ON QUANTUM GROUP SYMMETRIES OF CONFORMAL FIELD THEORIES

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Abstract

The appearance of quantum groups in conformal field theories is traced back to the Poisson-Lie symmetries of the classical chiral theory. A geometric quantization of the classical theory deforms the Poisson-Lie symmetries to the quantum group ones. This elucidates the fundamental role of chiral symmetries that quantum groups play in conformal models. As a byproduct, one obtains a more geometric approach to the representation theory of quantum groups.

1. Canonical structure of the chiral WZW models

Quantum Groups\(^1\)–\(^3\) (QGs) have entered into Conformal Field Theory (CFT) through the back door: it was discovered that the exchange properties of (some) CFT chiral vertex operators lead to the braid group representations related to QGs\(^4,5\), that the QG 6j symbols may be realized as braiding matrices of those operators\(^6–8\) and that the CFT fusion rules are related to the tensor product decomposition of quantum group representations at roots of unity\(^6,9\). In view of these relations it was becoming clear that QGs should play a role of new symmetries of chiral CFTs\(^10–19\). Since symmetries play a fundamental role in physics, it would be desirable to have an approach to CFTs which puts QG symmetries in the foreground. This is a report about an attempt at such an approach. The main idea we follow is to start at the classical level and to identify classical symmetries of the chiral CFTs which upon quantization become QG symmetries. This idea was pursued before in a series of papers started by Ref.\(^20\), or in the more direct sense, by the St. Petersburg school in Refs.\(^21,22\); see also Ref.\(^23,24\). The present exposition is based on a paper in preparation extending the results of Ref.\(^25\). Another (possibly more fundamental) approach has been proposed recently in Refs.\(^26–28\) were an

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an attempt was made to build a lattice version of CFT’s with local QG symmetry whose
global part survives the continuum limit. Although different from ours, this approach
also has some common points with our work.

We shall concentrate below on the Wess-Zumino-Witten (WZW) model\textsuperscript{29} of CFT
although our analysis may be easily extended to various coset theories. In that model
the classical field configurations are given by Lie group $G$-valued functions $g(x^0, x^1)$
of two-dimensional Minkowski space-time coordinates satisfying

\[ \partial_-(g\partial_+g^{-1}) = 0 \] (1)

where $\partial_\pm = (\partial_1 \pm \partial_0)$. On the cylinder $Z = \{ (x^0, x^1 \text{ mod } 2\pi) \}$, the classical solutions
are of the form

\[ g(x^0, x^1) = g_L(x^+) g_R(x^-)^{-1} \] (2)

where $x^\pm = x^1 \pm x^0$ and

\[ g_{L,R}(x + 2\pi) = g_{L,R}(x) \gamma_{L,R} \] (3)

with $\gamma_L = \gamma_R \in G$ (so that $g$ is periodic in $x^1$). Let us denote by $P$ the space of all
solutions of Eq. 1 on $Z$, i.e. the phase space of the complete WZW model. Introducing
the phase spaces $P_L$ ($P_R$) of left- (right-)movers as the spaces of smooth maps $g_L$ ($g_R$)
satisfying relation 3 and denoting by $\Delta$ the subset of $P_L \times P_R$ with equal monodromies
$\gamma_L = \gamma_R$, we have

\[ P = \Delta/G \] (4)

where $G$ acts by

\[ (g_L, g_R) \mapsto (g_L g_0, g_R g_0) \] (5)

and describes the ambiguity of representation 2. (Notice that under transformation 5,
$\gamma_{L,R} \mapsto Ad_{g_0^{-1}(\gamma_{L,R})}$). The canonical structure of the WZW theory is described by the
(uniquely determined) symplectic form $\Omega$ on phase space $P$ which may be written\textsuperscript{25} as

\[ \Omega_L - \Omega_R \] where
\[
\Omega_{L,R} = (4\pi)^{-1}k \int_0^{2\pi} \text{tr}(g_{L,R}^{-1}dg_{L,R}) \wedge \partial_x(g_{L,R}^{-1}dg_{L,R}) \\
+ (4\pi)^{-1}k \text{tr}(g_{L,R}^{-1}dg_{L,R})(0) \wedge (d\gamma_{L,R})^{-1}\gamma_{L,R}^{-1} \quad -(4\pi)^{-1}k \rho(\gamma_{L,R})
\]  

are 2-forms on \( P_{L,R} \). In Eq. 6, \( \rho \) is an arbitrary 2-form on \( G \). Such (and only such) an ambiguity arises because only the restriction of \( \Omega_L - \Omega_R \) to \( \Delta \subset P_L \times P_R \) enters in the determination of (unique) \( \Omega \).

\( \Omega_L \) seems to be a natural candidate for the 2-form defining the canonical structure for the left-movers (and similarly for the right-movers). Somewhat surprisingly, however, there are problems with such an interpretation. First, \( \Omega_L \) is not unique. Much worse, a straightforward calculation shows that

\[
d\Omega_L(g_L) = (12\pi)^{-1}k \text{tr}(\gamma_L^{-1}d\gamma_L)\wedge 3 - (4\pi)^{-1}k d\rho(\gamma_L)
\]

so that \( d\Omega_L \) can never be zero globally as \( \text{tr}(\gamma_L^{-1}d\gamma_L)^3 \) is not an exact form on (simple) \( G \). At least three possible ways out of the latter difficulty may be considered:

1. One may use ambiguity 5 to restrict \( P_L \) to maps with specific monodromies. For example, for compact \( G \) we may introduce \( P^{res}_L \) corresponding to monodromies \( \gamma_L \) in the Cartan subgroup \( T \subset G \). Similar choices may be made for non-compact groups. On \( P^{res}_L \), \( d\Omega_L = 0 \) whenever \( d\rho = 0 \) on \( T \). This is an approach parallel to that of Refs.\textsuperscript{12,20,21,30-32} which worked with the diagonal monodromy.

2. By choosing suitable \( \rho \) with \( d\rho = \text{tr}(\gamma_L^{-1}d\gamma_L)^3/3 \) on an open dense subset in \( G \), we may obtain a singular symplectic structure on \( P_L \) which leads however to a regular Poisson bracket (at the singularities of the symplectic form the Poisson structure ceases to be non-degenerate). We shall pursue this approach here.

3. One may interpret Eq. 7 as an obstruction to closeness of \( \Omega_L \) which would be reflected in the violation of the Jacobi identity for the Poisson bracket on \( P_L \). This leads to the appearance of classical counterparts of Drinfeld’s quasi-Hopf algebras\textsuperscript{33}. We shall discuss this approach elsewhere. See also Ref.\textsuperscript{34}.

The Poisson bracket induced by \( \Omega_L \) on \( P_L \) has the form

\[
\{g_L(x)_1, g_L(y)_2\} = -2\pi k^{-1} g_L(x)_1 g_L(y)_2 r^\pm(\gamma_L)
\]

in a shorthand notation where \( g_L(x)_1 = g_L(x) \otimes 1 \), \( g_L(y)_2 = 1 \otimes g_L(y) \) and \( r^\pm(\gamma_L) \in \mathcal{G}^C \otimes \mathcal{G}^C \) with \( \mathcal{G} \) the Lie algebra of \( G \) are all four treated as endomorphisms of \( V_1 \otimes V_2 \) with \( V_i \) representation spaces of \( G \). \( \pm \) sign in \( r^{\pm} \) is used depending on whether \( x < y \) or \( x > y \).
Poisson bracket 8 has, in general, one nasty feature: non-locality. The right hand side depends not only on the values of \( g_L \) at points appearing on the left hand side but also on the non-local monodromy of \( g_L \). It is then rather natural to ask if for certain choices of (singular) 2-form \( \rho \) the monodromy dependence of \( r^\pm \) disappears and Poisson bracket 8 becomes local\(^2\). Those choices would necessarily lead to matrices \( r^\pm \) satisfying the Classical Yang-Baxter Equation (CYBE) (without spectral parameter)

\[
[r^\pm_{12}, r^\pm_{13}] + [r^\pm_{12}, r^\pm_{23}] + [r^\pm_{13}, r^\pm_{23}] = 0
\]  

(9)
equivalent to the Jacobi identity for bracket 8 with constant \( r^\pm \). (Eq. 9 should be read as an equality between endomorphisms of \( V_1 \otimes V_2 \otimes V_3 \)). Moreover,

\[
r^\pm = r \pm \kappa
\]  

(10)
where \( r \in G^{\wedge 2} \) and \( \kappa = \sum t^a \otimes t^a \) is the quadratic Casimir of \( G \) (with the generators \( t^a \) normalized by \( \text{tr} t^a t^b = \delta^{ab}/2 \)).

Conversely, suppose that we are given a pair \( r^\pm \) satisfying relations 9 and 10. Consider, following Ref.\(^35\), the subspace

\[
G_r \equiv \{(r^+|v, r^-|v) \mid v \in G^* \} \subset G^C \oplus G^C .
\]  

(11)
Eqs. 9 and 10 imply that \( G_r \) is a complex Lie subalgebra of \( G^C \oplus G^C \), see Ref.\(^35\). Denote by \( G_r \) the corresponding Lie group \( \subset G^C \times G^C \) and by \( \iota \) the map

\[
G^C \times G^C \ni (\gamma_+, \gamma_-) \mapsto \gamma_+\gamma_-^{-1} \in G^C .
\]  

(12)
The restriction of \( \iota \) to \( G_r \) is a covering map onto an open dense subset \( G^C_0 \) in \( G^C \). Consider a (complex) 2-form \( \rho \) on \( G^C_0 \) defined in terms of (multivalued) coordinates \( (\gamma_+, \gamma_-) \) by

\[
\rho = \text{tr} \gamma_-^{-1} d\gamma_+ \wedge \gamma_- d\gamma_-^{-1}
\]  

(13)
The corresponding (singular) 2-form \( \Omega_L \) leads to Poisson bracket 8 which reproduces for \( |x-y|<2\pi \) the constant matrices \( r^\pm \) on the right hand side.

\(^2\) the chiral notion of locality should not be confused with physical locality in the complete theory holding in any case
Summarizing: there is an ambiguity in defining canonical structure of the chiral WZW theory. Possible local solutions are in one to one correspondence with pairs \( r^\pm \) of solutions of the CYBE.

The best known example of a pair \( r^\pm \) of solutions of the QYBE is obtained by taking in Eq. 10

\[
 r = \sum_{\alpha > 0} (e_\alpha \otimes e_{-\alpha} - e_{-\alpha} \otimes e_\alpha) / 2
\]

where the sum runs over the positive roots of \( \mathcal{G} \) and \( e_{\pm \alpha} \) are the corresponding nilpotent generators of \( \mathcal{G}^C \) normalized so that \( \text{tr} e_\alpha e_{-\alpha} = 1 \). The classification of solutions of the CYBE (without spectral parameter) may be found in Ref. 36.

2. Poisson-Lie symmetries.

Phase space \( P_L \) of the chiral WZW theory together with Poisson bracket 8 with monodromy independent \( r^\pm \) provide an (infinite-dimensional) example of a Poisson manifold \(^{37} \), i.e. a manifold supplied with a field of 2-covectors whose contraction with differentials of two functions on the manifold gives their Poisson bracket \(^{37} \). In our case, the Poisson structure of \( P_L \) comes from the inversion of a (complex) singular symplectic form \( \Omega_L \). The notion of a Poisson manifold is more general than that of a symplectic manifold. More importantly, it allows a natural generalization of the notion of symmetry. Conventionally, we would say that \( \Gamma \) is a symmetry group of the Poisson manifold \( \Pi \) if \( \Gamma \) acts (from left or right) on \( \Pi \) preserving its Poisson structure. The generalized (Poisson-Lie) symmetries involve the notion of a Poisson-Lie (PL) group i.e. a Lie group \( \Gamma \) provided with a Poisson structure compatible with the group multiplication \(^{38,35} \). As for Lie groups, there is a corresponding infinitesimal notion: that of a bialgebra i.e. of a Lie algebra \( \Upsilon \) (of \( \Gamma \)) together with a Lie algebra structure on the dual space \( \Upsilon^* \), both compatible in a suitable way. For each bialgebra, there is a dual bialgebra with the roles of \( \Upsilon \) and \( \Upsilon^* \) interchanged. This duality lifts to the (simply connected) PL groups which come in pairs \((\Gamma, \Gamma^*)\).

The simplest example of a PL group may be obtained by taking a Lie group \( \Gamma \) with the vanishing Poisson structure. The corresponding Lie algebra \( \Upsilon \) becomes a bialgebra with the vanishing Lie bracket on \( \Upsilon^* \) and the corresponding dual PL group is \( \Upsilon^* \) with addition as the group operation and with the Poisson bracket which to the linear functions on \( \Upsilon^* \) given by elements \( \tau, \sigma \in \Upsilon \) assigns the linear function given by \( [\tau, \sigma] \). The symplectic leaves of \( \Upsilon^* \) with this Poisson structure (i.e. connected components of common level sets of functions on \( \Upsilon^* \) with vanishing Poisson brackets with everybody else) are exactly the coadjoint orbits in \( \Upsilon^* \). As is well known, for large class of Lie groups (e.g. for the compact ones), the coadjoint orbits are related to irreducible representations of the group \(^{39} \).

\(^{3)} \) in fact the example is not quite conventional since the Poisson structure on \( P_L \) is complex
Another, less trivial example of a PL group is obtained by defining, following Sklyanin\textsuperscript{40}, a Poisson structure on a (complex) Lie group $\Gamma$ by putting

$$\{\gamma_1, \gamma_2\}_{\text{Skl}} = 2\pi k^{-1} [\gamma_1 \gamma_2, r^\pm]$$

in the notation of Eq. 8 ($\gamma$ is the matrix function on $\Gamma$ given by a representation, $\gamma_1 = \gamma \otimes 1$ etc.; both signs give the same Poisson bracket). The Lie algebra of the dual PL group $\Gamma^*$ may be identified with $\Upsilon_r$ defined as in 11 via the map $v \mapsto (r^+ | v, r^- | v)$ and $\Gamma^*$ itself with $\Gamma_r \subset \Gamma \times \Gamma$. The symplectic leaves of $\Gamma^*$ become then connected components of the preimages of the conjugacy classes in $\Gamma$ under the covering map $\iota$, see Ref.\textsuperscript{35}. They play a role in the representation theory of quantum groups.

We shall say that $\Gamma$ is a PL symmetry of Poisson manifold $\Pi$ if it is a PL group which acts on $\Pi$ so that the corresponding map $\Pi \times \Gamma \longrightarrow \Pi$ is Poisson i.e. preserves the Poisson brackets. In the case of $\Gamma$ with the vanishing Poisson structure this definition is equivalent to demanding that the action of $\Gamma$ preserves the Poisson structure of $\Pi$ so that the notion of a PL symmetry generalizes that of a standard (Lie) symmetry.

$P_L$ with the Poisson structure that we have introduced has several symmetries. First, there are conventional symmetries:

1. Loop group symmetry. $LG$ (the group of periodic maps $h(x)$ with values in $G$) acts on $P_L$ by $g_L \mapsto hg_L$ preserving $\Omega_L$ and the corresponding Poisson structure.

2. Conformal symmetry. Group $Diff_+(S^1)$ of orientation-preserving diffeomorphisms $D$ of the circle acts by $g_L \mapsto g_L \circ D$ again preserving $\Omega_L$.

But $P_L$ also has $G_{\text{Skl}}$ as a PL symmetry. Namely, $G$ acts on $P_L$ by

$$ (g_L, g_0) \mapsto g_Lg_0 , $$

and map 16 preserves the Poisson brackets if $G$ is taken with Sklyanin bracket 15 (for real $G$ this defines a complex Poisson structure on $G$ and the notion of a PL group should be extended accordingly).

Let us return to the general discussion of symmetries. If $\Gamma$ is a Lie symmetry of a symplectic manifold $\Pi$ one may often encode its action in the so called moment map\textsuperscript{41}

$$ m : \Pi \longrightarrow \Upsilon^* $$

such that if $\tau$ is in the Lie algebra $\Upsilon$ then the contraction of $m$ with $\tau$ gives a hamiltonian function on $\Pi$ generating the infinitesimal action of $\tau$. One also demands that the hamiltonian of $[\tau, \sigma]$ be the Poisson bracket of hamiltonians of $\tau$ and $\sigma$ or in other words that $m$ be a Poisson map if $\Upsilon^*$ is taken with the Poisson structure making it the dual
PL group to $\Gamma$ with the vanishing Poisson bracket (recall the discussion above). An example at hand is the Sugawara energy-momentum tensor $T = (2k)^{-1} \text{tr} J^2$ where the current $J = ik(\partial_x g_L) g_L^{-1}$. The quadratic differential $T$ may be viewed as a map

$$P_L \rightarrow Vect(S^1)^*$$

(18)

into the dual of the space of vector fields on the circle and is the moment map for the action of $Diff_+(S^1)$ on $P_L$.

In some situations there are obstructions to existence of the moment maps as defined above\textsuperscript{41}. For example, current $J$ could be viewed as a map of $P_L$ into $L\mathcal{G}^*$, the dual space to the Lie algebra of $L\mathcal{G}$ but as such would not provide a Poisson map because of the central term in the Poisson bracket of currents. Instead, one should consider a central extension $L\hat{\mathcal{G}} \rightarrow L\mathcal{G}$ of the loop algebra and treat $J$ as taking values in $L\hat{\mathcal{G}}^*$ which leads to the following diagram of the Poisson maps:

$$P_L \rightarrow L\hat{\mathcal{G}}^* \leftarrow L\mathcal{G}^*.$$  

(19)

The notion of a moment map extends to the case of PL symmetries\textsuperscript{42} where a moment map becomes an appropriate Poisson map\textsuperscript{43}

$$\Pi \rightarrow \Gamma^*.$$  

(20)

Again, there might exist obstruction to the existence of moment maps in strict sense. An example is provided by the case of PL symmetry of $P_L$ considered above. Instead of a map like 20, we find here a diagram of Poisson maps

$$P_L \rightarrow G \subset G^C \leftarrow G_r \equiv (G^C)^*.$$  

(21)

Above, the leftmost arrow is the map $g_L \mapsto \gamma_L(= g_L(0)^{-1}g(2\pi))$ and the right one is $\iota$ of 12. The Poisson structure on $G^C$ is given by

$$\{ \gamma_1, \gamma_2 \} = -2\pi k^{-1}(r^+\gamma_1\gamma_2 - r^-\gamma_1\gamma_2 - \gamma_2r^+\gamma_1 + \gamma_1\gamma_2r^+)$$

(22)

and the one on $G_r$ by

\textsuperscript{43} we imply here a slightly more restrictive notion of a moment map then the one of “momentum mapping” defined in Ref.\textsuperscript{42}
\begin{equation}
\{\gamma_+^1, \gamma_+^2\} = 2\pi k^{-1} [r^\pm, \gamma_+^1 \gamma_+^2], \\
\{\gamma_+^1, \gamma_-^2\} = 2\pi k^{-1} [r^+, \gamma_+^1 \gamma_-^2], \\
\{\gamma_-^1, \gamma_-^2\} = 2\pi k^{-1} [r^-, \gamma_-^1 \gamma_-^2].
\end{equation}

(23)

As we see, the monodromy plays the role of the (generalized) moment map for the PL
symmetry of the chiral phase space \(P_L\).

3. Classical vertex-IRF transformation

Let us assume for concreteness that \(G\) is a simple compact group. It is convenient
to parametrize \(g_L \in P_L\) writing

\[ g_L(x) = h(x) e^{i\tau} g_0^{-1} \]  

(24)

where \(h \in LG\), \(\tau\) is in the Cartan subalgebra \(T \subset G\) and \(g_0 \in G\). This parametrization
is not unique. First, \(\tau\) is determined up to the action of the affine Weyl group. We may
fix this ambiguity by taking \(\tau\) from the positive Weyl alcove \(A \subset T\). This will leave us
only with the possibility to multiply \(h\) and \(g_0\) on the right by the same element of the
Cartan subgroup \(T\). In parametrization 24,

\[ \Omega_L(g_L) = \Omega_{L1}(h, \tau) + \Omega_{L2}(g_0, \tau) \]  

(25)

where

\[ \Omega_{L1}(h, \tau) = (4\pi)^{-1} k \int_0^{2\pi} \text{tr} \left[ (h^{-1} dh) \land \partial_x (h^{-1} dh) + 2i\tau (h^{-1} dh)^\land 2 \\
- 2i(d\tau) \land (h^{-1} dh) \right] \]  

(26)

and

\[ \Omega_{L2}(g_0, \tau) = ki \text{tr} (d\tau) \land g_0^{-1} dg_0 + (4\pi)^{-1} k \text{tr} g_0^{-1} dg_0 \land Ad_{e^{2\pi i\tau}} (g_0^{-1} dg_0) \\
- (4\pi)^{-1} k \rho(g_0 e^{2\pi i\tau} g_0^{-1}) \]  

(27)

Symplectic form \(\Omega_{L1}(h, \tau)\) is equal to \(\Omega_{L1}\) of Eq. 6 restricted to the set of \(g_L\) with
monodromy in the Cartan subgroup \(T\), i.e. to \(P_L^{\text{res}}\) introduced above. \(P_L^{\text{res}}\) plays the role

8
of what has been called in Ref. 43 a “model space” for the Kac-Moody group \( \check{LG} \). It is a symplectic space which contains each (generic) coadjoint orbit of \( \check{LG} \) once. Indeed, if we fix \( \tau \) in the Weyl alcove \( A \), \( \Omega_{L1} \) becomes a (degenerate) 2-form on \( LG \) which coincides with the pullback of the symplectic form of the coadjoint orbit of \( LG \) labeled by \( \tau \) by the natural map of \( LG \) onto the orbit, see Ref. 44.

Similarly, \( \Omega_{L2}(g_0, \tau) \) may be viewed as the symplectic form on the “model space” for the PL group \( G \) with the Sklyanin Poisson structure. As was suggested above, for PL groups we should rather talk about the symplectic leaves of the dual group than about coadjoint orbits. For \( G \) (or \( G^C \)) with the Sklyanin Poisson structure, the dual group is isomorphic to \( G_r \) which covers by \( \iota \) (an open dense subset of) \( G^C \). Moreover, the symplectic leaves of the dual group correspond by \( \iota \) to the conjugacy classes in \( G^C \). Restricting to the compact group \( G \) and its conjugacy classes \( \equiv \{g_0 e^{2\pi i \tau g^{-1}} | g_0 \in G\} \) one may show that, in terms of \( g_0 \), the symplectic form of the symplectic leaves coincides with \( \Omega_{L2} \) at fixed \( \tau \).

As we see, the chiral phase space \( P_L \) may be realized as the fibered product

\[
M_{KM} \times_A M_{PL}
\]

of the Kac-Moody and the Poisson-Lie model spaces over the Weyl alcove \( A \), as summarized by Eq. 24 \[. \] The Poisson bracket of fields \( g_L^{res}(x) \equiv h(x) e^{i\tau x} \) on \( P_L^{res} \) has also form 8 but with \( \tau \)-dependent \( r^\pm \), the classical counterparts of the quantum group 6j symbols\[21,22. \] Eq. 24 establishes a relation between those fields and fields \( g_L \) with monodromy-independent \( r^\pm \) Poisson brackets. This is the classical version of the vertex-IRF transformation for the (WZW) CFTs\[10,11,16,17. \] It has similar flavor as field transformations described in Refs.\[20,12,21,31,32. \] with the important difference that there the vertex versions of the fields still live on the phase space with diagonal monodromy whereas our \( g_L \)’s are functionals on the bigger phase space \( P_L \) with general monodromy. As a result, contrary to Refs.\[21,31,32. \] we may obtain vertex fields with arbitrary solution \( r^\pm \) of the CYBE in the Poisson bracket. In particular, the standard r-matrix 14 may be used for any \( G \) whereas in Refs.\[31,32. \] for \( SU(N) \) with \( N \geq 3 \), different solutions were obtained.

4. Quantization

Let us briefly discuss how the preceding analysis may be extended to the quantum theory. We shall give a more complete account in a future publication. A good idea is to use geometric quantization\[45,46 \] which keeps track of the classical geometry. In view of presentation 28 of the chiral phase space, we may first quantize model spaces \( M_{KM} \) and \( M_{PL} \) separately and then impose condition \( \tau_{KM} = \tau_{PL} \) in the quantum theory.

The geometric quantization of the Kac-Moody model space is more or less standard. One takes the complex line bundle \( L_{KM} \) over \( M_{KM} \) with the hermitian connection of

\[ \]
curvature $\Omega_{L1}$ (this is possible for $k$ an integer) and polarization $\mathcal{P}_{KM}$ of $M_{KM}$ given by the (complex) tangent vectors annihilated by forms $d\tau$ and

$$
\frac{2\pi}{h} \int_0 h^{-1}dh(x)e^{inx}dx, \quad \text{tr}_\alpha e^{\frac{2\pi}{h} \int_0 h^{-1}dh(x)}dx, \quad \text{for } n < 0, \quad \alpha < 0.
$$

The space of quantum states is the homology of the sheaf of $\mathcal{P}_{KM}$-horizontal sections of $\mathcal{L}_{KM}$ (only $H^1$ contributes) or, equivalently, of distributional $\mathcal{P}_{KM}$-horizontal sections of $\mathcal{L}_{KM}$. The latter are supported by $k\tau$ in the weight lattice. For $\tau$ fixed at such a value, the problem reduces to the geometric quantization of the corresponding coadjoint orbit of $LG$ isomorphic to $LG/T$ on which the polarization induces the standard complex structure. Over $LG/T$, $\mathcal{L}_{KM}$ becomes a holomorphic line bundle and the quantum states its holomorphic sections. This way, for fixed $\tau$, we recover the Borel-Weil construction of the irreducible representation space $\hat{V}_{k,\lambda}$ of $\hat{L}G$ corresponding to the highest weight $\lambda = k\tau$ and level $k$, see Ref. 47. It is still better to use the improved geometric quantization where states are half-densities in which case, if we replace original $k$ by $k + h^\ast$ where $h^\ast$ is the dual Coxeter number of $G$, we end up, for fixed weight $\tau$, with $\hat{V}_{k,\lambda}$ for $\lambda + \rho = (k + h^\ast)\tau$ where here $\rho$ denotes the Weyl vector $\sum \alpha/2$. In any case the total spaces of states corresponding to $M_{KM}$ is

$$
V_{KM} = \bigoplus_{\text{integrable } \lambda} \hat{V}_{k,\lambda}
$$

where integrable weights satisfy $(\lambda + \rho)/(k + h^\ast) \in \mathbb{A}$ so that the direct sum runs through all reducible representations of $LG$ at level $k$.

Geometric quantization of the PL model space may be tried along the same lines. We take the complex line bundle $\mathcal{L}_{PL}$ over $M_{PL}$ with connection of curvature $\Omega_{L2}$ (since the latter is complex, the connection cannot be hermitian). Again $k\tau$ and the Cartan subgroup component of $g_0$ are canonically conjugate and if we take a polarization annihilated by $d\tau$, the states will be supported by $k\tau$ in the weight lattice. For fixed $\tau$, the problem reduces to the geometric quantization on conjugacy classes of $e^{2\pi i r}$ in $G$ (the condition that $\Omega_{L2}$ defines a Chern class of a line bundle over the conjugacy class is exactly that $k\tau$ be a weight). For $G = SU(2)$, the conjugacy classes are (generically) $\mathbb{CP}^1$ and their (complex) symplectic form induced by $\Omega_{L2}$ corresponding to $r$-matrix 14 is

$$
\omega_j = k(\pi i)^{-1}\sin(2\pi j/k)(|z|^2 + 1)^{-1}(e^{2\pi ij/k}|z|^2 + e^{-2\pi ij/k})^{-1}dzd\bar{z}
$$

$\tag{30}$

$^6$k\tau$ is canonically conjugate to the Cartan subgroup component of the zero mode of $h$.
for $k\tau = j\sigma^3$. $z$ is the standard complex coordinate of $\mathbb{C}P^1$. Notice that $\omega_j$ is a deformation of the symplectic structure of a coadjoint orbit of $SU(2)$ to which it tends in the classical limit $k \to \infty$. For $G = SU(2)$, the quantization of the conjugacy classes is simple since the complex structure of $\mathbb{C}P^1$ provides a polarization of $\omega_j$. We end up with the space $\mathcal{V}_{k,j}$ of holomorphic sections of $2j$’s power of the Hopf bundle over $\mathbb{C}P^1$ which may be represented as the space of polynomials in $z$ of degree $\leq 2j$ (if we use half-densities, we should replace $k$ by $k + 2$ and take $(k + 2)\tau = (j + 1/2)\sigma^3$). The complete space of states corresponding to $M_{PL}$ is then

$$V_{PL} = \bigoplus_{j=0,1/2,...,k/2} \mathcal{V}_{k,j} .$$

Usually, geometric quantization provides also prescriptions on how to assign quantum operators to (certain) classical physical quantities. These may be expressed in terms of a symbolic calculus using in the case of Kähler manifolds reproducing kernels or, in more physical terms, the formalism of coherent states. Although the present case of $\mathbb{C}P^1$ with form $\omega_j$ is not exactly of Kähler type ($\omega_j$ is complex), one may set up a symbolic calculus extending the coherent state formalism from standard $SU(2)$ to $SU(2)_{Skl}$ in such a way that the matrix elements of $\gamma_L = g_0e^{2\pi i \tau}g_0^{-1}$ treated as functions of $g_0T \in \mathbb{C}P^1$ become generators of the quantum deformation $U_q(SU(2))$ of the enveloping algebra of $SU(2)$ for $q = e^{\pi i/(k+2)}$. As a result, spaces $\mathcal{V}_{k,j}$ carry naturally spin $j$ representations of $U_q(SU(2))$ generated by quantizations of matrix elements of the monodromy $\gamma_L$ satisfying the commutation relations

$$R^+ \gamma_1 (R^-)^{-1} \gamma_2 = \gamma_2 R^+ \gamma_1 (R^-)^{-1} ,$$

the quantum counterpart of Eq. 22.

For more complicated groups than $SU(2)$, there are difficulties in applying the standard prescriptions of geometric quantization to the conjugacy classes of $G$ since the usual complex structure of $G/T$ does not give a polarization for the form induced by $\Omega_{L2}$ and there is no obvious replacement for it. It is rather clear that some aspects of non-commutative geometry have to be used if we want a systematic geometric procedure which produces representation spaces $\mathcal{V}_{k,\lambda}$ of $U_q(G)$ by quantizing the conjugacy classes in $G$ of $e^{2\pi i \lambda/k}$ (or rather of $e^{2\pi i (\lambda+\rho)/(k+\hbar^2)}$) taken with the symplectic form inherited from $\Omega_{L2}$ corresponding to $r$–matrix 14. Such a procedure could then be tried for non-standard solutions $r^\pm$ of the CYBE and could systematically produce their quantizations $R^\pm$ together with the corresponding quantum deformations of $G$ (see Ref. where one of such non-standard deformations was analyzed). In any case, at least for standard solution 14 of the CYBE, it is reasonable to take the quantum space of states for $M_{PL}$ as
\[ V_{PL} = \bigoplus_{\lambda \text{ integrable}} V_{k,\lambda} . \]  

Then the space of quantum states which corresponds to the fibered product \( M_{KM} \times_A M_{PL} \) clearly becomes

\[ V = \bigoplus_{\lambda \text{ integrable}} V_{k,\lambda} \otimes V_{k,\lambda} . \]

It remains still to quantize fields \( g_L(x) \). According to Eq. 24, they are built from fields \( g_{i\text{res}}^L(x) \equiv h(x)e^{ix\tau} \) living on \( M_{KM} \) and of matrix elements of \( g_0^{-1} \) defining functions on \( M_{PL} \). \( g_{i\text{res}}^L(x) \) may be quantized using e.g. free field realizations of the representations of Kac-Moody group \( LG \) (see Ref. 22 for the discussion of the \( SU(2) \) case). They become essentially chiral vertex operators of the WZW model. As for the matrix elements of \( g_0^{-1} \), we may quantize them by using symbolic calculus for \( SU(2) \) or, in general, by guesswork. They essentially play the role of quantum group vertex operators and may be expressed by the quantum Clebsch-Gordan coefficients. Combination 24 of both should produce quantum field \( g_L(x) \) acting in space 34 and exhibiting \( G \times \mathcal{U}_q(G) \) symmetries and \( R \)-matrix statistics:

\[ g_L(x)_1 g_L(y)_2 R^\pm = g_L(x)_2 g_L(y)_1 , \]

as discussed first in Ref. 10. It is known, however, that the program to construct operators \( g_L(x) \) with such properties (or its counterpart for the minimal models) meets difficulties due to the singular behavior of the quantum Clebsch-Gordan coefficients at integral \( k \) and has not been carried through completely yet. These difficulties seem to go back to the classical singularities of presentation 28 of the chiral phase space \( P_L \) which breaks down for \( \tau \) in Eq. 24 in the boundary of the Weyl alcove. An interesting open question is whether there exists a quantization of \( P_L \) which does not use separation 24 of the degrees of freedom but proceeds directly avoiding its problems.

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