Averaging principles for a class of non-autonomous slow-fast systems of SPDEs with jumps

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Abstract

In this paper, we aim to develop the averaging principle for a slow-fast system of stochastic reaction-diffusion equations driven by Poisson random measures. The coefficients of the equation are assumed to be functions of time, and some of them are periodic or almost periodic. Therefore, the Poisson term needs to be processed, and a new averaged equation needs to be given. For this reason, the existence of time-dependent evolution family of measures associated with the fast equation is studied, and proved that it is almost periodic. Next, according to the characteristics of almost periodic functions, the averaged coefficient is defined by the evolution family of measures, and the averaged equation is given. Finally, the validity of the averaging principle is verified by using the Khasminskii method.

Keywords. Non-autonomous; Averaging principles; Stochastic reaction-diffusion equations; Poisson random measures; Evolution families of measures

Mathematics subject classification. 70K70, 60H15, 60G51, 34K33

1. Introduction

The slow-fast systems are widely encountered in biology, ecology and other application areas. In this paper, we are concerned with the following non-autonomous slow-fast systems of stochastic partial differential equations (SPDEs) on a bounded domain \(\mathcal{O}\) of \(\mathbb{R}^d\) (\(d \geq 1\)):

\[
\begin{aligned}
\frac{\partial u}{\partial t}(t, \xi) &= \mathcal{A}_1(t) u(t, \xi) + b_1(t, \xi, u(t, \xi), v(t, \xi)) + f_1(t, \xi, u(t, \xi)) \frac{\partial \mathcal{Q}_1}{\partial t}(t, \xi) \\
&\quad + \int_{\mathbb{Z}} g_1(t, \xi, u(t, \xi), z) \frac{\partial \mathcal{N}_1}{\partial t}(t, \xi, dz), \\
\frac{\partial v}{\partial t}(t, \xi) &= \frac{1}{\epsilon} \left[ (\mathcal{A}_2(t) - \alpha) v(t, \xi) + b_2(t, \xi, u(t, \xi), v(t, \xi)) \right] \\
&\quad + \frac{1}{\sqrt{\epsilon}} f_2(t, \xi, u(t, \xi), v(t, \xi)) \frac{\partial \mathcal{Q}_2}{\partial t}(t, \xi) \\
&\quad + \int_{\mathbb{Z}} g_2(t, \xi, u(t, \xi), v(t, \xi), z) \frac{\partial \mathcal{N}_2}{\partial t}(t, \xi, dz), \\
\mathcal{N}_1 u(t, \xi) &= \mathcal{N}_2 v(t, \xi) = 0, \quad t \geq 0, \quad \xi \in \partial \mathcal{O},
\end{aligned}
\]

\(u(0, \xi) = x(\xi), \quad v(0, \xi) = y(\xi), \quad \xi \in \mathcal{O},\)

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where $\epsilon \ll 1$ is a positive parameter and $\alpha$ is a sufficiently large fixed constant. The operators $\mathcal{N}_1$ and $\mathcal{N}_2$ are boundary operators. The stochastic perturbations $\omega^{Q_1}, \omega^{Q_2}$ and $\bar{N}_1, \bar{N}_2$ are mutually independent Wiener processes and Poisson random measures on the same complete stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. All of them will be described in Section 2. For $i = 1, 2$, the operator $\mathcal{A}_i(t)$ and the functions $b_i, f_i, g_i$ depend on time, and we assume that the operator $\mathcal{A}_2(t)$ is periodic and the functions $b_1, b_2, f_2, g_2$ are almost periodic.

The goal of this paper is to establish an effective approximations for the slow equation of the original system (1.1) by using the averaging principle. The averaged equation is obtained as following

$$
\begin{align*}
\frac{\partial \bar{u}}{\partial t} (t, \xi) &= \mathcal{A}_1 (t) \bar{u} (t, \xi) + \bar{B}_1 (\bar{u} (t)) (\xi) + f_1 (t, \xi, \bar{u} (t, \xi)) \frac{\partial \omega^{Q_1}}{\partial t} (t, \xi) \\
&\quad + \int_{\mathbb{Z}} g_1 (t, \xi, \bar{u} (t, \xi), z) \frac{\partial \bar{N}_1}{\partial t} (t, \xi, dz), \\
\bar{u} (0, \xi) &= x (\xi), \quad \xi \in \mathcal{O}, \quad \mathcal{N}_1 \bar{u} (t, \xi) = 0, \quad t \geq 0, \quad \xi \in \partial \mathcal{O},
\end{align*}
$$

where $\bar{B}_1$ is the averaged coefficient, which will be given in equation (1.5). To demonstrate the validity of the averaging principle, we prove that for any $T > 0$ and $\eta > 0$, it yields

$$
\lim_{\epsilon \to 0} \mathbb{P} \left( \sup_{t \in [0, T]} \| u_{\epsilon} (t) - \bar{u} (t) \|_{L^2(\mathcal{O})} > \eta \right) = 0,
$$

where $\bar{u}$ is the solution of the averaged equation (1.2).

The theory of the averaging principle has a long history, originated by Laplace and Lagrange. It has been applied in celestial mechanics, oscillation theory, radiophysics and other fields. The firstly rigorous results for the deterministic case were given by Bogolyubov and Mitropolskii [1]. Moreover, Volosov [2] and Besjes [3] also promoted the development of the averaging principle. Then, great interests have appeared in its application to dynamical systems under random perturbations. An important contribution was that, in 1968, Khasminskii [4] originally proposed the averaging principle for stochastic differential equations (SDEs) driven by Brownian motion. Since then, the averaging principle has been an active area of research. Many studies on the averaging principle of SDEs have been presented, e.g., Givon [5], Freidlin and Wentzell [6], Duan [7], Thompson [8], Xu and his co-workers [9–11]. Recently, effective approximation for slow-fast SPDEs has been received extensive attention. Cerrai [12, 13] investigated the validity of the averaging principle for a class of stochastic reaction-diffusion equations with multiplicative noise. In addition, Wang and Roberts [14], Pei and Xu [15–17], Xu and Miao [18] also concerned the averaging principles for slow-fast SPDEs.

The above-mentioned papers mainly considered autonomous systems. For autonomous systems, as long as the initial value is given, the solution of which only depends on the duration of time, not on the selection of the initial time. However, if the initial time is different, the solution of non-autonomous equations with the same initial data will also be different. Therefore, compared with autonomous systems, the dynamic behavior of non-autonomous systems is more complex, which can portray more actual models. Chepyzhov and Vishik [19] studied the long time behavior of non-autonomous dissipative system. Carvalho [20] dealt with the theory of attractors for non-autonomous dynamical systems. Bunder and Roberts [21] considered the discrete modelling of non-autonomous PDEs.
In 2017, the validity of an averaging principle has been presented for non-autonomous slow-fast system of stochastic reaction-diffusion equations by Cerrai [22]. But, the system of this paper was driven by Gaussian noises, which is considered as an ideal noise source and can only simulate fluctuations near the mean value. Actually, due to the complexity of the external environment, random noise sources encountered in practical fields usually exhibit non-Gaussian properties, which may cause sharply fluctuations. It should be pointed out that Poisson noise, one of the most ubiquitous noise sources in many fields [23–25], can provide a good mathematical model to describe discontinuous random processes, some large moves and unpredictable events [26–28]. So, in this paper, we are devoted to developing the averaging principle for non-autonomous systems of reaction-diffusion equations driven by Wiener processes and Poisson random measures.

The key to using the averaging principle to analyze system (1.1) is the fast equation with a frozen slow component $x \in L^2(\mathcal{O})$:

$$
\begin{aligned}
&\frac{\partial v^{x,y}}{\partial t}(t,\xi) = [(A_2(t) - \alpha) v^{x,y}(t,\xi) + b_2(t,\xi, x(\xi), v^{x,y}(t,\xi))] \\
&\quad + f_2(t,\xi, x(\xi),v^{x,y}(t,\xi)) \frac{\partial Q_2}{\partial t}(t,\xi) \\
&\quad + \int_{\mathbb{R}} g_2(t,\xi, x(\xi),v^{x,y}(t,\xi), z) \frac{\partial N_2}{\partial t}(t,\xi,dz),
\end{aligned}
$$

(1.4)

By dealing with the Poisson terms, we prove that an evolution family of measures $(\mu^x_t; t \in \mathbb{R})$ on $L^2(\mathcal{O})$ for the fast equation (1.4) exists. Similar to study [22], assuming that $A_2(t)$ is periodic and $b_2,f_2,g_2$ are almost periodic, we prove that the evolution family of measures is almost periodic. With the aid of the theorem [29, Theorem 2.10], we prove that the family of functions

$$
\left\{ t \in \mathbb{R} \mapsto \int_{L^2(\mathcal{O})} B_1(t,x,y) \mu^x_t \, (dy) \right\}
$$

is uniformly almost periodic for any $x \in \mathbb{H}$, where $B_1(t,x,y)(\xi) = b_1(t,\xi, x(\xi), y(\xi))$ for any $x,y \in L^2(\mathcal{O})$ and $\xi \in \mathcal{O}$.

According to the characteristics of almost periodic function [22, Theorem 3.4], we define the averaged coefficient $\bar{B}_1$ as following

$$
\bar{B}_1(x) := \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{L^2(\mathcal{O})} B_1(t,x,y) \mu^x_t \, (dy) \, dt, \quad x \in L^2(\mathcal{O}).
$$

(1.5)

Finally, the averaged equation is obtained through the averaged coefficient $\bar{B}_1$. Using the classical Khasminskii method to the present situation, the averaging principle is effective.

The above-mentioned notations will be given in Section 2, and in this paper, $c > 0$ below with or without subscripts will represent a universal constant whose value may vary in different occasions.

2. Notations, assumptions and preliminaries

Let $\mathcal{O}$ be a bounded domain of $\mathbb{R}^d$ ($d \geq 1$) having a smooth boundary. In this paper, we denote $\mathbb{H}$ the separable Hilbert space $L^2(\mathcal{O})$, endowed with the usual scalar product

$$
\langle x, y \rangle_{\mathbb{H}} = \int_{\mathcal{O}} x(\xi) y(\xi) \, d\xi
$$
and with the corresponding norm $\| \cdot \|_\mathbb{H}$. The norm in $L^\infty (\mathcal{O})$ will be denoted by $\| \cdot \|_\infty$.

Furthermore, the subspace $\mathcal{D}((A)^\theta)$ [30–32] of the generator $A$ is dense in $\mathbb{H}$, and endowed with the norm

$$\| A \|_\theta = \| (A)^\theta A \|_\mathbb{H}, \quad A \in \mathcal{D}((A)^\theta),$$

for $0 \leq \theta < 1$, $0 < t \leq T$. According to [33, Theorem 6.13], there exists a $c_\theta > 0$, such that

$$\| (A)^\theta e^{At} \|_\mathbb{H} \leq c_\theta t^{-\theta}.$$ 

Denote by $B_b (\mathbb{H})$ the Banach space of the bounded Borel functions $\varphi : \mathbb{H} \to \mathbb{R}$, endowed with the sup-norm

$$\| \varphi \|_0 := \sup_{x \in \mathbb{H}} |\varphi (x)|,$$

and $\mathcal{L}_b (\mathbb{H})$ is the subspace of the uniformly continuous mappings.

We shall denote that $\mathcal{L} (\mathbb{H})$ is the space of the bounded linear operators in $\mathbb{H}$, and denote $\mathcal{L}_2 (\mathbb{H})$ the subspace of Hilbert-Schmidt operators, endowed with the norm

$$\| Q \|_2 = \sqrt{\text{Tr} [Q^*Q]}.$$ 

In the slow-fast system (1.1), the Gaussian noises $\partial \omega Q_1 / \partial t (t, \xi)$ and $\partial \omega Q_2 / \partial t (t, \xi)$ are assumed to be white in time and colored in space in the case of space dimension $d > 1$, for $t \geq 0$ and $\xi \in \mathcal{O}$. And, $\omega Q_i (t, \xi) (i = 1, 2)$ is the cylindrical Wiener processes, defined as

$$\omega Q_i (t, \xi) = \sum_{k=1}^{\infty} Q_i e_k (\xi) \beta_k (t), \quad i = 1, 2,$$

where $\{ e_k \}_{k \in \mathbb{N}}$ is a complete orthonormal basis in $\mathbb{H}$, $\{ \beta_k (t) \}_{k \in \mathbb{N}}$ is a sequence of mutually independent standard Brownian motion defined on the same complete stochastic basis $(\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P})$, $Q_i$ is a bounded linear operator on $\mathbb{H}$.

Next, we give the definitions of Poisson random measures $\tilde{N}_1 (dt, dz)$ and $\tilde{N}_2^\epsilon (dt, dz)$. Let $(\mathbb{Z}, \mathcal{B} (\mathbb{Z}))$ be a given measurable space and $v (dz)$ be a $\sigma$-finite measure on it. $D_{p_i^1}$, $i = 1, 2$ are two countable subsets of $\mathbb{R}_+$. Moreover, let $p_i^1, t \in D_{p_i^1}$ be a stationary $\mathcal{F}_t$-adapted Poisson point process on $\mathbb{Z}$ with the characteristic $v$, and $p_i^2, t \in D_{p_i^2}$ be the other stationary $\mathcal{F}_t$-adapted Poisson point process on $\mathbb{Z}$ with the characteristic $v/\epsilon$. Denote by $N_i (dt, dz), i = 1, 2$ the Poisson counting measure associated with $p_i$, i.e.,

$$N_i (t, A) := \sum_{s \in D_{p_i^1}, s \leq t} I_A (p_i^1), \quad i = 1, 2.$$ 

Let us denote the two independent compensated Poisson measures

$$\tilde{N}_1 (dt, dz) := N_1 (dt, dz) - v_1 (dz) dt$$

and

$$\tilde{N}_2^\epsilon (dt, dz) := N_2 (dt, dz) - \frac{1}{\epsilon} v_2 (dz) dt,$$
where \( v_1 (dz) dt \) and \( v_2 (dz) dt \) are the compensators.

Refer to [25, 34] for a more detailed description of the stochastic integral with respect to a cylindrical Wiener process and Poisson random measure.

For any \( t \in \mathbb{R} \), the operators \( \mathcal{A}_1 (t) \) and \( \mathcal{A}_2 (t) \) are second order uniformly elliptic operators, having continuous coefficients on \( \mathcal{O} \). The operators \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) are the boundary operators, which can be either the identity operator (Dirichlet boundary condition) or a first order operator (coefficients satisfying a uniform nontangentiality condition). We shall assume that the operator \( \mathcal{A}(t) \) has the following form

\[
\mathcal{A}_i (t) = \gamma_i (t) \mathcal{A}_i + \mathcal{L}_i (t), \quad t \in \mathbb{R}, \; i = 1, 2,
\]

where \( \mathcal{A}_i \) is a second order uniformly elliptic operator with continuous coefficients on \( \mathcal{O} \), which is independent of \( t \). In addition, \( \mathcal{L}_i (t) \) is a first order differential operator, has the form

\[
\mathcal{L}_i (t, \xi) u (\xi) = \langle l_i (t, \xi), \nabla u (\xi) \rangle_{\mathbb{R}^d}, \quad t \in \mathbb{R}, \; \xi \in \mathcal{O}.
\]

The realizations of the differential operators \( \mathcal{A}_i \) and \( \mathcal{L}_i \) in \( \mathbb{H} \) is \( A_i \) and \( L_i \). Moreover, \( A_1 \) and \( A_2 \) generate two analytic semigroups \( e^{tA_1} \) and \( e^{tA_2} \) respectively.

Now, we give the following assumptions:

(A1) (a) For \( i = 1, 2 \), the function \( \gamma_i : \mathbb{R} \to \mathbb{R} \) is continuous, and there exist \( \gamma_0, \gamma > 0 \) such that

\[
\gamma_0 \leq \gamma_i (t) \leq \gamma, \quad t \in \mathbb{R}.
\]

(b) For \( i = 1, 2 \), the function \( l_i : \mathbb{R} \times \mathcal{O} \to \mathbb{R}^d \) is continuous and bounded.

(A2) For \( i = 1, 2 \), there exist a complete orthonormal system \( \{ e_{i,k} \}_{k \in \mathbb{N}} \) in \( \mathbb{H} \) and two sequences of nonnegative real numbers \( \{ \alpha_{i,k} \}_{k \in \mathbb{N}} \) and \( \{ \lambda_{i,k} \}_{k \in \mathbb{N}} \) such that

\[
A_i e_{i,k} = -\alpha_{i,k} e_{i,k}, \quad Q_i e_{i,k} = \lambda_{i,k} e_{i,k}, \quad k \geq 1,
\]

and

\[
\kappa_i := \sum_{k=1}^{\infty} \lambda_{i,k}^\rho |e_{i,k}|^2_{\infty} < \infty, \quad \zeta_i := \sum_{k=1}^{\infty} \alpha_{i,k}^{-\beta_i} |e_{i,k}|^2_{\infty} < \infty,
\]

for some constants \( \rho_i \in (2, +\infty) \) and \( \beta_i \in (0, +\infty) \) such that

\[
[\beta_i (\rho_i - 2)]/\rho_i < 1.
\]

(A3) The mappings \( b_1 : \mathbb{R} \times \mathcal{O} \times \mathbb{R}^2 \to \mathbb{R} \), \( f_1 : \mathbb{R} \times \mathcal{O} \times \mathbb{R} \to \mathbb{R} \), \( g_1 : \mathbb{R} \times \mathcal{O} \times \mathbb{R} \times \mathbb{Z} \to \mathbb{R} \) are measurable, and the mappings \( b_1 (t, \xi, \cdot) : \mathbb{R}^2 \to \mathbb{R} \), \( f_1 (t, \xi, \cdot) : \mathbb{R} \to \mathbb{R} \), \( g_1 (t, \xi, \cdot, z) : \mathbb{R} \to \mathbb{R} \) are Lipschitz continuous and linearly growing, uniformly with respect to \( (t, \xi, z) \in \mathbb{R} \times \mathcal{O} \times \mathbb{Z} \). Moreover, for all \( p \geq 1 \), there exist positive constants \( c_1, c_2 \), such that for all \( x_1, x_2 \in \mathbb{R} \), we have

\[
\sup_{(t, \xi) \in \mathbb{R} \times \mathcal{O}} \int_{\mathbb{Z}} |g_1 (t, \xi, x_1, z)|^p \nu_1 (dz) \leq c_1 (1 + |x_1|^p),
\]

\[
\sup_{(t, \xi) \in \mathbb{R} \times \mathcal{O}} \int_{\mathbb{Z}} |g_1 (t, \xi, x_1, z) - g_1 (t, \xi, x_2, z)|^p \nu_1 (dz) \leq c_2 |x_1 - x_2|^p.
\]
(A4) The mappings $b_2 : \mathbb{R} \times \mathcal{O} \times \mathbb{R}^2 \to \mathbb{R}$, $f_2 : \mathbb{R} \times \mathcal{O} \times \mathbb{R}^2 \to \mathbb{R}$, $g_2 : \mathbb{R} \times \mathcal{O} \times \mathbb{R}^2 \times \mathbb{Z} \to \mathbb{R}$ are measurable, and the mappings $b_2 (t, \xi, \cdot) : \mathbb{R}^2 \to \mathbb{R}$, $f_2 (t, \xi, \cdot) : \mathbb{R}^2 \to \mathbb{R}$, $g_2 (t, \xi, \cdot, z) : \mathbb{R}^2 \to \mathbb{R}$ are Lipschitz continuous and linearly growing, uniformly with respect to $(t, \xi, z) \in \mathbb{R} \times \mathcal{O} \times \mathbb{Z}$. Moreover, for all $q \geq 1$, there exist positive constants $c_3, c_4$, such that for all $(x_i, y_i) \in \mathbb{R}^2$, $i = 1, 2$, we have

$$\sup_{(t, \xi) \in \mathbb{R} \times \mathcal{O}} \int_{\mathbb{Z}} |g_2 (t, \xi, x_1, y_1, z)|^q v_2 (dz) \leq c_3 (1 + |x_1|^q + |y_1|^q),$$

$$\sup_{(t, \xi) \in \mathbb{R} \times \mathcal{O}} \int_{\mathbb{Z}} |g_2 (t, \xi, x_1, y_1, z) - g_2 (t, \xi, x_2, y_2, z)|^q v_2 (dz) \leq c_4 (|x_1 - x_2|^q + |y_1 - y_2|^q).$$

**Remark 2.1.** For any $(t, \xi) \in \mathbb{R} \times \mathcal{O}$ and $x, y, h \in \mathbb{H}, z \in \mathbb{Z}$, we shall set

$$B_1 (t, x, y) (\xi) := b_1 (t, \xi, x (\xi), y (\xi)), \quad B_2 (t, x, y) (\xi) := b_2 (t, \xi, x (\xi), y (\xi)),
$$

$$[F_1 (t, x) h] (\xi) := f_1 (t, \xi, x (\xi)) h (\xi), \quad [F_2 (t, x) y h] (\xi) := f_2 (t, \xi, x (\xi), y (\xi)) h (\xi),$$

$$[G_1 (t, x, z) h] (\xi) := g_1 (t, \xi, x (\xi), z) h (\xi), \quad [G_2 (t, x, y, z) h] (\xi) := g_2 (t, \xi, x (\xi), y (\xi), z) h (\xi),$$

due to (A3) and (A4), for any fixed $(t, z) \in (\mathbb{R}, \mathbb{Z})$, the mappings

$$B_1 (t, \cdot) : \mathbb{H} \times \mathbb{H} \to \mathbb{H}, \quad B_2 (t, \cdot) : \mathbb{H} \times \mathbb{H} \to \mathbb{H},
$$

$$F_1 (t, \cdot) : \mathbb{H} \to \mathcal{L} (\mathbb{H}), \quad F_2 (t, \cdot) : \mathbb{H} \times \mathbb{H} \to \mathcal{L} (\mathbb{H}),$$

$$G_1 (t, \cdot, z) : \mathbb{H} \to \mathcal{L} (\mathbb{H}), \quad G_2 (t, \cdot, z) : \mathbb{H} \times \mathbb{H} \to \mathcal{L} (\mathbb{H}),$$

are Lipschitz continuous and linear growth conditions.

Now, for $i = 1, 2$, we define

$$\gamma_i (t, s) := \int_s^t \gamma_i (r) dr, \quad s < t,$$

and for any $\epsilon > 0$ and $\beta \geq 0$, set

$$U_{\beta, \epsilon, i} (t, s) = e^{\frac{1}{\epsilon} \gamma_i (t, s) A_i - \frac{\beta}{\epsilon} (t - s)}, \quad s < t.$$

For $\epsilon = 1$, we write $U_{\beta, i} (t, s)$, and for $\epsilon = 1$ and $\beta = 0$, we write $U_i (t, s)$.

Next, for any $\epsilon > 0, \beta \geq 0$ and for any $u \in \mathcal{C} ([s, t]; \mathbb{H}), r \in [s, t]$, we define

$$\psi_{\beta, \epsilon, i} (u; s) (r) = \frac{1}{\epsilon} \int_s^r U_{\beta, \epsilon, i} (r, \rho) \mathcal{L}_i (\rho) u (\rho) d\rho, \quad s < r < t.$$
Lemma 3.1. Under (A1)-(A4), for any $p \geq 1$ and $T > 0$, there exists a positive constant $c_{p,T}$, such that for any $x, y \in H$ and $\epsilon \in (0, 1]$, we have

$$\mathbb{E} \sup_{t \in [0,T]} \| u_{\epsilon} (t) \|_{H}^{p} \leq c_{p,T} (1 + \| x \|_{H}^{p} + \| y \|_{H}^{p}),$$

$$\int_{0}^{T} \mathbb{E} \| v_{\epsilon} (t) \|_{H}^{p} dt \leq c_{p,T} (1 + \| x \|_{H}^{p} + \| y \|_{H}^{p}).$$

Proof: For fixed $\epsilon \in (0, 1]$ and $x, y \in H$, for any $t \in [0,T]$, we denote

$$\Gamma_{1,\epsilon} (t) := \int_{0}^{t} U_{1} (t, r) F_{1} (r, u_{\epsilon} (r))dw^{Q_{1}} (r),$$

$$\Psi_{1,\epsilon} (t) := \int_{0}^{t} \int_{Z} U_{1} (t, r) G_{1} (r, u_{\epsilon} (r), z) \tilde{N}_{1} (dr, dz).$$
Set $A_{1,\epsilon}(t) := u_{\epsilon}(t) - \Gamma_{1,\epsilon}(t) - \Psi_{1,\epsilon}(t)$, we have
\[
\frac{d}{dt} A_{1,\epsilon}(t) = \gamma_1(t) A_{1,\epsilon}(t) + L_1(t) (A_{1,\epsilon}(t) + \Gamma_{1,\epsilon}(t) + \Psi_{1,\epsilon}(t)) + B_1(t, A_{1,\epsilon}(t) + \Gamma_{1,\epsilon}(t) + \Psi_{1,\epsilon}(t), v_{\epsilon}(t)), \quad A_{1,\epsilon}(0) = x.
\]

For any $p \geq 2$, because $B_1(\cdot)$ is Lipschitz continuous, using Young's inequality, we have
\[
\frac{1}{p} \frac{d}{dt} \| A_{1,\epsilon}(t) \|^p_H = \langle \gamma_1(t) A_{1,\epsilon}(t), A_{1,\epsilon}(t) \rangle + \langle L_1(t) (A_{1,\epsilon}(t) + \Gamma_{1,\epsilon}(t) + \Psi_{1,\epsilon}(t)), A_{1,\epsilon}(t) \rangle + \langle B_1(t, A_{1,\epsilon}(t) + \Gamma_{1,\epsilon}(t) + \Psi_{1,\epsilon}(t), v_{\epsilon}(t)) - B_1(t, \Gamma_{1,\epsilon}(t) + \Psi_{1,\epsilon}(t), v_{\epsilon}(t)), A_{1,\epsilon}(t) \rangle + \langle B_1(t, \Gamma_{1,\epsilon}(t) + \Psi_{1,\epsilon}(t), v_{\epsilon}(t)), A_{1,\epsilon}(t) \rangle - \langle B_1(t, \Psi_{1,\epsilon}(t), v_{\epsilon}(t)), A_{1,\epsilon}(t) \rangle \| A_{1,\epsilon}(t) \|^p_H
\]

\[
\leq c \| A_{1,\epsilon}(t) \|^p_H + c \| \Gamma_{1,\epsilon}(t) \|_H^p \| A_{1,\epsilon}(t) \|^p_H + c \| \Psi_{1,\epsilon}(t) \|_H \| A_{1,\epsilon}(t) \|^{p-1}_H + c \| B_1(t, \Psi_{1,\epsilon}(t), v_{\epsilon}(t)) \|_H \| A_{1,\epsilon}(t) \|^{p-1}_H
\]

\[
\leq c_p \| A_{1,\epsilon}(t) \|^p_H + c_p \left( 1 + \left\| \Gamma_{1,\epsilon}(t) \right\|_H^p + \left\| \Psi_{1,\epsilon}(t) \right\|_H^p + \left\| v_{\epsilon}(t) \right\|_H^p \right).
\]

This implies that
\[
\| A_{1,\epsilon}(t) \|^p_H \leq e^{c_p t} \| x \|^p_H + c_p \int_0^t e^{c_p (t-r)} \left( 1 + \left\| \Gamma_{1,\epsilon}(r) \right\|_H^p + \left\| \Psi_{1,\epsilon}(r) \right\|_H^p + \left\| v_{\epsilon}(r) \right\|_H^p \right) dr.
\]

According to the definition of $A_{1,\epsilon}(t)$, for any $t \in [0, T]$, we have
\[
\| u_{\epsilon}(t) \|^p_H \leq c_{p,T} \left( 1 + \| x \|^p_H + \sup_{r \in [0,T]} \left\| \Gamma_{1,\epsilon}(r) \right\|_H^p + \sup_{r \in [0,T]} \left\| \Psi_{1,\epsilon}(r) \right\|_H^p \right)
\]

\[
+ c_{p,T} \int_0^T \| v_{\epsilon}(r) \|^p_H dr,
\]

so
\[
\mathbb{E} \sup_{t \in [0,T]} \| u_{\epsilon}(t) \|^p_H \leq c_{p,T} \left( 1 + \| x \|^p_H + \mathbb{E} \sup_{t \in [0,T]} \left\| \Gamma_{1,\epsilon}(t) \right\|_H^p + \mathbb{E} \sup_{t \in [0,T]} \left\| \Psi_{1,\epsilon}(t) \right\|_H^p \right)
\]

\[
+ c_{p,T} \int_0^T \mathbb{E} \| v_{\epsilon}(r) \|^p_H dr.
\]

According to [12, Lemma 4.1] with $\theta = 0$, it is easy to prove that
\[
\mathbb{E} \sup_{t \in [0,T]} \left\| \Gamma_{1,\epsilon}(t) \right\|_H^p \leq c_{p,T} \int_0^T (1 + \mathbb{E} \| u_{\epsilon}(r) \|^p_H) dr.
\]

Due to (A3), using Kunita’s first inequality, we get
\[
\| \Psi_{1,\epsilon}(t) \|^p_H \leq c_p \left( \int_0^t \int_x \left\| e^{\gamma_1(t,r)} A_1(r, u_{\epsilon}(r), z) \right\|_H^2 v_{\epsilon}(dz) dr \right)^{\frac{p}{2}}
\]

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Let \( \Lambda \) so

According to the Gronwall inequality, we have

\[
\int_0^t \|e^\gamma(t,r)A_1 (r, u_\epsilon (r), z)\|_\mathbb{H}^p v_1 (dz) \, dr
\]

\[
= c_p \left( \int_0^t (1 + \|u_\epsilon (r)\|_\mathbb{H}^2) \, dr \right)^{\frac{p}{2}} + c_p \int_0^t (1 + \|u_\epsilon (r)\|_\mathbb{H}^p) \, dr;
\]

\[
\leq c_p, T \int_0^T (1 + \|u_\epsilon (r)\|_\mathbb{H}^p) \, dr,
\]

so

\[
\mathbb{E} \sup_{t \in [0,T]} \|\Psi_{1,\epsilon} (t)\|_\mathbb{H}^p \leq c_p, T \int_0^T (1 + \mathbb{E} \|u_\epsilon (r)\|_\mathbb{H}^p) \, dr.
\] (3.8)

Substituting (3.7) and (3.9) into (3.6), we yields

\[
\mathbb{E} \sup_{t \in [0,T]} \|u_\epsilon (t)\|_\mathbb{H}^p \leq c_p, T \left( 1 + \|x\|_\mathbb{H}^p + \int_0^T \mathbb{E} \|v_\epsilon (r)\|_\mathbb{H}^p \, dr \right) + c_p, T \int_0^T \mathbb{E} \sup_{r \in [0,T]} \|u_\epsilon (r)\|_\mathbb{H}^p \, dr.
\]

According to the Gronwall inequality, we get

\[
\mathbb{E} \sup_{t \in [0,T]} \|u_\epsilon (t)\|_\mathbb{H}^p \leq c_p, T \left( 1 + \|x\|_\mathbb{H}^p + \int_0^T \mathbb{E} \|v_\epsilon (r)\|_\mathbb{H}^p \, dr \right).
\] (3.10)

So, we have to estimate

\[
\int_0^T \mathbb{E} \|v_\epsilon (r)\|_\mathbb{H}^p \, dr.
\]

For any \( t \in [0,T] \), we set

\[
\Gamma_{2,\epsilon} (t) := \frac{1}{\sqrt{\epsilon}} \int_0^t U_{\alpha,\epsilon,2} (t,r) F_2 (r, u_\epsilon (r), v_\epsilon (r)) \, dw^{Q_2} (r),
\]

\[
\Psi_{2,\epsilon} (t) := \int_0^t \int_0^t U_{\alpha,\epsilon,2} (t,r) G_2 (r, u_\epsilon (r), v_\epsilon (r), z) \tilde{N}_2 (dr,dz).
\]

Let \( A_{2,\epsilon} (t) := v_\epsilon (t) - \Gamma_{2,\epsilon} (t) - \Psi_{2,\epsilon} (t) \), we have

\[
\frac{d}{dt} A_{2,\epsilon} (t) = \frac{1}{\epsilon} (\gamma_2 (t) A_2 - \alpha) A_{2,\epsilon} (t) + \frac{1}{\epsilon^2} L_2 (t) (A_{2,\epsilon} (t) + \Gamma_{2,\epsilon} (t) + \Psi_{2,\epsilon} (t))
\]

\[
+ B_2 (t, u_\epsilon (t), A_{2,\epsilon} (t), \Gamma_{2,\epsilon} (t) + \Psi_{2,\epsilon} (t)), \quad A_{2,\epsilon} (0) = y.
\]

For any \( p \geq 2 \), because \( \alpha > 0 \) is large enough, by proceeding as in equation (3.5), we can get

\[
\frac{1}{p} \frac{d}{dt} \|A_{2,\epsilon} (t)\|_\mathbb{H}^p \leq - \frac{\alpha}{2\epsilon} \|A_{2,\epsilon} (t)\|_\mathbb{H}^p + \frac{c_p}{\epsilon} \left( 1 + \|u_\epsilon (t)\|_\mathbb{H}^p + \|\Gamma_{2,\epsilon} (t)\|_\mathbb{H}^p + \|\Psi_{2,\epsilon} (t)\|_\mathbb{H}^p \right).
\]

According to the Gronwall inequality, we have

\[
\|A_{2,\epsilon} (t)\|_\mathbb{H}^p \leq \frac{c_p}{\epsilon} \int_0^t e^{-\frac{\alpha}{2\epsilon} (t-r)} \left( 1 + \|u_\epsilon (r)\|_\mathbb{H}^p + \|\Gamma_{2,\epsilon} (r)\|_\mathbb{H}^p + \|\Psi_{2,\epsilon} (r)\|_\mathbb{H}^p \right) \, dr + e^{-\frac{\alpha}{2\epsilon} t} \|y\|_\mathbb{H}^p.
\]
According to the definition of $A_{2, \epsilon} (t)$, for any $t \in [0, T]$, we yield

\[
\mathbb{E} \| v_\epsilon (t) \|_{H}^{p} \leq c_p \mathbb{E} \| \Gamma_{2, \epsilon} (t) \|_{H}^{p} + c_p \mathbb{E} \| \Psi_{2, \epsilon} (t) \|_{H}^{p} + c_p e^{-\frac{\alpha_2}{2} t} \| y \|_{H}^{p} + \frac{c_p}{\epsilon} \int_{0}^{t} e^{-\frac{\alpha_2}{2} (t-r)} \left( 1 + \mathbb{E} \| u_{\epsilon} (r) \|_{H}^{p} + \mathbb{E} \| \Gamma_{2, \epsilon} (r) \|_{H}^{p} + \mathbb{E} \| \Psi_{2, \epsilon} (r) \|_{H}^{p} \right) \, dr.
\]

Therefore, by integrating with respect to $t$, using Young’s inequality, we obtain

\[
\int_{0}^{t} \mathbb{E} \| v_\epsilon (r) \|_{H}^{p} \, dr \leq c_p (t) \left( 1 + \| y \|_{H}^{p} \right) + c_p \left( \int_{0}^{t} \mathbb{E} \| u_{\epsilon} (r) \|_{H}^{p} \, dr + \int_{0}^{t} \mathbb{E} \| \Gamma_{2, \epsilon} (r) \|_{H}^{p} \, dr \right) + \frac{c_p}{\epsilon} \int_{0}^{t} \left( 1 + \mathbb{E} \| u_{\epsilon} (r) \|_{H}^{p} + \mathbb{E} \| \Gamma_{2, \epsilon} (r) \|_{H}^{p} + \mathbb{E} \| \Psi_{2, \epsilon} (r) \|_{H}^{p} \right) \int_{0}^{t} e^{-\frac{\alpha_2}{2} r} \, dr \leq c_p (t) \left( 1 + \| y \|_{H}^{p} \right) + c_p \left( \int_{0}^{t} \mathbb{E} \| u_{\epsilon} (r) \|_{H}^{p} \, dr + \int_{0}^{t} \mathbb{E} \| \Gamma_{2, \epsilon} (r) \|_{H}^{p} \, dr \right) + \frac{c_p}{\epsilon} \int_{0}^{t} \mathbb{E} \| \Psi_{2, \epsilon} (r) \|_{H}^{p} \, dr \). \tag{3.11}
\]

According to the Burkholder-Davis-Gundy inequality, by proceeding as [12, Proposition 4.2], we can easily get

\[
\int_{0}^{t} \mathbb{E} \| \Gamma_{2, \epsilon} (r) \|_{H}^{p} \, dr \leq c_p \left( 1 + \mathbb{E} \| u_{\epsilon} (r) \|_{H}^{p} + \mathbb{E} \| v_\epsilon (r) \|_{H}^{p} \right) \, dr. \tag{3.12}
\]

Concerning the stochastic term $\Psi_{2, \epsilon} (t)$, using Kunita’s first inequality, we have

\[
\mathbb{E} \| \Psi_{2, \epsilon} (t) \|_{H}^{p} \leq c_p \mathbb{E} \left( \frac{1}{\epsilon} \int_{0}^{t} \int_{Z} \left| e^{-\frac{\alpha_2}{2} (t-r)} e^{\frac{\gamma_2 (t-r)}{\epsilon}} A_{2} G_{2} (r, u_{\epsilon} (r) , v_\epsilon (r) , z) \right|^{2} \, v_{2} (dz) \, dr \right)^{\frac{p}{2}} + \frac{c_p}{\epsilon} \mathbb{E} \left( \int_{0}^{t} \int_{Z} \left| e^{-\frac{\alpha_2}{2} (t-r)} e^{\frac{\gamma_2 (t-r)}{\epsilon}} A_{2} G_{2} (r, u_{\epsilon} (r) , v_\epsilon (r) , z) \right|^{p} \, v_{2} (dz) \, dr \right)^{\frac{1}{2}} \leq \frac{c_p}{\epsilon} \mathbb{E} \left( \int_{0}^{t} \int_{Z} e^{-\frac{\alpha_2}{2} (t-r)} e^{\frac{\gamma_2 (t-r)}{\epsilon}} \left| A_{2} G_{2} (r, u_{\epsilon} (r) , v_\epsilon (r) , z) \right|^{2} \, v_{2} (dz) \, dr \right)^{\frac{p}{2}} + \frac{c_p}{\epsilon} \mathbb{E} \left( \int_{0}^{t} \int_{Z} e^{-\frac{\alpha_2}{2} (t-r)} \left| G_{2} (r, u_{\epsilon} (r) , v_\epsilon (r) , z) \right|^{p} \, v_{2} (dz) \, dr \right)^{\frac{1}{2}} \leq \frac{c_p}{\epsilon} \mathbb{E} \left( \int_{0}^{t} e^{-\frac{\alpha_2}{2} (t-r)} \left( 1 + \| u_{\epsilon} (r) \|_{H}^{p} + \| v_\epsilon (r) \|_{H}^{p} \right) \, dr \right)^{\frac{p}{2}} + \frac{c_p}{\epsilon} \mathbb{E} \left( \int_{0}^{t} e^{-\frac{\alpha_2}{2} (t-r)} \left( 1 + \| u_{\epsilon} (r) \|_{H}^{p} + \| v_\epsilon (r) \|_{H}^{p} \right) \, dr \right)^{\frac{1}{2}} \].
\]

By integrating with respect to $t$ both sides and using Young’s inequality, we have

\[
\int_{0}^{t} \mathbb{E} \| \Psi_{2, \epsilon} (r) \|_{H}^{p} \, dr \leq \frac{c_p}{\epsilon} \int_{0}^{t} e^{-\frac{\alpha_2}{2} r} \left( 1 + \mathbb{E} \| u_{\epsilon} (r) \|_{H}^{p} + \mathbb{E} \| v_\epsilon (r) \|_{H}^{p} \right) \, dr \leq c_p (t) \left( 1 + \mathbb{E} \| u_{\epsilon} (r) \|_{H}^{p} + \mathbb{E} \| v_\epsilon (r) \|_{H}^{p} \right) \, dr. \tag{3.13}
\]
Substituting (3.12) and (3.13) into (3.11), we get
\[
\int_0^t \mathbb{E} \| v_\varepsilon (r) \|^p \mathbb{H} dr \leq c_p (t) \left( 1 + \| y \|^p \mathbb{H} + \int_0^t \mathbb{E} \| u_\varepsilon (r) \|^p \mathbb{H} dr \right) + c_p, T (t) \int_0^t \mathbb{E} \| v_\varepsilon (r) \|^p \mathbb{H} dr.
\]
As \( c_p (0) = 0 \) and \( c_p (t) \) is a continuous increasing function, we can fix \( t_0 > 0 \), such that for any \( t \leq t_0 \), we have \( c_p (t) \leq 1/2 \), so
\[
\int_0^t \mathbb{E} \| v_\varepsilon (r) \|^p \mathbb{H} dr \leq c_p (t) \left( 1 + \| y \|^p \mathbb{H} + \mathbb{E} \sup_{r \in [0, t]} \| u_\varepsilon (r) \|^p \mathbb{H} \right), \quad t \in [0, t_0]. \tag{3.14}
\]
Using this for (3.10), we have
\[
\mathbb{E} \sup_{r \in [0, t]} \| u_\varepsilon (r) \|^p \mathbb{H} \leq c_{p, T} (t) \left( 1 + \| x \|^p \mathbb{H} + \| y \|^p \mathbb{H} \right) + c_{p, T} (t) \mathbb{E} \sup_{r \in [0, t]} \| u_\varepsilon (r) \|^p \mathbb{H}, \quad t \in [0, t_0].
\]
Similarly, we also can fix \( 0 < t_1 \leq t_0 \), such that for any \( t \leq t_1 \), we have \( c_{p, T} (t) \leq 1/2 \), so
\[
\mathbb{E} \sup_{r \in [0, t]} \| u_\varepsilon (r) \|^p \mathbb{H} \leq c_{p, T} (t) \left( 1 + \| x \|^p \mathbb{H} + \| y \|^p \mathbb{H} \right), \quad t \in [0, t_1]. \tag{3.15}
\]
Substituting (3.15) into (3.14), it yields
\[
\int_0^t \mathbb{E} \| v_\varepsilon (r) \|^p \mathbb{H} dr \leq c_{p, T} (t) \left( 1 + \| x \|^p \mathbb{H} + \| y \|^p \mathbb{H} \right), \quad t \in [0, t_1]. \tag{3.16}
\]
For any \( p \geq 2 \), by repeating this in the intervals \( [t_1, 2t_1], [2t_1, 3t_1] \) etc., we can easily get (3.4). Substituting (3.4) into (3.10), we yield (3.3). Using the Hölder inequality, we can estimate (3.3) and (3.4) for \( p = 1 \).

Lemma 3.2. Under (A1)-(A4), there exists \( \bar{\theta} > 0 \), such that for any \( T > 0, p \geq 1, x \in \mathcal{D}((-A_1)^{\theta}) \) with \( \theta \in [0, \bar{\theta}) \) and \( y \in \mathbb{H} \), there exist a positive constant \( c_{p, \theta, T} > 0 \) such that
\[
\sup_{\varepsilon \in (0, 1]} \mathbb{E} \sup_{t \in [0, T]} \| u_\varepsilon (t) \|^p \theta \leq c_{p, \theta, T} (1 + \| x \|^p \theta + \| y \|^p \theta).
\tag{3.17}
\]

Proof: Assuming that \( x \in \mathcal{D}((-A_1)^{\theta})(\theta \geq 0) \), for any \( t \in [0, T] \), we have
\[
\begin{align*}
u_\varepsilon (t) &= U_1 (t, 0) x + \psi_1 (u_\varepsilon ; 0) (t) + \int_0^t U_1 (t, r) B_1 (r, u_\varepsilon (r), v_\varepsilon (r)) dr \\
&\quad + \int_0^t U_1 (t, r) F_1 (r, u_\varepsilon (r)) dr Q_1 (r) \\
&\quad + \int_0^t \int Z U_1 (t, r) G_1 (r, u_\varepsilon (r), z) \tilde{N}_1 (dr, dz).
\end{align*}
\]
Concerning the second term \( \psi_1 (u_\varepsilon ; 0) (t) \), we get
\[
\| \psi_1 (u_\varepsilon ; 0) (t) \|^p \theta \leq c_p \left\| \int_0^t (-A_1)^{\theta} e^{\gamma_1 (r, z) A_1 L_1 (r)} u_\varepsilon (r) dr \right\|^p.
\]
Due to (3.3) and the Hölder inequality, we get
\[
\|L_1(r)u_\varepsilon(r)\|_{\mathbb{H}}^p \leq c_{p,\varepsilon} \sup_{t \in [0,T]} \left\|u_\varepsilon(t)\right\|_{\mathbb{H}}^p \left(\int_0^t (t-r)^{-\theta} dr\right)^p
\]
\[
\leq c_{p,\varepsilon} \left(1 + \|x\|_{\mathbb{H}}^p + \|y\|_{\mathbb{H}}^p\right).
\]

For any \( p \geq 2 \), according to the proof of [12, Proposition 4.3], and thanks to (3.3) and (3.4), it is possible to show that there exists a \( \tilde{\theta} \geq 0 \), such that for any \( \theta \leq \tilde{\theta} \wedge 1/2 \), we have
\[
\mathbb{E} \sup_{t \in [0,T]} \left\| U_1(t, r) B_1(r, u_\varepsilon(r), v_\varepsilon(r)) dr \right\|_{\mathbb{H}}^p \leq c_{p,\varepsilon,T} \left(1 + \|x\|_{\mathbb{H}}^p + \|y\|_{\mathbb{H}}^p\right).
\]

Concerning the stochastic term \( \Psi_{1,\varepsilon}(t) \), using the factorization argument, we have
\[
\Psi_{1,\varepsilon}(t) = \int_0^t (t-r)^{\theta-1} e^{\gamma_1(t,r)A_1 \phi_{\varepsilon,\theta}(r)} dr,
\]
where
\[
\phi_{\varepsilon,\theta}(r) = \int_0^r \int_{\mathbb{Z}} (r-\sigma)^{-\theta} e^{\gamma_1(r,\sigma)A_1 G_1(\sigma, u_\varepsilon(\sigma), z) \tilde{N}_1(d\sigma, dz)}.
\]
Next, for any \( p \geq 2 \), let \( \tilde{\theta} = \frac{1}{2} - \frac{1}{2p} \wedge \frac{1}{2} \), for any \( \theta \leq \tilde{\theta} \), according to (A3) and Lemma 3.1, using Kunita’s first inequality and the Hölder inequality, we get
\[
\left\| \Psi_{1,\varepsilon}(t) \right\|_{\mathbb{H}}^p \leq c_\theta \left(\int_0^t (t-r)^{\theta-1} \left\| \phi_{\varepsilon,\theta}(r) \right\|_{\mathbb{H}}^p dr \right)^p \leq c_{p,\varepsilon,T} \left(1 + \|x\|_{\mathbb{H}}^p + \|y\|_{\mathbb{H}}^p\right).
\]
So, due to (3.3), we yield
\[
\mathbb{E} \sup_{t \in [0,T]} \left\| \Psi_{1,\varepsilon}(t) \right\|_{\mathbb{H}}^p \leq c_{p,\varepsilon,T} \left(1 + \|x\|_{\mathbb{H}}^p + \|y\|_{\mathbb{H}}^p\right).
\]
Hence, if we choose \( \tilde{\theta} := 1/8 \wedge \theta \wedge \hat{\theta} \), thanks to (3.18), (3.19), (3.20) and (3.21), for any \( p \geq 2 \) and \( \theta < \tilde{\theta} \), we get

\[
\mathbb{E} \sup_{t \in [0,T]} \| u_\epsilon (t) \|_\theta^p \leq c_{p,\theta,T} \left( 1 + \| x \|_\theta^p + \| y \|_{H^2}^p \right).
\]

Using the Hölder inequality, we can estimate (3.17) for \( p = 1 \). \( \square \)

**Lemma 3.3.** Under (A1)-(A4), for any \( \theta \in [0, \tilde{\theta}) \), there exists \( \beta(\theta) > 0 \), such that, for any \( T > 0, p \geq 1, x \in D((-A_1)^\theta), y \in H^2 \) and \( s, t \in [0, T] \), it holds

\[
\sup_{\epsilon \in (0,1]} \mathbb{E} \| u_\epsilon (t) - u_\epsilon (s) \|_{H^2}^p \leq c_{p,\theta,T} \left( \| t - s \|^{\beta(\theta)p} + \| t - s \| \right) \left( 1 + \| x \|_\theta^p + \| y \|_{H^2}^p \right).
\]

**Proof:** For any \( t \geq 0, 0 \leq h \leq 1 \), with \( t, t + h \in [0, T] \), we have

\[
u_\epsilon (t + h) - u_\epsilon (t) = (U_1(t + h,t) - I) u_\epsilon (t) + \psi_1(u_\epsilon (t) (t + h)
+ \int_t^{t+h} U_1(t + h,r)B_1(r, u_\epsilon (r),v_\epsilon (r))dr
+ \int_t^{t+h} U_1(t + h,r)F_1(r, u_\epsilon (r), h)dr
+ \int_t^{t+h} \int_Z U_1(t + h,r)G_1(r, u_\epsilon (r), z, \tilde{N}_1 (dr, dz)
\end{array}
\]

\[
\begin{array}{r}
:= \sum_{i=1}^5 I_i.
\end{array}
\]

By proceeding as the proof of [12, Proposition 4.4] and (3.18), fix \( \theta \in [0, \tilde{\theta}) \), for any \( p \geq 1 \), it is possible to show that

\[
\mathbb{E} \left| I_1 \right|_{H^2}^p \leq c_{p,T} T \theta^p \left( 1 + \| x \|_\theta^p + \| y \|_{H^2}^p \right),
\]

(3.24)

\[
\mathbb{E} \left| I_2 \right|_{H^2}^p \leq c_{p,T} T \theta^{p-1} \left( 1 + \| x \|_\theta^p + \| y \|_{H^2}^p \right),
\]

(3.25)

\[
\mathbb{E} \left| I_3 \right|_{H^2}^p \leq c_{p,T} T \theta \left( 1 + \| x \|_\theta^p + \| y \|_{H^2}^p \right)
\]

(3.26)

\[
\mathbb{E} \left| I_4 \right|_{H^2}^p \leq c_{p,T} T \theta \left( 1 + \| x \|_\theta^p + \| y \|_{H^2}^p \right).
\]

(3.27)

According to the proof of (3.8), using the Hölder inequality and (3.3), we have

\[
\mathbb{E} \left| I_5 \right|_{H^2}^p \leq c_p \mathbb{E} \left( \int_t^{t+h} \left( 1 + \| u_\epsilon (r) \|_{H^2}^2 \right)dr \right)^{\frac{p}{2}} + c_p \mathbb{E} \left( \int_t^{t+h} \left( 1 + \| u_\epsilon (r) \|_{H^2}^p \right)dr \right),
\]

\[
\leq c_p \left( \theta \frac{\beta_1 (p_1 - 2)}{2} + 1 \right) \left( \theta \frac{\beta_1 (p_1 - 2)}{2} + 1 \right) \left( 1 + \mathbb{E} \| u_\epsilon (r) \|_{H^2}^2 \right)dr
\]

\[
\leq c_{p,T} \left( \theta \frac{\beta_1 (p_1 - 2)}{2} + 1 \right) \left( 1 + \| x \|_{H^2}^p + \| y \|_{H^2}^p \right).
\]

(3.28)

Then, if we take \( \bar{p} > 1 \), such that

\[
\frac{\beta_1 (p_1 - 2)}{\rho_1} \frac{\bar{p}}{\bar{p} - 2} < 1,
\]

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we can get
\[
\mathbb{E} \|u_\epsilon(t+h) - u_\epsilon(t)\|_p^p \leq c_{p,T} \left( h^{(1-\frac{1}{p})p} + h^p + h^{\left(1 - \frac{1}{p} - \frac{\beta_1(p_1 - 2)}{2p_1^2}\right)p} + h^p \right) \times \left( 1 + \|x\|_p^p + \|y\|_p^p \right) + c_{p,T} h^p \left( 1 + \|x\|_p^p + \|y\|_p^p \right).
\]

As we are assuming \(|h| \leq 1\), (3.22) follows for any \(p \geq \bar{p}\) by taking
\[
\beta(\theta) := \min \left\{ \theta, 1 - \frac{1}{\bar{p}}, 1 - \frac{1}{\bar{p}} - \frac{\beta_1(p_1 - 2)}{2p_1^2}, \frac{1}{2} \right\}.
\]

From the Hölder inequality, we can estimate (3.22) for \(p < \bar{p}\),
\[
\mathbb{E} \|u_\epsilon(t+h) - u_\epsilon(t)\|_p^p \leq \mathbb{E} \|u_\epsilon(t+h) - u_\epsilon(t)\|_p^\bar{p}.
\]
so, we have (3.22). \(\Box\)

In view of the Garcia-Rademich-Rumsey theorem and the Arzelà-Ascoli theorem, we can infer that Lemma 3.2 and Lemma 3.3 imply the family \(\mathcal{L}(u_\epsilon)\}_{\epsilon \in (0,1]}\) is tight.

4. An evolution family of measures for the fast equation

For any frozen slow component \(x \in \mathbb{H}\), any initial condition \(y \in \mathbb{H}\), and any \(s \in \mathbb{R}\), we introduce the following problem
\[
dv(t) = \left( [A_2(t) - \alpha] v(t) + B_2(t, x, v(t)) \right) dt + F_2(t, x, v(t)) \, d\tilde{Q}_2(t)
+ \int_{\mathbb{Z}} G_2(t, x, v(t), z) \tilde{N}_2'(dt, dz),
\]
where
\[
w^{Q_2}(t) = \begin{cases} w_1^{Q_2}(t), & \text{if } t \geq 0, \\ w_2^{Q_2}(-t), & \text{if } t < 0, \end{cases}
\]
\[
\tilde{N}_2'(t, z) = \begin{cases} \tilde{N}_1'(t, z), & \text{if } t \geq 0, \\ \tilde{N}_3'(-t, z), & \text{if } t < 0, \end{cases}
\]
for two independent \(Q_2\)-Wiener processes \(w_1^{Q_2}(t), w_2^{Q_2}(t)\) and two independent compensated Poisson measures \(\tilde{N}_1'(dt, dz), \tilde{N}_3'(dt, dz)\) with the same Lévy measure are both defined as in Section 2.

According to the definition of the operator \(\psi_{\alpha,2}(:;s)\), we know that the mapping \(\psi_{\alpha,2}(:;s) : \mathcal{C}([s, T]; \mathbb{H}) \rightarrow \mathcal{C}([s, T]; \mathbb{H})\) is a linear bounded operator and it is Lipschitz continuous. Hence, we have that, for any \(x, y \in \mathbb{H}, p \geq 1\) and \(s < T\), there exists a unique mild solution [25] \(v^x(:;s; y)\) in the following form
\[
v^x(t; s, y) = U_{\alpha,2}(t, s) y + \psi_{\alpha,2}(v^x(:;s; y); s) \left( t \right) + \int_s^t U_{\alpha,2}(t, r) B_2(r, x, v^x(r; s, y)) dr
+ \int_s^t U_{\alpha,2}(t, r) F_2(r, x, v^x(r; s, y)) d\tilde{w}^{Q_2}(r)
+ \int_s^t \int_{\mathbb{Z}} U_{\alpha,2}(t, r) G_2(r, x, v^x(r; s, y), z) \tilde{N}_2'(dr, dz).
\]
Moreover, if the space $\mathcal{C}(\mathbb{R}; \mathbb{H})$ endowed with the topology of uniform convergence on bounded intervals, an $\{F_t\}_{t \in \mathbb{R}}$-adapted process $v^x$ is a mild solution of the equation

$$dv(t) = [(A_2(t) - \alpha) v(t) + B_2(t, x, v(t))] dt + F_2(t, x, v(t)) d\tilde{\omega}^{Q_2}(t) + \int_{\mathbb{H}} G_2(t, x, v(t), z) N'_p (dt, dz),$$

(4.2)

where $t \in \mathbb{R}$. Then, for every $s < t$, we have

$$v^x(t) = U_{\alpha, 2}(t, s) v^x(s) + \psi_{\alpha, 2} (v^x; s)(t) + \int_s^t U_{\alpha, 2}(t, r) B_2(r, x, v^x(r)) dr$$

$$+ \int_s^t U_{\alpha, 2}(t, r) F_2(r, x, v^x(r)) d\tilde{\omega}^{Q_2}(r)$$

$$+ \int_s^t \int_{\mathbb{H}} U_{\alpha, 2}(t, r) G_2(r, x, v^x(r), z) N'_p (dr, dz).$$

In what follows, for any $x \in \mathbb{H}$ and any adapted process $v$, we set

$$\Gamma_\alpha(v; s)(t) := \int_s^t U_{\alpha, 2}(t, r) F_2(r, x, v(r)) d\tilde{\omega}^{Q_2}(r), \quad t > s,$$

(4.3)

$$\Psi_\alpha(v; s)(t) := \int_s^t \int_{\mathbb{H}} U_{\alpha, 2}(t, r) G_2(r, x, v(r), z) N'_p (dr, dz), \quad t > s.$$

(4.4)

For any $0 < \delta < \alpha$ and any $v_1, v_2$ with $s < t$, by proceeding as in the proof of [36, Lemma 7.1], it is possible to show that there exists $\bar{p} > 1$, such that for any $p \geq \bar{p}$, we have

$$\sup_{r \in [s, t]} e^{\delta p(r-s)} \mathbb{E} \| \Gamma_\alpha(v_1; s)(r) - \Gamma_\alpha(v_2; s)(r) \|_{\mathbb{H}}^p \leq c_{p, 1} \frac{L_{f_2}^P}{(\alpha - \delta)^{p-2} p - 2} \sup_{r \in [s, t]} e^{\delta p(r-s)} \mathbb{E} \| v_1(r) - v_2(r) \|_{\mathbb{H}}^p,$$

(4.5)

where $L_{f_2}$ is the Lipschitz constant of $f_2$, and $c_{p, 1}, c_{p, 2}$ are two suitable positive constants independent of $\alpha > 0$ and $s < t$.

For the stochastic term $\Psi_\alpha(v; s)(t)$, using Kunita’s first inequality [23, Theorem 4.4.23], we get

$$\mathbb{E} \left\| \Psi_\alpha(v_1; s)(t) - \Psi_\alpha(v_2; s)(t) \right\|_{\mathbb{H}}^p \leq c_p \mathbb{E} \left( \int_s^t \int_{\mathbb{H}} \left\| e^{-\alpha(t-r)} e^{\gamma_2(t, r)} A_2 \left[ G_2(r, x, v_1(r), z) - G_2(r, x, v_2(r), z) \right] \right\|_{\mathbb{H}}^2 v_2'(dz) dr \right)^{\frac{p}{2}}$$

$$+ c_p \mathbb{E} \left( \int_s^t \int_{\mathbb{H}} \left\| e^{-\alpha(t-r)} e^{\gamma_2(t, r)} A_2 \left[ G_2(r, x, v_1(r), z) - G_2(r, x, v_2(r), z) \right] \right\|_{\mathbb{H}}^p v_2'(dz) dr \right)^{\frac{p}{2}}$$

$$\leq c_p L_{g_2}^p \left( \int_s^t e^{-2\alpha(t-r)} \| G_2(r, x, v_1(r), z) - G_2(r, x, v_2(r), z) \|_{\mathbb{H}}^2 v_2'(dz) dr \right)^{\frac{p}{2}}$$

$$+ c_p L_{g_2}^p \left( \int_s^t e^{-\alpha(r-s)} \| G_2(r, x, v_1(r), z) - G_2(r, x, v_2(r), z) \|_{\mathbb{H}}^p v_2'(dz) dr \right)^{\frac{p}{2}}$$

$$\leq c_p L_{g_2}^p \left[ \int_s^t e^{-2\alpha(t-r)} (t-r)^{\frac{p}{2}} dr \right] + \int_s^t e^{-\alpha(t-s)} (t-s)^{\frac{p}{2}} \sup_{r \in [s, t]} e^{\delta p(r-s)} \mathbb{E} \| v_1(r) - v_2(r) \|_{\mathbb{H}}^p,$$
\[ \leq c_{p,1} \frac{L_{g_2}^p}{p \alpha} e^{-\delta(p(t-s))} \sup_{r \in [s,t]} e^{\delta(p(r-s))} \mathbb{E} \left\| v_1(r) - v_2(r) \right\|_{\mathbb{H}}^p, \]

so

\[ \sup_{r \in [s,t]} e^{\delta(p(r-s))} \mathbb{E} \left\| \Psi_{\alpha} (v_1; s) (r) - \Psi_{\alpha} (v_2; s) (r) \right\|_{\mathbb{H}}^p \leq c_{p,1} \frac{L_{g_2}^p}{p \alpha} \sup_{r \in [s,t]} e^{\delta(p(r-s))} \mathbb{E} \left\| v_1(r) - v_2(r) \right\|_{\mathbb{H}}^p, \]  

(4.6)

where \( L_{g_2} \) is the Lipschitz constant of \( g_2 \), and \( c_{p,1}, c_{p,2} \) are two suitable positive constants independent of \( \alpha > 0 \) and \( s < t \).

Moreover, using (A4), we can show that

\[ \sup_{r \in [s,t]} e^{\delta(p(r-s))} \mathbb{E} \left\| \Gamma_{\alpha} (v; s) (r) \right\|_{\mathbb{H}}^p \leq c_{p,1} \frac{M_{f_2}^p}{(\alpha - \delta)^{p-1}} \sup_{r \in [s,t]} e^{\delta(p(r-s))} \left( 1 + \mathbb{E} \| v(r) \|_{\mathbb{H}}^p \right), \]  

(4.7)

\[ \sup_{r \in [s,t]} e^{\delta(p(r-s))} \mathbb{E} \left\| \Psi_{\alpha} (v; s) (r) \right\|_{\mathbb{H}}^p \leq c_{p,1} \frac{M_{g_2}^p}{(\alpha - \delta)^{p-1}} \sup_{r \in [s,t]} e^{\delta(p(r-s))} \left( 1 + \mathbb{E} \| v(r) \|_{\mathbb{H}}^p \right), \]  

(4.8)

where \( M_{f_2}, M_{g_2} \) are the linear growth constants of \( f_2, g_2 \), and \( c_{p,1}, c_{p,2} \) are two suitable positive constants independent of \( \alpha > 0 \) and \( s < t \).

For any fixed adapted process \( v \), let us introduce the problem

\[ dp(t) = (A_2(t) - \alpha) \rho(t) dt + F_2(t, x, v(t)) d\tilde{\omega}_2(t) \]

\[ + \int_{Z} G_2(t, x, v(t), z) \tilde{N}_2(dt, dz), \quad \rho(s) = 0. \]  

(4.9)

We denote that its unique mild solution is \( \rho_\alpha (v; s) \). This means that \( \rho_\alpha (v; s) \) solves the equation

\[ \rho_\alpha (v; s) (t) = \psi_{\alpha,2} (\rho_\alpha (v; s); s) (t) + \Gamma_{\alpha} (v; s) (t) + \Psi_{\alpha} (v; s) (t), \quad s < t < T. \]

Due to (4.5) and (4.6), using the same argument as the equation (5.8) in [22], it is easy to prove that for any process \( v_1, v_2 \) and \( 0 < \delta < \alpha \), we have

\[ \sup_{r \in [s,t]} e^{\delta(p(r-s))} \mathbb{E} \left\| \rho_\alpha (v_1; s) (r) - \rho_\alpha (v_2; s) (r) \right\|_{\mathbb{H}}^p \leq c_{p,1} \frac{L^p}{(\alpha - \delta)^{p-1}} \sup_{r \in [s,t]} e^{\delta(p(r-s))} \mathbb{E} \left\| v_1(r) - v_2(r) \right\|_{\mathbb{H}}^p, \]  

(4.10)

where \( L = \max \{ L_{g_2}, L_{f_2}, L_{g_2} \} \).

Similarly, thanks to (4.7) and (4.8), for any process \( v \) and \( 0 < \delta < \alpha \), we can prove that

\[ \sup_{r \in [s,t]} e^{\delta(p(r-s))} \mathbb{E} \left\| \rho_\alpha (v; s) (r) \right\|_{\mathbb{H}}^p \leq c_{p,1} \frac{M^p}{(\alpha - \delta)^{p-1}} \sup_{r \in [s,t]} e^{\delta(p(r-s))} \mathbb{E} \left( 1 + \| v(r) \|_{\mathbb{H}}^p \right), \]  

(4.11)

where \( M = \max \{ M_{g_2}, M_{f_2}, M_{g_2} \} \).

**Lemma 4.1.** Under (A1)-(A4), there exists \( \delta > 0 \), such that for any \( x, y \in \mathbb{H} \) and \( p \geq 1 \),

\[ \mathbb{E} \| v^x(t; s, y) \|_{\mathbb{H}}^p \leq c_p \left( 1 + \| x \|_{\mathbb{H}}^p + e^{-\delta(p(t-s))} \| y \|_{\mathbb{H}}^p \right), \quad s < t. \]  

(4.12)
**Proof:** We set \( A_\alpha (t) := v^x (t; s, y) - \rho_\alpha (t) \), where \( \rho_\alpha (t) = \rho_\alpha (v^x (\cdot; s, y); s) (t) \) is the solution of the problem (4.9) with \( v = v^x (\cdot; s, y) \). Using Young’s inequality, we have

\[
\frac{1}{p} \frac{d}{dt} \| A_\alpha (t) \|_{p,2}^p \leq \langle (A_2 (t) - \alpha) A_\alpha (t), A_\alpha (t) \rangle_{p,2} \| A_\alpha (t) \|_{p,2}^{p-2} \\
+ \langle B_2 (t, x, A_\alpha (t) + \rho_\alpha (t)) - B_2 (t, x, \rho_\alpha (t)), A_\alpha (t) \rangle_{p,2} \| A_\alpha (t) \|_{p,2}^{p-2} \\
+ \langle B_2 (t, x, \rho_\alpha (t)), A_\alpha (t) \rangle_{p,2} \| A_\alpha (t) \|_{p,2}^{p-2} \\
\leq -\alpha \| A_\alpha (t) \|_{p,2}^p + c \| A_\alpha (t) \|_{p,2}^p + c (1 + \| x \|_{p,2}^p + \| \rho_\alpha (t) \|_{p,2}^p) \| A_\alpha (t) \|_{p,2}^{p-1} \\
\leq -\alpha \| A_\alpha (t) \|_{p,2}^p + c_p \| A_\alpha (t) \|_{p,2}^p + c_p (1 + \| x \|_{p,2}^p + \| \rho_\alpha (t) \|_{p,2}^p) .
\]

Because \( \alpha \) is large enough, we can find \( \eta = \alpha - c_p > 0 \), such that

\[
\frac{d}{dt} \| A_\alpha (t) \|_{p,2}^p \leq -\eta p \| A_\alpha (t) \|_{p,2}^p + c_p (1 + \| x \|_{p,2}^p + \| \rho_\alpha (t) \|_{p,2}^p) .
\]

According to the Gronwall inequality, we have

\[
\| A_\alpha (t) \|_{p,2}^p \leq e^{-\eta p (t-s)} \| y \|_{p,2}^p + c_p \int_s^t e^{-\eta p (t-r)} \| \rho_\alpha (r) \|_{p,2}^p dr.
\]

So, for any \( p \geq 1 \),

\[
\| v^x (t; s, y) \|_{p,2}^p \leq c_p \| \rho_\alpha (t) \|_{p,2}^p + c_p e^{-\eta p (t-s)} \| y \|_{p,2}^p + c_p (1 + \| x \|_{p,2}^p) + c_p \int_s^t e^{-\eta p (t-r)} \| \rho_\alpha (r) \|_{p,2}^p dr.
\]

Fix \( 0 < \delta < \eta \), according to (4.11), we get

\[
e^{\delta p (t-s)} \mathbb{E} \| v^x (t; s, y) \|_{p,2}^p \leq c_p e^{\delta p (t-s)} \mathbb{E} \| \rho_\alpha (t) \|_{p,2}^p + c_p e^{(\delta-\eta) p (t-s)} \| y \|_{p,2}^p \\
+ c_p e^{\delta p (t-s)} (1 + \| x \|_{p,2}^p) + c_p \int_s^t e^{\delta p (r-s)} \mathbb{E} \| \rho_\alpha (r) \|_{p,2}^p dr \\
\leq c_{p,1} \frac{M_p}{(\alpha - \delta)^{p/2}} \sup_{r \in [s,t]} e^{\delta p (r-s)} (1 + \mathbb{E} \| v^x (r; s, y) \|_{p,2}^p) \\
+ c_{p,1} e^{\delta (\delta-\eta) (t-s)} \| y \|_{p,2}^p + c_p e^{\delta p (t-s)} (1 + \| x \|_{p,2}^p) \\
+ c_{p,1} \frac{M_p}{(\alpha - \delta)^{p/2}} \int_s^t \sup_{r \in [s,t]} e^{\delta p (r-s)} (1 + \mathbb{E} \| v^x (r; s, y) \|_{p,2}^p) dr.
\]

Taking \( \alpha_1 = (2c_{p,1} M_p)^{1/p^2} + \delta \), when \( \alpha \geq \alpha_1 \), we have

\[
\sup_{r \in [s,t]} e^{\delta p (r-s)} \mathbb{E} \| v^x (r; s, y) \|_{p,2}^p \leq c_p \| y \|_{p,2}^p + c_p e^{\delta p (t-s)} (1 + \| x \|_{p,2}^p) \\
+ \int_s^t \sup_{r \in [s,t]} e^{\delta p (r-s)} \mathbb{E} \| v^x (r; s, y) \|_{p,2}^p dr.
\]

Due to the Gronwall lemma, we have

\[
\sup_{r \in [s,t]} e^{\delta p (r-s)} \mathbb{E} \| v^x (r; s, y) \|_{p,2}^p \leq c_p \| y \|_{p,2}^p + c_p e^{\delta p (t-s)} (1 + \| x \|_{p,2}^p) .
\]

Hence, we get (4.12).  \( \square \)
Lemma 4.2. Under (A1)-(A4), for any $t \in \mathbb{R}$ and $x, y \in \mathbb{H}$, for all $p \geq 1$, there exists $\eta^x(t) \in L^p(\Omega; \mathbb{H})$ such that

$$\lim_{s \to -\infty} \mathbb{E} \|v^x(t; s, y) - \eta^x(t)\|_{\mathbb{H}}^p = 0. \quad (4.13)$$

Moreover, for any $p \geq 1$, there exists some $\delta_p > 0$, such that

$$\mathbb{E} \|v^x(t; s, y) - \eta^x(t)\|_{\mathbb{H}}^p \leq c_p e^{-\delta_p(t-s)} (1 + \|x\|^p_H + \|y\|^p_H). \quad (4.14)$$

Finally, $\eta^x$ is a mild solution in $\mathbb{R}$ of equation (4.2).

Proof: Fix $h > 0$ and define

$$\rho(t) = v^x(t; s, y) - v^x(t; s - h, y), \quad s < t.$$ We know that $\rho(t)$ is the unique mild solution of the problem

$$\begin{aligned}
d\rho(t) &= [(A_2(t) - \alpha) \rho(t) + B_2(t, x, v^x(t; s, y)) - B_2(t, x, v^x(t; s - h, y))] dt \\
&\quad + \left[ F_2(t, x, v^x(t; s, y)) - F_2(t, x, v^x(t; s - h, y)) \right] d\tilde{w}^{Q_2}(t) \\
&\quad + \int_{\mathbb{Z}} [G_2(t, x, v^x(t; s, y)) - G_2(t, x, v^x(t; s - h, y))] \tilde{N}_2'(dt, dz)
\end{aligned}$$

(4.15)

and

$$\rho(s) = y - v^x(s; s - h, y).$$

Multiply both sides of the above equation by $e^{\delta_p(t-s)}$. Because $\alpha$ large enough, according to [22, Lemma 2.4], we have

$$e^{\delta_p(t-s)} \mathbb{E} \|\rho(t)\|_{\mathbb{H}}^p \leq c_p e^{(\delta - \alpha)p(t-s)} \mathbb{E} \left\| e^{\gamma_2(t,s)A_2} (y - v^x(s; s - h, y)) \right\|_{\mathbb{H}}^p \leq$$

$$+ c_p \mathbb{E} \left\| \int_s^t e^{\gamma_2(t,r)A_2} e^{(\delta - \alpha)(t-r)} e^{\delta(r-s)} [B_2(r, x, v^x(r; s, y)) - B_2(r, x, v^x(r; s - h, y))] dr \right\|_{\mathbb{H}}^p$$

$$+ c_p \mathbb{E} \left\| \int_s^t e^{\gamma_2(t,r)A_2} e^{(\delta - \alpha)(t-r)} e^{\delta(r-s)} [F_2(r, x, v^x(r; s, y)) - F_2(r, x, v^x(r; s - h, y))] d\tilde{w}^{Q_2}(r) \right\|_{\mathbb{H}}^p$$

$$+ c_p \mathbb{E} \left\| \int_s^t \int_{\mathbb{Z}} e^{\gamma_2(t,r)A_2} e^{(\delta - \alpha)(t-r)} e^{\delta(r-s)} [G_2(r, x, v^x(r; s, y)) - G_2(r, x, v^x(r; s - h, y))] \tilde{N}_2'(dr, dz) \right\|_{\mathbb{H}}^p$$

$$:= \sum_{i=1}^4 I_i^t.$$
According to [12, Lemma 3.1], we know that for any $J \in \mathcal{L}(L^\infty(D), \mathbb{H}) \cap \mathcal{L}(\mathbb{H}, L^1(D))$ with $J = J^*$ and $s \geq 0$, we have

$$\|e^{sA_1}JQ_i\|^2 \leq K_1 s^{-\beta_i(p_i^2 - 2)} e^{-\alpha_i s} \|J\|^2_{\mathcal{L}(L^\infty(D), \mathbb{H})},$$

where

$$K_1 = (\beta_i/e)^{\rho_i} \zeta_i^{\rho_i} \kappa_i^{\beta_i}.\tag{4.16}$$

Taking $p > 1$, such that $\beta_i(p_i^2 - 2)p < 2$. Then, using the Burkholder-Davis-Gundy inequality and Kunita’s first inequality, we can get that for any $p \geq p$ and $0 < \delta < \alpha$, it yields

$$\mathcal{T}_1 \leq c_p L_p \sup_{r \in [s, t]} e^{\delta p(r-s)} \mathbb{E} \left\| \rho(r) \right\|^p_{\mathbb{H}} \left( \int_s^t e^{(\delta - \alpha)(t-r)} dr \right)^p$$

$$\leq c_p L_p \sup_{r \in [s, t]} e^{\delta p(r-s)} \mathbb{E} \left\| \rho(r) \right\|^p_{\mathbb{H}} \left( \int_s^t e^{2(\delta - \alpha)(t-r)} \gamma_2(t, r) \frac{\beta_2(p_2 - 2)}{p_2} e^{-\frac{\alpha(p_2 - 2)}{p_2} r} dr \right)^2$$

$$\leq c_p L_p \sup_{r \in [s, t]} e^{\delta p(r-s)} \mathbb{E} \left\| \rho(r) \right\|^p_{\mathbb{H}} \left( \int_s^t e^{2(\delta - \alpha)(t-r)} \gamma_2(t, r) \frac{\beta_2(p_2 - 2)}{p_2} e^{-\frac{\alpha(p_2 - 2)}{p_2} r} dr \right)^2$$

Hence, we have

$$\sup_{r \in [s, t]} e^{\delta p(r-s)} \mathbb{E} \left\| \rho(r) \right\|^p_{\mathbb{H}} \leq c_p \|y - v^x(s; s - h, y)\|^p_{\mathbb{H}} + c_p \frac{L_p}{(\alpha - \delta)^{\gamma_{p_2}}} \sup_{r \in [s, t]} e^{\delta p(r-s)} \mathbb{E} \left\| \rho(r) \right\|^p_{\mathbb{H}}.$$

Therefore, for $\alpha > 0$ large enough, we can find $0 < \bar{\delta}_p < \alpha$, such that

$$c_p \frac{L}{(\alpha - \bar{\delta}_p)^{\gamma_{p_2}}} < 1.$$
This implies that
\[ \sup_{r \in [s,t]} e^{\delta_p(r-s)} \mathbb{E} \| \rho (r) \|_p^p \leq c_p \| y - v^x (s; s-h, y) \|_p^p. \]

Let \( \delta_p = p \delta_p \), thanks to Lemma 4.1, we have
\[ \mathbb{E} \| v^x (t; s, y) - v^x (t; s-h, y) \|_p^p \leq c_p e^{-\delta_p(t-s)} \| y - v^x (s; s-h, y) \|_p^p \leq c_p e^{-\delta_p(t-s)} \left( 1 + \| x \|_p^p + \| y \|_p^p + e^{-\delta_p h} \| y \|_p^p \right). \]

Because \( L^p (\Omega; \mathbb{H}) \) is completeness, for any \( p \geq \bar{p}, \) let \( s \to -\infty \) in (4.17), there exists \( \eta^x (t) \in L^p (\Omega; \mathbb{H}) \) such that (4.13) hold. Then, if we let \( h \to \infty \) in (4.17), we obtain (4.14). Using the Hölder inequality, we can get (4.13) and (4.14) holds for any \( p < \bar{p} \).

If we take \( y_1, y_2 \in \mathbb{H}, \) use the same arguments for \( v^x (t; s, y_1) - v^x (t; y_2) \), \( s < t, \) we have
\[ \mathbb{E} \| v^x (t; s, y_1) - v^x (t; s, y_2) \|_p^p \leq c_p e^{-\delta_p(t-s)} \| y_1 - y_2 \|_p^p, \quad s < t. \]

Let \( s \to -\infty, \) this means that the limit \( \eta^x (t) \) does not depend on the initial condition \( y \in \mathbb{H}. \)

Finally, we prove that \( \eta^x (t) \) is a mild solution of equation (4.2). Due to the limit \( \eta^x (t) \) does not depend on the initial condition, we can let initial condition \( y = 0. \) For any \( s < t \) and \( h > 0, \) we have
\[ v^x (t; s-h, 0) = U_{\alpha,2} (t, s) v^x (s; s-h, 0) + \psi_{\alpha,2} (v^x (\cdot; s-h, 0) ; s) (t) \]
\[ + \int_s^t U_{\alpha,2} (t, r) B_2 (r, x, v^x (r; s-h, 0)) dr \]
\[ + \int_s^t U_{\alpha,2} (t, r) F_2 (r, x, v^x (r; s-h, 0)) dw^Q (r) \]
\[ + \int_s^t \int_{\mathbb{Z}} U_{\alpha,2} (t, r) G_2 (r, x, v^x (r; s-h, 0), z) \tilde{N}_{2'} (dr, dz). \]

Let \( h \to \infty \) on both sides, due to (4.13), we can get, for any \( s < t, \) have
\[ \eta^x (t) = U_{\alpha,2} (t, s) \eta^x (s) + \psi_{\alpha,2} (\eta^x (\cdot) ; s) (t) + \int_s^t U_{\alpha,2} (t, r) B_2 (r, x, \eta^x (r)) dr \]
\[ + \int_s^t U_{\alpha,2} (t, r) F_2 (r, x, \eta^x (r)) dw^Q (r) \]
\[ + \int_s^t \int_{\mathbb{Z}} U_{\alpha,2} (t, r) G_2 (r, x, \eta^x (r), z) \tilde{N}_{2'} (dr, dz). \]

This means that \( \eta^x (t) \) is a mild solution of equation (4.2).

For any \( t \in \mathbb{R} \) and \( x \in \mathbb{H}, \) we denote that the law of the random variable \( \eta^x (t) \) is \( \mu^x_t, \) and we introduce the transition evolution operator
\[ P_{s,t}^x \varphi (y) = \mathbb{E} \varphi (v^x (t; s, y)), \quad s < t, \quad y \in \mathbb{H}, \]
where \( \varphi \in B_b (\mathbb{H}). \)

Due to (4.12) and (4.13), for any \( p \geq 1, \) we have
\[ \sup_{t \in \mathbb{R}} \mathbb{E} \| \eta^x (t) \|_p^p \leq c_p \left( 1 + \| x \|_p^p \right), \quad x \in \mathbb{H}, \]
\[ \text{(4.19)} \]
so that
\[
\sup_{t \in \mathbb{R}} \int_{\mathbb{H}} \|y\|_\mathbb{H}^p \mu^x_t (dy) \leq c_p \left(1 + \|x\|_{\mathbb{H}}^p\right), \quad x \in \mathbb{H}. \tag{4.20}
\]

According to the above conclusion, by using the same arguments as \cite[Proposition 5.3]{22}, we know that the family \(\{\mu^x_t\}_{t \in \mathbb{R}}\) defines an evolution system of probability measures on \(\mathbb{H}\) for equation (4.1). This means that for any \(t \in \mathbb{R}\), \(\mu^x_t\) is a probability measure on \(\mathbb{H}\), and it holds that
\[
\int_{\mathbb{H}} P^x_{s,t} \varphi (y) \mu^x_t (dy) = \int_{\mathbb{H}} \varphi (y) \mu^x_s (dy), \quad s < t, \tag{4.21}
\]
for every \(\varphi \in C_b(\mathbb{H})\). Moreover, we also have
\[
\left| P^x_{s,t} \varphi (y) - \int_{\mathbb{H}} \varphi (y) \mu^x_t (dy) \right| \leq c e^{-\delta_1 (t-s)} \left(1 + \|x\|_{\mathbb{H}}\right). \tag{4.22}
\]

In order to get the averaged equation, we must ensure the existence of the averaged coefficient \(\bar{B}_1\). So, we need the evolution family of measures satisfying some nice properties. We give the following assumption.

(A5) (a) The functions \(\gamma_2 : \mathbb{R} \to (0, \infty)\) and \(l_2 : \mathbb{R} \times \mathcal{O} \to \mathbb{R}^d\) are periodic, with the same period.

(b) The families of functions
\[
\begin{align*}
B_{1,R} &:= \{ b_1 (\cdot, \xi, \sigma) : \xi \in \mathcal{O}, \sigma \in \mathcal{B}_{\mathbb{R}^2} (R) \}, \\
B_{2,R} &:= \{ b_2 (\cdot, \xi, \sigma) : \xi \in \mathcal{O}, \sigma \in \mathcal{B}_{\mathbb{R}^2} (R) \}, \\
F_R &:= \{ f_2 (\cdot, \xi, \sigma) : \xi \in \mathcal{O}, \sigma \in \mathcal{B}_{\mathbb{R}^2} (R) \}, \\
G_R &:= \{ g_2 (\cdot, \xi, \sigma, z) : \xi \in \mathcal{O}, \sigma \in \mathcal{B}_{\mathbb{R}^2} (R), z \in \mathbb{Z} \},
\end{align*}
\]
are uniformly almost periodic for any \(R > 0\).

**Remark 4.3.** Similar with the proof of \cite[Lemma 6.2]{22}, we get that under (A5), for any \(R > 0\), the families of functions
\[
\begin{align*}
\{ B_1 (\cdot, x, y) : (x, y) \in \mathcal{B}_{\mathbb{H} \times \mathbb{H}} (R) \}, & \quad \{ B_2 (\cdot, x, y) : (x, y) \in \mathcal{B}_{\mathbb{H} \times \mathbb{H}} (R) \}, \\
\{ F_2 (\cdot, x, y) : (x, y) \in \mathcal{B}_{\mathbb{H} \times \mathbb{H}} (R) \}, & \quad \{ G_2 (\cdot, x, y, z) : (x, y, z) \in \mathcal{B}_{\mathbb{H} \times \mathbb{H}} (R) \times \mathbb{Z} \},
\end{align*}
\]
are uniformly almost periodic.

Using the same argument as \cite[Lemma 6.4]{22}, we can prove that under the assumptions (A1)-(A4), there exists \(\theta > 0\), such that for any \(p \geq 1\) and \(x \in \mathbb{H}\)
\[
\sup_{t \in \mathbb{R}} \mathbb{E} \| \eta^x (t) \|_\theta^p \leq c_p \left(1 + \|x\|_{\mathbb{H}}^p\right). \tag{4.23}
\]

On the other hand, by preceding as in Lemma 4.2, we can get that under (A1)-(A4), for any fixed \(x_1, x_2 \in \mathbb{H}\), there exists \(c_p > 0\) such that
\[
\sup_{s < t} \mathbb{E} \| v^{x_1} (t; s, 0) - v^{x_2} (t; s, 0) \|_{\mathbb{H}}^p \leq c \|x_1 - x_2\|_{\mathbb{H}}^p, \tag{4.24}
\]
so, according to (4.13), we have
\[
\sup_{t \in \mathbb{R}} \mathbb{E} \| \eta^{x_1} (t) - \eta^{x_2} (t) \|_{\mathbb{H}}^p \leq c \|x_1 - x_2\|_{\mathbb{H}}^p. \tag{4.25}
\]
In view of the Garcia-Rademich-Rumsey theorem and the Arzelà-Ascoli theorem, equations (4.23) and (4.25) imply that the family of measures
\[ A := \{ \mu^x_t : t \in \mathbb{R}, x \in \mathbb{H} \} \]  
(4.26)
is tight in \( \mathcal{P}(\mathbb{H}) \).

As we know above, \( A_2(\cdot) \) is periodic, Remark 4.3 holds and the family of measures \( A \) is tight in \( \mathcal{P}(\mathbb{H}) \). By proceeding as [37], we can prove that the mapping \( t \in \mathbb{R} \mapsto \mu^x_t \in \mathcal{P}(\mathbb{H}) \) is almost periodic.

5. The averaged equation

Lemma 5.1. Under (A1)-(A5), for any \( x \in \mathbb{H} \), the family of functions
\[ \left\{ t \in \mathbb{R} \mapsto \int_{\mathbb{H}} B_1(t, x, y) \mu^x_t(dy) \right\} \]
is uniformly almost periodic.

Proof: Clearly, we have that \( B_1(t, x, \cdot) : \mathbb{H} \to \mathbb{H} \) is Lipschitz continuous and bounded for any fixed \( (t, x) \in \mathbb{R} \times \mathbb{H} \). Now, let us define
\[ \Phi(t, x) = \int_{\mathbb{H}} B_1(t, x, y) \mu^x_t(dy), \quad (t, x) \in \mathbb{R} \times \mathbb{H}. \]
Because the mapping \( t \in \mathbb{R} \mapsto \mu^x_t \in \mathcal{P}(\mathbb{H}) \) and the families of functions \( B_1(\cdot, x, y) \) are almost periodic, for any \( t \in \mathbb{R} \), we can find a \( \tau \in \mathbb{R} \) such that
\[
\| \Phi(t + \tau, x) - \Phi(t, x) \|_{\mathbb{H}} \leq \left\| \int_{\mathbb{H}} B_1(t + \tau, x, y) \mu^x_{t+\tau}(dy) - \int_{\mathbb{H}} B_1(t + \tau, x, y) \mu^x_t(dy) \right\|_{\mathbb{H}}
+ \left\| \int_{\mathbb{H}} B_1(t + \tau, x, y) \mu^x_t(dy) - \int_{\mathbb{H}} B_1(t, x, y) \mu^x_t(dy) \right\|_{\mathbb{H}}
\leq \sup_{y \in \mathbb{H}} \| B_1(t + \tau, x, y) \|_{\mathbb{H}} \left( \int_{\mathbb{H}} \| \mu^x_{t+\tau} - \mu^x_t \|_{\mathbb{H}}(dy) \right)
+ \int_{\mathbb{H}} \| B_1(t + \tau, x, y) - B_1(t, x, y) \|_{\mathbb{H}} \| \mu^x_t(dy) \|_{\mathbb{H}} < \epsilon,
\]
so, the function \( \Phi(\cdot, x) \) is almost periodic for any \( x \in \mathbb{H} \). Thanks to (A3) and (4.25), we can conclude that for any \( x_1, x_2 \in \mathbb{H} \)
\[
\| \Phi(t, x_1) - \Phi(t, x_2) \|_{\mathbb{H}} \leq \mathbb{E} \| B_1(t, x_1, \eta^{x_2}(t)) - B_1(t, x_2, \eta^{x_2}(t)) \|_{\mathbb{H}}
\leq c \left( \| x_1 - x_2 \|_{\mathbb{H}} + \mathbb{E} \| \eta^{x_2}(t) - \eta^{x_2}(t) \|_{\mathbb{H}} \right),
\leq c \| x_1 - x_2 \|_{\mathbb{H}}.
\]
This means that the family of functions \( \{ \Phi(\cdot, \cdot) : t \in \mathbb{R} \} \) is equicontinuous. The above conclusions indicate that the family \( \{ \Phi(\cdot, x) : x \in \mathbb{H} \} \) is uniformly almost periodic [29, Theorem 2.10]. \( \square \)

According to [22, Theorem 3.4], we define
\[ B_1(x) := \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{\mathbb{H}} B_1(t, x, y) \mu^x_t(dy) \, dt, \quad x \in \mathbb{H}, \]
(5.2)
and thanks to (A3) and (4.20), we have that
\[ \| B_1(x) \|_{\mathbb{H}} \leq c (1 + \| x \|_{\mathbb{H}}). \]
(5.3)
Lemma 5.2. Under (A1)-(A5), for any $T > 0, s \in \mathbb{R}$ and $x, y \in \mathbb{H}$,

$$
\mathbb{E}\left[\frac{1}{T} \int_s^{s+T} B_1 (t, x, v^x (t; s, y)) dt - \bar{B}_1 (x) \right] \leq \frac{c}{T} (1 + \|x\|_\mathbb{H} + \|y\|_\mathbb{H}) + \alpha (T, x),
$$

for some mapping $\alpha : [0, \infty) \times \mathbb{H} \rightarrow [0, \infty)$ such that

$$
sup_{T > 0} \alpha (T, x) \leq c (1 + \|x\|_\mathbb{H}), \quad x \in \mathbb{H},
$$

and

$$
\limsup_{T \to \infty} \alpha (T, x) = 0.
$$

Proof: We denote

$$
\psi^x B_1 (t, y) := B_1 (t, x, y) - \int_\mathbb{H} B_1 (t, x, w) \mu_t^x (dw).
$$

So

$$
\mathbb{E}\left(\frac{1}{T} \int_s^{s+T} B_1 (t, x, v^x (t; s, y)) \right)^2 \leq \frac{2}{T^2} \int_s^{s+T} \int_r^{s+T} \mathbb{E} \left[ \psi^x B_1 (r, v^x (r; s, y)) \psi^x B_1 (t, v^x (t; s, y)) \right] dt dr
$$

$$
= \frac{2}{T^2} \int_s^{s+T} \int_r^{s+T} \mathbb{E} \left[ \psi^x B_1 (r, v^x (r; s, y)) P^x_{r, t} \psi^x B_1 (r, v^x (r; s, y)) \right] dt dr
$$

$$
\leq \frac{2}{T^2} \int_s^{s+T} \int_r^{s+T} \left( \mathbb{E} |\psi^x B_1 (r, v^x (r; s, y))|^2 \right)^{\frac{1}{2}} \left( \mathbb{E} |P^x_{r, t} \psi^x B_1 (r, v^x (r; s, y))|^2 \right)^{\frac{1}{2}} dt dr.
$$

Due to (A3), (4.12) and (4.20), we have

$$
\mathbb{E} |\psi^x B_1 (r, v^x (r; s, y))|^2 \leq c \mathbb{E} \|B_1 (r, x, v^x (r; s, y))\|^2_{\mathbb{H}} + c \mathbb{E} \left( \int_\mathbb{H} \|B_1 (r, x, w)\|_{\mathbb{H}} \mu_t^x (dw) \right)^2
$$

$$
\leq c \left(1 + \|x\|^2_{\mathbb{H}} + \mathbb{E} \|v^x (r; s, y)\|^2_{\mathbb{H}} \right)
$$

$$
\leq c \left(1 + \|x\|^2_{\mathbb{H}} + e^{-\delta_2 (t-s)} \|y\|^2_{\mathbb{H}} \right),
$$

and according to (4.14), we get

$$
\mathbb{E} |P^x_{r, t} \psi^x B_1 (r, v^x (r; s, y))|^2 = \mathbb{E} \left[ P^x_{r, t} \left[ B_1 (r, x, v^x (r; s, y)) - \int_\mathbb{H} B_1 (r, x, w) \mu_t^x (dw) \right] \right]^2
$$

$$
= \mathbb{E} \left[ B_1 (t, x, v^x (t; s, y)) - B_1 (t, x, \eta^x (t)) \right]^2
$$

$$
\leq c \mathbb{E} \|v^x (t; s, y) - \eta^x (t)\|^2_{\mathbb{H}}
$$

$$
\leq ce^{-\delta_2 (t-s)} \left(1 + \|x\|^2_{\mathbb{H}} + \|y\|^2_{\mathbb{H}} \right).
$$

Let $\delta = \delta_2 / 2$, it follows

$$
\mathbb{E}\left[\frac{1}{T} \int_s^{s+T} B_1 (t, x, v^x (t; s, y)) dt - \bar{B}_1 (x) \right] \leq \frac{c\rho}{T} (1 + \|x\|_\mathbb{H} + \|y\|_\mathbb{H}) \left( \int_s^{s+T} \int_r^{s+T} e^{-\delta (t-s)} dt dr \right)^{\frac{1}{2}}
$$

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Next, we get that (5.1) is uniformly almost periodic, according to [22, Theorem 3.4], we get that the limit
\[
\lim_{T \to \infty} \frac{1}{T} \int_{s}^{s+T} B_{1}(t, x, w) \mu_{T}^{x} (dw) \ dt
\]
converges to \( B_{1}(x) \) uniformly with respect to \( s \in \mathbb{R} \) and \( x \in \mathbb{H} \). Therefore, if we define
\[
\alpha (T, x) = \left\| \frac{1}{T} \int_{s}^{s+T} B_{1}(t, x, w) \mu_{T}^{x} (dw) \ dt - B_{1}(x) \right\|_{\mathbb{H}},
\]
we get the conclusion.

Now, we introduce the averaged equation
\[
du (t) = \left[ A_{1}(t) u (t) + B_{1}(u (t)) \right] dt + F_{1}(t, u (t)) dw^{Q_{1}} (t)
+ \int_{\mathbb{Z}} G_{1}(t, u (t), z) \bar{N}_{1} (dt, dz), \quad u (0) = x \in \mathbb{H}.
\]

Due to the assumption (A3), we can easily get that the mapping \( B_{1} : \mathbb{H} \to \mathbb{H} \) is Lipschitz continuous. So, for any \( x \in \mathbb{H}, T > 0 \) and \( p \geq 1 \), equation (5.9) admits a unique mild solution \( \bar{u} [25] \).

### 6. Averaging principles

In this section, we prove that the slow motion \( u_{\epsilon} \) converges to the averaged motion \( \bar{u} \), as \( \epsilon \to 0 \).

**Theorem 6.1.** Under (A1)-(A5), fix \( x \in \mathcal{D}((-A_{1})^{0}) (\theta \in [0, \bar{\theta})) \) and \( y \in \mathbb{H} \), for any \( T > 0 \) and \( \eta > 0 \), we have
\[
\lim_{\epsilon \to 0} \mathbb{E} \left( \sup_{t \in [0, T]} \| u_{\epsilon} (t) - \bar{u} (t) \|_{\mathbb{H}} > \eta \right) = 0,
\]
where \( \bar{u} \) is the solution of the averaged equation (5.9).

**Proof:** For any \( h \in \mathcal{D}(A_{1}) \cap L^\infty (\mathcal{O}) \), we have
\[
\langle u_{\epsilon} (t), h \rangle_{\mathbb{H}} = \langle x, h \rangle_{\mathbb{H}} + \int_{0}^{t} \langle A_{1}(r) u_{\epsilon} (r), h \rangle_{\mathbb{H}} dr
+ \int_{0}^{t} \langle B_{1}(u_{\epsilon} (r)), h \rangle_{\mathbb{H}} dr
+ \int_{0}^{t} \langle F_{1}(r, u_{\epsilon} (r)) dw^{Q_{1}} (r), h \rangle_{\mathbb{H}}
+ \int_{0}^{t} \langle \int_{\mathbb{Z}} G_{1}(r, u_{\epsilon} (r), z) \bar{N}_{1} (dr, dz), h \rangle_{\mathbb{H}} + R_{\epsilon} (t),
\]
where
\[
R_{\epsilon} (t) := \int_{0}^{t} \langle B_{1}(r, u_{\epsilon} (r), \bar{u} (r)) - B_{1}(u_{\epsilon} (r)), h \rangle_{\mathbb{H}} dr.
\]
Because the family \( \{ \mathcal{L} (u_{\epsilon}) \}_{\epsilon \in (0, 1]} \) is tight, in order to prove Theorem 6.1, it is sufficient to prove
\[
\lim_{\epsilon \to 0} \mathbb{E} \sup_{t \in [0, T]} | R_{\epsilon} (t) | = 0.
\]
For any $\epsilon > 0$ and some deterministic constant $\delta_\epsilon > 0$, we divide the interval $[0, T]$ in subintervals of the size $\delta_\epsilon$. In each time interval $[k\delta_\epsilon, (k+1)\delta_\epsilon], k = 0, 1, \cdots, \lfloor T/\delta_\epsilon \rfloor$, we define the following auxiliary fast motion $\hat{\nu}_\epsilon$,

$$
\begin{align*}
  d\hat{\nu}_\epsilon (t) &= \frac{1}{\epsilon} \left[ (A_2 (t) - \alpha) \hat{\nu}_\epsilon (t) + B_2 (t, u_\epsilon (k\delta_\epsilon), \hat{\nu}_\epsilon (t)) \right] dt \\
  &+ \frac{1}{\sqrt{\epsilon}} F_2 (t, u_\epsilon (k\delta_\epsilon), \hat{\nu}_\epsilon (t)) d\omega^Q_2 (t) \\
  &+ \int_{Z} G_2 (t, u_\epsilon (k\delta_\epsilon), \hat{\nu}_\epsilon (t)) \tilde{N}_2 (dt, dz).
\end{align*}
$$

(6.2)

According to the definition of $\hat{\nu}_\epsilon$, we know that an analogous estimate to (3.4) holds. So, for any $p \geq 1$, we have

$$
\int_0^T \mathbb{E} \|\hat{\nu}_\epsilon (t)\|_H^p dt \leq c_{p,T} \left( 1 + \|x\|_B^p + \|y\|_H^p \right).
$$

(6.3)

**Lemma 6.2.** Under (A1)-(A5), fix $x \in \mathcal{D}(\mathbb{R}^d) (\theta \in [0, \bar{\theta})$ and $y \in \mathcal{H}$, there exists a constant $\kappa > 0$, such that if

$$
\delta_\epsilon = \epsilon \ln^{-\kappa},
$$

we have

$$
\lim_{\epsilon \to 0} \sup_{t \in [0, T]} \mathbb{E} \|\hat{\nu}_\epsilon (t) - v_\epsilon (t)\|_H^p = 0.
$$

(6.4)

**Proof:** Let $\epsilon > 0$ be fixed. For $k = 0, 1, \cdots, \lfloor T/\delta_\epsilon \rfloor$ and $t \in [k\delta_\epsilon, (k+1)\delta_\epsilon]$, let $\rho_\epsilon (t)$ be the solution of the problem

$$
\begin{align*}
  d\rho_\epsilon (t) &= \frac{1}{\epsilon} (A_2 (t) - \alpha) \rho_\epsilon (t) dt + \frac{1}{\sqrt{\epsilon}} K_\epsilon (t) d\omega^Q_2 (t) \\
  &+ \int_{Z} H_\epsilon (t, z) \tilde{N}_2 (dt, dz), \\
  \rho_\epsilon (k\delta_\epsilon) &= 0,
\end{align*}
$$

where

$$
\begin{align*}
  K_\epsilon (t) &:= F_2 (t, u_\epsilon (k\delta_\epsilon), \hat{\nu}_\epsilon (t)) - F_2 (t, u_\epsilon (t), v_\epsilon (t)), \\
  H_\epsilon (t, z) &:= G_2 (t, u_\epsilon (k\delta_\epsilon), \hat{\nu}_\epsilon (t), z) - G_2 (t, u_\epsilon (t), v_\epsilon (t), z).
\end{align*}
$$

We get

$$
\rho_\epsilon (t) = \psi_{\alpha, \epsilon, 2} (\rho_\epsilon, k\delta_\epsilon) + \Gamma_\epsilon (t) + \Psi_\epsilon (t), \\
\text{ where } \Gamma_\epsilon (t) = \frac{1}{\epsilon} \int_{k\delta_\epsilon}^t \int_{k\delta_\epsilon}^r U_{\alpha, \epsilon, 2} (t, r) K_\epsilon (r) d\omega^Q_2 (r),
$$

$$
\Psi_\epsilon (t) = \int_{k\delta_\epsilon}^t \int_{Z} U_{\alpha, \epsilon, 2} (t, r) H_\epsilon (r, z) \tilde{N}_2 (dr, dz).
$$

If we denote $\Lambda_\epsilon (t) := \hat{\nu}_\epsilon (t) - v_\epsilon (t)$ and $\tilde{\nu}_\epsilon (t) := \Lambda_\epsilon (t) - \rho_\epsilon (t)$, we have

$$
\begin{align*}
  d\tilde{\nu}_\epsilon (t) &= \frac{1}{\epsilon} \left[ (A_2 (t) - \alpha) \tilde{\nu}_\epsilon (t) + B_2 (t, u_\epsilon (k\delta_\epsilon), \hat{\nu}_\epsilon (t)) - B_2 (t, u_\epsilon (t), v_\epsilon (t)) \right] dt.
\end{align*}
$$

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Due to Lemma 3.3, for $\alpha > 0$ large enough, using Young’s inequality, we have

$$
\frac{1}{\epsilon} \langle (\gamma_2 (t) A_2 + L_2 (t) - 2) \partial_e (t) \partial_e (t) \rangle_{\mathbb{H}} \leq \frac{1}{\epsilon} \langle B_2 (t, \epsilon) \hat{\partial_e} (t) \hat{\partial_e} (t) \rangle_{\mathbb{H}} \leq \frac{1}{\epsilon} \langle (\gamma_2 (t) A_2 + L_2 (t) - 2) \partial_e (t) \partial_e (t) \rangle_{\mathbb{H}} \leq \frac{1}{\epsilon} \langle B_2 (t, \epsilon) \hat{\partial_e} (t) \hat{\partial_e} (t) \rangle_{\mathbb{H}}
$$

Using the Gronwall inequality, we get

$$
\left\| \partial_e (t) \right\|_{\mathbb{H}} \leq \frac{c_p}{\epsilon} \left( 1 + \left\| x \right\|_{\mathbb{H}}^p + \left\| y \right\|_{\mathbb{H}}^p \right) \left( \frac{\alpha}{\epsilon} \right) + \frac{c_p}{\epsilon} \int_{k\delta}^{t} \left\| \hat{\partial_e} (r) - \partial_e (r) \right\|_{\mathbb{H}}^p dr. \quad (6.5)
$$

By proceeding as [13, Lemma 6.3], we prove that

$$
\mathbb{E} \left\| I_e (t) \right\|_{\mathbb{H}}^p \leq \frac{c_p}{\epsilon} \left( 1 + \left\| x \right\|_{\mathbb{H}}^p + \left\| y \right\|_{\mathbb{H}}^p \right) \left( \frac{\alpha}{\epsilon} \right) + \frac{c_p}{\epsilon} \int_{k\delta}^{t} \mathbb{E} \left\| \hat{\partial_e} (r) - \partial_e (r) \right\|_{\mathbb{H}}^p dr. \quad (6.6)
$$

For the other stochastic term $\Psi_e (t)$, using Kunita’s first inequality and the Hölder inequality, we yield

$$
\mathbb{E} \left\| \Psi_e (t) \right\|_{\mathbb{H}}^p \leq \frac{c_p}{\epsilon} \left( 1 + \left\| x \right\|_{\mathbb{H}}^p + \left\| y \right\|_{\mathbb{H}}^p \right) \left( \frac{\alpha}{\epsilon} \right) + \frac{c_p}{\epsilon} \int_{k\delta}^{t} \mathbb{E} \left\| \hat{\partial_e} (r) - \partial_e (r) \right\|_{\mathbb{H}}^p dr. \quad (6.7)
$$

According to [22, Lemma 2.4] and equations (6.5), (6.6) and (6.7), we obtain

$$
\mathbb{E} \left\| \hat{v} e (t) - v_e (t) \right\|_{\mathbb{H}}^p \leq \frac{c_p}{\epsilon} \left( 1 + \left\| x \right\|_{\mathbb{H}}^p + \left\| y \right\|_{\mathbb{H}}^p \right) \left( \frac{\alpha}{\epsilon} \right) + \frac{c_p}{\epsilon} \int_{k\delta}^{t} \mathbb{E} \left\| \hat{v} e (r) - v_e (r) \right\|_{\mathbb{H}}^p dr. \quad (6.8)
$$
From the Gronwall lemma, this means
\[
E \| \hat{v}_\epsilon (t) - v_\epsilon (t) \|^p_H \leq c_p \left( \delta_\epsilon^{p+2} / \epsilon^2 + 1 / \epsilon \right) \left( \delta_\epsilon^{3(p+1)} + \delta_\epsilon^2 \right) e^{c_p ( \delta_\epsilon^{p+2} / \epsilon^2 + 1 / \epsilon) \delta_\epsilon} \left( 1 + \| x \|^p_\theta + \| y \|^p_\theta \right).
\]
For \( t \in [0, T] \), selecting \( \delta_\epsilon = \epsilon \ln^p \), then if we take \( \kappa < \frac{B_\theta \beta_\theta}{p+2} \wedge \frac{1}{1+2p+2} \), we have (6.4). □

**Lemma 6.3.** Under the same assumptions as in Theorem 6.1, for any \( T > 0 \), we have
\[
\lim_{\epsilon \to 0} \sup_{t \in [0, T]} | R_\epsilon (t) | = 0.
\] (6.8)

**Proof:** According to the definition of \( \bar{B}_1 \), we get that the mapping \( \bar{B}_1 : H \to H \) is Lipschitz continuous. Using assumption (A3), Lemma 3.3 and Lemma 6.2, we have
\[
\lim_{\epsilon \to 0} \sup_{t \in [0, T]} | R_\epsilon (t) |
\leq \lim_{\epsilon \to 0} \mathbb{E} \left[ \int_0^T | \langle B_1 (r, u_\epsilon (r), \hat{v}_\epsilon (r)) - B_1 (u_\epsilon ([r/\delta_\epsilon], \hat{v}_\epsilon (r)), h \rangle_H | dr 
+ \lim_{\epsilon \to 0} \sup_{t \in [0, T]} \mathbb{E} \left[ \int_0^T \langle B_1 (r, u_\epsilon ([r/\delta_\epsilon], \hat{v}_\epsilon (r)) - B_1 (u_\epsilon (r), h) \rangle_H dr \right] 
\leq \lim_{\epsilon \to 0} c_T \mathbb{E} \left[ \sup_{t \in [0, T]} \mathbb{E} \left[ \| u_\epsilon (r) - u_\epsilon ([r/\delta_\epsilon], \hat{v}_\epsilon (r)) \|_H \right] + \mathbb{E} \| \hat{v}_\epsilon (r) - v_\epsilon (r) \|_H \right] 
\leq \lim_{\epsilon \to 0} \mathbb{E} \left[ \int_0^T \langle B_1 (r, u_\epsilon (k \delta_\epsilon), \hat{v}_\epsilon (r)) - B_1 (u_\epsilon (k \delta_\epsilon), h) \rangle_H dr \right] 
+ \lim_{\epsilon \to 0} \mathbb{E} \left[ \int_0^T \langle B_1 (r, u_\epsilon (k \delta_\epsilon), \hat{v}_\epsilon (r)) - B_1 (u_\epsilon (k \delta_\epsilon), h) \rangle_H dr \right] 
\leq \lim_{\epsilon \to 0} c_T \mathbb{E} \left[ \sup_{t \in [0, T]} \mathbb{E} \left[ \| u_\epsilon (r) - u_\epsilon ([r/\delta_\epsilon], \hat{v}_\epsilon (r)) \|_H \right] + \mathbb{E} \| \hat{v}_\epsilon (r) - v_\epsilon (r) \|_H \right] 
\leq \lim_{\epsilon \to 0} \mathbb{E} \left[ \int_0^T \langle B_1 (r, u_\epsilon (k \delta_\epsilon), \hat{v}_\epsilon (r)) - B_1 (u_\epsilon (k \delta_\epsilon), h) \rangle_H dr \right] 
+ \lim_{\epsilon \to 0} \mathbb{E} \left[ \int_0^T \langle B_1 (r, u_\epsilon (k \delta_\epsilon), \hat{v}_\epsilon (r)) - B_1 (u_\epsilon (k \delta_\epsilon), h) \rangle_H dr \right] 
\leq \lim_{\epsilon \to 0} \mathbb{E} \left[ \int_0^T \langle B_1 (r, u_\epsilon (k \delta_\epsilon), \hat{v}_\epsilon (r)) - B_1 (u_\epsilon (k \delta_\epsilon), h) \rangle_H dr \right] 
\leq \lim_{\epsilon \to 0} \mathbb{E} \left[ \int_0^T \langle B_1 (r, u_\epsilon (k \delta_\epsilon), \hat{v}_\epsilon (r)) - B_1 (u_\epsilon (k \delta_\epsilon), h) \rangle_H dr \right].
\]

So, we have to show that
\[
\lim_{\epsilon \to 0} \sum_{k=0}^{[T/\delta_\epsilon]} \mathbb{E} \left[ \int_k \langle B_1 (r, u_\epsilon (k \delta_\epsilon), \hat{v}_\epsilon (r)) - B_1 (u_\epsilon (k \delta_\epsilon), h) \rangle_H dr \right] = 0. \quad (6.9)
\]
If we set \( \zeta_\epsilon = \delta_\epsilon / \epsilon \), we have
\[
\mathbb{E} \left[ \int_k \langle B_1 (r, u_\epsilon (k \delta_\epsilon), \hat{v}_\epsilon (r)) - B_1 (u_\epsilon (k \delta_\epsilon), h) \rangle_H dr \right] = \mathbb{E} \left[ \int_k \langle B_1 (k \delta_\epsilon + r, u_\epsilon (k \delta_\epsilon), \hat{v}_\epsilon (k \delta_\epsilon + r)) - B_1 (u_\epsilon (k \delta_\epsilon), h) \rangle_H dr \right] = \mathbb{E} \left[ \int_k \langle B_1 (k \delta_\epsilon + r, u_\epsilon (k \delta_\epsilon), \hat{v}_\epsilon (k \delta_\epsilon + r)) - B_1 (u_\epsilon (k \delta_\epsilon), h) \rangle_H dr \right] = \mathbb{E} \left[ \int_k \langle B_1 (k \delta_\epsilon + r, u_\epsilon (k \delta_\epsilon), \hat{v}_\epsilon (k \delta_\epsilon + r)) - B_1 (u_\epsilon (k \delta_\epsilon), h) \rangle_H dr \right] = \mathbb{E} \left[ \int_k \langle B_1 (k \delta_\epsilon + r, u_\epsilon (k \delta_\epsilon), \hat{v}_\epsilon (k \delta_\epsilon + r)) - B_1 (u_\epsilon (k \delta_\epsilon), h) \rangle_H dr \right] = \mathbb{E} \left[ \int_k \langle B_1 (k \delta_\epsilon + r, u_\epsilon (k \delta_\epsilon), \hat{v}_\epsilon (k \delta_\epsilon + r)) - B_1 (u_\epsilon (k \delta_\epsilon), h) \rangle_H dr \right].
\]
\[ = \delta \mathbb{E} \left[ \frac{1}{\epsilon} \int_0^\zeta \left\langle B_1 \left( k\delta + \epsilon r, u_\epsilon (k\delta), v_\epsilon (k\delta) \right), \hat{\gamma} u_\epsilon (k\delta), v_\epsilon (k\delta) \right\rangle dr \right], \]

where \( \hat{\gamma} u_\epsilon (k\delta), v_\epsilon (k\delta) \)(r) is the solution of the fast motion equation (4.1) with the initial datum given by \( u_\epsilon (k\delta) \) and the frozen slow component given by \( u_\epsilon (k\delta) \). In addition, the noises in (4.1) are independent of \( u_\epsilon (k\delta) \) and \( v_\epsilon (k\delta) \). According to the proof of Lemma 5.2, we get

\[
\mathbb{E} \left[ \int_{k\delta}^{(k+1)\delta} \left\langle B_1 \left( r, u_\epsilon (k\delta), \hat{v}_\epsilon (r) \right), \tilde{B}_1 (u_\epsilon (k\delta)), h \right\rangle dr \right] \leq \delta \mathbb{E} \left[ u_\epsilon (k\delta) \right] + \mathbb{E} \left[ \| u_\epsilon (k\delta) \|_\mathbb{H} \right] + \mathbb{E} \left[ \| h \|_\mathbb{H} \right] + \delta \| h \|_\mathbb{H} \mathbb{E} \alpha (\zeta, u_\epsilon (k\delta)).
\]

we get (6.9), so (6.8) holds.

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