Nonlinear superhorizon curvature perturbation in generic single-field inflation

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We develop a theory of nonlinear cosmological perturbations on superhorizon scales for generic single-field inflation. Our inflaton is described by the Lagrangian of the form $W(X, \phi) - G(X, \phi) \nabla \phi$ with $X = -\partial^\mu \phi \partial_\mu \phi/2$, which is no longer equivalent to a perfect fluid. This model is more general than k-inflation, and is called G-inflation. A general nonlinear solution for the metric and the scalar field is obtained at second order in gradient expansion. We derive a simple master equation governing the large-scale evolution of the nonlinear curvature perturbation. It turns out that the nonlinear evolution equation is deduced as a straightforward extension of the corresponding linear equation for the curvature perturbation on uniform $\phi$ hypersurfaces.

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I. INTRODUCTION

Non-Gaussianity in primordial fluctuations is one of the most powerful tools to distinguish different models of inflation (see, e.g., Ref. [1] and references therein). To quantify the amount of non-Gaussianity and clarify non-Gaussian observational signatures, it is important to develop methods to deal with nonlinear cosmological perturbations. Second-order perturbation theory [2–4] is frequently used for this purpose. Spatial gradient expansion is employed as well in the literature [5–27]. The former can be used to describe the generation and evolution of primordial perturbations inside the horizon, while the latter can deal with the classical nonlinear evolution after horizon exit. Therefore, it is important to develop both methods and to use them complementarily.

In this paper, we are interested in the classical evolution of nonlinear cosmological perturbations on superhorizon scales. This has been addressed extensively in the context of the separate universe approach [13] or, equivalently, the $\delta N$ formalism [6, 10, 11, 18]. The $\delta N$ formalism is the zeroth-order truncation of gradient expansion. However, higher-order contributions in gradient expansion can be important for extracting more detailed information about non-Gaussianity from primordial fluctuations. Indeed, it has been argued that, in the presence of a large effect from the decaying mode due to slow-roll violation, the second-order corrections play a crucial role in the superhorizon evolution of the curvature perturbation already at linear order [28–30], as well as at nonlinear order [22, 23]. In order to study time evolution of the curvature perturbation on superhorizon scales in the context of non-Gaussianity, it is necessary to develop nonlinear theory of cosmological perturbations valid up to second order in spatial gradient expansion.

The gradient expansion technique has been applied up to second order to a universe dominated by a canonical scalar field and by a perfect fluid with a constant equation of state parameter, $P/\rho = \text{const}$ [19, 20]. The formulae have been extended to be capable of a universe filled with a non-canonical scalar field described by a generic Lagrangian of the form $W(-\partial^\mu \phi \partial_\mu \phi/2, \phi)$, as well as a universe dominated by a perfect fluid with a general equation of state, $P = P(\rho)$ [21, 22]. Those systems are characterized by a single scalar degree of freedom, and hence one expects that a single master variable governs the evolution of scalar perturbations even at nonlinear order. By virtue of gradient expansion, one can indeed derive a simple evolution equation for an appropriately defined master variable $R_{uv}^{NL}$:

$$R_{uv}^{NL''} + 2z' R_{uv}^{NL'} + \frac{c_s^2}{4} (2) (R[R_{uv}^{NL'}]) = O(\epsilon^4), \quad (1.1)$$

where the prime represents differentiation with respect to the conformal time, $\epsilon$ is the small expansion parameter, and the other quantities will be defined in the rest of the paper. This equation is to be compared with its linear counterpart:

$$R_{uv}^{Lin''} + 2z' R_{uv}^{Lin'} - c_s^2 \Delta R_{uv}^{Lin} = 0, \quad (1.2)$$

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from which one notices the correspondence between the linear and nonlinear evolution equations. Gradient expansion can be applied also to a multi-component system, yielding the formalism “beyond δN” developed in a recent paper [23].

The purpose of this paper is to extend the gradient expansion formalism further to include a more general class of scalar-field theories obeying a second-order equation of motion. The scalar-field Lagrangian we consider is of the form $W(-\partial^\mu \phi \partial_{\mu} \phi/2, \phi) - G(-\partial^\mu \phi \partial_{\mu} \phi/2, \phi) \Box \phi$. Inflation driven by this scalar field is more general than k-inflation and is called G-inflation. It is known that k-inflation [31, 32] with the Lagrangian $W(-\partial^\mu \phi \partial_{\mu} \phi/2, \phi)$ is equivalently described by a perfect fluid cosmology. However, in the presence of $G \Box \phi$, the scalar field is no longer equivalent to a perfect fluid and behaves as an imperfect fluid [33, 34]. The authors of Ref. [20] investigated superhorizon conservation of the curvature perturbation from G-inflation at zeroth order in gradient expansion, and Gao worked out the zeroth-order analysis in the context of the most general single-field inflation model [21]. In this paper, we present a general analytic solution for the metric and the scalar field for G-inflation at second order in gradient expansion. By doing so we extend the previous result for a perfect fluid [21] and show that the nonlinear evolution equation of the form (1.1) is deduced straightforwardly from the corresponding linear result even in the case of G-inflation.

This paper is organized as follows. In the next section, we define the non-canonical scalar-field theory that we consider in this paper. In Sec. III we develop a theory of nonlinear cosmological perturbations on superhorizon scales and derive the field equations employing gradient expansion. We then integrate the relevant field equations to obtain a general solution for the metric and the scalar field in Sec. IV. The issue of defining appropriately the nonlinear curvature perturbation is addressed and the evolution equation for that variable is derived in Sec. V. Section VI is devoted to a summary of this paper and discussion.

II. G-INFLATION

In this paper we study a generic inflation model driven by a single scalar field. We go beyond k-inflation for which the Lagrangian for the scalar field is given by an arbitrary function of $\phi$ and $X := -g^{\mu \nu} \partial_{\mu} \phi \partial_{\nu} \phi/2$, $L = W(X, \phi)$, and consider the scalar field described by

$$I = \int d^4x \sqrt{-g} \left[ W(X, \phi) - G(X, \phi) \Box \phi \right],$$

(2.1)

where $G$ is also an arbitrary function of $\phi$ and $X$. We assume that $\phi$ is minimally coupled to gravity. Although the above action depends upon the second derivative of $\phi$ through $\Box \phi = g^{\mu \nu} \nabla_\mu \nabla_\nu \phi$, the resulting field equations for $\phi$ and $g_{\mu \nu}$ remain second order. In this sense the above action gives rise to a more general single-field inflation model than k-inflation, i.e., G-inflation [35–37]. The same scalar-field Lagrangian is used in the context of dark energy and called kinetic gravity braiding [33]. Interesting cosmological applications of the Lagrangian (2.1) can also be found, e.g., in [33, 34]. In fact, the most general inflation model with second-order field equations was proposed in [37] based on Horndeski’s scalar-tensor theory [40, 41]. However, in this paper we focus on the action (2.1) which belongs to a subclass of the most general single-field inflation model, because it involves sufficiently new and interesting ingredients while avoiding unwanted complexity. Throughout the paper we use Planck units, $M_{pl} = 1$, and assume that the vector $-g^{\mu \nu} \nabla_\nu \phi$ is timelike and future-directed. (The assumption is reasonable because we are not interested in a too inhomogeneous universe.)

The equation of motion for $\phi$ is given by

$$\nabla_\mu \left[ (W_X - G_\phi - G_X \Box \phi) \nabla^\mu \phi - G_X \nabla_\mu X \right] + W_\phi - G_\phi \Box \phi = 0,$$

(2.2)

where the subscripts $X$ and $\phi$ stand for differentiation with respect to $X$ and $\phi$, respectively. More explicitly, we have

$$W_X \Box \phi - W_X (\nabla_\mu \nabla_\nu \phi) (\nabla^\mu \phi \nabla^\nu \phi) - 2W_\phi X + W_\phi - 2(G_\phi - G_\phi X) \Box \phi$$

$$+ G_X \left[ (\nabla_\mu \nabla_\nu \phi) (\nabla^\mu \nabla^\nu \phi) - (\Box \phi)^2 + R_{\mu \nu} \nabla^\mu \phi \nabla^\nu \phi \right] + 2G_\phi X (\nabla_\mu \nabla_\nu \phi) (\nabla^\mu \phi \nabla^\nu \phi) + 2G_\phi X X - G_X (\nabla^\mu \nabla^\nu \phi - g^{\mu \nu} \Box \phi) (\nabla_\nu \phi \nabla_\lambda \phi) \nabla_\mu \phi = 0.$$

(2.3)

The energy-momentum tensor of the scalar field is given by

$$T_{\mu \nu} = W_X \nabla_\mu \phi \nabla_\nu \phi + W g_{\mu \nu} - (\nabla_\nu G \nabla_\mu \phi + \nabla_\mu G \nabla_\nu \phi) + g_{\mu \nu} \nabla_\lambda G \nabla^\lambda \phi - G_X \Box \phi \nabla_\mu \phi \nabla_\nu \phi.$$

(2.4)

It is well known that k-inflation allows for an equivalent description in terms of a perfect fluid, i.e., the energy-momentum tensor reduces to that of a perfect fluid with a four-velocity $u_\mu = \partial_\mu \phi / \sqrt{2X}$. However, as emphasized in [33, 34], for $G_X \neq 0$ the energy-momentum tensor cannot be expressed in a perfect-fluid form in general. This imperfect nature characterizes the crucial difference between G- and k-inflation.
III. NONLINEAR COSMOLOGICAL PERTURBATIONS

In this section we shall develop a theory of nonlinear cosmological perturbations on superhorizon scales, following Ref. [21]. For this purpose we employ the Arnowitt-Deser-Misner (ADM) formalism and perform a gradient expansion in the uniform expansion slicing and the time-slice-orthogonal threading.

A. The ADM decomposition

Employing the \((3 + 1)\)-decomposition of the metric, we write

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2 dt^2 + \hat{\gamma}_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{3.1} \]

where \(\alpha\) is the lapse function and \(\beta^i\) is the shift vector. Here, Latin indices run over \(1, 2, 3\). We introduce the unit vector \(n^\mu\) normal to the constant \(t\) hypersurfaces,

\[ n_\mu dx^\mu = -\alpha dt, \quad n^\mu \partial_\mu = \frac{1}{\alpha}(\partial_t - \beta^i \partial_i). \tag{3.2} \]

The extrinsic curvature \(K_{ij}\) of constant \(t\) hypersurfaces is given by

\[ K_{ij} = \nabla_i n_j = \frac{1}{2\alpha} \left( \partial_i \hat{\gamma}_{ij} - \dot{D}_i \beta_j - \dot{D}_j \beta_i \right), \tag{3.3} \]

where \(\dot{D}_i\) is the covariant derivative associated with the spatial metric \(\hat{\gamma}_{ij}\). The spatial metric and the extrinsic curvature can further be expressed in a convenient form as

\[ \hat{\gamma}_{ij} = a^2(t)e^{2\psi(t, x)}\gamma_{ij}, \tag{3.4} \]

\[ K_{ij} = a^2(t)e^{2\psi} \left( \frac{1}{3} K\gamma_{ij} + A_{ij} \right), \tag{3.5} \]

where \(a(t)\) is the scale factor of a fiducial homogeneous Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime, the determinant of \(\gamma_{ij}\) is normalised to unity, \(\det \gamma_{ij} = 1\), and \(A_{ij}\) is the trace-free part of \(K_{ij}\), \(\gamma^{ij} A_{ij} = 0\). The trace

\[ K := \hat{\gamma}^{ij} K_{ij} \]

is explicitly written as

\[ K = \frac{1}{\alpha} \left[ 3(H + \partial_t \psi) - \dot{D}_i \beta^i \right], \tag{3.6} \]

where \(H = H(t)\) is the Hubble parameter defined by \(H := d\ln a(t)/dt\). In deriving Eq. (3.6) \(\partial_t (\det \gamma_{ij}) = \gamma^{ij} \partial_t \gamma_{ij} = 0\) was used. Hereafter, in order to simplify the equations we choose the spatial coordinates appropriately to set

\[ \beta^i = 0. \tag{3.7} \]

We call this choice of spatial coordinates as the time-slice-orthogonal threading.

With \(\beta_i = 0\) all the independent components of the energy-momentum tensor \([24]\) are expressed as

\[ E := T_{\mu\nu} n^\mu n^\nu = (W_X - G_X \Box \phi)(\partial_{\perp} \phi)^2 - W - \partial_{\perp} G \partial_{\perp} \phi - \hat{\gamma}^{ij} \partial_t \partial_{\perp} \phi, \tag{3.8} \]

\[ -J_i := T_{\mu i} n^\mu = [(W_X - G_X \Box \phi) \partial_i \phi - \partial_t G \partial_{\perp} \phi - \partial_{\perp} G \partial_{\perp} \phi, \tag{3.9} \]

\[ S_{ij} := T_{ij}, \tag{3.10} \]

where \(\partial_{\perp} := n^\mu \partial_\mu\).

Let us now move on to the \((3 + 1)\)-decomposition of the Einstein equations. In the ADM Language, the Einstein equations are separated into four constraints (the Hamiltonian constraint and three momentum constraints) and six dynamical equations for the spatial metric. The constraints are

\[ \frac{1}{a^2} R[e^{2\psi} \gamma] + \frac{2}{3} K^2 - A_{ij} A^{ij} = 2E, \tag{3.11} \]

\[ \frac{2}{3} \partial_t K - e^{-3\psi} D_j \left( e^{3\psi} A^j_i \right) = J_i, \tag{3.12} \]
where $R[e^{2\phi}\gamma]$ is the Ricci scalar constructed from the metric $e^{2\phi}\gamma_{ij}$ and $D_i$ is the covariant derivative with respect to $\gamma_{ij}$. The spatial indices here are raised or lowered by $\gamma^{ij}$ and $\gamma_{ij}$, respectively. As for the dynamical equations, the following first-order equations for the spatial metric $(\dot{\psi}, \gamma_{ij})$ are deduced from the definition of the extrinsic curvature [33]:

$$\partial_\perp \psi = \frac{H}{\alpha} + \frac{K}{3},$$

$$\partial_\perp \gamma_{ij} = 2A_{ij}. \quad (3.13)$$

The dynamical equations for the extrinsic curvature $(K, A_{ij})$ are given by

$$\partial_\perp K = -\frac{K^2}{3} - A^{ij}A_{ij} + \frac{\dot{D}^2}{\alpha} - \frac{1}{2}(E + 3P),$$

$$\partial_\perp A_{ij} = -KA_{ij} + 2A_i^kA_{kj} + \left[\dot{D}_i\dot{D}_j\alpha\right]^{\text{TF}} - \frac{1}{a^2e^{2\phi}}[R_{ij}[e^{2\phi}\gamma] - S_{ij}]^{\text{TF}}, \quad (3.15)$$

where

$$P := \frac{1}{3}a^{-2}e^{-2\phi}\dot{\gamma}^{ij}S_{ij}, \quad (3.17)$$

$\dot{D}^2 := \dot{\gamma}^{ij}\dot{D}_i\dot{D}_j$, and $R_{ij}[e^{2\phi}\gamma]$ is the Ricci tensor constructed from the metric $e^{2\phi}\gamma_{ij}$. The trace-free projection operator $[\ldots]^{\text{TF}}$ is defined for an arbitrary tensor $Q_{ij}$ as

$$[Q_{ij}]^{\text{TF}} := Q_{ij} - \frac{1}{3}\gamma_{ij}\gamma^{kl}Q_{kl}. \quad (3.18)$$

For the purpose of solving the Einstein equations, the most convenient choice of the temporal coordinate is such that the expansion $K$ is uniform and takes the form:

$$K(t, x) = 3H(t). \quad (3.19)$$

Hereafter we call this gauge choice with [57] the uniform expansion gauge. Adopting this gauge choice, Eq. [3.13] reduces simply to

$$\partial_t \psi = H(\alpha - 1) =: H\delta\alpha(t, x). \quad (3.20)$$

From this, if we take the uniform expansion gauge, we can see that the time evolution of the curvature perturbation $\psi$ is caused by the inhomogeneous part of the lapse function $\delta\alpha(t, x)$ only. It is related to the non-adiabatic perturbation.

### B. Gradient expansion: basic assumptions and the order of various terms

In the gradient expansion approach, we introduce a flat FLRW universe characterized by $(a(t), \phi_0(t))$ as a background and suppose that the characteristic length scale $L = a/k$, where $k$ is a wavenumber of a perturbation, is longer than the Hubble length scale $1/H$ of the background, $HL \gg 1$. We use $\epsilon := 1/(HL) = k/(aH)$ as a small parameter to keep track of the order of various terms and expand the equations in terms of $\epsilon$, so that a spatial derivative acting on a perturbation raises the order by $\epsilon$.

The background flat FLRW universe characterized by $(a(t), \phi_0(t))$ satisfies the Einstein equations,

$$H^2(t) = \frac{1}{3}\rho_0, \quad \dot{H}(t) = -\frac{1}{2}(\rho_0 + P_0), \quad (3.21)$$

and the scalar-field equation of motion,

$$\ddot{\phi} + 3H\dot{\phi} = \left(W_\phi - 2XG_{\phi\phi} - 2G_{\phi,x}X\dot{\phi}\right)_0. \quad (3.22)$$

Here, an overdot (\dot) denotes differentiation with respect to $t$, and a subscript 0 indicates the corresponding background quantity, i.e., $W_0 := W(X_0, \phi_0)$, $(W_X)_0 := W_X(X_0, \phi_0)$, etc., where $X_0 := \phi_0^2/2$. The background energy density and pressure, $\rho_0$ and $P_0$, are given by

$$\rho_0 = \left[-W + 2X(W_X + 3HGX\dot{\phi} - G\phi)\right]_0, \quad (3.23)$$

$$P_0 = \left[-W - 2X(GX\ddot{\phi} + G\phi)\right]_0. \quad (3.24)$$
while $J_0$ is defined as
\[ J_0 = \left[ W_X \phi - 2G_\phi \phi + 6HG X \right]_0. \] (3.25)

If $W$ and $G$ do not depend on $\phi$, the right hand side of Eq. (3.22) vanishes and hence $J_0$ is conserved. In this case, $J_0$ is the Noether current associated with the shift symmetry $\phi \to \phi + c$. Note that the above quantities may be written in a different way as
\[ \rho_0 = J_0 \phi_0 - W_0 + 2(XG_\phi)_0, \] (3.26)
\[ \rho_0 + P_0 = J_0 \phi_0 - 2(G_X X \phi)_0. \] (3.27)

Note also that the scalar-field equation of motion (3.22) can be written as
\[ G \phi_0 + 3 \Theta J_0 + \mathcal{E}_\phi = 0, \] (3.28)
where
\begin{align*}
\mathcal{G}(t) &:= \mathcal{E}_X - 3\Theta(G_X \phi)_0, \\
\Theta(t) &:= H - (G_X X \phi)_0, \\
\mathcal{E}_\phi(t) &:= \left[ 2XW_X \phi - W_\phi + 6HG_\phi X \phi - 2XG_\phi \phi \right]_0, \\
\mathcal{E}_X(t) &:= \left[ W_X + 2XW_{XX} + 9HG_\phi X + 6HG_XX \phi - 2G_\phi - 2XG_\phi X \right]_0.
\end{align*}

These functions will also be used later.

Since the background FLRW universe must be recovered at zeroth order in gradient expansion, the spatial metric must take the locally homogeneous and isotropic form in the limit $\epsilon \to 0$. This leads to the following assumption:
\[ \partial_t \gamma_{ij} = O(\epsilon^2). \] (3.33)

This assumption is justified as follows [14, 19–25]. If $\partial_t \gamma_{ij}$ were $O(\epsilon)$, the leading term would correspond to homogeneous and anisotropic perturbations, which are known to decay quickly. We may therefore reasonably assume $\partial_t \gamma_{ij} = O(\epsilon^2)$ and not $\partial_t \gamma_{ij} = O(\epsilon)$. However, $\psi$ and $\gamma_{ij}$ (without any derivatives acting on them) are of order $O(1)$.

Using the assumption (3.33) made above and the basic equations derived in the previous subsection, one can now deduce the order of various terms in gradient expansion. First, from Eq. (3.14) we see that
\[ A_{ij} = O(\epsilon^2). \] (3.34)

Substituting Eq. (3.34) into Eq. (3.12), under the gauge condition (3.19), we obtain $J_i = O(\epsilon^3)$. Then, this condition combined with the definition (3.9) implies that $\partial_t \delta \phi = O(\epsilon^3)$, where $\delta \phi(t, x) := \phi(t, x) - \phi_0(t)$. The same equations also imply that $\partial_t G = O(\epsilon^3)$. By absorbing a homogeneous part of $\delta \phi$ into $\phi_0$ (and redefining $a(t)$ accordingly), we have
\[ \delta \phi = O(\epsilon^2). \] (3.35)

It is clear from the condition (3.34) and the Hamiltonian constraint (3.11) that
\[ \delta E := E(t, t') - \rho_0(t) = O(\epsilon^2). \] (3.36)

Since the definition (3.8) tells that $E - \rho_0 = \max\{O(\delta \phi), O(\partial_\perp \phi - \partial_\phi \phi_0)\}$, we see $\partial_t (\delta \phi) - \dot{\phi}_0 \delta \alpha = O(\epsilon^2)$, leading to
\[ \delta \alpha = O(\epsilon^2). \] (3.37)

Then, it follows immediately from Eq. (3.20) that
\[ \partial_t \psi = O(\epsilon^2). \] (3.38)

Similarly, for the spatial energy-momentum component we find
\[ \delta P := P(t, x) - P_0(t) = O(\epsilon^2). \] (3.39)

In summary, we have evaluated the order of various quantities as follows:
\[ \psi = O(1), \quad \gamma_{ij} = O(1), \quad \delta \alpha = O(\epsilon^2), \quad \delta \phi = O(\epsilon^2), \quad \delta E = O(\epsilon^2), \quad \delta P = O(\epsilon^2), \quad A_{ij} = O(\epsilon^2), \quad \partial_t \gamma_{ij} = O(\epsilon^2), \quad \partial_t \psi = O(\epsilon^2), \quad \beta^i = O(\epsilon^2), \quad \partial_t G = O(\epsilon^3), \quad [S_{ij}]^{TT} = O(\epsilon^6), \] (3.40)

where the assumptions made have also been included.
C. Field equations up to $O(\epsilon^2)$ in gradient expansion

Keeping the order of various terms $[3.40]$ in mind, let us derive the governing equations in the uniform expansion gauge. The Hamiltonian and momentum constraints are

$$R[e^{2\psi}] = 2\delta E + O(\epsilon^4),$$

$$e^{-3\psi}D_j\left(e^{3\psi}A^j\right) = -J_i + O(\epsilon^5).$$

(3.41) (3.42)

The evolution equations for the spatial metric are given by

$$\partial_t \psi = H \delta \alpha + O(\epsilon^4), \quad \partial_t \gamma_{ij} = 2A_{ij} + O(\epsilon^4),$$

(3.43)

while the evolution equations for the extrinsic curvature are

$$\partial_t A_{ij} = -3HA_{ij} - \frac{1}{a^2 e^{2\psi}} [R_{ij}[e^{2\psi}]]^T + O(\epsilon^4),$$

$$\frac{3}{\alpha} \partial_t H = -3H^2 - \frac{1}{2}(E + 3P) + O(\epsilon^4).$$

(3.44) (3.45)

Note that with the help of the background equations Eq. (3.45) can be recast into

$$\delta P + \frac{\delta E}{3} + (\rho_0 + P_0)\delta \alpha = O(\epsilon^4).$$

(3.46)

The components of the energy-momentum tensor are expanded as

$$E = 2XW_X - W + 6HG_X \partial_\perp \phi - 2XG_\phi + O(\epsilon^6),$$

$$P = W - \partial_\perp G \partial_\perp \phi + O(\epsilon^4),$$

$$-J_i = J_0 \partial_i(\delta \phi) - (G_X \delta \phi)_0 \partial_i(\delta X) + O(\epsilon^5),$$

(3.47) (3.48) (3.49)

where

$$X = (\partial_\perp \phi)^2/2 + O(\epsilon^6),$$

$$\delta X := X - X_0 = \dot{\phi}_0 \partial_i(\delta \phi) - 2X_0 \delta \alpha + O(\epsilon^4).$$

(3.50) (3.51)

Finally, noting that $\Box \phi = -\partial^2 \phi - 3H \partial_\perp \phi + O(\epsilon^4)$, the scalar-field equation of motion $[2.31]$ reduces to

$$W_X(\partial_\perp^2 \phi + 3H \partial_\perp \phi) + 2W_{X X} \partial_\perp^2 \phi + 2W_{\phi X} X - W_{\phi} = 2(G_\phi - G_{\phi X})(\partial_\perp^2 \phi + 3H \partial_\perp \phi) + 6G_X [\partial_\perp (H_X) + 3H^2 X] - 4XG_{\phi X} \partial_\perp^2 \phi - 2G_{\phi X} X + 6HG_{X X} \partial_\perp X = O(\epsilon^4).$$

(3.52)

This equation can also be written in a slightly simpler form as

$$\partial_\perp \mathcal{J} + 3H \mathcal{J} = W_\phi - 2XG_{\phi X} - 2XG_{\phi X} \partial_\perp^2 \phi + O(\epsilon^4),$$

(3.53)

where

$$\mathcal{J} = W_X \partial_\perp \phi - 2G_{\phi X} \partial_\perp + 6HG_X X.$$

(3.54)

It can be seen that Eq. (3.53) takes exactly the same form as the background scalar-field equation of motion $[3.43]$ under the identification $\partial_t \leftrightarrow \partial_\perp$. Now Eq. (3.45) can be written using $\mathcal{J}$ as

$$E = \mathcal{J} \partial_\perp \phi - W + 2XG_\phi.$$

(3.55)

From Eqs. (3.53) and (3.55) we find

$$\partial_\perp E = -3H(E + P) + O(\epsilon^4).$$

(3.56)

This equation is nothing but the conservation law, $n_\nu \nabla_\nu T^{\mu \nu} = 0$.

Combining Eq. (3.45) with Eq. (3.56), we obtain

$$\partial_t [a^2(\delta E)] = O(\epsilon^4).$$

(3.57)
We can expand Eq. (3.47) in terms of \( \delta \phi \) and \( \delta X \), and thus \( \delta E \) can be expressed as

\[
\delta E = \mathcal{E}_\phi(t) \delta \phi + \mathcal{E}_X(t) \delta X + \mathcal{O}(\epsilon^4).
\]

(3.58)

This equation relates \( \delta \phi \) and \( \delta X \) with a solution to the simple equation \( (3.57) \). With the help of Eq. (3.46), Eq. (3.48) can be regarded as an equation relating \( \delta \phi \) and \( \delta \alpha \). Similarly, one can express \( \delta P \) as

\[
\delta P = \frac{1}{a^3} \partial_t \left\{ a^3 \left[ J_0(\delta \phi) - (G_X \dot{\phi})_0(\delta X) \right] \right\} - (\rho_0 + P_0)(\delta \alpha) + \mathcal{O}(\epsilon^4).
\]

(3.59)

Using Eq. (3.46), one has

\[
\partial_t \left\{ a^3 \left[ J_0(\delta \phi) - (G_X \dot{\phi})_0(\delta X) \right] \right\} = \frac{a^3}{3} \delta E + \mathcal{O}(\epsilon^4),
\]

(3.60)

which can easily be integrated once to give another independent equation relating \( \delta \phi \) and \( \delta \alpha \). In the next section, we will give a general solution to the above set of equations.

### IV. GENERAL SOLUTION

Having thus derived all the relevant equations up to second order in gradient expansion, let us now present a general solution. First, since \( \psi = \mathcal{O}(1) \) and \( \partial_t \psi = \mathcal{O}(\epsilon^2) \), we find

\[
\phi = (0) C^\psi(x) + \mathcal{O}(\epsilon^2),
\]

(4.1)

where \( (0) C^\psi(x) \) is an integration constant which is an arbitrary function of the spatial coordinates \( x \). Here and hereafter, the superscript \( (n) \) indicates that the quantity is of order \( \epsilon^n \). Similarly, it follows from \( \gamma_{ij} = \mathcal{O}(1) \) and \( \partial_t \gamma_{ij} = \mathcal{O}(\epsilon^2) \) that

\[
\gamma_{ij} = (0) C^\gamma_{ij}(x) + \mathcal{O}(\epsilon^2),
\]

(4.2)

where \( (0) C^\gamma_{ij}(x) \) is a 3 \times 3 matrix with a unit determinant whose components depend only on the spatial coordinates. The evolution equations (3.50) can then be integrated to determine the \( \mathcal{O}(\epsilon^2) \) terms in \( \psi \) and \( \gamma_{ij} \) as

\[
\psi = (0) C^\psi(x) + \int_{t_*}^t H(t') \delta \alpha(t', x) dt' + \mathcal{O}(\epsilon^4),
\]

(4.3)

\[
\gamma_{ij} = (0) C^\gamma_{ij}(x) + 2 \int_{t_*}^t A_{ij}(t', x) dt' + \mathcal{O}(\epsilon^4),
\]

(4.4)

where \( t_* \) is some initial time and integration constants of \( \mathcal{O}(\epsilon^2) \) have been absorbed to \( (0) C^\psi(x) \) and \( (0) C^\gamma_{ij}(x) \).

Now Eq. (3.41) can be integrated to give

\[
A_{ij} = \frac{1}{a^3(t)} \left[ (2) F_{ij}(x) \int_{t_*}^t a(t') dt' + (2) C^\gamma_{ij}(x) \right] + \mathcal{O}(\epsilon^4),
\]

(4.5)

where

\[
(2) F_{ij}(x) := -\frac{1}{\epsilon^{2\psi}} \left[ R_{ij} [c^{2\psi} \gamma] \right]^{TF} \]

\[
= -\frac{1}{\epsilon^{2\psi}} \left[ (2) R_{ij} - \frac{1}{3} (2) R (0) C^\gamma_{ij} \right] + \left( \partial_i (0) C^\psi \partial_j (0) C^\psi - (0) D_i (0) D_j (0) C^\psi \right)
\]

\[
- \frac{1}{3} (0) C^\gamma_{kl} \left( \partial_k (0) C^\psi \partial_l (0) C^\psi - (0) D_k (0) D_l (0) C^\psi \right) (0) C^\gamma_{ij}.
\]

(4.6)

Here, \( (0) C^\gamma_{ij} \) is the inverse matrix of \( (0) C^\gamma_{ij} \), \( (2) R_{ij}(x) := R_{ij}[c^{2\psi}] \) and \( (2) R(x) := R[c^{2\psi}] \) are the Ricci tensor and the Ricci scalar constructed from the zeroth-order spatial metric \( (0) C^\gamma_{ij}(x) \), and \( (0) D \) is the covariant derivative.
associated with \((0) C_{ij}^x\). Note that \((0) C_{ij}^{x} (2) F_{ij} = 0\) by definition. The integration constant, \((2) C_{ij}^{x}(x)\), is a symmetric matrix whose components depend only on the spatial coordinates and which satisfies the traceless condition \((0) C_{ij}^{x} (2) C_{ij}^{x} = 0\). Substituting the above result to Eq. 4.3, we arrive at

\[
\gamma_{ij} = \frac{(0) C_{ij}^{x}(x)}{2} + \frac{(2) F_{ij}(x)}{a^3(t')} \int_{t}^{t'} a(t'')dt'' + \frac{(2) C_{ij}^{x}(x)}{a^3(t')} \int_{t}^{t'} \frac{dt'}{a^3(t')} + \mathcal{O}(\epsilon^4). \tag{4.7}
\]

Next, it is straightforward to integrate Eq. 4.57 to obtain

\[
\delta E = \frac{1}{a^2(t)} (2) K(x) + \mathcal{O}(\epsilon^4), \tag{4.8}
\]

where \((2) K(x)\) is an arbitrary function of the spatial coordinates. With this solution for \(\delta E\), Eqs. 4.58 and 4.60 reduce to

\[
\mathcal{E}_\phi(t)\delta \phi + \mathcal{E}_X(t)\delta X = \frac{1}{a^2(t)} (2) K(x) + \mathcal{O}(\epsilon^4), \tag{4.9}
\]

\[
\partial_t \left\{ a^3 \left[ \mathcal{J}_0(\delta \phi) - (G_X \dot{\phi})_0(\delta X) \right] \right\} = -\frac{1}{3 a^2(t)} (2) K(x) + \mathcal{O}(\epsilon^4), \tag{4.10}
\]

and the latter equation can further be integrated to give

\[
\mathcal{J}_0(\delta \phi) - (G_X \dot{\phi})_0(\delta X) = \frac{(2) C^x(x)}{a^3(t)} - \frac{(2) K(x)}{3a^2(t)} \int_{t}^{t'} a(t')dt' + \mathcal{O}(\epsilon^4), \tag{4.11}
\]

where we have introduced another integration constant \((2) C^x(x)\).

With the help of Eqs. 4.51, one can solve the system of equations 4.9 and 4.11, leading to

\[
\delta \phi = \frac{1}{A} \left\{ \left[ (G_X \dot{\phi})_0 - \frac{\mathcal{E}_X(t)}{3a(t)} \int_{t}^{t'} a(t')dt' \right] \frac{(2) K}{a^2(t)} + \frac{\mathcal{E}_X(t)}{a^3(t)} (2) C^x \right\} + \mathcal{O}(\epsilon^4), \tag{4.12}
\]

and

\[
\delta \alpha = \partial_t \left( \frac{\delta \phi}{\partial \phi} \right) - \frac{1}{2X_0 G(t)} \left\{ \left[ 1 - \frac{\Theta(t)}{a(t)} \int_{t}^{t'} a(t')dt' \right] \frac{(2) K}{a^2(t)} + \frac{3\Theta(t)}{a^3(t)} (2) C^x \right\} + \mathcal{O}(\epsilon^4), \tag{4.13}
\]

where we have introduced another integration constant \((2) C^x(x)\).

Finally, substituting Eq. 4.13 to Eq. 4.13, we obtain

\[
\psi = (0) C^w + \int_{t}^{t'} dt' H(t') \partial_{t'} \left( \frac{\delta \phi}{\partial \phi} \right) - \int_{t}^{t'} dt' \frac{H(t')}{2X_0 G(t')} \left\{ \left[ 1 - \frac{\Theta(t)}{a(t)} \int_{t}^{t'} a(t'')dt'' \right] \frac{(2) K}{a^2(t')} + \frac{3\Theta(t)}{a^3(t')} (2) C^x \right\} + \mathcal{O}(\epsilon^4)
\]

\[
= (0) C^w(x) + \frac{H \delta \phi}{\partial \phi} + \int_{t}^{t'} dt' \left( \frac{\rho_0 + P_0}{2} + \delta \phi \right) \frac{H(t')}{2X_0 G^2(t')} \left\{ \left[ 1 - \frac{\Theta(t)}{a(t')} \int_{t}^{t'} a(t'')dt'' \right] \frac{(2) K}{a^2(t')} + \frac{3\Theta(t)}{a^3(t')} (2) C^x \right\} + \mathcal{O}(\epsilon^4), \tag{4.14}
\]

where we performed integration by parts and used the background field equation.

So far we have introduced five integration constants, \((0) C^w(x)\), \((0) C_{ij}^{x}(x)\), \((2) C_{ij}^{x}(x)\), \((2) K(x)\), and \((2) C^x(x)\), upon solving the field equations up to \(O(\epsilon^2)\). Here, it should be pointed out that they are not independent. Indeed, Eqs. 4.11 and 4.12 impose the following constraints among the integration constants:

\[
(2) K(x) = \frac{(2) \dot{R}(x)}{2} + \mathcal{O}(\epsilon^4),
\]

\[
e^{-3(0) C^w(0)} C^{x} j^k(0) D_j \left[ e^{3(0) C^w(2) C_{kl}^{x}(x)} \right] = \partial_i (2) C^x(x) + \mathcal{O}(\epsilon^5),
\]

\[
e^{-3(0) C^w(0)} C^{x} j^k(0) D_j \left[ e^{3(0) C^w(2) F_{kl}(x)} \right] = -\frac{1}{6} \partial_i (2) \dot{R}(x) + \mathcal{O}(\epsilon^5), \tag{4.15}
\]
where \( \hat{R}(x) := R[e^{2(0)C^\psi} (0)C^\gamma] \) is the Ricci scalar constructed from the metric \( e^{2(0)C^\psi} (0)C^\gamma \). Here, \( \hat{R}(x) \) should not be confused with \( R(x) \). The latter is the Ricci scalar constructed from \( (0)C^\gamma \) and not from \( e^{2(0)C^\psi} (0)C^\gamma \). Explicitly,

\[
(2) \hat{R}(x) = \left[(2) R(x) - 2 \left( (2) D^2(0)C^\psi + (0)C^\gamma_{ij} \partial_i(0)C^\psi \partial_j(0)C^\psi \right) \right] e^{-2(0)C^\psi}.
\]

(4.16)

Note that the third equation is automatically satisfied provided that the last equation holds, as can be verified by using Eq. (4.16).

In summary, we have integrated the field equations up to second order in gradient expansion and obtained the following solution for generic single-field inflation:

\[
\delta E = \frac{(2) \hat{R}(x)}{2a^2} + \mathcal{O}(\epsilon^4),
\]

\[
\delta \phi = \frac{1}{a \hat{a}^2} \left[ \left( G_X \dot{\phi} \right)_0 - \frac{\xi_X}{3a} \int_{t_0}^t a(t') dt' \right] \frac{(2) \hat{R}(x)}{2} + \mathcal{O}(\epsilon^4),
\]

\[
\delta \alpha = \partial_i \left( \frac{\delta \phi}{\phi_0} \right) - \frac{1}{2X_0 \hat{a}^2} \left[ \left( 1 - \frac{\Theta}{a} \right) \int_{t_0}^t a(t') dt' \right] \frac{(2) \hat{R}(x)}{2} + \mathcal{O}(\epsilon^4),
\]

\[
\psi = (0)C^\psi(x) + \frac{H \delta \phi}{\phi_0} + \int_{t_0}^t dt' \left( \rho_0 + P_0 \right) \frac{\delta \phi}{\phi_0} \left[ \left( 1 - \frac{\Theta}{a} \right) \int_{t_0}^{t'} a(t'') dt'' \right] \frac{(2) \hat{R}(x)}{2} + \mathcal{O}(\epsilon^4),
\]

\[
A_{ij} = \frac{1}{a^3} \left[ (2) F_{ij}(x) \int_{t_0}^t a(t') dt' + (2) C^A_{ij}(x) \right] + \mathcal{O}(\epsilon^4),
\]

\[
\gamma_{ij} = (0)C^\gamma_{ij}(x) + 2 \left[ (2) F_{ij}(x) \int_{t_0}^t \frac{dt'}{a^3(t')} \int_{t_0}^{t'} a(t'') dt'' + (2) C^A_{ij}(x) \int_{t_0}^t \frac{dt'}{a^3(t')} \right] + \mathcal{O}(\epsilon^4).
\]

(4.17)

The x-dependent integration constants, \( (0)C^\psi, (0)C^\gamma_{ij}, (2)C^\chi \) and \( (2)C^A_{ij} \), satisfy the following conditions:

\[
(0)C^\gamma_{ij} = (0)C^\gamma_{ji}, \quad \det((0)C^\gamma_{ij}) = 1, \quad (2)C^A_{ij} = (2)C^A_{ji}, \quad (0)C^\gamma_{ij} (2)C^A_{ij} = 0,
\]

\[
e^{-3(0)C^\psi} (0)C^\gamma_{jk}(0) D_j \left( e^{3(0)C^\psi} (2)C^A_{ki} \right) = \partial_i (2)C^\chi.
\]

(4.18)

Before closing this section, we remark that the gauge condition \( 3.7 \) remains unchanged under a purely spatial coordinate transformation

\[
x^i \rightarrow \hat{x}^i = f^i(x).
\]

(4.19)

This means that the zeroth-order spatial metric \( (0)C^\gamma_{ij} \) contains three residual gauge degrees of freedom. Therefore, the number of degrees of freedom associated with each integration constant is summarized as follows:

\[
(0)C^\psi \quad \text{1 scalar growing mode = 1 component},
\]

\[
(0)C^\gamma_{ij} \quad \text{2 tensor growing modes = 5 components – 3 gauge},
\]

\[
(2)C^\chi \quad \text{1 scalar decaying mode = 1 component},
\]

\[
(2)C^A_{ij} \quad \text{2 tensor decaying modes = 5 components – 3 constraints}.
\]

(4.20)

V. NONLINEAR CURVATURE PERTURBATION

In this section, we will define a new variable which is a nonlinear generalization of the curvature perturbation up to \( \mathcal{O}(\epsilon^2) \) in gradient expansion. We will show that this variable satisfies a nonlinear second-order differential equation, and, as in Ref. 22, the equation can be deduced as a generalization of the corresponding linear perturbation equation. To do so, one should notice the following fact on the definition of the curvature perturbation: in linear theory the
curvature perturbation is named so because it is directly related to the three-dimensional Ricci scalar; $\psi$ may be called so at fully nonlinear order in perturbations and at leading order in gradient expansion; and, as pointed out in Ref. [22], $\psi$ is no longer appropriate to be called so at second order in gradient expansion. To define the curvature perturbation appropriately at $O(\epsilon^2)$, one needs to take into account the contribution from $\gamma_{ij}$. Let us denote this contribution as $\chi$. We carefully define the curvature perturbation as a sum of $\psi$ and $\chi$ so that the new variable reproduces the correct result in the linear limit. In what follows we remove the subscript 0 from the background quantities since there will be no danger of confusion.

A. Assumptions and definitions

As mentioned in the previous section, we still have residual spatial gauge degrees freedom, which we are going to fix appropriately. To do so, we assume that the contribution from gravitational waves to $\gamma_{ij}$ is negligible and consider the contribution from scalar-type perturbations only. We may then choose the spatial coordinates so that $\gamma_{ij}$ coincides with the flat metric at sufficiently late times during inflation,

$$\gamma_{ij} \rightarrow \delta_{ij} \quad (t \rightarrow \infty).$$

In reality, the limit $t \rightarrow \infty$ may be reasonably interpreted as $t \rightarrow t_{\text{late}}$ where $t_{\text{late}}$ is some time close to the end of inflation. Up to $O(\epsilon^2)$, this condition completely removes the residual three gauge degrees of freedom.

We wish to define appropriately a nonlinear curvature perturbation on uniform $\phi$ hypersurfaces ($\delta \phi(t, x) = 0$) and derive a nonlinear evolution equation for the perturbation. The nonlinear result on uniform $\phi$ hypersurfaces is to be compared with the linear result for G-inflation [23]. However, in the previous section the general solution was derived in the uniform expansion gauge. For our purpose we will therefore go from the uniform expansion gauge to the uniform $\phi$ gauge. In this case, it would be appropriate simply to define $\psi$ to be the nonlinear curvature perturbation. At second order in gradient expansion, however, this is not the correct way of defining the nonlinear curvature perturbation. We must extract the appropriate scalar part $\chi$ from $\gamma_{ij}$, which will yield an extra contribution to the total curvature perturbation, giving a correct definition of the nonlinear curvature perturbation at $O(\epsilon^2)$.

Let us use the subscripts $K$ and $u$ to indicate the quantity in the uniform $K$ and $\phi$ gauges, respectively, so that in what follows the subscript $K$ is attached to the solution derived in the previous section. First, we derive the relation between $\psi^u$ and $\psi_K$ up to $O(\epsilon^2)$. In general, one must consider a nonlinear transformation between different time slices. The detailed description on this issue can be found in Ref. [22]. However, thanks to the fact that $\delta \phi_K = O(\epsilon^2)$, one can go from the uniform $K$ gauge to the uniform $\phi$ gauge by the transformation analogous to the familiar linear gauge transformation. Thus, $\psi^u$ is obtained as

$$\psi^u = \psi_K - \frac{H}{\delta \phi_K} + O(\epsilon^3).$$

One might think that the shift vector $\beta^i_u$ appears in this new variable as a result of the gauge transformation, but $\beta^i$ can always be gauged away by using a spatial coordinate transformation. The general solution for $\psi^u$ valid up to $O(\epsilon^2)$ is thus given by the linear combination of the solution for $\psi_K$ and $\delta \phi_K$ displayed in Eq. (4.17). Note here that the spatial metric $\gamma_{ij}$ remains the same at $O(\epsilon^2)$ accuracy under the change from the uniform $K$ gauge to the uniform $\phi$ gauge:

$$\gamma_{ij} K = \gamma_{ij} u + O(\epsilon^4).$$

We now turn to the issue of appropriately defining a nonlinear curvature perturbation to $O(\epsilon^2)$ accuracy. Let us denote the linear curvature perturbation in the uniform $\phi$ gauge by $R^u_{\text{Lin}}$. In the linear limit, $\psi$ reduces to the longitudinal component $H^u_{\text{Lin}}$ of scalar perturbations, while $\chi$ to traceless component $H^u_{T \text{Lin}}$:

$$\psi \rightarrow H^u_{\text{Lin}}, \quad \chi \rightarrow H^u_{T \text{Lin}}.$$  

1 The gauge in which $\phi$ is uniform is sometimes called the unitary gauge. The unitary gauge does not coincide with the comoving gauge in G-inflation, as emphasized in [23].
The linear curvature perturbation is given by $\mathcal{R}^{\text{Lin}} = (H_0^{\text{Lin}} + H_0^{\text{Lin}}/3)Y$. Here, we have followed Ref. [48] and the perturbations are expanded in scalar harmonics $Y$ satisfying $\left(\partial_i \partial_i + k^2\right)Y_k = 0$, with the summation over $k$ suppressed for simplicity. The spatial metric in the linear limit is expressed as

$$\dot{\gamma}_{ij}^{\text{Lin}} = a^2 \left(\delta_{ij} + 2 H_0^{\text{Lin}} Y \delta_{ij} + 2 H_0^{\text{Lin}} Y_{ij}\right),$$

(5.5)

where $Y_{ij} = k^{-2} \left[\partial_i \partial_j - (1/3)\delta_{ij} \partial^2\right]Y_k$. Since $\psi$ corresponds to $H_0^{\text{Lin}}$, one can read off from the above expression that $\gamma_{ij} = \delta_{ij} + 2 H_0^{\text{Lin}} Y_{ij}$ in the linear limit. Thus, our task is to extract from $\gamma_{ij}$ the scalar component $\chi$ that reduces to $H_0^{\text{Lin}}$ in linear limit. It was shown in Ref. [22] that by using the inverse Laplacian operator on the flat background, $\Delta^{-1}$, one can naturally define $\chi$ as

$$\chi := -\frac{3}{4} \Delta^{-1} \left[\partial^i e^{-3\psi} \partial^j e^{3\psi} (\gamma_{ij} - \delta_{ij})\right].$$

(5.6)

In terms of $\chi$ defined above, the nonlinear curvature perturbation is defined, to $O(\epsilon^2)$, as

$$\mathcal{R}^{\text{NL}} := \psi + \frac{\chi}{3}.$$  

(5.7)

As is clear from Eq. (5.6), extracting $\chi$ generally requires a spatially nonlocal operation. However, as we will see in the next subsection, in the uniform $\phi$ gauge supplemented with the asymptotic condition on the spatial coordinates (5.1), it is possible to obtain the explicit expression for the nonlinear version of $\chi$ from our solution (4.17) without any nonlocal operation.

### B. Solution

We start with presenting an explicit expression for $\psi_u$. It follows from Eqs. (4.17) and (5.2) that

$$\psi_u = (0) C^\psi(x) + (2) C^\psi(x) + f_R(t) (2) \hat{R}(x) + f_\chi(t) (2) C^\chi(x) + O(\epsilon^4).$$

(5.8)

Note here that although the integration constant $\epsilon^2 C^\psi(x)$ was absorbed into the redefinition of $(0) C^\psi(x)$ in the previous section, we do not do so in this section for later convenience. The time-dependent functions $f_R(t)$ and $f_\chi(t)$ are defined as

$$f_R(t) := \int_{t_*}^t dt' \frac{(\rho + P)}{2 a^2} \left[G_X \phi - \frac{E_X}{3a} \int_{t_*}^{t'} a(t'')dt''\right] - \frac{H}{2X^2} \left(1 - \frac{\Theta}{a} \int_{t_*}^{t'} a(t'')dt''\right),$$

$$f_\chi(t) := \int_{t_*}^t dt' \frac{(\rho + P)E_X}{2 a^2} - \frac{3H\Theta}{2X^2}.$$  

(5.9)

Since $\gamma_{iju}$ coincides with $\gamma_{ij} K$ up to $O(\epsilon^2)$, it is straightforward to see

$$\gamma_{iju} = (0) C^\gamma_{ij}(x) + (2) C^\gamma_{ij}(x) + 2 g_F(t) (2) F_{ij}(x) + 2 g_A(t) (2) C^A_{ij}(x) + O(\epsilon^4),$$

(5.11)

where

$$g_F(t) := \int_{t_*}^t dt' a^3(t') \int_{t_*}^{t'} a(t'')dt'', \quad g_A(t) := \int_{t_*}^t dt' a^3(t').$$

(5.12)

The integration constants $(0) C^\gamma_{ij}$ and $(2) C^\gamma_{ij}$ are determined from the condition (5.1) as

$$(0) C^\gamma_{ij} = \delta_{ij},$$

$$(2) C^\gamma_{ij} = -2 g_F(\infty) (2) F_{ij} - 2 g_A(\infty) (2) C^A_{ij}.$$  

(5.13)

We now have $(0) C^\gamma_{ij} = \delta_{ij}$, and hence $(2) R_{ij}(x) = R_{ij}(0) C^\gamma = 0$. This simplifies the explicit expression for $(2) \hat{R}(x)$ and $(2) F_{ij}(x)$; they are given solely in terms of $(0) C^\psi$ and the usual derivative operator $\partial_i$.

Substituting Eq. (5.11) to the definition (5.6), we obtain

$$\frac{\chi_u}{3} = \frac{(2) \hat{R}(x)}{12} \left[g_F(t) - g_F(\infty)\right] - \frac{(2) C^\chi(x)}{2} \left[g_A(t) - g_A(\infty)\right] + O(\epsilon^4).$$

(5.14)
It is easy to verify that the linear limit of $\chi_u$ reduces consistently to $H Y Lin$. We then finally arrive at the following explicit solution for the appropriately defined nonlinear curvature perturbation in the uniform $\phi$ gauge:

$$
\mathcal{R}^{NL}_u = (0)C^{0}(x) + (2)C^{0}(x) + (2)\tilde{R}(x)\left[ f_{R}(t) + \frac{g_{F}(t)}{12} - \frac{g_{G}(\infty)}{12} \right] + (2)C^{0}(x)\left[ f_{\chi}(t) - \frac{g_{A}(t)}{2} + \frac{g_{A}(\infty)}{2} \right].
$$

(5.15)

Let us comment on the dependence of $\mathcal{R}^{NL}_u$ on the initial fiducial time $t_s$. One may take $t_s$ as the time when our nonlinear superhorizon solution is matched to the perturbative solution whose initial condition is fixed deep inside the horizon. Then, $\mathcal{R}^{NL}_u$ should not depend on the choice of $t_s$, though apparent dependences are found in the lower bounds of the integrals $f_R(t)$, $f_{\chi}(t)$, $g_{F}(t)$, and $g_{A}(t)$. Actually, in the same way as discussed in Ref. [22], one can check that $\mathcal{R}^{NL}_u$ is invariant under the infinitesimal shift $t_s \rightarrow t_s + \delta t_s$.

C. Second-order differential equation

Having obtained explicitly the solution $\mathcal{R}^{NL}_u$ in Eq. (5.15), now we are going to deduce the second-order differential equation that $\mathcal{R}^{NL}_u$ obeys at $\mathcal{O}(\varepsilon^2)$ accuracy. For this purpose, we rewrite $f_R(t)$ and $f_\chi(t)$ in terms of

$$
z := \frac{a\phi\sqrt{\Theta}}{\Theta}.
$$

(5.16)

This is a generalization of familiar $"z"$ in the Mukhanov-Sasaki equation [49], and reduces indeed to $a\sqrt{\rho + P}/Hc_s$ in the case of $k$-inflation. With some manipulation, it is found that $f_R(t)$ and $f_\chi(t)$ can be rewritten as

$$
f_R(\eta) = \int_{\eta_s}^{\eta} \frac{a(\eta')d\eta'}{z^2(\eta')} + \frac{1}{2} \int_{\eta_s}^{\eta} \frac{d\eta'}{z^2(\eta')} \int_{\eta_s}^{\eta'} a(\eta'')d\eta'' - \frac{1}{12} \int_{\eta_s}^{\eta} \int_{\eta_s}^{\eta'} \frac{a^2(\eta'')d\eta''}{z^2(\eta')}d\eta',
$$

$$
f_\chi(\eta) = \frac{1}{2} \int_{\eta_s}^{\eta} \frac{d\eta'}{z^2(\eta')} - 3 \int_{\eta_s}^{\eta} \frac{d\eta'}{z^2(\eta')} ,
$$

(5.17)

where the conformal time defined by $d\eta = dt/a(t)$ was used instead of $t$, and $\eta_s$ corresponds to the fiducial initial time. Further, it is convenient to express them in the form

$$
f_R(\eta) = F(\eta_s) - F(\eta) - \frac{1}{12}g_{F}(\eta), \quad f_\chi(\eta) = \frac{1}{2}g_{A}(\eta) + D(\eta_s) - D(\eta),
$$

(5.18)

where we defined

$$
D(\eta) = 3 \int_{\eta}^{0} \frac{d\eta'}{z^2(\eta')}, \quad F(\eta) = \frac{1}{2} \int_{\eta}^{0} \frac{d\eta'}{z^2(\eta')} \int_{\eta}^{\eta'} a^2(\eta'')d\eta'' - \frac{1}{2} \int_{\eta}^{0} \frac{a(\eta')d\eta'}{z^2(\eta')}.
$$

(5.19)

The functions $D(\eta)$ and $F(\eta)$ are defined so that $D, F \rightarrow 0$ as $\eta \rightarrow 0$. It is important to notice that $D(\eta)$ is the decaying mode in the long-wavelength limit, i.e., at leading order in gradient expansion, in the linear theory, satisfying

$$
D'' + 2\frac{\zeta'}{z} D' = 0 ,
$$

(5.20)

while $F(\eta)$ is the $\mathcal{O}(k^2)$ correction to the growing (constant) mode satisfying

$$
F'' + 2\frac{\zeta'}{z} F' + \epsilon^2_s = 0 ,
$$

(5.21)

where we assume that the growing mode solution is of the form $1 + k^2 F(\eta) + \mathcal{O}(k^4)$. In the above equations the prime stands for differentiation with respect to the conformal time and $\epsilon^2_s$ is the sound speed squared of the scalar fluctuations defined as

$$
\epsilon^2_s := \frac{F(t)}{\Theta(t)}, \quad F(t) := \frac{1}{X_0}(-\partial_t \Theta + \Theta G_X \phi_0).
$$

(5.22)

Using $D$ and $F$, Eq. (5.18) can be written as

$$
\mathcal{R}^{NL}_u(\eta) = (0)C^{0}(x) + (2)C^{0}(x) - (2)\tilde{R}(x)F(\eta) - (2)C^{0}(x)D(\eta) + \mathcal{O}(\epsilon^4),
$$

(5.23)
where time-independent terms of $O(\varepsilon^2)$ are collectively absorbed to $(2)\, C^{\text{NL}}(x)$. It turns out that the solution can be expressed simply in terms of the two time-dependent functions corresponding to the decaying mode and the $O(k^2)$ correction to the growing mode in the linear theory. This shows that, within $O(\varepsilon^2)$ accuracy in gradient expansion, the curvature perturbation $R_{\text{NL}}^u$ obeys the following nonlinear second-order differential equation:

$$R_{\text{NL}}^{u''} + 2\varepsilon' R_{\text{NL}}^{u'} + \frac{(2)^2}{4} R[\nabla^2 R_{\text{NL}}^u] = O(\varepsilon^4),$$  \hspace{1cm} (5.24)

where $(2)^2 R[\nabla^2 R_{\text{NL}}^u]$ is the Ricci scalar of the metric $\nabla^2$. Equation (5.24) is our main result. It is easy to see that in the linear limit Eq. (5.24) reproduces the previous result for the curvature perturbation in the unitary gauge $\text{Lin}$,

$$R_{\text{Lin}}^{u''} + 2\varepsilon' R_{\text{Lin}}^{u'} - \varepsilon^2 \nabla^2 R_{\text{Lin}}^u = 0,$$  \hspace{1cm} (5.25)

where $\Delta$ denotes the Laplacian operator on the flat background.

Equation (5.24) can be regarded as the master equation for the nonlinear superhorizon curvature perturbation at second order in gradient expansion. It must, however, be used with caution, since it is derived under the assumption that the decaying mode is negligible at leading order in gradient expansion. Moreover, if one sets the right-hand side of (5.24) set to exactly zero, this master equation becomes a closed equation, and it be a useful approximation to a full nonlinear solution on the Hubble horizon scales or even on scales somewhat smaller than the Hubble radius.

VI. SUMMARY AND DISCUSSION

In this paper, we have developed a theory of nonlinear cosmological perturbations on superhorizon scales for G-inflation, for which the inflaton Lagrangian is given by $W(X, \phi) = G(X, \phi) \square \phi$. In the case of $G_X = 0$, i.e., k-inflation, the energy-momentum tensor for the scalar field is equivalent to that of a perfect fluid. In the case of G-inflation, however, it can no longer be recast into a perfect fluid form, and hence its imperfect nature shows up when the inhomogeneity of the Universe is considered. We have solved the field equations using spatial gradient expansion in terms of a small parameter $\varepsilon := k/(aH)$, where $k$ is a wavenumber, and obtained a general solution for the metric and the scalar field up to $O(\varepsilon^2)$.

We have introduced an appropriately defined variable for the nonlinear curvature perturbation in the uniform $\phi$ gauge, $R_{\text{NL}}^u$. Upon linearization, this variable reduces to the previously defined linear curvature perturbation $R_{\text{Lin}}^u$ on uniform $\phi$ hypersurfaces. Then, it has been shown that $R_{\text{NL}}^u$ satisfies a nonlinear second-order differential equation (5.24), which is a natural extension of the linear perturbation equation for $R_{\text{Lin}}^u$. We believe that our result can further be extended to include generalized G-inflation, i.e., the most general single-field inflation model [37], though the computation required would be much more complicated.

The nonlinear evolution of perturbations, and hence the amount of non-Gaussianity, are affected by the $O(\varepsilon^2)$ corrections if, for example, there is a stage during which the slow-roll conditions are violated. Calculating the three point correlation function of curvature perturbations including the $O(\varepsilon^2)$ corrections will be addressed in a future publication. Finally we have comment on our method compared to the in-in formalism developed in the literature. Our formalism is valid on the classical evolution in superhorizon scales, while the in-in formalism can also calculate a quantum evolution on sub-horizon scale. So comparison with each other leads to picking out the quantum effect of non-Gaussianity directly. We have handled the curvature perturbation itself in our formalism, not the correlation function in the in-in one, then its time evolution is more clearly understood.

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[1] M. Sasaki and D. Wands, Classical and Quantum Gravity 27, 120301 (2010).
[2] J. M. Maldacena, JHEP 0305, 013 (2003), arXiv:astro-ph/0210603.
[3] V. Acquaviva, N. Bartolo, S. Matarrese and A. Riotto, Nucl. Phys. B 667, 119 (2003), arXiv:astro-ph/0209156.
[4] K. A. Malik and D. Wands, Class. Quant. Grav. 21, L65 (2004), arXiv:astro-ph/0307055.
[5] E. M. Lifshitz and I. M. Khalatnikov, Adv. Phys. 12, 185 (1963).
[6] A. A. Starobinsky, JETP Lett. 42, 152 (1985).
[7] D. S. Salopek and J. R. Bond, Phys. Rev. D42, 3936 (1990).
[8] N. Deruelle and D. Langlois, Phys. Rev. D52, 2007 (1995), arXiv:gr-qc/9411046.
[9] Nambu and A. Taruya, Class. Quant. Grav. 13, 705 (1996), arXiv:astro-ph/9911013.
[10] E. M. Lifshitz and I. M. Khalatnikov, Adv. Phys. 12, 185 (1963).
[11] A. A. Starobinsky, JETP Lett. 42, 152 (1985).
[12] D. S. Salopek and J. R. Bond, Phys. Rev. D42, 3936 (1990).
[13] N. Deruelle and D. Langlois, Phys. Rev. D52, 2007 (1995), arXiv:gr-qc/9411046.
[14] Y. Nambu and A. Taruya, Class. Quant. Grav. 13, 705 (1996), arXiv:astro-ph/9411013.
[15] M. Sasaki and T. Tanaka, Prog. Theor. Phys. 95, 71 (1996), arXiv:astro-ph/9507001.
[16] M. Sasaki and T. Tanaka, Prog. Theor. Phys. 99, 763 (1998), arXiv:gr-qc/9801017.
[17] H. Kodama and T. Hamazaki, Phys. Rev. D 57, 7177 (1998), arXiv:gr-qc/9712045.
[18] D. H. Lyth, K. A. Malik, D. H. Lyth, and A. R. Liddle, Phys. Rev. D62, 043527 (2000), arXiv:astro-ph/0003278.
[19] M. Sasaki and E. D. Stewart, Prog. Theor. Phys. 95, 71 (1996), arXiv:astro-ph/9507001.
[20] M. Sasaki and T. Tanaka, Prog. Theor. Phys. 99, 763 (1998), arXiv:gr-qc/9801017.
[21] H. Kodama and T. Hamazaki, Phys. Rev. D 57, 7177 (1998), arXiv:gr-qc/9712045.
[22] D. Wands, K. A. Malik, and M. Sasaki, JCAP 0505, 004 (2005), arXiv:astro-ph/0411220.
[23] D. H. Lyth, K. A. Malik, and M. Sasaki, JCAP 0505, 004 (2005), arXiv:astro-ph/0411220.
[24] G. I. Rigopoulos and E. P. S. Shellard, Phys. Rev. D 68, 123518 (2003), arXiv:astro-ph/0306620.
[25] G. I. Rigopoulos and E. P. S. Shellard, JCAP 0510, 006 (2005), arXiv:astro-ph/0405185.
[26] D. H. Lyth and Y. Rodriguez, Phys. Rev. D 71, 123508 (2005), arXiv:astro-ph/0502578.
[27] Y. Tanaka and M. Sasaki, Prog. Theor. Phys. 117, 633 (2007), arXiv:gr-qc/0612191.
[28] Y. Tanaka and M. Sasaki, Prog. Theor. Phys. 118, 455 (2007), arXiv:0706.0678.
[29] Y.-i. Takamizu and S. Mukohyama, JCAP 0901, 013 (2009), arXiv:0810.0746.
[30] Y.-i. Takamizu, S. Mukohyama, M. Sasaki, and Y. Tanaka, JCAP 1006, 019 (2010), arXiv:1004.1870.
[31] Y.-i. Takamizu and J. Yokoyama, Phys.Rev. D83, 043504 (2011), arXiv:1011.4566.
[32] Y.-i. Takamizu, S. Mukohyama, M. Sasaki, and Y. Tanaka, JCAP 1006, 019 (2010), arXiv:1004.1870.
[33] C. Armendariz-Picon, T. Damour and V. F. Mukhanov, Phys. Lett. B 458, 209 (1999), arXiv:hep-th/9904075.
[34] J. Garriga and V. F. Mukhanov, Phys. Lett. B 458, 219 (1999), arXiv:hep-th/9904176.
[35] C. Deffayet, O. Pujolas, I. Sawicki and A. Vikman, JCAP 1010, 026 (2010), arXiv:1008.0048.
[36] O. Pujolas, I. Sawicki and A. Vikman, JHEP 1111, 156 (2011), arXiv:1103.5360.
[37] T. Kobayashi, M. Yamaguchi and J. Yokoyama, Phys. Rev. Lett. 105, 231302 (2010), arXiv:1008.0603.
[38] T. Kobayashi, M. Yamaguchi and J. Yokoyama, Phys. Rev. D 83, 103524 (2011), arXiv:1103.1740.
[39] T. Kobayashi, M. Yamaguchi and J. Yokoyama, Prog. Theor. Phys. 126, 511 (2011), arXiv:1105.5723.
[40] D. A. Easson, I. Sawicki and A. Vikman, JCAP 1111, 021 (2011), arXiv:1109.1047.
[41] S. Mizuno and K. Koyama, Phys. Rev. D 82, 103518 (2010), arXiv:1009.0677.
[42] R. Kimura and K. Yamamoto, JCAP 1104, 025 (2011), arXiv:1011.2006.
[43] K. Kamada, T. Kobayashi, M. Yamaguchi and J. Yokoyama, Phys. Rev. D 83, 083515 (2011), arXiv:1012.4238.
[44] R. Kimura, T. Kobayashi and K. Yamamoto, JCAP 1104, 025 (2011), arXiv:1011.2006.
[45] C. Deffayet, Y. -F. Cai, D. A. Steer and G. Zahariade, JCAP 1208, 020 (2012), arXiv:1206.3879.
[46] J. Ohashi and S. Tsujikawa, JCAP 1205, 035 (2012), arXiv:1206.3879.
[47] C. Deffayet, X. Gao, D. A. Steer and G. Zahariade, Phys. Rev. D 84, 064039 (2011), arXiv:1103.3260.
[48] G. W. Horndeski, Int. J. Theor. Phys. 10 (1974) 363-384.
[49] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. 78 1 (1984).
[50] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rept. 215, 203 (1992).