Black hole entropy in modified gravity models

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Abstract

An analysis of some modified gravity models, based on the study of pure Schwarzschild and of Schwarzschild-de Sitter black holes, and involving the use of the Noether charge method, is carried out. Corrections to the classical Einsteinian black hole entropy appear. It is shown explicitly how the condition of positive entropy can be used in order to constrain the viability of modified gravity theories.

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I. INTRODUCTION.

Increasing interest is attracted by modified versions of general relativity \[1\]. They have been proposed as serious alternatives to Einstein’s theory of gravitation, and could be used to describe more accurately the observed accelerated expansion of our universe \[2, 3, 4, 5, 6\]. In addition, it has been shown \[2\] that it is actually possible to reconstruct the explicit form of the postulated curvature function \(f(R)\), from the universe expansion history.

It is quite well known that modified versions of general relativity are mathematically equivalent to scalar fields models (see e.g. \[1\]), meaning that a solution in a modified gravity model can always be mapped into a solution of the corresponding scalar field theory. In spite of this mathematical correspondence, physical equivalence does not always follow. In fact, two corresponding solutions of two equivalent theories can actually exhibit rather different physical behaviors. Furthermore, it is not necessary, in order to justify modified gravity, to do it by always using this relation with scalar field theories. Because of the new situation, in the following we will disregard this mathematical equivalence and do consider in our analysis modified gravity as an independent theory aiming directly at some measurable physical properties. What is more, in our treatment modified gravity will in fact be viewed just as a different classical theory of gravitation.

Although other models have been considered \[8\] with the Gauss-Bonnet scalar in the action, here we shall restrict our attention to pure \(f(R)\) models. As often discussed, there are limitations on the function \(f(R)\) when trying to construct a theory which is in agreement with the very precise solar system tests carried out so far, as well as with all the known cosmological bounds \[2\]. Recently, different models of that kind have been studied \[2, 7, 9, 11, 12\], the last three of them having been reported to pass all solar system tests; in addition, they exhibit a number of very interesting features. In \[12\] possible Newton law corrections to such models have been considered.

We present here an analysis of the models above based on the study of pure Schwarzschild and also Schwarzschild-de Sitter black holes (SBH, SdSBH), calculated through use of the Noether charge method. We start with a discussion of two examples considered in \[6, 14\] and go over to study more recent ones \[9, 11, 12\]. The direct confrontation of basic quantities, as the black hole entropy, with the well established, classical (Einsteinian) result can offer a further insight into the construction of a general \(f(R)\) theory. We start with a short review of modified \(f(R)\) gravity and the Noether charge method to compute the BH entropy and then calculate the BH entropy for the models in Refs. \[2, 7, 8\]. After that, we extend our analysis to the models \[2, 9, 11, 12\], taking due care of the sign of the BH entropy and also discussing the stability conditions as well as the existence of a Schwarzschild BH solution. We conclude by providing a brief comparison of the different models considered.

II. BLACK HOLE ENTROPY IN MODIFIED GRAVITY.

The action for \(f(R)\) gravity (see e.g. \[1\] for a review) is

\[
J = \frac{1}{k^2} \int d^4x \sqrt{-g} \mathcal{L} = \frac{1}{k^2} \int d^4x \sqrt{-g} (R + f(R)),
\]

with \(f(R)\) a generic function. As discussed in \[2\], in order to give rise to a realistic cosmology this function needs to fulfill some limiting conditions. However, for the moment we can ignore them, because they do not affect the considerations which will follow. The equations of motion for this theory, in the presence of matter, are

\[
(1 + f'(R)) R_{\mu\nu} - \frac{1}{2} (R + f(R)) g_{\mu\nu} + g_{\mu\nu} \Box f'(R) - \nabla_\mu \nabla_\nu f'(R) = k^2 T_{\mu\nu},
\]
where $T_{\mu\nu}$ is the matter stress energy tensor. Contracting the indices in the last equation, we obtain the relation
\[ R (f'(R) - 1) - 2f(R) + 3\Box f'(R) = k^2 T_{\mu\nu}. \] (3)

Since we are interested in the study of the Schwarzschild solution (in the case when there is a $R = 0$ solution) or either in Schwarzschild-De Sitter black holes, we restrict our reasoning to the case of metric tensors with constant scalar curvature in the vacuum. In that case, we simply have
\[ R_0 (f'(R_0) - 1) - 2f(R_0) = 0. \] (4)

For completeness, let us recall that, in order to build a realistic modified gravity, $f(R)$ needs satisfy the two conditions:
\[ \lim_{R \to \infty} f(R) = \text{const}, \quad \lim_{R \to 0} f(R) = 0, \] (5)

The first condition corresponds to the existence of an effective cosmological constant at high curvature. The second one allows for vacuum solutions, as for example Minkowski or Schwarzschild space-times. Then, although an effective cosmological constant exists, vacuum solutions are preserved and it is legitimate to study them also in a large scale universe with nonzero $R$. Moreover, in order to give rise to stable solutions (2), the following additional condition needs to be fulfilled:
\[ [1 + f'(R)]/f''(R) > R. \] (6)

We can now consider the Schwarzschild-De Sitter metric, a spherically symmetric solution of (1) with constant curvature $R_0$ (see [17])
\[ ds^2 = a(r)dt^2 - dr^2/a(r) - r^2d\Omega, \] (7)
where $a(r) = 1 - 2m/r - R_0r^2/12$ and $R_0 (f'(R_0) - 1) - 2f(R_0) = 0$.

We will use the Noether charge method, as discussed in [8, 18], in order to calculate the entropy for the Schwarzschild-De Sitter BH. The entropy formula reads
\[ S = 4\pi \int_{S^2} \sqrt{-g} \frac{\partial L}{\partial R}, \] (8)
and the integration is made on the external horizon of events surface. In the case of constant curvature, for a generic modified theory, the result is
\[ S = [1 + f'(R_0)]A_H/4G, \] (9)
where $A_H$ is the area of the BH horizon. This enables us to calculate the BH entropy for a generic $f(R)$ theory. We must here stress the fact that the requirement of positive black hole entropy simply avoids the appearance of ghost or tachyon fields in the corresponding scalar field theory. Then a negative entropy is simply a footprint of some instabilities in the Einstein frame. What is new in this picture is that we do not need to involve the (mathematical) equivalence of these models in order to give a physically meaningful interpretation of such constrain.

### III. BLACK HOLE ENTROPY FOR TWO MODIFIED GRAVITY MODELS WITH NO $R = 0$ SOLUTIONS.

In order to illustrate the method with explicit examples of entropy calculation, we analyze here two modified gravity models that appeared some time ago and which have been quite successful up to now. In those models no $R = 0$ solution occurs. The first one, introduced in [8], is given by
\[ f(R) = -a (R - \Lambda_1)^{-n} + b (R - \Lambda_2)^m, \] (10)
with $m, n, a, b > 0$. The condition to obtain a SdSBH (namely, $2f(R_0) = R_0 (f'(R_0) - 1)$) leads to
\[ R_0 \left[ an (R_0 - \Lambda_1)^{-n-1} + bm (R_0 - \Lambda_2)^{m-1} - 1 \right] = 2 \left[ -a (R_0 - \Lambda_1)^{-n} + b (R_0 - \Lambda_2)^m \right], \] (11)
so that we have for the entropy
\[ S = \frac{A_H}{4G} \left[ 1 + na (R_0 - \Lambda_1)^{-n-1} + mb (R_0 - \Lambda_2)^{m-1} \right]. \] (12)
Thus the SdSBH entropy is positive for all $R_0 > \Lambda_1, \Lambda_2$.

The second model, studied in [14], is defined by
\[ f(R) = \alpha \ln(R/\mu^2) + \beta R^m. \] (13)
This modified gravity model does not admit vacuum solutions, thus we can calculate the entropy for the SdSBH. The Ricci scalar is such that it satisfies the relation
\[ 2\alpha \ln(R_0/\mu^2) + \beta (2 - m)R_0^m - \alpha = 0. \] (14)
The SdSBH entropy is given by
\[ S = \frac{A_H}{4G} \left( 1 + \alpha/R_0 + \beta m R_0^{m-1} \right), \] (15)
and turns out to be positive for all values of $R_0 > 0$.

### IV. BLACK HOLE ENTROPY IN MODIFIED GRAVITY MODELS THAT COMPLY WITH THE SOLAR SYSTEM TESTS.

We now analyze three recent models [8, 11, 12] which have been proven to comply with the solar-system as well as with other cosmological parameter constraints. Their respective authors have given a complete discussion of each model, taking care to provide a range for the free parameters contained in the $f(R)$ function, and have also produced stable solutions. Here we just want to stress, with the help of these examples, how the corresponding BH entropy calculation offers a further tool in order to confront each of those modified gravity theories with Einstein’s general relativity, given the fact that the presence of spherically symmetric BH solutions is a necessary element of all local tests.
A. The Hu-Sawicki model.

In this model [9] (with $n, c_1, c_2 > 0$) we have

$$f(R) = -m^2 c_1 \left( \frac{R}{m^2} \right)^n c_2 \left( \frac{R}{m^2} \right) + 1.$$  

(16)

In [9], $m^2$ is chosen such that, at cosmological scale, $R >> m^2$ at the present epoch, and $f(R)$ satisfies the condition $f''(R) > 0$ for $R >> m^2$. This also ensures that solutions with $R >> m^2$ are stable. Moreover, the requirement that $c_1/c_2 \rightarrow 0$ at fixed $c_1/c_2$ gives a cosmological constant, in both cosmological and local tests of gravity. In spite of this fact, since $f(0) = 0$, this theory admits the Schwarzschild solution (i.e. $R = 0$). By the way we note that the stability condition for the vacuum solution is not satisfied unless $n = 1$ and $1 - c_1 > 0$. Therefore, except of this case, vacuum solutions (than also SBH) are unstable. Note also that $n = 1$ corresponds to a Lagrangian $L = R(1 - c_1)/k^2$ for small $R$, so it is associated with a correction to the gravitational coupling constant for small $R$, giving an effective $G_{eff} = G/(1 + f'(0))$ (see [10]).

The entropy formula gives for the SdS metric

$$S(R_0) = \frac{A_H}{4G} \left[ 1 - n c_1 \left( \frac{R_0}{m^2} \right)^n + 1 \right]^{2}. $$  

(17)

The entropy for the Schwarzschild solution is

$$S(0) = (1 - c_1) \frac{A_H}{4G}, \text{ for } n = 1;$$  

$$S(0) = \frac{A_H}{4G}, \text{ for } n > 1, \text{ as in the Einstein theory};$$  

$$S(0) = -\infty, \text{ for } 0 < n < 1, c_1 > 0.$$  

Then, in the only stable case, with $n = 1$ and $1 - c_1 > 0$, a correction to the classical Eistenin BH entropy is found. From [17] it also follows that, for SdS BH with $R_0 >> m^2$, the entropy is positive and corrections to its Einsteinian value are of order $(m^2/R_0)^{n+1}$.

B. The Starobinsky model.

In this model $f(R)$ is [11]

$$f(R) = \lambda C \left[ 1 + (R/C)^2 \right]^{-n} - 1,$$  

(19)

from where

$$f'(R) = -2n\lambda (R/C) \left[ 1 + (R/C)^2 \right]^{-n-1}. $$  

(20)

Note that in this case $f''(0) < 0$ an as thus, although this model admits a SBH solution, it is unstable together with all its vacuum solutions. In [11] the author limits his analysis to solutions that satisfy the following stability conditions

$$1 + f'(R) > 0, \quad f''(R) > 0, \quad 1 + f'(R) > R f''(R) $$  

(21)

We can therefore consider the SdSBH solutions, with curvature $R_0$ given by

$$R_0 \left\{ 2n\lambda (R_0/C) \left[ 1 + (R_0/C)^2 \right]^{-n-1} + 1 \right\} \quad = -2\lambda C \left[ (1 + (R_0/C)^2)^{-n} - 1 \right]. $$  

(22)

that satisfies also (21). In this case, the entropy is just

$$S = \frac{A_H}{4G} \left[ 1 - 2n\lambda (R_0/C) \left[ 1 + (R_0/C)^2 \right]^{-n-1} \right]. $$  

(23)

Therefore, in this case a non trivial correction of the SdSBH entropy is found. We just stress the fact that the SBH solutions have classical entropy but are unstable, and that the SdSBH ones have a modified entropy which, under the limitations stated in [11], is strictly positive.

C. The Appleby-Battye model.

Here $f(R)$ is given by [12]

$$f(R) = -R/2 + \log \left[ \cosh(aR) - \tanh(b) \sinh(aR) \right] / 2a$$  

(24)

and

$$f'(R) = [-1 + \tanh(aR - b)] / 2. $$  

(25)

This model admits a SBH solution. The entropy for the SdSBH is simply

$$S = [1 + \tanh(aR_0 - b)] A_H/8G. $$  

(26)

We can use the stability condition given in [12], $aR_0 - b >> 1$, to obtain

$$S \simeq A_H/4G. $$  

(27)

For the SBH, the stability condition is just $b << 0$. Moreover, being $f''(R) > 0$ for all $R$, in this model the vacuum solutions are always stable and there are no substantial corrections to the classical result.

D. Comparison of the behavior of $f(R)$ for the different models.

It is interesting to put together all three models and explicitly compare the behavior of the function $f(R)$, in particular, the stability of the Euclidean limit and the asymptotic behavior at large curvature. To simplify the comparison, we do not play with the values of the different parameters and set all coefficients equal to 1 and the curvature powers equal to 2 or 4. For the case of the Hu-Sawicki model, with

$$f_{HS}(x) = -\frac{x^4}{1 + x^4}, $$  

(28)
being \( x \equiv R/m^2 \), the corresponding plot is given in Fig. 1. For the case of the Starobinsky model, with

\[
f_{HS}(x) = -1 + \frac{1}{(1 + x^2)^2},
\]

being \( x \equiv R/R_0 \), we obtain Fig. 2. And for the case of the Appleby-Battye model, with

\[
f_{AB}(x) = -\frac{x}{2} + \frac{1}{2} \log(\cosh x + \sinh x),
\]

being \( x \equiv aR \) and \( b = 1/2 \), Fig. 3.

In a first comparison of these different models, we note that the one of Starobinsky, as remarked by the author himself \([11]\), has unstable vacuum solutions. This seems true also for the Hu-Sawicki model, except for the case when \( 0 < n < 1 \), that leads to a negative and infinite BH entropy. The Appleby-Battye model has the important property to possess stable vacuum solutions for a suitable range of the free parameters. It also yields an unmodified expression of the BH entropy.

V. BH ENTROPY IN A NEW MODEL THAT UNIFIES INFLATION AND COSMIC ACCELERATION.

Very recently, a modified gravity model has been published \([20]\), that unifies inflation and cosmic acceleration under the same picture and also complies with the solar system tests. It is

\[
f(R) = -f_0 \int_0^R \exp \left[ -\alpha \frac{R_{1n}^2}{(R_0 - R_1)^{2n}} - f_0 \frac{x}{\Lambda_i} \right] dx,
\]

where \( 0 < f < 1 \) and \( R_1 \) is a constant given by \( f_0 R_1 \int_0^1 e^{-\alpha/x^2} dx = R_{now}, \) and \( R_{now} \) is the Ricci scalar at present. The effective cosmological constant in the early universe is simply \( -f(-\infty) = \Lambda_i \) and the present cosmological constant is \( 2R_0 \). Because of the fact that \( f(0) = 0 \), this model allows for SBH. To be general, we first calculate the SdSBH entropy, which is given by

\[
S = \frac{A_H}{4G} \left( 1 - f_0 \exp \left[ -\alpha \frac{R_{1n}^2}{(R_0 - R_1)^{2n}} - f_0 \frac{R_0}{\Lambda_i} \right] \right),
\]

where \( R_0 \) is the SdSBH curvature that fulfills condition (4). Thus, black holes are less entropic than in Einstein’s theory. For the SBH, we have

\[
S = \frac{A_H}{4G} \left( 1 - f_0 e^{-\alpha} \right).
\]

Note that the stability condition is here \( f''(R) > 0 \), thus

\[
\frac{f_0}{\Lambda_i} > 2n \frac{R_{1n}^2}{(R_0 - R_1)^{2n+1}}.
\]

To have stability for SBH, we need that

\[
n < \frac{R_1 f_0}{2 \Lambda_i}.
\]

This can be considered, together with the condition \( n > 10 - 12 \) stated in \([21]\), to avoid Newton law corrections in our solar system and on the earth surroundings.
VI. CONCLUSIONS.

Comparison of BH entropy in modified gravity theories and in the usual Einsteinian gravity case have been carried out. We have here analyzed different suitable models, recently considered in the literature, and have shown explicitly how corrections to the ‘classical’ BH entropy can in fact appear. We have also argued that the condition of positive entropy can be used as an extra condition in order to constrain the viability of modified gravity theories. Of course this conditions is equivalent to the requirement that neither ghost nor tachyon fields appear in the equivalent scalar field models. Anyhow, if referred to the BH entropy, this condition has a direct interpretation in the framework of modified gravity, without needing to pass through the (mathematically but not physically equivalent) scalar field theories. We hope that this quite simple considerations may be useful for future analysis.

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