Equivariant Cohomology and Wall Crossing Formulas in Seiberg-Witten Theory

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Abstract

We use localization formulas in the theory of equivariant cohomology to rederive the wall crossing formulas of Li-Liu and Okonek-Teleman for Seiberg-Witten invariants.

One of the difficulties in the study of Donaldson invariants or Seiberg-Witten invariants for closed oriented 4-manifold with $b_2^+ = 1$ is that one has to deal with reducible solutions. There have been a lot of work in this direction in the Donaldson theory context (see Göttsche and the references therein). In the Seiberg-Witten theory, the $b_1 = 0$ case were discussed in Witten and Kronheimer-Mrowka. The general case was solved by Li-Liu. Very recently, Okonek-Teleman extended the definition of Seiberg-Witten invariants when $b_1 \neq 0$ and obtained a universal wall crossing formula for the invariants.

A common feature in such works is that the equations used to define the invariants depend on some parameters. The parameter spaces are divided into chambers by walls, where reducible solutions can occur. Within the same chamber, the invariants do not change. When the parameter changes smoothly from one chamber to another, the usual approach is to examine what happens when one cross the wall. The result is expressed as a wall crossing formula.

The above complications actually all come from one source: the configuration spaces are singular. They are quotients of contractible spaces by gauge groups, but the reducible solutions and irreducible solutions have different orbit types. This leads one to consider other cohomology theories for the configuration spaces. For example, Li-Tian have used intersection homology to study the wall crossing phenomenon addressed in Li-Liu. On the other hand, there have been many papers in physics literature defining topological quantum field theories by equivariant cohomology related to the action of gauge groups. Such cohomology theories are infinite dimensional in nature, since the gauge groups are infinite dimensional. In this paper, we use an essentially finite dimensional approach. A well-known procedure is to break the action of the gauge group into a free action by an infinite dimensional group, then followed by a finite dimensional compact group. Localization formulas in equivariant cohomology theory in the finite dimensional case can then be applied to study the wall crossing.

The rest of the paper is arranged as follows. §1 reviews the definition of equivariant cohomology. A localization formula due to Kalkman and two special cases are discussed in §2. §3 and §4 describe, respectively, how to use the localization formulas to derive the wall crossing formulas for Seiberg-Witten invariants due to Li-Liu and Okonek-Teleman.

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1 Equivariant cohomology

For simplicity of the presentation, we review only what we will use later about equivariant cohomology. For the general theory on equivariant cohomology, the reader is referred to [9, 1, 2]. We shall only consider the case of an $S^1$-action on a compact smooth manifold $W$, with fixed point set $F$. We allow $W$ to have boundary, but require that $F \cap \partial W = \emptyset$. The action of $S^1$ generates a vector field $X$ on $W$. In fact, for any $x \in W$, if we let $c(t) = \exp(\sqrt{-1}t) \cdot x$ then $X(x)$ is the tangent vector to $c(t)$ at $t = 0$. Denote by $\Omega^*_{S^1}(W)$ the space of differential forms on $W$ fixed under the $S^1$-action. Let $u$ be an indeterminate of degree 2 and consider the space $\Omega^*_{S^1}(W) \otimes \mathbb{R}[u]$. Define

$$d_{S^1} = d - u \cdot i_X : \Omega^*(W)^{S^1} \otimes \mathbb{R}[u] \to \Omega^*(W)^{S^1} \otimes \mathbb{R}[u]$$

as a derivation, whose action on $u$ is zero and $d_{S^1} \alpha = d\alpha - u i_X \alpha$ for an invariant form $\alpha \in \Omega^*(W)^{S^1}$. Now $d^2_{S^1} = -u(d i_X + i_X d) = -u L_X = 0$ on $\Omega^*(W)^{S^1} \otimes \mathbb{R}[u]$.

**Definition.** The equivariant cohomology of the $S^1$-space $W$ is defined by

$$H^*_{S^1}(W) = \ker d_{S^1} / \text{Im} d_{S^1}.$$ 

From this definition, it is clear that $H^*_{S^1}(W)$ is a $\mathbb{R}[u]$-module. Furthermore, if $S^1$ acts trivially on $W$, then $H^*_{S^1}(W) \cong H^*(W) \otimes \mathbb{R}[u]$, a trivial module.

Let $P$ be a connected closed oriented manifold, and $\pi : E \to P$ be a smooth complex vector bundle over $P$. Assume that there is an $S^1$-action on $E$ by bundle homomorphisms, which covers an $S^1$-action on $P$. Following Atiyah-Bott [1], one can define the equivariant Euler class as

$$\epsilon(E) = i^*i_* 1,$$

where $i : P \to E$ is the zero section, $i_*$ and $i^*$ are the push-forward and pullback homomorphisms in equivariant cohomology respectively. It is routine to verify that

$$\epsilon(E_1 \oplus E_2) = \epsilon(E_1) \epsilon(E_2)$$

for two $S^1$ bundles $E_1$ and $E_2$ over $P$. We will be concerned with the case when the action of $S^1$ on $P$ is trivial. In this case, by splitting principle [3], we can assume without loss of generality that $E$ has a decomposition as $S^1$ bundles

$$E = L_1 \oplus L_2 \oplus \cdots \oplus L_r,$$

where each $L_j$ is a line bundle, such that the action of $\exp(\sqrt{-1}t)$ on $L_j$ is multiplication by $\exp(\sqrt{-1}m_j t)$, for some weight $m_j \in \mathbb{Z}$. By formula (8.8) in Atiyah-Bott [4],

$$\epsilon(L_j) = m_j u + c_1(L_j).$$

Hence we have

$$\epsilon(E) = \prod_{j=1}^r (m_j u + c_1(L_j)).$$
2 Localization formulas

For an $S^1$-space $W$ with fixed point set $F$, let $\{P_k\}$ be the decomposition of $F$ into connected components. It is well-known that each $P_k$ is a smooth submanifold of $W$, hence $F$ has only finitely many components. The $S^1$-action on $W$ induces an action on the normal bundle $\nu_k$ of $P_k$ in $W$. The equivariant Euler class of $\nu_k$ can be computed as in §1. Now endow $W$ with an $S^1$-invariant metric. Define a 1-form $\theta$ on $W - F$ in the following way: $\theta(X) = 1, \theta|_{X^\perp} = 0$. Here we use $X^\perp$ to denote the orthogonal complement of $X$ in the tangent space. It is easy to see that $\theta$ is a connection on the principal bundle $W - F \to (W - F)/S^1$.

Following Kalkman \[5\], we define for any $\alpha = \sum \alpha_j u^j \in \Omega^*(M) \otimes \mathbb{R}[u]$,  
\[ r(\alpha) = \sum \alpha_j (d\theta)^j - \theta \wedge (i_X \alpha_j)(d\theta)^j. \]

It is easy to see that $r(\alpha)$ is $S^1$-invariant and $i_X r(\alpha) = 0$. So $r(\alpha)$ is the lifting of a form on $(W - F)/S^1$ via the projection $W - F \to (W - F)/S^1$. Notice that there is an operator  
\[ \int_M : \Omega^*(W)^{S^1} \otimes \mathbb{R}[u] \to \mathbb{R}[u] \]
induced by sending differential forms of degree $\dim(W)$ to its integral over $W$, and all other forms to zero. Now we can state a theorem due to Kalkman \[5\] (which can be also obtained by Witten’s localization principle \[11\]).

**Theorem 2.1** Let $W$ be an $S^1$-manifold with an invariant boundary $\partial W$, and fixed point set $F = \{P_k\}$, such that $F \cap \partial W = \emptyset$. Let $\alpha$ be an equivariant closed form on $M$ of total degree $\dim(W) - 2$. Then  
\[ \int_{\partial W/S^1} r(\alpha) = \sum_k \int_{P_k} \frac{\alpha u}{\epsilon(\nu_k)}. \]

We now give a construction of an equivariant closed form on $W$. Let $f : W \to \mathbb{R}$ be an $S^1$-invariant smooth function which vanishes near $F$, and $f \equiv 1$ outside a tubular neighborhood of $F$. Then $f \theta$ can be extended over $F$. It is straightforward to see that $d(f \theta) = u(-1 + i_X(f \theta)) = d(f \theta) - u(-1 + f)$ is an equivariant closed form. Assume now that $\dim(W) = 2(n + 1)$ and that $S^1$-action on the normal bundles $\nu_k$ all have weight 1. Let $\alpha = [d(f \theta) - u(-1 + f)]^n$, then near $\partial W$ we have $f \equiv 1$, and so  
\[ r(\alpha) = r((d\theta)^n) = (d\theta)^n. \]

Denote by $c$ the first Chern class of the principal $S^1$-bundle $\partial W \to \partial W/S^1$. Then in our normalization of $\theta$, $c = [-d\theta]$. An application of Theorem 2.1 then yields  
\[ \int_{\partial W/S^1} c^n = (-1)^n \sum_k \int_{P_k} \frac{u^{n+1}}{\sum_{j=1}^k c_{r_k-j}(\nu_k)u^j}, \quad (\ast) \]
where \( r_k \) is the complex rank of \( \nu_k \).

There is a slight generalization of the above formula. Let \( \dim(W) = d + 2 \), \( d \) is not necessarily even. Let \( k \) be a number between 1 and \( d \), which has the same parity as \( d \). Let \( \beta_1, \ldots, \beta_k \) be \( k \mathbb{S}^1 \)-invariant closed 1-forms on \( W \) such that for each \( j = 1, \ldots, k \), \( \beta_j|_{\partial W} \) is the pullback of a 1-form on \( \partial W \), which we still denote by \( \beta_j \). Now let \( l = \frac{1}{2}(d - k) \), and let

\[
\alpha = \beta_1 \wedge \cdots \wedge \beta_k \wedge [d(f\theta) - u(-1 + f)]^l.
\]

Then near \( \partial W \),

\[
r(\alpha) = r(\beta_1 \wedge \cdots \wedge \beta_k \wedge (d\theta)^l) = \beta_1 \wedge \cdots \wedge \beta_k \wedge (d\theta)^l.
\]

So by Theorem 2.1, we get

\[
\int_{\partial W/S^1} \beta_1 \wedge \cdots \wedge \beta_k \wedge c^l = \sum_k \int_{\nu_k} \frac{u^{l+1}\beta_1 \wedge \cdots \wedge \beta_k \wedge (d\theta)^l}{\sum_{j=1}^{r_k} c_{r_k-j}(\nu_k)u^j}.
\]

### 3 Applications to Seiberg-Witten theory: A simple case

Let \( X \) be a closed oriented 4-manifold. Given a Riemannian metric \( g \) and a \( Spin_c \) structure \( S \) on \( X \), there are associated hermitian rank 2 vector bundles \( V_+ \) and \( V_- \), and a bundle isomorphism

\[
\rho : \Lambda_+ \rightarrow su(V_+),
\]

where \( su(V_+) \) is the bundle of anti-Hermitian traceless maps on \( V_+ \). The Seiberg-Witten equations are for a pair \( (A, \Phi) \), where \( A \) is a unitary connection on \( L = \det(V_+) \), and \( \Phi \) a section of \( V_+ \). For any fixed \( \eta \in \Omega^+(X) \), the perturbed Seiberg-Witten equations are

\[
\begin{align*}
D_A \Phi &= 0 \\
\rho(iF_A^+ + \eta) &= (\Phi \otimes \Phi^*)_0
\end{align*}
\]

These equations have a huge degree of freedom. Let \( \mathcal{A} \) denote the set of all unitary connections on \( L \), \( \mathcal{G} \) the group \( Aut(L) = Map(X, S^1) \). \( \mathcal{G} \) is called the gauge group. There is an action of \( \mathcal{G} \) on \( \mathcal{A} \times \Gamma(V_+) \), which preserves the Seiberg-Witten equations. It is given by

\[
g \cdot (A, \Phi) = (A - 2g^{-1}dg, g\Phi).
\]

This action is not free and has two orbit types: if \( \Phi \neq 0 \), the stabilizer of \( (A, \Phi) \) is trivial; on the other hand, the stabilizer of \( (A, 0) \) is \( S^1 \). To fix this problem, we choose an arbitrary point \( x_0 \in X \) and let \( \mathcal{G}_0 = \{ g \in \mathcal{G} | g(x_0) = 1 \} \). Then \( \mathcal{G} = \mathcal{G}_0 \times S^1 \), and furthermore, the action of \( \mathcal{G}_0 \) on \( \mathcal{A} \times \Gamma(V_+) \) is free. We then get the residue \( S^1 \) action on the smooth \( (\mathcal{A} \times \Gamma(V_+))/\mathcal{G}_0 \).

Denote by \( M(S, g, \eta) \) and \( M^0(S, g, \eta) \) the quotients of the space \( M(S, g, \eta) \) of solutions to (1) by \( \mathcal{G} \) and \( \mathcal{G}_0 \) respectively. They have the following well-known properties [6]:

\[
\text{[Insert properties here]}
\]
• (a) $M(S, g, \eta)$ and $M^0(S, g, \eta)$ are compact in suitable topologies.

• (b) For a generic choice of $(g, \eta)$, $M^0(S, g, \eta)$ is a smooth manifold of dimension $d + 1 = 1 + \frac{1}{4}(c_1(L)^2 - 2\chi(X) - 3\tau(X))$, which can be oriented in a natural way.

• (c) For a generic choice of $(g, \eta)$ with $2\pi c_1^+(L) \neq \eta^b$, the harmonic part of $\eta$, $M^0(S, g, \eta)$ does not contain solutions with $\Phi = 0$ (called reducible solutions). The $S^1$-action on $M^0(S, g, \eta)$ then gives rise to a principal $S^1$-bundle $M^0(S, g, \eta) \to M(S, g, \eta)$. We call such a choice of $(g, \eta)$ a good choice. When $b^+_2(X) > 0$, there are good choices.

• (d) For two good choices $(g_0, \eta_0)$ and $(g_1, \eta_1)$, there is a path $(g_t, \eta_t)$ joining them, such that $M^0(S, g_t, \eta_t)$, $0 \leq t \leq 1$, form an oriented cobordism $W$ between $M^0(S, g_0, \eta_0)$ and $M^0(S, g_1, \eta_1)$. When $b^+_2(X) > 1$, it is possible to choose the path such that none of $M^0(S, g_t, \eta_t)$ admits a reducible solution.

For a good choice $(g, \eta)$, the Seiberg-Witten invariant is defined as follows: (a) if $d < 0$, $SW(S, g, \eta) = 0$; (b) if $d = 0$, then $M(S, g, \eta)$ is a finite union of signed points, and $SW(S, g, \eta)$ is the sum of the corresponding $\pm 1$'s; (c) if $d > 0$, the Seiberg-Witten invariant can be defined as the coupling of the fundamental class of $M(S, g, \eta)$ with the suitable power of the first Chern class of the principal $S^1$-bundle $M^0(S, g, \eta) \to M(S, g, \eta)$. So when $b^+_2(X) > 1$, $SW(S, g, \eta)$ does not depend on the good choice $(g, \eta)$ and is then a diffeomorphism invariant. However, if $b^+_2(X) = 1$, $c_1^+(L) = \eta^b$ defines a hypersurface in the space of $(g, \eta)$’s. It is called a “wall”, since it divides the space of $(g, \eta)$’s into two connected components, called “chambers”. For two good choices $(g_0, \eta_0)$ and $(g_1, \eta_1)$, the Seiberg-Witten invariants are the same. However, when they lie in different chambers the invariants may differ. A formula relating the Seiberg-Witten invariants for good choices in different chambers is called a wall crossing formula.

Since the invariant is nontrivial only if $d$ is odd, the above formula for $d$ shows that the only interesting case is when $b^+_2(X) = 1$, and $b_1(X)$ is even. The wall crossing formula of Seiberg-Witten invariants in the case $b^+_2 = 1$, $b_1 = 0$ and $d = 0$ was obtained by Witten [11] and Kronheimer-Mrowka [6]. The general wall crossing formula, proved by Li-Liu [3], can be stated as follows.

**Theorem 3.1** Let $X$ be a closed oriented 4-manifold with $b^+_2 = 1$ and $b_1$ even, $S$ a Spin$_c$ structure with $\det(V_+) = L$, such that $c_1(L)^2 - (2\chi(X) + 3\tau(X)) \geq 0$, then for any two good choices $(g_0, \eta_0)$ and $(g_1, \eta_1)$ in two different chambers, the Seiberg-Witten invariants $SW(S, g_0, \eta_0)$ and $SW(S, g_1, \eta_1)$ differ by

$$\pm \frac{1}{4} \int_{T^1} (\frac{1}{4} \Omega^2 \cdot c_1(L)[X])^{b_1/2}/(b_1/2)!,$$

where

$$\Omega = c_1(U) = \sum_i x_i \cdot y_i,$$

5
and \( U \) is the universal flat line bundle over \( T^{b_1} \times M \), \( \{ y_i \} \) is any basis of \( H^1(X; \mathbb{Z}) \) modulo torsion, and \( \{ x_i \} \) is the dual basis in \( H^1(T^{b_1}; \mathbb{Z}) \).

We will now reprove this theorem by the method described in §2. Take a path \(( g_t, \eta_t )\) that goes through the wall transversally once. Then the \( S^1 \)-action on the induced cobordism \( W \) has only one component in the fixed point set \( F \), namely the set of reducible solutions, which are parameterized by the torus \( T^{b_1} = H^1(X; \mathbb{R})/H^1(X; \mathbb{Z}) \). We shall assume that for each reducible solution \( (A, 0) \), \( \text{Coker}D_A = 0 \). (The general case can be modified by the method of Li-Liu [7], p. 808.) Under this assumption, the normal bundle of \( F \) in \( W \) is given by the index bundle \( \text{ind} \), whose fiber at each \((A, 0) \in F\) is given by \( \text{Ker}D_A \) (cf. Li-Liu [7]). It is clear that the \( S^1 \)-action on this normal bundle has only weight 1. Now we use formula (*) in §2 to get

\[
SW(S, g_1, \eta_1) - SW(S, g_0, \eta_0) = \int_{\partial W/S^1} c^{d/2}
\]

where \( r \) is the complex rank of \( \text{ind} \), so \( 2r + b_1 = d + 2 \). In the proof of Lemma 2.5 in [7], Li-Liu derived, by Atiyah-Singer family index theorem, that

\[
c_1(\text{ind}) = \frac{1}{4} \Omega^2 \cdot c_1(L)[X], \\
c_j(\text{ind}) = \frac{1}{j!} c_1(\text{ind})^j.
\]

Plugging the above equalities into (4), we see that the difference between the two Seiberg-Witten invariants is

\[
\pm \int_{T^{b_1}} \frac{u^{(d+2)/2}}{u^r \exp(c_1(\text{ind})/u)} \\
= \pm \int_{T^{b_1}} u^{-r+(d+2)/2} \exp(-c_1(\text{ind})/u) \\
= \pm \int_{T^{b_1}} u^{-r+(d+2)/2-b_1/2} c_1(\text{ind})^{b_1/2}/(b_1/2)! \\
= \pm \int_{T^{b_1}} c_1(\text{ind})^{b_1/2}/(b_1/2)!
\]

This completes the proof of Theorem 3.1.

4 Applications to Seiberg-Witten theory: The general case

Okonek-Teleman [8] extended the definition of Seiberg-Witten invariants. They also proved a wall crossing formula for such general Seiberg-Witten invariants.
In this section, we will give an equivalent definition of the general Seiberg-Witten invariants, which is along the line of the discussions in the preceding sections. We then reprove Okonek-Teleman’s formula by the localization formula (**).

We use the notations of §3. Let $L \rightarrow X$ be the Hermitian line bundle associated to a fixed $\text{Spin}_c$-structure $S$. For a good choice $(g, \eta)$, let $\pi_2 : \mathcal{M}(S, g, \eta) \times X \rightarrow X$ be the projection onto the second factor. Consider the pullback line bundle $\pi_2^* L$. The group $G$ acts freely on $\pi_2^* L$, which covers a free action of $G$ on $\mathcal{M}(S, g, \eta) \times X$. There is therefore a quotient line bundle

$$\mathcal{L} \rightarrow M(S, g, \eta) \times X.$$ 

We now define a group homomorphism $\mu : H_1(X; \mathbb{Z})/\text{Tor} \rightarrow H^1(M(S, g, \eta); \mathbb{R})$ by

$$\mu([A]) = \int_A c_1(\mathcal{L}),$$

where $A$ is a loop in $X$, and $[A]$ its homology class. It is easy to see that this is well-defined.

Let $d = \frac{1}{4}(c_1(L)^2 - 2\chi(X) - 3\tau(X))$. When $d < 0$, the Seiberg-Witten invariant $SW(S, g, \eta)$ is defined to be zero. When $d = 0$, it is defined as in §3. When $d > 0$, $SW(S, g, \eta)$ is defined as a linear map

$$\Lambda^*(H_1(M, \mathbb{Z})/\text{Tor}) \rightarrow \mathbb{R}.$$ 

More precisely, for $0 \leq k \leq \min\{b_1, d\}$, and $k$ has the same parity as $d$,

$$SW(S, g, \eta)([A_1], \cdots, [A_k]) = \pm \int_{M(S, g, \eta)} \mu([A_1]) \wedge \cdots \wedge \mu([A_k]) \wedge \Omega^l,$$

where $l = \frac{1}{2}(d - k)$ and $c$ is as in §3. For all other values of $k$, the invariant is defined to be zero. We remark that these invariants are actually integer-valued, even though we define them as integrals of differential forms.

Okonek-Teleman’s wall crossing formula can be stated as the following

**Theorem 4.1** For a fixed $\text{Spin}_c$-structure $S$ on a connected closed oriented 4-manifold $X$ with $b_2^+ = 1, d = \frac{1}{4}(c_1(L)^2 - 2\chi(X) - 3\tau(X)) \geq 0$, the Seiberg-Witten invariants for two good choices $(g_0, \eta_0)$ and $(g_1, \eta_1)$ in two different chambers are related by

$$SW(S, g_1, \eta_1)([A_1], \cdots, [A_k]) - SW(S, g_0, \eta_0)([A_1], \cdots, [A_k]) = \pm \int_{\mathcal{T}^{1^+}} \mu([A_1]) \wedge \cdots \wedge \mu([A_k]) \wedge \left( \frac{1}{4} \Omega^2 \cdot c_1(L)[X] \right)^{b_1 - k}/((b_1 - k)/2)!$$

where $[A_1], \cdots, [A_k]$ are homology classes in $H_1(X; \mathbb{Z})/\text{Tor}$, $\Omega$ is as in Theorem 3.1, $0 \leq k \leq \min\{b_1, d\}$, and $k$ has the same parity as $b_1$.

The proof is similar to the proof of Theorem 3.1. To start with, we give a parameterized version of the construction for $\mathcal{L}$. Take a path $(g_t, \eta_t)$ as in §3. Consider the infinite dimensional cobordism

$$\mathcal{W} = \bigcup_t \mathcal{M}(S, g_t, \eta_t).$$
Consider the pullback line bundle $\pi_2^*L$ on $W \times X$, where $\pi_2$ is again the projection onto the second factor. Modulo the action by $G_0$, we get a quotient line bundle

$$L_0 \to W \times X.$$ 

This is actually an $S^1$-bundle, since there is the residue action by $S^1 = G/G_0$. Since this $S^1$-action on $\partial W$ is free, it is straightforward to see that when restricted to $\partial W = M_0(S, g_0, \eta_0) \sqcup M_0(S, g_1, \eta_1)$, $L_0$ can be identified with the pullback of $L$ on $M_0(S, g_j, \eta_j)$ via the projection $M_0(S, g_j, \eta_j) \to M(S, g_j, \eta_j)$, for $j = 1, 2$. Endowing $L_0$ an $S^1$-invariant unitary connection, we then see that $c_1(L_0)$ is represented by an $S^1$-invariant closed 2-form. Define $\mu_0 : H_1(X; \mathbb{Z})/\text{Tor} \to H^1(W; \mathbb{R})$ by

$$\mu_0([A]) = \int_A c_1(L_0).$$

It is easy to see that when restricted to $\partial W$, $\mu_0([A])$ is the pullback of $\mu([A])$ on $M(S, g_0, \eta_0)$ and $M(S, g_1, \eta_1)$ respectively. Now let $\beta_j = \mu_0([A_j])$, for $j = 1, \cdots, k$, a computation similar to the one in §3 by formula (***) then proves Theorem 4.1.

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