**BG-Volterra Integral Equations and Relationship with BG-Differential Equations**

**BG-Volterra İntegral Denklemleri ve BG-Diferansiyel Denklemlerle İlişkisi**

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**Abstract**  
In this study, the Volterra integral equations are defined in the sense of bigeometric calculus by the aid of bigeometric integral. The main aim of the study is to research the relationship between bigeometric Volterra integral equations and bigeometric differential equations.

**Keywords**: Bigeometric Calculus, Bigeometric Differential Equations, Bigeometric Volterra Integral Equations

**Öz**  
Bu çalışmada, bigeometrik integral yardımıyla bigeometrik Volterra integral denklemleri tanımlanmıştır. Çalışmanın asıl amacı bigeometrik manada Volterra integral denklemleri ile bigeometrik manada diferansiyel denklemler arasındaki ilişkiyi araştırmaktır.

**Anahtar kelimeler**: Bigeometrik Hesap, Bigeometrik Diferansiyel Denklemler, Bigeometrik Volterra İntegral Denklemleri

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1. Introduction

Grossman and Katz have built a new calculus, called non-Newtonian calculus, between years 1967-1970 as an alternative to classic calculus. They have defined infinite family of calculus consisting of classic, geometric, harmonic and quadratic calculus, then they have created bigeometric, biharmonic and biquadratic calculus in this progress. Non-Newtonian calculus provides a wide application area in science, engineering and mathematics. Such as studied on can be expressed as theory of elasticity in the economy, the viscosity of the blood, computer science including image processing and artificial intelligence, biology, differential equations, functional analysis and probability theory. Non-Newtonian calculus is researched by various researchers such as Grossman (1979); Çakmak and Başar (2012, 2014a,b, 2015); Türkmen ve Başar (2012a,b); Tekin and Başar (2013); Kadak and Özlük (2014); Duyar and Oğur (2017); Duyar and Sağır (2017); Erdoğan and Duyar (2018); Sağır and Erdoğan (2019); Güngör (2020). One of the most popular non-Newtonian calculus, namely, bigeometric calculus which is investigated especially by Volterra and Hostinsky (1938); Grossmann (1983); Rybaczuk and Stopel (2000) investigated the fractal growth in material science by using bigeometric calculus. Aniszewska and Rybaczuk (2005) used bigeometric calculus on a multiplicative Lorenz System. Córdova-Lepe (2006) studied on measure of elasticity in economics by aid of bigeometric calculus. Boruah and Hazarika (2018a,b) named Bigeometric calculus as G-calculus and investigated basic properties of derivative and integral in the sense of bigeometric calculus and also applications in numerical analysis. Boruah et al. (2018) researched solvability of bigeometric differential equations by using numerical methods.

Integral equations have used for the solution of several problems in engineering, applied mathematics and mathematical physics since 18th century. The integral equations have begun to enter the problems of engineering and other fields because of the relationship with differential equations which have wide range of applications and so their importance has increased in recent years. The reader may refer for relevant terminology on the integral equations to Smithies (1958); Krasnov et al. (1971); Zarnan (2016); Brunner (2017); Maturi (2019).

In this paper, we define Volterra integral equations in bigeometric calculus by using the concept of bigeometric integral and called $BG$-Volterra integral equations. We prove Leibniz formula in the sense of bigeometric calculus and demonstrate the converting the $BG$-Volterra integral equations to bigeometric differential equations by aid of this formula. By defining the bigeometric linear differential equations with constant coefficients and variable coefficients, we demonstrate that they are converted to $BG$-Volterra integral equations.

A generator is one-to-one function $\alpha$ whose domain is $\mathbb{R}$ the set of real numbers and whose range is a subset of $\mathbb{R}$. It is indicated by $\mathbb{R}_\alpha=\{\alpha(x):x\in\mathbb{R}\}$ the range of generator $\alpha$. $\alpha$-arithmetic operations are described as indicated, below:

- $\alpha$ - addition: $x+y=\alpha\left[\alpha^{-1}(x)+\alpha^{-1}(y)\right]$
- $\alpha$ - subtraction: $x-y=\alpha\left[\alpha^{-1}(x)-\alpha^{-1}(y)\right]$
- $\alpha$ - multiplication: $x\times y=\alpha\left[\alpha^{-1}(x)\times\alpha^{-1}(y)\right]$
- $\alpha$ - division: $x/y=\alpha\left[\alpha^{-1}(x)/\alpha^{-1}(y)\right]$
- $\alpha$ - order: $x<\alpha\Leftrightarrow\alpha^{-1}(x)<\alpha^{-1}(y)$.

For $x,y\in\mathbb{R}_\alpha$. $(\mathbb{R}_\alpha,+\times)$ is complete field. In particular, the identity function $I$ generates classical arithmetic and the exponential function generates geometric arithmetic. The numbers $x>0$ are $\alpha$-positive numbers and the numbers $x<0$ are $\alpha$-negative numbers in $\mathbb{R}_\alpha$. $\alpha$-zero and $\alpha$-one numbers are denoted by $\alpha(0)=\hat{0}$ and $\alpha(1)=\hat{1}$, respectively. $\alpha$-integers are obtained by successive $\alpha$-addition of $\hat{1}$ to $\hat{0}$ and successive $\alpha$-subtraction of $\hat{1}$ from $\hat{0}$. Hence the $\alpha$-integers are as follows:

$$\ldots, \alpha(-2), \alpha(-1), \alpha(0), \alpha(1), \alpha(2), \ldots$$

For each integer $n$, we set $\hat{n}=\alpha(n)$. If $\hat{n}$ is an $\alpha$-positive integer, then it is $n$ times sum of $\hat{1}$ (Grossman and Katz, 1972).
Definition 1. The $\alpha$-absolute value of $x \in \mathbb{R}_\alpha$ determined by 

$$|x|_\alpha = \begin{cases} x & , x > 0 \\ 0 & , x = 0 \\ -x & , x < 0 \end{cases}$$

and this value is equivalent to $\alpha\left(\alpha^{-1}(x)\right)$. For $x \in \mathbb{R}_\alpha$, $x^{\alpha\varepsilon} = \alpha\left(\alpha^{-1}(x)\right)^{\varepsilon}$ and $\sqrt[\alpha]{x} = \alpha\left(\sqrt{\alpha^{-1}(x)}\right)$ (Grossman and Katz, 1972).

Definition 2. An open $\alpha$-interval on $\mathbb{R}_\alpha$ expressed with 

$$(r, s) = \{x \in \mathbb{R}_\alpha : r < x < s\} = \{x \in \mathbb{R}_\alpha : \alpha^{-1}(r) < \alpha^{-1}(x) < \alpha^{-1}(s)\} = \alpha\left(\alpha^{-1}(r), \alpha^{-1}(s)\right)$$

Similarly, a closed $\alpha$-interval on $\mathbb{R}_\alpha$ can be expressed (Grossman and Katz, 1972).

Definition 3. A point $a$ is said to be an interior point of the subset $A \subseteq \mathbb{R}_\alpha$ if there is an open $\alpha$-interval, contained entirely in the set $A$, which contains this point: 

$$a \in (r, s) \subseteq A \iff a \in \alpha^{-1}(A) \cap (r, s) = \alpha\left(\alpha^{-1}(r), \alpha^{-1}(s)\right)$$

According to this definition, $a$ is an interior point of the subset $A \subseteq \mathbb{R}_\alpha$ iff $\alpha^{-1}(a)$ is an interior point of the subset $\alpha^{-1}(A) \subseteq \mathbb{R}$. If a subset $A \subseteq \mathbb{R}_\alpha$ whose all points are interior points, it is called $\alpha$-open (Duyar and Oğur, 2017).

Definition 4. Let $\left(\mathbb{R}_\alpha, |x|_\alpha\right)$ be non-Newtonian metric space and $a \in \mathbb{R}_\alpha$. If $\left(\{a - \varepsilon, a + \varepsilon\} - \{a\}\right) \cap S \neq \emptyset$ for every $\varepsilon > 0$ where $S \subseteq \mathbb{R}_\alpha$, then the point $a$ is called $\alpha$-accumulation point of the set $S$. The set of all $\alpha$-accumulation points of $S$ is indicated by $S^{\alpha}$ ( Sağır and Erdoğan, 2019).

Definition 5. Let $(x_n)$ be sequence and $x$ be a point in the non-Newtonian metric space $\left(\mathbb{R}_\alpha, |x|_\alpha\right)$. If for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $|x_n - x|_\alpha < \varepsilon$ for all $n \geq n_0$, then it is said that the sequence $(x_n)$ $\alpha$-convergent and denoted by $\lim_{n \to \infty} x_n = x$ (Grossman and Katz, 1972; Çakmak and Başar, 2012).

Grossmann and Katz described the *-calculus with the help of two arbitrary selected generators. Let $\alpha$ and $\beta$ are arbitrarily chosen generators and * is the ordered pair of arithmetic ($\alpha$-arithmetic, $\beta$-arithmetic). The following notions are used.

| $\alpha$-arithmetic | $\beta$-arithmetic |
|---------------------|---------------------|
| Realm               | $A(\subseteq \mathbb{R}_\alpha)$ | $B(\subseteq \mathbb{R}_\beta)$ |
| Summation           | $\dagger$            | $\dagger$            |
| Subtraction         | $\dagger$            | $\dagger$            |
| Multiplication      | $\times$             | $\times$             |
| Division            | $\div$ (or $-\div$) | $\div$ (or $-\div$) |
| Order               | $\div$ (or $-\div$) | $\div$ (or $-\div$) |

If the generators $\alpha$ and $\beta$ are chosen as one of $I$ and $\exp$, the following special calculus are obtained.

| Calculus | $\alpha$ | $\beta$ |
|----------|-------------------|-------------------|
| Classic  | $I$               | $I$               |
| Geometric| $\exp$            | $I$               |
| Anageometric| $\exp$            | $I$               |
| Bigeometric| $\exp$            | $\exp$.            |
The \( \iota \) (iota) which is an isomorphism from \( \alpha \)-arithmetic to \( \beta \)-arithmetic uniquely satisfying the following three properties:

1. \( \iota \) is one to one,
2. \( \iota \) is on \( A \) and onto \( B \),
3. For any numbers \( u \) and \( v \) in \( A \),
   \[
   \iota(u+\nu) = \iota(u) + \iota(\nu), \\
   \iota(u \cdot \nu) = \iota(u) \cdot \iota(\nu), \\
   \iota(u \div \nu) = \iota(u) \div \iota(\nu), \\
   \iota(u \div i \nu) = \iota(u) \div i \iota(\nu), \\
   \iota(u\iota \nu) = \iota(u)\iota \iota(\nu), \\
   \iota(u \iota \nu) = \iota(u) \iota \iota(\nu). \]

It turns out that \( \iota(u) = \beta\iota(\alpha^{-1}(u)) \) for every \( u \) in \( A \). (Grossman and Katz, 1972).

2. Bigeometric Calculus

Throughout this study, we interest Bigeometric calculus that is the one of the family of non-Newtonian calculus. As mentioned above, the bigeometric calculus is the \( * \)-calculus for which \( \alpha = \beta = \exp \). That is to say, one uses geometric arithmetic on function arguments and values in the bigeometric calculus. Therefore, we begin with presenting the geometric arithmetic and its necessary properties.

If the function \( \exp \) from \( \mathbb{R} \) to \( \mathbb{R}^+ \) which gives \( \alpha^{-1}(x) = \ln x \) is selected as a generator, that is to say that \( \alpha \)-arithmetic turns into geometric arithmetic. The range of generator \( \exp \) is denoted by \( \mathbb{R}^\exp = \{ e^x : x \in \mathbb{R} \} \).

- **geometric addition**
  \[ x \oplus y = \alpha^{-1}(x) + \alpha^{-1}(y) = e^{\ln x + \ln y} = x \cdot y \]

- **geometric subtraction**
  \[ x \odot y = \alpha^{-1}(x) - \alpha^{-1}(y) = e^{\ln x - \ln y} = x / y, \ y \neq 0 \]

- **geometric multiplication**
  \[ x \odot y = \alpha^{-1}(x) \cdot \alpha^{-1}(y) = e^{\ln x \ln y} \]

- **geometric division**
  \[ x \oslash y = \alpha^{-1}(x) / \alpha^{-1}(y) = e^{\ln x / \ln y} = \sqrt{x} / y, \ y \neq 1 \]

- **geometric order**
  \[ x <_{\exp} y \iff \alpha^{-1}(x) = \ln x < \alpha^{-1}(y) = \ln y \]

\( (\mathbb{R}^\exp, \oplus, \odot) \) is a field with geometric zero \( 1 \) and geometric identity \( e \). The geometric positive real numbers and geometric negative real numbers are denoted by \( \mathbb{R}^+_\exp = \{ x \in \mathbb{R}^\exp : x > 1 \} \) and \( \mathbb{R}^-_\exp = \{ x \in \mathbb{R}^\exp : x < 1 \} \), respectively. Now, we will give some useful and necessary relations between geometric and classical arithmetic operations. The geometric absolute valued of \( x \in \mathbb{R}^\exp \) defined by

\[ |x|_{\exp} = \begin{cases} 
  x &, x > 1 \\
  1 &, x = 1 \\
  1/x &, x < 1 
\end{cases} \]

Thus \( |x|_{\exp} \geq 1 \). For all \( x, y \in \mathbb{R}^\exp \), the following relations hold:

\[ x \oplus y = x \cdot y \]
\[ x \odot y = x / y \]
\[ x \odot y = x^{\ln y} \]
\[ x \odot y = \sqrt{x} = e^{(\ln x) / 2} \]
\[ x \odot y = \frac{x}{y} = e^{\ln x} \]
\[ x \odot y = x^{\ln y} \]
\[ x \odot y = \sqrt{x} = |x|_{\exp} \]
\[ x \odot y = x \odot 1 = x \]

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The geometric factorial notation \( !_{\exp} \) denoted by \( n!_{\exp} = e^n \odot e^{n-1} \odot \ldots \odot e^2 \odot e = e^n! \) (Borah and Hazarika, 2018a). For example,
\[
0!_{\exp} = e^0 = 1 \\
1!_{\exp} = e! = e \\
2!_{\exp} = e^2! = e^2 \\
\ldots
definition 6. Let \( f : X \subset \mathbb{R}_{\exp} \to \mathbb{R}_{\exp} \) be a function and \( a \in X_{\exp} \), \( b \in \mathbb{R}_{\exp} \). If for every \( \varepsilon > e_{\exp} \) there is a number \( \delta = \delta(\varepsilon) > e_{\exp} \) such that \( |f(x) \odot b|_{e_{\exp}} < \varepsilon \) for all \( x \in X \) whenever \( 1 < |x \odot a|_{e_{\exp}} < \varepsilon \), then it is said that the \( BG \)-limit function \( f \) at the point \( a \) is \( b \) and it is indicated by \( BG \lim_{x \to a} f(x) = b \) or \( f(x) \to b \).

Remark 1. \( BG \lim_{x \to a} f(x) \) and \( \lim_{t \to \ln a} \ln f(t) \) coexist, and if they do exist \( BG \lim_{x \to a} f(x) = \exp \left\{ \lim_{t \to \ln a} \ln f(t) \right\} \).

Furthermore, \( f \) is \( BG \)-continuous at \( a \) iff \( \ln f \) is continuous at \( \ln a \) (Grossman and Katz, 1972; Grossman 1983).

Definition 9. Let \( f : (r,s) \subset \mathbb{R}_{\exp} \to \mathbb{R}_{\exp} \) be a function and \( a \in (r,s) \). If the following limit
\[
BG \lim_{h \to 0} f(x) \odot f(a)_{e_{\exp}} - \lim_{x \to a} \left[ f(x)_{e_{\exp}} \frac{1}{\ln x - \ln a} \right]_{e_{\exp}}
\]
exists, it is indicated by \( fBG(a) \) and called the \( BG \)-derivative of \( f \) at \( a \) and say that \( f \) is \( BG \)-differentiable. If the function \( f \) is \( BG \)-differentiable at all points of the \( \exp \)-open interval \( (r,s) \), then \( f \) is \( BG \)-differentiable on \( (r,s) \) and \( BG \)-derivative of \( f \) identified as
\[
BG \lim_{h \to 0} f(x) \odot f(h)_{e_{\exp}} - \lim_{h \to 0} \left[ f(hx)_{e_{\exp}} \frac{1}{\ln h} \right]_{e_{\exp}}
\]
for \( h \in \mathbb{R}_{\exp} \) and denoted by \( fBG(\ ) \) or \( \frac{dBG}{dx}f \) (Grossman and Katz, 1972; Grossman 1983; Borah and Hazarika, 2018a,b).
Remark 2. The derivatives \( f^{BG}(a) \) and \( (\ln f(\ln a))' \) coexist, and if they do exist
\[ f^{BG}(a) = \exp \left( \ln f(e^{\ln a}) \right)' = e^{f'(a)/f(a)}. \]
Therefore the relation between the \( BG \)-derivatives and classical derivatives can be written as follows:
\[ f^{BG}(x) = e^{f'(x)/f(x)} = e^{(\ln f(x))'}. \]
\((\text{Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018a,b}).\)

The second \( BG \)-derivative of \( f(x) \) is defined as
\[
\frac{d^2 f}{dx^2} = f^{(2BG)} = \lim_{h \to 0} \frac{f(x + h) \otimes f(x) - f(x)}{h} = \lim_{h \to 0} \frac{f^{BG}(x) - f^{BG}(x)_{h}^{BG}}{h} = e^{f'(x)/f(x)} - e^{x(f'(x)/f(x))'}.
\]
Similarly, the \( n^{th} \) order derivative is
\[
\frac{d^n f}{dx^n} = f^{(nBG)} = \lim_{h \to 0} \frac{f(x + h) \otimes f(x) - f(x)}{h} = \lim_{h \to 0} \frac{f^{BG}(x) - f^{BG}(x)_{h}^{BG}}{h} = e^{f'(x)/f(x)} - e^{x(f'(x)/f(x))'}. \]

Theorem 1. If \( f, g : (r,s) \subset \mathbb{R}^{exp} \rightarrow \mathbb{R}^{exp} \) are \( BG \)-differentiable functions and \( c \) is an arbitrary constant, then

1. \( (f(x) \circ g(x))^{BG} = f(x)^{BG} \otimes g(x)^{BG} \)
2. \( (f(x) \circ g(x))^{BG} = f(x)^{BG} \otimes g(x)^{BG} \)
3. \( (f(x)^c)^{BG} = (f(x)^c)^{BG} \)
\((\text{Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018a,b}).\)

Now, we will give some standard \( BG \)-derivatives:
\[
\frac{d}{dx}^{BG}(c) = 1, \quad \frac{d}{dx}^{BG}(\sin x) = e^{\cot x}, \quad \frac{d}{dx}^{BG}(\cot x) = e^{-\sec x \csc x},
\]
\[
\frac{d}{dx}^{BG}(\cos x) = e^{-\tan x}, \quad \frac{d}{dx}^{BG}(\sec x) = e^{\tan x}, \quad \frac{d}{dx}^{BG}(\csc x) = e^{-\cot x}.
\]
\((\text{Boruah and Hazarika, 2018a,b}).\)

Theorem 2. (Mean Value Theorem of \( BG \)-Calculus) If \( f \) is \( BG \)-continuous function on \([r,s] \subset \mathbb{R}^{exp} \) and \( BG \)-differentiable on \((r,s)\), there is \( r < c < s \) such that \( f^{BG}(c) = \frac{f(s) \circ f(r)}{s \circ r} \).
\((\text{Grossman and Katz, 1972; Grossman 1983; Kadak and Özlük, 2014}).\)

Definition 10. The \( BG \)-average of a \( BG \)-continuous positive function \( f \) on \([r,s] \subset \mathbb{R}^{exp} \) is defined as the exp-limit of the exp-convergent sequence whose \( n^{th} \) term is geometric average of \( f(a_1), f(a_2), \ldots, f(a_n) \).
where \( a_1, a_2, \ldots, a_n \) is the \( n \)-fold exp-partition of \([r, s]\) and denoted by \( M'_n f \). The \( BG \)-integral of a \( BG \)-continuous function \( f \) on \([r, s]\) is the positive number \( \int_{BG}^{BG} f \) and is denoted by

\[
\int_{BG}^{BG} f(x) \, dx = \exp \left( \int_{BG}^{BG} f(x) \, dx \right)
\]

(Grossman and Katz, 1972; Grossman 1983).

**Remark 3.** If \( f \) is \( BG \)-continuous on \([r, s] \subset \mathbb{R}_{exp}\), then \( \int_{BG}^{BG} f(x) \, dx = \exp \left( \int_{BG}^{BG} f(x) \, dx \right) \), i.e., the \( BG \)-integral of the function \( f \) is a \( BG \)-continuous positive function on \([r, s] \subset \mathbb{R}_{exp}\) is defined by

\[
\int_{BG}^{BG} f(x) \, dx = \exp \left( \int_{BG}^{BG} f(x) \, dx \right)
\]

(Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018b).

**Theorem 3.** If \( f \) and \( g \) are \( BG \)-continuous positive functions on \([r, s] \subset \mathbb{R}_{exp}\) and \( c \) is an arbitrary constant, then

1. \( \int_{BG}^{BG} (f(x) \pm g(x)) \, dx = \int_{BG}^{BG} f(x) \, dx \pm \int_{BG}^{BG} g(x) \, dx \)
2. \( \int_{BG}^{BG} f(x) \, dx = \int_{BG}^{BG} g(x) \, dx \)
3. \( \int_{BG}^{BG} f(x) \, dx = \int_{BG}^{BG} f(x) \, dx \)
4. \( \int_{BG}^{BG} f(x) \, dx = \int_{BG}^{BG} f(x) \, dx \)

where \( r \leq \exp t \leq \exp s \)

(Grossman and Katz, 1972; Grossman 1983; Boruah and Hazarika, 2018b).

Now, we will give some standard \( BG \)-integrations:

- \( \int_{BG}^{BG} \, dx = c \)
- \( \int_{BG}^{BG} e^{x} \, dx = e^{x} \)
- \( \int_{BG}^{BG} \sin x \, dx = \cos x \)
- \( \int_{BG}^{BG} \cos x \, dx = \sin x \)
- \( \int_{BG}^{BG} \tan x \, dx = \sec x \)
- \( \int_{BG}^{BG} \sec x \, dx = \tan x \)
- \( \int_{BG}^{BG} \csc x \, dx = \cot x \)
- \( \int_{BG}^{BG} \cot x \, dx = \csc x \)

(Boruah and Hazarika, 2018b).

**Theorem 4.** (First Fundamental Theorem of \( BG \)-calculus) If \( f \) is \( BG \)-continuous function on \([r, s] \subset \mathbb{R}_{exp}\) and \( g(x) = \int_{BG}^{BG} f(t) \, dt \) for every \( x \in [r, s] \), then \( g^{BG} = f \) on \([r, s] \)(Grossman and Katz, 1972; Grossman 1983).

**Theorem 5.** (Second Fundamental Theorem of \( BG \)-calculus) If \( f^{BG} \) is \( BG \)-continuous function on \([r, s] \subset \mathbb{R}_{exp}\), then

\[
\int_{BG}^{BG} f^{BG}(x) \, dx = f(s) - f(r)
\]

(Grossman and Katz, 1972; Grossman 1983).
Definition 11. An equation involving \( BG \)-differential coefficient is called a \( BG \)-differential equation, i.e., the \( n^{th} \) order \( BG \)-differential equation is defined as 
\[
G(x,y^{BG},...,y^{(n-1)BG},y^{nBG}(x)) = 1, \quad (x,y) \in \mathbb{R}_{exp} \times \mathbb{R}_{exp}
\]
(Boruaah et al., 2018).

3. \( BG \)-Volterra Integral Equations

The equation is called \( BG \)-integral equation where an unknown function appears under the \( BG \)-integral sign. The equation
\[
u(x) = f(x) \odot \left( \lambda \odot _{BG} \int_{a}^{x} K(x,t) \odot u(t) \, dt^{BG} \right)
\]
where \( f(x) \) and \( K(x,t) \) are known functions, \( \nu(x) \) is unknown function and \( \lambda \in \mathbb{R}_{exp} \), is said to be \( BG \)-Volterra linear integral equation of the second kind. The function \( K(x,t) \) is the kernel of \( BG \)-Volterra equation. If \( f(x) = 1 \) then the equation is reduced to the following form
\[
u(x) = \lambda \odot _{BG} \int_{a}^{x} K(x,t) \odot u(t) \, dt^{BG}
\]
and it is called homogeneous \( BG \)-Volterra linear integral equation of the second kind. The equation
\[
\lambda \odot _{BG} \int_{a}^{x} K(x,t) \odot u(t) \, dt^{BG} = f(x)
\]
where \( \nu(x) \) is unknown function is called \( BG \)-Volterra linear integral equation of the first kind.

Example 1. Demonstrate that \( \nu(x) = e^{x} \) is a solution of the \( BG \)-Volterra integral equation
\[
u(x) = (x \odot e) \odot _{BG} \left( x \odot _{BG} t \right) \odot u(t) \, dt^{BG}.
\]

Solution. Substituting the function \( e^{x} \) in place of \( \nu(x) \) into the right side of the equation, then
\[
(x \odot e) \odot _{BG} \left( x \odot _{BG} t \right) \odot u(t) \, dt^{BG} = (x \odot e) \odot _{BG} \left( x \odot _{BG} t \right) \odot e^{t} \, dt^{BG} = xe^{t} \cdot e^{t} = xe^{2t} = e^{x} = \nu(x).
\]

3.1. The Relationship with \( BG \)-Differential Equations

3.1.1. The Conversion of the \( BG \)-Volterra Integral Equations to \( BG \)-Differential Equations

In this section, we demonstrate the method of converting \( BG \)-Volterra integral equations into \( BG \)-differential equations. For this, we need Leibniz formula in Bigeometric calculus. Firstly, we will give some necessary definition and theorems.

Definition 12. Let \( f \) be a bipositive function with two variables. Then, we define its \( BG \)-partial derivatives as
\[
f_{x}^{BG}(x,y) = \frac{\partial^{BG} f(x,y)}{\partial x^{BG}} = \lim_{h \to 0^{+}} \frac{f(x+h,y) \odot f(x,y)}{h} \quad \text{exp}
\]
and
\[
f_{y}^{BG}(x,y) = \frac{\partial^{BG} f(x,y)}{\partial y^{BG}} = \lim_{h \to 0^{+}} \frac{f(x,y+h) \odot f(x,y)}{h} \quad \text{exp}.
\]

From the definition of \( BG \)-partial derivative, we find its relation with classical partial derivative, as follows:
Proof. Let \( \Omega \) be an \( \mathbb{R}^{\exp} \)-open set in \( \mathbb{R}^{\exp} \times \mathbb{R}^{\exp} \). Assume that \( f : \Omega \to \mathbb{R}^{\exp} \) be a function such that the \( \mathbb{B}G \)-partial derivative \( f^{2BG}_{xy} (x,y) \), \( f^{2BG}_{yx} (x,y) \) exists in \( \Omega \) and are \( \mathbb{B}G \)-continuous, then

\[
\frac{\partial^{BG}}{\partial x^{BG}} \left( \frac{\partial^{BG}}{\partial y^{BG}} f(x,y) \right) = \frac{\partial^{BG}}{\partial y^{BG}} \left( \frac{\partial^{BG}}{\partial x^{BG}} f(x,y) \right).
\]

Proof. Fix \( x \) and \( y \) and we define \( F(h,k) \) as

\[
F(h,k) = \frac{1}{h} \exp \circ \frac{1}{k} \exp \left[ f(x \oplus h, y \ominus k) \odot f(x \ominus h, y) \odot f(x, y \ominus k) \otimes f(x, y) \right].
\]

By using the mean value theorem in the sense of \( \mathbb{B}G \)-calculus, we obtain that
\[ F(h,k) = \frac{1}{h} \exp \left( \frac{1}{k} \exp \left[ \left( f(x \oplus h, y \oplus k) \circ f(x, y) \right) \circ \left( f(x \oplus h, y \oplus k) \circ f(x, y) \right) \right] \right) \]

\[ = \frac{1}{h} \exp \left( \frac{\partial_y^B G}{\partial_y^B G} (f(x \oplus h, y \oplus \lambda_1 \oplus k) \circ f(x, y \oplus \lambda_1 \oplus k)) \right) \]

\[ = \frac{\partial_y^B G}{\partial_y^B G} \left( f(x \oplus \lambda_2 \circ h, y \oplus \lambda_2 \circ k) \circ f(x \oplus \lambda_2 \circ h, y \oplus \lambda_2 \circ k) \right) \]...
\[
\frac{d^{BG}}{dx^{BG}} F(x,v(x)) = F^{BG}_x \left( x, v(x) \right) \oplus F^{BG}_{v(x)} \left( x, v(x) \right) \odot v_s^{BG} \tag{3}
\]

from \(BG\)-chain rule. Hence by using Theorem 7, we get
\[
\frac{d^{BG}}{dx^{BG}} \left[ \int_{u(x)}^{v(x)} f(x,t) dt \right]^{BG} = F^{BG}_x \left( x, v(x) \right) \oplus F^{BG}_{u(x)} \left( x, u(x) \right) \odot v_s^{BG}
\]

from the expressions (1), (2) and (3).

**Example 2.** Show that the \(BG\)-Volterra integral equation
\[
u(x) = \sin x \oplus_{BG} \int_{1}^{x} e^t \odot u(t) dt^{BG}
\]

can be transformed to \(BG\)-differential equation.

**Solution.** If we consider the equation and differentiate it by using \(BG\)-Leibniz formula, we obtain
\[
\frac{d^{BG}}{dx^{BG}} u(x) = \frac{d^{BG}}{dx^{BG}} \left[ \int_{1}^{x} u(t) \odot e^t dt^{BG} \right]
\]

\[
u^{BG}(x) = e^{x \cot x} \oplus_{BG} \int_{1}^{x} \frac{d^{BG}}{dx^{BG}} \left( u(t) \odot e^t \right) dt^{BG} \oplus u(x) \odot e^x \odot 1^{BG}_x
\]

\[
u^{BG}(x) = e^{x \cot x} \oplus_{BG} \int_{1}^{x} e dt^{BG} \oplus u(x) \odot e^x.
\]

Thus the \(BG\)-Volterra integral equation is equivalent to the \(BG\)-differential equation
\[
u^{BG}(x) \odot u(x) \odot e^x = e^{x \cot x} \odot x.
\]

### 3.1.2. The Conversion of the \(BG\)-Linear Differential Equations to \(BG\)-Volterra Integral Equations

In this part, we prove that it is converted to \(BG\)-Volterra integral equations by defining \(BG\)-linear differential equation with constant coefficients and variable.

**Definition 12.** The equation of the form
\[
y^{BG} \oplus a_1(x) \odot y^{(n-1)BG} \oplus \cdots \oplus a_n(x) \odot y = f(x)
\]

where \(f\) is a bipositive function, is called \(n^{th}\) order \(BG\)-linear differential equation. If the coefficients \(a_i(x)\) are constants, then the equation is called as \(BG\)-linear differential equation with constant coefficients; if not it is called \(BG\)-linear differential equation with variable coefficients.

**Theorem 9.** If \(n\) is a positive integer and \(a \in \mathbb{R}_{exp}^{a}\) with \(x \geq_{exp} a\), then we have
\[
\int_{u}^{x} \cdots \cdot (n) \cdots \cdot (1) \xi_{BG} \int_{u}^{x} dt^{BG} \odot u(t) dt^{BG} = \frac{e}{(n-1)!_{exp}} \exp_{BG} \int_{u}^{x} (x \odot t)^{(n-1)_{BG}} \odot u(t) dt^{BG}
\]

**Proof.** Take
\[ I_n = \mathcal{B}G \left( \frac{1}{x} \right)^{(n-1)}_{\infty} \circ u(t) \, dt. \]  
(4)

If it is taken \( F(x,t) = (x \circ t)^{(n-1)}_{\infty} \circ u(t) \), we write

\[
\frac{d^{BG} I_n}{dx^{BG}} = \frac{d^{BG}}{dx^{BG}} \left( \int_{a}^{x} F(x,t) \, dt^{BG} \right)
= \mathcal{B}G \left[ \frac{\partial^{BG}}{\partial x^{BG}} \left( F(x,t) \right) \right] \circ \left( F(x,\infty) \circ x^{BG} \right) \circ \left( F(x,a) \circ a^{BG} \right)
= \mathcal{B}G \int_{a}^{x} e^{F(x,t)} \, dt^{BG}
\]
(5)

by using \( BG \)-Leibniz rule. Since \( F(x,t) = (x \circ t)^{(n-1)}_{\infty} \circ u(t) \), then we write

\[
F_{x}^{\prime}(x,t) = \frac{1}{x(n-1)} F(x,t) \ln u(t) \left( \frac{x}{t} \right)^{(n-2)}. \]

Therefore we find

\[
\frac{d^{BG} I_n}{dx^{BG}} = \mathcal{B}G \int_{a}^{x} e^{F(x,t)} \, dt^{BG} = \mathcal{B}G \int_{a}^{x} \left( \frac{x}{t} \right)^{(n-1)}_{\infty} \circ u(t) \, dt^{BG} = \mathcal{B}G \int_{a}^{x} \left( \frac{x}{t} \right)^{(n-1)} \circ \left( \frac{x}{t} \right)^{(n-2)} \circ u(t) \, dt^{BG}
= \mathcal{B}G \int_{a}^{x} e^{(n-1)} \circ (x \circ t)^{(n-3)}_{\infty} \circ u(t) \, dt^{BG} = \left( I_{n-1} \right)^{(n-1)} = e^{n-1} \circ I_{n-1}
\]

from the equation (5). Hence we get

\[ \frac{d^{BG} I_n}{dx^{BG}} = \left( \int_{a}^{x} (x \circ t)^{(n-3)}_{\infty} \circ u(t) \, dt^{BG} \right)^{(n-1)} = e^{n-1} \circ I_{n-1} \]  
(6)

for \( n > 1 \). Since \( I_1 = \mathcal{B}G \int_{a}^{x} u(t) \, dt^{BG} \) for \( n = 1 \), then we write

\[ \frac{d^{BG} I_1}{dx^{BG}} = \frac{d^{BG}}{dx^{BG}} \left( \int_{a}^{x} u(t) \, dt^{BG} \right) = u(x). \]  
(7)

If it is taken \( BG \)-derivative of the equation (6) by using \( BG \)-Leibniz formula, then

\[
\frac{d^{2BG} I_n}{dx^{2BG}} = \frac{d^{BG}}{dx^{BG}} \left( \int_{a}^{x} e^{n-1} \circ (x \circ t)^{(n-3)}_{\infty} \circ u(t) \, dt^{BG} \right) = \mathcal{B}G \int_{a}^{x} e^{(n-1)} \circ (x \circ t)^{(n-3)}_{\infty} \circ u(t) \, dt^{BG}
\]

By proceeding similarly, we get

\[
\frac{d^{nBG} I_n}{dx^{nBG}} = e^{(n-1)} \circ e^{(n-2)} \circ \cdots \circ e^{1} \circ I_1 = e^{(n-1)!} \circ I_1 = (n-1)!_{\infty} \circ I_1.
\]

Hence we write

\[
\frac{d^{nBG} I_n}{dx^{nBG}} = (n-1)!_{\infty} \circ u(x)
\]
from the equation (7). Now, we will take $BG$-integral by considering the above relations. From the equation (7), $I_1(x) = \int_a^x u(x) \, dx_{BG}$. Also, we have

$$I_2(x) = e^t \bigodot_{BG} \int_a^x I_1(x) \, dx_{BG} = e^t \bigodot_{BG} \int_a^x u(x) \, dx_{BG} \, dx_{BG}$$

where $x_1$ and $x_2$ are parameters. By proceeding likewise, we get

$$I_n(x) = e^{(n-1)t} \bigodot_{BG} \int_a^x \int_a^x \cdots \int_a^x u(x) \, dx_{BG} \, dx_{BG} \cdots dx_{BG}$$

$$= (n-1)! \exp \bigodot_{BG} \int_a^x \int_a^x \cdots \int_a^x u(x) \, dx_{BG} \, dx_{BG} \cdots dx_{BG}$$

where $x_1, x_2, \ldots, x_n$ are parameters. If we write the equation (4) instead of the statement $I_n$, then we find

$$\int_a^x (x \bigotimes t)^{(n-1)\omega} \bigodot u(t) \, dt_{BG} = (n-1)! \exp \bigodot_{BG} \int_a^x \int_a^x \cdots \int_a^x u(t) \, dt_{BG} \cdots dt_{BG}.$$ 

If it is taken $x = x_1 = x_2 = \cdots = x_n$, then we obtain

$$\int_a^x (x \bigotimes t)^{(n-1)\omega} \bigodot u(t) \, dt_{BG} = (n-1)! \exp \bigodot_{BG} \int_a^x \cdots \int_a^x u(t) \, dt_{BG} \cdots dt_{BG}.$$ 

Therefore, we get

$$\int_a^x (x \bigotimes t)^{(n-1)\omega} \bigodot u(t) \, dt_{BG} = e \frac{1}{(n-1)!} \exp \bigodot_{BG} \int_a^x (x \bigotimes t)^{(n-1)\omega} \bigodot u(t) \, dt_{BG}$$

and this completes the proof.

Let $n^{th}$ order $BG$-linear differential equation

$$y^{(n)BG}(x) \bigoplus a_1(x) \bigodot y^{(n-1)BG}(x) + \cdots + a_{n-1}(x) \bigodot y^{(1)BG}(x) + a_n(x) \bigodot y = f(x) \quad (8)$$

given with the initial conditions

$$y(1) = c_0, y^{(1)BG}(1) = c_1, \ldots, y^{(n-3)BG}(1) = c_{n-1} \quad (9)$$

This $n^{th}$ order $BG$-linear differential equation can be reduced to a $BG$-Volterra integral equation. Hence the solution of (8)-(9) may be reduced to a solution of some $BG$-Volterra integral equation.

Taking $y^{(n)BG} = u(x)$, we can write

$$\frac{d^{BG}}{dx^{BG}} y^{(n-1)BG}(x) = u(x).$$

By $BG$-integrating both sides of this equality,

$$\int_1^x \frac{d^{BG}}{dx^{BG}} y^{(n-1)BG}(x) = \int_1^x u(t) \, dt_{BG}$$

$$y^{(n-1)BG}(x) \bigoplus y^{(n-1)BG}(1) = \int_1^x u(t) \, dt_{BG}$$

$$y^{(n-1)BG}(x) = c_{n-1} \bigoplus \int_1^x u(t) \, dt_{BG}$$

By proceeding similarly, we find
\[ BG \int_1^BG (y^{(n-2)BG}) = BG \int_1^BG \left( c_{n-1} \oplus BG \int_1^BG u(t)dtBG \right) dtBG \]

\[ y^{(n-2)BG} (x) \circ y^{(n-2)BG} (1) = BG \int_1^BG c_{n-1} dtBG \oplus BG \int_1^BG u(t)dtBG dtBG \]

\[ y^{(n-2)BG} (x) = c_{n-2} \oplus c_{n-1} \circ x \oplus BG \int_1^BG u(t)dtBG dtBG \]

\[ y^{BG} = c_1 \oplus c_2 \circ x \oplus \cdots \oplus c_{n-1} \circ x^{(n-3)BG} \oplus BG \int_1^BG \cdots (n-1) \cdots BG \int_1^BG u(t)dtBG \cdots dtBG. \]

Therefore, we obtain,

\[ y = c_0 \oplus c_1 \circ x \oplus c_2 \circ x^{(n-2)BG} \oplus \cdots \oplus c_{n-1} \circ x^{(n-1)BG} \oplus BG \int_1^BG \cdots (n-1) \cdots BG \int_1^BG u(t)dtBG \cdots dtBG. \]

If we take into account the above expressions, the \( BG \)-linear differential equation is written as follows

\[ u(x) + a_1(x) \circ \left( c_{n-1} \oplus BG \int_1^BG u(t)dtBG \right) \oplus a_2(x) \circ \left( c_{n-2} \oplus c_{n-1} \circ x \oplus BG \int_1^BG u(t)dtBG dtBG \right) \oplus \cdots \]

\[ \oplus a_n(x) \circ \left( c_0 \oplus c_1 \circ x \oplus c_2 \circ x^{(n-2)BG} \oplus \cdots \oplus c_{n-1} \circ x^{(n-1)BG} \oplus BG \int_1^BG \cdots (n-1) \cdots BG \int_1^BG u(t)dtBG \cdots dtBG \right) = f(x) \]

\[ u(x) \oplus a_1(x) \oplus BG \int_1^BG u(t)dtBG \oplus a_2(x) \oplus BG \int_1^BG u(t)dtBG \oplus \cdots \oplus a_n(x) \oplus BG \int_1^BG \cdots (n-1) \cdots BG \int_1^BG u(t)dtBG \cdots dtBG = \]

\[ f(x) \circ \left[ a_1(x) \oplus a_2(x) \oplus \cdots \oplus a_n(x) \oplus x^{(n-1)BG} \oplus c_{n-1} \oplus \left( a_1(x) \oplus a_2(x) \oplus \cdots \oplus a_n(x) \right) \oplus x^{(n-2)BG} \oplus c_{n-2} \oplus \cdots \oplus a_n(x) \right] \]

\[ F(x) = f(x) \circ f_{n-1}(x) \circ c_{n-1} \circ f_{n-2}(x) \circ c_{n-2} \cdots \circ f_0(x) \circ c_0 \]

Then, one can see that the equation (10) is in the following form:

\[ u(x) + a_1(x) + BG \int_1^BG u(t)dtBG \oplus a_2(x) + BG \int_1^BG u(t)dtBG \oplus \cdots \oplus a_n(x) + BG \int_1^BG \cdots (n-1) \cdots BG \int_1^BG u(t)dtBG \cdots dtBG = F(x). \]

By using Theorem 9, we get

\[ u(x) + a_1(x) + BG \int_1^BG u(t)dtBG \oplus a_2(x) + \frac{e}{1!exp} \circ a_2(x) \oplus (x \circ t)^y \oplus \cdots \oplus \frac{e}{(n-1)!exp} \circ a_n(x) \oplus (x \circ t)^{y(BG)} \circ u(t)dtBG = F(x). \]

If we edit this equation as

\[ u(x) + BG \int_1^BG [G(x) + \frac{e}{1!exp} \circ a_2(x) \oplus (x \circ t)^y \oplus \cdots \oplus \frac{e}{(n-1)!exp} \circ a_n(x)] \]

and set

\[ K(x,t) = a_1(x) + \frac{e}{1!exp} \circ a_2(x) \oplus (x \circ t)^y + \cdots + \frac{e}{(n-1)!exp} \circ a_n(x) \]

as the kernel function, then the equation (8) is turned into
\[ u(x) \odot_{BG} \int_1^x u(t) \odot_{BG} K(x,t) \, dt_{BG} = F(x) \]

which is a \( BG \)-Volterra integral equation of the second kind.

**Example 3.** Find \( BG \)-Volterra integral equation corresponding to the \( BG \)-differential equation:

\[ y^{2_{BG}} \odot_{BG} x \odot_{BG} y^{BG} = e^2 \]  

with the initial conditions \( y(1) = e \), \( y^{BG}(1) = 1 \).

**Solution.** Let \( \frac{d^{2_{BG}}}{dx^{2_{BG}}} y^{2_{BG}} = u(x) \). Since \( \frac{d^{2_{BG}}}{dx^{2_{BG}}} \frac{d^{BG}}{dx^{BG}} y^{BG} = u(x) \), then we write

\[ \int_1^x u(t) \, dt_{BG} \]

\[ y^{BG}(x) \odot_{BG} y^{BG}(1) = \int_1^x u(t) \, dt_{BG} \]

\[ y^{BG}(x) = \int_1^x u(t) \, dt_{BG} \]

Therefore, we find

\[ \int_1^x d^{BG}_{BG} y = \int_1^x \int_1^x u(t) \, dt_{BG} \, dt_{BG} \]

\[ y(x) \odot y(1) = e^{\exp_{BG} \int_1^x (x \odot t) \, dt_{BG}} \]

\[ y(x) = e \odot_{BG} \left( e \odot_{BG} \int_1^x (x \odot t) \, dt_{BG} \right) \]

If we replace the findings above into the given \( BG \)-differential equation, we obtain

\[ u(x) \odot x \odot_{BG} \int_1^x u(t) \, dt_{BG} \odot e \odot_{BG} \left( e \odot_{BG} \int_1^x (x \odot t) \, dt_{BG} \right) = e^2 \]

From this, we get \( BG \)-Volterra integral equation as

\[ u(x) = e \odot_{BG} \left( x^2 \odot t \right) \, dt_{BG} \].

**4. Conclusion**

In this paper, the Volterra integral equations are defined in the sense of bigeometric calculus by using the concept of bigeometric integral. The Leibniz formula is proved in bigeometric calculus and aid of this the bigeometric Volterra integral equations are converted to bigeometric differential equations. By defining the bigeometric linear differential equations with constant coefficients and variable coefficients, they are converted to bigeometric Volterra equations is proved.

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