REALISATION OF BENDING MEASURED LAMINATIONS
BY KLEINIAN SURFACE GROUPS

SHINPEI BABA AND KEN’ICHI OHISHIKA

ABSTRACT. For geometrically finite Kleinian surface groups, Bonahon
and Otal proved the existence part, and partly the uniqueness part of
the bending lamination conjecture. In this paper, we generalise the
existence part to general Kleinian surface groups including geometrically
infinite ones. Along the way, we also prove the compactness of the set
of Kleinian surface groups realising an arbitrarily fixed data of bending
laminations and ending laminations. Our proof is independent of that
of Bonahon and Otal.

1. Introduction

The simultaneous uniformisation theorem by Bers [2] gives a parametri-
sation of the quasi-Fuchsian space for a closed oriented surface $S$ by the
product of two Teichmüller spaces $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$, where $\bar{S}$ denotes $S$ with
its orientation reversed. This was generalised by the work of Kra, Maskit,
Marden and Sullivan, which shows that for any Kleinian surface group, or
more generally for any freely indecomposable Kleinian group $G$, its quasi-
conformal deformation space is parametrised by $\mathcal{T}(\Omega_G/G)$, where $\Omega_G$ is the
region of discontinuity of $G$ in the Riemann sphere.

In his lecture notes [24], Thurston considered the convex core of the quo-
tient hyperbolic 3-manifold $\mathbb{H}^3/G$ for a Kleinian (surface) group $G$. He
noticed that the boundary of the convex core has two pieces of information:
the hyperbolic structure and the bending lamination. In contrast to the pre-
vious work of Bers et al, these are obtained by just considering the quotient
hyperbolic manifolds, without looking at structures at infinity. He seems to
have conjectured that both hyperbolic structures and bending laminations on
the boundaries serve as other kinds of parametrisation of the quasi-Fuchsian
space, or more generally, the quasi-conformal deformation space of a freely
indecomposable Kleinian group.

As for the first of these two, the hyperbolic structures on the boundaries of
convex cores, Sullivan’s lemma (see e.g. [9]) shows that they are within uni-
versally bounded distance in the corresponding Teichmüller spaces from the
conformal structures at infinity on $\Omega_G/G$. Nevertheless, it is still unknown
if they really give a parametrisation of the deformation space.

As for bending laminations, Bonahon-Otal showed in [4] that every pair of
measured laminations on a closed orientable surface $S$, without homotopic
components and without compact leaves with weight larger than or equal to \( \pi \), can be realised as bending laminations of a quasi-Fuchsian group corresponding to \( S \). In particular, when both of the measured laminations are weighted multi-curves, it was proved that the realising quasi-Fuchsian group is unique (up to conjugation). They also showed the same result for quasi-conformal deformation spaces of general freely indecomposable geometrically finite groups. Their result was generalised to freely decomposable Kleinian groups by Lecuire \[12\].

In this paper, we prove a generalisation of the existence part of this result by Bonahon-Otal to general Kleinian surface groups including geometrically infinite ones (Theorem \[3.1\](1)), by adding ending laminations to the data. We shall furthermore prove the compactness of the set of representations (up to conjugacy) realising given bending laminations and ending laminations (Theorem \[3.1\](2)). The proofs of both are different from and independent of the results by Bonahon-Otal. On the other hand, we do not have the partial uniqueness result as was given by Bonahon-Otal, for we cannot invoke the theory of cone manifold deformation, whose generalisation to the case of geometrically infinite groups does not exist for the moment.

We shall prove the existence part and the compactness by showing the properness of the following composition of maps in Theorem \[3.3\]. For a Kleinian surface group \( G \), the theory of Bers-Kra-Maskit-Marden-Sullivan gives a parametrisation of the quasi-conformal deformation space \( q: \mathcal{T}(\Omega_G/G) \to \mathcal{QH}(G) \), where \( \Omega_G \) denotes the region of discontinuity of \( G \), \( \mathcal{T}(\Omega_G/G) \) the Teichmüller space of the Riemann surface \( \Omega_G/G \), and \( \mathcal{QH}(G) \) the space of quasi-conformal deformations of \( G \) modulo conjugacy. Sending each Kleinian group to its bending lamination, we get a map \( b: \mathcal{QH}(G) \to \mathcal{ML}(\Omega_G/G) \), where \( \mathcal{ML} \) denotes the space of measured laminations. Let \( D \subset \mathcal{ML}(\Omega_G/G) \) be the set of measured laminations evidently unrealisable, whose exact definition is given in Theorem \[3.1\]. In this setting, Theorem \[3.3\] states that \( q \circ b \) is a proper, degree-1 map to \( \mathcal{ML}(\Omega_G/G) \setminus D \). This in particular says that \( q \circ b \) is surjective to \( \mathcal{ML}(\Omega_G/G) \setminus D \), and hence we obtain the existence part of the main result. The compactness part is derived from the properness of the map.

Our proof of Theorem \[3.3\] is divided into two parts: we shall first show the properness of the map \( b \circ q \) in Section \[5\] and then that \( b \circ q \) has degree 1 in Section \[6\]. In the first part, relying on the analysis of geometric limits, as was given in Ohshika-Soma \[20\] and Ohshika \[19\], we shall show that any sequence going to infinity in \( \mathcal{T}(\Omega_G/G) \) has image under \( b \circ q \) which cannot stay in a compact set disjoint from \( D \). In the second part, we shall show that \( q \circ b \) can be properly homotoped to a local degree-1 map which is constructed using the earthquake map. Since the degree is invariant under a proper homotopy, this implies that \( b \circ q \) has also degree 1.

The authors would like to express their hearty gratitude to the referee for his/her careful reading and suggestions, due to which we could in particular remove some mistakes in the first version.
2. Preliminaries

2.1. Basics of Kleinian groups. A Kleinian group is a discrete subgroup of $\text{PSL}_2(\mathbb{C})$. In this paper, we only consider Kleinian groups isomorphic to the fundamental groups of closed orientable surfaces of genus greater than 1, which we call Kleinian surface groups. A Kleinian group acts on the Riemann sphere $\hat{\mathbb{C}}$ by linear fractional transformations and on the hyperbolic space $\mathbb{H}^3$ by orientation-preserving isometries. By considering the Poincaré model of $\mathbb{H}^3$, the Riemann sphere $\hat{\mathbb{C}}$ is regarded as the sphere at infinity of $\mathbb{H}^3$. The action on $\hat{\mathbb{C}}$ is a continuous extension of the action on $\mathbb{H}^3$ if we regard $\hat{\mathbb{C}}$ as the points at infinity in this way. For a Kleinian group $G$, its limit set $\Lambda_G$ is the closure of the set of fixed points of non-trivial elements of $G$. The complement of $\Lambda_G$ in $\hat{\mathbb{C}}$ is called the region of discontinuity of $G$, and is denoted by $\Omega_G$.

The smallest convex subset of $\mathbb{H}^3$ containing all geodesics both of whose endpoints at infinity lie on $\Lambda_G$ is called the Nielsen convex hull and is denoted by $H_N$. Since $H_N$ is a closed convex subset invariant under $G$, its quotient $H_N/G$ is a closed convex subset of $\mathbb{H}^3/G$, which is a 3-submanifold except for the case when $G$ is Fuchsian. The quotient $H_N/G$ is called the convex core of $\mathbb{H}^3/G$ and is denoted by $C(\mathbb{H}^3/G)$. The Kleinian group $G$ is said to be geometrically finite if $C(\mathbb{H}^3/G)$ has finite volume.

2.2. Geodesic and measured laminations. In this section, we consider an orientable surface $S$ which may have punctures but has no boundary. We fix a complete hyperbolic metric on $S$ which makes punctures cusps. A geodesic lamination $\lambda$ on $S$ is a closed subset consisting of disjoint simple geodesics which do not tend to cusps. A geodesic constituting $\lambda$ is called a leaf. A geodesic lamination is said to be minimal when it does not have a non-empty proper sublamination. Any geodesic lamination is decomposed into disjoint finitely many minimal sublaminations, which we call minimal components, and isolated leaves spiralling around minimal components. We say that a geodesic lamination is arational when every component of its complement is either simply connected or an annulus containing a cusp.

A measured lamination is a geodesic lamination equipped with a transverse invariant measure. The support of a measured lamination is a geodesic lamination having the property that the entire lamination coincides with the union of its minimal components. We denote the support of a measured lamination $\lambda$ by $|\lambda|$. Conversely, a geodesic lamination with this property always supports a transverse invariant measure. We always assume that the support of a measured lamination is the entire lamination.

For two measured laminations $\lambda$ and $\mu$, their intersection number $\iota(\lambda, \mu)$ is defined to be the integral of the product of the transverse measures of $\lambda$ and $\mu$ over the surface $S$. In particular, when $c$ is a simple closed curve, we regard $c$ as having the unit Dirac transverse measure and define $\iota(\lambda, c)$ as such. For two geodesic laminations $\lambda$ and $\lambda'$ on a hyperbolic surface $S$, ...
and a point \( p \in \lambda \cap \lambda' \), we can consider the angle formed by \( \lambda \) and \( \lambda' \) at \( p \) taking the value in \([0, \pi/2]\), which we denote by \( \angle_p(\lambda, \lambda') \). We define the angle between \( \lambda \) and \( \lambda' \) to be \( \sup_{p \in \lambda \cap \lambda'} \angle_p(\lambda, \lambda') \) and denote it by \( \angle_S(\lambda, \lambda') \).

A measured lamination \( \lambda \) is said to be uniquely ergodic if the transverse measure of \( \lambda \) is a unique transverse measure on its support up to scaling. A pair of geodesic (or measured) laminations \( \lambda_1 \) and \( \lambda_2 \) is said to fill up \( S \) if every geodesic lamination \( \mu \) on \( S \) intersects \( \lambda_1 \) or \( \lambda_2 \) transversely.

For a measured lamination \( \lambda \) or a geodesic lamination supporting a measured lamination on \( S \), its minimal supporting surface \( S(\lambda) \) is an incompressible compact subsurface containing \( \lambda \) and is minimal with respect to the inclusion, which is unique up to isotopy.

Geodesic laminations and measured laminations defined above depend on the hyperbolic metric given on \( S \). Still for two complete hyperbolic metrics \( m, n \) on \( S \), and a geodesic lamination \( \lambda \) on \((S, m)\), there is a unique geodesic lamination \( \lambda' \) on \((S, n)\) which is isotopic to \( \lambda \). By identifying \( \lambda \) and \( \lambda' \) as above, we can talk about geodesic laminations and measured laminations without specifying a hyperbolic metric. We note the intersection number does not depend on the choice of a hyperbolic metric whereas the angle does depend on it.

Thurston proved that the space of measured laminations with the weak topology with respect to the transverse measures is homeomorphic to the Euclidean space of dimension \( 6g - 6 + 2b \), where \( g \) is the genus and \( b \) is the number of punctures of \( S \). We denote this space by \( \text{ML}(S) \) and call it the measured lamination space of \( S \). There is a PL local chart of \( \text{ML}(S) \), which can be constructed using train tracks as in the next section.

2.3. Train tracks. We shall define basic terms on train tracks in this subsection. We refer the reader to Penner-Harer [22] for a more detailed account.

A train track \( \tau \) on \( S \) is a \( C^1 \)-graph (i.e. a graph whose edges are \( C^1 \)-arcs and tangent to each other at vertices) embedded in \( S \) whose edges are called branches and whose vertices are called switches, such that no component of \( S \setminus \tau \) is a disc with one corner or an annulus with \( C^1 \)-smooth boundary.

A weight system \( \omega \) on a train track \( \tau \) is a system of non-negative numbers, called weights, given on branches of \( \tau \) such that at each switch the sum of the weights on the incoming branches coincides with the sum of the weights on the outgoing branches.

A geodesic lamination \( \lambda \) is said to be carried by a train track \( \tau \), when it can be regularly homotoped to an immersion in \( \tau \). Any geodesic lamination has a train track carrying it. In particular, if a measured lamination \( \lambda \) is carried by a train track \( \tau \), it induces a weight system on \( \tau \), by defining the weight of a branch to be the total transverse measure of the leaves lying there (after a regular homotopy). We denote this weight system induced from \( \lambda \) by \( w(\lambda) \). Conversely, for any weight system \( \omega \) on a train track, we can construct a measured lamination \( \lambda \) such that \( w(\lambda) = \omega \).
A train track $\tau$ is said to be recurrent if it has a weight system which takes only positive values, and transversely recurrent if for each branch $b$ of $\tau$, there is a simple closed curve intersecting $\tau$ essentially (i.e. without cobounding a bigon) and transversely with non-empty intersection with $b$. Train tracks which are both recurrent and transversely recurrent are called bi-recurrent. Every measured lamination is carried by a bi-recurrent train track. For a bi-recurrent train track $\tau$, the set of measured laminations inducing weight systems with positive values on $\tau$ forms an open set in $ML(S)$, which we denote by $U(\tau)$. For an arational measured lamination $\lambda$, the open sets $U(\tau)$ for all bi-recurrent train tracks $\tau$ carrying $\lambda$ form a base of neighbourhoods of $\lambda$ in $ML(S)$.

### 2.4. Pleated surfaces

Let $M$ be a hyperbolic 3-manifold, and $F$ an orientable surface, which we assume to be either closed or the interior of a compact surface. A pleated surface $f: (F, m) \to M$, where $m$ is a complete hyperbolic metric on $F$, is a continuous map taking each cusp of $(F, m)$ to a cusp of $M$ such that for every point $x \in F$, there is at least one geodesic segment containing $x$ in its interior which is mapped isometrically to a geodesic segment in $M$ by $f$. The set of points on $F$ at which only one direction is mapped geodesically constitutes a geodesic lamination $\mu$ on $(F, m)$. We call $\mu$ the pleating locus of the pleated surface $f$. More generally, if a geodesic (or a measured) lamination $\lambda$ is mapped geodesically by a pleated surface $f$, we say that $f$ realises $\lambda$.

The boundary component of the convex core of a hyperbolic 3-manifold is an example of pleated surface. It has moreover a special property that the surface is bent only in one direction. The pleating locus of such a surface has a transverse measure coming from bending angles, and is called the bending lamination when it is regarded as a measured lamination.

### 2.5. Ending laminations

By Bonahon’s tameness theorem [3], it is known that for any faithful discrete representation $\phi: \pi_1(S) \to PSL_2(\mathbb{C})$, there is an orientation-preserving homeomorphism $\Phi: S \times (0, 1) \to \mathbb{H}^3/\phi(\pi_1(S))$ which induces $\phi$ between their fundamental groups.

For a hyperbolic 3-manifold $M = \mathbb{H}^3/\phi(\pi_1(S))$, its non-cuspidal part, denoted by $M_0$, is the complement of $\epsilon$-thin cusp neighbourhoods for some fixed positive number $\epsilon$ smaller than the three-dimensional Margulis constant. The boundary of $M_0$ consists of incompressible open annuli. (In the case of general Kleinian groups, incompressible tori may appear. We do not have such components since we only deal with Kleinian surface groups.) By the relative core theorem ([23, 13]), there is a compact submanifold $C$ of $M_0$, called a relative compact core of $M_0$, such that the inclusion is a homotopy equivalence and $C \cap \partial M_0$ is the union of core annuli of the components of $\partial M_0$. A core curve of each component of $C \cap \partial M_0$ represents a generator of a maximal parabolic subgroup of $\phi(\pi_1(S))$. We call these curves parabolic curves. A relative compact core $C$ of $M_0$ is always homeomorphic to
$S \times [0,1]$ in our setting. We fix orientations on $S$ and $\mathbb{H}^3$, and assume the identification of $S \times [0,1]$ with $C$ to preserve the orientations.

An end of $M_0$ is an inverse limit (with respect to the inclusion) of complementary components of compact sets in $M_0$. Each component $U$ of $M_0 \setminus C$ contains a unique end $e$ of $M_0$, and also its closure contains a unique component $\Sigma$ of $\text{Fr}_{M_0} C$, where $\text{Fr}_{M_0}$ denotes the frontier as a subspace of $M_0$. In this situation, we say that $\Sigma$ faces the end $e$. The end $e$ is said to be geometrically finite when it has a neighbourhood disjoint from any closed geodesic, and otherwise geometrically infinite. If $e$ is geometrically finite, there is a boundary component $F$ of the convex core $C(M)$ such that $F \cap M_0$ is isotopic to $\Sigma$. This component $F$ in turn corresponds to a component of $\Omega_G/G$ which is regarded as lying at infinity.

When $e$ is geometrically infinite, it was proved in [3] that there is a sequence of simple closed curves $c_i$ on $\Sigma$ which are homotopic in $U \cup \Sigma$ to closed geodesics $c_i^*$ tending to the end. Such an end is called simply degenerate. Regarding $c_i$ as a geodesic lamination on $\Sigma$, after fixing any hyperbolic metric on $\Sigma$, we consider the Hausdorff limit $c_\infty$ of $c_i$, which is a geodesic lamination. It was shown by Thurston [24] and Bonahon [3] that $c_\infty$ has only one minimal component $\lambda$, which is called the ending lamination of $e$, and that $S(\lambda) = \Sigma$. The geodesic lamination $\lambda$ is the support of a measured lamination which is a limit of $\{r_ic_i\}$ in the space of measured laminations, where $r_i$ is a positive scalar.

The notion of ending lamination was first introduced by Thurston using pleated surfaces as follows. Let $\Sigma$ be a subsurface of $S$ as above. If the end $e$ facing $\Sigma$ is geometrically infinite, there is a sequence of pleated surfaces $\{f_i\}$ homotopic to the inclusion of $\Sigma$ which tends to $e$. For instance, in the setting of the preceding paragraph, pleated surfaces realising the simple closed curves $c_i$ are such pleated surfaces. Thurston considered the Hausdorff limit of the geodesic laminations realised by such pleated surfaces, and proved that the limit has only one minimal component, which is defined to be the ending lamination of $e$. He also showed that the ending lamination thus defined does not depend on the choice of pleated surfaces.

Recall that the relative compact core $C$ is identified with $S \times [0,1]$. When a parabolic curve lies on $S_+ = S \times \{1\}$ (resp. $S_- = S \times \{0\}$), we call it an upper (resp. a lower) parabolic curve. When the end $e$ is above $C$, i.e., when $\Sigma$ lies on $S \times \{1\}$ (resp. $S \times \{0\}$), we say that the ending lamination $\lambda$ is an upper (resp. a lower) ending lamination. It was also proved in [24] and [3] that for each upper (resp. lower) ending lamination $\lambda$ of $\mathbb{H}^3/\phi(\pi_1(S))$, each boundary component of $S(\lambda)$ is an upper (resp. lower) parabolic curve. We call the union of the parabolic curve and the ending laminations regarded as lying on $S_- \sqcup S_+$ the $\text{qi}(\text{quasi-isometric})$-end invariant of $\phi$ (or $\mathbb{H}^3/\phi(\pi_1(S))$). In particular, the union of those lying on $S_+$ (resp. $S_-$) is called the upper (resp. lower) qi-end invariant.
2.6. Deformation spaces. The space of faithful discrete representations of \( \pi_1(S) \) into \( \text{PSL}_2(\mathbb{C}) \) modulo conjugacy is denoted by \( \text{AH}(S) \). We endow \( \text{AH}(S) \) with the topology induced from the representation space. Although each element of \( \text{AH}(S) \) is a conjugacy class of representations, by abusing notation, we denote it by its representative.

The interior of \( \text{AH}(S) \) is known to be the quasi-Fuchsian space \( \text{QF}(S) \). When \( \phi \in \text{AH}(S) \) is quasi-Fuchsian, letting \( G = \phi(\pi_1(S)) \), the conformal structure on \( \Omega_G/G \) induces a marked conformal structures at infinity, on \( S \times \{0\} \) and \( S \times \{1\} \). We note that the identification of \( S \times \{0\} \) to a component of \( \Omega_G/G \) is orientation-preserving, but that of \( S \times \{1\} \) is orientation-reversing. We use the symbol \( \mathcal{T}(S) \) to denote the Teichmüller space of \( S \) with its orientation reversed. Then the conformal structure on \( \Omega_G/G \) determines a point in \( \mathcal{T}(S) \times \mathcal{T}(\tilde{S}) \). Bers showed that this identification of the conformal structure on \( \Omega_G/G \) and a point in \( \mathcal{T}(S) \times \mathcal{T}(\tilde{S}) \) gives a parametrisation of \( \text{QF}(S) \), which we denote by \( q: \mathcal{T}(S) \times \mathcal{T}(\tilde{S}) \to \text{QF}(S) \).

For a general point \( \phi \in \text{AH}(S) \) and \( G = \phi(\pi_1(S)) \), let \( \lambda_- \) and \( \lambda_+ \) be the lower and upper qi-end invariants of \( \mathbb{H}^3/\phi(\pi_1(S)) \). We recall that \( \lambda_- \) (resp. \( \lambda_+ \)) has the property that for any component \( \lambda \) of \( \lambda_- \) (resp. \( \lambda_+ \)), every boundary component of \( S(\lambda) \) is contained in \( \lambda_- \) (resp. \( \lambda_+ \)). The quotient of the region of discontinuity \( \Omega_G/G \) is identified with the disjoint union of \( S_- \setminus S(\lambda_-) \) and \( S_+ \setminus S(\lambda_+) \), which we denote by \( \Sigma_- \) and \( \Sigma_+ \). The ending lamination theorem, proved by Minsky and Brock-Canary-Minsky [14, 17], shows that any Kleinian group in \( \text{AH}(S) \) having \( \lambda_- \) and \( \lambda_+ \) as lower and upper qi-end invariants is a quasi-conformal deformation of \( G \). Therefore, we denote the quasi-conformal deformation space of \( G \) by \( \text{QH}_{\lambda_-,-\lambda_+} \). The theory of Bers-Kra-Maskit-Marden-Sullivan shows that the conformal structures at infinity give a parametrisation \( q: \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \to \text{QH}_{\lambda_-,-\lambda_+} \).

2.7. Geometric limits. For a sequence of Kleinian groups \( \{G_i\} \), we say that \( \{G_i\} \) converges to a Kleinian group \( \Gamma \) geometrically if (i) every element \( \gamma \in \Gamma \) is a limit of some sequence \( \{g_i \in G_i\} \), and (ii) for every convergent subsequence \( \{g_{i_j} \in G_{i_j}\} \), its limit lies in \( \Gamma \). The geometric convergence is equivalent to the pointed Gromov-Hausdorff convergence of the corresponding hyperbolic 3-manifolds: fixing a basepoint \( x \in \mathbb{H}^3 \) and letting \( x_i \) and \( x_\infty \) be the projections of \( x \) to \( \mathbb{H}^3/G_i \) and \( \mathbb{H}^3/\Gamma \) respectively, the sequence of pointed hyperbolic 3-manifolds \( \{(\mathbb{H}^3/G_i, x_i)\} \) converges to \( (\mathbb{H}^3/\Gamma, x_\infty) \) in the sense of Gromov-Hausdorff if and only if \( \{G_i\} \) converges to \( \Gamma \) geometrically. Due to this fact, we also refer to Gromov-Hausdorff limits as geometric limits. The compactness of Gromov-Hausdorff topology shows that every sequence of non-elementary Kleinian groups has a geometric limit after passing to a subsequence. We recall that, by definition, if \( \{(\mathbb{H}^3/G_i, x_i)\} \) converges to \( (\mathbb{H}^3/\Gamma, x_\infty) \) in the sense of Gromov-Hausdorff, then there is a \( K_i \)-bi-Lipschitz diffeomorphism, which is called an approximate isometry between the \( R_i \)-ball around \( x_i \) and the \( K_i R_i \)-ball around \( x_\infty \), with \( K_i \to 1 \) and \( R_i \to \infty \).
Suppose that a sequence \( \{ \phi_i \} \) in \( \text{AH}(S) \) converges to \( \psi \in \text{AH}(S) \). In this situation, we always assume that we take representatives so that \( \{ \phi_i \} \) converges to \( \psi \) as genuine representations from \( \pi_1(S) \) into \( \text{PSL}_2(\mathbb{C}) \). Then, passing to a subsequence, \( \{ \phi_i(\pi_1(S)) \} \) converges to some Kleinian group \( \Gamma \). From the definition of geometric limits, it is easy to see that \( \Gamma \) contains \( \psi(\pi_1(S)) \). If \( \Gamma = \psi(\pi_1(S)) \), we say that \( \{ \phi_i \} \) converges to \( \psi \) strongly.

In Ohshika-Soma [20] a classification of geometric limits of Kleinian surface groups was given. An alternative description was also given in [19]. We shall now review some of the results there which will be used in the proof of the main theorem, in particular, in Section 5.2.

Let \( \{ \phi_i \} \) be a sequence in \( \text{AH}(S) \), and suppose that \( \{ \phi_i(\pi_1(S)) \} \) converges geometrically to a Kleinian group \( \Gamma \). The following is a paraphrase of a part of Theorem A in [20].

**Theorem 2.1.** The non-cuspidal part \( (\mathbb{H}^3/\Gamma)_0 \) is topologically embedded in \( S \times (0,1) \) in such a way that the following hold.

(a) Every end of \( (\mathbb{H}^3/\Gamma)_0 \) is mapped to a horizontal surface \( \Sigma \times \{ t \} \), where \( \Sigma \) is an incompressible subsurface of \( S \) and \( t \) lies in \( [0,1] \).

(b) Every geometrically finite end of \( (\mathbb{H}^3/\Gamma)_0 \) lies on either \( S \times \{ 0 \} \) or \( S \times \{ 1 \} \).

(c) Each boundary component of the convex core of \( (\mathbb{H}^3/\Gamma)_0 \) is a Gromov-Hausdorff limit of boundary components of convex cores of \( (\mathbb{H}^3/\phi_i(\pi_1(S)))_0 \) with some base points.

(d) Every geometrically infinite end is either simply degenerate or an accumulation set of countably many torus cusps or simply degenerate ends or both.

Identifying \( (\mathbb{H}^3/\Gamma)_0 \) with the image of its embedding in \( S \times (0,1) \) as above, we can talk about the horizontal direction and the vertical direction in \( (\mathbb{H}^3/\Gamma)_0 \).

The following lemma, which is [19] Lemma 4.13 and also can be found in [6] §3, will be used in Section 5.2.

**Lemma 2.2.** In the setting as above, suppose moreover that \( \{ \phi_i \} \) converges to \( \psi \in \text{AH}(S) \). Then the image of the inclusion of \( \psi(\pi_1(S)) \) into \( (\mathbb{H}^3/\Gamma)_0 \) is represented by an immersion \( f_\infty : S \to (\mathbb{H}^3/\Gamma)_0 \) which is horizontal except for disjoint annuli in \( S \) whose images wrap around torus cusps.

An immersion \( f_\infty \) as above is called an algebraic locus. We note that for an approximate isometry \( \rho_i \) from \( \mathbb{H}^3/\phi_i(\pi_1(S)) \) to \( \mathbb{H}^3/\Gamma \), the composition \( \rho_i^{-1} \circ f_\infty \) induces the same isomorphism as \( \phi_i \) between the fundamental groups for sufficiently large \( i \).

By Theorem 2.1 and Lemma 2.2, we can show the following.

**Corollary 2.3.** Suppose that \( \{ \phi_i \} \) in \( \text{AH}(S) \) converges to \( \psi \in \text{AH}(S) \). If \( (\mathbb{H}^3/\Gamma)_0 \) does not have a torus boundary component and every geometrically infinite end is homotopic into an algebraic locus, then \( \{ \phi_i \} \) converges to \( \psi \) strongly.
Proof. Since we assumed that \((\mathbb{H}^3/\Gamma)_0\) does not have a torus boundary component, the algebraic locus \(f_\infty\) in Lemma 2.2 cannot wrap around a boundary component, and hence is a horizontal surface. By assumption, every geometrically infinite end can be lifted to the algebraic limit, and hence simply degenerate. This implies that in the embedded image of \((\mathbb{H}^3/\Gamma)_0\) in \(S \times (0, 1)\), there is no other end on the side farther from \(f_\infty(S)\) of each geometrically infinite end. Therefore, we can isotope the embedding so that every geometrically infinite end lies on \(S \times \{0\} \cup S \times \{1\}\).

By Theorem 2.1(a), every geometrically finite also lies on \(\Sigma \times \{0, 1\}\). Each boundary component of \((\mathbb{H}^3/\Gamma)_0\), which is an open annulus, has both ends on the same level, either on \(S \times \{0\}\) or \(S \times \{1\}\), for it cannot pass through the algebraic locus. This means that \((\mathbb{H}^3/\Gamma)_0\) coincides with the complement of finitely many half solid tori lying on a neighbourhood of \(S \times \{0\}\) or \(S \times \{1\}\). Therefore it is homeomorphic to \(S \times (0, 1)\), and hence \(\psi(\pi_1(S)) = \Gamma\). □

As explained in [20, 19], if a sequence \(\{\phi_i\}\) in \(\text{AH}(S)\) converges geometrically to a Kleinian group \(\Gamma\), a geometric limit of (uniform) bi-Lipschitz model manifolds of \((\mathbb{H}^3/\phi_i(S))_0\) (due to Minsky [14]) serves as a bi-Lipschitz model manifold of \((\mathbb{H}^3/\Gamma)_0\). Suppose that \(\mathbb{H}^3/\phi_i(\pi_1(S))\) has a lower qi-end invariant \(\lambda_-\) and an upper qi-end invariant \(\lambda_+\). Let \(\Sigma_-\) and \(\Sigma_+\) be \(S \setminus S(\lambda_-)\) and \(S \setminus S(\lambda_+)\) respectively, and \(m_i^-\) and \(m_i^+\) the structures at infinity of \(\mathbb{H}^3/\phi_i(\pi_1(S))\) on \(\Sigma_-\) and \(\Sigma_+\) respectively. Let \(P_i^-\) and \(P_i^+\) be shortest pants decompositions (with respect to the hyperbolic length) of \((\Sigma_-, m_i^-)\) and \((\Sigma_+, m_i^+)\) respectively. By adding a shortest transversal simple closed curve to each component of \(P_i^-\) (resp. \(P_i^+\)) disjoint from all the other components, we get shortest markings \(M_i^-\) (resp. \(M_i^+)\). Minsky’s model manifold is constructed from the hierarchy of tight geodesics connecting the generalised markings \(M_i^- \cup \lambda_-\) and \(M_i^+ \cup \lambda_+\).

In this setting, we have the following lemmas.

Lemma 2.4. Suppose that \(\{\phi_i(\pi_1(S))\}\) converges geometrically to \(\Gamma\). Let \(l\) be a non-contractible simple closed curve on \(S\). We assume that the translation length of \(\phi_i(l)\) is bounded from above as \(i \to \infty\). Let \(\epsilon\) be a positive constant less than the three-dimensional Margulis constant.

1. If the \(\epsilon\)-Margulis tube around the closed geodesic representing \(\phi_i(l)\) converges geometrically to a torus cusp neighbourhood in \(\mathbb{H}^3/\Gamma\) as \(i \to \infty\), then both length\(_{m_i^+}(l)\) and length\(_{m_i^-}(l)\) (if one or both of them are defined) are bounded from below by a positive constant, and \(d_{\text{A}(l)}(M_i^- \cup \lambda_-, M_i^+ \cup \lambda_+)\) goes to \(\infty\). Here \(A(l)\) denotes an annular neighbourhood of \(l\) and \(d_{\text{A}(l)}\) the distance in the curve complex of \(A(l)\). Conversely if both length\(_{m_i^+}(l)\) and length\(_{m_i^-}(l)\) are bounded from below by a positive constant and \(d_{\text{A}(l)}(M_i^- \cup \lambda_-, M_i^+ \cup \lambda_+)\) goes to \(\infty\), then the \(\epsilon\)-Margulis tube around the closed geodesic representing \(\phi_i(l)\) converges geometrically to either a torus cusp neighbourhood or a \(\mathbb{Z}\)-cusp attached to a geometrically infinite end.
(2) Suppose moreover that \( \{\phi_i\} \) converges in \( \text{AH}(S) \), and let \( k \) be a simple closed curve intersecting \( l \) essentially. If the \( \epsilon \)-Margulis tube around the closed geodesic representing \( \phi_i(l) \) converges geometrically to a torus cusp lying above (resp. below) an algebraic locus as \( i \to \infty \), then \( d_{A(l)}(k, M_i^+ \cup \lambda_+) \) (resp. \( d_{A(l)}(k, M_i^- \cup \lambda_-) \)) goes to \( \infty \). In the case when an algebraic locus wraps around a torus cusp, we regard the torus cusp as lying both above and below the locus. Conversely, if \( d_{A(l)}(k, M_i^+ \cup \lambda_+) \) (resp. \( d_{A(l)}(k, M_i^- \cup \lambda_-) \)) goes to \( \infty \), then the \( \epsilon \)-Margulis tube around the closed geodesic representing \( \phi_i(l) \) converges geometrically to either a torus cusp neighbourhood or a \( \mathbb{Z} \)-cusp neighbourhood attached to a geometrically infinite end.

Proof. Recall that the model manifold of \( (\mathbb{H}^3/\Gamma)_0 \) can be taken to be a geometric limit of models constructed from hierarchies of tight geodesics. If \( \text{length}_{m_i}(l) \) or \( \text{length}_{\overline{m}_i}(l) \) goes to \( 0 \) after passing to a subsequence, then the corresponding geometrically finite block of the model manifold of \( (\mathbb{H}^3/\phi_i(\pi_1(S)))_0 \) splits along a simple closed curve representing \( l \) as \( i \to \infty \). This shows that the geometric limit of the Margulis tubes around the closed geodesic representing \( \phi_i(l) \) must be a \( \mathbb{Z} \)-cusp in \( \mathbb{H}^3/\Gamma \) in this case.

Now suppose that \( \text{length}_{m_i}(l) \) and \( \text{length}_{\overline{m}_i}(l) \) are bounded from below by a positive constant if one or both of them are defined. Then by [14] Lemma 9.4, we see that the Margulis tubes around the closed geodesic corresponding to \( \phi_i(l) \) converge geometrically to a torus cusp neighbourhood if and only if \( d_{A(l)}(M_i^- \cup \lambda_-, M_i^+ \cup \lambda_+) \) goes to \( \infty \). Thus we are done for the part (1).

A proof of the part (2) can be found in [19] Proof of Theorem 5.2. We summarise its argument here. Consider the situation where the Margulis tube \( V_l \) around the closed geodesic corresponding to \( \phi_i(l) \) converges to a torus cusp neighbourhood \( V_\infty \). First suppose that the algebraic locus does not wrap around \( V_\infty \) and that \( V_\infty \) lies above an algebraic locus. Recall that Minsky’s model manifold is constructed from a hierarchy \( H_i \) of tight geodesics in the curve complex of (subsurfaces of) \( S \). If the Margulis tube \( V_l \) converges to a torus cusp neighbourhood, then the annular neighbourhood \( A(l) \) supports a geodesic \( g_i \) in \( H_i \) whose length goes to \( \infty \). The last vertex of \( g_i \) is within bounded distance from \( \pi_{A(l)}(M_i^+ \cup \lambda_+) \) as \( i \to \infty \), where \( \pi_{A(l)} \) denotes the projection between the curve complexes \( \mathcal{CC}(S) \) and \( \mathcal{CC}(A(l)) \) induced by restricting curves to \( A(l) \). On the other hand, since the torus cusp lies above an algebraic locus, the initial vertex of \( g_i \) is within bounded distance from \( \pi_{A(l)}(k) \) as \( i \to \infty \). Thus, we have \( d_{A(l)}(k, M_i^+ \cup \lambda_+) \to \infty \).

The same argument works also in the case when \( V_\infty \) lies below an algebraic locus. (See [19] Claim 5.3 for more details.)

Next suppose that an algebraic locus wraps \( n \)-times around \( V_\infty \) with \( n \neq 0 \). Then there is a sequence of integers \( r(i) \) whose absolute values go to \( \infty \) such that the initial vertex of \( h_i \) is within bounded distance from \( \pi_{A(l)}(\tau_i^{nr(i)}(k)) \) and the last vertex is within bounded distance from
\[ \pi_{A(l)}(\tau_l^{(n+1)r(i)}(k)), \text{ where } \tau_l \text{ denotes the Dehn twist around } l. \] The initial vertex of \( h_i \) is also within bounded distance from \( \pi_{A(l)}(M_i^- \cup \lambda_-) \), and the last vertex of \( h_i \) is also within bounded distance from \( \pi_{A(l)}(M_i^+ \cup \lambda_+) \). Therefore both \( d_{A(l)}(k, M_i^- \cup \lambda_-) \) and \( d_{A(l)}(k, M_i^+ \cup \lambda_+) \) go to \( \infty \) in this case.

Finally, suppose that \( d_{A(l)}(k, M_i^- \cup \lambda_-) \) goes to \( \infty \). Then by \cite{14, Lemma 9.4} again, we see that the boundary of the Margulis tube \( \partial V_i \) either converges to the boundary of a torus cusp neighbourhood or diverges to give rise to an open annulus whose end is attached to a lower end of \( (\mathbb{H}^3/\Gamma)_0 \). Since \( d_{A(l)}(k, M_i^- \cup \lambda_-) \to \infty \), the curve \( l \) cannot be contained in the shortest pants decomposition \( M_i \). Therefore, the upper geometrically finite block of the model manifold \( \partial V_i \) cannot split along \( l \), and hence neither end of the geometric limit of \( \partial V_i \) cannot tend to \( S \times \{0\} \). It follows that the end of \( (\mathbb{H}^3/\Gamma)_0 \) to which an end of the geometric limit of \( \partial V_i \) tends must be geometrically infinite by Theorem 2.1-(b). Thus we have shown that the second part of (2) holds. \( \square \)

**Lemma 2.5.** Suppose that \( \{\phi_i\} \) converges in \( \text{AH}(S) \) to \( \psi \). Let \( \Sigma \) be an open incompressible subsurface of \( S \), with negative Euler characteristic. Then, \( \{P_i^+\Sigma\} \) (resp. \( \{P_i^-\Sigma\} \)) converges (in the Hausdorff topology) to a geodesic lamination containing a minimal component \( \lambda \) whose minimal supporting surface is \( \Sigma \) if and only if there is an upper (resp. a lower) simply degenerate end of \( (\mathbb{H}^3/\psi(\pi_1(S)))_0 \) whose ending lamination is \( \lambda \).

**Proof.** This is contained in \cite{19, Proposition 4.18, Theorem 5.2} and their generalisations explained in \cite{19, §10}, or alternatively in Brock-Bromberg-Canary-Lecuire \cite{6, Theorem 1.2}. \( \square \)

3. **Main theorems**

The purpose of this paper is to prove the following theorem, which is a generalisation of the existence part of the theorem of Bonahon-Otal \cite{4} to general, possibly geometrically infinite Kleinian surface groups. We do not have the uniqueness part of Bohon-Otal’s theorem (in the case when the bending laminations are multi-curves), but instead have compactness for the set of Kleinian groups realising given data of qi-end invariants and bending laminations.

**Theorem 3.1.** Let \( S \) be a closed oriented surface of genus greater than 1, and \( S_-, S_+ \) two copies of \( S \), where \( S_+ \) has the same orientation as \( S \) whereas \( S_- \) has the opposite orientation.

- Let \( \lambda_- \) and \( \lambda_+ \) be (possibly empty) geodesic laminations without non-compact isolated leaves on \( S_- \) and \( S_+ \) respectively, such that for every component \( \lambda \) of \( \lambda_- \) (resp. \( \lambda_+ \)), each boundary component of the minimal supporting surface \( S(\lambda) \) is isotopic to a closed geodesic contained in \( \lambda_- \) (resp. \( \lambda_+ \)).

- Let \( \mu_- \) and \( \mu_+ \) be measured laminations on \( S_- \) and \( S_+ \) such that
(a) $\lambda_\cap \mu_- = \emptyset$ and $\lambda_+ \cap \mu_+ = \emptyset$;
(b) neither $\mu_-$ nor $\mu_+$ contains a compact leaf of weight larger than or equal to $\pi$;
(c) $\lambda_- \sqcup \mu_-$ and $\lambda_+ \sqcup \mu_+$ fill up $S$ if we identify both $S_-$ and $S_+$ with $S$.

Then the following hold.

(1) There is $\rho \in \text{AH}(S)$ such that
   (i) the hyperbolic 3-manifold $\mathbb{H}^3/\rho(\pi_1(S))$ has $\lambda_-$ as its lower qi-end invariant and $\lambda_+$ as its upper qi-end invariant, and
   (ii) the hyperbolic 3-manifold $\mathbb{H}^3/\rho(\pi_1(S))$ realises $\mu_-$ and $\mu_+$ as the bending laminations on the lower and the upper boundaries respectively of its convex core $C(\mathbb{H}^3/\rho(\pi_1(S)))$.

(2) The set of all $\rho \in \text{AH}(S)$ satisfying the condition (1) is a compact subset of $QH_{\lambda_-,\lambda_+}$.

Remark 3.2. In the theorem above, the existence of $\mu_-, \mu_+$ satisfying (c) imposes on $\lambda_-, \lambda_+$ the condition that they share no minimal component.

This theorem is an immediate consequence of following Theorem 3.3, which states something a bit stronger.

Let $\lambda_-$ and $\lambda_+$ be geodesic laminations on $S_-$ and $S_+$ as in Theorem 3.1 which share no minimal component. Let $\Sigma_-$ and $\Sigma_+$ be the complements $S_- \setminus S(\lambda_-)$ and $S_+ \setminus S(\lambda_+)$ respectively. As we explained in Section 2.6 we have a parametrisation $q: \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \to QH_{\lambda_-,\lambda_+}$. Let $b: QH_{\lambda_-,\lambda_+} \to ML(\Sigma_-) \times ML(\Sigma_+)$ be the map taking $\rho \in QH_{\lambda_-,\lambda_+}$ to the bending lamination of $C(\mathbb{H}^3/\rho(\pi_1(S)))$.

**Theorem 3.3.** Let $D$ be the subset of $ML(\Sigma_-) \times ML(\Sigma_+)$ consisting of measured laminations which do not satisfy at least one of the conditions (b) and (c) in Theorem 3.1. Then the map $b \circ q: \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \to ML(\Sigma_-) \times ML(\Sigma_+) \setminus D$ is a proper, degree-1 map.

4. Bending laminations

We present some lemmas which will be useful for the proofs of our main theorems.

**Lemma 4.1.** For every $K > 0$, there is a positive constant $L(K)$ depending only on $K$ which goes to 0 as $K \to 0$ with the following property. For every $\rho \in \text{AH}(S)$, letting $M = \mathbb{H}^3/\rho(\pi_1(S))$, if $\Sigma$ is a boundary component of $C(M)$ with bending lamination $\lambda$ and $c$ is a simple closed curve on $\Sigma$ with length less than $K$, then $\iota(c, \lambda) < L(K)$.

**Proof.** This is a direct consequence of the boundedness of the average bending measure proved by Bridgeman [5].

As a consequence of this lemma, we have the following.
Corollary 4.2. Consider the situation in Theorem 3.1, and let \( \Sigma \) be a component of \( \Sigma_+ \cup \Sigma_- \). Let \( \{g_i\} \) be a sequence in \( \mathcal{F}(\Sigma_-) \times \mathcal{F}(\Sigma_+) \). Let \( \gamma_i \) be a shortest pants decomposition of \( g_i|\Sigma \). Then the sequence \( \{\iota(b \circ q(g_i), \gamma_i)\} \) is bounded.

Proof. By Bers’s lemma, there is a constant \( C \) depending only on \( S \) such that each component of the shortest pants decomposition of \( (\Sigma, g_i) \) has length less than \( C \). Therefore, by Lemma 4.1, \( \iota(b \circ q(g_i), \gamma_i) \) is bounded as \( i \rightarrow \infty \). \( \square \)

Lemma 4.3. For any \( \epsilon > 0 \), there is a positive constant \( \delta > 0 \) depending only on \( \epsilon \) satisfying the following.

Let \( \{\phi_i\} \) be a sequence in \( \mathcal{QH}_{\lambda_- \lambda_+} \), and \( \Sigma \) a component of \( \Sigma_+ \cup \Sigma_- \). Let \( \mu_i \) be the restriction of the bending lamination of the convex core of \( \mathbb{H}^3/\phi_i(\tau_1(S)) \) to (the component corresponding to) \( \Sigma \). We denote the hyperbolic metric on \( \Sigma \) as a boundary component of the convex core \( C(\mathbb{H}^3/\phi_i(\tau_1(M))) \) by \( \mu_i^\Sigma \).

Suppose

(i) that \( \{\mu_i\} \) converges to a measured lamination \( \mu_\infty \),

(ii) that \( \mu_\infty \) is decomposed into disjoint (possibly empty) sublaminations \( \mu_1^\Sigma \) and \( \mu_2^\Sigma \),

(iii) and that \( c \) is a simple closed curve on \( \Sigma \) such that \( \iota(c, \mu_\infty^1) < \delta \) and \( \angle_{(\Sigma, m_i^\Sigma)}(c, \mu_\infty^2) < \delta \) for large \( i \).

Let \( c_i \) be the closed geodesic on \( \Sigma \) with respect to \( m_i^\Sigma \) freely homotopic to \( c \), and \( c_i^* \) the closed geodesic in \( \mathbb{H}^3/\phi_i(\tau_1(S)) \) freely homotopic to \( \phi_i(c) \).

Then we have

\[
1 \leq \frac{\text{length}_{(\Sigma, m_i^\Sigma)}(c_i)}{\text{length}_{\mathbb{H}^3/\phi_i(\tau_1(S))}(c_i^*)} \leq 1 + \epsilon.
\]

Proof. Since the length of \( c_i^* \) is less than or equal to that of \( c_i \), the first inequality obviously holds.

Since the Hausdorff limit of the support \( |\mu_i| \) is a geodesic lamination containing \( \mu_\infty = \mu_1^\Sigma \cup \mu_2^\Sigma \), for sufficiently large \( i \), we can decompose \( c_i \) into two parts \( c_i^2 \) and \( c_i^1 \) such that \( \angle_{(\Sigma, m_i^\Sigma)}(c_i^2, \mu_i) < 2\delta \) and \( \iota(c_i^1, \mu_i) < 2\delta \). By applying [11, Proposition 4.1] for \( c_i^2 \) and [8, Corollary 4.6] for \( c_i^1 \), we obtain the second inequality. \( \square \)

5. Properness

In this section, we shall prove the properness of \( b \circ q \) in Theorem 3.3. The argument is by contradiction. Suppose that \( b \circ q \) is not proper. Then, there is a sequence \( \{(m_i^- , m_i^+)\} \subset \mathcal{F}(\Sigma_-) \times \mathcal{F}(\Sigma_+) \) without a convergent subsequence such that \( \{b \circ q(m_i^-, m_i^+)\} \) converges in \( \mathcal{M}(\Sigma_-) \times \mathcal{M}(\Sigma_+) \setminus D \) to a measured lamination \( \nu \) on \( \Sigma_+ \cup \Sigma_- \).

We are going to analyse a geometric limit of \( \{\mathbb{H}^3/q(m_i^-, m_i^+)\} \), making use of results in Ohshika-Soma [20] and Ohshika [19]. Let \( \phi_i : \tau_1(S) \rightarrow \text{PSL}_2(\mathbb{C}) \)
be a representative of $q(m_i^-, m_i^+)$ for each $i$. We take the $\phi_i$ to converge to some $\psi \in AH(S)$ if $\{q(m_i^-, m_i^+)\}$ converges in $AH(S)$.

We divide our argument into the following three cases:

(a) The case when $\{q(m_i^-, m_i^+)\}$ converges in $AH(S)$ strongly after passing to a subsequence.

(b) The case when $\{q(m_i^-, m_i^+)\}$ converges in $AH(S)$ after passing to a subsequence, but not strongly.

(c) The case when $\{q(m_i^-, m_i^+)\}$ diverges in $AH(S)$ (even after passing to a subsequence).

In the cases (a) and (b), we put a basepoint $\tilde{x}$ in $\mathbb{H}^3$, and by projecting it to $\mathbb{H}^3/\phi_i(\pi_1(S))$, we get a basepoint $x_i$. Taking the Gromov-Hausdorff limit of $(\mathbb{H}^3/\phi_i(\pi_1(S)), x_i)$, we obtain a geometric limit $(M_\infty, x_\infty)$ which is covered by the algebraic limit $\mathbb{H}^3/\psi(\pi_1(S))$.

Now we start to show that in each of the three cases, we get a contradiction.

5.1. Case (a). In this case, $M_\infty = \mathbb{H}^3/\psi(\pi_1(S))$. There are three possibilities for $\psi$:

(a)-(1) The case when $\psi$ is also contained in $QH_{\lambda_-, \lambda_+}$.

(a)-(2) The case when $(\mathbb{H}^3/\psi(\pi_1(S)))_0$ has a ‘new simply degenerate end’ not corresponding to a simply degenerate end of $(\mathbb{H}^3/\phi_i(\pi_1(S)))_0$.

(a)-(3) The case when $\mathbb{H}^3/\psi(\pi_1(S))$ does not have a new simply degenerate end, but has a ‘new $\mathbb{Z}$-cusp’ not corresponding to a $\mathbb{Z}$-cusp of $(\mathbb{H}^3/\phi_i(\pi_1(S)))_0$.

Since we are assuming that $\{(m_i^-, m_i^+)\}$ has no convergent subsequence, we can exclude the case (a)-(1).

We first assume that the condition (a)-(2) holds. Let $\Psi: S \times (0, 1) \to \mathbb{H}^3/\psi(\pi_1(S))$ be an orientation-preserving homeomorphism inducing $\psi$ between the fundamental groups. For brevity of description, we now assume that a new simply degenerate end is lower. The case when the new simply degenerate end is upper can be dealt with in the same way.

Let $\gamma_i$ be a shortest pants decomposition of $(\Sigma_-, m_i^-)$. Let $\gamma_\infty$ be the Hausdorff limit of the $\gamma_i$ regarded as geodesic laminations. Since we assumed that there is a new lower simply degenerate end, by Lemma 2.5, there is a minimal component $\ell$ of $\gamma_\infty$ which is the ending lamination of such an end. Let $\Sigma$ be the minimal supporting surface of $\ell$. Let $\lambda$ be a measured lamination supported on $\ell$. Let $\delta > 0$ be the positive constant given in Lemma 1.3 for $\gamma_i$. Now, take an essential simple closed curve $d$ on $\Sigma$ such that $\iota(\nu, d) < \delta$. Since we have $\iota(\lambda, d) > 0$ from the fact that $\Sigma = S(\lambda)$, and $\gamma_i$ converges to $\gamma_\infty$ containing $\ell = |\lambda|$ in the Hausdorff topology, the intersection number $\iota(\gamma_i, d)$ goes to $\infty$, which implies the length of $d$ with respect to $m_i^-$ also goes to $\infty$, for $\gamma_i$ is a shortest pants decomposition. On the other hand, $\{(b \circ q(m_i^-, m_i^+))\}$ converges to $\nu$ by our assumption. Then, by setting $\mu_\infty$
in the statement of Lemma 4.3 to be \( \nu \) and \( \mu_\infty^2 \) to be empty, it follows that the translation length of \( \phi_i(d) \) goes to \( \infty \) as \( i \to \infty \), contradicting the assumption that \( \{ \phi_i \} \) converges.

We now assume the condition (a)-(3) that \( \mathbb{H}^3/\psi(\pi_1(S)) \) has a new \( \mathbb{Z} \) cusp, represented by a non-contractible, non-peripheral simple closed curve \( c \) on \( \Sigma_\pm \), but does not have a new simply degenerate end. In the same way as in the preceding paragraph, we can assume that \( c \) lies on \( \Sigma_- \). Since we assumed that there is no new simply degenerate end, if the \( \mathbb{Z} \)-cusp represented by \( \psi(c) \) touches a simply degenerate end, it corresponds to a simply degenerate end of \( (\mathbb{H}^3/\phi_i(\pi_1(S))_0 \), which implies that \( \phi_i(c) \) is parabolic, contradicting our assumption that \( c \) is a new parabolic curve. Therefore, the lower \( \mathbb{Z} \)-cusp represented by \( c \) has geometrically finite ends on its both sides, which may coincide. Therefore, the lower boundary of the convex core of \( \mathbb{H}^3/\psi(\pi_1(S)) \) has bending angle \( \pi \) along \( c \). Since we assumed that \( \{ \phi_i \} \) converges to \( \psi \) strongly, the convex core of \( \mathbb{H}^3/\phi_i(\pi_1(S)) \) converges to that of \( \mathbb{H}^3/\psi(\pi_1(S)) \) (see [17, Proof of Theorem 5]), and hence the bending angle along \( c \subset \Sigma_- \) of the convex core of \( \mathbb{H}^3/\phi_i(\pi_1(S)) \) converges to \( \pi \). This means that \( \{ b \circ q(m_i^-, m_i^+) \} \) tends to a point in \( D \), contradicting our assumption. Thus, in every sub-case of the case (a), we have obtained a contradiction.

We state what we have proved in the last paragraph as a lemma to refer to it in the following section.

**Lemma 5.1.** Suppose that \( \{ \phi_i \} \) converges to \( \psi \) strongly, and that \( \mathbb{H}^3/\psi(\pi_1(S)) \) has a \( \mathbb{Z} \)-cusp whose core curve is a simple closed curve \( c \) on a component \( \Sigma \) of \( \Sigma_- \cup \Sigma_+ \). Suppose furthermore that the \( \mathbb{Z} \)-cusp has geometrically finite ends on its both sides. Then the bending angle on the boundary of the convex core of \( \mathbb{H}^3/\phi_i(\pi_1(S)) \) along \( c \) converges to \( \pi \).

**5.2. Case (b).** By Corollary 2.3, if \( \{ q(m_i^-, m_i^+) \} \) converges algebraically but not strongly, then the geometric limit either has a \( \mathbb{Z} \times \mathbb{Z} \)-cusp or a new simply degenerate end. We note that the following arguments only use the assumption that \( \nu \) satisfies the condition (c) of Theorem 3.1. Later in §6, we shall use the arguments again in the case when \( \nu \) satisfies (c) but not (b) of Theorem 3.1.

By Theorem 2.1, the geometric limit \( M_\infty \) is topologically embedded in \( S \times (0,1) \). We regard \( M_\infty \) as embedded in \( S \times (0,1) \) from now on, and we call the direction of \( S \times \{ t \} \) horizontal. By Lemma 2.2, there is an algebraic locus \( f_\infty : S \to M_\infty \) which can be lifted to an immersion into \( \mathbb{H}^3/\psi(\pi_1(S)) \) inducing \( \psi \) between the fundamental groups such that \( f_\infty(S) \) is horizontal except for the part where it goes around a torus cusp. We recall that the pull-back of \( f_\infty \) to \( \mathbb{H}^3/\phi_i(\pi_1(S)) \) by an approximate isometry between \( \mathbb{H}^3/\phi_i(\pi_1(S)) \) and \( M_\infty \) induces the isomorphism \( \phi_i \) between the fundamental groups.

A geometrically infinite end or a torus cusp is situated either above \( f_\infty(S) \) or below \( f_\infty(S) \), except for the case of a torus cusp around which \( f_\infty(S) \) goes.
When \( f_\infty(S) \) goes around a torus cusp \( T \), we regard \( T \) as being situated both above and below \( f_\infty(S) \). We say that a simply degenerate end or a torus cusp is nearest to \( f_\infty(S) \) when pleated surfaces tending to the end can be homotoped into \( f_\infty(S) \) in \( M_\infty \) for a simply degenerate end, and when a longitude can be homotoped into \( f_\infty(S) \) for a torus end. Since a wild geometrically infinite end is an accumulation of simply degenerate ends, it cannot be nearest to \( f_\infty(S) \).

**Claim 5.2.** Unless \( \{ \phi_i \} \) converges to \( \psi \) which has no new simply degenerate ends, there is either a new simply degenerate end or a torus cusp, which is nearest to \( f_\infty(S) \). If a simply degenerate end is nearest to \( f_\infty(S) \), then it can be lifted to the algebraic limit.

**Proof.** Since \( M_\infty \) is embedded in \( S \times (0,1) \), a simply degenerate end or a longitude of a torus end can be vertically homotoped into \( f_\infty(S) \) unless there is another simply degenerate end or a torus end which impedes this. Since ends or torus cusps can accumulate only into a horizontal surface containing an end, there cannot be ends which accumulate into \( f_\infty(S) \). Therefore, there must be a simply degenerate end or a torus cusp which can be homotoped into \( f_\infty(S) \) without being obstructed by other ends. This shows the first statement.

If a simply degenerate end is nearest to \( f_\infty(S) \), it is homotopic into \( f_\infty(S) \), and hence can be lifted to the algebraic limit. \( \square \)

5.2.1. Nearest new simply degenerate end. Suppose that there is a new simply degenerate end \( E \) which is nearest to \( f_\infty(S) \). We assume that \( E \) is situated above \( f_\infty(S) \). The case when \( E \) is situated below \( f_\infty(S) \) can be dealt with in the same way just by turning everything upside down. The end \( E \) has a neighbourhood of the form \( \Sigma \times (s,t) \) for an incompressible subsurface \( \Sigma \) of \( S \), where \( \Sigma \times \{ t \} \) corresponds to the end \( E \) (Figure 1). Let \( \delta \) be the constant given in Lemma 4.3 for \( \epsilon = 1 \). Take a simple closed curve \( d \) contained in \( \Sigma \) satisfying \( \iota(d, \nu) < \delta \). Let \( \lambda \) be a measured lamination on \( \Sigma \) whose support is the ending lamination of \( E \). Take a shortest pants decomposition \( C_i \) of \( (\Sigma_+, m_i^+) \). Then by Lemma 2.3 we see that \( C_i | \Sigma \) converges in the Hausdorff topology to a geodesic lamination whose only minimal component is \( \lambda \). Since \( \iota(d, \lambda) > 0 \), we have \( \iota(C_i, d) \to \infty \) and see that the length of \( d \) with respect to \( m_i^+ \) goes to \( \infty \). On the other hand, \( \{ \iota(d, b \circ q(m_i^-, m_i^+)) \} \) is bounded since \( b \circ q(m_i^-, m_i^+) \) converges to \( \nu \). We set \( \mu_\infty^1 \) in the statement of Lemma 4.3 to be this \( \nu \) and \( \mu_\infty^2 \) to be empty. Then we see that the translation length of \( \phi_i(d) \) goes to \( \infty \), contradicting our assumption that \( \{ \phi_i \} \) converges.

5.2.2. Nearest torus cusp. Let \( T \) be such a nearest torus cusp. Again, the case when there is a nearest torus cusp \( T \) below \( f_\infty(S) \) can also be dealt with in the same way, by turning everything upside down. Let \( l \) be a simple closed curve on \( \Sigma_+ \) whose image under \( f_\infty \) is homotopic to a longitude of \( T \).
Let $\delta > 0$ be the constant given in Lemma 4.3 for $\varepsilon = 1$, and take a simple closed curve $d$ on $\Sigma_+$ with $\iota(d, l) > 0$ and $\iota(d, \nu \setminus l) < \delta$. Since $l$ is homotopic to a longitude of a torus cusp in $M_\infty$, by Lemma 2.4-(1), the length of $l$ with respect to $m_i^+$ is bounded from below by a positive constant. Consider the shortest pants decomposition $P_i$ of $(\Sigma_+, m_i^+)$, and extend it to a shortest marking $M_i$ of $\Sigma_+$. Then by Lemma 2.4-(2), if we choose a simple closed curve $k$ on $S$ intersecting $l$ essentially, we have $d_A(l)(M_i, k) \rightarrow \infty$. This means that unless $P_i$ contains $l$ for all $i$ after passing to a subsequence, there is a component $a_i$ of $P_i$ which spirals around $l$ more and more as $i \rightarrow \infty$. Then, since $\iota(P_i, d) \geq \iota(a_i, d)$ and the right hand side goes to $\infty$, the length of $d$ with respect to $m_i^+$ goes to $\infty$.

In the case when $P_i$ contains $l$ for all $i$, there is a curve $t_i$ in $M_i$ with $\iota(l, t_i) > 0$ which is shortest among such curves. Since we are assuming that the length of $l$ with respect to $m_i^+$ is bounded away from 0, the length of $t_i$ with respect to $m_i^+$ is bounded above. Then by Lemma 2.4-(2), we see that $t_i$ spirals around $l$ more and more as $i \rightarrow \infty$. It follows that $d_A(t_i, d) \rightarrow \infty$. Since $t_i$ has bounded length and the length of $l$ with respect to $m_i^+$ does not go to 0, we see that the length of $d$ with respect to $m_i^+$ goes to $\infty$ also in this case.

By the observation above, the closed geodesic representing $d$ in $(\Sigma_+, m_i^+)$ spirals around $l$ more and more as $i \rightarrow \infty$. This implies that $\angle_{(\Sigma_+, m_i^+)}(d, l)$ goes to 0. We set $\mu^1_\infty$ to be $\nu \setminus l$ and $\mu^2_\infty$ to be $l$ with a positive weight given by $\nu$ if $l$ is contained in $|\nu|$, and $\mu^1_\infty$ to be $\nu$ and $\mu^2_\infty$ to be empty otherwise. Then we apply Lemma 4.3 and see that the translation length of $\phi_i(d)$ goes to $\infty$, contradicting our assumption that $\{\phi_i\}$ converges.

### 5.3. Case (c).

Suppose that $\{q(m_i^-, m_i^+)\}$ does not converge algebraically. By considering efficient pleated surfaces as in Thurston [26], the following can be proved. See Ohshika [18, Theorem 3.1] for a complete proof.

**Lemma 5.3.** There is a vertical codimension-1 lamination $L$ properly embedded in $S \times [0, 1]$, which is disjoint from both $\lambda_-$ and $\lambda_+$, such that for any sequence of weighted simple closed curves $s_i \gamma_i$ on $S$ converging to a
measured lamination \( \mu \), if \( \iota(L, \mu) > 0 \), then \( s_i \phi_i(\gamma_i) \to \infty \), where \( \phi_i(\gamma_i) \) denotes the length of the closed geodesic representing \( \phi_i(\gamma_i) \) in \( \mathbb{H}^3/\phi_i(\pi_1(S)) \). Indeed, there is a pleated surface called an ‘efficient’ pleated surface in \( \mathbb{H}^3/\phi_i(\pi_1(S)) \) on which the growth of the length of every measured lamination is comparable to that in \( \mathbb{H}^3/\phi_i(\pi_1(S)) \).

By the condition (c) of Theorem 3.1, which \( \nu \) must satisfy, there is a component \( \nu_0 \) of \( \nu \) intersecting a component \( L_0 \) of \( L \) essentially. Let \( \Sigma \) be a component of \( \Sigma_+ \backslash \Sigma_- \) containing \( \nu_0 \), and take a shortest pants decomposition \( P_i \) of \( (\Sigma, (m^-_i, m^+_i)) \Sigma \). Since \( P_1 \) is a shortest pants decomposition, by Sullivan’s lemma or [15, Proposition 2.1], \{length\( \phi_i(P_i) \}\} is bounded. By Lemma 5.3 we can apply the proof of [19, Lemma 5.7] using the length in \( \mathbb{H}^3/\phi_i(\pi_1(S)) \) instead of \( (m^-_i, m^+_i) \Sigma \), we see that the Hausdorff limit of \( P_i \) contains \( L_0 \cap \Sigma \). This implies in turn that there is a sequence of positive numbers \( r_i \) tending to 0 and a component \( P'_i \) of \( P_i \) such that \( r_i P'_i \) converges to a measured lamination containing \( L_0 \cap \Sigma \) on \( (\Sigma, (m^-_i, m^+_i)) \Sigma \) unless \( L_0 \cap \Sigma \) coincides with \( P'_i \) for every sufficiently large \( i \). Even in the latter case, unless the length of \( L_0 \cap \Sigma \) with respect to \( (m^-_i, m^+_i) \Sigma \) goes to 0, by letting \( P''_i \) be the shortest simple closed curve transverse to \( P'_i \), we can find \( s_i \) going to 0 such that \( s_i P''_i \to L_0 \cap \Sigma \). Since the length of \( P''_i \) with respect to \( (m^-_i, m^+_i) \Sigma \) is bounded, abusing the symbols, we denote \( s_i P''_i \) also by \( r_i P'_i \) in this case. The properties which we shall use below are that \( r_i \to 0 \) and that \{length\( (m^-_i, m^+_i) \Sigma \phi_i(P_i) \}\} is bounded.

If the length of \( L_0 \cap \Sigma \) with respect to \( (m^-_i, m^+_i) \Sigma \) goes to 0, the efficient pleated surface is pinched along \( L_0 \cap \Sigma \). This implies that \( \mathbb{H}^3/\phi_i(\pi_1(S)) \) is also pinched along \( L_0 \), and hence \( (\Sigma, (m^-_i, m^+_i) \Sigma ) \) is also pinched along \( L_0 \cap \Sigma \). Since \( \nu_0 \) intersects \( L_0 \), this contradicts Corollary 4.2.

It remains to consider the case when \( \{r_i P_i \} \) converges to \( L_0 \cap \Sigma \) with \( r_i \to 0 \). Since \( \iota(\nu_0, L_0) > 0 \) and \( r_i \) goes to 0, we see that \( \iota(P'_i, \nu_0) \to \infty \). This contradicts Corollary 4.2, and \{length\( (m^-_i, m^+_i) \Sigma \phi_i(P_i) \}\} is bounded. Thus we have completed the proof of the case (c).

We note that we did not use the assumption (b) of Theorem 3.1. We state what we have proved in the case (c) as a lemma for later use.

**Lemma 5.4.** Suppose that \{length\( (m^-_i, m^+_i) \Sigma \phi_i(P_i) \}\} converges to a measured lamination \( \nu \) satisfying the assumption (c) of Theorem 3.1. Then \{length\( (m^-_i, m^+_i) \Sigma \phi_i(P_i) \}\} converges in \( \text{AH}(S) \).

6. Homotopy to a degree-1 map

In this section, we shall prove that \( b \circ q \) is a degree-1 map to \( \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \backslash D \) by constructing a homotopy in the one-point compactification of \( \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \backslash D \) between the map induced from \( b \circ q \) and a degree-1 map. First, we shall define compactification where a homotopy will take place.
Definition 6.1. We let $\hat{T}$ be the one-point compactification of $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$, and $\hat{\mathcal{ML}}$ the one-point compactification of $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \setminus D$. Since $b \circ q$ is proper, it induces a continuous map $\hat{b} \circ q : \hat{T} \to \hat{\mathcal{ML}}$.

The following is immediate from the definition of one-point compactification.

Lemma 6.2. The map $b \circ q$ has degree 1 if and only if $\hat{b} \circ q$ has degree 1.

Therefore, we have only to show that $\hat{b} \circ q$ has degree 1. For that, we shall construct a homotopy in an open set from $b \circ q$ to a locally degree-1 map. To construct the latter map, we shall make use of the following homeomorphism derived from the earthquake introduced by Thurston (see Thurston [25] and Kerckhoff [11]).

Definition 6.3. Fix $g_- \in \mathcal{T}(\Sigma_-)$ and $g_+ \in \mathcal{T}(\Sigma_+)$. For $j = -, +$, we let $E_j : \mathcal{ML}(\Sigma_j) \to \mathcal{T}(\Sigma_j)$ be the left earthquake map, that is, a homeomorphism sending a measured lamination $\lambda \in \mathcal{ML}(\Sigma_j)$ to the marked hyperbolic structure obtained by the left earthquake along $\lambda$ on $(\Sigma_j, g_j)$. Then, we have a homeomorphism $E_- \times E_+ : \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \to \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$. We define $E : \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \to \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ to be the inverse of $E_- \times E_+$.

To construct a homotopy, we shall first define its support, which will be done by using an open neighbourhood of a point contained in the ‘corner’ of the product of the Thurston compactifications of the components of $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$. We now describe it more concretely. Let $\Sigma_1, \ldots, \Sigma_n$ be the components of $\Sigma_- \sqcup \Sigma_+$ that are not thrice-punctured spheres. We compactify each $\mathcal{T}(\Sigma_j)$ by attaching $\mathcal{PML}(\Sigma_j)$ as its boundary. We call their product $\prod_{j=1}^n (\mathcal{T}(\Sigma_j) \cup \mathcal{PML}(\Sigma_j))$ the Thurston compactification product of $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ and denote its boundary by $\mathcal{PML}$. A point of $\mathcal{PML}$ has a form $(x_j)_{j=1}^n$, where $x_j$ is either $\mathcal{T}(\Sigma_j)$ or $\mathcal{PML}(\Sigma_j)$ and at least one of the $x_j$ is contained in the boundary $\mathcal{PML}(\Sigma_j)$. We put the product topology on the compactification. We call the subset $\prod_{j=1}^n \mathcal{PML}(\Sigma_j)$ of the boundary the corner and denote it by $\mathcal{PML}_c$. A sequence $\{m_i\}$ of $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ converges to a point in $\mathcal{PML}_c$ after passing to a subsequence if $\{m_i|_{\Sigma_j}\}$ diverges for every component $\Sigma_j$ of $\Sigma_- \sqcup \Sigma_+$ that is not a thrice-punctured sphere.

Definition 6.4. Let $\Lambda$ be a point in $\mathcal{ML}(\Sigma_- \sqcup \Sigma_+)$ not contained in $D$ such that, for each component $\Sigma_j$ of $\Sigma_- \sqcup \Sigma_+$ that is not thrice-punctured sphere, the restriction $\Lambda \cap \Sigma_j$ is an arational uniquely ergodic measured lamination. By setting its $j$-th coordinate to be $[\Lambda|_{\Sigma_j}]$, we define $[\Lambda] \in \mathcal{PML}_c$.

We note that by the arationality, the condition that $\Lambda$ is not contained in $D$ is equivalent to, when $\lambda_- = \lambda_+ = \emptyset$, the condition that no two components of $\Lambda$ are homotopic, and otherwise, the condition that no component of the support of $\Lambda$ is homotopic to an ending lamination.
Definition 6.5. Let $\tau$ be a bi-recurrent train track on $\Sigma_- \sqcup \Sigma_+$ carrying $\Lambda$ by a weight system $\omega$ in such a way that $\omega$ takes a positive value on every branch of $\tau$. We call an arc connecting two measured laminations $\mu_1, \mu_2$ carried by $\tau$ a segment when it is a linear path with regard to the weight system. We note that this notion is independent of the choice of $\tau$ since the transition function between two weight systems is linear.

For two measured laminations $\lambda_1, \lambda_2$ carried by $\tau$ with weight systems $\omega_1, \omega_2$ respectively, we define $d_\tau(\lambda_1, \lambda_2)$ to be the sum of the differences of the weights of $\omega_1$ and $\omega_2$ on the branches of $\tau$.

From now on until the end of this section, we fix a train track $\tau$ as above.

In the same way as we did for $T(\Sigma_-) \times T(\Sigma_+)$, for each component $\Sigma_j$ ($j = 1, \ldots, n$) of $\Sigma_- \sqcup \Sigma_+$, we consider the ray compactification of $\mathcal{ML}(\Sigma_j)$ and regard $\mathcal{PML}(\Sigma_j)$ as its boundary at infinity, and define the ray compactification of $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ with boundary at infinity $\mathcal{PML}$ to be the product of the ray compactification of the components $\mathcal{ML}(\Sigma_1), \ldots, \mathcal{ML}(\Sigma_n)$. As before, we call $\mathcal{PML}_c$ the corner also in this ray compactification, and we see that a sequence $\{\lambda_i\}$ in $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ converges to a point in $\mathcal{PML}$ after passing to a subsequence if $\{\lambda_i|\Sigma_j\}$ diverges for every $j = 1, \ldots, n$.

Definition 6.6. We call a subset $U$ of $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ a truncated cone if $U$ consists of all measured laminations $\lambda$ that satisfy the following conditions.

(a) The train track $\tau$ carries $\lambda$.
(b) Each weight of $w(\lambda)$ is greater than a fixed positive constant $K$. (Recall from Definition 6.5 that we denote the weight system on $\tau$ corresponding to $\lambda$ by $w(\lambda)$.)

We say that a truncated cone $U$ is a truncated cone neighbourhood of $[\Lambda]$ (supported on $\tau$) when there is a neighbourhood $V$ of $[\Lambda]$ in the boundary at infinity $\mathcal{PML}$ such that the closure of $U$ in the ray compactification ($\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$) coincides with $V$.

The positive weight systems on $\tau$ form an open set in $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ since $\tau$ is bi-recurrent, and its ray compactification contains $[\Lambda]$ since each $\Lambda|\Sigma_j$ is arational and uniquely ergodic. Therefore we see that (the ray compactifications of) the truncated cone neighbourhoods form a basis of neighbourhoods of $[\Lambda]$ in the ray compactification ($\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$) $\cup \mathcal{PML}$.

The following is a well-known property of the earthquake map. (See Papadopoulos [21] for instance.)

Lemma 6.7. The map $E : T(\Sigma_-) \times T(\Sigma_+) \to \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ is a homeomorphism extending continuously to the identity on the boundary $\mathcal{PML}$.

We now show some lemmas and their corollaries which will be used in the main step of the proof of Theorem 3.3.

Lemma 6.8. Let $[\Lambda]$ is a projective lamination in $\mathcal{PML}_c$ as in Definition 6.4. Let $\{m_i\}$ be a sequence in $T(\Sigma_-) \times T(\Sigma_+)$ converging to $[\Lambda]$ in the Thurston
compactification product. Then \{b \circ q(m_i)\} also converges to [\Lambda] in the ray compactification (\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)) \cup \mathcal{PML}.

Proof. Let \Sigma_j be a component of \Sigma_- \cup \Sigma_+. Since \{m_i\} converges to [\Lambda] contained in the corner, the restriction \{m_i|\Sigma_j\} converges in the Thurston compactification to \[\Lambda|\Sigma_j\in \mathcal{PML}(\Sigma_j)\]. Let \kappa_i be a shortest simple closed geodesic in (\Sigma_j, m_i|\Sigma_j). Since \Lambda|\Sigma_j is arational and uniquely ergodic, there is a sequence of positive numbers \{s_i\} going to 0 such that \{s_i\kappa_i\} converges to the measured lamination \Lambda|\Sigma_j. By Corollary 6.9, \{\iota(b \circ q(m_i), \kappa_i)\} is bounded. By the continuity of intersection numbers combined with the arationality and the unique ergodicity of \Lambda|\Sigma_j, we see that either \{b \circ q(m_i)\} converges to \(s(\Lambda|\Sigma_j)\) for some positive scalar \(s\), or \{b \circ q(m_i)\} converges in the ray compactification to the point at infinity \([\Lambda|\Sigma_j]\in \mathcal{PML}(\Sigma_j)\).

It remains to show that the former case cannot happen. Suppose, seeking a contradiction, that \{b \circ q(m_i)\} converges to \(s(\Lambda|\Sigma_j)\). Then, by Lemma 4.3, we see that \(q(m_i)|\pi_1(\Sigma_j)\) converges algebraically. Let the constant \(\delta > 0\) given in Lemma 4.3 for \(\epsilon = 1\). We then take a simple closed curve \(c\) on \(\Sigma_j\) approximating \([\Lambda|\Sigma_j]\) such that \(\iota(c, s\Lambda) < \delta\). Since \{m_i|\Sigma_j\} converges to \([\Lambda|\Sigma_j]\) in the Thurston compactification, we see that length_{\Sigma_j}(c) goes to \(\infty\) as \(i \to \infty\). By Lemma 4.3, this implies that length_{\Sigma_j}(c) also goes to \(\infty\), contradicting the fact that \(q(m_i)|\pi_1(\Sigma_j)\) converges.

Thus we have shown that \{b \circ q(m_i)\} converges to \([\Lambda|\Sigma_j]\) for every \(j = 1, \ldots, n\), and hence \{b \circ q(m_i)\} converges to [\Lambda] in the ray compactification. \(\square\)

The lemma implies the following corollary.

**Corollary 6.9.** Let [\Lambda] be a projective measured lamination in \(\mathcal{PML}_c\) as given in Definition 6.4. For any truncated cone neighbourhood \(U\) of [\Lambda], there is a neighbourhood \(V\) of [\Lambda] in the Thurston compactification product of \(\mathcal{I}(\Sigma_-) \times \mathcal{I}(\Sigma_+)\) such that \(b \circ q(\mathcal{I}(\Sigma_-) \times \mathcal{I}(\Sigma_+))\) is contained in \(U\).

Proof. Consider a sequence \{m_i\} in \(\mathcal{I}(\Sigma_-) \times \mathcal{I}(\Sigma_+)\) converging to [\Lambda] in the Thurston compactification product. Then by Lemma 6.8, for any given truncated cone neighbourhood \(U\), the sequence \{b \circ q(m_i)\} is contained in \(U\) for sufficiently large \(i\). This implies the existence of a neighbourhood of [\Lambda] as desired. \(\square\)

In a more general case where we do not assume that the limit of \{m_i\} is [\Lambda], we have the following.

**Lemma 6.10.** Let \{m_i\} be a sequence in \(\mathcal{I}(\Sigma_-) \times \mathcal{I}(\Sigma_+)\) which converges to a point \((x_j)\in \mathcal{PML}\) in the Thurston compactification product. Suppose moreover that \{b \circ q(m_i)\} converges to a point \((y_j)\) in the ray compactification. For each \(x_j\) that lies on the boundary at infinity \(\mathcal{PML}(\Sigma_j)\), let \(\mu_j\) be a measured lamination with \([\mu_j] = x_j\). Then \(y_j\) lies either in \(\mathcal{ML}(\Sigma_j)\) and \(\iota(y_j, \mu_j) = 0\) or on the boundary at infinity \(\mathcal{PML}(\Sigma_j)\) and is represented by
a measured lamination $\nu_j$ with $\iota(\mu_j, \nu_j) = 0$. Moreover, for every $j$ such that $y_j$ lies in $\mathcal{PML}(\Sigma_j)$, the point $x_j$ also lies in $\mathcal{PML}(\Sigma_j)$.

Proof. Suppose that $x_j = [\mu_j]$ lies in $\mathcal{PML}(\Sigma_j)$. Let $\mu_j^0$ be a connected component of $\mu_j$. If $\mu_j^0$ is a simple closed curve, either (a) length$_{\mathbf{m}_j}(\mu_j^0) \to 0$ or (b) there are simple closed curves $d_i$ on $\Sigma_j$ with bounded length$_{\mathbf{m}_j}(d_i)$ and positive numbers $r_i \to 0$ such that $\{r_i d_i\}$ converges to a measured lamination $\tilde{\mu}_j$ containing $\mu_j^0$. The latter curve $d_i$ can be chosen to be a shortest simple closed curve on $\Sigma_j$ with respect to $\mathbf{m}_i$ intersecting $\mu_j^0$. Moreover, if $\mu_j^0$ is not a simple closed curve, the condition (b) always holds.

In the case (a), Lemma 4.1 implies that $\iota(\mu_j^0, b \circ q(\mathbf{m}_j)|\Sigma_j) \to 0$, which implies that $\iota(\mu_j^0, y_j) = 0$ when $y_j$ lies in $\mathcal{ML}(\Sigma_j)$ and $\iota(\mu_j^0, \nu_j) = 0$ when $x_j = [\nu_j]$ lies on the boundary at infinity, by the continuity of the intersection number. In the case (b), Lemma 4.1 implies that $\{\iota(d_i, b \circ q(\mathbf{m}_j))\}$ is bounded. Since $r_i$ tends to 0, by the continuity of the intersection number, we have $\iota(\tilde{\mu}_j, y_j) = 0$, and hence $\iota(\mu_j^0, y_j) = 0$ when $y_j$ lies in $\mathcal{ML}(\Sigma_j)$, and in the same way, $\iota(\mu_j^0, \nu_j) = 0$ when $x_j = [\nu_j]$ lies on the boundary at infinity. Thus, we have $\iota(\mu_j^0, y_j) = 0$ or $\iota(\mu_j^0, \nu_j) = 0$ for every connected component $\mu_j^0$ of $\mu_j$, which implies that $\iota(\mu_j, y_j) = 0$ or $\iota(\mu_j, \nu_j) = 0$.

To show the last statement, suppose that $x_j$ lies in $\mathcal{T}(\Sigma_j)$. Then, by Lemma 5.3 we see that $\{q(\mathbf{m}_j)|\pi_1(\Sigma_j)\}$ converges. Since $\{\mathbf{m}_j|\Sigma_j\}$ converges by assumption, the boundary component $\Sigma_j^i$ of the convex core $C(\mathbb{H}^3/q(\mathbf{m}_j))$ corresponding to $\Sigma_j$ converges geometrically to a boundary component $\Sigma_j^\infty$ of the convex core of the geometric limit, which is homotopic to an algebraic locus of $\Sigma_j$ (see the argument of [16], p.103)). Then the $j$-th component of $b \circ q(\mathbf{m}_j)$ converges to the bending lamination of $\Sigma_j^\infty$, and hence $y_j$ must be inside $\mathcal{ML}(\Sigma_j)$. This shows that if $y_j$ lies on the boundary at infinity, then so does $x_j$.

By a similar argument, we can also show the following proposition.

**Proposition 6.11.** Let $[\Lambda]$ be a projective lamination as in Definition 6.4. Let $\{\mathbf{m}_j\}$ be a sequence in $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ such that $\{b \circ q(\mathbf{m}_j)\}$ converges to a point at infinity represented by $[\Lambda]$ in the ray compactification. Then $\{\mathbf{m}_j\}$ converges to $[\Lambda]$ in the Thurston compactification product of $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$.}

Proof. We have only to show that any subsequence of $\{\mathbf{m}_j\}$ has a subsequence converging to $[\Lambda]$ in the Thurston compactification product. Passing to a subsequence, we can assume that $\{\mathbf{m}_j\}$ converges to either a point $\mathbf{n}$ in $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ or a projective lamination $(y_j) \in \mathcal{PML}$ in the Thurston compactification product. In the former case, by the continuity of the function $b$ due to [10], $\{b \circ q(\mathbf{m}_j)\}$ converges to $b \circ q(\mathbf{n})$, contradicting our assumption.

Suppose that $\{\mathbf{m}_j\}$ converges to $(y_j)$ in the Thurston compactification product. Since $[\Lambda]$ lies in the corner, and $\Lambda \cap \Sigma_j$ is arational and uniquely ergodic for every $j$, by Lemma 6.10 we have we have $y_j = \Lambda \cap \Sigma_j$, and hence $(y_j) = [\Lambda]$. This completes the proof. □
Since truncated cone neighbourhoods form a basis of neighbourhoods of \([\Lambda]\) as remarked before, we have the following corollary.

**Corollary 6.12.** Let \(V\) be a neighbourhood of \([\Lambda]\) in the Thurston compactification product of \(\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)\) for \([\Lambda]\) as in Definition 6.4. Then, there is a truncated cone neighbourhood \(U\) of \([\Lambda]\) in the ray compactification such that \((b \circ q)^{-1}(U)\) is contained in \(V \cap (\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+))\).

By combining these results, we obtain the following technical proposition, which constitutes an essential step for our construction of a homotopy.

**Proposition 6.13.** Let \(\Lambda\) be a measured lamination given in Definition 6.4. Then there are three nested truncated cone neighbourhoods \(U_0 \subset U_1 \subset U_2\) which satisfy the following.

1. Every measured lamination in \(U_2\) satisfies the condition (c) of Theorem 3.1 (with \(\lambda_-\) and \(\lambda_+\)).
2. Let \(V_1 = (b \circ q)^{-1}(U_1)\). Then there is an open set \(V_2\) containing the closure \(\overline{V_1}\) such that both \(b \circ q(V_2)\) and \(E(V_2)\) are contained in \(U_2\).
3. Neither \(E\) nor \(b \circ q\) maps a point outside \(V_1\) into \(U_0\).
4. For any point \(m \in V_2 \setminus V_1\), the segment connecting \(b \circ q(m)\) and \(E(m)\) is disjoint from \(U_0\).

**Proof.** If a measured lamination on \(\Sigma_- \sqcup \Sigma_+\) does not satisfy the condition (c) of Theorem 3.1 then either it has a component homotopic to a component of \(\lambda_-\) or \(\lambda_+\), or it has two components which are homotopic in \(S \times [0, 1]\). Since \(\Lambda\) is arational and is not contained in \(D\), every sufficiently small truncated cone neighbourhood of \([\Lambda]\) contains no such measured laminations. Therefore, by choosing a truncated cone neighbourhood \(U_2\) to be sufficiently small, the condition (1) is satisfied.

By Lemma 6.7 and Corollary 6.12, there is a neighbourhood \(V_2\) of \([\Lambda]\) in the Thurston compactification product such that both \(E(V_2)\) and \(b \circ q(V_2)\) are contained in \(U_2\). By Lemma 6.7 and Corollary 6.12 we can take a neighbourhood \(U_1\) of \([\Lambda]\) such that both \(E^{-1}(U_1)\) and \((b \circ q)^{-1}(U_1)\) are contained in \(V_2\). Again by Lemma 6.7 and Corollaries 6.9 and 6.12, we can take a neighbourhood \(V_1\) of \([\Lambda]\) in the Thurston compactification product such that \(V_1\) is contained in \(E^{-1}(U_1) \cap (b \circ q)^{-1}(U_1)\), and a truncated cone neighbourhood \(U_1\) of \([\Lambda]\) in the ray compactification such that both \(E^{-1}(U_1)\) and \((b \circ q)^{-1}(U_1)\) are contained in \(V_1\). These \(U_1, U_2, V_2\) and \(V_1 = (b \circ q)^{-1}(U_1)\) satisfy the condition (2).

Now, we shall show that we can take a truncated cone neighbourhood \(U_0\) of \([\Lambda]\) in the ray compactification satisfying (3) and (4). Let \(\{m_i\}\) be an arbitrary sequence in \(\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)\) lying outside \(V_1\). Since \(V_1\) is a neighbourhood of \([\Lambda]\) in the Thurston compactification product, Lemma 6.7 implies that \(\{E(m_i)\}\) cannot converge to \([\Lambda]\) in the ray compactification. Similarly, by Proposition 6.11 we see that \(\{b \circ q(m_i)\}\) cannot converge to \([\Lambda]\) in the ray compactification either. Therefore, if we take \(U_0\) to be a
Let \( \lambda_j \) a point on the segment connecting \( b \circ q(\mathbf{m}_2) \) and \( \mathcal{E}(\mathbf{m}_2) \) such that \( \{ \lambda_j \} \) converges to \([\Lambda]\) in the ray compactification.

Taking a subsequence, we can assume that either \( \{ \mathbf{m}_2 \} \) converges to a point \( \mathbf{n} \in \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \) or does not have a convergent sequence inside \( \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \), and converges to a point \( (w_j) \in \mathcal{PML} \) in the Thurston boundary.

The boundary point \((w_j) \in \mathcal{PML} \) in the latter case is distinct from \([\Lambda]\) since \( \{ \mathbf{m}_2 \} \) lies outside \( V_1 \). In the former case, passing to a subsequence \( \{ \lambda_j \} \) converges to a point inside \( \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \) which lies on a segment connecting \( \mathcal{E}(\mathbf{n}) \) and \( b \circ q(\mathbf{n}) \), contradicting our assumption.

In the latter case, by Lemma 6.7 we see that \( \{ \mathcal{E}(\mathbf{m}_2) \} \) converges to \((w_j') \in \mathcal{PML} \) such that \( w_j' = (w_j) \) if \( w_j \in \mathcal{T}(\Sigma_j) \) and \( w_j' = w_j \) otherwise. The sequence \( \{ b \circ q(\mathbf{m}_2) \} \) converges to a point \((y_j) \) in the ray compactification passing to a subsequence. If there is \( j \) such that \( w_j' \) lies in \( \mathcal{ML}(\Sigma_j) \), then \( y_j \) must lie in \( \mathcal{ML}(\Sigma_j) \). Then the limit of \( \lambda_j \) also lies in \( \mathcal{ML}(\Sigma_j) \) and contradicts the assumption that \([\Lambda]\) is a corner point.

Therefore, we have \((w_j') = (w_j) \) and \( w_j \in \mathcal{PML}(\Sigma_j) \). Let \( \mu \) be a measured lamination on \( \Sigma_- \cup \Sigma_+ \) representing \((w_j) \). By Lemma 6.10 we have \( \iota(y_j, \mu|\Sigma_j) = 0 \) if \( y_j \) lies inside \( \mathcal{ML}(\Sigma_j) \) and is represented by \( \mu_j' \in \mathcal{ML}(\Sigma_j) \) with \( y_j = [\mu_j'] \) and \( \iota(\mu|\Sigma_j, \mu_j') = 0 \) if \( y_j \) lies in \( \mathcal{PML}(\Sigma_j) \). Recall that \( \lambda_i \) lies on the segment between \( \mathcal{E}(\mathbf{m}_2) \) and \( b \circ q(\mathbf{m}_2) \). Therefore \( \lambda_j \) converges to a point \((z_j) \) in the ray compactification such that \( z_j \) is a weighted union of \( \mu|\Sigma_j \) and either \( y_j \) or \( \mu_j \). Therefore, we have \( \iota(\mu|\Sigma_j, z_j) = 0 \) if \( z_j \) lies inside \( \mathcal{ML}(\Sigma_j) \) and is represented by \( \nu_j \in \mathcal{ML}(\Sigma_j) \) with \( z_j = [\nu_j] \) and \( \iota(\nu_j, \mu|\Sigma_j) = 0 \) if \( z_j \) lies in \( \mathcal{PML}(\Sigma_j) \). Since \([\mu] \neq [\Lambda]\) and \( \Lambda|\Sigma_j \) is arational, we have \([z_j] \neq [\Lambda]\), and we are led to a contradiction. Thus we have shown that by taking \( U_0 \) to be sufficiently small, the condition (4) also holds. This completes the proof.

To define a locally degree-1 map \( \hat{F}: \hat{T} \to \hat{ML} \), and a homotopy from \( b \circ q \) to \( \hat{F} \), we shall first define a map \( F: \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \to \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \) which induces \( \hat{F} \).

**Definition 6.14.** Let \( U_0, U_1, U_2 \subset \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \) and \( V_1, V_2 \subset \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \) be open sets given in Proposition 6.13 and let \( \tau \) be the train track given in Definition 6.5. Let \( F: \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \to \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \) be a continuous map defined as follows:

(i) For \( x \) outside \( V_2 \), we define \( F(x) = b \circ q(x) \).

(ii) For \( x \in V_1 \), we define \( F(x) = \mathcal{E}(x) \).

(iii) For \( x \in V_2 \setminus V_1 \), letting \( t(x) \) be \( \frac{d_{r}(x, \overline{V_1})}{d_{r}(x, \overline{V_1}) + d_{r}(x, \overline{V_2})} \), we define \( F(x) \) to be the point dividing the segment connecting \( \mathcal{E}(x) \) and \( b \circ q(x) \) internally by \( t(x) : 1 - t(x) \) in the weight system coordinates of \( \tau \).
Lemma 6.15. For the map $F$ defined above, there is no sequence in $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ diverging to infinity, whose image under $F$ has a subsequence converging in $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \setminus D$.

Proof. Since both $\mathcal{E}$ and $b \circ q$ are proper, $\mathcal{E}$ as a map to $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ and $b \circ q$ as a map to $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \setminus D$, we have only to consider the case when $\{m_i\}$ lies in $V_2 \setminus V_1$. Let $\{m_i\}$ be a sequence in $V_2 \setminus V_1$ which does not have a convergent subsequence. We can assume that it converges in the Thurston compactification product to a point $(y_j)$ in $\mathcal{PM}\mathcal{L}$, passing to a subsequence. We need to show that $\{F(m_i)\}$ does not have a convergent subsequence in $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \setminus D$.

Recall from Proposition 6.13 that both $\{\mathcal{E}(m_i)\}$ and $\{b \circ q(m_i)\}$ lie in $U_2$. Since $\mathcal{E}$ is a proper map to $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$, the sequence $\{\mathcal{E}(m_i)\}$ diverges to infinity, necessarily within $U_2$. If $\{b \circ q(m_i)\}$ also diverges to infinity (within $U_2$), the segment connecting $\mathcal{E}(m_i)$ and $b \circ q(m_i)$ also diverges to infinity within $U_2$ as $i \to \infty$, and we are done. It remains to deal with the case when $\{b \circ q(m_i)\}$ converges to a point $\nu$ in $D$ after passing to a subsequence. By the part (1) of Proposition 6.13, $\nu$ must satisfy the condition (c) of Theorem 5.1 and hence contains a compact leaf with weight larger than or equal to $\pi$. We denote the union of all such components of $\nu$ by $\nu_0$. Let $\Sigma_k$ be a component of $\Sigma_- \sqcup \Sigma_+$ containing a component of $\nu_0$. Then $y_k$ lies in $\mathcal{PM}\mathcal{L}(\Sigma_k)$ since otherwise the component of the convex core boundary corresponding to $\Sigma_k$ converges geometrically without giving rise to a new parabolic curve, as argued in the previous section.

We first remark the following, which was just a restatement of Lemma 5.4

Claim 6.16. Let $\phi_i: \pi_1(S) \to \text{PSL}_2(\mathbb{C})$ be a representation corresponding to $q(m_i)$. Then, $\{\phi_i\}$ converges to some $\psi \in \text{AH}(S)$ as $i \to \infty$.

Let $P_i$ be a shortest pants decomposition of $(\Sigma_- \sqcup \Sigma_+, m_i)$. Let $P_\infty$ be the Hausdorff limit of $\{P_i\}$ (after passing to a subsequence), which is a geodesic lamination. Invoking an argument which we used in the preceding section, we can show the following.

Claim 6.17. Every minimal component of $P_\infty$ is a simple closed curve. The geometric limit $M_\infty$ has neither a new geometrically infinite end (i.e. one not corresponding to that of $\mathbb{H}^3/\phi_i(\pi_1(S))_0$) nor a torus cusp.

Proof. Suppose, seeking a contradiction, that $P_\infty$ has a minimal component $\varrho$ which is not a simple closed curve. Let $S(\varrho) \subset S$ be the minimal supporting surface of $\varrho$. Then $M_\infty$ must have a new geometrically infinite end which lifts to the algebraic limit. (This follows from the fact in Minsky’s model of $\mathbb{H}^3/\phi_i(\pi_1(S))$, the subsurface $S(\varrho)$ must support a tight geodesic whose length goes to $\infty$ in this case.)

Now, we put a basepoint on the algebraic locus $f_\infty(S)$, and consider a geometric limit $M_\infty$ containing the algebraic limit of $\{\phi_i\}$. As in Section 5.2.1 by taking a non-peripheral simple closed curve $d$ on $S(\varrho)$ with $\ell(d, \varrho) < \delta$
for the constant $\delta$ given in Lemma 4.3 with $\epsilon = 1$, we are led to a contradiction. Thus we have shown that every minimal component of $P_\infty$ is a simple closed curve. This argument also shows that $M_\infty$ cannot have a nearest simply degenerate end, for such an end can be lifted to the algebraic limit by Claim 5.2.

Next suppose that $M_\infty$ has a torus cusp. Since there is no nearest simply degenerate end for $M_\infty$, if there are torus cusps, we can take a nearest one by Claim 5.2. By repeating the arguments of Section 5.2.2, which can be applied also to our situation as remarked there, we get a contradiction.

Since $M_\infty$ has neither a nearest simply degenerate end nor a nearest torus cusp, by Claim 5.2 we see that $M_\infty$ does not have a new geometrically infinite end. □

We define a subset $P_\infty^0$ of $P_\infty$ to be the subset consisting of simple closed curves whose lengths with respect to $m_i$ go to 0 as $i \to \infty$.

Since $\{\phi_i\}$ converges to $\psi \in \text{AH}(S)$ as mentioned above, by Corollary 2.3, the claim above implies that the convergence is strong.

Claim 6.18. The multi-curve $P_\infty^0$ is contained in $\nu_0$.

Proof. Let $c$ be a component of $P_\infty^0$. We regard $c$ as lying on a compact core of $\mathbb{H}^3/\psi(\pi_1(S))$. Then length$_{m_i}(c)$ goes to 0 by the definition of $P_\infty^0$, and hence it represents a core curve of a $Z$-cusp neighbourhood lying outside the compact core. Since $\{\phi_i\}$ converges to $\psi$ strongly and $M_\infty$ does not have a new geometrically infinite end by Claim 6.17, the $Z$-cusp has geometrically finite ends on its both sides. We can apply Lemma 5.1 to see that the bending angle along $c$ converges to $\pi$, and hence $c$ is contained in $\nu_0$. □

We can further see the following.

Claim 6.19. Let $C$ be a minimal component of $P_\infty$ not contained in $P_\infty^0$. If $P_\infty$ does not contain any other minimal component homotopic to $C$ in $S \times [0,1]$, then the twisting parameter along $C$ of $m_i$ is bounded as $i \to \infty$. If $P_\infty$ has two distinct minimal components $C$ and $C'$ homotopic to each other in $S \times [0,1]$, then the difference of the twisting parameters of $m_i$ along $C$ and $C'$ is bounded as $i \to \infty$.

Proof. Suppose first that there is no other component in $P_\infty$ homotopic to $C$ in $S \times I$. If the twisting parameter of $m_i$ along $C$ goes to $\infty$ after passing to a subsequence, then by the part (1) of Lemma 2.4 the geometric limit $M_\infty$ has a corresponding torus cusp. This contradicts Claim 6.17. Also in the latter case when $C$ is homotopic to $C'$ in $S \times I$, by the same argument as above involving Lemma 2.4 we see that if the difference of twisting parameters of $m_i$ goes to $\infty$, then $M_\infty$ must have a torus cusp, contradicting Claim 6.17. □

This claim implies the following.

Claim 6.20. The projective lamination $y_k$ is supported in $P_\infty^0$. 
Proof. By the definition of the topology of the Thurston compactification product, we see that \(|y_k|\) is contained in \(P_\infty\). By Claim 6.17 every minimal component of \(P_\infty\) is a simple closed curve. If \(|y_j|\) has a component \(C\) contained in \(P_\infty\) but not in \(P_\infty^0\), then the twist parameter of \(m_i\) around \(C\) must go to either \(\infty\) or \(-\infty\). This contradicts Claim 6.19 unless there is another component of \(P_\infty\), homotopic to \(C\) in \(S \times I\). If there is such a component \(C'\), then the support of some coordinate \(y_l\) of \((y_j)\) must contain \(C'\) as a component by Claim 6.19 again. On the other hand, \((y_j)\) is contained in \(U_2\), and hence cannot have two components of projective laminations homotopic to each other in \(S \times I\). This is a contradiction.

Having proved these claims, we can now complete the proof of Lemma 6.15. Since \(|y_k|\) is contained in \(P_\infty^0\) by Claim 6.20, it lies in \(v_0\) by Claim 6.18. Therefore the segment connecting \(E(m_i)\) and \(b \circ q(m_i)\) converges uniformly on any compact set to a ray entirely lying in \(D\). This completes the proof.

Now we can define \(\tilde{F}\) which we mentioned before. By Lemma 6.15 \(F: \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \rightarrow \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)\) induces a unique continuous map \(\tilde{F}: \hat{\mathcal{T}} \rightarrow \hat{\mathcal{ML}}\) similarly to Definition 6.1. We shall show that \(\tilde{F}\) has degree 1. We use the symbols \(\hat{E}: \hat{\mathcal{T}} \rightarrow \hat{\mathcal{ML}}\) to denote the map induced by \(E\).

**Proposition 6.21.** The map \(\tilde{F}: \hat{\mathcal{T}} \rightarrow \hat{\mathcal{ML}}\) has degree 1.

**Proof.** Let \(\hat{U}_0\) be the open set in \(\hat{\mathcal{ML}}\) corresponding to \(U_0 \setminus D\), where \(U_0\) is the truncated cone neighbourhood in Proposition 6.13. We shall show that the restriction of \(\tilde{F}\) to \(\tilde{F}^{-1}(\hat{U}_0)\) has degree 1, which immediately implies that \(\tilde{F}\) has degree 1.

Since \(F|E^{-1}(U_0)\) coincides with \(E|E^{-1}(U_0)\), it is a homeomorphism to its image. We have only to show that there is no point outside \(E^{-1}(U_0)\) which is mapped into \(U_0\). Since \(F|V_1\) coincides with \(E|V_1\), there are no points in \(V_1 \setminus E^{-1}(U_0)\) mapped into \(U_0\) by \(F\). On the other hand, the conditions (3) and (4) in Proposition 6.13 guarantee that no points outside \(V_1\) are mapped into \(U_0\) by \(F\). This completes the proof.

The next step is to show that there is a homotopy between \(b \circ q\) and \(\tilde{F}\). We shall first construct a homotopy between \(b \circ q\) and \(F\).

We define a homotopy \(H: \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \times [0, 1] \rightarrow \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)\) from \(b \circ q\) to \(F\) as follows.

1. For any \(x\) outside \(V_2\) and any \(s \in [0, 1]\), \(H(x, s)\) is defined to be \(b \circ q(x)\).
2. For any \(x\) in \(V_2\) and any \(s \in [0, 1]\), \(H(x, s)\) is defined to be the point dividing the segment connecting \(b \circ q(x)\) and \(F(x)\) internally by \(s : 1 - s\).

**Lemma 6.22.** For any sequence \(\{t_i \in [0, 1]\}\) and any \(\{m_i\} \subset \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)\) diverging to infinity, \(\{H(m_i, t_i)\}\) cannot have a subsequence converging in \(\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)\) \(\setminus D\) as \(i \rightarrow \infty\).
Proof. The proof is quite similar to that of Lemma 6.15. We can assume that \( \{ m_i \} \) either lies outside \( V_2 \), in \( V_2 \setminus V_1 \), or in \( V_1 \), passing to a subsequence. In the case when \( \{ m_i \} \) lies outside \( V_2 \), the statement follows immediately from the properness of \( b \circ q \) as a map to \( \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \setminus D \). In the case when \( \{ m_i \} \) lies in either \( V_2 \setminus V_1 \) or \( V_1 \), the point \( H(m_i, t_i) \) lies in the segment connecting \( b \circ q(m_i) \) and \( \hat{E}(m_i) \). By the same argument as in the proof of Lemma 6.15 such a segment converges to a ray on \( D \) as \( i \to \infty \). This shows that \( \{ H(m_i, t_i) \} \) cannot have a subsequence converging in \( \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \setminus D \). □

This lemma immediately implies the following.

**Corollary 6.23.** The homotopy \( H \) induces a homotopy \( \hat{H} : \hat{T} \to \hat{\mathcal{ML}} \) between \( \hat{b} \circ \hat{q} \) to \( \hat{E} \).

Combining this corollary with Proposition 6.21, we conclude that \( \hat{b} \circ \hat{q} \) has degree 1, which in turn implies that \( b \circ q \) has degree 1. This completes the proof of the latter half of Theorem 3.3.

**References**

[1] Baba, S. 2π-grafting and complex projective structures, I. *Geom. Topol. 19*, 6 (2015), 3233–3287.

[2] Bers, L. Simultaneous uniformization. *Bulletin of the American Mathematical Society* 66, 2 (1960), 94–97.

[3] Bonahon, F. Bouts des variétés hyperboliques de dimension 3. *Annals of Mathematics* 124, 1 (1986), 71–158.

[4] Bonahon, F., and Otal, J.-P. Laminations mesurées de plissage des variétés hyperboliques de dimension 3. *Annals of Mathematics* 160, 3 (2004), 1013–1055.

[5] Bridgeman, M. Average bending of convex pleated planes in hyperbolic three-space. *Inventiones mathematicae* 132, 2 (1998), 381–391.

[6] Brock, J., Bromberg, K., Canary, R., and Lecuire, C. Convergence and divergence of Kleinian surface groups. *J. Topol.* 8, 3 (2015), 811–841.

[7] Brock, J. F., Canary, R. D., and Minsky, Y. N. The classification of Kleinian surface groups, II: The ending lamination conjecture. *Ann. of Math. (2)* 176, 1 (2012), 1–149.

[8] Epstein, D., Marden, A., and Markovic, V. Quasiconformal homeomorphisms and the convex hull boundary. *Annals of Mathematics* 159, 1 (2004), 305–336.

[9] Epstein, D. B. A., and Marden, A. Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces. In *Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984)*, vol. 111 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, Cambridge, 1987, pp. 113–253.

[10] Keen, L., and Series, C. Continuity of convex hull boundaries. *Pacific J. Math.* 168, 1 (1995), 183–206.

[11] Kerckhoff, S. P. The Nielsen realization problem. *Annals of Mathematics* 117, 2 (1983), 235–265.

[12] Lecuire, C. Plissage des variétés hyperboliques de dimension 3. *Invent. Math.* 164, 1 (2006), 85–141.
[13] McCullough, D. Compact submanifolds of 3-manifolds with boundary. *Quart. J. Math. Oxford Ser. (2) 37*, 147 (1986), 299–307.

[14] Minsky, Y. The classification of Kleinian surface groups. I. Models and bounds. *Ann. of Math. (2) 171*, 1 (2010), 1–107.

[15] Ohshika, K. Limits of geometrically tame Kleinian groups. *Inventiones mathematicae* 99, 1 (1990), 185–203.

[16] Ohshika, K. Geometric behaviour of Kleinian groups on boundaries for deformation spaces. *Quart. J. Math. Oxford Ser. (2) 43*, 169 (1992), 97–111.

[17] Ohshika, K. Strong convergence of Kleinian groups and Carathéodory convergence of domains of discontinuity. *Mathematical Proceedings of the Cambridge Philosophical Society* 112, 2 (1992), 297–307.

[18] Ohshika, K. Constructing geometrically infinite groups on boundaries of deformation spaces. *Journal of the Mathematical Society of Japan* 61, 4 (2009), 1261–1291.

[19] Ohshika, K. Divergence, exotic convergence and self-bumping in quasi-Fuchsian spaces. *Annales de la Faculté des sciences de Toulouse : Mathématiques* 29, 4 (2020), 805–895.

[20] Ohshika, K., and Soma, T. Geometry and Topology of Geometric Limits I. In *In the Tradition of Thurston*. Springer International Publishing, Cham, Dec. 2020, pp. 291–363.

[21] Papadopoulos, A. On Thurston’s boundary of Teichmüller space and the extension of earthquakes. *Topology Appl.* 41, 3 (1991), 147–177.

[22] Penner, R. C., and Harer, J. L. *Combinatorics of train tracks*, vol. 125 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, Princeton, 1992.

[23] Scott, G. P. Compact submanifolds of 3-manifolds. *J. London Math. Soc. (2) 7* (1973), 246–250.

[24] Thurston, W. P. Geometry and topology of three-manifolds. *lecture notes*, Princeton University.

[25] Thurston, W. P. Earthquakes in two-dimensional hyperbolic geometry. In *Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984)*, vol. 112 of *London Math. Soc. Lecture Note Ser*. Cambridge Univ. Press, Cambridge, 1986, pp. 91–112.

[26] Thurston, W. P. Hyperbolic Structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle. *arXiv.org* (Jan. 1998).

S.B.: Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka, Japan

Email address: baba@math.sci.osaka-u.ac.jp

K.O.: Department of Mathematics, Faculty of Science, Gakushuin University, Toshima-ku, Tokyo 171-8588, Japan

Email address: ohshika@math.gakushuin.ac.jp