Abstract categorial grammars with island constraints and effective decidability

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July 17, 2019

Abstract

A well-known approach to treating syntactic island constraints in the setting of Lambek grammars consists in adding specific bracket modalities to the logic.

We adapt this approach to abstract categorial grammars (ACG). Thus we define bracketed (implicational) linear logic, bracketed \( \lambda \)-calculus, and, eventually bracketed ACG based on bracketed \( \lambda \)-calculus. This allows us modeling at least simplest island constraints, typically, in the context of relativization.

Next we identify specific safely bracketed ACG which, just like ordinary (bracket-free) second order ACG generate effectively decidable languages, but are sufficiently flexible to model some higher order phenomena like relativization and correctly deal with syntactic islands, at least in simple toy examples.

1 Introduction

Abstract categorial grammars (ACG) [2] are a formalism for generating formal languages, similar to well-known Lambek grammars [10], but based on the ordinary (commutative) linear logic [4] and linear \( \lambda \)-calculus. Variants of this formalism are also known as \( \lambda \)-grammars [15] and linear grammars [11].

Unlike more traditional Lambek-style categorial grammars, ACG (and their siblings from [15], [11]) are not restricted to word-by-word processing of continuous strings and can easily manipulate discontinuous syntactic elements (i.e. tuples of strings). This gives them a remarkable flexibility and expressivity. From a certain point of view, ACG seem more simple and natural, being based on a more familiar and intuitive commutative logic.

However, as far as natural language modeling is concerned, ACG turn out to be too flexible and too expressive. If Lambek grammars generate precisely the class of context-free languages [16], which is probably too weak for a natural language, then ACG, in general, can generate NP-complete languages [19],
which is a catastrophe. It seems that the only large class of ACG known to
today to generate effectively decidable languages is second order ACG. But in
essence, second order ACG simply do not use any logic or any λ-calculus at all.
These grammars generate precisely the class of multiple context-free languages
[20], and it is questionable if using λ-terms in the notation adds something re-
really interesting to the much simpler original formalism of multiple context-free
grammars (MCFG) [21].

Unfortunately, potentially explosive parsing complexity is not the only draw-
back of ACG. For example, it was noticed that such grammars behave rather
poorly when modeling coordination [12]. Hybrid type logical grammars [9], which
combine commutative constructions of ACG with non-commutative Lambek-
style operations, might be a promising improvement (but see also [18]).

Another issue, and it is what will be discussed in this work, is how to deal
with syntactic islands, typical for natural languages.

From the point of view of logic and λ-calculus, syntactic island constraints
are restrictions on introducing λ-abstraction in terms and implication in types.
Islands present a problem for Lambek grammars as well, although in Lambek
grammars λ-abstraction is already restricted by very non-commutativity of the
calculus. In the context of non-commutative calculus, an approach to treating
islands was proposed in [13], [14]. It consists in adding to the underlying logic
specific bracket modalities in types and bracketed structure in sequents, which
essentially make the calculus partly non-associative. Types in a bracketed se-
quent are not allowed to move out of brackets and this precludes derivations
introducing unwanted implications. Languages generated by such bracketed
Lambek grammars turn out to be context-free [5], just as in the bracket-free
case. As for parsing such grammars, known algorithms so far are exponential
[8].

There are also proposals for modeling island constraints in the ACG setting.
In particular, see [17], where dependent types are used, and [7], where a general
technique for encoding different language phenomena is discussed.

In this paper, we adapt to ACG the bracket-modality approach of [13], [14].
Thus we define bracketed (implicational) linear logic, then bracketed λ-calculus,
which is typed with bracketed linear logic, and, eventually bracketed ACG based
on bracketed λ-calculus. This allows us modeling at least simplest island con-
straints, typically, in the context of relativization, essentially mimicking struc-
tures of [13].

Next we identify specific safely bracketed and second order safely bracketed
fragments of the logic, which satisfy certain bounded interpolation property. In
particular, any proof of a second order safely bracketed sequent with formulas of
some bounded complexity is equivalent to a one obtained from proofs of smaller
sequents with formulas of the same complexity using only the Cut rule. This
allows us reducing a second order safely bracketed ACG to a weakly equiva-
 lent ordinary (bracket-free) second order ACG, hence to an MCFG (just as any
Lambek grammar is reduced to a context-free grammar in [16]). It follows that
second order safely bracketed ACG of this paper generate effectively decidable
languages. Yet, unlike ordinary second order ACG, they turn out to be suffi-
ciently flexible to model some higher order phenomena like relativization and correctly deal with syntactic islands, at least in simple toy examples.

This effective decidability is the main interest of our approach compared to other proposals. We find quite remarkable that it is precisely the presence of island constraints that blocks explosive complexity of generated languages. Cannot this give a hint to the origin of island constraints (at least, some of them) in the natural language?

We should stress though that second order safely bracketed grammars are still very similar to ordinary second order ACG. For example, they require an excessive amount of atomic types compared to higher order formalisms. It can be said that, “morally”, second order ACG is not so much a categorial (logical) grammar as a generalized context-free formalism (see [6]), close to MCFG. From such a point of view, second order safely bracketed grammar to a large extent, also, is a generalized context-free formalism, but extended with some logical constructions. Well, why not?

Also we make no attempt to approach coordination, which is problematic in ACG. We hope, however, that bracket modalities and safe bracketing eventually can be combined with some hybrid constructions in the style of [9].

What is crucially missing at the moment is some concrete (denotational) model of bracketed logic and bracketed $\lambda$-calculus that would give good understanding of the system. All results so far are obtained by purely syntactic manipulations on terms and derivations, copying, whenever possible, constructions of bracketed Lambek calculus from [13]. It is not clear if the given axiomatic (basically copied from [13]) is indeed well-suited for the ACG setting and cannot be improved or what its possible extensions to other formalisms like [9] should be like. Understanding denotational semantics of bracketed linear logic is a subject of current work.

Finally, we do not propose any direct parsing algorithm. Brutal reduction of a second order safely bracketed ACG to an ordinary ACG and, eventually, to an MCFG is certainly exponential in the size of the original grammar. This subject is left for future study.

2 Bracketed linear logic

In this section we define bracketed linear logic that eventually will be the typing system for our grammars.

Given a set $N$ of atomic types or atomic formulas the set $Tp[N]$ of bracketed types or bracketed formulas is defined by induction:

- If $A \in N$ then $A \in Tp[N]$;
- if $A, B \in Tp[N]$ then

\[ A \rightarrow B, \Box A, \Box^{-1} A \in Tp[N]. \]
Formulas (types) not containing the □ and □⁻¹ connectives are familiar (linear) implicational formulas (types). We denote the set of implicational formulas (types) as $T p(N)$.

From now on we use the words “type” and “formula” as completely synonymous, preferably saying “type” when there are some λ-terms around, and “formula” otherwise.

We will consider specific bracketed sequents, which are defined using configurations of formulas.

A configuration (over a given set Φ) is defined by induction:

- if $A \in \Phi$ then $A$ is an elementary configuration;
- if $\Gamma_1, \ldots, \Gamma_n$, are elementary configurations, then the multiset $\Gamma_1, \ldots, \Gamma_n$ is a configuration;
- if $\Gamma \neq \emptyset$ is a configuration, then $[\Gamma]$ is an elementary configuration.

According to the above definition, the empty multiset is a configuration. In order to have consistent notation, we introduce the convention $[\emptyset] = \emptyset$.

A configuration without brackets is called a context.

A sequent (over $\Phi$) is an expression of the form $\Gamma \vdash A$, where $\Gamma$ is a configuration (over $\Phi$), and $A$ is a formula (from $\Phi$).

In the sequel, a Latin letter always stands for a configuration consisting of a single formula.

In order to formulate sequent calculus rules, we introduce notation for substituting a subconfiguration.

The expression $\Gamma(A)$ denotes a configuration with a selected occurrence of the formula $A$, and $\Gamma(\Delta)$ is the result of substituting the configuration $\Delta$ for $A$.

In details:

- if $\Gamma(A) = A$, then $\Gamma(\Delta) = \Delta$;
- if $\Gamma(A)$ is the multiset $\Gamma_1(A), \Gamma_2, \ldots, \Gamma_n$, then $\Gamma(\Delta) = \Gamma_1(\Delta), \Gamma_2, \ldots, \Gamma_n$;
- if $\Gamma(A) = [\Gamma'(A)]$, then $\Gamma(\Delta) = [\Gamma'(\Delta)]$.

**Definition 1** Bracketed implicational linear logic $\mathbb{LL}_{\preceq, \bot}$ is defined by the following sequent calculus rules.

- $A \vdash A$ (Id),
- $\Gamma \vdash A \quad \Gamma'(A) \vdash B$ \quad $\Gamma(\Gamma) \vdash B$ (Cut),
- $\Gamma, A \vdash B \quad \Gamma(\Gamma) \vdash A \rightarrow B$ \quad $\Gamma(\Gamma) \vdash A \rightarrow B$ (→ R),
- $\Gamma(\Gamma) \vdash A \quad \Gamma'(B) \vdash C$ \quad $\Gamma'(\Gamma, A \rightarrow B) \vdash C$ (→ L),
- $[\Gamma] \vdash \square A$ (□R),
- $\Gamma(\square A) \vdash B \quad [\Gamma(\Gamma)] \vdash B$ (□L),
- $[\Gamma] \vdash A \quad \Gamma(\square^{-1} A) \vdash B$ (□⁻¹ R),
- $\Gamma(\square^{-1} A) \vdash B \quad \Gamma(\square A) \vdash B$ (□⁻¹ L).
Lemma 1 The system $\text{LL}_{\rightarrow,[]}$ is cut-free.

Proof by routine and lengthy induction on derivation. □

Proving cut-elimination by induction on derivation amounts essentially to specifying a cut-elimination algorithm. In the sequel we assume that such an algorithm is indeed specified.

2.1 Natural deduction

We will consider bracketed logic as a typing system for a term calculus extending linear $\lambda$-calculus. Since it is traditional to formulate $\lambda$-calculus in the natural deduction format, we develop a natural deduction system for $\text{LL}_{\rightarrow,[]}$. 

Definition 2 The system $\text{NLL}_{\rightarrow,[]}\triangleq \text{LL}_{\rightarrow,[]} \cup \{\Box\}$ is defined by the following rules.

\[
\frac{}{A \vdash A} \quad \text{(Id)},
\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \quad \text{($\rightarrow$I)},
\frac{\Gamma \vdash A \quad \Gamma' \vdash A \rightarrow B}{\Gamma, \Gamma' \vdash B} \quad \text{($\rightarrow$E)},
\frac{\Gamma \vdash A}{[\Gamma] \vdash \Box A} \quad \text{($\Box$I)},
\frac{\Gamma [\Box][A] \vdash B}{\Gamma'([\Gamma]) \vdash B} \quad \text{($\Box$E)},
\frac{[\Gamma] \vdash A}{\Gamma \vdash \Box^{-1} A} \quad \text{($\Box^{-1}$I)},
\frac{\Gamma \vdash \Box^{-1} A \quad \Gamma'([A]) \vdash B}{\Gamma'([\Gamma]) \vdash B} \quad \text{($\Box^{-1}$E)}.
\]

Note 1 The set of sequents derivable $\text{NLL}_{\rightarrow,[]}\triangleq \text{LL}_{\rightarrow,[]} \cup \{\Box\}$ is closed under the Cut rule.

Proof Induction on derivation. □

Note 2 A sequent $\Gamma \vdash A$ is derivable in the natural deduction system $\text{NLL}_{\rightarrow,[]}\triangleq \text{LL}_{\rightarrow,[]} \cup \{\Box\}$ iff it is derivable in the sequent calculus $\text{LL}_{\rightarrow,[]}\triangleq \text{LL}_{\rightarrow,[]} \cup \{\Box\}$.

There are translations of sequent calculus proofs to natural deduction proofs, and of natural deduction proofs to sequent calculus proofs.

Proof Right introduction rules of sequent calculus proofs are the same as introduction rules of natural deduction. Left introduction rules of the sequent calculus are emulated in natural deduction using the Cut rule, which is admissible by the preceding note. Elimination rules of natural deduction are emulated in the sequent calculus similarly. □

It should be noted though that the two above translations are not mutual inverses.

In the next section we label bracketed sequents with terms and develop a typed term calculus.
3 Term assignment

We use a term language extending linear $\lambda$-calculus. We assume that the reader is familiar with basic notions of $\lambda$-calculus, in particular, $\alpha$-conversion, $\beta$- and $\eta$-reductions and normalization. See [1] for a reference.

Terms of the extended language are built using application, abstraction and four special constants $b, u, b^{-1}, u^{-1}$.

Let a countable set $X$ of variables not containing special constants be given. The set $\Lambda[(X)]$ of bracketed linear $\lambda$-terms over $X$ is defined by induction:

- any $x \in X$ is a term, and $FV(x) = \{x\}$;
- if $M, N$ are terms and $FV(M) \cap FV(N) = \emptyset$, then $K = (M \cdot N)$ is a term, and $FV(K) = FV(M) \cup FV(N)$;
- if $M$ is a term and $c$ is a special constant then $K = (c \cdot M)$ is a term, and $FV(K) = FV(M)$;
- if $M$ is a term and $x \in FV(M)$, then $K = \lambda x.M$ is a term, and $FV(K) = FV(M) \setminus \{x\}$.

For a term $K$, elements of the set $FV(K)$ are free variables of $K$. Variable occurrences in $K$ which are not free are bound. We identify terms differing by renaming bound variables.

We use the usual notational conventions: the outermost pair of brackets is dropped, the application symbol is omitted (i.e. $MN = M \cdot N$) application is left-associative (i.e. $((MN)K) = MNK$), nested abstractions are abbreviated $(\lambda x.(\lambda y.M)) = \lambda xy.M$, and the body of abstraction extends to the right as much as possible.

Terms without special constants are familiar linear $\lambda$-terms, we call them bracket-free. We denote the set of bracket-free terms as $\Lambda(X)$.

Labeled configurations are defined recursively, similarly to ordinary configurations.

- An expression of the form $x : A$, where $A$ is a type and $x$ is a variable, is an elementary labeled configuration;
- if $\Gamma_1, \ldots, \Gamma_n$ are elementary labeled configurations not having common variables, then the multiset $\Gamma_1, \ldots, \Gamma_n$ is a labeled configuration;
- if $\Gamma$ is a nonempty labeled configuration, then $[\Gamma]$ is an elementary labeled configuration.

A labeled configuration without brackets is a labeled context.

A typing judgement is an expression of the form $\Gamma \vdash M : A$, where $\Gamma$ is a labeled configuration, $M$ is a term and $A$ is a type.

If $\Gamma$ is a labeled configuration and $\overline{\Gamma}$ is a configuration obtained by erasing from $\Gamma$ all variables, we say that $\Gamma$ is a labeling of $\overline{\Gamma}$. 
Similarly, if a labeled configuration \( \Gamma \) is a labeling of \( \Gamma \), then any typing judgement \( \Gamma \vdash M : A \) is a labeling of the sequent \( \Gamma \vdash A \).

We now develop a calculus of typing judgements, which is a labeling of bracketed linear logic proofs.

### 3.1 Bracketed \( \lambda \)-calculus

**Definition 3** The system \( \lambda \text{NLL}_{\to,[]} \) is defined by the following rules

\[
\begin{align*}
x : A \vdash x : A \\ (\text{Id}), \\
\Gamma, x : A \vdash M : B & \quad \Gamma \vdash \lambda x. M : A \to B \\
(\to \text{I}), \\
\Gamma \vdash M : A & \quad \Gamma' \vdash N : A \to B \\
\Gamma, \Gamma' \vdash NM : B \\
(\to \text{E}), \\
\Gamma \vdash M : A & \quad [\Gamma] \vdash bM : \Box A \\
(\Box \text{I}), \\
\Gamma' \vdash N : B & \quad \Gamma'(\Gamma) \vdash N[x := b^{-1}M] : B \\
\Gamma' \vdash N : B & \quad \Gamma'[\Gamma] \vdash N[x := u^{-1}M] : B \\
(\Box \to \text{I}), \\
[\Gamma] \vdash uM : \Box^{-1}A & \quad \Gamma'[\Gamma] \vdash N[x := u^{-1}M] : B \\
(\Box^{-1} \text{I}), \\
\Gamma' \vdash N : B & \quad \Gamma'[\Gamma] \vdash N[x := u^{-1}M] : B \\
(\Box^{-1} \to \text{E}), \\
\end{align*}
\]

The fragment of the above system not involving \( \Box \) and \( \Box^{-1} \)-connectives in types, special constants in terms and brackets in configurations is the familiar linear typed \( \lambda \)-calculus. We denote it as \( \lambda \text{NLL}_{\to} \). Note, however, that, unlike \( \lambda \text{NLL}_{\to} \), the full system \( \lambda \text{NLL}_{\to,[]} \) does not satisfy the familiar property that any derivable typing judgement has unique derivation.

Now let \( \pi \) be a natural deduction proof of a sequent \( \Gamma \vdash A \), and let \( \tilde{\pi} \) be some labeling of \( \Gamma \). The following is immediate.

**Note 3** There is a unique term \( M \) and a derivation \( \tilde{\pi} \) of the labeled sequent \( \tilde{\Gamma} \vdash M : A \) in \( \lambda \text{NLL}_{\to,[]} \) such that \( \pi \) is obtained from \( \tilde{\pi} \) by erasing all terms. \( \Box \)

In notation as above we say that \( \tilde{\pi} \) is the labeling of \( \pi \) and \( \tilde{\Gamma} \vdash M : A \) is the labeling of the conclusion of \( \pi \) induced by \( \tilde{\Gamma} \).

Observe that Note 1 lifts to the labeled setting.

**Note 4** The system \( \lambda \text{NLL}_{\to,[]} \) is closed under the Substitution rule:

\[
\Gamma \vdash M : A \\
\Gamma'(x : A) \vdash N : B \\
\Gamma'(\Gamma) \vdash N[x := M] : B.
\]

### 3.2 In the sequent calculus format

We now assign terms to sequent calculus proofs as well.

**Definition 4** The system \( \lambda \text{LL}_{\to,[]} \) is defined by the following rules

\[
\begin{align*}
x : A \vdash x : A \\
(\text{Id}), \\
\Gamma \vdash M : A & \quad \Gamma'(x : A) \vdash N : B \\
\Gamma'(\Gamma) \vdash N[x := M] : B \\
(\text{Cut}), \\
\end{align*}
\]
\[ \Gamma, x : A \vdash M : B \]
\[ \Gamma \vdash \lambda x. M : A \rightarrow B \quad (\sim R), \]
\[ \Gamma \vdash M : A \quad \Gamma'(x : B) \vdash N[x := fM] : C \quad (\sim L), \]
\[ \Gamma \vdash M : A \quad \Gamma'(x : A) \vdash M : B \quad (\square R), \]
\[ \Gamma \vdash M : A \quad \Gamma'(x : A) \vdash M[x := b^{-1}x'] : B \quad (\square L), \]
\[ [\Gamma] \vdash M : A \quad \Gamma \vdash uM : \square^{-1}A \quad (\square^{-1} R), \]
\[ \Gamma \vdash \Gamma : A \quad \Gamma \vdash [x : A] \vdash M[x := b^{-1}x'] : B \quad (\square^{-1} L). \]

Obviously, just as in the case of natural deduction, given a sequent calculus proof \( \pi \), we can define a labeling of \( \pi \) in \( \lambda LL \). Then Note 2 lifts to the labeled setting.

**Note 5** A typing judgement \( \Gamma \vdash M : A \) is derivable in \( \lambda LL \) iff it is derivable in \( \lambda LL \).

We want, however, to have a one-to-one translation between natural deduction and sequent calculus formats. Thus, we are going to introduce an equivalence relation on derivations. The equivalence comes from \( \beta \)-equivalence of terms, which we define next.

### 3.3 Normalization

We define \( \beta \)-reductions of bracketed \( \lambda \)-terms, extending familiar \( \beta \)-reduction of ordinary \( \lambda \)-calculus.

**Definition 5** Binary relation \( \mapsto_{\beta} \) of one-step \( \beta \)-reducibility on terms is the smallest relation satisfying the properties

\[ (\lambda x. M)N \mapsto_{\beta} N[x := M] \]

\[ u^{-1}(uM) \mapsto_{\beta} M, \quad b^{-1}(bM) \mapsto_{\beta} M, \]

and if \( M \mapsto_{\beta} M' \) then

\[ MN \mapsto_{\beta} M'N, \quad NM \mapsto_{\beta} NM', \quad \lambda x. M \mapsto_{\beta} \lambda x. M', \]

and

\[ cM \mapsto_{\beta} cM' \]

for any special constant \( c \).

The relation \( \rightarrow_{\beta} \) of \( \beta \)-reducibility is the reflexive transitive closure of \( \mapsto_{\beta} \).

We say that a term \( M \) reduces to \( M' \) if \( M \rightarrow_{\beta} M' \).

A term is normal if it does not reduce to any term other than itself.

A term \( M' \) is a normal form of \( M \) if \( M \rightarrow_{\beta} M' \) and \( M' \) is normal.

**Lemma 2** If a typing judgement \( \Gamma \vdash M : A \) is derivable in \( \lambda LL \) then \( M \) has a unique normal form \( M' \) and the typing judgement \( \Gamma \vdash M' : A \) is derivable in \( \lambda LL \) as well.
Proof by induction on derivation. □

We say that a typing judgement $\Gamma \vdash M : A$ is normal if the term $M$ is normal.

Lemma 3 (Subformula property) If a normal typing judgement $\Gamma \vdash M : A$ is derivable in $\lambda NLL_{\rightarrow,[]}$, then all types occurring in its derivation are subformulas of types occurring in $\Gamma$ and $A$.

Proof By induction on derivation we establish that if

- $A = A_1 \rightarrow A_2$ and $M \neq \lambda x.M'$, or
- $A = \Box A'$ and $M \neq bM'$, or
- $A = \Box^{-1} A'$ and $M \neq uM'$,

then $A$ is a subformula of a type occurring in $\Gamma$.

Using the above, we prove the lemma, again by induction on derivation. □

We say that two terms are $\beta$-equivalent if they have the same normal form.

Similarly, we say that two derivable typing judgements $\Gamma \vdash M_i : A$, $i = 1, 2$, are $\beta$-equivalent if $M_1$ is $\beta$-equivalent to $M_2$.

Now let $\pi_1$, $\pi_2$ be natural deduction proofs of the same sequent $\Gamma \vdash A$.

Let $\tilde{\Gamma}$ be some labeling of $\Gamma$. We say that natural deduction proofs $\pi_1$, $\pi_2$ are equivalent if labelings of their conclusions induced by $\tilde{\Gamma}$ are $\beta$-equivalent. (See Note 3).

Obviously the above definition does not depend on a choice of $\tilde{\Gamma}$.

3.4 Relationship with cut-elimination

From the point of view of sequent calculus, normalization corresponds to cut-elimination, and term assignment is a way to define an equivalence relation identifying a sequent proof with its cut-free form.

Note 6 A typing judgement derivable in $\lambda LL_{\rightarrow,[]}$ without the Cut rule is normal. □

Note 7 Let a sequent calculus proof $\pi$ of the sequent $\Gamma \vdash A$ reduce to a proof $\pi'$ by cut-elimination.

If $\tilde{\Gamma}$ is a labeling of $\Gamma$, and $\tilde{\pi}$, $\tilde{\pi}'$ are labelings of, respectively, $\pi$, $\pi'$ induced by $\tilde{\Gamma}$, with conclusions, respectively, $\tilde{\Gamma} \vdash M : A$ and $\tilde{\Gamma} \vdash M' : A$, then $M$ $\beta$-reduces to $M'$. □

We define equivalence of sequent calculus proofs, just as for natural deduction proofs.

Namely two sequent proofs of $\Gamma \vdash A$ are equivalent if some (hence any) labeling $\tilde{\Gamma}$ of $\Gamma$ induces $\beta$-equivalent labelings of their conclusion.

Then we easily observe the following.
Note 8 Any sequent calculus proof is equivalent to its cut-free form.

The correspondence between sequent calculus proofs and natural deduction proofs is one-to-one up to equivalence. □

Note 9 Given a labeling $\tilde{\Gamma}$ of a configuration $\Gamma$, there is a one-to-one correspondence $\beta$-equivalence classes of derivable typing judgements of the form $\tilde{\Gamma} \vdash M : A$ and equivalence classes of sequent calculus proofs of $\Gamma \vdash A$. □

4 Commutation of rules

The material of this section is a digression and will not be used in the rest of the paper. Yet the question we are considering here certainly deserves attention.

Unlike the case of usual typed linear $\lambda$-calculus, in the system $\text{\textsc{NLL}}_{\text{\text{-}e,\text{-}v}}$, a derivable typing judgement may have different derivations. It seems natural to select some standard form of a derivation, if possible.

This is indeed possible by the following lemma.

Lemma 4 Let the typing judgement

$$\Gamma \vdash M : A$$

be derivable in $\text{\textsc{NLL}}_{\text{-e,\text{-}v}}$.

Then $M \neq b^{-1}M'$ and we have the following possibilities:

(i) $M = x$ is a variable and $\Gamma = x : A$, or

(ii) $M = NK$ with $N \neq b, u, b^{-1}, u^{-1}$, $\Gamma = \Gamma', \Gamma''$, and for some type $X$ we have derivable sequents

$$\Gamma' \vdash K : X, \quad \Gamma'' \vdash N : X \rightarrow A,$$

or

(iii) $M = \lambda x.M'$, $A = A_1 \rightarrow A_2$, and the sequent

$$\Gamma, x : A_1 \vdash M' : A_2$$

is derivable, or

(iv) $M = bM'$, $\Gamma = [\Gamma']$, $A = \Box A'$ and the sequent

$$\Gamma' \vdash M' : A'$$

is derivable, or

(v) $M = uM'$, $A = \Box^{-1}A'$ and the sequent

$$[\Gamma] \vdash M' : A'$$

is derivable, or
(vi) $M = u^{-1}M'$, $M' \neq b^{-1}M''$, $\Gamma = [\Gamma']$ and the sequent

$$\Gamma' \vdash M' : \square^{-1}A$$

is derivable, or

(vii) $M = u^{-1}(b^{-1}M')$, and the sequent

$$\Gamma \vdash M' : \square\square^{-1}A$$

is derivable.

**Proof** by induction on $M$.

A nontrivial case is when $M = u^{-1}M'$.

Then the last rule in the derivation of (1) must be $(\square^{-1}E)$ or $(\square E)$.

Assume that the last rule is $(\square^{-1}E)$.

Then we have representations

$$\Gamma = \Gamma_1([\Gamma_2]), \quad M' = N[x := u^{-1}K],$$

and for some type $X$ the sequents

$$\Gamma_1 \vdash K : \square^{-1}X, \quad \Gamma_2(x : X) \vdash N : A$$

are derivable.

If $N \neq x$, the statement follows from the induction hypothesis applied to $N$.

Otherwise, from the induction hypothesis applied to $N$ we get

$$X = A, \quad \Gamma_2(x : X) = x : A.$$ 

Hence $\Gamma = [\Gamma_1], K = M'$, and possibility (vi) holds. (Note that, by the induction hypothesis applied to $M' = K$, we have $M' \neq b^{-1}M''$.)

Assume now that the last rule in the derivation of (1) is $(\square E)$.

Then we have representations

$$\Gamma = \Gamma_1(\Gamma_2), \quad M' = N[x := b^{-1}K],$$

and for some type $X$ the sequents

$$\Gamma_1 \vdash K : \square X, \quad \Gamma_2([x : X]) \vdash u^{-1}N : A$$

are derivable.

If $N \neq x$, the statement follows from the induction hypothesis applied to $u^{-1}N$.

Otherwise, $N = u^{-1}x$, and by the induction hypothesis we have

$$X = \square^{-1}A, \quad \Gamma_2([x : X]) = [x : A].$$

Hence $\Gamma = \Gamma_1$ and $M = u^{-1}b^{-1}K$. Thus possibility (vii) holds. $\square$
A pleasant (and probably not surprising) corollary is that in a derivable typing judgement (or, by Note 9, in a sequent proof) all brackets can be erased without loss of any information.

If \( \Gamma \) is a (labeled) configuration, let us say that the underlying (labeled) context \( \Gamma_0 \) is the (labeled) context obtained from \( \Gamma \) by erasing all brackets.

Conversely, if \( \Gamma_0 \) is a (labeled) context we say that \( \Gamma \) is a bracketing of \( \Gamma_0 \), if \( \Gamma_0 \) is the underlying context of \( \Gamma \).

Lemma 4 immediately yields the following.

**Corollary 1** If \( M \) is a term, \( A \) is a type, and \( \Gamma_0 \) is a labeled context, such that for some bracketing \( \Gamma \) of \( \Gamma_0 \) the typing judgement \( \Gamma \vdash M : A \) is derivable, then this bracketing \( \Gamma \) is unique. \( \square \)

### 5 Safe bracketing and bounded interpolation

We now discuss a specific safely bracketed fragment, which will be used later to define effectively decidable safely bracketed grammars.

Let \( A \) be a \( \Box^{-1} \)-free formula.

We define the order \( \text{ord}(A) \) of \( A \) by induction:

- \( \text{ord}(A) = 1 \), if \( A \) is atomic;
- \( \text{ord}(\Box A) = \text{ord}(A) \);
- \( \text{ord}(A_1 \rightarrow A_2) = \max(\text{ord}(A_1) + 1, \text{ord}(A_2)) \).

For a general formula \( A \), we say that \( A \) is safely bracketed if

- any subformula \( \Box A' \) of \( A \) is \( \Box^{-1} \)-free;
- for any subformula \( A_1 \rightarrow A_2 \) of \( A \), the antecedent \( A_1 \) is \( \Box^{-1} \)-free.

We define the order \( \text{ord}(A) \) of a safely bracketed formula as the maximal order of its \( \Box^{-1} \)-free subformula.

We say that a configuration \( \Gamma \) is safely bracketed if all formulas occurring in \( \Gamma \) are safely bracketed.

We define the order \( \text{ord}(\Gamma) \) of a safely bracketed configuration \( \Gamma \) as

\[
\text{ord}(\Gamma) = \max_{Z \in \Gamma} \text{ord}(Z).
\]

We will be interested in the second order safely bracketed fragment. Its crucial property is specific bounded interpolation, which we are going to discuss now.
5.1 Bounded interpolation

We will need several parameters measuring complexity of formulas and sequents. For a formula \( A \), the bare size \(| A |\) of \( A \) is defined inductively by

- \(| p | = 1\) for \( p \) atomic;
- \(| A_1 \rightarrow A_2 | = | A_1 | + | A_2 |\);
- \(| □ A | = | □^{-1} A | = | A |\).

The \( □ \)-rank and \( □^{-1} \)-rank of \( A \), respectively, \( rk □ ( A ) \) and \( rk □^{-1} ( A ) \) are defined by

- \( rk □ ( p ) = rk □^{-1} ( p ) = 0 \) for \( p \) atomic;
- \( rk □ ( A_1 \rightarrow A_2 ) = \max ( rk □ ( A_1 ), rk □ ( A_2 ) ) \), \( rk □^{-1} ( A_1 \rightarrow A_2 ) = \max ( rk □^{-1} ( A_1 ), rk □^{-1} ( A_2 ) ) \);
- \( rk □ ( □ A ) = rk □ ( A ) + 1 \), \( rk □^{-1} ( □ A ) = rk □^{-1} ( A ) \);
- \( rk □ ( □^{-1} A ) = rk □ ( A ) \), \( rk □^{-1} ( □^{-1} A ) = rk □^{-1} ( A ) + 1 \).

For a configuration \( Γ \) the bare size and \( □ \), \( □^{-1} \)-ranks are defined, respectively, as

\[ | Γ | = \sum_{Z ∈ Γ} | Z |, \quad rk □ ( Γ ) = \max_{Z ∈ Γ} rk □ ( Z ), \quad rk □^{-1} ( Γ ) = \max_{Z ∈ Γ} rk □^{-1} ( Z ). \]

The bare size norm \(| | Γ | |\) is defined as

\[ | | Γ | | = \max_{Z ∈ Γ} | Z |. \]

The cardinality (i.e. number of formula occurrences) of \( Γ \) is denoted as \( \#( Γ )\).

Finally, given a configuration \( Γ \) with an occurrence of a formula \( A \), the degree \( deg_Γ ( A ) \) of \( A \) in \( Γ \) is defined by induction:

- \( deg_Γ ( A ) = 0 \);
- \( deg_{Γ_1,Γ_2} = deg_{Γ_1} ( A ) \), if \( A \) occurs in \( Γ_1 \);
- \( deg_{Γ} ( A ) = deg_{Γ} ( A ) + 1 \).

5.1.1 Interpolation in the safely bracketed case

Note 10 Let \( A, B \) be safely bracketed formulas, and let \( Γ \) be a safely bracketed configuration with an occurrence of \( B \), such that the sequent \( Γ ⊢ A \) is derivable. Then \( deg_Γ ( B ) ≥ \max rk □^{-1} ( B ) - rk □^{-1} ( A ) \).

Proof by induction on a cut-free derivation. □
Corollary 2 Let \( A, B \) be safely bracketed formulas with \( rk_{\square^{-1}}(B) \geq rk_{\square^{-1}}(A) \), and let \( \Gamma \) be a safely bracketed configuration with an occurrence of \( B' = \square^{-1}B \).

If \( \pi \) is a proof of \( \Gamma \vdash A \), then, up to equivalence of proofs, the last rule in \( \pi \) is \((\square^{-1}L)\) introducing \( B' \).

Proof by induction on a cut-free form of \( \pi \). (Using Note 10 when the last rule in \( \pi \) is \((\square^{-1}R)\).) \( \square \)

Corollary 3 Let \( A, B \) be safely bracketed formulas with \( rk_{\square^{-1}}(B) \geq rk_{\square^{-1}}(A) \), and let \( \Gamma \) be a safely bracketed configuration with an occurrence of \( B' = \square^{-1}B \).

If \( \pi \) is a proof of \( \Gamma \vdash A \), then there exists a representation \( \Gamma = \Gamma'([B']) \), and \( \pi \), up to equivalence, is obtained from proofs \( \pi_0, \pi' \) of, respectively, the sequents \([B'] \vdash B \) and \( \Gamma'(B) \vdash A \) using the Cut rule.

Proof Follows immediately from the preceding corollary. We take as \( \pi_0 \) the proof
\[
\frac{B \vdash B}{[B'] \vdash B \quad (\square^{-1}R)}
\]
corresponding to the derivable typing judgement \([x : B'] \vdash u^{-1}x : B \). \( \square \)

Corollary 4 Let \( A, B \) be safely bracketed formulas with \( rk_{\square^{-1}}(B) > rk_{\square^{-1}}(A) \), and let \( \Gamma \) be a safely bracketed configuration with an occurrence of \( B \).

Assume that \( \pi \) is a proof of \( \Gamma \vdash A \).

If \( B \) is not of the form \( \square^{-1}B' \), then \( B \) is of the form \( B = X \rightarrow \square^{-1}Y \), and there exists a representation
\[
\Gamma = \Gamma'([\Gamma_0, B])
\]
such that \( \pi \) is equivalent to a proof obtained from proofs \( \pi_0, \pi', \tilde{\pi} \) using the Cut rule as follows:
\[
\begin{array}{c}
\pi_0 \\
\Gamma_0 \vdash X \\
[\Gamma_0, B] \vdash Y \\
\end{array}
\frac{(\text{Cut}) \quad \pi'}{\Gamma \vdash A}
\\frac{\tilde{\pi}}{[\Gamma_0, B] \vdash Y}
\]
\[
\frac{Y \vdash Y}{X \vdash X \quad (\square^{-1}Y) \vdash E(\square^{-1}E) \quad (\rightarrow L)}
\]
corresponding to the derivable typing judgement
\[
[x : X, f : B] \vdash u^{-1}(fx) : Y.
\]
\( \square \)

For a configuration \( A \), let us denote the set of all subformulas occurring in \( \Gamma \) as \( Sf(\Gamma) \).
Lemma 5 Let $\pi$ be a proof of a safely bracketed sequent $\Gamma \vdash A$, where $A$ is $\square^{-1}$-free.

Then there exists a finite sequence of configurations

\[ \Gamma_1, \ldots, \Gamma_n \]

over $\text{Sf}(\Gamma, A)$ and formulas (interpolants)

\[ X_1, \ldots, X_n \in \text{Sf}(\Gamma, A), \]

where for each $i = 1, \ldots, n$, either $\Gamma_i$, $X_i$ are $\square^{-1}$-free or $\#(\Gamma_i) \leq 2$, such that $\pi$ is equivalent to a proof obtained from proofs $\pi_1, \ldots, \pi_n$ of, respectively,

\[ \Gamma_1 \vdash X_1, \ldots, \Gamma_n \vdash X_n \]

using only the Cut rule.

Proof by induction on the number of $\square^{-1}$ occurrences in $\Gamma$, using Corollaries 3 and 4. (When applying Corollary 4, it should be remembered that if a formula $B = X \rightarrow \square^{-1} Y$ is safely bracketed, then $X$ is $\square^{-1}$-free.) □

5.1.2 Interpolation in the $\square^{-1}$-free second order case

Lemma 6 Let $\Gamma$ be a $\square^{-1}$-free configuration and $A$, a $\square^{-1}$-free formula, such that $\text{ord}(\Gamma), \text{ord}(A) \leq 2$.

If the sequent $\Gamma \vdash A$ is derivable, then $|A| = |\Gamma| - 2\#(\Gamma) + 2$.

Proof by induction on a cut-free derivation. □

Corollary 5 Let $A, B, C$ be $\square^{-1}$-free formulas with $\text{ord}(A), \text{ord}(B) \leq 2$, $\text{ord}(C) = 1$.

If $\Gamma$ is a configuration whose underlying context is $B, C$, and the sequent $\Gamma \vdash A$ is derivable, then $|A| = |B| - 1$. □

Corollary 6 Let $A, X$ be $\square^{-1}$-free formulas with $\text{ord}(A), \text{ord}(X) \leq 2$.

Let $\Gamma$ be a $\square^{-1}$-free configuration with an occurrence of $X$, such that all formulas in $\Gamma$ except possibly $X$ are of second order.

If the sequent $\Gamma \vdash A$ is derivable then $|X| \leq |A|$.

Proof Observe that any second order formula has bare size greater than 1, so $|\Gamma| \geq 2\#(\Gamma) - 2 + |X|$. □

Corollary 7 Let $\Gamma$ be a first order $\square^{-1}$-free configuration with $\#(\Gamma) > 0$, and $A$ be a $\square^{-1}$-free formula with $\text{ord}(A) \leq 2$.

If the sequent $\Gamma \vdash A$ is derivable, then $\#(\Gamma) = 1$ and $\text{ord}(A) = 1$. □

Lemma 7 Let $\Gamma_1, \Gamma_2$ be $\square^{-1}$-free configurations with $\text{ord}(\Gamma_1) = 1$, $\text{ord}(\Gamma_2) = 2$, $\#(\Gamma_2) > 1$ and $\Gamma_1$ bracket-free.

Let $A$ be a $\square^{-1}$-free formula with $\text{ord}(A) \leq 2$.

If $\pi$ is a proof of $\Gamma \vdash A$, where $\Gamma = \Gamma_1, \Gamma_2$, then
(i) there exist a configuration \( \Gamma_0 \) with \(#(\Gamma_0) = 2 \), a representation

\[ \Gamma_2 = \Gamma'(\Gamma_0) \]

and a \( \square^{-1} \)-free formula (interpolant) \( X \) with

\[ \text{ord}(X) \leq 2, \quad \text{rk}_\square(X) \leq \text{rk}_\square(\Gamma, A), \]

such that, up to equivalence, \( \pi \) is obtained by the Cut rule from proofs of the sequents \( \Gamma_0 \vdash X \) and \( \Gamma_1, \Gamma'(X) \vdash A \);

(ii) moreover, if there is a first order formula \( C \) in \( \Gamma_2 \), then \( \Gamma_0 \) can be chosen such that \( C \in \Gamma_0 \).

Proof (sketch): Induction on a cut-free form of \( \pi \).

In any case, if \( #(\Gamma_2) = 2 \), then we write \( \Gamma_1 = A_1, \ldots, A_k \) and put \( \Gamma_0 = \Gamma_2 \), \( X = A_1 \to \ldots \to A_k \to \neg A \).

So assume that \( #(\Gamma_2) > 2 \)

When the last rule in \( \pi \) is \((\to R), (\square L) \) or \((\square R)\), the statement immediately follows from the induction hypothesis.

Assume that the last rule in \( \pi \) is \((\to L)\).

Then there is a representation \( \Gamma = \Gamma^r(\Gamma^l, S \to T) \) and \( \pi \) has the form

\[
\frac{\pi^l}{\Gamma^l \vdash S} \quad \frac{\pi^r}{\Gamma^r(T) \vdash A} \quad (\to L).
\]

It follows that there are representations

\[ \Gamma^l = \Gamma^l_1, \Gamma^l_2, \quad \Gamma^r = \Gamma^r_1, \Gamma^r_2(T), \quad \text{and} \quad \Gamma_2 = \Gamma_2^r(\Gamma^l_1, S \to T). \]

(Where \( \Gamma^l_1 \) can be nonempty only if \( \text{deg}(S \to T) = 0 \).)

Let us prove claim \((i)\) of the Lemma.

We assumed that \( #(\Gamma_2) > 2 \). This means that \( #(\Gamma^l_1) > 1 \) or \( #(\Gamma^r_2(T)) > 1 \).

If \( #(\Gamma^l_1) > 1 \), then, by Corollary 7, we have \( \text{ord}(\Gamma^l_1) = 2 \). Then we can use the induction hypothesis applied to \( \pi^l \), and the claim follows.

If \( #(\Gamma^r_2(T)) > 1 \), the reasoning is similar.

Now let us prove claim \((ii)\).

Assume that there is a first order formula \( C \) in \( \Gamma_2 \).

If \( C \in \Gamma^r_2(T) \), then \( C \neq T \), hence \( #(\Gamma^r_2(T)) > 1 \), and, again we use the induction hypothesis applied to \( \pi^r \) and deduce the claim.

If \( C \in \Gamma^l_1 \) and \( #(\Gamma^l_1) > 1 \), the reasoning is similar.

Finally, if \( C \in \Gamma^l_2 \) and \( #(\Gamma^l_2) = 1 \), so that \( \Gamma^l_2 \) contains no other formula, then \( \text{ord}(\Gamma^l_2) = 1 \), and, by Corollary 7, the configuration \( \Gamma^l_1 \) is empty. So \( \Gamma^l = C \).

Then we take \( \Gamma_0 = C, S \to T \) and \( X = T \). \( \square \)

Lemma 8 Let \( \Gamma \) be a \( \square^{-1} \)-free configurations with \( \text{ord}(\Gamma) \leq 2 \), \#(\Gamma) > 1, and \( A \) be a \( \square^{-1} \)-free formula with \( \text{ord}(A) \leq 2 \).
If $\pi$ is a proof of the sequent $\Gamma \vdash A$, then there exist a configuration $\Gamma_0$, a representation
$$\Gamma = \Gamma'(\Gamma_0), \quad \#(\Gamma_0) = 2$$
and a $\Box^{-1}$-free formula (interpolant) $X$ with
$$\text{ord}(X) \leq 2, \quad \text{rk}(X) \leq \text{rk}(\Gamma, A), \quad |X| \leq ||\Gamma, A||,$$
such that, up to equivalence, $\pi$ is obtained from proofs of the sequents $\Gamma_0 \vdash X$ and $\Gamma'(X) \vdash A$ by the Cut rule.

**Proof** By Corollary 7 we have ord($\Gamma$) = 2.

We put $\Gamma_2 = \Gamma$, $\Gamma_1 = \emptyset$, and apply Lemma 7.

This gives us a subconfiguration $\Gamma_0$ with a representation $\Gamma = \Gamma'(\Gamma_0)$ and an interpolant $X$.

Moreover, if there is a first order formula $C$ in $\Gamma$, we choose $\Gamma_0$ such that $C \in \Gamma$. Then we apply Corollary 5 to the derivable sequent $\Gamma_0 \vdash X$.

And if there is no first order formula in $\Gamma$, we apply Corollary 6 to the derivable sequent $\Gamma'(X) \vdash A$. $\square$

### 5.1.3 Interpolation in the second order safely bracketed case

Let $\Gamma$ be a configuration or a finite set of formulas.

We say that $Z$ is a generalized subformula of $\Gamma$, if
$$|Z| \leq ||\Gamma||, \quad \text{rk}(Z) \leq \text{rk}(\Gamma), \quad \text{rk}(\Box^{-1}(Z)) \leq \text{rk}(\Box^{-1}(\Gamma)),$$
(2)
and all atomic formulas occurring in $Z$ are occurring in $\Gamma$.

Let us denote the set of generalized subformulas of $\Gamma$ as $S^\text{gen}_f(\Gamma)$.

**Lemma 9** Let $\pi$ be a proof of a safely bracketed sequent $\Gamma \vdash A$, where $A$ is $\Box^{-1}$-free and ord($\Gamma$), ord($A$) $\leq 2$.

Then there exists a finite sequence of safely bracketed configurations $\Gamma_1, \ldots, \Gamma_n$ over $S^\text{gen}_f(\Gamma, A)$ and safely bracketed interpolants $X_1, \ldots, X_n \in S^\text{gen}_f(\Gamma, A)$, where for all $i = 1, \ldots, n$ the formula $X_i$ has the order not greater than 2, and the configuration $\Gamma_i$ has both the order and the cardinality not greater than 2, such that $\pi$ is equivalent to a proof obtained from proofs of, respectively,
$$\Gamma_1 \vdash X_1, \ldots, \Gamma_n \vdash X_n$$
using only the Cut rule.

**Proof** immediate from Lemmas 5 and 8. $\square$

Now we are prepared to discuss abstract categorial grammars.

### 6 Abstract categorial grammars

In this section we recall standard (bracket-free) abstract categorial grammars [2].
6.1 Linear signatures

**Definition 6** A linear signature, or, simply, signature \( \Sigma \) is a tuple \( \Sigma = (N, X, C, \tau) \), where \( N \) is a set of atomic types, \( X \) is a set of variables, \( C \) is a set of constants, \( C \cap X = \emptyset \) and

\[
\tau : C \rightarrow Tp(N)
\]

is a type assignment map.

We denote the set of linear bracket-free \( \lambda \)-terms built from \( X \) and \( C \) as \( \Lambda(X, C) \). That is \( \Lambda(X, C) = \Lambda(X \cup C) \).

Given a signature \( \Sigma = (N, X, C, \tau) \), the signature axioms of \( \Sigma \) are the labeled sequents

\[
\Gamma \vdash c : \tau(c), \text{ for } c \in C.
\]

A typing judgement is derivable in \( \Sigma \) (notation: \( \Gamma \vdash_{\Sigma} M : A \)) if it is derivable from signature axioms using rules of linear \( \lambda \)-calculus.

We say that a term \( M \) is typeable in \( \Sigma \) if there is a type \( A \) such that \( \vdash_{\Sigma} M : A \). In this case we say that \( A \) is the type of \( t \) in \( \Sigma \).

**Note 11** In notation as above, a typing judgement \( \Gamma \vdash M : A \) is derivable in \( \Sigma \) iff there exist constants

\[ c_1, \ldots, c_n \in C \]

and a term \( M' \) such that

\[
M = M'[x_1 := c_1, \ldots, x_n := c_n]
\]

and the typing judgement

\[
\Gamma, x_1 : \tau(c_1), \ldots, x_n : \tau(c_n) \vdash M' : S
\]

is derivable in \( \lambda LL_{\omega} \).

**Proof** by induction on derivation. \( \square \)

### 6.1.1 String signature

Let \( T \) be a finite alphabet.

The string signature \( Str_T \) over \( T \) has a single atomic type \( O \), the alphabet \( T \) as the set of constants and the typing assignment

\[
\tau(c) = O \rightarrow O \ \forall c \in T.
\]

We denote the type \( O \rightarrow O \) as \( str \), and we denote the set \( Tp(\{O\}) \) of string signature types as \( Tp(str) \).

We say that terms typeable in \( Str_T \) with the type \( str \) are string terms.

String terms represent words in the alphabet \( T \): a word \( a_1 \ldots a_n \) is represented as

\[
/ a_1 \ldots a_n / = (\lambda t . a_1 \ldots (a_n(t)) \ldots).
\] (3)
6.2 Grammars

Given two signatures $\Sigma_i = (N_i, X, C_i, \tau_i), i = 1, 2$, a map of signatures

$$\phi: \Sigma_1 \to \Sigma_2$$

is a pair $\phi = (F, G)$, where

- $F : T p(N_1) \to T p(N_2)$ is a function satisfying the type homomorphism property
  $$F(A \to B) = F(A) \to F(B),$$
  (4)
- $G : C_1 \to \Lambda(X, C_2)$ is a function such that for any $c \in C_1$ it holds that
  $$\vdash_{\Sigma_2} G(c) : F(\tau(c)).$$

The map $G$ above extends inductively to the map

$$G : \Lambda(X, C_1) \to \Lambda(X, C_2),$$

by

$$G(x) = x, \; x \in X, \; G(MN) = (G(M)G(N)), \; G(\lambda x.M) = (\lambda x.G(M)).$$

For economy of notation, we write $\phi(A)$ for $F(A)$ when $A \in T p(C_1)$, and we write $\phi(M)$ for $G(M)$ when $M \in \Lambda(X, C_1)$.

**Definition 7** A string abstract categorial grammar (string ACG) $G$ is a tuple $G = (\Sigma, T, S, \phi)$, where

- $\Sigma$ is a signature;
- $T$ is a finite alphabet
- $S$, the standard type, is an atomic type of $\Sigma$;
- $\phi : \Sigma \to Str_T$, the lexicon, is a map of signatures satisfying $\phi(S) = str$.

In the sequel, the term ACG always means string ACG.

In notation as above we say that $G$ is an ACG over the signature $\Sigma$. We say that $\Sigma$ is the abstract signature of $G$ and the set of signature $\Sigma$ axioms is the abstract vocabulary of $G$.

The language $L(G)$ generated by $G$ is the set of words over $T$ given by

$$L(G) = \{ w \in T^* \mid \exists M \vdash_{\Sigma} M : S \text{ and } \phi(M) =_{\beta\eta} /w/ \},$$

where the symbol $=_{\beta\eta}$ above denotes $\beta\eta$-equivalence of linear $\lambda$-terms.
6.2.1 Second order case

It is well known that, in general, an ACG can generate an NP-complete language [19]. However this does not apply to the case when the abstract vocabulary involves only second order types.

Let us say that a signature is of second order if all types in signature axioms are of second order.

ACG over second order signatures can be described as context-free grammars of λ-terms [6]. Following [6], we give a separate definition for second order ACG (which is not a particular case of the general Definition 7).

**Definition 8** A second order ACG $G$ is a tuple $G = (N, T, X, P, S, \phi)$, where

- $N$ is a finite set of atomic types or nonterminals;
- $T$ is a finite alphabet of terminals;
- $X$ is a countable set of variables;
- $P$ is a finite set of typing judgements, called productions, of the form
  \[ x_1 : A_1, \ldots, x_n : A_n \vdash M : A, \]
  \[ (5) \]
  where
  \[ x_1, \ldots, x_n \in X, \quad A_1, \ldots, A_n, A \in N \]
  and $M \in \Lambda(X, T)$;
- $\phi : Tp(N) \rightarrow Tp(str)$, the lexicon, is a function satisfying type homomorphism property (4), such that for any production of form (5) the typing judgement
  \[ x_1 : \phi(A_1), \ldots, x_n : \phi(A_n) \vdash M : \phi(A) \]
  is derivable in $Str_T$.

We say that a typing judgement $p$ of the form $\Gamma \vdash M : A$ is derivable in $G$ if $p$ is derivable from elements of $P$ and (Id) axioms using only the substitution rule. We write $\Gamma \vdash_G M : A$ in this case.

The language $L(G)$ generated by the above $G$ is the set

\[ L(G) = \{ w \in T^* \mid \exists M \vdash_G M : S \text{ and } M = \beta \eta /w/ \}. \]

Although second order ACG in the sense of Definition 8 are not, formally speaking, ACG in the standard sense of Definition 7, they are essentially equivalent to ACG (in the standard sense) over second order signatures. The equivalence is given by translating productions to signature axioms.

Let a second order signature $\Sigma = (N, X, C, \tau)$ and an ACG $G_1 = (\Sigma, T, S, \phi)$ be given.

For each $c \in C$ with

\[ \tau(c) = A_1 \rightarrow \ldots \rightarrow A_n \rightarrow A \]
we define a production \( p_c \) as the typing judgement
\[
x_1 : A_1, \ldots, x_n : A_n \vdash \phi_1(c)x_1 \ldots x_n : A,
\]
and put \( P = \{ p_c \mid c \in C \} \).

Then we define a second order ACG \( G_2 \) in the sense of Definition 8 as
\( G_2 = (N, T, X, P, S, \phi) \).

Conversely, let a second order ACG \( G_2 = (N, T, X, P, S, \phi) \) in the sense of
Definition 8 be given.

For each production \( p \in P \) of form (5) we introduce a new constant \( c_p \) and
assign a type
\[
\tau(c_p) = A_1 \rightarrow \ldots \rightarrow A_n \rightarrow A.
\]
We put then \( C = \{ c_p \mid p \in P \} \), which gives us a second order signature \( \Sigma = (N, X, C, \tau) \).

Next, we extend the lexicon \( \phi \) from types to constants by putting for each
\( p \in P \) of form (5)
\[
\phi(c_p) = \lambda x_1 \ldots x_n. M.
\]
This gives us an ACG \( G_1 = (\Sigma, T, S, \phi) \) in the sense of Definition 7.

**Note 12** In notation as above, given a labeled context \( \Gamma \) of the form
\[
\Gamma = x_1 : A_1, \ldots, x_k : A_k,
\]
where \( A_1, \ldots A_k, A \in N, \) and types
\[
B_1, \ldots B_n, B \in N,
\]
there exists a term \( M \) such that
\[
\Gamma \vdash_{G_1} M : B_1 \rightarrow \ldots \rightarrow B_n \rightarrow B \tag{6}
\]
iff there exists a term \( M' \) such that
\[
\Gamma, y_1 : B_1, \ldots, y_n : B_n \vdash_{G_2} M' : B \tag{7}
\]
and
\[
\lambda y_1 \ldots y_n. (M') =_{\beta\eta} \phi(M).
\]

**Proof** If \( M \) is a term in (6), then, by Note 11 and Lemma 2, we can replace
it with its \( \beta \)-normal form. Then we get \( M' \), using Lemma 3, induction on
derivation and Note 4. Given \( M' \) in (7), we get \( M \) by induction on derivation
and Note 4. \( \square \)

**Corollary 8** In notation as above \( L(G_1) = L(G_2) \). \( \square \)

Now, it is well known that second order ACG (or ACG over second order
signatures) generate precisely the class of multiple context-free languages [20].
And multiple context-free languages are effectively decidable (see [21]). Thus
we get the following.

**Theorem 1** [20] A language generated by a second order string ACG is effec-
tively decidable. \( \square \)
7 Adding brackets

We are going to add bracket modalities to ACG of the preceding section, which will allow us modeling at least some simplest higher order linguistic phenomena without losing effective decidability. We give a toy example closer to the end of the paper.

7.1 Bracketed signatures

We define bracketed signatures exactly as linear signatures with the only difference that the type assignment map can assign bracketed types.

Thus, a bracketed signature $\Sigma$ is a tuple $\Sigma = (N,X,C,\tau)$, where $N, X, C$ are as in Definition 6 with the additional condition that $C$ does not contain special constants, and $\tau$ is a map $\tau : C \to Tp_\| (N)$.

In the bracketed setting we consider sequents labeled with terms from the set $\Lambda_\| (X, C) = \Lambda_\| (X \cup C)$.

Signature axioms are defined in the same way as for the bracket-free case.

A labeled sequent is derivable in a bracketed signature if it is derivable from signature axioms using rules of $\lambda NLL_{\| \rightarrow \|}$.

**Note 13** In notation as above, a typing judgement $\Gamma \vdash M : A$ is derivable in $\Sigma$ iff there exist constants $c_1, \ldots, c_n \in C$,

some bracketing $\Delta$ of the context

$$\Delta_0 = x_1 : \tau(c_1), \ldots, x_n : \tau(c_n),$$

a configuration of the form $\Gamma' (\Delta)$ such that

$$\Gamma = \Gamma'(\emptyset)$$

and a term $M'$ such that

$$M = M'[x_1 := c_1, \ldots, x_n := c_n]$$

and the typing judgement $\Gamma' \vdash M' : S$ is derivable in $\lambda LL_{\rightarrow \| \rightarrow \|}$.

**Proof** by induction on derivation. \(\square\)

We say that a bracketed signature is safely bracketed if all types occurring in the signature axioms are safely bracketed.

We define the order of a safely bracketed signature as the maximal order of a type occurring in the signature axioms.
7.2 Bracketed ACG

Given a bracketed signature \( \Sigma_1 = (N_1, X, C_1, \tau_1) \) and a linear (bracket-free) signature \( \Sigma_1 = (N_1, X, C_1, \tau_1) \), we define degenerate signature map

\[ \phi : \Sigma_1 \to \Sigma_2 \]

as a pair \( \phi = (F, G) \), where

- \( F : T_{p[N_1]} \to T_{p[N_2]} \) is a function satisfying the property
  \[ F(A \rightarrow B) = F(A) \rightarrow F(B), \quad F(\square A) = F(\square^{-1} A) = F(A) \]

- \( G : C_1 \to \Lambda(X, C_2) \) is a function such that for any \( c \in C_1 \) it holds that
  \[ \vdash_{\Sigma_2} G(c) : F(\tau(c)). \] (8)

As in the bracket-free case, the map \( G \) extends inductively to a map of terms

\[ G : \Lambda[\square](X, C_1) \to \Lambda(X, C_2) \]

by

- \( G(x) = x, \ x \in X, \ G(MN) = (G(M)G(N)), \ G(\lambda x.M) = (\lambda x.G(M)) \)

- \( G(bM) = G(uM) = G(b^{-1}M) = G(u^{-1}M) = G(M) \)

As before, we write \( \phi(A) \) for \( F(A) \) when \( A \in T_{p[N_1]} \), and we write \( \phi(M) \) for \( G(M) \) when \( M \in \Lambda(X, C_1) \).

Also, if \( \Gamma \) is a labeled configuration over \( T_{p[N_1]} \) we write \( \phi(\Gamma) \) for the labeled context over \( T_{p[N_2]} \) obtained from the underlying context of \( \Gamma \) by replacing each type \( A \) with \( \phi(A) \).

**Note 14** If a typing judgement \( \Gamma \vdash M : A \) is derivable in \( \Sigma_1 \) then the typing judgement \( \phi(\Gamma) \vdash \phi(M) : \phi(A) \) is derivable in \( \Sigma_2 \). □

**Definition 9** A bracketed abstract categorial grammar (bracketed ACG) \( G \) is a tuple \( G = (\Sigma, T, \phi, S) \), where

- \( \Sigma \) is a bracketed signature;
- \( T \) is a finite alphabet
- \( \phi : \Sigma \to \text{Str}_T \), the lexicon, is a degenerate map of signatures;
- \( S \), the standard type, is an atomic type of \( \Sigma \), such that \( \phi(S) = \text{str} \).
Just as previously, we say that $G$ is a bracketed ACG over the signature $\Sigma$, that $\Sigma$ is the abstract signature of $G$, and that the set of signature $\Sigma$ axioms is the abstract vocabulary of $G$.

By Note 14, any term $M$ typeable in $\Sigma$ by the standard type $S$ translates under the lexicon map $\phi$ to a string term.

We define the language $L(G)$ generated by $G$ as the set of words over $T$ given by

$$L(G) = \{ w \in T^* | \exists t \vdash \Sigma M : S \text{ and } \phi(M) = \beta\eta /w/ \}.$$

**Note 15** In notation as above, a word $w \in T^*$ is in $L(G)$ iff there exist constants $c_1, \ldots, c_n \in C$ and a term $M$, such that for some bracketing $\Gamma$ of the context

$$\Gamma_0 = x_1 : \tau(c_1), \ldots, x_n : \tau(c_n)$$

the typing judgement $\Gamma \vdash M : S$ is derivable in $\Sigma$ and

$$\phi(M)[x_1 := \phi(c_1), \ldots, x_n = \phi(c_n)] \sim_{\beta\eta} /w/.$$

**Proof** follows from Note 13. □

### 7.3 Second order safely bracketed case

It turns out that bracketed ACG over safely bracketed second order signatures generate effectively decidable languages, just as ordinary ACG over second order signatures. In fact, the two classes of grammars generate the same class of languages.

We now proceed to proving this fact.

Let $\Phi$ be a finite set of types.

We define the set $Sf_{\text{gen safe}}(\Phi)$ as the set of all generalized subformulas of $\Phi$ (see (2)) that are safely bracketed formulas and of order not greater than 2.

**Lemma 10** Let $\Phi$ be a finite set of types.

There exists a finite set $\Pi(\Phi)$ of derivable typing judgements, whose types are in $Sf_{\text{safe gen}}(\Phi)$, such that any derivable typing judgement whose types are in $Sf_{\text{safe gen}}(\Phi)$ is $\beta$-equivalent to one obtained from elements of $\Pi(\Phi)$ using only the substitution rule.

**Proof** Since $\Phi$ is finite, the set $Sf_{\text{safe gen}}(\Phi)$ is finite as well.

By Note 9 we can identify $\beta$-equivalence classes of derivable typing judgements with equivalence classes of sequent proofs.

We take as $\Pi(\Phi)$ the set of all normal derivable typing judgements $\Gamma \vdash A$, such that $\#(\Gamma) \leq 2$ and all types occurring in $\Gamma, A$ belong to $Sf_{\text{safe gen}}(\Phi)$.

The statement follows then from Lemma 9. □

Now let $\Sigma = (N, X, C, \tau)$ be a second order safely bracketed signature, and $G = (\Sigma, T, \phi, S)$, a bracketed ACG.
We are going to construct a second order ACG $\hat{G}$ in the sense of Definition 8, which generates the same language.

Let $\Phi$ be the set of types occurring in the abstract vocabulary of $G$, and let $\Pi(\Phi)$ be the set from Lemma 10.

We define the set $\hat{N}$ of atomic types for the new grammar as

$$\hat{N} = \{ \hat{A} | A \in S^{\text{gen}}_{\text{safe}}(\Phi) \}.$$  

Then for any element $p \in \Pi(\Phi)$ of the form $\Gamma \vdash M : A$, where $\Gamma$ is some bracketing of the labeled context $x_1 : A_1, \ldots, x_n : A_n$, we define the typing judgement $\hat{p}$ as

$$x_1 : \hat{A}_1, \ldots, x_n : \hat{A}_n \vdash M : \hat{A}.$$  

The set of productions for the new grammar is

$$P = \{ \vdash \phi(c) : \tau(c) | c \in C \} \cup \{ \hat{p} | \pi \in \Pi(\Phi) \},$$

and the lexicon, $\hat{\phi}(\hat{A}) = \phi(A)$.

We define $\hat{G} = (\hat{N}, T, X, P, \hat{S}, \hat{\phi})$.

Lemma 10 immediately yields the following.

**Note 16** In notation as above, for all $A_1, \ldots, A_n, A \in S^{\text{gen}}_{\text{safe}}(\Phi)$ it holds that

$$x_1 : A_1, \ldots, x_n : A_n \vdash M : A,$$

iff

$$x_1 : \hat{A}_1, \ldots, x_n : \hat{A}_n \vdash_{\hat{G}} M : \hat{A}.$$  

**Proof** One direction follows from Note 15 and Lemma 10. The other direction follows from Notes 15 and 4 (using correspondence between derivable typing judgements and sequent proofs given by Note 9).

**Corollary 9** In notation as above, the grammars $G$ and $\hat{G}$ generate the same language.

Since a second order linear (bracket-free) signature can be seen in the obvious way as a second order safely bracketed signature, we get the following.

**Theorem 2** A language is generated by a bracketed ACG over a second order safely bracketed signature iff it is generated by a second order (bracket-free) ACG.

Combining the above with Theorem 1 we state the main result.

**Theorem 3** A language generated by a bracketed ACG over a second order safely bracketed signature is effectively decidable.
7.4 Example

We give a toy example of modeling simplest relative clause formation without violation of island constraints. In its essential structure it is a copy of a small fragment of bracketed Lambek grammar from [13], adapted to our formalism.

For brevity, we will use notation

\[ \vdash M_1, \ldots, M_n : A \]

as an abbreviation for the sequence of typing judgements

\[ \vdash M_1 : A, \ldots, \vdash M_n : A. \]

Also, in parallel with notation (3) for representing strings as \( \lambda \)-terms, we define notation for composition:

\[ M_1 \circ \ldots \circ M_n = (\lambda t. M_1(\ldots (M_n(t)) \ldots)), \quad id = \lambda t. t. \]

Now consider the following series of sentences.

\begin{align*}
\text{John loves Jane}, & \quad (9) \\
\text{John loves Jane that kisses Jim}, & \quad (10) \\
\text{Jane that loves John kisses Jim}, & \quad (11) \\
\text{Mary hates John that loves Jane that kisses Jim}. & \quad (12)
\end{align*}

We want to generate all of the above, without generating the ungrammatical

\begin{align*}
\ast \text{Mary hates Jim that John loves Jane that kisses}, & \quad (13) \\
\ast \text{Mary hates Jane that John loves that kisses Jim}. & \quad (14)
\end{align*}

This can be treated as follows.

Consider the following alphabet of terminal symbols

\{John, Jim, Jane, Mary, loves, hates, kisses, that\}

for the string signature and constants for the abstract signature

\{JOHN, JIM, JANE, MARY, LOVES, HATES, KISSES, THAT\}.

Let atomic types for the abstract signature be \{NP, S\}, and the abstract vocabulary be

\[ \vdash JOHN, JANE, JIM : NP, \quad \vdash LOVES, KISSES : NP \rightarrow NP \rightarrow S, \]

\[ \vdash THAT : (NP \rightarrow S) \rightarrow NP \rightarrow \square^{-1}NP. \]

In order to define a grammar we have to specify a lexicon \( \phi \). Let it be as follows:

\[ \phi(NP) = \Phi(S) = \text{str}, \]
\[ \phi(JOHN) = \text{John}, \quad \phi(JANE) = \text{Jane}, \quad \phi(JIM) = \text{Jim}, \]
\[ \phi(LOVES) = \lambda xy. (y \circ \text{loves} \circ x), \quad \phi(KISSES) = \lambda xy. (y \circ \text{kisses} \circ x), \]
\[ \phi(\text{THAT}) = \lambda fx. (x \circ \text{that} \circ (f \cdot \text{id})). \]

Note that \( \phi \) is indeed a homomorphism of signatures, i.e., property (8) holds. This gives us a second order safely bracketed grammar \( G \).

It is immediate that \( G \) generates (13).

For (9), it is easy to check that we can derive in \( G \) the following typing judgement:

\[ x : NP, [z : NP, y : NP] \vdash LOVES(u^{-1}(\text{THAT}(\lambda s.\text{KISSES} \cdot z \cdot s)y))x : S. \]  

(15)

The term on the right of the turnstile, informally speaking, corresponds to “\( x \) loves \( y \) that kisses \( z \)”.

Substituting \( \text{JOHN} \) for \( x \), \( \text{JANE} \) for \( y \) and \( \text{JIM} \) for \( z \) in (15) we get a term, which translates under \( \phi \) to a string term representing (10).

On the other hand, the variables \( y, z \) in (15) are confined in brackets and cannot be bound by \( \lambda \)-abstraction. This precludes generating (13) and (14), while (12) is still possible.

Sentence (11) is analyzed similarly.

We should note that the example above does not use the \( \Box \) connective. As is argued in [13], the \( \Box \)-connective might become useful when dealing with complex subject phrases in English (which are islands). We do not discuss this here and refer to [13].

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