A Multigraph Approach for Performing the Quantum Schur Transform

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Abstract

We take inspiration from the Okounkov–Vershik approach to the representation theory of the symmetric groups to develop a new way of understanding how the Schur–Weyl duality can be used to perform the Quantum Schur Transform. The Quantum Schur Transform is a unitary change of basis transformation between the computational basis of $(\mathbb{C}^d)^{\otimes n}$ and the Schur–Weyl basis of $(\mathbb{C}^d)^{\otimes n}$.

We describe a new multigraph, which we call the Schur–Weyl–Young graph, that represents both standard Weyl tableaux and standard Young tableaux in the same diagram. We suggest a major improvement on Louck’s formula for calculating the transition amplitudes between two standard Weyl tableaux appearing in adjacent levels of the Schur–Weyl–Young graph for the case $d = 2$, merely by looking at the entries in the two tableaux. The key theoretical component that underpins our results is the discovery of a branching rule for the Schur–Weyl states, which we call the Schur–Weyl branching rule. This branching rule allows us to perform the change of basis transformation described above in a straightforward manner for any $n$ and $d$.

1 Introduction

The Schur–Weyl duality is a famous result in representation theory relating, for any positive integer $d$ and any non-negative integer $n$, the irreducible representations appearing in the decomposition of the $n$-fold tensor product space $(\mathbb{C}^d)^{\otimes n}$ considered both as a representation of the symmetric group $S_n$, the group of permutations of $n$ objects, and as a representation of the special unitary group $SU(d)$, the Lie group of $d \times d$ unitary matrices with determinant one.

This result provides the key theoretical foundation for the existence of a unitary operator in quantum computing known as the Quantum Schur Transform. This operator is defined to be the unitary change of basis transformation which maps a quantum state expressed in the computational basis of $(\mathbb{C}^d)^{\otimes n}$ to the same quantum state expressed in the so-called Schur–Weyl basis of $(\mathbb{C}^d)^{\otimes n}$. Considerable effort has been devoted to constructing an efficient implementation of the Quantum Schur Transform. We review the history behind these efforts in Section 7. Notably, all of the implementations that have been discovered so far depend on a choice of either employing the representation theory of the symmetric groups or the representation theory of the unitary groups. In our approach, we make heavy use of the representation theory of the symmetric groups.

The representation theory of the symmetric groups has a long and rich history. The original version of the theory is credited to Young [5], who made his contributions in the early 1900s, although further, significant contributions were later made by luminaries such as von Neumann [20], Weyl [22] and James [8], amongst others. Young constructed the irreducible representations of the symmetric groups by introducing a class of diagrams originally called tableaux, but which are now commonly referred to as Young tableaux. In taking a combinatorial approach to these tableaux, Young proved that the only irreducible representations of the symmetric group $S_n$ (up to equivalence) are in bijective correspondence with the partitions of $n$. It was shown further that a branching rule exists for the irreducible representations of the symmetric groups. These irreducible representations can be represented in a graphical structure commonly known as the Young graph, consisting of levels indexed by the non-negative integers. The branching rule and the Young graph are major components of the theory; yet they only appear as a corollary at the end of Young’s formulation of the theory.
In 1996, a groundbreaking paper titled *A New Approach to the Representation Theory of the Symmetric Groups* was published in Russian by Okounkov and Vershik [18]. Its translation into English appeared in 2005 in the Journal of Mathematical Sciences [19]. Okounkov and Vershik took a rather different approach to Young. They saw that the symmetric groups naturally form a chain of groups, namely that $S_{n-1}$ is a subgroup of $S_n$, and they recognised that the branching rule and the graphical approach to representing the irreducible representations of this chain of groups was central to the theory. Hence they made this their starting point, and consequently only derived, as an “auxiliary element of the construction” [19], the Young tableaux. The authors suggest that their approach is the more natural one, given that the important combinatorial objects of the theory arise in a natural way, unlike in Young’s formulation, where they are introduced at the beginning without any motivation for their introduction.

In this paper, we take inspiration from the Okounkov–Vershik approach to the representation theory of the symmetric groups to develop a new way of understanding how the Schur–Weyl duality can be used to implement the Quantum Schur Transform. We define a new multigraph, which we call the Schur–Weyl–Young graph, to give a graphical depiction of the Schur–Weyl states. We also derive a branching rule for the Schur–Weyl states of $(\mathbb{C}^d)^\otimes n$ using this graph, which we call the Schur–Weyl branching rule, and consequently describe a procedure for performing the Quantum Schur Transform for any $n$ qudits in time polynomial in $n$ and $d$ using this branching rule. In addition, we show how we can dramatically speed up the calculation of the amplitudes involved in the Schur–Weyl branching rule for the case $d = 2$ merely by counting certain entries in the pairs of the Weyl tableaux that are involved.

This paper is organised as follows. In Section 2, we introduce the required background material from representation theory, namely the two major approaches to the representation theory of the symmetric group – the first formulated by Young, the second by Okounkov and Vershik – together with particular irreducible representations of $SU(d)$, namely the two major approaches to the representation theory of the symmetric group – the first formulated by Young, the second by Okounkov and Vershik – together with particular irreducible representations of $SU(d)$ that are of interest to us, before finishing with a description of the Schur–Weyl duality and the Schur–Weyl basis. In Section 3, we present our main theoretical results; in particular, we introduce the Schur–Weyl–Young graph and show how it leads to a branching rule for Schur–Weyl states. In Section 4, we present a simple set of rules for calculating the transition amplitudes between two standard Weyl tableaux appearing in adjacent levels of the Schur–Weyl–Young graph for the case $d = 2$, with a proof given in Appendix A. We use these rules to give examples of the Schur–Weyl branching rule in Section 5. We present our procedure for performing the Quantum Schur Transform in Section 6. In Section 7, we review the work existing in the literature that is related to ours, before concluding in Section 8.

## 2 Preliminaries

### 2.1 Remark on Terminology Used

In the literature, there are different vocabularies for describing the same ideas depending on whether one is a mathematician or a physicist. We aim, where possible, to use the mathematician’s vocabulary (tableaux, contents etc.), noting that it would not take much of a leap to convert these ideas into the physicist’s vocabulary (spin labels, spin components etc.). Furthermore, all representations in the following are considered to be over the field of complex numbers.

### 2.2 An Introduction to the Representation Theory of the Symmetric Groups

In this section, we introduce the key terminology and the major results from each of the formulations of the representation theory of the symmetric groups that were discussed in the Introduction. For more details on the Young formulation, see [15], and for more details on the Okounkov–Vershik formulation, see [4], [19].

#### 2.2.1 The Young Formulation

The Young formulation uses a combinatorial approach to develop the theory, beginning with partitions of a non-negative integer $n$ and associating to each one a so-called Young frame.

A partition $\lambda$ of $n$, denoted by $\lambda \vdash n$, is defined to be a tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$, $\sum_{i=1}^{k} \lambda_i = n$ and $\lambda_i \in \mathbb{Z}$ for all $i$. We say that such a partition consists of $k$ parts, and we define its length to be $k$. Note that in this definition, we allow partitions with zero parts, that is, some of the $\lambda_i$ can be 0.

To every partition $\lambda \vdash n$ we associate a Young frame of shape $\lambda$, that is, a diagram consisting of $k$ rows, where row $i$ consists of $\lambda_i$ boxes (by convention, the row number increases downwards). We will denote a Young frame of shape $\lambda$ also by $\lambda$, with the context making clear what is being referred to. Figure 1a) shows the Young frame for the partition $(4, 2, 2, 0) \vdash 8$. 

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**Figure 1a** shows the Young frame for the partition $(4, 2, 2, 0) \vdash 8$. 

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A Young frame is a graphical depiction of the shape of a partition. In his formulation, see [15], and for more details on the Okounkov–Vershik formulation, see [4], [19].
For each partition \( \lambda \vdash n \) we can assign to its corresponding Young frame of shape \( \lambda \) a set of coordinates, with the rows being numbered from top to bottom (with starting index 1) and the columns from left to right (also with starting index 1). We say that the box at coordinates \((i, j)\) is removable if there is no box at the locations \((i + 1, j)\) and \((i, j + 1)\). We say that the box at coordinates \((i, j)\) is addable if, when \(i = 1, j = \lambda_1 + 1\), else, when \(i > 1, \lambda_i = j - 1 < \lambda_{i-1}\). Figure 1b) shows the removable (X) and addable (*) boxes for the Young frame of shape \((4, 2, 2, 0) \vdash 8\).

Figure 1: a) The Young frame for the partition \((4, 2, 2, 0) \vdash 8\). b) X denotes the removable boxes, whereas * denotes the addable boxes.

For each partition \( \lambda \vdash n \), a Young tableau of shape \( \lambda \) is a bijection between the set \( \{1, 2, \ldots, n\} \) and the boxes of the Young frame of shape \( \lambda \). We represent such a bijection diagrammatically by filling in the boxes of the Young frame with the numbers \(1, 2, \ldots, n\) such that each number appears in exactly one box. Figure 2a) gives an example of a Young tableau for the partition \((4, 2, 2, 0) \vdash 8\). A Young tableau of shape \( \lambda \) is said to be standard if the numbers are increasing in each row (from left to right) and in each column (from top to bottom). Figure 2b) gives an example of a standard Young tableau for the partition \((4, 2, 2, 0) \vdash 8\). Note that the Young tableau given in Figure 2a) is not standard.

Figure 2: a) A Young tableau for the partition \((4, 2, 2, 0) \vdash 8\). It is not standard. b) A standard Young tableau for the partition \((4, 2, 2, 0) \vdash 8\).

Now for each partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n \), we obtain the following subgroup of \( S_n \), namely

\[
S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}
\]

This is called the Young subgroup for \( \lambda \), and we denote it by \( S_{\lambda} \). By taking the trivial representation of \( S_{\lambda} \) and inducing it up to \( S_n \), we obtain a representation of \( S_n \) which we denote by \( M^\lambda \). Notice that the representation is labelled by the partition. It is called the permutation module for \( \lambda \), and it has a basis consisting of the cosets of \( S_{\lambda} \) in \( S_n \) (there is an alternative formulation in terms of so-called \( \lambda \)-tabloids – see [15] for more details.)

\( M^\lambda \) itself may or may not be irreducible, but it can be shown that, as part of its decomposition into irreducible representations of \( S_n \), it has one copy of an irreducible representation which is denoted by \( S^\lambda \). \( S^\lambda \) is called a Specht module, and it can be shown that the set

\[
\{ S^\lambda \mid \lambda \vdash n \}
\]

is a complete set of irreducible representations for \( S_n \) (up to equivalence). That is, the irreducible representations of \( S_n \) (up to equivalence) are in bijective correspondence with the set of partitions of \( n \). It can further be shown that each \( S^\lambda \) has a basis which is in bijective correspondence with the set of standard Young tableaux of shape \( \lambda \).

It is well known that there is a branching rule for irreducible representations of \( S_n \). It exists in two versions, depending on whether we are restricting or inducing representations of \( S_n \). The restriction version states that for any partition \( \lambda \vdash n \), its accompanying irreducible representation of \( S_n \), \( S^\lambda \), can be expressed in the following form when restricted to be a representation of \( S_{n-1} \):

\[
S^\lambda \downarrow_{S_{n-1}}^{S_n} = \bigoplus_{\mu \vdash n-1, \mu \leftarrow \lambda} S^\mu
\]

(1)

where the direct sum is taken over all partitions \( \mu \vdash n - 1 \) such that \( \mu \) can be obtained by removing a box from the Young frame of \( \lambda \) (which is denoted by \( \mu \leftarrow \lambda \)).

The induction version states that for any partition \( \lambda \vdash n \), its accompanying irreducible representation of \( S_n \), \( S^\lambda \), can be expressed in the following form when induced to be a representation of \( S_{n+1} \):

\[
S^\lambda \uparrow_{S_n}^{S_{n+1}} = \bigoplus_{\mu \vdash n+1, \lambda \leftarrow \mu} S^\mu
\]

(2)
where the direct sum is taken over all partitions $\mu \vdash n + 1$ such that $\mu$ can be obtained by adding a box to the Young frame of $\lambda$ (which is denoted by $\lambda \leftarrow \mu$). These versions are, in fact, equivalent under the Frobenius Reciprocity Theorem.

The branching rule for irreducible representations of $S_n$ leads to a way of representing these irreducible representations in terms of a graph, which is known as the Young graph. Let $P(n)$ be the set of Young frames associated with the partitions of $n$, that is, $P(n) = \{ \lambda \mid \lambda \vdash n \}$.

The Young graph has a vertex set which is the disjoint union

$$\bigcup_{n \geq 0} P(n)$$

where the set $P(n)$ is called the $n$th level of the graph. We denote by $\emptyset$ the unique element of $P(0)$.

We say that two vertices $\mu \in P(n - 1)$ and $\lambda \in P(n)$ are joined by a single edge if and only if

$$\dim \text{Hom}_{S_n}(S_{\lambda} \uparrow S_n \downarrow S_{n-1}) = 1$$

that is, if and only if the multiplicity of $S^n_{\mu}$ in the restriction of $S^n_{\lambda}$ from $S_n$ to $S_{n-1}$ is 1. This is a direct consequence of the branching rule given in (1). Figure 3 shows the Young graph for the symmetric group $S_n$ up to and including $n = 3$.

2.2.2 The Okounkov–Vershik Formulation

We noted earlier that the Okounkov–Vershik formulation takes a rather different approach to the Young formulation. Okounkov and Vershik start by looking at the inductive chain of finite groups

$$\{1\} = G(0) \subset G(1) \subset G(2) \subset \ldots \subset G(n) \subset \ldots$$

and consider, for each $n$, the set of equivalence classes of irreducible representations of $G(n)$, which is denoted by $G(n)\wedge$.

From these sets, they form a multigraph called the Bratelli diagram. Its vertex set is the disjoint union

$$\bigcup_{n \geq 0} G(n)\wedge$$

where the set $G(n)\wedge$ is called the $n$th level of the diagram. They denote by $\emptyset$ the unique element of $G(0)\wedge$.

Writing $S^n_{\lambda}$ for the $G(n)$-module that corresponds to an irreducible representation $\lambda \in G(n)\wedge$, they say that two vertices $\mu \in G((n - 1)\wedge$ and $\lambda \in G(n)\wedge$ are joined by $k$ directed edges (from $\lambda$ to $\mu$) if

$$\dim \text{Hom}_{G(n-1)}(S^\mu, S^n_{\lambda} \uparrow S^n_{G(n-1)}) = k$$
A Multigraph Approach for Performing the Quantum Schur Transform

that is, if the multiplicity of $\mu$ in the restriction of $\lambda$ from $G(n)$ to $G(n-1)$ is $k$.

They write

$$\mu \leftarrow \lambda$$

if $\mu$ and $\lambda$ are connected by at least one edge in the Bratelli diagram, with corresponding notation

$$S^\mu \subset S^\lambda$$

for their respective modules.

They show that for the symmetric groups $G(n) = S_n$, the number of edges between any two vertices in adjacent levels of the Bratelli diagram is either 0 or 1. When any chain of finite groups has this property, they say that the multiplicities are simple or that the branching is simple.

Focusing now on the symmetric group $G(n) = S_n$, since the branching is simple, they obtain a decomposition

$$S^\lambda \uparrow_{S_n \uparrow S_{n-1}} S^\mu = \bigoplus_{\mu \in S_{n-1} \leftarrow \lambda} S^\mu$$

into a direct sum of irreducible $S_{n-1}$ modules. This is the restriction version of the branching rule for $S_n$.

Consequently, by using induction on $n$, they obtain a decomposition of $S^\lambda$ into irreducible $S_0$–modules

$$S^\lambda \uparrow_{S_0 \uparrow S_n} = \bigoplus_T S_T$$

where the direct sum is over all possible paths of irreducible representations

$$T = (\lambda_0 \leftarrow \lambda_1 \leftarrow \lambda_2 \leftarrow \cdots \leftarrow \lambda_{n-1} \leftarrow \lambda_n = \lambda)$$

with $\lambda_i \in S^\lambda_{n}$. Note that the $S_T$ are one–dimensional subspaces.

As there exists an $S_n$–invariant inner product for $S^\lambda$, they can choose a unit vector $v_T$ in each $S_T$ with respect to this inner product. Their union $\{v_T\}_T$ forms a basis of $S^\lambda$, which is called the Gelfand–Tsetlin basis, or GZ–basis, of $S^\lambda$.

Hence, by starting with the chain of symmetric groups

$$\{1\} = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_n \subset \ldots$$

and forming the Bratelli diagram, Okounkov and Vershik show that each path

$$T = (\lambda_0 \leftarrow \lambda_1 \leftarrow \lambda_2 \leftarrow \cdots \leftarrow \lambda_{n-1} \leftarrow \lambda_n = \lambda)$$

through this diagram bijectively corresponds to a chain of irreducible modules

$$\emptyset \subset S^{\lambda_1} \subset S^{\lambda_2} \subset \cdots \subset S^{\lambda_{n-1}} \subset S^{\lambda_n} = S^\lambda$$

which corresponds to a basis vector $v_T$ of $S^\lambda$ (that is unique up to a root of unity), all as a direct consequence of the branching being simple.

At this point there has been no mention of partitions or Young tableaux whatsoever in their formulation, yet they obtain many important results in the theory without them. In fact, Okounkov and Vershik suggest that, given how the Young formulation is developed, the correspondence between partitions and irreducible representations is unnatural, whereas they believe that their approach is more natural and directly leads to the same results. They go on to introduce partitions and Young tableaux rather late in their exposition, after they have introduced the Gelfand–Tsetlin algebra $GZ(n)$, the Young–Jucys–Murphy elements which generate it, and some sets which result from this algebra and the GZ-basis formed above. See [4], [19] for more details.

For our purposes, the only other results that we need from their formulation are that they prove that the Bratelli diagram for $S_n$ is precisely the Young graph; that each irreducible representation $\lambda_n \in S_n^\lambda$ is in bijective correspondence with a partition of $n$; and finally that each path (starting with $\lambda \in S_n^\lambda$) through the Young graph bijectively corresponds to a standard Young tableau with $n$ boxes. See Figure 8 for an example of the latter result. We will use and develop further all of these ideas to achieve a better understanding of the Schur–Weyl basis of $(C^d)^{\otimes n}$. 5
A Multigraph Approach for Performing the Quantum Schur Transform

\[
\begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & \end{array} \equiv \emptyset \leftarrow \begin{array}{c}
\end{array} \leftarrow \begin{array}{c}
\end{array} \leftarrow \begin{array}{c}
\end{array} \leftarrow \begin{array}{c}
\end{array} \leftarrow \begin{array}{c}
\end{array}
\end{array}
\]

Figure 4: An example showing how a standard Young tableau of shape \((3,2) \vdash 5\) corresponds bijectively to a path through the Young graph.

2.3 A Short Introduction to the Representation Theory of \(SU(d)\)

The representation theory of \(SU(d)\) has a rich history. Interested readers can see [6] for more details. Our interest here lies in particular irreducible representations of \(SU(d)\) discovered by Weyl [22], where exactly one exists (up to equivalence) for each partition \(\lambda \vdash n\), where \(n\) is any non-negative integer, such that \(\lambda\) has at most \(d\) non-zero parts. As each such irreducible representation corresponds bijectively to its partition \(\lambda\), we can denote them by \(W^\lambda\). To describe its basis, we need the following definitions.

For a given \(d\), a Weyl tableau of shape \(\lambda \vdash n\) consisting of at most \(d\) rows with content \((\mu_1, \mu_2, \ldots, \mu_d)\), where \(\mu_i \geq 0\) for all \(i\) and \(\sum_i \mu_i = n\), is defined to be a Young frame of shape \(\lambda\) filled out with entries from the alphabet \(\{1, \ldots, d\}\) hereafter written as \([d]\) - such that \(\mu_i\) boxes have entry \(i\), for each \(i = 1 \rightarrow d\). Such a tableau is said to be standard if the entries in the rows are weakly increasing (from left to right) and the entries in the columns are strictly increasing (from top to bottom). Figure [5a] shows an example of a Weyl tableau of shape \((4,2,2,0) \vdash 8\) with content \((2,3,1,2)\) that is not standard. Figure [5b] shows an example of a standard Weyl tableau of shape \((4,2,2,0) \vdash 8\) with content \((2,3,1,2)\).

An equivalent way of representing standard Weyl tableaux of some shape \(\lambda \vdash n\) consisting of at most \(d\) rows with entries from the alphabet \([d]\) in the following, it means that we are considering any Weyl tableau of shape \(\lambda\) consisting of at most \(d\) rows with any allowable \(d\)-length content.

\[
(1 2 3 2)
\]

\[
(1 4 2 4)
\]

Figure 5: a) A Weyl tableau for the partition \((4,2,2,0) \vdash 8\) with content \((2,3,1,2)\). It is not standard. b) A standard Weyl tableau for the partition \((4,2,2,0) \vdash 8\) with content \((2,3,1,2)\).

An equivalent way of representing standard Weyl tableaux of some shape \(\lambda \vdash n\) consisting of at most \(d\) rows with entries from the alphabet \([d]\) is through so-called Gelfand–Tsetlin patterns.

A Gelfand–Tsetlin pattern is an inverted triangle of numbers

\[
\begin{pmatrix}
m_{1,d} & m_{2,d} & \cdots & m_{d-1,d} & m_{d,d} \\
m_{1,d-1} & \cdots & m_{d-1,d-1} & m_{d,d} \\
\vdots & \ddots & \ddots & m_{d,d} \\
m_{1,2} & m_{2,2} & \cdots & m_{d-1,2} \\
m_{1,1} & m_{2,1} & \cdots & m_{d,1}
\end{pmatrix}
\]

consisting of some \(d\) levels where the numbers in the pattern satisfy the so-called in-betweenness condition

\[
m_{i,j} \geq m_{i,j-1} \geq m_{i+1,j} \quad 1 \leq i < j \leq d
\]

We denote by \([m]_i\) the numbers appearing of the \(i^{th}\) row of the pattern \((4)\), namely \(\{m_{1,i}, \ldots, m_{i,i}\}\), and denote by \((m)_{i}\) the subpattern consisting of the first \(i\) rows of the pattern \((4)\).

The top row in the pattern, \([m]_d\), gives the number of boxes in each row of the standard Weyl tableau to which the pattern corresponds, and so it corresponds to some partition \(\lambda\) of some non-negative integer \(n\).

We can convert a standard Weyl tableau of shape \(\lambda \vdash n\) of at most \(d\) rows with entries from the alphabet \([d]\) to a Gelfand–Tsetlin pattern with \(d\) levels by recognising that the number of boxes in row \(i\) of the tableau with the entry \(j\) is equal to \(m_{i,j} - m_{i,j-1}\). Hence we can use this to fill in the \(i^{th}\) diagonal of the pattern by starting from \(j = i\) and iterating through to \(j = d\) inclusive. Clearly then \([m]_d = \lambda\).

Conversely, to convert a Gelfand–Tsetlin pattern with \(d\) levels to a standard Weyl tableau of shape \([m]_d \vdash n\), we start with \([m]_1\) and place \(m_{1,1}\) boxes with entry 1 in the first row (of what will be our tableau). Then we take \([m]_2\), and add \(m_{1,2} - m_{1,1}\) boxes with entry 2 to the first row, and \(m_{2,2}\) boxes with entry 2 to the second row. Iterating through, we

\[
\begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & \end{array} \equiv \emptyset \leftarrow \begin{array}{c}
\end{array} \leftarrow \begin{array}{c}
\end{array} \leftarrow \begin{array}{c}
\end{array} \leftarrow \begin{array}{c}
\end{array} \leftarrow \begin{array}{c}
\end{array}
\end{array}
\]
take \([m]_j\), and add \(m_{1,j} - m_{1,j-1}\) boxes with entry \(j\) to the first row, \(m_{2,j} - m_{2,j-1}\) boxes with entry \(j\) to the second row, and so on, adding \(m_{j,j} - m_{j,j-1}\) boxes with entry \(j\) to the \(j^{th}\) row. This will result in the desired standard Weyl tableau.

In fact, these patterns were introduced by Gelfand and Tsetlin precisely for the purpose of labelling a basis of \(W^\lambda\), where \(\lambda\) is as described above. The basis of \(W^\lambda\) that we are interested in is in bijective correspondence with the set of all \(d\)-level Gelfand–Tsetlin patterns with the top level \([m]_d\) equal to \(\lambda\), or equivalently, with the set of all standard Weyl tableaux of shape \(\lambda\) with entries from the alphabet \([d]\).

We will see this basis appearing in the next section on the Schur–Weyl duality.

### 2.4 The Schur–Weyl Duality and the Schur–Weyl Basis

At its heart, the Schur–Weyl duality is a theorem in representation theory which describes a relationship between the irreducible representations of \(S_n\) and the irreducible representations of \(SU(d)\) appearing in the decomposition of \((\mathbb{C}^d)^\otimes n\) when considered as a representation of each group.

Recall that \(\mathbb{C}^d\) has a basis \([|k\rangle \mid k \in [n]]\) which is called the computational basis.

Hence \((\mathbb{C}^d)^\otimes n\) has as its computational basis
\[
\{(|v_1\rangle \otimes |v_2\rangle \otimes \cdots \otimes |v_n\rangle \mid v_i \in [d] \text{ for all } i = 1 \rightarrow n\}
\]

We know that \((\mathbb{C}^d)^\otimes n\) is a representation of the symmetric group \(S_n\) – written \(\rho : S_n \rightarrow GL((\mathbb{C}^d)^\otimes n)\) – under
\[
\rho(\sigma)(|v_1\rangle \otimes |v_2\rangle \otimes \cdots \otimes |v_n\rangle) = |v_{\sigma^{-1}(1)}\rangle \otimes |v_{\sigma^{-1}(2)}\rangle \otimes \cdots \otimes |v_{\sigma^{-1}(n)}\rangle
\]

where \(\sigma \in S_n\), \(|v_i\rangle\) is a computational basis state of \(\mathbb{C}^d\), and \(\sigma^{-1}(i)\) is the label resulting from the action of \(\sigma^{-1}\) on the index \(i\).

Similarly, \((\mathbb{C}^d)^\otimes n\) is also a representation of the special unitary group \(SU(d)\) – written \(\tau : SU(d) \rightarrow GL((\mathbb{C}^d)^\otimes n)\) – under
\[
\tau(U)(|v_1\rangle \otimes |v_2\rangle \otimes \cdots \otimes |v_n\rangle) = U |v_1\rangle \otimes U |v_2\rangle \otimes \cdots \otimes U |v_n\rangle
\]

where \(U \in SU(d)\) and, again, \(|v_i\rangle\) is a computational basis state of \(\mathbb{C}^d\). Clearly \(\tau(U) = U^\otimes n\).

As \(\rho\) and \(\tau\) are both representations of groups that admit a decomposition into irreducible representations, we get that
\[
(\mathbb{C}^d)^\otimes n \cong \bigoplus_{\lambda} C^{m^\lambda} \otimes S^\lambda
\]

as a representation of \(S_n\) and
\[
(\mathbb{C}^d)^\otimes n \cong \bigoplus_{\mu} W^{m^\mu} \otimes C^{n^\mu}
\]
as a representation of \(SU(d)\). Here, \(\lambda\) and \(\mu\) run over the set of partitions of \(n\) with at most \(d\) parts. Also, \(m^\lambda\) is the multiplicity of the irreducible \(S^\lambda\) in its corresponding decomposition, and, likewise, \(n^\mu\) is the multiplicity of the irreducible \(W^\mu\) in its corresponding decomposition.

However, by looking further into the structure of these representations, we can, in fact, say that \((\mathbb{C}^d)^\otimes n\) is a representation of the direct product group \(SU(d) \times S_n\), and that its decomposition into irreducible representations is given by
\[
(\mathbb{C}^d)^\otimes n \cong \bigoplus_{\lambda} W^\lambda \otimes S^\lambda
\]

where \(m^\lambda = \dim W^\lambda\) and \(n^\lambda = \dim S^\lambda\), and where \(\lambda\) runs over the set of partitions of \(n\) with at most \(d\) parts.

The isomorphism given in (10) is known as the Schur–Weyl duality. See [6] for more details.

From the duality we obtain the so-called Schur–Weyl basis of \((\mathbb{C}^d)^\otimes n\), which consists of a set of triplets of the form
\[
|\lambda t^y\rangle
\]

where \(\lambda\) is a partition of \(n\) with at most \(d\) parts, \(t\) is a standard Young tableau of shape \(\lambda\) with entries from the alphabet \([d]\) and \(y\) is a standard Young tableau of shape \(\lambda\).

This basis forms the focus of our attention for the rest of this paper.
3 The Schur–Weyl–Young Graph and the Schur–Weyl Branching Rule

We now present our main contributions. Specifically, we present a new way of understanding how the Schur–Weyl duality can be used to perform the Quantum Schur Transform. We achieve our results by taking inspiration from the Okounkov–Vershik formulation for the representation theory of the symmetric groups.

Consider the following chain of direct product groups

\[ SU(d) \times S_0 \subset SU(d) \times S_1 \subset SU(d) \times S_2 \subset \cdots \subset SU(d) \times S_n \]  

(11)

where we have fixed \( d \) and (in the following) have stopped the chain at level \( n \) for some \( n \).

We have, corresponding to this chain of groups, a chain of representations

\[ \emptyset \subset (\mathbb{C}^d)^{\otimes 1} \subset (\mathbb{C}^d)^{\otimes 2} \subset \cdots \subset (\mathbb{C}^d)^{\otimes n} \]  

(12)

as discussed in Section 2.

However, in following the Okounkov–Vershik formulation as inspiration, we are particularly interested in finding chains of irreducible representations for the chain of groups (11).

We can use the chain of representations given in (12) to construct chains of irreducible representations in the following way.

By the Okounkov–Vershik formulation, for every partition \( \lambda \) of \( n \) consisting of at most \( d \) non-zero parts, we can choose a path through the Young graph

\[ T = (\lambda_0 \leftarrow \lambda_1 \leftarrow \lambda_2 \leftarrow \cdots \leftarrow \lambda_{n-1} \leftarrow \lambda_n = \lambda) \]  

(13)

where each \( \lambda_i \) is a partition of \( i \) with at most \( d \) non-zero parts. In fact, for each such \( \lambda \), we get \( \dim S^\lambda \) many paths, which are in bijective correspondence with the standard Young tableaux of shape \( \lambda \).

The key point is that, for each partition \( \lambda_i \) appearing in the path \( T \), \( S^\lambda_i \) appears in the irreducible decomposition of \( (\mathbb{C}^d)^{\otimes n} \) when viewed as a representation of \( S_i \). This follows by the isomorphism given in \( \Box \). (This isomorphism is also why we only consider partitions of \( n \) with at most \( d \) non-zero parts in the above.)

We also know, by the same formulation, that the path given in (13) corresponds to a chain

\[ \emptyset \subset S^{\lambda_1} \subset S^{\lambda_2} \subset \cdots \subset S^{\lambda_{n-1}} \subset S^{\lambda_n} \]

of irreducible representations for the chain of symmetric groups

\[ \{1\} = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_n \]

Hence, by the Schur–Weyl duality (10), we can tensor on the accompanying irreducible representation of \( SU(d) \) (that is determined by the duality) to each irreducible representation in this chain, leading to the chain of irreducible representations

\[ \emptyset \subset W^{\lambda_1} \otimes S^{\lambda_1} \subset W^{\lambda_2} \otimes S^{\lambda_2} \subset \cdots \subset W^{\lambda_n} \otimes S^{\lambda_n} \]  

(14)

corresponding to the chain of direct product groups given in (11). Each \( W^{\lambda_i} \otimes S^{\lambda_i} \) appears in the decomposition of \( (\mathbb{C}^d)^{\otimes i} \) into irreducible representations when viewed as a representation of \( SU(d) \times S_i \).

This leads to the discovery of a new multigraph, which, to the best of our knowledge, has not been described previously in the literature. We modify the vertex and edge sets of the Young graph for the symmetric groups in the following way. Firstly, the vertex set: for a given \( d \), remove all Young frames that have more than \( d \) rows. Then, for each remaining Young frame, calculate all of the possible standard Weyl tableaux of the same shape using elements from the alphabet \( [d] \), and replace the Young frame by the set of all such Weyl tableaux. (Note that for \( d = 2 \) we will use the alphabet \( \{0, 1\} \) and not \( \{1, 2\} \) to align our notation with that for the computational basis of a qubit.) Then, for the edge set: for each (single) edge that appears between Young frames in the Young graph, we replace it with an edge between an element from each of the sets that have replaced them, that is, between standard Weyl tableaux, if the following holds.

Suppose that \( \mu \vdash n - 1 \) and \( \lambda \vdash n \) are partitions that had an edge between them in the Young graph. Let \( t^\mu \) be a standard Weyl tableau in the set replacing the Young frame of shape \( \mu \), and let \( t^\lambda \) be a standard Weyl tableau in the set replacing the Young frame of shape \( \lambda \). Then an edge exists between these tableaux if and only if \( t^\mu \) can be obtained from \( t^\lambda \) by removing an element of the alphabet \( [d] \) from it; equivalently, if \( t^\lambda \) can be obtained from \( t^\mu \) by adding an element of the alphabet \( [d] \) to it. We call such an edge a transition between standard Weyl tableaux. Clearly, between the sets appearing in this new graph, there may be multiple edges.

We will call this multigraph the Schur–Weyl–Young \( SU(d) - S_n \) graph (or the Schur–Weyl–Young graph when \( n \) and \( d \) are clearly understood). An example for \( d = 2 \), up to and including \( n = 3 \), is given in Figure 6.
We will use the Schur–Weyl–Young graph to calculate, for pairs of standard Weyl tableaux that have an edge between them, the transition amplitude (a term whose definition is delayed to Section 4).

What seems rather remarkable is that, as a direct consequence of starting with paths through the Young graph, we can construct a branching rule for the Schur–Weyl states.

Indeed, we have shown that for the chain of direct product groups (11)

$$SU(d) \times S_0 \subset SU(d) \times S_1 \subset SU(d) \times S_2 \subset \cdots \subset SU(d) \times S_n$$

we can form a chain of irreducible representations (14)

$$\emptyset \subset W^{\lambda_1} \otimes S^{\lambda_1} \subset W^{\lambda_2} \otimes S^{\lambda_2} \subset \cdots \subset W^{\lambda_n} \otimes S^{\lambda_n}$$

corresponding to a path (13)

$$T = (\lambda_0 \leftarrow \lambda_1 \leftarrow \lambda_2 \leftarrow \cdots \leftarrow \lambda_{n-1} \leftarrow \lambda_n)$$

through the Young graph, where each \( \lambda_i \) is a partition of \( i \) with at most \( d \) non-zero parts.

We noted that each \( W^{\lambda_i} \otimes S^{\lambda_i} \) appears in the decomposition of \((C^d)^{\otimes i}\) into irreducible representations when viewed as a representation of \( SU(d) \times S_i \).

Consequently, given the chain (12)

$$\emptyset \subset (C^d)^{\otimes 1} \subset (C^d)^{\otimes 2} \subset \cdots \subset (C^d)^{\otimes n}$$

we see that, as

$$(C^d)^{\otimes i-1} \otimes C^d = (C^d)^{\otimes i}$$

we must have, from (14), that

$$(W^{\lambda_{i-1}} \otimes S^{\lambda_{i-1}}) \otimes C^d = W^{\lambda_i} \otimes S^{\lambda_i} \quad (15)$$

In obtaining the equality given in (15), we have actually found a branching rule for the Schur–Weyl states that is intimately connected to the Schur–Weyl–Young diagram. It comes in two versions that are inverse operations of one another, depending on whether we start on the left-hand side or the right-hand side of the equality given in (15).
The left-to-right version takes a pair of quantum registers, the first a Schur–Weyl basis state of \((\mathbb{C}^d)^{\otimes i-1}\), the second a computational basis state of \(\mathbb{C}^d\), and gives an equivalent expression in terms of the Schur–Weyl basis of \((\mathbb{C}^d)^{\otimes i}\).

The right-to-left version is the inverse of this: it takes a Schur–Weyl basis state of \((\mathbb{C}^d)^{\otimes i}\) and gives its equivalent expression in terms of a superposition of pairs of quantum registers: in each pair, the first register is a Schur–Weyl basis state of \((\mathbb{C}^d)^{\otimes i-1}\) and the second register is a computational basis state of \(\mathbb{C}^d\).

Together, these versions make up what we shall call, from now on, the Schur–Weyl branching rule.

### 3.1 How does the Schur–Weyl branching rule work?

**Left-to-right version**

Take as input a pair of quantum registers, the first a Schur–Weyl basis state of \((\mathbb{C}^d)^{\otimes i-1}\), the second a computational basis state of \(\mathbb{C}^d\).

Therefore, the input is of the form \(|\mu t y\rangle|k\rangle\), where \(\mu \vdash i-1\) consists of at most \(d\) non-zero parts, \(t\) is a standard Weyl tableau of shape \(\mu\) with entries from the alphabet \([d]\), \(y\) is a standard Young tableau of shape \(\mu\), and \(k \in [d]\).

Consider now each possible addition of a box with entry \(i\) to the standard Young tableau \(y^*\) of some shape \(\lambda \vdash i\) consisting of at most \(d\) rows is formed. Clearly, \(y^*\) will also be standard. Each possibility \(y^*\) corresponds to elongating the path of partitions \(T\) that is determined by \(y\) with \(\lambda\), as suggested by equation (15), or, equivalently, to the induction version of the symmetric group branching rule.

Then, for each one of these new standard Young tableau \(y^*\), find all possible edges in the Schur–Weyl–Young graph between \(t\), the standard Weyl tableau of shape \(\mu\) given in the first register, and \(t^*\), a standard Weyl tableau of the new shape \(\lambda\), where the element \(k\) (given in the second register) has been added into \(t\) to give \(t^*\).

Each new Young tableau and each such edge between the standard Weyl tableaux (as described above) results in a Schur–Weyl state \(|\lambda t^* y^*\rangle\) of \((\mathbb{C}^d)^{\otimes i}\).

Form as output a superposition over all such states, with the amplitude for each state appearing in the superposition given by the transition amplitude between \(t\) and \(t^*\).

**Right-to-left version**

Take as input a Schur–Weyl basis state of \((\mathbb{C}^d)^{\otimes i}\), that is, \(|\lambda t y\rangle\), where \(\lambda \vdash i\) consists of at most \(d\) non-zero parts, \(t\) is a standard Weyl tableau of shape \(\lambda\) with entries from the alphabet \([d]\), and \(y\) is a standard Young tableau of shape \(\lambda\).

Let \(y^*\) be the standard Young tableau of shape \(\mu \vdash i-1\), where \(\mu\) is of the same shape as \(y\) but with the box with entry \(i\) removed. Note that we have used equation (15) here, since it tells us that the standard Young tableau appearing on the left-hand side is pre-determined by \(y\) (as \(y\) corresponds bijectively to a path of partitions \(T\) through the Young graph).

Now, for each \(k \in [d]\), find all possible edges in the Schur–Weyl–Young graph between \(t\), the standard Weyl tableau of shape \(\lambda\) given as input, and \(t^*\), a standard Weyl tableau of the new shape \(\mu\) with a single \(k\) removed.

Each such edge between the standard Weyl tableaux together with the new Young tableau \(y^*\) results in a Schur–Weyl basis state of the form \(|\mu t^* y^*\rangle\) of \((\mathbb{C}^d)^{\otimes i-1}\).

This state becomes the first register in a pair of quantum registers. Let the second register be \(|k\rangle\), where \(k\) is the entry removed from \(t\) to form \(t^*\).

Form as output a superposition of these new pairs of quantum registers, with the amplitude for each state appearing in the superposition given by the transition amplitude between \(t^*\) and \(t\).

### 3.1.1 Remarks

To end this section, we make clear how these new ideas have been inspired by the Okounkov–Vershik formulation for the representation theory of the symmetric groups.

In this formulation, we saw that a path beginning at some \(\lambda \vdash n\) through the Young graph corresponds bijectively to a standard Young tableau of shape \(\lambda\), which corresponds bijectively to a chain of irreducible representations relating to the chain of groups \(G(n) = S_n\). Also, the Young graph corresponds to a branching rule for irreducible representations of \(S_n\).
In the above, we have shown that, for a given \( d \), a pair of tableaux – the first a standard Weyl tableau of shape \( \lambda \vdash n \) consisting of at most \( d \) rows with entries from the alphabet \([d]\), the second a standard Young tableau of the same shape \( \lambda \) – corresponds to a multopath through the (newly formed) Schur–Weyl–Young graph, which corresponds to a chain of irreducible representations coming from the Schur–Weyl duality relating to the chain of groups \( G(n) = SU(d) \times S_n \). We did this by starting with paths through the Young graph – which correspond bijectively to standard Young tableaux – and modifying them to pair them with standard Weyl tableaux. As a result, we showed that the Schur–Weyl–Young graph corresponds to a branching rule for the Schur–Weyl basis states of \((C^d)^{\otimes n}\), a representation of the direct product group \( SU(d) \times S_n \).

The Schur–Weyl branching rule is important, because, as we will discuss further in Section 6, starting from a computational basis state of \((C^d)^{\otimes n}\) and iteratively applying the left-to-right version on each qudit – beginning at the left-most qudit – gives the same state expressed in terms of the Schur–Weyl basis of \((C^d)^{\otimes n}\) and, similarly, starting from a Schur–Weyl basis state of \((C^d)^{\otimes n}\) and iteratively applying the right-to-left version gives the same state expressed in terms of the computational basis of \((C^d)^{\otimes n}\).

We delay giving examples of these versions of the Schur–Weyl branching rule until Section 5, after we have described how to calculate the transition amplitudes between standard Weyl tableaux, which we do in the next section.

4 Simple Pattern Rules for the Calculation of Transition Amplitudes when \( d = 2 \)

As we have seen, an element of a vertex in the Schur–Weyl–Young graph at level \( n \) is a standard Weyl tableau \( t \) of some shape \( \lambda \vdash n \) consisting of at most \( d \) rows with entries from the alphabet \([d]\). This tableau has a number of connections to standard Weyl tableaux \( t^* \) in the \( n + 1 \)st level, where each connection is given by an edge in the graph. Each edge therefore describes a possible transition of \( t \) to one such \( t^* \) in the next level. Hence, in moving up a level (by the addition of an entry from the alphabet \([d]\) into \( t \) such that we obtain a standard Weyl tableau of some shape with \( n + 1 \) entries and at most \( d \) rows), the new Weyl tableau must be one of the \( t^* \). Consequently, each edge must come with a so-called transition amplitude whose square gives the probability that we will obtain the new standard Weyl tableau \( t^* \) out of all such possibilities.

Louck [12] has shown (independently of the existence of the Schur–Weyl–Young graph) that the transition amplitude, for general \( d \), between two standard Weyl tableaux that are in adjacent levels of the Schur–Weyl–Young diagram and are connected by an edge (denoted by \( \leftrightarrow \) in the following)

\[
\begin{pmatrix}
[m]_d \\
[m]_{d-1} \\
\vdots \\
[m]_k \\
(m)_{k-1}
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
[m]_d + e_d(\tau_d) \\
[m]_{d-1} + e_{d-1}(\tau_{d-1}) \\
\vdots \\
[m]_k + e_k(\tau_k) \\
(m)_{k-1}
\end{pmatrix}
\]

where each Weyl tableau has been expressed in its equivalent Gelfand–Tsetlin pattern form, is given by the multiplication of two terms, the first of which is

\[
\prod_{j=k+1}^{d} \text{sgn}(\tau_{j-1} - \tau_j) \left\{ \frac{\prod_{i=1, i \neq \tau_{j-1}}^{j-1} (p_{\tau_{j-1}, j - 1} - p_{i, j - 1}) \prod_{i=1, i \neq \tau_j}^{j} (p_{\tau_{j-1}, j - 1} - p_{i, j - 1} + 1)}{\prod_{i=1, i \neq \tau_{j-1}}^{j-1} (p_{\tau_{j-1}, j - 1} - p_{i, j - 1}) \prod_{i=1, i \neq \tau_j}^{j} (p_{\tau_{j-1}, j - 1} - p_{i, j - 1} + 1)} \right\}
\]

and the second of which is

\[
\prod_{i=1, i \neq \tau_k}^{k-1} (p_{\tau_k, k - 1} - p_{i, k - 1}) \prod_{i=1, i \neq \tau_k}^{k} (p_{\tau_k, k - 1} - p_{i, k - 1})
\]

unless \( k = 1 \), where the second term \(18\) in the multiplication is defined to be 1, and if \( k = d \), the first term \(17\) in the multiplication is defined to be 1.

Here, \( k \in [d] \) can be thought of either as the entry that is added to the first (lower level) Weyl tableau to form the second (higher level) Weyl tableau, or as the entry that is removed from the second Weyl tableau to form the first Weyl tableau.

The integer \( p_{i, j} \) appearing in \(17\) and \(18\) is called the partial hook, and it is defined to be equal to \( m_{i, j} + j - i \).

Finally, \( e_i(\tau_i) \) is a vector of length \( i \), with entry 1 in the position \( \tau_i \), and 0 otherwise. \( \tau_i \) itself is found by creating Weyl sub–tableaux formed from the entries \( 1 \rightarrow i \) inclusive appearing in each of the Weyl tableaux involved and then looking at which row in the sub–tableaux has an extra box in it. \( \tau_i \) is equal to this row number.
Whilst the above formula is true, it is rather involved, hence using it in practice to calculate the transition amplitude between standard Weyl tableaux is slow and fraught with danger. For the case $d = 2$, we have found a much simpler way of calculating these transition amplitudes that is based entirely on looking at the entries in the standard Weyl tableaux themselves. In fact, we have reduced the entire calculation to the application of four simple rules for this case. From now on, we will call these rules the Pattern Rules.

We give the Pattern Rules both in their Weyl tableau form and in their equivalent Gelfand–Tsetlin form below. A proof can be found in Appendix A.

### 4.1 The Pattern Rules for $d = 2$

Denote the transition amplitude between the standard Weyl tableaux (or Gelfand–Tsetlin patterns) by $\alpha$. In the following, the notation $\mu \leftrightarrow \lambda$ means that the transition is between standard Weyl tableaux of shapes $\mu$ and $\lambda$ respectively.

**Rule 1**

*In Weyl tableau form:* $(n - 1, 0) \leftrightarrow (n, 0)$

a) If the number of zeroes in each tableau are the same, then

$$\alpha = \sqrt{\frac{\text{Number of ones in the } (n, 0) \text{ tableau}}{n}}$$

(19)

b) Else

$$\alpha = \sqrt{\frac{\text{Number of zeroes in the } (n, 0) \text{ tableau}}{n}}$$

(20)

*In Gelfand–Tsetlin pattern form:*

$$\begin{pmatrix} n - 1 & m_{1,1}^{n-1} & 0 \\ m_{1,1}^{n-1} & m_{1,1}^{n} & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} n & m_{1,1}^{n} & 0 \\ m_{1,1}^{n} & m_{1,1}^{n-1} & 0 \end{pmatrix}$$

a) If $m_{1,1}^{n-1} = m_{1,1}^{n}$, then

$$\alpha = \sqrt{\frac{n - m_{1,1}^{n}}{n}}$$

(21)

b) Else

$$\alpha = \sqrt{\frac{m_{1,1}^{n}}{n}}$$

(22)

**Rule 2**

*In Weyl tableau form:* $(n - 1, 0) \leftrightarrow (n - 1, 1)$

a) If the number of zeroes in each tableau are the same, then

$$\alpha = \sqrt{\frac{\text{Number of zeroes in the } (n, 0) \text{ tableau}}{n}}$$

(23)

b) Else

$$\alpha = -\sqrt{\frac{\text{Number of ones in the } (n, 0) \text{ tableau}}{n}}$$

(24)

*In Gelfand–Tsetlin pattern form:*

$$\begin{pmatrix} n - 1 & m_{1,1}^{n-1} & 0 \\ m_{1,1}^{n-1} & m_{1,1}^{n} & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} n - 1 & m_{1,1}^{n} & 1 \\ m_{1,1}^{n} & m_{1,1}^{n-1} & 1 \end{pmatrix}$$

a) If $m_{1,1}^{n-1} = m_{1,1}^{n}$, then

$$\alpha = \sqrt{\frac{m_{1,1}^{n}}{n}}$$

(25)

b) Else

$$\alpha = -\sqrt{\frac{n - m_{1,1}^{n}}{n}}$$

(26)
Rule 3

In Weyl tableaux form: \((n - k - 1, k) \leftrightarrow (n - k, k)\)

Here the first \(k\) columns are the same in both Weyl tableaux. Delete all \(k\) such columns from both tableaux. Then apply Rule 1.

In Gelfand–Tsetlin pattern form:

\[
\begin{pmatrix}
{n-k-1} \\
{m_{1;1}^{-1}} \\
{k}
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
{n-k} \\
{m_{1;1}} \\
{k}
\end{pmatrix}
\]

Subtract \(k\) from every entry in both patterns. Then apply Rule 1.

Rule 4

In Weyl tableaux form: \((n - k, k - 1) \leftrightarrow (n - k, k)\)

Here the first \(k - 1\) columns are the same in both Weyl tableaux. Delete all \(k - 1\) such columns from both tableaux. Then apply Rule 2.

In Gelfand–Tsetlin pattern form:

\[
\begin{pmatrix}
{n-k} \\
{m_{1;1}^{-1}} \\
{k-1}
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
{n-k} \\
{m_{1;1}} \\
{k}
\end{pmatrix}
\]

Subtract \(k - 1\) from every entry in both patterns. Then apply Rule 2.

4.2 Examples

Example 1: \[
\begin{pmatrix}
0 & 0 \\
1 & 1
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
0 & 0 & 1 \\
1 & 1
\end{pmatrix}
\]

In Weyl tableaux form:

Apply rule 3 to the one shared column: \[
\begin{pmatrix}
0 & 0 \\
1 & 1
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
0 & 1 \\
1 & 1
\end{pmatrix}
\]

Now apply rule 1a) to these tableaux: \(\alpha = \frac{1}{\sqrt{2}}\).

In Gelfand–Tsetlin pattern form:

The transition is \(\begin{pmatrix}
2 \\
1
\end{pmatrix}
\leftrightarrow \begin{pmatrix}
3 \\
1
\end{pmatrix}\).

Apply rule 3. Subtracting 1 from every entry in both patterns gives \(\begin{pmatrix}
1 & 0 \\
1 & 0
\end{pmatrix}
\leftrightarrow \begin{pmatrix}
2 & 0 \\
1 & 0
\end{pmatrix}\).

Now apply rule 1a) to these patterns: \(\alpha = \frac{1}{\sqrt{2}}\).

Example 2: \[
\begin{pmatrix}
0 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
0 & 0 & 0 \\
1 & 1 & 1
\end{pmatrix}
\]

In Weyl tableaux form:

Apply rule 4 to the two shared columns: \[
\begin{pmatrix}
0 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}
\leftrightarrow \begin{pmatrix}
0 & 1 \\
1 & 1
\end{pmatrix}
\]

Now apply rule 2b) to these tableaux: \(\alpha = -\frac{1}{\sqrt{2}}\).

In Gelfand–Tsetlin pattern form:

The transition is \(\begin{pmatrix}
3 \\
2
\end{pmatrix}
\leftrightarrow \begin{pmatrix}
3 \\
3
\end{pmatrix}\).

Apply rule 4. Subtracting 2 from every entry in both patterns gives \(\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}
\leftrightarrow \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}\).

Now apply rule 2b) to these patterns: \(\alpha = -\frac{1}{\sqrt{2}}\).

Example 3: \[
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}
\]

In Weyl tableaux form:

Apply rule 4 to the four shared columns: \[
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}
\leftrightarrow \begin{pmatrix}
0 & 0 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

Now apply rule 2b) to these tableaux: \(\alpha = -\frac{\sqrt{2}}{2}\).
In Gelfand–Tsetlin pattern form:
The transition is \((7\ 4\ 4) \leftrightarrow (7\ 5\ 5)\).

Apply rule 4. Subtracting 4 from every entry in both patterns gives \((3\ 0\ 0) \leftrightarrow (3\ 1\ 1)\).

Now apply rule 2b) to these patterns:
\[\alpha = -\frac{\sqrt{3}}{2}.\]

It is easy to verify that we obtain the same transition amplitudes for each example by using Louck’s formula, the terms of which are given in (17) and (18).

5 Examples of the Schur–Weyl Branching Rule

In Sections 3 and 4 we have developed all of the theory that we need to be able to give examples of the Schur–Weyl branching rule.

Example 1: Left-to-right version
Consider the following quantum state of \((C^2)^\otimes 4\)
\[\lambda = (2, 1), t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, y = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\] |0\rangle
(27)

The only possible additions of a box with entry 4 to the standard Young tableau \(y\) such that we form a standard Young tableau \(y^*\) having at most two rows are given by \(\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}\) and \(\begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}\).

The only possible standard Weyl tableau \(t^*\) corresponding to these Young tableaux with a box with entry 0 added to \(t\) are given by \(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\) and \(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\) respectively.

Using the Pattern Rules given in subsection 4.1 we see that the transition amplitude for \(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\) equals \(\frac{1}{\sqrt{2}}\), and for \(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\) it equals \(-\frac{1}{\sqrt{2}}\).

Hence we have that (27) is equal to
\[\frac{1}{\sqrt{2}} |\lambda = (3, 1), t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, y = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\rangle - \frac{1}{\sqrt{2}} |\lambda = (2, 2), t = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, y = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\rangle\]
(28)
expressed in terms of the Schur–Weyl basis of \((C^2)^\otimes 4\).

Example 2: Right-to-left version
Consider the following quantum state of \((C^2)^\otimes 3\)
\[\lambda = (2, 1), t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, y = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\] |0\rangle
(29)

The only possible standard Weyl tableau with a single 0 removed of the shape \([1\ 2]\) (as the box containing entry 3 is to be removed from \(y\)) is given by \([1\ 2]\). Similarly, the only possible standard Weyl tableau with a single 1 removed of the shape \([1\ 2]\) is given by \([2\ 1]\).

As the transition amplitude for \([1\ 2]\) \leftrightarrow \([2\ 1]\) is \(-\frac{\sqrt{2}}{\sqrt{3}}\) and for \([2\ 1]\) \leftrightarrow \([1\ 2]\) is \(\frac{1}{\sqrt{3}}\), we have that (29) is equal to
\[-\frac{\sqrt{2}}{\sqrt{3}} |\lambda = (2, 0), t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, y = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\rangle + \frac{1}{\sqrt{3}} |\lambda = (2, 0), t = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, y = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\rangle|1\rangle\]
(30)
We see that the first register of each state in the superposition is a Schur–Weyl basis state of \((C^2)^\otimes 2\).

6 The Quantum Schur Transform
We mentioned at the end of Section 3 that the Schur–Weyl branching rule gives rise to a very simple procedure for performing the Quantum Schur Transform for any \(n\) qudits: that is, the unitary change of basis transformation that
maps the computational basis of \((\mathbb{C}^d)^\otimes n\) to the Schur–Weyl basis of \((\mathbb{C}^d)^\otimes n\) (with its adjoint giving the opposite transformation.)

It is now clear to see that we can perform this transformation merely by iteratively applying the left–to–right version of the Schur–Weyl branching rule on the input computational basis state(s), with the adjoint given by iteratively applying the right–to–left version on Schur–Weyl basis state(s). This procedure is polynomial in \(n\) and \(d\).

### 6.1 Examples of the Quantum Schur Transform with \(d = 2\)

We give concrete examples of the Quantum Schur Transform for the case \(d = 2\) in order to display the speed of our method – in particular, for performing the transformation with \(n\) large – whilst keeping in mind the brevity of our paper. However, we re-emphasise here for clarity that our method works for any \(d\) (and \(n\)).

**Example 1:** Express \(|0101\rangle\) in the Schur–Weyl basis of \((\mathbb{C}^2)^\otimes 4\).

We apply the left-to-right version of the Schur–Weyl branching rule four times, using the Pattern Rules to calculate the transition amplitudes. As

\[
|0101\rangle = |\lambda = (1, 0), t = 0\rangle, y = 1\rangle |101\rangle
\]

\[= \frac{1}{\sqrt{3}} |\lambda = (2, 0), t = 01, y = 12\rangle |01\rangle + \frac{1}{\sqrt{2}} |\lambda = (1, 1), t = 01, y = 12\rangle |01\rangle\]

\[= \frac{1}{\sqrt{3}} |\lambda = (3, 0), t = 001, y = 123\rangle |1\rangle - \frac{1}{\sqrt{6}} |\lambda = (2, 1), t = 001, y = 12\rangle |1\rangle\]

\[+ \frac{1}{\sqrt{2}} |\lambda = (2, 1), t = 001, y = 12\rangle |1\rangle\]

we have that

\[
|0101\rangle = \frac{1}{\sqrt{6}} |\lambda = (4, 0), t = 0001, y = 123\rangle + \frac{1}{6} |\lambda = (3, 1), t = 001, y = 123\rangle - \frac{1}{2\sqrt{3}} |\lambda = (3, 1), t = 001, y = 123\rangle - \frac{1}{2\sqrt{3}} |\lambda = (2, 2), t = 001, y = 123\rangle + \frac{1}{2} |\lambda = (2, 2), t = 001, y = 123\rangle
\]

**Example 2:** Express

\[
|\lambda = (2, 2), t = 0001, y = 123\rangle
\]

in the computational basis of \((\mathbb{C}^2)^\otimes 4\).

We apply the right-to-left Schur–Weyl branching rule four times, again using the Pattern Rules to calculate the transition amplitudes. As equation (31) equals

\[
\frac{1}{\sqrt{2}} |\lambda = (2, 1), t = 001, y = 123\rangle |1\rangle - \frac{1}{\sqrt{2}} |\lambda = (2, 1), t = 001, y = 123\rangle |0\rangle
\]

\[= \frac{1}{\sqrt{2}} |\lambda = (1, 1), t = 001, y = 12\rangle |01\rangle - |10\rangle\]

\[+ \frac{1}{2} |\lambda = (1, 0), t = 001, y = 12\rangle \langle 0101 | - \frac{1}{2} |\lambda = (1, 0), t = 001, y = 12\rangle \langle 0101 | - \frac{1}{2} \langle 0101 | - \frac{1}{2} \langle 0101 |\]

we have that equation (31) equals

\[
\frac{1}{2} |0101\rangle - \frac{1}{2} |0110\rangle - \frac{1}{2} |1001\rangle + \frac{1}{2} |1010\rangle
\]

### 7 Related Work

Bacon, Chuang and Harrow were the first to construct the Quantum Schur Transform for the \(n\)-fold tensor product \((\mathbb{C}^2)^\otimes n\) [1], which they extended to \((\mathbb{C}^d)^\otimes n\) in [2]. Their approach uses the representation theory of the unitary group,
namely by implementing Clebsch-Gordan transforms on a quantum computer and then by iteratively applying these<br>transforms to construct the Quantum Schur Transform. The quantum algorithm that they present in their extension<br>paper is polynomial in $n, d$ and $\log \frac{1}{\epsilon}$, where $\epsilon$ is the precision. Whilst our work will result in the same implementation<br>of the Quantum Schur Transform, it differs significantly in that we use the representation theory of the symmetric<br>group as the theoretical foundation for our approach. Consequently, we only consider unitary group representations<br>when we apply the Schur–Weyl duality to chains of irreducible representations of the symmetric group that form the<br>basis of our approach.

Krovi [11] presented a quantum algorithm for the Quantum Schur Transform that is polynomial in $n, \log d$ and $\log \frac{1}{\epsilon}$.<br>His approach uses the representation theory of the symmetric group but in a different way to ours, namely by block<br>diagonalising induced representations of the symmetric group known as permutation modules.

Kaczor and Jakubczyk [9] described a procedure for performing the Quantum Schur Transform that is based on the<br>Schensted insertion and the Robinson–Schensted–Knuth algorithm. Their approach is based on so-called fundamental<br>shift operators which can be used to find Clebsch–Gordan coefficients for the unitary group. This ultimately leads<br>to their being able to calculate the probability amplitudes for the Schur–Weyl basis states appearing in the output of<br>the Quantum Schur Transform. While the method that they use to construct the Schur–Weyl states – by introducing<br>a directed graph of Gel’fand–Tsetlin patterns – is similar in nature to the left-to-right version of the Schur–Weyl<br>branching rule that we described above, it differs significantly to ours in that it is derived from the representation<br>theory of the unitary group, and is given without proof, whereas our method is derived from the Okounkov–Vershik<br>formulation of the representation theory of the symmetric group, which leads not only to a left-to-right version but<br>also to a right-to-left version of the Schur–Weyl branching rule. Moreover, we provide a proof of all of our results.

8 Conclusion

In the preceding sections, we have shown how the Okounkov–Vershik formulation of the representation theory of the<br>symmetric groups $S_n$ can be adapted naturally to the Schur–Weyl duality, resulting in a new way of understanding<br>how the duality can be used to perform the Quantum Schur Transform.

In doing so, we have provided a different theoretical foundation for this transformation from those that have appeared<br>in the literature previously. In particular, we have constructed a new multigraph called the Schur–Weyl–Young graph,<br>and shown how this corresponds to a branching rule for the Schur–Weyl basis states of the $n$-fold tensor product space<br>$(\mathbb{C}^d)^\otimes n$.

We have also found some simple rules for calculating the transition amplitudes between standard Weyl tableaux in<br>adjacent levels of the Schur–Weyl–Young graph for the case $d = 2$, which are based entirely on looking at the entries<br>in the Weyl tableaux themselves. We have called these rules the Pattern Rules. It is left as an interesting challenge to<br>the reader to try to generalise the Pattern Rules to any $d$.

Our contributions now make it possible for anyone to change basis between the computational basis of $(\mathbb{C}^d)^\otimes n$ and<br>the Schur–Weyl basis $(\mathbb{C}^d)^\otimes n$ for any $d, n$ very quickly. Previously, this had been a slow and painstaking process.<br>As a result, the attention in quantum computing can now turn to working out which Schur–Weyl states are the most<br>desirable to prepare on a quantum computer, and to construct new quantum algorithms involving their use for the<br>purpose of solving information processing problems.

9 Acknowledgments

The author would like to thank his PhD supervisor Professor William J. Knottenbelt for being generous with his time<br>throughout the author’s period of research prior to the publication of this paper.

This work was funded by the Doctoral Scholarship for Applied Research which was awarded to the author under<br>Imperial College London’s Department of Computing Applied Research scheme. This work will form part of the<br>author’s PhD thesis at Imperial College London.

References

[1] D. Bacon, I. L. Chuang, and A. W. Harrow, Efficient Quantum Circuits for Schur and Clebsch-Gordan<br>Transforms, Physical Review Letters, 97 (2006).

[2] D. Bacon, I. L. Chuang, and A. W. Harrow, The Quantum Schur Transform: I. Efficient Qudit Circuits,<br>Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms, (2006).
[3] L. C. Biedenharn and J. D. Louck, *A Pattern Calculus for Tensor Operators in the Unitary Groups*, Communications in Mathematical Physics, 8 (1968), pp. 89–131.

[4] T. Ceccherini-Silberstein, F. Scarabotti, and F. Tolli, *Representation Theory of the Symmetric Groups*, Cambridge University Press, 2010.

[5] G. de Beauregard Robinson, *The Collected Papers of Alfred Young 1873-1940*, University of Toronto Press, 1977.

[6] R. Goodman and N. R. Wallach, *Symmetry, Representations and Invariants*, Springer, 2009.

[7] A. W. Harrow, *Applications of Coherent Classical Communication and the Schur Transform to Quantum Information Theory*, PhD thesis, Massachusetts Institute of Technology, 2005.

[8] G. James, *The Representation Theory of the Symmetric Group*, Cambridge University Press, 1981.

[9] M. Kaczor and P. Jakubczyk, *A new approach to the construction of Schur-Weyl states*, 2020.

[10] R. Kondor, *Group Theoretical Methods in Machine Learning*, PhD thesis, Columbia University, 2008.

[11] H. Krovi, *An efficient high dimensional quantum Schur transform*, Quantum, 3 (2019), p. 122.

[12] J. D. Louck, *Unitary Symmetry and Combinatorics*, World Scientific, 2008.

[13] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, 2010.

[14] D. J. Rowe, M. J. Carvalho, and J. Repka, *Dual pairing of symmetry and dynamical groups in physics*, Reviews of Modern Physics, 84 (2012), pp. 711–757.

[15] B. E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, Springer, 2000.

[16] E. Segal, *Group Representation Theory*. Course Notes for Imperial College London, 2014.

[17] S. Sternberg, *Group Theory and Physics*, Cambridge University Press, 1994.

[18] A. M. Vershik and A. Y. Okounkov, *A New Approach to the Representation Theory of the Symmetric Groups II*, Zapiski Seminarod POMI (In Russian) v.307, (1996).

[19] ———, *A New Approach to the Representation Theory of the Symmetric Groups II*, Journal of Mathematical Sciences, (2005).

[20] J. von Neumann, *Approximative Properties of Matrices of High Finite Order*, Portugal. Math., 3 (1942), pp. 1–62.

[21] M. Walter, *Symmetry and Quantum Information*. Course Notes for the University of Amsterdam, 2018.

[22] H. Weyl, *The Theory of Groups and Quantum Mechanics*, Dover Publications, 1950.

[23] P. Woit, *Quantum Theory, Groups and Representations: An Introduction*. Revised and Expanded Version, Under Construction, 2021.

[24] H. Zheng, Z. Li, J. Liu, S. Strelchuk, and R. Kondor, *Speeding up Learning Quantum States through Group Equivariant Convolutional Quantum Ansätze*, 2021.
Appendix A  Proof of the Pattern Rules for $d = 2$

We claimed in Section 4 that the Pattern Rules give the transition amplitude (denoted by $\alpha$ in the following) between standard Weyl tableaux of the following shapes:

Rule 1: $(n-1,0) \leftrightarrow (n,0)$

Rule 2: $(n-1,0) \leftrightarrow (n-1,1)$

Rule 3: $(n-k-1,k) \leftrightarrow (n-k,k)$

Rule 4: $(n-k,k-1) \leftrightarrow (n-k,k)$

Since these are the only possible transitions between shapes for the case $d = 2$ in the Schur–Weyl–Young graph, it is enough to show that our Pattern Rules give the same results as those coming from applying Louck’s formula (the terms of which are given in (17) and (18)).

We prove this by using the equivalent Gelfand–Tsetlin patterns for each rule.

Given that Louck’s formula considers the partial hooks for each entry of a Gelfand–Tsetlin pattern, we define, for convenience, the pattern of partial hooks for the Gelfand–Tsetlin pattern

\[
\begin{pmatrix}
m_{1,2}^{n-1} & m_{2,2}^{n-1} \\
m_{1,1}^{n-1} & m_{2,2}^{n-1}
\end{pmatrix}
\]

to be

\[
\begin{pmatrix}
p_{1,2} & p_{2,2} \\
p_{1,1} & p_{2,2}
\end{pmatrix}
= \begin{pmatrix}
m_{1,2}^{n-1} + 1 & m_{2,2}^{n-1} \\
m_{1,1}^{n-1} & m_{2,2}^{n-1}
\end{pmatrix}
\]

(32)

where $p_{i,j}$ is the partial hook, defined to be equal to $m_{i,j}^{n-1} + j - i$.

We now take each rule in turn.

**Rule 1:** We consider the edge between Gelfand–Tsetlin patterns of the form

\[
\begin{pmatrix}
n-1 \\
m_{1,1}^{n-1}
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
n \\
m_{1,1}^{n-1}
\end{pmatrix}
\]

a) If $m_{1,1}^{n-1} = m_{1,1}^{n}$, then, from (16), we have that $n = 2$, $k = 2$ and $t_2 = 1$, and so, as

\[
\begin{pmatrix}
p_{1,2} & p_{2,2} \\
p_{1,1} & p_{2,2}
\end{pmatrix}
= \begin{pmatrix}
n \\
m_{1,1}^{n-1}
\end{pmatrix}
\]

we see that Louck’s formula gives us that

\[
\alpha = \frac{|p_{1,2} - p_{1,1}|}{|p_{1,2} - p_{2,2}|} = \sqrt{\frac{n - m_{1,1}^{n-1}}{n}} = \sqrt{\frac{n - m_{1,1}^{n}}{n}}
\]

(33)

since $m_{1,1}^{n-1} = m_{1,1}^{n}$, This is the same result as (21).

b) Else we must have that $m_{1,1}^{n-1} + 1 = m_{1,1}^{n}$, and so, from (16), we have that $n = 2$, $k = 1$, $t_1 = 1$ and $t_2 = 1$. Hence, as

\[
\begin{pmatrix}
p_{1,2} & p_{2,2} \\
p_{1,1} & p_{2,2}
\end{pmatrix}
= \begin{pmatrix}
n \\
m_{1,1}^{n-1}
\end{pmatrix}
\]

we see that Louck’s formula gives us that

\[
\alpha = \sqrt{\frac{|p_{1,2} - p_{2,2} + 1|}{|p_{1,2} - p_{2,2}|}} = \sqrt{\frac{m_{1,1}^{n-1} + 1}{n}} = \sqrt{\frac{m_{1,1}^{n}}{n}}
\]

(34)

This is the same result as (22).

**Rule 2:** We consider the edge between Gelfand–Tsetlin patterns of the form

\[
\begin{pmatrix}
n-1 \\
m_{1,1}^{n-1}
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
n-1 \\
m_{1,1}^{n}
\end{pmatrix}
\]

a) If $m_{1,1}^{n-1} = m_{1,1}^{n}$, then, from (16), we have that $n = 2$, $k = 2$ and $t_2 = 2$, and so, as

\[
\begin{pmatrix}
p_{1,2} & p_{2,2} \\
p_{1,1} & p_{2,2}
\end{pmatrix}
= \begin{pmatrix}
n \\
m_{1,1}^{n-1}
\end{pmatrix}
\]
we see that Louck’s formula gives us that

$$\alpha = \sqrt{ \frac{p_{2,2} - p_{1,1}}{p_{2,2} - p_{1,2}}} = \sqrt{ \frac{m_{1,1}^{n-1}}{m_{1,1}^n}} = \sqrt{ \frac{m_{1,1}^n}{n}}$$

(35)

since $m_{1,1}^{n-1} = m_{1,1}^n$. This is the same result as (25).

b) Else we must have that $m_{1,1}^{n-1} + 1 = m_{1,1}^n$, and so, from (16), we have that $n = 2$, $k = 1$, $t_1 = 1$ and $t_2 = 2$. Hence, as

$$\left( \begin{array}{c} p_{1,2} \\ p_{1,1} \end{array} \right) = \left( \begin{array}{c} n \\ m_{1,1}^{n-1} \end{array} \right)$$

we see that Louck’s formula gives us that

$$\alpha = -\sqrt{ \frac{p_{1,1} - p_{1,2} + 1}{p_{2,2} - p_{1,2}}} = - \sqrt{ \frac{m_{1,1}^{n-1} - n + 1}{n}} = \sqrt{ \frac{n - m_{1,1}^n}{n}}$$

(36)

This is the same result as (26).

**Rule 3:** We consider the edge between Gelfand–Tsetlin patterns of the form

$$\left( \begin{array}{c} n-k-1 \\ m_{1,1}^{n-1} \end{array} \right) \leftrightarrow \left( \begin{array}{c} n-k \\ m_{1,1}^n \end{array} \right)$$

(37)

a) If $m_{1,1}^{n-1} = m_{1,1}^n$, then, from (16), we have that $n = 2$, $k = 2$ and $t_2 = 1$, and so, as

$$\left( \begin{array}{c} p_{1,2} \\ p_{1,1} \end{array} \right) = \left( \begin{array}{c} n-k \\ m_{1,1}^{n-1} \end{array} \right)$$

we see that Louck’s formula gives us that

$$\alpha = \sqrt{ \frac{p_{1,2} - p_{1,1}}{p_{2,2} - p_{1,2}}} = \sqrt{ \frac{n - k - m_{1,1}^{n-1}}{n - k - k}} = \sqrt{ \frac{(n - 2k) - (m_{1,1}^{n-1} - k)}{(n - 2k) - 0}}$$

(38)

This is the same result that we would have obtained had our edge been

$$\left( \begin{array}{c} n-2k-1 \\ m_{1,1}^{n-1} - k \end{array} \right) \leftrightarrow \left( \begin{array}{c} n-2k \\ m_{1,1}^n - k \end{array} \right)$$

(39)

with $m_{1,1}^{n-1} - k = m_{1,1}^n - k$, since, from (16), we have, for (39), that $n = 2$, $k = 2$ and $t_2 = 1$, and our pattern of partial hooks is

$$\left( \begin{array}{c} n-2k \\ m_{1,1}^{n-1} - k \end{array} \right)$$

Consequently this solution corresponds exactly to subtracting $k$ from every entry of our Gelfand–Tsetlin patterns (37) and applying Rule 1a) to (39).

As we have already verified this rule, it means that Louck’s formula and Rule 3a) give the same result.

b) Else we must have that $m_{1,1}^{n-1} + 1 = m_{1,1}^n$, and so, from (16), we have that $n = 2$, $k = 1$, $t_1 = 1$ and $t_2 = 1$. Hence, as

$$\left( \begin{array}{c} p_{1,2} \\ p_{1,1} \end{array} \right) = \left( \begin{array}{c} n-k \\ m_{1,1}^{n-1} \end{array} \right)$$

we see that Louck’s formula gives us that

$$\alpha = \sqrt{ \frac{p_{1,1} - p_{2,2} + 1}{p_{2,1} - p_{2,2}}} = \sqrt{ \frac{m_{1,1}^{n-1} - k + 1}{n - k - k}} = \sqrt{ \frac{(m_{1,1}^{n-1} - k) - 0 + 1}{(n - 2k) - 0}}$$

(40)

This is the same result that we would have obtained had our edge been

$$\left( \begin{array}{c} n-2k-1 \\ m_{1,1}^{n-1} - k \end{array} \right) \leftrightarrow \left( \begin{array}{c} n-2k \\ m_{1,1}^n - k \end{array} \right)$$

(41)

with $m_{1,1}^{n-1} - k + 1 = m_{1,1}^n - k$, since, from (16), we have, for (41), that $n = 2$, $k = 1$, $t_1 = 1$ and $t_2 = 1$, and our pattern of partial hooks is

$$\left( \begin{array}{c} n-2k \\ m_{1,1}^{n-1} - k \end{array} \right)$$
Consequently this solution corresponds exactly to subtracting $k$ from every entry of our Gelfand–Tsetlin patterns (37) and applying Rule 1b) to (41).

As we have already verified this rule, it means that Louck’s formula and Rule 3b) give the same result.

**Rule 4:** We consider the edge between Gelfand–Tsetlin patterns of the form

$$\left( \frac{n-k}{m_{n,1}^{k-1}} \right) \leftrightarrow \left( \frac{n-k}{m_{1,1}^{k}} \right)$$  \hspace{1cm} (42)

a) If $m_{1,1}^{n-1} = m_{1,1}^{1}$, then, from (16), we have that $n = 2, k = 2$ and $t_2 = 2$, and so, as

$$\left( p_{1,2} \quad p_{1,2} \right) = \left( \frac{n-k+1}{m_{1,1}^{n-1}} \right)$$

we see that Louck’s formula gives us that

$$\alpha = \sqrt{\frac{p_{2,2} - p_{1,1}}{p_{2,2} - p_{1,1}}} = \sqrt{\frac{(k-1) - m_{1,1}^{n-1}}{(k-1) - (n-k+1)}} = \sqrt{0 - (n^{n-1} - k + 1) \over 0 - (n - 2k + 2)}$$  \hspace{1cm} (43)

This is the same result that we would have obtained had our edge been

$$\left( \frac{n-2k+1}{m_{1,1}^{n-1}-k+1} \right) \leftrightarrow \left( \frac{n-2k+1}{m_{1,1}^{n-1}-k+1} \right)$$  \hspace{1cm} (44)

with $m_{1,1}^{n-1} - k + 1 = m_{1,1}^{1} - k + 1$, since, from (16), we have, for (44), that $n = 2, k = 2$ and $t_2 = 2$, and our pattern of partial hooks is

$$\left( \frac{n-2k}{m_{1,1}^{n-1}-k+1} \right)$$

Consequently this solution corresponds exactly to subtracting $k - 1$ from every entry of our Gelfand–Tsetlin patterns (42) and applying Rule 2a) to (44).

As we have already verified this rule, it means that Louck’s formula and Rule 4a) give the same result.

b) Else we must have that $m_{1,1}^{n-1} + 1 = m_{1,1}^{1}$, and so, from (16), we have that $n = 2, k = 1, t_1 = 1$ and $t_2 = 2$. Hence, as

$$\left( p_{1,2} \quad p_{1,2} \right) = \left( \frac{n-k+1}{m_{1,1}^{n-1}} \right)$$

we see that Louck’s formula gives us that

$$\alpha = -\sqrt{\frac{p_{1,1} - p_{1,2} + 1}{p_{2,2} - p_{1,2}}} = \sqrt{\frac{m_{1,1}^{n-1} - (n-k+1) + 1}{(k-1) - (n-k+1)}} = \sqrt{\frac{(m_{1,1}^{n-1} - k + 1) - (n - 2k + 2) + 1}{0 - (n - 2k + 2)}}$$  \hspace{1cm} (45)

This is the same result that we would have obtained had our edge been

$$\left( \frac{n-2k+1}{m_{1,1}^{n-1}-k+1} \right) \leftrightarrow \left( \frac{n-2k+1}{m_{1,1}^{n-1}-k+1} \right)$$  \hspace{1cm} (46)

with $m_{1,1}^{n-1} - k + 2 = m_{1,1}^{n} - k + 1$, since, from (16), we have, for (46), that $n = 2, k = 1, t_1 = 1$ and $t_2 = 2$, and our pattern of partial hooks is

$$\left( \frac{n-2k+2}{m_{1,1}^{n-1}-k+1} \right)$$

Consequently this solution corresponds exactly to subtracting $k - 1$ from every entry of our Gelfand–Tsetlin patterns (42) and applying Rule 1b) to (46).

As we have already verified this rule, it means that Louck’s formula and Rule 4b) give the same result.

Since we have shown that the Pattern Rules for $d = 2$ and Louck’s formula give the same transition amplitudes for all standard Weyl tableaux connected by an edge in the Schur–Weyl–Young graph, we have consequently proven our claim.