ON AN N-COMPONENT CAMASSA-HOLM EQUATION WITH PEAKONS

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Abstract. In this paper, we are concerned with N-Component CamassaHolm equation with peakons. Firstly, we establish the local well-posedness in a range of the Besov spaces by using Littlewood-Paley decomposition and transport equation theory. Secondly, we present a precise blowup scenario and several blowup results for strong solutions to that system, we then obtain the blowup rate of strong solutions when a blowup occurs. Next, we investigate the persistence property for the strong solutions. Finally, we consider the initial boundary value problem, our approach is based on sharp extension results for functions on the half-line and several symmetry preserving properties of the equations under discussion.

1. Introduction. In this paper, we consider the following N-Component Camassa-Holm equation with peakons

\[
\begin{align*}
    m_{1,t} + m_{1,x}u_1 + 2m_1u_{1,x} + & \left( m_1 \sum_{j \neq 1}^n u_j \right)_x + \sum_{j \neq 1}^n m_j u_{j,x} = 0, \\
    m_{2,t} + m_{2,x}u_2 + 2m_2u_{2,x} + & \left( m_2 \sum_{j \neq 2}^n u_j \right)_x + \sum_{j \neq 2}^n m_j u_{j,x} = 0, \\
    \vdots & \quad \vdots \\
    m_{n,t} + m_{n,x}u_n + 2m_nu_{n,x} + & \left( m_n \sum_{j \neq n}^n u_j \right)_x + \sum_{j \neq n}^n m_j u_{j,x} = 0,
\end{align*}
\]

(1)

where \( m_i = u_i - u_{i,xx}, i = 1, 2, \ldots, n \). A direct computation shows that the system (1) admits multipeakon solitons and \( H^1 \)-norm conservation laws. This system was proposed firstly by Fu etc. [30, 29, 31].

For \( n = 1 \), Eq. (1) becomes Camassa-Holm equation

\[
m_{1,t} + u_1m_{1,x} + 2u_{1,x}m_1 = 0, \quad m_1 = u_1 - u_{1,xx}
\]

(2)

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can itself be derived from the Korteweg-deVries equation by tri-Hamiltonian duality. The Camassa-Holm equation was originally proposed as a model for surface water waves in the shallow water regime (see [5, 20]) and has been studied extensively in the last twenty years because of its many remarkable properties. Camassa-Holm equation has traveling wave solutions of the form \( ce^{-|x-ct|} \), called peakons, which capture the main feature of the exact traveling wave solutions of greatest height of the governing equations (see [3, 19, 10]). Moreover, the shape of some peakons is stable under small perturbations, making these waves recognizable physically (see [22, 43]). It was shown in [21, 16, 9] that the inverse spectral or scattering approach was a powerful tool to handle Camassa-Holm equation and Eq. (2) is a completely integrable and The geometric formulations [18, 19, 42, 45], well-posedness and breaking waves, meaning solutions that remain bounded while its slope becomes unbounded in finite time [7, 13, 14, 15, 44] have been discussed. Moreover, the Camassa-Holm equation has global conservative solutions [3, 35] and dissipative solutions [4, 36]. An alternative modified Camassa-Holm equation was studied in [28]. Multi-component versions of the Camassa–Holm equation have been introduced and studied in [32, 33, 38, 39, 40, 17].

For \( n = 2 \), Eq. (1) becomes the two-component Camassa-Holm equation introduced in [30, 29]

\[
\begin{align*}
    m_{1,t} + m_{1,x} u_1 + 2 m_1 u_{1,x} + (m_1 u_2)_x + m_2 u_{2,x} &= 0, \\
    m_{2,t} + m_{2,x} u_2 + 2 u_2 u_{2,x} + (m_2 u_1)_x + m_1 u_{1,x} &= 0,
\end{align*}
\]

where \( m_i = u_i - u_{i,xx}, i = 1, 2 \). The Cauchy problem of the above two-component Camassa-Holm equation has been studied in [30, 29]. It has been shown that this system is locally well-posed for initial data \((u_0, v_0) \in H^s \times H^s \) with \( s > \frac{3}{2} \). Also, it has blowup solutions modeling wave breaking, moreover, an existence result for a class of local weak solutions was also given. In [50], the authors presented some new criteria on blow-up, global existence and blow-up rate of the solution. Moreover, they discussed persistence properties of this system. In [47], the authors studied the global conservative and dissipative solutions of system 1.2. In [45], the authors obtained the compact and bounded absorbing set and the existence of the global attractor for viscous system (3) with the periodic boundary condition in \( H^2 \) by uniform prior estimate. For \( n = 3 \), it was shown in [31] that the Eq. (1) has two peakon solitons. In [37], the authors establish the local well-posedness of the initial value problem for system (1) with \( n = 3 \) and present a precise blowup scenario and several blowup results for strong solutions to that system. Moreover, they determine the blowup rate of strong solutions to the system when a blowup occurs.

Motivated by the references above, we establish firstly the local well-posedness for the strong solutions to the Cauchy problem of system (1). The proof of the local well-posedness is inspired by the argument of approximate solutions by Danchin [23, 24, 25] in the study of the local wellposedness to the Camassa-Holm equation. Secondly, we then present a precise blowup scenario, several blowup results for strong solutions to that system, we also obtain the blowup rate of strong solutions to the system when a blowup occurs. Next, we investigate the persistence property for the strong solutions to (4) in \( L^\infty \) space which asymptotically exponential decay at infinity as their initial profiles. The idea comes from a recent work of Zhou and his collaborators [34] for the standard Camassa-Holm equation (for slower decay rate, we refer to [46] ). Finally, we study initial value boundary problems of the N-Component Camassa-Holm equation on the half-line subject to homogeneous
Dirichlet boundary conditions. Our approach is based on sharp extension results for functions on the half-line and several symmetry preserving properties of the equations under discussion.

The rest of this paper is organized as follows. In Section 2, we prove the local well-posedness of the Cauchy problem of system (1) in the Besov spaces. In Section 3, the blowup phenomena is considered. Section 4 is devoted to the study of the persistence property for the strong solutions. Finally, we consider the initial boundary value problem.

2. Local well-posedness in $B^s_{p,r}, p, r \in [1, \infty], s > \max\{\frac{3}{2}, 1 + \frac{1}{r}\}$. In this section, we shall establish local well-posedness of the initial value problem (1) in the Besov spaces. Note that $G(x) := \frac{1}{2} e^{-|x|}$ is the kernel of $(1 - \partial_x^2)^{-1}$. Then $(1 - \partial_x^2)^{-1} f = G * F$ for all $f \in L^2(\mathbb{R})$ and $G * u = u, i = 1, 2, \cdots, n$. Eq. (1) takes the form of a quasi-linear evolution equation of hyperbolic type

\[
\begin{cases}
  u_{1,t} + u_x \sum_{j=1}^{n} u_j + G \left( u_x \sum_{j \neq 1}^{n} u_{j,x} \right) + \partial_x G \\
  \quad + \left( u_1^2 + \frac{1}{2} \sum_{j \neq 1}^{n} u_j^2 \right) + \frac{1}{2} \left( u_x^2 - \sum_{j \neq 1}^{n} u_{j,x}^2 \right) + u_x \sum_{j \neq 1}^{n} u_j = 0, \\
  \quad t > 0, x \in \mathbb{R},
  \\
  u_{2,t} + u_x \sum_{j=1}^{n} u_j + G \left( u_x \sum_{j \neq 1}^{n} u_{j,x} \right) + \partial_x G \\
  \quad + \left( u_1^2 + \frac{1}{2} \sum_{j \neq 1}^{n} u_j^2 \right) + \frac{1}{2} \left( u_x^2 - \sum_{j \neq 1}^{n} u_{j,x}^2 \right) + u_x \sum_{j \neq 1}^{n} u_j = 0, \\
  \quad t > 0, x \in \mathbb{R},
  \\
  \vdots
  \\
  u_{n,t} + u_x \sum_{j=1}^{n} u_j + G \left( u_x \sum_{j \neq 1}^{n} u_{j,x} \right) + \partial_x G \\
  \quad + \left( u_1^2 + \frac{1}{2} \sum_{j \neq 1}^{n} u_j^2 \right) + \frac{1}{2} \left( u_x^2 - \sum_{j \neq 1}^{n} u_{j,x}^2 \right) + u_x \sum_{j \neq 1}^{n} u_j = 0, \\
  \quad t > 0, x \in \mathbb{R}, \quad j = 1, 2, \cdots, n.
\end{cases}
\]

Introducing the Fourier integral operators $P_1(D) = -(1 - \partial_x^2)^{-1}, P_2(D) = -\partial_x (1 - \partial_x^2)^{-1}$, then system (4) becomes

\[
\begin{cases}
  u_{1,t} + u_x \sum_{j=1}^{n} u_j = P_1(D) \left( u_x \sum_{j \neq 1}^{n} u_{j,x} \right) + P_2(D) \left( u_1^2 + \frac{1}{2} \sum_{j \neq 1}^{n} u_j^2 \right) \\
  \quad + \frac{1}{2} \left( u_x^2 - \sum_{j \neq 1}^{n} u_{j,x}^2 \right) + u_x \sum_{j \neq 1}^{n} u_j, \\
  \quad t > 0, x \in \mathbb{R},
  \\
  u_{2,t} + u_x \sum_{j=1}^{n} u_j = P_1(D) \left( u_x \sum_{j \neq 1}^{n} u_{j,x} \right) + P_2(D) \left( u_1^2 + \frac{1}{2} \sum_{j \neq 1}^{n} u_j^2 \right) \\
  \quad + \frac{1}{2} \left( u_x^2 - \sum_{j \neq 1}^{n} u_{j,x}^2 \right) + u_x \sum_{j \neq 1}^{n} u_j, \\
  \quad t > 0, x \in \mathbb{R},
  \\
  \vdots
  \\
  u_{n,t} + u_x \sum_{j=1}^{n} u_j = P_1(D) \left( u_x \sum_{j \neq 1}^{n} u_{j,x} \right) + P_2(D) \left( u_1^2 + \frac{1}{2} \sum_{j \neq 1}^{n} u_j^2 \right) \\
  \quad + \frac{1}{2} \left( u_x^2 - \sum_{j \neq 1}^{n} u_{j,x}^2 \right) + u_x \sum_{j \neq 1}^{n} u_j, \\
  \quad t > 0, x \in \mathbb{R}, \quad j = 1, 2, \cdots, n.
\end{cases}
\]

First, for the convenience of the readers, we recall some facts on the Littlewood-Paley decomposition and some useful lemmas.

**Notation.** $\mathcal{S}$ stands for the Schwartz space of smooth functions over $\mathbb{R}^d$ whose derivatives of all order decay at infinity. The set $\mathcal{S}'$ of temperate distributions is the dual set of $\mathcal{S}$ for the usual pairing. We denote the norm of the Lebesgue space $L^p(\mathbb{R})$ by $\| \cdot \|_{L^p}$ with $1 \leq p \leq \infty$, and the norm in the Sobolev space $H^s(\mathbb{R})$ with $s \in \mathbb{R}$ by $\| \cdot \|_{H^s}$. 
Proposition 2.1. (Littlewood-Paley decomposition [3]) Let $\mathcal{B} = \{ \xi \in \mathbb{R}^d, |\xi| \leq \frac{4}{3} \}$ and $\mathcal{C} = \{ \xi \in \mathbb{R}^d, \frac{4}{3} \leq |\xi| \leq \frac{8}{3} \}$. There exist two radial functions $\chi \in C_c^\infty(\mathcal{B})$ and $\varphi \in C_c^\infty(\mathcal{C})$ such that

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1, \forall \xi \in \mathbb{R}^d,$$

where $|q - q'| \geq 2 \Rightarrow \text{Supp} \varphi(2^{-q} \cdot) \cap \text{Supp} \varphi(2^{-q'} \cdot) = \varnothing$,

$$q \geq 1 \Rightarrow \supp \chi(\cdot) \cap \text{Supp} \varphi(2^{-q} \cdot) = \varnothing,$$

$$\frac{1}{3} \leq (\chi(\xi))^2 + \sum_{q \geq 0} (\varphi(2^{-q}\xi))^2 \leq 1, \forall \xi \in \mathbb{R}^d.$$

Furthermore, let $h \doteq \mathcal{F}^{-1} \varphi$ and $\tilde{h} \doteq \mathcal{F}^{-1} \chi$. Then for all $f \in S'(\mathbb{R}^d)$, the dyadic operators $\Delta_q$ and $S_q$ can be defined as follows

$$\Delta_q f \doteq \varphi(2^{-q} D) f = 2^{qd} \int_{\mathbb{R}^d} h(2^q y) f(x - y) dy \text{ for } q \geq 0,$$

$$S_q f \doteq \chi(2^{-q} D) f = \sum_{-1 \leq k \leq q - 1} \Delta_k = 2^{qd} \int_{\mathbb{R}^d} \tilde{h}(2^q y) f(x - y) dy,$$

$$\Delta_{-1} f \doteq S_0 f \text{ and } \Delta_q f \doteq 0 \text{ for } q \leq -2.$$

Hence,

$$f = \sum_{q \geq 0} \Delta_q f \text{ in } S'(\mathbb{R}^d),$$

where the right-hand side is called the nonhomogeneous Littlewood-Paley decomposition of $f$.

Lemma 2.1. (Bernstein’s inequality [24]) Let $\mathcal{B}$ be a ball with center 0 in $\mathbb{R}^d$ and $\mathcal{C}$ a ring with center 0 in $\mathbb{R}^d$. A constant $C$ exists so that, for any positive real number $\lambda$, any non-negative integer $k$, any smooth homogeneous function $\sigma$ of degree $m$ and any couple of real numbers $(a, b)$ with $b \geq a \geq 1$, there hold

$$\text{Supp} \hat{u} \subset \lambda \mathcal{B} \Rightarrow \sup_{|\alpha| = k} \| \partial^\alpha u \|_{L^a} \leq C^{k+1} \lambda^{b+1} \| u \|_{L^b},$$

$$\text{Supp} \hat{u} \subset \lambda \mathcal{C} \Rightarrow C^{-k-1} \lambda^k \| u \|_{L^a} \leq \sup_{|\alpha| = k} \| \partial^\alpha u \|_{L^a} \leq C^{k+1} \lambda^k \| u \|_{L^b},$$

$$\text{Supp} \hat{u} \subset \lambda \mathcal{C} \Rightarrow \| \sigma(D) u \|_{L^a} \leq C_{m,a} \lambda^{m+d(\frac{1}{b} - \frac{1}{a})} \| u \|_{L^b},$$

for any function $u \in L^a$.

Definition 2.1. (Besov space) Let $s \in \mathbb{R}, 1 \leq p, r \leq \infty$. The inhomogenous Besov space $B^s_{p,r}(\mathbb{R}^d)$ ($B^s_{p,r}$ for short) is defined by

$$B^s_{p,r} \doteq \{ f \in S'(\mathbb{R}^d) ; \| f \|_{B^s_{p,r}} < \infty \},$$

where

$$\| f \|_{B^s_{p,r}} \doteq \begin{cases} \left( \sum_{q \in \mathbb{Z}} 2^{qs} \| \Delta_q f \|_{L^p}^r \right)^{\frac{1}{r}} , & \text{ for } r < \infty, \\ \sup_{q \in \mathbb{Z}} 2^{qs} \| \Delta_q f \|_{L^p} , & \text{ for } r = \infty. \end{cases}$$

If $s = \infty$, $B^\infty_{p,r} \doteq \cap_{s \in \mathbb{R}} B^s_{p,r}$.

Proposition 2.2. (see [24]) Suppose that $s \in \mathbb{R}, 1 \leq p, r, r_i \leq \infty (i = 1, 2)$. We have
(1) Topological properties: $B_{p,r}^s$ is a Banach space which is continuously embedded in $S'$.

(2) Density: $C_c^\infty$ is dense in $B_{p,r}^s$ if $1 \leq p, r \leq \infty$.

(3) Embedding: $B_{p_1,r_1}^s \hookrightarrow B_{p_2,r_2}^{s-n\frac{1}{p_2}+\frac{1}{r_2}}$, if $p_1 \leq p_2$ and $r_1 \leq r_2$.

(4) Algebraic properties: $\forall s > 0$, $B_{p,r}^s \cap L^\infty$ is an algebra. Moreover, $B_{p,r}^s$ is an algebra, provided that $s > \frac{p}{p}$ or $s \geq \frac{n}{p}$ and $r = 1$.

(5) Complex interpolation:

\[ ||u||_{B_{p,r}^{s+\theta(1-s)\cdot 2}} \leq C ||u||_{B_{p,r}^s}^{1-\theta} ||u||_{B_{p,r}^{s+2}}^\theta, \quad \forall u \in B_{p,r}^{s+2} \cap B_{p,r}^s, \quad \forall \theta \in [0, 1]. \]

(6) Fatou lemma: If $(u_n)_{n \in \mathbb{N}}$ is bounded in $B_{p,r}^s$ and $u_n \rightarrow u$ in $S'$, then $u \in B_{p,r}^s$ and

\[ ||u||_{B_{p,r}^s} \leq \liminf_{n \rightarrow \infty} ||u_n||_{B_{p,r}^s}. \]

(7) Let $m \in \mathbb{R}$ and $f$ be an $S^m$-multiplier (i.e., $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is smooth and satisfies that $\forall \alpha \in \mathbb{N}^d$, there exists a constant $C_\alpha$, s.t. $|\partial^\alpha f(\xi)| \leq C_\alpha (1 + |\xi|^{m-|\alpha|})$ for all $\xi \in \mathbb{R}^d$). Then the operator $f(D)$ is continuous from $B_{p,r}^s$ to $B_{p,r}^{s-m}$.

Now we state some useful results in the transport equation theory, which are crucial to the proofs of our main theorems later.

**Lemma 2.2.** (see [23, 24]) Suppose that $(p, r) \in [1, +\infty]^2$ and $s > -\frac{d}{p}$. Let $v$ be a vector field such that $v\nu$ belongs to $L^1([0, T]; B_{p,r}^{s-1})$ if $s > 1 + \frac{d}{p}$ or to $L^1([0, T]; B_{p,r}^{s-2} \cap L^\infty)$ otherwise. Suppose also that $f_0 \in B_{p,r}^s$, $F \in L^1([0, T]; B_{p,r}^s)$ and that $f \in L^\infty(L^1([0, T]; B_{p,r}^s) \cap C([0, T]; S'))$ solves the $d$-dimensional linear transport equations

\[
\begin{align*}
\partial_t f + v \cdot \nabla f &= F, \\
\{f\}_{t=0} &= f_0.
\end{align*}
\]

Then there exists a constant $C$ depending only on $s, p$ and $d$ such that the following statements hold:

(1) If $r = 1$ or $s \neq 1 + \frac{d}{p}$, then

\[
||f||_{B_{p,r}^s} \leq ||f_0||_{B_{p,r}^s} + \int_0^t ||F(\tau)||_{B_{p,r}^s} d\tau + C \int_0^t V(\tau)||f(\tau)||_{B_{p,r}^s} d\tau,
\]

or

\[
||f||_{B_{p,r}^s} \leq e^{CV(t)} C \left( ||f_0||_{B_{p,r}^s} + \int_0^t e^{-CV(\tau)} ||F(\tau)||_{B_{p,r}^s} d\tau \right)
\]

hold, where $V(t) = \int_0^t ||\nabla v(\tau)||_{B_{p,r}^{s-1}} d\tau$ if $s < 1 + \frac{d}{p}$ and $V(t) = \int_0^t ||\nabla v(\tau)||_{B_{p,r}^{s-2} \cap L^\infty} d\tau$ otherwise.

(2) If $s \leq 1 + \frac{d}{p}$ and $\nabla f_0 \in L^\infty$, $\nabla f \in L^\infty([0, T] \times \mathbb{R}^d)$ and $\nabla F \in L^1([0, T]; L^\infty)$, then

\[
||f||_{B_{p,r}^s} + ||\nabla f||_{L^\infty}
\]

\[
\leq e^{CV(t)} \left( ||f_0||_{B_{p,r}^s} + ||\nabla f_0||_{L^\infty} + \int_0^t e^{-CV(\tau)} ||F(\tau)||_{B_{p,r}^s} + ||\nabla F(\tau)||_{L^\infty} d\tau \right)
\]

with $V(t) = \int_0^t ||\nabla v(\tau)||_{B_{p,r}^{s-1}} d\tau$. 

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Proposition 2.3. Let \( u \) and \( u(1) \) hold true. Moreover, \( r < f \rho > \). Let \( v \) be a time dependent vector field such that \( v \in L^p([0, T]; \mathbb{B}^{\rho}_{p,r}) \) for some \( \rho > 1, M > 0 \) and \( \nabla v \in L^1([0, T]; \mathbb{B}^{\rho}_{p,r} \cap \mathbb{L}^\infty) \) if \( s < 1 + \frac{4}{p^2} \) and \( \nabla v \in L^1([0, T]; \mathbb{B}^{\rho}_{p,r}^{-1}) \) if \( s > 1 + \frac{4}{p^2} \) or \( s = 1 + \frac{4}{p^2} \) and \( r = 1 \). Then the transport equations (T) has a unique solution \( f \in L^\infty([0, T]; \mathbb{B}^{r}_{p,r}) \cap (\mathbb{G} < s) \cap [0, T]; \mathbb{B}^{s}_{p,r}) \) and the inequalities in Lemma 2.2 hold true. Moreover, \( r < \infty \), then we have \( f \in C([0, T]; \mathbb{B}^{s}_{p,r}) \).

Lemma 2.4. (1-D Morse-type estimates \[23, 24\]) Assume that \( 1 \leq p, r \leq +\infty \), the following estimates hold:

(i) For \( s > 0 \),
\[ \|fg\|_{\mathbb{B}^{s}_{p,r}} \leq C(\|f\|_{\mathbb{B}^{s}_{p,r}}, \|g\|_{L^\infty} + \|g\|_{\mathbb{B}^{s}_{p,r}}, \|f\|_{L^\infty}); \]

(ii) \( \forall s_1 \leq r \leq s_2 \) \( s_2 \geq \frac{1}{r} \) if \( r = 1 \) and \( s_1 + s_2 > 0 \), we have
\[ \|fg\|_{\mathbb{B}^{s}_{p,r}} \leq C(\|f\|_{\mathbb{B}^{s}_{p,r}}, \|g\|_{\mathbb{B}^{s}_{p,r}}); \]

(iii) In Sobolev spaces \( H^s = \mathbb{B}^{s}_{2,2} \), we have for \( s > 0 \),
\[ \|f_\partial g\|_{H^r} \leq C(\|f\|_{H^{s+1}}, \|g\|_{L^\infty} + \|g\|_{H^r}, \|f\|_{L^\infty}), \]
where \( C \) is a positive constant independent of \( f \) and \( g \).

Definition 2.1. For \( T > 0, s \in \mathbb{R} \) and \( 1 \leq p \leq +\infty \), we set
\[ E^s_{p,r}(T) \triangleq C([0, T]; \mathbb{B}^{s}_{p,r}) \cap \mathcal{C}^1([0, T]; \mathbb{B}^{r}_{p,r}^{-1}) \text{ if } r \leq +\infty, \]
\[ E^s_{p,\infty}(T) \triangleq \mathbb{L}^\infty([0, T]; \mathbb{B}^{s}_{p,\infty}) \cap \mathcal{L}^1([0, T]; \mathbb{B}^{s}_{p,\infty}^{-1}) \]
and \( E^s_{p,r} \triangleq \cap_{T > 0} E^s_{p,r}(T) \).

We now have the following local well-posedness result.

Theorem 2.1. Let \( p, r \in [1, \infty] \) and \( s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\} \). Assume that \( (u_1, u_2, \cdots, u_n) \in (\mathbb{B}^{s}_{p,r})^n \). There exists a time \( T > 0 \) such that the initial-value problem (4) has a unique solution \( (u_1, u_2, \cdots, u_n) \in (E^s_{p,r}(T))^n \) and the map \( (u_1, u_2, \cdots, u_n) \mapsto (u_1, u_2, \cdots, u_n) \) is continuous from a neighborhood of \( (u_1, u_2, \cdots, u_n) \) in \( (\mathbb{B}^{s}_{p,r})^n \) into \( (C([0, T]; \mathbb{B}^{s}_{p,r}) \cap \mathcal{C}^1([0, T]; \mathbb{B}^{s}_{p,r}^{-1}))^n \) for every \( s' < s \) when \( r = \infty \) and \( s' = s \) whereas \( r < \infty \).

In the following, we denote \( C > 0 \) a generic constant only depending on \( p, r, s \). Uniqueness and continuity with respect to the initial data are an immediate consequence of the following result.

Proposition 2.3. Let \( 1 \leq p, r \leq +\infty \) and \( s \geq \max\{1 + \frac{1}{p}, \frac{3}{2}\} \). Suppose that \( (u_1(1), u_2(1), \cdots, u_n(1)) \) and \( (u(2, 1), u(2, 2), \cdots, u_n(2)) \) be two given solutions of the initial-value problem (4) with the initial data \( (u_1(1), u_2(1), \cdots, u_n(1)), (u(1, 1), u(1, 2), \cdots, u_n(1)) \in \mathbb{B}^{s}_{p,r} \), satisfying \( u_j(1), u_j(2) \in \mathbb{L}^\infty([0, T]; \mathbb{B}^{s}_{p,r}) \cap \mathcal{C}([0, T]; \mathcal{S}^j) \) for \( j = 1, 2, \cdots, n \). Then for
every \( t \in [0; T] \), we have

\[
\sum_{j=1}^{n} \| u_j^{(1)}(t) - u_j^{(2)}(t) \|_{B_{p,r}^{-1}} \leq \left( \sum_{j=1}^{n} \left\| u_j^{(1)} - u_j^{(2)} \right\|_{B_{p,r}^{-1}} \right) e^{C \int_{0}^{t} \sum_{j=1}^{n} \| u_j^{(1)}(\tau) \|_{B_{p,r}^{-1}} + \| u_j^{(2)}(\tau) \|_{B_{p,r}^{-1}}} d\tau. \tag{7}
\]

**Proof.** For \( s \neq 2 + \frac{1}{p} \), denote \( u_j^{(12)} \equiv u_j^{(2)} - u_j^{(1)} \), \( j = 1, 2, \cdots, n \). It is obvious that \( u_j^{(12)} \in L^\infty([0, T]; B_{p,r}^{-1}) \cap C([0, T]: S') \), \( j = 1, 2, \cdots, n \), which implies that \( u_j^{(12)} \in C([0, T]; B_{p,r}^{-1}) \), \( j = 1, 2, \cdots, n \), and \( (u_1^{(12)}, u_2^{(12)}, \cdots, u_n^{(12)}) \) solves the transport equations

\[
\left\{ \begin{array}{ll}
u_i^{(12)} + u_i^{(12)} \sum_{j=1}^{n} u_j^{(1)} = f_i, & i = 1, 2, \cdots, n, \\
u_i^{(12)} |_{t=0} = u_{i,0}^{(12)} = u_i^{(2)} - u_i^{(1)}, & i = 1, 2, \cdots, n,
\end{array} \right. \tag{8}
\]

with

\[
f_i = -u_i^{(2)} \sum_{j=1}^{n} u_j^{(12)} + P_1(D) \left( u_i^{(1)} \sum_{j \neq i} u_j^{(12)} + u_i^{(12)} \sum_{j \neq i} u_j^{(2)} \right) + P_2(D) \left( u_i^{(12)} \right) \left( u_i^{(12)} \right) + \frac{1}{2} \sum_{j \neq i} u_j^{(2)} \left( u_i^{(1)} + u_i^{(2)} \right) + \frac{1}{2} \left( u_i^{(12)} \left( u_i^{(12)} + u_i^{(12)} + u_i^{(2)} \right) \right) - \sum_{j \neq i} u_j^{(12)} \left( u_i^{(12)} + u_i^{(2)} \right).
\]

According to Lemma 2.2, we have

\[
e^{-C \int_{0}^{t} \| \partial_x (\sum_{j=1}^{n} u_j^{(1)}(\tau)) \|_{B_{p,r}^{-2}} d\tau} \| u_i^{(12)}(t) \|_{B_{p,r}^{-1}} \leq \| u_i^{(12)} \|_{B_{p,r}^{-1}} + C \int_{0}^{t} e^{-C \int_{0}^{\tau} \| \partial_x (\sum_{j=1}^{n} u_j^{(1)}(\tau')) \|_{B_{p,r}^{-2}} d\tau'} \| f_i \|_{B_{p,r}^{-1}} d\tau, \quad i = 1, 2, \cdots, n. \tag{9}
\]

For \( s > 1 + \frac{1}{p}, B_{p,r}^{-1} \subset L^\infty \) is an algebra according to Proposition 2.2, so we have

\[
\left\| \sum_{j=1}^{n} u_j^{(12)} \right\|_{B_{p,r}^{-1}} \leq C \left\| u_i^{(2)} \right\|_{B_{p,r}^{-1}} \sum_{j=1}^{n} \| u_j^{(12)} \|_{B_{p,r}^{-1}}, \quad i = 1, 2, \cdots, n.
\]

Since \( s > \max \left\{ \frac{4}{3}, 1 + \frac{1}{p} \right\} \), by Proposition 2.2 and Lemma 2.4, we have

\[
\left\| P_1(D) \left( u_i^{(1)} \sum_{j \neq i} u_j^{(12)} + u_i^{(12)} \sum_{j \neq i} u_j^{(2)} \right) \right\|_{B_{p,r}^{-1}} \leq C \sum_{j=1}^{n} \left( \| u_i^{(1)} u_j^{(12)} \|_{B_{p,r}^{-2}} + \| u_i^{(12)} u_j^{(2)} \|_{B_{p,r}^{-2}} \right) \leq C \left\| u_i^{(1)} \right\|_{B_{p,r}^{-1}} \sum_{j \neq i} \| u_j^{(12)} \|_{B_{p,r}^{-2}} + C \left\| u_i^{(12)} \right\|_{B_{p,r}^{-2}} \sum_{j \neq i} \| u_j^{(2)} \|_{B_{p,r}^{-2}}.
\]
\[
\begin{align*}
\| f_i \|_{B^{s-1}_{p,r}} &\leq C \left( \sum_{j=1}^{n} \| u_j^{(12)} \|_{B^{s-1}_{p,r}} \right) \left( \sum_{j=1}^{n} \left( \| u_j^{(1)} \|_{B^s_{p,r}} + \| u_j^{(2)} \|_{B^s_{p,r}} \right) \right), \quad i = 1, 2, \ldots, n.
\end{align*}
\]

Therefore, inserting the above estimates to (9) we obtain
\[
\begin{align*}
e^{-C \int_0^t \| \partial_x (\sum_{j=1}^{n} u_j^{(1)}) \|_{B^{s-2}_{p,r}} \, dt'} \left( \sum_{j=1}^{n} \| u_j^{(12)}(t) \|_{B^{s-1}_{p,r}} \right) &\leq \sum_{j=1}^{n} \| u_j^{(12)} \|_{B^{s-1}_{p,r}} + C \int_0^t e^{-C \int_0^{\tau} \| \partial_x (\sum_{j=1}^{n} u_j^{(1)}) \|_{B^{s-2}_{p,r}} \, d\tau'} \left( \sum_{j=1}^{n} \left( \| u_j^{(1)} \|_{B^s_{p,r}} + \| u_j^{(2)} \|_{B^s_{p,r}} \right) \right) d\tau.
\end{align*}
\]

Hence, applying the Gronwall inequality, we reach (7).

For the critical case \( s = 2 + \frac{1}{p} \), we here use the interpolation method to deal with it.

Now let us start the proof of Theorem 1.1, which is motivated by the proof of local existence theorem about the Camassa-Holm equation in [24]. Firstly, we shall use the classical Friedrichs regularization method to construct the approximate solutions to the Cauchy problem problem (4).
Lemma 2.5. Assume that \( u^{(0)}_j = 0, j = 1, 2, \ldots, n \). Let \( 1 \leq p, r \leq +\infty, s > \max\{\frac{3}{2}, 1 + \frac{1}{r}\} \) and \((u_{1,0}, u_{2,0}, \ldots, u_{n,0}) \in (B^s_{p,r})^n\). Then there exists a sequence of smooth functions \((u^{(l)}_1, u^{(l)}_2, \ldots, u^{(l)}_n)\) in \( (C(\mathbb{R}^+; B^s_{p,r}))^n \) solving the following linear transport equation by induction

\[
\begin{cases}
\left( \partial_t + \left( \sum_{j=1}^{n} u^{(l)}_j \right) \partial_x \right) u^{(l+1)}_i = P_1(D) \left( u^{(l)}_i \sum_{j=1}^{n} u^{(l)}_{j,x} \right) \\
+ P_2(D) \left( (u^{(l)}_i)^2 + \frac{1}{2} \sum_{j \neq i}^{n} (u^{(l)}_j)^2 + \frac{1}{2} \left( (u^{(l)}_{i,x})^2 - \sum_{j \neq i}^{n} (u^{(l)}_{j,x})^2 \right) + u^{(l)}_{i,x} \sum_{j \neq i}^{n} u^{(l)}_{j,x}, \right)
\end{cases}
\]

\( u^{(l+1)}_i |_{t=0} = u^{(l+1)}_i(0) = S_{n+1} u_{i,0}, i = 1, 2, \ldots, n. \)

\( \tag{10} \)

Moreover, there is a positive \( T \) such that the solutions satisfying the following properties

(i) \((u^{(l)}_1, u^{(l)}_2, \ldots, u^{(l)}_n)\) in \( (E^s_{p,r}(T))^n \).

(ii) \((u^{(l)}_1, u^{(l)}_2, \ldots, u^{(l)}_n)\) in \( (C([0, T]; B^s_{p,r}^{-1}))^n \).

Proof. Since all the data \( S_{n+1} u_{j,0}, j = 1, 2, \ldots, n \) belong to \( B^s_{p,r} \), Lemma 2.3 enables us to show by induction that for all \( l \in \mathbb{N} \), the equation (10) has a global solution which belongs to \( (C(R^+; B^s_{p,r}))^n \). And from Lemma 2.2 and the proof of Proposition 2.3 and the above inequality, we have the following inequality for all \( l \in \mathbb{N} \)

\[
e^{-C \int_0^t \| \partial_x \left( \sum_{j \neq i}^{n} u^{(l)}(\tau) \right) \|_{B^s_{p,r}^{-1}}^1 d\tau} \sum_{j=1}^{n} \| u^{(l+1)}_j(t) \|_{B^s_{p,r}} \leq \| u^{(l)}_i(t) \|_{B^s_{p,r}} + C \int_0^t e^{-C \int_0^{\tau} \| \partial_x \left( \sum_{j \neq i}^{n} u^{(l)}(\tau') \right) \|_{B^s_{p,r}^{-1}}^1 d\tau'} \left( \sum_{j=1}^{n} \| u^{(l)}_j(\tau) \|_{B^s_{p,r}} \right)^2 d\tau, \quad i = 1, 2, \ldots, n.
\]

Hence, we have

\[
e^{-C \int_0^t \| \partial_x \left( \sum_{j \neq i}^{n} u^{(l)}(\tau) \right) \|_{B^s_{p,r}^{-1}}^1 d\tau} \sum_{j=1}^{n} \| u^{(l+1)}_j(t) \|_{B^s_{p,r}} \leq \sum_{j=1}^{n} \| u^{(l)}_j(t) \|_{B^s_{p,r}} + C \int_0^t e^{-C \int_0^{\tau} \| \partial_x \left( \sum_{j \neq i}^{n} u^{(l)}(\tau') \right) \|_{B^s_{p,r}^{-1}}^1 d\tau'} \left( \sum_{j=1}^{n} \| u^{(l)}_j(\tau) \|_{B^s_{p,r}} \right)^2 d\tau.
\]

\( \tag{11} \)

Let us choose a \( T > 0 \) such that \( 2CT \sum_{j=1}^{n} \| u_{j,0} \|_{B^s_{p,r}} < 1 \), and suppose by induction that for all \( t \in [0, T] \)

\[
\sum_{j=1}^{n} \| u^{(l)}_j(t) \|_{B^s_{p,r}} \leq \frac{\sum_{j=1}^{n} \| u_{j,0}(t) \|_{B^s_{p,r}}}{1 - 2CT \sum_{j=1}^{n} \| u_{j,0}(t) \|_{B^s_{p,r}} t}, \quad \tag{12}
\]

One obtains from (12) that

\[
C \int_\tau^t \left( \sum_{j=1}^{n} \| u^{(l)}_j(\tau') \|_{B^s_{p,r}} \right) d\tau' \leq C \int_\tau^t \sum_{j=1}^{n} \| u^{(l)}_j(t) \|_{B^s_{p,r}} d\tau' \leq C \int_\tau^t \frac{\sum_{j=1}^{n} \| u_{j,0}(t) \|_{B^s_{p,r}}^1 d\tau'}{1 - 2CT \sum_{j=1}^{n} \| u_{j,0}(t) \|_{B^s_{p,r}} t} \left( \sum_{j=1}^{n} \| u^{(l)}_j(t) \|_{B^s_{p,r}} \right)^2 d\tau' = \frac{1}{2} \ln \left( 1 - 2CT \sum_{j=1}^{n} \| u_{j,0}(t) \|_{B^s_{p,r}} t \right).
\]
\[-\frac{1}{2}\ln \left( 1 - 2Ct \sum_{j=1}^{n} \|u_{j,0}(t)\|_{B^{s}_{p,r}} \right). \]  

(13)

And then inserting the inequalities (13) and (12) into (11) leads to

\[
\sum_{j=1}^{n} \|u_{j}^{(t+1)}(t)\|_{B^{s}_{p,r}} \leq \frac{\sum_{j=1}^{n} \|u_{j,0}(t)\|_{B^{s}_{p,r}}}{\left( 1 - 2Ct \sum_{j=1}^{n} \|u_{j,0}(t)\|_{B^{s}_{p,r}} \right)^{\frac{1}{2}}} + \frac{C}{\left( 1 - 2Ct \sum_{j=1}^{n} \|u_{j,0}(t)\|_{B^{s}_{p,r}} \right)^{\frac{1}{2}}} \left( 1 - 2Ct \sum_{j=1}^{n} \|u_{j,0}(t)\|_{B^{s}_{p,r}} \right)^{\frac{1}{2}} \left( \sum_{j=1}^{n} \|u_{j,0}(t)\|_{B^{s}_{p,r}} \right)^{\frac{1}{2}} \int_{0}^{t} \left( 1 - 2Ct \sum_{j=1}^{n} \|u_{j,0}(t)\|_{B^{s}_{p,r}} \right)^{\frac{1}{2}} d\tau.
\]

(14)

Hence, one can see that

\[
\sum_{j=1}^{n} \|u_{j}^{(t+1)}(t)\|_{B^{s}_{p,r}} \leq \frac{\sum_{j=1}^{n} \|u_{j,0}(t)\|_{B^{s}_{p,r}}}{1 - 2Ct \sum_{j=1}^{n} \|u_{j,0}(t)\|_{B^{s}_{p,r}}}
\]

which implies that \((u_{1}^{(t)}, u_{2}^{(t)}, \ldots, u_{n}^{(t)})_{t \in \mathbb{N}}\) is uniformly bounded in \((C([0; T]; B^{s}_{p,r}))^{n}\). Using the equation (10) and the similar argument in the proof Proposition 2.3, one can easily prove that \((\partial_{t}u_{1}^{(t)}, \partial_{t}u_{2}^{(t)}, \ldots, \partial_{t}u_{n}^{(t)})_{t \in \mathbb{N}}\) is uniformly bounded in \((C([0; T]; B^{s-1}_{p,r}))^{n}\). Hence, \((u_{1}^{(t)}, u_{2}^{(t)}, \ldots, u_{n}^{(t)})_{t \in \mathbb{N}}\) is uniformly bounded in \((E^{s}_{p,r}(T))^{n}\).

Now, it suffices to show that \((u_{1}^{(t)}, u_{2}^{(t)}, \ldots, u_{n}^{(t)})_{t \in \mathbb{N}}\) is a Cauchy sequence in \((C([0; T]; B^{s-1}_{p,r}))^{n}\). In fact, for all \(i, k \in \mathbb{N}\), from (10), we have

\[
\left( \partial_{t} + \left( \sum_{j=1}^{n} u_{j}^{(t)} \right) \partial_{x} \right) (u_{i}^{(t+k+1)} - u_{i}^{(t+1)}) = g_{i}, \quad i = 1, 2, \ldots, n
\]

with

\[
g_{i} = -u_{i,x}^{(t+1)} \sum_{j=1}^{n} (u_{j}^{(k+1)} - u_{j}^{(t)}) + P_{1}(D) \left( u_{i}^{(t)} \sum_{j \neq i} (u_{j,x}^{(k+1)} - u_{j,x}^{(t)}) + (u_{i}^{(k+1)} - u_{i}^{(t)}) \sum_{j \neq i} u_{j,x}^{(k+1)} \right) + P_{2}(D) \left( (u_{i}^{(k+1)} - u_{i}^{(t)})(u_{i}^{(t)} + u_{i}^{(k+1)}) + \frac{1}{2} \sum_{j \neq i} (u_{j}^{(k+1)} - u_{j}^{(t)})(u_{j}^{(t)} + u_{j}^{(k+1)}) \right) + \frac{1}{2} (u_{i,x}^{(k+1)} - u_{i,x}^{(t)})(u_{i,x}^{(t)} + u_{i,x}^{(k+1)}) - \frac{1}{2} \sum_{j \neq i} (u_{j,x}^{(k+1)} - u_{j,x}^{(t)})(u_{j,x}^{(t)} + u_{j,x}^{(k+1)}).
\]
Applying Lemma 2.2 again, then for every \( t \in [0,T] \), we obtain

\[
e^{-C \int_0^t \| \partial_x \left( \sum_{j=1}^n u_j^{(l)}(\tau) \right) \|_{B^{s-2}_{p,r}} d\tau} \| (u_j^{(l+1)}(t) - u_j^{(l)}(t))\|_{B^{s-1}_{p,r}} \leq \| u_j^{(k+1)}(t) - u_j^{(l)}(t) \|_{B^{s-1}_{p,r}}.
\]

\[
+ C \int_0^t e^{-C \int_0^\tau \| \partial_x \left( \sum_{j=1}^n u_j^{(l)}(\tau') \right) \|_{B^{s-2}_{p,r}} d\tau'} \| g_i^{(l)} \|_{B^{s-1}_{p,r}} d\tau, \quad i = 1, 2, \ldots, n.
\]

Similar to the proof of Proposition 2.3, in the case of \( s > \frac{1}{2} \) and solves the Cauchy problem (4). Since \( (u_1^{(l)}, u_2^{(l)}, \ldots, u_n^{(l)}) \in \mathbb{N} \) is uniformly bounded in \( (E_{p,r}^s(T))^n \) and

\[
u_{i,0}^{(l+1)} - u_{i,0}^{(l+1)} = S_{k+1} u_{i,0} - S_{k+1} u_{i,0} = \sum_{q=0}^{k+1} \Delta_q u_{i,0}, \quad i = 1, 2, \ldots, n,
\]

we get a constant \( C_T \) independent of \( l, k \) such that for all \( t \in [0,T] \)

\[
\sum_{j=1}^n \| (u_j^{(l+1)}(t) - u_j^{(l)}(t)) \|_{B^{s-1}_{p,r}} \leq C_T \left( 2^{-n} + \int_0^t \sum_{j=1}^n \| (u_j^{(l+1)}(\tau) - u_j^{(l)}(\tau)) \|_{B^{s-1}_{p,r}} d\tau \right).
\]

Arguing by induction with respect to the index \( l \), one can easily prove that

\[
\sum_{j=1}^n \| (u_j^{(k+1)}(t) - u_j^{(l+1)}(t)) \|_{L_p^\infty(B^{s-1}_{p,r})} \leq \frac{(TC_T)^{l+1}}{(l+1)!} \sum_{j=1}^n \| u_j^l \|_{L_p^\infty(B^{s-1}_{p,r})} + C_T \sum_{q=0}^l \frac{(TC_T)^q}{q!}.
\]

As \( \| u_i^{(k)} \|_{L_p^\infty(B^{s-1}_{p,r})}, i = 1, 2, \ldots, n \) and \( C \) are bounded independently of \( k \), there exists constant \( C_T' \) independent of \( l, k \) such that

\[
\sum_{j=1}^n \| (u_j^{(k+1)}(t) - u_j^{(l+1)}(t)) \|_{L_p^\infty(B^{s-1}_{p,r})} \leq C_T' 2^{-n}.
\]

Thus \( (u_1^{(l)}, u_2^{(l)}, \ldots, u_n^{(l)}) \in \mathbb{N} \) is a Cauchy sequence in \( (C([0,T]; B^{s-1}_{p,r}))^n \).

On the other hand, for the critical points \( s = 2 + \frac{1}{p} \), we can apply the interpolation method to show that \( (u_1^{(l)}, u_2^{(l)}, \ldots, u_n^{(l)}) \in \mathbb{N} \) is a Cauchy sequence in \( (C([0,T]; B^{s-1}_{p,r}))^n \) for this critical case. Therefore, we have completed the proof of Lemma 2.5.

**Proof of Theorem 2.1.** Thanks to Lemma 2.5, we obtain that \( (u_1^{(l)}, u_2^{(l)}, \ldots, u_n^{(l)}) \in \mathbb{N} \) is a Cauchy sequence in \( (C([0,T]; B^{s-1}_{p,r}))^n \), so it converges to some function \( (u_1^{(l)}, u_2^{(l)}, \ldots, u_n^{(l)}) \in (C([0,T]; B^{s-1}_{p,r}))^n \). We now have to check that \( (u_1, u_2, \ldots, u_n) \) belongs to \( (E_{p,r}^s(T))^n \) and solves the Cauchy problem (4). Since \( (u_1^{(l)}, u_2^{(l)}, \ldots, u_n^{(l)}) \in \mathbb{N} \) is uniformly bounded in \( (L^\infty(0,T; B^{s}_{p,r}))^n \) according to Lemma 2.5, the Fatou property for the Besov spaces (Proposition 2.2) guarantees that \( (u_1, u_2, \ldots, u_n) \) also belongs to \( (L^\infty(0,T; B^{s}_{p,r}))^n \).
On the other hand, as \((u^{(i)}_1, u^{(i)}_2, \cdots, u^{(i)}_n)\) converges to \((u_1, u_2, \cdots, u_n)\) in \((C([0, T]; B^{s-1}_{p,r})^n)\), an interpolation argument ensures that the convergence holds in \((C([0, T]; B^s_{p,r})^n)\), for any \(s' < s\). It is then easy to pass to the limit in the equation (10) and to conclude that \((u_1, u_2, \cdots, u_n)\) is indeed a solution to the Cauchy problem (4). Thanks to the fact that \((u_1, u_2, \cdots, u_n)\) belongs to \((L^\infty(0, T; B^s_{p,r}))^n\), the right-hand side of the following equations

\[
\begin{align*}
  u_{i,t} + u_{i,x} \sum_{j=1}^n u_j &= P_1(D) \left( u_1 \sum_{j \neq i}^n u_{j,x} \right) + P_2(D) \left( u_i^2 + \frac{1}{2} \sum_{j \neq i}^n u_j^2 \right. \\
  &+ \left. \frac{1}{2} \left( u_{i,x}^2 - u_{i,x}^{2,j} + \sum_{j \neq i}^n u_{j,x} \sum_{j \neq i}^n u_j \right) \right), i = 1, 2, \cdots, n
\end{align*}
\]

belong to \(L^\infty(0, T; B^s_{p,r})\). In particular, for the case \(r < \infty\), Lemma 2.3 enables us to conclude that \((u_1, u_2, \cdots, u_n)\) belongs to \((C([0, T]; B^{s'}_{p,r})^n)\) for any \(s' < s\). Finally, using the equation again, we see that \((u_1, u_2, \cdots, u_n)\) belongs to \((C([0, T]; B^{1}_{p,r})^n)\) if \(r < \infty\), and in \((L^\infty(0, T; B^{-1}_{p,r})^n)\) otherwise. Therefore, \((u_1, u_2, \cdots, u_n)\) belongs to \((E^{s}_{p,r}(T))\). Moreover, a standard use of a sequence of viscosity approximate solutions \((u_1, u_2, \cdots, u_{n,\varepsilon})\) for the Cauchy problem (4) which converges uniformly in \((C([0, T]; B^{s}_{p,r}), \cap C^1([0, T]; B^{s-1}_{p,r}))^n\) gives the continuity of the solution \((u_1, u_2, \cdots, u_n)\) in \((E^{s}_{p,r})^n\). The proof of Theorem is complete.

3. Blowup phenomena. In the section, we present a precise scenario that ensure strong solutions to Eq. (4) blowup in finite time. We first give two useful results which will be used in the sequel.

**Lemma 3.1.** Let \(z_0 = (u_1,0, u_2,0, \cdots, u_n,0) \in (H^s)^n, s > \frac{3}{2}\), and let \(T\) be the maximal existence time of the solution \(z = (u_1, u_2, \cdots, u_n)\) to (2.1) with the initial data \(z_0\). Then for all \(t \in [0, T)\), we have

\[
\begin{align*}
  \sum_{j=1}^n \|u(t, \cdot)\|_{H^1}^2 &= \sum_{j=1}^n \|u_{j,0}\|_{H^1}^2, \\
  \|u_t(t, \cdot)\|_{L^\infty}^2 &\leq \frac{1}{2} \sum_{j=1}^n \|u_{j,0}\|_{H^1}^2
\end{align*}
\]

**Proof.** Differentiating the equations in (4) with respect to \(x\), we get

\[
\begin{align*}
  u_{i,tx} &= -u_{i,x} \sum_{j=1}^n u_{j,x} - u_{i,x} \sum_{j=1}^n u_j - \partial_x p * \left( u_i \sum_{j \neq i}^n u_{j,x} \right) - \partial_x^2 p * h_i,
\end{align*}
\]

where

\[
\begin{align*}
  h_i &= u_i^2 + \frac{1}{2} \sum_{j \neq i}^n u_j^2 + \frac{1}{2} \left( u_{1,x}^2 - \sum_{j \neq i}^n u_{j,x}^2 \right) + u_{i,x} \sum_{j \neq i}^n u_{j,x}, i = 1, 2, \cdots, n.
\end{align*}
\]

Multiplying the No.1 equation in (4) by \(u_i\), multiplying the No.1 equation in (16) by \(u_{i,x}\), then adding them together, finally integrating by parts, we obtain

\[
\begin{align*}
  \frac{1}{2} \frac{d}{dt} \sum_{j=1}^n (\|u_j\|_{H^1}^2 + \|u_j\|_{H^1}^2) = \frac{d}{dt} \int_{\mathbb{R}} \sum_{j=1}^n (u_j(t, x) + u_{j,x}(t, x)) dx = 0.
\end{align*}
\]
Solving the above equation, we obtain (15).

In view of the above conservation law, we have

\[
2u_1^2(x, t) = \int_{-\infty}^{x} 2u_i u_{i,x} dx - \int_{x}^{\infty} 2u_i u_{i,x} dx \\
\leq \int_{\mathbb{R}} 2|u_i u_{i,x}| dx \leq \int_{\mathbb{R}} (u_i^2 + u_{i,x}^2) dx = \|u_i(t, \cdot)\|_{H^1} \\
\leq \sum_{j=1}^{n} \|u_j(t, \cdot)\|_{H^1} = \sum_{j=1}^{n} \|u_{j,0}\|_{H^1}.
\]

This completes the proof of Lemma 3.1.

Next, we present the precise blowup scenarios for solutions to Eq. (4). We first recall the following two lemmas.

**Lemma 3.2.** \([\text{III}]\). If \(s > 0\), then \(H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})\) is an algebra. Moreover

\[
\|fg\|_s \leq c \left( \|f\|_{L^\infty(\mathbb{R})}\|g\|_s + \|f\|_{L^\infty(\mathbb{R})}\|g\|_s \right),
\]

where \(c\) is a constant depending only on \(s\).

**Lemma 3.3.** \([\text{III}]\). If \(s > 0\), then

\[
\|\Lambda^{s} f\|_{L^2(\mathbb{R})} \leq c (\|\partial_x f\|_{L^\infty(\mathbb{R})}\|\Lambda^{-1} f\|_{L^2(\mathbb{R})} + \|\Lambda^{s} f\|_{L^2(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})}).
\]

**Theorem 3.1.** Let \(z_0 = (u_{1,0}, u_{2,0}, \cdots, u_{n,0}) \in (H^s)^n, s > \frac{3}{2}\), and let \(T\) be the maximal existence time of the solution \(z = (u_1, u_2, \cdots, u_n)\) to (4) with the initial data \(z_0\). If there exists \(M > 0\) such that

\[
\sum_{j=1}^{n} \|u_{j,x}(t, x)\|_{L^\infty} \leq M, \quad t \in [0, T),
\]

then the \((H^s)^n\) norm of the solution \(z\) does not blowup in finite time.

**Proof.** Let \(z\) be the solution to Eq. (4) with the initial data \(z_0 \in H^s, s > \frac{3}{2}\), and let \(T\) be the maximal existence time of the corresponding solution \(z\), which is guaranteed by Theorem 2.1. Throughout this proof, \(C > 0\) stands for a generic constant depending only on \(s\).

Applying the operator \(\Lambda^s\) to the equation in (4), multiplying by \(\Lambda^s u_i\) and integrating over \(\mathbb{R}\), we obtain

\[
\frac{d}{dt} \|u_i\|_s = -2(u_{i,x} \sum_{j=1}^{n} u_{j,0} u_i) + 2(u_i, f_{i1}(z)) + 2(u_i, f_{i2}(z)) + \Lambda^{1} A_{i1} + A_{i2} + A_{i3},
\]

where

\[
f_{i1} = -\Lambda^{-2} \left( u_i \sum_{j \neq i}^{n} u_{j,x} \right),
\]

\[
f_{i1} = -\partial_x \Lambda^{-2} \left( \frac{1}{2} \sum_{j \neq i}^{n} u_j^2 + \frac{1}{2} u_{i,x} - \sum_{j \neq i}^{n} u_{j,x} + u_{i,x} \sum_{j \neq i}^{n} u_{j,x} \right).
\]

Let us estimate \(A_{i1}, A_{i2}\) and \(A_{i3}\).

\[
|A_{i1}| = | -2(u_{i,x} \sum_{j=1}^{n} u_j)_x | = 2| (\Lambda^{s}(u_{i,x} \sum_{j=1}^{n} u_j), \Lambda^{s} u_0) |
\]
An application of Gronwall’s inequality and the assumption of the theorem yields

\[ 2\| (\sum_{j=1}^{n} u_j) u_{i,x} + (\sum_{j=1}^{n} u_j) u^2 \|_{L^2} \leq 2\| (\sum_{j=1}^{n} u_j) u^2 \|_{L^2} + \| (\sum_{j=1}^{n} u_j) u_{i,x} + (\sum_{j=1}^{n} u_j) u^2 \|_{L^2} \]

\[ \leq C \left( \sum_{j=1}^{n} \| u_{j,x} \|_{L^\infty} \right) \left( \sum_{j=1}^{n} \| u_j \|_s \right), \]

where we use Lemma 3.3 with \( r = s. \)

\[ |A_{12}| = |2(f_{12}(z), u_i)_s| \leq 2\| (\sum_{j \neq i}^{n} u_j) u_i \|_{L^2} \leq 2\| (\sum_{j \neq i}^{n} u_j) u_i \|_{L^2} \]

\[ \leq 2 \sum_{j=1}^{n} \| (\sum_{j \neq i}^{n} u_j) u_i \|_{L^2} \]

\[ \leq 2 \sum_{j=1}^{n} \| (\sum_{j \neq i}^{n} u_j) u_i \|_{L^2} \]

\[ \leq C \sum_{j=1}^{n} \left( \| u_{j,x} \|_{L^\infty} \| u_j \|_{s-1} + \| u_{j,x} \|_{L^\infty} \| u_j \|_{s-1} \right) \]

\[ \leq C \left( \sum_{j=1}^{n} \| u_{j,x} \|_{L^\infty} \right) \left( \sum_{j=1}^{n} \| u_j \|_s \right), \]

where we use Lemma 3.3 with \( r = s - 1. \)

\[ |A_{13}| = |2(f_{13}(z), u_i)_s| \leq 2\| f_{13}(z) \|_s \| u_i \|_s \]

\[ \leq C \| u_i \|_s \sum_{j=1}^{n} \| u_j \|_{L^\infty} \| u_j \|_{s-1} + \| u_{j,x} \|_{s-1} \]

\[ \leq C \| u_i \|_s \sum_{j=1}^{n} \| u_j \|_{L^\infty} \| u_j \|_{s-1} + \| u_{j,x} \|_{L^\infty} \| u_j \|_{s-1} \]

\[ \leq C \left( \sum_{j=1}^{n} \| u_{j,x} \|_{L^\infty} \right) \left( \sum_{j=1}^{n} \| u_j \|_s \right), \]

where we use Lemma 3.1 and Lemma 3.2 with \( r = s - 1. \)

Therefore

\[ \frac{d}{dt} \left( \sum_{j=1}^{n} \| u_j \|_s^2 \right) \leq C \left( \sum_{j=1}^{n} \| u_{j,x} \|_{L^\infty} \right) \left( \sum_{j=1}^{n} \| u_j \|_s \right), \]

An application of Gronwall’s inequality and the assumption of the theorem yields

\[ \sum_{j=1}^{n} \| u_j \|_s^2 \leq \exp(CMt) \left( \sum_{j=1}^{n} \| u_{j,0} \|_s^2 \right) \]

This completes the proof of the theorem.
Next, we present the precise blowup scenario.

**Theorem 3.2.** Let \( z_0 = (u_{1,0}, u_{2,0}, \cdots, u_{n,0}) \in (H^s)^n, s > \frac{3}{2}, \) and let \( T \) be the maximal existence time of the solution \( z = (u_1, u_2, \cdots, u_n) \) to (2.1) with the initial data \( z_0 \). Then the corresponding solution blows up in finite time if and only if there exists an \( i \ (i \in \{1, 2, \cdots, n\}) \) such that

\[
\liminf_{t \to T} \inf_{z \in \mathbb{R}} \{u_{i,x}(t, x)\} = -\infty.
\]

**Proof.** Assume that \( z_0 \in (H^s)^n \) for some \( s \in N, s \geq 2 \). Multiplying the No. \( i \) equation in (2.1) by \( m_i = u_i - u_{i,xx} \), integrating by parts, we have

\[
\frac{d}{dt} \int_{\mathbb{R}} \left( \sum_{j=1}^{n} m_i^2 \right) dx
= -2 \int_{\mathbb{R}} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} m_i m_{i,x} u_j + 2 \sum_{j=1}^{n} m_j^2 u_{j,x} + \sum_{i=1}^{n} \sum_{j \neq i}^{n} m_i m_j u_{i,x} u_{j,x} \right) dx
= -2 \int_{\mathbb{R}} \left( \sum_{i=1}^{n} m_i^2 u_{i,x} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_i^2 u_{j,x} + \sum_{i=1}^{n} \sum_{j \neq i}^{n} m_i m_j (u_{i,x} + u_{j,x}) \right) dx
= - \int_{\mathbb{R}} \left( \sum_{i=1}^{n} m_i^2 u_i + \sum_{i=1}^{n} \sum_{j \neq i}^{n} (u_{i,x} + u_{j,x})(m_i + n_j)^2 \right) dx.
\]

From the above equality, we see that if \( u_{i,x}, i = 1, 2, \cdots, n \) are bounded from below on \([0, T)\), i.e. there exist the positive constants \( M_i, i = 1, 2, \cdots, n \) such that \( u_{i,x} \geq -M_i \), then we have

\[
\frac{d}{dt} \int_{\mathbb{R}} \left( \sum_{j=1}^{n} m_i^2 \right) dx \leq \int_{\mathbb{R}} \left( \sum_{i=1}^{n} m_i^2 M_i + 2 \sum_{i=1}^{n} \sum_{j \neq i}^{n} (M_i + M_j)(m_i^2 + n_j^2) \right) dx
= 3n \left( \sum_{i=1}^{n} M_i \right) \int_{\mathbb{R}} \left( \sum_{j=1}^{n} m_i^2 \right) dx.
\]

By the definition of \( m_i, i = 1, 2, \cdots, n \), we have

\[
\int_{\mathbb{R}} m_i^2 dx = \int_{\mathbb{R}} (u - u_{xx})_i^2 dx = \int_{\mathbb{R}} (u_i^2 + 2u_{i,x}^2 + u_{i,xx}^2) dx.
\]

Hence, we obtain

\[
\|u\|_{H^2}^2 \leq \|m\|_{L^2}^2 \leq 2\|u\|_{H^2}^2
\]

By the above inequalities and Gronwall’s inequality, we get

\[
\sum_{j=1}^{n} \|u_j\|_{H^2}^2 \leq \int_{\mathbb{R}} \left( \sum_{j=1}^{n} m_i^2 \right) dx
\]
Differentiating Eq. (20) with respect to $z$ results in
$$\text{maximal existence time of the solution } z$$

The function $\Phi = 2 \sum_{j=1}^{n} u_{j,0}^{2} + \sum_{1<i<j}^{n} u_{i,j} + 2 \sum_{1<i<j}^{n} u_{i,x} u_{j,x} - \frac{1}{2} \sum_{j=1}^{n} u_{j,x}^{2}$

Differentiating Eq. (20) with respect to $x$ and using $\partial_{x}^{2} p * f = p * f - f$, we have
$$\Theta_{xx} + \Theta_{xx} + \Theta_{x}^{2} = \Phi - p * \Phi$$

Define $\Psi(t) = \sum_{j=1}^{n} u_{j,x}(t, \xi(t)) = \inf_{t \in \mathbb{R}} \left\{ \sum_{j=1}^{n} u_{j,x} \right\}$. Note that $\Theta_{xx}(t, \xi(t)) = 0$. Substituting $(t, \xi(t))$ into (19) and using Lemma 3.4, we have
$$\frac{d\Theta}{dt} = -\frac{3}{2} \sum_{j=1}^{n} u_{j,x}^{2} + 2 \sum_{j=1}^{n} u_{j,x}^{2} + \sum_{1<i<j}^{n} u_{i,j} - G * \Phi.$$
Noticing \( \sum_{j=1}^{n} u_{j,x}^2 \geq \left( \sum_{j=1}^{n} u_{j}^2 \right)^2 \) and \( G^* \left( 2 \sum_{j=1}^{n} u_{j}^2 + \sum_{1=i<j}^{n} u_{i}u_{j} \right) \geq 0 \), we have
\[
\frac{d\Theta}{dt} \leq - \frac{2}{n+1} \Theta^2 + 2 \sum_{j=1}^{n} u_{j}^2 + \sum_{1=i<j}^{n} u_{i}u_{j} - G^* \left( 2 \sum_{1=i<j}^{n} u_{i,x}u_{j,x} - \frac{1}{2} \sum_{j=1}^{n} u_{j,x}^2 \right).
\] (20)

Using Lemma 3.1, we have
\[
\left\| \sum_{j=1}^{n} u_{j}^2 + \sum_{1=i<j}^{n} u_{i}u_{j} \right\|_{L^\infty} \leq \frac{n+3}{2} \left\| \sum_{j=1}^{n} u_{j}^2 \right\|_{L^\infty} \leq \frac{n+3}{4} \sum_{j=1}^{n} \| u_{j,0} \|^2_{H^1}.
\] (21)

Note that \( \| G \|_{L^\infty} = \frac{1}{2} \) Using Young’s inequality and Lemma 3.1, we have
\[
\left\| G^* \left( 2 \sum_{1=i<j}^{n} u_{i,x}u_{j,x} - \frac{1}{2} \sum_{j=1}^{n} u_{j,x}^2 \right) \right\|_{L^\infty} \leq \left\| G^* \left( \frac{1}{2} \sum_{j=1}^{n} u_{j,x}^2 \right) \right\|_{L^\infty} \leq (n-1)\| G \|_{L^\infty} \left\| \sum_{j=1}^{n} u_{j,x}^2 \right\|_{L^1} + \frac{1}{2} \| G \|_{L^\infty} \| \sum_{j=1}^{n} u_{j,x}^2 \|_{L^1}
\] \leq \frac{2n-1}{4} \left( \sum_{j=1}^{n} \| u_{j,0} \|^2 \right). \quad (22)

By (20)-(22), we deduce that
\[
\frac{d\Theta}{dt} \leq - \frac{2}{n+1} \Theta^2 + \frac{3n+2}{4} \left( \sum_{j=1}^{n} \| u_{j,0} \|^2 \right).
\] (23)

Set \( K = \sqrt{\frac{(3n+2) \sum_{j=1}^{n} \| u_{j,0} \|^2}{2}} \). Then, we have
\[
\frac{d\Theta}{dt} \leq - \frac{2}{n+1} \Theta^2 + K^2.
\] (24)

Note that if \( \Theta(0) \leq - \frac{\sqrt{2(n+1)}}{2} K \), then \( \Theta(t) \leq - \frac{\sqrt{2(n+1)}}{2} K \), for all \( t \in [0, T) \). Therefore, we can solve the above inequality to obtain
\[
\Theta(0) + \frac{\sqrt{2(n+1)}}{2} K e^{\sqrt{2(n+1)} K t} - 1 \leq \frac{\Theta(0) + \sqrt{2(n+1)} K}{\Theta(0) - \frac{\sqrt{2(n+1)}}{2} K} < 0.
\]

Due to \( 0 < \frac{\Theta(0) + \sqrt{2(n+1)} K}{\Theta(0) - \frac{\sqrt{2(n+1)}}{2} K} < 1 \), then there exists \( T \), and \( 0 < T < \frac{1}{\sqrt{2(n+1)}} \ln \left( \frac{\Theta(0) + \sqrt{2(n+1)} K}{\Theta(0) - \frac{\sqrt{2(n+1)}}{2} K} \right) \), such that \( \lim_{t \to T} \Theta(t) = -\infty \). Applying Theorem 3.2, the solution \( z \) does not exist globally in time.
Using the oddness of the initial data and the symmetry of Eq. (4), we then give the second blowup result.

**Theorem 3.4.** Let \( z_0 = (u_{1,0}, u_{2,0}, \cdots, u_{n,0}) \in (H^s)^n, s > \frac{3}{2}, \) is odd. If

\[
\sum_{j=1}^{n} u'_{j,0}(x_0) < - \left[ \frac{(2n-1)(n+1)}{8} \sum_{j=1}^{n} \|u_{j,0}\|_{H^1}^2 \right]^{\frac{1}{2}},
\]

the corresponding solution to Eq. (4) blows up in finite time.

**Proof.** Let \( T > 0 \) be the maximal time of existence of the solution \( z \) to Eq. (4) with the initial data \( z_0 \). Define \( \Theta(t) = \sum_{j=1}^{n} u_{j,x}(t,0) \). Note that the symmetry \( u_i(t, x) \to -u(t, -x), i = 1, 2, \cdots, n \) enjoy by Eq. (4), so we obtain that if \( u_{i,0}, i = 1, 2, \cdots, n \) are odd, then \( u_i(t, x), i = 1, 2, \cdots, n \) are odd for any \( t \in [0, T] \). In particular, we have \( u_i(t, 0) = 0, u_{i,xx}(t,0) = 0, i = 1, 2, \cdots, n \).

Substituting \( (t, 0) \) into (20) and using (23), we have

\[
\frac{d\Theta}{dt} \leq - \frac{2}{n+1} \Theta^2 + \frac{2n-1}{4} \left( \sum_{j=1}^{n} \|u_{j,0}\|^2 \right).
\]

Set \( K = \sqrt{\frac{(2n-1)(n+1)}{2} \sum_{j=1}^{n} \|u_{j,0}\|^2} \). Then, we have

\[
\frac{d\Theta}{dt} \leq - \frac{2}{n+1} \Theta^2 + K^2.
\]

Note that if \( \Theta(0) \leq - \frac{\sqrt{2(n+1)}}{2} K \), then \( \Theta(t) \leq - \frac{\sqrt{2(n+1)}}{2} K \), for all \( t \in [0, T] \), Using the same argument in Theorem 3.4, we deduce that the solution \( z \) does not exist globally in time.

Next, we give more insight into the blow-up rate for the wave breaking solutions to Equation (17).

**Theorem 3.5.** Let \( z = (u_1, u_2, \cdots, u_n) \) be the solution to Eq. (4) with the initial data \( z_0 = (u_{1,0}, u_{2,0}, \cdots, u_{n,0}) \in (H^s)^n, s > \frac{3}{2} \) satisfying the assumption of Theorems 3.3 or 3.4, and let \( T > 0 \) be the maximal time of existence of the solution \( z \). If \( T < \infty \), we have

\[
\liminf_{t \to T} \inf_{x \in \mathbb{R}} \left( \sum_{j=1}^{n} u_{j,x}(t, x) \right)(T-t) = -\frac{n+1}{2}.
\]

**Proof.** By Lemma 3.1, we get the uniform bound of \( u_i, i = 1, 2, \cdots, n \), which implies that the solution remains uniformly bounded. For Theorem 3.3, we set \( \Psi(t) = \inf_{x \in \mathbb{R}} \left\{ \sum_{j=1}^{n} u_{j,x} \right\} \), while for Theorem 3.4, we set \( \Theta(t) = \sum_{j=1}^{n} u_{j,x}(t,0) \).

Using (3.6) and noticing the proofs of Theorems 3.3 and 3.4, we can find a constant \( K > 0 \) such that

\[
-K \leq \frac{d\Theta}{dt} + \frac{2}{n+1} \Theta^2 \leq K \quad \text{a.e., on} \quad (0, T).
\]

Choose \( \epsilon \in (0, \frac{1}{2}) \). Since

\[
\liminf_{t \to T} \inf_{x \in \mathbb{R}} \left( \sum_{j=1}^{n} u_{j,x}(t, x) \right) = -\infty.
\]
by Theorem 3.2, there is some $t_0 \in (0, T)$ with $g(t_0) + \lambda < 0$ and $\Theta(t_0) > \frac{K}{\epsilon}$. Let us first prove that

$$\Theta(t) > \frac{K}{\epsilon}, \quad t \in [t_0, T].$$

(28)

Since $\Theta$ is locally Lipschitz, there is some $\delta > 0$ such that

$$\Theta(t) > \frac{K}{\epsilon}, \quad t \in (t_0, t_0 + \delta).$$

(29)

Note that $\Theta$ is locally Lipschitz and therefore absolutely continuous. Integrating the previous relation on $(t_0, t_0 + \delta)$ yields that

$$\Theta(t_0 + \delta) \leq \Theta(t_0) < 0.$$

It follows from the above inequality that

$$\Theta(t_0 + \delta) \geq \Theta(t_0) > \frac{K}{\epsilon}.$$

The obtained contradiction completes the proof of the relation (28). By (28)-(29), we infer

$$\frac{2}{n+1} - \epsilon \leq -\frac{\Theta(t)}{\Theta^2} \leq \frac{2}{n+1} + \epsilon, \quad \text{a.e., on } (0, T)$$

(30)

For $t \in (t_0, T)$, integrating (30) on $(t, T)$ to get

$$\left(-\frac{2}{n+1} - \epsilon\right)(T-t) \leq \frac{1}{\Theta(t)} \leq \left(-\frac{2}{n+1} + \epsilon\right)(T-t), \quad t \in (t_0, T).$$

Since $\Theta(t) < 0$ on $[t_0, T)$, it follows that

$$-\frac{2}{n+1} + \epsilon \leq \Theta(t)(T-t) \leq -\frac{2}{n+1} - \epsilon, \quad t \in (t_0, T).$$

By the arbitrariness of $\epsilon \in (0, \frac{1}{2})$, the statement of the theorem follows.

4. Persistence properties. In this section, we shall investigate the following property for the strong solutions to (4) in $L^\infty$ space which asymptotically exponential decay at infinity as their initial profiles. The main idea comes from a recent work of Zhou and his collaborators [34] for the standard Camassa-Holm equation (for slower decay rate, we refer to [46]).

**Theorem 4.1.** Assume that $u_{i,0}(x) \in H^s$ with $s > \frac{3}{2}, i = 1, 2, \cdots, n$, satisfy that for some $\theta \in (0, 1]$

$$|u_{i,0}(x)|, |u_{i,0x}(x)| \sim O(e^{-\theta x}) \quad \text{as } x \uparrow \infty, i = 1, 2, \cdots, n.$$

Then the corresponding strong solution $(u_1(t,x), u_2(t,x), \cdots, u_n(t,x)) \in C([0,T); (H^s)^n)$ to (4) satisfies that

$$|u_{i}(t,x)|, |u_{i,x}(t,x)| \sim O(e^{-\theta x}) \quad \text{as } x \uparrow \infty, i = 1, 2, \cdots, n$$

uniformly in the time interval $[0, T)$.

**Notation.**

$$|f(x)| \sim O(|g(x)|) \quad \text{as } x \uparrow \infty \text{ if } \lim_{x \to \infty} \frac{|f(x)|}{|g(x)|} = L,$$
where $L$ is a nonnegative constant. In order to shorten the presentation in the sequel, we introduce

$$F_i = - u_i \sum_{j \neq i}^n u_{i, x} - G \ast \left( u_i \sum_{j \neq i}^n u_{j, x} \right) - \partial_x G \ast \left( u_i^2 + \frac{1}{2} \sum_{j \neq i}^n u_{j, x}^2 \right) + \frac{1}{2} \left( u_{i, x}^2 - u_i^2 \sum_{j \neq i}^n u_{j, x} + u_x \sum_{j \neq i}^n u_{j, x} \right), \quad i = 1, 2, \ldots, n. \quad (31)$$

**Proof.** In order to arrive at our result, we first introduce a weighted continuous function which is independent on $t$ as follows

$$\Phi_N(x) = \begin{cases} 1, & x \leq 1, \\ e^{\theta x}, & x \in (1, N), \\ e^{\theta N}, & x \geq N, \end{cases}$$

where $\theta \in (0, 1], N \in \mathbb{Z}^+, N > 2$. It is trivial that

$$0 \leq \Phi_N'(x) \leq \Phi_N(x), \quad \text{a.e., } x \in \mathbb{R},$$

where the derivative is with respect to the variable $x$. From the equations of $(4)$, we have

$$\partial_t (u_i \Phi_N) = \left( u_i \Phi_N \right) u_{i, x} + \Phi_N F_i, \quad i = 1, 2, \ldots, n. \quad (32)$$

Multiplying $(32)$ by $(u_i \Phi_N)^{2p-1}$ with $p \in \mathbb{Z}^+$ and integrating the result in the $x$-variable, we get

$$\int_{-\infty}^{+\infty} \partial_t (u_i \Phi_N) (u_i \Phi_N)^{2p-1} \, dx = \int_{-\infty}^{+\infty} (u_i \Phi_N) u_{i, x} (u_i \Phi_N)^{2k-1} \, dx + \int_{-\infty}^{+\infty} \Phi_N F_i (u_i \Phi_N)^{2p-1} \, dx,$$

from which we can deduce that

$$\frac{d}{dt} \| u_i \Phi_N \|_{L^{2p}} \leq \| u_{i, x} \|_{L^\infty} \| u_i \Phi_N \|_{L^{2p}} + \| \Phi_N F_i \|_{L^{2p}}.$$  

Denoting $M = \sup_{t \in [0, T]} \| (u_1(t), u_2(t), \ldots, u_n(t)) \|_{H^s}$ and by the Gronwall’s inequality, we obtain

$$\| u_i \Phi_N \|_{L^{2p}} \leq \left( \| u_{i, 0} \Phi_N \|_{L^{2p}} + \int_0^t \| \Phi_N F_i \|_{L^{2p}} \right) e^{Mt}. \quad (33)$$

Taking the limits in $(33)$, we get

$$\| u_{i, x} \Phi_N \|_{L^\infty} \leq \left( \| u_{i, 0} \Phi_N \|_{L^\infty} + \int_0^t \| \Phi_N F_i \|_{L^\infty} d\tau \right) e^{Mt}, \quad i = 1, 2, \ldots, n. \quad (34)$$

Next, differentiating the equations in $(4)$ in the $x$-variable produces the equations

$$u_{i, xx} = u_{i, x} u_{i, x} + u_{i, x}^2 + \partial_x F_i, \quad i = 1, 2, \ldots, n. \quad (35)$$

Using the weight function, we can rewrite $(35)$ as

$$\partial_t (u_i \Phi_N) = u_i u_{i, x} \Phi_N + (u_{i, x} \Phi_N) u_{i, x} + \Phi_N \partial_{i, x} F_i, \quad i = 1, 2, \ldots, n. \quad (36)$$
Multiplying (36) by \((u_{i,x}\Phi_N)^{2p-1}\) with \(p \in \mathbb{Z}^+\) and integrating the results in the \(x\)-variable, it follows that

\[
\int_{-\infty}^{+\infty} \partial_t (u_{i,x}\Phi_N)(u_{i,x}\Phi_N)^{2p-1} \, dx = \int_{-\infty}^{+\infty} u_i u_{i,xx}\Phi_N(u_{i,x}\Phi_N)^{2p-1} \, dx \\
+ \int_{-\infty}^{+\infty} (u_{i,x}\Phi_N)u_{i,x}(u_{i,x}\Phi_N)^{2p-1} \, dx + \int_{-\infty}^{+\infty} \Phi_N \partial F_i(u_{i,x}\Phi_N)^{2p-1} \, dx,
\]

\(i = 1, 2 \cdots, n.\) \hspace{1cm} (37)

For the first term on the right side of (37), we know

\[
\left| \int_{\mathbb{R}} (u_{i,x}\Phi_N)^{2p-1}\Phi_N u_i u_{i,xx} \, dx \right| \\
= \left| \int_{\mathbb{R}} (u_{i,x}\Phi_N)^{2p-1} u_i ((\Phi_N u_{i,x}) - \Phi_N') u_{i,x} \, dx \right| \\
= \left| \int_{\mathbb{R}} u_i \left( \frac{(u_{i,x}\Phi_N)^{2p}}{2p} \right) \, dx - \int_{\mathbb{R}} u_i (u_{i,x}\Phi_N)^{2p-1}\Phi_N' u_{i,x} \, dx \right| \\
\leq 2(\|u_{i,x}\|_\infty + \|u_i\|_\infty) \|u_{i,x}\Phi_N\|_{L^2}^2.
\]

Using the above estimates and Hölder inequality, we deduce that

\[
\frac{d}{dt}\|u_{i,x}\Phi_N\|_{L^2} \leq 5M \|u_{i,x}\Phi_N\|_{L^2} + \|\Phi_N \partial_x F_i\|_{L^2}.
\]

Thanks to the Gronwall’s inequality, it holds that

\[
\|u_{i,x}\Phi_N\|_{L^2} \leq (\|u_{i,0x}\Phi_N\|_{L^2} + \int_0^t \|\Phi_N \partial_x F_i\|_{L^2} \, d\tau) e^{5Mt}. \hspace{1cm} (38)
\]

Taking the limits in (38), we have

\[
\|u_{i,x}\Phi_N\|_{L^\infty} \leq (\|u_{i,0x}\Phi_N\|_{L^\infty} + \int_0^t \|\Phi_N \partial_x F_i\|_{L^\infty} \, d\tau) e^{5Mt}, \hspace{1cm} i = 1, 2, \cdots, n. \hspace{1cm} (39)
\]

Combining (34) and (39) together, it follows that

\[
\sum_{i=1}^n (\|u_{i,0}\Phi_N\|_{L^\infty} + \|u_{i,x}\Phi_N\|_{L^\infty}) \leq e^{5Mt} \sum_{i=1}^n (\|u_{0i}\Phi_N\|_{L^\infty} + \|u_{0x}\Phi_N\|_{L^\infty}) \\
+ e^{5Mt} \left( \int_0^t \sum_{i=1}^n (\|F_i\Phi_N\|_{L^\infty} + \|\Phi_N \partial_x F_i\|_{L^\infty}) \, d\tau \right).
\]

(40)

A simple calculation shows that there exists \(C_0 > 0\), depending only on \(\theta \in (0, 1)\), such that for any \(N \in \mathbb{Z}^+\),

\[
\Phi_N(x) \int_{-\infty}^{+\infty} e^{-|x-y|} \frac{1}{\Phi_N(y)} \, dy \leq C_0 = \frac{4}{1-\theta}.
\]

On the other hand, for a suitable function \(f\) and \(g\), one obtains,

\[
|\Phi_N * f(x)g(x)| \leq \frac{1}{2} \Phi_N(x) \int_{\mathbb{R}} e^{-|x-y|} \frac{1}{\Phi_N(y)} \Phi_N(y) f(y)g(y) \, dy \\
\leq \frac{1}{2} \|\Phi_N\|_\infty \|g\|_\infty \Phi_N(x) \int_{\mathbb{R}} e^{-|x-y|} \frac{1}{\Phi_N(y)} \, dy \\
\leq C_0 \|\Phi_N\|_\infty \|g\|_\infty.
\]
Similarly, we can get
\[ |\Phi_N \partial_x (p * f(x)g(x))| \leq C_p \| f \Phi_N \|_\infty \| g \|_\infty, \]
and
\[ |\Phi_N \partial_x^2 (p * f(x)g(x))| \leq C_p \| f \Phi_N \|_\infty \| g \|_\infty. \]

Thus, inserting the above estimates into (40), there exists a constant \( \bar{C} = \bar{C}(M, T, C_p) \) such that
\[
\sum_{i=1}^{n} \left( \| u_i \Phi_N \|_{L^\infty} + \| u_{i,x} \Phi_N \|_{L^\infty} \right)
\leq \sum_{i=1}^{n} \left( \| u_{i,0} \Phi_N \|_{L^\infty} + \| u_{i,0x} \Phi_N \|_{L^\infty} \right) + \bar{C} \int_{0}^{\tau} \sum_{i=1}^{n} \left( \| u_i \|_{L^\infty} + \| u_{i,x} \|_{L^\infty} + \| u_{i,xx} \|_{L^\infty} \right)
\times \sum_{i=1}^{n} \left( \| u_i \Phi_N \|_{L^\infty} + \| u_{i,xx} \Phi_N \|_{L^\infty} \right) d\tau
\leq \bar{C} \sum_{i=1}^{n} \left( \| u_{i,0} \Phi_N \|_{L^\infty} + \| u_{i,0x} \Phi_N \|_{L^\infty} \right) + \bar{C} \int_{0}^{\tau} \sum_{i=1}^{n} \left( \| u_i \Phi_N \|_{L^\infty} + \| u_{i,xx} \Phi_N \|_{L^\infty} \right) d\tau.
\]

(41)

Hence, for any \( N \in \mathbb{Z}^+ \) and any \( t \in [0, T] \), we have by Gronwall’s inequality
\[
\sum_{i=1}^{n} \left( \| u_i \Phi_N \|_{L^\infty} + \| u_{i,x} \Phi_N \|_{L^\infty} \right)
\leq \bar{C} \sum_{i=1}^{n} \left( \| u_{i,0} \Phi_N \|_{L^\infty} + \| u_{i,0x} \Phi_N \|_{L^\infty} \right)
\leq \bar{C} \sum_{i=1}^{n} \left( \| u_{i,0}(x) \max(1, e^{\theta x}) \|_{\infty} + \| u_{i,0x}(x) \max(1, e^{\theta x}) \|_{\infty} \right).
\]

(42)

Finally, passing limit as \( N \) goes to infinity in (42), we obtain
\[
\sum_{i=1}^{n} \left( \| u_i e^{\theta x} \|_{L^\infty} + \| u_{i,x} e^{\theta x} \|_{L^\infty} \right)
\leq \bar{C} \sum_{i=1}^{n} \left( \| u_{i,0}(x) \max(1, e^{\theta x}) \|_{\infty} + \| u_{i,0x}(x) \max(1, e^{\theta x}) \|_{\infty} \right).
\]

(43)

We complete the proof of Theorem 4.1.

\[ \square \]

5. Initial boundary value problem. In this section, we study the following initial boundary value problem of the N-Component Camassa-Holm equation with peakons on the interval \( \mathbb{R}_+ \), here \( \mathbb{R}_+ = [0, +\infty) \)

\[
\begin{align*}
m_{1,t} + m_{1,x}u_1 + 2m_{1,1}u_{1,x} + \left( m_1 \sum_{j\neq 1} u_j \right)_x + \sum_{j\neq 1} m_j u_{j,x} &= 0, & t > 0, x \in \mathbb{R}_+, \\
m_{2,t} + m_{2,x}u_2 + 2m_{2,2}u_{2,x} + \left( m_2 \sum_{j\neq 2} u_j \right)_x + \sum_{j\neq 2} m_j u_{j,x} &= 0, & t > 0, x \in \mathbb{R}_+, \\
\vdots & & \\
m_{n,t} + m_{n,x}u_n + 2m_{n,n}u_{n,x} + \left( m_n \sum_{j\neq n} u_j \right)_x + \sum_{j\neq n} m_j u_{j,x} &= 0, & t > 0, x \in \mathbb{R}_+, \\
u_1(0, x) &= u_{i,0}(x), i = 1, 2, \cdots, n, & x \in \mathbb{R}_+, \\
u_i(t, 0) &= 0, i = 1, 2, \cdots, n. & t \geq 0.
\end{align*}
\]

(43)
simplicity, we denote
\[ \tilde{T} = \text{Applying the Theorem 2.1 obtained in section 2, we have that there exist a maximal solution depends continuously on the initial data, i.e. the mapping} \]
\[ \tilde{z} \]}
\[ \text{Lemma 5.1.} \]
\[ \text{Given } s \in (\frac{1}{2}, \frac{5}{2}). \text{ Assume that } h \in H^s(\mathbb{R}_+) \text{ with } h(0) = 0. \text{ Let furthermore} \]
\[ \tilde{h}(x) = \begin{cases} h(x), & \text{if } x \geq 0, \\ -h(-x), & \text{if } x < 0. \end{cases} \]
\[ \text{Then } \tilde{h}(x) \in H^s(\mathbb{R}). \]
\[ \text{Proof of Theorem 5.1.} \text{ We first convert the initial boundary value problem of (43) into the cauchy problem of system (4). In order to do so, we extend the initial data} \]
\[ u_{i,0}(x), i = 1, 2, \ldots, n, \text{ defined on interval } \mathbb{R}_+ \text{ into an odd function defined on the line} \]
\[ \tilde{u}_{i,0}(x) = \begin{cases} u_{i,0}(x), & \text{if } x \geq 0, \\ -u_{i,0}(-x), & \text{if } x < 0, \end{cases} \]
\[ \text{Note that } u_{i,0}(x) \in H^s(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+), i = 1, 2, \ldots, n \text{ with } \frac{3}{2} < s < \frac{5}{2}. \text{ The above two relations and Lemma 5.1 show that} \]
\[ \text{We now can convert (43) into the } N \text{-component Camassa-Holm system on the whole line.} \]
\[ \begin{align*}
\bar{m}_{1,t} + \bar{m}_{1,x} \bar{u}_1 + 2 \bar{m}_1 \bar{u}_{1,x} + \left( \bar{m}_1 \sum_{j \neq 1}^n u_j \right)_x + \sum_{j \neq 1}^n \bar{m}_j \bar{u}_j, x = 0, & \quad t > 0, x \in \mathbb{R}_+, \\
\bar{m}_{2,t} + \bar{m}_{2,x} \bar{u}_2 + 2 \bar{m}_2 \bar{u}_{2,x} + \left( \bar{m}_2 \sum_{j \neq 2}^n u_j \right)_x + \sum_{j \neq 2}^n \bar{m}_j \bar{u}_j, x = 0, & \quad t > 0, x \in \mathbb{R}_+, \\
\vdots & \\
\bar{m}_{n,t} + \bar{m}_{n,x} \bar{u}_n + 2 \bar{m}_n \bar{u}_{n,x} + \left( \bar{m}_n \sum_{j \neq n}^n u_j \right)_x + \sum_{j \neq n}^n \bar{m}_j \bar{u}_j, x = 0, & \quad t > 0, x \in \mathbb{R}_+, \\
\bar{u}_i(0, x) = \tilde{u}_{i,0}(x), i = 1, 2, \ldots, n, & \quad x \in \mathbb{R}_+, \\
\end{align*} \]
\[ \text{Applying Theorem 2.1 obtained in section 2, we have that there exist a maximal } T = T((||\bar{z}_0||_{H^p}^*) > 0 \text{ and a unique solution} \]
\[ \tilde{z} = \tilde{z}(\cdot, z_0) \in C([0, T]; (H^s \cap H^1_0)^n) \cap C^1([0, T]; (H^{s-1} \cap H^1_0)^n). \text{ Moreover, the solution depends continuously on the initial data, i.e. the mapping} \]
\[ \bar{z}_0 \to \tilde{z}(\cdot, \bar{z}_0) : (H^s \cap H^1_0)^n \to C([0, T]; (H^s \cap H^1_0)^n) \cap C^1([0, T]; (H^{s-1} \cap H^1_0)^n) \text{ is continuous and} \]
\[ \text{the maximal time of existence } T > 0 \text{ can be chosen to be independent of } s. \text{ Here, for} \]
\[ \text{simpllicity, we denote } H^p(\mathbb{R}_+)(p = s, s - 1) \text{ and } H^1_0(\mathbb{R}_+) \text{ by } H^p \text{ and } H^1_0, \text{ respectively.} \]
\[ \text{Note that if } \tilde{z}(t, x) \text{ is a solution to (46), then it is easy to check that } \delta(t, x) = -\tilde{z}(t, -x), (t, x) \in [0, T) \times \mathbb{R} \text{ is also a solution of (43) with the initial data } z_0. \text{ By uniqueness we conclude that} \]
\[ \delta = \tilde{z} \text{ and } \tilde{z} \text{ is odd for any } t \in [0, T). \text{ Therefore, we have} \]
\[ \tilde{z}(t, 0) \equiv 0 \text{ for all } t \in [0, T). \]
\[ \text{Set } z(t, x) = \tilde{z}(t, x) \text{ for all } (t, x) \in [0, T) \times \mathbb{R}_+. \text{ Then we know that } z \in C([0, T]; (H^s \cap H^1_0)^n) \cap C^1([0, T]; (H^{s-1} \cap H^1_0)^n) \text{ is a solution to system (43), where} \]
Theorem 5.3. \( H^p(p = s, s - 1) \) and \( H_0^1 \) denote \( H^p(\mathbb{R}+) \) and \( H_0^1(\mathbb{R}+) \), respectively. On the other hand, if \( j(t, x) \) is a solution to (43) with initial data \( z_0(x) \), then

\[
\tilde{j}(t, x) = \begin{cases} 
    j(t, x), & \text{if } x \geq 0, \\
    -j(t, -x), & \text{if } x < 0,
\end{cases}
\]

is the unique solution to (46) with initial data \( \tilde{z}_0(x) \). By the uniqueness of \( \tilde{z}_0(x) \), we conclude that \( z(t, x) = j(t, x) \) is the unique solution to (43) with initial data \( z_0(x) \). Moreover, the solution \( z(t, x) \) depends continuously on the initial data \( z_0(x) \) and the maximal \( T \) is independent of \( s \). This completes the proof of the theorem. \( \square \)

Following the idea [27] [20], if \( s > \frac{3}{2} \), we can not deal with problem (43) by the above method. But, we may consider the following boundary value problem

\[
\begin{cases}
  m_{1,1} + m_{1,2}u_1 + 2m_{1,1}u_1 + \left( m_1 \sum_{j \neq 1} u_j \right)_x + \sum_{j \neq 1} m_j u_j, x = 0, & t > 0, x \in \mathbb{R}+, \\
  m_{2,1} + m_{2,2}u_2 + 2m_{2,1}u_2 + \left( m_2 \sum_{j \neq 2} u_j \right)_x + \sum_{j \neq 2} m_j u_j, x = 0, & t > 0, x \in \mathbb{R}+, \\
  \ldots \ldots \\
  m_{n,1} + m_{n,2}u_n + 2m_{n,1}u_n + \left( m_n \sum_{j \neq n} u_j \right)_x + \sum_{j \neq n} m_j u_j, x = 0, & t > 0, x \in \mathbb{R}+, \\
  u_1(0, x) = u_1(0, x), x = 1, 2, \ldots, n, & x \in \mathbb{R}+, \\
  u_1(0, x) = u_1(2k-2)(0) = \ldots = u_1(t, 0) = (0, 0), x = 1, 2, \ldots, n, & \text{if } t > 0.
\end{cases}
\]

For \( 2k + \frac{1}{2} < s < 2k + \frac{5}{2} \), we set

\[
D_k^s(\mathbb{R}_) = \{ h \in H^s(\mathbb{R}) \mid \tilde{h}^{(2k)}(0) = \tilde{h}^{(2k-2)}(0) = \ldots = h(0) = 0 \}.
\]

Lemma 5.2. Assume that \( h \in D_k^s(\mathbb{R}) \), where \( k = 0, 1, 2, \ldots \) and \( 2k + \frac{1}{2} < s < 2k + \frac{5}{2} \). Let furthermore

\[
\tilde{h}(x) = \begin{cases} 
    h(x), & \text{if } x \geq 0, \\
    -h(-x), & \text{if } x < 0.
\end{cases}
\]

Then \( \tilde{h}(x) \) belongs to

\[
D_k^s(\mathbb{R}) = \{ h \in H^s(\mathbb{R}) \mid \tilde{h}^{(2k)}(0) = \tilde{h}^{(2k-2)}(0) = \ldots = h(0) = 0 \}.
\]

By using Lemma 5.2, we have the following local well-posedness result

Theorem 5.2. Assume that \( (u_{1,0}, u_{2,0}, \ldots, u_{n,0}) \in (D_k^s(\mathbb{R})^n, \) where \( k = 0, 1, 2, \ldots \) and \( 2k + \frac{1}{2} < s < 2k + \frac{5}{2} \), then there exist a maximal \( T = T(\tilde{z}_0) > 0 \) and a unique solution \( z = (u_1, u_2, \ldots, u_n) \) to (48) such that \( z = (\tilde{z}, \tilde{z}_0) \in C([0, T]; (D_k^s(\mathbb{R}))^n) \cap C^1([0, T]; (D_k^{s-1}(\mathbb{R}))^n) \). Moreover, the solution depends continuously on the initial data, i.e. the mapping \( \tilde{z}_0 \rightarrow z(\tilde{z}, \tilde{z}_0) : (D_k^s(\mathbb{R}))^n \rightarrow C([0, T]; (D_k^s(\mathbb{R}))^n) \cap C^1([0, T]; (D_k^{s-1}(\mathbb{R}))^n) \) is continuous and the maximal time of existence \( T > 0 \) can be chosen to be independent of \( s \).

Proof. Following a similar argument of as in the proof of Theorem 5.1, we first extend the initial data \( z_0(x) \) defined on the interval \( \mathbb{R}+ \) into an odd function \( \tilde{z}_0(x) \) defined in (5.3) on the line. Since Lemma 5.2 and (45)-(46) show that \( \tilde{z}_0(x) \) is an odd function. Then, following the similar proof in Theorem 5.1, we can obtain the desired result of the theorem. This completes the proof. \( \square \)

We can also obtain a precise blow-up scenario, blow-up results and blow-up rate of (43) by using a similar method to [20].

Theorem 5.3. Let \( z_0 = (u_{1,0}, u_{2,0}, \ldots, u_{n,0}) \in (H^s(\mathbb{R}) \cap H^1_0(\mathbb{R}))^n \) with \( \frac{3}{2} < s < \frac{5}{2} \), and let \( T \) be the maximal existence time of the solution \( z = (u_1, u_2, \ldots, u_n) \)
to (4.3) with the initial data $z_0$. Then the corresponding solution blows up in finite time if and only if there exists an $i$ ($i \in \{1,2,\cdots,n\}$) such that

$$\liminf_{t \to T} \inf_{x \in \mathbb{R}} \{u_{i,x}(t,x)\} = -\infty.$$ 

**Theorem 5.4.** Let $z_0 = (u_{1,0},u_{2,0},\cdots,u_{n,0}) \in (H^s(\mathbb{R}_+) \cap H^1(\mathbb{R}_+))^n$ with $\frac{3}{2} < s < \frac{5}{2}$, and let $T$ be the maximal existence time of the solution $z = (u_1,u_2,\cdots,u_n)$ to (5.1) with the initial data $z_0$. If there is some $x_0 \in \mathbb{R}$ such that

$$\sum_{j=1}^{n} u'_{j,0}(x_0) < -\left[\frac{(3n+2)(n+1)}{8} \sum_{j=1}^{n} \|u_{j,0}\|_{H^2}^2\right]^{\frac{1}{2}},$$

then the corresponding solution to Eq. (4) blows up in finite time.

**Theorem 5.5.** Let $z_0 = (u_{1,0},u_{2,0},\cdots,u_{n,0}) \in (H^s(\mathbb{R}_+) \cap H^1(\mathbb{R}_+))^n$ with $\frac{3}{2} < s < \frac{5}{2}$, is odd. If

$$\sum_{j=1}^{n} u'_{j,0}(x_0) < -\left[\frac{(2n-1)(n+1)}{8} \sum_{j=1}^{n} \|u_{j,0}\|_{H^2}^2\right]^{\frac{1}{2}},$$

the corresponding solution to Eq. (4) blows up in finite time.

**Theorem 5.6.** Let $z = (u_1,u_2,\cdots,u_n)$ be the solution to Eq. (4) with the initial data $z_0 = (u_{1,0},u_{2,0},\cdots,u_{n,0}) \in (H^s(\mathbb{R}_+) \cap H^1(\mathbb{R}_+))^n$ with $\frac{3}{2} < s < \frac{5}{2}$ satisfying the assumption of Theorems 5.4 or 5.5, and let $T > 0$ be the maximal time of existence of the solution $z$. If $T < \infty$, we have

$$\liminf_{t \to T} \left(\inf_{x \in \mathbb{R}} \left(\sum_{j=1}^{n} u_{j,x}(t,x)\right) (T-t)\right) = -\frac{n+1}{2}.$$ 

**Remark 5.1.** Let $u_{i,t} \in D^s_k(\mathbb{R}_+)$, where $i = 1,2,\cdots,n$, $k = 0,1,2,\cdots$ and $2k+\frac{1}{2} < s < 2k+\frac{5}{2}$. Then Theorems 5.3-5.6 hold true for the corresponding solution $z$ to (50).

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**REFERENCES**

[1] M. Baouendi and C. Goulaouic, Remarks on the abstract form of nonlinear Cauchy-Kowalevski theorem, *Commun. Partial Differential Equation*, 2 (1977), 1151–1162.

[2] M. Baouendi and C. Goulaouic, Sharp estimates for analytic pseudodifferential operators and application to the Cauchy problem, *J. Differential Equations*, 48 (1983), 241–268.

[3] A. Bressan and A. Constantin, Global conservative solutions of the Camassa-Holm equation, *Arch. Ratton. Mech. Anal.*, 183 (2007), 215–239.

[4] A. Bressan and A. Constantin, Global dissipative solutions of the Camassa-Holm equation, *Anal. Appl.*, 5 (2007), 1–27.

[5] R. Camassa and D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, 71 (1993), 1661–1664.

[6] J. Chemin, Localization in Fourier space and Navier-Stokes system, *Phase Space Analysis of Partial Differential Equations*, Proceedings, CRM series, Pisa, 1 (2004), 53–135.

[7] A. Constantin, Existence of permanent and breaking waves for a shallow water equation: A geometric approach, *Ann. Inst. Fourier (Grenoble)*, 50 (2000), 321–362.
[9] A. Constantin, The trajectories of particles in Stokes waves, *Invent. Math.*, **166** (2006), 523–535.

[9] A. Constantin, On the inverse spectral problem for the Camassa-Holm equation, *J. Funct. Anal.*, **155** (1998), 352–363.

[10] A. Constantin and J. Escher, Particle trajectories in solitary water waves, *Bull. Amer. Math. Soc.*, **44** (2007), 423–431.

[11] A. Constantin and J. Escher, Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation *Commun. Pure Appl. Math.*, **51** (1998), 475–504.

[12] A. Constantin and J. Escher, Analyticity of periodic traveling free surface water waves with vorticity, *Ann. of Math.*, **173** (2011), 559–568.

[13] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.*, **181** (1998), 229–243.

[14] A. Constantin and J. Escher, On the blow-up rate and the blow-up set of breaking waves for a shallow water equation, *Math. Z.*, **233** (2000), 75–91.

[15] A. Constantin and J. Escher, Global existence and blow-up for a shallow water equation, *Ann. Scuola Norm. Sup. Pisa.*, **26** (1998), 303–328.

[16] A. Constantin and J. Escher, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, *Arch. Ration. Mech. Anal.*, **192** (2009), 165–186.

[17] A. Constantin and W. Strauss, Stability of peakons, *Comm. Pure Appl. Math.*, **53** (2000), 603–610.

[18] R. Danchin, A few remarks on the Camassa-Holm equation, *Differential Integral Equations*, **14** (2001), 953–988.

[19] R. Danchin, Fourier analysis methods for PDEs, Lecture Notes, 14 November, 2003.

[20] R. Danchin, A note on well-posedness for Camassa-Holm equation, *J. Differential Equations*, **192** (2003), 429–444.

[21] J. Escher and Z. Yin, Initial boundary value problems for nonlinear dispersive wave equations, *J. Funct. Anal.*, **256** (2009), 479–508.

[22] J. Escher and Z. Yin, Initial boundary value problems of the Camassa-Holm equation, *Comm. Partial Differential Equation*, **33** (2008), 377–395.

[23] G. Gui and Y. Liu, On the Cauchy problem for the integrable Camassa-Holm type equation with cubic nonlinearity, arXiv:1108.5368v2, 1–27.

[24] Y. Fu, G. Gui, Y. Liu and Z. Qu, Well-posedness and blow-up solution for a modified two-component periodic Camassa-Holm system with peakons, *Math. Ann.*, **348** (2010), 415–448.

[25] Y. Fu and C. Qu, Well-posedness and blow-up solution for a new coupled Camassa-Holm equations with peakons, *J. Math. Phys.*, **50** (2009), 012906, 25pp.

[26] Y. Fu and C. Qu, On a new Three-Component Camassa-Holm equation with peakons, *Comm. Theor. Phys.*, **53** (2010), 223–230.

[27] G. Gui and Y. Liu, Well-posedness and blow-up phenomena for a new three-component Camassa-Holm system with peakons, *J. Hyper. Differential Equations*, **9** (2012), 451–467.
[38] D. Holm and R. Ivanov, Multi-component generalizations of the CH equation: Geometrical aspects, peakons and numerical examples, J. Phys. A, 43 (2010), 492001, 20pp.

[39] D. Holm and R. Ivanov, Two-component CH system: Inverse scattering, peakons and geometry, Inverse Problems, 27 (2011), 045013, 19pp.

[40] D. Holm, L. Onarheim and C. Tronci, Singular solutions of a modified two-component Camassa-Holm equation, Phys. Rev. E., 79 (2009), 016601, 13pp.

[41] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, Comm. Pure Appl. Math., 41 (1988), 891–907.

[42] S. Kouranbaeva, The Camassa-Holm equation as a geodesic flow on the diffeomorphism group, J. Math. Phys., 40 (1999), 857–868.

[43] J. Lenells, A variational approach to the stability of periodic peakons, J. Nonlinear Math. Phys., 11 (2004), 151–163.

[44] Y. Li and P. Olver, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, J. Differential Equations, 162 (2000), 27–63.

[45] G. Misiolek, A shallow water equation as a geodesic flow on the Bott-Virasoro group, J. Geom. Phys., 24 (1998), 203–208.

[46] L. Ni and Y. Zhou, A new asymptotic behavior of solutions to the Camassa-Holm equation, Proc. Amer. Math. Soc., 140 (2012), 607–614.

[47] L. Tian, Y. Wang and J. Zhou, Global conservative and dissipative solutions of a coupled Camassa-Holm equations, J. Math. Phys., 52 (2011), 063702, 29pp.

[48] L. Tian and Y. Xu, Attractor for a viscous coupled Camassa-Holm equation, Adv. Differ. Equ., 2010 (2010), Art. ID 512812, 30 pp.

[49] J. F. Toland, Stokes waves, Topol. Methods Nonlinear Anal., 7 (1996), 1–48.

[50] M. Zhu and Blow-up, Global Existence and Persistence Properties for the Coupled Camassa-Holm equations, Math Phys. Anal Geom., 14 (2011), 197–209.

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