Take it in your stride: Do we need striding in CNNs?

Chen Kong
Simon Lucey
Carnegie Mellon University
{chenk, slucey}@andrew.cmu.edu

Abstract

Since their inception, CNNs have utilized some type of striding operator to reduce the overlap of receptive fields and spatial dimensions. Although having clear heuristic motivations (i.e. lowering the number of parameters to learn) the mathematical role of striding within CNN learning remains unclear. This paper offers a novel and mathematical rigorous perspective on the role of the striding operator within modern CNNs. Specifically, we demonstrate theoretically that one can always represent a CNN that incorporates striding with an equivalent non-striding CNN which has more filters and smaller size. Through this equivalence we are then able to characterize striding as an additional mechanism for parameter sharing among channels, thus reducing training complexity. Finally, the framework presented in this paper offers a new mathematical perspective on the role of striding which we hope shall facilitate and simplify the future theoretical analysis of CNNs.

1. Introduction

Convolutional Neural Networks (CNNs) [9, 10, 8] have facilitated a dramatic increase in the performance of perceptual tasks throughout various fields including image and signal processing [4, 16, 7, 2], speech recognition [1, 6, 11], and computer vision [3, 13, 5]. Almost all CNNs used in the above applications employ some sort of striding operator, but to date there has been limited theoretical analysis of its role other than as a heuristic for lowering the degrees of freedom within the network. For the purposes of this paper we shall refer to striding in the context of convolution, where the stride refers to the relative offset applied to filter kernel. Classical convolution implies a stride of one, however, non-unity stride values are commonly entertained within CNN literature (see Figure 1 for a visualization). Herein, when we state that a CNN does not employ striding we are actually implying that it is using only convolutional operators with unity stride.

The role of striding has been championed within CNN architectures as: (i) it can reduce spatial resolution, leading to computational benefits; and (ii) can reduce the overlap of receptive fields. Even though these two explanations provide some motivations to a certain degree, they are still largely superficial.

In this paper we offer a new mathematical tool to characterize theoretically the role of striding within modern CNNs, and why their employment has been so crucial for the empirical success of these networks across a myriad of perceptual tasks. To facilitate this characterization we ask a fundamental question: is striding necessary within a CNN? The short answer is no, when evaluating a pre-trained model. This paper demonstrates that any feedforward CNN utilizing non-unity strided convolution can be evaluated equivalently as a feedforward CNN that employs only unity strided convolution. This claim is based on a simple yet elegant observation that any non-unity strided convolution can be equivalently simplified to a classical conv-
volution with unity stride, but with the additional filters of smaller size. Figure 1 visualizes this insight through a toy example. One can see that the convolution between vectors $[1, 2, 3, 4]^T$ and $[1, 2, 3, 4, 5, 6, 7, 8]^T$ with stride two is equivalent to the summation of two convolutions with stride one. If one considers each smaller filter as a channel, then the summation of two convolutions can be reinterpreted as a two-channel convolution with unity stride. This insight is at the heart of our paper.

Striding, however, is still useful. We further demonstrate that our proposed simplification strategy actually increases the parameter space of the CNN implying an increase in the capacity of the network. Therefore during training, the striding operator reduces the parameter space by forcing parameter sharing among different channels and potentially helping the generalization properties of the network.

**Contributions:** We make the following contributions:

- We establish a clear mathematical definition of strided convolution and unveil an equivalence between multi-stride and multi-channel convolutions.

- Theoretically demonstrate that any feed forward CNN employing striding has a mathematically equivalent non-striding CNN architecture during evaluation.

- We reinterpret striding as a tool for sharing parameters along channels, and argue that this connection gives a more thorough and theoretical explanation for why striding is still an invaluable tool when designing CNN architectures.

2. Related Work

In the history of CNNs, some architectures, e.g. AlexNet [8], ZFNet [17], GoogLeNet [15], ResNet [5], etc., utilize large filters with non-unity stride, while some architectures, e.g. VGG net [13] and more, utilize smaller filters with unity stride. Despite of such common usage of striding, none of them explains the reasons for their designed stride size in a clear mathematical way. This makes striding most like a heuristic choice.

Recently, Springenberg et al. [14] were devoted to pursuing simpler CNN architectures and questioned the necessity of different components in the canonical pipeline. They found that max-pooling can simply be replaced by a convolutional layer with increased stride without loss in accuracy on several image recognition benchmarks. Based on this finding, they proposed a novel architecture that consists solely of convolution layers utilizing ReLU as the sole non-linearity. Although their work does not focus on the role of striding, the proposed replacement implies that striding possibly serves similar functionality to a certain type of pooling. Further, due to the linear essence of convolutions, convolutions with large stride is more preferable. This offers a novel perspective to understanding striding.

More recently, Papyan et al. [12] proposed to reinterpret the forward pass of CNNs as a thresholding pursuit of signals modeled through a novel Multi-Layer Convolutional Sparse Coding (ML-CSC) model. This reinterpretation gave a clear mathematical meaning, objective and model to CNNs, which can be in turn used to analyze the guarantees for the success of the forward pass. Specifically, the mutual coherence of a learned convolutional dictionary performs a major role in deciding the uniqueness of recovery of signals in ML-CSC model, and thus affecting the success of the forward pass. Based on this theory, they proposed that strides not only bring computational benefit, but also some theoretical benefits in terms of guarantees on uniqueness. On the one hand, striding lowers the mutual coherence of convolutional dictionary, thus leading to more non-zeros allowed per stripes. On the other hand, strides decrease the length of stripes. These twofold together encourage a sparse solutions to the ML-CSC model and thus a higher possibility of success that the forward pass will generate unique codes. However, due to the lack of sufficient experimental support, their interpretation of CNNs remains a theoretical hypothesis.

3. Prerequisite: Convolution as Tensor Product

Before considering strides, as prerequisite we first define tensor product and use it to represent convolutions. Convolution as a linear operator can always be represented as
a matrix or tensor multiplication. For example, in one-
dimensional space, convolution of two vectors \( h \) and \( x \) are
known to equal to the multiplication of a Toeplitz matrix
generated by \( h \) and the vector \( x \). Now, we show this equiv-
alence also holds for two-dimensional space.

**Definition 1 (Tensor product):** Given a four-dimensional
tensor \( \mathcal{H} \in \mathbb{R}^{m \times n \times d \times e} \) and a matrix \( X \in \mathbb{R}^{c \times d} \), the product of \( \mathcal{H} \) and \( X \) is a matrix with dimension \( m \times n \). Formally,

\[
\mathcal{H}X \in \mathbb{R}^{m \times n}, \quad \text{s.t.} \quad [\mathcal{H}X]_{i,j} = \sum_{k=1}^{d} \sum_{\ell=1}^{e} \mathcal{H}_{i,j,k,\ell}X_{\ell,k}. \quad (1)
\]

Further, inspired by Toeplitz matrices, we define a cer-
tain set of four-dimensional tensors that share a specific
structured pattern.

**Definition 2 (Convolutional Tensor):** Given an image filter
\( H \) of an image filter \( \mathcal{H} \), the convolutional tensor \( \mathcal{H} \) corresponding to \( H \) is defined as a four-dimensional tensor such that

\[
\mathcal{H}(i,j,i : i + a - 1, j : j + b - 1) = H, \quad (2)
\]

where \( i, j \in \mathbb{Z}^{+} \). The operator \( : \) denotes tensor slicing.

Similar to a Toeplitz matrix, a convolutional tensor actu-
ally covers all possible shifted version of the corresponding
image filter, shown in Figure 2. One may also notice that
the size of a convolutional tensor depends not only on the
size of corresponding image filter, but also on the size of
convolved image.

From the definition of the convolutional tensor, one can
derive the following three important properties. For the sake
of brevity, we omit the proof of them.

**Property 1** A convolutional tensor \( \mathcal{H} \) must satisfy

\[
\mathcal{H}_{i,j,k,\ell} = \mathcal{H}_{i+1,j,k+1,\ell} = \mathcal{H}_{i,j+1,k,\ell+1}, \quad (3)
\]

for any valid \( i, j, k, \ell \).

**Property 2** Any four-dimensional tensor \( \mathcal{H} \) satisfying

\[
\mathcal{H}_{i,j,k,\ell} = \mathcal{H}_{i+1,j+1,k+1,\ell+1} = \mathcal{H}_{i,j+1,k,\ell+1}, \quad (4)
\]

for any valid \( i, j, k, \ell \), must be a convolutional tensor.

This property is of important since it provides an addi-
tional support to characterize convolutional tensors. Note
that Property 1 and 2 together introduce the equality con-
straints in Equation 4 as a sufficient and necessary condition
to a convolutional tensor.

**Property 3 (Convolution as Tensor Product):** Convolution
of an image filter \( H \in \mathbb{R}^{e \times h} \) and an image \( X \) equals to the
tensor product of the corresponding convolutional tensor \( \mathcal{H} \)
and the image \( X \). Formally

\[
H \ast X = \mathcal{H}X. \quad (5)
\]

Figure 3: An example of \( \mathcal{T}^{(2)}_{m,n} \) for \( m, n \in \{1, 2\} \). In this case, each sampling function will return a \( 3 \times 3 \) matrix. Left visualizes the input matrix whose colors denote which sampling function will select it.

This property depicts the major role of a convolutional
tensor, representing a convolution as a tensor product. One
can prove this property trivially from the definition of con-
volutions and Definition 2. One advantage of representing
a convolution as a tensor product is to allow us to analyze
the parameters of convolutions, e.g. stride, pad, etc., by manip-
ulating the corresponding convolutional tensor. Specifically,
utilizing non-unity stride in convolution is equivalent to
applying a sub-sampling function to the associated con-
volutional tensor. Before digging into details, we need one
more tool to help, which is a sub-sampling function.

**Definition 3 (Sampling function of matrix) Define \( \mathcal{T}^{(s)}_{m,n} (\cdot) \) as a function with a matrix as input and its down-sampled sub-matrix as output. This function select the elements on the grid defined by \( m, n, s \), where \( (m, n) \) denotes the upper-left position of the grid, and \( s \) denotes the sub-sampling stride. Note that \( m, n \leq s \).

To make this definition clearer, Figure 3 shows an exam-
pling of \( \mathcal{T}^{(2)}_{m,n} (\cdot) \) applied to a \( 6 \times 6 \) matrix, for \( m = 1, 2 \)
and \( n = 1, 2 \) respectively. From this example, one might
notice that the elements in a matrix will be sampled and
only sampled once by the proposed sub-sampling functions,
without overlapping or double-sampling. Based on this
sub-sampling function of matrix, we further propose a sub-
sampling function of four-dimensional tensor. It is basically
the same operator but is conducted on the plane of two dis-
gnated dimension out of four.

**Definition 4 (Sampling function of tensor) Define \( \mathcal{S}^{(s)}_{p,q,s} (\cdot) \) as a function sampling the elements of a four-dimension ten-
sor. \( (i, j) \) denotes the dimension indexes forming a plane
on which elements are sampled. \( (p, q) \) denotes the upper-
left position for sub-sampling. \( s \) denotes the sub-sampling
stride size.

Note that if we treat a matrix as a four-dimensional ten-
sor with size one in third and forth dimension, then \( \mathcal{T}^{(s)}_{m,n} (\cdot) \)
can be represented equivalently as \( \mathcal{S}^{(s)}_{m,n} (\cdot) \).
After adding sub-sampling functions to our mathematic arsenal, we are now able to give and prove another property of convolutional tensors, which is of significant importance in the future analysis of strides.

**Property 4** For any positive integer \( p,q,m,n \leq s \), the sub-sampled tensor

\[
S_{p,q,s}^{1,2}(S_{m,n,s}(\mathcal{H}))
\]

is still a convolutional tensor if \( \mathcal{H} \) is a convolutional tensor. Further, this sub-sampled tensor covers all shifted version of a sub-sampled filter

\[
T_{p,q,s}(O_{m-1,n-1}(\mathcal{H})),
\]

where \( O_{m,n}(\cdot) \) denotes a padding operation which adds \( m \) zero-padding along the first dimension and \( n \) zero-padding along the second dimension.

**Proof:** From the Definition 4, it is trivial to show that the \((i,j,k,l)\)-th element in \( S_{p,q,s}^{1,2}(S_{m,n,s}(\mathcal{H})) \) equals to \( H(i-1)s+m+(j-1)s+n,(k-1)s+p+(\ell-1)s+q \).

Since \( \mathcal{H} \) is a convolutional tensor, from Property 1, it is implied that, for any valid \( i,j,k,l \),

\[
S_{p,q,s}^{1,2}(S_{m,n,s}(\mathcal{H}))_{i,j,k,l} = S_{p,q,s}^{1,2}(S_{m,n,s}(\mathcal{H}))_{i+1,j+1,k+1,l}
\]

and, similarly,

\[
S_{p,q,s}^{1,2}(S_{m,n,s}(\mathcal{H}))_{i,j,k,l} = S_{p,q,s}^{1,2}(S_{m,n,s}(\mathcal{H}))_{i,j+1,k,l+1}
\]

From Property 2, it is indicated that the sub-sampled tensor \( S_{p,q,s}^{1,2}(S_{m,n,s}(\mathcal{H})) \) must be a convolutional tensor. As we know the left-upper active area on \( H(1,1,:) \) is the corresponding filter to \( \mathcal{H} \), the filter corresponding to the sub-sampled convolutional tensor is \( T_{p,q,s}(O_{m-1,n-1}(\mathcal{H})) \).

**Theorem 1** Convolving an image filter \( \mathcal{H} \in \mathbb{R}^{a \times b} \) with an image \( X \in \mathbb{R}^{c \times d} \) in stride \( s \) equals to the summation of regular convolutions of filters \( \mathcal{H}_{p,q} \) and images \( X_{p,q} \) for \( p,q = 1,\ldots,s \). Each \( \mathcal{H}_{p,q} \) is sampled from \( \mathcal{H} \) and each \( X_{p,q} \) is sampled from \( X \). Formally,

\[
\mathcal{H} *_{s} X = \sum_{p=1}^{s} \sum_{q=1}^{s} \mathcal{H}_{p,q} * X_{p,q},
\]

where operator \( *_{s} \) denotes convolution with stride \( s \).

**Proof:** From the definition of stride, it is clear that one can consider convolution with stride size larger than one as convolution followed by a sub-sampling function. Formally,

\[
\mathcal{H} *_{s} X = T_{1,1}^{(s)}(\mathcal{H} * X).
\]

From Lemma 1 below, it is implied that

\[
T_{1,1}^{(s)}(\mathcal{H} * X) = \sum_{p=1}^{s} \sum_{q=1}^{s} T_{p,q}^{(s)}(\mathcal{H}) * T_{p,q}^{(s)}(X)
\]

By defining

\[
\mathcal{H}_{p,q} = T_{p,q}^{(s)}(\mathcal{H}), \quad X_{p,q} = T_{p,q}^{(s)}(X),
\]

Equation 11 is proven.

If one considers each sub-sampled feature map \( T_{p,q}^{(s)}(X) \) as a channel, then the summation of convolutions in Equation 12 can be reinterpreted as a multi-channel convolution. In this sense, Theorem 1 claims that a multi-stride convolution can always be represented equivalently by a multi-channel convolution. This insight inspires us to explore the simplicity of a CNN, hoping to find a more elegant and general architecture using solely multi-channel convolution with unity stride in each layer. We will show in the next section that this non-striding architecture exists for each feedforward all convolutional net and is mathematically proven to be able to achieve comparable results if not better.

Additionally, Theorem 1 leads to some empirical benefits. It potentially improves computational efficiency in a practical embedding when estimating non-unity strided convolution with larger filters by multi-channel convolution with smaller filters. We pause the analysis to the reasons for this computational efficiency, since it involves the low-level structure of data storage in memory, which is out of the interest of our paper.

Finally, this insight potentially offer a novel perspective to answer the question in signal processing: Why is stride commonly used in convolutional neural network but rarely found in convolutional sparse coding? That is caused by

**4. Eliminating stride from Convolution**

Armed with convolutional tensor and sub-sampling functions, we now demonstrate that in a two-dimensional convolution, non-unity stride can be easily reduced to unity stride by rearranging the element positions in images and image filters respectively.
the equivalence between multi-stride convolution and multi-channel convolution. Due to the common usage of multi-channel convolution in convolutional sparse coding, using non-unity stride barely bring any theoretical benefits. A more detailed analysis is out of the scope of our paper and is considered as the future work.

**Lemma 1** For any image filter $H$ and image $X$, it must be true that

$$T_{m,n}^{(s)}(H \ast X) = \sum_{p=1}^{s} \sum_{q=1}^{s} T_{p,q}^{(s)}(O_{m-1,n-1}(H)) \ast T_{p,q}^{(s)}(X).$$

(14)

**Proof:** From Definition 2, it is known that $H \ast X = \mathcal{H}X$. (15)

Further, from Definition 4 and 3, it is implied that

$$T_{m,n}^{(s)}(H \ast X) = S_{m,n,s}^{1,2}(H)X. \quad (16)$$

Since sampling functions $s_{p,q,s}^{(3,4)}(\cdot)$ for all $p, q = 1, \ldots, s$ will cover all elements and only cover once, it is indicated that

$$S_{m,n,s}^{1,2}(H)X = \sum_{p=1}^{s} \sum_{q=1}^{s} S_{p,q,s}^{3,4}(S_{m,n,s}^{1,2}(H)) T_{p,q}^{(s)}(X).$$

(17)

From the Property 4 above, $S_{p,q,s}^{3,4}(S_{m,n,s}^{1,2}(H))$ is a convolutional tensor and is generated by the image filter $T_{p,q}^{(s)}(O_{m-1,n-1}(H))$. Therefore, due to the equivalence between convolution and tensor product, Equation 14 is proven.

5. Eliminating Stride from CNN

Now, we propose a novel strategy to eliminate non-unity strided convolutions from CNN. From Theorem 1, we are allowed to replace a non-unity strided convolution by a unity strided convolution without loss of performance for evaluation. However, this replacement might not be trivial in a CNN architecture as any changes of the number of channels in one layer will affect the structures of other layers. To make this difficulty clearer and also give a direct insight, let us consider a simple three layer example. Note that, to be able to focus on convolution and stride, we will follow [14] to replace max-pooling by a convolutional layer with increased stride, and restrict ourselves to the analysis of all convolutional net in this section.

Suppose that we have a three layer CNN whose architecture is shown in Figure 4 (a). To eliminate non-unity strided convolutions from this example, one can replace the convolution layer Conv-2 by a new convolution layer Conv-2’ which utilizes unity stride and $Y$ by $Y’$ which rearranges the element positions of $Y$. From Theorem 1, it is guaranteed that this replacement will hold the same output as before. The new architecture is shown in Figure 4 (b). However, one may notice that this replacement ruins the connection from the previous layer Conv-1 to $Y’$, resulting in a failure of forward pass.

To solve the problem and reconnect the previous layer to the modified current layer, we need to revisit Lemma 1. As one observes, each channel of $Y’$ is a certain sub-sampling of the feature map in a channel of $Y$. Further each channel of $Y$ is the result of convolving $X$ with associated filters in Conv-1. Therefore, from Lemma 1, it is implied that each channel of $Y’$ is actually the summation of convolving all possible sub-sampled $X$ with sub-sampled filters in Conv-1. In this perspective, to connect two layers, one can replace Conv-1 by a multi-channel convolution layer, Conv-1’, whose filter has more channels but in smaller size, and at the same time $X$ by $X’$, where each channel of $X’$ is a certain sub-sampling of a channel of $X$. Since again the sub-sampling function will cover and only cover all elements once, $X’$ actually rearranges the element positions of $X$, sharing the same number and values of elements.

By this strategy, we derived a new architecture shown in Figure 4 (c) that only utilizes convolutions with unity stride and are guaranteed to have a complete forward pass. More importantly, this new architecture also hold the equivalence to the original multi-stride CNN in terms of the evaluations.
of forward pass, guaranteed by Lemma 1. To offer a deeper insight and prove this procedure more rigorously, we next propose and prove Theorem 2, summarizing the strategy above and generalize it to additional layers.

**Theorem 2** Any all convolutional net can be simplified to a non-striding architecture where only unity stride is utilized by each convolution layer, such that these two architectures are mathematically equivalent with respect to the evaluation of forward pass.

**Proof:** Suppose we have an all convolutional net with \( L \) layers, where each layer has \( C_\ell \) channels and utilizes \( s_\ell \) as stride size. Denote \( X^{(\ell,c)} \) as the feature map in \( \ell \)-th layer \( c \)-th channel and \( H^{(\ell,c)} \) is the corresponding filter. We now use mathematical induction to prove this theorem.

**The last layer (basis):** From the architecture, we have

\[
X^{(L,c)} = \eta\left\{ \sum_{k=1}^{C_{L-1}} H^{(L-1,k)} \ast_{s_{L-1}} X^{(L-1,k)} \right\},
\]

where function \( \eta(\cdot) \) is an element-wise non-linear activation function such as Rectified Linear Unit(ReLU). From Theorem 1, it is derived that

\[
X^{(L,c)} = \eta\left\{ \sum_{k=1}^{C_{L-1}} \sum_{p=1}^{s_{L-1}} \sum_{q=1}^{s_{L-1}} H^{(L-1,k)}_{p,q} \ast X^{(L-1,k)}_{p,q} \right\},
\]

where

\[
X^{(L-1,k)}_{p,q} = T^{(s_{L-1})}_{p,q}(X^{(L-1,k)}),
\]

\[
H^{(L-1,k)}_{p,q} = T^{(s_{L-1})}_{p,q}(H^{(L-1,k)}).
\]

For convenience, let us define \( \tilde{X}^{(L-1)} \) as a feature map with \( C_{L-1} \cdot s_{L-1} \cdot s_{L-1} \) channels. Each channel of it is \( X^{(L-1,k)}_{p,q} \) for \( p, q = 1, \ldots, s_{L-1} \) and \( k = 1, \ldots, C_{L-1} \). Similarly, define \( H^{(L-1,c)} \) as a filter with \( C_{L-1} \cdot s_{L-1} \cdot s_{L-1} \) channels. Each channel of it is \( H^{(L-1,k)}_{p,q} \) for \( p, q = 1, \ldots, s_{L-1} \) and \( k = 1, \ldots, C_{L-1} \). Then the right hand side of Equation 19 can be reinterpreted as a multi-channel convolution between \( \tilde{X}^{(L-1)} \) and \( H^{(L-1,c)} \). Formally,

\[
X^{(L,c)} = \eta\left\{ \tilde{X}^{(L-1)} \ast H^{(L-1,c)} \right\}.
\]

This basis step shows that one can always replace the last convolution layer by a multi-channel convolution with unity stride, such that the last layer feature map \( X^{(L,c)} \) stay the same. Next, we will show how to represent each channel of \( X^{(L-1)} \) by \( X^{(L-2)} \) and \( H^{(L-2)} \).

**Intermediate layer (inductive step):** Let us consider the \( \ell \)-th layer, where \( \ell < L - 1 \). Given any valid positive integer \( c \leq C_{\ell+1}, m \leq s, n \leq s \) and \( s \), one can show that a sub-sampling of activation layer is equivalent to sub-sampling before the activation layer as long as the activation layer is element-wise. Formally,

\[
T_{m,n}^{(s)}(X^{(\ell+1,c)}) = T_{m,n}^{(s)}(\eta\left\{ \sum_{k=1}^{C_\ell} H^{(\ell,k)} \ast_{s_\ell} X^{(\ell,k)} \right\})
\]

\[
= \eta\left\{ \sum_{k=1}^{C_\ell} T_{m,n}^{(s)}(H^{(\ell,k)} \ast X^{(\ell,k)}) \right\}.
\]

Further, from the definition of stride, one can replace the multi-stride convolution by a single-stride convolution followed by a sub-sampling function. Formally,

\[
T_{m,n}^{(s)}(X^{(\ell+1,c)}) = \eta\left\{ \sum_{k=1}^{C_\ell} T_{m,n}^{(s)}(H^{(\ell,k)} \ast X^{(\ell,k)}) \right\}
\]

\[
= \eta\left\{ \sum_{k=1}^{C_\ell} T_{m,n}^{(s)}(H^{(\ell,k)} \ast X^{(\ell,k)}) \right\}.
\]

It is clear to see that the composition of two sub-sampling function is also a sub-sampling function:

\[
T_{m,n}^{(s)} \circ T_{m,n}^{(s)} = T_{m,n}^{(s)},
\]

where, for the sake of brevity, we omit the derivation but give the results:

\[
m' = (m - 1)s_\ell + 1, n' = (n - 1)s_\ell + 1, s' = ss_\ell.
\]

Therefore, by combining the two sub-sampling functions, Equation 24 can be simplified as

\[
T_{m,n}^{(s)}(X^{(\ell+1,c)}) = \eta\left\{ \sum_{k=1}^{C_\ell} T_{m,n}^{(s')}(H^{(\ell,k)} \ast X^{(\ell,k)}) \right\}.
\]

From the Lemma 1, it is implied that

\[
T_{m,n}^{(s)}(X^{(\ell+1,c)}) = \eta\left\{ \sum_{k=1}^{C_\ell} \sum_{p=1}^{s'} \sum_{q=1}^{s'} H^{(\ell,k)}_{p,q,m',n'} \ast X^{(\ell,k)}_{p,q} \right\},
\]

where

\[
H^{(\ell,k)}_{p,q,m',n'} = T_{p,q}^{(s')}(O_{m'-1,n'-1}(H^{(\ell,k)})),
\]

\[
X^{(\ell,k)}_{p,q} = T_{p,q}^{(s')} (X^{(\ell,k)}).
\]

Let us define \( \tilde{X}^{(\ell)} \) as a feature map with \( C_\ell \cdot s' \cdot s' \) channels. Each channel of it is \( X^{(\ell,k)}_{p,q} \) for \( p, q = 1, \ldots, s' \),
and \( k = 1, \ldots, C_L \). Similarly, define \( \tilde{H}^{(\ell,c')} \) as a filter with \( C_L \cdot s' \cdot s' \) channels, where \( c' \) depends on \( (m,n,c) \). Each channel of it is \( H^{(\ell,k)}_{p,q,m',n'} \) for \( p,q = 1, \ldots, s_{L-1} \) and \( k = 1, \ldots, C_{L-1} \). Then the left hand side of Equation 28 can be considered as the \( c' \)-th channel of \( \tilde{X}^{(\ell+1)} \). The right hand side of Equation 28 can be reinterpreted as a multi-channel convolution between \( X^{(\ell)} \) and \( \tilde{H}^{(\ell,c')} \). Formally,

\[
\tilde{X}^{(\ell+1,c')} = \eta \left\{ \tilde{X}^{(\ell)} \ast \tilde{H}^{(\ell,c')} \right\}.
\]

This inductive step shows that one can always represent each channel of \( \tilde{X}^{(\ell+1)} \) as the result of a multi-channel convolution with unity stride of \( X^{(\ell)} \) and a multi-channel filter.

**Mathematical Induction:** In summary, in an all convolutional net, the followings are proven true:

- The last layer feature map \( X^{(L)} \) can always be represented by \( \tilde{X}^{(L-1)} \) with a unity strided convolution followed by a non-linear function.
- For any \( \ell < L - 1 \), the feature map \( \tilde{X}^{(\ell+1)} \) can always be represented by \( \tilde{X}^{(\ell)} \) with a unity strided convolution followed by a non-linear function.
- Since sub-sampling function \( T^{(s)}_{m,n} \) covers once all elements, the feature map \( X^{(\ell)} \) is a certain reshape of \( \tilde{X}^{(\ell)} \).

From mathematical induction, it is implied that, for any all convolutional net, one can reshape the feature map sequentially from the last layer to the first layer and thus derive a new architecture where only unity stride is utilized by each convolution layers. As a result of more elegant structures without non-unity stride, we refer to this new architecture as the non-striding architecture. Moreover, because the equality always holds during reshaping, shown in Equation 19 and Equation 28, the non-striding architecture must be equivalent to the original one with respect to the output of forward pass.

### 6. Discussion and Implication

The theory described so far guarantees the existence of the non-striding architecture for any all convolutional net and the proof of Theorem 2 further provides a methodology to recover it. However, one might wonder what we can benefit from eliminating stride. In this section, we will compare in detail the non-striding architecture against the original network, further establish theoretical and practical benefits, and also note a few practical limitations of our framework.

One notable difference between the non-striding and original architecture is the dimension of images fed into the network. The non-striding architecture is required to take \( X^{(0)} \) as input which is a certain reshape of normal images \( X^{(0)} \). This step can be conducted before training or evaluating the neural network by preprocessing images. Thus it potentially transfers some computation from neural network to preprocessing step.

Another significant difference is the size of filters between these two architectures. For example, in the architecture shown in Figure 4, one can see that the size of filter in Conv-1 is \( 3 \times 32 \times 5 \times 5 \), however in Conv-1’ it is \( 12 \times 128 \times 3 \times 3 \). More generally, by denoting \( H, W \) as the height and width of the filter and following the notations in Section 5, the size of \( H^{(\ell)} \) is \( C_L \cdot C_{L+1} \cdot W_L \cdot H_L \), while the size of \( \tilde{H}^{(\ell)} \) is \( s^2 \cdot C_L \cdot C_{L+1} \cdot W_L \cdot H_L \). This is because, in \( \tilde{H}^{(\ell)} \), the number of input channel becomes \( (s')^2 \) more and the number of output channel becomes \( s^2 \) more, but filter size only becomes \( (s')^2 \) smaller. This increased size of filters indicates that the non-striding architecture has higher capacity than the original one. Even though this augmented capacity is potentially to increase the performance, it also possibly ruins the training due to too large searching area.

The reason why this additional capacity is not observed during forward passing a pre-trained model is that the strategy we proposed naturally forces some parameter sharing in the non-striding architecture. Specifically, in Equation 29, it is clear that filters \( H^{(\ell,k)}_{p,q,m',n'} \) with different \( m' \) and \( n' \) are forced to share parameters since they are all sampled from the same filters \( \tilde{H}^{(\ell,k)} \). This finding unveils an important role of striding in a CNN, that is striding is an efficient practice to force parameters shared among different channels. Specifically, one can observe that using non-unity stride in the original architecture is equivalent to sharing parameters among different channels in the non-striding architecture. This decrease of degree of freedom leads to a lower expressibility of network but higher possibility of searching a good local minimum. This makes striding still an invaluable practice to help training in an empirical usage.

The final contribution of our work is to pursue the simplest architecture of a CNN. Following the work [14] suggesting to replace max-pooling by convolution with larger stride, our work proposes to replace non-unity strided convolutions by unity strided convolutions without loss of performance. This further simplifies the neural network, leading to a more elegant architecture with only non-linearity and classical convolution without the need of designing stride size. We hope this non-striding architecture shall facilitate and simplify the future theoretical analysis of CNNs.

### 7. A Toy Experiment

The theory described so far is very general and provides a guarantee to the existence of the non-striding architecture to arbitrary all convolutional net. To verify the theory and also provide a sense from a concrete neural network, we conducted a toy experiment applying a modified LeNet [10] to MNIST dataset. LeNet is a perfect target in our case.
Table 1: Model description of the two networks derived from LeNet. “Non-Unity Stride” denotes the architecture using non-unity strided convolution layer, while “unity stride” denotes the architecture using unity strided convolution layer. “Accuracy” refers as to the testing accuracy after independently training two architectures. As an example, “5 × 5 conv. 20” denotes as a convolution layer whose kernel size is 5 × 5 and the number of output channel is 20.

Table 2: Listing the dimension of each feature maps in the two architectures, utilizing channel × height × width format.

8. Conclusion

This paper, to our best knowledge, at the first time proposed to represent a non-unity strided convolution by a unity strided convolution with multi-channels. Based on this insight, we demonstrated that in any all convolutional net, non-unity strided convolution can be replaced by unity strided convolution without loss of performance for evaluations. Hereby, any feedforward CNN are proven to have a mathematically equivalent non-striding CNN architecture during evaluation such that it consists only of non-linearity and regular convolution with solely unity stride. We hope this non-striding architecture shall facilitate and simplify the future theoretical analysis of CNNs. Finally, by observing an increase in the capacity of the non-striding architecture, we demonstrated that striding reduces the number of trainable variables in convolutional filters by sharing parameters. This finding makes striding still a useful practice when designing CNN architectures.

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