Almost hypercomplex manifolds with Hermitian–Norden metrics and 4-dimensional indecomposable real Lie algebras depending on one parameter

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Abstract. We study almost hypercomplex structure with Hermitian–Norden metrics on 4-dimensional Lie groups considered as smooth manifolds. All the basic classes of a classification of 4-dimensional indecomposable real Lie algebras depending on one parameter are investigated. There are studied some geometrical characteristics of the respective almost hypercomplex manifolds with Hermitian–Norden metrics.

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Introduction

Almost hypercomplex structure $H$ on a $4n$-dimensional manifold $\mathcal{M}$ is a triad of anticommuting almost complex structures such that each of them is a composition of the other two structures. In [7,8], this structure $H$ is equipped with a metric structure of Hermitian–Norden type, generated by a pseudo-Riemannian metric $g$ of neutral signature. In this case, one (resp., the other two) of the almost complex structures of $H$ acts as an isometry (resp., act as anti-isometries) with respect to $g$ in each tangent fibre. The metric $g$ is Hermitian with respect to one of almost complex structures of $H$ and $g$ is a Norden metric regarding the other two almost complex structures of $H$. Then,

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there exist three (0,2)-tensors associated by $H$ to the metric $g$ — a Kähler form and two metrics of Hermitian–Norden type.

The derived manifold is called an almost hypercomplex manifold with Hermitian-Norden metrics. Its geometry is investigated in [7,8,10,11,13,14,17]. Let us remark that this type of manifolds are the only possible way to involve Norden-type metrics on almost hypercomplex manifolds. Similar structures and metrics on Lie groups considered as manifolds are studied in [2,3,15,18,21].

The present paper is organized as follows. In Sect. 1, we present some definitions and facts about the almost hypercomplex manifold with Hermitian–Norden metrics. In the next Sect. 2, we construct almost hypercomplex structures with Hermitian–Norden metrics on 4-dimensional Lie groups considered as smooth manifolds. Moreover, we determine the belonging of the constructed almost hypercomplex manifolds with Hermitian–Norden metrics to the certain classes of the respective classifications. The last Sect. 3 is devoted to the study of some geometric characteristics of the considered manifolds.

1. Preliminaries

A $4n$-dimensional differentiable manifold $\mathcal{M}$ is called an almost hypercomplex manifold if it is equipped with an almost hypercomplex structure $H = (J_1, J_2, J_3)$ with the following properties:

$$J_\alpha = J_\beta \circ J_\gamma = -J_\gamma \circ J_\beta, \quad J_\alpha^2 = -I,$$

for all cyclic permutations $(\alpha, \beta, \gamma)$ of $(1, 2, 3)$ and the identity $I$.

Let $g$ be a neutral metric on $(\mathcal{M}, H)$ having the properties

$$g(\cdot, \cdot) = \varepsilon_\alpha g(J_\alpha \cdot, J_\alpha \cdot),$$

where

$$\varepsilon_\alpha = \begin{cases} 1, & \alpha = 1; \\ -1, & \alpha = 2; 3. \end{cases}$$

The associated 2-form $g_1$ and the associated neutral metrics $g_2$ and $g_3$ are determined by

$$g_\alpha(\cdot, \cdot) = g(J_\alpha \cdot, \cdot) = -\varepsilon_\alpha g(\cdot, J_\alpha \cdot).$$

Let us remark that here and further, $\alpha$ runs over the range $\{1, 2, 3\}$ unless otherwise is stated.

The obtained structure $(H, G) = (J_1, J_2, J_3; g, g_1, g_2, g_3)$ on $\mathcal{M}$ is called an almost hypercomplex structure with Hermitian–Norden metrics and manifold $(\mathcal{M}, H, G)$ is called an almost hypercomplex manifold with Hermitian–Norden metrics [8].
According to [8], the fundamental tensors of such a manifold are the following three $(0,3)$-tensors

\[ F_\alpha (x, y, z) = g \left( (\nabla_x J_\alpha) y, z \right) = (\nabla_x g_\alpha) (y, z) , \quad (3) \]

where \( \nabla \) is the Levi–Civita connection generated by \( g \). These tensors have the following basic properties caused by the structures

\[ F_\alpha (x, y, z) = -\varepsilon_\alpha F_\alpha (x, z, y) = -\varepsilon_\alpha F_\alpha (x, J_\alpha y, J_\alpha z) . \quad (4) \]

The following relations between the tensors \( F_\alpha \) are valid

\[ F_1 (x, y, z) = F_2 (x, J_3 y, z) + F_3 (x, y, J_2 z) , \]
\[ F_2 (x, y, z) = F_3 (x, J_1 y, z) + F_1 (x, y, J_3 z) , \]
\[ F_3 (x, y, z) = F_1 (x, J_2 y, z) - F_2 (x, y, J_1 z) . \]

The corresponding Lee forms \( \theta_\alpha \) are determined by

\[ \theta_\alpha (\cdot) = g^{kl} F_\alpha (e_k, e_l, \cdot) \quad (5) \]

where \( \{e_1, e_2, \ldots, e_{4\alpha}\} \) is an arbitrary basis of \( T_p M, p \in M \) and \( g^{ij} \) are the corresponding components of the inverse matrix of \( g \).

Let us note that, according to (1), \((M, J_1, g)\) is an almost Hermitian manifold whereas the manifolds \((M, J_2, g)\) and \((M, J_3, g)\) are almost complex manifolds with Norden metric. The basic classes of these two types of manifolds are given in [6] and [4], respectively. In the case of the lowest dimension, \( \dim M = 4 \), the four basic classes of almost Hermitian manifolds with respect to \( J_1 \) are restricted to two: the class of the almost Kähler manifolds \( W_2 (J_1) \) and the class of the Hermitian manifolds \( W_1 (J_1) \), determined by:

\[ W_2 (J_1) : \bigcirc \{ F_1 (x, y, z) \} = 0; \]
\[ W_1 (J_1) : \quad F_1 (x, y, z) = \frac{1}{2} \left\{ g(x, y) \theta_1 (z) - g (x, J_1 y) \theta_1 (J_1 z) \right. \]
\[ \left. - g(x, z) \theta_1 (y) + g (x, J_1 z) \theta_1 (J_1 y) \right\} , \quad (6) \]

where \( \bigcirc \) is the cyclic sum by three arguments. The basic classes of the 4-dimensional almost Norden manifolds \( (\alpha = 2 \text{ or } 3) \) are determined as follows:

\[ W_2 (J_\alpha) : \bigcirc \{ F_\alpha (x, y, J_\alpha z) \} = 0, \quad \theta_\alpha = 0; \]
\[ W_3 (J_\alpha) : \bigcirc \{ F_\alpha (x, y, z) \} = 0. \quad (7) \]

The Nijenhuis tensor in terms of the covariant derivatives of \( J_\alpha \) and the corresponding \((0,3)\)-tensor for \( J_\alpha \) are defined by

\[ N_\alpha (x, y) = (\nabla_x J_\alpha) J_\alpha y - (\nabla_y J_\alpha) J_\alpha x + (\nabla_{J_\alpha x} J_\alpha) y - (\nabla_{J_\alpha y} J_\alpha) x, \]
\[ N_\alpha (x, y, z) = g (N_\alpha (x, y), z) . \quad (8) \]
Moreover, the following properties of $N_\alpha$ are valid [12]:

$$N_\alpha(x, y, z) = N_\alpha(x, J_\alpha y, J_\alpha z) = N_\alpha(J_\alpha x, y, J_\alpha z) = -N_\alpha(J_\alpha x, J_\alpha y, z).$$

Let $R = [\nabla, \nabla] - \nabla|.|$ be the curvature $(1,3)$-tensor of $\nabla$ and let the corresponding curvature $(0,4)$-tensor with respect to $g$ be denoted by the same letter:

$$R(x, y, z, w) = g(R(x, y)z, w).$$

The following properties of $R$ are well-known:

$$R(x, y, z, w) = -R(y, x, z, w) = -R(x, y, w, z),$$
$$R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0.$$  

The Ricci tensor $\rho$ and the scalar curvature $\tau$ for $R$ as well as their associated quantities $\rho^*_\alpha$, $\tau^*_\alpha$ and $\tau^{**}_\alpha$ are defined by:

$$\rho(y, z) = g^{ij} R(e_i, y, z, e_j), \quad \rho^*_\alpha(y, z) = g^{ij} R(e_i, y, z, J_\alpha e_j),$$
$$\tau = g^{ij} \rho(e_i, e_j), \quad \tau^*_\alpha = g^{ij} \rho^*(e_i, e_j), \quad \tau^{**}_\alpha = g^{ij} \rho^*(e_i, J_\alpha e_j).$$

Every non-degenerate 2-plane $\mu$ with a basis $\{x, y\}$ with respect to $g$ in $T_p\mathcal{M}$, $p \in \mathcal{M}$, has the following sectional curvature

$$k(\mu; p) = \frac{R(x, y, y, x)}{g(x, x)g(y, y) - g(x, y)^2}.$$  

A 2-plane $\mu$ is said to be holomorphic (resp., totally real) if the condition $\mu = J_\alpha \mu$ (resp., $\mu \perp J_\alpha \mu \neq \mu$ with respect to $g$) holds. The sectional curvature $k(\mu; p)$ of a holomorphic (resp., totally real) 2-plane is called holomorphic (resp., totally real) sectional curvature. Let $\{e_1, e_2, \ldots, e_{4n}\}$ be a basis of $T_p\mathcal{M}$. If $\mu$ has a basis $\{e_i, e_j\}$ ($i, j \in \{1, 2, \ldots, 4n\}, i \neq j$), then $\mu$ is called basic 2-plane and $k(\mu; p)$ – a basic sectional curvature.

2. Four-dimensional indecomposable real Lie algebras and almost hypercomplex structures with Hermitian-Norden metrics

Let

$L$ be a simply connected 4-dimensional real Lie group with corresponding Lie algebra $\mathfrak{l}$. A standard hypercomplex structure on $\mathfrak{l}$ is defined as in [20]:

$$J_1 e_1 = e_2, \quad J_1 e_2 = -e_1, \quad J_1 e_3 = -e_4, \quad J_1 e_4 = e_3;$$
$$J_2 e_1 = e_3, \quad J_2 e_2 = e_4, \quad J_2 e_3 = -e_1, \quad J_2 e_4 = -e_2;$$
$$J_3 e_1 = -e_4, \quad J_3 e_2 = e_3, \quad J_3 e_3 = -e_2, \quad J_3 e_4 = e_1,$$

where $\{e_1, e_2, e_3, e_4\}$ is a basis of $\mathfrak{l}$.
Table 1 Correspondence between the classes with parameters

| Classes   | [5] | [1]    | [16]  | [19]  |
|----------|-----|--------|-------|-------|
| 2 parameters | $\mathfrak{g}_{4,5}$ | $\tau_{4,a,b}$ | $\mathfrak{g}_{4,5}$ | $A_{4,5}^{a,b}$ |
|           | $\mathfrak{g}_{4,6}$ | $\tau'_{4,a,b}$ | $\mathfrak{g}_{4,6}$ | $A_{4,6}^{a,b}$ |
| 1 parameter | $\mathfrak{g}_{4,2}$ | $\tau_{4,a}$ | $\mathfrak{g}_{4,2}$ | $A_{4,2}^{a}$ |
|           | $\mathfrak{g}_{4,9}$ | $\mathfrak{d}_{4,1/1+b}$ | $\mathfrak{g}_{4,8}$ | $A_{4,9}^{a}$ |
|           | $\mathfrak{g}_{4,11}$ | $\mathfrak{d}'_{4,a}$ | $\mathfrak{g}_{4,9}$ | $A_{4,11}^{a}$ |

We introduce a pseudo-Euclidian metric $g$ of neutral signature for $x(x_1, x_2, x_3, x_4), y(y_1, y_2, y_3, y_4) \in l$:

$$g(x, y) = x^1 y^1 + x^2 y^2 - x^3 y^3 - x^4 y^4.$$ 

Note that, using the latter equality, it is valid that

$$g(e_1, e_1) = g(e_2, e_2) = -g(e_3, e_3) = -g(e_4, e_4) = 1,$$

$$g(e_i, e_j) = 0, \quad i \neq j \in \{1, 2, 3, 4\}.$$ (13)

According to (1) and (2), the metric $g$ generates an almost hypercomplex structure with Hermitian–Norden metrics on $l$. Then, $(L, H, G)$ is an almost hypercomplex manifold with Hermitian–Norden metrics.

A classification of real 4-dimensional indecomposable Lie algebras is given for instance in [16] and it can be found easily in [19] and [5]. In [1], it is given a classification of four dimensional solvable real Lie algebras and the authors establish the one-to-one correspondence between their classification and the classifications in [16] and [19]. The twelve basic classes of real 4-dimensional indecomposable Lie algebras are described by the non-zero Lie brackets with respect to $\{e_1, e_2, e_3, e_4\}$. Five of the basic classes are determined by real parameters – two classes use two parameters and three classes use one parameter. In Table 1, it is shown the correspondence between the classes which depend on parameters in the different classifications.

In the present work, we use the notation of the classes from [5].

Our purpose is to investigate how the basic geometrical properties of the manifolds under study depend on these parameters. In [9], we study both the basic classes of the considered classification depending on two parameters. Now, we focus our investigations on the following basic classes $\mathfrak{g}_{4,2}, \mathfrak{g}_{4,9}$ and $\mathfrak{g}_{4,11}$ which depend on one real parameter:

\[
\mathfrak{g}_{4,2} : [e_1, e_4] = ae_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_2 + e_3, \quad (a \neq 0); \\
\mathfrak{g}_{4,9} : [e_1, e_4] = (b + 1)e_1, \quad [e_2, e_3] = e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = be_3, \quad (-1 < b \leq 1); \\
\mathfrak{g}_{4,11} : [e_1, e_4] = 2ce_1, \quad [e_2, e_3] = e_1, \quad [e_2, e_4] = ce_2 - e_3, \quad [e_3, e_4] = e_2 + ce_3, \quad (c > 0). \]
Let us note that further, the indices $i, j, k, l$ run over the range $\{1, 2, 3, 4\}$.

### 2.1. The class $g_{4,2}$

Let us consider a manifold $(L, G, H)$ with corresponding Lie algebra from $g_{4,2}$. Using (1), (12), (14) and the Koszul equality for the considered basis

$$2g(\nabla_{e_1}e_j, e_k) = g([e_i, e_j], e_k) + g([e_k, e_i], e_j) + g([e_k, e_j], e_i),$$

we calculate the components of $\nabla$. The non-zero of them are:

$$\begin{align*}
\nabla_{e_1}e_1 &= ae_4, & \nabla_{e_1}e_4 &= ae_1, \\
\nabla_{e_2}e_2 &= e_4, & \nabla_{e_2}e_3 &= \frac{1}{2}e_4, & \nabla_{e_2}e_4 &= e_2 - \frac{1}{2}e_3, \\
\nabla_{e_3}e_2 &= \frac{1}{2}e_4, & \nabla_{e_3}e_3 &= -e_4, & \nabla_{e_3}e_4 &= \frac{1}{2}e_2 + e_3, \\
\nabla_{e_4}e_2 &= -\frac{1}{2}e_3, & \nabla_{e_4}e_3 &= -\frac{1}{2}e_2. 
\end{align*}$$

We obtain the basic components $(F_\alpha)_{ijk} = F_\alpha(e_i, e_j, e_k)$ of $F_\alpha$, using (3), (12), (13) and (17). The non-zero of them are determined by the following ones and properties (4)

$$\begin{align*}
(F_1)_{113} &= (F_2)_{112} = -\frac{1}{2} (F_3)_{111} = a, \\
(F_1)_{223} &= 2 (F_1)_{323} = 2 (F_1)_{413} = (F_2)_{314} = (F_2)_{322} = \frac{1}{2} (F_2)_{222} = 2 (F_2)_{223} = -2 (F_2)_{412} = (F_3)_{234} = (F_3)_{313} = -(F_3)_{422} = 2 (F_3)_{334} = -2 (F_3)_{213} = 1. 
\end{align*}$$

Using (5) and (18), we establish that the only non-zero basic components $(\theta_\alpha)_i = (\theta_\alpha)(e_i)$ of the corresponding Lee forms are

$$\begin{align*}
(\theta_1)_2 &= (\theta_2)_3 = - (\theta_3)_4 = 1, \\
(\theta_1)_3 &= -\frac{1}{2} (\theta_3)_1 = a + 1, & (\theta_2)_2 &= a + 3. 
\end{align*}$$

### 2.2. The class $g_{4,9}$

In this subsection we focus our investigations on a manifold $(L, G, H)$ with corresponding Lie algebra from $g_{4,9}$.

By similar way as in the previous subsection, we obtain the following results for $(L, G, H)$ in this case. The non-zero components of $\nabla$ are:

$$\begin{align*}
\nabla_{e_1}e_1 &= (b + 1)e_4, & \nabla_{e_1}e_2 &= \frac{1}{2}e_3, \\
\nabla_{e_1}e_3 &= \frac{1}{2}e_2, & \nabla_{e_1}e_4 &= (b + 1)e_1, \\
\nabla_{e_2}e_1 &= \frac{1}{2}e_3, & \nabla_{e_2}e_2 &= e_4, \\
\nabla_{e_2}e_3 &= \frac{1}{2}e_1, & \nabla_{e_2}e_4 &= e_2, \\
\nabla_{e_3}e_1 &= \frac{1}{2}e_2, & \nabla_{e_3}e_2 &= -\frac{1}{2}e_1, \\
\nabla_{e_3}e_3 &= -be_4, & \nabla_{e_3}e_4 &= be_3. 
\end{align*}$$
The non-zero of the components of $F$ are determined by the following ones and properties (4)

\[
(F_1)_{113} = (F_2)_{314} = (F_3)_{313} = b + \frac{1}{2},
\]
\[
(F_2)_{112} = b + \frac{3}{2}, \quad (F_3)_{111} = -2(b + 1),
\]
\[
\frac{1}{3}(F_1)_{223} = \frac{1}{2}(F_2)_{211} = \frac{1}{4}(F_2)_{222} = \frac{1}{2}(F_3)_{122} = -(F_3)_{212} = \frac{1}{2},
\]
\[
(\theta_1)_3 = b + 2, \quad (\theta_2)_2 = 2(b + 2), \quad (\theta_3)_1 = -3(b + 1).
\]

\[(21)\]

2.3. The class $g_{4,11}$

Now, we focus our investigations on a manifold $(L, G, H)$ with corresponding Lie algebra from $g_{4,11}$.

By similar way as in the previous two subsections, we obtain the following results for $(L, G, H)$ in the present case. The non-zero components of $\nabla$ are:

\[
\begin{align*}
\nabla e_1 e_1 &= 2ce_4, & \nabla e_1 e_2 &= \frac{1}{2}e_3, \\
\nabla e_1 e_3 &= \frac{1}{2}e_2, & \nabla e_1 e_4 &= 2ce_1, \\
\nabla e_2 e_1 &= \frac{1}{2}e_3, & \nabla e_2 e_2 &= ce_4, \\
\nabla e_3 e_1 &= \frac{1}{2}e_1 + e_4, & \nabla e_2 e_3 &= ce_2 - e_3, \\
\nabla e_3 e_2 &= \frac{1}{2}e_2, & \nabla e_3 e_3 &= -\frac{1}{2}e_1 + e_4, \\
\nabla e_3 e_4 &= -ce_4, & \nabla e_3 e_4 &= e_2 + ce_3.
\end{align*}
\]

\[(22)\]

The non-zero of the components of $F$ are determined by the following ones and properties (4)

\[
(F_1)_{223} = (F_2)_{314} = (F_3)_{313} = c + \frac{1}{2}, \quad (F_1)_{113} = 2c - \frac{1}{2},
\]
\[
(F_2)_{112} = 2c + \frac{1}{2}, \quad (F_3)_{234} = c - \frac{1}{2}, \quad (F_3)_{111} = -2(F_2)_{222} = -4c
\]
\[
(F_1)_{323} = (F_2)_{211} = (F_3)_{122} = (F_3)_{334} = 1,
\]
\[
(F_2)_{214} = -\frac{1}{2}(F_2)_{322} = (F_3)_{213} = -1,
\]
\[
(\theta_1)_2 = (\theta_2)_3 = -\frac{1}{2}(\theta_3)_4 = 1,
\]
\[
(\theta_1)_3 = 3c, \quad (\theta_2)_2 = 5c + 1, \quad (\theta_3)_1 = -6c.
\]

\[(23)\]

We generalize the obtained results in the three relevant classes by the following

**Theorem 2.1.** Let $(L, H, G)$ be a 4-dimensional almost hypercomplex manifold with Hermitian–Norden metrics. Then, the manifold $(L, H, G)$, corresponding to the different classes of 4-dimensional Lie algebras $g_{4,2}$, $g_{4,9}$ and $g_{4,11}$, belongs to a certain class regarding $J_\alpha$ given in the following table:

Moreover, we have:
| Lie algebra | Parameter | $J_1$       | $J_2$       | $J_3$       |
|------------|-----------|-------------|-------------|-------------|
| $\mathfrak{g}_{4,2}$ | $a = 1$ | $W_4$ | $W_1 \oplus W_2 \oplus W_3$ | $W_1 \oplus W_2 \oplus W_3$ | $a \neq 0; a \neq 1$ | $W_2 \oplus W_4$ | $W_1 \oplus W_2 \oplus W_3$ | $W_1 \oplus W_2 \oplus W_3$ |
| $\mathfrak{g}_{4,9}$ | $b = 1$ | $W_4$ | $W_1 \oplus W_2$ | $W_1 \oplus W_2$ | $-1 < b < 1$ | $W_2 \oplus W_4$ | $W_1 \oplus W_2$ | $W_1 \oplus W_2 \oplus W_3$ |
| $\mathfrak{g}_{4,11}$ | $c > 0$ | $W_2 \oplus W_4$ | $W_1 \oplus W_2 \oplus W_3$ | $W_1 \oplus W_2$ | $c > 0$ | $W_2 \oplus W_4$ | $W_1 \oplus W_2 \oplus W_3$ | $W_1 \oplus W_2$ |

- for each $a \neq 0$, $(L, H, G)$ does not belong to neither of $W_0, W_2$ for $J_1$; $W_0, W_1, W_2, W_3, W_1 \oplus W_2, W_1 \oplus W_3, W_2 \oplus W_3$ for $J_2$; $W_0, W_1, W_2, W_3, W_1 \oplus W_2, W_1 \oplus W_3, W_2 \oplus W_3$ for $J_3$;
- for each $-1 < b < 1$, $(L, H, G)$ does not belong to neither of $W_0, W_2$ for $J_1$; $W_0, W_1, W_2$ for $J_2$; $W_0, W_1, W_2, W_3, W_1 \oplus W_3, W_2 \oplus W_3$ for $J_3$;
- for each $c > 0$, $(L, H, G)$ does not belong to neither of $W_0, W_2, W_4$ for $J_1$; $W_0, W_1, W_2, W_3, W_1 \oplus W_2, W_1 \oplus W_3, W_2 \oplus W_3$ for $J_2$; $W_0, W_1, W_2, W_3$ for $J_3$.

**Proof.** Using the results in (18), (19), (21), (23) and the classification conditions (6), (7) for dimension 4, we establish the truthfulness of the assertion in each case. $\Box$

### 3. Some geometric characteristics of the considered manifolds

In this section we determine some geometric characteristics of the manifolds $(L, G, H)$ considered in the previous section and we investigate the corresponding geometric properties in relation with the real parameters of the considered three classes of Lie algebras.

Firstly, we consider a manifold $(L, G, H)$ with corresponding Lie algebra from $\mathfrak{g}_{4,2}$. Using (1), (8), (12) and (14), we obtain the basic components $(N_\alpha)_{ijk} = N_\alpha(e_i, e_j, e_k)$ of $N_\alpha$. The non-zero of them are determined by the following ones and properties (9)

\[(N_1)_{132} = (N_2)_{123} = a - 1, \quad (N_2)_{122} = (N_3)_{122} = 1.\]  

(24)

We calculate the basic components $R_{ijkl} = R(e_i, e_j, e_k, e_l)$ of $R$, using (1), (10), (12) and (14). The non-zero of them are determined by the following ones and properties (11)

\[R_{1212} = -R_{1313} = -a, \quad R_{1213} = -\frac{a}{2}, \quad R_{1414} = a^2,\]

\[R_{2323} = R_{2424} = \frac{5}{4}, \quad R_{2434} = 1, \quad R_{3434} = -\frac{1}{4}.\]  

(25)
We obtain the basic components $\rho_{jk} = \rho(e_j, e_k)$, $(\rho^*_\alpha)_{jk} = \rho^*_\alpha(e_j, e_k)$, as well as the values of $\tau, \tau^*_\alpha, \tau^{**}_\alpha$ and $k_{ij} = k(e_i, e_j)$ as follows:

$$
\rho_{11} = a(a + 2), \quad \rho_{22} = a + \frac{5}{2}, \quad \rho_{23} = \rho_{32} = \frac{a}{2} + 1, \quad \rho_{33} = \left(-a + \frac{3}{2}\right), \quad \rho_{44} = \left(-a^2 + \frac{3}{2}\right), \quad (\rho^*_1)_{12} = -(\rho^*_2)_{21} = (\rho^*_2)_{13} = (\rho^*_3)_{31} = (\rho^*_3)_{11} = a,
$$

$$(\rho^*_1)_{13} = -(\rho^*_2)_{31} = -(\rho^*_2)_{12} = -(\rho^*_2)_{21} = \frac{a}{2}, \quad (\rho^*_1)_{24} = -(\rho^*_1)_{42} = -(\rho^*_2)_{34} = -(\rho^*_2)_{43} = -1,$$

$$(\rho^*_1)_{34} = -(\rho^*_1)_{43} = \frac{1}{4}, \quad (\rho^*_2)_{24} = (\rho^*_2)_{42} = (\rho^*_3)_{23} = (\rho^*_3)_{32} = \frac{5}{4}, \quad (\rho^*_2)_{14} = (\rho^*_3)_{41} = -a^2, \quad (\rho^*_3)_{44} = -2,$$

$$\tau = 2a^2 + 4a + \frac{11}{2}, \quad \tau^*_1 = \tau^*_2 = 0, \quad \tau^*_3 = a + 2,$$

$$\tau^{**}_1 = 2a + \frac{1}{2}, \quad \tau^{**}_2 = 2a + \frac{5}{2}, \quad \tau^{**}_3 = 2a^2 + \frac{5}{2},$$

$$k_{12} = k_{13} = a, \quad k_{14} = a^2, \quad k_{23} = k_{24} = \frac{5}{4}, \quad k_{34} = \frac{1}{4}. \quad (26)$$

**Theorem 3.1.** Let $(L, H, G)$ be an almost hypercomplex manifold with Hermitian-Norden metrics and Lie algebra from the class $\mathfrak{g}_{4,2}$. The following characteristics of the manifold are valid:

1. $(L, H, G)$ is integrable for $J_1$ if and only if $a = 1$;
2. Every $(L, H, G)$ is non-flat;
3. Every $(L, H, G)$ has a positive scalar curvature;
4. Every $(L, H, G)$ is $\ast$-scalar flat w.r.t. $J_1$ and $J_2$;
5. $(L, H, G)$ is $\ast$-scalar flat w.r.t. $J_3$ if and only if $a = -2$;
6. $(L, H, G)$ is $\ast\ast$-scalar flat w.r.t. $J_1$ if and only if $a = -\frac{1}{4}$;
7. $(L, H, G)$ is $\ast\ast$-scalar flat w.r.t. $J_2$ if and only if $a = -\frac{3}{4}$;
8. Every $(L, H, G)$ has a positive $\ast\ast$-scalar curvature w.r.t. $J_3$;
9. $(L, H, G)$ has a positive basic holomorphic sectional curvatures w.r.t. $J_1$ (i.e. $k_{12}$ and $k_{34}$) and $J_2$ (i.e. $k_{13}$ and $k_{24}$) if and only if $a > 0$;
10. Every $(L, H, G)$ has a positive basic holomorphic sectional curvatures w.r.t. $J_3$ (i.e. $k_{14}$ and $k_{23}$);
11. $(L, H, G)$ has a positive basic totally real sectional curvatures w.r.t. $J_1$ (i.e. $k_{13}, k_{14}, k_{23}$ and $k_{24}$), $J_2$ (i.e. $k_{12}, k_{14}, k_{23}$ and $k_{34}$) and $J_3$ (i.e. $k_{12}, k_{13}, k_{24}$ and $k_{34}$) if and only if $a > 0$.

**Proof.** By virtue of (24), (25) and (26), we establish the truthfulness of the statements. \qed

Now we focus our study on a manifold $(L, G, H)$ with corresponding Lie algebra from $\mathfrak{g}_{4,9}$. By similar way as for $\mathfrak{g}_{4,2}$, we obtain the following results for
(L, G, H) in this case:

\[(N_1)_{132} = (N_3)_{132} = b - 1,\]
\[R_{1221} = R_{2323} = b + \frac{3}{4}, \quad R_{1234} = -\frac{b}{2},\]
\[R_{1313} = b^2 + b - \frac{1}{4}, \quad R_{1324} = \frac{1}{2}, \quad R_{1414} = (b + 1)^2,\]
\[R_{1423} = \frac{1}{2}(b + 1), \quad R_{2424} = 1, \quad R_{3443} = b^2,\]
\[\rho_{11} = 2b^2 + 4b + \frac{3}{2}, \quad \rho_{22} = 2b + \frac{5}{2},\]
\[\rho_{33} = -2b^2 - 2b - \frac{1}{2}, \quad \rho_{44} = -2b^2 - 2b - 2,\]
\[(\rho_1^*)_{12} = \frac{3}{2} \left( b + \frac{1}{2} \right), \quad (\rho_1^*)_{34} = b \left( b + \frac{1}{2} \right), \quad (\rho_2^*)_{13} = b^2 + 2b + \frac{1}{4},\]
\[(\rho_2^*)_{24} = b + \frac{3}{2}, \quad (\rho_3^*)_{14} = -\left( b^2 + \frac{5}{2}b + \frac{1}{2} \right), \quad (\rho_3^*)_{23} = \frac{3}{2} \left( b + \frac{1}{6} \right),\]
\[\tau = 6b^2 + 10b + \frac{13}{2}, \quad \tau_1^* = \tau_2^* = \tau_3^* = 0,\]
\[\tau_1^{**} = \tau_2^{**} = 2 \left( b^2 + b + \frac{3}{4} \right), \quad \tau_3^{**} = 2 \left( b^2 + 3b + \frac{7}{4} \right),\]
\[k_{12} = k_{23} = b + \frac{3}{4}, \quad k_{13} = b^2 + b - \frac{1}{4},\]
\[k_{14} = (b + 1)^2, \quad k_{24} = 1, \quad k_{34} = b^2\] (27)

and we calculate the rest nonzero components of \((N_\alpha)_{ijk}\) and \(R_{ijkl}\) using (9) and (11), respectively.

**Theorem 3.2.** Let \((L, H, G)\) be an almost hypercomplex manifold with Hermitian-Norden metrics and Lie algebra from the class \(g_{4,9}\). The following characteristics of the manifold are valid:

1. Every \((L, H, G)\) is integrable for \(J_2\);
2. Every \((L, H, G)\) is non-flat;
3. Every \((L, H, G)\) has a positive scalar curvature;
4. Every \((L, H, G)\) is \(\ast\)-scalar flat;
5. Every \((L, H, G)\) has a positive \(\ast\ast\)-scalar curvature w.r.t. \(J_1\) and \(J_2\);
6. \((L, H, G)\) is \(\ast\ast\)-scalar flat w.r.t. \(J_3\) if and only if \(b = \frac{\sqrt{2} - 3}{2}\);
7. \((L, H, G)\) has \(\ast\ast\)-scalar flat sectional curvatures w.r.t. \(J_1\) (i.e. \(k_{12}\) and \(k_{34}\)) and \(J_3\) (i.e. \(k_{14}\) and \(k_{23}\)) if and only if \(b \in \left( -\frac{3}{4}; 0 \right) \cup (0; 1]\);
8. \((L, H, G)\) has positive basic holomorphic sectional curvatures w.r.t. \(J_2\) (i.e. \(k_{13}\) and \(k_{24}\)) if and only if \(b \in \left( \frac{\sqrt{2} - 1}{2}; 1 \right]\);
9. \((L, H, G)\) has positive basic totally real sectional curvatures w.r.t. \(J_1\) (i.e. \(k_{13}, k_{14}, k_{23}\) and \(k_{24}\)) and \(J_3\) (i.e. \(k_{12}, k_{13}, k_{24}\) and \(k_{34}\)) if and only if \(b \in \left(\frac{\sqrt{3} - 1}{2}; 1\right]\); 

10. \((L, H, G)\) has positive basic totally real sectional curvatures w.r.t. \(J_2\) (i.e. \(k_{12}, k_{14}, k_{23}\) and \(k_{34}\)) if and only if \(b \in (-\frac{3}{4}; 0) \cup (0; 1]\).

\[\text{Proof.}\] By virtue of (27), we establish the truthfulness of the statements. \(\Box\)

Herein, we continue our study on a manifold \((L, G, H)\) with corresponding Lie algebra from \(\mathfrak{g}_{4,11}\). By similar way as previous ones, we obtain the following results for \((L, G, H)\) in this case:

\[
\begin{align*}
(N_1)_{132} &= (N_2)_{123} = c - 1, \\
(N_2)_{144} &= 3c, \quad (N_1)_{133} = -(N_2)_{122} = -1, \\
R_{2434} &= -R_{1213} = 2R_{1423} = -4R_{1234} = 4R_{1324} = 2c, \\
R_{1221} &= R_{1313} = 2c^2 - 1, \quad R_{2424} = -R_{3434} = c^2 - 1, \\
R_{1414} &= 4c^2, \quad R_{2323} = c^2 + \frac{7}{4}, \quad R_{1224} = R_{1334} = \frac{1}{2}, \\
\rho_{11} &= 8c^2 - \frac{1}{4}, \quad \rho_{22} = -\rho_{33} = 4c^2 + \frac{1}{2}, \\
\rho_{23} &= 4c, \quad \rho_{44} = -6c^2 + 2, \\
(\rho_1^*)_{12} &= 2c^2 + \frac{c}{2} - \frac{1}{4}, \quad (\rho_1^*)_{34} = c^2 + \frac{c}{2} - 1, \\
(\rho_2^*)_{13} &= (\rho_2^*)_{34} = 2c - \frac{1}{2}, \\
(\rho_2^*)_{13} &= 2c^2 + \frac{3c}{2} - \frac{1}{4}, \quad (\rho_2^*)_{24} = c^2 + \frac{3c}{2} - 1, \\
(\rho_2^*)_{24} &= (\rho_2^*)_{12} = -\left(2c + \frac{1}{2}\right), \quad (\rho_1^*)_{23} = c^2 + \frac{7}{4}, \\
(\rho_1^*)_{34} &= (\rho_1^*)_{34} = 4c, \quad (\rho_2^*)_{22} = (\rho_3^*)_{33} = -1, \\
\tau &= 18c^2 - \frac{3}{2}, \quad \tau_1^* = \tau_2^* = 0, \quad \tau_3^* = 8c, \\
\tau_1^{**} &= \tau_2^{**} = 6c^2 - \frac{5}{2}, \quad \tau_3^{**} = 10c^2 + \frac{7}{2}, \\
k_{12} &= k_{13} = 2c^2 - \frac{1}{4}, \quad k_{14} = 4c^2, \\
k_{23} &= c^2 + \frac{7}{4}, \quad k_{24} = k_{34} = c^2 - 1
\end{align*}\]

and we calculate the rest nonzero components of \((N_\alpha)_{ijk}\) and \(R_{ijkl}\) using (9) and (11), respectively.

**Theorem 3.3.** Let \((L, H, G)\) be an almost hypercomplex manifold with Hermitian-Norden metrics and Lie algebra from the class \(\mathfrak{g}_{4,11}\). The following characteristics of the manifold are valid:
1. Every \((L, H, G)\) is integrable for \(J_3\);
2. Every \((L, H, G)\) is non-flat;
3. \((L, H, G)\) is scalar flat if and only if \(c = \sqrt{3}/6\);
4. Every \((L, H, G)\) is \(*\)-scalar flat w.r.t. \(J_1\) and \(J_2\);
5. Every \((L, H, G)\) has a positive \(*\)-scalar curvature w.r.t. \(J_3\);
6. \((L, H, G)\) is \(\ast\)-scalar flat w.r.t. \(J_1\) and \(J_2\) if and only if \(c = \sqrt{15}/6\);
7. Every \((L, H, G)\) has a positive \(\ast\)-scalar curvature w.r.t. \(J_3\);
8. \((L, H, G)\) has positive (resp., negative) basic holomorphic sectional curvatures w.r.t. \(J_1\) (i.e. \(k_{12}\) and \(k_{34}\)) and \(J_2\) (i.e. \(k_{13}\) and \(k_{24}\)) if and only if \(c > 1\) (resp., \(0 < c < \sqrt{2}/4\));
9. Every \((L, H, G)\) has positive basic holomorphic sectional curvatures w.r.t. \(J_3\) (i.e. \(k_{14}\) and \(k_{23}\)));
10. \((L, H, G)\) has positive basic totally real sectional curvatures w.r.t. \(J_1\) (i.e. \(k_{13}, k_{14}, k_{23}\) and \(k_{24}\)) and \(J_2\) (i.e. \(k_{12}, k_{14}, k_{23}\) and \(k_{34}\)) if and only if \(c > 1\);
11. \((L, H, G)\) has positive (resp., negative) basic totally real sectional curvatures w.r.t. \(J_3\) (i.e. \(k_{12}, k_{13}, k_{24}\) and \(k_{34}\)) if and only if \(c > 1\) (resp., \(0 < c < \sqrt{2}/4\)).

Proof. By virtue of (28), we establish the truthfulness of the statements.  

Declaration

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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