A NOTE ON SINGULAR LIMITS TO HYPERBOLIC SYSTEMS

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Abstract. In this note we consider two different singular limits to hyperbolic system of conservation laws, namely the standard backward schemes for non linear semigroups and the semidiscrete scheme.

Under the assumption that the rarefaction curve of the corresponding hyperbolic system are straight lines, we prove the stability of the solution and the convergence to the perturbed system to the unique solution of the limit system for initial data with small total variation.

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1. Introduction

Consider a hyperbolic system of conservation laws

\[ \begin{cases} u_t + f(u)_x = 0 \\ u(0, x) = u_0(x) \end{cases} \]

where \( u \in \mathbb{R}^n \) and \( f \) is a smooth function from an open set \( \Omega \subseteq \mathbb{R}^n \) with values in \( \mathbb{R}^n \). Let \( K_0 \) be a compact set contained in \( \Omega \), and let \( \delta_1 \) sufficiently small such that the compact set

\[ K_1 \doteq \{ u \in \mathbb{R}^n : \text{dist}(u, K_0) \leq \delta_1 \} \]

is entirely contained in \( \Omega \).

We assume that the Jacobian matrix \( A = Df \) is uniformly strictly hyperbolic in \( K_1 \), i.e.

\[ \min_{i<j} \{ \lambda_j(u) - \lambda_i(u) \} \geq c > 0, \quad \forall u, v \in K_1, \]

where we denote by \( \lambda_i \) the eigenvalues of \( A \), \( \lambda_i < \lambda_j \). Let \( r_i, l_i \) be the its right, left eigenvectors.

In this setting it is well known that if \( u_0(-\infty) \in K_0 \) and \( \text{Tot.Var.}(u_0) \) is sufficiently small, there exists a unique “entropic” solution \( u : [0, +\infty) \to u_0 + L^1(\mathbb{R}, \mathbb{R}^n) \cap \text{BV}(\mathbb{R}, \mathbb{R}^n) \) in the sense of [4]. Moreover these solutions can be constructed as limits of wave front tracking approximations and they depend Lipschitz continuously on the initial data.

For a special class of systems, called in [6] Straight Line Systems, i.e. systems such that

\[ (Dr_i) r_i = 0, \]

very recently it has been proved that solutions to (1.1) can be constructed as \( L^1 \) limits of solutions to different singular approximation of the hyperbolic system:

- Vanishing viscosity approximation [3]. This is the limit as \( \epsilon \to 0 \) of the solution \( u^\epsilon(t) \) of the system
  \[ u_t + f(u)_x - \epsilon u_{xx} = 0. \]

- Relaxation approximation [2, 7]. While in case 1) the perturbation is parabolic, in this case we consider a hyperbolic perturbation, namely
  \[ u_t + f(u)_x = \epsilon (\Lambda^2 u_{xx} - u_{tt}), \]

where \( \Lambda \) is strictly bigger than all the eigenvalues of \( Df(u) \).

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Godunov scheme. This is a discrete scheme obtained from \((1.1)\) by considering differential ratio instead of derivatives:

\[
u(n+1, j+1) = u(n, j+1) + \frac{\Delta t}{\Delta x} \left[ f(u(n, j)) - f(u(n, j+1)) \right],
\]

where for stability condition it is assumed that \(0 < \lambda_1 < \cdots < \lambda_n < \Delta x/\Delta t\).

The main idea behind these approximations is to obtain uniform BV estimates for \(t \geq 0\), if the initial data \(u_0\) are of sufficiently small total variation.

This task is achieved by decomposing the equations satisfied by \(u_x, u_x\) and \(u_t\) in [2], or \(u(n, j) - u(n, j-1)\) in [6], as \(n\) scalar perturbed conservation laws, coupled by terms of higher order. These terms are then considered as the source of total variation. For the special case of straight line systems, a decomposition of \(u_x\) which makes the source terms integrable is the projection along the eigenvectors \(r_i\) of the Jacobian \(Df(u)\):

\[u_x = \sum_i v^i r_i(u)\]

Once it is proved that the \(L^1\) norm of the component \(v^i\) is bounded, by Helly’s theorem there exists a subsequence \(v^{\epsilon_k}\) converging to a weak solution \(\bar{u}(t)\) of \((1.1)\) as \(k \to \infty\).

To prove the uniqueness of the limit \(\bar{u}(t)\), one consider the equation for a perturbation \(h\) of the singular approximations. We observe that \(h = u_{\bar{u}}\) is a particular solution of such system. A generalization of the arguments used to prove an a priori bound on the total variation of \(u\) shows the boundedness of the \(L^1\) norm of the components \(h^i\), where

\[h = \sum_i h^i r_i(u)\]

By a standard homotopy argument [3], this yields the stability of all solutions of the approximating system. Since the Lipschitz continuous dependence on the initial data is uniform w.r.t. both \(\epsilon\) and \(t\), in the limit we obtain a uniform Lipschitz semigroup.

Finally it is well known that a uniform Lipschitz semigroup of solutions to \((1.1)\) is uniquely defined if we know the jumps conditions of the entropic shocks, see [3]. In this case, because of the condition \((1.4)\) and because in the scalar case the solution \(u^\epsilon\) converges to the entropic solution, an argument similar to the one in [3] implies that the jump conditions coincide with the scalar jumps along the eigenvectors \(r_i\).

Thus, under the assumption \((1.4)\), the limit semigroup is independent on the approximation and coincides with the solution constructed by wave front tracking using the classical Lax Riemann solver.

In this note we want to extend the previous approach to the following cases:

1. semigroup approximation. This is obtained as limit of the system

\[
u(t, x) - \frac{u(t, x) - u(t - \epsilon, x)}{\epsilon} + A(u(t, x))u(t, x)_x = 0.
\]

This is the standard backward scheme for non-linear semigroups.

2. Semi-discrete schemes. i.e. infinite dimensional ODE defined by

\[
\frac{\partial}{\partial t} u(t, x) + \frac{1}{\epsilon} \left( f(u(t, x)) - f(u(t, x - \epsilon)) \right) = 0.
\]

We will prove that as \(\epsilon \to 0\) the limits of the respective solutions converge to a unique solution to \((1.1)\), and that this limits defines a Lipschitz continuous semigroup \(S\) on the space of function with small TV.

Moreover this semigroup is perfectly defined by a Riemann solver which, as explained above, coincides with the classical one.

The same can be proved for quasilinear systems as in [3, 6], but for simplicity we consider only systems on conservation forms.

Without any loss of generality we assume that

\[
\min_i \{\lambda_i(u)\} = \kappa > 0, \quad \max_i \{\lambda_i(u)\} = K < 1,
\]

for all \(u\) in the compact set \(K_1\). The second condition is needed only in case 2).
2. Approximation by semigroup systems

We consider in this section the case 1) of Section 1, i.e. the following singular approximation to system of conservation laws:

\[ (2.1) \quad \frac{u(t, x) - u(t - \epsilon, x)}{\epsilon} + A(u(t, x))\frac{u(t, x)}{x} = 0, \]

where we recall that \( u \in \mathbb{R}^n \) and \( A(u) = Df(u) \). By the rescaling \( t \to t/\epsilon, \quad x \to x/\epsilon \) and setting for simplicity \( u_n(x) = u(n, x) \), we obtain the evolutionary equations

\[ (2.2) \quad u_n - u_{n-1} + A(u_n)u_{n,x} = 0. \]

It is easy to prove that if the BV norm of \( u_{n-1} \) is sufficiently small, then \( u_n \) exists: in fact the solution can be represented as

\[ u_n(x) = \int_0^\infty \exp \left\{ \int_x^\infty A^{-1}(u_n(z))\,dz \right\} A^{-1}(u_n(y))u_{n-1}(y)\,dy, \]

and since the eigenvalues of \( A \) are positive we have that

\[ \|u_n\|_\infty \leq CTot.Var.(u_{n-1}), \]

\( C \) being a uniform constant of \( \|A^{-1}\|_\infty \) in the compact set \( K_0 \).

2.1. Projection on rarefaction curves. We now start the procedure explained in Section 1. By projecting the derivative along the eigenvectors \( r_i(u_n) \) of \( A(u_n) \)

\[ (2.3) \quad u_{n,x} = \sum_i v^n_i r_i(u_n) = \sum_i v^n_i r_{i,n}, \]

the equations for the components \( v^n_i \) are

\[ \sum_i v^n_i r_{i,n} - \sum_i v^n_{i-1} r_{i,n-1} + \sum_j (\lambda_{ij} v^n_i v^n_j r_{i,n})_x = 0. \]

This can be rewritten as

\[ (2.4) \quad \sum_i \left( v^n_i - v^n_{i-1} + (\lambda_{ij} v^n_i)^2 \right) r_{i,n} = \sum_i v^n_{i-1} (r_{i,n-1} - r_{i,n}) - \sum_{ij} \lambda_{ij} v^n_i v^n_j (Dr_{i,n}) r_{j,n} \]

The left-hand side is in conservation form, and we consider the right-hand side as the source of total variation. If we assume as in the introduction that \( (Dr_{i,n})r_i(u) = 0 \), the function \( r_i(u) - r_i(v) \) is zero when \( u - v \) is parallel to \( r_i(u) = r_i(v) \). Thus we have

\[ (2.5) \quad r_i(u) - r_i(v) = \sum_{j \neq i} \alpha_j(u, v)\langle \beta^j(u), u - v \rangle, \]

where \( \alpha_j(u, v) = r_j(u) \). Using (2.3), the expansion (2.4) thus becomes

\[ (2.6) \quad v^n_i - v^n_{i-1} + (\lambda_{ij} v^n_i)^2 \]

\[ = \sum_{j \neq k} \lambda_{jk} v^n_{j-1} v^n_k \left( \alpha_j(u, u_{n-1}) \right) - v^n_i v^n_j \left( \beta^j_i(u), (Dr_{i,n})r_{j,n} \right) \]

\[ = \sum_{j \neq k} \lambda_{jk} v^n_{j-1} v^n_k + \sum_{j \neq k} K_{jk}(n) v_i v^n_j. \]

To estimate the source terms in (2.6), we first consider the case of two linear equations.

2.2. Analysis of the linear case. Consider a single linear equation

\[ (2.7) \quad v_n - v_{n-1} + \lambda v_{n,x} = 0, \quad \lambda > 0. \]

We can find the fundamental solution to the previous equation by means of Fourier transform: we have

\[ v_n(x) = \int_\mathbb{R} c(n, \xi)e^{-i\xi x}d\xi, \]

and substituting

\[ c(n, \xi) - c(n - 1, \xi) - i\lambda \xi c(n, \xi) = 0 \quad \Rightarrow \quad c(n, \xi) = \frac{c_0(\xi)}{(1 - i\lambda \xi)^n}. \]
In particular the fundamental solution has $c_0(\xi) \equiv 1/2\pi$, so that
\begin{equation}
(2.8)
\begin{align*}
v_n(x) &= \frac{1}{\lambda} \left( \frac{x}{\lambda} \right)^{n-1} e^{-x/\lambda} \frac{1}{(n-1)!} \chi_{[0, +\infty)}(x).
\end{align*}
\end{equation}

Consider two equations of the form (2.7).
\begin{align*}
(2.9)&
\begin{align*}
v_n - v_{n-1} + \lambda v_{n,x} &= 0 \\
z_n - z_{n-1} + \mu z_{n,x} &= 0
\end{align*}
\end{align*}
with initial data $v_0(x) = \delta(x)$ and $z_0(x) = \delta(x - x_0)$, and assume without any loss of generality that $\lambda > \mu > 0$. We can compute the intersection integrals: denoting with $d(n, \xi)$ the Fourier coefficients of $z_n(x)$ we have
\begin{equation}
(2.10)
\begin{align*}
\sum_{n=0}^{N} \int_{\mathbb{R}} v_n(x) z_n(x) dx &= \sum_{n=0}^{N} 2\pi \int_{\mathbb{R}} c(n, \xi) d(n, -\xi) e^{-i \xi x_0} d\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \sum_{n=0}^{N} \frac{1}{(1 - i \lambda \xi)(1 + i \mu \xi)} e^{-i \xi x_0} d\xi.
\end{align*}
\end{equation}

If $\xi$ is considered as a complex variable, we can let $N \to +\infty$ only in the region where
\begin{equation*}
Z \equiv \left\{ \xi \in \mathbb{R} : |(1 - i \lambda \xi)(1 + i \mu \xi)| < 1 \right\}
\end{equation*}
i.e. outside the regions depicted in Figure 1. Deforming the path to avoid the region $Z$, we can pass to the limit:
\begin{equation}
(2.11)
\begin{align*}
\sum_{n=0}^{+\infty} \int_{\mathbb{R}} v_n(x) z_n(x) dx &= \frac{1}{2\pi} \int_{\gamma} \frac{e^{-i \xi x_0}}{1 - \frac{1}{(1 - i \lambda \xi)(1 + i \mu \xi)}} d\xi \\
&= \frac{1}{2\pi} \int_{\gamma} \frac{(1 - i \lambda \xi)(1 + i \mu \xi)}{(1 - i \lambda \xi)(1 + i \mu \xi) - 1} e^{-i \xi x_0} d\xi.
\end{align*}
\end{equation}
By means of complex analysis we have finally that
\begin{equation}
P(x_0) = \sum_{n=0}^{+\infty} \int_{\mathbb{R}} v_n(x) z_n(x) dx = \begin{cases} 
1/\lambda \cdot \exp\left( (\lambda - \mu)/\lambda \mu \right) x_0 < 0 \\
1/(\lambda - \mu) 
\end{cases}
\end{equation}
In fact, depending on the sign of $x_0$, the integration along the line $\gamma$ is equivalent to the integration around the pole $0$ or the pole $P = i(\lambda - \mu)/\lambda \mu$. 

**Figure 1.** Integration path on the complex plane, where $P = i(\lambda - \mu)/\lambda \mu$. 

2.3. **BV estimates.** Now to prove that (2.3) has a solution with uniformly bounded total variation. Define the functional

\[(2.12) \quad Q(n) = Q(u_n, u_{n-1}) = \sum_{i < j} \int_{\mathbb{R}} P_0(x - y) \left\{ |v_n^i(x)v_n^j(y)| + |v_{n-1}^i(x)v_{n-1}^j(y)| + |v_n^i(x)v_{n-1}^j(y)| \right\} dx dy, \]

where \( P \) is computed substituting to \( \lambda - \mu \) the constant of separation of speeds \( c \), and taking the minimal value of the exponent \((\lambda - \mu)/\lambda\mu):\]

\[P_0(x) = \begin{cases} 1/c \cdot \exp\left(c/(K(K-c))x_0\right) & x_0 < 0 \\ 1/c & x_0 \geq 0 \end{cases} \]

We recall that \( c \) and \( Kappa \) are defined in the introduction.

Using the same analysis of (2.1), we see immediately that

\[(2.13) \quad Q(n) - Q(n - 1) = \sum_{i < j} \int_{\mathbb{R}} P_0(x - y) \left\{ |v_n^i(x)v_n^j(y)| - |v_{n-1}^i v_{n-1}^j| \right\} dx dy + \sum_{i < j} \int_{\mathbb{R}} P_0(x - y) \left\{ |v_{n-1}^i(x)v_n^j(y)| - |v_{n-2}^i v_{n-1}^j| \right\} dx dy + \sum_{i < j} \int_{\mathbb{R}} P_0(x - y) \left\{ |v_{n-1}^i(x)v_{n-1}^j(y)| - |v_{n-1}^i v_{n-2}^j| \right\} dx dy \leq - \left( 1 - C \max_{m=1,\ldots,n} \text{Tot.Var.}(u_m) \right) \left[ \sum_{j \neq k} |v_n^i v_n^j| + \sum_{j \neq k} |v_n^i v_n^j| \right], \]

where \( C \) is a constant depending only on \( H_{jk}^i, K_{jk}^i, \kappa, K \) and \( c \). Thus if \( \delta_0 \) is sufficiently small, using (2.13) we have

\[\text{Tot.Var.}(u_1) + C_0 Q(u_1, u_0) \leq \delta_1 \quad \text{and} \quad \frac{d}{dt} \left\{ \text{Tot.Var.}(u) + C_0 Q(u) \right\} \leq 0,\]

where the constant \( C_0 \) is big enough, independent on \( \delta_0 \). This proves that the solution \( u_n \) has uniformly bounded total variation for all \( n \in \mathbb{N} \).

2.4. **Stability estimates.** We now consider the stability estimates of (2.2). The equations for a perturbation \( u + \delta h \) as \( \delta \to 0 \) are

\[(2.14) \quad h_n - h_{n-1} + (A(u_n)h_n)_x = (DA(u_n)u_{n,x})h - (DA(u_n)h_n)u_{n,x}.\]

Using the same projection of (2.3), i.e.

\[h_n = \sum_i h_n^i r_{i,n},\]

we have that the equations for the components \( h_n^i \) are

\[(2.15) \quad h_n^i - h_{n-1}^i + (\lambda_{i,n} h_n^i)_x = \sum_{j \neq k} \left( \lambda_{k,n} h_{n-1}^j v_n^k (l_n^i, \alpha_j(u_n, u_{n-1})) - h_{n-1}^j v_n^k (l_n^i, (Dr_{i,n})r_{j,n}) \right) + \sum_{j \neq k} h_n^j v_n^k (l_n^i, (A(u_n)r_{j,n})r_{i,n} - (A(u_n)r_{i,n})r_{j,n}) \leq \sum_{j \neq k} H(n) h_{n-1}^j v_n^k + \sum_{j \neq k} K'(n) h_n^j v_n^k.\]

Using the same analysis of (3), it is easy to prove that a functional as in Section 2.3 gives the stability of the solution.
3. Approximation by semi-discrete scheme

We now consider the case 2) of Section 3, i.e. the following singular approximation to system of conservation laws:

\[
\frac{\partial}{\partial t} u(t, x) + \frac{1}{\epsilon} \left( f(u(t, x)) - f(u(t, x - \epsilon)) \right) = 0,
\]

where \( u \in \mathbb{R}^n \). By the rescaling \( t \to t/\epsilon, x \to x/\epsilon \), we obtain the evolutionary equations

\[
\dot{u}_n(t) + f(u_n(t)) - f(u_{n-1}(t)) = 0.
\]

The equation for the "derivative" \( v_n \equiv u_n - u_{n-1} \) are

\[
\dot{v}_n(t) + f(u_n(t)) - 2f(u_{n-1}(t)) + f(u_{n-2}(t)) = 0.
\]

3.1. Projection on rarefaction curves. The vector \( v_n \) is now decomposed along the eigenvectors \( r_{i,n} \) of the Riemann problem \( u_{n-1}, u_n \): we have

\[
\dot{u}_n(t) + \sum_i \lambda_{i,n} v_{i,n}^i r_{i,n} = 0,
\]

\[
\sum_i \left( \dot{v}_{i,n} + \lambda_{i,n} v_{i,n}^i - \lambda_{i,n-1} v_{i,n-1}^i \right) r_{i,n} = -\sum_{i,j} v_{i,n}^i v_{j,n}^j (Dr_{i,n}) r_{j,n} - \sum_{i,j} v_{i,n}^i v_{j,n-1}^j (Dr_{i,n}) r_{j,n-1} + \sum_i \lambda_{i,n-1} v_{i,n-1}^i (r_{i,n-1} - r_{i,n})
\]

\[
= -\sum_{i,j} v_{i,n}^i v_{j,n}^j (Dr_{i,n}) r_{j,n} - \sum_{i,j} v_{i,n}^i v_{j,n-1}^j (Dr_{i,n}) r_{j,n-1} + \sum_i \lambda_{i,n-1} v_{i,n-1}^i (r_{i,n-1} - r_{i,n})
\]

\[
+ \sum_i \lambda_{i,n-1} v_{i,n-1}^i (r_{i,n-1} - r_{i,n}),
\]

where \( \lambda_{i,n} \) and \( r_{i,n} \) are the eigenvalues and right eigenvectors of the average matrix

\[
A(u_n, u_{n-1}) = \int_0^1 Df(u_{n-1} + (u_n - u_{n-1}) s) ds.
\]

If we assume the condition (3.4), the functions \( (Dr_{i,n}) r_{j,n} \) and \( r_{i,n} - r_{i,n-1} \) are zero when \( u_n - u_{n-1} \) and \( u_{n-1} - u_{n-2} \) are parallel to \( r_{i,n} = r_{i,n-1} \). Thus we have

\[
\sum_{j \neq i} \alpha_{j,n} v_{j,n}^i,
\]

\[
\sum_{j \neq i} \beta_{j,n} v_{j,n}^i + \sum_{j \neq i} \gamma_{j,n-1} v_{j,n-1}^i,
\]

as in Section 2.1. Using (3.3), the expansion (3.4) thus becomes

\[
\dot{v}_{i,n} + \lambda_{i,n} v_{i,n}^i - \lambda_{i,n-1} v_{i,n-1}^i = \sum_{j \neq k} H_{i,n} v_{i,n}^j v_{k,n}^k + \sum_{j \neq k} G_{i,n} v_{j,n}^i v_{k,n-1}^k.
\]

To estimate the source terms in (3.6), we consider the case of two linear equations.

3.2. Analysis of the linear case. Consider a single linear equation

\[
\dot{v}_{i,n} + \lambda v_{i,n}^i - \lambda v_{i,n-1}^i = 0, \quad \lambda > 0.
\]

We can find the fundamental solution to the previous equation by means of Fourier transform: defining the periodic function

\[
c(t, x) \equiv \sum_n v_n(t) e^{inx},
\]
we have that the equation satisfied by $c$ is

$$c_t = \sum_n \dot{v}_n e^{inx} = \lambda \sum_n (v_{n-1} - v^n) e^{inx} = \lambda(e^{ix} - 1)c,$$

whose general solution is

$$c(t, x) = c(0, x) \exp(\lambda(e^{ix} - 1)t) .$$

In particular the fundamental solution starting at $n_0$ has $c(0, x) = \exp(in_0x)$, so that if $n_0 = 0$

$$v_n(t) = \left\{ \begin{array}{ll}
0 & x < 0 \\
(\lambda t)^n / n! \cdot \exp(-\lambda t) & x \geq 0
\end{array} \right. \quad (3.8)$$

If now we consider two equations of the form (3.7),

$$\dot{v}_n + \lambda(v_n - v_{n-1}) = 0, \quad \dot{z}_n + \mu(z_n - z_{n-1}) = 0. \quad (3.9)$$

we can compute the intersection integrals: denoting with $d(t, x)$ the Fourier transform of $z_n(t)$ and assuming that $\lambda > \mu > 0$, we have

$$\int_0^{+\infty} \sum_{n = -\infty}^{+\infty} v_n(t)z_n(t)dt = \int_0^{+\infty} \frac{1}{2\pi} \int_0^{2\pi} c(t, x)d(t, -x)e^{-inx}dxdt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{+\infty} \exp(\lambda(e^{ix} - 1)t + \mu(e^{-ix} - 1)t) e^{-inx}dxdt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} \lambda(e^{ix} - 1) + \mu(e^{-ix} - 1) dx$$

$$= \frac{1}{2\pi i} \oint_{\gamma} \frac{z^{-n_0}}{(z - 1)(\lambda z - \mu)} dz,$$

where $\gamma$ is the path represented in Figure 2.

By means of complex analysis we have that

$$P(n_0) = \int_0^{+\infty} \sum_{n = -\infty}^{+\infty} v_n(t)z_n(t)dx = \left\{ \begin{array}{ll} 1/(\lambda - \mu) \cdot (\lambda/\mu)^{n_0} & n_0 < 0 \\
1/(\lambda - \mu) & n_0 \geq 0 \end{array} \right. \quad (3.11)$$
3.3. BV estimates. Now to prove that (3.1) has a solution with uniformly bounded total variation. By defining the functional

\[ Q(u(t)) = \sum_{i \leq j} \sum_{n,m = -\infty}^{+\infty} P(n-m) \left\{ v^i_n(t)v^j_m(t) + v^i_n(t)v^j_{m-1}(t) + v^i_n(t)v^j_{m-1}(t) \right\} dxdy, \tag{3.12} \]

where \( P \) is computed using the constant of separation of speeds \( c \) instead of \( \lambda - \mu \) and \( 1 + c/K \) instead of \( \lambda/\mu \), since the left hand side of (3.6) is in conservation form, we conclude immediately that

\[ \text{Tot.Var.}(u(0)) + C_0 Q(u(0)) \leq \delta_1, \]

\[ \frac{d}{dt} \{ \text{Tot.Var.}(u) + C_0 Q(u) \} \leq 0. \]

This concludes the proof of bounded total variation.

3.4. Stability estimates. Finally we consider the stability estimates of (3.1). The equations for a perturbation \( u + \delta h \) as \( \delta \to 0 \) are

\[ \dot{h}_n(t) + Df(u_n)h_n - Df(u_{n-1}) = 0. \tag{3.13} \]

Considering the projection

\[ h_n(t) = \sum_i h^i_n(t)r_i(u_n), \]

we have that the equations for the components \( h^i_n \) are

\[ \dot{h}^i_n + \lambda_i(u_n)h^i_n - \lambda_i(u_{n-1})h^i_{n-1} = \sum_{j \neq k} H'(n)h^j_{n-1}v^k_n + \sum_{j \neq k} G'(n)h^j_nv^k_n. \]

At this point it is clear that a functional as in Section 2.3 proves the stability of the solution. This concludes the proof.

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