The Design of Dual Band Stacked Metasurfaces Using Integral Equations

Jordan Budhu, Member, IEEE, Eric Michielssen, Fellow, IEEE, and Anthony Grbic, Fellow, IEEE

Abstract—An integral equation-based approach for the design of dual-band stacked metasurfaces that are invariant in 1-D is presented. The stacked metasurface will generate collimated beams at the desired angles in each band upon reflection. The conductor-backed stacked metasurface consists of two metasurfaces (a patterned metallic cladding supported by a dielectric spacer) stacked one upon the other. The stacked metasurface is designed in three phases. First, the patterned metallic cladding of each metasurface is homogenized and modeled as an inhomogeneous impedance sheet. An electric field integral equation (EFIE) is written to model the mutual coupling between the homogenized elements within each metasurface and from metasurface to metasurface. The EFIE is transformed into matrix equations by the method of moments. The nonlinear matrix equations are solved at both bands iteratively resulting in dual-band complex-valued impedance sheets. In the second phase, optimization is applied to transform these complex-valued impedance sheets into purely reactive sheets suitable for printed circuit board fabrication by introducing surface waves. In the third phase, the metallic claddings of each metasurface are patterned for full-wave simulation of the dual-band stacked metasurface. Using this approach, two dual-band stacked metasurfaces are designed.

Index Terms—Antennas, dual band, metasurface, method of moments, multilayer, stacked.

I. INTRODUCTION

METASURFACES are the 2-D equivalents of metamaterials [1]. They are described by a boundary condition relating the average tangential electric and magnetic fields on the surface to the induced electric and magnetic surface currents flowing along it weighted by electric, magnetic, and magnetoelastic impedances or admittances. This boundary condition is called the generalized sheet transition condition (GSTC) [2], [3]. Metasurfaces engineered using these boundary conditions can perform various specified electromagnetic transformations, such as polarization rotation [4], [5], circular polarization conversion [6], collimation [7], beamforming [8], and anomalous reflection [9]–[13]. In cases where dual-band operation is desired, adding additional layers can result in operation at two different frequencies. When the surfaces are homogeneous, transfer matrix methods and network theory can be used to design the dual-band metasurfaces with great accuracy. In these cases, the periodic metasurface is reduced to a single unit cell employing periodic boundaries. Network theory concepts are applied to the transmission/reflection through/from the single unit cell. For example, in [14], an analytic technique based on these methods was used to design dual-band linear to circular polarizers for SatCom links. In [15]–[17], similar dual-band circular polarizers are presented for operation in reflection. In [18], an optical dual-band metasurface is designed for perfect optical antireflection at two terahertz frequencies. In [19], machine learning-based optimization is applied to design a unit-cell geometry for homogeneous dual-band frequency-selective surfaces and wideband polarizing metasurfaces. All of these homogeneous metasurfaces are well modeled using periodic expansions and network theory. Homogeneous metasurfaces, however, only allow for a limited set of dual-band field transformations typically involving polarization rotation, filtering, absorption, or antireflection.

Including inhomogeneity (nonperiodicity) allows for more complex dual-band field transformations, such as refraction [20], collimation [21]–[24], focusing [25], and beamforming [26]. Modeling the mutual coupling between elements of these metasurfaces using local periodicity assumptions is approximate and the finite metasurface dimensions are not accounted for. The analytic network theory-based techniques applied to a unit-cell geometry no longer provides accurate results since transverse coupling between cells is not modeled. Xu et al. [27] proposed several methods to circumvent these problems. One such solution introduces perfect electric conducting baffles to separate the inhomogeneous surface into noncoupling homogeneous cells. These surfaces contain many substrate vias that require advanced fabrication techniques and they cannot handle transverse magnetic polarization as perfect magnetic conductors are not available. Furthermore, if dual-band metasurfaces are designed using local periodicity assumptions, the frequencies of operation must be chosen so that operation at one band does not interfere with that at the other band. This places unnecessary restrictions on the selection of operating frequencies. Xi et al. [28] proposed several methods to circumvent this problem as well such as inserting a frequency-selective surface between the layers containing elements operating at distinct bands. This solution requires additional layers and does not model the mutual coupling accurately nor does it account for the finite metasurface dimensions.

Dual-band inhomogeneous metasurfaces can be designed without resorting to local periodicity assumptions or network-based theories by modeling them with integral
In Section II, a three-phase design approach for the design of dual-band metasurfaces is presented. In Section III, the design approach is applied to the design of two dual-band collimating metasurfaces. The collimated beams can be designed to scan to any far-field angle. The present scheme allows for great flexibility in the choice of operating frequencies and leads to the accurate prediction of the performance of dual-band inhomogeneous metasurfaces that are invariant in 1-D. The extension of the design approach to metasurfaces that vary in all three dimensions involves using the 3-D Green’s function rather than the 2-D version and also models correctly the mutual coupling between each of the unique elements in the inhomogeneous metasurface.

In Section II, a three-phase design approach for the design of dual-band metasurfaces is presented. In Section III, the design approach is applied to the design of two dual-band stacked metasurfaces. The first generates broadside beams at 2.4 and 5.1 GHz. The second generates a beam scanned to 30° off broadside when excited at 13.4 GHz and a broadside beam when excited at 35.75 GHz. This article is concluded in Section IV. An appendix provides numerical implementation details.

An $e^{j\omega t}$ time convention is used and suppressed throughout.

II. DESIGN OF DUAL-BAND STACKED METASURFACES

The dual-band finite width stacked metasurface geometry considered consists of two metasurfaces stacked one atop another (see Fig. 1). Each metasurface consists of a patterned metallic cladding layer supported by a dielectric spacer layer. The stack of metasurfaces is grounded by a finite width perfectly conducting backing layer. The stacked metasurface thus contains five total layers (see Fig. 2). The subwavelength texture of the patterned metallic claddings allows them to be homogenized. The patterned metallic claddings can therefore be modeled as inhomogeneous impedance sheets made of homogenized sheet impedance cells (see Fig. 3) according to the GSTC. In this case, the sheet impedances are assumed scalar, given that the problem is 2-D and only one polarization is considered. The dielectric spacers are modeled as volumetric distributions of electric polarization currents in free space following the volume equivalence principle. A system of electric field integral equations (EFIEs) is written to account for mutual coupling between the unique homogenized cells within each layer (intralayer) and from layer to layer (interlayer). The system of EFIE’s is transformed into a matrix equation by applying the method of moments discretization scheme.

The problem to solve is: given the incident and reflected fields, find the completely reactive sheet impedances necessary to obtain the desired field transformation. In the problem, both the sheet impedances and the induced surface currents on them are unknowns. As these quantities appear in the GSTC boundary condition as a product, the problem is inherently nonlinear. This is in contrast to the conventional forward problem of finding the reflected fields given the sheet impedances, which is a linear problem. The solution to the problem is completed in three phases.

Phase 1 involves solving the nonlinear matrix equations. A novel solution scheme has been developed to effectively linearize the matrix equations and solve them iteratively. The iterative scheme ping-pongs between frequencies at each iteration. Thus, the scheme designs the metasurface for operation at both frequencies simultaneously as it convergences (see Fig. 4 for preview). Consequently, there is great flexibility in the choice of each operating frequency used in design as elements operating at each band are not tuned independently but rather together. Phase 1 results in complex-valued sheet impedances (layers 1 and 3) since the specified transformations between the incident and reflected waves do not satisfy the conservation of local power density at the metasurface planes [9]–[13]. Implementation of the real part of the complex-valued sheet impedances (with positive and negative resistances) adds cost and complexity to the fabrication, given that active devices and attenuators are needed.
The metasurface considered consists of multiple subwavelength patterned metallic cladding layers.

Each of the subwavelength patterned claddings is homogenized and modeled as an inhomogeneous impedance sheet.

In phase 2, the real part is removed using an optimization approach originally introduced in [12]. The reactances (with resistances discarded) of the complex-valued sheet impedances (obtained in phase 1) are used as an initial point for gradient descent optimization. Since the convergence of gradient descent algorithms strongly depends on obtaining a good initial guess, the results of phase 1 are paramount to the design process. The optimization cost function is formulated as the root-mean-square difference between the far-field patterns of the purely reactive sheet of the optimization. Hence, the initial guess, the results of phase 1 are paramount to the descent optimization. Since the convergence of gradient descent algorithms strongly depends on obtaining a good initial guess, the results of phase 1 are paramount to the design process. At this stage, the problem is solved.

Although the specific implementation presented in this article is 2-D, the idea behind the approach can also be applied to 3-D structures.

A. Phase 1: Obtain Initial Metasurface Design Containing Complex-Valued Sheets

Consider a conductor-backed stack of two metasurfaces shown in Fig. 2. The top metasurface will be denoted metasurface 1 and will support surface current $J_T$ along its surface (layer 1). The bottom metasurface will be denoted metasurface 2 and will support surface current $J_B$ along its surface (layer 3). The stacked metasurface has width $w$ along $x$ and is invariant along $z$. The top (layer 2) and bottom (layer 4) dielectric layers have relative permittivities $\varepsilon_{r2}$ and $\varepsilon_{r4}$, and thicknesses $d_2$ and $d_4$, respectively. In what follows, the top impedance sheet (layer 1) will be denoted as $\eta_T$ and is assumed to reside in the $y = 0$ plane and the bottom impedance sheet (layer 3) will be denoted as $\eta_B$ and is assumed to reside at $y = -d_2$. Thus, the locations along the top (layer 1) and bottom (layer 3) impedance sheets are denoted by $\mathbf{\rho} = \mathbf{\rho}_T = x\hat{x}$ and $\mathbf{\rho} = \mathbf{\rho}_B = x\hat{x} - d_2\hat{y}$ with $-\omega/2 < x < \omega/2$. The stacked metasurface is illuminated by a $z$-directed line source with current $I_0$ Amps that resides at $\mathbf{\rho}_f = F\hat{y}$ meters above layer 1

$$E^{inc}(\mathbf{\rho}, \omega) = \frac{-I_0\omega\eta_0}{4c}H_0^{(2)}(\frac{\omega}{c}|\mathbf{\rho} - \mathbf{\rho}_f|)$$

where $\omega$ defines the angular frequency of excitation, $c$ is the speed of light in free space, $\eta_0$ is the wave impedance of free space, and $H_0^{(2)}(\cdot)$ denotes the Hankel function of the section kind of order zero.

The proposed algorithm determines the impedances $\eta_T(\varphi_T, \omega)$ and $\eta_B(\varphi_B, \omega)$ of the top and bottom impedance sheets that produce reflected beams with prescribed scan angle $\phi_{ sca}^T$ at frequency $\omega_a$ and $\phi_{ sca}^B$ at frequency $\omega_b$. These beams are realized by the desired scattered aperture fields

$$E_d^{sca}(\mathbf{\rho}_T, \omega) = \frac{2\eta_0S^{inc}(\mathbf{\rho}_T, \omega)}{\sin\phi_{ sca}^T}e^{j\psi_{ sca}(\omega)}$$

where $S^{inc}$ is the incident field power density [12], $\psi_{ sca}(\omega) = k_0\times\cos\phi_{ sca}^T$, and $k_0$ is the wavenumber in free space. It follows from (1) and (2) that the total aperture field at the design frequencies is

$$E_d^{total}(\mathbf{\rho}_T, \omega) = E^{inc}(\mathbf{\rho}_T, \omega) + E_d^{sca}(\mathbf{\rho}_T, \omega)$$

Let $J_T(\varphi_T, \omega)$ and $J_B(\varphi_B, \omega)$ denote electric current densities in the top and bottom impedance surfaces at frequency $\omega$. The total field in the bottom impedance sheet at $\omega$ is

$$E^{total}(\mathbf{\rho}_B, \omega) = \eta_B(\varphi_B, \omega)J_B(\varphi_B, \omega)$$

The current densities
Condition (3) at \( \omega = \omega_b \) is expressed as
\[
\begin{align*}
W_{T}^{\omega_b} &= [G_{TT}][I_{T}^{\omega_b}] + [G_{TB}][I_{B}^{\omega_b}] + [V_{T}^{\omega_b}] \\
[\eta_{T}^{\omega_b}]^{I_{T}^{\omega_b}} &= [G_{BT}][I_{T}^{\omega_b}] + [G_{BB}][I_{B}^{\omega_b}] + [V_{B}^{\omega_b}].
\end{align*}
\] (7)

The matrix definitions in (7) can be found in the Appendix. First, solve the system of equations (two equations two unknowns) (7) for the two unknown currents \([I_{T}^{\omega_b}]\) and \([I_{B}^{\omega_b}]\). Note that \([I_{T}^{\omega_a}]\) and \([I_{B}^{\omega_a}]\) are the complex coefficients vector of the expansions of \(J_{T}(\tilde{\rho}_{T}, \omega_b)\) and \(J_{B}(\tilde{\rho}_{B}, \omega_b)\) as in (A.3). Then, obtain the sheet impedances of the top metasurface from
\[
[\eta_{T}^{\omega_b}] = \frac{W_{T}^{\omega_b}}{I_{T}^{\omega_b}}.
\] (8)

It should be noted that the practical realization of \(\eta_{T}(\tilde{\rho}_{T}, \omega_b)\) fixes its value at all other frequencies, including \(\omega_a\). The full-wave simulation results for the frequency scaling of the patterned elements used in phase 3 show that they follow the usual frequency dependence relations for lumped inductive and capacitive reactances (see Fig. 6). Thus, next frequency-scale the top metasurfaces impedance from \(\omega_a\) to \(\omega_b\). The capacitive reactances are scaled down as
\[
X_{a}^{\omega_b} = \frac{\omega_a}{\omega_b} X_{a}^{\omega_a}
\] (9)

and the inductive reactances are scaled up as
\[
X_{b}^{\omega_a} = \frac{\omega_b}{\omega_a} X_{b}^{\omega_b}
\] (10)

if \(\omega_b > \omega_a\).

**Step 2: Compute \(\eta_{B}(\tilde{\rho}_{B}, \omega)\) by Enforcing Aperture Condition (3) at \(\omega_b\) Given \(\eta_{T}(\tilde{\rho}_{T}, \omega)\) From Step 1:**

Assuming that \(\eta_{T}(\tilde{\rho}_{T}, \omega_b)\) is known, we may compute \(J_{T}(\tilde{\rho}_{T}, \omega_b)\) from knowledge of the total aperture field (3) at \(\omega_b\) using
\[
[I_{T}^{\omega_b}] = \frac{W_{T}^{\omega_b}}{[\eta_{T}^{\omega_b}].
\] (11)

Once \(J_{T}(\tilde{\rho}_{T}, \omega_b)\) is known, an integral equation for \(J_{B}(\tilde{\rho}_{B}, \omega_b)\) is constructed by expressing the known total aperture field at \(\omega_b\) as the sum of fields produced by both currents
\[
W_{B}^{\omega_b} = [G_{TB}][I_{T}^{\omega_b}] + [G_{BB}][I_{B}^{\omega_b}] + [V_{B}^{\omega_b}].
\] (12)

After solving this equation for \(J_{B}(\tilde{\rho}_{B}, \omega_b)\), the total field on the impedance surface of the bottom metasurface (layer 3), \(E_{\text{tot}}(\tilde{\rho}_{B}, \omega_b)\), is computed as
\[
W_{B}^{\omega_b} = [G_{TB}][I_{T}^{\omega_b}] + [G_{BB}][I_{B}^{\omega_b}] + [V_{B}^{\omega_b}].
\] (13)

Knowledge of both \(E_{\text{tot}}(\tilde{\rho}_{B}, \omega_b)\) and \(J_{B}(\tilde{\rho}_{B}, \omega_b)\) finally allows \(\eta_{B}(\tilde{\rho}_{B}, \omega_b)\) to be computed via
\[
[\eta_{B}^{\omega_b}]^{I_{B}^{\omega_b}} = \frac{W_{B}^{\omega_b}}{[I_{B}^{\omega_b}].
\] (14)

The practical realization of \(\eta_{B}(\tilde{\rho}_{B}, \omega_b)\) fixes its value at all other frequencies, including \(\omega_a\). Thus, next frequency-scale \(\eta_{B}(\tilde{\rho}_{B}, \omega_b)\) to \(\eta_{B}(\tilde{\rho}_{B}, \omega_a)\). As before, the impedances follow the usual scaling laws for inductive and capacitive reactances. Thus, the capacitive reactances are scaled up as
\[
X_{b}^{\omega_a} = \frac{\omega_b}{\omega_a} X_{b}^{\omega_b}
\] (15)
and the inductive reactances are scaled down as

\[ X^{\omega_0} = \frac{\omega_0}{\Omega_0} X^{\omega_8} \]  

(16)

if \( \omega_0 > \omega_8 \).

Finally, update \( \eta_B(\hat{\rho}_B, \omega_a) \) by interpolating the new value with the previous one as

\[ \eta_B(\hat{\rho}_B, \omega_a) = s_1 \eta_B(\hat{\rho}_B, \omega_a)_{\text{new}} + s_2 \eta_B(\hat{\rho}_B, \omega_a)_{\text{previous}}, \]

where \( s_1 \) and \( s_2 \) are interpolation constants (\( s_1 = 0.2 \) and \( s_2 = 0.8 \)).

These two steps are repeated until convergence is achieved.

**B. Phase 2: Optimization of Initial Complex-Valued Sheets to Obtain Purely Reactive Sheets**

The real part of the converged complex-valued sheet impedances of (8) and (14) indicates the need for lossy (positive resistances) or active (negative resistances) elements. Implementation of lossy/active devices is costly and difficult to implement with standard printed circuit board fabrication techniques. A purely reactive metasurface is desired to simplify the realization. An optimization method based on gradient descent is next used to optimize the impedance sheets and obtain purely reactive sheet impedances. The optimization scheme is shown in Fig. 5. Given that the patterns produced by the stacked metasurface containing the complex-valued sheet impedances obtained in phase 1 represent the desired performance, the cost function for the optimization is

\[ f_{\text{cost}} = \text{RMS} \left( |E_{\text{far field}}^{\omega_0}(\phi)|_{\text{complex sheet}} - |E_{\text{far field}}^{\omega_0}(\phi)|_{\text{reactive sheet}} \right) \]

\[ + \text{RMS} \left( |E_{\text{far field}}^{\omega_0}(\phi)|_{\text{complex sheet}} - |E_{\text{far field}}^{\omega_0}(\phi)|_{\text{reactive sheet}} \right) \]  

(17)

where \( |E_{\text{far field}}^{\omega_0}(\phi)| \) and \( |E_{\text{far field}}^{\omega_0}(\phi)| \) are the magnitude of the scattered far-field pattern at frequencies \( \omega_0 \) and \( \omega_8 \), respectively. Note that rms is the root-mean-square value of the quantity in the brackets. The cost function attempts to minimize the difference between the far-field pattern radiated by the reactive sheet and the complex sheet. The optimization variables are the reactances of the elements themselves. Thus, the optimizer is seeded by throwing out the real part of the complex sheet impedances and keeping only the reactances. The optimization is constrained to reactances that are practically achievable through the patterning of subwavelength cells of metallic claddings. From Fig. 6, these limits are \( +j150 \Omega \) for inductive reactances and \( -j5000 \Omega \) for capacitive reactances. The optimization continues until (17) is minimized.

**C. Phase 3: Realization of Purely Reactive Sheets Through Patterning of the Metallic Claddings**

In phase 3, the optimized reactive sheets are realized through patterning of the metallic claddings. Five element topologies were utilized. They are shown in Fig. 5(a) and (b) and include a gap capacitor (Gap Cap), an interdigitated capacitor (IDC), an element made with parallel shunt LC resonators (Para Res), a straight-line inductor (Str Ind), and a meandered-line inductor (Meand). The Para Res element was designed to short at both frequencies simultaneously, so the elements follow the frequency scaling of (9), (10), (15), and (16) used in the design. The element unit cells are 0.0029 m (\( \lambda_B/20 \)) wide in the x-direction by 0.0059 m (\( \lambda_B/10 \)) tall in the z-direction. The metallization within the element cell is a maximum of 0.0026 m (0.9 \( \lambda_B/20 \)) wide in order to electrically isolate them from their nearest neighbors. There are two media to consider. The elements on layer 1 sit at an interface between a half-space of air and a half-space of dielectric. This case is shown in the inset of Fig. 6(b). On the other hand, the elements on layer 3 sit within a dielectric medium, as shown in the inset of Fig. 6(b). Extractions are run separately for each case. The element impedances at both frequencies are shown in Fig. 6 for both the elements on layer 1 [Fig. 6(a)] and the elements on layer 3 [Fig. 6(b)]. Although the design thus far (phases 1 and 2) was done without assuming any local periodicity, these impedances were extracted in a locally periodic environment by calculating the \( Z_{12} \) element of the impedance matrix. This produces a small error, which will be corrected through direct optimization of the cladding in full-wave simulation (see Section III). Fig. 6(a) and (b) shows the ideal scaling laws for inductive and capacitive reactances of (9), (10), (15), and (16) shown with black solid lines. The elements follow the assumed scaling laws closely.

**III. RESULTS**

**A. Design Example 1: Dual-Band Design for Wi-Fi Bands**

In this design, two popular Wi-Fi bands are serviced by the same antenna. The design operates at \( f_a = 2.4 \) GHz and \( f_b = 5.1 \) GHz. The metasurface has dimensions \( w = 0.2529 \) m. The line source was placed at a focal length of \( F = 0.25 \) m giving an F/D ratio of approximately 1. The dielectric spacers are Rogers 6010 substrates with \( \epsilon_r = 6.2, \epsilon_r = 10.7(1 - j0.0023) \). The dielectric spacer thicknesses are \( d_2 = d_4 = 1.27 \) mm. The integral equations given by (3) are solved by the method of moments (see the Appendix for details). \( N_1 = N_2 = N_3 = 86 \) basis and expansion functions were used in the moment method algorithm for the surfaces and \( N_2 = N_4 = 1500 \) basis and expansion functions for the dielectric volumes. A small gap was introduced between the elements to electrically isolate them by setting the surface current basis function width to 0.9 times the spacing between the basis centers.

The converged surface impedances computed in phase 1 for layers 1 and 3 are shown in Figs. 7(a) and 8(a). The surface
impedances at $\omega_a$ can be obtained through the assumed frequency scaling laws of (9), (10), (15), and (16) used during design. Convergence was achieved in less than 100 iterations. The far-field patterns resulting from the stacked metasurface containing the complex-valued sheets calculated using the method of moments code at each frequency are shown in Figs. 9 and 10 as the curves labeled “MoM Complex.” The imaginary part of Figs. 7(a) and 8(a) serve as the initial guess used in the gradient descent optimizer in phase 2. The far-field patterns labeled “MoM Complex” in Figs. 9 and 10 serve as $|E_{\text{far field}}(\phi)|_{\text{complex sheet}}$ and $|E_{\text{far field}}(\phi)|_{\text{complex sheet}}$ in (17). The optimized purely reactive sheet impedances of phase 2 are shown in Figs. 7(b) and 8(b). It should be noted...
that no inductive reactances exceed $+j150\Omega$ and no capacitive reactances exceed $-j5000\Omega$, as these are the limits imposed as constraints during the optimization.

In Figs. 11 and 12, the plane-wave spectrum of the total electric field along the top metasurface (at $\omega_a$ and $\omega_b$) and along the bottom metasurface (at $\omega_a$ and $\omega_b$) is obtained from the following:

$$\mathcal{E}_i^{\text{tot}}(k_s, \omega_a/b) = \Im \left\{ \mathcal{E}_i^{\text{tot}}(\varphi_i, \omega_a/b) \right\}
= \Im \left\{ \eta (\varphi_i, \omega_a/b) I_i(\varphi_i, \omega_a/b) \right\}. \quad (18)$$

In (18), $\Im$ is the Fourier transform operator. Note that $i$ indicates which metasurface can take values $i = T$ or $B$. The plane-wave spectrum is plotted for the cases of the complex sheet and the optimized reactive sheet. As can be seen, the optimization technique has introduced a significant evanescent spectrum. The evanescent spectrum is associated with surface waves that redistribute power along the metasurface planes in order to achieve passivity [12], [34].

In Figs. 9 and 10, the far-field patterns generated by the optimized reactive sheets are shown superimposed over those generated by the complex sheets. The patterns were calculated using both MoM code and the commercial finite-element electromagnetics solver COMSOL Multiphysics as a full-wave verification. The agreement is excellent.

In phase 3, the metallic claddings of both metasurfaces are patterned according to the impedances defined in Figs. 7(b) and 8(b). Because these impedances vary in a nonadiabatic way and because the extraction of the impedances associated with the patterned geometry was done in a periodic environment, the extracted $Z_{12}$ will be approximate and will require subsequent fine-tuning optimization of the patterned metallic cladding. The geometric parameter $l$ for each of the elements of the patterned metallic claddings [see Fig. 6(a) and (b)] was optimized using gradient descent optimization with the same cost function (17) except that the reactive sheets are replaced by the patterned claddings. The initial patterned metallic claddings built from Figs. 7(b) and 8(b) provide a very good initial guess, and hence, the optimization converges rapidly. The final patterned metallic claddings are shown in Fig. 13. In the figure, layer 3 is visible first with layer 1 shown behind it. The patterned stacked metasurface was placed within a parallel-plate waveguide (PPW) and simulated in COMSOL Multiphysics at both frequencies. The resulting far-field patterns are plotted superimposed over the method of moments results of the homogenized sheets version in Figs. 9 and 10. As shown in the figure, the patterned metasurfaces show excellent agreement with the theoretical results after the fine-tuning of the claddings, thus validating the design technique, element characterization, and patterning process.

B. Design Example 2: Dual-Band Design for Ka-Band and Ku-Band

This design operates at both $f_a = 13.4$ GHz and $f_b = 35.75$ GHz. The lower band beam is scanned to $30^\circ$ off broadside [$\psi^{\text{off}}(x) = k_0x\cos60^\circ$ in (2)], while the upper band beam is directed broadside [$\psi^{\text{off}}(x) = k_0x\cos90^\circ$ in (2)]. The metasurface has dimensions $w = 0.1791 \text{ m}$. The line source was placed at a focal length of $F = 0.0896 \text{ m}$ giving an F/D ratio of 0.5. The dielectric spacers are Rogers RT/Duroid 5880 substrates with $\epsilon_{r2} = \epsilon_{r4} = 2.2(1 - j0.0009)$. The dielectric
Fig. 13. Stacked metasurface patterned metallic claddings. Only layers 1 and 3 are shown (the dielectric spacers and ground plane are not shown). The perspective is shown in the inset. The metasurface is being viewed from behind layer 3 with layer 3 the first layer visible. These patterned layers are placed within a PPW in order to create an effective 2-D structure infinite in the z-direction.

Fig. 14. Sheet impedance for top metasurface at $\omega_b$. (a) Complex sheet impedance resulting from the direct application of the algorithm in Section II-A. (b) Purely reactive sheet impedance resulting from the optimization algorithm in Section II-B.

Fig. 15. Sheet impedance for bottom metasurface at $\omega_b$. (a) Complex sheet impedance resulting from the direct application of the algorithm in Section II-A. (b) Purely reactive sheet impedance resulting from the optimization algorithm in Section II-B.

The integral equations given by (3) were solved by the method of moments (see the Appendix for details). $N_1 = N_2 = N_3 = 214$ basis and expansion functions were used in the moment method algorithm for the surfaces and $N_2 = N_4 = 1070$ basis and expansion functions for the dielectric volumes.

The converged surface impedances computed in phase 1 for layers 1 and 3 at $\omega_b$ are shown in Figs. 14(a) and 15(a).

Fig. 16. Far-field pattern at frequency $\omega_a$. The far-field pattern resulting from the original complex sheet is shown in the blue curve (Complex). The far-field pattern resulting from the optimized purely reactive sheet is shown in the black curve (Opt. React).

Fig. 17. Far-field pattern at frequency $\omega_b$. The far-field pattern resulting from the original complex sheet is shown in the blue curve (Complex). The far-field pattern resulting from the optimized purely reactive sheet is shown in the black curve (Opt. React).
Convergence was achieved in less than 10 iterations. The far-field patterns calculated with the method of moments code for each frequency are shown in Figs. 16 and 17 as the curves labeled ‘Complex’.

The optimized purely reactive sheet impedances from phase 2 are shown in Figs. 14(b) and 15(b). The far-field patterns resulting from the optimized reactive sheets are shown superimposed in Figs. 16 and 17. The agreement is excellent. Since a patterned design was already shown, this second design will not be realized as a patterned metasurface. This design shows the versatility of the presented design algorithm to obtain scanned collimated beams at arbitrarily defined frequencies.

**IV. CONCLUSION**

A technique to design dual-band stacked metasurfaces was presented. The conductor-backed stacked metasurface considered consists of two metasurfaces (a patterned metallic cladding supported by a dielectric spacer) stacked one upon the other. The stacked metasurface is modeled using coupled volume surface integral equations. These integral equations are transformed into matrix equations by the method of moments. The matrix equations are nonlinear as the product of induced current density and unknown sheet impedance for the bottom metasurface cannot be related to a stipulated total field. An iterative design algorithm was developed to solve the nonlinear system of integral equations. The iterative design scheme allows for flexibility in the specification of the two operating frequencies. The stacked metasurfaces are designed in three phases. Phase 1 obtains a complex-valued impedance sheet design. In phase 2, the complex-valued sheet impedances are transformed into purely reactive sheet impedances through optimization. In phase 3, the purely reactive sheets are realized as patterned metallic claddings. Two design examples were reported. The first was a 2.4/5.1 GHz dual-band stacked metasurface. The second is a Ku-/Ka-band dual-band stacked metasurface. The far-field patterns of the optimized reactive design agree well with the original complex sheet design.

**APPENDIX**

**Method of Moment Solution Details**

**A. Construction of EFIEs and Method of Moment Matrix Equations**

Expanding (3) for the first layer (\(i = 1\)), the EFIE becomes:

\[
E^{inc}(x, 0) = \int_{-w/2}^{w/2} G(x, 0; x', 0)J_1(x', 0)dx' - \int_{-d_2/2}^{d_2/2} G(x, 0; x', y')J_2(x', y')dx'dy' - j\frac{\eta_0\omega}{c} \int_{-w/2}^{w/2} G(x, 0; x', -d_2)J_3(x', -d_2 - d_3)dx' + \eta_1(x, 0)J_1(x, 0)
\]

where \(J_i\) is the electric surface current density on layer \(i\) when \(i = 1, 3,\) and 5 or the electric volume current density within layer \(i\) when \(i = 2\) and 4. The 2-D Green’s function is given by:

\[
G(x, y; x', y') = \frac{1}{4j}H_0^{(2)}(\sqrt{c^2(x-x')^2 + (y-y')^2}).
\]

Note that the primed coordinates represent the source points and the unprimed coordinates represent the observation coordinates. There are two types of expansion functions for the two types of currents. For the surface currents, 1-D pulse basis functions are used. For the volumetric currents, 2-D pulse basis functions are used. The surface currents on layer \(i\) \((i = 1, 3,\) and 5) are expanded into \(N_i\) 1-D pulse basis functions each of width \(\Delta x = w_i/N_i\):

\[
J_i(x', y') = \sum_{n=0}^{N_i-1} I_n P_n(x', y')
\]

The volume currents on layer \(i\) \((i = 2\) and 4) are expanded into \(N_i\) 2-D pulse basis functions each area \(\Delta x \times \Delta y = w_i d_i / N_i\):

\[
J_i(x', y') = \sum_{n=0}^{N_i-1} I_n P_n(x', y')
\]

Substituting (A.2), (A.3), and (A.5) into (A.1) and using a Galerkin testing procedure, the integral equation in (A.1) can be discretized and written in the matrix form as:

\[
\begin{bmatrix} V_1 \end{bmatrix} = \begin{bmatrix} [Z_{11}] & [Z_{12}] \\ [Z_{21}] & [Z_{22}] \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} - \begin{bmatrix} [Z_{13}] \\ [Z_{23}] \end{bmatrix} \begin{bmatrix} [I_3] \\ [I_4] \end{bmatrix} = \begin{bmatrix} [\eta_1] \end{bmatrix}
\]

where the matrices \([Z_{ij}]\) are given by (A.8), as shown at the bottom of the next page. Here, \((x_m, y_m)\) and \((x_n, y_n)\) denote the coordinates of the centroid of testing function \(m\) and basis function \(n\), respectively. The matrix element \([Z_{ij}(m, n)]\) represents the mutual impedance between the basis function \(n\) in layer \(j\) and the test function \(m\) in layer \(i\). The self-impedance term occurs whenever \(i = j\) and \(m = n\) simultaneously in (A.8). In this case, the impedance matrix element is found from:

\[
Z_{ii}(n, n) = \frac{\eta_0\omega\Delta^2}{4c} + \frac{\eta_0\omega}{4\pi c} \Delta^2 \left[ 3 + \ln 4 - 2 \ln \left( \frac{\gamma \omega \Delta}{c} \right) \right] = \left[ \frac{\gamma^{1/2} \Delta}{\omega} \right] \Delta^2
\]
layers, the EFIEs are constructed throughout a volume. For example, the integral equation for layer 2 can be written as

\begin{equation}
E'_i(x, y) = \frac{j \eta_0 \omega}{c} \int_{-w/2}^{w/2} G(x, y; x', 0) J_i(x', 0) dx'
\end{equation}

\begin{equation}
- \frac{j \eta_0 \omega}{c} \int_{-d_2/2}^{d_2/2} \int_{-w/2}^{w/2} G(x, y; x', y') J_2(x', y') dx'dy'
\end{equation}

\begin{equation}
- \frac{j \eta_0 \omega}{c} \int_{-d_2/2}^{d_2/2} \int_{-w/2}^{w/2} G(x, y; x', y) J_2(x, y) dx'dy'
\end{equation}

\begin{equation}
- \frac{j \eta_0 \omega}{c} \int_{-w/2}^{w/2} G(x, y; x', y, -d_2) J_5(x', -d_2) dx'
\end{equation}

\begin{equation}
\eta_2(x, y) J_2(x, y)
\end{equation}

\begin{equation}
(A.10)
\end{equation}

where \(-d_2 < y < 0\). Substituting of (A.2), (A.3), and (A.5) into (A.10) and using a Galerkin testing procedure, the integral equation in (A.10) can be discretized and written in the matrix form as

\begin{equation}
[V_2] - [Z_2][I_1] - [Z_2][I_2] - [Z_3][I_3]
\end{equation}

\begin{equation}
- [Z_4][I_4] - [Z_5][I_5] = [\eta_2][I_2]
\end{equation}

\begin{equation}
(A.11)
\end{equation}

where in this case the matrices \([Z_{ij}]\) are given in (A.12), as shown at the bottom of the page. The self-impedance term occurs whenever \(i = j\) and \(m = n\) simultaneously in (A.12). For this case, the impedance matrix element is calculated using adaptive Gaussian quadrature numerical integrations routines as closed-form expressions are not available. The integral equation for the fourth layer is similar to (A.10).

Constructing the integral equations for the remaining layers produces five total coupled integral equations. Discretization of each of the integral equations generates the matrix equations given in the following:

\begin{equation}
\sum_{i=1}^{5} \sum_{j=1}^{5} [V_i] - [Z_{ij}][I_j] = [\eta_i][I_i]
\end{equation}

where \(i = 1, 2, 3, 4,\) and 5. The matrices \([\eta_i]\) are diagonal matrices. Note that, since the ground plane on layer 5 is perfectly conducting, the surface impedance \([\eta_5]\) can be set to zero. Equation (A.13) can be written in the block matrix form as (A.14), shown at the bottom of the next page.

The voltage vectors appearing in (A.14) are found either from

\begin{equation}
[V_i] = \int_{x_{mi} + \Delta x_i/2}^{x_{mi} + \Delta x_i/2} E_i^{inc}(x, y) dx
\end{equation}

where \(y\) is equal to 0, \(-d_2\), or \(-d_2 + d_4\) when \(i\) is 1, 3, or 5, respectively, or from

\begin{equation}
[V_i] = \int_{x_{mi} - \Delta x_i/2}^{x_{mi} + \Delta x_i/2} \int_{x_{mi} + \Delta x_i/2}^{x_{mi} + \Delta x_i/2} E^{inc}_i(x, y) dy dx
\end{equation}

\begin{equation}
(A.16)
\end{equation}

where \(-d_2 < y < 0\) or \(-d_4 < y < -d_2\) when \(i\) is 2 or 4, respectively. The vectors \(I_i\) are vectors of dimension \(N_i \times 1\) and represent the unknown coefficients of the expansions given in (A.3) and (A.5).

The desired total aperture electric field defined in (3) is converted to matrix form by

\begin{equation}
[W_T] = \int_{x_{mi} - \Delta x_i/2}^{x_{mi} + \Delta x_i/2} E_i^{tot}(\mathbf{r}, \omega) dx
\end{equation}

\begin{equation}
(A.17)
\end{equation}
B. Construction of Numerical Green’s Function

The numerical Green’s function appearing in (5) and (6) can be constructed from the matrix equation (A.14). In the formulations that follow, a superscript \( a \) indicates quantities calculated at frequency \( \omega_a \), and a superscript \( b \) indicates quantities calculated at frequency \( \omega_b \). Because the numerical Green’s function determines the electric field on the top metasurface (layer 1) or the bottom metasurface (layer 3) due to a known source on the top metasurface or the bottom metasurface radiating in the presence of the grounded dielectric spacers only, we may write (A.14) as

\[
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4 \\
V_5
\end{bmatrix} =
\begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} \\
Z_{21} & Z_{22} & Z_{23} & Z_{24} & Z_{25} \\
Z_{31} & Z_{32} & Z_{33} & Z_{34} & Z_{35} \\
Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} \\
Z_{51} & Z_{52} & Z_{53} & Z_{54} & Z_{55}
\end{bmatrix} \begin{bmatrix}
I_1^{a/b} \\
I_2^{a/b} \\
I_3^{a/b} \\
I_4^{a/b} \\
I_5^{a/b}
\end{bmatrix}
\times
\begin{bmatrix}
\frac{1}{j\omega_a c_0 (1 - \varepsilon_2)} \\
\frac{1}{j\omega_b c_0 (1 - \varepsilon_4)}
\end{bmatrix}
\]

To find \( E(\tilde{\rho}_T, J_T, \omega_{a/b}) \), we need to find the field on the top metasurface due to a known \( J_T \). Thus, we need to solve

\[
\begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} \\
Z_{21} & Z_{22} & Z_{23} & Z_{24} & Z_{25} \\
Z_{31} & Z_{32} & Z_{33} & Z_{34} & Z_{35} \\
Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} \\
Z_{51} & Z_{52} & Z_{53} & Z_{54} & Z_{55}
\end{bmatrix} \begin{bmatrix}
I_1^{a/b} \\
I_2^{a/b} \\
I_3^{a/b} \\
I_4^{a/b} \\
I_5^{a/b}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} \\
Z_{21} & Z_{22} & Z_{23} & Z_{24} & Z_{25} \\
Z_{31} & Z_{32} & Z_{33} & Z_{34} & Z_{35} \\
Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} \\
Z_{51} & Z_{52} & Z_{53} & Z_{54} & Z_{55}
\end{bmatrix} \begin{bmatrix}
I_1^{a/b} \\
I_2^{a/b} \\
I_3^{a/b} \\
I_4^{a/b} \\
I_5^{a/b}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Solve for \( [I_2^{a/b}], [I_4^{a/b}] \), and \( [I_5^{a/b}] \) due to \( [I_1^{a/b}] \)

\[
\begin{bmatrix}
I_2^{a/b} \\
I_4^{a/b} \\
I_5^{a/b}
\end{bmatrix} =
\begin{bmatrix}
I_1^{a/b} \\
1 \\
1
\end{bmatrix}
\begin{bmatrix}
Z_{12} & Z_{13} & Z_{14} & Z_{15} \\
Z_{22} & Z_{23} & Z_{24} & Z_{25} \\
Z_{32} & Z_{33} & Z_{34} & Z_{35} \\
Z_{42} & Z_{43} & Z_{44} & Z_{45} \\
Z_{52} & Z_{53} & Z_{54} & Z_{55}
\end{bmatrix}^{-1}
\begin{bmatrix}
-Z_{21}^{a/b} \\
-Z_{41}^{a/b} \\
-Z_{51}^{a/b}
\end{bmatrix}
\]

\[
(22)
\]

Now, we can find \( E(\tilde{\rho}_T, J_T, \omega_{a/b}) \), as shown at the bottom of the page.

The electric field on the bottom metasurface due to \( J_T \) can also be found as (A.25), shown at the top of the next page.

Similarly, one can find fields on the top metasurface (layer 1) and the bottom metasurface (layer 3) due to a known \( J_B \). This gives rise to definitions for \( [G_{TB}] \) and \( [G_{BB}] \).
Similarly, one can obtain an expression for $V_B^{a/b}$.

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Jordan Budhu (Member, IEEE) received the M.S. degree in electrical engineering from the California State University, Northridge, CA, USA, in 2010, and the Ph.D. degree in electrical engineering from the University of California at Los Angeles (UCLA), Los Angeles, CA, USA, in 2018. From 2011 to 2012, he was a Graduate Student Intern at the NASA Jet Propulsion Laboratory, Pasadena, CA, USA. In 2017, he was named a Teaching Fellow at UCLA. Since 2019, he has been a Lecturer with the Department of Electrical and Computer Engineering, University of Michigan. He is currently a Research Fellow with the Radiation Laboratory, University of Michigan, Ann Arbor, MI, USA. Starting in Fall 2022, he will be joining the Bradley Department of Electrical and Computer Engineering, Virginia Tech, Blacksburg, VA, USA, as an Assistant Professor. His research interests are in metamaterials and metasurfaces, computational electromagnetics algorithms for metamaterial and metasurface design, nanophotonics and metamaterials for the infrared, 3-D printed inhomogeneous lens design, CubeSat antennas, reflectarray antennas, scattering from inhomogeneous, anisotropic materials, and antenna theory.

Dr. Budhu’s awards and honors include the 2010 Eugene Cota Robles Fellowship from UCLA, the 2012 Best Poster Award at the IEEE Coastal Los Angeles Class-Tech Annual Meeting, the 2018 UCLA Henry Samueli School of Engineering and Applied Science Excellence in Teaching Award, and the First Place Award for the 2019 USNC-URSI Ernst K. Smith Student Paper Competition at the 2019 Boulder National Radio Science Meeting.

Eric Michielsen (Fellow, IEEE) received the M.S. degree (summa cum laude) in electrical engineering from the Katholieke Universiteit Leuven (KUL), Leuven, Belgium, in 1987, and the Ph.D. degree in electrical engineering from the University of Illinois at Urbana-Champaign (UIUC), Champaign, IL, USA, in 1992. From 1992 to 2005, he served on the faculty at UIUC. In 2005, he joined the University of Michigan (UM), Ann Arbor, MI, USA, where he is currently the Louise Ganiard Johnson Professor of Engineering and professor of electrical engineering and computer science. He has authored or coauthored over 200 journal articles and book chapters and over 400 papers in conference proceedings. His research interests include all aspects of theoretical and applied computational electromagnetics. His research focuses on the development of fast frequency and time-domain integral-equation-based techniques for analyzing electromagnetic phenomena, and the development of robust optimizers for the synthesis of electromagnetic/optical devices.

Dr. Michielsen is a member of URSI Commission B. He received the Belgian American Educational Foundation Fellowship in 1988. He was a recipient of the 1995 National Science Foundation CAREER Award, the 1998 Applied Computational Electromagnetics Society (ACES) Valued Service Award, the 2014 IEEE AP-S Chen-To-Tai Distinguished Educator Award, the 2017 IEEE APS Sergei A. Schelkunoff Transactions Prize Paper Award, the 2020 IEEE AP-S Harrington-Mitra Computational Electromagnetics Award, and the 2020 Eilleen Optical Society Award. In 1999, he was named a 1999 URSI United States National Committee Henry G. Booker Fellow and selected as a recipient of the URSI Koga Gold Medal. In 2003, he served as a Scholar at the Tel Aviv University Sackler Center for Advanced Studies.

Anthony Gribic (Fellow, IEEE) received the B.A.Sc., M.A.Sc., and Ph.D. degrees in electrical engineering from the University of Toronto, Toronto, ON, Canada, in 1998, 2000, and 2005, respectively. In 2006, he joined the Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI, USA, where he is currently a Professor. He has made pioneering contributions to the theory and development of electromagnetic metamaterials and metasurfaces: finely textured, engineered electromagnetic structures/surfaces that offer unprecedented wavefront control. His research interests include engineered electromagnetic structures (metamaterials, metasurfaces, electromagnetic bandgap materials, and frequency-selective surfaces), microwave circuits, antennas, plasmonics, wireless power transmission, and analytical electromagnetic/optics.

Dr. Gribic is serving on the IEEE Antennas and Propagation Society (AP-S) Field Awards Committee and the IEEE Fellow Selection Committee. Since 2018, he has been serving as a member for the Scientific Advisory Board, International Congress on Artificial Materials for Novel Wave Phenomena—Metamaterials. He was a recipient of the AFOSR Young Investigator Award and the NSF Faculty Early Career Development Award in 2008, the Presidential Early Career Award for Scientists and Engineers in 2008, the Outstanding Young Engineer Award from the IEEE Microwave Theory and Techniques Society, the Henry Russel Award from the University of Michigan, and the Booker Fellowship from the U.S. National Committee of the International Union of Radio Science in 2011. He was an inaugural recipient of the Ernest and Bettine Kuh Distinguished Faculty Scholar Award at the Department of Electrical and Computer Science, University of Michigan, in 2012. He received the 2018 University of Michigan Faculty Recognition Award for outstanding achievement in scholarly research, excellence as a teacher, advisor, and mentor, and distinguished service to the institution and profession in 2018. In 2021, he was selected as one of five finalists worldwide for the A. F. Harvey Engineering Research Prize, for his pioneering contributions to the field of electromagnetic metamaterials. He has been the Vice Chair of Technical Activities for the IEEE Antennas and Propagation Society, Chapter IV (Trident), IEEE Southeastern Michigan Section, from September 2007 to 2021. From July 2010 to July 2015, he was an Associate Editor for the rapid publication journal IEEE ANTENNAS AND WIRELESS PROPAGATION LETTERS. He was the Technical Program Co-Chair in 2012 and the Topic Co-Chair in 2016 and 2017 of the IEEE International Symposium on Antennas and Propagation and the USNC-URSI National Radio Science Meeting. He is a Distinguished Microwave Lecturer of the IEEE Microwave Theory and Techniques Society for the term 2022–2025.