Exploration of efficient numerical integration rule for wideband room-acoustics simulations by plane-wave-enriched finite-element method

Kanako Tamaru*, Takeshi Okuzono†, Shunichi Mukae‡ and Kimihiro Sakagami§

Environmental Acoustic Laboratory, Department of Architecture, Graduate School of Engineering, Kobe University, 1–1, Rokkodai-cho, Nada-ku, Kobe, 657–8501 Japan

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Abstract: In this study, we assessed the reduction in the computational costs of a room-acoustics solver by the partition of unity finite-element method (PUFEM), particularly addressing the element matrix construction process with numerical integration rules. The PUFEM enriches the approximation of sound fields by incorporating a general solution of the Helmholtz equation into shape functions. Plane wave enrichment is applied herein. In plane-wave-enriched FEM, the construction of element matrices using a high-order Gauss–Legendre quadrature is the main numerical operation with a long computational time. To reduce the computational time of the room-acoustics solver with plane-wave-enriched FEM, in this report, we describe our exploration of efficient integration rules via an ideal plane wave propagation problem in a duct. We present two integration rules: a well-used existing rule extended to the low-frequency range and another derived by the linear regression of the relationship between the number of wavelengths included in each element and the minimum number of integration points required for solution convergence. Numerical results revealed that both rules produce accurate frequency responses in a broad frequency range. However, the rule obtained by linear regression outperforms the extended rule.

Keywords: Numerical integration rule, Partition of unity finite-element method, Plane wave enrichment, Room acoustics, Wave-based method

1. INTRODUCTION

Wave-based numerical methods using the finite-element method (FEM) are expected to predict room acoustics accurately by solving a wave equation numerically. However, wave-based predictions require sufficient discretization in time and space to yield a reliable result. Such discretization entails high computational cost because room-acoustics simulations deal with large dimensions of architectural spaces, wide audible frequency ranges, and complex boundary conditions. Although recent developments in computer technology have led to some efficient wave-based methods [1,2], it remains difficult to predict room acoustics accurately with extended-reaction boundary conditions.

The partition of unity finite-element method (PUFEM) [3], which can achieve an accurate simulation by incorporating a general solution of governing equations into shape functions, has been formulated for acoustic simulations solving the Helmholtz equation in some studies [4–7]. Reportedly, the PUFEM has been found to markedly reduce the degrees of freedom (DOF) in FE meshes via elementary experiments [8]. In a PUFEM formulation, the sound pressure at each node is described using a set of plane waves propagating in various directions. They are incorporated into shape functions via the partition of the unity property. In doing so, sound fields are approximated up to high frequencies with a refinement approach called q-refinement by which a set of plane waves is gradually added at nodal points of a single mesh with increasing frequency. Herein, we designate this PUFEM formulation as plane-wave-enriched FEM. The principal advantage of plane-wave-enriched FEM is that this method enables us to use a coarse mesh irrespective of the frequency, unlike standard FEM. In fact, standard FEM with linear elements traditionally requires ten elements per wavelength, at least for spatial discretization where the discretization error must be reduced to within an acceptable level, but plane-wave-enriched FEM enables the use of elements of length many times greater than the wavelength of the analyzed frequency. In a recent study [9], the potential of plane-
wave-enriched FEM has been discussed via two-room acoustics problems in a single and a coupled room. Results of that study revealed that plane-wave-enriched FEM can predict wideband frequency responses accurately with only a hundredth of the degrees of freedom necessary for standard FEM. Therefore, plane-wave-enriched FEM can be a more efficient method than standard FEM for a room-acoustics solver. However, room-acoustics PUFEM remains under development. Various aspects remain to be studied for practical applications. For example, the following subjects should be studied further to enhance applicability, efficiency, and robustness: (1) increasing the efficiency of the element matrix construction process with domain and boundary integrals, which is the most computationally demanding part of plane-wave-enriched FEM; (2) creating a robust setup for plane wave enrichment, i.e., a way of adding how many plane waves at each nodal point; (3) implementing an extended-reacting sound absorber model for some sound-absorbing materials, e.g., microperforated panels and permeable membranes; (4) clarifying mesh size effects on the resulting accuracy for practical room acoustic problems; and (5) evaluating the performance for three-dimensional room-acoustics problems. The present paper addresses subject (1). We selected a high-order Gauss–Legendre rule as a general numerical integration scheme that can be applied to any type or shape of element.

This study was conducted to establish an efficient numerical integration rule suitable for wideband room acoustic simulations. First, the theory of plane-wave-enriched FEM is briefly described. Then, we explain why the efficient integration rule is required. Secondly, on the basis of the results of an examination of a plane wave propagation problem in a duct, we propose a numerical integration rule for wideband frequency analyses. Finally, we demonstrate the effectiveness of the proposed rule via two-dimensional real-scale office room problems in a single and a coupled room.

2. SOLVING ROOM ACOUSTIC PROBLEMS WITH PLANE-WAVE-ENRICHED FEM

2.1. Sound Propagation in a Closed Sound Field

Room-acoustics simulations generally solve the following Helmholtz equation in terms of sound pressure $p$ to predict sound propagation in a closed sound field $\Omega$.

$$\nabla^2 + k^2 p(\mathbf{r}, \omega) = 0, \quad \text{in } \Omega$$

(1)

Therein, $k$ stands for the wavenumber, $\omega$ expresses the angular frequency, and $\mathbf{r}$ denotes the position vector at an arbitrary point. For boundary $\Gamma$, we consider three basic boundary conditions: a rigid boundary $\Gamma_0$, a vibration boundary $\Gamma_v$, and an absorbing boundary $\Gamma_a$. They are

$$\frac{\partial p(\mathbf{r}, \omega)}{\partial n} = \begin{cases} 0 & \text{on } \Gamma_0 \\ -j\omega\rho_0 v_0(\mathbf{r}, \omega) & \text{on } \Gamma_v, \\ -jk\frac{1}{\varepsilon(\mathbf{r}, \omega)} p(\mathbf{r}, \omega) & \text{on } \Gamma_a, \end{cases}$$

(2)

where $\rho_0$, $v_0(\mathbf{r}, \omega)$, $\varepsilon(\mathbf{r}, \omega)$, and $j$ respectively represent the air density, vibration velocity, specific acoustic impedance ratio, and imaginary unit ($j^2 = -1$). The weak form for plane-wave-enriched FEM is

$$\int_{\Omega} (-\nabla \phi(\mathbf{r}, \omega) \nabla p(\mathbf{r}, \omega) + k^2 \phi(\mathbf{r}, \omega) p(\mathbf{r}, \omega)) d\Omega + \int_{\Gamma} \phi \frac{\partial p(\mathbf{r}, \omega)}{\partial n} d\Gamma = 0,$$

(3)

where $\phi(\mathbf{r}, \omega)$ is the arbitrary weight function.

2.2. Plane-wave-enriched Finite Elements for Two Dimensions

Standard FEM approximates sound pressure at arbitrary point $p(\mathbf{r}, \omega)$ within element $\Omega_e$ with the shape function $N_i(\xi)$ and nodal sound pressure $p_c^i$ as

$$p(\mathbf{r}, \omega) = \sum_{i=1}^{n} N_i(\xi) p_c^i,$$

(4)

where $\xi$ is the position vector in the local coordinate system. Plane-wave-enriched FEM incorporates a general solution of the Helmholtz equation into the shape function via the partition of unity property. For two-dimensional space, nodal sound pressure $p_c^i$ is given by a set of plane waves propagating in various directions as

$$p_c^i = \sum_{l=1}^{q} A_l^i e^{i(k_x \cos \theta_l + k_y \sin \theta_l)},$$

(5)

where $q$, $\theta_l$, and $A_l^i$ respectively represent the plane wave number, plane wave angle, the plane wave amplitude of propagation in direction $\theta_l$. Additionally, $(x, y)$ represents an arbitrary point in the two-dimensional global coordinate system. With Eq. (5), the sound pressure $p(x, y)$ in element $\Omega_e$ is approximated as

$$p(x, y) = \sum_{i=1}^{n} \sum_{l=1}^{q} N_i(\xi, \eta) e^{i(k_x \cos \theta_l + k_y \sin \theta_l)} A_l^i,$$

(6)

where $(\xi, \eta)$ is an arbitrary point in the two-dimensional local coordinate system. Defining the product of the shape function $N_i$ and plane wave $e^{i(k_x \cos \theta_l + k_y \sin \theta_l)}$ with a unit amplitude as a new shape function $P$, Eq. (6) can be expressed simply as

$$p(x, y) = \sum_{i=1}^{n} \sum_{l=1}^{m} P_{i-l} A_l^i.$$


\[ [K - k^2 M + jk C] A = -j \omega \rho_0 Q, \]

where the global stiffness matrix $K$, global mass matrix $M$, global dissipation matrix $C$, and external force vector $Q$ are respectively defined as

\[ K = \sum_e \int_{\Omega_e} \nabla \mathbf{P}^T \nabla d\Omega_e, \]

\[ M = \sum_e \int_{\Omega_e} \mathbf{P}^T \mathbf{P} d\Omega_e, \]

\[ C = \sum_e \int_{\Omega_e} \frac{1}{\rho_0 (r, \omega)} \int_{\Gamma_{ea}} \mathbf{P}^T \mathbf{P} d\Gamma_{ea}, \]

\[ Q = \sum_e \int_{\Omega_e} v_n(r, \omega) \int_{\Gamma_{ev}} \mathbf{P}^T d\Gamma_{ev}. \]

Here, $P$ and $A$ respectively represent the shape function and nodal amplitude vectors. Sound pressures in the domain are calculable by substituting the amplitude of plane waves $A_i^j$ obtained from Eq. (8) into Eq. (6) or Eq. (7).

2.4. Calculation of Element Matrices with a High-order Gauss–Legendre Rule

In plane-wave-enriched FEM, element matrix calculations are time-consuming compared with the solution process of the linear system of equations. Generally, the matrix calculations are performed using a high-order Gauss–Legendre rule, where the number of integration points changes depending on the analyzed frequency and element size. It differs from those in standard FEM, which uses a constant number of integration points irrespective of the frequency and element size. For standard FEM, the element construction process is trivial compared with the solution process.

As an example, for plane-wave-enriched four-node quadrilateral elements, the element stiffness matrix $K_e$ is calculated, using the Gauss–Legendre rule, as

\[ K_e = \sum_{i=1}^{n_g} \sum_{j=1}^{n_g} \omega_i \omega_j \nabla P(\xi_i, \eta_j)^T \nabla P(\xi_i, \eta_j) \text{det}(J), \]

where $(\xi_i, \eta_j)$ represents the local coordinate at integration points, $\omega_i$ and $\omega_j$ are weights, and $J$ denotes the Jacobian matrix. In the equation above, $n_g$ is the number of integration points that depends on the frequency and element size. For high-frequency analyses, a well-used rule exists [10]: around ten integration points per wavelength are contained within each element, as

\[ n_g = \text{int}(10n_w + 1), \]

where the function int() represents the conversion to an integer by rounding down and $n_w$ is defined as the ratio between wavelength $\lambda$ and element length $h_{max}$, i.e., $n_w = h_{max}/\lambda$. However, at present, no method has been established to ascertain the integration point in accordance with the analyzed frequency in order to perform an accurate and efficient wideband frequency response analysis including low frequencies. This report describes our exploration of the integration rule for the wideband frequency response analysis via investigations with an ideal one-dimensional problem.

3. EXPLORATION OF INTEGRATION POINTS FOR WIDEBAND ANALYSIS

We used a plane wave propagation problem in a duct to propose an efficient method of setting the integration points $n_g$ for wideband analyses. Figure 1 shows the duct model with the size of $1 \times 0.03 \text{m}$ including a vibration boundary $\Gamma_v$ at the tube inlet and an absorbing boundary $\Gamma_a$ at the tube outlet. The vibration velocity $v_n = 1 \text{ m/s}$ and normalized acoustic impedance $z_n = 1.0$ are respectively assigned to these boundary surfaces. The sound field in the duct is known to be completely describable by the superposition of two plane waves propagating in opposite directions. Therefore, this problem is useful for considering only the effect of $n_g$ on the resulting accuracy. We assumed the speed of sound to be $c_0 = 340 \text{ m/s}$ and the air density to be $\rho_0 = 1.205 \text{ kg/m}^3$. To investigate the effects of the length of elements, we used five meshes, each discretized with a different length of elements $h_{max} = 1 \text{ m}, 0.5 \text{ m}, 0.25 \text{ m}, 0.125 \text{ m}$ or $0.0625 \text{ m}$. For plane wave enrichment, two plane waves with $\theta = 0$ and $2\pi$ were respectively attached at nodal points.

To evaluate the numerical solution accuracy, we defined the relative error norm $e_2(f)$ from the theoretical solution as

\[ e_2(f) = \sqrt{\frac{\sum_{i=1}^{n} [p_{\text{em}}(f, x_i) - p_{\text{theory}}(f, x_i)]^2}{\sum_{i=1}^{n} [p_{\text{theory}}(f, x_i)]^2}}, \]

where $p_{\text{em}}(f, x_i)$ represents the sound pressure at point $x_i$ at frequency $f$ calculated by plane-wave-enriched FEM and $p_{\text{theory}}(f, x_i)$ stands for the corresponding theoretical sound pressure. The error was evaluated at $n = 101$ receiving points along the $x$-axis at $0.01 \text{ m}$ intervals.
3.1. Extension of Well-used Rule to Low-frequency Range

The definition of “high frequency” is not well established. In this section, we present an examination of the definition and extend the applicability of the well-used rule to the low-frequency range. It is noteworthy that, in an earlier work [9], this extended rule was used. However, we did not provide a detailed explanation of how the rule was derived. Herein, we describe the complete process of constructing the rule, which is a contribution beyond that in the previous work.

We calculated sound fields in the duct from 1 Hz to 5 kHz at 1 Hz intervals with the five meshes. Figure 2 presents relative error norms as a function of $n_w$ for each element length when using $n_g$ determined from Eq. (14). It is clear that Eq. (14) performs very well for $n_w > 1$ with an error magnitude of $10^{-4}$ to $10^{-3}$%, but shows significantly large errors when $n_w < 1$. This result suggests a definition of a frequency satisfying $n_w > 1$ as “high frequency.”

Using the above result, we endeavored to ascertain whether the increase in $n_g$ can reduce the error magnitude when $n_w < 1$. According to Eq. (14), $n_g$ is less than 10 at $n_w < 1$. Therefore, we increased the number of integration points $n_g$ from three to ten with one interval. Consequently, the error decreased from lower value of $n_w$ with increasing $n_g$. The relative error converged completely when $n_g = 10$, as depicted in Fig. 3. In addition, as an interesting property, the error became smaller when using larger elements. More specifically, the error magnitude became half that when using double-sized elements. On the basis of the results, we propose an extended rule for determining $n_g$ for a broad frequency range:

$$n_g = \begin{cases} 10 & (n_w < 1) \\ \text{int}(10n_w + 1) & (n_w \geq 1). \end{cases}$$  \hspace{1cm} (16)

In subsequent numerical experiments, we designate this rule as Rule 1.

![Fig. 2 Relative error norms for five element lengths: $n_g$ was determined from Eq. (14).](image1)

![Fig. 3 Relative error norms for five element lengths: $n_g$ was determined using Rule 1.](image2)

### Table 1
Analyzed frequencies at various element sizes and $n_w$ values.

| $n_w$ | $h_{\text{max}}, \text{m}$ | 1 | 0.5 | 0.25 | 0.125 | 0.0625 |
|-------|----------------|----|-----|------|-------|--------|
| 0.01  | 3               | 7  | 14  | 27   | 54    |
| 0.10  | 34              | 68 | 136 | 272  | 544   |
| 0.50  | 170             | 340| 680 | 1,360| 2,720 | 5,440  |
| 1.00  | 340             | 680| 1,360| 2,720| 5,440 |
| 2.00  | 680             | 1,360| 2,720| 5,440|
| 4.00  | 1,360           | 2,720| 5,440|
| 6.00  | 2,040           | 4,080|
| 8.00  | 2,720           | 5,440|
| 10.0  | 3,440           |     |

3.2. Exploration of a More Efficient Rule

To propose a more efficient integration rule based on $n_w$, we calculated sound fields in the duct at frequencies where $n_w = 0.01, 0.1, 0.5, 1, 2, 4, 6, 8$ and 10. According to $n_w = h_{\text{max}}/\lambda$, the analyzed frequencies $f$ are calculated as

$$f = \frac{c_0}{\lambda} = \frac{c_0 n_w h_{\text{max}}}{h_{\text{max}}}$$  \hspace{1cm} (17)

and shown with each element length in Table 1, but some frequencies are excluded because the plane wave condition $D < 0.5\lambda$ is not satisfied. Here, $D$ is the duct diameter, which is 0.03 m in this case.

Figures 4(a)–4(d) present relative error norms at $n_w = 0.01, 0.1, 1,$ and 10 when $n_g$ increases from 2 to 100. Errors at each $n_w$ converge with the same number of points irrespective of the mesh used. As an example, Fig. 4(a) shows that, at $n_w = 0.01$, errors converge with $n_g = 3$. Then, the relationship between $n_w$ and converged $n_g$ is presented in Fig. 5, including the line fitted by linear regression. We round up the slope and intercept of the regression function to integer values. The following formula is finally obtained:
Subsequently, we designate this rule as Rule 2.

4. EFFECTIVENESS OF PROPOSED INTEGRATION RULES

We compare the performance characteristics of the two proposed rules using four-node linear quadrilateral elements in two numerical experiments on the calculation of 2D sound fields in a single and a coupled rooms. Note that the two problems were used in the previous study [9]. We used them again for the performance comparison. Both sound fields generated by acoustic emission from a loudspeaker were computed from 20 Hz to 2.5 kHz at 1 Hz intervals. These problems have no analytical solution. Therefore, we used reference solutions calculated using fourth-order-accuracy FEM [11] with sufficiently fine meshes. The speed of sound $c_0 = 340 \text{ m/s}$ and the air density $\rho_0 = 1.205 \text{ kg/m}^3$ were assumed.

To evaluate the accuracy quantitatively, we defined the RMS error with respect to the spatial distribution of the sound pressure level (SPL) as

$$L_{\text{rms}}(f) = \sqrt{\frac{1}{N_p} \sum_{i=1}^{N_p} (L_{\text{em}}(f,i) - L_{\text{ref}}(f,i))^2}, \quad (19)$$

where $L_{\text{em}}(f,i)$ stands for the SPL in receiver $i$ at frequency $f$ calculated by plane-wave-enriched FEM. Also, $L_{\text{ref}}(f,i)$ denotes the SPL determined by the reference solution. $N_p$ stands for the number of received points. We applied 1/3 octave band averaging to the RMS error to capture the error behavior easily as

$$n_g = \text{int}(5n_w + 6). \quad (18)$$

Subsequently, we designate this rule as Rule 2.

Fig. 4  Relative error norms for four different $n_w$ values: (a) $n_w = 0.01$, (b) $n_w = 0.1$, (c) $n_w = 1.0$, and (d) $n_w = 10.0$.

Fig. 5  Relationship between $n_w$ and $n_g$ and line fitted by linear regression.
Here, $\bar{L}_{\text{rms}}(f_c)$ represents the RMS error at 1/3 octave band center frequency $f_c$, $N_l$ denotes the number of frequencies included within the 1/3 octave band, and $f_l$ and $f_u$ respectively denote the lower and upper limit frequencies.

4.1. Single-room Problem

Figure 6 shows the single-room model with an area of 39.92 m$^2$. The room’s boundary comprises a weakly absorbing boundary $\Gamma_{\text{wall}}$ and an absorbing boundary $\Gamma_{\text{MPP}}$ of a honeycomb-backed microperforated panel (MPP) absorber. The weakly absorbing boundary has a real-valued impedance corresponding to the normal-incidence absorption coefficient $\alpha_0 = 0.05$. The absorption characteristics of the MPP absorber are shown in Fig. 7. The absorption characteristics of the MPP absorber were computed theoretically [12] with a hole diameter of 0.5 mm, a panel thickness of 1 mm, a perforation ratio of 0.75%, and a surface density of 1.13 kg/m$^2$. The backing honeycomb core depth was 0.015 m. The vibration velocity $v = 1.0$ m/s was assigned to the vibration boundary $\Gamma_v$ assuming a speaker cone.

Figure 8 shows a FE mesh discretized with elements of lengths in the range of 0.2 to 0.4 m, which are three times larger than the wavelength at the upper-limit frequency of 2.5 kHz. The numbers of elements and nodes were 267 and 307, respectively. The number of plane waves $q$ for enrichment was determined with [6]

$$q = \text{round}[kh_{\text{max}} + C(kh_{\text{max}})^{1/3}] .$$

Here, the constant $C$ controls the resulting accuracy in the analysis. In this problem, $C = 13$ was used to ensure the highest accuracy [9]. In this case, the DOF changes from 2,149 to 16,271 at frequencies from 20 Hz to 2.5 kHz with the number of plane waves in the range from 7 to 53. The numbers of integration points were ascertained using Rules 1 and 2. Figure 9 shows the numbers of integration points in the two rules as a function of frequency. The numbers of integration points in the two rules become the same at $n_w = 1$, but the number in Rule 2 becomes less than that in Rule 1 when $n_w < 1$ and $n_w > 1$. In particular, the reduction rate becomes larger for higher frequencies.

The reference solution was calculated by fourth-order-accuracy FEM with two meshes. One was discretized with 0.01 m square elements (400,720 DOF) at frequencies of 20 Hz to 1.5 kHz. The spatial resolution was 22 elements per wavelength at 1.5 kHz. For higher frequencies, the other mesh was discretized with 0.005 m square elements (1,599,840 DOF). Its spatial resolution was 27 elements per wavelength at the upper-limit frequency.

Figure 10 shows RMS errors in plane-wave-enriched FEM with Rules 1 and 2, as well as the result of standard
FEM (0.01 m mesh) for reference. The errors were computed at the 15 receiving points shown in Fig. 6. The two integration rules perform very well with small errors of less than 1 dB at all frequencies. The error magnitude is lower than that in the standard FEM at frequencies higher than 200 Hz. It is noteworthy that the reason for the higher accuracy in standard FEM below 200 Hz is simple. At such frequencies, the mesh used has a markedly high spatial resolution that exceeds 170 elements per wavelength. Additionally, the RMS error of Rule 2 is almost the same as that of Rule 1 despite the use of fewer integration points, as shown in Fig. 9. Table 2 shows computational times in the matrix construction process for the two rules at various frequencies. The computational times of Rule 2 are less than those of Rule 1 at all frequencies. In particular, at 2 kHz and 2.5 kHz, the computational times of Rule 2 are less than half those of Rule 1. These results clearly indicated that Rule 2 enables a faster analysis of frequency responses than Rule 1 without loss of accuracy.

4.2. Coupled-room Problem

Figure 11 shows the coupled-room model with an area of 60.48 m², comprising four rooms. One is the single room described earlier. The other three are soundproof rooms. The room’s boundary comprises a weakly absorbing boundary $\Gamma_{\text{wall}}$ and a highly absorbing boundary $\Gamma_{\text{porous}}$ of a rigid-backed porous absorber. The weakly absorbing boundary has a real-valued impedance corresponding to normal-incidence absorption coefficient $\alpha_0 = 0.05$. The absorption characteristics of the porous absorber are shown in Fig. 12. The absorption characteristics of 32 kg/m³ of glass wool as the porous absorber were computed using an equivalent fluid model [13] with Miki’s empirical equation [14] for an air cavity thickness of 100 mm and a flow resistivity of 13,900 Pa s/m². The sound field in this problem has a large SPL difference between the largest

| $f$, Hz | Matrix calculation time, s | Rule 1 | Rule 2 |
|---------|---------------------------|--------|--------|
| 125     | 0.677                     | 0.309  |
| 250     | 1.095                     | 0.632  |
| 500     | 2.247                     | 1.548  |
| 1,000   | 6.139                     | 5.260  |
| 1,500   | 19.273                    | 11.987 |
| 2,000   | 46.053                    | 23.804 |
| 2,500   | 90.386                    | 41.114 |
room and the soundproof rooms. The vibration velocity \( v = 1.0 \, \text{m/s} \) was assigned to the vibration boundary \( \Gamma_v \).

Figure 13 shows a FE mesh discretized with elements of lengths in the range of 0.2 to 0.4 m. The numbers of elements and nodes were 413 and 493, respectively. The number of plane waves \( q \) was determined using Eq. (21) with \( C = 13 \). The DOF changes from 3,451 to 26,129.

The reference solution was calculated by fourth-order-accuracy FEM with two meshes. One was discretized with 0.01 m square elements (615,020 DOF) at frequencies of 20 Hz to 1.5 kHz. For higher frequencies, the other mesh was discretized with 0.005 m square elements (2,454,040 DOF).

Figure 14 shows RMS errors in plane-wave-enriched FEM with the two rules. The errors were computed at the 18 receiving points shown in Fig. 11. The result obtained for standard FEM (0.01 m mesh) is also shown for comparison. Plane-wave-enriched FEM with the proposed rules performs much better than standard FEM, showing lower errors at frequencies higher than 200 Hz. The reason for the high accuracy of FEM below 200 Hz is the same as in the case of the single-room problem, i.e., the very fine spatial resolution mesh in FEM. The two rules have almost the same accuracy. It is noteworthy that the numbers of integration points \( n_g \) for all frequencies are the same, as those shown in Fig. 9. Table 3 shows computational times in the matrix construction process at various frequencies. At all frequencies, Rule 2 shows faster calculations, further demonstrating its effectiveness. Finally, Fig. 15 shows SPL distributions of Rules 1 and 2, and the reference solution at 125 Hz and 2.5 kHz. Rules 1 and 2 show good agreement with the reference solution. This result also indicates that Rule 2 can predict the sound field accurately despite the use of fewer integration points than Rule 1 at both high and low frequencies.

### Table 3: Computational times of matrix construction process with Rules 1 and 2 for the coupled-room problem.

| \( f \), Hz | Rule 1    | Rule 2    |
|------------|-----------|-----------|
| 125        | 1.140     | 0.579     |
| 250        | 1.892     | 1.150     |
| 500        | 3.775     | 2.706     |
| 1,000      | 9.994     | 8.631     |
| 1,500      | 30.466    | 19.351    |
| 2,000      | 72.272    | 37.729    |
| 2,500      | 140.584   | 64.494    |

### 5. CONCLUSION

As described in this report, we assessed two integration rules for plane-wave-enriched FEM, that are suitable for performing wideband frequency response analyses accurately and efficiently using an ideal plane wave propagation problem. Because the presented Rule 1, which is an extension of a well-used rule to the low-frequency range, was used in the previous work [9] without explaining its
derivation process, we gave the complete explanation of the derivation of Rule 1. Thus, the derivation process shown here is a contribution beyond that of the previous work [9]. Another rule, which is the most important contribution of the present study, was obtained by the linear regression of the relationship between $n_w$ and the minimum number of integration points required for solution convergence. Furthermore, their performance characteristics were compared using two room-acoustics problems. The results demonstrated that the rule obtained by linear regression can analyze the frequency response more rapidly while maintaining accuracy than the extended rule because of the reduced number of integration points. Further examination will be undertaken to confirm the efficiency of the proposed rules in applications to problems that are more complicated.

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