$W_\infty$ GAUGE TRANSFORMATIONS AND THE ELECTROMAGNETIC INTERACTIONS OF ELECTRONS IN THE LOWEST LANDAU LEVEL

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Abstract

We construct a $W_\infty$ gauge field theory of electrons in the lowest Landau level. For this purpose we introduce an external gauge potential $A$ such that its $W_\infty$ gauge transformations cancel against the gauge transformation of the electron field. We then show that the electromagnetic interactions of electrons in the lowest Landau level are obtained through a non-linear realization of $A$ in terms of the $U(1)$ gauge potential $A^\mu$. As applications we derive the effective Lagrangians for circular droplets and for the $\nu = 1$ quantum Hall system.

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The electromagnetic interactions of electrons in the lowest Landau level have been obtained recently from the original Lagrangian by integrating out the components of electron field, which correspond to the higher Landau levels [1]. The resulting effective Lagrangian does not have the manifest gauge covariance that the original Lagrangian had. It appears that the gauge covariance is maintained in the effective Lagrangian through a non-linear realization of $W_\infty$ gauge transformations*. In this paper we wish to elaborate on this point and show that the effective Lagrangian can be derived to a certain extent as a non-linear realization of a $W_\infty$ gauge transformation.

Since the spin of the electrons is aligned in the strong magnetic field, we consider only a one component fermion field

$$\psi(x, y, t) = \sqrt{\frac{B}{2\pi}} e^{-\frac{1}{2}z^2} \sum_{n=0}^{\infty} \frac{\bar{z}^n}{\sqrt{n!}} C_n(t) = \sqrt{\frac{B}{2\pi}} e^{-\frac{1}{2}z^2} \sum_{n=0}^{\infty} \langle z|n\rangle C_n(t) ,$$

which obeys the lowest Landau level condition

$$\left( \partial_z + \frac{1}{2} \bar{z} \right) \psi(x, y, t) = 0 ,$$

where $z = \sqrt{\frac{B}{2}}(x + iy)$, $\bar{z} = \sqrt{\frac{B}{2}}(x - iy)$ and $|z\rangle$ is the coherent state basis of a pair of bosonic operators $\hat{a}$ and $\hat{a}^\dagger$. The modes $C_n$ satisfy the usual anticommutation relations

$$\{C_n, C_m^\dagger\} = \delta_{nm} .$$

Next we consider a time dependent unitary transformation in the space of $C_n$ [3]:

$$C_n(t) \rightarrow C'_n(t) = u_{nm} C_m(t) = \langle n|\hat{u}(t)|m\rangle C_m(t) .$$

An infinitesimal transformation is generated by a hermititian operator which we write as

$$\xi(\hat{a}, \hat{a}^\dagger, t) \xi^\dagger \xi$$

with the anti-normal order symbol, where $\xi$ is a real function when $\hat{a}$ and $\hat{a}^\dagger$ are replaced by $z$ and $\bar{z}$ respectively. Then using (1) we obtain the following infinitesimal

* This result could have been anticipated from the work of Shizuya [2], which deals with a general field theoretic formulation of electrons in a strong magnetic field.
transformation for $\psi$:

$$
\delta \psi(x, y, t) = -i \sqrt{\frac{B}{2\pi}} e^{-\frac{i}{2}|z|^2} \frac{1}{2} \xi(\partial_{\bar{z}}, \bar{z}, t) \sum_{n} \langle z|n \rangle C_n(t) = -i \xi(\partial_{\bar{z}} + \frac{1}{2} z, \bar{z}, t) \bar{\psi}(x, y, t). 
$$

(4)

where $\frac{1}{2}$ indicates that the derivatives are placed on the left of $z$ and $\bar{z}$. We call (4) the $W_\infty$ gauge transformation [3][4].

Introducing an external $W_\infty$ gauge potential $A$ we write a $W_\infty$ gauge invariant Lagrangian as follows*:

$$
L = \int dx dy \bar{\psi}(x, y, t) \left( i\partial_t - V(x, y) - A(x, y, t) \right) \psi(x, y, t),
$$

(5)

where we have separated a confining static potential $V$ from $A$. The confining potential is a strong static potential provided by other materials outside the system, and it confines the electrons to within the system. Using (4) and (5) we obtain the following expression for the gauge transformation of $A$:

$$
\delta A = \partial_t \xi + \frac{1}{B} \{ \xi, V \} + \frac{1}{B} \{ \xi, A \},
$$

(6)

where $\{ \ , \ \}$ is the Moyal bracket defined by

$$
\{ \xi_1, \xi_2 \} = iB \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (\partial^n_{\bar{z}} \xi_1 \partial^n_{\bar{z}} \xi_2 - \partial^n_{\bar{z}} \xi_1 \partial^n_{\bar{z}} \xi_2),
$$

(7)

which in the limit of large $B$ approaches to the Poisson bracket:

$$
\implies_{B \to \infty} \{ \xi_1, \xi_2 \}_{P.B.} = \epsilon_{0ij} \partial^i \xi_1 \partial^j \xi_2.
$$

(8)

The gauge transformation (6) for large $B$ is then given by

$$
\delta A(x, y, t) \approx n_{\mu} \partial^\mu \xi + \frac{1}{B} \epsilon_{0ij} \partial^i \xi \partial^j A,
$$

(9)

* To my knowledge this kind of gauge transformation was first considered and used by Das, Dhar, Mandal and Wadia in the context of $c = 1$ matrix model [5].
where
\[ n_0 = 1, \quad n_i = \frac{1}{B} \epsilon_{0ij} \partial^j V. \tag{10} \]

The space vector \( n_i \) is in general space dependent.

Let us imagine that a weak space time dependent electromagnetic field is further applied to the system. Let \( A^\mu \) be its vector potential. The Lagrangian of the system must be gauge invariant with respect to \( A^\mu \), irrespective of the strong background electric and magnetic fields. This implies that \( A \) and the \( W_\infty \) gauge transformations may be realized in terms of \( A^\mu \) and its gauge transformations. Because \( W_\infty \) gauge transformations (6) and (9) are of a non-abelian type, while the ordinary gauge transformations are abelian, non-linear realizations are the only possibility. We assume that the energy scale associated with \( A^\mu \) is much smaller than that of \( V \) and \( B \), so that we can neglect the higher derivatives of \( A^\mu \). Thus it is sufficient to consider the transformation (9), which we should realize by using
\[ \delta A^\mu = \partial^\mu \Lambda. \tag{11} \]

We notice first that since \( \Lambda \) and \( \xi \) are both infinitesimal \( \xi \) should be linear in \( \Lambda \). We expand \( A \) and \( \xi \) as power series of \( A^\mu \) and its derivatives. But by looking at the structure of equation (9) we conclude that the following expressions are sufficient:
\[ A = a_\mu A^\mu + \frac{1}{B} a_{\mu\nu} A^\mu A^\nu + \frac{1}{B} a_{\mu\nu\sigma} A^\mu \partial^\nu A^\sigma + \cdots \]
\[ \xi = \Lambda + \frac{1}{B} \xi_{\mu
u} \partial^\mu \Lambda \partial^\nu + \cdots \tag{12} \]
where \( a \)'s and \( \xi_{\mu\nu} \) are space dependent parameters that are to be determined. We compute \( \delta A \) by using (11) and (12), and then use (9) to determine \( a \)'s and \( \xi_{\mu\nu} \). The results are
\[ \xi_{[\mu\nu]} = \frac{1}{2} (\xi_{\mu\nu} - \xi_{\nu\mu}), \quad \xi_{\{\mu\nu\}} = \frac{1}{2} (\xi_{\mu\nu} + \xi_{\nu\mu}), \quad \xi_{[0i]} = 0, \quad \xi_{[ij]} = -\frac{1}{2} \epsilon_{0ij}, \]
\[ a^\mu = n^\mu + \kappa^\alpha \epsilon_{\alpha\mu\rho} \partial^\rho, \quad a_{\mu\nu} = \frac{1}{2} \epsilon_{0\mu\nu} \partial^j n^\nu + \frac{1}{2} n^\rho \partial^\mu \xi_{\rho\nu}, \]
\[ a_{000} = \xi_{\{00\}}, \quad a_{00i} = a_{i00} = \xi_{\{0i\}}, \quad a_{0i0} = \xi_{\{00\}} n^i, \]
\[ a_{ij0} = \epsilon_{0ij} + \xi_{\{0i\}} n^j, \quad a_{i0j} = -\frac{1}{2} \epsilon_{0ij} + \xi_{\{ij\}}, \]
\[ a_{0ij} = \xi_{\{0j\}} n^i, \quad a_{ijk} = \epsilon_{0ij} n^k - \frac{1}{2} \epsilon_{0ik} n^j + \xi_{\{ik\}} n^j, \tag{13} \]
\[ a_{00} = \frac{1}{2} \epsilon_{00} n^i, \quad a_{0} = \epsilon_{0} n^i, \quad a_{i} = \epsilon_{i} n^j, \quad a_{ij} = \epsilon_{ij} n^k, \quad a_{ijkl} = \epsilon_{ijkl} n^l. \]
where constant $\kappa^\alpha$ and $\xi_{\{\mu\nu\}}(x, y)$ are arbitrary. We remark that $n_i$’s are in general space dependent and $\epsilon_{0ij}\partial^j n_\nu = \epsilon_{0\nu j}\partial^j n_\mu$ because of (10). $A$ is then given by

$$A = n_\mu A^\mu + \frac{1}{B}\epsilon_{0ij} n_\mu A^i \partial^j A^\mu - \frac{1}{2B}\epsilon_{0ij} n_\mu A^i \partial^\mu A^j + \frac{1}{2B}\epsilon_{0ik}(\partial^k n_j) A^i A^j +$$

$$+ \epsilon_{\alpha\mu\nu}\kappa^\alpha F^{\mu\nu} + \frac{1}{2B}\epsilon_{0ij}\partial^\alpha(n_\alpha \xi_{\{\mu\nu\}} A^\mu A^\nu) \cdots \tag{14}$$

The first line of this expression does not contain the arbitrary constants. It is the minimum realization up to the second power of $A^\mu$. Let us denote it $A_{\text{min}}$. $A_{\text{min}}$ can be further simplified:

$$A_{\text{min}} = n_\mu A^\mu + \frac{1}{2B}\epsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho + \frac{1}{2B}\epsilon_{0ij}\partial^j (n_\mu n_\mu A^\mu) \tag{15}$$

In the case of uniform background electric field, $V = Ex$ and accordingly $n_i$ is a constant vector along the $y$ direction:

$$n_i = v\delta_{iy}, \quad v \equiv \frac{E}{B} \text{ for } V = Ex \tag{16}$$

$A_{\text{min}}$ is given by

$$A_{\text{min}} = A^0 + v A^y + \frac{1}{2B}\epsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho + \frac{1}{2B}\epsilon_{0ij}\partial^0 (A^i A^j) + \frac{v}{2B}(\partial^y(A^x A^y) - \partial^x(A^y)^2) \tag{17}$$

This is precisely the expression obtained in [1].

As applications we consider briefly 1) the edge fermions in a circular droplet (a dot in a strong magnetic field) and electromagnetic interactions 2) and the electromagnetic effective Lagrangian for a $\nu = 1$ quantum Hall system.

**Electromagnetic interaction of circular droplets (dots in a strong magnetic field)**

Let $V(x, y)$ be a confining potential:

$$V(x, y) = \frac{1}{2}\alpha[(x^2 + y^2) - R^2] \tag{18}$$

We compute the energy due to the confining potential. Using (1) we obtain

$$\int dxdy \psi^\dagger(x, y)V(x, y)\psi(x, y) = \frac{\alpha}{B} \sum_{n=0}^{\infty}(n + 1 - BR^2/2)C_n^\dagger C_n \tag{19}$$
The ground state is therefore obtained by filling the negative energy states:

$$|G\rangle = \prod_{0}^{N-1} C_{n}^\dagger C_{n}|0\rangle ,$$  \hspace{1cm} (20)

where the total number of electrons \( N \) is given by

$$N = \frac{BR^2}{2} - \frac{1}{2} .$$  \hspace{1cm} (21)

We keep only those electron operators that describe modes in the neighbourhood of the Fermi level. We define

$$C_{N+n-1/2} = b_{n}, \quad v_{\theta} = \alpha/B$$  \hspace{1cm} (22)

With respect to the new vacuum (20), \( b_{n} \) with positive \( n \) is an annihilation operator while with negative \( n \), a hole creation operator. (19) is then

$$\int dxdy\psi^\dagger(x,y)V(x,y)\psi(x,y) = \frac{\alpha}{B} \sum_{n} (n + \frac{1}{2} - N)C_{n}^\dagger C_{n} = v_{\theta} \sum_{n=\text{half integer}} n b_{n}^\dagger b_{n} .$$  \hspace{1cm} (23)

We restrict the range of sum to be \( n \ll N \), although eventually we will set \( -\infty < n < \infty \) after taking the large \( N \) limit.

We normal order the density operator with respect to the new vacuum. We obtain

$$\rho(x,y) \equiv \psi^\dagger(x,y)\psi(x,y) = \frac{B}{2\pi} \Theta(r,R) + \frac{B}{2\pi} e^{-|z|^2} \psi^\dagger(z)\psi(\bar{z}) : .$$  \hspace{1cm} (24)

where

$$\Theta(r,R) = e^{-|z|^2} \sum_{n=0}^{N-1} \frac{|z|^2}{n!} .$$  \hspace{1cm} (25)

In the limit of large \( B \) and \( N \) with \( R \) finite, we obtain

$$\lim_{B \to \infty, N \to \infty} \Theta(r,R) = \theta(R-r) .$$  \hspace{1cm} (26)

(see Appendix), where \( \theta(r) \) is the standard step function. Similarly we obtain

$$\lim_{B \to \infty, N \to \infty} \frac{B}{2\pi} e^{-|z|^2} : \psi^\dagger(z)\psi(\bar{z}) := \frac{1}{R} \delta(r-R) : \chi^\dagger(\theta)\chi(\theta) : .$$  \hspace{1cm} (27)
\begin{equation}
\chi(\theta) = \frac{1}{\sqrt{2\pi}} \sum_n e^{-in\theta} b_n, \quad z = |z|e^{i\theta}.
\end{equation}

Using (28) we can express (23) as

\begin{equation}
\int dx dy \psi^{\dagger}(x,y)V(x,y)\psi(x,y) = \int d\theta : \chi^{\dagger}(\theta) \left( -iv_\theta \partial_\theta \right) \chi(\theta) :
\end{equation}

Now it is straightforward to compute the effective Lagrangian (5). We obtain

\begin{equation}
L = L_{\text{bulk}} + L_{\text{boundary}},
\end{equation}

\begin{equation}
L_{\text{bulk}} = \frac{1}{2\pi} \int dxdy \left( -A^0 + \frac{1}{2\pi} b + \mathcal{L}_{C-S} \right),
\end{equation}

\begin{equation}
L_{\text{boundary}} = \oint d\theta \left[ \chi^{\dagger} \left( i\partial_t - A_0 \right) - v_\theta (i\partial_\theta - A_\theta) \right) \chi - \frac{v_\theta}{4\pi} A_\theta^2\right],
\end{equation}

where

\begin{equation}
\mathcal{L}_{C-S} = -\frac{1}{4\pi} \epsilon_{0ij} \left( A_i \partial^0 A^j - 2A_i^j \partial^j A^0 \right)
\end{equation}

and

\begin{equation}
b = \epsilon_{0ij} \partial^j A^i, \quad A_\theta = \epsilon_{0ij} x^j A^i.
\end{equation}

**Electromagnetic effective Lagrangian for \( \nu = 1 \) quantum Hall system**

Let us consider a rectangular quantum Hall system of size \( L_x \times L_y \), \( L_x \gg L_y \). Neglecting the edges of the \( x \) ends, we can represent this system by choosing a confining potential, which depends only on \( y \). The potential is flat in the middle and linearly rising at the both ends. Since in the limit of \( A^\mu = 0 \) the Hamiltonian is given by

\[ H = \int d^2z e^{-|z|^2} \bar{\psi}(z)\psi(z)V(y) \]

one can diagonalize it by choosing a \( y \) diagonal basis for the LLL electron field, which we simply write as \( \psi(y) \) (the normalization is fixed by the canonical anti-commutation relation): \( H = \int dy \bar{\psi}(y)\psi(y)V(y) \). This means that \( V(y) \) is the dispersion of this effective one dimensional system. We adjust the chemical potential such that the zero energy Fermi level intersects with \( V \) precisely at \( y = \pm \frac{L_y}{2} \). The ground
state is then the state with all the negative energy single particle levels filled by the LLL electrons. The low energy excitations are then the motion of the electrons near the Fermi level, i.e. electrons near the edges. These are edge excitations. We make the electron density operator normal ordered with respect to the ground state as before. After the normal ordering we keep only these degrees of freedom of the right and left electrons near the Fermi level. The calculations are entirely analogous as before. We obtain

$$\rho(x, y) \approx \frac{B}{2\pi} \left( \Theta(y; -\frac{L_y}{2}, \frac{L_y}{2}) + e^{-|z_R|^2} : \psi_R(z_R) \psi_R(\bar{z}_R) : + e^{-|z_L|^2} : \psi_L(z_L) \psi_R(\bar{z}_L) : \right),$$

(33)

where

$$\Theta(y; -\frac{L_y}{2}, \frac{L_y}{2}) = \sqrt{\frac{B}{\pi}} \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} dy' e^{-B(y'-y)^2} \rightarrow_{B \to \infty} \theta(y - \frac{L_y}{2}) \theta(y + \frac{L_y}{2}),$$

$$\psi_R(z_R) = \frac{B}{\pi} \left( \frac{1}{4} \int_{-\Delta}^{\Delta} dy' e^{-\frac{1}{4}(y'^2B - |z_R|^2) - \sqrt{2B} y' \bar{z}_R} \psi_R(y') \right),$$

(34)

$$z_R = \sqrt{\frac{B}{2}} (x + i(y + \frac{L_y}{2})), \quad \psi_R(y) = \psi(y + \frac{L_y}{2}).$$

Substituting these into the Lagrangian (5), where we keep only $A_{\text{min}}$, we obtain the following effective Lagrangian after some calculations:

$$L = L_{\text{bulk}} + L_{R\text{boundary}} + L_{L\text{boundary}},$$

(35)

$$L_{\text{bulk}} = \int_D \int dy \left( -A_0^0 + \frac{1}{2\pi} b + L_{C-S} \right),$$

(35a)

$$L_R = \int dx \left( \psi_R^\dagger(x,t) \left( i \partial_t - A_0^0_L \pm v_R(i \partial_x + A_0^R_L) \right) \psi_R (x,t) - \frac{1}{4\pi} v_R \left( A_R^x (x,t) \right)^2 \right),$$

(35b)

where we set $\Delta \to \infty$ and used $\psi_R^\dagger(x)$ for the Fourier transform of $\psi_R(y)$. When we use this Lagrangian, we must assume $\psi_R^\dagger(x)$ is a slowly varying function.

The $x$ component of the current density can be derived from (35) by taking a functional derivative with respect to $A^x$. It consists of a bulk current and the edge currents. The
bulk current is the contribution from the Chern-Simon term and only this contributes to the Hall current in the present formulation [6].

Finally, we should remark that the boundary Lagrangians (30b) and (35b) are the Lagrangians of chiral fermions interacting with an external electromagnetic field. The anomalies associated with these systems and their corresponding bosonization have been discussed by many authors, for example in [7]. Taking these anomalies into account one proves that the Lagrangians (30) and (35) are gauge invariant [8].

We also remark that calculations similar to these two applications discussed above have been done recently by the CERN group [9].

Appendix: Proof of (26) and (27)

Let $|z|^2 = x = \frac{B}{2} r^2$

$$\Theta(r; R) = F(x, N) \equiv e^x \sum_{n=0}^{N-1} \frac{x^n}{n!},$$  \hspace{1cm} (36)

$$\frac{\partial}{\partial x} F(x, N) = -F(x, N) + F(x, N - 1) = -e^{-x} x^{N-1}/(N-1)!.$$  \hspace{1cm} (37)

On the other hand

$$\psi(\bar{z}) = \sum_{n=-N+1/2}^{\infty} \frac{(z)^{N+n-1/2}}{\sqrt{(N+n-1/2)!}} b_n \approx \frac{|z|^{N-1/2}}{\sqrt{(N-1/2)!}} e^{-iN\theta} \sqrt{2\pi} \chi(\theta)$$  \hspace{1cm} (38)

Thus,

$$\frac{B}{2\pi} e^{-|z|^2} : \psi^\dagger(z)\psi(\bar{z}) : \approx Be^{-|z|^2} \frac{(|z|^2)^{(N-1/2)}}{(N-1/2)!} : \chi^\dagger(\theta)\chi(\theta) : \approx -\frac{\partial}{\partial x} F(x, N) B : \chi^\dagger(\theta)\chi(\theta) : .$$  \hspace{1cm} (39)

We evaluate $e^{-x} x^{N-1}$ by the saddle point method:

$$e^{-x} x^{N-1} = e^{-(x-(N-1)lnx)} \approx e^{-x_0 + x_0 \ln x_0} e^{-\frac{1}{2}(x-x_0)^2/x_0} \quad x_0 = N - 1.$$  

Using this and Sterling’s formula $(N - 1)! = x_0! \approx \sqrt{2\pi} e^{-x_0} (x_0)^{x_0+1/2}$ we obtain

$$-\frac{\partial}{\partial x} F(x, N) = -\frac{1}{\sqrt{2\pi x_0}} e^{-\frac{1}{2}(x-x_0)^2/x_0} \approx \frac{1}{BR} \sqrt{\frac{B}{\pi}} e^{-(r-R)^2} \implies_{B \to \infty} \frac{1}{BR} \delta(r - R),$$  \hspace{1cm} (40)
where we used (21). From (39) and (40), (27) follows. Notice \( F(0,N) = 1 \) from the definition (36). With this boundary condition we integrate (40) to obtain (26).

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