AN INTEGRABILITY OF THE PROBLEM ON MOTION OF CYLINDER AND VORTEX IN THE IDEAL FLUID

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In this paper we present the nonlinear Poisson structure and two first integrals in the problem on plane motion of circular cylinder and \( N \) point vortices in the ideal fluid. A priori this problem is not Hamiltonian. The particular case \( N = 1 \), i.e. the problem on interaction of cylinder and vortex, is integrable.

1. The equations of motion

Let's consider the problem on plane motion of cylindrical rigid body and \( n \) point vortices with circulations \( \Gamma_i \) in an unbounded volume of the ideal incompressible fluid motionless in infinity. We assume that the exterior force fields are absent, the surface of cylinder is ideally smooth, and the cylinder flow is circulating, i.e. the circulation along a closed contour encircling the cylinder is \( \Gamma \neq 0 \). The equations of motion of such system were almost simultaneously obtained by S. M. Ramodanov in \([1]\) and also in \([2]\). In \([1]\) it was assumed that \( n = 1 \) and in \([2]\) that \( \Gamma = 0 \). Paper \([3]\) is the expanded version of \([1]\) presenting the most general equations of motion of body and vortices (\( \Gamma \neq 0 \) and \( n \) is arbitrary). In the latter paper it is supposed that \( \Gamma = 0 \). In the further text we adhere to paper \([3]\).

Let's consider two frames of references: the motionless frame \( OXY \) and the one connected with the center of the cylinder \( Cxy \) performing a plane-parallel motion with respect to the first frame (Fig. 1). Let \( (v_1, v_2) = v \) be the projections of velocity of the center of cylinder onto the axes of \( Cxy \), \( (x_i, y_i) = r_i \) the projections of radius-vector connecting the center of cylinder with \( i \)-th vortex onto the same axes, \( \mu \) the mass of cylinder, \( R \) its radius. Paper \([3]\) shows that for the case of the inertial motion, equations for \( v, \ r \) are separated from the whole system and can be integrated independently. They have the following form

\[
\dot{r}_i = -v + \text{grad } \tilde{\varphi}_i|_{r=r_i}, \\
\mu \dot{v}_i = \lambda v_2 - \sum_{i=1}^{n} \lambda_i (\tilde{y}_i - \tilde{y}_i), \quad \mu \dot{v}_2 = -\lambda v_1 + \sum_{i=1}^{n} \lambda_i (\tilde{x}_i - \tilde{x}_i),
\]

where \( \tilde{r}_i = (\tilde{x}_i, \tilde{y}_i) \) is the inverse image of the \( i \)-th vortex conjugated with respect to the contour of cylinder by the rule \( \tilde{x}_i = \frac{R^2}{r^2} x_i, \ \tilde{y}_i = \frac{R^2}{r^2} y_i \) and function \( \tilde{\varphi}_i \) corresponds to the part of flow potential \( \varphi \) that has no singularities at \( r = r_i \). The flow potential outside the cylinder can be presented in the form

\[
\varphi(r) = -\frac{R^2}{r^2} (r, v) - \lambda \arctg \frac{y}{x} + \sum_{i=1}^{n} \lambda_i \left( \arctg \frac{y - \tilde{y}_i}{x - \tilde{x}_i} - \arctg \frac{y - \tilde{y}_i}{x - x_i} \right),
\]

where \( \lambda = \frac{1}{2\pi} \), \( \lambda_i = \frac{\Gamma}{2\pi} \).

It is easy to verify that the equations \((1.1)\) have an integral (similar to the energy integral if we use an analogy with classical mechanics) in the form

\[
H = \frac{1}{2} \mu (v_1^2 + v_2^2) + \frac{1}{2} \sum_i [\lambda_i^2 \ln (r_i^2 - R^2) - \lambda_i \lambda \ln r_i^2] + \frac{1}{2} \sum_{i<j} \lambda_i \lambda_j \ln \frac{R^4 - 2R^2 (r_i, r_j) + r_i^2 r_j^2}{|r_i - r_j|^2},
\]

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though the origin of this integral not obvious since (1.1) are not derived as usual from the Lagrangian formalism, but inherit some conservative properties from both the dynamics of rigid body and the motion of the ideal fluid. Note that (1.1) preserve the standard invariant measure.

2. A Hamiltonian structure of the equations of motion

The problem of existence of Hamiltonian form for (1.1) was considered in paper [2] and some results were obtained for the case $\Gamma = 0$, $\sum_{i=1}^{n} \Gamma_i = 0$. Here we present the Poisson structure for (1.1) in the general case. It is nonlinear with respect to the phase variables and nondegenerate. Its basis nonzero Poisson brackets are the following

$$
\{v_1, v_2\} = \frac{\lambda_1}{\mu^2} \sum \lambda_i \frac{r_i^4 - R^4}{r_i^4}, \quad \{v_1, x_i\} = \frac{1}{\mu} \frac{r_i^4 - R^2(x_i^2 - y_i^2)}{r_i^4},
$$

$$
\{v_2, x_i\} = -\frac{1}{\mu} \frac{2R^2x_iy_i}{r_i^4}, \quad \{v_2, y_i\} = \frac{1}{\mu} \frac{r_i^4 + R^2(x_i^2 - y_i^2)}{r_i^4}, \quad \{x_i, y_i\} = -\frac{1}{\lambda_i}. \tag{2.1}
$$

According to the Darboux theorem in the nondegenerate case we can always find the analytical transformation presenting (2.1) in the canonical form $\{p_i, q_i\} = \delta_{ij}$, $\{p_i, p_j\} = 0$, $\{q_i, q_j\} = 0$. However, the explicit form of this transformation is not required for our analysis.

3. An integrability of the system for the case of one vortex

Let’s consider the case $n = 1$, i.e. the simultaneous motion of vortex and cylinder. Denote $r_1 = r = (x, y)$. We can easily see that such system is invariant with respect to the rotation about the center of cylinder. Therefore, there is the integral

$$
F = \mu' v^2 + 2\mu \lambda_1 \left(1 - \frac{R^2}{r^2}\right)(xv_2 - yv_1) + \lambda_1 \left(\lambda_1 \left(\frac{r^2}{\mu} + \frac{R^4}{r^2}\right) - \lambda r^2\right), \tag{3.1}
$$

generating because of (2.1) the field of symmetries

$$
v_F = 2\lambda(v_2 - v_1, y_1 - x). \tag{3.2}
$$

The integral (3.1) is a generalization of the integral of moment from classical mechanics and using it we can integrate the system in quadratures. We can reduce the order of system down to one degree of freedom. This reduction is close to the Routh reduction. Let’s perform it explicitly.
As the variables of the reduced system we shall choose the integrals of the field of symmetries \(\mathbf{v}_F\) (see, for example, [3]). For example,

\[
\begin{align*}
 p_1 &= \mu(xv_1 + yv_2), \\
 p_2 &= \mu(xv_2 - yv_1), \\
 r_1 &= \mu^2(v_1^2 + v_2^2), \\
 r_2 &= x^2 + y^2.
\end{align*}
\]

Poisson brackets between these variables are the following

\[
\begin{align*}
\{p_1, p_2\} &= (\lambda - \lambda_1)r_2 + \frac{1}{\lambda_1}(p_1^2 + (p_2 - \lambda_1 R^2)^2), \\
\{r_1, r_2\} &= 4p_1\left(1 + \frac{R^2}{r^2}\right), \\
\{p_1, r_1\} &= 2(\lambda - \lambda_1)p_2 - 2p_1^2 + 2R^2 - \frac{2p_2}{r_2}, \\
\{p_1, r_2\} &= 2r_2 + \frac{2}{\lambda_1}(p_2 - \lambda_1 R^2), \\
\{p_2, r_1\} &= -2(\lambda - \lambda_1)p_1 + 2R^2 p_1 - \frac{2p_2}{r_2}, \\
\{p_2, r_2\} &= -2\frac{p_1}{\lambda_1}.
\end{align*}
\]

The Poisson structure [3,4] is degenerated, its rank is equal to two and therefore such reduced system has one degree of freedom. To obtain it in explicit form we shall exclude two variables from the set \((r_1, r_2, p_1, p_2)\) using the integral [3,1], the Casimir function of structure [3,1], and the obvious geometrical relation \(p_1^2 + p_2^2 - r_1 r_2 = 0\) following from [3,4].

The qualitative analysis of the reduced system would be an interesting study that could lead to some conclusions about the motion of the complete system [3,1].

4. The case of \(n\)-vortices

In the general case there is a generalization of integral [3,1]. It reads

\[
F = \mu v^2 + \sum_{i=1}^{n} \lambda_i \left(2\mu\left(1 - \frac{R^2}{r_i^2}\right)(x_i v_2 - y_i v_1) + (\lambda_i - \lambda) r_i^2 + \lambda_i \frac{R^4}{r_i^4}\right) + 2 \sum_{i<j} \lambda_i \lambda_j (r_i, r_j) \left(1 - \frac{R^2}{r_i^2}\right) \left(1 - \frac{R^2}{r_j^2}\right). \tag{4.1}
\]

Using this generalization we can also perform the reduction of the order of system to one degree of freedom. For the integrability of the obtained system we shall have \(n - 1\) involute integrals. In the general case, probably, they do not exist and the system of two vortexes interacting with the cylinder is already nonintegrable. However, it is not proved yet and there may exist the particular cases of integrability under additional restrictions on parameters of the system and on constants of the known integrals [1,2,14].

There is another interesting problem on a generalization of Poisson structure [1,2] and integrals [1,2,14] for the problem on interaction in fluid of two (or several) rigid bodies with the given circulations (certainly, in the plain formulation). Recently the equations of two cylinders in fluid (without circulations) were obtained by S. M. Ramodanov. In such formulation they are “plane” analog of the known Bjerkness problem on interaction of two balls in a flow of ideal fluid. These results for the first time were published in the collection. This particular problem is integrable in contrast to the more general situation when the flow of cylinders is circulating. The corresponding equations of motion are not obtained yet.

Let’s consider also another related problem, which is also integrable. It is the interaction of elliptic Kirchhoff vortex with point vortex in an approximation described by the second order theory of moments (see, for example, [3]). In contrast to the rigid body, the Kirchhoff vortex during the motion remains elliptic, but changes the lengths of semiaxes. The problems on interaction of two Kirchhoff vortexes or one Kirchhoff vortex with two point vortexes seem to be nonintegrable.

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