HIGHER AFFINE CONNECTIONS

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ABSTRACT. For a smooth manifold $M$, it was shown in [2] that every affine connection on the tangent bundle $T M$ naturally gives rise to covariant differentiation of multivector fields (MVFs) and differential forms along MVFs. In this paper, we generalize the covariant derivative of [2] and construct covariant derivatives along MVFs which are not induced by affine connections on $T M$. We call this more general class of covariant derivatives higher affine connections. Related notions of higher torsion and higher curvature are considered also.

1. Introduction

Let $M$ be a manifold. It was shown in [2] that every affine connection $\nabla$ on the tangent bundle $T M$ naturally gives rise to covariant differentiation of multivector fields (MVFs) and differential forms along MVFs. For covariant differentiation of MVFs along MVFs, the covariant derivative of [2] (which we will again denote as $\nabla$) satisfies

$$\nabla_{X \wedge Y} Z = (-1)^k X \wedge \nabla_Y Z + (-1)^{(k-1)} Y \wedge \nabla_X Z,$$

(1.1)

for $X \in \Gamma(\wedge^k TM), Y \in \Gamma(\wedge^l TM), \text{ and } Z \in \Gamma(\wedge^r TM)$. Any covariant derivative of MVFs along MVFs which satisfies (1.1) is necessarily induced by an affine connection on the tangent bundle. In this paper, we consider a class of covariant derivatives of MVFs along MVFs which satisfies all the main properties of those of [2] except possibly (1.1). We call such a covariant derivative a higher affine connection. Hence, the covariant differentiation of MVFs along MVFs from [2] can be seen as a special case of a higher affine connection.

For covariant differentiation of differential forms along MVFs, the covariant derivative of [2] has several nice properties which are summarized in Theorem 4.2 of [2]. To describe the approach of [2], we start with a decomposable $k$-vector field $X = X_1 \wedge \cdots \wedge X_k$ and an affine connection $\nabla$ on $T M$. For a differential form $\omega \in \Omega^\bullet(M)$, we define

$$\nabla_X \omega = \sum_{j=1}^k (-1)^{j+1} i_{X_1 \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_k} (\nabla_{X_j} \omega),$$

(1.2)

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where \( i_W \) is the right interior product defined by a MVF \( W \). Since \( \nabla \) is local in nature and every \( k \)-vector field is locally a finite sum of decomposable \( k \)-vector fields, the above definition extends to all \( X \in \Gamma(\wedge^k T M) \) by \( C^\infty(M) \)-linearity. Note that if \( \omega \) is an \( l \)-form and \( X \) a \( k \)-vector field, then \( \nabla X \omega \) is an \((l - k + 1)\)-form.

Since a higher connection is not completely defined by an affine connection on \( TM \), (1.2) is not directly applicable to higher connections. Even so, we show that higher connections do indeed allow for covariant differentiation of differential forms along MVFs. Moreover, the result has properties similar to those of [2].

The motivation for discarding condition (1.1) stems from two sources: generalized geometry [5] [6] and the basic idea at the root of string theory [4] [11] [12]. In generalized geometry, one replaces the tangent bundle with \( TM \oplus T^*M \) and the Lie bracket with the Courant bracket. In this setting, one looks for geometric structures on \( TM \oplus T^*M \) which are analogues of the familiar objects one encounters in differential geometry. In string theory, the notion of a point particle is replaced by one dimensional extended objects called strings. As a consequence of this, a worldline (i.e., the path that a point particle makes through spacetime) is generalized to a 2-dimensional worldsheet (i.e., the surface that a string sweeps out as it propagates). Let \( X : \Sigma \to M \) be a worldsheet map, where \( \Sigma \subset \mathbb{R}^2 \) has coordinates \((\tau, \sigma)\), and let \( p \) be a point on \( X \). One can think of \( X \) as a higher dimensional worldline with the following “tangent vectors” at \( p \):

\[
\partial_{\tau}X|_{(\tau_0, \sigma_0)}, \quad \partial_{\sigma}X|_{(\tau_0, \sigma_0)}, \quad (\partial_{\tau}X \wedge \partial_{\sigma}X)|_{(\tau_0, \sigma_0)},
\]

where \( X(\tau_0, \sigma_0) = p \) and \( \partial_{\tau}X := \frac{\partial X}{\partial \tau}, \quad \partial_{\sigma}X := \frac{\partial X}{\partial \sigma} \). In doing so, one regards \( \wedge^2 T M \) as part of an extended tangent bundle.

For MVFs, the natural analogue of the Lie bracket is the Schouten-Nijenhuis bracket (SNB) [9] [10]. The SNB of two multivector fields of degrees \( k \) and \( l \) is a multivector field of degree \( k + l - 1 \). With the SNB in the role of the Lie bracket, the idea of generalized geometry suggests that we must consider the full exterior bundle \( \wedge^* TM \) in the role of the tangent bundle as opposed to just \( TM \oplus \wedge^2 TM \). Our basic “philosophy” then is to treat \( \wedge^k T_p M \) (\( p \in M, \; k \geq 2 \)) as part of an extended tangent space on \( M \). Hence, a \( k \)-vector \( v \in \wedge^k T_p M \) should be regarded as a new kind of tangent vector. If we apply this viewpoint to the problem of covariant differentiation of MVFs along MVFs, one would expect \( X \wedge Y \) to play a more explicit role on the right side of (1.1). However, for this to be true, condition (1.1) must be discarded; the upshot of this is the notion of a higher affine connection.

The rest of the paper is organized as follows. In section 2, we review the basic machinery of MVFs and set up the notation we will use for the rest of the paper. In section 3, we introduce the notion of higher affine connections and prove a classification theorem for them (see Theorem 3.8). In addition, the notion of higher torsion is also introduced. In section 4, we define the covariant derivative of differential forms along MVFs in terms of higher
connections and examine the properties of this construction. In section 5, we propose two notions of curvature for higher connections. The first type, which we call the full higher curvature, is modeled after the classical curvature for an affine connection. However, unlike the classical case, the full higher curvature is not $C^\infty(M)$-linear in its arguments. The second type, which we call the stratified higher curvature, is $C^\infty(M)$-linear and takes advantage of the classification theorem for higher connections. In addition, we also show that the two notions of higher curvature are related to one another. Finally, in section 6, we conclude the paper with some open questions.

2. Preliminaries

Let $M$ be a smooth manifold. To simplify notation, set $A^k(M) := \Gamma(\wedge^k T M)$, $A(M) := \bigoplus_{k=0}^{\infty} A^k(M)$, and $A'(M) := \bigoplus_{k=1}^{\infty} A^k(M)$, where for a vector bundle $E \to M$, $\Gamma(E)$ denotes the space of sections of $E$. The space of $k$-forms on $M$ is denoted as $\Omega^k(M)$. For convenience, we set $A^k(M) = 0$ and $\Omega^k(M) = 0$ for $k < 0$.

Definition 2.1. Let $k \in \mathbb{N}$. A multiderivation of degree $k$ (or $k$-derivation) on a manifold $M$ is a $k$-linear map

$$\varphi : C^\infty(M) \times \cdots \times C^\infty(M) \to C^\infty(M)$$

over $\mathbb{R}$, which is totally antisymmetric and a derivation of $C^\infty(M)$ in each of its arguments, i.e.

(i) $\varphi(f_{\sigma(1)}, \ldots, f_{\sigma(k)}) = \text{sgn}(\sigma) \varphi(f_1, \ldots, f_k) \forall \sigma \in \mathfrak{S}_k$

(ii) $\varphi(f_1 g, f_2, \ldots, f_k) = \varphi(f_1, \ldots, f_k) g + f_1 \varphi(g, f_2, \ldots, f_k)$

for all $f_i, g \in C^\infty(M)$.

The next two results are well known in differential geometry and we state them without proof.

Proposition 2.2. Every $k$-vector field $X \in A^k(M)$ defines a $k$-derivation via

$$X(f_1, \ldots, f_k) := (df_1 \wedge \cdots \wedge df_k)(X). \quad (2.1)$$

Proposition 2.3. There is a one-one correspondence between the space of $k$-derivations and the space of $k$-vector fields. Specifically, every $k$-derivation $\varphi$ is given by

$$\varphi(f_1, \ldots, f_k) = X(f_1, \ldots, f_k), \quad f_i \in C^\infty(M), \; i = 1, \ldots, k,$n

for some unique $X \in A^k(M)$.

Definition 2.4. The Schouten-Nijenhuis bracket of multivector fields is the unique $\mathbb{R}$-bilinear map

$$[\cdot, \cdot] : A^k(M) \times A^l(M) \to A^{k+l-1}(M)$$

\footnote{1}{By manifold, we always mean one that is smooth, Hausdorff, and second countable.}

\footnote{2}{see Proposition 3.1 of [9]}
which satisfies the following conditions:

(i) For \( f, g \in \mathcal{C}^\infty(M) \), \([f, g] = 0\) for \( f, g \in \mathcal{C}^\infty(M) \), \([f, g] = 0\) is the Lie derivative of \( g \) with respect to \( f \).

(ii) For \( f, g \in \mathcal{A}(M) \), \([X, Q] = L_X Q\), the Lie derivative of \( Q \) with respect to \( X \).

(iii) For \( P, Q, X \in \mathcal{A}(M) \), \([P, Q] = (-1)^{(p-1)(q-1)} [Q, P] = (-1)^{(p-1)(q-1)}\) is a derivation of degree \( p - 1 \) for \( P \in \mathcal{A}^p(M) \) of the exterior product on \( \mathcal{A}(M) \), that is,

\[
\text{ad}_P(q \wedge R) = \text{ad}_P(q) \wedge R + (-1)^{(p-1)q} q \wedge \text{ad}_P(R)
\]

for \( q, R \in \mathcal{A}^q(M), \ R \in \mathcal{A}(M) \).

Definition 2.4 implies that \([9]\)

\[
(-1)^{(p-1)(r-1)} [P, [Q, R]] + (-1)^{(q-1)(p-1)} [Q, [R, P]]
\]

\[
+ (-1)^{(r-1)(q-1)} [R, [P, Q]] = 0
\]

for \( P \in \mathcal{A}^p(M), Q \in \mathcal{A}^q(M), \) and \( R \in \mathcal{A}^r(M) \). This together with (iii) of Definition 2.4 shows that \((\mathcal{A}(M), [\cdot, \cdot])\) is a graded Lie algebra if \( \text{deg} \mathcal{A}^p(M) := p - 1 \).

For \( X_1, X_2, \ldots, X_p, Y_1, \ldots, Y_q \in \mathcal{A}^1(M) \), the Schouten-Nijenhuis bracket is given explicitly by

\[
[X_1 \wedge \cdots \wedge X_p, Y_1 \wedge \cdots \wedge Y_q] = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_p \wedge Y_1 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_q.
\]

**Definition 2.5.** The interior product of a smooth function \( f \) with a \( k \)-derivation \( X \in \mathcal{A}^k(M) \) \((k \geq 1)\) is the \( k - 1 \)-derivation \( i_fX \) defined by

\[
i_fX(g_1, \ldots, g_{k-1}) := X(f, g_1, \ldots, g_{k-1}),
\]

for \( g_1, \ldots, g_{k-1} \in \mathcal{C}^\infty(M) \). For \( g \in \mathcal{A}^0(M) := \mathcal{C}^\infty(M) \), \( i_f g := 0 \).

**Proposition 2.6.** Let \( f, g \in \mathcal{C}^\infty(M), \ X \in \mathcal{A}^k(M), \) and \( Y, Y' \in \mathcal{A}(M) \). Then

(i) \( i_{f+g}X = i_fX + i_gX \)

(ii) \( i_f(Y + Y') = i_fY + i_fY' \)

(iii) \( i_{fg}X = gi_fX + fi_gX \)

(iv) \( i_f(X \wedge Y) = (i_fX) \wedge Y + (-1)^k X \wedge (i_f Y) \)

(v) \( [X, f] = (-1)^{k-1} i_fX \)

(vi) \( [fX, Y] = f[X, Y] - X \wedge i_fY \)

**Proof.** (i) and (ii) are immediate.

Note that (iii)-(v) is satisfied for the case when \( X \) is a smooth function. We now prove (iii)-(v) for the case when \( X \in \mathcal{A}^k(M) \) with \( k \geq 1 \).
(iii) is a direct consequence of the fact that $X$ is a derivation of $C^\infty(M)$ in each of its arguments. Specifically,

$$i_{fg}X := X(fg, \ldots, \cdot)$$

$$= X(f_1, \ldots, \cdot)g + fX(g, \ldots, \cdot)$$

$$= gi_fX + f i_gX.$$ 

For (iv), take $Y \in A^l(M)$ without loss of generality and set $g_1 := f$. Then

$$i_f(X \wedge Y)(g_2, \ldots, g_{k+1}) = (X \wedge Y)(g_1, g_2, \ldots, g_{k+1})$$

$$= \sum_{\sigma \in S(k,l)} \epsilon(\sigma)X(g_{\sigma(1)}, \ldots, g_{\sigma(k)})Y(g_{\sigma(k+1)}, \ldots, g_{\sigma(k+l)}),$$

(2.2)

where $\sigma \in S(k, l) \subset \mathcal{G}_{k+l}$ is a shuffle permutation, i.e., $\sigma$ satisfies

$$\sigma(1) < \cdots < \sigma(k), \quad \sigma(k + 1) < \cdots < \sigma(k + l)$$

and $\epsilon(\sigma) = +1$ ($-1$) if $\sigma$ is even (odd). (2.2) can be decomposed as

$$\sum_{\sigma \in S(k,l), \sigma(1)=1} \epsilon(\sigma)X(g_{\sigma(1)}, \ldots, g_{\sigma(k)})Y(g_{\sigma(k+1)}, \ldots, g_{\sigma(k+l)})$$

$$+ \sum_{\sigma \in S(k,l), \sigma(k+1)=1} \epsilon(\sigma)X(g_{\sigma(1)}, \ldots, g_{\sigma(k)})Y(g_{\sigma(k+1)}, \ldots, g_{\sigma(k+l)})$$

$$= \sum_{\sigma \in S(k-1,l)} \epsilon(\sigma)i_fX(\tilde{g}_{\sigma(1)}, \ldots, \tilde{g}_{\sigma(k-1)})Y(\tilde{g}_{\sigma(k)}, \ldots, \tilde{g}_{\sigma(l+k-1)})$$

$$+ (-1)^{kl} \sum_{\sigma \in S(l-1,k)} \epsilon(\sigma)i_fY(\tilde{g}_{\sigma(1)}, \ldots, \tilde{g}_{\sigma(l-1)})X(\tilde{g}_{\sigma(l)}, \ldots, \tilde{g}_{\sigma(k+l-1)}),$$

(2.3)

where $\tilde{g}_i = g_{i+1}$ for $i = 1, \ldots, k + l - 1$. (2.3) can then be rewritten as

$$(i_fX \wedge Y)(g_2, \ldots, g_{k+1}) + (-1)^{kl} (i_fY \wedge X)(g_2, \ldots, g_{k+l})$$

$$= (i_fX \wedge Y)(g_2, \ldots, g_{k+1}) + (-1)^{kl+k(l-1)} (X \wedge i_fY)(g_2, \ldots, g_{k+l})$$

$$= (i_fX \wedge Y)(g_2, \ldots, g_{k+1}) + (-1)^k (X \wedge i_fY)(g_2, \ldots, g_{k+l}).$$

For (v), let $X \in A^k(M)$. For $k = 0$, the result follows from Definition 2.4 and Definition 2.5. Now consider the case when $k \geq 1$. Condition (iv) of Definition 2.4 implies that the Schouten-Nijenhuis bracket is local in nature. Since $X$ is locally a finite sum of decomposable terms, it suffices to prove (v) of Proposition 2.6 for the case when $X$ is decomposable, i.e., $X = X_1 \wedge \cdots \wedge X_k$ where $X_i \in A^1(M)$ for $i = 1, \ldots, k$. For $k = 1$, $X \in A^1(M)$ and

$$[X, f] = L_X f = X f = i_fX = (-1)^{k-1}i_fX,$$

by (ii) of Definition 2.4. We now prove (v) by induction on $k$. Suppose then that (v) holds for $k$ (where $k \geq 1$) and let $X_{k+1} \in A^1(M)$. By (iii) and (iv)
of Definition 2.4, we have

\[
[X \wedge X_{k+1}, f] = (-1)^{k+1} [f, X \wedge X_{k+1}]
\]

\[
= (-1)^{k+1} ([f, X] \wedge X_{k+1} + (-1)^k X \wedge [f, X_{k+1}])
\]

\[
= (-1)^{k+1} ((-1)^k [X, f] \wedge X_{k+1} + (-1)^{k+1} X \wedge [X_{k+1}, f])
\]

\[
= (-1)^{k+1} (-i_f X \wedge X_{k+1} + (-1)^{k+1} X \wedge i_f X_{k+1})
\]

\[
= (-1)^k (i_f X \wedge X_{k+1} + (-1)^k X \wedge i_f X_{k+1})
\]

\[
= (-1)^k i_f (X \wedge X_{k+1}),
\]

where we used the induction hypothesis in the fourth equality and (iv) of Proposition 2.6 in the sixth equality.

For (vi), take \(Y \in A^l(M)\) without loss of generality. Then

\[
[fX, Y] = [f \wedge X, Y]
\]

\[
= -(-1)^{(k-1)(l-1)} [Y, f \wedge X]
\]

\[
= -(-1)^{(k-1)(l-1)} ([Y, f] \wedge X + f \wedge [Y, X])
\]

\[
= -(-1)^{(k-1)(l-1)} ((-1)^{(l-1)} i_f Y \wedge X + f[Y, X])
\]

\[
= f[X, Y] - (-1)^{(l-1)} i_f Y \wedge X
\]

\[
= f[X, Y] - X \wedge i_f Y,
\]

where we used (iii) and (iv) of Definition 2.4 in the second and third equalities respectively, and (v) of Proposition 2.6 was used in the fourth equality. This completes the proof. \(\square\)

**Definition 2.7.** Let \(X \in A^k(M), k \geq 1\). The **right interior product** by \(X\) \([9]\) is the \(C^\infty(M)\)-linear map \(i_X : \Omega^l(M) \to \Omega^{l-k}(M)\) defined by

\[
i_X \omega(Y) := \omega(X \wedge Y), \quad \forall Y \in A^{l-k}(M).
\]

**Remark 2.8.** For \(X \in A^1(M)\), one can show that \(i_X\) is a derivation of degree \(-1\), that is,

\[
i_X(\omega \wedge \eta) = (i_X \omega) \wedge \eta + (-1)^l \omega \wedge i_X \eta,
\]

for \(\omega \in \Omega^l(M), \eta \in \Omega^\bullet(M)\).

The right interior product is extended to all of \(A(M)\) by defining \(i_f \omega := f \omega\) for \(f \in A^0(M) := C^\infty(M)\).

**Proposition 2.9.** Let \(X \in A^k(M)\) and \(Y \in A^l(M)\). Then

(i) \(i_X \wedge Y = i_Y \circ i_X\)

(ii) \(i_Y \circ i_X = (-1)^{kl} i_X \circ i_Y\)
Proof. Let $\omega \in \Omega^p(M)$ with $p > k + l$ and let $Z \in A^{p-k-l}(M)$. For (i), we have

$$i_{X} \wedge Y \omega(Z) := \omega(X \wedge Y \wedge Z)$$

$$= (i_{X} \omega)(Y \wedge Z)$$

$$= i_{Y}(i_{X} \omega)(Z).$$

The case when $p = k + l$ is handled similarly.

For (ii), we have

$$i_{Y} \circ i_{X} = i_{X} \wedge Y = (-1)^{kl}i_{Y} \wedge X = (-1)^{kl}i_{X} \circ i_{Y}$$

where the first and third equality follows from part (i) of Proposition 2.9. □

We conclude this section by recalling the Lie derivative of a differential form $\omega$ along a MVF $X \in A^k(M)$ ($k \geq 1$) [3]:

$$L_{X} \omega := di_{X} \omega - (-1)^{k}i_{X}d\omega. \quad (2.6)$$

For $\omega \in \Omega^l(M)$, (2.6) implies that $L_{X} \omega \in \Omega^{l-k+1}(M)$. The next result summarizes the properties of the Lie derivative of differential forms along MVFs:

**Proposition 2.10.** [3] Let $X \in A^k(M)$, $Y \in A^l(M)$, and $\omega \in \Omega^*(M)$. Then

(i) $dL_{X} \omega = (-1)^{k-1}L_{X}d\omega$

(ii) $i_{[X,Y]} \omega = (-1)^{(k-1)l}L_{X}i_{Y} \omega - i_{Y}L_{X} \omega$

(iii) $L_{[X,Y]} \omega = (-1)^{(k-1)(l-1)}L_{X}L_{Y} \omega - L_{Y}L_{X} \omega$

(iv) $L_{X} \wedge \omega = (-1)^{l}i_{Y}L_{X} \omega + L_{Y}i_{X} \omega$

(vi) $L_{f}X = 0$ for $f \in C^\infty(M)$

**Proof.** See Proposition A.3 of [3]. □

3. Higher Affine Connections

**Definition 3.1.** A **higher affine connection** (or **higher connection**) on $M$ is a map

$$\nabla : \mathcal{A}(M) \times \mathcal{A}(M) \rightarrow \mathcal{A}(M), \quad (X,Y) \mapsto \nabla_{X}Y$$

such that

(i) $\nabla_{X}Y \in A^{k+l-1}(M)$ for $X \in A^k(M)$, $Y \in A^l(M)$

(ii) $\nabla_{fX+Y} = f\nabla_{X}Y + \nabla_{X}Y$ for $X, Y \in \mathcal{A}(M)$

(iii) $\nabla_{X}(Y + Y') = \nabla_{X}Y + \nabla_{X}Y'$ for $X, Y, Y' \in \mathcal{A}(M)$

(iv) $\nabla_{X}f = [X, f]$ for $X \in A^k(M)$, $f \in C^\infty(M)$

(v) $\nabla_{X}(Y \wedge Z) = (\nabla_{X}Y) \wedge Z + (-1)^{(k-1)l}Y \wedge \nabla_{X}Z$, for $X \in A^k(M)$, $Y \in A^l(M)$, $Z \in \mathcal{A}(M)$.

(vi) $\nabla_{f}X = 0$ for $f \in C^\infty(M)$
Remark 3.2. Note that axioms (iv) and (v) of Definition 3.1 are compatible. To see this, let \( X \in A^k(M) \). Then condition (iv) of the Schouten-Nijenhuis bracket (Definition 2.4) implies

\[
\nabla_X (fg) := [X, fg] = [X, f] \wedge g + f \wedge [X, g] = \nabla_X f \wedge g + f \wedge \nabla_X g = (\nabla_X f) g + f \nabla_X g.
\]

Corollary 3.3. Let \( \nabla \) be a higher connection on \( M \). Then

(i) \( \nabla_X fY = (-1)^{k-1} i_f X \wedge Y + f \nabla_X Y \) for \( X \in A^k(M), Y \in A(M), \) and \( f \in C^\infty(M) \); in particular, \( \nabla_X f = (-1)^{k-1} i_f X \)

(ii) the restriction of \( \nabla \) to \( A^1(M) \times A^1(M) \) is an affine connection on \( M \).

where the interior product \( i_f \) is given by Definition 2.5.

Proof. (i) of Corollary 3.3 follows from (iv) and (v) of Definition 3.1 and Proposition 2.6-(v). (ii) of Corollary 3.3 follows from Definition 3.1 and (i) of Corollary 3.3.

Proposition 3.4. Every manifold \( M \) admits a higher connection \( \nabla \) which satisfies

\[
\nabla_{X \wedge Y} Z = X \wedge \nabla_Y Z + (-1)^{kl} Y \wedge \nabla_X Z \tag{3.1}
\]

for \( X \in A^k(M), Y \in A^l(M), \) and \( Z \in A(M) \). In particular, any higher connection satisfying (3.1) is necessarily induced by an affine connection on \( TM \).

Proof. Let \( \{(U_\alpha, x^i_\alpha)\} \) be a collection of local coordinate systems which covers \( M \). Let \( Y \in A^l(M) \). Express \( Y \) in the \( x^i_\alpha \) coordinates as

\[
Y = Y^J \partial^j_\alpha,
\]

where \( Y^J \in C^\infty(U_\alpha) \), and for \( J = (j_1, \ldots, j_l) \), we define \( \partial^j_\alpha \) by

\[
\partial^j_\alpha := \frac{\partial}{\partial x^{j_1}_\alpha} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_l}_\alpha}.
\]

Define

\[
\nabla^\alpha_X Y := [X, Y^J] \wedge \partial^j_\alpha = (-1)^{k-1} i_y X \wedge \partial^j_\alpha \tag{3.2}
\]

\[
\nabla^\alpha_X f := [X, f], \quad f \in C^\infty(M). \tag{3.3}
\]

It's clear that \( \nabla^\alpha \) satisfies all the axioms of Definition 3.1 with the possible exception of axiom (v). To check axiom (v), let \( X \in A^k(M), Y \in A^l(M), \) and \( Z \in A(M) \). In local coordinates, write \( Y = Y^J \partial^j_\alpha \) and \( Z = Z^K \partial^K_\alpha \).
Then
\[ \nabla_X^\alpha (Y \wedge Z) = \nabla_X (Y^J Z^K \partial^\alpha_J \wedge \partial^\alpha_K) \]
\[ = (-1)^{k-1} i_{Y^J Z^K} X \wedge \partial^\alpha_J \wedge \partial^\alpha_K \]
\[ = (-1)^{k-1} (Z^K i_{Y^J} X + Y^J i_Z K X) \wedge \partial^\alpha_J \wedge \partial^\alpha_K \]
\[ = (-1)^{k-1} i_{Y^J} X \wedge \partial^\alpha_J \wedge Z + (-1)^{k-1} i_Z K X \wedge \partial^\alpha_K \]
\[ = \nabla_X^\alpha Y \wedge Z + (-1)^{k-1} (-1)^{(k-1)/l} Y \wedge i_Z K X \wedge \partial^\alpha_K \]
\[ = \nabla_X^\alpha Y \wedge Z + (-1)^{(k-1)/l} Y \wedge \nabla_X^\alpha Z, \]
where Proposition 2.6-(iii) was used in the third equality. This shows that \( \nabla^\alpha \) is a higher connection on \( U^\alpha \).

Since
\[ [X \wedge Y, f] = X \wedge [Y, f] + (-1)^{kl} Y \wedge [X, f] \] (3.4)
for \( X \in A^k(M), Y \in A^l(M), \) and \( f \in C^\infty(M), \) it follows that
\[ \nabla^\alpha_{X \wedge Y} Z = X \wedge \nabla^\alpha_Y Z + (-1)^{kl} Y \wedge \nabla^\alpha_X Z. \]

To obtain a higher connection on \( M \) which satisfies (3.1), let \( \{ \rho_\alpha \} \) be a partition of unity subordinate to \( \{ U_\alpha \} \). The desired higher connection is then
\[ \nabla_X Y = \rho_\alpha \nabla^\alpha_X Y. \]

The higher connection given by Proposition 3.4 is equivalent to the covariant derivative introduced in [2]. To see this, let \( \nabla \) be a higher connection given by Proposition 3.4 and define
\[ \tilde{\nabla}_X Y := (-1)^{k-1} \nabla_X Y. \]
for \( X \in A^k(M), Y \in A^l(M). \) Then a direct calculation shows
\[ \tilde{\nabla}_{X \wedge Y} Z = (-1)^{k} X \wedge \tilde{\nabla}_Y Z + (-1)^{(k-1)/l} Y \wedge \tilde{\nabla}_X Z, \]
which is precisely condition (1.1) from [2]. Equation (3.1) motivates the following definition:

**Definition 3.5.** A higher connection \( \nabla \) is called *induced* if it satisfies
\[ \nabla_{X \wedge Y} Z = X \wedge \nabla_Y Z + (-1)^{kl} Y \wedge \nabla_X Z \] (3.5)
for \( X \in A^k(M), Y \in A^l(M), \) and \( Z \in A(M). \)

If \( \nabla \) is an induced higher connection and \( X = X_1 \wedge \cdots \wedge X_k \) is a decomposable \( k \)-vector field, then for \( Y \in A(M), \) we have
\[ \nabla_X Y = \sum_{j=1}^{k} (-1)^{k-j} X_1 \wedge \cdots \wedge \widehat{X}_j \wedge \cdots \wedge X_k \wedge \nabla_X Y, \] (3.6)
where \( \widehat{X}_j \) denotes the omission of \( X_j. \)
Lemma 3.6. Let $\nabla$ and $\tilde{\nabla}$ be two higher connections on $M$ and let $F : \mathcal{A}(M) \times \mathcal{A}(M) \to \mathcal{A}(M)$ be given by $F(X,Y) := \nabla_X Y - \tilde{\nabla}_X Y$. Then $F$ is $C^\infty(M)$-linear in $X$ and $Y$. In particular, $F^{k,l} := F|_{A^k(M) \times A^l(M)}$ is a section of the bundle $\wedge^{k+l-1}TM \otimes \wedge^k T^*M \otimes \wedge^l T^*M$.

Proof. Let $X \in A^k(M)$, $Y \in \mathcal{A}(M)$, and $h \in C^\infty(M)$. It's clear that $F(hX,Y) = hF(X,Y)$. We now show that $F$ is $C^\infty(M)$-linear in $Y$: 

$$F(X,hY) = \nabla_X (hY) - \tilde{\nabla}_X (hY)$$

$$= (-1)^{k-1} i_h X \wedge Y + h\nabla_X Y - (-1)^{k-1} i_h X \wedge Y - h\tilde{\nabla}_X Y$$

$$= hF(X,Y).$$

□

Lemma 3.7. Let

$$\nabla : \mathcal{A}(M) \times (A^0(M) \oplus A^1(M)) \to \mathcal{A}(M), \quad (X,Y) \mapsto \nabla_X Y,$$

be a map which is

(i) $C^\infty(M)$-linear in the first argument

(ii) $\mathbb{R}$-linear in the second argument

and satisfies

$$\nabla_X f = [X,f], \quad \nabla_f Y = 0 \quad (3.7)$$

$$\nabla_X f Y = (-1)^{k-1} i_f X \wedge Y + f\nabla_X Y \quad (3.8)$$

for $X \in A^k(M)$, $Y \in A^l(M)$, and $f \in C^\infty(M)$. Then $\nabla$ extends uniquely to a higher connection on $M$.

Proof. Let $Y_j \in A^1(M)$ for $j = 1, \ldots, l$ and define

$$\nabla_X (Y_1 \wedge \cdots \wedge Y_l) := \sum_{j=1}^l (-1)^{(k-1)(j-1)+1} Y_1 \wedge \cdots \wedge \nabla_X Y_j \wedge \cdots \wedge Y_l \quad (3.9)$$

for $X \in A^k(M)$. Note that (3.8) implies that $\nabla$ is local. Since every $l$-vector field is locally a finite sum of decomposable $l$-vector fields, (3.9) extends $\nabla_X$ to all of $\mathcal{A}(M)$ by $\mathbb{R}$-linearity. Its clear that $\nabla$ satisfies all the axioms of Definition 3.1 with the possible exception of axiom (v). To check axiom (v), it suffices to consider the case where $Y = Y_1 \wedge \cdots \wedge Y_l$ and $Z = Z_1 \wedge \cdots \wedge Z_m$ are decomposable $l$ and $m$-vector fields respectively. First, we check that
\( \nabla_X(fY) \) satisfies axiom (v) for \( f \in C^\infty(M) \):
\[
\nabla_X(fY) = \nabla_X(Y_1 \wedge \cdots \wedge fY_t \wedge \cdots \wedge Y_l)
= (-1)^{(k-1)(t+1)}Y_1 \wedge Y_2 \wedge \cdots \wedge \nabla_X(fY_t) \wedge \cdots \wedge Y_l
+ \sum_{j \neq t} (-1)^{(k-1)(j+1)}Y_1 \wedge \cdots \wedge fY_t \wedge \cdots \wedge \nabla_XY_j \wedge \cdots \wedge Y_l
= (-1)^{(k-1)(t+1)}(-1)^{k-1}Y_1 \wedge \cdots \wedge Y_{t-1} \wedge i_fX \wedge Y_t \wedge \cdots \wedge Y_l
+ (-1)^{(k-1)(t+1)}fY_1 \wedge \cdots \wedge \nabla_XY_t \wedge \cdots \wedge Y_l
+ f \sum_{j \neq t} (-1)^{(k-1)(j+1)}Y_1 \wedge \cdots \wedge Y_t \wedge \cdots \wedge \nabla_XY_j \wedge \cdots \wedge Y_l
= (-1)^{k-1}i_fX \wedge Y + f \sum_{j=1}^l (-1)^{(k-1)(j+1)}Y_1 \wedge \cdots \wedge \nabla_XY_j \wedge \cdots \wedge Y_l
= (\nabla_Xf) \wedge Y + f\nabla_XY,
\]
where the last equality follows from Proposition 2.6(v). For \( \nabla_X(Y \wedge Z) \), we have
\[
\nabla_X(Y \wedge Z) = \nabla_X(Y_1 \wedge \cdots \wedge Y_l \wedge Z_1 \wedge \cdots \wedge Z_m)
= \sum_{j=1}^l (-1)^{(k-1)(j+1)}Y_1 \wedge \cdots \wedge \nabla_XY_j \wedge \cdots \wedge Y_l \wedge Z
+ \sum_{j=1}^m (-1)^{(k-1)(l+j+1)}Y \wedge Z_1 \wedge \cdots \wedge \nabla_XZ_j \wedge \cdots \wedge Z_m
= \nabla_XY \wedge Z + (-1)^{(k-1)l}Y \wedge \nabla_XZ.
\]
This shows that \( \nabla \) is a higher connection on \( M \). Lastly, note that if \( \nabla' \) is any higher connection which satisfies \( \nabla'_X Y = \nabla_X Y \) for all \( X \in \mathcal{A}^k(M), \ Y \in \mathcal{A}^1(M) \), then axiom (v) of Definition 3.1 implies \( \nabla' = \nabla \). \( \square \)

**Theorem 3.8.** Let \( \nabla \) be any higher connection on \( M \). Then there exists

(i) a unique affine connection \( \overline{\nabla} \) on \( TM \), and

(ii) a unique collection of sections \( F^k \) of \( \mathcal{E}^k := \wedge^k TM \otimes \wedge^k T^*M \otimes T^*M \)

for \( k = 2, \ldots, n := \dim M \)

such that for \( k = 1, \ldots, n \), \( \overline{\nabla} \) and \( \{ F^j \}_{j=2}^n \) satisfy

\[
\nabla_X Y = \overline{\nabla}_X Y + F^k(X, Y), \ \ \forall X \in \mathcal{A}^k(M), \ Y \in \mathcal{A}^1(M), \ \ \text{(3.10)}
\]

where we set \( F^1 \equiv 0 \) and \( \overline{\nabla}_X Y \) is understood to be the higher connection induced by the affine connection \( \overline{\nabla} \) according to \( (3.7) \). Conversely, any affine connection \( \overline{\nabla} \) on \( TM \) together with any collection of sections \( F^k \in \Gamma(\mathcal{E}^k) \) for \( k = 2, \ldots, n \) determines a unique higher connection on \( M \) which satisfies \( (3.10) \). In particular, there is a bijection between the space of all higher
connections and the set of all pairs of the form \((\tilde{\nabla}, \{F^j\})\), where \(\tilde{\nabla}\) is an affine connection on \(TM\) and \(F^j \in \Gamma(E^j)\) for \(j = 2, \ldots, n\).

**Proof.** Let \(\nabla\) be any higher connection on \(M\) and let \(\tilde{\nabla}\) be the affine connection on \(TM\) defined by \(\tilde{\nabla}_XY := \nabla_XY\) for \(X, Y \in A^1(M)\). Extend \(\tilde{\nabla}\) to a higher connection on \(M\) via (3.1). By Lemma 3.6

\[
F^k := (\nabla - \tilde{\nabla})|_{A^k(M) \times A^1(M)}: A^k(M) \times A^1(M) \to A^k(M)
\]

is a section of \(E^k := \Lambda^kTM \otimes \Lambda^kT^*M \otimes T^*M\) for \(k = 1, \ldots, n := \dim M\). (Note that \(F^1 \equiv 0\).) Hence, we have

\[
\nabla_XY = \tilde{\nabla}_XY + F^k(X, Y),
\]

for \(X \in A^k(M), Y \in A^1(M)\). To see that \((\tilde{\nabla}, \{F^j\})\) is unique, suppose that \((\tilde{\nabla}, \{G^j\})\) is another such pair which satisfies (3.10). Then for all \(X, Y \in A^1(M)\), we have

\[
\tilde{\nabla}_XY = \nabla_XY = \tilde{\nabla}_XY.
\]

Hence, \(\tilde{\nabla} = \tilde{\nabla}\). This fact together with (3.10) then implies that \(F^k = G^k\) for \(k = 2, \ldots, n\).

Conversely, suppose \(\tilde{\nabla}\) is an affine connection on \(TM\) (extended to a higher connection on \(M\) via (3.1)) and \(F^k\) is a section of the bundle \(E^k\) for \(k = 2, \ldots, n\). For convenience, set \(F^1 \equiv 0\). For \(X \in A^k(M), Y \in A^1(M)\), define \(\nabla_XY\) according to (3.10) and set \(\nabla_Xf := [X, f]\) and \(\nabla_f := 0\) for \(f \in C^\infty(M)\). Its clear that \(\nabla\) is \(C^\infty(M)\)-linear in the first argument and \(\mathbb{R}\)-linear in the second argument. In addition, for \(X \in A^k(M), Y \in A^1(M)\), and \(f \in C^\infty(M)\), we have

\[
\nabla_XfY = \tilde{\nabla}_X(fY) + F^k(X, fY)
= (-1)^{k-1}i_fX \wedge Y + f\tilde{\nabla}_X(Y) + fF^k(X, Y)
= (-1)^{k-1}i_fX \wedge Y + f\nabla_XY.
\]

By Lemma 3.7, \(\nabla\) extends uniquely to a higher connection on \(M\). Note that if \(\nabla'\) is any other higher connection satisfying (3.10), then axiom (v) of Definition 3.1 implies that \(\nabla' = \nabla\).

**Theorem 3.9.** Let \(\nabla\) be a higher connection and let \((\tilde{\nabla}, \{F^j\})\) be the unique pair associated to \(\nabla\) by Theorem 3.8. Then \(\nabla\) is an induced higher connection (i.e., one that satisfies (3.7)) iff \(F^j \equiv 0\) \(\forall j\).

**Proof.** Extend \(\tilde{\nabla}\) to a higher connection via (3.1).

\((\Rightarrow)\). Suppose that \(\nabla\) satisfies (3.1) and \(F^j \neq 0\) for some \(j \in \{2, \ldots, \dim M\}\). Let \(k \in \{2, \ldots, \dim M\}\) be the smallest value for which \(F^k \neq 0\). Since \(F^k \neq 0\), there exists some decomposable \(k\)-vector field \(X_1 \wedge \cdots \wedge X_k\) and some \(Y \in A^1(M)\) such that \(F^k(X, Y) \neq 0\). Let \(V = X_2 \wedge \cdots \wedge X_k\). By
Theorem 3.8, we have
\[ \nabla_{X_1 \wedge V} Y = \tilde{\nabla}_{X_1 \wedge V} Y + F^k(X_1 \wedge V, Y) \]
\[ = X_1 \wedge \tilde{\nabla}_V Y + (-1)^{k-1} V \wedge \tilde{\nabla}_{X_1} Y + F^k(X_1 \wedge V, Y) \]
\[ = X_1 \wedge \nabla_V Y + (-1)^{k-1} V \wedge \nabla_{X_1} Y + F^k(X_1 \wedge V, Y), \quad (3.12) \]
where we use the fact that \( F^j \equiv 0 \) for \( j < k \) in the last equality; this in turn implies that \( \tilde{\nabla}_V Y = \nabla_V Y \) and \( \tilde{\nabla}_{X_1} Y = \nabla_{X_1} Y \). On the other hand, since \( \nabla \) satisfies (3.1), we also have
\[ \nabla_{X_1 \wedge V} Y = X_1 \wedge \nabla_V Y + (-1)^{k-1} V \wedge \nabla_{X_1} Y. \quad (3.13) \]
Comparing (3.12) and (3.13) shows that \( F^k(X_1 \wedge V, Y) \equiv 0 \) which is a contradiction. Hence, \( F^j \equiv 0 \) for all \( j \).

(⇐). If \( F^j \equiv 0 \) for all \( j \), then Theorem 3.8 implies that \( \nabla = \tilde{\nabla} \) and the latter satisfies (3.1) by definition. This completes the proof. □

Example 3.10. Every closed string theory defined on a manifold \( M \) equips \( M \) with the following massless fields: gravity, the B-field, and the dilaton [11] [1] [7]. Theorem 3.8 provides a natural way of incorporating these fields into a higher connection. Mathematically speaking, gravity is identified with a (Lorentzian) metric \( g \), the B-field is a 2-form \( B \), and the dilaton is a scalar function \( \phi \). The low energy effective action for these fields is given by
\[ S_{\text{eff}} = \int dx^D \sqrt{-g} e^{-2\phi} (R + 4\partial\mu\partial^\mu \phi - \frac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu}), \]
where \( H := dB \) is the 3-form field strength of \( B \), \( R \) is the scalar curvature of \( g \), and \( D \) is the dimension of \( M \) (which is either 10 or 26).

Using Theorem 3.8 we associate to \( (M, g, B, \phi) \) the higher connection given by \( (\tilde{\nabla}, \{F^j\}) \) where \( \tilde{\nabla} \) is the Levi-Civita connection associated to \( g \) and
\[ F^2 = B^2 \otimes B \otimes d\phi, \quad F^3 = H^2 \otimes H \otimes d\phi, \quad F^k = 0 \quad \text{for} \quad k > 3. \quad (3.14) \]
In (3.14), \( B^2 \) and \( H^2 \) are, respectively, the 2 and 3-vector fields obtained by raising the indices of \( B \) and \( H \) using \( g \). The higher connection given by \( (\tilde{\nabla}, \{F^j\}) \) has the following feature: covariant differentiation along 2 and 3-vector fields depends upon all three fields while covariant differentiation along \( k \)-vector fields for \( k \neq 2, 3 \) depends on gravity alone. This example illustrates just one way these fields can be brought together in a higher connection. The key point, however, is that a higher connection has enough structure to accommodate such fields in the first place.

Let \( E^k \) be defined as in Theorem 3.8 and let \( F^k \in \Gamma(E^k) \). For later convenience, we now extend \( F^k \) to all of \( A^k(M) \times A(M) \) as follows: for
$X \in A^k(M), Y_i \in A^l(M), i = 1, \ldots, l$, and $f \in C^\infty(M)$, we define

$$F^k(X, f) := 0 \quad (3.15)$$

$$F^k(X, Y) := \sum_{j=1}^l (-1)^{(k-1)(j+1)} Y_1 \wedge \cdots \wedge F^k(X, Y_j) \wedge \cdots \wedge Y_l, \quad (3.16)$$

where $Y = Y_1 \wedge \cdots \wedge Y_l$. Since every $l$-vector field is locally a finite sum of decomposable $l$-vector fields, equations (3.15) and (3.16) extend $F^k$ to all of $A^k(M) \times A(M)$.

**Corollary 3.11.** Let $\nabla$ be a higher connection and let $(\tilde{\nabla}, \{F^j\})$ be the unique pair associated to $\nabla$ by Theorem 3.8. Then

$$\nabla_X Y = \tilde{\nabla}_X Y + F^k(X, Y) \quad (3.17)$$

for all $X \in A^k(M)$ and $Y \in A(M)$, where $F^k$ is extended to all of $A^k(M) \times A(M)$ via (3.15) and (3.16).

**Proof.** It suffices to prove (3.17) for the case when $Y = Y_1 \wedge \cdots \wedge Y_l$ is a decomposable $l$-vector field. Axiom (v) of Definition 3.1 then implies

$$\nabla_X Y = \sum_{j=1}^l (-1)^{(k-1)(j+1)} Y_1 \wedge \cdots \wedge \nabla_X Y_j \wedge \cdots \wedge Y_l$$

$$= \sum_{j=1}^l (-1)^{(k-1)(j+1)} Y_1 \wedge \cdots \wedge (\tilde{\nabla}_X Y_j + F^k(X, Y_j)) \wedge \cdots \wedge Y_l$$

$$= \sum_{j=1}^l (-1)^{(k-1)(j+1)} Y_1 \wedge \cdots \wedge \tilde{\nabla}_X Y_j \wedge \cdots \wedge Y_l$$

$$+ \sum_{j=1}^l (-1)^{(k-1)(j+1)} Y_1 \wedge \cdots \wedge F^k(X, Y_j) \wedge \cdots \wedge Y_l$$

$$= \tilde{\nabla}_X Y + F^k(X, Y),$$

where the second equality follows from Theorem 3.8. Lastly, note that if $Y = f \in C^\infty(M)$, then $\nabla_X f = \tilde{\nabla}_X f + 0 = [X, f]$. $\square$

We now introduce the notion of higher torsion for higher connections:

**Definition 3.12.** Let $\nabla$ be a higher connection. The higher torsion associated to $\nabla$ is defined by

$$T(X, Y) := \nabla_X Y - (-1)^{(k-1)(l-1)} \nabla_Y X - [X, Y] \quad (3.18)$$

for $X \in A^k(M), Y \in A^l(M)$. $\nabla$ is torsion-free if $T \equiv 0$.

**Proposition 3.13.** Let $\nabla$ be a higher connection with higher torsion $T$. For $X \in A^k(M), Y \in A^l(M)$, and $f \in C^\infty(M)$, $T$ satisfies

(i) $T(X, Y) = -(-1)^{(k-1)(l-1)} T(Y, X)$
(ii) \( T(fX, Y) = fT(X, Y) \)

**Proof.** Let \( s = (-1)^{(k-1)(l-1)} \). For (i), we have

\[
T(X, Y) := \nabla_X Y - s \nabla_Y X - [X, Y]
\]

\[
= -s(\nabla_Y X - s \nabla_X Y + s[X, Y])
\]

\[
= -s(\nabla_Y X - s \nabla_X Y + s(-s[Y, X]))
\]

\[
= -s(\nabla_Y X - s \nabla_X Y - [Y, X])
\]

\[
= -sT(Y, X).
\]

For (ii), we have

\[
T(fX, Y) = \nabla_{fX} Y - s \nabla_Y (fX) - [fX, Y]
\]

\[
= f \nabla_X Y - s((\nabla_Y f \wedge X + f \nabla_Y X) - (f[X, Y] - X \wedge i_f Y))
\]

\[
= f \nabla_X Y - s(\nabla_Y f \wedge X + f \nabla_Y X) - (f[X, Y] - X \wedge i_f Y)
\]

\[
= f \nabla_X Y - ((-1)^{(l-1)}i_f Y \wedge X + f \nabla_Y X) - (f[X, Y] - X \wedge i_f Y)
\]

\[
= f \nabla_X Y - f \nabla_Y X - s f \nabla_Y X - f[X, Y] + ((-1)^{(k-1)}i_f Y \wedge X
\]

\[
= fT(X, Y).
\]

where we used Proposition 2.6-(vi) and (v) in the second and fourth equalities respectively. \( \square \)

We conclude this section with the following two results which completely characterize all torsion-free higher connections.

**Proposition 3.14.** Let \( \tilde{\nabla} \) be an affine connection on \( TM \). Then the following statements are equivalent:

1. \( \tilde{\nabla} \) is torsion-free as an affine connection on \( TM \).
2. \( \tilde{\nabla} \) is torsion-free as an induced higher connection.

**Proof.** (i) \( \Longleftrightarrow \) (ii). Immediate.

(i) \( \Rightarrow \) (ii). Suppose that \( \tilde{\nabla} \) is torsion-free as an affine connection on \( TM \). Extend \( \tilde{\nabla} \) to an induced higher connection and let \( T \) denote its higher torsion. To prove that \( T \equiv 0 \), it suffices to show that \( T(X, Y) \equiv 0 \) for the case when \( X \) and \( Y \) are decomposable \( k \) and \( l \)-vector fields respectively. So, let \( X = X_1 \wedge \cdots \wedge X_k \) and \( Y = Y_1 \wedge \cdots \wedge Y_l \). To simplify things, write

\[
X[i] = X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k
\]

\[
Y[j] = Y_1 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_l,
\]

(3.19)
where $\tilde{X}_i$ and $\tilde{Y}_j$ denotes omission as usual. Using (3.6), we have

$$\tilde{\nabla}_X Y = \sum_{i=1}^{k} (-1)^{k-i} X[i] \land \tilde{\nabla}_X Y$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{l} (-1)^{k-i} (-1)^{j-l} X[i] \land \tilde{\nabla}_X Y \land Y[j]$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{l} (-1)^{i+j} \tilde{\nabla}_X Y \land X[i] \land Y[j].$$

Likewise,

$$\tilde{\nabla}_X Y = \sum_{i=1}^{k} \sum_{j=1}^{l} (-1)^{i+j} \tilde{\nabla}_Y X \land Y[j] \land X[i].$$

Recall that for $X, Y$ decomposable, the Schouten-Nijenhuis bracket is given by

$$[X, Y] = \sum_{i=1}^{k} \sum_{j=1}^{l} (-1)^{i+j} [X_i, Y_j] \land X[i] \land Y[j].$$

Putting everything together gives

$$T(X, Y) = \tilde{\nabla}_X Y - (-1)^{(k-1)(l-1)} \tilde{\nabla}_Y X - [X, Y]$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{l} (-1)^{i+j} (\tilde{\nabla}_X Y_j - \tilde{\nabla}_Y X_i - [X_i, Y_j]) \land X[i] \land Y[j]$$

$$= 0.$$

This completes the proof. \(\square\)

**Theorem 3.15.** Let $\nabla$ be a higher connection. Then $\nabla$ is torsion-free iff $\nabla$ is induced by a torsion-free affine connection on $TM$.

**Proof.** If $\nabla$ is induced by a torsion-free affine connection on $TM$, then $\nabla$ must also be torsion-free as a higher connection by Proposition 3.14.

Now suppose $\nabla$ is a torsion-free higher connection. Let $(\tilde{\nabla}, \{F^j\})$ be the unique pair associated to $\nabla$ by Theorem 3.8 and extend $F^j$ to $A^j(M) \times A(M)$ as in Corollary 3.11. Since $\nabla$ is a torsion-free higher connection, $\tilde{\nabla}$ must necessarily be torsion-free as an affine connection on $TM$. Proposition 3.14 then implies that $\tilde{\nabla}$ is also torsion-free as an induced higher connection. Consequently, for $X \in A^k(M)$ and $Y \in A^l(M)$, we have

$$T(X, Y) = \nabla_X Y - (-1)^{(k-1)(l-1)} \nabla_Y X - [X, Y]$$

$$= \tilde{\nabla}_X Y + F^k(X, Y) - (-1)^{(k-1)(l-1)} \tilde{\nabla}_Y X - (-1)^{(k-1)(l-1)} F^l(Y, X) - [X, Y]$$

$$= F^k(X, Y) - (-1)^{(k-1)(l-1)} F^l(Y, X).$$
Setting \( l = 1 \) and using the fact that \( T \equiv 0 \) gives \( F^k(X, Y) = 0 \) for all \( X \in A^k(M), Y \in A^1(M) \). Theorem 3.8 then implies that \( \nabla = \tilde{\nabla} \). This completes the proof. \( \square \)

4. Extension to Differential Forms

Let \( \nabla \) be a higher connection, \( \omega \in \Omega^l(M) \), and \( X \in A^k(M) \). For \( k > 0 \) and \( l - k + 1 \geq 0 \), we define \( \nabla_X \omega \in \Omega^{l-k+1}(M) \) via

\[
\nabla_X \omega(Y) := (-1)^{(k-1)(l-1)} L_X i_Y \omega - \omega(\nabla_X Y)
\]

for all \( Y \in A^{l-k+1}(M) \). (For \( \eta \in \Omega^0(M) \) and \( f \in A^0(M) := C^\infty(M) \), we understand \( \eta(f) \) to mean \( f \eta \).) Lastly, for \( k = 0 \) or \( l - k + 1 < 0 \), we set \( \nabla_X \omega := 0 \). We will now verify that \( \nabla_X \omega \) is indeed an \( (l - k + 1) \)-form. To do this, we need the following lemmas:

**Lemma 4.1.** Let \( X = X_1 \wedge \cdots \wedge X_k \) be a decomposable \( k \)-vector field and let \( \alpha \in \Omega^1(M) \) and \( \omega \in \Omega^*(M) \). Then

\[
i_X(\alpha \wedge \omega) = \sum_{j=1}^k (-1)^{j+1} \alpha(X_j) i_{X_1 \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_k} \omega + (-1)^k \alpha \wedge i_X \omega.
\]

**Proof.** We prove Lemma 4.1 by induction on \( k \). For \( k = 1 \), \( X \) is a vector field and (2.5) gives

\[
i_X(\alpha \wedge \omega) = \alpha(X) \omega - \alpha \wedge i_X \omega.
\]

Hence, Lemma 4.1 holds for \( k = 1 \). Now suppose that Lemma 4.1 holds for \( X = X_1 \wedge \cdots \wedge X_k \). Let \( X_{k+1} \in A^1(M) \). By Proposition 2.9, we have

\[
i_{X \wedge X_{k+1}}(\alpha \wedge \omega) = i_{X_{k+1}} i_X(\alpha \wedge \omega)
\]

\[
= i_{X_{k+1}} \left( \sum_{j=1}^k (-1)^{j+1} \alpha(X_j) i_{X_1 \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_k} \omega + (-1)^k \alpha \wedge i_X \omega \right)
\]

\[
= \sum_{j=1}^k (-1)^{j+1} \alpha(X_j) i_{X_{k+1} \wedge X_1 \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_k} \omega + (-1)^k \alpha \wedge i_{X_{k+1}} i_X \omega
\]

\[
= \sum_{j=1}^k (-1)^{j+1} \alpha(X_j) i_{X_1 \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_k} \wedge X_{k+1} \omega + (-1)^k \alpha \wedge i_{X_{k+1}} i_X \omega
\]

\[
= \sum_{j=1}^k (-1)^{j+1} \alpha(X_j) i_{X_1 \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_k} \wedge X_{k+1} \omega + (-1)^{k+2} \alpha(X_{k+1}) i_X \omega
\]

\[
+ (-1)^k \alpha \wedge i_{X \wedge X_{k+1}} \wedge X_{k+1} + (-1)^{k+1} \alpha \wedge i_X \wedge X_{k+1} \omega,
\]

which completes the proof.

\[\square\]
where we have used the induction hypothesis in the second equality, and Proposition 2.9 and equation (2.5) in the fourth equality. This completes the proof.

**Lemma 4.2.** Let \( X = X_1 \land \cdots \land X_k \) be a decomposable \( k \)-vector field and let \( f \in C^\infty(M) \). Then

\[
[X, f] = \sum_{j=1}^{k} (-1)^{k-j} (X_j f) X_1 \land \cdots \land \widehat{X_j} \land \cdots \land X_k
\]

**Proof.** We prove this by induction on \( k \). For \( k = 1 \), we have \( X = X_1 \) and \( [X_1, f] := X_1 f \). Hence, the lemma holds for \( k = 1 \). Suppose that the lemma holds for \( X = X_1 \land \cdots \land X_k \). Let \( X_{k+1} \in A^1(M) \). Then

\[
[X \land X_{k+1}, f] = \sum_{j=1}^{k} (-1)^{k-j} (X_j f) X_1 \land \cdots \land \widehat{X_j} \land \cdots \land X_k \land X_{k+1}
\]

where we have used the induction hypothesis in the sixth equality. This completes the proof.

**Lemma 4.3.** Let \( X \in A^k(M) \), \( f \in C^\infty(M) \), and \( \omega \in \Omega^*(M) \). Then

\[
L_X f \omega = f L_X \omega + i_{[X,f]} \omega.
\]

**Proof.** Without loss of generality, let \( X = X_1 \land \cdots \land X_k \) be a decomposable \( k \)-vector field. Then

\[
L_X f \omega = di_X f \omega - (-1)^k i_X df \omega
\]

where we have used the induction hypothesis in the second equality, and Proposition 2.9 and equation (2.5) in the fourth equality. This completes the proof.
where Lemma 4.1 is used in the third equality and Lemma 4.2 is used in the last equality.

\[ \square \]

**Proposition 4.4.** Let \( \nabla \) be a higher connection. Then \( \nabla X \omega = L_X \omega \in C^\infty(M) \) for \( X \in A^k(M) \) and \( \omega \in \Omega^{k-1}(M) \).

**Proof.** Let \( f \in C^\infty(M) \). Then

\[
f \nabla_X \omega = \nabla_X (f) \\
:= (-1)^{(k-1)(l-1)} L_X f \omega - \omega ([X, f]) \\
= f L_X \omega + i_{\omega} [X, f] \\
= f L_X \omega,
\]

where Lemma 4.3 is used in the third equality. This proves the proposition. \( \square \)

**Theorem 4.5.** Let \( \nabla \) be a higher connection, \( \omega \in \Omega(M) \), and \( X \in A^k(M) \) with \( k > 0 \). Then \( \nabla X \omega \in \Omega^{l-k+1}(M) \).

**Proof.** For \( l - k + 1 \geq 0 \), let \( Y \in A^{l-k+1}(M) \). It follows from (4.1) that \( \nabla X \omega(Y) \in C^\infty(M) \). To prove Theorem 4.5, it suffices to show

\[
\nabla_X \omega(fY) = f \nabla_X \omega(Y)
\]

for \( f \in C^\infty(M) \). We now verify (4.3):

\[
\nabla_X \omega(fY) = (-1)^{(k-1)(l-1)} L_X i_{fY} \omega - \omega ([X, f] \wedge Y) \\
= (-1)^{(k-1)(l-1)} L_X f i_Y \omega - \omega ([X, f] \wedge Y) - f \omega (\nabla_X Y) \\
= (-1)^{(k-1)(l-1)} f L_X i_Y \omega + (-1)^{(k-1)(l-1)} i_{[X, f]} i_Y \omega - \omega ([X, f] \wedge Y) - f \omega (\nabla_X Y) \\
= f \nabla_X \omega(Y) + (-1)^{(k-1)(l-1)} i_{[X, f]} i_Y \omega - \omega ([X, f] \wedge Y) \\
= f \nabla_X \omega(Y) + (-1)^{(k-1)(l-1)} i_Y \omega ([X, f]) - \omega ([X, f] \wedge Y) \\
= f \nabla_X \omega(Y) + (-1)^{(k-1)(l-1)} \omega (Y \wedge [X, f]) - \omega ([X, f] \wedge Y) \\
= f \nabla_X \omega(Y) + (-1)^{k(k-1)} \omega ([X, f] \wedge Y) - \omega ([X, f] \wedge Y) \\
= f \nabla_X \omega(Y),
\]

where Lemma 4.3 is used in the third equality. This completes the proof. \( \square \)

We now conclude this section with some of the properties of (4.1). Before doing so, we need a quick lemma:
Lemma 4.6. Let $X \in A^k(M)$, $\eta \in \Omega^*(M)$, and $f \in C^\infty(M)$. Then
$$L_fX\eta = df \wedge i_X\eta + fL_X\eta.$$  

Proof. From (2.6), we have
$$L_fX\eta = df i_X\eta - (-1)^k f i_X d\eta$$
$$= df i_X\eta - (-1)^k f i_X d\eta$$
$$= df \wedge i_X\eta + f d i_X\eta - (-1)^k f i_X d\eta$$
$$= df \wedge i_X\eta + fL_X\eta.$$  

\[\Box\]

Theorem 4.7. Let $\nabla$ be a higher connection. For $\omega \in \Omega^t(M)$, $X \in A^k(M)$ ($k > 0$), and $f \in C^\infty(M)$, $\nabla$ satisfies

(i) $\nabla f_X\omega = f \nabla_X\omega$

(ii) $\nabla_X f \omega = f \nabla_X\omega + i_{[X,f]}\omega$

(iii) $\nabla_X (i_W\omega) = (-1)^j(i_W \nabla_X\omega + i_{[X,W]}\omega)$ for $W \in A^l(M)$.

Proof. Let $l - k + 1 \geq 0$ and let $Y \in A^{l-k+1}(M)$. For (i), we have

$$\nabla f_X\omega(Y) = (-1)^{(k-1)(l-1)} L_fX i_Y\omega - \omega(\nabla f_X Y)$$
$$= (-1)^{(k-1)(l-1)} df \wedge i_Y i_X\omega + (-1)^{(k-1)(l-1)} fL_X i_Y\omega - f\omega(\nabla_X Y)$$
$$= (-1)^{(k-1)(l-1)} df \wedge i_Y \nabla_X\omega + f \nabla_X\omega(Y)$$
$$= f \nabla_X\omega(Y),$$

where Lemma 4.6 was used in the second equality, Proposition 2.9 was used in the third equality, and the last equality follows from the fact that $Y \wedge X \in A^{l+1}(M)$ and $\omega \in \Omega^t(M)$.

For (ii), we have

$$\nabla_X f \omega(Y) = (-1)^{(k-1)(l-1)} L_X i_Y(f\omega) - (f\omega)(\nabla_X Y)$$
$$= (-1)^{(k-1)(l-1)} L_X f i_Y\omega - (f\omega)(\nabla_X Y)$$
$$= (-1)^{(k-1)(l-1)} fL_X i_Y\omega + (-1)^{(k-1)(l-1)} i_{[X,f]} i_Y\omega - (f\omega)(\nabla_X Y)$$
$$= f \nabla_X\omega(Y) + (-1)^{(k-1)(l-1)} i_{[X,f]} i_Y\omega$$
$$= f \nabla_X\omega(Y) + i_{[X,f]} \omega(Y),$$

where Lemma 4.3 is used in the third equality and Proposition 2.9 is used in the fifth equality.

For (iii), note that for $j = 0$, (iii) follows directly from (ii) of Theorem 4.7. Now assume that $j > 0$. Let $t = l - j - k + 1 \geq 0$ and let $Z \in A^t(M)$.
Then
\[ \nabla_X (i_W \omega)(Z) = (-1)^{(l-j-1)(k-1)} L_X i_Z i_W \omega - i_W \omega(\nabla_X Z) \]
\[ = (-1)^{(l-1)(k-1)} (-1)^j(k-1) L_X i_W \wedge Z \omega - \omega(W \wedge \nabla_X Z) \]
\[ = (-1)^{(l-1)(k-1)} (-1)^j(k-1) L_X i_W \wedge Z \omega - (-1)^j(k-1) \omega(\nabla_X (W \wedge Z)) \]
\[ + (-1)^j(k-1) \omega(\nabla_X (W \wedge Z)) - \omega(W \wedge \nabla_X Z) \]
\[ = (-1)^j(k-1) \nabla_X \omega(W \wedge Z) \]
\[ + (-1)^j(k-1) \omega(\nabla_X W \wedge Z) + (-1)^j(k-1) \omega(W \wedge \nabla_X Z)) \]
\[ - \omega(W \wedge \nabla_X Z) \]
\[ = (-1)^j(k-1) \nabla_X \omega(W \wedge Z) + (-1)^j(k-1) \omega((\nabla_X W) \wedge Z) \]
\[ = (-1)^j(k-1)(i_W \nabla_X \omega(Z) + i \nabla_X W \omega(Z)), \]

where Proposition 2.9 is used in the second equality, and (ii) and axiom (v) of Definition 3.1 are used in the fourth equality. This completes the proof. \qed

**Proposition 4.8.** Let \( \nabla \) be a torsion-free higher connection. Then

\[ \nabla_{X \wedge Y} \omega = (-1)^j i_Y \nabla_X \omega + (-1)^{k-1} i_X \nabla_Y \omega \]

for \( X \in A^k(M), Y \in A^l(M), \) and \( \omega \in \Omega^m(M). \)

**Proof.** For \( m < k + l - 1, \) both sides of (4.4) are zero. In addition, for \( k = 0 \) or \( l = 0, \) (4.4) follows from Theorem 4.7(i). We now verify (4.4) for \( m \geq k + l - 1 \) with \( k, l > 0. \)

Let \( Z \in A^t(M) \) where \( t = m - k - l + 1. \) Then

\[ \nabla_{X \wedge Y} \omega(Z) = q L_{X \wedge Y} i_Z \omega - \omega(\nabla_{X \wedge Y} Z), \]

where \( q := (-1)^{(k+l-1)(m-1)}. \) Using (iv) and (ii) of Proposition 2.10 and (i) of Proposition 2.9, the first term in (4.5) can be decomposed as

\[ q L_{X \wedge Y} i_Z \omega = (-1)^{jl} L_X i_{Z \wedge Y} \omega - q(-1)^j i_Z [X, Y] \omega + q L_Y i_{Z \wedge X} \omega. \]

From (4.1), we have

\[ (-1)^{(k-1)(m-1)} L_X i_{Z \wedge Y} \omega = \nabla_X \omega(Z \wedge Y) + \omega(\nabla_X (Z \wedge Y)) \]

(4.7)

\[ (-1)^{(l-1)(m-1)} L_Y i_{Z \wedge X} \omega = \nabla_Y \omega(Z \wedge X) + \omega(\nabla_Y (Z \wedge X)). \]

(4.8)

Substituting (4.7) and (4.8) into (4.6) and using the fact that

\[ q = (-1)^{(k-1)(m-1)}(-1)^{l(m-1)} = (-1)^{(l-1)(m-1)}(-1)^{k(m-1)} \]

gives

\[ q L_{X \wedge Y} i_Z \omega = (-1)^{(l-1)(m-1)}(-1)^k \nabla_X \omega(Z \wedge Y) + \omega(\nabla_X (Z \wedge Y)) \]
\[ + (-1)^k ( \nabla_Y \omega(Z \wedge X) + \omega(\nabla_Y (Z \wedge X))) \]
\[ - q(-1)^j \omega(Z \wedge [X, Y]). \]

(4.9)
Using axiom (v) of Definition 3.1, (4.9) is further expanded as
\[ qL_{X \wedge Y} i_Z \omega = (-1)^{(m-1)}(-1)^k \nabla_X \omega (Z \wedge Y) \]
\[ + (-1)^{(m-1)}(-1)^k \omega ((\nabla_X Z) \wedge Y) \]
\[ + (-1)^{(m-1)}(-1)^l(-1)^{(l-1)}\omega (Z \wedge \nabla_X Y) \]
\[ + (-1)^{(k-1)}\nabla_Y \omega (Z \wedge X) \]
\[ + (-1)^{(m-1)}\omega ((\nabla_Y Z) \wedge X)) \]
\[ + (-1)^{(m-1)}(-1)^{(l-1)}\omega (Z \wedge \nabla_Y X) \]
\[ - q(-1)^{(l)}\omega (Z \wedge [X, Y]). \]  
(4.10)

Swapping all the wedge products in (4.10) with the appropriate signs gives
\[ qL_{X \wedge Y} i_Z \omega = (-1)^{l} \nabla_X \omega (Y \wedge Z) + (-1)^{kl} \omega (Y \wedge \nabla_X Z) \]
\[ + (-1)^{l} \omega ((\nabla_X Y) \wedge Z) + (-1)^{k(l-1)} \nabla_Y \omega (X \wedge Z) \]
\[ + \omega (X \wedge \nabla_Y Z) + (-1)^{k(l-1)}\omega ((\nabla_Y X) \wedge Z) \]
\[ - (-1)^{l} \omega ([X, Y] \wedge Z). \]  
(4.11)

(4.11) can be rewritten as
\[ qL_{X \wedge Y} i_Z \omega = (-1)^{l} i_Y (\nabla_X \omega)(Z) + (-1)^{kl} \omega (Y \wedge \nabla_X Z) \]
\[ + (-1)^{k(l-1)} i_X (\nabla_Y \omega)(Z) + \omega (X \wedge \nabla_Y Z) \]
\[ + (-1)^{l} \omega (T(X, Y) \wedge Z), \]  
(4.12)

where \( T \) is the higher torsion of \( \nabla \). Since \( \nabla \) is torsion-free, the last term vanishes and we obtain
\[ qL_{X \wedge Y} i_Z \omega = (-1)^{l} i_Y (\nabla_X \omega)(Z) + (-1)^{kl} \omega (Y \wedge \nabla_X Z) \]
\[ + (-1)^{k(l-1)} i_X (\nabla_Y \omega)(Z) + \omega (X \wedge \nabla_Y Z). \]  
(4.13)

Since \( \nabla \) is torsion-free, Theorem 3.15 implies that \( \nabla \) is induced. Hence, the second term in (4.5) decomposes as
\[ \omega (\nabla_X \wedge Y Z) = \omega (X \wedge \nabla_Y Z) + (-1)^{kl} \omega (Y \wedge \nabla_X Z). \]  
(4.14)

Substituting (4.13) and (4.14) into (4.5) proves (4.4) for the case when \( m \geq k + l - 1 \).

5. Higher Curvature

In this section, we propose two notions of curvature for higher connections. The first type, which we call the full higher curvature, is modeled after the classical curvature for an affine connection. However, unlike the classical case, the full higher curvature is not \( C^\infty(M) \)-linear in its arguments. The second type, which we call the stratified higher curvature, is \( C^\infty(M) \)-linear and takes advantage of the classification theorem for higher connections (see Theorem 3.8).
Definition 5.1. For $X \in A^k(M)$, $Y \in A^l(M)$, $Z \in A(M)$, the full higher curvature associated to a higher connection $\nabla$ is defined by

$$\mathcal{R}(X, Y)Z := \nabla_X \nabla_Y Z - (-1)^{(k-1)(l-1)}\nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z. \quad (5.1)$$

Note that for $X \in A^k(M)$, $Y \in A^l(M)$, $Z \in A^m(M)$, we have $\mathcal{R}(X, Y)Z \in A^{k+m+l-2}(M)$. In addition, the same argument used in Proposition 3.13 implies that

$$\mathcal{R}(X, Y)Z = -(-1)^{(k-1)(l-1)}\mathcal{R}(Y, X)Z \quad (5.2)$$

for $X \in A^k(M)$, $Y \in A^l(M)$, $Z \in A(M)$. The extent to which $\mathcal{R}$ fails to be $C^\infty(M)$ linear is given by the following proposition:

Proposition 5.2. Let $\nabla$ be a higher connection and let $\mathcal{R}$ be the associated full higher curvature. For $X \in A^k(M)$, $Y \in A^l(M)$, $Z \in A(M)$, and $f \in C^\infty(M)$, $\mathcal{R}$ satisfies

(i) $\mathcal{R}(fX, Y)Z = f\mathcal{R}(X, Y)Z + e_1(i_fY, X, Z)$ where

$$e_1(i_fY, X, Z) := (-1)^{k(l-1)}(\nabla_{i_fY \wedge X}Z - i_fY \wedge \nabla_X Z) \quad (5.3)$$

(ii) $\mathcal{R}(X, Y)(fZ) = f\mathcal{R}(X, Y)Z + e_2(X, Y, f)\wedge Z$ where

$$e_2(X, Y, f) := -(-1)^l(\nabla_X (i_fY) + (-1)^k\nabla_Y (i_fX) + (-1)^k i_f[X,Y]). \quad (5.4)$$

Proof. For (i), we have

$$\mathcal{R}(fX, Y)Z = \nabla f X \nabla_Y Z - (-1)^{(k-1)(l-1)}\nabla_Y \nabla f X Z - \nabla_{[fX,Y]}Z$$

$$= f\nabla_X \nabla_Y Z - (-1)^{(k-1)(l-1)}[(1)^{l-1}i_fY \wedge \nabla_X Z + f\nabla_Y \nabla_X Z]$$

$$- [f\nabla_{[X,Y]}Z - \nabla_{X \wedge i_fY}Z]$$

$$= f\mathcal{R}(X, Y)Z + \nabla_{X \wedge i_fY}Z - (-1)^{k(l-1)}i_fY \wedge \nabla_X Z$$

$$= f\mathcal{R}(X, Y)Z + (-1)^{k(l-1)}\nabla_{i_fY \wedge X}Z - (-1)^{k(l-1)}i_fY \wedge \nabla_X Z$$

$$= f\mathcal{R}(X, Y)Z + (-1)^{k(l-1)}[\nabla_{i_fY \wedge X}Z - i_fY \wedge \nabla_X Z]$$

where we used Corollary 3.3(i) and Proposition 2.6(vi) in the second equality.

For (ii), Corollary 3.3(i) implies

$$\mathcal{R}(X, Y)(fZ) = \nabla_X \nabla_Y (fZ) - (-1)^{(k-1)(l-1)}\nabla_Y \nabla_X (fZ) - \nabla_{[X,Y]}(fZ)$$

$$= \nabla_X [(-1)^{l-1}i_fY \wedge Z + f\nabla_Y Z]$$

$$- (-1)^{(k-1)(l-1)}\nabla_Y [(-1)^{k-1}i_fX \wedge Z + f\nabla_X Z]$$

$$- [(-1)^{k+l-2}i_f[X,Y] \wedge Z + f\nabla_{[X,Y]}Z]$$

$$= B_1 - (-1)^{(k-1)(l-1)}B_2 - B_3,$$
where
\[
\begin{align*}
B_1 & := \nabla_X[(-1)^{l-1}i_f Y \wedge Z + f\nabla_Y Z] \\
B_2 & := \nabla_Y[(-1)^{k-1}i_f X \wedge Z + f\nabla_X Z] \\
B_3 & := [(-1)^{k+l-2}i_f [X,Y] \wedge Z + f\nabla_{[X,Y]} Z].
\end{align*}
\]

Expanding \(B_1\) further gives
\[
\begin{align*}
B_1 & = (-1)^{l-1}[\nabla_X(i_f Y) \wedge Z + (-1)^{(l-1)(k-1)}i_f Y \wedge \nabla_X Z] + (-1)^{k-1}i_f X \wedge \nabla_Y Z \\
& \quad + f\nabla_X \nabla_Y Z
\end{align*}
\]

Swapping the \(X\)'s and \(Y\)'s and \(k\)'s and \(l\)'s, we obtain \(B_2\):
\[
\begin{align*}
B_2 & = (-1)^{k-1}\nabla_Y(i_f X) \wedge Z + (-1)^{(k-l)(k-1)}i_f X \wedge \nabla_Y Z + (-1)^{l-1}i_f Y \wedge \nabla_X Z \\
& \quad + f\nabla_Y \nabla_X Z.
\end{align*}
\]

Using the expanded \(B_1\) and \(B_2\), we have
\[
\mathcal{R}(X,Y)(fZ) = f\mathcal{R}(X,Y)Z
\]
\[
- (-1)^l(\nabla_X(i_f Y) + (-1)^k\nabla_Y(i_f X) + (-1)^k i_f[Z,Y]) \wedge Z.
\]

The following is an immediate consequence of Proposition 5.2.

**Corollary 5.3.** Let \(\nabla\) be a higher connection and let \(\mathcal{R}\) be the associated full higher curvature. Then
\[
\begin{align*}
(i) \quad & \mathcal{R}(fX,Y)Z = f\mathcal{R}(X,Y)Z \quad \text{for} \quad Y \in A^1(M), \quad X \in A^k(M) \\
(ii) \quad & \mathcal{R}(X,Y)(fZ) = f\mathcal{R}(X,Y)Z \quad \text{for} \quad X,Y \in A^1(M), \\
\end{align*}
\]

where \(f \in C^\infty(M)\) and \(Z \in \mathcal{A}(M)\).

**Remark 5.4.** Note that if \(\nabla\) is induced in the sense of Definition 3.5, \(e_1\) in Proposition 5.2 still does not vanish, but reduces to
\[
X \wedge \nabla_i Y Z.
\]

The behavior of \(\mathcal{R}\) in Proposition 5.2 along with Theorem 3.8 motivates the following definition:

**Definition 5.5.** Let \(\nabla\) be a higher connection and let \((\tilde{\nabla}, \{F^j\})\) be the unique pair associated to \(\nabla\) by Theorem 3.8. Extend \(F^j\) to \(A^j(M) \times \mathcal{A}(M)\) as in Corollary 3.11 and extend \(\tilde{\nabla}\) to an induced higher connection. The **stratified higher curvature** associated to \(\nabla\) is the pair \((R, \{Q^j\})\) where \(R\) is the ordinary curvature associated to the affine connection \(\tilde{\nabla}\), and
\[
Q^j : A^j(M) \times A^1(M) \times \mathcal{A}(M) \rightarrow \mathcal{A}(M)
\]
is defined by
\[
Q^j(X,Y)Z := F^j(X, \tilde{\nabla} Y Z) - \tilde{\nabla} Y F^j(X, Z) - F^j([X,Y], Z) + F^j(\tilde{\nabla}_X Y, Z),
\]
for $X \in A^j(M)$, $Y \in A^1(M)$, and $Z \in \mathcal{A}(M)$, where $\nabla_Y F^j(X,Z)$ means $\nabla_Y (F^j(X,Z))$.

**Remark 5.7.** Since the curvature operators associated to the $F^j$'s are $\mathcal{C}^\infty(M)$-linear, we interpret them as the curvature operators associated to the $F^j$'s.

The next result gives the relationship between the full higher curvature and the stratified higher curvature associated to $\nabla$ by Theorem 3.8. In addition, let $\mathcal{R}$ be the full higher curvature associated to $\nabla$, $\overline{\mathcal{R}}$ be the full higher curvature associated to $\nabla$, $\tilde{\mathcal{R}}$ be the full higher curvature associated to $\nabla$, $\tilde{\mathcal{R}}$.

**Proposition 5.8.** Let $\nabla$ be a higher connection and let $(R, \{Q^j\})$ be the stratified higher curvature associated to $\nabla$. In addition, let $(\tilde{\nabla}, \{F^j\})$ be the unique pair associated to $\nabla$ by Theorem 3.8. Then for $X \in A^j(M)$, $Y \in A^1(M)$, $Z \in \mathcal{A}(M)$, and $f \in \mathcal{C}^\infty(M)$, $Q^j$ satisfies

(i) $Q^j(fX,Y)Z = fQ^j(X,Y)Z$

(ii) $Q^j(X,Y)fZ = fQ^j(X,Y)Z$

(iii) $Q^j(X,fY)Z = fQ^j(X,Y)Z$.

**Proof.** Extend $F^j$ to $A^j(M) \times \mathcal{A}(M)$ as in Corollary 3.11. For (i), we have

$$Q^j(fX,Y)Z = F^j(fX, \tilde{\nabla}_Y Z) - \tilde{\nabla}_Y F^j(fX,Z) - F^j([fX,Y],Z) + F^j(\tilde{\nabla}_Y fX, Y)$$

$$= fF^j(X, \tilde{\nabla}_Y Z) - \tilde{\nabla}_Y fF^j(X,Z) - fF^j([X,Y],Z) + (Y f)F^j(X,Z)$$

$$+ fF^j(\tilde{\nabla}_X Y, Z)$$

$$= fF^j(X, \tilde{\nabla}_Y Z) - (Y f)F^j(X,Z) - f\tilde{\nabla}_Y fF^j(X,Z) - fF^j([X,Y],Z)$$

$$+ (Y f)F^j(X,Z) + fF^j(\tilde{\nabla}_X Y, Z)$$

$$= fQ^j(X,Y)Z,$$

where (vi) of Proposition 2.6 is used in the second equality. For (ii), we have

$$Q^j(X,Y)fZ = F^j(X, \tilde{\nabla}_Y fZ) - \tilde{\nabla}_Y F^j(X,fZ) - F^j([X,Y], fZ) + F^j(\tilde{\nabla}_Y fX, fZ)$$

$$= (Y f)F^j(X,Z) + fF^j(X, \tilde{\nabla}_Y Z) - (Y f)F^j(X,Z) - f\tilde{\nabla}_Y fF^j(X,Z)$$

$$- fF^j([X,Y],Z) + fF^j(\tilde{\nabla}_X Y, Z)$$

$$= fQ^j(X,Y)Z.$$

For (iii), we have

$$Q^j(X,fY)Z = F^j(X, \tilde{\nabla}_fY Z) - \tilde{\nabla}_fY F^j(X,Z) - F^j([X,fY],Z) + F^j(\tilde{\nabla}_X fY, Z)$$

$$= fF^j(X, \tilde{\nabla}_Y Z) - f\tilde{\nabla}_Y fF^j(X,Z) - fF^j([X,Y],Z) - F^j([X,f] \wedge Y, Z)$$

$$+ F^j([X,f] \wedge Y, Z) + fF^j(\tilde{\nabla}_X Y, Z)$$

$$= fQ^j(X,Y)Z,$$

where (v) and (vi) of Proposition 2.6 is used in the second equality. \qed

**Remark 5.7.** Since the $Q^j$’s are $\mathcal{C}^\infty(M)$-linear, we interpret them as the curvature operators associated to the $F^j$’s.
to $\nabla$ (as an induced higher connection), and let $(R, \{Q^j\})$ be the stratified higher curvature associated to $\nabla$. Then

$$R(X, Y)Z = \tilde{R}(X, Y)Z + Q^j(X, Y)Z - F^j(\tilde{\nabla}_X Y, Z)$$

for $X \in A^j(M)$, $Y \in A^1(M)$, and $Z \in A(M)$.

**Proof.** Extend $F^j$ to $A^j(M) \times A(M)$ as in Corollary 3.11. Using Corollary 3.11, we have

$$R(X, Y)Z = \nabla_X \nabla_Y Z - (-1)^{(j-1)/2} \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$= \tilde{\nabla}_X \tilde{\nabla}_Y Z + F^j(X, \tilde{\nabla}_Y Z) - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_Y F^j(X, Z)$$

$$- \tilde{\nabla}_{[X,Y]} Z - F^j([X,Y], Z)$$

$$= \tilde{R}(X, Y)Z + Q^j(X, Y)Z - F^j(\tilde{\nabla}_X Y, Z).$$

□

**Definition 5.9.** Let $\nabla$ be a higher connection and let $(R, \{Q^j\})$ be its associated stratified higher curvature. $\nabla$ is a flat higher connection if $R \equiv 0$ and $Q^j \equiv 0$ for all $j$.

To set up the next result, let $\nabla$ be a higher connection and let $(\tilde{\nabla}, \{F^j\})$ be the unique pair associated to $\nabla$ by Theorem 3.8. In addition, let $\Omega^p(M)$ be the space of $p$-forms. For $X \in A^j(M)$, $Z \in A^m(M)$, and $\omega \in \Omega^{j+m-1}(M)$, define

$$T_{j,m}^i(\omega, X, Z) := \omega(F^j(X, Z)) \in C^\infty(M).$$

(5.5)

**Theorem 5.10.** Let $\nabla$ be a higher connection and let $(\tilde{\nabla}, \{F^j\})$ be the unique pair associated to $\nabla$ by Theorem 3.8 and suppose that

(i) $\tilde{\nabla}$ is flat as an affine connection on $TM$,

(ii) $\tilde{\nabla}$ is torsion-free as an affine connection on $TM$,

(iii) $\tilde{\nabla}_Y T_{j,m}^i \equiv 0$ for all $Y \in A^1(M)$, where $T_{j,m}^i$ is defined by (5.5).

Then $\nabla$ is a flat higher connection.

**Proof.** Let $(R, \{Q^j\})$ be the stratified higher curvature associated to $\nabla$. By condition (i), $R \equiv 0$. We now show that (ii) and (iii) imply that $Q^j \equiv 0$. Let $X \in A^j(M)$, $Y \in A^1(M)$, and $Z \in A^m(M)$. Since $\tilde{\nabla}$ is torsion-free as an affine connection on $TM$, it is also torsion-free as an induced higher connection by Proposition 3.14. This implies

$$Q^j(X, Y)Z = F^j(X, \tilde{\nabla}_Y Z) - \tilde{\nabla}_Y F^j(X, Z) + F^j(\tilde{\nabla}_Y X, Z).$$

(5.6)
Let $\omega \in \Omega^{j+m-1}(M)$. Then

$$(\bar{\nabla}_Y T^{j,m})(\omega, X, Z) = Y(T^{j,m}(\omega, X, Z)) - T^{j,m}(\bar{\nabla}_Y \omega, X, Z) - T^{j,m}(\omega, \bar{\nabla}_Y X, Z)$$

$$= Y(\omega(F^j(X, Z))) - (\bar{\nabla}_Y \omega)(F^j(X, Z)) - \omega(F^j(\bar{\nabla}_Y X, Z))$$

$$= (\bar{\nabla}_Y \omega)(F^j(X, Z)) + \omega(\bar{\nabla}_Y F^j(X, Z)) - (\bar{\nabla}_Y \omega)(F^j(X, Z))$$

$$= \omega(\bar{\nabla}_Y F^j(X, Z)) - \omega(F^j(\bar{\nabla}_Y X, Z))$$

$$= -\omega(Q^j(X, Y)Z),$$

where the last equality follows from (5.6). Condition (iii) then implies that $Q^j(X, Y)Z \equiv 0$. This completes the proof. \(\square\)

**Corollary 5.11.** Let $\nabla$ be a higher connection and let $(\bar{\nabla}, \{F^j\})$ be the unique pair associated to $\nabla$ by Theorem 3.8. If $(\bar{\nabla}, \{F^j\})$ satisfies (ii) and (iii) of Theorem 5.10, then

$$R(X, Y)Z = \bar{R}(X, Y)Z - F^j(\bar{\nabla}_X Y, Z)$$

for $X \in A^j(M), Y \in A^1(M),$ and $Z \in A(M)$, where $R$ is the full higher curvature associated to $\nabla$ and $\bar{R}$ is the full higher curvature associated to $\bar{\nabla}$ (as an induced higher connection).

**Proof.** Let $(R, \{Q^j\})$ be the stratified higher curvature associated to $\nabla$. The proof of Theorem 3.10 shows that $Q^j \equiv 0$ if (ii) and (iii) of Theorem 5.10 are satisfied. Proposition 5.8 then implies the corollary. \(\square\)

### 6. Conclusion

In this paper, the notion of higher connections has been introduced as part of a program in differential geometry to extend the familiar constructions and operations for vector fields to multivector fields (MVF). The aforementioned program is motivated by generalized geometry and string theory, and is based on the idea of treating the full exterior bundle $\wedge^\bullet TM$ as an extended tangent bundle with the Schouten-Nijenhuis bracket playing the role of the Lie bracket of vector fields. Consequently, in the context of this program, a higher connection on the full exterior bundle $\wedge^\bullet TM$ is the analogue of an affine connection on the tangent bundle $TM$.

We now conclude the paper with the following open questions:

1. Is there a notion of “higher Riemannian manifolds” which extends the notion of Levi-Civita connection to (non-induced) higher connections?
2. If the answer to 1 is yes, is there a higher analogue of the Einstein field equations of general relativity which can be expressed in terms of the full or stratified higher curvatures?

3. What is the geometric meaning of the terms $e_1$ and $e_2$ in Proposition 5.2?

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