DEGENERATION OF CALABI-YAU MANIFOLD
WITH WEIL-PETERSSON METRIC

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Abstract. Koiso was first to introduce the Weil-Petersson metric in higher dimension. Tian showed that a moduli space of Calabi-Yau $n$-manifolds comes naturally with Weil-Petersson metric. In this paper we focus on determining for which degenerations the central fibre is at finite distance with respect to the Weil-Petersson metric. First we obtain a simple condition on the limiting mixed Hodge structure which is a necessary and sufficient condition for finite Weil-Petersson distance to the central fibre. This issue has been raised in the Physics literature but not extensively analyzed there. Then we combine the result with the canonical mixed Hodge structure of the central fibre and obtain a simple cohomological necessary and sufficient condition for the central fibre to be at finite distance. As a corollary, we prove that a central fibre with simple nodes is at finite distance.

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Introduction.

Koiso [Ks] first extended classical Weil-Petersson metric to higher dimension. Then Tian showed that a moduli space of Calabi-Yau $n$–manifolds comes naturally with the Weil-Petersson metric defined by using a generator of $H^{n,0}$, which first appeared in [Ti]. The problem of compactifying a global moduli space is not yet solved for higher dimensions, except for some examples in dimension 3 worked out with regards to Mirror Symmetry. Even so-called degeneration problems were not solved for $n \geq 3$. In this paper we complete the moduli space of Calabi-Yau $n$–folds with respect to the Weil-Petersson metric; i.e., to fill the central fibre of a degeneration which is at finite distance with respect to the Weil-Petersson metric. This constitutes a partial compactification of a moduli space of Calabi-Yau $n$–folds.

A degeneration of $n$–dimensional varieties is a proper flat holomorphic map from a variety $\mathcal{X}$ to a disk $\Delta$ of relative dimension $n$ such that the generic fibre is a smooth variety. In the case of Calabi-Yau $n$–folds, a one-parameter family of quintic $3$–folds in $\mathbb{C}P^4$ is a typical example. Any degeneration can be reduced to a semi-stable degeneration by a base change, thanks to Mumford’s semi-stable reduction theorem [Mum]. A semi-stable degeneration allows only a reduced divisor
with normal crossings for a central fibre, which is among all singularities the easiest to treat. A typical example arises from a full blow-up of a complete resolution of simple singularities such as simple nodal points. In fact by a glorious result of Hironaka, any singularity can be resolved into such a reduced variety with only normal crossings.

For lower dimensional cases corresponding to $K_{X_s} = 0$, the results are already known. In the case of the degeneration of complex tori the genus will strictly go down, i.e., the central fibre has to become a $\mathbb{C}P^1$. For K3 surfaces, there are 3 cases that were worked out recently by [PP] and [Kul]. Both results were obtained by strictly Algebro-Geometric methods.

On the other hand, the deformation theory of complex structures was originally developed by Kodaira and Spencer [KS]. Griffiths [Gr1] started the construction of the period map and later variation of Hodge structures to study moduli spaces of complex structures with fixed underlying topological manifolds. For complex varieties with $K_{X_s} = 0$, it is known that the deformation space of complex structures is unobstructed, by the theorem of Bogomolov-Tian-Todorov [Ti], [Tod]. As a result, one can look at the variation of Hodge structures associated to a given moduli space problem, in particular, a degeneration.

From a Hodge theoretic point of view, the study of a degeneration corresponds to the study of (limiting) mixed Hodge structures, which is in a sense a ‘degenerated’ Hodge structure of generic fibres. It should be noted that a ‘degenerated’ Hodge structure may no longer be a Hodge structure. Geometrically, a classifying space of Hodge structures is a subset of a product of Grassmannians. Since the automorphism group of this classifying space is transitive, it is also a homogeneous space. Hence, a representation of $\pi_1(\Delta)$, usually called the monodromy group, acts on the cohomology of generic fibres, and in the case of a semi-stable degeneration the monodromy operator is in fact unipotent by the Monodromy Theorem.

Independent of a variation of Hodge structures, any (possibly singular) algebraic variety carries a canonical mixed Hodge structure, as was first proved by Deligne [DeH2], [DeH3] in rather general language. So far, one of the few tools that works regardless of the dimension of a generic fibre is an exact sequence of cohomologies (and homologies) with underlying mixed Hodge structures, called the Clemens-Schmid exact sequence. This can connect the limiting mixed Hodge structure of a generic fibre with the canonical mixed Hodge structure of the central fibre.

It is usually impossible to compute a ‘limit’ of Hodge structures of generic fibres. Instead, we look at the nilpotent orbit of a limiting mixed Hodge structure. Schmid [Sch] proved that a nilpotent orbit behaves asymptotically as a real ‘limit’ of Hodge structures. Hence one can compute the distance to the central fibre using this nilpotent orbit.

We rely on the geometry of the classifying space, which, as a subspace of a product of Grassmannians, carries a canonical metric. Schmid’s nilpotent orbit theorem refers to the distance by this metric. However, the Weil-Petersson metric associated to any Calabi-Yau moduli space is bounded by this canonical metric.

Our main result is stated as the following Theorem;

**Theorem** A semistable degeneration of a Calabi-Yau $n$ dimensional compact complex variety is at finite distance with respect to the Weil-Petersson metric if and only if $H^{n,0}$ of one of the components of the central fibre has positive rank.
degeneration of Calabi-Yau $n$-folds and then define the Weil-Petersson metric and show it is essentially a direct summand of the canonical metric. Hence, the distance with respect to the Weil-Petersson metric is always bounded above by the distance with respect to the canonical metric. As the first step to prove the main theorem, we show the monodromy condition on the polarization of the limiting mixed Hodge structure of finite Weil-Petersson distance. In section 2 we investigate finite distance limiting mixed Hodge structures in more detail and obtain a monodromy condition. In section 3, we turn to the central fibre and look at the spectral sequence that is used to describe the canonical mixed Hodge structure of a central fibre. Finally, we combine the two constructions by means of the Clemens-Schmid exact sequence to obtain a rather simple cohomology condition on the central fibre as a criterion for finite distance. In section 4, we illustrate the result with several examples. In particular, we generalize a result appearing in the physics literature (eg. [CGH]) which says that a central fibre with simple nodes is at finite distance for dimension 3. We show that this holds for all dimensions greater than 2.

1. Degeneration with Weil-Petersson Metric

For our purposes, a Calabi-Yau manifold is a complex projective manifold of dimension $n$ with trivial canonical bundle. We will look at the degeneration space of Calabi-Yau $n$-folds over the unit disk $\Delta$.

Facts and proofs stated in the rest of this section can be found in the text [GrTA] Chapter 1 and Chapter 6.

**Definition 1.1** A degeneration is a proper flat holomorphic map $\pi: \mathcal{X} \longrightarrow \Delta$ of relative dimension $n$ such that $\mathcal{X}_s = \pi^{-1}(s)$ is a smooth complex variety for $s \neq 0$, and $\mathcal{X}$ is a Kähler manifold. A degeneration is semistable if the central fibre $\mathcal{X}_0$ is a divisor with (global) normal crossings; locally $\pi$ is defined by

$$s = x_1x_2x_3 \ldots x_k$$

i.e., $\mathcal{X}_0$ is written as a sum of irreducible components where each component $X_i$ is smooth and all components intersect transversely each other.

We further assume generic fibres to be projective and the total space to be pseudo-projective.

Let $f^*: \mathcal{X}^* \longrightarrow \Delta^*$ be the restriction to the punctured disk. Fix a smooth fiber $\mathcal{X}_s$. Since $f^*$ is a $C^\infty$ fibration, $\pi_1(\Delta^*)$ acts on the cohomology $H^m(\mathcal{X}_s)$. The map $T: H^m(\mathcal{X}_s) \longrightarrow H^m(\mathcal{X}_s)$ induced by the canonical generator of $\pi_1(\Delta^*)$ is called the Picard-Lefschetz transformation.

**Theorem (Monodromy Theorem)**

1. $T$ is quasi-unipotent, with index of unipotency at most $m$. In other words, there is some $k$ such that $(T^k - I)^{m+1} = 0$.
2. If $f: \mathcal{X} \longrightarrow \Delta$ is a semi-stable degeneration, then $T$ is unipotent (i.e., we take $k = 1$).

Proof [LA]
We are going to look at the variation of Hodge structures associated to a semi-stable degeneration of Calabi-Yau $n$-folds. Hence, by the Monodromy Theorem, we may assume $T$ is unipotent.

**Definition 1.2** A Hodge structure of weight $k$, denoted $\{H_Z, H^{p,q}\}$, is given by a lattice $H_Z$ of finite rank together with a decomposition on its complexification $H = H_Z \otimes \mathbb{C}$:

$$H = \bigoplus_{p+q=k} H^{p,q}$$

such that

$$H^{p,q} = H^{q,p}.$$

Since each fibre $\mathcal{X}_s$ is a complex projective manifold, by the Hodge Decomposition Theorem, the complex de Rham cohomology of $\mathcal{X}_s$ in each dimension can be written as

$$H^k_{DR}(\mathcal{X}_s, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(\mathcal{X}_s)$$

giving a Hodge structure.

Furthermore, since $\mathcal{X}$ is a pseudo-projective variety, the Kähler 1-1 form $\omega$ is rational and, in particular, its restriction to each fibre is rational.

We define $L: H^m(\mathcal{X}_s, \mathbb{C}) \rightarrow H^{m+2}(\mathcal{X}_s, \mathbb{C})$ by cup with $\omega$, and we define the primitive cohomology of $\mathcal{X}_s$ by

$$P^m(\mathcal{X}_s, \mathbb{C}) := \text{Ker} L^{n-m+1}.$$

This may be defined a priori up to a monodromy action; however, since the 1-1 form comes from the restriction of the cohomology of the total space, these are invariant under monodromy.

If we take the $H_Z$ lattice of Hodge structures as the intersection of the primitive cohomology and the integral cohomology, we can define a bilinear form

$$Q: H_Z \times H_Z \rightarrow \mathbb{Z},$$

called the **polarization** on the Hodge structure as follows:

$$Q(\phi, \psi) = (-1)^{k(k-1)/2} \int_{\mathcal{X}_s} \phi \wedge \psi \wedge \omega^{n-k},$$

where $k$ is the weight, $n$ the dimension of $\mathcal{X}_s$, and $\omega$ the Kähler form.

Now we can define a polarized Hodge structure.

**Definition 1.3** A polarized Hodge structure of weight $k$, denoted $\{H_Z, H^{p,q}, Q\}$, is given by a Hodge structure of weight $k$ together with a bilinear form

$$Q: H_Z \times H_Z \rightarrow \mathbb{Z},$$

which is symmetric for $k$ even and skew-symmetric for $k$ odd, satisfying the two **Hodge-Riemann** bilinear relations:

1. $Q(H^{p,q}, H^{p',q'}) = 0$ unless $p' = k - p$ and $q' = k - q$,
2. $\langle (\sqrt{-1})^{p-q} Q(\phi, \psi) \rangle > 0$ for any nonzero element $\phi$, in $H^{p,q}$.
Remark 1.4  Equivalently, we can replace the Hodge decomposition by a Hodge filtration, because a filtration varies holomorphically while a decomposition does not. Given a decomposition \( H = H^{p,q} \),

\[ F^p = H^{k,0} \oplus \ldots \oplus H^{k-p,p}, \quad \text{and} \quad H^{p,q} = F^p \cap \overline{F}^q. \]

\( \{F^p\} \) is called the **Hodge filtration** of \( H \). With this notation, the polarization satisfies:

1. \( Q(F^p, F^{k-p+1}) = 0 \)
2. \( Q(C\psi, \overline{\psi}) > 0 \) for any nonzero element \( \psi \) in \( H \), where \( C \) is the Weil operator of \( H \) defined by \( C|_{H^{p,q}} = (\sqrt{-1})^{p-q} \).

Hence, we have a polarized Hodge structure given by the Hodge decomposition of each de Rham cohomology group \( H^k_{DR}(X_s, C) \) with the canonical bilinear form as a polarization. From now on, we focus on the polarized Hodge structure of weight \( n \), i.e., \( H^n_{DR}(X_s, C) \). Since each \( X_s \) is a Calabi-Yau \( n \)-fold, the dimension of \( H^{n,0}(X_s) \) is always 1. Furthermore, the \( H^{n,0}(X_s) \) part of \( H^n_{DR}(X, C) \) always carries the polarization since \( H^{n,0}(X_s) \subset P^n(X_s) \), which can easily be seen to be primitive by the construction of the usual Hodge diamond.

We would like to see how the polarized Hodge structure of each fibre varies. Thus we need the notion of a variation of (polarized) Hodge structure. Here we use the construction done in [GrTA] Chapter 1. Essentially a variation of Hodge structures is a holomorphic map from a moduli space \( S \) to a classifying space \( D \), which is geometrically the open set of the subset of the product of Grassmannians, and the map has to satisfy certain tangential conditions.

The variation of Hodge structures of weight \( n \) associated to the Calabi-Yau degeneration is defined as the map \( \phi \) in the following commutative diagram where \( \mathfrak{h} \) denotes the upper half plane that is the universal cover of \( \Delta^* \):

\[ \begin{array}{ccc}
\mathfrak{h} & \xrightarrow{\phi} & D \\
\pi \downarrow & & \downarrow \\
\Delta^* & \xrightarrow{\phi} & D/\Gamma
\end{array} \]

and \( \phi \) is defined by associating to each \( X_s \) the Hodge decomposition of \( H^n_{DR}(X_s, C) \). This is the well-defined map from \( \Delta^* \) to \( D \) up to monodromy action by \( \Gamma \). Since

\[ \frac{\partial H^{p,q}}{\partial s} \subseteq H^{p,q} \oplus H^{p+1,q-1} \]

We have

\[ \frac{\partial F^p_s}{\partial s} \subseteq F^p_s \]

and, by conjugation,

\[ \frac{\partial F^p_s}{\partial s} \subseteq F^{p-1}_s \]

We can have such a commutative diagram since we are assuming that the degeneration is semi-stable and hence the monodromy is unipotent.
The polarization of weight $n$ variation of Hodge structures becomes simplified as following;

$$Q(\phi, \psi) = (-1)^{n(n-1)/2} \int_{\mathcal{X}_s} \phi \wedge \psi$$

**Remark 1.5** Let $M$ be a Calabi-Yau manifold. Then the map from the moduli space of $M$ to $D$ defined by the variation of Hodge structure is an immersion. Since by deformation theory, the space of infinitesimal deformation of complex structure is given by $H^1(\mathcal{X}_s, \Omega_s)$ and it is unobstructed by the theorem of Bogomolov-Tian-Todorov [Ti], [Tod].

Before we define the Weil-Petersson Metric we have to look at the canonical metric on the classifying space $D$. We will call this metric the VHS metric.

For $\tilde{\phi} : h \rightarrow D \subset \prod_{p=1}^{n} G(F_p, H)$, we have its differential

$$d\tilde{\phi} : T(h) \rightarrow T(D) \subset T(\prod_{p=1}^{n} G(F_p, H)).$$

Since this takes values in the horizontal subspace,

$$d\tilde{\phi}T(h) \subset \bigoplus_{p=1}^{n} \text{Hom}(H^{p,q}, H^{p-1,q+1}).$$

In other words, we have a following commutative diagram,

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(where \( i \) is the inclusion \( D \subset \prod_{p=1}^{n} G(F^p, H) \)) and \( v \in T(D) \), we have

\[
<v, v>_{VHS} = \sum_{p=1}^{n} tr A_{v_p} A_{v_p}^\dagger
\]

Since \( \dim H^{n,0} = 1 \), the summand for \( p = n \) is just

\[
tr(A_{v_n}^\dagger A_{v_n}) = \frac{<A_{v_n}^\dagger A_{v_n} \Omega, \Omega>_{n,0}}{<\Omega, \Omega>_{n,0}} = \frac{<A_{v_n} \Omega, A_{v_n} \Omega>_{n-1,1}}{<\Omega, \Omega>_{n,0}}.
\]

where \( 0 \neq \Omega \in H^{n,0} \).

Now we define the Weil-Petersson Metric. The classic Weil-Petersson metric was only defined on a moduli space of curves. Tian in [Ti] extended the definition to moduli spaces of Calabi-Yau \( n \)-folds.

Take a generator \( w \in H^{n,0}_s(\mathcal{X}, \mathcal{C}) \). We know \( H^{n,0} \) is a trivial line bundle and it has a natural Hermitian metric \((\sqrt{-1})^n Q(, )\) that satisfies the Hodge Riemann bilinear relation.

We define the Weil-Petersson metric to be a Kähler metric whose Kähler form is \( \frac{1}{2} \Theta(s) \), i.e.,

\[
R(s) = -\bar{\partial}s \partial s \log((\sqrt{-1})^n Q(w, \bar{w})) ds d\bar{s}.
\]

The definition does not depend on a choice of \( w \in H^{n,0}_s(\mathcal{X}, \mathcal{C}) \), since any \( w' \in H^{n,0}_s(\mathcal{X}, \mathcal{C}) \) is given by \( w' = hw \) where \( h \) is a holomorphic function.

The proof that it is indeed a metric on the moduli space is due to Tian [Ti].

**Proposition 1.6** The Weil-Petersson metric is a direct summand of the VHS-metric.

**Proof** See [Ti].

**Corollary 1.7** Since Weil-Petersson metric is essentially a summand of VHS metric, \( \rho_{wp} \) Weil-Petersson distance is always bounded by \( \rho_D \) VHS metric distance.

**Proof** It is clear by inequality of integrands of each distance integral.

Now we are ready to compute the distance associated to a degeneration of Calabi-Yau \( n \)-folds with respect to the Weil-Petersson metric. In order to do this, we need the nilpotent orbit originally defined on \( \tilde{D} \supset D \), which behaves asymptotically with the limit of Hodge filtrations on \( D \). The construction of the nilpotent orbit is given in Chapter 4 of [GrTA].

Let \( \phi: \Delta^* \to D/\Gamma \) be the variation of Hodge structures associated to the family \( \mathcal{X} \) of Calabi-Yau three-folds over the punctured disk. Since \( \Gamma \) has a representation in \( \pi_1(\Delta^*) = \mathbb{Z} \), we may call the generator \( T \). By the Monodromy Theorem, \((T - I)^{n+1} = 0 \). Let \( \mathfrak{h} = \{ w = u + vi | v > 0 \} \) be the upper half plane. Then we have a commutative diagram:

\[
\begin{array}{ccc}
\mathfrak{h} & \xrightarrow{\phi} & D \\
\pi \downarrow & & \downarrow \\
\Delta^* & \xrightarrow{\phi} & D/\Gamma
\end{array}
\]
where,
$$s = \pi(w) = \exp(2\pi iw) \quad \text{and} \quad \tilde{\phi}(w + 1) = T\tilde{\phi}(w).$$

Analytic continuation around $s = 0$ gives $H_{e^{2\pi is}} = TH_s$. Then

$$N := \log T = (T - I) - \frac{(T - I)^2}{2} + \frac{(T - I)^3}{3} \cdots \frac{(T - I)^n}{n}$$

Define
$$\tilde{\psi} : h \rightarrow \tilde{D} \supset D \quad \text{by} \quad \tilde{\psi}(w) = \exp(-wN)\tilde{\phi}(w)$$

Then
$$\tilde{\psi}(w + 1) = \exp(-(w + 1)N)\tilde{\phi}(w + 1) = \exp(-wN) \cdot \exp(-N) \cdot T\tilde{\phi}(w) = \exp(-wN) \cdot T^{-1} \cdot T\tilde{\phi}(w) = \exp(-wN)\tilde{\phi}(w) = \tilde{\psi}(w)$$

So, $\tilde{\psi}$ descends to a single-valued map $\psi : \Delta^* \rightarrow \tilde{D}$ given by
$$\psi(s) = \tilde{\psi}(w) = \exp(-\log s \frac{2\pi i}{N})\phi(s)$$

Unlike $\phi$, the map $\psi$ extends over $s = 0$ due to the following Theorem.

**Theorem (Cornalba and Griffiths)** The map $\psi : \Delta^* \rightarrow \tilde{D}$ extends across the origin to a map $\psi : \Delta \rightarrow \tilde{D}$.

**Proof** See [GrCo] or [GrSc].

In fact, the $\psi(0)$ has a special name.

**Definition 1.8** The filtration $\psi(0) \in \tilde{D}$ will be called the **limiting Hodge filtration** and will be denoted by $F^p_\infty$.

With the limiting Hodge filtration, we are ready to define the nilpotent orbit.

**Definition 1.9** The nilpotent orbit of a degenerating family over $\Delta^*$ is the map $O : h \rightarrow \tilde{D}$ given by $O(w) = \exp(wN)\psi(0)$.

Then
$$O(w + 1) = \exp((w + 1)N)\psi(0) = \exp(wN) \exp(N)\psi(0) = T\psi(w)$$

Schmid showed in the following theorem that the nilpotent orbit is a very good approximation of the period map.

**Theorem (Nilpotent orbit theorem)**

1. The nilpotent orbit is horizontal.
2. There is a non-negative number $\alpha$ such that, if $\text{Im}(w) > \alpha$, then $O(w)$ belong to $D$;
3. $O(w)$ is strongly asymptotic to $\tilde{\phi}(w)$ in the sense that $\rho_D(\tilde{\phi}(w), O(w)) \leq (\text{Im}w)^B e^{-2\pi |m w|}$ for some $B \geq 0$ and $\text{Im}w \geq A > 0$, where $\rho_D$ is the distance on $D(\subset \tilde{D})$ given by natural Hermitian Metric.
Proof See [Sch].

By Corollary 1.7 we immediately obtain Weil-Petersson version of Nilpotent orbit Theorem. Furthermore, even stronger version is true for Weil-Petersson metric.

**Lemma 1.10** The central fibre of a degeneration has finite Weil-Petersson metric distance if and only if the Weil-Petersson metric distance along the nilpotent orbit is finite.

Before we prove the lemma, first we state and prove the following result.

**Theorem 1.11** Suppose the limit filtration $\{F_\infty\}$ is given. Then the Weil-Petersson Metric distance of the corresponding nilpotent orbit is either 0 or infinite.

Furthermore the Weil-Petersson Metric distance of the corresponding nilpotent orbit is 0, if and only if $Q(\alpha, N^i\bar{\alpha}) = 0$ for all $i > 0$ where $\alpha \in F_\infty^n$.

Proof Let $O : h \rightarrow \tilde{D}$ be the nilpotent orbit, where

$$O(w) = e^{wN}F_\infty \quad \text{and} \quad O(w + 1) = TO(w).$$

Since we want to compute the Weil-Petersson Metric distance of the nilpotent orbit, we choose $\alpha \in F_\infty^n$ and set $\Omega(w) = e^{Nw}\alpha$. Then,

$$R(w) = -\partial w \partial w \log((\sqrt{-1})^n Q(\Omega(w), \Omega(w)))$$

We look at the argument of the logarithm and get

$$(i)^n Q(\Omega(w), \Omega(w)) = (i)^n Q(e^{Nw}\alpha, e^{N\bar{\alpha}}) \quad \text{since} \quad N \quad \text{is real}$$

$$= (i)^n Q(e^{N(w_0+iy)}\alpha, e^{N(w_0-iy)}\bar{\alpha})$$

$$= (i)^n Q(e^{Ny}\alpha, e^{-Ny}\bar{\alpha}) \quad \text{since} \quad e^{Nw_0} \quad \text{fixes the polarization}$$

$$= (i)^n Q(\alpha, e^{-2iyN}\bar{\alpha}) = (*)$$

By the monodromy theorem, $N^{n+1} = 0$. Hence

$$e^{-2iyN} = 1 - 2iyN - 2y^2N^2 + \frac{8}{3}iy^3N^3 + \ldots + \frac{(-2i)^n}{n!}y^nN^n$$

Then

$$(*) = (i)^n Q(\alpha, (1 - 2iyN - 2y^2N^2 + \frac{8}{3}iy^3N^3 + \ldots + \frac{(-2i)^n}{n!}y^nN^n)\bar{\alpha})$$

$$= (i)^n Q(\alpha, \bar{\alpha}) + (i)^n Q(\alpha, -2iyN\bar{\alpha}) - \ldots + (i)^n Q(\alpha, \frac{(-2i)^n}{n!}y^nN^n\bar{\alpha})$$

Let

$$(*) = p(y).$$

Then

$$R(w) = -(\frac{\partial^2}{\partial \alpha \partial \bar{\alpha}} + \frac{\partial^2}{\partial \bar{\alpha} \partial \alpha}) \log p(y).$$
Hence, the Weil-Petersson metric distance of the nilpotent orbit is given by

\[ \rho_{wp} = \int_{y_0}^{y} \sqrt{\frac{\partial^2}{\partial y^2} \log p(y)} dy \]

By the Nilpotent Orbit Theorem, for \( y >> 0 \), \( \mathcal{O}(w) \in D \). It follows that

\[ (i)^n Q(\Omega(w), \bar{\Omega}(w)) > 0, \]

and hence \( p(y) \) has to be positive. Then the imaginary part of the polynomial vanishes for \( y >> 0 \), i.e., it has to be identically zero.

Therefore, \( p(y) \) is real and the leading coefficient is positive. If we look at the leading term of \( p(y) \), that is, \( ay^k \), the argument of the square root of the integrand becomes

\[ \frac{-a^2(k-1)ky^{2k-2} + a^2(k-1)^2y^{2k-2}}{a^2y^{2k-2}} = \frac{a^2}{y^2} \]

As \( y \) approaches infinity, we may concentrate the effect of the leading term. The distance is approximated by

\[ \rho_{wp} = \lim_{y \to 0} \int_{y_0}^{y} \sqrt{\frac{a^2}{y^2}} dy \]

Hence the integral will be either infinite or zero, that is, when \( a = 0 \). The same argument works for the lower terms. Hence, \( \rho_{wp} \) is either infinite when all coefficients are zero. Now we go back to the definition of \( p(y) \). If we let

\[ Q(\alpha, \bar{\alpha}) = C_0, Q(\alpha, N\bar{\alpha}) = C_1, Q(\alpha, N^2\bar{\alpha}) = C_2, \ldots Q(\alpha, N^n\bar{\alpha}) = C_n, \]

then this coefficient condition is true if and only if

\[ C_1 = C_2 = C_3 = \ldots = C_n = 0. \]

Hence the Weil-Petersson distance is finite if and only if

\[ Q(\alpha, N^i\bar{\alpha}) = 0 \forall i > 0. \]

It remains to prove the Lemma 1.10.

**Proof of Lemma 1.10** Recall the map \( \tilde{\psi}(w) = \exp(-wN)(\tilde{\phi})(w) \) we described when we discussed the construction of the nilpotent orbit associated to the period map of the variation of Hodge Structure:

\[ \begin{align*}
\mathfrak{h} & \xrightarrow{\phi} D \\
\pi & \downarrow \quad \quad \downarrow \\
\mathbb{A}^1 & \xrightarrow{\phi} D/T
\end{align*} \]
$	ilde{\psi}(w)$ is periodic with period $2\pi i$ and it descends to the well-defined holomorphic map $\psi(s)$ from $\Delta^*$ to $\hat{D}$. By the theorem of Cornalba and Griffiths, this map extends across the origin and defined on $\Delta$.

In particular, $\Omega(s) = \exp(-wN)\hat{\Omega} = \exp(-\frac{\log s}{2\pi i}N)\hat{\Omega}(s)$ is a single-valued holomorphic section of the deepest level of the filtration $\exp(-wN)\tilde{\phi}(w)$ that extends over $\Delta$, where $\hat{\Omega}$ is a constant global flat section and $\hat{\Omega}(s)$ is the push-forward that is multi-valued. We may call $\Omega(s)$ a privileged section of the deepest level of the filtration.

Choose a frame $\{\sigma_i\}$ of privileged section of the Hodge bundle associated to the variation of Hodge structure. Since it is given by the horizontal displacement of a basis of the reference fibre, their values are constant. $\Omega(s)$ can be written as a convergent power series around the origin, of which coefficients $\Omega_k(s)$ are all privileged sections:

$$
\Omega(s) = \sum_{k=0}^{\infty} \Omega_k s^k \\
= \sum_{k=0}^{\infty} \left( \sum_{i=1}^{n} w_{ki} \sigma_i \right) s^k \quad \text{where} \quad w_{ki} \text{ are constant matrices} \\
= \sum_{i=1}^{n} \left( \sum_{k=0}^{\infty} w_{ki} s^k \right) \sigma_i
$$

We choose a norm that is given by the following norm:

$$<\sigma_i, \sigma_j> = \delta_{ij}$$

Notice with this norm $\|\Omega_k(s)\|$ is constant. Furthermore, by a fact in basic linear algebra, $Q(v, w) < K \|v\|\|w\| = K \sqrt{<v, v> \cdot <w, w>}$ for some $K$. Let $\tilde{\sigma}_i$'s be the multiple-valued descent of the global flat section that is used to define a frame of privileged sections, i.e., $\sigma_i = \exp(-wN)\tilde{\sigma}_i$. Then since $Q(, )$ is monodromy invariant $Q(\sigma_i, \sigma_j) = Q(\tilde{\sigma}_i, \tilde{\sigma}_j)$, where $\tilde{\sigma}_i$ are the pull-back to the half-plane $h$ from $\Delta^*$ by the abuse of notation. Hence $Q(\sigma_i, \sigma_j)$ is constant globally. Thus the bound on $Q(, )$ by $\|\|$ is uniform within the radius of convergence.

By the abuse of notation, $\Omega(s)$ pull-back to,

$$
\Omega(s) = \sum_{k=0}^{\infty} \Omega_k s^k \\
= \sum_{k=0}^{\infty} \Omega_k \exp(2\pi ikw)
$$

where $\Omega_k(w)$ is well-defind global section that is the pull-back of the privileded section $\Omega_k$.

Then the deepest filtration of $\tilde{\phi}(w)$ , that is $\Omega(w)$ can be written as:

$$
\Omega(w) = \exp(wN)\Omega(s) = \sum_{k=0}^{\infty} \exp(wN)\Omega_k \exp(2\pi ikw).
$$
Note that $\|\Omega_k\|$ is still constant, where $\|\|$ is the pull-back norm of the one we defined. Hence $\|\Omega_k\| \leq A^k$ for some $A > 0$.

Finally since the nilpotent orbit map is given by $\mathcal{O}(w) = \exp(wn) \tilde{\psi}(0)$, it's deepest filtration part, which by the abuse of notation denoted by $\mathcal{O}(w)$ is exactly,

$$\mathcal{O}(w) = \exp(wn)\Omega_0$$

that is the constant term of $\Omega(w)$. Let

$$\epsilon(w) = \Omega(w) - \mathcal{O}(w)$$

$$\delta(w) = \frac{\partial \Omega}{\partial w} - \frac{\partial \mathcal{O}}{\partial w}$$

In order to estimate the difference of two distance, first we estimate the bounds of $\|\epsilon(w)\|$ and $\|\delta(w)\|$. We write

$$\epsilon(w) = \sum_{k=1}^{\infty} \exp(2\pi ikw) \exp(Nw)\Omega_k$$

Then along a vertical ray,

$$\exp(Nw)\Omega_k = \exp(N(w_0 + iy))\Omega_k = \exp(Niy)\Omega_k$$

since $Nw_0$ fixes $\Omega_k$.

And $N^{n+1} = 0$ gives;

$$\exp(Nw)\Omega_k(w) = (1 + Niy + \cdots + \frac{(Niy)^n}{n!})\Omega_k(w)$$

Finally, since there exist $B > 0$, which only depends on the choice of norm, such that $\|\Omega_k\| \leq B$ for all $k \geq 1$ and $N$ raises norms by the bounded factor for all $k \geq 1$;

$$\|\exp(Nw)\|\Omega_k\| \leq O(y^k)B.$$ 

Thus,

$$\|\epsilon(w)\| = B \exp(-2\pi y)O(y^k) \text{ for some } k \leq n.$$ 

Similarly,

$$\delta(w) = \frac{\partial}{\partial w}(\Omega(w) - \mathcal{O}(w))$$

$$= \frac{\partial}{\partial w} \sum_{k=1}^{\infty} \exp(2\pi ikw) \exp(Nw)\Omega_k$$

$$= \frac{\partial}{\partial w} \sum_{k=1}^{\infty} \exp(2\pi ikw)(1 + Niy + \cdots + \frac{(Niy)^n}{n!})\Omega_k$$

$$= \sum_{k=1}^{\infty} 2\pi ik \exp(2\pi ikw)(1 + Niy + \cdots + \frac{(Niy)^n}{n!})\Omega_k$$

$$+ \exp(2\pi ikw)(-N - N^2iy + \cdots + \frac{N^n(i)^{n+1}y^{n-1}}{(n-1)!})\Omega_k$$

$$+ \exp(2\pi ikw)(1 + Niy + \cdots + \frac{(Niy)^n}{n!})(-N)\Omega_k$$
Hence,
\[ \| \delta(w) \| \leq O(y^k) \exp(-2\pi y)C \quad \text{for some } k \leq n \text{ and } C > 0. \]

We have prove that both \( \| \epsilon(w) \| \) and \( \| \delta(w) \| \) have exponential decay with respect to \( y = \text{Im}(w) \).

Next, we look at the difference between the Weil-Petetsson arc length integrands. Recall:
\[
R(\Omega(w)) = -\delta w \partial w \log(\sqrt{-1})^n Q(\Omega(w), \bar{\Omega}(w)) \quad \text{and} \\
R(\mathcal{O}(w)) = -\delta w \partial w \log(\sqrt{-1})^n Q(\mathcal{O}(w), \bar{\mathcal{O}}(w)).
\]

Note, from here on the form \( Q(\ , \ ) \) includes \( (\sqrt{1})^n \) So the difference of integrands is
\[
\sqrt{R(\Omega(w))} - \sqrt{R(\mathcal{O}(w))} = \frac{R(\Omega(w)) - R(\mathcal{O}(w))}{\sqrt{R(\Omega(w))} + \sqrt{R(\mathcal{O}(w))}} = (*)
\]
The numerator become
\[
R(\Omega(w)) - R(\mathcal{O}(w)) = - \frac{Q(\Omega, \bar{\Omega})Q(\partial \Omega, \partial \bar{\Omega}) - Q(\Omega, \partial \bar{\Omega})Q(\partial \Omega, \bar{\Omega})}{Q(\Omega, \Omega)^2} + \frac{Q(\mathcal{O}, \bar{\mathcal{O}})Q(\partial \mathcal{O}, \partial \bar{\mathcal{O}}) - Q(\mathcal{O}, \partial \bar{\mathcal{O}})Q(\partial \mathcal{O}, \bar{\mathcal{O}})}{Q(\mathcal{O}, \mathcal{O})^2} = (**) \]

where, \( \Omega = \Omega(w), \quad \mathcal{O} = \mathcal{O}(w). \)

By substituting \( \Omega(w) = \mathcal{O}(w) - \epsilon(w) \) and \( \partial \Omega(w) = \partial \mathcal{O}(w) - \delta(w) \), we get;
\[
(**) = - \frac{Q(\mathcal{O} + \epsilon, \bar{\mathcal{O}} + \bar{\epsilon})Q(\partial \mathcal{O} + \delta, \partial \bar{\mathcal{O}} + \bar{\delta}) - Q(\mathcal{O} + \epsilon, \partial \bar{\mathcal{O}} + \bar{\delta})Q(\partial \mathcal{O} + \delta, \bar{\mathcal{O}} + \epsilon)}{Q(\mathcal{O} + \epsilon, \bar{\mathcal{O}} + \bar{\epsilon})^2} + \frac{Q(\mathcal{O}, \bar{\mathcal{O}})Q(\partial \mathcal{O}, \partial \bar{\mathcal{O}}) - Q(\mathcal{O}, \partial \bar{\mathcal{O}})Q(\partial \mathcal{O}, \bar{\mathcal{O}})}{Q(\mathcal{O}, \mathcal{O})^2}
\]

After simplification we obtain;
\[
(**) = \frac{1}{Q(\mathcal{O}, \bar{\mathcal{O}})^2Q(\mathcal{O} + \epsilon, \bar{\mathcal{O}} + \bar{\epsilon})^2} F(w),
\]

where each terms in \( F(w) \) involves \( \epsilon, \delta, \) or both. Hence \( F(w) \) has a exponential decay. Finally we simplify the entire \( \sqrt{R(\Omega(w))} - \sqrt{R(\mathcal{O}(w))} \) and obtain;
\[
(*) = \frac{F(w)}{(Q(\mathcal{O}, \bar{\mathcal{O}}))^{3/2}Q(\mathcal{O}, \partial \mathcal{O})Q(\partial \mathcal{O}, \partial \bar{\mathcal{O}}) - Q(\mathcal{O}, \partial \bar{\mathcal{O}})Q(\partial \mathcal{O}, \bar{\mathcal{O}}) + G(w)},
\]

Where each term of \( G(w) \) involves \( \epsilon, \delta, \) or both. Since \( Q(\mathcal{O}, \bar{\mathcal{O}}) \) has the order of some power of \( y \), the entire term is dominated by exponential decay.

Thus we have showed that the difference between the two length is finite along any vertical ray.

Now suppose the nilpotent orbit has finite length along given parametric curve \( \alpha(t) : \mathbb{R} \rightarrow \mathfrak{h} \). Then the difference between the two length is given by;
\[
\lim_{t \to \infty} \left( \sqrt{R(\Omega(\alpha(t)))} - \sqrt{R(\mathcal{O}(\alpha(t)))} \right) \sqrt{(x'(t))^2 + (y'(t))^2} dt
\]
where \( \alpha(t) = x(t) + iy(t) \)

By assumption the total variation of

\[
\sqrt{(x'(t))^2 + (y'(t))^2}
\]

has to be finite. Hence the difference is finite and the Weil-Petersson metric distance of \( \tilde{\phi}(w) \) has to be finite.

On the other hand, if the nilpotent orbit has infinite length along given parametric curve \( \alpha(t) \), then the Weil-Petersson metric length of the nilpotent orbit along any given parametric curve is infinite. The difference of two integrand is,

\[
\sqrt{R(\mathcal{O}(w))} - \sqrt{R(\Omega(w))} = \frac{R(\mathcal{O}) - R(\Omega)}{\sqrt{R(\mathcal{O})} + \sqrt{R(\Omega)}} \leq \frac{R(\mathcal{O})}{\sqrt{R(\mathcal{O})} + \sqrt{R(\Omega)}}.
\]

Since \( \sqrt{R(\Omega)} \) is dominated by \( \sqrt{R(\mathcal{O})} \), the difference is dominated by a half of the nilpotent orbit distance integrand. Hence the distance along \( \phi(w) \) has to be infinite.

Thus we complete the proof.

2. Limiting mixed Hodge structure

Our next step is to determine for which limiting mixed Hodge structure the central fibre is at finite distance in the Weil Petersson metric. We have to define some terminology first. These definitions are found in Chapter 4 and Chapter 5 of [GrTA].

**Definition 2.1** Given a variation of Hodge structure of weight \( n \) over \( \Delta^* \), and a nilpotent monodromy \( N \) such that \( N^{n+1} = 0 \), there exists a unique ascending filtration \( \{W_i\} \) of \( H^Q \), called the *monodromy filtration*,

\[
0 \subset W_0 \subset W_1 \subset \ldots \subset W_{2n} = H^Q,
\]

satisfying

\[
N : W_i \to W_{i-2}
\]

\[
N^k : W_{n+k}/W_{n+k-1} \cong W_{n-k}/W_{n-k-1}.
\]

Note that the monodromy filtration is uniquely determined by these properties.

*Caution* All filtrations are on the complexification of the primitive rational cohomology in dimension \( n \). We are about to introduce a separate notion of primitivity related to the weight filtration. Any further reference to primitivity refers to newly introduced primitivity.

**Definition 2.2** Given a nilpotent element \( N \in g_{\mathbb{R}} \), such that \( N^{n+1} = 0 \) its monodromy weight filtration \( W = W(N) \), and each graded piece \( Gr^W_i \) of \( H^Q \), the *primitive subspaces* \( P_{n+j} \subset Gr^W_{n+j} \) are defined as follows:

\[
P_{n+j} = \ker\{N^{j+1} : Gr^W_{n+j} \to Gr^W_{n-j-2}\} \quad \text{if} \quad j \geq 0
\]

\[
P_{n+j} = 0 \quad \text{if} \quad j < 0.
\]
We have a Lefschetz decomposition:

\[ \text{Gr}_l^W = \bigoplus_{j \geq 0} N^j (P_{l+2j}) \]

Given \( n \), \( P_{n-1} = P_{n-2} = \ldots = P_1 = P_0 = 0 \), and

Next we define the mixed Hodge structure.

**Definition 2.3** A mixed Hodge structure \( \{ H_\mathbb{Q}, F^p, W_l \} \) of weight \( n \) consists of an ascending weight filtration, defined over \( \mathbb{Q} \),

\[
0 \subset W_0 \subset W_1 \subset W_2 \subset W_3 \subset \ldots \subset W_{2n} = H_\mathbb{C},
\]

such that the Hodge filtration induces the pure Hodge structure of weight \( m \) on the graded piece \( Gr_m = W_m/W_{m-1} \) of the weight filtration for each \( m = 0, \ldots, 2n \). The induced Hodge filtration on \( Gr_m \) is

\[
F^p(Gr_m) = (F^p \cap W_m)/(F^p \cap W_{m-1}).
\]

We have polarized versions of these definitions.

**Definition 2.4** A polarized mixed Hodge structure is a triple \( (W, F, N) \), where \( W \) is an increasing filtration of \( H \), \( F \in \check{D} \) and \( N \in \mathfrak{gR} \) is a rational nilpotent element such that \( N^{n+1} = 0 \), satisfying:

1. \( W \) is the monodromy weight filtration of \( N \).
2. \((W, F)\) is a mixed Hodge structure; that is, for every \( l \geq 0 \) the filtration induced by \( F \) on \( Gr_l^W \) is a Hodge structure of weight \( l \).
3. \( N(F^p) \subset F^{p-1} : \) for all \( 0 \leq p \leq n \).
4. For \( j \geq 0 \), the Hodge structure induced by \( F \) on the primitive subspace \( P_{n+j} \) is polarized by the bilinear form \( Q_j = Q(.,N^j) \).

We represent the limiting mixed Hodge structure of \( H^n \) pictorially as follows.

\[
\begin{array}{c}
H^{n,n} \\
H^{n,n-1}, H^{n-1,n} \\
\vdots \\
H^{n,0}, H^{n-1,1}, H^{n-2,2}, \ldots, H^{2,n-2}, H^{1,n-1}, H^{0,n} \\
\vdots \\
H^{2,0}, H^{1,1}, H^{0,2} \\
H^{1,0}, H^{0,1} \\
H^{0,0}
\end{array}
\]

Each row corresponds to the Hodge structure of each graded piece. Furthermore, we have isomorphisms

\[
N : Gr_{n+1} \cong Gr_{n-1}, \quad N^2 : Gr_{n+2} \cong Gr_{n-2}, \quad N^3 : Gr_{n+3} \cong Gr_{n-3}, \ldots,
\]

\[
N^{n-1} : Gr_{2n} \cong Gr_n, \quad N^n : Gr_{n} \cong Gr_n.
\]
Hence, there is a horizontal symmetry (by Hodge structure) and a vertical symmetry (by isomorphisms $N^i$, for $1 \leq i \leq n$) in this pictorial representation of the limiting mixed Hodge structure.

Since $F^n_{\infty}$ has rank 1, exactly one of the graded pieces $(Gr_{n+k}, 0 \geq k \geq n)$ is nonzero. There are $n+1$ cases.

Due to the following theorem of Schmid, the limiting filtration does induce the polarized limiting mixed Hodge structure.

**Theorem (Schmid)** Let $O(w) = e^{wN}F_{\infty}$ be a one variable nilpotent orbit where $F_{\infty}$ is a limit filtration. Then $(W(N), F_{\infty}, N)$ is a polarized mixed Hodge structure.

**Proof** See theorem 6.16 of [Sch]

Now we can compute the polarized mixed Hodge structure associated to the limit filtration and impose the condition of the finite distance case. Since our variation of Hodge structures comes from the degeneration space of a family of Calabi-Yau n-folds, the dimension of $F^n_{\infty}$ is 1.

**Theorem 2.5** Only one case satisfies the finite distance condition on $F^n_{\infty}$; i.e.,

$$Q(\alpha, N^i\bar{\alpha}) = 0, \text{ for all } i > 0$$

$F^n_{\infty}$ is at finite distance if and only if $F^nGr_n(H^n(X_s)) = H^{n,0}(X_s)$.

**Proof** Since $H^{n,0}(X_s)$ has rank 1, $F^n_{\infty} = F^nGr_{n+k}(H^n(X_s))$, is different from 0 for precisely one $k$, where $k$, $1 \geq k \geq n$. Then,

$$F^n_{\infty} \subset P_{n+k}, \text{ and } Q(F^n_{\infty}, N^kF^n_{\infty}) > 0,$$

by the definition of polarized mixed Hodge structure, where

$$F^n_{\infty} = F^nGr_{n+k}(H^n(X_s))$$

is polarized by $Q(\cdot, N^k\cdot)$ which satisfies HR-bilinear relations. Thus

$$N^{k+1}F^n_{\infty} = 0.$$ 

Hence $F^n_{\infty} = F^nGr_{n+k}$ is at finite distance if and only if $k = 0$, i.e.,

$$F^nGr_n(H^n(X_s)) = H^{n,0}(X_s) \neq 0.$$ 

**Corollary 2.6** The pictorial presentation of the limiting mixed Hodge structure at finite distance is the following. (See right after the proof.)

Furthermore, $N^{n-1} = 0$ for this limiting mixed Hodge structure.

**Proof** The picture is clearly determined once we set the non-zero part of $F^n_{\infty}$ to be $F^nGr_n(H^n(X_s)) = H^{n,0}(X_s)$. From the picture, each graded piece becomes

$$Gr_{2n} = P_{2n} = 0, Gr_{2n-1} = P_{2n-1} = 0$$

Hence the first non-zero primitive piece is

$$P_{2n-2} = \ker\{N^{n-2+1} : Gr_{n+n-2} \rightarrow Gr_{n-(n-2)-2}\}$$

$$= \ker\{N^{n-1} : Gr_{n-n} \rightarrow Gr_n\}.$$
Hence, by the Lefschetz decomposition, $N^{n-1}$ annihilates everything.

\[
\begin{array}{cccccc}
0 \\
0, 0 \\
0, X, 0 \\
0, Y, Y, 0 \\
\ldots \ldots \ldots \\
1, Z, \ldots \ldots \ldots Z, 1 \\
\ldots \ldots \\
0, Y, Y, 0 \\
0, X, 0 \\
0, 0 \\
0
\end{array}
\]

3. Central Fibre and Clemens-Schmid Exact Sequence

In order to interpret the finite distance condition of a Calabi-Yau degeneration we need to look at the central fibre. In fact, every variety carries a canonical functorial mixed Hodge structure; as proved in a more general setting by Deligne [DeH2] and [DeH3]. Before looking at the mixed Hodge structure of a central fibre, we need to understand its cohomology.

Remark 3.1 In the case of a semistable degeneration, one can construct a retraction from the total space to the central fibre; i.e., $r: \mathcal{X} \longrightarrow \mathcal{X}_0$, which induces isomorphisms

\[
r^*: H^m(\mathcal{X}_0, \mathbb{Q}) \rightarrow H^m(\mathcal{X}, \mathbb{Q}), \quad r_*: H^m(\mathcal{X}, \mathbb{Q}) \rightarrow H^m(\mathcal{X}_0, \mathbb{Q})
\]

This construction is due to Clemens in [Cl]. The cohomology of a central fibre is described by a Mayer-Vietoris type spectral sequence. We follow the construction given in Chapter 6 of [GrTA].

Construction 3.2 Let $\mathcal{X}_0 = \sum X_i$ where each $X_i$ is a smooth irreducible component of $\mathcal{X}_0$ and all components intersect transversely each other. Let $X_{i_0\ldots i_p}$ denote $X_{i_0} \cap X_{i_1} \cap \ldots \cap X_{i_p}$. We define the codimension $p$ stratum of $\mathcal{X}_0$ as

\[
\mathcal{X}^{[p]} = \bigsqcup_{i_0 < \ldots < i_p} X_{i_0\ldots i_p} \quad \text{(disjoint union)}.
\]

There is a natural inclusion map $i_p: \mathcal{X}^{[p]} \longrightarrow \mathcal{X}$.

By [Cl] we can choose an open cover $\mathcal{U}$ of a neighborhood of $\mathcal{X}_0$ in $\mathcal{X}$ satisfying the following condition:

1. For each $U \in \mathcal{U}$, the degeneration map $\pi$ restricted to $U$ is given by $s = x_1 x_2 x_3 \ldots x_k$ in suitable local coordinates.
2. $\check{H}^*(\mathcal{U} \cap \mathcal{X}_0, \mathbb{Q}) \cong \check{H}^*(\mathcal{X}_0, \mathbb{Q})$
3. $\check{H}^*(\mathcal{U}, \mathcal{O}) \cong \check{H}^*(\mathcal{X}, \mathcal{O})$.
Then we can define $E_0$ terms and differentials as followings;

$$E_0^{p,q} = \check{C}^q(\iota_p^{-1}(U), \mathbb{Q})$$

$d: E_0^{p,q} \to E_0^{p,q+1}$, the Cech coboundary,

$\delta: E_0^{p,q} \to E_0^{p+1,q}$, the combinatorial coboundary induced by,

$$\delta \phi(U \cap X_{j_0\ldots j_{p+1}}) = \sum_{\alpha} (-1)^{\alpha} \phi(U \cap X_{j_0\ldots j_{alpha}\ldots j_{p+1}}).$$

This defines a bigraded complex and we get an associated spectral sequence.

We have the following result of the convergence of this spectral sequence by Griffiths and Schmid in [GrSc].

**Theorem** The spectral sequence given above degenerates at $E_2$, and converges to $\check{H}^*(X_0, \mathbb{Q})$.

**Proof** See [GrSc].

Theorem comes from the de Rham analogue of our spectral sequence. Let

$$A^{p,q} := A^q(\mathcal{X}^{[p]})$$

be the complex of $C^\infty$ $q$-forms on $\mathcal{X}^{[p]}$, with two differentials,

$$d: A^{p,q} \to A^{p,q+1},$$

$$\delta: A^{p,q} \to A^{p+1,q},$$

the exterior differential, the combinatorial coboundary induced by,

$$\delta \phi_{j_0\ldots j_{p+1}} = \sum_{\alpha} (-1)^{\alpha} \phi_{j_0\ldots j_{alpha}\ldots j_{p+1}}|_{X_{j_0\ldots j_{p+1}}}.$$

Then this gives a bigraded complex and has an associated spectral sequence. It converges to $H^*_D(A^{*,*})$ and degenerates at $E_2$. By the de Rham Theorem for an algebraic variety with normal crossings, we have

$$H^*_D(A^{*,*}) \cong H^*(\mathcal{X}_0, \mathbb{C}).$$

At the $E_1$ level, we have a commutative diagram:

$$\begin{array}{ccc}
E_r^{p,q} & \xrightarrow{d_r} & E_r^{p,q+1} \\
\downarrow i & & \downarrow i \\
E_r^{p,q} \otimes \mathbb{C} & \xrightarrow{d_r} & E_r^{p,q+1} \otimes \mathbb{C} \\
\cong & & \cong \\
D^R E_r^{p,q} & \xrightarrow{d_r} & D^R E_r^{p,q+1}
\end{array}$$

This ensures that the topological spectral sequence degenerates at $E_2$.

Now we are ready to define the mixed Hodge structure on the cohomology of the central fibre by means of a spectral sequence. First, this spectral sequence gives a means to put a weight filtration on $H^*(\mathcal{X}_0, \mathbb{Q})$ as follows:
define

\[ W_k = \bigoplus_{q \leq k} E_{0,q}^*, \]

and let \( W_k(H^m) \) be the induced filtration. Then this will give an increasing filtration of length \( m \) on \( H^m \), and the graded pieces will be

\[ \text{Gr}_k(H^m) = E_{2}^{m-k,k} \quad \text{if} \quad 0 \leq k \leq m \quad \text{and} \]
\[ \text{Gr}_k(H^m) = 0 \quad \text{if} \quad k < 0 \quad \text{or} \quad k > m. \]

Next we put a filtration on the complexification of the original spectral sequence which is isomorphic to the usual de Rham sequence, i.e., at \( E_0 \) level,

\[ F^p(A^{k,l}) = \bigoplus_{r \geq p} A^{r,l-r}(X^k) \]

This induces filtrations on each \( E_r \) term, in particular, at the \( E_1 \) level we get the Hodge filtration:

\[ D^R E_1^{k,l} = H^l_{DR}(X^k) \]

Furthermore, \( d_1 \) is a morphism of Hodge structures. So, together with the weight filtration, \( H^*(X_0, \mathbb{C}) \) carries a mixed Hodge structure i.e., it carries a Hodge structure of weight \( k \) at each \( G_k \) piece.

To illustrate the filtrations we represent the \( E_1 \) level of the spectral sequence associated to an \( n \)-dimensional central fibre.

\[ E_1^{0,0} = H^{n,0}(X^0) \oplus H^{n-1,1}(X^0) \oplus \ldots \oplus H^{1, n-1}(X^0) \oplus H^{0, n}(X^0) \]
\[ E_1^{n-1,1} = H^{n-1,0}(X^1) \oplus H^{n-2,1}(X^1) \oplus \ldots \oplus H^{1, n-2}(X^1) \oplus H^{0, n-1}(X^1) \]
\[ E_1^{n-2,2} = H^{n-2,0}(X^2) \oplus H^{n-3,1}(X^2) \oplus \ldots \oplus H^{1, n-3}(X^2) \oplus H^{0, n-2}(X^2) \oplus \]

\[ \vdots \]

\[ E_1^{1,n-1} = H^{1,0}(X^{n-1}) \oplus H^{0,1}(X^{n-1}) \]
\[ E_1^{0,n} = H^{0,0}(X^n) \]

The \( E_2 \) terms give the lower half triangle of the picture of the mixed Hodge structure we used in Section 2. In other words, the mixed Hodge structure of the central fibre has all 0’s for the upper half.

For our purpose, we need to look at \( F^n \), the \( n \)th filtration of weight \( n \) Hodge filtration. Since \( d_1 \) respects the Hodge filtration, we get \( E_2^{n,0} \) by looking at

\[ 0 \rightarrow F^n E_1^{n,0} \rightarrow F^n E_1^{n,1} \rightarrow 0, \]

where

\[ F^n E_1^{n,0} \cong H^{n,0}(X^0) \]
\[ F^n E_1^{n,1} \cong H^{n,1}(X^1) \]
Notice, since $X^{[1]}$ is a first level stratum, its complex dimension is $n-1$, i.e., it can have at most $n-1$ non-trivial filtration depth. Hence,

$$F^n E_1^{n,1} = 0 \quad \text{and} \quad F^n E_2^{n,0} = F^n E_1^{n,0}$$

To interpret the finite distance condition, this is the crucial component of the mixed Hodge structure of a central fibre.

The final key construction to connect the mixed Hodge structure induced by the limiting filtration to the canonical mixed Hodge structure of a central fibre is the Clemens-Schmid exact sequence. The most elaborate description can be found in [Cl], and we again more or less follow the one given in [GrTA] Ch6. Since maps of the Clemens-Schmid exact sequence are not just maps of homology and cohomology groups but morphisms of mixed Hodge structures of appropriate types, we need first to define those morphisms precisely.

**Definition 3.3** A *weighted $Q$ vector space* is a $Q$ vector space $H$ together with an increasing filtration of $Q$ subspaces $0 \subset \ldots \subset W_k(H) \subset W_{k+1}(H) \subset \ldots \subset H$.

Then a morphism of weighted vector spaces of type $(r,r)$ is a linear map $\phi: H \to H'$ such that

$$\phi(W_k(H)) = W_{k+2r}(H') \cap \text{Im}\phi.$$ 

Both the monodromy weight filtration given in section 2 and the weight filtration given in this section are examples of weighted $Q$ vector spaces. The monodromy operator $N$ we have been looking at is an example of a type $(-1,-1)$ morphism from $H^m(X_0, Q)$ to itself.

We also need to recall the maps appearing in the Clemens-Schmid exact sequence.

1. Let $i: X_s \hookrightarrow X$ be a natural inclusion of the generic fiber into the total space. Since we know $H^m(X) \cong H^m(X_0)$, this induces

$$i^*: H^m(X_0) \to H^m(X_s).$$

2. Define $\alpha: H_{2n+2-m}(X_0) \to H^m(X_0)$ to be the composite

$$H_{2n+2-m}(X) \xrightarrow{P.D} H^m(X, \partial X) \to H^m(X)$$

where $P.D$ is the Poincaré duality map. Again, we identify cohomology of the central fibre with that of the total space.

3. Define $\beta: H^m(X_s) \to H_{2n-m}(X_0)$ to be the composite

$$H^m(X_s) \xrightarrow{P.D} H_{2n-m}(X_s) \xrightarrow{i} H_{2n-m}(X_s)$$

For further details see [Cl].

**Theorem of Clemens-Schmid exact sequence** The maps $\alpha, i^*, N, \beta$ are morphisms of weighted vector spaces of type $(n+1,n+1), (0,0), (-1,-1), (-n,-n)$, respectively, and the sequence

$$\to H_{2n+2-m}(X_0) \xrightarrow{\alpha} H^m(X_0) \xrightarrow{i^*} H^m(X_s) \xrightarrow{N} H^m(X_s) \xrightarrow{\beta}$$

$$\to H_{2n-m}(X_0) \to H^{m+2}(X_s)$$
is exact. Furthermore the morphisms in the Clemens-Schmid sequence are morphisms of mixed Hodge structures of the appropriate types.

Proof and construction  See [Cl].

For our purpose, we need to look at the exact sequence around $m = n$, i.e.,

$$
\to H_{2n+2-n}(X_0) \xrightarrow{\alpha} H^n(X_0) \xrightarrow{i^*} H^n(X_s) \xrightarrow{N} H^n(X_s) \xrightarrow{\beta} \to H_{2n-n} \xrightarrow{\alpha} H^{n+2} \to \ldots
$$

Furthermore, we can restrict the Clemens-Schmid exact sequence to the relevant filtered part of the graded pieces.

Here we need one more definition; we have to define the weight filtration on homology that gives the right definition of the graded pieces of homology. That is,

$$W_{-k}(H_m) = \text{Ann}(W_{k-1}(H^m)) = \{ h \in H_m | (W_{k-1}(H^m), h) = 0 \}.$$

Then,

$$Gr_k(H_m) \cong (Gr_{-k}(H^m))^* \quad \text{so that} \quad Gr_k(H_m) = 0 \quad \text{if} \quad k < -m \quad \text{or} \quad k > 0.$$

In our case, $H^n(X_0)$ carries the mixed Hodge structure we described in this section; that is, the canonical one of the central fibre, and $H^n(X_s)$ carries the mixed Hodge structure in section 2; that is, the limiting mixed Hodge structure. Hence,

1. $i^*: H^n(X_0) \to H^n(X_s)$ sends each piece of the mixed Hodge structure of a central fibre to the lower half of the limiting mixed Hodge structure without any filtration index shift; i.e.,

$$F^p(Gr_m H^n(X_0)) \to F^p(Gr_m H^n(X_s)) \quad \text{where} \quad 0 \geq p, \quad m \geq n.$$

2. $N: H^n(X_s) \to H^n(X_s)$ sends each piece of the limiting mixed Hodge Structure into itself but shifts the Hodge filtration index by $(-1, -1)$ and monodromy weight filtration index by $-1 - 1 = -2$; i.e.,

$$F^p(Gr_m H^n(X_s)) \to F^{p-1}(Gr_{m-2} H^n(X_s)).$$

Now we can restrict the Clemens-Schmid exact sequence to the following graded pieces.

$$F^{-1}Gr_{n-(2n+2)} H_{(2n+2)-n}(X_0) \xrightarrow{\alpha} F^n Gr_n H^n(X_0) \xrightarrow{i^*}$$

$$\to F^n Gr_n H^n(X_s) \xrightarrow{N} F^{n-1} Gr_{n-2} H^n(X_s) \xrightarrow{\beta}$$

and

$$F^{n-1} Gr_{n-2} H^n(X_s) = 0$$

since the $n - 2$ graded piece carries only up to $n - 2$ depth Hodge filtration and there will be no $n - 1$ deep filtration piece.
Furthermore, by a linear algebra fact about mixed Hodge structures (see [GrSc]),
\[ F^{-1}Gr_{n-2}H_{n+2}(\mathcal{X}_s) = Ann(F^2Gr_{n+2}H^{n+2}(\mathcal{X}_s)), \]
and it is 0 since we have only up to \( n \) -forms for Kähler manifolds.
Hence, we have
\[ F^nGr_nH^n(\mathcal{X}_s) \cong F^nGr_nH^n(\mathcal{X}_0). \]

Now we are ready to state and prove the main result.

**Theorem 3.4** A semistable degeneration of a Calabi-Yau \( n \) dimensional compact complex variety is at finite distance with respect to the Weil-Petersson metric if and only if \( H^{n,0} \) of one of the components of the central fibre has positive rank.

**Proof** Recall the result from this section, where we computed the \( E_1 \) terms of the spectral sequence of the central fibre.
\[ H^{n,0}(\mathcal{X}^{[0]}) = F^nE^{n,0}_1 \cong F^nE^{n,0}_2 \overset{(*)}{\cong} F^nGr_nH^n(\mathcal{X}_0) \]
where \( (*) \) is by definition.

On the other hand, a degeneration is at finite distance if and only if
\[ F^nGr_nH^n(\mathcal{X}_s) \neq 0, \quad \text{i.e.,} \]
\[ F^nGr_nH^n(\mathcal{X}_s) \cong F^nGr_nH^n(\mathcal{X}_0) \cong H^{n,0}(\mathcal{X}^{[0]}) \neq 0. \]
By definition, \( \mathcal{X}^{[0]} \) corresponds to the disjoint union of irreducible pieces of the central fibre of the degeneration. Hence, exactly one of the components of the central fibre has \( h^{n,0} \neq 0. \)

\[ \blacksquare \]

**Remark** Since the dimension of \( F_\infty \) is exactly one, actually \( h^{n,0} = 1. \)

### 4. Examples

Originally we were looking at only the Calabi-Yau 3-fold case, which is the main ingredient in the phenomena called ‘mirror symmetry’ in mathematical physics. Since the result applies in any dimension, we first check this with the known result of the low dimensional degeneration problem.

**Case 1.** Degeneration of Curves

Complex curves which have trivial canonical bundles are complex tori. It is a classical result that the genus of the central fibre of a semi-stable degeneration is strictly less than the genus of the generic fibre. Hence, each irreducible component of the central fibre has to consist of \( CP^1 \)'s and none of \( H^{1,0} \) is non-zero. Hence the degeneration has to be at infinite distance, which coincides with the known result.

**Case 2.** Degeneration of Surfaces

Smooth complex surfaces with trivial canonical bundles are K3 surfaces. The semi-stable degeneration of K3 surfaces is classified by the following theorem...
**Theorem (Kulikov [Kul], Persson and Pinkham [PP])** A semi-stable degeneration of K3 surfaces is birational to one for which the central fibre $X_0$ is one of the following three;

1. $X_0$ is smooth K3 surface.
2. $X_0 = X_0 \cup X_1 \cup \ldots \cup X_{k+1}$. $X_i$ meets only $X_{i\pm1}$, and each $X_i \cap X_{i+1}$ is an elliptic curve. $X_0$ and $X_{k+1}$ are rational surfaces, and for $1 \leq i \leq k$, $X_i$ is ruled with $X_i \cap X_{i\pm1}$ and $X_i \cap X_{i-1}$ sections of the ruling.
3. All components of $X_0$ are rational surfaces, $X_i \cap (\cup_{j\neq i} X_j)$ is a cycle of rational curves, and the dual graph of the configuration is homeomorphic to $S^2$.

Recall (cf. [GrHr]) both rational and elliptic ruled surfaces have Kodaira number $-1$, i.e.,

$$0 = h^0(X_i, \mathcal{O}(K_{X_i})) = h^{2,0}(X_i).$$

Hence only smooth central fibre gives finite distance.

**Case 3.** Nodal degeneration of Calabi-Yau $n$-folds, for $n \geq 3$

To get a semi-stable degeneration we have to blow-up all nodes, so that we will get a central fibre with normal crossings. Let $\pi : \mathcal{X} \to \Delta$ be a degeneration of Calabi-Yau $n$-folds. Suppose that the special fibre $X_0$ is a $n-$dimensional compact complex variety with $m$ nodes. Then the full blow-up of $m$ nodes, denoted $X_0'$, has the following configuration.

There are $m$ exceptional divisors of the total space $\mathcal{X}$, $\mathbb{C}P^n$, and a proper transform of $X_0$, $\tilde{X}_0$. Each $\mathbb{C}P^n$ intersects with $\tilde{X}_0$ once and only once; none of $\mathbb{C}P^n$ intersects each other. There are no self-intersections.

**Proposition 4.1** The blow-up of nodes are at finite distance with respect to Weil-Petersson metric. Hence the special fibre with nodes is at finite distance.

**Proof** Let $\pi' : \mathcal{X}' \to \Delta$ be the degeneration of the blow-up of nodes $\mathcal{X}'$, which is clearly semi-stable. If we denote $f : \mathcal{X}' \to \mathcal{X}$ to be the full blow-up map, then $\pi' = \pi \circ f$.

Since each generic fibre is a Calabi-Yau fibre, the canonical divisor of $\mathcal{X}$ is concentrated on the blow-up of the special fibre $X_0'$. Hence we have

\begin{equation}
K_{\mathcal{X}'} = kL_{\tilde{X}_0} + \Sigma k_iL_{D_i},
\end{equation}

where $L_{\tilde{X}_0}$ is the line bundle corresponding to $\tilde{X}_0$ as an effective divisor in $\mathcal{X}'$ and $L_{D_i}$ is the line bundle corresponding to each exceptional divisor $D_i \cong \mathbb{C}P^n$ in $\mathcal{X}'$.

Let $H_i$ denote the hyper-plane bundle over $D_i$. Then

\begin{align}
&L_{D_i}|_{D_i} = N^*_i = -H_i \in \text{Pic}(D_i), \\
&K_{D_i} = K_{\mathbb{C}P^n} = -(n+1)H_i.
\end{align}

**Claim**

$$L_{\tilde{X}_0} + 2\Sigma L_{D_i} = 0.$$

**Proof of Claim** Since $\mathcal{X}'$ is the full blow-up of nodes, each exceptional divisor has multiplicity 2, i.e.,

$$\tilde{X}_0 + 2\Sigma D_i = f^{-1}(X_i).$$
If we look at the trivial bundle of $X'$, it has a section given by
\[ x' \mapsto (x', \pi'(x')) = (x', \pi \circ f(x')). \]
Notice the zero locus of this section is exactly $\pi'(f^{-1}(X_0))$. Hence the corresponding line bundle is trivial.

Hence we have
\[ L_{\bar{X}_0} = -2\Sigma L_{D_i}. \]
We also get other equalities of line bundles:
\[ L_{\bar{X}_0}|_{D_i} = -2\Sigma L_{D_j}|_{D_i} = -2L_{D_i}|_{D_i} = 2H_i \]

since $L_{D_j}|_{D_i} = L_{D_j \cap D_i} = L_{\emptyset} = 0$,
and
\[ L_{\bar{X}_0}|_{\bar{X}_0} = -2\Sigma L_{D_i}|_{\bar{X}_0} = -2\Sigma L_{E_i} \in \text{Pic}(\bar{X}_0)_{\text{tag} 5}, \]
where $E_i = \bar{X}_0 \cap D_i$.

We are now ready to compute the canonical bundle of $\bar{X}_0$. By the adjunction formula for $D_i$ we have
\[ K_{D_i} = K_{X'}|_{D_i} + L_{D_i}|_{D_i}. \]
Notice
\[ K_{X'}|_{D_i} = kL_{\bar{X}_0}|_{D_i} + \Sigma k_j L_{D_j}|_{D_i} \quad \text{by (1)} \]
\[ = 2kH_i - k_i H_i \quad \text{by (2), (4)}. \]
Then we get
\[ -(n + 1)H_i = 2kH_i - k_i H_i - H_i \quad \text{by (3)} \]
\[ = (2k - k_i - 1)H_i. \]
Since the first Chern class gives the isomorphism:
\[ c_1 : \text{Pic}(CP^n) \rightarrow H^2(CP^n, \mathbb{Z}), \]
We get
\[ -(n + 1) = 2k - k_i - 1, \quad \text{i.e.,} \quad k_i = 2k + n. \]
Next, by the adjunction formula for $\bar{X}_0$,
\[ K_{\bar{X}_0} = K_{X'}|_{\bar{X}_0} + L_{\bar{X}_0}|_{\bar{X}_0}. \]
Then by (1) and $L_{\bar{X}_0} = -2\Sigma L_{D_i}$, we get
\[ K_{X'} = \Sigma k_i L_{D_i} - 2k \Sigma L_{D_i} \]
\[ = \Sigma (k_i - 2k) L_{D_i} \]
\[ = \Sigma (2k + n - 2k) L_{D_i} \]
\[ = \Sigma n L_{D_i}. \]
Hence

\[ K_{\tilde{X}_0} = \sum nL_{D_i}|_{\tilde{X}_0} + L_{\tilde{X}_0}|_{\tilde{X}_0} \]
\[ = n\sum L_{E_i} - 2\sum L_{E_i} \]
\[ = (n - 2)\sum L_{E_i}. \]

Recall \( E_i = \tilde{X}_0 \cap D_i \). So \( E_i \) is an effective divisor in \( \tilde{X}_0 \). Hence for \( n \geq 3 \), \( K_{\tilde{X}_0} \) has a non-trivial global holomorphic section, i.e., \( H^{n,0}(\tilde{X}_0) \neq 0 \). By Theorem 3.4 result follows.

\[ \square \]

Remark This result coincides with the lower dimensional cases. For surfaces, the special fibre has to be smooth. Therefore, it is also a K3 surface which has a trivial canonical bundle. If the fibres are curves, the canonical bundle cannot have a holomorphic global section.
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