Late time cosmic acceleration from vacuum Brans-Dicke theory in 5D

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Abstract

We show that the scalar-vacuum Brans-Dicke equations in 5D are equivalent to Brans-Dicke theory in 4D with a self-interacting potential and an effective matter field. The cosmological implication, in the context of FRW models, is that the observed accelerated expansion of the universe comes naturally from the condition that the scalar field is not a ghost, i.e., $\omega > -3/2$. We find an effective matter-dominated 4D universe which shows accelerated expansion if $-3/2 < \omega < -1$. We study the question of whether accelerated expansion can be made compatible with large values of $\omega$, within the framework of a 5D scalar-vacuum Brans-Dicke theory with variable, instead of constant, parameter $\omega$. In this framework, and based on a general class of solutions of the field equations, we demonstrate that accelerated expansion is incompatible with large values of $\omega$.

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1 Introduction

Currently, there is a general agreement among cosmologists that the expansion of the universe is speeding up, instead of slowing down. Evidence in favor of this is provided by observations of high-redshift supernovae Ia \[1\]-\[6\], the cosmic microwave background and galaxy power spectra \[7\]-\[13\].

Since the gravity of both matter and radiation is attractive, an accelerated expansion requires either modified Einstein equations or, in the context of general relativity, the presence of a mysterious form of matter (called dark energy), which accounts for 70% of the total content of the universe and remains unclustered on all scales where gravitational clustering of ordinary matter is seen.

Within the context of general relativity the simplest candidate for dark energy is the cosmological constant \[\Lambda\] \[14\]-\[15\]. However, to avoid the ‘cosmic coincidence problem’ \[16\] many researchers consider a time dependent cosmological term, or an evolving scalar field known as quintessence \[16\]-\[19\]. The main drawback in these models is that the scalar fields are introduced by ‘hand’, without explaining the origin of the fields.

In recent years there has been a renewed interest in scalar-tensor theories of gravity as viable alternatives to general relativity. In particular, some researchers resort to Brans-Dicke theory (BD) to explain the accelerated expansion of the universe from fundamental physics. The concept is that the scalar field in BD could lead to cosmological acceleration. However, it turns out that this is so only in very particular cases: for values of the coupling parameter \(\omega\) of the universe from fundamental physics. The concept is that the scalar field in BD could lead to cosmological relativity. In particular, some researchers resort to Brans-Dicke theory (BD) to explain the accelerated expansion is that the scalar fields are introduced by ‘hand’, without explaining the origin of the fields.

In our view the main deficiency, or controversial feature, in this approach is the equivalent to ordinary general relativity with two scalar fields, which can be used to explain the present accelerated expansion of the universe. In our view the main deficiency, or controversial feature, in this approach is the \(\alpha\) introduction of matter in 5D.

The present work is motivated by \[24\]. Here, our object is to study the question of whether we can obtain the observed accelerated expansion in 4D from Brans-Dicke theory in 5D without assuming the existence of higher-dimensional matter. We show that the answer to this question is positive. We find that the scalar-vacuum BD equations in 5D are equivalent to BD in 4D with a self interacting potential and an effective matter field. In our approach, the presence of a non-vanishing potential is crucial to recover a general BD theory in 4D; the attractive feature is that its shape is determined, up to a constant, by the reduction procedure. It turns out that the accelerated expansion of an effective dust universe comes naturally from the condition \(\omega > -3/2\), in the region \(-3/2 < \omega < -1\). This range of \(\omega\) is consistent with the one obtained in \[24\].

However, small values of \(|\omega|\) are in sharp contradiction with the solar system bound \(\omega > 600\). Therefore, as a possible way out of this problem, we consider a 5D scalar-vacuum BD theory with variable \(\omega\) and study the question of whether accelerated expansion is compatible with large values of \(\omega\). In 4D this approach has been suggested by Banerjee and Pavon \[20\]. Our analysis indicates that the answer to our question is negative. Namely, late time accelerated expansion is incompatible with large values of \(\omega\).

2 Dimensional reduction of Brans-Dicke theory in 5D

The Brans-Dicke theory of gravity in 5D is described by the action \[24\]

\[
S_{(5)} = \int d^5x \sqrt{\gamma^{(5)}} \left[ \phi R^{(5)} - \frac{\omega}{\phi} \gamma^{AB} (\nabla_A \phi) (\nabla_B \phi) \right] + 16\pi \int d^5x \sqrt{\gamma^{(5)}} L_m^{(5)}, \tag{1}
\]

where \(R^{(5)}\) is the curvature scalar associated with the 5D metric \(\gamma^{AB}\); \(\gamma^{(5)}\) is the determinant of \(\gamma^{AB}\); \(\phi\) is a scalar field; \(\omega\) is a dimensionless coupling constant; \(L_m^{(5)}\) represents the Lagrangian of the matter fields in 5D and does not depend on \(\phi\). We consider only scalar-vacuum configurations in 5D, i.e., put \(L_m^{(5)} = 0\).

The equations for the gravitational field in 5D derived from \[1\] read

\[
G^{(5)}_{AB} = R^{(5)}_{AB} - \frac{1}{2} \gamma_{AB} R^{(5)} = \frac{\omega}{\phi^2} \left[ (\nabla_A \phi) (\nabla_B \phi) - \frac{1}{2} \gamma_{AB} (\nabla^C \phi) (\nabla_C \phi) \right] + \frac{1}{\phi} \left( \nabla_A \nabla_B \phi - \gamma_{AB} \nabla^2 \phi \right). \tag{2}
\]
where $\nabla^2 \equiv \nabla^C \nabla_C$. The field equation for the scalar field $\phi$ is determined by (1) as
\[
\frac{2\omega}{\phi^2} \nabla^2 \phi - \frac{\omega}{\phi^2} (\nabla_A \phi) (\nabla^A \phi) + R^{(5)} = 0. \tag{3}
\]
Taking the trace of (2) we find
\[
R^{(5)} = \frac{\omega}{\phi^2} (\nabla_A \phi) (\nabla^A \phi) + \frac{8}{3\phi} \nabla^2 \phi. \tag{4}
\]
Consequently,
\[
\nabla^2 \phi = 0. \tag{5}
\]
In this work we use coordinates where the metric in 5D can be written as
\[
dS^2 = \gamma_{AB} dx^A dx^B = g_{\mu\nu}(x, y) dx^\mu dx^\nu + \epsilon \Phi^2(x, y) dy^2, \tag{6}
\]
in such a way that our 4D spacetime can be recovered by going onto a hypersurface $\Sigma_y : y = y_0 =$ constant, which is orthogonal to the 5D unit vector
\[
\hat{n}^A = \delta^A_4, \quad n_A n^A = \epsilon, \tag{7}
\]
along the extra dimension, and $g_{\mu\nu}$ can be interpreted as the metric of the spacetime. To maintain contact with (2) we assume that $\hat{n}^A$ is a Killing vector, which in practice means that the metric coefficients in (6) only depend on $t$.

The effective field equations (FE) in 4D are obtained from dimensional reduction of (2) and (5). To achieve such a reduction we note that
\[
\nabla_{\mu} \nabla_{\nu} \phi = D_{\mu} D_{\nu} \phi,
\]
\[
\nabla_4 \nabla_4 \phi = \epsilon \Phi (D_\alpha \Phi) (D^\alpha \phi),
\]
\[
\nabla^2 \phi = D^2 \phi + \frac{(D_\alpha \Phi) (D^\alpha \phi)}{\Phi}. \tag{8}
\]
where $D_\alpha$ is the covariant derivative on $\Sigma_y$, which is calculated with $g_{\mu\nu}$, and $D^2 \equiv D^\alpha D_\alpha$.

Using these expressions, the spacetime components ($A = \mu, B = \nu$) of the 5D field equations (2) can be written as
\[
G_{\mu\nu}^{(5)} = -g_{\mu\nu} (D_\alpha \Phi) (D^{\alpha} \phi) + \frac{\omega}{\phi^2} \left[ (D_{\mu} \phi) (D_{\nu} \phi) - \frac{1}{2} g_{\mu\nu} (D_\alpha \phi) (D^{\alpha} \phi) \right] + \frac{1}{\phi} \left( D_{\mu} D_{\nu} \phi - g_{\mu\nu} D^2 \phi \right). \tag{9}
\]

To construct the Einstein tensor in 4D we have to express $R^{(5)}_{\alpha\beta}$ and $R^{(5)}_{44}$ in terms of the corresponding 4D quantities. The Ricci tensor $R^{(4)}_{\mu\nu}$ of the metric $g_{\mu\nu}$ and the scalar field $\Phi$ are related to the Ricci tensor $R^{(5)}_{AB}$ of $\gamma_{AB}$ by
\[
R^{(5)}_{\alpha\beta} = R^{(4)}_{\alpha\beta} - \frac{D_\alpha D_\beta \Phi}{\Phi},
\]
\[
R^{(5)}_{44} = -\epsilon \Phi D^2 \phi. \tag{10}
\]
From (2)-(4) and the second equation in (10) we obtain the 4-dimensional equation for $\Phi$, viz.,
\[
\frac{D^2 \Phi}{\Phi} = -\frac{(D_\alpha \Phi) (D^{\alpha} \phi)}{\Phi \phi}. \tag{11}
\]

\footnote{Notation: $x^\mu = (x^0, x^1, x^2, x^3)$ are the coordinates in 4D and $y$ is the coordinate along the extra dimension. We use spacetime signature $(+, - , - , -)$, while $\epsilon = \pm 1$ allows for spacelike or timelike extra dimension.}
Substituting this into $R^{(5)} = \gamma^{AB} R_{AB}$ we find

$$R^{(5)} = R^{(4)} + \frac{2 (D_\alpha \Phi) (D^\alpha \phi)}{\Phi},$$

(12)

where $R^{(4)} = g^{\alpha\beta} R_{\alpha\beta}^{(4)}$ is the scalar curvature of the spacetime hypersurfaces $\Sigma_t$.

We are now ready to obtain the effective equations for gravity in 4\$D\$. With this aim we substitute the first equation in (10) and (12) into (9) and isolate $G^{(4)}_{\mu\nu} = R^{(4)}_{\mu\nu} - g_{\mu\nu} R^{(4)}/2$. The result can be written as

$$G^{(4)}_{\mu\nu} = \frac{8\pi}{\phi} T_{\mu\nu}^{(BD)} + \frac{\omega}{\phi^2} \left[ (D_\mu \phi) (D_\nu \phi) - \frac{1}{2} g_{\mu\nu} (D_\alpha \phi) (D^\alpha \phi) \right] + \frac{1}{\phi} (D_\mu D_\nu \phi - g_{\mu\nu} D^2 \phi) - g_{\mu\nu} \frac{V(\phi)}{2\phi}$$

(13)

where we have introduced the quantity $V(\phi)$, which (as we will see below) plays the role of an effective or induced scalar potential; and $T_{\mu\nu}^{(BD)}$ can be interpreted as an induced EMT for an effective BD theory in 4\$D\$. It is given by

$$\frac{8\pi}{\phi} T_{\mu\nu}^{(BD)} = \frac{D_\mu D_\nu \Phi}{\Phi} + \frac{g_{\mu\nu} V}{2\phi}.$$  

(14)

Taking the trace of (13) we obtain a simple relation between $R^{(4)}$ and $T^{(BD)} = g^{\mu\nu} T_{\mu\nu}^{(BD)}$, namely

$$R^{(4)} = -\frac{8\pi}{\phi} T^{(BD)} + \frac{\omega}{\phi^2} \left[ (D_\alpha \phi) (D^\alpha \phi) \right] + \frac{3D^2 \phi}{\phi} + \frac{2V}{\phi}.$$  

(15)

To obtain the equation of motion of $\phi$ in 4\$D\$, we substitute (12) and (15) into (3). After some manipulations we get

$$D^2 \phi = \frac{8\pi}{3 + 2\omega} T^{(BD)} + \frac{1}{3 + 2\omega} \left[ \frac{dV(\phi)}{d\phi} - 2V(\phi) \right],$$

(16)

where the potential is derived from the equation

$$\phi \frac{dV(\phi)}{d\phi} \equiv -2 (1 + \omega) \frac{(D_\alpha \Phi) (D^\alpha \phi)}{\Phi}.$$  

(17)

The above equations constitute the basis for our discussion. To an observer in 4\$D\$, who is not aware of the existence of an extra dimension, (13) and (16) are nothing but the Brans-Dicke FE in 4\$D\$ with a self interacting potential $V$ and an effective EMT given by (14). We note the importance of the potential: If $V = 0$, then the effective FE in 4\$D\$ only yield the BD theory with parameter $2\omega = -1$. Thus, although the introduction of $V(\phi)$ in (13) might at first glance look artificial, $V \neq 0$ is necessary to obtain a general BD theory in 4\$D\$ (a more detailed discussion is provided at the end of section 4). In what follows, to avoid the so-called ghost fields (fields with the “wrong” sign of the kinetic term), we assume $\omega > -3/2$.

3 Brans-Dicke cosmology in 5\$D\$

In cosmological applications, under the assumption of spatial isotropy and homogeneity, the line element in 5\$D\$ is taken to be an extended version of the conventional Friedmann-Robertson-Walker metric in 4\$D\$, namely

$$dS^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right] + e\Phi^2(t) dy^2,$$

(18)

where $k = 0, +1, -1$ and $(t, r, \theta, \phi)$ are the usual coordinates for a spacetime with spherically symmetric spatial sections. For this line element the vacuum ($T_{AB}^{(5)} = 0$) Brans-Dicke field equations in 5\$D\$ reduce as follows. From the 5\$D$ wave equation (3) we obtain

$$\ddot{\phi} + \dot{\phi} \left( \frac{3\dot{a}}{a} + \frac{\dot{\Phi}}{\Phi} \right) = 0.$$  

(19)

$^2$It is interesting to note that some string theories in the low energy limit also reduce to BD theory with $\omega = -1$ [21].
Using this expression in (2), the temporal component \( A = B = 0 \) becomes

\[
3 \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} + \frac{\Phi}{\phi} \right) + 3k \frac{\dot{a}}{a^2} = \frac{1}{\phi} \left( \dot{\phi} + \frac{\omega \phi^2}{2 \phi} \right).
\]

Again using (19), the spatial components \( A = B = 1, 2, 3 \) give

\[
2 \dddot{a} + \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} + 2 \frac{\Phi}{\phi} \right) + \frac{\dot{\phi}}{\phi} + k \frac{\dot{a}}{a} = \frac{\dot{\phi}}{\phi} \left( \frac{\dot{a}}{a} - \frac{\omega \phi}{2 \phi} \right).
\]

Similarly, the \( A = B = 4 \) component yields

\[
3 \left[ \frac{\dddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right] + 3k \frac{\dddot{a}}{a^2} = \frac{\dot{\phi}}{\phi} \left( \frac{\dot{\phi}}{\phi} - \frac{\omega \phi}{2 \phi} \right).
\]

After a simple integration, from (19) we obtain

\[
\dot{\phi} a^3 \Phi = c_3 = \text{constant} \neq 0.
\]

A similar expression, with \( \phi \leftrightarrow \Phi \), is obtained from (20)-(22) if we proceed as follows: (i) isolate \( k \) from (21); (ii) introduce \( k \) into (20) and isolate for \( \dddot{a} \); (iii) do the same for (22); (iv) equate the expressions for \( \dddot{a} \) obtained in the previous two steps; (v) use (19) to get rid of \( \dddot{\phi} \). The result is a simple second-order differential equation for \( \Phi \) whose first integral is given by

\[
\Phi_0 a^3 \Phi = c_2 = \text{constant} \neq 0.
\]

(We assume \( \Phi \neq \text{constant} \), otherwise the reduced 4D spacetime is empty, i.e., \( T^{(BD)}_{\mu \nu} = 0 \).) Combining (23) and (24) we find

\[
\phi \Phi_0 a^3 \phi = c_3 = \text{constant} \neq 0.
\]

(25)

It should be noted that the field equation (21) is identically satisfied by (26)-(27). Therefore, any solution to (28) gives rise to an exact solution to the FE (19)-(22).

For a detailed study of the space of solutions (which is not the object of this paper), it is convenient to write (28) in the form

\[
\alpha \left( \frac{\dot{a}}{a} \right)^2 + 4 \left( \beta \dot{a}^2 + k a \right) \left( \dot{\phi} a \dot{a} + \dddot{a} + k + a \dddot{a} \right) = 0,
\]

where

\[
\alpha = c_1 (\omega c_1 - 2c_2), \quad \beta = \left( \omega + \frac{3}{2} \right) c_1 c_2 + \frac{3c_2^2}{2}.
\]

It should be noted that the field equation (21) is identically satisfied by (26)-(27). Therefore, any solution to (28) gives rise to an exact solution to the FE (19)-(22).
For \( k \neq 0 \) the solutions to (28) can be expressed in terms of elementary functions only in some particular cases, which correspond to specific choices of the constants, e.g., \( c_1 = -c_2, \alpha = 0, \beta = 0 \), etc. However, for \( k = 0 \) we find that (28) admits a unique solution, which is

\[
a(t) = (C_1 t + C_2)^l,
\]

where \( C_1 \) and \( C_2 \) are constants of integration and \( l \) is a parameter that depends (in a very complicated way) on \( c_1, c_2 \) and \( \omega \). We note that astrophysical data from WMAP \[12\] and BOOMERANG \[25\], analyzed in the context of models based on GR, indicate that \( k = 0 \) around the present epoch. However, any other theory could give in principle very different answer.

In this work we restrict our attention to the BD cosmological models generated by the exact solution (30). From a practical viewpoint, the solution looks much simpler in terms of the new parameters \( l \) and \( m \) (instead of \( c_1 \) and \( c_2 \)), namely

\[
dS^2 = dt^2 - B^2 t^{2l} \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] + eC^2 t^{2m} dy^2,
\]

\[
\phi = D t (1 - m - 3l), \quad \omega = -\frac{12l^2 + 2 (m - 1) (3l + m)}{(1 - m - 3l)^2}.
\]

We note that \( \phi = \text{constant} \) and \( \omega \to \infty \) in the limit \( m \to (1 - 3l) \). In this limit, the field equations can only be satisfied if \( l = 1/2 \), i.e., \( m = -1/2 \). Therefore, we recover the 5D general-relativistic solution

\[
dS^2 = dt^2 - B^2 t \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] + \frac{eC^2}{t} dy^2,
\]

which is the unique solution to the Einstein field equations \( G_{AB} = 0 \) in 5D for the metric (18) with flat spatial sections (\( k = 0 \)).

### 4 Effective Brans-Dicke cosmology in 4D

We now proceed to study the effective 4D picture generated by the spatially flat 5D solutions discussed in the preceding section. For the line element (18), the non-vanishing components of the induced Brans-Dicke energy-momentum tensor (14) are

\[
\frac{8\pi}{\phi} T^{(BD)}_{00} = \frac{\ddot{\phi}}{\phi} + \frac{V}{2\phi},
\]

\[
\frac{8\pi}{\phi} T^{(BD)}_{11} = \frac{\ddot{\phi} \Phi}{a\Phi} + \frac{V}{a\Phi}.
\]

where \( V = V(\phi) \) should be determined from (17). We note that \( T^{(BD)}_{22} = T^{(BD)}_{33} = T^{(BD)}_{11} \), which means that the induced EMT looks like a perfect fluid with energy density \( \rho = T^{(BD)}_{00} \) and isotropic pressure \( p = -T^{(BD)}_{11} \).

We emphasize that as in the conventional 4D Brans-Dicke theory, in our models the (effective) EMT obeys the ordinary conservation law \( D\mu T^{(BD)}_{\mu} = 0 \) (the same as in Einstein’s theory), which in the cosmological realm yields the usual equation of motion

\[
\dot{\rho} + 3\frac{\dot{a}}{a} (\rho + p) = 0.
\]

Substituting (31) into (17) we obtain

\[
\Phi \left( \frac{dV}{d\phi} \right) = \frac{2m (m^2 + 3l^2 - 1)}{(m + 3l - 1) D^{2/(m+3l-1)}} \phi^{(m+3l+1)/(m+3l-1)}.
\]

\(^4\text{We disregard the “static” case where } l = 0.\)
Integrating this equation, and setting the integration constant equal to zero, we find

\[ V(\phi) = - \frac{2m \left( m^2 + 3l^2 - 1 \right)}{(m + 3l + 1) D^{2/(m+3l-1)}} \phi^{(m+3l+1)/(m+3l-1)}. \]  

(36)

The effective EMT \(^{(33)}\) is given by

\[ \frac{8\pi T^{0}_{\phi(BD)}}{\phi} = \frac{3ml (m - l - 1)}{t^2 (m + 3l + 1)}, \]  

\[ \frac{8\pi T^{-1}_{\phi(BD)}}{\phi} = - \frac{m (m + 1) (m - l - 1)}{t^2 (m + 3l + 1)}. \]  

(37)

Consequently, the equation of state is

\[ p = n\rho, \quad n = \frac{m + 1}{3l}. \]  

(38)

The parameters \( m \) and \( l \) can be expressed in terms of \( n \) and the acceleration parameter \( q = -\ddot{a}/\dot{a}^2 \) as

\[ m = \frac{3n}{q + 1} - 1, \quad \frac{1}{q + 1}. \]  

(39)

Since the present epoch of the universe is matter dominated we set \( m = -1 \). Thus

\[ \frac{8\pi T^{0}_{\phi(BD)}}{\phi} = \frac{l + 2}{t^2}, \quad \omega = - \frac{4 [3l (l - 1) + 1]}{2 - 3l^2}, \quad V = \frac{2l\phi^{3l/(3l-2)}}{D^{2/(3l-2)}}. \]  

(40)

For \( n = 0 \), the condition \( \omega > -3/2 \) restricts the range of \( l \) to be either \( l < 2 \left( 1 - \sqrt{2/3} \right) \approx 0.37 \) or \( l > 2 \left( 1 + \sqrt{2/3} \right) \approx 3.633 \). The former range requires \( q > (1 + \sqrt{6})/2 \approx 1.72 \), which is inapplicable to the present epoch. However, the latter range leads to accelerated cosmic expansion with \( q < (1 - \sqrt{6})/2 \approx -0.72 \). Thus, our reduced BD cosmological model expands in a way consistent with current observed measurements \( q = -0.67 \pm 0.25 \) \(^{[26]}\) in the context of models based on GR. Besides, for dust \( (m = -1) \) the extra dimension contracts while the spatial dimensions expand. A similar analysis can be easily done for any value of \( n \).

In summary, the reduced Brans-Dicke cosmological solution in 4\(D\), which is obtained from \(^{(31)}\), can be written as

\[ d\sigma^2 = dS_{\Sigma_y}^2 = dt^2 - B^2 t^{2l} \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \]

\[ \rho = \rho_0 \left( \frac{t_0}{t} \right)^{3l(n+1)}, \]

\[ \phi = \phi_0 \left( \frac{t}{t_0} \right)^{2-3l(n+1)}, \]

\[ V = V_0 \left( \frac{\phi}{\phi_0} \right)^{3l(n+1)/[3l(n+1)-2]}, \]  

(41)

with

\[ \phi_0 = \frac{8\pi (n + 1) \rho_0 t_0^2}{(l + 2 - 3ln)(1 - 3ln)}, \]

\[ V_0 = \frac{16\pi \rho_0 (l + 1) (3n^2 - 2n)}{2 + l(1 - 3n)}, \]  

(42)
where \( \rho_0 \) refers to the value of the energy density at some arbitrary fixed time \( t_0 \), and

\[
 l = l_{(+)} = \frac{6 (\omega + 1) + 3n (2\omega + 3) \mp \sqrt{9n^2 + 12 (1 + \omega) (3n - 1)}}{9 (\omega + 2)n^2 + 18 (\omega + 1)n + 3 (3\omega + 4)}.
\]  (43)

It is worth noticing that regardless of \( n \) for large values of \( |\omega| \) we get \( l = 2/3 (n + 1) \), which implies \( V = 0 \) and \( \phi = \) constant. Also, in this limit from (38) it follows that \( m = (n - 1)/(n + 1) \), which means that the extra dimension contracts, or compactifies, for any \(-1 < n < 1\). Consequently, formally for \( |\omega| \rightarrow \infty \) we recover the usual spatially flat FRW cosmology of ordinary general relativity.

Since the term inside of the root must be non-negative, we find \( \omega < -1 \) for dust. However, for radiation there are no restrictions on \( \omega \). For \( n = 1/3 \) we find \( l = l_{(+)} = 1/2 \), which is identical to the radiation-dominated epoch of general relativity, regardless of the specific value of \( \omega \). Therefore, the effective BD cosmology in 4D can give decelerated radiation era as well as accelerating matter dominated era.

For the sake of comparison, we note that the well-known spatially flat BD dust solutions [27] exist for any value of \( \omega \) and show accelerated expansion only for \(-2 \leq \omega \leq -3/2 \) (see [21] and references therein). However, our dust model requires \( \omega < -1 \) and the accelerated expansion occurs in the range \(-3/2 < \omega < -1 \), where the lower limit comes from the positive energy condition on the scalar field. The simultaneous occurrence of \( \omega > -3/2 \) and accelerated expansion is a consequence of the non-vanishing scalar potential.

If we assume \( V = 0 \) at the beginning, then we obtain a very limited class of BD cosmologies in 4D. In fact, in the case where \( V = 0 \) instead of (33) we have

\[
 \frac{8\pi T_0^{(BD)}}{\phi} = \frac{m (m - 1)}{t^2},
\]

\[
 \frac{8\pi T_1^{(BD)}}{\phi} = \frac{lm}{t^2}.
\]  (44)

Besides, from (17) it follows that \( \omega = -1 \). Then, from (21) we obtain \( m = \pm \sqrt{1 - 3l^2} \), which requires \(|l| \leq 1/\sqrt{3} \approx 0.577 \). The choice \( m = -\sqrt{1 - 3l^2} \) assures that \( p > 0 \) and \( p > 0 \). When \( l = 1/2 \) \((m = -1/2)\) we have \( \phi = \) constant and recover the spatially flat FRW solution of general relativity with \( p = \rho/3 \), which is the spacetime section of (22).

5 5D scalar-vacuum Brans-Dicke theory with variable \( \omega \)

We have seen that the scalar-vacuum Brans-Dicke cosmology in 5D yields accelerated expansion of an effective matter-dominated 4D universe if \(-3/2 < \omega < -1 \). This range of \( \omega \) is consistent with the one obtained from the reduced BD cosmologies with higher-dimensional matter discussed in [24]. However, these small values of \( |\omega| \) are in sharp contradiction with the solar system bound \( \omega > 600 \). This contradiction is consistently found when the ordinary BD theory in 4D is applied to cosmological problems like inflation and structure formation, not only to the late time cosmic acceleration (see e.g. [22] and references therein). As a possible way out of this problem, it has been suggested to consider a modified version of BD theory where the parameter \( \omega \) is a function of the scalar field rather than a constant [20].

In this section we study the question of whether accelerated expansion is compatible with large values of \( \omega \), within the context of a 5D scalar-vacuum BD theory with variable \( \omega \). With this aim we integrate the FE and obtain an explicit equation relating \( q \) and \( \omega \). Our analysis shows that the answer to our question is negative, namely, late time accelerated expansion is incompatible with large values of \( \omega \).

For a variable \( \omega \) the wave equation (15) becomes

\[
 \nabla^2 \phi = -\frac{3}{2 (3\omega + 4)} \frac{d\omega}{d\phi} (\nabla_A \phi) (\nabla^A \phi).
\]  (45)

Therefore, instead of (19) we now have

\[
 \ddot{\phi} + \frac{\dot{\phi}}{a} \left( \frac{3\dot{a}}{a} + \frac{\dot{\phi}}{\Phi} \right) = -\frac{3\dot{\omega} \dot{\phi}}{2 (3\omega + 4)}.
\]  (46)
This equation gives the first integral
\[ \dot{\phi} a^3 \Phi = \bar{c}_1 / \sqrt{3 \omega + 4}, \]  
(47)
where \( \bar{c}_1 \) is a constant of integration. In practice this is a definition of \( \omega \) in terms of the metric functions and the scalar field. It implies that \( \omega \) should obey the condition \( \omega \geq -4/3 \). In this regard we note that no such condition follows from the FE when \( \omega \) is constant.

Instead of the simplified FE (20)-(22) we now have
\[ 3 \frac{\ddot{a}}{a} \left( \frac{\dot{a}}{a} + \frac{\dot{\Phi}}{\Phi} \right) + \frac{3k}{a^2} = -\frac{\dot{\phi}}{\phi} \left( \frac{3\dot{a}}{a} + \frac{\dot{\Phi}}{\Phi} - \frac{\omega \dot{\phi}}{2\phi} \right). \]  
(48)
\[ \frac{2\ddot{a}}{a} + \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} + 2\dot{\Phi}/\Phi \right) + \frac{\dot{\Phi}}{\Phi} + \frac{k}{a^2} = -\frac{\ddot{\phi}}{\phi} \left( \frac{\dot{\phi}}{\phi} \right) \left( \frac{2\dot{a}}{a} + \frac{\dot{\Phi}}{\Phi} + \frac{\omega \dot{\phi}}{2\phi} \right). \]  
(49)
\[ 3 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right] + \frac{3k}{a^2} = -\frac{\ddot{\phi}}{\phi} \left( \frac{\dot{\phi}}{\phi} \right) \left( \frac{2\dot{a}}{a} + \frac{\dot{\Phi}}{\Phi} + \frac{\omega \dot{\phi}}{2\phi} \right). \]  
(50)

From (48) and (50) we eliminate \( \omega \) to obtain a first-order differential equation for \( \Phi \). This equation admits exact integration for \( k = 0 \), namely
\[ \Phi a^2 \left( 3\dot{\phi} + \dot{\phi} a \right) = \bar{c}_2. \]  
(51)
Similarly, from (49) and (50) we get a first-order differential equation for \( \phi \), which for \( k = 0 \) yields
\[ \phi a^2 \left( \dot{\phi} - \ddot{\phi} a \right) = \bar{c}_3. \]  
(52)

In the above expressions \( \bar{c}_2 \) and \( \bar{c}_3 \) are constants of integration.

An important result follows from \( a^3 \left( 3\dot{\phi} + \ddot{\phi} a \right) = 3 \bar{c}_2 - 3 \bar{c}_3 \) after substituting (47) into it. Specifically,
\[ \Phi a^3 \Phi = -\frac{\bar{c}_1}{3 \sqrt{3 \omega + 4}} + \frac{\bar{c}_2}{3} - \bar{c}_3. \]  
(53)
This equation says that a varying \( \omega \) demands \( \dot{\Phi} \neq 0 \), otherwise \( \omega = \text{constant} \).

Equation (51) divided by (52) yields a first-order differential equation which can be easily integrated to find
\[ \phi \propto \Phi^{-\bar{c}_2/\bar{c}_3} a^{(\bar{c}_2 - 3\bar{c}_3)/\bar{c}_3}, \quad \bar{c}_3 \neq 0, \]
\[ \Phi \propto a, \quad \bar{c}_3 = 0. \]  
(54)
Substituting these expressions back into either (51) or (52), and using (48)-(50), we obtain the general expressions for \( \Phi, \phi \) and \( \omega \) in terms \( a(t) \). There are three families of solutions listed below as I-III.

I:
\[
\Phi = A a F^{\bar{c}_3/(\bar{c}_3 - \bar{c}_2)}, \quad \bar{c}_2 - \bar{c}_3 \neq 0 \\
\phi = B a^{-3} F^{\bar{c}_2/(\bar{c}_2 - \bar{c}_3)}, \\
\omega = -\frac{2 \left( 6F^2 \dot{a}^2 - 4\bar{c}_2 F \dot{a} + \bar{c}_2 \bar{c}_3 \right)}{\left( 3F \dot{a} - \bar{c}_2 \right)^2},
\]  
(55)
where \( F \) is given by
\[ F \equiv C + (\bar{c}_2 - \bar{c}_3) f(t), \quad f(t) \equiv \int \frac{dt}{a(t)}, \]  
(56)
and \( A, B, C \) are arbitrary constants.
II:

\[
\Phi = A e^{-\bar{c}^2_{2}/C}, \quad \bar{c}_2 - \bar{c}_3 = 0, \quad \bar{c}_2 = \bar{c}_3 \neq 0, \\
\phi = (C/4)\bar{a}^{-3} e^{\bar{c}^2_{2}/C}, \\
\omega = -\frac{2 (6C^2\bar{a}^2 - 4\bar{c}_2 C \bar{a} + \bar{c}_2^2)}{(3\bar{a} - \bar{c}_2)^2}.
\]

(57)

III:

\[
\Phi = A \bar{a}, \quad \bar{c}_2 = \bar{c}_3 = 0, \\
\phi = B \bar{a}^{-3}, \\
\omega = -4/3.
\]

(58)

One can verify that (55)-(58) satisfy the field equations (48)-(50), as expected.

- We now substitute the general solution (55) into the “new” wave equation (46). The resulting equation allows us to express the deceleration parameter \( q = -\dot{a}/a^2 \) as a function of \( \omega \). For this we use

\[
\ddot{a} = -\frac{\bar{a}^2}{\alpha}, \\
\dot{a} = \dot{a}(\pm) = \frac{\bar{c}_2 \left[ \sqrt{3\omega + 4} \pm \sqrt{2(2 - 3\alpha)} \right]}{3F\sqrt{3\omega + 4}}, \quad \alpha = \frac{\bar{c}_3}{\bar{c}_2} \leq \frac{2}{3},
\]

(59)

where the second equation comes from the expression for \( \omega \) in (55), and \( \alpha \) is a dimensionless parameter. It should be noted that \( \dot{a}(\pm) \) represents an ever expanding (contracting) universe, while \( \dot{a}(\mp) \) changes its motion/sign for \( \omega = -2\alpha \).

After a long but straightforward calculation we find\(^5\)

\[
q(\pm)(\omega, \alpha) = \frac{3\sqrt{3\omega + 4} (1 - \alpha) \left[ 3\omega + 4 + \sqrt{3\omega + 4} \left( \sqrt{4 - 6\alpha} - 1 \right) - \sqrt{4 - 6\alpha} \right]}{(3\omega + 4) \left( 2\sqrt{4 - 6\alpha} - 1 \right) - (4 - 6\alpha) + \sqrt{3\omega + 4} \left[ 3\omega + 4 - 2\sqrt{4 - 6\alpha} + (4 - 6\alpha) \right]}. \\
q(-)(\omega, \alpha) = \frac{3\sqrt{3\omega + 4} (1 - \alpha) \left[ - (3\omega + 4) + \sqrt{3\omega + 4} \left( \sqrt{4 - 6\alpha} + 1 \right) - \sqrt{4 - 6\alpha} \right]}{(3\omega + 4) \left( 2\sqrt{4 - 6\alpha} + 1 \right) + (4 - 6\alpha) - \sqrt{3\omega + 4} \left[ 3\omega + 4 + 2\sqrt{4 - 6\alpha} + (4 - 6\alpha) \right]}. 
\]

(60)

(61)

where \( q(\pm) \) and \( q(-) \) correspond to the choice of positive or negative sign in (59), respectively. They are equal to each other only for \( \omega = -4/3 \) (\( q(\pm) = 0 \)); \( \alpha = 2/3 \) (\( q(-) = 1 \)), and in the limit \( \omega \to \infty \), namely

\[
\lim_{\omega \to \infty} q(\pm) = 3 (1 - \alpha) \geq 1,
\]

(62)

which follows from the requirement \( \alpha \leq 2/3 \). The denominator of \( q(\pm) \) vanishes at \( \omega = -1 \) and \( \omega = -2\alpha \). However, it is not difficult to verify that \( q(\pm) \) remains finite and positive for all values of \( \omega \) and \( \alpha \). In addition, for every \( \alpha \leq 2/3 \) we find that \( q(+) \) increases monotonically with \( \omega \) from zero at \( \omega = -4/3 \) to \( [3 (1 - \alpha)] \) as \( \omega \to \infty \).

The model generated by \( \dot{a}(-) \) is regular at \( \omega = -1 \). It represents a universe that expands (contracts) for \( \omega < -2\alpha \) and reverses its cycle for \( \omega > -2\alpha \) in such a way that \( q(-) \) diverges at \( \omega = -2\alpha \). Indeed, \( q(-) \) goes monotonically from 0 to \( -\infty \) as \( \omega \) increases from \( \omega = -4/3 \) to \( \omega = -2\alpha \). Conversely, \( q(-) \) goes monotonically from \( +\infty \) to \( [3 (1 - \alpha)] \) as \( \omega \) increases from \( \omega = -2\alpha \) to \( \omega \to \infty \).

Thus, in the framework under consideration, in this section we have used the general spatially-flat solution (55) to demonstrate that accelerated expansion \( q < 0 \) in an (ever)expanding universe is incompatible with large positive values of \( \omega \). The question of whether this result still holds for \( k \neq 0 \) remains open.

\(^5\)We observe that these expressions are obtained without the explicit knowledge of the function \( F \) (or \( f \)) introduced in (50).
6 Summary

In this work we have investigated the question of whether, we can get the observed late time accelerated universe from a scalar-vacuum Brans-Dicke theory in 5D. Thus, we have not introduced matter fields in 5D. Rather, after dimensional reduction we have seen that the effective equations in 4D can be regarded as those for the standard BD in 4D with a self interacting potential \( V = V(\phi) \) and a matter field. These appear in 4D as a consequence of the variation of \( \Phi \) with time, i.e., \( V = 0 \) and \( T^{(BD)}_{\mu\nu} = 0 \) when \( \Phi = \text{constant} \). In the case that \( \omega = -1 \), from (17) it follows that we can set \( V = 0 \) without loss of generality. However, if \( \omega \neq -1 \) (and \( \Phi \neq \text{constant} \)) we cannot assume \( V = 0 \). Thus, in our work the induced potential is not picked by hand. Instead, it is dictated by the geometry in 5D.

As a consequence, we have shown that the 5D scalar-vacuum BD solutions \([31]\) can be interpreted by an observer in 4D as a family of BD cosmological models \([41]-[43]\) with matter and an effective potential. These models can give decelerated radiation era as well as accelerating matter dominated era. Formally, in the limit \( |\omega| \to \infty \) they reduce to the spatially flat FRW cosmologies of general relativity.

Since small values of \( |\omega| \) are in contradiction with the solar system bound \( \omega > 600 \), here we have studied the question of whether accelerated expansion is compatible with large values of \( \omega \), within the framework of a 5D scalar-vacuum BD theory with variable \( \omega \). In this framework, and using the general class of spatially flat solutions (55), we have shown that the answer to this question is negative. Namely, accelerated expansion is incompatible with large positive values of \( \omega \). The origin of this incompatibility may be the straightforward transportation of \( \omega \) from large cosmological scales to the small scales of the solar system, without taking into consideration local inhomogeneities in astronomical scale. It is possible that such inhomogeneities can produce the large values of \( \omega \) observed in the solar system.

To finish the discussion, we should mention that, besides the obvious extension to cosmologies with \( k \neq 0 \), our work leaves a number of important questions open for future research. For example, we do not provide a description of how and when the BD universe goes from decelerated to accelerated expansion. This is because the assumption that our 4D spacetime can be recovered by going onto a hypersurface \( y = y_0 = \text{constant} \) only leads to effective matter that satisfies the barotropic equation of state (38). To obtain more general forms of effective matter, which can allow the study of the evolution of \( q \), we need a more “flexible” embedding approach\(^6\), e.g., the one considered in \([28]\). We have not discussed how matter fields in 5D can affect evolution of \( \omega \), nor have we considered whether or not matter fields in 4D can be incorporated in the discussion. However, in the context of the present work, the matter content of spacetime is an effective result of the reduction from 5D to 4D. To add matter in 4D we should follow an approach similar to the one used in braneworld models, i.e., identify our spacetime with some singular hypersurface (the brane where ordinary 4D matter fields are located) embedded in an empty 5D BD. This requires the consideration of more general solutions where the 5D metric and the scalar field are functions of the extra coordinate, similar to the ones recently considered in \([29]\). Our simplified model shows that the scalar field in higher dimensional BD theory alone can be responsible for the present cosmic accelerated expansion, which rules out dark energy dynamics. We are currently working on constructing more realistic models.

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\(^6\)The effective spacetime measured by an observer depends on her/his state of motion. The simplest physical scenario emerges in the rest (also called comoving) frame, which in the present case means \( dx^i = dy = 0 \). In such a frame, the spacetime is recovered by going onto some hypersurface \( \Sigma_y : y = y_0 = \text{constant} \), which is orthogonal to the unit 5D vector \( \xi \). A more “flexible” approach, that respects the spatial homogeneity and isotropy of FRW models, is to consider that 4D observers are at rest only in 3D \( (dx^i = 0) \), but moving in 5D, i.e., that our spacetime is recovered on a dynamical 4D hypersurface \( y = y(t) \) or \( t = t(\tau) \), \( y = y(\tau) \) in parametric form \([28]\).
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