Solution of Fokker-Planck equation for moderately coupled relativistic magnetoplasma having anisotropy in temperature

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Abstract-The Fokker-Planck equation is used to treat the large angle as well as small angle ion-electron collision in relativistic magnetoplasma having anisotropy in temperature. The relevant collision integrals are formulated with the introduction of an appropriate equilibrium distribution function. The kinetic equation is derived from first principle and Chapman-Enskog type of approximation is employed to derived the modified expressions for the first order perturbed distribution function $f_{11}$ and the corresponding diffusion drift velocity ($U_d$) exhibiting the effects of anisotropy in temperature.

I. Introduction:

Temperature anisotropy is an intrinsic phenomenon in space plasma. Recent advancement in space probe, discovery of high temperature astrophysical objects, black hole radiations and studies of high energy particles in accelerators and other confinement devices have generated interest in studying collisional diffusion and transport in a magnetized plasma where there is anisotropy in temperature. In particular, variations of thermal energy components in coronal surfaces of sun, solar bursts and pulsar winds have developed enthusiasm for understanding the effects of anisotropy in temperature on diffusion transport process in astrophysical as well as laboratory fusion plasma both theoretically [1,2] and experimentally [3,4].

Early works [2,5] have dealt with the collisional diffusion and transport across a magnetic field both for isotropic plasma and plasma including an anisotropy in energy momentum distribution. Mohanty and Baral [2] have derived analytical ready to use formulae for various transport coefficients of interest and have presented the numerical and graphical variations of transport coefficients with streaming parameter $V_0$. In addition, the issues concerning anisotropy in temperature in a non-relativistic magneto-plasma have also been addressed [6] with a view to understand the collisional kinetics of plasmas occurring in magneto spheric space, astrophysical situation and laboratory simulation diagnostics.

Recent studies [7,8] on collisional diffusion and transport relevant to various space probes warrant the consideration of anisotropic collisional kinetics when temperature across and along the magnetic field differs ($T_{11} \neq T_{\perp}$) relevant to pulsar winds and jets and particle acceleration in hypothetical cosmic pinches involving high temperature relativistic plasmas.
The present paper aims at unraveling the interesting features of temperature anisotropy involved in FP collisional kinetics of a strongly magnetized, relativistic, moderately coupled \([1 \leq \log 2/\theta_0 \leq 10]\) two components (electron, ion) plasma covering the non relativistic thermal regime where both the electrons and ions are classic; moderately relativistic thermal regime where the electrons are relativistic while ions remain classic and the ultra relativistic thermal regime where both the electrons and ions are relativistic in nature.

This paper is organized as follows. In section II we present the general theoretical formulation of the Fokker-Planck equation with the relevant collision integral for moderately coupled relativistic magneto plasma and introduce an appropriate Zero order equilibrium distribution function including an anisotropy in temperature \((T_e \neq T_i)\). In section III we solve the FP equation in Chapman Enskog method to obtain an expression for the perturbed distribution function \(f_{11}^+\) to first order in two small parameters \(\alpha\) and \(\beta\). The expression for diffusion drift velocity \(U_d\) is obtained taking velocity moment of \(f_{11}^+\) and significance of the temperature anisotropy \(T_e/T_i\) is emphasized. Finally a brief summary of these results is given in section IV and the application of the result for derivation of various diffusion transport coefficients covering non-relativistic thermal regimes, moderately relationship thermal regimes and ultra relationship thermal regimes are indicated.

### II. BASIC FORMULATIONS

The Fokker-Planck equation relevant to moderately coupled singly charged electron-ion plasma is given as [9].

\[
\partial f^\pm / \partial t + \mathbf{V}/\gamma_c \cdot \nabla f^\pm + \nabla \cdot [\mathbf{u}/m^2 \{ (\mathbf{E} + \mathbf{V} \times \mathbf{B})/\gamma_c \} f^\pm + \Gamma_c ] = 0
\]  

(1)

Where

\[
\Gamma_c = -2\pi e^4/m^2 \log(2/\theta_0) \left\{ \partial^2 f^\pm / \partial v_i \partial v_j \right\} + 2 \gamma_c \gamma_i \gamma_j / \log(2/\theta_0) \left\{ \partial^2 f^\pm / \partial v_i \partial v_j \right\} - 2f^\pm (v_i) \partial f^\pm / \partial v_i - 2f^\pm (v_i) \partial f^\pm / \partial v_i + \gamma_c \gamma_i \gamma_j / \log(2/\theta_0) \left\{ \partial^3 f^\pm / \partial v_i \partial v_j \partial v_k \right\}
\]

(2)

It is instructive to note that \(\Gamma_c\) represent the rate of change of distribution \(f^\pm\) due to collision with one another. Here \(f^+\) and \(f^-\) are defined as distribution function for ions and electrons respectively. \(\mathbf{v}\) is the momentum divided by the rest mass and the velocity \(\mathbf{u}=\mathbf{v}/\gamma, \gamma\) being the relativistic factor and \(\mathbf{U}=\mathbf{v}/\gamma - \mathbf{v}_i/\gamma_{i1}\), the relative velocity.

It is worthwhile to mention here that in the present context a moderately coupled relativistic magneto plasma having an anisotropy in temperature is dealt with in the regime \(2 \leq \log 2/\theta_0 \leq 10\) as exemplified by short pulse laser plasmas, inertial confinement fusion plasmas, X-ray laser plasmas, solar core and astrophysical plasmas like pulsar, quasar, and plasma occurring in the expansion of universe. The high density plasma necessitating quantum correction is excluded in this analysis.
In our collisional model, the plasma is considered to be an infinitely extended open ended system of fully ionized singly charged electron-ion relativistic plasma having a small density gradient ($\nabla N$) and temperature gradient ($\nabla (kT)$) in same direction. The plasma is treated as locally quasi-neutralized ($N_+ \approx N_- \approx 1/2N$) and isothermal ($T_+ = T_-$) plasma but having an anisotropy in temperature ($T_\perp \neq T_\parallel$). The mutually perpendicular magnetic field and induced electric field are assumed to be acting along the direction perpendicular to the direction of the gradients. The collision frequency ($\tau^{-1}$) is assumed to be much smaller than the cyclotron frequency ($\omega$) and the Larmor radius ($\lambda_G$) is assumed to be small with respect to the typical length scale.

As usual [5], the distribution function $f$ is expanded in terms of two small parameters $\alpha = \lambda_G / (N/|\nabla N|)$, the ratio of the radius of gyration $\lambda_G$ to the characteristic macroscopic distance or density in homogeneity scale length ($N/|\nabla N|$) and $\beta = (\tau \omega^2)^{-1}$, the ratio of the collision or frequency $\tau^{-1}$ to the gyration frequency $\omega^2$ as

$$f^\pm = f_{0}^\pm + f_{1}^\pm \quad \text{and} \quad f_{1}^\pm = f_{10}^\pm + f_{11}^\pm$$

Where $f_{0}^\pm$ is the assumed zero order equilibrium maxwellian distribution function having anisotropy in thermal energy.

$$f_{0}^\pm = Z_{0}^\pm \exp \left\{ - Z_{0}^\pm \frac{1}{\gamma} \right\}$$

Where $Z_{0}^\pm = (2/3) Z_{\perp}^\pm + (1/3) Z_{\parallel}^\pm$; \hfill (4)

$$Z_{\perp}^\pm = m^\pm c^2 / kT_{\perp} \quad \text{and} \quad Z_{\parallel}^\pm = m^\pm c^2 / kT_{\parallel}$$

The relativistic factor $\gamma = (1 - u^2/c^2)^{-1/2}$ and $K_n (z')$ is the Bessel function of second kind and order ‘n’ with argument $z'$. It is instructive to note that $T_\perp$ and $T_\parallel$ are the perpendicular and parallel components of temperature [6].

Following our early work [5] we assume that the uniform magnetic field $B$ is applied along $z$- direction and the electric field $E$ is induced owing to the motion of the diamagnetic plasma. In the present context $E$ is taken along $Y$ direction. Further more, there exist a small temperature gradient $\nabla (kT_{\perp})$ and small density gradient $\nabla N$ along the same $X$- direction.

The Zero order pressure is evaluated is the usual way [6] as

$$P = \sum m^\pm \int d^3v \left( v_i v_j / \gamma \right) f_{0}^\pm$$

Which on proper evaluation of the integral in $V$- space yields

$$P = (3/2) NkT_{\parallel} / (1 + T_{\perp}/2T_{\parallel})$$

Note that in the isotropic limit of temperature, $T_\perp = T_\parallel = T$, one can easily recover the well known expression,
To the first order is $\alpha$, the F.P. equation (1) can be written as

$$\frac{1}{\gamma} v_j (\partial f_0 / \partial x_i) + (e/m^* \gamma c) \epsilon_{ijk} v_k (\partial f_0 / \partial v_i) = -\nabla f \cdot (\Gamma^+_1 + \Gamma^+_1)$$

(9)

Multiplying both the sides by $m^* v_i$ and integrating over velocity space in the usual manner with the modified collision terms, it is straightforward to derive the equation of hydrostatic equilibrium,

$$\nabla P = (1/c) J \times B$$

(10)

Where the current density $J$ is given by

$$J = (1/2) Ne^2 (v_i^+ - v_i^-)$$

(11)

Following our early work [9] we introduce the modified linearized form of the collision integral to the first order in $\beta$ for relativistic moderately coupled plasma in the form

$$-\partial / \partial v_i (\Gamma_{10}^{++} + \Gamma_{10}^{+-} + \Gamma_{10}^{+\gamma} + \Gamma_{10}^{-+} ) = \partial / \partial v_i \{ \pi N e^4 / m^2 \log (2/\theta_0) (\gamma v_i v_j V / V_i) \partial / \partial v_i$$

$$f'(V) + 4\gamma / V_3 V_i f' + 2\gamma / V_3 (1/3 <u^2> + m^*/m <u^2>) \gamma / V_2 V_i f' / \partial v_i + 1/ \log (2/\theta_0)$$

$$+ (4\gamma / 3 V_i V_j V_k / V_j \delta_{ij} + V_i / V_j \delta_{jk} + V_j / V_i \delta_{ik} - V_i V_j V_k / V_j \delta_{ij} \partial^2 / \partial v_i \partial v_j) f' \}$$

(12)

where the mean velocities for the quasi-neutralized plasma species particles are defined as

$$u^\pm = (2/N) \int d^3 v (v_i / \gamma) f^\pm$$

(13)

It is worthwhile to note that for our fully ionized quasi-neutralized single charged electron-ion plasma the particles density $N$ is given as

$$N = N^+ + N^-, N^\equiv = N \equiv N/2$$

And $N^\equiv = \int d^3 v j^\equiv_0$ (14)

### III. DERIVATION OF DRIFT DIFFUSION VELOCITY

In order to account for the modified expression for the first order perturbed distribution function including an anisotropy in temperature ($T_\perp / T_\parallel \neq 1$), it is instructive to note here that for steady state solution $[\partial f^\pm / \partial t] = 0$ and to zero order in $\alpha, \partial N / \partial x_i = 0$.

Following our early analysis [9] we derive the appropriate Fokker Plank equation (1) to the first order in $\alpha$ as

$$1/\gamma_i V_j (\partial f_0 / \partial x_i) + (e/m^* \gamma c) E_j (\partial f_0 / \partial v_j) + (e/\gamma_m, m^* c) C_{ijk} V_i B_k \partial f_1 / \partial V_i = \partial / \partial V_i (\Gamma^+_1 + \Gamma^+_1)$$

(15)
The collision is treated as perturbation to zero order in $\beta$; then equation (15) yields,

\[
(1/\gamma_v) \nabla_i (\partial f_0^+ / \partial x_i) + (e/\gamma_v m^* c) C_{ijk} V_1 B_k (\partial f_0^+ / \partial V_i) = 0
\]  

(16)

The solution of equation (16) gives,

\[
f_{10}^+ = (1/\gamma_v \omega^*) f_0^+ v_i \partial / \partial x_i (\log f_0^+ )
\]  

(17)

Where $\omega^* = eB/\gamma_v m^* c$

The equation (17) is further simplified with the anistrophic Maxwellian distribution function equation (3) to yield

\[
f_{10}^+ = f_0^+ \phi^*(v)
\]  

(18)

Where $\phi^*(v) = (1/\gamma_v \omega^*) v_j ((1/N) (\partial N/\partial x) - [1/(1+ T_/2T_l)] - (2/3)(\gamma_v m^* c^2 / kT_l))$

\[
(1/kT_l) \partial / \partial x (kT_l)
\]  

(19)

Next, we apply the usual [9] Chapman Enskog analysis to simplify the collision integral equation(12) further to obtain the modified expression for the collision integral relevant to moderately coupled relativistic plasma with temperature anisotropy as

\[-\partial / \partial v_i (\Gamma_{10}^{++} + \Gamma_{10}^{+}) = \partial / \partial v_i [(4\pi Ne^2 \gamma^2 / m^2) \log (2/\theta_0) v_i ^2 ] + (1/(1+T_/2T_l)) - (2/3)(\gamma_v m^* c^2 / kT_l)]

\[-\partial / \partial v_i (\Gamma_{10}^{++} + \Gamma_{10}^{+}) = \partial / \partial v_i [(4\pi Ne^2 \gamma^2 / m^2) \log (2/\theta_0) v_i ^2 ] + (1/(1+T_/2T_l)) - (2/3)(\gamma_v m^* c^2 / kT_l)]

\[-\partial / \partial v_i (\Gamma_{10}^{++} + \Gamma_{10}^{+}) = \partial / \partial v_i [(4\pi Ne^2 \gamma^2 / m^2) \log (2/\theta_0) v_i ^2 ] + (1/(1+T_/2T_l)) - (2/3)(\gamma_v m^* c^2 / kT_l)]

On further simplification, the collision integral is reduced to the form

\[-\partial / \partial v_i (\Gamma_{10}^{++} + \Gamma_{10}^{+}) = \partial / \partial v_i [(4\pi Ne^2 \gamma^2 / m^2) \log (2/\theta_0) v_i ^2 ] + (1/(1+T_/2T_l)) - (2/3)(\gamma_v m^* c^2 / kT_l)]

\[-\partial / \partial v_i (\Gamma_{10}^{++} + \Gamma_{10}^{+}) = \partial / \partial v_i [(4\pi Ne^2 \gamma^2 / m^2) \log (2/\theta_0) v_i ^2 ] + (1/(1+T_/2T_l)) - (2/3)(\gamma_v m^* c^2 / kT_l)]

\[-\partial / \partial v_i (\Gamma_{10}^{++} + \Gamma_{10}^{+}) = \partial / \partial v_i [(4\pi Ne^2 \gamma^2 / m^2) \log (2/\theta_0) v_i ^2 ] + (1/(1+T_/2T_l)) - (2/3)(\gamma_v m^* c^2 / kT_l)]

Where $\gamma_v = (4\pi Ne^2 \gamma^2 / m^2) \log (2/\theta_0) [1+(1/\gamma_v \omega^*) (\partial / \partial x) \log N (2/3kT_l + 1/3kT_l)] + (2/3)(\gamma_v m^* c^2 / kT_l)$

\[-\partial / \partial v_i (\Gamma_{10}^{++} + \Gamma_{10}^{+}) = \partial / \partial v_i [(4\pi Ne^2 \gamma^2 / m^2) \log (2/\theta_0) v_i ^2 ] + (1/(1+T_/2T_l)) - (2/3)(\gamma_v m^* c^2 / kT_l)]

\[-\partial / \partial v_i (\Gamma_{10}^{++} + \Gamma_{10}^{+}) = \partial / \partial v_i [(4\pi Ne^2 \gamma^2 / m^2) \log (2/\theta_0) v_i ^2 ] + (1/(1+T_/2T_l)) - (2/3)(\gamma_v m^* c^2 / kT_l)]

\[-\partial / \partial v_i (\Gamma_{10}^{++} + \Gamma_{10}^{+}) = \partial / \partial v_i [(4\pi Ne^2 \gamma^2 / m^2) \log (2/\theta_0) v_i ^2 ] + (1/(1+T_/2T_l)) - (2/3)(\gamma_v m^* c^2 / kT_l)]

\[-\partial / \partial v_i (\Gamma_{10}^{++} + \Gamma_{10}^{+}) = \partial / \partial v_i [(4\pi Ne^2 \gamma^2 / m^2) \log (2/\theta_0) v_i ^2 ] + (1/(1+T_/2T_l)) - (2/3)(\gamma_v m^* c^2 / kT_l)]
To first order in α and first order in β the Fokker–Planck equation (1) assumes the form,

\[
\frac{\partial}{\partial t} f = \nabla \cdot (E_\parallel f) + (e/m^*) E \cdot \nabla f + (e/m^*) \gamma E_\perp \nabla T + \frac{1}{\gamma} \nabla \cdot \mathbf{j}.
\]

Where \( \Gamma_{10}^{++} \) and \( \Gamma_{10}^{+-} \) represent the corresponding ion–ion and ion–electron collision integrals respectively. Using eq. (21) the above equation is further simplified and solved to yield the expression for the first order perturbed function \( f_{11}^{++} \) as,

\[
f_{11}^{++} = \frac{1}{(2/3)N F_{10}^{++}} \frac{1}{E/B} \left| \frac{1}{v} \frac{\partial}{\partial v} \right| \frac{1}{v} \frac{\partial}{\partial v} \left( \frac{f_0}{v^3} \right), \quad \frac{\partial}{\partial v} \left( \frac{f_0}{v^3} \right)
\]

The drift diffusion velocity \( U_d \) is calculated by taking the velocity moment of \( f_{11}^{++} \) as

\[
U_d = \frac{1}{(2/3)N F_{10}^{++}} \frac{1}{E/B} \left| \frac{1}{v} \frac{\partial}{\partial v} \right| \frac{1}{v} \frac{\partial}{\partial v} \left( \frac{f_0}{v^3} \right)
\]

The complicated collision integrals can be evaluated in center of mass and relative velocity frame by substituting \( v = c \sin \theta \) is the usual manner [1].

It is worthwhile to mention here that our results contain terms involving \( 1/\log (2/\theta_0) \) term pertaining to moderately coupled plasma. However, in the limit of weakly coupled plasma we recover our earlier expression [10,11] by neglecting terms containing \( 1/\log (2/\theta_0) \) and taking the limit \( T_\perp = T_\parallel = T \).

In our subsequent work we propose to derive expression for various transport coefficients and emphasize the effects of temperature anisotropy (\( T_\perp / T_\parallel \neq 1 \)) on them in varying thermal regimes viz-non relativistic, moderately relativistic and ultra relativistic limits of temperatures.

### IV. SUMMARY AND DISCUSSIONS.

In this work we analytically investigate the effects of temperature anisotropy (\( T_\perp \neq T_\parallel \)) on the collisional diffusion and transport in a fully ionized relativistic magneto plasma on the basic of the Fokker Planck (FP) collision in the context of small angle binary encounters. This analysis is an extension of our early investigation [9] to include the anisotropy in temperature in moderately coupled relativistic plasma. The FP equation is solved with an appropriate anisotropic Maxwellian distribution function is the Chapman Enskog method. The modified FP collision term is developed from the first principle by relating the third order term in the Tyler expansion and the influence of temperature anisotropy is emphasized. The
important terms containing $1/\log(2/\theta_0)$ in the expression for the perturbed distribution function $f_{11}^+$ entails the modification is the moderately coupled plasma. The relevant expressions for weakly coupled plasma are recovered by neglecting the $1/\log(2/\theta_0)$ dependent terms.

It is also interesting to note that non realistic results[6] are recovered in the limit of temperature where ion temperature is less than $10^{13}$K and electron temperature is below $10^8$k such that $Z_{ii} Z_{11} >> 1$. The isotropic results of relativistic magneto plasma are also recovered in the temperature limit $T_i = T_e = T$. The modified expressions for drift diffusion velocity ($U_d$) are obtained taking the velocity moment of $f_{11}^+$. The process of evaluation of $U_d$ and the derivation of various transport coefficients are indicated to be presented in the subsequent work. The results are applicable to astrophysical laboratory plasmas and plasmas where an unstable state occurs to absence of directed flaxes [12].

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