SKELETONIZATIONS OF PHASE SPACE PATHS

John T. Whelan

University of Utah, Department of Physics
115 S 1400 E Room 201, Salt Lake City, UT 84112-0830

Abstract. Construction of skeletonized path integrals for a particle moving on a curved spatial manifold is considered. As shown by DeWitt, Kuchař and others, while the skeletonized configuration space action can be written unambiguously as a sum of Hamilton principal functions, different choices of the measure will lead to different Schrödinger equations. On the other hand, the Liouville measure provides a unique measure for a skeletonized phase space path integral, but there is a corresponding ambiguity in the skeletonization of a path through phase space. A family of skeletonization rules described by Kuchař and referred to here as geodesic interpolation is discussed, and shown to behave poorly under the involution process, wherein intermediate points are removed by extremization of the skeletonized action. A new skeletonization rule, tangent interpolation, is defined and shown to possess the desired involution properties.

1. SKELETONIZED PATH INTEGRALS

The path integral method seems to provide a generic recipe for producing a quantum theory given a classical action. This recipe is based on the principle that amplitudes are given by sums, over the relevant histories, of the complex exponential of the action. For example, a wavefunction \( \psi(x_i, t_i) \) is propagated to a later time \( t_f > t_i \) by

\[
\psi(x_f, t_f) = \int_{x_i} D^n x e^{iS[x]/\hbar} \psi(x_i, t_i),
\]

where \( D^n x \) denotes the phase space measure and \( S[x] \) is the classical action.
where the integral is over all paths ending at the argument \( x_f \). However, in order to provide a constructive definition for an expression like (1) one needs to give a meaning to formal concepts like an integral over paths, and it is in this step that decisions need to be made in the implementation of the quantum theory.

The system considered in this work is a free non-relativistic particle of unit mass moving on a curved \( n \)-dimensional spatial manifold, which is described by the action

\[
S[x] = \frac{1}{2} \int dt \, g_{ab}(x) \frac{dx^a}{dt} \frac{dx^b}{dt}.
\]

This is the simplest action which exhibits the ambiguities considered here, which correspond in an operator theory to questions of operator ordering. One can also modify the action to include potential terms, but previous work \( \text{[2]} \) suggests that the results will remain essentially the same.

### 1.1. Configuration Space

A formal path integral such as (1) can be given a concrete definition via the skeletonization process, in which the interval \( (t_i, t_f) \) is broken up by a series of \( N \) time instants \( \{t(I)\} \). The integral over all paths \( x(t) \) is realized, in the limit of an infinitely fine time slicing, as a product of integrals over the positions \( \{x(I)\} \) of the particle at the discrete times:

\[
\psi(x_f, t_f) = \lim_{N \to \infty} \left( \prod_{I=0}^{N} \int A(x(I+1)|t(I+1), x(I)|t(I)) \, dx(I) e^{iS(x(I+1)|t(I+1), x(I)|t(I))/\hbar} \right) \psi(x_i, t_i),
\]

where \( t(0) = t_i \) and \( t(N+1) = t_f \). This process replaces the action functional \( S[x] \) with a function \( \sum_{I=0}^{N} S(x(I+1)|t(I+1), x(I)|t(I)) \) which is constructed by summing contributions from the intervals between time slices. Since the equations of motion are second order, there is a preferred path between a pair of endpoints \( (x, t) \) and \( (x', t') \), namely, the classical path between them, which for the theory described by the action (2) is an affinely parametrized geodesic.\(^2\) The action functional evaluated on that piece of the path is the Hamilton principal function \( S(x'|t|x) \). On the other hand, the measure \( \prod_{I=0}^{N} \int A(x(I+1)|t(I+1), x(I)|t(I)) \, dx(I) \) for the path integral has no such natural definition in curved space. Feynman \( \text{[3]} \) showed that \( A(x'|t|x) = [2\pi i\hbar(t' - t)]^{-n/2} \) was the appropriate measure for flat space in Cartesian coordinates. The simplest generalization of the Feynman measure to curved space is \( \text{[3]} \)

\[
A_0(x'|t|x) = \frac{\sqrt{g}}{[2\pi i\hbar(t' - t)]^{n/2}},
\]

where \( g \) is the determinant of the metric \( g_{ab} \) at the point \( x \), which ensures that the path integral is invariant under coordinate changes. However, there are other measures which are similarly invariant and also reduce to the Feynman measure when the spatial manifold is flat. Expanding to second order in powers of the geodesic distance \( \sigma' \) between \( x \) and \( x' \) (which turns out to be sufficient to evaluate the path integral), one family of such measures is \( \text{[4]} \)

\[
A_\lambda(x'|t|x) = \frac{\sqrt{g}}{[2\pi i\hbar(t' - t)]^{n/2}} \left( 1 + \frac{\lambda}{3} R_{ab}^\prime y_a' y_b' + O(\sigma'^3) \right),
\]

\(^2\)Throughout this paper, the points in question, even when separated by a finite distance, are taken to be close enough that there is no crossing of geodesics.

\(^3\)Specific measures of this sort were discussed in \( \text{[2, 3]} \), and placed into the family (5) in \( \text{[3]} \).
where \( y'_a \) are the Riemann normal coördinates of the point \( x' \) with respect to the origin \( x \) and \( R^{ab} \) is the Ricci tensor at \( x \).

This ambiguity in defining the path integral measure has tangible consequences. Namely, if we use a member of the family (3) to define the measure in the skeletonized path integral (3), the wavefunction thus propagated satisfies a Schrödinger equation

\[
i\hbar \frac{\partial \psi(x, t)}{\partial t} = \left( -\frac{\hbar^2}{2} g^{ab} \nabla_a \nabla_b + \frac{1}{6} \frac{\lambda}{\hbar^2} R \right) \psi(x, t).
\]

(6)

The classical Hamiltonian \( g^{ab} p_a p_b \) is not quantized in the simplest way, by replacing it with an operator proportional to the covariant Laplacian, but also contains a measure-dependent term proportional to the scalar curvature. In particular, the straightforward generalization (4) of the Feynman measure (in which \( \lambda = 0 \)) leads to a Schrödinger equation with a curvature correction term.

1.2. Phase Space

One way to resolve the measure ambiguity is to start instead with a phase space path integral

\[
\psi(x_f, t_f) = \int_{x_f} D^n p D^n x e^{iS[p, x]/\hbar} \psi(x_i, t_i),
\]

(7)

in which the path integral measure is expressed as a product of Liouville measures \( d^n p d^n x / (2\pi\hbar)^n \) in the skeletonization

\[
\psi(x_f, t_f) = \lim_{N \to \infty} \left( \prod_{I=0}^{N} \int \frac{d^n p^{(I)} d^n x^{(I)}}{(2\pi\hbar)^n} e^{iS(x^{(I+1)}t^{(I+1)}|p^{(I)}, x^{(I)}t^{(I)})/\hbar} \right) \psi(x_i, t_i).
\]

(8)

Unfortunately, the ambiguity has now shifted into the definition of the contributions \( \{ S(x^{(I+1)}t^{(I+1)}|p^{(I)}, x^{(I)}t^{(I)}) \} \) to the skeletonized action. This is because there is in general no classical trajectory through phase space satisfying the conditions

\[
x(t) = x, \quad p(t) = p, \quad x(t') = x',
\]

(9)

since any two of these three conditions are sufficient to specify a classical phase space path.

Since there is no classical phase space path, we need some interpolation rule which associates a virtual path through phase space with the set of boundary conditions (3). (It should of course reproduce the classical path when the three boundary conditions are classically consistent.) A given interpolation rule defines a phase space principal function (PSPF) \( S(x'|p \ x t) \) whose value for a particular set of arguments will be the action functional evaluated along the virtual path which the rule specifies for those boundary conditions. The interpolation rule then defines a skeletonization, under which and the skeletonized canonical action appearing in (8) is a sum of PSPFs.

1.2.1. Geodesic Interpolation. One type of rule, introduced by Kuchař [1], prescribes the configuration space path to be the classical path from \( x \) to \( x' \), regardless of the value of \( p \). The momentum then “goes along for the ride,” being propagated in some linear way along the geodesic (Fig. [1a]). The resulting phase space principal function is

\[
S(x'|p \ x t) = p^a y'_a - \frac{t' - t}{2} G^{ab}(x'|x) p_a p_b ,
\]

(10)
with the functional form of $G^{ab}(x'|x)$ depending on the details of the propagation rule. We call this “geodesic interpolation” (GI) because the property of the classical trajectories which is retained by all of the virtual ones is that the configuration space projection of the path is a geodesic. This motivation for selecting rules in which the second order equation of motion, involving only the path $x(t)$, is obeyed independent of the path $p(t)$, comes from the invariance of the tensor formulation under point canonical transformations $[x \rightarrow X(x), p \rightarrow P(p,x)]$, in which the transformation of the $x$ components of a point in phase space is independent of its $p$ components, but not under general canonical transformations $[x \rightarrow X(p,x), p \rightarrow P(p,x)]$.

If the momentum $p$ is taken to be parallel-propagated along the geodesic, the tensor $G^{ab}(x'|x)$ in (11) becomes the inverse metric $g^{ab}$ at $x$. This is one of a family of propagation rules discussed by Kuchař which, for small intervals, have the expansion

$$G^{ab}(x'|x) = g^{ab} - \frac{\lambda}{3} R^{acbd} y_c y_d + O(\sigma^3),$$

(11)

where $R^{acbd}$ is the Riemann curvature tensor at $x$.

Performing the quadratic integrals over the momenta in the skeletonized path integral (8) induces a measure

$$A(x'|xt) = \frac{1}{\sqrt{[2\pi i \hbar (t' - t)]^n \det G^{ab}}}$$

(12)

on the remaining configuration space path integrals. The family of propagation rules given by (11) lead in this way to exactly the family of configuration space measures (5). This means that they produce quantum theories whose Hamiltonians contain curvature correction terms $\frac{1}{6} \hbar^2 R$; in particular, the parallel propagation rule ($\lambda = 0$) leads to

Figure 1. Geodesic interpolation (GI), illustrated in 1+1 dimensions, with the time direction running vertically and the space direction running horizontally. (a) Constructing a phase space path from the boundary conditions (9); the configuration space trajectory is the classical path, a straight line from $x$ to $x'$, while the momentum $p$ is held constant along the trajectory. In higher dimensions, if the spatial manifold is curved, the GI scheme also requires a rule for propagation of the momentum. (b) Failure of involution of paths for GI; using the values (19) of $x'$ and $p'$ which extremize the sum (18) of phase space principal functions, one constructs the two-step phase space trajectory, with the first step defined by $x$, $p$ and $x'$ and the second by $x'$, $p'$ and $x''$. Comparison to Fig. 1(a) shows that this is not the same path as would be constructed from $x$, $p$ and $x''$ by the GI prescription (although the two paths do have the same action when the space is flat).
the Feynman measure (4) and hence to a curvature-corrected quantum Hamiltonian. The case \( \lambda = 1 \), which Kuchař showed arose from a propagation rule based on the equation of geodesic deviation, leads to a Hamiltonian with no curvature correction term.

This work considers a criterion for selecting a phase space skeletonization rule, and hence a principal function. This criterion, as described in the following section, is based on a property of the PSPFs and paths themselves.

2. THE INVOLUTION PROPERTY

2.1. Configuration space

We will judge phase space skeletonizations by whether they obey a property analogous to the involution property satisfied by configuration space skeletonizations. The configuration space property is defined as follows: Consider the contribution to the skeletonized action from the intervals between three consecutive time instants, called \( t, t' \) and \( t'' \):

\[
S(x'' t'' | x' t') + S(x' t' | x t).
\] (13)

It is a function of three positions, \( x, x' \) and \( x'' \), and has associated with it a path made up of two geodesics connected at the point \( x' \). If we vary the location of that point \( x' \) while holding the endpoints \( x \) and \( x'' \) fixed, the expression (13) is minimized by the \( x' \) which lies at the appropriate point on the geodesic from \( x \) to \( x'' \). The path corresponding to that choice of \( x' \) is just the geodesic connecting the two endpoints \( x \) and \( x'' \), and its action is the Hamilton principal function \( S(x'' t'' | x t) \):

\[
S(x'' t'' | x' t') + S(x' t' | x t) \Rightarrow S(x'' t'' | x t).
\] (14)

This property is known as involution.

We say that the configuration space skeletonization procedure obeys involution of paths (IOP) because extremization of (13) with respect to the intermediate position \( x' \) leads to the classical path from \( x \) and \( x'' \), and involution of functions (IOF) because (14) holds. Clearly, the latter property is implied by the former.

2.2. Phase space

Given an interpolation rule, which defines a phase space principal function \( S(x' t' | p x t) \), we can define the analogous property for a phase space skeletonization by considering two consecutive intervals in the skeletonization. The five quantities \( x, p, x', p' \) and \( x'' \) determine a phase space path between times \( t \) and \( t'' \) according to the interpolation rule. The action for this path is

\[
S(x'' t'' | p' x' t') + S(x' t' | p x t).
\] (15)

Extremization with respect to the intermediate variables \( x' \) and \( p' \) leaves a function of \( x, p \) and \( x'' \); we say that involution of functions holds if that is the same PSPF with which we started:

\[
S(x'' t'' | p' x' t') + S(x' t' | p x t) \Rightarrow S(x'' t'' | p x t).
\] (16)
The stricter condition of involution of paths is satisfied if the phase space path defined by the extremizing values of \(x'\) and \(p'\) is the same one which the interpolation rule would have constructed \textit{a priori} from \(x\), \(p\) and \(x''\) without making reference to the intermediate time slice \(t'\).

These concepts are easily illustrated in the case where the spatial manifold is flat; in that case the preferred propagation rule associated with geodesic interpolation is simply to keep the momentum constant (as measured in a Cartesian coordinate system). This leads to a PSPF

\[
S(x't'|p\,x\,t) = p \cdot (x' - x) - \frac{t' - t}{2}p^2.
\]

The two-step action

\[
S(x''t''|p'x't') + S(x't'|p\,x\,t) = p' \cdot x'' + (p - p') \cdot x' - p \cdot x - \frac{t'' - t'}{2}p^2 - \frac{t' - t}{2}p^2
\]

is minimized by

\[
p' = p \quad \text{and} \quad x' = x'' - (t'' - t')p.
\]

Substituting these into (18) gives \(p \cdot (x'' - x) - \frac{t'' - t}{2}p^2 = S(x''t''|p\,x\,t)\), so the GI scheme in flat-space does satisfy IOF. On the other hand, it is easy to see that it does not satisfy IOP, since the path obtained by extremization has a configuration space projection which starts at \(x\) at time \(t\), follows a straight path to \(x'' - (t'' - t')p\) at \(t'\), and then makes a sharp turn before heading on to \(x''\) at \(t''\) (Fig. 1b). This is not the same as the path from \(x\) to \(x''\) which geodesic interpolation would dictate without the presence of the intermediate point, since that path is just a straight line from \(x\) to \(x''\).

If we look at the curved-space case in the limit that all of the distances are small, the zeroth-order results of course replicate the flat-space ones (including the failure of IOP). If we consider the family of PSPFs defined by (11), the behavior of the first correction terms will depend on the value of \(\lambda\). It turns out that for \(\lambda = -1\) (and for no other value) IOF continues to hold to the lowest non-trivial order.

The reason why IOF can hold even when IOP fails is this: Since a given PSPF \(S(x't'|p\,x\,t)\) corresponds to the action functional evaluated along a virtual, rather than classical, phase space path satisfying the boundary conditions (9), there are many different paths which yield the same function. We can use this fact to our advantage by defining a different skeletonization procedure which produces the same PSPF over small intervals as \(\lambda = -1\) geodesic interpolation, and thus also obeys the IOF property, but arises from a different family of paths, which satisfies IOP.

3. TANGENT INTERPOLATION

We will call the skeletonization procedure which respects IOP \textit{tangent interpolation} (TI). As opposed to the geodesic interpolation schemes, which require the configuration space part of the path to be the classical geodesic, the configuration space path in the TI scheme is not even continuous. However, the classical equation relating momentum to velocity is respected by the prescribed path, so that the momentum vector remains tangent to the configuration space trajectory. Hence the name “tangent interpolation.”

The phase space path (Fig. 2a) determined by a given set of boundary conditions (9) begins at \(x\) with the momentum \(p\) at time \(t\), jumps instantaneously to a point
\[ t' \quad x' \quad x \quad t \]

(a)

\[ t' \quad [p'] \quad [x' = \tilde{x}] \quad t'' \]

(b)

**Figure 2.** Tangent interpolation, illustrated in 1 + 1 dimensions. (a) The phase space path associated with the boundary conditions (9) has an initial configuration space jump from \( x \) to \( \tilde{x} \) and then follows the classical phase space trajectory from \( \tilde{x} \) to \( x' \); the point \( \tilde{x} \) is chosen so that the prescribed momentum \( p \) agrees with the initial momentum on the geodesic \( \tilde{x}x' \). (In higher-dimensional curved space, this comparison is made in Riemann normal coordinates based at \( x \).) (b) Involution of paths for TI. Using the \( x' \) and \( p' \) (19) which extremize the sum of phase space principal functions (18), one constructs a two-step phase space trajectory, first from \( t \) to \( t' \) and then from \( t' \) to \( t'' \). There is no discontinuity at \( t' \) for this choice of \( x' \) and \( p' \) (i.e., \( \tilde{x}' = x' \)), and the initial jump from \( x \) to \( \tilde{x} \) implied by the chosen \( x' \) at time \( t' \) is the same as the one implied by \( x'' \) at \( t'' \) (along with the initial \( p \) in either case). Thus the two-step phase space path is the same as we would get by TI in one step from \( t \) to \( t'' \). (Compare Fig. 2a.)

\[ \tilde{x} \] which is prescribed (in a manner detailed below) by the boundary conditions, and then follows the classical phase space trajectory from \( \tilde{x} \) at \( t \) to \( x' \) at \( t' \). The point \( \tilde{x} \) is chosen so that in a Riemann normal coordinate (RNC) system based at \( x \), there is no momentum discontinuity at the initial time \( t \), i.e., so that the classical momentum at the beginning of the geodesic from \( \tilde{x} \) to \( x' \) has the same covariant RNC components as \( p \). In flat space, this means that \( p = \frac{x' - \tilde{x}}{t' - t} \), or

\[ \tilde{x} = x' - p(t' - t), \quad (20) \]

although in curved space the relation \( \tilde{x}(x' t'|p x t) \) is only implicitly defined by the matching of momenta. However, the \( \tilde{x} \) which achieves the momentum matching condition can alternatively be found as the \( \tilde{x} \) which minimizes \( S(x' t'|\tilde{x} p x t) \), the action along such a path, for a given \( x, p \) and \( x' \). This action has two contributions, one from the initial discontinuity and one from geodesic from \( \tilde{x} \) to \( x' \), giving

\[ S(x' t'|\tilde{x} p x t) = p^a \dot{y}_a + S(x' t'|\tilde{x}t). \quad (21) \]

Extremization with respect to \( \tilde{x} \) gives the curved-space counterpart of (21), and defines the PSPF \( S(x' t'|p x t) \) for the tangent interpolation recipe. Note that while GI only defines a class of PSPFs in curved space, with a recipe for propagating the momentum needed to pick out a single PSPF, the TI procedure defines one unique path and thus only one PSPF.

In flat space, it is easy to verify that the TI prescription leads to the PSPF (17), by substituting (20) into the flat-space equivalent of (21),

\[ S(x' t'|\tilde{x} p x t) = p \cdot (\tilde{x} - x) + \frac{1}{2} \frac{(x' - \tilde{x})^2}{t' - t}. \quad (22) \]
The first correction term leads to the form (10,11) with the value $\lambda = -1$. This means, from the results of the previous section, that it satisfies IOF, at least to the first non-trivial order in the distances involved.

In fact, the TI PSPF satisfies the involution property to all orders, because the TI prescription obeys IOP, even over finite intervals. IOP can be verified for flat space as follows (Fig. 2b): Since the sum of the principal functions is given by (18), the values of $x'$ and $p'$ which extremize it are again given by (19). The auxiliary point $\tilde{x}'$ for the second interval is given, analogously to (20), by $\tilde{x}' = x'' - p'(t' - t) = x'$; since $\tilde{x}'$ and $x'$ coincide, there is no discontinuity in the path from $x'$ to $x''$, which is just a classical phase space path. The point $\tilde{x} = x' - p(t' - t) = x'' - p(t'' - t)$ to which the initial jump at time $t$ is made is in the same location whether the final condition is taken to be $x''$ at $t''$ or $x'$ [given by (19)] at $t'$.

4. CONCLUSIONS AND OUTLOOK

We have shown that the involution property described in Sec. 2 is obeyed by the phase space skeletonization procedure which we defined in Sec. 3 and named tangent interpolation. According to prior results, summarized in Sec. 1, using this skeletonization rule to construct a phase space path integral will induce a particular measure on the corresponding configuration space path integral. In this way, involution singles out a preferred realization of the skeletonized path integral. It is one which leads to a Schrödinger equation whose Hamiltonian differs from the simplest form, proportional to the covariant Laplacian, by a term $\frac{\hbar^2}{2}\bar{R}$ proportional to the scalar curvature of the spatial manifold. Understanding the implications of this result rests on two as yet open questions. First, is there any physical explanation for the modified Hamiltonian, and second, what is the significance of the involution property for the path integral prescription? Involution of the Hamilton principal function enables composition of propagators constructed via the configuration space path integral, for any choice of measure. Perhaps involution of the phase space principal function is related to composition of path integrals defining momentum space propagators. Further investigation is called for.

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