We discuss the quantum search algorithm using complex queries that has recently been published by Grover [4]. We recall the algorithm adding some details showing which complex query has to be evaluated. Based on this version of the algorithm we discuss its complexity.

I. INTRODUCTION

We assume that the reader is familiar with the paper Complex Quantum Queries/Quantum computers can search arbitrarily large databases by a single query [4].

A. Statement of the Problem

The algorithm of [4] solves the following problem:

**Problem:** Given a set $\mathcal{M}$ of $N$ items and a Boolean function $f: \mathcal{M} \to \{0, 1\}$, find an element $x \in \mathcal{M}$ with $f(x) = 1$ using the function

$$\hat{f}(T) = \left| \{ x \in T | f(x) = 1 \} \right| \mod 2.$$  

(1)

where $T \subseteq \mathcal{M}$ is an arbitrary subset of $\mathcal{M}$. We assume w.l.o.g. $N = 2^\nu$ and $\mathcal{M} = \{1, 2, \ldots, N\}$.

Grover considers $f(x)$ as an elementary query since only one item of $\mathcal{M}$ is involved, whereas the complex query $\hat{f}(T)$ depends on an arbitrary subset of $\mathcal{M}$. The function checks whether the number of elements of the subset $T$ satisfying the predicate $f$ is odd.

Another complex query is $\hat{f}(T) = \exists x \in T: f(x) = 1$ which can be used for binary searching.

B. Definitions

To formulate the quantum search algorithm it is helpful to consider the following auxiliary functions.

For $j = 1, \ldots, N$, the function

$$\chi_j : \{0, 1\}^\nu \to \{0, 1\}
\begin{array}{c}
(x_1, \ldots, x_\nu) \\
\mapsto \\
|\{ i : i \in \{1, \ldots, \nu\} | x_i = j \}| \mod 2
\end{array}$$

(2)

checks the parity of the number of $\nu$-bit-strings $x_i$ equal to $j$ in the $\nu$-tuple $(x_1, \ldots, x_\nu)$.

Furthermore, the subset $T$ used in the complex query $\hat{f}(T)$ is encoded by its incidence vector

$$\mathcal{X} = (\chi_T(1), \ldots, \chi_T(N))$$

where $\chi_T$ is the characteristic function of the subset $T$. Thus, the complex query can be considered as a Boolean function with $N$ inputs.

II. QUANTUM SEARCH ALGORITHM USING COMPLEX QUERIES

In this section we restate the algorithm of [4] and include some details showing how the necessary operations might be implemented.

Be $\nu$ a constant of order $N(\log N)^2$.

1. Prepare the following state on $\nu \eta + N + 1$ qubits:

$$|\psi_1\rangle = (|0\rangle \ldots |0\rangle) \otimes (|0\rangle \ldots |0\rangle) \otimes |0\rangle$$

$\nu \eta$
(a) Perform a Hadamard transform on the first $\nu\eta$ qubits and the last qubit, i.e.,

$$H = \frac{1}{\sqrt{2^{\nu\eta}}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes I_{2^N} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(b) Perform a $\sigma_z$ rotation on the last qubit.

This results in the state (to simplify the notation, normalization factors are omitted here and in the remainder of the paper):

$$|\psi_2\rangle = \left( \sum_{x=1}^{N} |x\rangle \right) \otimes (|0\rangle)^{\otimes N} \otimes (|0\rangle - |1\rangle)$$

$$= \sum_{(x_1,\ldots,x_\eta) \in \{1,\ldots,N\}^\eta} |x_1\rangle \otimes \ldots \otimes |x_\eta\rangle \otimes (|0\rangle \ldots |0\rangle \otimes (|0\rangle - |1\rangle).$$

3. For $j = 1,\ldots,\eta$ add the value of the function $\chi_j(x_1,\ldots,x_\eta) = |\{i : i \in \{1,\ldots,\eta\} | x_i = j\}| \mod 2$ to qubit $\nu\eta + j$ resulting in the state

$$|\psi_3\rangle = \sum_{(x_1,\ldots,x_\eta) \in \{1,\ldots,N\}^\eta} |x_1\rangle \otimes \ldots \otimes |x_\eta\rangle \otimes |\chi_1\rangle \otimes \ldots \otimes |\chi_\eta\rangle \otimes (|0\rangle - |1\rangle).$$

4. Add the value of the function $\tilde{f}(\mathcal{X})$ where $\mathcal{X} \subseteq \mathcal{M}$ is given by the support of the incidence vector $(\chi_1,\ldots,\chi_n)$ to the last qubit:

$$|\psi_4\rangle = \sum_{(x_1,\ldots,x_\eta) \in \{1,\ldots,N\}^\eta} |x_1\rangle \otimes \ldots \otimes |x_\eta\rangle \otimes |\chi_1\rangle \otimes \ldots \otimes |\chi_\eta\rangle \otimes \left( |0\rangle + \tilde{f}(\mathcal{X}) - |1\rangle + \tilde{f}(\mathcal{X}) \right)$$

$$= \sum_{(x_1,\ldots,x_\eta) \in \{1,\ldots,N\}^\eta} |x_1\rangle \otimes \ldots \otimes |x_\eta\rangle \otimes |\chi_1\rangle \otimes \ldots \otimes |\chi_\eta\rangle \otimes (-1)^{\tilde{f}(\mathcal{X})} (|0\rangle - |1\rangle).$$

5. Repeat step 3 to dis–entangle the states, i.e., for $j = 1,\ldots,\eta$ add the value of function $\chi_j(x_1,\ldots,x_\eta)$ to qubit $\nu\eta + j$ resulting in the state

$$|\psi_5\rangle = \sum_{(x_1,\ldots,x_\eta) \in \{1,\ldots,N\}^\eta} |x_1\rangle \otimes \ldots \otimes |x_\eta\rangle \otimes (|0\rangle)^{\otimes N} \otimes (-1)^{\tilde{f}(\mathcal{X})} (|0\rangle - |1\rangle)$$

$$= \left( \sum_{x=1}^{N} (-1)^{\tilde{f}(x)} |x\rangle \right) \otimes (|0\rangle)^{\otimes N} \otimes (|0\rangle - |1\rangle). \quad (3)$$

(As equality (3) is not obvious, it is proved separately in section III A.)

6. Apply the operator $D$ (inversion about average)

$$D = \frac{1}{\sqrt{2^n}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \otimes_{\nu\eta} \left( \begin{pmatrix} -1 \\ 1 \\ \ddots \end{pmatrix} \right) \otimes_{\eta} \left( \begin{pmatrix} 1 & 1 & \ldots \end{pmatrix} \right)$$

$$\otimes_{\eta} \frac{1}{\sqrt{2^n}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \otimes_{\nu\eta}$$

on each of the first $\eta$ registers of length $n$. After this step, the system is in a state that consists of $\eta$ (non–entangled) copies of the state after one step of the original quantum search algorithm:

$$|\psi_6\rangle = \left( \sum_{f(x) = 1} k_x |x\rangle + \sum_{f(x) = 0} l_x |x\rangle \right) \otimes (|0\rangle)^{\otimes N} \otimes (|0\rangle - |1\rangle) \quad (5)$$

with suitable values $k_x$ and $l_x$ as to be described in section III B.

7. By measuring each of the first $\eta$ registers of length $n$, a set $\mathcal{S} = \{\tilde{x}_1,\ldots,\tilde{x}_\eta\}$ of $\eta$ samples is obtained.

8. In a (classical) post–processing step, an element $x_0$ of the $\eta$ samples $\tilde{x}_i$ with maximal frequency is searched. This element $x_0$ is the output of the algorithm.
III. DISCUSSION

A. Proof of the Identity in Step 5

From the definition (1) of $\hat{f}(X)$ and the definition (2) of $\chi_j$ it follows that

$$\hat{f}(X) = \hat{f}(\chi_1, \ldots, \chi_N) = \sum_{i=1}^{\eta} f(x_i) \mod 2.$$  \hspace{1cm} (6)

This identity can be proved by the so-called method of double counting. For doing so note that each of the functions $\chi_j$ and thus $\chi_j(x)$ depend on $x = (x_1, \ldots, x_\eta)$. Thus,

$$\hat{f}(X(x)) = \hat{f}(\text{support}(\chi_1(x), \ldots, \chi_N(x))) = \{|j \in \{1, \ldots, N\}| \chi_j(x) = 1 \land f(j) = 1\} \mod 2 = \sum_{j=1}^{N} f(j) \mod 2 = \sum_{j=1}^{\eta} \sum_{i=1}^{N} \delta_{x_i,j} f(j) \mod 2 = \sum_{i=1}^{\eta} f(x_i) \mod 2.$$

Equation (6) implies

$$(-1)^{\hat{f}(X(x))} |x_1\rangle \otimes \ldots \otimes |x_\eta\rangle = \left((-1)^{f(x_1)} |x_1\rangle\right) \otimes \ldots \otimes \left((-1)^{f(x_\eta)} |x_\eta\rangle\right)$$

which proves the identity (3).

B. The Probability of Success

Be $t$ the number of elements of $\mathcal{M}$ that satisfy the predicate $f$. Then, the amplitudes $k_x$ and $l_x$ in (5) are

$$k_x = \left(3 - \frac{4t}{N}\right) \frac{1}{\sqrt{N}} \quad \text{and} \quad l_x = \left(1 - \frac{4t}{N}\right) \frac{1}{\sqrt{N}}.$$

Thus, the probabilities to measure an element $\hat{x}_i$ with $f(\hat{x}_i) = 1$ (or with $f(\hat{x}_i) = 0$) are

$$Pr[\hat{x}_i|f(\hat{x}_i) = 1] = \left(9 - \frac{24t}{N} + \left(\frac{4t}{N}\right)^2\right) \frac{1}{N}$$  \hspace{1cm} (7)

and

$$Pr[\hat{x}_i|f(\hat{x}_i) = 0] = \left(1 - \frac{8t}{N} + \left(\frac{4t}{N}\right)^2\right) \frac{1}{N}.$$  \hspace{1cm} (8)

The output of the algorithm is an element $x_0$ with maximal frequency in the sample $S$ of size $\eta$. Using (6) and (8), the probability that $x_0$ satisfies $f$, i.e., $Pr[f(x_0) = 1]$, might be calculated exactly given the values $N$, $t$, and $\eta$. Using the central limit theorem, in [4] it is shown that $Pr[f(x_0) = 1]$ approaches one for $\eta$ of order $N (\log N)^2$.

C. The Complexity of the Algorithm

In the following we consider the complexity of the steps of the algorithm as presented in section [1]. The number of elementary (two-bit) gates (cf. [1]) will be used as a measure for the complexity of the quantum operations.

- the number of Hadamard transforms:
  For the preparation of the equal superposition in step 1, $\nu\eta$ Hadamard transforms are needed. Twice that number is needed to apply the operator $D^{\otimes \eta} \otimes I$ for the inversion about average on the first $\nu\eta$ qubits.
• computation of $\chi_j$ in steps 3 and 5:
The function $\chi_j$ can be computed in the following manner: for $i = 1, \ldots, N$ add the value of the function $\delta_j(x_i) = \delta_{x_i,j}$ to qubit $\nu \eta + j$. In the language of \cite{1}, these are $\sigma_x$ rotations conditioned on $\nu$ qubits ($\bigwedge_\nu (\sigma_x)$) which can be achieved with $O(\nu)$ elementary operations.

This gives a total complexity of $O(\nu \eta)$ for steps 3 and 5.

Note that even though after step 5 the second register of $N$ qubits has been reset to $|0\rangle^{\otimes N}$, it was needed for the superimposed computation of $\tilde{f}(X)$ which indeed is a function $tildef(X(x_1, \ldots, x_\eta))$.

• inversion about average:
To compute the operator $D^{\otimes \eta} \otimes 1$ (inversion about average) besides the Hadamard transforms the conditional phase change (the diagonal matrix in \cite{1}) has to be applied $\eta$ times. These operations are $\sigma_z$ rotations conditioned on $\nu$ qubits ($\bigwedge_\nu (\sigma_z)$) and can be achieved with totally $O(\nu \eta)$ elementary operations.

• classical post–processing:
To find an element $x_0$ of maximal frequency in the sample $S$ the sequence of samples has to be sorted which has complexity $O(\eta \log \eta)$.

The complexity of the algorithm is dominated by the complexity of steps 3 and 5, possibly the evaluation of $\tilde{f}(X)$ in step 4, and of the (classical) post–processing in step 8. In summary, the complexity of the quantum search algorithm using complex queries is $O(\nu \eta) + O(\eta \log \eta)$ plus the complexity of one evaluation of the complex query $\tilde{f}$. For $\eta$ of order $N(\log N)^2$, the algorithm has a complexity of $O(N^2(\log N)^2)$.

IV. CONCLUSIONS

The main result of \cite{4} is that the complex query $\tilde{f}(T)$ has to be evaluated only once instead of $O(\sqrt{N/\nu})$ evaluations of the elementary query $f(x)$ in \cite{3}. Compared to the original quantum search algorithm \cite{3}, both the number of qubits and the number of quantum and classical operations to be performed are dramatically increased. Thus there is a trade–off between the number of elementary operations and the complexity of the complex query.

It has to be investigated in which situations it might be easier to evaluate the complex query once than to evaluate the elementary query many times.

Nevertheless, the quantum search algorithm \cite{4} proves that with a quantum computer only one query evaluation is needed whereas any classical algorithm will be limited by the information theoretic bound of at least logarithmic many queries.

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