Boosted Kaluza-Klein Magnetic Monopole

S. Sedigheh Hashemi and Nematollah Riazi∗
Department of Physics, Shahid Beheshti University, G.C., Evin, Tehran 19839, Iran

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Abstract

We consider a Kaluza-Klein vacuum solution which is closely related to the Gross-Perry-Sorkin (GPS) magnetic monopole. The solution can be obtained from the Euclidean Taub-NUT solution with an extra compact fifth spatial dimension within the formalism of Kaluza-Klein reduction. We study its physical properties as appearing in (3 + 1) spacetime dimensions, which turns out to be a static magnetic monopole. We then boost the GPS magnetic monopole along the extra dimension, and perform the Kaluza-Klein reduction. The resulting four-dimensional spacetime is a rotating stationary system, with both electric and magnetic fields. In fact, after the boost the magnetic monopole turns into a string connected to a dyon.

Keywords: Kaluza-Klein theory, magnetic monopole, extra dimensions

1 Introduction: Kaluza-Klein theory

The possibility that the universe is embedded in a (4 + d) dimensional world has gained the attention of many researches. In the Randall and Sundrum [1] theory the matter and fields are restricted to a four dimensional space-time known as brane which is embedded in a five dimensional spacetime (bulk). In Space-Time-Matter (STM) theory [2] all physical quantities such as matter density and pressure, gain a geometrical interpretation. Among these higher dimensional theories, the original Kaluza-Klein theory unifies gravity and electromagnetism [3], assuming that the fifth dimension is compact [4].

Kaluza’s idea was that the universe has four spatial dimensions, and the extra dimension is compactified to form a circle so small as to be unobservable [5]. Klein’s contribution was to make a reasonable physical basis for the compactification of the fifth dimension [6], [7]. This school of thinking later led to the eleven-dimensional supergravity theories in 1980s and to the “theory of everything” or ten-dimensional superstring theory [8].

With this unification the vacuum (4 + 1) dimensional Kaluza-Klein solutions that is \( \hat{G}_{AB} = 0 \) (the indices \( A, B, ... \) run over \( 0...4 \)), will reduce to the (3 + 1) Einstein field equations with effective matter and the curvature in (4 + 1) spacetime induces matter in (3 + 1) dimensional spacetime [9]. In the context of Kaluza-Klein, the Einstein tensor has the usual definition \( \hat{G}_{AB} \equiv R_{AB} - \frac{1}{2} \hat{R} \hat{g}_{AB} \), where \( \hat{R}_{AB} \) and \( \hat{R} = \hat{g}_{AB} \hat{R}^{AB} \) are the five-dimensional Ricci tensor and scalar, respectively, and \( \hat{g}_{AB} \) is the metric tensor in five dimensions [8]. The \( \mu \nu \) part of \( \hat{g}^{AB} \), which is \( \hat{g}^{\mu \nu} \) is the contravariant four dimensional metric tensor, and the electromagnetic potential and

∗Electronic address: n.riazi@sbu.ac.ir (Corresponding author).
the scalar field are given by $A^\mu$ and $\phi$, respectively. The general correspondence between the above components is given by

$$
\hat{g}^{AB} = \begin{pmatrix} g^{\mu\nu} & -\kappa A^\mu \\ -\kappa A^\nu & \kappa^2 A^\sigma A_\sigma + \phi^2 \end{pmatrix},
$$

(1)

where $\kappa$ is a coupling constant for the electromagnetic potential $A^\mu$ [10,11].

Many spherically symmetric solutions of Kaluza-Klein type are investigated in [12] and [13]. Among these solutions, the Gross-Perry-Sorkin (GPS) spacetime [4,14] is one of the exact vacuum solutions of Einstein field equations in five-dimensional gravity which is stationary and without event horizon representing a magnetic monopole which is usually called the Kaluza-Klein monopole [15,16]. As it is well known, the theory of magnetic monopole was formulated by Dirac in 1931 [17]. He showed that the electric charge quantization can be explained by the existence of a magnetic monopole. In addition to magnetic charge, monopoles are characterized by their peculiar topology. They carry one unit of Euler character, and consequently one can construct stationary dipole solutions from them [18].

The Kaluza-Klein monopole has an important role in M/String theory. As an example, a Ricci-flat eleven dimensional Lorentzian metric can be obtained from Kaluza-Klein monopole metric times six flat Euclidean dimensions, which when reduced to ten dimensional spacetime can be interpreted as a D6 brane solution of IIA string theory [15].

In this paper, we consider a vacuum solution of Kaluza-Klein theory in five-dimensional spacetime which is closely related to the Taub-NUT and GPS metric. The Taub-NUT solution has many interesting features; it carries a particular type of charge (NUT charge), which has topological origins and can be regarded as “gravitational magnetic charge” [19,20]. We boost the magnetic monopole along the fifth dimension and investigate its properties in the four-dimensional spacetime by using the Kaluza-Klein reduction. The monopole turns into a dyon connected to a magnetically charged string.

The plan of this paper is as follows. In section 2 we will review a Taub-NUT-like Kaluza-Klein solution and investigate its physical properties in four dimensions. In section 3 we will study the boosted Kaluza-Klein magnetic monopole, and explore its physical properties. In the last section we will draw our main conclusions.

## 2 Kaluza-Klein Magnetic Monopole and Taub-NUT Solution

In this section, we first review the main features of the Kaluza-Klein monopole before the boost. The Kaluza-Klein monopole, known also as Gross-Perry-Sorkin solution, is a generalization of the self-dual Euclidean Taub-NUT solution. The Taub-NUT solution was first discovered by Taub (1951), and subsequently by Newman, Tamburino and Unti (1963) as a generalization of the Schwarzschild spacetime [21,22]. This solution is a single, non-radiating and analytic extension of the Taub universe, the anisotropic but spatially homogeneous vacuum solution of Einstein field equations with topology $R^1 \times S^3$. The Taub metric is given by

$$
ds^2 = \frac{1}{V(t)}dt^2 + 4b^2V(t)(d\psi + \cos \theta d\phi)^2 + (t^2 + b^2)(d\theta^2 + \sin \theta^2d\phi^2),
$$

(2)

where $V(t) = -1 + 2(mt + b^2)(t^2 + b^2)^{-1}$, $m$ and $b$ are positive constants, $\psi, \phi, \theta$ are Euler angels with usual ranges [23]. The Taub-NUT solution is nowadays being involved in the context of
higher-dimensional theories of semi-classical quantum gravity \cite{21}. As an example, in the work by Gross and Perry \cite{4} and Sorkin \cite{25}, soliton solutions were obtained by embedding the Taub-NUT gravitational instanton inside the five dimensional Kaluza-Klein manifold \cite{21}. One such solution which obeys the Dirac quantization condition is considered in \cite{4}.

The Kaluza-Klein monopole of Gross-Perry-Sorkin is represented by the following metric \cite{4}

\[ ds^2 = -dt^2 + V(dx^5 + 4m(1 - \cos \theta)d\phi)^2 + \frac{1}{V}(dr^2 + r^2d\theta^2 + r^2\sin^2 \theta d\phi^2), \]  

where

\[ \frac{1}{V} = 1 + \frac{4m}{r}. \]  

The Taub-NUT instanton is obtained by putting \( dt = 0 \). For this solution the coordinate singularity is located at \( r = 0 \), which is called NUT singularity. This can be vanished if the extra coordinate \( x^5 \) is periodic with period \( 16\pi m = 2\pi R \), where \( R \) is the radius of the fifth dimension. Thus \( m = \sqrt{\frac{\pi G}{2e}} \) \cite{26}. The gauge field \( A_\phi \) is given by \( A_\phi = 4m(1 - \cos \theta) \), and the magnetic field is \( B = 4mr/r^3 \), which is clearly that of a monopole and has a Dirac string singularity in the range \( r = 0 \) to \( \infty \). The magnetic charge of this monopole is \( g = m/\sqrt{\pi G} \) which has one unit of Dirac charge. In this model, the total magnetic flux is constant. For this solution, the soliton mass is determined to be \( M^2 = m_p^2/16\alpha \) where \( m_p \) is the Planck mass and \( \alpha \) is the fine-structure constant.

In our previous work \cite{27}, we presented a metric which is a vacuum five dimensional solution, having some properties in common with the monopole of Gross-Perry-Sorkin, despite some differences. In this part, we will briefly review the results we obtained there (see \cite{27}), and will then extend the results in coming sections. The metric is given by

\[ ds^2_{(5)} = -dt^2 + (1 - \frac{2m}{r}) (dr^2 + r^2d\theta^2 + r^2\sin^2 \theta d\phi^2) + \left( \frac{4m^2}{1 - \frac{2m}{r}} \right) (d\psi + \cos \theta d\phi)^2, \]  

where, the extra coordinate is represented by \( \psi \). The coordinates take on the usual ranges \( r \geq 0, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi \) and \( 0 \leq \psi \leq 2\pi \). It should be noted that the metric \cite{3} can be obtained from \cite{3} by replacing \( dx^5 = 2m(d\psi + d\phi) \) and \( m \to -m/2 \). It should be noted that, for negative \( m \), we will still have a vacuum solution. In the next section, we will consider this case for some of our results.

The Killing vectors associated with metric \cite{3} are given by

\[ K_0 = (1, 0, 0, 0, 0), \quad K_1 = (0, 0, 0, 0, 1), \quad K_2 = (0, 0, 0, 1, 0), \]
\[ K_3 = (0, 0, -\sin \phi, -\cot \theta \cos \phi, \csc \theta \cos \phi), \]
\[ K_4 = (0, 0, \cos \phi, -\cot \theta \sin \phi, \csc \theta \sin \phi), \]  

which are the same as in the Taub-NUT metric discussed in \cite{28}, where the authors studied spinning particles in the Taub-NUT space.

The gauge field \( A_\mu \), and the scalar field \( \phi \) deduced from the metric \cite{11} with the help of \cite{11} are \( A_\phi = \cos \theta/\kappa \), and \( \phi^2 = 4m^2/(1 - \frac{2m}{r}) \), respectively. Moreover, the electromagnetic tensor is \( F_{\theta\phi} = -F_{\phi\theta} = -\sin \theta/\kappa \), which corresponds to a radial magnetic field \( B_\phi = 1/\kappa r^2 \) with a magnetic charge \( Q_M = 1/\kappa \). The total magnetic flux through any spherical surface centered at

\footnote{Note that not only the sign of \( m \) is taken differently from \cite{3}, but also the structure of the metric is different, leading to some essentially different results.}
the origin can be calculated via \([14]\) leading to the result \(\Phi_B = 2\pi/\kappa\), which is a constant (i.e. we have a point-like magnetic charge).

The four dimensional metric deduced from Eq. \([14]\) with the use of \([11]\) leads to the following asymptotically flat spacetime:

\[
ds^2_{(4)} = -dt^2 + \left(1 - \frac{2m}{r}\right)dr^2 + r^2 \left(1 - \frac{2m}{r}\right) (d\theta^2 + \sin^2 \theta d\phi^2).
\]

The four dimensional metric \([7]\) has two curvature singularities at \(r = 0\) and \(r = 2m\) unless \(m < 0\). If we calculate the surface area of a \(S^2\) hypersurface at constant \(t\) and \(r\), we see that the surface area \(A(r)\) becomes zero at \(r = 0\), as well as \(r = 2m\). This means that the \(r = 2m\) hypersurface is in fact a point (i.e. a sphere with zero surface area). For \(r > 2m\) the signature of the metric is proper \((- , + , + , +)\) but for the range \(0 < r < 2m\) the signature of the metric will be improper and non-Lorentzian \((- , - , - , -)\), thus the patch \(r < 2m\) is excluded from the physical spacetime. therefore, this spacetime is considered only in the range \(r \geq 2m\). Since the range \(0 < r < 2m\) is removed from the spacetime, there remains only one curvature singularity at \(r = 2m\).

By computing the components of the energy-momentum tensor for the metric \([7]\), one can show that the effective matter field around the singularity can not be considered as an ultra-relativistic quantum field (or radiation) in contrast to the Kaluza-Klein solitons described in \([29]\). On the other hand, the gravitational mass was derived in two ways and it was shown to vanish \((M_g = 0)\).

It turns out that the Kaluza-Klein monopole in isotropic coordinates gets the asymptotic form

\[
ds^2 = -(1 + \frac{2m}{r})^{-1/2}dt^2 + (1 + \frac{2m}{r})^{1/2}(dr^2 + r^2 d\Omega^2).
\]

If in the process of compactification from \((4 + 1)\) to \((3 + 1)\) dimensions we use the ansatz

\[
\hat{G}_{AB} = \phi^\beta \left( \begin{array}{cc}
g_{\mu\nu} + \phi A_\mu A_\nu & \phi A_\mu \\
\phi A_\nu & \phi 
\end{array} \right),
\]

then the choice of \(\beta\) for which \(\phi\) does not appear explicitly is called the Einstein frame. Using the above equation will lead to the four dimensional metric

\[
ds^2_{(4)} = \phi^{-\beta} \left[-dt^2 + \left(1 - \frac{2m}{r}\right)dr^2 + r^2 \left(1 - \frac{2m}{r}\right) d\Omega^2 \right].
\]

By choosing \(\phi^{-\beta} = (1 - \frac{2m}{r})^{-1/2}\), equation \([10]\) reduces to

\[
ds^2 = -(1 - \frac{2m}{r})^{-1/2}dt^2 + (1 - \frac{2m}{r})^{1/2}(dr^2 + r^2 d\Omega^2),
\]

which is the same as \([8]\) if we replace \(m\) by \(-m\).

### 3 The Boosted Kaluza-Klein Magnetic Monopole

In this section, we apply a boost to the Kaluza-Klein magnetic monopole which satisfies the vacuum Einstein field equations. The proposed boost is along the extra dimension \(\psi\) with the
boost parameter \( \alpha \). We consider metric (5) with coordinate renamed as \((t', r', \theta', \phi', \psi')\), and define the boosted coordinates as \((t, r, \theta, \phi, \psi)\). Then we apply the following transformations

\[
\begin{align*}
t' &= t \cosh \alpha - \psi \sinh \alpha, \\
\psi' &= \psi \cosh \alpha - t \sinh \alpha,
\end{align*}
\]

with the above transformations, the metric becomes

\[
ds^2 = -\left( \cosh^2 \alpha - \frac{4m^2 \sinh^2 \alpha}{1 - \frac{2m}{r}} \right) dt^2 + \left( 1 - \frac{2m}{r} \right) (dr^2 + r^2 d\theta^2) \\
+ \left( \frac{4m^2 \cos^2 \theta}{1 - \frac{2m}{r}} + r^2 \left( 1 - \frac{2m}{r} \right) \sin^2 \theta \right) d\phi^2 + \left( -\sin^2 \alpha + \frac{4m^2 \cos^2 \alpha}{1 - \frac{2m}{r}} \right) d\psi^2 \\
+ \sinh 2\alpha \left( 1 - \frac{4m^2}{1 - \frac{2m}{r}} \right) dt d\psi + \frac{8m^2}{1 - \frac{2m}{r}} \cos \theta \cosh \alpha d\phi d\psi - \frac{8m^2}{1 - \frac{2m}{r}} \cos \theta \sinh \alpha d\phi dt.
\]

which is no longer a static solution because of the \( d\phi dt \) term. The metric, however, remains stationary (i.e. \( \partial g_{AB} / \partial t = 0 \)). It should be stressed that for obtaining the main results of the present paper, which appear after the boost, it is not essential to choose a particular sign for \( m \) (we will consider both possibilities in what follows). Let us rewrite (14) in the form

\[
ds^2 = -\left( \cosh^2 \alpha - \frac{4m^2 \sinh^2 \alpha}{1 + \frac{2m}{r}} \right) dt^2 + \left( 1 + \frac{2m}{r} \right) (dr^2 + r^2 d\theta^2) \\
+ \left( \frac{4m^2 \cos^2 \theta}{1 + \frac{2m}{r}} + r^2 \left( 1 + \frac{2m}{r} \right) \sin^2 \theta \right) d\phi^2 + \left( -\sin^2 \alpha + \frac{4m^2 \cos^2 \alpha}{1 + \frac{2m}{r}} \right) d\psi^2 \\
+ \sinh 2\alpha \left( 1 - \frac{4m^2}{1 + \frac{2m}{r}} \right) dt d\psi + \frac{8m^2}{1 + \frac{2m}{r}} \cos \theta \cosh \alpha d\phi d\psi - \frac{8m^2}{1 - \frac{2m}{r}} \cos \theta \sinh \alpha d\phi dt.
\]

The transformed scalar and gauge fields from (14) are

\[
\phi^2 = \frac{4m^2 \cosh^2 \alpha}{1 - \frac{2m}{r}} - \sin^2 \alpha,
\]

\[
A_t = -\frac{(4m^2 r + 2m - r) \sinh 2\alpha}{2\kappa (4m^2 r \cosh^2 \alpha + (2m - r) \sinh^2 \alpha)},
\]

and

\[
A_\phi = \frac{4m^2 r \cosh \alpha \cos \theta}{\kappa (4m^2 r \cosh^2 \alpha + (2m - r) \sinh^2 \alpha)},
\]

which lead to the following electromagnetic field components

\[
F_{r\phi} = \frac{8m^3 \sinh^2 \alpha \cosh \alpha \cos \theta}{\kappa (4m^2 r \cosh^2 \alpha + (2m - r) \sinh^2 \alpha)^2} = -r \sin \theta B_\theta,
\]

\[
F_{rt} = \frac{4m^3 \sinh 2\alpha}{\kappa (4m^2 r \cosh^2 \alpha + (2m - r) \sinh^2 \alpha)^2} = E_r.
\]
\[ F_{\theta\phi} = -\frac{4m^2 r \cosh \alpha \sin \theta}{\kappa \left(4m^2 r \cosh^2 \alpha + (2m - r) \sinh^2 \alpha \right)} = r^2 \sin \theta B_r. \quad (21) \]

It can be seen that by setting \( \sinh \alpha = 0 \), the boosted solution will reduce to the previous metric \( \text{(5)} \). The results just obtained are valid for both signs of \( m \).

If we perform a Kaluza-Klein reduction, the four dimensional spacetime for the boosted metric \( \text{(14)} \) becomes

\[
\begin{align*}
\text{ds}^2_{(4)} &= -\frac{4m^2 r}{4m^2 r \cosh^2 \alpha + (2m - r) \sinh^2 \alpha} \, dt^2 + \left(1 - \frac{2m}{r}\right) \left(dr^2 + r^2 d\theta^2\right) \\
&\quad + r \left(-\sinh^2 \alpha \left(4m^2 \cos^2 \theta + (r - 2m)^2 \sin^2 \theta\right) - 4m^2 r (2m - r) \cosh^2 \alpha \sin^2 \theta\right) \left(d\phi^2 - \frac{4m^2 r \sinh \alpha \cos \theta}{4m^2 r \cosh^2 \alpha + (2m - r) \sinh^2 \alpha} \, dt d\phi\right). \quad (22)
\end{align*}
\]

The four dimensional solution \( \text{(22)} \) is singular at

\[ r_1 = -\frac{2m \sinh^2 \alpha}{4m^2 \cosh^2 \alpha - \sinh^2 \alpha}, \quad r_2 = 0. \quad (23) \]

The Ricci scalar for metric \( \text{(22)} \) can be calculated easily, and diverges at the following locations in addition to \( r_1 \), and \( r_2 \)

\[
\begin{align*}
 r_3 &= \frac{1}{7} \csc^2 \theta \left(6 \sin^2 \theta + \sqrt{\sin^2 \theta (64 - 43 \cos^2 \theta)} \right), \quad (24) \\
 r_4 &= \frac{1}{7} \csc^2 \theta \left(6 \sin^2 \theta - \sqrt{\sin^2 \theta (64 - 43 \cos^2 \theta)} \right), \quad (25) \\
 r_5 &= 0, \quad r_6 = 2, \quad r_7 = \frac{2}{7}. \quad (26)
\end{align*}
\]

in which \( r_3 \) to \( r_7 \) are simplified after arbitrarily setting \( \sinh \alpha = 1, \cosh \alpha = \sqrt{2} \), and \( m = 1 \). These points are curvature singularities since the Ricci scalar, and the nontrivial quadratic curvature invariant \( R^{\mu
u\alpha\beta} R_{\mu
u\alpha\beta} \) diverge. With these assumptions, \( r_1 \) equals \(-2/7\), therefore \( r_7 \) and \( r_1 \) are irrelevant since they become negative. It is also easy to see that \( r_4 \) is a negative function of coordinate \( \theta \). Therefore, only \( r_2 = r_5 = 0 \), and \( r_3(\theta) \) are relevant. It should be noted that the singularity \( r_3 \) can be removed by choosing suitable values for the parameters \( m \) and \( \alpha \) (e.g. \( m \simeq 5/100, \alpha \simeq \pi/6 \) lead to an imaginary value for \( r_3 \)). Also, the singularities for four dimensional metric deduced from \( \text{(15)} \) with arbitrarily setting \( \sinh \alpha = 1, \cosh \alpha = \sqrt{2}, \) and \( m = 1 \) are given by

\[
\begin{align*}
 r_1' &= -\frac{1}{7} \csc^2 \theta \left(6 \sin^2 \theta + \sqrt{\sin^2 \theta (64 - 43 \cos^2 \theta)} \right), \quad (27) \\
 r_2' &= \frac{1}{7} \csc^2 \theta \left(-6 \sin^2 \theta + \sqrt{\sin^2 \theta (64 - 43 \cos^2 \theta)} \right), \quad (28) \\
 r_3' &= \frac{2}{7}, \quad r_4' = 0 \quad r_5' = -2. \quad (29)
\end{align*}
\]

in which, \( r_1' \) is a negative function of \( \theta \), and \( r_5' \) is also negative, thus not physical.
In order to analyze the nature of the singularities, let us calculate the surface area of a $S^2$ hypersurface of constant $t$ and $r$ for metric \((22)\)

\[
A(r) = \int \sqrt{|g(2)|} \, dx^2 = 8\pi \sqrt{\frac{(2 - r)r^2}{7r + 2}} E\left(\frac{7r^2}{4} - 3r\right),
\]

where \(E\) stands for the elliptic integral. The result of the surface area is zero for the singularities \(r = 0,\) and \(r = 2\) (the integrand is simplified after setting \(\sinh \alpha = 1, \cosh \alpha = \sqrt{2}\)), which means that in our coordinate system \(r = 0,\) and \(r = 2\) are points. By using a coordinate transformation \(\tilde{r}^2 = r^2(1 - 2m/r),\) these two singularities will be transformed to \(\tilde{r} = 0.\) Furthermore, the determinant of the metric \((22)\) is positive in the range \(0 < r < 2m,\) and negative for the range \(r > 2m,\) thus the range \(0 < r < 2m\) is removed from the spacetime because of having an improper signature.

The infinite redshift surface \(r_{ir}\) for metric \((22)\) can exist if the condition \(r_{ir} > 0\) holds, i.e.

\[
4m^2 \cosh^2 \alpha - \sinh^2 \alpha < 0 \Rightarrow m^2 < \frac{1}{4} \tanh^2 \alpha.
\]

The Killing horizon can be obtained with the condition \(\xi^2 = 0,\) where \(\xi\) is a timelike Killing vector, that is \(\xi^\mu = \partial x^\mu / \partial t,\) therefore we have

\[
g_{\mu\nu} \xi^\mu \xi^\nu = -\frac{4m^2r}{4m^2r \cosh^2 \alpha + (2m - r) \sinh^2 \alpha} = 0,
\]

which gives \(r = 0.\) There are therefore no Killing or event horizons in the physical spacetime. The event horizon can be obtained by \(g^{rr} = 0,\) which also corresponds to \(r = 0.\)

For the four dimensional boosted Kaluza-Klein solution, we infer from \((20)\) and \((21)\) that the radial electric field \(E_r,\) and the radial magnetic field \(B_r\) do not vanish and consequently, one can find the net electric and magnetic fluxes through any two-dimensional surface.

The electric flux may be computed via \((30)\)

\[
Q_E = -\int_{\partial \Sigma} d^{n-2}z \sqrt{|\gamma^{\partial \Sigma}|} n_\mu \sigma_\nu F^{\mu\nu},
\]

where \(\Sigma\) is a hypersurface which is typically a hypersurface of constant \(t\) and \(r,\) \(|\gamma^{\partial \Sigma}|\) is the determinant of the induced metric on the boundary \(\partial \Sigma,\) \(n\) and \(\sigma\) are the unit normal vectors to boundary given by

\[
n^\mu = (1, 0, 0, 0), \quad \sigma^\mu = (0, 1, 0, 0),
\]

hence

\[
Q_E = \lim_{r \to \infty} \int_{S^2} E_r n^\mu \sigma^\nu g^{tt} g_{tt} r^2 \sin \theta \, d\theta \, d\phi.
\]

After calculating the integral for metric \((22),\) the result will be a function of \(r,\) which indicates that the charge is not point-like, but extended. If we take the limit \(r \to \infty,\) the electric flux approaches the constant value

\[
Q_E = \frac{4\pi}{\kappa} \frac{4m^3 \sinh 2\alpha}{(\cosh^2 \alpha (4m^2 - 1) + 1)^2},
\]
which for the typical choice $\sinh \alpha = 1$, $\cosh \alpha = \sqrt{2}$, and $m = 1$, takes the constant value

$$Q_E = \frac{32\pi \sqrt{2}}{49\kappa}. \quad (37)$$

The magnetic flux for the boosted Kaluza-Klein magnetic monopole turns out to be

$$\Phi_B = \oint s \frac{1}{2} \xi^{\alpha\beta} ds_{\alpha\beta} = \oint s \frac{1}{4} \xi^{\alpha\beta\mu\nu} F_{\mu\nu} ds_{\alpha\beta} = \oint s \frac{1}{2} |g^{(2)}| g^{\mu\nu} g^{\theta\theta} g^{\phi\phi} F_{\mu\nu} d\theta d\phi, \quad (38)$$

which gives

$$\Phi_B = \frac{2\pi}{\kappa} \int_{s^2} - \frac{r \sin \theta \cosh \alpha \left( \sinh^2 \alpha \left( 8m^2 + r \cos 2\theta (4m - r) - 4mr + r^2 \right) + 8m^2 r \sin^2 \theta (2m - r) \cosh^2 \alpha \right)^2}{8(2m - r) \left( 4m^2 r \sin^2 \theta (2m - r) \cosh^2 \alpha + \sinh^2 \alpha \left( 3m^2 \cos^2 \theta + \sin^2 \theta (r - 2m)^2 \right) \right)^2} d\theta, \quad (39)$$

which again, is a function of $r$. Taking the limit, $r \to \infty$ one obtains

$$\Phi_B = \frac{2\pi}{\kappa} \cosh \alpha. \quad (40)$$

The electric, and magnetic fluxes for the four dimensional metric deduced from (15) are valid for both signs of $m$.

We conclude that, the magnetic monopole gives rise to a dyon plus after the boost \[31\]. To see this, we convert the magnetic fields $B_r$ and $B_\theta$ from spherical coordinates to a cartesian one on the $(x, z)$ plane, that is

$$B_x = B_r \sin \theta + B_\theta \cos \theta, \quad (41)$$
$$B_z = B_r \cos \theta - B_\theta \sin \theta, \quad (42)$$

using $x = r \sin \theta$, $z = r \cos \theta$. The magnetic fields $B_x$ and $B_z$ are shown in Fig. (1). The vector field illustrates that there should be a dyon at the origin plus a string along the $z$ axis.

For the boosted Kaluza-Klein magnetic monopole, we can obtain the conserved quantities in the spacetime, in the presence of Killing vectors $\xi^\mu = \delta^\mu_t$ and $\xi^\mu = \delta^\mu_\phi$, which correspond to time translation and axial symmetry, respectively. To do this, we use the following integral for the conserved quantities \[32\]

$$I = \frac{1}{8\pi G} \int_s \nabla^m \xi^m d^2\Sigma_{mn}, \quad (43)$$

where $d^2\Sigma_{mn}$ should be taken over a two-dimensional surface located at the spatial infinity, that can be calculated via the metric components (22). If we consider the time translation invariance $\xi^\mu = \delta^\mu_t$, and substituting into the integral we get

$$M = \frac{m}{2G} \frac{\sinh 2\alpha}{4m^2 \cosh^2 \alpha - \sinh^2 \alpha}, \quad (44)$$

which is the total mass of the system. Note however, that in the case of axial symmetry, where the relevant Killing vector is $\xi^\mu = \delta^\mu_\phi$, the integral will give $J = 0$, where $J$ is the angular momentum. Also note that $M = 0$ for $\alpha = 0$, in agreement with the results of section\[2\] The boost, therefore generates mass and electric charge.
Figure 1: Vector plots for $B_x$ and $B_z$ in the case with $\cosh \alpha = \sqrt{2}$, $\sinh \alpha = 1$, $m = 1$ in the x-z plane, which corresponds to a dyon at the origin plus a magnetically charged string along the z axis.

4 Conclusion

Inspired by the Taub-NUT solution, we considered a Kaluza-Klein vacuum solution in five dimensions, which described a point-like magnetic monopole in four dimensional spacetime. The source supporting the four dimensional space-time was shown to differ from that of an ultra-relativistic fluid, in contrast to the solution of Wesson and Leon [29]. The pressure is anisotropic in both cases. We calculated the magnetic charge and showed that the total magnetic flux of the monopole through any spherical surface centered at the origin was constant, indicating that there is no extended magnetized source. The gravitational mass was derived in two ways and it was shown to vanish using both definitions. It was pointed out that the singularity which appears at finite $r$ is neither a horizon, nor a surface of finite, non-vanishing surface area. In a more appropriate coordinate system, this was shown to be a curvature singularity at $\tilde{r} = 0$. The main contribution of this paper was studying the properties of the boosted Kaluza-Klein magnetic monopole. It was shown that the boosted solution acquires significantly different physical properties in $(3+1)$ dimensions, including the appearance of a magnetically charged string attached to a dyon with extended electric and magnetic charges. We considered both signs $\pm$ for $m$ in our calculations, and showed that for obtaining the main results of the paper, which appear after boost, it was not essential to choose a particular sign for $m$.

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