A Hybrid Variance-Reduced Method for Decentralized Stochastic Non-Convex Optimization

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Abstract
This paper considers decentralized stochastic optimization over a network of $n$ nodes, where each node possesses a smooth non-convex local cost function and the goal of the networked nodes is to find an $\epsilon$-accurate first-order stationary point of the sum of the local costs. We focus on an online setting, where each node accesses its local cost only by means of a stochastic first-order oracle that returns a noisy version of the exact gradient. In this context, we propose a novel single-loop decentralized hybrid variance-reduced stochastic gradient method, called GT-HSGD, that outperforms the existing approaches in terms of both the oracle complexity and practical implementation. The GT-HSGD algorithm implements specialized local hybrid stochastic gradient estimators that are fused over the network to track the global gradient. Remarkably, GT-HSGD achieves a network topology-independent oracle complexity of $O(n^{-1}\epsilon^{-3})$ when the required error tolerance $\epsilon$ is small enough, leading to a linear speedup with respect to the centralized optimal online variance-reduced approaches that operate on a single node. Numerical experiments are provided to illustrate our main technical results.

1. Introduction
We consider $n$ nodes, such as machines or edge devices, communicating over a decentralized network described by a directed graph $G = (V, E)$, where $V = \{1, \cdots, n\}$ is the set of node indices and $E \subseteq V \times V$ is the collection of ordered pairs $(i, j)$, $i, j \in V$, such that node $j$ sends information to node $i$. Each node $i$ possesses a private local cost function $f_i : \mathbb{R}^p \rightarrow \mathbb{R}$ and the goal of the networked nodes is to solve, via local computation and communication, the following optimization problem:

$$\min_{x \in \mathbb{R}^p} F(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x).$$

This canonical formulation is known as decentralized optimization (Tsitsiklis et al., 1986; Nedić & Ozdaglar, 2009; Kar et al., 2012; Chen & Sayed, 2015) that has emerged as a promising framework for large-scale data science and machine learning problems (Lian et al., 2017; Assran et al., 2019). Decentralized optimization is essential in scenarios where data is geographically distributed and/or centralized data processing is infeasible due to communication and computation overhead or data privacy concerns. In this paper, we focus on an online and non-convex setting. In particular, we assume that each local cost $f_i$ is non-convex and each node $i$ only accesses $f_i$ by querying a local stochastic first-order oracle (SFO) (Nemirovskii et al., 2009) that returns a stochastic gradient, i.e., a noisy version of the exact gradient, at the queried point. As a concrete example of practical interest, the SFO mechanism applies to many online learning and expected risk minimization problems where the noise in SFO lies in the uncertainty of sampling from the underlying streaming data received at each node (Kar et al., 2012; Chen & Sayed, 2015). We are interested in the oracle complexity, i.e., the total number of queries to SFO required at each node, to find an $\epsilon$-accurate first-order stationary point $x^*$ of the global cost $F$ such that $\mathbb{E}[\|\nabla F(x^*)\|] \leq \epsilon$.

1.1. Related Work
We now briefly review the literature of decentralized non-convex optimization with SFO, which has been widely studied recently. Perhaps the most well-known approach is the decentralized stochastic gradient descent (DSGD) and its variants (Chen & Sayed, 2015; Kar et al., 2012; Vlaski & Sayed, 2019; Lian et al., 2017; Taheri et al., 2020), which combine average consensus and a local stochastic gradient step. Although being simple and effective, DSGD is known to have difficulties in handling heterogeneous data (Xin et al., 2020a). Recent works (Tang et al., 2018; Lu et al., 2019; Xin et al., 2020e; Yi et al., 2020) achieve robustness to heterogeneous environments by leveraging certain
decentralized bias-correction techniques such as EXTRA (type) (Shi et al., 2015; Yuan et al., 2020; Li & Lin, 2020), gradient tracking (Di Lorenzo & Scutari, 2016; Xu et al., 2015; Pu & Nedich, 2020; Xin et al., 2020; Nedich et al., 2017; Qu & Li, 2017; Xi et al., 2017), and primal-dual principles (Jakovetić, 2018; Li et al., 2019; Alghunaim et al., 2020; Xu et al., 2020). Built on top of these bias-correction techniques, very recent works (Sun et al., 2020) and (Pan et al., 2020) propose D-GET and D-SPIDER-SFO respectively that further incorporate online SARAH/SPIDER-type variance reduction schemes (Fang et al., 2018; Wang et al., 2019; Pham et al., 2020) to achieve lower oracle complexity when the SFO satisfies a mean-squared smoothness property. Finally, we note that the family of decentralized variance reduced methods has been significantly enriched recently, see, for instance, (Mokhtari & Ribeiro, 2016; Yuan et al., 2018; Xin et al., 2020a; Li et al., 2020b;a; Rajawat & Kumar, 2020; Xin et al., 2020b;c; Lü et al., 2020); however, these approaches are explicitly designed for empirical minimization where each local cost \( f_i \) is decomposed as a finite-sum of component functions, i.e., \( f_i = \frac{1}{m} \sum_{r=1}^{m} f_{i,r} \); it is therefore unclear whether these algorithms can be adapted to the online SFO setting, which is the focus of this paper.

1.2. Our Contributions

In this paper, we propose GT-HSGD, a novel online variance reduced method for decentralized non-convex optimization with stochastic first-order oracles (SFO). To achieve fast and robust performance, the GT-HSGD algorithm is built upon global gradient tracking (Di Lorenzo & Scutari, 2016; Xu et al., 2015) and a local hybrid stochastic gradient estimator (Liu et al., 2020; Tran-Dinh et al., 2020; Cutkosky & Orabona, 2019) that can be considered as a convex combination of the vanilla stochastic gradient returned by the SFO and a SARAH-type variance-reduced stochastic gradient (Nguyen et al., 2017). In the following, we emphasize the key advantages of GT-HSGD compared with the existing decentralized online (variance-reduced) approaches, from both theoretical and practical aspects.

Improved oracle complexity. A comparison of the oracle complexity of GT-HSGD with related algorithms is provided in Table 1, from which we have the following important observations. First of all, the oracle complexity of GT-HSGD is lower than that of DSGD, D2, GT-DSGD and D-PD-SGD, which are decentralized online algorithms without variance reduction; however, GT-HSGD imposes on the SFO an additional mean-squared smoothness (MSS) assumption that is required by all online variance-reduced techniques in the literature (Arjevani et al., 2019; Fang et al., 2018; Wang et al., 2019; Pham et al., 2020; Liu et al., 2020; Tran-Dinh et al., 2020; Cutkosky & Orabona, 2019; Sun et al., 2020; Pan et al., 2020; Zhou et al., 2020). Secondly, GT-HSGD further achieves a lower oracle complexity than the existing decentralized online variance-reduced methods D-GET (Sun et al., 2020) and D-SPIDER-SFO (Pan et al., 2020), especially in a regime where the required error tolerance \( \epsilon \) and the network spectral gap \((1 - \lambda)\) are relatively small. Moreover, when \( \epsilon \) is small enough such that \( \epsilon \leq \min \{ \lambda^{-4}(1 - \lambda)^3 n^{-1}, \lambda^{-1}(1 - \lambda)^{1.5} n^{-1} \} \), it can be verified that the oracle complexity of GT-HSGD reduces to \( O(n^{-1} \epsilon^{-3}) \), independent of the network topology, and GT-HSGD achieves a linear speedup, in terms of the scaling with the network size \( n \), compared with the centralized optimal online variance-reduced approaches that operate on a single node (Fang et al., 2018; Wang et al., 2019; Pham et al., 2020; Liu et al., 2020; Tran-Dinh et al., 2020; Zhou et al., 2020); see Section 3 for a detailed discussion. Sharp contrast, the speedup of D-GET (Sun et al., 2020) and D-SPIDER-SFO (Pan et al., 2020) is not clear compared with the aforementioned centralized optimal methods even if the network is fully connected, i.e., \( \lambda = 0 \).

More practical implementation. Both D-GET (Sun et al., 2020) and D-SPIDER-SFO (Pan et al., 2020) are double-loop algorithms that require very large minibatch sizes. In particular, during each inner loop they execute a fixed number of minibatch stochastic gradient type iterations with \( O(\epsilon^{-1}) \) oracle queries per update per node, while at every outer loop they obtain a stochastic gradient with mega minibatch size by \( O(\epsilon^{-2}) \) oracle queries at each node. Clearly, querying the oracles exceedingly, i.e., obtaining a large amount of samples, at each node and every iteration in online streaming data scenarios substantially jeopardizes the actual wall-clock time. This is because the next iteration cannot be performed until all nodes complete the sampling process. Moreover, the double-loop implementation may incur periodic network synchronizations. These issues are especially significant when the working environments of the nodes are heterogeneous. Conversely, the proposed GT-HSGD is a single-loop algorithm with \( O(1) \) oracle queries per update and only requires a large minibatch size with \( O(\epsilon^{-1}) \) oracle queries once in the initialization phase, i.e., before the update recursion is executed; see Algorithm 1 and Corollary 1 for details.

1.3. Roadmap and Notations

The rest of the paper is organized as follows. In Section 2, we state the problem formulation and develop the proposed GT-HSGD algorithm. Section 3 presents the main convergence results of GT-HSGD and their implications. Section 4 outlines the convergence analysis of GT-HSGD, while the detailed proofs are provided in the Appendix. Section 5 provides numerical experiments to illustrate our theoretical claims. Section 6 concludes the paper.

\[ A \text{ Hybrid Variance-Reduced Method for Decentralized Stochastic Non-Convex Optimization} \]
We adopt the following notations throughout the paper. We use lowercase bold letters to denote vectors and uppercase bold letters to denote matrices. The ceiling function is denoted as $\lceil \cdot \rceil$. The matrix $I_d$ represents the $d \times d$ identity; $I_d$ and $0_d$ are the $d$-dimensional column vectors of all ones and zeros, respectively. We denote $[x]_i$ as the $i$-th entry of a vector $x$. The Kronecker product of two matrices $A$ and $B$ is denoted by $A \otimes B$. We use $\| \cdot \|$ to denote the Euclidean norm of a vector or the spectral norm of a matrix. We use $\sigma(\cdot)$ to denote the $\sigma$-algebra generated by the sets and/or random vectors in its argument.

### 2. Problem Setup and GT-HSGD

In this section, we introduce the mathematical model of the stochastic first-order oracle (SFO) at each node and the communication network. Based on these formulations, we develop the proposed GT-HSGD algorithm.

#### 2.1. Optimization and Network Model

We work with a rich enough probability space $\{\Omega, \mathcal{P}, \mathcal{F}\}$. We consider decentralized recursive algorithms of interest that generate a sequence of estimates $\{x^t_i\}_{t \geq 0}$ of the first-order stationary points of $F$ at each node $i$, where $x^0_i$ is assumed constant. At each iteration $t$, each node $i$ observes a random vector $\xi^t_i$ in $\mathbb{R}^p$, which, for instance, may be considered as noise or as an online data sample. We then introduce the natural filtration (an increasing family of sub-$\sigma$-algebras of $\mathcal{F}$) induced by these random vectors observed sequentially by the networked nodes:

$$\mathcal{F}_0 := \{\Omega, \phi\}, \quad \mathcal{F}_t := \sigma (\{\xi^t_0, \xi^t_1, \cdots, \xi^t_{t-1} : i \in V\}), \quad \forall t \geq 1, \tag{1}$$

where $\phi$ is the empty set. We are now ready to define the SFO mechanism in the following. At each iteration $t$, each node $i$, given an input random vector $\xi \in \mathbb{R}^p$ that is $\mathcal{F}_t$-measurable, is able to query the local SFO to obtain a stochastic gradient of the form $g_i(x, \xi^t_i)$, where $g_i : \mathbb{R}^p \times \mathbb{R}^d \rightarrow \mathbb{R}^p$ is a Borel measurable function. We assume that the SFO satisfies the following four properties.

**Assumption 1 (Oracle).** For any $\mathcal{F}_t$-measurable random vectors $x, y \in \mathbb{R}^p$, we have the following: $\forall i \in V, \forall t \geq 0$:

- $\mathbb{E} [g_i(x, \xi^t_i)|\mathcal{F}_t] = \nabla f_i(x)$;
- $\mathbb{E} [\|g_i(x, \xi^t_i) - \nabla f_i(x)\|^2] \leq \nu_i^2, \quad \bar{\sigma}^2 := \frac{1}{n} \sum_{i=1}^n \nu_i^2$;
- the family $\{\xi^t_i : \forall t \geq 0, i \in V\}$ of random vectors is independent;
- $\mathbb{E} [\|g_i(x, \xi^t_i) - g_i(y, \xi^t_i)\|^2] \leq L^2 \mathbb{E} [\|x - y\|^2]$.

The first three properties above are standard and commonly used to establish the convergence of decentralized stochastic gradient methods. They however do not explicitly impose any structures on the stochastic gradient mapping $g_i$, other
than the measurability. On the other hand, the last property, the mean-squared smoothness, roughly speaking, requires that \( g_i \) is \( L \)-smooth on average with respect to the input arguments \( x \) and \( y \). As a simple example, Assumption 1 holds if \( f_i(x) = \frac{1}{2}x^T Q_i x \) and \( g_i(x, \xi_i) = Q_i x + \xi_i \), where \( Q_i \) is a constant matrix and \( \xi_i \) has zero mean and finite second moment. We further note that the mean-squared smoothness of each \( g_i \) implies, by Jensen’s inequality, that each \( f_i \) is \( L \)-smooth, i.e., \( \| \nabla f_i(x) - \nabla f_i(y) \| \leq L \| x - y \| \), and consequently the global function \( F \) is also \( L \)-smooth.

In addition, we make the following assumptions on \( F \) and the communication network \( G \).

**Assumption 2 (Global Function).** \( F \) is bounded below, i.e., \( F^* := \inf_{x \in \mathbb{R}^p} F(x) > -\infty \).

**Assumption 3 (Communication Network).** The directed network \( G \) admits a primitive and doubly-stochastic weight matrix \( \mathbf{W} = \{ w_{ij} \} \in \mathbb{R}^{n \times n} \). Hence, \( \mathbf{W} \mathbf{1}_n = \mathbf{W}^\top \mathbf{1}_n = \mathbf{1}_n \) and \( \Lambda := \| \mathbf{W} - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^\top \| \in [0,1) \).

The weight matrix \( \mathbf{W} \) that satisfies Assumption 3 may be designed for strongly-connected weight-balanced directed graphs (and thus for arbitrary connected undirected graphs). For example, the family of directed exponential directed graphs is weight-balanced and plays a key role in decentralized training (Assran et al., 2019). We note that \( \Lambda \) is known as the second largest singular value of \( \mathbf{W} \) and measures the algebraic connectivity of the graph, i.e., a smaller value of \( \Lambda \) roughly means a better connectivity. We note that several existing approaches require strictly stronger assumptions on \( \mathbf{W} \). For instance, \( D_2 \) (Tang et al., 2018) and \( D-PD-SGD \) (Yi et al., 2020) require \( \mathbf{W} \) to be symmetric and hence are restricted to undirected graphs.

### 2.2. Algorithm Development and Description

We now describe the proposed \( GT-HSGD \) algorithm and provide an intuitive construction. Recall that \( x_i^t \) is the estimate of an stationary point of the global cost \( F \) at node \( i \) and iteration \( t \). Let \( g_i(x_i^t, \xi_i^t) \) and \( g_i(x_{i-1}^t, \xi_i^t) \) be the corresponding stochastic gradients returned by the local SFO queried at \( x_i^t \) and \( x_{i-1}^t \) respectively. Motivated by the strong performance of recently introduced decentralized methods that combine gradient tracking and various reduction schemes for finite-sum problems (Xin et al., 2020b; Li et al., 2020a; Sun et al., 2020), we seek similar variance reduction for decentralized online problems with SFO. In particular, we focus on the following local hybrid variance reduced stochastic gradient estimator \( v_i^t \) introduced in (Liu et al., 2020; Tran-Dinh et al., 2020; Cutkosky & Orabona, 2019) for centralized online problems: for all \( t \geq 1 \),

\[
v_i^t = g_i(x_i^t, \xi_i^t) + (1 - \beta)(v_{i-1}^t - g_i(x_{i-1}^t, \xi_i^t)), \quad (2)
\]

for some applicable weight parameter \( \beta \in [0,1] \). This local gradient estimator \( v_i^t \) is fused, via a gradient tracking mechanism (Di Lorenzo & Scutari, 2016; Xu et al., 2015), over the network to update the global gradient tracker \( y_i^t \), which is subsequently used as the descent direction in the \( x_i^t \)-update. The complete description of \( GT-HSGD \) is provided in Algorithm 1. We note that the update (2) of \( v_i^t \) may be equivalently written as

\[
v_i^t = \beta \cdot g_i(x_i^t, \xi_i^t) + (1 - \beta) \cdot (g_i(x_i^t, \xi_i^t) - g_i(x_{i-1}^t, \xi_i^t) + v_{i-1}^t),
\]

which is a convex combination of the local vanilla stochastic gradient returned by the SFO and a SARA-type (Nguyen et al., 2017; Fang et al., 2018; Wang et al., 2019) gradient estimator. This discussion leads to the fact that \( GT-HSGD \) reduces to \( GT-DGD \) (Pu & Nedich, 2020; Xin et al., 2020; Lu et al., 2019) when \( \beta = 1 \), and becomes the inner loop of \( GT-SARAH \) (Xin et al., 2020b) when \( \beta = 0 \). However, our convergence analysis shows that \( GT-HSGD \) achieves its best oracle complexity and outperforms the existing decentralized online variance-reduced approaches (Sun et al., 2020; Pan et al., 2020) with a weight parameter \( \beta \in (0,1) \).

We then prove that neither \( GT-DGD \) nor the inner loop of \( GT-SARAH \), on their own, are able to outperform the proposed approach, making \( GT-HSGD \) a non-trivial algorithmic design for this problem class.

**Algorithm 1** \( GT-HSGD \) at each node \( i \)

**Require:** \( x_i^0 = x_0; \alpha; \beta; b_0; y_i^0 = 0_p; v_{i-1}^t = 0_p; T \).

1. Sample \( \{ \xi_{i,t} \}_{t=1}^{b_0} \) and \( v_0^1 = \frac{1}{b_0} \sum_{r=1}^{b_0} g_i(x_0^r, \xi_{0,r}) \);
2. \( y_i^t = \sum_{j=1}^{n} w_{ij} (y_i^{t-1} + v_i^t - v_i^{t-1}) \);
3. \( x_i^t = \sum_{j=1}^{n} w_{ij} (x_i^{t-1} - \alpha y_i^t) \);
4. for \( t = 1, 2, \ldots, T - 1 \) do
5. Sample \( \xi_i^t \);
6. \( v_i^t = g_i(x_i^t, \xi_i^t) + (1 - \beta)(v_{i-1}^t - g_i(x_{i-1}^t, \xi_i^t)) \);
7. \( y_{i+1}^t = \sum_{j=1}^{n} w_{ij} (y_i^t + v_i^t - v_{i-1}^t) \);
8. \( x_{i+1}^t = \sum_{j=1}^{n} w_{ij} (x_i^t - \alpha y_{i+1}^t) \);
9. end for

**output** \( \bar{x}_T \) selected uniformly at random from \( \{ x_i^t \}_{0 \leq t \leq T} \).

**Remark 1.** Clearly, each \( v_i^t \) is a conditionally biased estimator of \( \nabla f_i(x_i^t) \), i.e., \( \mathbb{E}[v_i^t] \neq \nabla f_i(x_i^t) \) in general. However, it can be shown that \( \mathbb{E}[v_i^t] = \mathbb{E}[v_i^t] \), meaning that \( v_i^t \) serves as a surrogate for the underlying exact gradient in the sense of total expectation.

### 3. Main Results

In this section, we present the main convergence results of \( GT-HSGD \) in this paper and discuss their salient features.
The formal convergence analysis is deferred to Section 4.

**Theorem 1.** If the weight parameter $\beta = \frac{48\lambda^2 \alpha^2}{n}$ and the step-size $\alpha$ is chosen as

$$0 < \alpha < \min \left\{ \frac{(1 - \lambda^2)^2}{90 \lambda^2}, \frac{\sqrt{n(1 - \lambda)}}{26 \lambda}, \frac{1}{4\sqrt{3}} \right\},$$

then the output $\bar{x}_T$ of GT-HSGD satisfies: $\forall T \geq 2,

$$E[\|\nabla F(\bar{x}_T)\|^2] \leq \frac{4F(x_0) - F^*}{\alpha T} + \frac{8\beta^2}{n} + \frac{4\beta^2}{n} + \frac{64\lambda^4 \|\nabla f(x_0)\|^2}{(1 - \lambda^2)^3 nT} + \frac{96\lambda^2 \beta^2}{(1 - \lambda^2)^3 b_0 T} + \frac{256\lambda^2 \beta^2 \tau^2}{(1 - \lambda^2)^3},$$

where $\|\nabla f(x_0)\|^2 = \sum_{i=1}^{n} \|\nabla f_i(x_0)\|^2$.

**Remark 2.** Theorem 1 holds for GT-HSGD with arbitrary initial minibatch size $b_0 \geq 1$.

Theorem 1 establishes a non-asymptotic bound, with no hidden constants, on the mean-squared stationary gap of GT-HSGD over any finite time horizon $T$.

**Transient and steady-state performance over infinite time horizon.** If $\alpha$ and $\beta$ are chosen according to Theorem 1, the mean-squared stationary gap $E[\|\nabla F(\bar{x}_T)\|^2]$ of GT-HSGD decays sublinearly at a rate of $O(1/T)$ up to a steady-state error (SSE) such that

$$\limsup_{T \to \infty} E[\|\nabla F(\bar{x}_T)\|^2] \leq \frac{8\beta^2}{n} + \frac{256\lambda^2 \beta^2 \tau^2}{(1 - \lambda^2)^3}.$$  \(3\)

In view of (3), the SSE of GT-HSGD is bounded by the sum of two terms: (i) the first term is in the order of $O(\beta^2)$ and the division by $n$ demonstrates the benefit of increasing the network size; (ii) the second term is in the order of $O(\beta^2)$ and reveals the impact of the spectral gap $(1 - \lambda)$ of the network topology. Clearly, the SSE can be made arbitrarily small by choosing small enough $\beta$ and $\alpha$. Moreover, since the spectral gap $(1 - \lambda)$ only appears in a higher order term of $\beta$ in (3), its impact reduces as $\beta$ becomes smaller, i.e., as we require a smaller SSE.

The following corollary is concerned with the finite-time convergence rate of GT-HSGD with specific choices of the algorithmic parameters $\alpha$, $\beta$, and $b_0$.

**Corollary 1.** Setting $\alpha = \frac{3^{2/3}}{8\sqrt{7}}$, $\beta = \frac{3^{1/3}}{4T^{2/3}}$, and $b_0 = \lfloor \frac{T^{1/3}}{n^{2/3}} \rfloor$ in Theorem 1, we have:

$$E[\|\nabla F(\bar{x}_T)\|^2] \leq \frac{32L(F(x_0) - F^*) + 12\beta^2}{(nT)^{2/3}} + \frac{64\lambda^4 \|\nabla f(x_0)\|^2}{(1 - \lambda^2)^3 nT} + \frac{240\lambda^2 \beta^2 \tau^2}{(1 - \lambda^2)^3 T^{4/3}},$$

for all $T > \max \{ \frac{1424\lambda^6 n^2}{(1 - \lambda^2)^3}, \frac{35A^3 n^{0.5}}{(1 - \lambda)^{1.5}} \}$. As a consequence, GT-HSGD achieves an $\epsilon$-accurate stationary point $x^*$ of the global cost $F$ such that $E[\|\nabla F(x^*)\|] \leq \epsilon$ with

$$H = \left( \max \{ H_{opt}, H_{net} \} \right) \text{iterations},$$

where $H_{opt}$ and $H_{net}$ are given respectively by

$$H_{opt} = \frac{(L(F(x_0) - F^*) + \tau^2)^{1.5}}{n\epsilon^3},$$

$$H_{net} = \max \left\{ \frac{\lambda^3 \|\nabla f(x_0)\|^2}{(1 - \lambda^2)^3 n\epsilon^2}, \frac{\lambda^3 n^{0.5} \tau^{1.5}}{(1 - \lambda^2)^2 \epsilon^{1.5}} \right\}.$$

The resulting total number of oracle queries at each node is thus $[H + H^{1/3} n^{-2/3}]$.

**Remark 3.** Since $H^{1/3} n^{-2/3}$ is much smaller than $H$, we treat the oracle complexity of GT-HSGD as $H$ for the ease of exposition in Table 1 and the following discussion.

An important implication of Corollary 1 is given in the following.

**A regime for network topology-independent oracle complexity and linear speedup.** According to Corollary 1, the oracle complexity of GT-HSGD at each node is bounded by the maximum of two terms: (i) the first term $H_{opt}$ is independent of the network topology and, more importantly, is $n$ times smaller than the oracle complexity of the optimal centralized online variance-reduced methods that execute on a single node for this problem class (Fang et al., 2018; Wang et al., 2019; Pham et al., 2020; Tran-Dinh et al., 2020; Liu et al., 2020); (ii) the second term $H_{net}$ depends on the network spectral gap $1 - \lambda$ and is in the lower order of $1/\epsilon$. These two observations lead to the interesting fact that the oracle complexity of GT-HSGD becomes independent of the network topology, i.e., $H_{opt}$ dominates $H_{net}$, if the required error tolerance $\epsilon$ is small enough such that $\epsilon \leq \min \{ \lambda^{-4} (1 - \lambda^2)^{-n^{-1}}, \lambda^{-1} (1 - \lambda)^{-1/3} n^{-1} \}$.

In this regime, GT-HSGD thus achieves a network topology-independent oracle complexity $H_{opt} = O(n^{-1} \epsilon^{-3})$, exhibiting a linear speedup compared with the aforementioned centralized optimal algorithms (Fang et al., 2018; Wang et al., 2019; Pham et al., 2020; Tran-Dinh et al., 2020; Liu et al., 2020; Zhou et al., 2020), in the sense that the total number of oracle queries required to achieve an $\epsilon$-accurate stationary point at each node is reduced by a factor of $1/n$.

**Remark 4.** The small error tolerance regime in the above discussion corresponds to a large number of oracle queries, which translates to the scenario where the required total number of iterations $T$ is large. Note that a large $T$ further implies that the step-size $\alpha$ and the weight parameter $\beta$ are small; see the expression of $\alpha$ and $\beta$ in Corollary 1.

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\(^2\)Since GT-HSGD computes $O(n)$ stochastic gradients in parallel per iteration across the nodes, the network size $n$ can be interpreted as the minibatch size of GT-HSGD.

\(^3\)The $O(\cdot)$ notation here does not absorb any problem parameters, i.e., it only hides universal constants.

\(^4\)This boundary condition follows from basic algebraic manipulations.
4. Outline of the Convergence Analysis

In this section, we outlines the proof of Theorem 1, while the detailed proofs are provided in the Appendix. We let Assumptions 1-3 hold throughout the rest of the paper without explicitly stating them. For the ease of exposition, we write the $x_t$- and $y_t$-update of GT-HSGD in the following equivalent matrix form: for all $t \geq 0$,

\begin{align*}
y_{t+1} &= W_y (y_t + v_t - y_{t-1}), \\
x_{t+1} &= W_x (x_t - \alpha y_{t+1}),
\end{align*}

(4a)

(4b)

where $W := W_y \otimes I_p$ and $x_t, y_t, v_t$ are square-integrable random vectors in $\mathbb{R}^{np}$ that respectively concatenate the local estimates $\{x^i_t\}_{i=1}^n$ of a stationary point of $F$, gradient trackers $\{y^i_t\}_{i=1}^n$, stochastic gradient estimators $\{v^i_t\}_{i=1}^n$. It is straightforward to verify that $x_t$ and $y_t$ are $\mathcal{F}_t$-measurable while $v_t$ is $\mathcal{F}_{t+1}$-measurable for all $t \geq 0$. For convenience, we also denote

$$\nabla f_i(x^i_t) := [\nabla f_1(x^1_t)^T, \ldots, \nabla f_n(x^n_t)^T]^T$$

and introduce the following quantities:

$$J := \left( \frac{1}{n} 1_n 1_n^T \right) \otimes I_p$$

$$x_t := \frac{1}{n} (1_n \otimes I_p) x_t, \quad y_t := \frac{1}{n} (1_n \otimes I_p) y_t,$$

$$v_t := \frac{1}{n} (1_n \otimes I_p) v_t, \quad \nabla F(x_t) := \frac{1}{n} (1_n \otimes I_p) \nabla f(x_t).$$

In the following lemma, we enlist several well-known results in the context of gradient tracking-based algorithms for decentralized stochastic optimization, whose proofs may be found in (Di Lorenzo & Scutari, 2016; Qu & Li, 2017; Xin et al., 2020d; Pu & Nedich, 2020).

**Lemma 1.** The following relationships hold.

(a) $\|W x - J x\| \leq \lambda \|x - J x\|, \forall x \in \mathbb{R}^{np}$.

(b) $y_{t+1} = v_t, \forall t \geq 0$.

(c) $\|\nabla F(x_t) - \nabla F(x_t)\| \leq \frac{L}{n} \|x_t - x_t\|^2, \forall t \geq 0$.

We note that Lemma 1(a) holds since $W$ is primitive and doubly-stochastic, Lemma 1(b) is a direct consequence of the gradient tracking update (4a) and Lemma 1(c) is due to the $L$-smoothness of each $f_i$. By the estimate update of GT-HSGD described in (4b) and Lemma 1(b), it is straightforward to obtain:

$$x_{t+1} = x_t - \alpha y_{t+1} = x_t - \alpha v_t, \quad \forall t \geq 0. \quad (5)$$

Hence, the mean state $x_t$ proceeds in the direction of the average of local stochastic gradient estimators $v_t$. With the help of (5) and the $L$-smoothness of $F$ and each $f_i$, we establish the following descent inequality which sheds light on the overall convergence analysis.

**Lemma 2.** If $0 < \alpha \leq \frac{1}{2L}$, then we have: for all $t \geq 0$,

$$\begin{align*}
\sum_{t=0}^T \|\nabla F(x_t)\|^2 &\leq \frac{2(F(x_0) - F^*)}{\alpha} - \frac{1}{2} \sum_{t=0}^T \|v_t\|^2 \\
&\quad + 2 \sum_{t=0}^T \|v_t - \nabla F(x_t)\|^2 + \frac{2L^2}{n} \sum_{t=0}^T \|x_t - J x_t\|^2.
\end{align*}$$

In light of Lemma 2, our approach to establishing the convergence of GT-HSGD is to seek the conditions on the algorithmic parameters of GT-HSGD, i.e., the step-size $\alpha$ and the weight parameter $\beta$, such that

$$\begin{align*}
- \frac{1}{2T} \sum_{t=0}^T \mathbb{E} \left[ \|v_t\|^2 \right] &+ \frac{2}{T} \sum_{t=0}^T \mathbb{E} \left[ \|v_t - \nabla F(x_t)\|^2 \right] \\
&\quad + \frac{2L^2}{nT} \sum_{t=0}^T \mathbb{E} \left[ \|x_t - J x_t\|^2 \right] = O \left( \frac{\alpha \beta}{b_0}, \frac{1}{T} \right), \quad (6)
\end{align*}$$

where $O(\alpha \beta, 1/b_0, 1/T)$ represents a nonnegative quantity which may be made arbitrarily small by choosing small enough $\alpha$ and $\beta$ along with large enough $T$ and $b_0$. If (6) holds, then Lemma 2 reduces to

$$\begin{align*}
\frac{1}{T+1} \sum_{t=0}^T \mathbb{E} \left[ \|\nabla F(x_t)\|^2 \right] &\leq \frac{2(F(x_0) - F^*)}{\alpha T} + O \left( \alpha \beta, \frac{1}{b_0}, \frac{1}{T} \right),
\end{align*}$$

which leads to the convergence arguments of GT-HSGD. For these purposes, we quantify $\sum_{t=0}^T \mathbb{E} \left[ \|v_t - \nabla F(x_t)\|^2 \right]$ and $\sum_{t=0}^T \mathbb{E} \left[ \|x_t - J x_t\|^2 \right]$ next.

4.1. Contraction Relationships

First of all, we establish upper bounds on the gradient variances $\mathbb{E} \left[ \|v_t - \nabla F(x_t)\|^2 \right]$ and $\mathbb{E} \left[ \|v_t - \nabla F(x_t)\|^2 \right]$ by exploiting the hybrid and recursive update of $v_t$.

**Lemma 3.** The following inequalities hold: for all $t \geq 1$,

$$\begin{align*}
\mathbb{E} \left[ \|v_t - \nabla F(x_t)\|^2 \right] &\leq (1 - \beta)^2 \mathbb{E} \left[ \|v_{t-1} - \nabla F(x_{t-1})\|^2 \right] \\
&\quad + \frac{6L^2 \alpha^2}{n} \mathbb{E} \left[ \|v_{t-1}\|^2 \right] + \frac{2\beta^2 \sigma^2}{n} \\
&\quad + \frac{6L^2}{n^2} (1 - \beta)^2 \mathbb{E} \left[ \|x_t - J x_t\|^2 + \|x_{t-1} - J x_{t-1}\|^2 \right], \quad (7)
\end{align*}$$

and for all $t \geq 1$,

$$\begin{align*}
\mathbb{E} \left[ \|v_t - \nabla F(x_t)\|^2 \right] &\leq (1 - \beta)^2 \mathbb{E} \left[ \|v_{t-1} - \nabla F(x_{t-1})\|^2 \right] \\
&\quad + 6nL^2 \alpha^2 (1 - \beta)^2 \mathbb{E} \left[ \|v_{t-1}\|^2 \right] + 2n \beta^2 \sigma^2 \\
&\quad + 6L^2 (1 - \beta)^2 \mathbb{E} \left[ \|x_t - J x_t\|^2 + \|x_{t-1} - J x_{t-1}\|^2 \right]. \quad (8)
\end{align*}$$
Remark 5. Since $v_t$ is a conditionally biased estimator of $\nabla f(x_t)$, (7) and (8) do not directly imply each other and need to be established separately.

We emphasize that the contraction structure of the gradient variances shown in Lemma 3 plays a crucial role in the convergence analysis. The following contraction bounds on the consensus errors $E[\|x_t - Jx_t\|^2]$ are standard in decentralized algorithms based on gradient tracking, e.g., (Pu & Nedich, 2020; Xin et al., 2020b); in particular, it follows directly from the $x_t$-update (4b) and Young’s inequality. 

**Lemma 4.** The following inequalities hold: $\forall t \geq 0$,

\[
\begin{align*}
\|x_{t+1} - Jx_{t+1}\|^2 &\leq \frac{1 + \lambda^2}{2} \|x_t - Jx_t\|^2 + \frac{2\alpha^2\lambda^2}{1 - \lambda^2} \|y_{t+1} - Jy_{t+1}\|^2 . \quad (9) \\
\|x_{t+1} - Jx_{t+1}\|^2 &\leq 2\lambda^2 \|x_t - Jx_t\|^2 + 2\alpha^2 \lambda^2 \|y_{t+1} - Jy_{t+1}\|^2 . \quad (10)
\end{align*}
\]

It is then clear from Lemma 4 that we need to further quantify the gradient tracking errors $E[\|y_t - Jy_t\|^2]$ in order to bound the consensus errors. These error bounds are shown in the following lemma.

**Lemma 5.** We have the following.

(a) $E[\|y_t - Jy_t\|^2] \leq \lambda^2 \|\nabla f(x_0)\|^2 + \lambda^2 n\beta^2 / b_0$. 

(b) If $0 < \alpha \leq \frac{1 - \lambda^2}{2\sqrt{\alpha^2} \lambda L}$, then $\forall t \geq 1$,

\[
\begin{align*}
E[\|y_{t+1} - Jy_{t+1}\|^2] &\leq \frac{3 + \lambda^2}{4} E[\|y_t - Jy_t\|^2] + \frac{21\lambda^2 n L^2 \alpha^2}{1 - \lambda^2} E[\|\nabla f(x_t)\|^2] \\
&+ \frac{63\lambda^2 L^2 x}{1 - \lambda^2} E[\|x_{t-1} - Jx_{t-1}\|^2] \\
&+ \frac{7\lambda^2 \beta^2}{1 - \lambda^2} E[\|v_{t-1} - \nabla f(x_{t-1})\|^2] + 3\lambda^2 n\beta^2 \beta^2 .
\end{align*}
\]

We note that establishing the contraction argument of gradient tracking errors in Lemma 5 requires a careful examination of the structure of the $v_t$-update.

### 4.2. Error Accumulations

To proceed, we observe, from Lemma 3, 4, and 5, that the recursions of the gradient variances, consensus, and gradient tracking errors admit similar forms. Therefore, we abstract out formulas for the accumulation of the error recursions of this type in the following lemma.

**Lemma 6.** Let $\{V_t\}_{t \geq 0}$, $\{R_t\}_{t \geq 0}$, and $\{Q_t\}_{t \geq 0}$ be non-negative sequences and $C \geq 0$ be some constant such that $V_t \leq qV_{t-1} + qR_{t-1} + Q_t + C, \forall t \geq 1$, where $q \in (0, 1)$. Then the following inequality holds: $\forall T \geq 1$,

\[
\sum_{t=0}^{T} V_t \leq \frac{V_0}{1 - q} + \frac{1}{1 - q} \sum_{t=0}^{T-1} R_t + \frac{1}{1 - q} \sum_{t=1}^{T} Q_t + \frac{CT}{1 - q} . \quad (11)
\]

Similarly, if $V_{t+1} \leq qV_t + R_t + C, \forall t \geq 1$, then we have $\forall T \geq 1$,

\[
\sum_{t=1}^{T} V_t \leq \frac{V_1}{1 - q} + \frac{1}{1 - q} \sum_{t=0}^{T-2} R_t + \frac{CT}{1 - q} . \quad (12)
\]

Applying Lemma 6 to Lemma 3 leads to the following upper bounds on the accumulated variances.

**Lemma 7.** For any $\beta \in (0, 1)$, the following inequalities hold: $\forall T \geq 1$,

\[
\sum_{t=0}^{T} E[\|v_t - \nabla f(x_t)\|^2] \leq \frac{\beta^2}{2b_0 n} + \frac{6L^2 \alpha^2 T-1}{n\beta} E[\|v_t\|^2] \\
+ \frac{12L^2}{n\beta} \sum_{t=0}^{T} E[\|x_t - Jx_t\|^2] + \frac{2\beta^2 T}{n^2} , \quad (13)
\]

and, $\forall T \geq 1$,

\[
\sum_{t=0}^{T} E[\|v_t - \nabla f(x_t)\|^2] \leq \frac{n^2 \beta^2}{2b_0} + \frac{6\lambda L^2 \alpha^2 T-1}{\beta} \sum_{t=0}^{T} E[\|v_t\|^2] \\
+ \frac{12L^2}{\beta} \sum_{t=0}^{T} E[\|x_t - Jx_t\|^2] + 2n\beta^2 T . \quad (14)
\]

It can be observed that (13) in Lemma 7 may be used to refine the left hand side of (6). The remaining step, naturally, is to bound $\sum_{t=0}^{T} E[\|x_t - Jx_t\|^2]$ in terms of $\sum_{t=0}^{T} E[\|v_t\|^2]$. This result is provided in the following lemma that is obtained with the help of Lemma 4, 5, 6, and 7.

**Lemma 8.** If $0 < \alpha \leq \frac{1 - \lambda^2}{2\sqrt{\alpha^2} \lambda L}$ and $\beta \in (0, 1)$, then the following inequality holds: $\forall T \geq 2$,

\[
\sum_{t=0}^{T} E[\|x_t - Jx_t\|^2] \leq \frac{20L^4 \alpha^2 T^2}{(1 - \lambda^2)^4} \sum_{t=0}^{T} E[\|v_t\|^2] + \frac{32L^2 \alpha^2 x}{(1 - \lambda^2)^3} + \left( \frac{7\beta}{1 - \lambda^2} + 1 \right) \frac{32\lambda^4 \beta^2 T^2}{(1 - \lambda^2)^3 b_0} + \left( \frac{14\beta}{1 - \lambda^2} + 3 \right) \frac{32\lambda^4 \beta^2 T^2}{(1 - \lambda^2)^3} .
\]

Finally, we note that Lemma 7 and 8 suffice to establish (6) and hence lead to Theorem 1; see the Appendix for details.
5. Numerical Experiments

In this section, we illustrate our theoretical results on the convergence of the proposed GT-HSGD algorithm with the help of numerical experiments.

**Model.** We consider a non-convex logistic regression model (Antoniadis et al., 2011) for decentralized binary classification. In particular, the decentralized non-convex optimization problem of interest takes the form

\[
\min_{x \in \mathbb{R}^p} F(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x) + r(x),
\]

where

\[
f_i(x) = \frac{1}{m} \sum_{j=1}^{m} \log \left( 1 + e^{-(x, \theta_{i,j})l_{i,j}} \right)
\]

and

\[
r(x) = R \sum_{k=1}^{p} \frac{|x|_k^2}{1 + |x|_k^2},
\]

where \(\theta_{i,j}\) is the feature vector, \(l_{i,j} \in \{-1, +1\}\) is the corresponding binary label, and \(r(x)\) is a non-convex regularizer.

To simulate the online SFO setting described in Section 2, each node \(i\) is only able to sample with replacement from its local data \(\{\theta_{i,j}, l_{i,j}\}_{j=1}^{m}\) and compute the corresponding (minibatch) stochastic gradient. Throughout all experiments, we set the number of the nodes to \(n = 20\) and the regularization parameter to \(R = 10^{-4}\).

**Data.** To test the performance of the applicable decentralized algorithms, we distribute the a9a, covertype, KDD98, MiniBooNE datasets uniformly over the nodes and normalize the feature vectors such that \(\|\theta_{i,j}\| = 1, \forall i, j\). The statistics of these datasets are provided in Table 2.

**Network topology.** We consider the following network topologies: the undirected ring graph, the undirected and directed exponential graphs, and the complete graph; see (Nedić et al., 2018; Xin et al., 2020f; Assran et al., 2019; Lian et al., 2017) for detailed configurations of these graphs. For all graphs, the associated doubly stochastic weights are set to be equal. The resulting second largest singular value \(\lambda\) of the weight matrices are 0.98, 0.75, 0.67, 0, respectively, demonstrating a significant difference in the algebraic connectivity of these graphs.

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**Figure 1.** A comparison of GT-HSGD with other decentralized online stochastic gradient algorithms over the undirected exponential graph of 20 nodes on the a9a, covertype, KDD98, and MiniBooNE datasets.
Performance measure. We measure the performance of the decentralized algorithms in question by the decrease of the global cost function value $F(\bar{x})$, to which we refer as loss, versus epochs, where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, with $x_i$ being the model at node $i$ and each epoch contains $m$ stochastic gradient computations at each node.

5.1. Comparison with the Existing Decentralized Stochastic Gradient Methods.

We conduct a performance comparison of GT-HSGD with GT-DSGD (Pu & Nedich, 2020; Lu et al., 2019; Xin et al., 2020e), D-GET (Sun et al., 2020), and D-SPIDER-SFO (Pan et al., 2020) over the undirected exponential graph of 20 nodes. Note that we use GT-DSGD to represent methods that do not incorporate online variance reduction techniques, since it in general matches or outperforms DSGD (Lian et al., 2017) and has a similar performance with D2 (Tang et al., 2018) and D-PD-SGD (Yi et al., 2020).

Parameter tuning. We set the parameters of GT-HSGD, GT-DSGD, D-GET, and D-SPIDER-SFO according to the following procedures. First, we find a very large step-size candidate set for each algorithm in comparison. Second, we choose the minibatch size candidate set for all algorithms as $B := \{1, 4, 8, 16, 32, 64, 128, 256, 512, 1024\}$: the minibatch size of GT-DSGD, the minibatch size of GT-HSGD at $t = 0$, the minibatch size of D-GET and D-SPIDER-SFO at inner- and outer-loop are all chosen from $B$. Third, for D-GET and D-SPIDER-SFO, we choose the inner-loop length candidate set as $\{m, \frac{m}{2}, \frac{m}{4}, \ldots, \frac{m}{2^k}, \ldots, \frac{2m}{b}, \frac{2m}{2b}, \ldots, \frac{20m}{b}\}$, where $m$ is the local data size and $b$ is the minibatch size at the inner-loop. Fourth, we iterate over all combinations of parameters for each algorithm to find its best performance. In particular, we find that the best performance of GT-HSGD is attained with a small $\beta$ and a relatively large $\alpha$ as Corollary 1 suggests.

The experimental results are provided in Fig. 1, where we observe that GT-HSGD achieves faster convergence than the other algorithms in comparison on those four datasets. This observation is coherent with our main convergence results that GT-HSGD achieves a lower oracle complexity than the existing approaches; see Table 1.

5.2. Topology-Independent Rate of GT-HSGD

We test the performance of GT-HSGD over different network topologies. In particular, we follow the procedures described in Section 5.1 to find the best set of parameters for GT-HSGD over the complete graph and then use this parameter set for other graphs. The corresponding experimental results are presented in Fig. 2. Clearly, it can be observed that when the number of iterations is large enough, that is to say, the required error tolerance is small enough, the convergence rate of GT-HSGD is not affected by the underlying network topology. This interesting phenomenon is consistent with our convergence theory; see Corollary 1 and the related discussion in Section 3.

6. Conclusion

In this paper, we investigate decentralized stochastic optimization to minimize a sum of smooth non-convex cost functions over a network of nodes. Assuming that each node has access to a stochastic first-order oracle, we propose GT-HSGD, a novel single-loop decentralized algorithm that leverages local hybrid variance reduction and gradient tracking to achieve provably fast convergence and robust performance. Compared with the existing online variance-reduced methods, GT-HSGD achieves a lower oracle complexity with a more practical implementation. We further show that GT-HSGD achieves a network topology-independent oracle complexity, when the required error tolerance is small enough, leading to a linear speedup with respect to the centralized optimal methods that execute on a single node.

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A. Proof of Lemma 2

We recall the standard Descent Lemma (Nesterov, 2018), i.e., \( \forall x, y \in \mathbb{R}^p \),

\[
F(y) \leq F(x) + \langle \nabla F(x), y - x \rangle + \frac{L}{2} \| y - x \|^2 ,
\]

(15)

since the global function \( F \) is \( L \)-smooth. Setting \( y = x_{t+1} \) and \( x = x_t \) in (15) and using (5), we have: \( \forall t \geq 0 \),

\[
F(x_{t+1}) \leq F(x_t) - \langle \nabla F(x_t), x_{t+1} - x_t \rangle + \frac{L}{2} \| x_{t+1} - x_t \|^2
\]

\[
\leq F(x_t) - \alpha \langle \nabla F(x_t), v_t \rangle + \frac{L \alpha^2}{2} \| v_t \|^2 .
\]

(16)

Using \( \langle a, b \rangle = \frac{1}{2} (\| a \|^2 + \| b \|^2 - \| a - b \|^2) \), \( \forall a, b \in \mathbb{R}^p \), in (16) gives: for \( 0 < \alpha \leq \frac{1}{2L} \) and \( \forall t \geq 0 \),

\[
F(x_{t+1}) \leq F(x_t) - \frac{\alpha}{2} \| \nabla F(x_t) \|^2 - \left( \frac{\alpha}{2} - \frac{L \alpha^2}{2} \right) \| v_t \|^2 + \frac{\alpha}{2} \| v_t - \nabla F(x_t) \|^2 ,
\]

\[
\leq F(x_t) - \frac{\alpha}{2} \| \nabla F(x_t) \|^2 - \left( \frac{\alpha}{2} - \frac{L \alpha^2}{2} \right) \| v_t \|^2 + \alpha \| v_t - \nabla \tilde{F}(x_t) \|^2 + \alpha \| \nabla \tilde{F}(x_t) - \nabla F(x_t) \|^2 ,
\]

\[
\leq F(x_t) - \frac{\alpha}{2} \| \nabla F(x_t) \|^2 - \frac{\alpha}{4} \| v_t \|^2 + \alpha \| v_t - \nabla \tilde{F}(x_t) \|^2 + \frac{\alpha L^2}{n} \| x_t - Jx_t \|^2 ,
\]

(17)

where \( (i) \) is due to Lemma 1(c) and that \( \frac{L \alpha^2}{2} \leq \frac{\alpha}{2} \) since \( 0 < \alpha \leq \frac{1}{2L} \). Rearranging (17), we have: for \( 0 < \alpha \leq \frac{1}{2L} \) and \( \forall t \geq 0 \),

\[
\| \nabla F(x_t) \|^2 \leq \frac{2(F(x_t) - F(x_{t+1}))}{\alpha} - \frac{1}{2} \| v_t \|^2 + 2 \| v_t - \nabla \tilde{F}(x_t) \|^2 + \frac{2L^2}{n} \| x_t - Jx_t \|^2 .
\]

(18)

Taking the telescoping sum of (18) over \( t \) from 0 to \( T \), \( \forall T \geq 0 \) and using the fact that \( F \) bounded below by \( F^* \) in the resulting inequality finishes the proof.

B. Proof of Lemma 3

B.1. Proof of Eq. (7)

We recall that the update of each local stochastic gradient estimator \( v^i_t, \forall t \geq 1 \), in (2) may be written equivalently as follows:

\[
v^i_t = \beta g_i(x^i_t, \xi^i_t) + (1 - \beta) \left( g_i(x^i_t, \xi^i_t) - g_i(x^i_{t-1}, \xi^i_t) + v^i_{t-1} \right) ,
\]

where \( \beta \in (0, 1) \). We have: \( \forall t \geq 1 \) and \( \forall i \in \mathcal{V} \),

\[
v^i_t - \nabla f_i(x^i_t) = \beta g_i(x^i_t, \xi^i_t) + (1 - \beta) \left( g_i(x^i_t, \xi^i_t) - g_i(x^i_{t-1}, \xi^i_t) + v^i_{t-1} \right) - \beta \nabla f_i(x^i_t) - (1 - \beta) \nabla f_i(x^i_t)
\]

\[
= \beta \left( g_i(x^i_t, \xi^i_t) - \nabla f_i(x^i_t) \right) + (1 - \beta) \left( g_i(x^i_{t-1}, \xi^i_t) - g_i(x^i_{t-1}, \xi^i_t) + v^i_{t-1} - \nabla f_i(x^i_t) \right)
\]

\[
= \beta \left( g_i(x^i_t, \xi^i_t) - \nabla f_i(x^i_t) \right) + (1 - \beta) \left( g_i(x^i_{t-1}, \xi^i_t) - g_i(x^i_{t-1}, \xi^i_t) + \nabla f_i(x^i_{t-1}) - \nabla f_i(x^i_t) \right)
\]

\[
+ (1 - \beta) \left( v^i_{t-1} - \nabla f_i(x^i_{t-1}) \right) .
\]

(19)

In (19), we observe that \( \forall t \geq 1 \) and \( \forall i \in \mathcal{V} \),

\[
E \left[ g_i(x^i_t, \xi^i_t) - \nabla f_i(x^i_t) \right] = 0_p ,
\]

(20)

\[
E \left[ g_i(x^i_t, \xi^i_t) - g_i(x^i_{t-1}, \xi^i_t) + \nabla f_i(x^i_{t-1}) - \nabla f_i(x^i_t) \right] = 0_p ,
\]

(21)
by the definition of the filtration $\mathcal{F}_t$ in (1). Averaging (19) over $i$ from 1 to $n$ gives: $\forall t \geq 0$,

$$\nabla_t - \nabla f_t(x_t) = (1 - \beta)\left(\nabla_{t-1} - \nabla f_{t-1}(x_{t-1})\right) + \beta \cdot \frac{1}{n} \sum_{i=1}^{n} \left(g_i(x_i^t, \xi_i^t) - \nabla f_i(x_i^t)\right) =: z_t$$

$$+ (1 - \beta) \cdot \frac{1}{n} \sum_{i=1}^{n} \left(g_i(x_i^t, \xi_i^t) - g_i(x_i^t_{t-1}, \xi_i^t) + \nabla f_i(x_i^t_{t-1}) - \nabla f_i(x_i^t)\right).$$

(22)

Note that $E[z_t, \mathcal{F}_t] = E[x_t, \mathcal{F}_t] = 0_p$ by (20) and (21). In light of (22), we have: $\forall t \geq 1$,

$$E\left[\left\|\nabla_t - \nabla f_t(x_t)\right\|^2 | \mathcal{F}_t\right] = (1 - \beta)^2 \left\|\nabla_{t-1} - \nabla f_{t-1}(x_{t-1})\right\|^2 + E\left[\left\|\beta s_t + (1 - \beta)z_t\right\|^2 | \mathcal{F}_t\right]$$

$$+ 2E\left[\left\langle (1 - \beta) \left(\nabla_{t-1} - \nabla f_{t-1}(x_{t-1})\right), \beta s_t + (1 - \beta)z_t \right\rangle | \mathcal{F}_t\right]$$

$$\overset{(i)}{=} (1 - \beta)^2 \left\|\nabla_{t-1} - \nabla f_{t-1}(x_{t-1})\right\|^2 + E\left[\left\|\beta s_t + (1 - \beta)z_t\right\|^2 | \mathcal{F}_t\right]$$

$$\leq (1 - \beta)^2 \left\|\nabla_{t-1} - \nabla f_{t-1}(x_{t-1})\right\|^2 + 2\beta^2 E\left[\left\|s_t\right\|^2 | \mathcal{F}_t\right] + 2(1 - \beta)^2 E\left[\left\|z_t\right\|^2 | \mathcal{F}_t\right],$$

(23)

where $(i)$ is due to

$$E\left[\left\langle (1 - \beta) \left(\nabla_{t-1} - \nabla f_{t-1}(x_{t-1})\right), \beta s_t + (1 - \beta)z_t \right\rangle | \mathcal{F}_t\right] = 0,$$

since $E[s_t | \mathcal{F}_t] = E[z_t | \mathcal{F}_t] = 0_p$ and $(\nabla_{t-1} - \nabla f_{t-1}(x_{t-1}))$ is $\mathcal{F}_t$-measurable. We next bound the second and the third term in (23) respectively. For the second term in (23), we observe that $\forall t \geq 1$,

$$E\left[\left\|s_t\right\|^2\right] = \frac{1}{n^2} \sum_{i=1}^{n} E\left[\left\|g_i(x_i^t, \xi_i^t) - \nabla f_i(x_i^t)\right\|^2\right] + \frac{1}{n^2} \sum_{i \neq j} E\left[\left\langle g_i(x_i^t, \xi_i^t) - \nabla f_i(x_i^t), g_j(x_j^t, \xi_j^t) - \nabla f_j(x_j^t)\right\rangle\right]$$

$$\overset{(i)}{=} \frac{1}{n^2} \sum_{i=1}^{n} E\left[\left\|g_i(x_i^t, \xi_i^t) - \nabla f_i(x_i^t)\right\|^2\right]$$

$$\leq \frac{\nu^2}{n},$$

(24)

We note that $(i)$ in (24) uses that whenever $i \neq j$,

$$E\left[\left\langle g_i(x_i^t, \xi_i^t) - \nabla f_i(x_i^t), g_j(x_j^t, \xi_j^t) - \nabla f_j(x_j^t)\right\rangle | \mathcal{F}_i\right]$$

$$\overset{(ii)}{=} E\left[\left\langle E\left[g_i(x_i^t, \xi_i^t) | \mathcal{F}_i\right] - \nabla f_i(x_i^t), g_j(x_j^t, \xi_j^t) - \nabla f_j(x_j^t)\right\rangle | \mathcal{F}_i\right]$$

$$\overset{(iii)}{=} E\left[\left\langle E\left[g_i(x_i^t, \xi_i^t) | \mathcal{F}_i\right] - \nabla f_i(x_i^t), g_j(x_j^t, \xi_j^t) - \nabla f_j(x_j^t)\right\rangle | \mathcal{F}_i\right]$$

$$= 0,$$

(25)

where $(ii)$ is due to the tower property of the conditional expectation and $(iii)$ uses that $\xi_i^t$ is independent of $\{\xi_j^t, \mathcal{F}_j\}$ and $x_i^t$ is $\mathcal{F}_t$-measurable. Towards the third term (23), we define for the ease of exposition, $\forall t \geq 1$,

$$\hat{\nabla}_t := \nabla f_i(x_i^t) - \nabla f_i(x_i^t_{t-1})$$
and recall that $E \left[ g_i(x_i^t, \xi_i^t) - g_i(x_{i-1}^t, \xi_i^t)]_F \right] = \hat{\nabla}^i_t$. Observe that $\forall t \geq 1$,

$$
\begin{align*}
E \left[ \|z_t\|^2 | F_t \right] &= E \left[ \left\| \frac{1}{n} \sum_{i=1}^n \left( g_i(x_i^t, \xi_i^t) - g_i(x_{i-1}^t, \xi_i^t) - \hat{\nabla}^i_t \right) \right\|_2^2 | F_t \right] \\
&= \frac{1}{n^2} \sum_{i=1}^n E \left[ \left\| g_i(x_i^t, \xi_i^t) - g_i(x_{i-1}^t, \xi_i^t) - \hat{\nabla}^i_t \right\|_2^2 | F_t \right] \\
&+ \frac{1}{n^2} \sum_{i \neq j}^n E \left[ \left( g_i(x_i^t, \xi_i^t) - g_i(x_{i-1}^t, \xi_i^t) - \hat{\nabla}^i_t, g_j(x_i^t, \xi_i^t) - g_j(x_{i-1}^t, \xi_i^t) - \hat{\nabla}^j_t \right) | F_t \right] \\
&\leq (i) \frac{1}{n^2} \sum_{i=1}^n E \left[ \left\| g_i(x_i^t, \xi_i^t) - g_i(x_{i-1}^t, \xi_i^t) - \hat{\nabla}^i_t \right\|_2^2 | F_t \right], \\
&\leq (ii) \frac{1}{n^2} \sum_{i=1}^n E \left[ \left\| g_i(x_i^t, \xi_i^t) - g_i(x_{i-1}^t, \xi_i^t) \right\|^2 | F_t \right],
\end{align*}
$$

(26)

where $(i)$ follows from a similar line of arguments as (25) and $(ii)$ uses the conditional variance decomposition, i.e., for any random vector $a \in \mathbb{R}^p$ consisted of square-integrable random variables,

$$
E \left[ \left\| a - E[a | F_t] \right\|^2 | F_t \right] = E \left[ \left\| a \right\|^2 | F_t \right] - \left\| E[a | F_t] \right\|^2.
$$

(27)

To proceed from (26), we take its expectation and observe that $\forall t \geq 1$,

$$
\begin{align*}
E \left[ \|z_t\|^2 \right] &\leq \frac{1}{n^2} \sum_{i=1}^n E \left[ \left\| g_i(x_i^t, \xi_i^t) - g_i(x_{i-1}^t, \xi_i^t) \right\|^2 \right] \\
&\leq (i) \frac{L^2}{n^2} \sum_{i=1}^n E \left[ \left\| x_i^t - x_{i-1}^t \right\|^2 \right] \\
&= \frac{L^2}{n^2} \sum_{i=1}^n E \left[ \left\| x_i^t - x_{i-1}^t \right\|^2 \right] \\
&= \frac{L^2}{n^2} \sum_{i=1}^n E \left[ \left\| x_i^t - Jx_i^t + Jx_i^t - Jx_{i-1} + Jx_{i-1} - x_{i-1} \right\|^2 \right] \\
&\leq \frac{3L^2}{n^2} \sum_{i=1}^n E \left[ \left\| x_i^t - Jx_i^t \right\|^2 + n \left\| x_i^t - x_{i-1} \right\|^2 + \left\| x_{i-1} - Jx_{i-1} \right\|^2 \right] \\
&= (ii) \frac{3L^2 \alpha^2}{n} \sum_{i=1}^n E \left[ \left\| v_{i-1} \right\|^2 \right] + \frac{3L^2}{n^2} \left( E \left[ \left\| x_i^t - Jx_i^t \right\|^2 + \left\| x_{i-1} - Jx_{i-1} \right\|^2 \right] \right),
\end{align*}
$$

(28)

where $(i)$ uses the mean-squared smoothness of each $g_i$ and $(ii)$ uses the update of $x_i$ in (5). The proof follows by taking the expectation (23) and then using (24) and (28) in the resulting inequality.

**B.2. Proof of Eq. (8)**

We recall from (19) the following relationship: $\forall t \geq 1$,

$$
\begin{align*}
v_i^t - \nabla f_i(x_i^t) = \beta \left( g_i(x_i^t, \xi_i^t) - \nabla f_i(x_i^t) \right) + (1 - \beta) \left( g_i(x_i^t, \xi_i^t) - g_i(x_{i-1}^t, \xi_i^t) + \nabla f_i(x_{i-1}^t) - \nabla f_i(x_i^t) \right) \\
+ (1 - \beta) \left( v_{i-1}^t - \nabla f_i(x_{i-1}^t) \right).
\end{align*}
$$

(29)
We recall that the conditional expectation of the first and the second term in (29) with respect to $\mathcal{F}_t$ is zero and that the third term in (29) is $\mathcal{F}_t$-measurable. Following a similar procedure in the proof of (23), we have: $\forall t \geq 1$,

\[
\mathbb{E} \left[ \| v^i_t - \nabla f_i(x^i_t) \|^2 | \mathcal{F}_t \right] \leq (1 - \beta)^2 \| v^i_{t-1} - \nabla f_i(x^i_{t-1}) \|^2 + 2\beta^2 \mathbb{E} \left[ \| g_i(x^i_t, \xi^i_t) - \nabla f_i(x^i_t) \|^2 | \mathcal{F}_t \right] + 2(1 - \beta)^2 \mathbb{E} \left[ \| g_i(x^i_t, \xi^i_t) - g_i(x^i_{t-1}, \xi^i_{t-1}) - (\nabla f_i(x^i_t) - \nabla f_i(x^i_{t-1})) \|^2 | \mathcal{F}_t \right] \]

\[
\leq (1 - \beta)^2 \| v^i_{t-1} - \nabla f_i(x^i_{t-1}) \|^2 + 2\beta^2 \mathbb{E} \left[ \| g_i(x^i_t, \xi^i_t) - \nabla f_i(x^i_t) \|^2 | \mathcal{F}_t \right] + 2(1 - \beta)^2 \mathbb{E} \left[ \| g_i(x^i_t, \xi^i_t) - g_i(x^i_{t-1}, \xi^i_{t-1}) \|^2 | \mathcal{F}_t \right] \]

\[
(30)
\]

where $(i)$ uses the conditional variance decomposition (27). We then take the expectation of (30) with the help of the mean-squared smoothness and the bounded variance of each $g_i$ to proceed: $\forall t \geq 1$,

\[
\mathbb{E} \left[ \| v^i_t - \nabla f_i(x^i_t) \|^2 \right] \leq (1 - \beta)^2 \mathbb{E} \left[ \| v^i_{t-1} - \nabla f_i(x^i_{t-1}) \|^2 \right] + 2\beta^2 \mathbb{E} \left[ \| x^i_t - x^i_{t-1} \|^2 \right] + 2(1 - \beta)^2 \mathbb{E} \left[ \| \nabla x^i_t - \nabla x^i_{t-1} \|^2 \right] + 6(1 - \beta)^2 \mathbb{E} \left[ \| x^i_t - x^i_{t-1} \|^2 + \| x^i_t - x^i_{t-1} \|^2 \right] + 6(1 - \beta)^2 \mathbb{E} \left[ \| \nabla x^i_{t-1} \|^2 \right] + 6(1 - \beta)^2 \mathbb{E} \left[ \| \nabla x^i_{t-1} \|^2 \right],
\]

\[
(31)
\]

where the last line uses the $x^i_t$-update in (5). Summing up (31) over $i$ from 1 to $n$ completes the proof.

C. Proof of Lemma 5

C.1. Proof of Lemma 5(a)

Recall the initialization of GT-HSGD that $v_{-1} = 0_{np}$, $y_0 = 0_{np}$, and $v^0_0 = \frac{1}{b_0} \sum_{r=1}^{b_0} g_i(x^i_0, \xi^i_{0,r})$. Using the gradient tracking update (24a) at iteration $t = 0$, we have:

\[
\mathbb{E} \left[ \| y_1 - Jy_1 \|^2 \right] = \mathbb{E} \left[ \| W (y_0 + v_0 - v_{-1}) - J W (y_0 + v_0 - v_{-1}) \|^2 \right]
\]

\[
\overset{(i)}{=} \mathbb{E} \left[ \| (W - J) v_0 \|^2 \right] \]

\[
\overset{(ii)}{=} \lambda^2 \mathbb{E} \left[ \| v_0 - \nabla f(x_0) + \nabla f(x_0) \|^2 \right] \]

\[
= \lambda^2 \sum_{i=1}^{n} \mathbb{E} \left[ \| v^i_0 - \nabla f_i(x^i_0) \|^2 \right] + \lambda^2 \| \nabla f(x_0) \|^2 \]

\[
\overset{(iii)}{=} \lambda^2 \sum_{i=1}^{n} \mathbb{E} \left[ \left( \frac{1}{b_0} \sum_{r=1}^{b_0} (g_i(x^i_0, \xi^i_{0,r}) - \nabla f_i(x^i_0)) \right) \right] + \lambda^2 \| \nabla f(x_0) \|^2 \]

\[
\overset{(iv)}{=} \lambda^2 \sum_{i=1}^{n} \sum_{r=1}^{b_0} \mathbb{E} \left[ \| g_i(x^i_0, \xi^i_{0,r}) - \nabla f_i(x^i_0) \|^2 \right] + \lambda^2 \| \nabla f(x_0) \|^2, \quad (32)
\]

where $(i)$ uses $JW = J$ and the initial condition of $v_{-1}$ and $y_0$, $(ii)$ uses $\| W - J \| = \lambda$, $(iii)$ is due to the initialization of $v^i_0$, and $(iv)$ follows from the fact that $\{ \xi^i_{0,1}, \xi^i_{0,2}, \cdots, \xi^i_{0,b_0} \}, \forall i \in V$, is an independent family of random vectors, by a similar line of arguments in (24) and (25). The proof then follows by using the bounded variance of each $g_i$ in (32).
C.2. Proof of Lemma 5(b)

Following the gradient tracking update (4a), we have: \( \forall t \geq 1, \)
\[
\| y_{t+1} - Jy_{t+1} \|^2 = \| W (y_t + v_t - v_{t-1}) - JW (y_t + v_t - v_{t-1}) \|^2 \\
= \| W y_t - Jy_t + (W - J) (v_t - v_{t-1}) \|^2 \\
\leq \lambda^2 \| y_t - Jy_t \|^2 + 2 \langle W y_t - Jy_t, (W - J) (v_t - v_{t-1}) \rangle + \lambda^2 \| v_t - v_{t-1} \|^2 , \tag{33}
\]
where (i) uses \( JW = J \) and (ii) is due to \( \| W - J \| = \lambda \). In the following, we bound \( A_t \) and the last term in (33) respectively. We recall the update of each local stochastic gradient estimator \( v_i^t \) in (2): \( \forall t \geq 1, \)
\[
v_i^t = g_i(x_i^t, \xi_i^t) + (1 - \beta)v_i^{t-1} - (1 - \beta)g_i(x_{i-1}^t, \xi_i^t).
\]
We observe that \( \forall t \geq 1 \) and \( \forall i \in V, \)
\[
v_i^t - v_i^{t-1} = g_i(x_i^t, \xi_i^t) - \beta v_i^{t-1} - (1 - \beta)g_i(x_{i-1}^t, \xi_i^t) \\
= g_i(x_i^t, \xi_i^t) - g_i(x_{i-1}^t, \xi_i^t) - \beta v_i^{t-1} + \beta g_i(x_{i-1}^t, \xi_i^t) \\
= g_i(x_i^t, \xi_i^t) - g_i(x_{i-1}^t, \xi_i^t) - \beta (v_i^{t-1} - \nabla f_i(x_i^{t-1})) + \beta (g_i(x_{i-1}^t, \xi_i^t) - \nabla f_i(x_{i-1}^t)). \tag{34}
\]
Moreover, we observe from (34) that \( \forall t \geq 1, \)
\[
E [ v_t - v_{t-1} | F_t ] = \nabla f(x_t) - \nabla f(x_{t-1}) - \beta (v_{t-1} - \nabla f(x_{t-1})). \tag{35}
\]
Towards \( A_t \), we have: \( \forall t \geq 1, \)
\[
E [ A_t | F_t ] \overset{(i)}{=} 2 \langle W y_t - Jy_t, (W - J) E [ v_t - v_{t-1} | F_t ] \rangle \\
\overset{(ii)}{=} 2 \langle W y_t - Jy_t, (W - J) \left( \nabla f(x_t) - \nabla f(x_{t-1}) - \beta (v_{t-1} - \nabla f(x_{t-1})) \right) \rangle \\
\overset{(iii)}{\leq} 2 \lambda \| y_t - Jy_t \| \cdot \lambda \left\| \nabla f(x_t) - \nabla f(x_{t-1}) - \beta (v_{t-1} - \nabla f(x_{t-1})) \right\| \\
\overset{(iv)}{\leq} \frac{1 - \lambda^2}{2} \| y_t - Jy_t \|^2 + \frac{2 \lambda^2}{1 - \lambda^2} \left\| \nabla f(x_t) - \nabla f(x_{t-1}) - \beta (v_{t-1} - \nabla f(x_{t-1})) \right\|^2 , \\
\overset{(v)}{\leq} \frac{1 - \lambda^2}{2} \| y_t - Jy_t \|^2 + \frac{4 \lambda^2 L^2}{1 - \lambda^2} \| x_t - x_{t-1} \|^2 + \frac{4 \lambda^2 \beta^2}{1 - \lambda^2} \| v_{t-1} - \nabla f(x_{t-1}) \|^2 , \tag{36}
\]
where (i) is due to the \( F_t \)-measurability of \( y_t \), (ii) uses (35), (iii) is due to the Cauchy-Schwarz inequality and \( \| W - J \| = \lambda \), (iv) uses the elementary inequality that \( 2ab \leq \frac{\eta a^2 + b^2}{\eta} \), with \( \eta = \frac{1 - \lambda^2}{2 \lambda^2} \) for any \( a, b \in \mathbb{R} \), and (v) holds since each \( f_i \) is \( L \)-smooth. Next, towards the last term in (33), we take the expectation of (34) to obtain: \( \forall t \geq 1 \) and \( \forall i \in V, \)
\[
E \left[ \| v_i^t - v_i^{t-1} \|^2 \right] \leq 3E \left[ \| g_i(x_i^t, \xi_i^t) - g_i(x_{i-1}^t, \xi_i^t) \|^2 \right] + 3\beta^2 E \left[ \| v_i^{t-1} - \nabla f_i(x_{i-1}^t) \|^2 \right] \\
+ 3\beta^2 E \left[ \| g_i(x_{i-1}^t, \xi_i^t) - \nabla f_i(x_{i-1}^t) \|^2 \right] \\
\leq 3L^2 E \left[ \| x_i^t - x_{i-1}^t \|^2 \right] + 3\beta^2 E \left[ \| v_i^{t-1} - \nabla f_i(x_{i-1}^t) \|^2 \right] + 3\beta^2 \nu_i^2 , \tag{37}
\]
where (37) is due to the mean-squared smoothness and the bounded variance of each \( g_i \). Summing up (37) over \( i \) from 1 to \( n \) gives an upper bound on the last term in (33): \( \forall t \geq 1, \)
\[
\lambda^2 E \left[ \| v_t - v_{t-1} \|^2 \right] \leq 3\lambda^2 L^2 E \left[ \| x_t - x_{t-1} \|^2 \right] + 3\lambda^2 \beta^2 E \left[ \| v_{t-1} - \nabla f(x_{t-1}) \|^2 \right] + 3\lambda^2 n \beta^2 \nu^2 . \tag{38}
\]
We now use (36) and (38) in (33) to obtain: \( \forall t \geq 1, \)
\[
E \left[ \|y_{t+1} - Jy_{t+1}\|^2 \right] \leq \frac{1 + \lambda^2}{2} E \left[ \|y_t - Jy_t\|^2 \right] + \frac{7\lambda^2 L^2}{1 - \lambda^2} E \left[ \|x_t - x_{t-1}\|^2 \right]
+ \frac{7\lambda^2 \beta^2}{1 - \lambda^2} E \left[ \|v_{t-1} - \nabla f(x_{t-1})\|^2 \right] + 3\lambda^2 n\beta^2 \gamma^2.
\] (39)

Towards the second term in (39), we use (10) to obtain: \( \forall t \geq 1, \)
\[
\|x_t - x_{t-1}\|^2 = \|x_t - Jx_t + Jx_t - Jx_{t-1} + Jx_{t-1} - x_{t-1}\|^2 \leq 3 \|x_t - Jx_t\|^2 + 3n\alpha^2 \|v_{t-1}\|^2 + 3 \|x_{t-1} - Jx_{t-1}\|^2,
\] (40)

where (i) uses the \( x_t \)-update in (5). Finally, we use (40) in (39) to obtain: \( \forall t \geq 1, \)
\[
E \left[ \|y_{t+1} - Jy_{t+1}\|^2 \right] \leq \left( \frac{1 + \lambda^2}{2} + \frac{42\lambda^4 L^2 \alpha^2}{1 - \lambda^2} \right) E \left[ \|y_t - Jy_t\|^2 \right] + \frac{21\lambda^2 nL^2 \alpha^2}{1 - \lambda^2} E \left[ \|v_{t-1}\|^2 \right]
+ \frac{63\lambda^2 L^2}{1 - \lambda^2} E \left[ \|x_{t-1} - Jx_{t-1}\|^2 \right] + \frac{7\lambda^2 \beta^2}{1 - \lambda^2} E \left[ \|v_{t-1} - \nabla f(x_{t-1})\|^2 \right] + 3\lambda^2 n\beta^2 \gamma^2.
\]

The proof is completed by the fact that \( \frac{1 + \lambda^2}{2} + \frac{42\lambda^4 L^2 \alpha^2}{1 - \lambda^2} \leq \frac{3 + \lambda^2}{4} \) if \( 0 < \alpha \leq \frac{1 - \lambda^2}{2\sqrt{42\lambda^4 L}} \).

### D. Proof of Lemma 6

#### D.1. Proof of Eq. (11)

We recursively apply the inequality on \( V_t \) from \( t = 0 \) to \( t = T \) to obtain: \( \forall t \geq 1, \)
\[
V_t \leq qV_{t-1} + qR_{t-1} + Q_t + C
\leq q^2V_{t-2} + (q^2R_{t-2} + qR_{t-1}) + (qQ_{t-1} + Q_t) + (qC + C)
\]
\[\vdots\]
\[
\leq q^tV_0 + \sum_{i=0}^{t-1} q^{t-i}R_i + \sum_{i=1}^{t} q^{i-1}Q_i + C \sum_{i=0}^{t-1} q^i.
\] (41)

Summing up (41) over \( t \) from \( t = 1 \) to \( T \) gives: \( \forall T \geq 1, \)
\[
\sum_{t=0}^{T} V_t \leq V_0 \sum_{t=0}^{T} q^t + \sum_{t=1}^{T} q^{t-1}R_t + \sum_{t=1}^{T} \sum_{i=1}^{t} q^{i-1}Q_t + C \sum_{i=0}^{T-1} \sum_{t=1}^{i} q^i
\leq V_0 \sum_{t=0}^{\infty} q^t + \sum_{t=1}^{T-1} \left( \sum_{i=0}^{\infty} q^i \right) R_t + \sum_{t=1}^{T} \left( \sum_{i=0}^{\infty} q^i \right) Q_t + C \sum_{t=1}^{T} \sum_{i=0}^{\infty} q^i,
\]
and the proof follows by \( \sum_{i=0}^{\infty} q^i = (1 - q)^{-1} \).

#### D.2. Proof of Eq. (12)

We recursively apply the inequality on \( V_t \) from \( t = 1 \) to \( t = T \) to obtain: \( \forall t \geq 1, \)
\[
V_{t+1} \leq qV_t + R_{t-1} + C
\leq q^2V_{t-1} + (q^2R_{t-2} + qR_{t-1}) + (qC + C)
\]
\[\vdots\]
\[
\leq q^tV_1 + \sum_{i=0}^{t-1} q^{t-i}R_i + C \sum_{i=0}^{t-1} q^i.
\] (42)
We sum up (42) over $t$ from 1 to $T - 1$ to obtain: $\forall T \geq 2$,

\[
\sum_{t=0}^{T-1} V_{t+1} \leq V_1 \sum_{t=0}^{T-1} q^t + \sum_{t=1}^{T-1} \sum_{i=0}^{t-1} q^{i+1} R_t + C \sum_{i=0}^{T-1} \sum_{t=1}^{t-i} q^i \\
\leq V_1 \sum_{t=0}^{T-1} q^t + \sum_{t=0}^{T-2} \left( \sum_{i=0}^{\infty} q^i \right) R_t + C \sum_{i=0}^{T-1} \sum_{t=1}^{t-i} q^i,
\]

and the proof follows by $\sum_{i=0}^{\infty} q^i = (1 - q)^{-1}$.

E. Proof of Lemma 7

E.1. Proof of Eq. (13)

We first observe that $\frac{1}{1-(1-\beta)^2} = \frac{1}{\beta}$ for $\beta \in (0, 1)$. Applying (11) to (7) gives: $\forall T \geq 1$,

\[
\sum_{t=0}^{T} \mathbb{E} \left[ \left\| \nabla_t - \nabla T(x_t) \right\|^2 \right] \\
\leq \frac{\mathbb{E} \left[ \left\| \nabla_0 - \nabla T(x_0) \right\|^2 \right]}{\beta} + 6L^2 \alpha^2 \sum_{t=0}^{T-1} \mathbb{E} \left[ \left\| \nabla_t \right\|^2 \right] + \frac{6L^2}{n^2 \beta} \sum_{t=0}^{T-1} \mathbb{E} \left[ \left\| x_{t+1} - Jx_{t+1} \right\|^2 + \left\| x_t - Jx_t \right\|^2 \right] + \frac{2\beta \sigma^2 T}{n},
\]

\[
\leq \frac{\mathbb{E} \left[ \left\| \nabla_0 - \nabla T(x_0) \right\|^2 \right]}{\beta} + \frac{12L^2}{n^2 \beta} \sum_{t=0}^{T-1} \mathbb{E} \left[ \left\| x_t - Jx_t \right\|^2 \right] + \frac{2\beta \sigma^2 T}{n}. \quad (43)
\]

Towards the first term in (43), we observe that

\[
\mathbb{E} \left[ \left\| \nabla_0 - \nabla T(x_0) \right\|^2 \right] = \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^{\infty} \frac{1}{b_0} \sum_{r=1}^{b_0} \left( \mathbf{g}_i(x_0^i, \xi_{0,r}) - \nabla f_i(x_0^i) \right) \right\|^2 \right] \\
\overset{(i)}{=} \frac{1}{n^2 b_0^2} \sum_{i=1}^{\infty} \sum_{r=1}^{b_0} \mathbb{E} \left[ \left\| \mathbf{g}_i(x_0^i, \xi_{0,r}) - \nabla f_i(x_0^i) \right\|^2 \right] \leq \frac{\sigma^2}{nb_0}, \quad (44)
\]

where (i) follows from a similar line of arguments in (25). Then (13) follows from using (44) in (43).

E.2. Proof of Eq. (14)

We apply (11) to (8) to obtain: $\forall T \geq 1$,

\[
\sum_{t=0}^{T} \mathbb{E} \left[ \left\| \nabla_t - \nabla T(x_t) \right\|^2 \right] \\
\leq \frac{\mathbb{E} \left[ \left\| \nabla_0 - \nabla T(x_0) \right\|^2 \right]}{\beta} + \frac{6nL^2 \alpha^2}{\beta} \sum_{t=0}^{T-1} \mathbb{E} \left[ \left\| \nabla_t \right\|^2 \right] + \frac{6L^2}{n^2 \beta} \sum_{t=0}^{T-1} \mathbb{E} \left[ \left\| x_{t+1} - Jx_{t+1} \right\|^2 + \left\| x_t - Jx_t \right\|^2 \right] + 2n\beta \sigma^2 T \\
\leq \frac{\mathbb{E} \left[ \left\| \nabla_0 - \nabla T(x_0) \right\|^2 \right]}{\beta} + \frac{12L^2}{n^2 \beta} \sum_{t=0}^{T-1} \mathbb{E} \left[ \left\| x_t - Jx_t \right\|^2 \right] + 2n\beta \sigma^2 T. \quad (45)
\]

In (45), we observe that

\[
\mathbb{E} \left[ \left\| \nabla_0 - \nabla T(x_0) \right\|^2 \right] = \sum_{i=1}^{\infty} \mathbb{E} \left[ \left\| \frac{1}{b_0} \sum_{r=1}^{b_0} \left( \mathbf{g}_i(x_0^i, \xi_{0,r}) - \nabla f_i(x_0^i) \right) \right\|^2 \right] \\
\overset{(i)}{=} \frac{1}{b_0^2} \sum_{i=1}^{\infty} \sum_{r=1}^{b_0} \mathbb{E} \left[ \left\| \mathbf{g}_i(x_0^i, \xi_{0,r}) - \nabla f_i(x_0^i) \right\|^2 \right] \leq \frac{n \sigma^2}{b_0}, \quad (46)
\]

where (i) follows from a similar line of arguments in (25). Then (14) follows from using (46) in (45).
F. Proof of Lemma 8

We recall that \( \|x_i - Jx_i\| = 0 \), since it is assumed without generality that \( x_i^0 = x_j^0 \) for any \( i, j \in \mathcal{V} \). Applying (11) to (9) yields: \( \forall T \geq 1, \)

\[
\sum_{t=0}^{T} \|x_t - Jx_t\|^2 \leq \frac{4\lambda^2n^2}{(1 - \lambda^2)^2} \sum_{t=1}^{T} \|y_t - Jy_t\|^2. \tag{47}
\]

To further bound \( \sum_{t=1}^{T} \|y_t - Jy_t\|^2 \), we apply (12) in Lemma 5(b) to obtain: if \( 0 < \alpha \leq \frac{1 - \lambda^2}{2\sqrt{\lambda^2L}} \), then \( \forall T \geq 2, \)

\[
\sum_{t=1}^{T} \mathbb{E} \left[ \|y_t - Jy_t\|^2 \right] \leq \frac{4\mathbb{E} \left[ \|y_1 - Jy_1\|^2 \right]}{1 - \lambda^2} + \frac{84\lambda^2nL^2\alpha^2}{(1 - \lambda^2)^2} \sum_{t=0}^{T-2} \sum_{t=0}^{T-2} \mathbb{E} \left[ \|\nu_t\|^2 \right] + \frac{252\lambda^2L^2\alpha^2}{(1 - \lambda^2)^2} \sum_{t=0}^{T-2} \mathbb{E} \left[ \|x_t - Jx_t\|^2 \right] + \frac{28\lambda^2\beta^2}{(1 - \lambda^2)^2} \sum_{t=0}^{T-2} \mathbb{E} \left[ \|v_t - \nabla f(x_t)\|^2 \right] + \frac{12\lambda^2\beta^2\nu^2T}{1 - \lambda^2} \leq \frac{84\lambda^2nL^2\alpha^2}{(1 - \lambda^2)^2} \sum_{t=0}^{T-2} \sum_{t=0}^{T-2} \mathbb{E} \left[ \|\nu_t\|^2 \right] + \frac{252\lambda^2L^2\alpha^2}{(1 - \lambda^2)^2} \sum_{t=0}^{T-2} \mathbb{E} \left[ \|x_t - Jx_t\|^2 \right] + \frac{28\lambda^2\beta^2}{(1 - \lambda^2)^2} \sum_{t=0}^{T-2} \mathbb{E} \left[ \|v_t - \nabla f(x_t)\|^2 \right] + \frac{12\lambda^2\beta^2\nu^2T}{1 - \lambda^2} + \frac{4\lambda^2\|\nabla f(x_0)\|^2}{1 - \lambda^2} + \frac{4\lambda^2\nu^2}{(1 - \lambda^2)b_0}, \tag{48}
\]

where the last inequality is due to Lemma 5(a). To proceed, we use (14), an upper bound on \( \sum_t \mathbb{E} \left[ \|v_t - \nabla f(x_t)\|^2 \right] \), in (48) to obtain: if \( 0 < \alpha \leq \frac{1 - \lambda^2}{2\sqrt{\lambda^2L}} \) and \( \beta \in (0, 1) \), then \( \forall T \geq 2, \)

\[
\sum_{t=1}^{T} \mathbb{E} \left[ \|y_t - Jy_t\|^2 \right] \leq \frac{252\lambda^2nL^2\alpha^2}{(1 - \lambda^2)^2} \sum_{t=0}^{T-2} \sum_{t=0}^{T-2} \mathbb{E} \left[ \|\nu_t\|^2 \right] + \frac{588\lambda^2L^2\alpha^2}{(1 - \lambda^2)^2} \sum_{t=0}^{T-1} \mathbb{E} \left[ \|x_t - Jx_t\|^2 \right] + \frac{28\lambda^2\beta^2}{(1 - \lambda^2)^2} \sum_{t=0}^{T-2} \mathbb{E} \left[ \|v_t - \nabla f(x_t)\|^2 \right] + \frac{56\lambda^2\beta^2\nu^2T}{1 - \lambda^2} + \frac{12\lambda^2\beta^2\nu^2T}{1 - \lambda^2} + \frac{4\lambda^2\|\nabla f(x_0)\|^2}{1 - \lambda^2} + \frac{4\lambda^2\nu^2}{(1 - \lambda^2)b_0} = \frac{252\lambda^2nL^2\alpha^2}{(1 - \lambda^2)^2} \sum_{t=0}^{T-2} \sum_{t=0}^{T-2} \mathbb{E} \left[ \|\nu_t\|^2 \right] + \frac{588\lambda^2L^2\alpha^2}{(1 - \lambda^2)^2} \sum_{t=0}^{T-1} \mathbb{E} \left[ \|x_t - Jx_t\|^2 \right] + \frac{28\lambda^2\beta^2}{(1 - \lambda^2)^2} \sum_{t=0}^{T-2} \mathbb{E} \left[ \|v_t - \nabla f(x_t)\|^2 \right] + \frac{56\lambda^2\beta^2\nu^2T}{1 - \lambda^2} + \frac{12\lambda^2\beta^2\nu^2T}{1 - \lambda^2} + \frac{4\lambda^2\|\nabla f(x_0)\|^2}{1 - \lambda^2} + \frac{4\lambda^2\nu^2}{(1 - \lambda^2)b_0} + \left( \frac{7\beta}{1 - \lambda^2} + 1 \right) \frac{4\lambda^2\nu^2}{(1 - \lambda^2)b_0} + \left( \frac{14\beta}{1 - \lambda^2} + 3 \right) \frac{4\lambda^2\beta^2\nu^2T}{1 - \lambda^2} + \frac{4\lambda^2\|\nabla f(x_0)\|^2}{1 - \lambda^2} \tag{49}
\]

Finally, we use (49) in (47) to obtain: \( \forall T \geq 2, \)

\[
\sum_{t=0}^{T} \mathbb{E} \left[ \|x_t - Jx_t\|^2 \right] \leq \frac{1008\lambda^4nL^2\alpha^4}{(1 - \lambda^2)^4} \sum_{t=0}^{T-2} \sum_{t=0}^{T-2} \mathbb{E} \left[ \|\nu_t\|^2 \right] + \frac{2352\lambda^4L^2\beta^2T}{(1 - \lambda^2)^2} \sum_{t=0}^{T-1} \mathbb{E} \left[ \|x_t - Jx_t\|^2 \right] + \left( \frac{7\beta}{1 - \lambda^2} + 1 \right) \frac{16\lambda^4\beta^2\nu^2T}{(1 - \lambda^2)^3} + \left( \frac{14\beta}{1 - \lambda^2} + 3 \right) \frac{16\lambda^4\beta^2\nu^2T}{1 - \lambda^2} + \frac{16\lambda^4\|\nabla f(x_0)\|^2}{1 - \lambda^2}\alpha^2 + \frac{16\lambda^4\nu^2}{(1 - \lambda^2)^3}b_0, \tag{50}
\]

which may be written equivalently as

\[
\left( \frac{1 - 2352\lambda^4L^2\beta^2T}{1 - \lambda^2} \right) \sum_{t=0}^{T} \mathbb{E} \left[ \|x_t - Jx_t\|^2 \right] \leq \frac{1008\lambda^4nL^2\alpha^4}{(1 - \lambda^2)^4} \sum_{t=0}^{T-2} \sum_{t=0}^{T-2} \mathbb{E} \left[ \|\nu_t\|^2 \right] + \left( \frac{7\beta}{1 - \lambda^2} + 1 \right) \frac{16\lambda^4\beta^2\nu^2T}{(1 - \lambda^2)^3} + \left( \frac{14\beta}{1 - \lambda^2} + 3 \right) \frac{16\lambda^4\beta^2\nu^2T}{1 - \lambda^2} + \frac{16\lambda^4\|\nabla f(x_0)\|^2}{1 - \lambda^2}\alpha^2 + \frac{16\lambda^4\nu^2}{(1 - \lambda^2)^3}b_0. \tag{50}
\]

We observe in (50) that \( \frac{2352\lambda^4L^2\beta^2T}{(1 - \lambda^2)^2} \leq \frac{1}{2} \) if \( 0 < \alpha \leq \frac{(1-\lambda^2)^2}{2\lambda^2L} \), and the proof follows.
G. Proof of Theorem 1

For the ease of presentation, we denote $\Delta_0 := F(x_0) - F^*$ in the following. We apply (13) to Lemma 2 to obtain: if $0 < \alpha \leq \frac{1}{2\rho_T}$, then $\forall T \geq 1,$

\[
\frac{T}{n} \sum_{t=0}^{T} \mathbb{E} \left[ \|\nabla F(x_t)\|^2 \right] \leq \frac{2\Delta_0}{\alpha} - \frac{1}{2} \sum_{t=0}^{T} \mathbb{E} \left[ \|\nabla x_t\|^2 \right] + \frac{2L^2}{n} \frac{\sum_{t=0}^{T} \mathbb{E} \left[ \|x_t - Jx_t\|^2 \right]}{n} + \frac{2\rho_T}{\beta b_0 n} + \frac{12L^2 \alpha^2}{n^2 \beta} \sum_{t=0}^{T} \mathbb{E} \left[ \|\nabla x_t\|^2 \right] + \frac{4\rho_T^2 T}{n} \leq \frac{2\Delta_0}{\alpha} - \frac{1}{4} \sum_{t=0}^{T} \mathbb{E} \left[ \|\nabla x_t\|^2 \right] + \frac{2L^2}{n} \left( 1 + \frac{12}{n \beta} \right) \sum_{t=0}^{T} \mathbb{E} \left[ \|x_t - Jx_t\|^2 \right] + \frac{2\rho_T^2}{\beta b_0 n} + \frac{4\rho_T^2 T}{n}.
\]

(51)

Therefore, if $0 < \alpha < \frac{1}{4\sqrt{3}L}$ and $\frac{48L^2 \alpha^2}{n} \leq \beta < 1$, i.e., $\frac{1}{4} - \frac{12L^2 \alpha^2}{n^2 \beta} \geq 0$, we may drop the last term in (51) to obtain: $\forall T \geq 1,$

\[
\frac{T}{n} \sum_{t=0}^{T} \mathbb{E} \left[ \|\nabla F(x_t)\|^2 \right] \leq \frac{2\Delta_0}{\alpha} - \frac{1}{4} \sum_{t=0}^{T} \mathbb{E} \left[ \|\nabla x_t\|^2 \right] + \frac{2L^2}{n} \left( 1 + \frac{12}{n \beta} \right) \sum_{t=0}^{T} \mathbb{E} \left[ \|x_t - Jx_t\|^2 \right] + \frac{2\rho_T^2}{\beta b_0 n} + \frac{4\rho_T^2 T}{n}.
\]

(52)

Moreover, we observe: $\forall T \geq 1,$

\[
\frac{1}{n} \sum_{t=1}^{n} \sum_{t=0}^{T} \mathbb{E} \left[ \|\nabla F(x_t)\|^2 \right] \leq \frac{2\Delta_0}{\alpha} - \frac{1}{2} \sum_{t=0}^{T} \mathbb{E} \left[ \|\nabla x_t\|^2 \right] + \frac{6L^2}{n} \left( 1 + \frac{8}{n \beta} \right) \sum_{t=0}^{T} \mathbb{E} \left[ \|x_t - Jx_t\|^2 \right] + \frac{4\rho_T^2}{\beta b_0 n} + \frac{8\rho_T^2 T}{n}.
\]

(53)

where the last line uses the $L$-smoothness of $F$. Using (52) in (53) yields: if $0 < \alpha < \frac{1}{4\sqrt{3}L}$ and $48L^2 \alpha^2/n \leq \beta < 1$, then

\[
\frac{1}{n} \sum_{t=1}^{n} \sum_{t=0}^{T} \mathbb{E} \left[ \|\nabla F(x_t)\|^2 \right] \leq \frac{4\Delta_0}{\alpha} - \frac{1}{2} \sum_{t=0}^{T} \mathbb{E} \left[ \|\nabla x_t\|^2 \right] + \frac{6L^2}{n} \left( 1 + \frac{1}{6L^2 \alpha^2} \right) \sum_{t=0}^{T} \mathbb{E} \left[ \|x_t - Jx_t\|^2 \right] + \frac{4\rho_T^2}{\beta b_0 n} + \frac{8\rho_T^2 T}{n}.
\]

(54)

According to (54), if $0 < \alpha < \frac{1}{4\sqrt{3}L}$ and $\beta = 48L^2 \alpha^2/n$, we have: $\forall T \geq 1,$

\[
\frac{1}{n} \sum_{t=1}^{n} \sum_{t=0}^{T} \mathbb{E} \left[ \|\nabla F(x_t)\|^2 \right] \leq \frac{4\Delta_0}{\alpha} - \frac{1}{2} \sum_{t=0}^{T} \mathbb{E} \left[ \|\nabla x_t\|^2 \right] + \frac{6L^2}{n} \left( 1 + \frac{1}{6L^2 \alpha^2} \right) \sum_{t=0}^{T} \mathbb{E} \left[ \|x_t - Jx_t\|^2 \right] + \frac{4\rho_T^2}{\beta b_0 n} + \frac{8\rho_T^2 T}{n},
\]

(55)

where the last line is due to $6L^2 \alpha^2 < 1/8$. To simplify $\Phi_T$, we use Lemma 8 to obtain: if $0 < \alpha \leq \frac{(1-\lambda^2)}{70\lambda^2 \lambda^2}$, then $\forall T \geq 2$,

\[
\Phi_T \leq - \frac{1}{2} \left( 1 - \frac{8064 \lambda^2 \alpha^2}{(1 - \lambda^2)^4} \right) \frac{T}{n} \sum_{t=0}^{T} \mathbb{E} \left[ \|\nabla x_t\|^2 \right] + \frac{64 \lambda^4}{(1 - \lambda^2)^3} \frac{\|\nabla F(x_0)\|^2}{n} + \left( \frac{7\beta}{1 - \lambda^2} + 1 \right) \frac{64 \lambda^4 \rho_T^2}{(1 - \lambda^2)^3 b_0} + \left( \frac{14\beta}{1 - \lambda^2} + 3 \right) \frac{64 \lambda^4 \beta^2 \rho_T^2 T}{(1 - \lambda^2)^3}.
\]

(56)
In (56), we observe that if \( 0 < \alpha \leq \frac{(1-\lambda^2)^2}{90\lambda^2} \), then \( 1 - \frac{8064\lambda^4L^2\alpha^2}{(1-\lambda^2)^2} \geq 0 \) and thus the first term in (56) may be dropped; moreover, if \( 0 < \alpha \leq \frac{\sqrt{n(1-\lambda^2)}}{26\lambda L} \), then \( \beta = \frac{48L^2\alpha}{n} \leq \frac{1-\lambda^2}{14\lambda^2} \). Hence, if \( 0 < \alpha \leq \min \left\{ \frac{(1-\lambda^2)^2}{90\lambda^2}, \frac{\sqrt{n(1-\lambda^2)}}{26\lambda L} \right\} \frac{1}{L} \), then (56) reduces to: \( \forall T \geq 2 \),

\[
\Phi_T \leq \frac{64\lambda^4}{(1-\lambda^2)^3} \frac{\|\nabla f(x_0)\|^2}{n} + \frac{96\lambda^2\beta^2}{(1-\lambda^2)^3} b_0 + \frac{256\lambda^2\beta^2\nu^2 T}{(1-\lambda^2)^3}. \tag{57}
\]

Finally, we use (57) in (55) to obtain: if \( 0 < \alpha < \min \left\{ \frac{1}{4\sqrt{3}}, \frac{(1-\lambda^2)^2}{90\lambda^2}, \frac{\sqrt{n(1-\lambda^2)}}{26\lambda L} \right\} \frac{1}{L} \), we have: \( \forall T \geq 2 \),

\[
\frac{1}{n(T+1)} \sum_{t=0}^{T} \sum_{i=1}^{n} \mathbb{E} \left[ \|\nabla F(x_i^t)\|^2 \right] \leq \frac{4\Delta_0}{\alpha T} + \frac{4\nu^2}{\beta b_0 n T} + \frac{8\beta^2}{n} + \frac{64\lambda^4}{(1-\lambda^2)^3} \frac{\|\nabla f(x_0)\|^2}{n} + \frac{96\lambda^2\beta^2}{(1-\lambda^2)^3} b_0 T + \frac{256\lambda^2\beta^2\nu^2}{(1-\lambda^2)^3}. \tag{58}
\]

The proof follows by (58) and that \( \mathbb{E}[\|\nabla F(\bar{x}_T)\|^2] = \frac{1}{n(T+1)} \sum_{t=1}^{n} \sum_{i=1}^{T} \mathbb{E}[\|\nabla F(x_i^t)\|^2] \) since \( \bar{x}_T \) is chosen uniformly at random from \( \{x_i^t : \forall i \in \mathcal{V}, 0 \leq t \leq T\} \).