The Stokes Paradox in Inhomogeneous Elastostatics

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Abstract  We prove that the displacement problem of inhomogeneous elastostatics in a two-dimensional exterior Lipschitz domain has a unique solution with finite Dirichlet integral $u$, vanishing uniformly at infinity if and only if the boundary datum satisfies a suitable compatibility condition (Stokes paradox). Moreover, we prove that it is unique under the sharp condition $u = o(\log r)$ and decays uniformly at infinity with a rate depending on the elasticities. In particular, if these last ones tend to a homogeneous state at large distance, then $u = O(r^{-\alpha})$, for every $\alpha < 1$.

Keywords  Inhomogeneous elasticity · Two–dimensional exterior domains · Existence and uniqueness theorems · Stokes paradox

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1 Introduction

Let $\Omega$ be an exterior Lipschitz domain of $\mathbb{R}^2$. The displacement problem of plane elastostatics in exterior domains is to find a solution to the equations

\[
\begin{align*}
\text{div } C [\nabla u] &= 0 \quad \text{in } \Omega, \\
u &= \hat{u} \quad \text{on } \partial\Omega, \\
\lim_{r \to +\infty} u(x) &= 0,
\end{align*}
\]

(1)

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where $\mathbf{u}$ is the (unknown) displacement field, $\hat{\mathbf{u}}$ is an (assigned) boundary displacement, $\mathbf{C} = [C_{ijkl}]$ is the (assigned) elasticity tensor, i.e., a map from $\Omega \times \text{Lin} \to \text{Sym}$, linear on $\text{Sym}$ and vanishing in $\Omega \times \text{Skw}$. We shall assume $\mathbf{C}$ to be symmetric, i.e., $C_{ijkl} = C_{klij}$ and positive definite, i.e.,

$$
\mu_0 |E|^2 \leq \mathbf{E} \cdot \mathbf{C} [\mathbf{E}] \leq \mu_\varepsilon |E|^2, \quad \forall \mathbf{E} \in \text{Sym}, \quad \text{a.e. in } \Omega.
$$

By appealing to the principle of virtual work and taking into account that $\mathbf{\varphi} \in C^\infty_0 (\Omega)$ is an admissible (or virtual) displacement, we say that $\mathbf{u} \in W^{1,2}_\text{loc} (\Omega)$ is a weak solution (variational solution for $q = 2$) to (1) if

$$
\int_{\Omega} \nabla \mathbf{\varphi} \cdot \mathbf{C} [\nabla \mathbf{u}] = 0, \quad \forall \mathbf{\varphi} \in C^\infty_0 (\Omega).
$$

A weak solution to (1) is a weak solution to (1) which satisfies the boundary condition in the sense of the trace in Sobolev’s spaces and tends to zero at infinity in a generalized sense. If $\mathbf{u} \in W^{1,q}_\text{loc} (\Omega)$ is a weak solution to (1) the traction field on the boundary

$$
\mathbf{s} (\mathbf{u}) = \mathbf{C} [\nabla \mathbf{u}] n
$$

exists as a well defined field of $W^{-1/q,q} (\partial \Omega)$ and for $q = 2$ the following generalized work and energy relation [9] holds

$$
\int_{\Omega_R} \nabla \mathbf{u} \cdot \mathbf{C} [\nabla \mathbf{u}] = \int_{\partial \Omega} \mathbf{u} \cdot \mathbf{s} (\mathbf{u}) + \int_{\partial S_R} \mathbf{u} \cdot \mathbf{s} (\mathbf{u}),
$$

for every $R$ such that $S_R \supset \bar{\Omega}^c$, where with abuse of notation by $\int_{\Sigma} \mathbf{u} \cdot \mathbf{s} (\mathbf{u})$ we mean the value of the functional $\mathbf{s} (\mathbf{u}) \in W^{-1/2,2} (\Sigma)$ at $\mathbf{u} \in W^{1/2,2} (\Sigma)$ and $n$ is the unit outward (with respect to $\Omega$) normal to $\partial \Omega$. It will be clear from the context when we shall refer to an ordinary integral or to a functional.

It is a routine to show that under assumption (2), (1) has a unique solution $\mathbf{u} \in D^{1,2} (\Omega)$, we shall call $D$–solution (for the notation see at the end of this section). Moreover, it exhibits more regularity provided $\mathbf{C}$, $\partial \Omega$ and $\hat{\mathbf{u}}$ are more regular. In particular, the following well–known theorem holds [8, 12].

**Theorem 1** Let $\Omega$ be an exterior Lipschitz domain of $\mathbb{R}^2$ and let $\mathbf{C}$ satisfy (2). If $\hat{\mathbf{u}} \in W^{1/2,2} (\partial \Omega)$, then (1) has a unique $D$–solution $\mathbf{u}$ which is locally Hölder continuous in $\Omega$. Moreover, if $\Omega$ is of class $C^k$, $\mathbf{C} \in C^{k-1}_\text{loc} (\bar{\Omega})$ and $\hat{\mathbf{u}} \in W^{k-1/q,q} (\partial \Omega)$ ($k \geq 1$, $q \in (1, +\infty)$), then $\mathbf{u} \in W^{k,q}_\text{loc} (\bar{\Omega})$.

The main problem left open by Theorem 1 is to establish the behavior of the variational solution at large distance: does $\mathbf{u}$ converge to a constant vector at infinity and, if so, does (or under what conditions and in what sense) $\mathbf{u}$ satisfy (1)? For constant $\mathbf{C}$ (homogeneous elasticity) the situation is well understood (see, e.g., [18, 20]), at least in its negative information. Indeed, a solution to (1) is expressed by a simple layer potential

$$
\mathbf{u} (x) = \mathbf{v} (\mathbf{\psi} (x)) + \kappa,
$$

1For constant $\mathbf{C}$ (homogeneous elasticity) it is sufficient to assume that $\mathbf{C}$ is strongly elliptic, i.e., there is $\lambda_0 > 0$ such that $\lambda_0 |a|^2 |b|^2 \leq a \cdot \mathbf{C} [a \otimes b] b$, for all $a, b \in \mathbb{R}^2$. Springer
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for some \( \psi \in W^{-1/2,2}(\partial \Omega) \), where

\[
v[\psi](x) = \int_{\partial \Omega} U(x - y) \psi(y) ds_y
\]

is the simple layer with density \( \psi \) such that

\[
\int_{\partial \Omega} \psi = 0
\]

is the simple layer with density \( \psi \) such that

\[
\int_{\partial \Omega} \psi = 0
\]

and

\[
U(x - y) = \Phi_0 \log |x - y| + \Phi(x - y),
\]

with \( \Phi_0 \in \text{Lin} \) and \( \Phi : \mathbb{R}^2 \setminus \{0\} \to \text{Lin} \) homogeneous of degree zero, is the fundamental solution to equations (1) (see, e.g., [10]). The space \( C = \{ \psi \in L^2(\partial \Omega) : v[\psi]_{|\partial \Omega} = \text{constant} \} \) has dimension two and if \( \{ \psi_1, \psi_2 \} \) is a basis of \( C \), then \( \int_{\partial \Omega} \psi_1, \int_{\partial \Omega} \psi_2 \) is a basis of \( \mathbb{R}^2 \); (3) assures that \( u - \kappa = O(r^{-1}) \), where the constant vector \( \kappa \) is determined by the relation

\[
\int_{\partial \Omega} (\hat{u} - \kappa) \cdot \psi' = 0 \quad \forall \psi' \in C.
\]

Hence it follows

**Theorem 2** Let \( \Omega \) be an exterior Lipschitz domain of \( \mathbb{R}^2 \) and let \( C \) be constant and strongly elliptic. If \( \hat{u} \in W^{1/2,2}(\partial \Omega) \), then (1) has a unique \( D \)–solution, analytic in \( \Omega \), if and only if

\[
\int_{\partial \Omega} \hat{u} \cdot \psi' = 0, \quad \forall \psi' \in C \iff \int_{\partial \Omega} s(u) = 0. \tag{4}
\]

Moreover, \( u \) is unique in the class

\[
\{ u \in W^{1,2}_{\text{loc}}(\Omega) : u = o(\log r) \}
\]

and modulo a field \( v[\psi'] - v[\psi']_{|\partial \Omega}, \psi' \in C \), in the class

\[
\{ u \in W^{1,2}_{\text{loc}}(\Omega) : u = o(r) \}.
\]

An immediate consequence of (4) is nonexistence of a solution to (1) corresponding to a constant boundary datum. This phenomenon for the Stokes’ equations

\[
\mu \Delta u - \nabla p = 0,
\]

\[
\text{div } u = 0,
\]

is popular as Stokes paradox and goes back to the pioneering work of G.G. Stokes (1851) on the study of the (slow) translational motions of a ball in an incompressible viscous fluid of viscosity \( \mu \) (see [7] and Ch. V of [6]). Clearly, as it stands, Stokes paradox can be read only as a negative result, unless we are able to find an analytic expression of the densities of \( C \). As far as we know, this is possible only for the ellipse of equation \( f(\xi) = 1 \). Indeed, in this case it is known that \( C = \text{spn}\{ e_1/|\nabla f|, e_2/|\nabla f| \} \) (see, e.g., [23]) and Theorem 2 reads
Theorem 3 Let $\Omega$ be the exterior of an ellipse of equation $f(\xi) = 1$ and let $C$ be constant and strongly elliptic. If $\hat{u} \in W^{1/2,2}(\partial \Omega)$, then (1) has a unique solution expressed by a simple layer potential, with a density $\psi \in W^{-1/2,2}(\partial \Omega)$ satisfying (3), if and only if
\[
\int_{\partial \Omega} \frac{\hat{u}}{\sqrt{|\nabla f|}} = 0.
\]

The situation is not so clear in inhomogeneous elasticity. In fact, in such a case it is not known whether $u$ converges at infinity and even the definition of the space $C$ needs to be clarified.

The purpose of this paper is to show that results similar to those stated in Theorem 2 hold in inhomogeneous elasticity, at least in its negative meaning.

By $\mathcal{M}$ we shall denote the linear space of variational solutions to
\[
\text{div } C[\nabla h] = 0 \quad \text{in } \Omega,
\]
\[
h = 0 \quad \text{on } \partial \Omega,
\]
\[
h \in BMO.
\]

We say that $C$ is regular at infinity if there is a constant elasticity tensor $C_0$ such that
\[
\lim_{|x| \to +\infty} C(x) = C_0.
\]

The following theorem holds.

Theorem 4 (Stokes Paradox of inhomogeneous elastostatics) Let $\Omega$ be an exterior Lipschitz domain of $\mathbb{R}^2$ and let $C$ satisfy (2). It holds:

(i) $\dim \mathcal{M} = 2$ and if $\{h_1, h_2\}$ is a basis of $\mathcal{M}$, then $\int_{\partial \Omega} s(h_1), \int_{\partial \Omega} s(h_2)$ is a basis of $\mathbb{R}^2$.

(ii) If $\hat{u} \in W^{1/2,2}(\partial \Omega)$, then system (1) has a unique $D$–solution $u$ if and only if
\[
\int_{\partial \Omega} \hat{u} \cdot s(h) = 0, \quad \forall h \in \mathcal{M}.
\]

(iii) $u$ is unique in the class (5) and modulo a field $h \in \mathcal{M}$ in the class
\[
\{u \in W_{loc}^{1,2}(\Omega) : u = o(r^{\gamma/2})\},
\]
where
\[
\gamma = \frac{\mu_0}{\mu_0 + 2\mu_e}
\]
(iv) there is a positive $\alpha$ depending on the elasticities such that
\[
u = O(r^{-\alpha})
\]
Moreover, if $C$ is regular at infinity then (8) holds for all $\alpha < 1$.

Clearly, (i)–(ii) imply in particular that if $\hat{u}$ is constant, then (1) has no solution in $D^{1,2}(\Omega)$. Similar results have been recently proved for the Stokes system in [19] (Stokes paradox).
Remark 1 (5) is a sharp uniqueness class. As far as we are aware, the largest uniqueness class known was \( \{ u \in W^{1,2}_\text{loc}(\Omega) : u = O(\log^{(1+\beta)/2} r) \} \), with \( \beta > 1 \) depending on \( \partial \Omega \) (see [21]).

Also, for more particular tensor \( \mathbf{C} \) we prove

**Theorem 5** Let \( \Omega \) be an exterior Lipschitz domain of \( \mathbb{R}^2 \) and let \( \mathbf{C} : \Omega \times \text{Lin} \to \text{Lin} \) satisfy

\[
\lambda |\mathbf{E}|^2 \leq \mathbf{E} : \mathbf{C}[\mathbf{E}] \leq \Lambda |\mathbf{E}|^2, \quad \forall \mathbf{E} \in \text{Lin}.
\]

A variational solution to the system

\[
\begin{align*}
\text{div} \mathbf{C}[\nabla u] &= 0 \quad \text{in } \Omega, \\
u &= \hat{u} \quad \text{on } \partial \Omega,
\end{align*}
\]

is unique in the class

\[
\{ u : u = o(r^{1/\sqrt{L}}) \} / \mathcal{M}, \quad L = \Lambda / \lambda,
\]

and if \( u \) belongs to \( D^{1,2}(\mathcal{L}S_{R_0}) \), then

\[
u - u_0 = O(r^{-1/\sqrt{L}}),
\]

for all positive \( \varepsilon \), where \( u_0 \) is the constant vector defined by

\[
\int_{\partial \Omega} (\hat{u} - u_0) \cdot s(h) = 0, \quad \forall h \in \mathcal{M}.
\]

Theorems 4, 5 are proved in Sect. 3. In Sect. 2 we collect the main tools we shall need to prove them and in Sect. 4, by means of a counter-example, we observe that the exponent in (10) is sharp: uniqueness fails in the class defined by (10) with \( O \) instead of \( o \).

**Notation** Unless otherwise specified, we will essentially use the notation of the classical monograph [9] of M.E. Gurtin. In indicial notation (\( \text{div} \mathbf{C}[\nabla u] \)) \(i = \partial_j (C_{ijhk} \partial_k u_h) \). Lin is the space of second–order tensors (linear maps from \( \mathbb{R}^2 \) into itself) and Sym, Skw are the spaces of the symmetric and skew elements of Lin respectively. As is customary, if \( \mathbf{E} \in \text{Lin} \) and \( \mathbf{v} \in \mathbb{R}^2 \), \( \mathbf{E} \mathbf{v} \) is the vector with components \( E_{ij} v_j \) and \( \hat{\mathbf{v}} \), \( \tilde{\mathbf{v}} \) denote respectively the symmetric and skew parts of \( \nabla \mathbf{u} \). \( \Omega = \mathbb{R}^2 \setminus \overline{\Omega'} \), with \( \Omega' \) bounded; \( (o, (e_i))_{i=1,2}, o \in \Omega' \), is the standard orthonormal reference frame; \( x = x - o, r = |x|, S_R = \{ x \in \mathbb{R}^2 : r < R \} , T_R = S_R \setminus S_{2R}, \Omega_R = \Omega \cap S_R ; R_0 \) is a large positive constant such that \( S_{R_0} \supset \overline{\Omega'} ; e_r = x/r \), for all \( x \neq o \). \( W^{k,q}(\Omega) \) \((k \in \mathbb{N}_0, q \in (1, +\infty))\) denotes the ordinary Sobolev’s space [8]; \( W^{k,q}_\text{loc}(\Omega) \) and \( W^{k,q}_\text{loc}(\overline{\Omega}) \) are the spaces of all \( \varphi \in W^{k,q}(K) \) such that \( \varphi \in W^{k,q}_\text{loc}(K) \) for every compact \( K \subset \Omega \) and \( K \subset \overline{\Omega} \) respectively. \( W^{1,q}_\text{loc}(\partial \Omega) \) is the trace space of \( D^{1,q}(\Omega) = \{ \varphi \in L^{1,q}_\text{loc}(\Omega) : ||\nabla \varphi||_{L^q(\Omega)} < +\infty \} \) \((q > 1)\) and \( W^{−1,q}_\text{loc}(\partial \Omega) \) is its dual space. \( BMO = \text{BMO}(\mathbb{R}^2) = \{ \varphi \in L^1_\text{loc}(\mathbb{R}^2) : \sup_{K \in \mathbb{R}_2} \frac{1}{|K|} \int_{SR} |\varphi - \varphi_{SR}| < +\infty \} \). \( H^1(\mathbb{R}^2) \) is the Hardy space. As is usual, if \( f(x) \) and \( \phi(r) \) are functions defined in a neighborhood of infinity \( \mathcal{L}S_{R_0} \), then \( f(x) = o(\phi(r)) \) and \( f(x) = O(\phi(r)) \) mean respectively that \( \lim_{r \to +\infty} f/\phi = 0 \) and \( f/\phi \) is bounded in \( \mathcal{L}S_{R_0} \). To alleviate notation, we do not distinguish between scalar, vector and second–order tensor space functions; \( c \) will denote a positive constant whose numerical value is not essential to our purposes; also we let \( c(\varepsilon) \) denote a positive function of \( \varepsilon > 0 \) such that \( \lim_{\varepsilon \to 0^+} c(\varepsilon) = 0 \).
2 Preliminary Results

Let us collect the main tools we shall need to prove Theorem 4 and 5 and that have some interest in themselves. By \( \mathcal{I} \) we shall denote the exterior of a ball \( S_{R_0} \supseteq \mathbb{C} \Omega \).

Lemma 1 ([5, 12]) Let \( u \in D^{1,q}(\mathcal{I}), q \in (1, +\infty) \). If \( q > 2 \) then \( u/r \in L^q(\mathcal{I}) \) and if \( q < 2 \), then there is a constant vector \( u_0 \) such that
\[
\int_{\mathcal{I}} \frac{|u - u_0|^q}{r^q} \leq c \int_{\mathcal{I}} |\nabla u|^q \quad \text{Hardy's inequality.}
\]
Moreover, if \( u \in D^{1,q}(\mathcal{I}) \) for all \( q \) in a neighborhood of 2, then \( u = u_0 + o(1) \).

The following classical result is due to N.G. Meyers [13].

Lemma 2 Let \( \Omega, C \) and \( \hat{u} \) satisfy the hypotheses of Theorem 1 and let \( u \) be the D–solution to system \((1)_{1,2}\). Then, there exists \( \bar{q} > 2 \) depending on \( C \) such that
\[
|u|_{W^{1,q}_{\text{loc}}(\Omega)} \leq c \quad \forall q \in (1, \bar{q}).
\]

Lemma 3 If \( u \) is a variational solution to \((1)_{1}\) in \( S_R \), then for all \( 0 < \rho < R \leq \tilde{R} \),
\[
\int_{S_{\rho}} |\nabla u|^2 \leq c \left( \frac{\rho}{R} \right)^{\gamma} \int_{S_R} |\nabla u|^2, \quad \gamma = \frac{\mu_0}{\mu_0 + 2\mu_e}.
\]

Proof Assume first that \( u \) is regular. Taking into account that
\[
|\hat{\nabla} u|^2 - |\tilde{\nabla} u|^2 = \nabla u \cdot \nabla u^\top = \text{div}[(\nabla u)u - (\text{div} u)1]u + |\text{div} u|^2,
\]
a simple computation yields
\[
\mu_0 G(R) = \mu_0 \int_{S_R} |\nabla u|^2 \leq \mu_0 \int_{S_R} (|\nabla u|^2 + |\text{div} u|^2)
\]
\[
= -\mu_0 \int_{\partial S_R} e_R \cdot [\nabla u - (\text{div} u)1]u + 2\mu_0 \int_{S_R} |\hat{\nabla} u|^2 \leq 2 \int_{\partial S_R} u \cdot C[\nabla u]e_R - \mu_0 \int_{\partial S_R} e_R \cdot [\nabla u - (\text{div} u)1]u.
\]

\[\text{It is worth recalling that for } q = 2 \text{ Hardy’s inequality takes the form}
\]
\[
\int_{\mathcal{I}} \frac{|u|^2}{r^2 \log^2 r} \leq 4 \int_{\mathcal{I}} |\nabla u|^2 + \frac{2 \pi}{\log R_0} \int_{0}^{2\pi} |u|^2(R_0, \theta).
\]

\[\text{By virtue of } [14] \text{ } \bar{q} \text{ cannot be too large.}\]
Since
\[
\int_{0}^{2\pi} \left[ (\nabla u)^\top e_R - (\text{div } u)e_R \right](R, \theta) d\theta = 0,
\]
\[
\int_{0}^{2\pi} C[\nabla u]e_R(R, \theta) d\theta = 0,
\]
by Schwarz’s inequality, Wirtinger’s inequality and bearing in mind that 
\[|\nabla u - (\text{div } u)1| = |\nabla u|,\]
and taking into account that by the basic calculus
\[
G'(R) = \int_{\partial S_R} |\nabla u|^2, \]
(13) yields
\[
\gamma G(R) \leq R G'(R).
\]
Hence (11) follows by a simple integration. The above argument applies to a variational solution by a classical approximation argument (see, e.g., footnote 1 in [14]).

**Remark 2** If \(u\) is a variational solution to (1), vanishing on \(\partial \Omega\) and such that \(\int_{\partial \Omega} s(u) = 0\), then by repeating the steps in the proof of Lemma 3, it follows
\[
\int_{\Omega_R} |\nabla u|^2 \leq c \left( \frac{\rho}{R} \right)^{\gamma} \int_{\Omega} |\nabla u|^2.
\]

**Lemma 4** If \(u\) is a variational solution to
\[
\text{div } C[\nabla u] + f = 0 \quad \text{in } \Omega, 
\]
with \(f\) having compact support, then
\[
\int_{\Omega_R} |\nabla u|^2 \leq c \left\{ \frac{1}{R^2} \int_{\Omega} |u|^2 + \sigma(u) \right\},
\]
(16)
where\footnote{By abuse of notation, when $f$ is a distribution the last integral is understood as a duality pairing.}
\[
\sigma(u) = 2 \int_{\partial\Omega} u \cdot s(u) - \mu_0 \int_{\Omega} n \cdot (\nabla u - (\text{div} u)1) + 2 \int_{\Omega} f \cdot u.
\]

**Proof** Let
\[
g_R(r) = \begin{cases} 
0, & r > 2R, \\
1, & r < R, \\
R^{-1}(2R - r), & r \in [R, 2R].
\end{cases}
\] (17)
with $S_R \supset \text{supp} f$. A standard calculation similar to (13) yields
\[
\mu_0 \int_{\Omega} g_R^2 (|\nabla u|^2 + |\text{div} u|^2) \leq 2 \int_{\Omega} g_R \nabla g_R \cdot (\mu_0(\nabla u - (\text{div} u)1) - 2C[\nabla u])u
\]
\[+ 2 \int_{\partial\Omega} u \cdot s(u) - \mu_0 \int_{\partial\Omega} n \cdot (\nabla u - (\text{div} u)1) + 2 \int_{\Omega} f \cdot u.
\] (18)
By a simple application of Cauchy’s inequality (18) implies
\[
\int_{\Omega} g_R^2 |\nabla u|^2 \leq c \left( \int_{\Omega} |\nabla g_R|^2 |u|^2 + \sigma(u) \right).
\]
Hence (16) follows by the properties of the function $g_R$. \hfill \Box

**Remark 3** Under the stronger assumption $u$ is a $D$-solution, we can repeat the previous argument to obtain instead of (16) the following inequality
\[
\int_{\subset S_R} |\nabla u|^2 \, dx \leq \frac{c}{R^2} \int_{T_R} |u|^2.
\] (19)
In such case instead of the function $g_R$ we have to consider the function
\[
\eta_R(r) = \begin{cases} 
0, & r < R, \\
1, & r > 2R, \\
R^{-1}(r - R), & r \in [R, 2R].
\end{cases}
\] (20)
and the thesis follows similarly.

**Lemma 5** Let $u$ be a variational solution to (1) such that
\[
\int_{\partial\Omega} s(u) = 0.
\] (21)
If
\[
u(x) = o(r^{\gamma/2}),
\] (22)
then $\nabla u \in L^2(\Omega)$ and

$$\int_{\Omega} \nabla u \cdot C[\nabla u] = \int_{\partial \Omega} u \cdot s(u).$$  (23)

**Proof** Let $\eta_{\tilde{R}}$ be the function defined in (20). The field

$$v = \eta_{\tilde{R}} u$$  (24)

is a variational solution to

$$\text{div} \ C[\nabla v] + f = 0 \quad \text{in } \mathbb{R}^2,$$  (25)

with

$$f_i = -C_{ijhk} \partial_k u_h \partial_j \eta_{\tilde{R}} - \partial_j (C_{ijhk} \partial_k u_h \partial_j \eta_{\tilde{R}}).$$  (26)

Let $v_1$ and $v_2$ be the variational solutions to the systems

$$\text{div} \ C[\nabla v_1] = 0 \quad \text{in } S_R,$$

$$v_1 = v \quad \text{on } \partial S_R,$$  (27)

and

$$\text{div} \ C[\nabla v_2] + f = 0 \quad \text{in } S_R,$$

$$v_2 = 0 \quad \text{on } \partial S_R,$$  (28)

respectively, with $R > 2 \tilde{R}$. By (11)

$$\int_{\tilde{S}_R} |\nabla v_1|^2 \leq c \int_{\tilde{S}_R} \gamma |\nabla v_1|^2.$$  (29)

A simple computation and the first Korn inequality

$$\|\nabla v_2\|_{L^2(S_R)} \leq \sqrt{2}\|\hat{\nabla} v_2\|_{L^2(S_R)}$$

yield

$$\mu_0 \int_{S_R} |\nabla v_2|^2 \leq 2 \int_{\tilde{T}_R} \partial_j v_{2i} C_{ijhk} \partial_k u_h \partial_j \eta_{\tilde{R}} - 2 \int_{\tilde{T}_R} v_{2i} C_{ijhk} \partial_k u_h \partial_j \eta_{\tilde{R}} = J_1 + J_2.$$

By Schwarz’s inequality

$$|J_1|^2 \leq c \int_{S_R} |\nabla v_2|^2 \int_{\tilde{T}_R} |u|^2 \leq c \int_{S_R} |\nabla v_2|^2,$$

and since by (21) $\int_{\tilde{T}_R} C_{ijhk} \partial_k u_h \partial_j \eta_{\tilde{R}} = 0$,

$$|J_2|^2 \leq c \int_{\tilde{T}_R} \left| \frac{1}{|\tilde{T}_R|} \int_{\tilde{T}_R} v_2 \right|^2 \int_{\tilde{T}_R} |C[\nabla u]|^2 \leq c \int_{S_R} |\nabla v_2|^2.$$
Hence

\[ \int_{S_R} |\nabla v_2|^2 \leq c_0. \]  

By uniqueness \( v = v_1 + v_2 \) in \( S_R \). Therefore, putting together (29), (30), using the inequality \(|a + b|^2 \leq 2|a|^2 + 2|b|^2 \) and Lemma 4, we get

\[
\int_{S_\rho} |\nabla v|^2 \leq 2 \int_{S_\rho} (|\nabla v_1|^2 + |\nabla v_2|^2) \leq c \left( \frac{\rho}{R} \right)^{\gamma} \int_{S_R} |\nabla v_1|^2 + c_0
\]

\[
\leq c \left( \frac{\rho}{R} \right)^{\gamma} \int_{S_R} |\nabla v|^2 + c_0 \leq c(\rho) R^{2 + \gamma} \int_{\mathcal{T}_R} |u|^2 + c.
\]  

Hence, taking into account (22), letting \( R \to +\infty \), we obtain \( \nabla u \in L^2(\Omega) \).

Let consider now the function (17). Multiplying (1) \text{1} scalarly by \( g_R u \) and integrating by parts, we get

\[ \int_{\Omega} g_R \nabla u \cdot \mathbf{C}[\nabla u] = \int_{\partial \Omega} u \cdot s(u) - \int_{\mathcal{T}_R} \nabla g_R \cdot \mathbf{C}[\nabla u] u. \]  

From (21) it follows that \( \int_{\mathcal{T}_R} \mathbf{C}[\nabla u] e_r = 0 \), so that by applying Schwarz’s inequality and Poincaré’s inequality

\[ \left| \int_{\Omega} \nabla g_R \cdot \mathbf{C}[\nabla u] u \right| \leq \frac{c}{R} \left( \int_{\mathcal{T}_R} |u - u_{\mathcal{T}_R}|^2 \right)^{1/2} \left( \int_{\mathcal{T}_R} |\nabla u|^2 \right)^{1/2} \leq c \int_{\mathcal{T}_R} |\nabla u|^2. \]

Therefore, (23) follows from (32) by letting \( R \to +\infty \) and taking into account the properties of \( g_R \) and that \( \nabla u \in L^2(\Omega) \). \( \square \)

\textbf{Remark 4} In the previous Lemma we proved, in particular, that a variational solution which satisfies (21) and (22) is a \( D \)-solution. Another sufficient condition to have a \( D \)-solution is to assume (21) and \( u \in D^{1,q}(S_{R_0}) \), for some \( q \in \left( 2, \frac{4}{2 - \gamma} \right) \). Indeed, by reasoning as in (31) and applying Hölder’s inequality we obtain

\[ \int_{S_\rho} |\nabla v|^2 \leq c \left( \frac{\rho}{R} \right)^{\gamma} \int_{S_R} |\nabla v|^2 + c_0 \leq c(\rho) R^{2 - 2q} \left( \int_{S_{R_0}} |\nabla v|^q \right)^{2/q} + c_0. \]

Then we get \( \nabla u \in L^2(\Omega) \) on letting \( R \to +\infty \).

\textbf{Remark 5} From Lemma 5 it follows that up to a constant the homogeneous traction problem

\[
\begin{align*}
\text{div } \mathbf{C}[\nabla u] &= 0 \quad \text{in } \Omega, \\
s(u) &= 0 \quad \text{on } \partial \Omega, \\
u(x) &= o(r^{\gamma/2}),
\end{align*}
\]

has only the trivial solution.
Lemma 6  A $D$–solution $u$ to (1) satisfies (21) and for all $R > \rho \gg R_0$,
\[
\int_{\mathcal{S}_R} |\nabla u|^2 \leq c \left( \frac{\rho}{R} \right)^\gamma \int_{\mathcal{S}_\rho} |\nabla u|^2.
\] (33)

Proof  As in the proof of Lemma 3, it is sufficient to assume $u$ regular. Multiplying (1) by the function (17) and integrating over $\Omega$, we have
\[
\int_{\partial \Omega} s(u) = \int_{\mathcal{T}_R} \mathbf{C}[\nabla u] \nabla \mathbf{g}_R.
\]
Hence (21) follows taking into account that by Schwarz’s inequality
\[
\left| \int_{\partial \Omega} s(u) \right| = \left| \int_{\mathcal{T}_R} \mathbf{C}[\nabla u] \nabla \mathbf{g}_R \right| \leq \frac{1}{R} \left\{ \int_{\mathcal{T}_R} |\nabla u|^2 \right\}^{1/2} \left\{ \int_{\mathcal{T}_R} \right\}^{1/2} \leq c \| \nabla u \|_{L^2(\mathcal{T}_R)},
\]
and letting $R \to +\infty$.

A standard computation similar to (13) yields, for $\varrho > R$,
\[
\mu_0 \int_{\mathcal{S}_R} g_\varrho (|\nabla u|^2 + |\text{div} u|^2) \leq -\int_{\partial \mathcal{S}_\varrho} \mathbf{e}_R \cdot [2\mathbf{C}[\nabla u] - \mu_0(\nabla u - (\text{div} u) \mathbf{1})] u + \frac{1}{\varrho} \int_{\mathcal{T}_\varrho} \mathbf{e}_\varrho \cdot [2\mathbf{C}[\nabla u] - \mu_0(\nabla u - (\text{div} u) \mathbf{1})] u.
\]
Hence, since by (14), Schwarz’s inequality and Wirtinger’s inequality
\[
\left| \frac{1}{\varrho} \int_{\mathcal{T}_\varrho} \mathbf{e}_\varrho \cdot [2\mathbf{C}[\nabla u] - \mu_0(\nabla u - (\text{div} u) \mathbf{1})] u \right| \leq c \| \nabla u \|_{L^2(\mathcal{T}_\varrho)}^2,
\]
letting $\varrho \to +\infty$, it follows
\[
\mu_0 \int_{\mathcal{S}_R} (|\nabla u|^2 + |\text{div} u|^2) \leq -\int_{\partial \mathcal{S}_R} \mathbf{e}_R \cdot [2\mathbf{C}[\nabla u] - \mu_0(\nabla u - (\text{div} u) \mathbf{1})] u. \] (34)

Now proceeding as we did in the proof of Lemma 3, (34) yields
\[
\gamma Q(R) = \gamma \int_{\mathcal{S}_R} |\nabla u|^2 \leq R \int_{\partial \mathcal{S}_R} |\nabla u|^2. \] (35)

Since by the basic calculus
\[
Q'(R) = -\int_{\partial \mathcal{S}_R} |\nabla u|^2,
\]
(33) follows from (35) by a simple integration. \qed
Lemma 7 There is $\epsilon = \epsilon(\gamma) > 0$ such that every $D$-solution $u$ to $(1)_1$ belongs to $D^{1,q}(\mathcal{I})$ for all $q \in (2 - \epsilon, 2 + \epsilon)$. Moreover, if $C$ is regular at infinity, then $u \in D^{1,q}(\mathcal{I})$ for all $q \in (1, +\infty)$.

Proof To prove the lemma we follow a standard argument (see, e.g., [1] p. 92). Let $\eta_{\bar{R}}$ be the function defined by (20). The field $v = \eta_{\bar{R}}u$ is a variational solution to
\[ \text{div } C_0[\nabla v] + \text{div}(C - C_0)[\nabla v] + f = 0 \quad \text{in } \mathbb{R}^2, \]
where $f$ is defined by (26) and $C_0$ is a constant elasticity tensor. Let $U(x - y)$ be the fundamental solution to the operator $\text{div } C_0[\nabla \cdot]$, the integral transform
\[ Q[v](x) = \nabla \int_{\mathbb{R}^2} U(x - y)(C - C_0)[\nabla v](y)dy \]
maps $D^{1,q}$ into itself for every $q \in (1, +\infty)$. Set
\[ v_f(x) = \int_{\mathbb{R}^2} U(x - y)f(y)dy \in D^{1,q}(\mathbb{R}^2), \quad \forall q \in (1, +\infty), \]
and consider the integral equation in $D^{1,q}$
\[ v'(x) = v_f(x) + Q[v](x). \quad (36) \]
Choose
\[ C_{0,ijkh} = \mu_0 \delta_{ih} \delta_{jk}. \]
Since [1]
\[ \|Q[v]\|_{D^{1,q}} \leq c(q) \frac{\mu_0 - \mu_e}{\mu_e} \|v\|_{D^{1,q}} \]
and
\[ \lim_{q \to 2} c(q) = 1, \]
there is $\epsilon > 0$ such that (36) is a contraction in $D^{1,q}$, $q \in (2 - \epsilon, 2 + \epsilon)$. Therefore it has a unique fixed point which must coincide with $v$; but $v = u$ at large distance and so $u \in D^{1,q}(\mathcal{I})$. If $C$ is regular at infinity, then, choosing $\bar{R}$ large as we want, we can make $|C(x) - C_0|$ arbitrarily small and, as a consequence, $\|Q[v]\|_{D^{1,q}} \leq \beta \|v\|_{D^{1,q}}$, for every small positive $\beta$ and this is sufficient to conclude the proof. \qed

Extend $C$ to the whole of $\mathbb{R}^2$ by setting $C = \tilde{C}$ in $\mathring{\Omega}$ (say), with $\tilde{C}$ constant and positive definite. Clearly, the new elasticity tensor (we denote by the same symbol) satisfies (2) (almost everywhere) in $\mathbb{R}^2$.

The Hölder regularity of variational solutions to $(1)_1$ is sufficient to prove the unique existence of a fundamental (or Green) function $G(x, y)$ to $(1)_1$ in $\mathbb{R}^2$ (see [2, 4, 11, 22]), which satisfies
\[ \varphi(x) = \int_{\mathbb{R}^2} \nabla \varphi(y) \cdot C[\nabla G(x, y)]dy, \]
for all $\varphi \in C_0^\infty(\mathbb{R}^2)$. It is a variational solution to (1) in $x$ [resp. in $y$] in every domain not containing $y$ [resp. $x$]. Moreover, $G(x, y) = G^T(y, x)$ and for $f \in \mathcal{H}^1(\mathbb{R}^2)$ the field

$$u(x) = \int_{\mathbb{R}^2} G(x, y) f(y) \, da_y \in D^{1,2}(\mathbb{R}^2) \cap C^{0,\mu}_{\text{loc}}(\mathbb{R}^2)$$

is the unique variational solution to

$$\text{div} \mathbf{C}[\nabla u] + f = 0 \quad \text{in} \ \mathbb{R}^2. \quad (37)$$

$G(x, \cdot)$ belongs to the John–Nirenberg space $BMO(\mathbb{R}^2)$ (see, e.g., [8]) and has a logarithm singularity at $x$ and at infinity. Set $w(x) = G(x, o) e$, with $e$ constant vector. Let us show that $\nabla w/ \in L^2(\partial \mathbb{S}_R^0)$ and $\nabla w/ \in L^q(\partial \mathbb{S}_R^0)$ for all $q$ in a right neighborhood of 2. Indeed, if $w \in D^{1,2}(\mathbb{S}_R^0)$, then, by applying (19) and Hölder’s inequality, we get

$$\int_{\mathbb{S}_R} |\nabla w|^2 \leq C R^{-4/q} \left\{ \int_{\mathbb{S}_R} |w|^q \right\}^{2/q}, \quad q > 2.$$

Therefore, from (33) it follows

$$\int_{\mathbb{S}_R} |\nabla w|^2 \leq \frac{c_p \gamma^{-4/q}}{R^\gamma} \int_{\mathbb{S}_R} |w|^q.$$

Hence, choosing $q > 4/\gamma$, letting $\rho \to 0$ and taking into account that $w \in L^q_{\text{loc}}(\mathbb{R}^2)$, we have the contradiction $\nabla w = 0$. The field $v = \eta_R w$ is a solution to (37) where $\eta_R$ and $f$ are defined by (20), (26), respectively. By well–known estimates [22] and (16) for large $R$, we have

$$\left( \int_{\mathbb{S}_R} |\nabla v|^q \right) \frac{1}{q} \leq c \left\{ R^{-1+2/q} \left( \int_{\mathbb{S}_R} |\nabla v|^2 \right) \right\}^{\frac{1}{2}} + c_f \right\}$$

$$\leq c \left\{ R^{-2+2/q} \left( \int_{\mathbb{S}_R} |w|^2 \right) + c_f \right\},$$

for $q \in (2, \bar{q})$, where $\bar{q} > 2$ is given in Lemma 2 and $c_f$ is a constant depending on $f$.

Hence, letting $R \to +\infty$ and bearing in mind the behavior of $w$ at large distance, it follows that $\nabla w \in L^q(\partial \mathbb{S}_R^0)$. Collecting the above results we can say that the fundamental function satisfies:

(i) $G(x, y) \notin D^{1,2}(\mathbb{S}_R(x))$ for all $R > 0$;

(ii) $G(x, y) \in D^{1,q}(\mathbb{S}_R(x))$, for all $q \in (2, \bar{q})$, with $\bar{q} > 2$ depending on $\mathbf{C}$. 

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3 Proof of Theorems 4, 5

Proof of Theorem 4 (i) – If \( h(\neq 0) \in \mathcal{M} \), then \( \int_{\partial \Omega} s(h) \neq 0 \), otherwise, bearing in mind that \( h \in BMO \), Caccioppoli’s inequality writes

\[
\int_{S_{R/2}} |\nabla h|^2 \leq \frac{c}{R^2} \int_{S_R} \left| h - \frac{1}{|S_R|} \int_{S_R} h \right|^2 \leq c,
\]

for some \( c \) independent of \( R \). Hence \( h \) should be a \( D \)-solution and so by uniqueness \( h = 0 \).

Let \( u_i \in D^{1,2}(\Omega) \) \((i = 1, 2)\) be the solutions to (1)1,2 with \( \hat{u}_i = -G(x, o)e_i \) and set \( h_i = u_i + G(x, o)e_i \). If \( \alpha_i h_i = 0 \), then \( \alpha_i G(x, o)e_i \in D^{1,2}(\Omega) \) and this is possible if and only if \( \alpha_i e_i = 0 \), i.e. \( \alpha_i = 0 \) and the system \( \{h_1, h_2\} \) is linearly independent. Therefore \( \dim \mathcal{M} \geq 2 \). Clearly, for every \( \{h_1, h_2, h_3\} \subset \mathcal{M} \) the system \( \{\int_{\partial \Omega} s(h_i)\}_{i \in \{1, 2, 3\}} \) is linearly dependent.

Hence it follows that \( h = 0 \), with \( h = \alpha_i h_i \). Since this implies that \( h = 0 \), we conclude that \( \dim \mathcal{M} = 2 \). It is obvious that if \( \{h_1, h_2\} \) is a basis of \( \mathcal{M} \), then \( \{\int_{\partial \Omega} s(h_1), \int_{\partial \Omega} s(h_2)\} \) is a basis of \( \mathbb{R}^2 \).

(ii) – Multiply (1)1 scalarly by \( g_R h \), with \( h \in \mathcal{M} \). Integrating by parts we get

\[
\int_{\partial \Omega} (\hat{u} - u_0) \cdot s(h) = -\frac{1}{R} \int_{T_R} (u - u_0) \cdot C[\nabla h]e_R + \frac{1}{R} \int_{T_R} h \cdot C[\nabla u]e_R.
\]

Choosing \( s(< 2) \) very close to 2 we have

\[
\frac{1}{R} \int_{T_R} (u - u_0) \cdot C[\nabla h]e_R \leq c \left\{ \int_{T_R} \frac{|u - u_0|^1}{r^s} \right\}^{1/s} \left\{ \int_{\Omega} |\nabla h|^s \right\}^{1/s'}
\]

\[
\frac{1}{R} \int_{T_R} h \cdot C[\nabla u]e_R \leq c \left\{ \int_{T_R} \frac{|h|^s}{r^{s'}} \right\}^{1/s'} \left\{ \int_{\Omega} |\nabla u|^s \right\}^{1/s}.
\]

Therefore, letting \( R \to +\infty \) in (38), in virtue of Lemma 1 and 7 and the properties of \( G \), we see that

\[
\int_{\partial \Omega} (\hat{u} - u_0) \cdot s(h) = 0, \quad \forall h \in \mathcal{M}.
\]

Hence it follows that \( u_0 = 0 \) if and only if \( \hat{u} \) satisfies (7).

(iii) – If \( u = o(r^{\gamma / 2}) \) is a nonzero variational solution to (1)1, vanishing on \( \partial \Omega \), then there are scalars \( \alpha_1 \) and \( \alpha_2 \) such that

\[
\int_{\partial \Omega} s(u) = \alpha_1 \int_{\partial \Omega} s(h_1) + \alpha_2 \int_{\partial \Omega} s(h_2).
\]
where \( \{ h_1, h_2 \} \) is a basis of \( \mathfrak{M} \). Therefore, by (15) and (16) the field \( v = u - \alpha_1 h_1 - \alpha_2 h_2 \) satisfies
\[
\int_{\Omega} |\nabla v|^2 \leq \frac{c \rho^\gamma}{R^{2+\gamma}} \int_{T_R} |v|^2.
\]
Hence, letting \( R \to +\infty \), it follows that \( u \in \mathfrak{M} \). Clearly, if \( u(x) = o(\log r) \), then \( u = 0 \).

(iv) – Let \( R < |x| < 2R, R \gg R_0 \), let \( \mathcal{A} \) be a neighborhood of \( x \). By Hölder’s inequality and Sobolev’s inequality
\[
\int_{\mathcal{A}} |u|^2 \leq c \left\{ \int_{\mathcal{A}} |u|^{2q/(2-q)} \right\}^{(2-q)/q} \leq c \left\{ \int_{\mathcal{C}_{SR}} |\nabla u|^q \right\}^{2/q},
\]
for \( q \in (2 - \epsilon(\gamma), 2) \). Hence by the classical convexity inequality
\[
\| \nabla u \|_{L^q(\mathcal{C}_{SR})} \leq \| \nabla u \|^\theta_{L^q(\mathcal{C}_{SR})} \| \nabla u \|^{1-\theta}_{L^2(\mathcal{C}_{SR})},
\]
with \( 2 - \epsilon(\gamma) < s < q, \theta = s(2-q)/q(2-s) \), taking into account Lemma 6 and 7, it follows
\[
\int_{\mathcal{A}} |u|^2 \leq c R^{(\theta-1)\gamma}.
\]
Putting together (33) and (39), we have
\[
\int_{\mathcal{A}} |u|^2 + \frac{1}{\rho^{2+\gamma}} \int_{\mathcal{S}_{\rho}(x)} |u - u_{S\rho}(x)|^2 \leq c R^{(\theta-1)\gamma}.
\]
Hence (8) follows taking into account well–known results of S. Campanato (see, e.g., [8] Theorem 2.9) and that \( \theta \to 0 \) for \( q \to 2 \).

Let now \( \mathbf{C} \) satisfy (6) and let \( u', u'' \) be the variational solutions to the systems
\[
\text{div } \mathbf{C}_0[\nabla u'] = 0 \quad \text{in } S_R(x),
\]
\[
u' = u \quad \text{on } \partial S_R(x),
\]
and
\[
\text{div } \mathbf{C}_0[\nabla u''] + \text{div} (\mathbf{C} - \mathbf{C}_0)[\nabla u] = 0 \quad \text{in } S_R(x),
\]
\[
u'' = 0 \quad \text{on } \partial S_R(x),
\]
respectively. Applying Poincaré’s and Caccioppoli’s inequalities we have
\[
\int_{S_R(x)} |u''|^2 \leq R^2 \int_{S_R(x)} |\nabla u''|^2 \leq c(\epsilon) R^2 \int_{S_R(x)} |\nabla u|^2 \leq c(\epsilon) \int_{T_R(x)} |u|^2.
\]
Hence, taking into account that
\[
\int_{S_{\rho}(x)} |u'|^2 \leq c \left( \rho^\gamma \right)^2 \int_{S_{\rho}(x)} |u'|^2,
\]
it follows [1] (see also [17] where the contact problem [16] is examined)

\[ \int_{S_{\rho}(x)} |u|^2 \leq c \left( \frac{\rho}{R} \right)^{2-\epsilon} \int_{S_R(x)} |u|^2. \]  

(40)

Putting together (40) and Hölder’s inequality

\[ \int_{S_R(x)} |u|^2 \leq c R^{2(s-2)/s} \left\{ \int_{S_R(x)} |u|^s \right\}^{2/s}, \]

for \( s > 2 \), we get

\[ \int_{S_{\rho}(x)} |u|^2 + \frac{1}{\rho^{4-\epsilon}} \int_{S_{\rho}(x)} |u - u_{S_{\rho}(x)}|^2 \leq \frac{c}{R^{2-\epsilon}} \int_{S_R(x)} |\nabla u|^2 \]

\[ +c R^{\epsilon-2+2(s-2)/s} \left\{ \int_{S_R(x)} |u|^s \right\}^{2/s}. \]  

(41)

Since we can choose \( s(> 2) \) near to 2 as we want, (41) yields

\[ |u(x)| \leq \frac{c}{|x|^{1-\epsilon}}, \]

for all positive \( \epsilon \).

□

**Proof of Theorem 5** If \( \mathbf{C} \) satisfies the stronger assumption (9), by the argument in [15] one shows that a variational solution to \( \text{div} \mathbf{C}[\nabla u] = 0 \) in \( S_R(x) \) satisfies

\[ \int_{S_{\rho}(x)} |\nabla u|^2 \leq c \left( \frac{\rho}{R} \right)^{2/\sqrt{L}} \int_{S_R(x)} |\nabla u|^2, \]

for every \( \rho \in (0, R] \) and the Lemmas hold with \( \gamma \) replaced by \( 2/\sqrt{L} \). Hence the desired results follow by repeating the steps in the proof of Theorem 4.

□

**4 A Counter–Example**

The following slight modification of a famous counter–example by E. De Giorgi [3] assures that the uniqueness class in Theorem 5 and the rates of decay are sharp.

Let \( \tilde{\mathbf{C}} \) be the symmetric elasticity tensor defined by

\[ \tilde{\mathbf{C}}[\mathbf{L}] = \text{sym} \mathbf{L} + 4\xi^{-2}(e_r \otimes e_r)(e_r \cdot Le_r), \quad \xi \neq 0, \ \mathbf{L} \in \text{Lin}. \]

Clearly, \( \tilde{\mathbf{C}} \) is bounded on \( \mathbb{R}^2 \) and \( C^\infty \) on \( \mathbb{R}^2 \setminus \{0\} \). Since

\[ \mathbf{L} \cdot \tilde{\mathbf{C}}[\mathbf{L}] = 4\xi^{-2}|e_r \cdot Le_r|^2 + |\mathbf{L}|^2, \quad \forall \mathbf{L} \in \text{Sym}, \]

\[ \hat{\mathbf{C}}[\mathbf{L}] = \text{sym} \mathbf{L} + 4\xi^{-2}(e_r \otimes e_r)(e_r \cdot Le_r), \quad \xi \neq 0, \ \mathbf{L} \in \text{Lin}. \]
\( \tilde{C} \) satisfies (2) with \( \mu_0 = 1 \) and \( \mu_e = 1 + 4\xi^{-2} \). A simple computation \cite{DeGiorgi68} shows that the equation

\[
\text{div} \tilde{C}[\nabla u] = 0
\]

admits the family of solutions

\[
u' = (c_1r^\epsilon + c_2r^{-\epsilon})e_r,
\]

with

\[
\epsilon = \frac{|\xi|}{\sqrt{4 + \xi^2}}.
\]

for every \( c_1, c_2 \in \mathbb{R} \). Of course, for \( c_1 = 1, c_2 = -1 \), \( u' = 0 \) on \( \partial S_1 \) and \( u' \in D^{1,q}(\tilde{C}S_1) \) for \( q > 2/(1-\epsilon) \), \( u' \notin D^{1,q}(\tilde{C}S_1) \) for \( q \leq 2/(1-\epsilon) \) so that, in particular, bearing in mind the properties of \( G \), \( u' \notin M \). For differential systems satisfying the stronger assumption (9) the above example shows that the decay \( u - u_0 = o(r^{1/\sqrt{L}}) \) is optimal for \( D \)-solutions and the class \( \{ u : u = o(r^{1/\sqrt{L}}) \} \) is borderline for uniqueness of the variational solution to the Dirichlet problem up to a field of \( M \).

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