When Does an Ensemble of Matrices with Randomly Scaled Rows Lose Rank?

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Abstract—We consider the problem of determining rank loss conditions for a concatenation of full-rank matrices, such that each row of the composing matrices is scaled by a random coefficient. This problem has applications in wireless interference management and recommendation systems. We determine necessary and sufficient conditions for the design of each matrix, such that the random ensemble will almost surely lose rank by a certain amount. The result is proved by converting the problem to determining rank loss conditions for the union of some specific matroids, and then using tools from matroid and graph theories to derive the necessary and sufficient conditions. As an application, we discuss how this result can be applied to the problem of topological interference management, and characterize the linear symmetric degrees of freedom for a class of network topologies.

I. INTRODUCTION

We consider the problem in Figure 1 in which $K$ users design matrices $B_1, \ldots, B_K$, where $B_i$ is an $n \times m_i$ matrix, designed by the $i^{th}$ user, with full column rank $m_i \leq n$. A destination obtains the matrix $B_D = [\Lambda_1 B_1 \ A_2 B_2 \ \cdots \ A_K B_K]$, where $\Lambda_i$ is an $n \times n$ diagonal matrix with random diagonal entries, and the set of all random coefficients is drawn from a continuous joint distribution. It is easy to see that $R = \min \left( \sum_{i=1}^{K} m_i, n \right)$ is the maximum rank that the matrix $B_D$ can have. However, by a careful design, one can reduce the rank of this matrix. In particular, the question that we address in this work is: Under what conditions on the design of the matrices $B_1, \ldots, B_K$ will the matrix $B_D$ lose rank by $\tau$ almost surely, i.e., rank($B_D$) $\leq R - \tau$, while each of the matrices $B_1, \ldots, B_K$ has full column rank?

![Fig. 1: Multiple users, each one having a matrix $B_i$, and another user receiving a concatenation of these matrices with randomly-scaled rows.](image)

The aforementioned problem arises naturally when understanding the fundamentals of interference management in wireless networks. In particular, consider the problem of topological interference management (e.g., [1], [2]), where no information about the channel state is assumed to be available at the transmitters, except for the information about the network topology. In this setting, the $K$ users in Figure 1 represent transmitters that are interfering at a single receiver. The matrix $B_i$ represents the beamforming matrix at the $i^{th}$ transmitter and the $j^{th}$ diagonal element of the matrix $\Lambda_i$ represents the channel coefficient in the $j^{th}$ time slot. The column span of the matrix $B_D$ represents the space of the received interference. The aforementioned problem is that of determining the conditions on the beamforming matrices that result in an alignment of the interference caused by users 1 to $K$ in a subspace whose dimension is at most $R - \tau$.

Another motivating example arises in recommendation systems, where for a fixed set of items, each matrix $B_i$ represents the ratings given by the $i^{th}$ group of users. Each row represents one user’s ratings for the set of items, and each diagonal element of $\Lambda_i$ represents a random scaling factor that reflects the user’s own bias. For example, one user can give a rating of 10 to his most favorite movie and 5 for his least favorite while another user can have a highest rating of 8 and a lowest rating of 4. The problem in this setting would be to understand conditions on the ratings of each user group that result in a rank loss of the ensemble. Understanding conditions of rank loss in this application is useful for completion algorithms that recover missing ratings. In other words, the answer of our question in this setting specifies the structure of matrix entries which suffice to complete the entire rating matrix.

As a special case of this problem, one can consider each $B_i$ being only a column vector. This is equivalent to the case where instead of each row, each individual element is scaled by a random coefficient. In this case, the problem can be shown to have a combinatorial structure, depending on the position of zero/non-zero elements of $B_i$’s. In particular, there is a well-known classic result in combinatorics which mentions that in this case, the ensemble matrix $B_D$ will lose rank by $\tau$ (i.e., rank($B_D$) $\leq R - \tau$) if and only if there does not exist a matching of size greater than $R - \tau$ between rows and columns of $B_D$, where a row is connected to a column if and only if the element with the corresponding row and column indices is non-zero (see e.g., [3]–[5] and also the Zippel-Schwartz Lemma [6], [7]). However, once $B_i$’s are not column vectors, the problem becomes much more complicated since there is a structure in the random scaling; elements of the same row in each matrix are scaled by the same coefficient.

In this paper, we characterize necessary and sufficient conditions for rank-loss of the matrix $B_D$. In particular, we determine under what condition on $B_i$’s, the matrix $B_D$ loses rank by $\tau$ almost surely. We first connect the rank of $B_D$ to the rank of the union of certain matroids. We then use the matroid union theorem to derive a linear-algebraic condition on the
structure of $B_1,...,B_K$. We finally simplify this condition by constructing appropriate bipartite graphs whose vertices represent bases for the column span of each of the matrices and using Hall’s marriage theorem to reach a final condition expressed through matching sizes on the constructed graphs.

As an application of the derived rank-loss condition, we utilize the result in the context of the topological interference management problem by solving a set of previously open problems [2], where we characterize the linear symmetric degree of freedom (DoF) of a class of network topologies. In this setting, the model in Figure 1 can be seen as a set of $K$ transmitters causing interference at a destination, and it is desired to understand conditions on the design of the beamforming matrices that result in reducing the dimension of the subspace occupied by interference at the receiver.

II. Problem Formulation and Main Result

Consider $K$ matrices $B_1,...,B_K$, where each matrix $B_i, i \in [K]$ has size $m \times n$, $m \leq n$ (we use $[K]$ to denote the set $\{1,...,K\}$ for any positive integer $K$). Without loss of generality, we assume that each matrix is full-column rank, since the linearly-dependent columns can be removed from each matrix. Furthermore, consider $K$ diagonal matrices $\Lambda_1,...,\Lambda_K$, each of size $n \times n$, where their diagonal elements are drawn from a joint continuous distribution.

In this paper, the problem under consideration is the rank loss of the matrix $B_D = [\Lambda_1B_1 \Lambda_2B_2 \ldots \Lambda KB_K]$. To be precise, we aim to find an equivalent condition for when

$$\text{rank}([\Lambda_1B_1 \Lambda_2B_2 \ldots \Lambda KB_K]) \overset{a.s.}{\leq} R - \tau, \quad (1)$$

where $R = \min \left(\sum_{i=1}^{K} m_i, n\right)$ denotes the maximum possible rank of $[\Lambda_1B_1 \Lambda_2B_2 \ldots \Lambda KB_K]$ and $\tau \in \mathbb{Z}^+.$

**Notation:** For a matrix $B \in \mathbb{R}^{m \times n}$, we use calligraphic $B$ to denote the subspace in $\mathbb{R}^n$ spanned by the columns of $B$. Also, for any $X \subseteq [n]$ and $Y \subseteq [m]$, $B_{X,Y}$ denotes the submatrix of $B$ created by removing the rows with indices outside $X$ and removing the columns with indices outside $Y$. Furthermore, for any $X \subseteq [n], P_X^K$ denotes the set of all partitions of $X$ to $K$ disjoint subsets (each one possibly empty). Finally, for any $J \subseteq [n]$, $J^c$ denotes the complement of $J$ in $[n]$ (i.e., $[n] \setminus J$) and $S_J$ denotes the subspace of $\mathbb{R}^n$ spanned by the columns of the $n \times n$ identity matrix with indices in $J$. In other words, $S_J$ is the subspace of $\mathbb{R}^n$ which includes all the vectors that have zero entries in $J^c$. We call $S_J$ the sparse subspace of the set $J$.

We now state the main result in the following theorem.

**Theorem 1.** The following two statements are equivalent.

$$\text{rank}([\Lambda_1B_1 \Lambda_2B_2 \ldots \Lambda KB_K]) \overset{a.s.}{\leq} R - \tau, \quad (C1)$$

\forall i \in [K], \text{ s.t. } \sum_{i=1}^{K} |Y_i| = R, \quad \exists J \subseteq [n]: \sum_{i=1}^{K} \text{dim}(S_J \cap B_{I_i,Y_i}) \geq |J| + \tau. \quad (C2)

**Example 1.** As a simple example, consider the case where $K = 2$ and $m_1 = m_2 = \frac{n}{2}$. In this case, $B_1$ and $B_2$ will each be of size $n \times \frac{n}{2}$ and the only possible choice for $Y_1$ and $Y_2$ will be $\left[\frac{n}{2}\right]$. Theorem (C1) implies that $\text{rank}([\Lambda_1B_1 \Lambda_2B_2]) \overset{a.s.}{\leq} n - \tau$ if and only if there exists a set $J \subseteq [n]$ such that,

$$\text{dim}(S_J \cap B_1) + \text{dim}(S_J \cap B_2) \geq |J| + \tau. \quad (2)$$

**Example 2.** Continuing Example (C1) assume $n = 2m_1 = 2m_2 = 4$ and suppose we fix $B_1 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}^T$. In this case, Theorem (C1) implies that the only way for the matrix $[\Lambda_1B_1 \Lambda_2B_2]$ to lose rank by $\tau = 1$ almost-surely is that both columns of $B_2$ have zeros in their 4th entries. The set $J \subseteq [4]$ satisfying (2) would be $\{1,2,3\}$ in this case.

**Remark 1.** The significance of the above result lies in finding a condition on the structure of the matrices $B_i, i \in [K]$, such that the statistical rank loss condition is met. Checking the structural condition of (C2) does not involve statistical analysis and relies only on the combinatorial structure of $B_i$’s.

**Remark 2.** When each element of the matrices $B_1,...,B_K$ is scaled by a random coefficient, (C1) corresponds to the size of a bipartite graph matching representing rows and columns of $B_D$ as the two partite sets. This condition will be similar to (C2), but instead of $\text{dim}(S_J \cap B_i)$, the number of columns in $B_i$ inside the subspace $S_J$ is considered. This is intuitive as in the considered setting, each row is scaled by the same coefficient, and hence, the angles between column vectors are preserved. Our result can be seen as a generalization of the classical rank-loss results for random matrices.

**Remark 3.** We show in Section IV how the result of Theorem (C1) can be used to characterize the linear degrees of freedom for the topological interference management problem of a class of networks that had been considered as an open problem before.

III. Proof of the Main Result

The proof of Theorem (C1) is composed of four steps, each of which we present a condition equivalent to condition (C1).

A. Step 1: Expansion of Determinant

We begin by the following equivalence lemma.

**Lemma 1.** Condition (C1) is equivalent to the following.

\forall X \subseteq [n]: |X| > R - \tau, \quad \forall Y_i \subseteq [m_i], i \in [K] \text{ s.t. } \sum_{i=1}^{K} |Y_i| = |X|, \quad \forall (I_1, I_2, ..., I_K) \in P_X^K \text{ s.t. } |I_i| = |Y_i| = |X|, \quad \text{det}(B_{I_i,Y_i}) = 0. \quad (C3)

**Proof:** By the definition of matrix rank, the condition in (C1) is equivalent to the fact that any square submatrix of $B_D$ with size greater than $(R - \tau) \times (R - \tau)$ should have a zero determinant almost-surely. This means that for any subset of rows $X \subseteq [n]: |X| > R - \tau$ and any subsets of columns $Y_i \subseteq [m_i], i \in [K] \text{ s.t. } \sum_{i=1}^{K} |Y_i| = |X|,$

$$\text{det}([\Lambda_{1,X} B_{1,Y_1} \ldots \Lambda_{K,X} B_{K,Y_K}]) \overset{a.s.}{=} 0.$$
It can be shown that this determinant is composed of monomials in the channel gains whose coefficients are in the form of \( \prod_{i=1}^{K} \det(B_{i,1,Y_i}) \) for some \((I_1, I_2, ..., I_K) \in P_X^K \) s.t. \(|I_i| = |Y_i|\). By the Zippel-Schwartz Lemma \([6, 7]\), for the whole multivariate polynomial to be equal to zero for almost all values of the channel gains, each of these coefficients should be equal to zero, which gives \((C3)\).

**Example 3.** Consider the case where \(K = 2, n = 3, m_1 = 1, m_2 = 2\), and we have \(B_1 = \begin{bmatrix} a_{11} & a_{31} \end{bmatrix}^T, B_2 = \begin{bmatrix} a_{11}' & a_{31}' \end{bmatrix}^T \). Also, let \(\lambda_i, \lambda'_i, i \in \{1, 2, 3\}\), be the \(i\)th diagonal element of \(A_1, A_2\), respectively. Then we have,

\[
B_D = \begin{bmatrix} \lambda_1 a_{11} & \lambda_1' a_{11}' & \lambda_1 a_{12} \\
\lambda_2 a_{21} & \lambda_2' a_{21}' & \lambda_2 a_{22} \\
\lambda_3 a_{31} & \lambda_3' a_{31}' & \lambda_3 a_{32} \end{bmatrix}.
\]

Now, \(\text{rank}(B_D) \leq 2\) is equivalent to \(\det(B_D) = 0\). Note that,

\[
\det(B_D) = \lambda_1 \lambda_2' \lambda_3' \left[ a_{11} \det \left( \begin{bmatrix} a_{21}' & a_{22}' \\ a_{31}' & a_{32}' \end{bmatrix} \right) \right] - \lambda_2 \lambda_1' \lambda_3' \left[ a_{21} \det \left( \begin{bmatrix} a_{11}' & a_{12}' \\ a_{31}' & a_{32}' \end{bmatrix} \right) \right] + \lambda_3 \lambda_1' \lambda_2' \left[ a_{31} \det \left( \begin{bmatrix} a_{11}' & a_{12}' \\ a_{21}' & a_{22}' \end{bmatrix} \right) \right].
\]

Therefore, the Zippel-Schwartz Lemma implies that this determinant is almost-surely equal to zero if and only if each of the products inside the brackets is equal to zero.

**B. Step 2: Sparse Subspaces**

We now express the condition \((C3)\) in terms of sparse subspaces. Recall that a sparse subspace is defined by column vectors that have zero entries for a specific set of rows.

**Lemma 2.** Condition \((C3)\) is equivalent to the following.

\[
\forall X \subseteq [n] : |X| > R - \tau, \quad \forall Y_i \subseteq [m_i], i \in [K] \text{ s.t. } \sum_{i=1}^{K} |Y_i| = |X|, \\
\forall (I_1, I_2, ..., I_K) \in P_X^K \text{ s.t. } |I_i| = |Y_i|, \forall i \in [K] : \\
\sum_{i=1}^{K} \dim(S_{I_i} \cap B_{i,*Y_i}) > 0.
\]

**Proof:** First assume that the condition in \((C3)\) holds. Then for some \(i \in [K]\) there exists a linear combination of the columns in \(B_{i,1,Y_i}\) which is equal to the zero vector. Therefore, if we apply this linear combination to the entire matrix \(B_{i,*Y_i}\), we end up with a vector in the sparse subspace \(S_{I_i}\) since \(B_{i,*Y_i}\) is full-column rank. This implies \((C4)\).

If the condition in \((C4)\) holds, then for some \(i \in [K]\), there exists a linear combination of the columns in \(B_{i,*Y_i}\) which is zero in the coordinates in \(I_i\). This means that for any \(I_i : |I_i| = |Y_i|, \det(B_{i,1,Y_i}) = 0\), implying that \((C3)\) holds.

**C. Step 3: Connection to Rank of Union of Matroids**

In this step, we show how to represent condition \((C4)\) in terms of the rank loss of the union of certain matroids. To this end, for any choice of \(X \subseteq [n]\) and \(Y_i \subseteq [m_i], \forall i \in [K]\) which satisfy \(\sum_{i=1}^{K} |Y_i| = |X| > R - \tau\) and

\[
\dim(S_{X^c} \cap B_{i,*Y_i}) = 0, \forall i \in [K],
\]

define \(\mathcal{I}_{i,X,Y_i}\) as

\[
\mathcal{I}_{i,X,Y_i} = \{ I \subseteq X : \dim(S_{I \cup X^c} \cap B_{i,*Y_i}) = 0 \}, i \in [K].
\]

We have the following claim that is proved in Appendix A.

**Claim 1.** \(M_{i,X,Y_i} = (X, \mathcal{I}_{i,X,Y_i})\) is a matroid with rank function \(r_{i,X,Y_i}(J) = |J| - \dim(S_{J \cup X^c} \cap B_{i,*Y_i}), i \in [K].\)

For any matroid \(M_{i,X,Y_i}\), the dual matroid \(M_{i,X,Y_i}'\) is defined as \(M_{i,X,Y_i}' = (X, \mathcal{I}_{i,X,Y_i})\) whose basis sets are the complements of the basis sets of \(M_{i,X,Y_i}\) \([8]\). Using Theorem 39.2 in \([8]\) and Claim 1 the rank of this dual matroid is

\[
r_{i,X,Y_i}'(J) = |J| - r_{i,X,Y_i}(X) + r_{i,X,Y_i}(X \setminus J) = |J| - (|X| - |Y_i|) + (|X| - |J| - \dim(S_{J \cup X^c} \cap B_{i,*Y_i})) = |Y_i| - \dim(S_{J \cup X^c} \cap B_{i,*Y_i}).
\]

(5)

Now, consider the union of the dual matroids \(M_{i,X,Y_i}', i \in [K]\), denoted by \(\bigvee_{i=1}^{K} M_{i,X,Y_i}' = (X, \bigvee_{i=1}^{K} \mathcal{I}_{i,X,Y_i})\), where

\[
\bigvee_{i=1}^{K} \mathcal{I}_{i,X,Y_i} = \bigcup_{i=1}^{K} \mathcal{I}_{i,X,Y_i} = \{ I_i : I_i \in \mathcal{I}_{i,X,Y_i}, i \in [K]\}.
\]

Let \(r_{X,Y_1,...,Y_K}'(\cdot)\) denote the rank of \(\bigvee_{i=1}^{K} M_{i,X,Y_i}'\). Then we have the following lemma whose proof is in Appendix B.

**Lemma 3.** For any \(X \subseteq [n], Y_i \subseteq [m_i], i \in [K] \text{ s.t. } \sum_{i=1}^{K} |Y_i| = |X| > R - \tau\) and \((C3)\) is satisfied, the following are equivalent

\[
\forall (I_1, I_2, ..., I_K) \in P_X^K \text{ s.t. } |I_i| = |Y_i| : \\
\sum_{i=1}^{K} \dim(S_{I_i} \cap B_{i,*Y_i}) > 0.
\]

(6)

\[
r_{X,Y_1,...,Y_K}'(X) < |X|.
\]

(7)

Note that using this lemma, we can now replace the last two lines of condition \((C4)\) by simply \(r_{X,Y_1,...,Y_K}'(X) < |X|\).

**D. Step 4: Matroid Union Theorem**

In this step of the proof, we make use of the Matroid Union Theorem \([8]\) to characterize an equivalent condition to \((7)\). We start by stating the Matroid Union Theorem.

**Theorem 2.** (Matroid Union Theorem \([8, Chapter 42]\)) Let \(M_i = (E_i, \mathcal{I}_i), i \in [K],\) be \(K\) matroids with rank functions \(r_i(\cdot)\). Then \(\bigvee_{i=1}^{K} M_i\) is also a matroid with rank function

\[
r(U) = \min_{T \subseteq U} \left( |U \setminus T| + \sum_{i=1}^{K} r_i(T \cap E_i) \right).
\]

For a matroid \(M = (X, T), J \subseteq X\) is called a basis set if \(J \in T\) and there is no \(J' \in T\) such that \(J \subset J' \subseteq X\) \([8\ Ch. 39]\).
Equipped with the above theorem, we can now state the following equivalence lemma.

**Lemma 4.** Condition \((C4)\) is equivalent to the following.
\[
\forall X \subseteq [n] : |X| > R - \tau, \forall Y_i \subseteq [m_i], i \in [K] \text{ s.t. } \sum_{i=1}^{K} |Y_i| = |X|, \exists J \subseteq X : \sum_{i=1}^{K} \dim(S_{J \cup X} \cap B_{i,Y_i}) > |J|. \tag{C5}
\]

**Proof:** If for a specific choice of \(X\) and \(Y_i\)'s, \((\mathbf{3})\) is not satisfied (i.e., \(\dim(S_X \cap B_{i,Y_i}) > 0\) for some \(i \in [K]\)), then it is clear that both statements in \((\mathbf{4})\) and \((\mathbf{5})\) hold. Hence, w.l.o.g. we assume that \((\mathbf{5})\) is satisfied. For any \(X \subseteq [n], Y_i \subseteq [m_i], i \in [K] \text{ s.t. } \sum_{i=1}^{K} |Y_i| = |X| > R - \tau\) and \((\mathbf{3})\) is satisfied, we will get from Lemma 3 that the condition in \((\mathbf{4})\) is equivalent to,
\[
r^r_{x,y_1,\ldots,y_K}(X) < |X|
\]
\[
\Leftrightarrow \min_{J \subseteq X} \left( |X| - |J| + \sum_{i=1}^{K} (|Y_i| - \dim(S_{J \cap B_{i,Y_i}})) \right) < |X|
\]
\[
\Leftrightarrow |X| + \sum_{i=1}^{K} |Y_i| - \max_{J \subseteq X} \left( |J| + \sum_{i=1}^{K} \dim(S_{J \cap B_{i,Y_i}}) \right) < |X|
\]
\[
\Leftrightarrow \max_{J \subseteq X} \left( |J| + \sum_{i=1}^{K} \dim(S_{J \cap B_{i,Y_i}}) \right) > |X|
\]
\[
\Leftrightarrow \exists J \subseteq X : \sum_{i=1}^{K} \dim(S_{J \cap B_{i,Y_i}}) > |X| - |J|
\]
\[
\Leftrightarrow \sum_{i=1}^{K} \dim(S_{J \cup X} \cap B_{i,Y_i}) > |J|
\]
where (a) follows from \((\mathbf{3})\) and Theorem 2, (b) follows from the assumption that \(\sum_{i=1}^{K} |Y_i| = |X|\) and (c) follows by changing \(J\) to \(X \setminus J\). This completes the proof. \[\blacksquare\]

**E. Step 5: Hall’s Marriage Theorem**

In the final step of the proof, we prove the equivalence between \((\mathbf{C})\) and \((\mathbf{2})\) by constructing an appropriate bipartite graph. One partite set represents the column vectors of a carefully chosen set of bases for the subspaces \(B_{i,Y_i}\) and the other partite set has \(n\) elements, each corresponding to a row. A column vector vertex is connected to a row vertex if and only if the vector has a non-zero entry in the corresponding row. We then use a variation of Hall’s marriage theorem to complete the proof by representing the final condition \((\mathbf{2})\) in terms of a matching size on the constructed bipartite graph. The complete proof is available in Appendix D.

**IV. APPLICATION TO TOPOLOGICAL INTERFERENCE MANAGEMENT**

In this section, we study an application of Theorem 1 to characterize the linear symmetric degrees of freedom for a class of topological interference management problems as defined next.

**A. Topological Interference Management: System Model**

We consider \(K\)-user interference networks composed of \(K\) transmitter nodes \(\{T_i\}_{i=1}^{K}\) and \(K\) receiver nodes \(\{D_i\}_{i=1}^{K}\). Each transmitter \(T_i\) intends to deliver a message \(W_i \in \mathcal{W}_i\) to its corresponding receiver \(D_i\). We assume that each receiver is subject to interference only from a specific subset of the other transmitters and the interference power that it receives from the other transmitters is below the noise level. This leads to a network topology indicating the network interference pattern.

Each transmitter \(T_i\) intends to send a vector \(w_i \in \mathbb{R}^{m_i}\) of \(m_i\) symbols to its desired receiver \(D_i\) over \(n\) time slots. This message is encoded to the transmit vector \(x_i = B_i w_i\), where \(B_i\) denotes the linear beamforming \(\text{preoding} \) matrix of transmitter \(i\), which is of size \(n \times m_i\). The received signal of receiver \(j\) over \(n\) time slots is given by,
\[
y_{j} = (\Lambda_{jj}B_j)w_{j} + \sum_{i \in I_j} (\Lambda_{ij}B_i)w_{i} + z_{j},
\]
where \(I_j\) is the set of transmitters interfering at receiver \(j\), \(\Lambda_{jj}\) is the \(n \times n\) diagonal matrix with the \(j^{th}\) diagonal element being equal to the value of the channel coefficient between transmitter \(i\) and receiver \(j\) in time slot \(k\), and \(z_{j}\) is the noise vector at receiver \(j\) where each of its elements is an i.i.d. \(\mathcal{N}(0, N)\) random variable, \(N\) being the noise variance. The channel gain values are assumed to be identically distributed and drawn from a continuous distribution at each time slot. We assume that transmitters have no knowledge about the realization of the channel gains except for the topology of the network. However, the receivers have full channel state information. We refer to this assumption as no CSIT (channel state information at the transmitters) beyond topology. Each precoding matrix \(B_i\) is an \(n \times m_i\) matrix that can only depend on the knowledge of topology. At receiver \(j\), the interference subspace denoted by \(I_j\) can be written as
\[
I_j = \bigcup_{i \in I_j} \text{colspan}(\Lambda_{ij}B_i).
\]
In order to decode its desired symbols, receiver \(j\) projects its received signal subspace given by \(\text{colspan}(\Lambda_{jj}B_j)\) onto the subspace orthogonal to \(I_j\), and its successful decoding condition can be expressed as
\[
\dim\left(\text{proj}_{I_j} \text{colspan}(\Lambda_{jj}B_j)\right) = m_j. \tag{9}
\]
If the above decodability condition is satisfied at all the receivers \(\{D_i\}_{i=1}^{K}\) for almost all realizations of channel gains, then the linear degrees of freedom (DoF) tuple \((m_1/n, \ldots, m_K/n)\) is achievable under the aforementioned linear scheme. The linear symmetric DoF \(d_{sym}\) is defined as the supremum \(d\) for which the linear DoF tuple \((d, \ldots, d)\) is achievable.

**B. Characterizing the Linear Symmetric DoF**

Consider a coding scheme achieving a linear symmetric DoF \(d\) over \(n\) time slots. At each receiver \(j\), the decodability condition \((\mathbf{9})\) implies that \(\dim(I_j) \leq n(1-d)\). Using Theorem 1,
we obtain the following equivalent condition on the design of beamforming matrices corresponding to interfering signals,

\[ \forall Y_i \subseteq [nd], i \in I_j \text{ s.t. } \sum_{i \in I_j} |Y_i| = \min (nd |I_j|, n), \]

\[ \exists J \subseteq [n]: \sum_{i \in I_j} \dim(S_j \cap B_i, Y_i) \geq |J| + \tau, \quad (10) \]

where \( \tau = n(3d - 1) \). Using the condition in (10) to reach a converse for the achievable linear symmetric DoF of arbitrary network topologies is a difficult problem, as it is not clear what the required number of time slots \( n \) is to achieve \( d_{\text{sym}} \). Further, for each value \( n \), reducing the complexity of the search for the optimal design of the beamforming matrices by converting it to a problem of combinatorial optimization does not seem straightforward. However, under the restriction to a certain class of topologies, the task becomes easier as it reduces to a problem independent of the value of \( n \), and can be described directly in terms of the interference conflict pattern between network users. We introduce the properties of the considered network topologies through the interference sets \( I_j, j \in [K] \).

We consider topologies where the following properties hold,

- **(P1) Maximum Degree:** For all \( j \in [K], |I_j| \leq 2 \).
- **(P2) Non-overlapping Interference Sets:** For all \( j, k \in [K] \) such that \( \max(|I_j|, |I_k|) = 2 \), \( I_j \cap I_k = \emptyset \). The property (P1) simplifies the problem because for any network with at least one interfering link, \( d_{\text{sym}} \leq \frac{3}{2} \). (P1) then implies that \( nd |I_j| \leq n \). Therefore, (10) reduces to,

\[ \exists J \subseteq [n]: \sum_{i \in I_j} \dim(S_j \cap B_i) \geq |J| + \tau, \quad (11) \]

Moreover, (P2) simplifies the problem as in this case we can conclude from (11) that there is no loss in generality in assuming that \( S_j \subseteq B_i, \forall i \in I_j \) (see Appendix D). The problem then becomes that of finding sparse subspaces for each interference set of size 2 such that the subspaces corresponding to conflicting interference sets do not overlap. This is captured through the chromatic number of a reduced conflict graph that captures only conflicts between interference sets.

Let \( G' \) be the reduced conflict graph as introduced in [9]. An edge \( i \rightarrow j \) exists in \( G' \) if and only if transmitter \( i \) is interfering at receiver \( j \) and there exists a transmitter \( s \) and a receiver \( k \) such that \( k \neq s \) and both transmitters \( i \) and \( s \) are causing interference at receiver \( k \). We now state the result on the linear symmetric DoF for the considered class of topologies and defer the proof to Appendix D. We call any network topology with at least one interference link an interference network topology.

**Theorem 3.** For any interference network topology satisfying (P1) and (P2), the linear symmetric DoF is given by,

\[ d_{\text{sym}} = \min \left( \frac{1}{2}, \frac{\chi(G')}{3\chi(G')} \right), \quad (12) \]

where \( \chi(G') \) is the chromatic number of the graph \( G' \).

**Remark 4.** In [9], it was proved that \( d_{\text{sym}} = \frac{1}{2} \) if and only if \( G' \) is bipartite, i.e., \( \chi(G') = 2 \). The above result of Theorem 3 can be then considered as a generalization of the result in [9] for the considered class of topologies.

**Remark 5.** In [2], several examples of topologies have been discussed for which the converse for \( d_{\text{sym}} \) has remained open. Using Theorem 3, we can now characterize the linear symmetric DoF of all those topologies which satisfy (P1) and (P2).

![Figure 2](image-url)  

**V. Conclusion**

We characterized necessary and sufficient conditions for the rank loss of a concatenation of full rank matrices with randomly scaled rows. We then used the result to characterize the linear symmetric DoF for a class of topological interference management problems that was previously studied in [3]. In general, our result solves an underlying fundamental problem in topological interference management; we are considering for future work how it can be used as a cornerstone to characterize the linear degrees of freedom region for arbitrary network topologies. We also plan to study the impact of this result in other applications: of particular interest is the recommendation systems application introduced in this paper. One can use our result in this setting to show how rough information about the user preferences suffices to provide performance guarantees for matrix completion algorithms.

**References**

[1] S. A. Jafar, “Topological Interference Management through Index Coding,” IEEE Transactions on Information Theory, vol. 60, no. 1, pp. 529-568, January 2014.

[2] N. Naderializadeh and A. S. Avestimehr, “Interference networks with no CSIT: Impact of topology,” IEEE Transactions on Information Theory, vol. 61, no. 2, pp. 917-938, February 2015.

[3] W. H. Cunningham and J. F. Geelen, “The Optimal Path-Matching Problem,” Combinatorica, 17 (3) (1997), 315-337.

[4] L. Lovasz, “On determinants, matchings and random algorithms,” in Fundamentals of Computation Theory, Berlin, 1979, 565-574.

[5] W. T. Tutte, “The factorization of linear graphs,” J. London Math. Soc., 22(1947), 107-111.

[6] R. Zippel, “Probabilistic Algorithms for Sparse Polynomials,” in Proceedings of the International Symposium on Symbolic and Algebraic Computation, pp. 216-226, 1979.

[7] J. T. Schwartz, “Fast Probabilistic Algorithms for Verification of Polynomial Identities,” Journal of the ACM, 27(4):701-717, 1980.

[8] A. Schrijver, Combinatorial optimization: polyhedra and efficiency, vol. 24, Springer, 2003.
APPENDIX A
PROOF OF CLAIM [4]
Without loss of generality, we consider the case of $i = 1$. We first show that $M_{1,X,Y}$ is a matroid. To this end, we need to prove the following properties.

- If $I \subseteq J$ and $J \in I_1,X,Y_1$, then $I \in I_1,X,Y_1$: This is clear since $S_{\cup X} \subseteq S_{\cup X}$, and we therefore have
  
  $$\dim(S_{\cup X} \cap B_{1+,Y}) \leq \dim(S_{\cup X} \cap B_{1+,Y}) = 0$$

- If $I \in I_1,X,Y_1$, $J \in I_1,X,Y_1$, and $|I| < |J|$, then $\exists j \in J \setminus I$ such that $I \cup \{j\} \in I_1,X,Y_1$. Assume $I = \{i_1, \ldots, i_k\}$ and $J = \{i_1, \ldots, k' + j, \ldots, j', \ldots, j\}$ where $k' \leq k$ and $k' + l > k$. Using this notation, we have that $J \setminus I = \{j_1, \ldots, j_l\}$.
  
  Suppose that this property is not true. This implies that adding any of the $e_{i_j}$'s to $I$ will force it to lie outside the set $I_1,X,Y_1$, where $e_{i_j}$ denotes the $i$th standard basis vector. If $X^c = \{x_1, \ldots, x_m\}$, this is equivalent to the fact that there exist coefficients $\lambda_{ij}$, $i \in [l], j \in [k]$, $\beta_{ij}$, $i \in [l], j \in [m]$, such that

  $$\lambda_{11} e_{i_1} + \cdots + \lambda_{1k} e_{i_k} + \beta_{11} e_{x_{i_1}} + \cdots + \beta_{1m} e_{x_{m}} +$$

  $$\mu_1 e_{i_j} = v_1$$

  (13)

  $$\lambda_{21} e_{i_1} + \cdots + \lambda_{2k} e_{i_k} + \beta_{21} e_{x_{i_1}} + \cdots + \beta_{2m} e_{x_{m}} +$$

  $$\mu_2 e_{i_j} = v_2$$

  (14)

  $$\vdots$$

  $$\lambda_{l1} e_{i_1} + \cdots + \lambda_{lk} e_{i_k} + \beta_{l1} e_{x_{i_1}} + \cdots + \beta_{lm} e_{x_{m}} +$$

  $$\mu_{l1} e_{i_j} = v_l$$

  (15)

  where $v_1, \ldots, v_l$ belong to $B_{1+,Y_1}$. Now, consider the following vectors

  $$\lambda_i^T = [\lambda_i, k' + 1, \lambda_i, k' + 2, \ldots, \lambda_i, k], i \in [l].$$

  (16)

  There are $l$ of these vectors in $\mathbb{R}^{k' - k}$ and since we assumed that $k' + l > k$, these vectors should be linearly dependent. This implies that there exist non-all-zero coefficients $a_i$, $i \in \{1, \ldots, l\}$ such that $\sum_{i=1}^l a_i \lambda_i = 0$. Multiplying each $a_i$ by the corresponding equation in (13), 15 and then adding the resulting equations yields

  $$\sum_{n=1}^{k'} \left( \sum_{i=1}^l a_i \lambda_{in} \right) e_{i_n} + \sum_{n=1}^m \left( \sum_{i=1}^l a_i \beta_{in} \right) e_{x_{m}} +$$

  $$\sum_{i=1}^l (a_i \mu_i) e_{i_j} = \sum_{i=1}^l a_i v_i,$$

  (17)

  where the vectors in (17) are non-zero since the coefficients $a_i$ are not all zeros and all the coefficients $\mu_i$ are non-zero. The RHS of (17) is a vector in $B_{1+,Y_1}$, which implies that there exists a linear combination of the bases of $S_{\cup X}$ which lies in $B_{1+,Y_1}$, therefore contradicting the fact that $J \in I_1,X,Y_1$. Hence, this property is also true.

  Having proven the above properties, it is now verified that $M_{1,X,Y} = \{X, I_1,X,Y_1\}$ is a matroid.

  To complete the proof of Claim [4], we need to show that the rank function of $M_{1,X,Y}$ is equal to $r_{1,X,Y}(J) = |J| - \dim(S_{\cup X} \cap B_{1+,Y_1}), J \subseteq X$. We do so through the following two steps.

  - For any $I \subseteq J$ such that $\dim(S_{\cup X} \cap B_{1+,Y_1}) = 0$, we have that $\dim(S_{\cup X} \cap B_{1+,Y_1} + S_{\cup X} \cap B_{1+,Y_1}) = 0$. Therefore, since $\dim(S_{\cup X} \cap B_{1+,Y_1}) = 0$, we will get $\dim(S_{\cup X} \cap S_{\cup X} \cap B_{1+,Y_1}) \leq \dim(S_{\cup X} \cap B_{1+,Y_1})$. But $S_{\cup X} \cap S_{\cup X} = S_{\cup X}$. Hence, $\dim(S_{\cup X} \cap S_{\cup X} \cap B_{1+,Y_1}) \leq \dim(S_{\cup X} \cap B_{1+,Y_1})$, implying that $r_{1,X,Y}(J) = |J| - \dim(S_{\cup X} \cap B_{1+,Y_1})$.

  - Now, only need to show that there exists some $K \subseteq J$ for which $|K| = |J| - \dim(S_{\cup X} \cap B_{1+,Y_1})$. Since $\dim(S_{\cup X} \cap B_{1+,Y_1}) = 0$, the vectors in $\{f_1, \ldots, f_d\} \cup \{e_i : i \in [X^c]\}$ form a basis for the space $S_{\cup X}$. Then, by the Steinitz exchange lemma, there exists a subset $K \subseteq J$ such that the vectors in $\{f_1, \ldots, f_d\} \cup \{e_i : i \in K \cup X^c\}$ form a basis for the space $S_{\cup X}$. This implies that $S_K$ has no intersection with $S_{\cup X} \cap B_{1+,Y_1} + S_X$ and therefore with $B_{1+,Y_1}$ (except for the zero vector) and $|K| = |J \cup X^c| - \dim(S_{\cup X} \cap B_{1+,Y_1} + S_X) = |J| - \dim(S_{\cup X} \cap B_{1+,Y_1})$, suggesting that $K$ is the actual desired subset of $J$.

  The above two steps imply that $r_{1,X,Y}(J) = |J| - \dim(S_{\cup X} \cap B_{1+,Y_1}), J \subseteq X$. The same arguments hold for any $M_{1,X,Y} = \{X, I_1,X,Y_1\}, i \in [K]$ as well. This completes the proof.

APPENDIX B
PROOF OF LEMMA [5]
Condition (6) is equivalent to the following for any valid choice of $X$ and $Y_i$'s:

$$\hat{(I_1, I_2, \ldots, I_K)} \in P_X^K :$$

$$|I_i| = |Y_i| \quad \text{and} \quad |X \setminus I_i| = |I_i| \quad \text{for all} \quad i \in [K].$$

(18)

Now, let us focus on a specific $i \in [K]$ and the corresponding set $I_i$ whose size satisfies $|I_i| = |Y_i|$, $|X \setminus I_i| = |X| - |Y_i| = |X| - \dim(S_{\cup X} \cap B_{1+,Y_i}) = r_{1,X,Y_i}(X)$.

(19)

For two subspaces $U$ and $V$, $U + V$ stands for $\text{span}(U \cup V)$. 
where (19) follows from Claim 1. By the definition of the rank of a matroid (see, e.g., [8] Ch. 39), (19) implies that all the members of \( I_i, X, Y \), i.e., the independent sets of \( M_i, X, Y \), are of size at most \( |X \setminus I_i| \). This means that if \( X \setminus I_i \in I_i, X, Y \), then it is a basis for \( M_i, X, Y \). This, in turn, is equivalent to \( I_i \) being a basis for the dual matroid \( M^*_i, X, Y \).

On the other hand, from (5) we have that the rank of the dual matroid \( M^*_i, X, Y \) is upper bounded by \(|Y_i|\), implying that all the members of \( I_i, X, Y \), have size at most \(|Y_i|\). Thus, under the constraint \(|I_i| = |Y_i|\), \( I_i \) being a basis for the dual matroid \( M^*_i, X, Y \), is equivalent to \( I_i \in I_i, X, Y \). Consequently, (18) (hence (4)) is equivalent to the following for any valid choice of \( X \) and \( Y_i \):

\[
\mathscr{H}(I_1, I_2, \ldots, I_K) \in P_X^K : \quad |I_i| = |Y_i| \text{ and } I_i \in I_i, X, Y_i, \forall i \in [K]. \tag{20}
\]

Finally, it is easy to verify that (20) is equivalent to \( r_{X, Y_i, \ldots, Y_k}(X) < |X| \), since there cannot exist any \((I_1, I_2, \ldots, I_K) \in P_X^K\) such that \( I_i \in I_i, X, Y \), at the same time for all \( i \in [K] \). This completes the proof.

**Appendix C**

**Proof of equivalence of (C5) and (C2)**

We use the following variant of Hall’s marriage theorem.

**Theorem 4.** Let \( G = (A \cup B, E) \) be a bipartite graph. \( G \) has a matching of size \( k \) if and only if the size of the neighboring set \( |N(I)| \geq |I| - |B| + k \) for any \( I \subseteq B \).

We first show that (C2) \( \Rightarrow \) (C5). Assume that (C2) holds and fix \( Y_i \subseteq [m_i], i \in [K] : \sum_{i=1}^K |Y_i| = R \). Let \( J^* \subseteq [n] \) be such that \( \sum_{i=1}^K \dim(S_j \cap B_i, X, Y_i) \geq |J^*| + \tau \). For each \( i \in [K] \), let \( c_1^{(i)}, \ldots, c_{|Y_i|}^{(i)} \) be \( |Y_i| \) column vectors that form a basis for \( B_i, X, Y_i \) and a subset of these vectors form a basis for \( S_j \cap B_i, X, Y_i \). Now, let \( C_i = \{c_1^{(i)}, \ldots, c_{|Y_i|}^{(i)}\} \) and construct a bipartite graph \( G = (A \cup C, E) \) where \( A = \{1, 2, \ldots, n\} \) and \( C = \{C_1, \ldots, C_K\} \) is a multisets consisting of the elements of the sets \( C_i, i \in [K] \). A and \( C \) are the two partite vertex sets of the graph \( G \). For any \( i \in [n] \) and \( c \in C, (i, c) \in E \) if and only if the vector \( c \) has a non-zero entry in the \( i \)-th position, i.e., if \( c = (c_1, \ldots, c_n) \) then \( a_i \in E \Leftrightarrow c_i \neq 0 \). Now, note that all the vertices in \( C \) that correspond to vectors in \( S_j \), can only be connected to vertices in \( A \) that correspond to the set \( J^* \). Since \( \sum_{i=1}^K \dim(S_j \cap B_i, X, Y_i) \geq |J^*| + \tau \), it follows that there is a subset of the partite set \( C \) of size at least \( |J^*| + \tau \) whose neighboring set has size at most \( |J^*| + \tau \). It follows from Theorem 4 that there is no matching of size \( R - \tau + 1 \) in the bipartite graph \( G \). It follows that for any \( X \subseteq [n] : |X| > R - \tau \) and any \( K \) sets \( Y_i' \subseteq C, i \in [K] : \sum_{i=1}^K |Y_i'| = |X| \), there is no matching between the vertices corresponding to the set \( X \) in the partite set \( A \) and the vertices in \( \{Y_1', \ldots, Y_K'\} \), and hence, there is a set \( J \subseteq X \) such that the vertices corresponding to the set \( X \setminus J \) are connected to less than \(|X \setminus J| \) vertices in \( \cup_{i=1}^K Y_i' \). It follows that for this choice of the set \( J \), there are more than \(|J| \) vertices in \( \cup_{i=1}^K Y_i' \) that are not connected to any vertex in the set \( X \setminus J \) in the partite set \( A \). Now, if \( \forall i \in [K], Y'_i \cap Y_i \neq \emptyset \), then \( \sum_{i=1}^K \dim(S_j \cup B_i, X, Y'_i) > |J| \). We now use the above argument to prove that (C5) holds as follows. For all \( Y'_i \subseteq [m_i], i \in [K] \) such that \( \sum_{i=1}^K |Y'_i| > R - \tau \), we find \( Y_i, i \in [K] \) such that \( Y'_i \subseteq Y_i, \forall i \in [K] \) and \( \sum_{i=1}^K |Y_i| = R \), and then use the above argument to show that,

\[
\exists J \subseteq [n] : \sum_{i=1}^K \dim(S_j \cap B_i, X, Y'_i) > |J| + \tau \Rightarrow \forall X \subseteq [n] : |X| = \sum_{i=1}^K |Y'_i|, \\exists J \subseteq X : \sum_{i=1}^K \dim(S_j \cup B_i, X, Y'_i) > |J|,
\]

and hence, (C5) follows.

We now show that (C5) \( \Rightarrow \) (C2) by contradiction. Suppose that (C2) does not hold and fix \( Y_i \subseteq [m_i], i \in [K] : \sum_{i=1}^K |Y_i| = R \) such that for any \( J \subseteq [n], \sum_{i=1}^K \dim(S_j \cap B_i, X, Y_i) < |J| + \tau \). For each \( i \in [K] \), let \( c_1^{(i)}, \ldots, c_{|Y_i|}^{(i)} \) be \( |Y_i| \) column vectors that form a basis for \( B_i, X, Y_i \) and define \( C_i = \{c_1^{(i)}, \ldots, c_{|Y_i|}^{(i)}\} \). Now, we construct a bipartite graph \( G = (A \cup C, E) \) where \( A = \{1, 2, \ldots, n\} \) and \( C = \{C_1, \ldots, C_K\} \) is a multisets consisting of the elements of the sets \( C_i, i \in [K] \). Also, for any \( i \in [n], c \in C, (i, c) \in E \Leftrightarrow c_i \neq 0 \).

From the selection of the sets \( Y_i, i \in [K] \), we have that for any \( I \subseteq C \), the neighboring set \( N(I) \subseteq A \) has size \( |N(I)| > |I| - \tau \). It follows from Theorem 4 that there is a matching in \( G \) of size \(|C| - \tau + 1 \). Such a matching will include \( R - \tau + 1 \) edges incident on \( R - \tau + 1 \) nodes in \( A \) (which we denote by \( X \)) and \( R - \tau + 1 \) nodes in \( C \) (which we denote by \( Y' \)). For each \( i \in [K] \), let \( Y'_i = \{j : c_j^{(i)} \in Y'\} \). Now, in order to show that (C5) is not true, it suffices to show that for the specific choice of \( X, Y'_i, i \in [K] \) mentioned above, the following holds:

\[
\forall J \subseteq X : \sum_{i=1}^K \dim(S_j \cup B_i, X, Y'_i) \leq |J| \tag{21}
\]

Fix any \( J \subseteq X \). We know that there exist \( |X \setminus J| \) nodes in \( Y' \) that are connected to the vertices corresponding to elements in \( X \setminus J \) through the edges in the matching. This implies that there exist \( |X \setminus J| \) columns in \( \{B_{1, i, X'}, \ldots, B_{K, i, X'}\} \) which have at least one non-zero entry in \( X \setminus J \) and therefore cannot belong to \( S_{J, X'} \). This means that \( \sum_{i=1}^K \dim(S_j \cup B_i, X, Y'_i) \) cannot be greater than the dimension of the span of the remaining columns, which is at most \( \sum_{i=1}^K |Y'_i| - |X \setminus J| = |J| \), verifying (21). Hence (C5) does not hold.

**Appendix D**

**Proof of Theorem 3**

We know from [1] that for any interference network topology, \( d_{sym} \leq \frac{1}{2} \). For the case where the reduced conflict graph topology does not exist, i.e., \( \chi(G') = 1, d_{sym} = \frac{1}{2} \) is achievable by having each of the column vectors in each matrix \( B_i, i \in [K] \) to have no zero entries. Hence, we consider the case where
\( \chi(G') \geq 2 \) and show that \( d_{\text{sym}} = \frac{1}{3}(\chi(G')+1) \) in this case. We first show that for any receiver with two or more interfering signals, the sparse subspace \( S_J \) of \((\mathbb{C}^2) \) is fully occupied by the interference, almost surely. More precisely, we prove the following corollary of the equivalent condition of Theorem 1.

**Lemma 5.** If \((\mathbb{C}^2) \) holds, then for a minimal set \( J \) satisfying \((\mathbb{C}^2) \), \( S_J \subseteq B_D \). More precisely,

\[
\exists Y_i \subseteq [m_i], i \in [K] \text{ s.t. } \sum_{i=1}^{K} |Y_i| = \min \left( \sum_{i=1}^{K} m_i \right), \quad J \subseteq [n] : \sum_{i=1}^{K} \dim(\mathcal{S}_J \cap B_{i,x,Y_i}) \geq |J| + x, x \geq \tau, \quad \wedge \exists L : J \subseteq [n] : \sum_{i=1}^{K} \dim(\mathcal{S}_L \cap B_{i,x,Y_i}) \geq |L| + x, \quad \Rightarrow S_J \subseteq \text{colspan}(\{A_1 B_1 \ldots A_K B_K\}). \quad (22)
\]

**Proof:** Let \( \mathcal{J}^* \) be a set satisfying the condition in \((\mathbb{C}^2) \). For each \( i \in [K] \), let \( c_i^{(1)}, \ldots, c_i^{(n_i)} \) be \( n_i \) vectors that form a basis for \( S_{J_i} \cap B_{i,x,Y_i} \), where \( \sum_{i=1}^{K} n_i = |\mathcal{J}^*| + x \) and let \( \mathcal{C} = \{c_1^{(1)}, \ldots, c_K^{(n_K)} \} \). Let \( \mathcal{G} = \{C_1, \ldots, C_K\} \) be the multiset consisting of the elements of \( \mathcal{C} \), \( i \in [K] \) and let \( G = (\mathcal{J}^* \cup C, E) \) be the bipartite graph whose left partite set consists of vertices corresponding to elements in \( \mathcal{J}^* \) and right partite set consists of vertices corresponding to the elements in \( \mathcal{C} \), and \( \forall i \neq \mathcal{J}^* \), \( c_i \neq 0 \). Since \( \exists L : \sum_{i=1}^{K} \dim(\mathcal{S}_L \cap B_{i,x,Y_i}) \geq |L| + x + 1 \), we know that for any subset of vertices \( I \subseteq C \), the neighboring set \( N(I) \) satisfies the condition \( |N(I)| \geq |I| - x \). It follows from Theorem 4 in Appendix C that there is a matching in \( G \) of size \( |C| - x \). Also, since \( |\mathcal{J}^*| = |C| - x \), we know that there is a matching in \( G \) covering all elements of the left partite set. For each \( i \in [K] \), let \( c_i^{(1)}, \ldots, c_i^{(n_i)} \), be the elements of \( \{c_i^{(1)}, \ldots, c_i^{(n_i)} \} \) in the matching in the right partite set \( C \), where \( \sum_{i=1}^{K} n_i = |\mathcal{J}^*| \). For each \( i \in [K] \), let \( C_i = \{c_i^{(1)}, \ldots, c_i^{(n_i)} \} \). Let \( \mathcal{C}^* = \{C_1, \ldots, C_K\} \) and consider the bipartite graph \( G^* = (\mathcal{J}^* \cup \mathcal{C}^*, E^*) \), where \( \forall i \in \mathcal{J}^*, c_i \in \mathcal{C}^*, (i,b) \in E^* \Leftrightarrow c_i \neq 0 \). Since \( G^* \) has a perfect matching, it follows that \( J \subseteq \mathcal{J}^* : \sum_{i=1}^{K} \dim(\mathcal{S}_J \cap B_{i,x,Y_i}) \geq |J| \), and hence, from \((\mathbb{C}^2) \) we know that \( \{A_1 B_1, \ldots, A_K B_K\} \) is full rank almost surely. Now, since colspan \( \{A_1 B_1, \ldots, A_K B_K\} \) has \(|\mathcal{J}^*| \) linearly independent column vectors almost surely, it follows that \( S_J \subseteq \text{colspan}(\{A_1 B_1, \ldots, A_K B_K\}) \), and hence, \( S_J \subseteq B_D \).

From the condition in \((\mathbb{C}^2) \), Lemma 5 and the decodability condition \((\mathbb{C}^3) \), we obtain the following condition,

\[
\forall r \in [K] : I_r = \{r_1, r_2\}, \exists J_r \subseteq [n] : \dim(S_{J_r} \cap B_{r_1}) + \dim(S_{J_r} \cap B_{r_2}) \geq |J_r| + \tau, \\
\text{dim}(S_{J_r} \cap B_r) = 0. \quad (23)
\]

We now use the following lemma to restrict our attention to a simpler condition for the considered class of topologies.

**Lemma 6.** For the case where interference sets do not overlap, if there exist \( B_1, \ldots, B_K \) such that \((\mathbb{C}^2) \) holds, then there exist \( B_1, \ldots, B_K \) such that condition holds,

\[
\forall r \in [K] : I_r = \{r_1, r_2\}, \exists J_r \subseteq [n] : \\
|J_r| = \tau, \dim(S_{J_r} \cap B_{r_1}) + \dim(S_{J_r} \cap B_{r_2}) = 0. \quad (24)
\]

**Proof:** Fix a design for the transmit beamforming matrices \( B_1, \ldots, B_K \) such that \((\mathbb{C}^2) \) holds. For each \( r \in [K] : |I_r| = 2 \), fix a subset \( J_r \subseteq [n] \) such that the condition in \((\mathbb{C}^3) \) is satisfied for the chosen subsets; it is easy to verify that \( |J_r| \geq \tau \) for any selected subset. We then set \( J_r = \phi \) for every \( r \in [K] \) such that \( |I_r| < 2 \).

For each \( r \in [K] \) such that \( I_r = \{r_1, r_2\} \), we choose the new beamforming matrices \( B_{i,r}^{(\text{new})} \) and \( B_{r}^{(\text{new})} \) as follows, let \( J_r = J_r \setminus (J_{r_1} \cup J_{r_2}) \) and \( S_{J_r}^{(\text{new})} = S_{J_r} \setminus (S_{J_{r_1}} \cup S_{J_{r_2}}) \); since \((\mathbb{C}^3) \) is satisfied, we know that the following holds,

\[
\dim(S_{J_r}^{(\text{new})} \cap B_{r_1}) + \dim(S_{J_r}^{(\text{new})} \cap B_{r_2}) \geq |J_r| + \tau, \quad (25)
\]

and hence, we also know that \(|J_r| \geq \tau \). For \( i \in \{1, 2\} \), let \( B_i \) be an \( n \times n \) matrix with \( n \) columns forming a basis for \( B_{r_i} \) and a subset of these columns form a basis for \( S_{J_r}^{(\text{new})} \) and \( B_{r}^{(\text{new})} \); we then fix \( J_r^{(\text{new})} \subseteq J_r : |J_r^{(\text{new})}| = \tau \) and replace the columns that form a basis for \( S_{J_r}^{(\text{new})} \) with an equal number of linearly independent columns in \( S_{J_r}^{(\text{new})} \) and \( \tau \) of the new columns form a basis for \( S_{J_r}^{(\text{new})} \) to construct the matrix \( B_{r_i}^{(\text{new})} \) from \( B_{r_i} \).

After performing the above step, it is straightforward to verify that the following holds.

\[
\forall r \in [K] : I_r = \{r_1, r_2\}, \\
S_{J_r}^{(\text{new})} \subseteq B_{r_1}^{(\text{new})} \cap B_{r_2}^{(\text{new})}, \quad \dim(S_{J_r}^{(\text{new})} \cap B_{r}^{(\text{new})}) = 0. \quad (26)
\]

and hence, the new beamforming matrices satisfy the condition in \((\mathbb{C}^4) \).

We now complete the proof by arguing that there exists a design of the beamforming matrices that satisfies \((\mathbb{C}^4) \) if and only if \( \tau \leq \frac{1}{1/\sqrt{\tau}} \). This follows directly by observing that the decodability condition in \((\mathbb{C}^4) \) \((\dim(S_{J_r} \cap B_r) = 0) \) is satisfied if and only if there is no overlap between sparse subspaces corresponding to conflicting interference sets; more precisely,

\[
r, d \in [K] : r \neq d, |I_r| = |I_d| = 2 \Rightarrow J_r \cap J_d = \phi. \quad (27)
\]