Massless hook field in $AdS_{d+1}$ from the holographic perspective

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Abstract

We systematically consider the AdS/CFT correspondence for a simplest mixed-symmetry massless gauge field described by hook Young diagram. We introduce the radial gauge fixing and explicitly solve the Dirichlet problem for the hook field equations. Solution finding conveniently splits in two steps. We first define an incomplete solution characterized by a functional freedom and then impose the boundary conditions. The resulting complete solution is fixed unambiguously up to boundary values. Two-point correlation function of hook primary operators is found via the corresponding boundary effective action computed separately in even and odd boundary dimensions. In particular, the higher-derivative action for boundary conformal hook fields is identified with a singular part of the effective action in even dimensions. The bulk/boundary symmetry transmutation within the Dirichlet boundary problem is explicitly studied. It is shown that traces of boundary fields are Stueckelberg-like modes that can be algebraically gauged away so that boundary fields are traceless.

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Elementary particles in higher-dimensional $AdS_{d+1}$ spacetime with $d \geq 4$ are characterized by more than one spin number. Such mixed-symmetry particles, both (partially)-massless and massive, naturally appear in the spectra of two important classes of theories, string theory (in particular, strings on the $AdS_5 \times S^5$ background geometry) and higher spin theory [1, 2, 3]. Studying mixed-symmetry field dynamics brings to light many interesting and still purely understood issues and one of them is related to the $AdS_{d+1}/CFT_d$ correspondence for the bulk massless higher spin fields and their dual conformal description on the boundary.

Among all massless fields of general symmetry type there is a distinguished subset of fields of particular "hook" symmetry type described by Young diagrams with one row of arbitrary length $s$ and one column of height $p \leq \frac{d}{2}$. These are fields that appear in $d$-dimensional extension of original Flato-Frønsdal theorem [4] proved recently by Vasiliev: a tensor product of two spinor singletons decomposes into an infinite direct sum of bosonic massless fields of mixed-symmetry hook type and a finite set of massive totally antisymmetric fields including a massive scalar [5]. By analogy with totally symmetric fields arising in the tensor product of two scalar singletons, the Flato-Frønsdal theorem for two spinor singletons provides a group-theoretical foundation of the $AdS_{d+1}/CFT_d$ correspondence in the mixed-symmetry case as well. In particular, one may develop a general formulation of (not necessarily conformal) hook type conserved currents in $d$-dimensional flat space and establish a precise correspondence between mixed-symmetry gauge fields in the bulk and particular class of conserved currents built of two spinors living on the boundary [6].

From the field-theoretical perspective, a consideration of mixed-symmetry fields via the Gubser-Klebanov-Polyakov and Witten procedure [7] is available now for massless mixed-symmetry fields in $AdS_5$ [8] and for massive mixed-symmetry fields in $AdS_{d+1}$ [9]. Massless mixed-symmetry fields in $AdS_{d+1}$ spaces of generic dimension had not been considered holographically and one of the main goals of the present paper is to initiate the study.

In doing this we consider the metric-like quadratic action for a simplest $s = 2, p = 2$ massless hook field in $AdS_{d+1}$ proposed in [1] and apply the Gubser-Klebanov-Polyakov and Witten procedure. Our consideration is similar to those done in the lower spin cases in [7, 17, 18, 19, 20, 21]. In particular, we use Euclidean version of $AdS_{d+1}$ spacetime which symmetry algebra is $o(d + 1, 1)$ and introduce the infrared cutoff parameter $\epsilon$. The cutoff displaces the boundary into the bulk thereby allowing to formulate the Dirichlet boundary problem. Then, in order to solve the field equations it is more convenient to consider theory in momentum boundary space so that all equations become algebraic with respect to boundary momenta. Finally, we define and compute the boundary effective action.

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1 This statement is a generalization of the theorem established by Flato and Frønsdal in $d = 3$ case [4]. However, in this case the tensor product of two singletons (spinor and/or scalar) contain totally symmetric fields only.

2 From the holographic perspective, totally symmetric fields of arbitrary spin were originally considered in [2, 10], and recently in [11]. See also recent papers on the holographic description of totally symmetric fields within the unfolded formulation [12, 13, 14, 15].

3 Contrary to the frame-like formulation that generalizes the frame formulation of the gravity to higher spin theories (for review see, e.g., [16]) we work with higher spin metric-like fields which generalize the standard gravitational metric field.
While the general strategy is similar to the previously considered cases, there is one ingredient that is definitely new for mixed-symmetry dynamics. Indeed, considering the duality for massless mixed-symmetry fields one faces an important peculiarity which can be generally described as the Brink-Metsaev-Vasiliev (BMV) mechanism \[\text{[1]}\]. It says that an irreducible massless mixed-symmetry \(AdS_{d+1}\) field decomposes in the flat limit into a collection of irreducible mixed-symmetry Minkowski \(\mathbb{R}^{d,1}\) fields. For instance, a simplest massless mixed-symmetry \(AdS_{d+1}\) field of hook symmetry type decomposes into a massless hook field and a massless graviton-like field in \(\mathbb{R}^{d,1}\). In other words, the BMV mechanism implies that a given spin mixed-symmetry massless field in \(AdS_{d+1}\) has more PDoF than its the same spin massless cousin in \(\mathbb{R}^{d,1}\), or, equivalently, less gauge symmetries. \[\text{[2]}\]

Let \(Y_{o(m)}(s, p-1)\) be an irreducible module of \(o(m)\) algebra described by Young diagram of hook symmetry type, where \(s\) and \(p\) are length and height. Recalling that a little Wigner algebra in \(\mathbb{R}^{d,1}\) is \(o(d-1)\) we conclude that PDoF carried by the simplest hook field \(s = 2\) and \(p = 2\) are described as the direct sum,

\[
Y_{o(d-1)}(2, 1) \oplus Y_{o(d-1)}(2, 0).
\]  

(1.1)

Note that for totally symmetric spin-\(s\) fields (and, more generally, for arbitrary rectangular diagrams) the PDoF are described by a single representation \(Y_{o(d-1)}(s, 0)\), so that the BMV mechanism is absent in this case.

It follows that due to BMV mechanism an application of the standard holographic prescription is a somewhat delicate procedure. This happens because tensors describing PDoF are identified with initial values for propagating fields in the bulk and the boundary problem makes this identification manifest. Indeed, original Lorentz \(o(d, 1)\) bulk mixed-symmetry hook field can be decomposed into \(o(d-1, 1)\) components that is convenient from the boundary perspective because their initial values are Lorentz fields in \(d\)-dimensional Minkowski space. For the simplest hook field there appear several \(o(d-1, 1)\) irreducible components including those described by \(Y_{o(d-1,1)}(2, 1)\) and \(Y_{o(d-1,1)}(2, 0)\) diagrams. In their turn each of these two components contains a smaller component described by \(Y_{o(d-1)}(2, 0)\) diagram. However, having not solved the field equations explicitly, it is problematic to say which of \(o(d-1, 1)\) components (or their linear combination) of the original bulk field contains an \(o(d-1)\) graviton-like component to be identified with the second term in the PDoF decomposition \[\text{[3]}\]. The final answer obtained in this paper is that PDoF described by \([\text{1}]\) are contained in a hook \(o(d-1, 1)\) component of the original bulk field, while other symmetry type components are proportional to the hook initial values.

A related question concerns two-point correlation functions that follow from the holographic effective action: though conformal dimensions of the boundary conformal operators are entirely fixed by the group theory of \(o(d-1, 2)\) algebra, a set of shadow fields (which are complementary to conformal primary fields) involved in the effective boundary action crucially depend on a bulk formulation. For instance, the study of \(AdS_{5}\) massless mixed-symmetry fields in Ref. \[\text{[3]}\] gives a set of shadow fields being packaged in a single generating

\footnote{Of course, one can always introduce auxiliary algebraic symmetries to obtain a formulation analogous to Stueckelberg formulation of massive electrodynamics (here the role of the cosmological constant is played by the mass). A Stueckelberg like formulation possesses an equal number of physical degrees of freedom before and after the flat limit is taken.}
scalar function so that the resulting effective action yields a collection of two-point correlators with conformal dimensions varying in a range. However, revealing a prescribed conformal dimension guaranteed by the symmetry arguments is not manifest at all and requires an additional analysis. In our case, there is a single shadow field described by an $o(d - 1, 1)$ traceless hook tensor so that respective two-point correlation function takes the standard form [1] and conformal dimension is manifest and equals the vacuum energy of the bulk hook field [22].

The organization of the paper is as follows. Since a detailed consideration rapidly becomes highly technical, in particular, solution finding and computing the effective action, we have relegated most technicalities to the appendices. The body of the paper mainly discusses final results and contains almost no intermediate computations. This is why we describe sections and appendices mixed up together.

In Section 2 we briefly recall a few basic facts about $AdS_{d+1}$ hook field dynamics and describe the quadratic action for a massless hook $AdS_{d+1}$ field proposed in [1]. In particular, in Section 2.1 we introduce a gauge invariant total derivative term so that the full quadratic action contains two free parameters [23]. Then, in Section 2.2 we obtain the hook field equations that follow from the action and derive their differential consequences to be referred further to as constraints. These constraints are analogous to the Lorentz gauge in the massive electrodynamics though in the present case the hook field system is more complicated because of non-trivial gauge symmetry. Also, in Section 2.3 we evaluate the on-shell value of the action.

In Appendix B we rewrite the field equations and constraints in the form suitable for solution finding. To this end, in Section 3 we decompose original Lorentz $o(d, 1)$ hook gauge field into $o(d)$ components and impose gauge fixing conditions that generalize the radial gauge in the gravitation. Then, in Section 4 all fields are Fourier transformed and rescaled in the way all covariant derivatives in the field equations are replaced by comma derivatives and all contractions of indices are performed with respect to Euclidean metrics. It turns out that this trick greatly simplifies the whole analysis. The resulting field equations and constraints are given in Section B.1.

In Section 4.1 we propose to view solution finding as split into two parts: incomplete solution followed by complete solution. The incomplete solution solves a part of the equation system. As a result, it fixes $z$-dependence only (here $z$ is the radial direction in the Poincare coordinates) and contains a functional freedom. The complete solution is obtained by solving the remaining part of the equation system. Also, the complete solution involves particular boundary conditions which fix completely all arbitrary functions entering the incomplete solution up to boundary values of fields. The incomplete solution and complete solution are analyzed in Appendices C and D, respectively.

In section 5 we formulate the Dirichlet boundary problem: all fields are required to take fixed values at the conformal boundary displaced at small but finite distance $z = \epsilon$ from its true position at $z = 0$. The problem has a unique solution parameterized by a single hook traceless $o(d)$ boundary tensor. All other components of the bulk field either vanish identically on the boundary or tend to zero asymptotically at $\epsilon \to 0$. It is interesting to note that a trace of the boundary hook component can be non-zero but the leftover gauge symmetry analysis done in Section 7.2 shows that the trace can be gauged away by
a Stueckelberg-like symmetry transformation. The final form of the complete solution is given in Section 5.1, while its boundary behavior is analyzed in Section 5.2.

In Section 6 we consider the effective action and associated two-point correlation function. In particular, we explicitly analyze $\epsilon$-decomposition of the on-shell value of the bulk action and identify the two-point correlation function with its non-local part. The analysis is done separately for two cases of even and odd boundary dimensions. In particular, for even boundary dimensions we regularize the ill-defined kernel of the effective action and find the higher-derivative action for conformal fields previously known in the literature [24].

In Section 7 we analyze the bulk/boundary global and local symmetry transmutation. We find that boundary hook tensor is a shadow field with a correct conformal dimension, $\Delta_s = 0$, so that a complementary conformal primary field has a dimension $\Delta_p = d$. Moreover, we analyze the leftover gauge symmetry transformation and find out that its derivative part is accompanied by Stueckelberg-like terms that can be used to gauge away the traces of the boundary value tensor.

In Appendix A we collect main formulae related to Poincaré parametrization of $\text{AdS}_{d+1}$ spacetime. Appendix E contains discussion of modified Bessel functions and their small argument decompositions for (non-)integer orders. In Appendix G we discuss Fourier transformations for certain type functions in momentum space. In particular, we derive $d$-dimensional generalization of 4$d$ differential regularization scheme proposed in [25].

2 Action for spin-$(2, 1)$ field in $\text{AdS}_{d+1}$

The Euclidean version of $\text{AdS}_{d+1}$ spacetime is globally described as an upper-half space with the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = R^2 \frac{dz dz + dx^i dx^i}{z^2}, \quad (2.1)$$

where $x^\mu = (x^0 \equiv z, x^i)$ are Poincaré coordinates, $\mu = 0, 1, ..., d$ and $i = 1, ..., d$, while $R$ is a radius of $\text{AdS}_{d+1}$ (starting from Section 2.3 we set $R = 1$). The conformal boundary is isomorphic to $d$-dimensional sphere being a compactification of the plane $z = 0$ by the single point $z = \infty$. In what follows, we displace the boundary $z = 0$ into the bulk and introduce the cutoff parameter $\epsilon$ so that a radial coordinate runs $z \in [\epsilon, \infty)$.

The global symmetries of Euclidean $\text{AdS}_{d+1}$ space are organized into $o(d+1, 1)$ algebra. We consider those modules of $o(d+1, 1)$ that correspond to unitary modules $\mathcal{D}(E_0| s, p - 1)$ of $o(d, 2)$ isometry algebra of the standard $\text{AdS}_{d+1}$ spacetime (with Lorentz signature) [22]. The modules have spins $s$ and $p$ represented as Young diagrams $Y_{o(d)}(s, p - 1)$ with one row of length $s$ and one column of height $p$, and the vacuum energy

$$E_0 = s + d - 2. \quad (2.2)$$

Note that the same value of the energy holds for totally symmetric fields because the general formula for $E_0$ depends on the length of the uppermost rectangular block only. As a consequence, the energies of length-$s$ hook and totally symmetric fields are the same. In

\footnote{Christoffel symbols as well as Riemann, Ricci and scalar curvatures associated with the metric (2.1) are given in Appendix A.}
particular, for the spin $s = 2$, $p = 2$ hook particle we obtain the energy $E_0 = d$, which value coincides with that of the $AdS_{d+1}$ graviton field.

Let $\varphi_{\mu\nu,\rho}(x)$ be $o(d,1)$ Lorentz tensor field with index permutation symmetry corresponding to the hook Young diagram with two cell in the first row and a single cell in the second row, i.e., $\varphi_{\mu\nu,\rho}(x) = \varphi_{\nu\mu,\rho}(x)$ and $\varphi_{\mu\nu,\rho}(x) + \varphi_{\mu\rho,\nu}(x) + \varphi_{\nu\rho,\mu}(x) \equiv 0$. Its gauge symmetry transformation is given by

$$\delta \varphi_{\mu\nu,\rho}(x) = \nabla_\mu \chi_{\nu\rho}(x) + \nabla_\nu \chi_{\mu\rho}(x),$$  

(2.3)

where $\nabla_\mu$ is a covariant derivative evaluated with respect to the background metric (2.1) (see Appendix A), and $\chi_{\nu\rho}(x)$ is antisymmetric gauge parameter, $\chi_{\nu\rho}(x) = -\chi_{\rho\nu}(x)$. The hook gauge field with the transformation law (2.3) corresponds to unitary module $\mathcal{D}(d|2,1)$ discussed above.

### 2.1 Quadratic action

The gauge invariant action for the hook field has been originally formulated by Brink, Metsaev and Vasiliev within the metric-like approach [1]. Up to total derivative terms discussed below the action is given by

$$S_0 = \frac{h_0}{2} \int d^{d+1}x \sqrt{g} \left( \nabla_\lambda \varphi_{\mu\nu,\rho} \nabla_\lambda \varphi_{\mu\nu,\rho} - \frac{3}{2} \nabla_\lambda \varphi_\mu \nabla_\lambda \varphi_\mu - 2 \nabla_\mu \varphi_{\mu\nu,\rho} \nabla_\lambda \varphi_{\lambda\nu,\rho} + \right.$$ 

$$- \nabla_\rho \varphi_{\mu\nu,\rho} \nabla_\lambda \varphi_{\mu\nu,\lambda} + 3 \nabla_\nu \varphi_\mu \nabla_\rho \varphi_{\nu\rho,\mu} + \frac{3}{2} \nabla_\mu \varphi_\mu \nabla_\nu \varphi_\nu + \frac{3}{2} \nabla_\nu \varphi_\nu \nabla_\rho \varphi_{\rho\mu,\nu} + \frac{3}{2} \varphi_{\mu\nu,\rho} \varphi_{\rho\mu,\nu} - \frac{3}{2} \varphi_{\mu\nu,\rho} \varphi_{\rho\mu,\nu} + \right.$$ 

$$\left. 3 \frac{d-4}{2R^2} \varphi_{\mu\nu,\rho} \varphi_{\rho\mu,\nu} \right),$$  

(2.4)

where the field $\varphi_\rho = g^{\mu\nu} \varphi_{\mu\nu,\rho}$ denotes the trace, and $h_0$ is an arbitrary dimensionless normalization constant.

Action (2.4) is invariant under gauge transformations (2.3). It is remarkable that in the flat limit $R \to \infty$ action (2.4) exhibits the gauge symmetry enhancement. In addition to (2.3) it becomes invariant with respect to new gauge transformations,

$$\delta \varphi_{\mu\nu,\rho}(x) = 2 \partial_\rho S_{\mu\nu}(x) - \partial_\mu S_{\nu\rho}(x) - \partial_\nu S_{\mu\rho}(x),$$  

(2.5)

with traceful symmetric parameter, $S_{\mu\nu}(x) = S_{\nu\mu}(x)$. It follows that in the flat limit action (2.4) reproduces the action for a hook massless field obtained by Curtright [26].

Allowing for total derivative terms, one observes that the most general form of quadratic action for the hook field reads

$$S = S_0 + S_1 \equiv S_0 + \frac{h_1}{2} \int d^{d+1}x \sqrt{g} \mathcal{O}, \quad \mathcal{O} = \nabla_\lambda U^\lambda,$$

(2.6)
where $S_0$ is given by (2.4), while vector $U^\nu$ and its divergence are 28

$$U^\lambda = \varphi_\mu \nabla_\nu \varphi^{\lambda,\mu} + 2\varphi_\mu \nabla_\rho \varphi^{\mu,\rho} \text{ , } \quad \mathcal{O} = \nabla_\lambda \varphi_\mu \nabla_\rho \varphi^{\lambda,\mu} + 2\nabla_\lambda \varphi_\mu \nabla_\rho \varphi^{\mu,\rho} + \frac{3}{2} \varphi_\mu \varphi^\mu .$$

(2.7)

Coefficient $h_1$ is an arbitrary dimensionless constant. It is worth noting that adding total derivative $\mathcal{O}$ keeps all gauge invariances intact, both on the AdS background and in the flat limit.

### 2.2 Equations of motion and constraints

Equations of motion that follow from action (2.4) can be represented as

$$\frac{\delta S_0}{\delta \varphi_{\mu\nu,\rho}} \equiv \mathcal{E}_{\mu\nu,\rho} = 0 ,$$

(2.8)

where variation $\mathcal{E}_{\mu\nu,\rho} = \mathcal{E}_{\mu\nu,\rho}(\varphi, \partial \varphi)$ denotes a resulting second order combination of hook fields. One can also consider its trace, $\mathcal{E}_\rho \equiv g^{\mu\nu} \mathcal{E}_{\mu\nu,\rho}$.

From the discussion of the flat space symmetry enhancement (2.5) in the previous section it follows that the hook field theory in $AdS_{d+1}$ resembles the massive electrodynamics in $\mathbb{R}^{d,1}$. Namely, taking the massless limit in the Proca theory one observes the gauge symmetry enhancement so that the resulting theory is the Maxwell electrodynamics. It implies that in the massive regime one obtains the Lorentz condition $m^2 \partial_\mu A^\mu = 0$ just by taking the divergence on the field equations. In the massless limit the Lorentz condition turns into the Noether identity for the Maxwell gauge symmetry. The parameter of mass in the electrodynamics is to some extent analogous to the cosmological constant in the hook field theory. It follows that there are constraints on massless hook fields in $AdS_{d+1}$ analogous to the Lorentz condition on massive vector fields in $\mathbb{R}^{d,1}$.

Indeed, taking the divergence on the field equations $\nabla^\rho \mathcal{E}_{\mu\nu,\rho}$ one obtains the following differential constraints

$$2\nabla^\lambda \varphi_{\mu\nu,\lambda} + \nabla_\mu \varphi_\nu + \nabla_\nu \varphi_\mu = 0 , \quad \nabla^\lambda \varphi_\lambda = 0 .$$

(2.9)

Here the later condition is the trace of the former one. One may explicitly check that the above equations are gauge invariant.

Using trace equations $\mathcal{E}_\rho = 0$ and constraints (2.9), the original fields equation $\mathcal{E}_{\mu\nu,\rho} = 0$ (2.8) can be simplified so that the resulting equations take the form,

$$\nabla^2 \varphi_{\mu\nu,\rho} - \nabla_\mu \nabla^\lambda \varphi_{\lambda\nu,\rho} - \nabla_\nu \nabla^\lambda \varphi_{\lambda\mu,\rho} + \frac{1}{2} \nabla_\mu \nabla_\nu \varphi_\rho + \frac{1}{2} \nabla_\nu \nabla_\mu \varphi_\rho +$$

$$+ \frac{3}{R^2} \varphi_{\mu\nu,\rho} - \frac{1}{R^2} (2g_{\mu\nu} \varphi_\rho - g_{\mu\rho} \varphi_\nu - g_{\nu\rho} \varphi_\mu) = 0 ,$$

(2.10)

where $\nabla^2 = \nabla^\lambda \nabla_\lambda$. Taking the trace $g_{\mu\nu}$ of (2.10) yields

$$\nabla^2 \varphi_\gamma - \nabla^\alpha \nabla_\alpha \varphi_{\alpha\beta,\gamma} + \frac{3 - 2d}{2R^2} \varphi_\gamma = 0 .$$

(2.11)
Young symmetry combination of the field equations (2.10) is proportional to the first derivative of the first constraint in (2.9). Therefore, denoting the field equations (2.10) as \( E_{\mu \nu, \rho} \) and constraints (2.9) as \( T_{\mu \nu} \) and \( T \), we have \( E_{\mu \nu, \rho} + E_{\mu \rho, \nu} + E_{\nu \rho, \mu} = \nabla_\mu T_{\nu \rho} + \nabla_\nu T_{\rho \mu} + \nabla_\rho T_{\mu \nu} \approx 0 \). Also, \( g^{\mu \nu} T_{\mu \nu} = 4 T \).

The system of field equations (2.10) supplemented with constraints (2.9) correctly describes the hook field dynamics in \( AdS_{d+1} \) spacetime. The form of dynamical equations obtained in this section is the starting point of further analysis.

### 2.3 On-shell value of the action

Total action (2.6) admits an equivalent representation

\[
S = \frac{1}{2} \int d^{d+1} x \sqrt{g} \left( h_0 \nabla_\lambda V^\lambda + h_1 \nabla_\lambda U^\lambda + \varphi^{\mu \nu, \rho} E_{\mu \nu, \rho} \right),
\]

where \( E_{\mu \nu, \rho} \) is the left-hand-side of the field equations (2.8), vector \( U^\lambda \) is given in (2.7) and vector \( V^\lambda \) is given by

\[
V^\lambda = \varphi^{\mu \nu, \rho} \nabla_\lambda \varphi_{\mu \nu, \rho} - \frac{3}{2} \varphi_\mu \nabla_\lambda \varphi^\mu - 2 \varphi^{\lambda \nu, \rho} \nabla_\mu \varphi_{\mu \nu, \rho} +
\]

\[
- \varphi^{\nu \rho, \lambda} \nabla_\rho \varphi_{\mu \nu, \rho} + \frac{3}{2} \varphi_\mu \nabla_\rho \varphi^{\rho \mu, \nu} + \frac{3}{2} \varphi^{\lambda \rho, \mu} \nabla_\rho \varphi_\mu + \frac{3}{2} \varphi^{\rho \nu} \nabla_\rho \varphi^\nu.
\]

It follows that on-shell value of action (2.12) is given by

\[
S \approx \frac{1}{2} \int d^{d+1} x \sqrt{g} \left( h_0 \nabla_\nu V^\nu + h_1 \nabla_\nu U^\nu \right) = \frac{\epsilon^{-d}}{2} \int d^d x \left( h_0 V_\mu + h_1 U_\mu \right) n^\mu \bigg|_{z=\epsilon}
\]

\[
= - \frac{\epsilon^{-d+1}}{2} \int d^d x \left( h_0 V_0 + h_1 U_0 \right) \bigg|_{z=\epsilon},
\]

where \( \approx \) means on-shell equality, while the boundary is placed at \( z = \epsilon \) and \( n^\mu = (-z, 0, ..., 0) \) is orthogonal to the boundary unit vector pointing outward. Expression (2.14) will be used in computing the effective boundary action and associated two-point correlation functions in Section 6.

### 3 Radial gauge conditions

In order to implement the Gubser-Klebanov-Polyakov and Witten prescription [7] one should find a solution to the field equations provided certain boundary conditions imposed on the bulk fields. For lower spin gauge fields formulated within the metric-like approach the most straightforward way to achieve the goal is to use a gauge fixing similar to the radial gauge \( h_{0 \mu} = 0 \) in the gravitation theory, see, e.g., [17, 18, 19, 20]. In what follows, we show that using the radial gauge fixing for hook fields is still an efficient tool though for higher spin fields of arbitrary shape it may get complicated.
To impose the radial gauge one decomposes $o(d, 1)$ tensor field $\varphi_{\mu\nu,\rho}(x)$ into $o(d)$ components. The resulting list contains the following components:

| Component Type   | Field Expression |
|------------------|------------------|
| Hook component   | $\varphi_{ij,k}$ |
| Symmetric component | $\varphi_{ij,0} \equiv \varphi_{ij}$ |
| Antisymmetric component | $\varphi_{0[i,j]} = \varphi_{0i,j} - \varphi_{0j,i}$ |
| Vector component  | $\varphi_{00,j}$ |

where $i, j = 1, ..., d$. The hook $\varphi_{ij,k}$ and symmetric $\varphi_{ij}$ components from the list (3.1) are traceful. Their traces will be denoted as

$$g^{ij} \varphi_{ij,k} \equiv \varphi_k, \quad g^{ij} \varphi_{ij} \equiv \varphi_0,$$

where metric tensor $g^{ij}$ stands for $o(d)$ part of the original metric $g^{\mu\nu}$, i.e., $g^{ij} = z^2 \delta^{ij}$ and $g_{ik}g^{kj} = \delta_i^j$. Recalling that the number of independent components for a hook tensor in $n$ dimensions equals $n(n^2 - 1)/3$ one can easily check the above decomposition just by summing up and comparing respective dimensions.

Now, using gauge transformations with antisymmetric gauge parameters (2.3) one imposes the radial gauge condition,

$$\varphi_{0[\mu,\nu]} = 0, \quad \text{or, equivalently,} \quad \varphi_{0[i,j]} = \varphi_{00,j} = 0.$$  

Obviously, the number of imposed gauge conditions is equal to the total number of independent components of antisymmetric and vector $o(d)$ tensors (3.1). The remaining gauge fields are hook $\varphi_{ij,k}$ and symmetric $\varphi_{ij}$ tensor fields as well as their traces $\varphi_k$ and $\varphi_0$, four different $o(d)$ tensor fields in total. It is worth noting that the radial gauge fixing (3.3) is incomplete so that there exists a leftover gauge symmetry analyzed in Section 7.2.

### 4 Solving the equations of motion

Besides imposing convenient gauge fixing conditions a proper treatment of the equations also involves factoring out $z^2$-factors from the $\partial^\mu$-derivatives and fields, along with using momentum representation with respect to $x^i$-coordinates. Indeed, to simplify consideration it is convenient to split covariant derivative $\nabla_\mu$ into a comma derivative $\partial_\mu$ and Christoffel coefficients $\Gamma_\mu$. Then, one re-scales derivatives $\partial^\mu = g^{\mu\nu} \partial_\nu$ by $z^2$-factor contained in the metric so that indices are lowered and raised via $\delta_{\mu\nu}$. It follows that using the radial gauge (3.3) the remaining fields are redefined as

$$\varphi_{ij,k}(z, x) \rightarrow \varphi_{ij,k}(z, x), \quad \varphi_{ij}(z, x) \rightarrow \varphi_{ij}(z, x), \quad \varphi_i(z, x) \rightarrow z^2 \varphi_i(z, x), \quad \varphi_0(z, x) \rightarrow z^2 \varphi_0(z, x),$$

while

$$\partial_z = \frac{\partial}{\partial z}, \quad \partial^i = \partial_i = \frac{\partial}{\partial x^i}, \quad \text{and} \quad \Box = \partial^i \partial_i.$$
Now, denoting rescaled fields (4.1) as \( \tilde{\Phi}(z, x) \) (suppressing the indices) we introduce their Fourier transform images as

\[
\tilde{\Phi}(z, x) = \int \frac{d^dx}{(2\pi)^d} e^{-ik\cdot x}\Phi(z, k), \quad \Phi(z, k) = \int d^dxe^{ik\cdot x} \tilde{\Phi}(z, x), \quad (4.3)
\]

where contraction \( k \cdot x = \delta_{ij}k^i x^j \) is evaluated with respect to Euclidean metric. In other words, interpreting Euclidean \( AdS_{d+1} \) spacetime as a stack of Euclidean spaces \( \mathbb{R}^d \) of varying size, we perform standard Fourier transformation for a given fixed value of \( z \)-coordinate that otherwise is not applicable in global \( AdS_{d+1} \) spacetime.

### 4.1 Incomplete and complete solutions

Detailed analysis of the field equations and constraints starts with rewriting all equations according to index splitting \( \mu, \nu, \ldots \) to 0 and \( i, j, \ldots \). All tensor fields are Fourier transformed \( \Phi = \Phi(z, k) \) (4.3) and comma derivatives are given by \( \partial_m = ik_m \). The resulting component equations (B.1) - (B.13) given in Appendix B form the system of 1st and 2nd order ordinary differential equations in \( z \)-variable. With respect to momenta \( k^i \) all equations are algebraic.

While a part of these equations (both 1st and 2nd order equations) are directly solved via elementary functions, treating another part (2nd order equations for symmetric and hook components) is more tricky. Indeed, there are 2nd order equations which are Bessel equations with non-vanishing right-hand-side part. Using collective notation \( \Phi(z, k) \) one may represent those equations as follows,

\[
\left[ \partial_z^2 + \frac{\alpha}{z} \partial_z + \frac{\beta}{z^2} + \gamma \right] \Phi(z, k) = \Upsilon(z, k), \quad (4.4)
\]

where coefficients \( \alpha, \beta, \gamma \) define the Bessel equation on the left-hand-side, while the right-hand-side is given by some tensor function \( \Upsilon \) expressed via various tensor components. It is suggested to search for a solution to such equations via the traceless and transverse (TT) decomposition,

\[
\Phi(z, k) = \tilde{\Phi}(z, k) + \Xi\left(\Phi(z, k), k^m, \frac{1}{k^2}\right), \quad (4.5)
\]

where the first term is a TT component, while the second term is a non-analytical at zero momenta combination involving original field \( \Phi(z, k) \) and momenta \( k^m \). Explicit computation shows that a TT component solves the homogeneous Bessel equation,

\[
\left[ \partial_z^2 + \frac{\alpha}{z} \partial_z + \frac{\beta}{z^2} + \gamma \right] \Phi(z, k) = 0. \quad (4.6)
\]

Changing variables one finds that depending on coefficients in (4.4) a TT component is the modified Bessel function of a given order \( \tilde{\Phi}(z, k) = z^\mu K_\nu(\mu k) \bar{\Phi}(k) \), where \( \Phi(k) \) is some boundary TT tensor, and \( \mu \) and \( \nu \) are fixed numbers.

When solving the hook equation system we use the following strategy.

- Firstly, we consider a part of the equation system and find solutions for each of tensor components \( \varphi_{ij,k}, \varphi_{ij} \) and their traces \( \varphi_i, \varphi_0 \) (4.1) modulo some coefficients that...
are arbitrary functions of momenta. Such a solution completely fixes $z$-dependence but $k^m$-dependence is mainly ambiguous. Since the resulting solution contains a functional freedom it will be referred to as an incomplete solution. It is explicitly considered in Appendix C.

Singling out the incomplete solution is preliminary to finding the complete solution. Its most important feature is that boundary conditions are not yet imposed.

- Secondly, using a remaining part of the equation system we find the final solution provided particular boundary conditions. Namely, we formulate the Dirichlet boundary problem. It follows that all undetermined functions entering the incomplete solution are expressed via a single boundary tensor of hook symmetry type, see Section 5.

In particular, it follows that arbitrary functions parameterizing the incomplete solution may get dependent on extra parameters which characterize particular boundary conditions. For instance, introducing the cutoff $\epsilon$ is irrelevant to finding incomplete solution but it appears later on when setting the Dirichlet problem.

5 Boundary value problem

A list of incomplete solutions found in Appendix C reads ($\nu = d/2$)

$$\varphi_0(z, k) = \frac{C(\epsilon, k)}{z^2}, \quad \varphi_i(z, k) = \frac{A_i(\epsilon, k)}{z^2} + \frac{C(\epsilon, k)}{z^2},$$

$$\varphi_{ij}(z, k) = z^{\nu-2} K_{ij}(z k) F_{ij}(\epsilon, k) + \ldots,$$

$$\varphi_{ij,k}(z, k) = z^{\nu-3} K_{ij}(z k) G_{ij,k}(\epsilon, k) + \ldots,$$  

(5.1)

where expressions for traces are exact, while those for symmetric and hook components display only TT parts, and the dots denote a non-analytical contribution of the type (4.5). Other field components listed in (3.1) vanish by virtue the radial gauge fixing (3.3). In order to find exact values of unspecified boundary tensors $A_i(\epsilon, k), B_i(\epsilon, k), C(\epsilon, k),$ and TT boundary tensors $F_{ij}(\epsilon, k), G_{ij,k}(\epsilon, k)$ (along with not displayed in (5.1) arbitrary tensor $a_{ij}(\epsilon, k)$ entering a non-analytical part in the hook component, see (C.16)) one have to fix one or another set of boundary conditions. In what follows, we impose the Dirichlet conditions.

Namely, we claim that hook component $\varphi_{ij,k}(z, k)$ is to be fixed on the boundary while other fields are required not blowing up at $\epsilon \to 0$. Then, the only consistent way to fulfill this boundary condition is to require that boundary values of all components are expressed in terms of unconstrained boundary value of the hook component,

$$\varphi_{ij,k}(\epsilon, k) \equiv \pi_{ij,k}(k), \quad \text{with vanishing trace} \quad \delta^{ij} \pi_{ij,k}(k) = 0.$$  

(5.2)

One can also leave the boundary value of the trace non-vanishing. However, as discussed in the end of section 7.2 there exists the leftover gauge symmetry on the boundary of Stueckelberg type that allows one to gauge the trace away.
5.1 Complete solution

The final answer for various tensor components computed in Appendix D is the following.

- Trace components, formulas (D.3), (D.14):
  \[ \varphi_0(z, k) = 0, \quad \varphi_i(z, k) = \frac{\epsilon^3}{4(d-2)} \left[ \frac{1}{z} - \frac{\epsilon^2}{z^3} \right] k^m k^n \pi_{mn,i}(k). \] (5.3)

- Symmetric component, formula (D.10):
  \[ \varphi_{ij}(z, k) = -i \left( \frac{\epsilon}{z} \right)^2 \frac{\epsilon K_\nu(zk)}{W(\epsilon k) K_\nu(\epsilon k)} \left( k^m \pi_{ij,m}(k) + \frac{k^m k^n}{2k^2} (k_i \pi_{mn,j}(k) + k_j \pi_{mn,i}(k)) \right) \]
  \[ - \frac{i\epsilon}{2(d-2)} \left( \frac{\epsilon}{z} \right)^2 \frac{k^m k^n}{k^2} (k_i \pi_{mn,j}(k) + k_j \pi_{mn,i}(k)), \] (5.4)
where functions $K_\nu$ and $W$ defined in (E.2) are combinations of modified Bessel functions.

- Hook component, formula (D.16):
  \[ \varphi_{ij,k}(z, k) = \left( \frac{\epsilon}{z} \right)^3 \frac{K_\nu(zk)}{K_\nu(\epsilon k)} \tilde{\varphi}_{ij,k}(k) + \cdots, \] (5.5)
where $\tilde{\varphi}_{ij,k}(k)$ is a TT part (D.13) of the boundary hook tensor $\pi_{ij,k}(k)$, while the dots stand for the terms (a couple of dozens of them) that define a non-analytical contribution of the type (4.5).

5.2 Behavior at the boundary

Let us examine the complete solution in the limit $\epsilon \to 0$.

- The boundary value of the trace component (5.3) vanishes identically,
  \[ \varphi_i(\epsilon, k) = 0, \] (5.6)
what agrees with tracelessness of the hook field boundary value $\varphi_{ij,k}(\epsilon, k) = \pi_{ij,k}(k)$. A scalar component of the trace is zero for any values of coordinate $z$, see (5.3).

- From (5.4) we find that the symmetric component tends to zero on the boundary,
  \[ \varphi_{mn}(\epsilon, k) = \frac{i\epsilon}{(d-2)} k^l \pi_{mn,l}(k) + \cdots \bigg|_{\epsilon=0} = 0. \] (5.7)
Note that its trace vanishes as well.
Substituting $z = \epsilon$ into the hook component solution (5.3) and using definition (D.13) one reproduces the original boundary condition (5.2),

$$\varphi_{ij,k}(\epsilon, k) \equiv \pi_{ij,k}(k).$$

(5.8)

All other components of the bulk hook field are set to zero by virtue of the radial gauge fixing (3.3).

The above analysis answers questions about PDoF discussed in Introduction. So, the boundary behavior of the hook field manifestly demonstrates that PDoF are carried by the hook component $\varphi_{ij,k}(x, k)$ only. It follows that a graviton-like mode arising via the BMV mechanism is an $o(d-1)$ symmetric component of $o(d)$ boundary tensor $\pi_{ij,k}(k)$. Put differently, formula (5.7) says that another candidate to contain a graviton-like mode vanishes on the boundary. In particular, it implies that the boundary effective action is expressed in terms of boundary hook tensors only.

6 Effective action and two-point correlation function

Let us consider on-shell quadratic action written in terms of the Fourier transform,

$$S \approx \int d^dx d^dy \int \frac{d^dk}{(2\pi)^d} e^{-ik(x-y)} I_{\epsilon,k}(x, y),$$

(6.1)

where the kernel $I_{\epsilon,k}(x, y)$ is some bi-local function in position space parameterized by momenta and the cutoff parameter. In general, the kernel is given by the Laurent series in the cutoff parameter, while expansion coefficients are not necessarily analytical functions at zero momenta. It can be represented as a sum of two groups of terms,

$$I_{\epsilon,k}(x, y) = N_{\epsilon,k}(x, y) + \text{Loc}_{\epsilon,k}(x, y),$$

(6.2)

where $\text{Loc}_{\epsilon,k}(x, y)$ denote all local terms that under Fourier transform give contact delta-function terms. Such terms can be consistently removed through the holographic renormalization procedure [37]. In other words, local terms given in momentum space are represented as $\sim (k^2)^m$, where orders $m$ are positive integers.

Explicit computation shows that all poles in the cutoff parameter have vanishing coefficients so that only the regular part of $N_{\epsilon,k}(x, y)$ remains starting from the term proportional to $\epsilon^6$. Therefore, we define the effective action as follows,

$$S_{\text{eff}} = \frac{1}{\epsilon^6} \int d^dx d^dy \int \frac{d^dk}{(2\pi)^d} e^{-ik(x-y)} N_{\epsilon,k}(x, y) \bigg|_{\epsilon \to 0}.$$  

(6.3)

The power of the cutoff parameter here indicates how quickly quantity $N_{\epsilon,k}(x, y)$ grows as the boundary surface shifts to the infinity.

Now consider the surface term in the quadratic action (2.14) defined by radial vector components $V_0$ (2.13) and $U_0$ (2.7). Using the gauge fixing (3.3) and the boundary behavior of the solution discussed in section 5.2 one finds that all terms in $U_0$ contribute to
$\text{Loc}, k_p x, y_q$, while the only term in $V_0$ that can contain non-local contributions is given by the normal derivative of the hook component,

$$S \approx -\frac{h_0 \epsilon^{-2\nu+7}}{2} \int d^d x d^d y \int \frac{d^d k}{(2\pi)^d} e^{-ik(x-y)} \varphi_{ij,k}(\epsilon, k) \partial_z \varphi_{ij,k}(z, k) \bigg|_{z=\epsilon}, \quad \nu = d/2 . \quad (6.4)$$

All other terms in $V_0$ belong to $\text{Loc}, k_p x, y_q$. In particular, it follows that overall constant $h_1$ entering the total action (2.6) falls out the resulting effective action.

### 6.1 Two-point correlation function for hook field

We assume that the effective action (6.3) inherited from (6.4) has the form

$$S_{\text{eff}} = C \int d^d x d^d y \left[ \pi^{ij,k}(x) \langle O_{ij,k}(x) O_{mn,l}(y) \rangle \pi_{mn,l}(y) \right], \quad (6.5)$$

where $C$ is a normalization constant, and two-point function $\langle O \rangle$

$$\langle O_{ij,k}(x) O_{mn,l}(0) \rangle = \frac{\Pi_{ij,kmn,l}(x)}{|x|^{2\Delta}}, \quad \Delta = d , \quad (6.6)$$

is contracted with two boundary traceless hook tensor fields $\pi^{ij,k}(x)$ being initial values for the bulk fields, and projector $\Pi_{ij,kmn,l}(x)$ is given by

$$\Pi_{ij,kmn,l} = (I_{im} I_{jn} + I_{in} I_{jm}) I_{kl}$$

$$-\frac{1}{2} (I_{im} I_{kn} + I_{in} I_{km}) I_{jl} - \frac{1}{2} (I_{jm} I_{kn} + I_{jn} I_{km}) I_{il} - \frac{2}{d-1} T_{ij,kmn,l} , \quad (6.7)$$

where matrix

$$I_{mn} = \delta_{mn} - 2 \frac{x_m x_n}{x^2} , \quad (6.8)$$

is proportional to Jacobi matrix for the inversion transformation $x^i \rightarrow x^i/x^2$ in $\mathbb{R}^d$, while trace part $T_{ij,kmn,l}$ is given by

$$T_{ij,kmn,l} = \delta_{ij} \delta_{mn} I_{kl} - \frac{1}{2} \delta_{mn} \left( \delta_{ik} I_{jl} + \delta_{jk} I_{il} \right)$$

$$-\frac{1}{2} \delta_{ij} \delta_{ml} I_{kn} + \frac{1}{4} \delta_{ml} \left( \delta_{ik} I_{jn} + \delta_{jk} I_{im} \right)$$

$$-\frac{1}{2} \delta_{ij} \delta_{nl} I_{km} + \frac{1}{4} \delta_{nl} \left( \delta_{ik} I_{jm} + \delta_{jk} I_{im} \right) . \quad (6.9)$$

The form of the two-point function (6.6) is completely fixed by conformal symmetry algebra $o(d+1,1)$ realized by up-to-scale isometries of the Euclidean metric in $\mathbb{R}^d$ space. Now we should justify the appearance of this function as a boundary value of the action. To this end, we discuss transformation properties of the bulk and boundary values of fields and action, see Section 7.
6.2 Odd boundary dimensions \( d \)

Consider first the case of odd dimensions \( d \) that corresponds to non-integer values of the modified Bessel function order \( \nu = d/2 \). In order to find the on-shell value of action (6.4) we have to compute a normal derivative \( \partial_z \varphi_{ij,k}(z,k)|_{z=\epsilon} \). Straightforward computation yields formulas collected in Appendix \( F \). As a result, we find that the normal derivative decomposes in degrees of the cutoff \( \epsilon \) as follows

\[
\partial_z \varphi_{ij,k}(z,k)|_{z=\epsilon} = \epsilon^{-1} \Lambda^{(-1)}_{ij,k}(k) + \epsilon \Lambda^{(1)}_{ij,k}(k) + \ldots \\
+ \epsilon^{2\nu-1} k^{2\nu} \Lambda^{(2\nu-1)}_{ij,k}(k) + \ldots,
\]

(6.10)

where the dots in the first and second lines stand for higher order terms, \( \mathcal{O}(\epsilon^3 k^4) \) and \( \mathcal{O}(\epsilon^{2\nu+1} k^{2\nu+2}) \), respectively. Functions \( \Lambda^{(\nu)}_{ij,k}(k) \) are some combinations of the boundary hook tensor and momenta. Note that the first term in (6.10) comes from differentiating ratios \( \epsilon/z \) raised to different powers which appear inside the solution (5.3). The second and third terms along with their higher order cousins come from differentiating \( \mathcal{O}(\epsilon^m k^{2n+1}) \) and \( \mathcal{O}(\epsilon^{2\nu+1} k^{2\nu+2}) \) terms in the decomposition of the logarithmic derivatives of various functions inside (D.10) computed using the modified Bessel function decomposition (E.3). Therefore, we have three types of terms that should be analyzed separately.

The reason for singling out the first three terms in (6.10) is twofold. First, these are lowest order terms in \( \epsilon \) inside their groups and therefore higher order contributions are negligible in the limit \( \epsilon \to 0 \). Second, the first two terms contain contributions singular in \( k \) as well as those proportional to even degrees of momenta, \( k^{2n} \), while all members of the third term contains odd degrees of momenta only, \( k^{d+2l} \) (recall \( \nu = d/2 \) and \( d \) is odd). Therefore, when considering decomposition (6.10) we control \( \epsilon \)-smallness and non-analyticity at zero momenta.

From expressions (F.3) - (F.11) one can read off the lowest order functions to obtain

\[
\Lambda^{(-1)}_{ij,k}(k) = 0, \quad \Lambda^{(1)}_{ij,k}(k) = 0,
\]

(6.11)

while \( k^{2\nu} \Lambda^{(2\nu-1)}_{ij,k}(k) \) defines the lowest order non-local contribution to the on-shell value of the action and thereby produce the searched-for two-point correlation function.

The final expression for the effective action (6.3) obtained by substituting coefficient \( \Lambda^{(2\nu-1)}_{ij,k}(k) \) (6.10) into on-shell action (6.4) is given by

\[
S_{\text{eff}} = \kappa_o \int d^dxd^dy \int \frac{d^dk}{(2\pi)^d} e^{-ik(x-y)} k^d \mathcal{Z}(k|\pi(x),\pi(y)), \quad \kappa_o = \hbar \frac{d}{2d+1} \frac{\Gamma(1-\nu/2)}{\Gamma(1+\nu/2)},
\]

(6.12)

where the bi-local kernel function reads

\[
\mathcal{Z}(k|\pi(x),\pi(y)) = \left( \pi_{ij,k}(x)\pi_{ij,k}(y) - 2\pi_{mi,j}(x)\pi_{mi,j}(y) \frac{k^m k^n}{k^2} - \frac{d+1}{d-2} \pi_{ij,k}(x)\pi_{ij,n}(y) \frac{k^m k^n}{k^2} + \frac{3}{2} \pi_{ij,k}(x)\pi_{mn,k}(y) \frac{k^m k^n k^m k^n}{k^4} \right).
\]

(6.13)
Note that $\Gamma(1 - d/2)$ in (6.12) does not have zeros for odd $d$ considered in this section.

Then, using the Fourier transform formula (G.2) one obtains two-point function in position space,

$$\tilde{S}_{\text{eff}} = \tilde{\kappa}_0 \int d^d x d^d y \frac{\tilde{Z}(\pi(x), \pi(y))}{|x - y|^{2d}}, \quad \tilde{\kappa}_0 = -\frac{h_0}{2\pi^{d/2}} \frac{1}{(d - 2) \Gamma(d/2)}, \quad (6.14)$$

with the bi-local kernel given by

$$\tilde{Z}(\pi(x), \pi(y)) = \left(\pi_{ij,k}(x)\pi_{ij,k}(y) - 4\pi_{mi,j}(x)\pi_{ni,j}(y)\frac{z^m z^n}{z^2} - 2\pi_{ij,k}(x)\pi_{ij,m}(y)\frac{z^k z^m}{z^2} + 6\pi_{ij,k}(x)\pi_{mn,k}(y)\frac{z^i z^j z^m z^n}{z^4}\right),$$

(6.15)

where we introduced notation $z^i = x^i - y^i$ and $z^2 = z^i z_i$. Using simple algebra, one shows that bi-local function $\tilde{Z}(\pi(x), \pi(y))$ coincides with the tensor (6.7) contracted with $\pi_{ij,k}$ as follows

$$\tilde{Z}(\pi(x), \pi(y)) = \frac{1}{3} \pi_{ij,k}(x)\Pi_{ij,kmn,l}(x - y)\pi_{mn,l}(y).$$

(6.16)

In particular, from (6.10) we obtain that the effective action does reproduce a two-point function of two primary mixed-symmetry operators with conformal dimensions (vacuum energies) $\Delta = E_0 = d (2.2)$.

### 6.3 Even boundary dimensions $d$

All the results obtained in the previous section can obviously be generalized to any non-integer values of parameter $\nu = d/2$. To elaborate the case of integer $\nu$ or even dimensions $d$ one has to redo the above computation starting form a small argument decomposition of the modified Bessel function of integer order. Repeating computation is necessary because the decomposition for integer $\nu$ contains a logarithmic contribution, see formula (E.6). It follows that computing a two-point correlation function in momentum space yields a result different from non-integer $\nu$ case discussed above. In position space the final result coincides with (6.14) that can be explained by absence of the gamma function zeros in the overall coefficient (6.14) contrary to zeros in (6.12).

Quite analogously to expression (6.10) one finds that $\epsilon$-decomposition of the normal derivative for even $d$ is given by

$$\partial_z \varphi_{ij,k}(z, k) \big|_{z=\epsilon} = \epsilon^{-1} \Lambda_{ij,k}^{(-1)}(k) + \epsilon \Lambda_{ij,k}^{(1)}(k) + \ldots$$

$$+ \epsilon^{2\nu - 1} k^{2\nu} \ln \frac{\epsilon k}{2} \Lambda_{ij,k}^{(2\nu - 1)}(k) + \ldots,$$

(6.17)

where the dots in the first and second lines stand for higher order terms, $O(\epsilon^3 k^4)$ and $O(\epsilon^{2\nu + 1} k^{2\nu + 2})$, respectively. A direct computation yields relations $\Lambda_{ij,k}^{(-1)}(k) = \Lambda_{ij,k}^{(1)}(k) = 0$, which obviously coincide with (6.11) because first groups of terms in the modified Bessel function decompositions (E.3) and (E.6) are the same. The difference appears in the last
term containing the logarithm. It follows that the renormalized value of the on-shell action in momentum space is equal to

$$S_{\text{eff}} = \kappa_e \int d^d x d^d y \int \frac{d^d k}{(2\pi)^d} e^{-ik(x-y)} \kappa^d \ln \frac{ek}{2} Z(k|\pi(x),\pi(y)),$$

(6.18)

where the bi-local kernel function is given by (6.13), while an overall coefficient is different,

$$\kappa_e = (-)^d \frac{h_0 d}{2^d \Gamma(d/2) \Gamma(d/2 + 1)}.$$

(6.19)

To perform a Fourier transformation one uses formula (G.19) that gives rise to the following expression,

$$\tilde{S}_{\text{eff}} = \tilde{\kappa}_e \int d^d x d^d y \tilde{Z}(\pi(x),\pi(y)) \frac{1}{|x-y|^{2d}}, \quad \tilde{\kappa}_e = -\frac{h_0}{2^d \Gamma(d/2) \Gamma(d/2)} \frac{1}{\Gamma(d/2)}.$$

(6.20)

which is obviously identical to the previously obtained on-shell action in odd dimensions $d (6.14)$.

7 Bulk/boundary symmetry transmutation

Bulk action (2.6) is invariant under global and local transformations,

$$\delta \varphi_{\mu\nu,\rho}(x) = \delta^L x \varphi_{\mu\nu,\rho}(x) + \delta^G x \varphi_{\mu\nu,\rho}(x),$$

(7.1)

where local part is given by gauge transformations (2.3), while global part is the Lie derivative with respect to Killing vectors of the background $AdS_{d+1}$ metric (see below). Having found explicit solution to the Dirichlet problem for the hook field equations it is useful to examine the behavior of symmetries (7.1).

In our case global symmetries of Euclidean $AdS_{d+1}$ spacetime form $o(d+1,1)$ algebra which describes conformal invariance of the boundary effective action. Since quadratic action (2.3) is invariant under global $AdS_{d+1}$ symmetries it is natural to expect that the invariance is kept unbroken for the boundary action. Indeed, in Section 6 we have explicitly shown that definite parts of the on-shell action are identified with renormalized two-point correlation functions which are conformally invariant. It shows that the $\epsilon$-regularization used in practice is consistent with global conformal symmetry at least for the Dirichlet boundary conditions.

7.1 Global symmetries

Consider now global symmetry transformations for both bulk and boundary fields. Our aim here is to demonstrate the bulk/boundary symmetry transmutation occurring in the limit $z = \epsilon \to 0$. To this end, one finds Killing vectors of the Euclidean $AdS_{d+1}$ metric (2.4). These are given by $o(d)$ scalar and vector components of $o(d+1)$ Killing vector $\xi^\mu = (\xi^0(z,x),\xi^m(z,x))$, namely,

$$\xi^0(z,x) = (D - 2(Kx))z,$$

(7.2)
\[ \xi^m(z, x) = \Lambda^m x_i + P^m + D x^m + K^m x^2 - 2x^m(Kx) + K^m(z^2 - \epsilon^2), \] (7.3)

where \( \Lambda^m = -\Lambda^m, P^m, K^m, D \) are \( o(d) \) constant tensors that parameterize algebra \( o(d + 1, 1) \). This particular parameterization turns out to be convenient from the boundary geometry perspective because \( \Lambda^m \) and \( P^m \) are then identified with parameters of Lorentz boosts and translations in \( \mathbb{R}^d \) space, while \( K^m \) and \( D \) with those of special conformal and scale transformations.

Global \( o(d + 1, 1) \) transformations of the bulk hook field \( \varphi_{\mu\nu, \rho}(x) \) are defined in a standard fashion,

\[
\delta^\xi \varphi_{\mu\nu, \rho}(x) = L_\xi \varphi_{\mu\nu, \rho}(x) = \xi^\gamma \partial_\gamma \varphi_{\mu\nu, \rho}(x) + \partial_\mu \xi^\gamma \varphi_{\gamma\nu, \rho}(x) + \partial_\nu \xi^\gamma \varphi_{\mu\gamma, \rho}(x) + \partial_\rho \xi^\gamma \varphi_{\mu\nu, \gamma}(x),
\]
(7.4)

as the Lie derivative with respect to the Killing field \( \xi^\mu \) given by (7.2) and (7.3).

In Section 5.2 we established that almost all \( o(d + 1) \) components of the original field have vanishing boundary values. The only component with non-vanishing boundary value is given by hook component \( \varphi_{ij,k}(z, x) \), see (5.2). It follows that this particular component transformation law read off from (7.4) takes the following form

\[
\delta^\xi \varphi_{mn, l}(z, x) = L_\xi \varphi_{mn, l}(z, x) + \xi^0 \partial_z \varphi_{mn, l}(z, x) - \frac{1}{2} \partial_m \xi^0 \varphi_{nl}(z, x) - \frac{1}{2} \partial_n \xi^0 \varphi_{ml}(z, x) + \partial_l \xi^0 \varphi_{mn}(z, x),
\]
(7.5)

where \( L_\xi \) denotes the Lie derivative evaluated with respect to Killing vector components (7.3). Using (D.10) and (D.14) one finds that in the limit \( \epsilon \to 0 \) last three terms tend to zero, see (5.7) and (5.6). Then, using Killing vector \( z \)-component (7.2) in momentum space we find that the leading contribution to

\[
\xi^0 \partial_z \varphi_{ij,k}(z, k) \bigg|_{z = \epsilon} \sim \epsilon \partial_z \varphi_{ij,k}(z, k) \bigg|_{z = \epsilon} = \Lambda^{(-1)}_{ij,k}(k) = 0
\]
(7.6)

vanishes identically by virtue of (3.10), (3.11). As a result, transformation law (7.4) induced on the boundary in the limit \( \epsilon \to 0 \) takes the form

\[
\delta^\xi \pi_{ij,k}(x) = L_\xi \varphi_{ij,k}(x).
\]
(7.7)

It implies that boundary tensor field \( \pi_{ij,k}(x) \) has a scale dimension \( \Delta_s = 0 \) (for discussion of transformation properties of mixed-symmetry primary fields in the present context see 5]). Recalling that scale dimensions of conformal partners in \( d \) dimensions are related as \( \Delta_s + \Delta_p = d \), one immediately obtains that the conformal partner for the initial value has scale dimension \( \Delta_p = d \).

On the other hand, action (2.6) is invariant under global \( o(d + 1, 1) \) symmetry transformations and this invariance transmutes into global invariance of the on-shell action (1.5). Then dual scale dimension \( \Delta_s = 0 \) guarantees that conformal partners for boundary values \( \pi_{ij,k}(k) \) are primary conformal hook fields \( \mathcal{O}^{ij,k}(k) \) of critical dimensions \( \Delta_p = d \), while their two-point correlation functions are given by (5.6). As expected, this particular conformal dimension exactly coincides with the vacuum energy of massless hook field in \( AdS_{d+1} \).
Relations (7.6) are literally valid for non-integer $\nu = d/2$ only. In even dimensions the normal derivative evaluated in momentum space contains logarithmic contribution (6.17). However, using Fourier transform back to position space based on the differential regularization (G.19) one restores regular $\epsilon$-behavior so for even boundary dimensions formula (7.7) remains formally valid in the limit $\epsilon \to 0$.

### 7.2 Local symmetries

Radial gauge fixing (3.3) gives rise to residual gauge transformations. To find their explicit form one solves equations which express the fact that local variations of the gauge fixing conditions are to be compensated by global variations,

$$
\delta_G \varphi_{00,i}(x) = \delta_L \varphi_{00,i}(x) , \quad \delta_G \varphi_{0[i,j]}(x) = \delta_L \varphi_{0[i,j]}(x) ,
$$

(7.8)

where local variations are read off from the gauge transformation law (2.3). Then, it follows that the gauge fixing conditions remain intact against transformations (7.1).

To solve conditions (7.8) one represents transformation law (2.3) in the following form

$$
\delta_L \varphi_{\mu \nu, \rho}(x) = \partial_\mu \chi_{\nu \rho}(x) - \partial_\nu \chi_{\mu \rho}(x) - 2 \Gamma^\gamma_{\mu \nu} \chi_{\gamma \rho}(x) - \Gamma^\gamma_{\mu \rho} \chi_{\nu \gamma}(x) - \Gamma^\gamma_{\nu \rho} \chi_{\mu \gamma}(x) ,
$$

(7.9)

where Christoffel symbols are given by (A.1). Decomposing original $o(d,1)$ gauge parameter into $o(d)$ components as $\chi_{mn}(x) = -\chi_{nm}(x)$ and $\chi_{m}(x) \equiv \chi_{0m}(x)$, one obtains that vector and antisymmetric parameters transform as

$$
\delta_L \varphi_{00,m}(z, x) = \frac{2}{z} \left( N_z + 2 \right) \chi_m(z, x) , \quad N_z = z \frac{d}{dz} ,
$$

(7.10)

On the other hand, using (7.4) one obtains that global transformations for these components are given by

$$
\delta_G \varphi_{00,m}(z, x) = -2zK^n \varphi_{nm}(z, x) ,
$$

(7.11)

$$
\delta_G \varphi_{0[m,n]}(z, x) = 2zK^l \varphi_{[m,n]}(z, x) ,
$$

(7.12)

where symmetric and hook field components (more precisely, their Fourier transforms) are given by solutions (D.10) and (D.16), and $K^m$ is a constant parameter of the special conformal transformation, see (7.3).

Consider now conditions (7.8). The first equation in (7.8) is inhomogeneous linear equation,

$$
(N_z + 2) \chi_m(z, x) = -z^2 K^n \varphi_{nm}(z, x) ,
$$

(7.13)

and its general solution is given by

$$
\chi_n(z, x) = -\int_0^1 dt t^3 z^2 K^m \varphi_{mn}(tz, x) + \frac{g_n(\epsilon, x)}{z^2} ,
$$

(7.14)
where \( g_n(z, x) \) is an arbitrary vector function representing the general solution to homogeneous equation (7.12). The second equation in (7.8) is

\[
(N_z + 3) \chi_{mn}(z, x) = z^2 K^l \varphi_{[m,n]}(z, x) - \frac{z}{2} \partial_{[m} \chi_{n]}(z, x), \tag{7.14}
\]

and its general solution is given by

\[
\chi_{mn}(z, x) = \int_0^1 dt \int_0^1 dp \int_0^1 dp' z^2 K^l \varphi_{[m,n]}(z, x) - \frac{z}{2} \partial_{[m} \chi_{n]}(z, x) + \sum f_{mn}(\epsilon, x), \tag{7.15}
\]

where \( f_{mn}(\epsilon, x) \) is an arbitrary antisymmetric function representing the general solution to homogeneous equation (7.14). We conclude that there is a solution for gauge and global parameters that leaves the gauge fixing conditions (3.3) intact.

Let us now discuss the boundary behavior of parameters (7.13) and (7.15). Discussion in section 5.2 allows one to conclude that

\[
\chi_{m}^p(z, x) = \epsilon^{-2} g_m(\epsilon, x), \quad \chi_{mn}(z, x) = \epsilon^{-3} f_{mn}(\epsilon, x). \tag{7.16}
\]

To avoid poles in \( \epsilon \) one fixes \( \epsilon \)-dependence of functions \( g_m(\epsilon, x) \) and \( f_{mn}(\epsilon, x) \) as follows

\[
g_m(\epsilon, x) = \epsilon^3 \vartheta_i(x) + \ldots, \quad f_{mn}(\epsilon, x) = \epsilon^3 \zeta_{mn}(x) + \ldots, \tag{7.17}
\]

where the dots stand for higher order terms in \( \epsilon \), and \( \vartheta_m(x) \) and \( \zeta_{mn}(x) \) are arbitrary boundary tensors. The first decomposition in (7.17) is explained by the form of the gauge transformation for the hook component read off from (7.9),

\[
\delta_L \varphi_{mn,k}(z, x) = \partial_m \chi_{nk}(z, x) + \partial_n \chi_{mk}(z, x) - \frac{1}{z} \left( 2 \delta_{mn} \chi_k(z, x) - \delta_{mk} \chi_n(z, x) - \delta_{nk} \chi_m(z, x) \right), \tag{7.18}
\]

where gauge parameters are given by (7.13) and (7.15). We see that the right hand side of the above expression is not singular iff parameter \( g_m(z, x) \) has asymptote (7.17) in the limit \( \epsilon \to 0 \): this is why the first decomposition in (7.17) starts from \( \epsilon^3 \) and not from \( \epsilon^2 \) as one might conclude from (7.16).

It is remarkable that apart from a derivative part defined by gauge parameter \( \zeta_{mn}(x) \) the transformation law (7.18) restricted to the boundary contains also an algebraic (Stueckelberg) part defined by an independent parameter \( \vartheta_m(x) \),

\[
\delta_L \varphi_{mn,k}(x) = \partial_m \zeta_{nk}(x) + \partial_n \zeta_{mk}(x) - \left( 2 \delta_{mn} \vartheta_k(x) - \delta_{mk} \vartheta_n(x) - \delta_{nk} \vartheta_m(x) \right). \tag{7.19}
\]

One observes that Stueckelberg parameters are sufficient to gauge away the trace of boundary hook tensor \( \varphi_{ij,k}(x) \). This is why in Section 5 we have chosen the boundary value of the hook component to be traceless, see (5.2).

This phenomenon has not been discussed earlier in the literature despite the fact that Stueckelberg-like transformations for tensors on the boundary directly arise from covariant derivatives contained in original bulk gauge transformations, cf. formula (7.9). It follows
that Stueckelberg-like transformations allow one to make boundary value tensors traceless. In particular, for the massless spin-2 case considered [17, 18, 20] the trace of symmetric boundary tensor can be shifted to zero by virtue of the Stueckelberg-like transformation and does not need to be analyzed at all.

Then, it follows that the leftover gauge transformations projected on the boundary hook traceless tensor \( \pi_{mn,l}(x) \) are given by

\[
\delta^L \pi_{mn,l}(x) = \partial_m \zeta_{nl}(x) + \partial_n \zeta_{ml}(x) - \frac{1}{d-1} \left[ 2 \delta_{mn} \partial^k \zeta_{kl}(x) - \delta_{nl} \partial^k \zeta_{km}(x) - \delta_{ml} \partial^k \zeta_{kn}(x) \right],
\]

where \( \zeta_{mn}(x) \) is antisymmetric leftover gauge parameter \( \zeta_{mn} = -\zeta_{nm} \), and both side of (7.20) have hook symmetry and vanishing traces.

8 Action for conformal gauge hook fields

Following totally symmetric field analysis of [17, 38] we regularize the effective action which is ill-defined in even dimensions and identify the prefactor of its singular part with the gauge invariant action for conformal hook fields \( \pi_{mn,k}(x) \). In this way we reconstruct the action known previously in the literature [24] (see formula (8.3) below).

In even dimensions functions \( 1/x^{d+2m} \), where \( m = 0, 1, 2, \ldots \), are ill-defined as distributions (see, e.g., [39]). In our case parameter \( m \) takes values \( m = d/2, d/2 - 1, d/2 - 2, \ldots \), and the main idea to treat such distributions is to use formula (8.1) in order to power down \( 1/x^{d+2m} \) to \( 1/x^d \). In its turn, function \( 1/x^d \) can be treated according to one or another regularization scheme. For instance, one can employ the dimensional regularization and represent \( 1/x^d \sim \frac{1}{\sigma} \delta(x) + \text{finite part} \), where \( d - [d] = -2\sigma \), and \([d]\) is integer. The finite part can be defined using the differential regularization described in Appendix G. The singular part is given by

\[
\frac{1}{|x - y|^{d+2m}} \bigg|_{\sigma = 0} \tau_m \bigg[ \frac{s_d}{2\sigma} \delta(x - y) \bigg], \tag{8.1}
\]

where coefficient \( \tau_m \) is given by (G.4), and \( s_d \) is a surface area of a unit \( S^{d-1} \) sphere. Also, one can obtain regularization formulas for functions of the type \( x^{-d-2m} x_i \ldots x_{ij} \) appearing in the two-point correlator (6.14), (6.15) just by successively acting with derivatives \( \partial_i \) on the basic regularization relation (8.1).

Applying the above regularization scheme one finds that modulo a non-vanishing normalization, the coefficient in the singular part of the effective action kernel (7.20) is identified with the following action

\[
S_{\text{conf}} = \int d^d x \mathcal{L}_{\text{conf}}, \tag{8.2}
\]

where the Lagrangian is given by

\[
\mathcal{L}_{\text{conf}} = \pi_{ij,k} \partial^2 \pi^{ij,k} + 2 \partial^m \pi_{mi,j} \partial^{d-1} \partial_n \pi^{ni,j} + \frac{d+1}{d-2} \partial^m \pi_{ij,m} \partial^{d-1} \partial_n \pi^{ij,n} + \frac{3}{2} \partial^i \partial^j \pi_{ij,k} \partial^{d-2} \partial_m \partial_n \pi^{mn,k}. \tag{8.3}
\]

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Conformal invariance of original effective action (6.20) and critical conformal dimension guarantee that this Lagrangian is conformally invariant and gauge invariant under the gauge transformation (7.20) that can also be checked by a direct computation.

To conclude this section it is worth to comment that the above consideration suggests that gauge invariant actions for conformal fields of arbitrary mixed-symmetry type in Minkowski spacetime of even dimensions built in [24] can be systematically reconstructed by the regularization procedure applied to two-point correlation functions of mixed-symmetry primary fields. Indeed, it can be shown that two-point correlators of two primaries are entirely expressed via matrix (6.8) just by maintaining Young symmetry and trace properties (see [6] for detailed discussion of the case of mixed-symmetry primary fields described by hook Young diagrams of arbitrary length and height). By construction, such a correlator contracted with two shadow fields is conformally invariant being in fact the effective boundary action. Then, applying the regularization procedure described above one directly obtains conformally invariant quadratic action for shadow fields. For critical conformal dimensions shadow fields are gauge fields so respective action is also gauge invariant.

9 Conclusions

In this paper we have explicitly considered free mixed-symmetry field dynamics from the holographic perspective. The analysis has been performed for the simplest hook field within the metric-like formulation. In particular, we have defined the radial gauge fixing that turns out to be convenient when solving the Dirichlet boundary problem. Having explicit solution to the Dirichlet problem allowed us to analyze subtleties related to the BMV mechanism inherent to mixed-symmetry field dynamics. As the main outcome, we have found the set of initial values to be identified with shadow fields living on the boundary and computed corresponding effective action thereby constructing two-point correlation functions. As a by-product, we have analyzed the bulk/boundary symmetry transmutation and identified the singular part of the effective action in even boundary dimensions with gauge invariant action for conformal hook fields known in the literature.

The results obtained in this paper could be extended along the following lines. First, it would be interesting to consider the case of $AdS_5$ massless hook field dynamics which is relevant in the context of $\mathcal{N} \geq 2$-extended 5d higher spin theories, see, e.g., [10, 24]. In this case the metric-like quadratic action can be represented in first order form [11] so that the holographic analysis should resemble consideration of massive 2-form fields in $AdS_5$ [12]. Second, the present consideration can be potentially extended beyond the free field level. Indeed, up to now some consistent interaction vertices of massless mixed-symmetry fields between themselves and with the gravity are known in the cubic approximation [34, 35]. Third, one can use the incomplete solution found in Section C to explore other boundary conditions, including Neumann conditions and other non-standard cases.

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A  \textit{AdS}_{d+1} \text{ spacetime in Poincare coordinates}

The Christoffel symbols associated with the \textit{AdS}_{d+1} geometry defined by the metric in Poincare form (2.11) are given by

\[
\Gamma_{\mu
u}^\lambda = -\frac{1}{z}(\delta^0_{\mu}\delta_{\nu}^\lambda + \delta^0_{\nu}\delta_{\mu}^\lambda - \delta^0_{\mu\nu}) , \quad (A.1)
\]

\[
\Gamma_{ij}^k = 0 , \quad \Gamma_{0j}^i = -\frac{1}{z}\delta_{ij} , \quad \Gamma_{ij}^0 = \frac{1}{z}\delta_{ij} , \quad \Gamma_{00}^0 = -\frac{1}{z} , \quad (A.2)
\]

and satisfy the following relations

\[
\partial_\mu \Gamma_{\beta\gamma}^\alpha = -\frac{1}{z}\delta_{0\mu} \Gamma_{\beta\gamma}^\alpha , \quad g^{\gamma\rho} \Gamma_{\alpha\gamma}^\rho = \frac{(d-1)z}{R^2} \delta^0_{\alpha} , \quad \Gamma_{\gamma\alpha}^\gamma = -\frac{d+1}{z} \delta_{\alpha}^0 . \quad (A.3)
\]

By definition, a covariant derivative is \( \nabla_\mu T_\nu = \partial_\mu T_\nu - \Gamma^\rho_{\mu\nu} T_\rho \), where \( T_\nu \) is a covector. The Riemann curvature \( R^\rho_{\lambda\mu\nu} = \partial_\mu \Gamma_{\lambda\nu}^\rho - ... \) has the form

\[
R^\rho_{\lambda\mu\nu} = \frac{1}{R^2} (\delta^\rho_{\nu} g_{\lambda\mu} - \delta^\rho_{\mu} g_{\lambda\nu}) , \quad (A.4)
\]

while the Ricci tensor, scalar curvature, and the cosmological constant are

\[
R_{\mu\nu} = -\frac{d}{R^2} g_{\mu\nu} , \quad \mathcal{R} = -\frac{d(d+1)}{R^2} , \quad \Lambda = \frac{d(d-1)}{R^2} . \quad (A.5)
\]

Here we used the \textit{AdS}_{d+1} background Einstein equations of motion \( R_{\mu\nu} + \frac{\Lambda}{d-1} g_{\mu\nu} = 0 \).

Using explicit form of Riemann curvature and following standard definitions \([\nabla_\mu, \nabla_\nu] T_\rho = -T_\lambda R^\lambda_{\rho\mu\nu} \) and \([\nabla_\mu, \nabla_\nu] T^\rho = T^\lambda R^\rho_{\lambda\mu\nu} \), one obtains

\[
[\nabla_\mu, \nabla_\nu] T_\rho = -\frac{1}{R^2} (g_{\rho\mu} T_\nu - g_{\rho\nu} T_\mu) , \quad [\nabla_\mu, \nabla_\nu] T^\rho = -\frac{1}{R^2} (\delta^\rho_\mu T_\nu - \delta^\rho_\nu T_\mu) . \quad (A.6)
\]

Since \([\partial_\mu, \partial_\nu] = 0 \) and \( \partial^\mu = g^{\mu\nu} \partial_\nu \) it follows that

\[
[\partial^\mu, \partial^\nu] = \frac{2}{z^2} g^{\mu0} \partial^\nu - \frac{2}{z} g^{\nu0} \partial^\mu , \quad [\partial^i, \partial^z] = -\frac{2}{z} \partial^i . \quad (A.7)
\]

B  Component form of the field equations

B.1 Main equations of motion

Below we rewrite equations of motion (2.10), (2.11) and constraints (2.9) according to index splitting \( \mu, \nu, ... \) to 0 and \( i, j, ... \). All tensor fields are Fourier transformed \( \Phi = \Phi(z, k) \) (4.3) and comma derivatives are given by \( \partial_m = ik_m \).
Sector $\mu = i, \nu = j, \rho = k$. Equations (2.10) take the form
\[
E_{ij,k} = \left[ \frac{d^2}{z^2} - \frac{d - 7}{z} \frac{\partial_z^2}{z} - k^2 \right] \varphi_{ij,k} + k_i k_j^m \varphi_{m,j,k} + k_j k_i^m \varphi_{m,i,k} + \frac{i}{2} \left( k_i \partial_z \varphi_{jk} + k_j \partial_z \varphi_{ik} \right) + k_i k_j \varphi + i \frac{6 - d}{2z} k_i \varphi_{jk} + i \frac{6 - d}{2z} k_j \varphi_{ik} + \frac{2i}{z} k_i \varphi_{ij} - i \frac{d}{z} \delta_{ij} k^m \varphi_{mk} + i \frac{i}{z} \delta_{ik} k^m \varphi_{mj} + i \frac{i}{z} \delta_{jk} k^m \varphi_{mi} - \frac{i}{z} \delta_{jk} k_i \varphi_0 - \frac{i}{z} \delta_{ik} k_j \varphi_0 - \frac{1}{z} \delta_{ij} \partial_z \varphi_k - \frac{(9 - 3d)}{z^2} \varphi_{ij,k} - \frac{3}{z^2} \delta_{ij} \varphi_k = 0 .
\] (B.1)

Here $k^2 = \delta_{ij} k^i k^j$ is the momentum space realization of the D’Alambertian operator $\nabla^2$.

By contracting the above equation with $\delta_{ij}$ one obtains
\[
E_{i,k} = \left[ \frac{d^2}{z^2} + \frac{7 - 2d}{z} \frac{\partial_z^2}{z} - 2k^2 \right] \varphi_k + 2k^m \varphi_{mn,k} + ik^m \partial_z \varphi_{mk} - \frac{4}{z} \varphi_k + i \frac{8 - 2d}{z} k^m \varphi_{mk} + \frac{9 - 6d}{z^2} \varphi_k = 0 .
\] (B.2)

Sector $\mu = i, \nu = j, \rho = 0$. Equations (2.10) take the form
\[
E_{ij,0} = \left[ \frac{d^2}{z^2} + \frac{8 - d}{z} \frac{\partial_z^2}{z} - k^2 \right] \varphi_{ij} + k_i k_j^m \varphi_{m,i} + k_j k_i^m \varphi_{m,j} + k_i k_j \varphi_0 + \frac{3i}{z} k^m \varphi_{ij,m} + \frac{3i}{2z} (k_j \varphi_j + k_i \varphi_i) - \frac{1}{z} \delta_{ij} \partial_z \varphi_0 + \frac{16 - 5d}{z^2} \varphi_{ij} - \frac{5}{z^2} \delta_{ij} \varphi_0 = 0 .
\] (B.3)

By contracting the above equation with $\delta_{ij}$ one obtains
\[
E_{i,0} = \left[ \frac{d^2}{z^2} + \frac{8 - 2d}{z} \frac{\partial_z^2}{z} - 2k^2 \right] \varphi_0 + \frac{6i}{z} \varphi_{m} + 2k^m \varphi_{mn} + \frac{16 - 10d}{z^2} \varphi_0 = 0 .
\] (B.4)

Sector $\mu = 0, \nu = 0, \rho = i$. Equations (2.10) take the form
\[
E_{0,i} = \partial_z^2 \varphi_i + \frac{5}{z} \partial_z \varphi_i + ik^m \partial_z \varphi_{mi} + \frac{2i}{z} k^m \varphi_{mi} + \frac{3}{z^2} \varphi_i = 0 .
\] (B.5)

Sector $\mu = 0, \nu = 0, \rho = [i, j]$. In what follows, antisymmetrization comes with a unit weight.

Equations (2.10) take the form
\[
E_{0[j,k]} = -\frac{1}{2} \left( k_j k^m \varphi_{mk} - k_k k^m \varphi_{mj} \right) + i \left( k^m \partial_z \varphi_{mk,j} - k^m \partial_z \varphi_{mj,k} \right) - \frac{3i}{z} \left( k^m \varphi_{mj,k} - k^m \varphi_{mk,j} \right) + \frac{3i}{z} (k_j \varphi_k - k_k \varphi_j) + i \left( k_j \partial_z \varphi_k - k_k \partial_z \varphi_j \right) = 0 .
\] (B.6)

Sector $\mu = 0, \nu = 0, \rho = 0$. In this case equations (2.10) reduce to
\[
E_{0,0} = \partial_z^2 \varphi_0 + \frac{6}{z} \partial_z \varphi_0 + \frac{6}{z^2} \varphi_0 = 0 .
\] (B.7)

and the left-hand-side of this relation vanishes by virtue of constraints, see discussion after formula (2.11).
B.2 Trace equations

Scalar and vector components of trace equation (2.11) are given by

\[ E_0 \equiv \left[ \varphi_0^2 + \frac{7 - d}{z} \varphi_0 + \frac{3i}{z} k^m \varphi_m + k^m k^n \varphi_{mn} + \frac{11 - 5d}{z^2} \varphi_0 \right] = 0 , \quad (B.8) \]

\[ E_i \equiv \left[ \varphi_i^2 + \frac{5}{z} \varphi_i \right] + \frac{k^m}{z} k^m \varphi_{mi} + i k^m \varphi_{mi} - \frac{2i}{z} k_i \varphi_0 + 5 \frac{d}{z} k^m \varphi_{mi} + 3 \frac{6 - 3d}{z^2} \varphi_i = 0 . \quad (B.9) \]

As a consistency check one may make sure that trace relations valid in \( o(d, 1) \) notation are rewritten in \( o(d, q) \) notation as

\[ E_{ii,0} \equiv E_{00,0} = 2 E_0, \]

\[ E_{ii,k} \equiv E_{00,k} = 2 E_k, \]

\[ E_0 = \left[ \varphi_0^2 + \frac{7 - d}{z} \varphi_0 + \frac{3i}{z} k^m \varphi_m + k^m k^n \varphi_{mn} + \frac{11 - 5d}{z^2} \varphi_0 \right] = 0 , \quad (B.8) \]

\[ E_i = \left[ \varphi_i^2 + \frac{5}{z} \varphi_i \right] + \frac{k^m}{z} k^m \varphi_{mi} + i k^m \varphi_{mi} - \frac{2i}{z} k_i \varphi_0 + 5 \frac{d}{z} k^m \varphi_{mi} + 3 \frac{6 - 3d}{z^2} \varphi_i = 0 . \quad (B.9) \]

B.3 Differential constraints

Component form of constraints (2.9) is given by

\[ \frac{1}{2} T_{00} \equiv (\varphi_0 + \frac{2}{z}) \varphi_0 = 0 , \quad (B.10) \]

\[ T_{0i} \equiv \left( \frac{-i k^m \varphi_{mi} + \varphi_i + i k_i \varphi_0 + \frac{3}{z} \varphi_i}{} \right) = 0 , \quad (B.11) \]

\[ \frac{1}{2} T_{ij} \equiv \varphi_{ij} + i k^m \varphi_{ij,m} + \frac{i}{2} (k_i \varphi_j + k_j \varphi_i) - \frac{1}{z} \delta_{ij} \varphi_0 - \frac{d - 4}{z} \varphi_{ij} = 0 , \quad (B.12) \]

and

\[ T \equiv \varphi_0 + i k^m \varphi_m - \frac{d - 3}{z} \varphi_0 = 0 . \quad (B.13) \]

It is obvious that adding up traces yields \( T_{00} + T_{ii} = 4T \).

C Incomplete solution to equations and constraints

C.1 Trace components

Solution to constraints (B.10) and (B.13) can be represented in the form,

\[ \varphi_0(z, k) = \frac{C(\epsilon, k)}{z^2}, \quad i k^m \varphi_m(z, k) = \frac{d - 1}{z} \varphi_0(z, k) , \quad (C.1) \]

where \( C(\epsilon, k) \) is an arbitrary function of momenta. Recalling that the boundary is displaced into the bulk, we observe that function \( C(\epsilon, k) \) depends also on the cutoff parameter \( \epsilon \) which is a boundary value for the \( z \)-variable. In this case it regulates how the function approaches
the boundary plane \( z = \epsilon \). From expressions (C.1) it follows that \( k^m \varphi_m \) is a homogeneous function: \((z \partial_z + 3)k^m \varphi_m = 0\). Then, using constraint (B.11) one finds the relation

\[
    ik^m \varphi_{mi} = \frac{z \partial_z + 3}{z} \varphi_i + ik_i \varphi_0 .
\]  

(C.2)

Taking the divergence \( \partial^i T_{0i} = 0 \) and using homogeneity of \( \partial^m \varphi_m \) one finds that symmetric component \( \varphi_j \) satisfies the relation

\[
    k^m k^n \varphi_{mn} = k^2 \varphi_0 .
\]  

(C.3)

Taking the above relations into account one shows that divergence \( k^m E_{00,m} \) of equation (B.5) vanishes identically.

Substituting (C.1) and (C.2) back into the equation (B.5) one finds

\[
    B_z^2 z \varphi_i' = 3 \varphi_i .
\]  

(C.4)

The general solution is searched for in the form \( \sim z^\alpha \), where \( \alpha \) is an index of power. Associated quadratic equation has roots \( \alpha = \{-1, -3\} \) so that one obtains the solution,

\[
    \varphi_i(z, k) = \frac{A_i(\epsilon, k)}{z} + \frac{B_i(\epsilon, k)}{z^3} ,
\]  

(C.5)

where \( A_i(\epsilon, k) \) and \( B_i(\epsilon, k) \) are arbitrary functions of momenta.

We see that the trace of \( \varphi_{\mu\nu\rho}(z, k) \) is a simple function of \( z \). It is instructive to compare \( z \)-dependence of \( \varphi_\rho(z, k) \) (recall that its \( o(d) \) components are \( \varphi_i(z, k) \) and \( \varphi_0(z, k) \)) with those of other fields considered previously. So the authors of [17] claimed a trace of massless graviton field \( h_{\mu\nu} \) diverges on the boundary and simply set it to zero. However, in [18] it has been noted that the trace behavior is more subtle and requires additional analysis. Indeed, in [20] it has been explicitly shown that the trace depends on \( z \) quadratically. On the other hand, in the massive graviton case considered in [21] the trace is proportional to the non-trivial ratio of modified Bessel functions similar to solution of a scalar field equation in \( AdS_{d+1} \).

To conclude this paragraph we write down several expressions useful for the further analysis. Using (C.2) and taking the divergence \( \partial^i T_{ij} = 0 \) of constraint (B.12) one obtains

\[
    k^m k^n \varphi_{mn,i} = k^2 \varphi_i + k_i (k^m \varphi_m) + \frac{2i}{z} k_i \varphi_0 + \frac{2(d - 2)}{z} k^m \varphi_{mi} .
\]  

(C.6)

In particular, one shows that the right-hand-side of the above relation satisfies Young symmetry condition \( k^m k^n k^i \varphi_{mn,i}(z, k) = 0 \). Substituting (C.6) and the previously obtained relations into equation (B.9) yields identity. Constraint (B.12) gives rise to

\[
    ik^m \varphi_{ij,m} = -\partial_z \varphi_{ij} - \frac{i}{2} (k_i \varphi_j + k_j \varphi_i) + \frac{1}{z} \delta_{ij} \varphi_0 + \frac{d - 4}{z} \varphi_{ij} .
\]  

(C.7)
C.2 Symmetric component

Let us study the \( z \)-dependence of the symmetric component \( \varphi_{ij}(z, k) \). To this end, taking into account the trace solutions we reconsider equation (B.3). It can be cast into the form,

\[
\hat{E}_s \varphi_{ij} = Y_{ij} ,
\]

(C.8)

where (in)homogeneous parts are defined by

\[
\hat{E}_s = \left[ \frac{\partial^2}{z^2} + \frac{5 - d}{z} \partial_z + \frac{4 - 2d}{z^2} - k^2 \right] , \quad Y_{ij} = -k_j k^m \varphi_{im} - k_i k^m \varphi_{mj} + k_i k_j \varphi_0 .
\]

(C.9)

For a given tensor field we find a traceless and transverse (TT) decomposition \cite{18},

\[
\varphi_{ij} = \bar{\varphi}_{ij} + \frac{k_i k^m}{k^2} \varphi_{mj} + \frac{k_j k^m}{k^2} \varphi_{mi} - \frac{k_i k_j k^m k^n}{k^4} \varphi_{mn} +
\]

\[
\quad + \frac{1}{d-1} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \left( \varphi_0 - \frac{k^m k^n}{k^2} \varphi_{mn} \right) ,
\]

(C.10)

where \( \bar{\varphi}_{ij} \) is a TT tensor, i.e., it satisfies \( k^l \bar{\varphi}_{ij} = 0 \) and \( \delta^{ij} \bar{\varphi}_{ij} = 0 \).

The following lemma holds.

**Lemma C.1.** Traceless and transverse symmetric tensor \( \bar{\varphi}_{ij} \) defined by (C.10) satisfies homogeneous differential equation

\[
\hat{E}_s \bar{\varphi}_{ij}(z, k) = 0 ,
\]

(C.11)

where operator \( \hat{E}_s \) is given by (C.9).

The proof of the lemma is straightforward. A combination \( \varphi_{ij} - \bar{\varphi}_{ij} \) read off from the TT-decomposition is a particular integral of (C.8), while a general integral of the homogeneous equation is given by \cite{18}

\[
\varphi_{ij}(z, k) = z^{\nu - 2} K_\nu(zk) F_{ij}(\epsilon, k) , \quad \nu = \frac{d}{2} ,
\]

(C.12)

where \( F_{ij}(\epsilon, k) \) is some symmetric TT tensor on the boundary and \( K_\nu(y) \) is modified Bessel function of the second kind (see Appendix E). Using relations (C.2) and (C.3) along with the TT-decomposition (C.10) one obtains the \( z \)-dependence of the symmetric component

\[
\varphi_{ij}(z, k) = z^{\nu - 2} K_\nu(zk) F_{ij}(\epsilon, k) - \frac{3}{z} \left( k_i \varphi_{ij}(z, k) + k_j \varphi_{ji}(z, k) \right) + \frac{k_i k_j}{k^2} \varphi_0(z, k) .
\]

(C.13)

\(^7\) Second order equation (C.11) has two linearly independent solutions, modified Bessel functions of first and second kinds. The reason for choosing modified function of second order \( K_\nu(z) \) is that it exponentially decays for \( z \to 0 \), while another branch exponentially blows up for \( z \to 0 \), and therefore is discarded.
C.3 Divergence relations

To establish $z$-dependence of the hook component one proceeds along the same lines as with the symmetric component. For this purpose, one needs to find an expression for the divergence $k^m\varphi_{mi,j}(z,k)$ that enters the inhomogeneous part of the equation (B.1).

To treat a non-symmetric tensor with two indices one decomposes it into symmetric and antisymmetric parts in a standard fashion,

$$k^m\varphi_{mi,j} = \frac{1}{2}(k^m\varphi_{mi,j} + k^m\varphi_{mj,i}) + \frac{1}{2}(k^m\varphi_{mi,j} - k^m\varphi_{mj,i}) = \frac{1}{2}S_{ij} + \frac{1}{2}A_{ij}. \quad (C.14)$$

Compute first the antisymmetric part. To this end, using (C.2) we cast equation (B.6) into the following form,

$$(z\partial_z + 3)\tilde{A}_{ij} = 0 , \quad \tilde{A}_{ij} \equiv A_{ij} - \frac{3}{2}H_{ij} , \quad H_{ij} \equiv k_i\varphi_j - k_j\varphi_i. \quad (C.15)$$

This differential equation is solved as $\tilde{A}_{ij}(z,k) = z^{-3}a_{ij}(\epsilon,k)$, where $a_{ij}(\epsilon,k)$ is some arbitrary antisymmetric boundary tensor. Therefore, the antisymmetric part of $k^m\varphi_{mi,j}(z,k)$ is given by

$$A_{ij}(z,k) = \frac{a_{ij}(\epsilon,k)}{z^3} + \frac{3}{2}(k_i\varphi_j(z,k) - k_j\varphi_i(z,k)). \quad (C.16)$$

In order to find symmetric part $S_{ij}$ we note that by virtue of Young symmetry properties of $\varphi_{ij,k}$ it is equal to $S_{ij}(z,k) = -k^m\varphi_{ij,m}(z,k)$. This combination follows from (C.7),

$$iS_{ij}(z,k) = -\partial_z\varphi_{ij}(z,k) - \frac{i}{2}(k_i\varphi_j(z,k) + k_j\varphi_i(z,k)) + \frac{1}{z}\delta_{ij}\varphi_0(z,k) + \frac{d-4}{z}\varphi_{ij}(z,k). \quad (C.17)$$

Gathering everything together, one obtains the final expression for divergence (C.14),

$$k^m\varphi_{mi,j}(z,k) = \frac{1}{2}\frac{a_{ij}(\epsilon,k)}{z^3} + \frac{3}{4}(k_i\varphi_j(z,k) - k_j\varphi_i(z,k))$$

$$+ \frac{i}{2}\partial_z\varphi_{ij}(z,k) - \frac{1}{4}(k_i\varphi_j(z,k) + k_j\varphi_i(z,k)) - \frac{i}{2z}\delta_{ij}\varphi_0(z,k) - \frac{i}{2z}\varphi_{ij}(z,k). \quad (C.18)$$

C.4 Hook component

In this paragraph we analyze a TT-decomposition for the hook component and substitute it into equation (B.1). To this end, one represents (B.1) as follows

$$\hat{E}_h \varphi_{ij,k} = Y_{ij||k}, \quad (C.19)$$

where homogeneous part is defined by

$$\hat{E}_h = \left[ \partial_z^2 - \frac{d-7}{z}\partial_z + \Box + \frac{(9-3d)}{z^2} \right], \quad (C.20)$$
while the inhomogeneous part $Y_{ij|k}$ can be easily read off from original equation (B.1). Note that analogously to the symmetric component case, $Y_{ij|k}$ is obtained by using the trace solutions. Next, we find the following TT-decomposition for the hook tensor,

$$
\varphi_{ij,k} = \bar{\varphi}_{ij,k} + \frac{1}{k^2} \left( k_i k^m \varphi_{mj,k} + k_j k^m \varphi_{im,k} + k_k k^m \varphi_{ij,m} \right) - \frac{1}{k^4} \left( k_i k_j k^m k^n \varphi_{mn,k} - \frac{1}{2} k_i k_k k^m k^n \varphi_{mn,j} - \frac{1}{2} k_j k_k k^m k^n \varphi_{mn,i} \right) + \frac{1}{d-2} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \left( \varphi_k - \frac{1}{k^2} k^m k^n \varphi_{mn,k} - \frac{k_k}{k^2} k^m \varphi_m \right) - \frac{1}{2(d-2)} \left( \delta_{ik} - \frac{k_i k_k}{k^2} \right) \left( \varphi_j - \frac{1}{k^2} k^m k^n \varphi_{mn,j} - \frac{k_j}{k^2} k^m \varphi_m \right) - \frac{1}{2(d-2)} \left( \delta_{jk} - \frac{k_j k_k}{k^2} \right) \left( \varphi_i - \frac{1}{k^2} k^m k^n \varphi_{mn,i} - \frac{k_i}{k^2} k^m \varphi_m \right),
$$

(C.21)

where $\bar{\varphi}_{ij,k}$ is a TT tensor, i.e., it satisfies $k^i \bar{\varphi}_{ij,k} = 0$ and $\delta^{ij} \bar{\varphi}_{ij,k} = 0$.

**Lemma C.2.** Traceless and transverse hook tensor $\bar{\varphi}_{ij,k}$ defined by (C.21) satisfies homogeneous differential equation

$$
\hat{E}_h \bar{\varphi}_{ij,k}(z, k) = 0,
$$

(C.22)

where operator $\hat{E}_h$ is given by (C.20).

The proof of the lemma is straightforward but technically cumbersome. A combination $\varphi_{ij,k} - \bar{\varphi}_{ij,k}$ read off from the TT-decomposition is a particular integral of (C.19), while a general integral of the homogeneous equation is given by

$$
\varphi_{ij,k}(z, k) = z^{\nu-3} K_\nu(z k) G_{ij,k}(\epsilon, k), \quad \nu = \frac{d}{2},
$$

(C.23)

where $G_{ij,k}(\epsilon, k)$ is some hook TT tensor on the boundary and $K_\nu(y)$ is the modified Bessel function of the second kind.

Using various relations obtained in the previous paragraphs one derives the on-shell version of TT decomposition for the hook component,

$$
\varphi_{ij,k} \approx \bar{\varphi}_{ij,k} + \frac{1}{k^2} \left( k_i k^m \varphi_{mj,k} + k_j k^m \varphi_{im,k} + k_k k^m \varphi_{ij,m} - k_i k_j \varphi_k + \frac{1}{2} k_i k_k \varphi_j + \frac{1}{2} k_j k_k \varphi_i \right) - 2(d-2) \frac{k_i k_j}{k^4} Z_k + (d-2) \frac{k_k k_j}{k^4} Z_i + (d-2) \frac{k_k k_i}{k^4} Z_j - \frac{2}{k^2} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) Z_k + \frac{1}{k^2} \left( \delta_{ik} - \frac{k_i k_k}{k^2} \right) Z_j + \frac{1}{k^2} \left( \delta_{jk} - \frac{k_j k_k}{k^2} \right) Z_i,
$$

(C.24)

where notation $Z_i = z^{-2}(z \partial_z + 3)\varphi_i$ is introduced, and $\approx$ means an on-shell equality. TT tensor $\bar{\varphi}_{ij,k}(z, k)$ is given by (C.23), the trace $\varphi_k(z, k)$ is given by (C.3), and the divergences are given by (C.7), (C.18).
D  Complete solution to equations and constraints

D.1 Useful notation

We use several combinations of the master boundary tensor (D.2) that define boundary tensor structure of various components and their divergences,

\[ L_{ij}(k) = k^m \pi_{ij,m}(k), \quad L_i(k) = k^m k^m \pi_{mn,i}(k), \]

\[ Y_{ij}(k) = k_i L_j(k) - k_j L_i(k), \quad S_{ij}(k) = k_i L_j(k) + k_j L_i(k), \]

\[ T_{ij}(k) = k^m \pi_{mi,j}(k) - k^m \pi_{mj,i}(k), \quad M_{ij}(k) = 2L_{ij}(k) + \frac{1}{k^2} S_{ij}(k), \quad (D.1) \]

\[ M_{ij,k}(k) = 2k_i k_j L_k(k) - k_i k_k L_j(k) - k_j k_k L_i(k), \]

\[ Z_{ij,k}(k) = 2\delta_{ij} L_k(k) - \delta_{ik} L_j(k) - \delta_{jk} L_i(k). \]

These tensors have the following properties. These are either symmetric, \( S_{[mn]} = 0, M_{[mn]} = 0, \) or antisymmetric \( Y_{(mn)} = 0, T_{(mn)} = 0, \) or hook symmetric, \( M_{(mn,k)} = 0, Z_{(mn,k)} = 0. \) There are also some obvious contractions useful in practice, \( k^m L_m = 0, 2k^m L_m = -L_n, 2k^m T_{mn} = 3L_n, k^m S_{mk} = k^2 L_k, \delta^m S_{mn} = 0, \delta^m M_{mn} = 0, \) and \( k^m M_{mn} = 0. \)

D.2 Trace components, part I

As the master boundary tensor is traceless (D.2) one observes that arbitrary functions defining the traces are expressed in terms of boundary tensors (D.1) in the following way,

\[ A_i(\epsilon, k) = a(\epsilon, k) L_i(k), \quad B_i(\epsilon, k) = b(\epsilon, k) L_i(k), \quad C(\epsilon, k) = 0, \quad (D.2) \]

where \( L_i(k) \) is given by (D.4), and \( a(\epsilon, k), b(\epsilon, k) \) are new unknown functions. Note that by virtue of Young symmetry of the master boundary tensor one derives constraints \( k^m A_m = k^m B_m \equiv 0, \) which, however, admit local solutions (D.2). Function \( C \) vanishes since it is impossible to build a scalar from traceless \( \pi_{ij,k} \) via contracting it with momenta. Therefore, a scalar component of the trace is zero,

\[ \varphi_0(z, k) = 0. \quad (D.3) \]

D.3 Symmetric component

Consider now symmetric component (C.13). Using (D.1) one can represent TT tensor \( F_{ij}(\epsilon, k) \) as proportional to a TT part of \( L_{ij}(k) = k^m \pi_{ij,m}(k) \) in the following way,

\[ F_{ij}(\epsilon, k) = F(\epsilon, k) \bar{\varphi}_{ij}(k), \quad (D.4) \]

where \( F(\epsilon, k) \) is a new unknown function, and the combination,

\[ \bar{\varphi}_{ij}(k) = L_{ij}(k) + \frac{1}{2k^2} S_{ij}(k), \quad (D.5) \]
is traceless and transverse. Then, solution (C.13) can be cast into the form

$$\varphi_{ij}(z, k) = \frac{1}{z^2} K_\nu(zk) F(\epsilon, k) \varphi_{ij}(k) - 2i \frac{a(\epsilon, k)}{z^2 k^2} S_{ij}(k), \quad (D.6)$$

where new modified Bessel function $K_\nu$ is defined by (E.2). Using vanishing trace (D.3) one finds that symmetric divergence relation (C.17) takes the form

$$i k^m \varphi_{ij,m}(z, k) = - \frac{W(zk) K_\nu(zk)}{z^3} F(\epsilon, k) \varphi_{ij}(k) - \frac{2i(d - 2)a(\epsilon, k)}{z^2 k^2} S_{ij}(k) - \frac{ia(\epsilon, k)}{2z} S_{ij}(k) - \frac{ib(\epsilon, k)}{2z^3} S_{ij}(k), \quad (D.7)$$

where function $W(zk)$ is given by (E.2).

The divergence relation (D.7) is required to be well-defined when $z$ is approaching the boundary. For small $\epsilon \to 0$ the left-hand-side of this relation is finite and equals $i L_{ij}(k)$, while finiteness of the right-hand-side requires $F(\epsilon, k) = \epsilon^3 \tilde{F}(k) + \ldots$ and $a(\epsilon, k) = \epsilon^3 \tilde{a}(k) + \ldots$, where tildes denote some $\epsilon$-independent fixed tensors on the boundary and the dots stand for the higher order terms in the cutoff parameter, cf. formula (E.3). In particular, such a boundary behavior implies that symmetric component tends to zero at $\epsilon \to 0$.

Requiring identical tensor structures on both sides of divergence relation (D.7) in the point $z = \epsilon$ yields the following constraints,

$$F(\epsilon, k) = - \frac{i \epsilon^3}{W(\epsilon k) K_\nu(\epsilon k)}, \quad (D.8)$$

$$\frac{k^2}{\epsilon^3} b(\epsilon, k) + \left( \frac{k^2}{\epsilon} + \frac{4(d - 2)}{\epsilon^3} \right) a(\epsilon, k) - 1 = 0, \quad \text{or,} \quad a(\epsilon, k) = \frac{\epsilon^3 - b(\epsilon, k) k^2}{\epsilon^2 k^2 + 4(d - 2)}. \quad (D.9)$$

Indeed, using decomposition (D.5) one shows that adjusting the boundary values of both sides in (D.7) yields the coefficient (D.8) in front of $L_{ij}(k)$, while the group of terms proportional to $S_{ij}(k)$ is to vanish that produces constraint (D.9). We see that these functions conform to the pattern of the small $\epsilon$ behavior discussed above.

Writing down the resulting expression for the symmetric component,

$$\varphi_{ij}(z, k) = -i \left( \frac{\epsilon^2}{z} \right) \frac{\epsilon}{W(\epsilon k) K_\nu(\epsilon k)} \varphi_{ij}(k) - \frac{2i a(\epsilon, k)}{z^2 k^2} S_{ij}(k), \quad (D.10)$$

and using (D.3) we find its boundary value

$$\varphi_{ij}(\epsilon, k) = \frac{i}{2 W(\epsilon k)} \left( k_{i} k_{m} \pi_{ij,m} + \frac{k_{i} k_{m} k_{n} \pi_{mn,j} + k_{j} k_{m} k_{n} \pi_{mn,i}}{k^2} \right) - \frac{2i}{\epsilon^2} \left( k_{i} k_{m} k_{n} \pi_{mn,j} + k_{j} k_{m} k_{n} \pi_{mn,i} \right) a(\epsilon, k). \quad (D.11)$$
We assume that function \( a(\epsilon, k) \) is analytical with respect to \( k \) so the idea is to find such \( a(\epsilon, k) \) that all poles in momenta cancel each other. Since the leading term in the small \( \epsilon \) decomposition of function \( W (E.2) \) is given by
\[
W(\epsilon k) = -(d - 2) + \ldots ,
\]
(D.12)
it follows that all poles in (D.11) disappear provided that
\[
a(\epsilon, k) = \frac{\epsilon^3}{4(d - 2)} , \quad b(\epsilon, k) = -\frac{\epsilon^5}{4(d - 2)} .
\]
(D.13)
Note that the above procedure fixes function \( a(\epsilon, k) \) while function \( b(\epsilon, k) \) is determined by equation (D.9).

D.4 Trace components, part II

Substituting (D.13) into (C.5) and (D.2), one finds the final solution for a vector component of the trace,
\[
\varphi_i(z, k) = \frac{\epsilon^3}{4(d - 2)} \left[ \frac{1}{z} - \frac{\epsilon^2}{z^3} \right] L_i(k) .
\]
(D.14)
Recall that a scalar trace component vanishes, (D.3).

D.5 Hook component

Using the boundary condition (C.2) one finds that the boundary value of TT-component \( \bar{\varphi}_{ij,k} = \bar{\varphi}_{ij,k}(k) \) read off from (C.21) is given by
\[
\bar{\varphi}_{ij,k}(z, k) = \left( \frac{\epsilon}{z} \right)^3 \frac{K_{ij}(z)}{K_{ij}(\epsilon k)} \bar{\varphi}_{ij,k}(k) + \frac{1}{k^2} \left[ k_i k^m \varphi_{mj,k} + k_j k^m \varphi_{mi,k} + k_k k^m \varphi_{ij,m} \right] +
\]
\[
+ \frac{1}{k^4} \left( k_i k_j \pi_k - \frac{1}{2} k_i k_k \pi_j - \frac{1}{2} k_j k_k \pi_i \right) + \frac{1}{2(d - 2) k^2} \frac{1}{k^2} \left( 2\Pi_{ij} \pi_k - \Pi_{ik} \pi_j - \Pi_{jk} \pi_i \right) ,
\]
(D.15)
where we introduced projector \( \Pi_{ij} = \delta_{ij} - k_i k_j / k^2 \). Then, the form (with some terms written implicitly) of the hook component read off from the on-shell version of the TT decomposition (C.24), (C.23) is
\[
\varphi_{ij,k}(z, k) = \left( \frac{\epsilon}{z} \right)^3 \frac{K_{ij}(z k)}{K_{ij}(\epsilon k)} \varphi_{ij,k}(k) + \frac{1}{k^2} \left( k_i k^m \varphi_{mj,k} + k_j k^m \varphi_{mi,k} + k_k k^m \varphi_{ij,m} \right)(z, k)
\]
\[
- \frac{a(\epsilon, k)}{2k^2} \left( \frac{1}{z} + \frac{4(d - 3)}{k^2} - \epsilon^2 \right) M_{ij,k}(k) - \frac{2}{z^3} \frac{a(\epsilon, k)}{k^2} Z_{ij,k}(k) ,
\]
(D.16)
where \( L_i(k) \), \( M_{ij,k}(k) \), and \( Z_{ij,k}(k) \) are given by (D.11). To derive (D.16) we used the trace solution obtained in the previous section. Explicit expressions for divergences occurring in the right-hand-side are given by
\[
k^m \varphi_{ij,m}(z, k) = \frac{1}{2} \left( \frac{\epsilon}{z} \right)^3 \frac{W(z k) K_{ij}(z k)}{W(\epsilon k) K_{ij}(\epsilon k)} M_{ij,k}(k) - \left[ \frac{2(d - 2)}{z^3} \frac{a(\epsilon, k)}{k^2} + \frac{b(\epsilon, k)}{2z^3} + \frac{a(\epsilon, k)}{2z} \right] S_{ij}(k) ,
\]
(D.17)
and
\[ k^m \varphi_{m[i,j]}(z, k) = \left( \frac{\epsilon}{z} \right)^3 T_{ij}(k) + \frac{3}{2} \left( a(\epsilon, k) - \frac{\epsilon^2 a(\epsilon, k)}{z^3} \right) Y_{ij}(k), \] (D.18)

so that the full non-symmetric divergence combination is given by
\[ k^m \varphi_{mi,j}(z, k) = \frac{1}{4} \left( \frac{\epsilon}{z} \right)^3 \frac{W(zk) K_\nu(zk)}{W(\epsilon k) K_\nu(\epsilon k)} M_{ij}(k) - \frac{1}{2} \left[ \frac{2(d-2) a(\epsilon, k)}{k^2} + \frac{b(\epsilon, k)}{2z} + \frac{a(\epsilon, k)}{2z} \right] S_{ij}(k) \] (D.19)

Functions \( a(\epsilon, k) \) and \( b(\epsilon, k) \) are determined by (D.13). Substituting above divergences into expression (D.16) will give the final answer for the hook component. However, we do not gather all constituents together because the resulting expressions is exceedingly lengthy. Fortunately, in order to compute the boundary effective action it suffices to have implicit expression (D.16) supplemented with divergence formula (D.19), see Section 6.

E Ascending series for the modified Bessel functions

We use the following definition of modified Bessel function,
\[ K_\nu(u) = \frac{\pi}{2 \sin(\pi \nu)} \left( I_{-\nu}(u) - I_\nu(u) \right), \] (E.1)

where the prefactor can be expressed via \( \Gamma(\alpha) \Gamma(1-\alpha) = \pi/\sin(\pi \alpha) \), and \( I_\nu(u) \) is a Bessel function of arbitrary order \( \nu \in \mathbb{R} \). "The right of this equation is replaced by its limiting value if \( \alpha \) is an integer or zero" (quoted from [13], eq. 9.6.2). In our case order \( \nu = d/2 \) that is either integer or half-integer so these two cases are to be considered separately. Actually, being decomposed in two groups of (non)-analytical terms the modified Bessel functions of integer order contain additional logarithm \( \ln u \) in front of the analytical part.

We also introduce the following combinations of the modified Bessel functions useful in practice,
\[ K_\nu(u) = u^\nu K_\nu(u), \quad W(u) = \frac{2 K_\nu(z) - K_{\nu+1}(u)}{K_\nu(u)}. \] (E.2)

Non-integer orders \( \nu \). Consider non-integer orders \( \nu \) which in our case correspond to odd boundary dimensions \( d \). From the definition of modified Bessel functions (E.1) and (E.2) one finds
\[ K_\nu(u) = 2^{\nu-1} \Gamma(\nu) \Gamma(1-\nu) \left[ \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(1-\nu+n)} \left( \frac{u}{2} \right)^{2n} - \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(1+\nu+n)} \left( \frac{u}{2} \right)^{2\nu+2n} \right]. \] (E.3)
For the present problem variable $u$ is identified with the square root of $d$-dimensional D’Alambertean $\sqrt{\Box}$ so the local part of of $\mathcal{K}_\nu(\epsilon \sqrt{\Box})$ is provided by terms with even powers of $u$ (the first sum in (E.3)) while odd powers give non-local contribution (the second sum in (E.3)). First few terms read off from the above decomposition are

$$
\mathcal{K}_\nu(u) = 2^{\nu-1} \Gamma(\nu) \left[ 1 + \frac{u^2}{4(1-\nu)} - \frac{1}{2^{2\nu}} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} u^{2\nu} + \ldots \right]. \tag{E.4}
$$

Also, we find the decomposition

$$
W(u)\mathcal{K}_\nu(u) = 2\mathcal{K}_\nu(u) - \mathcal{K}_{\nu+1}(u) =
$$

$$
= 2^{\nu}(1-\nu)\Gamma(\nu) \left[ 1 + \frac{1}{4(1-\nu)^2} u^2 - \frac{1}{2^{2\nu}(1-\nu)} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} u^{2\nu} + \ldots \right]. \tag{E.5}
$$

**Integer orders $\nu$.** For integer orders $\nu$ that correspond to even boundary dimensions $d$ the modified Bessel function $\mathcal{K}_\nu(u)$ (E.2) can be represented as the following series

$$
\mathcal{K}_\nu(u) = 2^{\nu-1} \Gamma(\nu) \sum_{n=0}^{\nu-1} \frac{1}{n! \Gamma(1-\nu+n)} \left( \frac{u}{2} \right)^{2n} +
$$

$$
+ \frac{(-)^{\nu+1}}{2^{2\nu-1} \Gamma(\nu) \Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{2^{\nu-1}}{n! \Gamma(1+\nu+n)} \left( \frac{u}{2} \right)^{2\nu+2n} \left( 2 \ln \frac{u}{2} - \psi(n+1) - \psi(\nu + n + 1) \right), \tag{E.6}
$$

where $\psi(k)$ is the logarithmic derivative $\psi(k) = \Gamma'(k)/\Gamma(k)$. All terms in the above series have even powers of $u$ except for those proportional to the logarithm which therefore define a non-local part of $\mathcal{K}_\nu(\epsilon \sqrt{\Box})$. The decompositions analogous to (E.4) and (E.5) read

$$
\mathcal{K}_\nu(u) = 2^{\nu-1} \Gamma(\nu) \left[ 1 + \frac{u^2}{4(1-\nu)} + \frac{(-)^{\nu+1}}{2^{2\nu-1} \Gamma(\nu) \Gamma(\nu+1)} u^{2\nu} \ln \frac{u}{2} + \ldots \right], \tag{E.7}
$$

and

$$
W(u)\mathcal{K}_\nu(u) = 2\mathcal{K}_\nu(u) - \mathcal{K}_{\nu+1}(u) =
$$

$$
= 2^\nu(1-\nu)\Gamma(\nu) \left[ 1 + \frac{1}{4(1-\nu)^2} u^2 + \frac{(-)^{\nu+1}}{2^{2\nu-1}(1-\nu) \Gamma(\nu) \Gamma(\nu+1)} u^{2\nu} \ln \frac{u}{2} + \ldots \right]. \tag{E.8}
$$

**F Details of computation of the effective action**

To simplify computations we use the following obvious identity,

$$
\partial_z \left[ \left( \frac{z}{\epsilon} \right)^\gamma \frac{F(z\epsilon)}{F(\epsilon k)} \right]_{z=\epsilon} = \epsilon^{-1} \left( \gamma + (k\epsilon) \frac{d}{d(\epsilon k)} \ln F(\epsilon k) \right), \tag{F.1}
$$

where $F(u)$ is an arbitrary function and $\gamma$ is a real constant. The expression on the left-hand-side is motivated by the form of solution for the hook component (D.16).
Let us represent a normal derivative of the hook component as a sum of six terms,

\[
\partial_z \varphi_{ij,k} \bigg|_{z=\epsilon} = \sum_{n=1}^{3} A_n + B + C + D ,
\]

where using solution (D.16) and omitting explicit indices we obtain the following derivatives

\[
D = \partial_z \left[ \left( \frac{z}{\epsilon} \right)^{-3} \frac{K_{\nu}(z \epsilon)}{K_{\nu}(\epsilon)} \right] \varphi_{ij,k} \bigg|_{z=\epsilon} ,
\]

and

\[
A_1 = \frac{k_i}{k^2} \partial_z (k^m \varphi_{mj,k}) \bigg|_{z=\epsilon} , \quad A_2 = \frac{k_j}{k^2} \partial_z (k^m \varphi_{mi,k}) \bigg|_{z=\epsilon} , \quad A_3 = \frac{k_k}{k^2} \partial_z (k^m \varphi_{ij,m}) \bigg|_{z=\epsilon} ,
\]

\[
B = \partial_z \left[ -\frac{a}{2k^2} \left( \frac{1}{z^3} + \frac{1}{z^3} \left( \frac{d-3}{k^2} - \epsilon^2 \right) \right) M_{ij,k} \right] \bigg|_{z=\epsilon} , \quad C = \partial_z \left[ -\frac{2a}{k^2} \frac{1}{z^2} Z_{ij,k} \right] \bigg|_{z=\epsilon} .
\]

Evaluating \( z \)-derivatives in quantities \( A_1, A_2, A_3 \), and \( B, C, D \) given by formulas (F.3)-(F.5) yields the following expressions,

\[
D = \left[ -\frac{3}{\epsilon} + \frac{\epsilon k^2}{2(1-\nu)} - \frac{2\nu (1-\nu)}{2\nu (1+\nu)} \epsilon^{2\nu-1} k^2 + \ldots \right] \varphi_{ij,k} ,
\]

\[
A_1 = \frac{3}{4} \frac{k_i}{k^2} M_{jk} - \frac{1}{2} \left[ \frac{6(d-2)}{\epsilon^4} \frac{a}{k^2} + \frac{3}{2} \frac{b}{\epsilon^4} + \frac{a}{2\epsilon^2} \right] \frac{k_i}{k^2} S_{jk} - \frac{3}{2\epsilon^2} \frac{k_i}{k^2} T_{jk} - \frac{3}{4} \left( \frac{a}{\epsilon^2} + \frac{3b}{\epsilon^4} \right) \frac{k_i}{k^2} Y_{jk} - \frac{1}{2} \Theta(\epsilon, k) \frac{k_i}{k^2} M_{jk} ,
\]

\[
A_2 = \frac{3}{4} \frac{k_j}{k^2} M_{ik} - \frac{1}{2} \left[ \frac{6(d-2)}{\epsilon^4} \frac{a}{k^2} + \frac{3}{2} \frac{b}{\epsilon^4} + \frac{a}{2\epsilon^2} \right] \frac{k_j}{k^2} S_{ik} - \frac{3}{2\epsilon^2} \frac{k_j}{k^2} T_{ik} - \frac{3}{4} \left( \frac{a}{\epsilon^2} + \frac{3b}{\epsilon^4} \right) \frac{k_j}{k^2} Y_{ik} - \frac{1}{2} \Theta(\epsilon, k) \frac{k_j}{k^2} M_{ik} ,
\]

\[
A_3 = \frac{3}{2\epsilon^2} \frac{k_k}{k^2} M_{ij} + \left[ \frac{6(d-2)}{\epsilon^4} \frac{a}{k^2} + \frac{3}{2} \frac{b}{\epsilon^4} + \frac{a}{2\epsilon^2} \right] \frac{k_k}{k^2} S_{ij} + \Theta(\epsilon, k) \frac{k_k}{k^2} M_{ij} ,
\]

\[
B = \frac{a}{2k^2} \left( \frac{1}{\epsilon^2} + \frac{3}{\epsilon^4} \frac{A(d-3)}{k^2} - \epsilon^2 \right) M_{ij,k} , \quad C = \frac{6a}{\epsilon^4 k^2} Z_{ij,k} ,
\]

where quantities \( a \) and \( b \) are given by (D.13), and

\[
\Theta(\epsilon, k) = \left( \frac{1}{4} \frac{2-\nu}{(1-\nu)^2} \epsilon k^2 - \frac{\nu}{2^{2\nu}(1-\nu)} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \epsilon^{2\nu-1} k^2 + \ldots \right) .
\]

Quantities \( A_2 \) and \( A_3 \) are equal to each over under permutation of indices \( i \) and \( j \).
**G Fourier transforms and differential regularization**

Non-integer values $d/2$. To perform a Fourier transformation of a two-point function defined in momentum space to position space one calculates integrals of the following type

\[
\int \frac{d^d k}{(2\pi)^d} e^{-i k x^\alpha} \frac{2^\alpha \Gamma(d+\alpha)}{\Gamma(\frac{d}{2})} \frac{1}{x^{d+\alpha}}.
\]  \(\text{(G.1)}\)

For positive integer $\alpha/2$ the gamma function in the denominator has zeros so the Fourier transform is ill-defined, see the next paragraph. By taking $x$-derivatives one obtains

\[
(-i)^\beta \int \frac{d^d k}{(2\pi)^d} e^{-i k x^\alpha} k_{m_1} \cdots k_{m_\beta} \Gamma_c(m_1) \cdots \Gamma_c(m_\beta) \frac{1}{x^{d+\alpha}}.
\]  \(\text{(G.2)}\)

Integer values $d/2$. To find Fourier transforms of logarithmic functions multiplied by polynomial functions we generalize the differential regularization scheme for $4d \varphi^4$ theory proposed in [25] to any even dimensions $d$.

The idea is to identically rewrite function $1/x^{2\rho}$, where $x \in \mathbb{R}^d/\{0\}$ and $\rho \in \mathbb{Z}_+$, in the following way

\[
\frac{1}{x^{2\rho}} = \Box G(x^2),
\]  \(\text{(G.3)}\)

where $\Box = \partial^i \partial_i$ and $x^2 = x^i x_i$ are evaluated with respect to Euclidean metric $\delta_{ij}$. It is convenient to change $w = x^2$. Then equation \(\text{(G.3)}\) takes the form

\[
4 w^{\rho - \frac{d}{2} + 1} \frac{d}{dw} \left[ w^\frac{d}{2} \frac{d}{dw} G(w) \right] = 1.
\]  \(\text{(G.4)}\)

The solution to this second-order ODE at $\rho = d/2$ for even $d$ is given by

\[
G(x^2) = -\frac{1}{2d-4} \ln(x M)^2 + C, \quad x \neq 0,
\]  \(\text{(G.5)}\)

where $M$ and $C$ are dimensionful integration constants. In what follows, we set $C = 0$. For $d = 4$ the above formula reproduces result obtained previously in [25]. Therefore, we conclude that aside from the singularity, function $1/x^{2\rho}$ admits an equivalent representation with a logarithm and a dimensionful constant so the name differential regularization. It is worth noting that $1/x^{2\rho}$ is a homogeneous function while $G(x^2)$ is not. However, identity \(\text{(G.3)}\) holds for any values of constant $M$ so scale transformation $x^m \rightarrow tx^m$ define an equivalence between functions $G(x^2)$ with $M$ and $tM$.

Other functions $1/x^{d+2m}$, $m = 0, 1, 2, ...$ can be obtained by differentiating with $\Box$, e.g.,

\[
\Box x^{-d} = 2 d x^{-d-2}. \quad \text{In particular, one has}
\]

\[
\frac{1}{x^{d+2m}} = \tau_m \Box^m \frac{1}{x^d}, \quad \tau_m = \frac{\Gamma(d/2)}{4^m \Gamma(m+1) \Gamma(d/2+m)}, \quad m = 0, 1, 2, ...,
\]  \(\text{(G.6)}\)

\footnote{See also Ref. [44], where these integrals are evaluated by inserting a test function $e^{-\mu k^2}$ and taking the limit $\mu \rightarrow 0$.}
so that function $x^{-2d}$ can be cast into the form

$$\frac{1}{x^{2d}} = \frac{2^{-d+1}}{\Gamma(d+1)} x^{d+1} \ln(xM)^2 \bigg/ x^{d-2}.$$  

(G.7)

By construction, $d/2 + 1$ is an integer positive number, so we conclude that dimensions $d$ must be even. In odd dimensions $d$ the above representation is not valid; in particular, it becomes non-local. The above regulated form of $x^{-2d}$ is convenient to Fourier transform. To obtain this formula one notices that i) Fourier transforms of power law functions are easy to obtain ii) power law functions can be represented as logarithmic series,

$$(Mx)^{2a} = \sum_{n=0}^{\infty} \frac{a^n}{n!} \ln^n(xM)^2 = 1 + 2a \ln(xM) + \ldots , \quad \text{(G.8)}$$

where $a$ is an indeterminant variable, and then finds its Fourier transform,

$$\int d^d x e^{i k x} \frac{(Mx)^{2a}}{x^{d-2}} = \frac{4\pi^{\frac{d}{2}}}{k^2} \left( \frac{2M}{k} \right)^{2a} \frac{\Gamma(1+a)}{\Gamma(d/2-a-1)} . \quad \text{(G.9)}$$

Representing gamma functions via formula 8.342.2 from [45]

$$\ln \Gamma(1+a) = \frac{1}{2} \ln \frac{\pi a}{\sin \pi a} - \zeta(2) a - \sum_{n=1}^{\infty} \frac{a^{2n+1}}{2n+1} \zeta(2n+1) , \quad \text{(G.10)}$$

where $\zeta(n)$ is the zeta-function, and

$$\Gamma(d/2-a-1) = (d/2 - 2 - a)(d/2 - 3 - a)...(1-a)\Gamma(1-a) , \quad \text{(G.11)}$$

decomposing with respect to small $a$ one finds

$$\frac{\Gamma(1+a)}{\Gamma(d/2 - a - 1)} = (1 + aH_{d/2-2})(1 - 2\zeta(2) a) + \mathcal{O}(a^2) , \quad \text{(G.12)}$$

where $H_{d/2-2}$ are harmonic numbers ($H_0 = 0$, $H_1 = 1$, $H_2 = 3/2$, ...). Terms linear in $a$ in the right-hand-side of (G.9) are then

$$\mathcal{O}(a^2) = \frac{4\pi^{\frac{d}{2}}}{\Gamma(d/2 - 1)} \frac{1}{k^2} \left[ 1 - 2a \ln \left( \frac{kH_{d/2-2}}{2M} \right) \right] + \mathcal{O}(a^2) , \quad \text{(G.13)}$$

where $\ln H_{d/2-2} = \zeta(2) - H_{d/2-2}/2$. It follows that

$$\int d^d x e^{i k x} \frac{1}{x^{d-2}} = \frac{4\pi^{\frac{d}{2}}}{\Gamma(d/2 - 1)} \frac{1}{k^2} , \quad \text{(G.14)}$$

$$\int d^d x e^{i k x} \frac{\ln(Mx)}{x^{d-2}} = - \frac{4\pi^{\frac{d}{2}}}{\Gamma(d/2 - 1)} \frac{1}{k^2} \ln \frac{k}{M} , \quad \text{(G.15)}$$

where $\tilde{M} = 2M/H_{d/2-2}$. Note that (G.14) coincides with (G.1) evaluated at $\alpha = -2$. Higher orders of $a$ in (G.9) produce different powers of the logarithm.
Now, using the Fourier transform inverse to (G.15) and acting with $\Box$ one obtains
\[
\int \frac{d^d k}{(2\pi)^d} e^{-ikx} k^{2\alpha-2} \ln \frac{k}{M} = (-)^{\alpha+1} \frac{\Gamma(d/2 - 1)}{4\pi^{d/2}} \Box \ln(Mx),
\] (G.16)
where $\alpha = 0, 1, 2, \ldots$. Using formula (G.6) at $n = \alpha - 1$ allows to represent the above integral in the form analogous to (G.1),
\[
\int \frac{d^d k}{(2\pi)^d} e^{-ikx} k^{2\alpha-2} \ln \frac{k}{M} = (-)^{\alpha} \frac{2^{d-3}}{\pi^{d/2}} \Gamma(d/2 + \alpha - 1) \frac{1}{x^{d+2\alpha-2}},
\] (G.17)
where $\alpha = 1, 2, \ldots$. By taking $x$-derivatives in the integral (G.16) one obtains
\[
i^\beta \int \frac{d^d k}{(2\pi)^d} e^{-ikx} k_{m_1} \cdots k_{m_\beta} k^{2\alpha-2} \ln \frac{k}{M} = (-)^{\alpha+\beta} \frac{\Gamma(d/2 - 1)}{4\pi^{d/2}} \Box^{\alpha} \partial_{m_1} \cdots \partial_{m_\beta} \ln(Mx) x^{d-2},
\] (G.18)
where $\alpha, \beta = 0, 1, 2, \ldots$. Finally, one arrives at the formula analogous to (G.2),
\[
i^\beta \int \frac{d^d k}{(2\pi)^d} e^{-ikx} k_{m_1} \cdots k_{m_\beta} k^{2\alpha-2} \ln \frac{k}{M} =
\]
\[
= (-)^{\alpha+\beta} \frac{2^{d-3}}{\pi^{d/2}} \Gamma(d/2 + \alpha - 1) \partial_{m_1} \cdots \partial_{m_\beta} \frac{1}{x^{d+2\alpha-2}},
\] (G.19)
where $\alpha = 1, 2, \ldots$ and $\beta = 0, 1, 2, \ldots$.

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