A renewal scheme for non-uniformly hyperbolic semiflows

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November 2014

Abstract

We investigate a renewal scheme for non-uniformly hyperbolic semiflows that closely resembles the renewal scheme developed in the discrete time case, in order to obtain sharp estimates for the correlation function. Also, the involved observables are supported on a flow-box of unbounded length. The present abstract setting does not require the use of Markov structure. However, the classes of examples covered here are rather restrictive. In these examples, it is easier to exploit the full force of the method and get optimal results for observables supported on finite length flow-boxes.

1 Introduction

Mixing is a delicate phenomenon for flows. Exponential decay of correlations for Hölder observables has been established for Anosov flows with $C^1$ stable and unstable foliations in [7] and for contact Anosov flows in [13]. Building upon the techniques developed in these works, exponential decay of correlations has been later established for ‘less smooth’ or Markov systems (see, for instance, [5, 4, 3]).

The situation for superpolynomial decay of correlations (rapid mixing) is somewhat better. The work [8] established rapid mixing for (nontrivial) basic sets for typical Axiom A flows. This was extended in [16] to non-uniformly hyperbolic flows given by a suspension over a Young tower with exponential tails [23].

The recent work [20] develops an operator renewal theory framework for flows and applies this to the study of mixing properties of (non-uniformly hyperbolic flows that can be modeled as) suspension semiflows over Gibbs Markov maps. For this class of continuous time systems, [20] obtains: a) upper and lower bounds for polynomial decay of correlation in the finite measure preserving case and b) sharp mixing rates in the infinite measure preserving case.

The results obtained in [20] for infinite measure preserving suspension semiflows over Gibbs Markov maps (satisfying certain assumptions among which regular variation for the roof function is a must) are the direct analogue of the results in [19]. Polynomial upper bounds on the correlation for semiflows over Gibbs Markov maps (e.g. the class of flows that can be modeled as suspensions over maps with indifferent fixed points as in [14]) have previously been established in [17]. In [20], the authors show that for a large class of systems considered in [17], the established mixing rates are sharp; namely using operator

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renewal theory techniques they obtain lower and upper bounds. Although the method of proof is significantly different from the discrete time scenario, the results in [20] on decay of correlation for finite measure preserving suspension semiflows over Gibbs Markov maps (with polynomial roof function) are again the direct analogue of the results in the discrete time set-up [10, 21].

In the setting of Gibbs Markov semiflows the results of [20] are optimal, so this paper cannot improve on them. Instead, the aim of this paper is to investigate a different renewal scheme for semiflows that more closely resembles the renewal scheme developed in the discrete time case. Contrary to the results obtained in [20], this method allows us to study decay of correlation for observables that are supported on a flow-box of unbounded length. While the abstract framework developed here does not require the use of Markov structure and allows us to optimal results for observables supported on a flow-box of unbounded length, the set of examples covered here is rather restrictive (see below).

We provide an abstract framework similar in structure to the ones developed for discrete time systems. In Section 4 we list a set of hypotheses (H0)-(H6), with versions for the finite and infinite measure setting, under which the main theorems in Section 5, namely Theorems 5.1 and 5.2 for the finite and infinite measure setting respectively, give optimal bounds for the correlation function. The abstract set-up does not require the use of Markov structure (so, neither Gibbs Markov) for the underlying map.

The main ingredients are:
(I) The type of renewal equation established in [20] (see Proposition 3.1), or more precisely, the argument used in establishing a renewal equation for flows in [20].

(II) A new inducing scheme which resembles the inducing scheme employed in the discrete time scenario, namely we induce to a hyperbolic map with exponential decay, see Section 2. The inducing scheme used in [20] involves observables supported on a flow-box of unit length, and the action of the inducing scheme in the flow direction is somewhat trivial. In contrast to inducing to a thickened Poincaré section as in [20], we induce to a flow-box \( \tilde{Y} \) with in principle unbounded flow-time. We induce in such a way that the induced version of the semiflow is a uniformly hyperbolic map \( \Phi \), acting non-trivially in all dimensions, by forcing expansion in the flow direction.

The choice for the present inducing scheme creates certain technical complications that are overcome by introducing scaled versions of the measures and observables. Although at first this looks counter-intuitive and considerable complicates the formula for the induce time \( \varphi \), this choice ensures that given \( R \) is the transfer operator associated with \( \Phi \), its twisted version \( R(e^{-s\varphi}) \), \( \Re s \geq 0 \), has the spectral properties required in (H2) and (H3).

(III) We notice that twisted transfer operators can be related to proper Laplace transforms of non delta functions. More precisely, the twisted version \( R(e^{-s\varphi}v) \) of the transfer operator associated with \( \Phi \), can be related to \( \int_0^\infty R_t v e^{-s\varphi} dt \), where \( R_t v = R(1_{(t<\varphi<t+1)} v) \). For details we refer to Section 3. This makes it possible to show that many techniques/calculations from the discrete time scenario [21, 10, 19] carry over to the continuous case.

(I)-(III) above allow us to develop an abstract framework based on assumptions on

(H0,1) properties of the region \( \tilde{Y} \) and tail estimates of the induce time \( \varphi \),

(H2,3) functional analytic assumptions for the map \( \Phi \), in some appropriate Banach space with norm \( \| \| \).
(H4,5) the asymptotic behavior of the integral \( \int_t^\infty \| R_\sigma \| \, d\sigma \) for the finite reps. infinite measure case, and

(H6) a Dolgopyat-type inequality.

When applying the abstract framework to concrete examples, one needs to choose a norm that is adapted to the indicator function \( 1_{\{t < \varphi < t + 1\}} \) appearing in \( R_t \), in order to verify assumption (H4). We construct such a norm/Banach space in the setting of suspension semiflows over Markov maps. This newly constructed norm combines integration over well-chosen curves in \( \hat{Y} \) with the usual H"older space, but in order to have this Banach space embedded in \( L^\infty \) as well, we need to restrict to analytic maps and observables: see Section 10. However, in the infinite measure setting, Theorem 5.2 gives a relaxed version of condition (H4), see Remark 4.1, and for observables supported on a flow-box of unit length, this version is easy to check: see Section 9. We believe that such a relaxation is possible for the finite measure setting too, but we will not address this at the moment.

To ensure that (H6) holds, we further assume a Diophantine ratio condition (see Subsection 9.3 for details) for the return time \( \varphi \), which is natural in this class; see \([8, 16, 20])\)

**Notation:** We will write \( a(t) \ll b(t) \) or \( a(t) = O(b(t)) \) if there is a constant \( C > 0 \) such that \( a(t) \leq C b(t) \) for all \( t \). Similarly, \( a(t) = o(b(t)) \) means that \( \lim_{t \to 0} a(t)/b(t) = 0 \).

## 2 A general inducing scheme for flows

### 2.1 Inducing to a semiflow over an expanding base map

Let \( g_t \) be a \( C^2 \) semiflow on a manifold \( \mathcal{M} \). Let \( Y \times \{0\} \) be a section transversal to \( g_t \), and \( \hat{Y} = \bigcup_{y \in Y} \{y\} \times [0, \hat{h}(y)) \) be a flow-box where the coordinates \( \hat{y} = (y, u) \) are chosen such that within \( \hat{Y} \) the flow becomes parallel and of unit speed:

\[
g_t(y, u) = (y, u + t) \quad \text{for } 0 \leq u, u + t \leq \hat{h}(y).
\]

Let

\[
\varphi_0 = \min\{t > 0 : g_t(y, 0) \in Y \times \{0\}\}
\]

be the first return time to the section. The function \( \varphi_0 \) can be in \( L^1(\mu) \) (the finite case), or not (infinite case); in either case we will put some tail conditions on \( \varphi_0 \).

We assume that the Poincaré map \( F = g_{\varphi_0} \) is a uniformly hyperbolic map with partition \( \mathcal{P} \), and it preserves a probability measure \( \mu \). This means that the total mass of \( \mathcal{M} \) is \( \int_Y \varphi_0 \, d\mu \).

We also assume that \( F \) is uniformly expanding and has bounded distortion, see (9.1)

In the above, the height function \( \hat{h} : Y \to (0, \infty) \) is defined and \( \hat{h}(y) \leq \frac{1}{2} \varphi_0(y) \) \( \mu \)-a.e. We will assume that \( \hat{h} \in L^p(\mu) \) for some \( p > 1 \), that \( \inf_{y \in Y} \hat{h}(y) \geq 1 \), and that \( \hat{h} \) is \( C^2 \) smooth on each \( Z \in \mathcal{P} \).

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1Instead of a Diophantine condition, one could work with assumptions as in [3, 5] and as such obtain a better exponent \( \alpha \), namely \( \alpha \in (0, 1) \), in assumption (H6). By working with this sort of assumptions one can establish optimal bounds for the correlation function \( \int_Y vw \circ f \, d\tilde{\mu} \) for \( C^m \)-smooth \( w \), where \( m > \alpha \), but not arbitrarily large. We do not consider this sort of assumptions here because we do not exploit the advantage of a smaller \( m \) in the proofs of the present abstract results. For a future use of this type of assumption we refer to Remark 6.12.
The corresponding suspension semiflow on $\tilde{Y}$ is

$$
\tilde{g}_t(y,u) = \begin{cases} 
(y, u + t) & \text{if } 0 \leq u, u + t < \tilde{h}(y), \\
(Fy, 0) & \text{if } t = \tilde{h}(y) - u,
\end{cases}
$$

and then continued for $t > \tilde{h}(y) - u$ by the usual group property of a flow.

**Remark 2.1** If $(\mathcal{M}, g_t)$ is itself a suspension flow over some base map $f : X \to X$ with roof function $h$, then we can take $Y \subset X$, $F = f^\tau : Y \to Y$ is the induced map with induce time $\tau$, and $\varphi_0(y) = \sum_{i=0}^{\tau-1} h \circ f^i(y)$. For example, suspension flows over interval maps with a neutral fixed point (see Section 9 and 10) fit in this framework.

We define a return map to $\tilde{Y}$ which complements $F = g_{\varphi_0}$ with an artificial hyperbolic part (the doubling map) in the $u$-direction. Set

$$
K(y) := \frac{2\tilde{h}(Fy)}{\tilde{h}(y)}. \quad (2.1)
$$

Then taking $\Phi_1(y,u) := g_{\varphi(y,u)}(y,u)$ results in

$$
\Phi_1(y,u) = \begin{cases} 
(Fy, K(y)u) & \text{if } u < \tilde{h}(y)/2, \\
(Fy, K(y)u - \tilde{h}(Fy)) & \text{if } u \geq \tilde{h}(y)/2,
\end{cases} \quad (2.2)
$$

for

$$
\varphi_1(y,u) = \varphi_0(\tilde{g}) + \begin{cases} 
(K(y) - 1)u & \text{if } u < \tilde{h}(y)/2, \\
(K(y) - 1)u - \tilde{h}(Fy) & \text{if } u \geq \tilde{h}(y)/2.
\end{cases} \quad (2.3)
$$

**2.2 Remetrize to make $\Phi$ uniformly expanding**

The idea behind $\Phi_1$ is that it maps the flow-line $\{y\} \times [0, \tilde{h}(y)]$ as a piecewise expanding map onto $\{Fy\} \times [0, \tilde{h}(Fy)]$. This is non-injective: for every $0 \leq u < \tilde{h}(y)/2$, there is another $u' := u + \tilde{h}(y)/2$ such that $\Phi_1(y,u) = \Phi(y,u')$. Yet, $\Phi_1$ is ergodic with respect to $\frac{1}{h} \, d\mu \, du$ and produces a mixing product map if $(Y,F,\mu)$ is mixing.

However, if $K(y) = \frac{2h(Fy)}{h(y)} \leq 1$, then $\Phi_1$ is still not expanding in the vertical $u$-direction. This can be remedied by a change of coordinates

$$
\zeta : \tilde{Y} \to Y \times [0,1], \quad (y,u) \mapsto (y, \frac{u}{h(y)}),
$$

Then $\Phi := \zeta \circ \Phi_1 \circ \zeta^{-1}$ is precisely the doubling map in the vertical direction: $\Phi(y,u) = (Fy, 2u \mod 1)$, and hence $\Phi$ is uniformly expanding. In these new coordinates the formulas are

$$
\Phi(y,u) = f_{\varphi(y,u)}(y,u) = (Fy, 2u \mod 1), \quad (2.4)
$$

where

$$
\varphi(y,u) = \varphi_0(y) + \begin{cases} 
(2\tilde{h}(Fy) - \tilde{h}(y))u & \text{if } 0 \leq u < \frac{1}{2}; \\
(2\tilde{h}(Fy) - \tilde{h}(y))u - \tilde{h}(Fy) & \text{if } \frac{1}{2} \leq u < 1.
\end{cases} \quad (2.5)
$$
Figure 1: The flows \( g_t : \mathcal{M} \to \mathcal{M} \), \( \tilde{g}_t : \tilde{Y} \to \tilde{Y} \) and map \( \Phi_1(\tilde{y}) = g_{\varphi(\tilde{y})}(\tilde{y}) \) acting as doubling map on the vertical coordinate. On the right is the image under the change of coordinate \((y, u^*) = \zeta(y, u)\) make the map \( \Phi \) uniformly expanding.

Using the \( F \)-invariance of \( \mu \), it is straightforward to check that \( \int_{\tilde{Y}} (\varphi - \varphi_0) d\mu_\Phi = 0 \). Hence,

\[
\int_{\tilde{Y}} \varphi d\mu_\Phi = \int_Y \varphi_0 d\mu. \tag{2.6}
\]

The change of coordinates \( \zeta \) comes at the price that the semiflow \( f_t = \zeta \circ g_t \circ \zeta^{-1} \), although parallel, is not of constant speed:

\[
f_t(y, u) = (y, u + t/\tilde{h}(y)) \quad \text{for } 0 \leq u < 1, \quad 0 \leq u + t/\tilde{h}(y) < 1.
\]

That is, the speed is constant on each flow-line, but differs from flow-line to flow-line. Therefore \( f_t \) and \( \Phi \) don’t preserve the same measure; they preserve \( \tilde{\mu} \) and \( \mu_\Phi \) respectively, and these measures are equivalent via the scaling

\[
\tilde{\mu} = \tilde{h} d\mu du = \tilde{h} d\mu_\Phi. \tag{2.7}
\]

3 Operator renewal equation

Let \( L_t : L^1(\tilde{\mu}) \to L^1(\tilde{\mu}) \) be the transfer operator for the flow \( f_t \) defined by \( \int_{\tilde{Y}} L_t v w d\tilde{\mu} = \int_{\tilde{Y}} v w \circ f_t d\tilde{\mu} \) for all \( w \in L^\infty(\tilde{\mu}) \). Through-out, we write \( v \in L^1(\tilde{\mu}) \), \( w \in L^\infty(\tilde{\mu}) \) and \( v^* \in L^1(\mu_\Phi) \), \( w^* \in L^\infty(\mu_\Phi) \). In particular, we note that \( v \in L^1(\tilde{\mu}) \) can be written as \( v = \frac{v^*}{\tilde{h}} \) where \( v^* \in L^1(\mu_\Phi) \).

Define \( T_t, U_t : L^1(\tilde{\mu}) \to L^1(\tilde{\mu}) \) by

\[
T_t v = 1_{\tilde{Y}} L_t (1_{\tilde{Y}} v), \quad U_t v = 1_{\tilde{Y}} L_t (1_{\{t<\varphi\}} v). \tag{3.1}
\]
For $s \in \mathbb{C}$, we define the following Laplace transforms:

$$
\hat{T}(s) := \int_{0}^{\infty} T_{t} e^{-st} dt, \quad \hat{U}(s) := \int_{0}^{\infty} U_{t} e^{-st} dt.
$$

Let $R : L^{1}(\mu_{\Phi}) \to L^{1}(\mu_{\Phi})$ be transfer operator associated with $\Phi$ defined by

$$
\int_{Y} R v^{*} w^{*} d\mu_{\Phi} = \int_{Y} v^{*} w^{*} \circ \Phi d\mu_{\Phi}
$$

for all $w^{*} \in L^{\infty}(\mu_{\Phi})$. For $s \in \mathbb{C}$, we define the following twisted/perturbed transfer operator

$$
\hat{R}(s)v^{*} := R(e^{-s\varphi}v^{*}).
$$

Clearly, $\hat{R}, \hat{T}, \hat{U}$ are analytic on $\mathbb{H} = \{\Re s > 0\}$ and $\hat{R}$ is well-defined on $\overline{\mathbb{H}} = \{\Re s \geq 0\}$.

**Proposition 3.1** The following holds $\hat{\mu}$-a.e. on $\tilde{Y}$ for all $s \in \mathbb{C}$:

$$
\hat{T}(s)(I - \hat{R}(s)) = \hat{U}(s).
$$

**Proof** Apart from converting $\hat{\mu}$ to $\mu_{\Phi}$, where required, the argument below goes exactly as the [20, Proof of Theorem 3.2]. Recall $d\hat{\mu} = \hat{h} d\mu_{\Phi}$. Note that $v \in L^{1}(\hat{\mu})$ can be written as $v = \frac{v}{\hat{h}}$ where $v^{*} \in L^{1}(\mu_{\Phi})$. By direct computation:

$$
\int_{\tilde{Y}} \hat{T}(s)\hat{R}(s)v \cdot w \, d\hat{\mu} = \int_{\tilde{Y}} \int_{0}^{\infty} 1_{\tilde{Y}} L_{t}(1_{\tilde{Y}} R(e^{-s\varphi}v)) e^{-st} w \, dt \, d\hat{\mu}
$$

$$
= \int_{0}^{\infty} \int_{\tilde{Y}} \hat{h} R(e^{-s\varphi}v) \cdot w \circ f_{t} \, d\mu_{\Phi} \, e^{-st} dt
$$

$$
= \int_{0}^{\infty} \int_{\tilde{Y}} v^{*} \cdot w \circ f_{s-t} e^{-s(\varphi+t)} \, d\mu_{\Phi} \, dt
$$

$$
= \int_{0}^{\infty} \int_{\tilde{Y}} v \cdot w \circ f_{s-t} e^{-s(\varphi+t)} \, d\hat{\mu} \, dt
$$

$$
= \int_{\tilde{Y}} \int_{\varphi}^{\infty} v \cdot w \circ f_{t} e^{-st} \, dt \, d\hat{\mu} = \int_{\tilde{Y}} \int_{\varphi}^{\infty} L_{t} v \cdot w e^{-st} \, dt \, d\hat{\mu}
$$

$$
= \int_{\tilde{Y}} \left( \int_{0}^{\infty} L_{t} v \cdot w e^{-st} \, dt \, d\hat{\mu} - \int_{0}^{\varphi} L_{t} v \cdot w e^{-st} \, dt \, d\hat{\mu} \right) \, d\hat{\mu}
$$

$$
= \int_{\tilde{Y}} \hat{T}(s)v \cdot w \, d\hat{\mu} - \int_{\tilde{Y}} \hat{U}(s)v \cdot w \, d\hat{\mu}.
$$

In the sequel we will repeatedly convert integration with respect to $\hat{\mu}$ to integration with respect to $\mu_{\Phi}$. As an example, we note that for $v \in L^{1}(\hat{\mu})$, and thus, $v^{*} = \hat{h} v \in L^{1}(\mu_{\Phi})$, and $w \in L^{\infty}(\hat{\mu})$,

$$
\int_{\tilde{Y}} \hat{U}(s)v \cdot w \, d\hat{\mu} = \int_{0}^{\infty} \int_{\tilde{Y}} 1_{\{\varphi>0\}} L_{t} v w e^{-st} \, dt \, d\hat{\mu} = \int_{0}^{\infty} \int_{\tilde{Y}} 1_{\{\varphi>0\}} v w \circ f_{t} e^{-st} \, d\hat{\mu}
$$

$$
= \int_{0}^{\infty} \int_{\tilde{Y}} 1_{\{\varphi>0\}} v^{*} w \circ f_{t} e^{-st} \, d\mu_{\Phi} = \int_{0}^{\infty} \int_{\tilde{Y}} 1_{\{\varphi>0\}} L_{t} v^{*} w e^{-st} \, d\mu_{\Phi}
$$

$$
= \int_{\tilde{Y}} \hat{U}(s)v^{*} w \, d\mu_{\Phi}.
$$

(3.2)
3.1 Relating the twisted transfer operator $\hat{R}(s)$ with a Laplace transform of non-delta functions

Although in the sequel we will not view $\hat{R}(s)$ as a Laplace transform (as noticed in [20]), any twisted transfer operator $\hat{R}(s)v^* := \hat{R}(e^{-s\varphi}v^*)$ can be written as $\hat{R}(s) := \int_0^\infty \hat{R}(\varphi - t)e^{-st}dt$, we will sometimes make use of the following representation:

**Lemma 3.2** For $s \in \mathbb{R}$, let $a \in \mathbb{R}_+$ such that $e^{sa} \neq 1$. Set $R_{t,a}v^* = R(1_{t<\varphi<t+a})v^*$ and define $\hat{R}_a(s) = \int_0^\infty R_{t,a}e^{-st}dt$. Then

$$\hat{R}(s) = \frac{s}{e^{sa} - 1} \hat{R}_a(s), \quad \hat{R}(0) = \hat{L}_1(0) = R.$$

**Proof** Compute that

$$\int_Y \hat{R}_a(s)v^*w^*d\mu_\varphi = \int_Y \int_0^\infty R(1_{t<\varphi<t+a})v^*w^*e^{-st}dt \, d\mu_\varphi = \int_Y \int_{\varphi-a}^\varphi Rv^*w^*e^{-st}dt \, d\mu_\varphi$$

$$= \int_Y R(e^{-s\varphi}v^*)w^* \cdot \int_{\varphi-a}^{\varphi} e^{-s(t-\varphi)}dt \, d\mu_\varphi = \int_Y R(e^{-s\varphi}v^*)w^* \, d\mu_\varphi \cdot \int_0^a e^{st}dt$$

$$= \int_0^a e^{st}dt \cdot \int_Y R(e^{-s\varphi}v^*)w^* \, d\mu_\varphi = e^{sa} - 1 \int_Y \hat{R}(s)v^*w^* \, d\mu_\varphi.$$

So, $\hat{R}(s) = s(e^{sa} - 1)^{-1} \hat{R}_a(s)$, as required.

For the second equality, set $a = 1$ and note that $s(e^s - 1)^{-1} = 1 + O(s)$, as $s \to 0$. ■

4 Abstract set-up

We assume the setting and notation introduced in Section 2. Throughout, we assume that one of the two tail conditions holds:

**(H0)**

i) Finite case: $\mu_\varphi((y,u) \in \tilde{Y} : \varphi(y,u) > t) = O(t^{-\beta})$, $\beta > 1$.

ii) Infinite case: $\mu_\varphi((y,u) \in \tilde{Y} : \varphi(y,u) > t) = \ell(t)t^{-\beta}$ where $\ell$ is slowly varying and $\beta \in (1/2, 1)$.

We require that

**(H1)** $\inf_{y \in Y} \tilde{h}(y) \geq 1$ and that $\tilde{h} = \varphi_\gamma^\beta$, where

i) Finite case. Under (H0) i), we assume $\gamma \in (0,1)$.

ii) Infinite case. Under (H0) ii), we assume $\gamma \in (0, \min\{\frac{2\beta-1}{1-\beta}, \frac{1-\beta}{2\beta-1}, \beta\})$.

Among others, as will be made clear by Lemma 4.4 below, assumption (H1) ensures that the tails $\mu(y \in Y : \varphi_0(y) > t)$ and $\mu_\varphi((y,u) \in \tilde{Y} : \varphi(y,u) > t)$ are of the same order. The assumption $\gamma < \beta$ ensures that in both cases of (H0), $\int_Y \tilde{h}^p \, d\mu < \infty$ for all $1 \leq p < \beta/\gamma$ and that $\mu_\varphi(\tilde{Y}) < \infty$.

In this paper, we work with (H1) above, but simplifications for the case $\tilde{h}$ bounded from above and below will be pointed throughout the paper.
4.1 Functional analytic assumptions

We require that $\Phi$ satisfies the functional analytic assumptions listed below. We assume that there exists a Banach space $\mathcal{B}$, with norm $\|\cdot\|_\mathcal{B}$ such that

(H2) i) The space $\mathcal{B}$ contains constant functions and $\mathcal{B} \subset L^\infty(\mu_\Phi)$.

ii) $1$ is a simple eigenvalue for $R$, isolated in the spectrum of $R$.

Recall that $\hat{R}(s)v^* = R(e^{-s\varphi}v^*)$ is the twisted transfer operator associated with the map $\Phi$. By (H2) ii), $1$ is an isolated eigenvalue in the spectrum of $\hat{R}(0)$. In addition to (H2) ii), we require

(H3) The spectral radius of $\hat{R}(s)$ is strictly less than 1 for $s \in \mathbb{H} - \{0\}$ and is equal to 1 for $s = 0$.

For $s \in \mathbb{H}$, let $a \in \mathbb{R}_+$ such that $e^{sa} \neq 1$. Set $R_{t,a}v^* = R(1_{\{t<\varphi<t+a\}}v^*)$ and define $\hat{L}_a(s) = \int_0^\infty R_{t,a}e^{-st}dt$. By Lemma 3.2, $\hat{R}(s) = s(e^s - 1)^{-1}\hat{L}_a(s)$. Given $a > 0$, we make certain assumptions on $\|R_{t,a}\|$, which in the sequel will be used to obtain appropriate continuity properties for $\hat{R}$.

(H4) Finite case. Under (H0) i), we require that for any $\tau < \beta$, the following upper bound hold uniformly in $a \in [1, 2]$:

$$\int_0^\infty \sigma^\tau \|R_{\sigma,a}\|_\mathcal{B} d\sigma < \infty,$$

(H5) Infinite case. Under (H0) ii), we require that there exists a Banach space $\mathcal{B}_0$ such that $\mathcal{B} \subset \mathcal{B}_0 \subset L^\infty$ such that

i) There exists constants $C_1 > 0$, $C_2 < 1$ and some $\theta \in (0, 1)$ such that

$$\|\hat{R}(s)v\|_\mathcal{B} \leq C_1\theta^n\|v\|_\mathcal{B} + C_2\|v\|_{\mathcal{B}_0}, \quad \|\hat{R}(s)v\|_{\mathcal{B}_0} \leq \|v\|_{\mathcal{B}_0}.$$

ii) The following upper bound holds uniformly in $a \in [1, 2]$,

$$\int_0^\infty \sigma^\tau \|R_{\sigma,a}\|_{\mathcal{B} \to \mathcal{B}_0} d\sigma < \infty,$$

for $\max\{1 - \beta, 2\beta - 1\} < \tau < \frac{\beta}{1+\gamma}$.

Remark 4.1 Assumption (H4) is very strong: it does not hold in standard Banach spaces such as Hölder or BV, unless we make further very restrictive assumptions on the return time $\varphi$ (such as piecewise constant on partition elements of the $\Phi$-partition). However, as we show in Section 10, it can be verified for a Banach space of analytic functions, which also puts restrictions on the map $\Phi$.

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\textsuperscript{2} The assumption $\mathcal{B} \subset L^\infty(\mu_\Phi)$ can be relaxed to $\mathcal{B} \subset L^2(\mu_\Phi)$. Because the main results are cumbersome to state under the weaker assumption (and require more elaborated arguments), we do not pursue this issue here.
Assumption (H5) ii) is rather mild. As we show in Section 9, under the assumption \( \tilde{h} \) bounded from above, (H5) ii) holds for typical suspension flows over Markov maps with indifferent fixed points, for \( \mathcal{B}_0 = L^\infty(\mu_\phi) \). In this case (H5) is easy to check (see Remark 9.3). In view of Remark 4.2 below, in Section 9 we construct a variant \( \mathcal{B}_0 \) of the Hölder space \( \mathcal{B} \), with \( \mathcal{B} \subset \mathcal{B}_0 \subset L^\infty(\mu_\phi) \), such that both \( (\mathcal{B}, \mathcal{B}_0) \) and \( (\mathcal{B}_0, L^\infty) \) satisfy (H5) (see Remark 9.7).

Assumption (H5) ii) makes the proofs slightly more difficult. In particular, one has to estimate several operators in the \( \| \cdot \|_{\mathcal{B} \to \mathcal{B}_0} \) norm. We can pursue this issue in the proof of Theorem 5.2 along some arguments in [15], which deals with similar estimates in the context of discrete time hyperbolic infinite measure preserving systems; in the present set up the arguments in [15] are greatly simplified by the fact that \( \mathcal{B} \subset \mathcal{B}_0 \subset L^\infty \).

**Remark 4.2** We believe that the argument we provide below for the proof of Theorem 5.1 under the strong (H4) can be adapted to work with an assumption of the type (H5) with appropriate \( \tau \); more precisely, we would assume that there exists a space \( \mathcal{B}_0 \) such that both \( (\mathcal{B}, \mathcal{B}_0) \) and \( (\mathcal{B}_0, L^\infty) \) satisfy the appropriate finite case version of (H5). However, because the involved argument is rather complicated, here we reduce the analysis to the case where (H4) holds. However, see Remarks 6.12 and 6.22 for an outline of future work using a weak form of (H4).

**Remark 4.3** It is easy to see that (H4), (H5) ii) above implies that \( \int_t^{\infty} \| R_{\sigma, a} \| \, d\sigma \ll t^{-\tau}, \) for \( \tau \) as in (H4) and (H5) ii), respectively.

### 4.2 Assumptions (H0): analogy with the discrete time scenario

The first result below shows that assumption (H0) can be verified by estimating the tail \( \mu(\varphi_0 > t) \), which is easier to verify. In a large class of examples the tail \( \mu(\varphi_0 > t) \) can be estimated based on knowledge about \( \mu(\tau > n) \): see [20].

**Lemma 4.4** Assume \( \tilde{h} = \varphi_0^\gamma, \gamma \in (0, 1) \). Then for any \( 0 < \delta < 1, \)

\[
\mu_\phi(\varphi > t) = \mu(\varphi_0 > t(1 - t^{-\delta})) + O(\mu(\varphi_0 > t^{1-\delta/\gamma})).
\]

**Proof** Fix \( 0 < \delta < 1 \). We argue considering each of two formulas for \( \varphi \) in (2.5). For \( u \in [0, \frac{1}{2}] \) we have \( \varphi(y, u) = \varphi_0 + (2\tilde{h}(Fy) - \tilde{h}(y))u \). Since we also know \( \tilde{h} = \varphi_0^\gamma, \gamma \in (0, 1), \varphi \geq t \) implies that \( \varphi_0 > t(1 - t^{-\delta}) \) or \( 2\tilde{h}(Fy) > t^{1-\delta} \). Thus,

\[
\mu(\varphi > t) \leq \mu(\varphi_0 > t(1 - t^{-\delta})) + \mu(2\tilde{h} \circ F > t^{1-\delta})
= \mu(\varphi_0 > t(1 - t^{-\delta})) + \mu(2\tilde{h} > t^{1-\delta}),
\]

by \( F \)-invariance of \( \mu \). The conclusion for \( u \in [0, \frac{1}{2}] \) follows.

For \( u \in [\frac{1}{2}, 1] \), the argument is similar since \( \varphi(y, u) = \varphi_0 + (2\tilde{h}(Fy) - \tilde{h}(y))u - \tilde{h}(Fy) \).

Again since \( \tilde{h} = \varphi_0^\gamma, \gamma \in (0, 1), \varphi \geq t \) implies that \( \varphi_0 > t(1 - t^{-\delta}) \) or \( \tilde{h}(Fy) > t^{1-\delta} \). From here on the argument goes exactly same as in the case \( u \in [0, \frac{1}{2}] \).

Also in analogy with the discrete time case, we note that
Proposition 4.5 Assume $\varphi \in L^1(\mu_\Phi)$. Let $\frac{d}{d(-s)}\hat{R}(s)\bigg|_{s=0}$ be the derivative of $\hat{R}(s)$ in $-s$ evaluated at 0. Then

$$\int_{Y} \frac{d}{d(-s)}\hat{R}(s)\bigg|_{s=0} 1_{\varphi} \, d\mu_\Phi = \int_{Y} \varphi_0 \, d\mu.$$ 

Proof Using the pointwise formula for the twisted transfer operator, we write

$$\hat{R}(s)1_{\varphi} = R(e^{-s\varphi}1_{\varphi}) = \sum_{\Phi(y',u')=(y,u)} e^{p(y')} e^{-s\varphi(y',u')} \varphi(y',u'),$$

where $e^{p(y')}$ is the potential associated with the hyperbolic map $\Phi$. Therefore,

$$\frac{d}{d(-s)}\hat{R}(s) = \sum_{\Phi(y',u')=(y,u)} e^{p(y')} e^{-s\varphi(y',u')} \varphi(y',u').$$

Evaluating at 0,

$$\frac{d}{d(-s)}\hat{R}(s) \bigg|_{s=0} 1_{\varphi}(y,u) = \sum_{\Phi(y',u')=(y,u)} e^{p(y')} \varphi(y',u') = R\varphi(y,u).$$

Thus,

$$\int_{Y} \frac{d}{d(-s)}\hat{R}(s)\bigg|_{s=0} 1_{\varphi} \, d\mu_\Phi = \int_{Y} R\varphi \cdot 1 \, d\mu_\Phi = \int_{Y} \varphi \, d\mu_\Phi.$$ 

The conclusion follows from the above together with (2.6).

4.3 Assumptions required in the continuous time case: Dolgopyat type inequality

We recall that $\hat{\mu}$ is $f_1|_{\hat{Y}}$ invariant. In the finite measure case we normalize the measure $\hat{\mu}$ such that $\hat{\mu} = \hat{\mu}/\varphi_0$ with $\varphi_0 = \int_{Y} \varphi_0 \, d\mu$, is $f_1|_{\hat{Y}}$ a probability measure. In the infinite measure case we let $\hat{\mu} = \hat{\mu}$.

For appropriate $v, w$, we want to estimate the correlation function

$$\rho_t(v, w) = \int_{Y} T_t vw \, d\hat{\mu}.$$ 

Let $\hat{\rho}(s)(v, w) = \int_{0}^{\infty} \rho_t(v, w)e^{-st} \, dt$ be its corresponding Laplace transform. By Proposition 3.1, hypotheses (H2) and (H3), for all $s \in \mathbb{H} - \{0\}$

$$\hat{\rho}(s)(v, w) = \int_{Y} \hat{T}(s)vw \, d\hat{\mu} = \int_{Y} \hat{U}(s)(I - \hat{R}(s))^{-1}vw \, d\hat{\mu}.$$ 

Hypothesis (H4) gives a good control of $(I - \hat{R}(a + ib))^{-1}$ for $a \geq 0$ and $|b| < 1$. To be able to estimate the inverse Laplace transform $\rho_t(v, w)$ of $\hat{\rho}(s)$, we need a good understanding of the asymptotics of $(I - \hat{R}(a + ib))^{-1}$, for $a \geq 0$ and large values of $b$. In this sense we assume
(H6) Dolgopyat type inequality. There exist $C > 0$ and $\alpha > 0$ such that for all $|b| \geq 1$
\[ \|(I - \hat{R}(ib))^{-1}\|_B \leq C|b|^\alpha. \]

Sometimes in the sequel we will need the following form of (H6):
\[(H6') \text{ There exist } C_0 > 0, \theta \in (0, 1) \text{ and some } \alpha_0 > 0 \text{ such that for all } |b| \geq 1 \text{ and } k > (1 + \log C_0|b|)/\log \theta, \]
\[ \|\hat{R}(a + ib)^k\|_B \leq 1 - |b|^{-\alpha_0}. \]

**Remark 4.6** By a standard argument, one checks that (H5') implies (H6). More precisely, by (H2) and (H3), $\|\hat{R}(ib)\| \leq C_1$ for some constant $C_1$ independent of $j$. Together with (H5') this implies that
\[ \|(I - \hat{R}(ib))^{-1}\|_B = \|(I + \hat{R}(ib) + \cdots + \hat{R}^{k-1}(ib))(I - \hat{R}^k(ib))^{-1}\|_B \]
\[ \leq kC_1b^{\alpha_0} \leq C_1\frac{1 + \log C_0 + \log |b|}{\log \theta}b^{\alpha_0} \leq Cb^{\alpha}, \]
for some $\alpha > \alpha_0$ and $C$ depending only on $C_0$, $C_1$, $\theta$ and $\alpha - \alpha_0$.

### 4.4 Partitions of $\tilde{Y}$ and $w$ observables

Let $P$ be the partition of $Y$ into domains of continuity of $F$, and for $n \geq 1$, let $P_n = P \lor F^{-1}P \lor \cdots \lor F^{-(n-1)}P$ be the $n$-th joint of this partition. On $\tilde{Y}$, in the vertical direction, let $Q$ be defined as the partition of $\tilde{Y}$ into the complementary domains of the line $\{(y,1/2) : y \in Y\}$, and the $n$-th joint $Q_n$ as the partition of $\tilde{Y}$ into the complementary domains of the lines $\{(y,j2^{-n}) : y \in Y\}$ for the integers $0 < j < 2^n$. Then $\Phi$ is continuous on each element of the product partition $\tilde{P}_n := P_n \times Q_n$.

Let $w^* : \tilde{Y} \to C$, $C^m$ smooth (for some $m \geq 0$ to be specified below) in the (vertical) $u$-direction and piecewise continuous (or smooth) in the (horizontal) $y$-direction. Assume also that
\[ \frac{\partial^j w^*}{\partial y^j}(y,0) = \frac{\partial^j w^*}{\partial y^j}(y,\tilde{h}(y)) \]
\[ \text{for all } y \in Y \text{ and } j = 0, \ldots, m. \]

Note that for each $n$, $w^* \circ \Phi^n(y)$ is discontinuous at the lines $\{(y,j2^{-n}) : y \in Y\}$ for $0 < j < 2^n$, but the left and right limits of $\lim_{\varepsilon \to \pm 0} \Phi^n(y,u + \varepsilon)$ equal $(F^n y,0)$ resp. $(F^n y,\tilde{h}(F^n y))$ due to the Markov property in the $u$-direction.

By our assumption, the function values and partial derivatives of $w^*$ are identical at these points. Therefore, the partial derivatives $\frac{\partial}{\partial y^j}w^* \circ \Phi^n(y,u)$ in the $u$-direction exist at all points and they depend continuously on $y$ within the domains of $P_n$. Hence, we can assume $\|\frac{\partial}{\partial y^j}w^* \circ \Phi^n(y,u)\|_{L^\infty(\mu_\Phi)} < \infty$, for all $j = 0, \ldots, m$.

In what follows, we let $C^m(\tilde{Y},\mu_\Phi)$ be the class of functions $w^*$ that satisfy (4.1) and such that $\|\frac{\partial}{\partial y^j}w^* \circ \Phi^n(y,u)\|_{L^\infty(\mu_\Phi)} < \infty$, for all $j = 0, \ldots, m$, and set
\[ C^m(\tilde{Y},\tilde{\mu}) = \{w : \tilde{Y} \to C, w = \frac{w^*}{h} \text{ with } w^* \in C^m(\tilde{Y},\mu_\Phi)\}. \]
Proposition 4.7 Let \( m \geq 1 \). Suppose that \( v \in L^1(\mu) \) and \( w \in C^m(\tilde{Y}, \tilde{\mu}) \). Then
\[
\hat{\rho}(s)(v, w) = \sum_{j=1}^{m} \rho_{v, \partial^j w}(0) s^{-j} + s^{-m} \hat{\rho}_{v, \partial^m w}(s),
\]
where \( \partial^j w \) indicates the \( j \)-th partial derivative w.r.t. the second variable.

Proof We recall the short argument for convenience (see for instance [20]). Note that \( \rho_t(v, w) \) is \( m \)-times differentiable and \( \rho_{v,w}^{(j)} = \rho_{v, \partial^j w} \) for \( j = 0, \ldots, m \). By Taylor’s Theorem,
\[
P_m(t) = \sum_{j=0}^{m-1} \frac{1}{j!} \rho_{v,w}^{(j)}(0) t^j, \quad H_m(t) = \int_0^t g(t - \tau) \rho_{v,w}^{(m)}(\tau) d\tau, \quad g(t) = \frac{t^{m-1}}{(m-1)!}.
\]
Hence \( \hat{\rho}(s)(v, w) = \sum_{j=0}^{m-1} \rho_{v, \partial^j w}(0) s^{-(j+1)} + \hat{H}_m(s) \), where \( \hat{H}_m(s) = \hat{g}(s) \hat{\rho}_{v, \partial^m w}(s) = s^{-m} \hat{\rho}_{v, \partial^m w}(s) \).

5 Main results in the abstract set-up

In contrast to the discrete time operator renewal theory which is concerned with estimating the operators \( T_t \) in the norm of some appropriate Banach space, here we follow the strategy in [20]. Namely, we adapt renewal theory techniques to estimate the correlation function
\[
\rho_t(v, w) = \int_{\tilde{Y}} vw \circ f_t d\tilde{\mu},
\]
where \( d\tilde{\mu} = \frac{d\mu}{\varphi_0} \) for \( \varphi_0 = \int_Y \Phi d\mu \) in the finite case (under (H0 i)) and \( d\tilde{\mu} = d\tilde{\mu} \) in the infinite case (under (H0 ii)).

For the statement of the main results, we recall that \( C^m(\tilde{Y}, \tilde{\mu}) \) is the class of observables defined in (4.2). Recall that \( \mathcal{B} \) is the Banach space defined by (H2) and (H3) and that the corresponding norm is denoted by \( \| \cdot \|_\mathcal{B} \).

5.1 Finite case

Under (H0 i), we let \( \epsilon > 0 \) and define
\[
\eta(t) = \frac{1}{\varphi_0} \int_{-t}^{\infty} \mu_\phi(\varphi > \tau) d\tau, \quad \xi_{\beta,\epsilon}(t) = \begin{cases} t^{-(\beta-\epsilon)}, & \beta \geq 2, \\ t^{-(2\beta-2)}, & 1 < \beta < 2. \end{cases}
\]
(5.1)

With these specified we state:

Theorem 5.1 (Finite measure) Assume (H0) i), (H1) i), (H2), (H3), (H4) and (H6). Set \( \alpha \) such that (H6) holds. Let \( v = \frac{v^*}{T} \), with \( v^* \in \mathcal{B} \). Let \( w \in C^m(\tilde{Y}, \tilde{\mu}) \). The following hold for all \( m \in \mathbb{N} \) such that \( m \geq 3 + \alpha(\beta + 1) \) and for any \( \epsilon > 0 \).
(a) Let $\eta$ and $\xi_{\beta-\epsilon}$ be as defined in (5.1). Then,

$$\rho_t(v,w) - \frac{1}{\varphi_0} \int_{\tilde{Y}} v \, d\tilde{\mu} \int_{\tilde{Y}} w \, d\tilde{\mu} = \eta(t) \int_{\tilde{Y}} v \, d\tilde{\mu} \int_{\tilde{Y}} w \, d\tilde{\mu} + O(\|v^*\|_B \|w\|_{C^m(\tilde{Y},\tilde{\mu})} \xi_{\beta,\epsilon}(t)).$$

(b) Suppose further that $\int v \, d\tilde{\mu} = 0$. Then,

$$\rho_t(v,w) = O(\|v^*\|_B \|w\|_{C^m(\tilde{Y},\tilde{\mu})} t^{-(\beta-\epsilon)}).$$

5.2 Results in the infinite case

Set $d_\beta = \frac{1}{\pi} \sin \pi \beta$. With this specified we state:

**Theorem 5.2 (Infinite measure)** Assume (H0) ii), (H1) ii), (H2), (H3), (H5) and (H6). Set $\alpha$ such that (H6) holds. The following hold for all $m \in \mathbb{N}$ such that $m \geq 2(\alpha + 1)$.

Let $v = v^* \tilde{h}$, with $v^* \in B$. Let $w \in C^m(\tilde{Y},\tilde{\mu})$. Then

$$\ell(t)t^{1-\beta} \rho_t(v,w) \to d_\beta \int_{\tilde{Y}} v \, d\tilde{\mu} \int_{\tilde{Y}} w \, d\tilde{\mu}.$$

**Remark 5.3** The results for the case $\beta = 1$ and higher order asymptotics of $\rho_t(v,w)$ obtained in [20] can be also obtained in this framework. To simplify the exposition we omit these issues here.

6 Arguments for the finite case: proof of Theorem 5.1

Let $\hat{B}(s) = s(I - \hat{R}(s))^{-1}$, $s \in \mathbb{H}$. Note that

$$\hat{\rho}(s)(v,w) = \frac{1}{s} \int_{\tilde{Y}} \hat{U}(s) \hat{B}(s)vw \, d\tilde{\mu}. \quad (6.1)$$

The first result below on the asymptotic behavior of $\hat{B}$ will be essential in the proof of Theorem 5.1. Before its statement we establish the following

**Notation** Because some of our result below have a direct analogue among the results in [20] we use the same notation here.

a) Let $A$ be a general Banach space. Suppose that $S : [0, \infty) \to A$ lies in $L^1$ with Laplace transform $\hat{S} : \mathbb{H} \to A$. In what follows we write $\hat{S} \in \mathcal{R}_A(a(t))$ if $\|S(t)\| \leq Ca(t)$ for all $t \geq 0$. When $A = L^1$ we simply write $\hat{S} \in \mathcal{R}(a(t))$.

b) Let $s \mapsto \hat{S}(s)$ be an analytic family of Banach space $A$-valued operators, $s \in \mathbb{H}$, such that the family extends continuously to $\mathbb{H}$. If $p \geq 0$ is an integer, define

$$d_p \hat{S}(ib) = \max_{j=0,\ldots,p} \|\hat{S}^{(j)}(ib)\|_A.$$ 

If $p > 0$ is not an integer, define

$$d_p \hat{S}(ib) = d_{[p]} \hat{S}(ib) + \sup_{h \neq 0} \|\hat{S}^{(p)}(i(b + h)) - \hat{S}^{(p)}(ib)\|/|h|^{p-[p]}.$$
We recall that \( P : L^1(\mu_\Phi) \to L^1(\mu_\Phi) \) is the spectral projection associated with the eigenvalue 1 with \( Pv^* = \int v^* d\mu_\Phi \). Let \( P_{\hat{\phi}_0} = (\hat{\phi}_0)^{-1}P \). We also recall that by Lemma 3.2, \( \hat{R}(s) = s(e^s - 1)^{-1}\hat{L}_1(s) \) for all \( s \in \mathbb{H} \) with \( e^s \neq 1 \), where \( \hat{L}_1(s) = \int_0^\infty R_{t,1} e^{-st} dt \).

**Proposition 6.1** Assume \((H0) i\), \((H1) i\), \((H2)\), \((H3)\), \((H4)\) and \((H6)\). Then

\[
s^{-1}\hat{B}(s) = s^{-1}P_{\hat{\phi}_0} + s^{-1}P_{\hat{\phi}_0} \left( \int_t^\infty R_{\sigma,1} d\sigma \right) P_{\hat{\phi}_0} + \hat{E}(s),
\]

where \( \hat{E}(s) \) is as follows

a) There exists \( 0 < r < 1 \) such that for any \( C^\infty \) function \( \psi : \mathbb{R} \to [0,1] \) with \( \text{supp} \psi \subset [-r,r] \),

\[
\psi(b)\hat{E}(ib) \in \mathcal{R}_B(\xi_{\beta,i}(t)).
\]

b) Write \( s = a + ib \), for \( a \geq 0 \) and \( b \in \mathbb{R} \). Then for all \( b \in \mathbb{R} \) with \( |b| \geq 1 \),

\[
\|\hat{E}(s)\| \ll |b|^\alpha,
\]

where \( \alpha \) is as in \((H6)\).

**Remark 6.2** Item b) of the above result is not used as such in this work. It is an immediate consequence of item a) and \((H6)\); we provide it here only for a complete description of \( \hat{E} \).

**Proposition 6.3** Assume the setting and notation of Proposition 6.1 a). Let \( v^* \in \mathcal{B} \) and assume that \( P v^* = 0 \). Then \( \psi(b)b^{-1}\hat{B}(ib) \in \mathcal{R}_{(t^{-\beta-\epsilon})} \).

The proof of Proposition 6.1 is postponed to Section 6.2. Using equation (6.1) and Proposition 6.1 (which is new and required for the proof of Theorem 5.1 in our abstract setting) we can proceed to the proof of Theorem 5.1, following the main steps in [20]. First we notice that

\[
\hat{\rho}(s)(v,w) = \frac{1}{s} \int_{\hat{Y}} \hat{U}(s)P_{\hat{\phi}_0}vwd\hat{\mu} + \frac{1}{s} \int_{\hat{Y}} \hat{U}(s)P_{\hat{\phi}_0} \left( \int_t^\infty R_{\sigma,1} d\sigma \right) P_{\hat{\phi}_0}vwd\hat{\mu} + \int_{\hat{Y}} \hat{U}(s)\hat{E}(s)vwd\hat{\mu}
\]

\[
= \frac{1}{\hat{\phi}_0} \frac{1}{s} \int_{\hat{Y}} \hat{U}(s)P_{\hat{\phi}_0}vwd\hat{\mu} + \frac{1}{\hat{\phi}_0} \frac{1}{s} \int_{\hat{Y}} \hat{U}(s)P_{\hat{\phi}_0} \left( \int_t^\infty R_{\sigma,1} d\sigma \right) P_{\hat{\phi}_0}vwd\hat{\mu}
\]

\[
+ \frac{1}{\hat{\phi}_0} \int_{\hat{Y}} \hat{U}(s)\hat{E}(s)vwd\hat{\mu}
\]

\[
= \hat{\rho}_1(s)(v,w) + \hat{\rho}_2(s)(v,w) + \hat{\rho}_3(s)(v,w).
\]

Hence, it suffices to estimate the inverse Laplace transforms \( \rho_1(t), \rho_2(t), \rho_3(t) \) of \( \hat{\rho}_1(s), \hat{\rho}_2(s), \hat{\rho}_3(s) \). For this purpose, we collect some technical estimates.

**Lemma 6.4** Assume the setting of Theorem 5.1. Then the inverse Laplace transform \( \rho_1(t) \) of \( \hat{\rho}_1(s) \) is given by

\[
\rho_1(t)(v,w) = \int_{\hat{Y}} v d\hat{\mu} \int_{\hat{Y}} w d\hat{\mu} + E(t),
\]

where \( |E(t)| = O(t^{-(\beta-\epsilon)}\|v^*\|_{L^\infty(\mu_\Phi)}\|w^*\|_{L^\infty(\mu_\Phi)}) \) for any \( \epsilon > 0 \).
Proof By Lemma 8.3 and the definition of $\hat{\rho}_1$,

$$\rho_1(t) = \frac{1}{\varphi_0} \int_Y P_{\tilde{\phi}_0} \left( \int_0^u \tilde{\eta}(y) v(y, \tau) \, d\tau \right) w \, d\tilde{\mu} + \frac{1}{\varphi_0} \int_Y P_{\tilde{\phi}_0} \left( \tilde{\eta}(y) \int_u^1 v(y, \tau) \, d\tau \right) w^* \circ \Phi \, d\mu + E(t),$$

where $w = \frac{w^*}{h}$ with $w^* \in L^\infty(\mu_{\Phi})$ and $|E(t)| = O(t^{-\beta} \|v^*\|_{L^\infty(\mu_{\Phi})} \|w^*\|_{L^\infty(\tilde{\mu})})$. Compute that

$$P_{\tilde{\phi}_0} \left( \int_0^u (\tilde{\eta}(y) v(y, \tau) \, d\tau \right) = \frac{1}{\varphi_0} \int_Y \tilde{\eta}(y) \int_0^u v(y, \tau) \, d\tau \, d\mu + \frac{1}{\varphi_0} \int_Y \int_0^u v(y, \tau) \, d\tau \, d\tilde{\mu}$$

and

$$P_{\tilde{\phi}_0} \left( \tilde{\eta}(y) \int_u^1 v(y, \tau) \, d\tau \right) = \frac{1}{\varphi_0} \int_Y \tilde{\eta}(y) \int_u^1 v(y, \tau) \, d\tau \, d\mu + \frac{1}{\varphi_0} \int_Y \int_u^1 v(y, \tau) \, d\tau \, d\tilde{\mu}.$$

Thus,

$$\rho_1(t) = \frac{1}{\varphi_0} \int_Y \int_0^u v(y, \tau) \, d\tau \, d\tilde{\mu} \int_Y w \, d\tilde{\mu} + \frac{1}{\varphi_0} \int_Y \int_0^1 v(y, \tau) \, d\tau \, d\mu \int_Y w^* \circ \Phi \, d\mu.$$

But,

$$\int_Y w^* \circ \Phi \, d\mu = \int_Y w^* \, d\mu = \int_Y w \, d\tilde{\mu},$$

for $w = w^*/h \in L^\infty(\tilde{\mu})$. Altogether,

$$\rho_1(t) = \frac{1}{\varphi_0} \left( \int_Y \int_0^u v(y, \tau) \, d\tau \, d\tilde{\mu} + \int_Y \int_0^1 v(y, \tau) \, d\tau \, d\mu \right) \int_Y w \, d\tilde{\mu} + E(t)$$

$$= \frac{1}{\varphi_0} \left( \int_Y v \, d\tilde{\mu} \int_Y w \, d\tilde{\mu} + E(t) \right) = \int_Y v \, d\tilde{\mu} \int_Y w \, d\tilde{\mu} + E(t).$$

Lemma 6.5 Assume the setting of Theorem 5.1. Then the inverse Laplace transforms $\hat{\rho}_2(t)$ of $\hat{\rho}_2(s)$ is given by

$$\rho_2(t)(v, w) = \eta(t) \int_Y v \, d\tilde{\mu} \int_Y w \, d\tilde{\mu} + E(t),$$

where $\eta(t)$ is as defined in (5.1) and $|E(t)| = O(t^{-\beta} \|v^*\|_{L^\infty(\mu_{\Phi})} \|w^*\|_{L^\infty(\mu_{\Phi})}).$

Proof By definition $\hat{\rho}_2(s) = \frac{1}{\varphi_0} \int_Y \frac{U(s)}{s} P_{\tilde{\phi}_0} \left( \int_t^\infty R_{\sigma} \, d\sigma \right) P_{\tilde{\phi}_0} v \omega \, d\tilde{\mu}$. Recall $v = \frac{w^*}{h}$, $v^* \in B \subset L^\infty(\mu_{\Phi})$ and compute that

$$P_{\tilde{\phi}_0} \left( \int_t^\infty R_{\sigma} \, d\sigma \right) P_{\tilde{\phi}_0} v = \frac{1}{\varphi_0} \int_Y v \, d\mu \int_Y \int_t^\infty R_{\sigma} \, d\sigma \, d\mu = \frac{1}{\varphi_0} \eta(t) \int_Y v \, d\mu.$$
Together with Lemma 8.3, this implies
\[
\rho_2(t) = \frac{1}{\varphi_0} \eta(t) \int_{\tilde{Y}} \tilde{h}(y) \left( \int_{0}^{u} v(y, \tau) d\tau \right) d\mu_{\Phi} \int_{\tilde{Y}} w \, d\tilde{\mu}
+ \frac{1}{\varphi_0} \eta(t) \int_{\tilde{Y}} \tilde{h}(y) \left( \int_{u}^{1} v(y, \tau) d\tau \right) d\mu_{\Phi} \int_{\tilde{Y}} w^* \circ \Phi \, d\mu_{\Phi} + E(t),
\]
where \(E(t)\) is as in the statement of the lemma. As in the proof of Lemma 6.4 we note that
\(\int_{\tilde{Y}} w^* \circ \Phi \, d\mu_{\Phi} = \int_{\tilde{Y}} w \, d\tilde{\mu}\), for \(w = w^*/\tilde{h} \in L^{\infty}(\tilde{\mu})\) and that
\[
\rho_2(t) = \frac{1}{\varphi_0} \eta(t) \int_{\tilde{Y}} \left( \int_{0}^{1} \tilde{h}(y) v(y, \tau) d\tau \right) d\mu_{\Phi} \int_{\tilde{Y}} w \, d\tilde{\mu} + E(t)
= \frac{1}{\varphi_0} \eta(t) \int_{\tilde{Y}} v \, d\tilde{\mu} \int_{\tilde{Y}} w \, d\tilde{\mu} + E(t) = \eta(t) \int_{\tilde{Y}} v \, d\tilde{\mu} \int_{\tilde{Y}} w \, d\tilde{\mu} + E(t).
\]

Following the strategy in [20], the inverse Laplace transform \(\hat{\rho}_3(t)\) of \(\hat{\rho}_3(s)\) will be computed by moving the contour of integration to the imaginary axis (the functions in question are nonsingular on \(\mathbb{H}\)). Hence we deal with inverse Fourier transforms. In this sense, we enlarge the definition of \(R(a(t))\) to include functions defined on the imaginary axis with inverse Fourier transform dominated by \(a(t)\).

We state the result on \(\rho_3(t)\) below and postpone the proof to Subsection 6.1.

**Lemma 6.6** Assume the setting of Theorem 5.1. Let \(\psi : \mathbb{R} \rightarrow [0, 1]\) be a \(C^{\infty}\) function with \(\text{supp} \psi \subset [-3, 3]\). Then, \((1 - \psi(b))\hat{\rho}(ib) \in R(||v^*||_{B}||w||_{C^{\infty}_{w}(Y)}(1/t^{b+\epsilon}))\), for any \(\epsilon > 0\).

**Lemma 6.7** Assume the setting and notation of Lemma 6.6. Then for all \(p > 0\),
\[
(1 - \psi(b))(\hat{\rho}_1(ib) + \hat{\rho}_2(ib)) \in R(||v^*||_{B}||w||_{L^{\infty}(\tilde{\mu})}(1/t^{p})�).
\]

**Proof** By definition,
\[
\hat{\rho}_1(s)(v, w) + \hat{\rho}_2(s)(v, w) = \frac{1}{s} \int_{\tilde{Y}} \tilde{U}(s) \left( P + P \left( \int_{t}^{\infty} R_{\sigma,1} \, d\sigma \right) P \right) v \, w \, d\tilde{\mu}
= \frac{1}{s} \tilde{U}(s) \left( P + P \left( \int_{t}^{\infty} R_{\sigma,1} \, d\sigma \right) P \right) v^* \, w \, d\mu_{\Phi}.
\]

By Lemma 8.5 we know that \(\tilde{U}(s) : B \rightarrow L^{1}(\mu_{\Phi})\) is bounded. Thus, for all \(|b| > 1, \|\frac{1}{t} \int_{\tilde{Y}} \tilde{U}(ib) \, d\mu_{\Phi}\| \leq C/|b|\), for some \(C > 0\). Hence, \(|(1 - \psi(b))(\hat{\rho}_1(ib) + \hat{\rho}_2(ib))| \leq C/|b|\). This together with Lemma 6.9(b) implies the desired conclusion.

As an immediate consequence of Lemma 6.6 and Lemma 6.7, we have

**Corollary 6.8** Assume the setting and notation of Lemma 6.6. Then the following holds for any \(\epsilon > 0\)
\[
(1 - \psi(b))\hat{\rho}_3(ib) \in R(||v^*||_{B}||w||_{L^{\infty}(\tilde{\mu})}(1/t^{b+\epsilon}))�.
\]
We can now complete

**Proof of Theorem 5.1** Item a) follows by Proposition 6.1 a), Lemma 6.4, Lemma 6.5 and Corollary 6.8. Item b) follows by Proposition 6.3 a) and and Corollary 6.8.

### 6.1 Proof of Lemma 6.6

We start by collecting a few technical estimates.

**Lemma 6.9** [20, Proposition 14.1]

(a) Suppose that the family \( b \mapsto \hat{S}(ib) \) is \( C^p \) for some \( p > 0 \) and that there is a constant \( C > 0 \) such that \( d_p \hat{S}(ib) \leq C|b|^{-2} \) for \( |b| > 1 \). Then \( \hat{S} \in \mathcal{R}(1/t^p) \).

(b) Suppose that \( g: \mathbb{R} \to \mathbb{R} \) is \( C^\infty \), such that \( g \equiv 0 \) in a neighborhood of 0, and \( g(b) \equiv 1 \) for \( |b| \) sufficiently large. Let \( m \geq 1 \). Then \( g(b)/b^m \in \mathcal{R}(1/t^p) \) for all \( p > 0 \).

The next two results can be viewed as the analogue of [20, Propositions 2.1 and 12.2] in our abstract framework.

**Lemma 6.10** Let \( \mathcal{B} \) be the Banach space defined by (H2) and (H3). Assume (H4). Then, for any \( \epsilon > 0 \), viewed as a family of operators on \( \mathcal{B}, b \mapsto \tilde{R}(ib) \) is \( C^{\beta-\epsilon} \) and \( d_{\beta-\epsilon} \tilde{R}(ib) \leq C(1 + |b|) \) for all \( b \in \mathbb{R} \).

**Proof** Let \( |b| \leq 1 \). By Lemma 3.2, \( \tilde{R}(ib) = -ib(e^{-ib} - 1)^{-1}\hat{L}_1(ib) \), where \( \hat{L}_1(ib) = \int_0^\infty R_{t,1}e^{ibt} \, dt \). It is easy to verify that for any \( p > 0 \),

\[
\frac{-ib}{e^{-ib} - 1} = 1 + O(b), \quad \text{as } b \to 0.
\]

Also, by Lemma 3.2, \( \tilde{R}(ib) = -ib(e^{-ib} - 1)^{-1}\hat{L}_a(ib) \) for any \( |b| > 1 \) such that \( e^{-iba} \neq 1 \), where \( \hat{L}(ib) = \int_0^\infty R_{t,a}e^{ibt} \, dt \). Given \( b \in \mathbb{R} \), fix \( a \in \mathbb{R}_+ \) such that \( |e^{-iba} - 1| > 1 \). Then, there exists some constant \( C > 0 \) such that

\[
\frac{-ib}{e^{-iba} - 1} \leq C|b|.
\]

Hence, it suffices to show that for all \( b \in \mathbb{R} \) and appropriate \( a \in \mathbb{R}_+ \), there exists \( C > 0 \) such that for any \( \epsilon > 0 \), \( d_{\beta-\epsilon} \hat{L}_a(ib) \leq C \).

Under (H4), let \( \tau < \beta \). Put \( \epsilon_0 = \tau - [\tau] \). By (H4),

\[
\|i^{-[\tau]}d_{[\tau]}\hat{L}_a(ib)\| = \left\| \int_0^\infty t^{[\tau]} R_{t,a}e^{ibt} \, dt \right\| \leq \int_0^\infty t^{[\tau]} \| R_{t,a} \| \, dt < \infty.
\]

Moreover, there exists \( C > 0 \) such that

\[
\| \hat{L}_a^{(\tau)}(ib + h)) - \hat{L}_a^{(\tau)}(ib) \| \leq \left\| \int_0^\infty t^{[\tau]} R_{t,a}e^{ibt}(e^{ih} - 1) \, dt \right\|
\leq Ch^{\epsilon_0} \int_0^\infty t^{[\tau]+\epsilon_0} \| R_{t,a} \| \, dt = Ch^{\epsilon_0} \int_0^\infty t^\tau \| R_{t,a} \| \, dt \ll h^{\epsilon_0},
\]

where the last inequality holds given (H4). The conclusion follows since (H4) holds for any \( \tau < \beta \).
Corollary 6.11 Assume the setting of Lemma 6.10. Suppose that $(H6)$ holds and fix $\alpha$ accordingly. Then, viewed as a family of operators on $\mathcal{B}$, $b \mapsto (I - \hat{R}(ib))^{-1}$ is $C^{\beta - \epsilon}$ and there exists $C > 0$ such that $d_{\beta - \epsilon}(I - \hat{R}(ib))^{-1} \leq C|b|^{\alpha(\beta - \epsilon + 1) + 1}$ for all $|b| > 1$.

Proof As in the proof of [20, Proposition 12.2], we give the details for $\beta - \epsilon$ not an integer. A straightforward induction argument shows that $\frac{d^j}{db^j}(I - \hat{R}(ib))^{-1}$ is a finite linear combination of factors

$$\hat{M}_k \in \{(I - \hat{R})^{-(k+1)}, \ d_k \hat{R} \}, \ k = 1, \ldots, j,$$

for each $j \in \mathbb{N}$, $j \leq \beta - \epsilon$. Also, by (H6) and Lemma 6.10, $\max_{k=1, \ldots, j} \|\hat{M}_k(ib)\| \ll |b|^{\alpha(\beta - \epsilon + 1) + 1}$, and $\max_{k=1, \ldots, j} d_k \hat{M}_k(ib) \ll |b|^{\alpha(\beta - \epsilon + 1) + 1}$. The required estimate follows. ■

Remark 6.12 We believe that replacing (H4) with an assumption of the form (H5) (i.e., such that both $(\mathcal{B}, \mathcal{B}_0)$ and $(\mathcal{B}_0, L^\infty)$ satisfy (H5) with $\tau$ as appropriate) one can show that $\|\frac{d^j}{db^j}(I - \hat{R}(ib))^{-1}\|_{\mathcal{B} \times \mathcal{B}_0} \ll C|b|^{\alpha(\beta - \epsilon + 1) + 1}$ for all $|b| > 1$: a) obtaining $\alpha \in (0, 1)$ as suggested in footnote 1; b) exploiting the type of arguments used in the proof of Lemma 7.7.

Proof of Lemma 6.6 By Proposition 4.7, $\hat{\rho}(s)(v, w) = \hat{P}_m(s) + \hat{H}_m(s)$, where $\hat{P}_m(s)$ is a linear combination of $s^{-j}$, $j = 1, \ldots, m$, and $\hat{H}_m(s) = s^{-m} \hat{\rho}_{v, \partial_t^m w}(s)$.

By the argument used in the proof of [20, Proposition 3.7], $(1 - \psi(b))\hat{P}_m(ib) \in \mathcal{R}(1/t^p)$ for all $p > 0$. Note that

$$\hat{\rho}(s)(v, \partial_t^m w) = \frac{1}{\varphi_0} \int_{\mathbb{R}} \hat{U}(s)(I - \hat{R})^{-1}(s)v \partial_t^m w \ d\mu.$$

Recall $v = \frac{\omega}{h}$ with $v^* \in \mathcal{B}$ and $d\mu = h \ d\mu_\phi$. Proceeding as in (3.2),

$$\hat{H}_m(s) = s^{-m} \int_{\mathbb{R}} \hat{U}(s)(I - \hat{R})^{-1}(s)v^* \partial_t^m w \ d\mu_\phi.$$

By Lemma 8.4, we know that $\hat{U}(s) : \mathcal{B} \to L^1(\mu_\phi)$ lies in $\mathcal{R}_{\mathcal{B} \times L^1(\mu_\phi)}(1/t^\beta)$. Since $\mathcal{B} \subset L^\infty(\mu_\phi)$ it remains to show that $Q(ib) = b^{-m}(1 - \psi(b))(I - \hat{L}(ib))^{-1}(s)$ lies in $\mathcal{R}_B(1/t^\beta)$.

By Lemma 6.10, $\hat{R}(ib)$ is $C^{\beta - \epsilon}$, for any $\epsilon > 0$. Hence, $(I - \hat{R})^{-1}(ib)$ is $C^{\beta - \epsilon}$ on $\mathbb{R} \setminus \{0\}$ and $Q(ib)$ is $C^{\beta - \epsilon}$ on $\mathbb{R}$. Moreover, by Corollary 6.11, for all $|b| > 1$ there exists $C > 0$ such that $d_{\beta - \epsilon}(I - \hat{R}(ib))^{-1} \leq C|b|^{\alpha(\beta - \epsilon + 1) + 1}$. Hence for all $|b| > 1$ and all $m - \alpha(\beta - \epsilon + 1) + 3$, $d_{\beta - \epsilon}Q(ib) \ll |b|^{-2}$. Together with Lemma 6.9(a), we obtain that $Q \in \mathcal{R}_B(1/t^{\beta - \epsilon})$, as required.

6.2 Several technical results required in the proof of Proposition 6.1

As in [20], a main step in proving a result of the form of Theorem 5.1 is based on the following continuous time version of [10, 21, First Main Lemma]. In our abstract set-up, we state:

Lemma 6.13 A version of [20, Lemma 13.1] Assume $(H0)$ i), (H2), (H3) and (H4). Let $\psi : \mathbb{R} \to [0, 1]$ be $C^\infty$ with supp $\psi \subset [-r, r]$ where $r \in (0, 1)$ is sufficiently small and such that $\psi \equiv 1$ in a neighborhood of $0$. Then, for any $\epsilon > 0$, $\psi \hat{B} \in \mathcal{R}_B(1/t^{\beta - \epsilon})$. 

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The next result is required in the proof of Lemma 6.13.

**Lemma 6.14** Assume (H0) i), (H2), (H3) and (H4). For all $C^\infty$ functions $\psi : \mathbb{R} \to [0,1]$ with $\text{supp} \psi \subset [-3,3]$ and for any $\epsilon > 0$,

$$
\psi(b) \frac{\hat{R}(ib) - \hat{R}(0)}{b} \in \mathcal{R}_B(1/t^{\beta-\epsilon}).
$$

**Proof** We first prove a). Recall that $R_{t,1}v = R(1_{t<\sigma<t+1}v)$ and $\hat{L}_1(s) = \int_0^\infty R_{t,1}e^{-st}dt$, $s \in \mathbb{R}$. By Lemma 3.2, for all $|b| < 1$,

$$
\frac{\hat{R}(ib) - \hat{R}(0)}{b} = \hat{L}_1(ib) - \hat{L}_1(0) + \frac{1 - e^{-ib} - ib}{b(e^{-ib} - 1)} \hat{L}_1(ib).
$$

From the proof of Lemma 6.10 we know that $\hat{L}_1(ib)$ is $C^{\beta-\epsilon_0}$, for some small $\epsilon_0 > 0$ and $d_{\beta-\epsilon} \hat{L}_1(ib) \leq C$, for some constant $C > 0$. Also, it is easy to verify that for any $p > 0$, $\psi(b)\frac{1 - e^{-ib} - ib}{b(e^{-ib} - 1)}$ is $C^p$ and $d_p\left(\psi(b)\frac{1 - e^{-ib} - ib}{b(e^{-ib} - 1)}\right) \leq C$, for some constant $C > 0$. It follows that $d_{\beta-\epsilon}\left(\psi(b)\frac{1 - e^{-ib} - ib}{b(e^{-ib} - 1)} \hat{L}_1(ib)\right) \leq C$, for some constant $C > 0$. Note that the inverse Fourier transform $S(t)$ of $\psi(b)\frac{1 - e^{-ib} - ib}{b(e^{-ib} - 1)} \hat{L}_1(ib)$ is given by

$$
S(t) = \int_{-3}^3 \psi(b)\frac{1 - e^{-ib} - ib}{b(e^{-ib} - 1)} \hat{L}_1(ib)e^{ibt}db.
$$

Integration by parts gives

$$
\|S(t)\| \ll t^{-(\beta-\epsilon)} \int_{-3}^3 \left| d_{\beta-\epsilon}\left(\psi(b)\frac{1 - e^{-ib} - ib}{b(e^{-ib} - 1)} \hat{L}_1(ib)\right)\right| db \ll t^{-(\beta-\epsilon)}.
$$

Thus, the second term of (6.2) lies in $\mathcal{R}_B(1/t^{\beta-\epsilon})$.

It remains to deal with the first term of (6.2). Compute that

$$
\hat{L}_1(ib) - \hat{L}_1(0) = \int_0^\infty R_{t,1}e^{-ibt} - 1 dib = \int_0^\infty R_{t,1}(\int_0^t e^{-ib\sigma}d\sigma)dt = \int_0^\infty (\int_0^\infty R_{\sigma,1}d\sigma)e^{-ibt}dt.
$$

The above equation together with (H4) (or more precisely, Remark 4.3) implies that $\psi(b)\frac{\hat{L}(ib) - \hat{L}_1(0)}{b} \in \mathcal{R}_B(1/t^{\beta-\epsilon})$, which ends the proof.

**Remark 6.15** For later use (in the proof of Proposition 6.1), we note that continuing from (6.2) (with $s = -a + ib$, $a \geq 0$, $|b| < 1$ instead of $ib$), we obtain that

$$
\frac{\hat{R}(s) - \hat{R}(0)}{s} = \frac{\hat{L}_1(s) - \hat{L}_1(0)}{s} - \hat{L}_1(s) + \hat{F}(s),
$$
where $\hat{F}(s) = g(s)\hat{L}_1(s)$ with $g(s) = O(s)$, as $s \to 0$ and $g(s)$ belongs to $C^p$ for any $p > 0$. By the argument used in the proof of Lemma 6.14, $\psi(b)\hat{F}(ib) \in \mathcal{R}_B(1/t^{\beta-\epsilon})$. Together with (6.3) (with $s$ instead of $ib$), the above equation implies that
\[
\frac{\hat{R}(s) - \hat{R}(0)}{s} = \int_0^\infty R_{t,1} \int_0^t (e^{-s\sigma} - 1) \, d\sigma + \hat{F}(s), \tag{6.4}
\]
where $\|\hat{F}(s)\| = O(s)$ as $s \to 0$.

The rest of the proof of Lemma 6.13 goes exactly as the proof of [20, Lemma 13.1] with the norm $\|\cdot\|_B$ of our function space $B$ replacing the norm $\|\cdot\|_\theta$ in [20]. This is possible due to Lemma 6.14. We provide the main steps for the reader’s convenience.

The following result has been established in [20]. The corresponding proof in [20] builds upon the strategy in [10]. Roughly, it establishes the existence of an operator $\hat{R}$ that is identical to $\hat{R}$ in a neighborhood of 0 and whose eigenvalue $\hat{\lambda}$ is well defined on the imaginary axis. So, one can speak of the inverse Laplace transform of $(1 - \hat{\lambda}(b))/b$. Furthermore, the result below establishes that $(1 - \lambda(b))/b$ is different from zero on a compact interval $[-r, r]$ for some $r > 0$, and one can speak of the inverse Laplace transform of $b(1 - \lambda(b))^{-1}$.

**Proposition 6.16** [20, Proposition 13.4, Proposition 13.5] Assume (H0 i), (H2), (H3) and (H4). Let $\delta > 0$. For any $\epsilon > 0$ and for all $r > 0$ sufficiently small, there exists a $C^{\beta-\epsilon}$ family $b \mapsto \hat{R}(ib)$ with a $C^{\beta-\epsilon}$ family of simple eigenvalues $\hat{\lambda}(b) \in \{z \in \mathbb{C} : |z - 1| < \delta\}$ such that

(a) $\hat{R}(ib) \equiv \hat{R}(ib)$ for $|b| \leq r$.

(b) $\hat{R}(ib) \equiv \hat{R}(0)$ and $\hat{\lambda}^*(b) \equiv 1$ for $|b| \geq 2$.

(c) $\|\hat{R}(ib) - \hat{R}(0)\|_B < \delta$ for all $b \in \mathbb{R}$.

(d) For all $b \in \mathbb{R}$, the spectrum of $\hat{R}(ib)$ consists of $\hat{\lambda}(b)$ together with a subset of $\{z : |z - 1| \geq 3\delta\}$.

(e) $(1 - \lambda(b))/b$ is bounded away from zero on $[-r, r]$.

(f) $(1 - \lambda(b))/b \in \mathcal{R}(1/t^{\beta-\epsilon})$.

(g) Let $\hat{B}(ib) = b(I - \hat{R}(ib))^{-1}$. Then, for $b \in [-r, r]$, $\hat{B}(ib) = P + \hat{D}(ib)$, where $\hat{D}(ib) \in \mathcal{R}_B(1/t^{\beta-\epsilon})$.

**Remark 6.17** The proof of Proposition 6.16 goes word by word as the proofs of [20, Proposition 13.4, Proposition 13.5] with Lemma 6.13 above replacing [20, Lemma 13.3].

**Proof of Lemma 6.13** By Proposition 6.16(a), $\psi\hat{B} = \hat{B}\psi$ where $\hat{B}(ib) = b(I - \hat{R}(ib))^{-1}$. Let $\hat{P}(b)$ be the spectral projection associated with $\hat{\lambda}(b)$. By definition, $\hat{B}(ib) = ((1 - \hat{\lambda}(b))/b)^{-1}\hat{P}(b) + b(I - \hat{R}(ib))^{-1}(I - \hat{P}(b))$.

The second term is $C^{\beta-\epsilon}$, for any $\epsilon > 0$. Hence, it lies in $\mathcal{R}_B(1/t^{\beta-\epsilon})$ when multiplied by $\psi$. By the argument used in the proof of [20, Lemma 13.1] (which applies to our setting because of Lemma 6.14), we have $\psi(b)((1 - \lambda(b))/b)^{-1}\hat{P}(b) \in \mathcal{R}_B(1/t^{\beta-\epsilon})$. The conclusion follows.
6.3 A step in the proof of Proposition 6.1 analogous to the discrete time setting

In the present and next sections we show that Proposition 6.1 a), which can be viewed as the continuous time version of [10, Theorem 1] (a generalization of [21, Theorem 1]), can be proved by adapting the techniques developed in [10, 21]. A variant of Proposition 6.1 a) is implicit in [20], which restricts the analysis to suspension flows over Gibbs Markov maps. The argument used in the proof Proposition 6.1 a) is essentially different from the type of arguments used in [20]. This is, of course, required since this result is formulated in the abstract setting of Section 4 as opposed to the setting of Gibbs Markov maps in [20]. As we explain below, due to Lemma 3.2, we are able to adapt the main steps in [10, 21, Proof of Theorem 1] from discrete to continuous time systems and as such offer a transparent proof of Theorem 5.1.

We start by constructing a continuous time version of the polynomial operator valued function used in [10, 21]. Recall from Lemma 3.2 that \( \hat{R}(s) = \frac{s}{e^s - 1} \hat{L}_1(s) \) for all \( s = -a + ib \), \( a \geq 0, |b| < 1 \), \( \hat{L}_1(s) = \int_0^\infty R_{t,1} e^{-st} \, dt \). To simplify notation, throughout this section we let \( \hat{L}(s) := \hat{L}_1(s) = \int_0^\infty R_{t,1} e^{-st} \, dt \).

For \( N \geq 0 \) define

\[
\hat{L}_N(s) = \int_0^N R_{t,1} e^{-st} \, dt + \int_N^\infty R_{t,1} dt + (-e^{-s} - 1) \int_N^\infty t R_{t,1} dt
\]

\[
\hat{R}_N(s) = \frac{s}{e^s - 1} \hat{L}_N(s), \quad \hat{B}_N(s) = s(I - \hat{R}_N(s))^{-1}.
\]  

Throughout this section we assume (H0 i), (H2), (H3) and (H4).

The first result below can be viewed as the continuous time version of [21, Step 1, Proof of Lemma 5].

**Lemma 6.18** There exists \( \delta > 0 \) such that for all \( s \in \mathbb{H} \) with \( b \in B_\delta(0) \)

\[
\hat{B}_N(s) = P_{\varphi_a} + (1 - e^{-s}) \hat{D}_N(s),
\]

where \( \hat{D}_N(s) \) is analytic.

**Proof** By definition, \( \hat{L}_N(0) = \hat{L}_1(0) = \hat{R}(0) \). Hence, \( \hat{R}_N(0) = \hat{L}_1(0) \). Denote by \( \frac{d}{d(-s)} \hat{L}_N(s) \) the derivative in \( -s \) and note that \( \frac{d}{d(-s)} \hat{L}_N(s) \bigg|_{s=0} = \frac{d}{d(-s)} \hat{L}_a(s) \bigg|_{s=0} \). Since

\[
\frac{d}{d(-s)} \hat{R}_N(s) = \left( \frac{s}{e^s - 1} \right)' \hat{L}_N(s) + \frac{s}{e^s - 1} \frac{d}{d(-s)} \hat{L}_N(s),
\]

we have \( \frac{d}{d(-s)} \hat{R}_N(s) \bigg|_{s=0} = \frac{d}{d(-s)} \hat{R}(s) \bigg|_{s=0} \). Since \( s \to \hat{R}_N(s) \) is differentiable in \( \mathbb{H} \), there exists \( \delta_0 > 0 \) such that \( \hat{R}_N(s) \) has an eigenvalue \( \lambda_N(s) \) in \( B_{\delta_0}(0) \) such that

1. Since \( \hat{R}_N(0) = \hat{L}_1(0) = R \), \( \lambda_N(0) \) is simple and isolated in the spectrum of \( \hat{R}_N(0) \) and \( \lambda_N(0) = 1 \). (Recall that by (H2 ii), 1 is an isolated simple eigenvalue in the spectrum of \( R \).)
2. The rest of the spectrum of $\hat{R}_N(s)$ is contained in \{\(\lambda \in \mathbb{C} : |\lambda| < 1\)\}.

Recall that $P$ is the spectral projection associated with the eigenvalue 1 of $\hat{R}(0) = R$. Let $P_N(s)$ be the spectral projection associated with $\lambda_N(s)$ and note that $P_N(0) = P$.

To obtain the expansion of $\lambda_N(s)$ as $s \to 0$, we follow [21]. More precisely, starting from $\hat{R}_N(s)P_N(s) = \lambda_N(s)P_N(s)$, differentiating (in $-s$) and applying $P_N(s)$ to both sides, $P_N(s)\frac{d}{d(-s)}\hat{R}_N(s)P_N(s) = \lambda_N(s)P_N(s)\frac{d}{d(-s)}\hat{R}_N(s) + \frac{d}{d(-s)}\lambda_N(s)P_N(s)$. Next, by Proposition 4.5, $P\frac{d}{d(-s)}\hat{R}(s)\big|_{s=0} = \bar{\varphi}_0 \neq 0$. Combined with the previous equation, we obtain that $s \to 0$,

$$\lambda_N(s) = 1 + s \cdot \left[ P\frac{d}{d(-s)}\hat{R}(s)\big|_{s=0} + o(s) \right] = 1 + s\bar{\varphi}_0 + o(s). \tag{6.6}$$

Let $Q_N(s) = I - P_N(s)$ be the complementary spectral projection. Putting the above together, we have that there exists $0 < \delta < \delta_0$ such that for all $s \in B_\delta(0)$,

$$\hat{B}_N(s) = \left( \frac{1 - \lambda_N(s)}{s} \right)^{-1} P_N(s) + (e^{-s} - 1)(I - \hat{R}_N(s))^{-1}Q_N(s),$$

with $||(I - \hat{R}_N(s))^{-1}Q_N(s)|| \leq C$ for some constant $C > 0$. By (6.6), $s = 0$ is the only zero of $1 - \lambda_N(s)$. This together with the above equation and the analyticity of $\hat{R}_N$ ends the proof.

The next result is the analogue of Lemma 6.14 for $\hat{R}_N$.

\textbf{Lemma 6.19} For all $C^\infty$ functions $\psi : \mathbb{R} \to [0,1]$ with $\text{supp} \psi \subset [-3,3]$,

$$\psi(b)\frac{\hat{R}(ib) - \hat{R}_N(ib)}{b} \in \mathcal{R}_B(1/t^{\beta-\epsilon}).$$

\textbf{Proof} Write

$$\frac{\hat{R}(ib) - \hat{R}_N(ib)}{b} = \frac{\hat{R}(ib) - \hat{R}(0)}{b} - \frac{\hat{R}_N(ib) - \hat{R}(0)}{b}$$

$$= \frac{\hat{L}(ib) - \hat{L}(0)}{b} + \frac{1 - e^{-ib} - ib}{b(e^{-ib} - 1)}\hat{L}(ib) + \frac{\hat{L}_N(ib) - \hat{L}(0)}{b} + \frac{1 - e^{-ib} - ib}{b(e^{-ib} - 1)}\hat{L}_N(ib)$$

$$= \frac{\hat{L}(ib) - \hat{L}(0)}{b} + \frac{\hat{L}_N(ib) - \hat{L}(0)}{b} + \frac{1 - e^{-ib} - ib}{b(e^{-ib} - 1)}(\hat{L}(ib) - \hat{L}_N(ib)). \tag{6.7}$$

By the argument used in the proof of Lemma 6.14 (which relies on Remark 4.3), when multiplied by $\psi$, each term in (6.7) lies in $\mathcal{R}_B(1/t^{\beta-\epsilon})$. The conclusion follows.

Based on the spectral properties of $\hat{R}_N$ mentioned in the proof of Lemma 6.18, we can obtain a continuous time version of [21, Second main Lemma].

\textbf{Lemma 6.20} For $s = a + ib$, $a \geq 0$, $|b| \leq 1$, put $\hat{B}_N(s) = P\bar{\varphi} + (e^{-s} - 1)\hat{D}_N(s)$.

Choose $\psi$ such that the conclusion of Lemma 6.13 holds. Then for any $p > 0$,

$$\psi(b)\hat{D}_N(ib) \in \mathcal{R}_B(t^{-p}).$$
**Proof** First, we note that \( \psi(b)\hat{B}_N(ib) \) is well defined. Let \( B(t) \) be the inverse Fourier transform of \( \psi B \). By Lemma 6.13, \( \psi B \in \mathcal{R}_B(1/t^{\beta-\epsilon}) \). Hence, there exists \( C_0 \) such that
\[
\int_0^\infty \|B(t)\| dt \leq C_0. \tag{6.8}
\]
Recall supp \( \psi \subset [-r, r] \) where \( r \in (0, 1) \) is sufficiently small. For \( |b| \leq r \), write
\[
\hat{B}_N(ib) = \hat{B}(ib) \left( I - \frac{\hat{R}(ib) - \hat{R}_N(ib)}{b} \hat{B}(ib) \right)^{-1}.
\]
Continuing from equation (6.7),
\[
\frac{\hat{R}(ib) - \hat{R}_N(ib)}{b} = \int_N^\infty \left( \int_t^\infty R_{\tau,1} d\tau \right) e^{-ibt} dt
+ \frac{1 - e^{-ib} - ib}{b(e^{-ib} - 1)} \int_N^\infty R_{t,1} e^{-ibt} dt + \frac{e^{-ib} - 1}{b} \int_N^\infty t R_{t,1} dt.
\]
Clearly, \( \frac{1 - e^{-ib} - ib}{b(e^{-ib} - 1)} = 1 + O(|b|) \), as \( b \to 0 \). Together with (H4), this implies that for any \( \tau < \beta \) and for all \( |b| \leq 1 \),
\[
\left\| \frac{\hat{R}(ib) - \hat{R}_N(ib)}{b} \right\| \leq CN^{-(\tau-1)}, \tag{6.9}
\]
for some positive finite constant \( C \). Choosing \( N \leq (1/2)C_0 \) (with \( C_0 \) as in (6.8)), we have that \( \psi(b)\hat{B}_N(ib) \) is well defined.

Reasoning as in [21, Step 1 of Proof of Lemma 6], for any \( b_0 \in \text{supp } \psi \), there exists \( \delta_0 > 0 \) such that \( b \to \hat{B}_N(ib) \) is analytic in \( B_{\delta_0}(0) \). Also, by Lemma 6.18, there exists \( \delta > 0 \) such for all \( b \in B_{\delta}(0) \), \( \hat{B}_N(ib) = P + b\hat{D}_N(ib) \), where \( \hat{D}_N \) is analytic. It follows that for any \( p > 0 \), \( \psi(b)\hat{D}_N(ib) \) is \( C^p \) with \( d_p(\psi(b)\hat{D}_N(ib)) \leq C \), for some constant \( C > 0 \). By the argument used in Lemma 6.14 (in estimating \( S(t) \) there), we have \( \psi(b)\hat{D}_N(ib) \in \mathcal{R}_B(t^{-p}) \), which ends the proof.

### 6.4 Proofs of Proposition 6.1 and Proposition 6.3

At this point in the exposition we can summarize the rest of the argument and emphasize on the analogy with the discrete time situation.

Put \( \hat{C}(s) = s^{-1}(\hat{R}(s) - \hat{R}_N(s)) \). By equation (6.9), \( \|\hat{C}(s)\| \leq CN^{-(\tau-1)} \), for all \( s \in \mathbb{H} \) with \( |b| < 1 \), for some positive finite constant \( C \). By Lemma 6.20, \( \|\hat{C}(s)\| \leq C \), for some positive finite constant \( C \). Hence, we can choose \( N \) such that \( (I - \hat{C}(s)\hat{B}_N(s))^{-1} \) is well defined for all \( s \in \mathbb{H} \) with \( |b| < 1 \).

By the resolvent equality, \( \hat{B}(s) = \hat{B}_N(s)(I - \hat{C}(s)\hat{B}_N(s))^{-1} \), for all \( s \in \mathbb{H} \) with \( |b| < 1 \) and
\[
\hat{B}(s) = \hat{B}_N(s) + \hat{B}_N(s)\hat{C}(s)\hat{B}_N(s) + [\hat{B}_N(s)\hat{C}(s)]^2 \hat{B}(s).
\]
Hence,
\[
s^{-1}\hat{B}(s) = s^{-1}\hat{B}_N(s) + s^{-1}\hat{B}_N(s)\hat{C}(s)\hat{B}_N(s) + s^{-1}[\hat{B}_N(s)\hat{C}(s)]^2 \hat{B}(s). \tag{6.10}
\]
A discrete time version of the above identity has been used in [10, 21].

As already mentioned, in contrast to the discrete time scenario, following the strategy in [20] we only estimate the correlation function $\rho_t(v, w)$. For such a strategy it suffices to estimate the inverse Fourier transform (in the operator norm) of $\psi(b)B_N(ib)$, $\psi(b)\hat{B}_N(ib)\hat{C}(ib\hat{B}_N(ib))$, and $\psi(b)(\hat{B}_N(ib)\hat{C}(ib))]^2\hat{B}(ib)$, with $\psi$ chosen as in Lemma 6.13.

The inverse Fourier transform of the first term is dealt with in Lemma 6.20. In what follows we estimate the inverse Fourier transform (in the operator norm) of $\psi(b)B_N(ib)\hat{C}(ib\hat{B}_N(ib))$, and $\psi(b)(\hat{B}_N(ib)\hat{C}(ib))]^2\hat{B}(ib)$ by adapting the techniques in [10, 21] (including those steps in [10] needed to deal with the case $\beta > 1$) to the continuous time scenario. Again, the basic observation that makes this possible is Lemma 3.2.

**Lemma 6.21** Assume (H0) i), (H1) and (H2), (H3) and (H4). Choose $\psi$ such that the conclusion of Lemma 6.13 holds. Then, the following hold for any $\epsilon > 0$ and any $p > 0$.

a) $s^{-1}\hat{B}_N(s) = s^{-1}P_\varphi + \hat{D}_N(s)$, where $\psi(b)\hat{D}_N(ib) \in \mathcal{R}_B(t^{-p})$.

b) $\hat{B}_N(s)\hat{C}(s)\hat{B}_N(s) = s^{-1}P_{\varphi}(\int_t^\infty R_{\sigma,1}d\sigma)P_{\varphi} + \hat{A}(s)$, where $\psi(b)\hat{A}(ib) \in \mathcal{R}_B(t^{-(\beta - \epsilon)})$.

c) $\psi(b)b^{-1}[\hat{B}_N(ib)\hat{C}(ib)]^2\hat{B}(ib) \in \mathcal{R}_B(a(t))$, where

$$a(t) \leq \begin{cases} t^{-(\beta - \epsilon)} & \text{if } \beta > 2; \\ t^{-(2 - \epsilon)} & \text{if } \beta = 2, \text{ for any } \epsilon' > \epsilon; \\ t^{-(2\beta - 2)} & \text{if } \beta < 2. \end{cases}$$

**Proof** Item a) follows immediately from Lemma 6.20 i). As seen below, the proof of b) is straightforward. By Remark 6.15 and (6.3),

$$\hat{C}(s) = \int_0^\infty \left( \int_t^\infty R_{\sigma,1}d\sigma \right) e^{-st} dt + \hat{F}(s),$$

where $\psi(b)\hat{F}(ib) \in \mathcal{R}_B(t^{-(\beta - \epsilon)})$. Together with item a), this implies that

$$s^{-1}\hat{B}_N(s)\hat{C}(s)\hat{B}_N(s) = s^{-1}P_{\varphi}(\int_t^\infty R_{\sigma,1}d\sigma)P_{\varphi} + \hat{A}(s),$$

where $\hat{A}(s)$ is a sum of products, all of them including the factors $\hat{D}_N(s)$ and $\hat{F}(s)$. Hence, $\psi(b)\hat{A}(ib) \in \mathcal{R}_B(t^{-(\beta - \epsilon)})$.

We continue with the proof of c). We first note that by (6.4)

$$\hat{C}(s) = \int_0^\infty R_{t,1} \int_0^t (e^{-s\sigma} - 1) d\sigma dt + \hat{F}(s),$$

where $\|\hat{F}(s)\| = O(s)$ as $s \to 0$. Also, for any $0 < \delta \leq 1$ and $s$ small,

$$\int_0^\infty \|R_{t,1}\| \int_0^t |e^{-s\sigma} - 1| d\sigma dt \leq |s|^\delta \int_0^\infty \|R_{t,1}\| \int_0^t \sigma^\delta d\sigma dt = |s|^\delta \int_0^\infty t^{\delta + 1} \|R_{t,1}\| dt.$$
Since by (H4), \( \int_0^\infty t^{\delta+1} \| R_{t,1} \| dt < \infty \), we have
\[
\int_0^\infty R_{t,1} \int_0^t (e^{-s\sigma} - 1) d\sigma dt \to 0, \text{ as } s \to 0
\]
and as a consequence, \( \hat{C}(s) \to 0 \) as \( s \to 0 \).

Put \( \hat{G}(ib) = \psi(b) \hat{B}_N(ib) \hat{C}(ib) \). By item a) and Lemma 6.14, \( \hat{G}(ib) \in \mathcal{R}_B(t^{-(\beta-\e)}) \). Clearly, \( \hat{G}(s) \to 0 \) as \( s \to 0 \) and \( \hat{G}(0) = 0 \). Writing \( \hat{G}(s) = \int_0^\infty G(t)e^{-st} dt \), we have \( \int_0^\infty G(t) dt = 0 \) and \( \|G(t)\| \ll t^{-(\beta-\e)} \). Thus,
\[
\frac{\hat{G}(ib)}{-ib} = \frac{\hat{G}(ib) - \hat{G}(0)}{-ib} = \int_0^\infty G(t) \frac{e^{-ibt} - 1}{-ib} dt = \int_0^\infty (\int_t^\infty G(\sigma) d\sigma) e^{-ibt} dt.
\]
Since \( \|G(t)\| \ll t^{-(\beta-\e)} \), we have \( \int_t^\infty \|G(\sigma)\| d\sigma \ll t^{-(\beta+1-\e)} \) and hence \( b^{-1} \hat{G}(ib) \in \mathcal{R}_B(t^{-(\beta-1-\e)}) \).

Next, put \( \hat{E}(ib) = b^{-1} \hat{G}(ib)^2 \hat{B}(ib) \). We want to estimate the inverse Fourier transform of \( \hat{E}(ib) \). Write \( \hat{E}(ib)' := \frac{d}{db} \hat{E}(ib) \). To obtain the required estimates, we proceed as in the discrete time scenario [10, 21] by estimating the inverse Fourier transform of \( \hat{E}(ib)' \) and then integrate. Let \( \hat{B}' \) and \( \hat{G}' \) denote the first derivative of \( \hat{B}, \hat{G} \) in \( b \). Compute that
\[
\hat{E}(ib)' = -\left( \frac{\hat{G}(ib)}{b} \right)^2 \hat{B}(ib) + i \left[ \frac{\hat{G}(ib)}{b} \hat{G}(ib)' + \frac{\hat{G}(ib)}{b} \hat{G}(ib) \hat{G}(ib)' \right] \hat{B}(ib)
+ i \hat{G}(ib) \left( \frac{\hat{G}(ib)}{b} \right) \hat{B}(ib)'.
\]
By Lemma 6.13, \( \psi(b) \hat{B}(ib) \in \mathcal{R}_B(1/t^{\beta-\e}) \). It follows that \( \psi(b) \hat{B}' \in \mathcal{R}_B(1/t^{\beta+1-\e}) \). Due to these estimates and the fact that \( b^{-1} \hat{G}(ib) \in \mathcal{R}_B(t^{-(\beta-1-\e)}) \), the rest of the argument goes exactly like in the discrete time case [10]. We recall the main elements. First, the statement and proof of [10, Lemma 4.3] on convolutions goes exactly the same as in the discrete time case (with of course, sums replaced by integrals). As a consequence we obtain,
\[
b^{-2} \hat{G}(ib)^2 \hat{B}(ib) \in \mathcal{R}_B(b(t)),
\]
where
\[
b(t) \leq \begin{cases} t^{-(\beta-1-\e)} & \text{if } \beta > 2; \\
 t^{-(1-\e)} & \text{if } \beta = 2, \text{ for any } \epsilon' > \epsilon; \\
 t^{-(2\beta-3)} & \text{if } \beta < 2.
\end{cases}
\]
Also, based on the continuous time version of [10, Lemma 4.3] and the fact that \( \psi \hat{B}' \in \mathcal{R}_B(1/t^{\beta-1-\e}) \), one obtains similar estimates for the other terms of \( \hat{E}(ib)' \). Integrating, we obtain that \( \hat{E}(ib) \in \mathcal{R}_B(a(t)) \), with \( a(t) \) as in the statement of item c), as required.

Proposition 6.1 follows immediately from equation (6.10) and Lemma 6.21. It remains to complete the

**Proof of Proposition 6.3** Recall \( Pv = 0 \). Continuing from (6.10),
\[
s^{-1} \hat{B}(s) = s^{-1} \hat{B}_N(s) + s^{-1} \hat{B}_N(s) \hat{C}(s) \hat{B}_N(s) + s^{-1} [\hat{B}_N(s) \hat{C}(s)]^2 + s^{-1} [\hat{B}_N(s) \hat{C}(s)]^3 \hat{B}(s).
\]
By Proposition 6.1 a), \( s^{-1} \hat{B}_N(s)v = \hat{D}_N(s)v \), where \( \psi(b)\hat{D}_N(ib) \in \mathcal{R}_B(t^{-p}) \), for any \( p > 0 \). By Proposition 6.1 b), \( \hat{B}_N(s)\hat{C}(s)\hat{B}_N(s)v = \hat{A}(s)v \), where \( \psi(b)\hat{A}(ib) \in \mathcal{R}_B(t^{-(\beta-\epsilon)}) \). Thus,

\[
s^{-1}[\hat{B}_N(s)\hat{C}(s)]^2v = \hat{A}(s)\hat{C}(s)\hat{D}_N(s)v
\]

and

\[
s^{-1}[\hat{B}_N(s)\hat{C}(s)]^3\hat{B}(s)v = \hat{A}(s)\hat{C}(s)\hat{D}_N(s)\hat{C}(s)\hat{B}(s).
\]

By Lemma 6.19, \( \psi\hat{C}(ib) \in \mathcal{R}_B(1/t^{\beta-\epsilon}) \). By Lemma 6.13, \( \psi\hat{B}(ib) \in \mathcal{R}_B(1/t^{\beta-\epsilon}) \). Hence, under the additional assumption \( P\psi = 0 \), \( \psi b^{-1}[\hat{B}_N(ib)\hat{C}(ib)]^2 \in \mathcal{R}_B(1/t^{\beta-\epsilon}) \) and \( \psi b^{-1}[\hat{B}_N(ib)\hat{C}(ib)]^3\hat{B}(ib) \in \mathcal{R}_B(1/t^{\beta-\epsilon}) \). The conclusion follows by putting the above estimates together.

**Remark 6.22** Showing that the estimates provided by Lemma 6.21 hold with \( \| \|_{\mathcal{B}} \) replaced by \( \| \|_{\mathcal{B},\mathcal{B}_0} \) under a weakened (H4) (that is, such that both \( (\mathcal{B},\mathcal{B}_0) \) and \( (\mathcal{B}_0,\mathcal{L}_\infty) \) satisfy (H5) with \( \tau \) as appropriate) brings up several complications. Among these, we note that a) one needs to work with an appropriate version of Lemma 6.13; b) the last term of equation (6.10) is a complicated product, so its inverse Laplace transform cannot be easily estimated under a weakened (H4).

We believe that for a) one could exploit a decomposition of \( \psi\hat{B} \) into the scalar part given by \( \hat{\lambda} \) (of which inverse Laplace transform can be estimated under a weaker (H4)) and the rest. Also, we believe that this route for dealing with a) can be further combined with repeated applications of the type of arguments and abstract results of [12] (or rather the improved version of the mentioned abstract result in [11]) to solve b). This type argument for dealing with a) and b) above constitutes the subject of work in progress.

7 Arguments in the infinite case: proof of Theorem 5.2

As in Section 6, in this section we make transparent that the present abstract set-up allows us to show that main part of the techniques developed for the discrete time scenario (namely, [19] with some required generalizations in [15]) carry over to the continuous time case. Due to (H5) ii), in part of the arguments we follow the steps in [15], which exploits an analogue of (H5) ii) in the discrete time setting. Equally important, as in Section 6, some techniques/calculations required to deal with continuous time infinite measure preserving systems are directly borrowed from [20].

7.1 Estimates for \( (I - \hat{R})^{-1} \)

By (H2), (H3) there exist \( \delta > 0 \) and a continuous family \( \lambda(ib) \) of simple eigenvalues of \( \hat{R}(ib) \) such that \( \lambda(ib) \) is well defined for all \( |b| < \delta \) and \( \lambda(0) = 1 \). In what follows, we let \( P(ib) \) be the associated spectral projection, \( v(ib) \) be the associated eigenfunction and set \( Q(ib) = I - P(ib) \).

The continuity properties of \( \hat{R}(ib) \), \( b \in \mathbb{R} \), are obtained via Lemma 3.2 and assumption (H4).
**Lemma 7.1** Assume (H2), (H3) and (H5) ii). Let \( \tau \) be as defined by (H4). Then there exists \( C > 0 \) such that for any \( h > 0 \),

\[
\| \hat{R}(i(b+h)) - \hat{R}(ib) \|_{\mathcal{B}\to\mathcal{B}_0} \leq C \max\{1,|b|\} h^\tau.
\]

**Proof** By Lemma 3.2, for all \( b \in \mathbb{R} \) and \( a > 0 \) such that \( e^{-iba} \neq 1 \), \( \hat{R}(ib) = g_a(ib)\hat{L}_a(ib) \) with \( \hat{L}_a(ib) = \int_0^\infty R_{t,a}e^{-ait} dt \) and \( g_a(ib) = \frac{a}{e^{-rab}-1} \).

A standard calculation shows that \( |g_1(i(b+h)) - g_1(ib)| \leq h \), for all \( |b| < 1 \) and \( |g_1(b)| \ll 1 \). Next, given \( b \in \mathbb{R} \), fix \( a > 0 \) such that \( |e^{-iba} - 1| > 1 \). Then, there exists some constant \( C > 0 \) such that \( |g_a(i(b+h)) - g_a(ib)| \leq Ch \) and \( |g_a(b)| \leq C|b| \).

It remains to show that \( \| \hat{L}_a(i(b+h)) - \hat{L}_a(ib) \|_{\mathcal{B}\to\mathcal{B}_0} \ll h^\tau \), for all \( b \in \mathbb{R} \) and appropriate \( a > 0 \). This is an immediate consequence of (H5) ii):

\[
\| \hat{L}_a(ib_1) - \hat{L}_a(ib_2) \|_{\mathcal{B}\to\mathcal{B}_0} = \| \int_0^\infty \int_0^\infty R_{t,a}(e^{-i(b+h)t} - e^{-ibt}) \|_{\mathcal{B}\to\mathcal{B}_0} \ll h^\tau \int_0^\infty t^\tau \| R_{t,a} \|_{\mathcal{B}\to\mathcal{B}_0} dt \ll h^\tau.
\]

To estimate \( \| P(ib) - P \|_{\mathcal{B}\to\mathcal{B}_0} \) for \( b \) close to zero (and as such \( \| Q(ib) - Q \|_{\mathcal{B}\to\mathcal{B}_0}, \| v(ib) - v(0) \|_{\mathcal{B}\to\mathcal{B}_0} \) we recall the following abstract result of [12], which due to Lemma 7.1 applies to our setting with no modification of the involved proof (except adjusting the corresponding labeling of the quantities and parameters used there).

**Lemma 7.2** [12, Corollary 1] Assume (H2), (H3) and (H5). Then, there exists \( \delta_0 > 0 \) and some constant \( C > 0 \) such that for all \( b \in \text{spec}(\hat{R}(ib)) \cap \mathcal{B}_0, (0) \), for all \( h \leq |b| \) and for any \( \epsilon > 0 \),

\[
\| P(ib) - P \|_{\mathcal{B}\to\mathcal{B}_0} \leq C|b|^{\tau - \epsilon}, \quad \| P(i(b+h)) \|_{\mathcal{B}\to\mathcal{B}_0} \leq C h^{\tau - \epsilon}.
\]

Moreover, the same estimates hold for the families \( Q(z) \) and \( v(z) \).

The result below is a consequence of (H0) ii), Lemma 7.1 and Lemma 7.2.

**Lemma 7.3** Assume (H0) ii), (H2), (H3), (H5) i) and (H5) ii) with \( \max\{2\beta - 1, 1 - \beta\} < \tau < \beta - \gamma \). Fix \( \delta_0 > 0 \) such that Lemma 7.2 holds.

Let \( c_\beta = i \int_0^\infty e^{-is} s^{-\beta} ds \). Then, for all \( |b| < \delta_0 \) and for any \( \epsilon > 0 \),

\[
(1-\lambda(ib))^{-1} = c_\beta^{-1} \ell(1/|b|)^{-1} b^{-\beta} + O(|b|^{2\tau - \epsilon}), \quad (I-\hat{R}(ib))^{-1} = c_\beta^{-1} \ell(1/|b|)^{-1} b^{-\beta} (P+E(ib)),
\]

where \( E(ib) \) is a family of operators satisfying \( \| E(ib) \|_{\mathcal{B}\to\mathcal{B}_0} = O(|b|^{\tau - \epsilon}) \).

**Proof** The argument is standard. For similar arguments we refer to, for instance, [2, 19, 18, 20]). Due to our assumption (H5) ii), we need to use Lemma 7.2 (for a similar use of Lemma 7.2 in a different set up we refer [15]).
Recall \( \hat{R}(ib)v = R(e^{-ib\varphi}v) \). Following the formalism in [11] (a simplification of [2]), we write

\[
\lambda(ib) = \int_{\mathcal{Y}} \lambda(ib)v(ib) \, d\mu_\Phi = \int_{\mathcal{Y}} R(e^{-ib\varphi}v(ib)) \, d\mu_\Phi = \int_{\mathcal{Y}} e^{-ib\varphi} \, d\mu_\Phi + V(ib),
\]

where \( V(ib) = \int_{\mathcal{Y}} (\hat{R}(ib) - R(0))(v(ib) - v(0)) \, d\mu_\Phi \).

By the argument in \([9]\), \( 1 - \int_{\mathcal{Y}} e^{-ib\varphi} \, d\mu \sim c_\beta \ell(1/b)\beta \) as \( b \to 0^+ \).

By Lemma 7.1 and Corollary 7.2, the families \( \hat{R}(ib) : \mathcal{B} \to \mathcal{B}_0 \) and \( v(ib) : \mathcal{B} \to \mathcal{B}_0 \) are \( C^r \) and \( C^{r-\epsilon_0} \), respectively, in \( \hat{B}_\delta(0) \). Since \( \mathcal{B} \subset \mathcal{B}_0 \subset L^\infty(\mu_\Phi) \), \( |V(ib)| = O(b^{2r-\epsilon}) \). Thus,

\[
1 - \lambda(ib) = c_\beta \ell(1/b)\beta + O(|b|^{2r-\epsilon}).
\]

Next, for all \( |b| < \delta_0 \),

\[
(I - R(ib))^{-1} = (1 - \lambda(ib))^{-1}P - (1 - \lambda(ib))^{-1}(P(ib) - P) + (I - R(ib))^{-1}Q(ib).
\]

By (H3), \( \| (I - R(ib))^{-1}Q(ib) \|_{\mathcal{B}} = O(1) \). By Corollary 7.2, \( \| P(ib) - P(0) \|_{\mathcal{B}\to\mathcal{B}_0} \ll b^{r-\epsilon_0} \).

Set

\[
E(ib) = P(ib) - P(0) + (1 - \lambda(ib))(I - R(ib))^{-1}Q(ib)
\]

and note that \( \| E(ib) \|_{\mathcal{B}\to\mathcal{B}_0} = O(|b|^{r-\epsilon}) \).

Thus,

\[
(I - \hat{R}(ib))^{-1} = (1 - \lambda(ib))^{-1}(P + E(ib)) = c_\beta^{-1} \ell(1/b)^{-1}b^{-\beta}(P + E(ib)),
\]

as required.

An immediate consequence of Lemma 7.3 is:

**Corollary 7.4** Assume the setting and notation of Lemma 7.3. Then

i) \( \| (I - \hat{R}(ib))^{-1} \|_{\mathcal{B}\to\mathcal{B}_0} \ll \ell(1/b)^{-1}|b|^{-\beta} \) for all \( 0 < |b| < \delta_0 \).

ii) There exists \( C > 0 \) such that \( \| (I - \hat{R}(ib))^{-1} \|_{\mathcal{B}\to\mathcal{B}_0} \leq C \), for all \( \delta_0 < |b| < 1 \).

Using Lemma 7.1, (H5) i) and the type of arguments in [12] we obtain

**Corollary 7.5** Assume the setting and notation of Lemma 7.3. Then, there exists \( C > 0 \) such that \( \| (I - \hat{R}(ib))^{-1} - (I - \hat{R}(i(b + h)))^{-1} \|_{\mathcal{B}\to\mathcal{B}_0} \leq C h^\tau \log(1/h) \), for all \( \delta_0 < |b| < 1 \) and \( h > 0 \).

**Proof** By the resolvent equality,

\[
(I - \hat{R}(ib))^{-1} - (I - \hat{R}(i(b + h)))^{-1} = (I - \hat{R}(ib))^{-1}(\hat{R}(ib) - \hat{R}(i(b + h))(I - \hat{R}(i(b + h)))^{-1} := (I - \hat{R}(ib))^{-1}A(b,h).
\]

Next, by (H2), \( \| (I - \hat{R}(ib))^{-1} \|_{\mathcal{B}} \leq C \), for some \( C > 0 \) for all \( \delta_0 < |b| < 1 \). Together with (H5) i) and the type of arguments in [12], this implies that for any \( v \in \mathcal{B} \),

\[
\| (I - \hat{R}(ib))^{-1}A(b,h)v \|_{\mathcal{B}_0} \leq |n| \| \hat{R}(ib)^{j}A(b,h)v \|_{\mathcal{B}_0} + \| \hat{R}(ib)v(I - \hat{R}(ib))^{-1}A(b,h)\|_{\mathcal{B}_0} \\
\leq n \| A(b,h)v \|_{\mathcal{B}_0} + C_1 \theta^n \| (I - \hat{R}(ib))^{-1}A(b,h)v \|_{\mathcal{B}} + C_2 \| (I - \hat{R}(ib))^{-1}A(b,h)v \|_{\mathcal{B}_0}.
\]
By Lemma 7.1, for all $\delta_0 < |b| < 1$, $\|A(b, h)v\|_{B_0} \leq h^\tau \|v\|_B$. Recalling $C_2 < 1$ and using the last displayed inequality,
\[
(1 - C_2)\|(I - \hat{R}(ib))^{-1}A(b, h)v\|_{B_0} \leq C_1\theta^n\|v\|_B + nh^\tau \|v\|_B.
\]
The conclusion follows by taking $n = \lceil \tau \log(h) \log(\theta)^{-1} \rceil$ (so, $\theta^n \ll h^\tau$ and $n \ll \log(1/h)$).

The next result provides the required continuity estimates for $\lambda(ib)$ and $(I - \hat{R}(ib))^{-1}$ under (H5) ii).

**Lemma 7.6** [15, Proposition 3.7] Assume the setting and notation of Lemma 7.3. Then the following hold for any $0 < h \leq |b| < \delta^*$ and any $\epsilon > 0$.

i) $|\lambda(i(b + h)) - \lambda(ib)| \ll h^\beta \ell(1/h) + h^\tau - \epsilon |b|^\beta$.

ii) Also,
\[
\|(I - \hat{R}(ib))^{-1} - (I - \hat{R}(i(b+h)))^{-1}\|_{B \rightarrow B_0} \ll \ell(1/|b|)^{-2}h^{\tau - \epsilon}|b|^{-\beta} + \ell(1/|b|)^{-2}h^\beta |b|^{-2\beta}.
\]

**Proof** The proof goes word by word as the proof of [22, Proposition 4.2], [15, Proposition 3.8] (by setting $u = 0$, $\theta = b$ and relabeling the parameters used there), with the change that in the present set-up $B \subset B_0 \subset L^\infty(\mu_\Phi)$. Because the proof of item i) is short we provide below for completeness. As in [15, Proposition 3.8], item ii) follows from item i) together with Lemma 7.2.

Put $\Delta_\lambda = |\lambda(i(b + h)) - \lambda(ib)|$. Using eq. (7.1) we write
\[
\Delta_\lambda = \int_Y (\lambda(i(b + h)) - \lambda(ib)) d \mu_\Phi + \int_Y (\hat{R}(i(b + h)) - \hat{R}(ib))(\lambda(i(b + h)) - \lambda(ib)) d \mu_\Phi
\]
\[
= \int_Y (\lambda(i(b + h)) - \lambda(ib)) d \mu_\Phi + V_1(ib) + V_2(ib).
\]

Using the fact that $B \subset B_0 \subset L^\infty(\mu_\Phi)$, we get
\[
|V_1(ib)| \ll \|v(i(b + h)) - v(0)\|_{B \rightarrow B_0} \int_Y (\hat{R}(i(b + h)) - \hat{R}(ib)) d \mu_\Phi
\]
\[
= \|v(i(b + h)) - v(0)\|_{B \rightarrow B_0} \int_Y (e^{-i(b+h)\varphi} - e^{-ib\varphi}) d \mu_\Phi
\]
and
\[
|V_2(ib)| \ll \|v(i(b + h)) - v(ib)\|_{B \rightarrow B_0} \int_Y (\hat{R}(i(b + h)) - \hat{R}(0)) d \mu_\Phi
\]
\[
= \|v(i(b + h)) - v(ib)\|_{B \rightarrow B_0} \int_Y (e^{-i(b+h)\varphi} - 1) d \mu_\Phi.
\]

By Corollary 7.2, for any $\epsilon > 0$, $\|(v(i(b + h)) - v(ib))\|_{B \rightarrow B_0} \ll h^{\tau - \epsilon}$. Similarly, $\|v(i(b + h)) - v(0)\|_{B \rightarrow B_0} \ll |b + h|^{\tau - \epsilon} \ll |b|^{\tau - \epsilon}$.

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Next, let $G(x) = \mu_\varphi(\varphi \leq x)$. It is easy to see that $\int_0^\infty e^{-ibx}(e^{-ihx} - 1)dG(x)$. The estimate $|\int_0^\infty e^{-ibx}(e^{-ihx} - 1)dG(x)| \ll h^\beta \ell(1/h)$ follows by the argument used in the proof of [9, Lemma 3.3.2]. Similarly, $|\int_Y(e^{-i(b+h)\varphi} - e^{-ib\varphi})d\mu_\varphi| \ll |b+h|^\beta \ll b^\beta$.

Putting the above together and using that $0 < h \leq |b|$, $|V_1(ib)| \ll |b + h|^\tau h^\beta \ll |b|^\tau - \epsilon h^\beta \ll h^\tau \epsilon |b|^\beta$ and $|V_2(ib)| \ll h^\tau \epsilon |b|^\beta$. Since $|\int_Y(e^{-i(b+h)\varphi} - e^{-ib\varphi})d\mu_\varphi| \ll h^\beta \ell(1/h)$, the conclusion follows.

\begin{lemma}
Assume (H0) ii) and (H5) ii). Assume (H6’) and choose $\alpha$ such that (H6) holds. Assume $|b| \geq 1$. Then the following holds as $h \to 0$,

$$
\|(I - R(ib))^{-1} - (I - R(i(b + h)))^{-1}\|_{\mathcal{B} \to \mathcal{B}_0} \ll h^\tau \log(|b|)|b|^{2\alpha + 1}.
$$

\end{lemma}

\textbf{Proof} Write $(I - R(ib))^{-1} - (I - R(i(b + h)))^{-1} = (I - R(ib))^{-1} A(b, h)$, with $A(b, h) = (\tilde{R}(ib) - \tilde{R}(i(b + h)))(I - R(i(b + h)))^{-1}$. With the notation of Remark 4.6, let

$$
\frac{k}{h \log \theta} \approx \frac{1 + \log C_0 + \log |b|}{h \log \theta}
$$

and note that

$$
\|(I - \tilde{R}(ib))^{-1} A(b, h)v\|_{\mathcal{B}_0} \leq \sum_{j=1}^k \|\tilde{R}(ib)^j A(b, h)v\|_{\mathcal{B}_0} + \|\tilde{R}(ib)^k (I - \tilde{R}(ib))^{-1} A(b, h)v\|_{\mathcal{B}_0}
$$

Using (H6’), (H6) and the fact that $\|A(b, h)v\|_{\mathcal{B}_0} \leq |b|^\alpha + 1 h^\tau \|v\|_{\mathcal{B}}$ (by Lemma 7.1),

$$
|b|^{-\alpha} \|(I - \tilde{R}(ib))^{-1} A(b, h)v\|_{\mathcal{B}_0} \leq C \log(|b|) \|A(b, h)v\|_{\mathcal{B}_0} \leq C \log(|b|)|b|^\alpha + 1 h^\tau \|v\|_{\mathcal{B}},
$$

for $C > 0$. The conclusion follows.

\section{Estimates for $\hat{T} = \hat{U}(I - \hat{R})^{-1}$}

In this subsection, we combine our estimates for $(I - \hat{R})^{-1}$ obtained in Subsection 7.1 with the estimates for $\hat{U}$ obtained in Subsection 8.2. Recall that $c_\beta = i \int_{0}^{\infty} e^{i\sigma - \beta \sigma} d\sigma$.

Given that $\hat{T} = \hat{U}(I - \hat{R})^{-1}$ is a product, in the result below we speak of $\|\hat{U}(I - \hat{R})^{-1}\|_{(\mathcal{B} \to \mathcal{B}_0) \to L^1(\mu_\varphi)} \leq \|\hat{U}\|_{(\mathcal{B}_0 \to L^1(\mu_\varphi))} \|(I - \hat{R})^{-1}\|_{\mathcal{B} \to \mathcal{B}_0}$.

\begin{lemma}
Assume (H0) ii), (H1) ii), (H2), (H3) and (H5). Assume (H6) and choose $\alpha$ accordingly. Let $v = \frac{v}{h}$ with $v^* \in \mathcal{B}$ and $w = \frac{w^*}{h}$ with $w^* \in L^\infty(\mu_\varphi)$. Let $\tau$ be given as in (H4). Then,

\begin{enumerate}
\item (a) $\int_Y \hat{T}(ib)vw \, d\mu = c_\beta^{-1} \ell(1/|b|)^{1-\beta} b^{-\beta} \int_Y v \, d\mu \int_Y w \, d\mu + o(1))$.
\item (b) $\|\hat{T}(ib)\|_{(\mathcal{B} \to \mathcal{B}_0) \to L^1(\mu_\varphi)} \ll \left\{
\begin{aligned}
\ell(1/|b|)^{1-\beta} \, |b|^{-\beta}, & \quad 0 < |b| < 1, \\
|b|^\alpha, & \quad |b| \geq 1.
\end{aligned}
\right.$
\item (c) For any $0 < h < |b| < 1$ and for any $\epsilon > 0$,

$$
\|\hat{T}(i(b + h)) - \hat{T}(ib)\|_{(\mathcal{B} \to \mathcal{B}_0) \to L^1(\mu_\varphi)} \ll \ell(1/|b|)^{-2} h^{\tau - \epsilon} |b|^{-\beta} + \ell(1/|b|)^{-2} \ell(1/h) h^\beta |b|^{-2\beta}.
$$
\end{enumerate}

\end{lemma}
(d) For $|b| > 1$ and any $0 < h < 1$,
\[ \| \hat{T}(i(b + h)) - \hat{T}(i(b)) \|_{(B \to B_0) \to L^1(\mu_\Phi)} \ll |b|^{2 \alpha + 1} h^{\gamma} (\log |b|) |b|^{2 \alpha + 1}. \]

**Proof** (a) By Lemma 8.7(b), $\hat{U}$ is continuous. Thus,
\[ \lim_{b \to 0^+} \ell(1/b) b^3 \int_\gamma \hat{T}(ib)v w d\hat{\mu} = \lim_{b \to 0^+} \int_\gamma \hat{U}(ib) \ell(1/b) b^3 (I - \hat{R}(ib))^{-1} v w d\hat{\mu} \]
\[ = \int_\gamma \hat{U}(0) \lim_{b \to 0^+} \ell(1/b) b^3 (I - \hat{R}(ib))^{-1} v w d\hat{\mu}. \]

Combined with Lemma 8.7(a), the above equality yields
\[ \lim_{b \to 0^+} \ell(1/b) b^3 \int_\gamma \hat{T}(ib)v w d\hat{\mu} = \int_\gamma \hat{h} \int_0^u \lim_{b \to 0^+} \ell(1/b) b^3 (I - \hat{R}(ib))^{-1} v(y, \sigma) d\sigma w^*(y, u) d\mu_\Phi \]
\[ + \int_\gamma \hat{h} \int_0^u \| R \| \ell(1/b) b^3 (I - \hat{R}(ib))^{-1} v(y, \sigma) d\sigma w^*(y, u) d\mu_\Phi \]
\[ = \int_\gamma \hat{h} \int_0^u \ell(1/b) b^3 (I - \hat{R}(ib))^{-1} v(y, \sigma) d\sigma w^*(y, u) d\mu_\Phi \]
\[ + \int_\gamma \hat{h} \int_0^u \| \ell(1/b) b^3 (I - \hat{R}(ib))^{-1} v(y, \sigma) d\sigma w^*(y, u) \circ \Phi d\mu_\Phi. \]

Recall $w = \frac{w^*}{h}$. Hence, by Lemma 7.3,
\[ \lim_{b \to 0^+} \ell(1/b) b^3 \int_\gamma \hat{T}(ib)v w d\hat{\mu} = c_\beta^{-1} \int_\gamma \hat{h} \int_0^u v(y, \sigma) d\sigma d\mu_\Phi \int_\gamma w d\hat{\mu} \]
\[ + c_\beta^{-1} \int_\gamma \hat{h} \int_0^1 v(y, \sigma) d\sigma d\mu_\Phi \int_\gamma w^* \circ \Phi d\mu_\Phi. \]

Since $\int_\gamma w^* \circ \Phi d\mu_\Phi = \int_\gamma w^* d\mu_\Phi = \int_\gamma w d\hat{\mu},$
\[ \lim_{b \to 0^+} \ell(1/b) b^3 \int_\gamma \hat{T}(ib)v w d\hat{\mu} = c_\beta^{-1} \int_\gamma \hat{h} \left( \int_0^u v(y, \sigma) \right) + \int_0^1 v(y, \sigma) d\sigma \right) d\mu_\Phi \int_\gamma w d\hat{\mu} \]
\[ = c_\beta^{-1} \int_\gamma \hat{h} v d\mu_\Phi \int_\gamma \hat{w} d\mu = c_\beta^{-1} \int_\gamma v \hat{d} d\mu \int_\gamma \hat{w} d\hat{\mu}, \]
which ends the proof of (a).

(b) Recall $\hat{T}(ib) = \hat{U}(ib)(I - \hat{R}(ib))^{-1}$ and note that
\[ \| \hat{T}(ib) \|_{(B \to B_0) \to L^1(\mu_\Phi)} \ll \| \hat{U}(ib) \|_{B_0 \to L^1(\mu_\Phi)} \| (I - \hat{R}(ib))^{-1} \|_{B \to B_0}. \]

When $0 < |b| < 1$, the conclusion follows by Corollary 7.4 and Remark 8.6 (which ensures $\| \hat{U}(ib) \|_{B_0 \to L^1(\mu_\Phi)} \leq C$, for some positive constant $C$). When $|b| > 1$, the conclusion follows from (H6) and Lemma 8.4.

(c) Note that
\[ \| \hat{T}(ib) - \hat{T}(i(b - h)) \|_{(B \to B_0) \to L^1(\mu_\Phi)} \ll \| \hat{U}(ib) - \hat{U}(i(b - h)) \|_{B_0 \to L^1(\mu_\Phi)} \| (I - \hat{R}(ib))^{-1} \|_{B \to B_0} \]
\[ + \| \hat{U}(ib) \|_{B \to L^1(\mu_\Phi)} \| (I - \hat{R}(ib))^{-1} - (I - \hat{R}(i(b - h)))^{-1} \|_{B \to B_0} \]
\[ = D(b, h) + E(b, h). \tag{7.2} \]
Recall $0 < h < |b| < 1$ and $\tau$ is such that (H5) ii) holds. By Lemma 8.7(b), $\|\hat{U}(ib) - \hat{U}(i(b-h))\|_{B \to L^1(\mu_b)} \ll h^\tau$. By Corollary 7.4, $\|(I - R'(ib))^{-1}\|_{B \to B_0} \ll \ell(1/|b|)^{-1}|b|^{-\beta}$. Hence,

$$D(b, h) \ll \ell(1/|b|)^{-1}|b|^{-\beta} h^\tau.$$  

To deal with $E(b, h)$ we consider two cases: i) $0 < h < |b| < \delta_0$ and ii) $\delta_0 < |b| < 1$ and $h > 0$. In both cases, $\delta_0$ is fixed such that the conclusion of Lemma 7.3 holds.

In case i), Lemma 7.6 ii) gives that for any $\epsilon > 0$,

$$\|(I - R'(i(b-h)))^{-1} - (I - R'(ib))^{-1}\|_{B \to B_0} \ll \ell(1/|b|)^{-2} h^{\tau-\epsilon}|b|^{-\beta} + \ell(1/|b|)^{-2} \ell(1/h) h^{\beta}|b|^{-2\beta}.$$  

By Remark 8.6, $\|\hat{U}(ib)\|_{B_0 \to L^1(\mu_b)} \leq C$, for some positive constant $C > 0$. Hence,

$$E(b, h) \ll \ell(1/|b|)^{-2} h^{\tau-\epsilon}|b|^{-\beta} + \ell(1/|b|)^{-2} \ell(1/h) h^{\beta}|b|^{-2\beta}.$$  

In case ii), by Corollary 7.5 we know that $\|(I - R'(i(b-h)))^{-1} - (I - R'(ib))^{-1}\|_{B \to B_0} \leq C h^\tau$, for some constant $C > 0$. We already know that $\|\hat{U}(ib)\|_{B_0 \to L^1(\mu_b)} \leq C$, for some positive constant $C > 0$. Hence, $E(b, h) \ll C h^\tau$.

The conclusion follows by putting together the estimates for $D(b, h)$ and $E(b, h)$.

(d) We continue from (7.2), recalling that we consider the case $|b| > 1$ and $0 < h < 1$. By Lemma 8.7(b), $\|\hat{U}(i(b+h)) - \hat{U}(ib)\|_{B_0 \to L^1(\mu_b)} \ll h^\tau$. By (H6), $\|(I - R'(ib))^{-1}\|_{B} \ll |b|^\alpha$. Hence,

$$D(b, h) \ll |b|^\alpha h^\tau.$$  

By Lemma 8.4, $\|\hat{U}(ib)\|_{B_0 \to L^1(\mu_b)} \leq C$, for some positive constant $C$. By Lemma 7.7,

$$\|(I - R'(i(b-h)))^{-1} - (I - R'(ib))^{-1}\|_{B \to B_0} \ll h^{\tau}(\log |b|)|b|^{2\alpha+1}. $$  

Hence,

$$E(b, h) \ll h^{\tau}(\log |b|)|b|^{2\alpha+1}.$$  

The conclusion follows by putting together the estimates for $D(b, h)$ and $E(b, h)$.

7.3 Proof of Theorem 5.2

Recall that $\rho_t(v, w) = \int_Y vw \circ f_\tau d\mu$ and for $s \in \mathbb{H}$, set

$$\hat{\rho}(s)(v, w) = \int_0^\infty \rho_t(v, w)e^{-st}dt = \int_Y \hat{T}(s)vw d\bar{\mu}. \quad (7.3)$$

The result below has been established in the set-up of [20]. It applies to the present set-up with no modification.

Proposition 7.9 [20, Proposition 6.2] The analytic function $\hat{\rho}(s)(v, w), \Re s > 0$, extends to a continuous function on $\{\Re s \geq 0\} \setminus \{0\}$. Suppressing the dependence on $v, w$, the correlation function is given by

$$\rho_t = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ibt}\hat{\rho}(ib)\, db = \frac{1}{\pi} \Re \left( \int_0^\infty e^{ibt}\hat{\rho}(ib)\, db \right).$$
The next three results provides an estimate for \( \int_c^d e^{ibt} \hat{\rho}(ib) \, db \) for different regimes of \( 0 < c < d < \infty \), as specified below.

**Lemma 7.10** For any \( a > 0 \),
\[
\lim_{t \to \infty} t^{1-\beta} \int_0^{a/t} e^{ibt} \hat{\rho}(i\beta, w) \, db = c_{\beta}^{-1} \int_0^a e^{i\sigma \beta} \, d\sigma \int_Y \rho \, d\mu \int_Y w \, d\mu.
\]

**Proof** The follows by the argument of [20, Proposition 6.2] with Lemma 7.8 (a) replacing [20, Corollary 5.10]. For the analogous argument in the discrete time scenario we refer to [19, Lemma 5.2].

**Lemma 7.11** Let \( \beta' \in (\frac{1}{2}, \beta) \). Let \( \tau \) be given as in (H4) and let \( \tau' \in (1 - \beta, \tau) \). Then for all \( a \in (\pi, t) \),
\[
\int_{a/t}^1 e^{ibt} \hat{\rho}_{v,w}(ib) \, db \ll \ell(t)^{1-\beta} \ell^{-\tau(1-\beta+\tau')},
\]
\[
\int_{a/t}^1 e^{ibt} \hat{\rho}_{v,w}(ib) \, db \ll \ell(t)^{1-\beta} \ell^{-\tau(1-\beta+\tau')} a^{1-\beta'}.
\]

**Proof** The argument below is similar to the argument used in the proofs of [20, Proposition 6.4], [19, Lemma 5.1], with obvious adaptation due to the estimates above.

Let \( 0 < b < 1 \). By Corollary 7.4,
\[
|\hat{\rho}(ib)| \ll \ell(1/b)^{-1} b^{-\beta} \|v\|_B \|w^*\|_{L^{\infty}(\mu_B)}.
\]

By Lemma 7.8(b,c) with \( h = \pi/t \), we have that for any \( \epsilon > 0 \)
\[
|\hat{\rho}(ib) - \hat{\rho}(i(b - \pi/t))| \ll \ell(1/b)^{-2} \ell(1/h) b^{-2\beta} \ell^{-\tau(1-\beta+\tau')} a^{1-\beta'}.
\]

From here on we suppress the factor \( \|v^*\|_B \|w^*\|_{L^{\infty}(\mu_B)} \). Write
\[
I = \int_{a/t}^1 e^{ibt} \hat{\rho}(ib) \, db = - \int_{(a+\pi)/t}^{(a+\pi)/t} e^{ibt} \hat{\rho}(i(b - \pi/t)) \, db.
\]

Then \( 2I = I_1 + I_2 + I_3 \), where
\[
I_1 = - \int_{1}^{(a+\pi)/t} e^{ibt} \hat{\rho}(i(b - \pi/t)) \, db, \quad I_2 = \int_{a/t}^{(a+\pi)/t} e^{ibt} \hat{\rho}(ib) \, db,
\]
\[
I_3 = \int_{(a+\pi)/t}^{1} e^{ibt} (\hat{\rho}(ib) - \hat{\rho}(i(b - \pi/t))) \, db.
\]

Clearly \( I_1 = O(t^{-1}) \), and by Potter’s bounds (see, for instance, [6]),
\[
|I_2| \ll \int_{a/t}^{(a+\pi)/t} \ell(1/b)^{-1} b^{-\beta} \, db = \ell(t)^{-1} t^{-(1-\beta)} \int_a^{a+\pi} \ell(t/\ell(\sigma))^{-1} \, d\sigma
\]
\[
\ll \ell(t)^{-1} t^{-(1-\beta)} \int_a^{a+\pi} \sigma^{-\beta'} \, d\sigma \ll \ell(t)^{-1} t^{-(1-\beta)} a^{-\beta'}.
\]
Next,

\[ |I_3| \ll \ell(t)^{-1} \int_{a/t}^{t} \ell(1/b)^{-2} b^{-2\beta} \, db + t^{-(\tau - \epsilon_0)} \int_{a/t}^{1} \ell(1/b)^{-2} b^{-\beta} \, db = I_{3,1} + I_{3,2}. \]

By Potter’s bounds,

\[ I_{3,1} = \ell(t)^{-1} t^{\beta - 1} \int_{a/t}^{t} \left( \frac{\ell(t)}{\ell(t/\sigma)} \right)^{2} \sigma^{-2\beta} \, d\sigma \ll t^{(1 - \beta, \tau)} \int_{a/t}^{\infty} \sigma^{-2\beta} \, d\sigma \ll t^{(1 - \beta)} \int_{a/t}^{1} \sigma^{-(2\beta - 1)}. \]

Also, \( I_{3,2} = \ell(t)^{-1} t^{\beta - 1} \int_{a/t}^{1} \left( \frac{\ell(t)}{\ell(1/b)} \right)^{2} b^{-\beta} \, db. \) By Potter’s bounds, for any \( \epsilon > 0, \) \( \left( \frac{\ell(t)}{\ell(1/b)} \right)^{2} \ll t^{\epsilon} b^{-\epsilon}. \) Hence,

\[ I_{3,2} = \ell(t)^{-1} t^{\beta - 1} \int_{a/t}^{1} b^{-\beta - \epsilon} \, db. \]

Since \( \epsilon \) can be taken arbitrarily small, \( I_{3,2} \ll t^{-(\tau - \epsilon_0)} a^{1 - \beta - \epsilon}, \) for any \( \epsilon' > \epsilon. \) The conclusion follows since \( \tau' \in (1 - \beta, \tau) \) and \( \beta' \in (1/2, \beta). \)

For the next result we recall that \( C^{m}(\hat{Y}, \hat{\mu}) \) is the class of observables defined in (4.2).

**Lemma 7.12** Assume \((H4)\) \(\langle ii \rangle.\) Assume \((H6)\) and choose \(\alpha\) accordingly. Choose \(m > 2\alpha + 2.\) Let \(w \in C^{m}(\hat{Y}, \hat{\mu}).\) Assume \((H4)\) with \(\tau > 1 - \beta\) as given there. Then,

\[ |\int_{-1}^{\infty} e^{ibt} \hat{\rho}(ib)(v, w) \, db| \ll t^{-\tau} \|v\|_{B} \|w^{*}\|_{L^{\infty}(\mu_{\Phi})}. \]

**Proof** The argument below adapts the proof of [20, Proposition 6.5] to the present context.

By Proposition 4.7, \( \hat{\rho}(s)(v, w) = \hat{P}_{m}(s) + \hat{H}_{m}(s), \) where \( \hat{P}_{m}(s) \) is a linear combination of \( s^{-j}, j = 1, \ldots, m, \) and \( \hat{H}_{m}(s) = s^{-m} \hat{\rho}_{v, \hat{\rho}^{m}_{w}}(s). \)

By the argument used in the proof of [20, Proposition 6.5],

\[ |\int_{-1}^{\infty} e^{ibt} P_{m}(ib) \, db| \ll t^{-1}. \]

By Lemma 7.8(d) with \( h = \pi/t, \) there exists some constant \( C > 0 \) such that

\[ |\hat{H}_{m}(ib) - \hat{H}_{m}(i(b - \pi/t))| \leq C b^{-(m - 2\alpha - 1)} (\log |b|) t^{\tau} \|v^{*}\|_{B} \|\partial_{t}^{m} w\|_{\infty}. \]

Suppressing the term \( \|v^{*}\|_{B} \|\partial_{t}^{m} w\|_{\infty}, \)

\[ |2 \int_{1}^{\infty} e^{ibt} \hat{H}_{m}(ib) \, db| \leq \int_{1}^{\infty} |\hat{H}_{m}(ib) - \hat{H}_{m}(i(b - \pi/t))| \, db + \int_{1}^{1 + \pi/t} |\hat{H}_{m}(i(b - \pi/t))| \, db \ll t^{-\tau} \int_{1}^{\infty} (\log |b|) b^{-(m - 2\alpha - 1)} \, db + O(t^{-1}) = O(t^{-\tau}), \]

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where in the last inequality we have used that $m > 2\alpha + 2$. 

We can now complete the

**Proof of Theorem 5.2** By Lemma 7.10, Lemma 7.11 and Lemma 7.12, we obtain that for all $a \in (\pi, t)$,

$$
\lim_{t \to \infty} t^{1-\beta} f(t) \int_0^\infty e^{ibt} \rho(ib)(v, w) \, db = c_\beta^{-1} \int_0^a e^{i\sigma \beta} d\sigma \int_\gamma v \, d\tilde{\mu} \int_\gamma w \, d\tilde{\mu} + O(a^{-2\beta'} - 1).
$$

Because $\beta' > 1/2$ and $\tau' > 1 - \beta'$,

$$
\lim_{t \to \infty} t^{1-\beta} \int_0^\infty e^{ibt} \rho(ib)(v, w) \, db = c_\beta^{-1} \int_0^\infty e^{i\sigma \beta} d\sigma \int_\gamma v \, d\tilde{\mu} \int_\gamma w \, d\tilde{\mu}.
$$

Since $\Re(c_\beta^{-1} \int_0^\infty e^{i\sigma \beta} d\sigma) = \sin \beta \pi$, the result follows from Proposition 7.9.

8 **Several estimates related to the Laplace transform $\hat{U}$**

In this section we study the asymptotic behavior of the Laplace transform $\hat{U}(s)$ defined before Proposition 3.1. To some extent, our calculations below follow the strategy of obtaining similar estimates for $\hat{U}$ as defined in [20]. Under the present assumption (H1) (so, $\tilde{h}$ unbounded) the calculations require more work. Thus, we cannot simply cite the somewhat similar arguments in [20], but need to carry out the new calculations. However, we mention under the assumption that $\tilde{h}$ is bounded (limiting to the present (H2), that is $B \in L^\infty(\mu\Phi)$), all the arguments in [20] used to study the asymptotic behavior of $\hat{U}(s)$, simply go through.

We start by obtaining precise formula for the inverse Laplace transform $U_t$.

**Proposition 8.1** Assume that $\tilde{h}$ is bounded away from zero and $\varphi \geq \tilde{h}$. Let $v \in L^1(\mu\tilde{h})$ and $w = w^*/h \in L^\infty(\mu\tilde{h})$ with $w^* \in L^\infty(\mu\Phi)$. If $t \leq \tilde{h}(y) u$ then

$$
\int_\gamma (U_t v) w \, d\tilde{\mu} = \int_\gamma 1_{[t/\tilde{h}, 1]}(u) v(y, u - t/\tilde{h}); w^* \, d\mu\Phi.
$$

Also, if $t > \tilde{h}(y) u$, then

$$
\int_\gamma (U_t v) w \, d\tilde{\mu} = \int_\gamma R(v_t) w^* \, d\mu\Phi,
$$

where $v_t(y, u) = 1_{\{t < \varphi < t + \tilde{h}(y)(1-u)\}} v(y, u + \varphi - t/\tilde{h}(y))$.

**Remark 8.2** By (3.2), $\int_\gamma (U_t v) w \, d\tilde{\mu} = \int_\gamma (U_t v^*) w \, d\mu\Phi$, for $v = \varphi / \tilde{h}$, $v^* \in L^1(\mu\Phi)$. Together with the Proposition 8.1, this provides a $\mu\Phi$-a.e. formula for $U_t$. In what follows we do not use such a pointwise formula (rather counter-intuitive due to presence of $v^*$ on the RHS and $v$ on the LHS), but stick to the integral formula provided by Proposition 8.1.
Proof  First we deal with \( t < \bar{h}(y)u \). The assumption \( \varphi > \bar{h} \) implies \( 1_{\{\varphi > t\}} = 1_{\bar{Y}} \tilde{\mu} \)-a.e. on \( \bar{Y} \), and therefore, for \( w \in L^\infty(\tilde{\mu}) \),

\[
\int_{\bar{Y}} (U_t v) w \, d\tilde{\mu} = \int_{\bar{Y}} 1_{\{\varphi > t\}} v w \circ f_t \, d\tilde{\mu} = \int_{\bar{Y}} \int_0^{1-t/\bar{h}(y)} v(y, u) w(y, u + t/\bar{h}(y)) \bar{h}(y) du \, d\mu \\
= \int_{\bar{Y}} \int_{t/\bar{h}(y)}^1 v(y, u - t/\bar{h}(y)) w(y, u) \bar{h}(y) du \, d\mu \\
= \int_{\bar{Y}} 1_{[t/\bar{h}(y), 1]} v(y, u - t/\bar{h}(y)) w(y, u) d\tilde{\mu} = \int_{\bar{Y}} 1_{[t/\bar{h}(y), 1]}(u) v(y, u - t/\bar{h}(y)) w^* d\mu_{\Phi},
\]

where in the last equality we used that \( w = w^*/\bar{h} \in L^\infty(\tilde{\mu}) \) with \( w^* \in L^\infty(\mu_{\Phi}) \).

For the case \( t \geq \bar{h}(y)u \), we need to work with the transfer operator \( R \) of \( \Phi \). Recall \( d\mu_{\Phi} = \frac{1}{\bar{h}} d\tilde{\mu} \). So,

\[
\int_{\bar{Y}} (U_t v) w \, d\tilde{\mu} = \int_{\bar{Y}} 1_{\{\varphi > t\}} \tilde{h} v \cdot \left( \frac{w^*}{\bar{h}} \right) \circ f_t \, d\mu_{\Phi}.
\]

With these specified we compute that

\[
\int_{\bar{Y}} (U_t v) w \, d\tilde{\mu} = \int_{\bar{Y}} 1_{\{\varphi > t\}} \tilde{h} v \cdot \left( \frac{w^*}{\bar{h}} \right) \circ f_t \circ \varphi \circ \Phi \, d\mu_{\Phi} \\
= \int_{\bar{Y}} \int_0^{1/2} \tilde{h}(y) 1_{\{\varphi > t\}} v(y, u) \left( \frac{w^*}{\bar{h}} \right) (Fy, 2u + \frac{t - \varphi}{\bar{h}(Fy)}) du \, d\mu \\
+ \int_{\bar{Y}} \int_{1/2}^1 \tilde{h}(y) 1_{\{\varphi > t\}} v(y, u) \left( \frac{w^*}{\bar{h}} \right) (Fy, 2u - 1 + \frac{t - \varphi}{\bar{h}(Fy)}) du \, d\mu \\
= \int_{\bar{Y}} \int_0^{1/2} \tilde{h}(y) (Fy) 1_{\{\varphi > t\}} v(y, u) w^* (Fy, \frac{t - \varphi(y) + \tilde{h}(y) u}{\bar{h}(Fy)}) du \, d\mu \\
+ \int_{\bar{Y}} \int_{1/2}^1 \tilde{h}(y) (Fy) 1_{\{\varphi > t\}} v(y, u) w^* (Fy, \frac{t - \varphi(y) + \tilde{h}(y) u}{\bar{h}(Fy)}) du \, d\mu \\
= I_1 + I_2.
\]

For \( I_1 \) we use the change of coordinates \( 2u' = \frac{t + \tilde{h}(y)u - \varphi(y)}{\bar{h}(Fy)} \) for \( u' \in [0, \frac{1}{2}) \), so \( du = 2\tilde{h}(Fy)/\bar{h}(y) du' = K(y) du' \) as in (2.1). This gives

\[
I_1 = \int_{\bar{Y}} \int_0^{\frac{1}{2}} 1_{\{\varphi > t\}} \, 2 v(y, u' + \frac{\varphi(y) - t}{\bar{h}(y)}) w^* \circ \Phi_1(y, u') \, du' \, d\mu,
\]

where \( \Phi_1 : Y \times [0, \frac{1}{2}) \rightarrow \bar{Y} \) is the first branch of the map \( \Phi \). Since \( v \) is supported on \( \bar{Y} \), the second argument \( u' + \frac{\varphi(y) - t}{\bar{h}(y)} \) needs to belong to \( [0, 1] \) to give a nonzero value. We emphasize this by the indicator function \( 1_{\{\varphi > t + \tilde{h}(y)u' < t + \bar{h}(y)\}} \), which combined with \( 1_{\{\varphi > t\}} \) gives \( 1_{\{t < \varphi < h(y)(1-u')\}} \). Hence

\[
I_1 = \int_{\bar{Y}} \int_0^{\frac{1}{2}} 1_{\{t < \varphi + \tilde{h}(y)(1-u')\}} 2 v(y, u' + \frac{\varphi(y) - t}{\bar{h}(y)}) w^* \circ \Phi_1(y, u') \, du' \, d\mu.
\]
For $I_2$ we use the change of coordinates $2u' - 1 = \frac{t + \tilde{h}(y)u - \varphi_0}{h(F_y)}$, for $u' \in \left[\frac{1}{2}, 1\right)$ so $du = K(y)du'$ with $K(y)$ again. The definition of $\varphi$ on this range gives

$$\varphi(y, u') + \tilde{h}(y)u' = \varphi_0(y) + (2\tilde{h}(F_y) - \tilde{h}(y))u' - \tilde{h}(F_y) = \varphi_0(y) + (2\tilde{h}(F_y) - \tilde{h}(y))(u' - \frac{1}{2}),$$

so the indicator function $1_{\{t < \varphi + \tilde{h}(y)u' < t + \tilde{h}(y)\}}$ preserves the same form, as does $1_{\{\varphi > t\}}$. Also

$$u = \frac{1}{2} \left( \frac{(2u' - 1)\tilde{h}(F_y) + \varphi - t}{\tilde{h}(y)} \right) = \frac{2u'h(F_y) + \varphi - t}{\tilde{h}(y)}.$$ Therefore, denoting by $\Phi_2 : Y \to \left[\frac{1}{2}, 1\right) \to \tilde{Y}$ the second branch of $\Phi$,

$$I_2 = \int_{\tilde{Y}} \int_{\frac{1}{2}}^{1} 1_{\{t < \varphi + \tilde{h}(y)(1 - u')\}} 2v(y, u' + \frac{\varphi - t}{\tilde{h}(y)}) w^* \circ \Phi_2(y, u') \, du' \, d\mu$$

Note that Lebesgue measure is invariant in the $u$-direction. Therefore, using the pointwise formula for the transfer operator on both branches (in the $u$-direction),

$$I_1 + I_2 = \int_Y \int_0^1 1_{\{t < \varphi + \tilde{h}(y)(1 - u)\}} v(y, u + \frac{\varphi - t}{\tilde{h}(y)}) (2w^* \circ \Phi_1 + 2w^* \circ \Phi_2) \, du \, d\mu$$

$$= \int_{\tilde{Y}} \int_{\tilde{Y}} R(1_{\{t < \varphi + \tilde{h}(y)(1 - u)\}} v(y, u + \frac{\varphi - t}{\tilde{h}(y)}) w^* \circ \Phi_1) \, d\mu_\Phi.$$ Thus

$$\int_{\tilde{Y}} (U_tw) \, d\tilde{\mu} = \int_{\tilde{Y}} R(1_{\{t < \varphi + \tilde{h}(y)(1 - u)\}} v(y, u + \frac{\varphi - t}{\tilde{h}(y)}) w^* \circ \Phi_1) \, d\mu_\Phi,$$

as required.

### 8.1 Estimates for $\tilde{U}$ required in the finite case

We first estimate the inverse Laplace transform of $s^{-1} \int_{\tilde{Y}} \tilde{U}(s)vw \, d\tilde{\mu}$, as follows

**Lemma 8.3** Assume (H0) i), and (H1) i). Let $B$ the Banach space defined by (H2). Let $v = \frac{\varphi}{\tilde{h}}$ with $v^* \in B$ and $w = \frac{w^*}{\tilde{h}}$ with $w^* \in L^\infty(\mu_\Phi)$. Define $\tilde{r}(s) := s^{-1} \int_{\tilde{Y}} \tilde{U}(s)vw \, d\tilde{\mu}$. Let $r(t)$ be the inverse Laplace transform of $\tilde{r}(s)$. Then

$$r(t) = \tilde{h}(y) \int_0^u v(y, \tau) \, d\tau \, w \, d\tilde{\mu} + \int_{\tilde{Y}} \tilde{h}(y) \left( \int_0^1 v(y, \tau) \, d\tau \right) w^* \circ \Phi \, d\mu_\Phi + E(t),$$

where $|E(t)| \leq \|v^*\|_{L^\infty(\mu_\Phi)} \|w^*\|_{L^\infty(\mu_\Phi)} t^{-\beta}$.

**Proof** Write $\tilde{r}(s) = s^{-1} \int_0^\infty \int_{\tilde{Y}} U_tw \, d\tilde{\mu} e^{-st} \, dt$. Using Proposition 8.1,

$$\int_{\tilde{Y}} s^{-1}(\tilde{U}(s)v)(y, u) w(y, u) \, d\tilde{\mu} = \int_{\tilde{Y}} s^{-1} \int_0^{\tilde{h}(y)u} e^{-st} v(y, u - \frac{\tau}{\tilde{h}}) 1_{\left[\frac{1}{2}, 1\right]}(u) \, d\tau w(y, u) \, d\tilde{\mu} + \int_{\tilde{Y}} s^{-1} \int_{\tilde{h}(y)u}^\infty e^{-s\tau} (R v_\tau)(y, u) \, d\tau \, w^* \circ \Phi \, d\mu_\Phi.$$
with inverse Laplace transform

$$\int_{Y} \int_{0}^{\tilde{h}(y)u} 1_{[\frac{\tau}{\tilde{h}}, \infty)}(t)v(y, u - \tau) d\tau w(y, u) d\tilde{\mu} + \int_{Y} \int_{\tilde{h}(y)u}^{\infty} 1_{[\tau, \infty)}(t)(R\nu_{\tau})(y, u) d\tau w^{*}d\mu_{\Phi}.$$ 

Hence \( r(t) = r_{1}(t) + r_{2}(t) \) where

$$r_{1}(t) = \int_{Y} \int_{0}^{\tilde{h}(y)u} 1_{[\frac{\tau}{\tilde{h}}, \infty)}(t)v(y, u - \frac{\tau}{\tilde{h}}) d\tau w d\tilde{\mu}, \quad r_{2}(t) = \int_{Y} \int_{\tilde{h}(y)u}^{\infty} 1_{[\tau, \infty)}(t)(R\nu_{\tau})(y, u) d\tau w^{*}d\mu_{\Phi}.$$ 

Changing coordinates \( u - \tau / \tilde{h} \to \tau \) gives

$$r_{1}(t) = \int_{Y} \tilde{h} \int_{0}^{u} 1_{[\frac{\tau}{\tilde{h}}, \infty)}(t)v(y, \tau) d\tau w d\tilde{\mu} = \int_{Y} \tilde{h} \int_{0}^{u} v(y, \tau) d\tau w d\tilde{\mu} + E_{1}(t),$$

where

$$|E_{1}(t)| \leq \int_{Y} \tilde{h} \int_{0}^{u} 1_{[\frac{\tilde{h}(u - \tau)}{\tilde{h}}, \infty)}(t)\left|v(y, \tau)\right| d\tau |w(y, u)| d\tilde{\mu} \leq \|w^{*}\|_{L^{1}(\mu_{\Phi})} \|w^{*}\|_{L^{1}(\mu_{\Phi})} \mu(t < \tilde{h}),$$

and because \( \tilde{h} = \varphi_{0}^{*} \), using (H0) i) and Lemma 4.4, we get \( \mu(t < \tilde{h}) \ll e^{-\beta/\tau}. \)

For \( r_{2}(t) \), recall \( v_{1}(y, u) = 1_{\{t < \varphi < t + \tilde{h}(y)(1 - u)\}} v(y, u + \tilde{h} / \tilde{h}(y)) \). For \( t > \tilde{h}(y)u \),

$$r_{2}(t) = \int_{Y} \left( \int_{\varphi + \tilde{h}u - \tilde{h}}^{\varphi} 1_{[\tau, \infty)}(t)\nu_{\tau}(y, u + \frac{\varphi - \tau}{\tilde{h}(y)})d\tau \right) w^{*}d\mu_{\Phi}$$

$$= \int_{Y} \left( \int_{\varphi + \tilde{h}u - \tilde{h}}^{\varphi} 1_{[\tau, \infty)}(t)v(y, u + \frac{\varphi - \tau}{\tilde{h}(y)})d\tau \right) w^{*} \circ \Phi d\mu_{\Phi}$$

$$= \int_{Y} \tilde{h} \left( \int_{0}^{1} 1_{[\frac{\varphi - \tau}{\tilde{h}(y)}, \infty)}(t)v(y, \tau)d\tau \right) w^{*} \circ \Phi d\mu_{\Phi}$$

$$= \int_{Y} \tilde{h} \left( \int_{0}^{1} v(y, \tau)d\tau \right) w^{*} \circ \Phi d\mu_{\Phi} - E_{2}(t),$$

where

$$|E_{2}(t)| = \int_{Y} \tilde{h} \left( \int_{0}^{1} 1_{\{\varphi > \tilde{h}t + \tau - \tilde{u}h\}}\left|v(y, \tau)\right|d\tau \right) w^{*} \circ \Phi d\mu_{\Phi}$$

$$\leq \|w^{*}\|_{L^{1}(\mu_{\Phi})} \int_{Y} \tilde{h} \left( \int_{0}^{1} 1_{\{\varphi > \tilde{h}t + \tau - \tilde{u}h\}}\left|v(y, \tau)\right|d\tau \right) d\mu_{\Phi}. \quad (8.1)$$

The set of the indicator function can be rewritten as

$$\{\varphi > \tilde{h}t + \tau - \tilde{u}h\} = \begin{cases} \{\varphi_{0}(y) > \tilde{h}(y)t + \tau - 2\tilde{h}F(y)u\} & \text{if } u < \frac{1}{2}, \\ \{\varphi_{0}(y) > \tilde{h}(y)t + \tau - 2\tilde{h}F(y)u - \tilde{h}F(y)\} & \text{if } u \geq \frac{1}{2}. \end{cases}$$

In either case, we get the inclusion

$$\{\varphi > \tilde{h}t + \tau - \tilde{u}h\} \subset \{\varphi_{0} > \tilde{h}t - \tilde{h} \circ F\} \subset \{\varphi_{0} > \tilde{h}t/2\} \cup \{\tilde{h} \circ F > t/2\}. \quad (8.2)$$
Recall \( v = \frac{w}{h} \) with \( v^* \in \mathcal{B} \), \( \tilde{h} = \varphi_0^\gamma \), \( \gamma < 1 \) and compute that
\[
|E_2(t)| \leq \|v^*\|_{L^\infty(\mu_\Phi)} \|w^*\|_{L^\infty(\mu_\Phi)} \left( \mu(\varphi_0^{1-\gamma} > t) + \mu(\varphi_0^\gamma \circ F > t) \right).
\]
Using the \( \mu \) invariance under \( F \), \( \mu(\varphi_0^\gamma \circ F > t) = \mu(\varphi_0^\gamma > t) \). This together with (H0) i) and Lemma 4.4 implies that \( \mu(\varphi_0^{1-\gamma} > t) \ll t^{-\frac{\beta}{2}} < t^{-\beta} \), since by (H1) i, \( \gamma < 1 \). Also, \( \mu(\varphi_0^{1-\gamma} > t) \ll t^{-\frac{\beta}{2}} < t^{-\beta} \). The conclusion follows by combining \( E_1 \) and \( E_2 \).

**Lemma 8.4** Assume either case i) or ii) of (H0). Let \( \mathcal{B} \) be the Banach space defined by (H2). Then, the function \( \bar{U}(s) : \mathcal{B} \to L^1(\mu_\Phi) \) lies in \( \mathcal{R}_{\mathcal{B} \to L^1(\mu_\Phi)}/(1/t^\beta) \)

**Proof** Let \( v^* \in \mathcal{B} \). By definition (see (3.2)), \( \int_{Y} \bar{U}_t v^* w d\mu_\Phi = \int_{Y} 1_{\{\varphi_0 > t\}} v^* \cdot w \circ f_t \, d\mu_\Phi \). Hence, \( \|\bar{U}_t\|_{L^1(\mu_\Phi)} \leq \|v^*\|_{L^\infty(\mu_\Phi)} \mu_\Phi(\varphi_0 > t) \). The conclusion follows since \( \mathcal{B} \subset L^\infty(\mu_\Phi) \).

**Lemma 8.5** Assume either form of (H0) and (H1). Let \( \mathcal{B} \) the Banach space defined by (H2). Then, the family of linear operators \( \bar{U}(s) : \mathcal{B} \to L^1(\mu_\Phi) \), \( s \in \mathbb{H}, b \in \mathbb{R} \), is uniformly bounded; that is, there exists \( C > 0 \) such that \( \|\bar{U}(s)\|_{\mathcal{B} \to L^1(\mu_\Phi)} \leq C \) for all \( s \in \mathbb{H} \).

**Proof** Let \( v = \frac{w}{h} \) with \( v^* \in \mathcal{B} \). Using (3.2) and Proposition 8.1
\[
|\bar{U}(ib)v|_{L^1(\mu_\Phi)} = \int_{Y} |\bar{U}(s)v| \, d\mu_\Phi \leq \int_{Y} \int_{0}^{h(y)u} 1_{[0,h(y)u]}(u) |v(y,u - \frac{t}{h(y)})| \, dt \, d\mu_\Phi
\]
\[
+ \int_{Y} \int_{h(y)u}^{\infty} 1_{[t - \varphi < t + \tilde{h}(y)(1-u)]} |v(y,u + \frac{\varphi - t}{\tilde{h}(y)})| \, dt \, d\mu_\Phi
\]
\[
= I_1 + I_2.
\]
Using the change of coordinates \( \tau = u - t/\tilde{h}(y) \),
\[
I_1 = \int_{Y} \int_{0}^{1} \int_{0}^{u} |v^*(y,\tau)| \, d\tau \, du \, d\mu.
\]
Similarly,
\[
I_2 = \int_{Y} \int_{0}^{1} \int_{\varphi + \tilde{h}(y)(1-u)}^{\varphi} |v^*(y,u + \frac{\varphi - t}{\tilde{h}(y)})| \, dt \, du \, d\mu
\]
\[
= \int_{Y} \int_{0}^{1} \int_{u}^{1} |v^*(y,\tau)| \, d\tau \, du \, d\mu
\]
Thus, \( I_1 + I_2 \leq \int_{Y} |v^*(y,u)| \, d\mu_\Phi \leq \|v^*\|_{L^\infty(\mu_\Phi)} \mu_\Phi(Y) \), as required.

**Remark 8.6** Let \( \mathcal{B}_0 \subset L^\infty(\mu_\Phi) \) as defined in (H4). By the argument of Lemma 8.5, we have \( \|\bar{U}(s)\|_{\mathcal{B}_0 \to L^1(\mu_\Phi)} \leq C \) for all \( s \in \mathbb{H} \).
8.2 Estimates for $\hat{U}$ required in the infinite case

**Lemma 8.7** (a) Assume either form of (H0) and (H1). Let $v \in L^1(\mu)$ and $w = \frac{w^*}{h}$ with $w^* \in L^\infty(\mu_\Phi)$. Then

$$\int_Y \hat{U}(0)v(y,u)w(y,u) \, d\mu = \int_Y \int_0^u \frac{\tilde{h}(y)v(y,\tau)}{h(y)} \, d\tau \, w^*(y,u) \, d\mu_\Phi$$

$$+ \int_Y \int_u^1 R(\tilde{h}(y)v(y,\tau)) \, d\tau \, w^*(y,u) \, d\mu_\Phi.$$

(b) Assume (H0) ii) and (H1) ii). Let $B$ be a Banach space defined by (H2). Let $B_0$ be a Banach space with $B \subset B_0 \subset L^\infty(\mu_\Phi)$. Let $\tau$ be as defined in (H4). Then,

$$\|\hat{U}(ib_2) - \hat{U}(ib_1)\|_{B_0 \to L^1(\mu_\Phi)} \ll |b_1 - b_2|^\tau.$$

**Proof** (a) By Proposition 8.1,

$$\int_Y \hat{U}(0)vwd\tilde{\mu} = \int_Y \int_0^{\frac{t}{h(y)}} 1_{\{\frac{t}{h(y)} \leq \tau \leq 1\}}(u)v(y,u + \frac{t}{h(y)}) w^*(y,t) \, dt \, d\mu_\Phi$$

$$+ \int_Y \int_{\frac{t}{h(y)}}^\infty R(1_{\{\frac{t}{h(y)} \leq \tau \leq 1\}}(u)v(y,u + \frac{\phi - t}{h(y)}) w^*(y,t) \, dt \, d\mu_\Phi$$

$$= I_1 + I_2.$$

We compute the first integral with the change of coordinates $\tau = u - \frac{t}{h(y)}$:

$$I_1 = \int_Y \int_0^{\frac{t}{h(y)}} v(y,u - \frac{t}{h(y)}) w^*(y,t) \, dt \, d\mu_\Phi$$

$$= \int_Y \int_0^u \tilde{h}(y)v(y,\tau) \, d\tau \, w^*(y,t) \, d\mu_\Phi.$$

We continue with the estimate for $I_2$. Under (H1), $\phi > \tilde{h}$. Thus, the bounds of the $t$-integral become $\phi - \tilde{h}(y)(1 - u) < t < \phi$:

$$I_2 = \int_Y \int_{h(y)}^{\infty} R(1_{\{\frac{\phi - t}{h(y)} \leq \tau \leq 1\}}(u)v(y,u + \frac{\phi - t}{h(y)}) w^*(y,u) \, dt \, d\mu_\Phi$$

$$= \int_Y \int_{\phi - \tilde{h}(y)(1 - u)}^{\phi} R(v(y,u + \frac{\phi - t}{h(y)}) w^*(y,u) \, dt \, d\mu_\Phi$$

$$= \int_Y \int_{\phi - \tilde{h}(y)(1 - u)}^{\phi} v(y,u + \frac{\phi - t}{h(y)}) w^*(y,u) \, dt \, d\mu_\Phi$$

$$= \int_Y \int_0^1 v(y,\tau) \, d\tau \, w^*(y,u) \, d\mu_\Phi$$

$$= \int_Y \int_0^1 R(v(y,\tau)) \, d\tau \, w^*(y,u) \, d\mu_\Phi.$$
For (b), we write \( \psi(y,u,t) = 1\{t < y < t + \tilde{h}(y)(1-u)\} \) and \( B(b_1,b_2,t) = |e^{ib_2t} - e^{ib_1t}| \). By Proposition 8.1,

\[
|\tilde{U}(ib_2)v - \tilde{U}(ib_1)v|_{L^1(\mu_b)} = \int \tilde{h}(y)|\tilde{U}(ib_2)v - \tilde{U}(ib_1)v| \, d\mu
\]

\[
\leq \int \tilde{h}(y) \int_0^{\tilde{h}(y)} B(b_1,b_2,t) 1_{\{t/\tilde{h}(y),1\}}(u)|v(y,u - \frac{t}{\tilde{h}(y)})| \, dt \, d\mu
\]

\[
+ \int \tilde{h}(y) \int_{\tilde{h}(y)}^\infty \psi(y,u,t)B(b_1,b_2,t)|v(y,u + \frac{\varphi - t}{\tilde{h}(y)})| \, dt \, d\mu
\]

\[
= I_1 + I_2.
\]

For \( I_1 \) we use the change of coordinates \( \tau = u - \frac{t}{\tilde{h}(y)} \). This gives

\[
I_1 \leq |b_1 - b_2| \int \tilde{h}(y) \int_0^{\tilde{h}(y)} (u - \tau)|v(y,\tau)| \, d\tau \, d\mu
\]

\[
\leq |b_1 - b_2| \int \tilde{h}(y) \int_0^{\tilde{h}(y)} u \int_0^u |v(y,\tau)| \, d\tau \, du \, d\mu
\]

\[
\leq |b_1 - b_2| \int \tilde{h}(y)|v(y,u)| \, d\mu. \tag{8.3}
\]

Recall that \( v \in L^1(\tilde{\mu}) \). So, \( v = \frac{\varphi}{\tilde{h}} \) with \( v^* \in L^1(\mu_b) \). Let \( v^* \in B_0 \subset L^\infty(\mu) \) and recall that \( \tilde{h} \in L^1(\mu_b) \). Hence,

\[
|I_1| \leq |b_1 - b_2| \int \tilde{h} \cdot v^*(y,u) \, d\mu_b \leq |b_1 - b_2||v^*||_{L^\infty(\mu_b)} \int \tilde{h} \, d\mu,
\]

as required.

For \( I_2 \), we first mimic the argument used in the proof of Lemma 8.3 in estimating \( r_2(t) \) there. Namely, we use the change \( t \to \sigma \tilde{h} \) and compute that

\[
I_2 = \int \tilde{h} \int_0^\infty \int_0^\infty \psi(y,u,t)B(b_1,b_2,t)|v(y,u + \frac{\varphi - t}{\tilde{h}(y)})| \, dt \, d\mu
\]

\[
= \int \tilde{h} \int_0^\infty \psi(y,u,\sigma \tilde{h})B(b_1,b_2,\sigma \tilde{h})|v(y,u + \frac{\varphi}{\tilde{h}} - \sigma)| \, d\sigma \, d\mu
\]

\[
= \int \tilde{h} \int_0^\infty B(b_1,b_2,\sigma \tilde{h})|v(y,u + \frac{\varphi}{\tilde{h}} - \sigma)| \, d\sigma \, d\mu.
\]

Using the change of coordinates \( u + \frac{\varphi}{\tilde{h}} - \sigma \to \sigma \) and the fact that \( B(b_1,b_2,\sigma) \leq |b_1 - b_2|^\tau |\sigma^\tau \), with \( \tau \) as in (H4), we have

\[
I_2 = \int \tilde{h} \int_0^1 B(b_1,b_2,u \tilde{h} + \varphi - \sigma \tilde{h})|v(y,\sigma)| \, d\sigma \, d\mu
\]

\[
\leq |b_1 - b_2|^\tau \int \tilde{h} \int_0^1 (\varphi + \tilde{h}(u - \sigma))^\tau |v(y,\sigma)| \, d\sigma \, d\mu
\]

\[
\leq |b_1 - b_2|^\tau \int \tilde{h} \int_0^1 \varphi^\tau |v(y,\sigma)| \, d\sigma \, d\mu. \tag{8.4}
\]
Since \( v = \frac{v^*}{h} \) with \( v^* \in B_0 \subset L^\infty(\mu) \) and \( h \in L^1(\mu_\Phi) \),

\[
|I_2| \leq |b_1 - b_2|^\tau \|v^*\|_{L^\infty(\mu_\Phi)} \int_Y h \cdot \varphi^\tau d\mu.
\]

By definition, \( \varphi(y) \leq \varphi_0(y) + \tilde{h}(Fy) \) and by (H1), \( \tilde{h} = \varphi_0^\gamma \). Hence,

\[
\tilde{h} \varphi^\tau \leq \varphi_0^{\tau + \gamma} + \varphi_0^\gamma (h \circ F)^\tau.
\]

Using the \( \mu \) invariance under \( F \),

\[
\int_Y \tilde{h} \cdot \varphi^\tau d\mu \leq \int_Y \varphi_0^{\tau + \gamma} d\mu + \int_Y \varphi_0^{\tau(1 + \gamma)} d\mu < 2 \int_Y \varphi_0^{\tau(1 + \gamma)} d\mu.
\]

By (H0) ii) and Lemma 4.4, \( \mu(\varphi_0 > t) \ll t^{-(\beta - \epsilon)} \), for any small \( \epsilon > 0 \). By (H4), \( \tau(1 + \gamma) < \beta \).

Thus, \( \int_Y \tilde{h} \cdot \varphi^\tau d\mu < \infty \). As a consequence, \( |I_2| \leq |b_1 - b_2|^\tau \|v^*\|_{L^\infty(\mu_\Phi)} \). The conclusion follows.

9 Semiflows over Markov maps: bounded \( \tilde{h} \)

Markov maps represent a class of examples where the conditions of the abstract setting are likely to hold, except that condition (H4) is problematic for standard norms. However, in the infinite measure case, we are allowed pass to a weaker space \( B_0 \) (see Theorem 5.2) in which to check (H5). Contrary to (H1), we stipulate that \( \tilde{h} \) is bounded (we can assume without loss of generality that \( \tilde{h} \equiv 1 \)). We let \( \mathcal{B} \) to be the space of \( \theta \)-Hölder functions, embedded in \( B_0 \) which we can choose to be \( L^\infty(\mu_\Phi) \). However, in view of the finite measure case, it may be helpful to choose a stronger space for \( B_0 \), which is embedded in \( L^\infty(\mu_\Phi) \) but has extra properties.

9.1 The set-up

Recall the partitions \( \mathcal{P}, \mathcal{P}_n \) and \( \mathcal{P}, \mathcal{P}_n \) from Section 4.4. For \( y_1, y_2 \in Y \), define the separation time \( s(y_1, y_2) \) as the smallest integer \( n \geq 0 \) such that \( F^n y_1 \) and \( F^n y_2 \) lie in different elements of \( \mathcal{P} \). Similarly for \( \bar{y}_1, \bar{y}_2 \in \bar{Y} \), let \( \bar{s}(\bar{y}_1, \bar{y}_2) \) be the smallest integer \( n \geq 0 \) such that \( \Phi^n \bar{y}_1 \) and \( \Phi^n \bar{y}_2 \) lie in different elements of \( \mathcal{P} \).

For given \( \theta \in (0, 1) \), let \( \mathcal{B} (\bar{Y}) \) be the Banach space of function \( v \) supported on \( \bar{Y} \), with norm \( \|v\|_\theta = |v|_\theta + \|v\|_\infty \), where \( \|v\|_\infty = \|v\|_{L^\infty(\mu_\Phi)} \) and the seminorm \( |v|_\theta \) is defined as

\[
|v|_\theta = \sup_{\bar{y}_1 \neq \bar{y}_2 \in \bar{Y}} \theta^{-\bar{s}(\bar{y}_1, \bar{y}_2)} |v(\bar{y}_1) - v(\bar{y}_2)|.
\]

Let \( f : X \rightarrow X \) be a non-uniformly expanding map with a single indifferent fixed point, say at \( 0 \in X \). Consider a suspension flow over \( f \) with continuous roof function \( h \) and assume that \( h \) is bounded and bounded away from zero. Assume that \( F : Y \rightarrow Y \) is an induced map over \( f \), with the following properties:

1. \( F \) is full-branched, i.e., \( F(Z) = Y \) for every \( Z \) in the Markov partition \( \mathcal{P} \), and the induced time \( \tau_F : Y \rightarrow \mathbb{N} \) such that \( F = f^{\tau_F} \) is constant on each \( Z \in \mathcal{P} \).
(2) $F$ is expanding and there is a distortion constant $C_{dis}$ such that
\[
\frac{|DF^k(y_1)|}{|DF^k(y_2)|} \leq C_{dis},
\] (9.1)
for all $k \geq 0$, $Z \in \mathcal{P}_k$ and $y_1, y_2 \in Z$. This condition implies that $F$ preserves a measure $\mu$, absolutely continuous w.r.t. Lebesgue, such that $\frac{1}{C_\mu} \leq \frac{d\mu}{dx} \leq C_\mu$ for some $C_\mu > 0$.

Define the potential $p : Y \to \mathbb{R}$, $p(y) = \log \frac{d\mu}{dx}$; we assume that there is a constant $C_p$ such that
\[
e^{p_n(y)} \leq C_p \mu(Z) \quad \text{and} \quad |e^{p_n(y_1)} - e^{p_n(y_2)}| \leq C_p \mu(Z) \theta^n(F^n(y_1), F^n(y_2)),
\] (9.2)
for every $y, y_1, y_2 \in Z$, $Z \in \mathcal{P}_n$.

(3) The roof function of the induced system $F : Y \to Y$ is $\varphi_0 = \sum_{i=0}^{\tau_F - 1} h \circ f^i \leq \tau_F \sup h$. We assume that there exists $C_{\varphi_0} > 2$
\[
|\varphi_0(y_1) - \varphi_0(y_2)| \leq C_{\varphi_0} \theta^n(y_1, y_2),
\] (9.3)
for all $y_1, y_2 \in Z$, $Z \in \mathcal{P}$.

If there is only one $Z \in \mathcal{P}$ with $\tau_F(Z) = n$, the following is immediate: There is $h_n = h(0) n + o(n)$ such that
\[
|\varphi_0(y) - h_n| \leq C_{\varphi_0}
\] (9.4)
for all $y \in Z$, $Z \in \mathcal{P}$ with $\tau_F(Z) = n$. Let us write $\eta : \mathbb{R} \to \mathbb{N}$ for the asymptotic inverse of $h_n$ in the sense that $\eta(t)$ is minimal such that $h_\eta(t) \geq t$. For example, for the case $\beta \in (0, 1)$, if the roof function $h$ is differentiable near $0$, and the branch of $f$ with the indifferent fixed point is $x \mapsto x + x^{1+1/\beta}$, then $h_n = h(0) n + \frac{h'(0)}{1-\beta} n^{1-\beta} + o(n^{1-\beta})$, and $\eta(t) = t/h(0) + O(t^{1-\beta})$.

(4) The induce time $\tau_F$ satisfies the tail condition
\[
\mu(y \in Y : \tau_F(y) \geq n) = O(n^{-\beta}).
\] (9.5)

By the argument [20, Proposition 2.6], the same tail condition holds for $\mu(y \in Y : \varphi_0(y) \geq h_n)$.

9.2 The space $\mathcal{B}_0$ and norm $|| \cdot ||_{\mathcal{B}_0}$

The standard $\theta$-Hölder norm on $\mathcal{B}(\tilde{Y})$ does not work well with the sets
\[
S_{t,a} = \{(y, u) \in \tilde{Y} : t < \varphi(y, u) < t + a\}
\] (9.6)
of the indicator function involved in $R_{t,a}$. Since $S_{t,a}$ is not aligned with $\tilde{P}_n$, we get $||R_{t,a} v||_\theta = \infty$ for most $v \in \mathcal{B}(\tilde{Y})$. In this section we define a version of the $\theta$-Hölder seminorm and the $\infty$-norm, where we first integrate over a one-dimensional curve.

Let $\mathcal{G}_0$ be the collection of piecewise linear curves $G_0 = \{y, u(y)\}_{y \in Y}$ such that
Figure 2: Schematic picture of two curves $G_1$ and $G_2$, and their $\Phi^r$-preimages $G'_1$ and $G'_2$.

(a) $\left| \frac{\partial u}{\partial y} \right| = 1/|Y|$ wherever the derivative is defined.

(b) For Lebesgue a.e. $u \in [0, 1)$, there is exactly one $y \in Y$ such that $(y, u) \in G_0$.

Next let $\mathcal{G} = \bigcup_{j \geq 0} \Phi^{-j}(G_0)$, see Figure 2. Hence $\Phi^{-1}(\mathcal{G}) \subset \mathcal{G}$ and for every multivalued curve $G = \{(y, u(y))\}_{y \in Y} \in \mathcal{G}$ we can take $r = r(G) \geq 0$ such that $G \in \Phi^{-r}(G_0)$. Then we have $r(\Phi^{-1}G) = r(G) + 1$ and

(c) For all $Z \in \mathcal{P}_r$ and Lebesgue a.e. $u \in [0, 1)$, there is exactly one $y \in Z$ such that $(y, u) \in G$.

(d) For all $Z \in \mathcal{P}_r$ and $y \in Z$, there are exactly $2^r$ values $u_j \in [0, 1)$ such that $(y, u_j) \in G$. (The notation $u(y) = \{u_j(y), 0 \leq j < 2^r\}$ is our shorthand for this.)

For $r = r(G)$, let

$$\int_Y v(y, u(y)) \, d\mu(y) = \int_Y \frac{1}{2^r} \sum_{j=0}^{2^r-1} v(y, u_j(y)) \, d\mu(y)$$

be our notation for the weighted integral of $v$ over the multivalued curve $G$. In the sequel, we will usually estimate a single integral in this sum.

Given $G \in \mathcal{G}$ with $r(G) = r$, let $G(y_1)$ denote the multivalued curve translated in the $u$-direction mod 1 so that $(y_1, 0) \in G(y_1)$, and similar for $G(y_2)$. Let $(y, u_1(y))$ and $(y, u_2(y))$, $y \in Y$ parametrize the multivalued curves $G(y_1)$ and $G(y_2)$, respectively. Due to property (a), $G(y_1)$ and $G(y_2)$ are vertical translation of each other by

$$u_{1,2} = \frac{|y_1 - y_2|}{2^r |Y|}. \quad (9.8)$$

Take $\theta_0 \in (0, 1)$ such that

$$\theta_0^{-1} \leq \inf_{Z \in \mathcal{P}_r} \inf_{y_1 \neq y_2 \in Z} \frac{|Fy_1 - Fy_2|}{|y_1 - y_2|} \cdot (9.9)$$
and define the seminorm for $v \in \mathcal{B}(\bar{Y})$:

$$|v|_{B_0}^* = \sup_{G_1, G_2 \in \mathcal{G}} \theta_0^{-s(y_1, y_2)} \int_Y |v(y, u_1(y)) - v(y, u_2(y))| \, d\mu(y),$$

(9.10)

where it will always be assumed that $r(G_1) = r(G_2)$. Our Banach space $B_0$ will be the completion of $\mathcal{B}$ w.r.t. the norm

$$\|v\|_{B_0} = |v|_{B_0}^* + \|v\|_\infty.$$  

(9.11)

**Lemma 9.1** For all $\theta \in (0, \frac{1}{2})$ and $\theta_0$ satisfying (9.9), there is a constant $K$ such that $|v|_{B_0}^* \leq K|v|_{B_0}$ for every $\theta$-Hölder function $v$ on $\bar{Y}$, hence $\mathcal{B}(\bar{Y})$ is embedded in $B_0(\bar{Y})$.

**Proof** Let $G_1, G_2 \in \mathcal{G}$ be arbitrary, with base points $y_1 < y_2$ and parametrizations $(y, u_1(y))$ and $(y, u_2(y))$. We say that $y \in E_n$ if $n$ is the smallest integer such that $k2^{-n} \in (u_2(y), u_1(y))$ for some odd integer $k$. Note that the part of $G_1$ near $Y \times \{0\}$ where $G_2$ is close to $Y \times \{1\}$ counts for $F_1$. Therefore $E_1$ consists two intervals and $E_n$ for $n \geq 2$ consist of $2^{n-1}$ intervals, all of width $|y_2 - y_1|$. On $E_n$, the separation time $\tilde{s}((y, u_1(y)), (y, u_2(y))) = n$, and therefore, using the fact that $\frac{d\mu}{dx} \leq C_\mu$, we obtain

$$\int_{E_n} |v(y, u_1(y)) - v(y, u_2(y))| \, d\mu(y) \leq 2^n \theta^n |v|_\theta C_\mu |y_2 - y_1|.$$  

Dividing by $\theta_0^{-s(y_1, y_2)} \geq |y_2 - y_1|$ and summing over $n \geq 1$, we find

$$\theta_0^{-s(y_1, y_2)} \int_Y |v(y, u_1(y)) - v(y, u_2(y))| \, d\mu(y) \leq \sum_{n \geq 1} 2^n \theta^n C_\mu |v|_\theta \leq \frac{2\theta C_\mu}{1 - 2\theta} |v|_\theta,$$

proving the lemma. \hfill $\blacksquare$

### 9.3 The result for bounded $\tilde{h}$

The following Diophantine condition below plays the role (A2) in [20] (namely that there exists periodic points $y_1, y_2 \in Y$ such that the ratio $\varphi_0(y_1)/\varphi_0(y_2)$ is Diophantine)

(♠) There exist two periodic points $\tilde{y}_1, \tilde{y}_2 \in \tilde{Y}$ such that the ratio $\varphi(\tilde{y}_1)/\varphi(\tilde{y}_2)$ is Diophantine.

**Proposition 9.2** Every system satisfying conditions (♠) and (1)-(4) in Section 9.1 for the above spaces ($\mathcal{B}, \|\|_B$) and ($B_0, \|\|_{B_0}$) satisfies the conclusion of Theorem 5.2.

**Remark 9.3** In view of Remark 4.1, we note that this proposition can also be stated for $B_0 = L^\infty(\mu_\theta)$. The relevant proofs for checking (H5), (see Lemma 9.4 and Proposition 9.8 below, where only Step I needs to be considered) only become easier.

The proof consists of verifying the conditions required for Theorem 5.2, that is, verifying that assumptions (H0)--(H6) are satisfied. Condition (H0) is supplied by (9.5) and Lemma 4.4, and it does not rely on the Markov structure. Condition (H1) can freely be assumed since $h$ is bounded away from zero.
9.4 Verifying (H2), (H3) and (H6) with bounded $\tilde{h}$

Since $\Phi$ is Gibbs Markov, (H2) is satisfied with the space $\mathcal{B}$ as described above (see, for instance, [20, Lemma 4.1]. Using the Diophantine assumption (♣) condition (H3) follows as in [20, Proposition 3.5 (a)]. Also, given that $\Phi$ is Gibbs Markov and (♣), (H6) can be verified as in [16, Lemma 3.13] (see also subsection 10.5 where we recall the relevant details for the reader convenience).

9.5 Verifying (H5) with bounded $\tilde{h}$

We start with the Lasota-Yorke inequality for the spaces $\mathcal{B}$ and $\mathcal{B}_0$.

**Lemma 9.4 (LY($\| \|_\mathcal{B}$, $\| \|_{\mathcal{B}_0}$))** Assume that $R \geq 0$ and that $\epsilon > 0$ is so small that $\phi \in L^1(\mu_\Phi)$. There exists constants $K_1, K_2, K_3 > 0$ such that

$$|\hat{R}^n(s)v|_\theta \leq K_1 \theta^n |v|_{\mathcal{B}_0} + K_2 (1 + |s|^\epsilon) \|v\|_{\mathcal{B}_0} \quad \text{and} \quad \|\hat{R}^n(s)v\|_{\mathcal{B}_0} \leq K_3 \|v\|_{\mathcal{B}_0},$$

(9.12)

for all $v \in \mathcal{B}$ and $n \in \mathbb{N}$.

**Proof** We have the pointwise formula for the transfer operator $R$:

$$(Rv)(y,u) = \sum_{(y',u')=(y,u)} \frac{1}{2} e^{p(y')}v(y',u'),$$

(9.13)

with potential $p(y) = \log \frac{du}{d\mu_F}$, and obviously the factor $\frac{1}{2}$ comes from the constant expansion 2 in the $u$-direction. Since $\tilde{h} \equiv 1$, we have by (2.5)

$$\varphi(y,u) = \varphi_0(y) + \begin{cases} u, & u \in [0, \frac{1}{2}); \\ u - 1, & u \in [\frac{1}{2}, 1). \end{cases}$$

(9.14)

The first inequality of (9.12) is part (b) of Lemma 4.1 in [20], where we used (9.2), (9.3) and also that $\|v\|_\infty \leq \|v\|_{\mathcal{B}_0}$. The only difference is that in our case $\varphi(y,u) = \varphi_0(y) + \psi(y,u)$, where $\varphi_0(y)$ plays the role of the roof function in [20, Lemma 4.1]. Since $\psi$ is linear and continuous on partition elements of $\hat{P}_n$, there are no additional difficulties here.

Now to show the second inequality of (9.12), we decompose the ergodic sum $\varphi_n(y,u) = \varphi_0_n(y) + \psi_n(y,u)$ and note that $|\psi_n(y,u_1) - \psi_n(y,u_2)| = |u_1 - u_2| \leq 2^{-r} |Y|^{-1} |y_1 - y_2| \leq$
2^{-r} |Y|^{-1} \theta^{s(y_1, y_2)}. This gives

$$|\hat{R}^n(s)v|_0^* = \sup_{G_1, G_2 \in \mathcal{G}} \theta_0^{-s(y_1, y_2)} \sum_{W \in \mathcal{P}_n} \frac{1}{2^n} \int_{\mathcal{Z}} e^{P_n(y') - s \varphi_0, n(y')} \left| e^{-s \psi_n(y', \varphi_1(y')) v(y', u_1(y')) - e^{-s \psi_n(y', \varphi_2(y')) v(y', u_2(y'))} \right| d\mu(y)$$

$$\leq \sup_{\Phi^{-n}G_1, \Phi^{-n}(G_2) \in \mathcal{G}} \theta_0^{-s(y_1, y_2)} \int_{Y} \left| e^{-s \psi_n(y', \varphi_1(y')) v(y', u_1(y')) - e^{-s \psi_n(y', \varphi_2(y')) v(y', u_2(y'))} \right| d\mu(y)$$

$$\leq \sup_{\Phi^{-n}G_1, \Phi^{-n}(G_2) \in \mathcal{G}} \theta_0^{-s(y_1, y_2)} \int_{Y} \left| v(y', u_1(y')) - v(y', u_2(y')) \right| d\mu(y)$$

$$+ \left| e^{-s \psi_n(y', \varphi_1(y')) v(y', u_1(y')) - s \psi_n(y', u_1(y')) \right| d\mu(y)$$

$$\leq \theta_0^{n} |v|_0^* + C_\mu \|v\|_{L^\infty(\mu_\phi)}.$$
**Proposition 9.8** Assume tail condition (4) and \( \bar{h} \equiv 1 \). Fix \( \theta \in (0, 1) \) and let \( \mathcal{B}_0 \) be the Banach spaces as above. Then for \( \tau < \beta \),

\[
\int_0^\infty \sigma^\tau \| R_{\sigma, a} \|_{\mathcal{B} \rightarrow \mathcal{B}_0} \ d\sigma < \infty.
\]

**Remark 9.9** The proof below also shows that \( \int_0^\infty \sigma^\tau \| R_{\sigma, a} \|_{\mathcal{B}_0 \rightarrow L^\infty(\mu, \phi)} \ d\sigma < \infty \).

**Proof** Assumption (9.4) implies that \( |t - h_n| \leq |\varphi - \varphi_0| + a + |\varphi_0 - h_n| \leq C_{\varphi_0} + 1 + a \), so there is \( C' = C'(a) \) such that

\[
|\eta(t) - C'| \leq n < |\eta(t) + C'|, \tag{9.15}
\]

where we recall from property (3) that \( \eta \) is the asymptotic inverse of \( h_n \).

**Step I: Estimates for \( \| R_{t, a} v \|_\infty \).** Using (9.13), we have

\[
R_{t, a} v = \sum_{W \in \tilde{P}} \frac{1}{2} e^{p(y_W')} 1_{S_{t, a}}(y_W', u_W') v(y_W', u_W'),
\]

where \( (y_W', u_W') = \Phi^{-1}(y, u) \cap W \). Each \( W \in \tilde{P} \) has the form \( Z \times [0, \frac{1}{2}) \) or \( Z \times [\frac{1}{2}, 1) \) for \( Z \in \mathcal{P} \). This gives by (9.2)

\[
\| R_{t, a} v \|_\infty \leq \sum_{W \in \tilde{P}, W \cap S_{t, a} \neq \emptyset} \frac{1}{2} e^{p(y_W')} \| v \|_\infty \leq \sum_{|h_{\tau_F}(y) - t| \leq C_{\varphi_0} + 1 + a} C_p \mu(Z) \| v \|_\infty \leq C_p \| v \|_\infty \mu(y \in Y : \tau_F(y) - \eta(t) \leq C') \tag{9.16}
\]

Since \( \eta(t) = t/h(0) + o(t) \), we obtain

\[
\int_0^\infty \sigma^\tau \| R_{\sigma, a} \|_\infty \ d\sigma \ll 2C' C_p h(0)^{\tau + 1} \sum_n n^{\tau} \mu(y \in Y : \tau_F(y) = n) \ll 2C' C_p h(0)^{\tau + 1} \sum_n n^{\tau - 1} \mu(y \in Y : \tau_F(y) \geq n) < \infty,
\]

by assumption (9.5) and since \( \tau < \beta \).

**Step II: Estimates for \( |R_{t, a} v|_{\bar{g}_0}^\ast \).** Using \( e^{p(y_Z')} \leq C_p \mu(Z) \) from (9.2), we can estimate

\[
|R_{t, a} v|_{\bar{g}_0}^\ast \leq \sup_{G_1, G_2 \in \mathcal{G}} \theta_0^{-s(y_1, y_2)} \sum_{k=0}^1 \sum_{Z \in \tilde{P}_k} C_p \mu(Z) T_k(G_1, G_2, Z),
\]

where the sum over \( k = 0, 1 \) refers to the two \( W \in \tilde{P} \) stacked over the same \( Z \in \mathcal{P} \), and

\[
T_k(G_1, G_2, Z) = \int_Y |1_{S_{t, a}} v(y_Z', u'_1(y_Z')) - 1_{S_{t, a}} v(y_Z', u'_2(y_Z'))| \ d\mu(y).
\]

Here \( ((y'_Z), u'_j(y'_Z)), j = 1, 2, \) are the preimage points \( \Phi^{-1}_W(y, u_j(y)) \) where we suppressed the dependence on \( k \), and we will also suppress the dependence on \( Z \).
We need to compare the integral of \( v \) over preimages of \( G_1 \) and \( G_2 \) which are the vertical translation of each other by \( \frac{1}{2} u_{1,2} \) (as in (9.8)). These translations are naturally small if \( s(y_1, y_2) \) is large. To deal with intersections of these curves with \( S_{t,a} \), it helps to divide into subcases, see Figure 3:

**Case A:** Both \((y', \underline{u}_1(y'))\) and \((y', \underline{u}_2(y')) \notin S_{t,a}\). There is no contribution to the integral, so we can ignore this case.

**Case B:** Both \((y', \underline{u}_1(y'))\) and \((y', \underline{u}_2(y')) \in S_{t,a}\), so \(1_{S_{t,a}} v(y', \underline{u}_1(y')) - 1_{S_{t,a}} v(y', \underline{u}_2(y')) = v(y', \underline{u}_1(y')) - v(y', \underline{u}_2(y'))\). Now we argue as in the proof of Lemma 9.1, saying that \( y \in E_n \) if \( \underline{u}_1(y) \) and \( \underline{u}_2(y) \) are separated by \( k2^{-n} \) for some odd integer \( k \). Equivalently, \( \underline{u}_1'(y') \) and \( \underline{u}_2'(y') \) are separated by \( k'2^{-(n+1)} \) for some odd \( k' \), so the separation time increases by 1. Therefore “the B part” of \( T_k(G_1, G_2, Z) \) is

\[
T_k(G_1, G_2, Z) = \sum_{n \geq 1} \int_{E_n} |v(y', \underline{u}_1(y')) - v(y', \underline{u}_2(y'))| d\mu(y)
\]

\[
\leq \sum_{n \geq 1} 2^n \theta^{n+1} C_\mu |y_1 - y_2| |v||_\theta \leq \frac{2\theta^2}{1 - 2\theta} C_\mu \theta_0^{s(y_1, y_2)} |v||_\theta.
\]

**Case C:** Only one of \((y', \underline{u}_1(y'))\) and \((y', \underline{u}_2(y')) \in S_{t,a}\). This applies to at most two intervals (one “on either side” of Case B), of length \(|y_2 - y_1|\). Therefore “the C part” of \( T_k(G_1, G_2, Z) \) can be estimated as

\[
2||v||_\infty |y_2 - y_1| \leq 2||v||_\infty \theta_0^{s(y_1, y_2)}.
\]

![Figure 3: Schematic picture of cases A-C for \( S_{t,a} \) intersecting a cylinder \( W \).](image)

The set \( S_{t,a} \cap W \) is bounded above and below by curves

\[
\begin{aligned}
\{ u_+(y) & = t + a - \varphi_0(y), \\
u_-(y) & = t - \varphi_0(y). \\
\}
\end{aligned}
\]

Combining cases B and C we find

\[
\theta_0^{-s(y_1, y_2)} T_k(G_1, G_2, Z) \leq 2||v||_\infty + \frac{2\theta^2}{1 - 2\theta} C_\mu |v||_\theta \leq M ||v||_\theta,
\]

for some \( M = M(\theta, C_\mu) \).

The indicator function \( 1_{S_{t,a}} \) means that it suffices to sum over \( Z \in \mathcal{P} \) with \(|h_{\tau_F(Z)} - t| < C\varphi_0 + 1 + a\). This gives, following the computation of (9.16) that

\[
|\mathcal{R}_{t,a} v|_{\theta_0}^* \leq \sum_{|h_{\tau_F(Z)} - t| < C\varphi_0 + 1 + a} C_p \mu(Z) M ||v||_\theta
\]

\[
\leq C_p M ||v||_\theta \mu(y \in Y : |\tau_F(y) - \eta(t)| \leq C'),
\]

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so by (9.5) and combining with the estimate from Step I, we get
\[
\int_0^\infty \sigma^T \|R_{\sigma,a}\|_{\mathcal{B} \to \mathcal{B}_0} \, d\sigma \ll 2C'C_p h(0)^T (M+1) \sum_n n^T \mu(y \in Y : \tau_F(y) = n) < \infty,
\]
as required.

10 Example: Semiflows over analytic maps: unbounded \(\tilde{h}\)

The difficulty in checking condition (H4) for unbounded \(\tilde{h}\) is that on the one hand \(\|R_{t,a}\|\) needs to be proportional to \(\mu_\Phi(S_{t,a})\) where \(S_{t,a}\) as in (9.6), which we address by letting \(\|\|_{\mathcal{B}}\) involve integrals over “diagonal multivalued curves”; on the other hand \(\mathcal{B}\) needs to be embedded in \(L^\infty(\mu_\Phi)\) (and hence \(\|v\|_\infty \ll \|v\|_{\mathcal{B}}\)). Therefore we resort to piecewise analytic Markov maps with a class of observables \(v\) that are complex analytic in both directions, i.e., there is \(\rho > 0\) and complex \(\rho\)-neighborhoods \(Y_\rho\) in \(C\) of the real interval \(Y\), and \([0,1)_{\rho}\) in \(C\) of \([0,1),\) such that for each \(u \in [0,1),\) \(v(\cdot,u)\) is complex analytic on \(Y_\rho\) and vice versa for each \(y \in [0,1),\) \(v(y,\cdot)\) is complex analytic on \([0,1)_{\rho}\). We call this class \(\mathcal{B} = \mathcal{B}(Y_\rho)\), defining its norm in Section 10.1 below. In addition to properties (1)-(3)\(^3\) in Section 9.1, we pose the following extra conditions:

(4') \(\tau_F\) satisfies the tail condition

\[
\mu(y \in Y : \tau_F(y) = n) = \begin{cases} 
O(n^{-(\beta+1)}), & \text{if } 1 < \beta, \\
\ell(n)n^{-(\beta+1)}, & \text{if } 0 < \beta \leq 1,
\end{cases}
\]

(10.1)

for some slowly varying function \(\ell\). By the argument [20, Proposition 2.6], the same tail condition holds for \(\mu(y \in Y : \varphi_0(y) \in [h_n, h_n+1])\).

(5) To make \(\mathcal{B}(Y_\rho)\) invariant under the transfer operator associated with \(\Phi\), the inverse branches of \(F : Y \to Y\) are assumed to be complex analytic as well. That is, for each \(Z \in \mathcal{P}\) we can extend the inverse branches \(F_{Z}^{-1} : Y \to Z\) to complex analytic maps \(F_{Z}^{-1} : Y_\rho \to Z_\rho\), where the \(Z_\rho\) are appropriate neighborhoods of \(Z\) in \(C\). (In the \(u\)-direction, \(\Phi^{-1}\) is clearly analytic, so we don’t need extra assumptions there.)

(6) The parameter \(\theta \in (0,1)\) is such that

\[
\theta^{-1/\epsilon'} \leq \inf_{Z \in \mathcal{P}} \inf_{y_1 \neq y_2 \in Z} \frac{|Fy_1 - Fy_2|}{|y_1 - y_2|}
\]

for some \(\epsilon' \in (0,1 - \frac{1}{p})\), where \(p > 1\) is such that \(\tilde{h} = \varphi_0^\gamma \in L^p(\mu_\Phi)\) as in (H1).

(7) We restrict \(\mathcal{B}(Y_\rho)\) to those \(v\) which satisfy

\[
v(y,0) = v(y,1) \quad \text{for all } y \in Y.
\]

\(^3\)but the second half of (9.2) and (9.3) are not necessary
10.1 The space $\mathcal{B}(\tilde{Y}_\rho)$ with norm $\|\|_B = ||^*_\rho + ||^*_\infty$.

We define $\mathcal{G}_0$ as in Section 9.2 (except that the second parts of equations (9.2) and (9.3) are not needed in this section. Let $\mathcal{G} = \bigcup_{r \geq 1} \mathcal{G}_r := \bigcup_{r \geq 1} \Phi^{-r}(\mathcal{G}_0)$ where we emphasize that the union starts at $r \geq 1$, so $r(G) \geq 1$ for all $G \in \mathcal{G}$. Define the seminorm for $v \in \mathcal{B}(\tilde{Y}_\rho)$

$$|v|^*_{\rho} = \sup_{G_1, G_2 \in \mathcal{G}} \theta^{-s(y_1,y_2)} \int_{Y} |v(y, u_1(y)) - v(y, u_2(y))| d\mu(y),$$

and weak norm

$$\|v\|_\infty^* = \sup_{G \in \mathcal{G}} \int_{Y} |v(y, u)| d\mu(y).$$

The norm $\|v\|_B = |v|^*_{\rho} + \|v\|_\infty^*$ will then make $\mathcal{B}(\tilde{Y}_\rho)$ into a Banach space. The choice of bi-analytic functions ensures that $\|v\|_\infty^*$ is actually equivalent to $\|v\|_\infty$.

**Lemma 10.1** There is $C_\rho$ such that for all $v \in \mathcal{B}(\tilde{Y}_\rho)$

$$\frac{1}{C_\rho} \|v\|_\infty \leq \|v\|_\infty \leq C_\rho \|v\|_\infty^*$$

and

$$\|\partial v / \partial u\|_\infty \leq \frac{1}{\rho} \|v\|_\infty$$

for all $v \in \mathcal{B}$.

**Proof** Formula (10.6) follows directly from the Cauchy formula $\frac{\partial v(y,u)}{\partial u} = \frac{1}{2\pi i} \int_{\Gamma} \frac{v(y,\zeta)}{(\zeta-u)^3} d\zeta$ by taking $\Gamma$ a circle of radius $\rho$ around $u$.

The first inequality of (10.5) follows by taking $C_\rho = |Y|$.

The other inequality means roughly that $v$ and is not disproportionately large on small sets. To prove the inequality, let $(y_0, u_0)$ be such that $|v(y_0, u_0)| = \|v\|_\infty$. If $|y-y_0| \leq A := \frac{\|v\|_\infty}{\frac{1}{3} \|v\|_\infty}$, then $|v(y_0, u_0) - v(y, u_0)| \leq |y_0 - y| \|\partial v / \partial y\|_\infty \leq \frac{1}{3} \|v\|_\infty$. Similarly, if $|u-u_0| \leq B := \frac{\|v\|_\infty}{\frac{1}{3} \|v\|_\infty}$, then $|v(y, u_0) - v(y, u)| \leq |u_0 - u| \|\partial v / \partial u\|_\infty \leq \frac{1}{3} \|v\|_\infty$.

This implies that $|v(y, u)| \leq \frac{1}{3} \|v\|_\infty$ for all $(y, u)$ in the rectangle $([y_0-A, y_0+A] \cap Y) \times ([u_0-B, u_0+B] \cap [0, 1])$. If $G = \{(y, u(y)) \} \in Y \in \mathcal{G}$ is a curve through $(y_0, u_0)$, then (using Lemma 10.3 below)

$$\|v\|_\infty^* \geq \int_{Y} |v(y, u(y))| d\mu(y) \geq \frac{1}{3C_\mu} \|v\|_\infty \min(A, B / C_{\text{dis}}).$$

Using the Cauchy formula again, $A, B \geq \rho/3$. This gives $\|v\|_\infty \leq \frac{9C_\mu C_{\text{dis}}}{\rho} \|v\|_\infty^*$. 

10.2 The result for unbounded $\tilde{h}$

**Proposition 10.2** Every system satisfying conditions (♠) and (1)-(3) in Section 9.1 and $(\frac{4}{3})$-(7) in Section 10 for the above space $(\mathcal{B}(\tilde{Y}_\rho), \|\|_B)$ satisfies the conclusion of Theorem 5.2. and Theorem 5.2.
The proof consists of verifying the conditions required for Theorem 5.1. Condition (H0) is supplied by Lemma 4.4, and it does not rely on the Markov structure. Condition (H1) can freely be assumed since $h$ is bounded away from zero. Using this Diophantine assumption (♣) condition (H3) follows as in [20, Proposition 3.5 (a)]. The verification of (H2), (H4) and (H6) takes some more work; this will be carried out in the following subsections.

### 10.3 Verifying (H4)

Since computing the norm of $R_{t,a}$ will involve integration over preimage curves in $G$, the following property about the slope of multivalued curves $G \in G$ is necessary.

**Lemma 10.3** For every $r \geq 0$, $Z \in \mathcal{P}_r$ and $G = \{(y,u_j(y))_{y \in Y, 0 \leq j < 2^r} \in \mathcal{G}_r,$
\[ \frac{1}{2^r |Z| C_{\text{dis}}} \leq \left| \frac{du_j(y)}{dy} \right| \leq \frac{C_{\text{dis}}}{2^r |Z|}, \quad j = 0, \ldots, 2^r - 1, \]
whenever $\frac{du_j(y)}{dy}$ is defined at $y \in Z$.

**Proof** This is a consequence of distortion condition (9.1), which, combined with the Mean Value Theorem, implies that $|Y|/C_{\text{dis}} \leq |DF^r(y')||Z| \leq C_{\text{dis}}|Y|$. Indeed, if $G = \Phi^{-r}(G_0)$ is parametrized as $\{(y',u_j(y'))_{y' \in Y, j = 0, \ldots, 2^r - 1} \}$ and $y = F^r(y')$ is used to parametrize $G_0 = \{(y,u(y))_{y \in Y} = \{(F^r(y'),2^ru_j(y') \mod 1)_{y' \in Z}, \text{then we have} \}
\[ \frac{1}{|Y|} = \left| \frac{du(y)}{dy} \right| = 2^r \left| \frac{du_j(y')}{dy'} \right| \left| \frac{dy'}{dy} \right|. \]
This gives
\[ \left| \frac{du_j(y')}{dy'} \right| = \frac{|DF^r(y')|}{2^r |Y|} \leq \frac{C_{\text{dis}}}{2^r |Z|}, \]
and the lower bound follows in the same way. \hfill \blacksquare

**Proposition 10.4** Let $\mathcal{B}(\tilde{Y}_\rho)$ be the Banach space be equipped with the norm $\| \cdot \|_\mathcal{B} = \| \cdot \|_\rho^* + \| \cdot \|_\infty^*$ from (10.3) and (10.4). Assume (9.1) and tail condition (10.1), and let $0 < \epsilon' < \min\{1 - 1/p, 1 - \gamma\}$ as in property (6). Then
\[ \| R_{t,a} \|_\mathcal{B} \ll t^{-(1+\beta-\epsilon')} . \]

**Proof** We divide the 2-cylinders in $\tilde{Y}$ into three groups, and estimate $\| R_{t,a} v \|_\infty^*$, splitting the involved integrals according to these cases. The final estimate for $\| R_{t,a} v \|_\infty^*$ brings these three together in the form of the sum of convolutions. To estimate $| R_{t,a} v |^*_\mathcal{B}$, we split the involved integrals according to the same three cases, leading again to a final sum of convolutions. However, since we need compare the integrals along different (parallel) multivalued curves, the way these multivalued curves intersect $S_{t,a}$ requires a further subdivision into cases A, B and C.
Step I: Subdividing into Cases (1)-(3). Recall from (2.5) that \( \varphi(y, u) = \varphi_0(y) + \psi(y, u) \), where

\[
\psi(y, u) = \begin{cases} 
(2\tilde{h} \circ F(y) - \tilde{h}(y))u, & u \in [0, \frac{1}{2}); \\
(2\tilde{h} \circ F(y) - \tilde{h}(y))u - \tilde{h} \circ F(y), & u \in [\frac{1}{2}, 1].
\end{cases}
\]  

(10.8)

A 1-cylinder \( Z \) with \( \tau_F(Z) = n \) only contributes to the estimate for \( R_{t,a} \) if \( \psi(y, u) + h_n \approx t \) for some \( y \in Z, \ u \in [0, 1) \). On 2-cylinders \( W \subseteq \tilde{P}_2 \), \( 2\tilde{h}(Fy) - \tilde{h}(y) \) vary no more than \( 2C_{\varphi_0} \) due to (9.4) and (H1), i.e., \( \tilde{h} = \varphi_0^\gamma \). It helps to split the set \( \hat{Y} \) into three regions, each coming with a certain range of “allowed \( n = \tau_F(Z) \)” that contribute to the estimates for \( R_{t,a} \) for particular ranges for the value of \( 2\tilde{h}(Fy) - \tilde{h}(y) \), and we first take the range \( 0 \leq t < \frac{1}{2} \). Recall constant \( C_{\varphi_0} > a \) from property (3) in Section 9.

Case (1) \( |2\tilde{h}(Fy) - \tilde{h}(y)| \leq 4C_{\varphi_0} \). Here \( |t - h_n| \leq |\varphi - \varphi_0| + a + |\varphi_0 - h_n| \leq 3C_{\varphi_0} + a \), so there is \( C' = C'(a) \) such that \( |\eta(t) - C'| \leq n < |\eta(t) + C'| \).

Case (2) \( 2\tilde{h}(Fy) - \tilde{h}(y) < -4C_{\varphi_0} \). Hence there is a region \( W_0 \) of \( (y, u) \in W \) where \( \psi(y, u) < -C_{\varphi_0} \), and there \( t - h_n \leq (\varphi - \varphi_0) + (\varphi_0 - h_n) < 0 \). On the other hand, recalling from (H1) that \( \tilde{h} = \varphi_0^\gamma \), we have the lower bound \( t - h_n \geq (\varphi - \varphi_0) - C_{\varphi_0} - a \geq -\tilde{h}(y) - C_{\varphi_0} - a \geq -2h^\gamma_n \). Therefore \( h_n - 2h_n^\gamma \leq t < h_n \), so there is \( C' \) such that \( \eta(t) \leq n \leq |\eta(t) + C'^\gamma| \).

Case (3) \( 2\tilde{h}(Fy) - \tilde{h}(y) > 4C_{\varphi_0} \). Since now \( \psi(y, u) \) has no upper bound, the range of allowed \( n \) will be \( 1 \leq n \leq \eta(t) \).

For the range \( \frac{1}{2} \leq u < 1 \), we can make similar computations, but the effect will be that we replace \( \eta(t) \) in the above formulas by the larger value \( \eta(t + \tilde{h}(Fy)) \). This means that the estimates will improve compared to the range \( 0 \leq u < \frac{1}{2} \), so we will omit the computations for \( \frac{1}{2} \leq u < 1 \).

Step II: Estimates for \( \|R_{t,a}\|_\infty \). By (9.13), we have

\[
R_{t,a}v = \sum_{W \in \tilde{P}} \frac{1}{2} e^{p(y_W')} 1_{S_{t,a}}(y_W', u_W') v(y_W', u_W'),
\]

where \( (y_W', u_W') = \Phi^{-1}(y, u) \cap W \). Each \( W \in \tilde{P} \) has the form \( Z \times [0, \frac{1}{2}) \) or \( Z \times [\frac{1}{2}, 1) \) for \( Z \in \mathcal{P} \) with \( \tau_F(Z) = n \). So two such \( Ws \) are stacked vertically above the same \( Z \).

This means for the \( \| \cdot \|_\infty \)-norm

\[
\|R_{t,a}v\|_\infty = \sup_{G \subseteq \tilde{G}} \sum_{W \in \tilde{P} \cap G} \int_{Y} \frac{1}{2} e^{p(y_W')} 1_{S_{t,a}}(y_W', u_W') |v(y_W', u_W')| \, d\mu(y)
\]

\[
\leq \sup_{G \subseteq \tilde{G}} \sum_{Z \in \mathcal{P} \cap G} \int_{Z} \frac{1}{2} e^{p(y_Z')} 1_{S_{t,a}}(y_Z', u_Z') |v(y_Z', u_Z')| \, d\mu(y)
\]

\[
+ \frac{1}{2} \int_{Z} |v(y_Z', u_Z') + \frac{1}{2}| \, d\mu(y) \leq \|v\|_\infty \sup_{G \subseteq \tilde{G}} \sum_{Z \in \mathcal{P} \cap G} \int_{Z} \left(1_{S_{t,a}}(y_Z', u_Z') + \frac{1}{2}\right) \, d\mu(y),
\]

(10.9)
and by (10.5), we can bound \( \|v\|_{\infty} \leq C_{\mu}\|v\|_{*\infty}^* \). To estimate \( \int_Z 1_{S_{t,a}}(y', u(y')) \, d\mu(y') \) (and the estimate of \( \int_Z 1_{S_{t,a}}(y', u_1(y')) + \frac{1}{2} \, d\mu(y') \) goes along the same argument), we use the announced case distinction:

**Case (1)** For those \( Z' \in \mathcal{P}_2 \) contained \( Z \in \mathcal{P} \) with \( \tau_F(Z) = n \) satisfying \( |n - \eta(t)| \leq C' \), it suffices to estimate \( \sum_{Z' \in \mathcal{P}_2} \int_{Z'} 1_{S_{t,a}}(y', u(y')) \, d\mu(y') \leq \mu(Z) \).

**Case (2)** We need to consider those regions \( W_0 \subset Z' \times [0, \frac{1}{2}] \), \( Z' \in \mathcal{P}_2 \) contained in \( Z \) on which \( \inf_{(y,u) \in W_0} \psi(y,u) < -C_{\varphi_0} \). For fixed \( t \) and any such \( Z' \), we have \( |t - h_n| \leq |\varphi - \varphi_0| + |\varphi_0 - h_n| \leq \frac{1}{2} |2\tilde{h}(F_y) - \tilde{h}(y)| + C_{\varphi_0} \). This gives \( |2\tilde{h}(F_y) - \tilde{h}(y)| \geq |t - h_n| \). The sets \( S_{t,a} \cap (Z' \times [0,1]) \) are contained in horizontal strips of height \( \leq a(2\tilde{h}(F_y) - \tilde{h}(y) - 1) \leq a|t - h_n|^{-1} \). Since the preimage curve \( G' \) of \( G \) has \( r(G') \geq 2 \), \( G' \) intersects this strip transversally with a slope \( \geq 1/4C_{\text{dis}} |Z'| \) by (10.7). Therefore

\[
\int_{\cup_{\text{Case 2}, Z' \subset Z'}} 1_{S_{t,a}}(y', u(y')) \, d\mu(y') \leq a|t - h_n|^{-1} 4C_{\text{dis}} \mu(Z).
\]

**Case (3)** Consider those \( Z' \in \mathcal{P}_2 \) contained in \( Z \in \mathcal{P} \) with \( \tau_F(Z) = n \), on which \( 2\tilde{h}(F_y) - \tilde{h}(y) > 4C_{\varphi_0} \). For fixed \( t \), we have \( t < \varphi(y,u) = \varphi_0(y) + \psi(y,u) \leq h_n + C_{\varphi_0} + \tilde{h}(F_y) \), so that \( \tilde{h}(t - h_n) \leq (t - h_n) - C_{\varphi_0} \leq \tilde{h}(F_y) \).

Due to the distortion control of \( F \) of property (2) and the tail estimates of \( \varphi_0 = \tilde{h}^{1/\gamma} \) given in property (4), we can find a constant \( \tilde{C} \) such that

\[
\mu(y \in Z : k \leq \tilde{h}(F_y) < k + 1) \leq \mu(y \in Y : k \leq \tilde{h}(y) < k + 1) \mu(Z) \\
\leq \mu(y \in Y : k|1/\gamma | \leq \varphi_0(y) < (k + 1)^{1/\gamma}) \mu(Z) \\
\leq \tilde{C}((k^{1/\gamma} (1 + 1/\gamma)) \mu(Z).
\]

(10.10)

This implies that the 2-cylinders \( Z' \in \mathcal{P}_2 \), contained in \( Z \), on which \( k \leq \tilde{h}(F_y) < k + 1 \) with \( \lceil \frac{1}{2}(t - h_n) \rceil = k \), have combined measure \( \leq \tilde{C}|t - h_n|^{-(1 + 1/\gamma)} \mu(Z) \). Thus we obtain

\[
\int_{\cup_{\text{Case 3}, Z'}} 1_{S_{t,a}}(y', u(y')) \, d\mu(y') \leq \tilde{C}|t - h_n|^{-(1 + 1/\gamma)} \mu(Z).
\]

Combining Cases (1)-(3) and taking into account the allowed ranges of \( n = \tau_F(Z) \), gives

\[
\|R_{1_{S_{t,a} \cap (Z \times [0,1])}} v \|_{*\infty}^* \leq \left( \sum_{n = |\eta(t) + C'|}^{n(\eta(t) + C')} \ell(n)n^{-(\beta + 1)} \right) \mu(Z) \\
+ \left( \sum_{n = |\eta(t) + C'|}^{n(\eta(t) + C') + 1} \ell(n)n^{-(\beta + 1)} |t - h_n|^{-1} \right) \mu(Z) \\
+ \sum_{n = |\eta(t)|}^{n(\eta(t) + 1)} \ell(n)n^{-(\beta + 1)} \left( |t - h_n|^{-(\beta + 1)/\gamma} + \mu(Z) \right) \|v\|_{*\infty}^*
\]

and \( \|v\|_{\infty} \leq C_{\text{dis}} \|v\|_{*\infty}^* \) by (10.5). Recall from (10.1) that \( \mu(\cup_{\tau_F(Z) = n} Z) = \ell(n)n^{-(\beta + 1)} \). Therefore, summing over all \( Z \in \mathcal{P} \) gives the convolutions:

\[
\|R_{t,a} v\|_{*\infty}^* \leq \left( \sum_{n = |\eta(t) + C'|}^{n(\eta(t) + C') + 1} \ell(n)n^{-(\beta + 1)} + \sum_{n = |\eta(t)| + 1}^{n(\eta(t) + 1)} \ell(n)n^{-(\beta + 1)} |t - h_n|^{-1} \right) \|v\|_{*\infty}^*
\]

\[
+ \sum_{n = |\eta(t)|}^{\eta(t)} \ell(n)n^{-(\beta + 1)} \left( |t - h_n|^{-(\beta + 1)/\gamma} + \mu(Z) \right) \|v\|_{*\infty}^*.
\]
Step III: Estimates for $|R_{t,a}v|^*_p$. For the $||^*_p$-seminorm, we need to compare the integral of $v$ over preimage multivalued curves $G(y'_1)$ and $G(y'_2)$, which are the vertical translation of each other by $\frac{1}{2}u_{1,2}$. (Recall from (9.8) that $u_{1,2}$ is small if $s(y_1,y_2)$ is large.) We have to consider the intersections of these multivalued curves with $S_{t,a}$, and therefore it makes sense to subdivide the Cases (1)-(3) into subcases, see Figure 4.

![Figure 4: Schematic picture of cases A-C for $S_{t,a}$ intersecting a cylinder $W$.](image)

The set $S_{t,a} \cap W$ is bounded above and below by curves

$$
\begin{align*}
\{ u_+(y) &= \frac{t+a-\varphi_0(y)}{2h(Fy)-h(y)} \\
u_-(y) &= \frac{t-\varphi_0(y)}{2h(Fy)-h(y)}
\end{align*}
$$

provided the preimages $y'_1, y'_2 \in Z$. (Note that, assuming $y'_1 < y'_2$, the bottom piece of the multivalued curve $G(y'_1)$ may have to be paired to the top piece of the multivalued curve $G(y'_2)$). Due to our assumption (10.2), this pairing doesn’t create discontinuity problems.

Case A: Both $(y',u_1(y'))$ and $(y',u_2(y')) \notin S_{t,a}$. There is no contribution to the integral, so we can ignore this case.

Case B: Both $(y',u_1(y'))$ and $(y',u_2(y')) \in S_{t,a}$. In this case, by Lemmas 10.1 and 10.3,

$$
|v(y',u_1(y')) - v(y',u_2(y'))| \leq \left| \frac{\partial v}{\partial u} \right|_{\infty} \frac{u_{1,2}}{2} \leq \frac{C_p C_{dis}}{2\rho} \|v\|^*_p |y_1 - y_2|
$$

provided the preimages $y'_1, y'_2 \in Z$. (Note that, assuming $y'_1 < y'_2$, the bottom piece of the multivalued curve $G(y'_1)$ may have to be paired to the top piece of the multivalued curve $G(y'_2)$). Due to our assumption (10.2), this pairing doesn’t create discontinuity problems.

Case C: Only one of $(y',u_1(y'))$ and $(y',u_2(y')) \in S_{t,a}$. This applies to two intervals (one "on either side" of Case B) of length at most

$$
|y'_1 - y'_2| \leq C_\mu \mu(Z) |y_1 - y_2|,
$$

provided the preimages $y'_1, y'_2 \in Z$.

The $||^*_p$-seminorm of $R_{t,a}v$ takes a form similar to (10.9). Changing coordinates $y \to y' = F_{Z}^{-1}(y)$ gives:

$$
\begin{align*}
|R_{t,a}v|^*_p &= \sup_{y_1,y_2 \in Y} \theta^{-s(y_1,y_2)} \sup_{G_1,G_2 \in G} \sum_{W \in \mathcal{P}} \int_Y \frac{1}{2} \left| e^{p(y'_W)} 1_{S_{t,a}}(y'_W,u_1(y'_W))v(y'_W,u_1(y'_W)) - e^{p(y'_W)} 1_{S_{t,a}}(y'_W,u_2(y'_W))v(y'_W,u_2(y'_W)) \right| \, d\mu(y) \\
&\leq \sup_{y_1,y_2 \in Y} \theta^{-s(y_1,y_2)} \sup_{G_1,G_2 \in G} \frac{1}{2} \sum_{k=0}^{\infty} \sum_{Z \in \mathcal{P}} \int_Z \left| 1_{S_{t,a}}(y',u_1(y'))v(y',u_1(y')) - 1_{S_{t,a}}(y',u_2(y'))v(y',u_2(y')) \right| \, d\mu(y'),
\end{align*}
$$

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where the sum over \( k = 0, 1 \) refers to the two cylinders \( W \) stacked above the same \( Z \). (For simplicity of notation, we suppress this \( k \)-dependence in the parametrizations \( y \) of the multivalued curves.)

We will use the division into cases B and C for the \( \| \cdot \|_\theta \)-norm confined to each \( Z \in \mathcal{P} \) separately.

**Case (1)** For those \( Z' \in \mathcal{P} \) contained in \( Z \in \mathcal{P} \) with \( \tau_F(Z) = n \) and \( |n - \eta(t)| \leq C' \), we have by (10.11) and (10.12):

\[
\int_{\cup Z' \cap \text{Case B}} + \int_{\cap Z' \cap \text{Case C}} |1_{S_{t,a}}(y', \tilde{u}_1(y'))v(y', \tilde{u}_1(y')) - 1_{S_{t,a}}(y', \tilde{u}_2(y'))v(y', \tilde{u}_2(y'))| \, d\mu(y') \\
\leq \frac{C_p C_{dis}}{2 \rho} \|v\|_\infty^s |y_1 - y_2| \mu(Z) + 2C'_\mu \mu(Z)|y_1 - y_2| \|v\|_\infty \\
\leq \left( \frac{C_{dis}}{2 \rho} + 2C'_\mu \right) \|v\|_\infty^s \theta^{s(y_1, y_2)} \mu(Z).
\]

**Case (2)** Again, we consider those regions \( W_0 \subset Z' \times [0, \frac{1}{2}] \), \( Z' \in \mathcal{P}_2 \) contained in \( Z \), on which \( \inf_{(y,u) \in W_0} \psi(y,u) < -C_{\psi} \). As before, the sets \( S_{t,a} \cap (Z' \times [0,1]) \) are contained in horizontal strips of height \( \leq a[2\tilde{h}(Fy) - \tilde{h}(y)]^{-1} \leq a|t - h_n|^{-1} \).

For the “Case B part” in \( Z' \), the multivalued curves \( G'(y') \) and \( G(y_2) \) cross \( S_{t,a} \cap (Z' \times [0,1]) \) with slope \( \geq 1/(2C_{dis}|Z'|) \) by Lemma 10.3. This means that \( Z' \cap \text{Case B} \) is an interval of length \( \leq 2C_{dis}|Z'|a|t - h_n|^{-1} \) and \( |Z'| \leq C'_\mu \mu(Z) \). Thus by (10.11), the integral over \( Z' \) results in

\[
\int_{Z' \cap \text{Case B}} |v(y', \tilde{u}_1(y')) - v(y', \tilde{u}_2(y'))| \, d\mu(y') \leq \int_{Z' \cap \text{Case B}} \frac{C_p C_{dis}}{2 \rho} \|v\|_\infty^s |y_1 - y_2| \, d\mu(y') \\
\leq \frac{aC'_\mu C_{dis}}{\rho} \|v\|_\infty^s |t - h_n|^{-1} \theta^{s(y_1, y_2)} \mu(Z').
\]

For the “Case C part”, the integration within \( Z' \) is over at most two separate curves with slope \( \geq 1/(2C_{dis}|Z'|) \) inside \( S_{t,a} \). Hence, we need to integrate \( |v(y', \tilde{u}_1(y'))| \) or \( |v(y', \tilde{u}_2(y'))| \) over intervals of length at most

\[
\min\{C'_\mu \mu(Z')|y_1 - y_2|, 2C_{dis}|Z'|a|t - h_n|^{-1}\}
\leq C'_\mu (1 + 2C_{dis})\mu(Z')|y_1 - y_2|^{-1} a|t - h_n|^{-1} \theta^{(s(y_1, y_2))}
\leq C'_\mu (1 + 2C_{dis})\mu(Z') \theta^{s(y_1, y_2)} a|t - h_n|^{-1} \theta^{(s(y_1, y_2))},
\]

where \( \epsilon' \) is given in property (6). By (10.5),

\[
\int_{Z' \cap \text{Case C}} |1_{S_{t,a}}(y', \tilde{u}_1(y'))v(y', \tilde{u}_1(y')) - 1_{S_{t,a}}(y', \tilde{u}_2(y'))v(y', \tilde{u}_2(y'))| \, d\mu(y') \\
\leq C'_\mu C_p (1 + 2C_{dis})\|v\|_\infty^s a|t - h_n|^{-1} \theta^{(s(y_1, y_2))} \mu(Z').
\]

Summing up over all \( Z' \in \mathcal{P}_2 \) of this type contained in \( Z \) of Case (2), we get

\[
\int_{Z} |1_{S_{t,a}}(y', \tilde{u}_1(y'))v(y', \tilde{u}_1(y')) - 1_{S_{t,a}}(y', \tilde{u}_2(y'))v(y', \tilde{u}_2(y'))| \, d\mu(y') \\
\leq C'_\mu C_p (1 + 2C_{dis}) a|t - h_n|^{-1} \theta^{(s(y_1, y_2))} \|v\|_\infty^s \mu(Z).
\]
Case (3) Consider those $Z' \in \mathcal{P}_2$ contained in $Z \in \mathcal{P}$ with $\tau_F(Z) = n$, on which $2\tilde{h}(Fy) - \tilde{h}(y) > 4C_{\varphi_0}$. For fixed $t$, we have again $t - h_n \leq \tilde{h}(Fy)$. As in the argument leading up to (10.10), the 2-cylinders $Z'$ where $k \leq \tilde{h}(Fy) < k + 1$ have combined measure $\leq C't - h_n|^{-\beta(1+\gamma)/\gamma} \mu(Z)$. For the “Case B part” of the integral, we therefore obtain by (10.11)

$$
\int_{Z' \cap \text{Case B}} |v(y', \underline{u}_1(y')) - v(y', \bar{u}_2(y'))| \, d\mu(y') \leq \int_{Z' \cap \text{Case B}} \frac{C_{\rho}C_{\text{dis}}}{2^\rho} \|v||_{\infty} |y_1 - y_2| \, d\mu(y') \leq \frac{C_{\rho}C_{\text{dis}}}{2^\rho} \|v||_{\infty} \tilde{C}|t - h_n|^{-1+\beta/\gamma} \theta^{\gamma(y_1,y_2)} \mu(Z).
$$

For the Case C part, we need to integrate $|v(y', \underline{u}_1(y'))|$ or $|v(y', \bar{u}_2(y'))|$ over at most two separate intervals of length at most

$$
\min\{C_{\mu} \mu(Z') |y_1 - y_2|, \tilde{C} \mu(Z') |t - h_n|^{-1+\beta/\gamma}\} \leq \max\{C_{\mu}, \tilde{C}\} \mu(Z') |y_1 - y_2|^{-\beta/\gamma}(1-\epsilon') \leq \max\{C_{\mu}, \tilde{C}\} \mu(Z') \theta^{\gamma(y_1,y_2)} |t - h_n|^{-1+\beta/\gamma}(1-\epsilon').
$$

Therefore

$$
\int_{Z' \cap \text{Case C}} |1_{\{t, \alpha\}}(y', \underline{u}_1(y'))v(y', \underline{u}_1(y')) - 1_{\{t, \alpha\}}(y', \bar{u}_2(y'))v(y', \bar{u}_2(y'))| \, d\mu(y') \leq \max\{C_{\mu}, \tilde{C}\} C_{\rho} \|v||_{\infty} |t - h_n|^{-1+\beta/\gamma}(1-\epsilon') \theta^{\gamma(y_1,y_2)} \mu(Z).
$$

Finally, combining Cases (1)-(3) with all the constants $a, C_{\text{dis}}, \ldots, C_{\rho}$ replaced by the notation $\ll$, and taking into account the allowed ranges of $n = \tau_F(Z)$, we obtain

$$
|R_{1_{\{t, \alpha\}}(Z \times \{0,1\})^2} \|_{\bar{O}} \ll \left(1_{\{n = \eta(t)\leq e^{-\gamma'}\}} \mu(Z) + 1_{\{n = \eta(t) < \eta(t) + C'\gamma\}} C'|t - h_n|^{-(1-\epsilon')} \mu(Z) + 1_{\{1 \leq n \leq \eta(t)\}} C'|t - h_n|^{-1+\beta/\gamma(1-\epsilon')} \mu(Z) \right) \|v||_{\infty}.
$$

Recall that $\mu(\cup_{\tau_F(Z) = n} Z) = \ell(n)n^{-(\beta+1)}$. Therefore, summing over all $Z \in \mathcal{P}$ gives:

$$
|R_{t, \alpha} v \|_{\bar{O}} \ll \left( \sum_{n = \eta(t) + C'} \ell(n)n^{-(\beta+1)} + \sum_{n = \eta(t) + 1} \ell(n)n^{-(\beta+1)} |t - h_n|^{-(1-\epsilon')} \right) \|v||_{\infty} + \sum_{n = 1}^{\eta(t)} \ell(n)n^{-(\beta+1)} |t - h_n|^{-1+\beta/\gamma(1-\epsilon')} \|v||_{\infty}.
$$

Combining the above estimates for $\|v||_{\bar{O}}$ and $\|v||_{\infty}$ gives the bound $\|R_{t, \alpha} v \|_{\bar{O}} \ll t^{-(1+\beta-\epsilon')}$ as required.
10.4 Verifying (H2)

First we show that the twisted transfer operator \( \hat{R}(s)v = R(e^{-s\varphi}v) \), \( \Re s \geq 0 \), satisfies the Lasota-Yorke inequality. The difficult part is the behavior of \( R \) under \( \|v\|_\theta^* \), and the discontinuities in the twist \( e^{-s\varphi} \) that it comes with.

Lemma 10.5 (LY(1 \( \|B, \|_{L^\infty(\mu_B)} \))) Assume that \( \Re s \geq 0 \). There exists constants \( K_1, K_2 > 0 \) such that

\[
|\hat{R}^n(s)v|_\theta^* \leq K_1 \theta^n|v|_\theta^* + K_2 |s| \|v\|_\infty, \quad \text{and} \quad \|\hat{R}(s)v\|_\infty^* \leq \|v\|_\infty^*,
\]

(10.13)

for all \( v \in B \) satisfying (10.5) and \( n \in \mathbb{N} \).

Proof

Let \( W = Z \times [j2^{-n}, (j + 1)2^{-n}) \in \hat{P}_n \), with \( Z \in \mathcal{P}_n \) defined in Section 10.1.

Let multivalued curve \( G \in \mathcal{G} \) with \( r = r(G) \) be given. The translated multivalued curves \( G(y_1), G(y_2) \) are parametrized as \( (y, u_1(y)) \) and \( (y, u_2(y)) \). For \( (y, u_j(y)) \subset \hat{Y}, \ j = 1, 2, \) and \( W \in \hat{P}_n \), we will use \( (y_W, u_j(y_W)) \) to denote the points in \( \Phi^{-n}(y, u_j(y)) \cap W \). Also let \( \varphi_n = \sum_{j=0}^{n-1} \varphi \circ \Phi^j \) and analogously \( \varphi_{0,n} = \sum_{j=0}^{n-1} \varphi_0 \circ F^j \) and \( \psi_n = \sum_{j=0}^{n-1} \psi \circ \Phi^j \), where \( \psi = \varphi - \varphi_0 \) as in (10.8).

With this notation, we obtain

\[
|\hat{R}^n(s)v|_\theta^* = \sup_{G_1,G_2 \in \mathcal{G}} \theta^{-s} \int_Y \sum_{W \in \mathcal{P}_n} \frac{1}{2^n} |e^{p_n(y_W)} - s\varphi_{0,n}(y_W)|\left| e^{-s\psi_n(y_W, u_1(y_W))}v(y_W, u_1(y_W)) - e^{-s\psi_n(y_W, u_2(y_W))}v(y_W, u_2(y_W)) \right| d\mu(y)
\]

\[
\leq \sup_{G_1,G_2 \in \mathcal{G}} \theta^{-s} \int_Y \sum_{W \in \mathcal{P}_n} \frac{1}{2^n} |e^{p_n(y_W)}| \left| e^{-s\psi_n(y_W, u_1(y_W))} - e^{-s\psi_n(y_W, u_2(y_W))} \right| d\mu(y)
\]

\[
+ |e^{-s\varphi_n(y_W, u_1(y_W))} - e^{-s\varphi_n(y_W, u_2(y_W))}| \left| v(y_W, u_1(y_W)) - v(y_W, u_2(y_W)) \right| d\mu(y)
\]

\[
= \sup_{G_1,G_2 \in \mathcal{G}} \theta^{-s} \int_Y (I_1 + I_2).
\]

To estimate \( I_1 \), we majorize \( |e^{-s\varphi_n(y_W, u_1(y_W))}| \) by 1 (possible because \( \Re s \geq 0 \)). Using the change of coordinates \( y \rightarrow y' = y_W \) (so \( e^{p_n(y_W)} d\mu(y) = d\mu(y') \)) we obtain

\[
I_1 = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \sum_{Z \in \mathcal{P}_n} \int_Z \left| e^{-s(\psi_n(y', u_1(y')) - \psi_n(y', u_2(y')))} - 1 \right| \left| v(y', u_1(y')) \right| d\mu(y')
\]

\[
\leq |s| \|v\|_\infty \frac{1}{2^n} \sum_{k=0}^{2^n-1} \sum_{Z \in \mathcal{P}_n} \int_Z \left| \psi_n(y', u_1(y')) - \psi_n(y', u_2(y')) \right| d\mu(y'),
\]

(10.14)

where the sum over \( k \) refers to the \( 2^n \) cylinders \( W \in \hat{P}_n \) stacked over a single \( Z \in \mathcal{P}_n \). (We suppress this \( k \)-dependence in our notation \( u_1(y') \) and \( u_2(y') \).)
To estimate this integral, we pair pieces $Q_1$ of $G(y'_1)$ with pieces $Q_2$ of $G(y'_2)$ if they are vertical translations of one another by $2^{-n}u_{1,2}$. The discontinuity of $\psi_n$ (or rather the discontinuity of $\psi$ appearing at $\{u = \frac{1}{2}\}$, where there is a jump of $\tilde{h} \circ F$) causes some complications, which we will deal with below. But if $\Phi^j(Q_1)$ and $\Phi^j(Q_2)$ are not separated by the line $\{u = \frac{1}{2}\}$, then

$$|\psi \circ \Phi^j(y', u_1(y')) - \psi \circ \Phi^j(y', u_2(y'))| \leq |2\tilde{h} \circ F^{j+1}(y') - \tilde{h} \circ F^j(y')|2^{-n}u_{1,2}.$$  

This gives an estimate or the “continuous part” of (10.14), using (9.7) with $r = r(G)$, as

$$\frac{1}{2^n} \sum_{k=0}^{2^n-1} \sum_{Z \in P_n} \int_{Z \cap \text{continuous}} |\psi_n(y', u_1(y')) - \psi_n(y', u_2(y'))| \, d\mu(y')$$

$$\leq 2^n \sum_{k=0}^{2^n-1} \sum_{Z \in P_n} \sum_{j=0}^{n-1} \int_Z |2\tilde{h} \circ F^{j+1}(y') - \tilde{h} \circ F^j(y')|2^{-n}u_{1,2} \, d\mu(y')$$

$$\leq 2^n \sum_{j=0}^{n-1} 2^{-n}u_{1,2} \int_Y (2\tilde{h} \circ F^{j+1}(y') + \tilde{h} \circ F^j(y')) \, d\mu(y') \leq \frac{3\|\tilde{h}\|_{L^1(\mu)} \theta(y_1,y_2)}{|Y|},$$

by $F$-invariance of $\mu$ and property (a) to bound $u_{1,2} \leq 2^{-r}|Y|^{-1}|y_1 - y_2| \leq |Y|^{-1} \theta(y_1,y_2)$.

Discontinuities in $\psi_n(y,u)$ occur when $\Phi^j(y,u)$ lies on the horizontal line $\{u = \frac{1}{2}\}$ for some $j \leq n$, and the jump in the value of $\psi \circ \Phi^j$ is $\tilde{h} \circ F(F^j(y))$. There is also a difference in value between $\psi \circ \Phi^j(y,0)$ with $\psi \circ \Phi^j(y,1)$. Therefore, to estimate the “discontinuous part” of (10.14), we need to integrate over all pieces $Q_1$ of $G(y'_1)$ and pieces $Q_2$ of $G(y'_2)$ that:

(i) touch a line $\{u = a2^{-(j+1)}\}$, $0 \leq j < n$ and odd integer $a$, from opposite sides (because here the discontinuity line $\{u = \frac{1}{2}\}$ is reached after $j$ iterates), or

(ii) $Q_1$ touches $\{u = 0\}$ while $Q_2$ touches $\{u = 1\}$ (because these are the only pieces of $G(y'_1)$ and $G(y'_2)$ that are not the vertical translations of each other by $2^{-n}u_{1,2}$).

There are precisely $2^k$ points $u = a2^{-(k+1)}$, $a$ odd, such that $2^ku \mod 1 = \frac{1}{2}$. For each of these, the pieces $Q_1, Q_2$, touching the line $\{u = a2^{-(k+1)}\}$ contribute to the discontinuous part for iterate $j = k, \ldots, n-1$, namely for $n-k$ iterates.

Let $U = [y_1, y_2]$ and $U_j = F^{-j}(U)$; these are unions of intervals of combined measure $\mu(U)$. Since $\tilde{h} \in L^p(\mu)$, the Hölder inequality implies that the integral of $\tilde{h} \circ F$ over any set of measure $\mu(U)$ is at most $\|\tilde{h}\|_{L^p(\mu)} \mu(U)^{1 - \frac{1}{p}} \leq \|\tilde{h}\|_{L^p(\mu)} C_\mu^{1 - \frac{1}{p}} |y_2 - y_1|^{1 - \frac{1}{p}} \leq \|\tilde{h}\|_{L^p(\mu)} C_\mu^{1 - \frac{1}{p}} \theta(y_1,y_2)$ by the choice of $\theta$ in property 6. This gives

$$\int_{U_{n,j}} \tilde{h} \circ F^{j+1} \, d\mu(y') = \int_Y (1_{U_{n,j}} \tilde{h} \circ F) \circ F^j(y') \, d\mu(y')$$

$$= \int_{U_{n,j}} \tilde{h} \circ F(y) \, d\mu(y) \leq \|\tilde{h}\|_{L^p(\mu)} C_\mu^{1 - \frac{1}{p}} \theta(y_1,y_2).$$
Therefore, the “discontinuous part” of (10.14) is bounded as
\[
\frac{1}{2^n} \sum_{k=0}^{2^n-1} \sum_{Z \in \mathcal{P}_n} \int_{Z \cap \text{discontinuous}} |\psi_n(y', u_1(y')) - \psi_n(y', u_2(y'))| \, d\mu(y') \leq \sum_{j=0}^n \frac{1}{2^n} ((n - j)2^j + n) \int_{U_n} \tilde{h} \circ F^{j+1}(y') \leq 2 \|\tilde{h}\|_{L^p(\mu)} C_{\mu}^{-\frac{1}{p}} \theta^s(y_1, y_2).
\]
Combining the two, we find by (10.5)
\[
I_1 \leq |s| C_{\rho} \|v\|_\infty^s \left( 3 \|\tilde{h}\|_{L^1(\mu)}/|Y| + 2 \|\tilde{h}\|_{L^p(\mu)} C_{\mu}^{-\frac{1}{p}} \right) \theta^s(y_1, y_2).
\]
To estimate $I_2$, we majorize $|e^{-s\varphi_n(y_W, u_2(y_W))}|$ by 1, and then we have the difference of the integrals of $v$ taken over the preimage curves $\Phi^{-n}(G(y_1))$ and $\Phi^{-n}(G(y_2))$. By the definition of $\|v\|_\theta$, this is less than $\theta^{n+s(y_1, y_2)} |v|_\theta^s$.

The $\|v\|_\infty^\ast$-norm poses no problem:
\[
\|\tilde{R}(s)v\|_\infty^\ast = \sup_{G \in \mathcal{G}} \sum_{W \in \mathcal{P}} \frac{1}{2} \int Z |e^{p(y') - s\varphi(y', u(y'))}| |v(y', u(y'))| \, d\mu(y) \leq \sup_{\Phi^{-1}G \in \mathcal{G}} \sum_{y' \in Y} |v(y', u(y'))| \, d\mu(y') \leq \|v\|_\infty^\ast,
\]
because the sum of integrals over all $W \in \tilde{\mathcal{P}}$, after the change of coordinates $y \rightarrow y' = y_W$, amounts to integrating over a single $\Phi$-preimage multivalued curve as in (9.7).

\begin{remark}
For every $\theta \in (\frac{1}{2}, 1)$ and using (10.6), we have
\[
\sup_{u_1, u_2 \in [0,1]} \frac{|v(y, u_1) - v(y, u_2)|}{\theta^s((y, u_1), (y, u_2))} \leq \sup_{u_1, u_2 \in [0,1]} \frac{|v(y, u_1) - v(y, u_2)|}{|u_1 - u_2|} \leq \|\frac{\partial v}{\partial u}\|_\infty \leq \frac{1}{\rho} \|v\|_\infty,
\]
uniformly in $y$, where the separation time $\tilde{s}((y, u_1), (y, u_2))$ is taken w.r.t. $\tilde{\mathcal{P}}$. Together with (10.5), this means that the norm $\|v\|_\theta = |v|_\theta + \|v\|_\infty$ is equivalent to $\|v\|_B$.

The following can be proved directly for the norms ($\|\cdot\|_B$, $\|\cdot\|_\infty^\ast$) by means of the Arzela-Ascoli Theorem. But passing to the equivalent pair of norms ($\|\cdot\|_\theta$, $\|\cdot\|_\infty$), we can also refer to known results (for instance [20, Proposition 3.5(b)]) to conclude that the Theorem of Ionescu-Tulcea and Marinescu applies. That is, there is a uniform constant $K$ such that
\[
\|\tilde{R}(s)(v - \int v \, d\mu_\Phi)\|_B \leq K \theta^n \|v\|_B,
\]
and in particular, $\tilde{R}(s)$ acts quasi-compactly on $(B, \|\cdot\|_B)$. Since $(\tilde{\mathcal{Y}}, \tilde{\mathcal{P}}, \mu_\Phi)$ is ergodic, the eigenvalue 1 of $R = \tilde{R}(0)$ is simple. This verifies (H2) ii).
10.5 Verifying (H6): the Dolgopyat type inequality

For the verification of hypothesis (H6) we refer to [20, Lemma 5.2]. However, let us sketch the argument for obtaining the weak form of the Dolgopyat type inequality. For details we refer to [16] (see also [20] for a different setting of the arguments in [16]).

For \( b \in \mathbb{R} \), define \( M_b : L^\infty(\mu_\Phi) \to L^\infty(\mu_\Phi), M_b v = e^{ib\psi} v \circ \Phi \). We say that there are \textit{approximate eigenfunctions} on a subset \( \mathcal{Z} \subset \tilde{Y} \) if there exist constants \( \alpha > 0 \) arbitrarily large, \( \beta > 0 \) and \( C \geq 1 \), and sequences \( |b_k| \to \infty, \psi_k \in [0, 2\pi) \), \( \theta \)-Hölder \( u_k \) with \( |u_k| \equiv 1 \), such that setting \( n_k = \lfloor \beta \ln |b_k| \rfloor \),

\[
|M_{n_k} u_k (\tilde{y}) - e^{i\psi_k} u_k (\tilde{y})| \leq C |b_k|^{-\alpha},
\]

for all \( \tilde{y} \in \mathcal{Z} \) and all \( k \geq 1 \). A subset \( \mathcal{Z}_0 \subset \tilde{Y} \) is called a \textit{finite subsystem} if \( \mathcal{Z}_0 = \bigcap_{n \geq 0} \Phi^{-n} \mathcal{Z} \) where \( \mathcal{Z} \) is a finite union of partition elements \( W \in \mathcal{P} \).

By [8, Section 13], the Diophantine condition \((\clubsuit)\) ensures that there exists a finite subsystem such that there are no approximate eigenfunctions on \( \mathcal{Z}_0 \). Together with [16, Lemma 3.13], which can be applied to our setting because \( \|\|_B \) and \( \|\|_\theta \) are equivalent (see Remark 10.6) and because the technical estimates in [20, Lemma 4.1] hold with exactly same proof since \( \Phi \) is Gibbs Markov, this implies that (H6) holds.

Acknowledgments. We would like to thank Ian Melbourne for pointing out in an early version of this manuscript that any version of (H4) cannot hold in standard Banach spaces without restrictive assumptions on the induce time function as well as for useful suggestions on a late version. We would also like to thank Carlangelo Liverani for useful discussions on the topic of the paper in a very early stage of this project.

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