Casimir forces in a T operator approach

Oded Kenneth\(^1\) and Israel Klich\(^2\)

\(^1\) Department of Physics, Technion, Haifa 32000 Israel
\(^2\) Department of Physics, California Institute of Technology, MC 114-36 Pasadena, CA 91125

We explore the scattering approach to Casimir forces. Our main tool is the description of Casimir energy in terms of transition operators. The approach is valid for the scalar fields as well as electromagnetic fields. We provide several equivalent derivations of the formula presented in Kenneth and Klich [Phys. Rev. Lett. 97, 160401(2006)]. We study the convergence properties of the formula and how to utilize it, together with scattering data to compute the force. Next, we discuss the form of the the formula in special cases such as the simplified form obtained when a single object is placed next to a mirror. We illustrate the approach by describing the force between scatterers in one dimension and three dimensions, where we obtain the interaction energy between two spherical bodies at all distances. We also consider the cases of scalar Casimir effect between spherical bodies with different radii as well as different dielectric functions.

I. INTRODUCTION

The Casimir force\(^1\) is one of the fundamental predictions of quantum physics. It explores the interplay between a quantum field and external "classical" like objects such as boundary conditions, background dielectric bodies or space-time metric. While the classical objects modify the behavior of the field due to their presence, the field, in turn, acts on the objects, typically by exerting forces. Much work has been devoted to understanding the effect, as it appears in varied branches of physics: from condensed matter (interaction between surfaces in fluids) to gravitation and cosmology.

The first precise measurement of the effect by Lambreux\(^2\), signaled a new age of Casimir force measurements, and led to a revived interest in the theory behind the effect. In recent years, the force between various objects (such as two plates, plate and a sphere, corrugated plate and sphere, etc \(^2, 3, 4\)) was measured. Moreover the dependence on various properties of the materials used, such as corrections due to finite conductivity and temperature \(^5, 6\), as well as on geometry has been investigated. There is excellent agreement between the experiments and the theoretical predictions, which is being constantly improved. For an introduction to the subject as well as reviews of progress see e.g.\(^7, 8, 9, 10\).

Different variants (both material and geometric) of the force have been proposed, discussed and motivated by pure theoretical interest as well as by potential eventual application in nano-mechanical structures \(^11, 12\).

The original method used by Casimir, that of mode summation, has led to a large body of work on the effect in simple geometries, where the modes may be exactly computed. For more general cases one has to use other available approaches such as the Green’s function approach or the path-integral approach. Significant progress in utilizing these techniques numerically has been reported lately \(^13, 14\).

In the 1D case, scattering approach to Casimir physics has proved very useful. Indeed, many of the calculations of Casimir interaction between bodies are based on scattering theory, as the photon spectrum in an open geometry is continuous and it’s description requires scattering.

In this paper, we explore a scattering approach to Casimir effect in higher dimensions. The approach is based on analysis of a determinant formula for Casimir interactions obtained in Ref \(^15\), and may be viewed as a generalization of previous formulas, especially related to scattering, such as the Lifshitz formula \(^14\), and the results of Balian and Duplantier \(^17\). Within this approach, the Casimir energy is encoded in a determinant of the operator \(1 - T_A G_0 T_B G_0\) where \(T_A, T_B\) are Lippmann-Schwinger \(T\) operators associated with bodies \(A\) and \(B\) and \(G_0\) is the photons Green’s function; we shall therefore refer to the formula as the \(TGTG\) formula.

In \(^13\) it was shown how general results regarding the direction of the force between bodies related by reflection can be obtained from the \(TGTG\) formula. For example, the sign problem of interaction between two hemispheres was resolved. This result was subsequently extended to a large class of interacting fields possessing the "reflection positivity" property \(^19\) (See also \(^20\), where use is made of reflection positivity arguments to infer attraction between vortices and anti-vortices in a frustrated XY model). In \(^21\) an alternative derivation of the formula was presented.

The paper is organized as follows. In section (II) we start with a derivation of the determinant formula, as well as supply alternative derivations in terms of Green’s functions and the \(T\) operator of a pair of perturbations. Section (III) illustrates how one obtains the appropriate formula in the vector (electromagnetic (EM)) case. Sections (IV) and (V) cover simplified cases: the special case of a body placed next to a perfect mirror, and the dilute limit, which deals with very weak dielectrics by expanding round \(e = 1\).
We then proceed to show how the formula is to be applied in actual calculations. We explain how the formula is to be used together with partial wave expansions of the scattered states (sections (VI) and (VII)). In 1D where only two modes (left and right movers) exist at each \( \omega \) this leads to a known closed form formula for the Casimir energy in terms of reflection coefficients (see, e.g. [22, 23]).

In Section (VIII) we use spherical waves to obtain an explicit expansion for the Casimir interaction between compact bodies. We demonstrate this by computing the force between two spheres at all distances, thereby generalizing the approach of [24] to spheres beyond Dirichlet boundary conditions, and going beyond the proximity force approximation. We also consider cases of spheres with unequal radii, as well as as spheres with arbitrary dielectric function. In section (IX) results are extended to dielectrics can be written as Alternative derivations of Eq (11) are elaborated in the following subsections.

The action of a real massless scalar field in the presence of dielectrics can be written as

\[
S[\phi] = \frac{1}{2} \int d^{d}r \int \frac{d\omega}{2\pi} \phi_{\omega}^* (\nabla^{2} + \omega^{2} \epsilon(x, \omega)) \phi_{\omega}
\]

where \( \phi_{\omega} = \phi_{-\omega} \), and \( \epsilon(\omega, x) = 1 + \chi(x, \omega) \) is the dielectric function (we use units \( \hbar = c = 1 \)). This action is the simplest action which yields the scalar analog of the Maxwell equation

\[
\nabla \times \nabla \times \vec{A} - \frac{\omega^{2}}{c^{2}} \epsilon(\omega, x) \vec{A} = 0
\]

for the vector potential in the radiation gauge. Alternatively this action can be derived by coupling a scalar field to an auxiliary field living on the regions of space where \( \epsilon \neq 1 \), and then integrating out these fields, as done, e.g. in [27].

Formally, the free energy of the system is obtained from the partition function \( Z \) given by:

\[
Z = \int D \phi e^{iS[\phi]}
\]

Performing the Gaussian integration, one finds that the change in energy due to introduction of \( \chi \) in the system is

\[
E_C = E_\chi - E_{\chi=0} = \left( \sum_{j} |\lambda_{j}| \right)^{1} \left( 1 + \omega^{2} \chi(x, \omega)(\nabla^{2} + \omega^{2} + i0)^{-1} \right)
\]

At this point, we encounter one of the main features of Casimir physics - the need to properly isolate the physically relevant part of the energy out of a formally ill defined expression. A determinant (such as in Eq (11)) is mathematically well defined only if it has the form \( \det(1 + A) \), where \( A \) is a "trace class" operator, i.e. \( \sum_{j} |\lambda_{j}| \) is finite. If \( A \) is not a trace class operator, one may obtain different or infinite results for the determinant, depending on the order in which the eigenvalues of \( 1 + A \) are multiplied. The expression above is not of the required form. To see this note that \( A' \) in this case is given by:

\[
\omega^{2} \chi(x, \omega)(\nabla^{2} + \omega^{2} + i0)^{-1}
\]

This is an operator of the form \( g(x)f(\nabla) \). If such an operator is "trace class" then it’s trace is known to be given by the Birman-Solomyak result [28]:

\[
\text{Tr}(g(x)f(i\nabla)) = \int d^{d}x g(x) \int d^{d}k f(k)
\]

in our case we have \( \int d^{d}x \chi(x) < \infty \), however \( \int d^{d}k (-k^{2} + \omega^{2} + i0)^{-1} \) diverges, and is indicating that the operator involved doesn’t have a well defined trace [30].

II. THE TGTG FORMULA: CASIMIR INTERACTION AS A REGULAR DETERMINANT.

In this section, we explain how the part of the free energy of a Gaussian theory that depends on distance between bodies, and as such is responsible for the Casimir force, may be expressed in terms of a regular determinant, and discuss some of its properties. Some of the material covered here appeared in the literature, however, as far as we know, the final formula was never written in this general form; furthermore, it’s mathematical properties where not rigorously addressed previously. We note, however that an elaborate and rigorous analysis of a related problem involving impenetrable discs was carried out in [25].

We start by presenting the derivation of the determinant formula (11) in the path integral approach [22, 26, 27]. We first treat the case of a scalar field and explain later how the result is extended to the EM field.
As such, the expression \([11]\) only has meaning when specifying physical cutoffs. Removing physical cutoffs will leave us with an ill-defined expression and so we keep in mind cutoffs at high momenta in the notation \(\text{det}_A\).

At high frequencies, \(\chi(\omega, x) \to 0\) provides a physical frequency cutoff. For \(\text{Re } \omega, \text{Im } \omega > 0\) both \(\chi(\omega)\) and \((\nabla^2 + \omega^2 + i0)^{-1}\) are analytic, justifying Wick-rotation of the integration to the imaginary axis \(i \omega\) ending up with:

\[
E_C = \int_0^{\infty} \frac{d\omega}{2\pi} \log \text{det}_A (1 + \omega^2 \chi(x, i\omega) G_0(x, x'))
\]  

(7)

Where \(G_0(x, x') = \langle x | \frac{1}{\chi} | x' \rangle\). Restricting the operator \((1 + \omega^2 \chi G_0)\) to the support of \(\chi\) (more precisely to \(L^2(\text{Supp}(\chi))\)) clearly does not affect its determinant. Note that Eq. (7) is still ill defined if one removes the cutoff, as can be immediately seen from the argument based on Eq. [3].

We now consider the case depicted in Fig. 1 of two bodies \(A, B\) immersed in vacuum. \(\chi\) is assumed nonzero only inside the volumes of the two dielectrics \(A, B\) and we therefore consider in the following \((1 + \omega^2 \chi G_0)\) as an operator on \(H_A \oplus H_B \to H_A \oplus H_B\) where \(H_A = L^2(A)\) and \(H_B = L^2(B)\) [37]. It is then convenient to write

\[
(1 + \omega^2 \chi G_0)|_{H_A \oplus H_B} = \begin{pmatrix} 1_A + \omega^2 \chi_A G_{0,AA} & \omega^2 \chi_B G_{0,AB} \\ \omega^2 \chi_A G_{0,BA} & 1_B + \omega^2 \chi_B G_{0,BB} \end{pmatrix},
\]

(8)

where \(G_{0,\alpha\beta}\) is \(G_0\) restricted to \(H_\alpha \to H_\beta\) (equivalently, \(G_{0,\alpha\beta} = P_\alpha G_0 P_\beta\), where \(P_A = 1 \oplus 0\) and \(P_B = 0 \oplus 1\), are projections on \(H_A, H_B\) respectively). It turns out that the part of the energy that depends on mutual position of the bodies, and as such is responsible for the force, is a well-defined quantity, which is independent of the cutoffs. To see this, we subtract contributions which do not depend on relative positions of the bodies \(A, B\):

\[
E_C = E_C(A \cup B) - E_C(A) - E_C(B)
\]

(9)

As in Ref. [27], this amounts to subtracting the diagonal contributions to the determinant, which are not sensitive to the distance between the bodies, (i.e. only contributes to their self energies). This yields

\[
E_C = \int_0^{\infty} \frac{d\omega}{2\pi} \text{log det}_A \begin{pmatrix} 1_A + \omega^2 \chi_A G_{0,AA} & \omega^2 \chi_A G_{0,AB} \\ \omega^2 \chi_B G_{0,BA} & 1_B + \omega^2 \chi_B G_{0,BB} \end{pmatrix}
\]

\[
- \text{log det}_A \begin{pmatrix} 1_A & 1-BG_{0,AB} \\ 0 & 1_B + \omega^2 \chi_B G_{0,BB} \end{pmatrix}
\]

(10)

\[
= \int_0^{\infty} \frac{d\omega}{2\pi} \text{log det}_A \begin{pmatrix} 1_A & T_{AB} \\ T_{BA} & 1_B \end{pmatrix},
\]

where \(T_\alpha = \frac{1}{1 + \omega^2 \chi_{\alpha,\alpha}}\chi_\alpha\). Finally, using the relation

\[
\text{det} \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix} = \text{det}(1 - YX),
\]

which holds for block matrices we have:

\[
E_C(a) = \int_0^{\infty} \frac{d\omega}{2\pi} \text{log det}(1 - T_A G_{0,AB} T_B G_{0,BA}).
\]

(11)

Henceforth, we refer to (11) as the TGTG formula throughout the paper. Up to Wick rotation, the operators \(T_\alpha\) are exactly the \(T\) operators appearing in the Lippmann-Schwinger equation, as will be discussed in the next subsections. The Wick rotation \(T(\omega) \rightarrow T(i\omega)\) has the effect of turning \(T_\alpha\) into hermitian operators as well as of avoiding potential singularities (which may occur at real frequencies).

In Eq. (11), we disposed of the cutoff \(\Lambda\) as the expression is well defined in the continuum limit. In practical terms this means that replacing the infinite dimensional matrix of \(1 - T_A G_{0,AB} T_B G_{0,BA}\) by its upper-left \(n \times n\) block with \(n\) large enough and calculating the resulting ordinary determinant, gives an arbitrarily good approximation to a (finite) quantity which we call \(\text{det}(1 - T_A G_{0,AB} T_B G_{0,BA})\). This point is discussed in detail in appendix [B] where we prove some mathematical properties of the operators involved. The details are not essential for understanding the applications of the formula, so a reader not interested in mathematical rigor may skip them.

**Dirichlet and Neumann boundary conditions**

In many cases, and indeed in the original presentation by Casimir, one is interested in sharp boundary conditions, such as Dirichlet or Neumann. Sharp boundary conditions result in singular energy density at the surface, as field modes are required to vanish for all momentum scales. Typically, the local energy density diverges as the inverse fourth power of the distance from the boundary [29].

It is important to point out that the above considerations also describe the conducting case with minor changes. Following [25], assume conducting boundary conditions are given over a surface \(\Sigma\), parameterized by
internal coordinate \( u \) and by the embedding in \( \mathbb{R}^3 \) given by \( x(u) \). One may describe a simple metal by taking \( \chi(\omega) = \frac{\omega^2}{\Delta} \) on \( \Sigma \), and letting \( \Sigma \) have a thickness of a few skin depths \( l/\omega_p \), \( l \sim O(1) \), here \( \omega_p \) is the plasma frequency (proportional to the effective electron density in the metal). In the limit of large \( \omega_p \) one retains the same expression as [11], with the following substitutions:

\[
E_C(u) = \frac{1}{2\pi} \int_0^\infty d\omega (\log \det (1 - \mathcal{M}_{BA} \frac{\mathcal{M}_A}{1 + \mathcal{M}_A} \mathcal{M}_{AB} \frac{1}{1 + \mathcal{M}_B}))
\]

where in the Dirichlet case \( \mathcal{M} \) is given by:

\[
M^{(D)}(u, u'; \omega) = l\omega_p \sqrt{g(u)G_0(x(u), x(u'))} \sqrt{g(u')} \tag{13}
\]

and acting on the surfaces \( \Sigma \). Similarly Neumann boundary conditions may be treated in the path integral method by taking [30]:

\[
M^{(N)}(u, u'; \omega) = \sqrt{g(u)g(u')\delta_{n(u)}\delta_{n(u')}G_0(x(u), x(u'))} \tag{14}
\]

in [12]. We remark, that rigorous discussion of the formula in the Neumann case requires further analysis which we did not pursue in this paper (see remarks after Eq. [10]).

**Derivation using Green’s functions and T operators**

To make contact with Green’s function approach we supply in this section an alternative derivation of the TGTG formula. Most of the derivation is standard and may be skipped by readers interested only in new results. However, we point out that our approach where the T operator of combined scatterers is utilized seems new. Here, we use the Green’s function in order to express the density of states (DOS) of a differential operator with background, and then perform the mode summation by integrating over energies.

We briefly remind the reader some of the required material. The standard discussion of this is usually done in the context of non-relativistic quantum mechanics. The retarded/advanced \( G^\pm \) are then defined by:

\[
(E \pm is - \mathcal{H})G^\pm(E) = I \tag{15}
\]

This equation should be understood as an operator identity. If \( \mathcal{H} \) is a differential operator, for example \( \mathcal{H} = -\Delta \) then it is the operator form of the differential equation:

\[
(E \pm is + \Delta)G(x, x') = \delta(x - x') \tag{16}
\]

Using the representation \( \langle n|G^\pm(E)|n' \rangle = \lim_{s \to 0} \frac{\delta_{n'n}}{E \pm is - E_n} \), one then finds that the DOS is given by:

\[
\frac{1}{\pi} \text{Im Tr} G^\pm(E) = \mp \sum_n \delta(E - E_n) = \mp \rho(E). \tag{17}
\]

Noting the identity

\[
\partial_E \log |E \pm is - E_n| = \frac{1}{E \pm is - E_n}, \tag{18}
\]

one can rewrite this as

\[
\rho(E) = \pm \frac{1}{\pi} \text{Im} \partial_E \text{Tr} \log G^\pm(E). \tag{19}
\]

We are more interested in the relativistic version of this. (Indeed the Casimir force vanishes in the non-relativistic limit, as the exchange of very massive virtual particles is suppressed.) In the relativistic context the Feynman propagator \( G \) is defined by a similar formula to that of \( G^\pm \):

\[
H(\omega^2 + is)G = I \tag{20}
\]

For example, the action [1] corresponds to \( H = -\Delta - \omega^2 \epsilon \). In free space \( \epsilon = 1 \) we obtain the same equation as Eq. [16] apart from the substitution \( E \to \omega^2 \). (There is also a not very interesting conventional overall minus sign, which is the reason some signs in the following equations are different from what the reader may remember.) In the presence of nontrivial background (e.g. dielectric) the \( \omega \) dependence of \( H \) can take quite an arbitrary form, a fact that slightly complicates the derivation of the DOS. One may take advantage of the relation

\[
\text{Im} \frac{F'(x \pm is)}{F(x \pm is)} = \mp \pi \sum_n \delta(x - x_n)
\]

where \( F(x) \) is any real function having simple zeroes at the points \( \{x_n\} \). Indeed, away from the zeroes \( \{x_n\} \) the fact that \( F \) is real guarantees vanishing of the l.h.s while near the zero \( x_n \) we have \( \text{Im} \frac{F'(x \pm is)}{F(x \pm is)} = \text{Im} \frac{F'(x_n)}{F(x_n)} = \mp \pi \delta(x - x_n) \). Generalizing the relation from real functions \( F(x) \) to hermitian operators \( H(\omega^2) \) [38] allows writing

\[
\text{Im} \partial_\omega \text{Tr} \log G(\omega) = -\text{Im} \text{Tr} H'(\omega)G(\omega) = \pi \rho(\omega) \tag{21}
\]

which is the obvious analog of Eq. [19]. (Note however that similar generalization of Eq. [17] would usually be false.) In [21] we implicitly assumed \( \omega > 0 \) to avoid an extra \( \text{sign}(\omega) \) factor.

Now, assume that \( G_0 \) is known for \( H_0 \) and we add a perturbation \( V \), i.e.

\[
(H_0(\omega^2 + s) + V(\omega^2 + is))G = I \tag{22}
\]

The change in DOS due to introduction of the potential \( V \) is formally:

\[
\Delta \rho = \frac{1}{\pi} \text{Im} \partial_\omega \text{Tr} \log GG_0^{-1} \tag{23}
\]

We will be interested in the change in energy due to change in the distance \( a \) between two separated potentials \( V_A \) and \( V_B \), which make up \( V \). So we take \( V = V_A + V_B \).
Thus,\
\[ \partial_\omega \Delta \rho (\omega) = \frac{1}{\pi} \text{Im} \partial_\omega \partial_\alpha \text{Tr} \log (G G_0^{-1}) = \frac{1}{\pi} \text{Im} \partial_\omega \partial_\alpha \text{Tr} \log (I + G_0 V)^{-1}. \] (24)

Defining the \( T \) matrix by\
\[ T = V (I + G_0 V)^{-1}, \] (25)
we may also write\
\[ \partial_\omega \Delta \rho (\omega) = \frac{1}{\pi} \text{Im} \partial_\omega \partial_\alpha \text{Tr} \log (I - G_0 T). \] (26)

Alternatively, formally writing "\( \partial_\omega \text{det}(V_A + V_B) = 0 \)" since \( V_A, V_B \) act in different subspaces one can write that\
\[ \partial_\omega \Delta \rho (\omega) = \frac{1}{\pi} \text{Im} \partial_\omega \partial_\alpha \text{Tr} \log T. \] (27)

this last formal expression, however, should be handled with care, and so we avoid using it.

The \( T \) matrix satisfies:
\[ G(\omega) = G_0(\omega) - G_0(\omega)T(\omega)G_0(\omega) \] (28)

(here \( \omega > 0 \) is actually \( \omega + i\delta \)), and frequently appears in scattering theory. Also note \( T = V - V G V \).

The operator \( T \) appears in the Lippmann-Schwinger equation as follows. Given a solution \( \phi \) of the free equation, without a potential \( H_0(\omega)\phi = 0 \), one constructs a solution \( \psi \) of the eigenvalue equation \( H_0(\omega)\psi = 0 \) having the same incoming part \( \psi_{in} = \phi_{in} \). Formally, this is done by looking for a solution of:
\[ \psi = \phi - G_0 V \psi, \]
which is the Lippmann-Schwinger equation. It follows that \( \psi = (I + G_0 V)^{-1} \phi = (I - G_0 T)\phi \), thus we may build a new solution \( \psi \) from a solution \( \phi \) of the free equation. For example, choosing \( \phi \) to be a plane wave solution, one obtains
\[ \psi_k = e^{ikx} - \int dk' G_0(k') \langle k'|T|k \rangle e^{ik'x}. \] (29)

Note that our relativistic normalization convention implies that \( T \) is related to the scattering matrix via \( S = 1 - 2 \pi i \delta (\omega^2 - H_0) T \).

We now address the case of two potentials \( V_A, V_B \). We assume for simplicity that cutoffs are in place, and so work with the \( T \) operators as matrices. We compute the joint transition matrix for both perturbations \( T_{A \cup B} \), and show that the part independent on "self energy" is exactly Eq. (11).

By using the formula (28) as \( G_i = G_0 - G_0 T_i G_0 \) (with \( i = A, B \)), together with the definition of \( T \) (25), and straightforward algebraic manipulations we obtain:
\[ \frac{1}{1 + \epsilon_0 (V_A + V_B)} = (1 - G_0 T_A) - \epsilon_0 G_0 T_A (1 - G_0 T_B) \] (30)

and so the joint \( T \) operator of a pair of perturbations may be factored as
\[ T_{A \cup B} = (V_A + V_B) \frac{1}{1 + \epsilon_0 (V_A + V_B)} = (1 - G_0 (V_A + V_B))^{-1} \]
\[ (V_A + V_B) (1 - G_0 T_A) \frac{1}{1 - \epsilon_0 T_A G_0 T_B} (1 - G_0 T_B). \] (31)

The important feature of this expression is the observation that the only part of the expression which directly mixes between the \( A \) and \( B \) is the factor \( 1 - T_B G_A T_A \). Indeed, plugging Eq. (30) in Eq. (24) we see that the contribution of frequency \( \omega \) to the force is now given by:
\[ \partial_\omega \Delta \rho (\omega) = \frac{1}{\pi} \text{Im} \partial_\omega \partial_\alpha \text{Tr} \log (I + G_0 V_A + V_B))^{-1} = \frac{1}{\pi} \text{Im} \partial_\omega \partial_\alpha \log \text{det}(1 - G_0 T_A) + \log \text{det}(1 - G_0 T_B) - \log \text{det}(1 - G_0 T_A G_0 T_B) = -\frac{1}{\pi} \text{Im} \partial_\omega \partial_\alpha \log \text{det}(1 - G_0 T_A G_0 T_B) \] (32)

leading again to our expression for the energy Eq. (11). Alternatively one may simply verify the correctness of Eq. (11) by noting that:
\[ 1 - G_0 T_A G_0 T_B = 1 - G_0 V_A 1 - \epsilon_0 T_A G_0 V_B \frac{1}{1 + \epsilon_0 V_B} \]
\[ = \frac{1}{1 + \epsilon_0 V_A} \left( 1 + G_0 V_A \right) \frac{1}{1 + \epsilon_0 V_B} \left[ 1 + G_0 (V_A + V_B) \right] \frac{1}{1 + \epsilon_0 V_B} \]
and using Eq. (24).

### III. THE ELECTROMAGNETIC CASE

Here, we follow the approach of [31]. The statistical properties of the electromagnetic field in a medium are described by the appropriate photonic Green’s function. The electromagnetic fields are derived from the electromagnetic potentials \( A^\alpha, \alpha = 0, \ldots, 3 \). (It is convenient to work in the gauge \( A^0 = 0 \).) The retarded Green’s function \( D_{ik} \) is defined by:
\[ D_{ik}(X_1, X_2) = \begin{cases} (A_i(X_1) A_k(X_2) - A_k(X_2) A_i(X_1)) & t_1 < t_2 \\ 0 & \text{otherwise} \end{cases} \] (33)

where \( X_1, X_2 \) are 4-vectors \( X = (X^0, X^3) \) and \( k, i = 1, \ldots, 3 \). The angular brackets denote averaging with respect to the Gibbs distribution.

The interaction of the electromagnetic field with a classical current \( \mathbf{J} \) put in the medium is given by
\[ V = -\frac{1}{c} \int \mathbf{J} \cdot \mathbf{A}. \]

Kubo’s formula allows us to treat this interaction within linear response. By Kubo’s formula the mean value \( \mathbf{A}_i \) in presence of a current \( \mathbf{J} \) satisfies:
\[ \mathbf{A}_i(r) = -\frac{1}{hc} \int D_{ik}^R (\omega; r, r') \mathbf{J}_k(r') d^3r', \] (34)
where
\[ D^R_{ik}(\omega; \mathbf{r}, \mathbf{r}') = \int_0^\infty e^{i\omega t} D^R_{ik}(t; \mathbf{r}, \mathbf{r}') dt \] (35)

The function \( D \) is sometimes referred to as the generalized susceptibility of the system [31].

From Maxwell’s equations it follows that in a medium with a given permittivity tensor \( \varepsilon_{ij} \), permeability tensor \( \mu_{ij} \), and current \( J \), the vector potential \( A \) is given by\[ (\nabla \times (\mu^{-1}\nabla \times - \frac{\omega^2}{c^2} A) - c^2 \mu J) = \frac{4\pi}{c} \mathbf{J}. \] (36)

Substituting Eq. (34) in Eq. (36), we see that \( D \) is a Green’s function for the equation:
\[ \nabla \times \mu^{-1} \nabla \times D - \frac{\omega^2}{c^2} \varepsilon D = -4\pi i\hbar \delta(\mathbf{r} - \mathbf{r}') \] (37)

where \( I \) is the three dimensional unit matrix. In the following we shall work in units where \( c = \hbar = 1 \).

The Green’s function \( D \) is then used to obtain the well known expression Eq. (80.8) in Lifshitz and Pitaevskii [31] for the change in free energy due to variation of the dielectric function \( \varepsilon \) at a temperature \( T \):
\[ \delta F = \delta F_0 + \frac{1}{2} T \sum_{n=-\infty}^{\infty} \omega_n^2 \text{Tr}(D\delta \varepsilon). \] (38)

Here \( F_0 \) is the free energy due to material properties not related to long wavelength photon field, and \( \omega_n = \frac{2\pi n}{T} \) are Matsubara frequencies. \( D \) is the temperature Green’s function of the long wave photon field given by
\[ D(x, x'; \omega)_{ij} = \langle \mathbf{A}(x) | \mathbf{A}(x') \rangle_{ij} \]

may be written as \( \delta F = \delta F_0 + \delta F_C \) where
\[ F_C = \frac{1}{2} T \sum_{n=-\infty}^{\infty} [\log \det(\nabla \times \nabla \times + \omega_n^2 \varepsilon(x, \mathbf{x}_n)) - \log \det(\nabla \times \nabla \times + \omega^2_n)] \]
\[ = \frac{1}{2} T \sum_{n=-\infty}^{\infty} \log \det(1 + \omega_n^2 \varepsilon(x, \mathbf{x}_n) D(0)(\mathbf{x}_n)) \]

Here \( D(0)(\mathbf{x}, \mathbf{x'}; \omega_n)_{ij} = \langle \mathbf{A}(\mathbf{x}) | \mathbf{A}(\mathbf{x'}) \rangle_{ij} \). Note that \( F_C \) is exactly the same as (4), with the scalar propagator \( G_0 \) replaced by the vector propagator \( D_0 \). For later reference we write here the explicit expression for \( D_0 \):
\[ D_{0ij}(k, i\omega) = \frac{-4\pi}{k^2 + \omega^2} (\delta_{ij} + \frac{k_i k_j}{k^2}) \] (39)

Thus, starting with this expression, one repeats Eq. (9) and Eq. (10) to get Eq. (11), replacing \( G_0 \) by \( D_0 \) everywhere (including in the definition of the \( T \) operators). The analysis of the determinant now proceeds exactly as in the scalar case.

Alternatively, the EM case may similarly be derived starting from the functional determinant corresponding to an EM action analogous to Eq. (1). In the axial gauge \( A_0 = 0 \) this action takes the form:
\[ S = \frac{1}{2} \int d^3r \int \frac{d\omega}{2\pi} \frac{1}{2} \nabla^2 \chi_A(x, \omega) + \omega^2 \varepsilon(x, \omega) \nabla \cdot \mathbf{A}. \] (40)

A permeable body may similarly be described within our approach by replacing the dielectric interaction term \( \chi_A(x, \mathbf{A}) \) in the Lagrangian by a magnetic term: \( \mathbf{A} \nabla \times (1 - \frac{1}{\mu}) \nabla \times \mathbf{A} \). One may then go on through our derivation using the (differential) operator \( \nabla \times (1 - \frac{1}{\mu}\nabla) \nabla \times \mathbf{A} \) instead of \( -\omega^2 \chi_A(x) \) everywhere. One major difference between the two cases is worth noting: whereas the dielectric term is always described (after Wick rotation) by a positive operator, the operator in the magnetic term turns out to be negative (for \( \mu > 1 \)). This fact can be related to the known Casimir electric-magnetic repulsion. Moreover, the ideal \( \mu \to \infty \) limit is seen to correspond to a Lagrangian in which the term \( (\nabla \times \mathbf{A})^2 \) is missing (inside the body) which makes a highly irregular lagrangian.

Analogously with a scalar field satisfying Neumann boundary conditions suggests that this situation may be described by dropping the \( (\nabla \phi)^2 \) term inside the Neumann body. There are also other arguments in favor of that approach [32], however we did not bring these argument to a completely rigorous form.

IV. DIELECTRIC IN FRONT OF A MIRROR

A somewhat simplified, but useful in practice, version of our formula is obtained in the case of a body placed close to a mirror. Consider the body \( A \) to the left of a Dirichlet mirror \( B \) located at \( x_n = a/2 \). It is well known (using the image method) that the effect of the Dirichlet mirror is to replace the free propagator \( G_0 \) by
\[ G_B(x, x') = G_0(x, x') - G_0(x, J(x')) \] (41)

where \( J(x, x') = (x, a-x', x') \) denotes reflection through the mirror plane. This may be written as \( G_B - G_0 = -G_0 J \) where \( J \) is the operator defined by \( J \psi(x) = \psi(J(x)) \). Noting the standard relation \( G_B = G_0 - G_0 T_B G_0 \) between the Green function in the presence of scatterer \( B \) to its \( T \) matrix one concludes \( G_0 T_B G_0 = G_0 J \) which when substituted in (11) gives
\[ E_C(a) = \int_0^{\infty} \frac{d\omega}{2\pi} \log \det(1 - G_0 J T_A). \] (42)

An alternative (though closely related) approach is to note that by complete analogy to Eq. (7) the energy it costs to place a body \( A \) near a mirror \( B \) is
\[ E_C = \int_0^{\infty} \frac{d\omega}{2\pi} \log \det(1 + \omega^2 \chi_A(x, i\omega) G_B(x, x')). \]

Subtracting the energy \( E_C = \int_0^{\infty} \frac{d\omega}{2\pi} \log \det(1 + \omega^2 \chi_A(x, i\omega) G_0(x, x')) \) it cost to put \( A \) in vacuum then gives the Casimir interaction energy. Using the relation
\[ (1 + G_B V_A)/(1 + G_0 V_A) = \frac{1}{1 + (G_B - G_0) T_A} = 1 - G_0 J T_A \] (43)
leads again to \( (2) \).

Yet another way of obtaining the same result is by substituting \( \chi_B = \lambda \delta(x_n - a/2) \) in the definition of \( T_B \) and doing the algebra. One then finds

\[
G_0 T_B G_0 = \int \frac{d^2k}{(2\pi)^2} e^{ik_\perp (x-x')_\perp} \frac{\chi^2}{2i(\omega^2 - 2q)} e^{-|q|x_n - |q|x'_n} \bigg|_{q = \sqrt{\omega^2 + k_\perp^2}}.
\]

which in the limit \( \lambda \to \infty \) reduces, as expected, to the expression \( G_0J \) obtained through the image method.

We now address the case of a Neumann mirror. Note, that the Green function in the presence of a Neumann condition (corresponding to \( \omega = 1 \) in the above results) would lead in a similar limit to a Neumann mirror. (See however remark at the end of the previous section).

A similar treatment is applicable in the more physically relevant EM case. The boundary conditions \( E\parallel = 0 \) may be enforced by requiring the vector potential to satisfy \( JA - A \) where \( J \) is defined to act on vectors as \( JA(x) = (A\parallel(J(x)), -A\perp(J(x))) \) (Here \( A\parallel, A\perp \) denote the components of \( A \) parallel and normal to the mirror surface. The temporal component is considered as a parallel component though in practice we usually choose a gauge where it vanishes.)

The EM Casimir interaction between a dielectric and a mirror is then given by a formula similar to Eq. \( (1) \) with \( G_0, J \) replaced by the EM propagator \( D_0 \) and the vectorial \( J \) defined above.

It is interesting to also consider an ideal permeable mirror (having \( \mu \to \infty, \epsilon = 1 \)). This corresponds to the boundary condition \( B\parallel = 0 \) which may be enforced by requiring the vector potential to satisfy \( JA = +A \). Thus, the Casimir interaction of body \( A \) with such a mirror will be given by an expression involving the determinant \( \det(1 + TA D_0 J) \).

\section{DILUTE LIMIT}

In the following sections we consider strategies of using the \( TGTG \) formula in actual calculations. A particularly simple case is when \( \chi \) is small, which is commonly referred to as the "dilute" case (and sometimes as "low contrast"). Here we briefly sketch how to best use the formula in this limit. As shown in the appendix (Theorem \( (B) \), one always have \( ||\sqrt{GTGT} \sqrt{G}|| < 1 \), therefore we may expand the log \( \det(1-...) \) expression \( (1) \) in powers:

\[
E_C = -\int \frac{d\omega}{2\pi} \sum_{m} \frac{1}{m} \text{Tr}(T_A G_0 T_B G_0)^m. \tag{45}
\]

In the dilute limit \( \chi < 1 \), so one may also substitute the expansion

\[
T_\alpha = -\sum_{n=0}^{\infty} (-\omega^2\chi_\alpha G_0)^n \omega^2 \chi_\alpha \tag{46}
\]

in Eq. \( (15) \) and compute the involved integrals to desired order. This expansion is the continuous equivalent to summation of two body forces, and is equal to the Born series appearing, for example, in Ref. \( (31) \).

\section{SCATTERING APPROACH}

As remarked above, the operator \( T_A G_0 T_B G_0 \) appearing in our formula is closely related to scattering data. The purpose of this section is to clarify this relation and make it more explicit. In order to keep better touch with conventions used in scattering theory, we usually avoid in the following sections using Wick rotation and thus we work in Lorentzian rather than Euclidean space with real rather than imaginary frequency and with the Feynman rather than the Euclidean propagator.

As mentioned above, the arguments of \( G_0 \) in Eq. \( (11) \) never coincide, implying that when \( G_0(x_a, x_b) \) is considered as a function of \( x_b \) alone it is a solution of the (homogeneous) free wave equation. Thus one may expand \( G_0(x_a, x_b) \) in the form \( \sum c_{\alpha\beta} \phi_\alpha^*(x_a) \phi_\beta(x_b) \) where \( \{\phi_\alpha(x_a)\}, \{\phi_\beta(x_b)\} \) are some sets of free wave solutions of energy \( \omega \). There is, of course, great freedom in choosing the sets \( \{\phi_\alpha(x_a)\}, \{\phi_\beta(x_b)\} \). In practice one would choose these in a way that makes subsequent calculations easier. Since we consider \( T_A G_0 T_B G_0 \) as acting only on the volume of object \( A \), these considerations also apply to the propagator on the right of this expression.

The Lippmann-Schwinger operator \( T(\omega) \) is related to the S-matrix by \( (33) \)

\[
S = 1 - 2\pi i \delta(\omega^2 - \omega'^2) T(\omega). \tag{47}
\]

Therefore, \( T(\omega) \) has the property that its matrix element \( \langle \alpha | T | \beta \rangle \) between a pair of free states \( \alpha, \beta \) having energy \( \omega \) is equal to the corresponding matrix element of the transition matrix. Since the operator \( T_B \) in \( T_A G_0 T_B G_0 \) is sandwiched between a pair of free Feynman propagators corresponding to energy \( \omega \), we may identify it with the corresponding transition matrix. Due to the cyclicity of the determinant \( \det(1 - T_A G_0 T_B G_0) \) the same is true of \( T_A \).

Substituting the expansion \( G_0(x_a, x_b) = \sum c_{\alpha\beta} \phi_\alpha^*(x_a) \phi_\beta(x_b) \) we arrive at

\[
T_A G_0 T_B G_0 = \sum_{\alpha'\beta'} T_A |\alpha\rangle c_{\alpha\beta} \langle \beta | T_B | \beta' \rangle c_{\alpha',\beta'} |\alpha'\rangle \tag{48}
\]
The Casimir interaction will then be given explicitly by
\[ E = \int_0^\infty \frac{d\omega}{2\pi} \log \det(1 - K(i\omega)). \]  
Even here \( K \) is responsible to the cancelling of the factor \( B \).

VII. PARTIAL WAVES EXPANSION

In the following sections, we consider strategies of using the representation \([15]\) by restricting the \( K \) matrix to a finite subspace which gives the dominant contribution to the force. Indeed, in many cases of interest only a few partial waves are significantly scattered; the best example for this is when objects are far apart, and from a large distance look point like. At this limit one expects significant contribution only from \( s \)-wave scattering. In the more general case, \( K \) may be approximated by a finite dimensional matrix corresponding to several partial waves. In order to see how this works in practice we consider below a few simple cases.

One dimensional systems

A particularly simple case occurs when the system is one-dimensional. Consider, e.g., a scalar field in 1D. All states of energy \( \omega \) are then spanned by two modes: left and right movers \( |L\rangle, |R\rangle = \frac{1}{\sqrt{\omega}} e^{\pm i\omega x} \). Hence, in this case, the determinant Eq. \([11]\) can easily be evaluated. To see how this is done, we write the Feynman propagator explicitly as
\[ G_0 = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx}}{\omega^2 - k^2 + i0} = -\frac{i}{2\omega} e^{i|\omega|x} \]  
We consider a pair of scatterers \( A, B \) such that \( A \) is on the left of \( B \). This immediately implies that we have \( x_a < x_b \) and therefore
\[ G_{0BA}(x_b, x_a) = -\frac{i}{2\omega} e^{i\omega(x_b-x_a)} = -\frac{2\pi}{2\omega} |R\rangle \langle L| \]  
Similarly, we also have \( G_{0AB} = -\frac{2\pi}{2\omega} |L\rangle \langle R| \). Using this we see that the operator \( K \) in Eq. \([18]\) turns into the e-number
\[ K = -\frac{2\pi i}{2\omega} i^2 |R\rangle \langle T_A |L\rangle \langle L|T_B |R| = \hat{r}_A(\omega)r_B(\omega). \]  
Here \( r_B(\hat{r}_A) \) is the reflection coefficient for a wave hitting scatterer \( B \) from the left (\( A \) from the right) to be reflected back. Note that the normalization of \( T \) implied by Eq.\([47]\) is responsible to the cancelling of the factor \( -\frac{2\pi i}{2\omega} \). (Had we used relativistic normalization for \( |L, R\rangle \) the factor \( 2\omega \) would not have appeared.) We thus conclude
\[ \det(1 - T_A G_0 T_B G_0) = 1 - \hat{r}_A(\omega)r_B(\omega). \]

The tilde on \( r_A \) serves to remind us that it is the reflection coefficient from the right side of \( A \).

We remark that \( \hat{r}_A(\omega)r_B(\omega) \) implicitly depends on the distance between \( A, B \) through the (phase) dependence of \( r_A, r_B \) on the scatterers locations. To make this explicit, note that moving a scatterer a distance \( a \) affects the reflection coefficients as \( r \rightarrow e^{-2ia\omega}, \hat{r} \rightarrow e^{2ia\omega\hat{r}} \).

Moving the scatterers a distance \( a \) apart therefore results in
\[ \det(1 - T_A G_0 T_B G_0) \rightarrow (1 - e^{2ia\omega\hat{r}_A(\omega)r_B(\omega)}). \]
Substituting in \([18]\) we obtain the familiar formula for 1d Casimir interaction between scatterers \([22, 23, 33]\).

Multi-component field in 1d

The considerations used above for a single scalar field in one dimension extend to a situation where \( \phi = (\phi_1, \phi_2, ..., \phi_n) \) is an \( n \) component field. In this case, the reflection coefficients \( r_{AB} \) turn into \( n \times n \) matrices and one finds det\((1 - T_A G_0 T_B G_0) = \det(1 - \hat{r}_A(\omega)r_B(\omega)) \) where the determinant on the right is of a usual \( n \times n \) matrix.

Plane wave expansion.

In physical three dimensional space, there are many different possible ways to expand the propagator \( G_0(x_a, x_b) = \sum C_{\alpha\beta} \phi_\alpha^*(x_a)\phi_\beta(x_b) \) in terms of free wave solutions \( \{\phi_\alpha(x_a)\}, \{\phi_\beta(x_b)\} \). In the next section we describe the expansion in spherical waves (which is probably the most useful expansion), and we demonstrate its use to calculating the Casimir force between compact object. However, for the sake of simplicity we first describe here a plane wave expansion which is the immediate generalization of Eq. \([50]\). A simple heuristic way to arrive at this generalization is to formally think of the field \( \phi \) in three dimensions as one dimensional field having infinitely many components labelled by its transverse momenta. Indeed, such point of view has been successfully used in describing transport in quasi 1D conductors in mesoscopic physics, whereby each transverse component corresponds to a scattering channel (see for example Ref. \([34]\)). This suggests splitting \( k \) into its \( z \)-component \( k_z \) and its transverse components \( k\parallel = (k_x, k_y) \). The 3d propagator may then be written as:
\[ G_0 = -\int \frac{d^2k\parallel}{(2\pi)^2} \frac{ie^{i|k\parallel \cdot r_\parallel x_\parallel}}{2k_z} |k_z = \sqrt{\omega^2 - k^2 + i0} \]  
Here \( \sqrt{\omega^2 - k^2 + i0} \) may be either real and positive (for \( \omega^2 > k^2 \)) or pure imaginary (for \( \omega^2 < k^2 \)) in which case the \( i0 \) prescription implies that it must be chosen on the
positive imaginary axis. Assuming that $A$ is located to the left of $B$ along the z-axis it follows that

$$T_A G_0 T_B G_0 \equiv \int \frac{dk_x dk_y dk_z}{(2\pi)^3} T_A \langle (q_x, q_y, -q_z) \rangle \times \frac{1}{2q_z} \langle (k_x, k_y, k_z) \rangle \frac{1}{2q_z} \langle (k_x, k_y, k_z) \rangle,$$

where $q_z = \sqrt{\omega^2 - q_x^2 - q_y^2 + i0}$ and $k_z = \sqrt{\omega^2 - k_x^2 - k_y^2 + i0}$.

When considering only the terms satisfying $\omega^2 > q_x^2 + q_y^2$, Eq. (52) indeed looks like a straightforward generalization of the 1d result. However, as this expression shows, to get the correct result one must also include the contribution of evanescent waves ($q_z^2 > \omega^2$). Upon Wick rotation, however, the distinction between ordinary and evanescent waves disappears. It may also be noted that (since in general $q_z \neq k_z$) the variation of the $\langle (q_x, q_y, -q_z) \rangle T_B \langle (k_x, k_y, k_z) \rangle$ matrix elements upon moving $B$ along the z-axis is considerably more complicated than in the 1d case.

The above representation may be helpful in problems where the scatterers $A, B$ have exact or approximate planar geometry (e.g. corrugated plates). Though the theorem guaranteeing finite trace does not apply for infinite plates, one may show that dividing by the plate area leads to finite result. We remark that actual calculation of the determinant requires discretizing $k_\parallel$ which corresponds to assuming large but finite plates. Alternatively one may use Eq. [45] with continuous $k_\parallel$.

VIII. SPHERICAL WAVES EXPANSION

When describing interaction between two compact bodies, often it is convenient to represent the transition matrices $T$ in a spherical wave basis. To do so, we choose two points $P_A, P_B$ inside bodies $A, B$ respectively. We parameterize the points of body $A$ by the radius vector $\mathbf{r} = \mathbf{r}_A$ measured from the point $P_A$ and the points of $B$ by the radius vector $\mathbf{r} = \mathbf{r}_B$ measured from the point $P_B$. The vector connecting $P_A$ and $P_B$ will be denoted by $\hat{a}$ (Fig 2). In the scalar case, the free spherical waves centered at $P_A, P_B$ are given by

$$|lm\rangle_{A,B} = \sqrt{\frac{2\omega^2}{\pi}} j_l(\omega r_{A,B}) Y_l^m(\hat{r}_{A,B}),$$

with the normalization $\langle \omega' l'm' | \omega lm \rangle = \delta_{\omega \omega'} \delta_{mm'} \delta(l - l')$. To use Eq. (45), the scalar 3d Green function $G_0 = -\frac{\omega^2}{4\pi r^2}$, is expanded in terms of the spherical harmonic functions centered around $P_A$ and those centered around $P_B$.

$$G_\omega = \sum_{lm,l'm'} |lm\rangle_B C_{lm:l'm'} \langle l'm'|A\rangle$$

FIG. 2: Coordinate system used for the partial wave approach

where (See appendix A for a proof of the following equations.)

$$C_{lm;l'm'}(\omega) = -\frac{\alpha}{2\pi} \sum_{l''m''} C \left( \begin{array}{ccc} l & l' & l'' \\ m & m' & m'' \end{array} \right) \frac{i}{2\pi} d\Omega Y_{lm}^* Y_{l'm''} Y_{l''m''} = (-1)^m \sqrt{\frac{4\pi(2l+1)(2l'+1)(2l''+1)}{\pi}} \times \left( \begin{array}{ccc} l & l' & l'' \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} l & l' & l'' \\ m & -m' & -m'' \end{array} \right)$$

have known expressions in terms of the $3j$ symbol or as an integral of spherical functions:

$$C \left( \begin{array}{ccc} l & l' & l'' \\ m & m' & m'' \end{array} \right) = \frac{4\pi}{\omega} \sum_{l''m''} C \left( \begin{array}{ccc} l & l' & l'' \\ m & m' & m'' \end{array} \right) \sqrt{\frac{2}{\pi \omega^2}} K_{l''(l+l')/2}(\omega a) Y_{l''m''}(\hat{a})$$

In actual computations, it is often more convenient to use the Wick-rotated expression. This may be expressed as

$$C_{lm;l'm'}(i\omega) = -\frac{\alpha}{2\pi} \sum_{l''m''} g_{lm;l'm'} \left( \begin{array}{ccc} l & l' & l'' \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} l & l' & l'' \\ m & -m' & -m'' \end{array} \right)$$

are real. Here $K_{l''+l}(\omega a)$ are modified Bessel functions of the second kind. Equations (50) and (57) may be somewhat simplified by choosing the $z$-axis along $\hat{a}$.

The above expansion of $G_\omega$ allows expressing $T_A G_0 T_B G_0$ in terms of matrix elements $\langle l'm'|T|m\rangle$ of the transition matrices of the two scatterers. The Casimir interaction may then be written as in Eq. (45) where

$$K_{lm;l'm'} = \langle -1 \rangle^{l+l'+l+2} \delta_{l_1 l_2 l_3 l_4} \delta_{m_1 m_2 m_3 m_4} \delta_{m_1 m_2 m_3 m_4}$$

Here $C_{lm;l'm'}$ is given by Eq. (50) or Eq. (57), summation over $l_1, m_1, l_2, m_2, l_3, m_3$ is implied and we note that the extra sign resulted from $C_{lm;l'm'}(\hat{a}) \equiv (-1)^{l+l'} C_{lm;l'm'}(\hat{a}) = C_{lm;l'm'}(\hat{a})$. If we assume that only waves having $l \leq l_0$ are significantly scattered then $K$ will turn into a finite
$$(l_0 + 1)^2 \times (l_0 + 1)^2$$ matrix (since the dimension of the subspace $l \leq l_0$ is $\sum_{l=0}^{l_0} (2l + 1) = (l_0 + 1)^2$). We stress that this argument does not require us to assume spherical symmetry of the scatterers.

When $A, B$ are very far apart, the interaction between them is governed by waves of very low frequency and therefore also low $l$. At this limit the leading contribution comes from the $s$-wave scattering transition matrix element $(l = 0 | T_{A,B} | l = 0) \approx 2\omega^2 \lambda_{A,B}/\pi$, where $\lambda$ is the scattering length.

The matrix $K$ then reduces to the scalar $K = -\omega^2 \lambda_A \lambda_B \left( h_0^{(1)}(\omega a) \right)^2 = 4\pi \frac{\lambda_A \lambda_B}{a} e^{2i\omega}$. Doing the integral [43] one arrives at

$$E_C = -\frac{\lambda_A \lambda_B}{a^3}.$$

This limit corresponds to the scalar version of the well known Casimir-Polder interaction. Our formalism, however, allows calculating corrections to it up to any desirable finite order in $\frac{1}{a}$. For example, for two Dirichlet spheres of radii $R_1, R_2$ at distance $a$ between their centers the expansion gives:

$$E = \frac{R_1 R_2}{4\pi a^3} - \frac{R_1 R_2 (R_1 + R_2)}{8\pi a^3} \left( \frac{\omega R_1}{\pi} \right)^2 + \frac{R_1 R_2 (34 R_1^2 + 9 R_1 R_2 + 34 R_2^2)}{48 \pi a^5} - \frac{R_1 R_2 (R_1 + R_2) (2 R_1^2 + 23 R_1 R_2 + 2 R_2^2)}{36 \pi a^7} + \ldots$$

### Spherical scatterers

Significant simplification is possible whenever $A, B$ have spherical symmetry. First, the $T$-matrices are diagonal in angular momentum basis and so may be expressed as

$$\langle l' m' | T_{A,B} | l m \rangle = \delta_{l' l} \delta_{m' m} \frac{2i\omega}{2\pi} \left( e^{2i\delta_{l',l}} - 1 \right),$$

where the normalization factor $\frac{2i\omega}{2\pi}$ follows from Eq[47]. A second consequence is that rotation around $\hat{a}$ (which from now on we take as coinciding with $\hat{z}$ axis) is a symmetry of the whole system. The determinant therefore factors as a product of terms corresponding to different values of the azimuthal number $m$. The energy turns into a sum of the corresponding terms

$$E = \sum_m \int \frac{d\omega}{2\pi} \log \det(1 - K^{(m)}(i\omega)).$$

The matrices $\{K^{(m)}_{l l'}\}_{l' - |m|}^{\infty}$ defined for each $m \in \mathbb{Z}$ (actually $K^{(-m)} = K^{(m)}$) are infinite dimensional but may be approximated in numerical calculations by finite matrices corresponding to $l, l' \leq$ some $l_0$. The operator $K^{(m)}$ may be written explicitly as

$$K^{(m)}_{l l'} = \sum_j g^{(m)}_{l j} t^{(A)}_{j l} g^{(m)}_{l' j} t^{(B)}_{j l'},$$

where we used the notation:

$$t_j = \frac{1}{2} \left( -1 \right)^j (e^{2i\delta_j} - 1),$$

$$g^{(m)}_{l j} (i\omega) = (-1)^{m+1} \sqrt{(2l_1 + 1)(2l_2 + 1)} \sum_k (2l + 1) \sqrt{\frac{2}{\pi^4} K_{l + \frac{1}{2}}(i\omega)} \left( \begin{array}{ccc} l_2 & l_1 & l \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} l_2 & l_1 & l \\ m & -m & 0 \end{array} \right)$$

Note that both $g^{(m)}_{l j} (i\omega)$ and $t_j (i\omega)$ are real.

### Dirichlet Spheres

The simplest example for which the above may be applied is the interaction of two hard (Dirichlet) spheres. The $T$-matrix elements are well known in this case, and are given by $t_j (i\omega) = \left( -1 \right)^{j+1} \frac{\omega R}{\pi a}$. The numerical calculation of the two determinants det$(1 \pm K)$ is then somewhat easier than direct calculation of det$(1 - K)$. Moreover, comparison with section [14] shows that the two determinants det$(1 \pm K)$ (actually with $K_{l l'} = (-1)^m g_{l l'} t_{l l'}$) correspond to the Casimir interaction energies $E_{D,N}$ of a single hard sphere and a Dirichlet/Neumann mirror a distance $a/2$ away. The symmetric two hard sphere system then has the energy $E_S = E_D + E_N$. (One may also understand this in terms of decomposition into even and odd modes.)

We have done the calculation including partial waves of $l \leq l_0$ for different values of $l_0$ and considered the $l_0$ dependence of the results as a test for convergence. Most calculations included modes up to $l_0 = 10$, but for small values of sphere separation $a$, we used larger $l_0$ even up to $l_0 = 72$ for $a/R = 2.1$. Since we expected the error to behave roughly as $E_C - E(l_0) \sim O(e^{-c \omega})$, we tried to fit the results with this assumed asymptotics. The numerical calculation included partial waves of $l \leq l_0$ for different values of $l_0$ and considered the $l_0$ dependence of the results as a test for convergence. Most calculations included modes up to $l_0 = 10$, but for small values of sphere separation $a$, we used larger $l_0$ even up to $l_0 = 72$ for $a/R = 2.1$. Since we expected the error to behave roughly as $E_C - E(l_0) \sim O(e^{-c \omega})$, we tried to fit the results with this assumed asymptotics.
bit crude since our numerics is consistent with \( c \) being a slowly growing function of \( l_0 \) (which might be due to sub-leading asymptotics). By matching our results with the assumed asymptotics, one can obtain a corrected estimate for \( E_C \). Comparison of this estimate with results obtained by increasing \( l_0 \) gave good agreement.

| \( a/R \) | \( c \) | \( L(1\%) \) |
|---|---|---|
| 2  | 0  | ∞  |
| 2.1 | 0.1831 | |
| 2.2 | 0.3416  | |
| 2.35 | 0.579  | |
| 2.5  | 0.787  | |
| 2.75 | 1.065  | |
| 3    | 1.3334  | |
| 3.5  | 1.7823  | |
| 4    | 2.142  | |
| 5    | 2.751  | |
| 7    | 3.441  | |

The following table (63) and Fig. 3 show the results for the Casimir energy itself (measured in units of \( \frac{\hbar c}{2\pi} \)). \( E_D \) denotes the energy of Dirichlet-mirror+(Dirichlet)sphere system, \( E_N \) denotes the energy of Neumann-mirror+(Dirichlet)sphere system, \( E_S \) denotes the energy of the symmetric two hard sphere configuration (having \( E_S = E_D + E_N \)). The result for \( E_D \) are in perfect agreement with a similar calculation done in Ref. [24].

\[
\begin{array}{|c|c|c|c|}
\hline
\( a/R \) & \( E_D \) & \( E_N \) & \( E_S \) \\
\hline
2.1 & -8.75 & 7.66 & -1.0939 \\
2.2 & -2.2129 & 1.9382 & -0.27477 \\
2.35 & -0.739 & 0.6488 & -0.0902822 \\
2.5 & -0.3688 & 0.3245 & -0.0443005 \\
2.75 & -0.1679 & 0.1483 & -0.0195891 \\
3 & -0.09703 & 0.08613 & -0.0108937 \\
3.5 & -0.044981 & 0.0403034 & -0.0046768 \\
4 & -0.0261973 & 0.0236767 & -0.00252067 \\
5 & -0.0123048 & 0.0112853 & -0.00101948 \\
7 & -0.00477708 & 0.00447243 & -0.000304649 \\
10 & -0.00199796 & 0.00190445 & -0.0000935083 \\
13 & -0.0011022 & 0.00106165 & -0.0000405423 \\
16 & -0.000700129 & 0.000678957 & -0.0000211723 \\
\hline
\end{array}
\]

FIG. 3: The calculated Casimir energy of: (a) Two Dirichlet spheres of radius \( R \) at distance \( a \) between their centers. (b,c) A Dirichlet sphere of radius \( R \) whose center is at a distance \( a/2 \) from a Dirichlet/Neumann mirror. The graphs show \( E/E_0 \) as a function of \( (a/R - 1)^{-1} \) where \( E_0 \) is the large distance asymptotic expression of it. Specifically: (a) \( E_0^S = -\frac{\hbar^2}{4\pi R} \frac{a^2}{(a-2R)^2} \), (b,c) \( E_0^{D,N} = \frac{\hbar^2}{2\pi R} \frac{1}{(a-2R)^2} \). At short distances \( E/E_0 \) approach the PFA prediction (a) \( a/2R \sim 0.27 \), (b) \( a/2R \sim 0.54 \), (c) \( \frac{\hbar^2}{2\pi R} \sim 0.47 \). The black curve shows the calculated exact result for \( l_0 \rightarrow \infty \). We extrapolated it to \( a = 2R \) and \( a = \infty \) using the known asymptotics. The colored graphs show the result of including partial waves of \( l \leq l_0 \) where: \( l_0 = 0 \) (red), \( l_0 = 1 \) (sky blue), \( l_0 = 2 \) (green), \( l_0 = 10 \) (blue).

We would like to mention two points regarding the actual implementation of the numerical calculation. (Our earlier numerical attempts failed because we were not fully aware of these points.) The expressions of \( g_{\text{tr}}(i\omega) \), \( t_{\text{tr}}(i\omega) \) may attain at small \( \omega \)’s very large/small values respectively in such a way that only their product remain finite. At large \( \omega \)’s similar phenomena occur with \( t_{\text{tr}}(i\omega) \) large and \( g_{\text{tr}}(i\omega) \) small.
small. Thus to avoid computer overflow it is much better to “renormalize” these two quantities redefining
\[ \tilde{g}_l = \frac{z_l z_l}{\tilde{t}_l}, \]
\[ \tilde{t}_l = \frac{t_l}{z_l^2} \]
with \( z_l \sim (R\omega)^{l+1/2} e^{R\omega} \).

A second important point is that one should make sure that the computer program doing the calculation does not use the expansion of \( I_{l+1/2}(x) \) in terms of elementary functions. In MATHEMATICA (which we used) this expansion is an automatic default whenever the index of the Bessel function is half integer. However this expansion is known to be numerically unstable (except for very small \( l \)) and using it would lead to errors.

The general formula works well for \( R_1 \neq R_2 \). For example taking \( R_1 = R_0, R_2 = 2R_0 \) and measuring \( E \) in units of \( \frac{\hbar}{R_0^3} \) we found

### Interaction energy between a sphere of radius \( R_0 \) and a sphere of radius \( 2R_0 \)

| \( a/R_0 \) | \( E \) |
|---|---|
| 3.1 | -1.4554 |
| 3.2 | -0.367535 |
| 3.3 | -0.164591 |
| 3.4 | -0.0931057 |
| 3.5 | -0.0598295 |
| 3.67 | -0.0334525 |
| 3.83 | -0.021821 |
| 4 | -0.01501511 |
| 5 | -0.00362365 |
| 6 | -0.0015196954 |
| 8 | -0.00047970126 |
| 10 | -0.00021536976316 |
| 14 | -0.0000696380241 |
| 18 | -0.00003103693506 |
| 22 | -0.0000164921322 |

### Dielectric Spheres

The formula also works well for finite dielectric constant. For example the numerical results for \( R_1 = R_0; R_2 = 2R_0; a = 4R_0 \) as a function of \( \epsilon_1 = \epsilon_2 \) are given by the following table and Fig. 4

| \( \epsilon \) | \( E \) |
|---|---|
| 64 | -0.003092 |
| 100 | -0.003927 |
| 900 | -0.00829 |
| 10^3 | -0.0084835 |
| 10^4 | -0.01184 |
| 10^5 | -0.01364 |
| 10^6 | -0.01447 |
| 10^7 | -0.01483 |
| 10^8 | -0.01495 |
| \( \infty \) | -0.01501511 |

FIG. 4: The calculated Casimir energy of two (scalar) Dielectric spheres of radii \( R_2 = 2R_1 \) at centers distance \( a = 4R_1 \) depicted as a function of their dielectric constant \( \epsilon_1 = \epsilon_2 \). The energy \( E \) was normalized by the Dirichlet spheres result \( E_0 \) so that at \( \epsilon \to \infty \) we obtain \( E/E_0 = 1 \).

The calculation may easily be repeated for any given \( R_1, R_2, \epsilon_1, \epsilon_2, a \).

### IX. ELECTROMAGNETIC FIELD

To extend the ideas of the previous section from the scalar to the EM case one needs to present the EM propagator in a form analogous to eqs. (54)-(56). The required representation of the EM propagator (derived in appendix A) is

\[
\overline{D}_0 = |(j\alpha) B| \mathcal{C}_{j\alpha,m\alpha'}(j' m' \alpha')_A | (66)\
\]

where \( \alpha, \alpha' \) can take the two values 0, 1 corresponding to the TE (magnetic multipole) or TM (electric multipole) modes respectively. The \( \mathcal{C} \) coefficients are given by

\[
\mathcal{C}_{j\alpha,m\alpha'} = \sum \mathcal{Y}^\dagger_{jm}(\omega a)Y^\star_{jm'}(\omega a)Y^\star_{jm'}(\omega a)\]
where $\tilde{Y}^{(a)}$ may be defined in terms of vectorial spherical harmonics as

$$Y_{jm}^{(0)} = \tilde{Y}_{jm}, \quad (68)$$

$$Y_{jm}^{(1)} = \sqrt{\frac{j+1}{2j+1}} Y_{j,j-1,m} + \sqrt{\frac{j}{2j+1}} Y_{j,j+1,m}. \quad (69)$$

These functions satisfy $Y_{jm}^{(1)} = \tilde{r} \times Y_{jm}^{(0)}$, $iY_{jm}^{(0)} = \tilde{r} \times Y_{jm}^{(1)}$.

After Wick rotating, we obtain $c_{jma;j'm'a'}(i\omega) = i^{j'-j+a-a'} \frac{\pi}{\sqrt{2g_{jma;j'm'a'}}}$ where the coefficients

$$g_{jma;j'm'a'}(i\omega) = \sqrt{\frac{2\pi}{\omega}} \quad (70)$$

$$\times \sum'_{\mu''} K_{\mu'+\frac{1}{2}}(\omega a) Y_{\mu''m''}(\hat{a}) \int d\Omega Y_{\mu'm'}(Y_{j,m}'^{(a)*)} \cdot Y_{j,m}'^{(a')})$$

are real.

The integrals $\int d\Omega Y_{j,m3}(Y_{j,m1}'^{(a)*}, Y_{j,m2}^{(a)})$ appearing in (67 70) can be expressed explicitly in terms of 3j-symbols as follows. For $\alpha = \alpha'$ it is given by

$$\frac{1}{2\sqrt{j(j+1)(2j+1)}} \frac{1}{\sqrt{4\pi}} \sqrt{\frac{1}{x}} \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ -m_1 & m_2 & m_3 \end{array} \right)$$

(71)

(wich vanishes unless $j_1 + j_2 + j_3 \equiv 0 \mod 2$. For $\alpha \neq \alpha'$ the integral is nonzero only provided $j_1 + j_2 + j_3 \equiv 1 \mod 2$ in which case it is given by

$$(1)^{m_1} \frac{1}{\sqrt{\frac{1}{x}}} \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ 1 & -1 & 0 \end{array} \right) \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ -m_1 & m_2 & m_3 \end{array} \right)$$

(72)

Eq.(71) may be derived from Eq.(60) by using the identity

$$\sqrt{j(j+1)}Y_{jm}^{(0)} = \tilde{L}Y_{jm} \quad (73)$$

(where $L$ is the angular momentum operator) and integration by parts. The relation(72)

was found with the help of Eq. (18) in Ref. [3].

**Spherical scatterers**

Assuming spherically symmetric scatterers, one may define phase shifts $\delta_{TE}^{(a)}(\omega), \delta_{TM}^{(a)}(\omega)$ (by parity these two channels do not mix). Similarly to the scalar case we use the notation $t_{j,0} = \frac{1}{2} (-1)^{j+1} e^{2\delta^{(a)}(\omega) - 1}$.

Choosing the z-axis along $\hat{a}$ the operator $K = TGTG$ splits to independent blocks $K^{(m)}$ corresponding to the values of the azimuthal number $m$. In a given block the $g$-matrix elements become

$$g_{jma;j'm'a'}^{(m)} = \sum'_{\mu''} K_{\mu'+\frac{1}{2}}(\omega a) \int d\Omega Y_{j,m1}'^{(a)*} \cdot Y_{j,m2}^{(a')}$$

The matrix $K^{(m)}(i\omega)$ is then written explicitly as:

$$K_{jma;j'm'a'}^{(m)}(i\omega) = t_{j,0}^{(A)} g_{jma;j'm'a'}^{(m)} t_{j,0}^{(B)} g_{jma;j'm'a'}^{(m)}$$

In the particular case of a perfectly conducting sphere of radius $R$ one has:

$$t_{TE}^{(A)}(i\omega) = \frac{\pi}{2} \frac{J_{1}(\omega R)}{\omega R J'_{1}(\omega R)}, \quad (74)$$

$$t_{TM}^{(B)}(i\omega) = -\frac{\pi}{2} \frac{\omega}{\omega R} \left( \sqrt{\pi} J_{1}(x) \right), \quad (75)$$

Using this we numerically calculated the electromagnetic Casimir energy for a pair of conducting spheres at distance $a$ between their centers. As in the scalar case writing $K = K^2$ and considering det$(1 \pm K)$ separately allowed us to also find the interaction energies $E_e, E_m$ of a sphere near a conducting/infinitely permeable mirror placed $a/2$ from its center. The two spheres energy is then the sum $E_e + E_m$. Most of the calculations where done by including modes having $j \leq 10$, however for the shortest distances $a = 2.35, 2.2, 2.1$ we considered convergence slower, we extended the retained modes up to $j = 20, 40, 60$ respectively. The results are shown in the following table (76) (written in units where $R = 1$) and in Fig. 5.

| $a$ | $E_e$ | $E_m$ | $E_s$ |
|-----|-------|-------|-------|
| 2.1 | -16.15 | 14.5 | -1662 |
| 2.2 | -3.82 | 3.48 | -0.37635 |
| 2.35 | -1.157 | 1.073 | -8.356 \cdot 10^{-2} |
| 2.5 | -0.53 | 0.50 | -3.18 \cdot 10^{-2} |
| 2.75 | -0.211 | 0.201 | -9.595 \cdot 10^{-3} |
| 3 | -0.1074 | 0.1036 | -3.787 \cdot 10^{-3} |
| 3.5 | -3.97 \cdot 10^{-2} | 3.88 \cdot 10^{-2} | -8.917 \cdot 10^{-4} |
| 4 | -1.89 \cdot 10^{-2} | 1.86 \cdot 10^{-2} | -2.864 \cdot 10^{-4} |
| 5 | -6.24 \cdot 10^{-3} | 6.19 \cdot 10^{-3} | -4.887 \cdot 10^{-5} |
| 7 | -1.38 \cdot 10^{-3} | 1.37 \cdot 10^{-3} | -3.965 \cdot 10^{-6} |
| 10 | -3.06 \cdot 10^{-4} | 3.06 \cdot 10^{-4} | -3.032 \cdot 10^{-7} |
| 13 | -1.04 \cdot 10^{-4} | 1.04 \cdot 10^{-4} | -4.703 \cdot 10^{-8} |
| 16 | -4.47 \cdot 10^{-5} | 4.47 \cdot 10^{-5} | -1.085 \cdot 10^{-8} |

The numerical results seem to converge as $j_0 \to \infty$ at roughly an exponential rate. The graph in Fig. 6 shows how the speed of convergence depends on the distance between the bodies. It is interesting to note that the results obtained in section VIII for the scalar case give almost the same graph. Also, one can easily check that the results for $E_s$ are basically the same as the ones obtained in [21] taking into account we chose to normalize the energy in comparison with the large distance asymptotic expression for the energy.
correspond to our results for two conducting spheres (where $\vec{r}$ behaving roughly as $a/R$ as a function of the separation distance $d$). The colored graphs show the result of including partial waves of $j \leq j_0$ where: $j_0 = 1$ (red), $j_0 = 2$ (sky blue), $j_0 = 4$ (green), $j_0 = 10$ (blue).

**APPENDIX A: PROOF OF THE GREEN’S FUNCTION EXPANSIONS EQNS (54,66)**

**Scalar case**

Suppose $\vec{R} = \vec{a} + \vec{r}'$ then obviously $e^{i\vec{R} \cdot \vec{a}} = e^{i\vec{r}' \cdot \vec{a}} e^{i\vec{a} \cdot \vec{R}}$. Inserting the well known expansion

$$e^{i\vec{R} \cdot \vec{a}} = 4\pi \sum_{l=0}^\infty i^l Y^*_{lm}(\vec{a}) Y_{lm}(\vec{r}') j_l(kr)$$

We get

$$\sum_l i^l Y^*_{lm}(\vec{R}) j_l(kR) =$$

$$4\pi \left( \sum_{l' m'} i^{l'} Y^*_{lm'}(\vec{R}) Y_{l'm'}(\vec{a}) j_{l'}(kr) \right)$$

$$\times \left( \sum_{l'' m''} i^{l''} Y_{l''m''}(\vec{r}') j_{l''}(kr') \right)$$

Multiplying both sides by $Y_{lm}(\vec{R})$ and integrating $\int d\Omega_k$ we find

$$j_l(kR)Y_{lm}(\vec{R}) =$$

$$4\pi \sum_{l' m'} i^{l'} Y^*_{lm'}(\vec{R}) Y_{l'm'}(\vec{a}) j_{l'}(kr) Y_{l'm'}(\vec{r}')$$

Concentrating on the case $R, a > r'$, it makes sense to separate the ingoing and outgoing parts in the last equation. This amounts to replacing the bessel functions $j_l(kR), j_l(ka)$ by hankel functions $h_l(kR), h_l(ka)$ of the first or second type corresponding to outgoing or ingoing waves. Since this argument may seem as hand-waving, we will return and elaborate on it more at the end of the proof. Equating the outgoing parts we have:

$$h_l^{(1)}(kR)Y_{lm}(\vec{R}) =$$

$$4\pi \sum_{l' m'} i^{l'} Y^*_{lm'}(\vec{R}) Y_{l'm'}(\vec{a}) j_{l'}(kr) Y_{l'm'}(\vec{r}')$$

$$\times i^{l''} j_{l''}(kr') Y_{l''m''}(\vec{r}')$$

It is well known that for $R > r$ the free propagator may be expanded as

$$\frac{-1}{4\pi |\vec{R} - \vec{r}|} e^{ik|\vec{R} - \vec{r}|} = -i k \sum_l j_l(\omega r) h_l^{(1)}(kR) Y_{lm}(\vec{r}) Y_{lm}(\vec{R})$$

Substituting here Eq. (A.2) we finally get

$$G(\vec{r}, \vec{R}) =$$

$$-4\pi i\omega \sum_{l' m'} i^{l'} Y^*_{lm'}(\vec{R}) Y_{l'm'}(\vec{a}) j_{l'}(kr) Y_{l'm'}(\vec{r}')$$

Which is exactly Eq. (54).

Let us now return to the derivation of Eq. (A.2) from Eq. (A.1). We first note that the function $h_l^{(1)}(kR)Y_{l'mo}(\vec{R})$ with $\vec{R} = \vec{a} + \vec{r}'$ being a solution of the free wave equation may be expanded around $\vec{r}' = 0$ in the form

$$h_l^{(1)}(kR)Y_{l'mo}(\vec{R}) = \sum_{lm} \left( c_l^{(1)} h_l^{(1)}(kr) + c_l^{(2)} h_l^{(2)}(kr) \right) Y_{lm}(\vec{r})$$

for some ($\vec{a}$ dependent) constants $c_l^{(1)}, c_l^{(2)}$. To be more precise $h_l^{(1)}(kR)Y_{l'mo}(\vec{R})$ is a solution only for $\vec{r}' \neq -\vec{a}$ (i.e. $\vec{R} \neq \vec{R}'$) therefore one has two separate expansions: one for $r < a$ and another for $r > a$. We concentrate on the latter.

Since $h_l^{(1)}(kR)Y_{l'mo}(\vec{R})$ is a purely outgoing wave it is clear that the expansion in terms of $\vec{r}'$ must also contain only outgoing waves i.e. $c_l^{(1)} \equiv 0$. This claim is based on “physical intuition”. A more rigorous mathematical argument may be constructed by considering first pure imaginary $k = iq$ with $q > 0$. One then note that $h_l^{(1)}(iqR)$ is exponentially decreasing as $R \rightarrow \infty$ which imply that the same must hold for the r.h.s. Since the
$Y_{lm}$’s are linearly independent this require all the $c_{lm}^{(1)}$’s to vanish.

A similar expansion obviously exists also for $h^{(2)}$:

$$h_{l_0}^{(2)}(kr)Y_{lm_0}(\vec{r}) = \sum c_{lm}^{(2)} h_{l_0}^{(2)}(kr)Y_{lm}(\vec{r})$$

Summing the two expansions we have

$$j_{l_0}(kr)Y_{lm_0}(\vec{r}) \equiv \frac{1}{2} \left( h_{l_0}^{(1)}(kr) + h_{l_0}^{(2)}(kr) \right) Y_{lm}(\vec{r}) =$$

$$= \frac{1}{2} \sum \left( c_{lm}^{(1)} h_{l_0}^{(1)}(kr) + c_{lm}^{(2)} h_{l_0}^{(2)}(kr) \right) Y_{lm}(\vec{r})$$

However such an expansion is clearly unique. Therefore it must be the same as the expansion in Eq. (A.1). Comparing the two (and using $j_l \equiv \frac{1}{2}(h^{(1)} + h^{(2)})$) we deduce

$$c_{lm}^{(1)} = c_{lm}^{(2)} =$$

$$4\pi \sum_{l',m'} i^{l''} + l - l_0 \left( \int d\Omega Y_{lm_0} Y_{lm}^* Y_{l'm'}^* \right) \times j_{l''}(k\alpha) Y_{l'm'}(\vec{a})$$

which proves Eq. (A.2).

**The electromagnetic case**

To derive the EM expansion we similarly start by using the identity

$$e^{ikr} \times \hat{\mathbf{r}} =$$

$$4\pi \sum i^{|j|} j_l(kr) \tilde{Y}_{jlm}(\hat{\mathbf{k}}) \otimes \tilde{Y}_{jlm}(\hat{\mathbf{r}})$$

Repeating the same steps as for the scalar we then find that

$$\tilde{\mathbf{G}}_0 = -\frac{e^{i\omega r}}{4\pi r} \times \hat{\mathbf{r}}$$

may be expanded as

$$\tilde{\mathbf{G}}_\omega = |(jlm)_{AB}| C_{jlm;j'l'm'}^{(j'l'm')_A}$$

where

$$|(jlm)_{AB}| = \sqrt{\frac{2\omega^2}{\pi}} j_l(\omega r_{A,B}) \tilde{Y}_{jlm}(\hat{r}_{A,B})$$

are the free vectorial spherical wave functions centered at $P_A, P_B$. The $C$ coefficients may be written as

$$C_{jlm;j'l'm'} = -\frac{e^{i\omega r}}{2\omega} \sum_{l''m''} \left[ C_{jlm;j'l'm'}^{(j'l'm'_{10})} \right]$$

Here $\tilde{Y}_{jlm}$ are vectorial spherical harmonics, $Y_{lm}$ are the usual scalar spherical harmonics, and $j_l, h_l$ are spherical Bessel and Hankel functions. The coefficients

$$\tilde{C}_{j'j''} \left( \begin{array}{ccc} l & l' & l'' \\ m & m' & m'' \end{array} \right)$$

are found to be expressed as the following integral of spherical functions:

$$\tilde{C}_{j'j''} \left( \begin{array}{ccc} l & l' & l'' \\ m & m' & m'' \end{array} \right) =$$

$$4\pi \int d\Omega (\tilde{Y}_{j'lm} \cdot \tilde{Y}_{j''l'm'}^* Y_{l'm''}^*)$$

The radiation gauge propagator $D_0$ is given by the transverse part of $\tilde{\mathbf{G}}_0$. In Eq. (A.6) each $j, m$ correspond to three different spherical function $|jlm\rangle$ (having $l = j - 1, j, j + 1$). These may be decomposed in terms of the TE and TM modes and a nonphysical longitudinal mode.

$$|TE| = |jjm\rangle$$

$$|TM| = \sqrt{\frac{j + 1}{2j + 1}} |jjm\rangle - \sqrt{\frac{j}{2j + 1}} |jm\rangle$$

$$|L| = \sqrt{\frac{j + 1}{2j + 1}} |jjm\rangle + \sqrt{\frac{j}{2j + 1}} |jm\rangle$$

To obtain the required expansion of the radiation gauge propagator $D_0$ we need to rewrite Eq. (A.5) in terms of these three modes and drop the parts containing the longitudinal mode. This can be done quite straightforwardly leading to the results (66-69).

**APPENDIX B: ANALYTICAL PROPERTIES OF THE $T_A G_0 T_B G_0$-OPERATOR:**

Having established the form (11) for the energy, we turn here to discuss the properties of this expression. The main aim of this appendix is to rigorously show that the object log det$(1 - T_A G_0 T_B G_0)$ is well defined and finite. The main mathematical notions and theorems which we use here, are briefly reviewed in appendix C.

As already remarked in the introduction it is well known that det$(1 - M)$ is well defined whenever $M$ is a trace class operator (definition C.4). We would like to show that for a large class of situations (including a pair of disjoint finite bodies $A, B$, separated by a finite distance) the operator $T_A G_0 T_B G_0 : H_A \rightarrow H_A$ is trace class in the continuum limit, and so prove that indeed the expression (11) is finite and well defined.
Indeed, by theorem C.3 the mere fact that $G_0(x,y)$ is a smooth function for $x \neq y$ is sufficient to guarantee that for any pair of compact volumes $A, B \in \mathbb{R}^3$ at finite mutual distance the operator $G_{0\text{AB}}$ is trace class. To deduce that $T_A G_{0\text{AB}} T_B G_{0\text{BA}}$ is trace class (and by similar argument also $1 - G_0 J_T A$ appearing in Eq. (32)) it is then enough (proposition C.6) to make sure $T_{A,B}(i\omega)$ are bounded (definition C.2).

In the context of dielectric interaction, it is particularly easy to show that $T(i\omega)$ is bounded. In physical systems at equilibrium, it follows from causality properties of the dielectric function $\chi$, that $\chi(i\omega, x) \geq 0$. We then have the following

**Lemma B.1.** For $\chi(i\omega, x) > 0$, the $T$ operators are positive and bounded.

**Proof:** Since $G_0, \chi > 0$ (definition C.3) one may write $T = \sqrt{\chi + \omega^2 \chi_{0\text{AB}}} \sqrt{T}$ from which it is seen that $T > 0$ and that in the operator norm $\|T\| \leq \omega^2 \|\chi\|$.

In fact, this holds also for nonlocal $\chi$ as long as $f(x) \mapsto \int_A \chi(i\omega, x, x') f(x') dx'$ is a bounded positive operator $H_A \rightarrow H_A$. In the context of more general type of interactions which may not be positive, one needs to use some assumption on the stability of the system to guarantee that $T(i\omega)$ is bounded. We do not elaborate on this here.

An alternative approach to proving the trace class property of $T_A G_{0\text{AB}} T_B G_{0\text{BA}}$ is based on the notion of a Hilbert-Schmidt operator (definition C.7 also denoted H.S.). Here the frequently used strategy in operator analysis is to use the following fact: if $U \in \text{H.S. and } V \in \text{H.S.}, then UV \in \text{t.c.}$ The advantage of this approach is that it is very easy to check if an operator is Hilbert Schmidt. Since the Hilbert Schmidt norm is $\|A\|_{\text{H.S.}} = \text{Tr}(A^\dagger A)$, one may evaluate it directly, (e.g. by computing $\int |A(x, x')|^2$).

**Theorem B.2.** For any two bodies $A, B$ such that $\int_{A \times B} dx dy |G_0(x, y)|^2 < \infty$, $T_A G_{0\text{AB}} T_B G_{0\text{BA}}$ is trace class.

**Proof:** First we show that $T_A G_{0\text{AB}}$ and $T_B G_{0\text{BA}}$ are Hilbert Schmidt operators. This can be verified in the following way. We have just seen that $T_A, T_B$ are bounded operators. Now note that $G_{0\text{AB}}$ is Hilbert-Schmidt, since

$$\|G_{0\text{AB}}\|_{\text{H.S.}} = \int_{A \times B} dx dy |G_{0\text{AB}}(x, y)|^2,$$

which is finite under the condition above. Now the inequality $\|T_A G_{0\text{AB}}\|_{\text{H.S.}} \leq \|T_A\| \|G_{0\text{AB}}\|_{\text{H.S.}}$ implies that $T_A G_{0\text{AB}}$ is Hilbert Schmidt. Finally using $U, V \in \text{H.S.} \Rightarrow UV \in \text{t.c.}$ we see that $T_A G_{0\text{AB}} T_B G_{0\text{BA}} \in \text{t.c.}$

**Corollary B.3.** For any finite bodies $A, B$, such that distance$(A, B) > 0$, and any Green’s function which is finite away from the diagonal, $T_A G_{0\text{AB}} G_0 \in \text{t.c.}$

**Example B.4.** For the scalar field discussed above, $G_0(x, y) = \frac{e^{-|x-y|/\omega}}{4\pi|x-y|}$, the condition is satisfied. In the same way it is satisfied for the electromagnetic field (one has to take into account also matrix indices but these discrete indices do not change finiteness of the integrals)

**Remark B.5.** The $\omega$ integration in Eq. (31), is convergent. To see this note that $G_0$ decays exponentially with $\omega$ therefore, $\|G_0\|_{\text{H.S.}}$ decays exponentially, also the $\|T\|$’s do not grow more then quadratically in $\omega$.

In the EM case one may also worry due to the factor $\frac{1}{\sqrt{x^2}}$ appearing in $D_{0\text{AB}}(x, y) = (\delta_{ij} - \frac{1}{\sqrt{\pi}} \nabla_i \nabla_j G_0(x, y))$, about convergence for $\omega \sim 0$.

One may also show that $G_{0\text{AB}}$ are t.c. themselves by using H.S. properties. The bodies are assumed not to touch, thus we can choose a $G_0^{\infty}$ (compactly supported and infinitely smooth) function $f_A$, such that $P_A f_A = P_A$, and $P_B f_A = 0$ where $P_A, P_B$ are the projections on $L^2(A), L^2(B)$ (i.e. $f_A(x) = 1$ for $x \in A$, and it then smoothly goes to 0, before reaching body $B$ see Fig[7]).

Writing:

$$G_{0\text{AB}}^{\infty} = L_1 L_2$$

$$L_1 = P_A \frac{1}{1 + \omega^2 p^2} ; \quad L_2 = (p^2 + \omega^2) f_A G_0 P_B,$$

we see that if $4\alpha > d$,

$$\|L_1\|^2_{\text{H.S.}} = \text{Tr}(P_A \frac{1}{1 + \omega^2 p^2} (P_A \frac{1}{1 + \omega^2 p^2})^\dagger) = \text{Vol}(A) \int d^4 p |\frac{1}{1 + \omega^2 p^2}| < \infty$$

and so $L_1$ is Hilbert Schmidt. Next, we check that $L_2 \in \text{H.S.}$ To see this last point, note that

$$< x | L_2 | x' >= = < x | (p^2 + \omega^2) f_A G_0 P_B | x' > =$$

$$(\Delta_x + \omega^2) f_A(x) G_0(x-x') P_B(x')$$

Since $G_0(x-x')$ is smooth away from $x = x'$, where the expression is anyway zero because $f_A P_B = 0$, and since $(x | L_2 | x')$ has compact support (for integer $\alpha$) we see that $\|L_2\|^2_{\text{H.S.}} = \int d^d x |L_2|^2 < \infty$. Thus, $G_{0\text{AB}}$ can be written as a product of two H.S. operators, and as such is trace class.

Finally, we have that

**FIG. 7:** The support of the function $f_A$
Theorem B.6. (Eigenvalues of TGTG) For \( \chi > 0 \), all eigenvalues \( \lambda \) of the (compact) operator \( T_A G_0 A_B T_B G_0 B_A \) appearing in (11) satisfy \( 1 > \lambda \geq 0 \).

Proof: We will use repeatedly that for bounded operators \( X, Y \) the nonzero eigenvalues of \( XY \) and \( YX \) are the same. Note first that \( G_0, \chi \geq 0 \) (as operators) implies

\[
\text{spec}(\chi G_0) \setminus \{0\} = \text{spec}(\sqrt{G_0}X\sqrt{G_0}) \setminus \{0\} \subset [0, \infty).
\]

Writing \( T_A G_0 = 1 - \frac{1}{1 + \omega^2 \chi_0^2} \) as an operator on \( L^2(\mathbb{R}^3) \) it is then clear that its spectrum lies in \([0,1]\). The same conclusion then applies to the operator \( \sqrt{G_0 T_A \omega \sqrt{G_0}} \) but since it is hermitian one concludes also \( \| \sqrt{G_0 T_A \omega \sqrt{G_0}} \| < 1 \) from which it follows \( \| \sqrt{G_0 T_A \omega \sqrt{G_0}} \| < 1 \) and hence \( \lambda < 1 \). Similarly \( \sqrt{G_0 T_A \omega \sqrt{G_0}} \geq 0 \) imply \( \lambda \geq 0 \).

**APPENDIX C: SOME PROPERTIES OF (INFINITE DIMENSIONAL) OPERATORS**

Here we recall some mathematical notions that we have used in describing the trace class properties of Eq. (11).

**Definition C.1.** For an operator \( B : H \rightarrow H \), the operator norm of \( \|B\| \) is defined as

\[
\|B\| = \sup_{\psi \in H, \psi \neq 0} \frac{|\langle B\psi | \psi \rangle|}{|\langle \psi | \psi \rangle|}.
\]

**Definition C.2.** An operator \( B \) is bounded if \( \|B\| < \infty \).

**Definition C.3.** An operator \( A : H \rightarrow H \) is called a positive operator (denoted \( A > 0 \)) iff \( \langle \psi | A \psi \rangle \geq 0 \) for every \( \psi \in H \).

This implies that \( A \) is hermitian and its spectrum nonnegative. If \( A : H \rightarrow H \) is a positive operator then there exist a unique positive operator \( B : H \rightarrow H \) satisfying \( A = B^2 \). \( B \) is called the square root of \( A \) and denoted \( \sqrt{A} \).

**Definition C.4.** An operator \( A : H_1 \rightarrow H_2 \) is called trace class (and denoted \( A \in \mathcal{J} \)) iff

\[
\sum |\langle \psi_n | A \psi_n \rangle| < \infty \quad \text{where} \quad \{\psi_n\}_{n=1}^{\infty} \text{is some orthonormal basis of } H_1.
\]

It can be shown that this condition does not depend on the choice of the orthonormal basis. (Note that the definition makes sense even when \( H_1 \neq H_2 \).

If \( A : H \rightarrow H \) is trace class then for any orthonormal basis \( \{\psi_n\}_{n=1}^{\infty} \) of \( H \) the sum \( \sum \langle \psi_n | A \psi_n \rangle \) converges to the same (finite) value which is denoted \( \text{tr}(A) \) and called the trace of \( A \). One then also have \( \text{tr}(A) = \sum \lambda_n \) where \( \{\lambda_n\} \) are the eigenvalues of \( A \) (Lidskii’s theorem).

If \( A : H \rightarrow H \) is trace class then the determinant \( \det(1 + A) \) may also be rigorously defined and one has \( \det(1 + A) = \prod (1 + \lambda_n) \).

The following theorem may be proved using the well known fact that the Fourier coefficients of a smooth \( K(x,y) \) decay faster then any power. (Note that these coefficient also serve as the matrix elements with respect to Fourier basis of the operator defined by \( K \).)

**Theorem C.5.** Consider an operator \( A : L^2(D_1) \rightarrow L^2(D_2) \) where \( D_1, D_2 \) are some domains in \( \mathbb{R}^n \) which is given explicitly as an integral \( A\psi(x) = \int_{D_1} K(x,y)\psi(y)dy \). A sufficient condition for \( A \) to be trace class is that \( D_1, D_2 \) are compact and \( K(x,y) \) is smooth in a neighborhood of \( D_1 \times D_2 \).

**Proposition C.6.** If \( A \) is trace class and \( B \) bounded then \( AB \) and \( BA \) are also trace class and \( \text{Tr}(AB) = \langle 0 | \text{Tr}(BA) | 0 \rangle \).

**Definition C.7.** \( M \) is a Hilbert Schmidt operator (denoted \( M \in \mathcal{H.S.} \)) if \( ||M||_{\mathcal{H.S.}} = \text{Tr}(M^\dagger M) < \infty \).

In particular we mention that the product of two Hilbert Schmidt operators always give a trace class operator.

* Electronic address: \texttt{klich@caltech.edu}

[1] H. B. G. Casimir, Proc. Koninkl. Ned. Akad. Wet. 51, 793 (1948).
[2] S. K. Lamoreaux, Phys. Rev. Lett. 78 5-8(1997);
[3] U. Mohideen and A. Roy Phys. Rev. Lett. 81 4549 (1998).
[4] G. Bressi, G. Carugno, R. Onofrio and G. Ruoso, Phys. Rev. Lett. 88, 041804 (2002).
[5] B. Geyer, G. L. Klimchitskaya, and V. M. Mostepanenko, Phys. Rev. A 67, 062102 (2005).
[6] I. Pirozhkenko, A. Lambrecht, V. B. Svetovoy, New Journal of Physics 8, 238 (2006).
[7] M. Kardar and R. Golestanian, Rev. Mod. Phys. 71, 1233 (1999).
[8] M. Bordag M., U. Mohideen and V.M. Mostepanenko, Phys.Rept.353:1-205,2001.
[9] K A Milton, The Casimir Effect: Physical Manifestations of Zero-Point Energy, World Scientific, 2001.
[10] P. W. Miloni, The Quantum Vacuum, An Introduction to Quantum Electrodynamics, (Academic Press, San Diego 1994).
[11] A. Ashourvan, M. Miri, R. Golestanian, Phys. Rev. Lett. 98, 140801 (2007).
[12] T. Emig, Phys. Rev. Lett. 98,160801 (2007).
[13] H. Gies and K. Klingmuller, Phys.Rev.Lett.97, 220405 (2006); Phys.Rev. D74, 045002(2006).
[14] A. Rodriguez, M. Ibañescu, D. Iannuzzi, J. D. Joannopoulos, S. G. Johnson, Phys. Rev. A, 76, 032106 (2007). preprint arXiv:0705.3661.
[15] O. Kenneth and I. Klich, Phys. Rev. Lett. 97, 160401 (2006).
[16] E. M. Lifshitz, Sov. Phys. JETP 2, 73 (1956).
[17] R. Balian and D. Duplantier, Ann. Phys.(N.Y.) 112, 165, (1978).
[18] M. Reed and B. Simon, Methods of mathematical physics 3: Scattering Theory, Academic Press 1979.
[19] C P Bachas 2007 J. Phys. A: Math. Theor. 40 9089(2007). quant-ph/0611082.
[20] Z. Nussinov, cond-mat/0107339 (Appendix A and footnote [35] therein).
[21] T. Emig, N. Graham, R. L. Jaffe and M. Kardar, Phys.Rev.Lett.99,170403 (2007), arXiv:0707.1862v2
[22] O. Kenneth, preprint hep-th/9912102.
[23] C. Genet, A. Lambrecht and S. Reinaud, Phys. Rev. A67, 043811 (2003); M. T. Jaekel and S. Reinaud, J. Phys. I 1, 1395 (1991). quant-ph/0101067.

[24] A. Bulgac, P. Magierski and A. Wirzba, Phys. Rev. D73, 025007 (2006).

[25] A. Wirzba, Phys. Rept. 309, 1-116 (1999).

[26] H. Li and M. Kardar, Phys. Rev. Lett. 67, 3275 (1991); H. Li and M. Kardar, Phys. Rev. A46, 6490 (1992).

[27] J. Feinberg, A. Mann and M. Revzen, Annals of Physics (New York) 288, 103-136 (2001).

[28] B. Simon. Trace ideals and their applications, LMS vol 35. Cambridge University Press, New York, 1979.

[29] P. Candelas and D. Deutsch, Phys. Rev. D20, 3063 (1979).

[30] R. Bscher, T. Emig Phys. Rev. A69, 062101 (2004).

[31] E. M. Lifshitz and L. P. Pitaevskii, Statistical Physics, Pt. 2, Pergamon, Oxford, 1984.

[32] O. Kenneth and S.Nussinov, Phys. Rev. D65, 085014 (2002).

[33] N. G. van Kampen, B. R. A. Nijboer and K. Schram, Phys. Lett. 26A, 7, 307 (1968).

[34] P. A. Mello and D. Stone, Phys. Rev. B44, 3559 (1991).

[35] J.S. Dowker Class. Quantum Grav. 7, 1241-1251 (1990).

[36] This argument may be made precise by choosing a series of functions for which the sum \( \sum_{n=1}^{N} \langle \phi_n | A | \phi_n \rangle \) diverges as \( N \to \infty \).

[37] Generalization to more than two dielectrics is straightforward.

[38] This may be justified by expanding in an (\( \omega \)-dependent) eigenstates basis of \( H(\omega) \) and noting that terms containing \( \frac{d}{d\omega} | n(\omega) \rangle \) cancel.

[39] Most standard textbooks discuss the non-relativistic case and therefore include a factor \( \delta(\omega - \omega') \) instead of \( \delta(\omega^2 - \omega'^2) \). Writing the delta function in terms of momentum the two cases reduce to the same expression: \( \delta(k^2 - k'^2) \).