A NON-HYPERELLIPTIC CURVE WITH TORSION CERESA CYCLE MODULO ALGEBRAIC EQUIVALENCE

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ABSTRACT. We exhibit a non-hyperelliptic curve $C$ of genus 3 such that the class of the Ceresa cycle $[C] - [C^-]$ in $JC$ modulo algebraic equivalence is torsion.

1. INTRODUCTION

Let $C$ be a complex curve of genus $g \geq 2$, and $p$ a point of $C$. We embed $C$ into its Jacobian $J$ by the Abel-Jacobi map $x \mapsto [x] - [p]$; we denote by $C^-$ the image of $C$ under the involution $(-1)_{} : a \mapsto -a$ of $J$. The Ceresa class is the class $\cj(C) := [C] - [C^-]$ in the group $A_1(J)$ of 1-cycles on $J$ modulo algebraic equivalence (it is independent of the choice of $p$). Since $(-1)_{}$ acts trivially on $H^0(J,\mathbb{Z})$ for $p$ even, $\cj(C)$ belongs to the Griffiths group $G(J)$, the kernel of the cycle class map $A_1(J) \to H^{2g-2}(J,\mathbb{Z})$.

Ceresa classes have played a prominent role in the study of Griffiths groups, especially in the development of techniques for showing that a given element is non-zero [C, C-P, H]. In addition they played an important role in showing that $G(J)$ can have infinite rank [N]. As the conjectures of Bloch and Beilinson were developed and are studied $\cj(C)$ appears repeatedly [Bl, BST], [Z, §1.5], always as an element of infinite order.

When $C$ is hyperelliptic, $\cj(C) = 0$; in fact $C - C^-$ is zero as a cycle when $p$ is a Weierstrass point. In this note we will exhibit what we believe to be the first example of a non-hyperelliptic curve $C$ with $\cj(C) = 0$ in $A_1(J) \otimes \mathbb{Q}$. The curve $C$ has genus 3, and admits an automorphism $\sigma$ of order 9, such that the quotient variety $J/\langle \sigma \rangle$ is uniruled. This implies that the Griffiths group of a resolution of $J/\langle \sigma \rangle$ is torsion; going back to $J$ gives the result.

2. MAIN RESULT

Theorem. Let $C \subset \mathbb{P}^2$ be the genus 3 curve defined by $X^4 + XZ^3 + Y^3Z = 0$. Then $\cj(C) = 0$ in $A_1(J) \otimes \mathbb{Q}$.

Proof: Let $\zeta$ be a primitive 9-th root of unity. We consider the automorphism $\sigma$ of $C$ defined by $\sigma(X, Y, Z) = (X, \zeta^2 Y, \zeta^3 Z)$. We use the fixed point $p = (0, 0, 1)$ to embed $C$ in its Jacobian $J$, so that the action of $\sigma$ on $J$ preserves $C$ and $C^-$. We denote by $V$ the quotient variety $J/\langle \sigma \rangle$, and by $\pi : J \to V$ the quotient map. Let $F \subset J$ be the subset of elements with nontrivial stabilizer; the singular locus $\Sing V$ of $V$ is $\pi(F)$. We put $J^o := J \setminus F$ and $V^o := V \setminus \Sing V$.

Lemma 1. $\Sing V$ is finite; the points $\pi(x)$ for $x \in \Ker(1, -\sigma)$ are non-canonical singularities.

Proof: The space $T_0(J)$ is canonically identified with $H^0(C, K_C)^*$. The elements of $H^0(C, K_C)$ are of the form $L \frac{X}{Y^2 Z} \frac{dZ - ZdX}{dY^2 Z}$, with $L \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ [FK, §9.3, Corollary of Theorem 1]. It follows that the eigenvalues of $\sigma$ on $H^0(C, K_C)$ are $\zeta^5, \zeta^7, \zeta^8$, and those on $T_0(J) = H^0(C, K_C)^*$ are $\zeta, \zeta^2, \zeta^4$.

Thanks to E. Colombo and B. van Geemen for their crucial input. The second named author thanks S. Katz and M. Reid for helpful discussions.
Therefore $\ker(1_J - \sigma^d)$ is finite for $0 < d < 9$, so $F$ is finite. Since $1 + 2 + 4 < 9$, Reid's criterion [K] Theorem 3.1] implies that the singular points $\pi(x)$ for $x \in \ker(1_J - \sigma)$ are not canonical.

Lemma 2. Let $V$ be a resolution of singularities; it suffices to prove that $V$ has Kodaira dimension $-\infty$ [M]. Suppose this is not the case: there exist an integer $r \geq 1$ and a nonzero section of $K_V^r$. By restriction to $\rho^{-1}(V^o) \cong V^o$, we get a section $\omega$ of $K_{V^o}^r$, whose pull back under $\pi$ is a nonzero section of $K_{\tilde{V}}^r$; therefore $\omega$ is a generator of $K_{\tilde{V}}^r$, hence extends to a generator of the reflexive sheaf $K_{\tilde{V}}^{[r]}$ (with the notation of [R]). By construction this generator remains regular on $\tilde{V}$, which means that the singularities of $\tilde{V}$ are canonical [K] Proposition 1.2, contradicting Lemma 1.

Lemma 3. Let $X$ be a uniruled smooth projective threefold. The Griffiths group $G(X)$ is torsion.

Proof: Let $\rho : \tilde{V} \to V$ be a resolution of singularities; it suffices to prove that $\tilde{V}$ has Kodaira dimension $-\infty$. Suppose this is not the case: there exist an integer $r \geq 1$ and a nonzero section $\omega$ of $K_{\tilde{V}}^r$. By restriction to $\rho^{-1}(V^o) \cong V^o$, we get a section $\omega$ of $K_{V^o}^r$, whose pull back under $\pi$ is a nonzero section of $K_{\tilde{V}}^r$; therefore $\omega$ is a generator of $K_{\tilde{V}}^r$, hence extends to a generator of the reflexive sheaf $K_{\tilde{V}}^{[r]}$ (with the notation of [R]). By construction this generator remains regular on $\tilde{V}$, which means that the singularities of $\tilde{V}$ are canonical [K] Proposition 1.2, contradicting Lemma 1.

Remark. One can actually deduce from [BLS] Theorem 1 (ii) that $G(X) = 0$ – but we will not need this fact.

Proof of the Theorem: We can choose the resolution $\rho : \tilde{V} \to V$ so that $E := \rho^{-1}(\text{Sing} V)$ is a normal crossing divisor, whose irreducible components are smooth and rational [Fu] Corollary of Theorem 1.

Let $C$ and $C^-$ be the images in $V$ of $C$ and $C^-$, and let $\tilde{C}$ and $\tilde{C}^-$ be their proper transforms in $\tilde{V}$. We have $[\tilde{C}] - [\tilde{C}^+] = \frac{1}{\pi_*}([C] - [C^-]) = 0$ in $H^4(V^o, \mathbb{Q})$. Now we have an exact sequence [D Corollaire 8.2.8]

$$H^2(\tilde{E}, \mathbb{Q}) \xrightarrow{i_*} H^4(\tilde{V}, \mathbb{Q}) \to H^4(V^o, \mathbb{Q}),$$

where $\tilde{E}$ is the normalization of $E$, and $i$ the composition $\tilde{E} \to E \to \tilde{V}$. Therefore we have $[\tilde{C}] - [\tilde{C}^-] = i_*z$ in $H^4(\tilde{V}, \mathbb{Q})$ for some class $z \in H^2(\tilde{E}, \mathbb{Q})$. Since the components of $\tilde{E}$ are rational, $z$ is the class of an element $\tilde{z}$ of $A_1(\tilde{E}) \otimes \mathbb{Q}$. Then $[\tilde{C}] - [\tilde{C}^-] - i_*\tilde{z} \in A_1(\tilde{V}) \otimes \mathbb{Q}$ is homologous to zero, hence equal to zero by Lemma 3. Restricting to $\tilde{V} - E \cong V^o$, we get $[\tilde{C}] - [\tilde{C}^-] = 0$ in $A_1(V^o \otimes \mathbb{Q})$, hence $[\tilde{C}] - [\tilde{C}^-] = \pi^*(\tilde{C} - \tilde{C}^-) = 0$ in $A_1(J^o) \otimes \mathbb{Q}$. But the restriction map $A_1(J) \to A_1(J^o)$ is an isomorphism [Fu] Example 10.3.4, hence the Theorem.

3. COMPLEMENTS

Corollary 1. Let $\Theta$ be a Theta divisor on $J$. We have $[C] = \frac{[\Theta]^2}{2}$ in $A_1(J) \otimes \mathbb{Q}$ (Poincaré formula).

Proof: Indeed for any genus 3 curve $C$ we have $[\Theta]^2 = [C] + [C^-]$ in $A_1(J) \otimes \mathbb{Q}$ (if $p, q$ are two distinct points of $C$, the intersection of $\Theta$ with its translate by $[p] - [q]$ is the union of a translate of $C$ and a translate of $C^-$ — see for instance [Mu] Lecture IV]). Thus the corollary is equivalent to the theorem.

Recall that the modified diagonal cycle $\Gamma(C, p)$, first considered in [G-S], is the element $\Gamma(C, p) = A_1(C^o)$ defined as follows. We denote by $[x, x, x], [x, x, p], [x, p, p]$ etc. the classes in $A_1(C \times C \times C)$ of the image of $C$ by the maps $x \mapsto (x, x, x), x \mapsto (x, x, p), x \mapsto (x, p, p)$ etc. Then:

$$\Gamma(C, p) := [x, x, x] - [x, x, p] - [x, p, p] - [p, p, p] + [x, x, p] + [x, p, p] + [p, p, p] + [p, p, x].$$

By [FLV] Remark 3.4, we have
Corollary 2. \( \Gamma(C, p) = 0 \) in \( A_1(C^3) \otimes \mathbb{Q} \).

Finally let us mention the result of \([B3]\): the class of \( [C] - [C^-] \) in the intermediate Jacobian \( J_1(J) \) is torsion. It can be also deduced from our theorem, though the proof in \([B3]\) is more direct.

In \([BLLS]\) the authors construct a genus 7 curve with the same property, and suggest that the corresponding Ceresa cycle should be torsion modulo algebraic equivalence (Remark 1.2).

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