New $R$-matrices from Representations of Braid-Monoid Algebras based on Superalgebras

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Abstract

In this paper we discuss representations of the Birman-Wenzl-Murakami algebra as well as of its dilute extension containing several free parameters. These representations are based on superalgebras and their baxterizations permit us to derive novel trigonometric solutions of the graded Yang-Baxter equation. In this way we obtain the multiparametric $R$-matrices associated to the $U_q[sl(r|2m)^{(2)}]$, $U_q[osp(r|2m)^{(1)}]$ and $U_q[osp(r = 2n|2m)^{(2)}]$ quantum symmetries. Two other families of multiparametric $R$-matrices not predicted before within the context of quantum superalgebras are also presented. The latter systems are indeed non-trivial generalizations of the $U_q[D_n^{(2)}]$ vertex model when both distinct edge variables statistics and extra free-parameters are admissible.

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1 Introduction

The Yang-Baxter equation is undoubtedly the cornerstone of the theory of two-dimensional integrable systems of statistical mechanics and quantum field theory. It is frequently viewed as an operator relation for a matrix $R_{ab}(x)$ defined on the tensor product of two $N$-dimensional vectors spaces $V_a$ and $V_b$, which reads

$$R_{12}(x_1)R_{13}(x_1x_2)R_{23}(x_2) = R_{23}(x_2)R_{13}(x_1x_2)R_{12}(x_1), \quad (1)$$

where $x_i = e^{\lambda_i}$ are arbitrary multiplicative spectral parameters.

The elements of the $R$-matrix $R_{ab}(x)$ can be thought of either as the Boltzmann weights of vertex models in statistical mechanics [1] or as factorizable scattering amplitudes between particles in relativistic field theories [2]. Therefore, the search for solutions of the Yang-Baxter equation is indeed a central issue in the field of exactly solvable models. Unfortunately, a complete classification of the solutions of the Yang-Baxter equation is so far beyond our reach. An important class of solutions is denominated trigonometric $R$-matrices which contain an extra free parameter $q$ besides the spectral parameter. An approach to derive such $R$-matrices has its roots on the possibility of performing an appropriate $q$-deformation in a given classical Lie algebra $G$ [3, 4, 5]. This method, the $U_q[G]$ quantum group framework, permits us in principle to associate a fundamental trigonometric $R$-matrix to each Lie algebra [6, 7] or Lie superalgebra [8]. In particular, the $R$-matrices expressions in terms of the standard Weyl matrices have been known since two decades ago for all non-exceptional Lie algebras [6]. Similar statement can not be made for superalgebras since the most general results are still concentrated on the $U_q[sl(n|m)^{(1)}]$ symmetry [9]. Representative examples for other superalgebras have been investigated for particular supergroup symmetries and for instance can be found in refs. [10, 11]. In fact, attempts to systematically carry on the above program for superalgebras [12, 13] have encountered serious technical obstacles to be overcome before explicit expressions could be written down.

In spite of these difficulties, some progresses have recently been made towards to the presentation of explicit expressions for the $R$-matrices based on general classes of superalgebras
We have for instance exhibited [14] the $R$-matrices associated to the $U_q[sl(r|2m)^{(2)}]$, $U_q[osp(r|2m)^{(1)}]$ and $U_q[osp(r = 2n|2m)^{(2)}]$ quantum superalgebras in terms of the Weyl matrices. This step makes possible the statistical mechanics interpretation of these systems and has offered suitable expressions to perform the corresponding transfer matrices diagonalization. These results, however, have been obtained for a specific grading of the Grassmann parities by means of brute force analysis of the respective $U_q[G]$ intertwiner operators.

The purpose of this paper is to elaborate further on our previous results [14] by first unveiling the algebraic structure that is behind the $U_q[sl(r|2m)^{(2)}]$, $U_q[osp(r|2m)^{(1)}]$ and $U_q[osp(r = 2n|2m)^{(2)}]$ $R$-matrices. We will argue that these two-dimensional solvable lattice models are intimately connected with the representations of the so-called Birman-Wenzl-Murakami algebra [16, 17]. This relationship allows us to generalize these $R$-matrices for a more general class of gradings and with a considerable amount of free-parameters related to possible multiparametric extension of quantum algebras [18]. We believe that this study throw new light on the classification problem of the fundamental trigonometric vertex models having both bosonic and fermionic degrees of freedom. In fact, it makes possible the derivation of novel families of such solvable models whose existence have not even been predicted before by means of the quantum group framework [8, 12, 13].

This paper has been organized as follows. In section 2 we introduce multiparametric representations of the braid algebra motived on the structure of the $U_q[sl(r|2m)^{(2)}]$, $U_q[osp(r|2m)^{(1)}]$ and $U_q[osp(r = 2n|2m)^{(2)}]$ $R$-matrices previously obtained by us [14]. These generalized representations are shown to satisfy the Birman-Wenzl-Murakami algebra for a variety of grading choices. We reintroduce the spectral parameter via the baxterization procedure [19, 20] and the multiparametric $R$-matrices associated to the $U_q[sl(r|2m)^{(2)}]$, $U_q[osp(r|2m)^{(1)}]$ and $U_q[osp(r = 2n|2m)^{(2)}]$ superalgebras are presented. This permits us in section 3 to find new representations of the dilute version of the Birman-Wenzl-Murakami algebra [21]. The study of the corresponding baxterization leads us to two novel families of $R$-matrices such that each of them produces two distinct vertex models branches whose edge variables can be of bosonic or fermionic types. These systems turn out to be highly non-trivial extensions of the $U_q[D_{n+1}^{(2)}]$
vertex model [9, 14]. This is the case even in the absence of fermionic edge variables because the free-parameters produce by themselves a generalized structure for the Boltzmann weights. Our conclusions are summarized in section 4. In Appendix A we describe a special form of the additional free parameters that is helpful to make connections to Lie superalgebras. In appendix B we present the crossing matrices of a $R$-matrix exhibited in section 3.

2 The braid-monoid algebra

The braid-monoid algebra [17] is generated by the identity $I$, the braid operator $b_i^+$ and its inverse $b_i^-$ as well as the monoid $E_i$. The index $i$ represents for instance the $i$-th site of a one-dimensional lattice of length $L$. As usual the braid operators $b_i^\pm$ obey the Artin braid group algebra [22],

\[
    b_i^+ b_i^- = b_i^- b_i^+ = I \]
\[
    b_i^+ b_j^+ = b_j^+ b_i^+ \quad \text{for } |i - j| \geq 2
\]
\[
    b_i^+ b_{i+1}^+ b_i^- = b_{i+1}^+ b_i^- b_{i+1}^+.
\]

(2)

On the other hand the monoid $E_i$ is a Temperley-Lieb operator [23] subjected to the relations,

\[
    E_i E_j = E_j E_i \quad \text{for } |i - j| \geq 2
\]
\[
    E_i^2 = Q E_i
\]
\[
    E_i E_{i+1} E_i = E_i,
\]

(3)

where $Q$ is a complex parameter.

The braid group and the Temperley-Lieb algebra can be combined together into a single two parameters algebra provided the following additional relations are satisfied,

\[
    b_i^+ E_i = E_i b_i^+ = \omega E_i
\]
\[
    b_i^+ E_j = E_j b_i^+ \quad \text{for } |i - j| \geq 2
\]
\[
    b_{i\pm1}^+ b_i^+ E_{i\pm1} = E_i b_{i\pm1}^+ b_i^+ = E_i E_{i+1},
\]

(4)
where $\omega$ is another complex parameter.

In what follows we shall argue that the multiparametric $U_q[sl(r|2m)^{(2)}]$, $U_q[osp(r|2m)^{(1)}]$ and $U_q[osp(r = 2n|2m)^{(2)}]$ $R$-matrices can be derived from the representations of a quotient of the braid-monoid algebra denominated Birman-Wenzl-Murakami algebra [16]. The idea is first to investigate suitable asymptotic limits of the corresponding $R$-matrices given by us for a specific grading [14]. This study reveals us that the main structure of the braid representations can indeed be generalized to accommodate additional free-parameters and many distinct $Z_2$ grading possibilities.

In order to do that lets us briefly recollect some basic definitions. We start by recalling that the vector spaces of these $R$-matrices are constituted of $r$ bosonic and $2m$ fermionic degrees of freedom. A given $\alpha$-th degree of freedom is distinguished by its Grassmann parity $p_\alpha$:

$$p_\alpha = \begin{cases} 0 & \text{for } \alpha \text{ bosonic} \\ 1 & \text{for } \alpha \text{ fermionic} \end{cases}$$

In this situation the relationship between braid algebra (2) and the Yang-Baxter equation is made with the help of the following graded permutator,

$$P = \sum_{\alpha,\beta=1}^{N} (-1)^{p_\alpha p_\beta} \epsilon_{\alpha\beta} \otimes \epsilon_{\beta\alpha},$$

where $N = r + 2m$ and $\epsilon_{\alpha\beta}$ denotes the standard $N \times N$ Weyl matrices.

In fact, by defining a new matrix $\tilde{R}_{ab}(x) = P_{ab}R_{ab}(x)$ one can rewrite Eq.(1) in a form that is not only insensitive to grading, namely

$$\tilde{R}_{12}(x_1)\tilde{R}_{23}(x_1x_2)\tilde{R}_{12}(x_2) = \tilde{R}_{23}(x_2)\tilde{R}_{12}(x_1x_2)\tilde{R}_{23}(x_1),$$

but also with a striking similarity with the braid algebra (2).

The braid representation can now be obtained from a given $\tilde{R}_{ab}(x)$ by considering appropriate limits of the spectral parameter $x$ such that Eq.(7) becomes asymptotically independent of the variables $x_i$. In our case this can be achieved by taking the following limits,

$$b^{\pm(l)} = \lim_{\lambda \to \pm \infty} \left[ \theta_{\pm}(x = e^\lambda)\tilde{R}_{12}(x = e^\lambda) \right]$$

4
where $\theta_\pm(x)$ are appropriate normalizations. The upper index $l$ in $b^{\pm(l)}$ anticipates the existence of two possible classes of braids to be described below. Furthermore, the braid operator $b_i^{+(l)}$ and its inverse $b_i^{-(l)}$ follow directly from $b^{\pm(l)}$ by the standard construction,

$$b_i^{\pm(l)} = \bigotimes_{j=1}^{i-1} I_N b_i^{\pm(l)} \bigotimes_{j=i+2}^L I_N$$

where $I_N$ is the $N \times N$ identity matrix.

It turns out that the most general braid representations $b_i^{\pm(l)}$ we have found, that are compatible with the $R$-matrices presented by us [14], have the following form,

$$b_i^{+(l)} = \sum_{\alpha \neq \alpha'} (-1)^{p_\alpha} q^{-1+2p_\alpha} e_{\alpha\alpha} \otimes e_{\alpha\alpha} + \sum_{\alpha,\beta=1}^N (-1)^{p_\alpha p_\beta} e_{\beta\alpha} \otimes e_{\alpha\beta} + \left( q - \frac{1}{q} \right) \sum_{\alpha,\beta=1}^N \sum_{\alpha < \beta, \alpha \neq \beta'} e_{\alpha\alpha} \otimes e_{\beta\beta} + \sum_{\alpha,\beta=1}^N a^{+(l)}_{\alpha\beta} e_{\alpha'\beta} \otimes e_{\alpha\beta'}$$

(10)

and

$$b_i^{-(l)} = \sum_{\alpha \neq \alpha'} (-1)^{p_\alpha} q^{-1-2p_\alpha} e_{\alpha\alpha} \otimes e_{\alpha\alpha} + \sum_{\alpha,\beta=1}^N (-1)^{p_\alpha p_\beta} e_{\beta\alpha} \otimes e_{\alpha\beta} + \left( q - \frac{1}{q} \right) \sum_{\alpha,\beta=1}^N \sum_{\alpha < \beta, \alpha \neq \beta'} e_{\beta\beta} \otimes e_{\alpha\alpha} + \sum_{\alpha,\beta=1}^N a^{-(l)}_{\alpha\beta} e_{\alpha\beta} \otimes e_{\alpha'\beta'},$$

(11)

where $\alpha' = N + 1 - \alpha$.

An interesting feature of the above proposal is that there exists some freedom in fixing the coefficients $a_{\alpha\beta}^{\pm(l)}$ for several choices of the Grassmann parities. These grading possibilities are those consonant with the many possible $U(1)$ symmetries implicitly assumed in our construction (10,11) of the braids. More specifically, the parities $p_\alpha$ are required to satisfy the following reflexion condition,

$$p_\alpha = p_{\alpha'}.$$  

(12)

Taking condition (12) into account we found that the coefficients $a_{\alpha\beta}^{\pm(l)}$ can be represented
in terms of the following general forms,

\[
\begin{align*}
a^{(l)}_{\alpha\beta} &= 
\begin{cases}
\left(\frac{1}{q} - q\right) \left[\frac{\epsilon^{(l)}_{\alpha\beta}}{q^{\alpha} - q^{\beta}} - \delta_{\alpha\beta}\right] & \alpha > \beta \\
0 & \alpha < \beta \\
1 & \alpha = \beta = \beta' \\
(-1)p_{\alpha}q^{-1 + 2p_{\alpha}} & \alpha = \beta \neq \beta'
\end{cases} \\
a_{\alpha\beta} &= 
\begin{cases}
\left(\frac{1}{q} - \frac{1}{q}\right) \left[\frac{\epsilon^{(l)}_{\alpha\beta}}{q^{\alpha} - q^{\beta}} - \delta_{\alpha\beta}\right] & \alpha < \beta \\
0 & \alpha > \beta \\
1 & \alpha = \beta = \beta' \\
(-1)p_{\alpha}q^{1 - 2p_{\alpha}} & \alpha = \beta \neq \beta'
\end{cases}
\end{align*}
\]

(13)

The remarked possibility of two different series of braids is therefore encoded in the variables \(\epsilon^{(l)}_{\alpha}\) and \(t^{(l)}_{\alpha}\). The first family is defined for any integer value of \(N\) and the respective parameters \(\epsilon^{(1)}_{\alpha}\) and \(t^{(1)}_{\alpha}\) satisfy the relations

\[
\epsilon^{(1)}_{\alpha} = (-1)p_{\alpha}\epsilon^{(1)}_{\alpha'} \quad \text{and} \quad t^{(1)}_{\alpha} = t^{(1)}_{\alpha'} - 2 \left[ p_{\alpha} + \frac{N}{2} - \alpha - 2 \sum_{\beta=\alpha}^{\left[\frac{N+1}{2}\right]} p_{\beta} \right],
\]

(14)

where \(\alpha\) can take values on the interval \(1 \leq \alpha < \left[\frac{N+1}{2}\right]\). We recall that \(\left[\frac{N+1}{2}\right]\) denotes the largest integer less than \(\frac{N+1}{2}\).

The second family of braid representations is valid only for \(N\) even and the respective variables \(\epsilon^{(2)}_{\alpha}\) and \(t^{(2)}_{\alpha}\) are then given by

\[
\epsilon^{(2)}_{\alpha} = -(-1)p_{\alpha}\epsilon^{(2)}_{\alpha'} \quad \text{and} \quad t^{(2)}_{\alpha} = t^{(2)}_{\alpha'} - 2 \left[ p_{\alpha} + \frac{N}{2} + 1 - \alpha - 2 \sum_{\beta=\alpha}^{\frac{N}{2}} p_{\beta} \right],
\]

(15)

where in this case \(1 \leq \alpha \leq \frac{N}{2}\).

From expressions (14,15) we conclude that each set of variables \(\epsilon^{(l)}_{\alpha}\) and \(t^{(l)}_{\alpha}\) provides us the number of \(\left[\frac{N+1}{2}\right]\) free parameters. This freedom is expected for braids related to representations of quantum algebras because Hopf algebras can accommodate suitable multiparametric extensions [18]. However, the explicit construction of universal \(R\)-matrices with such additional free parameters is by no means a simple task, specially for superalgebras. Here we
conjecture that the braids $b^{+1}$ and $b^{+2}$ are in direct correspondence with the multiparametric $U_q[osp(r|2m)^{(1)}]$ and $U_q[osp(r = 2n|2m)^{(2)}]$ universal $R$-matrices, respectively. An evidence supporting this conjecture is discussed in Appendix A. Before proceeding we remark that such type of braids that mix both bosonic and fermionic variables have early been referred as “non-standard” braid group representations [24, 25]. To our knowledge, however, the expressions for the braids in terms of the standard Weyl matrices [10] for general $N$ and with many arbitrary variables $\epsilon^{(l)}_\alpha$ and $t^{(l)}_\alpha$ as well as their relationship with multiparametric $R$-matrices invariant by superalgebras are novel results in the literature.

We now turn our attention to the study of the eigenvalues structure of the braids (10-15). In order to establish the connection with the representations of the Birman-Wenzl-Murakami algebra is important that such braids have at most three distinct eigenvalues [20]. By direct inspection we conclude that $b^{+l}$ indeed satisfies the following cubic relation,

$$
\left( b^{+l} + \frac{1}{q} I_N \otimes I_N \right) \left( b^{+l} - q I_N \otimes I_N \right) \left( b^{+l} - \sigma_l I_N \otimes I_N \right) = 0,
$$

(16)

where the third eigenvalue $\sigma_l$ is given by

$$
\sigma_l = \begin{cases} 
q^{1-r+2m} & \text{for } l = 1 \\
-q^{-1-2n+2m} & \text{for } l = 2 .
\end{cases}
$$

(17)

The next step in order to close Birman-Wenzl-Murakami algebra is to assure that the corresponding Temperley-Lieb operator $E^{(l)}$ is related to the braid $b^{\pm l}$ through the following identity,

$$
E^{(l)} = I_N \otimes I_N + \frac{b^{+l} - b^{-l}}{q^{-1} - q} .
$$

(18)

By substituting expressions [10] into Eq. (18) we can therefore determine the explicit form of the respective monoid operator, namely

$$
E^{(l)} = \sum_{\alpha,\beta} \frac{\epsilon^{(l)}_\alpha}{\epsilon^{(l)}_\beta} q^{t^{(l)}_\beta - t^{(l)}_\alpha} \epsilon^{(l)}_{\alpha^\prime \beta} \otimes \epsilon^{(l)}_{\alpha \beta^\prime} .
$$

(19)

Our next remaining task is to verify whether or not the braid $b^{+l}$ [10, 15] and the respective monoid $E^{(l)}$ [19] satisfy the braid-monoid relations (25). This indeed occurs provided the
complex parameters $Q$ and $\omega$ are set to assume the values
\[ Q = 1 + \frac{\sigma_l - \sigma_l^{-1}}{q^{-1} - q} \quad \text{and} \quad \omega = \sigma_l. \quad (20) \]

Let us now discuss how to introduce the spectral parameter $x$ into these braids representations so as to construct the corresponding solution $\hat{R}_{ab}(x)$ of the Yang-Baxter equation. The Birman-Wenzl-Murakami algebra is known to provide us the sufficient conditions to perform the baxterization procedure \[20\]. It turns out that for each representation of this braid-monoid algebra one can find two independent matrices $\hat{R}^{(l,1)}(x)$ and $\hat{R}^{(l,2)}(x)$ satisfying Yang-Baxter equation (7). These solutions are
\[ \hat{R}^{(l,k)}(x) = \left( \frac{1}{q} - q \right)x(x - \xi_l^{(k)}) I_N \otimes I_N + (x - 1)(x - \xi_l^{(k)}) b^{+(l)} + (q - \frac{1}{q})x(x - 1) E^{(l)}, \quad (21) \]
where $\xi_l^{(k)}$ is given by
\[ \xi_l^{(k)} = \begin{cases} -\frac{q}{\sigma_l} & \text{for } k = 1 \\ \frac{1}{q\sigma_l} & \text{for } k = 2 \end{cases}. \quad (22) \]

From the first sight one would think that these two types of baxterization, when $N$ even and $N$ odd are considered separately, would in principle lead us to six different solvable vertex models. This, however, is not the case because the $R$-matrices $\hat{R}^{(1,1)}(x)$ and $\hat{R}^{(2,2)}(x)$ for $N$ even coincide, after gauge transformations are performed. Therefore, we have altogether five different $R$-matrices that can be obtained by direct substitution of Eqs. (10, 19) into Eq. (21). After some cumbersome simplifications, their expressions in terms of the standard Weyl matrices are given by
\[ \hat{R}^{(l,k)}(x) = \sum_{\alpha=1}^{N} \sum_{\alpha \neq \alpha'} (x - \xi_l^{(k)})(x^{1-p_\alpha} - q^2 x^{p_\alpha}) e_{\alpha\alpha} \otimes e_{\alpha\alpha} + q(x - 1)(x - \xi_l^{(k)}) \sum_{\alpha,\beta=1}^{N} (-1)^{p_\alpha p_\beta} e_{\alpha\beta} \otimes e_{\alpha\beta} \]
\[ + \left( 1 - q^2 \right) (x - \xi_l^{(k)}) \left[ x \sum_{\alpha,\beta=1}^{N} e_{\alpha\alpha} \otimes e_{\beta\beta} + \sum_{\alpha,\beta=1}^{N} e_{\alpha\alpha} \otimes e_{\beta\beta} \right] + \sum_{\alpha,\beta=1}^{N} d^{(l,k)}_{\alpha,\beta}(x) e_{\alpha'\beta} \otimes e_{\alpha\beta'}, \quad (23) \]
such that the functions $d^{(l,k)}_{\alpha,\beta}(x)$ are

$$
\begin{cases}
q(x - 1)(x - \xi^{(k)}_l) + x(q^2 - 1)(\xi^{(k)}_l - 1) & \alpha = \beta = \beta' \\
(x - 1)[(x - \xi^{(k)}_l)(-1)^{p_\alpha}q^{2p_\alpha} + x(q^2 - 1)] & \alpha = \beta \neq \beta' \\
(q^2 - 1)\left[\xi^{(k)}_l(x - 1)\frac{\alpha^{(l)}}{\epsilon^{(l)}_\beta}q^{\alpha^{(l)} - t^{(l)}_\beta} - \delta_{\alpha,\beta'}(x - \xi^{(k)}_l)\right] & \alpha < \beta \\
(q^2 - 1)\left[\epsilon^{(l)}_\beta(x - 1)\frac{\alpha^{(l)}}{\epsilon^{(l)}_\beta}q^{\alpha^{(l)} - t^{(l)}_\beta} - \delta_{\alpha,\beta'}(x - \xi^{(k)}_l)\right] & \alpha > \beta
\end{cases}
$$

At this point we stress that expressions (23,24) are valid in the general situation when the variables $\epsilon^{(l)}_\alpha$ and $t^{(l)}_\alpha$ fulfill the relations (14,15) and for the variety of gradings satisfying condition (12). The possible relationship between such $R$-matrices and the corresponding underlying quantum superalgebras is proposed in Table 1. This matching has been done by comparing Eqs. (23,24) with our previous $R$-matrices results [14]. This comparison has also taken into account a symmetrical form for the variables $\epsilon^{(l)}_\alpha$ and $t^{(l)}_\alpha$ described in Appendix A.

| $R$-matrix | Superalgebra          | $\xi^{(k)}_l$ |
|------------|-----------------------|--------------|
| $\tilde{R}^{(1,1)}(x)$ | $U_q[sl^{(2)}(r|2m)]$ | $-q^{r-2m}$ |
| $\tilde{R}^{(1,2)}(x)$ | $U_q[osp^{(1)}(r|2m)]$ | $q^{r-2m-2}$ |
| $\tilde{R}^{(2,1)}(x)$ | $U_q[osp^{(2)}(r = 2n|2m)]$ | $q^{2n-2m+2}$ |

Table 1: The relationship between $\tilde{R}^{(l,k)}(x)$ and superalgebras. The expressions for the parameters $\xi^{(k)}_l$ are also given.

We believe that these $R$-matrices together with Table 1 extend in a significative way our earlier results [14] for solvable models based on superalgebras. In next section we shall see that we can profit even more from the approach described here.

## 3 The dilute braid-monoid algebra

The dilute braid-monoid algebra [21] turns out to be an interesting special case of the two colour generalization of the braid-monoid algebra [26]. The later algebra is generated by
coloured braid $b_i^{±(a,b)}$ and monoid $E_i^{(a,b)}$ operators such that the label $a, b = 1, 2$ denotes the two possible colours. The other elements are the projectors $P_i^{(a)}$ which project onto the $a$-th colour at the $i$-th site of a chain of size $L$. They satisfy the standard projectors relations given by

$$ P_i^{(a)} P_i^{(b)} = \delta_{ab} P_i^{(a)} , \quad \sum_{a=1}^{2} P_i^{(a)} = I . \quad (25) $$

In the dilute case one of the colours plays the role of vacancy of a string in the usual braid-monoid diagrams \cite{21}. This means that the corresponding representation of the subalgebra generated by the elements related to this colour is one-dimensional. Choosing the second colour $a = 2$ as an empty string the non-trivial braids entering in the dilute braid-monoid algebra are $b_i^{±(1,1)}, b_i^{±(1,2)}$ and $b_i^{±(2,1)}$. They satisfy the following generalized braid group relations \footnote{Here it has been assumed that $b_i^{+(a,b)} = b_i^{-(a,b)}$ for $a \neq b$ and $b_i^{±(2,2)} = E_i^{(2,2)} = p_i^{(2,2)}$.}

$$ b_i^{-(a,b)} b_i^{+(b,a)} = b_i^{+(a,b)} b_i^{-(b,a)} = p_i^{(b,a)} , $$

$$ b_i^{+(a,b)} b_{i+1}^{+(c,b)} b_i^{+(c,a)} = b_{i+1}^{+(c,a)} b_i^{+(c,b)} b_{i+1}^{+(b,a)} , \quad (26) $$

where the $p_i^{(a,b)}$ are composed projectors defined as $p_i^{(a,b)} = P_i^{(a)} P_{i+1}^{(b)}$.

By the same token the corresponding coloured monoid operators $E_i^{(1,1)}, E_i^{(1,2)}$ and $E_i^{(2,1)}$ are subjected to extended Temperley-Lieb relations,

$$ E_i^{(a,b)} E_i^{(c,a)} = Q^{(a)} E_i^{(c,b)} , $$

$$ E_i^{(a,b)} E_{i+1}^{(a,a)} E_i^{(c,a)} = E_i^{(c,b)} p_{i+1}^{(c,a)} , $$

$$ E_i^{(a,b)} E_{i-1}^{(a,a)} E_i^{(c,a)} = E_i^{(c,b)} p_{i-1}^{(a,c)} . \quad (27) $$

As usual it is assumed that any two of such generators acting at positions $i$ and $j$ with $|i - j| \geq 2$ commute. Besides that, an additional set of relations among the braid and monoid generators are required to be satisfied, namely

$$ b_i^{+(a,a)} E_i^{(b,a)} = \omega^{(a)} E_i^{(b,a)} , $$

$$ E_i^{+(a,b)} b_i^{(a,a)} = \omega^{(a)} E_i^{(a,b)} , $$

$$ b_{i+1}^{+(c,b)} b_i^{+(c,b)} E_i^{(a,b)} = E_i^{(a,b)} b_{i+1}^{+(c,a)} b_i^{+(c,a)} = E_i^{(c,b)} E_i^{(a,c)} , $$

$$ b_{i-1}^{+(b,c)} b_i^{+(b,c)} E_{i-1}^{(a,b)} = E_{i-1}^{(a,b)} b_{i-1}^{+(a,c)} b_i^{+(a,c)} = E_i^{(c,b)} E_{i-1}^{(a,c)} . \quad (28) $$
In analogy to section 2 the dilute Birman-Wenzl-Murakami algebra emerges as a quotient of the dilute braid monoid algebra [23, 28]. As before this quotient demands further restrictions between the braids $b_i^{\pm(1,1)}$ and the Temperley-Lieb operators $E_i^{(1,1)}$ such as the analog of the cubic relation [16] given by,

$$
(b_i^{+(1,1)} + \frac{1}{q} p_i^{(1,1)}) (b_i^{+(1,1)} - q p_i^{(1,1)}) (b_i^{+(1,1)} - \omega^{(1)} p_i^{(1,1)}) = 0.
$$

(29)

as well as the following polynomial relation for the monoid $E_i^{(1,1)}$,

$$
E_i^{(1,1)} = p_i^{(1,1)} + \frac{b_i^{+(1,1)} - b_i^{-(1,1)}}{q^{-1} - q}.
$$

(30)

A relevant feature of such quotient of the dilute braid-monoid algebra is that the operators related to the first colour $b_i^{\pm(1,1)}$, $E_i^{(1,1)}$ and $p_i^{(1,1)}$ close a subalgebra of Birman-Wenzl-Murakami type. This suggests therefore that the braid-monoid operators constructed in section 2 can be used as the starting point to obtain representations of the dilute version of the Birman-Wenzl-Murakami algebra. More precisely, these representations can be found from our previous results by first adding one extra bosonic degree of freedom, corresponding to the second colour, to the original local space of states. As a consequence of that the action of a given operator $\hat{O}_i$ at the $i$-th site is now given by

$$
\hat{O}_i = \bigotimes_{j=1}^{i-1} I_{N+1} \hat{O} \bigotimes_{j=i+1}^{L} I_{N+1}.
$$

(31)

The fact that the second colour has been chosen to be trivially represented leads us to the following general expressions for the projectors [21],

$$
P^{(1)} = \sum_{\alpha=1}^{N} \bar{e}_{\alpha \alpha}, \quad P^{(2)} = \bar{e}_{N+1 \times N+1}.
$$

(32)

where $\bar{e}_{\alpha \beta}$ are $(N + 1) \times (N + 1)$ Weyl matrices.

The respective representations for the braids $b^{\pm(l(1,1)}$ and monoids $E^{(l(1,1)}$ can then formally be taken from Eqs. [10, 11, 19], where once again the upper index $l$ takes account of two possible classes of representations. More specifically, these operators are now $(N + 1)^2 \times (N + 1)^2$
matrices whose explicit expressions are,

\begin{align*}
\mathbf{b}^{+(l|1,1)} &= \sum_{\alpha \neq \alpha'} (-1)^{p_{\alpha}} q^{1-2p_{\alpha}} \bar{e}_{\alpha \alpha} \otimes \bar{e}_{\alpha \alpha} + \sum_{\alpha, \beta = 1 \atop \alpha \neq \beta, \beta'} (-1)^{p_{\alpha} p_{\beta}} \bar{e}_{\beta \alpha} \otimes \bar{e}_{\alpha \beta} \\
&+ \left( q - \frac{1}{q} \right) \sum_{\alpha, \beta = 1 \atop \alpha < \beta, \alpha \neq \beta'} \bar{e}_{\alpha \alpha} \otimes \bar{e}_{\beta \beta} + \sum_{\alpha, \beta = 1 \atop \alpha \neq \beta, \beta'} \alpha_{\alpha \beta}^{(+)} \bar{e}_{\alpha \beta} \otimes \bar{e}_{\alpha \beta'}, \\
\mathbf{b}^{-|(l|1,1)} &= \sum_{\alpha \neq \alpha'} (-1)^{p_{\alpha}} q^{-1+2p_{\alpha}} \bar{e}_{\alpha \alpha} \otimes \bar{e}_{\alpha \alpha} + \sum_{\alpha, \beta = 1 \atop \alpha \neq \beta, \beta'} (-1)^{p_{\alpha} p_{\beta}} \bar{e}_{\beta \alpha} \otimes \bar{e}_{\alpha \beta} \\
&+ \left( \frac{1}{q} - q \right) \sum_{\alpha, \beta = 1 \atop \alpha < \beta, \alpha \neq \beta'} \bar{e}_{\beta \beta} \otimes \bar{e}_{\alpha \alpha} + \sum_{\alpha, \beta = 1 \atop \alpha \neq \beta, \beta'} a_{\alpha \beta}^{(-)} \bar{e}_{\alpha \beta'} \otimes \bar{e}_{\alpha \beta'}, \\
\mathbf{E}^{(l|1,1)} &= \sum_{\alpha, \beta = 1 \atop \alpha \neq \beta} \epsilon_{\alpha \beta}^{(l)} q_{\alpha \beta}^{(l)} \bar{e}_{\alpha \beta} \otimes \bar{e}_{\alpha \beta'}.
\end{align*}

At this point it should be emphasized that the operators (33-35) together with the projector \( p^{(1,1)} \) close the dilute Birman-Wenzl-Murakami subalgebra as long as the parameters \( Q^{(1)} = Q, \omega^{(1)} = \omega \) and \( Q^{(2)} = Q^{(2)} = 1 \). The expressions for the mixed braids \( \mathbf{b}^{+(l|1,2)} \) and \( \mathbf{b}^{+(l|2,1)} \) follows almost directly from the definition of the projectors, namely

\begin{align*}
\mathbf{b}^{+(l|1,2)} &= \sum_{\alpha = 1}^{N} \bar{e}_{\alpha \alpha} \otimes \bar{e}_{\alpha \alpha} \\
\mathbf{b}^{+(l|2,1)} &= \sum_{\alpha = 1}^{N} \bar{e}_{\alpha \alpha} \otimes \bar{e}_{\alpha \alpha}.
\end{align*}

In order to obtain the mixed Temperley-Lieb operators some extra amount of work is however necessary. It turns out that they are given by

\begin{align*}
\mathbf{E}^{(l|2,1)} &= \sum_{\alpha = 1}^{N} \frac{\epsilon_{\alpha}^{(l)}}{\epsilon_{\alpha}^{(l)}} q_{\alpha, \alpha}^{(l)} \bar{e}_{\alpha \alpha} \otimes \bar{e}_{\alpha \alpha}, \\
\mathbf{E}^{(l|1,2)} &= \sum_{\alpha = 1}^{N} \frac{\epsilon_{\alpha}^{(l)}}{\epsilon_{\alpha}^{(l)}} q_{\alpha, \alpha}^{(l)} \bar{e}_{\alpha \alpha} \otimes \bar{e}_{\alpha \alpha'},
\end{align*}

where \( \epsilon_{\alpha}^{(l)} \) and \( q_{\alpha, \alpha}^{(l)} \) are arbitrary additional parameters associated to the dilution.

Next we turn to the problem of constructing spectral parameter dependent \( R \)-matrices from the above realizations of the dilute Birman-Wenzl-Murakami algebra. As before, every
representation of this algebra can be baxterized to yield a solution of the Yang-Baxter equation \[21\]. The corresponding expression for \( \tilde{R}^{(l)}(x) \) is

\[
\tilde{R}^{(l)}(x) = \left( \frac{1}{q - q} \right) \eta p^{(1,1)} + (x - \frac{1}{x}) \left( \frac{x}{\tau_l} p^{+(l|1,1)} - \frac{x}{\tau_l} b^{-(l|1,1)} \right) + \left( \frac{1}{q - q} \right) \left( \frac{\tau_l}{x} - \frac{x}{\tau_l} \right) \left( p^{(1,2)} + p^{(2,1)} \right) - \kappa_1 \left( x - \frac{1}{x} \right) \left( \frac{\tau_l}{x} - \frac{x}{\tau_l} \right) \left( b^{+(l|2,1)} + b^{-(l|2,1)} \right) + \kappa_2 \left( \frac{1}{q - q} \right) (x - \frac{1}{x}) \left( E^{(l|1,2)} + E^{(l|2,1)} \right) + \left[ \eta \left( \frac{1}{q - q} \right) - (x - \frac{1}{x}) \left( \frac{\tau_l}{x} - \frac{x}{\tau_l} \right) \right] p^{(2,2)},
\]

(39)

where \( \tau_l^2 = \sigma_l, \eta_l = \tau_l - \tau_l^{-1} \) and \( \kappa_{1,2} = \pm 1 \) are arbitrary signs.

Direct substitution of the dilute representations \([32, 33]\) in Eq.\([39]\) leads us to expressions for the \( R \)-matrices whose expected underlying \( U(1) \) symmetries are difficult to be recognized at first sight. These charge conservations can however be made more explicit by means of suitable unitary transformations that preserve the Yang-Baxter equation, namely

\[
\tilde{R}^{(l)}(x) = (S \otimes S)^{-1} \tilde{R}^{(l)}(x) (S \otimes S),
\]

(40)

where \( S \) is an invertible \((N + 1) \times (N + 1)\) matrix.

For even \( N = 2n + 2m \) we have found that the appropriate matrix \( S_{\text{even}} \) is given by

\[
S_{\text{even}} = \sum_{\alpha=1}^{n+m} \tilde{e}_{\alpha \alpha} + \sum_{\alpha=n+m+1}^{2n+2m} \tilde{e}_{\alpha \alpha+1} + \tilde{e}_{2n+2m+1 \alpha+m+1},
\]

(41)

and that the corresponding transformed \( R \)-matrices \( \tilde{R}^{(1)}(x) \) and \( \tilde{R}^{(2)}(x) \) are closely related to those of the \( U_q[osp^{(1)}(2n + 1|2m)] \) and \( U_q[sl^{(2)}(2n + 1|2m)] \) superalgebras given in section 2, respectively. In fact, they can be made equivalent by spectral parameter dependent gauge transformations and therefore they do not produce new vertex models.

The situation for odd \( N = 2n + 1 + 2m \) is fortunately much more interesting. In this case the matrix \( S_{\text{odd}} = S_1 S_2 \) where \( S_1 \) and \( S_2 \) are given by,

\[
S_1 = \sum_{\alpha=1}^{n+m+1} \tilde{e}_{\alpha \alpha} + \sum_{\alpha=n+m+2}^{2n+2m+1} \tilde{e}_{\alpha \alpha+1} + \tilde{e}_{2n+2m+2 \alpha+m+2}
\]

(42)

\[
S_2 = \sum_{\alpha=1}^{n+m} \tilde{e}_{\alpha \alpha} + \sum_{\alpha=n+m+3}^{2n+2m+2} \tilde{e}_{\alpha \alpha} + \frac{1}{\sqrt{2}} \left( \tilde{e}_{n+m+1 \alpha+m+1} + \tilde{e}_{n+m+1 \alpha+m+2} + \tilde{e}_{n+m+2 \alpha+m+1} - \tilde{e}_{n+m+2 \alpha+m+2} \right).
\]
The associated $R$-matrix $\hat{\mathcal{R}}^{(1)}(x)$ for $N$ odd is indeed novel as compared to that of the vertex models described in section 2. The explicit form of such $R$-matrix turns out to be

\[
\hat{\mathcal{R}}^{(1)}(x) =
\sum_{\alpha \neq \bar{n}+1} \frac{q(x^2 - \zeta^2)}{x^2(1 - \rho_\alpha)} \sum_{\alpha \neq \bar{n}+1} \frac{q(x^2 - 1)(x^2 - \zeta^2)}{x^2(1 - \rho_\alpha)} \sum_{\alpha \neq \bar{n}+1} \frac{(1 + \kappa_1) (\bar{e}_\alpha \otimes \bar{e}_\alpha \otimes \bar{e}_\beta \otimes \bar{e}_\beta)}{\beta = \bar{n}+1, \bar{n}+2}
\]

\[+ \frac{1}{2} q(x^2 - 1)(x^2 - \zeta^2) \sum_{\alpha \neq \bar{n}+1} \frac{q(x^2 - 1)(x^2 - \zeta^2)}{x^2(1 - \rho_\alpha)} \sum_{\alpha \neq \bar{n}+1} \frac{(1 + \kappa_1) (\bar{e}_\alpha \otimes \bar{e}_\alpha \otimes \bar{e}_\beta \otimes \bar{e}_\beta)}{\beta = \bar{n}+1, \bar{n}+2}
\]

\[+ (1 - \kappa_1) (\bar{e}_\beta \otimes \bar{e}_\alpha \otimes \bar{e}_\beta \otimes \bar{e}_\beta + \bar{e}_\beta \otimes \bar{e}_\alpha \otimes \bar{e}_\alpha \otimes \bar{e}_\beta]
\]

\[+ \sum_{\alpha \neq \bar{n}+1} \frac{g_{\alpha \beta}(x) \bar{e}_\alpha \otimes \bar{e}_\alpha \otimes \bar{e}_\beta \otimes \bar{e}_\beta}{\alpha \neq \bar{n}+1, \bar{n}+2}
\]

\[= (x^2 - \zeta^2) \left[ \sum_{\alpha \neq \bar{n}+1} \frac{q(x^2 - 1)(x^2 - \zeta^2)}{x^2(1 - \rho_\alpha)} \sum_{\alpha \neq \bar{n}+1} \frac{(1 + \kappa_1) (\bar{e}_\alpha \otimes \bar{e}_\alpha \otimes \bar{e}_\alpha \otimes \bar{e}_\alpha)}{\beta = \bar{n}+1, \bar{n}+2}
\]

\[+ (x - 1) \left[ \sum_{\alpha \neq \bar{n}+1} \frac{g_{\alpha \beta}(x) \bar{e}_\alpha \otimes \bar{e}_\alpha \otimes \bar{e}_\beta \otimes \bar{e}_\beta}{\alpha \neq \bar{n}+1, \bar{n}+2}
\]

\[+ \frac{1}{2} \sum_{\alpha \neq \bar{n}+1} \frac{b^+_{\alpha}(x) \bar{e}_\alpha \otimes \bar{e}_\alpha \otimes \bar{e}_\beta \otimes \bar{e}_\beta + \bar{b}^+_{\alpha}(x) \bar{e}_\beta \otimes \bar{e}_\alpha \otimes \bar{e}_\alpha \otimes \bar{e}_\beta + \bar{c}^+_{\alpha}(x) \bar{e}_\alpha \otimes \bar{e}_\alpha \otimes \bar{e}_\alpha \otimes \bar{e}_\alpha}{\beta = \bar{n}+1, \bar{n}+2}
\]

\[+ \frac{1}{2} \kappa_2 \zeta (q^2 - 1)x(x^2 - 1) \mathcal{F}_x \sum_{\alpha \neq \bar{n}+1} \frac{(1 - \alpha^2 - \beta^2)}{x^2(1 - \rho_\alpha)} \sum_{\alpha \neq \bar{n}+1} \frac{(-1)^{\alpha - \beta} (\bar{e}_\alpha \otimes \bar{e}_\alpha \otimes \bar{e}_\alpha \otimes \bar{e}_\alpha)}{\beta = \bar{n}+1, \bar{n}+2}
\]

where $\alpha'' = N + 2 - \alpha$, $\bar{n} = n + m$ and $\zeta = q^{n-m}$.

The Boltzmann weights $g_{\alpha \beta}(x)$, $b^+_{\alpha}(x)$, $\bar{b}^+_{\alpha}(x)$, $c^+_{\alpha}(x)$ and $d^+_{\alpha}(x)$ are given by,

\[
g_{\alpha \beta}(x) = \begin{cases} (x^2 - 1) \frac{(x^2 - \zeta^2)(-1)^{\rho_\alpha} q^{2\rho_\alpha} + x^2(q^2 - 1)}{(q^2 - 1)^2} & \alpha = \beta \\
\frac{q^2 - 1}{(q^2 - 1)^2} \frac{(x^2 - 1) \frac{\alpha \beta}{\alpha \beta} q^{2\alpha \beta - l^2 - \delta_{\alpha \beta}} - \delta_{\alpha \beta}}{x^2(q^2 - 1)} & \alpha < \beta \\
\frac{q^2 - 1}{(q^2 - 1)^2} \frac{(x^2 - 1) \frac{\alpha \beta}{\alpha \beta} q^{2\alpha \beta - l^2 - \delta_{\alpha \beta}} - \delta_{\alpha \beta}}{x^2(q^2 - 1)} & \alpha > \beta \end{cases}
\]
\[ b_\alpha^\pm (x) = \begin{cases} \pm \frac{\alpha^{(1)}}{\epsilon_{N+1}} q^{(1)}_\alpha (q^2 - 1)(x^2 - 1)(x \kappa_2 \mp \frac{\alpha^{(1)}}{\epsilon_{N+1}} q^{(1)}_\kappa (\alpha^{(1)} - \alpha^{(1)} - t^{(1)}_\alpha) \zeta) & \alpha < \bar{n} + 1 \\ (-1)^{p_\alpha} \frac{\alpha^{(1)}}{\epsilon_{\bar{n}+1}} q^{(2)}_\alpha (q^2 - 1)(x^2 - 1)x (x \mp \kappa_2 \frac{\alpha^{(1)}}{\epsilon_{N+1}} q^{(1)}_\kappa (\alpha^{(1)} - t^{(1)}_\alpha) \zeta) & \alpha > \bar{n} + 2 \end{cases} \quad \] (45)

\[ \bar{b}_\alpha^\pm (x) = \begin{cases} \mp (-1)^{p_\alpha} \frac{\alpha^{(1)}}{\epsilon_{\bar{n}+1}} q^{(2)}_\alpha (q^2 - 1)(x^2 - 1)x (x \mp \kappa_2 \frac{\alpha^{(1)}}{\epsilon_{N+1}} q^{(1)}_\kappa (\alpha^{(1)} - t^{(1)}_\alpha) \zeta) & \alpha < \bar{n} + 1 \\ \frac{\alpha^{(1)}}{\epsilon_{\bar{n}+1}} q^{(2)}_\alpha (q^2 - 1)(x^2 - 1)x (x \mp \kappa_2 \frac{\alpha^{(1)}}{\epsilon_{N+1}} q^{(1)}_\kappa (\alpha^{(1)} - t^{(1)}_\alpha) \zeta) & \alpha > \bar{n} + 2 \end{cases} \quad (46) \]

\[ c^\pm (x) = \pm \frac{1}{2} (q^2 - 1)(\zeta + \kappa_2 \mathcal{F}_+) x (x \mp 1) \left[ x \kappa_2 \frac{\zeta \mathcal{F}_+ - \kappa_2}{(\zeta + \kappa_2 \mathcal{F}_+)} \pm \zeta \right] + \frac{1}{2} q(x^2 - 1)(x^2 - \zeta^2) \quad (47) \]

\[ d^\pm (x) = \pm \frac{1}{2} (q^2 - 1)(\zeta - \kappa_2 \mathcal{F}_+) x (x \pm 1) \left[ x \kappa_2 \frac{\zeta \mathcal{F}_+ - \kappa_2}{(\zeta - \kappa_2 \mathcal{F}_+)} \pm \zeta \right] + \frac{1}{2} q(x^2 - 1)(x^2 - \zeta^2) \quad (48) \]

The auxiliary variables \( \mathcal{F}_\pm, \bar{\epsilon}_\alpha, \bar{t}_\alpha, \bar{t}_\alpha^{(1)} \) and \( \bar{t}_\alpha^{(2)} \) entering in the above weights definition depend directly on the additional parameters \( \epsilon_\alpha^{(1)} \) and \( t_\alpha^{(1)} \) as follows,

\[ \mathcal{F}_\pm = -\frac{1}{2} \left[ \frac{\epsilon_{n+1}^{(1)}}{\epsilon_{N+1}^{(1)}} q^{(1)}_{n+1} - t^{(1)}_{n+1} \pm \frac{\epsilon_{N+1}^{(1)}}{\epsilon_{\bar{n}+1}^{(1)}} q^{(1)}_{\bar{n}+1} - t^{(1)}_{\bar{n}+1} \right] \quad (50) \]

\[ \bar{\epsilon}_\alpha = \begin{cases} \epsilon_\alpha^{(1)} & 1 \leq \alpha \leq \bar{n} \\ \epsilon_{\alpha-1}^{(1)} & \bar{n} + 3 \leq \alpha \leq N + 1 \end{cases} \quad (51) \]

\[ \bar{t}_\alpha = \begin{cases} t_\alpha^{(1)} & 1 \leq \alpha \leq \bar{n} \\ t_{\alpha-1}^{(1)} & \bar{n} + 3 \leq \alpha \leq N + 1 \end{cases} \quad (52) \]

and

\[ \bar{t}_\alpha^{(1)} = \begin{cases} t_\alpha^{(1)} - t_{N+1}^{(1)} + n - m & 1 \leq \alpha \leq \bar{n} \\ t_\alpha^{(1)} - t_{\bar{n}+1}^{(1)} + 2\alpha - 5 - 2\bar{n} - 2\bar{p}_\alpha - 4 \sum_{\beta=\bar{n}+3}^{\alpha-1} \bar{p}_\beta & \bar{n} + 3 \leq \alpha \leq N + 1 \end{cases} \quad (53) \]

\[ \bar{t}_\alpha^{(2)} = \begin{cases} t_{N+1}^{(1)} - t_\alpha^{(1)} + 2\alpha - (n - m + 1) - 2\bar{p}_\alpha - 4 \sum_{\beta=1}^{\alpha-1} \bar{p}_\beta & 1 \leq \alpha \leq \bar{n} \\ t_{\bar{n}+1}^{(1)} - t_\alpha^{(1)} & \bar{n} + 3 \leq \alpha \leq N + 1 \end{cases} \quad (54) \]
Finally, the renormalized parities $\bar{p}_\alpha$ are related to that of the section 2 by the following expression,

\[
\bar{p}_\alpha = \begin{cases} 
p_{\alpha} & 1 \leq \alpha \leq \bar{n} \\
0 & \alpha = \bar{n} + 1 \\
0 & \alpha = \bar{n} + 2 \\
p_{\alpha-1} & \bar{n} + 3 \leq \alpha \leq N + 1
\end{cases}
\] (55)

From Eqs. (43-55) one clearly notes that the above $R$-matrix has in fact two possible branches governed by the discrete parameter $\kappa_1 = \pm 1$. This gives origin to two distinct vertex models since the structure of some of the Boltzmann weights depend drastically on the sign of $\kappa_1$. On the other hand, the parameter $k_2$ apparently does not play such a relevant role in the $R$-matrix (43). Indeed, the transformation $\kappa_2 \rightarrow -\kappa_2$ followed by similar reflexion in the variable $\zeta$ leaves the whole $R$-matrix (43) invariant apart from a trivial sign change on the weights $b_{\alpha}^\pm(x)$ and $\overline{b}_{\alpha}^\pm$.

To the best of our knowledge the general multiparametric structure of the $R$-matrix (43-55) is new even when the fermionic degrees of freedom are absent. Though the basic form of $\tilde{R}^{(1)}(x)$ for $\kappa_1 = 1$ with all $p_{\alpha} = 0$ resembles that of the $U_q[D_{n+1}^{(2)}]$ $R$-matrix given by Jimbo [6] there exists essential differences among these $R$-matrices. A direct comparison reveals that our $R$-matrix presents extra relevant Boltzmann weights as compared to Jimbo’s $U_q[D_{n+1}^{(2)}]$ vertex model [6] such as the last term of Eq.(43). Besides that, the spectral parameter dependence of some of the weights depends strongly on the additional variables $\epsilon_{\alpha}^{(1)}$ and $t_{\alpha}^{(1)}$. In fact, it is only for a fine tuning between these extra parameters that all the above mentioned differences are canceled out. This appears to indicate that Jimbo’s $U_q[D_{n+1}^{(2)}]$ $R$-matrix is a particular case and probably does not capture the most general structure admissible in the $U_q[D_{n+1}^{(2)}]$ quantum group deformations. Other indication of this fact occurs when one tries to solve the Jimbo’s $U_q[D_{n+1}^{(2)}]$ vertex model by means of the quantum inverse scattering method [27]. One notices that the first nested Bethe ansatz for $n \geq 2$ is already governed by a multiparametric $R$-matrix having more Boltzmann weights entries than that of the Jimbo’s $U_q[D_{n-1}^{(2)}]$ $R$-matrix. Therefore, a consistent algebraic Bethe ansatz solution of these systems will require the class of the multiparametric $R$-matrix exhibited here from the very beginning.
Yet another interesting property was found in the course of an explicit Yang-Baxter verification of Eqs. (43–55). We observed that there exists a second integrable family differing from that defined by Eqs. (43–55) only in respect to the Boltzmann weights \( c^\pm(x) \) and \( d^\pm(x) \). In other words, the whole structure of the \( R \)-matrix (43) as well as the form of the weights \( g_{\alpha,\beta}(x) \), \( b_\alpha^\pm(x) \) and \( \bar{b}_\alpha^\pm(x) \) are kept unchanged except the \( c^\pm(x) \) and \( d^\pm(x) \) weights. For such second family the spectral parameter dependence of the respective \( c^\pm(x) \) and \( d^\pm(x) \) weights are

\[
c^\pm(x) = \pm \frac{1}{2}(q^2 - 1)(\zeta + \kappa_2 F_+) x(x \mp 1) \left[ x\kappa_2 \left( \frac{\zeta F_+ + \kappa_2}{\zeta + \kappa_2 F_+} \right) \pm \zeta \right] + \frac{1 - \kappa_1}{2} q(x^2 - 1)(x^2 - \zeta^2)
\]

\[
d^\pm(x) = \pm \frac{1}{2}(q^2 - 1)(\zeta - \kappa_2 F_+) x(x \pm 1) \left[ x\kappa_2 \left( \frac{\zeta F_+ - \kappa_2}{\zeta - \kappa_2 F_+} \right) \pm \zeta \right] + \frac{1 + \kappa_1}{2} q(x^2 - 1)(x^2 - \zeta^2).
\]

Interesting enough we note that the weights (56, 57) are related to the previous one through the reflexion \( \kappa_1 \to -\kappa_1 \). Here we stress that this transformation applies only for such specific weights subset. Therefore, one expects that Eq. (43) with weights \( c^\pm(x) \) and \( d^\pm(x) \) given by either Eqs. (48–49) or Eqs. (56–57) would provide us different \( R \)-matrices. In fact, we have verified for some values of \( L \) that the spectrum of the transfer matrices built from these two integrable families are indeed unrelated.

In order to emphasize the extension of our results concerning the presence of fermionic degrees of freedom, it is convenient to present the \( R \)-matrix (43–55) for special choices of the additional parameters \( \epsilon^{(1)}_\alpha \) and \( t^{(1)}_\alpha \) such that Jimbo’s \( U_q[D^{(2)}_{n+1}] \) \( R \)-matrix is recovered when all \( p_\alpha = 0 \). This occurs by choosing the variables \( \epsilon^{(1)}_\alpha \) and \( t^{(1)}_\alpha \) for \( 1 \leq \alpha \leq N \) as described in Appendix A as well as by setting \( \epsilon^{(1)}_{N+1} = -1 \) and \( t^{(1)}_{N+1} = n + m + 1 \). After carrying on the corresponding simplifications in Eqs. (43–55) we find that the \( R \)-matrix (43) can be rewritten as follows,

\[
\tilde{R}^{(1)}(x) =
\]
\[
\sum_{\alpha \neq \bar{n}+1} (x^2 - \zeta^2) \left[ x^{2(1-\beta_{\alpha})} - q^2 x^{2\beta_{\alpha}} \right] \bar{\epsilon}_{\alpha} \alpha \otimes \bar{\epsilon}_{\alpha} \alpha + q(x^2 - 1)(x^2 - \zeta^2) \sum_{\alpha \neq \beta, \beta' \neq \bar{n}+1, \bar{n}+2} (-1)^{\beta_{\alpha} \beta_{\beta'}} \bar{\epsilon}_{\beta} \alpha \otimes \bar{\epsilon}_{\beta} \beta \\
+ \frac{1}{2} q(x^2 - 1)(x^2 - \zeta^2) \sum_{\alpha, \beta \neq \bar{n}+1, \bar{n}+2} [(1 + \kappa_1) (\bar{\epsilon}_{\beta} \alpha \otimes \bar{\epsilon}_{\alpha} \beta + \bar{\epsilon}_{\alpha} \beta \otimes \bar{\epsilon}_{\beta} \alpha)] + (1 - \kappa_1) (\bar{\epsilon}_{\beta} \alpha \otimes \bar{\epsilon}_{\alpha'} \beta + \bar{\epsilon}_{\alpha'} \beta \otimes \bar{\epsilon}_{\beta} \alpha) + \sum_{\alpha, \beta \neq \bar{n}+1, \bar{n}+2} \bar{g}_{\alpha \beta}(x) \bar{\epsilon}_{\alpha} \alpha \otimes \bar{\epsilon}_{\alpha} \beta
\]

\[
- (q^2 - 1)(x^2 - \zeta^2) \left[ \sum_{\alpha < \beta, \alpha' \neq \beta', \beta' \neq \bar{n}+1, \bar{n}+2} + x^2 \sum_{\alpha > \beta, \alpha' \neq \beta', \beta' \neq \bar{n}+1, \bar{n}+2} \right] \bar{\epsilon}_{\beta} \alpha \otimes \bar{\epsilon}_{\alpha} \alpha
\]

\[
- \frac{1}{2} (q^2 - 1)(x^2 - \zeta^2)[(x + 1) \left( \sum_{\alpha < \bar{n}+1} + x \sum_{\alpha > \bar{n}+2} \right) (\bar{\epsilon}_{\beta} \alpha \otimes \bar{\epsilon}_{\alpha} \alpha + \bar{\epsilon}_{\alpha} \alpha \otimes \bar{\epsilon}_{\beta} \beta' \beta'')]
\]

\[
+ (x - 1) \left( - \sum_{\alpha < \bar{n}+1} + x \sum_{\alpha > \bar{n}+2} \right) (\bar{\epsilon}_{\beta} \beta' \alpha \otimes \bar{\epsilon}_{\alpha} \alpha + \bar{\epsilon}_{\alpha} \alpha \otimes \bar{\epsilon}_{\beta} \beta' \beta'')
\]

\[
+ \frac{1}{2} \sum_{\alpha \neq \bar{n}+1, \bar{n}+2} \left[ \bar{b}_{\alpha}^+ (x) (\bar{\epsilon}_{\alpha} \alpha \otimes \bar{\epsilon}_{\alpha} \alpha + \bar{\epsilon}_{\alpha} \alpha \otimes \bar{\epsilon}_{\alpha} \alpha) + \bar{b}_{\alpha}^- (x) (\bar{\epsilon}_{\alpha} \alpha \otimes \bar{\epsilon}_{\alpha} \alpha + \bar{\epsilon}_{\alpha} \alpha \otimes \bar{\epsilon}_{\alpha} \alpha) \right]
\]

\[
+ \sum_{\alpha = \bar{n}+1, \bar{n}+2} \left[ \bar{c}_{\alpha}^+ (x) \bar{\epsilon}_{\alpha} \alpha \otimes \bar{\epsilon}_{\alpha} \alpha + \bar{c}_{\alpha}^- (x) \bar{\epsilon}_{\alpha} \alpha \otimes \bar{\epsilon}_{\alpha} \alpha + \bar{d}_{\alpha}^+ (x) \bar{\epsilon}_{\alpha} \alpha \otimes \bar{\epsilon}_{\alpha} \alpha + \bar{d}_{\alpha}^- (x) \bar{\epsilon}_{\alpha} \alpha \otimes \bar{\epsilon}_{\alpha} \alpha \right].
\]

(58)

The respective Boltzmann weights \( \bar{g}_{\alpha \beta}(x) \), \( \bar{b}_{\alpha}^\pm(x) \), \( \bar{c}_{\alpha}^\pm(x) \) and \( \bar{d}_{\alpha}^\pm(x) \) are now given by

\[
\bar{g}_{\alpha \beta}(x) = \begin{cases} 
(x^2 - \zeta^2) [(x^2 - \zeta^2)(-1)^{\beta_{\alpha} q^2 \beta_{\alpha} x^2(q^2 - 1)] & \alpha = \beta \\
(q^2 - 1) \left[ \zeta^2 (x^2 - 1) \bar{\epsilon}_{\beta} \alpha \alpha \otimes \bar{\epsilon}_{\beta} \alpha \alpha - \delta_{\alpha} \beta' (x^2 - \zeta^2) \right] & \alpha < \beta \\
(q^2 - 1) x^2 \left[ (x^2 - 1) \bar{\epsilon}_{\beta} \alpha \alpha \otimes \bar{\epsilon}_{\beta} \alpha \alpha - \delta_{\alpha} \beta' (x^2 - \zeta^2) \right] & \alpha > \beta
\end{cases} \]

(59)

\[
\bar{b}_{\alpha}^\pm(x) = \begin{cases} 
\pm \bar{\epsilon}_{\alpha} q^2 \alpha (q^2 - 1)(x^2 - 1)(x \bar{\kappa} \pm \zeta) & \alpha < \bar{n} + 1 \\
\bar{\epsilon}_{\alpha} q^2 \alpha (q^2 - 1)(x^2 - 1)x(x \bar{\kappa} \pm \zeta) & \alpha > \bar{n} + 2
\end{cases}
\]

(60)

\[
\bar{c}_{\alpha}^\pm(x) = \pm \frac{1}{2} (q^2 - 1)(\zeta + \kappa_2)x(x \mp 1)(x \bar{\kappa} \pm \zeta) + \frac{1 + \nu_1}{2} q(x^2 - 1)(x^2 - \zeta^2)
\]

(61)

\[
\bar{d}_{\alpha}^\pm(x) = \pm \frac{1}{2} (q^2 - 1)(\zeta - \kappa_2)x(x \pm 1)(x \bar{\kappa} \pm \zeta) + \frac{1 - \nu_1}{2} q(x^2 - 1)(x^2 - \zeta^2)
\]

(62)

where the lower index \( \nu = \pm 1 \) in the weights \( \bar{c}_{\alpha}^\pm(x) \) and \( \bar{d}_{\alpha}^\pm(x) \) indicates the two possible families
of models discussed previously. The explicit expressions for the variables $\tilde{\epsilon}_\alpha$, $\tilde{t}_\alpha$ and $\tilde{t}_\alpha$ are

$$
\tilde{\epsilon}_\alpha = \begin{cases}
(\frac{-1}{2})^{m+1} & 1 \leq \alpha \leq \bar{n} \\
1 & \alpha = \bar{n} + 1 \\
1 & \alpha = \bar{n} + 2 \\
(\frac{-1}{2})^{m+1} & \bar{n} + 3 \leq \alpha \leq N + 1
\end{cases}
$$

(63)

$$
\tilde{t}_\alpha = \begin{cases}
\alpha + \left[1 - \bar{p}_\alpha + 2 \sum_{\beta=1}^{\bar{p}} \bar{p}_\beta \right] & 1 \leq \alpha \leq \bar{n} \\
\bar{n} + \frac{1}{2} & \alpha = \bar{n} + 1 \\
\bar{n} + \frac{3}{2} & \alpha = \bar{n} + 2 \\
\alpha - \left[1 - \bar{p}_\alpha + 2 \sum_{\beta=\bar{n}+3}^{\alpha} \bar{p}_\beta \right] & \bar{n} + 3 \leq \alpha \leq N + 1
\end{cases}
$$

(64)

$$
\tilde{t}_\alpha = \begin{cases}
\alpha - \left[\frac{1}{2} - \bar{p}_\alpha + 2 \sum_{\beta=1}^{\bar{p}} \bar{p}_\beta \right] & 1 \leq \alpha \leq \bar{n} \\
\alpha - \left[\bar{n} + \frac{5}{2} - \bar{p}_\alpha + 2 \sum_{\beta=\bar{n}+3}^{\alpha} \bar{p}_\beta \right] & \bar{n} + 3 \leq \alpha \leq N + 1
\end{cases}
$$

(65)

Now it is not difficult to recognize that expressions (63) for the branch $\kappa_1 = 1$ and $\nu = 1$ with all $p_\alpha = 0$ indeed reproduce the $U_q[D_{n+1}^{(2)}]$ $R$-matrix. This means that in general the $R$-matrix (58) should be considered as a non-trivial generalization of Jimbo’s $U_q[D_{n+1}^{(2)}]$ vertex model when the respective edge variables admit both bosonic and fermionic statistics. To our knowledge such interesting possibility has not been predicted before even in the realm of a powerful method such as the quantum supergroup formalism [8, 12, 13]. To shed some light on the construction of the $R$-matrix (58) in the context of quantum superalgebras one can study its respective $q \to 1$ limit. By performing this analysis we found that the classical limit of the $R$-matrix (58) with $\kappa_1 = \kappa_2 = \nu = 1$ turns out to be the rational $osp(2n+2|2m)$ $R$-matrices [23]. Therefore it is plausible to suppose that the $R$-matrix (58) could be derived as a quantum deformation of the $osp(2n+2|2m)$ Lie superalgebra with a given automorphism. It remains however the precise identification of the order of the corresponding automorphism and this step has eluded us so far.
We would like to close this section by discussing useful properties satisfied by the $R$-matrix $R^{(1)}(x) = P \tilde{R}^{(1)}(x)$ where $\tilde{R}^{(1)}(x)$ refers to the matrix given in Eq. (58). Besides regularity and unitarity this $R$-matrix satisfies the so-called $PT$ symmetry given by

$$P_{12} R^{(1)}_{12}(x) P_{12} = [R^{(1)}_{12}]^{st_1 st_2}(x),$$

where the symbol $st_k$ denotes the supertransposition in the space with index $k$. Yet another property is the crossing symmetry, namely

$$R^{(1)}_{12}(x) = \frac{\rho(x)}{\rho(\zeta/x)} V_1 [R^{(1)}_{12}]^{st_2}(\zeta/x) V^{-1}_1,$$

where $\rho(x)$ is a convenient normalization and $V$ is an anti-diagonal matrix. The explicit expressions for these quantities have been collected in Appendix B.

4 Conclusions

In this paper we have presented explicit representations of the Birman-Wenzl-Murakami algebra as well as of its dilute generalization. The representations contain a considerable amount of free parameters and the respective degrees of freedom can be of bosonic of fermionic type. We argued that the corresponding braids should be related to the multiparametric universal $R$-matrices associated to the $U_q[osp(r|2m)]^{(1)}$ and $U_q[osp(r = 2n|2m)]^{(2)}$ symmetries.

The baxterization of the representations of the Birman-Wenzl-Murakami algebra produced solutions of the Yang-Baxter equation invariant by the $U_q[osp(r|2m)]^{(1)}$, $U_q[sl(r|2m)]^{(2)}$ and $U_q[osp(r = 2n|2m)]^{(2)}$ quantum superalgebras. The dilute baxterization has leaded us to two other families of $R$-matrices not previously foreseen by the framework of quantum supergroups. These systems can in fact be considered as rather non-trivial extensions of Jimbo’s $U_q[D^{(2)}_{n+1}]$ $R$-matrix. This occurs even when the fermionic variables are absent because the presence of extra parameters produces us the multiparametric $U_q[D^{(2)}_{n+1}]$ $R$-matrix whose general structure was not known before. We noted that this knowledge is essential to implement the algebraic Bethe ansatz for these vertex models in a consistent way.
Besides that, our study also pave the way to build the representations of the two-colour Birman-Wenzl-Murakami algebra \[26\]. From any representation of this type one can in principle construct another $R$-matrices via the baxterization procedure. In view to what has been discussed above one expects that new solvable vertex models could them be derived. It would be interesting to know the type of lattice models with both bosonic and fermionic degrees of freedom that are obtained from this construction. We hope to report on this problem in a future publication.

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Appendix A: The symmetrical gauge

In this appendix we briefly describe a symmetrical form for the variables $\epsilon_{\alpha}^{(l)}$ and $t_{\alpha}^{(l)}$. This choice of variables ensures us that the respective $R$-matrix becomes $PT$ invariant. The idea is to explore the arbitrariness of Eqs.\((14,15)\) by fixing in a convenient way some of these variables.

For the first family an appropriate choice of these variables on the interval $1 \leq \alpha \leq \frac{N+1}{2}$ will lead us to the following symmetrical structure,

$$
\begin{align*}
\epsilon_{\alpha}^{(1)} &= \begin{cases} 
(1) \frac{-p_{\alpha}}{2} & 1 \leq \alpha < \frac{N+1}{2} \\
1 & \frac{N+1}{2} < \alpha \leq N
\end{cases} \\
t_{\alpha}^{(1)} &= \begin{cases} 
\frac{N+1}{2} & 1 \leq \alpha < \frac{N+1}{2} \\
\alpha - \left[ \frac{1}{2} - p_{\alpha} + 2 \sum_{\beta=\frac{N+1}{2}+1}^{\alpha} p_{\beta} \right] & \frac{N+1}{2} < \alpha \leq N
\end{cases}
\end{align*}
$$

(A.1)

From our previous work [14] we see that this is exactly the form of the corresponding variables appearing in the $U_q[osp(r|2m)^{(1)}]$ $R$-matrix.

Similarly, a suitable choice of the variables $\epsilon_{\alpha}^{(2)}$ and $t_{\alpha}^{(2)}$ for $1 \leq \alpha \leq \frac{N}{2}$ produces us the form

$$
\begin{align*}
\epsilon_{\alpha}^{(2)} &= \begin{cases} 
(1) \frac{-p_{\alpha}}{2} & 1 \leq \alpha \leq \frac{N}{2} \\
-1 & \frac{N}{2} + 1 \leq \alpha \leq N
\end{cases} \\
t_{\alpha}^{(2)} &= \begin{cases} 
\alpha - \left[ \frac{1}{2} + p_{\alpha} - 2 \sum_{\beta=\alpha}^{\frac{N}{2}} p_{\beta} \right] & 1 \leq \alpha \leq \frac{N}{2} \\
\alpha + \left[ \frac{1}{2} + p_{\alpha} - 2 \sum_{\beta=\frac{N}{2}+1}^{\frac{N}{2}+1} p_{\beta} \right] & \frac{N}{2} + 1 \leq \alpha \leq N
\end{cases}
\end{align*}
$$

(A.3)

(A.4)

which is just that related with the $U_q[osp(r=2n|2m)^{(2)}]$ $R$-matrix given in ref.\([14]\).

The above results strongly suggest that $b^{+\,(1)}$ and $b^{+\,(2)}$ should be associated to the multi-parametric $U_q[osp(r|2m)^{(1)}]$ and $U_q[osp(r=2n|2m)^{(2)}]$ universal $R$-matrices, respectively.
Appendix B: Crossing symmetry

The purpose here is to present the explicit expressions for the normalization function \( \rho(x) \) and the crossing matrix \( V \). The normalization is

\[
\rho(x) = q(x^2 - 1)(x^2 - \zeta^2),
\]

while the only non-null entries of the matrix \( V \) are the anti-diagonal elements \( V_{\alpha \alpha''} \), namely

\[
V_{\alpha \alpha''} = \begin{cases} 
(-1)^{\frac{\bar{n} - 1}{2}} q^{\frac{\alpha - 1 - \bar{p}_{\alpha} - 2}{2} \sum_{\beta=2}^{\alpha-1} \bar{p}_{\beta}} & \text{if } \alpha = 1 \\
(-1)^{\frac{\bar{n} - 1}{2}} q^{\frac{\alpha - 1 - \bar{p}_{\alpha} - 2}{2} \sum_{\beta=2}^{\alpha} \bar{p}_{\beta}} & 1 < \alpha < \bar{n} + 1 \\
(-1)^{\frac{\bar{n} - 1}{2}} q^{\frac{\alpha - 1 - \bar{p}_{\alpha} - 2}{2} \sum_{\beta=2}^{\bar{n}} \bar{p}_{\beta}} & \alpha = \bar{n} + 1 \\
(-1)^{\frac{\bar{n} - 1}{2}} q^{\frac{\alpha - 3 - \bar{p}_{\alpha} - 2}{2} \sum_{\beta=2}^{\bar{n}} \bar{p}_{\beta}} & \alpha = \bar{n} + 2 \\
(-1)^{\frac{\bar{n} - 1}{2}} q^{\frac{\alpha - 1 - \bar{p}_{\alpha} - 2}{2} \sum_{\beta=2}^{\bar{n} + 2} \bar{p}_{\beta}} & \bar{n} + 2 < \alpha \leq N + 1
\end{cases}
\]

References

[1] R.J. Baxter, “Exactly Solved Models in Statistical Mechanics”, Academic Press, New York, 1982.

[2] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. 120 (1979) 253.

[3] L.D. Faddeev, N.Y. Reshetikhin and L.A. Takhtajan, Algebra Analysis 1 (1987) 178.

[4] V.G. Drinfeld, J. Sov. Math. 41 (1988) 898; Proc. Int. Math., AMS Providence 1987, vol. 1, pg. 798.

[5] M. Jimbo, Lett. Math. Phys. 10 (1985) 63.

[6] M. Jimbo, Comm.Math.Phys. 102 (1986) 247.

[7] V.V. Bazhanov, Phys. Lett. B 159 (1985) 321.

[8] V.V. Bazhanov and A.G. Shadrikov, Theor.Math.Phys. 73 (1987) 1302; R.B. Zhang, A.J. Bracken and M.D. Gould, Phys.Lett.B 257 (1991) 133.
[9] M. Chaichian and P.P. Kulish, Phys.Lett.B 234 (1990) 72; T. Deguchi and Y. Akutsu, J.Phys.A:Math.Gen. 23 (1990) 1861

[10] P.P. Kulish and N.Yu. Reshetikhin, Lett.Math.Phys. 18 (1989) 143; H. Saleur, Nucl.Phys.B 336 (1990) 363.

[11] T. Deguchi, A. Fujii and K. Ito, Phys.Lett.B 238 (1990) 242; M.D. Gould, J.R. Links, Y.Z. Zhang and I. Tsohantjis, J.Phys.A:Math.Gen. 30 (1997) 4313; A.J. Bracken, G.W. Delius, M.D. Gould and J.R. Links and Y.-Z. Zhang, J.Phys.A:Math.Gen. 27 (1994) 6551; Z. Maassarani, J.Phys.A:Math.Gen. 28 (1995) 1305.

[12] G.W. Delius, M.D. Gould, J.R. Links and Y.Z. Zhang, Int.J.Mod.Phys. A 10 (1995) 3259, J.Phys.A.Math.Gen. 28 (1995) 6203.

[13] G.W. Delius, M.D. Gould, and Y.Z. Zhang, Int.J.Mod.Phys. A 11 (1996) 3415; M.D. Gould and Y.Z. Zhang, Nucl.Phys.B 566 (2000) 529.

[14] W. Galleas and M.J. Martins, Nucl. Phys. B 699 (2004) 455.

[15] K.A. Dancer, M.D. Gould and J. Links, math.QA/0504313.

[16] J. Birman and H. Wenzl, Trans. Am. Math. Soc. 313 (1989) 249; J. Murakami, Osaka J. Math. 24 (1987) 745.

[17] M. Wadati, T. Deguchi and Y. Akutsu, Phys. Rep. 180 (1989) 247.

[18] N. Reshetikhin, Lett.Math.Phys. 20 (1990) 331.

[19] V.F.R. Jones, Comm. Math. Phys. 125 (1989) 459; Int. J. Mod. Phys. B 4 (1990) 701.

[20] Y. Cheng, M.L. Ge and X. Xue, Comm. Math. Phys. 136 (1991) 195; M.L. Ge, Y.S. Wu and K. Xue, Int. J. Mod. Phys. A 6 (1991) 3735.

[21] V. Grimm, J. Phys. A: Math. Gen. 217 (1994) 5897; Lett. Math. Phys. 32 (1994) 183; U. Grimm and S.O. Warnar, Nucl. Phys. B 435 (1995) 482.
[22] E. Artin, *Ann.Math.* 48 (1947) 101.

[23] H.V.N. Temperley and E.H. Lieb, *Proc.R.Soc. (London) A* 322 (1971) 251.

[24] M. Couture, M.L. Ge, H.C. Lee and N.C. Schneing, *J. Phys. A: Math. Gen.* 23 (1990) 4751; M. Couture, M.L. Ge and H.C. Lee, *J. Phys. A: Math. Gen.* 23 (1990) 4765; M. Couture, Y. Cheng, M.L. Ge and K. Xue, *Int. J. Mod. Phys. A* 6 (1991) 359.

[25] M.L. Ge and K. Xue, *J. Math. Phys.* 32 (1991) 1301.

[26] U. Grimm and P.A. Pearce, *J. Phys. A: Math. Gen.* 26 (1993) 7435; U. Grimm and S.O. Warnaar, *J.Phys.A* 28 (1995) 7197.

[27] M.J. Martins, *Phys.Rev.E.* 59 (1999) 7220.

[28] P.P. Kulish, *J.Sov.Math.* 35 (1986) 2648; M.J. Martins and P.B. Ramos, *Nucl.Phys.B* 500 (1997) 579.