiclassical symbol of order $m \in \mathbb{R}$ on $T^*\mathbb{T}^d$ if (2.9) holds uniformly in $h$ with $a(x,\xi)$ replaced by $a(x,\xi;h)^\ell$.

The space of symbols of order $m$, either classical or semiclassical, is denoted as $S^m$, which is short for $S^m(T^*\mathbb{T}^d)$.

The topology on the space $S^m$ is generated by the seminorms $\{\mathcal{N}_{\alpha,\beta,m}(\cdot)\}_{\alpha,\beta \in \mathbb{N}_0^d}$ given by (2.9). For any $k \in \mathbb{N}_0$, we denote $\mathcal{N}_{k,m}(a) = \sup_{|\alpha|+|\beta| \leq k} \mathcal{N}_{\alpha,\beta,m}(a)$. We often write $\mathcal{N}_k(a)$ for short if the order of $a$ is clear.

Note that if $a \in S^m$ and $b \in S^{m'}$, then $a + b \in S^{\max\{m,m'\}}$, and $ab \in S^{m+m'}$. Also, for any $a \in S^m$ and $a \in \mathbb{N}_0^d$, $\partial_\alpha^\alpha a \in S^m$ and $\partial_\beta^\beta a \in S^{m-|\alpha|}$. For these operations, we have corresponding seminorm relations, for instance, $\mathcal{N}_k(\partial_\alpha^\alpha a) \leq \mathcal{N}_{k+|\alpha|}(a)$.

**Remark 2.4.** The standard toroidal symbols are defined on $\mathbb{T}^d \times \mathbb{R}^d$. Here we use symbols $a(x,\xi)$ defined on $T^*\mathbb{T}^d \cong \mathbb{T}^d \times \mathbb{R}^d$ since we need to use $a(x,h\xi)$ for $h \in (0,1]$ as well.

\[\text{In the general theory of semiclassical analysis, } h \ll 1 \text{ is the Planck’s constant parametrizing the whole family of symbol functions and thus the symbol calculus for corresponding semiclassical pseudo-differential operators.}\]
(see definitions \text{2.5} and \text{2.6}). However, the two different domains give the same family of symbols, because a symbol $\tilde{a} \in S^m(\mathbb{T}^d \times \mathbb{Z}^d)$ is a toroidal symbol if and only if there exists a symbol $a \in S^m(\mathbb{T}^d \times \mathbb{R}^d)$ such that $\tilde{a} = a|_{\mathbb{T}^d \times \mathbb{Z}^d}$ (see e.g. [38], theorem 4.5.3).

\textbf{2.2.2. Pseudo-differential operators.} Let $\hbar \in (0, 1]$.

\textbf{Definition 2.5.} Given a symbol $a \in S^m$, the linear operator $\text{Op}_\hbar(a) : \mathcal{D}(\mathbb{T}^d) \to \mathcal{D}(\mathbb{T}^d)$ defined by

$$\text{Op}_\hbar(a) \varphi(x) = \int_{\mathbb{T}^d} a(x, \xi) e^{2\pi i \frac{1}{\hbar} (x - \xi) y} \varphi(y) dy$$

$$= \int_{\mathbb{T}^d} a(x, \hbar \xi) e^{2\pi i \frac{1}{\hbar} (x - \xi) y} \varphi(y) dy d\xi$$

(2.10)

is called a (toroidal) $\hbar$-scaled pseudo-differential operator (PDO) of order $m$ corresponding to the symbol $a \in S^m$. We denote the space of $\hbar$-scaled PDOs of order $m$ by $\text{OP}_\hbar S^m$.

It is easy to check that (2.10) implies that $\text{Op}_\hbar(a)(C^\infty_0(\mathbb{T}^d)) \subset \mathcal{D}(\mathbb{T}^d)$, and thus $\text{Op}_\hbar(a)(D(\mathbb{T}^d)) \subset \mathcal{D}(\mathbb{T}^d)$ by the compactness of $\mathbb{T}^d$. Hence $\text{Op}_\hbar(a)$ is a well-defined operator from $\mathcal{D}(\mathbb{T}^d)$ to itself.

By standard duality argument, we extend $\text{Op}_\hbar(a) : \mathcal{D}'(\mathbb{T}^d) \to \mathcal{D}'(\mathbb{T}^d)$. That is, if we let $\text{Op}_\hbar(a)' : \mathcal{D}'(\mathbb{T}^d) \to \mathcal{D}'(\mathbb{T}^d)$ be the dual operator of $\text{Op}_\hbar(a)$, then for any $\varphi \in \mathcal{D}'(\mathbb{T}^d)$, $\text{Op}_\hbar(a)' \varphi$ is defined by the formula

$$\langle \text{Op}_\hbar(a)' \varphi, \psi \rangle = \langle \varphi, \text{Op}_\hbar(a) \psi \rangle \quad \forall \psi \in \mathcal{D}(\mathbb{T}^d).$$

Note that $\varphi(\psi) = \int_{\mathbb{T}^d} \varphi(x) \psi(x) dx$ if $\varphi \in \mathcal{D}(\mathbb{T}^d)$ is regarded as an element in $\mathcal{D}'(\mathbb{T}^d)$. It is then not hard to obtain

$$\text{Op}_\hbar(a)' \varphi(x) = \int_{\mathbb{T}^d} a(y, h\xi) e^{2\pi i \frac{1}{\hbar} (x - \xi) y} \varphi(y) dy \quad \forall \varphi \in \mathcal{D}(\mathbb{T}^d),$$

which implies that the dual operator $\text{Op}_\hbar(a)'$ preserves the space $\mathcal{D}(\mathbb{T}^d)$. Therefore, by the duality, for any $\psi \in \mathcal{D}'(\mathbb{T}^d)$, $\text{Op}_\hbar(a) \psi \in \mathcal{D}'(\mathbb{T}^d)$ can be defined by

$$\langle \text{Op}_\hbar(a) \psi, \varphi \rangle = \langle \varphi, \text{Op}_\hbar(a)' \psi \rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{T}^d).$$

Moreover, for any $s \in \mathbb{R}$ and any symbol $a \in S^m$, the $\hbar$-scaled PDO $\text{Op}_\hbar(a) : H^s(\mathbb{T}^d) \to H^{s-m}(\mathbb{T}^d)$ is a bounded operator. This fact is proven for $\mathbb{R}^d$ in proposition 5.5 of [40], for which the proof can be easily modified to $\mathbb{T}^d$.

The formula with $\hbar = 1$ in (2.10) gives the definition of classical pseudo-differential operator $\text{Op}(a) = \text{Op}_1(a)$. We denote $\text{OPS}^m = \text{OP}_1 S^m$. In this way, the $\hbar$-scaled PDO with semiclassical symbol $a \in S^m$ can be regarded as the classical PDO with symbol $a_\hbar \in S^m$, that is, $\text{Op}_\hbar(a) = \text{Op}(a_\hbar)$, where $a_\hbar(x, \xi) = a(x, h\xi; \hbar)$.

We see by (2.4), that if $a(x, \xi) = 1$, then $\text{Op}(a) = \text{Id}$; and if $a(x, \xi) = i2\pi \xi_k$ for some $k = 1, 2, \ldots, d$, then $\text{Op}(a) = \frac{\partial}{\partial x_k}$.

\textbf{2.2.3. Fourier integral operators.}

\textbf{Definition 2.6.} A (toroidal) $\hbar$-scaled Fourier integral operator (FIO) $\Phi_\hbar : \mathcal{D}(\mathbb{T}^d) \to \mathcal{D}(\mathbb{T}^d)$ with amplitude function $a \in S^m$ and a real-valued phase function $S \in S^1$ is of the form
\[ \Phi_h \varphi(x) = \Phi_h(a, S) \varphi(x) = \int_{T^* \mathbb{R}^d} a(x, \xi) e^{2\pi i \frac{S(x, \xi)}{\hbar} - y} \varphi(y) dy d\xi, \]

where the phase function \( S(x, \xi) \) satisfies the following conditions:

1. there are \( c_1, c_2 > 0 \) such that \( \left| \frac{\partial S(x, \xi)}{\partial \xi} \right| \geq c_1 |\xi| \) for all \( (x, \xi) \) with \( |\xi| \geq c_2; \)
2. \( S(x, \xi) \) is strongly non-degenerate, i.e. there is \( c_3 > 0 \) such that

\[ \left| \det \left( \frac{\partial^2 S(x, \xi)}{\partial x \partial \xi} \right) \right| \geq c_3 \quad \text{for any } (x, \xi) \in T^* \mathbb{R}^d. \]

Note that the classical Fourier integral operator \( \Phi = \Phi_1 \) is the one with \( h = 1. \)

By standard duality argument, we can extend \( \Phi_h : \mathcal{D}'(\mathbb{T}^d) \to \mathcal{D}'(\mathbb{T}^d). \) Further, \( \Phi_h : \mathcal{H}^s(\mathbb{T}^d) \to \mathcal{H}^{-m}(\mathbb{T}^d) \) is a bounded operator if its amplitude \( a \in S^m. \) This fact is proven for \( \mathbb{R}^d \) in proposition 3.1 of [37], for which the proof can be easily modified to \( \mathbb{T}^d. \)

**Remark 2.7.**

(i) Hörmander’s definition of phase functions usually assumes the homogeneity of degree one in \( \xi. \) Following Egorov [16], we replace the homogeneity by that \( S \in S^1 \) and condition (1).

(ii) If we take \( S(x, \xi) = x \cdot \xi, \) then \( \Phi_h(a, S) \) becomes an \( h \)-scaled pseudo-differential operator with the symbol \( a. \)

**Definition 2.8.** The canonical transformation associated to an \( h \)-scaled FIO with phase \( S \) is the transformation \( F_h : T^* \mathbb{R}^d \to T^* \mathbb{R}^d \) which sends \( (x, h\xi) \) to \( (y, h\eta), \) such that

\[ y = \frac{\partial S(x, \eta)}{\partial \eta}, \quad \xi = \frac{\partial S(x, \eta)}{\partial x}. \quad (2.11) \]

In other words, the phase function \( S \) serves as the generating function of the canonical transformation.

In the classical case when \( h = 1, \) we write \( F = F_1. \)

### 2.3. Some theorems.

Now we introduce some important theorems in semiclassical analysis and write them in a form that we will use to prove propositions 3.1 and 3.2.

#### 2.3.1. The symbol calculus.

If \( m < m', \) then \( S^m \subset S^{m'} \) and \( \text{OP}_h S^m \subset \text{OP}_h S^{m'}. \) Set \( S^{-\infty} = \bigcap_{m \in \mathbb{R}} S^m. \) If \( a \in S^{-\infty}, \) then \( \text{OP}_h(a) \) is a smoothing (and hence compact) operator.

Given two classical symbols \( a, b \in S^m, \) if the difference \( a - b \in S^{m'} \) for some \( -\infty \leq m' < m, \) we write \( a = b \pmod{S^{m'}}. \)

Given two semiclassical symbols \( a = a(x, \xi; h) \) and \( b = b(x, \xi; h) \) in \( S^m, \) if

\[ a(x, \xi; h) - b(x, \xi; h) = h^{m'-m} r(x, \xi; h) \]

for some \( r \in S^{m'}, \) we shall denote as \( a = b \pmod{h^{m'-m} S^{m'}} \) for short.

For classical PDOs, the following results are standard.
Theorem 2.9.

(1) Adjoint: If \( A \in \text{OPS}^m \) has a symbol \( a \), then the adjoint operator \( A^* \in \text{OPS}^m \) has a symbol \( a^* = \overline{a} \pmod{S^{m-1}} \).

(2) Composition: If \( A \in \text{OPS}^m \) has a symbol \( a \) and \( B \in \text{OPS}^m \) has a symbol \( b \), then the compositions \( A \circ B \in \text{OPS}^{m+m'} \) has a symbol \( a \# b = ab \pmod{S^{m+m'-1}} \).

(3) Inverse: If \( A \in \text{OPS}^m \) has an elliptic symbol \( a \) and is invertible, then \( A^{-1} \in \text{OPS}^{-m} \) has a symbol \( a^{-1} \pmod{S^{-m-1}} \).

Moreover, the \( \mathcal{N}_k \)-seminorm of all the remainders in the above modulo class only depends on the \( \mathcal{N}_k \)-seminorm of the original symbols.

Recall that \( \text{OP}_h(a) \) can be regarded as \( \text{OP}(a_h) \), where \( a_h(x, \xi) = a(x, h\xi) \). As a direct consequence of the above theorem and standard symbol calculus, we have the following rules of the symbol calculus for the \( h \)-scaled PDOs.

Theorem 2.10.

(1) Adjoint: If \( A \in \text{OP}_h S^m \) has a symbol \( a \), then the adjoint operator \( A^* \in \text{OP}_h S^m \) has a symbol \( a^* = \overline{a} \pmod{hS^{m-1}} \).

(2) Composition: If \( A \in \text{OP}_h S^m \) has a symbol \( a \) and \( B \in \text{OP}_h S^m \) has a symbol \( b \), then the compositions \( A \circ B \in \text{OP}_h S^{m+m'} \) has a symbol \( a \# b = ab \pmod{hS^{m+m'-1}} \).

(3) Inverse: If \( A \in \text{OP}_h S^m \) has an elliptic symbol \( a \) and is invertible, then \( A^{-1} \in \text{OP}_h S^{-m} \) has a symbol \( a^{-1} \pmod{hS^{-m-1}} \).

Moreover, if \( hr \) is one of the remainders in the above modulo class, then the seminorm \( \mathcal{N}_k(r) \) only depends on \( \mathcal{N}_{k+2}(a) \).

2.3.2. Egorov’s theorem. Let \( \Omega \) be an open domain in \( \mathbb{T}^d \). We say that a symbol \( a \in S^m \) is supported in \( \Omega \times \mathbb{R}^d \) if \( a(x, \xi) = 0 \) for any \( (x, \xi) \in (\mathbb{T}^d \setminus \Omega) \times \mathbb{R}^d \). The class of such symbols is denoted by \( S^m(\Omega \times \mathbb{R}^d) \).

We first state the original version of classical Egorov’s theorem in [16] for the invertible case.

Theorem 2.11. Let \( A \in \text{OPS}^m \) with symbol \( a \in S^m(\Omega \times \mathbb{R}^d) \), and \( \Phi \) be a classical FIO with amplitude \( b \in S^0 \) and phase \( S \). Let \( F(x, \xi) = (y, \eta) \) be the canonical transformation associated to \( \Phi \) (as defined in (2.11) with \( h = 1 \)), and \( \Omega' \) be the image of \( \Omega \times \mathbb{R}^d \) under the first \( d \) components of \( F \). We assume that \( F : \Omega \times \mathbb{R}^d \to \Omega' \times \mathbb{R}^d \) is a bijective map. Then the operator \( \Phi^* A \Phi \in \text{OPS}^m \) has a symbol \( \tilde{a} \in S^m(\Omega' \times \mathbb{R}^d) \) such that

\[
\tilde{a}(y, \eta) = \tilde{a}(F(x, \xi)) = a(x, \xi)|b(x, \xi)|^2 \left| \det \left( \frac{\partial^2 S}{\partial x \partial \xi} \right) \right|^{-1} \pmod{S^{m-1}}.
\]

Remark 2.12. From the proof in [16], it is easy to see that the \( \mathcal{N}_k \)-seminorm of the remainder in the above modulo class only relies on the \( \mathcal{N}_{k+2} \)-seminorms of \( a, b \) and the \( \mathcal{N}_k \)-seminorm of \( S \).

For our purpose, we need the following version of Egorov’s theorem.

\footnote{A symbol \( s \in S^m \) is elliptic if there are \( c > 0 \) and \( R > 0 \) such that \( |a(x, \xi)| \geq c|\xi|^m \) for all \( (x, \xi) \) with \( |\xi| > R \).}
Theorem 2.13. Let \( A \in \text{OPS}^m \) with symbol \( a \in S^m \), and \( \Phi \) be a classical FIO with amplitude \( b \in S^0 \) and phase \( S \). Let \( \mathcal{F}(x, \xi) = (y, \eta) \) be the canonical transformation associated to \( \Phi \). We assume that \( \mathcal{F} \) is a surjective local diffeomorphism of \( T^*T^d \) with finite inverse branches. Moreover, for each \( x \in \mathbb{T}^d \), the map \( \xi \mapsto \mathcal{F}(x, \xi) \) is bijective. Then the operator \( \Phi^*A\Phi \in \text{OPS}^m \) has a symbol \( \tilde{a} \) such that

\[
\tilde{a}(y, \eta) = \sum_{\mathcal{F}(x, \xi) = (y, \eta)} a(x, \xi) |b(x, \xi)|^2 \left| \det \left( \frac{\partial^2 S}{\partial x \partial \xi} \right) \right|^{-1} \quad (\text{mod } S^{m-1}).
\]

Moreover, the \( N_k \)-seminorm of the remainder in the above modulo class only relies on the \( N_{k+2} \)-seminorms of \( a, b \) and the \( N_{k+4} \)-seminorm of \( S \).

Proof. By the properties of the canonical map \( \mathcal{F} \), we can choose an finite open cover \( \{\Omega_i\} \) of \( \mathbb{T}^d \) such that each \( \Omega_i \times \mathbb{R}^d \) is strictly inside an inverse branch of \( \mathcal{F} \). By partition of unity, there are \( \chi_i \in C_0^\infty(\Omega_i; [0, 1]) \) such that \( \sum \chi_i = 1 \). We define symbols \( a_i \in S^m(\Omega_i \times \mathbb{R}^d) \) by \( a_i(x, \xi) = \chi_i(x)a(x, \xi) \), and set \( A_i = \text{Op}(a_i) \). By theorem 2.11, each \( \Phi^*A_i\Phi \in \text{OPS}^m \) has a symbol \( \tilde{a}_i \) such that

\[
\tilde{a}_i(y, \eta) = \chi_i(x)a(x, \xi) |b(x, \xi)|^2 \left| \det \left( \frac{\partial^2 S}{\partial x \partial \xi} \right) \right|^{-1} \quad (\text{mod } S^{m-1})
\]

for any \( (y, \eta) \in \mathcal{F}(\Omega_i \times \mathbb{R}^d) \) with the only pre-image \( (x, \xi) \) in \( \Omega_i \times \mathbb{R}^d \); and \( \tilde{a}_i(y, \eta) = 0 \) if \( (y, \eta) \not\in \mathcal{F}(\Omega_i \times \mathbb{R}^d) \). Therefore,

\[
\Phi^*A\Phi = \Phi^*\left( \sum_i A_i \right)\Phi = \sum_i \Phi^*A_i\Phi \in \text{OPS}^m,
\]

and its symbol \( \tilde{a}(y, \eta) \) is given by

\[
\sum_i \tilde{a}_i(y, \eta) = \sum_{\mathcal{F}(x, \xi) = (y, \eta)} \sum_i \chi_i(x)a(x, \xi) |b(x, \xi)|^2 \left| \det \left( \frac{\partial^2 S}{\partial x \partial \xi} \right) \right|^{-1} \quad (\text{mod } S^{m-1})
\]

\[
= \sum_{\mathcal{F}(x, \xi) = (y, \eta)} a(x, \xi) |b(x, \xi)|^2 \left| \det \left( \frac{\partial^2 S}{\partial x \partial \xi} \right) \right|^{-1} \left( \sum_{i \in \Omega_i} \chi_i(x) \right) \quad (\text{mod } S^{m-1})
\]

\[
= \sum_{\mathcal{F}(x, \xi) = (y, \eta)} a(x, \xi) |b(x, \xi)|^2 \left| \det \left( \frac{\partial^2 S}{\partial x \partial \xi} \right) \right|^{-1} \quad (\text{mod } S^{m-1}).
\]

The seminorm dependence of the remainder is straightforward by remark 2.12. \( \square \)

Remark 2.14. We can easily adapt the proof of the above two theorems for the \( h \)-scaled situation and show that if \( A \in \text{OP}_bS^m \) has a symbol \( a \in S^m \) and \( \Phi_b \) is the \( h \)-scaled FIO with amplitude \( b \in S^0 \) and phase \( S \), then the symbol of \( \Phi_b^*A\Phi_b \in \text{OPS}^m \) is still given by (2.12) but with \( \mathcal{F}(x, \xi) = (y, \eta) \) replaced by \( \mathcal{F}_b(x, h\xi) = (y, h\eta) \), and \( (\text{mod } S^{m-1}) \) replaced by \( (\text{mod } h^2 S^{m-1}) \). Moreover, if \( h \) is the remainder in the modulo class, then \( N_k(r) \) only depends on the \( N_{k+2} \)-seminorms of \( a, b \) and the \( N_{k+4} \)-seminorm of \( S \).

2.3.3. \( L^2 \)-continuity. The following result is the classical Calderon–Vainlancourt theorem (see theorem 2.8.1 in [31] for instance).
Theorem 2.15 (Calderon–Vaillancourt). Let \( a(x, \xi) \in S^0 \), then \( \text{Op}(a) : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d) \) is a bounded operator such that
\[
\|\text{Op}(a)\|_{L^2 \to L^2} \leq M_1 \|a\|_{C^k} \leq M_1 \hat{N}_k(a),
\]
for some \( M_1 > 0 \) and \( k_1 \in \mathbb{N} \) that only depend on the dimension \( d \).

To get finer \( L^2 \)-norm estimates, we first state a version of \( L^2 \)-continuity for a classical PDO of order \( 0 \) established in [20].

Theorem 2.16. If \( a(x, \xi) \in S^0 \), then \( \text{Op}(a) : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d) \) is a bounded operator. Moreover, for any \( \varepsilon > 0 \), there is a decomposition
\[
\text{Op}(a) = K(\varepsilon) + R(\varepsilon)
\]
such that \( K(\varepsilon) : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d) \) is a compact operator and
\[
\|R(\varepsilon)\|_{L^2 \to L^2} \leq \sup_{x, |\xi| \to \infty} \sup_x |a(x, \xi)| + \varepsilon = \sup_{x, |\xi| \to \infty} |a_0(x, \xi)| + \varepsilon,
\]
where \( a_0 \in S^0 \) is such that \( a = a_0 \pmod{S^{-1}} \).

For an \( h \)-scaled PDO of order \( 0 \), we need a version of Calderon–Vaillancourt theorem, which applies not only for \( h \to 0 \) but for arbitrary \( h \in (0, 1] \). See similar statements in [44], theorems 4.23 or 5.1 in the formulation of Weyl quantization.

Theorem 2.17. If \( a(x, \xi) \in S^0 \), then \( \text{Op}_h(a) : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d) \) is a bounded operator. Moreover, if \( a = a_0 + hr \) for some \( a_0 \in S^0 \) and \( r \in S^{-1} \), then for any \( \varepsilon > 0 \), we have
\[
\|\text{Op}_h(a)\|_{L^2 \to L^2} \leq \sup_{(x, \xi) \in \mathbb{R}^d} |a_0(x, \xi)| + \varepsilon + hC_{k_2}(\varepsilon, a_0, r),
\]
where the constant \( C_{k_2}(\varepsilon, a_0, r) \) only depends on \( \varepsilon \) and the \( \hat{N}_k \)-seminorms of \( a_0 \) and \( r \), for some \( k_2 \in \mathbb{N} \) that only depends on the dimension \( d \).

Proof. Recall that \( \text{Op}_h(a) = \text{Op}(a_h) \), where \( a_h(x, \xi) = a(x, h\xi; h) \in S^0 \), by Calderon–Vaillancourt theorem stated above, \( \text{Op}_h(a) \) is a bounded operator on \( L^2(\mathbb{T}^d) \). For the operator norm estimate, we mimic the proof in section 7.5 of [40], which is originally due to Hörmander, and which is called “square root trick”.

For any \( \varepsilon > 0 \), we set \( M = \sup_{(x, \xi) \in \mathbb{R}^d} |a_0(x, \xi)| + \varepsilon \), and \( b = \sqrt{M^2 - |a_0|^2} \in S^0 \). By theorem 2.10, the operator \( \text{Op}_h(a)^*\text{Op}_h(a) \in \text{OP}_hS^0 \) has a symbol
\[
a^* \# a = \bar{a}a \pmod{hS^{-1}} = |a_0|^2 + 2hR(a_0r) + h^2|r|^2 \pmod{hS^{-1}} = M^2 - b^2 \pmod{hS^{-1}},
\]
that is, \( a^* \# a = M^2 - b^2 + hr_1 \), for some \( r_1 \in S^{-1} \). Similarly, the operator \( \text{Op}_h(b)^*\text{Op}_h(b) \in \text{OP}_hS^0 \) has a symbol \( b^* \# b = b^2 \pmod{hS^{-1}} \), that is, \( b^* \# b = b^2 + hr_2 \), for some \( r_2 \in S^{-1} \). Therefore, for any \( \varphi \in L^2(\mathbb{T}^d) \).
\[ \|\text{Op}_h(a)\|_{L^2}^2 = \langle \text{Op}_h(a) \rangle \|\text{Op}_h(a)\varphi, \varphi\rangle_{L^2} \]
\[ = M^2\|\varphi\|_{L^2}^2 - \langle \text{Op}_h(b^2)\varphi, \varphi\rangle_{L^2} + h\langle \text{Op}_h(r_1)\varphi, \varphi\rangle_{L^2} \]
\[ = M^2\|\varphi\|_{L^2}^2 - \|\text{Op}_h(b)\|_{L^2}^2 + h\langle \text{Op}_h(r_1 + r_2)\varphi, \varphi\rangle_{L^2} \]
\[ \leq (M^2 + h\|\text{Op}_h(r_1 + r_2)\|_{L^2 \to L^2})\|\varphi\|_{L^2}^2 \]
\[ \leq (M + h)\|\text{Op}_h(r_1 + r_2)\|_{L^2 \to L^2}^2\|\varphi\|_{L^2}^2. \]

It remains to show that \( \|\text{Op}_h(r_1 + r_2)\|_{L^2 \to L^2} \) has an upper bound that is related to \( \varepsilon, a_0 \) and \( r \).
Indeed, by Calderon–Vaillancourt theorem,
\[ \|\text{Op}_h(r_1 + r_2)\|_{L^2 \to L^2} = \|\text{Op}((r_1 + r_2)h)\|_{L^2 \to L^2} \leq M_1 [\mathcal{N}_k(r_1) + \mathcal{N}_k(r_2)]. \]

By the construction of \( r_1 \), we have that \( \mathcal{N}_k(r_1) \) depends only on the \( \mathcal{N}_{k-2} \)-semimodules of \( a_0, r \), and thus only of \( a_0, r \), since \( \mathcal{N}_{k+2}(a) = \mathcal{N}_{k+2}(a_0 + hr) \leq \mathcal{N}_{k+2}(a_0) + \mathcal{N}_{k+2}(r) \). Similarly, \( \mathcal{N}_k(r_2) \) depends only on \( \varepsilon \) and the \( \mathcal{N}_{k-2} \)-semimodules of \( b \), and thus only of \( \varepsilon, a_0 \) and \( r \). In other words, let \( k_2 = k_1 + 2 \), then
\[ \frac{\|\text{Op}_h(r_1 + r_2)\|_{L^2 \to L^2}}{2M} \leq M_1 [\mathcal{N}_k(r_1) + \mathcal{N}_k(r_2)] \frac{\sup_{(\varepsilon, x, \xi)} |a_0(x, \xi)| + 2\varepsilon}{2\sup_{(\varepsilon, x, \xi)} |a_0(x, \xi)| + 2\varepsilon} \leq C_{k_3}(\varepsilon, a_0, r), \]

where \( C_{k_3}(\varepsilon, a_0, r) \) is a constant that only depends on \( \varepsilon \) and the \( \mathcal{N}_{k_3} \)-semimodules of \( a_0 \) and \( r \). This finishes the proof of the theorem.

3. Spectral gap and coboundary: proof of the theorems

3.1. Decomposition of Koopman operator

Let \( s < 0 \) be an arbitrary negative order. Recall that \( \mathcal{W} = H(T^d) \otimes H^{-s}(T^d) \) is defined by (2.7), and the Koopman operator \( \tilde{F} \) acts on \( \mathcal{W} \).

We shall decompose the Koopman operator \( \tilde{F} \) according to fiberwise Fourier expansion. More precisely, given \( \phi \in \mathcal{W} \), we write the Fourier series expansion along \( T^d \)-direction as
\[ \phi(x, y) = \sum_{\nu \in \mathbb{Z}^d} \phi_{\nu}(x)e^{2\pi i \nu \cdot y}, \]
where the Fourier coefficients are defined by
\[ \phi_{\nu}(x) = \int_{T^d} \phi(x, y)e^{-2\pi i \nu \cdot y}dy \in H^s(T^d), \quad \nu \in \mathbb{Z}^d. \]

It is clear that the family of functions \( \{e^{2\pi i \nu \cdot y}\}_{\nu \in \mathbb{Z}^d} \) forms an orthogonal basis of \( H^{-s}(T^d) \), and
\[ \|e^{2\pi i \nu \cdot y}\|_{H^{-s}(T^d)} = |\nu|^{-s}. \]
Therefore, the Hilbert inner product on \( \mathcal{W} \) is given by
\[ \langle \phi_1, \phi_2 \rangle_{\mathcal{W}} = \sum_{\nu \in \mathbb{Z}^d} |\nu|^{-2s}\langle \phi_{\nu}, \phi_{\nu} \rangle_{H^{-s}(T^d)}, \quad \text{for any} \ \phi_1, \phi_2 \in \mathcal{W}. \]

We thus consider the \( |\nu|^{-1} \)-scaled \( s \)-inner product \( \langle \cdot, \cdot \rangle_{s, \nu^{-1}} \) on \( H^s(T^d) \), or \( \langle \cdot, \cdot \rangle_{s, \nu} \) for short. That is, \( \phi, \psi \in H^s(T^d) \).
\begin{equation}
\langle \varphi, \psi \rangle_{\nu, \nu} = (\nu)^{-2} (\langle \varphi, \psi \rangle)_s = \sum_{\xi \in \mathbb{Z}^d} (\nu)^{-2} \langle \xi \rangle^2 \varphi(\xi) \overline{\psi(\xi)}.
\end{equation}

and we denote by \( H^s_\nu(\mathbb{T}^d) \) the space of \( s \)-order Sobolev functions on \( \mathbb{T}^d \) endowed with the new inner product \( \langle \cdot, \cdot \rangle_{\nu, \nu} \). Note that \( H^s_\nu(\mathbb{T}^d) = H^s(\mathbb{T}^d) \) as spaces of Sobolev functions, although they are equipped with different but equivalent inner products. In this way, we obtain an orthogonal decomposition
\[
\mathcal{W}^\nu = H^s(\mathbb{T}^d) \oplus H^{-s}(\mathbb{T}^d),
\]
such that the inner product of two functions \( \phi^j(x, y) = \sum_{\nu \in \mathbb{Z}^d} \phi^j_{\nu}(x) e^{i2\pi \nu \cdot y} \in \mathcal{W}^\nu, j = 1, 2 \), is given by
\[
\langle \phi^1, \phi^2 \rangle_{\mathcal{W}^\nu} = \sum_{\nu \in \mathbb{Z}^d} \langle \phi^1_{\nu}, \phi^2_{\nu} \rangle_{H^s_\nu(\mathbb{T}^d)}.
\]
Also this decomposition is \( \hat{F} \)-invariant, since for each Fourier mode \( \nu \in \mathbb{Z}^d \),
\[
\hat{F}(\phi_{\nu}(x) e^{i2\pi \nu \cdot y}) = [\phi_{\nu}(Tx) e^{i2\pi \nu \cdot y}] e^{i2\pi \nu \cdot y},
\]
and it can be shown that \( \phi_{\nu}(Tx) e^{i2\pi \nu \cdot y} \in H^s(\mathbb{T}^d) \cong H^s_\nu(\mathbb{T}^d) \). Correspondingly, we decompose \( \hat{F} = \bigoplus_{\nu \in \mathbb{Z}^d} \hat{F}_{\nu} \), where each \( \hat{F}_{\nu} \cong \hat{F}_0 \) acts by
\[
\hat{F}_{\nu} \varphi(x) = \varphi(Tx) e^{i2\pi \nu \cdot y}, \quad \varphi \in H^s_\nu(\mathbb{T}^d).
\]
Note that \( H^s_\nu(\mathbb{T}^d) \) is the dual space of \( H^{-s}_\nu(\mathbb{T}^d) \), and the dual operator \( \hat{F}_0^* \) of the operator \( \hat{F}_0 \) has the form
\[
\hat{F}_0^* \psi(y) = \sum_{\nu \in \mathbb{Z}^d} e^{i2\pi \nu \cdot y} \phi(\nu), \quad \psi \in H^{-s}_\nu(\mathbb{T}^d).
\]
In other words, \( \hat{F}_0^* H^{-s}_\nu(\mathbb{T}^d) \) is the RPF transfer operator over \( T : \mathbb{T}^d \to \mathbb{T}^d \) for the complex potential function \(-\log |\text{Jac}(T)| + i2\pi \nu \cdot \tau \). In the case when \( \nu = 0 \), we have that \( \hat{F}_0^* h = h \), that is, the density function \( h(x) \) of the smooth invariant measure \( \mu \) w.r.t. \( dx \) is provided by the eigenvector corresponding to the leading simple eigenvalue 1 of \( \hat{F}_0^* \). See [36] for more details.

In the study of the RPF transfer operators, we often need to normalize \( \hat{F}_0 \) into a new operator \( \mathcal{L} \) such that \( \mathcal{L} 1 = 1 \). To do so, we use the fact \( \hat{F}_0^* h = h \) in the following particular form:
\[
\sum_{T^n y = x} \mathcal{A}(y) = 1, \quad \text{for all } x \in \mathbb{T}^d,
\]
where
\[
\mathcal{A}(y) = \frac{1}{|\text{Jac}(T)(y)|} \frac{h(y)}{h(Ty)}.
\]
Then the normalized transfer operator is defined by \( \mathcal{L} \psi(x) = \sum_{T^n y = x} \mathcal{A}(y) \psi(y) \). Similarly, we have for all \( n \in \mathbb{N} \),
\[
\sum_{T^n y = x} \mathcal{A}_n(y) = 1, \quad \text{for all } x \in \mathbb{T}^d,
\]
This fact is easy to show for \( s \in \mathbb{N} \cup \{0\} \), and hence is also true when \( s \) is a negative integer by duality. For the general case, treat \( H^s \) as the interpolation between \( H^{|s|} \) and \( H^{|s|+1} \). See section 4.2 in [39] for details.
where
\[ \mathcal{A}_n(y) = \frac{1}{\text{Jac}(T^n)(y)} h(y). \] (3.5)

Then the iterates of \( \mathcal{L} \) is given by
\[ \mathcal{L}^n \psi(x) = \sum_{T^n y = x} \mathcal{A}_n(y) \psi(y). \]

In this paper, although we do not directly study \( \mathcal{L} \), we shall see that the factors \( \mathcal{A}(y) \) and \( \mathcal{A}_n(y) \) would appear in the formulas of symbols related to the Koopman operators.

3.2. Spectral gap

Recall that the notion of spectral gap is given by definition 1.3. According to the decomposition of \( \hat{F} : \mathcal{W}^s \to \mathcal{W}^s \), the spectral gap property for \( \hat{F} \) follows from the following propositions. The proof of the propositions will be given in the next section, using semiclassical analysis.

**Proposition 3.1.** Let \( s < 0 \). There are \( C_1 > 0 \) and \( \rho_1 \in (0, 1) \) such that for any \( \nu \in \mathbb{Z}^\ell \),
\[ \hat{F}_\nu : H^s_\nu(T^d) \to H^s_\nu(T^d) \] can be written as
\[ \hat{F}_\nu = K_\nu + R_\nu, \]
where \( K_\nu \) is a compact operator and
\[ \| R_\nu^n \|_{H^s_\nu(T^d)} \leq C_1 \rho_1^n, \quad n \in \mathbb{N}. \] (3.6)

**Proposition 3.2.** Let \( s < 0 \) and assume that \( \tau \) is not an essential coboundary. There are \( C_2 > 0, \rho_2 \in (0, 1) \) and \( \nu_1 > 0 \) such that for any \( \nu \in \mathbb{Z}^\ell \) with \( |\nu| \geq \nu_1 \),
\[ \| \hat{F}_\nu^n \|_{H^s_\nu(T^d)} \leq C_2 \rho_2^n, \quad n \in \mathbb{N}. \]

**Remark 3.3.**

(i) The quasi-compactness property is well known for Ruelle–Perron–Frobenius transfer operator on Hölder function spaces over expanding maps. Proposition 3.1 can be regarded as its dual version. The estimate in (3.6) shows that the essential spectral radius of \( \hat{F}_\nu \) is no more than \( \rho_1 \). See (4.6) for the definition of \( \rho_1 \), which depends on the Sobolev order \( s \) and the minimal expansion rate \( \gamma \) given by (1.1).

(ii) Proposition 3.2 shows that the operator \( \hat{F}_\nu \) is essentially a contraction when the Fourier mode \( \nu \) is very large, and the spectral radius of \( \hat{F}_\nu \) is no more than \( \rho_2 \). See (4.13) for the definition of \( \rho_2 \).

3.3. Proof of theorem 2

Recall that the space \( \mathcal{W}^s = H^s(T^d) \otimes H^{-s}(T^d) \) is defined in (2.7), where \( s < 0 \).

**Lemma 3.4.** The spectral radius \( \text{Sp}(\hat{F}_\nu|_{H^s_\nu(T^d)}) \leq 1 \) for \( \nu \in \mathbb{Z}^\ell \).

**Proof.** The proof is similar as in [20], as we sketch here.

It follows from proposition 3.1 that the essential spectral radius of \( \hat{F}_\nu \) is no more than \( \rho_1 \in (0, 1) \), and thus \( \hat{F}_\nu \) has a discrete spectrum outside the circle \( \{ z \in \mathbb{C} : |z| = \rho_1 \} \).
Let $\text{Spec}(\hat{F}_\nu)$ the spectrum of the operator $\hat{F}_\nu$. We can choose $\rho_3 \in (\rho_1,1)$ such that $\text{Spec}(\hat{F}_\nu) \cap \{z : |z| = \rho_3\} = \emptyset$ and $\text{Spec}(\hat{F}_\nu) \cap \{z : |z| > \rho_3\}$ consists of finitely many points. Let $\Pi_\nu$ be the spectral projection of $\hat{F}_\nu$ inside the circle $\{z : |z| = \rho_3\}$, that is, 

$$
\Pi_\nu = \frac{1}{2\pi i} \int_{|z|=\rho_3} (z\text{Id} - \hat{F}_\nu)^{-1}dz,
$$

then it is well known that $\Pi_\nu$ is a projection and it commutes with $\hat{F}_\nu$, i.e. $\Pi_\nu \hat{F}_\nu = \hat{F}_\nu \Pi_\nu$ (see e.g. [29]). Moreover, if we set $K_\nu^1 = (\text{Id} - \Pi_\nu) \hat{F}_\nu$ and $R_\nu^1 = \Pi_\nu \hat{F}_\nu$, then we have $\hat{F}_\nu = K_\nu^1 + R_\nu^1$ such that $K_\nu^1 R_\nu^1 = R_\nu^1 K_\nu^1 = 0$, and 

$$
\text{Spec}(K_\nu^1) = \text{Spec}(\hat{F}_\nu) \cap \{z : |z| > \rho_3\},
$$

$$
\text{Spec}(R_\nu^1) = \text{Spec}(\hat{F}_\nu) \cap \{z : |z| < \rho_3\}.
$$

In other words, $K_\nu^1$ has finite rank and the spectral radius of $R_\nu^1$ is less than $\rho_3$. To prove that $\text{Sp}(\hat{F}_\nu|_{H_s^\nu(T^d)}) \leq 1$ for $\nu \in \mathbb{Z}^d$, it is then sufficient to show that all eigenvalues of $K_\nu^1$ are of modulus no more than 1.

The general Jordan decomposition of $K_\nu^1$ can be written 

$$
K_\nu^1 = \sum_{i=1}^k \left( \lambda_i \begin{pmatrix} d_i & \nu_1 \otimes w_1 + \nu_d \otimes w_{d(j+1)} \end{pmatrix} \right),
$$

where $d_i$ is the dimension of the Jordan block associated with the eigenvalue $\lambda_i$, with $\nu_j \in H_0^\nu(T^d)$ and $w_j \in H_\nu^s(T^d)$. We arrange eigenvalues such that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_k|$. Now if $|\lambda_1| > 1$, we can choose $\varphi, \psi \in D(T^d)$ such that $\psi_1(\psi) \neq 0$ and $w_{11}(\varphi) \neq 0$ since $D(T^d)$ is dense in both $H_0^\nu(T^d)$ and $H_\nu^s(T^d)$. On one hand, 

$$
\left| \langle \psi, \hat{F}_\nu^\alpha \varphi \rangle_{H_\nu^s, H_0^\nu} \right| \leq \int_{T^d} |\psi||\varphi| \, dx \leq \|\psi\|_{C^0} \|\varphi\|_{C^0}.
$$

On the other hand, 

$$
\left| \langle \psi, \hat{F}_\nu^\alpha \varphi \rangle_{H_\nu^s, H_0^\nu} \right| \geq \left| \langle \psi, (K_\nu^1)^n \varphi \rangle_{H_\nu^s, H_0^\nu} \right| - \left| \langle \psi, (R_\nu^1)^n \varphi \rangle_{H_\nu^s, H_0^\nu} \right|.
$$

The second term converges to 0 since $\left\| (R_\nu^1)^n \right\|_{H_\nu^s(T^d)} = O(\rho_3^n)$, while the first term 

$$
\left| \langle \psi, (K_\nu^1)^n \varphi \rangle_{H_\nu^s, H_0^\nu} \right| \leq \sum_{i=1}^k \min(n,d_i-1) \begin{pmatrix} n \end{pmatrix} \lambda_i^{n-d_i} \sum_{j=1}^{d_i} w_j(\psi) w_{ij+1}(\varphi)
$$

has a leading growth $|\lambda_1|^n |\psi_1(\psi)| |w_{11}(\varphi)| \to \infty$ as $n \to \infty$, which is a contradiction. Therefore, all eigenvalues of $K_\nu^1$ are of modulus no more than 1.

\textbf{Lemma 3.5.} If $\tau$ is not an essential coboundary over $T$, then the spectral radius $\text{Sp}(\hat{F}_\nu|_{H_s^\nu(T^d)}) < 1$ for $\nu \in \mathbb{Z}^d \setminus \{0\}$. Moreover, $I$ is the only eigenvalue of $\hat{F}_0$ on the unit circle, which is simple with eigenspace of constant functions.
Proof. Note that the essential spectral radius of $\tilde{F}_\nu|H^s_H(\mathbb{T}^d)$ is no more than $\rho_1$ by (3.6). For any $\rho_3 \in (\rho_1, 1)$, by lemma 3.4, the spectrum of $\tilde{F}_\nu|H^s_H(\mathbb{T}^d)$ in \{z $\in \mathbb{C}: \rho_3 \leq |z| \leq 1$\} consists of finitely many isolated eigenvalues of finite multiplicity. Consequently, the spectral radius $Sp(\tilde{F}_\nu|H^s_H(\mathbb{T}^d))$ equals to the largest modulus of its eigenvalues.

Let $\lambda$ be an eigenvalue of $\tilde{F}_\nu$ with modulus 1, and $\varphi \in H^s_H(\mathbb{T}^d) \subset H^{s-\frac{d}{2}-1}_H(\mathbb{T}^d)$ be a corresponding eigenvector such that $\tilde{F}_\nu \varphi = \lambda \varphi$. To prove this lemma, it is sufficient to show that $\nu = 0$, $\lambda = 1$, and $\varphi$ is a constant function.

The following argument is essentially due to Pollicott [34]. By duality, there is $\psi \in H^{\nu+\frac{d}{2}+1}_H(\mathbb{T}^d) \subset C(\mathbb{T}^d)$ such that $\tilde{F}_\nu^* \psi = \lambda \psi$. Let $\lambda = e^{2\pi i c}$ for some $c \in \mathbb{R}$, and set $h(x) = \psi \frac{\psi}{|\psi|}$, where $h(x)$ is the density function of $\mu$ w.r.t. $dx$. By the definition of $\tilde{F}_\nu^*$ in (3.3) and $A(y)$ in (3.4), we have

$$\sum_{i=1}^n A(y) e^{2\pi i \nu \cdot \tau(y)} |\psi(y)| = \overline{\psi}( Ty).$$

(3.7)

Now choose $z$ such that $|\overline{\psi}(z)|$ obtains maximum. Since $\sum_{y=1}^n A(y) = 1$, we must have $|\overline{\psi}(y)| = |\overline{\psi}(z)|$ for all $y \in T^{-1}(z)$. By induction, we get that $|\overline{\psi}(y)| = |\overline{\psi}(z)|$ for all $y \in \bigcup_{n=1}^\infty T^{-n}(z)$. Since $T$ is mixing, the set $\bigcup_{n=0}^\infty T^{-n}(z)$ is dense in $\mathbb{T}^d$, and hence $|\overline{\psi}(x)| = |\overline{\psi}(z)|$ is constant. Thus (3.7) is a convex combination of points of a circle which lies on the circle. From this we deduce that

$$e^{2\pi i \nu \cdot \tau(y)} |\psi(y)| = \overline{\psi}( Ty)$$

for all $y \in \mathbb{T}^d$, and hence (adjust $c$ by an integer value if needed),

$$\nu \cdot \tau(y) = c - \frac{1}{2\pi} \arg \overline{\psi}(y) + \frac{1}{2\pi} \arg \overline{\psi}( Ty).$$

Since $\tau$ is not an essential coboundary over $T$, we must have $\nu = 0$. By integrating the last equation w.r.t. $d\mu$, we also have that $c = 0$ and thus $\lambda = 1$. Further, $\arg \overline{\psi}$ constant since it is $T$-invariant, and thus $\overline{\psi} \equiv$ constant, which implies that $\psi = h\overline{\psi} = h\overline{\psi}$ is a constant multiple of $h$. Therefore, the space $\{ \psi : \tilde{F}_0 \psi = \psi \}$ is one-dimensional, so is the space $\{ \varphi : \tilde{F}_0 \varphi = \varphi \}$ by duality. Since $\tilde{F}_0 1 = 1$, we must have that $\varphi$ is a constant function.

Now we are ready to prove theorem 2.

Proof of theorem 2. We assume that $\tau$ is not an essential coboundary. Hence, the results of proposition 3.2 can be applied.

By lemmas 3.4 and 3.5, we have the following:

(i) When $\nu = 0$, the spectrum $\text{Spec}(\tilde{F}_0) = \{ 1 \} \cup \mathcal{K}_0$, where 1 is a simple eigenvalue of $\tilde{F}_0$ and $\mathcal{K}_0$ is a compact subset of the open unit disk $\mathbb{D}$. Choose $r_0 \in [0, 1)$ such that $\mathcal{K}_0$ is strictly contained inside the circle $\{ z : |z| = r_0 \}$. Let $\Pi_0$ be the spectral projection of $\tilde{F}_0$ inside $\{ z : |z| = r_0 \}$, and note that $\ker(\Pi_0)$ is the one-dimensional subspace consisting of constant functions. Write $V_{\text{const}} = \ker(\Pi_0)$ and $V_0 = \text{Im}(\Pi_0)$. It is clear that

$$H^s_0(\mathbb{T}^d) = V_{\text{const}} \oplus V_0.$$

Moreover, both subspaces are preserved by $\tilde{F}_0$ since $\Pi_0 \tilde{F}_0 = \tilde{F}_0 \Pi_0$, and the spectra of $\tilde{F}_0$ restricted to these two subspaces are given by
Therefore, there is $M_0 > 0$ such that $\|\hat{F}_n|\nu\| \leq M_0 r_n^s$ for all $n \in \mathbb{N}$.
(ii) For all $\nu \in \mathbb{Z} \setminus \{0\}$, the spectrum $\text{Spec}(\hat{F}_\nu)$ is strictly inside the open unit disk $\mathbb{D}$.
Equivalently, there are $M_\nu > 0$ and $r_\nu \in [0, 1)$ such that $\|\hat{F}_\nu |H_\nu(T^d)\| \leq M_\nu r_\nu^n$ for all $n \in \mathbb{N}$.
And by proposition 3.2.
(iii) When $|\nu| \geqslant n_1$, $\|\hat{F}_\nu |H_\nu(T^d)\| \leq C_2 r_\nu^n$ for all $n \in \mathbb{N}$.

We set the direct sum

$$V = V_0 \oplus \left( \bigoplus_{\nu \in \mathbb{Z} \setminus \{0\}} H_\nu(T^d) \right), \quad (3.8)$$

then $\mathcal{W} = V_{\text{Const}} \oplus V$. Furthermore, let $C_4 := \max\{C_2, \max_{|\nu| < n_1} \{M_\nu\}\}$ and $\rho_4 := \max\{r_2, \max_{|\nu| < n_1} \{r_\nu\}\}$, then we have $\|\hat{F}_n|v\| \leq C_4 \rho_4^n$ for all $n \in \mathbb{N}$. In other words, $\hat{F} = \oplus_{\nu \in \mathbb{Z}} \hat{F}_\nu$ has spectrum

$$\text{Spec}(\hat{F}) = \{1\} \cup \mathcal{K},$$

where $\mathcal{K} = \text{Spec}(\hat{F}|V) \subset \{z \in \mathbb{C} : |z| \leq \rho_4\}$, and 1 is the only leading simple eigenvalue with eigenvectors being constant functions. So $\hat{F} : \mathcal{W} \to \mathcal{W}$ has spectral gap. \hfill \Box

3.4. Proof of theorem 3

Now we use theorem 2 to prove theorem 3. What we need to do is to show that if $\hat{F} : \mathcal{W} \to \mathcal{W}$ has a spectral gap, then it is exponentially mixing. In the proof we regard the Hölder continuous observables $\phi$ and $\psi$ as elements in $\mathcal{W}^s$ and $\mathcal{W}^s$ respectively.

**Proof of theorem 3.** Since $\hat{F} : \mathcal{W} \to \mathcal{W}$ has a spectral gap, we can write

$$\hat{F} = \hat{F}|V_{\text{Const}} + \hat{F}|V =: \mathcal{P} + \mathcal{N},$$

where $V$ is defined in (3.8). From the proof of theorem 2, we know that

(a) $\mathcal{P}$ is a 1-dimensional projection, i.e. $\mathcal{P}^2 = \mathcal{P}$.
(b) $\mathcal{N}$ is a bounded operator with spectral radius $\text{Sp}(\mathcal{N}) \leq \rho_4 < 1$. In fact, $||\mathcal{N}|n|| \leq C_4 \rho_4^n$ for all $n \in \mathbb{N}$.
(c) $\mathcal{P} \mathcal{N} = \mathcal{N} \mathcal{P} = 0$.

Furthermore, 1 is the greatest simple eigenvalue for $\hat{F}_0$ with eigenvector 1 as well as for $\hat{F}_0$ with eigenvector $h$. It means that the bilinear form associated to $\mathcal{P}$ on $\mathcal{W}^{s-1} \times \mathcal{W}^{s}$ is generated by $1 \otimes h \in \mathcal{W}^s \otimes \mathcal{W}^{s}$, that is, for any $\psi \in \mathcal{W}^{s-1}$ and $\phi \in \mathcal{W}^{s}$,

$$\langle \psi, \mathcal{P}(\phi) \rangle_{\mathcal{W}^{s-1}, \mathcal{W}^{s}} = \langle \psi, (1 \otimes h)(\phi) \rangle_{\mathcal{W}^{s-1}, \mathcal{W}^{s}}.$$

Suppose $\phi, \psi \in C^0(T^{d+\ell})$ are given. Pick $s \in [-\alpha, 0)$ and let $\mathcal{W}^{s-1} = H^{-s}(T^d) \otimes H^{s}(T^d)$. Then the dual space of $\mathcal{W}^{s-1}$ is $\mathcal{W}^{s} = H^{s}(T^d) \otimes H^{-s}(T^d)$. Note that $C^0(T^{d+\ell})$ is contained in both $\mathcal{W}^{s-1}$ and $\mathcal{W}^{s}$, and thus $\phi \in \mathcal{W}^{s}$ and $\phi h \in \mathcal{W}^{s-1}$, where $h \in C^{\infty}(T^d)$ is the density func-
tion of $\mu$ w.r.t. $dx$. Hence,

$$
\int (\phi \circ F^n)\psi dA = \int (\phi \circ F^n)\psi h \, dx dy \\
= (\psi h, F^n(\phi))_{W^{1-1},W^0} \\
= (\psi h, \mathcal{P}(\phi))_{W^{1-1},W^0} + (\psi h, N^\alpha(\phi))_{W^{1-1},W^0} \\
= (\psi h, (1 \otimes h)(\phi))_{W^{1-1},W^0} + (\psi h, N^\alpha(\phi))_{W^{1-1},W^0} \\
= \left( \int \psi h dx dy \right) \cdot 1_{W^{1-1},W^0} + (\psi h, N^\alpha(\phi))_{W^{1-1},W^0} \\
= \int \psi dA \int \phi dA + (\psi h, N^\alpha(\phi))_{W^{1-1},W^0}.
$$

That is, the correlation function

$$
C_{\phi, \psi} = \int (\psi h, N^\alpha(\phi))_{W^{1-1},W^0} \leq \|N^\alpha\| \|\psi h\|_{W^{1-1}} \|\phi\|_{W^0} \leq C_{\phi, \psi} \rho_4^n
$$

where $C_{\phi, \psi} = C_4 \|\psi h\|_{W^{1-1}} \|\phi\|_{W^0}$.

\[ \square \]

**Remark 3.6.** Using some Sobolev inequalities, it is not hard to get that $\|\psi h\|_{W^{1-1}} \leq C_5 \|\psi\|_{C^\alpha} \|h\|_{C^\alpha}$ and $\|\phi\|_{W^0} \leq C_6 \|\phi\|_{C^\alpha}$, and hence $C_{\phi, \psi} \leq C_7 \|\phi\|_{C^\alpha} \|\psi\|_{C^\alpha}$.

### 3.5. Proof of theorem 4

Now we show the characters of the non-mixing skew products $F_\tau$, that is, $\tau_1, \tau_2, \ldots, \tau_\ell$ are integrally dependent mod $\mathcal{B}$.

**Proof of theorem 4.**

(i) $\Rightarrow$ (ii). Suppose $\tau_1, \tau_2, \ldots, \tau_\ell$ are integrally dependent mod $\mathcal{B}$, that is, there are $\bar{c} \in \mathbb{R}$ and $u \in C^\infty(\mathbb{T}^d, \mathbb{R})$ such that

$$
\bar{c} \cdot \tau(x) = c + u(x) - u(Tx).
$$

For any $(x, y) \in \mathbb{T}^d \times \mathbb{T}^d$, the set

$$
\mathcal{L}(x, y) = \{(x', y') \in \mathbb{T}^d \times \mathbb{T}^d : \bar{c} \cdot y' + u(x') = \bar{c} \cdot y + u(x) \pmod{\mathbb{Z}}\}
$$

is well-defined. Moreover, since $u$ is a smooth map, $\mathcal{L}(x, y)$ is a smooth $(d + \ell - 1)$-dimensional manifold, and $\{\mathcal{L}(x, y) : (x, y) \in \mathbb{T}^d \times \mathbb{T}^d\}$ form a foliation of $\mathbb{T}^d \times \mathbb{T}^d$. It is clear that for any fixed $x \in \mathbb{T}^d$,

$$
\mathcal{L}(x, y)\big|_{\{x\} \times \mathbb{T}^d} = \{(x, y') \in \{x\} \times \mathbb{T}^d : \bar{c} \cdot (y - y') = 0 \pmod{\mathbb{Z}}\}.
$$

It implies that the leaves of $\mathcal{L}(x, y)\big|_{\{x\} \times \mathbb{T}^d}$ are normal to $\bar{c}$.

For $(x', y') \in \mathcal{L}(x, y)$, the definition of $F$ gives

$$
F(x, y) = (Tx, y + \tau(x)) \quad \text{and} \quad F(x', y') = (Tx', y' + \tau(x')),
$$

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then \( \nu \cdot (y + \tau(x)) + u(Tx) = \nu \cdot y + \nu \cdot \tau(x) + u(Tx) = \nu \cdot y + c + u(x) \pmod{\mathbb{Z}} \)

and similarly \( \nu \cdot (y' + \tau(x')) + u(Tx') = \nu \cdot y' + c + u(x') \pmod{\mathbb{Z}} \). Hence we obtain \( \nu \cdot (y' + \tau(x')) + u(Tx') = \nu \cdot (y + \tau(x)) + u(Tx) \pmod{\mathbb{Z}} \).

By definition of \( \mathcal{L} \), we get \( F(x', y') \in \mathcal{L}(F(x, y)) \), that is, the foliation is \( F \)-invariant.

(ii) \( \Rightarrow \) (iii). Restricted to \( \{ p \} \times \mathbb{T}^d \) the leaves of the foliation \( \mathcal{L} \) become \((\ell - 1)\) dimensional tori because the leaves are normal to \( \nu \). Hence the quotient space \( \mathbb{T}^d \times \sim \) is a circle \( \mathbb{T} \), where \( y \sim y' \) if \( y \) and \( y' \) are in the same leave of \( \mathcal{L}|_{\{ p \} \times \mathbb{T}^d} \). Let \( \pi|_{\{ p \} \times \mathbb{T}^d} : \{ p \} \times \mathbb{T}^d \to \{ p \} \times \mathbb{T} \) be the quotient map. Clearly \( \pi|_{\{ p \} \times \mathbb{T}} \) is continuous. Since \( F : \{ p \} \times \mathbb{T}^d \to \{ p \} \times \mathbb{T}^d \) is a rotation given by \( F(p, y) = (p, y + \tau(p)) \) and preserves the leaves, it induces an rotation \( G|_{\{ p \} \times \mathbb{T}} : \{ p \} \times \mathbb{T} \to \{ p \} \times \mathbb{T} \) such that \( \pi|_{\{ p \} \times \mathbb{T}} \circ F|_{\{ p \} \times \mathbb{T}} = G|_{\{ p \} \times \mathbb{T}} \circ \pi|_{\{ p \} \times \mathbb{T}} \). We also denote by \( R \) the rotation, where \( c \in \mathbb{T} \).

\( \pi|_{\{ p \} \times \mathbb{T}} \) and \( G|_{\{ p \} \times \mathbb{T}} \) can be extended to map \( \pi : \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{T}^d \times \mathbb{T} \) and \( G : \mathbb{T}^d \times \mathbb{T} \to \mathbb{T}^d \times \mathbb{T} \) in a natural way. That is, for any \( (x, y) \in \mathbb{T}^d \times \mathbb{T}^d \), let \( y' \in \mathcal{L}(x, y) \cap \{ \{ p \} \times \mathbb{T}^d \} \), and define \( \pi(x, y) = (x, \pi|_{\{ p \} \times \mathbb{T}}(y')) \); and for any \( (x, y) \in \mathbb{T}^d \times \mathbb{T} \) define \( G(x, y) = (Tx, G|_{\{ p \} \times \mathbb{T}}(y)) = (Tx, R_{y}(y)) \). Note that for all \( x \in \mathbb{T}^d \), the leaves of \( \mathcal{L}(x, y)|_{\{ x \} \times \mathbb{T}^d} \) are normal to \( \nu \). It is easy to check that \( \pi \circ F = G \circ \pi \).

(iii) \( \Rightarrow \) (iv). Weak mixing property does not hold for the circle rotation \( R \), let alone the extension \( F \).

(iv) \( \Rightarrow \) (i). It follows from the results by Parry and Pollicott [33], and also by Field and Parry [26].

4. Spectrums of \( \hat{F}_\nu \): proof of propositions 3.1 and 3.2

In this section, we shall prove the two main propositions—propositions 3.1 and 3.2. The main scheme of the proofs and constructions are originated from Faure [18] and many other related works. More precisely, we shall use semiclassical analysis to prove propositions 3.1 and 3.2. The flexibility of the parameter \( h \) allows us to deal with the operators \( \hat{F}_\nu : \mathcal{H}_\nu(\mathbb{T}^d) \to \mathcal{H}_\nu(\mathbb{T}^d) \), \( \nu \in \mathbb{Z}^d \), in two different ways. To be precise, for any fixed frequency \( \nu \in \mathbb{Z}^d \), we take \( h = 1 \) and treat \( \hat{F}_\nu \) as a classical FIO (up to a smoothing operator) in the proof of proposition 3.1; while for proposition 3.2, we set \( h = 1/\max \{ 1, |\nu| \} \) and regard \( \hat{F}_\nu \) as an \( h \)-scaled FIO (up to an \( h \)-scaled smoothing operator).

4.1. The Sobolev spaces with non-standard inner products

We first construct a particular symbol on \( T^* \mathbb{T}^d \) as follows. Choose

\[
R > \max \left\{ 1, \frac{\max \{ 1, 2||D\tau|| \} }{\gamma - 1} \right\}
\]  

(4.1)
where $\gamma$ is given in (1.1), and $\|D\tau\| = \sup_{x \in \mathbb{R}^d}|D_x \tau|$. Let $g_0 \in C^\infty(\mathbb{R}^+)$ be such that

$$g_0(t) = \begin{cases} 1, & t \leq R; \\ t, & t \geq \frac{\gamma + 1}{2} R, \end{cases} \tag{4.2}$$

and for $t \in [R, \frac{\gamma + 1}{2} R)$, $g_0(t)$ is strictly increasing and $1 \leq g_0(t) \leq t$. Set $g(\xi) = g_0(|\xi|)$ for $\xi \in \mathbb{R}^d$. Given $s < 0$, define an elliptic symbol

$$\lambda_s(x, \xi) = h(x) \frac{1}{2} g(\xi)^t \in S^t,$$ \tag{4.3}

where $h(x)$ is the density function of $\mu$ w.r.t. dx. Further, given $\nu \in \mathbb{Z}^d$, define

$$\lambda_{s, \nu}(x, \xi) = \lambda_s \left( x, \frac{\xi}{|\nu|} \right) \in S^t,$$ \tag{4.4}

where $[\nu] := \max\{1, |\nu|\}$.

Denote $\Lambda_{s, \nu} = \text{Op}(\lambda_{s, \nu}) \in \text{OPS}^s$, and define an inner product on $H^s(\mathbb{T}^d)$ by

$$\langle \varphi, \psi \rangle_{\Lambda_{\nu, s}} = \langle \Lambda_{s, \nu} \varphi, \Lambda_{s, \nu} \psi \rangle_{L^2}. \quad \varphi, \psi \in H^s(\mathbb{T}^d).$$

When equipped with $\langle \cdot, \cdot \rangle_{\Lambda_{s, \nu}}$, $H^s(\mathbb{T}^d)$ is denoted by $H^s_{\Lambda_{s, \nu}}(\mathbb{T}^d)$ instead. The Sobolev space $H^s_{\Lambda_{s, \nu}}$ is unitarily equivalent to the $L^2$ space, that is,

$$H^s_{\Lambda_{s, \nu}}(\mathbb{T}^d) \cong L^2(\mathbb{T}^d), \quad \text{or } H^s_{\Lambda_{s, \nu}}(\mathbb{T}^d) \cong \Lambda_{s, \nu}^{-1} L^2(\mathbb{T}^d).$$

We claim that the spaces $H^s_{\Lambda_{s, \nu}}(\mathbb{T}^d)$ and $H^s_{\nu}(\mathbb{T}^d)$, which are identical as the set of $s$-order Sobolev functions, have comparable inner products in the following sense: there is $C_1 = C_1(d, s) > 0$ such that

$$\frac{1}{C_1} |\langle \varphi, \psi \rangle_{\Lambda_{\nu, s}}| \leq |\langle \varphi, \psi \rangle_{\Lambda_{s, \nu}}| \leq C_1 |\langle \varphi, \psi \rangle_{\Lambda_{s, \nu}}|, \quad \text{for any } \varphi, \psi \in H^s(\mathbb{T}^d). \tag{4.5}$$

To see this, recall that $H^s_{\nu}(\mathbb{T}^d)$ is equipped with the $(\nu)^{-1}$-scaled $s$-inner product $\langle \cdot, \cdot \rangle_{s, \nu}$ given by (3.1). Alternatively, we have

$$\text{Op}(\nu^{-1}\langle \cdot \rangle^t_{\nu}) H^s_{\nu}(\mathbb{T}^d) \cong L^2(\mathbb{T}^d).$$

Then the comparability of inner products simply follows from that $\lambda_{s, \nu}(x, \xi) \propto (\nu)^{-t}(\xi)^s$, i.e. there is $C_0 = C_0(d, s) > 0$ such that for any $\nu \in \mathbb{Z}^d$,

$$\frac{1}{C_0} \leq \frac{\lambda_{s, \nu}(x, \xi)}{(\nu)^{-t}(\xi)^s} \leq C_0, \quad \text{for any } (x, \xi) \in T^* \mathbb{T}^d.$$

### 4.2. Proof of proposition 3.1

Recall that $F_{\nu} : H^s_{\nu}(\mathbb{T}^d) \to H^s_{\nu}(\mathbb{T}^d)$ is defined in (3.2). Switching to the inner product $\langle \cdot, \cdot \rangle_{s, \nu}$, we mainly study the operator $\hat{F}_{\nu} : H^s_{\Lambda_{s, \nu}}(\mathbb{T}^d) \to H^s_{\Lambda_{s, \nu}}(\mathbb{T}^d)$ instead.

**Proof of proposition 3.1.** Let $s < 0$ and $\nu \in \mathbb{Z}^d$ be fixed. By the formula of $\hat{F}_{\nu}$ in (3.2), the Fourier transform (2.2) and inverse transform (2.3), we rewrite

$$\hat{\hat{F}_{\nu}}(x) = \sum_{\xi \in \mathbb{Z}^d} \int_{\mathbb{T}^d} e^{i 2 \pi \nu \cdot \tau(s)} e^{i 2 \pi |\xi - y| \cdot \tau(y)} \psi(y) dy.$$
In the above formula, \( \hat{F}_\nu \) defines a classical (i.e. \( \hbar = 1 \)) toroidal Fourier series operator (FSO) with amplitude \( a^\nu(x, \xi) = e^{i2\pi \nu \tau(\xi)} \in S^0 \) and phase \( S(x, \xi) = Tx \cdot \xi \in S^1 \). For more details on FSO, we refer the readers to [38], section 4.13.

Let \( \Phi_1(a^\nu, S) \) be the classical toroidal FIO given by definition 2.6 with the same amplitude and phase. It is shown in [38] that the difference operator \( \hat{F}_\nu - \Phi_1(a^\nu, S) \) is smoothing and thus compact. Therefore, the result in proposition 3.1 holds for \( \hat{F}_\nu \) if and only if it holds for \( \Phi_1(a^\nu, S) \). In the rest of the proof, we shall analyze the classical toroidal FIO \( \Phi_1(a^\nu, S) \), but still denote it by \( \hat{F}_\nu \) for notational convenience.

The canonical transformation \( F : (x, \xi) \mapsto (y, \eta) \) associated to \( \hat{F}_\nu \) is given by

\[
y = Tx, \quad \eta = [(D, T)^{-1}] \xi,
\]

which is irrelevant to \( \nu \) since the phase function \( S \) is independent of \( \nu \).

We have the following commutative diagram

\[
\begin{array}{ccc}
H_{\Lambda_{\nu}, \nu}(\mathbb{T}^d) & \xrightarrow{\hat{F}_\nu} & H_{\Lambda_{\nu}, \nu}(\mathbb{T}^d) \\
\Lambda_{\nu, \nu} & \downarrow & \Lambda_{\nu, \nu} \\
L^2(\mathbb{T}^d) & \xrightarrow{Q_\nu} & L^2(\mathbb{T}^d),
\end{array}
\]

where \( Q_\nu = \Lambda_{\nu, \nu} \hat{F}_\nu \Lambda_{\nu, \nu}^{-1} \). We then set

\[
P_\nu = Q_\nu^* Q_\nu = (\Lambda_{\nu, \nu})^{-1} \left[ \hat{F}_\nu^* (\Lambda_{\nu, \nu}^* \Lambda_{\nu, \nu}) \hat{F}_\nu \right] \Lambda_{\nu, \nu}^{-1}
= (\text{Op}(\lambda_{\nu, \nu})^{-1})^* \left[ \Phi(a^\nu, S)^* (\text{Op}(\lambda_{\nu, \nu})^* \text{Op}(\lambda_{\nu, \nu})) \Phi(a^\nu, S) \right] \text{Op}(\lambda_{\nu, \nu})^{-1}.
\]

By the symbol calculus (theorem 2.9) and the Egorov’s theorem (theorem 2.13), the operator \( P_\nu \) is a classical PDO of order 0. Denote by \( p_\nu(x, \xi) \) the symbol of \( P_\nu \). By the \( L^2 \)-continuity theorem (theorem 2.16), \( P_\nu : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d) \) is a bounded operator such that for any \( \epsilon > 0 \), we can write \( P_\nu = K_\nu^0(\epsilon) + R_\nu^0(\epsilon) = : K_\nu^0 + R_\nu^0 \), where \( K_\nu^0 \) is compact; and moreover, by lemma 4.1 below, the definition of \( g \) in section 4.1 and the definition \( \gamma \) in (1.1), we get

\[
\|R_\nu^0\|_{L^2 \to L^2} \leq \sup_x \sup_{|\xi| \to \infty} |p_\nu(x, \xi)| + \epsilon
= \sup_x \limsup_{|\xi| \to \infty} \sum_{y} A(y) \left( \frac{g((D, T)^{-1}(\xi/|\nu|))}{g(\xi/|\nu|)} \right)^{2s} + \epsilon
\leq \sup_x \sum_{y} A(y) \limsup_{|\xi| \to \infty} \left( \frac{|(D, T)^{-1}\xi|}{|\xi|} \right)^{2s} + \epsilon
\leq \sup_x \sum_{y} A(y) \gamma^{2s} + \epsilon = \gamma^{2s} + \epsilon.
\]

Choose \( \epsilon > 0 \) small enough such that

\[
\rho_1 := \sqrt{\gamma^{2s} + \epsilon} < 1.
\]
By the polar decomposition, \( Q_\nu = \sqrt{T_\nu^*} U_\nu \) for some partial isometry \( U_\nu : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d) \), and thus there is also a decomposition \( Q_\nu = K_\nu^* + R_\nu^* \) such that \( K_\nu^* \) is compact and \( \| R_\nu^* \|_{L^2 \to L^2} \leq \rho_\nu \).

By unitary equivalence between \( Q_\nu : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d) \) and \( F_\nu : H_{\Lambda,\nu}(\mathbb{T}^d) \to H_{\Lambda,\nu}(\mathbb{T}^d) \), there is a similar decomposition \( F_\nu = K_\nu + R_\nu \) such that \( K_\nu \) is compact, and \( \| R_\nu \|_{H_{\Lambda,\nu}(\mathbb{T}^d) \to \mathcal{B}(\mathbb{C}^n)} \leq \rho_\nu \).

By the choice of the constant \( C_1 \) in \( (4.5) \), we get

\[
\| P_\nu^{\text{op}} \|_{H^0_{\Lambda,\nu}(\mathbb{T}^d) \to \mathcal{B}(\mathbb{C}^n)} \leq C_1 \| P_\nu \|_{H_{\Lambda,\nu}(\mathbb{T}^d) \to \mathcal{B}(\mathbb{C}^n)} \leq C_1 \| R_\nu \|_{H_{\Lambda,\nu}(\mathbb{T}^d) \to \mathcal{B}(\mathbb{C}^n)} \leq C_1 \rho_\nu.
\]

This completes the proof of proposition 3.1.

**Lemma 4.1.** \( P_\nu \in \text{OPS}^0 \) has a symbol

\[
p_\nu(x, \xi) = \sum_{x = Tx_t} A(y) \left( \frac{g((D_1T)^t(\xi/\nu))}{g(\xi/\nu)} \right)^{2s} \left( \text{mod } S^{-1} \right),
\]

where \( A(y) \) is defined in \( (3.4) \).

**Proof.** Note that \( \Lambda_{x,\nu} \in \text{OPS}^0 \) has a symbol \( \lambda_{x,\nu} \) given by \( (4.4) \). By theorem \( 2.9 \), \( \Lambda_{x,\nu}^* \in \text{OPS}^0 \) has a symbol \( \lambda_{x,\nu}^* \) \( \mod S^{-1} \); \( \Lambda_{x,\nu}^* \in \text{OPS}^{-s} \) has a symbol \( \lambda_{x,\nu}^{-s} \) \( \mod S^{-1} \), and so does \( (\Lambda_{x,\nu}^*)^* \in \text{OPS}^{-s} \). Further, \( \Lambda_{x,\nu}^* \Lambda_{x,\nu} \in \text{OPS}^{2s} \) has a symbol \( \lambda_{x,\nu}^{2s} \) \( \mod S^{2s-1} \). Then by Egorov’s theorem \( 2.13 \), \( \tilde{F}_\nu(\Lambda_{x,\nu}^* \Lambda_{x,\nu}) F_\nu \in \text{OPS}^{2s} \) has a symbol

\[
\tilde{a}(y, \eta) = \sum_{x = Tx_t} \lambda_{x,\nu}^2(x, \xi) \left( g^{2s-(\xi/\nu)} \right)^{2s} \left( \text{det}(D_1T)^t \right)^{2s-1} \left( \text{mod } S^{2s-1} \right)
\]

Use the composition rule and recall the definition of \( \lambda_{x,\nu} \) in \( (4.4) \), we have \( P_\nu \in \text{OPS}^0 \) with a symbol

\[
p_\nu(x, \eta) = \sum_{x = Tx_t} \lambda_{x,\nu}^2(x, (D_1T)^t(\eta/\nu)) \frac{1}{|\text{Jac}(T)(x)|} \left( \text{mod } S^{-1} \right)
\]

\[
= \sum_{x = Tx_t} \frac{1}{|\text{Jac}(T)(x)|} \left( \text{mod } S^{-1} \right)
\]

\[
= \sum_{x = Tx_t} A(x) \left( \frac{g((D_1T)^t(\eta/\nu))}{g(\eta/\nu)} \right)^{2s} \left( \text{mod } S^{-1} \right).
\]

This is what we need.

**4.3. Proof of proposition 3.2**

To prove proposition 3.2, we relate the semiclassical parameter \( h \) with a given frequency \( \nu \in \mathbb{Z}^d \) by setting \( h = 1/|\nu| \). In this way, we study the operator \( \tilde{F}_\nu \) as an \( h \)-scaled toroidal
FSO, and hence separate the dependence of $\hat{F}_\nu^\omega$ on the frequency $\nu$ into two parts: the dependence on the modulus $|\nu| = 1/\hbar$ and that on the direction vector $n_\nu := \nu/|\nu|$. The key step of the proof is the estimate stated and proved in lemma 5.1 in the next section.

**Proof of proposition 3.2.** Let $s < 0$ and assume that $\tau$ is not an essential coboundary over $T : \mathbb{T}^d \to \mathbb{T}^d$. Given $\nu \in \mathbb{Z}^d$, set $\hbar = 1/|\nu|$, where $|\nu| = \max\{1,|\nu|\}$. Also denote by $n_\nu = \nu/|\nu|$ the direction vector of $\nu$, and note that either $n_\nu = 0$ (only if $\nu = 0$) or $n_\nu$ lies on the $(\ell - 1)$-dimensional unit sphere $\mathbb{S}^{\ell-1}$.

For any $n \in \mathbb{N}$, the operator $\hat{F}_\nu^\omega$ can then be rewritten as

$$
\hat{F}_\nu^\omega \cdot (x) = \varphi(T^nx) e^{i2\pi \nu \cdot \sum_{k=0}^{n-1} \tau(T^kx)}
= \sum_{\xi \in \mathbb{Z}^d} \int_{T^d} e^{i2\pi \left((T^nx + \xi + \nu \cdot \sum_{k=0}^{n-1} \tau(T^kx)) - y \cdot \xi\right)} \varphi(y) dy
= \sum_{\xi \in \mathbb{Z}^d} \int_{T^d} e^{i2\pi \frac{1}{\hbar} \left((T^nx + \nu \cdot \sum_{k=0}^{n-1} \tau(T^kx)) - y \cdot \xi\right)} \varphi(y) dy.
$$

That is, $\hat{F}_\nu^\omega$ is regarded as an $\hbar$-scaled toroidal Fourier series operator (FSO) with amplitude $a = 1$ and phase

$$
S_{n_\nu}(x, \xi) = T^nx \cdot \xi + n_\nu \cdot \sum_{k=0}^{n-1} \tau(T^kx).
$$

Please see [38], section 4.13 for details about FSO.

Let $\Phi_{\hbar}(1, S_{n_\nu})$ be the $\hbar$-scaled toroidal FIO given by definition 2.6. It can be shown that the difference $\hat{F}_\nu^\omega - \Phi_{\hbar}(1, S_{n_\nu}) = R$ for some smoothing operator $R$. Therefore, as $\hbar = 1/|\nu| \to 0$, the result in proposition 3.2 holds for $\hat{F}_\nu^\omega$ if and only if it holds for $\Phi_{\hbar}(1, S_{n_\nu})$. In the rest of the proof, we shall analyze the $\hbar$-scaled toroidal FIO $\Phi_{\hbar}(1, S_{n_\nu})$, but still denote it by $\hat{F}_\nu^\omega$ for notational convenience.

Note that the canonical transformation $\mathcal{F}_h : (x, h\xi) \mapsto (y, h\eta)$ associated to $\hat{F}_\nu^\omega$ is given by

$$
y = T^nx, \quad \eta = \left[(D_x T^x)^r\right]^{-1}[\xi - W_{n_\nu}(x)],
$$

where

$$
W_{n_\nu}(x) = W_n(x)n_\nu \quad \text{and} \quad W_n(x) = \sum_{k=0}^{n-1}(D_x T^x)^r(D_{y^r} T^y)^r.
$$

By (4.3), (4.4) and that $\hbar = 1/|\nu|$, we rewrite $\lambda_{s, \nu}(x, \xi) = \lambda_s(x, h\xi)$, and hence $\Lambda_{x, \nu} = \text{Op}(\lambda_{x, \nu}) = \text{Op}_h(\Lambda_x) \in \text{OP}_h(S^d)$. The following commutative diagram

$$
\begin{array}{c}
H_{\Lambda_{s, \nu}} (\mathbb{T}^d) \xrightarrow{\hat{F}_\nu^\omega} H_{\Lambda_{s, \nu}} (\mathbb{T}^d) \\
\Lambda_{x, \nu} \downarrow \quad \downarrow \Lambda_{x, \nu}
\end{array}
$$

$$
L^2(\mathbb{T}^d) \xrightarrow{\hat{Q}_{\nu, -}} L^2(\mathbb{T}^d)
$$

suggests that we should instead study the operator
\[ \tilde{P}_{\nu, n} = \tilde{Q}_{\nu, n} \tilde{Q}_{\nu, n} = (\Lambda_{\nu, n}^{-1})^* \left( \left( \tilde{F}_{\nu}^n \right)^* \left( \Lambda_{\nu, n}^* \Lambda_{\nu, n} \tilde{F}_{\nu}^n \right) \Lambda_{\nu, n}^{-1} \right) \]

\[ = (\text{Op}_h(\lambda_n)^{-1})^* \left( \Phi_h(1, S_{\nu, n})^* \Phi_h(1, S_{\nu, n}) \right) \text{Op}_h(\lambda_n)^{-1}. \quad (4.9) \]

By the $h$-scaled symbol calculus (theorem 2.10) and the $h$-scaled version of Egorov’s theorem (theorem 2.13), we have that $\tilde{P}_{\nu, n} \in \text{Op}_h S_0$, and it has a symbol of the form $\tilde{p}_{\nu, n} + h \tilde{r}_{\nu, n}$ given by lemma 4.2 below. Hence, by the $h$-scaled $L^2$-continuity theorem (theorem 2.17), we may choose a sufficiently small $\varepsilon_0 > 0$ such that

\[ \|\tilde{P}_{\nu, n}\|_{L^2(T^d)} \leq \sup_{(x, \xi) \in T^* T^d} \tilde{p}_{\nu, n}(x, \xi) + \varepsilon_0 + h C_{\tilde{F}} (\varepsilon_0, \tilde{p}_{\nu, n}, \tilde{r}_{\nu, n}). \]

Moreover, by part (2) in lemma 4.2, we get that there exists $L(n) > 0$ independent of $\nu$ such that $C_{\tilde{F}} (\varepsilon_0, \tilde{p}_{\nu, n}, \tilde{r}_{\nu, n}) \leq L(n)$. So we obtain

\[ \|\tilde{P}_{\nu, n}\|_{L^2(T^d)} \leq \sup_{(x, \xi) \in T^* T^d} \tilde{p}_{\nu, n}(x, \xi) + \varepsilon_0 + hL(n). \quad (4.10) \]

By sublemma 5.2 (1) in the next section, we have that for any $n \in \mathbb{N}$,

\[ \|\tilde{P}_{\nu, n}\|_{L^2(T^d)} \leq 2 + L(n), \]

and hence

\[ \|\tilde{F}_{\nu}^n|H_{\Lambda_{\nu, n}}(T^d)\| = \sqrt{\|\tilde{P}_{\nu, n}\|_{L^2(T^d)}} \leq \sqrt{2 + L(n)}. \quad (4.11) \]

Furthermore, by (4.10) and our key lemma—lemma 5.1 in the next section, there are $n_0 \in \mathbb{N}$ and $\tilde{p}_0 < 1$ such that for all $\nu \in \mathbb{Z} \setminus \{0\}$,

\[ \|\tilde{P}_{\nu, n_0}\|_{L^2(T^d)} \leq \sup_{(x, \xi) \in T^* T^d} \tilde{p}_{\nu, n_0}(x, \xi) + \varepsilon_0 + hL(n_0) \]

\[ \leq \tilde{p}_0 + \varepsilon_0 + \frac{L(n_0)}{|\nu|}. \quad (4.12) \]

Note that we are allowed to make $\varepsilon_0$ sufficiently small such that $\tilde{p}_0 + \varepsilon_0 < 1$ (although the function $L(\cdot)$ may be enlarged). Then we choose $\nu_1 > 0$ such that

\[ \rho_2 := \left( \tilde{p}_0 + \varepsilon_0 + \frac{L(n_0)}{\nu_1} \right)^{1/n_0} < 1. \quad (4.13) \]

By (4.12), we have for all $\nu \in \mathbb{Z} \setminus \{0\}$ with $|\nu| = |\nu| \geq \nu_1$,

\[ \|\tilde{F}_{\nu}^n|H_{\Lambda_{\nu, n}}(T^d)\| = \sqrt{\|\tilde{P}_{\nu, n_0}\|_{L^2(T^d)}} \leq \sqrt{\tilde{p}_0 + \varepsilon_0 + \frac{L(n_0)}{|\nu|}} \leq \rho_2. \quad (4.14) \]

Now for any $n \in \mathbb{N}$, we write $n = kn_0 + j$, where $k \in \mathbb{N}$ and $0 \leq j < n_0$. Then by (4.11) and (4.14), we have for all $\nu \in \mathbb{Z}$ with $|\nu| \geq \nu_1$,

\[ \|\tilde{F}_{\nu}^n|H_{\Lambda_{\nu, n}}(T^d)\| \leq \|\tilde{F}_{\nu}^n|H_{\Lambda_{\nu, n}}(T^d)\| \leq \rho_2^{k \nu_0} \sqrt{2 + L(j)} \leq C_2 \rho_2^k. \]
where we set
\[ C'_2 := \max_{1 \leq j < n} \rho_j^2 \sqrt{2 + L(j)}. \]

Switch back to the inner product \( \langle \cdot, \cdot \rangle_{s, \nu} \), and recall the choice of \( C_1 \) in (4.5). We take \( C_2 = C_1 C'_2 \), then for all \( \nu \in \mathbb{Z}^d \) with \( \| \nu \| \geq \nu_1 \),
\[ \| \tilde{F}_\nu^n H^n_{s, \nu}(T)^d \| \leq C_2 \rho_n^2. \]

This completes the proof of proposition 3.2. \( \square \)

**Lemma 4.2.** Given \( \nu \in \mathbb{Z}^d \), let \( h = 1/\| \nu \| \). For any \( n \in \mathbb{N} \), \( \tilde{P}_{\nu, n} \in \text{OP}_h S^0 \) has a symbol of the form \( \tilde{p}_{n, \nu} + h\tilde{r}_{n, \nu} \), where \( \tilde{p}_{n, \nu} \in S^0 \) and \( \tilde{r}_{n, \nu} \in S^{-1} \), such that

(1) \( \tilde{p}_{n, \nu} \) is positive, and is given by
\[ \tilde{p}_{n, \nu}(x, \xi) = \sum_{s = T^n y} A_n(y) \left( \frac{g(D_s T^n)^t \xi + W_{n, \nu}(y)}{g(\xi)} \right)^{2s}, \quad (4.15) \]
where \( A_n(y) \) is given in (3.5) and \( W_{n, \nu}(y) \) is given in (4.8);

(2) Let \( C_{k_3}(\cdot, \cdot) \) be as introduced in theorem 2.17. There is \( L(n) = L(n; d, T, \tau) \) independent of \( \nu \) such that
\[ C_{k_3}(\tilde{p}_{n, \nu}, \tilde{r}_{n, \nu}) \leq L(n). \]

**Proof.** Recall that \( \Lambda_{\nu, \nu} = \text{OP}_h(\lambda_\nu) \in \text{OP}_h S^0 \). By theorem 2.10, \( \Lambda_{\nu, \nu}^* \in \text{OP}_h S^0 \) has a symbol \( \lambda_\nu \) (mod \( hS^{-1} \)), and \( \Lambda_{\nu, \nu}^{-1} \in \text{OP}_h S^{-1} \) both have a symbol \( \lambda_\nu^{-1} \) (mod \( hS^{-1} \)). Further, \( \Lambda_{\nu, \nu}^* \Lambda_{\nu, \nu} \in \text{OP}_h S^{2t} \) has a symbol \( \lambda_\nu^2 \) (mod \( hS^{2t-1} \)). By the \( h \)-scaled version of the Egorov’s theorem (see theorem 2.13 and remark 2.14), \( F_{\nu, n}(\Lambda_{\nu, \nu}^* \Lambda_{\nu, \nu}) \tilde{F}_{\nu, n} \in \text{OP}_h S^{2t} \) has a symbol
\[ \tilde{a}_n(x, \xi) = \sum_{\nu = T^n y} \lambda_\nu^2 (y, \eta) \cdot 1^2 \cdot \det(D_s T^n)^t \xi \cdot \det(D_s T^n) \xi = \sum_{\nu = T^n y} \lambda_{2, \nu}^2 y, D_s T^n \xi + W_{n, \nu}(y) ) \mid \text{Jac}(T^n)(y) \mid \quad (\text{mod } hS^{2t-1}). \]

By composition rule again, \( \tilde{p}_{n, \nu} \in \text{OP}_h S^0 \) has a symbol
\[ \tilde{p}_{n, \nu}(x, \xi) = \sum_{\nu = T^n y} \lambda_{2, \nu}^2 y, D_s T^n \xi + W_{n, \nu}(y) ) \mid \text{Jac}(T^n)(y) \mid \frac{1}{\lambda_{2, \nu}(x, \xi)} \quad (\text{mod } hS^{-1}) \]
\[ = \sum_{\nu = T^n y} \frac{1}{\text{Jac}(T^n)(y)} \frac{h(y) g((D_s T^n)^t \xi + W_{n, \nu}(y))^{2s}}{h(x) g(\xi)^{2s}} \quad (\text{mod } hS^{-1}) \]
\[ = \sum_{\nu = T^n y} A_n(y) \left( \frac{g((D_s T^n)^t \xi + W_{n, \nu}(y))}{g(\xi)} \right)^{2s} \quad (\text{mod } hS^{-1}). \]

This finishes the proof of the first part.
Since all the above modulo terms are calculated from the symbol \( \lambda \), and the phase \( S_{n_0, n} \), which depends on \( n_0 \) but not \( [\nu] \), we can write the full symbol of \( \tilde{\nu}_{\nu, n} \) by \( \tilde{p}_{n_0, n} + h\tilde{r}_{n_0, n} \) for some \( \tilde{r}_{n_0, n} \in S^{-1} \).

By theorem 2.17, \( C_0(\tilde{p}_{n_0, n}, \tilde{r}_{n_0, n}) \) is bounded by a constant which only depends on the \( \mathcal{N}_2 \)-seminorms of \( \tilde{p}_{n_0, n} \) and \( \tilde{r}_{n_0, n} \). To prove the second part of this lemma, it is sufficient to show that the \( \mathcal{N}_2 \)-seminorms of \( \tilde{p}_{n_0, n} \) and \( \tilde{r}_{n_0, n} \) are bounded by a term that does not depend on \( \nu \). Indeed, by the formula of \( \tilde{p}_{\nu, n} \) in (4.9), the \( h \)-scaled symbol calculus and Egorov’s theorem, we find that the \( \mathcal{N}_2 \)-seminorm of \( \tilde{p}_{n_0, n} \) depends on \( \mathcal{N}_2 \)-seminorm of \( \lambda_\nu \) and \( \mathcal{N}_2 \)-seminorm of \( S_{n_0, n} \); while the \( \mathcal{N}_2 \)-seminorm of \( \tilde{r}_{n_0, n} \) depends on \( \mathcal{N}_2 \)-seminorm of \( \lambda_\nu \) and \( \mathcal{N}_2 \)-seminorm of \( S_{n_0, n} \). Clearly the \( \mathcal{N}_2 \)- and \( \mathcal{N}_2 \)-seminorms of \( \lambda_\nu \) are independent of \( \nu \). Note that \( S_{n_0} \) is given by (4.7), and then we have

\[
\mathcal{N}_{k+2}(S_{n_0, n}) \leq \mathcal{N}_{k+2}(T^n x \cdot \xi) + |n_\nu| \mathcal{N}_{k+2} \left( \sum_{k=0}^{n-1} \tau(T^k x) \right)
\]

\[
\leq \mathcal{N}_{k+2}(T^n x \cdot \xi) + \mathcal{N}_{k+2} \left( \sum_{k=0}^{n-1} \tau(T^k x) \right),
\]

which implies that the \( \mathcal{N}_{k+2} \)-seminorm of \( S_{n_0, n} \) is independent of \( \nu \). Similarly, \( \mathcal{N}_{k+4} \)-seminorm of \( S_{n_0, n} \) is also independent of \( \nu \). This is what we need.

5. Estimates of \( \tilde{p}_{n_0, n} \): proof of lemma 5.1

5.1. Lemma 5.1 and its proof

The estimates given in lemma 5.1 in this section is the most important step to prove proposition 3.2.

**Lemma 5.1.** If \( \tau(x) \) is not an essential coboundary over \( T \), then there exists \( n_0 \in \mathbb{N} \) such that

\[
\tilde{p}_0 := \sup_{\nu \in \mathbb{Z}} \sup_{(x, \xi) \in T \times \mathbb{R}^d} \tilde{p}_{n_0, n}(x, \xi) < 1.
\]

**Proof.** Given \( n \in \mathbb{Z}^{d-1} \) and \( x \in T^d \), we consider the affine map \( F_{n, x} : \mathbb{R}^d \to \mathbb{R}^d \) given by

\[
F_{n, x}(\xi) = (D_x T)^n \xi + (D_x T)^n x \text{ for any } \xi \in \mathbb{R}^d,
\]

and the \( n \)th iterates

\[
F_{n, x}^n(\xi) = \prod_{k=0}^{n-1} F_{n, x} T^k(\xi) = (D_x T^n)^n \xi + W_{n, x}(x) \text{ for any } n \in \mathbb{N},
\]

where \( W_{n, x}(x) = W_{n}(x) n \), and \( W_{n}(x) \) is given by (4.8). Conventionally, we set \( W_{n, 0}(x) = 0 \) and \( F_{n, x}^0 = \text{id} \). We also define

\[
\tilde{p}_{n, x}(x, \xi) = \sum_{r=1}^{\infty} A_\nu(y) \left[ \frac{g \left( F_{n, x}^n(\xi) \right)}{g(\xi)} \right]^{2r},
\]
which extends the formula of \( \tilde{p}_{n,0}(x, \xi) \) given by (4.15) to all \( n \in \mathbb{S}^{l-1} \). For any fixed \( n \in \mathbb{N} \), it is easy to check that the function \( (n, x, \xi) \mapsto \tilde{p}_{n,0}(x, \xi) \) is of class \( C^\infty \). Further, we set

\[
\tilde{p}_{n,0} := \sup_{(x, \xi) \in T^{l-1}} \tilde{p}_{n,0}(x, \xi).
\]  

(5.5)

The properties of \( \tilde{p}_{n,0} \) will be given by sublemma 5.3 below.

Recall that \( R \) is given by (4.1). By sublemma 5.4 below, for any \( n \in \mathbb{S}^{l-1} \), there exists \( n_0(n) \in \mathbb{N} \) such that for any \( (x, \xi) \in T^{l-1} \), we have \( |\mathcal{F}^{n_0(n)}_{n}(\xi)| > 2R \) for some \( y \in T_{-n_0(n)} \). Then by (5.3) and (4.8), we have for any \( n' \in \mathbb{S}^{l-1} \),

\[
|\mathcal{F}^{n_0(n)}_{n'}(\xi)| \geq |\mathcal{F}^{n_0(n)}_{n}(\xi)| - |\mathcal{F}^{n_0(n)}_{n'}(\xi)| \geq 2R - |\mathcal{F}^{n_0(n)}_{n'}(\xi)|.
\]

Hence there is \( \varepsilon(n) > 0 \) such that \( |\mathcal{F}^{n_0(n)}_{n'}(\xi)| > R \) whenever \( |n' - n| < \varepsilon(n) \). By sublemma 5.2 (2), we get \( \tilde{p}_{n',n_0(n)}(x, \xi) < 1 \). Since \( \tilde{p}_{n',n_0(n)}(x, \xi) \) is continuous with \( x \) and \( \xi \), we get that for any compact set \( U \subset T^{l-1} \), \( \max_{(x, \xi) \in U} \tilde{p}_{n',n_0(n)}(x, \xi) < 1 \). Together with sublemma 5.2 (3), we obtain

\[
\tilde{p}_{n',n_0(n)} = \max \left\{ \sup_{(x, \xi) \in U_0} \tilde{p}_{n',n_0(n)}(x, \xi), \sup_{(x, \xi) \in T^{l-1} \setminus U_0} \tilde{p}_{n',n_0(n)}(x, \xi) \right\}
\]

\[
\leq \max \left\{ \max_{(x, \xi) \in U_0} \tilde{p}_{n',n_0(n)}(x, \xi), \left( \frac{\gamma + 1}{2} \right)^{|x|} \right\} < 1,
\]

where \( U_0 := \{ (x, \xi) : |\xi| \leq R \} \) is a compact subdomain in \( T^{l-1} \), and \( \gamma \) is given by (4.1). Moreover, by sublemma 5.3 (2), \( \tilde{p}_{n',n} \leq \tilde{p}_{n',n_0(n)} < 1 \) for any \( n \geq n_0(n) \).

To sum up, for any \( n \in \mathbb{S}^{l-1} \), there are \( n_0(n) \in \mathbb{N} \) and \( \varepsilon(n) > 0 \) such that \( \tilde{p}_{n',n} < 1 \) for all \( n' \in B(n, \varepsilon(n)) \) and \( n \geq n_0(n) \), where \( B(n, \varepsilon(n)) \) denotes the open ball in \( \mathbb{S}^{l-1} \) with center at \( n \) and of radius \( \varepsilon(n) \). Since \( \mathbb{S}^{l-1} \) is compact, there are \( n_1, n_2, \ldots, n_k \in \mathbb{S}^{l-1} \) such that the finite collection of open balls \( \{ B(n_i, \varepsilon(n_i)) \}_{1 \leq i \leq k} \) covers \( \mathbb{S}^{l-1} \). Therefore, if we set

\[
n_0 = \max \{ n_0(n_1), \ldots, n_0(n_k) \},
\]

then \( \tilde{p}_{n_0_0} < 1 \) for all \( n \in \mathbb{S}^{l-1} \). By sublemma 5.3 (3), we know that the function \( n \mapsto \tilde{p}_{n_0_0} \) is continuous, then \( \sup_{n \in \mathbb{S}^{l-1}} \tilde{p}_{n_0_0} = \max_{n \in \mathbb{S}^{l-1}} \tilde{p}_{n_0_0} < 1 \), from which (5.1) follows.

\[
5.2. \text{Sublemmas and proofs}
\]

Recall that \( R \) and \( \gamma \) are given by (4.1) and (1.1) respectively.

\[
5.2. \text{Sublemma 5.2.} \quad \text{Let} \ n \in \mathbb{S}^{l-1}. \text{ Then}
\]

1. \( \tilde{p}_{n_0}(x, \xi) \leq 1 \) for all \( n \in \mathbb{N} \) and \( (x, \xi) \in T^{l-1} \);
2. \( \tilde{p}_{n_0}(x, \xi) < 1 \) if and only if there is \( y \in T^{l-1} \) such that \( |\mathcal{F}^{n_0}_{n_0}(\xi)| > R \);
3. \( \tilde{p}_{n_0}(x, \xi) \leq \left( \frac{\gamma + 1}{2} \right)^{|x|} < 1 \) for all \( n \in \mathbb{N} \) and \( (x, \xi) \in T^{l-1} \) with \( |\xi| > R \).
Proof. The key observation is the following: for any \( n \in \mathbb{N}, y \in \mathbb{T}^d \) and \( \xi \in \mathbb{R}^d \),

\[
|F_{n,y}^n(\xi)| > \frac{\gamma + 1}{2}|\xi| \quad \text{if} \quad |\xi| > R. \tag{5.6}
\]

Indeed, by the choice of \( R \) in (4.1), we have \( |\xi| > R > \frac{2\|D\tau\|}{\gamma - 1} \). So by (5.2),

\[
|F_{n,y}(\xi)| \geq |(D,T)\xi| - |(D,T)n| \geq |\gamma| |\xi| - \|D\tau\| \geq |\gamma| |\xi| - \frac{\gamma - 1}{2}|\xi| = \frac{\gamma + 1}{2}|\xi|.
\]

Hence by induction, we have for all \( n \geq 1 \),

\[
|F_{n,y}^n(\xi)| \geq \left( \frac{\gamma + 1}{2} \right)^n |\xi| \geq \frac{\gamma + 1}{2}|\xi|.
\]

Consequently, for any \( n \in \mathbb{N}, y \in \mathbb{T}^d \) and \( \xi \in \mathbb{R}^d \),

- if \( |F_{n,y}^n(\xi)| \leq R \), then \( |F_{n,y}^n(\xi)| \leq R \) for all \( 0 \leq k \leq n \), and in particular, we must have \( |\xi| \leq R \);
- if \( |F_{n,y}^n(\xi)| > R \), then \( |F_{n,y}^n(\xi)| > |\xi| \) no matter whether \( |\xi| > R \) or not.

By the definition of \( g(\xi) \) given by (4.2), the quotient

\[
\left[ \frac{g \left( F_{n,y}^n(\xi) \right)}{g(\xi)} \right]^{2^s} = \begin{cases} 1 & \text{if} \quad |F_{n,y}^n(\xi)| \leq R; \\ < 1 & \text{otherwise}. \end{cases}
\]

In either case, we always get

\[
0 < \left[ \frac{g \left( F_{n,y}^n(\xi) \right)}{g(\xi)} \right]^{2^s} \leq 1. \tag{5.7}
\]

Recall that \( \sum_{x \in \mathbb{T}^d} A_n(y) = 1 \), where \( A_n(y) \) is positive and defined by (3.5). Therefore, for any \( n \in \mathbb{N} \) and \((x,\xi) \in T^\ast \mathbb{T}^d\),

\[
\tilde{p}_{n,n}(x,\xi) = \sum_{x \in \mathbb{T}^d} A_n(y) \left[ \frac{g \left( F_{n,y}^n(\xi) \right)}{g(\xi)} \right]^{2^s} \leq \sum_{x \in \mathbb{T}^d} A_n(y) = 1.
\]

Clearly, we have that \( \tilde{p}_{n,n}(x,\xi) < 1 \) if and only if \( |F_{n,y}^n(\xi)| > R \) for some \( y \in T^{-n}x \). Moreover, if \( |\xi| > R \), then by (5.6), we have that

\[
\left[ \frac{g \left( F_{n,y}^n(\xi) \right)}{g(\xi)} \right]^{2^s} \leq \left( \frac{\gamma + 1}{2} \right)^{2^s} \quad \text{for all} \quad y \in T^{-n}x,
\]

and hence \( \tilde{p}_{n,n}(x,\xi) \leq \left( \frac{\gamma + 1}{2} \right)^{2^s} \). \( \square \)

**Sublemma 5.3.** Let \( \tilde{p}_{n,n} \) be defined as in (5.5).

(1) For any \( n \in \mathbb{N} \) and \( n \in \mathbb{N} \), we have that \( 0 < \tilde{p}_{n,n} \leq 1 \).
For any $n \in S^{f-1}$, the sequence $\{\tilde{p}_{n,n}\}_{n \in \mathbb{N}}$ is non-increasing.

(3) For any $n \in \mathbb{N}$, the function $n \mapsto \tilde{p}_{n,n}$ is continuous.

**Proof.** As shown by sublemma 5.2 (1), $0 < \tilde{p}_{n,n}(x,\xi) \leq 1$ for any $n \in S^{f-1}$, $n \in \mathbb{N}$ and $(x, \xi) \in T^*\mathbb{T}^d$, and thus $0 < \tilde{p}_{n,n} \leq 1$.

Let $n \in S^{f-1}$ be fixed. For any $(x, \xi) \in T^*\mathbb{T}^d$ and $m, n \in \mathbb{N}$, by (5.7) we have

$$\tilde{p}_{n,n+m}(x,\xi) = \sum_{s=T+my} A_{n+m}(y) \left[ \frac{g(F^n_{n+m}(\xi))}{g(\xi)} \right]^{2s} \leq \sum_{s=T+my} A_n(z) A_m(y) \left[ \frac{g(F^n_m(\xi))}{g(\xi)} \right]^{2s} \leq \sum_{s=T+my} A_n(z) \tilde{p}_{n,m}(z,\xi) \leq \tilde{p}_{n,m} \sum_{s=T+my} A_n(z) = \tilde{p}_{n,n}.$$  

Hence, $\tilde{p}_{n,n+m} = \sup_{(x,\xi) \in T^*\mathbb{T}^d} \tilde{p}_{n,n+m}(x,\xi) \leq \tilde{p}_{n,n}$. This proves that the sequence $\{\tilde{p}_{n,n}\}_{n \in \mathbb{N}}$ is non-increasing.

Let $n \in \mathbb{N}$ be fixed. To show that the function $n \mapsto \tilde{p}_{n,n}$ is continuous, it suffices to show that the family

$$\{n \mapsto \tilde{p}_{n,n}(x,\xi) : (x, \xi) \in T^*\mathbb{T}^d\} \subset C^0(S^{f-1})$$

is uniformly bounded and equicontinuous. The uniform boundedness is already given by sublemma 5.2 (1). The equicontinuity follows from that

$$\sup_{(x,\xi) \in T^*\mathbb{T}^d} \left| \frac{\partial}{\partial n} \tilde{p}_{n,n}(x,\xi) \right| = \sup_{(x,\xi) \in T^*\mathbb{T}^d} \sum_{s=T+my} A_n(y) 2s \left[ \frac{g(F^n_m(\xi))}{g(\xi)} \right]^{2s-1} \left| Dg \left( F^n_{n+m}(\xi) \right) \right| W_n(y) |T^{s+y}|.$$  

$$\leq 2|s| \sup_{(x,\xi) \in T^*\mathbb{T}^d} \sum_{s=T+my} A_n(y) \left[ \frac{g(F^n_m(\xi))}{g(\xi)} \right]^{2s} \left| Dg \right| \left| DT \right| \tilde{p}_{n,n} \leq 2n|s| \left| Dg \right| \left| DT \right| < \infty.$$  

Here we have used (5.3), (5.4), (4.8) and the following properties of $g(\xi)$ (see (4.2)): $g(\xi) \geq 1$ for any $\xi \in \mathbb{R}^d$ and $\left| Dg \right| = \sup_{\xi \in \mathbb{R}^d} |Dg(\xi)| < \infty$.  

**Sublemma 5.4.** Suppose $\tau(x)$ is not an essential coboundary over $T$. For any $n \in S^{f-1}$, there is $n_0(n) \in \mathbb{N}$ such that for any $(x, \xi) \in T^*\mathbb{T}^d$,

$$|F^n_{n_0(n)}(\xi)| > 2R$$

for some $y \in T^{-n_0(n)}(x)$.

**Proof.** Let us argue by contradiction. If this sublemma does not hold for some $n^* \in S^{f-1}$, then for any $n \in \mathbb{N}$, there is $(x_0, \xi_0) \in T^*\mathbb{T}^d$ such that $|F^n_{n^*}(\xi_0)| \leq 2R$ for any $y \in T^{-n}(x_0)$. In fact, we further have

$$387$$
| |  | |
|---|---|---|
| $|f_k^{\mathbf{n}, \omega} (\xi_n)| \leq 2R$ for any $y \in T^{-n}(x_n)$ and $0 \leq k \leq n$, | (5.8) |

since otherwise if $|f_k^{\mathbf{n}, \omega} (\xi_n)| > 2R$ for some $0 \leq k < n$, then by (5.6), $|f_k^{\mathbf{n}, \omega} (\xi_n)| = |f_{n-k}^{\mathbf{n}, \omega} (f_k^{\mathbf{n}, \omega} (\xi_n))| \geq 2^{|\mathbf{n}|} |f_k^{\mathbf{n}, \omega} (\xi_n)| > 2R$. Note that in particular, $|\xi_n| \leq 2R$.

For $y \in T^{-n}(x_n)$ and $0 \leq k \leq n$, denote

$$
\tilde{W}_{n, k} (y) = [(D_y T^k)^{-1} W_{n, k} (y) = \sum_{j=0}^{k-1} [(D_T T^{-j})^j]^{-1} [D_T T^{-j}]^*] \textbf{n}^*.
$$

(5.9)

Using (5.3) and then (5.8), we can write

$$
\left| (D_y T^k)^j (\xi_n + \tilde{W}_{n, k} (y)) = \left| (D_y T^k)^j \xi_n + W_{n, k} (y) \right| = |f_k^{\mathbf{n}, \omega} (\xi_n)| \leq 2R.
$$

By (1.1), we have

$$
\left| \xi_n + \tilde{W}_{n, k} (y) \right| \leq \frac{2R}{\gamma}, \quad \text{for any } y \in T^{-n}(x_n), \ 0 \leq k \leq n.
$$

(5.10)

We would like to rewrite $\tilde{W}_{n, k} (y)$ in terms of $x_n$ as follows. Suppose the degree of the expanding endomorphism $T : \mathbb{T}^d \to \mathbb{T}^d$ is $N$. We denote

$$
\Sigma_N = \{ i = (i_1, i_2, \ldots, i_N) : i_j = 0, 1, \ldots, N-1, \ \ 1 \leq i \leq N \leq \infty. \}
$$

Let $T_0^{-1}, T_1^{-1}, \ldots, T_{N-1}^{-1}$ be the inverse branches of $T$. Given $x \in \mathbb{T}^d$ and $i \in \Sigma_N$, we denote $T_i^{-1} x = T_{i_1}^{-1} \cdots T_{i_N}^{-1} x$, which is well-defined whenever $0 \leq j \leq n \leq \infty$ and $j$ is finite. We then define

$$
V_{n, k} (i, x) := \sum_{j=1}^{k} D_j \left[ \tau (T_j^{-1} (x)) \cdot \textbf{n}^* \right] = \sum_{j=1}^{k} [(D_{T_j} T^{-j})^j]^{-1} (D_{T_j} T^{-j})^* \textbf{n}^*. \quad (5.11)
$$

for any $1 \leq k \leq n \leq \infty$ and $k$ is finite.

Note that

$$
\sum_{j=m}^{\infty} \left| [(D_{T_j} T^{-j})]^{-1} (D_{T_j} T^{-j})^* \textbf{n}^* \right| \leq \|D_T\| \sum_{j=m}^{\infty} \gamma^{-j} \to 0 \quad \text{as } m \to \infty, \quad (5.12)
$$

and the convergence is uniform. That is, the sequence $\{V_{n, k} (i, x)\}_{\infty}$ is uniform Cauchy. Hence $V_{n, \infty} (i, x)$ is well-defined as in (5.11) for all $x \in \mathbb{T}^d$ and $i \in \Sigma_N$. Denote $V_{n} (i, x) = V_{n, \infty} (i, x)$. We have

$$
\lim_{k \to \infty} V_{n, k} (i, x) = V_{n} (i, x) \quad \text{for any } x \in \mathbb{T}^d, \ i \in \Sigma_N, \quad (5.13)
$$

and the convergence is uniform. Moreover, by (5.12), for any $n > 0$,

$$
|V_{n, k} (i, x) - V_{n} (i, x)| \leq \frac{\|D_T\|}{\gamma^n (1-\gamma)}. \quad (5.14)
$$

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Comparing (5.9) and (5.11), we see $\tilde{W}_{n^*, n}(y) = V_{n^*, n}(i, x)$ whenever $y = T_i^{-n}(x)$. Hence by (5.10) and (5.14), for any $n \in \mathbb{N}$,

$$|\xi_n + V_{n^*, n}(i, x_n)| \leq |\xi_n + V_{n^*, n}(i, x_0)| + |V_{n^*, n}(i, x_n) - V_{n^*}(i, x_n)| \leq \frac{2R}{\gamma^n} + \frac{||D\tau||}{\gamma^n(1 - \gamma)}.$$  

(5.15)

Since the sequence $\{(x_n, \xi_n)\}$ lies in the compact subdomain $\mathcal{U}_{2R} := \{(x, \xi) : |\xi| \leq 2R\}$ of $T^*\mathbb{T}^d$, there is an accumulation point $(\xi^*, x^*)$. Choosing subsequences if necessary, we take $n \to \infty$ and obtain from (5.15) that $V_{n^*}(i, x^*) = -\xi^*$, regardless of the choice for $i \in \Sigma_\infty^N$.

For any $x \in \mathbb{T}^d$, take $w \in \{0, 1, \ldots, N - 1\}$ such that $x = T_w^{-1}(Tx)$. For any $i \in \Sigma_\infty^N$, we can directly check (5.11) (when $k = \infty$) to get

$$(D_i T)^j V_{n^*}(wi, Tx) = V_{n^*}(i, x) + (D_i \tau)^j n^*.$$  

(5.16)

By claim 1 below, we know that $V_{n^*}(i, x)$ is independent of $i$ for any $x \in \mathbb{T}^d$. Hence, we can define a function $V_{n^*} : \mathbb{T}^d \to \mathbb{R}$ by

$$V_{n^*}(x) = V_{n^*}(i, x), \text{ for any } i \in \Sigma_\infty^N,$$

and thus (5.16) is rewritten as

$$(D_i T)^j V_{n^*}(Tx) = V_{n^*}(x) + (D_i \tau)^j n^*.$$  

(5.17)

By claim 2 below, which asserts that the 1-form on $\mathbb{T}^d$ given by $V_{n^*}(x) \cdot dx$ is exact, there is a potential function $u$ such that $\nabla_x u = V_{n^*}(x)$. Alternatively, we can define the function $u : \mathbb{T}^d \to \mathbb{R}$ by

$$u(x) = \int_{\Gamma_0 x} V_{n^*}(z) \cdot dz, \text{ } x \in \mathbb{T}^d,$$

where $\Gamma_0 x$ is any smooth path in $\mathbb{T}^d$ from $0 = (0, 0, \ldots, 0)$ to $x$.

On both sides of (5.17), we replace $x$ by $tx$, take the dot product with $x$ and integrate with respect to $t$ from 0 to 1, then we get

$$\int_{\Gamma_0 x} V_{n^*}(z) \cdot dz = \int_{\Gamma_0 x} V_{n^*}(z) \cdot dz + \int_{\Gamma_0 x} (D_i \tau)^j n^* \cdot dz,$$

where $\Gamma_0^0 := \{tx : 0 \leq t \leq 1\}$, and $\Gamma_0^1, \Gamma_0^T, \Gamma_0^{T_0} := \{T(t)x : 0 \leq t \leq 1\}$. In other words, we have $u(Tx) - u(T0) = u(x) - u(0) + n^* \cdot \tau(x) - n^* \cdot \tau(0)$.

Note that $u(0) = 0$. Let $c = n^* \cdot \tau(0) - u(T0)$, then we get

$$n^* \cdot \tau(x) = c - u(x) + u(Tx),$$

which contradicts to the fact that $\tau(x)$ is not an essential coboundary over $T$. \hfill \Box

**Remark 5.5.** We would like to mention that Faure and Weich constructed a function similar to (5.11) in [24], proposition 4.9, which plays an important role in the study of asymptotic spectral gap for open partially expanding systems.
Claim 1. Suppose that there exists a point \((x^*, \xi^*)\) such that \(V_n^\ast(i, x^*) = -\xi^*\), regardless of the choice for \(i \in \Sigma_n^\infty\). Then for any \(x \in \mathbb{T}^d\), \(V_n^\ast(i, x)\) is independent of \(i\), that is, \(V_n^\ast(i, x) = V_n^\ast(i', x)\) for all \(i, i' \in \Sigma_n^\infty\).

Proof. Taking \(x = T_w^{-1} x^*\) in (5.16) for some \(w \in \{0, 1, \ldots, N - 1\}\), we get
\[
V_n^\ast(i, T_w^{-1} x^*) = -(D_{T_w^{-1} x^*} T)^\prime \xi^* - [D_{T_w^{-1} x^*} \tau] ' n^*.
\]
The right hand side is independent of \(i\), and hence \(V_n^\ast(i, T_w^{-1} x^*) = V_n^\ast(0, T_w^{-1} x^*)\), where \(0 = (0, 0, \ldots) \in \Sigma_n^\infty\).

Inductively, one can show that \(V_n^\ast(i, x) = V_n^\ast(0, x)\) for all \(x \in \bigcup_{n=1}^\infty T^{-n}(x^*)\) and thus for all \(x \in \mathbb{T}^d\), since the set \(\bigcup_{n=1}^\infty T^{-n}(x^*)\) is dense in \(\mathbb{T}^d\).

Claim 2. The 1-form on \(\mathbb{T}^d\) given by
\[
V_n^\ast(x) \cdot dx = V_n^{l1}(x)dx_1 + \cdots + V_n^{ld}(x)dx_d
\]
is exact.

Proof. We first show that \(V_n^\ast(x) \cdot dx\) is a closed 1-form, which is equivalent to showing that for any \(x \in \mathbb{T}^d\),
\[
\frac{\partial}{\partial x_j} V_n^{l_i}(x) = \frac{\partial}{\partial x_i} V_n^{l_j}(x), \quad 1 \leq i \leq j \leq d.
\] (5.18)

Indeed, by (5.13) and claim 1, \(V_n^{l_i}(i, x)\) converges uniformly to \(V_n^{l_i}(i, x) = V_n^\ast(x)\) as \(k \to \infty\). By similar calculation as in (5.12), we have that \(\frac{\partial}{\partial x_j} V_n^{l_i}(i, x)\) converges uniformly as \(k \to \infty\), and hence \(\frac{\partial}{\partial x_j} V_n^{l_i}(i, x) = \lim_{k \to \infty} \frac{\partial}{\partial x_j} V_n^{l_i}(i, x)\). We see from (5.11) that for each \(k \in \mathbb{N}\) and any \(i \in \Sigma_n^\infty\), the 1-form \(V_n^{l_i}(i, x) \cdot dx = d \left(\sum_{j=1}^k \tau(T_i^j(x)) \cdot n^*\right)\) is exact and hence closed. Thus,
\[
\frac{\partial}{\partial x_i} V_n^{l_i}(i, x) = \frac{\partial}{\partial x_i} V_n^{l_j}(i, x), \quad 1 \leq i \leq j \leq d,
\]
from which (5.18) follows by taking \(k \to \infty\).

Now we show that \(V_n^\ast(x) \cdot dx\) is exact. Since \(V_n^\ast(x) \cdot dx\) is closed, it is sufficient to prove that for any \(x = (x_1, x_2, \ldots, x_d) \in \mathbb{T}^d\) and \(1 \leq k \leq d\),
\[
\int_0^1 V_n^\ast(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d) \, dt = 0. \quad (5.19)
\]
To see this, by (5.11) and claim 1, we rewrite for arbitrary \(M \in \mathbb{N}\),

\[
\int_0^1 V_n^\ast(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d) \, dt = 0.
\]
\[ V^{\star}(x) = \frac{1}{NM} \sum_{i \in \Sigma^M_N} V^{\star}(i0^\infty, x) \]

\[ = \frac{1}{NM} \sum_{i \in \Sigma^M_N} V^{\star,M}(i0^\infty, x) + \frac{1}{NM} \sum_{i \in \Sigma^M_N} [V^{\star}(i0^\infty, x) - V^{\star,M}(i0^\infty, x)] \]

\[ = \frac{1}{NM} \sum_{j=1}^{M} \sum_{i \in \Sigma^J_N} D_x \left[ \tau(T_{ij}^{-1} \cdots T_{in}^{-1}) \cdot n^\star \right] + \frac{1}{NM} \sum_{i \in \Sigma^M_N} [V^{\star}(i0^\infty, x) - V^{\star,M}(i0^\infty, x)] \]

\[ = : I^{\star}(x) + J^{\star}(x). \]

Here we denote \( i0^\infty = (i_1, i_2, \ldots, i_n, 0, 0, \ldots) \in \Sigma^\infty_N \) for any \( i = (i_1, i_2, \ldots, i_n) \in \Sigma^M_N \).

On one hand, let \( I^{\star}_k \) be the \( k \)th component of \( I^{\star} \), then

\[ \int_0^1 I^{\star}_k(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d) \, dt \]

\[ = \frac{1}{NM} \sum_{j=1}^{M} \sum_{i \in \Sigma^J_N} \int_0^1 \frac{\partial}{\partial x_k} \left[ \tau(T_{ij}^{-1} \cdots T_{in}^{-1}(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)) \cdot n^\star \right] \, dt \]

\[ = \frac{1}{NM} \sum_{j=1}^{M} n^\star \cdot \sum_{i \in \Sigma^J_N} \left[ \tau(T_{ij}^{-1} \cdots T_{in}^{-1}(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_d)) - \tau(T_{ij}^{-1} \cdots T_{in}^{-1}(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_d)) \right] \]

\[ = \frac{1}{NM} \sum_{j=1}^{M} n^\star \cdot \left[ \sum_{i \in \Sigma^J_N} \tau(T_{ij}^{-1} \cdots T_{in}^{-1}(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_d)) - \tau(T_{ij}^{-1} \cdots T_{in}^{-1}(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_d)) \right] = 0. \]

The last term must vanish since \( \{T_{ij}^{-1} \cdots T_{in}^{-1}(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_d) : i \in \Sigma^J_N \} \) and \( \{T_{ij}^{-1} \cdots T_{in}^{-1}(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_d) : i \in \Sigma^J_N \} \) are just two representations for the set of all \( j \)th pre-images of the point \( (x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_d) = (x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_d) \) in \( \mathbb{T}^d \).

On the other hand, by (5.12) and (5.13), the convergence \( V^{\star,M}(i0^\infty, x) \to V^{\star}(i0^\infty, x) \) is uniform in \( i \) and \( x \) as \( M \to \infty \). By choosing \( M \) large enough, the integral of the \( k \)th component of \( J^{\star}(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d) \) with respect to \( t \) from 0 to 1 is arbitrary small and hence 0. It follows that (5.19) holds.

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References

[1] Arnoldi J F 2012 Fractal Weyl law for skew extensions of expanding maps Nonlinearity 25 1671–93
[2] Arnoldi J F, Faure F and Weich T 2017 Asymptotic spectral gap and Weyl law for Ruelle resonances of open partially expanding maps Ergod. Theor. Dynam. Syst. 37 1–58
[3] Avila A, Gouëzel S and Yoccoz J C 2006 Exponential mixing for the Teichmüller flow Publ. Math. Inst. Hautes Études Sci. 104 143–211
[4] Baladi V, Demers M F andLiverani C 2018 Exponential decay of correlations for finite horizon Sinai billiard flows Inventiones Math. 211 39–177
[5] Baladi V and Gouëzel S 2010 Banach spaces for piecewise cone-hyperbolic maps J. Mod. Dyn. 4 91–137
[6] Baladi V and Liverani C 2012 Exponential decay of correlations for piecewise cone hyperbolic contact flows Commun. Math. Phys. 314 689–773
[7] Baladi V and Tsujii M 2007 Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms Ann. Inst. Fourier 57 127–54
[8] Baladi V and Tsujii M 2008 Spectra of differentiable hyperbolic maps Traces in Number Theory, Geometry, Quantum Fields (Aspects of Mathematics) vol E38 (Wiesbaden: Vieweg) pp 1–21
[9] Baladi V and Vallée B 2005 Exponential decay of correlations for surface semi-flows without finite Markov partitions Proc. Am. Math. Soc. 133 865–74
[10] Burns K and Wilkinson A 1999 Stable ergodicity of skew products Ann. Sci. École Norm. Sup. 32 859–89
[11] Butterley O and Eslami P 2017 Exponential mixing for skew products with discontinuities Trans. Am. Math. Soc. 369 783–803
[12] Dolgopyat D 1998 On decay of correlations in anosov flows Ann. Math. 147 357–90
[13] Dolgopyat D 2002 On mixing properties of compact group extensions of hyperbolic systems Isr. J. Math. 130 157–205
[14] Dyatlov S and Guillarmou C 2016 Pollicott–Ruelle resonances for open systems Ann. Henri Poincaré 17 3089–146
[15] Dyatlov S and Zworski M 2016 Dynamical zeta functions for Anosov flows via microlocal analysis Ann. Sci. École Norm. Supér. 49 543–77
[16] Egorov Y V 1986 Linear Differential Equations of Principal Type (Contemporary Soviet Mathematics) (New York: Consultants Bureau) (translated from the Russian by Dang Prem Kumar)
[17] Faure F 2007 Prequantum chaos: resonances of the prequantum cat map J. Mod. Dyn. 1 255–85
[18] Faure F 2011 Semiclassical origin of the spectral gap for transfer operators of a partially expanding map Nonlinearity 24 1473–98
[19] Faure F and Roy N 2006 Ruelle–Pollicott resonances for real analytic hyperbolic maps Nonlinearity 19 1233–52
[20] Faure F, Roy N and Sjöstrand J 2008 Semi-classical approach for Anosov diffeomorphisms and Ruelle resonances Open Math. J. 1 35–81
[21] Faure F and Tsujii M 2013 Band structure of the Ruelle spectrum of contact Anosov flows C. R. Math. Acad. Sci. Paris 351 385–91
[22] Faure F and Tsujii M 2014 Resonances for geodesic flows on negatively curved manifolds Proc. of the Int. Congress of Mathematicians—Seoul vol III (Seoul: Kyung Moon Sa) pp 683–97
[23] Faure F and Tsujii M 2014 Semiclassical approach for the Ruelle–Pollicott spectrum of hyperbolic dynamics Analytic and Probabilistic Approaches to Dynamics in Negative Curvature (Springer INdAM Series) vol 9 (Berlin: Springer) pp 65–135
[24] Faure F and Weich T 2017 Global normal form and asymptotic spectral gap for open partially expanding maps Commun. Math. Phys. 356 755–822
[25] Field M, Melbourne I and Török A 2005 Stable ergodicity for smooth compact Lie group extensions of hyperbolic basic sets Ergod. Theor. Dynam. Syst. 25 517–51
[26] Field M and Parry W 1999 Stable ergodicity of skew extensions by compact Lie groups Topology 38 167–87
[27] Guedes Bonthoneau Y and Weich T 2018 Pollicott–Ruelle resonances for manifolds with hyperbolic cusps (in preparation) (arXiv: 1712.07832)
[28] Jenkinson O 2002 Smooth cocycle rigidity for expanding maps, and an application to Mostow rigidity Math. Proc. Camb. Phil. Soc. 132 439–52
[29] Kato T 1995 Perturbation Theory for Linear Operators (Classics in Mathematics) (Berlin: Springer) (reprint of the 1980 edition)
[30] Liverani C 2004 On contact Anosov flows Ann. Math. 159 1275–312
[31] Martinez A 2002 An Introduction to Semiclassical and Microlocal Analysis (Universitext) (New York: Springer)
[32] Nakano Y, Tsujii M and Wittsten J 2016 The partial captivity condition for U(1) extensions of expanding maps on the circle Nonlinearity 29 1917–25
[33] Parry W and Pollicott M 1997 Stability of mixing for toral extensions of hyperbolic systems Tr. Mat. Inst. Steklova 216 354–63 (www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=t m&paperid=1017&option_lang=eng)
[34] Pollicott M 1984 A complex Ruelle–Perron–Frobenius theorem and two counterexamples Ergod. Theor. Dynam. Syst. 4 135–46
[35] Pollicott M 1999 On the mixing of Axiom A attracting flows and a conjecture of Ruelle Ergod. Theor. Dynam. Syst. 19 535–48
[36] Ruelle D 2004 The mathematical structures of equilibrium statistical mechanics Thermodynamic Formalism (Cambridge Mathematical Library) 2nd edn (Cambridge: Cambridge University Press) (https://doi.org/10.1017/CBO9780511617546)
[37] Ruzhansky M and Sugimoto M 2006 Global calculus of Fourier integral operators, weighted estimates, and applications to global analysis of hyperbolic equations Pseudo-Differential Operators, Related Topics (Operator Theory: Advances and Applications vol 164) (Basel: Birkhäuser) pp 65–78
[38] Ruzhansky M and Turunen V 2010 Background analysis and advanced topics Pseudo-Differential Operators and Symmetries (Pseudo-Differential Operators. Theory and Applications vol 2) (Basel: Birkhäuser) (https://doi.org/10.1007/978-3-7643-8514-9)
[39] Taylor M E 2011 Partial Differential Equations I. Basic Theory (Applied Mathematical Sciences vol 115) 2nd edn (New York: Springer) (https://doi.org/10.1007/978-1-4419-7055-8)
[40] Taylor M E 2011 Partial Differential Equations II. Qualitative Studies of Linear Equations (Applied Mathematical Sciences vol 116) 2nd edn (New York: Springer) (https://doi.org/10.1007/978-1-4419-7052-7)
[41] Tsujii M 2008 Decay of correlations in suspension semi-flows of angle-multiplying maps Ergod. Theor. Dynam. Syst. 28 291–317
[42] Tsujii M 2010 Quasi-compactness of transfer operators for contact Anosov flows Nonlinearity 23 1495–545
[43] Zhang Z 2018 On the smooth dependence of SRB measures for partially hyperbolic systems Commun. Math. Phys. 358 45–79
[44] Zworski M 2012 Semiclassical Analysis (Graduate Studies in Mathematics vol 138) (Providence, RI: American Mathematical Society) (https://doi.org/10.1090/gsm/138)