Decidability of the Equivalence of Multi-Letter Quantum Finite Automata

Daowen Qiu\textsuperscript{a,b,c} \textsuperscript{†}, Xiangfu Zou\textsuperscript{a,d}, and Lvzhou Li\textsuperscript{a}

\textsuperscript{a}Department of Computer Science, Zhongshan University, Guangzhou 510275, China
\textsuperscript{b}SQIG–Instituto de Telecomunicações, IST, Av. Rovisco Pais 1049-001, Lisbon, Portugal
\textsuperscript{c}The State Key Laboratory of Computer Science, Institute of Software, Chinese Academy of Sciences, Beijing 100080, China
\textsuperscript{d}Department of Mathematics and Physics, Wuyi University, Jiangmen 529020, China

Abstract

Multi-letter \textit{quantum finite automata} (QFAs) were a new one-way QFA model proposed recently by Belovs, Rosmanis, and Smotrovs (LNCS, Vol. 4588, Springer, Berlin, 2007, pp. 60-71), and they showed that multi-letter QFAs can accept with no error some regular languages \((a+b)^*b\) that are unacceptable by the one-way QFAs. In this paper, we study the decidability of the equivalence of multi-letter QFAs, and the main technical contributions are as follows: (1) We show that any two automata, a \(k_1\)-letter QFA \(A_1\) and a \(k_2\)-letter QFA \(A_2\), over the same input alphabet \(\Sigma\) are equivalent if and only if they are \((n^2m^{k-1} - m^{k-1} + k)\)-equivalent, where \(m = |\Sigma|\) is the cardinality of \(\Sigma\), \(k = \max(k_1, k_2)\), and \(n = n_1 + n_2\), with \(n_1\) and \(n_2\) being the numbers of states of \(A_1\) and \(A_2\), respectively. When \(k = 1\), we obtain the decidability of equivalence of measure-once QFAs in the literature. (2) However, if we determine the equivalence of multi-letter QFAs by checking all strings of length not more than \(n^2m^{k-1} - m^{k-1} + k\), then the worst time complexity is exponential, i.e., \(O(n^6m^{n^2m^{k-1} - m^{k-1} + 2k-1})\). Therefore, we design a polynomial-time \(O(m^{2k-1}n^8 + km^kn^6)\) algorithm for determining the equivalence of any two multi-letter QFAs. Here, the time complexity is concerning the number of states in the multi-letter QFAs, and \(k\) is thought of as a constant.

\textbf{Keywords:} Quantum computing; Multi-letter finite automata; Quantum finite automata; Equivalence; Decidability

\footnotesize
\textsuperscript{†}This work was supported by the National Natural Science Foundation under Grant 60573006 and Grant 60873055, the Research Foundation for the Doctorial Program of Higher School of Ministry of Education under Grant 20050558015, and Program for New Century Excellent Talents in University (NCET) of China.

\textsuperscript{†}Corresponding author (D.W. Qiu): issqdw@mail.sysu.edu.cn
1 Introduction

Quantum finite automata (QFAs) are a kind of theoretical models of quantum computers with finite memory. This kind of computing machines was first studied by Moore and Crutchfield [15], as well as by Kondacs and Watrous [11] independently. Then it was dealt with in depth by Ambainis and Freivalds [2], Brodsky and Pippenger [5], and other authors (see, e.g., the references in [7,18]). The study of QFAs is mainly divided into two directions: one is one-way quantum finite automata (1QFAs) whose tape heads only move one cell to the right at each computation step (1QFAs have been extensively studied in [4]), and the other is two-way quantum finite automata (2QFAs), in which the tape heads are allowed to move towards the right or left, or to be stationary [11]. (Notably, Amano and Iwama [1] dealt with a decidability problem concerning an intermediate form called 1.5QFAs, whose tape heads are allowed to move right or to be stationary; Hirvensalo [9] investigated a decidability problem related to one-way QFAs.) Furthermore, by considering the number of times the measurement is performed in a computation, 1QFAs have two different forms: measure-once 1QFAs (MO-1QFAs) proposed by Moore and Crutchfield [15], and, measure-many 1QFAs (MM-1QFAs) studied first by Kondacs and Watrous [11].

MM-1QFAs are strictly more powerful than MO-1QFAs [2,4] (Indeed, $a^n b^n$ can be accepted by MM-1QFAs with bounded error but not by any MO-1QFA with bounded error). Due to the unitarity of quantum physics and finite memory of finite automata, both MO-1QFAs and MM-1QFAs can only accept proper subclasses of regular languages with bounded error (see, e.g., [11,2,5,4]). Indeed, it was shown that the regular language $(a+b)^*b$ cannot be accepted by any MM-1QFA with bounded error [11].

Recently, Belovs, Rosmanis, and Smotrovs [3] proposed a new one-way QFA model, namely, multi-letter QFAs, which can be thought of as a quantum counterpart of more restricted classical one-way multi-head finite automata (see, e.g., [10]). Roughly speaking, a $k$-letter QFA is not limited to seeing only one, the just-incoming input letter, but can see several earlier received letters as well. That is, the quantum state transition which the automaton performs at each step depends on the last $k$ letters received. For the other computing principle, it is similar to the MO-1QFAs as described above. Indeed, when $k=1$, it reduces to an MO-1QFA. By $L(QFA_k)$ we denote the class of languages accepted with bounded error by $k$-letter QFAs. Any given $k$-letter QFA can be simulated by some $(k+1)$-letter QFA. Qiu and Yu [19] have proved that $L(QFA_k) \subset L(QFA_{k+1})$ for $k=1,2,...$, where the inclusion $\subset$ is proper. Therefore, $(k+1)$-letter QFAs are computationally more powerful than $k$-letter QFAs. Belovs et al. [3] have already showed that $(a+b)^*b$ can be accepted by a 2-letter QFA but, as proved in [11], it cannot be accepted by any MM-1QFA with bounded error. Therefore, multi-letter QFAs can accept some regular languages that cannot be accepted by any MM-1QFA and MO-1QFA.
As we know, determining the equivalence for computing models is an important issue in the theory of computation (see, e.g., [16, 22, 8]). Two computing models over the same input alphabet $\Sigma$ are $n$-equivalent if and only if their accepting probabilities are equal for the input strings of length not more than $n$, while they are equivalent if and only if their accepting probabilities are equal for all input strings.

Concerning the problem of determining the equivalence for QFAs, there exist some works [5, 15] that deals with the simplest case—MO-1QFAs. For quantum sequential machines (QSMs), Qiu and Li [17, 12] gave a method for determining whether or not any two given QSMs are equivalent. This method applies to determining the equivalence between any two MO-1QFAs and also is different from the previous ones. For the equivalence problem of MM-1QFAs, inspired by the works of [22] and [4], Li and Qiu [13] presented a polynomial-time algorithm for determining whether or not any two given MM-1QFAs are equivalent. Recently, Qiu and Yu [19] proved that any two automata, a $k_1$-letter QFA $A_1$ and a $k_2$-letter QFA $A_2$ over the same input alphabet $\Sigma = \{\sigma\}$ are equivalent if and only if they are $(n_1 + n_2)^2 + k_1 - 1$-equivalent, where $n_1$ and $n_2$ are the numbers of states of $A_1$ and $A_2$, respectively, and $k = \max(k_1, k_2)$.

In this paper, we study the equivalence of multi-letter QFAs for input alphabet $\Sigma$ that has arbitrary number of elements, say $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$. More specifically, we prove that any two automata, a $k_1$-letter QFA $A_1$ and a $k_2$-letter QFA $A_2$ over the input alphabet $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, are equivalent if and only if they are $((n_1 + n_2)^2 m^{k-1} - m^{k-1} + k)$-equivalent, where $n_1$ and $n_2$ are the numbers of states of $A_1$ and $A_2$, respectively, $k = \max(k_1, k_2)$. As a corollary, when $k = 1$, we obtain that any two given MO-1QFAs $A_1$ and $A_2$ over the input alphabet $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, are equivalent if and only if they are $(n_1 + n_2)^2$-equivalent. In addition, when $m = 1$, we have that any two automata, a $k_1$-letter QFAs $A_1$ and a $k_2$-letter $A_2$, over the input alphabet $\Sigma = \{\sigma\}$ are equivalent if and only if they are $((n_1 + n_2)^2 + k_1 - 1)$-equivalent.

However, if we determine the equivalence of multi-letter QFAs by checking all strings $x$ with $|x| \leq n^2 m^{k-1} - m^{k-1} + k$, then the time complexity is exponential, i.e., $O(n^6 m^n m^{k-1} - m^{k-1} + 2k - 1)$, where $|x|$ is the length of $x$, and $n = n_1 + n_2$. Therefore, in this paper, we design a polynomial-time $O(m^{2k-1} n^8 + km^k n^6)$ algorithm for determining the equivalence of any two multi-letter QFAs.

The remainder of the paper is organized as follows. In Section 2, we recall the definition of multi-letter QFAs and other related definitions, and some related results are reviewed. In Section 3, we give a condition (mentioned before) for whether two multi-letter QFAs are equivalent. Also, some corollaries are obtained, and those corollaries correspond to results found in literature. Then, in Section 4, we design a polynomial-time $O(m^{2k-1} n^8 + km^k n^6)$ algorithm for determining the equivalence of any two multi-letter QFAs. Finally, in Section 5 we address some related issues for further consideration.
We would mention that the technical process in this paper is new and much more complicated than that in the literature \cite{15, 5, 12, 13, 14} regarding the equivalence of 1QFAs. In general, symbols will be explained when they first appear. In this paper, for a matrix $A$, we use $A^\ast$ and $A^\dagger$ to denote its conjugate and conjugate transpose, respectively.

2 Preliminaries

In this section, we briefly review some definitions and related properties that will be used in the consequent sections. For the details, we refer to \cite{3, 19}. First we recall $k$-letter deterministic finite automata ($k$-letter DFAs).

Definition 1 (\cite{3}). A $k$-letter deterministic finite automaton ($k$-letter DFA) is defined by a quintuple $(Q, Q_{acc}, q_0, \Sigma, \gamma)$, where $Q$ is a finite set of states, $Q_{acc} \subseteq Q$ is the set of accepting states, $q_0 \in Q$ is the initial state, $\Sigma$ is a finite input alphabet, and $\gamma$ is a transition function that maps $Q \times T^k$ to $Q$, where $T = \{\Lambda\} \cup \Sigma$ and letter $\Lambda \notin \Sigma$ denotes the empty letter, and $T^k \subset T^*$ consists of all strings of length $k$.

We describe the computing process of a $k$-letter DFA on an input string $x$ in $\Sigma^*$, where $x = \sigma_1\sigma_2\cdots\sigma_n$, and $\Sigma^*$ denotes the set of all strings over $\Sigma$. The $k$-letter DFA has a tape which contains the letter $\Lambda$ in its first $k-1$ positions followed by the input string $x$. The automaton starts in the initial state $q_0$ and has $k$ reading heads which initially are on the first $k$ positions of the tape (clearly, the $k$th head reads $\sigma_1$ and the other heads read $\Lambda$). Then the automaton transfers to a new state as current state and all heads move right a position in parallel. Now the $(k-1)$th and $k$th heads point to $\sigma_1$ and $\sigma_2$, respectively, and the others, if any, to $\Lambda$. Subsequently, the automaton transfers to a new state and all heads move to the right. This process does not stop until the $k$th head has read the last letter $\sigma_n$. The input string $x$ is accepted if and only if the automaton enters an accepting state after its $k$th head reading the last letter $\sigma_n$.

Clearly, $k$-letter DFAs are not more powerful than DFAs. Indeed, the family of languages accepted by $k$-letter DFAs, for $k \geq 1$, is exactly the family of regular languages. A group finite automaton (GFA) \cite{5} is a DFA whose state transition function, say $\delta$, satisfies that for any input symbol $\sigma$, $\delta(\cdot, \sigma)$ is a one-to-one map on the state set, i.e., a permutation on the state set.

Definition 2 (\cite{3}). A $k$-letter DFA $(Q, Q_{acc}, q_0, \Sigma, \gamma)$ is called a $k$-letter group finite automaton ($k$-letter GFA) if and only if for any string $x \in T^k$ the function $\gamma_x(q) = \gamma(q, x)$ is a bijection from $Q$ to $Q$.

Now we recall the definition of multi-letter QFAs \cite{3}. 

4
Definition 3 (\[3\]). A $k$-letter QFA $\mathcal{A}$ is defined by a quintuple $\mathcal{A} = (Q, Q_{acc}, |\psi_0\rangle, \Sigma, \mu)$ where $Q$ is a set of states, $Q_{acc} \subseteq Q$ is the set of accepting states, $|\psi_0\rangle$ is the initial unit state that is a superposition of the states in $Q$, $\Sigma$ is a finite input alphabet, and $\mu$ is a function that assigns a unitary transition matrix $U_w$ on $\mathbb{C}^{|Q|}$ for each string $w \in (\{\Lambda\} \cup \Sigma)^k$, where $|Q|$ is the cardinality of $Q$.

The computation of a $k$-letter QFA $\mathcal{A}$ works in the same way as the computation of an MO-1QFA \[15, 5\], except that it applies unitary transformations corresponding not only to the last letter but the last $k$ letters received (like a $k$-letter DFA). When $k = 1$, it is exactly an MO-1QFA as pointed out before. According to \[3\], all languages accepted by $k$-letter QFAs with bounded error are regular languages for any $k$.

Now we give the probability $P_{A}(x)$ for $k$-letter QFA $\mathcal{A} = (Q, Q_{acc}, |\psi_0\rangle, \Sigma, \mu)$ accepting any input string $x = \sigma_1\sigma_2\cdots\sigma_m$. From the definition we know that, for any $w \in (\{\Lambda\} \cup \Sigma)^k$, $\mu(w)$ is a unitary matrix. By the definition of $\mu$, we can define the unitary transition for each string $x = \sigma_1\sigma_2\cdots\sigma_m \in \Sigma^*$. By $\overline{\mu}$ (induced by $\mu$) we mean a map from $\Sigma^*$ to the set of all $|Q| \times |Q|$ unitary matrices. Specifically, $\overline{\mu}$ is induced by $\mu$ in the following way: For $x = \sigma_1\sigma_2\cdots\sigma_m \in \Sigma^*$,

$$\overline{\mu}(x) = \begin{cases} \mu(\Lambda^{k-1}\sigma_1)\mu(\Lambda^{k-2}\sigma_1\sigma_2)\cdots\mu(\Lambda^{k-m}\sigma_m), & \text{if } m < k, \\ \mu(\Lambda^{k-1}\sigma_1)\mu(\Lambda^{k-2}\sigma_1\sigma_2)\cdots\mu(\Lambda^{k-m+1}\sigma_{m-k+1}\sigma_{m-k+2}\cdots\sigma_m), & \text{if } m \geq k, \end{cases}$$

(1)

which implies the computing process of $\mathcal{A}$ for input string $x$.

We identify the states in $Q$ with an orthonormal basis of $\mathbb{C}^{|Q|}$, and let $P_{acc}$ denote the projection operator on the subspace spanned by $Q_{acc}$. Then we define that

$$P_{A}(x) = ||\langle \psi_0 | \overline{\mu}(x) P_{acc} \rangle ||^2.$$  

(2)

3 Determining the equivalence between multi-letter quantum finite automata

For any given $k_1$-letter QFA $\mathcal{A}_1$ and $k_2$-letter QFA $\mathcal{A}_2$ over the same input alphabet $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, our purpose is to determine whether they are equivalent. We give the definition of equivalence between two multi-letter QFAs.

Definition 4. A $k_1$-letter QFA $\mathcal{A}_1$ and another $k_2$-letter QFA $\mathcal{A}_2$ over the same input alphabet $\Sigma$ are said to be equivalent (resp. $t$-equivalent) if $P_{A_1}(w) = P_{A_2}(w)$ for any $w \in \Sigma^*$ (resp. for any input string $w$ with $|w| \leq t$).

We introduce some notation.. $\mathbb{C}^n$ denotes the Euclidean space consisting of all $n$-dimensional complex vectors. For a subset $S$ of $\mathbb{C}^n$, $\text{span}S$ represents the minimal subspace spanned by
We denote $F$ for any $l \in \mathbb{N}$.

For any $w \in \Sigma^*$, let $\mathbb{G}(l, w) = \text{span}\{\langle \psi_0 | \pi(x) : x \in \Sigma^*, |x| \leq l \}$.

Then there exists an integer $i_0 \leq (n - 1)m^{k-1} + k$ such that, for any $i \geq i_0$, $F(i) = F(i_0)$, where $n$ is the number of states of $Q$.

**Proof.** We denote

$$\Sigma^{(k-1)} = \{x : x \in \Sigma^*, |x| = k - 1\};$$

and

$$\mathbb{G}(l, w) = \text{span}\{\langle \psi_0 | \pi(xw) : x \in \Sigma^*, |x| \leq l \}$$

for any $w \in \Sigma^{(k-1)}$ and any $l \in \{0, 1, 2, \cdots\}$. In addition, we denote

$$\mathbb{H}(l) = \oplus_{w \in \Sigma^{(k-1)}} \mathbb{G}(l, w)$$

for any $l \in \{0, 1, 2, \cdots\}$, where $\oplus_{w \in \Sigma^{(k-1)}} \mathbb{G}(l, w)$ is the direct sum of $\mathbb{G}(l, w)$ for all $w \in \Sigma^{(k-1)}$. It is clear that

$$\mathbb{G}(l, w) \subseteq \mathbb{G}(l + 1, w), \forall w \in \Sigma^{(k-1)} \text{ and } \forall l \in \{0, 1, 2, \cdots\},$$

and

$$\mathbb{H}(l) \subseteq \mathbb{H}(l + 1), \forall l \in \{0, 1, 2, \cdots\}.$$ (7)

Since $\mathbb{G}(l, w)$ is a subspace of $\mathbb{C}^n$, we obtain that

$$1 \leq \dim(\mathbb{G}(l, w)) \leq n,$$ (8)

for any $w \in \Sigma^{(k-1)}$ and any $l \in \{0, 1, 2, \cdots\}$, where $\dim(\mathbb{G}(l, w))$ denotes the dimension of $\mathbb{G}(l, w)$. Furthermore, by the definition of direct sum, we have

$$m^{k-1} \leq \dim(\mathbb{H}(l)) \leq nm^{k-1}$$ (9)

for any $l \in \{0, 1, 2, \cdots\}$. Therefore, by Eq. (7) there exists $l_0 \leq (n - 1)m^{k-1} + 1$ such that

$$\mathbb{H}(l_0) = \mathbb{H}(l_0 + 1).$$ (10)

Equivalently,

$$\mathbb{G}(l_0, w) = \mathbb{G}(l_0 + 1, w), \forall w \in \Sigma^{(k-1)}.$$ (11)

Let $i_0 = l_0 + (k - 1) \leq (n - 1)m^{k-1} + k$. Now, we prove by induction that, for any $i \geq i_0$, $F(i) = F(i_0)$.

**Base step.** When $i = i_0$, it is clear that $F(i) = F(i_0)$.
Induction step. Suppose $F(j) = F(i_0)$, for some $j \geq i_0$. Our purpose is to prove that $F(j + 1) = F(i_0)$. For any given $w \in \Sigma^{(j+1)}$, we denote $w = \sigma_1 \sigma_2 \cdots \sigma_{i_0} \sigma_{i_0+1} \cdots \sigma_{i_0} \sigma_{i_0+1} \cdots \sigma_{j} \sigma_{j+1}$ and let $u_0 = \sigma_{i_0+2} \cdots \sigma_{i_0} \sigma_{i_0+1}$. Clearly, $u_0 \in \Sigma^{(k-1)}$, and $\langle \psi_0|\overline{M}(\sigma_1 \sigma_2 \cdots \sigma_{i_0} \sigma_{i_0+1}) \rangle = \langle \psi_0|\overline{M}(\sigma_1 \sigma_2 \cdots \sigma_{i_0} \sigma_{i_0+1}w_0) \rangle \in G(l_0 + 1, w_0).

Due to $\mathbb{H}(l_0) = \mathbb{H}(l_0 + 1)$, i.e., $\mathbb{G}(l_0, w) = \mathbb{G}(l_0 + 1, w)$ for any $w \in \Sigma^{(k-1)}$, we obtain that $\langle \psi_0|\overline{M}(\sigma_1 \sigma_2 \cdots \sigma_{i_0} \sigma_{i_0+1}w_0) \rangle \in G(l_0, w_0)$. Therefore, $\langle \psi_0|\overline{M}(\sigma_1 \sigma_2 \cdots \sigma_{i_0} \sigma_{i_0+1}w_0) \rangle$ can be linearly represented by the vectors of $\mathbb{G}(l_0, w_0)$. As a result, there exist a finite index set $\Gamma$ and $x_\gamma \in \{x : x \in \Sigma^*, |x| \leq l_0\}$ as well as complex numbers $p_\gamma$ with $\gamma \in \Gamma$, such that

$$
\langle \psi_0|\overline{M}(\sigma_1 \sigma_2 \cdots \sigma_{i_0} \sigma_{i_0+1}w_0) \rangle = \sum_{\gamma \in \Gamma} p_\gamma \langle \psi_0|\overline{M}(x_\gamma w_0) \rangle.
$$

Therefore, we have

$$
\langle \psi_0|\overline{M}(w) \rangle = \langle \psi_0|\overline{M}(\sigma_1 \sigma_2 \cdots \sigma_{i_0} \sigma_{i_0+1}w_0) \rangle = \sum_{\gamma \in \Gamma} p_\gamma \langle \psi_0|\overline{M}(x_\gamma w_0) \rangle \mu(w_0 \sigma_{i_0+2}) \cdots \mu(\sigma_{j-k+2} \cdots \sigma_{j+1}) = \sum_{\gamma \in \Gamma} p_\gamma \langle \psi_0|\overline{M}(x_\gamma \sigma_{i_0} \sigma_{i_0+1} \cdots \sigma_{j} \sigma_{j+1}) \rangle \in F(j).
$$

Consequently, $\langle \psi_0|\overline{M}(w) \rangle \in F(j)$, and we get that $F(j + 1) \subseteq F(j)$. On the other hand, $F(j) \subseteq F(j + 1)$ always holds. Hence, we obtain that $F(j + 1) = F(j) = F(i_0)$. The proof is completed.

With the same method of proof as that for Lemma 1, we can obtain the following lemma.

**Lemma 2.** For $\Sigma = \{\sigma_i : i = 1, 2, \ldots, m\}$ and a $k$-letter QFA $A = (Q, Q_{acc}, |\psi_0\rangle, \Sigma, \mu)$, there exists an integer $i_0 \leq (n^2 - 1)m^k - 1 + k$ such that, for any $i \geq i_0$, $E(i) = E(i_0)$, where $n$ is the number of states of $Q$, $E(j) = \text{span}\{\langle \psi_0| \otimes (|\psi_0\rangle^* \overline{\nu}(x) : x \in \Sigma^*, |x| \leq j\}$ for $j = 1, 2, \ldots$, and $\overline{\nu}(x) = \overline{\mu}(x) \otimes \overline{\mu}(x)^*$, where $*$ denotes the conjugate operation.

**Proof.** It is exactly similar to the proof of Lemma 1. \(\square\)

To prove the equivalence of multi-letter QFAs, we further present a lemma. For $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, a $k_1$-letter QFA $A_1 = (Q_1, Q_{acc}^{(1)}, |\psi_0^{(1)}\rangle, \Sigma, \mu_1)$ and another $k_2$-letter QFA $A_2 = (Q_2, Q_{acc}^{(2)}, |\psi_0^{(2)}\rangle, \Sigma, \mu_2)$, let $P_{acc}^{(1)}$ and $P_{acc}^{(2)}$ denote the projection operators on the subspaces spanned by $Q_{acc}^{(1)}$ and $Q_{acc}^{(2)}$, respectively. For any string $x \in \Sigma^*$, we set $\overline{\nu}(x) = \overline{\mu}_1(x) \oplus \overline{\mu}_2(x)$ and $P_{acc} = P_{acc}^{(1)} \oplus P_{acc}^{(2)}$, $Q_{acc} = \{\eta_1 \oplus \eta_2 : |\eta_1| \in Q_{acc}^{(1)} \text{ and } |\eta_2| \in Q_{acc}^{(2)}\}$. In addition, we denote $|\eta_1\rangle = |\psi_0^{(1)}\rangle \oplus 0_2$ and $|\eta_2\rangle = 0_1 \oplus |\psi_0^{(2)}\rangle$, where $0_1$ and $0_2$ represent column zero vectors of $n_1$ and $n_2$ dimensions, respectively.
Lemma 3. $A_1$ and $A_2$ above are equivalent if and only if

$$((\eta_1)(\eta_1)^* - \eta_2((\eta_2)^*)\overline{\nu}(x) \sum_{p_j \in Q_{acc}} (|p_j\rangle(|p_j\rangle)^*) = 0 \quad (13)$$

for all strings $x \in \Sigma^+$, where $\overline{\nu}(x) = \overline{\mu}(x) \otimes \overline{\mu}(x)^*$.

Proof. Denote that, for any string $x \in \Sigma^*$,

$$P_{\eta_1}(x) = \|\langle \eta_1 | \overline{\mu}(x) \overline{P}_{acc} \rangle \|^2 \quad (14)$$

and

$$P_{\eta_2}(x) = \|\langle \eta_2 | \overline{\mu}(x) \overline{P}_{acc} \rangle \|^2. \quad (15)$$

Indeed, we further have that

$$P_{\eta_1}(x) = \|\langle \eta_1 | \overline{\mu}(x) \overline{P}_{acc} \rangle \|^2$$

$$= \langle \eta_1 | \overline{\mu}(x) \overline{P}_{acc} \overline{P}_{acc} \overline{\mu}(x)^\dagger | \eta_1 \rangle$$

$$= \langle \eta_1 | \overline{\mu}(x) \overline{P}_{acc} \overline{\mu}(x)^\dagger | \eta_1 \rangle$$

$$= \langle \psi^{(1)}_0 | \overline{\mu}(x) P_{acc}(x)^\dagger | \psi^{(1)}_0 \rangle$$

$$= P_{A_1}(x). \quad (16)$$

Similarly,

$$P_{\eta_2}(x) = P_{A_2}(x). \quad (17)$$

Therefore,

$$P_{A_1}(x) = P_{A_2}(x) \quad (18)$$

holds if and only if

$$P_{\eta_1}(x) = P_{\eta_2}(x) \quad (19)$$

for any string $x \in \Sigma^*$. On the other hand, we have that

$$P_{\eta_1}(x) = \|\langle \eta_1 | \overline{\mu}(x) \overline{P}_{acc} \rangle \|^2$$

$$= \sum_{p_j \in Q_{acc}} |\langle \eta_1 | \overline{\mu}(x)|p_j\rangle|^2$$

$$= \sum_{p_j \in Q_{acc}} \langle \eta_1 | \overline{\mu}(x)|p_j\rangle(\langle \eta_1 | \overline{\mu}(x)|p_j\rangle)^*$$

$$= \sum_{p_j \in Q_{acc}} \langle \eta_1 |((\langle \eta_1 \rangle)^* (\overline{\mu}(x) \otimes (\overline{\mu}(x))^*)|p_j\rangle(|p_j\rangle)^*$$

$$= \langle \eta_1 |((\langle \eta_1 \rangle)^* (\overline{\mu}(x) \otimes (\overline{\mu}(x))^*) \sum_{p_j \in Q_{acc}} |p_j\rangle(|p_j\rangle)^*. \quad (20)$$

Similarly,

$$P_{\eta_2}(x) = \langle \eta_2 |((\langle \eta_2 \rangle)^* (\overline{\mu}(x) \otimes (\overline{\mu}(x))^*) \sum_{p_j \in Q_{acc}} |p_j\rangle(|p_j\rangle)^*. \quad (21)$$

8
Therefore, Eq. (19) holds if and only if

\[ \langle \eta_1 | ((\langle \eta_1 |)^* (\overline{\rho}(x) \otimes (\overline{\rho}(x))^*) ) \sum_{p_j} (|p_j\rangle (|p_j\rangle)^*) = \langle \eta_2 | ((\langle \eta_2 |)^* (\overline{\rho}(x) \otimes (\overline{\rho}(x))^*) ) \sum_{p_j} (|p_j\rangle (|p_j\rangle)^* \]  

(22)

for any string \( x \in \Sigma^* \). Denote

\[ \overline{\rho}(x) = \overline{\rho}(x) \otimes (\overline{\rho}(x))^* \]  

(23)

Clearly \( \overline{\rho}(x) \) is an \( n^2 \times n^2 \) complex square matrix. Then the equivalence between \( A_1 \) and \( A_2 \) depends on whether or not the following equation holds for any string \( x \in \Sigma^* \):

\[ \langle \eta_1 | ((\langle \eta_1 |)^* \overline{\rho}(x) ) \sum_{p_j} (|p_j\rangle (|p_j\rangle)^* = \langle \eta_2 | ((\langle \eta_2 |)^* \overline{\rho}(x) ) \sum_{p_j} (|p_j\rangle (|p_j\rangle)^* \]  

(24)

i.e.,

\[ ((\langle \eta_1 |)^* - \langle \eta_2 |)^* \overline{\rho}(x) ) \sum_{p_j} (|p_j\rangle (|p_j\rangle)^* = 0. \]  

(25)

With the above lemmas we are ready to prove the main theorem.

**Theorem 4.** \( A_1 \) and \( A_2 \) above are equivalent if and only if they are \((n^2m^{k-1} - m^{k-1} + k)\)-equivalent, where \( k = \max(k_1, k_2) \) and \( n = n_1 + n_2 \), with \( n_i \) being the number of states in \( Q_i \), \( i = 1, 2 \).

**Proof.** Denote \( \mathbb{D}(j) = \text{span}\{ (\langle \eta_1 |)^* - \langle \eta_2 |)^* \overline{\rho}(x) : x \in \Sigma^*, |x| \leq j \} \) for \( j = 1, 2, \cdots \). Here \( \mathbb{D}(j) \) is a subspace of \( \mathbb{C}^{n^2} \). By Lemma 2 we can readily obtain that there exists an \( i_0 \leq (n^2 - 1)m^{k-1} + k \) such that

\[ \mathbb{D}(i) = \mathbb{D}(i_0) \]  

(26)

for all \( i \geq i_0 \). Eq. (26) implies that, for any \( x \in \Sigma^* \) with \( |x| > (n^2 - 1)m^{k-1} + k \), \( (\langle \eta_1 |)^* - \langle \eta_2 |)^* \overline{\rho}(x) \) can be linearly represented by some vectors in \( \{ (\langle \eta_1 |)^* - \langle \eta_2 |)^* \overline{\rho}(y) : y \in \Sigma^* \text{ and } |y| \leq (n^2 - 1)m^{k-1} + k \} \).

Consequently, if Eq. (13) holds for all \( x \) with \( |x| \leq (n^2 - 1)m^{k-1} + k \), then so does it for all \( x \) with \( |x| > (n^2 - 1)m^{k-1} + k \). We have proved this theorem.

From Theorem 4 we can get a sufficient and necessary condition for the equivalence of MO-1QFAs \((k = 1)\), and we describe it by the following corollary, which was also presented by Li and Qiu 14.
Corollary 5. For $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, an MO-1QFA $A_1 = (Q_1, Q_{\text{acc}}^{(1)}, |\psi_0^{(1)}\rangle, \Sigma, \mu_1)$ and another MO-1QFA $A_2 = (Q_2, Q_{\text{acc}}^{(2)}, |\psi_0^{(2)}\rangle, \Sigma, \mu_2)$ are equivalent if and only if they are $(n_1 + n_2)^2$-equivalent, where $n_i$ is the number of states of $Q_i$, $i = 1, 2$.

Remark 1. We analyze the complexity of computation in Theorem 4. As in [22], we assume that all the inputs consist of complex numbers whose real and imaginary parts are rational numbers and that each arithmetic operation on rational numbers can be done in constant time. Again we denote $n = n_1 + n_2$. Note that in time $O(n^4)$ we check whether or not Eq. (13) holds for $x \in \Sigma^*$ with $|x| = i$. Because the length of $x$ to be checked in Eq. (13) is at most $(n^2 - 1)m^{k-1} + k$, the time complexity for checking whether the two multi-letter QFAs are equivalent is $O(n^4(m + 2m^2 + \ldots + ((n^2 - 1)m^{k-1} + k)n^2m^{k-1-k+1+k}))$, that is $O(n^6m^{n^2m^{k-1-k+1+2k-1}})$.

From Theorem 4, we can obtain a sufficient and necessary condition for the equivalence of multi-QFAs over the same single input alphabet, and we describe it by the following corollary, which has been studied by Qiu and Yu [19].

Corollary 6. For $\Sigma = \{\sigma\}$, a $k_1$-letter QFA $A_1 = (Q_1, Q_{\text{acc}}^{(1)}, |\psi_0^{(1)}\rangle, \Sigma, \mu_1)$ and another $k_2$-letter QFA $A_2 = (Q_2, Q_{\text{acc}}^{(2)}, |\psi_0^{(2)}\rangle, \Sigma, \mu_2)$ are equivalent if and only if they are $(n^2 + k - 1)$-equivalent, where $k = \max(k_1, k_2)$ and $n = n_1 + n_2$, with $n_i$ being the number of states of $Q_i$, $i = 1, 2$.

4 A polynomial-time algorithm for determining the equivalence between multi-letter QFAs

According to our analysis of the complexity of computation in Remark 1, we need exponential time for checking whether or not two multi-letter QFAs are equivalent if Eq. (13) is checked for all strings $x \in \Sigma^*$ with $|x| \leq (n^2 - 1)m^{k-1} + k$. In this section, our purpose is to design a polynomial-time algorithm for determining the equivalence between any two multi-letter QFAs.

We still use the symbols from Lemma 3 and Theorem 4 and its proof. In addition, denote $\langle \eta \rangle = \langle \eta_1 \rangle \langle \eta_2 \rangle = \langle \eta_2 \rangle \langle \eta_1 \rangle$ and $|P_{\text{acc}}\rangle = \sum_{p_j \in Q_{\text{acc}}} |p_j\rangle \langle p_j \rangle$. Then, Eq. (13) is equivalent to

$$\langle \eta | \mathcal{P}(x) | P_{\text{acc}} \rangle = 0. \quad (27)$$

For $w \in \Sigma^{(k-1)}$, denote $G(l, w) = \text{span}\{\langle \eta | \mathcal{P}(xw) : x \in \Sigma^*, |x| \leq l \}$. Then, from the proof of Lemma 1 we know that there exists $l_0 \leq (n^2 - 1)m^{k-1} + 1$ such that, for any $w \in \Sigma^{(k-1)}$ and any $l \geq l_0$, $G(l, w) = G(l_0, w)$. Since $G(l_0, w)$ is a subspace of $\mathbb{C}^{n^2}$, $\dim(G(l_0, w)) \leq n^2$.

If within polynomial time we can find a basis, denoted by $\mathcal{B}(w)$, of the subspace $G(l_0, w)$, for all $w \in \Sigma^{(k-1)}$, then any element in $\cup_{w \in \Sigma^{(k-1)}} G(l_0, w)$ can be linearly represented by these
elements in $\bigcup_{w \in \Sigma^{(k-1)}} B(w)$. Therefore, to determine whether or not Eq. \((\text{13})\) holds, it suffices to check Eq. \((\text{13})\) for all elements in $\bigcup_{w \in \Sigma^{(k-1)}} B(w) \bigcup \{\langle \eta | \overline{\varphi}(x) : x \in \bigcup_{l=0}^{k-2} \Sigma^{(l)}\}$. 

### 4.1 A polynomial time algorithm for finding out a base of $G(l_0, w)$

First, we give some useful definitions. For $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, we define a strict order \textsuperscript{[20]} “$<$” on $\Sigma^+$ ($\Sigma^+ = \Sigma^* \setminus \{\epsilon\}$) by, $\forall x_1, x_2 \in \Sigma^+$,

$$x_1 < x_2 \text{ iff } |x_1| < |x_2| \text{ or } (|x_1| = |x_2|, x_1 = y\sigma_i z_1, x_2 = y\sigma_j z_2 \text{ and } i < j),$$

where $\sigma_i, \sigma_j \in \Sigma$ and $y, z_1, z_2 \in \Sigma^*$. Based on the strict order $<$, we define a partial order \textsuperscript{[20]} “$\leq$” on $\Sigma^+$ by, $\forall x_1, x_2 \in \Sigma^+$,

$$x_1 \leq x_2 \text{ iff } x_1 < x_2 \text{ or } x_1 = x_2.$$  \hspace{1cm} (29)

The partial order $\leq$ can be expanded to $\Sigma^*$ if we set $\epsilon \leq x$ for all $x \in \Sigma^*$. Clearly, $\Sigma^*$ is a well-ordered set \textsuperscript{[20]}, i.e., every nonempty subset of $\Sigma^*$ contains a least element.

In the following we design a polynomial time algorithm to search for $B(w)$ for all $w \in \Sigma^{(k-1)}$, which is described by the following Theorem.

**Theorem 7.** For a $k_1$-letter QFA $A_1 = (Q_1, Q_1^{(1)}|\psi_0^{(1)}\rangle, \Sigma, \mu_1)$ and another $k_2$-letter QFA $A_2 = (Q_2, Q_2^{(2)}|\psi_0^{(2)}\rangle, \Sigma, \mu_2)$, where $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, there exists a polynomial time $O(m^{2k-1}n^8 + km^k n^6)$ algorithm finding $\bigcup_{w \in \Sigma^{(k-1)}} B(w)$, where, as above, $B(w)$ is a base of subspace $\mathbb{G}(l_0, w), \langle \eta \rangle = \langle \eta_1 \rangle^* - \langle \eta_2 \rangle^*$, and $\overline{\varphi}(x) = (\overline{\varphi}_1(x) \oplus \overline{\varphi}_2(x)) \oplus (\overline{\varphi}_1(x) \oplus \overline{\varphi}_2(x))^*$, $l_0 \leq (n^2 - 1)m^{k-1} + 1$.

**Proof.** In this algorithm, we need a few symbols. First, $\Sigma^{(k)}$ is defined as $\Sigma^{(k)}$ but it is further required that the elements of $\Sigma^{(k)}$ are arranged according to the order $<$ from the least $\sigma_k^1$ to the biggest $\sigma_k^m$, where $\sigma_k^j$ means the string $\underbrace{\sigma \ldots \sigma}_{k}$. In addition, denote $x\Sigma_{<} = \{x\sigma_1, x\sigma_2, \ldots, x\sigma_m\}$ for $x \in \Sigma^*$.

We outline the process for searching for $B(w)$, for all $w \in \Sigma^{(k-1)}$. Initially, we let $\langle \eta | \overline{\varphi}(w) \rangle \in B(w)$ for $w \in \Sigma^{(k-1)}$, and let queue be $\Sigma^{(k)}_{<}$. Then take an element $x$ from queue and determine $x = yw_i$ for some $y \in \Sigma^*$ and $w_i \in \Sigma^{(k-1)}$, and further check whether or not $\langle \eta | \overline{\varphi}(x) \rangle \in \text{span} B(w_i)$. If $\langle \eta | \overline{\varphi}(x) \rangle \notin \text{span} B(w_i)$, then add $\langle \eta | \overline{\varphi}(x) \rangle$ to $B(w_i)$ and add the elements of $x\Sigma_{<}$ to queue from small element to large one sequentially. By repeating the above process, we take another element $x'$ from queue and then determine $x' = y'w_j$ for some $y' \in \Sigma^*$ and $w_j \in \Sigma^{(k-1)}$, and further check whether or not $\langle \eta | \overline{\varphi}(x') \rangle \in \text{span} B(w_j)$. Continue this process, until queue is empty. Since the number of the elements in $\bigcup_{w \in \Sigma^{(k-1)}} B(w)$ is at most $n^2m^{k-1}$, the number of the elements in queue is at most $n^2m^k$. In addition, from the proof of Lemma 1, we know that $\mathbb{G}(l_0, w) = \mathbb{G}(l_0 + i, w)$ for any $i \geq 0$. Therefore, if $x$ belongs
to the queue, then $k - 1 \leq |x| \leq l_0 + k - 1 \leq (n^2 - 1)m^{k-1} + k$. Therefore, the above process will always end.

We describe the process by Figure 1 and then we will analyze its time complexity.

We prove that $B(w)$ found out in the algorithm is exactly a base of $G(l_0, w)$. By the algorithm, the vectors in $B(w)$ are linearly independent. Next, we only need to prove that, for every $w \in \Sigma^{(k-1)}$, all vectors in $G(l_0, w)$ can be linearly expressed by the vectors in $B(w)$. It suffices to prove that, for any $x \in \Sigma^*$ with $x = x_0w$, $\langle \eta | \varphi(x) \rangle$ can be linearly expressed by the vectors in $B(w)$. We proceed by induction on the partial order $\leq$ on $\Sigma^*$.

(1) Basis.

When $x_0 = \epsilon$, we know that $\langle \eta | \varphi(w) \rangle \in B(w)$ by the beginning of the algorithm. Therefore, $\langle \eta | \varphi(x) \rangle$ can be linearly expressed by the vectors in $B(w)$.

(2) Induction.

Assume that, for some $y > \epsilon$, when $x_0 < y$, $\langle \eta | \varphi(x_0w) \rangle$ can be linearly expressed by the vectors in $B(w)$. Now, we prove that $\langle \eta | \varphi(yw) \rangle$ can be linearly expressed by the vectors in $B(w)$ as well.

(i) If $x = yw$ appears in queue, then we can clearly know that $\langle \eta | \varphi(x) \rangle$ can be linearly expressed by the vectors in $B(w)$ by the algorithm.

(ii) If $x = yw$ does not appear in queue, then we know that there exists a $w' \in \Sigma^{(k-1)}$ and $x_1, x_2 \in \Sigma^*$ with $|x_2| \geq 1$ such that

$$x = yw = x_1 w' x_2,$$

(30)
and
\[ \langle \eta | \mathcal{P}(x_1 w') = \sum_{\gamma \in \Gamma} p_{\gamma} \langle \eta | \mathcal{P}(y_{\gamma} w'), \] (31)
for some set of indices \( \Gamma \), where \( p_{\gamma} \in \mathbb{C} \), \( y_{\gamma} \in \Sigma^* \), and \( y_{\gamma} < x_1 \) for all \( \gamma \in \Gamma \). Therefore,
\[ \langle \eta | \mathcal{P}(x) = \sum_{\gamma \in \Gamma} p_{\gamma} \langle \eta | \mathcal{P}(y_{\gamma} w' x_2) \] (32)
and \( y_{\gamma} w' x_2 < x_1 w' x_2 = x = yw \). Let \( y_{\gamma} w' x_2 = y'_{\gamma} w \). Then \( y'_{\gamma} < y \). Therefore, by the inductive assumption, we know that \( \langle \eta | \mathcal{P}(y'_{\gamma} w) \) can be linearly expressed by the vectors in \( \mathcal{B}(w) \). Consequently, by Eq. (32) we obtain that \( \langle \eta | \mathcal{P}(x) \) can be linearly expressed by the vectors in \( \mathcal{B}(w) \).

(3) Conclusion.

For any \( \langle \eta | \mathcal{P}(x) \in \mathcal{G}(l_0, w) \), \( \langle \eta | \mathcal{P}(x) \) can be linearly expressed by the vectors in \( \mathcal{B}(w) \). Hence, \( \mathcal{B}(w) \) is a base of \( \mathcal{G}(l_0, w) \), for every \( w \in \Sigma^{(k-1)} \).

**Complexity of the algorithm.** First we assume that all the inputs consist of complex numbers whose real and imaginary parts are rational numbers and that each arithmetic operation on rational numbers can be done in constant time. Note that with time \( O(|x|^4) \) we compute \( \langle \eta | \mathcal{P}(x) \). Then we need to recall that to verify whether a set of \( n \)-dimensional vectors is linearly independent needs time \( O(n^3) \) \cite{6}. The time complexity to check whether or not \( \langle \eta | \mathcal{P}(x) \in \text{span}\mathcal{B}(w) \) is \( O(n^6) \).

Because the basis \( \mathcal{B}(w) \) has at most \( n^2 \) elements, \( \bigcup_{w \in \Sigma^{(k-1)}} \mathcal{B}(w) \) has at most \( n^2 m^{k-1} \) elements. Every element produces at most \( m \) valid children nodes. Therefore, we visit at most \( O(n^2 m^k) \) nodes in *queue*. At every visited node \( x \) the algorithm may do three things: (i) calculating \( \langle \eta | \mathcal{P}(x) \), which needs time \( O(|x|^4) \); (ii) finding the string \( w \) composed of the last \( k - 1 \) letters of \( x \), which needs time \( O(k) \); (iii) verifying whether or not the \( n^2 \)-dimensional vector \( \langle \eta | \mathcal{P}(x) \) is linearly independent of the set \( \mathcal{B}(w) \), which needs time \( O(n^6) \) according to the result in \cite{8}. In addition, from the proof of Lemma 1, we know that \( \mathcal{G}(l_0, w) = \mathcal{G}(l_0 + i, w) \) for any \( i \geq 0 \) and any \( w \in \Sigma^{(k-1)} \). Therefore, if \( x \) belongs to the *queue*, then \( k \leq |x| \leq l_0 + k - 1 \leq (n^2 - 1)m^{k-1} + k \).

Hence, the worst time complexity is \( O(n^2 m^k(n^6 m^{k-1}+kn^4+k+n^6)) \), that is, \( O(m^{2k-1}n^8+km^k n^6) \). \( \square \)

### 4.2 A polynomial time algorithm for determining the equivalence between multi-letter QFAs

By virtue of Algorithm (I), we can determine the equivalence between any two multi-letter QFAs in polynomial time.
**Theorem 8.** Let $\Sigma$, $k_1$-letter QFA $A_1$, and $k_2$-letter QFA $A_2$ be the same as Theorem 7. Then there exists a polynomial time $O(m^{2k-1}n^8 + km^kn^6)$ algorithm to determine whether or not $A_1$ and $A_2$ are equivalent.

**Proof.** Denote $B_0 = \{ \langle \eta | \overline{\sigma}(x) : x \in \bigcup_{i=0}^{k-2} \Sigma^{(i)} \}$. By using Algorithm (I), in time $O(m^{2k-1}n^8 + km^kn^6)$ we can find $\bigcup_{w \in \Sigma^{(k-1)}} B(w)$. Since $B(w)$ is a base of $G(l_0,w)$, we know that every vector in $\{ \langle \eta | \overline{\sigma}(x) : x \in \Sigma^*, |x| \geq 1 \}$ can be linearly expressed by a finite number of vectors in $\bigcup_{w \in \Sigma^{(k-1)}} B(w) \bigcup B_0$. Therefore, we have the following algorithm.

Figure 2. Algorithm (II) for determining the equivalence between multi-letter QFAs.

```
Input: $A_1 = (Q_1, Q^{(1)}_{acc}, |\psi^{(1)}_0\rangle, \Sigma, \mu_1, k_1)$ and $A_2 = (Q_2, Q^{(2)}_{acc}, |\psi^{(2)}_0\rangle, \Sigma, \mu_2, k_2)$.
Step 1:
By means of Algorithm (I), find out $B(w)$ for all $w \in \Sigma^{(k-1)}$; compute $B_0$;
Step 2:
If $\forall \langle \psi | \in \bigcup_{w \in \Sigma^{(k-1)}} B(w) \bigcup B_0, \langle \psi |P_{acc} = 0$ then return $(A_1$ and $A_2$ are equivalent)
else return the $x$ for which $\langle \eta | \overline{\sigma}(x) |P_{acc} \neq 0;
```

**Complexity of the algorithm.** We compute $B_0$ in time $O(m^{k-2}n^4)$. Therefore, by Algorithm (I), in Step 1, we need time $O(m^{2k-1}n^8 + km^kn^6)$.

Provided $\langle \eta | \overline{\sigma}(x)$ has been computed, in time $O(n^2)$ we can decide whether $\langle \eta | \overline{\sigma}(x)P_{acc} \neq 0$. Since the number of elements in $B_0 \bigcup \bigcup_{w \in \Sigma^{(k-1)}} B(w)$ is at most $O(m^{k-1}n^2)$, the worst time complexity in Step 2 is $O(m^{k-1}n^4)$.

In summary, the worst time complexity in Algorithm (II) is $O(m^{2k-1}n^8 + km^kn^6)$.

**5 Concluding remarks**

In this paper, we have solved the decidability of equivalence of multi-letter QFAs. More exactly, we have proved that any two automata, a $k_1$-letter QFAs $A_1$ and a $k_2$-letter QFA $A_2$ over the input alphabet $\Sigma = \{ \sigma_1, \sigma_2, \ldots, \sigma_m \}$, are equivalent if and only if they are $(n^2m^{k-1} - m^{k-1} + k)$-equivalent, where $n = n_1 + n_2$, $n_1$ and $n_2$ are the numbers of states of $A_1$ and $A_2$, respectively, and $k = \max(k_1, k_2)$. By this result, we have obtained the decidability of equivalence of multi-letter QFAs over the same single input alphabet $\Sigma = \{ \sigma \}$ and the decidability of equivalence of MO-1QFAs.

However, if we determine the equivalence of multi-letter QFAs by checking all strings $x$
with \(|x| \leq n^2m^{k-1} - m^{k-1} + k\), then the time complexity is exponential. Therefore, we have
designed a polynomial-time \(O(m^{2k-1}n^8 + km^kn^6)\) algorithm for determining the equivalence
of any two multi-letter QFAs.

It is worth pointing out that although we have given an upper bound on the length
of strings to be verified for determining whether two multi-letter QFAs are equivalent, the
optimality of the upper bound has not been discussed, and this is worthy of further consider-
ation. An open issue concerns the state complexity of multi-letter QFAs compared with
the MO-1QFAs for accepting some languages (for example, unary regular languages [21, 23]).
Also, recalling the relation between MM-1QFAs and MO-1QFAs, the power of measure-many
multi-letter QFAs is worth being clarified. Whether or not measure-many multi-letter QFAs
can recognize non-regular languages may also be considered in the future.

References

[1] M. Amano, K. Iwama, Undecidability on Quantum Finite Automata, in: Proceed-
ings of the 31st Annual ACM Symposium on Theory of Computing, Atlanta, Geor-
gia, USA, 1999, pp. 368-375.

[2] A. Ambainis, R. Freivalds, One-way quantum finite automata: strengths, weak-
nesses and generalizations, in: Proceedings of the 39th Annual Symposium on Foun-
dations of Computer Science, IEEE Computer Society Press, Palo Alto, California,
USA, 1998, pp. 332-341. Also quant-ph/9802062, 1998.

[3] A. Belovs, A. Rosmanis, and J. Smotrovs, Multi-letter Reversible and Quantum
Finite Automata, in: Proceedings of the 13th International Conference on Develop-
ments in Language Theory (DLT’2007), Lecture Notes in Computer Science, Vol.
4588, Springer, Berlin, 2007, pp. 60-71.

[4] A. Bertoni, C. Mereghetti, B. Palano, Quantum Computing: 1-Way Quantum
Automata, in: Proceedings of the 9th International Conference on Developments in Language Theory (DLT’2003), Lecture Notes in Computer Science, Vol. 2710,
Springer, Berlin, 2003, pp. 1-20.

[5] A. Brodsky, N. Pippenger, Characterizations of 1-way quantum finite automata,
SIAM Journal on Computing 31 (2002) 1456-1478. Also quant-ph/9903014, 1999.

[6] D.K. Faddeev, V.N. Faddeeva, Computational Methods of Linear Algebra, Freeman,
San Francisco, 1963.

[7] J. Gruska, Quantum Computing, McGraw-Hill, London, 1999.
[8] V. Halava, M. Hirvensalo, and R. de Wolf, Marked PCP is decidable, *Theoretical Computer Science* 255 (2001) 193-204.

[9] M. Hirvensalo, Improved Undecidability Results on the Emptiness Problem of Probabilistic and Quantum Cut-Point Languages, in: SOFSEM’2007, Lecture Notes in Computer Science, Vol. 4362, Springer, Berlin, 2007, pp. 309-319.

[10] J. Hromkovič, One-way multihead deterministic finite automata, *Acta. Informatica* 19 (1983) 377-384.

[11] A. Kondacs, J. Watrous, On the power of finite state automata, in: Proceedings of the 38th IEEE Annual Symposium on Foundations of Computer Science, Miami Beach, Florida, USA, 1997, pp. 66-75.

[12] L.Z. Li, D.W. Qiu, Determination of equivalence between quantum sequential machines, *Theoretical Computer Science* 358 (2006) 65-74.

[13] L.Z. Li, D.W. Qiu, Determining the equivalence for one-way quantum finite automata, *Theoretical Computer Science* 403 (2008) 42-51.

[14] L.Z. Li, D.W. Qiu, A note on quantum sequential machines, *Theoretical Computer Science* 410 (2009) 2529-2535.

[15] C. Moore, J.P. Crutchfield, Quantum automata and quantum grammars, *Theoretical Computer Science* 237 (2000) 275-306. Also quant-ph/9707031 1997.

[16] A. Paz, Introduction to Probabilistic Automata, Academic Press, New York, 1971.

[17] D.W. Qiu, Characterization of Sequential Quantum Machines, *International Journal of Theoretical Physics* 41 (2002) 811-822.

[18] D.W. Qiu, L.Z. Li, An overview of quantum computation models: quantum automata, *Frontiers of Computer Science in China* 2 (2)(2008) 193-207.

[19] D.W. Qiu, S. Yu, Hierarchy and equivalence of multi-letter quantum finite automata, *Theoretical Computer Science*, 410 (2009) 3006-3017. Also arXiv: 0812.0852, 2008.

[20] S. Roman, Lattices and ordered sets, Springer, New York, 2008.

[21] G. Rozenberg, A. Salomaa (Eds.), Handbook of Formal Languages, Vol. 1, Springer-Verlag, Berlin, 1997.

[22] W.G. Tzeng, A Polynomial-time Algorithm for the Equivalence of Probabilistic Automata, *SIAM Journal on Computing* 21 (2) (1992) 216-227.

[23] S. Yu, Regular Languages, In: G. Rozenberg, A. Salomaa (Eds.), Handbook of Formal Languages, Springer-Verlag, Berlin, 1998, pp. 41-110.