BOUNDS ON THE SPECTRUM AND REDUCING SUBSPACES
OF A J-SELF-ADJOINT OPERATOR

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ABSTRACT. Given a self-adjoint involution $J$ on a Hilbert space $H$, we consider a $J$-self-adjoint operator $L = A + V$ on $H$ where $A$ is a possibly unbounded self-adjoint operator commuting with $J$ and $V$ a bounded $J$-self-adjoint operator anti-commuting with $J$. We establish optimal estimates on the position of the spectrum of $L$ with respect to the spectrum of $A$ and we obtain norm bounds on the operator angles between maximal uniformly definite reducing subspaces of the unperturbed operator $A$ and the perturbed operator $L$. All the bounds are given in terms of the norm of $V$ and the distances between pairs of disjoint spectral sets associated with the operator $L$ and/or the operator $A$. As an example, the quantum harmonic oscillator under a $PT$-symmetric perturbation is discussed. The sharp norm bounds obtained for the operator angles generalize the celebrated Davis-Kahan trigonometric theorems to the case of $J$-self-adjoint perturbations.

1. INTRODUCTION

Let $H$ be a Hilbert space and $J$ a self-adjoint involution on $H$, that is, $J^* = J$ and $J^2 = I$, where $J \neq I$ and $I$ denotes the identity operator. A linear operator $L$ on $H$ is called $J$-self-adjoint if the product $JL$ is a self-adjoint operator on $H$, that is, $(JL)^* = JL$.

In this paper we consider a $J$-self-adjoint operator $L$ of the form $L = A + V$ where $A$ is a (possibly unbounded) self-adjoint operator commuting with $J$ and $V$ a bounded $J$-self-adjoint operator anti-commuting with $J$. Since the involution $J$ is both unitary and self-adjoint, its spectrum consists of the two points $+1$ and $-1$ and hence

$$J = E_J(\{+1\}) - E_J(\{-1\}),$$

(1.1)

where $E_J(\{\pm1\})$ denote the corresponding spectral projections of $J$. Thus, the involution $J$ induces a natural decomposition of the Hilbert space $H$ into the sum

$$H = H_0 \oplus H_1$$

(1.2)

of the complementary orthogonal subspaces

$$H_0 = \text{Ran} \ E_J(\{+1\}), \quad H_1 = \text{Ran} \ E_J(\{-1\}).$$

(1.3)

Our assumptions on the operators $A$ and $V$ imply that they are nothing but the diagonal and off-diagonal parts of $L$ with respect to the decomposition (1.2):

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}, \quad \text{Dom}(A) = \text{Dom}(A_0) \oplus \text{Dom}(A_1) \subset H_0 \oplus H_1,$$

(1.4)

$$V = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}, \quad \text{Dom}(V) = H_0 \oplus H_1;$$

(1.5)
here the entries \( A_0 = A|_{\mathcal{H}_0} \) and \( A_1 = A|_{\mathcal{H}_1} \) are self-adjoint operators on \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \), respectively, and \( B = V|_{\mathcal{H}_1} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0) \) is bounded. Thus the operator \( L \) may be viewed as an off-diagonal bounded \(-J\)-self-adjoint perturbation of the block diagonal self-adjoint operator matrix \( A \).

A powerful tool to study operators \( L \) admitting a block operator matrix representation with respect to a self-adjoint involution \( J \) is furnished by indefinite inner product spaces. This idea was first used in [26] to prove a general theorem on block-diagonalizability for a \(-J\)-accretive operator \( A \) and a self-adjoint perturbation \( V \), with application to Dirac operators. The main ingredient of this approach is to show that the perturbed reducing subspaces are maximal uniformly positive and negative, respectively, with respect to the indefinite inner product. As a consequence, they admit graph representations by angular operators which measure the deviation between the unperturbed and the perturbed reducing subspaces.

In the situation considered in the present paper, the self-adjoint involution \( J \) induces an indefinite inner product by means of the formula

\[
[x, y] = (Jx, y), \quad x, y \in \mathcal{H}.
\]

The Hilbert space \( \mathcal{H} \) equipped with the indefinite inner product (1.6) is a Krein space which we denote by \( \mathcal{K} \), assuming that \( \mathcal{K} \) stands for the pair \( \{\mathcal{H}, J\} \). Note that every \(-J\)-self-adjoint operator on \( \mathcal{H} \) is a self-adjoint operator on the Krein space \( \mathcal{K} \); in particular, the operators \( A, V, \) and \( L = A + V \) are self-adjoint operators on \( \mathcal{K} \). In the Krein space \( \mathcal{K} \), a (closed) subspace \( \mathcal{L} \subset \mathcal{K} \) is said to be uniformly positive if there exists a \( \gamma > 0 \) such that

\[
[x, x] \geq \gamma \|x\|^2 \quad \text{for every} \quad x \in \mathcal{K}, \ x \neq 0,
\]

where \( \| \cdot \| \) denotes the norm on \( \mathcal{K} \). The subspace \( \mathcal{L} \) is called maximal uniformly positive if it is not a proper subset of another uniformly positive subspace of \( \mathcal{K} \). Uniformly negative and maximal uniformly negative subspaces of \( \mathcal{K} \) are defined in a similar way, replacing the inequality in (1.7) by \( [x, x] \leq -\gamma \|x\|^2 \). Direct sums of subspaces of \( \mathcal{K} \) (or \( \mathcal{H} \)) that are \(-J\)-orthogonal (i.e. orthogonal with respect to the inner product \([\cdot, \cdot]\)) are denoted with “\([\cdot]^{\perp}\)”.

Further definitions related to Krein spaces and linear operators therein may be found, e.g., in [23], [11], [16], or [7].

The subspaces \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \), which simultaneously reduce \( A \) and \( J \), are maximal uniformly positive and maximal uniformly negative, respectively, with respect to the inner product (1.6) induced by \( J \). Throughout this paper, we assume that also the perturbed operator \( L = A + V \) possesses a maximal uniformly positive invariant subspace \( \mathcal{H}_0' \). Then the complementary \(-J\)-orthogonal subspace \( \mathcal{H}_1' = \mathcal{H}_0'|^{\perp} \) is invariant for \( L \) as well; hence both \( \mathcal{H}_0' \) and \( \mathcal{H}_1' \) are automatically reducing subspaces for \( L \) and the spectrum of \( L \) is purely real (see, e.g., Corollary 2.12 below).

The main goal of this paper is to establish bounds on the position of the reducing subspaces \( \mathcal{H}_0' \) or \( \mathcal{H}_1' \) of the perturbed operator \( L = A + V \) relative to the subspaces \( \mathcal{H}_0 \) or \( \mathcal{H}_1 \). The bounds are given in terms of the norm of the perturbation \( V \) and of the distances between the unperturbed spectra

\[
\sigma_0 = \text{spec}(A|_{\mathcal{H}_0}) \quad \text{and} \quad \sigma_1 = \text{spec}(A|_{\mathcal{H}_1})
\]

of \( A \) and/or the perturbed spectra

\[
\sigma_0' = \text{spec}(L|_{\mathcal{H}_0'}) \quad \text{and} \quad \sigma_1' = \text{spec}(L|_{\mathcal{H}_1'})
\]

of \( L \) in their respective maximal uniformly definite reducing subspaces.

We describe the mutual geometry of the maximal uniformly definite reducing subspaces of the unperturbed and perturbed operators \( A \) and \( L = A + V \) by using the concept of operator angles between two subspaces of a Hilbert space (for a discussion of this concept and references see,
e.g., [18]). Recall that the operator angle $\Theta(\mathcal{S}_i, \mathcal{S}'_i)$ between $\mathcal{S}_i$ and $\mathcal{S}'_i$ measured relative to $\mathcal{S}_i$ is given by (see, e.g., [19])

$$\Theta_i = \Theta(\mathcal{S}_i, \mathcal{S}'_i) = \arcsin \sqrt{I_{\mathcal{S}_i} - P_{\mathcal{S}_i}P_{\mathcal{S}'_i}}, \quad i = 0, 1,$$

(1.10)

where $I_{\mathcal{S}_i}$ denotes the identity operator on $\mathcal{S}_i$, and $P_{\mathcal{S}_i}$ and $P_{\mathcal{S}'_i}$ stand for the orthogonal projections in $\mathcal{S}_i$ onto $\mathcal{S}_i$ and $\mathcal{S}'_i$, respectively. By definition, the operator angle $\Theta(\mathcal{S}_i, \mathcal{S}'_i)$ is a non-negative operator on $\mathcal{S}_i$ and

$$\|\Theta(\mathcal{S}_i, \mathcal{S}'_i)\| = \max \text{spec}(\Theta(\mathcal{S}_i, \mathcal{S}'_i)) \leq \pi/2.$$

The main tool we use to estimate the operator angles $\Theta_i = \Theta(\mathcal{S}_i, \mathcal{S}'_i)$ is their relation to solutions of the operator Riccati equation

$$KA_0 - A_1 K + KBK = -B^*,$$

(1.11)

where the coefficients $A_0$, $A_1$, and $B$ are the entries of the block matrix representations (1.4) and (1.5) of the operators $A$ and $V$. In fact, given a maximal uniformly positive reducing subspace $\mathcal{S}'_0$ of $L = A + V$, there exists a unique uniformly contractive solution $K$ ($\|K\| < 1$) to the Riccati equation (1.11) such that $\mathcal{S}'_0$ is the graph of $K$; the maximal uniformly negative reducing subspace $\mathcal{S}'_1$ of $L$, which is $J$-orthogonal to $\mathcal{S}'_0$, is the graph of the adjoint of $K$. Since $\|K\| < 1$ and $|K| = \tan \Theta(\mathcal{S}_i, \mathcal{S}'_i)$ (see Remark 2.6 and Lemma 2.8 below), the operator angle always satisfies the two-sided inequality

$$0 \leq \Theta_i < \pi/4, \quad i = 0, 1.$$

(1.12)

By establishing tighter norm bounds on the uniformly contractive solution $K$ of (1.11), we thus obtain tighter norm bounds for the operator angles (1.10).

Sufficient conditions guaranteeing the existence of maximal uniformly definite reducing subspaces for the operator $L = A + V$ may be found, e.g., in [6] and [34, 35]. In particular, one of the main results of [6] is as follows. Here and in the sequel, by $\text{conv}(\sigma)$ we denote the convex hull of a Borel set $\sigma \subset \mathbb{R}$.

**Theorem 1.1** ([6], Theorem 5.8 (ii)). Assume that the spectral sets $\sigma_0$ and $\sigma_1$ are disjoint, i.e.

$$d := \text{dist}(\sigma_0, \sigma_1) > 0,$$

(1.13)

that one of these sets lies in a finite or infinite gap of the other one, i.e.

$$\text{conv}(\sigma_i) \cap \sigma_{1-i} = \emptyset \quad \text{or} \quad \sigma_i \cap \text{conv}(\sigma_{1-i}) = \emptyset \quad \text{for some} \ i = 0, 1,$$

and that \(\|V\| < d/2\). Then

$$\text{spec}(L) = \sigma'_0 \cup \sigma'_1,$$

where the (disjoint) sets $\sigma'_0 \subset \mathbb{R}$ and $\sigma'_1 \subset \mathbb{R}$ lie in the closed $\|V\|/2$-neighbourhoods of the sets $\sigma_0$ and $\sigma_1$, respectively. The complementary spectral subspaces $\mathcal{S}'_0$ and $\mathcal{S}'_1$ of $L$ associated with the spectral sets $\sigma'_0$ and $\sigma'_1$ are maximal uniformly positive and maximal uniformly negative, respectively, and satisfy the sharp norm bound

$$\tan \Theta_i \leq \tanh \left( \frac{1}{2} \arctanh \frac{2\|V\|}{d} \right), \quad i = 0, 1,$$

(1.14)

or, equivalently,

$$\|\sin 2\Theta_i\| \leq \frac{2\|V\|}{d}, \quad i = 0, 1.$$

(1.15)
The bound (1.14) relies on the disjointness of the spectral sets \( \sigma_0 \) and \( \sigma_1 \) of the unperturbed operator \( A \) and involves the distance between \( \sigma_0 \) and \( \sigma_1 \). Therefore, this bound (as well as the other bounds from [6, Theorem 5.8]) is an \textit{a priori estimate}. In the present paper, we establish bounds on the operator angles \( \Theta_j \) that involve at least one of the perturbed spectral sets \( \sigma'_0 \) and \( \sigma'_1 \). In general, for these new bounds to hold, the disjointness (1.13) of the sets \( \sigma_0 \) and \( \sigma_1 \) is not required at all.

Our first main result is a \textit{semi-a posteriori bound} on the operator angles \( \Theta_j \) involving the distances \( \text{dist}(\sigma_0, \sigma'_1) \) and/or \( \text{dist}(\sigma_1, \sigma'_0) \) between one unperturbed and one perturbed spectral set.

**Theorem 1.2.** Suppose that \( L \) has a maximal uniformly positive reducing subspace \( \mathcal{H}_0' \) in the Krein space \( \mathcal{K} = \{ \mathcal{H}, J \} \).

(i) If for some \( i = 0, 1 \) the spectral sets \( \sigma_i \) and \( \sigma'_{1-i} \) are disjoint, i.e.

\[
\delta_i := \text{dist}(\sigma_i, \sigma'_{1-i}) > 0, \tag{1.16}
\]

then the operator angles \( \Theta_j \) satisfy the bound

\[
\| \tan \Theta_j \| \leq \frac{\pi}{2} \frac{||V||}{\delta_i} \quad \text{for both } j = 0, 1. \tag{1.17}
\]

(ii) If, in addition, one of the sets \( \sigma_i \) and \( \sigma'_{1-i} \) satisfying (1.16) lies in a finite or infinite gap of the other one, i.e.

\[
\text{conv}(\sigma_i) \cap \sigma'_{1-i} = \emptyset \quad \text{or} \quad \sigma_i \cap \text{conv}(\sigma'_{1-i}) = \emptyset, \tag{1.18}
\]

then we have the stronger estimate

\[
\| \tan \Theta_j \| \leq \frac{||V||}{\delta_i} \quad \text{for both } j = 0, 1. \tag{1.19}
\]

Our second main result is a completely \textit{a posteriori estimate} since it only involves the distance between the spectral sets \( \sigma'_0 \) and \( \sigma'_1 \) associated with the perturbed operator \( L \).

**Theorem 1.3.** Suppose that \( L \) has a maximal uniformly positive reducing subspace \( \mathcal{H}_0' \) in the Krein space \( \mathcal{K} = \{ \mathcal{H}, J \} \).

(i) If the spectral sets \( \sigma'_0 \) and \( \sigma'_1 \) are disjoint, i.e.

\[
\hat{\delta} := \text{dist}(\sigma'_0, \sigma'_1) > 0, \tag{1.20}
\]

then the operator angles \( \Theta_j \) satisfy the estimate

\[
\| \tan \Theta_j \| \leq \frac{\pi}{2} \frac{||V||}{\hat{\delta}} \quad \text{for both } j = 0, 1. \tag{1.21}
\]

(ii) If, in addition, for some \( i = 0, 1 \) the set \( \sigma'_i \) is bounded and lies in a finite or infinite gap of \( \sigma'_{1-i} \), i.e.

\[
\text{conv}(\sigma'_i) \cap \sigma'_{1-i} = \emptyset, \tag{1.22}
\]

then we have the stronger estimate

\[
\| \tan \Theta_j \| \leq \frac{||V||}{\sqrt{\hat{\delta}^2 + ||V||^2}} \quad \text{for both } j = 0, 1. \tag{1.23}
\]
(iii) Furthermore, if both spectral sets $\sigma_0'$ and $\sigma_1'$ are bounded and subordinated, i.e.

$$\text{conv}(\sigma_0') \cap \text{conv}(\sigma_1') = \emptyset,$$

(1.24)

then we have the even stronger estimate

$$\|\tan 2\Theta_j\| \leq \frac{2\|V\|}{\delta} \quad \text{for both } j = 0, 1.$$  

(1.25)

The bounds (1.19) and (1.25) as well as the bound (1.23) in the case of a finite gap are optimal (see Remarks 4.1-4.3). Moreover, the sharp a priori bound (1.14) turns out to be a corollary either to Theorem 1.2(ii) or to Theorem 1.3(iii) (see Theorem 6.4 and Remark 6.8 respectively).

The semi-a posteriori bounds of Theorem 1.2 and the completely a posteriori ones of Theorem 1.3 complement the a priori norm bounds on the variation of spectral subspaces for $J$-self-adjoint operators proved in [6, Theorem 5.8]. The sharp norm bounds of these theorems represent analogues of the celebrated trigonometric estimates for self-adjoint operators known as Davis-Kahan sin $\Theta$, sin $2\Theta$, tan $\Theta$, and tan $2\Theta$ theorems (see [13] and the subsequent papers [5, 19, 20, 21, 31]): the bound (1.15) may be called the a priori sin $2\Theta$ theorem for $J$-self-adjoint operators; the bounds (1.19) and (1.23) may be called the semi-a posteriori and completely a posteriori tan $\Theta$ theorems, respectively; the bound (1.25) may be called the a posteriori tan $2\Theta$ theorem.

The plan of the paper is as follows. In Section 2 we give necessary definitions and recall some basic results on the block diagonalization of $J$-self-adjoint $2 \times 2$ block operator matrices. In Section 3 we establish several semi-a posteriori and completely a posteriori norm bounds on uniformly contractive solutions to operator Riccati equations of the form (1.11). Using these results, we prove both Theorems 1.2 and 1.3 in Section 4. Assuming that the spectral sets (1.8) do not intersect and $\|V\| < 4 \text{dist}(\sigma_0, \sigma_1)$, in Section 5 we obtain sharp estimates on the position of the isolated components of the spectrum of $L = A + V$ confined in the closed $\|V\|$-neighbourhoods of the sets $\sigma_0$ and $\sigma_1$. In this section, we also establish bounds on the spectrum for more general $2 \times 2$ block operator matrices that need not be $J$-self-adjoint. In Section 6, we combine Theorems 1.2 and 1.3 with the spectral estimates of Section 5 and discuss the emerging a priori norm bounds on variation of the spectral subspaces of a self-adjoint operator on a Hilbert space under $J$-self-adjoint perturbations. Finally, in Section 7 we apply some of the bounds obtained to the Schrödinger operator describing an $N$-dimensional isotropic harmonic oscillator under a $\mathcal{P}$-$\mathcal{P}$-symmetric perturbation (see e.g. [3]); here the parity operator $\mathcal{P}$ plays the role of the self-adjoint involution $J$ (see [2, 6, 27]).

The following notations are used throughout the paper. By a subspace of a Hilbert space we always mean a closed linear subset. The identity operator on a subspace (or on the whole Hilbert space) $\mathcal{M}$ is denoted by $I_{\mathcal{M}}$; if no confusion arises, the index $\mathcal{M}$ is often omitted. The Banach space of bounded linear operators from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{H}'$ is denoted by $\mathcal{B}(\mathcal{H}, \mathcal{H}')$ and by $\mathcal{B}(\mathcal{H})$ if $\mathcal{H} = \mathcal{H}'$. The symbol $\cup$ is used for the union of two disjoint sets. By $O_r(\Sigma)$, $r \geq 0$, we denote the closed $r$-neighbourhood of a Borel set $\Sigma$ in the complex plane $\mathbb{C}$, i.e. $O_r(\Sigma) = \{z \in \mathbb{C} \mid \text{dist}(z, \Sigma) \leq r\}$. By a finite gap of a closed Borel set $\sigma \subset \mathbb{R}$, $\sigma \neq \emptyset$, we understand an open interval $(a, b)$, $-\infty < a < b < \infty$, such that $\sigma \cap (a, b) = \emptyset$ and $a, b \in \sigma$; an infinite gap of $\sigma$ is a semi-infinite interval $(a, b)$ such that $\sigma \cap (a, b) = \emptyset$ and either $a = -\infty$, $|b| < \infty$, and $b \in \sigma$ or $|a| < \infty$, $a \in \sigma$, and $b = \infty$.
2. Preliminaries

In this section we recall some results on the block diagonalization of $J$-self-adjoint operator matrices in terms of solutions to the related operator Riccati equations and on norm bounds for solutions to operator Sylvester equations. We also recall a couple of statements on maximal uniformly definite subspaces of a Krein space. For notational setup we adopt the following

**Assumption 2.1.** Let $J$ be a self-adjoint involution on a Hilbert space $\mathcal{H}$, $J \neq 1$, and let $\mathcal{H}_0$ and $\mathcal{H}_1$ be the spectral subspaces $^1\mathcal{H}$ of $J$. Also assume that $\mathcal{A}$ is a (possibly unbounded) self-adjoint operator on $\mathcal{H}$ diagonal with respect the decomposition $^2\mathcal{H}$, which means that $\mathcal{H}_0$ and $\mathcal{H}_1$ are the reducing subspaces of $\mathcal{A}$ and the representation $^3\mathcal{H}$ holds with $\mathcal{A}_0$ and $\mathcal{A}_1$ the self-adjoint operators on $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. Let $\mathcal{V}$ be a bounded operator on $\mathcal{H}$ admitting, relative to $^2\mathcal{H}$, the representation $^4\mathcal{H}$ where $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$. Finally, let

$$L = A + V = \begin{pmatrix} A_0 & B \\ -B^* & A_1 \end{pmatrix}, \quad \text{Dom}(L) = \text{Dom}(A).$$

(2.1)

With a block operator matrix $L$ of the form (2.1) we associate the operator Riccati equation $^1\mathcal{H}$ where $K$ is a linear operator from $\mathcal{H}_0$ to $\mathcal{H}_1$. There are different concepts of solutions to such an equation; here we recall the notion of weak and strong solutions (see $^3\mathcal{H}$).

**Definition 2.2.** Assume that Assumption 2.1 is satisfied. A bounded operator $K \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)$ is said to be a weak solution to the Riccati equation (1.11) if

$$(KA_0x, y) - (Kx, A_1^*y) + (KBk, y) = -(B^*x, y) \quad \text{for all } x \in \text{Dom}(A_0), \ y \in \text{Dom}(A_1^*).$$

A bounded operator $K \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)$ is called a strong solution to the Riccati equation (1.11) if

$$\text{Ran}(K|_{\text{Dom}(A_0)}) \subset \text{Dom}(A_1)$$

(2.2)

and

$$KA_0x - A_1Kx + KBk = -B^*x \quad \text{for all } x \in \text{Dom}(A_0).$$

(2.3)

**Remark 2.3.** Obviously, every strong solution $K \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)$ to the Riccati equation (1.11) is also a weak solution. In fact, the two notions are equivalent by $^3\mathcal{H}$ Lemma 5.2: every weak solution $K \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)$ of the operator Riccati equation (1.11) is also a strong solution.

**Remark 2.4.** With the block operator matrix (2.1), one can also associate the operator Riccati equation

$$K'A_1 - A_0K' - K'B^*K' = B$$

(2.4)

where $K'$ is a linear operator from $\mathcal{H}_1$ to $\mathcal{H}_0$. From Definition 2.2 it immediately follows that $K \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)$ is a weak (and hence strong) solution to (1.11) if and only if $K' = K^*$ is a weak (and hence strong) solution to (2.4).

**Definition 2.5.** Let $\mathcal{N}$ be a subspace of the Hilbert space $\mathcal{H}$, $\mathcal{N}^\perp = \mathcal{H} \ominus \mathcal{N}$ its orthogonal complement, and $K$ a bounded linear operator from $\mathcal{N}$ to $\mathcal{N}^\perp$. Denote by $P_{\mathcal{N}}$ and $P_{\mathcal{N}^\perp}$ the orthogonal projections in $\mathcal{H}$ onto the subspaces $\mathcal{N}$ and $\mathcal{N}^\perp$, respectively. The set

$$\mathcal{G}(K) = \{ x \in \mathcal{H} \mid P_{\mathcal{N}^\perp}x = KP_{\mathcal{N}}x \}$$

is called the graph subspace associated with the operator $K$.

**Remark 2.6.** If a subspace $\mathcal{G} \subset \mathcal{H}$ is a graph $\mathcal{G} = \mathcal{G}(K)$ of a bounded linear operator $K \in \mathcal{B}(\mathcal{N}, \mathcal{N}^\perp)$, then $K$ is called the angular operator for the (ordered) pair of subspaces $\mathcal{N}$ and $\mathcal{G}$; the usage of this term is explained by the equality (see $^8\mathcal{H}$; cf. $^9\mathcal{H}$ and $^10\mathcal{H}$)

$$|K| = \tan \Theta(\mathcal{N}, \mathcal{G}),$$

(2.5)
where $|K|$ is the modulus of $K$, $|K| = \sqrt{K^*K}$, and $\Theta(\mathcal{M}, \mathcal{G})$ is the operator angle between the subspaces $\mathcal{M}$ and $\mathcal{G}$ measured relative to the subspace $\mathcal{M}$ (see definition (1.10)).

It is well known that strong solutions to the Riccati equations (1.11) and (2.4) determine invariant subspaces for the operator matrix $L$ by means of their graph subspaces (see, e.g., [3] and [24]). More precisely, the following correspondences hold (see, e.g., [6, Lemma 2.4]).

**Lemma 2.7.** Assume that Assumption 2.1 holds. Then the graph $\mathcal{G}(K)$ of an operator $K \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)$ satisfying (2.2) is an invariant subspace for the operator matrix $L$ if and only if $K$ is a strong solution to the operator Riccati equation (1.11). Similarly, the graph $\mathcal{G}(K')$ of an operator $K' \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$ is an invariant subspace for $L$ if and only if $K'$ is a strong solution to the Riccati equation (2.4).

The next two statements are well-known facts in the theory of spaces with indefinite metric (see, e.g., [7, Section I.8, in particular, Corollaries I.8.13 and I.8.14]).

**Lemma 2.8.** A subspace $\mathcal{L}$ is a maximal uniformly positive subspace of the Krein space $\mathcal{K}$ if and only if there is a uniform contraction $K \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)$ (i.e. $\|K\| < 1$) such that $\mathcal{L}$ is the graph $\mathcal{G}(K)$ of the contraction $K$. Similarly, a subspace $\mathcal{L}'$ is a maximal uniformly negative subspace of the Krein space $\mathcal{K}$ if and only if $\mathcal{L}'$ is the graph $\mathcal{G}(K')$ of a uniform contraction $K' \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$.

**Lemma 2.9.** Let $\mathcal{L}$ be a maximal uniformly positive subspace of the Krein space $\mathcal{K}$. Then the orthogonal complement $\mathcal{L}^\perp$ of $\mathcal{L}$ in $\mathcal{K}$ is a maximal uniformly negative subspace. If $\mathcal{L}$ is a graph subspace, $\mathcal{L} = \mathcal{G}(K)$ with $K \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)$, then $\mathcal{L}^\perp$ is the graph of the adjoint of $K$, i.e. $\mathcal{L}^\perp = \mathcal{G}(K^*)$, and $\mathcal{L} = \mathcal{L}^\perp$.

Many more details on Krein spaces and linear operators therein may be found in [22], [23], [11], [16] or [7].

The following sufficient condition for a $J$-self-adjoint block operator matrix of the form (2.1) to be similar to a self-adjoint operator on $\mathcal{H}$ was proved in [6]; for the particular case where the spectra of the entries $A_0$ and $A_1$ are subordinated, say $\text{max spec}(A_0) < \text{min spec}(A_1)$, closely related results may be found in [11, Theorem 4.1] and [30, Theorem 3.2].

**Theorem 2.10 ([6, Theorem 5.2]).** Assume that $L = A + V$ satisfies Assumption 2.1. Suppose that the Riccati equation (1.11) has a weak (and hence strong) solution $K \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)$ such that $\|K\| < 1$. Then:

(i) The operator matrix $L$ has purely real spectrum and it is similar to a self-adjoint operator on $\mathcal{H}$. In particular, the equality

$$L = TAT^{-1}$$

holds, where $T$ is a bounded and boundedly invertible operator on $\mathcal{H}$ given by

$$T = \begin{pmatrix} I & K^* \\ K & I \end{pmatrix} \left( I - K^*K \right)^{-1/2} \left( I - K^*K \right)^{-1/2}$$

and $A$ is a block diagonal self-adjoint operator on $\mathcal{H}$,

$$A = \text{diag}(A_0, A_1), \quad \text{Dom}(A) = \text{Dom}(A_0) \oplus \text{Dom}(A_1),$$

whose entries

$$A_0 = (I - K^*K)^{1/2}(A_0 + BK)(I - K^*K)^{-1/2},$$

$$\text{Dom}(A_0) = \text{Ran}(I - K^*K)^{1/2}|_{\text{Dom}(A_0)}.$$

(2.9)

(2.10)
and
\[
\Lambda_1 = (I - KK^*)^{1/2}(A_1 - B^*K^*)(I - KK^*)^{-1/2},
\]
\[
\text{Dom}(\Lambda_1) = \text{Ran}(I - KK^*)^{1/2}\left|\text{Dom}(A_1)\right|
\]
are self-adjoint operators on the corresponding Hilbert space components \(\mathcal{H}_0\) and \(\mathcal{H}_1\), respectively.

(ii) The graph subspaces \(\mathcal{G}_0 = \mathcal{G}(K)\) and \(\mathcal{G}_1 = \mathcal{G}(K^*)\) are invariant under \(L\), mutually orthogonal with respect to the indefinite inner product \((1.6)\), and
\[
\mathcal{G} = \mathcal{G}_0 \perp \mathcal{G}_1,\]

The subspace \(\mathcal{G}_0\) is maximal uniformly positive, while \(\mathcal{G}_1\) is maximal uniformly negative. The restrictions of \(L\) onto \(\mathcal{G}_0\) and \(\mathcal{G}_1\) are \(\mathcal{G}\)-unitary equivalent to the self-adjoint operators \(\Lambda_0\) and \(\Lambda_1\), respectively.

**Remark 2.11.** The requirement \(\|K\| < 1\) is sharp in the sense that if there is no uniformly contractive solution to the Riccati equation \((1.11)\), then the operator matrix \(L\) need not be similar to a self-adjoint operator at all; this can be seen, e.g., from \([6, \text{Example 5.5}]\).

An elementary consequence of Theorem 2.10 is the following property of maximal uniformly definite subspaces of \(J\)-self-adjoint operators \(L = A + V\) with self-adjoint \(A\) and bounded \(V\).

**Corollary 2.12.** Assume that \(L = A + V\) satisfies Assumption 2.7. Suppose that \(L\) has a maximal uniformly positive (resp. negative) invariant subspace \(\mathcal{K}_0\) of \(\mathcal{K} = \{\mathcal{K}, J\}\). Then \(\mathcal{K}_0 = \mathcal{K}_0^{-1}\) is also an invariant subspace of \(L\), which is uniformly maximal negative (resp. positive); the restrictions of \(L\) to \(\mathcal{K}_0\) and \(\mathcal{K}_0^{-1}\) are \(\mathcal{K}\)-unitary equivalent to self-adjoint operators on the Hilbert spaces \(\mathcal{H}_0\) and \(\mathcal{H}_1\), respectively.

**Proof.** We give the proof for the case where \(\mathcal{K}_0\) is a maximal uniformly positive subspace; the proof for maximal uniformly negative \(\mathcal{K}_0\) is analogous.

By Lemma 2.8, \(\mathcal{K}_0\) is the graph of a uniform contraction \(K : \mathcal{H}_0 \to \mathcal{H}_1\). Since \(\mathcal{K}_0\) is invariant under \(L\), Lemma 2.7 shows that \(K\) is a uniformly contractive strong solution to the Riccati equation \((1.11)\). Now all claims follow immediately from Theorem 2.10. \(\square\)

Riccati equations are closely related to operator Sylvester equations (also called Kato-Rosenblum equations). In this paper we use the following well known result on sharp norm bounds for strong solutions to operator Sylvester equations (cf. \([6, \text{Theorem 4.9}]\)).

**Theorem 2.13.** Let \(A_0\) and \(A_1\) be (possibly unbounded) self-adjoint operators on the Hilbert spaces \(\mathcal{H}_0\) and \(\mathcal{H}_1\), respectively, and \(Y \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)\). If the spectra \(\text{spec}(A_0)\) and \(\text{spec}(A_1)\) are disjoint, i.e.
\[
d := \text{dist}(\text{spec}(A_0), \text{spec}(A_1)) > 0,
\]
then the operator Sylvester equation
\[
XA_0 - A_1X = Y
\]
has a unique strong solution \(X \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)\); the solution \(X\) satisfies the norm bound
\[
\|X\| \leq \frac{\pi}{2} \frac{\|Y\|}{d}; \quad (2.11)
\]
if, in addition, one of the sets \(\text{spec}(A_0)\) and \(\text{spec}(A_1)\) lies in a finite or infinite gap of the other one, then \(X\) satisfies the stronger norm bound
\[
\|X\| \leq \frac{\|Y\|}{d}. \quad (2.12)
\]
Remark 2.14. The fact that the constant $\pi/2$ in the estimate (2.11) for the generic disposition of the sets $\text{spec}(A_0)$ and $\text{spec}(A_1)$ is best possible is due to R. McEachin [29]. The existence of the bound (2.12) for the particular case where one of the sets $\text{spec}(A_0)$ and $\text{spec}(A_1)$ lies in a finite or infinite gap of the other one may be traced back to E. Heinz [15] (also see [9, Theorem 3.2] and [6, Theorem 3.4]). For more details and references we refer the reader to [3, Remark 2.8] and [6, Remark 4.10].

In the proofs of several statements below we will use the following elementary result, the proof of which is left to the reader.

Lemma 2.15. Let $\varphi$ be a scalar analytic function of a complex variable $z$ whose Taylor series

$$\varphi(z) = \sum_{k=0}^{\infty} a_k z^k, \quad a_k = \frac{1}{k!} \frac{d^k \varphi(0)}{dz^k}, \quad k = 1, 2, \ldots, \tag{2.13}$$

is absolutely convergent on the open disc $\{z \in \mathbb{C} : |z| < r\}$ for some $r > 0$. Let $M \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$ and $N \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)$ be bounded operators with $\|MN\| < r$ and $\|NM\| < r$. Then

$$M \varphi(NM) = \varphi(NM)M, \tag{2.14}$$

where for a bounded linear operator $T$ on a Hilbert space $\mathcal{H}$ with $\|T\| < r$ the value of $\varphi(T)$ is defined by the series

$$\varphi(T) = \sum_{k=0}^{\infty} a_k T^k.$$

We also need the following auxiliary statement.

Lemma 2.16. Assume that Assumption (2.7) holds and suppose that the Riccati equation (1.11) has a weak (and hence strong) solution $K \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$ such that $\|K\| < 1$. Then

$$\text{Ran}(K|_{\text{Dom}(A_0)}) \subset \text{Dom}(A_1) \tag{2.15}$$

and

$$KA_0 y - A_1 Ky = -(I - KK^\ast)^{1/2}B^\ast (I - K^\ast K)^{1/2} y \quad \text{for all } y \in \text{Dom}(A_0), \tag{2.16}$$

where $A_0$ and $A_1$ are the self-adjoint operators given by (2.9) and (2.10), respectively.

Proof. It is straightforward to verify that if $K$ is a strong solution to the Riccati equation (1.11), then

$$KZ_0 x - Z_1 K x = -B^\ast (I - K^\ast K) x \quad \text{for all } x \in \text{Dom}(A_0). \tag{2.17}$$

Since $K$ is assumed to be a uniform contraction, Theorem (2.10)(i) applies and yields

$$K(I - K^\ast K)^{-1/2}A_0(I - K^\ast K)^{1/2} x - (I - KK^\ast)^{-1/2}A_1(I - KK^\ast)^{1/2}K x = -B^\ast (I - K^\ast K) x \quad \text{for all } x \in \text{Dom}(A_0). \tag{2.18}$$

Applying $(I - KK^\ast)^{1/2}$ from the left to both sides of (2.18) and choosing $x = (I - K^\ast K)^{-1/2} y$ with $y \in \text{Dom}(A_0)$, we arrive at the Sylvester equation

$$XA_0 y - A_1 X y = Y y \quad \text{for all } y \in \text{Dom}(A_0), \tag{2.19}$$

where

$$X = (I - KK^\ast)^{1/2}K(I - K^\ast K)^{-1/2}, \tag{2.20}$$

$$Y = - (I - KK^\ast)^{1/2}B^\ast (I - K^\ast K)^{1/2}. \tag{2.21}$$
Note that we have $\text{Ran}(I - K^*K)^{-1/2}|_{\text{Dom}(A_0)} = \text{Dom}(A_0)$ by (2.9), $\text{Ran}(K|_{\text{Dom}(A_0)}) \subset \text{Dom}(A_1)$ by (2.2), and thus, by (2.10),
\begin{equation}
\text{Ran}(X|_{\text{Dom}(A_0)}) \subset \text{Dom}(A_1).
\end{equation}
Hence $X$ is a strong solution to the Sylvester equation (2.19).

Furthermore, the Taylor series (2.13) of the function $\varphi(z) = (1 - z)^{1/2}$ is absolutely convergent on the disc $\{z \in \mathbb{C} : |z| < 1\}$. Since $\|K\| < 1$, Lemma 2.15 applies and yields that $(I - KK^*)^{1/2}K = K(I - K^*K)^{1/2}$. Therefore, (2.20) simplifies to nothing but the identity $X = K$. Now the claims follow from the inclusion (2.22) and the identities (2.19), (2.21). □

3. Bounds on Uniformly Contractive Solutions to the Riccati Equations

Assuming Assumption 2.1 in this section we prove several norm bounds on uniformly contractive solutions $K$ to the Riccati equation (1.11) (provided such solutions exist). These bounds are obtained under the hypothesis that either the spectra of the operators $Z_0 = A_0 + BK$ and $A_1$ or the spectra of $Z_0$ and $Z_1 = A_1 - B^*K^*$ are disjoint. Note that, by Theorem 2.10(i), the assumption $\|K\| < 1$ implies that the spectra of $Z_0$ and $Z_1$ are both real, that is, $\text{spec}(Z_0) \subset \mathbb{R}$ and $\text{spec}(Z_1) \subset \mathbb{R}$.

Throughout this section we use the following notations:
\begin{align*}
\delta_{Z_0,A_1} &:= \text{dist}(\text{spec}(Z_0), \text{spec}(A_1)), \\
\delta_{Z_0,Z_1} &:= \text{dist}(\text{spec}(Z_0), \text{spec}(Z_1)).
\end{align*}

3.1. Semi-a posteriori bounds. First, we establish norm bounds on $K$ that only contain the norm of $B$ and the distance $\delta_{Z_0,A_1}$. Therefore, these bounds may be viewed as semi-a posteriori estimates on $K$ since the set $\text{spec}(Z_0) = \text{spec}(A_0)$ corresponds to the perturbed operator $L = A + V$ (see Theorem 2.10), while the other set, $\text{spec}(A_1)$, is part of the spectrum of the unperturbed operator $A$.

**Theorem 3.1.** Assume that Assumption 2.1 holds and suppose that the Riccati equation (1.11) has a weak (and hence strong) solution $K \in \mathcal{B}(\delta_1, \delta_0)$ such that $\|K\| < 1$. Then:

(i) If the spectra of the operators $Z_0 = A_0 + BK$, $\text{Dom}(Z_0) = \text{Dom}(A_0)$, and of $A_1$ are disjoint, i.e.
\begin{equation}
\delta_{Z_0,A_1} > 0,
\end{equation}
then the solution $K$ satisfies the inequality
\begin{equation}
\|K\| \leq \frac{\pi}{2} \frac{\|B\|}{\delta_{Z_0,A_1}}.
\end{equation}

(ii) If, in addition, one of the sets $\text{spec}(Z_0)$ or $\text{spec}(A_1)$ lies in a finite or infinite gap of the other one, i.e.
\begin{equation}
\text{conv} \left( \text{spec}(Z_0) \right) \cap \text{spec}(A_1) = \emptyset
\end{equation}
or
\begin{equation}
\text{spec}(Z_0) \cap \text{conv} \left( \text{spec}(A_1) \right) = \emptyset,
\end{equation}
then the solution $K$ satisfies the stronger inequality
\begin{equation}
\|K\| \leq \frac{\|B\|}{\delta_{Z_0,A_1}}.
\end{equation}
Proof. The assumption that $K$ is a strong solution to the Riccati equation (1.11) is equivalent to
\[ \text{Ran}(K |_{\text{Dom}(Z_0)}) = \text{Ran}(K |_{\text{Dom}(A_0)}) \subseteq \text{Dom}(A_1) \text{ and} \]
\[ KZ_0x - A_1Kx = -B^*x \quad \text{for all } x \in \text{Dom}(A_0) = \text{Dom}(Z_0). \]  
(3.8)

Since $K$ is a uniform contraction, $\|K\| < 1$, we can use Theorem 2.10(i) to rewrite (3.8) as
\[ K(I - K^*K)^{-1/2}A_0(I - K^*K)^{1/2}x - A_1Kx = -B^*x \quad \text{for all } x \in \text{Dom}(A_0), \]
(3.9)
where $A_0$ is the self-adjoint operator defined by (2.9). If we choose $x = (I - K^*K)^{-1/2}y$ with
\[ y \in \text{Dom}(A_0), \]
we can write (3.9) as
\[ K(I - K^*K)^{-1/2}A_0y - A_1K(I - K^*K)^{-1/2}y = -B^*(I - K^*K)^{-1/2}y \]
(3.10)
for all $y \in \text{Dom}(A_0)$;

note that $\text{Ran}K(I - K^*K)^{-1/2}|_{\text{Dom}(A_0)} \subseteq \text{Dom}(A_1)$ since $\text{Ran}(I - K^*K)^{1/2}|_{\text{Dom}(A_0)} = \text{Dom}(A_0)$
(see (2.9)) and $\text{Ran}(K |_{\text{Dom}(A_0)}) \subseteq \text{Dom}(A_1)$. Equality (3.10) means that the operator
\[ X = K(I - K^*K)^{-1/2} \]
(3.11)
is a strong solution to the operator Sylvester equation
\[ XA_0 - A_1X = Y \]
(3.12)
with $Y = -B^*(I - K^*K)^{-1/2}$. Obviously, for the norm of $Y$ we have
\[ \|Y\| \leq \frac{\|B\|}{\sqrt{1 - \|K\|^2}}. \]
(3.13)

If $|K| = \sqrt{K^*K}$ denotes the modulus of $K$, then the modulus $|X| = \sqrt{X^*X}$ of the operator $X$
defined in (3.11) is given by
\[ |X| = |K|(I - |K|^2)^{1/2}. \]
Taking into account that $\||X|| = \|X\|$ and $\||K|| = \|K\|$, the spectral theorem implies that
\[ \|X\| = \frac{\|K\|}{\sqrt{1 - \|K\|^2}}. \]
(3.14)

Due to the similarity (2.9) of the operators $A_0$ and $Z_0$, we have $\text{spec}(A_0) = \text{spec}(Z_0)$ and hence, by (3.3),
\[ \text{dist}(\text{spec}(A_0), \text{spec}(A_1)) = \delta_{Z_0,A_1}. \]
(3.15)

Applying Theorem 2.13 and using (3.13) as well as (3.14), we readily arrive at
\[ \frac{\|K\|}{\sqrt{1 - \|K\|^2}} = \|X\| \leq c \frac{\|Y\|}{\delta_{Z_0,A_1}} \leq c \frac{\|B\|}{\delta_{Z_0,A_1} \sqrt{1 - \|K\|^2}} \]
where $c = \pi/2$ in case (i) and $c = 1$ in case (ii) so that, in both cases,
\[ \|K\| \leq c \frac{\|B\|}{\delta_{Z_0,A_1}}. \]
□

Remark 3.2. In order to compete with the hypothesis $\|K\| < 1$, the bounds (3.4) and (3.7) are
of interest only if $\|B\| < 2\delta_{Z_0,A_1}/\pi$ in case (i) and $\|B\| < \delta_{Z_0,A_1}$ in case (ii).

Remark 3.3. For all spectral dispositions such that (3.5) or (3.6) holds and $\|B\| < \delta_{Z_0,A_1}$, the
bound (3.7) is sharp in the sense that given an arbitrary $\beta > 0$ and arbitrary $\delta > \beta$ one can
always find $A$ and $V$ such that $\|V\| = \beta$, $\delta_{Z_0,A_1} = \delta$, and $\|K\| = \beta/\delta$. 
The following examples illustrate the sharpness of (3.7) and Remark 3.3.

**Example 3.4.** Let $\mathcal{H}_0 = \mathbb{C}^2$ and $\mathcal{H}_1 = \mathbb{C}$. Assume that $b, d \in \mathbb{R}$ are such that $0 \leq b < d/2$ and let

$$A_0 = \begin{pmatrix} -d & 0 \\ 0 & d \end{pmatrix}, \quad A_1 = 0, \quad B = \begin{pmatrix} 0 \\ b \end{pmatrix}.$$ 

For this choice of $A_0, A_1,$ and $B$, the Riccati equation (1.11) has a unique uniformly contractive solution of the form $K = (0 - \kappa)$ where $\kappa$ is given by

$$\kappa = \frac{b}{\frac{d}{2} + \sqrt{\frac{d^2}{4} - b^2}}.$$ 

(3.16)

Hence

$$Z_0 = A_0 + BK = \begin{pmatrix} -d & 0 \\ 0 & \frac{d}{2} + \sqrt{\frac{d^2}{4} - b^2} \end{pmatrix}$$

and the set $\text{spec} (A_1) = \{0\}$ lies in the gap $(-d, d/2 + \sqrt{\frac{d^2}{4} - b^2})$ of $\text{spec} (Z_0)$, so that (3.6) holds. Altogether we have

$$\| K \| = \frac{\| B \|}{\delta_{Z_0, A_1}},$$

(3.17)

i.e. equality in (3.7).

**Example 3.5.** Let $\mathcal{H}_0 = \mathbb{C}$ and $\mathcal{H}_1 = \mathbb{C}^2$. Assume that $b, d \in \mathbb{R}$ are such that $0 \leq b < d$ and set

$$A_0 = 0, \quad A_1 = \begin{pmatrix} -d & 0 \\ 0 & d \end{pmatrix}, \quad B = \begin{pmatrix} b/\sqrt{2} \\ b/\sqrt{2} \end{pmatrix}.$$ 

By inspection, one can verify that the $2 \times 1$ matrix

$$K = \begin{pmatrix} -\frac{b}{\sqrt{2d}} \\ \frac{b}{\sqrt{2d}} \end{pmatrix}$$

is a solution to the operator Riccati equation (1.11). Clearly,

$$\| B \| = b, \quad \| K \| = \frac{b}{d}$$

(3.18)

and

$$Z_0 = A_0 + BK = 0, \quad Z_1 = A_1 - B^* K^* = \begin{pmatrix} -d + \frac{b^2}{2d} & -\frac{b^2}{2d} \\ \frac{b^2}{2d} & d - \frac{b^2}{2d} \end{pmatrix}.$$ 

Obviously, the set $\text{spec} (Z_0) = \{0\}$ lies within the gap $(-d, d)$ of the set $\text{spec} (A_1) = \{-d, d\}$. Furthermore, $\delta_{Z_0, A_1} = d$ and hence, by (3.18),

$$\| K \| = \frac{\| B \|}{\delta_{Z_0, A_1}},$$

(3.19)

For later reference, we note that $\text{spec} (Z_1) = \{-\sqrt{d^2 - b^2}, \sqrt{d^2 - b^2}\}$ so that $\delta_{Z_0, Z_1} = \sqrt{d^2 - b^2}$ and thus

$$\| K \| = \frac{\| B \|}{\sqrt{\delta_{Z_0, Z_1}^2 + b^2}} = \frac{\| B \|}{\sqrt{\delta_{Z_0, Z_1}^2 + \| B \|^2}}.$$ 

(3.20)
Example 3.6. Let \( \mathcal{H}_0 = \mathcal{H}_1 = \mathbb{C} \). Assume that \( b, d \in \mathbb{R} \) are such that \( 0 < b < d/2 \) and set

\[
A_0 = -d/2, \quad A_1 = d/2, \quad B = b.
\]

Then the Riccati equation (1.11) appears to be the numeric quadratic equation \( bK^2 + Kd = -b \). The only solution \( K = \kappa \in \mathbb{R} \) with norm \( \|K\| = |\kappa| < 1 \) where \( \kappa \) is again given by (3.16). One immediately verifies that

\[
Z_0 = A_0 + Bk = -\sqrt{\frac{d^2}{4} - b^2}, \quad Z_1 = A_1 - B^*K^* = \sqrt{\frac{d^2}{4} - b^2}.
\]

(3.21)

Here the sets \( \text{spec}(Z_0) \) and \( \text{spec}(A_1) \) are even subordinated to each other, so that both (3.5) and (3.6) hold. Together with (3.16) and (3.21), we again obtain the equality

\[
\|K\| = \frac{|B|}{\delta_{Z_0, A_1}}.
\]

(3.22)

For later reference, we also observe that

\[
\frac{\|K\|}{1 - \|K\|^2} = \frac{|B|}{\delta_{Z_0, Z_1}}.
\]

(3.23)

3.2. Completely a posteriori bounds. In this subsection we consider the case where the spectra of the operators \( Z_0 = A_0 + BK \) and \( Z_1 = A_1 - B^*K^* \) are disjoint. The bounds on \( K \) obtained here depend only on \( \|B\| \) and on the distance \( \delta_{Z_0, Z_1} \) between the subsets \( \text{spec}(Z_0) = \text{spec}(A_0) \) and \( \text{spec}(Z_1) = \text{spec}(A_1) \) of the spectrum of the perturbed operator \( L = A + V \) (see Theorem 2.10). Therefore, they may be viewed as a posteriori bounds on \( K \).

We begin with the most general result where nothing is known on the mutual position of \( \text{spec}(Z_0) \) and \( \text{spec}(Z_1) \) except that they do not intersect.

Theorem 3.7. Assume that \( L = A + V \) satisfies Assumption 2.7 and suppose that the Riccati equation (1.11) has a weak (and hence strong) solution \( K \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0) \) such that \( \|K\| < 1 \). If the spectra of the operators \( Z_0 = A_0 + BK \), \( \text{Dom}(Z_0) = \text{Dom}(A_1) \), and \( Z_1 = A_1 - B^*K^* \), \( \text{Dom}(Z_1) = \text{Dom}(A_1) \), do not intersect, that is,

\[
\delta_{Z_0, Z_1} > 0,
\]

then

\[
\|K\| \leq \frac{\pi}{2} \frac{\|B\|}{\delta_{Z_0, Z_1}}.
\]

(3.24)

Proof. By Lemma 2.16 the Riccati equation (1.11) can be written in the form (2.16). For the term \( Y = -(I - KK^*)^{1/2}B^*(I - K^*K)^{1/2} \) on the right-hand side of (2.16), we have \( \|Y\| \leq \|B\| \) since both \( K^*K \) and \( KK^* \) are non-negative and, in addition, \( \|K\|\|K^*\| < 1 \). Since \( \text{spec}(A_0) = \text{spec}(Z_0) \) and \( \text{spec}(A_1) = \text{spec}(Z_1) \), we have dist \( \text{spec}(\Lambda_0), \text{spec}(\Lambda_1) \) = \( \delta_{Z_0, Z_1} \). To complete the proof, it remains to apply the bound (2.11) from Theorem 2.13 to (2.16). \( \square \)

Remark 3.8. Under the stronger assumption that one of the spectral sets \( \text{spec}(Z_0) \) and \( \text{spec}(Z_1) \) lies in a finite or infinite gap of the other one, i.e. if

\[
\text{conv}(\text{spec}(Z_i)) \cap \text{spec}(Z_{1-i}) = \emptyset \quad \text{for some} \quad i = 0, 1,
\]

Theorem 2.13 also yields the estimate

\[
\|K\| \leq \frac{\|B\|}{\delta_{Z_0, Z_1}}.
\]

(3.25)
This estimate, however, is of no interest in the case where the corresponding operator $Z$ is bounded: the bound (3.27) in the following theorem is stronger than (3.25).

**Theorem 3.9.** Assume that $L = A + V$ satisfies Assumption 2.7 and suppose, in addition, that $A_0$ is bounded. Let the Riccati equation (1.11) have a weak (and hence strong) solution $K \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$ such that $\|K\| < 1$. If the spectrum of the (bounded) operator $Z_0 = A_0 + BK$ lies in a finite or infinite gap of the spectrum of the operator $Z_1 = A_1 - B^*K^*$, Dom($Z_1$) = Dom($A_1$), that is,

$$\text{conv}\left(\text{spec}(Z_0) \cap \text{spec}(Z_1)\right) = \emptyset,$$

then

$$\|K\| \leq \frac{\|B\|}{\sqrt{\delta^2_{Z_0, Z_1} + \|B\|^2}}.$$  

(3.26)

**Proof.** Throughout the proof we assume that $B \neq 0$ and, thus, necessarily

$$K \neq 0.$$  

(3.28)

Let $U$ be the partial isometry in the polar decomposition $K = U|K|$ of $K$. If we adopt the convention that $U$ is extended to Ker($K$) = Ker($|K|$) by

$$U|\text{Ker}(K)| = 0,$$  

(3.29)

then $U$ is uniquely defined on the whole space $\mathcal{H}_0$ (see [10] Theorem 8.1.2 or [11] §VI.7.2) and

$$U$$

is an isometry on Ran($|K|$) = Ran($K^*$).  

(3.30)

First we apply Lemma 2.16 and transform the Riccati equation (1.11) to the form (2.16). Since the operator $A_0$ is bounded, $A_0 \in \mathcal{B}(\mathcal{H}_0)$, we may then rewrite (2.16) as

$$K \Lambda_0 - \Lambda_1 U|K| = - (I - KK^*)^{1/2} B^* (I - |K|^2)^{1/2} = -\tilde{B}^* (I - |K|^2)^{1/2}$$  

(3.31)

where we have set

$$\tilde{B} = B(I - KK^*)^{1/2}.$$  

(3.32)

We tackle the cases where the gap of spec($\Lambda_1$) containing the set spec($\Lambda_0$) is finite or infinite in a slightly different manner. If this gap is finite, we may assume without loss of generality that it is centered at zero, i.e. it is of the form $(-a, a)$ with $a > 0$; otherwise, we simply replace $\Lambda_0$ and $\Lambda_1$ in (3.31) by $\Lambda_0' = \Lambda_0 - \lambda_1 I$ and $\Lambda_1' = \Lambda_1 - \lambda_1 I$, respectively, where $\lambda_1$ is the center of the gap. Then

$$0 \in \rho(\Lambda_1), \quad \|\Lambda_1^{-1}\| < \frac{1}{a}, \quad \text{and} \quad \|\Lambda_0\| \leq a - \delta_{Z_0, Z_1}.$$  

(3.33)

If the gap of spec($\Lambda_1$) containing the spec($\Lambda_0$) is infinite, we may assume without loss of generality that the interval $[\min\text{spec}(\Lambda_0), \max\text{spec}(\Lambda_0)]$ is centered at zero and that the spectrum of $\Lambda_1$ lies either in the interval $(-\infty, -a]$ where $a = -\max\text{spec}(\Lambda_1)$ or in the interval $[a, \infty)$ where $a = \min\text{spec}(\Lambda_1)$. Then, again, all three statements of (3.33) hold.

In the following, we may thus treat the two above cases together. Since $0 \notin \text{spec}(\Lambda_1)$, we further rewrite (3.31) in the form

$$U|K| = \Lambda_1^{-1} \left( K \Lambda_0 + \tilde{B}^* (I - |K|^2)^{1/2} \right)$$  

(3.34)

and set $\kappa = \max\text{spec}(|K|) = \|K\|$. By assumption (3.28), we have $\kappa > 0$.

If $\kappa$ is an eigenvalue of $|K|$ and $x$ a corresponding eigenvector with $\|x\| = 1$, then, by applying both sides of (3.34) to $x$, we immediately arrive at

$$\kappa Ux = \Lambda_1^{-1} \left( K \Lambda_0 x + \sqrt{I - \kappa^2} \tilde{B}^* x \right).$$  

(3.35)
By (3.32), we have \( \| \tilde{B}x \| \leq \| \tilde{B}^* \| \leq \| B \| (1 - KK)^{1/2} \leq \| B \| \). Then, by (3.33) and (3.35), we obtain
\[
\kappa \leq \frac{1}{a} \left( \kappa(a - \delta_{Z_0,Z_1}) + \sqrt{1 - \kappa^2 \| B \|} \right),
\]
(3.36)
taking into account that \( x \in \text{Ran}(|K|) \) and thus \( \| Ux \| = \| x \| = 1 \) by (3.30).

If \( \kappa \) is not an eigenvalue of \( |K| \), then it belongs to the essential spectrum of \( |K| \). Hence we obtain a singular sequence \( \{ x_n \}_{n=1}^\infty \) of \( |K| \) at \( \kappa \) by choosing arbitrary
\[
x_n \in \text{Ran} E_{|K|}(\kappa(1 - 1/n, \kappa)), \quad \| x_n \| = 1, \quad n = 1,2,\ldots;
\]
(3.37)
here \( E_{|K|} \) denotes the spectral measure of \( |K| \) and, at the same time, the (right-continuous) spectral function of \( |K| \), that is, \( E_{|K|}(\mu) = E_{|K|}(\mu) \). Obviously, \( |K|x_n = \kappa x_n + \epsilon_n \) with
\[
\| \epsilon_n \| = \left\| \int_{\kappa(1 - \frac{1}{n}, \kappa)}^\kappa dE_{|K|}(\mu) \right\| \leq \frac{1}{n}.
\]
(3.38)
Similarly, \((I - |K|^2)^{1/2}x_n = \sqrt{1 - \kappa^2}x_n + \zeta_n\) with
\[
\| \zeta_n \| = \left\| \int_{\kappa(1 - \frac{1}{n}, \kappa)}^\kappa dE_{|K|}(\mu) \left( \sqrt{1 - \mu^2} - \sqrt{1 - \kappa^2} \right) x_n \right\|
\leq \sup_{\mu \in (\kappa(1 - 1/n), \kappa)} \left| \sqrt{1 - \mu^2} - \sqrt{1 - \kappa^2} \right|
= \left( 1 - \kappa^2 \left( 1 - \frac{1}{n} \right)^2 \right)^{1/2} - (1 - \kappa^2)^{1/2} < \frac{1}{n} \frac{\kappa^2}{\sqrt{1 - \kappa^2}}.
\]
(3.39)
Applying both sides of equality (3.34) to \( x_n \), we arrive at
\[
\kappa Ux_n = \Lambda_1^{-1}(K\Lambda_0 x_n + \sqrt{1 - \kappa^2} \tilde{B}^* x_n) + \beta_n
\]
(3.40)
with
\[
\beta_n = \Lambda_1^{-1} \tilde{B}^* \zeta_n - U \epsilon_n \to 0 \quad \text{as} \quad n \to \infty;
\]
(3.41)
here we have used that \( \epsilon_n = |K|x_n - \kappa x_n \to 0 \) and \( \zeta_n = (I - |K|^2)^{1/2}x_n - \sqrt{1 - \kappa^2}x_n \to 0 \) as \( n \to \infty \) by (3.33) and (3.39), respectively. Since \( x_n \in \text{Ran}(|K|) \) and thus \( \| Ux_n \| = \| x_n \| = 1 \) by (3.30), the relation (3.40) implies that
\[
\kappa \leq \frac{1}{a} \left( \kappa(a - \delta_{Z_0,Z_1}) + \sqrt{1 - \kappa^2 \| B \|} \right) + \| \beta_n \|, \quad n = 1,2,\ldots,\infty,
\]
(3.42)
which, by (3.41), turns into the bound (3.36) after taking the limit \( n \to \infty \). Solving inequality (3.36) for \( \kappa \) and recalling that \( \kappa = \| K \| \), we conclude the estimate (3.27).

**Remark 3.10.** The bound (3.27) is sharp. In fact, equality (3.20) in Example 3.5 shows that, for the spectral dispositions (3.26) where \( \text{spec}(Z_0) \) lies in a finite gap of \( \text{spec}(Z_1) \), equality prevails in (3.27).

The strongest a posteriori bound for the solution \( K \) is obtained under the assumptions that the spectra of \( Z_0 \) and \( Z_1 \) are subordinated, i.e.
\[
\max \text{spec}(Z_0) < \min \text{spec}(Z_1) \quad \text{or} \quad \max \text{spec}(Z_1) < \min \text{spec}(Z_0),
\]
(3.43)
and that \( A_0 \) and \( A_1 \) are bounded.
Theorem 3.11. Assume that $L = A + V$ satisfies Assumption 2.1 and suppose, in addition, that the operators $A_0$, $A_1$ (and hence $A$ and $L$) are bounded. Let the Riccati equation 1.11 have a weak (and hence strong) solution $K \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_0)$ such that $\|K\| < 1$. If the spectra of the operators $Z_0 = A_0 + BK$ and $Z_1 = A_1 - B^*K^+$ are subordinated, that is,

$$\max \text{spec}(Z_0) < \min \text{spec}(Z_1) \quad \text{or} \quad \max \text{spec}(Z_1) < \min \text{spec}(Z_0), \quad (3.44)$$

then

$$\|K\| \leq \tan \left( \frac{1}{2} \arctan \frac{2\|B\|}{\delta_{Z_0, Z_1}} \right). \quad (3.45)$$

Proof. As in the proof of Theorem 3.9 we apply Lemma 2.16 and rewrite the Riccati equation in the form (2.16); note that the self-adjoint operators $\Lambda_0$ and $\Lambda_1$ on the left-hand side of (2.16) are bounded by Theorem 2.10 since $A_0$ and $A_1$ are bounded, $B$ is bounded, and $K$ is a uniform contraction. Assuming that $B \neq 0$, we again have $K \neq 0$ (cf. (3.28)).

Let $U$ be the partial isometry in the polar decomposition $K = U|K|$ of $K$ (see the proof of Theorem 3.9). By Lemma 2.15 with $\varphi(z) = (1 - z)^{1/2}$, $M = U^*$, and $N = |K|^2U^*$, we obtain

$$U^*(I - KK^*)^{1/2} = U^*(I - U|K|^2U^*)^{1/2}$$

$$= (I - U^*|K|^2)^{1/2}U^*$$

Here, in the last step, we have used the property that $U$ is an isometry on $\text{Ran}(|K|) = \text{Ran}(K^*)$ by (3.30) so that $U^*U|K| = |K|$ and $U^*U|K|^2 = |K|^2$.

If we apply the operator $U^*$ to both sides of (2.16) from the left, we arrive at an equation that only involves $|K|$, but not $K$ and $K^*$ themselves:

$$|K|\Lambda_0 - U^*\Lambda_1 U|K| = -(I - |K|^2)^{1/2}U^*B^*(I - |K|^2)^{1/2}. \quad (3.46)$$

Let $\kappa = \max \text{spec}(|K|)$. Clearly, $0 < \kappa = |K| < 1$. If $\kappa$ is an eigenvalue of $|K|$ and $x \in \mathcal{H}_0$, $\|x\| = 1$, is an eigenvector of $|K|$ at $\kappa$, that is, $|K|x = \kappa x$, then (3.46) immediately implies that

$$\kappa \left( \frac{1}{1 - \kappa^2} (\Lambda_0 x, x) - (\Lambda_1 Ux, Ux) \right) = -(U^*B^*x, x). \quad (3.47)$$

Since $x \in \text{Ran}(|K|)$, by (3.30) we have $\|Ux\| = \|x\| = 1$ so that

$$(\Lambda_1 Ux, Ux) \geq \min \text{spec}(\Lambda_1) = \min \text{spec}(Z_1), \quad (3.48)$$

$$(\Lambda_0 x, x) \leq \max \text{spec}(\Lambda_0) = \max \text{spec}(Z_0). \quad (3.49)$$

Because the spectra of $Z_0$ and $Z_1$ are subordinated by assumption (3.44), the inequalities (3.48) and (3.49) yield that

$$|(\Lambda_0 x, x) - (\Lambda_1 Ux, Ux)| \geq \delta_{Z_0, Z_1} > 0.$$ 

This and (3.47) imply the inequality

$$\frac{\kappa}{1 - \kappa^2} \leq \frac{\|B\|}{\delta_{Z_0, Z_1}}. \quad (3.50)$$

If $\kappa$ is not an eigenvalue of $|K|$, it belongs to the essential spectrum of $|K|$. We introduce a singular sequence $\{x_n\}_{n=1}^\infty$ of $|K|$ at $\kappa$ as in (3.37); in particular, $\|x_n\| = 1$. With this choice of $x_n$, (3.46) implies that

$$\frac{\kappa}{1 - \kappa^2} (\Lambda_0 x_n, x_n) - (\Lambda_1 Ux_n, Ux_n) = -(U^*B^*x_n, x_n) + \alpha_n \quad (3.51)$$
where 
\[
\alpha_n = \frac{(U^*A_1Ux_n, x_n) - (A_0x_n, x_n)}{1 - \kappa^2} - \frac{(U^*B^*\zeta_n, x_n) + (U^*Bx_n, \zeta_n)}{\sqrt{1 - \kappa^2}}.
\]
Because of \(x_n \in \text{Ran}(|K|)\) and (3.30), we have \(\|Ux_n\| = \|x_n\| = 1\). Now the same reasoning as in (3.48) and (3.49) yields that
\[
\big| (A_0x_n, x_n) - (A_1Ux_n, Ux_n) \big| \geq \delta_{Z_0,Z_1} > 0.
\]
Hence (3.51) shows that
\[
\frac{\kappa}{1 - \kappa^2} \leq \frac{\|B\|}{\delta_{Z_0,Z_1}} + \frac{|\alpha_n|}{\delta_{Z_0,Z_1}}, \quad n = 1, 2, \ldots, \infty.
\] (3.52)
As \(\|x_n\| = 1\) and both \(A_0\) and \(A_1\) are bounded operators, (3.38) and (3.39) show that \(\alpha_n \to 0\) for \(n \to \infty\). Taking the limit \(n \to \infty\) in (3.52), we again arrive at inequality (3.50).

To complete the proof it remains to notice that, by the formula for double arguments of the tangent function, the left-hand side of (3.50) may be written as 
\[
\frac{\kappa}{1 - \kappa^2} = \frac{1}{2} \tan(2\arctan \kappa),
\] (3.53)
and to recall that \(\kappa = \|K\|\).

**Remark 3.12.** The bound (3.45) is optimal. This may be seen from Example 3.6 where \(\delta_0 = \delta_1 = \mathbb{C}\) and \(A_0 = -d/2, A_1 = d/2, B = b\) with \(b, d \in \mathbb{R}, 0 < b < d/2\). In fact, by (3.53), equality (3.23) therein is equivalent to
\[
\|K\| = \tan \left( \frac{1}{2} \arctan \frac{2\|B\|}{\delta_{Z_0,Z_1}} \right).
\]

4. PROOFS OF THEOREMS 1.2 AND 1.3

Using the results of Section 3 we are now able to prove our main results, Theorems 1.2 and Theorem 1.3, which were formulated in the introduction. In particular, Theorem 1.2 appears to be a corollary to Theorem 3.1 in Section 3.

**Proof of Theorem 1.2** According to the definitions (1.8), (1.9) and Theorem 2.10 we have
\[
\sigma_i = \text{spec}(A_i), \quad \sigma_i' = \text{spec}(Z_i), \quad i = 0, 1,
\]
with \(Z_0 = A_0 + BK, \text{Dom}(Z_0) = \text{Dom}(A_0)\) and \(Z_1 = A_1 - B^*K^*, \text{Dom}(Z_1) = \text{Dom}(A_1)\) as above.

We prove the theorem in the case \(\text{dist}(\sigma_0, \sigma_1) = \delta_{Z_0,A_1} > 0\); the case \(\text{dist}(\sigma_0', \sigma_0) > 0\) may be reduced to the first case by replacing the involution \(J\) with \(J' = -J\) and making the corresponding index changes in the notations of Assumption 2.1.

By assumption, \(\delta_0'\) is a maximal uniformly positive subspace of the Krein space \(\mathcal{K} = \{\delta_0, J\}\). Thus Lemma 2.8 implies that \(\delta_0'\) is the graph of a uniform contraction \(K : \delta_0 \to \delta_1\), i.e. \(\delta_0' = \mathcal{G}(K)\). By assumption, \(\delta_0'\) is also a reducing and hence invariant subspace of \(L\). Now Lemma 2.7 yields that \(K\) is a strong solution to the operator Riccati equation (1.11). By Lemma 2.9 and Theorem 2.10 we know that \(\delta_1 = \mathcal{G}(K^*) = \mathcal{G}(K)^{[-1]} = \delta_0^{[-1]}\) and hence, by (2.5) and definition (1.10),
\[
\| \tan \Theta_0 \| = \Theta(\delta_0, \delta_0') = \|K\| = \|K^*\| = \Theta(\delta_1, \delta_1') = \| \tan \Theta_1 \|.
\] (4.1)
Now both claims (i) and (ii) are immediate consequences of the respective statements (i) and (ii) of Theorem 3.1. \(\square\)
Remark 4.1. Under the natural assumption that \( \| V \| < \delta_i \) for some \( i \in \{0, 1\} \), the bound (1.19) is optimal with respect to the mutual positions of the sets \( \sigma_i \) and \( \sigma_i' \) described in condition (1.18). This follows from Remark 3.3 and the subsequent examples together with the equalities (4.1).

Theorems 3.7, 3.9, and 3.11 enable us to prove Theorem 1.3.

Proof of Theorem 1.3. By definition (1.9) and Theorem 2.10 we have
\[
\sigma_i' = \text{spec}(Z_i), \quad i = 0, 1,
\]
with \( Z_0 = A_0 + BK, \text{Dom}(Z_0) = \text{Dom}(A_0) \) and \( Z_1 = A_1 - B^*K^* \), \( \text{Dom}(Z_1) = \text{Dom}(A_1) \). Hence assumption (1.20) implies that \( \delta_{Z_0Z_1} = \text{dist}(\sigma_0', \sigma_1') = \hat{\delta} > 0 \) by (3.2).

As in the proof of Theorem 1.2 we conclude that \( \mathcal{S}'_0 \) is the graph \( \mathcal{G}(K) \) of a uniformly contractive strong solution \( K \) to the operator Riccati equation (1.11), while \( \mathcal{S}'_1 \) is the graph \( \mathcal{G}(K^*) \) of the adjoint of \( K \).

For claim (i), the bound (1.24) follows from estimate (3.24) in Theorem 3.7 using relation (4.1).

For claim (ii), the bound (1.23) for \( i = 0 \) follows from estimate (3.27) in Theorem 3.9 again using relation (4.1); for \( i = 1 \) it follows from the case \( i = 0 \) by passing from \( J \) to the new involution \( J' = -J \).

For claim (iii), the bound (1.25) follows from estimate (3.45) in Theorem 3.11 if we use (4.1) and the facts that \( \| \tan \Theta_j \| = \| \Theta_j \| \) and, by (1.12), \( \| \tan 2\Theta_j \| = 2 \| \Theta_j \|, j = 0, 1. \)

Remark 4.2. The bound (1.23) is sharp whenever the gap of \( \sigma_i'-\sigma_i \) containing \( \sigma_i' \) is finite. For \( i = 0 \) this follows from the fact that the estimate (3.27) in Theorem 3.9 is sharp by Example 3.5 (see Remark 3.10) together with the identity (4.1); for \( i = 1 \) it follows using the involution \( J' = -J \) instead of \( J \).

Remark 4.3. The bound (1.25) is best possible. This follows from the fact that the estimate (3.45) in Theorem 3.11 is sharp by Example 3.6 (see Remark 3.12 together with (4.1) and (4.1)).

5. Estimates for the Perturbed Spectra

In the next section we want to use the operator angle bounds of Theorems 1.2 and 1.3 to prove a priori bounds on the variation of spectral subspaces of the self-adjoint operator \( A \) under an off-diagonal \( J \)-self-adjoint perturbation \( V \). To this end, we establish some tight enclosures for the spectral components of the perturbed operator \( L = A + V \) in the present section.

We assume that the initial spectra \( \sigma_0 = \text{spec}(A_0) \) and \( \sigma_1 = \text{spec}(A_1) \) of the block diagonal entries \( A_0 \) and \( A_1 \) of \( A \) (see (1.4)) are disjoint, i.e.
\[
d = \text{dist}(\sigma_0, \sigma_1) > 0.
\]
Then the subspaces \( \mathcal{S}_0 \) and \( \mathcal{S}_1 \) introduced in Assumption 2.1 are the spectral subspaces of \( A \) associated with the spectral components \( \sigma_0 \) and \( \sigma_1 \), respectively.

In the following, our aim is to find certain bounds for the perturbation \( V \) and a constant \( r_V \geq 0 \) such that
\[
\text{dist}(\sigma_i', \sigma_i') > 0 \quad \text{and} \quad \sigma_i' \subset O_{r_V}(\sigma_i) \cap \mathbb{R}, \quad i = 0, 1.
\]
This yields the lower bounds
\[
\delta_i = \text{dist}(\sigma_i, \sigma_{i-1}) \geq d - r_V, \quad i = 0, 1, \quad \text{and} \quad \hat{\delta} = \text{dist}(\sigma_0', \sigma_1') \geq d - 2r_V.
\]
Together with the estimates in Theorems 1.2 and 1.3 respectively, they will give us the desired a priori estimates for \( \tan \Theta_j \), depending only on the initial distance \( d \) of the unperturbed spectra and on the norm of \( V \).

For completely arbitrary (i.e. not necessarily off-diagonal) perturbations \( V \) of the self-adjoint operator \( A \), it is well-known that the assumption

\[
\| V \| < \frac{d}{2} \tag{5.4}
\]

guarantees that (5.2) holds with

\[
r_V = \| V \| \tag{5.5}
\]

(see, e.g., [17, Section V.4]) and hence, by (5.3),

\[
\delta_i \geq d - \| V \| > \frac{d}{2}, \quad i = 0, 1, \quad \hat{\delta} \geq d - 2\| V \| > 0. \tag{5.6}
\]

For off-diagonal perturbations \( V \), earlier results in [19], [20] for self-adjoint \( V \) and in [6] for non-symmetric \( V \) show that the constant \( r_V \) may be improved considerably. In the following we extend these results under the sole assumption (5.1) that the spectral components \( \sigma_0 \) and \( \sigma_1 \) of \( A \) are disjoint.

Unlike the previous sections, we do not always require \( A \) to be self-adjoint and \( V \) to be \( J \)-self-adjoint; here we use the following more general setting.

**Assumption 5.1.** Let \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) be complementary orthogonal subspaces of the Hilbert space \( \mathcal{H} \). Assume that \( A \) is a closed operator on \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \) diagonal with respect to this decomposition, i.e.

\[
A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}, \quad \text{Dom}(A) = \text{Dom}(A_0) \oplus \text{Dom}(A_1),
\]

where \( A_0 \) and \( A_1 \) are closed operators on \( \mathcal{H}_0 \) and \( \mathcal{H}_0 \), respectively. Suppose that \( V \) is an off-diagonal bounded operator on \( \mathcal{H} \), i.e.

\[
V = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}
\]

with \( B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0), \ C \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1) \), and let

\[
L = A + V = \begin{pmatrix} A_0 & B \\ C & A_1 \end{pmatrix}, \quad \text{Dom}(L) = \text{Dom}(A). \tag{5.7}
\]

The following two elementary auxiliary results are used in the proofs below.

**Lemma 5.2.** Assume that \( L = A + V \) satisfies Assumption 5.1 and define the Schur complement \( S_0 \) of \( A \) by

\[
S_0(\lambda) := A_0 - \lambda - B(A_1 - \lambda)^{-1}C, \quad \text{Dom}(S_0(\lambda)) := \text{Dom}(A_0),
\]

for \( \lambda \in \rho(A_1) \). Then

(i) the resolvent set \( \rho(S_0) := \{ \lambda \in \rho(A_1) : S_0(\lambda) \text{ is bijective} \} \) of \( S_0 \) satisfies

\[
\rho(S_0) = \rho(L) \cap \rho(A_1);
\]

(ii) for \( \lambda \in \rho(A_0) \cap \rho(A_1) \) we have

\[
\| B(A_1 - \lambda)^{-1}C(A_0 - \lambda)^{-1} \| < 1 \implies \lambda \in \rho(L). \tag{5.8}
\]
Proof. Both claims (i) and (ii) are well known (see, e.g., [32]); we recall the short proofs for the convenience of the reader.

(i) It is easy to check that, for arbitrary \( f \in \mathcal{H}_0, \ g \in \mathcal{H}_1 \) and \( x \in \text{Dom}(A_0), \ y \in \text{Dom}(A_1) \),

\[
(L - \lambda)(x, y) = \begin{pmatrix} f \\ g \end{pmatrix} \iff \begin{cases} S_0(\lambda)x = f - B(A_1 - \lambda)^{-1}g, \\ y = (A_1 - \lambda)^{-1}(g - Cx), \end{cases}
\]

which proves the claim.

(ii) For \( \lambda \in \rho(A_0) \cap \rho(A_1) \), one can write

\[
S_0(\lambda) = (I - B(A_1 - \lambda)^{-1}C(A_0 - \lambda)^{-1})(A_0 - \lambda).
\]

Thus a Neumann series argument together with (i) proves (5.8). \( \square \)

**Lemma 5.3.** Let \( a, b, v \in \mathbb{R} \) be such that \( \delta := b - a > 0 \) and \( 0 \leq v < \delta / 2 \). Then

\[
(t - a)(b - t) > v^2 \iff a + r < t < b - r
\]

where

\[
r = \frac{\delta}{2} - \sqrt{\frac{\delta^2}{4} - v^2} = v \tan \left( \frac{1}{2} \arcsin \frac{2v}{\delta} \right).
\]

Proof. The claims are obvious; for the last equality, observe the formula for double arguments of the sine function in terms of the tangent function. \( \square \)

In the following theorem, we consider the case where the diagonal entries \( A_0 \) and \( A_1 \) of the block operator matrix (5.7) are self-adjoint and their spectra do not intersect; the bounded perturbation \( V \) need not have any symmetry here.

**Theorem 5.4.** Assume that \( L = A + V \) satisfies Assumption 5.1 and let \( A_0 \) and \( A_1 \) be self-adjoint. Assume, in addition, that their spectra \( \sigma_0 = \text{spec}(A_0) \) and \( \sigma_1 = \text{spec}(A_1) \) are disjoint, i.e.

\[
d = \text{dist}(\sigma_0, \sigma_1) > 0,
\]

and let the entries \( B \) and \( C \) of \( V \) be such that

\[
\sqrt{\|B\|\|C\|} < \frac{d}{2}.
\]

Then

\[
\text{spec}(L) = \sigma'_0 \cup \sigma'_1, \quad \sigma'_i \subset O_{r_V}(\sigma_i) \cap \mathbb{R}, \ i = 0, 1,
\]

where \( r_V \) is given by

\[
r_V = \sqrt{\|B\|\|C\|} \tan \left( \frac{1}{2} \arcsin \frac{2\sqrt{\|B\|\|C\|}}{d} \right) < \sqrt{\|B\|\|C\|};
\]

in particular, if \( V \) is J-self-adjoint and hence \( C = -B^* \), then

\[
r_V = \|V\| \tan \left( \frac{1}{2} \arcsin \frac{2\|V\|}{d} \right) < \|V\|.
\]

Proof. Throughout the proof, we assume that \( \lambda \in \mathbb{C} \) is such that

\[
\text{dist}(\lambda, \sigma_0 \cup \sigma_1) > r_V;
\]
hence, in particular, \( \lambda \in \rho(A_0) \cap \rho(A_1) \). Since \( A_0 \) and \( A_1 \) are assumed to be self-adjoint, we have \( \|(A_i - \lambda)^{-1}\| = 1 / \text{dist}(\lambda, \sigma_i), i = 0, 1 \), and thus
\[
\|B(A_1 - \lambda)^{-1}C(A_0 - \lambda)^{-1}\| \leq \frac{\|B\|\|C\|}{\text{dist}(\lambda, \sigma_0)\text{dist}(\lambda, \sigma_1)}. \tag{5.15}
\]

First we consider the case that \( \lambda \) lies in a strip of the form
\[
\{ z \in \mathbb{C} \mid a + r \nu < \Re z < b - r \nu \} \tag{5.16}
\]
where \((a, b)\) is a finite gap of the spectrum of \( A \) with \( a \in \sigma_0 \) and \( b \in \sigma_1 \); the case \( a \in \sigma_1 \) and \( b \in \sigma_0 \) is analogous. Then we have \( b - a \geq \delta \) and hence, by assumption (5.10) and Lemma 5.3, we obtain
\[
\frac{\|B\|\|C\|}{\text{dist}(\lambda, \sigma_0)\text{dist}(\lambda, \sigma_1)} = \frac{\|B\|\|C\|}{|\lambda - a|\|\lambda - b|} \leq \frac{\|B\|\|C\|}{(\Re \lambda - a)(b - \Re \lambda)} < 1. \tag{5.17}
\]

Now (5.15) and (5.17) together with Lemma 5.2(ii) show that \( \lambda \in \rho(L) \).

If \( \lambda \) does not belong to a strip of the form (5.16) with \( a \in \sigma_i \) and \( b \in \sigma_{1 - i} \) for \( i = 0 \) or \( i = 1 \), then it is not difficult to check that, by (5.14), either for \( i = 0 \) or \( i = 1 \)
\[
\text{dist}(\Re \lambda, \sigma_i) \geq d - r \nu, \quad \text{dist}(\lambda, \sigma_{1 - i}) > r \nu. \tag{5.18}
\]

Combining (5.15), (5.18) together with the definition (5.12) of \( r \nu \), we arrive at the estimate
\[
\|B(A_1 - \lambda)^{-1}C(A_0 - \lambda)^{-1}\| < \frac{\|B\|\|C\|}{r \nu(d - r \nu)} = 1.
\]

Hence Lemma 5.2(ii) again shows that \( \lambda \in \rho(L) \). \( \square \)

**Remark 5.5.** Theorem 5.4 improves the spectral bounds given in [6] Remarks 4.6 and 4.13. In fact, the bound \( r \) in [6] (4.11) can be written equivalently as
\[
r = \sqrt{\|B\|\|C\|} \tanh \left( \frac{1}{2} \arctanh \frac{2\sqrt{\|B\|\|C\|}}{\delta} \right) = \sqrt{\|B\|\|C\|} \tan \left( \frac{1}{2} \arcsin \frac{2\sqrt{\|B\|\|C\|}}{\delta} \right) \tag{5.19}
\]
and applies whenever \( \|B\|\|C\| < \delta / 2 \). Here, by [6] Theorem 4.11, we have \( \delta = 2d / \pi < d \) in the general case (5.1) and \( \delta = d \) if one additionally assumes that
\[
\text{conv}(\sigma_0) \cap \sigma_1 = \emptyset \quad \text{or} \quad \sigma_0 \cap \text{conv}(\sigma_1) = \emptyset. \tag{5.20}
\]
This shows that, under the additional assumption (5.20), the spectral bound in Theorem 5.4 coincides with the one in [6] Remarks 4.6, whereas in the general case (5.1) Theorem 5.4 holds with the weaker norm bound \( \|B\|\|C\| < d / 2 \) on \( V \) and the bound \( r \nu \) is strictly smaller than the bound \( r \) in [6] (4.11).

**Remark 5.6.** The spectral bound (5.11) with \( r \nu \) given by (5.12) is optimal. This may be seen from [6] Examples 4.15, 4.16.

In the next two theorems we drop the assumption that \( A \) is self-adjoint. Instead we impose conditions to ensure that the components \( A_0 \) and \( A_1 \) of \( A \) satisfy certain resolvent estimates.

To this end, we use the *numerical range* \( W(T) \) of a linear operator \( T \) with domain \( \text{Dom}(T) \) in a Hilbert space, defined as
\[
W(T) = \{(Tx, x) \mid x \in \text{Dom}(T), \|x\| = 1\}.
\]
Recall that the numerical range is always convex and that \( \text{spec}(T) \subset \overline{W(T)} \) if every (of the at most two) connected components of \( \mathbb{C} \setminus W(T) \) contains at least one point of \( \rho(T) \) (see \cite[Theorems V.3.1 and V.3.2]{17}); in this case, 

\[
\|(T - \lambda)^{-1}\| \leq \frac{1}{\text{dist}(\lambda, W(T))}, \quad \lambda \notin \overline{W(T)}.
\]

First we consider the case where the spectra and the numerical ranges of \( A_0 \) and \( A_1 \) are separated by a vertical strip.

**Theorem 5.7.** Assume that \( L = A + V \) satisfies Assumption 5.1 and that there exist \( a, b \in \mathbb{R} \) such that \( d = b - a > 0 \),

\[
\text{Re} W(A_0) \leq a < b \leq \text{Re} W(A_1) \quad \text{or} \quad \text{Re} W(A_1) \leq a < b \leq \text{Re} W(A_0),
\]

and

\[
\{ z \in \mathbb{C} \mid a < \text{Re} z < b \} \cap \rho(A_0) \cap \rho(A_1) \neq \emptyset.
\]

If

\[
\sqrt{\|B\| \|C\|} < d/2,
\]

and \( r_V \) is defined as in (5.12), then

\[
\{ z \in \mathbb{C} \mid a + r_V < \text{Re} z < b - r_V \} \subset \rho(L).
\]

**Proof.** Without loss of generality, we suppose that \( \text{Re} W(A_0) \leq a < b \leq \text{Re} W(A_1) \). In this case the assumptions (5.21), (5.22) imply that (see \cite[Theorem V.3.2]{17})

\[
\text{spec}(A_0) \subset W(A_0) \subset \{ z \in \mathbb{C} \mid \text{Re} z \leq a \}, \quad \text{spec}(A_1) \subset W(A_1) \subset \{ z \in \mathbb{C} \mid \text{Re} z \geq b \}. \quad (5.23)
\]

Let \( \lambda \in \mathbb{C} \) be such that \( a + r_V < \text{Re} \lambda < b + r_V \) and hence \( \lambda \in \rho(A_0) \cap \rho(A_1) \). Then, by (5.23),

\[
\|(A_0 - \lambda)^{-1}\| \leq \frac{1}{\text{dist}(\lambda, W(A_0))} \leq \frac{1}{\text{Re} \lambda - a}, \quad \|(A_1 - \lambda)^{-1}\| \leq \frac{1}{\text{dist}(\lambda, W(A_1))} \leq \frac{1}{b - \text{Re} \lambda}.
\]

Thus Lemma 5.3 shows that

\[
\|B(A_1 - \lambda)^{-1}C(A_0 - \lambda)^{-1}\| \leq \frac{\|B\| \|C\|}{(\text{Re} \lambda - a)(b - \text{Re} \lambda)} < 1,
\]

and hence \( \lambda \in \rho(L) \) by Lemma 5.2(ii). \( \square \)

Next we consider the case where the spectra and the numerical ranges of \( A_0 \) and \( A_1 \) (and hence of \( A \)) lie in one half-plane and the perturbation \( V \) is \( J \)-self-adjoint, i.e. \( C = -B^* \).

While all previous theorems were of perturbational character, the following theorem is not. In fact, we prove implications of the form

\[
\text{Re} \text{spec}(A) \leq \text{Re} W(A) \leq a \quad \Rightarrow \quad \text{Re} \text{spec}(L) \leq a,
\]

independently of the norm of \( V \).

This type of results relies on the **quadratic numerical range** \( W^2(L) \) of the operator \( L \) with respect to the block representation (5.7). The set \( W^2(L) \) is defined as (see \cite[(2.2)]{25} and also \cite[Definition 3.1]{33} or \cite[Definition 2.5.1]{32})

\[
W^2(L) = \bigcup_{x \in \text{Dom}(A_0), \ y \in \text{Dom}(A_1)} \{ \text{spec}(L_{x,y}) \}_{|x| = |y| = 1}.
\]
where $L_{x,y} \in M_2(\mathbb{C})$ is a $2 \times 2$ matrix given by

$$L_{x,y} := \begin{pmatrix} A_{0,x,x} & (A_{y,x}) \\ (B_{y,x}) & (A_{y,y}) \end{pmatrix}, \quad x \in \text{Dom}(A_0), y \in \text{Dom}(A_1), \|x\| = \|y\| = 1.$$

The quadratic numerical range is not convex and may consist of at most two connected components. It is always contained in the numerical range, $W^2(L) \subset W(L)$, and the inclusion $\text{spec}(L) \subset W^2(L)$ holds if every connected component of $\mathbb{C} \setminus W^2(L)$ contains at least one point of $\rho(L)$ (see \cite[Theorem 2.1]{25}, \cite[Proposition 3.2, Theorems 4.2 and 4.7]{33} or \cite[Theorem 2.5.3, Theorems 2.5.10 and 2.5.15]{32}).

**Theorem 5.8.** Assume that $L = A + V$ satisfies Assumption \ref{A} with $C = -B^*$ and let $a, b \in \mathbb{R}$.

(i) If $\text{Re} W(A) \leq a$ and $\rho(A) \cap \{z \in \mathbb{C} \mid \text{Re} z > a\} \neq \emptyset$, then $\text{Re} \text{spec}(L) \leq a$.

(ii) If $\text{Re} W(A) \geq b$ and $\rho(A) \cap \{z \in \mathbb{C} \mid \text{Re} z < b\} \neq \emptyset$, then $\text{Re} \text{spec}(L) \geq b$.

In particular, if $A$ is self-adjoint, we have

$$\inf \text{spec}(A) \leq \text{Re} \text{spec}(L) \leq \sup \text{spec}(A).$$

**Proof.** We prove (i); the proof of (ii) is completely analogous. By the assumption on $A$ and since $V$ is bounded, it is obvious that $\{z \in \mathbb{C} \mid \text{Re} z > a\} \cap \rho(L) \neq \emptyset$ for $L = A + V$. Hence $\text{spec}(L) \subset W^2(L)$ and so it suffices to show that $\text{Re} W^2(L) \leq a$.

Suppose, to the contrary, that there exists a $\lambda \in W^2(L)$ with $\text{Re} \lambda > a$. By the definition of $W^2(L)$, there are $x \in \text{Dom}(A_0), y \in \text{Dom}(A_1), \|x\| = \|y\| = 1$, such that $\lambda \in \text{spec}(L_{x,y})$, i.e.

$$0 = \det (L_{x,y} - \lambda) = (A_{0,x,x} - \lambda)(A_{y,y} - \lambda) + \|B_{y,x}\|^2,$$

where we have used that $C = -B^*$. Splitting into real and imaginary parts, we conclude that

$$0 = \text{Re} (A_{0,x,x} - \lambda) \text{Im} (A_{1,y,y} - \lambda) + \text{Im} (A_{0,x,x} - \lambda) \text{Re} (A_{1,y,y} - \lambda),$$

$$-\|B_{y,x}\|^2 = \text{Re} (A_{0,x,x} - \lambda) \text{Re} (A_{1,y,y} - \lambda) - \text{Im} (A_{0,x,x} - \lambda) \text{Im} (A_{1,y,y} - \lambda).$$

Solving the first equation for $\text{Im} (A_{1,y,y} - \lambda)$ and inserting into the second equation, we find

$$-\|B_{y,x}\|^2 = \left( (\text{Re} (A_{0,x,x}) - \lambda)^2 + (\text{Im} (A_{0,x,x}) - \lambda)^2 \right) \frac{\text{Re} (A_{1,y,y}) - \lambda}{\text{Re} (A_{0,x,x}) - \lambda}.$$

By the assumption on $A$, we have $\text{Re} \lambda > a \geq \text{Re} W(A) = \text{Re} (W(A_0) \cup W(A_1))$ and hence both the first and the second factor on the right hand side of (5.24) are positive, a contradiction.

6. A priori bounds on variation of spectral subspaces

In this section we use the (semi-) a posteriori norm bounds for the operator angles from Theorems \ref{1.2} and \ref{1.3} together with the spectral estimates from Section \ref{5} to derive a priori estimates for the variation of the spectral subspaces of the self-adjoint operator $A$ under a $J$-self-adjoint off-diagonal perturbation $V$.

To ensure that solutions of the corresponding Riccati equations exist, we use some results of \cite{6} and \cite{35}. They provide sufficient conditions on the perturbation $V$ and spectral sets $\sigma_0$ and $\sigma_1$ guaranteeing that the perturbed operator $L = A + V$ is similar to a self-adjoint operator on a Hilbert space and that the spectral subspaces of $L$ associated with the perturbed spectral sets $\sigma_0'$ and $\sigma_1'$, both being real in this case, are maximal uniformly definite in the Krein space $\mathbb{K}$. 


In order to formulate the conditions from [35], we need to specify all those finite gaps of the spectrum of \( A \) that separate the subsets \( \sigma_0 \) and \( \sigma_1 \). We denote these gaps by \( \Delta_n \), where \( n \in \mathbb{Z} \) runs from \( N_- \) through \( N_+ \) with \(- \infty \leq N_- \leq 0, 0 \leq N_+ \leq +\infty \), and let\[ \Delta_n = (a_n, b_n), \quad -\infty < \ldots < a_{n-1} < b_{n-1} \leq a_n < b_n \leq a_{n+1} < b_{n+1} \leq \ldots \leq +\infty, \]assigning the value of \( n = 0 \), say, to the gap that is closest to the origin \( z = 0 \). For every such gap, we have \( \Delta_n \cap \sigma_0 = \emptyset, \Delta_n \cap \sigma_1 = \emptyset \), and \( a_n \in \sigma_i, \ b_n \in \sigma_{1-i} \) where either \( i = 0 \) or \( i = 1 \).

If the total number \( N \) of the gaps between \( \sigma_0 \) and \( \sigma_1 \) is finite, then both \( N_- \), \( N_+ \) are finite and \( N = |N_-| + N_+ + 1 \). Otherwise, at least one of \( N_- \) and \( N_+ \) is infinite.

The next theorem is an immediate consequence of [6] Theorem 5.8] and [35] Theorem 3 and Corollary 4], combined with Theorem [5.4]

**Theorem 6.1.** Assume that \( L = A + V \) satisfies Assumption [2.7] and let the spectra \( \sigma_0 = \text{spec}(A_0) \) and \( \sigma_1 = \text{spec}(A_1) \) be disjoint, i.e.
\[ d = \text{dist}(\sigma_0, \sigma_1) > 0. \]

Assume, in addition, that one of the following holds:

\( i \) \( \|V\| < \frac{d}{\pi} \),

\( ii \) \( \|V\| < \frac{d}{2} \) and \( \sum_{n=N_-}^{N_+} \frac{1}{b_n - a_n} < \infty. \)

Then
\[ \text{spec}(L) \subset \sigma_0' \cup \sigma_1', \quad \sigma_i' \subset O_{r_V}(\sigma_i) \cap \mathbb{R}, \quad i = 0, 1, \]
with \( r_V \) given by (5.19). The operator \( L \) is similar to a self-adjoint operator on \( \tilde{\mathcal{H}} \). The spectral subspaces \( \tilde{\mathcal{H}}_0' \) and \( \tilde{\mathcal{H}}_1' \) of \( L \) associated with the sets \( \sigma_0' \) and \( \sigma_1' \) are mutually orthogonal in the Krein space \( \{ \tilde{\mathcal{H}}, J \} \) and maximal uniformly positive resp. negative therein.

**Remark 6.2.** In case (i) Theorem 6.1 follows from [6] Theorem 5.8 (i) and Remark 5.11], together with Theorem 5.4]. In case (ii) the spectral projections \( E_L(\sigma_i') \) of \( L \) associated with its isolated spectral components \( \sigma_i' \), \( i = 0, 1, \) are well-defined (see [35] Corollary 4]), the operator \( J(E_L(\sigma_0') - E_L(\sigma_1')) \) is self-adjoint, and there exists a \( \gamma \in (0, 1] \) such that
\[ J(E_L(\sigma_0') - E_L(\sigma_1')) \geq \gamma I. \] (6.2)

The latter was established (under much more general assumptions on \( V \) than (5.4]) in the proof of [35] Theorems 1 and 3]. Using inequality (6.2], one easily verifies that the spectral subspaces \( \tilde{\mathcal{H}}_0' = \text{Ran} E_L(\sigma_0') \) and \( \tilde{\mathcal{H}}_1' = \text{Ran} E_L(\sigma_1') \) are maximal uniformly positive and maximal uniformly negative, respectively. In fact, it suffices to show the uniform definiteness of one spectral subspace (see Corollary 2.12).

**Remark 6.3.** The lengths of the gaps \( (a_n, b_n) \) of \( \text{spec}(A) \) separating the sets \( \sigma_0 \) and \( \sigma_1 \) have to be uniformly bounded from below. Apart from this, condition (i) imposes no further restriction on the behaviour of the lengths, whereas condition (ii) requires that \( b_n - a_n \) tends to \( \infty \) faster than \( |n| \) as \( |n| \to \infty \).

The following a priori bound on the operator angles \( \Theta(\tilde{\mathcal{H}}_i, \tilde{\mathcal{H}}_i') \) between the unperturbed and the perturbed spectral subspaces \( \tilde{\mathcal{H}}_i \) of \( A \) and \( \tilde{\mathcal{H}}_i' \) of \( L = A + V \) improves the corresponding bound derived in [6] Theorem 5.8 (i)] (see Remark 6.6 below).
Suppose that \( L = A + V \) satisfies the assumptions of Theorem 6.1. Let \( \Sigma_i, \Sigma'_i \) be the spectral subspaces of \( A \) corresponding to \( \sigma_i \) and of \( L \) corresponding to \( \sigma'_i \), respectively, \( i = 0, 1 \). Then the operator angles \( \Theta_i = \Theta(\Sigma_i, \Sigma'_i) \), \( i = 0, 1 \), satisfy the estimate

\[
\tan \Theta_i \leq \frac{\pi}{2} \tan \left( \frac{1}{2} \arcsin \frac{2\|V\|}{d} \right), \quad i = 0, 1;
\]

if, in addition,

\[
\text{conv}(\sigma_0) \cap \sigma_1 = \emptyset \quad \text{or} \quad \sigma_0 \cap \text{conv}(\sigma_1) = \emptyset,
\]

then

\[
\tan \Theta_i \leq \frac{1}{2} \arcsin \frac{2\|V\|}{d}, \quad i = 0, 1.
\]

Proof. By Theorem 6.1, the perturbed operator \( L \) is similar to a self-adjoint operator and its disjoint spectral components \( \sigma'_i \subset \mathbb{R} \), \( i = 0, 1 \), satisfy the inclusions (5.2) with \( r_V \) given by (5.19). The latter implies that, for \( i = 0, 1 \),

\[
\delta_i = \text{dist}(\sigma_i, \sigma_{1-i}) \geq d - r_V = \sqrt{\frac{d^2}{4} - \|V\|^2}
\]

and hence, by (5.9),

\[
\frac{\|V\|}{\delta_i} \leq \frac{\|V\|}{d - r_V} = \tan \left( \frac{1}{2} \arcsin \frac{2\|V\|}{d} \right).
\]

In addition, Theorem 6.1 also shows that the spectral subspaces \( \Sigma'_i \) associated with \( \sigma'_i \) are maximal uniformly definite. Thus all assumptions of Theorem 1.2 (i) (1.2 (ii), respectively) are satisfied and the claimed bound (6.3) (6.5), respectively follows from (1.17) (1.19, respectively) together with (6.6).

Remark 6.5. The operators \( \tan \Theta_i \), \( i = 0, 1 \), in Theorem 6.4 coincide with the moduli \( |K| \) or \( |K'| \) of uniformly contractive solutions to the Riccati equations (1.11) and (2.4), respectively (see Remark 2.6 and Lemma 2.9); hence we always have \( \tan \Theta_i < 1 \).

If \( L = A + V \) satisfies condition (i) in Theorem 6.1 i.e. \( \|V\| < d/\pi \), then the bound (6.3) is always less than 1; if \( L \) satisfies condition (ii) in this theorem and hence \( \|V\| < d/2 \), then for the bound (6.3) to be less than 1 we need to have \( d \) and \( V \) such that

\[
\|V\| < \frac{d}{2} \sin \left( \frac{1}{2} \arctanh \frac{2\|V\|}{\pi} \right) = \frac{d}{2} \frac{4\pi}{4 + \pi^2} \approx \frac{d}{2} \times 0.9060367012.
\]

Remark 6.6. The bound (6.3) is stronger than the previously known bound

\[
\tan \Theta_i \leq \frac{\pi}{2} \arctanh \left( \frac{\|V\|}{d} \right) = \frac{\pi}{2} \arcsin \frac{\pi\|V\|}{d}, \quad i = 0, 1,
\]

from [6] Theorem 5.8 (i)] and extends it to perturbations \( V \) that do not satisfy the condition \( \|V\| < d/\pi \) required therein. The former is a consequence of the trigonometric inequality

\[
\frac{\pi}{2} \arcsin (2t) < \frac{\pi}{2} \arctanh (\pi t), \quad t \in (0, 1/\pi).
\]

Note that, if \( \|V\| < d/\pi \), then the bound (6.7) may be written equivalently as

\[
\sin 2\Theta_i \leq \frac{\pi\|V\|}{d}, \quad i = 0, 1.
\]

For the particular case (6.4), the bound (6.5) coincides with the previously known bound (5.14) from [6] Theorem 5.8 (ii)] since then the corresponding spectral bounds \( r_V \) and \( r \) (defined in (5.13) and (5.19), respectively) coincide (see Remark 5.5).
Theorem 1.3 (iii) results exactly in the a priori sharp norm bound (1.15) from Theorem 1.1.

Remark 6.8. If \( \| V \| < d/2 \) and the spectral sets \( \sigma_0 \) and \( \sigma_1 \) are bounded and subordinated, i.e. \( \text{conv}(\sigma_0) \cap \text{conv}(\sigma_1) = \emptyset \), then combining inequality (6.9) with the estimate (1.25) from Theorem 1.3 (iii) results exactly in the a priori sharp norm bound (1.15) from Theorem 1.1.

7. Quantum harmonic oscillator under a \( \mathcal{PT} \)-symmetric perturbation

In this section we apply the results of the previous sections to the \( N \)-dimensional isotropic quantum harmonic oscillator under a \( \mathcal{PT} \)-symmetric perturbation.

Let \( \mathcal{H} = L_2(\mathbb{R}^N) \) for some \( N \in \mathbb{N} \). Assuming that the units are chosen such that \( \hbar = m = \omega = 1 \), the Hamiltonian of the isotropic quantum harmonic oscillator is given by

\[
(Af)(x) = -\frac{1}{2} \Delta f(x) + \frac{1}{2} |x|^2 f(x), \quad \text{Dom}(A) = \left\{ f \in W^2_0(\mathbb{R}^N) \left| \int_{\mathbb{R}^N} dx |x|^4 |f(x)|^2 < \infty \right. \right\},
\]

where \( \Delta \) is the Laplacian and \( W^2_0(\mathbb{R}^N) \) stands for the Sobolev space of \( L^2(\mathbb{R}^N) \)-functions that have their second partial derivatives in \( L^2(\mathbb{R}^N) \).

It is well-known that the Hamiltonian \( A \) is a self-adjoint operator in \( L_2(\mathbb{R}^N) \) and its spectrum consists of eigenvalues of the form

\[
\lambda_n = n + N/2, \quad n = 0, 1, 2, \ldots,
\]

whose multiplicities \( \mu_n \) are given by the binomial coefficients (see, e.g., [28] and the references therein)

\[
\mu_n = \binom{N + n - 1}{n}, \quad n = 0, 1, 2, \ldots.
\]

For \( n \) even, the corresponding eigenfunctions \( f(x) \) are symmetric with respect to space reflection \( x \mapsto -x \) (i.e. \( f(-x) = f(x) \)). For \( n \) odd, the eigenfunctions are anti-symmetric (i.e. \( f(-x) = -f(x) \)). Hence if we partition the spectrum \( \text{spec}(A) = \sigma_0 \cup \sigma_1 \) with

\[
\sigma_0 = \{ n + N/2 \mid n = 0, 2, 4, \ldots \}, \quad \sigma_1 = \{ n + N/2 \mid n = 1, 3, 5, \ldots \},
\]

then the subspaces

\[
\mathcal{H}_0 = L_{2,\text{even}}(\mathbb{R}^N), \quad \mathcal{H}_1 = L_{2,\text{odd}}(\mathbb{R}^N)
\]

of symmetric and anti-symmetric functions are the complementary spectral subspaces of \( A \) corresponding to the spectral components \( \sigma_0 \) and \( \sigma_1 \), respectively. Obviously,

\[
d = \text{dist}(\sigma_0, \sigma_1) = 1.
\]
Let $\mathcal{P}$ be the parity operator on $L_2(\mathbb{R}^N)$, $(\mathcal{P}f)(-x) = f(-x)$, and $\mathcal{T}$ the (antilinear) operator of complex conjugation, $(\mathcal{T}f)(x) = \overline{f(x)}$, $f \in L_2(\mathbb{R}^N)$. An operator $V$ on $L_2(\mathbb{R}^N)$ is called $\mathcal{P}\mathcal{T}$-symmetric if it commutes with the product $\mathcal{P}\mathcal{T}$, i.e.,

$$\mathcal{P}\mathcal{T}V = V\mathcal{P}\mathcal{T}.$$  \hfill (7.5)

Clearly, the parity operator $\mathcal{P}$ is a self-adjoint involution on $L_2(\mathbb{R}^N)$ whose spectral subspaces

$$\text{Ran} \, \mathcal{P}(\{+1\}) = L_2_{\text{even}}(\mathbb{R}^N), \quad \text{Ran} \, \mathcal{P}(\{-1\}) = L_2_{\text{odd}}(\mathbb{R}^N)$$

coincide with the respective spectral subspaces (7.4) of the Hamiltonian (7.1).

From now on, let $V$ be the multiplication operator by a function of the form

$$V(x) = ib(x), \quad x \in \mathbb{R}^N,$$

where $b \in L_\infty(\mathbb{R}^N)$ is real-valued and anti-symmetric, i.e. $b(x) \in \mathbb{R}$ and $b(-x) = -b(x)$ for a.e. $x \in \mathbb{R}^N$. Such an operator $V$ is not only $\mathcal{P}\mathcal{T}$-symmetric on $L_2(\mathbb{R}^N)$ (see, e.g., [12, Section 3]) but also $J$-self-adjoint with respect to the involution $J = \mathcal{P}$. Moreover, it is anticommuting with $\mathcal{P}$ which means that such a $V$ is off-diagonal with respect to the decomposition $\mathcal{F}_j = \mathcal{F}_{j_0} \oplus \mathcal{F}_{j_1}$.

By [12] Theorem 3.2 the spectrum of the perturbed Hamiltonian $L = A + V$ given by

$$(Lf)(x) = -\frac{1}{2} \Delta f(x) + \frac{1}{2} |x|^2 f(x) + ib(x)f(x), \quad \text{Dom}(L) = \text{Dom}(A),$$

remains real (and discrete) whenever $||V|| = ||b||_\infty < 1/2$. Furthermore, [12, Proposition 3.5] implies that for $||V|| < 1/2$ the closed $O(||V||)\overline{\{\lambda_n\}}$-neighbourhood of the eigenvalue (7.2) of $A$ contains exactly $\mu_n$ (real) eigenvalues $\lambda_{n,k}$, $k = 1, 2, \ldots, \mu_n$, of $L$, counted with multiplicities, where $\mu_n$ is given by (7.3).

Combining [12] Theorem 3.2 and Proposition 3.5 with Theorem 5.4 ensures that, in fact, the eigenvalues $\lambda_{n,k}$ satisfy the estimates

$$|\lambda_{n,k} - (n + N/2)| < r_V, \quad n = 0, 1, 2, \ldots, \quad k = 1, 2, \ldots, \mu_n,$$

where

$$r_V = ||b||_\infty \tan \left(\frac{1}{2} \arcsin(2||b||_\infty)\right) < ||b||_\infty.$$

Further, assume that the stronger inequality $||V|| = ||b||_\infty < 1/\pi$ holds. In this case it follows from [6, Theorem 5.8 (i)] that $L$ is similar to a self-adjoint operator. At the same time, Theorem 6.4 implies the following bound on the variation of the spectral subspaces (7.4): \hfill (7.6)

$$\tan \Theta_j \leq \frac{\pi}{2} \tan \left(\frac{1}{2} \arcsin(2||b||_\infty)\right) < 1, \quad j = 0, 1,$$

where $\Theta_j = \Theta(\mathcal{F}_j, \mathcal{F}'_j)$ denotes the operator angle between the subspace $\mathcal{F}_j$ and the spectral subspace $\mathcal{F}'_j$ of $L$ associated with the spectral subset $\sigma'_j = \text{spec}(L) \cap O_{r_V}(\sigma_j)$, $j = 0, 1$. The estimate (7.6) improves the corresponding bound for the one-dimensional case obtained in [6, Section 6] (cf. Remark 6.6).

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