Dilatonic interpolation between Reissner–Nordström and Bertotti–Robinson spacetimes with physical consequences

S Habib Mazharimousavi, M Halilsoy, I Sakalli and O Gurtug

Department of Physics, Eastern Mediterranean University, G. Magusa, north Cyprus, Mersin-10, Turkey
E-mail: habib.mazhari@emu.edu.tr, mustafa.halilsoy@emu.edu.tr, izzet.sakalli@emu.edu.tr and ozay.gurtug@emu.edu.tr

Received 26 August 2009, in final form 22 February 2010
Published 16 April 2010
Online at stacks.iop.org/CQG/27/105005

Abstract
We give a general class of static, spherically symmetric, non-asymptotically flat and asymptotically non-(anti) de Sitter black hole solutions in Einstein–Maxwell–Dilaton (EMD) theory of gravity in four dimensions. In this general study we couple a magnetic Maxwell field with a general dilaton potential, while double Liouville-type potentials are coupled with gravity. We show that the dilatonic parameters play a key role in switching between the Bertotti–Robinson and Reissner–Nordström spacetimes. We study the stability of such black holes under a linear radial perturbation, and in this sense we find exceptional cases where the EMD black holes are unstable. In continuation, we give a detailed study of the spin-weighted harmonics in the dilatonic Hawking radiation spectrum and compare our results with previously known ones. Finally, we investigate the status of resulting naked singularities of our general solution when probed with quantum test particles.

PACS numbers: 04.20.−q, 04.70.−s

1. Introduction
We revisit the four-dimensional Einstein–Maxwell–Dilaton (EMD) theory and show that there is still plenty of room available to contribute to the subject. The double Liouville potential and general dilaton coupling are considered to obtain more general solutions with extra parameters and diagonal metric in the theory. From the outset we recall that, depending on the relative parameters, the double Liouville potential has the advantage of admitting local extrema and critical points. The Higgs potential also shares such features, whereas a single Liouville potential lacks these properties. Double Liouville-type potentials also arise when higher dimensional theories are compactified to four-dimensional spacetimes and expectedly bring in further richness. All known solutions to date can be obtained [1–3] as particular limits...
of our general solution, and it contains new solutions as well. In the most general form, our solution covers Reissner–Nordstrom (RN)-type black holes and Bertotti–Robinson (BR) spacetimes interpolated within the same metric. The interpolation of two different solutions in general relativity is not a new idea [4]. Particular limits of the dilatonic parameter yield the RN and BR spacetimes. In between the two, lies the linear dilaton black hole (LDBH) for the specific choice of the parameters. It is well known that the near-horizon geometry of the extremal RN black hole yields the BR electromagnetic universe. The latter [5] is important for various reasons: it is a singularity-free non-black hole solution which admits maximal symmetry and finds application in conformal field theory correspondence (i.e. AdS/CFT). Particles in the BR universe move with uniform acceleration in a conformally flat background. These features are mostly valid not only in \( N = 4 \) but also in higher dimensions \( (N > 4) \). The topological structure of the BR spacetime is still \( \text{AdS}_2 \times S^{N-2} \) in \( N \)-dimensions with the radius of \( S^{N-2} \) depending on the dimension of space. Recently, we have extended the Maxwell field in the BR spacetime to the Yang–Mills (YM) field and obtained common features that match with the Maxwell field [6]. The dilatonic black hole solution involved in the general solution obtained in this paper is non-asymptotically flat; therefore, we expressed it in terms of the quasilocal mass (\( M_{QL} \)) [7]. The metric is regular at horizons with only available singularity at \( r = 0 \). Another feature is the asymptotic \( (r \to \infty) \) absence of (anti-)de-Sitter property which was also discovered within the context of different models [3]. Our general solution has been tested for stability against the radial, linear perturbations. We found that the presence of dilaton can trigger instability in the RN black hole which is stable otherwise. Our analysis proves that the BR sector remains manifestly stable against such perturbations. Thermodynamic stability has also been discussed briefly by considering the specific heat of the metric. Divergence in the specific heat for specific values of the parameters signals phase transition in our thermodynamic system, i.e. topology changes in the spacetime.

Next, we concentrate ourselves on the LDBH case and analyze the Hawking temperature both from the semi-classical and standard surface gravity methods [8, 9]. We point out the contrasts between the two methods when there are single and double horizons. The high frequency limit of the semi-classical radiation spectrum method (SCRSRM) does not agree with Hawking’s result. It is observed, as an interesting contribution in this work, that the coupling between the scalar field charge and the magnetic charge of the spacetime gives rise to spin-weighted spheroidal harmonics which plays a dominant role in the difference. In the absence of such coupling, when the scalar field is assumed chargeless for instance, similar analysis was carried out previously and we had recovered the same results easily. It turns out that the very existence of a spin-weighted spheroidal harmonics in the theory transforms a divergent temperature spacetime to a finite one. We argue that such a behavior may play a leading role in the detection of such LDBHs.

In the final section of the paper we appeal once more to the test scalar field equation, but this time with the purpose of investigating the quantum nature of the naked singularities. We identify first the particular solution that yields horizonless naked singularity at \( r = 0 \). By invoking the Horowitz–Marolf [10] criterion on the quantum nature of classical singularities we explore under which set of parameters classically singular but quantum mechanically regular metrics can occur in our general solution.

The organization of our paper is as follows. In section 2 we introduce our action, field equations and obtain the general solution. Section 3 singles out the linear dilaton case and investigates the stability of our general solution. Application of the SCRSRM and its connection with the Hawking temperature is employed in section 4. Section 5 discusses the status of naked singularities from the quantum picture. We summarize our results in section 6.
2. Field equations and the metric ansatz for EMD gravity

The four-dimensional action in the EMD theory is given by
\[ S = \int d^4x \sqrt{-g} \left( \frac{1}{2} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) - \frac{1}{2} W(\phi) (F_{\lambda\sigma}) F^{\lambda\sigma} \right), \]  
where
\[ V(\phi) = V_1 e^{\beta_1 \phi} + V_2 e^{\beta_2 \phi}, \quad W(\phi) = \lambda_1 e^{-2\gamma_1 \phi} + \lambda_2 e^{-2\gamma_2 \phi}. \]

\( \phi \) refers to the dilaton scalar potential and \( \gamma_i \) denotes the dilaton parameter, \( \lambda_i \) is a constant and \( V(\phi) \) is a double Liouville-type potential. We note that we exclude the simultaneous values \( \beta_1 = \beta_2 \) and \( \gamma_1 = \gamma_2 \) in general, since these particular values lead to the already known cases.

Let us remark that although the double Liouville potential in \( V(\phi) \), which renders local minima, is necessary for construction of possible vacuum states, the similar choice for \( W(\phi) \) seems less appealing. It will be justified from the exact solutions below, however, that there are asymptotics which remain inaccessible by the choice of a single Liouville term in \( W(\phi) \). Unless stated otherwise, at both asymptotes of \( r = 0 \) and \( r = \infty \) (or \( \tilde{r} = 0 \) and \( \tilde{r} = \infty \) for LDBH) dilatonic coupling to the magnetic field becomes much stronger. Choosing a single Liouville potential simply loses strength at one end of the range. Besides, it is all a matter of choice to set \( \lambda_1 = 0 \) (or \( \lambda_2 = 0 \)), which makes the dilatonic coupling asymptotically free. In the LDBH case as it will be proved, if we set \( \lambda_1 = 0 \), we remove the possibility of an inner (Cauchy) horizon which justifies the advantages and motivation for choosing the double Liouville-type potential in \( W(\phi) \). In (1) \( R \) is the usual Ricci scalar and \( F = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu \) is the Maxwell 2-form (with \( \wedge \) indicating the wedge product) given by

\[ F = dA, \]

for \( A = A_\mu \, dx^\mu \), the potential 1-form. Our pure magnetic potential with charge \( Q \), which is given by

\[ A = -Q \cos \theta \, d\phi \]
leads to

\[ F = Q \sin \theta \, d\theta \wedge d\phi. \]

Let us note that with the present choice of \( W(\phi) \) the electric–magnetic symmetry that exists in the standard dilatonic coupling, i.e. \( \lambda_1(\lambda_2) = 0 \), is no longer valid. Our choice in this paper relies entirely on the magnetic choice. Variations of the action with respect to the gravitational field \( g_{\mu\nu} \) and the scalar field \( \phi \) lead, respectively, to the EMD field equations

\[ R_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + V(\phi) g_{\mu\nu} + W(\phi) \left( 2 F_{\mu\lambda} F^\lambda_\nu - \frac{1}{2} F_{\sigma\nu} F^{\lambda\sigma} g_{\mu\nu} \right), \]

\[ \nabla^2 \phi - V'(\phi) - \frac{1}{2} W'(\phi) (F_{\sigma\nu} F^{\lambda\sigma}) = 0, \quad \left( r \equiv \frac{d}{d\phi} \right), \]

where \( R_{\mu\nu} \) is the Ricci tensor. Variation with respect to the gauge potential \( A \) yields the Maxwell equation

\[ d(W(\phi)^* F) = 0, \]

where the Hodge star \( * \) means duality.
2.1. Ansatz and the solutions

Our ansatz line element for EMD gravity is chosen to be
\[ ds^2 = -f(r) \, dt^2 + \frac{1}{f(r)} \, dr^2 + R(r)^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \]
(9)

with \( f(r) \) and \( R(r) \) being the only functions of \( r \) while the Maxwell invariant takes the form
\[ F_{\lambda\sigma} F^{\lambda\sigma} = \frac{2Q^2}{R^4}. \]
(10)

The Maxwell equation (8) is satisfied automatically and the field equations become
\[ \nabla^2 \phi := \frac{1}{R^2} (R^2 f \phi')' = V'(\phi) + \frac{1}{2} W'(\phi) (F_{\lambda\sigma} F^{\lambda\sigma}), \]
(11)
\[ R^r_r := -\left( f R R' \right) R - \frac{(f R R')'}{2R^2} = f \phi'^2 + V(\phi) - \frac{W(\phi)}{2} (F_{\lambda\sigma} F^{\lambda\sigma}), \]
(12)
\[ R^\theta_\theta = R^\phi_\phi := 1 - \frac{V(\phi)}{2} \left( 1 - \frac{1}{\alpha^2} \right) A^4 - \frac{W(\phi)}{2} (F_{\lambda\sigma} F^{\lambda\sigma}), \]
(13)

where a prime stands for derivative with respect to the argument of the function. We start with an ansatz for \( R(r) \) as
\[ R(r) = A e^{\eta \phi}, \]
(15)
where \( A \) and \( \eta \) are found to be constants. Substitution in (12) and (13) implies
\[ \phi(r) = \frac{2\eta}{2\eta^2 + 1} \ln r. \]
(16)

Finally by putting these results into equations (11) and (14) one finds that by setting
\[ \eta = -\frac{1}{\alpha \sqrt{2}}, \]
(17)
and
\[ \gamma_1 = -\frac{\alpha}{\sqrt{2}}, \quad \gamma_2 = \frac{1}{\alpha \sqrt{2}}, \]
\[ \beta_1 = \sqrt{2} \alpha, \quad \beta_2 = \frac{\sqrt{2}}{\alpha}, \]
(18)
a general solution for \( f(r) \) reads
\[ f(r) = (1 + \alpha^2)^2 \left[ \frac{Q^2 \lambda_1 r^{\frac{2\alpha}{1+\alpha}}}{{(1+\alpha^2)A^4}} + \left( \frac{Q^2 \lambda_2}{A^4} - V_2 \right) \frac{r^{\frac{2\alpha}{1+\alpha}}}{(1+\alpha^2)\alpha^2} - \frac{V_1 r^{\frac{2\alpha}{1+\alpha}}}{3-\alpha^2} - Mr^{-\frac{\alpha}{1+\alpha}} \right], \]
(19)

with the constraint condition
\[ -V_2 (1-\alpha^2) A^4 - \alpha^2 A^2 + \lambda_2 Q^2 (1+\alpha^2) = 0. \]
(20)

Herein, \( M \) is a mass-related integration constant and \( \alpha \) and \( A \) are constants that will serve to parametrize the solution. We note that in the case that we are interested in we must choose, when \( \alpha = 0 = \lambda_2 = V_2 \) and \( r \to \infty \), the Newtonian limit \( M \to 2M \), so that \( M \) represents the
Newtonian mass. Therefore, the dilatonic function $\phi$, Liouville potential $V$ and $W$ in terms of $\alpha$ become

$$
\phi(r) = -\frac{\alpha}{1 + \alpha^2} \ln r, \quad R(r) = Ar^{\frac{1}{1 + \alpha^2}}, \\
V = V_1 r^{\frac{2}{1 + \alpha^2}} + V_2 r^{\frac{2}{1 + \alpha^2}}, \quad W = \lambda_1 r^{\frac{2}{1 + \alpha^2}} + \lambda_2 r^{\frac{2}{1 + \alpha^2}}.
$$

We remark that solution (16–20) is the general diagonal solution that covers all particular solutions of this kind known so far. For arbitrary value of $\alpha$, other than 0, 1 and $\infty$, yields a new solution in accordance with our ansatz. It is also observed that our metric and potentials are invariant under $\alpha \to -\alpha$, whereas $\phi \to -\phi$. The asymptotic behavior of the metric function $f(r)$ and other limiting cases can be summarized as follows:

$$
\lim_{r \to \infty} f(r) \to \begin{cases} 
(1 + \alpha^2) \left( -\frac{V_1 r^{\frac{2}{1 + \alpha^2}}}{3 - \alpha^2} \right) & 0 \leq \alpha^2 < 1 \\
2 \left( \frac{Q^2 \lambda_2}{A^2} - V_1 - V_2 \right) & \alpha^2 = 1 \\
(1 + \alpha^2) \left( \frac{Q^2 \lambda_2}{A^2} - V_2 \right) r^{\frac{2}{1 + \alpha^2}} & 1 < \alpha^2
\end{cases}
$$

$$
\lim_{r \to 0^+} f(r) \to (1 + \alpha^2) \left( \frac{Q^2 \lambda_1}{A^2} r^{\frac{2}{1 + \alpha^2}} \right).
$$

The case $\alpha^2 = 1$ will be studied separately, while the case $\alpha^2 = 0$, with the choice of $\lambda_2 = 0$, $V_2 = 0$ and $A = 1 = \lambda_1$ leads to

$$
f(r) = 1 - \frac{V_1}{3} r^2 - \frac{M}{r} + \frac{Q^2}{r^2}, \quad R(r) = r, \quad \phi = 0,
$$

which corresponds to the action

$$
S_{\alpha^2=0} = \int d^4x \sqrt{-g} \left( \frac{1}{2} R - V_1 - \frac{1}{2} (F_{\mu \nu} F^{\mu \nu}) \right).
$$

This is recognized as the four-dimensional action in the EM theory with the solution representing an RN black hole with a cosmological constant. Another limiting case of interest consists of the case with $\alpha^2 \to \infty$, $\lambda_2 = 1$, with the action

$$
S_{\alpha^2=\infty} = \int d^4x \sqrt{-g} \left( \frac{1}{2} R - V_2 - \frac{1}{2} (F_{\mu \nu} F^{\mu \nu}) \right),
$$

leading to the solution

$$
f(r) = \left( \frac{Q^2}{A^2} - V_2 \right) r^2 - \tilde{M} r, \quad A^2(V_2 A^2 + 1) = Q^2, \quad R(r) = A, \quad \phi(r) = 0,
$$

where $\tilde{M}$ is the mass related integration constant. Here also we have a four-dimensional action in the EM theory with a cosmological constant but the metric function represents a BR spacetime.

By looking at the asymptotic behaviors of the general solution one finds that $0 \leq \alpha^2 < 1$ and $1 < \alpha^2$ correspond to the cases of RN and BR solutions, respectively. Here $\alpha^2 = 1$ acts much like a phase transition which changes the structure of spacetime from RN into BR. The thermodynamic instability from the expression of specific heat capacity $C_Q$, (equation (58) given below) justifies this fact. It is quite interesting to see what will be the answer if one chooses $\alpha^2 = 1$. In the next section we concentrate on this critical value for $\alpha^2$.
3. The linear dilaton

From the asymptotic behavior of the metric function, one may see that $\alpha^2 = 1$ is a critical value and the behavior of spacetime changes. In this chapter, we only concentrate on the specific value for $\alpha^2$, and will be referred to as linear dilaton. The general solution after this setting reads

\[-\gamma_1 = \gamma_2 = \frac{1}{\sqrt{2}}, \quad \beta_1 = \beta_2 = \sqrt{2} \],

\[\phi(r) = -\frac{1}{\sqrt{2}} \ln r, \quad R(r) = A\sqrt{r},\] (27)

\[V = \frac{\tilde{V}}{r}, \quad W = \frac{\lambda_1}{r} + \lambda_2 r,\]

\[A^2 = 2\lambda_2 Q^2, \quad (\lambda_2 > 0)\]

\[\phi(r) = -\frac{1}{\sqrt{2}} \ln A\sqrt{r}, \quad R(r) = A\sqrt{r},\] (28)

where $\tilde{V} = V_1 + V_2$.

In order to explore the physical properties of the linear dilaton case, we perform the transformation $R(r) = A\sqrt{r} \rightarrow \tilde{r}$. This transforms the metric into,

\[ds^2 = -f(\tilde{r}) \, dt^2 + \frac{4\tilde{r}^2}{A^2 f(\tilde{r})} \, d\tilde{r}^2 + \tilde{r}^2 d\Omega^2,\] (29)

where

\[f(\tilde{r}) = \frac{1}{\tilde{r}^2} \left( \left( \frac{1}{A^2} - 2\tilde{V} \right) \frac{\tilde{r}^2}{A^2} - M_{QL} \tilde{r}^2 + \frac{\lambda_1}{\lambda_2} \right),\] (30)

where the mass $M_{QL}$ denotes the quasilocal mass whose general definition is given below in equation (46). Other related parameters transform into the following forms:

\[\phi(\tilde{r}) = \sqrt{2} \ln \left( \frac{A}{\tilde{r}} \right), \quad V(\tilde{r}) = \frac{A^2}{\tilde{r}^2} (V_1 + V_2), \quad W(\tilde{r}) = \frac{\lambda_1 A^2}{\tilde{r}^2} + \frac{\lambda_2 A^2}{\tilde{r}^2}.\] (31)

The location of horizons can be found if we set the metric function $g_{tt} = 0$. The solution is

\[\tilde{r}_h = \frac{1}{2a} \sqrt{M_{QL} \pm \sqrt{M_{QL}^2 - 4ac}},\] (32)

where

\[a = \left( \frac{1}{A^2} - 2\tilde{V} \right) \frac{1}{A^2}, \quad c = \frac{\lambda_1}{\lambda_2}.\] (33)

The linear dilaton solution admits single or double horizons if the parameters are chosen appropriately. Another possible case is the extremal limit that occurs if $M_{QL}^2 = 4ac$. The horizon in this particular case is given by $\tilde{r}_h = \sqrt{\frac{M_{QL}}{2a}}$. The double horizon case occurs if the parameters simultaneously satisfy $M_{QL} > \sqrt{M_{QL}^2 - 4ac}$ and $M_{QL}^2 > 4ac$. This choice leads to the horizons

\[\tilde{r}_+ = \sqrt{\frac{M_{QL} + \sqrt{M_{QL}^2 - 4ac}}{2a}}, \quad \tilde{r}_- = \sqrt{\frac{M_{QL} - \sqrt{M_{QL}^2 - 4ac}}{2a}}.\] (34)

The relations between the parameters and double Liouville-type potentials in the formation of black holes become evident if one looks for the critical case. This is the case when
\( M_{QL} = \sqrt{M_{QL}^2 - 4ac} \), which follows from \( 2\tilde{V} = \frac{1}{\tilde{r}} \). Hence, if \( 2\tilde{V} < \frac{1}{\tilde{r}} \), no horizon forms and the central singularity \( \tilde{r} = 0 \) becomes a \textit{naked} singularity. It can be easily seen that for \( \lambda_1 = 0 \), or for the single Liouville-type potential in \( W(\phi) \), we have automatically single, outer event horizon alone. Another interesting property is in the behavior of the curvature scalar \( R \).

The curvature scalar for the metric function (29) is

\[
R = -\frac{4\tilde{r}^2(aA^4 - 1) - A^4(\tilde{r}^2 - M_{QL}\tilde{r}^2 + c)}{2\tilde{r}^6}.
\]

Note that the curvature scalar is finite at the location of horizons. Furthermore, when \( \tilde{r} \to \infty \), the Kretschmann and curvature scalars, the Liouville-type potentials and the coupling term of dilaton with Maxwell field all vanish. The mass and charge are finite and the dominant field is the gravity with finite curvature. Consequently, the solution given in equation (29) is well behaved. However, the \( Q = 0 \) limit does not exist.

3.1. Linear stability analysis of the general solution

By employing a similar method used by Yazadjiev [11] we investigate the stability of the possible EMD solution, in terms of a linear, radial perturbation. To do so, we assume that our dilatonic scalar field \( \phi(r) \) changes into \( \phi_\circ(r) + \psi(t, r) \), in which \( \psi(t, r) \) is very weak compared to the original dilaton field \( \phi_\circ(r) \) and we call it the perturbed term. As a result we choose our perturbed metric as

\[
\text{d}s^2 = -f(r) e^{\chi(t, r)} \text{d}t^2 + e^{\chi(t, r)} \text{d}r^2 + R(r)^2 \text{d}\Omega_2^2.
\]

One should note that, since our gauge potentials are magnetic, the Maxwell equations (equation (8)) are satisfied. The linearized version of the field equations (11)-(14) plus one extra term for \( R_{tr} \) are given by

\[
R_{\tau r} : \frac{\chi(t, r)R'(r)}{R(r)} = \partial_t \phi_\circ(r) \partial_r \psi(t, r),
\]

\[
\nabla_\tau^2 \psi - \chi \nabla_\tau^2 \phi_\circ + \frac{1}{2} (\Gamma - \chi) \phi_\circ f - \partial_\phi^2 W(\phi_\circ) \psi = \frac{Q^2}{R(r)^2} \partial_\phi^2 W(\phi_\circ) \psi,
\]

\[
R_{\theta \theta} : (1 - R_{\theta \theta}) \chi - \frac{1}{2} R R' f (\Gamma - \chi) \tau = \left( R^2 \partial_\phi^2 W(\phi_\circ) + \frac{Q^2}{R^2} \partial_\phi W(\phi_\circ) \right) \psi.
\]

where a lower index \( \circ \) represents the quantity in the unperturbed metric. The first equation in this set implies

\[
\chi(t, r) = \frac{1}{\eta} \psi(t, r),
\]

which after making substitutions in the two latter equations and eliminating \( (\Gamma - \chi) \), becomes

\[
\nabla_\tau^2 \psi(t, r) - U(r) \psi(t, r) = 0,
\]

where

\[
U(r) = \frac{2}{r \omega^2} \left\{ \frac{\alpha^2}{A^2} + \left( \frac{Q^2 \lambda_2}{A^2} + V_2 \right) \left( \frac{1 - \alpha^4}{\alpha^2} \right) \right\}.
\]

To get these results we have implicitly used constraint (20) on \( A \). Again by imposing the same constraint, one can show that \( U(r) \) is positive. It is not difficult to apply the separation method on (41) to get

\[
\psi(t, r) = e^{\psi(t)} \xi(r), \quad \nabla_\tau^2 \xi(r) - U_{\text{eff}}(r) \xi(r) = 0, \quad U_{\text{eff}}(r) = \left( \frac{\epsilon^2}{r^2} + U(r) \right),
\]
where $\epsilon$ is a constant. If one shows that the effective potential $U_{\text{eff}}(r)$ is positive for any real value for $\epsilon$, it means that there exists a solution for $\zeta(r)$ which is not bounded. In other words by the linear perturbation our black hole solution is stable for any value of $\epsilon$.

But in our case one must be careful. For instance let us go back to the general solution (19) and set $V_1 = 0$,

$$f(r) = \frac{(1 + \alpha^2)}{r} \left\{ \left( \frac{Q^2 \lambda_2}{A^4} - V_2 \right) \frac{r^2}{\alpha^2} - M (1 + \alpha^2)r + \frac{Q^2 \lambda_1}{A^4} \right\}. \quad (44)$$

This solution may have double horizons, single horizon (extremal) or no horizon. These depend on the values of the parameters. One may note that this solution is a non-asymptotically flat metric and therefore the ADM mass is not defined in general. Following the quasilocal mass formalism introduced by Brown and York [7] it is known that a spherically symmetric $N$-dimensional metric solution such as

$$ds^2 = -F(R)^2dt^2 + \frac{dR^2}{G(R)^2} + R^2 d\Omega_{N-2}^2 \quad (45)$$

admits a quasilocal mass $M_{\text{QL}}$ defined by [6, 7]

$$M_{\text{QL}} = \frac{N - 2}{2} R_{\text{B}}^{N-3} F(R_{\text{B}}) G_{\text{ref}}(R_{\text{B}}) - G(R_{\text{B}})). \quad (46)$$

Here $G_{\text{ref}}(R)$ is an arbitrary non-negative reference function, which yields the zero of the energy for the background spacetime, and $R_{\text{B}}$ is the radius of the space-like hypersurface boundary. Applying this formalism to solution (44), one obtains the horizon $M$ in terms of $M_{\text{QL}}$ as

$$M = \frac{2}{(1 + \alpha^2)A^2} M_{\text{QL}}, \quad (47)$$

after which the metric function becomes

$$f(r) = \frac{(1 + \alpha^2)}{r} \left\{ \left( \frac{Q^2 \lambda_2}{A^4} - V_2 \right) \frac{r^2}{\alpha^2} - \frac{2M_{\text{QL}}}{A^2}r + \frac{Q^2 \lambda_1}{A^4} \right\}. \quad (48)$$

Indeed, since we wish to cover all known solutions in the literature of this kind, we consider $\lambda_i, M_{\text{QL}} \geq 0$ and

$$\left( \frac{Q^2 \lambda_2}{A^4} - V_2 \right) \geq 0. \quad (49)$$

This condition together with equation (20) gives a transparent view of $U_{\text{eff}}(r)$. In other words, after simplification, one can rewrite $U(r)$ as

$$U(r) = \frac{2}{r^2} \left\{ \left( \frac{Q^2 \lambda_2}{A^4} - V_2 \right) + \left( \frac{Q^2 \lambda_2}{A^4} + V_2 \right) \frac{1}{\alpha^2} \right\}, \quad (50)$$

which reveals for $-\frac{Q^2 \lambda_2}{A^4} \leq V_2 \leq \frac{Q^2 \lambda_2}{A^4}$, $U(r)$ and then $U_{\text{eff}}(r)$ are positive, which means that the corresponding metric is stable. But for $V_2 < -\frac{Q^2 \lambda_2}{A^4}$, if $\alpha^2 < \alpha^2_{\text{critical}}$ where

$$\alpha^2_{\text{critical}} = \frac{|V_2| - \frac{Q^2 \lambda_2}{A^4}}{|V_2| + \frac{Q^2 \lambda_2}{A^4}}, \quad (51)$$

then $U(r)$ gets a negative value and therefore our solution faces an instability condition. Here it is interesting to note that $\alpha^2_{\text{critical}} < 1$ belongs to the RN-type black hole solutions, i.e. a BR-type solution is automatically stable for any value of $\alpha^2$. 

8
The general solution reveals another interesting case after we set $V_1 = 0$ and $\lambda_2 = 0$ i.e.

$$f(r) = \frac{1 + \alpha^2}{r} \left( -\frac{V_2}{\alpha^2} r^2 - \frac{2 M_{QL}}{A^2} r + \frac{Q^2 \lambda_1}{A^2} \right). \quad (52)$$

Upon choosing $V_2 < 0$ ($V_2 > 0$) this admits the effective potential

$$U(r) = \frac{2|V_2|}{r} \left( \frac{\alpha^2 - 1}{\alpha^2} \right) \quad (53)$$

which clearly from (20), for $\alpha^2 < 1 (\alpha^2 > 1)$, manifests an unstable black hole solution. As a result we observe that a stable RN black hole becomes unstable under certain conditions in the presence of a dilaton and a Liouville potential.

### 3.2. Thermodynamic stability

Concerning solution (19), we set the parameters $\lambda_1 = \lambda_2 = 1$ and $V_1 = 0$ to get

$$f(r) = \frac{1 + \alpha^2}{r} \left( \left( \frac{Q^2}{A^4} - V_2 \right) r^2 - \frac{2 M_{QL}}{A^2} r + \frac{Q^2}{A^4} \right), \quad (54)$$

where in terms of the radius of horizon $r_h$ one finds the quasilocal mass as

$$M_{QL} = \frac{r_h^2 (Q^2 - V_2 A^4)}{2 A^2 \alpha^2 r_h}. \quad (55)$$

The Hawking temperature

$$T_H = \frac{f'(r_h)}{4\pi} = \frac{(1 + \alpha^2) \left[ r_h^2 (A^2 - 2 Q^2) - Q^2 (1 - \alpha^2) \right]}{4 (1 - \alpha^2) A^4 \pi r_h^3}, \quad (56)$$

and the Bekenstein–Hawking entropy

$$S = \frac{a}{4} = \pi r_h^2, \quad (57)$$

where $a$ is the area of the black hole, together leads to the heat capacity $C_Q$ for constant $Q$ as

$$C_Q = T_H \left( \frac{\partial S}{\partial T_H} \right)_Q = \frac{(\alpha^2 + 1) \left[ r_h^2 (2 - \frac{A}{Q})^2 + 1 - \alpha^2 \right]}{(\alpha^2 - 1) \left[ r_h^2 (2 - \frac{A}{Q})^2 + 3 + \alpha^2 \right]} 2\pi r_h^2. \quad (58)$$

Our black hole solution becomes thermodynamically stable/unstable depending on $C_Q > 0/C_Q < 0$ which is not difficult to test from this expression. For $(\frac{A}{Q})^2 < 2$ and $\alpha^2 < 1$, as an example, our black hole becomes thermodynamically unstable. Also for $(\frac{A}{Q})^2 = 2$ one gets

$$C_Q = -\frac{\alpha^2 + 1}{3 + \alpha^2} 2\pi r_h^2,$$

which shows an instability independent of the values of $\alpha$. Table 1 illustrates the stable and unstable regions in terms of $\alpha^2$ and $x = r_h^2 \left( 2 - \left( \frac{A}{Q} \right)^2 \right)$.

Equation (58) also reveals that $\alpha^2 = 1$ (i.e. the linear dilaton) is a phase transition point; however, there may be other possible transition points following a solution for $\alpha$ in the quadratic equation

$$r_h^2 \left( 2 - \left( \frac{A}{Q} \right)^2 \right) + 3 + \alpha^2 = 0. \quad (59)$$
4. Application of the SCRSM and Hawking temperature

In this section, we shall attempt to make a more precise temperature calculation for the non-extreme LDBHs, $\alpha^2 = 1$ given in equation (28), by using a method of semi-classical radiation spectrum, which has been recently designated as SCRSM [9]. The main difference between our present work and others [8, 9 (and references therein)] is that the considered non-extreme LDBHs possess two horizons, due to having a magnetic charge, instead of one.

Here, we first consider a massless scalar field $\Psi$ with a charge $q$ obeying the covariant Klein–Gordon equation in the LDBH geometry. Namely, we look for the exact solution of the following equation:

$$\square \Psi = 0,$$  \hspace{1cm} (60)

where the d’ Alembertian operator $\square$ is given by

$$\square = \frac{1}{\sqrt{-g}}D_\mu(\sqrt{-g}g^{\mu\nu}D_\nu),$$  \hspace{1cm} (61)

where $D_\mu$ symbolizes the covariant gauge differential operator as being

$$D_\mu = \partial_\mu - iqA_\mu.$$  \hspace{1cm} (62)

The scalar wavefunction $\Psi$ of equation (60) can be separated into the angular and radial equations by letting

$$\Psi = Z(r)\xi(\theta)e^{i(m\phi-\omega t)}.$$  \hspace{1cm} (63)

The separated angular equation can be found as

$$\xi'' + \cot \theta \xi' + \left[\lambda - \frac{(m + p \cos \theta)^2}{\sin^2 \theta}\right] \xi = 0,$$  \hspace{1cm} (64)

where $p = qQ$ and $\lambda$ is a separation constant. (From now on, a prime denotes the derivative with respect to its argument.) After setting the eigenvalue $\lambda = l(l + 1) - p^2$ in equation (64), one can see that solutions to the angular part, $\xi(\theta)e^{i(m\phi-\omega t)}$, are the spin-weighted spheroidal harmonics $pY_{lm}(\theta, \phi)$ with spin-weighted $p$ [13].

On the other hand, before proceeding to the radial equation, one may rewrite the metric function $f(r)$ in equation (28) as

$$f(r) = \frac{b}{r} (r - r_2)(r - r_1).$$  \hspace{1cm} (65)

where $r_2$ and $r_1$ denote the outer and inner horizons of the LDBHs, respectively. In the new form of the metric function (equation (65)), the physical parameters are

$$b = \frac{1}{A^2} - 2\tilde{V}, \quad r_2 = \frac{1}{2b} (c + \sqrt{c^2 - 4ab}), \quad r_1 = \frac{1}{2b} (c - \sqrt{c^2 - 4ab}),$$  \hspace{1cm} (66)
where
\[ c = \hat{M} = 4M \quad \text{and} \quad a = \frac{\lambda_1}{\lambda_2 A^2}. \] (67)

Since the algorithm in the calculations of the SCRSM cover only the outer region of the black hole \((r > r_2)\), we must impose a condition in order to keep \(f(r)\) positive i.e. \(b > 0\). Henceforth, one can derive the following radial equation as
\[ b(r - r_2)(r - r_1)Z'' + b(2r - r_2 - r_1)Z' + \left( \frac{r^2 \omega^2}{b(r - r_2)(r - r_1)} - \frac{\lambda}{A^2} \right) Z = 0. \] (68)

The above equation can be solved in terms of hypergeometric functions. Here, we give the final result as
\[ Z(r) = C_1(r - r_2)^{i\bar{\omega}r_z}(r - r_1)^{-i\bar{\omega}r_z} F\left[ \hat{a}, \hat{b}; \hat{\bar{c}}; \frac{r_2 - r}{r_2 - r_1} \right] + C_2(r - r_2)^{-i\bar{\omega}r_z}(r - r_1)^{-i\bar{\omega}r_z} F\left[ \hat{\bar{a}} - \hat{\bar{c}} + 1, \hat{\bar{b}} - \hat{\bar{c}} + 1; 2 - \hat{\bar{c}}; \frac{r_2 - r}{r_2 - r_1} \right]. \] (69)

The parameters of the hypergeometric functions are
\begin{align*}
\hat{a} &= \frac{1}{2} + i \left( \frac{\omega}{b} + \sigma \right), & \hat{b} &= \frac{1}{2} + i \left( \frac{\omega}{b} - \sigma \right) \quad \text{and} \quad \hat{\bar{c}} = 1 + 2i\bar{\omega}r_z, \\
\sigma &= \frac{1}{b} \sqrt{\omega^2 - \frac{\lambda b}{A^2} - \left( \frac{b}{2} \right)^2}, & \hat{\omega} &= \omega \bar{\eta} \quad \text{and} \quad \bar{\eta} = \frac{1}{b(r_2 - r_1)}. \tag{70}
\end{align*}

Here, \(\sigma\) is assumed to have real values. Furthermore, setting
\[ r - r_2 = \exp \left( \frac{x}{\eta r_2} \right), \] (72)

one gets the behavior of the partial wave near the outer horizon \((r \to r_2)\) as
\[ \Psi \simeq C_1 e^{i\omega (r - r_2)} + C_2 e^{-i\omega (r - r_2)}. \] (73)

One may infer the constants \(C_1\) and \(C_2\) as being the amplitudes of the near-horizon outgoing and ingoing waves, respectively.

In the literature, there exists a useful feature of the hypergeometric functions, which is a transformation of the hypergeometric functions of any argument (say \(z\)) to the hypergeometric functions of its inverse argument \((1/z)\). The relevant transformation is given by [14]
\[ F(\tilde{a}, \tilde{b}; \tilde{\bar{c}}; z) = \frac{\Gamma(\tilde{c})\Gamma(\tilde{b} - \tilde{a})}{\Gamma(\tilde{b})\Gamma(\tilde{c} - \tilde{a})} (-z)^{-\tilde{a}} F(\tilde{a}, \tilde{a} + 1 - \tilde{c}; \tilde{a} + 1 - \tilde{b}; 1/z) \]
\[ + \frac{\Gamma(\tilde{c})\Gamma(\tilde{b} - \tilde{a})}{\Gamma(\tilde{a})\Gamma(\tilde{c} - \tilde{b})} (-z)^{-\tilde{b}} F(\tilde{\bar{b}}, \tilde{\bar{b}} + 1 - \tilde{c}; \tilde{\bar{b}} + 1 - \tilde{a}; 1/z). \] (74)

The above transformation easily leads us to obtain the asymptotic behavior of the partial wave. After applying the transformation to general solution (69), we obtain the partial wave near infinity as follows:
\[ \Psi \simeq \frac{(r - r_1)^{-i\bar{\omega}r_z}}{\sqrt{r - r_2}} \left[ B_1 \exp i \left( \frac{x}{\eta r_2} (\sigma + \omega \bar{\eta} r_1 - \omega t) \right) + B_2 \exp i \left( -\frac{x}{\eta r_2} (\sigma + \omega \bar{\eta} r_1 - \omega t) \right) \right]. \] (75)

On the other hand, since we consider the case of \(r \to \infty\), the overall-factor term
\[ (r - r_1)^{-i\bar{\omega}r_z} \equiv \exp i \left( -\frac{x \omega r_1}{r_2} \right), \] (76)
whence the partial wave (75) reduces to
\[ \Psi \sim \frac{1}{\sqrt{r - r_+}} \left[ B_1 \exp \left( \frac{x}{\eta r_+} \sigma - \omega t \right) + B_2 \exp \left( - \frac{x}{\eta r_+} \sigma - \omega t \right) \right], \quad (77) \]
where \( B_1 \) and \( B_2 \) correspond to the amplitudes of the asymptotic outgoing and ingoing waves, respectively. One can derive the relations between \( B_1, B_2 \) and \( C_1, C_2 \) as follows:
\[ B_1 = C_1 \frac{\Gamma(\tilde{\sigma}) \Gamma(\tilde{\sigma} - \tilde{b})}{\Gamma(\tilde{c} - \tilde{b})} + C_2 \frac{\Gamma(2 - \tilde{\sigma}) \Gamma(\tilde{a} - \tilde{b})}{\Gamma(\tilde{c} - \tilde{b} + 1) \Gamma(1 - \tilde{b})}, \]
\[ B_2 = C_1 \frac{\Gamma(\tilde{c}) \Gamma(\tilde{b} - \tilde{a})}{\Gamma(\tilde{b}) \Gamma(\tilde{c} - \tilde{a})} + C_2 \frac{\Gamma(2 - \tilde{\sigma}) \Gamma(\tilde{b} - \tilde{a})}{\Gamma(\tilde{b} - \tilde{c} + 1) \Gamma(1 - \tilde{a})}. \quad (78) \]

Hawking radiation can be considered as the inverse process of scattering by the black hole such that the outgoing mode at the spatial infinity should be absent \([8]\). Briefly \( B_1 = 0 \), and it naturally yields the coefficient for reflection by the black hole as
\[ R = \frac{|C_1|^2}{|C_2|^2} = \frac{\left| \Gamma(\tilde{c} - \tilde{b}) \right|^2 \left| \Gamma(\tilde{a}) \right|^2}{\left| \Gamma(1 - \tilde{b}) \right|^2 \left| \Gamma(\tilde{a} - \tilde{c} + 1) \right|^2}, \quad (79) \]
which is equivalent to
\[ R = \frac{\cosh \pi \left( \sigma - \frac{\omega}{\tilde{b}} \left( \frac{\ln r_+}{r_+} \right) \right) \cosh \pi \left( \sigma - \frac{\omega}{\tilde{b}} \right)}{\cosh \pi \left( \sigma + \frac{\omega}{\tilde{b}} \left( \frac{\ln r_+}{r_+} \right) \right) \cosh \pi \left( \sigma + \frac{\omega}{\tilde{b}} \right)} \]. \quad (80)

Thus, the resulting radiation spectrum is
\[ N = (e^\tau - 1)^{-1} = \frac{R}{1 - R} \rightarrow T = \frac{\omega}{\ln \left( \frac{1}{R} \right)}, \quad (81) \]
and finally, one can read the more precise value of the temperature as
\[ T = \frac{\omega}{\ln \left[ \frac{\cosh \pi \left( \sigma + \frac{\omega}{\tilde{b}} \left( \frac{\ln r_+}{r_+} \right) \right) \cosh \pi \left( \sigma + \frac{\omega}{\tilde{b}} \right)}{\cosh \pi \left( \sigma - \frac{\omega}{\tilde{b}} \left( \frac{\ln r_+}{r_+} \right) \right) \cosh \pi \left( \sigma - \frac{\omega}{\tilde{b}} \right)} \right]}, \quad (82) \]
This must be considered as the equilibrium temperature of the quantum field at the vacuum state, valid for all frequencies. In the limit of ultrahigh frequencies \((\sigma \simeq \frac{\omega}{b})\), equation (82) reduces to
\[ T_{\text{high}} \simeq \lim_{\omega \rightarrow 1} T \simeq \frac{\omega}{\ln \left( \exp \left( \frac{2\pi \omega}{b} \right) \right)} \simeq \frac{b}{4\pi}, \quad (83) \]
which smears out the \( \omega \)-dependence and results in a pure thermal spectrum. One can immediately observe that equation (83) is independent from the horizons of the non-extreme LDBHs similar to the other four-dimensional LDBH solutions \([8, 9]\) possessing one horizon. But, contrary to the others \([8, 9]\), the resulting high frequency temperature \(T_{\text{high}}\) (equation (83)) differs from the standard Hawking temperature \(T_H\) \([15]\), which is computed as usual by dividing the surface gravity by \(2\pi\),
\[ T_H = \frac{\kappa}{2\pi} = \frac{f'}{4\pi} \bigg|_{r=r_+} = \frac{b}{4\pi} \left( 1 - \frac{r_+}{r_1} \right), \quad (84) \]
Let us note that the same result for the \(T_H\) can be obtained from Hawking’s period argument of the Euclideanized line element. For this purpose we complexify the time in (29) by \(t \rightarrow i\tau\) and rearrange the terms so that the line element reads in the form \(R^2 \times S^2\), given by
\[ ds^2 \sim \left( \frac{d\rho}{\Sigma_{\rho}} \right)^2 + (\rho d\tau)^2 + \rho_+^2 (d\theta^2 + \sin^2 \theta \ d\phi^2). \quad (85) \]
Here $r_{0,\omega}$ stands for the value of the radial coordinate on the outer horizon (when it exists) and the constant $\Sigma$, reads

$$
\Sigma_ω = \frac{1}{2} \left( \frac{1}{Aω} - 2\bar{V} \right) \left[ 1 - \frac{4\ell_{\omega}}{\lambda_2 A^2} \sqrt{M + M^2 - \frac{4\omega}{\lambda_2 A^2} \left( \frac{1}{\pi} - 2\bar{V} \right)} \right],
$$

which relates to the period of the angle $\tau$, upon the overall multiplication by $\Sigma_ω$. Since $T_H$ is the inverse of the period we obtain

$$
T_H = \frac{1}{2\pi} \Sigma_ω,
$$

which is identical to (84), valid for double-horizon Hawking temperature. In order to find the vacuum ‘in’ and ‘out’ states for the scalar field we have to choose the metric such that both the surface gravity and mass of the black hole vanish. This can be done from (65) and (66), by choosing $c^2 = 4ab$ first, to make an extremal LDBH (with zero temperature), and next, to let $c \to 0$ to also make the mass zero. These conditions cast our LDBH metric into

$$
dx^2 = -bd\tau^2 + \frac{dr^2}{br} + A^2r^d dΩ^2.
$$

By simple arrangement this vacuum metric transforms into

$$
dx^2 = \rho^2(-d\tau^2 + dx^2 + dΩ^2)
$$

where

$$r = e^{\beta ω}, \quad t = \frac{β}{β} τ, \quad ρ = A e^{\frac{β}{ω} τ}, \quad β = A\sqrt{b}.
$$

The massless Klein–Gordon equation $\nabla^2 \Psi = 0$, with $\Phi = \frac{1}{ρ} \Psi$ takes the form

$$
\frac{1}{ρ^2} \left[ \frac{∂}{ω} - \frac{β^2}{4} + \ell(\ell + 1) \right] \Psi = 0.
$$

The vacuum ‘in’ and ‘out’ solutions for the scalar field are

$$
\Phi_{in} \sim \frac{1}{\sqrt{T}} e^{-iβ(σ - x)} \quad \Phi_{out} \sim \frac{1}{\sqrt{T}} e^{iβ(σ - x)}
$$

where $σ$ has the meaning from (71). Once these states propagate from vacuum they turn into thermal states as described above.

So the question arises here as in which case does the temperature equation (82) match with the value of $T_H$ in equation (84). The answer is absolutely related to the value of physical parameter, $σ$. Let us assume that the value of the parameter $σ$ is so great that it predominates the term $\frac{σ}{ω} \left( \frac{r_{0,\omega}}{r_\omega} \right)$ (but $\frac{ω}{ω} \left( \frac{r_{0,\omega}}{r_\omega} \right)$ is still comparable with $σ$) in the expression of temperature (82). Unless this assumption is not violated, the corresponding limit of $T$ will be $T_H$. In summary,

$$
T_H \approx \lim_{σ \to \frac{ω}{ω} \left( \frac{r_{0,\omega}}{r_\omega} \right) > 1} T \simeq \omega \ln \left\{ \exp \frac{σ}{ω} \left( \frac{r_{0,\omega}}{r_\omega} \right) \exp \frac{π}{ω} \left( \frac{r_{0,\omega}}{r_\omega} \right) \right\} \exp \frac{π}{ω} \left( \frac{r_{0,\omega}}{r_\omega} \right)
$$

$$
\approx \frac{b}{4\pi} \left( 1 - \frac{r_\omega}{r_{0,\omega}} \right).
$$

Another question may immediately come out: How does $σ$ maintain its predomination against $\frac{ω}{ω} \left( \frac{r_{0,\omega}}{r_\omega} \right)$? To clarify the question, one can check equation (71) in order to see that the predomination of $σ$ strictly depends on the negative values of $λ$. However, this is possible only with the case of $p^2 = q^2 Q^2 > l(l + 1)$. Hence, a significant remark is revealed that obtaining
Figure 1. Temperature $T$ as a function of $\omega$ for the non-extreme LDBHs in the case of $r_1, r_2 \neq 0$. The plots are governed by equation (82). Different line styles belong to different $|p|$-values: a dotted line corresponds to $|p| = 0.5$ (as an example of low $|p|$-values) and a solid line is for $|p| = 10$ (as an example for high $|p|$-values). The physical parameters in equation (82) are chosen as follows: $\ell = 1, b = 1, A = 1, r_1 = 0.5$ and $r_2 = 1$.

$T_H$ of the non-extreme LDBHs from the SCRSM, the only possibility is to consider charged scalar waves instead of chargeless ones.

Furthermore, we want to show the most intriguing figures about the spectrum temperature equation (82). To this end, first we plot $T$ versus frequency $\omega$ of non-extreme LDBHs with $r_1, r_2 \neq 0$ for low and high $|p|$-values, and display all graphs in figure 1. As can be seen from figure 1, in the high frequencies the thermal behaviors of the LDBHs with different $|p|$-values exhibit similar behaviors in which their temperatures approach to $T_{\text{high}}$ while $\omega \to \infty$. The plot with a low $|p|$-value in figure 1 does not behave like the Hawking temperature. On the other hand, the other plot in figure 1, which has a high $|p|$-value represents the Hawking temperature $T_H$ in the low frequencies ($\omega > 0$). Beside this, once the parameter $\sigma$ loses its predomination against $\frac{\omega}{\frac{\pi}{2} (r_2-r_1)}$, the latter plot increases to reach the $T_{\text{high}}$ with increasing frequency as well. In the case of the non-extreme LDBHs with $r_1 = 0$, there is no difference between $T_{\text{high}}$ and $T_H$ because of equation (84), and at the low $|p|$-values the temperature $T$ exhibits similar behavior as in the case $r_1, r_2 \neq 0$, which is the well-known thermal character in the EMD theory [8]. By the way, one should exclude $\omega = 0$ during the plotting of the temperature, because it causes uncertainty for the temperature equation (82) and physically this case is not acceptable since we consider the propagation of scalar waves. Figure 2 presents the $T$ versus frequency $\omega$ graph of non-extreme LDBHs with $r_1 = 0$ at a high $|p|$ value. In this figure, it is illustrated that by increasing the frequency from $0^+$, the temperature first starts from a constant value, which is $T_H$ and then makes a peak (not much higher than $T_H$), and then decreases back to $T_H$ while $\omega \to \infty$. Rousingly, one can observe that the
behavior of the graph in figure 2 is very similar to the graph obtained from the well-known Planck radiation formula, see for instance [16]. Besides, both figure 1 and figure 2 show us that whenever high $|p|$-values are present, the frequency of the scalar wave needed to detect the temperature of the LDBHs to be the Hawking temperature $T_H$ can either be very high (only for the $r_1 = 0$ case, which is already known before [8]) or low. The latter information about the relationship between $T_H$ and low frequencies is completely new to us, and may play a crucial role for the thermal detection of the LDBHs in the future.

5. Singularity analysis

In section 2, we present a solution in four-dimensional static spherically symmetric EMD theory that incorporates two Liouville-type potential terms coupled with gravity together with magnetically charged dilatonic parameters. We have clarified that the solution possess a central singularity which is a characteristic feature for spherically symmetric systems. In the solutions that admit black holes this singularity is clothed by horizons. However, there are cases in which this singularity is not hidden behind horizons. In such cases the singularity is called a naked singularity.

In classical general relativity, singularities are described as incomplete geodesics. This simply means that the evolution of time-like or null geodesics is not defined after a finite proper time. There is a general consensus that removal of classical singularities is not only important for quantum gravity but also for other fundamental theories. In view of this consensus, we are aiming to analyze whether these classical naked singularities that occur in the general solution
Figure 3. (a) Penrose diagram for the no-horizon case, $\tilde{M}^2 - 4ab < 0$ in which $r = 0$ is a naked singularity. (b) Penrose diagram of the LDBH with two distinct horizons $r_1 \neq r_2$, where $r = 0$ is a time-like singularity. (c) Penrose diagram of the LDBH with a single horizon at $r = r_0$. Singular nature of $r = 0$ is not affected.
and Marolf (HM) [10]. HM have proposed a criterion to test the classical singularities with quantum test particles that obey the Klein–Gordon equation for static spacetimes having time-like singularities. The criterion of HM has been applied successfully for several spacetimes [17–19] within the context of quantum mechanical concepts. Among the others, HM have already analyzed the quantum singularity for the extreme case of the charged dilatonic black hole in the absence of Liouville-type potentials. They confirmed that for a specific interval of dilaton parameter, the singularity is quantum mechanically regular. The brief review of the criterion is as follows.

A scalar quantum particle with mass $m$ is described by the Klein–Gordon equation

$$ \nabla^\mu \nabla_\mu - m^2 \psi = 0. $$

This equation can be written by splitting the temporal and spatial portion into

$$ \frac{\partial^2 \psi}{\partial t^2} = -A \psi, $$

where $f = -\xi^\mu \xi_\mu$ with $\xi^\mu$ the time-like Killing field, while $D_i$ is the spatial covariant derivative defined on the static slice $\Sigma_1$. Then, the Klein–Gordon equation for a free relativistic particle satisfies

$$ i \frac{\partial \psi}{\partial t} = \sqrt{A} \psi, $$

with the solution

$$ \psi(t) = \exp(i t \sqrt{A}) \psi(0). $$

If the extension of the operator $A$ is not essentially self-adjoint, the future time evolution of the wavefunction is ambiguous. Then, the HM criterion defines the spacetime quantum mechanically singular. However, if there is only one self-adjoint extension, the operator $A$ is said to be essentially self-adjoint and the quantum evolution $\psi(t)$ is uniquely determined by the initial condition. According to the HM criterion, this spacetime is said to be quantum mechanically regular. Consequently, a sufficient condition for the operator $A$ to be essentially self-adjoint is to investigate the solutions satisfying the following equation (see [20] for a detailed mathematical background):

$$ A \psi \pm i \psi = 0. \tag{93} $$

This equation admits a separable solution and hence the radial part becomes

$$ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{f R^2} \frac{\partial (f R^2)}{\partial r} \frac{\partial \phi}{\partial r} - \frac{l(l+1)}{f r^2} \phi - \frac{m^2}{f} \phi \pm \frac{i \phi}{f} = 0, \tag{94} $$

where $l(l+1) \geq 0$ is the eigenvalue of the Laplacian on the 2-sphere. The necessary condition for the operator $A$ to be essentially self-adjoint is that at least one of the solutions to this equation fails to be of finite norm when $r \to 0$. In summary, the self-adjointness of the operator $A$ implies the well-posedness of the initial value problem. Therefore, the suitable norm $\|\phi\|$ for this case is the Sobolev norm which is used first time within this context by Ishibashi and Hosoya [20] defined by

$$ \|\phi\|^2 = \frac{q^2}{2} \int R^2 f^{-1} |\phi|^2 \, d\mu \, dr + \frac{1}{2} \int R^2 f \left| \frac{\partial \phi}{\partial r} \right|^2 \, d\mu \, dr, \tag{95} $$

where $q^2$ is a positive constant and $d\mu$ is the volume element on the unit 2-sphere. The regularity of the central singularity at $r = 0$ in the quantum mechanical sense requires that the squared norm of the solutions of equation (94) should be divergent for each $l(l+1)$ and each sign of imaginary term. The norm $\|\phi\|$ is divergent for $l(l+1) > 0$ if it is for $l = 0$, so essential self-adjointness will be examined for the $l = 0$ ($S$-wave) case. This implies essential self-adjointness for the operator $A$. Furthermore, we assume a massless case (i.e. $m = 0$), and ignore the term $\pm \frac{i \phi}{f}$ (since it is negligible near the origin).
5.1. A more general case

The general solution for any value of $\alpha^2$ which is related to the dilaton parameters $\gamma_1$ and $\gamma_2$ is given in equation (18). Since this solution is complicated enough for integrability, we consider the specific values of $\alpha^2 = 3$ and $V_1 = 0$. Hence, the general solution becomes

$$f(r) = \frac{16\alpha_2}{\sqrt{r}} (r - r_2)(r - r_1),$$

where

$$\tilde{r}_{1,2} = \frac{M \pm \sqrt{M^2 - 4\alpha_1\alpha_2}}{2\alpha_2}, \quad a_1 = \frac{Q^2\lambda_1}{4A^4}, \quad a_2 = \frac{1}{12} \left( \frac{Q^2\lambda_2}{A^4} - V_2 \right).$$

The extreme case occurs when $M^2 = 4\alpha_1\alpha_2$. In this case there is one horizon only and it is given by $r_h = \frac{M}{2\alpha_2}$. If $M > \sqrt{M^2 - 4\alpha_1\alpha_2}$ and $M^2 > 4\alpha_1\alpha_2$, this particular case admits two horizons given by $\tilde{r}_{1,2}$. However, if $M^2 - 4\alpha_1\alpha_2 < 0$, no black hole forms and hence, the singularity at $r = 0$ becomes naked.

As a requirement of the HM criterion, the singularity at $r = 0$ must have a time-like character. This can be checked if one introduces a tortoise coordinate defined by $r^* = \int \frac{dr}{f}$ and takes its limit as $r \to 0$. We found that the limit is finite. Therefore, the singularity is time-like. The solution for equation (94) is

$$\phi(r) = \frac{A^{-\frac{1}{2}}}{16\alpha_2} \ln \left| \frac{r - \tilde{r}_2}{r - \tilde{r}_1} \right|.$$  

The first and the second terms of the squared norm defined in equation (95) are finite. Therefore the spacetime is quantum mechanically singular. For the double-horizon case, $M^2 - 4\alpha_1\alpha_2 > 0$, which implies $r_1 \neq r_2 \neq r_1$, the time-like singularity at $r = 0$ is not naked, and its Penrose diagram is depicted in figure 3(a). However, for a special case $\lambda_1 = 0$, the solution to equation (94) is

$$\phi(r) = \frac{A^2}{bM} \ln \left| \frac{r - \tilde{r}_h}{r} \right|.$$  

5.2. The linear dilaton case

The metric function for the linear dilaton case can be written as (from equation (65))

$$f(r) = \frac{b}{r}(r - r_2)(r - r_1),$$

where

$$r_{1,2} = \frac{\tilde{M} \pm \sqrt{\tilde{M}^2 - 4ab}}{2b}, \quad a = \frac{\lambda_1}{A^2\lambda_2}, \quad b = \left( \frac{1}{A^2} - 2\tilde{V} \right).$$

The naked singularity occurs when $\tilde{M}^2 - 4ab < 0$. The tortoise coordinate $r^* = \int \frac{dr}{f}$ is finite and indicates a time-like character at $r = 0$. The Penrose diagram of this particular case is shown in figure 3(a). The radial part of the separable equation (94) has a solution for the linear dilaton case as

$$\phi(r) = \frac{1}{bA^2(r_2 - r_1)} \ln \left| \frac{r - r_2}{r - r_1} \right|.$$  

The first and the second terms of the squared norm defined in equation (95) are finite. Therefore the spacetime is quantum mechanically singular. For the double-horizon case, $\tilde{M}^2 - 4ab > 0$, which implies $r_1 \neq r_2 \neq 0 \neq r_1$, the time-like singularity at $r = 0$ is not naked, and its Penrose diagram is depicted in figure 3(b). However, for a special case $\lambda_1 = 0$, the solution to equation (94) is

$$\phi(r) = \frac{A^2}{b\tilde{M}} \ln \left| \frac{r - \tilde{r}_h}{r} \right|.$$
where $r_h = \frac{M}{\rho}$. The first term of the squared norm (95) is finite, whereas the second term behaves as
\[
\sim (\ln |r|)|_{r=0} \rightarrow \infty. \tag{101}
\]
Hence, under the condition $\lambda_1 = 0$, the central classical singularity becomes quantum mechanically non-singular. When we have a single-horizon, with the choice $\lambda_1 = 0$, for example, the singularity $r = 0$, is shown in the Penrose diagram (figure 3(c)).

5.3. Near horizon behaviors

In order to study the global behavior of our solution, at least for specific choices of parameters, and to be able to sketch the Penrose diagrams, we cast the metric into the form apt for near horizons. With the choice $V_1 = V_2 = 0$ our metric function $f(r)$ takes the form
\[
f(r) = \frac{1}{A^2 r^{1+\delta}} \left( r^2 - \frac{4MQL}{1 + \delta} r + \frac{\lambda_1}{\lambda_2} \frac{1 - \delta}{1 + \delta} \right), \tag{102}
\]
where
\[
\delta = \frac{1 - \alpha^2}{1 + \alpha^2}, \quad -1 < \delta < 1 \tag{103}
\]
and
\[
A^2 = \frac{2}{1 - \delta} \lambda_2 Q^2. \tag{104}
\]
Upon the choice of parameters involved, we can have double, single or no-horizon cases. By a redefinition for time, our line element reads, in brief,
\[
d\tilde{s}^2 = A^2 d\tilde{s}^2 \tag{105}
\]
where
\[
d\tilde{s}^2 = -\frac{(r - r_{-})(r - r_{+})}{r^{1+\delta}} dr^2 + \frac{r^{1+\delta}}{(r - r_{-})(r - r_{+})} dr^2 + r^{1+\delta} d\Omega^2 \tag{106}
\]
and
\[
r_{\pm} = \frac{2MQL}{1 + \delta} \left( 1 \pm \sqrt{1 - \frac{\lambda_1}{\lambda_2} \frac{1 - \delta^2}{4M^2QL^2}} \right). \tag{107}
\]
We note that the global structure of $d\tilde{s}^2$ is same with $ds^2$, and therefore we analyze $d\tilde{s}^2$. The singularity structure of (106) can be seen from the Kretchmann scalar $K$, which reads
\[
\lim_{r \rightarrow 0} K \sim \begin{cases} \frac{r^{-2(3+\delta)}}{\delta \neq \pm 1}, & \delta \neq \mp 1 \\ \frac{r^{-3}}{\delta = \pm 1} \end{cases} \tag{108}
\]
\[
\lim_{r \rightarrow \infty} K \sim \begin{cases} \frac{r^{-2(3+\delta)}}{\delta \neq \pm 1}, & \delta \neq \mp 1 \\ \text{constant}, & \delta = \pm 1 \end{cases} \tag{109}
\]
We concentrate now on the near horizon geometry by the following reparametrization, with new coordinates $(\tilde{r}, \tilde{t})$:
\[
\tilde{r}_- = r_-, \quad \tilde{r}_+ = r_+ + \epsilon b_0, \quad r = r_0 + \epsilon \tilde{r}, \quad t = \frac{1}{\epsilon} \tilde{t} \tag{110}
\]
where $r_0$ and $b_0$ are constants and $\epsilon \rightarrow 0$, is a small parameter. We obtain, upon relabeling $\tilde{r} = r$ and $\tilde{t} = t$,
\[
d\tilde{s}^2 = -\frac{r(r - b_0)}{r^{1+\delta}} dr^2 + \frac{r^{1+\delta}}{(r - b_0)} dr^2 + r^{1+\delta} d\Omega^2. \tag{111}
\]
In figure 4 we plot the Penrose diagrams for the specific cases \( b_0 = 0 \) and \( b_0 > 0 \). The case \( b_0 < 0 \) does not differ from the case of \( b_0 = 0 \), and as a matter of fact this particular case corresponds to the BR limit, which is known to correspond to the near extreme geometry of the RN black hole. The more standard BR is obtained from the present one by the inversion \( r \rightarrow \frac{1}{r} \).

6. Conclusion

We have shown that a dilaton field with Liouville’s potential interpolates between the RN black hole and non-black hole BR solution. The general solution for the metric function suggests that the dilatonic presence induces significant changes in the solutions; for example, asymptotically flat black holes become non-asymptotically flat. It is shown, through radial linear perturbation, that dilatons can add instabilities to the otherwise stable RN black hole.
whereas BR remains stable. Also from the thermodynamic point of view, by invoking specific heat the system can be tested against stability and phase transition. In the non-extreme LDBH case, which is a particular solution of our general solution the statistical and the standard Hawking temperatures are compared and plotted. It has been pointed out that with charged scalar waves and spin-weighted coupling the two results match for the case of double horizons. We recall that in the single horizon case, in spite of the existence of a linear dilaton such a discrepancy does not arise. It is remarkable that the spin-weighted spheroidal harmonics serve to convert the diverging temperature spectrum into a finite one. The presence of dilatons makes the spacetime highly singular at $r = 0$. Whether these singularities are also quantum mechanically singular or not, we send a quantum test particle and apply the criterion due to Horowitz and Marolf. We find that under a certain choice of our parameters the naked singularities create an infinite repulsive quantum potential so that the particle feels a regular spacetime.

References

[1] Gibbons G and Maeda K 1988 Nucl. Phys. B 298 741
Yazadjiev S S 2005 Class. Quant. Grav. 22 3875
[2] Garfinkle D, Horowitz G t and Strominger A 1991 Phys. Rev. D 43 3140
[3] Chan K C K, Horne J H and Mann R B 1995 Nucl. Phys. B 447 441
Bose S and Lohiya D 1999 Phys. Rev. D 59 044019
Kryiakopoulos E 2006 Class. Quantum Grav. 23 7591
[4] Halilsoy M 1993 Gen. Rel. Grav. 25 275
Clément G and Leygnac C 2004 Phys. Rev. D 70 084018
[5] Bertotti B 1959 Phys. Rev. 116 1131
Robinson I 1959 Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. 7 351
[6] Mazharimousavi S H and Halilsoy M 2008 J. Cosmol. Astropart. Phys. JCAP12(2008)005
Mazharimousavi S H, Halilsoy M and Amirabi Z 2010 Gen. Rel. Grav. 42 261
[7] Brown J D and York J W 1993 Phys. Rev. D 47 1407
Brown J D, Creighton J and Mann R B 1994 Phys. Rev. D 50 6394
[8] Clément G, Fabris J C and Marques G T 2007 Phys. Lett. B 651 54
[9] Mazharimousavi S H, Sakalli I and Halilsoy M 2009 Phys. Lett. B 672 177
[10] Horowitz G T and Marolf D 1995 Phys. Rev. D 52 5670
[11] Yazadjiev S S 2005 Phys. Rev. D 72 044006
[12] Wald R M 1980 J. Math. Phys. 21 2802
[13] Goldberg J N, Macfarlane A J, Newman E T, Rohrlich F and Sudarshan E C G 1967 J. Math. Phys. 8 2155
[14] Abramowitz M and Stegun I A 1965 Handbook of Mathematical Functions (New York: Dover)
[15] Wald R M 1984 General Relativity (Chicago and London: The University of Chicago Press)
[16] Serway R A, Moses C J and Moyer C A 1989 Modern Physics (Orlando: Saunders)
[17] Heliwell T M, Konkowski D A and Arndt V 2003 Gen. Rel. Grav. 35 79
[18] Pitelli J P M and Letelier P S 2007 J. Math. Phys. 48 092501
[19] Pitelli J P M and Letelier P S 2008 Phys. Rev. D 77 124030
[20] Ishibashi A and Hosoya A 1999 Phys. Rev. D 60 104028