EXPONENTIAL FRAMES ON UNBOUNDED SETS

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Abstract. For every set $S$ of finite measure in $\mathbb{R}$ we construct a discrete set of real frequencies $\Lambda$ such that the exponential system $\{\exp(i\lambda t), \lambda \in \Lambda\}$ is a frame in $L^2(S)$.

1. Introduction

This note can be viewed as a continuation of our previous paper [NOU]. In [NOU] we constructed "good" sampling sets for the Paley–Wiener spaces $PW_S$ of entire $L^2(\mathbb{R})$–functions with bounded spectrum $S$ in $\mathbb{R}$. This construction is based on a result in [BSS] on existence of well-invertible sub-matrices of large orthogonal matrices. Recently, an important progress in the latter area has been made in [MSS]. Based on this, we prove existence of exponential frames in $L^2(S)$, for every unbounded set $S$ in $\mathbb{R}$ of finite measure.

Recall that a system of vectors $E = \{u_j\}$ is a frame in a Hilbert space $H$ if there are positive constants $a, A$ such that

$$a \|h\|^2 \leq \sum_{u_j \in E} |\langle h, u_j \rangle|^2 \leq A \|h\|^2 \quad \forall h \in H.$$ 

The numbers $a$ and $A$ above are called frame bounds.

Given a discrete set $\Lambda$ in $\mathbb{R}$, we denote by

$$E(\Lambda) := \{e^{i\lambda t}\}_{\lambda \in \Lambda}$$

the system of exponentials with frequencies in $\Lambda$.

Exponential frames $E(\Lambda)$ in $L^2(S)$ (equivalently, stable sampling sets $\Lambda$ for $PW_S$) have been carefully studied from different points of view. There is a large number of results in the area. In the classical case when $S$ is an interval, such systems were essentially characterized by Beurling [B] in terms of the so-called "lower uniform density" of $\Lambda$. A complete description of exponential frames for intervals is given by Ortega–Cerdà and Seip [OS]. However, the problem of existence of exponential frames for unbounded sets remained open. The following result fills this gap by showing that for every set $S$ of finite measure, the space $L^2(S)$ admits an exponential frame:

**Theorem 1.** There are positive constants $c, C$ such that for every set $S \subset \mathbb{R}$ of finite measure there is a discrete set $\Lambda \subset \mathbb{R}$ such that $E(\Lambda)$ is a frame in $L^2(S)$ with frame bounds $c|S|$ and $C|S|$.

Here by $|S|$ we denote the measure of $S$. 

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Remark 1. Theorem 1 and its proof hold also in higher dimensions (e.g. in \( \mathbb{R}^d \) for any positive integer \( d \)). We present the theorem in dimension one in order to simplify notation.

Remark 2. The frame bounds are essential in many contexts, since they characterize the “quality” of frame decompositions. Assume that an exponential system \( E(\Lambda) \) forms an orthogonal basis in \( L^2(S) \). One can easily check that in this case \( E(\Lambda) \) is a frame in \( L^2(S) \) with frame bounds \( a = A = |S| \). These are, in a sense, the “optimal” frame bounds. In general, there may be no exponential orthogonal basis in \( L^2(S) \). However, Theorem 1 shows that an exponential frame in \( L^2(S) \) always exists with “almost” (up to fixed multiplicative constants) optimal frame bounds.

Remark 3. A result similar to Theorem 1 regarding the existence of complete exponential systems \( E(\Lambda) \) in \( L^2(S) \) (equivalently, existence of uniqueness sets \( \Lambda \) for \( PW_S \)) is obtained in \([OU]\) by an effective direct construction. That is not the case here, since the proof of Theorem A below in \([MSS]\) involves stochastic elements.

Remark 4. Assume that \( S \) lies on an interval of length \( 2\pi d, d > 0 \). It follows from Lemma 10 below that a set \( \Lambda \) satisfying the conclusion of Theorem 1 can be chosen satisfying \( \Lambda \subset (1/d)\mathbb{Z} \).

Remark 5. Assume that \( \Lambda \) satisfies the conclusions of Theorem 1. Then there are two absolute constants \( k, K \) such that the inequalities

\[
k|S| < \frac{|\Lambda \cap \Omega|}{|\Omega|} < K|S|
\]

hold whenever \( \Omega \) is a sufficiently long interval in \( \mathbb{R} \). In fact, one can choose any numbers \( k < 1/2\pi \) and \( K > 4C \), where \( C \) is the constant in Theorem 1. Then, as it was shown by Landau \([L]\) (for a more elementary proof see \([NO]\)), the left hand side inequality above follows from the frame property of \( E(\Lambda) \). The right hand side inequality follows from Lemma 6 (ii) below.

2. Well-invertible sub-matrices

Our construction is based on the following result by Marcus, Spielman and Srivastava from \([MSS]\):

**Theorem A.** Let \( \epsilon > 0 \), and \( u_1, \ldots, u_m \in \mathbb{C}^n \) such that \( \|u_i\|^2 \leq \epsilon \) for all \( i = 1, \ldots, m \), and

\[
\sum_{i=1}^{m} |\langle w, u_i \rangle|^2 = \|w\|^2 \quad \forall w \in \mathbb{C}^n.
\]

Then there exists a partition of \( \{1, \ldots, m\} \) into \( S_1 \) and \( S_2 \), such that for each \( j = 1, 2 \),

\[
\sum_{i \in S_j} |\langle w, u_i \rangle|^2 \leq \frac{(1 + \sqrt{2}\epsilon)^2}{2} \|w\|^2 \quad \forall w \in \mathbb{C}^n.
\]

Observe that, clearly, \((1 + \sqrt{2}\epsilon)^2 \leq 1 + 5\sqrt{\epsilon}\) when \( \epsilon < 1 \).

**Remark 6.** Let \( \epsilon < 1 \). Since

\[
\sum_{i \in S_1} |\langle w, u_i \rangle|^2 = \|w\|^2 - \sum_{i \in S_2} |\langle w, u_i \rangle|^2,
\]

we have
estimate (1) shows that the two-sided estimate holds for each \( j = 1, 2 \):
\[
\frac{1 - 5\sqrt{\epsilon}}{2} \|w\|^2 \leq \sum_{i \in S_j} |\langle w, u_i \rangle|^2 \leq \frac{1 + 5\sqrt{\epsilon}}{2} \|w\|^2 \quad \forall w \in \mathbb{C}^n.
\]

The following corollary (see Corollary F.2 in [HO]) gives a reformulation of Theorem A in a form well prepared for an induction process:

**Corollary B.** Let \( v_1, \ldots, v_k \in \mathbb{C}^n \) be such that \( \|v_i\|^2 \leq \delta \) for all \( i = 1, \ldots, k \). If
\[
\alpha \|w\|^2 \leq \sum_{i=1}^{k} |\langle w, v_i \rangle|^2 \leq \beta \|w\|^2 \quad \forall w \in \mathbb{C}^n,
\]
with some numbers \( \alpha > \delta \) and \( \beta \), then there exists a partition of \( \{1, \ldots, k\} \) into \( S_1 \) and \( S_2 \) such that for each \( j = 1, 2 \),
\[
\frac{1 - 5\sqrt{\delta/\alpha}}{2} \|w\|^2 \leq \sum_{i \in S_j} |\langle w, v_i \rangle|^2 \leq \frac{1 + 5\sqrt{\delta/\alpha}}{2} \beta \|w\|^2 \quad \forall w \in \mathbb{C}^n.
\]

For the sake of completeness, we reproduce the proof.

Let \( M : \mathbb{C}^n \to \mathbb{C}^n \) be the operator defined by \( Mw = \sum_{i=1}^{k} \langle w, v_i \rangle v_i \). Observe that \( M \) is positive and that
\[
\alpha \|w\|^2 \leq \|M^{1/2}w\|^2 \leq \beta \|w\|^2 \quad \forall w \in \mathbb{C}^n.
\]

Set \( u_i = M^{-1/2}v_i \). Then \( \|u_i\|^2 \leq \|v_i\|^2/\alpha \leq \delta/\alpha \). Further, for all \( w \in \mathbb{C}^n \),
\[
\sum_{i=1}^{k} \langle w, u_i \rangle u_i = M^{-1/2} \sum_{i=1}^{k} \langle M^{-1/2}w, v_i \rangle v_i = M^{-1/2} MM^{-1/2} w = w.
\]

We see that \( u_i \) satisfy the assumptions of Theorem A with \( m = k \) and \( \epsilon = \delta/\alpha < 1 \). Hence, there is a partition of \( \{1, \ldots, k\} \) into two sets \( S_1 \) and \( S_2 \) satisfying [2]. Using the right hand side of (2) we get
\[
\sum_{i \in S_j} |\langle w, v_i \rangle|^2 = \sum_{i \in S_j} |\langle M^{1/2}w, u_i \rangle|^2 \leq \frac{1 + 5\sqrt{\epsilon}}{2} \|M^{1/2}w\|^2
\]
\[
\leq \frac{1 + 5\sqrt{\delta/\alpha}}{2} \beta \|w\|^2 = \frac{1 + 5\sqrt{\delta/\alpha}}{2} \beta \|w\|^2.
\]

The proof of the left hand side of (3) is similar.

We will use an elementary lemma:

**Lemma 1.** Let \( 0 < \delta < 1/100 \), and let \( \alpha_j, \beta_j, j = 0, 1, \ldots, \) be defined inductively
\[
\alpha_0 = \beta_0 = 1, \quad \alpha_{j+1} := \alpha_j \frac{1 - 5\sqrt{\delta/\alpha_j}}{2}, \quad \beta_{j+1} := \beta_j \frac{1 + 5\sqrt{\delta/\alpha_j}}{2}.
\]
Then there exist a positive absolute constant \( C \) and a number \( L \in \mathbb{N} \) such that
\[
\alpha_j \geq 100\delta \quad \text{for } j \leq L,
\]
and
\[
25\delta \leq \alpha_{L+1} < 100\delta, \quad \beta_{L+1} < C\alpha_{L+1}.
\]
Proof. Clearly, if \( a_j \geq 100\delta \), then
\[
\frac{\alpha_j}{4} \leq \alpha_{j+1} < \frac{\alpha_j}{2}.
\]

Denote by \( L \geq 1 \) the greatest number such that \( a_L \geq 100\delta \), and set \( \gamma_j := \frac{5\sqrt{\delta/a_j}}{2} \), \( j \leq L \). Then \( \gamma_{L-j} < 2^{-1-j/2} \). It follows that
\[
\prod_{j=0}^{L} \frac{1 + \gamma_j}{1 - \gamma_j} < C := \prod_{j=0}^{\infty} \frac{1 + 2^{-1-j/2}}{1 - 2^{-1-j/2}}.
\]
This gives \( b_{L+1} < Ca_{L+1} \), and the lemma follows. \( \square \)

**Lemma 2.** Assume the hypothesis of Theorem A is fulfilled and that \( \|u_i\|^2 = n/m \), \( i = 1, \ldots, m \). Then there is a subset \( J \subset \{1, \ldots, m\} \) such that
\[
(4) \quad c_0 \frac{n}{m} \|w\|^2 \leq \sum_{i \in J} \langle w, u_i \rangle^2 \leq C_0 \frac{n}{m} \|w\|^2 \quad \forall w \in \mathbb{C}^n,
\]
where \( c_0 \) and \( C_0 \) are some absolute positive constants.

Proof. If \( n/m \geq 1/100 \), then (4) holds with \( J = \{1, \ldots, m\} \) and \( C_0 = c_0 = 100 \).

Assume \( \delta := n/m < 1/100 \). Let \( \alpha_j \) and \( \beta_j \) be as defined in Lemma 1. Then the vectors \( v_i = u_i \) satisfy the assumptions of Corollary B with \( \alpha_0 = \beta_0 = 1 \). Hence, a set \( J_1 \subset \{1, \ldots, m\} \) exists such that
\[
\alpha_1 \|w\|^2 \leq \sum_{i \in J_1} \langle w, u_i \rangle^2 \leq \beta_1 \|w\|^2 \quad \forall w \in \mathbb{C}^n.
\]

Since \( \alpha_1 \geq \alpha_L > 100\delta \), we may apply Corollary B the second time to get a set \( J_2 \subset J_1 \) such that the two-sided inequality above holds with \( \alpha_2 \) and \( \beta_2 \), and so on. Since \( \alpha_L > 100\delta \), Corollary B can be applied \( L \) times. We thus obtain a set \( J_{L+1} \subset \{1, \ldots, m\} \) for which the two-sided inequality holds with \( \alpha_{L+1} \) and \( \beta_{L+1} \). From Lemma 1 it follows that (4) is true with \( J = J_{L+1} \). \( \square \)

We now reformulate Lemma 2 in terms more convenient for our application. Given a matrix \( A \) of order \( m \times n \) and a subset \( J \subset \{1, \ldots, m\} \), we denote by \( A(J) \) the sub-matrix of \( A \) whose rows belong to the index set \( J \).

**Lemma 3.** There exist positive constants \( c_0, C_0 > 0 \), such that whenever \( A \) is an \( m \times n \) matrix which is a sub-matrix of some \( m \times m \) orthonormal matrix, and such that all of its rows have equal \( l^2 \) norm, one can find a subset \( J \subset \{1, \ldots, m\} \) such that
\[
(5) \quad c_0 \frac{n}{m} \|w\|^2 \leq \|A(J)w\|^2 \leq C_0 \frac{n}{m} \|w\|^2 \quad \forall w \in \mathbb{C}^n.
\]

3. Auxiliary results

In what follows we write \( F = \hat{f} \), where \( f \) is the Fourier transform of \( F \):
\[
f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-itx} F(t) \, dt.
\]

Given a discrete set \( \Lambda \), we denote by \( d(\Lambda) \) its separation constant
\[
d(\Lambda) := \inf_{\lambda, \lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'|.
\]

Given a sequence of sets \( \Lambda_j \) satisfying \( d(\Lambda_j) \geq d > 0 \) for all \( j \), a set \( \Lambda \) is called the weak limit of \( \Lambda_j \) if for every \( \epsilon > 0 \) and for every interval \( \Omega = (a, b), a, b \notin \Lambda \),
both inclusions $\Lambda_j \cap \Omega \subset (\Lambda \cap \Omega) + (-\epsilon, \epsilon)$ and $\Lambda \cap \Omega \subset (\Lambda_j \cap \Omega) + (-\epsilon, \epsilon)$ hold for all but a finite number of $j$’s. The standard diagonal procedure implies that if $\Lambda_j$ satisfy $d(\Lambda_j) \geq d > 0$ for all $j$, then there is a subsequence which weakly converges to some (maybe, empty) set $\Lambda$ satisfying $d(\Lambda) \geq d$.

Recall that the Paley–Wiener space $PW_S$ is defined as the space of all functions $f \in L^2(\mathbb{R})$ such that $\hat{f}$ vanishes a.e. outside $S$. When the measure of $S$ is finite, we have

$$\int_S |F(t)| \, dt \leq \|F\| \sqrt{|S|} \quad \forall F \in L^2(S).$$

Here $\|F\|$ means the $L^2$–norm of $F$. Hence, $\hat{f} \in L^1(\mathbb{R})$ for every $f \in PW_S$, and so every function $f \in PW_S$ is continuous.

Sometimes it will be more convenient for us to work with the Paley–Wiener space $PW_S$, rather than $L^2(S)$. In this connection we observe that by taking the Fourier transform, Theorem 1 is equivalent to the following statement:

**There exist positive constants $c, C$ such that for every set $S \subset \mathbb{R}, |S| < \infty$, there is a discrete set $\Lambda \subset \mathbb{R}$ such that**

$$c|S|\|f\|^2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq C|S|\|f\|^2 \quad \forall f \in PW_S. \tag{6}$$

We will prove (6) with the constants $C = C_0$ and $c = c_0/(36C_0)$, where $c_0$ and $C_0$ are the constants in Lemma 3.

We will need the following Bessel’s inequality (see [Y], Ch. 4.3): Given a set $\Lambda$ satisfying $d(\Lambda) > 0$ and a bounded set $S$, there is a constant $K$ which depends only on $d(\Lambda)$ and the diameter of $S$ such that

$$\sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq K\|f\|^2 \quad \forall f \in PW_S.$$

The proof of Theorem 1 below uses three auxiliary lemmas:

**Lemma 4.** Let $S$ be a bounded set of positive measure and let $\Lambda_k \subset \mathbb{R}$ be a sequence of sets satisfying $d(\Lambda_k) > \delta > 0, k = 1, 2, \ldots$, which converges weakly to some set $\Lambda$. Then

$$\lim_{k \to \infty} \sum_{\lambda \in \Lambda_k} |f(\lambda)|^2 = \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \quad \forall f \in PW_S.$$

**Proof.** Take any function $f \in PW_S$, and pick up a point $x_l \in [l\delta - \delta/2, l\delta + \delta/2]$ such that

$$|f(x_l)| = \max_{|x - l\delta| \leq \delta/2} |f(x)| \quad \forall l \in \mathbb{Z}.$$

Since $x_{l+2} - x_l \geq \delta$, the sequence $x_l$ is a union of two sets each having separation constant $\geq \delta$. By Bessel’s inequality, we see that

$$\sum_{l \in \mathbb{Z}} |f(x_l)|^2 < \infty.$$

Let $R > 0$, and write

$$| \sum_{\lambda \in \Lambda_k} |f(\lambda)|^2 - \sum_{\lambda \in \Lambda} |f(\lambda)|^2 |$$

$$\leq \sum_{\lambda \in \Lambda_k, |\lambda| < R} |f(\lambda)|^2 - \sum_{\lambda \in \Lambda, |\lambda| < R} |f(\lambda)|^2 + 2 \sum_{|k| \geq R/\delta} |f(x_k)|^2.$$
The first term in the right hand side tends to zero as $k \to \infty$ whenever $\pm R \not\in \Lambda$, while the second one tends to zero as $R \to \infty$. This proves the lemma. □

Lemma 5. Let $S_1 \subseteq S_2 \subseteq \ldots$ be an increasing sequence of bounded sets in $\mathbb{R}$ with $S = \bigcup_j S_j$ being a set of finite measure. Let $\Lambda \subset \mathbb{R}$ and positive $k, K$ be such that the inequalities

\begin{equation}
(k \|f_j\|^2 \leq \sum_{\lambda \in \Lambda} |f_j(\lambda)|^2 \leq K\|f_j\|^2 \quad \forall f_j \in PW_{S_j})
\end{equation}

hold for every $j$. Then

\begin{equation}
(k \|f\|^2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq K\|f\|^2 \quad \forall f \in PW_S).
\end{equation}

Proof. Given a function $f \in PW_S$, let $f_j \in PW_{S_j}$ be the Fourier transform of the function $\hat{f} \cdot \mathbb{1}_{S_j}$, where $\mathbb{1}_{S_j}$ is the indicator function of $S_j$. Then the $L^1$-norm of $\hat{f} - \hat{f}_j$ tends to zero as $j \to \infty$, and so the functions $f_j(x)$ converge uniformly to $f(x)$.

For every $R > 0$ we have,

$$\sum_{\lambda \in \Lambda, |\lambda| < R} |f_j(\lambda)|^2 \leq K\|f_j\|^2.$$

Taking the limit as $j \to \infty$, we obtain

$$\sum_{\lambda \in \Lambda, |\lambda| < R} |f(\lambda)|^2 \leq K\|f\|^2.$$

By letting $R \to \infty$, we obtain the right hand side inequality in (8). Using this inequality, we get

$$\left(\sum_{\lambda \in \Lambda} |f(\lambda)|^2\right)^{1/2} \geq \left(\sum_{\lambda \in \Lambda} |f_j(\lambda)|^2\right)^{1/2} - \left(\sum_{\lambda \in \Lambda} |f - f_j(\lambda)|^2\right)^{1/2}$$

$$\geq k^{1/2}\|f_j\| - K^{1/2}\|f - f_j\|^2.$$

Taking the limit as $j \to \infty$, we prove the left hand side inequality in (8). □

Lemma 6. Assume that the inequality

\begin{equation}
\sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq C|S|\|f\|^2 \quad \forall f \in PW_S
\end{equation}

is true for some $C > 0, S \subset \mathbb{R}, |S| < \infty$, and $\Lambda \subset \mathbb{R}$. Then

(i) There is a constant $\eta > 0$ which depends only on $S$ such that

$$\#(\Lambda \cap \Omega) \leq 9C,$$

for every interval $\Omega \subset \mathbb{R}, |\Omega| = \eta$.

(ii) There is a constant $K > 0$ which depends only on $S$ such that

$$\frac{\#(\Lambda \cap \Omega)}{|\Omega|} \leq 4C|S|,$$

for every interval $\Omega \subset \mathbb{R}, |\Omega| \geq K$. 

Proof. (i) Denote by $h \in PW_S$ the Fourier transform of the indicator function $1_S$. Then $h(x)$ is continuous,

$$h(0) = \frac{|S|}{\sqrt{2\pi}}, \quad \|h\|^2 = \|1_S\|^2 = |S|.$$

Choose $\eta > 0$ so small that $|h(x)| > |S|/3, |x| \leq \eta/2$. Then, applying (9) for $f = h$, we see that the statement (i) of Lemma 6 holds for $\Omega = [-\eta/2, \eta/2]$. To complete the proof, it suffices to observe that every function $h(x - x_0), x_0 \in \mathbb{R}$, belongs to $PW_S$.

(ii) Take any function $g \in PW_S$ satisfying $\|g\| = 1$, and choose a number $R$ such that

$$\int_{-R}^R |g(x)|^2 \, dx \geq \frac{1}{2}.$$

Assume $K > 2R$. We now apply (9) to the function $f(x) := g(x - s)$ and integrate over $(-K, K)$ with respect to $s$:

$$\int_{-K}^K \sum_{\lambda \in \Lambda} |g(\lambda - s)|^2 \, ds \leq 2KC|S|.$$

When $|\lambda| < K/2$, we have

$$\int_{-K}^K |g(\lambda - s)|^2 \, ds \geq \int_{-R}^R |g(s)|^2 \, ds \geq \frac{1}{2}.$$

We conclude that

$$\frac{|\Lambda \cap (-K/2, K/2)|}{2} \leq \int_{-K}^K \sum_{\lambda \in \Lambda} |g(\lambda - s)|^2 \, ds \leq 2KC|S|.$$

This proves statement (ii). \qed

4. Proof of Theorem 1

The proof of Theorem 1 will consist of a series of lemmas.

Lemma 7. Let $n, m \in \mathbb{N}, n < m$. For every set

$$S = \bigcup_{r \in I} \left[ \frac{2\pi r}{m}, \frac{2\pi (r + 1)}{m} \right], \quad I \subset \{0, \ldots, m - 1\}, \#I = n,$$

there is a set $\Lambda \subset \mathbb{Z}$ such that

$$c_0 |S| \|f\|^2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq C_0 |S| \|f\|^2 \quad \forall f \in PW_S,$$

where $c_0, C_0$ are the constants in Lemma 3.

Proof. Observe that $|S| = 2\pi n/m$, and denote by

$$\mathcal{F}_I := (e^{2\pi i r \omega / m})_{r \in I, \omega = 0, \ldots, m - 1}$$

the sub-matrix of the Fourier matrix $\mathcal{F}$ whose columns are indexed by $I$. Since the matrix $(\sqrt{m})^{-1}\mathcal{F}$ is orthonormal, by Lemma 3 there exists $J \subset \{0, \ldots, m - 1\}$ such that

$$c_0 n \|w\|^2 \leq \|\mathcal{F}_I(J)w\|_{l^2(I)}^2 \leq C_0 n \|w\|^2, \quad w \in l^2(I).$$
Observe that every function $F \in L^2(S)$ can be written as

$$F(t) = \sum_{r \in I} F_r(t - \frac{2\pi r}{m}),$$

where $F_r \in L^2(0, \frac{2\pi}{m})$ is defined by

$$F_r(t) := F(t + \frac{2\pi r}{m}) 1_{[0, \frac{2\pi}{m}]}(t).$$

Therefore, every function $f \in PW_S$ admits a representation

$$f(x) = \sum_{r \in I} e^{i \frac{2\pi r x}{m}} f_r(x), \quad f_r \in PW_{[0, \frac{2\pi}{m}]},$$

where the functions $e^{i \frac{2\pi r x}{m}} f_r(x)$ are orthogonal in $L^2(\mathbb{R})$. We note that for every function $h \in PW_{[0, \frac{2\pi}{m}]}$ we have

$$\frac{2\pi}{m} \|h\|^2 = \sum_{\lambda \in m\mathbb{Z}} |h(\lambda)|^2.$$

We now verify that the sequence

$$\Lambda := \{j + km : j \in J, k \in \mathbb{Z}\}$$

satisfies (10). Take any function $f \in PW_S$. Then

$$\sum_{j \in J} \sum_{k \in \mathbb{Z}} |f(j + km)|^2 = \sum_{j \in J} \sum_{k \in \mathbb{Z}} \left| \sum_{r \in I} e^{i \frac{2\pi r j}{m}} f_r(j + km) \right|^2.$$

For every $j \in J$ we apply (12) to the function $\sum_{r \in I} e^{i \frac{2\pi r x}{m}} f_r(x)$. We find that the last expression is equal to

$$\frac{2\pi}{m} \sum_{j \in J} \int_{\mathbb{R}} \left\| \sum_{r \in I} e^{i \frac{2\pi r x}{m}} f_r(x) \right\|^2 \, dx = \frac{2\pi}{m} \int_{\mathbb{R}} \| \mathcal{F}_I(J)(f_r(x))_{r \in I} \|^2_{L^2(\mathbb{R})} \, dx. $$

By inequality (11) we have on one hand,

$$\sum_{\lambda \in \Lambda} |f(\lambda)|^2 \geq c_0 \frac{n}{m} \int_{\mathbb{R}} \sum_{r \in I} |f_r(x)|^2 \, dx$$

$$= c_0 \frac{n}{m} \int_{\mathbb{R}} \sum_{r \in I} e^{i \frac{2\pi r x}{m}} f_r(x)^2 \, dx = c_0 \frac{n}{m} \int_{\mathbb{R}} \sum_{r \in I} e^{i \frac{2\pi r x}{m}} f_r(x)^2 \, dx$$

$$= c_0 \frac{n}{m} \int_{\mathbb{R}} |f(x)|^2 \, dx,$$

while on the other hand, applying the same computation, we get

$$\sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq C_0 \frac{n}{m} \int_{\mathbb{R}} \sum_{r \in I} |f_r(x)|^2 \, dx = C_0 \frac{n}{m} \int_{\mathbb{R}} |f(x)|^2 \, dx.$$

This completes the proof. □

**Lemma 8.** For every compact set $S \subset [0, 2\pi]$ of positive measure there is a set $\Lambda \subset \mathbb{Z}$ such that (10) holds.

This follows immediately from Lemma 7, since every such set $S$ can be covered by a set from Lemma 7 whose measure is arbitrarily close to $|S|$. 

**Lemma 9.** For every set $S \subset [0, 2\pi]$ of positive measure there is a set $\Lambda \subset \mathbb{Z}$ such that \(14\) holds.

**Proof.** It suffices to prove Lemma 9 for open sets. Let $S$ be such a set and let $S_1 \subset S_2 \subset \ldots$ be an increasing sequence of compact sets such that $S = \bigcup_j S_j$. By Lemma 8, there exist sets $\Lambda_j \subset \mathbb{Z}$ such that

\[
c_0|S||f_j|^2 \leq \sum_{\lambda \in \Lambda_j} |f_j(\lambda)|^2 \leq C_0|S||f_j|^2 \quad \forall f_j \in PW_{S_j},
\]

where $c_0, C_0$ are the constants in Lemma 3. Since $PW_{S_j} \subset PW_{S_k}$, $k > j$, we have

\[
c_0|S_k||f_j|^2 \leq \sum_{\lambda \in \Lambda_k} |f_j(\lambda)|^2 \leq C_0|S_k||f_j|^2 \quad \forall f_j \in PW_{S_j}.
\]

We may assume that $\Lambda_k$ converge weakly to some set $\Lambda \subset \mathbb{Z}$. Using Lemma 4, we take the limit as $k \to \infty$:

\[
c_0|S||f_j|^2 \leq \sum_{\lambda \in \Lambda} |f_j(\lambda)|^2 \leq C_0|S||f_j|^2 \quad \forall f_j \in PW_{S_j}.
\]

Now, the result follows from Lemma 5. \(\square\)

**Lemma 10.** For every bounded set $S$ of positive measure there is a set $\Lambda \subset (1/d)\mathbb{Z}$ such that \(14\) holds, where $d$ is any positive number such that $S$ lies on an interval of length $2\pi d$.

Observe that the translations of $S$ change neither the frame property of $E(\Lambda)$ nor the frame constants. So, it suffices to assume that $S \subset [0, 2\pi d]$. Then the result follows from Lemma 9 by re-scaling. \(\square\)

**Proof of Theorem 1.** We may assume that $S$ is an unbounded set of finite measure.

Let $S_1 \subset S_2 \subset \ldots$ be any sequence of bounded sets satisfying $S = \bigcup_j S_j$. By Lemma 10, there exist discrete sets $\Lambda_j$ such that \(13\) is true. Since $PW_{S_j} \subset PW_{S_k}$, $j < k$, we see that \(14\) holds for all $j < k$.

By Lemma 6(i), there is a number $\eta > 0$ and an integer $r$ which depends only on the constant $C_0$ in \(13\) (it is easy to check that one may take $r \leq 36C_0$) such that every set $\Lambda_k$ can be split up into $r$ subsets $\Lambda_k^{(l)}$ satisfying $\eta \cdot l = 1, \ldots, r$.

By taking an appropriate subsequence, we may assume that each $\Lambda_k^{(l)}$ converges weakly to some set $\Lambda^{(l)}$ as $k \to \infty$. By Lemma 4, we may take limit in \(14\) as $k \to \infty$:

\[
c_0|S||f_j|^2 \leq \sum_{l=1}^r \sum_{\lambda \in \Lambda^{(l)}} |f_j(\lambda)|^2 \leq C_0|S||f_j|^2 \quad \forall f_j \in PW_{S_j}.
\]

Set $\Lambda := \bigcup_{l=1}^r \Lambda^{(l)}$. It may happen that the sets $\Lambda^{(l)}$ have common points. Anyway, we have

\[
\sum_{\lambda \in \Lambda} |f_j(\lambda)|^2 \leq \sum_{l=1}^r \sum_{\lambda \in \Lambda^{(l)}} |f_j(\lambda)|^2 \leq r \sum_{\lambda \in \Lambda} |f_j(\lambda)|^2.
\]

From the latter inequalities, it readily follows that

\[
\frac{c_0}{r} |S||f_j|^2 \leq \sum_{\lambda \in \Lambda} |f_j(\lambda)|^2 \leq C_0|S||f_j|^2 \quad \forall f_j \in PW_{S_j}.
\]

Theorem 1 now follows easily from Lemma 5. \(\square\)
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