An Improved Discrete Hardy Inequality

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Abstract. In this note, we prove an improvement of the classical discrete Hardy inequality. Our improved Hardy-type inequality holds with a weight \( w \) which is strictly greater than the classical Hardy weight \( w_H(n) := 1/(2n)^2 \), where \( n \in \mathbb{N} \).

In 1921, Landau wrote a letter to Hardy including a proof of the inequality
\[
\sum_{n=1}^{\infty} a_n^p \geq \left( \frac{p-1}{p} \right)^p \sum_{n=1}^{\infty} \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)^p,
\]
where \( \{a_n\}_{n=1}^{\infty} \) is any sequence of nonnegative real numbers and \( 1 < p < \infty \). This inequality was stated before by Hardy, and therefore, it is called a Hardy inequality (see, for example, [1] and also [3] for a marvelous description on the prehistory of the celebrated Hardy inequality). Since then Hardy-type inequalities have received an enormous amount of attention.

Let \( C_c(\mathbb{N}) \) be the space of finitely supported functions on \( \mathbb{N} := \{1, 2, 3, \ldots\} \). Inequality (1) is equivalent to the following inequality:
\[
\sum_{n=1}^{\infty} |\varphi(n) - \varphi(n-1)|^p \geq \left( \frac{p-1}{p} \right)^p \sum_{n=1}^{\infty} \frac{|\varphi(n)|^p}{n^p},
\]
for all \( \varphi \in C_c(\mathbb{N}) \), where we set \( \varphi(0) = 0 \) as a “Dirichlet boundary condition.”

The goal of this note is to prove the following improvement of the classical Hardy inequality (2) for the case \( p = 2 \).

**Theorem 1.** Let \( \varphi \in C_c(\mathbb{N}) \) and \( \varphi(0) = 0 \). Then
\[
\sum_{n=1}^{\infty} \left( \varphi(n) - \varphi(n-1) \right)^2 \geq \sum_{n=1}^{\infty} w(n)\varphi(n)^2,
\]
where \( w \) is a strictly positive function on \( \mathbb{N} \) given by
\[
w(n) = \sum_{k=1}^{\infty} \frac{(4k)}{2k} \frac{1}{(4k-1)2^{4k-1}} \frac{1}{n^{2k}} = \frac{1}{4n^2} + \frac{5}{64n^4} + \frac{21}{512n^6} + \cdots
\]
for \( n \geq 2 \) and \( w(1) = 2 - \sqrt{2} \). In particular, \( w(n) \) is strictly greater than the classical Hardy weight \( w_H(n) := 1/(2n)^2 \), \( n \in \mathbb{N} \).
Remark. In [2], we show that (3) cannot be improved and is optimal in the following three aspects:

- There is no function \( \tilde{w} \geq w \) (where \( w \) is given by (4)) such that (3) holds with \( \tilde{w} \) instead of \( w \) (“Criticality”).
- The ground state is not an eigenfunction (“Null-criticality”).
- For any \( \lambda > 0 \), the Hardy inequality outside of any compact set fails for the weight \( (1 + \lambda)w \) (“Optimality near infinity”).

A nonzero mapping \( u : \mathbb{N} \to \mathbb{R} \) taking nonnegative values is called positive and we write \( u \geq 0 \) in this case. If \( u(n) > 0 \) for all \( n \in \mathbb{N} \), we call \( u \) a weight and we set \( u(0) = 0 \). For weights \( u \) we write \( \ell^2(\mathbb{N}, u) \) for the space of all functions \( f : \mathbb{N} \to \mathbb{R} \) such that \( \sum_{n=1}^{\infty} f(n)^2 u(n) < \infty \). Clearly, \( \ell^2(\mathbb{N}, u) \) equipped with the scalar product

\[
\langle f, g \rangle_u := \sum_{n=0}^{\infty} f(n)g(n)u(n) \quad f, g \in \ell^2(\mathbb{N}, u)
\]

is a Hilbert space with the induced norm \( \| \cdot \|_u \). We clearly have:

**Lemma 1.** For any weight \( u : \mathbb{N} \to (0, \infty) \), the operator \( T_u : \ell^2(\mathbb{N}, u^2) \to \ell^2(\mathbb{N}) \), given by \( (T_u\varphi)(n) := u(n)\varphi(n) \) is unitary.

We consider \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \) as the standard graph on the nonnegative integers. Note that here 0 is added as a “boundary point” of \( \mathbb{N} \). We say that \( n, m \in \mathbb{N}_0 \) are connected by an edge (and we write \( n \sim m \)) if \( |n - m| = 1 \).

Given a weight \( u \), the combinatorial Laplacian \( \Delta_u \) associated with \( u \) is given by

\[
\Delta_u \varphi(n) := \frac{1}{u(n)^2} \sum_{m \sim n} u(n)u(m)(\varphi(n) - \varphi(m)) \quad n \geq 1,
\]

where \( \varphi \) is an arbitrary function on \( \mathbb{N} \) and we set \( \varphi(0) = 0 \) (as we set \( u(0) = 0 \) above). This can be understood as a “Dirichlet boundary condition” at \( n = 0 \). The operator \( \Delta := \Delta_1 \) is the standard discrete combinatorial Laplace operator corresponding to the constant weight \( u = 1 \).

**Lemma 2 (Ground state transform [2]).** Let \( u \) be a weight, and let \( w \) be a function on \( \mathbb{N} \). If \( (\Delta - w)u = 0 \) on \( \mathbb{N} \), then

\[
T_u^{-1}(\Delta - w)T_u = \Delta_u \quad \text{on } C_c(\mathbb{N}).
\]

**Proof.** We compute for \( n \geq 1 \),

\[
\Delta T_u \varphi(n) = \sum_{m \sim n} \left( u(n)\varphi(n) - u(m)\varphi(m) \right)
= (\Delta - w)u(n)\varphi(n) + u(n)w(n)\varphi(n) + \sum_{m \sim n} u(m)(\varphi(n) - \varphi(m))
= 0 + w(n)T_u \varphi(n) + (T_u \Delta_u \varphi)(n).
\]

\[\square\]
For a weight $u$ on $\mathbb{N}$, we define a quadratic form $h_u$ on the space $C_c(\mathbb{N})$, by

$$h_u(\varphi) := \frac{1}{2} \sum_{n \in \mathbb{N}} \sum_{m \sim n} u(n) u(m) (\varphi(n) - \varphi(m))^2 \quad \varphi \in C_c(\mathbb{N}),$$

again with the “Dirichlet boundary condition” $\varphi(0) = 0$ and $u(0) = 0$. For the constant weight $u = 1$, we write $h := h_1$. A direct calculation connects the form $h_u$ and $\Delta u$ through the inner product on $\ell^2(\mathbb{N}, u^2)$.

**Lemma 3 (Green formula).** For a weight $u$ on $\mathbb{N}$, we have

$$\langle \Delta \varphi, \varphi \rangle_{u^2} = h_u(\varphi) \geq 0 \quad \varphi \in C_c(\mathbb{N}).$$

**Proof.** The proof follows by a direct calculation. Indeed,

$$\langle \Delta \varphi, \varphi \rangle_{u^2} = \sum_{n \in \mathbb{N}} u^2(n) \Delta_u \varphi(n) \varphi(n)$$

$$= \sum_{n \in \mathbb{N}} \sum_{m \sim n} u(n) u(m) (\varphi(n) - \varphi(m))^2$$

$$+ \sum_{n \in \mathbb{N}} \sum_{m \sim n} u(n) u(m) (\varphi(n) - \varphi(m)) \varphi(m)$$

$$= 2h_u(\varphi) - \langle \Delta u \varphi, \varphi \rangle_{u^2}. \quad \blacksquare$$

**Proposition 4.** Let $u$ be a weight and suppose that $w \geq 0$ on $\mathbb{N}$. If $(\Delta - w)u = 0$ on $\mathbb{N}$, then $h(\varphi) \geq \|\varphi\|_w^2$ for all $\varphi \in C_c(\mathbb{N})$ with $\varphi(0) = 0$.

**Proof.** Using Lemmas 1, 2, and 3 we obtain for all $\varphi \in C_c(\mathbb{N}),$

$$\langle (\Delta - w)\varphi, \varphi \rangle = \langle T_u^{-1}(\Delta - w)T_u^{-1} \varphi, T_u^{-1} \varphi \rangle_{u^2}$$

$$= \langle \Delta_u \varphi, \varphi \rangle_{u^2}$$

$$= h_u(\varphi) \geq 0. \quad \blacksquare$$

We are now in a position to prove the main theorem of this note.

**Proof of Theorem 1.** We define the function $w : \mathbb{N} \to \mathbb{R}$ as

$$w(n) := \frac{\Delta_n^{1/2}}{n^{1/2}} \quad n \geq 1.$$

Applying the definition of the combinatorial Laplacian, we arrive at

$$w(n) = 2 - \left( \left(1 + \frac{1}{n}\right)^{1/2} + \left(1 - \frac{1}{n}\right)^{1/2} \right)$$

for $n \geq 1$. The Taylor expansion of the square root at 1 gives, for $n \geq 2$,

$$\left(1 \pm \frac{1}{n}\right)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} \left(\pm \frac{1}{n}\right)^k = 1 \pm \frac{1}{2n} - \frac{1}{8n^2} \pm \frac{1}{16n^3} - \frac{5}{128n^4} \pm \cdots. \quad (5)$$

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and thus, (5) leads to the claimed series representation for \( w(n) \), i.e.,

\[
 w(n) = - \sum_{k=1}^{\infty} \left( \frac{1}{2k} \right) \frac{1}{n^{2k}} = \sum_{k=1}^{\infty} \left( \frac{4k}{2k} \right) \frac{1}{(4k-1)^2} \frac{1}{n^{2k}} \quad n \geq 2. \tag{6}
\]

Note that \( w(1) = 2 - \sqrt{2} \). So, \( w > w_H > 0 \) on \( \mathbb{N} \). By construction, we have \((\Delta - w)u = 0\) on \( \mathbb{N} \) for the weight \( u(n) := n^{1/2} \), \( n \geq 1 \). The validity of the Hardy inequality now follows from Proposition 4. 

\[\blacksquare\]

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