Features of forming a parallel staging launch vehicle mathematical model using the method of d' Alembert-Kane

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Abstract. The article discusses features of dynamic characteristics in a parallel-staging launch vehicle’s elastic design. In contrast to traditional forms of body transverse vibrations in rockets with tandem staging, the parallel-staging scheme features vibrational forms of spatial character which are associated with longitudinal and torsional vibrations of side blocks of the rocket. The attention in the paper is drawn to the simplicity and clarity of drawing up general equations of dynamics in the form of d’Alembert-Kane, as well as to the features of projections of overload forces and the main engines thrust onto the related and inertial axes.

Key words: dynamic design, vibrations of an elastic body, parallel-staging rocket, launch-vehicle, natural frequencies, liquid propellant, blocks

At the beginning of the last century N E Zhukovskij considered the problem of the motion of a rigid body with cavities completely filled with a liquid. He introduced the concept of the inertia tensor of an equivalent body and obtained a number of analytic expressions for the now-known Zhukovsky potentials [1]. In connection with the instability of the flight of the first missiles in the 50s of the 20th century, a dynamic scheme of the body with cavities partially filled with liquid was developed, taking into account the wave oscillations of its free surface in the field of overload (Moiseev N N, Narimanov G S, Okhotsimsky D E, Rabinovich B I). Somewhat later, the phenomenon of unstable flight of missiles, caused by elastic oscillations of the construction, required the introduction of corresponding additional degrees of freedom into the dynamic scheme.

The hydrodynamic parameters of the dynamic scheme are determined by the solution of a homogeneous boundary value problem having a countable set of eigenvalues. Long-term experience shows that it is enough to take into account one basic tone of oscillations of the free surface of the liquid in each fuel tank of the launch vehicle (LV). This tone of fluid oscillations can be described by a pendulum model.

As a rule, the design of large-sized LV has a package layout. A feature of the LV package design is the excitation in flight of spatial flexural-longitudinal-torsional vibrations of its design. In this general case, the equations of motion of the LV, corresponding to all the control channels, are interconnected.

The order of the system of equations for the study of the stability of the motion of the LV significantly increases. The presence in this system of close or multiple oscillation frequencies of carrier rocket complicates the analysis of the stability of its controlled motion.

Elastic characteristics of the dynamic scheme are calculated from the solution of the problems of elasticity theory for the construction of LV with a “frozen” liquid. At present, finite-element methods
for determining the frequencies and forms of elastic vibrations in the construction of LV with the use of software packages Nastran, Abaqus, and others have been widely used.

In the compilation of differential equations of motion of the LV, the Lagrangian equations of the second kind are usually used. However, for this purpose it is better to use the simpler method of d’Alembert-Kane [2], in which the physical clarity of the d’Alembert principle is combined with the procedure for automatically eliminating the reactions of bonds, which is especially important in the case of nonholonomic systems.

As an illustration of this method, let us consider the procedure for constructing a dynamic scheme using the example of the design of a LV representing a package of four blocks. Each unit has an oxidizer tank, a fuel tank and a propulsion system (PS). We introduce the coupled coordinate system \( O_{xyz} \), whose axis \( Ox \) coincides with the longitudinal axis of the LV. The axis \( Oy \) coincides with the pitch plane, the axis \( Oz \) with the yaw plane. The longitudinal axes of the side blocks are parallel to \( Ox \) and have coordinates \( y_{ok}, z_{ok} \) \((k = 1 - 4)\).

The associated coordinate system \( O_{xyz} \) deviates from the trajectory coordinate system \( O_{xyz}^\dagger \) by the amount of normal and lateral drift \( Y(t) \) and \( Z(t) \) turns to the yaw \( \psi(t) \), pitch \( \theta(t) \) and roll \( \gamma(t) \) angles. In the \( k \)-th tank, the point of suspension of the pendulum has coordinates \( C_k(x_{Ck}, y_{ok}, z_{ok}) \). In the pendulum model of fluid oscillations, the length of the pendulum \( \ell_S \) is represented, and the angles of its deviation from the vertical axis are equal \((0, S_{\psi k}, -S_{\theta k})\). There are four engines that can be deflected in two planes with the help of 8 drives through the channels of pitch and yaw. Their rotation is determined by the coordinates \( \delta_{\psi k}(t), \delta_{\theta k}(t) \). These motors are used for control and in the roll channel. The point of application of the tractive force \( P_k \) is located on the longitudinal axis of each block \( B_k(x_{Bk}, y_{ok}, z_{ok}) \). Each final element of the construction is displaced by \( \Delta \bar{R} \) and rotated by a spatial angle \( \Delta \Phi \). This displacement and rotation are factorized in a series according to the forms of elastic oscillations

\[
\Delta \bar{R} = \sum_{j=1}^{j_q} (U_j, V_j, W_j) q_j(t) , \quad \Delta \Phi = \sum_{j=1}^{j_q} (\phi_j, \psi_j, \chi_j) q_j(t) .
\]

The speed of an arbitrary final element of the construction is composed of the velocity of the center of mass \( \vec{V}_0 = (\dot{X}, \dot{Y}, \dot{Z}) \); from the velocity due to the angular velocity of the LV with respect to its center of mass \( \bar{O} \)

\[
\Delta \vec{V} = \bar{\partial} \times \bar{R} , \quad \text{where} \quad \bar{\partial} = (\dot{\gamma}, \dot{\psi}, \dot{\theta}) , \quad \bar{R} = (x, y, z) .
\]

In addition, there is a relative speed associated with the elastic vibrations of the construction, with fluctuations in liquid fuel, with fluctuations in the propulsion system.

The absolute velocity of the center of mass \( \bar{D} \) of the rocking part of the \( k \)-th motor rotating with respect to the point \( B_k \) is

\[
\bar{V}_D^k = (\dot{X}, \dot{Y}, \dot{Z}) + ((\dot{\psi} y_{ok} - \dot{\theta} y_{ok}), (\dot{\theta} x_{ok} - \dot{\gamma} z_{ok}), (\dot{\gamma} y_{ok} - \dot{\psi} x_{ok})) +

+ \sum_{j=1}^{j_q} (U_j^k, V_j^k, W_j^k) |_{x_B} \dot{q}_j(t) + \ell_D ((\dot{\psi}_{\psi k}^* + \dot{\theta}_{\theta k}^* + \dot{\gamma}_{\gamma k}^*) - \dot{\psi}_{\psi k}^* \dot{\theta}_{\theta k}^* \dot{\gamma}_{\gamma k}^*) .
\]

The length of the pendulum is \( \ell_D = x_B - x_D \). The angle of the absolute rotation of the pendulum is represented as the sum.
The partial velocities, that is, the partial derivatives of the velocity (2) with relation to \( \dot{\gamma}, \dot{\psi}, \dot{\theta}, \dot{\Theta}, \dot{\varphi}, \dot{\psi}_k \), have the form

\[
\begin{align*}
\frac{\partial \mathbf{V}_D^k}{\partial \dot{\gamma}} &= -z_{0k} \mathbf{j} + y_{0k} \mathbf{k}, \\
\frac{\partial \mathbf{V}_D^k}{\partial \dot{\psi}} &= z_{0k} \mathbf{i} - x_D \mathbf{k}, \\
\frac{\partial \mathbf{V}_D^k}{\partial \dot{\theta}} &= -y_{0k} \mathbf{i} + x_D \mathbf{j}, \\
\frac{\partial \mathbf{V}_D^k}{\partial \dot{\Theta}} &= \mathbf{J}, \\
\frac{\partial \mathbf{V}_D^k}{\partial \dot{\varphi}} &= k, \\
\frac{\partial \mathbf{V}_D^k}{\partial \dot{\psi}_k} &= \left( (\mathbf{U}_j^k \mathbf{i} + (V_j^k - \ell_D \mathbf{X}_j^k) \mathbf{j} + (W_j^k + \ell_D \mathbf{Y}_j^k) \mathbf{k}) \right)_{sb}, \\
\frac{\partial \mathbf{V}_D^k}{\partial \delta_{sk}} &= (i \delta_{sk} - j) \ell_D, \\
\frac{\partial \mathbf{V}_D^k}{\partial \delta_{\psi k}} &= (i \delta_{\psi k} + k) \ell_D.
\end{align*}
\]

(4)

Acceleration of the center of mass of the \( k \)-th engine will be equal to

\[
\mathbf{a}_D^k = (\dot{X}, \dot{Y}, \dot{Z}) + \left( (\mathbf{J} z_{0k} - \mathbf{J} y_{0k}), (\mathbf{J} x_D - \mathbf{J} z_{0k}), (\mathbf{J} y_{0k} - \mathbf{J} x_D) \right) +
\]

\[
+ \sum_{j=1}^{I_k} \left( (V_j^k, W_j^k, W_j^k) \right)_{sb} \dot{q}_j(t) + (0, -\ell_D \delta_{sk}, \ell_D \delta_{\psi k}) +
\]

\[
+ \sum_{j=1}^{I_k} \left( (0, -\ell_D \mathbf{X}_j^k, \ell_D \mathbf{Y}_j^k) \right)_{sb} \dot{q}_j(t).
\]

Similarly, it is possible to obtain expressions for the partial velocities and for accelerating the pendulum corresponding to the movable part of the liquid mass in the \( k \)-th tank, which will have a similar structure.

The main force acting on the LV is the thrust \( \mathbf{P}_k \) created by the engines on each block. It is simultaneously a tracking and controlling force. In the stabilization planes, the vector of thrust directed from the tail to the spout of the LV is, with the axis \( \mathbf{Ox}^* \), the angles

\[
\delta^k_{\psi} = \varphi + \sum_{j=1}^{I_k} \chi_j^k q_j + \delta_{sk}, \quad \psi^k_{\psi} = \psi + \sum_{j=1}^{I_k} \mathbf{Y}_j^k q_j + \delta_{\psi k}.
\]

The vector of force generated by the thrust of the \( k \)-th engine, according to the rotation matrix, will take the following form: \( \mathbf{P}_k = \mathbf{P}_k \left( 1, \delta^k_{\psi}, -\psi^k_{\psi} \right) \).

Therefore, the projections of the generalized force created by the propulsion system LV, on the plane of pitch and on the yaw plane, will be equal

\[
\begin{align*}
F_{x_k} &= \sum_{k=1}^{4} P_k, \\
F_{y_k} &= \varphi \sum_{k=1}^{4} P_k + \sum_{k=1}^{4} \sum_{j=1}^{I_k} P_k \chi_j^k \left| \mathbf{b}_k q_j + \sum_{k=1}^{4} P_k \delta_{sk} \right|, \\
F_{z_k} &= -\psi \sum_{k=1}^{4} P_k - \sum_{k=1}^{4} \sum_{j=1}^{I_k} P_k \psi_j^k \left| \mathbf{b}_k q_j - \sum_{k=1}^{4} P_k \delta_{\psi k} \right|.
\end{align*}
\]

If the forces from the thrust of the engines are projected on the axes of the associated coordinate system, then in the linear approximation we obtain
\[
\begin{align*}
F_x^P &= \sum_{k=1}^{4} P_k, & F_y^P &= \sum_{k=1}^{4} F_{y,k}^P, & F_z^P &= \sum_{k=1}^{4} F_{z,k}^P, \\
F_{y,k}^P &= P_k(\delta_{y,k} + \sum_{j=1}^{j_B} X_j^k |s_B q_j|), & F_{z,k}^P &= -P_k(\delta_{z,k} + \sum_{j=1}^{j_B} \psi_j^k |s_B q_j|).
\end{align*}
\]

According to (4), the static moment from the traction force is
\[
M^P = \sum_{k=1}^{4} (y_{0,k} F_{x,k}^P - z_{0,k} F_{z,k}^P), \quad M^P_y = \sum_{k=1}^{4} ((z_{0,k} + \sum_{j=1}^{j_B} W_j^k |s_B q_j|) P_k - x_B F_{z,k}^P),
\]
\[
M^P_z = \sum_{k=1}^{4} (x_B F_{y,k}^P - (y_{0,k} + \sum_{j=1}^{j_B} V_j^k |s_B q_j|) P_k), \quad x_D = x_B - l_D.
\]

Here it is taken into account that the axial line of the \(k\)-th block is in the deformed state, and the value of \(P_k\) is not a small value.

In the equation of elastic vibrations, the static component of the thrust of the motors is
\[
Q_j^P = \sum_{k=1}^{4} \left( P_k U_j^k |s_B + F_{y,k}^P (V_j^k - \ell_D X_j^k) |s_B + F_{z,k}^P (W_j^k + \ell_D \psi_j^k) |s_B \right).
\]

The missile in flight is affected by the quasistatic force created by the overload field. Each element \(d m\) has a force
\[
d \vec{F}_g = -\vec{n}_s n_s g d m,
\]
which is directed along the axis \(\Omega x^*\) of the trajectory coordinate system. The projections of this force on the axis of the coupled coordinate system will be equal to
\[
d \vec{F}_g = -n_s g d m (1, -\vartheta, \psi).
\]

Since the equations of forces are projected on the axis of the trajectory coordinate system, the projections of this force on the axis \(\Omega^* y^*\) and \(\Omega^* z^*\) are equal to zero. These forces are of decisive importance for the frequency of fluid vibrations, the approximation of which is represented by a mathematical pendulum. The point of suspension of the pendulum has coordinates \(C(x_i + \ell_S i, y_{0,k}, z_{0,k})\). Taking into account the deformation of the LV housing, the pendulum rotation relative to the overload vector will be determined by the angles
\[
S^*_{\vartheta} = S_{\vartheta} - \sum_{j=1}^{j_B} \vartheta_j |x_j q_j - \vartheta, \quad S^*_{\psi_i} = S_{\psi_i} + \sum_{j=1}^{j_B} \psi_j |x_j q_j + \psi.
\]

By analogy with (2) - (4), the partial velocities for fluid oscillators are partial derivatives of its velocity.

\[
\begin{align*}
\frac{\partial V_{\vartheta}}{\partial \vartheta j} &= -(z_{0,k} + \ell_S S^*_{\vartheta_j}) j + (y_{0,k} + \ell_S S^*_{\theta_j}) k, \\
\frac{\partial V_{\vartheta}}{\partial \psi} &= (z_{0,k} + \ell_S S^*_{\psi_i}) i - (x_{ci} - \ell_S) k, \\
\frac{\partial V_{\vartheta}}{\partial \vartheta} &= -(y_{0,k} + \ell_S S^*_{\vartheta}) i + (x_{ci} - \ell_S) j.
\end{align*}
\]
The static moment due to the overload created by the \( i \)-th oscillator is determined by multiplying the refined partial velocity (12) by the vector of the overload force (10)

\[
\Delta M_x^c = 0, \quad \Delta M_y^c = -n_x g \sum_{i=1}^{i_s} m_i (z_{0i} + \ell_{S_{qi}}) S_{\theta_i}^* + \sum_{j=1}^{j_q} \chi_{j} \psi_{ji} S_{\psi_j}^* + \sum_{j=1}^{j_q} \psi_{ji} S_{\psi_j}^* + \sum_{j=1}^{j_q} \psi_{ji} S_{\psi_j}^*.
\]

In the equations of elastic vibrations, static additives will appear

\[
\Delta Q_j^c = -n_x g \sum_{i=1}^{i_s} m_i (U_j + \ell_{S_{qi}} (\psi_j S_{\psi_j}^* - \chi_{j} S_{\psi_j}^* + \chi_{j} S_{\psi_j}^*)),
\]

Similar additions will appear in the equations of fluid oscillations

\[
\Delta S_{\theta_i} = -n_x g \sum_{i=1}^{i_s} m_i \ell_{S_{qi}} (S_{\theta_i} - \sum_{j=1}^{j_q} \chi_{j} q_{ji} - \theta),
\]

To determine the inertial forces, multiply the acceleration vector (5) of the element \( dm \) by the first partial velocity (4), which is equal to the unit vector \( j \), and integrate over the entire volume of the LV structure. Then multiply by the partial velocity equal to the unit vector \( k \), and also integrate over the volume of LV. Carrying out the addition of inertial forces with the active forces of thrust PS, static forces of the overload, as well as implicitly defined aerodynamic forces (index "a"), we obtain the equation of forces along the pitch channel

\[
-(m \ddot{Y} + \sum_{i=1}^{i_s} m_i \ell_{S_{qi}} S_{\theta_i}) + \theta \sum_{k=1}^{4} P_k + \sum_{k=1}^{4} \sum_{j=1}^{j_q} P_k \chi_{j} S_{\psi_j}^* + \sum_{k=1}^{4} \sum_{j=1}^{j_q} P_k \psi_{ji} S_{\psi_j}^* = 0.
\]

A similar equation of forces is obtained in the yaw channel

\[
-(m \ddot{Z} + \sum_{i=1}^{i_s} m_i \ell_{S_{qi}} S_{\psi_i}) - \psi \sum_{k=0}^{4} P_k - \sum_{k=0}^{4} \sum_{j=1}^{j_q} P_k \psi_{ji} S_{\psi_j}^* = 0.
\]

To obtain the moment equation over the roll channel, we must use the first partial velocity (4) with respect to the angle \( \gamma \), by multiplying it by the acceleration of all the elements of the LV, integrating in volume and adding the moments of the active forces.

\[
-(m \ddot{P} + \sum_{i=1}^{i_s} m_i \ell_{S_{qi}} S_{\psi_i}) - \psi \sum_{k=0}^{4} P_k - \sum_{k=0}^{4} \sum_{j=1}^{j_q} P_k \psi_{ji} S_{\psi_j}^* = 0.
\]
Similarly, it is possible to write the equations of moments in the yaw channel.

\[
-J_y \ddot{\psi} + \sum_{i=1}^{k_i} m_i \ell_{x_i} (-x_i \dot{S}_{\psi_i}) + \sum_{k=1}^{4} m^k_D \ell_D (z_{0_k} \delta_{\psi_k} + y_{0_k} \dot{S}_{\psi_k}) +
\]

\[
+ \sum_{k=1}^{4} \left( -z_{0_k} P_k (\delta_{\psi_k} + \sum_{j=1}^{j_q} \chi^k_j |_{s_B} x_j) - y_{0_k} P_k (\psi_{\psi_k} + \sum_{j=1}^{j_q} \psi^k_j |_{s_B} q_j) \right) + M_{x_\alpha} + M_{y_0} = 0.
\]

The equation of moments in the pitch channel has a similar form

\[
-J_z \ddot{\theta} + \sum_{i=1}^{k_i} m_i \ell_{x_i} (-x_i \dot{S}_{\theta_i}) + \sum_{k=1}^{4} m^k_D \ell_D (x_{0_k} \delta_{\theta_k} + \ell_D \delta_{\theta_k}) +
\]

\[
+ \sum_{k=1}^{4} \left( x_{0_k} P_k (\delta_{\theta_k} + \sum_{j=1}^{j_q} \chi^k_j |_{s_B} q_j) + y_{0_k} P_k (\psi_{\theta_k} + \sum_{j=1}^{j_q} \psi^k_j |_{s_B} q_j) \right) +
\]

\[
+ n_x g \sum_{i=1}^{k_i} m_i (y_{0_k} + \ell_{x_i} S_{\theta_i}) - n_x g \sum_{k=1}^{4} m^k_D (y_{0_k} + \ell_D \delta_{\theta_k}) + M_{z_\alpha} + M_{z_0} = 0.
\]

The equations of elastic oscillations have a spatial form

\[
- \mu_j \ddot{q}_j + \sum_{i=1}^{k_i} m_i \ell_{x_i} \left( V^i_j \dot{S}_{\delta_i} + W^i_j \dot{S}_{\psi_i} \right) -
\]

\[
- \sum_{k=1}^{4} m^k_D \left( -\ell_D (V^k_j |_{x_s} - \ell_D \chi^k_j |_{x_s}) \delta_{\delta_k} + \ell_D (W^k_j |_{x_s} + \ell_D \psi^k_j |_{x_s}) \delta_{\psi_k} \right) +
\]

\[
+ \sum_{k=1}^{4} \left( P_k U^k_j |_{x_s} + P_k (\delta_{\delta_k} + \sum_{j=1}^{j_q} \chi^k_j |_{x_s} q_j) (V^k_j |_{x_s} + \ell_D \psi^k_j |_{x_s}) -
\]

\[
- P_k (\delta_{\psi_k} + \sum_{j=1}^{j_q} \psi^k_j |_{x_s} q_j) (W^k_j |_{x_s} - \ell_D \chi^k_j |_{x_s}) \right) -
\]

\[
- n_x g \sum_{k=1}^{4} m^k_D \left( U^k_j |_{x_s} + \ell_D (\psi^k_j |_{x_s} \delta_{\psi_k} + \chi^k_j \delta_{\psi_k}) \right) -
\]

\[
- n_x g \sum_{k=1}^{4} m_i \left( U^k_j |_{x_s} + \ell_{x_i} (\psi^k_j |_{x_s} \delta_{\psi_k} + \chi^k_j \delta_{\psi_k}) \right) + Q^0_j = 0.
\]

The equations of fluid oscillations are divided into two groups.
The dynamic scheme in the presented form (16) - (22) corresponds to the spatial motion of the LV. In the equations (21) - (22), there are connections between the angular coordinates in pitch, yaw and roll channels. In the particular case, it is possible to split the general system of equations of motion into subsystems describing isolated motions in the pitch plane, in the yawing plane, isolated rotation with respect to the longitudinal axis. This is possible in the case when the design of the LV 1) has two planes of symmetry, 2) the distribution of liquid masses in the lateral blocks, and also the stiffness characteristics, is strictly symmetric, and 3) the propulsion systems operate simultaneously with the same thrust. In this case, the spectrum of elastic vibrations of its construction breaks up into independent subspectra characterized by a certain type of deformation of the central block. Forms of elastic vibrations should be transformed to a symmetrical form. Instead of four lateral blocks, one "generalized" block is introduced in the mathematical model of the LV. For generalized coordinates of fluid oscillations and angular coordinates of control motors, a linear transformation is carried out. For example, instead of the four original coordinates, two new

\[ s_{g} = s_{\varphi 1} + s_{\varphi 2} + s_{\varphi 3} + s_{\varphi 4} , \quad s_{g,\varphi} = s_{\varphi 1} - s_{\varphi 2} + s_{\varphi 3} - s_{\varphi 4} \]

Then the general spatial system of equations of motion breaks up into three subsystems along the control channels (pitch, yaw, roll) [3]

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