Conformal Dilaton Gravity and Warped Spacetimes in 5D

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Abstract. An exact time-dependent solution of a black hole is found in conformally invariant gravity on a warped Randall-Sundrum spacetime, by writing the metric $g_{\mu\nu} = \omega^{\frac{4}{n-2}} \tilde{g}_{\mu\nu}$. Here $\tilde{g}_{\mu\nu}$ represents the "un-physical" spacetime and $\omega$ the dilaton field, which will be treated on equal footing as any renormalizable scalar field. In the case of a five-dimensional warped spacetime, we write $(\tilde{g}_{\mu\nu})^{(4)} = \bar{\omega}^{2(4)} \tilde{g}_{\mu\nu}$. Both $\omega$ and $\bar{\omega}$ can be used to describe the different notion the in-going and outside observers have of the Hawking radiation by using different conformal gauge freedom. The disagreement about the interior of the black hole is explained by the antipodal map of points on the horizon. The free parameters of the solution can be chosen in such a way that $\tilde{g}_{\mu\nu}$ is singular-free and topologically regular, even for $\omega \to 0$. It is remarkable that the 5D and 4D effective field equations for the metric components and dilaton fields can be written in general dimension $n = 4, 5$. It is conjectured that, in context of quantization procedures in the vicinity of the horizon, unitarity problems only occur in the bulk at large extra-dimension scale. The subtraction point in an effective theory will be in the UV only in the bulk, because the use of a large extra dimension results in a fundamental Planck scale comparable with the electroweak scale.

Keywords: conformal invariance, dilaton field, black hole complementarity, brane world models

1. Introduction

Our universe could be a 4-surface ("brane") embedded in a 5-dimensional spacetime, where the extra dimension represents the "bulk". We will focus on the "warped variant of these so-called brane-worlds gravity models, originally introduced by Randall and Sundrum (RS) and extended by Shiromizu, et al. See also the overview work of Maartens. At the same time, Arkani-Hamed, et al. (ADD) considered also higher dimensional models which put forward the idea that a large volume for the compact extra dimensions will lower the Planck scale. Their models are not restricted to 5 dimensions,
as is the case in the RS model with one "preferred" extra dimension. The advantage of these models is that they could solve the hierarchy problem, i.e., why there is such a huge gap between the electroweak and Planck scale. It has also promising results for achieving late-time acceleration of the universe without dark energy or matter (DGP models). However, one should be careful in changing general relativity theory (GRT). If one adapt that Nature will choose the most simple Einstein-Hilbert (EH) action, then, for example, the cosmological constant should not be included in the EH action. One avoids in this way the huge discrepancy between the contribution from zero-point fluctuations in quantum field theory and the predicted value in GRT (some 120 orders of magnitude).

Another unsolved issue, is the complementarity near a black hole. In the vicinity of the horizon, curvature will be huge, so quantum effects will become important. Up till now, no convincing theory of quantum gravity is available. Many attempts were made in order to make a renormalizable and unitary quantum gravity model. I refer to the standard works on these subjects, for example Parker, et al and the fundamental work of 't Hooft, et al. A well know method in constructing a renormalizable model is to add fourth order derivative terms in the curvature tensor (Euler-term). However, one looses unitarity. Also the "old" effective field theory (EFT) has its problems. One ignores what is going on at high energy. In order to solve the anomalies one encounters in calculating the effective action, one can apply the so-called conformal dilaton gravity (CDG) model. CDG is a promising route to tackle the problems arising in quantum gravity model, such as the loss of unitarity close to the horizon. One assumes local conformal symmetry, which is spontaneously broken (for example by a quartic self-coupling of the Higgs field). Changing the symmetry of the action was also successful in the past, i.e., in the SM of particle physics. A numerical investigation of the coupling was recently done. The key feature in CDG, is the splitting of the metric tensor $g_{\mu\nu} = \omega^{\frac{4}{n-2}}(x)\tilde{g}_{\mu\nu}$, with $\omega$ the dilaton field. Applying perturbation techniques (and renormalization/dimensional regularization), in order to find the effective action and its divergencies, one first integrate over $\omega$ (shifted to the complex contour), considered as a conventional renormalizable scalar field and afterwards over $\tilde{g}_{\mu\nu}$ and matter fields. The dilaton field is locally unobservable. It is fixed when we choose the global spacetime and coordinate system. If one applies this principle to a black hole spacetime, then the energy-momentum tensor of $\omega$ determines the Hawking radiation. We shall see that it can be calculated in our model. When $\tilde{g}_{\mu\nu}$ is flat, then the handling of the anomalies simplifies considerably. When $\tilde{g}_{\mu\nu}$ is non-flat, the problems are more deep-seated.

In this context, the modification of GRT by an additional spacetime dimension could be an alternative compromise, because Einstein gravity on the brane will be modified by the very embedding itself and opens up a possible new way to address the dark energy problem. These models can be applied to the standard Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime and the modification on the Friedmann equations can be investigated. Recently, Maldacena, et al., applies the RS model
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to two black hole spacetimes and could construct a traversable macroscopic wormhole solution by adding only a 5D U(1) gauge field (see also Maldacena\textsuperscript{27}). However, an empty bulk would be preferable. In stead, one can investigate the contribution of the projected 5D Weyl tensor on the 4D brane. It carries information of the gravitational field outside the brane. If one writes the 5D Einstein equation in CDG setting, it could be possible that an effective theory can be constructed without an UV cutoff, because the fundamental scale $M_5$ can be much less than the effective scale $M_{Pl}$ due to the warp factor. The physical scale is therefore not determined by $M_{Pl}$.

In section 2 we review the 5D RS model. In section 3 we outline the principles of CDG. In section 4 we apply the model to a black hole spacetime. In section 5 we present a physical interpretation. All the calculations were done by computer-aided software.\textsuperscript{‡}

2. Brane-world models

2.1. The Randall-Sundrum model

In the brane-world scenario under consideration (the RS1 model), the 4-dimensional world is described by a domain wall $\left((^4M, ^4g_{\mu\nu})\right)$ in a 5-dimensional spacetime $\left((^5M, ^5g_{\mu\nu})\right)$. All the standard model fields are confined to the brane. On the other hand, gravity can propagate in the full spacetime. The Planck scale is now an effective coupling constant, describing gravity on scales much larger than the extra dimension ($d$) and related to the fundamental scale via the volume of the extra dimension $M_{pl}^2 \sim M_{4+d}^2$, with $L$ the the length scale of the extra dimension. If $L \sim \frac{1}{M_{pl}}$, then $M_{4+d} \sim M_{pl}$. If the extra dimensional volume is far above the Planck scale, then the true fundamental scale $M_{4+d}$ can be much less than the effective scale $\sim 10^{19}$GeV. From a different point of view, one could say that the dynamics of the higher dimensional gravitational fields influence the brane dynamics. We used the notations of Shiromizu, et al\textsuperscript{3} and Maartens\textsuperscript{4} The 5-dimensional Einstein equations are

\begin{equation}
(5)G_{\mu\nu} = -\Lambda_5 (5)g_{\mu\nu} + \kappa_5^2 (5)T_{\mu\nu},
\end{equation}

with $\kappa_5^2 = 8\pi G_5 = \frac{8\pi}{M_5^2}$. The negative bulk tension $\Lambda_5$ is offset by the positive brane tension $\lambda$ (vacuum energy). One writes the 5D metric

\begin{equation}
(5)g_{\mu\nu} = (4)g_{\mu\nu} + n_\mu n_\nu \quad \text{(5)ds}^2 = (4)g_{\mu\nu}(x^\alpha, y)dx^\mu dx^\nu + \Gamma(y)dy^2,
\end{equation}

with $n^\mu$ the unit normal to the brane and $y$ the extra dimension. The extrinsic curvature of $y = \text{const}$ surfaces is given by

\begin{equation}
K_{\mu\nu} = (4)g_{\alpha\nu} (5)\nabla_\alpha n_\nu.
\end{equation}

The Gauss-Codazzi equation determines the change of $K_{\mu\nu}$ along $y = \text{const}$.

\begin{equation}
(4)\nabla_\mu K^\mu_{\nu} - (4)\nabla_\nu K = (5)R_{\mu\sigma} (4)g_{\nu}^\mu n_\sigma
\end{equation}

\textsuperscript{‡} the interested reader can obtain from the author the Maple file with all the calculations used in this manuscript.
The relation between the 4D and 5D Ricci tensors is given by the Gauss equation

\[ (4)R_{\mu\nu} = (5)R_{\rho\sigma}^{(4)}g_\rho^{(4)}g_\sigma^{(4)} - (5)R^{\alpha\beta\gamma\delta}_{\rho\sigma}n_\alpha^{(4)}g_\mu^{(4)}n_\gamma^{(4)}g_\nu^{(4)} + KK_{\mu\nu} - K_{\mu}^{\alpha}K_{\nu\alpha}. \]  (5)

The second term on the right-hand side is Eq. (1) is zero, if there are no matter fields in the bulk \((5)T_{\mu\nu} = 0\). Using the Israel’s junction conditions and by imposing \(Z_2\)-symmetry, one obtains the effective induced Einstein equations on the brane

\[ (4)G_{\mu\nu} = -\Lambda_{\text{eff}}^{(4)}g_{\mu\nu} + \kappa_4^2T_{\mu\nu} + \frac{2}{3}\kappa_5^2F_{\mu\nu} - E_{\mu\nu}, \]  (6)

with \((4)T_{\mu\nu}\) the energy-momentum tensor on the brane, \(S_{\mu\nu}\) the quadratic contribution of the energy-momentum tensor \((4)T_{\mu\nu}\) arising from the extrinsic curvature terms in the projected Einstein tensor,

\[ S_{\mu\nu} = \frac{1}{12}(4)T^{(4)}T_{\mu\nu} - \frac{1}{4}(4)T_{\mu\alpha}^{(4)}T_{\nu}^{\alpha} + \frac{1}{24}(4)g_{\mu\nu}\left(3(4)T^{(4)}T_{\alpha\beta}^{(4)}T^{\alpha\beta} - (4)T^2\right). \]  (7)

and \(F_{\mu\nu}\) is the correction term from the bulk energy-momentum tensor,

\[ F_{\mu\nu} = (5)T_{\alpha\beta}^{(4)}g_\mu^{(4)}g_\nu^{(4)} + (4)g_{\mu\nu}\left(5T_{\alpha\beta}^{(4)}n^{\alpha}n^{\beta} - \frac{1}{4}(5)T\right). \]  (8)

In the RS model, the effective cosmological constant is \(\Lambda_{\text{eff}} = \frac{1}{2}(\Lambda_5 + \frac{\lambda^2}{6})\), with \(\lambda\) the brane tension, \(\sim \frac{M_5^2}{l^2}\) and \(l\) the effective size of the extra dimension. Because there is no deviation of Newton’s law down to \(O(10^{-1})\), we have \(\lambda > (1TeV)^4, M_5 > 10^5TeV\). The 4 dimensional cosmological constant is zero, when the RS balance between \(\Lambda_5\) and \(\lambda\) is broken. The fundamental gravity scale is lowered, possibly even down to the electroweak scale \(\sim TeV\). For an empty bulk, \(F_{\mu\nu} = 0\). The last term in Eq. (6) represents the projected Weyl term and is a part of the 5D Weyl tensor that carries information of the gravitational field outside the brane.

\[ E_{\mu\nu} = (5)C_\beta^{\alpha\mu}n_\alpha n^{\rho(4)}g_\mu^{(4)}g_\nu^{(4)}. \]  (9)

Note that \((4)T_{\mu\nu}\) is the energy-momentum tensor of the particles and fields confined to the brane (plus the term \(\lambda(4)g_{\mu\nu}\)). The extrinsic curvature of the brane can then be expressed as

\[ K_{\mu\nu} = -\frac{1}{2}\kappa_5^2\left((4)T_{\mu\nu} - \frac{1}{3}(4)T\right). \]  (10)

The 4D contracted Bianchi identity becomes (for empty bulk)

\[ (4)\nabla^{\mu}E_{\mu\nu} = \kappa_4^2(4)\nabla^{\mu}T_{\mu\nu} + \kappa_5^4\nabla^{\mu}S_{\mu\nu}. \]  (11)

It relates \((4)\nabla^{\mu}E_{\mu\nu}\) to the several “matter terms” on the right-hand side of Eq. (11) and expresses the fact that 4D spacetime variation in the matter-radiation can source KK-modes. We shall see in an application in section 4, that this identity will lead
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to a third-order partial differential equation, which forms with the effective Einstein equations, a closed system of pde’s. Further, Eq.(4) becomes

\begin{equation}
(4) \nabla_\mu K^\mu - (4) \nabla_\nu K^\nu = \kappa_5^2 (4) T_{\rho\sigma}^\nu n^\sigma (4) g^\rho_\nu. \tag{12}
\end{equation}

One must realize that in the conformal dilaton application, \((4) T_{\mu\nu}\) contains also a contribution from the dilaton field \(\omega\), which is a "scale"-term by the splitting \(g_{\mu\nu} = \omega^2 \tilde{g}_{\mu\nu}\) (section 3). The dilaton will play the role of a "scalar field". In general, the covariant splitting which is applied in the warped spacetime model, has the advantage that it shows a clear insight into the interplay between the 4D and 5D gravitational fields. However, one needs to boundary conditions on the brane for a consistent dynamical evolution.

2.2. Application to the FLRW spacetime

In a former study we investigated the brane-world model of Randall and Sundrum on the warped 5 dimensional spacetime\[23\]

\begin{equation}
ds^2 = W(t, r, y)^2 \left[ e^{2\gamma(t,r)-2\psi(t,r)} \left( -dt^2 + dr^2 \right) + e^{2\psi(t,r)} dz^2 + r^2 \left( d\phi + N(t, r) dt \right)^2 \right] + \Gamma(y) dy^2. \tag{13}
\end{equation}

In first instance, we will take \(\Gamma(y) = 1\). From the 5D Einstein equations

\begin{equation}
(5) G_{\mu\nu} + \Lambda_5 (5) g_{\mu\nu} = 0, \tag{14}
\end{equation}

one obtains for the warpfactor

\begin{equation}
W(t, r, y) = W_1(t, r) W_2(y), \tag{15}
\end{equation}

with

\begin{equation}
W_2(y) = e^{\sqrt{-\frac{\Lambda_5}{\kappa_5}} (y-y_0)} \tag{16}
\end{equation}

and

\begin{equation}
W_1(t, r)^2 = \pm \frac{1}{\tau r} \left( d_1 e^{\sqrt{2\tau} r} - d_2 e^{-\sqrt{2\tau} r} \right) \left( d_3 e^{\sqrt{2\tau} r} - d_4 e^{-\sqrt{2\tau} r} \right), \tag{17}
\end{equation}

where \(\tau\) and \(d_i\) are constants. One recognizes the well-known RS warpfactor \(W_2(y)\). It turns out that \(W_1\) cannot be isolated from the effective 4D brane equations.

2.3. Application to a black hole spacetime

Let us now apply the equation on the spacetime (a 4D BTZ mode\[24,25,29-31\])

\begin{equation}
ds^2 = W(t, r, y)^2 \left[ -N(t, r)^2 dt^2 + \frac{1}{N(t, r)^2} dr^2 + dz^2 + r^2 (d\phi + N(t, r) dt)^2 \right] + dy^2. \tag{18}
\end{equation}

Again, from the 5D Einstein equations, one obtains for \(W(t, r, y) = W_1(t, r) W_2(y)\), with \(W_2(y)\) the same expression, i.e., Eq.(16). However, \(W_1(t, r)\) cannot be isolated. One needs the 4D effective Einstein equations. It will become clear, that \(W_1\) is related
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...to the scale of spacetime of the geometry near horizon region. In order to make the connection with the CDG model, one would like to write the metric as

$$ds^2 = W(t, r, y)^2 \left[ -N(t, r)^2 dt^2 + \frac{1}{N(t, r)^2} dr^2 + dz^2 + r^2 (d\varphi + N_\varphi(t, r) dt)^2 + dy^2 \right]. \quad (19)$$

From the 5D Einstein equations, however, we then obtain that $W_2(y) = \text{const.}$

3. Conformal dilaton gravity

3.1. The basics

Where Einstein's theory of general relativity treats the large-scale behavior of gravity, the Standard Model (SM) of particle physics is the theoretical framework where, on small-scale, the other three forces of nature are explained. Combining these two theories leads to fundamental difficulties regarding the nature of spacetime. It is one of the greatest challenges of theoretical physics. It is believed that the unification has taken place close to the Planck scale, $1.6 \times 10^{-35} m,$ so far out of reach of experimental verification. It is conjectured, however, that close to the horizon of a black hole, quantum gravity effects will be manifest. So observations on black holes could be a way to collect quantum-gravitational information. It is well known that Einstein gravity, when quantized, is unitary, but nonrenormalizable. In general, there is no obvious way to attain renormalizability without violating unitarity. Higher order derivative theories turn out to be renormalizable at the one-loop quantum level, but lacks unitarity of the S-matrix. In the low energy domain (well below the UV cutoff), one can construct an effective field theory that can be quantized and is renormalizable. However, just as in Standard Model (SM), one needs gauge-fixing terms as well as Faddeev-Popov ghost fields. However, this route runs into problems when one tries to incorporate dark mass, dark energy and singularities. CDG could offer an alternative. Conformal gravity is part of the covariant approach to quantum gravity. There are several methods to attack the consequences of conformal symmetry at the quantum level. We follow the method initiated by 't Hooft. See also Alvarez, et al. and Codello, et al. The fundamental calculation of the divergences of the action involving the gravitational field, where already done long time ago. See also the overview work of Parker. Let us summarize the CDG idea. One can split the spacetime (in the 5D case)

$$(^5g_{\mu\nu} = \omega^{4/3(5)} \tilde{g}_{\mu\nu}, \quad (20)$$

where $\omega$ is a dilaton metascalar, encoding all the scale dependencies. One can call the ”un-physical” $\tilde{g}_{\mu\nu}$ a metatensor, i.e., it transforms as a tensor, but contains powers of the Jacobian of the coordinate transformation. One can easily prove that the action (in n dimensions and without matter terms)

$$S = \int d^n x \sqrt{-\tilde{g}} \left[ \frac{1}{2} \xi \omega^2 \tilde{R} + \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \omega \partial_\nu \omega + \Lambda_\kappa \frac{4}{n-2} \xi \frac{n^2}{n^2-2} \omega \frac{2n}{n-2} \right] \quad (21)$$
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is conformal invariant under
\[ g_{\mu\nu} \rightarrow \Omega^{\frac{4}{n-2}} g_{\mu\nu}, \quad \omega \rightarrow \Omega^{-\frac{n-2}{2}} \omega. \]  
(22)

The coupling constant $\Lambda$ is dimensionless in any dimension. It fulfills also the off-shell Ward identity. The constant $\xi = \frac{n-2}{4(n-1)}$. One re-scaled $\omega \rightarrow \kappa \sqrt{\xi} \omega$, with $\frac{1}{\kappa^2} = \frac{M_{pl}^{n-2}}{16\pi G_N}$. So Newton’s constant has disappeared completely. Variation of the action leads to the field equations
\[ \xi \omega \tilde{R} - \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \omega - \frac{2n}{n-2} \Lambda \kappa \frac{4}{n-2} \xi \frac{n}{n-2} \omega^{\frac{n-2}{2}} = 0 \]  
(23)

and
\[ \omega^2 \tilde{G}_{\mu\nu} = T^\omega_{\mu\nu} - \Lambda \tilde{g}_{\mu\nu} \kappa \frac{4}{n-2} \xi \frac{n}{n-2} \omega^{\frac{2n}{n-2}}, \]  
(24)

with
\[ T^\omega_{\mu\nu} = \tilde{\nabla}_\mu \tilde{\nabla}_\nu \omega^2 - \tilde{g}_{\mu\nu} \tilde{\nabla}^2 \omega^2 + \frac{1}{\xi} \left( \frac{1}{2} \tilde{g}_{\alpha\beta} \tilde{g}_{\mu\nu} - \tilde{g}_{\mu\alpha} \tilde{g}_{\nu\beta} \right) \partial^\alpha \omega \partial^\beta \omega. \]  
(25)

The covariant derivatives are taken with respect to $\tilde{g}_{\mu\nu}$.

3.2. Notes on anomalies

In general, one conjectures that conformal invariance is an exact symmetry, comparable with the Higgs mechanism. The effective action will stay locally conformal invariant after the $\omega$ integration. The integration over $\tilde{g}_{\mu\nu}$ (and matter terms) will not be performed before imposing conformal constraints. The problem is how to handle the functional integration over $\tilde{g}_{\mu\nu}$ (the "curved background anomaly"). On a flat background, the conformal anomalies can be handled properly. The choice of $\omega$ is then unique. $\omega$ is a renormalizable field which could create the spacetime twofold: an infalling and outside observer use different ways to fix the conformal gauge in order to overcome unitarity problems. For non-flat $\tilde{g}_{\mu\nu}$, standard QM does not apply, because there is no notion of energy in a conformal invariant theory. Further, if one adds matter sources to the Lagrangian, for example a scalar gauge field ($\Phi, A_\mu$), it will break the conformal invariance, i.e., it spoils the tracelessness of the energy momentum tensor. However, if one shifts $\omega$ to the complex contour ($\omega^2 \rightarrow -\frac{\omega^2}{\kappa^2}$), then the scalar field and dilaton field can be handled on equal footing (in order to obtain the same unitarity properties).

Using dimensional regularization and renormalization, it was found that the scalar field Lagrangian
\[ \mathcal{L} = \sqrt{-g} \left( -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} R \phi^2 \right), \]  
(26)
generates an effective action, whose divergent part is
\[ \Gamma^{\text{div}} \sim \frac{\sqrt{-g}}{4-n} \left( R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right). \]  
(27)

\[ \frac{\kappa}{16\pi G_N} \approx \frac{\sqrt{-g}}{4-n} \left( R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right). \]  
(22)

\[ R \rightarrow \frac{\kappa}{16\pi G_N} R - \frac{\kappa}{16\pi G_N} \nabla^\nu \partial_\nu \Omega. \]  
(22)

\[ \frac{\kappa}{16\pi G_N} \approx \frac{\sqrt{-g}}{4-n} \left( R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right). \]  
(27)

\[ \frac{\kappa}{16\pi G_N} \approx \frac{\sqrt{-g}}{4-n} \left( R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right). \]  
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(27)

\[ \frac{\kappa}{16\pi G_N} \approx \frac{\sqrt{-g}}{4-n} \left( R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right). \]  
(27)
One can apply this result directly to the Lagrangian Eq.\((21)\), where \(R\) and \(g\) are replaced by \(\tilde{R}\) and \(\tilde{g}\). So the effective action becomes

\[
S_{\text{eff}} = C \int d^n x \frac{\sqrt{-\tilde{g}}}{8\pi^2 (4 - n)} \left( \tilde{R}^{\mu\nu} \tilde{R}_{\mu\nu} - \frac{1}{3} \tilde{R}^2 \right). \tag{28}
\]

We will treat now \(\tilde{g}_{\mu\nu}\) classical, so we will have no unitarity problem. Further, we have still the infinite coefficient in front of the effective action. It is conjectured\(^\text{11}\) that no counter term is needed in order to make the integral finite. The divergent coefficient in front of \(S_{\text{eff}}\) will not be a problem when \(\tilde{g}_{\mu\nu}\) is treated classical: \(N\) and \(N^\phi\) can take large values and in the limit that \(\frac{C}{4 - n} \to \infty\), quantum fluctuations would vanish and only the classical parts would remain. Only \(\omega\) (and matter fields) are described by renormalizable Lagrangians. This means that \(\tilde{g}_{\mu\nu}\) is flat beyond the Planck scale, but gets renormalized by \(\omega\) and matter fields at much lower scales.

Quantum amplitudes are obtained by integrating the exponent of the entire action over all the components of \(g_{\mu\nu}\), now written as \(\omega^2 \tilde{g}_{\mu\nu}\) (or, in the 5D case, \(\tilde{g}_{\mu\nu} = \omega^{4/3(5)} \tilde{g}_{\mu\nu}\)). The integration over \(\omega\) must be along a complex contour\(^\text{11}\) Further, we consider the brane spacetime \(\tilde{g}_{\mu\nu} = (4) \tilde{g}_{\mu\nu} + n_\mu n_\nu\) with now \((4) \tilde{g}_{\mu\nu} = \tilde{\omega}^{2(4)} \tilde{g}_{\mu\nu}\). Our \(\tilde{g}_{\mu\nu}\) is given by the up-lifted BTZ black hole spacetime. The evolution equation for \(N\) will contain no intrinsic mass or length scale. (for \(\Lambda = 0\)) and reflects the fact that we are dealing here with gravitational waves only. The scale is determined by \(\omega\) and \(\tilde{\omega}\). One says that \(\omega\) is part of the gravitational sector. It fixes the scale at which quantum gravity effects becomes important.

It is obvious, that close to the horizon and ergo-sphere of the spacetime Eq.\((26)\), the curvature is high and quantum effects will become important. We already mentioned that the Lagrangian Eq.\((21)\) with \(\Lambda = 0\) contains no intrinsic mass or length scale. This is comparable with the classical Lagrangian of the SM, if one omits the Higgs mass term. So it is challenging to apply a spontaneous symmetry breaking mechanism of conformal invariance by adding a mass term \(-\frac{1}{2} \sqrt{-\bar{g}} m^2 \bar{\phi}^2\) (through the quartic self-coupling of the Higgs field) and other contributions\(^\text{13}\) just as the Higgs mechanism in SM. Newton’s constant then reappears by the re-scaling \(\omega \to \kappa \sqrt{\xi \omega}\). One conjectures that the effective action (with matter fields and on the black hole spacetime)) will be locally spontaneously broken\(^\text{15}\)

Now the RS idea can be applied: the renormalized counter terms in \(S_{\text{eff}}\) will cause unitarity problems only in the bulk at the large extra-dimension-scale \((l)\), but are invisible on the brane. This means that the subraction point will be in the UV region as seen by an observer in the bulk. Note that this is possible, because we have in the induced 4D Einstein equations the KK-contribution from the bulk, i.e., the electric

\(^\text{11}\) For the renormalization problems caused by the mass term, we refer to the literature. It is conjectured that it will not spoil conformal invariance of the effective action. The application to our model is currently under investigation.
part of the Weyl tensor $E_{\mu\nu}$. From the brane observer, these KK corrections are non-local, since they incorporate 5D gravity wave modes. Note that if the extra dimensional volume is significantly above the Planck scale, than the true fundamental scale $M_{4+d}$ can be much less than the effective scale $M_{pl} \sim 10^{19}$ GeV. In the RS-1 model, $M_{3}^5 = \frac{M_{pl} l}{M}$, with $l$ the effective length scale scale of the infinite large extra dimension $y$. The warp factor (Eq.(16)) then causes a finite contribution to the 5D volume

$$\int d^5x \sqrt{-g} \sim \int d^4x \int dy W_2 \sim l \int d^4x.$$  (29)

So the effective size of the extra dimension probed by the 5D graviton, is $l$. Note that the negative bulk tension is taken $\Lambda_5 \sim -\frac{l}{2} \sim -\mu^2$ in the RS-1 model, with $\mu$ the energy scale. So for low energy scale it prevents gravity leaking into the extra dimension. As by-product, the model solves the hierarchy problem.

In order to solve the conformal anomaly, caused by the interaction of the matter fields and gravity, one calculates the beta-functions (note that the trace is proportional to the beta functions), which has to be zero (these finite quantum theories are scale invariant). For a renormalizable theory, there must be as many beta functions as adjustable parameters, thus allowing to completely determine the theory. Another problem is the dynamics of $\tilde{g}_{\mu\nu}$. If the behavior is classical, then there is an interaction with the quantum fields, i.e., the dilaton field. On a black hole spacetime, one has to deal with Hawking radiation, even in the absent of matter terms, by outgoing gravitational waves. This is represented by the dynamical equation for the dilaton field, which decouples from $\tilde{g}_{\mu\nu}$. It is clear that $\tilde{\omega}$ will not decouple when a scalar field is incorporated.\[19\]

We shall see in section 4 that $\tilde{\omega}$ determines the evolution of $\tilde{g}_{\mu\nu}$ (i.e., $N$ and $N^r$).

4. The field equations of the warped CDG model

4.1. The 5D equations

Let us apply the equations of section 3 to the 5D spacetime $(\tilde{g}_{\mu\nu} = \omega^{4/3} \tilde{g}_{\mu\nu})$

$$ds^2 = \omega(t, r, y)^{4/3} \left[ -N(t, r)^2 dt^2 + \frac{1}{N(t, r)^2} dr^2 + dz^2 + r^2(d\varphi + N_\varphi(t, r)dt)^2 + dy^2 \right].$$  (30)

This metric is used for the 3D BTZ black hole solution (where the $dz^2$ is removed) and can be up-lifted to 4D\[19\][25]. From the 5D Einstein equations Eq.(24), one obtains of course again $\omega(t, r, y) = \omega_2(y)\omega_1(t, r)$, with $\omega_2(y) = l = \text{constant} = \text{length scale of the extra dimension}$.

The set of PDE’s for $\omega$ and $N$ now becomes (we omit the subscript 1 in $\omega$)

$$\ddot{\omega} = -N^4 \omega'' + \frac{5}{3\omega} (N^4 \omega'^2 + \omega^2)$$  (31)

$$\ddot{N} = \frac{3N^2}{N} - N^4 \left( N'' + \frac{3N'}{r} + \frac{r^2}{N} \right) - \frac{2}{\omega} \left[ N^5 \left( \omega'' - \frac{5\omega'^2}{3\omega} + \frac{\omega'}{r} \right) + N^4 \omega'N' + \dot{\omega} \dot{N} \right]$$  (32)
The equation for $N$ can also be presented without $\omega''$ on the right hand side. Then there is an extra constraint equation in $\omega''$:

$$\omega'' = -\frac{\Lambda \omega^{7/3} \kappa^{4/3} \sqrt{18} l}{16 N^2} - \frac{\omega' N'}{N} - \frac{\omega'}{2r} + \frac{4 \omega^2}{3 \omega N^4} - \frac{\dot{\omega} \dot{N}}{N^5}. \quad (33)$$

A non-trivial solution for $\omega$ becomes (compare with the warped spacetime solution of Eq.(17))

$$\omega = \left[ \frac{c_1}{(r + c_2)t + c_3 r + c_4} \right]^{3/2} \quad (34)$$

(or $r$ and $t$ interchanged). For $c_2 c_3 = c_4$, we find the exact solution for $N$

$$N^2 = \frac{1}{5 r^2} \frac{10 c_2^3 r^2 + 20 c_2^2 r^3 + 15 c_2 r^4 + 4 r^5 + D_1}{D_2 (c_3 + t)^4 + D_3}, \quad (35)$$

with $c_i, D_i$ constants. Note that $N$ can be written as $N = N_1(r)N_2(t)$. The dilaton equation Eq.(23) is superfluous. One easily checks that the trace of Eq.(24) is zero and that $\nabla^\mu (T^\omega_{\mu\nu} - \Lambda g_{\mu\nu} \kappa^{4/3} \xi^{2/3} \omega^{2/3}) = 0$. Also, the Ward identity is fulfilled. The equations for $N^\phi$ are

$$N^\phi' = -\frac{2 \omega N^\phi'}{\omega} \quad \text{(36)}$$

$$N^\phi'^2 = -\frac{4 N}{r^3} \left( N' + \frac{N \omega'}{\omega} \right) \quad \text{(37)}$$

$$N^\phi'' = -N^\phi' \left( \frac{2 \omega'}{\omega} + \frac{3}{r} \right) \quad \text{(38)}$$

with solution

$$N^\phi = F_1(t) + \int \frac{1}{r^3 \omega^2} dr \quad (39)$$

Figure 1. Example of an exact solution of $\tilde{g}_{tt}$ (left) and $\omega^{4/3} \tilde{g}_{tt}$.
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with \( F(t) \) an arbitrary function in \( t \). A special solution of \( N^\varphi \) is

\[
N^\varphi = \frac{(t + c_3)^3}{2r^2} \left( 6c_2r^2 \ln(r) - c_2^3 - 6c_2^2 r + 2r^3 \right)
\]  

(40)

Further, we have the constraint

\[
N' = -\frac{N\omega'}{\omega} + \frac{e_1}{N\omega^4 r^3}
\]

with \( e_1 \) a constant. This constraint can be used as boundary condition in the numerical solution.

In figure 1 we plotted a typical exact solutions of \( \omega^4 g_{tt} \) and \( g_{tt} \). The Killing horizon is where the ergo region begins. Note however, that the general exact solution for \( \omega \) has one more free parameter \( c_4 \).

4.2. The effective 4D Einstein equations on the brane

We now write

\[
\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + n_{\mu}n_{\nu}
\]

(42)

with \( n_\mu = [0, 0, 0, 0, \sqrt{l}] \) the unit normal to the brane. Ones again, we define

\[
\bar{g}_{\mu\nu} = \bar{\omega}^2 \tilde{g}_{\mu\nu}
\]

(43)

The effective Einstein equations on the brane are

\[
\bar{G}_{\mu\nu} = \frac{1}{\bar{\omega}^2} \left(-\Lambda_{eff} \bar{g}_{\mu\nu} - \Lambda \kappa^2 (\bar{g}_{\mu\nu}\bar{\omega}^4 + \bar{T}_{\mu\nu}(^{(\bar{\omega})}) - \mathcal{E}_{\mu\nu},
\]

(44)

with \( \mathcal{E}_{\mu\nu} \) the electric part of the Weyl tensor, given by

\[
\mathcal{E}_{\mu\nu} = (5) C_{\alpha\beta\gamma\delta} \delta^{(4)} \tilde{g}_{\alpha\mu} \tilde{g}_{\delta\nu}
\]

(45)

Note that \( \mathcal{E}_{\mu\nu} \) is trace free. For the ”un-physical” metric \( \tilde{g}_{\mu\nu} \) we have now

\[
ds^2 = -N(t, r)^2 dt^2 + \frac{1}{N(t, r)^2} dr^2 + dz^2 + r^2 (d\varphi + N\varphi(t, r) dt)^2.
\]

(46)

Again, an equation for \( \bar{\omega} \) can be isolated (we take from now on \( \Lambda_{eff} = 0 \), i.e.,

\[
\ddot{\bar{\omega}} = -N^4 \ddot{\bar{\omega}} + \frac{2}{\bar{\omega}} (N^4 \dot{\bar{\omega}}^2 + \dot{\bar{\omega}}^2)
\]

(47)

\[
\ddot{N} = \frac{3N^2}{N} - N^4 \left( N'' - \frac{3N'}{r} + \frac{N'' r^3}{N} \right)
- \frac{3}{\bar{\omega}} \left[ N^5 \left( \ddot{\bar{\omega}} - \frac{2\dot{\bar{\omega}}^2}{\bar{\omega}} + \frac{\dot{\bar{\omega}}'}{r} \right) + N^4 \bar{\omega}' N' + \dot{\bar{\omega}} \dot{N} \right]
\]

(48)

Now the non-trivial solution of \( \bar{\omega} \) becomes (compare with Eq. (34))

\[
\bar{\omega} = \left[ \frac{b_1}{(r + b_2) t + b_3 r + b_4} \right],
\]

(49)
with $b_i$ constants. The solution for $N$ is the same as Eq.(35) (with $c_i$ replaced by $b_i$ and $D_i$ by $C_i$)

$$N^2 = \frac{1}{5r^2} \frac{10b_2^3r^2 + 20b_2^3 + 15b_2 r^4 + 4r^5 + C_1}{C_2(b_3 + t)^4 + C_3},$$

(50)

The constraint for $\ddot{\omega}''$ becomes now

$$\ddot{\omega}'' = -\frac{\Lambda \omega}{9 N^2} - \frac{\omega' N'}{N} - \frac{\omega'}{2r} + \frac{2 \dot{\omega}^2}{\omega N^4} - \frac{\dot{\omega} \dot{N}}{N^5}$$

(51)

The equations for $N^\phi$ are

$$\dot{N}^\phi = -\frac{3 \dot{\omega} N^{\phi'}}{\omega}$$

(52)

$$N^{\phi'2} = -\frac{2N}{r^3} \left(2N' + 3 \frac{N \omega'}{\omega}\right)$$

(53)

$$N^\phi'' = -3N^{\phi'} \left(\frac{\omega'}{\omega} + \frac{1}{r}\right)$$

(54)

with solution

$$N^\phi = F_2(t) + \int \frac{1}{r^3 \omega^3} dr$$

(55)

with $F_2(t)$ an arbitrary function in $t$. A typical solution is also given by Eq(40)!

Further, we have the constraint

$$N' = -\frac{3N \omega'}{2 \omega} + \frac{e_2}{N \omega^6 r^3}$$

(56)

with $e_2$ a constant. In figure 2 we plotted $g_{tt}$ for $\bar{g}_{\mu \nu}$ and $g_{\mu \nu}$.

Again, the dilaton equation is superfluous and the trace of Eq.(24) again zero. The conservation equations now become

$$\bar{\nabla}^\mu E_{\mu \nu} = \bar{\nabla}^\mu \left[ \frac{1}{\omega^2} \left( -\Lambda \kappa^2(4) g_{\mu \nu} \omega^4 + (4) T_{\mu \nu} (\omega) \right) \right]$$

(57)

which yields differential equations for $\ddot{N}'$ and $\dot{N}$ as boundary conditions at the brane. It can be described as the non-local conservation equation. In the high energy case close to the horizon, one must include the $S_{\mu \nu}$ term. So the divergence of $E_{\mu \nu}$ is constrained. In the non-conformal case, Eq.(37) contains on the right hand side also the quadratic correction $S_{\mu \nu}$ of the matter fields on the brane. The effective field equations, Eq.(44), are then not a closed system. One needs the Bianchi equations. In fact, $E_{\mu \nu}$ encodes corrections from the 5D graviton effects and are for the brane observer non-local. In our model under consideration, we have only the $T_{\mu \nu}^{(\omega)}$ term and no source terms (apart from the 5D $\Lambda_5$). But it still source the KK modes. The dilaton $\omega$ plays the role of a ”scalar field”. But we don’t need the 5D equations themselves, because the solution for $N$ is the same! It is only the $\omega^{4/3}$ which represents the 5D contribution. There is no exchange of energy-momentum between the bulk and brane. If one applies the model to a FLRW model,

then the evolution equations are very complicated. Then inhomogeneous and anisotropic effects from the 4D matter radiation distribution on the brane are source for the 5D Weyl tensor, $E_{\mu \nu}$ then causes non-local back-reaction on the brane. One needs an approximation scheme in order to find the missing evolution equation for $E_{\mu \nu}$. 
Figure 2. Left: typical plot of $\tilde{g}_{tt}$ for the 4D induced spacetime. Right: typical plot of $\omega^{4/3} \tilde{g}_{tt}$. We observe that the ergo sphere fades away as time increases.

4.3. The 4D and 5D equations written in dimension $n$

One can write the differential equations for $n = 5, 4$ as

$$\ddot{\omega} = -N^4 \omega'' + \frac{n}{\omega(n-2)} \left( N^4 \omega'^2 + \dot{\omega}^2 \right)$$

(58)

$$\dot{N} = \frac{3N^2}{N} - N^4 \left( N'' + \frac{3N'}{r} + \frac{N'^2}{N} \right)$$

$$- \frac{n-1}{(n-3)\omega} \left[ N^5 \left( \omega'' + \frac{\omega'}{r} + \frac{n}{2-n} \frac{\omega'^2}{\omega} \right) + N^4 \omega'N' + \dot{\omega}N \right]$$

(59)

with solution

$$\omega = \left( \frac{a_1}{(r + a_2)t + a_3r + a_4} \right)^{\frac{1}{2}n-1}, N^2 = \frac{1}{5r^2} \frac{10a_2^3r^2 + 20a_2^3r^3 + 15a_2r^4 + 4r^5 + C_1}{C_2(a_3 + t)^4 + C_3}$$

(60)

with $a_i$ some constants. The constraint becomes

$$\ddot{\omega}'' = -2n \frac{\Lambda l \kappa^{n-2} \xi^{3(n-1)} \omega^{n+2}}{N^2} - \frac{\omega'N'}{N} - \frac{\omega'}{2r} + \frac{4}{n-2} \frac{\dot{\omega}^2}{\omega N^4} - \frac{\dot{\omega}N}{N^5}$$

(61)

Note that the differential equations are constraint by an equation for $\omega''$, which contains the $\Lambda$-term. We have now a closed system of the 4D and 5D equations.
4.4. Contribution from the bulk

If one omits the contribution from the bulk, i.e., \( \mathcal{E}_{\mu\nu} \), one obtains the solution (\( d_i \) constants)

\[
\omega = \frac{1}{(t + d_3)r + d_2t + d_2d_3}, \quad N^2 = \frac{1}{4r^2} \frac{6d_2r^2 + 8d_2r^3 + 3r^4 + D_1}{D_2(t + d_3)^3 + D_3}.
\]

(62)

The solution for \( N^\varphi \) becomes

\[
N^\varphi = \frac{(t + d_3)^2}{2r^2} \left( 2r^2 \ln(r) - d_2(d_2 + 4r) \right).
\]

(63)

If we compare this solution with Eq.(50), we observe that the highest power of the radial coordinate in \( N^2 \) is now \( r^2 \) in stead of \( r^3 \). Further, for \( D_1 = 0 \), \( N^2 \) has, for any \( d_2 \), no real roots, so it has naked singularities. So the bulk contribution generates at least one horizon (see next section).

5. Location of the horizon’s, Hawking radiation and antipodal identification

Let us inspect more closely the behavior of the induced spacetime on the brane, found in the preceding section. It turns out, by inspection of \( \bar{g}_{\mu\nu} \), that the spacetime possesses horizons (coordinate singularities, i.e., 0, 1 or 2) and ergo-sphere (Killing horizon). \( N^2 \) becomes singular for \( t = t_H = -b_3 + \sqrt{-\frac{C_3}{C_2}} \). However, \( \bar{g}_{\mu\nu} \) can be made regular everywhere and singular free by suitable choices of the parameters \( b_i, c_i \) and \( C_i \). For \( C_1 = 0 \), \( \bar{g}_{\mu\nu} \) has one real zero \( r_H = \sim |1.606b_2| \) and two complex zero’s \( \sim (0.178 \pm 0.638i)b_2 \). In figure 3 we plotted \( N_1(r)^2 \) for several values of \( C_1 \) and \( b_2 \).

However, there is no restriction on the angular velocity of the horizon: the equation for \( N^\varphi \) decouples. Further, one must remark that we don’t deal here with the 3D BTZ (with the obligatory cosmological constant). There are no gravitational degrees of freedom in 3D Einstein gravity. Still this 3D model has interesting applications, such as the connection with the asymptotic AdS3. It possesses local defects, black holes and asymptotic symmetries. In our uplifted model we don’t need a cosmological constant. If one omits the \( dz^2 \) in Eq.(16), one needs a different approach. We shall see that the properties of our \( \bar{g}_{\mu\nu} \) can be compared with the Kerr black hole.

![Figure 3. Four possible plots of \( N_1(r)^2 \).](image-url)
5.1. The "classical" picture.

In order to investigate the features of the time-dependent gravitational field of our rotating object and the resulting production of Hawking radiation, one should take into account the complementarity of different observers. The annihilation operators of particles at late time can be expressed as superpositions of annihilation and creation operators of particles at early times (Bogolubov transformation). The created particles form a thermal spectrum determined by the surface gravity, i.e., the gravitational acceleration at the horizon. There will be also a back-reaction on the spacetime. This emission is caused by the formation of an event horizon. Finally the black hole will evaporate. During the evaporation, the surface gravity and temperature increases.

The treatment of the rotating black hole thermodynamics can be found in Wald. \cite{Wald}

The mass loss can be expressed as

$$\delta M = \frac{\kappa}{8\pi} \delta A + \Omega_H \delta J_H.$$  \hspace{1cm} (64)

with $\kappa$ the surface gravity of the black hole (see section 5.5), $A$ the area of the event horizon, $J_H$ the angular momentum and

$$\Omega_H = \frac{d\varphi}{dt} = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = N^{\varphi}$$  \hspace{1cm} (65)

the coordinate angular velocity (the angular velocity of null generators of the event horizon). $N^{\varphi}$ given by Eq.\((40)\), evaluated at $r = r_H$. It blows-up $\sim t^3$. Massless quanta created by the black hole are governed by null geodesics that begin far outside the collapsing body at early times, move inwards and become outgoing (radially) null geodesics and escape from the collapsing body. They reach future null infinity at arbitrarily late times.

To describe the spectrum of the Hawking particles, one usually defines geodesics $x^\mu(\lambda)$, with $\lambda$ an affine parameter. The geodesic is defined by $\nabla^\lambda(dx^\mu/d\lambda)$, with the covariant derivative along the geodesic. In null coordinates $(u, v)$ one has for the affine separation between two outgoing null geodesics that escape from the collapsing body just before the formation of the event horizon, $u(v) = -\frac{1}{r_H} \log \left(\frac{v_0 - v}{K}\right)$, with $K$ a constant.

In the Boyer-Lindquist coordinates, an incoming null geodesic take an infinite coordinate time to reach the event horizon and it will winds an infinite number of times around the $z$-axes.

5.2. Changing coordinates

If we define the coordinates, $dr^* \equiv \frac{1}{N_1(t)^2} dr$ and $dt^* \equiv N_2(t)^2 dt$, then our induced spacetime of Eq.\((46)\) can be written as

$$ds^2 = \omega^{A/3} \omega^2 \left[ \frac{N_1^2}{N_2} \left( -dt^*^2 + dr^*^2 \right) + dz^2 + r^2 (d\varphi + \frac{N^{\varphi}}{N_2} dt^*)^2 \right].$$  \hspace{1cm} (66)
with
\[
N_1^2 = \frac{10b_2^3 r^2 + 20b_2^3 r^3 + 15b_2 r^4 + 4r^5 + C_1}{5r^2}, \quad N_2^2 = \frac{1}{C_2(t + b_3)^4 + C_3}
\] (67)

and
\[
\begin{align*}
& r^* = \frac{1}{4} \sum_{r_i^H} \frac{r_i^H \log(r - r_i^H)}{(r_i^H + b_2)^3}, \\
& t^* = \frac{1}{4C_2} \sum_{t_i^H} \frac{\log(t - t_i^H)}{(t_i^H + b_3)^2}.
\end{align*}
\] (68)

The sum it taken over the roots of \((10b_2^3 r^2 + 20b_2^3 r^3 + 15b_2 r^4 + 4r^5 + C_1)\) and \(C_2(t + b_3)^4 + C_3\), i.e., \(r_i^H\) and \(t_i^H\). This coordinate system is comparable with a Kruskal-Szekeres coordinate system. Further, one can define the azimuthal angular coordinate \(\varphi^* \equiv \frac{(d\varphi + N\varphi)}{N^2 dt^*}\), which can be used when an incoming null geodesic falls into the event horizon. \(\varphi^*\) is the azimuthal angle in a coordinate system rotating about the z-axis relative to the Boyer-Lindquist coordinates.

5.3. Penrose diagram

If we introduce outgoing and ingoing radial null coordinates, \(u = t^* - r^*\), \(v = t^* + r^*\), then \(u = \text{cst.}\), and \(v = \text{cst.}\), describe the outgoing and ingoing null curves. We then have
\[
ds^2 = \omega^{4/3} \omega^2 \left[ \frac{N_1^2}{N_2^2} dudv + dz^2 + r^2 d\varphi^{*2} \right].
\] (69)

Next we apply the usual Penrose compactification,
\[
u = \tan U, \quad v = \tan V.
\] (70)

We then obtain
\[
ds^2 = \omega^{4/3} \omega^2 \left[ \frac{N_1^2}{N_2^2} \frac{dU dV}{\cos^2 U \cos^2 V} + dz^2 + r^2 d\varphi^{*2} \right].
\] (71)

Further, from Eq. (68) we have
\[
\begin{align*}
& r^* = \frac{1}{2}(v - u) = \frac{1}{2}(\tan V - \tan U) = \frac{1}{4} \sum_{r_i^H} \frac{r_i^H \log(r - r_i^H)}{(r_i^H + b_2)^3}, \\
& t^* = \frac{1}{2}(v + u) = \frac{1}{2}(\tan V + \tan U) = \frac{1}{4C_2} \sum_{t_i^H} \frac{\log(t - t_i^H)}{(t_i^H + b_3)^3}.
\end{align*}
\] (72)

The Penrose diagram bears strong resemblance with the 3D BTZ black hole solution. However, there are differences. First of all, we have no cosmological constant and we have have a physical 4D spacetime. Further, the factor \(\frac{N_1^2}{N_2^2}\) is quite different. \(\frac{N_1^2}{N_2^2}\) can have no roots, so no horizons. Even for \(r = 0\), \(\bar{g}_{\mu\nu}\) can be regular and is conformally flat (\(z, \varphi\) suppressed). This can also be seen, when one solves from Eq. (72) (for the case of one horizon \(r_H \sim 1.606b_2\)).
\[
\begin{align*}
& r = r_H + \exp \left[ 2(v - u)(r_H + b_2)^3/rH \right], \\
& t = t_H + \exp \left[ 2C_2(v + u)(t_H + b_3)^2 \right].
\end{align*}
\] (73)
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So \( \bar{g}_{\mu\nu} \) can be used to locate the light cones.

We can write the spacetime \( \bar{g}_{\mu\nu} \) in the form

\[
\bar{ds}^2 = (C_2(t + b_3)^4 + C_3) \left[ \frac{10b_3^2r^2 + 20b_2r^3 + 15b_2r^4 + 4r^5 + C_1}{5r^2} dudv \right] + dz^2 + r^2d\phi^2,
\]

or

\[
\bar{ds}^2 = \Theta^2(U,V) dU dV + dz^2 + r^2d\phi^2
\]

\[
\Theta^2 = C_2 e^{6(tanV - tanU)(r_H + b_3)^3/r_H + 8C_2(tanV + tanU)(t_H + b_3)^3}
\]

Finally, we transform to coordinates \( T, T = U + V, R = U - V \), in order to plot the Penrose diagram of figure 4. We took \( C_1 = C_3 = 0 \) and one horizon \( r_H \). We then have

\[
u = \tan U = \log \frac{(t - t_H)\beta}{(r - r_H)\alpha}, \quad v = \tan V = \log(t - t_H)^\beta(r - r_H)^\alpha.
\]

with

\[
\alpha = \frac{r_H}{4(r_H + b_2)^3}, \quad \beta = \frac{1}{4C_2(t_H + b_3)^3}
\]

Note that we have the "scale-factor" \( \Theta^2 \) in front of \((-dT^2 + dR^2)\), which causes no harm in construction the diagram. For \( t = t_H, \Theta \to \infty! \) For \( t \to \infty, \Theta \to \infty \) OR \( r \to r_H \) Note that although \( r = 0 \) has disappeared as singular point, it is in the line element Eq. (74) for \( C_1 \neq 0 \).

If one makes the antipodal identification \( U \to -U, V \to -V \), one obtains \( \alpha \to -\alpha \) and \( \beta \to -\beta \). This is only possible if \( (b_2, b_3) \to (-b_2, -b_3) \) and \( r_H \) and \(-r_H \) are identified!

Following Strauss, et. al,\[38\] we can use a slightly different coordinate system, in order to study quantum effects more clearly (again 1 horizon and \( C_1 = C_3 = 0 \)). We define

\[
U = e^{\kappa(r^* - t^*)}, \quad V = e^{\kappa(r^* + t^*)} \quad r > r_H
\]

\[
U = -e^{\kappa(r^* - t^*)}, \quad V = -e^{\kappa(r^* + t^*)} \quad r < r_H
\]

Further,

\[
\tilde{U} = \tanh U, \quad \tilde{V} = \tanh V
\]

The spacetime can then be written as

\[
ds^2 = \omega^{4/3} \omega^2 \left[ H(\tilde{U}, \tilde{V}) d\tilde{U} d\tilde{V} + dz^2 + r^2d\phi^2 \right]
\]

with

\[
H = \frac{N_1^2}{N_2^2 \kappa^2 \arctanh \tilde{U} \arctanh \tilde{V} \arctanh \tilde{V}(1-\tilde{U}^2)(1-\tilde{V}^2)}
\]

We can write \( r \) and \( t \) as

\[
r = r_H + \left( \arctanh \tilde{U} \arctanh \tilde{V} \right)^{\frac{1}{\kappa \pi}}, \quad t = t_H + \left( \frac{\arctanh \tilde{V}}{\arctanh \tilde{U}} \right)^{\frac{1}{\kappa \pi}}
\]
In figure 5 we plotted the Penrose diagram in \((\bar{U}, \bar{V})\)-coordinates. $\bar{g}_{\mu\nu}$ is regular everywhere and conformally flat. The "scale-term" $H$ is consistent with the features of the Penrose diagram.

5.4. The role of the dilaton field.

Let us now return to $g_{\mu\nu} = \omega^{4/3} \bar{\omega}^2 \bar{g}_{\mu\nu}$. In the CDG setting, the evaporation is expressed by the complementarity transformation of $\omega$ between the in-going and outside observer. Only the outside observer will notice the outgoing Hawking radiation. The local observer notice only the in-going radiation.\footnote{1}

In order to apply quantum mechanics in the vicinity of horizon\footnote{crossing the horizon in this antipodal mapping, also interchanges creation and annihilation operators and reverse time. So there will be an entanglement between positive energy particle with the positive energy antiparticles at the antipodes. Moreover, one can deal with pure quantum states in stead of} one can apply

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Kruskal diagram for $\bar{g}_{\mu\nu}$ in $(T, R)$-coordinates.}
\end{figure}
Figure 5. Kruskal diagram for $\tilde{g}_{\mu\nu}$ in $(\tilde{U}, \tilde{V})$-coordinates. The antipodal map between region I and II is quite clear here. If one approaches the horizon from the outside and passes the horizon, one approaches from the "otherside" the horizon.

an antipodal identification, a boundary condition in order to remove the inside of the horizon. In this approach one connects the regions I and II in the Kruskal diagram of figure 5 in order to solve the unitarity and the entanglement problems.\textsuperscript{33} This procedure shed some inside how particles transmit the information they carry, across the horizon. The horizon changes from a 2-cylinder into a projected one. Every point $(r, t)$ corresponds with two points $(u, v), (-u, -v)$. If one identifies $(-u, -v, z, \varphi) \equiv (u, v, -z, \varphi + \pi)$ (i.e., $t^* \rightarrow -t^*, r^* \rightarrow -r^*, z \rightarrow -z, \varphi \rightarrow \varphi + \pi$), then region I and III correspond to the same black hole, so there is no inside.

For $b_1 = b_3 b_2$, we have now

$$ds^2 = \omega^{4/3} \omega^2 \left[ (C_2(t + b_3)^4 + C_3) \left( \frac{10b_3^2 r^2 + 20b_2^2 r^3 + 15b_2 r^4 + 4r^5 + C_1}{N_5^2 dt^*} \right) \right].$$

Different observers use different $\omega$ (disagreement about the vacuum state), i.e., the scale, by adjusting the constants. The local observer will not notify the new boundary condition of the antipodal map. Only the outside observer will become aware of the thermodynamical mixed states. This resolves the information problem.\textsuperscript{33}
mapping. We see that
\[
\frac{\omega^4 \bar{\omega}^2}{(r + c_2)^2(t + c_3)^2}(r + b_2)^2(t + b_3)^2,
\]
approaches zero for coordinate time \( t \to \infty \), so \( g_{\mu\nu} \) shrinks to zero. We observe that the time derivative of Eq. (84) is non-zero, so we have a non-zero \( T^\omega \) (see Eq. (25)). This is the energy-momentum tensor of the Hawking particles. By suitable choices of the constants one can make it asymptotically regular.

5.5. Surface gravity

In order to calculate the surface gravity (needed for the Hawking radiation and the notion of the quasi-local mass), one must realize that we are dealing here with a non-stationary black hole. For stationary Kerr black holes, one defines the Killing vector \( \chi^\mu = \xi^\mu_t + \Omega_H \xi^\mu_\phi \), normal to the Killing horizon. \( \Omega_H \) is the angular velocity of the horizon, given by Eq. (65) and \( \xi^\mu_t \xi^\nu_t = g_{tt}, \xi^\mu_\phi \xi^\nu_\phi = g_{\phi\phi} \). The surface gravity is then defined as
\[
\kappa^2 = -\frac{1}{2} \nabla^\mu \chi^\nu \nabla_\mu \chi^\nu,
\]
evaluated at the horizon \( r = r_H \). An equivalent form is \( \chi^\mu \nabla_\mu \chi^\nu = 2\kappa \chi^\nu \). We say that \( \chi^\mu \) is geodesic on the horizon and \( \kappa \) measures the extent to which \( \chi^\mu \) fails to be affinely parameterized.

In our non-stationary case, the calculations complicates, because we have an evolving horizon.\(^{32,34} \)

We need now null generators of the local non-Killing horizon and a suitable normalization (parameterization) of the surface gravity. This is related to non-affinity of null geodesics on the horizon. For a Killing field, it is easy, because the Killing equation \( \nabla_\mu l_\nu + \nabla_\nu l_\mu = 0 \) possesses a gauge freedom, i.e., a constant factor which can be used to re-scale \( l_\mu \) and make \( l_\mu l^\mu = -1 \) at spacelike infinity. The Lie derivative then tells us how a spacetime tensor changes when one takes an infinitesimal step along along the vector field. For a Killing field \( \chi^\mu \) we have \( \mathcal{L}_\chi g_{\mu\nu} = 0 \).

Let us define outgoing and ingoing null normals for \( \bar{g}_{\mu\nu}, \bar{l}^\mu \) and \( \bar{m}^\mu \) (\( \bar{l}^\mu \bar{l}_\mu = \bar{m}^\mu \bar{m}_\mu = 0 \), no Killing fields), such that \( \bar{l}^\mu \bar{m}_\mu = -1 \):
\[
\bar{l}^\mu = (1, N \sqrt{N^2 - r^2 N^2}, 0, 0), \quad \bar{m}^\mu = \left(-\frac{1}{2r^2 N^2 - 2N^2}, -\frac{N}{2 \sqrt{N^2 - r^2 N^2}}, 0, 0\right)
\]

The surface gravity is then\(^{35,36} \)
\[
\kappa = -\bar{m}^\nu \bar{l}_\mu \nabla^\mu \bar{l}_\nu.
\]
* The Killing vector is a non-affinely parameterized geodesic on the horizon
\( \dagger \) An approved method is the use of Eddington-Finkelstein coordinates. An other possible way out is to apply the transformation \( t \to iz, z \to it \) and follow the method of Medved, et al.\(^{37} \)
One can evaluate Eq. (87) at the horizon, because we have exact expressions for $N$ and $N^\varphi$. It turns out for suitable parameters, that at $t = -b_3$, $\kappa \to \infty$.

The value of $\bar{l}^\mu \bar{m}^\nu \nabla^\nu \bar{m}_\mu$ depends on the normalization for $l^\mu$. One should take

$$\bar{l}^\mu \bar{m}^\nu \nabla^\nu \bar{m}_\mu = 0,$$

which yields an equation for $\partial_t N^\varphi$

$$\partial_t N^\varphi = \frac{-1}{r^2 N^\varphi N}(r^2 N^\varphi - 2N^2)\partial_t N.$$  (89)

This determines the function of $F_2(t)$ in Eq (56). This choice is also necessary, in order to obtain the correct expression for the Lie derivative $\mathcal{L}_l \bar{g}_{\mu\nu}$

The surface gravity then becomes

$$\kappa = 2N \left( \partial_r (\sqrt{N^2} - \frac{r^2 N^\varphi}{2}) + \partial_t \left( \frac{1}{N} \right) \right) = 2N \left( \partial_t \sqrt{\bar{g}_{rr}} - \partial_r \sqrt{\bar{g}_{tt}} \right).$$

(90)

This is consistent with the metric definition of $\kappa$ for the time independent case.

5.6. The choice of $\Omega$

If $v^\mu$ is a vector tangent to a geodesic with affine parameter $\lambda$, then

$$\partial_\lambda (v^\mu l_\mu) = v^\nu \nabla_\nu (l_\mu v^\mu) = v^\nu v^\mu \nabla_\nu l_\mu = \frac{1}{4} v^\nu v_\nu \nabla^\mu l_\mu,$$

(91)

which vanishes for $v^\nu v_\nu = 0$. So $l^\mu v_\mu$ is conserved along null geodesics. Remember that we still have the freedom of Eq. (22). One can verify that $v^\nu \nabla_\nu v^\mu$ transforms as

$$v^\nu \nabla_\nu v^\mu \rightarrow v^\nu \nabla_\nu v^\mu + \frac{1}{\Omega} (2v^\nu v^\mu \nabla_\nu \Omega - v^\nu v_\nu \nabla^\mu \Omega).$$

(92)

For $v^\mu v_\mu = 0$ we obtain

$$v^\nu \nabla_\nu v^\mu \rightarrow v^\nu \nabla_\nu v^\mu + \frac{2}{\Omega} v^\mu v_\mu \nabla^\nu \Omega.$$  (93)

Now $v^\mu$ is a tangent vector to a geodesic if $v^\nu \nabla_\nu v^\mu = \alpha v^\mu$. If $\alpha = 0$, we have an affinely parameterized geodesic. Conformal transformations preserve null geodesics. So conformal transformations preserve affinely parameterized geodesics, if we take

$$\frac{2}{\Omega} v^\mu \nabla_\nu \Omega = 0,$$

or

$$\partial_\lambda \Omega = \pm N \sqrt{N^2} - \frac{r^2 N^\varphi}{2} \partial_\lambda \Omega,$$

(94)

which is consistent with the null condition for spacetime $\bar{g}_{\mu\nu}$. It makes the surface gravity also conformal invariant (note that $\nabla_\nu \Omega^2$ is non-singular at the horizon).

We conjecture that this constraint can be used for the outside observer. Can one construct from $\bar{l}$ and $\bar{m}$ a conformal Killing vector? (note that we have $\mathcal{L}_l \bar{g}_{\mu\nu} \neq 0$ and $\mathcal{L}_m \bar{g}_{\mu\nu} \neq 0$) If so, then one can construct a true Killing field for the conformally related metric $\Omega^2 \bar{g}_{\mu\nu}$ and a conformal invariant surface gravity of a conformal Killing horizon.
Remember, we have the conformal Killing equation for a conformal Killing vector (not null)
\[ \nabla_\mu \bar{l}_\nu + \nabla_\nu \bar{l}_\mu = \frac{1}{4} \nabla_\sigma \bar{l}_\sigma \bar{g}_{\mu\nu}. \] (95)

The Lie derivative transforms as
\[ \mathcal{L}_{\bar{l}} \bar{g}_{\mu\nu} \rightarrow \mathcal{L}_{\bar{l}} \log \Omega^2 \bar{g}_{\mu\nu}. \] (96)

Suppose that \( \hat{g}_{\mu\nu} = \Omega^2 \bar{g}_{\mu\nu} \). A spacetime with a conformal killing field \( \bar{l} \) is conformal to a spacetime for which \( \bar{l} \) is a true Killing field. That is, if \( \mathcal{L}_{\bar{l}} \hat{g}_{\mu\nu} = 2k \hat{g}_{\mu\nu} \), then \( \mathcal{L}_{\bar{l}} \Omega^2 = 0 \), if
\[ \mathcal{L}_{\bar{l}} \Omega^2 + 2k \Omega^2 = 0. \] (97)

The solution is given along integral curves of \( \bar{l} \) by \( \log \Omega = -\int k dv \), with \( \bar{l}^\mu \nabla_\mu v = 1 \).

The black hole complementarity is now formulated as follows: both observers uses different ways to fix the conformal gauge \( \Omega \). They disagree about the vacuum state. In our simplified model, with exact solutions for \( \omega \) and \( \bar{g}_{\mu\nu} \), we conjecture that these gauges can be found by constructing conformal killing fields and a conformal invariant surface gravity.

5.7. Flux density

We can also calculate the flux density \( \hat{G}_{tr} \).
\[ \frac{2(t + b_3)^3}{r \left[ (t + b_3)^4 + \frac{c_3}{c_2} \right]}. \] (98)

Then the flux across the horizon \( r = r_H \) in an amount of proper time can be found by integrating the expression
\[ \int_{t_{\text{init}}}^{t_{\text{final}}} \frac{4\pi r^2}{r \left[ (t + b_3)^4 + \frac{c_3}{c_2} \right]} 2N(t + b_3)^3 dt, \] (99)
which becomes infinite at \( t_H \).

6. Numerical solutions

We can also solve the PDE’s numerically. In figure 6 we plotted a typical solution for the 5D case. The numerical solution of the 4D counterpart case doesn’t differ significantly. With the numerical ”back-up” solution, one can investigate how the constants of the exact solution are related to the boundary conditions and the \( \Lambda \omega^4 \) term.
Figure 6. Example of a regular numerical vacuum solution in case of the 5D spacetime of section 4.1. We have only gravitational waves. \( c_1 = 1; c_2 = 0.15; c_3 = -0.2; c_4 = 1 \).

We observe that an ergo-sphere \((g_{tt} = 0)\) is dynamically formed at some \( r \) and \( t \). For the boundary condition for \( N \), we used Eq. (41). Note that the angular momentum is depicted without an arbitrary function in \( t \) (see Eq. (39)). It must be chosen such that \( N^\phi \) doesn’t change sign. We also see that \( \omega \) shrinks gradually to zero as the \( t \rightarrow \infty \) (see also the exact solution).

7. Conclusions

We investigated the conformal invariance of the Einstein-Hilbert action on a dynamical warped 5D black hole spacetime. By solving both Einstein and dilaton equations, we find an exact solution for the dilaton field and the metric components of the "un-physical" spacetime \( \bar{g}_{\mu\nu} \), where the real spacetime is written as \( \bar{g}_{\mu\nu} = \omega^{4/3} \bar{g}_{\mu\nu} \) and \( \omega^{(4)\bar{g}_{\mu\nu}} = \bar{\omega}^{2(4)\bar{g}_{\mu\nu}} \). In our model, \( \omega \) can be seen as the contribution of gravitational waves from the bulk (KK-modes), while \( \bar{\omega} \) is the brane component. There is no exchange of energy-momentum between the bulk and brane in our vacuum model.

It is conjectures that the different conformal gauge freedom, \( \Omega \), the in-going and outside observers possesses, can be calculated by demanding a conformal invariant surface gravity. This means that the complementarity is expressed by the different notion of the vacuum state. They disagree about the interior of the black hole. The solution guarantees regularity of the action when \( \omega \rightarrow 0 \). We don’t need a Weyl term in the action (generates negative metric states). In stead, we have a contribution from the bulk, i.e., the electric part of the 5D Weyl tensor. It is remarkable that the 5D field equations and the effective 4D equations can be written for general dimension \( n \), with \( n = 4, 5 \). The energy-momentum tensor of the time-dependent dilaton, determining the
Hawking radiation, can be calculated exactly. By suitable choice of the parameters, the spacetime $\bar{g}_{\mu\nu}$ can be regular and singular free.

In context of quantization procedures, counter terms in an effective action will cause problems, only in the bulk spacetime of the "large" extra dimension and not for the brane spacetime. When the extra dimensional volume is significantly above the Planck scale, then the true fundamental scale can be much less than the effective scale $10^{19}$ GeV. This means that no UV cut-off is necessary on the brane. This exact solution, nonetheless without mass terms, can be used to tackle the deep-seated problem of the black hole complementarity: the infalling and outside observer experience different $\omega$ by the choice of $\Omega$.

The solution fits also very well in the antipodal mapping, when crossing the horizon. The Penrose diagram for $\bar{g}_{\mu\nu}$, in suitable Kruskal coordinates, shows the features of the antipodal map of region I on region II: the inside of the black hole is removed. The in-going observer, when crossing the horizon, turns up at "the other side" of the horizon.

The next task is to incorporate mass into our model and investigate the dilaton-scalar field interaction. The conformal invariance will then spontaneously be broken. It is conjectured that the parameters of the model will be calculable in the final version. This is currently under investigation by the author.

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