TANNAKA DUALITY, COCLOSED CATEGORIES AND RECONSTRUCTION FOR NONARCHIMEDEAN BIALGEBRAS

ANTON LYUBININ

Abstract. The topic of this paper is a generalization of Tannaka duality to coclosed categories. As an application we prove reconstruction theorems for coalgebras (bialgebras, Hopf algebras) in categories of topological vector spaces over a nonarchimedean field $K$. In particular, our results imply reconstruction and recognition theorems for categories of locally analytic representations of compact $p$-adic groups, which was the major motivation for this work. Also, as an example, we discuss a certain (trivial) extension of the geometric Satake correspondence.

Introduction

The first results on reconstruction seem to be the Pontryagin duality theorems for abelian locally compact groups, followed by the T. Tannaka reconstruction theorem [Ta] and M. Krein recognition theorem [Kr] for general compact groups. In short, Tannaka-Krein duality states that one can reconstruct the group from its category of finite-dimensional representations. It was then extended to pro-algebraic groups and Hopf algebras in [D] [DM] [S], where the concepts of rigid objects and Tannakian categories were introduced. Around the same time reconstruction theorems for infinite-dimensional representations (in vector space over the field of complex numbers) of (locally) compact groups were obtained in several papers, among which we would like to mention [K1] [K2], followed by [K3], where theorems on reconstruction with infinite-dimensional representations were obtained with the use of a certain generalization of rigid categories (called reflexive monoidal categories), though the recognition problem remained unconsidered. Among the huge number of works, that followed, on reconstruction using rigid categories, we would like to mention [PAR2] (and [PAR1]), which seems to provide the most general context for reconstruction in monoidal categories (we do not consider extensions to higher algebraic structures) and on which we build heavily. This list is by no means exhaustive and is very subjective.

In reconstruction theory, one can reconstruct the group $G$ from the forgetful functor $F : \text{Rep}(G) \to \text{Vect}_{K}^{fd}$ to the category of finite-dimensional vector spaces as the group of natural monoidal automorphisms of $F$ (in the case of algebraic groups, this is a simple consequence of the Yoneda lemma). For the reconstruction of a Hopf algebra (of functions on $G$) one reconstruct it as a predual object, the coendomorphism object $\text{coend}(F)$. Since the coalgebra theory is simpler than the theory of algebras, the second approach turn out to be easier and is followed by most authors for the reconstruction of coalgebras, bialgebras and Hopf algebras. In a general rigid monoidal category $\mathcal{C}_0$ one can construct $\text{coend}(F)$ for a functor $F : \mathcal{B} \to \mathcal{C}_0$ as a colimit of a diagram of objects of the form $F(B)^{\vee} \otimes_{\mathcal{C}} F(B)$, which requires us to assume the existence of a cocomplete category $\mathcal{C} \supset \mathcal{C}_0$ and to assume
either $\mathcal{B}$ or $\mathcal{C}_0$ to be small for the above mentioned colimit to exist. These conditions require us to consider fiber functors into categories of objects that satisfy some finiteness conditions (finite-dimensional vector spaces, finitely generated projective modules, etc.). One can also define coend $\text{coend}(F)$ as a representing object for the functor $\text{Nat}(F, F \otimes \mathcal{C}^{-}) : \mathcal{C} \to \text{Sets}$ of monoidal natural transformations, assuming that $\mathcal{C}$ is locally small (otherwise $\text{Nat}(F, F \otimes \mathcal{C}^{-})$ is not a functor to $\text{Sets}$). While this definition works in general, to prove the existence of coend $\text{coend}(F)$ one still need a direct construction of it and thus is bound to the case of functors $F : \mathcal{B} \to \mathcal{C}_0$ into a rigid category.

The starting point (and the essence) for this paper is the observation that for the computation of the coend $\text{coend}(F)$ of the fiber functor $F$ one does not need the category $\mathcal{C}_0$ to be rigid, but rather it only has to be coclosed, i.e. the tensor product functor $Y \otimes \mathcal{C}^{-}$ has to have a left adjoint functor $\text{cohom}_{\mathcal{C}}(\mathcal{C}^{-}, Y)$. According to the theorem of Eilenberg and Watts, any such functor for categories of modules $\text{Mod}_R$ over a ring $R$ is itself a tensor product functor $Y' \otimes \mathcal{C}^{-}$ and one can take $Y'$ as a dual of $Y$, recovering the rigid structure from the adjunction. Thus in the case of categories of modules the coclosedness requirement is equivalent to requiring rigidity. There is a conjecture that Eilenberg-Watts theorem holds in any abelian monoidal category, so it is no wonder that the fact that left adjoints can be used in Tannaka reconstruction was either not noticed or not given importance by researchers. Although the author is not aware of the status of this conjecture for abelian categories, in nonabelian case it is false (and this paper contains a counter-example). Thus the use of cohomoms allows us to compute coend $\text{coend}(F)$ in categories that are not rigid, i.e. without usual finiteness assumptions on objects of the category $\mathcal{C}_0$, and thus one can do reconstruction in a more general setting. Another feature of this work is that we never require preservation of arbitrary colimits by tensor products, which is common for all previous works. While in algebra this requirement is usually not restrictive (since algebraic tensor product functor is a left adjoint), topological tensor products seldom enjoy this property and thus in the topological setting one needs reconstruction theorems that do not require preservation of arbitrary colimits.

Let us briefly outline the content of the paper.

In section 1 we review the definition and some basic properties of cohomomorphisms. We define coend $\text{coend}(F)$ in coclosed categories and work out its basic properties by giving direct proofs, without using $\text{Nat}(F, F \otimes \mathcal{C}^{-})$. We think it is useful for understanding how these properties work, and also it allows to apply these constructions to the case when the category $\mathcal{C}$ is not locally small (though in our applications it is locally small). We use the framework of $\mathcal{C}$-categories, introduced by B. Pareigis for reconstruction problems, which we briefly review in preliminaries, along with some results of [PAR2]. We also introduce the concept of $\mathcal{C}$-cowedge, corresponding to $\mathcal{C}$-natural transformations ([PAR2]) and give direct construction of $\text{coend}_{\mathcal{C}}(F)$. This allows us to state reconstruction and recognition theorems as a consequence of the results of [PAR2], though we slightly relax the assumptions, to adapt them for our applications. We also give the conditions (in section 1.9) when $F : \mathcal{B} \to \mathcal{C}_0 \subset \mathcal{C}$ gives an equivalence of categories $\mathcal{B}$ and $\text{Comod}_{\mathcal{C}_0} \subset \text{coend}_{\mathcal{C}_0}(F)$, which is stated in [SCH, Chapter 2] for general $\mathcal{C}$-categories as an open question and proved there only for vector spaces.
In section 2 we apply our results to state reconstruction and recognition theorems in the categories of topological vector spaces over nonarchimedean field $K$, which was our original motivation to study this question. The category $LS$ of $LS$-spaces, that we consider, is the category of spaces of locally analytic representation theory, and thus our results can be viewed as (a part of) Tannaka-Krein duality for compact $p$-adic groups. We also prove reconstruction theorems for a bialgebra of compact type from its category of Banach comodules.

Most of our results are formulated in their simplest versions, but the construction is very flexible and can be adapted to different settings, in which one might need to do reconstruction. We show this in section 3, where we adapt out results for reconstruction in the category of Banach spaces $Ban_K$. The usual problem with $Ban_K$ is that it is not cocomplete (and not complete), thus one cannot take a colimit to construct coend $(F)$. We overcome this problem by putting a restriction on cowedges and natural transformations that we call boundedness (one will see that this name is natural) and show that the results of section 1 survive this change. As an example, we prove an extension of the geometric Satake correspondence to the Hopf algebra of rigid analytic functions. This result, although being trivial, may serve as an indication that there exists a more interesting version of the geometric Satake correspondence for $p$-adic groups, which is the subject of [L2].

Acknowledgments. Although the question, answered in this paper, was raised quite a while ago, the active phase of this work was completed during my stay in the University of Science and Technology of China as a post-doc. I would like to thank the institution, the department, the Wu Wenjun CAS Key Laboratory of Mathematics and its director professor Sen Hu, and especially professor Yun Gao for hospitality, stimulation, support and excellent research conditions. Needless to say that all possible errors and inaccuracies in this paper are solely my fault. I also thank Peter Schauenburg for explaining me one vague point in [SCH].

I first started thinking about possible version of Tannaka duality in nonarchimedean setting when I was taking a course of Alex Rosenberg on reconstruction theorems. Unfortunately, the key points of the present work became clear to me only recently, when it was already too late to discuss it with him. This paper is devoted to his memory.

Preliminaries

Throughout this paper $(C, \otimes_C)$ is a monoidal category.

0.1. $C$-categories. We briefly review some notions and results from [PAR2]. For any unknown term with no reference one should look there for explanations.

Definition 0.1. Let $(C, \otimes_C)$ be a monoidal category.

- A category $B$ together with a bifunctor $\otimes_{CB} : C \times B \to B$ and coherent natural isomorphisms $\beta : (X \otimes_C Y) \otimes_{CB} P \to X \otimes_C (Y \otimes_{CB} P)$ (for $X, Y \in C, P \in B$) and $\pi : I \otimes_{CB} P \to P$ will be called a (left) $C$-category. $C$ is called control category.

- Let $B$ and $B'$ be $C$-categories. A functor $F : B \to B'$ together with a coherent natural isomorphism $\xi : F(X \otimes_{CB} P) \to X \otimes_{CB'} F(P)$ is called a $C$-functor. Note that if $B'$ is also a monoidal category and $M \in B'$ then $F \otimes_{B'} M$ is also a $C$-functor.
Let $\mathcal{C} = B \to B'$ and $\mathcal{F} : B \to B'$ be $\mathcal{C}$-functors. A $\mathcal{C}$-natural transformation (or a $\mathcal{C}$-morphism) $\phi : \mathcal{F} \to \mathcal{F}'$ is a natural transformation, compatible with the natural isomorphisms $\xi$ and $\xi'$. The collection of $\mathcal{C}$-natural transformations from $\mathcal{F}$ to $\mathcal{F}'$ will be denoted by $\text{Nat}_\mathcal{C}(\mathcal{F}, \mathcal{F}')$. The set of all natural transformations $\text{Nat}(\mathcal{F}, \mathcal{F}')$ corresponds to the one-element control category.

**Definition 0.4.** Let $\mathcal{A}$ be a $\mathcal{C}$-monoidal category.

- A $\mathcal{C}$-monoidal category $\mathcal{A}$ is called $\mathcal{C}$-braided if the braidings in $\mathcal{A}$ and $\mathcal{C}$ are coherent (but braiding on $\mathcal{A}$ is not a $\mathcal{C}$-morphism).
- An object $P$ in a $\mathcal{C}$-monoidal category $\mathcal{A}$ is called $\mathcal{C}$-central if $P \otimes_A X = P$ for all $X \in \mathcal{A}$. The set of all $\mathcal{C}$-central objects in $\mathcal{A}$ is denoted by $\mathcal{A}_c$.

**Proposition 0.3.** [PAR2 2.10] Let $\mathcal{A}$ be $\mathcal{C}$-braided $\mathcal{C}$-monoidal category. Let $B$ be a $\mathcal{C}$-central bialgebra in $\mathcal{A}$, $C$ be a coalgebra in $\mathcal{A}$ and $z : C \to B$ be a coalgebra morphism. Then

1. $\text{Comod}_\mathcal{A} - C$ is a $\mathcal{C}$-category
2. $\text{Comod}_\mathcal{A} - B$ is a $\mathcal{C}$-monoidal category
3. $\mathcal{F}^* : \text{Comod}_\mathcal{A} - C \to \text{Comod}_\mathcal{A} - B$ is a $\mathcal{C}$-functor
4. $\mathcal{F}^*$ is the forgetful functor $\mathcal{F} : \text{Comod}_\mathcal{A} - C \to \mathcal{A}$ is a $\mathcal{C}$-functor
5. If $C$ is a $\mathcal{C}$-central bialgebra and $z : C \to B$ is a bialgebra morphism then $\mathcal{F}^*$ is a $\mathcal{C}$-monoidal functor
6. The forgetful functor $\mathcal{F} : \text{Comod}_\mathcal{A} - B \to \mathcal{A}$ is a $\mathcal{C}$-monoidal functor

Recall the (slightly changed) notion of $\mathcal{C}_0$-generated coalgebra [PAR2 2.6].

**Definition 0.4.** Let $\mathcal{C}$ be a monoidal category, $\mathcal{C}_0 \subset \mathcal{C}$ be a full monoidal subcategory and $I$ be a poset. The coalgebra $C \in \mathcal{C}$ is called (right) $\mathcal{C}_0 - I$-generated if the following holds:

1. $C$ is a colimit in $\mathcal{C}$ of an $I$-diagram of objects $C_i \in \mathcal{C}_0$;
2. all morphisms $\text{id}_X \otimes_C j_i \otimes_M : X \otimes_C C_i \otimes_M \to X \otimes_C C \otimes_M$ are monomorphisms in $\mathcal{C}$, where $X \in \mathcal{C}_0$, $M \in \mathcal{C}$ and $j_i : C_i \to C$ are the monomorphisms from the colimit diagram;
3. every $C_i$ is a subcoalgebra of $C$ via $j_i : C_i \to C$;
4. If $(P, \rho_P)$ is a (right) comodule over $C$ and $P \in \mathcal{C}_0$ then $\exists i$ and $\rho_{P, i} : P \to P \otimes_C C_i$ such that $\rho_P = (\text{id}_P \otimes_C j_i) \circ \rho_{P, i}$. 

If we don’t want to specify $I$ or it’s choice is clear we say that $C$ is $C_0$-generated.

**Remark 0.5.** For a $C_0-I$-generated coalgebra $C \in \mathcal{C}$ and any subcoalgebra $C' \hookrightarrow C$ such that $C' \in C_0$ the identity $\text{id}_C = (\epsilon_C \otimes \text{id}_C) \circ \Delta_C$ and 0.4.4 imply that we have a monomorphism $C' \hookrightarrow C_i$ for some $i \in I$.

**Example 0.6.** Fundamental theorem of coalgebras: for any field $K$ every coassociative coalgebra in the category $\text{Vect}_K$ of all vector spaces is $\text{Vect}^{fd}_K$-generated, where $\text{Vect}^{fd}_K$ is the category of all finite-dimensional vector spaces.

**Example 0.7.** Let $K$ be a locally compact nonarchimedean field and $G$ be a compact locally $K$-analytic group. Then the coalgebra $C_{la}(G, K)$ of locally analytic functions on $G$ is $\text{Ban}_K - \mathbb{N}$-generated in the category $\text{LCTVS}$ of locally convex topological vector spaces or in the category $\text{LB}$ of LB-spaces.

0.2. **Coendomorphism of bifunctors** $\mathcal{C} \times \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$. Let $\mathcal{F} : \mathcal{C} \times \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ be a bifunctor.

**Definition 0.8.** A dinatural transformation $\mu : \mathcal{F} \Rightarrow d$ from $\mathcal{F}$ to a constant bifunctor with value $d \in \text{Ob}(\mathcal{D})$ is a family of morphisms $\mu_c : \mathcal{F}(c, c) \rightarrow d$ such that for any morphism $f : c \rightarrow c'$ the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{F}(c, c') & \xrightarrow{\mathcal{F}(\text{id}_c, f^{\text{op}})} & \mathcal{F}(c, c) \\
\downarrow & & \downarrow_{\mu_c} \\
\mathcal{F}(f, \text{id}_{c'}) & & \mathcal{F}(c', c') \xrightarrow{\mu_{c'}} d
\end{array}
$$

Similarly one defines a dinatural transformation from a constant to $\mathcal{F}$.

We will also call dinatural transformation from a constant to $\mathcal{F}$ a wedge over $\mathcal{F}$. A dinatural transformation from $\mathcal{F}$ to a constant will be a cowedge over $\mathcal{F}$. The definition of a morphism of (co)wedges is clear. (Co)wedges over $\mathcal{F}$ form a category.

**Definition 0.9.** $\text{end}(\mathcal{F})$ is a terminal object in the category of wedges over $\mathcal{F}$. $\text{coend}(\mathcal{F})$ is an initial object in the category of cowedges over $\mathcal{F}$.

0.3. **Subdivision category, existence of coend.** [MAC IX.5]. To a category $\mathcal{C}$ we associate its subdivision category $\mathcal{C}^\$ in the following way: we set $\text{Ob}(\mathcal{C}^\$) = $\text{Ob}(\mathcal{C}) \cup \text{Mor}(\mathcal{C})$ (note that $c \in \mathcal{C}$ and $\text{id}_c$ give different objects $c^\$ and $\text{id}_c^\$ in $\mathcal{C}^\$). The morphisms are the identity morphisms for the above mentioned objects and also if $f : c \rightarrow d$ is the morphism in $\mathcal{C}$ then it gives rise to the two morphisms $c^\$ $\rightarrow f^\$ and $f^\$ $\rightarrow d^\$ in $\mathcal{C}^\$.

For each bifunctor $\mathcal{F} : \mathcal{C} \times \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ we define a functor $\mathcal{F}^\$ : $\mathcal{C}^\$ $\rightarrow$ $\mathcal{D}$ via the diagram

$$
\begin{array}{ccc}
c^\$ & \xrightarrow{\mathcal{F}(f, \text{id}_c)} & f^\$ \\
\downarrow & & \downarrow \\
\mathcal{F}(c, c) & \xrightarrow{\mathcal{F}(\text{id}_c, f^{\text{op}})} & \mathcal{F}(d, c) \\
\downarrow & & \downarrow \\
\mathcal{F}(d, c) & \xrightarrow{\mathcal{F}(\text{id}_d, f^{\text{op}})} & \mathcal{F}(d, d)
\end{array}
$$
A (co)cone over $F$ is exactly a (co)wedge over $F$ and vice versa. Thus a (co)limit of $F$ exists if and only if a (co)end of $F$ exists ([MAC] IX.5.1) and

$$\lim F = \text{end } F, \quad \varinjlim F = \text{coend } F.$$  

Thus if $C$ is small and $D$ is complete (cocomplete) then end $F$ (coend $(F)$) exists ([MAC] IX.5.2).

If $C$ or $D$ is small, one can also compute coend $(F)$ as the coequalizer

$$\bigcup_{f \in \text{Mor}(C)} F(\text{dom}(f), \text{codom}(f)) \xrightarrow{p} \bigcup_{q \in \text{Ob}(C)} F(c, c) \rightarrow \text{coend } (F),$$

where the equalized pair is defined by equations

$$p \circ i_f = i_{\text{dom}(f)} \circ F(id_{\text{dom}(f)}, f^{op}),$$

$$q \circ i_f = i_{\text{codom}(f)} \circ F(f, id_{\text{codom}(f)}).$$

0.4. **Functor** $\text{Nat}_C(F, F \otimes_A -)$. Let $A, B$ be $C$-categories and $F : B \rightarrow A$ be a $C$-functor. If $B$ is small and $A$ is locally small and $C$-monoidal, then the $C$-natural transformations form a functor

$$\text{Nat}_C(F, F \otimes_A -) : A \rightarrow \text{Sets}.$$

Suppose $\text{Nat}_C(F, F \otimes_A -)$ is representable. By analogy with the case of all natural transformations, we denote the representing object by $\text{coend}_C(F)$ with the universal $C$-morphism $\delta : F \rightarrow F \otimes_A \text{coend}_C(F)$.

$F^2 := F \otimes_A F : B \times B \rightarrow A$ is $C$-bifunctor, and so is $F^2 \otimes_A M$ for all $M \in A$. The $C$-natural transformations $\text{Nat}_C(F^2, F^2 \otimes_A M)$ form a set and thus define a functor

$$\text{Nat}_C(F^2, F^2 \otimes_A -) : A \rightarrow \text{Set}.$$

If $\text{coend}_C(F)$ is $C$-central (always true if $C = A$ and $A$ is symmetric) then

$$\delta_2 = (id_F \otimes_A \tau \otimes_A id_{\text{coend}_C(\delta)}) \circ (\delta \otimes_A \delta) : F^2 \rightarrow F^2 \otimes_A \text{coend}_C(F)^2$$

is a $C$-bimorphism. Similar statement holds for $\text{Nat}_C(F^n, F^n \otimes_A -)$ and $\delta_n$.

**Definition 0.10.** ([PAR2] 3.5) We will call $\text{Nat}_C(F, F \otimes_A -)$ $n$-representable if $\text{coend}_C(F)$ is $C$-central and $\text{Nat}_C(F^n, F^n \otimes_A -)$ is representable with $\text{coend}_C(F^n)$ and $\delta_n$, and multirepresentable if it is $n$-representable for all $n$.

In the following propositions we summarize some results from [PAR2].

**Proposition 0.11.** The functor $\text{Nat}_C(F, F \otimes_A -)$ has the following properties ([PAR2] 3.3, 3.6]):

- $\text{Nat}_C(F, F \otimes_A M) \cong A(\text{coend}_C(F), M)$;
- (equivalent to representability) for every $C$-morphism $\phi : F \rightarrow F \otimes_A M$ there exists a morphism $\psi : \text{coend}_C(F) \rightarrow M$ such that the diagram

$$\begin{array}{ccc}
F(X) & \xrightarrow{\delta_X} & F(X) \otimes_A \text{coend}_C(F) \\
\phi_X \downarrow & & \downarrow \text{id}_{F(X) \otimes_A \psi} \\
F(X) \otimes_A M & \overset{\psi}{\longrightarrow} & F(X) \otimes_A M
\end{array}$$

commutes for every $X \in B$;
coend\(_C(F)\) is a coalgebra, unique up to isomorphism of coalgebras;*

\[\forall P \in B: F(P)\text{ is a coend}\(_C(F)\)-comodule, every } F(f)\text{ is a comodule morphism;}\]

\[\forall P \in B, \forall X \in C: F(X \otimes_{CB} P)\text{ is isomorphic to } X \otimes_{CA} F(P)\text{ as coend}\(_C(F)\)-comodule;\]

Now let \(A\) be \(C\)-braided \(C\)-monoidal, \(B\) be \(C\)-monoidal and \(F : B \to A\) be \(C\)-monoidal functor.

- if \(\text{Nat}_C(F, F \otimes_A -)\) is multirepresentable then coend\(_C(F)\) is a bialgebra in \(A\). It is unique up to isomorphism of bialgebras;
- if \(F\) factors through the full subcategory \(A_0\) of rigid objects of \(A\), then coend\(_C(F)\) is Hopf algebra in \(A\);
- if \(C \subseteq C'\) then there is a coalgebra epimorphism coend\(_C(F)\) \(\twoheadrightarrow\) coend\(_{C'}(F)\) \([\text{PAR2}\ 5.2]\). In particular, there exists an epimorphism coend\(_C(F)\) \(\twoheadrightarrow\) coend\(_C(F)\).

**Proposition 0.12.** (Reconstruction theorem) Let \(C\) be a braided monoidal category and \(C_0\) be a full braided monoidal subcategory.

- Let \(C\) be a \(C_0\)-generated coalgebra in \(C\) and \(\text{Nat}_{C_0}(F, F \otimes_C -)\) is representable by \(C\) \([\text{PAR2}\ 4.3]\);
- Let \(C\) also be cocomplete and let \(\otimes\) preserve colimits in both variables. Let \(C\) also be a \(C_0\)-central bialgebra. Then \(\text{Nat}_{C_0}(F, F \otimes_C -)\) is multirepresentable by \(C\) \([\text{PAR2}\ 4.5]\).

**Proposition 0.13.** (Recognition theorem \([\text{PAR2}\ 4.7]\)) Let \(C\) be a cocomplete braided monoidal category and \(C_0\) be a (locally small) full braided monoidal subcategory of rigid objects. Assume that \(\otimes\) preserve colimits in both variables. Let \(B\) be a small category and \(F : B \to C_0 \subset C\) be a functor.

- the functor \(\text{Nat}(F, F \otimes_C -)\) is multirepresentable;
- if \(B\) be a \(C_0\)-category and \(F : B \to C_0\) be a \(C_0\)-functor, then \(\text{Nat}_{C_0}(F, F \otimes_C -)\) is representable;
- if coend\(_{C_0}(F)\) is \(C_0\)-central then \(\text{Nat}_{C_0}(F, F \otimes_C -)\) is multirepresentable.

1. **Coclosed categories.**

**Definition 1.1.** Let \(C\) be a monoidal category and \(X, Y \in C\). And object cohom\(_C(X, Y)\) (or simply cohom\((X, Y)\)) (with the morphism \(\text{coev}_{X, Y} : X \to Y \otimes_C \text{cohom}_C(X, Y)\)) is called (right) cohomoemorphism object for \(X\) and \(Y\) if for every \(Z \in C\) and every morphism \(\phi : X \to Y \otimes_C Z\) there is a unique morphism \(\text{coact}_{X, Y, Z}(\phi) : \text{cohom}_C(X, Y) \to Z\) satisfying the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\text{coev}_{X, Y}} & Y \otimes_C \text{cohom}_C(X, Y) \\
\downarrow^{\phi} & & \downarrow^{\text{id}_Y \otimes_C \text{coact}_{X, Y, Z}(\phi)} \\
Y \otimes_C Z
\end{array}
\]
If it exists, $\text{cohom}_{\mathcal{C}}(X,Y)$ is unique up to an isomorphism.

Instead of $\text{coact}_{X,Y,Z}(\phi)$ we might write just $\text{coact}(\phi)$ or $\bar{\phi}$.

**Definition 1.2.** We say that the category $\mathcal{C}$ is *right coclosed* if for all $X, Y \in \mathcal{C}$ there exists a right cohomomorphism object. We also say that a subcategory $\mathcal{C}_0 \subset \mathcal{C}$ is *right coclosed in $\mathcal{C}$* if $\text{cohom}_{\mathcal{C}}(X,Y) \in \mathcal{C}_0$ exists for any $X, Y \in \mathcal{C}_0$.

Similarly one can define left cohomomorphism objects and left coclosed categories. In this paper we will always consider right coclosed categories, unless we explicitly mention the opposite.

In a (right) coclosed category $\mathcal{C}$ a map $\phi : X \to Z \otimes_{\mathcal{C}} C$ induce the map

$$
\Delta \equiv \text{coact} \left( (\text{coev}_{Z,Y} \otimes_{\mathcal{C}} \text{id}_{\text{cohom}_{\mathcal{C}}(X,Z)}) \circ \phi \right) :
\text{cohom}_{\mathcal{C}}(X,Y) \to \text{cohom}_{\mathcal{C}}(Z,Y) \otimes_{\mathcal{C}} C
$$

via diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\phi} & Z \otimes_{\mathcal{C}} C \\
\text{coev}_{X,Y} \downarrow & & \downarrow \text{coev}_{Z,Y} \otimes_{\mathcal{C}} \text{id}_{\text{cohom}_{\mathcal{C}}(X,Z)} \\
Y \otimes_{\mathcal{C}} \text{cohom}_{\mathcal{C}}(X,Y) & \xrightarrow{\text{id}_Y \otimes_{\mathcal{C}} \Delta} & Y \otimes_{\mathcal{C}} \text{cohom}_{\mathcal{C}}(Z,Y) \otimes_{\mathcal{C}} C
\end{array}
$$

If we take $\phi := \text{coev}_{X,Z} : X \to Z \otimes_{\mathcal{C}} \text{cohom}_{\mathcal{C}}(X,Z)$ it gives us cocomposition map $\Delta_{X,Y,Z} : \text{cohom}_{\mathcal{C}}(X,Y) \to \text{cohom}_{\mathcal{C}}(Z,Y) \otimes_{\mathcal{C}} \text{cohom}_{\mathcal{C}}(X,Z)$.

One can see that $\text{cohom}_{\mathcal{C}}(-,-)$ defines a bifunctor $\text{Cohom} : \mathcal{C} \times \mathcal{C}^\text{op} \to \mathcal{C}$. The functor $\text{Cohom}(-,-)$ is the left adjoint to the functor $Y \otimes_{\mathcal{C}} -$, i.e. we have an isomorphism

$$\mathcal{C}(\text{cohom}_{\mathcal{C}}(X,Y), Z) \cong \mathcal{C}(X,Y \otimes_{\mathcal{C}} Z).$$

**Lemma 1.3.** For any object $T \in \mathcal{C}$ we have the following identities

1. $\mathcal{C}(\text{cohom}_{\mathcal{C}}(X,Y), Z), T) = \mathcal{C}(\text{cohom}_{\mathcal{C}}(X,Y), Z \otimes_{\mathcal{C}} T) = \mathcal{C}(X, Y \otimes_{\mathcal{C}} Z \otimes_{\mathcal{C}} T) = \mathcal{C}(\text{cohom}_{\mathcal{C}}(X,Y), Z);$
2. $\mathcal{C}(\text{cohom}_{\mathcal{C}}(X,Y), I) \cong \mathcal{C}(X, Y \otimes_{\mathcal{C}} I) \cong \mathcal{C}(X,Y);$
3. $\text{cohom}_{\mathcal{C}}(X,I) \cong X$ with $\text{coev}_{X,I} = \text{id}_X$

**Example 1.4.** Let $\mathcal{C} = \text{vec}_K$ be the category of finite-dimensional vector spaces over the field $K$ with the tensor structure given by the tensor product of vector spaces. For $X, Y \in \text{vec}$ let us compute $\text{cohom}_{\text{vec}_K}(X,Y)$ (without using hom-tensor adjunction).

Let $\{x_i\}_{i=1,...,n}$ and $\{y_j\}_{j=1,...,m}$ be the bases of $X$ and $Y$ correspondingly. Let $X^*$ is the linear dual space of $Y$ and $\{y_j\}_{j=1,...,m}$ be its basis, dual to $\{x_i\}_{i=1,...,n}$. For any $K$-linear map $\phi : X \to Y \otimes Z_\phi$ with $Z_\phi \in \text{vec}_K$ denote by $z_{ij}^\phi = (y_j \otimes \text{id}_{Z_\phi}) \circ \phi(x_i)$. $\phi$ is defined via its values on $x_i$, which can be written as $\phi(x_i) = \sum y_jz_{ij}^\phi$. Without loss of generality we can say that $\{z_{ij}^\phi\}$ span the whole $Z_\phi$. The largest such $Z_\phi$ one can get is $Z_\Phi := K^{nm}$, i.e. when all $z_{ij}^\phi$ are linearly independent and all other cases are quotients $\pi_\phi : Z_\Phi \to Z_\phi$ by the corresponding relations $R_\phi := \ker(\pi_\phi)$ between $z_{ij}^\phi$. For any two maps $\phi_n : X \to Y \otimes Z_{\phi_n}$, $n = 1, 2$, the transition map $\pi_{1,2} : Z_{\phi_1} \to Z_{\phi_2}$ such that $\phi_2 = (\text{id}_Y \otimes \pi_{1,2}) \circ \phi_1$, if exists, is unique, since it defined
by the values \( z_{ij}^k \). Thus we have a diagram \( \{ Z_{ij} \} \) with limit \( Z_\phi \). Since this diagram is finite, it is preserved by the tensor product and \( Y \otimes Z_{ij} \) exists. Thus we get the map \( \Phi : X \to Y \otimes Z_{ij} \).

This means that \( \text{cohom}_{\text{vec}}(X,Y) \cong K^{nm} \) and \( \text{coev}_{X,Y} = \Phi \). Since \( \text{cohom} \) is covariant in \( X \) and contravariant in \( Y \), we write it as \( \text{cohom}_{\text{vec}}(X,Y) \cong Y^* \otimes X \).

**Remark 1.5.** The above argument shows that in \( C = \text{Vec}_K \) for finite-dimensional \( X \) and \( Y \) we have \( \text{cohom}_{\text{vec}}(X,Y) \cong Y^* \otimes X \). It also explains why, if either \( X \) or \( Y \) has infinite dimension, \( \text{cohom}_{\text{vec}}(X,Y) \) does not exist, since in this case the diagram \( \{ Z_{ij} \} \) is (filtered) infinite and its limit is not preserved by tensoring with \( Y \).

### 1.2. Coendomorphisms.

**Definition 1.6.** We define the coendomorphism object of \( X \in C \) as \( \text{coend}_C(X) := \text{cohom}_C(X,X) \). We will also write \( \text{coev}_X \) for \( \text{coev}_{X,X} \).

**Lemma 1.7.** Let \( X \in C \) and suppose \( \text{coend}_C(X) \) exists. Then

1. \( \text{coend}_C(X) \) is a coalgebra in \( C \) with the comultiplication
   \[
   \Delta_{\text{coend}_C(X)} := \text{coact} \left( (\text{coev}_X \otimes_C \text{id}_{\text{coend}_C(X)}) \circ \text{coev}_X \right)
   \]
   and the counit \( \epsilon_{\text{coend}_C(X)} := \text{coact}(\text{id}_X) \);

2. \( X \) is a right comodule over \( \text{coend}_C(X) \) via \( \rho_X := \text{coev}_X \).

**Proof.** The check of coalgebra axioms for \( (\text{coend}_C(X), \Delta_{\text{coend}_C(X)}, \epsilon_{\text{coend}_C(X)}) \) is straightforward.

The check of coassociativity for \( \rho_X \) is also straightforward. From the relation \( \text{id}_X = (\text{id}_X \otimes_C \epsilon_{\text{coend}_C(X)}) \circ \text{coev}_X \) we get that \( X \) is a comodule over \( \text{coend}_C(X) \). \( \square \)

**Remark 1.8.** Let \( \phi : X \to X \otimes_C C \) be a morphism in \( C \). Then

- one has a morphism
  \[
  \rho_{\phi} := \text{coact} \left( (\text{coev}_X \otimes_C \text{id}_C) \circ \phi \right) : \text{coend}_C(X) \to \text{coend}_C(X) \otimes_C C;
  \]
- if \( C \) is a coalgebra and \( \phi \) is a comodule coaction, then
  - \( \rho_{\phi} \) is a comodule coaction;
  - \( \text{coact}(\phi) : \text{coend}_C(X) \to C \) is a coalgebra morphism;
- one can reconstruct the coalgebra morphism \( \text{coact}(\rho_{\phi}) : \text{coend}_C(X) \to C \) as
  \[
  \text{coact}(\rho_{\phi}) = (\epsilon_{\text{coend}_C(X)} \otimes_C \text{id}_C) \circ \rho_{\phi}.
  \]

### 1.3. Rigid tensor categories.

In the treatment of rigid monoidal categories we will follow [PAR1].

**Definition 1.9.** Let \( X \in C \). \( (X^*, \text{ev} : X^* \otimes_C X \to I) \) is a (left) dual of \( X \) if there exists \( db : I \to X \otimes_C X^* \), s.t.

\[
(\text{id}_X \otimes_C \text{ev}) \circ (db \otimes_C \text{id}_X^*) = \text{id}_X
\]

and

\[
(\text{ev} \otimes_C \text{id}_X) \circ (\text{id}_X \otimes_C db) = \text{id}_X.
\]

\( C \) is (left) rigid if every object has a (left) dual.

Dual objects have the following properties:
• \((X^*, ev)\) is a left dual for \(X\) iff \(- \otimes \mathcal{C} X : \mathcal{C} \to \mathcal{C}\) is a left adjoint to \(- \otimes \mathcal{C} X^*\), i.e.

\[ \mathcal{C} (- \otimes \mathcal{C} X, -) \cong \mathcal{C} (-, - \otimes \mathcal{C} X^*) \]

iff

\[ X^* \otimes \mathcal{C} - : \mathcal{C} \to \mathcal{C}\]

is a left adjoint to \(X \otimes \mathcal{C} -\) [\text{PAR} 3.3.3-5];

• For all \(X, X^*\) and \(d\) are unique;

• \(X \otimes \mathcal{C} X^*\) is an algebra [\text{PAR} 3.3.14];

• \(X^* \otimes \mathcal{C} X\) is a coalgebra [\text{PAR} 3.3.14];

• For all \(X, Y \in \mathcal{C}\) \(Y^* \otimes \mathcal{C} X \cong \text{cohom}_{\mathcal{C}}(X, Y)\) [\text{PAR} 3.3.5], i.e. left rigid categories are right coclosed;

• duality operation forms a functor \((-)^* : \mathcal{C} \to \mathcal{C}^{op}\) (follows from [\text{PAR} 3.3.8]).

From the above properties it is clear that in a braided rigid category \(\text{Cohom}\) is a tensor bifunctor, i.e.

\[
\text{cohom}_{\mathcal{C}}(X \otimes \mathcal{C} U, Y \otimes \mathcal{C} V) \cong (Y \otimes \mathcal{C} V)^* \otimes \mathcal{C} (X \otimes \mathcal{C} U) \cong V^* \otimes \mathcal{C} Y^* \otimes \mathcal{C} X \otimes \mathcal{C} U \cong \\
\cong Y^* \otimes \mathcal{C} X \otimes \mathcal{C} V \otimes \mathcal{C} U \cong \text{cohom}_{\mathcal{C}}(X, Y) \otimes \mathcal{C} \text{cohom}_{\mathcal{C}}(U, V).
\]

**Lemma 1.10.** Let \(\mathcal{C}\) be a braided right coclosed monoidal category such that \(\text{Cohom}\) is a tensor bifunctor. Then \(\mathcal{C}\) is a left rigid category.

**Proof.** We have the following isomorphism

\[
\text{cohom}_{\mathcal{C}}(X, Y) \cong \text{cohom}_{\mathcal{C}}(X \otimes \mathcal{C} I, I \otimes \mathcal{C} Y) \cong \\
\text{cohom}_{\mathcal{C}}(X, I) \otimes \mathcal{C} \text{cohom}_{\mathcal{C}}(I, Y) \cong \text{cohom}_{\mathcal{C}}(I, Y) \otimes \mathcal{C} X.
\]

By the definition of \(\text{cohom}_{\mathcal{C}}(X, Y)\), the functor \(\text{cohom}_{\mathcal{C}}(I, Y) \otimes \mathcal{C} -\) is the left adjoint to the functor \(Y \otimes \mathcal{C} -\). By [\text{PAR} 3.3.5] \(\text{cohom}_{\mathcal{C}}(I, Y)\) is the left dual of \(Y\). \(\square\)

### 1.4. Coendomorphism coalgebra of a functor.

In this subsection let \(\mathcal{C}\) be a monoidal category, \(\mathcal{C}_0 \subset \mathcal{C}\) be a subcategory, coclosed in \(\mathcal{C}\), and \(\mathbb{F} : \mathcal{D} \to \mathcal{C}_0 \subset \mathcal{C}\) be a functor. Then we have a functor \(\mathbb{F} \times \mathbb{F}^{op} : \mathcal{D} \times \mathcal{D}^{op} \to \mathcal{C}_0 \times \mathcal{C}_0^{op}\) and we can form a bifunctor

\[
\text{Cohom}_{\mathcal{C}} \circ (\mathbb{F} \times \mathbb{F}^{op}) : \mathcal{D} \times \mathcal{D}^{op} \to \mathcal{C}.
\]

**Definition 1.11.** We define \(\text{coend}(\mathbb{F}) := \text{coend}(\text{Cohom}_{\mathcal{C}} \circ (\mathbb{F} \times \mathbb{F}^{op}))\).

Clearly, under the assumptions above, \(\text{coend}(\mathbb{F})\) exists if \(\mathcal{C}\) is cocomplete and either \(\mathcal{C}_0\) or \(\mathcal{D}\) is small.

**Remark 1.12.** If \(\mathcal{C}\) is a rigid category, for a functor \(\mathbb{F} : \mathcal{D} \to \mathcal{C}\) we have the notion of \(\text{coend}(\mathbb{F}) := \text{coend}(\mathbb{F} \otimes \mathcal{C} \mathbb{F}^*)\), where \(\mathbb{F}^*\) is the composition \((-)^* \circ \mathbb{F}\). Since for all \(X, Y \in \mathcal{C}\) \(X \otimes \mathcal{C} Y^* \cong \text{coend}_{\mathcal{C}}(X, Y)\), this definition of \(\text{coend}\) is a special case of ours.

**Proposition 1.13.** Let \(\mathbb{F} : \mathcal{D} \to \mathcal{C}_0 \subset \mathcal{C}\) be a functor and suppose that \(\text{coend}(\mathbb{F})\) exists. Then

1. \(\text{coend}(\mathbb{F})\) is a coalgebra in \(\mathcal{C}\);
2. \(\mathbb{F}(X)\) is a comodule over \(\text{coend}(\mathbb{F})\) for every \(X \in \text{Ob}\(\mathcal{D}\)\);
3. \(\mathbb{F}(\phi)\) is a \(\text{coend}(\mathbb{F})\)-comodule morphism between \(\mathbb{F}(X)\) and \(\mathbb{F}(Y)\) for every \(\phi \in \mathcal{D}(X, Y)\);
4. the transformation \(\delta_{\mathbb{F}} : \mathbb{F} \to \mathbb{F} \otimes \mathcal{C} \text{coend}(\mathbb{F})\), with \(\delta_{\mathbb{F}}(X)\) being the comodule structure from \(\mathbb{F}\) is natural;
(5) \( \delta_F \) is universal, i.e. any transformation \( F \to F \otimes C M \) factors through \( \delta_F \) via morphism \( \text{coend}(F) \to M \).

**Proof.** (4) can be proven via diagram

\[
\begin{array}{cccccc}
\delta_F(X) \\
F(X) \xrightarrow{\text{coev}_F(X)} F(X) \otimes_C \text{coend}_C(F(X)) \xrightarrow{\text{id}_F(X) \otimes_C i_X} F(X) \otimes_C \text{coend}(F) \\
& | & | \\
& F(\phi) \otimes_C \text{coend}(F) & F(\phi) \otimes_C \text{id}_{\text{coend}(F)} \\
F(Y) \xrightarrow{\text{coev}_F(Y)} F(Y) \otimes_C \text{coend}_C(F(Y)) \xrightarrow{\text{id}_F(Y) \otimes_C i_Y} F(Y) \otimes_C \text{coend}(F) \\
& | & | \\
& \delta_F(Y) & \\
\end{array}
\]

with \( \alpha = \text{id}_F(Y) \otimes_C \text{cohom}_C(F(X), F(Y)) \) and \( \beta = \text{id}_F(Y) \otimes_C \text{cohom}_C(F(\phi), \text{id}_F(Y)) \). Similarly one can prove that

\[
\delta_F^2 := \delta_F \circ \delta_F : F \to F \otimes_C \text{coend}(F) \otimes_C \text{coend}(F)
\]

is also a natural transformation. Since every \( F(X) \) is a comodule over \( \text{coend}_C(F(X)) \), this imply that the diagrams

\[
\begin{array}{ccc}
\text{coend}_C(F(Y)) \xrightarrow{\Delta_{\text{coend}_C(F(Y))}} \text{coend}_C(F(Y)) \otimes_C \text{coend}_C(F(Y)) \\
& \downarrow \text{id}_Y & \downarrow \text{id}_Y \otimes_C i_Y \\
\text{cohom}_C(F(X), F(Y)) \xrightarrow{\Delta_{\text{coend}(F)}} \text{coend}(F) \otimes_C \text{coend}(F) \\
& \downarrow \text{id}_X & \downarrow \text{id}_X \otimes_C i_X \\
\text{coend}_C(F(X)) \xrightarrow{\Delta_{\text{coend}_C(F(X))}} \text{coend}_C(F(X)) \otimes_C \text{coend}_C(F(X)) \\
\end{array}
\]

and
The diagrams of dinatural transformations (note that \( \text{coend}(F) \otimes \text{coend}(F) \) does not have to be the colimit of \( \text{coend}(F(X)) \otimes \text{coend}(F(X)) \)). Thus \( \Delta_{\text{coend}(F)} \) and \( \epsilon_{\text{coend}(F)} \) and coalgebra structure is induced by the one of \( \text{coend}(F(X)) \), which proves (1). The check of the coalgebra axioms is straightforward.

For (2), the \( \text{coend}(F) \)-comodule structure on \( F(X) \) is given by

\[
\delta_{F(X)} := (\text{id}_{F(X)} \otimes i_X) \circ \text{coev}_{F(X)}.
\]

Again, the check of the axioms is straightforward. (3) is equivalent to (4). (5) follows from the universal property of \( \text{coend}(F) \).

1.5. **Relation to functor** \( \text{Nat}(F, F \otimes -) \).

Now let \( C \) be a locally small category. Then natural transformations \( \text{Nat}(F, F \otimes_C M) \) form a functor \( \text{Nat}(F, F \otimes_C -) : C \to \text{Set} \). In [PAR2] for a functor \( F : D \to C \) of one variable \( \text{coend}(F) \) is defined as a representing object of \( \text{Nat}(F, F \otimes_C -) \). This definition is more general than 1.11 since one might not be able to form the bifunctor \( \text{Cohom}_C \circ (F \times F \text{op}) \). The following results show that when one can - the two definitions agree. Thus one might think of the definition 1.11 as of the way to compute \( \text{coend}(F) \) when the target category is coclosed.

**Proposition 1.14.** There is a one-to-one correspondence between elements of \( \text{Nat}(F, F \otimes_C M) \) and cowedges from \( \text{Cohom}_C \circ (F \times F \text{op}) \) to \( M \).

**Proof.** Since we have adjunction \( \text{cohom}_C (-, -) \dashv (- \otimes -) \), from correspondences

\[
\mathcal{C}(\text{cohom}_C(F(X), F(X)), M) \leftrightarrow \mathcal{C}(F(X), F(X) \otimes_C M)
\]

one has a one-to-one correspondence between the families of morphisms

\[
\{\mu_X : F(X) \to F(X) \otimes_C M\}
\]

and the families of morphisms

\[
\{\mu'_X : \text{cohom}_C(F(X), F(X)) \to M\}.
\]

One can check directly that naturality condition on \( \{\mu_X\} \) implies dinaturality condition on \( \{\mu'_X\} \) and vice versa via the diagram.
Corollary 1.15. Suppose coend \((\mathcal{F})\) exists. Then \(Nat(\mathcal{F}, \mathcal{F} \otimes \mathcal{C} \rightarrow)\) is representable by \(\text{coend}(\mathcal{F})\).

Proof. Follows from the universal property of \(\text{coend}(\mathcal{F})\). \(\square\)

1.6. \(\mathcal{C}\)-cowedges of a \(\mathcal{C}\)-functor and \(\text{coend}_\mathcal{C}(\mathcal{F})\). Let \(\mathcal{A}\) and \(\mathcal{B}\) be \(\mathcal{C}\)-categories, \(\mathcal{B}_0 \subset \mathcal{B}\) coclosed in \(\mathcal{B}\) and \(\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}_0 \subset \mathcal{B}\) be a \(\mathcal{C}\)-functor. Similar to the proof of the proposition \(\text{1.14}\) for a \(\mathcal{C}\)-natural transformation \(\mu : \mathcal{F} \rightarrow \mathcal{F} \otimes M\) we can expand the diagram, expressing it’s \(\mathcal{C}\)-naturality, to the diagram

which motivates the following definition. Denote the induced map

\[\lambda_{\mathcal{C},X} : \text{cohom}_\mathcal{B}(\mathcal{F}(\mathcal{C} \otimes_{\mathcal{A}} X), C \otimes \mathcal{B}\mathcal{F}(X)) \rightarrow \text{coend}_\mathcal{B}(\mathcal{F}(X)).\]

Definition 1.16. We say that a cowedge \(\mu' : \text{cohom}(\mathcal{F}, \mathcal{F}^{op}) \rightarrow d\) is a \(\mathcal{C}\)-cowedge over \(\mathcal{F}\) if for all \(X \in \mathcal{A}\), \(C \in \mathcal{C}\) the following diagram

\[\begin{array}{ccc}
\text{id}_{\mathcal{B}(Y)} \otimes C \text{cohom}_\mathcal{C}(\text{id}_{\mathcal{B}(X)}, \mathcal{F}(f)^{op}) & \xrightarrow{\mathcal{F}(f) \otimes C \text{id}_{\text{coend}_\mathcal{C}(\mathcal{F}(X))}} & \mathcal{F}(X) \otimes C \text{coend}_\mathcal{C}(\mathcal{F}(X)) \\
\mathcal{F}(X) \otimes C \text{coend}_\mathcal{C}(\mathcal{F}(X)) & \xrightarrow{\text{id}_{\mathcal{B}(Y)} \otimes C \mu_X} & \mathcal{F}(Y) \otimes C \text{coend}_\mathcal{C}(\mathcal{F}(X)) \\
\mathcal{F}(Y) \otimes C \text{coend}_\mathcal{C}(\mathcal{F}(X)) & \xrightarrow{\text{id}_{\mathcal{B}(Y)} \otimes C \mu'_X} & \mathcal{F}(Y) \otimes C \text{coend}_\mathcal{C}(\mathcal{F}(X))
\end{array}\]
is commutative.

Similar to the section 0.2, $C$-cowedges form a subcategory of the category of cowedges over $F$.

**Definition 1.17.** Similar to the definition 0.9 we define $\text{coend}_C (F)$ as the initial object in this category.

The diagram in the beginning of this section proves the following result.

**Lemma 1.18.** Let $F : A \to B \in B$ be a $C$-functor. Then under the correspondence from the proposition 1.14 $C$-cowedges correspond precisely to $C$-natural transformations.

**Corollary 1.19.** If $B$ is cocomplete and either $A$ or $B$ is small then $\text{coend}_C (F)$ exists for every $C$-functor $F$ (similar to section 0.3).

Similar to the section 0.3, one can construct $\text{coend}_C (F)$ explicitly as a coequalizer. Namely,

$$F_1 \coprod_{q} F_2 \xrightarrow{p} \coprod_{X \in \text{Ob}(A)} \text{coend}_B (F(X)) \to \text{coend}_B (F),$$

where

$$F_1 := \coprod_{f \in \text{Mor}(A)} \text{cohom}_B (F (\text{dom} (f)), F (\text{codom} (f)))$$

and

$$F_2 := \coprod_{X \in \text{Ob}(A), C \in \text{Ob}(C)} \text{cohom}_B (F (C \otimes_C A X), C \otimes_C B F (X)).$$

The equalized pair is defined by equations

$$p \circ i_f = i_{\text{dom}(f)} \circ \text{cohom}_B (\text{id}_{F(\text{dom}(f))}, F (f)^{op}),$$

$$q \circ i_f = i_{\text{codom}(f)} \circ \text{cohom}_B (F (f), \text{id}_{F(\text{codom}(f))})$$

and by

$$p \circ i_{C,X} = i_{C \otimes_C A X} \circ \left( \text{cohom}_B (\text{id}_{F(C \otimes_C A X)}, \zeta^{op}) \right),$$

$$q \circ i_{C,X} = i_X \circ \lambda_{C,X}.$$

**Corollary 1.20.** Suppose $\text{coend}_C (F)$ exists. Then

1. $ Nat_C (F, F \otimes_B -)$ is representable by $\text{coend}_C (F)$;
2. $\delta_{F,C} : F \to F \otimes_B \text{coend}_C (F)$ is universal, i.e. any $C$-transformation $F \to F \otimes_B M$ factors though $\delta_{F,C}$ via morphism $\text{coend}_C (F) \to M$. 
Remark 1.21. If \( C \in \mathcal{C} \) is a coalgebra and \( \mathcal{F} : \text{Comod}_C - C \to \mathcal{C} \) is a forgetful functor from the category of right comodules over \( C \), then \( \text{Nat}_C (\mathcal{F}, \mathcal{F} \otimes \mathcal{B} -) \) is multirepresentable (similar to [PAR1, 3.8.6]).

1.7. Reconstruction theorems. We slightly reformulate the “restricted” reconstruction theorems.

Proposition 1.22. Let \( \mathcal{C} \) be a braided monoidal category and \( \mathcal{C}_0 \) be a full braided monoidal subcategory.

- Let \( \mathcal{C} \) be a \( \mathcal{C}_0 - I \)-generated coalgebra in \( \mathcal{C} \) and \( \mathcal{F} : \text{Comod}_{\mathcal{C}_0} - \mathcal{C}_0 \subset \mathcal{C} \) be the forgetful functor. Then we have an isomorphism of coalgebras \( \text{coend}_{\mathcal{C}_0} (\mathcal{F}) \cong \mathcal{C} \) [PAR2, 4.3];
- Let \( \mathcal{C} \) also have colimits of \( I \)-diagrams and let \( \otimes \) preserve those colimits in both variables (and thus preserve \( I - I \)-colimits). Let \( \mathcal{C} \) also be a \( \mathcal{C}_0 \)-central bialgebra (and thus \( \text{Comod}_{\mathcal{C}_0} - \mathcal{C} \) is a monoidal category). Then \( \text{Nat}_{\mathcal{C}_0} (\mathcal{F}, \mathcal{F} \otimes \mathcal{C} -) \) is multirepresentable by \( \mathcal{C} \) and \( \text{coend}_{\mathcal{C}_0} (\mathcal{F}) \cong \mathcal{C} \) is the isomorphism of bialgebras [PAR2, 4.5].

Remark 1.23. It is often possible to reconstruct the bialgebra structure without additional assumptions on \( \mathcal{C} \) and \( \otimes \) (see 2.1.2).

1.8. Recognition theorem. From our previous results we know that for a \( \mathcal{C}_0 \)-functor \( \mathcal{F} : \mathcal{B} \to \mathcal{C}_0 \subset \mathcal{C} \) if the category \( \mathcal{C} \) is cocomplete, \( \mathcal{C}_0 \subset \mathcal{C} \) is coclosed in \( \mathcal{C} \) and either \( \mathcal{B} \) or \( \mathcal{C}_0 \) is small, then \( \text{coend} (\mathcal{F}) \) exists. Together with the proposition 1.25 this solves the recognition problem for coalgebras. For the recognition of bialgebras we can simply use the corresponding part of the proposition 0.11.

We combine these statements in the following proposition (compare with [PAR2, 4.7]).

Proposition 1.24. Let \( \mathcal{A}, \mathcal{B} \) be \( \mathcal{C} \)-categories, \( \mathcal{A} \) be locally small, cocomplete and \( \mathcal{C} \)-monoidal, \( \mathcal{A}_0 \subset \mathcal{A} \) be coclosed in \( \mathcal{A} \), \( \mathcal{F} : \mathcal{B} \to \mathcal{A}_0 \subset \mathcal{A} \) be a \( \mathcal{C} \)-functor. Let also \( \mathcal{A}_0 \) or \( \mathcal{B} \) be small. Then

1. \( \mathcal{A} := \text{coend}_C (\mathcal{F}) \) exists and \( \mathcal{F} \) factors as \( \mathcal{F} = \mathbb{I} \circ \mathbb{I} \), with \( \mathbb{I}_\mathcal{B} : \mathcal{B} \to \text{Comod}_{\mathcal{A}_0} - \mathcal{A} \), \( \mathcal{U} : \text{Comod}_{\mathcal{A}_0} - \mathcal{A} \to \mathcal{A}_0 \) being the forgetful functor;
2. let \( \mathcal{A} \) be \( \mathcal{C} \)-braided \( \mathcal{C} \)-monoidal, \( \mathcal{B} \) is \( \mathcal{C} \)-monoidal, \( \mathcal{F} \) is \( \mathcal{C} \)-monoidal functor and \( \text{Nat}_C (\mathcal{F}, \mathcal{F} \otimes \mathcal{C} -) \) be multirepresentable. Then \( \text{coend}_C (\mathcal{F}) \) is a bialgebra in \( \mathcal{A} \), \( \text{Comod}_{\mathcal{A}_0} - \mathcal{A} \) has natural \( \mathcal{C} \)-monoidal structure and \( \mathbb{I}_\mathcal{B} \) is a \( \mathcal{C} \)-monoidal functor;

1.9. Equivalence of categories. Let \( \mathcal{C} \) be a monoidal category, \( \mathcal{C}_0 \subset \mathcal{C} \) be a full monoidal subcategory and \( \mathcal{C}_0 \subset \mathcal{C} \) be a coalgebra.

Let \( \mathcal{A} \) be a \( \mathcal{C}_0 \)-category and \( \mathcal{F} : \mathcal{A} \to \mathcal{C}_0 \) be a \( \mathcal{C}_0 \)-functor. If \( \mathcal{F} = \text{coend}_{\mathcal{C}_0} (\mathcal{F}) \) exists then we have a \( \mathcal{C}_0 \)-functor

\[
\mathbb{I}_\mathcal{A} : \mathcal{A} \to \text{Comod}_{\mathcal{C}_0} - \mathcal{C}
\]

and \( \mathcal{F} \) factors as \( \mathcal{F} = \mathbb{U} \circ \mathbb{I}_\mathcal{A} \), with \( \mathbb{U} : \text{Comod}_{\mathcal{C}_0} - \mathcal{C} \to \mathcal{C}_0 \) being the forgetful functor. If \( \mathcal{F} \) is faithful then \( \mathbb{I}_\mathcal{A} \) is an embedding of categories. Let’s check when it is a category equivalence.

First note that, since \( \mathbb{I}_\mathcal{A} \) is a \( \mathcal{C}_0 \)-functor, we always have an isomorphism

\[
\mathbb{I}_\mathcal{A} (\mathcal{A}) \otimes_{\mathcal{C}} \mathbb{I}_\mathcal{A} (\mathcal{A}) \cong \mathbb{I}_\mathcal{A} (\mathbb{I}_\mathcal{A} (\mathcal{A}) \otimes_{\mathcal{C}} \mathcal{A}).
\]
**Theorem 1.25.** Suppose \( C \) is \( C_0 \)-\( I \)-generated by a filtered system \( \{I_A(A_i)\}_{i \in I} \) of coalgebras \( \text{I}_A(A_i) \), such that \( \Delta_{i_A(A_i)} \in \text{I}_A(A_i, \text{I}_A(A_i) \otimes_{\text{C}_0 A} A_i) \). Let also \( \mathcal{A} \) have equalizers.

If \( F \) is faithful and preserves equalizers then \( \text{I}_A \) is a category equivalence.

**Proof.** Let \( (M, \rho_M) \in \text{Comod}_{C_0} - C \). Since \( C \) is \( C_0 \)-\( I \)-generated, \( (M, \rho_M) \in \text{Comod}_{C_0} - \text{I}_A(A_i) \) for some \( i \in I \). Since \( M \) is a comodule, it is an equalizer of the pair

\[
M \otimes C \text{I}_A(A_i) \xrightarrow{\rho_M \otimes_C \text{id}_{I_A(A_i)}} M \otimes C \text{I}_A(A_i) \otimes_C \text{I}_A(A_i).
\]

Since \( F \) is \( C_0 \)-functor, then so is \( \text{I}_A \). Thus the top morphism is actually

\[
\text{I}_A(M \otimes_{C_0 A_i} A_i) \xrightarrow{\text{I}_A(\rho_M \otimes_{C_0 A_i} \text{id}_{I_A(A_i)})} \text{I}_A((M \otimes_{C_0} \text{I}_A(A_i)) \otimes_{C_0 A} A_i)
\]

and the bottom one is

\[
\text{I}_A(M \otimes_{C_0 A_i} A_i) \xrightarrow{\text{id}_M \otimes_{C_0 A_i} \Delta_{I_A(A_i)}} \text{I}_A(M \otimes_{C_0 A} (\text{I}_A(A_i) \otimes_{C_0 A} A_i)).
\]

Since \( \mathcal{A} \) has equalizers, there exists an object \( \tilde{M} \in \mathcal{A} \) satisfying the diagram

\[
\tilde{M} \xrightarrow{\tilde{\rho}_M} M \otimes_{C_0 A_i} A_i \xrightarrow{\rho_M \otimes_{C_0 A_i} \text{id}_{I_A(A_i)}} M \otimes_{C_0 A_i} (\text{I}_A(A_i) \otimes_{C_0 A} A_i).
\]

Since \( F \) and the forgetful functor \( U : \text{Comod}_{C_0} - C \to C_0 \) both preserve equalizers (\( U \) creates finite limits), then so is \( \text{I}_A \) and thus \( \text{I}_A(\tilde{M}) \cong M \). Thus \( F \) is essentially surjective.

\( \text{I}_A \) is faithful since \( F \) is. Let \( M, N \in \text{Comod}_{C_0} - C \) and let \( f : M \to N \) be a \( C \)-comodule morphism. Since \( C \) is \( C_0 \)-\( I \)-generated and the system \( \{\text{I}_A(A_i)\}_{i \in I} \) is filtered, \( M, N \in \text{Comod}_{C_0} - \text{I}_A(A_i) \) for some \( i \in I \) and we have a diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\rho_M} & M \otimes_C \text{I}_A(A_i) \\
\downarrow{f} & & \downarrow{f \otimes_C \text{id}_{I_A(A_i)} \rho_M \otimes_C \text{id}_{I_A(A_i)}} \\
N & \xrightarrow{\rho_N} & N \otimes_C \text{I}_A(A_i) \xrightarrow{\text{id}_N \otimes_C \Delta_{I_A(A_i)}} N \otimes_C \text{I}_A(A_i) \otimes_C \text{I}_A(A_i)
\end{array}
\]

where \( (f \otimes_C \text{id}_{I_A(A_i)}) \circ \rho_M \) equalizes the kernel pair. Since \( \text{I}_A \) is faithful, in the diagram
\[
\begin{array}{c}
\tilde{\mu} \\
\| \| \\
\tilde{\rho}_N \\
\| \| \\
\tilde{\mu} M & \to M \otimes_{C_0} A_i \\
\| \| \\
\| \| \\
\| \| \\
N & \to N \otimes_{C_0} A_i \xrightarrow{\rho_N \otimes_{C_0} id_{A_i}} N \otimes_{C_0} \Delta_{A_i} \xrightarrow{id_N \otimes_{C_0} \tilde{\Delta}_{A_i}} N \otimes_{C_0} (\mathbb{I}_A (A_i) \otimes_{C_0} A_i).
\end{array}
\]

\((f \otimes_{C_0} id_{A_i}) \circ \tilde{\rho}_M\) also equalizes the kernel pair. This implies the existence of the map \(\tilde{f} : \tilde{M} \to \tilde{N}\) and thus \(\mathbb{I}_A\) is full and is a category equivalence. \(\square\)

**Corollary 1.26.** Let \(C_0\) be a rigid category, \(A\) and \(C_0\) have finite colimits, \(A\) or \(C_0\) be small and \(F\) be right exact. Then \(\mathbb{I}_A\) is a category equivalence.

**Proof.** All we need to show is that \(C\) is \(C_0\)-\(I\)-generated by a filtered system \(\{\mathbb{I}_A (A_i)\}_i\) of coalgebras \(\mathbb{I}_A (A_i)\), such that \(\Delta_{\mathbb{I}_A (A_i)} \in \mathbb{I}_A (A_i, \mathbb{I}_A (A_i) \otimes_{C_0} A_i)\).

As \(\text{coend}_{C_0} (F)\), \(C\) is a colimit of the diagram, composed from pieces of the form

\[
\begin{array}{ccc}
F (C \otimes_{C_0} X)^* \otimes_C F (C \otimes_{C_0} X) & \xrightarrow{\xi^* \otimes_C \text{id}_F (C \otimes_{C_0} X)} & F (X)^* \otimes_C F (X) \\
\| & & \| \\
\| & & \| \\
\| & & \| \\
\| & & \| \\
\| & & \| \\
F (X)^* \otimes_C C^* \otimes_C F (C \otimes_{C_0} X) & \xrightarrow{\text{id}_{F (X)^*} \otimes_C \text{id}_F (C \otimes_{C_0} X)} & F (X)^* \otimes_C F (X) \\
\| & & \| \\
\| & & \| \\
\| & & \| \\
\| & & \| \\
\| & & \| \\
(\text{id}_{F (X)^*} \otimes_C \text{ev}) \otimes_C \text{id}_X & \xrightarrow{\text{id}_{F (X)^*} \otimes_C \text{id}_X} & F (X)^* \otimes_C F (X)
\end{array}
\]

which, since \(F\) is a \(C_0\)-functor, is the image of the diagram in \(A\), made of pieces of the form

\[
\begin{array}{ccc}
F (C \otimes_{C_0} X)^* \otimes_{C_0} (C \otimes_{C_0} X) & \xrightarrow{\xi^* \otimes_{C_0} \text{id}_F (C \otimes_{C_0} X)} & F (X)^* \otimes_{C_0} X \\
\| & & \| \\
\| & & \| \\
\| & & \| \\
\| & & \| \\
\| & & \| \\
(\text{id}_{F (X)^*} \otimes_{C_0} \text{ev}) \otimes_{C_0} \text{id}_X & \xrightarrow{\text{id}_{F (X)^*} \otimes_{C_0} \text{id}_X} & F (X)^* \otimes_{C_0} X
\end{array}
\]

For any diagram it’s colimit is a filtered colimit of colimits of it’s finite subdiagrams. Thus we can consider the diagrams, composed of finite number of pieces of the above form. This will give us our filtered system \(I\) and their colimits \(A_i\) and \(\mathbb{I}_A (A_i)\). One can check that the coalgebra structure

\[
\Delta := \text{id} \otimes_C db \otimes_C \text{id} : F (X)^* \otimes_C F (X) \to (F (X)^* \otimes_C F (X)) \otimes_C (F (X)^* \otimes_C F (X))
\]
TANNAKA DUALITY, COCLOSED CATEGORIES AND RECONSTRUCTION FOR NONARCHIMEDEAN BIALGEBRAS

induce a cocone

\[
\begin{array}{ccc}
F(C \otimes_C A)^* \otimes_C F(C \otimes_C A) & \xrightarrow{\Delta} & (F(C \otimes_C A)^* \otimes_C F(C \otimes_C A)) \otimes_C (F(C \otimes_C A)^* \otimes_C F(C \otimes_C A)) \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quito
Let $\text{CHLCTVS}$ be the category of complete Hausdorff locally convex topological vector spaces. It is a symmetric monoidal category, with monoidal structure given by complete projective tensor product $\otimes_{K,π}$. LS-spaces form a monoidal subcategory $\text{LS}$ of $\text{CHLCTVS}$. Since, in general, inductive and projective tensor product topologies do not coincide, the above identity shows that in $\text{CHLCTVS}$ two LS-spaces $U$ and $V$ have a cohomomorphism object $\text{cohom}_{\text{CHLCTVS}}(U,V) \cong U \otimes_{K,π} V'$, although $U$ and $V$ might not be rigid objects. Also note that the category $\text{CHLCTVS}$ is cocomplete, with the colimit $\widehat{\lim} V_i$ being the forgetful functor; denote the category $\text{LS}$-generated modules over a commutative ring $\mathbb{C}$ by $\text{LS}_{\mathbb{C}}$.

Before considering reconstruction theorems, lets recall that in the algebraic case, i.e. when $C$ is a full monoidal subcategory of the category $\text{R-mod}_{fg}$ of finitely generated modules over a commutative ring $R$, every natural transformation $φ$ of $C$-functors $F, F': A → C$ is $C$-natural ([PAR2, 6.4]). This imply that in this case the coendomorphism object $\text{coend}_{C}(F)$ does not depend on the control category $C$ and we may simply consider $\text{coend}(F)$. The categories we consider share the same property.

**Lemma 2.1.** Let $C$ be a full monoidal subcategory of $\text{Ban}_K$ or $\text{LCTVS}$ and $A$ be a $C$-category. Let $F, F': A → C$ be $C$-functors. Then every natural transformation $φ: F → F'$ is $C$-natural.

**Proof.** Similar to [PAR2, 6.4] one can prove that the identity, required for $φ$ to be $C$-natural, holds on elementary tensors $x \otimes a ∈ C \otimes F(A), C ∈ C, A ∈ A$. Since all maps involved there are continuous, the identity holds in general. □

2.1. **Reconstruction.** Let $C$ denote the category $\text{CHLCTVS}$ and $\text{LS}$ denote the category of LS-spaces.

2.1.1. **Full reconstruction and recognition.** For any two monoidal subcategories $C ⊂ D ⊂ \text{CHLCTVS}$ the category $D$ is naturally a $C$-category. If $A ∈ D$ is a coalgebra in $D$, then the categories $\text{Comod}_D - A$ and $\text{Comod}_C - A$ are naturally $C$-categories. Proposition [122] can be applied to produce the following result.

**Proposition 2.2.** (Reconstruction theorem)

- Let $C$ be a coalgebra in $\text{LS}$ and $F: \text{Comod}_\text{LS} - C → \text{LS}$ be the forgetful functor. Then we have an isomorphism of coalgebras $\text{coend}(F) ≅ C$;
- Let $C$ be a bialgebra (and thus $\text{Comod}_\text{LS} - C$ is a monoidal category). Then $\text{Nat}(F, F ⊗ C)$ is multirepresentable by $C$ and $\text{coend}(F) ≅ C$ is the isomorphism of bialgebras.

The recognition theorem in full setting will have the following form (one of few possible).

**Proposition 2.3.** Let $B$ be an $\text{LS}$-category and $F: B → \text{LS}$ be an $\text{LS}$-functor. Then

1. $A := \text{coend}(F)$ exists in $\text{CHLCTVS}$ and $F$ factors as $F = U \circ B$, with $B : B → \text{Comod}_\text{LS} - A, U : \text{Comod}_\text{LS} - A → \text{LS}$ being the forgetful functor;
2. let $B$ be $\text{LS}$-monoidal, $F$ is $\text{LS}$-monoidal functor and $\text{Nat}_\text{LS}(F, F ⊗ -)$ be multirepresentable. Then $A$ is a bialgebra in $\text{CHLCTVS}$, $\text{Comod}_\text{LS} - A$ has natural $\text{LS}$-monoidal structure and $B$ is a $\text{LS}$-monoidal functor;
(3) if $A$ is an LS-space and $A \in \mathcal{F}(\mathcal{B})$ with $\Delta_A \in \mathcal{F}(\text{Mor}(\mathcal{B}))$, $\mathcal{B}$ has equalizers and $\mathcal{F}$ is faithful and preserve them, then $\mathbb{N}$ is a category equivalence.

**Proof.** Direct application of the recognition theorem [L24] and equivalence theorem 2.25. The only thing one needs to prove is that the category $\mathcal{LS}$ is small. This follows from the facts that LS-spaces are nuclear ([PGS 11.3.5.ix]) and that all nuclear spaces are isomorphic to a subspace of some power of the “universal nuclear space” ([PGS 8.8.3]).

**Remark 2.4.** One can also argue directly that, since any LS-space $V$ is of countable type, if we consider $X = K^N$ then we have an embedding of sets $V \subset X$ and for the topology $\tau_V$ of $V$ we have $\tau_V \subset 2^X$.

### 2.1.2. Banach reconstruction for coalgebras of compact type

Recall that we call $C \in \mathcal{LS}$ (in terminology of [L1]) a coalgebra of compact type (or CT-coalgebra) if it is a compact inductive limit of an inductive sequence of Banach $\mathcal{S}$-coalgebras. Denote by $\text{Ban}_K \subset \text{CHLCTVS}$ the image of the $\text{Ban}_K$ under the forgetful functor into the category $\text{CHLCTVS}$ (forgets the norm, but remembers topology). Another reconstruction theorem we would like to consider is the reconstruction of $C$ from the category $\text{Comod}_{\text{Ban}_K} = C$.

If $C \cong \lim_{\rightarrow} C_i$ and $(\mathcal{V}, \rho_V) \in \text{Comod}_{\text{Ban}_K} = C$ then the coaction $\rho_V : V \rightarrow V \mathcal{S}C$ send $V$ into the space $V \mathcal{S}C$. Since $V \mathcal{S}C \cong \lim_{\rightarrow} (V \mathcal{S}C_i)$ [L1 3.4.6] is a regular LB-space (follows from splitting lemma [L1 5.11]), $\rho_V$ must factor through $V \mathcal{S}C_i$ for some $i$, which means $(V, \rho_V) \in \text{Comod}_{\text{Ban}_K} = C$. This imply that $C$ is Ban$_K - N$-generated in $\text{CHLCTVS}$ and thus we get the reconstruction for coalgebra structure.

**Proposition 2.5.** Let $C$ be a CT-$\mathcal{S}$-coalgebra and $\mathcal{F} : \text{Comod}_{\text{Ban}_K} = C \rightarrow \text{Ban}_K \subset \text{CHLCTVS}$ be the forgetful functor. Then we have an isomorphism of coalgebras coend $(\mathcal{F}) \cong C$.

For the reconstruction for bialgebra structure, since the tensor product in $\text{CHLCTVS}$ does not satisfy the conditions of the proposition [L24], we cannot apply it here. Instead, we first need to prove the following lemma.

**Lemma 2.6.** Let $C$ be a CT-$\mathcal{S}$-coalgebra and $\mathcal{F} : \text{Comod}_{\text{Ban}_K} = C \rightarrow \text{Ban}_K \subset \text{CHLCTVS}$ be the forgetful functor. Then the functor $\text{Nat}_{\text{Ban}_K} (\mathcal{F}, \mathcal{F}\mathcal{S}_K, \mathcal{F})$ is multirepresentable, i.e. $\text{coend} (\mathcal{F}^n) \cong C^n$.

**Proof.** Similar to [PAR1 3.8.6]. Let $C = \text{coend} (\mathcal{F})$, $\delta : \mathcal{F} \rightarrow \mathcal{F}\mathcal{S}C$ be the universal transformation and denote $\delta_n := \tau \circ (\delta^\otimes_n) : \mathcal{F}^\otimes \rightarrow \mathcal{F}\mathcal{S}\mathcal{S}C^\otimes$ the transformation

$$\delta_n : \mathcal{F}(X_1) \mathcal{S}\mathcal{F}(X_2) \mathcal{S} \cdots \mathcal{F}(X_n) \rightarrow \mathcal{F}(X_1) \mathcal{S}\mathcal{F}(X_2) \mathcal{S} \cdots \mathcal{F}(X_n) \mathcal{S}C$$

$$\cong \mathcal{F}(X_1) \mathcal{S}\mathcal{F}(X_2) \mathcal{S} \cdots \mathcal{F}(X_n) \mathcal{S}C^\otimes$$

Let $M \in \text{CHLCTVS}$ and $\phi : \mathcal{F}^\otimes \rightarrow \mathcal{F}\mathcal{S}_M$ be a natural transformation. Since $C$ is a CT-$\mathcal{S}$-coalgebra, there exists a system $\{C_i\}_{i \in N}$ of Banach $\mathcal{S}$-coalgebras, such that $C \cong \lim_{\rightarrow} C_i$ is a compact inductive limit. Each $C_i$ is a right Banach $\mathcal{S}$-comodule over $C$ via canonical injections $C_i \rightarrow C$. For every multi-index $i$ define a map

$$\phi_i : C_{i_1} \mathcal{S}C_{i_2} \mathcal{S} \cdots \mathcal{S}C_{i_n} = \mathcal{F}(C_{i_1}) \mathcal{S}\mathcal{F}(C_{i_2}) \mathcal{S} \cdots \mathcal{F}(C_{i_n}) \rightarrow M$$

$$\phi_i := \left(C^\otimes \mathcal{S}\text{id}_M\right) \circ \phi (C_{i_1} \mathcal{S}C_{i_2} \mathcal{S} \cdots \mathcal{S}C_{i_n})$$.
Since the projective tensor product with Banach spaces preserve compact limits, we have an isomorphism
\[ \lim C_{i_1} \otimes C_{i_2} \otimes \ldots \otimes C_{i_n} \cong C^{\otimes n} \]
and, via the universal property of limits, \( \phi_i \) induce the map \( \tilde{\phi} : C^{\otimes n} \to M \). The rest of the proof goes exactly as in [PAR1, 3.8.6]. \( \square \)

Now we can apply proposition 3.11 to prove reconstruction for bialgebras.

**Proposition 2.7.** Let \( C \) be a CT-\( \otimes \)-bialgebra (and thus \( \text{Comod}_{\text{Ban}_K} C \) is a monoidal category). Then \( \text{Nat}_{\text{Ban}_K} (F, F \otimes -) \) is multirepresentable by \( C \) and \( \text{coend}(F) \cong C \)
is the isomorphism of bialgebras.

### 3. Reconstruction for Banach coalgebras

If one wants to reconstruct a Banach coalgebra, one has to restrict to finite-dimensional comodules. The reason is because only for finite-dimensional Banach spaces \( X,Y \in \text{Ban}_K \) a cohomomorphism object \( \text{cohom}_{\text{Ban}_K}(X,Y) \) exists (same reason as for \( K \)-vector spaces). Another problem is that the category \( \text{Ban}_K \) is not cocomplete, so for a functor \( F \) into finite-dimensional Banach spaces \( \text{coend}(F) \) might not exist. We will modify our previous construction to handle this situation.

#### 3.1. Relative limits and colimits.

Recall that
- for a small category \( J \) a \( J \)-diagram in a category \( C \) is a functor \( \mathbb{D} : J \to C \);
- for a \( c \in C \) the constant functor \( \Delta(c) \in \text{Funct}(J,C) \) is a functor with values \( \Delta(c)(j) := c \) and \( \Delta(c)(j \to j') := \text{id}_c \);
- a cone over \( \mathbb{D} \) is a natural transformation \( \phi : \Delta(c) \to \mathbb{D} \) for some \( c \in C \);
- a cocone over \( \mathbb{D} \) is a natural transformation \( \psi : \mathbb{D} \to \Delta(c) \);
- a morphism of (co)cones is natural transformation of constant functors \( \Delta(c) \to \Delta(c') \), i.e. a morphism \( c \to c' \) in \( C \);
- cones and cocones over \( \mathbb{D} \) form categories \( \text{Cones}(\mathbb{D}) \) and \( \text{Cocones}(\mathbb{D}) \) correspondingly;
- a limit \( \text{lim} \mathbb{D} \) of \( \mathbb{D} \) is a terminal object in \( \text{Cones}(\mathbb{D}) \) (if it exists);
- a colimit \( \text{colim} \mathbb{D} \) of \( \mathbb{D} \) is an initial object in \( \text{Cocones}(\mathbb{D}) \).

Now let \( \mathcal{E} - \text{cones}(\mathbb{D}) \subset \text{Cones}(\mathbb{D}) \) (\( \mathcal{E} - \text{cocones} \subset \text{Cocones}(\mathbb{D}) \)) be a subcategory, which belongs to a certain collection \( \mathcal{E} \subset \text{Mor} (\text{Funct}(J,C)) \).

**Definition 3.1.** An \( \mathcal{E} \)-limit of \( \mathbb{D} \) is a terminal object in \( \mathcal{E} - \text{cones}(\mathbb{D}) \), which we denote by \( \mathcal{E} - \text{lim} \mathbb{D} \).

Similarly, an \( \mathcal{E} \)-colimit of \( \mathbb{D} \) is an initial object in \( \mathcal{E} - \text{cocones}(\mathbb{D}) \), which we denote by \( \mathcal{E} - \text{colim} \mathbb{D} \).

**Remark 3.2.** One can give more general definitions, which we omit here for clarity.

**Example 3.3.** Let \( \mathcal{C} = \text{Ban}_K \) and let \( F,G \in \text{Ob}(\text{Funct}(J,\text{Ban}_K)) \). We say that a natural transformation \( \phi : F \to G \) is bounded if \( \exists C_0 > 0 \) such that \( |\phi(j)| < C_0 \) for all morphisms \( j \in J \). Denote the set of bounded natural transformations \( F \to G \) by \( B\text{Nat}(F,G) \) and the collection of all bounded natural transformations by \( B\text{Nat} \subset \text{Mor}(\text{Funct}(J,\text{Ban}_K)) \).
Clearly identity transformation id* : F → F is bounded for every F ∈ Ob(Funct (J,BanK)) and the composition of bounded natural transformations is bounded. It is also clear that any natural transformation between constant functors is bounded. Thus for any diagram D : J → BanK we can form categories of bounded cones BCônes(D) and bounded cocones BCôcones(D). The corresponding relative limits and colimits will be called Banach limit and colimit and denoted as BlimD and BlimD.

Consider some examples.

For every functor F ∈ Ob(Funct (J,BanK)) we first consider the Banach direct product

\[ \prod F := \left\{ f ∈ \prod F(j) | \sup_j \|f_j\| < \infty \right\} \]

with projections πj : \( \prod F \rightarrow F(j) \). \( \prod F \) is a Banach space w.r.t. supremum-norm. The intersection of all kernels ker (πj − F (j') → πj'), for all morphisms \( j' → j \), is a closed subspace of \( \prod F \) (possibly a zero subspace) and it satisfies the universal property of BlimD.

To form Banach colimit of F we first consider the Banach direct sum

\[ \sum F := \left\{ f ∈ \prod F | \forall \epsilon > 0 : \text{card} \left\{ j : \|f_j\| > \epsilon \right\} < \infty \right\} , \]

(completion of the algebraic direct sum \( \bigoplus_{j∈J} F(j) \) in \( \prod F \)) with injections \( φ_j : F(j) \rightarrow \sum F \). \( \sum F \) is a closed subspace of \( \prod F \). Let I be the closure in \( \sum F \) of the span of the elements \( φ_j′(x) − φ_j \circ F(j' → j)(x) \) for all morphisms \( j' → j \) and for all \( x ∈ F(j') \). The quotient \( \sum F/I \) is a Banach colimit of F.

3.2. **Bounded coends.** In this section we show that one can do reconstruction in the category of Banach spaces if one work in relative setting. First we need to make the corresponding definitions.

**Definition 3.4.** Let \( B \) be a category.

- Let G : \( B × B^{op} \rightarrow \text{Ban}_K \) be a bifunctor. A (co)wedge μ is called *bounded* if \( ∃C_μ > 0 \) such that \( \|μ_c\| < C_μ \) for all \( c ∈ C \).
- \( \text{Bend}(G) \) is a terminal object in the category of bounded wedges over \( G \).
- \( \text{Bcoend}(G) \) is an initial object in the category of bounded coedges over \( G \).
- For a functor \( F : \mathcal{B} → \text{Ban}_{K}^{fd} ⊂ \text{Ban}_{K} \) valued in finite-dimensional Banach spaces \( \text{Ban}_{K}^{fd} \), we define

\[ \text{Bcoend}(F) := \text{Bcoend}(\text{Cohom} ∘ (F × F^{op})). \]

**Lemma 3.5.** Similar to the case of Nat and coend one can prove the following results.

- there is a one-to-one correspondence between elements of BNat \( (F,F ⊗_C M) \) and bounded coedges from \( \text{Cohom} ∘ (F × F^{op}) \) to \( M \);
- if Bcoend \( (F) \) exists, we have an isomorphism

\[ \text{BNat}(F,F ⊗_C M) \cong \text{Ban}_K(\text{Bcoend}(F), M); \]
- under the isomorphism above, from the identity morphism \( \text{id}_{\text{Bcoend}(F)} \) we get a bounded universal natural transformation \( δ_F : F → F ⊗ F^{op}_{\text{Bcoend}(F)} \).

**Remark 3.6.** Let \( F : \mathcal{B} → \text{Ban}_K \) be a functor. If \( \mathcal{B} \) is small and the objects cohom \( (F(X), F(X)) \) exist (i.e. if \( F(X) \) are finite dimensional) , then the Bcoend \( (F) \)
will exist and represent $BNat(F, F\overline{\otimes} -)$. So one can proceed similar to [PAR2] and define $\text{Bcoend}(F)$ as the representing object for $BNat(F, F\overline{\otimes} -)$.

**Proposition 3.7.** Let $\text{Bcoend}(F)$ exist. Then

1. $\text{Bcoend}(F)$ is a Banach $\overline{\otimes}$-coalgebra:
   - the comultiplication $\Delta_{\text{Bcoend}(F)}$ corresponds to the transformation $(\delta_F \otimes \text{id}_M) \circ \delta_F$;
   - the counit $\epsilon_{\text{Bcoend}(F)}$ corresponds to the transformation $\text{id}_F : F \to F \overline{\otimes} K \cong F$;
2. $F(X)$ is a right Banach $\overline{\otimes}$-comodule over $\text{Bcoend}(F)$ via $\rho_{F(X)} := \delta_F(X)$ for every $X \in B$;
3. $F(\phi)$ is a morphism of $\text{Bcoend}(F)$-$\overline{\otimes}$-comodules for every morphism $\phi \in \text{Mor}(B)$;
4. if $B$ is monoidal, $F$ is also be monoidal and $BNat(F, F\overline{\otimes} -)$ is multirepresentable, then $\text{Bcoend}(F)$ is a Banach $\overline{\otimes}$-bialgebra; Now let $\forall X \in B : \dim F(X) < \infty$. Then
5. if $B$ is monoidal and $F$ is also be monoidal then $\text{Bcoend}(F)$ is a Banach $\overline{\otimes}$-bialgebra;
6. if $B$ is rigid then $\text{Bcoend}(F)$ is a Banach Hopf $\overline{\otimes}$-algebra.

**Definition 3.8.** Let $C_0 \subset \text{Ban}_K$ be a full monoidal subcategory and $I$ be a poset. The Banach $\overline{\otimes}$-coalgebra $C$ is called $C_0 - I$-generated if the following holds:

1. $C$ is a Banach colimit of an $I$-diagram of objects $C_i \in C_0$;
2. all morphisms $X \overline{\otimes} j_i : X \overline{\otimes} C_i \overline{\otimes} M \to X \overline{\otimes} C \overline{\otimes} M$ are monomorphisms in $C$, where $X \in C_0$, $M \in \text{Ban}_K$ and $j_i : C_i \to C$ are the monomorphisms from the colimit diagram;
3. every $C_i$ is a Banach $\overline{\otimes}$-subcoalgebra of $C$ via $j_i : C_i \to C$;
4. if $(P, \rho_P)$ is a (right) Banach $\overline{\otimes}$-comodule over $C$ and $P \in C_0$ then $\exists i$ and $\rho_{P,i} : P \to P \overline{\otimes} C_i$ such that $\rho_P = (\text{id}_P \overline{\otimes} j_i) \circ \rho_{P,i}$.

**Proposition 3.9.** (Reconstruction theorem) Let $C_0$ be a full braided monoidal subcategory of $\text{Ban}_K$.

1. Let $C$ be a $C_0 - I$-generated Banach $\overline{\otimes}$-coalgebra and $F : \text{Comod}_{C_0} - C \to C_0 \subset \text{Ban}_K$ be the forgetful functor. Then we have an isomorphism of coalgebras $\text{Bcoend}(F) \cong C$;
2. Let $C$ also be a Banach $\overline{\otimes}$-bialgebra (and thus $\text{Comod}_{C_0} - C$ is a monoidal category). Then $BNat(F, F\overline{\otimes} -)$ is multirepresentable by $C$ and $\text{Bcoend}(F) \cong C$ is the isomorphism of bialgebras.

*Proof.* Let $\{C_i\}_{i \in I}$ be the generating diagram for $C \cong \lim C_i$ and $\phi : F \to F \overline{\otimes} M$ be a bounded natural transformation. Define the maps $\tilde{\phi}_i : C_i \to M$ as $\phi_i := (\epsilon_{C_i} \overline{\otimes} \text{id}_M) \circ \phi(C_i)$. Since the transformation $\phi$ is bounded, the maps $\{\tilde{\phi}_i\}_{i \in I}$ form a bounded cocone over $\{C_i\}_{i \in I}$, and thus define a map $\bar{\phi} : C \to M$. The rest of the proof of (1) goes as in [PAR1] 3.8.4.

The multirepresentability of $BNat(F, F\overline{\otimes} -)$ is proven similar to lemma [PAR2] 2.3.7. Then the bialgebra structure is reconstructed in the standard way (see [SCH1] 2.3.7). □

Since $\text{cohom}_{\text{Ban}_K}(X, Y)$ exists only for finite-dimensional $X, Y \in \text{Ban}_K$, the recognition theorem is essentially repeats the one in rigid setting.
Proposition 3.10. Let \( \mathcal{B} \) be \( \text{Ban}_K^{fd} \)-category and \( \mathcal{F} : \mathcal{B} \to \text{Ban}_K^{fd} \subset \text{Ban}_K \) be a \( \text{Ban}_K^{fd} \)-functor. Then

(1) \( A = B\text{coend}(\mathcal{F}) \) exists and \( \mathcal{F} \) factors as \( \mathcal{F} = U \circ I_\mathcal{B} \), with \( I_\mathcal{B} : \mathcal{B} \to \text{Comod}_{\text{Ban}_K^{fd}} - A, U : \text{Comod}_{\text{Ban}_K^{fd}} - A \to \text{Ban}_K^{fd} \) being the forgetful functor;

(2) if \( \mathcal{B} \) is \( K \)-linear and abelian, and \( \mathcal{F} \) is \( K \)-linear, exact and faithful, then \( I_\mathcal{B} \) is a category equivalence.

3.3. Example: Geometric Satake correspondence. Let \( K \) be a spherically complete extension of \( \mathbb{Q}_p \subset K \subset \mathbb{Q}_p \), \( G_K \) be a smooth split semi-simple simply connected group scheme over \( K \) and \( \hat{G}_K \) be the reductive group scheme corresponding to the dual root datum. The geometric Satake correspondence from [MirV, 14.1] states that there is a monoidal equivalence of rigid abelian categories

\[
P_{G(\mathcal{O})} \left( Gr_{\mathcal{T}_{\mathcal{O}}}, K \right) \cong \text{Rep} \left( \hat{G}_K \right),
\]

where \( Gr_{\mathcal{T}_{\mathcal{O}}} \) is the affine Grassmannian over \( \mathcal{T}_{\mathcal{O}} \), \( \mathcal{O} \) is the ring of power series \( \mathcal{T}_{\mathcal{O}}[[x]] \), \( P_{G(\mathcal{O})} \left( Gr_{\mathcal{T}_{\mathcal{O}}}, K \right) \) is the category of \( G(\mathcal{O}) \)-equivariant perverse sheaves on \( Gr_{\mathcal{T}_{\mathcal{O}}} \) with coefficients in \( K \) and \( \text{Rep} \left( \hat{G}_K \right) \) is the category of finite-dimensional representations of \( \hat{G}_K \). Since finite-dimensional representations of \( \hat{G}_K \) are equivalent to finite-dimensional comodules over \( K \left[ \hat{G}_K \right] \), one can reformulate this equivalence as

\[
P_{G(\mathcal{O})} \left( Gr_{\mathcal{T}_{\mathcal{O}}}, K \right) \cong \text{Comod}_{\text{Vect}_K^{fd}} - K \left[ \hat{G}_K \right].
\]

Let \( K \{ \hat{G}_K \} \) be the affinoid Hopf \( \hat{\mathcal{O}} \)-algebra, corresponding to the generic fiber of the formal completion of \( \hat{G}_K \) at identity (equivalently, algebra of affinoid functions on the maximal open compact subgroup the locally analytic group \( \mathbb{G} \) of \( K \)-points of \( \hat{G}_K \)). We would like to extend the above correspondence to equivalence

\[
P_{G(\mathcal{O})} \left( Gr_{\mathcal{T}_{\mathcal{O}}}, K \right) \cong \text{Comod}_{\text{Ban}_K^{fd}} - K \{ \hat{G}_K \}.
\]

Since any finite-dimensional comodule over \( K \{ \hat{G}_K \} \) is, in fact, algebraic comodule over \( K \left[ \hat{G}_K \right] \), the upper equivalence is the trivial consequence of the geometric Satake correspondence. We will show how to reconstruct \( K \{ \hat{G}_K \} \) as bounded coend of the “fiber functor”.

In order to do so, we compose the equivalence \( T : P_{G(\mathcal{O})} \left( Gr_{\mathcal{T}_{\mathcal{O}}}, K \right) \to \text{Comod}_{\text{Vect}_K^{fd}} - K \left[ \hat{G}_K \right] \) with the functor \( \mathbb{F} : \text{Comod}_{\text{Vect}_K^{fd}} - K \left[ \hat{G}_K \right] \to \text{Ban}_K \). To define \( \mathbb{F} \) we recall that

- simple \( K \left[ \hat{G}_K \right] \)-comodules (i.e. irreducible representations of \( \hat{G}_K \)) are finite-dimensional and parametrized by dominant integral weights \( \lambda \in P^* \);
- the algebra \( K \left[ \hat{G}_K \right] = \bigoplus_{\lambda \in P^*} m_{\lambda} L(\lambda) \) as a \( K \)-vector space is isomorphic to the direct sum of irreducible representations \( L(\lambda) \) with multiplicities \( m_{\lambda} \);
- each \( K \left[ \hat{G}_K \right] \)-comodule \( V \simeq \bigoplus_{\lambda \in P^*} m_{V,\lambda} L(\lambda) \) is semi-simple.

Since \( K \left[ \hat{G}_K \right] \) is dense in \( K \{ \hat{G}_K \} \), \( K \{ \hat{G}_K \} \) is equal to the Banach direct sum of \( m_{\lambda} L(\lambda) \) and each \( L(\lambda) \) has the norm \( \| \cdot \|_{L(\lambda)} \), induced by the norm on \( K \{ \hat{G}_K \} \). Thus for every \( V \simeq \bigoplus_{\lambda \in P^*} m_{V,\lambda} L(\lambda) \) we have the direct sum norm \( \| \cdot \|_V \) on \( V \): for
\( v \in V, \sum v_\lambda, v_\lambda \in L(\lambda) \), we set \( \|v\|_V := \max \|v_\lambda\|_{L(\lambda)} \), and our functor \( F \) maps \( V \) to \( (V, \|\cdot\|_V) \).

The functor \( F \circ T : P_{G(\mathcal{O})}(Gr_{\mathcal{Q}_p}, K) \to \text{Ban}_K \) satisfies the assumption of our recognition theorem and \( B\text{coend} (F \circ T) = K \{G_K\} \).

**Appendix**

**Comodule structure on** \( \text{cohom}_C(X,Y) \). In applications reconstructing bialgebra structure might be sufficient due to the uniqueness of the antipode. However in the recognition theorem one would like to have conditions, which define Hopf algebra structure on \( \text{coend}_C \). Our conjecture is that one also can reconstruct Hopf algebra structure under assumptions weaker than rigidity. Currently we cannot prove it in full generality and this is the reason why in the title of this paper we only put “reconstruction for bialgebras”.

Suppose we have a Hopf algebra \( H \) in a category \( \mathcal{C} \) and \( \text{Comod}_{\mathcal{C}_0} - H \) be the category right \( H \)-comodules, which are the objects of the rigid subcategory \( \mathcal{C}_0 \subset \mathcal{C} \). Then for every \( X \in \text{Comod}_{\mathcal{C}_0} - H \) we have a comodule structure on \( X^* \) via the antipode of \( H \) and this gives the rigid structure on \( \text{Comod}_{\mathcal{C}_0} - H \). Thus the category \( \text{Comod}_{\mathcal{C}_0} - H \) is coclosed and for any \( X,Y \in \text{Comod}_{\mathcal{C}_0} - H \) we have an equality in \( \mathcal{C}_0 \)

\[
\text{coend}_{\text{Comod}_{\mathcal{C}_0} - H}(X,Y) = \text{coend}_{\mathcal{C}_0}(X,Y) = Y^* \otimes X.
\]

We anticipate that the first part of this equality holds for general coclosed categories. Here we only explain how to give \( \text{cohom}_C(X,Y) \) a comodule structure.

Let now \( \mathcal{C}_0 \subset \mathcal{C} \) be a subcategory coclosed in \( \mathcal{C} \).

Let \( C \in \mathcal{C} \) be a coalgebra and \( X \in \mathcal{C}_0 \) is a right \( C \)-comodule. Then the coaction \( \rho_X : X \to X \otimes_C C \) induces the map

\[
\rho_{\text{cohom}_C(X,Y)} := \Delta_{X,Y} : \text{cohom}_C(X,Y) \to \text{cohom}_C(X,Y) \otimes_C C
\]

for every \( Y \in \mathcal{C}_0 \) via diagram

\[
\begin{array}{ccc}
X \otimes_C C & \overset{\text{coev}_{X,Y} \otimes_C \text{id}_C}{\longrightarrow} & Y \otimes_C \text{cohom}_C(X,Y) \\
\rho_X \downarrow & & \downarrow \text{id}_Y \otimes_C \rho_{\text{cohom}_C(X,Y)} \\
X & \overset{\text{coev}_{X,Y}}{\longrightarrow} & Y \otimes_C \text{cohom}_C(X,Y)
\end{array}
\]

One can check that \( \rho_{\text{cohom}_C(X,Y)} \) satisfy the axioms of the right \( C \)-comodule coaction. In a similar way the coaction \( \rho_Y : Y \to Y \otimes_C C \) induces the map

\[
\rho_{\text{cohom}_C(X,Y)} : \text{cohom}_C(X,Y) \to C \otimes_C \text{cohom}_C(X,Y)
\]

via diagram
One can check that $\tilde{\rho}_l^{\text{cohom}}(X,Y)$ satisfy the axioms of the left $C^\text{cop}$-comodule coaction. In case $C = H$ is a Hopf algebra in $\mathcal{C}$ we can turn $\tilde{\rho}_l^{\text{cohom}}(X,Y)$ into a right $H$-comodule coaction

$$\rho_l^{\text{cohom}}(X,Y) := \left(\text{id}_C \otimes_S H\right) \circ \tilde{\rho}_l^{\text{cohom}}(X,Y).$$

Combining $\rho_l^{\text{cohom}}(X,Y)$ and $\rho_r^{\text{cohom}}(X,Y)$ (similar to the tensor product of $X^*$ and $X$ in a rigid category), we define the map

$$\rho_{\text{cohom}}(X,Y) := \left(\text{id}_C \otimes_C m_H\right) \circ \left(\rho_l^{\text{cohom}}(X,Y) \otimes_C \text{id}_H\right) \circ \rho_r^{\text{cohom}}(X,Y)$$

which satisfy the axioms of the right $H$-comodule coaction. Under this comodule structure, the coevaluation map

$$\text{coev}_{X,Y} : X \to Y \otimes_C \text{cohom}\mathcal{C}(X,Y)$$

becomes a morphism of right $H$-comodules.

References

[DM] Deligne, P., Milne, J.S.: Tannakian categories. LNM 900 (1982).

[D] Deligne, P.: Categories Tannakiennes. Grothendieck Festschrift 1. Birkhäuser, 1990.

[EM] Emerton, M.: Locally analytic vectors in representations of locally analytic $p$-adic groups, Draft: September 19, 2011, available at www.math.uchicago.edu/~emerton.

[Kr] Krein, M.G.: A principle of duality for bicompact groups and quadratic block algebras. DAN SSSR 69 (1949), 725–728. (in Russian)

[L1] Lyubinin, A.: Nonarchimedean coalgebras and coadmissible modules. p-adic Numbers Ultrametric Anal. Appl. 6 (2014), no. 2, 105–134.

[L2] Lyubinin, A.: Geometric Satake correspondence for compact $p$-adic groups. in preparation.

[MAC] Mac Lane, S.: Categories for the working mathematician. Second edition. Graduate Texts in Mathematics, 5. Springer-Verlag, New York, 1998.

[MirV] Mirković, I.; Vilonen, K.: Geometric Langlands duality and representations of algebraic groups over commutative rings. Ann. of Math. (2) 166 (2007), no. 1, 95–143.

[NFA] Schneider, P.: Nonarchimedean functional analysis. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2002.

[PAR1] Pareigis, B.: Lectures on quantum groups and non-commutative geometry. Available at: http://www.mathematik.uni-muenchen.de/~pareigis/Vorlesungen/02SS/QGandNCG.pdf.

[PAR2] Pareigis, B.: Reconstruction of hidden symmetries. J. Algebra 183 (1996), no. 1, 90–154.

[PGS] Perez-Garcia, C.; Schikhof, W.: Locally convex spaces over non-Archimedean valued fields. Cambridge Studies in Advanced Mathematics, 119. Cambridge University Press, Cambridge, 2010.

[R1] Rosenberg, A. L.: Duality theorems for groups and Lie algebras. Russian Math. Surveys 26, 36 (1971), 253–254.

[R2] Rosenberg, A. L.: Duality and representations of groups. Proc. of XI Natnl. (USSR) Algebraic Colloquium (1971) (in Russian).

[R3] Rosenberg, A. L.: Reconstruction of groups. Selecta Math. (N.S.) 9 (2003), no. 1, 101–118.

[S] Saavedra, R.N.: Categories Tannakiennes. LNM 265 (1972).

[SCH] Schauenburg, P.: Tannaka duality for arbitrary Hopf algebras. Algebra Berichte [Algebra Reports], 66. Verlag Reinhard Fischer, Munich, 1992.
[Ta] Tannaka, T.: Über den Dualitätssatz der nichtkommutativen Gruppen. Tohoku Math. J. 45 (1938), no. 1, 1–12.

Department of Mathematics, School of Mathematical Sciences,
University of Science and Technology of China, Hefei, Anhui, PRC
E-mail address: anton@lyubinin.kiev.ua