Critical Collapse of Skyrmions

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We study first order phase transitions in the gravitational collapse of spherically symmetric skyrmions. Static sphaleron solutions are shown to play the role of critical solutions separating black-hole spacetimes from no-black-hole spacetimes. In particular, we find a new type of first order phase transition where subcritical data do not disperse but evolve towards a static regular stable solution. We also demonstrate explicitly where subcritical data do not disperse but evolve towards a separating black-hole spacetimes from no-black-hole spacetimes.

In order to solve the initial value problem numerically we use the polar time slicing and the areal radial coordinate. The two coupling constants \( \alpha \) and \( e \) are hereafter set to one, which amounts to using \( f/e \) and \( 1/e^2 \) as the units of mass and length, respectively.

For the chiral field we assume the hedgehog ansatz \( \mathbf{u} \), where, for some initial data, the fundamental Bartnik-McKinnon sphaleron (static soliton with one unstable mode) plays the role of a critical solution. In this case there is a gap in the spectrum of black-hole masses which is reminiscent of the first order phase transition.

Surprisingly, none of the models studied so far had a stable static solution (a "star") and consequently the possible endpoints of evolution were basically reduced to the collapse/ dispersion alternative. As far as we know, the only exception is the collapse/oscillatior alternative recently found in the collapse of massive scalar field [1]. If a model has a star-like solution, then there will be an open set of initial data which evolve towards this solution (basin of attraction) and it is natural to expect the occurrence of critical behaviour at the boundary of this set.

The Einstein-Skyrme model which possesses the whole zoo of static regular solutions, both stable and unstable, seems to be a suitable testing ground to address this issue. In this letter we focus our attention on a new type of first order phase transition we found in this model for baryon-number-one data. We also briefly discuss a novel feature of mass gap in the baryon-number-zero sector. A detailed description of both first and second order phase transitions will be published elsewhere [2].

We consider the Einstein-Skyrme system, so the matter in our model is an \( SU(2) \)-valued scalar function \( U(x) \) (called a chiral field) with dynamics given by the Lagrangian

\[
L = \frac{f^2}{4} Tr(\nabla_a \nabla^a U^{-1}) + \frac{1}{32e^2} Tr(\nabla_a U)^{-1} \nabla_b U \nabla_b U^{-1}],
\]

where \( \nabla_a \) is the covariant derivative with respect to the spacetime metric. The two coupling constants \( f \) and \( e \) are hereafter set to one, which amounts to using \( f/e \) and \( 1/e^2 \) as the units of mass and length, respectively.

We specialize to spherical symmetry. For the metric we use the polar time slicing and the areal radial coordinate

\[
d s^2 = -e^{-2\delta(r,t)} N(r,t) d t^2 + N^{-1}(r,t) d r^2 + r^2 d \Omega^2.
\]

For the chiral field we assume the hedgehog ansatz \( U = \exp(i \sigma \cdot \vec{r} F(r,t)) \), where \( \vec{\sigma} \) is the vector of Pauli matrices. Using overrots and primes to denote \( \partial / \partial t \) and \( \partial / \partial r \) respectively, we introduce an auxiliary variable \( P = u e^\delta N^{-1} F \), where \( u = r^2 + 2 \sin^2 F \). Then, the Einstein-Skyrme equations reduce to

\[
\dot{F} = e^{-\delta} N \frac{P}{u},
\]

\[
\dot{P} = (e^{-\delta} N u F')' + \sin(2F) e^{-\delta} \left( N \left( \frac{P^2}{u^2} - F'^2 \right) - \frac{u}{2r^2} - \frac{1}{2} \right),
\]

\[
\dot{N} = -\frac{2\alpha}{r} e^{-\delta} N^2 P F',
\]

\[
\delta' = -\frac{\alpha u}{r} \left( \frac{P^2}{u^2} + F'^2 \right),
\]

\[
N' = \frac{1 - N}{r} - \frac{\alpha}{r} \left( 2 \sin^2 F + \frac{\sin^4 F}{r^2} + u N \left( \frac{P^2}{u^2} + F'^2 \right) \right).
\]

Here \( \alpha = 4\pi G f^2 \) is the dimensionless coupling constant. In order to solve the initial value problem numerically we...
have implemented a free evolution scheme using a second order finite difference method on uniform grid. The conservation of the hamiltonian constraint (7) was used only to check the accuracy of the code. To ensure regularity at the center we impose the boundary condition $F(r, t) = O(r)$ for $r \to 0$. To minimize reflections from the outer boundary of the grid we impose a sort of outgoing wave condition there. Asymptotic flatness requires that $F(r, 0) = Br + O(1/r^2)$ at infinity, where the integer $B$, called the baryon number, is equal to the topological degree of the chiral field. As long as no horizon forms, the coupling constant $\alpha$ equals $\alpha_1$, and disappear for $\alpha > \alpha_1$. For given $\alpha$ the mass $m_u$ of the unstable soliton is larger than the mass $m_s$ of the stable soliton (for example, for $\alpha = 0.02$ we have $m_s \simeq 68.05$ and $m_u \simeq 80.70$). As $\alpha \to 0$, $m_u$ tends (from below) to the mass of the flat-space skyrmion ($\simeq 72.92$), whereas $m_u \to \infty$, which is the manifestation of a nonperturbative character of the solution $X^u$.

Since for $B = 1$ the dispersion of the chiral field to infinity is forbidden, generic initial data end up either as black holes or as the stable soliton $X^s$. The respective basins of attraction of these two final states depend on the coupling constant $\alpha$. This is illustrated in Fig. 1 for a typical time-symmetric kink-type initial data. Notice that for $\alpha < \alpha_1$ there are two critical configurations along the one-parameter family $F_p$ (the points $A$ and $B$ on the dashed line segment on Fig. 1). Since the parameter $p$ (the width of a kink) is inversely proportional to the ”condensation” of initial mass, the transition at the point $A$ could be interpreted as a ”weak/strong” field transition. However at the point $B$ it is the ”weaker” configuration that collapses, simply because it is unable to get rid of the excess mass by the time it shrinks to size of the stable soliton. This behaviour indicates that it is difficult to formulate a criterion for collapse in terms of initial data alone.

The critical initial data lying on the boundary of the basin of attraction of $X^s$ asymptote to the unstable solution $X^u$. In other words, the codimension one stable manifold of the solution $X^u$ divides (locally) the phase space into collapsing and non-collapsing data. Near-critical data approach $X^u$, stay in its vicinity for some time, and eventually collapse to a black hole or decay into $X^s$. This is shown in Fig. 2 for a marginally subcritical data (lying slightly below the point $B$ on Fig. 1).

During the final stage of the evolution of a marginally supercritical solution (i.e. when the solution runs away from the vicinity of the critical solution and forms a black hole) almost no energy is radiated away to infinity. Therefore the mass gap in this first order phase transition equals $m_u$ (up to 0.1%).

For subcritical data the process of settling down to the stable soliton seems to have a universal late-time be-
haviour which is dominated by a fundamental quasinormal mode. The parameters of this mode depend strongly on $\alpha$. Having convinced ourselves that these parameters are indeed universal, that is do not depend on the excitation, we have determined them from the evolution of finite perturbations of $X^*$ (see Fig. 3). The details of such perturbation are washed out very rapidly and a characteristic exponentially damped "ringing" dominates the late time evolution.

The above results are in perfect agreement with linear \cite{3} and nonlinear \cite{5} stability analysis of gravitating skyrmions. In the intermediate asymptotics the evolution of near-critical data is well approximated by the linearization about $X^u$

$$X_p(r, t) \approx X^u(r) + C(p - p^*)e^{\lambda t} \delta X^u(r) + \text{decaying modes},$$

where $\delta X^u(r)$ is the single unstable eigenmode associated with a positive eigenvalue $\lambda$. Depending on whether $p > p^*$ or $p < p^*$, the solution $X_p$ eventually collapses to a black hole or decays to $X^*$. The "lifetime" $T$ of a near-critical solution staying in the vicinity of $X^u$ is determined by the time in which the amplitude of the unstable mode grows to a finite size: $|p - p^*|e^{\lambda T} \sim O(1)$, which gives $T \sim -\lambda^{-1} \ln |p - p^*|$. Thus, the larger $\lambda$, the better fine-tuning is required to see the solution $X^u$ clearly pronounced as the intermediate attractor.

We have verified the formula (8) by comparing the unstable eigenmode $\delta X^u$ with the snapshots of a near-critical solution departing from the vicinity of $X^u$. This is shown in Fig. 4 for the same initial data as in Fig. 2. For $t = 24$ the profile of $\dot{F}(r, 24)$ is practically indistinguishable from the (suitably normalized) unstable eigenmode $\delta F^u(r)$. A slight deviation of $\dot{F}(r, 23)$ from $\delta F^u(r)$ for large $r$ is due to the fact that by $t = 22$ the decaying modes in (8) have not yet died away completely. On the other hand, for $t = 25$ the deviation for small $r$ signals the onset of nonlinear regime.

Using (8), we have also computed the eigenvalue $\lambda$ directly from the nonlinear evolution by monitoring $F_p(r, 24)$ at several discrete radii $r_i$, $i = 1, ..., n$, and evaluating the averaged quantity

$$\frac{1}{\lambda T} \sum \ln \left( \frac{\dot{F}_p(r_i, 24 + \Delta t)/\dot{F}_p(r_i, 24)}{\delta F^u(r_i)} \right)$$

for some small $\Delta t$. The result agrees up to three decimal places with the eigenvalue obtained via linear stability analysis.

$B = 0$: In this topological sector, the Eqs.(3-7) have a pair of static regular solutions for $\alpha < \alpha_0 \simeq 0.00147$. One of them has two unstable modes, hence it plays no role in the evolution of generic initial data. The second solution, call it $Y^u$, has one unstable mode, so it is a candidate for the critical solution. Indeed, analysing the ingoing "generalized Gaussian" profile $F(r, 0) = Ar^3 e^{-(r-r_0)^2/\Delta^2}$, we have found $Y^u$ as the intermediate attractor at the border between collapse and dispersion. Since in this case the critical behaviour is analogous to the first order phase transition in the EYM model \cite{3}, here we discuss only the issue of mass gap in this transition. Analysing near-supercritical solutions we found that, in contrast to the previously studied cases, the mass gap is not equal to the mass of the sphaleron. In order to determine the mass gap more precisely we adopted the following strategy. Rather than improving the accuracy of fine-tuning (which is computationally time-consuming because the eigenvalue of the unstable eigenmode is rather large), we began the evolution with specially prepared initial data of the form of the static solution $Y^u$ plus the unstable eigenmode with a small positive amplitude: $Y(r, 0) = Y^u(r) + \epsilon \delta Y^u(r)$. In other words, we pushed the static solution along its unstable manifold and let it collapse. It turned out that the mass

![FIG. 3. Quasinormal ringing about the stable soliton for $\alpha = 0.03$. We plot $dN = |N(r_0, t) - N^*(r_0)|$ from the evolution of initial data $F(r, 0) = \pi \tanh(r)$ (dots). Here $r_0 = 0.8$. The least squares fit for the times $t > 10$ (represented by the solid line) gives the frequency $\omega \simeq 0.494$ and the damping rate $\tau \simeq 11.69$.](image1)

![FIG. 4. The snapshots of $\dot{F}(r, t)$ (solid lines) departing from the vicinity of the critical solution $X^u$. On each snapshot we superimpose (dashed lines) the unstable eigenmode $\delta F^u(r)$ computed via linear perturbation analysis. For $t = 24$ the amplitude of $\delta F^u(r)$ is normalized to the maximum of $F(r, 24)$. The corresponding amplitudes for $t = 23$ and $t = 25$ are rescaled by the factors $e^{-}\tau$ and $e^{\lambda}$, respectively.](image2)
gap is considerably smaller than the mass of the critical solution (which of course sets the upper bound for a mass gap). This is illustrated in Fig. 5, where one can clearly see how a substantial amount of the initial mass is being radiated away to infinity. The resulting black hole has the mass $\simeq 253.1$, which is 75.6% of the mass of $Y^\nu$.

![Graph showing mass aspect $\mu(r, t)$](image)

FIG. 5. The mass aspect $\mu(r, t)$ (defined by $N = 1 - 2\mu/r$) from the evolution of the static solution $Y^\nu$ perturbed along the unstable direction (for $\alpha = 0.00145$). The mass inside radius $r$ at time $t$ measured in units of $f/e$ is given by $m(r, t) = \frac{4\pi}{\alpha} \mu(r, t)$. The plateau of $\mu(r, t)$ determines the mass gap.

Concluding, we believe that the results presented here have clarified certain aspects of first order phase transitions in gravitational collapse and, in combination with the results of [3] and [4], they help to understand which features of this phenomenon are generic.

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