ON THE PRICING OF ASIAN OPTIONS WITH GEOMETRIC AVERAGE OF AMERICAN TYPE WITH STOCHASTIC INTEREST RATE: A STOCHASTIC OPTIMAL CONTROL APPROACH

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Abstract. In this work, through stochastic optimal control in continuous time the optimal decision making in consumption and investment is modeled by a rational economic agent, representative of an economy, who is a consumer and an investor adverse to risk; this in a finite time horizon of stochastic length. The assumptions of the model are: a consumption function of HARA type, a representative company that has a stochastic production process, the agent invests in a stock and an American-style Asian put option with floating strike equal to the geometric average subscribed on the stock, both modeled by controlled Markovian processes; as well as the investment of a principal in a bank account. The model is solved with dynamic programming in continuous time, particularly the Hamilton-Jacobi-Bellman PDE is obtained, and a function in separable variables is proposed as a solution to set the optimal trajectories of consumption and investment. In the solution analysis is determined: in equilibrium, the process of short interest rate that is driven by a square root process with reversion to the mean; and through a system of differential equations of risk premiums, a PDE is deduced equivalent to the Black-Scholes-Merton but to value an American-style Asian put option.

1. Introduction. The financial world is a complicated combination of random phenomena and unpredictable events. The growing participation of the agents that operate in this world and the complexity in the modern financial market have prompted the development of financial derivatives that satisfy the requirements of financial risk management and the establishment of hedging strategies in positions against possible price variations, arbitrage operations and other methods. That is why innovation in the negotiation of exotic derivatives such as Asian options represents an alternative in times of extreme volatility, with accessible transaction costs. Asian options, actively traded in the currency markets, are derivatives dependent on the trajectory of the underlying interest rates and commodities, whose payment

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depends explicitly on the average prices of the underlying items. The main advantage of these investments is that they diminish the effect of possible manipulations of the market that can happen close to the expiration date. In general, the longer the average period, the smoother the trajectory will be. In the exchange market, these options provide the treasury with a hedging instrument for a series of flows whose exercise is settled in cash. For commodities, such as industrial metals, fuels or grains, among others, the average is also useful as a resource to eliminate the extreme sensitivity of the options expiration value to the underlying cash price on that particular day. Eydeland and Wolyniec [7] give an example of how Asian options play an important role in price risk management by local distribution companies in the gas market. Also, in the oil markets, this type of derivative is frequently issued to stabilize cash flows that arise from the fulfillment of obligations with customers. Fanelli, Maddalena and Musti [8] develop a general equilibrium model for the electricity market by means of a theory of interest rates and Asian options of arithmetic average for purposes of forecasting future electricity demand and price adjustment as an effect of the market efficiency.

From a theoretical point of view, Asian options with arithmetic average have received growing interest due to the complexity of the problems related to the determination of a closed formula for the price of these options. For example, in the standard model of Black and Scholes (BS), when the underlying asset is driven by geometric Brownian motion, the distribution of the arithmetic average is not lognormal, and it is a challenge to characterize such a distribution analytically. A first approach was proposed in the seminal work of Yor ([24], [25]) but with limited practical use in the pricing of Asian options. Dassios and Nagaradjasarma [5] provide closed formulas for Asian options with fixed exercise price with arithmetic average, with the performance of the underlying driven by a square root process. In addition, they obtain results on the distribution of the square root process and its temporal average, including analytical formulas for their joint density and moments.

Other numerical approaches in the pricing of Asian options in the context of the BS model are the following: For Monte Carlo simulation, Kenna and Vorst [14], Jourdain and Sbai [13], and Guasoni and Robertson [11] have contributed work. Kim and Wee [15] obtain closed formulas for the price of an Asian option with fixed and floating exercise prices with a geometric average, with the assumption that the price of the underlying asset follows the stochastic volatility model of Heston. Dingç, Sak and Hörmann [6] propose a general scheme with variance reduction technique in three cases: Levý processes, stochastic volatility and regime switching. Weiping and Su [26] obtain a formula for the Asian option price with arithmetic average using the Edgeworth series with its sensitivities and hedging. With regard to American-style Asian options, Gounen and O’Hara [10] develop an analytical formula for the price of an Asian floating-type option with interesting results on Asian American options with fixed and floating exercise price. Russo and Staino [21] develop a pricing model for Asian options with arithmetic average and stochastic volatility, with correlation between the price of the underlying asset and its volatility.

On the other hand, in recent decades there has been an increase in the use of optimal stochastic control as an effective tool that allows to adequately solve some complex problems in financial economics. Within such problems, are stochastic dynamic general equilibrium models, from which it is possible to model and solve, among others, models of representative agent, which has a limited budget and
needs to optimize its utility function for consumption, as well as your investment in a portfolio of assets. Merton’s seminal work [19] examines the optimal portfolio choice when investment opportunities vary over time, with several extensions in Merton [20], Chen and Weiyn [3], Farhadi, Erjaee and Salehi [9], Yan [23] to name a few.

A review of the literature on the pricing of European options with control can be found, for example, in Ben-Ameur, Breton and L’Ecuyer [1], in which a numerical procedure for the price of American-Asian options of Bermuda style is developed based on dynamic programming combined with polynomial approximation by finite element parts of the value function, and the theoretical properties of the value function and the optimal exercise strategy are established. In Marcozzi [16], the problem of optimal control associated with ultradiffusion processes is solved as a stochastic differential equation restricted to the expected performance of the system on the set of feasible trajectories. In particular, these authors use the finite element method of lines to approximate the function of the price of a European call option in a market subject to liquidity risk of the assets (including limit orders) and brokerage fees. Martínez, Martínez and Venegas [18] develop a consumption model of a rational risk-adverse agent with a limited budget that solves the problem of allocating its initial wealth between consumption and investment in a portfolio of assets over a finite time horizon of stochastic length to maximize expected total utility. In Martínez, Ortiz and Martínez [17], the agent can invest in an American put option in an underlying with stochastic volatility, with an extension to an Asian option with a floating exercise price equal to the arithmetic average. It is worthwhile to mention that, unlike the latter, this research is carried out in a stochastic scenario with more robust assumptions, given that in addition to the geometric average assumption, two stopping time functions, stochastic interest rate are included in the model and a company representative of an economy whose production function is also stochastic. Hence, an optimal stochastic control model is generated that is more complex in its structure and therefore in its analysis; and at the application level, much more adapted to the contingent dynamics of financial markets.

In the framework of the research described above, the main contributions of this paper are: 1) The pricing of an Asian option of American type with floating exercise price equal to the geometric average is made based on assumptions of economic rationality and through a theoretical approach of optimal stochastic control in continuous time, and 2) in equilibrium, the process of the short rate of type Cox-Ingersoll-Ross [4] (CIR) is determined endogenously.

This paper is organized as follows: The following section describes the theoretical framework of this research. In the third section, the assets of the investment portfolio are defined. The fourth section deals with the agent’s intertemporal budget constraint. The fifth section describes the production process, through which the process of the short rate will be established in section 10. The sixth section describes the stopping time process that prevents the agent from falling into bankruptcy, while providing the continuous exercise of the option and the optimal stochastic control problem to be solved. In sections seven and eight, the solution to the problem stated in the previous section is carried out: The Hamilton-Jacobi-Bellman partial differential equation and the controls that optimize the decision problem are obtained. In the ninth section, the option pricing formula for the Asian put option with geometric average price of American type is modeled, and the process for the short rate of CIR type is established. Finally, conclusions are presented.
2. Description of the problem. In this section, through dynamic stochastic general equilibrium a model is established of a rational economic agent that is a consumer and an investor, representative of a small and closed economy in which only one good is produced and consumed.

Consider an economic agent who has a time horizon, represented by the interval \([0, T]\), where \(T\) is stochastic (and will be determined in the sixth section). At time \(t = 0\), the agent is endowed with an initial wealth \(W_0\) and faces the decision how to allocate its wealth between consumption and investment, with the objectives of maximizing its total expected utility discounted from consumption of a generic good and determining optimum proportions of investment in a portfolio of assets while avoiding bankruptcy.

For this purpose, it is assumed that there is a banking system in which the agent can borrow at a continuous rate of interest \(r > 0\), whose process will be determined endogenously in the equilibrium. In this system, short sales are allowed and unlimited. It is also assumed that the agent can invest in three assets: a principal, such as savings in a bank; a stock; and an Asian put option of American type. The prices of the assets are expressed in real terms\(^1\).

It is assumed that the agent’s expected utility function is given by:

\[
E \left[ \int_0^T U(t, c_t) dt | \mathcal{F}_0 \right]
\]

where \(U\) is the discounted utility function for consumption and \(\mathcal{F}_0\) is the relevant available information at time \(t = 0\).

3. Assets and returns. The agent invests in the bank a principal \(L_0 = L(0)\) which pays a continuously capitalizable interest rate \(r > 0\) and whose process will be determined in the ninth section. The rate of change in the value of the investment is given by:

\[
dR_{L_t} = \frac{dL_t}{L_t} = r dt.
\]  

The second asset is a stock whose price dynamics in real terms is modeled by the following stochastic differential equation:

\[
dR_{X_t} = \frac{dX_t}{X_t} = \mu dt + \sigma B_t,
\]

where \(B_t\) is a Wiener process or Brownian motion defined over a fixed probability space with augmented filtration \(\left(\Omega, \mathcal{F}, (\mathcal{F}_t^B)_{t \in [0, T]}, \mathbb{P}\right)\), where \(\mu \in \mathbb{R}\) and \(0 < \sigma \in \mathbb{R}\), representing the drift parameter and the instantaneous volatility of the asset, respectively.

The third asset is an Asian put option of American type subscribed on the stock that has the process of real prices defined in (2). For an Asian put option with an average exercise price equal to the geometric average, the intrinsic value of the contract on the maturity date \(T\) is equal to \(\text{max} \left(K_{t,T} - X_T, 0\right)\). The process of the geometric mean in continuous time\(^2\), is defined by:

\[\text{Geometric average is defined as: } \bar{K}_t = \frac{1}{T-t} \int_t^T \ln(X_u) du = \lim_{n \to \infty} \frac{1}{n} \sqrt[n]{\prod_{i=1}^{n} \ln(X_{t_i})},\]  

where \(t_i, i = 1, \ldots, n\), represent a partition of \([t, T]\), Venegas-Martínez [22].
From last equation let:

\[ M_{gt} = \int_{0}^{T} \ln(X_u) du \Rightarrow M_{gt} = \ln(X_t) dt. \]  

(4)

Based on (2), (3) and (4) the change in the price of the Asian put option is given by the function

\[ O_A = O_A(X_t, M_{gt}, t) \]  

applying Ito’s Lemma leads to:

\[ dO_A = \left( \frac{\partial O_A}{\partial t} + \frac{\partial O_A}{\partial X_t} \mu X_t + \frac{\partial O_A}{\partial M_{gt}} \ln(X_t) + \frac{1}{2} \frac{\partial^2 O_A}{\partial X_t^2} \sigma^2 X_t^2 \right) dt + \frac{\partial O_A}{\partial X_t} \sigma X_t dB_t. \]  

(5)

The option’s yield is given by the percentage change of the premium, i.e.:

\[ dR_{O_A} \equiv \frac{dO_A}{O_A}. \]  

(6)

If from (5) is denoted to:

\[ \mu_{O_A} = \frac{1}{O_A} \left( \frac{\partial O_A}{\partial t} + \frac{\partial O_A}{\partial X_t} \mu X_t + \frac{\partial O_A}{\partial M_{gt}} \ln(X_t) + \frac{1}{2} \frac{\partial^2 O_A}{\partial X_t^2} \sigma^2 X_t^2 \right) \]  

\[ \sigma_{O_A} = \frac{1}{O_A} \left( \frac{\partial O_A}{\partial X_t} \sigma X_t dB_t \right) \]  

(7)

It follows that:

\[ dO_A = O_A \mu_{O_A} dt + O_A \sigma_{O_A} dB_t. \]  

(8)

The proportions relative to the investment portfolio at time \( t \), which the agent assigns to the stock and Asian put option, are denoted by \( \phi_{1t} \) and \( \phi_{2t} \), respectively. The proportion \( 1 - \phi_{1t} - \phi_{2t} \) is designated for savings, and \( c_t \) denotes the consumption rate, which is requested \( c_t \geq 0 \) \( \forall t \). Additionally it is assumed that the agent trades in a market that operates continuously there are no commissions to brokers or tax payments to tax authorities.

4. **Wealth equation.** Based on the assumptions stated, it is now assumed that \( W_t \) represents the real wealth of the consumer at time \( t \). Thus the dynamics of the wealth process are given by:

\[ dW_t = W_t \phi_{1t} dR_{X_t} + W_t \phi_{2t} dR_{O_A} + W_t (1 - \phi_{1t} - \phi_{2t}) dR_{MGt} - c_t dt \]  

(9)

That is:

\[ \frac{dW_t}{W_t} = \left( r + \phi_{1t} (\mu - r) + \phi_{2t} (\mu_{O_A} - r) - \frac{c_t}{W_t} \right) dt + (\phi_{1t} \sigma + \phi_{2t} \sigma_{O_A}) dB_t, \]  

(10)

and equivalently:

\[ \mu_W = \left( r + \phi_{1t} (\mu - r) + \phi_{2t} (\mu_{O_A} - r) - \frac{c_t}{W_t} \right) \quad \text{and} \quad \sigma_W = (\phi_{1t} \sigma + \phi_{2t} \sigma_{O_A}), \]  

(11)

The equation of wealth is rewritten as:

\[ \frac{dW_t}{W_t} = \mu_W dt + \sigma_W dB_t, \]  

(12)
5. Production function. It is assumed that the representative company in this economy has a production function \( z_t = Q_t X_t \). During the productive process, the marginal product of capital, \( Q_t \), is modeled by the following stochastic differential equation:

\[
dQ_t = \alpha (Q_t) \, dt + \beta (Q_t) \, dV_t,
\]

where:

\[
\alpha (Q_t) = \kappa (\theta - Q_t) \quad \text{and} \quad \beta (Q_t) = \nu \sqrt{Q_t}
\]

with \( \kappa, \theta \) and \( \nu \) positive constants and \( \{V_t\}_{t \geq 0} \) a one-dimensional Brownian motion defined over a fixed probability space with augmented filtration denoted by \((\Omega^V, \mathcal{F}^V, (\mathcal{F}^V_t)_{t \in [0, T]}, \mathbb{P}^V)\). For purposes of making the equations analytically tractable, it will be assumed that the Wiener processes are independent, i.e. \( \text{Cov}(dB_t, dV_t) = 0 \).

6. Stopping time and problem statement. Given the assumption that short sales are allowed and unlimited, note that if the economic agent presents an anxious behavior for consumption, it could fall into bankruptcy, simultaneously punishing the banking system. To deal financially with this problem and to ensure that the problem does not degenerate mathematically, the restricted domain is defined as \( D = [0, T] \times \{ x \mid x > 0 \} \), and the function

\[
\xi = \min \{ \inf \{ t > 0 \mid W_t = 0 \}, T \},
\]

so when the stochastic process of wealth is zero, then the problem ends.

In summary, the formal approach to the consumer utility maximization problem as an optimal stochastic control problem is:

\[
\begin{align*}
\text{Maximize} & \quad E \left[ \int_0^\xi U(t, c_t) \, dt \mid \mathcal{F}_0 \right] \\
dW_t &= W_t \mu_W \, dt + W_t \sigma_W \, dB_t \\
W_0 &= w_0 \\
\phi_{1t} + \phi_{2t} &= 1 \\
c_t &\geq 0, \forall \, t \geq 0.
\end{align*}
\]

7. Dynamic programming: Partial differential equation (PDE) of Hamilton Jacobi Bellman (HJB). To solve the problem in (15) and find the proportions that optimize the investment portfolio and consumption, the value function of the problem is defined as follows

\[
\begin{align*}
H(W_t, Q_t, M_{Gt}, t) &= \max_{\phi_{1t}, \phi_{2t} \in \mathbb{R}, 0 \leq c_t \mid [t, \xi]} \mathbb{E} \left[ \int_t^\xi U(c_s, s) \, ds \mid \mathcal{F}_t \right] \\
&= \max_{\phi_{1t}, \phi_{2t} \in \mathbb{R}, 0 \leq c_t \mid [t, \xi]} \mathbb{E} \left[ \int_t^{t+dt} U(c_s, s) \, ds + \int_t^\xi U(c_s, s) \, ds \mid \mathcal{F}_t \right]
\end{align*}
\]

Applying the mean value theorem of integral calculus to the first term and using a recursive relationship in the second term obtains

\[
H(W_t, Q_t, M_{Gt}, t) = \max_{\phi_{1t}, \phi_{2t} \in \mathbb{R}, 0 \leq c_t \mid [t, t+dt]} \mathbb{E} \left\{ U(c_t, t) \, dt + o(dt) \right\}
\]
+ H (W_t + dW_t, Q_t + dQ_t, M_{Gt} + dM_{Gt}, t + dt | F_t \}.

and applying a Taylor series expansion to the second summand gives

\begin{align*}
H (W_t, Q_t, M_{Gt}, t) &= \max_{\phi_{11}, \phi_{22} \in \mathbb{R}, \phi_{ii} \leq c_i, t, t + dt} E \left\{ U (c_i, t) dt + o (dt) \\
+ H (W_t, Q_t, M_{Gt}, t) + dH (W_t, Q_t, M_{Gt}, t) + o (dt) | F_t \right\}
\end{align*}

accordingly

\begin{align*}
0 &= \max_{\phi_{11}, \phi_{22} \in \mathbb{R}, \phi_{ii} \leq c_i, t, t + dt} E \left\{ U (c_i, t) dt + o (dt) + dH (W_t, Q_t, M_{Gt}, t) | F_t \right\}.
\end{align*}

Applying Ito’s lemma to \(dH(W_t, Q_t, M_{Gt}, t)\), it follows that

\begin{align*}
0 &= \max_{\phi_{11}, \phi_{22} \in \mathbb{R}, \phi_{ii} \leq c_i, t, t + dt} E \left\{ U (c_i, t) dt + o (dt) + \left[ \frac{\partial H (W_t, Q_t, M_{Gt}, t)}{\partial t} \right]ight.
+ \frac{\partial H (W_t, Q_t, M_{Gt}, t)}{\partial W_t} W_t \mu_W + \frac{\partial H (W_t, Q_t, M_{Gt}, t)}{\partial Q_t} Q_t \alpha
+ \frac{\partial H (W_t, Q_t, M_{Gt}, t)}{\partial M_{Gt}} \ln (X_t) + \frac{1}{2} \frac{\partial^2 H (W_t, Q_t, M_{Gt}, t)}{\partial Q_t^2} Q_t^2 \beta^2
+ \frac{1}{2} \frac{\partial^2 H (W_t, Q_t, M_{Gt}, t)}{\partial W_t^2} W_t^2 \sigma_W^2 \right\} dt + \frac{\partial H (W_t, Q_t, M_{Gt}, t)}{\partial Q_t} Q_t \beta dV_t
+ \frac{\partial H (W_t, Q_t, M_{Gt}, t)}{\partial M_{Gt}} \ln (X_t) + \frac{1}{2} \frac{\partial^2 H (W_t, Q_t, M_{Gt}, t)}{\partial Q_t^2} Q_t^2 \beta^2
+ \frac{1}{2} \frac{\partial^2 H (W_t, Q_t, M_{Gt}, t)}{\partial W_t^2} W_t^2 \sigma_W^2 \right\} | F_t \}.
\end{align*}

Taking the expected value of last equation and since \(dB_t \sim \mathcal{N} (0, dt)\) and \(dV_t \sim \mathcal{N} (0, dt)\), then stochastic terms are eliminated. Also, dividing the expression between \(dt\) and taking the limit when \(dt \to 0\), the following results

\begin{align*}
0 &= \max_{\phi_{11}, \phi_{22} \in \mathbb{R}, \phi_{ii} \leq c_i, t, t + dt} E \left\{ U (c_i, t) + \left[ \frac{\partial H (W_t, Q_t, M_{Gt}, t)}{\partial t} \right]ight.
+ \frac{\partial H (W_t, Q_t, M_{Gt}, t)}{\partial W_t} W_t \mu_W + \frac{\partial H (W_t, Q_t, M_{Gt}, t)}{\partial Q_t} Q_t \alpha
+ \frac{\partial H (W_t, Q_t, M_{Gt}, t)}{\partial M_{Gt}} \ln (X_t) + \frac{1}{2} \frac{\partial^2 H (W_t, Q_t, M_{Gt}, t)}{\partial Q_t^2} Q_t^2 \beta^2
+ \frac{1}{2} \frac{\partial^2 H (W_t, Q_t, M_{Gt}, t)}{\partial W_t^2} W_t^2 \sigma_W^2 \right\} | F_t \}.
\end{align*}

To the last equation the corresponding boundary conditions must be imposed, which results in the PDE of HJB

\begin{align*}
0 &= \max_{\phi_{11}, \phi_{22} \in \mathbb{R}, \phi_{ii} \leq c_i, t, t + dt} E \left\{ U (c_i, t) + \left[ \frac{\partial H (W_t, Q_t, M_{Gt}, t)}{\partial t} \right]ight.
+ \frac{\partial H (W_t, Q_t, M_{Gt}, t)}{\partial W_t} W_t \mu_W + \frac{\partial H (W_t, Q_t, M_{Gt}, t)}{\partial Q_t} Q_t \alpha
\end{align*}
7.1. Utility function. It is assumed that the utility function has the functional form \( U(c_t) = e^{-\rho t} F(c_t) \), where \( F(c_t) \) is a member of the HARA utility functions family (Merton [20] and Hakansson [12]). Specifically, the following consumption function was chosen

\[
U(c_t) = e^{-\rho t} \frac{c_t^\gamma}{\gamma}, \quad 0 < \gamma < 1.
\]

7.2. First order conditions. Assuming an interior maximum in (17) and making the appropriate substitutions corresponding to the utility function and from (11)

\[
0 = \left\{ e^{-\rho t} \frac{c_t^\gamma}{\gamma} + \frac{1}{2} \frac{\partial^2 H(W_t, Q_t, M_{GT}, t)}{\partial W_t^2} W_t^2 (\phi_{1t} \sigma + \phi_{2t} \sigma_{O_A})^2 \right\}
+ \frac{1}{2} \frac{\partial^2 H(W_t, Q_t, M_{GT}, t)}{\partial Q_t^2} Q_t^2 \beta^2 + \frac{\partial H(W_t, Q_t, M_{GT}, t)}{\partial t}
+ \frac{\partial H(W_t, Q_t, M_{GT}, t)}{\partial W_t} W_t \left( r + \phi_{1t} (\mu - r) + \phi_{2t} (\mu_{O_A} - r) - \frac{c_t}{W_t} \right)
+ \frac{\partial H(W_t, Q_t, M_{GT}, t)}{\partial Q_t} Q_t \alpha + \frac{\partial H(W_t, Q_t, M_{GT}, t)}{\partial M_{GT}} \ln (X_t) \right\} \mathcal{F}_t.
\]

The objective is to optimize the PDE of HJB for the controls \( \phi_{1t}, \phi_{2t} \) and \( c_t \), and hence the first order conditions are

\[
c_t^{\gamma-1} = \left[ \frac{\partial H(W_t, Q_t, M_{GT}, t)}{\partial W_t} \right] e^{\rho t}.
\]

\[
\phi_{1t} = - \frac{\partial H(W_t, Q_t, M_{GT}, t)}{\partial W_t} W_t (\mu - r) + \frac{\partial^2 H(W_t, Q_t, M_{GT}, t)}{\partial W_t^2} W_t^2 \sigma \phi_{2t} \sigma_{O_A}
- \frac{\partial^2 H(W_t, Q_t, M_{GT}, t)}{\partial W_t^2} W_t^2 \sigma^2,
\]

\[
\phi_{2t} = - \frac{\partial H(W_t, Q_t, M_{GT}, t)}{\partial W_t} W_t (\mu_{O_A} - r) + \frac{\partial^2 H(W_t, Q_t, M_{GT}, t)}{\partial W_t^2} W_t^2 \sigma \phi_{1t} \sigma_{O_A}
- \frac{\partial^2 H(W_t, Q_t, M_{GT}, t)}{\partial W_t^2} W_t^2 \sigma_{O_A}^2.
\]

8. Optimal controls. In order to select the function \( H(W_t, Q_t, M_{GT}, t) \) that satisfies PDE of HJB, the following separable product function is proposed

\[
H(W_t, Q_t, M_{GT}, t) = e^{-\rho t} g(M_{GT}, t) \left[ \frac{Q_t W_t}{\gamma (\gamma - 1)} \right],
\]

where \( \gamma, \sigma, \mu_{O_A}, \mu \) are constants.
Along with \( g(M_{GT}, T) = 0 \), due to the boundary conditions of the HJB equation. Given \( H \)

\[
\frac{\partial H}{\partial t}(W_t, Q_t, M_{GT}, t) = e^{-rt} [W_t Q_t]^y \frac{\partial g(M_{GT}, t)}{\partial t} + \rho e^{-rt} [W_t Q_t]^y \frac{\partial g(M_{GT}, t)}{\partial M_{GT}}
\]

\[
\frac{\partial H}{\partial M_{GT}}(W_t, Q_t, M_{GT}, t) = e^{-rt} [W_t Q_t]^y \frac{\partial g(M_{GT}, t)}{\partial M_{GT}}
\]

\[
\frac{\partial H}{\partial W_t}(W_t, Q_t, M_{GT}, t) = W_t e^{-rt} g(M_{GT}, t)
\]

\[
\frac{\partial^2 H}{\partial W_t^2}(W_t, Q_t, M_{GT}, t) = (\gamma - 1) W_t e^{-rt} g(M_{GT}, t)
\]

Substituting the values of (24) in (20), (21) and (22), it follows that

\[
\frac{\partial H}{\partial Q_t}(W_t, Q_t, M_{GT}, t) = W_t e^{-rt} g(M_{GT}, t)
\]

\[
\frac{\partial^2 H}{\partial Q_t^2}(W_t, Q_t, M_{GT}, t) = 0.
\]

9. Pricing Asian put option of American type. From the system of equations in (28)

\[
\lambda = \varphi \lambda_{OA},
\]

Substituting in (29) \( \mu_{OA} \) and \( \sigma_{OA} \) given in (7)

\[
\varphi = \frac{\sigma}{\sigma_{OA}}, \lambda = -\frac{\mu - r}{(\gamma - 1)\sigma_{OA}}^\gamma, \text{ and } \lambda_{OA} = -\frac{\mu_{OA} - r}{(\gamma - 1)\sigma_{OA}}^\gamma.
\]

In this system, risk premiums are a linear combination of each other, which is identified by the determinant \( D = 0 \). This is logical because the option inherits properties of the underlying process.

The above equation is a partial differential equation of second order, which is equivalent to the Black-Scholes-Merton equation, with the difference that (30) has one more term that represents the average of the underlying. Through this equation the price of the Asian put option of American style with geometric average exercise
price will be obtained. The appropriate boundary conditions in order to obtain a unique solution:

\[
\frac{\partial O_A}{\partial t} + \frac{\partial O_A}{\partial M_{GT}} X_t \ln (X_t) + \frac{1}{2} \frac{\partial^2 O_A}{\partial X_t^2} \sigma^2 X_t^2 + \frac{\partial O_A}{\partial X_t} X_t r - r O_A = 0, \\
O_A (X_t, M_{GT}, T) \geq \max \left[ e^{\frac{M_{GT}}{T}} - X_T, 0 \right],
\]

\( t \leq \xi. \)

where \( \xi = \min \{ \xi, \hat{\xi} \} \), and \( \hat{\xi} \) is the stopping time where \( \max \left[ e^{\frac{M_{GT}}{T}} - X_T, 0 \right] \) is reached. To discover the dynamics of the short rate, consider a corner solution \( \hat{\phi}_1 t = 1 \) and \( \hat{\phi}_2 t = 0 \), which leads from equation (26) to

\( r = \mu - (1 - \gamma) \sigma^2. \)

Now, from the equation of the productive process in (13) let \( \mu = \tilde{\mu} Q_t \) and \( (1 - \gamma) \sigma^2 = (1 - \gamma) \tilde{\sigma}^2 Q_t \), where together with (32) results in

\( r_t = \eta Q_t, \)

with \( \eta = \tilde{\mu} - (1 - \gamma) \tilde{\sigma}^2 \). Hence, from (13) and (14)

\[
d r_t = \eta d Q_t \\
= \eta [\alpha (Q_t) dt + \beta (Q_t) d V_t] \\
= \eta \kappa (\theta - Q_t) dt + \eta \nu \sqrt{Q_t} d V_t \\
= \kappa (\eta \theta - r_t) dt + \sqrt{\eta \nu} \sqrt{r_t} d V_t,
\]

setting \( a = \kappa, b = \eta \theta \) and \( \delta = \sqrt{\eta \nu}, \) the dynamics of the short rate are given by

\[
d r_t = a (b - r_t) dt + \delta \sqrt{r_t} d V_t,
\]

with \( a > 0, b > 0, \) and \( \delta > 0. \) The properties of \( r_t \) are described by Cox et al [4] and Brigo and Mercurio [2], among others.

10. **Conclusions.** This research proposes and resolves, under arguments of economic rationality, a microeconomic model of stochastic dynamic general equilibrium on portfolio decisions for investment and consumption assets, of an agent adverse to risk, representative of a small and closed economy, subject to market risk of the assets of the portfolio and endowed with a limited budget, this in a finite time horizon but with stochastic terminal date. In the solution process, the partial differential equation of HJB is obtained and the problem of stochastic optimization is reduced to solving a deterministic differential equation that simplifies the problem. By solving the partial differential equation of HJB, the optimal trajectories of consumption and investment are obtained.

The valuation model of the Asian option of American type with stochastic interest rate is established based on assumptions of economic rationality and is expressed by a partial differential equation from the market risk premiums of the stock and the option. The process of short interest rate is of type CIR and is determined endogenously in the equilibrium. This model is a contribution to the state of the art on the subject.

The fact that the functional form for the investment assets and the equations that feed the model are established as controlled Markovian processes, and that the interest rate is modeled by a mean reversion process, together with a HARA utility function, made it possible, on the one hand, to transform the stochastic formulation...
of the solution process into a deterministic partial differential equation of Hamilton Jacobi Bellman and, on the other hand, to obtain the partial differential equation to value the Asian option of the American type. It is worth mentioning that the stopping time established in the model made it possible to obtain the American style of the option.

Finally, possible extensions are different functional forms for the assets of the portfolio so that the movements of the prices in the markets are modeled more precisely, as well as the use of optimal stochastic control combined with fractional Brownian motion.

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