GENERAL CONDITION FOR EXPONENTIAL STABILITY OF THERMOELASTIC BRESSE SYSTEMS WITH CATTANEO’S LAW

PEDRO ROBERTO DE LIMA*
Departamento de Matemática
Universidade Estadual do Centro-Oeste
Guarapuava, PR, CEP 85040-167, Brazil

HUGO D. FERNÁNDEZ SARE
Departamento de Matemática
Universidade Federal de Juiz de Fora
Juiz de Fora, MG, CEP 36036-900, Brazil.

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Abstract. In this paper, we give a new and more general sufficient condition for exponential stability of thermoelastic Bresse systems with heat flux given by Cattaneo’s law acting in shear and longitudinal motion equations. This condition, which we also prove to be necessary in some special cases, is given by a relation between the constants of the system and generalizes the well-known equal wave speed condition.

1. Introduction. The Bresse system describes the behaviour of a thin curved beam with length ℓ in terms of three variables: the vertical displacement ϕ, the angle of rotation of the cross-section ψ and the horizontal displacement w, which are functions of time t ∈ [0, ∞) and position x ∈ [0, ℓ].

The system is given by

\[
\begin{align*}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi + lw)_x - lk_0(w_x - l\varphi) &= 0 \quad \text{(vertical motion)} \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) &= 0 \quad \text{(shear motion)} \\
\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) &= 0 \quad \text{(longitudinal motion)}
\end{align*}
\]

(1.1)

where all coefficients are positive constants. It is important to note here that, from the physical point of view, k_0 is completely determined by ρ₁, ρ₂ and b. More precisely,

\[ k_0 = \frac{b\rho_1}{\rho_2} \]  

(1.2)

which means that the shear and longitudinal motions have the same wave speeds (for details, we refer the reader to the derivation of the model in [9, 8] as well as to the discussion in the Introduction of [2]). On this system, the standard approach followed in the literature is to study (1.1) without any restriction on k_0, which

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*Corresponding author.
gives general results particularly valid in the case (1.2). Here, we will follow the same approach but, noting that the relationship between the coefficients plays a fundamental role in the asymptotic properties of the system, we will keep (1.2) in mind recalling it when appropriate. In this sense, following the terminology used in [2], we will refer to the general system (without any restriction on \( k_0 \)) as the mathematical system and to the particular case where (1.2) holds as the physical system.

On asymptotic stability of Bresse systems, note that (1.1) is conservative but different kinds of damping can be introduced. For example, for thermal damping with heat flux given by Cattaneo’s law, the system takes the form

\[
\begin{aligned}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi + lw)_x - l[k_0(w_x - l\varphi) - \gamma \theta] &= 0 \\
\rho_2 \psi_{tt} - [b\psi_x - \gamma \vartheta]_x + k(\varphi_x + \psi + lw) &= 0 \\
\rho_1 w_{tt} - [k_0(w_x - l\varphi) - \gamma \theta]_x + kl(\varphi_x + \psi + lw) &= 0 \\
\vartheta_t + k_1 p_x + m\psi_{xt} &= 0 \\
\varsigma_p t + \delta_p + \vartheta_x &= 0 \\
\theta_t + k_1 q_x + m(w_{xt} - l\varphi_t) &= 0
\end{aligned}
\]

(1.3)

where \( \theta \) and \( \vartheta \) are temperature deviations and all coefficients are positive constants.

We are interested specifically in the study the thermoelastic Bresse model (1.3) with boundary conditions

\[
\varphi = \psi = w = \vartheta = \theta = 0 \quad \text{on} \quad [0, \infty) \times \{0, \ell\}
\]

(1.4)

and initial conditions

\[
\begin{aligned}
\varphi &= \varphi_0, \quad \psi = \psi_0, \quad w = w_0, \quad \vartheta = \vartheta_0, \quad \theta = \theta_0 \\
\varphi_t &= \varphi_1, \quad \psi_t = \psi_1, \quad w_t = w_1, \quad p = p_0, \quad q = q_0
\end{aligned}
\]

(1.5)

Before formulating our results, let us consider three variants of system (1.3)-(1.5) which have been studied in the literature and are the main references of the present paper.

1. If we take \( \tau = \varsigma = 0 \), that is, when the heat flux is governed by the Fourier law, system (1.3) reduces to

\[
\begin{aligned}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi + lw)_x - l[k_0(w_x - l\varphi) - \gamma \theta] &= 0 \\
\rho_2 \psi_{tt} - [b\psi_x - \gamma \vartheta]_x + k(\varphi_x + \psi + lw) &= 0 \\
\rho_1 w_{tt} - [k_0(w_x - l\varphi) - \gamma \theta]_x + kl(\varphi_x + \psi + lw) &= 0 \\
\vartheta_t - k_1 p_x + m\psi_{xt} &= 0 \\
\varsigma_p t + \delta_p + \vartheta_x &= 0 \\
\theta_t - k_1 q_x + m(w_{xt} - l\varphi_t) &= 0
\end{aligned}
\]

(1.6)

In [10], the authors proved that the physical system (1.6) with boundary conditions (1.4) and initial conditions

\[
\begin{aligned}
\varphi &= \varphi_0, \quad \psi = \psi_0, \quad w = w_0, \quad \vartheta = \vartheta_0 \\
\varphi_t &= \varphi_1, \quad \psi_t = \psi_1, \quad w_t = w_1, \quad \theta = \theta_0
\end{aligned}
\]

on \( \{0\} \times (0, \ell) \) is exponentially stable if and only if

\[
b - \frac{k \rho_2}{\rho_1} = 0.
\]

(1.7)
2. Neglecting the effects of $\theta$, system (1.3) reduces to
\[
\begin{align*}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l \varphi) &= 0 \\
\rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi + lw) + \gamma \theta_x &= 0 \\
\rho_1 w_{tt} - k_0 (w_x - l \varphi)_x + kl(\varphi_x + \psi + lw) &= 0 \\
\rho_1 w_t + k_1 p_x + m \psi_{xx} &= 0, \quad \varsigma p_t + \delta p + \vartheta_x = 0.
\end{align*}
\] (1.8)

For this system, in [3] the author proved that the mathematical system (1.8) with boundary conditions $\varphi = \psi = w_x = \vartheta = 0$ on $[0, \infty) \times \{0, \ell\}$ and initial conditions
\[
\begin{align*}
\varphi = \varphi_0, \quad \psi = \psi_0, \quad w = w_0, \quad \vartheta = \vartheta_0 \\
\varphi_t = \varphi_1, \quad \psi_t = \psi_1, \quad w_t = w_1, \quad p = p_0
\end{align*}
\] on $\{0\} \times (0, \ell)
\]
is exponentially stable if and only if
\[
\left( \varsigma - \frac{k_1 \rho_1}{k} \right) \left( b - \frac{k \rho_2}{\rho_1} \right) + \varsigma \gamma m = 0 \quad \text{and} \quad k_0 - k = 0. \quad (1.9)
\]

3. If we neglect the effects of $\vartheta$, then (1.3) reduces to
\[
\begin{align*}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l \varphi) + l \gamma \theta &= 0 \\
\rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi + lw) &= 0 \\
\rho_1 w_{tt} - k_0 (w_x - l \varphi)_x + kl(\varphi_x + \psi + lw) + \gamma \theta_x &= 0 \\
\theta_t + k_1 q_x + m(w_{xx} - l \varphi_t) &= 0, \quad \tau q_t + \delta q + \theta_x = 0.
\end{align*}
\] (1.10)

In [2], the authors proved that the mathematical system (1.10) with boundary conditions $\varphi = \psi = w_x = \theta = 0$ on $[0, \infty) \times \{0, \ell\}$ and initial conditions
\[
\begin{align*}
\varphi = \varphi_0, \quad \psi = \psi_0, \quad w = w_0, \quad \theta = \theta_0 \\
\varphi_t = \varphi_1, \quad \psi_t = \psi_1, \quad w_t = w_1, \quad q = q_0
\end{align*}
\] on $\{0\} \times (0, \ell)
\]
is exponentially stable if and only if
\[
b - \frac{k \rho_2}{\rho_1} = 0 \quad \text{and} \quad \left( \tau - \frac{k_1 \rho_1}{k} \right)(k_0 - k) + \tau \gamma m = 0. \quad (1.11)
\]

It is important to remark here that conditions (1.7), (1.9) and (1.11) cannot be satisfied from the physical point of view. In fact, for conditions (1.7) and (1.11), this happens because the equality $b - \frac{k \rho_2}{\rho_1} = 0$ is incompatible with the physical condition
\[
b = 2(1 + \nu) \frac{k \rho_2}{\rho_1}, \quad (1.12)
\]

which results from the relation $E = 2G(1 + \nu)$ between the Young’s modulus $E$, the shear modulus $G$ and the Poisson’s ratio $\nu$. For condition (1.9), this happens because, for the physical system, we have $k_0 - k = 0 \iff b = \frac{k \rho_2}{\rho_1}$ and thus in this case we also obtain an incompatibility with (1.12).

Now, returning to the context of system (1.8), adding another thermal dissipation acting in the longitudinal motion equation, the resulting system is exponentially stable even in the case where the second equality of (1.9) is dropped. In other words, if
\[
\left( \varsigma - \frac{k_1 \rho_1}{k} \right) \left( b - \frac{k \rho_2}{\rho_1} \right) + \varsigma \gamma m = 0, \quad (1.13)
\]
then system (1.3)-(1.5) is exponentially stable. This result was proved in [5] and, to the best of our knowledge, it is the unique known result on exponential stability of (1.3)-(1.5) providing only a sufficient condition. In the present paper, we will prove
a similar result but starting our analysis studying system (1.10) instead of (1.8). In fact, adding another thermal damping in system (1.10) now acting in the shear motion equation, we will prove that the resulting system is exponentially stable even in the case where the first equality of (1.11) is dropped. In other words, if

\[\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau \gamma m = 0,\]  

(1.14)

then system (1.3)-(1.5) is exponentially stable. Note that in both cases, after adding the other damping, we obtain the same system (1.3). It implies that the equalities (1.13) and (1.14) are two different sufficient conditions for the exponential stability of (1.3)-(1.5). So, combining these two results, we conclude that a new and more general sufficient condition for the exponential decay of system (1.3)-(1.5) is given by

\[\left(\varsigma - \frac{k_1\rho_1}{k}\right)\left(b - \frac{k\rho_2}{\rho_1}\right) + \varsigma \gamma m \right] \left[\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau \gamma m\right] = 0.\]  

(1.15)

Summarizing our previous analysis, in the present paper we study system (1.3)-(1.5) and prove that it is exponentially stable provided that equality (1.15) holds. In addition, we prove that, for some special cases, condition (1.15) is also a necessary condition for the exponential stability of system (1.3)-(1.5) which, as far as we know, is an open problem in the literature. Observe that (1.15) generalizes the standard equal wave speed condition (1.7). Additionally, observe that the number

\[\left(\varsigma - \frac{k_1\rho_1}{k}\right)\left(b - \frac{k\rho_2}{\rho_1}\right) + \varsigma \gamma m \right] \left[\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau \gamma m\right] = 0,\]  

(1.15)

obtained when we multiply (1.15) by \((-\frac{\rho_1}{k})^2\), generalizes the stability number of the Timoshenko system with Cattaneo’s law given in [12].

Now, let us explain the relevance of our result from the physical point of view. First, compared with system (1.6), system (1.3) considers thermal effects governed by the Cattaneo’s law which removes the paradox of infinite propagation speed inherent in the Fourier’s law. Additionally, compared with systems (1.8) and (1.10), system (1.3) is more realistic because it does not neglect the effects of the temperature in any direction. Second, for the physical system, condition (1.15) reduces to

\[\left(\varsigma - \frac{k_1\rho_1}{k}\right)\left(b - \frac{k\rho_2}{\rho_1}\right) + \varsigma \gamma m \right] \left[\left(\tau - \frac{k_1\rho_1}{k}\right)(k_0 - k) + \tau \gamma m\right] = 0,\]  

which is not incompatible with (1.12), contrary to what happens with conditions (1.7), (1.9) and (1.11).

Finally but not less important, we observe that there exists a quite large number of references in the literature studying Bresse systems. Here we have cited only a few of them which we believe are the most important for the understanding of our contributions. For a brief and most comprehensive survey on Bresse systems with thermal damping we refer the reader to the Introduction of [2]. Additionally to that survey, we can mention the recent papers [1] and [6]. In [1], the authors studied a variant of (1.10) for different mixed boundary conditions. In such variant, comparing with the model described above, the corresponding coupling terms are different: the term \(\theta\) in the first equation and the term \(\varphi_t\) in the coupling of the heat equation were omitted. In this case, the authors proved that condition (1.11) is no longer equivalent to the exponential decay and an extra restriction on the curvature
is needed. Under the same condition on the curvature, polynomial stability was also obtained even in the case where (1.11) is not satisfied. In [6], the author obtained similar results for the same model with Cattaneo’s law replaced by thermal effects of types I and III. In this case, condition (1.11) is replaced by the usual equal wave speeds condition, the same restriction on the curvature is needed.

The remainder of this paper is organized as follows. In Section 2, we formulate the problem (1.3)-(1.5) as an abstract Cauchy problem and prove existence and uniqueness of solution. For this, we will apply a well-known consequence of the Lumer-Philips Theorem. In Section 3, we study the exponential stability of the system and in Section 4 we discuss its non-exponential decay. For this, in both sections we will apply the Gearhart-Prüss Theorem.

2. Abstract setting and well-posedness. Let us start by remembering the following well-known consequence of the Lumer-Philips Theorem:

Theorem 2.1 (see [11]). Let $\mathcal{H}$ be a Hilbert space and $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined linear operator. If $A$ is dissipative and $0 \in \rho(A)$, then $A$ is the infinitesimal generator of a $C_0$-semigroup of contractions on $\mathcal{H}$.

Now, in order to apply this result, we will include our problem in the context of semigroups. For this, consider the phase space

$$
\mathbb{H} = H^1_0(0, \ell) \times L^2(0, \ell) \times H^1_0(0, \ell) \times L^2(0, \ell) \times L^2(0, \ell) \times L^2(0, \ell),
$$

where

$$
L^2_0(0, \ell) = \left\{ u \in L^2(0, \ell) \mid \int_0^{\ell} u(x) \, dx = 0 \right\}
$$

and $H^1_0(0, \ell) = H^1(0, \ell) \cap L^2_0(0, \ell)$, equipped with norm

$$
\|U\|_{\mathbb{H}}^2 = \rho_1 \|\Phi\|_{L^2}^2 + \rho_2 \|\Psi\|_{L^2}^2 + \rho_1 \|W\|_{L^2}^2 + b \|\psi\|_{L^2}^2 + k \|\varphi\|_{L^2}^2 + \psi + lw^2 + k_0 \|w_x - l \varphi\|_{L^2}^2 + \frac{\gamma_m}{m} \left( \|\vartheta\|_{L^2}^2 + \|\varphi\|_{L^2}^2 \right) + \frac{\gamma_k}{m} \left( \|\varphi\|_{L^2}^2 + \tau \|q\|_{L^2}^2 \right),
$$

where $U = (\varphi, \Phi, \psi, \Psi, w, W, \vartheta, p, \theta, q)$. As usual, under the assumption that $\ell \ell$ is not a multiple of $\pi$, the space $\mathbb{H}$ is a Hilbert space and $\| \cdot \|_{\mathbb{H}}$ is equivalent to the usual norm of $\mathbb{H}$. Otherwise, if $\ell \ell$ is multiple of $\pi$, then $\| \cdot \|_{\mathbb{H}}$ is not a norm because there are non-zero vectors $U \in \mathbb{H}$ satisfying $\|U\|_{\mathbb{H}} = 0$, for example, $U = \left( \sin(lx), 0, 0, 0, -\cos(lx), 0, 0, 0, 0, 0 \right) \in \mathbb{H}$. Therefore, here and thereafter, we suppose that

$$
\ell \ell \neq n \pi, \quad \forall \, n \in \mathbb{N}.
$$

For convenience of the reader, we present below a complete proof of the completeness of $\mathbb{H}$ and the equivalence of norms (which will be tacitly used many times along the paper).

Proposition 1. Suppose that condition (2.2) holds. Then, the space $\mathbb{H}$ equipped with the norm $\| \cdot \|_{\mathbb{H}}$ given by (2.1) is complete. As a consequence, $\| \cdot \|_{\mathbb{H}}$ is equivalent to the usual norm $\| \cdot \|_{\mathbb{H}}$ of $\mathbb{H}$ defined by

$$
\|U\|_{\mathbb{H}}^2 = \|\varphi\|_{H^1}^2 + \|\Phi\|_{L^2}^2 + \|\psi\|_{H^1}^2 + \|\Psi\|_{L^2}^2 + \|w\|_{H^1}^2 + \|W\|_{L^2}^2 + \|\vartheta\|_{L^2}^2 + \|p\|_{L^2}^2 + \|\theta\|_{L^2}^2 + \|q\|_{L^2}^2.
$$

(2.3)
Proof. Take a Cauchy sequence \((U_n)_{n \in \mathbb{N}}\) in \(\mathbb{H}\). Then, writing

\[ U_n = (\varphi(n), \Phi(n), \psi(n), \Psi(n), w(n), W(n), g(n), p(n), q(n)), \]

we conclude that

1. \((\Phi(n))_{n \in \mathbb{N}}, (\psi(n))_{n \in \mathbb{N}}, (w(n) - l\varphi(n))_{n \in \mathbb{N}}, (\psi(n))_{n \in \mathbb{N}}\) and \((\theta(n))_{n \in \mathbb{N}}\) are Cauchy sequences in \((L^2(0, \ell), \| \cdot \|_{L^2})\).

2. \((\Psi(n))_{n \in \mathbb{N}}, (W(n))_{n \in \mathbb{N}}, (\varphi(n) + \psi(n) + lw(n))_{n \in \mathbb{N}}, (p(n))_{n \in \mathbb{N}}\) and \((q(n))_{n \in \mathbb{N}}\) are Cauchy sequences in \((L^2(0, \ell), \| \cdot \|_{L^2})\).

Since \((L^2(0, \ell), \| \cdot \|_{L^2})\) and \((L^2(0, \ell), \| \cdot \|_{L^2})\) are complete spaces, it follows that there exist \(\Phi, F, G, \vartheta, \theta \in L^2(0, \ell)\) and \(\Psi, W, H, p, q \in L^2(0, \ell)\) such that

\[
\begin{align*}
\|\Phi(n) - \Phi\|_{L^2} & \xrightarrow{n \to \infty} 0, \\
\|w(n) - l\varphi(n) - G\|_{L^2} & \xrightarrow{n \to \infty} 0, \\
\|\theta(n) - \theta\|_{L^2} & \xrightarrow{n \to \infty} 0, \\
\|\Psi(n) - \Psi\|_{L^2} & \xrightarrow{n \to \infty} 0, \\
\|\varphi(n) + \psi(n) + lw(n) - H\|_{L^2} & \xrightarrow{n \to \infty} 0, \\
\|p(n) - p\|_{L^2} & \xrightarrow{n \to \infty} 0.
\end{align*}
\] (2.4)

Let \(\psi\) be the solution in \(H^1_0(0, \ell)\) of the equation

\[ \psi_x = F \] (2.5)

and, for this \(\psi\), let \((\varphi, w)\) be the solution in \(H^1_0(0, \ell) \times H^1_0(0, \ell)\) of the system

\[
\begin{align*}
w_x - lw &= G, \\
\varphi_x + lw &= H - \psi.
\end{align*}
\] (2.6)

Then, \(U := (\varphi, \Phi, \psi, \Psi, w, W, \vartheta, \theta, q)\) belongs to \(\mathbb{H}\) and, substituting (2.5)-(2.6) into (2.4), we conclude that \((U_n)_{n \in \mathbb{N}}\) converges to \(U\) in \(\mathbb{H}\). This shows that \(\mathbb{H}\) is complete.

It is clear that there exists a constant \(C > 0\) such that \(\|U\|_{\mathbb{H}} \leq C\|U\|_{\mathbb{H}}\) for all \(U \in \mathbb{H}\). Therefore, as a consequence of the Open Mapping Theorem (see Corollary 4.4.2 in [7]), we conclude that \(\| \cdot \|_{\mathbb{H}}\) and \(\| \cdot \|_{\mathbb{H}}\) are equivalent. \(\square\)

Now, define \(\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \to \mathbb{H}\) by

\[
\mathcal{A}U = \left( \begin{array}{c} 
\frac{k}{\rho_1}(\varphi_x + \psi + lw)_x + \frac{k_0}{\rho_1}(w_x - l\varphi) - \frac{l\gamma}{\rho_1} \Phi \\
\frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi + lw) - \frac{\gamma}{\rho_2} \vartheta_x \\
k_0(w_x - l\varphi)_x - \frac{k_1}{\rho_1}(\varphi_x + \psi + lw) - \frac{\gamma}{\rho_1} \vartheta_x \\
-\frac{\delta}{\varsigma}p - \frac{1}{\varsigma} \vartheta_x \\
-\frac{\delta}{\tau}q - \frac{1}{\tau} \vartheta_x
\end{array} \right)
\]

with domain

\[ D(\mathcal{A}) = \{ U \in \mathbb{H} \mid \varphi \in H^2(0, \ell), \ \Phi, \psi, w_x, \vartheta, \theta \in H^1_0(0, \ell), \ \Psi, W, p, q \in H^1(0, \ell) \}. \]
Remark 1. It is not difficult to show that if \((U_n)_{n \in \mathbb{N}}\) is bounded in \(D(A)\), with respect to the graph norm \(\| \cdot \|_{D(A)}\), then it is bounded in the space
\[(H^2 \cap H^1_0) \times H^1 \times (H^2 \cap H^1_0) \times H^1 \times (H^2 \cap H^1_0) \times H^1 \times [H^1 \times H^1]_2^2,\]
with respect to its usual norm, which is compactly embedded in \((\mathbb{H}, | \cdot |_\mathbb{H})\) by the standard compactness embedding theorems. Therefore, \((D(A), \| \cdot \|_{D(A)})\) is compactly embedded in \((\mathbb{H}, \| \cdot \|_{\mathbb{H}})\), which implies that all elements of \(\sigma(A)\) are eigenvalues of \(A\) (see Proposition II.4.25 and Corollary IV.1.19 in [4]).

Under this setting, problem (1.3)-(1.5) can be written as
\[
\begin{cases}
U_t = AU, & t > 0 \\
U(0) = U_0
\end{cases}
\]
(2.7)
where \(U_0 := (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, \theta_0, p_0, \theta_0, q_0)\) and we can present the main result of this section.

Theorem 2.2 (Existence and uniqueness). The operator \(A\) is the infinitesimal generator of a \(C_0\)-semigroups of contractions on \(\mathbb{H}\) and thus, for each initial data \(U_0 \in D(A)\), the problem (2.7) has a unique classical solution \(U \in C([0, \infty); D(A)) \cap C^1([0, \infty); \mathbb{H})\). Also, for each initial data \(U_0 \in \mathbb{H}\), the said problem has a unique mild solution \(U \in C([0, \infty); \mathbb{H})\).

Proof. We verify the hypotheses of Theorem 2.1.

- \(D(A)\) is dense in \(\mathbb{H}\). We have
\[
\mathbb{H} = Y_1 \times Y_2 \times Y_3 \times Y_4 \times Y_5 \times Y_6 \times Y_7 \times Y_8 \times Y_9 \times Y_{10},
\]
(2.8)
and
\[
D(A) = S_1 \times S_2 \times S_3 \times S_4 \times S_5 \times S_6 \times S_7 \times S_8 \times S_9 \times S_{10}
\]
(2.9)
and
\[
\|(y_1, y_2, \ldots, y_{10})\|^2_\mathbb{H} = \|y_1\|^2_{Y_1} + \|y_2\|^2_{Y_2} + \cdots + \|y_{10}\|^2_{Y_{10}},
\]
(2.10)
where
\[
S_i = \begin{cases}
H^2 \cap H^1_0, & \text{if } i = 1 \\
H^1_0, & \text{if } i = 2, 7, 9 \\
\{u \in H^1_0 | u_x \in H^1_0\}, & \text{if } i = 3, 5 \\
H^1_0, & \text{if } i = 4, 6, 8, 10
\end{cases}
\]
and
\[
Y_i = \begin{cases}
H^2, & \text{if } i = 1 \\
L^2, & \text{if } i = 2, 7, 9 \\
H^1, & \text{if } i = 3, 5 \\
L^2, & \text{if } i = 4, 6, 8, 10.
\end{cases}
\]

In view of (2.8)-(2.10), it is enough to show that \(S_i\) is a dense subspace of \((Y_i, \| \cdot \|_{Y_i})\) for \(i = 1, 2, 3, 4\). For this, let us remember that
\[
C^\infty_0(0, \ell) \subset [H^2(0, \ell) \cap H^1_0(0, \ell)],
\]
(2.11)
\[
C^\infty_0(0, \ell) \text{ is dense in } H^1_0(0, \ell),
\]
(2.12)
\[
C^\infty_0(0, \ell) \subset H^1_0(0, \ell),
\]
(2.13)
\[
C^\infty_0(0, \ell) \text{ is dense in } L^2(0, \ell).
\]
(2.14)
From (2.11) and (2.12), \(S_1\) is dense in \(Y_1\). From (2.13) and (2.14), \(S_2\) is dense in \(Y_2\). From (2.14), given \(f \in Y_3\), there exists \((z^{(n)})_{n \in \mathbb{N}}\) in \(C^\infty_0(0, \ell)\) which converges to \(f_x\) in \(L^2(0, \ell)\). Therefore there exist a sequence \((g^{(n)})_{n \in \mathbb{N}}\) in \(S_3\), defined by
\[
g^{(n)}(x) = \int_0^x z^{(n)}(s) \, ds - \frac{1}{\ell} \int_0^\ell \int_0^y z^{(n)}(s) \, ds \, dy,
\]
which converges to \( f \) in \( Y_3 \). In other words: \( S_3 \) is dense in \( Y_3 \). From (2.14) again, given \( f \in Y_4 \), there exists \((z^{(n)})_{n \in \mathbb{N}} \) in \( C_0^\infty(0, \ell) \) which converges to \( f \) in \( L_2(0, \ell) \). Therefore there exist a sequence \((g^{(n)})_{n \in \mathbb{N}} \) in \( S_4 \), defined by

\[
g^{(n)}(x) = z^{(n)}(x) - \frac{1}{\ell} \int_0^\ell z^{(n)}(s) \, ds,
\]

which converges to \( f \) in \( Y_3 \). In other words: \( S_4 \) is dense in \( Y_4 \).

- **\( \mathcal{A} \) is dissipative.** A straightforward computation shows that

\[
\mathfrak{R}(\mathcal{A}U, U)_{\mathcal{H}} = -\frac{\gamma \delta k_1}{m} (\|p\|_2^2 + \|q\|_2^2), \quad \forall \ U \in D(\mathcal{A})
\]

and this proves the dissipativity of \( \mathcal{A} \).

- \( 0 \notin \rho(\mathcal{A}) \). Suppose that \( 0 \notin \rho(\mathcal{A}) \). Then, by Remark 1, \( 0 \) in an eigenvalue of \( \mathcal{A} \). Therefore, there exist \( U \neq 0 \) in \( D(\mathcal{A}) \) satisfying equation \( \mathcal{A}U = 0 \), which in terms of its components can be written as

\[
\begin{align*}
\Phi &= 0, \quad \Psi = 0, \quad W = 0 \\
k(\varphi_x + \psi + lw)_x + k_0 l(w_x - l \varphi) - l \gamma \vartheta_x &= 0 \\
b \psi_{xx} - k(\varphi_x + \psi + lw) - \gamma \vartheta_x &= 0 \\
k_0 (w_x - l \varphi)_x - k l(\varphi_x + \psi + lw) - \gamma \theta_x &= 0 \\
- k_1 p_x - m \Psi_x &= 0, \quad - k_1 q_x - m (W_x - l \Phi) = 0, \\
- \delta p - \vartheta_x &= 0, \quad - \delta q - \theta_x &= 0.
\end{align*}
\]

From (2.16), it follows that \( \Phi = \Psi = W = 0 \). Substituting into (2.20), it follows that \( p_x = q_x = 0 \), which implies \( p = q = 0 \), because \( p, q \in H_0^1 \). Substituting this into (2.21), it follows that \( \vartheta_x = \theta_x = 0 \). Since \( \vartheta, \theta \in H_0^1 \), we conclude that \( \vartheta = \theta = 0 \). Substituting this into (2.17), (2.18), (2.19), it follows that \((\varphi, \psi, w)\) satisfies

\[
B((\varphi^*, \psi^*, w^*), (\varphi, \psi, w)) = 0, \quad \forall \ (\varphi^*, \psi^*, w^*) \in H_0^1 \times H_0^1 \times H_*^1
\]

where \( B \) is defined by

\[
B((\varphi^*, \psi^*, w^*), (\varphi, \psi, w)) = k \int_0^\ell (\varphi^*_x + \psi^* + lw^*)(\varphi_x + \psi + lw) \, dx + k_0 \int_0^\ell (w^*_x - lw^*) (w_x - l \varphi) \, dx + b \int_0^\ell \psi^*_x \psi_x \, dx.
\]

Since \( B \) is a continuous coercive sesquilinear form on \( H_0^1 \times H_0^1 \times H_*^1 \), it follows from the Lax-Milgram Theorem that (2.22) has a unique solution and thus \( \varphi = \psi = w = 0 \). This shows that \( U = 0 \), which is a contradiction. Therefore, \( 0 \notin \rho(\mathcal{A}) \). \( \Box \)

3. **Exponential stability.** In this section, we prove that condition (1.15) is sufficient for exponential stability. This result will be a consequence of the following result.

**Theorem 3.1** (Gearhart-Prüss, see [11]). **A bounded \( C_0 \)-semigroup \( \{e^{kA}\}_{k \geq 0} \) on a Hilbert space is exponentially stable if and only if both conditions below hold.**

(a) \( \mathbb{R} \subset \rho(A) \).

(b) \( \lim \sup_{|\beta| \to \infty} \|i\beta I - A\|_\mathcal{L}^{-1} < \infty \).

We start by proving some lemmas about a sequence \((\beta_n)_{n \in \mathbb{N}} \subset \mathbb{R} \) and a sequence \( U_n := (\varphi^{(n)}, \Phi^{(n)}, \psi^{(n)}, \Psi^{(n)}, w^{(n)}, W^{(n)}, \theta^{(n)}, p^{(n)}, \theta^{(n)}, q^{(n)}) \in D(\mathcal{A}) \)
such that
\[ \beta_n \stackrel{n \to \infty}{\to} \infty, \quad \langle U_n \rangle_{n \in \mathbb{N}} \text{ is bounded in } \mathbb{H} \quad \text{and} \quad \| (i \beta_n - A) U_n \|_\mathbb{H} \stackrel{n \to \infty}{\to} 0. \quad (3.1) \]

Before to formulate the lemmas we observe that, using the equivalence between the norm \( \| \cdot \|_\mathbb{H} \) given by (2.1) it follows from (3.1) that
\[
\begin{align*}
&i \beta_n \phi^{(n)} - \Phi^{(n)} \to 0 \quad (3.2) \\
&i \beta_n \phi_x^{(n)} - \Phi_x^{(n)} \to 0 \quad (3.3) \\
&\rho_1 i \beta_n \Phi^{(n)} - k(\varphi_x^{(n)} + \psi^{(n)} + l w^{(n)})_x - k_0 l (w_x^{(n)} - l \varphi^{(n)}) + l \gamma \theta^{(n)} \to 0 \quad (3.4) \\
&i \beta_n \psi^{(n)} - \Psi^{(n)} \to 0 \quad (3.5) \\
&i \beta_n \varphi^{(n)} - \Phi^{(n)} \to 0 \quad (3.6) \\
&\rho_2 i \beta_n \varphi^{(n)} - b \psi^{(n)}_x + k(\varphi_x^{(n)} + \psi^{(n)} + l w^{(n)}) + \gamma \theta^{(n)} \to 0 \quad (3.7) \\
&i \beta_n w^{(n)} - W^{(n)} \to 0 \quad (3.8) \\
&i \beta_n w_x^{(n)} - W_x^{(n)} \to 0 \quad (3.9) \\
&\rho_1 i \beta_n W^{(n)} - k_0 (w_x^{(n)} - l \varphi_x^{(n)}) + k l (\varphi_x^{(n)} + \psi^{(n)} + l w^{(n)}) + \gamma \theta_x^{(n)} \to 0 \quad (3.10) \\
&i \beta_n \varphi^{(n)} + k_1 p_x^{(n)} + m \psi^{(n)} \to 0 \quad (3.11) \\
&\gamma \beta_n \rho^{(n)} + \delta p^{(n)} + \psi_x^{(n)} \to 0 \quad (3.12) \\
&i \beta_n \varphi^{(n)} + k_1 p_x^{(n)} + m (W_x^{(n)} - l \Phi^{(n)}) \to 0 \quad (3.13) \\
&\tau i \beta_n q^{(n)} + \delta q^{(n)} + \theta^{(n)} \to 0 \quad (3.14)
\end{align*}
\]

with all convergences in the sense of \( L^2(0, \ell) \), as \( n \to \infty \).

**Lemma 3.2.** Assume (3.1). Then, the following sequences are bounded in \( L^2(0, \ell) \):
\( \)
(a) \( \langle \Phi^{(n)} \rangle_{n \in \mathbb{N}}, \langle \Psi^{(n)} \rangle_{n \in \mathbb{N}}, \langle W^{(n)} \rangle_{n \in \mathbb{N}}, \langle \varphi_x^{(n)} \rangle_{n \in \mathbb{N}}, \langle \psi^{(n)} + l w^{(n)} \rangle_{n \in \mathbb{N}}, \langle w_x^{(n)} - l \varphi^{(n)} \rangle_{n \in \mathbb{N}}, \langle \varphi^{(n)}_x + \psi^{(n)} + l w^{(n)} \rangle_{n \in \mathbb{N}} \).
(b) \( \langle \varphi_x^{(n)} \rangle_{n \in \mathbb{N}}, \langle w_x^{(n)} \rangle_{n \in \mathbb{N}} \).
(c) \( \left( \frac{\Phi^{(n)}}{i \beta_n} \right)_{n \in \mathbb{N}}, \left( \frac{\Psi^{(n)}}{i \beta_n} \right)_{n \in \mathbb{N}}, \left( \frac{W^{(n)}}{i \beta_n} \right)_{n \in \mathbb{N}} \).
(d) \( \left( \frac{\varphi^{(n)}_x}{i \beta_n} \right)_{n \in \mathbb{N}}, \left( \frac{w_x^{(n)} - l \varphi^{(n)}_x}{i \beta_n} \right)_{n \in \mathbb{N}} \).

**Proof.** Since \( \langle U_n \rangle_{n \in \mathbb{N}} \) is bounded in \( \mathbb{H} \), it follows from (2.1) that the following sequences are bounded in \( L^2(0, \ell) \):
\[
\langle \Phi^{(n)} \rangle_{n \in \mathbb{N}}, \langle \Psi^{(n)} \rangle_{n \in \mathbb{N}}, \langle W^{(n)} \rangle_{n \in \mathbb{N}}, \langle \varphi_x^{(n)} \rangle_{n \in \mathbb{N}}, \langle \psi^{(n)} + l w^{(n)} \rangle_{n \in \mathbb{N}}, \langle w_x^{(n)} - l \varphi^{(n)} \rangle_{n \in \mathbb{N}}, \langle \varphi^{(n)}_x + \psi^{(n)} + l w^{(n)} \rangle_{n \in \mathbb{N}}.
\]

Then, the sequences of (a) are bounded in \( L^2(0, \ell) \). The boundedness of the sequences in (b) are a direct consequence of the equivalence of the norms (2.1) and (2.3). For (c), note that all sequences in (3.2)–(3.14) are bounded in \( L^2(0, \ell) \). In particular, multiplying by the bounded sequence \( \frac{1}{i \beta_n} \) \( n \in \mathbb{N} \), the following sequences are also bounded:
\[
\left( \frac{\Phi^{(n)}}{i \beta_n} \right)_{n \in \mathbb{N}} = \langle \varphi_x^{(n)} \rangle_{n \in \mathbb{N}} - \left( \frac{\Phi^{(n)}}{i \beta_n} \right)_{n \in \mathbb{N}}.
\]
which proves (c). Additionally, we have the boundedness of the following sequences:

\[
\begin{align*}
\left( b \frac{\psi_x^{(n)}}{i\beta_n} - \frac{\gamma \varphi_x^{(n)}}{i\beta_n} \right)_{n \in \mathbb{N}} &= \left( \rho_2 \Psi_x^{(n)} + k \frac{\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}}{i\beta_n} \right)_{n \in \mathbb{N}}, \\
\left( k_0 \frac{(w_x^{(n)})_x - l \varphi_x^{(n)}}{i\beta_n} - \frac{\varphi_x^{(n)}}{i\beta_n} \right)_{n \in \mathbb{N}} &= \left( \rho_1 W^{(n)} + kl \frac{\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}}{i\beta_n} \right)_{n \in \mathbb{N}}, \\
\left( \frac{\psi_x^{(n)}}{i\beta_n} \right)_{n \in \mathbb{N}} &= \left( \rho_2 \Psi_x^{(n)} + k \frac{\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}}{i\beta_n} \right)_{n \in \mathbb{N}}, \\
\left( \frac{\varphi_x^{(n)}}{i\beta_n} \right)_{n \in \mathbb{N}} &= \left( \rho_1 W^{(n)} + kl \frac{\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}}{i\beta_n} \right)_{n \in \mathbb{N}}, \\
\left( \frac{\psi_x^{(n)}}{i\beta_n} \right)_{n \in \mathbb{N}} &= \left( \rho_2 \Psi_x^{(n)} + k \frac{\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}}{i\beta_n} \right)_{n \in \mathbb{N}}, \\
\left( \frac{\varphi_x^{(n)}}{i\beta_n} \right)_{n \in \mathbb{N}} &= \left( \rho_1 W^{(n)} + kl \frac{\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}}{i\beta_n} \right)_{n \in \mathbb{N}},
\end{align*}
\]

which completes the proof of (d).

Lemma 3.3. Assume (3.1). Then, \( q^{(n)} \to 0 \) and \( p^{(n)} \to 0 \).

Proof. Let us write \( G_n := i^2 \beta_n (U_n - \mu U_n) \). Then, multiplying by \( U \),

\[
i^2 \beta_n \| U_n \|_{\mathbb{H}}^2 - \left( \mu U_n, U_n \right)_{\mathbb{H}} = \left( G_n, U_n \right)_{\mathbb{H}}.
\]

Now, taking the real part and recalling (3.1) together with equation (2.15), we conclude that

\[
\| q^{(n)} \|^2_{L^2} + \| p^{(n)} \|^2_{L^2} \leq \frac{m}{\gamma \delta k_1} \| G_n \|_{\mathbb{H}} \| U_n \|_{\mathbb{H}} = \frac{m}{\gamma \delta k_1} \| i^2 \beta_n (U_n - \mu U_n) \|_{\mathbb{H}} \| U \|_{\mathbb{H}} \to 0
\]

which proves the result.

Lemma 3.4. Assume (3.1). Then,

(a) \( \frac{1}{i^2 \beta_n} (W_x^{(n)}, \theta^{(n)})_{L^2} \to 0 \), \( \frac{1}{i^2 \beta_n} (q^{(n)}, \theta_x^{(n)})_{L^2} \to 0 \).

(b) \( \frac{1}{i^2 \beta_n} (\Psi_x^{(n)}, \theta^{(n)})_{L^2} \to 0 \), \( \frac{1}{i^2 \beta_n} (p^{(n)}, \theta_x^{(n)})_{L^2} \to 0 \).
Proof. From (3.14) and Lemma 3.3 we deduce that $\frac{1}{i\beta_n}\theta_i(n) \to 0$. Then, multiplying by $W(n)$ and $q(n)$ in $L^2(0, \ell)$, we obtain (a). Analogously, using equation (3.12) and Lemma 3.3 we have $\frac{1}{i\beta_n}\varphi_i(n) \to 0$. Then, multiplying by $\Psi(n)$ and $p(n)$ in $L^2(0, \ell)$, we deduce (b).

**Lemma 3.5.** Assume (3.1). Then, $\theta(n) \to 0$ and $\varphi(n) \to 0$.

**Proof.** From (3.13) we have

$$\theta(n) + \frac{k_1}{i\beta_n} q_x(n) + \frac{m}{i\beta_n} W_x(n) \to 0.$$  

Then, multiplying by $\theta(n)$ in $L^2(0, \ell)$,

$$\|\theta(n)\|^2_{L^2} - \frac{k_1}{i\beta_n}(q(n), \theta_x(n))_{L^2} + \frac{m}{i\beta_n}(W(n), \theta(n))_{L^2} \to 0,$$

which implies, using Lemma 3.4, that $\theta(n) \to 0$. Analogously, from (3.11) we have

$$\varphi(n) + \frac{k_1}{i\beta_n} q_x(n) + \frac{m}{i\beta_n} \Psi(n) \to 0.$$  

Then, multiplying by $\varphi(n)$ in $L^2(0, \ell)$ we obtain

$$\|\varphi(n)\|^2_{L^2} - \frac{k_1}{i\beta_n}(q(n), \varphi_x(n))_{L^2} + \frac{m}{i\beta_n}(\Psi(n), \varphi(n))_{L^2} \to 0,$$

which implies, using Lemma 3.4, that $\varphi(n) \to 0$.

**Lemma 3.6.** Assume (3.1). Then, $w_x(n) - l\varphi(n) \to 0$ and $\psi_x(n) \to 0$.

**Proof.** From (3.13), we deduce that

$$\theta(n) + \frac{k_1}{i\beta_n} q_x(n) + \frac{m}{i\beta_n}(W(n) - l\Phi(n)) \to 0.$$  

On the other hand, from (3.2) and (3.9) we can obtain

$$i\beta_n(w_x(n) - l\varphi(n)) - (W(n) - l\Phi(n)) \to 0.$$  

Multiplying this by $\frac{m}{i\beta_n}$ and adding to the first convergence, we obtain

$$\theta(n) + \frac{k_1}{i\beta_n} q_x(n) + m(w_x(n) - l\varphi(n)) \to 0.$$  

Finally, multiplying by the bounded sequence $(w_x(n) - l\varphi(n))$ in $L^2(0, \ell)$ we deduce that

$$(\theta(n), w_x(n) - l\varphi(n))_{L^2} + k_1 \left(\frac{q(n), (w_x(n) - l\varphi(n))_x}{i\beta_n}\right)_{L^2} + m\|w_x(n) - l\varphi(n)\|^2_{L^2} \to 0.$$  

Note that the first term goes to zero by Lemma 3.5, and the second also goes to zero by Lemmas 3.2 and 3.3. This implies that $w_x(n) - l\varphi(n) \to 0$. Analogously, from (3.11),

$$\varphi(n) + \frac{k_1}{i\beta_n} q_x(n) + \frac{m}{i\beta_n} \Psi(n) \to 0.$$  

From (3.6), multiplying by $\frac{m}{i\beta_n}$, we have $m\psi_x(n) - \frac{m}{i\beta_n} \psi_x(n) \to 0$. Then, adding the last convergences, we obtain

$$\varphi(n) + \frac{k_1}{i\beta_n} q_x(n) + m\psi_x(n) \to 0.$$
which implies, multiplying by $\psi^{(n)}_x$ in $L^2(0, l)$, that
\[
(\psi^{(n)}_x, \psi^{(n)}_x)_{L^2} + k_1 \left( p^{(n)}(x), \frac{\psi^{(n)}_{xx}}{13n} \right)_{L^2} + m\|\psi^{(n)}_x\|_{L^2}^2 \to 0.
\]
Again, the first term goes to zero by Lemma 3.5 and the second by Lemmas 3.2 and 3.3. This implies that $\psi^{(n)}_x \to 0$.

**Lemma 3.7.** Assume (3.1). Then, $W^{(n)} \to 0$ and $\Psi^{(n)} \to 0$.

**Proof.** Multiplying (3.10) by $\frac{1}{13n}$, which goes to zero, and using the boundedness of $(\varphi^{(n)}_x + \psi^{(n)} + lw^{(n)})$, which comes from Lemma 3.2, we deduce
\[
W^{(n)} - \frac{k_0}{\rho_1 13n} (w^{(n)} - l\varphi^{(n)}_x)_{L^2} + \frac{\gamma}{\rho_1 13n} (\theta^{(n)}_x, W^{(n)})_{L^2} \to 0.
\]
Then, multiplying by $W^{(n)}$ in $L^2(0, l)$,
\[
\|W^{(n)}\|_{L^2}^2 = \frac{k_0}{\rho_1} \left( w^{(n)} - l\varphi^{(n)}_x, \frac{W^{(n)}_x}{13n} \right)_{L^2} + \frac{\gamma}{\rho_1 13n} (\theta^{(n)}_x, W^{(n)})_{L^2} \to 0.
\]
Note that the second term goes to zero by Lemmas 3.2 and 3.6. Also, the last term goes to zero by Lemmas 3.2 and 3.4. These convergences imply the convergence $W^{(n)} \to 0$. Analogously, multiplying (3.7) by $\frac{1}{13n}$ and using the boundedness of $(\varphi^{(n)}_x + \psi^{(n)} + lw^{(n)})$ again, we obtain
\[
\Psi^{(n)}_x - \frac{b}{\rho_2 13n} \psi^{(n)} + \frac{\gamma}{\rho_2 13n} \theta^{(n)}_x \to 0,
\]
which implies, multiplying by $\Psi^{(n)}_x$ in $L^2(0, l)$,
\[
\|\Psi^{(n)}_x\|_{L^2}^2 = \frac{b}{\rho_2} \left( \psi^{(n)}_x, \frac{\Psi^{(n)}_x}{13n} \right)_{L^2} + \frac{\gamma}{\rho_2 13n} (\theta^{(n)}_x, \Psi^{(n)})_{L^2} \to 0.
\]
Again, the second term goes to zero by Lemmas 3.2 and 3.6. Also, the last term goes to zero by Lemmas 3.2 and 3.4. This implies the convergence $\Psi^{(n)} \to 0$.

**Lemma 3.8.** Assume (3.1). Then,
(a) $\tau_0 (q^{(n)}, \Phi^{(n)}_x)_{L^2} - (\theta^{(n)}_x, \varphi^{(n)}_x)_{L^2} \to 0$.
(b) $\zeta (p^{(n)}, \Phi^{(n)}_x)_{L^2} - (\theta^{(n)}_x, \varphi^{(n)}_x)_{L^2} \to 0$.

**Proof.** From Lemmas 3.2 and 3.5,
\[
(\theta^{(n)}_x, \psi^{(n)} + lw^{(n)})_{L^2} = - (\theta^{(n)}_x, \psi^{(n)}_x)_{L^2} - l(\theta^{(n)}_x, w^{(n)}_x)_{L^2} \to 0,
\]
which implies, multiplying (3.14) by the bounded sequence $(\varphi^{(n)}_x + \psi^{(n)} + lw^{(n)})$ in $L^2(0, l)$, that
\[
\tau_0 \beta_n (\varphi^{(n)}_x + \psi^{(n)} + lw^{(n)}) + (\theta^{(n)}_x, \varphi^{(n)}_x)_{L^2} \to 0.
\]
On the other hand, from (3.3), (3.5) and (3.8) we obtain
\[
13n (\varphi^{(n)}_x + \psi^{(n)} + lw^{(n)}) - (\Phi^{(n)}_x + \Psi^{(n)} + IW^{(n)}) \to 0,
\]
which implies, multiplying by $q^{(n)}$ in $L^2(0, l)$ and using Lemmas 3.2 and 3.3,
\[
-13n (q^{(n)}_x, \varphi^{(n)}_x + \psi^{(n)} + lw^{(n)})_{L^2} - (q^{(n)}_x, \Phi^{(n)}_x)_{L^2} \to 0.
\]
Finally, multiplying by $\tau$ and adding to (3.15) we get the convergence (a). Analogously, from Lemmas 3.2 and 3.5,

$$(\varphi_x^{(n)}, \psi^{(n)})_{L^2} - (\varphi_x^{(n)}, \psi^{(n)})_{L^2} - l(\varphi^{(n)}, w_x^{(n)})_{L^2} \rightarrow 0,$$

which implies, multiplying (3.12) by $(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})$ in $L^2(0, \ell)$, that

$$\zeta \beta_n (p^{(n)} + \varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{L^2} + (\varphi_x^{(n)}, \varphi_x^{(n)})_{L^2} \rightarrow 0. \quad (3.17)$$

Moreover, multiplying (3.16) by $p^{(n)}$ in $L^2(0, \ell)$ and using Lemmas 3.2 and 3.3,

$$-i \beta_n (p^{(n)} + \varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{L^2} - (p^{(n)} + \Phi^{(n)})_{L^2} \rightarrow 0.$$

Then, multiplying by $\zeta$ and adding to (3.17) we obtain the convergence (b).

**Lemma 3.9.** Assume (3.1) and condition (1.15). Then,

$$\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)} \rightarrow 0 \quad \text{and} \quad \Phi^{(n)} \rightarrow 0.$$

**Proof.** Note that condition (1.15) is equivalent to

$$\left( \zeta - \frac{k_1 \rho_1}{k} \right) \left( b - \frac{k \rho_2}{\rho_1} \right) \equiv \chi = 0 \quad \text{or} \quad \left( \tau - \frac{k_1 \rho_1}{k} \right) \equiv \chi = 0. \quad (3.18)$$

First, let us assume that $\chi = 0$. From Lemmas 3.2 and 3.5,

$$(\varphi_x^{(n)}, \psi^{(n)} + lw^{(n)})_{L^2} = -(\varphi_x^{(n)}, \psi^{(n)} + lw^{(n)})_{L^2} - l(\varphi^{(n)}, w_x^{(n)})_{L^2} \rightarrow 0. \quad (3.19)$$

Multiplying (3.10) by $(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})$ in $L^2(0, \ell)$ and using (3.19),

$$\rho_1 i \beta_n (W_x^{(n)} + \varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{L^2} + k_0 (w_x^{(n)} - l \varphi_x^{(n)}, (\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{x})_{L^2} + kl \| \varphi_x^{(n)} + \psi^{(n)} + lw^{(n)} \|_{L^2} + \gamma \| \varphi_x^{(n)} + \varphi_x^{(n)} \|_{L^2} \rightarrow 0. \quad (3.20)$$

On the other hand, from (3.3), (3.5) and (3.8) we obtain

$$i \beta_n (\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)}) - (\Phi^{(n)} + \Psi^{(n)} + lW^{(n)}) \rightarrow 0. \quad (3.21)$$

Multiplying this by $W^{(n)}$ in $L^2(0, \ell)$ and using Lemma 3.7, we deduce

$$-i \beta_n (W^{(n)}, \varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{L^2} - (W^{(n)}, \Phi^{(n)})_{L^2} \rightarrow 0.$$

Then, multiplying by $\rho_1$ and adding to (3.20),

$$- \rho_1 (W^{(n)}, \Phi^{(n)})_{L^2} + k_0 (w_x^{(n)} - l \varphi_x^{(n)}, (\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{x})_{L^2} + kl \| \varphi_x^{(n)} + \psi^{(n)} + lw^{(n)} \|_{L^2} + \gamma \| \varphi_x^{(n)} + \varphi_x^{(n)} \|_{L^2} \rightarrow 0. \quad (3.22)$$

On the other hand, from (3.4), Lemmas 3.5 and 3.6, we deduce that

$$\rho_1 i \beta_n \Phi^{(n)} - k(\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{x} \rightarrow 0, \quad (3.23)$$

which implies, multiplying by $w_x^{(n)} - l \varphi^{(n)}$ in $L^2(0, \ell)$, that

$$- \rho_1 i \beta_n (w_x^{(n)} - l \varphi^{(n)}, \Phi^{(n)})_{L^2} - k(w_x^{(n)} - l \varphi^{(n)}, (\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)})_{x})_{L^2} \rightarrow 0. \quad (3.24)$$

Also, from (3.2) and (3.9),

$$i \beta_n (w_x^{(n)} - l \varphi^{(n)}) - (W_x^{(n)} - l \Phi^{(n)}) \rightarrow 0.$$

Multiplying this by $\rho_1 \Phi^{(n)}$ in $L^2(0, \ell)$,

$$\rho_1 i \beta_n (w_x^{(n)} - l \varphi^{(n)}, \Phi^{(n)})_{L^2} - \rho_1 (W_x^{(n)} - l \Phi^{(n)}, \Phi^{(n)})_{L^2} \rightarrow 0,$$
which implies, adding to (3.24), that
\[ \rho_1(W^{(n)}, \Phi_x^{(n)})_{L^2} + \rho_1 l\|\varphi_x^{(n)}\|_{L^2}^2 - k(w_x^{(n)} - l\varphi_x^{(n)}, (\varphi_x^{(n)} + \psi_x^{(n)} + l w^{(n)} x)_L^2 \to 0. \]

Multiplying the last expression by \( \frac{k_0}{k} \), adding to \( \gamma x (a) \), where \( (a) \) is the first convergence of Lemma 3.8, and finally adding to (3.22), we obtain
\[ \rho_1 \left( \frac{k_0}{k} - 1 \right) (W^{(n)}, \Phi_x^{(n)})_{L^2} + kl\|\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)} \|_{L^2}^2 \]
\[ + \frac{k_0 \rho_1 l}{k}\|\Phi^{(n)}\|_{L^2}^2 + \gamma (q^{(n)}, \Phi_x^{(n)})_{L^2} \to 0. \] (3.25)

On the other hand, multiplying (3.13) by \( \Phi \) in \( L^2(0, \ell) \),
\[ i_\beta_n (\theta^{(n)}, \Phi^{(n)})_{L^2} - k_1 (q^{(n)}, \Phi_x^{(n)})_{L^2} - m(W^{(n)}, \Phi_x^{(n)})_{L^2} - ml\|\Phi^{(n)}\|_{L^2}^2 \to 0. \] (3.26)

Also, multiplying (3.23) by \( \theta \) in \( L^2(0, \ell) \) and using (3.19), we obtain
\[ -\rho_1 i_\beta_n (\theta^{(n)}, \Phi^{(n)})_{L^2} + k(\theta^{(n)}, \varphi_x^{(n)})_{L^2} \to 0. \]

Then, doing \( k x (a) \), with \( (a) \) of Lemma 3.8, and adding to the last expression, we conclude that
\[ -\rho_1 i_\beta_n (\theta^{(n)}, \Phi^{(n)})_{L^2} + \tau k (q^{(n)}, \Phi_x^{(n)})_{L^2} \to 0. \] (3.27)

Now, in order to apply the same strategy of [3], let us define
\[ \sigma_1 := \left( 1 - \frac{k_1 \rho_1}{\tau k} \right) \neq 0, \text{ because } \chi = 0. \]

In addition, \( \chi = 0 \) implies \( \frac{\rho_1}{k \sigma_1} \chi = 0 \), that is,
\[ \rho_1 \left( \frac{k_0}{k} - 1 \right) + \frac{\rho_1 \gamma m}{k \sigma_1} = 0. \] (3.28)

Multiplying (3.26) by \( -\frac{\rho_1 \gamma}{k \sigma_1} \), we have
\[ -\frac{\rho_1 \gamma}{k \sigma_1} i_\beta_n (\theta^{(n)}, \Phi^{(n)})_{L^2} + \frac{\rho_1 \gamma k_1}{k \sigma_1} (q^{(n)}, \Phi_x^{(n)})_{L^2} \]
\[ + \frac{\rho_1 \gamma m}{k \sigma_1} (W^{(n)}, \Phi_x^{(n)})_{L^2} + \frac{\rho_1 \gamma ml}{k \sigma_1} \|\Phi^{(n)}\|_{L^2}^2 \to 0. \]

Multiplying (3.27) by \( -\frac{\gamma}{k \sigma_1} \) and adding to the last expression,
\[ \left( \frac{\rho_1 \gamma k_1}{k \sigma_1} - \frac{\gamma}{\sigma_1} \right) (q^{(n)}, \Phi_x^{(n)})_{L^2} + \frac{\rho_1 \gamma m}{k \sigma_1} (W^{(n)}, \Phi_x^{(n)})_{L^2} + \frac{\rho_1 \gamma ml}{k \sigma_1} \|\Phi^{(n)}\|_{L^2}^2 \to 0, \]

which, added to (3.25) and using (3.28), yields
\[ kl\|\varphi_x^{(n)} + \psi^{(n)} + lw^{(n)} \|_{L^2}^2 + \left( \frac{\rho_1 \gamma ml}{k \sigma_1} + \frac{k_0 \rho_1 l}{k} \right) \|\Phi^{(n)}\|_{L^2}^2 \]
\[ + \left( \frac{\rho_1 \gamma k_1}{k \sigma_1} - \frac{\gamma}{\sigma_1} + \gamma \tau \right) (q^{(n)}, \Phi_x^{(n)})_{L^2} \to 0 \]

which implies the desired result because, again by (3.28),
\[ \left( \frac{\rho_1 \gamma k_1}{k \sigma_1} - \frac{\gamma}{\sigma_1} + \gamma \tau \right) = \frac{\gamma}{k \sigma_1} (\rho_1 k_1 - k \tau + k \sigma_1 \tau) = 0. \]
Second, let us assume that \( \xi = 0 \). From Lemmas 3.2 and 3.5,
\[
(\varphi_x^{(n)}, \psi^{(n)})_{L^2} + w^{(n)}_{L^2} = -(\varphi_x^{(n)}, \psi^{(n)})_{L^2} + l(\varphi^{(n)}, w^{(n)}_{L^2}) \to 0,
\]
(3.29)
Multiplying (3.7) by \((\varphi_x^{(n)} + \varphi^{(n)} + l w^{(n)})\) in \(L^2(0, \ell)\) and using (3.29),
\[
\rho_2 i \beta_n (\Phi^{(n)}, \varphi_x^{(n)} + \psi^{(n)} + l w^{(n)})_{L^2} + b(\psi^{(n)}, (\varphi_x^{(n)} + \psi^{(n)} + l w^{(n)}))_{L^2} + k||\varphi_x^{(n)} + \psi^{(n)} + l w^{(n)}||^2_{L^2} + \gamma(\varphi_x^{(n)}, \varphi_x^{(n)})_{L^2} \to 0.
\]
(3.30)
Multiplying (3.21) by \(\Psi^{(n)}\) in \(L^2(0, \ell)\) and using Lemma 3.7,
\[
-i \beta_n (\Psi^{(n)}, \varphi_x^{(n)} + \psi^{(n)} + l w^{(n)})_{L^2} - (\Psi^{(n)}, \Phi_x^{(n)})_{L^2} \to 0,
\]
which, multiplied by \(\rho_2\) and added to (3.30), implies
\[
-\rho_2 (\Psi^{(n)}, \Phi_x^{(n)})_{L^2} + b(\psi^{(n)}, (\varphi_x^{(n)} + \psi^{(n)} + l w^{(n)}))_{L^2} + k||\varphi_x^{(n)} + \psi^{(n)} + l w^{(n)}||^2_{L^2} + \gamma(\varphi_x^{(n)}, \varphi_x^{(n)})_{L^2} \to 0.
\]
(3.31)
Also, multiplying (3.23) by \(\psi_x^{(n)}\) in \(L^2(0, \ell)\), we have
\[
-\rho_1 i \beta_n (\psi_x^{(n)}, \Phi^{(n)})_{L^2} - k(\psi_x^{(n)}, (\varphi^{(n)} + \psi^{(n)} + l w^{(n)}))_{L^2} \to 0.
\]
Multiplying (3.6) by \(\rho_1 \Phi^{(n)}\) in \(L^2(0, \ell)\),
\[
\rho_1 i \beta_n (\psi_x^{(n)}, \Phi^{(n)})_{L^2} - \rho_1 (\Psi^{(n)}, \Phi_x^{(n)})_{L^2} \to 0.
\]
Then, adding the last two convergences, we obtain
\[
-\rho_1 (\Psi^{(n)}, \Phi_x^{(n)})_{L^2} - k(\psi_x^{(n)}, (\varphi^{(n)} + \psi^{(n)} + l w^{(n)}))_{L^2} \to 0,
\]
which, multiplied by \(\frac{b}{k}\), added to \(\gamma \times (b)\), where \(b\) is the second convergence of Lemma 3.8, and finally added to (3.31), implies that
\[
\left(\frac{\rho_1 b}{k} - \rho_2\right) (\Psi^{(n)}, \Phi_x^{(n)})_{L^2} + k||\varphi_x^{(n)} + \psi^{(n)} + l w^{(n)}||^2_{L^2} + \gamma(\psi^{(n)}, \Phi_x^{(n)})_{L^2} \to 0.
\]
(3.32)
On the other hand, multiplying (3.11) by \(\Phi^{(n)}\) in \(L^2(0, \ell)\),
\[
i \beta_n (\varphi_x^{(n)}, \Phi^{(n)})_{L^2} - k_1(\varphi_x^{(n)}, \Phi_x^{(n)})_{L^2} - m(\Psi^{(n)}, \Phi_x^{(n)})_{L^2} \to 0.
\]
(3.33)
Multiplying (3.23) by \(\varphi^{(n)}\) in \(L^2(0, \ell)\) and using (3.29),
\[
-\rho_1 i \beta_n (\varphi_x^{(n)}, \Phi^{(n)})_{L^2} + k(\varphi_x^{(n)}, \varphi_x^{(n)})_{L^2} \to 0,
\]
which, added to \(k \times (b)\), with \(b\) of Lemma 3.8, implies
\[
-\rho_1 i \beta_n (\varphi_x^{(n)}, \Phi^{(n)})_{L^2} - \zeta k(\varphi_x^{(n)}, \Phi_x^{(n)})_{L^2} \to 0.
\]
(3.34)
As before, let us define
\[
\sigma_2 := \left(1 - \frac{k_1 \rho_1}{\varsigma k}\right) \neq 0, \text{ because } \xi = 0.
\]
In addition, \(\xi = 0\) implies \(\frac{\rho_1}{k \sigma_2} \xi = 0\), that is,
\[
\left(\frac{\rho_1 b}{k} - \rho_2\right) + \frac{\rho_1 \gamma m}{k \sigma_2} = 0.
\]
(3.35)
Multiplying (3.33) by \(-\frac{\rho_1 \gamma}{k \sigma_2}\), we have
\[
-\frac{\rho_1 \gamma}{k \sigma_2} i \beta_n (\varphi_x^{(n)}, \Phi^{(n)})_{L^2} + \frac{\rho_1 \gamma k_1}{k \sigma_2} (\varphi_x^{(n)}, \Phi_x^{(n)})_{L^2} + \frac{\rho_1 \gamma m}{k \sigma_2} (\Psi^{(n)}, \Phi_x^{(n)})_{L^2} \to 0.
\]
Multiplying (3.34) by $-\gamma/k\sigma^2$ and adding to the last expression, 

$$\left(\frac{\rho_1\gamma k_1}{k\sigma^2} - \frac{\gamma}{\sigma^2}\right)(p^{(n)}, \Phi_x^{(n)})_{L^2} + \frac{\rho_1\gamma m}{k\sigma^2}(\Psi^{(n)}, \Phi_x^{(n)})_{L^2} \to 0,$$

which, added to (3.32) and using (3.35), implies 

$$k\|\varphi^{(n)} + \psi^{(n)} + lw^{(n)}\|_{L^2} + \left(\frac{\rho_1\gamma k_1}{k\sigma^2} - \frac{\gamma}{\sigma^2} + \gamma\varsigma\right)(p^{(n)}, \Phi_x^{(n)})_{L^2} \to 0,$$

which yields the first desired convergence because 

$$\left(\frac{\rho_1\gamma k_1}{k\sigma^2} - \frac{\gamma}{\sigma^2} + \gamma\varsigma\right) = \frac{\gamma}{k\sigma^2}(\rho_1 k_1 - k\varsigma + k\sigma_2\varsigma) = 0.$$ 

Then, multiplying (3.4) by $\frac{1}{\rho_0(n)}\Phi^{(n)}$ in $L^2(0,\ell)$, applying integration by parts and using the previous Lemmas, we obtain the second desired convergence. \hfill \Box

Now, we are ready to prove the main results of this section.

**Theorem 3.10 (Exponential decay).** Suppose that condition (1.15) is true, that is,

$$\left(\varsigma - k_1\rho_1 k\right)\left(b - \frac{k_2}{\rho_1}\right) + \varsigma\gamma m = 0.$$ \hspace{1cm} (3.36)

Then, the semigroup generated by $A$ is exponentially stable.

**Proof.** As mentioned earlier, condition (3.36) is equivalent to (3.18). As seen in the Introduction, in [5] it was proved that $\xi = 0$ implies exponential stability. So, it remains to show that the semigroup generated by $A$ is exponentially stable provided that $\chi = 0$. Nevertheless, as our argument is different, we give a complete proof. To this purpose, let us verify the conditions of Theorem 3.1. We emphasize, however, that the imaginary axis is always contained in $\rho(A)$, no matter if (3.36) is satisfied or not.

• $i\mathbb{R} \subset \rho(A)$. Let us assume, by contradiction, that the inclusion is not valid. Then, there exists $\lambda \in i\mathbb{R}$ such that $\lambda \in \sigma(A)$, with $\lambda \neq 0$ because $0 \in \rho(A)$ (as seen in the proof of Theorem 2.2). By Remark 1, $\lambda$ is an eigenvalue of $A$. Therefore, there exists $U \neq 0$ in $D(A)$ satisfying the resolvent equation $AU = \lambda U$, which in terms of its components can be written as

$$\lambda \varphi - \Phi = 0 \hspace{1cm} (3.37)$$
$$\rho_1 \lambda \Phi - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) + l\gamma \theta = 0 \hspace{1cm} (3.38)$$
$$\lambda \psi - \Psi = 0 \hspace{1cm} (3.39)$$
$$\rho_2 \lambda \Psi - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma \vartheta_x = 0 \hspace{1cm} (3.40)$$
$$\lambda w - W = 0 \hspace{1cm} (3.41)$$
$$\rho_1 \lambda W - k_0 (w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) + \gamma \theta_x = 0 \hspace{1cm} (3.42)$$
$$\lambda \theta + k_1 p_x + m\Psi_x = 0 \hspace{1cm} (3.43)$$
$$\varsigma \lambda p + \delta p + \vartheta_x = 0 \hspace{1cm} (3.44)$$
$$\lambda \theta + k_1 q_x + m(W_x - l\Phi) = 0 \hspace{1cm} (3.45)$$
$$\tau \lambda q + \delta q + \vartheta = 0 \hspace{1cm} (3.46)$$
Then, multiplying by $U \in D(A)$, it follows from the dissipative property (2.15) that $p = q = 0$. Now, substituting into (3.44), (3.46) we obtain

$$\vartheta_x = \theta_x = 0 \text{ in } L^2(0, L) \Rightarrow \vartheta = \theta = 0 \text{ in } L^2(0, L).$$

Therefore, by (3.43), (3.45), we have $\Psi_x = W_x - i\Phi = 0$. Then, from (3.37), (3.39) and (3.41) it follows that $\psi_x = w_x - l\varphi = 0$, which implies $\psi = \Psi = 0$. Finally, from (3.38), (3.40), (3.42) and applying all identities obtained, we conclude that

$$\rho_1\lambda\Phi - k(\varphi_x + lw)_x = 0, \quad k(\varphi_x + lw) = 0, \quad \rho_1\lambda W + kl(\varphi_x + lw) = 0$$

which implies, substituting the second equality in the others, that $\Phi = W = 0$ and thus $\varphi = w = 0$. This shows that $U = 0$, which is a contradiction. Therefore, the inclusion $\mathbb{R} \subset \rho(A)$ is valid.

$\bullet$ lim sup$_{\beta \to \infty} \| (i\beta I - A)^{-1} \| < \infty$. For this limit, it is sufficient to prove that there exist constants $C, \beta_0 > 0$ such that

$$\| (i\beta I - A)^{-1} \| \leq C, \quad \forall \beta \geq \beta_0. \quad (3.47)$$

In fact, by contradiction, let us assume that (3.47) is not true. Then, given any $n \in \mathbb{N}$, it is not true that $\| (i\beta I - A)^{-1} \| \leq n$ for all $\beta > n$. Consequently, there exists a sequence $\beta_n > n$ such that $\| (i\beta_n - A)^{-1} \| > n$. The last inequality implies that there exists a sequence $(F_n)_{n \in \mathbb{N}} \subset \mathbb{H}$ such that

$$\| (i\beta_n - A)^{-1} F_n \| > n \| F_n \|.$$

Then, defining $U_n = \frac{(i\beta_n - A)^{-1} F_n}{\| (i\beta_n - A)^{-1} F_n \|}$ we have

$$\| (i\beta_n - A) U_n \| = \frac{\| F_n \|}{\| (i\beta_n - A)^{-1} F_n \|} < \frac{1}{n}.$$

The last inequality shows that, if (3.47) does not hold, then there exist a sequence $(\beta_n)_{n \in \mathbb{N}}$ of positive real numbers and a sequence $(U_n)_{n \in \mathbb{N}} \subset D(A)$ such that

$$\beta_n \xrightarrow{n \to \infty} \infty, \quad \| U_n \| = 1, \quad \| (i\beta_n - A) U_n \| \xrightarrow{n \to \infty} 0. \quad (3.48)$$

Therefore, recalling assumption (3.36), we see that the hypothesis of Lemmas 3.2-3.9 are satisfied. Thus, from the convergences in Lemmas 3.3, 3.5, 3.6, 3.7 and 3.9, we conclude that $U_n \to 0$ in $\mathbb{H}$ which contradicts (3.48). Then, the second condition of Theorem 3.1 holds.

4. **Non-exponential decay.** In this section, we want to discuss the converse of Theorem 3.10. For this, assume that

$$\left[ \left( \zeta - \frac{k_1 \rho_1}{k} \right) \left( b - \frac{k \rho_2}{\rho_1} \right) + \gamma \frac{m}{\rho_1} \right] \left[ \left( \frac{\tau - \frac{k_1 \rho_1}{k}}{k_0} - k \right) \left( \frac{k_0}{k} \right) + \gamma m \right] \neq 0, \quad (4.1)$$

which is equivalent to

$$\xi \neq 0 \text{ and } \chi \neq 0. \quad (4.2)$$

In view of Theorem 3.1, to show lack of exponential decay, it is enough to show that there exist a sequence $(\beta_n)_{n \in \mathbb{N}}$ of positive real numbers such that $\beta_n \xrightarrow{n \to \infty} \infty$ and a bounded sequence $(F_n)_{n \in \mathbb{N}}$ in $\mathbb{H}$ such that

$$\| (i\beta_n I - A)^{-1} F_n \| \xrightarrow{n \to \infty} \infty. \quad (4.3)$$
Let us write $c_n = \frac{n\pi}{T}$ and define

$$
\beta_n = \sqrt{\frac{1}{\rho_1}(kc_n^2 + k_0l^2)}, \quad F_n = (0, \rho_1^{-1}\sin(c_nx), 0, 0, 0, 0).
$$

Then $(F_n)_{n \in \mathbb{N}}$ is a bounded sequence in $\mathbb{H}$. In addition,

$$
\|(i\beta_n - A)^{-1}F_n\|_{\mathbb{H}}^2 = \rho_1\|\Phi^{(0)}\|_{L^2}^2 + \rho_2\|\Psi^{(0)}\|_{L^2}^2 + \rho_1\|W^{(0)}\|_{L^2}^2 + \rho_1\|\psi^{(0)}\|_{L^2}^2
+ k\|\varphi^{(n)} + \psi^{(n)} + lw^{(n)} + k_0u^{(n)} - l\varphi^{(n)}\|_{L^2}^2
+ \frac{\gamma}{m}(\|\theta^{(n)}\|_{L^2}^2 + \|\varphi^{(n)}\|_{L^2}^2)
+ \frac{k_1}{m}(\|p^{(n)}\|_{L^2}^2 + \|\varphi^{(n)}\|_{L^2}^2),
$$

where $(\varphi^{(n)}, \psi^{(n)}, \Psi^{(n)}, w^{(n)}, W^{(n)}, \theta^{(n)}, p^{(n)}, \hat{W}(n), q^{(n)}) := U_n$ is the unique solution in $D(A)$ of the resolvent equation

$$
(i\beta_n - A)U_n = F_n.
$$

By ansatz, we suppose that

$$
\varphi^{(n)}(x) = A_n \sin(c_nx), \quad \psi^{(n)}(x) = B_n \cos(c_nx),
$$

$$
w^{(n)}(x) = C_n \cos(c_nx), \quad \theta^{(n)}(x) = D_n \sin(c_nx),
$$

$$
p^{(n)}(x) = \hat{E}_n \cos(c_nx), \quad \hat{\theta}^{(n)}(x) = \hat{D}_n \sin(c_nx),
$$

Then, substituting into the resolvent equation (4.5), we conclude that

$$
U_n = (\varphi^{(n)}, \psi^{(n)}, \Psi^{(n)}, w^{(n)}, W^{(n)}, \theta^{(n)}, p^{(n)}, \hat{W}(n), q^{(n)})
$$

given by (4.6) with

$$
\Phi^{(n)} = i\beta_n \varphi^{(n)}, \quad \Psi^{(n)} = i\beta_n \psi^{(n)}, \quad W^{(n)} = i\beta_n w^{(n)}
$$

is the solution of (4.5) if and only if the coefficients $A_n, B_n, C_n, \hat{D}_n, \hat{E}_n, D_n,$ and $E_n$ satisfy the linear system

$$
\begin{align*}
\begin{cases}
(\rho_1(\beta^2_n)^2 + kc_n^2 + k_0l^2)A_n + kc_nB_n + l(k + k_0)c_nC_n + l\gamma D_n = 1 \\
k_0c_nA_n + (\rho_2(\beta^2_n)^2 + kc_n^2 + k)B_n + klC_n + \gamma c_n \hat{D}_n = 0 \\
l(k + k_0)c_nA_n + klB_n + (\rho_1(\beta^2_n)^2 + k_0c_n^2 + kl^2)C_n + \gamma c_n \hat{D}_n = 0 \\
-lmi\beta_n c_nB_n + i\beta_n \hat{D}_n - k_1c_n \hat{E}_n = 0 \\
(i\beta_n + \delta)\hat{E}_n + c_n \hat{D}_n = 0 \\
-lmi\beta_n A_n - mi\beta_n C_n + i\beta_n \hat{D}_n - k_1c_n E_n = 0 \\
(i\beta_n + \delta) \hat{E}_n + c_n \hat{D}_n = 0
\end{cases}
\end{align*}
$$

which can be written as

$$
\begin{align*}
\begin{bmatrix}
\hat{E}_n \\
\hat{D}_n \\
\beta_n \\
\beta_n \\
\beta_n \\
\beta_n \\
\beta_n
\end{bmatrix}
= M_n
\begin{bmatrix}
A_n \\
B_n \\
C_n \\
\hat{D}_n \\
\hat{D}_n
\end{bmatrix}
\end{align*}
$$

where $M_n$ is a matrix that depends on $\beta_n$.
where
\[ p_n^{(1)} = \rho_1 (i\beta_n)^2 + k c_n^2 + k_0 l^2, \quad p_n^{(2)} = \rho_2 (i\beta_n)^2 + b c_n^2 + k, \]
\[ p_n^{(3)} = \rho_1 (i\beta_n)^2 + k_0 c_n^2 + k l^2, \quad p_n^{(4)} = i \beta_n + \frac{k_1 c_n^2}{\beta_n + \delta}, \quad p_n^{(5)} = i \beta_n + \frac{k_1 c_n^2}{\tau_1 \beta_n + \delta}. \]

Now, using the definition of \( \beta_n \), we have
\[ p_n^{(1)} = 0, \quad p_n^{(2)} = \left( b - \frac{\rho_2 k}{\rho_1} \right) c_n^2 - \frac{\rho_2 k_0 l^2}{\rho_1} + k = O(n^2), \]
\[ p_n^{(3)} = (k_0 - k)c_n^2 + (k - k_0)l^2 = O(n^2), \]
\[ p_n^{(4)} = i \sqrt{\frac{1}{\rho_1}} (k c_n^2 + k_0 l^2) + \frac{k_1 c_n^2}{\sqrt{\frac{1}{\rho_1}} (k c_n^2 + k_0 l^2) + \delta} = O(n), \]
\[ p_n^{(5)} = i \sqrt{\frac{1}{\rho_1}} (k c_n^2 + k_0 l^2) + \frac{k_1 c_n^2}{\tau_1 \sqrt{\frac{1}{\rho_1}} (k c_n^2 + k_0 l^2) + \delta} = O(n), \]
which implies that
\[ \Delta_n := \det(\tilde{M}_n) \]
\[ = \tilde{p}_n^{(1)} \left[ -k^2 c_n^4 - l^2 (k + k_0)^2 c_n^2 p_n^{(2)} - 2l^2 (k + k_0) \gamma m (i\beta_n) c_n^2 p_n^{(2)} \right] \]
\[ + \tilde{p}_n^{(3)} \left[ l^2 \gamma m (i\beta_n) p_n^{(2)} p_n^{(2)} - k^2 \gamma m (i\beta_n) c_n^2 \right] + O(n^5). \]

For the moment, let us assume that
\[ \tilde{\Delta}_n \neq 0 \]
for all \( n \in \mathbb{N} \) sufficiently large. Then, we conclude that system (4.8) has a unique solution \([A_n, B_n, C_n, \tilde{D}_n, D_n]^T\) given by the Cramer’s Rule. Consequently, system (4.7) has a unique solution \([A_n, B_n, C_n, \tilde{D}_n, \tilde{E}_n, D_n, E_n]^T\) which implies that the solution of the resolvent equation (4.5) is given by (4.6), for all \( n \in \mathbb{N} \) sufficiently large. Therefore, we can estimate (4.4) by
\[ \| (i\beta_n - \hat{A})^{-1} F \|_{L^2} \geq \rho_1 \| \Phi^{(n)} \|_{L^2} = \rho_1 \beta_n^2 |A_n|^2 \int_0^\ell | \sin(c_n x) |^2 dx = \rho_1 \frac{\ell}{2} \beta_n^2 |A_n|^2, \]
where \( A_n \) is given by
\[ A_n = \frac{\tilde{A}_n}{\Delta_n}, \]
with \( \tilde{A}_n \) defined as
\[ \tilde{A}_n := \det \begin{bmatrix}
1 & kc_n & l(k + k_0)c_n & 0 & l\gamma \\
0 & p_n^{(2)} & kl & \gamma c_n & 0 \\
0 & kl & p_n^{(3)} & 0 & \gamma c_n \\
0 & -mi\beta_n c_n & 0 & \tilde{p}_n^{(4)} & 0 \\
0 & 0 & -mi\beta_n c_n & 0 & p_n^{(4)}
\end{bmatrix} \]
\[ = -k^2 l^2 p_n^{(4)} \tilde{p}_n^{(4)} + \gamma^2 m^2 (i\beta_n)^2 c_n^4 + \gamma m i\beta_n p_n^{(3)} p_n^{(4)} c_n^2 \]
\[ + \gamma m i\beta_n p_n^{(2)} p_n^{(4)} c_n^2 + p_n^{(2)} \tilde{p}_n^{(4)} p_n^{(4)} \]
\[ = \left( \gamma m i\beta_n c_n^2 + p_n^{(3)} \tilde{p}_n^{(4)} \right) \left( \gamma m i\beta_n c_n^2 + p_n^{(2)} \tilde{p}_n^{(4)} \right) + O(n^2). \]
Here, using the convergences
\[
\frac{p_n^{(2)}}{n^2} \xrightarrow{n \to \infty} \left(b - \frac{\rho_2 k}{\rho_1}\right) \frac{\pi^2}{\ell^2}, \quad \frac{p_n^{(3)}}{n^2} \xrightarrow{n \to \infty} (k_0 - k) \frac{\pi^2}{\ell^2}, \quad \frac{\beta_n}{n} \xrightarrow{n \to \infty} \frac{\pi}{\ell} \sqrt{\frac{k}{\rho_1}},
\]
we deduce
\[
\frac{\hat{A}_n}{n^6} \xrightarrow{n \to \infty} \left(\gamma m \left[\pi \frac{k}{\rho_1} \left(1 - \frac{k_1 \rho_1}{\xi k}\right)\right] + \left((k_0 - k) \frac{\pi^2}{\ell^2}\right) \left[\frac{\pi}{\ell} \sqrt{\frac{k}{\rho_1}} \left(1 - \frac{k_1 \rho_1}{\xi k}\right)\right]\right)
\times \left(\gamma m \left[\frac{\pi}{\ell} \sqrt{\frac{k}{\rho_1}} \left(1 - \frac{k_1 \rho_1}{\xi k}\right)\right] + \left((b - \frac{\rho_2 k}{\rho_1}) \frac{\pi^2}{\ell^2}\right) \left[\frac{\pi}{\ell} \sqrt{\frac{k}{\rho_1}} \left(1 - \frac{k_1 \rho_1}{\xi k}\right)\right]\right) =: L_1.
\]
Analogously we can deduce that
\[
\frac{\hat{A}_n}{n^6} \xrightarrow{n \to \infty} L_2,
\]
for some constant \(L_2 \in \mathbb{C}\). Then, using (4.2), we conclude that
\[
L_1 = -\frac{\pi^6}{\ell^6} \frac{k}{\rho_1} \frac{1}{\tau \xi} \chi \xi \neq 0
\]
which implies
\[
|A_n| = \left|\frac{\hat{A}_n}{\Delta_n}\right| = \left|\frac{\hat{A}_n}{n^6} \frac{\Delta_n}{n^6}\right| \xrightarrow{n \to \infty} L = \begin{cases} \frac{|L_1|}{|L_2|} \neq 0, & \text{if } L_2 \neq 0, \\ \infty, & \text{if } L_2 = 0. \end{cases}
\]
Since \(\beta_n \xrightarrow{n \to \infty} \infty\), it follows from (4.10) that
\[
\|(i\beta_n - A)^{-1} F\|_{\infty}^2 \geq \rho_1 \frac{\ell}{2} \beta_n |A_n|^2 \xrightarrow{n \to \infty} \infty
\]
which implies (4.3).

By now, we have proved the converse of Theorem 3.10 provided that condition (4.9) is justified, which is easy to do in some spacial cases but difficult (if possible) in general cases. For example, under condition (4.1), the lack of exponential stability holds provided that:

1. \(L_2\) in (4.11) is nonzero because, in this case, \(\hat{A}_n\) (and thus \(\Delta_n\)) is nonzero for all \(n\) large enough. In particular, we have non-exponential decay provided that
\[
\text{Re}(L_2) = \frac{\pi^6}{\ell^6} \frac{k}{\rho_1} \left[k^2 \frac{1}{\tau} \left(1 - \frac{k_1 \rho_1}{\xi k}\right) \chi + \left((k_0 + k) \frac{1}{\tau k}\right) \left(1 - \frac{k_1 \rho_1}{\xi k}\right) + (3k + k_0) \gamma m\right] \tau^2 \xi \neq 0
\]
which happens, for example, if \(1 - \frac{k_1 \rho_1}{\xi k} = 1 - \frac{k_1 \rho_1}{\xi \frac{\Gamma}{k}} = 0\).

2. \(k_0 \leq k\) because, in this case, \(\text{Im}(\Delta_n)\) (and thus \(\Delta_n\)) is nonzero for all \(n\) large enough, since, as a straightforward calculation shows,
\[
\text{Im}(\tilde{\Delta}_n) = \left(a_0 + a_2 n^2 + a_4 n^4 + a_6 n^6\right) \frac{k_1 \delta c_0^2 \beta_n}{\left(\frac{a_0}{\rho_1} + \frac{a_2}{\rho_1} n + \frac{a_4}{\rho_1} n^2 + \frac{a_6}{\rho_1} n^4\right)^2 + \frac{k_1 \delta c_0^2 \beta_n}{\left(\frac{a_0}{\rho_1} + \frac{a_2}{\rho_1} n + \frac{a_4}{\rho_1} n^2 + \frac{a_6}{\rho_1} n^4\right)^2} + \frac{k_0 \delta c_0^2 \beta_n}{\left(\frac{a_0}{\rho_1} + \frac{a_2}{\rho_1} n + \frac{a_4}{\rho_1} n^2 + \frac{a_6}{\rho_1} n^4\right)^2}} = 0.
\]
where $a_0, a_2, a_4, a_6$ are constants with

$$a_0 = \left[ \left( k - \frac{\rho_2}{\rho_1} k_0 l^2 \right) (k - k_0) l^2 - k^2 l^2 \right] \left( \delta^2 + \frac{\tau^2}{\rho_1} k_0 l^2 \right) l^2 \gamma m. \quad \text{if } k_0 \leq k$$

In particular, taking $k_0 = k$, we conclude that the physical system (1.3)-(1.5) is not exponentially stable under the usual equal wave speed condition.

Consequently, as we do not have complete control of constant $L_2$, we can guarantee that condition (4.1) is necessary for the exponential decay for all $k_0 \leq k$. The other case ($k_0 > k$) remains as an open problem.

5. **Final remarks.** We have considered $\varsigma > 0, \tau > 0$. However, an analogous argument can be applied in general and the same result obtained in Section 3 is valid for the other cases ($\varsigma > 0, \tau = 0$ or $\varsigma = 0, \tau > 0$ or $\varsigma = \tau = 0$). More precisely, under the boundary conditions (1.4):

1. For the case $\varsigma > 0, \tau = 0$, the mathematical system
   \[
   \begin{align*}
   \rho_1 \varphi_{tt} - k(\varphi_x + \psi + lw)_x - l[k_0(w_x - l\varphi) - \gamma \theta] &= 0 \\
   \rho_2 \psi_{tt} - [b\psi_x - \gamma \theta]_x + k(\varphi_x + \psi + lw) &= 0 \\
   \rho_1 w_{tt} - [k_0(w_x - l\varphi) - \gamma \theta]_x + kl(\varphi_x + \psi + lw) &= 0 \\
   \vartheta_t + k_1 p_x + m\varphi_{xt} &= 0, \quad \varphi_t + \delta p + \vartheta_x = 0 \\
   \theta_t - \frac{k_1}{\delta} \vartheta_{xx} + m(w_{xt} - l\varphi_t) &= 0
   \end{align*}
   \]
   is exponentially stable if
   \[
   \left( \varsigma - \frac{k_1 \rho_1}{k} \right) \left( b - \frac{k \rho_2}{\rho_1} \right) + \varsigma \gamma m \right](k_0 - k) = 0.
   \]

2. For the case $\varsigma = 0, \tau > 0$, the mathematical system
   \[
   \begin{align*}
   \rho_1 \varphi_{tt} - k(\varphi_x + \psi + lw)_x - l[k_0(w_x - l\varphi) - \gamma \theta] &= 0 \\
   \rho_2 \psi_{tt} - [b\psi_x - \gamma \theta]_x + k(\varphi_x + \psi + lw) &= 0 \\
   \rho_1 w_{tt} - [k_0(w_x - l\varphi) - \gamma \theta]_x + kl(\varphi_x + \psi + lw) &= 0 \\
   \vartheta_t - \frac{k_1}{\delta} \vartheta_{xx} + m\varphi_{xt} &= 0 \\
   \theta_t + k_1 q_x + m(w_{xt} - l\varphi_t) &= 0, \quad \tau q_t + \delta q + \theta_x = 0
   \end{align*}
   \]
   is exponentially stable if
   \[
   \left( b - \frac{k \rho_2}{\rho_1} \right) \left( \tau - \frac{k_1 \rho_1}{k} \right) (k_0 - k) + \tau \gamma m \right] = 0.
   \]

3. For the case $\varsigma = 0, \tau = 0$, the mathematical system
   \[
   \begin{align*}
   \rho_1 \varphi_{tt} - k(\varphi_x + \psi + lw)_x - l[k_0(w_x - l\varphi) - \gamma \theta] &= 0 \\
   \rho_2 \psi_{tt} - [b\psi_x - \gamma \theta]_x + k(\varphi_x + \psi + lw) &= 0 \\
   \rho_1 w_{tt} - [k_0(w_x - l\varphi) - \gamma \theta]_x + kl(\varphi_x + \psi + lw) &= 0 \\
   \vartheta_t - \frac{k_1}{\delta} \vartheta_{xx} + m\varphi_{xt} &= 0, \quad \theta_t - \frac{k_1}{\delta} \theta_{xx} + m(w_{xt} - l\varphi_t) &= 0
   \end{align*}
   \]
   (5.1)
is exponentially stable if

$$\left( b - \frac{k_0}{\rho_1} \right) (k_0 - k) = 0. \quad (5.2)$$

In particular, if $k_0 = \frac{b\rho_1}{\rho_2}$, then equality (5.2) reduces to $b - \frac{k_0}{\rho_1} = 0$. As discussed in the Introduction, this result was obtained in [10], where the authors studied the physical system (5.1).

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E-mail address: pedrorobertodelima@gmail.com
E-mail address: hugo.sare@ice.ufjf.br