Topological complexity of collision-free multi-tasking motion planning on orientable surfaces

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Abstract
We compute the higher topological complexity of ordered configuration spaces of orientable surfaces, thus extending Cohen-Farber’s description of the ordinary topological complexity of those spaces.

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1 Introduction
The configuration space of $n$ distinct ordered points of a space $X$, $\text{Conf}(X, n)$, is the subspace of the $n$-fold cartesian power $X^{\times n}$ given by

$$\text{Conf}(X, n) = \{(x_1, \ldots, x_n) \in X^{\times n} : x_i \neq x_j \text{ whenever } i \neq j\}.$$

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These spaces play a central role in a number of settings in mathematics, as well as in other areas of science. Our interest lies in topological robotics, where $\text{Conf}(X,n)$ arises as the model for the state space of a system consisting of $n$ distinct particles moving without collisions on $X$. We focus on the case $X = \Sigma_g$, an orientable surface of genus $g$. Farber’s topological complexity (TC) of $\text{Conf}(\Sigma_g, n)$ has been described in [2]. The purpose of this work is to extend Cohen-Farber’s results by describing (in Theorem 1.1 below) the higher topological complexity of $\text{Conf}(\Sigma_g, n)$.

In preparation for the bulk of the paper, we now recall Farber’s notion of topological complexity, together with Rudyak’s generalization, the so-called higher topological complexity.

The concept of topological complexity (TC) of a space $X$ was introduced early this millennium by Michael Farber as a way to utilize techniques from homotopy theory in order to model and study, from a topological perspective, the motion planning problem in robotics. If $P(X)$ stands for the space of free paths in $X$, then $\text{TC}(X)$ is the reduced Schwarz genus (also known as sectional category) of the fibration $e: P(X) \rightarrow X \times X$ given by $e(\gamma) = (\gamma(0), \gamma(1))$. We refer the reader to the book [5] and the references therein for a discussion of the meaning, relevance, and basic properties of Farber’s concept.

The idea was generalized a few years later by Rudyak, who defined in [9] the $s$-th topological complexity of $X$, $\text{TC}_s(X)$, as the reduced Schwarz genus of the $s$-th fold evaluation map $e_s: P(X) \rightarrow X^{s \times s}$ given by

$$e_s(\gamma) = \left( \gamma(0), \gamma \left( \frac{1}{s - 1} \right), \gamma \left( \frac{2}{s - 1} \right), \ldots, \gamma \left( \frac{s - 2}{s - 1} \right), \gamma(1) \right).$$

In particular $\text{TC} = \text{TC}_2$. Rudyak’s higher topological complexity has been studied systematically in [1]. The term “higher” comes by considering the base space $X^s$ of $e_s$ as the space of sequences of prescribed stages in the motion planning of a robot with state space $X$, while Farber’s original definition (with $s = 2$) deals only with the space $X \times X$ of initial-final stages of the robot.

We now state our main result.

**Theorem 1.1.** The $s$-th topological complexity of $\text{Conf}(\Sigma_g, n)$ is given by

$$\text{TC}_s(\text{Conf}(\Sigma_g, n)) = \begin{cases} 
s, & \text{if } g = 0 \text{ and } n \leq 2; \\
sn - 3, & \text{if } g = 0 \text{ and } n \geq 3; \\
s(n+1) - 2, & \text{if } g = 1 \text{ and } n \geq 1; \\
sn + 1, & \text{if } g \geq 2 \text{ and } n \geq 1. 
\end{cases}$$

The case $n = 1$ in Theorem 1.1 has been noted in previous works; see [1, Corollary 3.12] for the case $g \leq 1$, [1, Example 16.4] for case $g \geq 2$ with $s = 2$, and [7, Proposition 5.1] for
the case \( g \geq 2 \) with \( s \geq 3 \). This also covers the case \( g = 0 \) with \( n = 2 \) since \( \text{Conf}(S^2, 2) \) has the homotopy type of \( S^2 \). Indeed, by the Gram-Schmidt process, \( S^2 \) sits inside \( F(S^2, 2) \) via the map \( x \mapsto (x, -x) \) as a strong deformation retract. Therefore, in what follows we restrict ourselves to the case \( n \geq 2 \), and in fact \( n \geq 3 \) if \( g = 0 \).

2 Upper bounds

**Genus 0.** For \( n \geq 3 \) the ordered configuration space of \( n \) distinct points on the 2-dimensional sphere \( S^2 \) admits a homotopy decomposition

\[
\text{Conf}(S^2, n) \simeq \text{SO}(3) \times \text{Conf}(\mathbb{R}^2 - Q_2, n - 3)
\]

where \( Q_2 \) is a set of two fixed points on \( \mathbb{R}^2 \) (see [2, Theorem 3.1], for instance). The higher topological complexity of both factors is known: The topological group \( \text{SO}(3) \simeq \mathbb{RP}^3 \) has

\[
\text{TC}_s(\text{SO}(3)) = \text{cat}(\mathbb{RP}^3^{s-1}) = (s-1)\text{cat}(\mathbb{RP}^3) = 3(s-1)
\]

in view of [3], whereas [6, Theorem 1.3] gives

\[
\text{TC}_s(\text{Conf}(\mathbb{R}^2 - Q_2, n - 3)) = s(n - 3).
\]

Then [6, Theorem 1.3] gives \( \text{TC}_s(\text{Conf}(S^2, n)) \leq sn - 3 \).

**Genus 1.** Since \( T = S^1 \times S^1 \) is a group, there is a topological decomposition

\[
\text{Conf}(T, n) \cong T \times \text{Conf}(T - Q_1, n - 1)
\]

where \( Q_1 \) is a fixed point in \( T \). It has been noted that

\[
\text{TC}_s(T) = 2(s-1).
\]

On the other hand, \( \text{Conf}(T - Q_1, n - 1) \) has the homotopy type of a cell complex of dimension \( n - 1 \) (see [2, proof of Theorem 4.1]). So [1, Theorem 3.9] gives

\[
\text{TC}_s(\text{Conf}(T - Q_1, n - 1)) \leq s(n - 1),
\]

and we get \( \text{TC}_s(\text{Conf}(T, n)) \leq s(n + 1) - 2 \).

**Genus at least 2.** As noted in the proof of [2, Theorem 5.1], \( \text{Conf}(\Sigma_g, n) \) has the homotopy type of a cell complex of dimension \( n + 1 \). We thus immediately obtain \( \text{TC}_s(\text{Conf}(\Sigma_g, n)) \leq s(n + 1) \).

The central goal of the paper is to show, by homological methods, that the three upper bounds described in this section are in fact equalities.
3 Zero divisors via the Totaro spectral sequence

For a (graded-)commutative unital algebra $A$ over a field $F$, let $\mu_s: A \otimes s \to A$ denote the iterated multiplication map determined by $\mu_s(a_1 \otimes \cdots \otimes a_s) = a_1 \cdots a_s$. (All tensor products are taken over $F$.) Elements in the kernel of $\mu_s$ are called $s$-th zero divisors of $A$, and the $s$-th zero-divisors cup-length of $A$, denoted by $zcl_s(A)$, is the maximal number of $s$-th zero-divisors of $A$ having a non-trivial product. Then, for a field $F$, the $s$-th zero-divisors cup-length of a space $X$ is $zcl_s,F(X) := zcl_s(H^\ast(X;F))$. In other words, $zcl_s,F(X)$ is the maximal number of elements in the kernel of the morphism $\Delta^\ast_s: H^\ast(X \times^s;R) \to H^\ast(X;R)$ having a non-trivial product, where $\Delta_s: X \to X \times^s$ stands for the iterated diagonal.

**Proposition 3.1** ([9, Proposition 3.4]). For any field $F$, $\text{TC}_s(X) \geq zcl_s,F(X)$.

We use Proposition 3.1 to show that each of the upper bounds described in the previous section are sharp. The simplest situation, i.e. that for $S^2$, is based on the obvious observation that, for algebras $A'$ and $A''$ as above, $A := A' \otimes A''$ is a (graded-)commutative unital algebra with multiplication

$$(a'_1 \otimes a''_1)(a'_2 \otimes a''_2) := (-1)^{\deg(a''_1)\deg(a'_2)}a'_1a'_2 \otimes a''_1a''_2$$

and, in these conditions,

$$zcl_s(A) \geq zcl_s(A') + zcl_s(A'').$$

For instance, (1) yields

$$zcl_{s,F}(\text{Conf}(S^2,n)) \geq zcl_{s,F}(\mathbb{R}P^3) + zcl_{s,F}(\text{Conf}(\mathbb{R}^2 - Q_2, n - 3)).$$

(4)

**Proof of Theorem 1.1 for $g = 0$ and $n \geq 3$.** In view of the proof of [6, Theorem 5.1], the assertion in (3) can be strengthened to

$$\text{TC}_s(\text{Conf}(\mathbb{R}^2 - Q_2, n - 3)) = s(n - 3) = zcl_{s,Z_2}(\text{Conf}(\mathbb{R}^2 - Q_2, n - 3)), $$

whereas the corresponding equality

$$\text{TC}_s(\mathbb{R}P^3) = 3(s - 1) = zcl_{s,Z_2}(\mathbb{R}P^3),$$

extending (2), is an easy exercise. Together with (4) and Proposition 3.1 we then get

$$\text{TC}_s(\text{Conf}(S^2,n)) \geq sn - 3,$$

which completes the proof in view of the upper bound given in Section 2 for $g = 0$ and $n \geq 3$. 

\hfill $\square$
Let $D \leq E$ only the subalgebra to the relations where the subindex $i$ given by the set $\beta_{p, q}$ for any $a \in M$ the spectral sequence is particularly amenable when $M$ is an orientable manifold. As shown by Cohen-Taylor (3) and Totaro (10), the spectral sequence is particularly amenable when $M$ is a complex projective manifold (e.g. $M = \Sigma_g$). We do not need the whole spectral sequence $\{E(g)_{i, q}\}_{i \geq 2}$ for $M = \Sigma_g$, only the subalgebra $E(g)_{i, q}$ of $H^\ast(\text{Conf}(\Sigma_g, n))$ detected on the base axis of the spectral sequence, which is described next.

Recall that the rational cohomology algebra $H^\ast(\Sigma_g)$ is the polynomial ring on $2g$ generators $a(p), b(p) \in H^1(\Sigma_g)$ with $1 \leq p \leq g$, and an additional generator $\omega \in H^2(\Sigma_g)$ subject to the relations

$$a(p)a(q) = b(p)b(q) = 0, \quad \text{and} \quad a(p)b(q) = \begin{cases} \omega, & p = q; \\ 0, & p \neq q. \end{cases}$$

for any $p, q \in \{1, \ldots, g\}$. Consequently, $H^\ast(\Sigma_g^\times n)$ is generated by 1-dimensional classes $a_i(p)$ and $b_i(p)$ $(1 \leq i \leq n$ and $1 \leq p \leq g)$ and by 2-dimensional classes $\omega_i$ $(1 \leq i \leq n)$, where the subindex $i$ indicates the cartesian factor where the classes come from, subject to the relations

$$a_i(p)a_i(q) = b_i(p)b_i(q) = 0 \quad \text{and} \quad a_i(p)b_i(q) = \begin{cases} \omega_i, & p = q; \\ 0, & p \neq q, \end{cases}$$

for $p, q \in \{1, \ldots, g\}$ and $i \in \{1, \ldots, n\}$. In particular, an additive basis for $H^\ast(\Sigma_g^\times n)$ is given by the set $\beta_1$ consisting of the (tensor) products $u = u_1 \cdots u_n$ satisfying

$$u_i \in \{1, a_i(p), b_i(p), \omega_i: 1 \leq p \leq g\}, \quad \text{for each} \ i \in \{1, \ldots, n\}. \quad \text{(7)}$$

Let $D_g$ be the ideal of $H^\ast(\Sigma_g^\times n)$ generated by the elements

$$\omega_i + \omega_j + \sum_{p=1}^g (b_i(p)a_j(p) - a_i(p)b_j(p))$$

for $1 \leq i < j \leq n$. In the spectral sequence, $H^\ast(\Sigma_g^\times n)$ corresponds to the base $E^\ast_2$, and $D_g$ corresponds to the image of the only differentials landing on the base. Therefore:

\[ zcl_{s, Q}(\text{Conf}(\Sigma_g, n)) \geq \begin{cases} s(n + 1) - 2, & g = 1; \\ s(n + 1), & g \geq 2. \end{cases} \]
**Lemma 3.2 ([10] Theorem 4]).** The quotient $E(g)^{*,0}_\infty = H^*(\Sigma_g^{\times n})/D_g$ is a subalgebra of $H^*(\text{Conf}(\Sigma_g,n))$.

In particular, (5) will follow once we prove

$$zcl_s(E(g)^{*,0}_\infty) \geq \begin{cases} s(n + 1) - 2, & g = 1; \\ s(n + 1), & g \geq 2. \end{cases} \tag{9}$$

Actually, a more explicit statement (in terms of a suitably large non-trivial product of s-th zero-divisors of $E(g)^{*,0}_\infty$) is given in Theorem 3.4 below, which requires some preparatory notation.

For $1 \leq i \leq n$ and $1 \leq p \leq g$, consider the elements $x_i(p), y_i(p) \in E(g)^{*,0}_\infty$ defined by

- $x_i(p) = a_i(p)$ and $y_i(p) = b_i(p)$, if $p \geq 2$, or if $p = 1$ with $i = 1$;
- $x_i(1) = a_i(1) - x_1(1)$ and $y_i(1) = b_i(1) - y_1(1)$, if $i \geq 2$.

In order to simplify notation, it will be convenient to write $x_i$ and $y_i$ as alternatives for $x_i(1)$ and $y_i(1)$, respectively. Likewise, $a_i$ and $b_i$ will be used as substitutes of $a_i(1)$ and $b_i(1)$, respectively.

**Example 3.3.** The relations (6) do not hold in $H^*(\Sigma_g^{\times n})$ if the letters $a$ and $b$ are replaced, respectively, by the letters $x$ and $y$. For instance, $a_j(p)a_j(1) = 0$, but if $j, p \geq 2$,

$$x_j(p)x_j(1) = a_j(p)(a_j(1) - a_1(1)) = a_j(p)a_1(1) \neq 0.$$  

Likewise, $a_j(1)b_j(1) = \omega_j$, while for $2 \leq j \leq n$,

$$x_j(1)y_j(1) = (a_j(1) - a_1(1))(b_j(1) - b_1(1)) = \omega_j + \omega_1 + b_1(1)a_j(1) - a_1(1)b_j(1) \tag{10}$$

$$= \omega_j + \omega_1 + y_1(1)(x_j(1) + x_1(1)) - x_1(1)(y_j(1) + y_1(1))$$

$$= \omega_j + \omega_1 + y_1(1)x_j(1) - \omega_1 - x_1(1)y_j(1) - \omega_1$$

$$= \omega_j - \omega_1 + y_1(1)x_j(1) - x_1(1)y_j(1). \tag{11}$$

We are now in position to define the s-th zero-divisors of $E(g)^{*,0}_\infty$ we need. In fact, we start by describing four types of s-zero-divisors of $H^*(\Sigma_g^{\times n})$.

(I) For an element $u \in H^*(\Sigma_g^{\times n})$ of positive degree (so $u^2 = 0$), consider the product $\bar{u} \in H^*(\Sigma_g^{\times n})^{\otimes s}$ given by

$$\bar{u} := \prod_{\ell=2}^{s} (u \otimes 1 \otimes \cdots \otimes 1 \otimes 1 - 1 \otimes \cdots \otimes 1 \otimes u \otimes 1 \otimes \cdots \otimes 1)$$

$$= \sum_{\ell=1}^{s} \pm u \otimes u \otimes \cdots \otimes 1 \otimes u \otimes \cdots \otimes u.$$
Here, the index on top of a tensor factor indicates the coordinate where such a factor appears. Note that \( \bar{u} \) is a product of \( s-1 \) \( s \)-th zero-divisors. We are interested in the product

\[
\prod_{i=1}^{n} x_i = \sum \pm \prod_{i,j=1}^{s} x_{J_i} \otimes \cdots \otimes x_{J_s},
\]

where the sum is taken over all subsets \( J_1, J_2, \ldots, J_s \subseteq \{1, \ldots, n\} \) with the property that every \( i \in \{1, \ldots, n\} \) belongs to exactly \( s-1 \) subsets \( J_k \) \((1 \leq k \leq s)\), and where

\[
x_{J_i} := \prod_{i \in J_i} x_i
\]

for \( t \in \{1, \ldots, s\} \).

(II) For \( i \in \{1, \ldots, n\} \), consider the \( s \)-th zero-divisor

\[
\bar{y}_i := y_i \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes 1 \otimes y_i \in H^*(\Sigma_g^{\times n})^\otimes s
\]

and the product

\[
\prod_{i=1}^{n} \bar{y}_i = \sum \pm y_{J^c} \otimes 1 \otimes \cdots \otimes 1 \otimes y_J,
\]

where \( J^c \) stands for the complement of \( J \) in \( \{1, \ldots, n\} \).

(III) For \( i \in \{2, \ldots, s-1\} \), consider the \( s \)-th zero-divisor

\[
y_{1,i} := y_1 \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes 1 \otimes y_1 \otimes 1 \otimes \cdots \otimes 1 \in H^*(\Sigma_g^{\times n})^\otimes s
\]

and the product

\[
\prod_{i=2}^{s-1} y_{1,i} = \sum_{(\varepsilon_1, \ldots, \varepsilon_{s-1}) \in M_s} \pm y_1^{\varepsilon_1} \otimes y_1^{\varepsilon_2} \otimes \cdots \otimes y_1^{\varepsilon_{s-1}} \otimes 1,
\]

where \( M_s := \{(\varepsilon_1, \ldots, \varepsilon_{s-1}) : \exists ! j \in \{1, \ldots, s-1\} \text{ with } \varepsilon_j = 0 \text{ and } \varepsilon_i = 1 \text{ for } i \neq j\} \).

(IV) If \( g \geq 2 \), consider the \( s \)-th zero divisors \( c, d \in H^*(\Sigma_g^{\times n})^\otimes s \) given by

\[
c = a_1(2) \otimes 1 \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes a_1(2) \otimes 1 \otimes \cdots \otimes 1,
\]

\[
d = \begin{cases} b_1(2) \otimes 1 - 1 \otimes b_1(2), & \text{if } s = 2; \\ b_1(2) \otimes 1 \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes 1 \otimes b_1(2) \otimes 1 \otimes \cdots \otimes 1, & \text{if } s \geq 3. \end{cases}
\]

The inequality in (9) and, therefore, Theorem 1.1 for \( g > 0 \) are immediate consequences of the following result, whose proof is the central goal in the reminder of the paper.

**Theorem 3.4.** (i) The image of \( \left( \prod_{i=2}^{s-1} y_{1,i} \right) \cdot \left( \prod_{i=1}^{n} (\bar{x}_i \bar{y}_i) \right) \) in \( E(1)^{s,0}_\infty \) is non-zero.

(ii) If \( g \geq 2 \), the image of \( c \cdot d \cdot \left( \prod_{i=2}^{s-1} y_{1,i} \right) \cdot \left( \prod_{i=1}^{n} (\bar{x}_i \bar{y}_i) \right) \) in \( E(g)^{s,0}_\infty \) is non-zero.
4 A subquotient of the cohomology of $\text{Conf}(\Sigma_g, n)$

The proof of the non-vanishing of the products indicated in Theorem 3.4 is greatly simplified by actually working on the quotient of $E(g)^{\ast,0}$ obtained by moding out by the ideal generated by the elements

$$x_i(p)x_j(q), \quad x_i(p)y_j(q), \quad y_i(p)y_j(q)$$

(15)

with $p, q \in \{2, \ldots, g\}$ and $i, j \in \{1, \ldots, n\}$, $i \neq j$, and by the elements

$$x_i y_j$$

(16)

with $i, j \in \{2, \ldots, n\}$, $i \neq j$. Our strategy has two main steps:

S1. We first get a full additive description of the quotient $A_g$ of $H^\ast(\Sigma \times n)$ by the ideal generated by the elements in (15).

S2. Then we prove that the products indicated in Theorem 3.4 are in fact non-trivial in the quotient $B_g$ of $A_g$ by the $A_g$-ideal generated by the elements in (8) and (16).

Furthermore, when dealing with the second step, and in view of the relations coming from (15), the elements in (8) can safely be replaced by the elements

$$\omega_i + \omega_j + b_i(1)a_j(1) - a_i(1)b_j(1)$$

(17)

for $1 \leq i < j \leq n$. It follows that the identity maps on generators induce ring morphisms $B_1 \to B_2 \to B_3 \to \cdots$. In particular, item (i) in Theorem 3.4 becomes a direct consequence of the proof of item (ii) in Theorem 3.4 sketched in steps S1 and S2 above. Accordingly, we assume $g \geq 2$ in the remainder of the section.

Step S1 above is accomplished in either of the next two results.

**Lemma 4.1.** An additive basis of $A_g$ is given by the set $\beta_2$ consisting of the $A_g$-images of the monomials $u_1 \cdots u_n \in H^\ast(\Sigma_g \times n)$ satisfying the following two conditions:

(i) For each $i \in \{1, \ldots, n\}$, the factor $u_i$ belongs to $\{1, a_i(p), b_i(p), \omega_i : 1 \leq p \leq g\}$.

(ii) At most one of $u_1, \ldots, u_n$ belongs to $\{a_i(p), b_i(p), \omega_i : 1 \leq i \leq n$ and $2 \leq p \leq g\}$.

**Proof.** Recall the additive basis $\beta_1$ of $H^\ast(\Sigma_g \times n)$ consisting of the products $u = u_1 \cdots u_n$ satisfying (7)—i.e. condition (6) of the lemma. In these terms, the defining relations for $A_g$ coming from the elements in (15) take the form

$$a_i(p)a_j(q) = a_i(p)b_j(q) = b_i(p)b_j(q) = 0$$

(18)

for $p, q \in \{2, \ldots, g\}$ and $i, j \in \{1, \ldots, n\}$ with $i \neq j$. The fact that $a_i(p)b_i(p) = \omega_i$ for any pair $(i, p)$ then implies that a basis element $u_1 \cdots u_n \in \beta_1$ vanishes in $A_g$ whenever
condition \[\text{[13]}\] of the lemma fails. The result follows since, for any \(u = u_1 \cdots u_n \in \beta_1\), the \(u\)-multiple (in \(H^*(\Sigma_g^n)\)) of any of the elements

\[
a_i(p)a_j(q), \quad a_i(p)b_j(q), \quad b_i(p)b_j(q)
\]

as in \(\text{[13]}\) either vanishes or, else, reduces (up to a sign) to an element of \(\beta_1\) for which \(\text{[13]}\) fails. For instance, in the case of an element of the form \(a_i(p_0)b_j(q_0)\) with \(p_0, q_0 \geq 2\) and \(i \neq j\),

\[
u_1 \cdots u_n \cdot a_i(p_0)b_j(q_0) = \pm u_1 \cdots \tilde{u}_i \cdots \tilde{u}_j \cdots u_n (u_i a_i(p_0)) (u_j b_j(q_0))
\]

which, in view of \(\text{[13]}\), is either zero or, else, an element of \(\beta_1\) not satisfying \(\text{[13]}\).

\[\square\]

**Corollary 4.2.** An additive basis of \(A_g\) is given by the set \(\beta_2'\) consisting of the \(A_g\)-images of the monomials \(v_1 \cdots v_n \in H^*(\Sigma_g^n)\) satisfying the following two conditions:

(i) For each \(i \in \{1, \ldots, n\}\), the factor \(v_i\) belongs to \(\{1, x_i(p), y_i(p), \omega_i : 1 \leq p \leq g\}\).

(ii) At most one of \(v_1, \ldots, v_n\) belongs to \(\{x_i(p), y_i(p), \omega_i : 1 \leq i \leq n\text{ and } 2 \leq p \leq g\}\).

**Proof.** Since \(\beta_2'\) and \(\beta_2'\) have the same cardinality, it is enough to prove that (the obvious preimage in \(H^*(\Sigma_g^n)\)) of each element of \(\beta_2\) can be expressed, modulo the \(H^*(\Sigma_g^n)\)-ideal \(\mathcal{I}\) generated by elements in \(\text{[15]}\), in terms of (the obvious preimages of) elements of \(\beta_2'\).

Fix \(u_1 \cdots u_n \in \beta_2\) and let \(J \subseteq \{1, \ldots, n\}\) be the set of indices \(i\) for which \(u_i \in \{a_i, b_i\}\). Thus \(u_i = 1\) for all \(i \in \{1, \ldots, n\} - J\) with the possible exception of a single element \(i_0 \in \{1, \ldots, n\} - J\) for which \(u_{i_0} \in \{a_{i_0}(p), b_{i_0}(p), \omega_{i_0} : 2 \leq p \leq g\}\). In what follows we let

\[
z_i = \begin{cases} x_i, & \text{if } u_i = a_i; \\ y_i, & \text{if } u_i = b_i; \end{cases} \quad \text{and} \quad u_i' = \begin{cases} x_1, & \text{if } u_i = a_i; \\ y_1, & \text{if } u_i = b_i; \end{cases}
\]

for \(i \in J\).

**Case 1.** Assume \(1 \in J\), and that the possible exceptional \(i_0\) does not hold. Then

\[
u_1 \cdots u_n = \prod_{i \in J} u_i = u_1 \prod_{i \in J - \{1\}} (z_i + u_i') = z_1 \left( \prod_{i \in J - \{1\}} z_i + \sum_{j \in J - \{1\}} \left( \pm u_j' \prod_{i \in J - \{j\}} z_i \right) \right)
\]

which, as an element of \(H^*(\Sigma_g^n)\), is a linear combination of elements in \(\beta_2'\). (Note that some terms in the latter summation may vanish as the corresponding factor \(z_1 u_j'\) may be trivial.)

**Case 2.** Assume \(1 \notin J\), and that the possible exceptional \(i_0\) does not hold. Then

\[
u_1 \cdots u_n = \prod_{i \in J} u_i = \prod_{i \in J} (z_i + u_i')
\]

\[
= \prod_{i \in J} z_i + \sum_{j \in J} \left( \pm u_j' \prod_{i \in J - \{j\}} z_i \right) + \sum_{j_1, j_2 \in J, j_1 < j_2} \left( \pm u_{j_1}' u_{j_2}' \prod_{i \in J - \{j_1, j_2\}} z_i \right)
\]

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again a linear combination in $H^*(\Sigma^n_{x})$ of elements in $\beta'_2$. (As in the previous case, some elements in the latter summation may vanish as the corresponding factor $u'_j u'_i$ may be trivial.)

**Case 3.** Assume $1 \in J$, and that the possible exceptional case $i_0$ holds ($i_0 \neq 1$ is forced). Then

$$u_1 \cdots u_n = \pm u_1 u_{i_0} \prod_{i \in J \setminus \{1\}} (z_i + u'_i) = \pm z_1 u_{i_0} \left( \prod_{i \in J \setminus \{1\}} z_i + \sum_{j \in J \setminus \{1\}} \left( \pm u'_j \prod_{i \in J \setminus \{j\}} z_i \right) \right)$$

$$= \pm u_{i_0} \left( \prod_{i \in J} z_i + \sum_{j \in J \setminus \{1\}} \left( \pm u'_j \prod_{i \in J \setminus \{j\}} z_i \right) \right).$$

This time the term $u_{i_0} \prod_{i \in J} z_i$ lies in $\beta'_2$ (for $a_{i_0}(p) = x_{i_0}(p)$ and $b_{i_0}(p) = y_{i_0}(p)$ if $2 \leq p \leq g$), while each of the terms

$$u_{i_0} u'_j \prod_{i \in J \setminus \{j\}} z_i \quad (19)$$

(with $j \in J \setminus \{1\}$) vanishes modulo $\mathcal{I}$. In fact, the factor $u'_j z_1$ in (19) is 0 or $\omega_1$; either way $u_{i_0} u'_j z_1 \equiv 0 \pmod{\mathcal{I}}$ since, after all, $\omega_t = a_t(2)b_t(2) = x_t(2)y_t(2)$ for $t = 1, i_0$ (c.f. the assertion following (18) in the proof of Lemma 4.1).

**Case 4.** Assume that the exceptional $i_0$ holds with $i_0 = 1$ (so $1 \notin J$). Then

$$u_1 \cdots u_n = \pm u_{i_0} \prod_{i \in J} u_i = \pm u_{i_0} \prod_{i \in J} (z_i + u'_i) = \pm u_{i_0} \left( \prod_{i \in J} z_i + \sum_{j \in J} \pm u'_j \prod_{i \in J \setminus \{j\}} z_i \right).$$

As in Case 3, the term $u_{i_0} \prod_{i \in J} z_i$ lies in $\beta'_2$. But now, each of the terms $u_{i_0} u'_j \prod_{i \in J \setminus \{i\}} z_i$ (with $j \in J$) vanishes directly in $H^*(\Sigma^n_{x})$. Indeed, $u_{i_0} u'_j = 0$ as $u'_j \in \{a_1, b_1\}$ and $u_{i_0} \in \{a_1(p), b_1(p), \omega_1 : 2 \leq p \leq g\}$.

**Case 5.** Assume $1 \notin J$, and that the exceptional $i_0$ holds with $i_0 \neq 1$. Then

$$u_1 \cdots u_n = \pm u_{i_0} \prod_{i \in J} (z_i + u'_i) = \pm u_{i_0} \left( \prod_{i \in J} z_i + \sum_{j \in J} \pm u'_j \prod_{i \in J \setminus \{j\}} z_i + \sum_{j_1, j_2 \in J \setminus \{j\}} \pm u'_j u'_{j_2} \prod_{i \in J \setminus \{j, j_1, j_2\}} z_i \right).$$

Now the term $u_{i_0} \prod_{i \in J} z_i$ as well as the terms $u_{i_0} u'_j \prod_{i \in J \setminus \{j\}} z_i$ (with $j \in J$) lie in $\beta'_2$. The rest of the terms in the previous displayed equation vanish modulo $\mathcal{I}$ just as in Case 3 above.

We now start working toward the completion of step S2. Recall that $B_g$ is the quotient of $A_g$ by the ideal generated by the elements in (16) and (17). As noted in (10), the case
1 = i < j \leq n of the latter generators is given by \( x_j y_j \), whereas for the case \( 2 \leq i < j \leq n \) we have

\[
\omega_i + \omega_j + b_i a_j - a_i b_j = (a_i - a_j)(b_i - b_j) = (x_i - x_j)(y_i - y_j) = x_i y_i + x_j y_j - x_i y_j - x_j y_i.
\]

Consequently we will work with the simplified presentation

\[
B_g = A_g / J_g
\]

where \( J_g \) is the \( A_g \)-ideal generated by the products \( x_i y_j \) with \( i, j \in \{2, \ldots, n\} \).

A key ingredient for step S2 is given by the next result, whose proof is deferred to the next section of the paper.

**Theorem 4.3.** The \( B_g \)-images of the two elements \( \omega_1 x_2 \cdots x_n, \omega_1 y_2 \cdots y_n \in H^*(\Sigma^x_g) \) are distinct and, in fact, linearly independent.

*Proof of item (ii) of Theorem 3.4 for \( s = 2 \).* As advertised at the beginning of this section, it suffices to work in \( B_g \). Direct calculation gives \( cdx_1 y_1 = 2 \omega_1 \otimes \omega_1 \) and (by induction on \( n \geq 2 \), keeping in mind the relations in \( B_g \) coming from the ideal \( J_g \))

\[
 cd \left( \prod_{i=1}^{n} (\bar{x}_i \bar{y}_i) \right) = 2 \omega_1 \otimes \omega_1 \left( \pm x_2 \cdots x_n \otimes y_2 \cdots y_n \pm y_2 \cdots y_n \otimes x_2 \cdots x_n \right),
\]

which is non-zero in \( B_g \) in view of Theorem 4.3. (Note that the factor (14) degenerates to 1.)

*Proof of item (ii) of Theorem 3.4 for \( s \geq 3 \).* Up to a sign, the product under consideration, \( cd \left( \prod_{i=2}^{s-1} y_{1,i} \right) \left( \prod_{i=1}^{s-1} (\bar{x}_i \bar{y}_i) \right) \), is a sum running over the subsets \( J, J_1, J_2, \ldots, J_s \) of \( \{1, \ldots, n\} \) specified in (12) and (13), over the tuples \( (\epsilon_1, \epsilon_2, \ldots, \epsilon_{s-1}) \in M_s \) specified in (14), and over the pairs \( (\alpha_1, \alpha_2) \) and \( (\beta_1, \beta_3) \) satisfying \( \{\alpha_1, \alpha_2\} = \{0, 1\} = \{\beta_1, \beta_3\} \). The term \( T \) corresponding to such a data takes the form indicated below, depending on the value of \( s \).

- **If \( s \geq 5 \),** \( \pm a_1(2)^{\alpha_1} b_1(2)^{\beta_1} y_1^t y_{j,x} x_{J_1} \otimes a_1(2)^{\alpha_2} y_1^t x_{J_2} \otimes b_1(2)^{\beta_2} y_1^t x_{J_2} \otimes y_1^t x_{J_3} \otimes \cdots \otimes y_1^{s-1} x_{J_{s-1}} \otimes y_{J,J_2}. \)
• If $s = 4$,

$$\pm a_1(2)\alpha b_1(2)\beta y_1^t y_{j^c x_j} \otimes a_1(2)\alpha y_1^f x_{j_2} \otimes b_1(2)\beta y_3 x_j y_{j^c x_j}.$$  

• If $s = 3$,

$$\pm a_1(2)\alpha b_1(2)\beta y_1^t y_{j^c x_j} \otimes a_1(2)\alpha y_1^f x_{j_2} \otimes b_1(2)\beta y_3 x_j y_j x_{j_3}.$$  

In either case, such a term $T$ vanishes in $B_g$ unless each of the following conditions holds:

1. $J = \{1\}$ or $J = \{1, \ldots, n\}$.  
   (Indeed, if $1 \notin J$, then the non-triviality of $T$ in $B_g$ forces $\alpha_1 = \beta_1 = \epsilon_1 = 0$, so that $\alpha_2 = \beta_3 = 1$ and $\epsilon_i = 1$ for $2 \leq i \leq s - 1$, which is impossible since $a_1(2)y_1 = 0$. Thus $1 \in J$ must hold. Furthermore, $2$ lies in $s - 1$ of the sets $J_1, \ldots, J_s$ so, in particular, $x_2$ shows up either in the first tensor factor of $T$ (where $y_{j^c}$ appears), or in the last tensor factor of $T$ (where $y_j$ appears). Therefore, the reduced form of the defining relations in $B_g$ and the non-triviality of $T$ in $B_g$ force either $J - \{1\} = \emptyset$, or $J - \{1\} = \{2, \ldots, n\}$.)

2. $1 \notin J_1$, so that $1 \in J_i$ for $2 \leq i \leq s$.
   (Indeed, if $1 \in J_1$, the non-triviality of $T$ in $B_g$ forces $\alpha_1 = 0 = \beta_1$, so $\alpha_2 = 1 = \beta_3$. But this is incompatible with the non-triviality of $T$ in $B_g$ and the fact that $1$ must lie in either $J_2$ or $J_3$.)

3. $\alpha_2 = 0 = \beta_3$, so that $\alpha_1 = 1 = \beta_1$.  
   (For we have just noted that $1 \in J_2 \cap J_3$.)

4. $\epsilon_1 = 0$, so that $\epsilon_i = 1$ for $2 \leq i \leq s - 1$.  
   (For we have just noted that $\alpha_1 = 1 = \beta_1$.)

Further, when $J = \{1\}$, the term $T$ vanishes in $B_g$ unless $J_1 = \emptyset$ (the inclusion $J_1 \subseteq \{1\}$ follows by looking at the first tensor factor of $T$ and the relations defining $B_g$, whereas the actual equality $J_1 = \emptyset$ follows from condition 2 above) and, therefore, $J_i = \{1, \ldots, n\}$ for $2 \leq i \leq s$. Thus, the only such $T$ with (potentially) non-vanishing image in $B_g$ is, up to a sign,

$$a_1(2)b_1(2)y_2 \cdots y_n \otimes y_1 x_1 \cdots x_n \otimes \cdots \otimes y_1 x_1 \cdots x_n$$

$$= \pm \omega_1 y_2 \cdots y_n \otimes \omega_1 x_2 \cdots x_n \otimes \cdots \otimes \omega_1 x_2 \cdots x_n.$$ (21)

Likewise, when $J = \{1, \ldots, n\}$, the term $T$ vanishes in $B_g$ unless $J_s = \{1\}$ (the inclusion $J_s \subseteq \{1\}$ follows by looking at the last tensor factor of $T$ and the relations defining $B_g$, whereas the actual equality $J_s = \{1\}$ follows from condition 2 above) and $J_i = \{2, \ldots, n\}$ while $J_i = \{1, \ldots, n\}$ for $2 \leq i \leq s - 1$ (in view of condition 2 above and the properties of
the $J_i$’s). Thus, the only such $T$ with (potentially) non-vanishing image in $B_g$ is, up to a sign,

$$a_1(2)b_1(2)x_2 \cdots x_n \otimes y_1 x_1 \cdots x_n \otimes \cdots \otimes y_1 x_1 \cdots x_n \otimes y_1 \cdots y_n = \pm \omega_1 x_2 \cdots x_n \otimes \cdots \otimes \omega_1 x_2 \cdots x_n \otimes \omega_1 y_2 \cdots y_n. \tag{22}$$

Consequently, the image in $B_g$ of the product under consideration is the sum of the term in (21) and the term in (22), which is non-zero by Theorem 4.3.

$$\square$$

5 Proof of Theorem 4.3

In view of the particularly simple presentation (20) of $B_g$, it might be tempting to guess the form of an additive basis for $B_g$ which, in addition, could easily imply Theorem 4.3. However, a few unexpected relations holding in $B_g$ are hidden in $J_g$. It is the purpose of this section to uncover, in the most efficient way (for the purpose of proving Theorem 4.3), some of these unexpected relations.

Recall the additive basis $\beta'_2$ of $A_g$ in Corollary 4.2, that is, the set of products $v_1 \cdots v_n$ satisfying the two conditions:

(i) For each $i \in \{1, \ldots, n\}$, the factor $v_i$ belongs to $\{1, x_i(p), y_i(p), \omega_i : 1 \leq p \leq g\}$.

(ii) At most one of $v_1, \ldots, v_n$ belongs to $\{x_i(p), y_i(p), \omega_i : 1 \leq i \leq n$ and $2 \leq p \leq g\}$.

The verification of the following two lemmas is a straightforward and, thus, omitted task.

**Lemma 5.1.** Let $2 \leq j \leq n$. For $v_1 \cdots v_n \in \beta'_2$, the product $v_1 \cdots v_n \cdot x_j y_j$ vanishes in $A_g$ provided either one of the following conditions holds:

(i) $v_j \in \{x_j(p), y_j(p), \omega_j : 1 \leq p \leq g\}$.

(ii) $v_1 \in \{x_1(p), y_1(p), \omega_1 : 2 \leq p \leq g\}$.

(iii) $v_1 \in \{x_1, y_1\}$ and $v_k \in \{x_k(p), y_k(p), \omega_k : 2 \leq p \leq g\}$ for some $k \notin \{1, j\}$.

Furthermore, the following relations hold in $A_g$:

(iv) $x_1 \cdot x_j y_j = x_1 \omega_j + \omega_1 x_j$.

(v) $y_1 \cdot x_j y_j = y_1 \omega_j + \omega_1 y_j$.

(vi) $z_k \cdot x_j y_j = z_k y_1 x_j - z_k x_1 y_j$, for $z_k \in \{x_k(p), y_k(p), \omega_k : 2 \leq p \leq g\}$ with $k \notin \{1, j\}$.

**Lemma 5.2.** Let $i, j \in \{2, \ldots, n\}$ with $i \neq j$. Then, in $A_g$:
(1) The only non-trivial products \( z_i \cdot x_i y_j \) with \( z_i \in \{ x_i(p), y_i(p), \omega_i : 1 \leq p \leq g \} \) are

(i) \( y_i \cdot x_i y_j = -\omega_i y_j + \omega_1 y_j - y_1 x_i y_j + x_1 y_i y_j \).

(ii) \( z_i \cdot x_i y_j = -z_i x_i y_j \), for \( z_i \in \{ x_i(p), y_i(p), \omega_i : 2 \leq p \leq g \} \).

(2) The only non-trivial products \( z_j \cdot x_i y_j \) with \( z_j \in \{ x_j(p), y_j(p), \omega_j : 1 \leq p \leq g \} \) are

(iii) \( x_j \cdot x_i y_j = -x_i \omega_j + x_i \omega_1 + y_1 x_i x_j - x_1 x_i y_j \).

(iv) \( z_j \cdot x_i y_j = -z_j x_i y_j \), for \( z_j \in \{ x_j(p), y_j(p), \omega_j : 2 \leq p \leq g \} \).

(3) The only non-trivial product \( z_i z_j \cdot x_i y_j \) with \( z_i \) and \( z_j \) as in (1) and (2) above is

(v) \( y_i x_j \cdot x_i y_j = y_i \omega_i x_j + y_1 x_i \omega_j - x_i \omega_1 y_j - x_1 y_i \omega_j + \omega_1 y_i x_j - \omega_1 x_i y_j \).

(4) The only non-trivial products \( z_1 z_i \cdot x_i y_j \) with \( z_1 \in \{ x_1(p), y_1(p), \omega_1 : 1 \leq g \leq p \} \) and \( z_i \) as in (1) above are

(vi) \( x_1 y_i \cdot x_i y_j = -x_1 \omega_i y_j - \omega_1 x_i y_j \).

(vii) \( y_1 y_i \cdot x_i y_j = -y_1 \omega_i y_j - \omega_1 y_i y_j \).

(5) The only non-trivial products \( z_1 z_j \cdot x_i y_j \) with \( z_j \) and \( z_1 \) as in (2) and (4) above are

(viii) \( x_1 x_j \cdot x_i y_j = -x_1 x_i \omega_j + \omega_1 x_i x_j \).

(ix) \( y_1 x_j \cdot x_i y_j = -y_1 x_i \omega_j + \omega_1 x_i y_j \).

(6) All products \( z_1 z_i z_j \cdot x_i y_j \) with \( z_i, z_j \) and \( z_1 \) as in (1), (2) and (4) vanish.

Set \( \gamma_2 = \beta_2' - \gamma_1 \), where \( \gamma_1 \subseteq \beta_2' \) consists of the products \( v_1 \cdots v_n \) satisfying either one of the following two conditions:

(iii) There is a unique \( i \in \{1, \cdots, n\} \) for which \( v_i = \omega_i \) and \( v_j = x_j \) for \( j \neq i \).

(iv) There is a unique \( i \in \{1, \cdots, n\} \) for which \( v_i = \omega_i \) and \( v_j = y_j \) for \( j \neq i \).

There is an obvious additive splitting \( A_g = C_{g,1} \oplus C_{g,2} \), where \( C_{g,\epsilon} \) is the additive span of \( \gamma_\epsilon \) \( (\epsilon = 1, 2) \). The final technical task in this paper, the proof of Theorem 43, will be accomplished below by arguing first that the ideal \( J_g \) defining \( B_g \) preserves the above splitting, i.e. by giving an additive decomposition

\[
J_g = J_{g,1} \oplus J_{g,2},
\]

(23)

where \( J_{g,\epsilon} \) is a vector subspace of \( C_{g,\epsilon} \) \( (\epsilon = 1, 2) \), and then by giving a description of the (additive structure of the) quotient \( C_{g,1}/J_{g,1} \), for which a basis will clearly be given by the two elements in the statement of Theorem 43.

In what follows, an element \( v = v_1 \cdots v_n \in \beta_2' \), will be denoted as
• $v(0)$ to indicate that $v_k \in \{1, x_k, y_k\}$ for all $k = 1, \ldots, n$;

• $v(i_1, \ldots, i_t)$, for $i_1, \ldots, i_t \in \{1, \ldots, n\}$, to indicate that $v_k = 1$ for $k \in 1, \ldots, t$.

These two conventions will also be combined. For instance, by writing $v(0, 1, j)$ we mean that the element $v \in \beta_2'$ satisfies $v_k \in \{1, x_k, y_k\}$ for all $k = 1, \ldots, n$, as well as $v_1 = v_j = 1$.

Proof of Theorem 4.3. A set of additive generators of $J_q$ is given by the products $v \cdot r$ with $v = v_1 \cdots v_n \in \beta_2'$ and $r \in \{x_iy_j : i, j \in \{2, \ldots, n\}\}$. The additive decomposition (23) will follow once we check that

\[\text{the expression of each such product } v \cdot r = v_1 \cdots v_n \cdot x_iy_j \text{ (in terms of the basis } \beta_2') \text{ involves either only elements of } \gamma_1 \text{ or, else, only elements of } \gamma_2.\]  

(24)

Case $i = j \geq 2$. By Lemma 5.1(i), we only need to consider products $v(j) \cdot x_iy_j$. Recalling from (11) that $x_iy_j = \omega_j - \omega_i + y_1x_j - x_1y_j$, it is clear that (24) holds, with $\gamma_2$ being the relevant basis, if $v = v(1, j)$. In checking this type of assertions, the reader might find it convenient to consider first the case $v = v(0, 1, j)$. Thus, by Lemma 5.1(ii) and (iii), we can assume $v_1 \in \{x_1, y_1\}$ and $v = v(0)$. In other words, it remains to consider products of the form

\[x_1v(0, 1, j) \cdot x_iy_j \text{ and } y_1v(0, 1, j) \cdot x_iy_j.\]

It is clear from Lemma 5.1(iv) and (v) that (24) holds true for the two types of products just described, and that the only such products whose expression in terms of the basis $\beta_2'$ involves only elements from $\gamma_1$ can actually be written, up to a sign, as

\[\omega_1x_2 \cdots x_n + (-1)^j x_1x_2 \cdots x_{j-1}\omega_jx_{j+1} \cdots x_n\]  

(25)

and

\[\omega_1y_2 \cdots y_n + (-1)^j y_1y_2 \cdots y_{j-1}\omega_jy_{j+1} \cdots y_n.\]  

(26)

Case $i, j \in \{2, \ldots, n\}$ with $i \neq j$. It is obvious that (24) holds, with $\gamma_2$ being the relevant basis, provided $v = v(i, j)$. The rest of the possibilities can be analyzed on a term-by-term basis, depending on the values of $z_i$ and $z_j$ in a product $z_iz_jv(i, j) \cdot x_iy_j$, where $z_i \in \{1, x_i(p), y_i(p), \omega_i : 1 \leq p \leq g\}$. Actually, by Lemma 5.2, the only factors involved in the expression of any $z_iz_j \cdot x_iy_j$ can come from the coordinates 1, $i$ and $j$. Therefore it is convenient to split the analysis by considering the products

\[z_iz_jv(1, i, j) \cdot x_iy_j \text{ and } z_1z_iz_jv(1, i, j) \cdot x_iy_j.\]  

(27)

Lemma 5.2 describes the expression of the corresponding factors $z_iz_j \cdot x_iy_j$ and $z_1z_iz_j \cdot x_iy_j$ in terms of the basis $\beta_2'$. In all such cases one checks, by direct inspection, that

• (24) holds true for all products in (27),
• the only products in (27) whose expression in terms of $\beta'_2$ involves elements from $\gamma_1$ are those arising from instances (vii) and (viii) of Lemma 5.2 in which case
• the resulting expressions in terms of the basis $\beta'_2$ coincide with those in (25) and (26) —note that signs in items (vii) and (viii) of Lemma 5.2 are important here!

The proof is complete since the above considerations imply that the decomposition (23) holds in such a way that an additive basis for the resulting additive summand $C_{g,1}/J_{g,1}$ of $B_g$ is given by the two elements in the statement of Theorem 4.3.

6 The case $s = 2$

As mentioned in the introduction, the case $s = 2$ in Theorem 1.1 reduces to Theorem A in [2]. We have given full proof details for that case too because we believe that there are a couple of weak points and, most critically, at least one flawed argument in the homological part of Cohen-Farber’s argument. This section describes such potential problems. The reader is assumed to be familiar with the notation in [2].

The main problem happens at the end of the fourth paragraph of the proof of [2, Theorem 5.1], where the authors assert that the proof of the case for genus $g \geq 2$ can be reduced to the consideration of the $g = 2$ case by “annihilating all generators of the form $1 \times \cdots \times u \times \cdots \times 1$ where $u \in \{a(q), b(q) : 3 \leq q \leq g\}$”. (Note the typo “$3 \leq q \leq n$” in [2].) Such an argument does not work because if, for instance, we set $a(3) = 0$ in the $i$-th axis, then $w = a(3)b(3)$ would also be zero in that axis. But this interferes (for $i = 1$) with Cohen and Farber’s later calculation using the non-triviality of $\omega_1$ (see the last displayed formula in the proof of [2, Theorem 5.1]).

In addition, we believe that a weak argument arises at the end of the proof of [2, Theorem 5.1], where the authors assert that

$$\text{the non-zero term } \pm 2\omega_1y_2y_3\cdots y_n \otimes x_1x_2x_3\cdots x_n \text{ arises in the expansion of the product } \bar{a}_1\bar{b}_1\bar{c}_1\bar{d}_1 \prod_{j=2}^{n} \bar{x}_j\bar{y}_j \text{ in such a way that no other summand in the expansion involves this (non-zero) tensor product.} \tag{28}$$

The (apparently implicit) argument supporting (28) is based on two facts noted in earlier parts of Cohen-Farber’s paper:

(I) On the one hand, as indicated at the end of the proof of [2, Theorem 4.1] (i.e. when dealing with the algebra $A_T$ in the genus-1 case), the expansion (in terms of basis elements) of $\prod_{j=1}^{n} \bar{x}_j\bar{y}_j$ uses (with coefficient $\pm 1$) the basis element $y_1y_2y_3\cdots y_n \otimes x_1x_2x_3\cdots x_n$.

(II) On the other hand, near the bottom of page 656 of [2], it is observed (without further explanation, though) that “The subalgebra of $B_\Sigma$ generated by $\{a_i, b_i : 1 \leq i \leq n\}$ is isomorphic to the subalgebra $A_T$ arising in the genus one case”.

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The problem is that the latter two facts do not really support \((28)\) for, although \(A_T\) were a honest subalgebra of \(B_\Sigma\), nothing is said about the (potential) injectivity of the obvious map \((2\omega_1 \otimes \omega_1) \cdot A_T \rightarrow B_\Sigma\). In the Cohen-Farber approach, fixing these problems requires, in principle, an explicit description of additive bases for the subquotient algebras they deal with. Such a task tends to become combinatorially involved, specially in the case of Rudyak’s higher TC. We have greatly simplify the job by working in a much smaller subquotient—small enough to detect just the minimal needed information.

Also worth remarking is what appears to the authors of this paper to be a weak statement of item (ii) in [2, Lemma 2.1], namely, the assertion that an epimorphic image \(B\) of an algebra \(A\) over a field has \(\text{zcl}(A) \geq \text{zcl}(B)\). The verification of such a property is left as a “straightforward exercise” in [2] and, as in the case of the dual statement in item (i), its proof should naturally start by picking zero-divisors \(b_1, \ldots, b_t \in B \otimes B\) with \(b_1 \cdots b_t \neq 0\). With these conditions it is certainly obvious that, for any choice of preimages \(a_i \in A \otimes A\) of each \(b_i\), the product \(a_1 \cdots a_n\) is forced to be non-zero. But the point is to make sure that each \(a_i\) can be chosen to be a zero-divisor in \(A\), which does not seem to be accomplishable in the stated generality. Nonetheless, what can certainly be done (and has been done in this paper) is to argue the non-triviality of some given product of zero-divisors in \(A \otimes A\) by exhibiting the non-triviality of the image of the product in \(B \otimes B\).

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