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Conservation of angular momentum of light in single scattering

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Abstract: In this paper we discuss the conservation of angular momentum of light in single scattering of circularly polarized light from a spherical, non-absorbing particle. We show that the angular momentum carried by the incident wave is distributed in the scattered waves between terms related to polarization or spin and to orbital angular momentum, respectively. We also show that, in all scattering directions, a constant ratio exists between the flux density of the total angular momentum and the intensity.

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1. Introduction

In elastic scattering from a non absorbing spherical particle, two parameters of the electromagnetic field are conserved: the energy and the angular momentum component along the propagation direction. Linear momentum as a whole is also conserved, of course, but some of it is transferred to the particle leading to radiation pressure. The conservation of energy (which is a scalar quantity) leads to a normalization condition for the integrated energy flux density, which is further used in defining the scattering cross section[1, 2]. Similarly, the conservation of angular momentum should be related to the angular momentum flux density. The continuity conditions for the angular momentum density can be described by three equations (one for each component of the vector) or one equation for a tensor[3].
We are interested in calculating the angular moment flux density of the electromagnetic field which results from scattering of a circularly polarized wave from a non-absorbing spherical particle. It was demonstrated that scattering of circularly polarized wave does not exert torque on the particle and that transfer of angular momentum from the field to the particles is mediated only by absorption[4]. Therefore, the angular momentum of the field is preserved and it should be interesting to know how the flux density of the angular momentum is distributed between the spin or polarization term which will be henceforth designated by \( s \) and the orbital angular momentum (OAM), which will be denoted by \( l \). The total angular momentum flux density, \( j \), is given by the sum of these two terms. The problem of torques applied to particles was treated extensively in the context of particle manipulation or "optical tweezers" [5]. In this paper we will limit ourself to cases in which no torque is applied and we will discuss the angular momentum carried by the scattered electromagnetic wave.

2. Scattering effects on spin angular momentum

Let us consider an incident plane wave which is monochromatic (angular frequency \( \omega \)), left circularly polarized, has an amplitude \( E_0 \) and propagates in the direction \( \hat{z} \) (the ‘hat’ denotes a unit vector). This is the simplest example of a paraxial wave which has the general form

\[
E(x, y, z) = \exp(i\mathbf{k}\cdot\mathbf{r}) F(x, y, z)
\]

(where \( F(x, y, z) \) is a slowly varying spatial envelope). For such a wave one can employ expressions for angular momentum flux density of a paraxial wave[6, 7, 8] and write the \( z \) component of the angular momentum flux density can be written as

\[
j_z(x, y, z) = \frac{c}{2i\omega} \mathbf{E}^\times \cdot \nabla | \frac{c\mathbf{E}_0}{2i\omega} \mathbf{E}^\times \times \mathbf{E} | _z
\]

\[
= \frac{c\mathbf{E}_0}{2i\omega} F_k^* \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) F_k + \frac{c\mathbf{E}_0}{2i\omega} (F_y^* F_y - F_y F_y)^*.
\]

The first term relates to the transverse distribution of the field and is the OAM term. The second term reflects the angular momentum carried by circular polarization is zero for linear polarization and. For a class of paraxial beams for which the transverse field distribution can be written as

\[
F(r, \phi) = u(r) \exp(i m \phi)
\]

and recognizing that \( \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = \frac{\partial}{\partial \phi} \) we find that

\[
j_z(r, \phi) = \frac{c \mathbf{E}_0}{2i\omega} (m + \sigma) |u(r)|^2 = (m + \sigma) \frac{l}{\omega}.
\]
incident field carries a spin angular momentum of \( +\hbar \). The rate at which the angular momentum is removed from the incident field can be derived from the scattered power and is given by \( \sigma_{sc} I_0 / \omega \hat{z} \). This is the source term for the angular momentum of the scattered field and it should be recovered by integrating over all directions the angular momentum flux density of the scattered field.

From general scattering theory it is known that the scattered electric field in the far zone can be written as

\[
E(\theta) = \begin{pmatrix} E_L(\theta) \\ E_R(\theta) \end{pmatrix} = \frac{1}{r} \exp(ikr) \begin{pmatrix} S_{LL}(\theta) & S_{RL}(\theta) \\ S_{LR}(\theta) & S_{RR}(\theta) \end{pmatrix} \begin{pmatrix} E_0 \\ 0 \end{pmatrix},
\]

(5)

where the matrix \( \mathcal{S} \) is the scattering matrix in the circular polarization basis, the \( L \) and \( R \) subscripts designate the left and right circular polarizations. In Eq (5), \( k \) is the wave number and \( r \) is the distance from the center of the scattering particle which is both the origin of the coordinates frame and the reference point for the angular momentum calculations. The scattering matrix in the circular base can be related to the more usual amplitude scattering matrix given in terms of the parallel and perpendicular electric field components with respect to the scattering plane. For a spherical particle, this scattering matrix is diagonal with elements \( S_2(\theta), S_1(\theta) \) and the relation is[1]

\[
\begin{pmatrix} S_{LL}(\theta) & S_{RL}(\theta) \\ S_{LR}(\theta) & S_{RR}(\theta) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} S_2(\theta) + S_1(\theta) & S_2(\theta) - S_1(\theta) \\ S_2(\theta) - S_1(\theta) & S_2(\theta) + S_1(\theta) \end{pmatrix}.
\]

(6)

The notation used in this paper is illustrated in Figure 1.

It is important to remember that scattered field components are given in terms of a coordinate frame which is rotated such that the local scattered field is given in terms of component is a plane which is perpendicular to the scattering direction. However, in the following it is necessary to express all the scattered waves in the same reference frame such that spatial derivatives can be made consistently. The most convenient system of coordinates is the one associated with
the incident wave and, in this case, it is found that the rotation introduces a phase term \( \exp(i\phi) \) where \( \phi \) is the azimuth angle [2]. We note here that in the case of an incident wave which is right circularly polarized, the phase factor is \( \exp(-i\phi) \).

The scattered field can now be written as

\[
E(\theta, \phi) = \frac{\exp(ikr)}{r}E_0 \exp(i\phi) \left\{ S_{LL}(\theta) \hat{L} + S_{LR}(\theta) \hat{R} \right\}
\]

and expanding further the left and right circular unit vectors in terms of the locally transverse unit vectors

\[
\hat{L} = \frac{1}{\sqrt{2}} \left( \hat{\theta} + i \hat{\phi} \right)
\]
\[
\hat{R} = \frac{1}{\sqrt{2}} \left( \hat{\theta} - i \hat{\phi} \right)
\]

one finally obtains

\[
E(\theta, \phi) = \frac{\exp(ikr)}{r}E_0 \exp(i\phi) \left\{ S_{LL}(\theta) \hat{\theta} + S_{LR}(\theta) \hat{\phi} \right\}
\]

where the notation

\[
S_{\theta}(\theta) = |S_{LL}(\theta) + S_{LR}(\theta)| / \sqrt{2} \quad \text{and} \quad S_{\phi}(\theta) = i[S_{LL}(\theta) - S_{LR}(\theta)] / \sqrt{2}
\]

has been used.

Having found \( E \), we can now proceed to calculate the angular momentum flux density through a radially oriented infinitesimal area in the radiation zone. When the wave is approximated locally as a plane wave the spin term can be found using Eq. (2):

\[
s(\theta) = \varepsilon_0 c \frac{E_s^2}{2 \omega} \left[ S_{\theta}(\theta) S_{\phi}(\theta) - S_{\phi}(\theta) S_{\theta}(\theta) \right] \hat{r}
\]

This result is physically reasonable: the spin is simply the difference of intensities of the two radially outgoing orthogonal circular polarization components divided by the angular frequency of radiation. The common phase term \( \exp(i\phi) \) does not play a roll in this calculation.

Due to axial symmetry, all the components of \( s \) average to zero when an integration over the angles is performed except for the \( z \) component which is

\[
s_z(\theta) = \varepsilon_0 c \frac{E_0^2}{2 \omega} \left[ |S_{LL}(\theta)|^2 - |S_{LR}(\theta)|^2 \right] \cos(\theta).
\]

For a Rayleigh scatterer, one can immediately find that the expression in Eq. (11) reduces to

\[
s_z(\theta) = \frac{3\varepsilon_0 c}{16\pi \omega} \frac{E_0^2}{r^2} \sigma_{sc} \cos^2(\theta).
\]

In order to calculate the total scattered flux, the spin flux density must be integrated over a sphere of radius \( r \) to obtain
\[ \bar{\Sigma}_z = 1 + \frac{\varepsilon_0 c^2}{4\omega} \sigma_{sc} \delta \bar{I}_0 = \frac{1}{2} \sigma_{sc} \frac{I_0}{\omega}. \]  

(13)

The bar superscript in Eq. (13) indicates integration over a sphere of arbitrary radius, in the far field. Notably, one can see that only half of the angular momentum flux removed from the incident wave is contained in the spin term. Of course, this conclusion is similar to the results obtained for the case of a field radiated by a rotating dipole[9].

In order to evaluate the more complex case of a Mie scatterer let us rewrite Eq. (11) as

\[ s_z(\theta) = \frac{1}{\omega} [I_L(\theta) - I_R(\theta)] \cos(\theta) = \frac{1}{\omega} V(\theta) \cos(\theta), \]  

(14)

where \( I_L(\theta), I_R(\theta) \) are the scattered intensities with left and right hand circular polarization respectively and \( V(\theta) \) is the fourth Stokes parameter. In our case the incident wave is characterized by a Stokes vector of the form \([1, 0, 0, 1]\) while the scattered wave Stokes vector is \([F_{11}(\theta), F_{21}(\theta), F_{34}(\theta), F_{44}(\theta)]\) where \( \vec{F}(\theta) \) is the angle dependant \(4 \times 4\) scattering matrix\[1\] which is block diagonal for spherical particles. Since we obtained that for a left hand circular incident wave \( V(\theta) = F_{44}(\theta) \), we can calculate how much of the angular momentum flux density is contained in the spin term, normalized by the scattered angular momentum:

\[ \frac{\omega \bar{\Sigma}_z}{\sigma_{sc} \bar{I}_0} \sim \frac{2\pi \omega \int \sin(\theta) d\theta F_{44}(\theta) \cos(\theta)}{\sigma_{sc} \bar{I}_0}. \]  

(15)

It is worth mentioning that this expression is similar to the asymmetry factor defined by MacKintosh and John[10] (designated there as \(A\)) but note that their definition lacks the \( \cos(\theta) \) in the numerator and therefore it reflects the overall helicity flip rather then the \( z \) component of the spin.

Numerical evaluations of Eq. (15) based on the Mie theory suggest that a good approximation of this ratio is

\[ \frac{\omega \bar{\Sigma}_z}{\sigma_{sc} \bar{I}_0} \sim 1 + g^2, \]  

(16)

where \( g = \langle \cos(\theta) \rangle \) is the so-called scattering asymmetry parameter. For highly forward scattering which is helicity preserving, this ratio is close to 1 meaning that the total angular momentum is concentrated in the polarization term. Some examples are presented in Fig. 2.

The calculations were made for several relative indices of refraction: 1.09 which is comparable to the case of silica spheres in water, 1.18 which represents polystyrene spheres in water and a hypothetical higher contrast material with a relative index of refraction of 1.25. One can observe that the agreement with the dependence suggested in Eq. (16) is excellent for \( g \) below 0.1 and above 0.75. A higher order polynomial which will be better in the intermediate range is of course possible. It is also interesting to examine the mean helicity (the Mackintosh-John parameter) of the scattered field which is illustrated in Fig 3. One can note that up to \( g = 0.7 \) there is a good linear dependence with a slope of about 1.3.

3. Scattering effects on orbital angular momentum

Let us now turn our attention to the more complex issue of the OAM term which is[8, 9]

\[ I(\theta) = \frac{\varepsilon_0 c}{i2\omega} \mathbf{E}^* (\hat{r} \times \nabla) \mathbf{E}. \]  

(17)

In order to evaluate this expression we employ the spherical coordinates form of the gradient operator and recall that the unit vectors have to be differentiated as well; for example \( \frac{\partial}{\partial \phi} = \)
cos (θ) \hat{\phi} \text{ and } \frac{\partial \hat{\phi}}{\partial \phi} = (\hat{z} \cos (\theta) - \hat{r}) / \sin (\theta). \text{ Also note that the common phase term, } \exp (i \phi), \text{ is important in this case and should be considered in the differentiation. Accounting only for the terms that contribute to the } z \text{ component, we obtain after some algebra that}

\[ l_z(\theta) = \frac{\varepsilon_0 c^2 E_0^2}{2 \omega r^2} \left( |S_{LL}(\theta)|^2 + |S_{LR}(\theta)|^2 - \left[ |S_{LL}(\theta)|^2 - |S_{LR}(\theta)|^2 \right] \cos (\theta) \right) \, . \] (18)

This result, together with the one expressed in Eq. (11) indicate that \( s_z(\theta) \) and \( l_z(\theta) \) sum up to an expression which is proportional to the scattered intensity. For each \( \theta \), the total angular momentum flux density is simply the intensity divided by the angular frequency of the radiation. Integrating over a spherical surface in the far field leads indeed to a manifestation of the conservation of angular momentum flux:

\[ \bar{j}_z = \bar{s}_z + \bar{l}_z = \sigma_{sc} \frac{l_0}{\omega} \hat{z}. \] (19)

Moreover, we note again that at each \( \theta \), the ratio \( j_z(\theta) / I(\theta) \) is constant. In the quantum description one may say that a photon scattered in any direction carries the same angular momentum as an incident photon, but, in different directions it is distributed differently between the spin and OAM terms. It is now evident that in the forward and backward directions the angular momentum is carried only by the polarization. In other words the helicity is fully preserved in the forward direction and it is fully reversed in the backward direction. For some angles, which for a Rayleigh scatterer is only 90°, on the other hand, the scattered light is linearly polarized and therefore does not carry any spin angular momentum. At these angles the angular momentum is carried only by OAM term.

Figure 4 illustrates the three dimensional distribution of the spin term (relative to the scattered intensity) plotted as a function of scattering direction for several values of the size parameter \( x = 2 \pi a / \lambda \), where \( a \) is the radius of the particle and \( \lambda \) is the wavelength. The orbital angular
momentum distribution is the complementary one (as it will be given by one minus the spin). It is interesting to note that for small size parameters the changes in the normalized distribution are small. For large particles on the other hand, in which Mie resonances are dominant, one can observe that there are directions in which the spin term is negative- meaning that the normalized orbital term is larger then 1. The transition to such behavior happens for particles with a size parameter of about $\pi$ (i.e. a particle diameter which is approximately equal to the wavelength).

In order to illustrate the distinction between spin and orbital angular momenta one may use the following "gedanken" experiment. Let us consider a small dielectric sphere which is slightly absorbing, and is placed in the far field of a Rayleigh scatterer. In the exact forward direction the particle will rotate about itself due to the absorption of circularly polarized light. At of 90° with respect to the direction of incidence, on the other hand, the test particle will "orbit" the scattering particle due to the phase gradient in the scattered field. We would like to emphasize that this effect is not due to absorption and that, of course, the test particle will experience a radial force due to radiation pressure.

Finally, we note that the orbital angular momentum of scattered light can also be interpreted as the result of a slight direction dependant shift of the apparent origin of the scattered waves, a shift which introduces an "impact parameter" of the order of $\lambda/2\pi$. The origin of this shift can be identified in the classical electromagnetic theory where it is known that the Poynting vector of radiation from a rotating dipole (equivalent to a Rayleigh scatterer illuminated by a circularly polarized light) spirals in the near field. As a result the far field radiation seems to be emerging not from the center of the dipole but from a shifted position[11] as illustrated qualitatively in Figure 5. In this figure the incident light direction is into the page and the scattered wave is observed in a transverse plane.

The angle $\alpha$ is approximated by $\lambda/2\pi r$ and the angular momentum at the point $r$ can then be calculated to be $k r \alpha = 1$. This means of course that the orbital angular momentum carries all the angular momentum. In our gedanken experiment, the slight tilt of the k-vector induces a rotating motion of the test particle placed at $r$. This is the mechanism which couples angular
Fig. 4. Three-dimensional angular distributions of the normalized spin term (angular momentum content of the spin term divided by the intensity) for several size parameters. The relative index of refraction was 1.09. Left circularly polarized light is incident along the $z$ axis (the direction is indicated by the red arrow). The color bars represent the spin and a complementary illustration can be obtained for the OAM.

Fig. 5. Qualitative illustration of the Poynting vector direction for radiation scattered by a Rayleigh scatterer illuminated by a circularly polarized light which is perpendicular to the plane of the figure as indicated. The spiraling of the Poynting vector results in an apparent angular shift of the light at the far field.

4. Conclusions

In this paper we elucidated the mechanism by which the scattering from a spherical non-absorbing particle is transferring angular momentum carried by an electromagnetic field from the spin term to the orbital angular momentum term. The relative part of the angular momentum contained in the spin term can be calculated exactly and it ranges from 0.5 for a Rayleigh scatterer to 1 in the case of a highly forward scatterer. In any scattering direction the ratio be-
tween the total angular momentum flux density carried and the intensity, is constant. Moreover, this ratio has the same value as the corresponding one evaluated for the incident wave. The orbital angular momentum complements the spin angular momentum and in some directions may carry all the angular momentum. The orbital angular momentum carried by the scattered field mediates the transfer of angular momentum to the medium.