Semiuniform semigroups and convolution

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Abstract
Semiuniform semigroups provide a natural setting for the convolution of generalized finite measures on semigroups. A semiuniform semigroup is said to be ambitable if each uniformly bounded uniformly equicontinuous set of functions on the semigroup is contained in an ambit. In the convolution algebras constructed over ambitable semigroups, topological centres have a tractable characterization.

1 Semiuniform products and semiuniform semigroups

Functional analysis on topological and semitopological semigroups is well developed [1]. However, concrete semigroups often carry not only a topology but also a natural uniform structure. That suggests that it is worthwhile to investigate semigroups endowed with compatible uniform structures.

An observation at the end of the author’s paper [10] points out that the main results in that paper, proved there for topological groups, hold more generally for semiuniform semigroups. Moreover, the definition and basic properties of ambitable topological groups [11], and their connection to topological centres in convolution algebras, are easily generalized to semiuniform semigroups. These generalizations are described in the current paper.

Several theorems below are obtained by simple modifications of the proofs for topological groups in the author’s previous papers [10][11]. The modified proofs are included here for the sake of completeness.

Most of the notation used here is defined in [10]. All uniform structures are assumed to be Hausdorff, and all linear spaces over the field \( \mathbb{R} \) of reals.

Following Isbell [5], we denote by \( X*Y \) the semiuniform product of uniform spaces \( X \) and \( Y \). By definition, a semigroup \( X \) with a uniform structure is a semiuniform semigroup if the semigroup operation \( (x,y) \mapsto xy \) is uniformly continuous from the semiuniform product \( X*X \) to \( X \). In other words, \( X \) is a semiuniform semigroup if and only if

- the set \( \{ x \mapsto xy \mid y \in X \} \) of mappings from \( X \) to \( X \) is uniformly equicontinuous; and
- for each \( x \in X \), the mapping \( y \mapsto xy \) from \( X \) to \( X \) is uniformly continuous.
Hindman and Strauss [4] used these two conditions in extending a semigroup operation to the uniform compactification.

A more precise term would be a right semiuniform semigroup, since we may also define a left semiuniform semigroup by requiring the mapping \((x, y) \mapsto yx\) to be uniformly continuous from \(X^*X\) to \(X\). However, in this paper we only study right semiuniform semigroups, and omit the qualifier right.

The class of semiuniform semigroups includes several familiar classes of semigroups with additional structure:

**Discrete semigroups.** Every semigroup with the discrete uniformity is a semiuniform semigroup.

**Topological groups.** Every topological group with its right uniformity is a semiuniform semigroup ([10], Lemma 4.1).

**Uniform semigroups.** Every uniform semigroup in the sense of Marxen [9] is a semiuniform semigroup.

**Semigroups of uniform endomorphisms.** Let \(Y\) be a uniform space, and let \(U(Y, Y)\) be the uniform space of uniformly continuous mappings from \(Y\) to itself, as defined by Isbell ([5], Ch. III). With the composition operation, \(U(Y, Y)\) is a semiuniform semigroup.

**Balls in normed algebras.** Let \(A\) be a normed algebra and \(r\) a real number, \(0 < r \leq 1\). Let \(B_r\) and \(\overline{B}_r\) be the open and the closed ball with diameter \(r\) in \(A\). Then \(B_r\) and \(\overline{B}_r\) with the algebra multiplication and the uniformity defined by the norm in \(A\) are semiuniform semigroups.

## 2 Convolution

As in [10], if \(X, Y\) and \(Z\) are sets and \(p\) is a mapping from \(X \times Y\) to \(Z\) then define \(\lambda_x p(x, y)\) to be the mapping \(x \mapsto p(x, y)\) from \(X\) to \(Z\) and \(\lambda_y p(x, y)\) to be the mapping \(y \mapsto p(x, y)\) from \(Y\) to \(Z\).

When \(X\) is a uniform space, denote by \(\mathcal{UP}(X)\) the set of all uniformly continuous pseudometrics on \(X\). Recall [10][11] that for a uniform space \(X\),

- \(\mathcal{U}_b(X)\) is the space of bounded uniformly continuous real-valued functions on \(X\) with the sup norm \(\|\cdot\|\);
- for a pseudometric \(\Delta\) on \(X\),
  \[
  \operatorname{Lip}(\Delta) = \{ f : X \to [-1, 1] \mid |f(x) - f(x')| \leq \Delta(x, x') \text{ for all } x, x' \in X \}
  \]
  \[
  \operatorname{Lip}^+(\Delta) = \{ f \in \operatorname{Lip}(\Delta) \mid f(x) \geq 0 \text{ for all } x \in X \}
  \]
  and both \(\operatorname{Lip}(\Delta)\) and \(\operatorname{Lip}^+(\Delta)\) are always considered with the topology of pointwise convergence on \(X\);
• \( M(X) \) is the norm dual of \( U_b(X) \);

• the \textit{weak* topology} on \( M(X) \) is the weak topology of the duality \( \langle M(X), U_b(X) \rangle \);

• the \textit{UEB topology} on \( M(X) \) is the topology of uniform convergence on the sets \( \text{Lip}(\Delta) \), where \( \Delta \) ranges over all uniformly continuous pseudometrics on \( X \); equivalently, it is the topology of uniform convergence on uniformly equicontinuous bounded subsets of \( U_b(X) \).

• the \textit{UEB uniformity} is the corresponding translation-invariant uniformity on \( M(X) \);

• when \( p \) is a uniformly continuous mapping from \( X \) to a uniform space \( Y \), the linear mapping \( M(p) : M(X) \to M(Y) \) is defined by \( M(p)(\mu)(f) = \mu(f \circ p) \) for \( f \in U_b(Y) \);

• the subspace \( M_u(X) \) of \( M(X) \) is defined as follows: \( \mu \in M_u(X) \) iff \( \mu \) is continuous on \( \text{Lip}(\Delta) \) for each \( \Delta \in U_P(X) \) (or, equivalently, \( \mu \) is continuous on \( \text{Lip}^+(\Delta) \) for each \( \Delta \in U_P(X) \));

• if \( X \) and \( Y \) are uniform spaces, \( \mu \in M(X) \), \( \nu \in M(Y) \), then the \textit{direct product} \( \mu \otimes \nu \) is an element of \( M(X \times Y) \) defined by \( \mu \otimes \nu(f) = \mu(\{ x \nu(f(x, y)) \}) \) for \( f \in U_b(X \times Y) \);

• for \( x \in X \), \( \delta_x \in M(X) \) is defined by \( \delta_x(f) = f(x) \) for \( f \in U_b(X) \);

• \( \textbf{Mol}(X) \) is the linear subspace of \( M(X) \) generated by the set \( \{ \delta_x \mid x \in X \} \);

• the mapping \( \delta : x \mapsto \delta_x \) is a topological embedding of \( X \) to \( M(X) \) with the weak* topology;

• \( \overline{X} \), the weak* closure of \( \delta(X) \) in \( M(X) \), is a \textit{uniform compactification of} \( X \);

• \( \widehat{X} = \overline{X} \cap M_u(X) \) with the UEB uniformity is a completion of \( X \);

• we define

\[
\begin{align*}
U_b^+(X) &= \{ f \in U_b(X) \mid f(x) \geq 0 \text{ for } x \in X \} \\
M^+(X) &= \{ \mu \in M(X) \mid \mu(f) \geq 0 \text{ for } f \in U_b(X) \}
\end{align*}
\]

and similarly for \( \textbf{Mol}^+(X) \) and \( M_u^+(X) \).

Let \( X \) be a semiuniform semigroup and \( \mu, \nu \in M(X) \). Then \( \mu \otimes \nu \in M(X \times X) \). Since the mapping \( m : (x,y) \mapsto xy \) is uniformly continuous from \( X \times X \) to \( X \), the image \( M(m)(\mu \otimes \nu) \) of \( \mu \otimes \nu \) is well defined as an element of \( M(X) \). We denote \( \mu \star \nu = M(m)(\mu \otimes \nu) \) and call \( \mu \star \nu \) the \textit{convolution} of \( \mu \) and \( \nu \). Expanding the definition, we obtain

\[
\mu \star \nu(f) = \mu(\{ x \nu(f(x,y)) \})
\]

for \( \mu, \nu \in M(X) \), \( f \in U_b(X) \). Thus \( \mu \star \nu \) is the \textit{evolution} of \( \mu \) and \( \nu \) in the terminology of Pym \cite{Pym1973}\cite{Pym1974}. On \( \overline{X} \), the convolution \( \star \) coincides with the operation defined by Hindman and Strauss \cite{Hindman1998}, 21.43.

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More generally, let $X$ be a semigroup acting semiuniformly on a uniform space $Y$. By that we mean that $X$ is a semigroup endowed with a uniformity and there is a uniformly continuous mapping $m : X \times Y \to Y$ such that $m(s, m(s', y)) = m(ss', y)$ for all $s, s' \in X$, $y \in Y$. In that case, define the convolution operation from $M(X) \times M(Y)$ to $M(Y)$ by $\mu \star \nu = M(m)(\mu \otimes \nu)$ for $\mu \in M(X)$ and $\nu \in M(Y)$. The definition of $\mu \star \nu$ for a semigroup $X$ and $\mu, \nu \in M(X)$ is obtained as a special case for $X = Y$.

The proofs in this section are completely analogous to those in section 4 in [10], where the same results are proved for topological groups.

**Theorem 2.1** Let $X$ be a semiuniform semigroup acting semiuniformly on a uniform space $Y$. Let $\mu, \mu' \in M(X)$, $\nu \in M(Y)$. Then $(\mu \star \mu') \star \nu = \mu \star (\mu' \star \nu)$.

**Proof.** Apply Lemma 3.7 in [10].

**Theorem 2.2** Let $X$ be a semigroup acting semiuniformly on a uniform space $Y$. Let $r \in \mathbb{R}$, $\mu, \mu' \in M(X)$, $\nu, \nu' \in M(Y)$. Then

\[
\begin{align*}
(r\mu) \star \nu &= \mu \star (r\nu) = r(\mu \star \nu) \\
(\mu + \mu') \star \nu &= (\mu \star \nu) + (\mu' \star \nu) \\
\mu \star (\nu + \nu') &= (\mu \star \nu) + (\mu \star \nu')
\end{align*}
\]

**Proof.** Apply Lemma 3.6 in [10].

**Theorem 2.3** Let $X$ be a semigroup acting semiuniformly on a uniform space $Y$. Let $\Phi$ be one of the functors $M^+$, Mol, Mol$, M_u$, $M_u^+$, $\chi_X \overline{X}$ and $\chi_X X$. If $\mu \in \Phi(X)$ and $\nu \in \Phi(Y)$ then $\mu \star \nu \in \Phi(Y)$.

**Proof.** Apply Lemma 3.5, Theorem 3.13 and Lemma 6.1 in [10].

**Theorem 2.4** Let $X$ be a semigroup acting semiuniformly on a uniform space $Y$.

1. $\|\mu \star \nu\| \leq \|\mu\| \cdot \|\nu\|$ for any $\mu \in M(X)$, $\nu \in M(Y)$.

2. Let $B \subseteq M(Y)$ be a set bounded in the $\|\|\|$ norm, $\mu_0 \in M_u(X)$, and $\nu_0 \in B$. When $M(X)$ and $B$ are endowed with their UEB topology, the mapping $(\mu, \nu) \mapsto \mu \star \nu$ from $M(X) \times B$ to $M(Y)$ is jointly continuous at $(\mu_0, \nu_0)$.

3. The mapping $(\mu, \nu) \mapsto \mu \star \nu$ is jointly sequentially continuous from $M_u(X) \times M_u(Y)$ to $M_u(Y)$ when $M_u(X)$ and $M_u(Y)$ are endowed with their weak* topology.

4. Let $\mu_0 \in M_u^+(X)$ and $\nu_0 \in M^+(Y)$. When $M^+(X)$ and $M^+(Y)$ are endowed with their weak* topology, the mapping $(\mu, \nu) \mapsto \mu \star \nu$ from $M^+(X) \times M^+(Y)$ to $M^+(Y)$ is jointly continuous at $(\mu_0, \nu_0)$.
5. If \( \mu \in \mathcal{M}_u(X) \) then the mapping \( \nu \mapsto \mu \star \nu \) from \( \mathcal{M}(Y) \) to itself is weak* continuous.

**Proof.** All statements follow from results in section 3 of [10]: Part 1 from Lemma 3.4, Part 2 from Theorem 3.8, part 3 from Corollary 3.9, part 4 from Theorem 3.10 and part 5 from Theorem 3.12. \( \square \)

As a corollary we obtain that for any semiuniform semigroup \( X \) the spaces \( \mathcal{M}(X) \) and \( \mathcal{M}_u(X) \) with the operations \( \star \) and + and the norm \( \| \cdot \| \) are Banach algebras.

For the special case of topological groups, it is noted in section 4 of [10] that algebraic identities are inherited from \( \text{Mol}(X) \) to \( \mathcal{M}_u(X) \). In view of Theorem 2.4, the same is true for semiuniform semigroups. In particular, if a semiuniform semigroup \( X \) is commutative then so is \( \mathcal{M}_u(X) \).

### 3 Ambitable semigroups and topological centres

The following definitions and questions are a straightforward generalization of those for ambitable topological groups [11].

Let \( X \) be a semigroup, \( f \) a real-valued function on \( X \) and \( x \in X \). The *right translation* of \( f \) by \( x \) is the function \( \rho^x(f) = z \mapsto f(zx) \). The set \( \text{orb}(f) = \{ \rho^x(f) \mid x \in X \} \) is the *right orbit* of \( f \), and \( \text{orb}(f) \) is the closure of \( \text{orb}(f) \) in the product space \( \mathbb{R}^X \).

Say that a semiuniform semigroup \( X \) is *ambitable* if for every \( \Delta \in \text{UP}(X) \) there exists \( f \in \text{U}_{\mathbb{R}}(X) \) such that \( \text{Lip}^+(\Delta) \subseteq \text{orb}(f) \).

**Question 1** Which semiuniform semigroups are ambitable?

This question is motivated by Questions 2 and 3 below, in the same way as for topological groups [11].

If \( S \) is a semigroup with a topology, define its *(right) topological centre* by

\[
\Lambda(S) = \{ x \in S \mid \text{the mapping } y \mapsto xy \text{ is continuous on } S \}.
\]

Now let \( X \) be a semiuniform semigroup. Consider the semigroup \( \mathcal{M}(X) \) with the convolution operation \( \star \) and the weak* topology, and its subsemigroup \( \overline{X} \) (uniform compactification of \( X \)). By part 3 of Theorem 2.4, \( \mathcal{M}_u(X) \subseteq \Lambda(\mathcal{M}(X)) \), and therefore also \( \hat{X} \subseteq \Lambda(\overline{X}) \).

The results of Lau [6], Lau and Pym [7], Lashkarizadeh Bami [8], Ferri and Neufang [3], and Dales, Lau and Strauss [2], among others, lead to the following questions.

**Question 2** Which semiuniform semigroups \( X \) satisfy \( \mathcal{M}_u(X) = \Lambda(\mathcal{M}(X)) \)?

**Question 3** Which semiuniform semigroups \( X \) satisfy \( \hat{X} = \Lambda(\overline{X}) \)?

If \( X \) is precompact then \( \mathcal{M}_u(X) = \mathcal{M}(X) \) and \( \hat{X} = \overline{X} \), and therefore \( \mathcal{M}_u(X) = \Lambda(\mathcal{M}(X)) \) and \( \hat{X} = \Lambda(\overline{X}) \).

Next we repeat the proofs in section 5 of [11] to derive positive answers to Questions 2 and 3 for ambitable semigroups.
Lemma 3.1 (cf. 5.1 in [11]) Let $X$ be a semiuniform semigroup and $f \in U_b(X)$.

1. The mapping $\varphi : \nu \mapsto \chi \nu(\chi f(xy))$ is continuous from $X$ to the product space $\mathbb{R}^X$.

2. $\varphi(X) = \text{orb}(f)$.

Proof. 1. As noted above, $\delta_x \in \Lambda(X)$ for each $x \in X$, and thus the mapping $\nu \mapsto \delta_x * \nu$ is weak* continuous from $X$ to itself. Since $\delta_x * \nu(f) = \nu(\chi f(xy))$, this means that the mapping $\nu \mapsto \chi \nu(\chi f(xy))$ from $X$ to $\mathbb{R}$ is continuous for each $x \in X$, and therefore the mapping $\nu \mapsto \chi \nu(\chi f(xy))$ is continuous from $X$ to $\mathbb{R}^X$.

2. $\varphi(\delta_x) = \rho^x(f)$ for all $x \in X$, and therefore $\varphi(\delta(X)) = \text{orb}(f)$. The mapping $\varphi$ is continuous by part 1, $X$ is compact, and $\delta(X)$ is dense in $X$. It follows that $\varphi(X) = \text{orb}(f)$. \qed

Lemma 3.2 (cf. 5.2 in [11]) Let $X$ be any semiuniform semigroup, $\mu \in M(X)$ and $f \in U_b(X)$. If the mapping $\nu \mapsto \mu * \nu$ from $X$ to $M(X)$ is weak* continuous then $\mu$ is continuous on $\text{orb}(f)$.

Proof. As in Lemma 3.1 define $\varphi(\nu) = \chi \nu(\chi f(xy))$ for $\nu \in X$.

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi} & \text{orb}(f) \\
\nu \mapsto \mu * \nu(f) & \downarrow & \mu \\
\downarrow & & \downarrow \\
\mathbb{R} & & \mathbb{R}
\end{array}
$$

By the definition of convolution, $\mu * \nu(f) = \mu(\chi \nu(\chi f(xy))) = \mu(\varphi(\nu))$. Thus $\mu \circ \varphi$ is continuous from $X$ to $\mathbb{R}$.

By Lemma 3.1 $\varphi$ is continuous from $X$ to $\text{orb}(f)$, and $\varphi(X) = \text{orb}(f)$. Since $X$ is compact, it follows that $\mu$ is continuous on $\text{orb}(f)$. \qed

Theorem 3.3 (cf. 5.3 in [11]) If $X$ is an ambitable semiuniform semigroup, $S \subseteq M(X)$, and $S$ with the $*$ operation is a semigroup such that $\overline{X} \subseteq S$, then $\Lambda(S) = M_u(X) \cap S$.

Proof. As is noted above, $M_u(X) \subseteq \Lambda(M(X))$. Therefore $M_u(X) \cap S \subseteq \Lambda(S)$ for every semigroup $S \subseteq M(X)$.

To prove the opposite inclusion, take any $\mu \in \Lambda(S)$ and any $\Delta \in \mathcal{U}P(X)$. Since $\overline{X} \subseteq S$, the mapping $\nu \mapsto \mu * \nu$ from $X$ to $M(X)$ is weak* continuous by the definition of $\Lambda(S)$. Since $X$ is ambitable, $\text{Lip}^+(\Delta) \subseteq \text{orb}(f)$ for some $f \in U_b(X)$. By Lemma 3.2 $\mu$ is continuous on $\text{orb}(f)$ and therefore also on $\text{Lip}^+(\Delta)$. Thus $\mu \in M_u(X)$. \qed

Corollary 3.4 (cf. 5.4 in [11]) If $X$ is an ambitable semiuniform semigroup then $M_u(X) = \Lambda(M(X))$ and $\overline{X} = \Lambda(X)$.

Proof. Apply 3.3 with $S = M(X)$ and with $S = \overline{X}$. \qed
4 Discrete semigroups

If \( X \) is a discrete uniform space then \( \mathcal{U}_b(X) = \ell_\infty(X) \), \( \hat{X} = X \), \( \overline{X} = \beta X \) is the Stone-Čech compactification of \( X \), and \( M_u(X) = \ell_1(X) \). For discrete semigroups, Questions 2 and 3 of the previous section have been extensively studied, often in the following equivalent form:

- For which semigroups \( X \) is \( \ell_1(X) \) left strongly Arens irregular? (\(^2\), 2.24)
- Which semigroups are left strongly Arens irregular? (\(^2\), 6.11)

Dales, Lau and Strauss \(^2\) give an example of a countable infinite abelian semigroup \( X \) for which \( \ell_1(X) \) is not left strongly Arens irregular but \( X \) is (loc. cit., 12.21). In other words, \( M_u(X) \neq \Lambda(M(X)) \) and \( \hat{X} = \Lambda(X) \). By Corollary 3.4, such \( X \) is not ambitable.

This section offers a partial answer to Question 1 for discrete semigroups. The approach is similar to that in section 4 of \(^{11}\), which in turn is a variant of the factorization method in \(^3\).

When \( P \) and \( R \) are subsets of a semigroup \( X \), write \( R^{-1}P = \{ x \in X \mid rx \in P \text{ for some } r \in R \} \).

For any semigroup \( X \), define the metric \( \Delta_X \) on \( X \) by \( \Delta_X(x, y) = 1 \) for \( x \neq y \). Then \( \text{Lip}^+(\Delta_X) = [0, 1]^X \), and \( \text{Lip}^+(\Delta_X) \supseteq \text{Lip}^+(\Delta) \) for any pseudometric \( \Delta \) on \( X \).

The cardinality of a set \( A \) is denoted \( |A| \). When \( X \) is an infinite semigroup, consider the following two properties:

1. If \( F \subseteq X \) is finite then \( |\{ z \in X \mid xz \neq yz \text{ for all } x, y \in F, x \neq y \}| = |X| \).
2. If \( P \subseteq X \), \( |P| < |X| \) and \( x \in X \) then \( |\{ x \}^{-1}P| < |X| \).

Note that (1) holds in any infinite right cancellative semigroup (in fact in any infinite near right cancellative semigroup as defined in \(^2\)), and (2) holds in any infinite weakly left cancellative semigroup.

The following equivalent form of (2) will be used below:

(2a) If \( F, P \subseteq X \), \( F \) is finite and \( |P| < |X| \) then \( |F^{-1}P| < |X| \).

Lemma 4.1 Let \( X \) be an infinite semigroup satisfying (1) and (2). Let \( A \) be a set such that \( |A| \leq |X| \), and for each \( \alpha \in A \) let \( F_\alpha \) be a non-empty finite subset of \( X \). Then there exist elements \( x_\alpha \in X \) for \( \alpha \in A \) such that

(i) the mapping \( x \mapsto xx_\alpha \) is injective on \( F_\alpha \) for every \( \alpha \in A \);

(ii) \( F_\alpha x_\alpha \cap F_\beta x_\beta = \emptyset \) for all \( \alpha, \beta \in A \), \( \alpha \neq \beta \).

Proof. Without loss of generality, assume that \( A \) is the set of ordinals of cardinality \( |X| \). The construction of \( x_\alpha \) proceeds by transfinite induction. For \( \gamma \in A \), let \( S(\gamma) \) be the statement “for all \( \alpha \leq \gamma \) there exist elements \( x_\alpha \in X \) such that (i) and (ii) hold for \( \alpha, \beta \leq \gamma \).”
To prove $S(0)$, observe that (1) implies

\[ | \{ z \in X \mid xz \neq yz \text{ for all } x, y \in F_0, x \neq y \} | = |X| \]

and therefore there is $x_0 \in X$ such that the mapping $x \mapsto xx_0$ is injective on $F_0$.

Now assume that $\gamma \in A$ and $S(\gamma')$ is true for all $\gamma' < \gamma$. We want to prove $S(\gamma)$.

As in the base step, we get

\[ | \{ z \in X \mid xz \neq yz \text{ for all } x, y \in F_{\gamma}, x \neq y \} | = |X| \]

Let $P = \bigcup_{\alpha < \gamma} F_{\alpha}x_{\alpha}$. Since $|P| < |X|$, from (2a) we get $|F_{\gamma}^{-1}P| < |X|$. Thus there exists $x_{\gamma} \in X \setminus F_{\gamma}^{-1}P$ such that the mapping $x \mapsto xx_{\gamma}$ is injective on $F_{\gamma}$. Since $x_{\gamma} \notin F_{\gamma}^{-1}P$, we have $F_{\alpha}x_{\alpha} \cap F_{\gamma}x_{\gamma} = \emptyset$ for all $\alpha < \gamma$. □

**Theorem 4.2** Every infinite discrete semigroup $X$ satisfying (1) and (2) is ambitable.

Thus, in particular, every infinite discrete group is ambitable. That follows also from the general results in [11].

**Proof.** The topology of the compact space $\text{Lip}^+(\Delta X) = [0, 1]^X$ has an open basis $O$ such that $|O| = |X|$ and every set $U \in O$ is a basic neighbourhood of the form

\[ U = \{ f \in \text{Lip}^+(\Delta X) \mid |f(x) - h_U(x)| < \varepsilon_U \text{ for } x \in F_U \} \]

where $F_U \subseteq X$ is a finite set, $h_U \in \text{Lip}^+(\Delta X)$, and $\varepsilon_U > 0$.

By Lemma [4.4] with $O$ in place of $A$, there are $x_U \in X$ for $U \in O$ such that

(i) the mapping $x \mapsto xx_U$ is injective on $F_U$ for every $U \in O$;

(ii) $F_Ux_U \cap F_Vx_V = \emptyset$ for all $U, V \in O$, $U \neq V$.

Define the function $f : X \to [0, 1]$ by

\[ f(y) = \begin{cases} h_U(x) & \text{if } y = xx_U, x \in F_U \\ 0 & \text{if } y \notin \bigcup_{U \in O} F_Ux_U \end{cases} \]

In view (i) and (ii), $f$ is well defined. Since $\rho^{xy}(f)(x) = h_U(x)$ for $x \in F_U$, orb($f$) intersects every set in $O$. Thus orb($f$) is dense in $\text{Lip}^+(\Delta X) = [0, 1]^X$. □

By Corollary [3.3] we obtain the following partial answers to Questions 2 and 3.

**Corollary 4.3** If $X$ is an infinite semigroup satisfying (1) and (2) then $\ell^1(X) = \Lambda(\ell^\infty(X))$ and $X = \Lambda(X)$. In other words, if $X$ is an infinite semigroup satisfying (1) and (2) then $\ell^1(X)$ and $X$ are left strongly Arens irregular. This is a variant of Corollary 12.16 in [2].
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