AMENABLE OF UNIVERSAL 2-GRIGORCHUK GROUP

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Abstract. We consider the universal Grigorchuk 2-group, i.e., the group such that every Grigorchuk 2-group is a quotient. We show that this group has a nice universal representation in the group of all functions \( \text{Func}(\{0,1,2\}^\mathbb{N}, \text{Aut}(T_2)) \), where \( T_2 \) is a group of automorphisms of the binary tree. Finally, we prove that this universal Grigorchuk 2-group is amenable. The proof is an application of the “Münchhausen trick” developed by V. Kaimanovich.

1. Introduction.

Consider a class of Grigorchuk 2-groups \( G_\omega \). These groups are parametrized by infinite sequences \( \omega \in \{0,1,2\}^\mathbb{N} \). They have subexponential growth. They have a nice generating set, i.e., they are all generated by 4-elements \( a_\omega, b_\omega, c_\omega, d_\omega \). In this short note I would like to consider a universal Grigorchuk 2-group \( Gr_2 \), which is a quotient of the free group \( F_4 = \langle a, b, c, d \rangle \) by a set of words in \( a, b, c, d \) such that these word are trivial for every Grigorchuk group \( G_\omega \).

Let \( \pi_\omega : F_4 \to G_\omega \) be a map sending generator \( a, b, c, d \) to \( a_\omega, b_\omega, c_\omega, d_\omega \). Then \( Gr_2 \) is defined as

\[
Gr_2 = F_4 / (\cap_\omega \text{Ker}(\pi_\omega))
\]

I will show that \( Gr_2 \) has a nice representation in the group of all functions from the set \( \{0,1,2\}^\mathbb{N} \) to the group of automorphisms of the binary tree \( \text{Aut}(T_2) \). This group is self-similar. It is self-similar with respect to an isomorphic group. Because of this technicality I will introduce the notion of a nested structure. This is essentially an embedding of \( Gr_2 \) to the matrix group on the group algebra of \( Gr_2 \).

The main result of this note is

Theorem 1.1. \( Gr_2 \) is amenable

To prove this I invoke a method suggested by V. Kaimanovich [K04] called “Münchhausen trick”. The proof is quite simple and follows along the line of [K04].

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3. Nested structures.

Let $H$ be a group and $X$ be a set. Then the permutation wreath product of $H$ and $X$ is $H^X \ltimes Sym(X)$, where the product is defined as

$$(g^x, \sigma)(h^y, \tau) = ((gh^y)^x, \sigma \tau).$$

**Definition:** We say $G$ is nested inside $H$ with finite set $X$ if there exists a group homomorphism $\phi : G \to \tilde{G} = H^X \ltimes Sym(X)$, such that the image of the induced map $P_\phi : G \to \tilde{G} \to Sym(X)$ is irreducible. We denote this structure $(G, H, X, \phi)$.

Now if $G$ is nested in $H$ with set $X$, I abuse the notation and identify $G$ with its image and write $g = (h_g, \sigma_g) \in \tilde{G}$, where $h_g \in H^X$ and $\sigma_g \in Sym(X)$.

4. Random walks on groups with nested structures.

Next two sections are taken directly from [K04]. However I reproduce them here for the reader’s convenience, and because I will need to make only a few tiny modifications.

4.1. Preliminaries. Recall that the (right) random walk on a countable group $G$ determined by a probability measure $\mu$ is the Markov chain with the state space $G$ and the transition probabilities

$$\pi_g(k) = \mu(g^{-1}k)$$

which are equivariant with respect to the left action of the group on itself. In other words, for a point $g$ the random walk moves at the next moment of time to the point $gk$, where the random increment from $k$ is chosen according to the distribution $\mu$. We shall use for this description of transition probabilities of the random walk $(G, \mu)$ the notation

$$g_k \sim \mu.$$ (1)

Thus, if the random walk starts at the moment 0 from a point $g_0$, then its position at time $n$ is

$$g_n = g_0k_1k_2 \ldots k_n,$$

where $k_i \in G$ is a Bernoulli sequence of independent $\mu$-distributed increments.
For a nested structure \((G, H, X, \phi)\) and a probability measure \(\mu = \mu_G\) on \(G\) denote by
\[
\mu_X = P_\phi(\mu) \quad \mu_\tilde{G} = \phi(\mu),
\]
the images of \(\mu\) under the homomorphisms \(\phi : G \to \tilde{G}\) and \(P_\phi : G \to Sym(X)\).

Then the sample paths of the random walk \((\tilde{G}, \mu_\tilde{G})\) (the image of the random walk \((G, \mu)\) under the embedding \(\phi : G \to \tilde{G}\)) starting from the identity are
\[
(g_n, \sigma_n) = (h_1, \tau_1)(h_2, \tau_2) \ldots (h_n, \tau_n),
\]
where \((h_i, \tau_i)\) are the \(\mu_\tilde{G}\)-distributed increments of the random walk, so that by definition,
\[
(2) \quad g_{n+1} = g_n h_{n+1}^{\sigma_n}, \quad \sigma_{n+1} = \sigma_n \tau_{n+1}.
\]

### 4.2. Reduction to a random walk with internal degrees of freedom.

In particular, formula (2) implies that for any fixed \(x \in X\)
\[
(3) \quad g_{n+1}^x = g_n^x (h_{n+1}^{\sigma_n})^x = g_n^x h_{n+1}^{x, \sigma_n},
\]
so that at any given time \(n\) the transition law from \(g^x\) to \(g_{n+1}^x\) is determined just by the values \(g^x\) and \(\sigma_n(x)\). Therefore the image
\[
(4) \quad (g_n^x, x, \sigma_n)
\]
of the original random walk \((\tilde{G}, \mu_\tilde{G})\) under projection
\[
(5) \quad \Pi_x : \tilde{G} \to H \times X, \quad (g, \sigma) \mapsto (g^x, x, \sigma)
\]
is also a Markov chain.

Formula (3) shows that in the notation (1) transition of the quotient chains (2) on \(H \times X\) are
\[
(6) \quad (g, z) \xrightarrow{h, \tau} (g h^x, z, \tau).
\]

Therefore,

(i) The transition probabilities (6) of the chains (2) are the same for all \(x \in X\);

(ii) These transition probabilities are equivariant with respect to the action of the group \(H\) on itself on the left.

Thus the chains (2) are random walks with internal degrees of freedom (RWIDF) on \(H\) (the space of these degrees being \(X\)) [KSS3]. In other terminologies they are matrix-valued random walks [CW89] or covering Markov chains with the deck transformation group \(H\) and the quotient space \(X\) [K95].

Recall that the transition probabilities
\[
\pi_{g,x}((gh, y)) = \mu_{x,y}(h)
\]
of a general RWIDF are determined by a \(d \times d\) matrix
\[
M = (\mu_{x,y})_{x,y \in X},
\]
(where \( d = \text{card} X \) is the cardinality of the space of internal degrees of freedom \( X \)) of sub-probability measure on the group with
\[
\sum_y \| \mu_{x,y} \| = 1,
\]
where \( \| \mu_{x,y} \| \) denotes the total mass of the measure \( \mu_{x,y} \). \( \mu_{x,y} \) can be treated as an element of the group algebra. In the case of a usual random walk this matrix is \( 1 \times 1 \) and consists just of a single probability measure on the group which determines the random walk.

The image of the RWIDF \((H, M)\) under the map \((g, x) \rightarrow x\) is the quotient Markov chain on \( X \) with the transition matrix
\[
P = (p_{x,y}) = aM, \quad p_{x,y} = \| \mu_{x,y} \|, \tag{7}
\]
which is the image of the matrix \( M \) under the augmentation homomorphism \( a : g \rightarrow 1 \).

In our situation, since \( \mu_\tilde{G} \) is the image of the measure \( \mu \) under the map \( \phi \), the matrix \( M \) is
\[
M = M^\mu = \sum \mu(g)M^g, \tag{8}
\]
where \( M^g \) are the matrices determined as
\[
M^g_{x,y} = \begin{cases} 
  h_g^x, & \text{if } y = x.\sigma_g \\
  0, & \text{otherwise}
\end{cases}.
\]

**Remark:** The map \( g \rightarrow M^g \) is in fact an embedding of the group \( G \) into the algebra \( \text{Mat}(d, \ell^1(H, \mathbb{R})) \) of \( d \times d \)-matrices over the group algebra of \( H \).

For the RWIDF \((H, M)\) determined by the matrix \( M \) \( \mathbb{N} \) the quotient chain on \( X \) has the transition probabilities \((7)\)
\[
z \xrightarrow{(h, \tau) \sim \mu_\tilde{G}} z.\tau,
\]
or, equivalently
\[
z \xrightarrow{\tau \sim \mu_X} z.\tau. \tag{9}
\]

4.3. **Nested reductions.** Since the Markov chain \((9)\) is obtained from a random walk on the group of permutation \( \text{Sym}(X) \), any state \( x \in X \) is recurrent. Therefore, the corresponding set \( H \times \{ x \} \) is recurrent for the RWIDF \((6)\). Recall that the stopping a Markov chain at the times when it visits a certain recurrent subset of the state space gives a new Markov chain on this recurrent subset (the trace of the original Markov chain). In our case the transition probabilities of the induced chain on \( H \times \{ x \} \) are obviously equivariant with respect to the left action of the group \( H \) on itself (because the original RWIDF also has this property). Therefore, the induced chain on \( H \times \{ x \} \) is actually the usual random walk determined by a certain probability measure \( \mu^x \) on \( H \).
Theorem 4.1. The measures $\mu^x$, $x \in X$ can be expressed in terms of the matrix $M$ as

\begin{equation}
\mu^x = \mu_{x,x} + M_{x,x}(I + M_{x,x} + M^2_{x,x} + \ldots)M_{x,x} = \mu_{x,x} + M_{x,x}(I - M_{x,x})^{-1}M_{x,x},
\end{equation}

where $M_{x,x}$ (resp., $M_{x,x}$ denotes the row $(\mu_{x,y})_{y\neq x}$ (resp., the column $(\mu_{y,x})_{y\neq x}$) of the matrix $M$ with the removed element $\mu_{x,x}$ and $M_{x,x}$ is the $(d-1) \times (d-1)$ matrix (where $d = \text{card}X$) obtained from $M$ by removing its $x$-th row and column. The multiplication above is understood in the usual matrix sense in the group algebra.

The proof is the same as in [K04]

5. Entropy Estimates.

Recall that the entropy of a probability measure $\theta = \{\theta_i\}$ is defined as

$$H(\theta) = -\sum \theta_i \log(\theta_i).$$

If $\mu$ is a probability measure on a countable group $G$ with $H(\mu) < \infty$, then the (asymptotic) entropy of random walk $(G, \mu)$ is defined as the limit

$$h(G, \mu) = \lim_{n \to \infty} \frac{1}{n} H(\mu_n),$$

where $\mu_n$ denotes the $n$-fold convolution of the measure $\mu$, i.e., the distribution of the position at time $n$ of the random walk $(G, \mu)$ issued from the identity of the group $G$.

The following theorem can be easily deduced from Theorem 3.3 [K04].

Theorem 5.1. Let $(G, H, X, \phi)$ be a nested structure. Let $\mu$ be a probability measure on $G$ with finite entropy $H(\mu)$. Then for all $x \in X$

$$h(G, \mu) \leq h(H, \mu^x).$$

5.1. Entropy and amenability. Let me quickly recall an amenability criteria. There are many definitions of amenability; we will use the Folner condition. Let $G$ be a countable group. If for a given finite set $K$ and $\epsilon > 0$ there exists a finite subset $A \subset G$ such that

$$|Ag \Delta A| \leq \epsilon |A|, \forall g \in K,$$

then we say that $G$ is amenable.

Another equivalent condition of amenability is an existence of a right-hand invariant mean on $G$.

The fundamental fact relating the asymptotic entropy with amenability of the group $G$ is that $h(G, \mu) = 0$ if and only if the Poisson boundary of the random walk $(G, \mu)$ is trivial. This in turn implies amenability (as the action of the group on Poisson boundary is amenable). Moreover $n$-fold convolution of the measure $\mu$ converges to invariant mean [KV83]. Therefore, if a group $G$ carries a non-degenerate random walk with vanishing asymptotic entropy, then it must be amenable.
6. GRIGORCHUK 2-GROUPS.

In this section I describe a construction of Grigorchuk 2-group. For complete description and further results see [Gr84]. Let \( \Omega = \{0, 1, 2\}^\mathbb{N} \) and \( \sigma : \Omega \to \Omega \) is left side shift. For every \( \omega \in \Omega \), there exists the Grigorchuk 2-group \( G_\omega \) which is a subgroup of the group of automorphisms of the binary tree \( Aut(T_2) \). The group \( G_\omega \) is generated by 4 elements \( a, b_\omega, c_\omega, d_\omega \in Aut(T_2) \).

To describe the group I need to specify the action of each element on the binary tree. Each vertex of the tree can be represented as a word in the 2-letter alphabet \( \{0, 1\} \). The action on the tree is easy to recover from that definition.

Define \( t_i : \{0, 1, 2\} \to Aut(T_2) \) as
\[
t_i(j) = \begin{cases} 
1 & \text{if } i = j \\
a & \text{if } i \neq j 
\end{cases}
\]

For \( x \in \{0, 1\}^\mathbb{N} \) I have
\[
a(0x) = 1x, \quad a(1x) = 0x \\
b_\omega(0x) = 0t_0(\omega_1)(x), \quad b_\omega(1x) = 1b_{\sigma(\omega)}(x), \\
c_\omega(0x) = 0t_1(\omega_1)(x), \quad c_\omega(1x) = 1c_{\sigma(\omega)}(x), \\
d_\omega(0x) = 0t_2(\omega_1)(x), \quad d_\omega(1x) = 1d_{\sigma(\omega)}(x),
\]
where \( \omega = (\omega_1, \omega_2, \omega_3, \ldots) \).

Let \( \varepsilon : \{0, 1\} \to \{0, 1\} \) is defined as \( \varepsilon(i) = 1 - i \). Observe that there exists \( \pi_i : G_\omega \to G_{\sigma^i \omega} \) for \( i \in \{0, 1\} \), where \( \pi_i \) are defined as \( g(\varepsilon^i(0)x) = \varepsilon^{l(g)}(i)\pi_i(g)(x) \), where \( l(g) \in \{0, 1\} \). This proves

Lemma 6.1. The embedding
\[
\phi_\omega : G_\omega \to G_{\sigma^0 \omega}^{\{0, 1\}} \times Sym(\{0, 1\}),
\]
defined as \( g \to (\pi_0(g), \pi_1(g), \varepsilon^{l(g)}) \), induces the nested structure
\[
(G_\omega, G_{\sigma \omega}, \{0, 1\}, \phi_\omega).
\]

Let me give another description of \( G_\omega \) that is easier is to visualize. It is the original Grigorchuk description of \( G_\omega \). The space of ends is "almost" the interval \( [0, 1] \).

Let \( \Delta \) be an interval. Denote by \( I \) an identity transformation on \( \Delta \) and by \( T \) a transposition of two halves of \( \Delta \).

For each \( \omega \in \Omega \) define a \( 3 \times \infty \) matrix \( \overline{\omega} \) by replacing \( \omega_i \) with columns \( \overline{\omega_i} \), where
\[
\bar{0} = \begin{pmatrix} T \\ T \\ I \end{pmatrix}, \quad \bar{1} = \begin{pmatrix} T \\ I \\ T \end{pmatrix}, \quad \bar{2} = \begin{pmatrix} I \\ T \\ T \end{pmatrix}
\]
By $U^\omega = (u_1^\omega, u_2^\omega, \ldots), V^\omega = (v_1^\omega, v_2^\omega, \ldots), W^\omega = (w_1^\omega, w_2^\omega, \ldots)$ denote the rows of $\omega$. Think of them as of infinite words in the alphabet $\{T, I\}$.

Define transformations $a_\omega, b_\omega, c_\omega, d_\omega$ of an interval $\Delta = [0, 1] \setminus \mathbb{Q}$ as follows:

\[
\begin{align*}
a_\omega &: 0 \quad T \quad 1 \\
b_\omega &: 0 \quad u_1^\omega \quad u_2^\omega \quad 1 \\
c_\omega &: 0 \quad v_1^\omega \quad v_2^\omega \quad \ldots \quad 1 \\
d_\omega &: 0 \quad w_1^\omega \quad w_2^\omega \quad \ldots \quad 1
\end{align*}
\]

Observe that $a_\omega$ is independent of $\omega$, and will be further denoted by $a$. Let $G_\omega$ be a group of transformations of the interval $\Delta$ generated by $a, b_\omega, c_\omega, d_\omega$.

It was proved by Grigorchuk [Gr84] that if $\omega$ does not become constant, (i.e., there does not exist $N$ such that $\omega_{N+1} = \omega_{N+2} = \ldots$) the group $G_\omega$ has subexponential growth. In case when there exists $N$ such that $\omega_N \neq \omega_{N+1} = \omega_{N+2} = \ldots$, $G_\omega$ has a subgroup of finite index isomorphic to the free abelian group of rank $2^N$.

Before I proceed to the next section I mention the result proved by Grigorchuk [Gr84] (see Theorem 7.1)

**Theorem 6.1.** Suppose that function $\rho(n)$ grows more slowly than any exponential function, i.e., $\rho(n) = o(2^{\epsilon n})$ for all $\epsilon > 0$. Then there exist $\omega \in \Omega$ and $N$ such that $|B_\omega(N)| > \rho(N)$, where $B_\omega(n)$ is a ball of radius $n$ in $G_\omega$ with respect to generators $a, b_\omega, c_\omega, d_\omega$.

**Remark:** This is not the exact theorem 7.1 from [Gr84]. But it is a simplified version of it.

7. **Constructing the universal group**

Let $X$ be a set and $(G, \odot)$ be a group. Denote by $(G^X, \star)$ be the group of all maps from $X$ to $G$ with pointwise composition.

**Proposition 7.1.** Let $\sigma$ be a surjective map from $X$ to $X$. Then

$$\hat{\sigma} : G^X \to G^X$$

defined as $f \to f \circ \sigma$ is an injective homomorphism.

Before I proceed I will need a simple lemma.

**Lemma 7.1.** Let $f : X \to X$ be a surjective map. Assume $g : X \to Y$ be some map. Then $g \equiv \text{const}$ if and only if $g \circ f \equiv \text{const}$

\[\square\] Since $f$ is surjective, there exists right inverse, i.e., $h : X \to X$ such that $f \circ h = Id$. Therefore if $g \circ f \equiv \text{const}$ then $g = (g \circ f) \circ h \equiv \text{const} \circ h = \text{const}$.

**Proof of Proposition:** Let $f, g \in G^X$, then $(\hat{\sigma}(f \star g))(x) = (f \star g)(\sigma(x)) = f(\sigma(x)) \circ g(\sigma(x)) = \hat{\sigma}(f)(x) \odot \hat{\sigma}(g)(x) = (\hat{\sigma}(f) \star \hat{\sigma}(g))(x)$ for
all \( x \in X \). Since \( Id \in G^X \) is a constant map, \( \hat{\sigma}(Id) = Id \). Thus \( \hat{\sigma} \) is a homomorphism.

Now if \( f \in \ker(\hat{\sigma}) \) then \( f \circ \sigma = Id \). By the above lemma \( f = Id \). Thus \( \hat{\sigma} \) is injective. \( \blacksquare \)

Fix \( \phi_1, \phi_2, \ldots, \phi_n \in G^X \). Then for each \( \omega \in X \), I define

\[
H_{\omega} = \langle \phi_1(\omega), \ldots, \phi_n(\omega) \rangle \leq G.
\]

It is easy to observe that \( H = \langle \phi_1, \phi_2, \ldots, \phi_n \rangle \) is the smallest group such that for every \( \omega \in X \), \( H_{\omega} \) is a quotient of \( H \).

8. Universal Grigorchuk 2-group.

Let \( X = \Omega = \{0, 1, 2\} \mathbb{Z}^+ \), \( H = \text{Aut}(T_2) \), where \( T_2 \) is a binary tree, and \( \sigma \) is a shift on \( \Omega \).

I am going to consider a subgroup of \( H^\Omega \) generated by maps \( A = a \), where \( a \) is a flip at the top vertex. \( B(\omega) = b_\omega \), \( C(\omega) = c_\omega \), \( D(\omega) = d_\omega \), where \( b_\omega, c_\omega, d_\omega \) are element of the Grigorchuk group corresponding to \( \omega \).

Let \( Gr_2 = \langle A, B, C, D \rangle \). I will call \( Gr_2 \) the universal Grigorchuk 2-group.

It is clear that for every \( \omega \in \Omega \), the evaluation map at \( \omega \) is a surjective map from \( G \) to \( G_\omega \).

Lemma 8.1. \( \text{The group } Gr_2 \text{ has exponential growth.} \)

\( \Box \) Let \( \rho(n) \) be the growth function with respect to generators \( A, B, C, D \). Assume toward a contradiction, that \( Gr_2 \) does not have exponential growth. Then \( \lim_{n \to \infty} \frac{\log(\rho(n)))}{n} = 0 \). Thus \( \rho(n) \) has subexponential growth, i.e., \( \rho(n) \leq e^{\epsilon n} \) for every \( \epsilon > 0 \). By Theorem 7.1 in [Gr84], there exists \( \omega \) and \( N \) such that \( B_\omega(n) > \rho(n) \), where \( B_\omega(n) \) is the size of the ball of radium \( n \) inside the group \( G_\omega \) with respect to generators \( a_\omega, b_\omega, c_\omega, d_\omega \).

However \( G_\omega \) is a quotient of \( G \) and generators \( A, B, C, D \) maps to \( a_\omega, b_\omega, c_\omega, d_\omega \). Thus for every \( n \), \( \rho(n) \geq B_\omega(n) \). But for \( N \), I have \( \rho(N) > B_\omega(N) \). Contradiction. \( \blacksquare \)

Lemma 8.2. \( \text{The map } \phi : Gr_2 \to \sigma(Gr_2)^{\{0, 1\}} \times \text{Sym}(\{0, 1\}) \) defined by

\[
\phi(A) = (Id, Id, \varepsilon),
\phi(B) = (\tilde{t}_0, B \circ \sigma, Id),
\phi(C) = (\tilde{t}_1, C \circ \sigma, Id),
\phi(D) = (\tilde{t}_2, D \circ \sigma, Id)
\]

induces a nested structure \( (Gr_2, \sigma(Gr_2), \{0, 1\}, \phi) \), where

\[
\tilde{t}_i(\omega) = \begin{cases} 
Id & \text{if } \omega_1 = i \\
\alpha & \text{if } \omega_1 \neq i
\end{cases}.
\]
The proof is an easy consequence of Lemma 6.1. The hard part is to show that it is an embedding. If \( W(A, B, C, D) \) is a word such that \( \phi(W(A, B, C, D)) = (\text{Id}, \text{Id}, \text{Id}) \), then \( \phi_\omega(W(A, B, C, D)(\omega)) = (\text{Id}, \text{Id}, \text{Id}) \) for all \( \omega \in \Omega \). But since \( \phi_\omega \) is an embedding I have
\[
W(A(\omega), B(\omega), C(\omega), D(\omega)) = \text{Id}
\]
for every \( \omega \in \Omega \). This proves the lemma. □

9. Amenability of Universal Grigorchuk 2-group.

In this section I complete the proof of Theorem 1.1. Let \( Gr_2 = \langle A, B, C, D \rangle \) then \( \hat{\sigma}(Gr_2) \cong \langle A, B \circ \sigma, C \circ \sigma, D \circ \sigma \rangle \), as \( A \circ \sigma = A \).

Since we have a nested structure, we can write everything in matrix form.
\[
A \leftrightarrow \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix}, \quad B \leftrightarrow \begin{pmatrix} \tilde{t}_0 & 0 \\ 0 & B \circ \sigma \end{pmatrix}, \quad C \leftrightarrow \begin{pmatrix} \tilde{t}_1 & 0 \\ 0 & C \circ \sigma \end{pmatrix}, \quad D \leftrightarrow \begin{pmatrix} \tilde{t}_2 & 0 \\ 0 & D \circ \sigma \end{pmatrix}.
\]

A trivial observation is that \( \tilde{t}_1 + \tilde{t}_2 + \tilde{t}_3 = 2A + 1 \in R(G) \subset R(H^X) \) in the group algebras. Consider a random walk on \( Gr_2 \) generated by \( \mu_0 = \alpha \text{Id} + \beta A + m(B + C + D) \). This corresponds to
\[
\mu_0 \leftrightarrow \begin{pmatrix} \alpha \text{Id} + m(2A + \text{Id}) & \beta \text{Id} \\ \beta \text{Id} & \alpha \text{Id} + m(B \circ \sigma + C \circ \sigma + D \circ \sigma) \end{pmatrix}.
\]

Consider an induced random walk on the left subtree. (I will denote induced measure with superscript \( \text{ind} \).) It will be supported on \( A, B \circ \sigma, C \circ \sigma, D \circ \sigma \) and using formula (10) from Theorem 4.1 we have
\[
\mu_1 = \mu_0^{\text{ind}} = (\alpha + m) \text{Id} + 2mA + \beta^2 \sum_{i=0}^{\infty} (\alpha \text{Id} + m(B \circ \sigma + C \circ \sigma + D \circ \sigma))^i = \\
(\alpha + m + x) \text{Id} + 2mA + \frac{\beta^2 - x}{3} (B \circ \sigma + C \circ \sigma + D \circ \sigma),
\]
where \( x \) is some non-negative number depending on \( \beta \) and \( m \), coming from the term
\[
\beta^2 \sum_{i=0}^{\infty} (\alpha \text{Id} + m(B \circ \sigma + C \circ \sigma + D \circ \sigma))^i,
\]
and using the fact that \( B, C, D \) commute and \( B^2 = C^2 = D^2 = BCD = \text{Id} \).

Remark: I have used that \( \mu_0 \) and \( \mu_1 \) are probabilities measures to make the calculations.

Now recall that \( \hat{\sigma}(Gr_2) \cong Gr_2 \). Therefore, this gives a random walk on the initial group. It is possible to calculate \( \mu_1 \) explicitly and then find \( \mu_0 \) that satisfies the equation \( \mu_0 = \delta \text{Id} + (1 - \delta)\mu_1 \) for some \( 0 < \delta < 1 \). In this case the problem would be reduced to the method of V. Kaimanovich in [K04].
I would like to use a small trick in order to avoid calculations of $x$. We can induce a random walk on the left subtree, again giving rise to a new measure

$$\mu_2 = \mu_1^{\text{ind}} = (\alpha + m + x + \frac{\beta - x}{3} + y)\text{Id} + 2\frac{\beta - x}{3}A + \frac{2m - y}{3}(B \circ \sigma^2 + C \circ \sigma^2 + D \circ \sigma^2),$$

Now it is easy to observe that the coefficients near $A, B, C, D$ have shrunk by a factor at least $2/3$. Since $\mu_n$ are probability measures, $\mu_n \to \text{Id}$ as $n \to \infty$, where $\mu_{n+1} = \mu_n^{\text{ind}}$. As $h(\mu) \leq H(\mu)$, and $h(\mu_i) \leq h(\mu_{i+1})$ it follows that

$$h(\mu_0) \leq \limsup_{n \to \infty} h(\mu_n) \leq \lim_{n \to \infty} H(\mu_n) = 0.$$

This proves that $Gr_2$ is amenable. ■

Remark: In the last part of the proof, it is possible to actually calculate $x$ and then construct a self-similar measure as in [K04]. I did not do that in order to show another way to estimate the entropy of the random walk.

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