Introducing Time Dependence into the Static Maxwell Equations

Avraham Gal

Racah Institute of Physics, The Hebrew University,
Jerusalem 91904, Israel
(October 24, 2018)

Using to a minimum extent special relativity input, and relying on the Lorentz-force expression for the force acting on a charged particle in motion under the influence of electric (\(E\)) and magnetic (\(B\)) fields, the Maxwell curl equations are shown to follow from the covariance of the two Maxwell divergence equations \(\nabla \cdot E = 4\pi\rho\) and \(\nabla \cdot B = 0\).

I. INTRODUCTION

The Maxwell equations in vacuum provide the theoretical framework for studying electromagnetism. Most of the undergraduate introductory-physics textbooks on Electricity and Magnetism (EM) start by introducing the basic observations made, prior to Maxwell, regarding electric charges and currents. The Maxwell equations are then ‘derived’ by unifying these observations into physical laws governing the space- and time evolution of the electric field \(E\) and the magnetic field \(B\) (for a non exhaustive list of representative textbooks, see Refs. [1–6]). The equations which one usually finds easier to motivate than others are the two divergence equations

\[
\nabla \cdot E = 4\pi\rho ,
\]

(1)

and

\[
\nabla \cdot B = 0 .
\]

(2)

Eq. (1) is the differential form of Gauss’ law, relating the electric field \(E\) to its charge sources, where \(\rho\) is the charge density. Eq. (2), for the magnetic field \(B\), expresses the non observation, and consequently the non existence of magnetic charges. One then proceeds to explain how magnetism is generated by currents rather than by charges, resulting in the magnetic-field Maxwell curl equation

\[
\nabla \times B = \frac{4\pi}{c} J ,
\]

(3)

whereas the electric field is curl free:

\[
\nabla \times E = 0 .
\]

(4)

The constant \(c\) in Eq. (3) is the speed of light in vacuum. Equations (3) and (4) apply to static situations, where any time variation of the underlying charge density \(\rho\) and current density \(J\), and hence of \(E\) and \(B\) respectively, is excluded. Allowing for time dependence, the above Maxwell curl equations become

\[
\nabla \times B = \frac{4\pi}{c} J + \frac{1}{c} \frac{\partial E}{\partial t} ,
\]

(5)
\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}. \] (6)

However, it turns out to be more problematic, and certainly less transparent, to motivate Eqs. (5,6) by the appropriate phenomena, e.g. adding the displacement current for ‘deriving’ Eq. (5) and electromagnetic induction experiments for ‘deriving’ Eq. (6). In particular, Galili and collaborators [7–9] have recently drawn attention to the subtleties involved in relating Faraday’s law to Maxwell’s equation (3).

The purpose of this note is to show that one may derive Eqs. (5,6) directly from Eqs. (1,2) using a minimum input of EM (the Lorentz force) and special relativity physics material that is within the scope of normal electromagnetism curriculum, by requiring that Eqs. (1,2), due to charge conservation, hold in any inertial coordinate frame. Specifically, we will require (i) a knowledge of and familiarity with the Lorentz force acting on a charge \( q \) moving at an instantaneous velocity \( \mathbf{v} \); and (ii) a knowledge of what a Lorentz four vector means, particularly for space-time \((\mathbf{r}, ct)\) and for momentum-energy \((\mathbf{p}, E/c)\), but also for the current-charge density four vector \((\mathbf{J}, c\rho)\), in terms of a Lorentz boost from one inertial frame to any other inertial one. Once the Maxwell equations (3,4) are derived from (1,2), the physics contents and phenomenology of these former equations can be easily and systematically unfolded within a regular undergraduate introductory physics course.

II. AUXILIARY MATERIAL

A. Lorentz transformation

A boost with velocity \( \mathbf{V} \) is applied in the \( x \) (longitudinal) direction, without affecting the \( y, z \) transverse directions:

\[ x' = \gamma (x - \beta ct), \quad ct' = \gamma (ct - \beta x), \] (7)

where \( \beta = \frac{V}{c}, \gamma = \left(1 - \beta^2\right)^{-1/2} \). From (7) one shows that the derivatives transform as follows:

\[ \frac{\partial}{\partial x'} = \gamma \left( \frac{\partial}{\partial x} + \beta \frac{1}{c} \frac{\partial}{\partial t} \right), \quad \frac{1}{c} \frac{\partial}{\partial t'} = \gamma \left( \frac{1}{c} \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial x} \right). \] (8)

The Lorentz transformation rule (7) applies also to the four momentum \((\mathbf{p}, E/c)\) and to the four current density \((\mathbf{J}, c\rho)\). In particular, for the latter one:

\[ J'_x = \gamma \left( J_x - \beta c \rho \right), \quad c \rho' = \gamma \left( c \rho - \beta J_x \right). \] (9)

In order to motivate the (perhaps less familiar) transformation rule (9), one recalls the continuity equation

\[ \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0, \] (10)

which is usually taught within the context of charge conservation in EM courses [1] and which can be written in a covariant form \( \partial_\mu J^\mu = 0 \), implying that \( J^\mu (\mu = 1, 2, 3, 4) \) is a four vector.
B. Lorentz force

This is the basic dynamic law which governs the motion of charged particles under the influence of electromagnetic fields:

$$\mathbf{F} = q \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right).$$

(11)

In many respects, Eq. (11) may be regarded as the operational definition of the electromagnetic fields [1]. Below, we will use this Lorentz-force expression in order to determine the Lorentz transformation properties of $\mathbf{E}$ and $\mathbf{B}$. A caveat of the present approach, as generally is the case when relying on Eq. (11), is that the constant $c$ is the same speed of light constant as the one appearing in the Lorentz transformation (7).

C. Transformation properties of $\mathbf{E}$ and $\mathbf{B}$

It is shown in the Appendix that the form of the Lorentz force, Eq. (11), leads to the following Lorentz transformation properties of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$:

$$E'_x = E_x, \quad E'_y = \gamma (E_y - \beta B_z), \quad E'_z = \gamma (E_z + \beta B_y),$$

(12)

$$B'_x = B_x, \quad B'_y = \gamma (B_y + \beta E_z), \quad B'_z = \gamma (B_z - \beta E_y).$$

(13)

III. THE MAXWELL EQUATIONS

Here we show, using the auxiliary material of Sect. 1, how the Maxwell equation (1) leads to the Maxwell equation (5), and how the Maxwell equation (2) leads to the Maxwell equation (6). The key assumption is that the laws of electromagnetism expressed by Eqs. (1) and (2) (essentially charge conservation) have the same form in any inertial frame of reference. We start with Eq. (1) in the primed coordinate system,

$$\nabla' \cdot \mathbf{E}' = 4\pi \rho',$$

(14)

and express the primed quantities in terms of the unprimed ones, using Eqs. (8) (2). One obtains:

$$\gamma \left( \frac{\partial}{\partial x} + \beta \frac{1}{c} \frac{\partial}{\partial t} \right) E_x + \frac{\partial}{\partial y} \gamma (E_y - \beta B_z) + \frac{\partial}{\partial z} \gamma (E_z + \beta B_y) = \frac{4\pi}{c} \gamma (c\rho - \beta J_x).$$

(15)

Identifying the terms on the l.h.s. which combine to $\gamma \nabla \cdot \mathbf{E}$, and on the r.h.s. the term $\gamma 4\pi \rho$, we make use of Eq. (11) in the unprimed coordinate system to cancel out these terms, resulting in

$$\frac{1}{c} \frac{\partial}{\partial t} E_x - \left( \frac{\partial}{\partial y} B_z - \frac{\partial}{\partial z} B_y \right) = -\frac{4\pi}{c} J_x,$$

(16)
(\nabla \times \mathbf{B})_x = \frac{4\pi}{c} J_x + \frac{1}{c} \frac{\partial}{\partial t} E_x , \quad (17)

which in view of the arbitrariness in choosing the boost (x) direction leads to the curl equation (3)

\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial}{\partial t} \mathbf{E} . \quad (18)

It is worth noting that applying the divergence operation to both sides of the Maxwell equation (18) which we herewith proved starting from the Maxwell equation (4), and making use of the latter, one immediately recovers the continuity equation (10) which provides the full (non static) content of the underlying assumption of charge conservation.

Similarly, starting with Eq. (2) in the primed coordinate system,

\nabla' \cdot \mathbf{B}' = 0 , \quad (19)

and expressing the primed quantities in terms of the unprimed ones using Eqs. (8,13), one obtains:

\gamma \left( \frac{\partial}{\partial x} + \beta \frac{1}{c} \frac{\partial}{\partial t} \right) B_x + \frac{\partial}{\partial y} \gamma (B_y + \beta E_z) + \frac{\partial}{\partial z} \gamma (B_z - \beta E_y) = 0 . \quad (20)

The terms on the l.h.s. combining to \gamma \nabla \cdot \mathbf{B} give rise to zero by Eq. (3), while the remaining terms lead to

\frac{1}{c} \frac{\partial}{\partial t} B_x + \left( \frac{\partial}{\partial y} E_z - \frac{\partial}{\partial z} E_y \right) = 0 , \quad (21)

which upon permuting cyclically the x, y, z directions produces the curl equation (8):

\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{B} . \quad (22)

### IV. DISCUSSION AND CONCLUSION

We have shown that accepting the Lorentz force, plus a minimal amount of special relativity material, it becomes possible to derive each of the (vector) Maxwell curl equations (3,8) from one of the (scalar) Maxwell divergence equations (4,2), respectively. This suggests a more economical way of getting into the unified framework of the Maxwell equations than is done in most of the other approaches of teaching an introductory EM course. In order to understand how come, on two separate occasions, one divergence equation gives rise to three equations which are combined into a curl equation, we note that the Maxwell equations (4) and (8) can be lumped covariantly together into the form

\frac{\partial F_{\alpha \beta}}{\partial x^\beta} + \frac{4\pi}{c} j^\alpha = 0 , \quad (23)
where $F^{\alpha\beta}$ is the $4 \times 4$ antisymmetric field tensor and the greek indices run over the four space-time components $[0,1,2,3]$. $J^\alpha$ is the current density four vector, see Eq. (1). If the form (23) is satisfied for one of these four-vector components $\alpha$, by covariance it is also satisfied for the other ones, since no direction in the four-dimensional space has preference over the other ones. Eq. (1) corresponds to the time-like component, whereas the vector Eq. (5) corresponds to the three space-like components. A corollary of Eq. (23), due to the antisymmetry of $F^{\alpha\beta}$, is the continuity equation (10), in agreement with the discussion in Sect. III following Eq. (18).

Similarly, the Maxwell equations (2) and (6) can be lumped covariantly together into the form

$$\varepsilon^{\alpha\beta\gamma\delta} \frac{\partial F_{\gamma\delta}}{\partial x^\beta} = 0,$$

where $\varepsilon^{\alpha\beta\gamma\delta}$ is the fully antisymmetric unit tensor in four dimensions $[0,1,2,3]$. Eq. (2) corresponds to the time-like component, whereas the vector Eq. (6) corresponds to the three space-like components.

Obviously, if students were familiar with the antisymmetric field tensor $F^{\mu\nu}$, and once Eqs. (1,2) are shown to be equivalently written as the time-like components of Eqs. (23, 24) respectively, then it would have been straightforward to argue for the validity of Eqs. (5,6) respectively, using only the argument of Lorentz covariance. However, since most introductory EM courses do not introduce $F^{\mu\nu}$, the right thing to do in such a case would be to follow the explicit construction outlined in the present note which emphasizes the frame independence of charge conservation.

**ACKNOWLEDGMENTS**

Stimulating discussions with, and useful advice from, Igal Galili and Issachar Unna are gratefully acknowledged.

**APPENDIX: DERIVATION OF EQS. (12, 13) FOR BOOSTING E AND B**

Let $K$ denote the laboratory coordinate frame, where a charged particle (charge $q$) moves with an instantaneous velocity $\mathbf{v}$ which defines the $x$ axis. The Lorentz force acting on it is given by

$$\mathbf{F} = q \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right),$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields. Define the instantaneous rest frame of the particle, where

$$\mathbf{F'} = q\mathbf{E'}.$$  

The frame $K'$ moves with velocity $\mathbf{V} = \mathbf{v}$ with respect to $K$. We assume a knowledge of the force transformation law which is derived by making use how the momentum $\mathbf{p}$ and time $t$ transform and recalling that $\mathbf{F} = d\mathbf{p}/dt$. Thus, $F'_x = F_x$, leading to
\[ E'_x = E_x + \left( \frac{\mathbf{v}}{c} \times \mathbf{B} \right)_x = E_x , \]  

(27)

since \( \mathbf{v} \) is parallel to the \( x \) direction. Similarly, \( F'_y = \gamma F_y \), leading to

\[ E'_y = \gamma E_y + \gamma \left( \frac{\mathbf{v}}{c} \times \mathbf{B} \right)_y = \gamma (E_y - \beta B_z) , \]

(28)

and \( F'_z = \gamma F_z \), leading to

\[ E'_z = \gamma E_z + \gamma \left( \frac{\mathbf{v}}{c} \times \mathbf{B} \right)_z = \gamma (E_z + \beta B_y) . \]

(29)

Eqs. (27-29) give \( \mathbf{E}' \) in terms of \( \mathbf{E} \) and \( \mathbf{B} \), coinciding with Eq. (12) of the main text.

In order to proceed with the derivation of Eq. (13) for \( \mathbf{B}' \) in terms of \( \mathbf{E} \) and \( \mathbf{B} \), we invert Eq. (28) by noting that interchanging unprimed and primed quantities amounts to just changing the sign of \( \beta \):

\[ E_y = \gamma (E'_y + \beta B'_z) . \]

(30)

Substituting \( E'_y \) from Eq. (28), one gets

\[ E_y = \gamma \left[ \gamma (E_y - \beta B_z) + \beta B'_z \right] , \]

(31)

and since \( (1 - \gamma^2) = -\beta^2 \gamma^2 \), the final outcome is

\[ B'_z = \gamma (B_z - \beta E_y) . \]

(32)

Manipulating similarly on Eq. (29), one gets

\[ B'_y = \gamma (B_y + \beta E_z) . \]

(33)

Eqs. (32-33) for the transverse components of \( \mathbf{B} \) agree with the corresponding expressions in Eq. (13). Further arguments have to be invoked for showing that \( B'_x = B_x \), thus completing the derivation of Eq. (13). To spare a technically more complicated procedure, we note that the transverse components of \( \mathbf{B} \) in Eq. (13) are obtained from those of \( \mathbf{E} \) in Eq. (12) by interchanging \( \mathbf{E} \) and \( \mathbf{B} \) plus reversing the sign of \( \beta \). Equation (27) above for the longitudinal component of \( \mathbf{E} \), namely \( E'_x = E_x \), then suggests a similar rule \( B'_x = B_x \) for the longitudinal component of \( \mathbf{B} \). It is useful to record at this stage, as follows from Eqs. (12,13), that \( \mathbf{E} \cdot \mathbf{B} \) and \( \mathbf{E}^2 - \mathbf{B}^2 \) are invariant under Lorentz transformations [1], leaving however the demonstration of it to a later stage of the curriculum.

[1] E.M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1985).
[2] H.C. Ohanian, *University Physics* (Norton, New York, 1989).
[3] M. Alonso and E.J. Finn, *Physics* (Addison-Wesley, Reading, MA, 1992).
[4] R. Chabay and B. Sherwood, *Electric and Magnetic Interactions* (Wiley, New York, 1995).
[5] H.D. Young and R.A. Freedman, *University Physics* (Addison-Wesley, Reading, MA, 1996).
[6] D. Halliday, R. Resnick and J. Walker, *Fundamentals of Physics* (Wiley, New York, 2001).
[7] I. Galili and D. Kaplan, “Changing approach in teaching electromagnetism in a conceptually oriented introductory physics course”, Am. J. Phys. 65(7), 657-667 (1997).
[8] I. Galili and Y. Lehavi, “Faraday’s law of electromagnetic induction revisited”, submitted to Am. J. Phys. (Dec. 2000).
[9] I. Galili and D. Kaplan, “Using relativistic approach in teaching electromagnetic induction”, submitted to Am. J. Phys. (Feb. 2001).
[10] J.D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).
[11] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, UK, 1975).