Models of Bounded Arithmetic and variants of Pigeonhole Principle

Mykyta Narusevych*

Faculty of Mathematics and Physics
Charles University†

Abstract

We give elementary proof that theory $T_{1}^{2}(R)$ augmented by the weak pigeonhole principle for all $\Delta^{b}_{1}(R)$-definable relations does not prove the bijective pigeonhole principle for $R$. This can be derived from known more general results but our proof yields a model of $T_{1}^{2}(R)$ in which $ontoPHP_{n+1}^{m+1}(R)$ fails for some nonstandard element $n$ while $PHP_{m}^{m+1}$ holds for all $\Delta^{b}_{1}(R)$-definable relations and all $m \leq n^{1-\epsilon}$, where $\epsilon > 0$ is a fixed standard rational parameter. This can be seen as a step towards solving an open question posed by M. Ajtai in [2, Page 3].

1 Introduction

This paper studies first-order bounded arithmetic theories in a language extended by an unspecified relation symbol $R$ (such theories are often called relativized). The main theories we are to inspect all contain theory $T_{2}(R)$, a subtheory of a theory $T_{3}(R)$, and are augmented by instances of (variants of) the pigeonhole principle. Several separations among bounded arithmetic theories and unprovability of various combinatorial principles are known (see [7, Chapter 10]). What is not known is whether the consecutive theories $T_{2}^{i}(R)$ and $T_{3}^{i+1}(R)$ (subtheories of $T_{2}(R)$ with induction restricted to $\Sigma^{i}_{1}(R)$-formulas and $\Sigma^{i+1}_{1}(R)$-formulas, respectively) can be separated by principles of a fixed quantifier complexity independent of $i$. This problem relates to the depth $d$ versus $d+1$ problem in proof complexity (see [8, Problem 14.3.2]). It is only known that if we weaken the theories by excluding the smash function $x \# y$ (the resulting theories are denoted $T_{2}^{i}(R)$) such a separation is possible (see [5]).

We think that to make progress on this problem it may be useful to have stand-alone and elementary proofs of various separation results that can be obtained by more general methods aiming at a large variety of stronger theories.

*Supported by the Charles University project PRIMUS/21/SCI/014, Charles University Research Centre program No. UNCE/24/SCI/022 and GA UK project No. 246223
†Sokolovská 83, Prague, 186 75, The Czech Republic
For example, the random restriction method using the PHP switching lemma [1] applies to the whole theory $T_2(R)$ and in a sense blurs the distinction between its fragments.

The main technical result is a new and elementary proof of the following statement:

$$T_2^1(R) + WPHP_{m}^{2m}(\Delta^b_1(R)) \nvdash \text{ontoPHP}^{n+1}_n(R),$$

where $\text{ontoPHP}^{n+1}_n(R)$ denotes the bijective pigeonhole principle stated for a binary relation symbol $R$ and $WPHP_{m}^{2m}(\Delta^b_1(R))$ denotes the weak pigeonhole principle stated for all $\Delta^b_1(R)$-definable relations.

The qualification “elementary” means that the proof does not refer to any general frameworks aiming at a larger class of theories or a too-strong method aiming at stronger theories. For example, we can note that (1) can be shown indirectly by the following argument. Firstly, it is known that $T_2(R)$ (and even $T_2^2(R)$) proves $WPHP_{m}^{2m}(R)$ (see [13]). This can be used to derive $T_2(R) \vdash WPHP_{m}^{2m}(\Delta^b_1(R))$ (the same is true even for $T_2^2(R)$). Secondly, using the restrictions technique and applying the already mentioned PHP switching lemma one can show that $T_2(R) \nvdash \text{ontoPHP}^{n+1}_n(R)$ (see [1], [9] and [14]). Combining these results we can derive the mentioned statement. Another general method is that of [10] which extends an original construction of [15]. The statement (1) is then an instance of a more general theorem showing, in short, that over $T_2(R)$ any weak combinatorial principle cannot prove a strong one (see [10] for details).

Our method has the advantage that its simple variant can be used to construct a model of:

$$T_2^1(R) + \exists n(\forall m \leq n^{1-\epsilon} PHP_{m}^{n+1}(\Delta^b_1(R)) + \neg\text{ontoPHP}^{n+1}_n(R)), \quad (2)$$

with $\epsilon > 0$ being an arbitrarily small standard rational parameter. Neither of the two mentioned general methods that can be used to prove (1) seem to be modifiable to prove (2).

As mentioned in the abstract, (2) can be seen as a step towards an open question posed by M. Ajtai in [2, Page 3]. The full problem can be stated as follows.

**Problem** (M. Ajtai [2]). Does the theory:

$$T_2(R) + \forall m \leq n/2(\text{PHP}_{m}^{n+1}(\Sigma^b_{\infty}(R)))$$

prove $\text{PHP}^{n+1}_n(R)$?

### 2 Preliminaries

The family of theories we are studying in this paper are all sound axiomatizations of true arithmetic over a first-order language denoted simply as $L$. This language consists of constant symbols 0 and 1, unary operations $\lfloor \frac{x}{2} \rfloor$ and $|x|$, and...
binary operations \(x + y\), \(x \cdot y\) and \(x \# y\), and binary relation \(x < y\). The intended meanings of 0, 1, \(\left\lfloor \frac{x}{2} \right\rfloor\), \(x + y\), \(x \cdot y\), \(x < y\) are obvious. Symbol \(|x|\) denotes bit-length of \(x\), i.e. \(|x| = \left\lfloor \log_2 x \right\rfloor + 1\), and \(x \# y\) stands for binary smash function which is standardly interpreted as \(2^{x|y|}\).

The main theory for \(L\) is the \emph{bounded arithmetic} \(T_2\) which is axiomatized by a finite universal theory \(BASIC\) together with the induction scheme for all \emph{bounded formulas}. Recall that a formula is said to be bounded if all quantifiers occurring in it are of the form \(\forall x < t\) or \(\exists x < t\) with an \(L\)-term \(t\) not containing \(x\) (see \([3]\) or \([8]\) for a comprehensive treatment of \(T_2\) and its fragments).

To define theories \(S_2^1\) and \(T_2^1\) and \(\Delta_1^b\)-formulas we need to recall the class \(\Sigma_1^b\) of bounded \(L\)-formulas. These are formulas formed from atomic ones using connectives, any \emph{sharply bounded} quantifiers, and bounded existential quantifiers at the front. The class \(\Pi_1^b\) is defined dually using bounded universal quantifiers. Sharply bounded quantifiers are of the form \(\forall x < |t|\) and \(\exists x < |t|\) with an \(L\)-term \(t\) not containing \(x\).

Theory \(S_2^1\) extends the already mentioned \(BASIC\) by the scheme of polynomial induction \(PIND\) of the form:

\[
(\phi(0) \land \forall x(\phi(\lfloor \frac{x}{2} \rfloor)) \rightarrow \phi(x)) \rightarrow \forall x\phi(x),
\]

for all \(\Sigma_1^b\)-formulas, while \(T_2^1\) has ordinary induction \(IND\) for all \(\Sigma_1^b\)-formulas. The induction formulas may contain, as is usual, other free variables than the induction variable.

A \(\Sigma_1^b\)-formula \(\phi(x)\) is said to be \(\Delta_1^b\) in \(S_2^1\) iff there is a \(\Pi_1^b\)-formula \(\psi(x)\) so that \(S_2^1 \vdash \phi(x) \equiv \psi(x)\). We often leave out the reference “in \(S_2^1\)” and consider it automatically included. It can be shown that \(\Delta_1^b\)-definable sets on \(\mathbb{N}\) are exactly those decidable by polynomial-time algorithms \([3]\) page 64, Theorem 2).

Already the weakest of the mentioned theories \(S_2^1\) is strong enough to talk about bounded sets or sequences, i.e. this theory is sequential (consult \([7]\) Lemma 5.1.5)).

This paper focuses on the unprovability of a particular combinatorial statement in a \emph{relativized} version of \(T_2^1\) denoted as \(T_2^1(R)\) (and in a particular extension of the mentioned theory). To obtain such a theory we first expand the language \(L\) by a single binary relation \(R(x, y)\). One can naturally define classes \(\Sigma_1^b(R)\) and \(\Pi_1^b(R)\). \(T_2^1(R)\) is axiomatized by \(BASIC\) together with the induction scheme for all \(\Sigma_1^b(R)\)-formulas. Theories \(S_2^1(R), T_2^1(R)\) and the class \(\Delta_1^b(R)\) in \(S_2^1(R)\) are defined analogously. Similarly as before, one can show that \(\Delta_1^b(R)\)-definable sets on \(\mathbb{N}\) are exactly those decidable by polynomial time algorithms with oracle access to \(R\).

It is a well-known fact that the \emph{least number principle} axiom scheme is equivalent to induction when stated for all formulas. The same is true for \(\Sigma_1^b\)-formulas, i.e. theories \(LNP(\Sigma_1^b)\) and \(T_2^1\) are equivalent (see \([7]\) Lemma 5.2.7)). The situation is the same in the relativized case.

**Definition 2.1.** For parameters \(n, m\) we define \(PHP_m^n(R)\) as a disjunction of the following formulas:
• $\exists a < n \forall b < m : \neg R(a, b)$,
• $\exists a \neq a' < n \exists b < m : R(a, b) \land R(a', b)$,
• $\exists a < n \exists b \neq b' < m : R(a, b) \land R(a, b')$.

We define $ontoPHP_n^m(R)$ as a disjunction of $PHP_n^m(R)$ with the formula:

• $\exists b < m \forall a < n : \neg R(a, b)$.

We denote $PHP^{2m}_m(R)$ as $WPHP^{2m}_m(R)$ for cosmetic reasons. $PHP_n^m(\Delta^b_1(R))$ stands for a set of instances of the above disjunction with $R$ substituted for all $\Delta^b_1(R)$-formulas. As usual, those formulas may contain additional free variables than $a$ and $b$. Similarly, we denote $PHP^{2m}_m(\Delta^b_1(R))$ as $WPHP^{2m}_m(\Delta^b_1(R))$.

3 Forcing

We show the statement (1) by constructing a model of:

$$T_2(R) + \forall m WPHP^m_m(\Delta^b_1(R)) + \exists n \neg ontoPHP^{n+1}_n(R),$$

employing forcing.

Forcing is a generic name for a class of methods and its various formal variants for bounded arithmetic were developed, cf. [12, 1, 7, 3, 15, 10]. We want to keep the presentation elementary and as self-contained as possible and hence we do not refer to any of the general expositions. Our approach is pretty much the same as the one in [12] or [15].

Throughout the following two sections, $M$ is a countable non-standard model of $Th(N)$ in the $L$-language, and $n$ is a non-standard number from $M$. We denote the set $\{m \in M \mid m < n\}$ as $\llbracket n \rrbracket$.

We let $I$ be a substructure of $M$ with the domain $I$ defined as $\{m \mid m < 2^{\lfloor n \rfloor} \text{ for some standard } c\}$. $I$ will eventually play the role of the universe of a structure where $R$ will be suitably interpreted, although all the arguments through the following sections are done inside $M$. The main reason behind choosing such $I$ is that numbers of magnitude $2^n$ are absent from $I$, while they do exist in $M$.

Since all the formulas representing various combinatorial principles discussed in the previous section may contain free variables, whenever we mention $L(R)$-formulas or $L(R)$-sentences in the current context we allow them to contain parameters from $I$.

**Definition 3.1.** Let $P$ be the set of all partial injective functions in $M$ of size $\leq \lfloor n \rfloor^c$ for standard $c$ between the sets $\lfloor n + 1 \rfloor$ and $\lfloor n \rfloor$. We call members of $P$ conditions and we assume $P$ is partially ordered by inclusion. The relation $\sigma \supseteq \rho$ is also denoted $\sigma \leq \rho$.

We say that $\sigma$ and $\tau$ from $P$ are compatible (denoted $\sigma \parallel \tau$), if there is $\delta \in P$ so that $\delta \leq \sigma$ and $\delta \leq \tau$, otherwise $\sigma$ and $\tau$ are called incompatible (denoted $\sigma \perp \tau$).

4
We say that a set \( D \subseteq P \) is \( P \)-definable, if it is definable in \((\mathcal{M}, P)\), the expansion of \( \mathcal{M} \) by predicate for \( P \). We say that \( D \subseteq P \) is dense, if for all \( \sigma \in P \) there is \( \tau \in D \) so that \( \tau \leq \sigma \).

Note that conditions \( \sigma \) and \( \tau \) are compatible iff \( \sigma \cup \tau \in P \).

**Definition 3.2.** A \( G \subseteq P \) is a filter, if all the conditions from \( G \) are pairwise compatible and for any \( \sigma \in G \) and \( \tau \in P \) so that \( \tau \geq \sigma \) it holds that \( \tau \in G \) (i.e. \( G \) is upwards closed).

We say that a filter \( G \) is generic if for any dense and \( P \)-definable \( D \) it holds that \( G \cap D \neq \emptyset \).

The following statement is well-known.

**Proposition 3.3.** For any \( \sigma \in P \) there exists a generic filter \( G \) containing \( \sigma \).

**Proposition 3.4.** Let \( G \) be any generic filter. Then \( \bigcup G \) is a bijective function between the sets \([n+1]\) and \([n]\). We will denote such function as \( R_G \).

**Proof.** It is clear that for any filter \( F \) the set \( \bigcup F \) is a partial injective function between the sets \([n+1]\) and \([n]\). To show the totality of \( R_G \) let \( a \in [n+1] \) and let \( D_a \) be the set of all conditions defined on \( a \). Such \( D_a \) is dense and \( P \)-definable and so \( R_G \) is defined on \( a \). One then shows the surjectivity of \( R_G \) in a similar way by considering the set \( D_b \) of all conditions having \( b \in [n] \) in their range. \( \square \)

For \( R_G \) as above we denote the \( L(R) \)-structure with the domain \( \mathbb{I} \) where \( R_G \) interprets \( R \) as \((\mathbb{I}, R_G)\).

**Definition 3.5.** Let \( \sigma \in P \) and \( \phi \) be an \( L(R) \)-sentence. We say that \( \sigma \) forces \( \phi \) or \( \phi \) is forced by \( \sigma \) (denoted as \( \sigma \models \phi \)) if for any generic filter \( G \) containing \( \sigma \) it holds that \((\mathbb{I}, R_G) \models \phi \).

Note that if sigma forces \( \phi \) and \( \tau \leq \sigma \), then \( \tau \) forces \( \phi \) too.

**Corollary 3.6.** Any condition \( \sigma \) (or, equivalently, the empty condition \( \sigma = \emptyset \)) forces \( \neg \text{ontoPHP}^{n+1}(R) \).

We now discuss the forcing of various types of formulas. What is needed for the main results can be summarized as follows.

**Theorem 3.7.** Let \( \sigma \in P \). Then the following holds:

1. for \( a \in [n+1] \) and \( b \in [n] \) it holds that \( \sigma \models R(a, b) \) iff \( \{(a, b)\} \subseteq \sigma \);
2. for \( a \in [n+1] \) and \( b \in [n] \) it holds that \( \sigma \models \neg R(a, b) \) iff \( \{(a, b)\} \perp \sigma \);
3. for any two \( L(R) \)-sentences \( \phi \) and \( \theta \) it holds that \( \sigma \models \phi \land \theta \) iff \( \sigma \models \phi \) and \( \sigma \models \theta \);
4. for any \( L(R) \)-formula \( \phi(x) \) it holds that \( \sigma \models \forall x \phi(x) \) iff \( \forall a \in \mathbb{I} : \sigma \models \phi(a) \);
5. for any sharply bounded \( L(R) \)-sentence \( \phi \) it holds that \( \sigma \models \neg \phi \) iff \( \forall \tau \leq \sigma \) it holds that \( \tau \not\models \phi \).
6. for any two sharply bounded \( L(R) \)-sentences \( \phi \) and \( \theta \) it holds that \( \sigma \) forces \( \phi \lor \theta \) if and only if \( \forall \tau \leq \sigma \exists \rho \leq \tau : \rho \vDash \phi \lor \theta \);

7. for sharply bounded formula \( \phi(\overline{x}) \) it holds that \( \sigma \) forces \( \exists \overline{x} \phi(\overline{x}) \) if and only if \( \forall \tau \leq \sigma \exists \rho \leq \tau \exists \overline{x} \phi(\overline{x}) \) implies \( \rho \vDash \phi(\overline{x}) \).

We split the proof of the above result into separate lemmas.

**Definition 3.8.** We say that \( L(R) \)-sentences \( \phi \) and \( \theta \) are forcing-equivalent (or, simply \( f \)-equivalent) if for any \( \sigma \in \mathbb{P} \) it holds that \( \sigma \vDash \phi \) iff \( \sigma \vDash \theta \).

Logical equivalence implies \( f \)-equivalence. So we may safely assume all the \( L(R) \)-formulas we are discussing are in prenex normal forms. Furthermore, evaluating all the \( L \)-subformulas and all the \( L \)-subterms in \( \mathbb{M} \) of some \( L(R) \)-sentence leads to an \( f \)-equivalent sentence and so we may further assume that all the atomic subformulas are of the form \( R(a, b) \) or \( \neg R(a, b) \) for some \( a, b \in \mathbb{M} \) (actually in \( \mathbb{I} \) since we allow parameters from \( \mathbb{I} \) only).

**Lemma 3.9.** Let \( \sigma \in \mathbb{P} \), \( a \in [n+1] \) and \( b \in [n] \). Then, \( \sigma \vdash R(a, b) \) iff \( \{ (a, b) \} \subseteq \sigma \) and \( \sigma \vdash \neg R(a, b) \) iff \( \{ (a, b) \} \not\subseteq \sigma \).

**Proof.** Assuming \( \{ (a, b) \} \subseteq \sigma \) it is clear that \( \sigma \vdash R(a, b) \). So assume that \( \{ (a, b) \} \not\subseteq \sigma \). We can extend \( \sigma \) to \( \tau \) so that \( (a', b), (a, b) \in \tau \) for \( a' \neq a \) and \( b' \neq b \). Such \( \tau \) forces \( \neg R(a, b) \) implying \( \sigma \nvdash R(a, b) \).

The equivalence for \( \neg R(a, b) \) is done similarly. \( \square \)

**Lemma 3.10.** Let \( \sigma \in \mathbb{P} \). For any two \( L(R) \)-sentences \( \phi \) and \( \theta \) it holds that \( \sigma \vdash \phi \land \theta \) iff \( \sigma \vdash \phi \) and \( \sigma \vdash \theta \). For any \( L(R) \)-formula \( \psi(x) \) it holds that \( \sigma \vdash \forall x \psi(x) \) iff for all \( a \in I : \sigma \vdash \psi(a) \).

**Proof.** Both equivalences follow immediately from a more general statement that \( \sigma \vdash \bigwedge_i \phi_i \) iff \( \forall i \sigma \vdash \phi_i \). The statement itself easily follows from the definitions. \( \square \)

**Lemma 3.11.** Let \( \sigma \in \mathbb{P} \) and \( \phi \) be a sharply bounded \( L(R) \)-sentence. Then \( \sigma \vdash \neg \phi \) iff \( \forall \tau \leq \sigma \) it holds that \( \tau \nvDash \phi \).

**Proof.** The left-to-right direction is clear. Assume now that \( \sigma \nvDash \neg \phi \). The statement then follows from an auxiliary observation: given a structure \( (\mathbb{I}, R_G) \) satisfying \( \phi \) one can find for a suitable standard \( c \) a \( |n|^c \)-large subset \( \tau \) in \( \mathbb{M} \) of \( R_G \) which forces \( \phi \). This, in turn, can be established by induction on logical complexity of \( \phi \). Below we show a sketch of the proof.

For atomic or negated atomic \( \phi \) the claim follows from [3.9]. Cases for conjunction and disjunction are shown in the exact same way as the ones for sharply bounded quantifiers below.

Let \( \phi \) be of the form \( \forall x \leq |n|^c \psi(x) \). By induction hypothesis we find \( \tau_x \) for each \( x \leq |n|^c \) so that \( \tau_x \vdash \psi(x) \). Note that all \( \tau_x \) are pairwise compatible and so we can define \( \tau = \bigcup_x \tau_x \) which satisfies \( |\tau| \leq |n|^c \) for a suitable \( c' \) and \( \tau \vdash \forall x \leq |n|^c \psi(x) \).

Finally, let \( \phi \) be of the form \( \exists x \leq |n|^c \psi(x) \). Similarly as above, we pick particular \( x \) so that \( \tau_x \vdash \psi(x) \) and let \( \tau = \tau_x \). \( \square \)
Lemma 3.12. Let $\sigma \in \mathcal{P}$. For any two sharply bounded $L(R)$-sentences $\phi$ and $\theta$ it holds that $\sigma \models \phi \lor \theta$ iff $\forall \tau \leq \sigma \exists \rho \leq \tau : \rho \models \phi$ or $\rho \models \theta$.

Proof. Let $\sigma$ force $\phi \lor \theta$ and let $\tau \leq \sigma$. Assume for contradiction that $\forall \rho \leq \tau \rho \nvdash \phi$ and $\rho \nvdash \theta$. Since both $\phi$ and $\theta$ are sharply bounded we use Lemma 3.11 to derive $\tau \models \neg \phi$ and $\tau \models \neg \theta$ which is equivalent to $\tau \models \neg \phi \land \neg \theta$ by Lemma 3.10. This, however, contradicts the fact that $\sigma \models \phi \lor \theta$ and $\tau \leq \sigma$.

Now assume $\sigma \nvdash \phi \lor \theta$. Note that Lemma 3.11 can be formulated as $\pi \nvdash \psi$ iff $\exists \delta \leq \pi$ so that $\delta \models \neg \psi$ for $\psi$ a sharply bounded formula. Since $\phi \lor \theta$ is sharply bounded, we can use the above statement to extend $\sigma$ to $\tau$ so that $\tau \models \neg \phi \land \neg \theta$. Such $\tau$ cannot be extended to $\rho$ forcing $\phi$ or $\theta$. 

Before we prove the last part of Theorem 3.7 we state the following proposition which is readily established by induction on logical complexity together with the previously proved lemmas.

Proposition 3.13. Let $\phi(\overline{x})$ be a sharply bounded formula. The set of all pairs $(\sigma, \overline{c})$ so that $\sigma \models \phi(\overline{c})$ is $\mathbb{P}$-definable.

Lemma 3.14. For any sharply bounded $L(R)$-formula $\phi(\overline{x})$ it holds that $\sigma \models \exists \overline{x} \phi(\overline{x})$ iff $\forall \tau \leq \sigma \exists \rho \leq \tau \exists \overline{c} \in \mathbb{I} : \rho \models \phi(\overline{c})$.

Proof. Assume $\sigma \models \exists \overline{x} \phi(\overline{x})$ and let $\tau \leq \sigma$. Note that $\tau \models \exists \overline{x} \phi(\overline{x})$, too. Let $G$ be a generic filter containing $\tau$ such that it holds $(I, R_G) \models \exists \phi(\overline{c})$. We can then find $\overline{c}$ so that $(I, R_G) \models \phi(\overline{c})$ and as in the proof of Lemma 3.11 we can take a $|n|^\rho$-large subset $\rho$ in $M$ of $R_G$ so that $\rho \leq \tau$ and $\rho \models \phi(\overline{c})$.

Conversely, assume $\forall \tau \leq \sigma \exists \rho \leq \tau \exists \overline{c} \in \mathbb{I} : \rho \models \phi(\overline{c})$. Note that this means the set $D$ of all conditions $\rho$ for which there is $\overline{c}$ so that $\rho \models \phi(\overline{c})$ is dense relative to $\sigma$, i.e. the density condition is satisfied for all $\tau$ extending $\sigma$.

By Proposition 3.13 it follows that the set $D$ is also $\mathbb{P}$-definable. Thus any generic filter $G$ containing $\sigma$ must contain some condition $\rho$ forcing $\phi(\overline{c})$ for some $\overline{c}$ proving $\sigma \models \exists \overline{x} \phi(\overline{x})$. The latter is true since any generic filter containing $\sigma$ must intersect any $\mathbb{P}$-definable set $D$ which is dense relative to $\sigma$. This can be shown by considering the set $D' = D \cup \{\rho \mid \rho \perp \sigma\}$ which is dense and $\mathbb{P}$-definable and so for any generic $G$ there is some $\rho \in G \cap D'$. But if $G$ contains $\sigma$, such $\rho$ must be compatible with $\sigma$ and so is necessary a member of $D$.

The following theorem is a fundamental fact regarding $T^3_2(R)$ and forcing construction. The first version of the theorem appeared in [12] for closely related theory $T^3_1(R)$ and its modification for $T^3_2(R)$ was shown in [15]. We skip the proof, the reader can find details in [7], pp. 273 - 274.

Theorem 3.15. For any sharply bounded $L(R)$-sentence $\phi(x, \overline{y})$ it holds that:

$$\emptyset \models LNP(\exists y \phi(x, y))$$

Corollary 3.16. For any generic filter $G$ it holds that:

$$(I, R_G) \models T^3_2(R) + \exists n \neg \text{ontoPH} P^{n+1}_n(R).$$
4 **WPHP-Arrays**

This whole section is devoted to the proof of:

\[ \emptyset \models WPHP^2_m(\Delta^b_1(R)), \]

where \( m \in \mathbb{I} \).

The argument is as follows: assuming \( \emptyset \not\models WPHP^2_m(\phi(x, y)) \) for a \( \Delta^b_1(R) \)-formula \( \phi(x, y) \) we construct a certain combinatorial structure. Then, by using two different arguments we establish upper and lower bounds on the size of such structure resulting in a contradiction, thus showing that \( \emptyset \models WPHP^2_m(\Delta^b_1(R)) \).

The following concepts are defined in the ground model \( \mathbb{M} \).

**Definition 4.1.** Let \( \sigma \in \mathbb{P} \) and \( m, k \in \mathbb{M} \) with \( k \leq |n|^c \) for standard \( c \). We define an \( (m, k, \sigma) \)-WPHP-array (or just \( A_{k, \sigma} \)-array) as an array \( A \) indexed by \([2^m] \times [m] \) so that each entry is a set of conditions of sizes bounded from above by \( k \) and satisfying the following properties:

1. \( \forall a \neq a' \in [2^m] \forall b \neq b' \in [m] : A(a,b) \cap A(a',b) = A(a,b) \cap A(a,b') = \emptyset; \)
2. \( \forall a, a' \in [2^m] \forall b \in [m] \forall \tau \in A(a,b) \forall \tau' \in A(a',b) : \tau \not= \tau' \rightarrow \tau \perp \tau'; \)
3. \( \forall a \in [2^m] \forall b, b' \in [m] \forall \tau \in A(a,b) \forall \tau' \in A(a,b') : \tau \not= \tau' \rightarrow \tau \perp \tau'; \)
4. \( \forall \rho \leq \sigma \forall a \in [2^m] \exists b \in [m] \exists \tau \in A(a,b) : \tau \parallel \rho. \)

We require that each \( \tau \in A(a,b) \) is compatible with \( \sigma \) and does not intersect \( \sigma \).

**Proposition 4.2.** Let \( A \) be \( A^k_{m, \sigma} \)-array. Define the size of \( A \) as the number \( N = \sum_{a,b} |A(a,b)| \), where \( |A(a,b)| \) is a number of conditions in \( A(a,b) \). Then:

\[ N = \sum_{a} |\bigcup_{b} A(a,b)| = \sum_{b} |\bigcup_{a} A(a,b)|. \]

**Proof.** This follows from the first property of \( A^k_{m, \sigma} \)-arrays. \( \square \)

It is hard to compute non-trivial lower and upper bounds on the size of a general \( A^k_{m, \sigma} \)-array \( A \). The way to proceed is to create a related array \( A' \) with nicer combinatorial properties.

**Definition 4.3.** An \( A^k_{m, \sigma} \)-array \( A \) is called \( k' \)-uniform (or just uniform in case \( k' = k \)), if all the conditions from the set \( \bigcup_{a,b} A(a,b) \) are of the size \( k' \).

As we will show later, uniformity alone is enough to compute non-trivial upper bounds on the size of a \( A^k_{m, \sigma} \)-array. However, it is not enough to calculate meaningful lower bounds.

**Definition 4.4.** Let \( D \subseteq [n + 1] \) and \( R \subseteq [n] \). A PHP-tree over \( D \) and \( R \) is defined inductively as follows:

- a single node (a root) is a PHP-tree over any \( D \) and \( R \);
for every \( a \in D \) the following is a PHP-tree over \( D \) and \( R \):

- at the root the tree branches according to all \( b \in R \), labeling the corresponding edge \( (a, b) \);
- at the end-point of the edge labeled by \( (a, b) \) the tree continues as a PHP-tree over \( D \setminus \{a\} \) and \( R \setminus \{b\} \);

for every \( b \in R \) the following is a PHP-tree over \( D \) and \( R \):

- at the root the tree branches according to all \( a \in D \), labeling the corresponding edge \( (a, b) \);
- at the end-point of the edge labeled by \( (a, b) \) the tree continues as a PHP-tree over \( D \setminus \{a\} \) and \( R \setminus \{b\} \).

We further label each node of a PHP-tree by the set of pairs labeling edges leading to the given node from the root (the root itself is labeled as \( \emptyset \)).

For \( \sigma \in \mathbb{P} \) we define \( PHP^\sigma \)-tree as a PHP-tree over \([n+1] \setminus D_\sigma \) and \([n] \setminus R_\sigma \), where \( D_\sigma \) is the domain of \( \sigma \) and \( R_\sigma \) is the range of \( \sigma \).

For \( \sigma, \tau \in \mathbb{P} \) which are compatible we define \( PHP^\tau_\sigma \)-tree as a PHP-tree with the property that for each \( (a, b) \in \tau \setminus \sigma \) every leaf in the given tree is labeled by a set either containing \( (a, b) \) or containing \( (a', b) \) and \( (a, b') \) for \( a \neq a' \) and \( b \neq b' \).

We say that a PHP-tree is \( k \)-uniform if all its maximal paths (starting from the root) have the same length equal to \( k \).

For a detailed discussion about PHP-trees and their applications consult [8, Chapter 15.1].

Since each \( \sigma \in \mathbb{P} \) is an element of \( \mathbb{M} \) we can talk about the size of \( \sigma \) (denoted as \( |\sigma| \)) which may be non-standard.

**Proposition 4.5.** Let \( \sigma \in \mathbb{P} \) and \( P \) a \( k \)-uniform \( PHP^\sigma \)-tree. The number of maximal paths (i.e. the number of leaves) in \( P \) is lower-bounded by \( \frac{(n-|\sigma|)!}{(n-|\sigma|-k)!} \).

**Proof.** The minimal number of paths is achieved for a \( PHP^\sigma \)-tree with nodes labeled by elements of the larger set (initially \([n+1] \setminus D_\sigma \) of size \( n + 1 - |\sigma| \)) and branching according to elements from the smaller set (initially \([n] \setminus R_\sigma \) which has size \( n - |\sigma| \)).

**Definition 4.6.** We say that a set of conditions \( T \in \mathbb{M} \) is realized by a PHP-tree if there is a PHP-tree \( P \) so that \( T \) equals the set of labels of leaves of \( P \). Similarly, we define realizations by \( PHP^\sigma \)-tree, \( PHP^\tau_\sigma \)-tree and uniform versions of the previous notions.

**Corollary 4.7.** If \( T \in \mathbb{M} \) is realized by a \( k \)-uniform \( PHP^\sigma \)-tree, then \( |T| \geq \frac{(n-|\sigma|)!}{(n-|\sigma|-k)!} \).

**Proposition 4.8.** Let \( T \in \mathbb{M} \) be realized by a \( PHP^\sigma \)-tree. It then holds:

1. \( \forall \tau, \tau' \in T : \tau \neq \tau' \rightarrow \tau \perp \tau' \).
2. \( \forall \rho \leq \sigma \exists \tau \in T : \tau \parallel \rho. \)

Proof. The first property follows from the fact that any two labelings of the leaves in any PHP-tree are incompatible. The second property follows from the definitions.

We now state a lemma which allows us to glue together various PHP-trees to get a bigger one. The idea is rather simple, every PHP-tree can be extended to a deeper one by simply appending another PHP-tree to a leaf in the original tree, assuming that the tree we are inserting is compatible with the path leading to such a leaf. Of course, one needs to ensure that the tree we get is definable in \( M \). An immediate corollary is that every PHP-tree can be extended to a \( k \)-uniform one for a suitable \( k \).

Lemma 4.9. Let \( P \) be a PHP\( _\sigma^\tau \)-tree. Let \( (S_\lambda)_\lambda \in \mathbb{M} \) a collection of PHP-trees where \( \lambda \) ranges over all labels of the leaves of \( P \). For any such \( \lambda \) we assume \( S_\lambda \) is a PHP\( _\lambda^\sigma \)-tree.

We define a tree \( P \triangleleft S \) as an extension of \( P \) by \( (S_\lambda)_\lambda \) where we append \( S_\lambda \) to the leaf of \( P \) labeled by \( \lambda \). It

Then, \( P \triangleleft S \) is a PHP\( _\sigma^\tau \)-tree with the property that each maximal path of \( P \triangleleft S \) extends a unique maximal path of \( P \).

Proof. \( P \triangleleft S \) is definable in \( M \), since both \( P \) and \( (S_\lambda)_\lambda \) are in \( M \).

The fact that each \( S_\lambda \) is a PHP\( _\lambda^\sigma \)-tree implies \( P \triangleleft S \) is a PHP-tree.

Since \( P \) is a PHP-tree over \( [n+1] \setminus D_\sigma \) and \( [n] \setminus R_\sigma \) and \( S_\lambda \) is a PHP-tree over \( [n+1] \setminus D_\sigma \) and \( [n] \setminus R_\sigma \) it follows that \( P \triangleleft S \) is PHP-tree over \( [n+1] \setminus D_\sigma \) and \( [n] \setminus R_\sigma \) and thus is a PHP\( _\sigma^\tau \)-tree.

Each maximal path of \( P \triangleleft S \) extends a unique maximal path of \( P \) by the definition of extension.

Finally, since any label \( \pi \in \mathcal{P} \) of a leaf of \( P \triangleleft S \) is a superset of the label of a leaf of \( P \) it follows that \( \pi \) does contain \((a, b)\) or \((a', b)\) and \((a, b')\) with \( a \neq a' \) and \( b \neq b' \) for all \((a, b) \in \tau \setminus \sigma \). This implies \( P \triangleleft S \) is a PHP\( _\sigma^\tau \)-tree.

In general, one can consider appending PHP-trees to leaves of another PHP-tree the above assumptions, namely that each \( S_\lambda \) is a PHP\( _\lambda^\sigma \)-tree (we are ignoring the initial condition \( \sigma \) here). In such a case one needs to ensure that the resulting tree is a PHP-tree by first removing all branches of \( S_\lambda \) which are incompatible with \( \lambda \). We then shrink all nodes of the resulting tree which have only a single edge going outwards.

In the above Lemma 4.9 we overcome this complication by assuming that \( S_\lambda \) already takes into account the label of the leaf it is being appended to which is formally expressed as \( S_\lambda \) being a PHP\( _\lambda^\sigma \)-tree.

When \( P' \) is a tree obtained from a tree \( P \) by the above process of appending trees to leaves we say that \( P' \) extends \( P \).

Corollary 4.10. For every \( k' \geq k \in \mathbb{M} \) with \( k' \leq n + 1 \) a PHP\( _\sigma^\tau \)-tree of depth \( k \) can be extended to a \( k' \)-uniform PHP\( _\tau \)-tree.
In the next theorem, we identify $PHP$-trees with the labelings of their leaves or, in other words, with the labelings of their maximal paths.

**Theorem 4.11.** Let $A$ be an $A^k_{m,\sigma}$-array for $\sigma \in \mathbb{P}$. There exists $k' \geq k$ enjoying $k' \leq |n|^c$ for a suitable standard $c$ and a $k'$-uniform $A^k_{m,\sigma}$-array of size $N \geq 2m \cdot \frac{(n-|\sigma|)!}{(n-|\sigma|-k)!}$.

**Proof.** We construct an $A^k_{m,\sigma}$-array $A'$ in two stages. Firstly, we specify the contents of each row. Then, we specify how to distribute created content among the columns so that the result is an $A^k_{m,\sigma}$-array.

Let $A_0 = \bigcup_b A(a, b)$ for $a \in [2m]$. We construct $A'_0$ (i.e. the contents of the $a$-th row of $A'$) by the following procedure which takes $k$ steps.

At first, pick $\tau \in A_0$. Note that $\tau \cap \sigma = \emptyset$ and $|\tau| \leq k$, implying existence of a $PHP^{\tau}$-tree of depth $\leq 2k$. Extend such a tree to a uniform $PHP^{\tau}$-tree $P_1$ of depth exactly $2k$ which is achieved utilizing Corollary 4.10.

Note that $P_1$ satisfies the following properties:

- $\forall \rho \in P_1 \exists \pi \in A_0 : \pi |\rho|$
- $\forall \rho \in P_1 \forall \pi \in A_0 : \pi |\rho \rightarrow |\pi \cap \rho| \geq 1$.

The first property follows from the fact that $A$ is an $A^k_{m,\sigma}$-array and any $\rho$ from $P_1$ is compatible with $\sigma$.

The second property is proved by cases. Assuming $\pi = \tau$ it follows that $\pi |\rho \rightarrow \pi \geq \rho$ and so $|\pi \cap \rho| = |\pi| \geq 1$. So let $\pi \neq \tau$. In such a case it follows that $\pi \perp \tau$ and so must contain a pair $(a, b)$ such that $\tau$ contains either $(a', b)$ or $(a, b')$ for $a \neq a'$ and $b \neq b'$. In any case a branch $\rho$ of $P_1$ compatible with $\pi$ must also contain $(a, b)$, proving $|\pi \cap \rho| \geq |\{(a, b)\}| = 1$. This finishes the first step of the construction.

At the $i$-th step we are given a uniform $PHP^{\tau}$-tree $P_{i-1}$ of depth $2(i-1)k$ satisfying the following properties:

- $\forall \rho \in P_{i-1} \exists \pi \in A_0 : \pi |\rho|$.
- $\forall \rho \in P_{i-1} \forall \pi \in A_0 : (\pi |\rho \land |\pi| \geq (i-1)) \rightarrow |\pi \cap \rho| \geq (i-1)$.

Using the first property of $P_{i-1}$ pick $\pi_\rho \in A_0$ for each $\rho \in P_{i-1}$ so that $\pi_\rho |\rho$. For such $\pi_\rho$ create a $PHP^{\sigma \cup \rho}$-tree of depth $\leq 2k$ and extend it to a $2k$-uniform $PHP^{\sigma \cup \rho}$-tree $P_i^{\rho}$.

Let $P_i = P_{i-1} \cup \{P_i^{\rho} \mid \rho \in P_{i-1}\}$. By Lemma 4.3 we see that $P_i$ is a $2(i \cdot k)$-uniform $PHP^{\tau}$-tree.

Clearly:

- $\forall \rho \in P_i \exists \pi \in A_0 : \pi |\rho$.

To prove the second desired property let $\rho \in P_i$ and $\pi \in A_0$ so that $\pi |\rho$ and $|\pi| \geq i$. Let $\rho_\pi$ be the unique maximal path of $P_{i-1}$ so that $\rho_\pi \geq \rho$. Note that $\pi |\rho_\pi$ and so $|\pi \cap \rho_\pi| \geq |\pi \cap \rho| \geq (i-1)$. Assume $|\pi \cap \rho_\pi| = i-1$. Let $\pi_\rho_\pi \in A_0$ be the condition chosen to create $P_i^{\rho_\pi}$.\]
In case \( \pi_{\rho_*} = \pi \) it follows that \( |\pi \cap \rho| = |\pi| \geq i \). So assume \( \pi \neq \pi_{\rho_*} \).

Then, \( \pi \perp \pi_{\rho_*} \). But since both \( \pi \) and \( \pi_{\rho_*} \) are compatible with \( \rho_* \) it follows that
\[
(\pi \setminus \rho_*) \perp (\pi_{\rho_*} \setminus \rho_*) \text{ and so } |(\pi \setminus \rho_*) \cap (\rho \setminus \rho_*)| \geq 1, \text{ implying } |\pi \cap \rho| \geq i.
\]

So, after the end of the \( k \)-th step we get a \( 2k^2 \)-uniform PHP\( \rho \)-tree \( P_k \) satisfying the following properties:

- \( \forall \rho \in P_k \exists \pi \in A_a : \pi \| \rho \);
- \( \forall \rho \in P_k \forall \pi \in A_a : \pi \| \rho \rightarrow \pi \geq \rho \).

By Corollary 4.7 the number of leaves of \( P_k \) is bounded from below by \( \frac{(n - |\sigma|)!}{(n - |\sigma| - 2k^2)!} \).

Note that each \( \rho \in P_k \) is compatible with \( \sigma \) and does not intersect it. Furthermore, by Proposition 4.8 any two different \( \rho \) and \( \rho' \) from \( P_k \) are incompatible, and for any \( \pi \leq \sigma \) there is a \( \rho \in P_k \) compatible with it. This implies, assuming the contents of each row of \( A' \) are created by the process described above, the resulting \( A' \) satisfies all of the properties of uniform \( A_{m,\sigma}^{2k^2} \)-array, assuming the content of each row is correctly distributed among the columns to satisfy property 3 of (4.1).

The distribution proceeds as follows: for any \( \rho \in P_k \) pick \( \tau \in A_a \) so that \( \rho \leq \tau \). Note that such \( \tau \) is unique and is uniquely placed into one of the columns of \( A \). Let \( b \in [m] \) so that \( \tau \in A(a, b) \). Put \( \rho \) into \( A'(a, b) \).

The above gives us a well-defined array \( A' \) which, as we have already seen, satisfies all of the properties of the \( \{1, \ldots, m\} \) up to the third one.

Let \( b, b' \in [m] \) be different and let \( \tau \in A'(a, b), \tau' \in A'(a, b') \). Let \( \pi \) and \( \pi' \) be conditions from \( A(a, b) \) and \( A(a, b') \), respectively, so that \( \tau \leq \pi \) and \( \tau' \leq \pi' \). Since \( \pi \perp \pi' \) it follows that \( \tau \perp \tau' \). This shows that the third property is satisfied, as well.

Finally, by Proposition 4.2 the size of the array \( A' \) is bounded from below by
\[
2m \cdot \min_a \left| \bigcup_b A'(a, b) \right| \geq 2m \cdot \frac{(n - |\sigma|)!}{(n - |\sigma| - 2k^2)!}
\]
with the inequality following from the fact that all the rows of \( A' \) are \( 2k^2 \)-uniform PHP\( \rho \)-trees.

By a slightly more careful construction, one can get \( k' \) from the above proof to be equal to \( k^2 + k \) instead of \( 2k^2 \). This, however, does not make a major difference.

**Theorem 4.12.** Let \( A \) be a \( k \)-uniform \( A_{m,\sigma}^k \)-array for \( \sigma \in \mathbb{P} \). Then, the size of \( A \) is bounded from above by \( m \cdot \frac{(n + 1 - |\sigma|)!}{(n + 1 - |\sigma| - k)!} \).

**Proof.** The proof is reminiscent of the famous Erdős–Ko–Rado theorem as in [6]. By Proposition 4.2 we know that the size of \( A \) is bounded from above by \( m \cdot \max_a |\bigcup_a A(a, b)| \).

12
Let \( A^b \) denote \( \bigcup_a A(a,b) \). In particular, each \( \tau \in A^b \) is of size \( k \) and different 
\( \tau, \tau' \in A^b \) are incompatible.

We view each \( \tau \) as a \( k \)-large matching of the \( K_{n+1-|\sigma|,n-|\sigma|} \) (a complete 
bipartite graph with partitions of sizes \( n+1-|\sigma| \) and \( n-|\sigma| \)). We will now prove that for any \( k, c, d \in \mathbb{M} \) enjoying \( k \leq c \leq d \) it holds that the size of 
any family \( \mathcal{M} \) of pairwise-incompatible \( k \)-large matchings (i.e. union of any two 
matchings of \( \mathcal{M} \) is not a matching) of \( K_{d,c} \) is bounded from above by \( \frac{d!}{(d-k)!} \).

Note that any two different submatchings of some fixed \( c \)-large matching of \( K_{d,c} \) are compatible and so \( \mathcal{M} \) can contain at most one submatching of 
such \( M \). There are \( k! \binom{d}{c} \binom{c}{k} \) different \( k \)-large matchings of \( K_{d,c} \) and each such 
matching can be extended to \( (c-k)! \binom{d-k}{c-k} \) different \( c \)-large matchings of \( K_{d,c} \).

Any two \( c \)-large matchings extending different \( k \)-large matchings from \( \mathcal{M} \) must 
necessarily be incompatible and, in particular, different.

It follows that:

\[
|\mathcal{M}| \leq c! \binom{d}{c} \cdot \frac{1}{(c-k)! \binom{d-k}{c-k}} = \frac{d!}{(d-k)!}.
\]

\[\square\]

**Corollary 4.13.** For any \( \sigma \in \mathbb{P}, m \in \mathbb{M} \) and \( k \leq |n|^c \) for standard \( c \) there does 
not exist a \( A_{m,\sigma}^b \)-array .

**Proof.** Assume \( A \) is a \( A_{m,\sigma}^b \)-array . By Theorem 4.11 we can assume it is \( k \)-
uniform and has size \( N \geq 2m \cdot \frac{(n-|\sigma|)!}{(n-|\sigma|-k)!} \).

By Theorem 4.12 it follows that \( N \leq m \cdot \frac{(n+1-|\sigma|)!}{(n+1-|\sigma|-k)!} \).

Combining both lower and upper bounds we derive:

\[
2 \leq \frac{n+1-|\sigma|}{n+1-|\sigma|-k},
\]

which is a contradiction, since both \( |\sigma| \) and \( k \) are bounded by \( |n|^c \) for a suitable 
standard \( c \).

\[\square\]

To get the desired contradiction assuming that \( (I, R_G) \not\models WPHP_m^2(\Delta^k_1(R)) \) 
we prove the following and key theorem.

**Theorem 4.14.** Assume \( \emptyset \not\models WPHP_m^2(\Delta^k_1(R)) \) for \( m \in I \). Then, there exists 
an \( A_{m,\sigma}^b \)-array for some \( \sigma \in \mathbb{P} \) and \( k \leq |n|^c \) for a suitable standard \( c \).

**Proof.** Let \( \phi(x,y) \) be \( \Delta^k_1(R) \)-formula with parameters from \( I \) so that \( \emptyset \) does not 
force \( WPHP_m^2(\phi(x,y)) \).

Let \( D \subseteq \mathbb{P} \) contain such \( \rho \in \mathbb{P} \) that one of the below holds:

- \( \exists a \neq a' \in [2m] \exists b \in [m] : \rho \models \phi(a,b) \land \phi(a',b); \)
- \( \exists a \in [2m] \exists b \neq b' \in [m] : \rho \models \phi(a,b) \land \phi(a,b'); \)

13
• \[ \exists a \in [2m] \forall b \in [m] : \rho \models \neg \phi(a, b). \]

We want to show that \( D \) is \( \mathbb{P} \)-definable. Since \( \phi(x, y) \) is a \( \Delta^1_0(R) \)-formula and \( \Delta^1_0(R) \subseteq \Sigma^1_0(R) \) we can assume \( \phi(x, y) \) is of the form:

\[
\exists \pi \leq t(x, y) \psi(x, y, \pi),
\]

where \( t \) is an \( L \)-term with parameters from \( \mathbb{I} \) and \( \psi(x, y, \pi) \) is sharply bounded. It is now enough to use Theorem 3.7 to show \( \mathbb{P} \)-definability of \( D \) by writing it as a union of three sets, one for each condition defining \( D \), and then applying different parts of (3.7) depending on the structure of \( \phi(x, y) \).

As an example, we show that the set of all conditions forcing \( \phi(a, b) \) for particular \( a, b \in \mathbb{I} \) is \( \mathbb{P} \)-definable. By (3.7, 7) we get:

\[
\rho \models \exists \pi \leq t(a, b) \psi(a, b, \pi)
\]

iff

\[
\forall \tau \leq \rho \exists \pi \leq \tau \exists \pi \leq t(a, b) : \pi \models \psi(a, b, \pi).
\]

So it is enough to show \( \mathbb{P} \)-definability of the set of all conditions \( \pi \) forcing \( \psi(a, b, \pi) \). This follows immediately from Proposition [3.13]

It is clear that any condition in \( D \) forces \( \text{WPHP}^{2m}(\phi(x, y)) \). Thus, according to the assumptions, \( D \) is not dense. Let \( \sigma \in \mathbb{P} \) be a condition not extendable to any condition from \( D \).

As a next step, we define an array \( A \in \mathbb{M} \) which should satisfy the properties of the \( A_{m, \sigma} \)-array for a standard \( c \). We start by writing the formula \( \phi(x, y) \) as \( \phi'(x, y, \bar{v}) \) so that \( \bar{v} \in \mathbb{I} \) and \( \phi'(x, y, \bar{v}) \) is a \( \Delta^1_0(R) \)-formula without parameters.

We shall use the following statement that follows readily from well-known facts.

**Fact.** There is standard \( c \geq 1 \) depending just on \( \phi' \) and parameters \( \bar{v} \) and \( m \) from \( \mathbb{I} \) such that for any \( a \in [2m], b \in [m] \) there is a depth \( |n|^c \) \( \text{PHP}^{2m} \)-tree \( T_{a, b} \in \mathbb{M} \) that decides the truth value of \( \phi(a, b) \) in \( (\mathbb{I}, R_G) \).

The quickest (although not the most elementary) way to see this is to use the already mentioned fact that \( \phi'(x, y, \bar{v}) \) defines a relation computable by a standard polynomial-time Turing machine \( U \) definable in \( S^1_2(R) \) and querying oracle about relation \( R \) (see [7] Sections 6 and 7)). When \( \bar{v} \) and \( m \) from \( \mathbb{I} \) (i.e. of bit-size polynomial in \( |n| \)) are fixed and inputs \( a, b \) are bounded by fixed \( m \), the running time of \( U \) is bounded by \( |n|^c \) for some standard \( c \) and this number bounds also the number of queries \( U \) makes on any input. We already know that for generic \( G \) the structure \( (\mathbb{I}, R_G) \) is a model of \( T^1_2(R) \models S^1_2(R) \) and so \( U \) works there as well.

Given \( a, b \), the computation develops according to oracle answers and all possible computations can be represented by a decision tree \( T'_{a, b} \) of depth \( \leq |n|^c \) that asks about truth values of atomic sentences \( R(u, v) \), for some \( u \in [n + 1] \) and \( v \in [n] \).
To represent this by a PHP-tree $T_{a,b}$ we simply replace query “$R(u,v)?$” in $T^*_{a,b}$ by a PHP-query “$u \mapsto ?$” (i.e. query asking for a $w$ satisfying $R(u,w)$); if the answer is $v$, then the original query in $T^*_{a,b}$ is answered “yes”, in all other cases it is answered “no”. Because we know that the generic $R_G$ violates $ontoPHP^{n+1}_n(R)$, we can delete from $T_{a,b}$ all paths that are not partial one-to-one maps, getting the desired tree $T_{a,b}$.

As before, we identify $T_{a,b}$ with its maximal paths which, in turn, are represented by partial one-to-one maps between $[n+1]$ and $[n]$ of size $\leq |n|$, i.e. as elements of $\mathbb{P}$. Let $T^+_{a,b}$ be all such maps for which the corresponding maximal paths in $T_{a,b}$ lead to acceptance of $\phi(a,b)$.

We define $A$ as $A(a,b) = T^+_{a,b}$. Since $T_{a,b}$ is a PHP$_\sigma$-tree two different conditions from $A(a,b)$ are incompatible and all such conditions are compatible with $\sigma$ while their intersections with $\sigma$ are empty.

It is now enough to show that $A$ satisfies all four properties of (4.1). The first three properties follow immediately from the fact that $\sigma$ is not extendable to a condition which either forces $\phi(a,b) \land \phi(a',b)$ or $\phi(a,b) \land \phi(a,b')$ for $a \neq a'$ and $b \neq b'$.

Assume the fourth property is not satisfied, i.e.:

$$\exists \rho \leq \sigma \exists a \in [2m] \forall b \in [m] \forall \tau \in A(a,b) : \tau \perp \rho.$$ 

It is enough to show $\rho \Vdash \neg \phi(a,b)$ for $a$ as above and all $b \in [m]$ to derive a contradiction. Since $\phi(a,b)$ is $\exists \overline{m} \leq t(a,b) \psi(a,b,\overline{m})$ for a sharply bounded $\psi(a,b,\overline{m})$, we get:

$$\rho \Vdash \neg \phi(a,b) \iff \rho \Vdash \forall \overline{m} \leq t(a,b) \neg \psi(a,b,\overline{m}),$$

where by (3.7) 4) the latter is equivalent to:

$$\forall \overline{m} \leq t(a,b) \rho \Vdash \neg \psi(a,b,\overline{m}).$$

Since $\psi$ is sharply bounded we use (3.7) 7) to derive:

$$\rho \Vdash \neg \psi(a,b,\overline{m}) \iff \forall \pi \leq \rho \pi \nmid \psi(a,b,\overline{m}).$$

Pick any $\pi$ extending $\rho$ and assume it forces $\psi(a,b,\overline{m})$. Thus, $\pi \Vdash \phi(a,b)$ which implies $\pi$ is compatible with some $\tau \in A(a,b)$. However, this implies $\rho |\tau$ since $\pi \leq \rho$ contradicting our initial assumption that $\rho$ is incompatible with every $\tau$ from $A(a,b)$ for every $b \in [m]$.

\[\Box\]

Corollary 4.15.

$$\emptyset \Vdash WPHP^{2m}_m(\Delta^b_1(R)),$$

implying that for any generic filter $G$ it holds that:

$$(\mathbb{I}, R_G) = T^1_2(R) + \forall m WPHP^{2m}_m(\Delta^b_1(R)) + \exists n \neg ontoPHP^{n+1}_n(R).$$
As an application of our proof methods, we derive the following result already advertised in the abstract.

**Theorem 4.16.** There exists a model of:

\[ T_2^1(R) + \exists n(\forall m \leq n^{1-\epsilon}(PHP_m^{m+1}(\Delta^b_1(R))) + \neg PHP_n^{n+1}(R)), \]

where \( \epsilon \) is an arbitrarily small standard rational.

**Proof.** The model is actually the same \((I, R_G)\) as in 4.15.

Assuming:

\[ \emptyset \not\subseteq \forall m \leq n^{1-\epsilon}(PHP_m^{m+1}(\Delta^b_1(R))) \]

it follows that one can construct \((n^{1-\epsilon}, |n|^c, \sigma)\)-PHP-array defined analogously as in [4,11] with the only difference being that the indexing set is \([n^{1-\epsilon}+1] \times [n^{1-\epsilon}].\)

Theorems 4.11 and 4.12 still apply implying:

\[
\begin{align*}
(n^{1-\epsilon} + 1) \cdot \frac{(n - |\sigma|)!}{(n - |\sigma| - |n|^c)!} & \leq n^{1-\epsilon} \cdot \frac{(n + 1 - |\sigma|)!}{(n + 1 - |\sigma| - |n|^c)!}.
\end{align*}
\]

The above results in:

\[
\frac{n^{1-\epsilon} + 1}{n^{1-\epsilon}} \leq \frac{n + 1 - |\sigma|}{n + 1 - |\sigma| - |n|^c},
\]

which is a contradiction, since \(|\sigma| \leq |n|^c\) and \(c', c, \epsilon\) are standard. \(\square\)

**Acknowledgements**

The work first appeared in [11] supervised by Jan Krajíček. I am indebted to my supervisor for detailed comments and suggestions.

**References**

[1] M. Ajtai. The complexity of the pigeonhole principle. In *Proceedings of the IEEE 29th Annual Symposium on Foundations of Computer Science*, pages 346 – 355, 1988.

[2] M. Ajtai. Parity and the Pigeonhole Principle. In S. R. Buss and P. Scott, editors, *Feasible mathematics*, Progress in Computer Science and Applied Logic, pages 1 – 24. Birkhäuser, 1990.

[3] A. Atserias and M. Müller. Partially definable forcing and bounded arithmetic. *Archive for Mathematical Logic*, 54:1–33, 2015.

[4] S. R. Buss. *Bounded Arithmetic*. PhD thesis, Princeton University, 1985.
[5] R. Impagliazzo and J Krajíček. A Note on Conservativity Relations Among Bounded Arithmetic Theories. *Mathematical Logic Quarterly*, 48(3):375–377, 2002.

[6] G. O. H. Katona. A simple proof of the Erdős-Chao Ko-Rado Theorem. *Journal of Combinatorial Theory, Series B*, 13(2):183 – 184, 1972.

[7] J. Krajíček. *Bounded Arithmetic, Propositional Logic and Complexity Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1995.

[8] J. Krajíček. *Proof Complexity*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2019.

[9] J. Krajíček, P. Pudlák, and A. Woods. Exponential lower bound to the size of bounded depth Frege proofs of the pigeonhole principle. *Random Structures and Algorithms*, 7:15 – 39, 1995.

[10] M. Müller. Typical forcings, NP search problems and an extension of a theorem of Riis. *Annals of Pure and Applied Logic*, 172(4):1029–30, 2021.

[11] M. Narusevych. Models of bounded arithmetic. Master’s thesis, Charles University, Prague, 2022.

[12] J. Paris and A. J. Wilkie. Counting problems in bounded arithmetic. pages 332 – 334, 1985.

[13] J. Paris, A. J. Wilkie, and A. R. Woods. Provability of the pigeonhole principle and the existence of infinitely many primes. *J. Symbolic Logic*, 53:1235 – 1244, 1988.

[14] T. Pitassi, P. Beame, and R. Impagliazzo. Exponential lower bounds for the pigeonhole principle. *Random Structures and Algorithms*, 7:15 – 39, 1995.

[15] Søren Riis. Making infinite structures finite in models of second order bounded arithmetic. *Arithmetic, proof theory and computational complexity*, pages 289–319, 1993.