First passage time law for some Lévy processes with compound Poisson: existence of a conditionnal density with incomplete observation

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Abstract

We study the default risk in incomplete information. That means, we model the value of a firm by one Lévy process which is the sum of brownian motion with drift and compound Poisson process. This Lévy process can not be observed completely and we let an other process which representes the available information on the firm. We obtain an equation satisfied by the conditional density of the default time given the available information and closed form expression for the density.

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1 Introduction

In our work, we study a first passage time of a level $x > 0$ by a jump diffusion process $X$ which respectively models default and assets of a firm. We investigate the behavior of the default time under incomplete observation of assets. Such a study is very important when failure of a large industrial company or some great political decision taken by the parlement can affect the dividend policy of the issuing firms. In the litterature, we find some papers in relation to this topic. Duffie and Lando in [8] suppose that bond investors cannot observe the issuer’s assets directly and receive instead only periodic and imperfect reports. For a setting in which the assets of the firm are a geometric Brownian motion until informed equityholders optimally liquidate, they derive the conditional distribution of the assets, given accounting data and survivorship. Dorobantu [2] has the intensity function of the default time. That is very important for investors, but the information brought by this intensity is low. Volpi et al [18] prove that the Laplace transform of the random triple (first passage time, overshoot, undershoot) satisfies some kind of integral equation and after normalization of the first passage time, they show under some assumptions that the triple random converges in distribution as $x$ goes to $\infty$. In [9], the authors study a model of a financial market in which the dividend rates of two risky assets change their initial values to other constant ones at the times at which certain unobservable external events occur. The asset price dynamics are described by geometric Brownian motion with random drift rates switching at exponential random times which are independent of each other and of the constantly correlated driving Brownian motion. They obtain closed expressions for rational values of European contingent claims through the filtering estimates of the occurrence of

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switching times and their conditional probability density derived given the filtration generated by the underlying asset price process. Coutin and Dorobantu in [6] prove that the law of the default time has a density (defective when \( \mathbb{E}(X_1) < 0 \)) with respect to the Lebesgue measure. We extend this approach by using some filtering theory. The purpose of our paper is to add to these studies the behavior of the conditional law of a first passage time by a Lévy process with compound Poisson process given partial information. The paper is organized as follow: In Section 2, we recall the model and the filtering framework. In Section 3, we show the existence of the density and in Section 4, we show that the conditional density satisfies some kind of integro-differential equation. To finish, we give an appendix.

2 Models and filtering framework

In this section, we recall the model and the results of Coutin-Dorobantu [6] and present the filtering framework of Pardoux [14] or Coutin [5].

2.1 Model

Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) be a filtered probability space. Let \( \tilde{X} \) be a Brownian motion with drift \( m \in \mathbb{R} \) and for \( z > 0, \tilde{\tau}_z = \inf\{t \geq 0, \tilde{X}_t \geq z \} \). By (5.12) page 197 of [12], \( \tilde{\tau}_z \) has the following law on \( \mathbb{R}_+ \):

\[
\tilde{f}(u, z)du + \mathbb{P}(\tilde{\tau}_z = \infty)\delta_{\infty}(du)
\]

where

\[
\tilde{f}(u, z) = \frac{|z|}{\sqrt{2\pi u^3}} \exp\left[-\frac{1}{2u}(z - mu)^2\right]1_{[0, +\infty]}(u) \quad \text{and} \quad \mathbb{P}(\tilde{\tau}_z = \infty) = 1 - e^{mz - |mz|}.
\]

The function \( \tilde{f}(\cdot, z) \) is \( C^\infty \) on \( [0, +\infty[ \), and all its derivatives admits 0 as right limit at 0 and then it is \( C^\infty \). Let \( X \) be a Lévy process defined as following

\[
X_t = mt + W_t + \sum_{i=1}^{N_t} Y_i \quad t \in \mathbb{R}^+.
\]

Here \( (W_t)_{t \geq 0} \) is a brownian motion, \( m \in \mathbb{R} \), \( (N_t)_{t \geq 0} \) is a counting Poisson process with intensity \( \lambda \) and \( (Y_i)_{i \in \mathbb{N}^*} \) a sequence of identically and independent (iid) random variables with distribution \( F_Y \). All this object are independent. The process \( X \) models the firm value. The default is modeled by the hitting time of level \( x > 0 \). That means

\[
\tau_x = \inf\{t \geq 0 : X_t \geq x\}.
\]

We recall from Coutin-Dorobantu [6] that \( \tau_x \) has a density with respect to Lebesgue measure possibly defective which is defined by

\[
f(t, x) = \begin{cases} 
\lambda \mathbb{E}(1_{\tau_x > t}(1 - F_Y)(x - X_t)) + \mathbb{E}(1_{\tau_x > T_{N_t}} \tilde{f}(t - T_{N_t}, x - X_{T_{N_t}})) & \text{if } t > 0 \\
\frac{\lambda}{2}(2 - F_Y(x) - F_Y(x_-)) + \frac{\lambda}{4}(F_Y(x) - F_Y(x_-)) & \text{if } t = 0.
\end{cases}
\]
2.2 Filtering framework

Instead of observing perfectly the process $X$, we observe a process $Q$ defined by

$$Q_t = \int_0^t h(X_s)ds + B_t, \quad t \in \mathbb{R}_+$$

where $h$ is a Borel and bounded function, $B$ a Brownian motion which is independent of $W, N,$ and $Y$. This is a filtering problem and we introduce the framework as in [5] and [1]. Let $(\Omega^Q, F^Q, (F^Q_t, t \geq 0), \mathbb{P}^Q)$ (respectively $(\Omega^W, F^W, (F^W_t, t \geq 0), \mathbb{P}^W)$) be a measured space on which $Q$ (respectively $W$) is a $\mathbb{R}$-valued Brownian motion.

Let $(\Omega^M, F^M, (F^M_t, t \geq 0), \mathbb{P}^M)$ be a measured space on which $(Y_i, i \in \mathbb{N}^*)$ is a sequence of i.i.d random variables with distribution function $F_Y$ and $M$ a Poisson random measure with intensity $\Pi(dt, A) = \lambda \int_A F_Y(dy)dt, \quad \lambda > 0$. We define

$$\Omega^\circ = \Omega^Q \times \Omega^W \times \Omega^M, \quad F = F^Q \otimes F^W \otimes F^M, \quad \mathbb{P}^\circ = \mathbb{P}^Q \otimes \mathbb{P}^W \otimes \mathbb{P}^M.$$

All used filtrations are c\’adl\’ag and complete. Under $\mathbb{P}^\circ$, we consider the processes $Q$ and $X$ defined by

$$X_t = mt + W_t + \sum_{i=1}^{N_t} Y_i, \quad t \in \mathbb{R}_+.$$

Remarks 1.

1. The processes $(W, Q)$ is a $\mathbb{R}^2$-valued $(\mathbb{P}^\circ, F)$-Brownian motion.

2. The compensated measure $\tilde{N}$ has $(\mathbb{P}^\circ, F)$-intensity $\Pi(dt, A) = \lambda \int_A F_Y(dy)dt$.

Since the function $h$ is bounded the (Novikov) condition $\mathbb{E}\left(e^{\frac{1}{2} \int_0^t h^2(X_s)ds}\right) < \infty$ is satisfied and we define the following exponential martingale by

$$L_t = \exp\left(\int_0^t h(X_s)dQ_s - \frac{1}{2} \int_0^t h^2(X_s)ds\right), \quad t \in \mathbb{R}_+.$$

For a fixed maturity $T > 0$, the process $(L_t \wedge T, t \in \mathbb{R}_+)$ is a uniformly integrable $(\mathbb{P}^\circ, F)$-martingale.

Définition 1. The probability $\mathbb{P}$, called observation probability, is defined as follow

$$\frac{d\mathbb{P}}{d\mathbb{P}^\circ} |_{\mathcal{F}_T} = L_T.$$

The probability measures $\mathbb{P}$ and $\mathbb{P}^\circ$ are equivalent. Then using Girsanov theorem, the signal $X$ and the observation $Q$ are represented under $\mathbb{P}$ by

$$X_t = mt + W_t + \sum_{i=1}^{N_t} Y_i,$$

$$Q_t = \int_0^t h(X_s)ds + B_t, \quad t \in \mathbb{R}.$$

Remarks 2. The processes $(W, B)$ is a $\mathbb{R}^2$-valued $(\mathbb{P}, F)$-Brownian motion.

Under $\mathbb{P}$ as under $\mathbb{P}^\circ$, we use the same notations $m, W, N, Y$ and $Q$.  

3
3 Existence of the conditional density

Proposition 1. For all \( t > 0 \), on the set \( \{ \tau_x > t \} \), the law of \( \tau_x \) has the following form

\[
\tilde{f}(t, x) dr = \left[ 1 - \mathbb{E}(e^{m(x-X_t) - |m(x-X_t)|}) | \mathcal{G}_t \right] \delta_\infty(dr) \tag{4}
\]

where

\[
\tilde{f}(t, x) := \mathbb{E}[f(r-t, x-X_t) | \mathcal{G}_t]
\]

We start with lemma

Lemma 1. There exists some constantes \( \tilde{C} \) and \( C \) such that \( \forall t > 0, \ x > 0, \)

\[
\tilde{f}(t, x) \leq \tilde{C}(\frac{1}{t} + \frac{1}{\sqrt{t}}) \quad \text{and} \quad f(t, x) \leq C(1 + \frac{1}{\sqrt{t}} + \frac{1}{t^2}). \tag{5}
\]

Proof. On one hand, we have:

\[
\tilde{f}(t, x) = \frac{x}{\sqrt{2\pi t^3}} \exp\left[-\frac{(x-\sqrt{t})^2}{2t}\right]
\]

\[
= \frac{x-\sqrt{t}}{\sqrt{2\pi t^3}} \exp\left[-\frac{(x-\sqrt{t})^2}{2t}\right] + \frac{\sqrt{t}}{\sqrt{2\pi t^3}} \exp\left[-\frac{(x-\sqrt{t})^2}{2t}\right]
\]

\[
\leq \frac{x-\sqrt{t}}{\sqrt{2\pi t^3}} + \frac{|\sqrt{t}|}{\sqrt{2\pi t}} \exp\left[-\frac{(x-\sqrt{t})^2}{2t}\right]
\]

Let \( C = \sup_{u \in \mathbb{R}} \|ue^{-u^2/2}\| \) and it follows that

\[
\tilde{f}(t, x) \leq \frac{C}{t\sqrt{2\pi}} + \frac{|t|}{\sqrt{2\pi t}}, \quad t \in \mathbb{R}_+, x \in \mathbb{R}_+.
\]

On other hand, according to \([6]\),

\[
f(t, x) \leq \lambda + \mathbb{E}(1_{\tau_x > T_{N_t}} \tilde{f}(t-T_{N_t}, x-X_{T_{N_t}})),
\]

where

\[
\mathbb{E}(1_{\tau_x > T_{N_t}} \tilde{f}(t-T_{N_t}, x-X_{T_{N_t}})) = \mathbb{E}\left(1_{\{x-mT_{N_t} - \sum_{i=1}^{N_t} Y_i - \sqrt{T_{N_t}} B_1 > 0\}} \tilde{f}(t-T_{N_t}, x-mT_{N_t} - \sum_{i=1}^{N_t} Y_i - \sqrt{T_{N_t}} B_1)\right)
\]

and \( B_1 \) is a Gaussian random variable independent of \( N, Y, i \in \mathbb{N}^* \). According to the lemma 3.1 of the appendix of \([6]\), we get

\[
f(t, x) \leq \lambda + \mathbb{E}\left(1_{\{x-mT_{N_t} - \sum_{i=1}^{N_t} Y_i - \sqrt{T_{N_t}} G > 0\}} \frac{|x-mT_{N_t} - \sum_{i=1}^{N_t} Y_i - \sqrt{T_{N_t}} B|}{\sqrt{2\pi(t-T_{N_t})^3}} \exp\left[\frac{(x-mt-\sum_{i=1}^{N_t} Y_i - \sqrt{T_{N_t}} B)^2}{2(t-T_{N_t})}\right]\right)
\]

\[
\leq \lambda + \mathbb{E}\left(\frac{|x-mT_{N_t} - \sum_{i=1}^{N_t} Y_i - \sqrt{T_{N_t}} B|}{\sqrt{2\pi(t-T_{N_t})^3}} \exp\left[\frac{(x-mt-\sum_{i=1}^{N_t} Y_i - \sqrt{T_{N_t}} B)^2}{2(t-T_{N_t})}\right]\right).
\]

According to lemma.. of the appendix and letting \( t = t - T_{N_t}, \sigma = \sqrt{T_{N_t}} \), it follows that

\[
f(t, x) \leq \lambda + \frac{C}{t} t^{\frac{1}{2}} + \frac{C'}{t} \mathbb{E}\left(\frac{T_{N_t}}{t-T_{N_t}}\right).
\]

This achieve the proof. \( \square \)
Now, we prove the proposition.

Proof. First note that, using the Markov property at point $t$,
\[
\mathbb{E}(1_{\tau_x=\infty}|G_t) = 1_{\tau_x>t}\mathbb{P}(\tau_x-X_t = \infty).
\]
Second, for all $b \geq t$ the Markov property at point $t$ ensures equality
\[
\mathbb{E}(1_{t<\tau_x<b}|G_t) = \mathbb{E}
\left[1_{\tau_x>t} \int_t^b f(r-t, x-X_t) dr | G_t \right].
\]
According to (5), for $t < a < b$,
\[
\mathbb{E}
\left[1_{\tau_x>t} \int_a^b f(r-t, x-X_t) dr \right] < \infty
\]
and
\[
\mathbb{E}
\left[1_{\tau_x>t} \int_a^b f(r-t, x-X_t) dr | G_t \right] = \int_a^b \mathbb{E}[1_{\tau_x>t}f(r-t, x-X_t)|G_t] dr \ \forall b \ \text{p.s.}
\]
Since
\[
M_1 : b \rightarrow \mathbb{E}
\left[1_{\tau_x>t} \int_t^b f(r-t, x-X_t) dr | G_t \right]
\]
and
\[
M_2 : b \rightarrow \int_t^b \mathbb{E}[1_{\tau_x>t}f(r-t, x-X_t)|G_t] dr
\]
are increasing, they are submartingales with respect to the filtration $\tilde{G}_b = G_t \ \forall b \geq t$. Note that
\[
\mathbb{E}
\left[1_{\tau_x>t} \int_a^b f(r-t, x-X_t) dr \right] = \int_a^b \mathbb{E}[1_{\tau_x>t}f(r-t, x-X_t)] dr
\]
then, $b \rightarrow \mathbb{E}(M_1(b))$ and $b \rightarrow \mathbb{E}(M_2(b))$ are continuous. Using the 2.9 p 61 of Revuz-Yor [17], they have same càdlàg modification for all b. But almost surely, $M_1(b) = M_2(b)$, then almost surely $\forall b, M_1(b) = M_2(b)$. That means almost surely for all $b > a > t$,
\[
1_{\tau_x>t}\mathbb{E}(1_{a<\tau_x\leq b}|G_t) = 1_{\tau_x>t} \int_a^b \mathbb{E}[1_{\tau_x>t}f(r-t, x-X_t)|G_t] dr.
\]
Taking $a = t + \frac{1}{n}$ and letting $n$ going to infinity yields that, $\mathbb{P} - p.s \ \forall \ b$,
\[
\mathbb{E}(1_{t<\tau_x\leq b}|G_t) = \int_t^b \mathbb{E}[1_{\tau_x>t}f(r-t, x-X_t)|G_t] dr.
\]

4 Mixed filtering-Integro-differential equation for conditional density

In this section, we give one of main results of our work. For this purpose, let $(G_t)_{t \geq 0}$ be the filtration information available on both the firm and default occurrence for investors at time $t$ defined by $G_t = \sigma(Q_u; u \leq t, 1_{\tau_x \leq u}; u \leq t)$. 

\[
\]


**Theorem 1.** Let \( t > 0 \) be a real number. For any \( r > t \), on the set \( \{ \tau_x > t \} \), the conditional density of \( \tau_x \) given \( \mathcal{G}_t \) is given by

\[
\hat{f}(r, t, x) = \frac{1}{\mathbb{P}(\tau_x > t)} f(r, x) + \int_0^t \frac{\mathbb{E}(1_{\tau_x > t} h(X_u) f(r - u, x - X_u) | \mathcal{G}_u)}{\mathbb{E}(1_{\tau_x > t} G(t - u, x - X_u) | \mathcal{G}_u)} dQ_u
\]

\[
- \int_0^t \frac{\mathbb{E}(1_{\tau_x > t} h(X_u) G(t - u, x - X_u) | \mathcal{G}_u)}{\mathbb{E}(1_{\tau_x > t} G(t - u, x - X_u) | \mathcal{G}_u)^2} dQ_u
\]

\[
+ \int_0^t \frac{\mathbb{E}(1_{\tau_x > t} h(X_u) G(t - u, x - X_u) | \mathcal{G}_u)^2}{\mathbb{E}(1_{\tau_x > t} G(t - u, x - X_u) | \mathcal{G}_u)^3} du
\]

\[- \int_0^t \frac{\mathbb{E}(1_{\tau_x > t} h(X_u) f(r - u, x - X_u) | \mathcal{G}_u) \mathbb{E}(1_{\tau_x > t} h(X_u) G(t - u, x - X_u) | \mathcal{G}_u) dQ_u}{\mathbb{E}(1_{\tau_x > t} G(t - u, x - X_u) | \mathcal{G}_u) dQ_u}.
\]

where \( G(t, x) = \mathbb{P}(\tau_x > t) \).

The next lemma is inspired of Jeanblanc [11] and Dorobantu [7].

**Lemma 2.** For all \( t \in \mathbb{R}_+ \), for all \( a \) and \( b \) such that \( t < a < b \),

\[
\mathbb{E}(1_{\tau_x > t} | \mathcal{F}_t^Q) > 0 \quad \text{and} \quad \mathbb{E}(1_{a < \tau_x < b} | \mathcal{G}_t) = 1_{\tau_x > t} \frac{\mathbb{E}^0(\mathbb{1}_{a < \tau_x < b} | \mathcal{F}_t^Q)}{\mathbb{E}^0(1_{\tau_x > t} | \mathcal{F}_t^Q)}.
\]

**Proof.** Assume that there exists \( t_0 \) such that \( \mathbb{P}(\tau_x > t_0) = 0 \). Then for all \( t \geq t_0 \), \( \mathbb{P}(\tau_x \leq t_0) = 1 \). It follows that the density function of \( \tau_x \) is the zero function on \([t_0, +\infty[\). This means

\[
\lambda \mathbb{E}(1_{\tau_x > t}(1 - F_Y)(x - X_t)) + \mathbb{E}(1_{\tau_x > T_N} (f(t - T_N, x - X_{T_N})) = 0 \quad \mathbb{P} - a.s \quad \forall t \in [t_0, +\infty[.
\]

Then, \( \mathbb{P}(\tau_x \leq t) = 1 \) implies that \( \mathbb{E}(1_{\tau_x > t}(1 - F_Y)(x - X_t)) = 0 \).

Thus \( \mathbb{E}(1_{\tau_x > T_N} (f(t - T_N, x - X_{T_N})) = 0 \). But we have \( t - T_N > 0 \) \( \mathbb{P} - a.s \) and on the set \( \{ \tau_x > T_N \} \), \( x - X_{T_N} > 0 \). Therefore, \( f(t - T_N, x - X_{T_N}) > 0 \) for all \( t \geq t_0 \). Hence, we obtain \( 1_{\tau_x > T_N} = 0, \forall t \geq t_0 \) what is not possible. Then \( \mathbb{E}(1_{\tau_x > t} | \mathcal{F}_t^Q) > 0 \).

On the set \( \{ \tau_x > t \} \), any \( \mathcal{G}_t \)– measurable random variable coincides with some \( \mathcal{F}_t^Q \)– measurable random variable, for more detail, see Jeanblanc in [11] pp 18. Then there exists \( \mathcal{F}_t^Q \)– measurable random variable \( Z \) such that

\[
\mathbb{E}(1_{\tau_x > t} Y | \mathcal{G}_t) = 1_{\tau_x > t} Z.
\]

Taking the conditional expectation with respect to \( \mathcal{F}_t^Q \), we get

\[
\mathbb{E}(1_{\tau_x > t} Y | \mathcal{F}_t^Q) = \mathbb{E}(\tau_x > t | \mathcal{F}_t^Q).
\]

This implies that

\[
\mathbb{E}(1_{\tau_x > t} Y | \mathcal{G}_t) = 1_{\tau_x > t} \frac{\mathbb{E}(Y 1_{\tau_x > t} | \mathcal{F}_t^Q)}{\mathbb{E}(1_{\tau_x > t} | \mathcal{F}_t^Q)}.
\]

For \( Y = 1_{a < \tau_x < b} \quad (a > t) \) using Kallianpur-Striebel formula (see Pardoux [14]) we obtain

\[
\mathbb{E}(1_{a < \tau_x < b} | \mathcal{G}_t) = 1_{\tau_x > t} \frac{\mathbb{E}^0(L_b 1_{a < \tau_x < b} | \mathcal{F}_t^Q)}{\mathbb{E}^0(1_{\tau_x > t} | \mathcal{F}_t^Q)}.
\]
This lemma is found in [5].

**Lemma 3.** The family

\[ S_t = \left\{ S_t = \exp \left( \int_0^t \rho_s dQ_s - \frac{1}{2} \int_0^t \rho_s^2 ds \right), \rho \in L^2([0, T], \mathbb{R}) \right\} \]

is total in \( L^2(\Omega, \mathcal{F}_t^Q, \mathbb{P}^0) \).

**Lemma 4.** Let \( \{U_t, t \geq 0\} \) be an \( \mathcal{F}_t^{W,N,Y} \)-progressively measurable process such that for all \( t \geq 0 \), we have

\[ \mathbb{E}^0 \left[ \int_0^t U_s^2 ds \right] < +\infty. \]

Then

\[ \mathbb{E}^0 \left[ \int_0^t U_s dQ_s | \mathcal{F}_t^{W,N,Y} \right] = 0. \]  \hfill (7)

**Proof.** As the previous lemma (lemma 3), the family

\[ \mathcal{R}_t = \left\{ r_t = \mathcal{E} \left[ \int_0^t \gamma_s dW_s + \int_0^t \int_A (e^{\beta_s(x) - 1}) \tilde{N}(ds dx), \gamma \in L^2([0, T], \mathbb{R}), \beta \in L^\infty([0, T] \times A, \mathbb{R}) \right] \right\} \]

is total in \( L^2(\Omega, \mathcal{F}_t^{W,N,Y}, \mathbb{P}^0) \), where \( \tilde{N} \) is a compensated Poisson random measure on \( \mathbb{R} \times \mathbb{R} \) and \( A \subset \mathbb{R} \) is a borel set. Therefore, since \( r_t = 1 + \int_0^t r_s \gamma_s dW_s + \int_0^t \int_A r_s (e^{\beta_s(x) - 1}) \tilde{N}(ds dx) \), by Itô’s formula, we have

\[ \mathbb{E}^0 \left[ r_t \mathbb{E}^0 \left[ \int_0^t U_s dQ_s | \mathcal{F}_t^{W,N,Y} \right] \right] = \mathbb{E}^0 \left[ \int_0^t r_t \int_0^t U_s dQ_s \right] \]

\[ = \mathbb{E}^0 \left[ \int_0^t r_s \gamma_s dW_s + \int_0^t \int_A r_s (e^{\beta_s(x) - 1}) \tilde{N}(ds dx) \right] \]

\[ + \mathbb{E}^0 \left[ \int_0^t \int_A r_s U_s (e^{\beta_s(x) - 1}) d < \tilde{N}, Q > \right] = 0. \]

The equality is obtained from the fact that \( < Q, W > = < Q, \tilde{N} > = 0 \) by independence. \( \square \)

**Proposition 2.** For all \( t < a < b \), we have \( \mathbb{P}^0 - a.s \)

\[ \mathbb{E}^0 (L_b 1_{a < \tau_x < b} | \mathcal{F}_t^Q) = \mathbb{P}^0 (a < \tau_x < b) + \int_0^t \mathbb{E}^0 (1_{\tau_x > u} L_u h(X_u) | G(a - u, x - X_u) - G(b - u, x - X_u)) | \mathcal{F}_u^Q) dQ_u. \]  \hfill (8)

**Proof.** Let \( S_t \in \mathcal{S}_t \). Since \( X \) and \( Q \) are independent under \( \mathbb{P}^0 \) and using Itô formula,

\[ \mathbb{E}^0 (1_{a < \tau_x < b} L_b S_t) = \mathbb{P}^0 (a < \tau_x < b) + \mathbb{E}^0 \left( 1_{a < \tau_x < b} \int_0^t L_u S_u \rho_u h(X_u) du \right). \]

Markov property at point \( u \) gives us

\[ \mathbb{E}^0 (1_{a < \tau_x < b} L_b S_t) = \mathbb{P}^0 (a < \tau_x < b) + \mathbb{E}^0 \left( \int_0^t L_u S_u \rho_u h(X_u) 1_{\tau_x > u} [G(a - u, x - X_u) - G(b - u, x - X_u)] du \right) \]
Conditioning by $\mathcal{F}_u^Q$, it follows that
\[
\mathbb{E}^0(1_{a<\tau_x<b} L_b S_t) = \mathbb{P}^0(a < \tau_x < b) + \mathbb{E}^0 \left( \int_0^t S_u \rho_u \mathbb{E}^0 \left( 1_{\tau_x > u} L_u h(X_u) | \mathbb{F}_u^Q \right) (a - u, x - X_u) - G(b - u, x - X_u) \right) du.
\]

Using the Itô’s formula in reverse, it follows that
\[
\mathbb{E}^0(1_{a<\tau_x<b} L_b S_t) = \mathbb{E}^0 \left( S_t \left[ \mathbb{P}^0(a < \tau_x < b) + \int_0^t \mathbb{E}^0 \left( 1_{\tau_x > u} L_u h(X_u) | \mathbb{F}_u^Q \right) (a - u, x - X_u) - G(b - u, x - X_u) \right] du \right).
\]

Since $\mathcal{S}$ is dense in $L^2(\Omega, \mathcal{F}, \mathbb{P})$,
\[
\mathbb{E}^0(\mathbb{I}_{a<\tau_x<b} \mathcal{F}_t^Q) = \mathbb{P}^0(a < \tau_x < b) + \int_0^t \mathbb{E}^0 \left( 1_{\tau_x > u} L_u h(X_u) \left( G(T - u, x - X_u) - G(b - u, x - X_u) \right) \right) du.
\]

**Corollary 1.** For $t \leq T$,
\[
\mathbb{E}^0(1_{\tau_x > T} \mathcal{L}T | \mathcal{F}_t^Q) = \mathbb{P}^0(\tau_x > T) + \int_0^t \mathbb{E}^0 \left( 1_{\tau_x > u} L_u h(X_u) \left( G(T - u, x - X_u) - G(b - u, x - X_u) \right) \right) du.
\]  

Let us now find an expression for conditional probability density process defined from the representation
\[
\mathbb{E}(\mathbb{I}_{a<\tau_x<b}) = \int_a^b \tilde{f}(r, t, x) dr \text{ for some } a > t.
\]

Applying lemma 4, it follows that
\[
\mathbb{E}(\mathbb{I}_{a<\tau_x<b} | G_t) = \mathbb{E}^0 \left( \frac{\mathbb{E}^0(\mathbb{L}_t \mathbb{I}_{a<\tau_x<b} | \mathcal{F}_t^Q)}{\mathbb{E}^0(\mathbb{1}_{\tau_x > t} L_t | \mathcal{F}_t^Q)} \right).
\]

By previous lemma 4, we show that
\[
\mathbb{E}^0(1_{\tau_x > t} L_t | \mathcal{F}_t^Q) = \mathbb{P}^0(\tau_x > t) + \int_0^t \int_t^{+\infty} \mathbb{E}^0 \left( 1_{\tau_x > u} L_u h(X_u) f(r - u, x - X_u) \mathcal{F}_u^Q \right) dr du.
\]

But, since the inequality $\int_0^t \mathbb{E}^0(f^2(t - u, x - X_u))) du < \infty$ is not satisfied, we are not able to prove that $\mathbb{E}^0(1_{\tau_x > t} L_t | \mathcal{F}_t^Q)$ is a semimartingale (e.g. see theorem 65 of Protter [15]). This leads us to consider for $t \leq T < t + 1$, the expression $\mathbb{E}^0(1_{\tau_x > T} \mathcal{L}T | \mathcal{F}_t^Q)$. In the next proposition, we deal with the term $\frac{\mathbb{E}^0(\mathbb{L}_t \mathbb{I}_{a<\tau_x<b} | \mathcal{F}_t^Q)}{\mathbb{E}^0(1_{\tau_x > t} L_t | \mathcal{F}_t^Q)}$, and we take the limit when $T$ goes to $t$. 

\[8\]
Proposition 3. For any $0 < t < a < b$, we have on the set $\{\tau_x > t\}$,

$$
\begin{align*}
\mathbb{E}(1_{a < \tau_x < b}|\mathcal{G}_t) &= \frac{\mathbb{E}^0(a < \tau_x < b)}{\mathbb{P}^0(\tau_x > t)} \\
& + \int_0^t \frac{\mathbb{E}^0(1_{\tau_x > u}L_u h(X_u)[G(a - u, x - X_u) - G(b - u, x - X_u)]|\mathcal{F}_u^Q)}{\mathbb{P}^0(1_{\tau_x > t}L_u|\mathcal{F}_u^Q)} dQ_u \\
& - \int_0^t \frac{\mathbb{E}^0(L_u 1_{a < \tau_x < b}|\mathcal{F}_u^Q)\mathbb{E}^0(1_{\tau_x > u}L_u h(X_u)G(t - u, x - X_u)|\mathcal{F}_u^Q)}{[\mathbb{E}^0(1_{\tau_x > t}L_u|\mathcal{F}_u^Q)]^2} dQ_u \\
& + \int_0^t \frac{\mathbb{E}^0(L_u 1_{a < \tau_x < b}|\mathcal{F}_u^Q)[\mathbb{E}^0(1_{\tau_x > u}L_u h(X_u)G(t - u, x - X_u)|\mathcal{F}_u^Q)]^2}{[\mathbb{E}^0(1_{\tau_x > t}L_u|\mathcal{F}_u^Q)]^2} dQ_u \\
& - \int_0^t \frac{\mathbb{E}^0(1_{\tau_x > u}L_u h(X_u)[G(a - u, x - X_u) - G(b - u, x - X_u)]|\mathcal{F}_u^Q)}{[\mathbb{E}^0(1_{\tau_x > t}L_u|\mathcal{F}_u^Q)]^2} dQ_u \\
& \times \frac{\mathbb{E}^0(1_{\tau_x > u}L_u h(X_u)G(t - u, x - X_u)|\mathcal{F}_u^Q)}{[\mathbb{E}^0(1_{\tau_x > t}L_u|\mathcal{F}_u^Q)]^2} dQ_u.
\end{align*}
$$

Proof. We first apply Itô’s formula to $\frac{\mathbb{E}^0(L_t 1_{a < \tau_x < b}|\mathcal{F}_u^Q)}{\mathbb{P}^0(1_{\tau_x > t}L_t|\mathcal{F}_u^Q)}$. Second, we take the limit when $T$ goes to $t$.

$$
\begin{align*}
\mathbb{E}^0(L_t 1_{a < \tau_x < b}|\mathcal{F}_u^Q) &= \mathbb{E}^0(a < \tau_x < b) \\
& + \int_0^t \frac{\mathbb{E}^0(1_{\tau_x > u}L_u h(X_u)[G(a - u, x - X_u) - G(b - u, x - X_u)]|\mathcal{F}_u^Q)}{\mathbb{P}^0(1_{\tau_x > t}L_u|\mathcal{F}_u^Q)} dQ_u \\
& - \int_0^t \frac{\mathbb{E}^0(L_u 1_{a < \tau_x < b}|\mathcal{F}_u^Q)\mathbb{E}^0(1_{\tau_x > u}L_u h(X_u)G(T - u, x - X_u)|\mathcal{F}_u^Q)}{[\mathbb{E}^0(1_{\tau_x > t}L_u|\mathcal{F}_u^Q)]^2} dQ_u \\
& + \int_0^t \frac{\mathbb{E}^0(L_u 1_{a < \tau_x < b}|\mathcal{F}_u^Q)[\mathbb{E}^0(1_{\tau_x > u}L_u h(X_u)G(T - u, x - X_u)|\mathcal{F}_u^Q)]^2}{[\mathbb{E}^0(1_{\tau_x > t}L_u|\mathcal{F}_u^Q)]^2} dQ_u \\
& - \int_0^t \frac{\mathbb{E}^0(1_{\tau_x > u}L_u h(X_u)[G(a - u, x - X_u) - G(b - u, x - X_u)]|\mathcal{F}_u^Q)}{[\mathbb{E}^0(1_{\tau_x > t}L_u|\mathcal{F}_u^Q)]^2} dQ_u \\
& \times \frac{\mathbb{E}^0(1_{\tau_x > u}L_u h(X_u)G(T - u, x - X_u)|\mathcal{F}_u^Q)}{[\mathbb{E}^0(1_{\tau_x > t}L_u|\mathcal{F}_u^Q)]^2} dQ_u.
\end{align*}
$$

Now, we let $T$ goes to $t$. For this end, assume that $t < T \leq t + 1$. It is enough to show that

$$
\mathbb{E}^0\left(\int_0^t \left[\frac{(Z_u^i)^i - (Z_u^{i+1})^i}{(Z_u^i Z_u^{i+1})^i}\right]^{2+\varepsilon} d\mu\right) < \infty,
$$

where $Z_u^t = \mathbb{E}^0(1_{\tau_x > t}L_u|\mathcal{F}_u^Q)$ and $i \in \{1, 2, 3\}$. But, since $t \mapsto \frac{1}{Z_u^t}$ is increasing, we have

$$
\mathbb{E}^0\left(\int_0^t \left[\frac{(Z_u^i)^i - (Z_u^{i+1})^i}{(Z_u^i Z_u^{i+1})^i}\right]^{2+\varepsilon} d\mu\right) \leq \mathbb{E}^0\left(\int_0^t \left[\frac{(Z_u^i)^i - (Z_u^{i+1})^i}{(Z_u^i Z_u^{i+1})^i}\right]^{2+\varepsilon} d\mu\right).
$$
The fact that \(|(Z_u^t)^i - (Z_u^{t+1})^i| \leq (Z_u^1)^i|\) leads us to write
\[
\mathbb{E}^0 \left( \int_0^t \left[ \frac{(Z_u^i)^i - (Z_u^{t+1})^i}{(Z_u^t)^i(Z_u^{t+1})^i} \right]^{2+\varepsilon} du \right) \leq \mathbb{E}^0 \left( \int_0^t \frac{du}{(Z_u^t)^i(2+\varepsilon)} \right)
\]
\[
\leq \mathbb{E}^0 \left( \int_0^t \mathbb{E}^0 \left[ \frac{1}{L_u 1_{\tau > t+1} |F_u^Q|} \right]^{i(2+\varepsilon)} du \right)
\]
\[
\leq \mathbb{E}^0 \left( \int_0^t \mathbb{E}^0 \left[ \frac{1}{L_u 1_{\tau > t+1} |F_u^Q|} \right]^{(2+\varepsilon)} du \right) \text{ by Jensen inequality .}
\]
\[
\leq \int_0^t \mathbb{E}^0 \left[ \left( \frac{1}{L_u 1_{\tau > t+1}} \right)^{(2+\varepsilon)} \right] du < +\infty.
\]

We use the fact that \(G(T - u, x - X_u)\) goes to \(G(t - u, x - X_u)\) when \(T\) goes to \(t\) and Lebesgue dominated convergence theorem and it follows that
\[
\lim_{T \to t} \frac{\mathbb{E}^0(L_t 1_{a < \tau < b} |F_t^Q|)}{\mathbb{E}^0(1_{\tau > T} L_t |F_t^Q|)} = \frac{\mathbb{E}^0(a < \tau_x < b)}{\mathbb{E}^0(\tau_x > t)}
\]
\[
+ \int_0^t \mathbb{E}^0(1_{\tau_x > u} L_u h(X_u)|G(a - u, x - X_u) - G(b - u, x - X_u)| |F_u^Q|) \, dQ_u
\]
\[
- \int_0^t \mathbb{E}^0(1_{\tau_x > u} L_u h(X_u) G(t - u, x - X_u)|F_u^Q|) \, dQ_u
\]
\[
+ \int_0^t \mathbb{E}^0(1_{\tau_x > u} L_u h(X_u)|G(a - u, x - X_u) - G(b - u, x - X_u)| |F_u^Q|)^2 \, dQ_u
\]
\[
\times \frac{\mathbb{E}^0(1_{\tau_x > u} L_u h(X_u) G(t - u, x - X_u)|F_u^Q|)}{\mathbb{E}^0(1_{\tau_x > u} L_u |F_u^Q|)^2} \, du.
\]

Now, since \(t < a \leq r\), the stochastic Fubini’s theorem can be used because from lemma II
\[
\mathbb{E} \left[ \left( \int_0^t f(r, u) \, du \right)^{\frac{3}{2}} \right] < \infty.
\]
To obtain the result under \(\mathbb{P}\), each fraction under the integral is multiplied and divided by the same term \(\mathbb{E}^0(1_{\tau_x > u} L_u |F_u^Q|)\). Therefore, on the set \(\{\tau_x > t\}\),
\[
\mathbb{E}(1_{a < \tau_x < b} |G_t|) = \mathbb{E} \left( \int_a^b f(r, x) \, dr \right) + \mathbb{E} \left( \int_a^b \frac{1_{\tau_x > u} h(X_u)f(r - u, x - X_u)|G_u|}{\mathbb{E}(1_{\tau_x > u} G(t - u, x - X_u)|G_u|)} \, dQ_u \right)
\]
\[
- \int_a^b \int_0^t \frac{f(r, u)}{\mathbb{E}(1_{\tau_x > u} G(t - u, x - X_u)|G_u|)^2} \, dQ_u \, dr
\]
\[
+ \int_a^b \int_0^t \frac{f(r, u)}{\mathbb{E}(1_{\tau_x > u} G(t - u, x - X_u)|G_u|)^3} \, du \, dr
\]
\[
- \int_a^b \int_0^t \frac{\mathbb{E}(1_{\tau_x > u} h(X_u)f(r - u, x - X_u)|G_u|)}{\mathbb{E}(1_{\tau_x > u} G(t - u, x - X_u)|G_u|)^2} \, dQ_u \, dr.
\]
After some calculation, we obtain $\mathbb{A}$ submartingale with increasing expectation. This yields Lemma 5.

### 5 Appendix

**Remarks 3.**

Indeed, $Y_u = \mathbb{E}(1_{\tau_u > u} G(t-u, x-X_u) | \mathcal{G}_u)$ is a martingale and then $(Y^2_u)_{u \geq 0}$ is submartingale with increasing expectation. This yields

$$\mathbb{E} \left( \int_0^t \frac{du}{\mathbb{E}(1_{\tau_u > u} G(t-u, x-X_u) | \mathcal{G}_u)^2} \right) < \infty.$$ 

**Lemma 5.** Let $G$ be a Gaussian random variable $N(0,1)$ and let $m, \mu \in \mathbb{R}, t, \sigma \in \mathbb{R}_+$. Then

$A(\mu, \sigma, m, t) := \mathbb{E} \left( \left[ \frac{\mu - \sigma G + mt}{\sqrt{2\pi t^3}} \right] e^{-\frac{(\mu - \sigma G)^2}{2t}} \right)$

satisfies

$$A(\mu, \sigma, m, t) \leq \frac{C_1}{\sigma^2 + t^2} + \frac{C_2}{\sqrt{\sigma^2 + t}} + \frac{\sigma C_3}{(\sigma^2 + t)^{\frac{3}{2}}}$$

whith $C_1, C_2,$ and $C_3$ positive constantes.

**Proof.** We use the law of $G$ and it follows that

$$A(\mu, \sigma, m, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[ \frac{\mu - \sigma y + mt}{\sqrt{2\pi t^3}} \right] e^{-\frac{(\mu - \sigma y)^2}{2t}} dy.$$ 

Since $\frac{(\mu - \sigma y)^2}{t} + y^2 = \sigma^2 + t\left(\frac{y - \frac{\mu \sigma}{\sigma^2 + t}}{t}\right)^2 + \frac{\mu^2}{t}$, then

$$A(\mu, \sigma, m, t) = \frac{1}{\sqrt{2t}} \int_{\mathbb{R}} \frac{\mu - \sigma y + mt}{\sqrt{2\pi t^3}} \exp \left[ -\sigma^2 t - \frac{2t}{\mu \sigma} \left( y - \frac{\mu \sigma}{\sigma^2 + t} \right)^2 - \frac{\mu^2}{2(\sigma^2 + t)} \right] dy.$$ 

By a change of variable $z = y - \frac{\mu \sigma}{\sigma^2 + t}$, we have

$$A(\mu, \sigma, m, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\mu - \sigma(z + \frac{\mu \sigma}{\sigma^2 + t}) + mt}{\sqrt{2\pi t^3}} \exp \left[ -\sigma^2 t - \frac{2t}{\mu \sigma} z^2 - \frac{\mu^2}{2(\sigma^2 + t)} \right] dz.$$ 

A new change of variable $y = z \sqrt{\frac{\sigma^2 + t}{\mu \sigma}}$ leads us to

$$A(\mu, \sigma, m, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[ \frac{\mu}{\sqrt{\sigma^2 + t}} + m \sqrt{\sigma^2 + t} \right] \Phi \left( \frac{\mu}{\sqrt{\sigma^2 + t}} + m \sqrt{\sigma^2 + t} \right) - \sigma y \right) + \exp \left[ -\frac{y^2}{2} - \frac{\mu^2}{2(\sigma^2 + t)} \right] dy.$$ 

After some calculation, we obtain

$$A(\mu, \sigma, m, t) = \frac{\exp\left( -\frac{\mu^2}{2(\sigma^2 + t)} \right)}{\left( \frac{\sigma}{2(\sigma^2 + t)} \right) \Phi \left( \frac{\mu}{\sqrt{\sigma^2 + t}} + m \sqrt{\sigma^2 + t} \right)}$$ 

$$+ \frac{1}{4\pi(\sigma^2 + t)\sqrt{t}} \exp \left[ -\frac{t}{2\sigma^2} \left( \frac{\mu}{\sqrt{\sigma^2 + t}} + m \sqrt{\sigma^2 + t} \right) \right]$$
where Φ is the Gaussian distribution function which is bounded by 1. Now, we get

\[
A(\mu, \sigma, m, t) \leq \frac{C_1}{(\sigma^2 + t)^{\frac{3}{2}}} + \frac{C_2}{\sqrt{\sigma^2 + t}} + \frac{\sigma C_3}{(\sigma^2 + t)\sqrt{t}}
\]  

(12)

whith \(C_1, C_2,\) and \(C_3\) positive constants.
References

[1] D. Applebaum *Lévy processes and stochastic calculus*, second edition, Cambridge university press, 2009

[2] A. BAIN and D. CRISAN *Fundamentals of stochastic filtering*, Stochastic Modeling and Applied Probability, Vol. 60, Springer, New York 2009

[3] V. BERNYK, R. C. DALANG, G. PESKIR *The law of the supremum of a stable Lévy Process with no negative jumps* ANN. Probab. Vol 36, Number 5, 2008, pp 1777-1789

[4] R. CONT and P. TANKOV *Financial modeling with jump processes* Chapman & Hall/CRC Financial mathematics series 2004

[5] L. COUTIN *Filtrage d’un système càd-làg: Application du calcul des variations stochas- tiques à l’existence d’une densité*, Stochastics and Stochastics Reports, 1996, Vol. 58, pp 209-243

[6] L. COUTIN and D. DOROBANTU *First passage time law for some Lévy process with compound Poisson: existence of a density*, Bernoulli, Vol. 17, number 4, pp 1127-1135

[7] D. DOROBANTU *Modélisation de risque de défaut en entreprise*, thèse de l’Université de Toulouse 3, 2007 DI

[8] D. DUFFIE and D. LANDO *Term Structure of Credit Spreads with Incomplete Accounting Information*, Econometrica, Vol 69, 2001, pp 633-664

[9] P. GAPEEV V. and M. JEANBLANC *Princing and filtering in two-dimensional dividend switching model* International Journal of Theoretical and Applied Finance (2010) 13(7) pp 1001-1017

[10] X. GUO, R. JARROW and Y. ZENG *Credit model with incomplete information, (earlier version Information reduction in credit risk models)* Mathematics of operations research, 34(2): 320-332, 2009

[11] M. JEANBLANC, RUTKOWSKI *Modeling of default risk Mathematical tools*, Mathematical Finance ; theory and practice. Modern Mathematics Series, High education Press, 2000

[12] I. KARATZAS, S. E. SHREVE *Brownian Motion and Stochastic Calculus*, Second Edition, Springer-Verlag, New York, 1991

[13] S. G. KOU and H. WANG *First passage time of a jump diffusion process* Adv. Appl. Prob. 35, 2003, pp 504-531

[14] E. PARDOUX *Filtrage non linéaire et équations aux dérivées partielles stochastiques associées*, Ecole d’Été de Probabilités de Saint-Flour-1989, Lecture Notes in Mathematics1464, Springer Verlag-Heidelberg- New York 1991

[15] P. PROTTER *Stochastic integration and differential equation* Springer, Berlin, Second edition 2003
[16] P. PROTTER Volterra equations driven by semimartingale, The annals of Probability, Vol 13, No 2 (Mai 1985), pp. 518 – 530

[17] D. REVUZ M. YOR Continuous Martingales and Brownian Motion, second edition, Springer, Berlin

[18] B. ROYNETTE, P. VALLOIS, A. VOLPI, Asymptotic behavior of the passage time, overshoot and undershoot for some Lévy processes ESAIM PS Vol. 12, 2008, pp 58-93

[19] K. Sato Lévy processes and infinitely divisible distribution, volume 68 of Cambridge studies in advanced Mathematics, Cambridge University Press, Cambridge, 1999, Translated from the 1990 Japanaise original, Revised by the author