SIMPLE POLYADIC GROUPS

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Abstract. The main aim of this article is to establish a classification of simple polyadic groups in terms of ordinary groups and their automorphisms. We give two different definitions of simpleness for polyadic groups, from the point of views of universal algebra, UAS (universal algebraically simpleness), and group theory, GTS (group theoretically simpleness). We obtain the necessary and sufficient conditions for a polyadic group to be UAS or GTS.

1. Introduction

Let \( G \) be a non-empty set and \( n \) be a positive integer. If \( f : G^n \to G \) is an \( n \)-ary operation, then we use the compact notation \( f(x^n) \) for the elements \( f(x_1, \ldots, x_n) \). In general, if \( x_i, x_{i+1}, \ldots, x_j \) is an arbitrary sequence of elements in \( G \), then we denote it as \( x^j_i \). In the special case, when all terms of this sequence are equal to a constant \( x \), we show it by \( x(t) \), where \( t \) is the number of terms. During this article, we assume that \( n \geq 2 \). We say that an \( n \)-ary operation is associative, if for any \( 1 \leq i < j \leq n \), the equality

\[
    f(x_i^{n-i-1}, x^n_{n+i}, x^{2n-1}_{n+i}) = f(x_j^{n+j-1}, x^n_{n+j}, x^{2n-1}_{n+j})
\]

holds for all \( x_1, \ldots, x_{2n-1} \in G \). An \( n \)-ary system \((G, f)\) is called an \( n \)-ary group or a polyadic group, if \( f \) is associative and for all \( a_1, \ldots, a_n, b \in G \) and \( 1 \leq i \leq n \), there exists a unique element \( x \in G \) such that

\[
    f(a_i^{n-1}, x, a_{i+1}^n) = b.
\]

It is proved that the uniqueness assumption on the solution \( x \) can be dropped, see [5] for details. Clearly, the case \( n = 2 \) is just the definition of ordinary groups.

For a review of basic notions, we introduce some materials. First of all, there is the classical paper of E. Post [19], which is one of the first articles published on the subject. In this paper, Post proves his well-known Coset theorem. Many basic properties of polyadic groups are studied in this paper. The articles [3] and [15] are among the first materials written on the polyadic groups. Reader who knows Russian, can use the book of Galmak [11], for an
almost complete description of polyadic groups and as we understand from its English abstract, some of our results are also proved in that book by a different method. The articles [1], [3], [12], and [13] can be used for study of axioms of polyadic groups as well as their varieties.

Note that an \( n \)-ary system \((G,f)\) of the form \( f(x_1^n) = x_1x_2\ldots x_nb \), where \((G,\cdot)\) is a group and \( b \) a fixed element belonging to the center of \((G,\cdot)\), is an \( n \)-ary group. Such an \( n \)-ary group is called \( b \)-derived from the group \((G,\cdot)\) and it is denoted by \( \text{der}_b(G,\cdot) \). In the case when \( b \) is the identity of \((G,\cdot)\) we say that such a polyadic group is reduced to the group \((G,\cdot)\) or derived from \((G,\cdot)\) and we use the notation \( \text{der}(G,\cdot) \) for it. But for every \( n > 2 \), there are \( n \)-ary groups which are not derived from any group. An \( n \)-ary group \((G,f)\) is derived from some group if and only if it contains an element \( a \) (called an \( n \)-ary identity) such that

\[
f(\underbrace{a,\ldots,a}_{(i-1)} ,x, \underbrace{a,\ldots,a}_{(n-i)}) = x
\]

holds for all \( x \in G \) and for all \( i = 1, \ldots, n \).

From the definition of an \( n \)-ary group \((G,f)\) we can directly see that for every \( x \in G \) there exists only one \( y \in G \) satisfying the equation

\[
f(\underbrace{x,\ldots,x}_{(n-1)} ,y) = x.
\]

This element is called skew to \( x \) and it is denoted by \( \overline{x} \). As Dörnte proved (see a [3]), the following identities hold for all \( x,y \in G \), \( 2 \leq i,j \leq n \) and \( 1 \leq k \leq n \)

\[
f(\underbrace{x,\overline{x},x,\ldots,x}_{(i-2)} ,y) = f(\underbrace{y,\overline{x},\ldots,x}_{(j-2)}) = y,
\]

\[
f(\underbrace{x,\overline{x},x,\ldots,x}_{(k-1)} ) = x.
\]

Suppose \((G,f)\) is a polyadic group and \( a \in G \) is a fixed element. Define a binary operation

\[
x * y = f(\underbrace{x,\ldots,x}_{(n-2)},y).
\]

Then \((G,*)\) is an ordinary group, called the retract of \((G,f)\) over \( a \). Such a retract will be denoted by \( \text{ret}_a(G,f) \). All retracts of a polyadic group are isomorphic, see [8]. The identity of the group \((G,*)\) is \( \overline{a} \). One can verify that the inverse element to \( x \) has the form

\[
y = f(\overline{x},\overline{x} ,\ldots,\overline{x}).
\]

One of the most fundamental theorems of polyadic group is the following, now known as Hosszú-Gloskin’s theorem. We will use it frequently in this article and the reader can use [6], [7], [14] and [21] for detailed discussions.

**Theorem 1.1.** Let \((G,f)\) be an \( n \)-ary group. Then

1. on \( G \) one can define an operation \( \cdot \) such that \((G,\cdot)\) is a group,
2. there exist an automorphism \( \theta \) of \((G,\cdot)\) and \( b \in G \), such that \( \theta(b) = b \),
3. \( \theta^2(x) = bx^{-1}b^{-1} \), for every \( x \in G \),
4. \( f(x_1^n) = x_1\theta(x_2)\theta^2(x_3)\ldots\theta^{n-1}(x_n)b \), for all \( x_1,\ldots,x_n \in G \).
According to this theorem, we use the notation $\text{der}_{\theta,b}(G,f)$ for $(G,f)$ and we say that $(G,f)$ is $(\theta,b)$-derived from the group $(G,\cdot)$. During this paper, we will assume that $(G,f) = \text{der}_{\theta,b}(G,f)$.

Varieties of polyadic groups and the structure of congruences on polyadic groups are studied in [1], [4] and [22]. It is proved that all congruences on polyadic groups are commute and so the lattice of congruences is modular. In this article we will give a very simple proof for this fact. Among other issues, investigated on polyadic groups, is the representation theory. In [9] the representation theory of polyadic groups is studied and [20] contains generalizations of some important theorems of character theory of finite groups to the case of polyadic groups.

We established some fundamental results on the structure of homomorphisms of polyadic groups in [16]. We will use the main result of that article here as a very strong tool. Suppose $(H,\ast)$ is an ordinary group and $a \in H$. In the following theorem, we denote the inner automorphism $x \mapsto a \ast x \ast a^{-1}$ by $I_a$. Also $R_a$ denotes the map $x \mapsto x \ast a$.

**Theorem 1.2.** Suppose $(G,f) = \text{der}_{\theta,b}(G,\cdot)$ and $(H,h) = \text{der}_{\eta,c}(H,\ast)$ are two polyadic groups. Let $\psi : (G,f) \rightarrow (H,h)$ be a homomorphism. Then there exists $a \in H$ and an ordinary homomorphism $\phi : (G,\cdot) \rightarrow (H,\ast)$, such that $\psi = R_a \phi$. Further $a$ and $\phi$ satisfy the following conditions;

$$h^{(n)}(a) = \phi(b) \ast a \quad \text{and} \quad \phi \theta = I_a \eta \phi.$$

Conversely, if $a$ and $\phi$ satisfy the above two conditions, then $\psi = R_a \phi$ is a homomorphism $(G,f) \rightarrow (H,h)$.

Clearly, using the above theorem, we can determine when two polyadic groups $(G,f) = \text{der}_{\theta,b}(G,\cdot)$ and $(H,h) = \text{der}_{\eta,c}(H,\ast)$ are isomorphic, and also, it can be applied for a complete description of polyadic representations, see [16] for details. Using this theorem, we will determine the structure of polyadic subgroups in the section 4.

Before going to explanation of the motivations for the recent work, we recall the definition of normal polyadic subgroups from [9]. An $n$-ary subgroup $H$ of a polyadic group $(G,f)$ is called normal if

$$f(\overline{x}, (n-3) \uparrow x, h, x) \in H$$

for all $h \in H$ and $x \in G$. If every normal subgroup of $(G,f)$ is singleton or equal to $G$, then we say that $(G,f)$ is group theoretically simple or it is GTS for short. If $H = G$ is the only normal subgroup of $(G,f)$, then we say it is strongly simple in the group theoretic sense or GTS* for short. For any normal subgroup $H$ of an $n$-ary group $(G,f)$, we define the relation $\sim_H$ on $G$, by

$$x \sim_H y \iff \exists h_1, \ldots, h_{n-1} \in H : y = f(x, h_1^{n-1}).$$
Now, it is easy to see that such defined relation is an equivalence on $G$. The equivalence class of $G$, containing $x$ is denoted by $xH$ and is called the left coset of $H$ with the representative $x$. On the set $G/H = \{xH : x \in G\}$, we introduce the operation

$$f_H(x_1H, x_2H, \ldots, x_nH) = f(x^n_1)H.$$ 

Then $(G/H, f_H)$ is an $n$-ary group derived from the group $ret_H(G/H, f_H)$, see [9]. One of the main aims of this article is to classify all $GTS$ polyadic groups. We will give a necessary and sufficient condition for a polyadic group $(G, f)$ to be $GTS$ in terms of the ordinary group $(G, \cdot)$ and the automorphism $\theta$. It is possible to define another kind of simpleness for polyadic groups, universal algebraically simpleness. Note that an equivalence relation $R$ over $G$ is said to be a congruence, if

1. $\forall i: x_iRy_i \Rightarrow f(x^n_1)Rf(y^n_1)$,
2. $xRy \Rightarrow xRy$.

For example, if $H$ is a normal polyadic subgroup of $(G, f)$, then $R = \sim_H$ is a congruence, see [9]. We say that $(G, f)$ is universal algebraically simple or $UAS$ for short, if the only congruence is the equality and $G \times G$. We also, will give a classification of $UAS$ polyadic groups and we will prove that $UAS \subseteq GTS$.

Our motivation to study of simple polyadic groups came from $n$-Lie algebras (or Filippov algebras). A vector space $L$ over a field $\mathbb{F}$ is an $n$-ary Lie algebra, if it is equipped with an alternative $n$-linear map $[-, \ldots, -] : L^n \rightarrow L$ such that the following Jacobi identity holds

$$\sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, [x_i, y_1, \ldots, y_{n-1}], x_{i+1}, \ldots, x_n] = 0.$$ 

The case $n = 2$ is ordinary Lie algebra. The notions such as, subalgebra and ideal can be defined as usual, see [10]. For $n > 2$, it is proved that there is only one simple $n$-ary Lie algebra and the dimension of this unique simple Lie algebra is $n + 1$, see [2] for details. This fact is a large difference between ordinary and $n$-ary Lie algebras, because there are lots of simple ordinary Lie algebras. The case of $n$-ary Lie algebras, may suggest that similarly there are a few simple polyadic groups for $n > 2$. But as we shall see, this is not true and there are many simple $n$-ary groups (both $UAS$ and $GTS$), even more than the ordinary simple groups. This fact, suggests us that there might be a more general definition of $n$-ary Lie algebras which is not discovered until today. In our opinion, the recent definition of Filippov for $n$-ary Lie algebras is only a small piece of an unknown algebraic structure. It may be logically true way if we look for these general notion of real $n$-Lie algebras via studying polyadic Lie groups.
2. Elementary notions

We assume that our polyadic group has the form \((G, f) = \text{der}_\theta b(G, f)\). The identity element of \((G, \cdot)\) will be denoted by \(e\). First we express the skew element \(\overline{x}\) in terms of \(x\) and \(\theta\).

**Lemma 2.1.** We have
\[
\overline{x} = b^{-1}\theta^{n-2}(x^{-1}) \cdots \theta^2(x^{-1})\theta(x^{-1}).
\]

**Proof.** Suppose \(y = b^{-1}\theta^{n-2}(x^{-1}) \cdots \theta^2(x^{-1})\theta(x^{-1})\). Then we have
\[
f(y, (n-1) x) = y\theta(x)\theta^2(x) \cdots \theta^{n-1}(x)b
= b^{-1}\theta^{n-1}(x)b
= b^{-1}bxb^{-1}b
= x.
\]
This shows that \(y = \overline{x}\). \(\square\)

**Definition 2.2.** Suppose \(H \leq (G, f)\) and for all \(x \in G\) and \(h \in H\)
\[
f(\overline{x}, (n-3) x, h, x) \in H.
\]
Then \(H\) is called a normal polyadic subgroup of and it is denoted by \(H \trianglelefteq (G, f)\).

**Lemma 2.3.** Assume that \(H \leq (G, f)\). Then \(H\) is normal, iff for all \(x \in G\) and \(h \in H\), we have \(\theta^{-1}(x^{-1}h)x \in H\).

**Proof.** Note that
\[
f(\overline{x}, (n-3) x, h, x) = b^{-1}\theta^{n-2}(x^{-1}) \cdots \theta(x^{-1})\theta(x) \cdots \theta^{n-3}(x)\theta^{n-2}(h)bx
= b^{-1}\theta^{n-2}(x^{-1})\theta^{n-2}(h)bx
= b^{-1}\theta^{n-2}(x^{-1}h)bx
= b^{-1}\theta^{n-1}(\theta^{-1}(x^{-1}h))bx
= b^{-1}b\theta^{-1}(x^{-1}h)b^{-1}bx
= \theta^{-1}(x^{-1}h)x,
\]
and the lemma is proved. \(\square\)

**Definition 2.4.** An equivalence relation \(R\) over \(G\) is said to be a congruence, if
1. \(\forall i : x_i Ry_i \Rightarrow f(x^n_i)Rf(y^n_i)\),
2. \(xRy \Rightarrow \overline{x}R\overline{y}\).

We denote the set of all congruences of \((G, f)\) by \(\text{Cong}(G, f)\). The following theorem is proved in \(9\).

**Theorem 2.5.** Suppose \(H \leq (G, f)\) and define \(\sim_H\) by
\[
x \sim_H y \iff \exists h_1, \ldots, h_{n-1} \in H : y = f(x, h_1^{n-1}).
\]
Then $R$ is a congruence and if we let $xH = [x]_R$, (the equivalence class of $x$), then the set $G/H = \{xH : x \in G\}$ is an $n$-ary group with the operation

$$f_H(x_1H, \ldots, x_nH) = f(x^n_1)H.$$ 

Further we have

$$(G/H, f_H) = \text{der}(\text{ret}_H(G/H, f_H)),$$

and so it is reduced.

**Definition 2.6.** A polyadic group $(G, f)$ is group theoretically simple, or GTS, if

$$H \trianglelefteq (G, f) \Rightarrow |H| = 1 \text{ or } H = G.$$ 

We say that $(G, f)$ is universal algebraically simple, or UAS, if

$$R \in \text{Cong}(G, f) \Rightarrow R = \Delta \text{ or } R = G \times G,$$

where $\Delta$ denotes equality.

**Proposition 2.7.** Every UAS is also GTS.

**Proof.** Suppose $(G, f)$ is UAS and $H \trianglelefteq (G, f)$. Let $R = \sim_H$. So we have $R = \Delta$ or $G \times G$. If $R = \Delta$, then the following implication is true,

$$(\exists h_1, \ldots, h_{n-1} \in H : y = f(x, h_1^{n-1})) \Rightarrow x = y.$$ 

Assume that $|H| > 1$, so there exist distinct elements $h_1, h_2 \in H$. But then there is $h_3 \in H$ such that $h_2 = f(h_1, h_3, h_1^{(n-2)})$. Therefore $h_1 = h_2$, a contradiction. Hence $|H| = 1$. On the other hand, if $R = G \times G$, then for any $x, y \in G$, we have

$$\exists h_1, \ldots, h_{n-1} \in H : y = f(x, h_1^{n-1}).$$ 

So, if we let $x \in H$, then for any $y \in G$, $y = f(x, h_1^{n-1}) \in H$. This shows that $H = G$. \hfill $\square$

A polyadic group $(G, f)$ is strongly GTS, if $H = G$ is its only normal polyadic subgroup. The class of such polyadic groups will be denoted by $\text{GTS}^*$. The next proposition shows that $\text{GTS}^*$ is the more important part of $\text{GTS}$.

**Proposition 2.8.** If $(G, f)$ has a singleton normal polyadic subgroup, then it is reduced.

**Proof.** Suppose $H = \{u\}$ is a polyadic normal subgroup of $(G, f)$. For any $x \in G$, the coset $xH$ is also singleton and so $xH = \{x\}$. The map $\delta : G \to G/H$ defined by $\delta(x) = \{x\}$ is then an isomorphism and hence

$$(G,f) \cong (G/H, f_H) = \text{der}(\text{ret}_H(G/H, f_H)),$$

and therefore $(G, f)$ is reduced. \hfill $\square$
By the above proposition, if \((G, f)\) is GTS but not strong, then \((G, f) = \text{der}(G, \cdot)\), where \((G, \cdot)\) is an ordinary simple group. Hence it remains to determine polyadic groups which are GTS*. This will be done in the section 4.

3. UAS and the lattice of congruences

In this section we determine the structure of congruences of polyadic groups and we give a necessary and sufficient condition for a polyadic group to be UAS. Note that the binary group \(G \times G\) has an automorphism \((x, y) \mapsto (\theta(x), \theta(y))\) which we denote it by the same symbol \(\theta\). As in the previous section, \(\text{Cong}(G, f)\) is the set of all congruences of \((G, f)\). This set is a lattice under the operations of intersection and product (composition). We also denote by \(\text{Eq}(G)\) the set of all equivalence relations of \(G\).

**Theorem 3.1.** \(R \in \text{Cong}(G, f)\) iff \(R \in \text{Eq}(G)\) and \(R\) is a \(\theta\)-invariant subgroup of \(G \times G\).

**Proof.** Let \(R \in \text{Con}(G, f)\). We prove that \(R\) is a \(\theta\)-invariant subgroup of \(G \times G\). Suppose \(xRy\) and

\[
x_1 = x, \; y_1 = y, \; x_2 = y_2 = e, \; \ldots, \; x_n = y_n = e.
\]

Now, since \(f(x_1^n)Rf(y_1^n)\), we must have \(xbRyb\). A similar argument with

\[
x_1 = x, \; y_1 = b^{-2}, \; x_2 = y_2 = \cdots x_{n-1} = y_{n-1} = e, \; x_n = x, \; y_n = y,
\]

shows that \(b^{-1}xRb^{-1}y\). Now, let \(xRy\) and assume that

\[
x_1 = y_1 = e, \; x_2 = x, \; y_2 = y, \; x_3 = y_3 = e, \; \ldots, \; x_{n-1} = y_{n-1} = e, \; x_n = y_n = b.
\]

Then \(f(x_1^n)Rf(y_1^n)\), which means that \(\theta(x)R\theta(y)\). This shows that \(\theta(R) \subseteq R\). Note that the converse is also true, i.e. if \(\theta(x)R\theta(y)\), then \(xRy\). This is true, since \(\theta(x)R\theta(y)\) implies \(\theta^{-1}(x)R\theta^{-1}(y)\). So we have \(bxb^{-1}Rbyb^{-1}\) and therefore \(xRy\) by the above argument. Summarizing what we proved above, we have

\[
xRy \iff \theta(x)R\theta(y).
\]

Now, assume that \(xRy\) and \(uRv\). Let \(u' = \theta^{-1}(u)\), and \(v' = \theta^{-1}(v)\). So \(u'Rv'\) and again a similar argument shows that \(x\theta(u')Ry\theta(v')\), hence \(xuRyv\). This shows that \(R\) is closed under the binary operation of \(G \times G\). Finally we show that \(xRy\) implies \(x^{-1}Ry^{-1}\). Note that \(xRy\) implies \(x\overline{R}y\). But

\[
\overline{x} = b^{-1}\theta^{-2}(x^{-1})\cdots \theta^{2}(x^{-1})\theta(x^{-1})
\]

\[
\overline{y} = b^{-1}\theta^{-2}(y^{-1})\cdots \theta^{2}(y^{-1})\theta(y^{-1}).
\]

Therefore

\[
b^{-1}\theta^{-2}(x^{-1})\cdots \theta^{2}(x^{-1})\theta(x^{-1})Rb^{-1}\theta^{-2}(y^{-1})\cdots \theta^{2}(y^{-1})\theta(y^{-1}),
\]

and hence

\[
\theta(\theta^{-3}(x^{-1})\cdots \theta(x^{-1})x^{-1})R\theta(\theta^{-3}(y^{-1})\cdots \theta(y^{-1})y^{-1}).
\]
Therefore, we conclude that
\[ \theta^{n-3}(x^{-1}) \cdots \theta(x^{-1})x^{-1}R\theta^{n-3}(y^{-1}) \cdots \theta(y^{-1})y^{-1}. \]
Continuing this argument, we obtain finally \( x^{-1}Ry^{-1} \). So, we proved that \( R \) is a \( \theta \)-invariant subgroup of \( G \times G \). Note that, clearly the converse is also true, so we proved the assertion. \( \square \)

**Proposition 3.2.** Let \( H_R = \{ x \in G : xRe \} = [e]_R \). Then \( H_R \) is a \( \theta \)-invariant normal subgroup of \( G \) and it is a normal polyadic subgroup of \( (G, f) \) only in the case \( bRe \).

**Proof.** Let \( x, y \in H_R \). Then \( xRe \) and \( yRe \) and so \( xy^{-1}Re \). Also if \( x \in H_R \) and \( a \in G \) then \( axa^{-1}Re \) and therefore \( H_R \) is a normal subgroup of \( (G, \cdot) \). Moreover if \( xRe \) then \( \theta(x)Re \) and hence \( H_R \) is \( \theta \)-invariant. Note that if \( x_1, \ldots, x_n \in H_R \), then \( f(x^n)Rb \) and hence \( H_R \leq (G, f) \) iff \( bRe \). \( \square \)

Now, we are ready to give the necessary and sufficient condition for a polyadic group to be UAS.

**Theorem 3.3.** \((G, f)\) is UAS iff the only normal \( \theta \)-invariant subgroups of \((G, \cdot)\) are trivial subgroups.

**Proof.** Let \((G, \cdot)\) be \( \theta \)-simple (i.e. it has no non-trivial normal \( \theta \)-invariant subgroup) and \( R \in Con(G, f) \). Since \( H_R \) is a \( \theta \)-invariant normal subgroup of \((G, \cdot)\), so \( H_R = 1 \) or \( G \). This shows that \( R = \Delta \) or \( G \times G \). So \((G, f)\) is UAS. Conversely, suppose that \((G, f)\) is UAS, and let \( H \) be a \( \theta \)-invariant normal subgroup of \((G, \cdot)\). Define
\[ R = \{ (x, y) : x^{-1}y \in H \}. \]
It is easy to see that \( R \in Con(G, f) \) and so, \( R = \Delta \) or \( G \times G \) which implies that \( H = 1 \) or \( G \). \( \square \)

In the previous section, we saw that \( H \trianglelefteq (G, f) \) implies that \((G/H, f_H)\) is reduced, but this is not true for \( G/R \), when \( R \) is an arbitrary congruence. To determine its structure, note that, since \( H_R \) is \( \theta \)-invariant, so we can define a new automorphism
\[ \theta_R : \frac{G}{H_R} \rightarrow \frac{G}{H_R} \]
by \( \theta_R([x]) = [\theta(x)] \). Let \( b_R = [b] \). Then we have
\[ f_R([x_1], \ldots, [x_n]) = [f(x^n)] \]
\[ = [x_1\theta(x_2) \cdots \theta^{n-1}(x_n)b] \]
\[ = [x_1][\theta_R([x_2]) \cdots \theta_R^{n-1}([x_n])b_R]. \]
This shows that
\[ \frac{G}{R} = der_{\theta_R, b_R}(\frac{G}{H_R}). \]

**Lemma 3.4.** Let \( R \leq G \times G \). Then \( R \in Eq(G) \) iff \( \Delta \subseteq R \).
Proof. One side is trivial. So assume that $\Delta \subseteq R$. Let $(x, y) \in R$. We show that $(y, x) \in R$. Note that

$$(y, x) = (x, x)(x^{-1}, y^{-1})(y, y) \in R.$$ 

So $R$ is symmetric. Now suppose $(x, y), (y, z) \in R$. Then

$$(x, z) = (x, y)(y^{-1}, y^{-1})(y, z) \in R.$$ 

This completes the proof. \qed

**Corollary 3.5.** We have $\text{Cong}(G, f) = \{ R \leq_\theta G \times G : \Delta \subseteq R \}$.

**Proposition 3.6.** Let $R, Q \in \text{Cong}(G, f)$. Then the following assertions are true.

1. as subgroups of $G \times G$, we have $RQ = QR$.
2. we have $R \circ Q = RQ$.
3. we have $R \circ Q = Q \circ R$, so $\text{Cong}(G, f)$ is a modular lattice.
4. we have $H_{RQ} = H_RH_Q$ and $H_{R\cap Q} = H_R \cap H_Q$.

Proof. We know that $H_Q \leq G$, so $1 \times H_Q \leq G \times G$. Let $(x, y) \in R$ and $(u, v) \in Q$. Since $R$ is a congruence, so $(xu, yu) \in R$ and also

$$(x, y)(u, v) = (x, y)(u, u)(u^{-1}, u^{-1})(u, v) = (xu, yu)(e, u^{-1}v).$$

Now, $(e, u^{-1}v) \in Q$, so $u^{-1}v \in H_Q$ and hence $(e, u^{-1}v) \in 1 \times H_Q$. But, we have $R(1 \times H_Q) = (1 \times H_Q)R$, so

$$(x, y)(u, v) = (e, w)(x', y'),$$

for some $(e, w) \in Q$ and $(x', y') \in R$. This shows that $RQ = QR$. To prove 2, note that

$$R \circ Q = \{(x, y) \in G \times G : \exists u, \ (x, u) \in Q \text{ and } (u, y) \in R\}.$$ 

So, if $(x, y) \in R \circ Q$, then $(x, y) = (x, u)(u^{-1}, u^{-1})(u, y) \in RQ$. Hence $R \circ Q \subseteq RQ$. The converse is also true. The proofs are 3 and 4 are now clear. \qed

A congruence $R \in \text{Cong}(G, f)$ is called normal if there exists a normal polyadic group $H \leq (G, f)$ such that $R \sim_H$. In what follows, we determine the normal elements of $\text{Cong}(G, f)$.

**Proposition 3.7.** Let $R$ be a normal congruence and $H$ be the normal polyadic subgroup corresponding to $R$. Then there exists an element $a \in G$ such that $H = aH_R$.

Proof. Suppose $x, y \in H$ and $h_1, \ldots, h_{n-2} \in H$ are arbitrary elements. We know that there is $h_{n-1} \in H$ such that $y = f(x, h_1^{n-1})$. Hence $x \sim_H y$ and therefore there exists $a \in G$ such that $x, y \in [a]_R$. Hence $H \subseteq aH_R$. Now, suppose $u \in aH_R$. Then there is $x \in H$ such that $u \sim_H x$, so there are $h_1, \ldots, h_{n-1} \in H$ such that $u = f(x, h_1^{n-1}) \in H$. Hence $aH_R \subseteq H$. \qed
Theorem 3.8. Let $R$ be a congruence. Then $R$ is normal iff there exists $a \in G$ such that
1. $aRa$,
2. for all $x \in G$, we have $f(\overline{x}, (n-3), a, x) Ra$.

Proof. Suppose $R$ satisfies 1 and 2. We first show that $H = aH_R$ is a normal polyadic subgroup. Suppose $x_i \in H$ for $1 \leq i \leq n$. Then for all $i$, we have $x_iRa$ and since $aRa$, hence $f(x_i^n)Ra = f(\overline{x_i^n}(a^{-1}))$. This shows that $f(x_i^n)Ra$ and therefore $f(x_i^n) \in H$. On the other hand, if $x \in H$, then $xRa$ and so $\overline{x}Ra$. Now, since $aRa$, so $\overline{x}Ra$, which proves that $\overline{x} \in H$. Therefore $H$ is a polyadic subgroup. Now, let $x \in G$ and $h \in H$. Then $hRa$ and hence, $f(\overline{x}, (n-3), h, x)Ra = f(\overline{x}, (n-3), a, x)$. But by 2, we have $f(\overline{x}, (n-3), a, x)Ra$, so $f(\overline{x}, (n-3), h, x) \in H$, proving that $H$ is a normal.

Now, we prove that $R = \sim_H$. Suppose $x \sim_H y$. Then there exist $h_1, \ldots, h_{n-1} \in H$, such that $y = f(x, h_1^{n-1})$. But, then every $h_i$ is of the form $ah'_i$, with $h'_i \in H_R$. Remember from 3.2 that $H_R$ is a $\theta$-invariant normal subgroup of $(G, \cdot)$. So we have $y = f(x, ah'_1, \ldots, ah'_{n-1}) = f(x, a)h'$ for some $h' \in H_R$. Note that, $f(x, a) = x(\overline{a})^{-1}a$ by 2.1. On the other hand, $(\overline{a})^{-1}a \in H_R$ by 1. Hence

$$y = f(x, (n-1), a)h' \in xH_R,$$

which shows that $xRy$. Hence, we proved that $x \sim_H y$ implies $xRy$. The converse is also true, showing that $R = \sim_H$. Therefore $R$ is a normal congruence.

Now suppose $R$ is normal. So $R = \sim_H$ for some normal polyadic subgroup $H$. By 3.7, there is $a \in G$ such that $H = aH_R$. We prove that $a$ satisfies 1 and 2 above. Let $x_1, \ldots, x_n \in H$. Then for all $i$, we have $x_iRa$, and hence $f(x_i^n)Ra = f(a)$. Since $H$ is a polyadic subgroup, so $f(x_i^n)Ra$ and hence $aRa$. Now, using 2.1, it is easy to see that $f(a) = a(\overline{a})^{-1}a$. Therefore, $f(a)a^{-1} \in H_R$ implies that $a(\overline{a})^{-1}aa^{-1} \in H_R$. This shows that $a(\overline{a})^{-1} \in H_R$. So $aRa$, proving 1. The proof of 2 is similar.

Form the above theorem, one can deduce that if $R, Q \in Cong(G, f)$ with $R$ normal and $R \subseteq Q$, then $Q$ is also normal. We can restate the above theorem as in the following form.

Corollary 3.9. A congruence $R$ is normal iff there exists an element $a \in G$ such that
1. $aRa$,
2. for all $x \in G$, $\theta^{-1}(x^{-1}a)xRa$. 

4. GTS and normal polyadic subgroups

This section is devoted to GTS polyadic groups. Again, we assume that \((G, f) = \text{der}_{\theta, b}(G, f)\) is an n-ary group. For \(u \in G\), define a new binary operation on \(G\) by \(x \ast y = xu^{-1}y\). Then \((G, \ast)\) is an isomorphic copy of \((G, \cdot)\) and the isomorphism is the map \(x \mapsto xu\). We denote this new group by \(G_u\). Its identity is \(u\) and the inverse of \(x\) is \(ux^{-1}u\), which we denote it by \(x^{-u}\). We define an automorphism of \(G_u\) by \(\psi_u(x) = u\theta(x)\theta(u^{-1})\). It can be easily checked that this is actually an automorphism of \(G_u\).

Theorem 4.1. We have \(H \trianglelefteq (G, f)\) iff there exists an element \(u \in H\) such that

1. \(H\) is a \(\psi_u\)-invariant normal subgroup of \(G_u\),
2. for all \(x \in G\), we have \(\theta^{-1}(x^{-1}u)x \in H\).

Proof. We first determine the structure of polyadic subgroups, using 1.2. Suppose \(H \leq (G, f)\). We denote the restriction of \(f\) to \(H\) by \(f\), so there is a binary operation on \(H\), say \(\ast\), an automorphism \(\psi\) and an element \(c \in H\), such that \((H, f) = \text{der}_{\psi, c}(H, \ast)\).

The inclusion map \(j : H \rightarrow G\) is a polyadic homomorphism, hence by 1.2, there is an element \(u \in G\) and an ordinary homomorphism \(\phi : (H, f) \rightarrow (G, f)\), with the properties

i- \(j = R_u\phi\),
ii- \(f^{(n)}(u) = \phi(c)u\),
iii- \(\phi\psi = I_u\theta\phi\).

From i, we deduce that for any \(x \in H\), \(\phi(x) = xu^{-1}\), and so by ii, we have \(f^{(n)}(u) = c\). Moreover, since \(\phi\) is an ordinary homomorphism, so using \(\phi(x \ast y) = \phi(x)\phi(y)\) and \(\phi(x) = xu^{-1}\), we obtain \(x \ast y = xu^{-1}y\). Finally, by iii, we have \(\psi(x) = u\theta(x)\theta(u^{-1})\), and therefore we must have \((H, \ast) \leq G_u\).

Further, \(H\) is invariant under \(\psi_u\) and hence \(\psi = \psi_u|_H\). So, we proved that \(H\) is a polyadic subgroup of \((G, f)\) iff there exists an element \(u\) such that \(H\) is a \(\psi_u\)-invariant subgroup of \(G_u\). Now, suppose such an \(H\) is normal in \((G, f)\). We show that \(H \trianglelefteq G_u\), equivalently

\[x^{-u} \ast h \ast x \in H\]

for all \(x \in G_u\) and \(h \in H\). Note that this last statement is equivalent to \(ux^{-1}hu^{-1}x \in H\). Since \(H\) is a normal polyadic subgroup, so by 2.3,

\[\theta^{-1}(x^{-1}h)x, \theta^{-1}(x^{-1}u)x \in H.\]

But \(H\) is \(\psi_u\)-invariant, so

\[\psi_u(\theta^{-1}(x^{-1}h)x), \psi_u(\theta^{-1}(x^{-1}u)x) \in H.\]
Therefore the following element also belongs to $H$,
\[
\psi_u(\theta^{-1}(x^{-1} h)x) \ast \psi_u(\theta^{-1}(x^{-1} u)x)^{-u} = \begin{align*}
wx^{-1}h\theta(x)\theta(u^{-1})u^{-1}u & (ux^{-1}u\theta(x)\theta(u^{-1}))^{-1}u \\
wx^{-1}hu^{-1}x.
\end{align*}
\]
This shows that $H \trianglelefteq G_u$. Note that the following is also automatically holds:
\[
\forall x \in G : \theta^{-1}(x^{-1}u)x \in H.
\]
Conversely, suppose there is a $u \in G$ such that $H$ is a $\psi_u$-invariant normal subgroup of $G_u$ and
\[
\forall x \in G : \theta^{-1}(x^{-1}u)x \in H.
\]
We show that $H$ is a normal polyadic subgroup. The equality
\[
\psi_u(\theta^{-1}(x^{-1}h)x) \ast \psi_u(\theta^{-1}(x^{-1}u)x)^{-u} = x^{-u} \ast h \ast x
\]
shows that $\psi_u(\theta^{-1}(x^{-1}h)x) \in H$, and since $H$ is invariant under $\psi_u$, so $\theta^{-1}(x^{-1}h)x \in H$. Therefore, $H$ is a normal polyadic subgroup. \hfill $\square$

**Lemma 4.2.** Suppose $u \in G$ is an arbitrary element. Then $H$ is a $\theta$-invariant normal subgroup of $(G, \cdot)$ iff $Hu$ is $\psi_u$-invariant normal subgroup of $G_u$.

**Proof.** Suppose $H \trianglelefteq_{\theta} G$. Then it can be checked that $Hu$ is a subgroup of $G_u$. For any $x \in G$ and $h \in H$ we have
\[
x^{-u} \ast hu \ast x = (ux^{-1}u)^{-1}(hu)u^{-1}x = wx^{-1}hx,
\]
which is clearly an element of $Hu$. So $Hu \trianglelefteq G_u$. Also
\[
\psi_u(hu) = w\theta(hu)\theta(u^{-1}) = w\theta(u),
\]
which shows that $Hu$ is $\psi_u$-invariant. Conversely, suppose $H$ is a $\psi_u$-invariant normal subgroup of $G_u$. We show that $Hu^{-1}$ is a $\theta$-invariant normal subgroup of $G$. We have
\[
xhu^{-1}x^{-1} = (xu) \ast h \ast (xu)^{-1}u^{-1},
\]
which belongs to $Hu^{-1}$. So $Hu^{-1} \trianglelefteq G$. Similarly, $\theta(hu^{-1}) = u^{-1}\psi_u(h)$ belongs to $u^{-1}H = e \ast H = H \ast e = Hu^{-1}$. Hence $Hu^{-1}$ is $\theta$-invariant. \hfill $\square$

Suppose $K$ is a $\theta$-invariant normal subgroup of $(G, \cdot)$. Then $\theta$ induces an automorphism of $G/K$ which we denote it by $\theta_K$ in what follows.

**Corollary 4.3.** Let $H \trianglelefteq (G, f)$. Then there exists an element $u$ such that $K = H \cdot u^{-1}$ is a $\theta$-invariant normal subgroup of $G$ and $\theta_K$ is an inner automorphism. The converse is also true.

**Proof.** First, we notice that $H \cdot u^{-1}$ is not a polyadic coset, but it is the set \{hu^{-1} : h \in H\}. Suppose $H \trianglelefteq (G, f)$. By 4.1, there is an element $u \in H$ such that $H$ is a $\psi_u$-invariant normal subgroup of $G_u$ and for any $x \in G$, we have $\theta^{-1}(x^{-1}u)x \in H$. Let $K = H \cdot u^{-1}$. By the above lemma $K$ is a $\theta$-invariant
normal subgroup of $G$. Now, $\theta^{-1}(x^{-1}u)x \in H$ and $H$ is $\psi_u$-invariant, so $x^{-u} \ast \psi_u(x) \in H$. Therefore $\psi_u(x) \ast H = x \ast H$. Since $H$ is normal in $G_u$, we have $H \ast \psi_u(x) = H \ast x$ and this is equivalent to $K \psi_u(x) = Kx$. Now $K$ is a normal subgroup of $G$, and hence $\psi_u(x)K = xK$. So $\theta(xu^{-1})K = u^{-1}xK$. If we put $y = xu^{-1}$, then $\theta(y)K = u^{-1}yuK$ and this proves that $\theta_K$ is an inner automorphism.

Conversely, suppose $K$ is a $\theta$-invariant normal subgroup of $G$ and $\theta_K = I_{aK}$. Then by a similar argument one can prove that $H = Ku^{-1} \trianglelefteq (G,f)$. □

We saw before that if $(G,f)$ is a non-strong GTS, then it is reduced. So we determine when a polyadic group belongs to the class GTS$^*$.

**Theorem 4.4.** A polyadic group $(G,f)$ is GTS$^*$ iff whenever $K$ is a $\theta$-invariant normal subgroup of $(G,\cdot)$ with $\theta_K$ inner, then $K = G$.

**Proof.** First let $(G,f)$ be GTS$^*$ and $K$ be a $\theta$-invariant normal subgroup of $G$ with $\theta_K$ inner. Then by the above corollary, there is a $u$ such that $H = Ku^{-1} \trianglelefteq (G,f)$. Hence $Ku^{-1} = G$ and so $K = G$. To prove the converse, suppose $H \trianglelefteq (G,f)$. So there is a $u$ such that $K = H \cdot u^{-1}$ satisfies our hypothesis. Therefore $K = G$ and hence $H = G$, proving that $(G,f)$ is GTS$^*$. □

As a final remark, the reader most notice that the binary groups $G_u$ which we used in this section, are in fact retract of $(G,f)$ in the case when $u$ is the skew for some other element. Suppose $u = \overline{a}$. In the group $\text{ret}_a(G,f)$, as we said in the introduction, the identity is $\overline{a}$ and if we define $c = f(\overline{a})$ and $\phi(x) = f(\overline{a},x,\overline{a}^{(n-2)})$, then by [21], we have

$$(G,f) = \text{der}_{\phi,c}(\text{ret}_a(G,f),\ast).$$

It is easy to see that $x \ast y = x(\overline{a})^{-1}y$ and hence $G_u = \text{ret}_a(G,f)$. But, note that in general, it is false to say that every $u$ is equal to the skew element of some $a$; there are polyadic groups in which the skew elements of any $x$ and $y$ are equal. So in the general case $G_u$ is not equal to any retract.

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