Renormalization group approach to multiple-arc random matrix models

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Abstract

We study critical and universal behaviors of unitary invariant non-gaussian random matrix ensembles within the framework of the large-$N$ renormalization group. For a simple double-well model we find an unstable fixed point and a stable inverse-gaussian fixed point. The former is identified as the critical point of single/double-arc phase transition with a discontinuity of the third derivative of the free energy. The latter signifies a novel universality of large-$N$ correlators other than the usual single arc type. This phase structure is consistent with the universality classification of two-level correlators for multiple-arc models by Ambjørn and Akemann. We also establish the stability of the gaussian fixed point in the multi-coupling model.

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1 Introduction

The universal behaviors in random matrix theories are key ingredients in their level-statistical application, and have been discussed extensively. It is known that there are several universal quantities which do not explicitly depend on the probability distribution of a matrix ensemble. Because of the universality we may calculate such quantities from a simple gaussian ensemble which shares the same symmetries as the physical system in concern. Recently, Ambjørn and Akemann pointed out a universality classification of matrix ensembles with respect to the two-level correlators [1]. The large-$N$ (or “smoothed” [2]) connected two-level correlator depends solely on the support configuration of the level density $\rho(\lambda)$. For example, this correlator in a matrix ensemble with a single-arc level density is identical to that calculated in the gaussian ensemble.

In this letter, we study the phase structure of a non-gaussian matrix ensemble with a multiple-arc level density in terms of the large-$N$ renormalization group [3]. The general algorithm of the large-$N$ renormalization group enables us to find out the exact location of the critical point as a fixed point and to specify the order of the phase transition exactly [4]. Moreover, its renormalization group flow clarifies the global phase structure by classifying the parameter space into phases separated by critical surfaces emanating from unstable fixed points [5]. Along this line, a conjectural account for the branched-polymeric behavior of $c > 1$ bosonic strings was recently provided [6].

We first concern ourselves on the merging phase transition where two energy bands coalesce into a single band. This merging transition has recently attracted attention, for instance, in the application of random matrix theory to the level statistics of QCD Dirac operator [7]; there the level density around $\lambda = 0$ is identified with the chiral condensate of quarks, and thus its vanishing is related to the chiral symmetry restoration at a finite temperature [8]. More specifically, in order to study the phase structure involving different arc configurations, we investigate the simplest non-gaussian model with a $U(N)$ and $\mathbb{Z}_2$-invariant distribution $P(\phi) \propto \exp \left[ -N \text{tr} \left( -\frac{1}{2} \phi^2 + \frac{4}{3} \phi^4 \right) \right]$, which can take either single or double-arc level density depending on the parameter $g$. It is known that at a merging point $g = 1/4$, the third derivative of the free energy has a discontinuity due to the switchover of the functional form of $\rho(\lambda)$ [9]. We shall correctly identify this critical point as an unstable fixed point of the renormalization group, from which the renormalization group flow is emitted. As a straightforward exercise we derive the explicit form of the free energy by integrating the renormalization group equation.

Next we shall discuss the off-critical, universal behavior of the model. It is argued to be determined by the nature around the stable fixed point which is a final destination of the renormalization group flow [4]. To proceed we need to define the approximate $\beta$-function with required properties. The phase diagram described by such a $\beta$-function is in accord with the above mentioned universality
classification by the large-$N$ two-level correlator. Finally we establish the stability of the gaussian point by calculating all the associated scaling exponents in multicoupling models.

2 Renormalization group approach

We consider an ensemble of random $N \times N$ hermitian matrices $\phi$ with a weight

$$P_N(\phi) = Z_N(g_j)^{-1} \exp[-N \, \text{tr} \, V(\phi)], \quad V(\phi) = \sum_{k=2}^{m} \frac{g_k}{k} \phi^k. \quad (1)$$

Hereafter the coupling constant $g_2$ is fixed either to $-1$ (in sect.3, 4, 5) or to $+1$ (in sect.6), and $g_2$-dependences will not be indicated explicitly. The expectation value with respect to the weight is denoted by $\langle \cdots \rangle$. The partition function $Z_N(\phi)$ is defined so that $\langle \mathbf{1} \rangle = 1$. It is written as an integral over eigenvalues

$$Z_N(\phi) = c_N \int \prod_{\ell=1}^{N} d\lambda_{\ell} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^2 \exp \left[-N \sum_{k=1}^{N} V(\lambda_k) \right] \quad (2)$$

after an integration over angular part of $\phi$, which leaves us the volume $c_N$ of $U(N)$. The free energy is defined as

$$F(N, g_j) = -\frac{1}{N^2} \log \left[ \frac{Z_N(\phi)}{Z_N(\phi = 0)|_{g_2=1}} \right] . \quad (3)$$

We briefly recall the renormalization group approach to random matrix models developed in refs.[1, 2, 3]. In the spirit of the renormalization group, we integrate a part of the degrees of freedom, namely the $(N+1)$-th eigenvalue $\lambda_{N+1} \equiv \lambda$, in $Z_{N+1}(g_j)$ in order to relate it to $Z_N(g_j + \delta g_j)$,

$$Z_{N+1}(g_j) = \frac{c_{N+1}}{c_N} \int d\phi \, e^{-(N+1) \, \text{tr} \, V(\phi)} \int d\lambda \, e^{-(N+1) \, V(\lambda) + 2 \, \text{tr} \, \log |\lambda - \phi|}$$

$$= \frac{c_{N+1}}{c_N} Z_N(g_j) \exp \left[-\langle \text{tr} \, V(\phi) \rangle - NV(\langle \lambda_s \rangle) + 2\langle \text{tr} \, \log |\langle \lambda_s \rangle - \phi| \rangle \right] . \quad (4)$$

In the second line, we have evaluated the $\lambda$-integration by the large-$N$ saddle point, and used the factorization property of the correlation functions. It leads to the saddle point equation for the expectation value $\langle \lambda_s(g_j, \phi) \rangle$:

$$V'(\langle \lambda_s \rangle) = 2 \left\langle \frac{1}{N} \, \text{tr} \, \frac{1}{\langle \lambda_s \rangle - \phi} \right\rangle . \quad (5)$$

By the use of the loop (Schwinger-Dyson) equation, eq.(5) is shown to be equivalent to the condition

$$\rho(\langle \lambda_s \rangle) = 0, \quad \left( \rho(\lambda) := -\frac{1}{\pi} \text{Im} \left\langle \frac{1}{N} \, \text{tr} \, \frac{1}{\lambda + i\epsilon - \phi} \right\rangle \right) . \quad (6)$$
which means that the \((N+1)\)-th eigenvalue settles down to one of the edges of the arcs of \(\rho(\lambda)\). Expanding eq. (4) into a series in \(1/N\), we have the renormalization group equation obeyed by the free energy:

\[
\left[N \frac{\partial}{\partial N} + 2\right] F(N, g_j) = -\frac{3}{2} + \langle \frac{1}{N} \text{tr} V(\phi) \rangle + V(\langle \lambda_s \rangle) - 2 \left( \frac{1}{N} \text{tr} \log |\langle \lambda_s \rangle - \phi| \right) + O\left(\frac{1}{N}\right)
\]

\[= G\left(g_j, \frac{\partial F}{\partial g_j}\right). \tag{7}\]

In the right hand side the coupling constants \(g_3, \ldots, g_m\) and the one-point functions \(a_j := \frac{\partial F}{\partial g_j} = \frac{1}{N} \langle \text{tr} \phi^j \rangle\) \((3 \leq j \leq m)\) are regarded as independent variables. Other one-point functions appearing in the right hand side are understood as functions of \(g_j\) and \(a_j\) \((3 \leq j \leq m)\) via the Schwinger-Dyson equations. In the same way, \(\langle \lambda_s \rangle\) is also regarded as a function of \(g_j, a_j\) \((3 \leq j \leq m)\). The explicit form of \(G\) is given by

\[G(g_j, a_j) = -1 + \sum_{j=3}^{m} \left( 1 - \frac{j}{2} \right) g_j a_j + V(\langle \lambda_s \rangle) \tag{8}\]

\[-2 \int_{\pm \infty}^{\langle \lambda_s \rangle} dz \left( \langle \frac{1}{N} \text{tr} \frac{1}{z - \phi} \rangle - \frac{1}{z} \right) - 2 \log \langle \lambda_s \rangle.\]

Let us turn to the process of extracting singular behavior of \(F\) from the renormalization group equation (7). For simplicity, we consider the case with a single coupling constant \(g\). The extension to multi-coupling constant case is straightforward. We assume that the free energy in the large-\(N\) limit has a fractional power singularity

\[F(g, N) = \sum_{k=0}^{\infty} a_k (g - g^*)^k + \sum_{k=0}^{\infty} b_k (g - g^*)^{k+\gamma} + O(N^{-2}) \quad (\gamma \in \mathbb{R}_+ \setminus \mathbb{Z}, b_0 \neq 0). \tag{9}\]

The critical point \(g^*\), critical exponent \(\gamma\) and coefficients \(a_1, a_2\) are determined by solving the following closed set of equations

\[0 = G_{,a}(g^*, a_1) \tag{10a}\]

\[2 = G_{,g,a}(g^*, a_1) + 2a_2 G_{,a,a}(g^*, a_1) \tag{10b}\]

\[2a_1 = G_{,g}(g^*, a_1) \tag{10c}\]

\[2a_2 = \frac{1}{2} G_{,g,g}(g^*, a_1) + 2a_2 G_{,g,a}(g^*, a_1) + 2(a_2)^2 G_{,a,a}(g^*, a_1) \tag{10d}\]

where \(,g\) and \(,a\) denote partial derivatives. These condition can easily be established by substituting (7) into the large-\(N\) renormalization group equation

\[2F(N, g) = G\left(g, \frac{dF}{dg}\right). \tag{11}\]
By comparing coefficients of \((g - g_*)^k\) and \((g - g_*)^{k+\gamma}\) in the both sides we obtain a family of equations for \(a_j, b_j, g_*, \) and \(\gamma\). The coefficients of \((g - g_*)^{k+\gamma}\) contain both \(a_j\) and \(b_j\), whereas those of \((g - g_*)^k\) do only \(a_j\) (thus they can be solved for \(a_j\)). Since these equations are homogeneous in \(b_0\) the overall normalization of the singular part is left undetermined.

The above procedure can as well be used to detect the discontinuity of a derivative of the free energy. We now assume that the \(\gamma\)-th derivative of the free energy is discontinuous:

\[
F(g, N) = \sum_{k=0}^{\infty} a_k (g - g_*)^k + \sum_{k=0}^{\infty} b_k (g - g_*)^{k+\gamma} \text{sgn}(g - g_*) + O(N^{-2}) \quad (12)
\]

By substituting \((12)\) into the large-\(N\) renormalization group equation and comparing coefficients of \((g - g_*)^k\) and \((g - g_*)^{k+\gamma} \text{sgn}(g - g_*)\), we can easily show that eqs.\((10)\) hold.

3 Merging transition fixed point

The simplest model which undergoes the merging transition is the one with quartic potential with negative quadratic term \[9\]

\[
V(\phi) = -\frac{1}{2} \phi^2 + \frac{g}{4} \phi^4. \tag{13}
\]

Though a parameter \(g_3\) is missing compared to \(\square\), we can set \(g_{\text{odd}} = a_{\text{odd}} = 0\) consistently because of the \(\mathbb{Z}_2\) symmetry \[\square\].

We present the explicit form of the \(G\)-function:

\[
G(g, a) = -1 - ga - \frac{1}{2} \langle \lambda_s \rangle^2 + \frac{g}{4} \langle \lambda_s \rangle^4 - 2 \log \langle \lambda_s \rangle - \int_{\pm \infty} \langle \lambda_s \rangle dz \left( -z + g z^3 - \sqrt{(-z + g z^3)^2 - 4(-1 + g z^2 - g + 4g^2a) - \frac{2}{z}} \right) \quad (14)
\]

which is supplemented by the saddle point equation:

\[
(-\langle \lambda_s \rangle + g \langle \lambda_s \rangle^3)^2 - 4(-1 + g \langle \lambda_s \rangle^2 - g + 4g^2a) = 0 \tag{15}
\]

determining \(\langle \lambda_s \rangle\) as a function of \(g\) and \(a\).

\[\square\]

So far we have implicitly assumed that the powers of the singular parts do not contribute to the regular parts. It is true for \(\gamma \notin \mathbb{Q}\) or \(\gamma = p/q\) with \(p - q \geq 3\) and \(p, q\) are co-prime (fractional power singularity) and for \(\gamma \geq 3\) (discontinuity).

\[\square\]

Here we have excluded the possibility of spontaneous breaking of the \(\mathbb{Z}_2\) symmetry \[\square\]. The linearized RG flow for the broken \(\mathbb{Z}_2\) phase is readily found in Fig.3 of ref.\[\square\] by restricting it onto the curve \(2g_3^2 - 9g_4 = 0\).
One can show analytically that $g_* = 1/4, \gamma = 3, a_1 = 5, a_2 = -18$ solves eqs.(10). The integer value of the $\gamma$ exponent suggests that around $g_* = 1/4$ the free energy is of the latter form (12) and has a discontinuity in its third derivative. This is in accord with the large-$N$ solution found in [9]; at $g \downarrow 1/4$, the double-well potential becomes well separated so that the level density splits into two disconnected arcs. We stress that our renormalization group equation can actually detect not only the fractional power behavior of $F(g)$ but also its discontinuity due to the switchover of the functional form of $\rho(\lambda)$.

The readers should notice that, in the previous procedure we have not made use of any kinds of multiple-arc ansatz, which is used in obtaining large-$N$ solutions [10]. If we were interested only in a particular arc configuration, we could assume a factorized form of the discriminant of the square root in the $G$-function (14). However our method, which uses the discriminant containing a variable $a$ as it is, can represent any possible arc configurations. Thus by exploring all solutions to eqs.(10) in the two-parameter space $(g, a)$, we can find solutions with arbitrary arc configurations respecting the $Z_2$ symmetry.

4 Free energy

To obtain the free energy, one usually reads off the level density $\rho(\lambda)$ from the expectation value of the resolvent and then takes the average of $V(\lambda) - \mathcal{P} \log |\lambda - \lambda'|$ with respect to it [10]. Here we show that the explicit form of the free energy can as well be obtained by integrating the renormalization group equation (7). Our calculation avoids computing cumbersome principal-part integration.

For this practical use, it suffices to choose a particular arc configuration from the outset and to use the fact that the saddle point $\langle \lambda_s \rangle$ falls on an edge of the arcs due to the level repulsion. Then $\langle \lambda_a \rangle$ is expressed as a function of $g$ only and the renormalization group equation takes a simple linear form.

In $g > g_*$ case, assuming a single arc configuration at $[-A, A]$ and the asymptotics $N^{-1} \langle \text{tr} (z - \phi)^{-1} \rangle \sim 1/z \ (|z| \to \infty)$, we obtain

$$\left\langle \frac{1}{N} \text{tr} \frac{1}{z - \phi} \right\rangle = \frac{1}{2} \left[-z + gz^3 - \left(-1 + \frac{g}{2} A^2 + g z^2\right) \sqrt{z^2 - A^2}\right].$$ (16)

where $A^2 = \frac{2}{3g} (1 + \sqrt{1 + 12g})$. Knowing that the saddle point is given by the edge of the arc $\langle \lambda_a \rangle = \pm A$, the $G$-function (14) then takes the form:

$$G \left(g, \frac{dF}{dg}\right) = -g \frac{dF}{dg} - \frac{A^2}{2} - \frac{A^2}{8} - \log \frac{A^2}{4}.$$ (17)

In the leading order of $1/N$, the renormalization group equation (7) becomes

$$2F(g) = G \left(g, \frac{dF}{dg}\right).$$ (18)
The solution to this differential equation with the G-function (17) is
\[ F(g) = -\frac{3}{8} - \frac{5A^2}{48} - \frac{A^4}{384} - \frac{1}{2} \log \frac{A^2}{4} + \frac{C_1}{g^2}, \]  
where \( C_1 \) is an integration constant.

In \( g < g_* \) case, we assume the double arc configuration at \([-A_2, -A_1] \cup [A_1, A_2]\),

\[ \left\langle \frac{1}{N} \text{tr} \frac{1}{z - \phi} \right\rangle = \frac{1}{2} \left( -z + g z^3 - g z \sqrt{(z^2 - A_1^2)(z^2 - A_2^2)} \right), \]  
where \( A_1^2 = \frac{(1 - 2\sqrt{g})}{g} \) and \( A_2^2 = \frac{(1 + 2\sqrt{g})}{g} \). By requiring the continuity from the \( g > g_* \) case, we are obliged to choose the outer edge \( \pm A_2 \) as the relevant saddle point \( \langle \lambda_s \rangle \). Then the G-function (14) takes the form:
\[ G \left( g, \frac{dF}{dg} \right) = -g \frac{dF}{dg} - \frac{1}{2} + \frac{1}{2} \log g - \frac{1}{4g}. \]  
The solution to the renormalization group equation (18) with this G-function becomes
\[ F(g) = -\frac{3}{8} + \frac{1}{4} \log g - \frac{1}{4g} + \frac{C_2}{g^2} \]  
with an integration constant \( C_2 \). The integration constants \( C_1 \) and \( C_2 \) must be the same because of the continuity of \( F(g) \) at \( g = g_* \). On the other hand, \( C_2 \) must vanish since the free energy should be dominated by the minima of the potential in the small-\( g \) limit,
\[ F(g) \sim V \left( \pm \frac{1}{\sqrt{g}} \right) = -\frac{1}{4g} \quad (g \searrow 0). \]  
This result (19) and (22) with \( C_1 = C_2 = 0 \) is identical to that obtained in ref.\[9\] using BIPZ method.

5 Inverse-gaussian fixed point

The concept of flow is indispensable in the renormalization group analysis. However, the nonlinearity of our exact renormalization group equation (7) with respect to \( \partial F / \partial g \) makes an idea of flow ambiguous. Thus we wish to define an approximate flow which visualizes the phase structure correctly. Indeed there is such a sensible definition of flow.

We define an approximate \( \beta \)-function expanded around \( \bar{g} \) as
\[ \beta(g) := \left. \frac{\partial G(g, a)}{\partial a} \right|_{a = a_1 + 2a_2(g - \bar{g})}, \]  
\[ \text{See its free energies err: } E_B \text{ misses a constant } \frac{1}{2} \log 2; \ E_C \text{ should read (in our convention (13))} \]
\[ -\frac{3}{8} + \frac{-1 + \sqrt{1 - 15g}}{675 g^2} + \frac{-33 - 10 \sqrt{1 - 15g}}{180 g} - \frac{1}{2} \log \frac{1 - \sqrt{1 - 15g}}{15 g}. \]
where $\bar{a}_1 = \frac{dE}{dg}(\bar{g})$, $\bar{a}_2 = \frac{1}{2} \frac{d^2 E}{d^2 g}(\bar{g})$, and shall check the validity of this definition. Let us first note that

$$\beta^{\text{exact}}(g) := \frac{\partial G(g, a)}{\partial a} \bigg|_{a = 4\frac{dE}{dg}(g)}$$

is the ‘exact’ beta function in the sense that the free energy $F$ obeys a linear equation

$$\left[ N \frac{\partial}{\partial N} - \beta^{\text{exact}}(g) \frac{\partial}{\partial g} + 2 \right] F(N, g) = r(g). \quad (26)$$

The above ‘exact’ beta function is practically useless because it requires the full information of the free energy. It is desired to extract correct qualitative information (the phase structure of the model or the universality classification of phase transitions) by the use of the renormalization group equation (7) in case the exact form of the free energy is not available. On the other hand, the approximated $\beta$-function (24) is obtained by replacing $dF/dg$ with its truncated expansion around $\bar{g}$. We can determine the coefficients $a_1$ and $a_2$ by solving (10) even if full information of $F$ is not available. Thus we can calculate $\beta(g)$ for arbitrary $g$ only if eqs.(10) are solved at a single point $\bar{g}$.

Since we have kept the linear term in $g$ in the power series expansion of $dF/dg$, the zeroth and the first derivatives of $G$ at $g = \bar{g}$ are exactly reproduced. Therefore $\beta(g)$ enjoys the following property$^\S$: If $\bar{g}$ is a fixed point of the flow (i.e. solves eqs.(10)), then

$$\beta(\bar{g}) = 0, \quad (27a)$$

$$\frac{\partial \beta}{\partial g}(\bar{g}) = \frac{2}{\gamma}. \quad (27b)$$

Even if $\bar{g}$ does not solve the equations (10), the definition (24) makes sense and $\beta(g)$ approximates the character of the renormalization group transformation in the vicinity of $\bar{g}$. However, expansion around such a $\bar{g}$ is not very useful because of the lack of algorithm to determine $a_1(\bar{g})$.

The definition (24) is easily extended to multi-coupling case as:

$$\beta_k(g_j) := \frac{\partial G(g_j, a_j)}{\partial a_k} \bigg|_{a_j = \bar{a}_j + 2 \sum \bar{a}_{j\ell}(g_{\ell} - \bar{g}_{\ell})}$$

where $\bar{a}_j = a_j(\bar{g}_k)$. Notations $a_j$ and $a_{j\ell}$ are those used in ref.[5].

In Fig.10, the $\beta$-function for the model (13) expanded around the merging fixed point $g_\ast = 1/4$ is plotted. We observe that the approximate $\beta$-function is smooth

$^\S$Previously we have proposed in [6] a linearized $\beta$-function

$$\beta^{\text{linear}}(g) := \frac{\partial G}{\partial a} \bigg|_{a = \bar{a}_1}.$$
everywhere even though the matrix integral is ill-defined for $g \leq 0$. It vanishes at the very vicinity of the origin with a negative exponent. This indicates that the origin is an attractive fixed point. The level density becomes two infinitely separated semi-circles when approaching this point.

It is argued by Brézin and Zee [2], although not rigorously proven, that the universality of the large-$N$ connected two-level correlator be attributed to the presence of attractive gaussian fixed point of the renormalization group transformation. If we accept this argument, then the attractive inverse-gaussian fixed point in the model (13) plays the same role for the new universality for the multiple-arc matrix model recently found in ref. [1].

6 Stability of the gaussian fixed point

To strengthen our assertion that each fixed point corresponds to a universality class of large-$N$ two-level correlators, we present here the analysis of stability of the gaussian potential in the model

\[
V(\phi) = \sum_{k=2}^{\infty} \frac{g_k}{k} \phi^k, \quad g_2 = +1.
\]  

\[
\text{(29)}
\]

\footnote{A renormalization group analysis of the stability of the gaussian potential has been done by Y. Morita et al. (Phys. Rev. B52 (1995) 4716) based on mapping from matrix models to vector models. They have imposed an arbitrary restriction to a smaller subspace of coupling constants which is inconsistent with the renormalization group flow, yielding incorrect values of the exponents.}
We evaluate the derivatives of the $G$-function (8) at the gaussian point $g_k = 0$ ($k = 3, 4, \cdots$). We immediately find that

$$\left. \frac{\partial}{\partial a} \ell \langle \lambda_a \rangle \right|_{g_j=0} = 0$$

implying

$$G_{a_i}(g_j = 0, a_n) = G_{a_i, a_k}(g_j = 0, a_n) = 0$$

for arbitrary $a_n$. Thus the gaussian point is always a fixed point.

To obtain the exponents on the gaussian fixed point, we need to calculate $G_{g_k a_\ell}$. The second term in (8) contributes $(1 - k/2)\delta_{\ell k}$ to it. It can also be shown that

$$\left. \frac{\partial^2}{\partial g_k \partial a_\ell} \left( \frac{1}{N} \text{tr} \frac{1}{z - \phi} \right) \right|_{g_j=0} = 0 \quad \text{if} \quad \ell + 2 > k.$$  

Therefore the scaling exponent matrix $G_{\ell k} := G_{g_k a_\ell} + 2a_k r G_{a_\ell, a_r}$ is upper triangular with eigenvalues

$$G_{kk}(0, 0, \ldots) = 1 - \frac{k}{2} \quad (k = 3, 4, \cdots).$$

The negativity of the exponents indicates the stability of the gaussian fixed point in the entire coupling constant space (29) in accord with the universality of the large-$N$ two-level correlator.

In the $g_2 = -1$ case, the model at the inverse-gaussian point is ill-defined and therefore the general algorithm cannot extract any critical exponents around this point. Nevertheless, the model is well-defined except at the inverse-gaussian point. The inverse-gaussian point is a fixed point in the sense of the limiting procedure

$$\lim_{g \to 0} G_{a}(g, a) = 0.$$  

The stability of this fixed point is guaranteed by the negative definiteness of the extended $\beta$-function $G_{a}$ for $0 < g < \frac{1}{4}$.

7 Summary

This letter is devoted mainly to the application of the method of large-$N$ renormalization group to a matrix model with a double-well potential. The advantage of our method is that the arc configuration need not be specified from the outset, but rather a single renormalization group equation (14) (and its approximated $\beta$-function) encompasses all possible arc configurations. As a direct consequence of it, we have extracted exact values of the critical (fixed) point and exponent associated with the splitting/merging of the arc(s). Furthermore, by the use of approximate $\beta$-function we have found a novel inverse-gaussian fixed point, whose
stability may guarantee the large-$N$ universality of double-arc matrix ensemble. We have also calculated the scaling exponents around the gaussian fixed point for a matrix model with a generic potential. Their negativity, i.e. the stability of the gaussian point against perturbation of potential preserving the single-arc configuration, supports the large-$N$ universality of (usual single arc) matrix ensembles.

We wish to utilize our results to the random matrix analysis of disordered physical systems.

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