The Order of the Unitary Subgroups of Group Algebras

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Abstract
Let $FG$ be the group algebra of a finite $p$-group $G$ over a finite field $F$ of positive characteristic $p$. Let $\otimes$ be an involution of the algebra $FG$ which is a linear extension of an anti-automorphism of the group $G$ to $FG$. If $p$ is an odd prime, then the order of the $\otimes$-unitary subgroup of $FG$ is established. For the case $p = 2$ we generalize a result obtained for finite abelian 2-groups. It is proved that the order of the $\ast$-unitary subgroup of $FG$ of a non-abelian 2-group is always divisible by a number which depends only on the size of $F$, the order of $G$ and the number of elements of order two in $G$. Moreover, we show that the order of the $\ast$-unitary subgroup of $FG$ determines the order of the finite $p$-group $G$.

Keywords: group algebras; unit group of group algebras; unitary subgroups

1. Introduction and main results
Let $FG$ be the group algebra of a finite $p$-group $G$ over a finite field $F$ of positive characteristic $p$. Let

$$V(FG) = \{ x = \sum_{g \in G} \alpha_g g \in FG \mid \chi(x) = \sum_{g \in G} \alpha_g = 1 \}$$

be the group of normalized units of $FG$, where $\chi(x)$ is the augmentation map (see [3, Chapters 2-3, p. 194-196]). In this case, the order of the group $V(FG)$ is equal to $|F|^{|G|-1}$, so the order of $V(FG)$ can be very large even for...
a small group $G$. Note that, studying the structure of the group $V(FG)$ is a rather difficult task (for more details see the survey [5]).

Let $\boxdot$ be an involution of the algebra $FG$. We say that the involution $\boxdot$ arises from the group $G$, if $\boxdot$ is a linear extension of an anti-automorphism of $G$ to $FG$. An example for such kind of involution is the canonical involution that is the linear extension of the anti-automorphism of $G$ which sends each element of $G$ to its inverse. This involution is usually denoted by $\ast$.

An element $u \in V(FG)$ is called $\boxdot$-unitary, if $u^{\boxdot} = u^{-1}$ with respect to the involution $\boxdot$ of $FG$. The set $V_\boxdot(FG)$ of all $\boxdot$-unitary units forms a subgroup of $V(FG)$ which is called $\boxdot$-unitary subgroup. Interest in the structure of unitary subgroups arose in algebraic topology and unitary $K$-theory (see Novikov’s papers [17] and Bovdi’s paper [8]). Let $L$ be a finite Galois extension of $F$ with Galois group $G$, where $F$ is a finite field of characteristic two. Serre [18] identified an interesting relation between the self-dual normal basis of $L$ over $F$ and the $\ast$-unitary subgroup of $FG$. This relationship also makes the study of the unitary subgroups timely.

The unitary subgroups have been proven to be very useful subgroups in several studies (see [2, 3, 6, 7, 10, 11, 13, 14, 15] and [16]). However, we know very little about their structure, as even finding their order is a challenging problem. The first results in this area were published in the 1980’s. For finite abelian $p$-groups $G$ the order of $V_\ast(FG)$ was given in [6].

**Proposition 1. ([9, Theorem 2])** Let $G$ be a finite abelian 2-group. If $F$ is a finite field of characteristic 2, then the order $|V_\ast(FG)|$ is divisible by $|F|^\frac{1}{2}(|G|+|G\{2\}|) - 1$, that is,

$$|V_\ast(FG)| = \Theta \cdot |F|^\frac{1}{2}(|G|+|G\{2\}|) - 1,$$

where $\Theta = |G^2\{2\}|$ and $|S|$ denotes the size of a finite set $S$.

It follows that the number $\Theta$ does not depend on the size of the field $F$. The following breakthrough result was proved for certain non-abelian 2-groups by Bovdi and Roza.

**Proposition 2. ([12, Corollary 2])** If $|F| = 2^m \geq 2$, then:

$$|V_\ast(FG)| = \Theta \cdot |F|^\frac{1}{2}(|G|+|G\{2\}|) - 1,$$

where
(i) \( \Theta = 1 \) if \( G \) is a dihedral 2-group;

(ii) \( \Theta = 4 \) if \( G \) is a generalized quaternion 2-group.

In [1], the value of the number \( \Theta \) was given for all non-abelian groups of order \( 2^4 \). Although, for these groups, the number \( \Theta \) is not equal to \(|G^2\{2\}|\) it does not depend on the field \( F \). Wang and Liu [19] evaluated \( \Theta \) in the case when \( G \) is a non-abelian 2-group given by a central extension of the form

\[
1 \rightarrow C_{2^m} \rightarrow G \rightarrow C_2 \times \cdots \times C_2 \rightarrow 1,
\]

in which \( m \geq 1 \) and \(|G'| = 2\). At present, the question of whether the quotient \( \Theta \) depends on the field \( F \) is still open.

Our main results are the following.

**Theorem 1.** Let \( G \) be a finite \( p \)-group, where \( p \) is an odd prime and let \( F \) be a finite field of characteristic \( p \). If \( \otimes \) is an involution of \( FG \) that arises from the group \( G \), then

\[
|V_{\otimes}(FG)| = |F|^\frac{1}{2}(|G|-|G_{\otimes}|),
\]

where \( G_{\otimes} = \{ g \mid g = g^{\otimes} \} \).

Let \( \xi(G) \) denote the center of the group \( G \) and \( \xi(G)\{2\} \) denote the set of elements of order two in \( \xi(G) \).

**Theorem 2.** Let \( G \) be a finite 2-group. If \( F \) is a finite field of characteristic two, then

\[
|V_*(FG)| = \Theta \cdot |F|^\frac{1}{2}(|G|+|G\{2\})-1
\]

for some integer \( \Theta \). Moreover, if the set \( T_c = \{ g \in G \mid g^2 = c \} \) is commutative for some \( c \in \xi(G)\{2\} \), then \( \Theta \) does not depend on the field \( F \).

By combining Theorem 1 and Theorem 2 we have the following.

**Corollary 1.** Let \( G \) be a finite \( p \)-group. If \( F \) is a finite field of characteristic \( p \), then the order of the \(*\)-unitary subgroup of \( FG \) determines the order of \( G \).
2. Proofs

Let $G$ be a finite $p$-group, let $F$ be a finite field of $\text{char}(F) = p > 2$ and let $\otimes$ be an involution of $FG$ which arises from $G$. An element $x \in FG$ is called skew-symmetric under the involution $\otimes$ if $x^{\otimes} = -x$. Let $FG_{\otimes}$ denote the set of all skew-symmetric elements of $FG$.

**Proof of Theorem 1.** Let $z \in FG$ such that $1 + z$ is invertible. Clearly, $1 - z$ and $1 + z$ commute, therefore $1 - z$ and $(1 + z)^{-1}$ also commute.

Let $Q = \{x \in FG \mid 1 + x \text{ is invertible in } FG\}$. Let us define the map $f : Q \to FG$ by

$$f(x) = (1 - x)(1 + x)^{-1}.$$  

If $y \in FG_{\otimes}$, then $\chi(y) = 0$, so

$$\chi(1 + y) = \chi(1 - y) = \chi(1 + y^{\otimes}) = 1$$

and $1 + y, 1 - y, 1 + y^{\otimes}$ are normalized units. Hence

$$f(y)f(y^{\otimes}) = (1 - y)(1 + y)^{-1}(1 + y^{\otimes})^{-1}(1 - y^{\otimes})$$
$$= (1 - y)(1 + y)^{-1}(1 - y)^{-1}(1 + y)$$
$$= (1 + y)^{-1}(1 - y)(1 - y)^{-1}(1 + y)$$
$$= 1.$$

Consequently, $f(y) \in V_{\otimes}(FG)$ and $f : FG_{\otimes} \to V_{\otimes}(FG)$ is a surjection.

Let $x \in V_{\otimes}(FG)$. Evidently, $1 + x, 1 + x^{\otimes}$ and $1 + x^{-1}$ are invertible, because $\chi(1 + x) = \chi(1 + x^{\otimes}) = \chi(1 + x^{-1}) = 2$ is invertible in $F$. Therefore $V_{\otimes}(FG)$ is a subset of $Q$. Let $y$ denote the element $f(x)$. Then

$$y^{\otimes} = f(x)^{\otimes} = (1 + x^{\otimes})^{-1}(1 - x^{\otimes}) = (1 + x^{-1})^{-1}(1 - x^{-1})$$
$$= (x^{-1}(x + 1))^{-1}x^{-1}(x - 1) = -(1 + x)^{-1}xx^{-1}(1 - x)$$
$$= -y.$$

Therefore $f : V_{\otimes}(FG) \to FG_{\otimes}$ is a surjection.

Similar computation shows that

$$f(f(x)) = (1 - (1 - x)(1 + x))^{-1}(1 + (1 - x)(1 + x))^{-1}$$
$$= ((1 + x) - (1 - x))(1 + x)^{-1}
(1 + (1 + x) + (1 - x))^{-1}$$
$$= ((1 + x) - (1 - x))(1 + x)^{-1}
((1 + x) + (1 - x))^{-1}$$
$$= ((1 + x) - (1 - x))(1 + x + (1 - x))^{-1}$$
$$= x$$

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for every $x \in V_\circ(G)$, so $f$ is a bijection between $FG_\circ$ and $V_\circ(G)$.

Since $FG_\circ$ is a linear space over $F$ with basis $\{g - g^\circ \mid g \in G \setminus G_\circ\}$,

$$|V_\circ(G)| = |FG_\circ| = |F| \frac{|F[G]|}{|G_\circ|}. \quad \square$$

Let $H$ be a normal subgroup of $G$ and let $I(H) := \langle 1 + h \mid h \in H \rangle_{FG}$ be an ideal of $FG$ generated by the set $\{1 + h \mid h \in H\}$. Clearly,

$$FG/I(H) \cong FG/\ker(\Psi) \cong F[G/H],$$

where $\Psi : FG \to FG/I(H)$ is the natural homomorphism.

Let us denote by $V_\ast(FG)\ast$ the *-unitary subgroup of the factor algebra $FG/I(H)$, where $G\ast = G/H$. It is easy to check that the set

$$N_\Psi^\ast = \{x \in V(FG) \mid \Psi(x) \in V_\ast(FG)\ast\}$$

forms a subgroup in $V(FG)$. Let $I(H)^+ = \{1 + x \mid x \in I(H)\}$. The subgroup $I(H)^+$ is normal in $V(FG)$ and $S_H = \{xx^\ast \mid x \in N_\Psi^\ast\}$ is a subset of $I(H)^+$, because $xx^\ast \in 1 + \ker(\Psi) = I(H)^+$ for all $x \in N_\Psi^\ast$.

First, we need the following.

**Lemma 1.** Let $H$ be a normal subgroup of a finite 2-group $G$. Set $G\ast = G/H$. If $|F| = 2^m \geq 2$, then

$$|V_\ast(FG)| = |F|^{\frac{|G\ast|}{|H|}} \cdot \frac{|V_\ast(FG)\ast|}{|S_H|}. \quad (1)$$

**Proof.** Let $\Phi : V(FG) \to V(FG)$ be a map such that $\Phi(x) = xx^\ast$ for every $x \in V(FG)$. The sets $\Phi(x)$ and $\Phi(y)$ coincide if and only if $y \in x \cdot V_\ast(FG)$.

Indeed, if $y \in x \cdot V_\ast(FG)$, then $y = xv$ for some $v \in V_\ast(FG)$. Therefore

$$\Phi(y) = yy^\ast = xv(xv)^\ast = xvv^\ast x^\ast = xx^\ast = \Phi(x).$$

Assume that $\Phi(x) = \Phi(y)$ for some $x, y \in V(FG)$. Then $xx^\ast = yy^\ast$, or equivalently, $y^{-1}x = y^*(x^*)^{-1}$. Therefore

$$(x^{-1}y)^{-1} = y^{-1}x = y^*(x^*)^{-1} = (x^{-1}y)^\ast$$

which confirms that $x^{-1}y \in V_\ast(FG)$. 

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Since \( [N^*_q : V_*(FG)] = |S_H| \),

\[
|V_*(FG)| = \frac{|N^*_q|}{|S_H|} = |I(H)^+| \cdot \frac{|V_*(FG)|}{|S_H|}.
\]

We should note that \( V_*(FG) \) is usually not a normal subgroup of \( N^*_q \).

The ideal \( I(H) \) can be considered as a vector space over \( F \) with the following basis \( \{ u(1 + h) \mid u \in T(G/H), \ h \in H \} \), where \( T(G/H) \) is a complete set of left coset representatives of \( H \) in \( G \). Consequently,

\[
|I(H)^+| = |I(H)| = |F|^{\frac{|G|}{|H|}}.
\]

\[ \Box \]

**Proof of Theorem 2.** Let \( G \) be a 2-group of order \( 2^n \) and let \( H \) be a subgroup of \( G \) generated by a central element \( c \) of order two. Evidently, the set \( S_H = \{ xx^* \mid x \in N^*_q \} \) is a subset of \( I(H)^+ \cap V_*(FG) \) and every \( y \in S_H \) is a symmetric element. Moreover, the support of \( y \) does not contain elements of order two by [1, Lemma 2.5]. Thus, every \( y \in S_H \) can be written as

\[
y = 1 + \sum_{g \in G \setminus (G \{2\} \cup T_c)} \alpha_g (g + g^{-1}) \hat{H} + \sum_{g \in T_c} \beta_g g \hat{H}, \quad (\alpha_g, \beta_g \in F)
\]

where \( T_c = \{ g \in G \mid g^2 = c \} \) and \( \hat{H} = 1 + c \). This yields that

\[
|S_H| \leq |F|^{\frac{1}{4}|(G|-(G \{2\})|)+\frac{1}{4}|T_c|} = |F|^{\frac{1}{4}|(G|-(G \{2\})|)+\frac{1}{4}|T_c|}. \tag{2}
\]

Let us prove that if \( T_c \) is a commutative set, then

\[
|F|^{\frac{1}{4}|(G|-(G \{2\})|)+\frac{1}{4}|T_c|} \cdot 2^{-\frac{1}{2}|T_c|} \leq |S_H|. \tag{3}
\]

Let \( N_1 \) be a group generated by the elements \( 1 + \alpha_g (g + g^{-1}) \hat{H} \), in which \( g^2 \not\in H \) and \( \alpha_g \in F \). Evidently, \( N_1 \) is an elementary abelian subgroup of \( I(H)^+ \). Since \( g^2 \not\in H \) (equivalently \( g \in G \setminus (G \{2\} \cup T_c) \)),

\[
1 + \alpha_g (g + g^{-1}) \hat{H} = 1 + \alpha_g g^{-1}(1 + g^2) \hat{H} \neq 1
\]

and

\[
1 + \alpha_g (g + g^{-1}) \hat{H} = (1 + \alpha_g \hat{H})(1 + \alpha_g g \hat{H}) \in S_H.
\]

If \( z \in N_1 \), then

\[
z = \prod_{g^2 \not\in H} (1 + \alpha_g (g + g^{-1}) \hat{H}) = \prod_{g^2 \not\in H} (1 + \alpha_g \hat{H})(1 + \alpha_g g \hat{H}),
\]

and

\[
\prod_{g^2 \not\in H} \alpha_g \hat{H} = \prod_{g^2 \not\in H} \alpha_g g \hat{H}
\]

is a subset of \( P \). Thus, \( P \) is a normal subgroup of \( N^*_q \).

Since \( |N^*_q : P| = |S_H| \),

\[
|P| = \frac{|N^*_q|}{|S_H|} = |I(H)^+| \cdot \frac{|N^*_q|}{|S_H|}.
\]

Therefore, \( P \) is a normal subgroup of \( N^*_q \) and \( P \subset S_H \) is a normal subgroup of \( N^*_q \). Consequently,

\[
|P| = |N^*_q : S_H| = |I(H)^+| \cdot \frac{|N^*_q|}{|S_H|}.
\]
so \((1 + \alpha_g \hat{H}) \in I(H)^+\) and

\[
z = \left( \prod_{g^2 \in H} (1 + \alpha_g \hat{H}) \right) \cdot \left( \prod_{g^2 \in H} (1 + \alpha_g \hat{H}) \right)^* \in S_H.
\]

Thus \(N_1\) is a subgroup in the set \(S_H\) and \(|N_1| = |F|^{\frac{1}{2}(|G|+|G(2)|+|T_c|)}\).

The map \(\tau : F \to F\) defined by \(\tau(\alpha) = \alpha + \alpha^2\) \((\alpha \in F)\) is a homomorphism on the additive group of the field \(F\) with kernel \(\ker(\tau) = \{0, 1\}\) (see [1, Lemma 10]). Therefore \(|\text{im}(\tau)| = \frac{|F|^2}{2}\).

Suppose that \(T_c\) is a commutative set. Let \(N_2\) be a group generated by the elements \(1 + \alpha_g \hat{H}\), in which \(g \in T_c\) and \(\alpha_g \in F\). Since

\[
(1 + \omega g + \omega g^2)(1 + \omega g + \omega g^2)^* = 1 + (\omega + \omega^2)g \hat{H}
\]

we have \(1 + \alpha_g \hat{H} \in S_H\) for every \(\alpha_g \in \text{im}(\tau)\). The group \(N_2\), being \(T_c\) commutative, is a subgroup in \(S_H\) and

\[
|N_2| = |\text{im}(\tau)|^{\frac{1}{2}|T_c|} = |F|^{\frac{1}{2}|T_c|} \cdot 2^{-\frac{1}{2}|T_c|}.
\]

Let \(x \in N_1\) and \(y \in N_2\). There exist \(x_1 \in I(H)^+\) and \(y_1 \in N_2^\ast\) such that \(x_1x_1^* = x\) and \(y_1y_1^* = y\). Since \(I(H)^+\) is an elementary 2-group, the element \(y \in N_1\) commutes with \(x_1\) and

\[
yx = yx_1x_1^* = x_1yx_1^* = x_1y_1y_1^* = x_1y_1)(x_1y_1)^* \in S_H.
\]

Therefore \(N_1 \times N_2\) is a subgroup in \(S_H\) and

\[
|N_1 \times N_2| = |F|^{\frac{1}{2}(|G|+|G(2)|+|T_c|)} \cdot 2^{-\frac{1}{2}|T_c|}.
\]

Consequently,

\[
|F|^{\frac{1}{2}(|G|+|G(2)|+|T_c|)} \cdot 2^{-\frac{1}{2}|T_c|} \leq |S_H|.
\]

Now, we are ready to prove the theorem. If \(n = 3\), then the theorem is true by Propositions [1] and [2]. Suppose that \(n > 3\). In the factor group \(\overline{G} = G/H\) the element \(\overline{g}\) has order two if and only if either \(g \in G\{2\}\) or \(g \in T_c\). Therefore, \(|\overline{G}\{2\}| = \frac{|G(2)|+|T_c|}{2}\). According to the inductive hypothesis

\[
|V_+(\overline{G})| = 2^s \cdot |F|^{\frac{1}{2}(|G|+|G(2)|+|T_c|)} - 1 = 2^s \cdot |F|^{\frac{1}{2}(|G|+|G(2)|+|T_c|)} - 1
\]

The number of subgroups of \(S_H\) is 

\[
|S_H| = |F|^{\frac{1}{2}(|G|+|G(2)|+|T_c|)} \cdot 2^{-\frac{1}{2}|T_c|}.
\]
for some $s \geq 0$. Using Lemma 1 and equation (2) we obtain that

$$|V_s(FG)| = |F|^{\frac{|G|}{2}} \cdot \frac{|V_s(FG)|}{|S_H|}$$

$$\geq |F|^{\frac{|G|}{2}} \cdot 2^{s \cdot \frac{1}{2} \cdot \frac{(|G|+|G(2)|)+|T_c|}{(|G|-|G(2)|)+|T_c|}}$$

$$= 2^s \cdot |F|^{\frac{1}{4}(|G|+|G(2)|)-1}.$$ 

Therefore

$$2^s \cdot |F|^{\frac{1}{4}(|G|+|G(2)|)-1} \leq |V_s(FG)|$$

(4)

and $|V_s(FG)|$ is divisible by $|F|^{\frac{1}{4}(|G|+|G(2)|)-1}$.

Similarly, using Lemma 1 and (3), we obtain that

$$|V_s(FG)| = |F|^{\frac{|G|}{2}} \cdot \frac{|V_s(FG)|}{|S_H|}$$

$$\leq |F|^{\frac{|G|}{2}} \cdot 2^{s \cdot \frac{1}{2} \cdot \frac{(|G|+|G(2)|)+|T_c|}{(|G|-|G(2)|)+|T_c|}}$$

$$= 2^{s+\frac{1}{2}|T_c|} \cdot |F|^{\frac{1}{4}(|G|+|G(2)|)-1}.$$ 

The size of the set $T_c$ does not depend on the field $F$. Since $2^s$ does not depend on the field $F$ by the inductive hypothesis, the proof is complete. □

We should remark that $2^s$ in the inequality (4) is usually not inherited via the factorization. For example, for the semidihedral group $D_{16}$ of order 16 we have that

$$|V_s(FD_{16})| = 2 \cdot |F|^{\frac{1}{4}(|D_{16}|+|D_{16}(2)|)-1}$$

by [1, Lemma 3.4]. However, $D_8 \cong D_{16}/H$, where $H$ is the center of $D_{16}$ and $D_8$ is the dihedral group of order 8 and $|V_s(FD_8)| = |F|^{\frac{1}{4}(|D_8|+|D_8(2)|)-1}$ by Proposition 2.

Proof of Corollary 7. If $p = 2$, then Theorem 2 implies that

$$|F|^{\frac{|G|}{2}-1} \leq |F|^{\frac{1}{4}(|G|+|G(2)|)-1} \leq |V_s(FG)| \leq |F|^{\frac{|G|}{2}-1}.$$ 

If $p$ is an odd prime, then $|V_s(FG)| = |F|^{\frac{1}{4}(|G|-1)}$ by Theorem 1. Hence, $|F|^{\frac{|G|}{2}-1} \leq |V_s(FG)| \leq |F|^{\frac{|G|}{2}-1}$, which confirms that the order of $V_s(FG)$ determines the order of $G$ for every finite $p$-groups. □
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