ON FORMAL LOCAL COHOMOLOGY AND CONNECTEDNESS

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ABSTRACT. Let \( a \) denote an ideal of a local ring \((R, \mathfrak{m})\). Let \( M \) be a finitely generated \( R \)-module. There is a systematic study of the formal cohomology modules \( \lim_{\leftarrow} H^i_m(M/a^nM), i \in \mathbb{Z} \). We analyze their \( R \)-module structure, the upper and lower vanishing and non-vanishing in terms of intrinsic data of \( M \), and its functorial behavior. These cohomology modules occur in relation to the formal completion of the punctured spectrum \( \text{Spec } R \setminus V(\mathfrak{m}) \).

As a new cohomological data there is a description on the formal grade \( \text{grade}(a, M) \) defined as the minimal non-vanishing of the formal cohomology modules. There are various exact sequences concerning the formal cohomology modules. Among them a Mayer-Vietoris sequence for two ideals. It applies to new connectedness results. There are also relations to local cohomological dimensions.

1. INTRODUCTION

Let \( a \) denote an ideal of a local ring \((R, \mathfrak{m})\). For a finitely generated \( R \)-module \( M \) let \( H^i_a(M), i \in \mathbb{N} \), denote the local cohomology module of \( M \) with respect to \( a \) (cf. [11] for the basic definitions). There are the following integers related to these local cohomology modules

\[
\text{grade}(a, M) = \inf \{ i \in \mathbb{Z} : H^i_a(M) \neq 0 \}, \quad \text{cd}(a, M) = \sup \{ i \in \mathbb{Z} : H^i_a(M) \neq 0 \},
\]

called the grade (resp. the cohomological dimension) of \( M \) with respect \( a \) (cf. Section 2.2). In general we have the bounds \( \text{height}_M a \leq \text{cd}(a, M) \leq \dim M \). In the case of \( \mathfrak{m} \) the maximal ideal it follows that \( \text{grade}(\mathfrak{m}, M) = \text{depth } M \) and \( \text{cd}(\mathfrak{m}, M) = \dim M \).

Here we consider the asymptotic behavior of the family of local cohomology modules \( \{ H^i_a(M/a^nM) \}_{n \in \mathbb{N}} \) for an integer \( i \in \mathbb{Z} \). By the natural homomorphisms these families form a projective system. Their projective limit \( \lim_{\leftarrow} H^i_m(M/a^nM) \) is called the \( i \)-th formal local cohomology of \( M \) with respect to \( a \). Not so much is known about these modules. In the case of a regular local ring they have been studied by Peskine and Szpiro (cf. [17, Chapter III]) in relation to the vanishing of local cohomology modules. Another kind of investigations about formal cohomology has been done by Faltings (cf. [5]).

Moreover, \( \lim_{\leftarrow} H^i_m(M/a^nM) \) occurs as the \( i \)-th cohomology module of the \( a \)-adic completion of the Čech complex \( \check{C}_\mathfrak{m} \otimes M \) (cf. Section 3), where \( \mathfrak{x} \) denotes a system of elements of \( R \) such that \( \text{Rad } \mathfrak{x} R = \mathfrak{m} \).

The main subject of the paper is a systematic study of the formal local cohomology modules. Above all we are interested in the first resp. last non-vanishing of the formal cohomology. As an easy result of this type the following result is proved:

**Theorem 1.1.** Let \( a \) denote an ideal of a local ring \((R, \mathfrak{m})\). Then

\[
\dim M/aM = \sup \{ i \in \mathbb{Z} : \lim_{\leftarrow} H^i_m(M/a^nM) \neq 0 \}
\]

for a finitely generated \( R \)-module \( M \).
The description of \( \inf \{ i \in \mathbb{Z} : \lim_{n \to \infty} H^i_m(M/\mathfrak{a}^n M) \neq 0 \} \) in terms of intrinsic data seems to be not obvious. Following the intention of Peskine and Szpiro (cf. [17, Chapter III]) we define the formal grade as

\[
\text{fgrade}(\mathfrak{a}, M) = \inf \{ i \in \mathbb{Z} : \lim_{n \to \infty} H^i_m(M/\mathfrak{a}^n M) \neq 0 \}
\]

for an ideal \( \mathfrak{a} \) and a finitely generated \( R \)-module \( M \). Since the formal cohomology does not change by passing to the completion of \( R \) (cf. [3,3]) we may assume – without loss of generality – the existence of a dualizing complex \( D_R \) for \( R \). So we may express the formal cohomology in terms of the local cohomology of the dualizing complex.

**Theorem 1.2.** Let \( (R, \mathfrak{m}) \) denote a local ring possessing a (normalized) dualizing complex \( D_R \). Let \( \mathfrak{a} \) denote an ideal of \( R \). For a finitely generated \( R \)-module \( M \) it follows

\[
\begin{align*}
&\text{(a) } \lim_{n \to \infty} H^i_m(M/\mathfrak{a}^n M) \simeq \text{Hom}_R(H^i_\mathfrak{a}(\text{Hom}_R(M, D_R)), E), \text{ for all } i \in \mathbb{Z}, \\
&\text{(b) } \text{fgrade}(\mathfrak{a}, M) = \inf \{ i - \text{cd}(\mathfrak{a}, K^i(M)) : i = 0, \ldots, \dim M \}.
\end{align*}
\]

Here \( K^i(M) = H^{-i}(\text{Hom}(M, D_R)), i = 0, \ldots, \dim M, \) denotes the \( i \)-th module of deficiency (cf. Section 2.3).

Another result concerns the vanishing of the formal cohomology \( \lim_{n \to \infty} H^i_m(M/\mathfrak{a}^n M) \) and the dimension of the associated prime ideals of the underlying module.

**Theorem 1.3.** Let \( \mathfrak{a} \) denote an ideal of a local ring \( (R, \mathfrak{m}) \). Let \( M \) be a finitely generated \( R \)-module. Then

\[
\begin{align*}
&\text{(a) } \text{fgrade}(\mathfrak{a}, M) \leq \dim M - \text{cd}(\mathfrak{a}, M), \\
&\text{(b) } \text{fgrade}(\mathfrak{a}, M) \leq \dim \hat{R}/\mathfrak{p} - \text{cd}(\mathfrak{a}\hat{R}, \hat{R}/\mathfrak{p}) \quad \text{for all } \mathfrak{p} \in \text{Ass } \hat{M},
\end{align*}
\]

where \( \hat{M} \) denotes the \( \mathfrak{m} \)-adic completion of \( M \).

In Section 5 there is a Mayer-Vietoris sequence for the formal cohomology, analogous to the corresponding sequence for the local cohomology. As in the case of local cohomology this applies to connectedness results of certain subsets of \( \text{Spec } R \). To this end let \( c(R/\mathfrak{c}) \) denote the connectedness dimension of \( V(\mathfrak{c}) \) for an ideal \( \mathfrak{c} \) (cf. [5,7]).

**Theorem 1.4.** Let \( \mathfrak{a} \) be an ideal of \( (R, \mathfrak{m}) \). For a finitely generated \( R \)-module \( M \) there are the estimates:

\[
\begin{align*}
&\text{(a) } \text{fgrade}(\mathfrak{a}, M) - 1 \leq c(\hat{R}/(\mathfrak{a}\hat{R}, \mathfrak{p})) \quad \text{for all } \mathfrak{p} \in \text{Ass } \hat{M}, \\
&\text{(b) } \text{Ass } \hat{M} = \text{Ass } \hat{M} = \text{Ass } \hat{M} = \text{Ass } \hat{M} \text{ and } H^d_m(\hat{R}/\text{Ann } M), d = \dim M, \text{ is an indecomposable } \\
&\text{R-module. Then } \text{fgrade}(\mathfrak{a}, M) - 1 \leq c(M/\mathfrak{a}M).
\end{align*}
\]

In particular, when \( \lim_{n \to \infty} H^i_m(M/\mathfrak{a}^n M) = 0 \) for \( i = 0, 1 \), then \( V(\mathfrak{a}\hat{R}, \mathfrak{p}) \setminus V(\hat{\mathfrak{m}}) \) is connected for all \( \mathfrak{p} \in \text{Ass } \hat{M} \).

In Section 2 of the paper we start with some preliminaries about notation, local cohomology, dualizing complexes, and commutative algebra. Section 3 is devoted to the definitions and basic results about formal cohomology, its relation to duality, as well as exact sequences for various situations. In Section 4 there are vanishing and non-vanishing results about formal cohomology. This Section contains also the results about the formal grade. In Section 5 there is the Mayer-Vietoris sequence for formal cohomology and the connectedness properties. In addition, there are also results about the connectedness and the local cohomology.
2. Preliminary Results

2.1. Notation. In the present paper $(R, m, k)$ denotes a local Noetherian ring with its residue field $k = R/m$. In the following let $a, b, \ldots$ denote ideals of $R$. Let $M$ be an $R$-module. By $X: \ldots \to X^n \xrightarrow{d_n} X^{n+1} \to \ldots$ we denote a complex of $R$-modules.

Let $x = x_1, \ldots, x_n$ be a system of elements of $R$. Then $K(x; X)$ and $K^*(x; X)$ are the Koszul complexes of $X$ with respect to $x$ (cf. [22] for the definition of Koszul complexes and basic facts about homological algebra).

For an arbitrary $R$-complex $X$ there is a complex $I$ of injective $R$-modules (resp. a complex $F$ of flat $R$-modules) and a quasi-isomorphism $X \sim I$ (resp. $F \sim X$) (cf. [24] or [2] for the construction). We call $I$ (resp. $F$) an injective (resp. a flat) resolution of $X$.

For an $R$-complex $X$ and an integer $m \in \mathbb{Z}$ define the shifted complex $X[m]$ by $X[m]^n = X^{m+n}, n \in \mathbb{Z}$, and $d_X[m] = (-1)^m d_X$, where $d$ denotes the boundary map.

2.2. Local cohomology. Let $x = x_1, \ldots, x_n$ be a system of elements of the ring $R$ and let $a = (x_1, \ldots, x_n)R$ the ideal generated by these elements. The local cohomology $R\Gamma_a(X)$ of $X$ with respect to $a$ in the derived category is defined by $R\Gamma_a(I)$, where $X \sim I$ denotes the injective resolution (cf. [13] resp. [9]). For an integer $i \in \mathbb{Z}$ define $H^i_a(X) = H^i(R\Gamma_a(I))$. Note that up to isomorphisms it is independent on $I$.

Moreover let $\check{C}_x$ denote the Čech complex with respect to $x$ (cf. [19] or [20]). Then there is a canonical isomorphism $R\Gamma_a(I) \simeq \check{C}_x \otimes I$ for a complex of injective $R$-modules $I$ (cf. [20]). Because $\check{C}_x$ is a bounded $R$-complex of flat $R$-modules it induces the following isomorphism $\check{C}_x \otimes X \simeq \check{C}_x \otimes I$. That is, the local cohomology $H^i_a(X), i \in \mathbb{Z}$, may be computed as the cohomology $H^i(\check{C}_x \otimes X)$.

For a finitely generated $R$-module $M$ there is the following characterization

$$\text{grade}(a, M) = \inf \{i \in \mathbb{Z} : H^i_a(M) \neq 0\}$$

for the grade($a, M$) of the $R$-module $M$ with respect to the ideal $a$. For the supremum of the non-vanishing there is the following definition

$$\text{cd}(a, M) = \sup \{i \in \mathbb{Z} : H^i_a(M) \neq 0\},$$

where $\text{cd}(a, M)$ is called the cohomological dimension of $M$ with respect to $a$. Recall that $\text{cd}(a, M) \leq \dim_R M$ with equality in the case $\text{Rad} a = m$ (cf. [11]). Moreover $\text{height}_R a \leq \text{cd}(a, M)$, where $\text{height}_R a = \text{height}(a, \text{Ann}_R M)/\text{Ann}_R M$. In general it is a difficult problem to calculate the cohomological dimension $\text{cd}(a, R)$ of an ideal.

We need here another preliminary result about cohomological dimensions. It was invented by Divaani-Aazar, Naghipour and Tousi (cf. [7]). For sake of completeness we include a proof.

Lemma 2.1. Let $a$ denote an ideal of a local ring $(R, m)$. Let $M, N$ be two finitely generated $R$-modules such that $\text{Supp} N \subseteq \text{Supp} M$. Then $\text{cd}(a, N) \leq \text{cd}(a, M)$.

Proof. It will be enough to show that $H^i_a(N) = 0$ for all integers $\text{cd}(a, M) < i \leq \dim M + 1$. The proof will be shown by an descending induction on $i$.

First note that the claim is true for $i = \dim M + 1$. (cf. [11]). Now let $i \leq \dim M$. We proceed by a trick invented by Delfino and Marley (cf. the proof of [6 Proposition 1]). By the
assumption we have $\text{Supp} \ N \subseteq \text{Supp} \ M$, and therefore $\text{Rad} \ \text{Ann} \ R \ N \supseteq \mathfrak{c}$, where $\mathfrak{c} = \text{Ann} \ R \ M$. Whence there is an $n \in \mathbb{N}$ such that $\mathfrak{c}^n N = 0$. Thus $N$ possesses a filtration

$$0 = \mathfrak{c}^n N \subset \mathfrak{c}^{n-1} N \subset \ldots \subset \mathfrak{c} N \subset N,$$

such that $\mathfrak{c}^{i-1} N/\mathfrak{c}^i N, i = 1, \ldots n$, is a finitely generated $R/\mathfrak{c}$-module.

By Gruson’s theorem (cf. [25, Theorem 4.1]) a finitely generated $R/\mathfrak{c}$-module $T$ admits a filtration

$$0 = T_0 \subset T_1 \subset \ldots \subset T_k = T,$$

such that $T_j/T_{j-1}, j = 1, \ldots k$, is a homomorphic image of finitely many copies of $M$.

We prove now the vanishing of $H^i_a(T)$. By using short exact sequences and induction on $k$ it suffices to prove the case when $k = 1$. Thus, there is an exact sequence

$$0 \rightarrow K \rightarrow M^m \rightarrow T \rightarrow 0$$

for some positive integer $m$. It induces an exact sequence

$$\ldots \rightarrow H^i_a(K) \rightarrow H^i_a(M)^m \rightarrow H^i_a(T) \rightarrow H^{i+1}_a(K) \rightarrow \ldots .$$

By the inductive hypothesis $H^{i+1}_a(K) = 0$, so that $H^i_a(T) = 0$.

Finally we prove that $H^i_a(N) = 0$. By the use of short exact sequences and induction on $n$, it suffices to prove the case when $n = 1$, which is obviously true by the aid of the previous argument. □

As a corollary of the previous Lemma 2.1 it follows that the cohomological dimension of a finitely generated $R$-module $M$ is determined by the cohomological dimension of its minimal associated prime ideals. To this end let $\text{Min} \ M$ denote the minimal elements of $\text{Supp} \ M$, where $M$ denotes an $R$-module.

**Corollary 2.2.** Let $M$ be a finitely generated $R$-module. Then

$$\text{cd}(a, M) = \text{cd}(a, R/ \text{Ann} \ R \ M) = \max \{ \text{cd}(a, R/\mathfrak{p}) : \mathfrak{p} \in \text{Min} \ M \}$$

for any ideal $a$ of $R$.

**Proof.** The fist equality is clear because of $V(\text{Ann} \ R \ M) = \text{Supp} \ M$ (cf. [2.1]). For the proof of the second define $N = \bigoplus_{\mathfrak{p} \in \text{Min} \ M} R/\mathfrak{p}$. Then it follows that

$$\text{cd}(a, N) = \max \{ \text{cd}(a, R/\mathfrak{p}) : \mathfrak{p} \in \text{Min} \ M \}.$$

Remember that the local cohomology commutes with direct sums. Furthermore we have $\text{Supp} \ M = \text{Supp} \ N$. So the statement is a consequence of Lemma 2.1. □

As another preliminary result we need the behavior of the cohomological dimension of an $R$-module with respect to an ideal $a$ by passing to $(a, xR)$.

**Lemma 2.3.** Let $a$ denote an ideal of a local ring $(R, \mathfrak{m})$. Let $M$ be a finitely generated $R$-module. Then

$$\text{cd}((a, xR), M) \leq \text{cd}(a, M) + 1$$

for any element $x \in \mathfrak{m}$.
Proof. With the notation of the lemma there is the short exact sequence
\[ 0 \to H^1_{xR}(H^1_a(M)) \to H^{i+1}_{(a,xR)}(M) \to H^0_{xR}(H^{i+1}_a(M)) \to 0 \]
for all \( i \in \mathbb{Z} \) (cf. for instance [20, Corollary 3.5]). Now put \( c = \text{cd}(a, M) \). Then by the definition of the cohomological dimension the short exact sequence implies that \( H^{i+1}_{(a,xR)}(M) = 0 \) for all \( i > c \). In other words \( \text{cd}((a, xR), M) \leq c + 1 \), which finishes the proof.

2.3. Dualizing complexes. In this subsection let \((R, m)\) denote a local ring possessing a dualizing complex \( D_R \). That is a bounded complex of injective \( R \)-modules whose cohomology modules \( H^i(D_R) \), \( i \in \mathbb{Z} \), are finitely generated \( R \)-modules. We refer to [13, Chapter V, §2] or to [19, 1.2] for basic results about dualizing complexes.

By the result of T. Kawasaki (cf. [16]) \( R \) possesses a dualizing complex if and only if \( R \) is the factor ring of a Gorenstein ring.

Note that the natural homomorphism of complexes
\[ M \to \text{Hom}_R(\text{Hom}_R(M, D_R), D_R) \]
induces an isomorphism in cohomology for any finitely generated \( R \)-module \( M \). Moreover there is an integer \( l \in \mathbb{Z} \) such that
\[ \text{Hom}_R(k, D_R) \simeq k[l], \]
where \( k = R/m \) denotes the residue field of \( R \). As follows by a shifting we may always assume without loss of generality assume that \( l = 0 \). Then the dualizing complex \( D_R \) is called normalized. In the following let us always assume that a dualizing complex is normalized.

Then a dualizing complex has the following structure
\[ D_R^i \simeq \bigoplus_{p \in \text{Spec} R, \dim R/p = i} E_R(R/p), \]
where \( E_R(R/p) \) denotes the injective hull of \( R/p \) as \( R \)-module. Therefore \( D_R^i = 0 \) for \( i < -\dim R \) and \( i > 0 \).

Proposition 2.4. Let \((R, m)\) denote a local ring with the dualizing complex \( D_R \).

(a) \( D_R \otimes R_p \simeq D_{R_p}[\text{dim} R/p] \) for \( p \in \text{Spec} R \).

(b) (Local duality) There is a canonical isomorphism
\[ H^i_m(M) \simeq \text{Hom}_R(H^{-i}(\text{Hom}_R(M, D_R)), E), \quad E = E_R(R/m), \]
for a finitely generated \( R \)-module \( M \) and all \( i \in \mathbb{Z} \).

The proof is well-known (cf. [13] resp. [19]). For a certain application remember the definition of the modules of deficiencies of an \( R \)-module \( M \) (cf. [19, Section 1.2]).

Definition 2.5. Let \( M \) denote a finitely generated \( R \)-module and \( d = \dim M \). For an integer \( i \in \mathbb{Z} \) define
\[ K^i(M) := H^{-i}(\text{Hom}_R(M, D_R)). \]
The module \( K(M) := K^d(M) \) is called the canonical module of \( M \). For \( i \neq d \) the modules \( K^i(M) \) are called the modules of deficiency of \( M \). Note that \( K^i(M) = 0 \) for all \( i < 0 \) or \( i > d \).
By the local duality theorem there are the canonical isomorphisms
\[ H^i_m(M) \simeq \text{Hom}_R(K^i(M), E), i \in \mathbb{Z}, \]
where \( E = E_R(R/m) \) denotes the injective hull of the residue field. Remember that all of the \( K^i(M), i \in \mathbb{Z}, \) are finitely generated \( R \)-modules. Moreover \( M \) is a Cohen-Macaulay module if and only if \( K^d(M) = 0 \) for all \( i \neq d \). Whence the modules of deficiencies of \( M \) measure the deviation of \( M \) from being a Cohen-Macaulay module. Here is a summary about results we use in the sequel.

**Proposition 2.6.** Let \( M \) denote a \( d \)-dimensional \( A \)-module. Let \( k \in \mathbb{N} \) an integer. Then the following results are true:

(a) \( \dim K^i(M) \leq i \) for all \( 0 \leq i < d \) and \( \dim K(M) = d. \)

(b) \( \text{Ass} \ K(M) = (\text{Ass} \ M)_d. \)

(c) \( (\text{Ass} \ K^i(M))_i = (\text{Ass} \ M)_i \) for all \( 0 \leq i < d. \)

(d) \( K(M) \) satisfies \( S_k. \)

(e) \( M \) satisfies \( S_k \) if and only if \( \dim K^i(M) \leq i - k \) for all \( 0 \leq i < \dim M. \)

For a finitely generated \( R \)-module \( X \) let \( (\text{Ass} \ X)_i = \{ p \in \text{Ass} X : \dim R/p = i \} \) for an integer \( i \in \mathbb{Z}. \) Cf. [19] Section 1 for the details of the proof of Proposition 2.6.

2.4. **On commutative algebra.** Let \( M \) be a finitely generated \( R \)-module, \( R \) a commutative Noetherian ring. Let \( \text{Ass}_R M = \{ p_1, \ldots, p_s \} \) denote the set of associated prime ideals. Let
\[ 0 = Z(p_1) \cap \ldots \cap Z(p_t) \]

denote a minimal primary decomposition of \( M. \) That is, \( M/Z(p_i), i = 1, \ldots, t, \) is a non-zero \( p_i \)-coprimary \( R \)-module.

Next we want to prove a constructive version of a result of N. Bourbaki (cf. [3] Ch. IV, §2, Prop. 6]).

**Lemma 2.7.** With the previous notation let \( S = \{ p_1, \ldots, p_s \} \) denote a subset of \( \text{Ass}_R M \) for a certain numeration of the associated prime ideals of \( M. \) Put \( U = \cap_{i=1}^s Z(p_i). \) Then
\[ \text{Ass}_R M/U = S \quad \text{and} \quad \text{Ass}_R U = \text{Ass}_R M \setminus S. \]

**Proof.** Let \( \text{Ass}_R M = \{ p_1, \ldots, p_t \} \) and \( 0 = Z(p_1) \cap \ldots \cap Z(p_t) \) a minimal primary decomposition. First it is clear that \( \text{Ass}_R M/U = S. \) Remember that \( U = \cap_{i=1}^s Z(p_i) \) is a reduced minimal primary decomposition. Define \( V = \cap_{i=s+1}^t Z(p_i). \) In order to show the second part of the claim it will be enough to prove that \( \text{Ass}_R U = \{ p_{s+1}, \ldots, p_t \}. \)

First note that \( U \cong U + V/V \subseteq M/V. \) Therefore \( \text{Ass}_R U \subseteq \{ p_{s+1}, \ldots, p_t \} \) as easily seen. Now let \( p \in \{ p_{s+1}, \ldots, p_t \} \) be a given prime ideal. Then \( U/U \cap Z(p) \cong U + Z(p)/Z(p) \) is a non-zero \( p \)-coprimary module. Since \( U \cap Z(p) \) is part of a minimal reduced primary decomposition of \( 0 \) in \( U \) it follows that \( p \in \text{Ass}_R U, \) as required. \( \square \)

3. **On the definition of formal cohomology**

3.1. **The basic definitions.** Let \( (R, m, k) \) be a local Noetherian ring. Let \( \underline{x} = x_1, \ldots, x_r \) denote a system of elements of \( R \) and \( b = \text{Rad}(\underline{x}R). \) Let \( \mathcal{C}_R \) denote the Čech complex of \( R \) with respect to \( \underline{x}. \) For an \( R \)-module \( M \) and an ideal \( \mathfrak{a} \) the projective system of \( R \)-modules \( \{ M/\mathfrak{a}^nM \}_{n \in \mathbb{N}} \)
induces a projective system of $R$-complexes $\{ \hat{C}_x \otimes M/a^nM \}$. Its projective limit $\varprojlim (\hat{C}_x \otimes M/a^nM)$ is the main object of our investigations.

**Definition 3.1.** For an integer $i \in \mathbb{Z}$ the cohomology module $H^i(\varprojlim \hat{C}_x \otimes M/a^nM)$ is called the $i$-th $a$-formal cohomology with respect to $b$. In the case of $b = m$ we speak simply about the $i$-th $a$-formal cohomology. By abuse of notation we say also formal cohomology in case there will be no doubt on $a$.

In the following let $\Lambda^a = \varprojlim (\cdot \otimes R/a^n)$ denote the $a$-adic completion. For an $R$-module $M$ it turns out that the complex $\varprojlim (\hat{C}_x \otimes M \otimes R/a^n)$ is isomorphic to $\Lambda^a(\hat{C}_x \otimes M)$. In the derived category this complex is isomorphic to $\Lambda^a(\Gamma_b(I))$, where $M \sim I$ denotes an injective resolution of $M$. For further results in this direction see [22].

As a first result here there is a relation of the formal cohomology with respect to the projective limits of certain local cohomology modules.

**Proposition 3.2.** With the previous notation there is the following short exact sequence

$$0 \to \varprojlim H^1_b(M/a^nM) \to H^i(\varprojlim (\hat{C}_x \otimes M/a^nM)) \to \varprojlim H^1_b(M/a^nM) \to 0$$

for all $i \in \mathbb{Z}$. In the case of $b = m$ and a finitely generated $R$-module $M$ it provides isomorphisms

$$H^i(\varprojlim (\hat{C}_x \otimes M/a^nM)) \simeq \varprojlim H^i_m(M/a^nM)$$

for all $i \in \mathbb{Z}$.

**Proof.** The Čech complex $\hat{C}_x$ is a complex of flat $R$-modules. Whence the natural epimorphism $M/a^{n+1}M \to M/a^nM$, $n \in \mathbb{N}$, induces an $R$-morphism of $R$-complexes

$$\hat{C}_x \otimes M/a^{n+1}M \to \hat{C}_x \otimes M/a^nM$$

which is degree-wise an epimorphism. By the definition of the projective limit there is a short exact sequence of complexes

$$0 \to \varprojlim (\hat{C}_x \otimes M/a^nM) \to \prod (\hat{C}_x \otimes M/a^nM) \to \prod (\hat{C}_x \otimes M/a^nM) \to 0$$

(cf. e.g. [22]). Now the long exact cohomology sequence provides the first part of the claim. To this end break it up into short exact sequences and take into account that homology commutes with direct products.

For the proof of the second part remember that $H^i_m(M/a^nM)$, $i \in \mathbb{Z}$, is an Artinian $R$-module whenever $M$ is a finitely generated $R$ (cf. [11], Section 6)). So the corresponding projective system satisfies the Mittag-Leffler condition. That is, $\varprojlim H^1$ vanishes on the projective system of Artinian $R$-modules. The proof is now a consequence of the first part. \hfill \Box

Let $(\hat{R}, \hat{m})$ denote the $m$-adic completion of $(R, m)$. An Artinian $R$-module $A$ has a natural structure of an $\hat{R}$-module such that the natural homomorphisms $A \to \hat{A}$ and $A \to \hat{A} \otimes R$ are isomorphisms.

**Proposition 3.3.** Let $M$ be a finitely generated $R$-module. Then $\varprojlim H^i_m(M/a^nM)$, $i \in \mathbb{Z}$, has a natural structure as an $\hat{R}$-module and and there are isomorphisms

$$\varprojlim H^i_m(M/a^nM) \simeq \varprojlim H^i_m(\hat{M}/a^n\hat{M})$$

for all $i \in \mathbb{Z}$. 
Proof. Let \( N \) be a finitely generated \( R \)-module. Then it is known that \( H^i_m(N), i \in \mathbb{Z}, \) is an Artinian \( R \)-module (cf. e.g. [11, Section 6]). Because of the previous remarks and the flatness of \( \hat{R} \) over \( R \) there are \( R \)-isomorphisms \( H^i_m(N) \simeq H^i_m(N) \) for all \( i \in \mathbb{Z} \). Now take \( N = M/a^nM \) and pass to the projective limit. Then this proves the claim. \( \Box \)

The previous result has the advantage that one might assume the existence of a dualizing complex in order to consider the formal cohomology. Note that by the Cohen Structure theorem \( \hat{R} \) is the factor ring of a regular local ring.

Let \( U = \text{Spec } R \setminus \{m\} \). Let \( (\hat{U}, \mathcal{O}_U) \) denote the formal completion of \( U \) along \( V(a) \setminus \{m\} \) (cf. [5] and [17] for the details). For an \( R \)-module \( M \) let \( \mathcal{F} \) denote the associated sheaf on \( U \). Let \( \hat{\mathcal{F}} \) denote the coherent \( \mathcal{O}_U \)-sheaf associated to \( \lim \mathcal{M}/\mathcal{M}^aM \). Let \( \mathcal{M}^a \) denote the \( a \)-adic completion of \( M \). Moreover \( \mathcal{J} \) denotes the ideal sheaf of \( a \) on \( (U, \mathcal{O}_U) \). Then there is the following relation to the formal local cohomology (cf. also [17]).

**Lemma 3.4.** Let \( M \) denote a finitely generated \( R \)-module. With the previous notation there are an exact sequence
\[
0 \to \lim H^0_m(M/a^nM) \to \hat{M}^a \to H^0(\hat{U}, \hat{\mathcal{F}}) \to \lim H^1_m(M/a^nM) \to 0
\]
and isomorphisms
\[
H^i(\hat{U}, \hat{\mathcal{F}}) \simeq \lim H^{i+1}_m(M/a^nM)
\]
for all \( i \geq 1 \).

**Proof.** Let \( n \in \mathbb{N} \) denote an integer. First remember that there is a functorial exact sequence
\[
0 \to H^0_m(M/a^nM) \to M/a^nM \xrightarrow{\phi_n} H^0(U, \mathcal{F}/\mathcal{J}^n\mathcal{F}) \to H^1_m(M/a^nM) \to 0
\]
and isomorphisms \( H^i(U, \mathcal{F}/\mathcal{J}^n\mathcal{F}) \simeq H^{i+1}_m(M/a^nM) \) for all \( i \in \mathbb{Z} \) (cf. e.g. [11]). The family of \( R \)-modules \( \{\text{Im } \phi_n\}_{n \in \mathbb{N}}, \) as a surjective system, and the families \( \{H^i_m(M/a^nM)\}_{n \in \mathbb{N}}, i \in \mathbb{N}, \) as families of Artinian \( R \)-modules, both satisfy the Mittag-Leffler condition. Therefore, the above exact sequence induces – by passing to the projective limit – an exact sequence
\[
0 \to \lim H^0_m(M/a^nM) \to \hat{M}^a \to H^0(\hat{U}, \hat{\mathcal{F}}) \to \lim H^1_m(M/a^nM) \to 0,
\]
which proves the first part of the claim.

The above isomorphisms provide an isomorphism
\[
\lim H^i(U, \mathcal{F}/\mathcal{J}^n\mathcal{F}) \simeq \lim H^{i+1}_m(M/a^nM)
\]
for all \( i \in \mathbb{Z} \). Now the natural homomorphism \( H^i(\hat{U}, \hat{\mathcal{F}}) \to \lim H^i(U, \mathcal{F}/\mathcal{J}^n\mathcal{F}), i \in \mathbb{Z}, \) yields an isomorphism (cf. [17, Ch. III, Prop.2.1]). This finishes the proof of the statement. \( \Box \)

3.2. **On duality.** In this subsection let \( (R, m) \) denote a local ring possessing a dualizing complex \( D_R \). The main goal of the considerations here is an expression of the formal cohomology in terms of a certain local cohomology of the dualizing complex. To be more precise the following result holds.

**Theorem 3.5.** Let \( M \) be a finitely generated \( R \)-module. For an ideal \( a \) of \( R \) there are isomorphisms
\[
\lim H^i_m(M/a^nM) \simeq \text{Hom}_R(H^i_a(\text{Hom}_R(M, D_R)), E),
\]
for all \( i \in \mathbb{Z} \), where \( E = E_R(R/m) \) denotes the injective hull of the residue field \( k \).
**Proof.** Let $n \in \mathbb{N}$ be an integer. By virtue of the Local Duality Theorem (cf. [2.4]) there are the isomorphisms

$$H^i_m(M/a^n M) \simeq \text{Hom}_R(H^{-i}(\text{Hom}_R(M/a^n M, D_R)), E)$$

for all $i \in \mathbb{Z}$. By passing to the projective limit there are isomorphisms

$$\lim_{\leftarrow} H^i_m(M/a^n M) \simeq \text{Hom}_R(H^{-i}(\lim_{\leftarrow} \text{Hom}_R(M/a^n M, D_R)), E)$$

for all $i \in \mathbb{Z}$. To this end remember that the injective limit commutes with cohomology and is transformed into a corresponding projective system by Hom in the first place. Now the proof turns out because $\lim_{\leftarrow} \text{Hom}(M/a^n M, D_R) \simeq \Gamma_a(\text{Hom}_R(M, D_R))$ as easily seen. \(\square\)

**Remark 3.6.** In the case the local ring \((R, m)\) possesses a dualizing complex it is a quotient of a local Gorenstein ring \((S, n)\) (cf. [16]). Therefore, we may use

$$D_R = \text{Hom}_S(R, I_S)[-n], \ n = \text{dim} S,$$

as the (normalized) dualizing complex, where $I_S$ denotes the minimal injective resolution of $S$ as an $S$-module. By the local duality (cf. [2.4])

$$\lim_{\leftarrow} H^i_m(M/a^n M) \simeq \text{Hom}_R(\lim_{\leftarrow} \text{Ext}^{n-i}_S(M/a^n M, S), E)$$

for all $i \in \mathbb{Z}$, where $E$ denotes the injective hull of the residue field. In his unpublished habilitation (cf. [14]) Herzog introduced

$$H^i_a(M, N) = \lim_{\leftarrow} \text{Ext}^i_R(M/a^n M, N), \ i \in \mathbb{Z},$$

for two $R$-modules $M, N$ and an ideal $a \subset R$ as the generalized local cohomology with respect to $a$. With the previous notation there are isomorphisms

$$\lim_{\leftarrow} H^i_m(M/a^n M) \simeq \text{Hom}_R(H^{n-i}_{a_S}(M, S), E), \ i \in \mathbb{Z},$$

where $M$ is considered as an $S$-module. So, the $i$-th $a$-formal cohomology $\lim_{\leftarrow} H^i_m(M/a^n M)$ is isomorphic to the Matlis dual of $H^{n-i}_{a_S}(M, S)$ equipped with its natural $R$-module structure.

The previous result has as a consequence a non-vanishing behavior of the formal cohomology, important for the subsequent considerations.

**Corollary 3.7.** Let $p$ denote a prime ideal and $i \in \mathbb{Z}$ be such that $\lim_{\leftarrow} H^i_{p R_p}(M_p/a^n M_p) \neq 0$. Then $\lim_{\leftarrow} H^{i+\text{dim} R/p}_m(M/a^n M) \neq 0$.

**Proof.** By virtue of Matlis’ duality for the local ring $R_p$ it follows that $H^{-i}_{a R_p}(\text{Hom}(M_p, D_{R_p}))$ does not vanish (cf. [3.5]). Now there is an isomorphism of complexes

$$\text{Hom}(M_p, D_{R_p}) \simeq \text{Hom}_R(M, D_R)[-\text{dim} R/p] \otimes R_p$$

(cf. [2.4] and remember that $M$ is a finitely generated $R$-module). But this provides the isomorphisms

$$H^{-j}_{a R_p}(\text{Hom}(M_p, D_{R_p})) \simeq H^{-j-\text{dim} R/p}(\text{Hom}(M, D_R)) \otimes R_p$$

for all $j \in \mathbb{Z}$. Therefore $H^{-i}_{a R_p}(\text{Hom}(M_p, D_{R_p})) \neq 0$. By Matlis’ duality this implies the non-vanishing of $\lim_{\leftarrow} H^{i+\text{dim} R/p}_m(M/a^n M)$ (cf. [3.5]). This completes the proof. \(\square\)
We conclude this subsection with the proof of the fact that equivalent ideal topologies define isomorphic formal cohomology modules. Here \(\{M_n\}_{n \in \mathbb{N}}\) is called a decreasing family of submodules provided \(M_{n+1} \subseteq M_n\) for all \(n \in \mathbb{N}\).

**Lemma 3.8.** Let \(M\) be a finitely generated \(R\)-module. Let \(\{M_n\}_{n \in \mathbb{N}}\) be a decreasing family of submodules of \(M\). Suppose that their topology is equivalent to the \(a\)-adic topology on \(M\). Then there are isomorphisms

\[
\lim H^i_m(M/a^nM) \simeq \lim H^i_m(M/M_n)
\]

for all \(i \in \mathbb{Z}\).

**Proof.** Let \(\check{C}_x\) denote the Čech complex of \(R\) with respect to a system of elements \(x\) such that \(\operatorname{Rad}_x R = m\). For any flat \(R\)-module \(F\) there is an isomorphism \(\lim F \otimes (M/M_n) \simeq \lim F \otimes M/a^nM\). To this end remember that \(F \otimes (M/N) \simeq (F \otimes M)/(F \otimes N)\) for any submodule \(N \subseteq M\). Moreover, \(\{F \otimes M_n\}\) is equivalent to the \(a\)-adic topology on \(F \otimes M\).

Since \(\check{C}_x\) is a bounded complex of flat \(R\)-modules this isomorphism extends to an isomorphism \(\lim \check{C}_x \otimes M/M_n \simeq \lim \check{C}_x \otimes M/a^nM\) of \(R\)-complexes. Therefore, it will be enough to show that

\[
H^i(\lim \check{C}_x \otimes (M/M_n)) \simeq \lim H^i(\check{C}_x \otimes M/M_n), \quad i \in \mathbb{Z},
\]

(cf. [3.2]). Since \(H^i(\check{C}_x \otimes (M/M_n)) \simeq H^i_m(M/M_n), \quad i \in \mathbb{Z}\), is an Artinian \(R\)-module this follows by the Mittag-Leffler arguments as in the proof of the second part of [3.2] \(\square\)

As a first structure result on the formal cohomology modules \(\lim H^i_m(M/a^nM), \ i \in \mathbb{Z}\), for a finitely generated \(R\)-module \(M\) we consider their behavior with respect to the \(a\)-adic completion. Let \(L_i \Lambda^a, \ i \in \mathbb{Z}\), denote the left derived functors of the \(a\)-adic completion functor \(\lim (- \otimes R/a^n)\) (cf. [10], [23] for the basic results for modules and [22] for an extension to complexes). An extensive consideration of the functors \(L_i \Lambda^a, \ i \in \mathbb{Z}\), has been done in the fundamental work [1].

**Theorem 3.9.** Let \(a\) denote an ideal of an arbitrary local ring \((R, m)\). Let \(M\) be a finitely generated \(R\)-module. For an integer \(j \in \mathbb{Z}\) there are the following isomorphisms

\[
L_i \Lambda^a(\lim H^i_m(M/a^nM)) \simeq \begin{cases} 0 & \text{for } i \neq 0, \\ (\lim H^i_m(M/a^nM))^a & \text{for } i = 0. \end{cases}
\]

Moreover, \(\lim H^j_m(M/a^nM)\) is an \(a\)-adic complete \(R\)-module, i.e. \(\lim H^j_m(M/a^nM))^a \simeq \lim H^j_m(M/a^nM)\).

**Proof.** Without loss of generality we may assume that \((R, m)\) admits a dualizing complex \(D_R\) (cf. [3.3]). For simplicity of notation put \(X^j := \lim H^j_m(M/a^nM), \ j \in \mathbb{Z}\). Then there is the following isomorphism \(X^j \simeq \operatorname{Hom}(H^j, E)\), where \(H^j := H^j_a(\operatorname{Hom}(M, D_R))\) (cf. [3.5]).

Let \(X\) denote an \(R\)-module. For the computation of \(L_i \Lambda^a(X), \ i \in \mathbb{Z}\), there is the following short exact sequence

\[
0 \to \lim \limits^1 \operatorname{Tor}^{R+1}_i(R/a^n, X) \to L_i \Lambda^a(X) \to \lim \operatorname{Tor}^R_i(R/a^n, X) \to 0
\]

(cf. [10] Prop. 1.1) or [22]). Thus, for the first part of our claim it will be enough to prove that \(\lim \limits^1 \operatorname{Tor}^{R+1}_i(R/a^n, X) = 0\) for all \(i \in \mathbb{Z}\) and \(\lim \operatorname{Tor}^R_i(R/a^n, X) = 0\) for all integers \(i \neq 0\).
To this end consider $H^i_a(H^j) \simeq \varprojlim \text{Ext}^i(R/a^n, H^j)$. Because of $\text{Supp } H^j \subseteq V(a)$ clearly $H^i_a(H^j) = 0$ for all $i \neq 0$ and $H^0_a(H^j) \simeq H^j$. By the definition of the direct limit there is the following, canonical exact sequence

$$0 \to \bigoplus_{n \in \mathbb{N}} \text{Ext}^i(R/a^n, H^j) \xrightarrow{\Phi_i} \bigoplus_{n \in \mathbb{N}} \text{Ext}^i(R/a^n, H^j) \to H^i_a(H^j) \to 0,$$

where $\Phi_i$ is defined by the definition of the direct limit. Now apply the Matlis duality functor $\text{Hom}(\cdot, E)$. Because of

$$\text{Hom}(\text{Ext}^i(R/a^n, H^j), E) \simeq \text{Tor}_i(R/a^n, X^j)$$

for all $i, j \in \mathbb{Z}$ and all $n \in \mathbb{N}$, it transforms the direct system $\{\text{Ext}^i(R/a^n, H^j)\}_{n \in \mathbb{N}}$ into the inverse system $\{\text{Tor}_i(R/a^n, X^j)\}_{n \in \mathbb{N}}$. Moreover it provides the short exact sequences

$$0 \to \text{Hom}(H^i_a(H^j), E) \to \prod_{n \in \mathbb{N}} \text{Tor}_i(R/a^n, X^j) \xrightarrow{\Psi_i} \prod_{n \in \mathbb{N}} \text{Tor}_i(R/a^n, X^j) \to 0$$

for all $i, j \in \mathbb{Z}$. By the definition of the homomorphism $\Psi_i$ it follows that

$$\text{Coker } \Psi_i \simeq \varprojlim^{-1} \text{Tor}^R_i(R/a^n, X) \quad \text{and} \quad \text{Ker } \Psi_i \simeq \varprojlim \text{Tor}^R_i(R/a^n, X).$$

By the vanishing of the local cohomology of $H^j$ this provides the vanishing results of $\varprojlim^{-1}$ and $\varprojlim$ of the Tor’s as claimed above. Moreover, for $i = 0$ it yields the isomorphisms

$$X^j \simeq \text{Ker } \Psi_0 \simeq (\varprojlim H^i_a(M/a^nM))^a.$$

To this end remember that $X^j \simeq \text{Hom}(H^j, E)$, as mentioned above. This finally completes the proof of the result.

The class $C_a$ of $R$-modules $X$ such that $L_i \Lambda^a(X) = 0$ for $i > 0$ and $L_0 \Lambda^a(X) = X^a$ has been introduced by Simon (cf. [23, 5.2]). Therefore, the $a$-formal cohomology modules of a finitely generated $R$-module $M$ belong to $C_a$.

As a corollary there is the following Nakayama type criterion about the vanishing of the $a$-formal cohomology.

**Corollary 3.10.** Let $M$ denote a finitely generated $R$-module. Let $j \in \mathbb{Z}$. Suppose that $\varprojlim H^j_a(M/a^nM) = a(\varprojlim H^j_a(M/a^nM))$. Then $\varprojlim H^j_a(M/a^nM) = 0$.

**Proof.** For simplicity of notation put $\varprojlim H^j_a(M/a^nM) = X$. The assumption provides $X = a^nX, n \in \mathbb{N}$, as follows by an induction. Therefore

$$0 = \varprojlim X/a^nX = X^a.$$

By the Theorem [3.9] $X$ is $a$-adically complete. Therefore $X = X^a$ and $X = 0$, as required. □

3.3. **Exact sequences.** First of all we want to relate the behavior of the formal cohomology with respect to short exact sequences of $R$-modules. This is a technical tool that simplifies arguments in further considerations.

**Theorem 3.11.** Let $(R, m)$ denote a local ring. Let $0 \to A \to B \to C \to 0$ denote a short exact sequence of finitely generated $R$-modules. For an ideal $a$ of $R$ there is a long exact sequence

$$\ldots \to \varprojlim H^j_a(A/a^nA) \to \varprojlim H^j_a(B/a^nB) \to \varprojlim H^j_a(C/a^nC) \to \varprojlim H^{j+1}_a(A/a^nA) \to \ldots$$
Proof. For any finitely generated $R$-module $M$ the formal cohomology of $M$ and $\hat{M}$ coincide (cf. [3,2]). So we may assume the existence of a dualizing complex $D_R$. Let $\check{C}_\underline{\varepsilon}$ denote the Čech complex of $R$ with respect to a system of elements $\underline{\varepsilon}$ such that $\text{Rad} \underline{\varepsilon} R = \text{Rad} \underline{a}$. The short exact sequence $0 \to A \to B \to C \to 0$ induces a short exact sequence of $R$-complexes

$$0 \to \check{C}_\underline{\varepsilon} \otimes \text{Hom}(C, D_R) \to \check{C}_\underline{\varepsilon} \otimes \text{Hom}(B, D_R) \to \check{C}_\underline{\varepsilon} \otimes \text{Hom}(A, D_R) \to 0.$$ 

Remember that $D_R$ resp. $\check{C}_\underline{\varepsilon}$ is a bounded complex of injective resp. flat $R$-modules. By passing to the Matlis dual and taking the long exact cohomology sequence this proves the claim.

Remark 3.12. One might ask for a corresponding result for a short exact sequence $0 \to A \to B \to C \to 0$, where the $R$-modules are not necessarily finitely generated. It is not clear whether this will be true.

An alternative proof of [3,11] works as follows. The short exact sequence induces a projective system of short exact sequences

$$0 \to \check{C}_\underline{\varepsilon} \otimes A/B \cap a^n A \to \check{C}_\underline{\varepsilon} B/a^n B \to \check{C}_\underline{\varepsilon} \otimes C/a^n C \to 0$$

for all $n \in \mathbb{N}$. Because $\check{C}_\underline{\varepsilon}$ is a complex of flat $R$-modules and because the maps $A/B \cap a^{n+1} A \to A/B \cap a^n A$ are surjective it follows that the projective system of $R$-complexes $\{\check{C}_\underline{\varepsilon} \otimes A/B \cap a^n A\}_{n \in \mathbb{Z}}$ satisfies degree-wise the Mittag-Leffler condition. Therefore the projective limit provides a short exact sequence of complexes

$$0 \to \lim_{\leftarrow} \check{C}_\underline{\varepsilon} \otimes A/B \cap a^n A \to \lim_{\leftarrow} \check{C}_\underline{\varepsilon} \otimes B/a^n B \to \lim_{\leftarrow} \check{C}_\underline{\varepsilon} \otimes C/a^n C \to 0.$$ 

By view of the long exact cohomology sequence it follows (cf. the definition and [3,2]) that there a long exact sequence

$$\ldots \to \lim_{\leftarrow} H^i_m(A/B \cap a^n A) \to \lim_{\leftarrow} H^i_m(B/a^n B) \to \lim_{\leftarrow} H^i_m(C/a^n C) \to \ldots .$$

In the case $\{B \cap a^n A\}$ is equivalent to the $a$-adic topology on $A$ this yields another proof of the exact sequence in [3,11] (cf. [3,8]). By the Artin-Rees Lemma (cf. [3] Ch. III, §3, Cor. 1) this is true in case $B$ is a finitely generated $R$-module.

As an application let us consider the behavior of the formal cohomology by factoring out the $m$-torsion.

Corollary 3.13. Let $(R, m)$ denote a local ring. For a finitely generated $R$-module $M$ let $N \subseteq M$ be an $R$-module such that $\text{Supp} N \cap V(\underline{a}) \subseteq V(m)$. Put $\hat{M} = M/N$. Then there is a short exact sequence

$$0 \to N^a \to \lim_{\leftarrow} H^0_m(M/a^n M) \to \lim_{\leftarrow} H^0_m(\hat{M}/a^n \hat{M}) \to 0$$

and isomorphisms $\lim_{\leftarrow} H^i_m(M/a^n M) \simeq \lim_{\leftarrow} H^i_m(\hat{M}/a^n \hat{M})$ for all $i \geq 1$.

Proof. There is the following short exact sequence $0 \to N \to M \to \hat{M} \to 0$. Then there is the long exact sequence

$$\ldots \to \lim_{\leftarrow} H^i_m(N/a^n N) \to \lim_{\leftarrow} H^i_m(M/a^n M) \to \lim_{\leftarrow} H^i_m(\hat{M}/a^n \hat{M}) \to \lim_{\leftarrow} H^{i+1}_m(N/a^n N) \to \ldots$$
(cf. 3.11). By view of the assumption $\text{Supp} \, N \cap V(a) \subseteq V(m)$ it follows that $N/a^nN$ is an $R$-module of finite length for all $n \in \mathbb{N}$. That is, $H^i_m(N/a^nN) = 0$ for $i > 0$ and all $n \in \mathbb{N}$. Moreover $H^0_m(N/a^nN) \simeq N/a^nN$ and therefore $\lim H^i_m(N/a^nN) \simeq N^n$. So the above long exact sequence provides the short exact sequence and the isomorphisms of the claim. 

In the subsequent section there is a generalization of 3.13. In fact there is a precise computation of the 0-th formal cohomology.

**Theorem 3.14.** Let $M$ be a finitely generated $R$-module. Choose $x \in \mathfrak{m}$ an element such that $x \notin p$ for all $p \in \text{Ass}_R M \setminus \{m\}$. Then there are short exact sequences

$$0 \rightarrow H_0(x; \lim H^i_m(M/a^nM)) \rightarrow \lim H^i_m(M'/a^nM') \rightarrow H_1(x; \lim H^{i+1}_m(M/a^nM)) \rightarrow 0$$

for all $i \in \mathbb{Z}$, where $M' = M/xM$.

**Proof.** By the choice of $x$ it follows that $0 :_M x$ is an $R$-module of finite length. Moreover the multiplication by $x$ induces an exact sequence

$$0 \rightarrow 0 :_M x \rightarrow M \xrightarrow{x} M \rightarrow M' \rightarrow 0$$

breaks into two short exact sequences $0 \rightarrow N \rightarrow M \rightarrow \bar{M} \rightarrow 0$, where $N = 0 :_M x$ and $\bar{M} = M/N$, and $0 \rightarrow \bar{M} \xrightarrow{x} M \rightarrow M' \rightarrow 0$.

The first of these sequences induces isomorphisms $\lim H^i_m(M/a^nM) \simeq \lim H^i_m(\bar{M}/a^n\bar{M})$ for all $i > 0$ and a short exact sequence

$$0 \rightarrow N \rightarrow \lim H^0_m(M/a^nM) \rightarrow \lim H^0_m(\bar{M}/a^n\bar{M}) \rightarrow 0$$

(cf 3.13). The second sequence induces a long exact sequence for the formal cohomology modules

$$\ldots \rightarrow \lim H^i_m(\bar{M}/a^n\bar{M}) \xrightarrow{x} \lim H^i_m(M/a^nM) \rightarrow \lim H^i_m(M'/a^nM') \rightarrow \ldots$$

(cf 3.11). With the isomorphisms above this proves the claim for $i > 0$. To this end one has to break up the long exact sequence into short exact sequences.

For the proof in the case $i = 0$, the only remaining case, consider the composite of the above short exact sequence with the previous one for $i = 0$. Then this completes the proof for $i = 0$. 

Another short exact sequence relates the $a$-formal cohomology to the $(a, xR)$-formal cohomology for any element $x \in \mathfrak{m}$. To be more precise:

**Theorem 3.15.** Let $x \in \mathfrak{m}$ denote an element of $(R, \mathfrak{m})$. For an ideal $a$ and a finitely generated $R$-module $M$ there is the long exact sequence

$$\ldots \rightarrow \text{Hom}(R_x, \lim H^i_m(M/a^nM)) \rightarrow \lim H^i_m(M/a^nM) \rightarrow \lim H^i_m(M/(a, x)^nM) \rightarrow \ldots$$

for all $i \in \mathbb{Z}$.

**Proof.** Without loss of generality (cf. 3.3) we may assume that $R$ admits a dualizing complex $D_R$. The Čech complex $\hat{C}_x$ of the single element $x$ is the fibre of the natural homomorphism $R \rightarrow R_x$. So there is a split exact sequence

$$0 \rightarrow R_x[-1] \rightarrow \hat{C}_x \rightarrow R \rightarrow 0.$$
Let \( \underline{x} \) denote a system of elements of \( \hat{R} \) such that \( \text{Rad} \ a = \text{Rad} \ \underline{x}R \). By tensoring the above short exact sequence of flat \( \hat{R} \)-modules with \( \hat{C}_\underline{x} \otimes \text{Hom}(M, D_{\hat{R}}) \) it provides an exact sequence of \( \hat{R} \)-complexes

\[
0 \to \hat{C}_\underline{x} \otimes \text{Hom}(M, D_{\hat{R}}) \otimes R_x[-1] \to \hat{C}_{\underline{x}, x} \otimes \text{Hom}(M, D_{\hat{R}}) \to \hat{C}_\underline{x} \otimes \text{Hom}(M, D_{\hat{R}}) \to 0.
\]

Notice that the above short exact sequence of complexes is split exact. Taking the long exact cohomology sequence it provides an exact sequence

\[
\cdots \to H^j_{(a,x\hat{R})}(\text{Hom}(M, D_{\hat{R}})) \to H^j_a(\text{Hom}(M, D_{\hat{R}})) \to H^j_a(\text{Hom}(M, D_{\hat{R}})) \otimes R_x \to \cdots
\]

for all \( j \in \mathbb{Z} \). By applying Matlis’ duality it provides the exact sequence of the statement (cf. 3.5). \( \square \)

As an application of Theorem 3.15 there is an exact sequence for the formal cohomology with respect to an ideal generated by a single element.

**Corollary 3.16.** Let \( x \in m \) denote an element. Let \( M \) be a finitely generated \( \hat{R} \)-module. Then there is a short exact sequence

\[
\cdots \to \text{Hom}(R_x, H^i_m(M)) \to H^i_m(M) \to \lim_{\longleftarrow} H^i_m(M/\hat{a}^nM) \to \cdots
\]

for all \( i \in \mathbb{Z} \).

**Proof.** The corollary is a consequence of Theorem 3.15 with the particular case \( a = 0 \). \( \square \)

### 4. Vanishing results

#### 4.1. On the 0-th formal cohomology.
Let \((R, m)\) denote a local ring. Let \( M \) be a finitely generated \( R \)-module. For an \( R \)-submodule \( N \) of \( M \) denote by \( N : M \langle m \rangle \) the ultimate constant \( R \)-module \( N : M \langle m \rangle \), \( n \) large.

Let \( 0 = \bigcap_{p \in \text{Ass} M} Z(p) \) denote a minimal primary decomposition of \( 0 \) in \( M \). Moreover, let \( a \) denote an ideal of \( \hat{R} \). Then define

\[
T_a(M) = \{ p \in \text{Ass} R M : \dim R/(a, p) = 0 \}.
\]

Furthermore, put

\[
u_M(a) = \bigcap_{p \in \text{Ass} R M \setminus T_a(M)} Z(p).
\]

Now it will be shown that \( u_M(a) \) plays an important rôle in order to understand the 0-th formal cohomology module. To this end denote by \( \hat{R} \) the completion of \( R \) and \( \hat{M} \simeq M \otimes \hat{R} \) the completion of the finitely generated \( R \)-module \( M \).

**Lemma 4.1.** With the previous notation we have:

(a) \( \bigcap_{n \geq 1} (a^n M : M \langle m \rangle) = u_M(a) \).

(b) \( \text{Ass}_R(u_M(a)) = T_a(M) \).

(c) \( \lim_{\longleftarrow} H^0_m(M/\hat{a}^nM) \simeq u_M(a \hat{R}) \).

**Proof.** The proof of (a) is easily seen because of

\[
\bigcap_{n \geq 1} (a^n M : M \langle m \rangle) = \bigcap_{p \in \text{Supp} M/\hat{a}^nM} \ker(M \to M_p)
\]

(cf. [18, (2.1)] for the details). Then the statement in (b) is a consequence of (a) (cf. 2.7).
In order to proof (c) first note that one may assume $M = \hat{M}$ and $R = \hat{R}$ as follows by passing to the completion (cf. [3.3]). But now $H^0_m(M/\mathfrak{a}^n M) \simeq a^n M :_M \langle m \rangle / \mathfrak{a}^n M$. So there is a short exact sequence of inverse systems

$$0 \to \{a^n M\}_{n \in \mathbb{N}} \to \{a^n M :_M \langle m \rangle\}_{n \in \mathbb{N}} \to \{H^0_m(M/\mathfrak{a}^n M)\}_{n \in \mathbb{N}} \to 0.$$ 

By passing to the projective limit it provides an injection

$$0 \to \bigcap_{n \geq 1} (a^n M :_M \langle m \rangle) \xrightarrow{\phi} \lim_{\leftarrow} H^0_m(M/\mathfrak{a}^n M).$$

In order to finish it will be enough to prove that $\phi$ is surjective. To this end let

$$\{y_n + \mathfrak{a}^n M\} \in \lim_{\leftarrow} H^0_m(M/\mathfrak{a}^n M),$$

where $y_n \in a^n M :_M \langle m \rangle$ for all $n \in \mathbb{N}$. This sequence defines an element $z \in \lim_{\leftarrow} M/\mathfrak{a}^n M = M$. Note that $M$ as an $m$-adically complete module is also $\mathfrak{a}$-adically complete (cf. [26, Ch. VIII]). That is, for every $n \in \mathbb{N}$ there exists an $n_0 \geq n$ such that $z - y_m \in \mathfrak{a}^n M$ for all $m \geq n_0$. Therefore $z \in \cap_{n \geq 1} (a^n M :_M \langle m \rangle)$, as required.

By view of [4.1] there is the following vanishing result for the 0-th formal cohomology.

**Corollary 4.2.** With the previous notation we have that $\lim_{\leftarrow} H^0_m(M/\mathfrak{a}^n M) = 0$ if and only if $\dim \hat{R}/(\mathfrak{a}\hat{R}, \mathfrak{p}) > 0$ for all $\mathfrak{p} \in \text{Ass}_{\hat{R}}\hat{M}$.

In particular, the vanishing $\lim_{\leftarrow} H^0_m(M/\mathfrak{a}^n M) = 0$ implies that $\text{depth } M > 0$.

**Proof.** It turns out that $\lim_{\leftarrow} H^0_m(M/\mathfrak{a}^n M) = 0$ if and only if $\text{Ass}_{\hat{R}}(u_{\hat{M}}(\mathfrak{a}\hat{R})) = \emptyset$. But this is equivalent to the statement (cf. [4.1]). In particular, $\lim_{\leftarrow} H^0_m(M/\mathfrak{a}^n M) = 0$ implies that $\hat{m} \notin \text{Ass}_{\hat{R}}\hat{M}$, whence $\text{depth } M > 0$. 

Next we want to extent the statement in [3.13]

**Corollary 4.3.** Let $(R, \mathfrak{m})$ denote a complete local ring. For a finitely generated $R$-module $M$ put $U = u_M(\mathfrak{a})$ and $\hat{M} = M/U$. Then:

(a) $\lim_{\leftarrow} H^0_m(M/\mathfrak{a}^n M) \simeq U$ and $\lim_{\leftarrow} H^0_m(\hat{M}/\mathfrak{a}^n \hat{M}) = 0$.

(b) $\lim_{\leftarrow} H^i_m(M/\mathfrak{a}^n M) \simeq \lim_{\leftarrow} H^i_m(\hat{M}/\mathfrak{a}^n \hat{M})$ for all $i \geq 1$.

**Proof.** For the proofs of the statements in (a) see [4.1]. Now observe that

$$\text{Supp } U \cap V(\mathfrak{a}) = (\cup_{\mathfrak{p} \in T(\mathfrak{m})}V(\mathfrak{p})) \cap V(\mathfrak{a}) \subseteq V(\mathfrak{m}).$$

By virtue of [3.13] this proves the isomorphisms in (b).

**4.2. A non-vanishing result.** The aim of this subsection will be to determine the integer

$$\sup\{ i \in \mathbb{Z} : \lim_{\leftarrow} H^i_m(M/\mathfrak{a}^n M) \neq 0 \}.$$ 

Here let $M$ be a finitely generated $R$-module. Let $\mathfrak{a}$ denote an ideal in the local ring $(R, \mathfrak{m})$. We start with an almost trivial observation.

**Proposition 4.4.** Let $\mathfrak{a}$ be an ideal such that $\dim M/\mathfrak{a} M = 0$. Then

$$\lim_{\leftarrow} H^i_m(M/\mathfrak{a}^n M) \simeq \begin{cases} 
0 & \text{for } i \neq 0 \\
M^\mathfrak{a} & \text{for } i = 0.
\end{cases}$$
Proof. It follows that $H^i_m(M/\mathfrak{a}^nM) = 0$ for all $i \neq 0$. Notice that $M/\mathfrak{a}M$ is an $R$-module of finite length. Furthermore, it provides $H^0_m(M/\mathfrak{a}^nM) \simeq M/\mathfrak{a}^nM$. Passing to the projective limit finishes the proof. \hfill \Box

Now the preparation for the first non-vanishing result is finished.

**Theorem 4.5.** Let $\mathfrak{a}$ denote an ideal of $(R, \mathfrak{m})$. Then
\[
\dim_R M/\mathfrak{a}M = \sup \{i \in \mathbb{Z} : \lim H^i_m(M/\mathfrak{a}^nM) \neq 0\}
\]
for a finitely generated $R$-module $M$.

**Proof.** Because of $\dim M/\mathfrak{a}^nM = \dim M/\mathfrak{a}M$ for all $n \in \mathbb{N}$ we first note that $H^i_m(M/\mathfrak{a}^nM)$ vanishes for all $i > \dim_R M/\mathfrak{a}M$ (cf. e.g. [11, Proposition 1.12]). Therefore
\[
\dim_R M/\mathfrak{a}M \geq \sup \{i \in \mathbb{Z} : \lim H^i_m(M/\mathfrak{a}^nM) \neq 0\}.
\]
Second note that we may assume the existence of a dualizing complex (cf. [3, 3]).

In order to prove the equality take $\mathfrak{p} \in \text{Supp}_R M \cap V(\mathfrak{a})$ such that $\dim R/\mathfrak{p} = \dim_R M/\mathfrak{a}M$. Then $\lim H^0_{\mathfrak{p}^nA}(M/\mathfrak{a}^nM_\mathfrak{p}) \neq 0$ (cf. [4, 4]). Observe that $M_\mathfrak{p}/\mathfrak{a}M_\mathfrak{p}$ is a zero-dimensional $R_\mathfrak{p}$-module. Therefore $\lim H^d_m R/\mathfrak{p}(M/\mathfrak{a}^nM) \neq 0$ (cf. [3, 7]). \hfill \Box

**Remark 4.6.** Another proof for the non-vanishing of $\lim H^d_m(M/\mathfrak{a}^nM)$, $d = \dim M/\mathfrak{a}M$, can be seen as follows. First note $\dim M/\mathfrak{a}^nM = d$ for all $n \in \mathbb{N}$. Then the short exact sequence
\[
0 \rightarrow \mathfrak{a}^nM/\mathfrak{a}^{n+1}M \rightarrow M/\mathfrak{a}^{n+1}M \rightarrow M/\mathfrak{a}^nM \rightarrow 0
\]
induces an epimorphism $H^d_m(M/\mathfrak{a}^{n+1}M) \rightarrow H^d_m(M/\mathfrak{a}^nM) \rightarrow 0$, of non-zero $R$-modules for all $n \in \mathbb{N}$. Remember that $\dim \mathfrak{a}^nM/\mathfrak{a}^{n+1}M \leq d$ and therefore $H^{d+1}_m(\mathfrak{a}^nM/\mathfrak{a}^{n+1}M) = 0$. Whence the inverse limit $\lim H^d_m(\mathfrak{a}^nM/\mathfrak{a}^nM)$ is not zero.

### 4.3. The formal grade.

Let $M$ denote a finitely generated $R$-module, where $(R, \mathfrak{m})$ is a local ring. For an ideal $\mathfrak{a}$ it is shown that $\sup \{i \in \mathbb{Z} : \lim H^i_m(M/\mathfrak{a}^nM) \neq 0\}$ is equal to $\dim_R M/\mathfrak{a}M$ (cf. [4, 5]). Now we start to investigate the infimum for the non-vanishing.

**Definition 4.7.** For an ideal $\mathfrak{a}$ of $R$ define the formal grade, $f\text{grade}(\mathfrak{a}, M)$, by
\[
f\text{grade}(\mathfrak{a}, M) = \inf \{i \in \mathbb{Z} : \lim H^i_m(M/\mathfrak{a}^nM) \neq 0\}.
\]
Note that the ordinary grade is defined by $\text{grade}(\mathfrak{a}, M) = \inf \{i \in \mathbb{Z} : H^i_\mathfrak{a}(M) \neq 0\}$ (cf. [11]).

The notion of formal grade was introduced by Peskine and Szpiro (cf. [17]). Not so much is known about it. We continue here with a few more investigation on the formal grade. In the following lemma (cf. [4, 3]) there is a summary of basic results.

**Lemma 4.8.** Let $\mathfrak{a}$ denote an ideal of $(R, \mathfrak{m})$. Let $M$ be a finitely generated $R$-module.

(a) $f\text{grade}(\mathfrak{a}, M/\mathfrak{x}M) \geq f\text{grade}(\mathfrak{a}, M) - 1$, provided $\mathfrak{x} \notin \mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}_R M \setminus \{\mathfrak{m}\}$.

(b) $f\text{grade}(\mathfrak{a}, M) \leq \min \{\text{depth}_RM, \dim M/\mathfrak{a}M\}$.

(c) Suppose that $R$ possesses a dualizing complex. Then
\[
f\text{grade}(\mathfrak{a}, M) \leq f\text{grade}(\mathfrak{a}, R_{\mathfrak{p}}, M_{\mathfrak{p}}) + \dim R/\mathfrak{p}
\]
for all $\mathfrak{p} \in \text{Supp} M \cap V(\mathfrak{a})$.

(d) Suppose that $R$ is a Gorenstein ring. Then $f\text{grade}(\mathfrak{a}, R) + \text{cd}(\mathfrak{a}, R) = \dim R$. 


Proof. By virtue of the short exact sequences in [3,14] it follows that $\text{fgrade}(a, M/xM) \geq \text{fgrade}(a, M) - 1$. That is, the statement (a) is shown.

In order to prove (b) first note $\text{fgrade}(a, M) \leq \dim M/aM$ (cf. [4,5]). Next we prove $\text{fgrade}(a, M) \leq \text{depth}_R M$ by an induction on $t = \text{fgrade}(a, M)$. In case $t = 0$ the claim holds trivially. So let $t \geq 1$. Then $\lim H^0_0(M/a^nM) = 0$ by the definition of the formal grade. Therefore there is an $M$-regular element $x \in m$ (cf. [4,2]). Whence

$t - 1 \leq \text{fgrade}(a, M/xM) \leq \text{depth}_M M = \text{depth}_M M - 1$

by the aid of (a) and the induction hypothesis. So the proof of (b) is complete.

For the proof of (c) let $t = \text{fgrade}(aR_p, M_p)$. Then $\text{fgrade}(a, M) \leq t + \dim R/p$, (cf. [3,7]).

Let $R$ be a Gorenstein ring. Then $R[\dim R] \simeq D_R$ for the dualizing complex $D_R$ (cf. [13]). Therefore $\lim H^0_0(R/a^n) \simeq \text{Hom}(H^{\dim R}_a(R), E)$ (cf. [3,5]), which proves (d). \hfill \Box

It is a difficult problem to determine the cohomological dimension $\text{cd}(a, R)$. So the above result (d) in [4,8] illustrates the difficulty in order to calculate $\text{fgrade}(a, R)$. In the next result there is a generalization of [4,8] (d) for an arbitrary finitely generated module $R$-module $M$.

Theorem 4.9. Let $(R, m)$ denote a local ring with a dualizing complex $D_R$. Let $a$ denote an ideal of $R$. Then

$$\text{fgrade}(a, M) = \inf \{i - \text{cd}(a, K^i(M)) : i = 0, \ldots, \dim M\}$$

for a finitely generated $R$-module $M$.

Proof. By the definition of the formal grade and Theorem [3,5] there is the equality

$$\text{fgrade}(a, M) = -\sup \{i \in \mathbb{Z} : H^i_a(\text{Hom}(M, D_R)) \neq 0\}.$$

Let $X$ denote an arbitrary complex of $R$-modules. Put $s(X) = \sup \{i \in \mathbb{Z} : H^i(X) \neq 0\}$.

Let $x = x_1, \ldots, x_r$ denote a system of elements of $R$ generating the ideal $a$. Let $\check{C}_x$ denote the corresponding Čech complex. Then

$$H^i_a(\text{Hom}(M, D_R)) \simeq H^i(\check{C}_x \otimes \text{Hom}(M, D_R))$$

for all $i \in \mathbb{Z}$ (cf. [20 Theorem 3.2]). Therefore, it will be enough to compute

$$s(\check{C}_x \otimes \text{Hom}(M, D_R)).$$

Since $\text{Hom}(M, D_R)$ is a bounded complex with finitely generated cohomology modules and $\check{C}_x$ is a bounded complex of flat $R$-modules it follows that

$$s(\check{C}_x \otimes \text{Hom}(M, D_R)) = \sup \{s(\check{C}_x \otimes H^i(\text{Hom}(M, D_R))) + i : i \in \mathbb{Z}\}$$

(cf. [8 Proposition 2.5]). Because of $K^{-i}(M) = H^i(\text{Hom}(M, D_R)), i \in \mathbb{Z}$, it turns out that $s(\check{C}_x \otimes H^i(\text{Hom}(M, D_R))) = \text{cd}(a, K^{-i}(M))$ by the definition of the cohomological dimension. Whence the claim is shown to be true. \hfill \Box

There is an expression of the cohomological dimension in terms of the cohomological dimension of the minimal primes (cf. Corollary [2,2]. One might expect a similar result for the formal grade expressing $\text{fgrade}(a, M)$ in terms of the minimum of $\text{fgrade}(a, R/p)$, where the minimum is taken over all $p \in \text{Min} M$ or $p \in \text{Ass} M$. This is not the case as the following example shows.
Example 4.10. Let \((R, m)\) denote a \(d\)-dimensional complete local domain such that \(H^i_m(R) = 0\) for all \(i \neq 1, d, H^1_m(R) \simeq k\) and \(d \geq 4\). Such rings exist. Let \(D\) denote the global transform of \(R\). Then \(D\) is a finitely generated \(R\)-module with \(H^i_m(D) = 0\) for all \(i \neq d\). Then \(K(R) \simeq K(D)\) as easily seen. Now choose \(\{x, y\}\) a \(K(D)\)-regular sequence and \(a = (x, y)R\). It follows that \(\text{fgrade}(a, D) = d - 2, \text{fgrade}(a, R) = 1\) (cf. \[4.9\]), while \(\text{Ass} R = \text{Ass} D = \{(0)\}\).

Moreover the example also shows that there are local rings such that \(\text{fgrade}(a, R) \neq \dim R - \text{cd}(a, K(R))\). But in any case there is the following bound for the formal grade.

Corollary 4.11. Let \(a\) be an ideal of the local ring \((R, m)\). Then

\[
\text{fgrade}(a, M) \leq \dim M - \text{cd}(a, M)
\]

for a finitely generated \(R\)-module \(M\).

Proof. By Corollary \[2.2\] there exists a prime ideal \(p \in \text{Ass}_R M\) such that \(\text{cd}(a, M) = \text{cd}(a, R/p)\). Moreover, it follows that \(p \in \text{Ass} K^i(M)\) for a certain \(0 \leq i \leq \dim M\), (cf. Proposition \[2.6\]). But this implies \(\text{cd}(a, R/p) \leq \text{cd}(a, K^i(M))\) as it is again a consequence of Corollary \[2.2\]. By Theorem \[4.9\] this implies that

\[
\text{fgrade}(a, M) \leq \dim M - \text{cd}(a, K^i(M)) \leq \dim M - \text{cd}(a, M),
\]

as required. \(\square\)

Because \(\text{height}_M a \leq \text{cd}(a, M)\) it follows that the bound in Corollary \[4.11\] is in fact an improvement of the inequality \(\text{fgrade}(a, M) \leq \dim M/aM\) (cf. Theorem \[4.5\]).

Another difficulty about the formal grade is to characterize the equality in \[4.8\](a). This has to do with a lack of information about the \(R\)-module structure of \(\varprojlim H^i_m(M/a^nM), i \in \mathbb{Z}\).

Theorem 4.12. Let \(M\) be a finitely generated \(R\)-module. Then

\[
\dim \hat{R}/(a\hat{R}, p) \geq \text{fgrade}(a, M)
\]

for all \(p \in \text{Ass} \hat{M}\).

Proof. Without loss of generality one may assume that \(R = \hat{R}\) (cf. \[3.3\]). We proceed by induction on \(t = \text{fgrade}(a, M)\). First consider the case of \(t = 1\). By our assumption

\[
\text{Supp}_R u_M(a) = \emptyset
\]

(cf. \[4.1\]). But \(\text{Supp}_R u_M(a) = \bigcup_{p \in \text{Supp}(M)} V(p)\) (cf. \[4.1\]). This implies that \(\dim \hat{R}/(a, p) \geq 1\) for all \(p \in \text{Ass}_R M\).

Now let \(t > 1\), i.e. in particular \(\dim \hat{R}/(a, p) \geq 1\) for all \(p \in \text{Ass}_R M\). By prime avoidance arguments one may choose an element \(x \in m\) which forms a parameter for all \(R\)-modules \(R/(a, p)\), where \(p \in \text{Ass} M\).

The long exact sequence

\[
\ldots \rightarrow \text{Hom}(R_x, \varprojlim H^i_m(M/a^nM)) \rightarrow \varprojlim H^i_m(M/a^nM) \rightarrow \varprojlim H^i_m(M/(a, x)^nM) \rightarrow \ldots
\]

(cf. \[3.15\]) provides that \(\varprojlim H^i_m(M/(a, x)^nM) = 0\) for all \(i < t - 1\). Therefore

\[
\dim \hat{R}/(a, xR, p) \geq t - 1\]

for all \(p \in \text{Ass} M\) as a consequence of the the inductive hypothesis.

By the choice of \(x \in m\) as a parameter for all \(R/(a, p), p \in \text{Ass} M\), this proves that \(\dim \hat{R}/(a, p) \geq t\) for all \(p \in \text{Ass} M\). This completes the inductive step. \(\square\)
In general the equality in Theorem 4.12 does not hold. In fact, this has to do with certain connectedness properties studied in more detail in the next section.

Example 4.13. Let $R = k[[x_1, x_2, x_3, x_4]]$ denote the formal power series ring in four variables over a field $k$. Put $\mathfrak{c} = (x_1, x_2)R \cap (x_3, x_4)R$. Then $\text{fgrade}(\mathfrak{c}, R) = 1$ (cf. Example 5.2), while $\dim R/\mathfrak{c} = 2$.

We will continue here with another estimate of the formal grade related to the cohomological dimension of certain associated prime ideals.

**Theorem 4.14.** Let $(R, m)$ be a local ring. Let $M$ denote a finitely generated $R$-module. Then

$$\dim \hat{R}/p \geq \text{cd}(a \hat{R}, \hat{R}/p) + \text{fgrade}(a, M)$$

for all $p \in \text{Ass}_{\hat{R}} \hat{M}$.

**Proof.** As mentioned above we may assume $R = \hat{R}$ as follows by passing to the completion (cf. 3.3). Now let $p \in \text{Ass} M$ be an associated prime ideal with $\dim R/p = i$ for a certain $0 \leq i \leq \dim M$. That is

$$p \in (\text{Ass} M)_i = (\text{Ass} K^i(M))_i$$

(cf. 2.6). Moreover, it follows that $\text{Supp} R/p \subseteq \text{Supp} K^i(M)$. Therefore (cf. 2.1) we see that $\text{cd}(a, R/p) \leq \text{cd}(a, K^i(M))$.

By the assumption and the conclusion above it follows

$$i - \text{cd}(a, R/p) \geq i - \text{cd}(a, K^i(M)) \geq \text{fgrade}(a, M)$$

(cf. 4.9). Because of $i = \dim R/p$ this finishes the proof. \hfill $\square$

As $\hat{R}/p$ is a complete local domain it is a catenary ring and therefore

$$\dim \hat{R}/p = \dim \hat{R}/(a \hat{R}, p) + \text{height}(a \hat{R}, p)/p.$$

Moreover $\text{height}(a \hat{R}, p)/p \leq \text{cd}(a \hat{R}, \hat{R}/p)$. So, Theorem 4.14 is in fact a sharpening of Theorem 4.12.

5. Connectedness properties

5.1. **The Mayer-Vietoris sequence.** As it is well-known (cf. e.g. [4, Section 19], [15] and [20]) the Mayer-Vietoris sequence in local cohomology is an important tool for connectedness phenomena. Here we want to continue with a variant of the Mayer-Vietoris sequence for formal cohomology.

**Theorem 5.1.** Let $a, b$ two ideals of a local ring $(R, m)$. For a finitely generated $R$-module $M$ there is the long exact sequence

$$\ldots \rightarrow \lim H^i_m(M/(a \cap b)^nM) \rightarrow \lim H^i_m(M/a^nM) \oplus \lim H^i_m(M/b^nM) \rightarrow \lim H^i_m(M/(a, b)^nM) \rightarrow \ldots,$$

where $i \in \mathbb{Z}$. 
Proof. Let \( n \in \mathbb{Z} \) denote an integer. Then there is the following natural exact sequence
\[
0 \to M/(a^n M \cap b^n M) \to M/a^n M \oplus M/b^n M \to M/(a^n, b^n)M \to 0.
\]
Now the long exact local cohomology sequence provides by passing to the projective limit the following long exact cohomology sequence
\[
\ldots \to \lim H^i_m(M/(a^n M \cap b^n M)) \to \lim H^i_m(M/a^n M) \oplus \lim H^i_m(M/b^n M) \to \lim H^i_m(M/(a^n, b^n)M) \to \ldots.
\]
Notice that the projective limit on projective systems of Artinian modules is exact.

Now we observe that the \((a, b)\)-adic filtration is equivalent to the filtration \( \{(a^n, b^n)M\}_{n \in \mathbb{N}} \).

In order to finish the proof we have to show that the \((a \cap b)\)-adic filtration on \( M \) is equivalent to the filtration \( \{(a \cap b^n)M\}_{n \in \mathbb{N}} \) (cf. \([3, \text{Ch. III, §3, Cor. 1}]\)).

To this end first note that \((ab)^n M \subseteq (a^n \cap b^n)M \subseteq a^n M \cap b^n M \) for all \( n \in \mathbb{N} \). Let \( m \in \mathbb{N} \) denote a given integer. By the Artin-Rees Lemma (cf. \([3, \text{Ch. III, §3, Cor. 1}]\)) there exists an \( k \in \mathbb{N} \) such that \( a^{n+k} b^{m-k} N \subseteq a^{n-k} b^{-m+n} N \) for all \( n \geq k \). Since the \(ab\)-adic and the \( a \cap b \)-adic topology on \( M \) are equivalent this finishes the proof.

The above result (cf. \([5, \text{I}]\)) provides an example related to the supports of formal cohomology.

Example 5.2. Let \( k \) be a field. Let \( R = k[[x_1, x_2, x_3, x_4]] \) denote the formal power series ring in four variables over \( k \). Put \( a = (x_1, x_2)R \) and \( b = (x_3, x_4)R \). Then the Mayer-Vietoris sequence provides the following two isomorphisms
\[
R \simeq \lim H^1_m(R/(a \cap b)^n) \quad \text{and} \quad \lim H^2_m(R/(a \cap b)^n) \simeq \lim H^2_m(R/a^n) \oplus \lim H^2_m(R/b^n).
\]
To this end remark that \((a, b)\) is the maximal ideal of the complete local ring \( R \). Therefore \( \text{Supp} H^1_m(R/(a \cap b)^n) = \text{Spec} R \), while \( \dim R/a \cap b = 2 \).

Note that the example was introduced by Hartshorne (cf. \([12]\)). In the following we want to extend these considerations to a more subtle investigation.

5.2. On the connectedness. Next let us summarize a few technical preparations for the connectedness results. Let \((R, m)\) denote a local ring.

Lemma 5.3. Let \( M \) be a finitely generated \( R \)-module. Let \( a, b \) denote two ideals of \( R \). Suppose that \( \lim H^1_m(M/(a \cap b)^n M) = 0 \). Then \( T_{aR}(M) \cup T_{bR}(M) = T_{(a, b)R}(M) \).

Proof. First remember that we may assume that \((R, m)\) is a complete local ring (cf. \([3, \text{C}3]\)). With the notation introduced in Section 4.1 it is clear that the left hand side of the statement is contained in the right hand side.

In order to prove the reverse containment relation the Mayer-Vietoris sequence (cf. \([5, \text{I}]\)) provides an epimorphism
\[
u_M(a) \oplus u_M(b) \to u_M(a, b) \to 0
\]
(use Lemma \([4, \text{I}]\)). Now let \( p \in \text{Ass } u_M(a, b) \), i.e. \( p \in \text{Ass } M \) and \( \dim R/(p, a, b) = 0 \). In particular it follows that \( p \in \text{Supp } u_M(a, b) \) and therefore \( p \in \text{Supp } u_M(a) \oplus u_M(b) \). Without loss of generality we may conclude that \( p \in \text{Supp } u_M(a) \). So there exists a prime ideal \( q \in \text{Ass } u_M(a) \) with \( q \subseteq p \). Whence \( q \in \text{Ass } M \) and \( \dim R/(q, a) = 0 \) (cf. Lemma \([4, \text{I}]\)). Because of \( p \in \text{Ass } M \) and \( q \subseteq p \) this implies \( p \in \text{Ass } u_M(a) \), which finishes the proof. \( \square \)
As another consequence of the Mayer-Vietoris sequence there is the following connectedness result. To this end an $R$-module $M$ is called indecomposable whenever $M = N_1 \oplus N_2$ implies either $M = N_1$ and $N_2 = 0$ or $N_1 = 0$ and $M = N_2$.

**Lemma 5.4.** Let $\hat{M}$ denote an indecomposable $\hat{R}$-module. Suppose that $\text{fgrade}(a, M) \geq 2$ for an ideal $a$ of $R$. Then $\text{Supp}_R \frac{M}{a\hat{M} \setminus \{\hat{m}\}}$ is connected.

**Proof.** Because of $\text{fgrade}(a, M) = \text{fgrade}(a\hat{R}, \hat{M})$ (cf. [3,3]) we may assume that $R$ is a complete local ring. Now suppose that $\text{Supp}_R \frac{M}{aM \setminus \{m\}}$ is disconnected. Then there are two ideals $b, c$ of $\hat{R}$ satisfying the following properties
1. $\text{Rad}(a, \text{Ann} M) = \text{Rad}(b \cap c),$
2. $(b, c)$ is an $m$-primary ideal, and
3. neither $b$ nor $c$ is an $m$-primary ideal.

Then the Mayer-Vietoris sequence (cf. [5,1]) provides an isomorphism
$$\varprojlim H^0_m(M/b^n M) \oplus \varprojlim H^0_m(M/c^n M) \simeq \varprojlim H^0_m(M/(b, c)^n M).$$

But $(b, c)$ is an $m$-primary ideal and therefore $\varprojlim H^0_m(M/(b, c)^n M) \simeq M$ (cf. [4,4]). By the indecomposability of $M$ it follows – say –
$$\varprojlim H^0_m(M/b^n M) \simeq M \quad \text{and} \quad \varprojlim H^0_m(M/c^n M) = 0.$$

Therefore, by [4,1] it turns out that $\dim R/(p, b) = 0$ for all $p \in \text{Ass} M$. Because of
$$m = \bigcap_{p \in \text{Ass} M} \text{Rad}(b, p) = \text{Rad} b$$
it yields that $b$ is an $m$-primary ideal. This is a contradiction. □

One might observe that for the proof of $\text{Rad} b = m$ it will be enough to consider only the minimal prime ideals $p \in \text{Ass} M$. This is a corner stone for a generalization in the next subsection.

The indecomposibility of $M$ in [5,4] is essential as the following example shows.

**Example 5.5.** With the notation of Example [5,2] put $M = R/a \oplus R/b$. Let $c = a \cap b$. Then $\text{fgrade}(c, M) = \text{depth} M = 2$, while $\text{Supp}_R M/cM \setminus \{m\}$ is not connected. Recall that $c = \text{Ann} M$.

We apply the previous Lemma in order to derive a corresponding connectedness result related to the cohomological dimension. To this end we introduce the notion $\text{Assh} M = \{p \in \text{Ass}_M : \dim R/p = \dim M\}$ for a finitely generated $R$-module.

**Theorem 5.6.** Let $(R, m)$ denote a local ring. Let $a$ be an ideal of $R$. Suppose that
(a) $\text{Ass} \hat{R} = \text{Assh} \hat{R},$
(b) $H^\text{dim} R_m(R)$ is indecomposable,
(c) $\text{cd}(a, R) \leq \dim R - 2.$

Then $V(a\hat{R}) \setminus V(m\hat{R})$ is connected.

**Proof.** Because of $\text{cd}(a, R) = \text{cd}(a\hat{R}, \hat{R})$ one may assume that $R$ possesses a dualizing complex (cf. [3,3]). Observe that $H^d_m(R) \simeq H^d_m(\hat{R}), d = \dim R = \dim \hat{R}$.
Let \( \mathbb{Q}(R) \) denote the total ring of quotients of \( R \). Then there exists a birational extension ring \( R \subset S \subset \mathbb{Q}(R) \) such that \( S \) is a finitely generated \( R \)-module and satisfies the condition \( S_2 \) (cf. [21 5.3]). To this end we have to use (a). Whence it follows that

\[
\text{cd}(a, K^i(S)) \leq \dim K^i(S) \leq i - 2
\]

for all \( 0 \leq i < \dim S = d \) (cf. 2.6). Moreover, the short exact sequence

\[
0 \to R \to S \to S/R \to 0
\]

provides the vanishing \( H^i_a(S) = 0 \) for all \( i > d - 2 \). To this end observe that \( \dim S/R \leq d - 2 \) (cf. [21 5.3]) and that \( H^i_a(R) = 0 \) for \( i > d - 2 \). Therefore \( \text{cd}(a, S) \leq \dim S - 2 \). Since \( \text{Supp} S = \text{Supp} K(S) \) we obtain \( \text{cd}(a, S) = \text{cd}(a, K(S)) \) (cf. 2.2). But then it follows that

\[
\text{fgrade}(a, S) = \min\{i - \text{cd}(a, K^i(S)) : i = 0, \ldots, \dim S \} \geq 2
\]

(cf. 4.9). In order to apply 5.4 we show that \( S \) as an \( R \)-module is indecomposable.

Assume the contrary, i.e. \( S \simeq S_1 \oplus S_2 \) for two non-zero \( R \)-modules \( S_i, i = 1, 2 \). Clearly \( \dim S_i = d, i = 1, 2 \). This follows since \( S \) has the property that \( \dim S/p = \dim S \) for all \( p \in \text{Supp} R \) (cf. [21]).

By considering the local cohomology modules we see that

\[
H^d_m(R) \simeq H^d_m(S) \simeq H^d(S_1) \oplus H^d_m(S_2), \quad \text{and} \quad H^d_m(S_i) \neq 0, \; i = 1, 2.
\]

Notice that \( \dim S/R \leq d - 2 \). Because \( H^d_m(R) \) is supposed to be indecomposable by condition (b) this is a contradiction.

So, the previous result (cf. 5.4) finally implies that

\[
\text{Supp}_R S/\mathfrak{a}S \setminus V(\mathfrak{m}) = V(\mathfrak{a}) \setminus V(\mathfrak{m})
\]

is connected. To this end remember that \( \text{Supp}_R S = \text{Spec} R \). \( \square \)

We note that Theorem 5.6 extends [19 2.27], where the condition \( S_2 \) is assumed for \( R \) in order to derive the connectedness property. Note that the indecomposability of \( H^\dim R_m(R) \) was studied by Hochster and Huneke (cf. [15 Theorem 4.1]).

5.3. The connectedness dimension. Next let us summarize a few technical preparations for further connectedness results. Let \( (R, \mathfrak{m}) \) denote a local ring.

**Definition 5.7.** For an \( R \)-module \( M \) define

\[
c(M) = \min\{\dim R/\mathfrak{c} : V(\mathfrak{c}) \subseteq \text{Supp} M \text{ and } \text{Supp} M \setminus V(\mathfrak{c}) \text{ is disconnected}\}.
\]

We refer to [4 Section 19] for more details about the definition. Here we notice that \( c(M) \leq \dim M \) with equality provided \( \text{Supp} M \) is irreducible. Moreover \( c(M) \geq 0 \).

Now let \( M \) be a finitely generated \( R \)-module. Let \( \mathfrak{p}_1, \ldots, \mathfrak{p}_r \) denote the distinct minimal prime ideals of \( \text{Supp} M = V(\text{Ann}_RM) \).

Let \( S(r) \) denote the set of all ordered pairs \( (A, B) \) of non-empty subsets of \( \{1, \ldots, r\} \) such that \( A \cup B = \{1, \ldots, r\} \).

**Lemma 5.8.** Let \( M \) be a finitely generated \( R \)-module. Then

\[
c(M) = \min\{\dim R/((\cap_{i \in A}\mathfrak{p}_i), (\cap_{j \in B}\mathfrak{p}_j)) : (A, B) \in S(r)\}.
\]

**Proof.** The result is a module theoretic version of [4 19.1.15]. For the details of the proof we refer to [4 19.1.15 and 19.2.5]. To this end observe that \( \text{Supp} M = V(\text{Ann} M) \). \( \square \)
Next we want to continue with a the behavior of the connectedness dimension by a generic hyperplane section. To be more precise:

**Lemma 5.9.** Let \( M \) denote a finitely generated \( R \)-module with \( c(M) > 0 \). Then there exists an element \( x \in m \) such that \( c(M) \geq c(M/xM) + 1 \).

**Proof.** Let \( p_1, \ldots, p_r \) denote the distinct minimal prime ideals of \( V(\text{Ann}_R M) \). Then \( c(M) = \dim R/e > 0 \) for an ideal \( e = (\cap_{i \in A} p_i, \cap_{j \in B} p_j) \) with a certain pair \( (A, B) \in S(r) \) (cf. Lemma 5.8). Now choose \( x \in m \) as a parameter of \( R/e \), i.e. \( c(M) - 1 = \dim R/(xR, e) \).

Next observe that \( V(\langle x, a \cap b \rangle = V(\langle x, a \rangle \cup V(\langle x, b \rangle = V(\langle x, a \rangle \cap \langle x, b \rangle) \) for two ideals \( a, b \) of \( R \). Then there are the following equalities for the radical ideals

\[
\text{Rad}(xR, e) = \text{Rad}(\cap_{i \in A} p_i, xR), \cap_{j \in B} (p_j, xR)) = \text{Rad}(\cap_{i \in A} \text{Rad}(p_i, xR), \cap_{j \in B} \text{Rad}(p_j, xR))
\]

as easily seen. Let \( \mathcal{P}_1, \ldots, \mathcal{P}_s \) denote the distinct minimal prime ideals of \( V(xR, \text{Ann}_R M) \). By easy computations it follows that

\[
V(\text{Ann}_R M, xR) = V(\cap_{i=1}^s p_i) \cap V(xR) = V(\cap_{i=1}^s (p_i, xR)).
\]

Whence, the set of prime ideals \( \mathcal{P}_1, \ldots, \mathcal{P}_s \) coincides with the set of minimal prime ideals of the ideal \( \cap_{i=1}^s (p_i, xR) \) and

\[
\cap_{i=1}^s \mathcal{P}_i = \text{Rad}(\cap_{i=1}^s (p_i, xR)) = \text{Rad}(\text{Ann}_R M, xR).
\]

By avoiding redundant components in \( \cap_{i \in A} \text{Rad}(p_i, xR) \) and \( \cap_{j \in B} \text{Rad}(p_j, xR) \) resp. we derive a representation

\[
\cap_{i \in A} \text{Rad}(\langle p_i, xR \rangle) = \cap_{i \in \mathcal{A}} \mathcal{P}_i \quad \text{and} \quad \cap_{j \in B} \text{Rad}(\langle p_j, xR \rangle) = \cap_{j \in \mathcal{B}} \mathcal{P}_j
\]

for an ordered pair \( (\mathcal{A}, \mathcal{B}) \in S(s) \). This means that

\[
c(M) - 1 = \dim R/(xR, e) = \dim R/(\cap_{i \in \mathcal{A}} \mathcal{P}_i, \cap_{j \in \mathcal{B}} \mathcal{P}_j) \geq c(M/xM),
\]

as required. Note that the dimension does not change by passing to the radical. \( \square \)

As a consequence of the Lemmas 5.9 and 2.3 one has the following result, relating the connectedness dimension of \( \tilde{R}/a \) and the cohomological dimension.

**Corollary 5.10.** Let \( a \) be an ideal of a local ring \( (R, m) \). Suppose that \( H^d_m(R) \) is indecomposable and \( \text{Ass } \tilde{R} = \text{Assh } \hat{R} \). Then \( c(\tilde{R}/a\hat{R}) \geq \dim \tilde{R} - \text{cd}(a, R) - 1 \).

**Proof.** First note that we may assume that \( R = \hat{R} \), that is \( R \) is complete (cf. 3.3). For the proof we proceed by an induction on \( c(R/a) \). In the case of \( c(R/a) = 0 \) the result is a consequence of 5.6. So assume that \( c(R/a) > 0 \). Then there exists an element \( x \in m \) such that \( c(R/a) \geq c(R/(a, xR)) + 1 \) (cf. 5.9). By the inductive hypothesis

\[
c(R/a) - 1 \geq c(R/(a, xR)) \geq \dim R - \text{cd}((a, xR), R) - 1.
\]

On the other hand \( \text{cd}((a, xR), R) \leq \text{cd}(a, R) + 1 \) (cf. 2.3). Now this completes the inductive step by putting together these inequalities. \( \square \)

In their paper [7] Theorem 3.4] the authors claimed the validity of 5.10 without the condition that \( H^d_m(R) \) is indecomposable. This is not correct as follows by Example 5.2. To this end let \( c = a \cap b \). Then \( \text{cd}(c, R/c) = 0 \), \( \dim R/c = 2 \), \( c(R/c) = 0 \). Moreover \( H^2_m(R/c) \simeq H^2_m(R/a) \oplus H^2_m(R/b) \) and both of the direct summands do not vanish.
5.4. Formal cohomology and connectedness. In this subsection we relate the vanishing of the formal cohomology to the connectedness properties.

Theorem 5.11. Let \( a \) denote an ideal of a local ring \((\hat{R}, m)\). Let \( M \) be a finitely generated \( R \)-module. Then \( c(\hat{R}/(a\hat{R}, p)) \geq \text{fgrade}(a, M) - 1 \) for all \( p \in \text{Ass}_{\hat{R}} \hat{M} \).

Proof. First of all we note that Corollary 5.10 applied to \( a \in \hat{R}/p, p \in \text{Ass}_{\hat{R}} \hat{M} \), provides the following inequality
\[
c(\hat{R}/(a\hat{R}, p)) \geq \dim \hat{R}/p - \text{cd}(a\hat{R}, \hat{R}/p) - 1.
\]
To this end we have to prove that \( H^i_m(\hat{R}/p), i = \dim \hat{R}/p \) is indecomposable. By local duality it will be enough to prove that the canonical module \( K(\hat{R}/p) \) is an indecomposable \( \hat{R}/p \)-module. Since \( \hat{R}/p \) is a domain and since \( K(\hat{R}/p) \) is a torsion-free \( \hat{R}/p \)-module of rank 1, it is indecomposable.

On the other hand (cf. 4.12) it follows that
\[
\dim \hat{R}/p - \text{cd}(a\hat{R}, \hat{R}/p) \geq \text{fgrade}(a, M).
\]
Putting together both of the estimates the desired inequality is shown to be true. \( \Box \)

As a particular case of Theorem 5.11 there is the following corollary.

Corollary 5.12. Let \( a \) denote an ideal of a local ring \((R, m)\). Let \( M \) be a finitely generated \( R \)-module. Suppose that \( \lim H^i_m(M/a^nM) = 0 \) for \( i \leq 1 \). Then \( V(a\hat{R}, p) \setminus V(m) \) is connected for all \( p \in \text{Ass}_{\hat{R}} \hat{M} \).

Proof. As follows by the definitions the claim is a particular case of 5.11. To this end recall that \( \text{fgrade}(a, M) \geq 2 \).

It is noteworthy to remark that the converse of the previous results are not true.

Example 5.13. With the notion of 5.2 put \( M = R/c, c = a \cap b \). Then \( V(c, p) \setminus V(m) \) is connected for all \( p \in \text{Ass} M \), while
\[
\lim H^1_m(M/c^nM) \simeq H^1_m(M) \simeq R/m,
\]
as it is easily seen.

As further application of the results of this and the previous subsection there is another estimate of the formal grade, more in the sense of Theorem 5.4

Corollary 5.14. Let \( M \) denote a finitely generated \( R \)-module, where \((R, m)\) is a local ring. Suppose that

(a) \( \text{Ass}_{\hat{R}} \hat{M} = \text{Assh}_{\hat{R}} \hat{M} \) and
(b) \( H^d_m(R/\text{Ann}_R M), d = \dim M, \) is indecomposable.

Then \( c(\hat{M}/a\hat{M}) \geq \text{fgrade}(a, M) - 1 \).

Proof. Without loss of generality we may assume that \( R \) is a complete local ring (cf. 3.3). Moreover, by the definition it follows that \( c(M/aM) = c(R/(a, \text{Ann}_R M)) \). The assumption (a) implies that \( \text{Ass} R/\text{Ann}_R M = \text{Assh} R/\text{Ann}_R M \). Because \( H^d_m(R/\text{Ann}_R M) \) is indecomposable we may apply 5.4 so that
\[
c(R/(a, \text{Ann}_R M)) \geq \dim R/\text{Ann}_R M - \text{cd}(a, R/\text{Ann}_R M) - 1.
\]
But now $\dim M = \dim R/\Ann_R M$. Furthermore $\cd(a, M) = \cd(a, R/\Ann_R M)$ (cf. 2.2). Because of $\dim M - \cd(a, M) \geq \fgrade(a, M)$ (cf. 4.11) this finishes the proof. □

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