DAHA-JONES POLYNOMIALS OF TORUS KNOTS

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Abstract. DAHA-Jones polynomials of torus knots $T(r, s)$ are studied systematically for reduced root systems and in the case of $C\vee C_1$. We prove the polynomiality and evaluation conjectures from the author's previous paper on torus knots and extend the theory by the color exchange and further symmetries. DAHA-Jones polynomials for $C\vee C_1$ depend on 5 parameters. Their surprising connection to the DAHA-superpolynomials (type $A$) for the knots $T(2p + 1, 2)$ is obtained, a remarkable combination of the color exchange conditions and the author's duality conjecture (justified by Gorsky and Negut). The DAHA-superpolynomials for symmetric and wedge powers (and torus knots) conjecturally coincide with the Khovanov-Rozansky stable polynomials, those originated in the theory of BPS states and the superpolynomials defined via rational DAHA in connection with certain Hilbert schemes, though not much is known about such connections beyond the HOMFLYPT and Kauffman polynomials. We also define certain arithmetic counterparts of DAHA-Jones polynomials for the absolute Galois group instead of torus knots in the case of $C\vee C_1$.

Key words: double affine Hecke algebra; Jones polynomials; Khovanov-Rozansky homology; torus knots; Macdonald polynomials; Askey-Wilson polynomials; Verlinde algebra; absolute Galois group; Chern-Simons theory.

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0. Introduction

This paper is a systematic exposition of the new theory of DAHA-Jones polynomials of torus knots $T(r,s)$ for reduced root systems and in the case of $C^\vee C_1$. We prove the (key) Polynomiality Conjecture from [Ch4] and the Evaluation Conjecture there, also extending the theory by the important Color Exchange, the symmetry $(r,s) \mapsto (s,r)$ and further properties of DAHA-Jones polynomials and superpolynomials. We mention that the justification of the Stabilization Conjecture in type $A$ is provided in [GN]. It was announced as proven in [Ch4] (with a link to [SV]); it justifies the existence of DAHA-superpolynomials.

Superpolynomials. The DAHA-superpolynomials of $T(r,s)$ conjecturally coincide with the Poincarè polynomials of stable triply graded Khovanov-Rozansky homology of torus knots for $sl_{n+1}$ as $n \to \infty$ [KhR1, KhR2, Ras]; see also [Web, Rou] and references there concerning the categorification. Only reduced Jones and other polynomials (with value 1 at the unknot) are considered in this paper, as well as in [Ch4]. Such Connection Conjecture was checked for HOMFLYPT and Kauffman polynomials [Ch4, Ste] and when the Khovanov-Rozansky polynomials are available; there is also a link to the Heegard-Floer homology. Our color exchange has something in common with the so-called colored differentials (see [GGS]), but there are significant differences and the existence of the latter is a conjecture.

Relying on such a connection, DAHA provide a self-consistent powerful approach to torus knots (arbitrary root systems and weights). It is relatively simple. Only the positivity for the rectangles and the stabilization-duality in types $B–C–D$ remain open among the intrinsic properties of the DAHA-polynomials conjectured in [Ch4]. The $B–C–D$ systems are more natural to address within the $C^\vee C_n$–theory, to be considered in author’s further works. The system $C^\vee C_n$, the most general classical one, can be managed following the present paper. We note that counterparts of the properties of DAHA-superpolynomials are generally difficult challenges in the Khovanov-Rozansky theory.

Let us briefly mention other approaches to superpolynomials. The (physics) theory of the BPS states is used, which is not mathematically rigorous but fruitful; see e.g. [DGR, AS, FGS, DMS]. We mention that the duality for symmetric and wedge powers was conjectured in [GS]. The rational DAHA can be employed here (in connection...
with certain Hilbert schemes); see [GORS, GN] and references there. The superpolynomials obtained via rational DAHA perfectly match our DAHA-superpolynomials, but this approach is restricted to symmetric or wedge powers so far. We do not discuss this direction and related important geometric developments; the link to our superpolynomials has now no explanation (in spite of using DAHA in both theories).

The $C^\vee C_1$–case. The DAHA-Jones polynomials in this case depend on 5 parameters and are related to those of type $A_1$ in nontrivial ways. Their surprising connection to the DAHA-superpolynomials (type $A$) for $T(2p+1, 2)$ is discovered; its proof is a remarkable combination of the color exchange and the Duality Conjecture from [Ch4], justified in [GN]. We will outline our approach to the proof of the latter based on the new $q, t$–level-rank duality, which deserves a separate paper.

The relation of $C^\vee C_1$ to superpolynomials requires special parameters. Geometric meaning of the whole set of parameters is unclear at the moment. This restricts potential value of the general $C^\vee C_n$–theory, though there are solid reasons to follow this avenue.

We note that the $C^\vee C_1$–theory somewhat deviates from that in [Ch4] (any reduced root systems). For instance, the symmetry $(r, s) \mapsto (s, r)$ holds in this case only if $rs$ is even. Also, the DAHA-Jones polynomials for $(2p+1, 1)$ become nontrivial for odd $p > 1$, which are exactly the ones resulting in the $A$–type superpolynomials for $T(2p+1, 2)$. For any $s$ and odd $r$, the DAHA-Jones polynomials for $(r, s)$ in type $C^\vee C_1$ coincide with those of type $A_1$ for $T(r, 2s)$ under certain specialization of the parameters, which is, by the way, related to the connection between the continued fraction of $2s/r$ and $s/r$ (Hurwitz).

Our formula for the superpolynomials of torus knots $T(2p+1, 2)$ (conjectured in [DMS, FGS]) is of independent interest due to the connection to (half-differentials of) modular forms via the WRT and Kashaev invariants; see [Hi]. Hopefully it will clarify and generalize the modularity properties of the Jones polynomials. Its proof is a convincing demonstration of the efficiency of the DAHA-based approach.

At the end of the paper, counterparts of DAHA-Jones polynomials in type $C^\vee C_1$ are defined for the absolute Galois group instead of torus knots, which is based on the rigidity of (“major”) perfect DAHA-modules at roots of unity in the terminology of [Ch1]. We follow [Ch5], though do not have the classification of such perfect modules for $C^\vee C_1$. See [OS] concerning the rigidity for generic $q$ in the case of $C^\vee C_1$. 
We would like to mention paper [BS], which approaches the skein modules (for sl$_2$) via DAHA of type $C_1^\vee C_1$. The knots $T(r,2)$ for odd $r$ are among the main examples there, but not only torus knots are considered in this paper. We see no connection with our work at the moment. Actually, such a connection is more likely at the level of $A$–polynomials (see e.g., [FGS]), which are beyond the present work.

1. Reduced root systems

1.1. Affine Weyl group. Let $R = \{\alpha\} \subset \mathbb{R}^n$ be a root system of type $A, B, C, D, E, F, G$ with respect to a euclidean form $(z, z')$ on $\mathbb{R}^n \ni z, z'$, $W$ the Weyl group generated by the reflections $s_\alpha$, $R_+$ the set of positive roots corresponding to fixed simple roots $\alpha_1, \ldots, \alpha_n$, $R_- = -R_+$. The form is normalized by the condition $(\alpha, \alpha) = 2$ for short roots. The root lattice and the weight lattice are:

$$Q = \oplus_{i=1}^n \mathbb{Z}\alpha_i \subset P = \oplus_{i=1}^n \mathbb{Z}\omega_i,$$

where $\{\omega_i\}$ are fundamental weights: $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ for the coroots $\alpha^\vee = 2\alpha/((\alpha, \alpha))$. Replacing $\mathbb{Z}$ by $\mathbb{Z}^+ = \{m \in \mathbb{Z}, m \geq 0\}$, we obtain $Q^+, P^+$. Here and further see e.g., [Bo] or [Ch1].

Setting $\nu_\alpha \overset{\text{def}}{=} (\alpha, \alpha)/2$, the vectors $\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \mathbb{R}^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$ for $\alpha \in R, j \in \mathbb{Z}$ form the twisted affine root system $\tilde{R} \supset R$ ($z \in \mathbb{R}^n$ are identified with $[z, 0]$). We add $\alpha_0 \overset{\text{def}}{=} [-\vartheta, 1]$ to the simple roots for the maximal short root $\vartheta \in R_+$. The corresponding set $\tilde{R}_+$ of positive roots is $R_+ \cup \{[\alpha, \nu_\alpha j], \alpha \in R, j > 0\}$.

The set of the indices of the images of $\alpha_0$ by all automorphisms of the affine Dynkin diagram will be denoted by $O$ ($O = \{0\}$ for $E_8, F_4, G_2$). Let $O' \overset{\text{def}}{=} \{r \in O, r \neq 0\}$. The elements $\omega_r$ for $r \in O'$ are minuscule weights, defined by the inequalities $(\omega_r, \alpha^\vee) \leq 1$ for all $\alpha \in R_+$. We set $\omega_0 = 0$ for the sake of uniformity.

Given $\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \tilde{R}$, $b \in P$, let

$$s_\tilde{\alpha}(\tilde{z}) = \tilde{z} - (z, \alpha^\vee)\tilde{\alpha}, \quad b'(\tilde{z}) = [z, \zeta - (z, b)]$$

for $\tilde{z} = [z, \zeta] \in \mathbb{R}^{n+1}$. The affine Weyl group $\tilde{W} = \langle s_\tilde{\alpha}, \tilde{\alpha} \in \tilde{R}_+ \rangle$ is the semidirect product $W \ltimes Q$ of its subgroups $W = \langle s_\alpha, \alpha \in R_+ \rangle$ and $Q$, where $\alpha$ is identified with

$$s_\alpha s_{[\alpha, \nu_\alpha]} = s_{[-\alpha, \nu_\alpha]} s_\alpha \quad \text{for} \quad \alpha \in R.$$
The extended Weyl group \( \hat{W} \) is \( W \ltimes P \), where the corresponding action in \( \mathbb{R}^{n+1} \) is
\[
(wb)([z,\zeta]) = [w(z),\zeta - (z,b)] \quad \text{for} \quad w \in W, b \in P.
\]

It is isomorphic to \( \hat{W} \ltimes \Pi \) for \( \Pi \defeq P/Q \). The latter group consists of \( \pi_0 = \text{id} \) and the images \( \pi_r \) of minuscule \( \omega_r \) in \( P/Q \).

The group \( \Pi \) is naturally identified with the subgroup of \( \hat{W} \) of the elements of the length zero; the length is defined as follows:
\[
l(\hat{w}) = |\lambda(\hat{w})| \quad \text{for} \quad \lambda(\hat{w}) \defeq \hat{R}_+ \cap \hat{w}^{-1}(-\hat{R}_+).
\]

One has \( \omega_r = \pi_r u_r \) for \( r \in O' \), where \( u_r \) is the element \( u \in W \) of minimal length such that \( u(\omega_r) \in -P_+ \).

Setting \( \hat{w} = \pi_r \hat{w} \in \hat{W} \) for \( \pi_r \in \Pi, \hat{w} \in \hat{W} \), \( l(\hat{w}) \) coincides with the length of any reduced decomposition of \( \hat{w} \) in terms of the simple reflections \( s_i, 0 \leq i \leq n \). We will also use the partial lengths \( l_{\text{sht}}, l_{\text{lng}} \), which count correspondingly short and long \( s_i \) in reduced decompositions.

1.2. Definition of DAHA. We follow [Ch4, Ch1]. Let \( m, b \) be the least natural number such that \( (P,P) = (1/m)\mathbb{Z} \). Thus \( m = |\Pi| \) unless \( m = 2 \) for \( D_{2k} \) and \( m = 1 \) for \( B_{2k}, C_k \).

The double affine Hecke algebra, \( \text{DAHA} \), depends on the parameters \( q, t_\nu (\nu \in \{\nu_\alpha\}) \) and will be defined over the ring \( Z_{q,t} \defeq \mathbb{Z}[q^{\pm 1/m}, t_\nu^{\pm 1/2}] \) formed by polynomials in terms of \( q^{\pm 1/m} \) and \( \{t_\nu^{1/2}\} \). Note that the coefficients of the Macdonald polynomials will belong to \( \mathbb{Q}(q,t_\nu) \).

For \( \alpha = [\alpha,\nu_\alpha] \in \hat{R}, \ 0 \leq i \leq n \), we set
\[
t_{\alpha} = t_\alpha = t_{\nu_\alpha} = q_{\alpha_i}^{k_\nu}, \quad q_{\alpha} = q^{\nu_\alpha}, \quad t_i = t_{\alpha_i}, \quad q_i = q_{\alpha_i},
\]

Also, using here and below \( sht, lng \) instead of \( \nu \), let
\[
\rho_k \defeq \frac{1}{2} \sum_{\alpha > 0} k_\alpha \alpha = k_{\text{sht}} \rho_{\text{sht}} + k_{\text{lng}} \rho_{\text{lng}}, \quad \rho_\nu = \frac{1}{2} \sum_{\nu_\alpha = \nu} \alpha = \sum_{\nu_i = \nu, j > 0} \omega_i.
\]

For pairwise commutative \( X_1, \ldots, X_n \),
\[
X_\tilde{b} \defeq \prod_{i=1}^n X_i^{l_i} q^{j} \quad \text{if} \quad \tilde{b} = [b,j], \ \hat{w}(X_\tilde{b}) = X_{\hat{w}(\tilde{b})},
\]

where \( b = \sum_{i=1}^n l_i \omega_i \in P, \ j \in \frac{1}{m} \mathbb{Z}, \ \hat{w} \in \hat{W}. \)
For instance, $X_0 \overset{\text{def}}{=} X_{\alpha_0} = qX_{\phi}^{-1}$.

We use that $\pi_r^{-1}$ is $\pi_{r(i)}$, where $i$ is the standard involution of the nonaffine Dynkin diagram, induced by $\alpha_i \mapsto -w_0(\alpha_i)$; $w_0$ is the longest element in $W$. We set $m_{ij} = 2, 3, 4, 6$ when the number of links between $\alpha_i$ and $\alpha_j$ in the affine Dynkin diagram is $0, 1, 2, 3$. Recall that $\omega_r = \pi_r u_r$ for $r \in O'$ (see above).

**Definition 1.1.** The double affine Hecke algebra $\mathcal{H}$ is generated over $\mathbb{Z}_{q,t}$ by the elements $\{T_i, 0 \leq i \leq n\}$, pairwise commutative $\{X_b, b \in P\}$ satisfying (1.3) and the group $\Pi$, where the following relations are imposed:

- (o) $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$, $0 \leq i \leq n$;
- (i) $T_i T_j T_i \ldots = T_j T_i T_j \ldots$, $m_{ij}$ factors on each side;
- (ii) $\pi_r T_i \pi_r^{-1} = T_j$ if $\pi_r(\alpha_i) = \alpha_j$;
- (iii) $T_i X_b = X_b X_{\alpha_i^{-1} T_i^{-1}}$ if $(b, \alpha_i) = 1$, $0 \leq i \leq n$;
- (iv) $T_i X_b = X_i T_i$ if $(b, \alpha_i) = 0$ for $0 \leq i \leq n$;
- (v) $\pi_r X_b \pi_r^{-1} = X_{\pi_r(b)} = X_{w^{-1}_r(b) q^{(\omega_r(b),b)}}$, $r \in O'$.

Given $\tilde{w} \in \tilde{W}, r \in O$, the product

$$T_{\pi_r \tilde{w}} \overset{\text{def}}{=} \pi_r T_i \ldots T_i$$

where $\tilde{w} = s_{i_1} \ldots s_{i_l}$ for $l = l(\tilde{w})$, does not depend on the choice of the reduced decomposition. Moreover,

$$T_{\tilde{v}} T_{\tilde{w}} = T_{\tilde{v} \tilde{w}}$$

whenever $l(\tilde{v} \tilde{w}) = l(\tilde{v}) + l(\tilde{w})$ for $\tilde{v}, \tilde{w} \in \tilde{W}$.

In particular, we arrive at the pairwise commutative elements:

$$Y_b \overset{\text{def}}{=} \prod_{i=1}^n Y_i^{l_i} \text{ if } b = \sum_{i=1}^n l_i \omega_i \in P, Y_i \overset{\text{def}}{=} T_{\omega_i}, b \in P.$$

**1.3. The automorphisms.** We will begin with the anti-involution

$$X_b^* = X_b^{-1}, \quad Y_b^* = Y_b^{-1}, \quad T_i^* = T_i^{-1},$$

$$t_i^v \mapsto t_i^{-v}, \quad q^v \mapsto q^{-v}, \quad 0 \leq i \leq n,$$

where $v \in \mathbb{Q}$ ($v \in \frac{1}{2m}\mathbb{Z}$ will, actually, be sufficient). See [Ch1], (3.2.18). This is the group inversion for any products of the generators $X_b, Y_b, T_i$, as well as for $q^{1/m}, t_i^{1/2}$, and therefore commutes with all automorphisms and anti-automorphisms below.
The following maps can be (uniquely) extended to an automorphism of $\mathcal{H}$, where $q^{1/(2m)}$ must be added to $\mathbb{Z}_{q,t}$ (see [Ch1], (3.2.10)-(3.2.15)):

\begin{align}
1.8) \quad & \tau_+ : X_b \mapsto X_b, \ T_i \mapsto T_i (i > 0), \ Y_r \mapsto Y_r q^{\frac{(q^r-q^r)}{2}}, \\
& \tau_+ : T_0 \mapsto q^{-1} X_{\theta} T_0^{-1}, \ \pi_r \mapsto q^{\frac{(q^r-q^r)}{2}} X_r \pi_r (r \in O'), \\
1.9) \quad & \tau_- : Y_b \mapsto Y_b, \ T_i \mapsto T_i (i \geq 0), \ X_r \mapsto Y_r X_r q^{\frac{(q^r-q^r)}{2}}, \\
& \tau_- (X_{\theta}) = q T_0 X_{\theta}^{-1} T_{\theta}^{-1}; \ \sigma \overset{\text{def}}{=} \tau_+ \tau_- \tau_+ = \tau_- \tau_+ \tau_-^{-1}, \\
1.10) \quad & \sigma (X_b) = Y_b^{-1}, \ \sigma (Y_b) = T_{w_0}^{-1} X_b^{-1} T_{w_0}, \ \sigma (T_i) = T_i (i > 0).
\end{align}

These automorphisms fix $t_\nu, q$ and their fractional powers, as well as the following anti-involution:

\begin{equation}
1.11) \quad \phi : X_b \mapsto Y_b^{-1}, \ Y_b \mapsto X_b^{-1}, \ T_i \mapsto T_i (1 \leq i \leq n).
\end{equation}

Extending the standard involution \( \iota \) of the nonaffine Dynkin diagram used above, let $\iota(b) = b'$ if $b \in P$ and the longest element $w_0 \in W$. The relations

\begin{equation}
1.12) \quad \iota(X_b) = X_{\iota(b)}, \ \iota(Y_b) = Y_{\iota(b)}, \ T_i' = T_{\iota(i)}
\end{equation}

can be uniquely extended the automorphism $H \mapsto \iota(H) \in \mathcal{H}$. Proposition 3.2.2 from [Ch1] states that

\begin{equation}
1.13) \quad \sigma^2 (H) = T_{w_0}^{-1} H' T_{w_0} \quad \text{for} \quad H \in \mathcal{H}.
\end{equation}

We will also need the involutions $\varepsilon \overset{\text{def}}{=} \varphi \cdot \ast = \ast \cdot \varphi$ (here $\cdot$ is used for the composition) and $\eta \overset{\text{def}}{=} \varepsilon \sigma = \sigma^{-1} \varepsilon$:

\begin{align}
1.14) \quad & \varepsilon : X_b \mapsto Y_b, \ Y_b \mapsto X_b, \ T_i \mapsto T_i^{-1} (1 \leq i \leq n), \\
1.15) \quad & \eta : T_i \mapsto T_i^{-1}, \ X_b \mapsto X_b^{-1}, \ \pi_r \mapsto \pi_r (0 \leq i \leq n),
\end{align}

where $b \in P, \ r \in O'$.

Both “conjugate” $t, q$; namely, $t_\nu' \mapsto t_\nu^{-\nu}, q^\nu' \mapsto q^{-\nu}$. The involution $\eta$ extends the Kazhdan–Lusztig involution in the affine Hecke theory; see [Ch1], (3.2.19-22). Note that $\varepsilon \tau_{\pm} = \tau_{\pm} = \varphi \tau_{\pm} \varphi = \sigma \tau_{\pm}^{-1} \sigma^{-1}, \ \eta \tau_{\pm} \eta = \tau_{\pm}^{-1}, \ \varphi \sigma \varphi = \sigma^{-1} = \eta \sigma \eta$.

Also, $\varphi \varepsilon = \varepsilon \varphi$ and $\varphi \eta \varphi = \eta \sigma^{-2} = \sigma^2 \eta$. Let us list the matrices corresponding to the automorphisms and anti-automorphisms above upon the natural projection onto $GL_2(\mathbb{Z})$, corresponding to $t_\nu' = 1 = q^\nu$.
for any rational \( v \) \((v \in \frac{1}{2m} \mathbb{Z} \text{ is sufficient})\). The matrix \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) will represent the map \( X_b \mapsto X_b^\alpha Y_b^\gamma, Y_b \mapsto X_b^\beta Y_b^\delta \) for \( b \in P \). One has:

\[
\tau_+ \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \tau_- \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma^2 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
\varepsilon \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \varphi \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \eta \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \phi \sigma^2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

We define the projective \( GL_2^\wedge(\mathbb{Z}) \) as the group generated by \( \tau_\pm, \eta \) subject to the relations \( \tau_+ \tau_-^{-1} \tau_+ = \tau_-^{-1} \tau_+ \tau_-^{-1}, \quad \eta^2 = 1 \) and \( \eta \tau_\pm \eta = \tau_\pm^{-1} \). The span of \( \tau_\pm \) is the projective \( PSL_2^\wedge(\mathbb{Z}) \) (due to Steinberg), which is isomorphic to the braid group \( B_3 \).

### 1.4. Macdonald polynomials

Following [Ch1], we use the PBW theorem to express any \( H \in \mathcal{HH} \) in the form \( \sum_{a,w,b} c_{a,w,b} X_a T_w Y_b \) for \( w \in W, \ a, b \in P \) (this presentation is unique). Then we substitute:

\[
(1.16) \quad \{ \} \mapsto q^{-(\rho_k, a)}, \quad Y_b \mapsto q^{(\rho_k, b)}, \quad T_i \mapsto t_i^{1/2}.
\]

By construction, the resulting functional \( \mathcal{HH} \ni H \mapsto \{ H \}_ev \) acts via the projection \( H \mapsto H(1) \) of \( \mathcal{HH} \) onto the polynomial representation \( V \), which is the \( \mathcal{HH} \)–module induced from the one-dimensional character \( T_i(1) = t_i^{-1/2} = Y_i(1) \) for \( 1 \leq i \leq n \) and \( T_0(1) = t_0^{-1/2} \); recall that \( t_0 = t_{\text{sh}} \). Explicitly, \( \{ H \}_ev = H(1)(q^{-\rho_k}) \); see [Ch1, Ch4].

In detail, the polynomial representation \( V \) is isomorphic to \( \mathbb{Z}_{q,t}[X_b] \) as a vector space and the action of \( T_i(0 \leq i \leq n) \) there is given by the Demazure-Lusztig operators:

\[
(1.17) \quad T_i = t_i^{1/2} s_i + (t_i^{1/2} - t_i^{-1/2})(X_{a_i} - 1)^{-1}(s_i - 1), \quad 0 \leq i \leq n.
\]

The elements \( X_b \) become the multiplication operators and \( \pi_r(r \in O') \) act via the general formula \( \hat{w}(X_a) = X_{\hat{w}(b)} \) for \( \hat{w} \in \hat{W} \). Note that \( \tau_- \) and \( \eta \) naturally act in the polynomial representation. For the latter,

\[
(1.18) \quad \eta(f) = f^*, \quad \text{where} \quad X_b^* = X_{-b}, \quad (q^v)^* = q^{-v}, \quad (t^v)^* = t^{-v} \quad \text{for} \quad v \in \mathbb{Q}.
\]

One has the following relations:

\[
(1.19) \quad \{ \varphi(H) \}_ev = \{ H \}_ev, \quad \{ \iota(H) \}_ev = \{ H \}_ev,
\]

\[
\{ \eta(H) \}_ev = \{ H \}^*_ev, \quad \{ \sigma^2(H) \}_ev = \{ H \}_ev.
\]
The Macdonald polynomials $P_b(X)$ (due to Kadell for the classical root systems) are uniquely defined as follows. For $b \in P_+$,

$$
P_b = \sum_{\nu' \in W(b)} X_{\nu'} \in \oplus_{b',c \in b+Q_+} \mathbb{Q} q_{a,t}^d X_c,$$

and $CT(P_b X_{\mu}(X; q, t)) = 0$,

$$
c_+ \in W(c) \cap P_+, \quad \mu(X; q, t) \overset{\text{def}}{=} \prod_{\alpha \in R^+} \prod_{j=0}^{\infty} \frac{(1-X_\alpha q_\alpha^j)(1-X^{-1}_\alpha q_\alpha^{j+1})}{(1-X_\alpha t_\alpha q_\alpha^j)(1-X^{-1}_\alpha t_\alpha q_\alpha^{j+1})}.
$$

Here $CT$ is the constant term; $\mu$ is considered a Laurent series of $X_b$ with the coefficients expanded in terms of positive powers of $q$. The coefficients of $P_b$ belong to the $\mathbb{Q}(q, t)$ (see also below). One has:

$$
(1.20) \quad P_b(X^{-1}) = P_{b'}(X) = P_b(X), \quad P_b(q^{-\rho_b}) = P_b(q^\rho_b)
$$

$$
(1.21) \quad (P_b(q^{-\rho_b}))^* = q^{-(\rho_b, b)} \prod_{\alpha > 0} \prod_{j=0}^{(\alpha', b)-1} \frac{1 - q_\alpha^j t_\alpha X_\alpha(q^\rho_b)}{1 - q_\alpha^j X_\alpha(q^\rho_b)}.
$$

DAHA provides an important alternative (operator) approach to $P$-polynomials; namely, they satisfy the defining relations

$$
(1.22) \quad L(f(P_b)) = f(q^{\rho_b+b}) P_b, \quad L \overset{\text{def}}{=} f(X_a \mapsto Y_a)
$$

for any symmetric ($W$-invariant) polynomial $f \in \mathbb{C}[X_a, a \in P]^W$.

These polynomials are $t$-symmetrizations of the nonsymmetric Macdonald polynomials $E_b \in \mathcal{V}$, which are defined for $b \in P_+$ from the relations $Y_a(E_b) = q^{-(a, b+w_b(\rho_b))} E_b$ for all $a$. Here $w_b$ is the element of maximal length in the centralizer of $b$ in $W$. The normalization here is by the condition that the coefficient of $X_b$ in $E_b$ is 1. More exactly, $E_b - X_b \in \oplus_{b',c \in b+Q_+} \mathbb{Q}(q, t) X_c$. See [Mac] and (6.14) from [Ch2] or (3.3.14) from [Ch1] (the differential version is due to Opdam):

$$
(1.23) \quad E_b(q^{-\rho_b}) = q^{-(\rho_b, b)} \prod_{\alpha > 0} \prod_{j=1}^{(\alpha', b)-1} \frac{1 - q_\alpha^j t_\alpha X_\alpha(q^\rho_b)}{1 - q_\alpha^j X_\alpha(q^\rho_b)} \quad \text{for} \quad b \in P_+.
$$

For any $b \in P$, we define $E$-polynomials as follows:

$$
Y_a(E_b) = q^{-(a, b+w_b(\rho_b))} E_b, \quad \text{where the coefficient of } X_b \in E_b \text{ is 1,}
$$

and $w_b \in W$ is of maximal possible length such that $b \in w_b(P_+)$.

The technique of intertwining operators for the $E$-polynomials results in the following existence criterion. Assuming that $q$ is not a root of unity, the products $den_b \cdot E_b$, $den_b \cdot P_b$ and $num_b \cdot E_b/E_b(q^{-\rho_b})$
for $b \in P_+$ are well defined for the numerator $\text{num}_b$ and the denominator $\text{den}_b$ of the expression for $E_b(q^{-\rho_k})$ from (1.23). Note that all binomials in $\text{num}_b$, $\text{den}_b$ involve $q$ and $t$ (both). Also:

$$P_b(q^{-\rho_k}) = \Pi_R^b E_b(q^{-\rho_k}) \quad \text{for} \quad b \in P_+, \quad \Pi_R^b = \prod_{\alpha>0,(\alpha,b)>0} \frac{1 - t_\alpha X_\alpha(q^{\rho_k})}{1 - X_\alpha(q^{\rho_k})}.$$  

The product $\Pi_R^b$ becomes the Poincaré polynomial $\Pi_R$ for generic $b \in P_+$. See formulas (6.33), (7.15) from [Ch2] and (3.3.45) from [Ch1].

Setting $E_b^\circ = E_b/E_b(q^{-\rho_k})$ for any $b$ and $P_b^\circ = P_b/P_b(q^{-\rho_k})$ for $b \in P_+$, the following relation holds for any $b \in P$:

$$(1.24) \quad P_{b+}^\circ = (\Pi_R)^{-1} \mathcal{P}(E_b^\circ) \quad \text{for} \quad \mathcal{P}_+ \overset{\text{def}}{=} \sum_{w \in W} t_{\text{shl}}^{1/2} t_{\text{lng}}^{(w)/2} T_w,$$

where $t_{\text{shl}}, t_{\text{lng}}$ count correspondingly the number of $s_i$ for short and long $\alpha_i$ in any reduced decomposition $w = s_{i_1} \cdots s_{i_l}$. We check that the right-hand side here is proportional to $P_{b+}$ and then evaluate at $q^{-\rho_k}$ by applying the general formula $\{T_i(f)\}_{\text{ev}} = t_i^{1/2} \{f\}_{\text{ev}}$ for $i > 0$.

The following particular case of (1.24) will be mainly used:

$$\prod_{\alpha>0,(\alpha,b)=0} \frac{1 - t_\alpha X_\alpha(q^{\rho_k})}{1 - X_\alpha(q^{\rho_k})} P_b = \mathcal{P}_+(E_b) \quad \text{for} \quad b \in P_+.$$  

1.5. DAHA-Jones polynomials. The following theorem is the first part of Conjecture 2.1 from [Ch4] extended as follows. We enlarge $\text{PSL}_2^\wedge(\mathbb{Z})$ used there to $\text{GL}_2^\wedge(\mathbb{Z})$, consider in detail the symmetry $(r,s) \mapsto (s,r)$ and the mirror images in (1.27), and add important color-exchange relations (1.29); last but not least, the justifications are provided.

We represent a given torus knot $T_r(s)$ by a matrix $\gamma_{r,s} \in \text{GL}_2(\mathbb{Z})$ such that its first column is $(r,s)^{tr}$ ($tr$ is the transposition), assuming that $\text{gcd}(r,s) = 1$; $r, s$ can be 0 or negative. Then $\hat{\gamma}_{r,s} \in \text{GL}_2^\wedge(\mathbb{Z})$ is by definition a pullback of $\gamma_{r,s}$.

Note that any $(r,s)$ can be obviously lifted to $\gamma$ of determinant 1 and, accordingly, to $\hat{\gamma}$ from the subgroup $\text{PSL}_2^\wedge$ generated by $\{r\pm\}$, i.e. without using $\eta$. This is sufficient for the construction of the DAHA-Jones polynomials below; using $\text{GL}_2(\mathbb{Z})$ instead of $\text{PSL}_2(\mathbb{Z}) \subset \text{GL}_2(\mathbb{Z})$ is equivalent to the conjugation relation from (1.27).

For a polynomial $F$ in terms of positive and negative fractional powers of $q$ and $t_\nu$, the tilde-normalization $\tilde{F}$ will be the result of division
by the lowest $q,t_{w}$-monomial (assuming that it is well defined); $\tilde F$ contains only non-negative powers of $q,t_{w}$. In the following theorem, the lowest monomials always exist and the powers are integral.

**Theorem 1.2.**  

(i) **Polynomiality.** Given $T(r,s)$, a root system $R$ and a weight $b \in P_{+}$, we set

$$
J_{D_{r,s}}^{R}(b; q, t) = \frac{\tilde{\gamma}_{r,s}(P_{b})/P_{b}(q^{-\rho_{b}}) }{E_{c}(q^{-\rho_{c}}) } \text{ for } c \in W(b),
$$

$$
J_{D_{r}}(b; q, t) = \frac{\tilde{\gamma}_{r,s}(E_{c})/E_{c}(q^{-\rho_{c}}) }{E_{c}(q^{-\rho_{c}}) } \text{ for } c \in W(b),
$$

$$
J_{D_{r,s}}^{R}(b; q, t) = \{ \tilde{\gamma}_{r,s}(P_{b})/P_{b}(q^{-\rho_{b}}) \}_{ev}.
$$

Then $J_{D_{r,s}}^{R}(b; q, t)$, the tilde-normalization of the (reduced) DAHA-Jones polynomial, does not depend on the particular choice of $\gamma_{r,s} \in GL_{2}(\mathbb{Z})$ and $\tilde{\gamma}_{r,s} \in GL_{2}(\mathbb{Z})$. It is a polynomial in terms of $q, \{ t_{w} \}$. Moreover, using the conjugation involution $\ast$ from (1.18),

$$
J_{D_{r,s}}^{R}(b; q, t) = \{ \tilde{\gamma}_{r,s}(P_{b})/P_{b}(q^{-\rho_{b}}) \}_{ev} \ast \text{ for } c \in W(b),
$$

$$
\tilde{J}_{D_{r,1}}(b; q, t) = 1, \quad J_{D_{r,s}}^{R}(b; q, t) = (J_{D_{r,s}}^{R}(b; q, t))^{\ast},
$$

(1.27)  

$$
J_{D_{r,s}}^{R}(b; q, t) = J_{D_{s,r}}(b; q, t) = J_{D_{s,-r}}(b; q, t).
$$

(ii) **Color Exchange.** Let us assume that for $b,c \in P_{+}$ and certain $\nu \notin w \in W$,

$$
q^{(b+\rho_{b}-w(\rho_{c})-w(c), \alpha)} = q^{(b-w(c), \alpha')}_{sht} (\rho_{w}, \alpha')_{sht} (\rho_{w}, \alpha')_{ling}^{\ast}
$$

for any $\alpha \in R_{+}$, where $q$ is not a root of unity and $\rho_{b}^{w} \overset{\text{def}}{=} w(\rho_{b}) - \rho_{b}$. For instance, taking $P_{+} \ni b = w(\rho_{k}) - \rho_{k} + w(c)$ for a given $c \in P_{+}$ is sufficient. The product in the right-hand side is in terms of integral powers of $q_{c}$ and $t_{w}^{\nu}$, where $\rho_{b}^{w} \in l_{\nu}P$ for $l_{\nu} \in \mathbb{Z}_{+}$. Then

$$
J_{D_{r,s}}^{R}(b; q, t) = J_{D_{r,s}}^{R}(c; q, t) \text{ for such } q, \{ t_{w} \} \text{ and any } r,s.
$$

In particular, this holds for $w = s_{i}(i > 0), c \in P_{+}$ and for

$$
b = c - (k_{i} + (c, \alpha_{j}^{\vee})) \alpha_{i} \text{ provided } 2k_{i} + (c, \alpha_{j}^{\vee}) \in -\mathbb{Z}_{+} \text{ and } (c, \alpha_{j}^{\vee}) + \nu(k_{i} + (c, \alpha_{j}^{\vee})) \in \mathbb{Z}_{+} \text{ for all } j > 0 \text{ such that } (\alpha_{i}, \alpha_{j}) < 0,
$$

where $\nu = \nu_{i}$ for short $\alpha_{j}$ and $\nu = 1$ otherwise (thus $k_{i} \in -\mathbb{Z}_{+}$ unless for $A_{1}$ and if $\alpha_{i}$ is long for $C_{n\geq 2}$). Another particular case of (1.28) is $w = w_{0}$ with $2k_{\nu} \in -\mathbb{Z}_{+}$, where

$$
b = -2\rho_{k} + w_{0}(c) \text{ provided } (c, \alpha_{i}^{\vee}) \leq -2k_{i} \text{ for all } i > 0.
$$
(iii) **Evaluation.** Relation (1.28) results in (1.29) for any \( q \), including roots of unity, due to the polynomiality of \( JD_{r,s}(b; q,t) \). Upon \( q \to 1 \), one has \( E_{\omega_0}(b) = P_b \) and \( P_{b+c} = P_b P_c \) for \( b, c \in P_+ \). Accordingly, 

\[
(1.32) \quad \tilde{JD}_{r,s}(\sum_{i=1}^n b_i \omega_i ; q = 1, t) = \prod_{i=1}^n \tilde{JD}_{r,s}(\omega_i ; q = 1, t)^{b_i} \quad \text{for any } r, s.
\]

**Proof.** By construction, \( \tilde{JD}_{r,s}^R(b; q,t) \) depends only on the first column of the matrix \( \gamma = \gamma_{r,s} \). The switch from \( P_b^\circ = P_b / P_b(q^{-\rho_k}) \) to \( E_b^\circ = E_b / E_b(q^{-\rho_k}) \) is straightforward, since one can perform the \( t \)-symmetrization inside (1.26) using formula (1.24). Let us deduce the symmetries in (1.26) from (1.19). First of all, (1.12) and (1.19) result in \( \iota \cdot \gamma = \gamma \cdot t \) for any \( \gamma \in GL_2(\mathbb{Z}) \) and therefore provide 

\[
(1.33) \quad JD_{r,s}^R(\iota(b) ; q,t) = JD_{r,s}^R(b ; q,t).
\]

Then, using the \( \varphi \)-invariance of \( \{ \cdot \}_e \), 

\[
(1.34) \quad \{ \varphi(\cdots \tau_+^\beta \tau_-^\alpha(P_b^\circ)) \}_e = \{ \cdots \tau_-^\beta \tau_+^\alpha(P_b^\circ(Y^{-1})) \}_e
\]

\[
= \{ \cdots \tau_-^\beta \tau_+^\alpha(P_b^\circ(X)) \}_e = \{ \cdots \tau_-^\beta \tau_+^\alpha \sigma(P_b^\circ) \}_e,
\]

which proves that \( JD_{r,s}(b; q,t) = JD_{-s,-r}(b; q,t) \). Similarly, 

\[
\{ \eta(\cdots \tau_+^\beta \tau_-^\alpha(P_b^\circ)) \}_e = \{ \cdots \tau_-^\beta \tau_+^\alpha((P_b^\circ)^*) \}_e
\]

\[
= \{ \cdots \tau_-^\beta \tau_+^\alpha(P_{i(b)}^\circ) \}_e = \{ \cdots \tau_-^\beta \tau_+^\alpha(P_b^\circ) \}_e = \{ \cdots \tau_-^\beta \tau_+^\alpha(P_b^\circ)^* \}_e,
\]

which results in \( JD_{r,s}(b; q,t) = JD_{r,-s}(b; q,t)^* \). Combining this symmetry with the previous one, 

\[
JD_{r,s}(b; q,t) = JD_{-s,-r}(b; q,t)^* = JD_{-r,s}(b; q,t)^* = JD_{r,-s}(b; q,t),
\]

The latter symmetry can be established directly following (1.34):

\[
\{ \varphi(\cdots \tau_+^\beta \tau_-^\alpha(P_b^\circ)) \}_e = \{ \cdots \tau_-^\beta \tau_+^\alpha(P_b^\circ(Y^{-1})) \}_e
\]

\[
= \{ \cdots \tau_-^\beta \tau_+^\alpha \sigma^{-1}(P_{i(b)}^\circ(X)) \}_e = \{ \cdots \tau_-^\beta \tau_+^\alpha \sigma^{-1}(P_b^\circ(X)) \}_e,
\]

where we use (1.13) and (1.33).

Finally for \( b \in P_+ \), 

\[
JD_{r,1}(b; q,t) = JD_{1,r}(b; q,t) = \{ \tau_-(P_b^\circ) \}_e = \{ \tau_-^r \cdot P_b^\circ \cdot \tau_-^{-r}(1) \}_e
\]

\[
= \{ \tau_-^r(P_b^\circ) \}_e = \{ q^{-r(b,b)/2-r(b,\rho_k)} P_b^\circ \}_e = q^{-r(b,b)/2-r(b,\rho_k)},
\]

where \( \tau_-^{\pm r}(-) \}_e \) means the action in \( \mathcal{V} \); see formula (1.37) below.
Polynomiality of JD. We will use the radical, Rad, of the evaluation pairing defined as follows:
\[
\{E,F\}_{cv} = E(Y^{-1})(F(X))(t^{-\rho}), \quad E,F \in \mathcal{V}.
\]

Theorem 11.8 from [Ch2] gives the necessary and sufficient conditions for \( \text{Rad} \neq \{0\} \) for generic \( q \). For instance (though we do not really need this), the radical is nonzero in the case of \( ADE \) if and only if
\[
t = q^{-\frac{l}{m_n}} \zeta_l^j \quad \text{for} \quad 1 \leq i \leq n, \quad 0 \leq j, j' \leq m_i, \quad j+j'>0, \quad l \in \mathbb{Z}_+,
\]
where \( t = t_{\text{sh}t}, \zeta_l \) are primitive \((m_i+1)\)th roots of unity for the classical exponents \( m_i \) (see [Bo]). This is Theorem 11.1 from [Ch2].

If \( JD_{r,s}(b;q,t) \) is not a polynomial for \( b \in P_+ \) and admissible \( r,s \), then \( E^\circ_b = E_b/E_b(q^{-\rho_0}) \) has a pole at \( \epsilon = 0 \) of order \( l > 0 \) for \( \epsilon \) equal to one of the binomials \( (1 - q_\rho^j t_\alpha^p) \) in the numerator \( \text{num}_b \) of (1.23) for certain \( j > 0, p > 0 \). Recall that the inequality \( j > 0 \) here is due to using \( E \)-polynomials; see (1.26). Here and further we will localize and complete the ring of constants \( \mathbb{Z}_{q,t} = \mathbb{Z}[q^{1/m}, t^{1/2}] \) with respect to \( \epsilon \), i.e. for the principle ideal \( (1 - q_\rho^j t_\alpha^p) \); the notations will be \( \mathbb{Z}_{q,t}^{(\epsilon)}, \mathcal{V}^{(\epsilon)} \).

For the sake of definiteness, we will take a maximal binomial here, i.e. such that \( (1 - q_\rho^j t_\alpha^p) \) is not in \( \text{num}_b \) for any \( \mathbb{Z} \ni v > 1 \). Accordingly, we introduce the following filtration of submodules of \( \mathcal{V} \):
\[
(1.35) \quad \text{Rad}_{\epsilon,t} = \{ F \in \mathcal{V} \mid \{ F, \mathcal{V} \}_{cv} \in \epsilon^\ell \mathbb{Z}_{q,t} \} \quad \text{for} \quad \ell \in \mathbb{Z}_+.
\]

setting \( \text{Rad}(\epsilon = 0) = \text{Rad}_{\epsilon,\infty} \); we add \( q^{1/(2m)} \) to \( \mathbb{Z}_{q,t}^{(\epsilon)} \) here and below.

Note that it is generally not impossible that such \( (1 - q_\rho^j t_\alpha^p) \) coincides with one of the binomials in the denominator \( \text{den}_b \) of (1.23). As a matter of fact, this is not the case due to the consideration below, but we do not really need this fact.

The polynomials \( E'_b = \epsilon^\ell E_b^\circ, P'_b = \epsilon^\ell P_b^\circ \in \mathcal{V}^{(\epsilon)} \) are eigenfunctions respectively for \( \{ Y_a \} \) and \( \{ L_f \} \) from (1.22). Moreover, \( E'_b(q^{-\rho_0}) \in \epsilon^\ell \mathbb{Z}_{q,t}^{(\epsilon)} \supset P'_b(q^{-\rho_0}) \). Therefore \( E'_b \in \text{Rad}_{\epsilon,t} \supset P'_b \); see e.g., Lemmas 11.4-5 from [Ch2]. Thus \( \text{Rad}(\epsilon = 0) \neq \{0\} \) and, for instance, Theorem 11.8 there implies that \( (1 - q_\rho^j t_\alpha^p) \) does not coincide with the binomials in \( \text{den}_b \) from (1.23) (as it was claimed above).

Let us now switch from \( \text{Rad} \) to
\[
(1.36) \quad \text{RAD}_{\epsilon,t} \overset{\text{def}}{=} \{ H \in \mathcal{H} \mid \{ \mathcal{H}, H, \mathcal{H} \}_{cv} \in \epsilon^\ell \mathbb{Z}_{q,t} \} \quad \text{for} \quad \ell \in \mathbb{Z}_+.
\]
We set $RAD(\epsilon = 0) = RAD_{\epsilon,\infty}$. Equivalently, one has:

$$RAD_{\epsilon,t} = \{ H \in \mathcal{H} \mid H(\mathcal{V}) \subset Rad_{\epsilon,t} \},$$

since $Rad_{\epsilon,t} = \{ F \in \mathcal{V} \mid \{ \mathcal{H}(F) \}_{ev} \in \epsilon^t\mathbb{Z}_{q,t}^{(\epsilon)} \}$; see Lemma 11.3, [Ch2].

Here $q$ is not a root of unity. Therefore Proposition 3.2 from [Ch3] states that any $Y -$invariant submodule of $\mathcal{V}$ is invariant with respect to the natural action of $\tau_-$ in $\mathcal{V}$. For instance,

$$\tau_-(E_b) = q^{-(b, b)/2 - (b, \rho_k)} E_b \quad \text{and} \quad \tau_-(P_b) = q^{-(b, b)/2 - (b, \rho_k)} P_b,$$

assuming that $E_b$ for $b \in P_+$ is well defined. See also Proposition 3.3.4 from [Ch1].

Therefore $\psi$ and $\tau_-$ preserve $RAD_{\epsilon,r}$ for any $r \in \mathbb{Z}_+$ (and generic $q$), as well as $\eta$. Thus the whole $GL_n^{(e)}(\mathbb{Z})$ fixes each $RAD_{\epsilon,r}$. This implies that $\tilde{\gamma}(P_b^\circ) \in RAD_{\epsilon,t}$ and $\{ \tilde{\gamma}(P_b^\circ) \}_{ev}$ is divisible by $\epsilon^t$. Hence $\widetilde{JD}_{r,s}(b; q, t)$ has no singularity at $\epsilon = 0$, which contradiction is sufficient to claim polynomiality of $\widetilde{JD}_{r,s}(b; q, t)$ for any $b \in P_+$ and $r, s$.

**Parts** (ii, iii). The justification of Part (ii) of the theorem is quite similar. Let $\epsilon$ be 1 minus the right-hand side of (1.28) and $l \in \mathbb{Z}_+$ is minimal such that $\epsilon^l P_b^\circ$ and $\epsilon^l P_c^\circ$ are regular in $\mathcal{V}^{(\epsilon)}$; as above, $\mathcal{V}^{(\epsilon)}$ is defined over $\mathbb{Z}_{q,t}^{(\epsilon)}$. The conditions from (1.28) are necessary and sufficient for these polynomials to have coinciding eigenvalues for any $L_f$ from (1.22). This coincidence results in the relations $\epsilon^l(P_b^\circ - P_c^\circ) \in Rad_{\epsilon,t+1}$, which gives the required.

Note that given $c, q, t_w$, there always exists $b \in P_+$ satisfying (1.28) such that the polynomial $P_b^\circ$ is regular at $\epsilon = 0$. The regularity of $JD_{r,s}(b; q, t)$ from (i) is granted automatically for such $b$. Moreover one such $b$ can be canonically constructed in terms of the right Bruhat ordering defined for the root subsystem of $\tilde{R}$ associated with the stabilizer of $\rho_k$ in $\hat{W}$. It was called primary in [Ch2]; see formula (9.2) from Section 9.1 there.

It is not really necessary to assume in (1.28) that $q$ is not a root of unity, since we have the polynomiality of the $JD-$polynomials. Let us make $q = 1$. Then the technique of intertwining operators of $E-$polynomials readily results in the formula $E_{\omega_0(b)} = P_b$ for $b \in P_+$ and the multiplicative property $P_b P_c = P_{b+c}$ for $b, c \in P_+$. We use that $E_{s_i(b)} = (s_i + 1) E_b$ for $b \in P$ assuming that $(\alpha_i, b) > 0$ for $i > 0$. Also, $E_{b+u \vartheta} = X_\vartheta s_\vartheta(E_b) + E_b$ if $u = 1 - (\vartheta, b) > 0$ and $E_{b+u \omega_r} = X_{\omega_r} \pi_r(E_b)$
for \( r \in O' \). See e.g., Proposition 3.3.5 from [Ch1]. Therefore

\[
\hat{\gamma}(P_b) = \prod_{i=1}^{n} \hat{\gamma}(P_{\omega_i})^{b_i} \text{ for } b = \sum_{i=1}^{n} b_i \omega_i \in P_.
\]

We need to apply this operator to \( 1 \in \mathcal{V} \). Firstly, \( L_f \) is a constant, which is \( \{ f \} = f(t_{\text{sh}t} \rho_0 \rho_{\text{ling}}) \), when acting on symmetric polynomial \( F \in \mathcal{V}_W \) for any symmetric \( f \) due to \( q = 1 \); see (1.22). Secondly, the elements \( f(X) \) and \( L_f \) are central in \( \mathcal{H} \) if \( q^{1/m} = 1 \), as well as any operators \( \hat{\gamma}(f(X)) \) for symmetric \( f \) and any \( \gamma \in GL^2(\mathbb{Z}) \). See Theorem 3.8.5 from [Ch1]. The action of \( GL^2(\mathbb{Z}) \) in the center of \( \mathcal{H} \) at \( q = 1 \) and at the roots of unity is important in DAHA theory and have geometric applications. For instance, we have that \( f(Y) = \{ f \} \) in the whole \( \mathcal{V} \) for symmetric \( f \) and \( q^{1/m} = 1 \). Thirdly and finally,

\[
\hat{\gamma}(P_b)(F \in \mathcal{V}_W) = \prod_{i=1}^{n} (\hat{\gamma}(P_{\omega_i})(F))^{b_i} \text{ for } \gamma = \gamma_{r,s} \text{ and }
\]

\[
JD_{r,s}(b; q=1, t) = \left\{ (\hat{\gamma}(P_b)(1 \in \mathcal{V}_W) \right\} = \prod_{i=1}^{n} \left\{ (\hat{\gamma}(P_{\omega_i})(1))^{b_i} \right\}.
\]

We mention that Part (ii) leads to resonance \( \tilde{\mathcal{D}} \)-polynomials, which are the limits of linear combinations of \( \tilde{\mathcal{D}}_{r,s}(b; q,t) \) for the same \( q,t \) and for \( b \) from (ii) corresponding to a given \( c \). We divide such linear combinations by the leading powers of \( \epsilon \) before taking the limit \( \epsilon \to 0; \epsilon \) is 1 minus the right-hand side of (1.28). They are related to the nonsemisimple Macdonald polynomials from [Ch2] in the spherical normalization, but we will not touch this upon in the present paper.

1.6. Superpolynomials. Theorem 1.2 has the following extension to the DAHA- superpolynomials in type \( A_n \). The following stabilization theorem was announced in [Ch4]; it was mentioned there (Section 2.4.1, “Confirmations”) that its justification is similar to Lemma 4.3 and formula (4.1) from [SV]. The complete proof is published in [GN] (following [SV]). We switch to \( A_n \) and naturally set \( t = t_{\text{sh}t} = q^k \).

**Theorem 1.3.** (i) For the root system \( A_n \), let us consider \( P_+ \ni b = \sum_{i=1}^{n} b_i \omega_i \) as a (dominant) weight for any \( A_N \) with \( N \geq n - 1 \), where we set \( \omega_n = 0 \) upon the restriction to \( A_{n-1} \). Then given \( T(r,s) \), there exists \( \mathcal{H}_{r,s}(b; q,t,a) \), a polynomial in terms of \( a, q, t^{\pm 1} \), such that its
coefficient of $a^0$ is tilde-normalized (i.e. in the form $\sum_{u,v \geq 0} C_{u,v} q^u t^v$ with $C_{0,0} = 1$) and the following specializations hold:

\[(1.39) \quad \mathcal{H}_{r,s}(b; q, t, a = -t^{N+1}) = \tilde{\mathcal{D}}_{r,s}^{A_N}(b; q, t) \quad \text{for any} \quad N \geq n - 1.\]

(ii) Imposing the Color Exchange relation (1.28), we will consider $w$ there as an element of $S_{N+1}$ for every $N \geq n$ (the Weyl group for $A_N$) naturally acting in the corresponding $\mathcal{P}$. Then given $r, s$ and up to a proper power $q^* t^*$,

\[(1.40) \quad \mathcal{H}_{r,s}(b; q, t, a) = q^* t^* \mathcal{H}_{r,s}(c; q, t, a) \quad \text{for such } q, \{t_\nu\}.\]

In particular, let $w = s_i = (i, i + 1)$ with $i < n$. Then for a dominant $c$ and $b = c - (k + (c, \alpha_i))\alpha_i$, the components of $b$ are

\[(1.41) \quad b_i = -2k - c_i, \quad b_j = c_j + c_i + k \quad \text{for } j = i \pm 1 > 0, \quad b_j = c_j \quad \text{otherwise}.\]

Here $k \in -\mathbb{Z}_+$ satisfies the relations $c_i/2 \leq -k \leq c_i + \min\{c_i+1 > 0\}$, which are necessary and sufficient for $b \in \mathcal{P}_+$. 

(iii) Making $q = 1$, one has:

\[(1.42) \quad \mathcal{H}_{r,s}(b; q = 1, t, a) = \prod_{i=1}^{n} \mathcal{H}_{r,s}(\omega_i; q = 1, t, a)^{b_i} \quad \text{for } b = \sum_{i=1}^{n} b_i \omega_i.\]

This gives that the $a$-degree $\deg_a(\mathcal{H}_{r,s}(b; q, t, a))$ of $\mathcal{H}_{r,s}(b; q, t, a)$ equals $\min(|r|, |s|)$ times the number of boxes in the Young diagram $\lambda_b$ associated with $b \in \mathcal{P}_+$. These claims are from Conjecture 2.6 of [Ch4]. □

The conjectural relation of DAHA-superpolynomials to the stable Khovanov-Rozansky polynomials for $sl_{n+1}$ is as follows: $a \mapsto t^{n+1} \sqrt{t/q}$. Note that this is not the substitution from (1.39), but they are linked. See Section 2.3 in [Ch4] for discussion and some references; in particular, formula (2.12) there connects our (DAHA-based) parameters with the standard ones. Such relations match the conjectural links between the Khovanov-Rozansky homology [KhR1, KhR2, Ras] and the superpolynomials introduced via the BPS states [DGR, AS, FGS, GGS] as well as those obtained in terms of rational DAHA, which are deeply related to certain Hilbert schemes [GORS, GN].

The coincidence was confirmed when the stable Khovanov-Rozansky polynomials are available and for $\mathcal{H}_{r,s}(b; q, q, -a)$, which do coincide with the corresponding HOMFLYPT polynomial. See Proposition 2.3.
from [Ch4], [Ste] and [RJ] (concerning the Jones polynomials and Quantum groups) and references there. When \( n + 1 = 0 \), the relation to the Heegard- Floer homology of torus knots is expected.

**Color exchange combinatorially.** We associate with \( c = \sum_{i=1}^{n} c_i \omega_i \) in Part (ii) the Young diagram

\[
\lambda_c = \{ m_1 = c_1 + \ldots + c_n, m_2 = c_1 + \ldots + c_{n-1}, \ldots, m_n = c_n, 0, 0, \ldots \}
\]

Then we switch to \( \lambda'_c = \{ m'_i = m_i - k(i - 1) \} \), apply \( w \in W \) to \( \lambda'_c \) and finally obtain

\[
\lambda_b = \{ m'_{w(i)} + k(i - 1), \} = \{ m_{w(i)} + k(i - w(i)) \}.
\]

Here \( w \) transforms the rows of \( \lambda \) and we set \( w\{m_1,m_2,\ldots,m_n\} = \{m_{w(1)},m_{w(2)},\ldots,m_{w(n)}\} \). Given \( k < 0 \) (it can be fractional), \( \lambda_b \) must be a Young diagram, which determines the set of all \( w \) that can be used. Note that when \( k = +1 \) (formally), the procedure \( \lambda_c \mapsto \lambda_b \) is actually from the Jacobi-Trudi formula.

Let us briefly discuss the Duality Conjectures 2.5 from [Ch4]. It states that

\[
H_{r,s}(\lambda; q, t, a) = q^* t^* H_{r,s}(\lambda^t; t^{-1}, q^{-1}, a),
\]

where we switch from weights to the corresponding Young diagrams and \( \lambda^t \) is the transposition of \( \lambda \). This was justified in [GN] using the modified Macdonald polynomials. Let us outline a direct proof (which is expected to work for \( \mathbb{C}^N \mathbb{C}_n \)). We use the perfect representation at \( t = q^{-(s+1)/(n+1)} \), which is defined as \( V/\text{Rad} \) for \( s \in \mathbb{Z}_+ \) provided \( \gcd(s+1,n+1) = 1 \). More generally, the Coxeter number must be used here instead of \( (n+1) \). See [Ch1]. Then \( a = -t^{n+1} = -(q^{-1})^{s+1} \) and we can identify the left-hand side of (1.44) for \( A_n \) with the right-hand side for \( A_s \). This identification goes via the theory at roots of unity; we compare the zeros in both sides of (1.44).

**More on Part (iii).** Formula (1.42) follows directly from (1.32). Let us mention that the formulas for \( H_{r,s}(\omega_i; q = 1, t, a) \) are not difficult to calculate for simple knots, but generally they are involved (even for \( q = 1 \)). Note that (1.32) combined with the duality results in

\[
H_{r,s}(\lambda; q, t = 1, a) = \prod_{i=1}^{n} H_{r,s}(m_i \omega_1; q, t = 1, a)^{m_i},
\]

where \( m_i \) is the number of boxes in the \( i \)th row of \( \lambda \). A direct justification of (1.45) without using the duality is not clear at the moment.
Concerning the $a$–degrees there, it is relatively straightforward to justify that $\deg_a(JD_{r,s}(\lambda; q, t, a)) \leq \min(|r|, |s|) \deg(\lambda)$; see e.g. [GN]. Then the evaluation formula at $q = 1$ (or even that for $q = 1 = t$) gives that we actually have the equality.

1.7. Examples of color exchange. To begin with, let $i = 1, k = -2, c = \omega_1 + \omega_2, b = 3\omega_1$ in Part (ii) of the theorem. According to Section 3.1 from [Ch4],

$$\mathcal{H}_{3,2}(\omega_1 + \omega_2; q, t, a) =$$

$$1 + \frac{a^3 q^6}{q - t} + 2qt - qt^2 + 2q^2 t^2 + q^3 t^2 - q^2 t^3 + 2q^3 t^3 - q^3 t^4 + 2q^4 t^4 + q^5 t^5$$

$$+ a(2q^2 + q^3 + \frac{2}{q} - q^2 t + 3q^3 t + q^4 t - q^4 t^2 + 3q^4 t^2 + q^5 t^2 - q^4 t^3 + 2q^5 t^3 + q^6 t^4)$$

$$+ a^2(q^4 + q^5 + \frac{q^5}{q} + \frac{q^5}{q} - q^4 t + q^5 t + q^6 t + q^6 t^2).$$

This is, by the way, the simplest example of a DAHA-superpolynomial with negative coefficients. Conjecturally, the negative terms are not present only for rectangle Young diagrams; see [Ch4]. This is in obvious contrast with formula (4.14) from [GGS] for the 3-hook. It resembles our one (both have the same number of terms), but has no negative terms. The $t$–powers are odd and even there (they are even in our one upon using the standard parameters $q, t, a$). Formula (4.14) is of course a suggestion, not the result of a formal calculation.

The second superpolynomial is

$$\mathcal{H}_{3,2}(3\omega_1; q, t, a) =$$

$$1 + a^3 q^{12} + q^3 t + q^4 t + q^5 t + q^6 t^2 + q^7 t^2 + q^8 t^2 + q^9 t^3$$

$$+ a(q^3 + q^4 + q^5 + q^6 t + 2q^7 t + 2q^8 t + q^9 t + q^{10} t^2 + q^{11} t^2)$$

$$+ a^2(q^7 + q^8 + q^9 + q^{10} t + q^{11} t + q^{12} t).$$

Combinatorially, the procedure is as follows:

$$\lambda_c = \begin{array}{c|c|c|c|c|c|c} \hline & + & + & k\rho' & \hline \hline + & + & \hline \end{array} \quad \rightarrow \quad \begin{array}{c|c|c|c|c|c|c} \hline & + & + & \hline \hline + & + & \hline \end{array} \quad \rightarrow \quad \begin{array}{c|c|c|c|c|c|c} \hline & + & + & \hline \hline + & + & \hline \end{array} = \lambda_b.$$

Here $-\rho'$ is a stable re-normalization of $-\rho$ given by the diagram \{0, 1, 2, 3, \cdots\}. The boxes added due to the translation by $k\rho' = -2\rho$ are marked by +. Generally, we can add fractional boxes here if $k$ is fractional (recall that it is always negative).

We obtain that

$$q^4 \mathcal{H}_{3,2}(\omega_1 + \omega_2; q, t, a) = \mathcal{H}_{3,2}(3\omega_1; q, t, a) \quad \text{for} \quad t = q^{-2}.$$
The corresponding resonance $JD$–polynomial is
\[
\lim_{\epsilon \to 0} \left( t^q \mathcal{H}_{3,3}(\omega_1 + \omega_2; q, t, a) - \mathcal{H}_{3,3}(3\omega_1; q, t, a) \right)/\epsilon \quad \text{for} \quad \epsilon = 1 - tq^2.
\]

Since $\mathcal{H}_{3,3}(\omega_1 + \omega_2; q, t, a)$ contains negative terms, a remarkable cancelation of monomials occurs in this polynomials upon $t = q^{-2}$. Relation (1.46) holds for any $r, s$ with a proper coefficient of proportionality, which is $q^{12}$ for the knot $T(3, 4)$.

Let us take now $c = \omega_1 + \omega_2, w = s_1s_2s_1, k = -1/2$. Then $b = \omega_3$ and we obtain that $\tau^* \mathcal{H}_{r,s}(\omega_1 + \omega_2; q, t, a) = \mathcal{H}_{r,s}(\omega_3; q, t, a)$ for $q = t^{-2}$; the coefficient of proportionality is $t^4$ for $T(3, 2)$.

The corresponding combinatorial transformation is as follows:

\[
\lambda_c \rightarrow \begin{array}{c} +1 \\ k \end{array} \rightarrow \begin{array}{c} +1 \\ +1 \\ +1 \\ +1 \end{array} \rightarrow \begin{array}{c} +1 \\ \downarrow \end{array} \rightarrow \begin{array}{c} +1 \\ +1 \\ +1 \\ +1 \end{array} \rightarrow \lambda_b.
\]

The output coincides with (1.46) if the duality (1.44) is employed.

The following two examples deal with more involved $w$. Let us take $w^{(1)} = s_1s_2 \cdots s_p$ and $w^{(2)} = s_p \cdots s_2s_1$ for $k = -\ell$ and the corresponding weights $c^{(1)} = \ell\omega_{p+1}, c^{(2)} = \ell p \omega_{p+1}$. Then one finds that

\[
b^{(1)} = \ell(p+1)\omega_1 \quad \text{and} \quad b^{(2)} = \ell(p+1)\omega_p.
\]

We apply here the general relation $\rho - w(\rho) = \sum_{\alpha \in R_\rho \cap w(R_\rho)} \alpha$ and formulas $\alpha_i = 2\omega_i - \omega_{i-1} - \omega_{i+1}$, which hold in any $A_n$ if $n > i$ ($\omega_0 = 0$). See below the combinatorial interpretation of this calculation.

Thus we obtain that

\[(1.47) \quad \mathcal{H}_{r,s}(c^{(i)}; q, t=q^{-\ell}, a) = q^i \mathcal{H}_{r,s}(b^{(i)}; q, t=q^{-\ell}, a) \quad \text{for} \quad i = 1, 2.\]

When $\ell = 1$, the Young diagrams for $\{\omega_{p+1}, (p+1)\omega_1\}$ and those for $\{p\omega_{p+1}, (p+1)\omega_p\}$ are transposed to each other. Therefore (1.47) follow from (1.44) for $\ell = 1$; the relation $tq = 1$ is obviously preserved by the duality transformation $t \leftrightarrow q^{-1}$.

Combining (1.47) for $\ell = 1$ and $i = 1$ with its restriction to $A_n$ for $n = p+1$, where $\omega_n$ and $\omega_1$ result in coinciding $\widehat{JD}$–polynomials,

\[
\mathcal{H}_{r,s}(\omega_1; q, t = q^{-1}, a = -\ell^{n+1}) = q^i \mathcal{H}_{r,s}(n\omega_1; q, t = q^{-1}, a = -\ell^{n+1}).
\]
Here we see some links to the so-called colored differentials; see e.g., [DGR, GS, FGS, GGS]. However our color exchange from (1.40) preserves the number of boxes of Young diagrams, in contrast to the colored differentials from [GGS] and other papers. Also, the differentials there do not seem mathematically rigorous (the differentials from [KhR1, KhR2, Ras] are) and it is surprising to us that the colored differentials are introduced only for symmetric and wedge powers.

**Fat hooks.** Generalizing, let us consider a fat hook corresponding to $c = v_1 \omega_{u_1 + u_2} + v_2 \omega_{u_1}$ for $u_1, u_2, v_1, v_2 \in \mathbb{Z}_+$. Let $k = -\ell \in -\mathbb{Z}_+$ (can be fractional) subject to the following relations:

\[
(1.48) \quad v_1 + v_2 \geq u_2\ell \in \mathbb{Z}_+ \ni (u_1 + u_2)\ell \geq v_2.
\]

We set $b = (v_1 + v_2 - u_2\ell)\omega_{u_1 + u_2} + ((u_1 + u_2)\ell - v_2)\omega_{u_2}$ and claim that

\[
(1.49) \quad \mathcal{H}_{r,s}(c; q, t = q^{-\ell}, a) = q^r \mathcal{H}_{r,s}(b; q, t = q^{-\ell}, a) \quad \text{for any } r, s.
\]

Combinatorially, $u_1, u_2, v_1, v_2$ are the number of lines and columns in the Young diagram for $c$. This diagram and the corresponding Young block-diagram for $b$ are as follows:

\[
\lambda_c = \begin{array}{c} u_1 \\ u_2 \end{array} \begin{array}{c} v_1 \\ v_2 \end{array}, \quad \lambda_b = \begin{array}{c} u_2 \\ u_1 \end{array} \begin{array}{c} v_1 + v_2 - u_2\ell \\ (u_1 + u_2)\ell - v_2 \end{array}.
\]

The permutation $w$ transposes the two block-lines of the diagram $\lambda_c$, i.e. it moves the first $u_1$ lines of $\lambda_c$ down and the last $u_2$ lines up (without changing their relative positions within the corresponding block-lines). The integrality condition for $k = \ell$ is $\gcd(u_1, u_2)\ell \in \mathbb{Z}_+$. For instance, $\ell = 1/u$ is possible if $u_1 = u = u_2$; in this case, the inequalities from (1.48) become $v_1 + v_2 \geq 1, v_2 \leq 2$ and the pair $v_1, v_2$ will be transformed to $v_1 + v_2 - 1, 2 - v_2$.

Note that when $\ell = 1$ (i.e. $k = -1$) and $u_1 = v_2, u_2 = v_1$, the diagram $\lambda_b$ is the transposition of $\lambda_c$. Then (1.49) follows from the duality.

**Rectangles.** This construction has the following application to the Young diagrams that are rectangles. Setting $v_1 = 0 = v_1 + v_2 - u_2\ell = v_2 - u_2\ell$, one has $c = v_2\omega_{u_1} = u_2\ell\omega_{u_1}$ and $b = u_1\ell\omega_{u_2}$. Thus the rectangles $\lambda_c = \{u_2\ell \times u_1\}$ and $\lambda_b = \{u_1\ell \times u_2\}$ satisfy (1.49). Here $u_1$ and $u_2$ are arbitrary positive, $\gcd(u_1, u_2)\ell$ must be integral. Note that one can set here $v_2 = 0 = v_1 + v_2 - u_2\ell = v_1 - u_2\ell$. The corresponding
weights become $c = u_2 \ell \omega_{u_1 + u_2}$ and $b = (u_1 + u_2) \ell \omega_{u_2}$; thus we arrive at a particular case of the previous relation (use $u_1 + u_2 > u_2$). Finally:

\begin{align}
(1.50) \quad q^* \mathcal{H}_{r,s}(u \omega_{v}; q, q^{-u}, a) &= q^* \mathcal{H}_{r,s}(u \omega_{v}; q, q^{-u}, a), \\
i^* \mathcal{H}_{r,s}(u \omega_{v}; t^{-v}, t, a) &= i^* \mathcal{H}_{r,s}(u \omega_{v}; t^{-v}, t, a),
\end{align}

where the second formula is due to the duality (1.44). Actually it can be justified directly (similarly to the proof of the first formula).

For instance, $i^* \mathcal{H}_{3,2}(2 \omega_2; t^{-2}, t, a) = \mathcal{H}_{3,2}(\omega_4; t^{-2}, t, a)$ for

\[
\mathcal{H}_{3,2}(2 \omega_2; q, t, a) = 1 + \frac{q^2}{q^3 + q^5 + q^7 + q^{10} + q^{12} + q^{16}} + \frac{q^6}{q^9 + q^{10} + q^{12} + q^{14}} + \frac{q^{12}}{q^{15} + q^{18} + q^{20} + q^{22} + q^{24}} + \frac{q^{18}}{q^{21} + q^{24} + q^{26} + q^{28} + q^{30}} + \frac{q^{24}}{q^{27} + q^{30} + q^{32} + q^{34} + q^{36}} + \frac{q^{30}}{q^{33} + q^{36} + q^{38} + q^{40} + q^{42}} + \frac{q^{36}}{q^{39} + q^{42} + q^{44} + q^{46} + q^{48}} + \frac{q^{42}}{q^{45} + q^{48} + q^{50} + q^{52} + q^{54}} + \frac{q^{48}}{q^{51} + q^{54} + q^{56} + q^{58} + q^{60}}.
\]

**Color exchange as a recovery tool.** The positivity conjecture for rectangles (the last from [Ch4] in type $A$ that remains open) states that the coefficients of $\mathcal{H}_{r,s}(u \omega_{v}; q, t, a)$ are all positive. In spite of the color-exchange connection, the number of monomials in terms of $q, t, a$ (ignoring their numerical coefficients) is different for $\mathcal{H}_{3,2}(2 \omega_2; q, t, a)$ and $\mathcal{H}_{3,2}(\omega_4; q, t, a)$ (66 and 60). Both result in 35 different monomials after the substitution $q = t^{-2}$ (a significant reduction).

We claim that the positivity of $\mathcal{H}_{3,2}(2 \omega_2; q, t, a)$, its self-duality, knowing the above specialization at $q = t^{-2}$ and the evaluation at $q = 1$ uniquely determine this polynomial (up to a multiplier $q^* t^*$).

Let us provide some details. Assuming that we know the $q, t$–monomials that occur in $\mathcal{H}_{3,2}(2 \omega_2; q, t, a)$, the duality restricts the 66-dimensional space of the corresponding undetermined coefficients.
to the space $V$ of dimension 36. Decomposing such $V$ with respect to $a^i$, one has $V = \bigoplus_{i=0}^{4} V_i$, where the corresponding dimensions are $\{\dim_0 = 9, 12, 10, 4, \dim_4 = 1\}$. Then the dimensions of the kernels of the specialization map $q = t^{-2}$ are $\{1, 3, 2, 0, 0\}$. The evaluation $q = 1$ has the corresponding kernels of dimensions $\{2, 4, 3, 1, 0\}$. There will be no common kernel if these 2 maps are combined.

Here we do not use the positivity, but assume that the monomials are those from the actual $H_{3, 2}(2\omega_2; q, t, a)$. The positivity gives that any other set of coefficients will be obtained from the actual one when at least 8 monomials (for one of $V_i$) are replaced by some other 8 (with positive coefficients) and the corresponding difference belongs to the ideal $((1-q)(1-t)(t-q^{-2})(q-t^{-2}))$. This appeared impossible.

Moreover, one can involve here the reduction $t = q$ (which gives essentially the HOMFLYPT polynomial of $T(3, 2)$ for the weight $2\omega_2$ and can be assumed known), as well as $H_{3, 2}(2\omega_2; q, t, a = -t^2) = 1$. The resulting system of (linear) equations for the coefficients of $H_{3, 2}(2\omega_2; q, t, a)$ will then become significantly overdetermined.

However the “next” polynomial $H_{3, 2}(2\omega_3; q, t, a)$ can not be recovered from the color-exchange relations (1.50) combined with the evaluations at $t = 1$ and $q = 1$; here $t = q^{-2}$ for the reduction to $6\omega_1$ and $q = t^{-3}$ for that to $\omega_6$. As above, we do not use the positivity, but assume that the monomials can be only those actually present in $H_{3, 2}(2\omega_3; q, t, a)$ (with undetermined coefficients).

### 2. DAHA of type $C^\vee C_1$

#### 2.1. Main definitions.

Double affine Hecke algebra of type $C^\vee C_1$, denoted by $\mathcal{H} = \mathcal{H}_{q, u, v}$ in this paper, is generated by $U_1, U_0, V_1, V_0$ subject to the relations

\[
(U_i - u_i^{1/2})(U_i + u_i^{-1/2}) = 0, \quad (V_i - v_i^{1/2})(V_i + v_i^{-1/2}) = 0,
\]

\[
q^{1/4} V_i V_0 U_0 U_1 = 1, \quad i = 0, 1 \text{ here and below; we set:}
\]

\[
X \overset{\text{def}}{=} V_1^{-1} U_1^{-1} = q^{1/4} V_0 U_0, \quad Y \overset{\text{def}}{=} U_0 U_1 = q^{-1/4} V_0^{-1} V_1^{-1}.
\]

The natural definition ring is $\mathbb{Z}_{q, u, v} \overset{\text{def}}{=} \mathbb{Z}[q^{\pm 1/4}, u_i^{\pm 1/2}, v_i^{\pm 1/2}]$, though we will mainly use its field of fractions $\mathbb{Q}_{q, u, v}$ when the Askey-Wilson polynomials are needed. Note that $V_1, X, Y$ obviously generate $\mathcal{H}_{q, u, v}$.

Here and below we closely follow [NS], modifying the generators and
parameters from (2.2) there as follows:

\[(2.2) \quad V_i \mapsto U_i^{-1}, \quad V_i' \mapsto V_i^{-1} (i = 0, 1), \quad g_{ns} \mapsto q^{-\frac{1}{2}}, \quad k_i \mapsto u_i^{-\frac{1}{2}}, \quad u_i \mapsto v_i^{-\frac{1}{2}}, \]

\[X \overset{\text{def}}{=} V_0^{-1}(V_0')^{-1} = q_{ns}^{1/2} V_0 V_0' \mapsto X^{-1} = U_1 V_1 = q^{1/4} U_0^{-1} V_0^{-1}, \]

\[Y \overset{\text{def}}{=} V_1 V_0 = q_{ns}^{-1/2}(V_1')^{-1}(V_0')^{-1} \mapsto \forall^{-1} = U_1^{-1} U_0^{-1} = q^{1/4} V_1 V_0.\]

These arrows mean that $V_i$ from [NS] is our $U_i^{-1}$, their $g_{ns}$ is our $q^{-1/2}$, their $X$ is our $X$ and so on. The relation $q_{ns}^{1/2} V_1 V_0 V_0' = 1$ becomes that from (2.1) in our notations.

Such changes are convenient to establish the relations to DAHA of type $A_1$ and to the superpolynomials for $T(2p + 1, 2)$ (a surprising application of the $C^\vee C$-theory). A convenient way of obtaining our relations and formulas from their ones (or those from [Ob]) is simply by transposing all terms $(AB \cdots C \mapsto C \cdots BA)$ followed by the substitutions $V \mapsto U, V' \mapsto V, g_{ns} \mapsto q^{1/2}, X \mapsto X, Y \mapsto Y$.

**Automorphisms.** We redefine the automorphisms $\sigma, \tau, \eta$ from [Ob], Proposition 1.3 as follows: $\sigma \mapsto \sigma^{-1}$, $\tau \mapsto \tau_+^{-1}$; $\eta$ remains unchanged (this is an extension of the Kazhdan-Lusztig involution).

In full detail, the following maps can be uniquely extended to automorphisms of the whole $\mathcal{H}_{q,u,v}$:

\[(2.3) \quad V_1 \mapsto V_1, \quad X \mapsto X, \quad Y \mapsto V_0 U_1 = q^{-1/4} V_1^{-1} Y^{-1} U_1, \quad u_0 \leftrightarrow v_0, \]

\[(2.4) \quad U_0 \mapsto U_0, \quad Y \mapsto Y, \quad X \mapsto V_0^{-1} U_1^{-1} = q^{1/4} U_0 X^{-1} U_1^{-1}, \quad v_0 \leftrightarrow v_1, \]

\[(2.5) \quad X \mapsto X^{-1}, \quad Y \mapsto U_1 Y^{-1} U_1^{-1}, \quad u_i \mapsto u_i^{-1}, \quad q \mapsto q^{-1}, \quad v_i \mapsto v_i^{-1}.\]

The parameter $q$ and respectively $u_1, v_1$ and $u_0, u_1$ remain unchanged under the action of $\tau_\pm$. We set $\sigma \overset{\text{def}}{=} \tau_+ \tau_- \tau_+ = \tau_- \tau_+ \tau_-^{-1}$; this automorphism transposes $t_0$ and $v_1$ and corresponds to $\mu^{-1}$ from [NS], 8.5. One has:

\[(2.6) \quad \sigma : U_0 \mapsto V_1, \quad V_0 \mapsto V_1 V_0 V_0^{-1}, \quad V_1 \mapsto U_1^{-1} U_0 U_1, \quad U_1 \mapsto U_1, \]

\[X \mapsto Y^{-1}, \quad Y \mapsto U_1^{-1} X^{-1} U_1, \quad u_0 \leftrightarrow v_1, \quad u_1 \leftrightarrow u_1, \quad v_0 \mapsto v_0.\]
In the notations from [NS], formulas (2.3)-(2.5) are as follows:

- \( \tau_+ : V_0 \mapsto V_0^\vee = q_{\text{ns}}^{-1/2} V_0^{-1} X, \ V_0^\vee \mapsto (V_0^\vee)^{-1} V_0 V_0^\vee, \ V_1 \mapsto V_1, \)

- \( V_1^\vee \mapsto V_1^\vee, \ X \mapsto X, \ Y \mapsto V_1 V_0^\vee = q_{\text{ns}}^{-1/2} V_1 Y^{-1} (V_1^\vee)^{-1}, \ k_0 \mapsto u_0, \)

- \( \tau_- : V_1^\vee \mapsto V_0^\vee = q_{\text{ns}}^{-1/2} V_0^{-1} X, \ V_0^\vee \mapsto V_0^\vee V_1 (V_0^\vee)^{-1}, \ V_1 \mapsto V_1, \)

\[ V_0 \mapsto V_0, \ Y \mapsto Y, \ X \mapsto V_1^{-1} (V_0^\vee)^{-1} = q_{\text{ns}}^{1/2} V_1^{-1} X^{-1} V_0, \ u_0 \mapsto u_1, \]

- \( \eta : V_1 \mapsto V_1^{-1}, \ V_0 \mapsto V_0^{-1}, \ V_1^\vee \mapsto V_1^{-1} (V_1^\vee)^{-1} V_1, \ V_0^\vee \mapsto V_0 (V_0^\vee)^{-1} V_0^{-1}, \)

\[ X \mapsto X^{-1}, \ Y \mapsto V_1^{-1} Y^{-1} V_1, \ k_i, u_i, q \mapsto k_i^{-1}, u_i^{-1}, q^{-1} (i = 1, 2). \]

Next, we will need the anti-involution \( \nu = \varphi_{\text{ns}} \) from 8.5 in [NS], which corresponds to our \( \varphi \):

- (2.7) \( \varphi : U_0 \mapsto V_1, \ U_1 \mapsto U_1, \ V_0 \mapsto V_0, \ X \leftrightarrow Y^{-1}, \ u_0 \leftrightarrow v_1, \)

- \( \varphi_{\text{ns}} : V_0 \leftrightarrow V_1^\vee, \ V_1 \leftrightarrow V_1^\vee, \ V_0^\vee \leftrightarrow V_0^\vee, \ X \leftrightarrow Y^{-1}, \ k_0 \leftrightarrow u_1. \)

All relations of the reduced theory in Section 1.3 holds for \( \tau_{\pm}, \sigma, \varphi \). For instance,

\[ \tau_\pm = \varphi \tau_{\pm} \varphi = \sigma \tau_{\pm}^{-1} \sigma^{-1}, \ \eta \tau_{\pm} = \tau_{\pm}^{-1}, \ \varphi \sigma \varphi = \sigma^{-1} = \eta \sigma \eta, \]

(2.8) \( \varphi \eta \varphi = \eta \sigma^{-2} = \sigma^2 \eta, \ \sigma^2 (H) = U_1^{-1} (H) U_1 \) for \( H \in \mathcal{H}_{q,t,u}. \)

Also, due to the group nature of the definition of \( \mathcal{H}_{q,u,v} \), we have the inversion anti-involution \( * \), sending all generators of \( \mathcal{H}_{q,u,v} \) and parameters \( q, u_i, v_i \) (and their products) to the corresponding inversions.

Finally, let us define the sign-automorphisms of \( \mathcal{H}_{q,t,u} \):

- (2.9) \( \varsigma_x : V_0 \mapsto -V_0, \ v_0^1/2 \mapsto -v_0^1/2, \ V_1 \mapsto -V_1, \ v_1^1/2 \mapsto -v_1^1/2, \)

- \( \varsigma_y : U_0 \mapsto -U_0, \ u_0^1/2 \mapsto -u_0^1/2, \ V_0 \mapsto -V_0, \ v_0^1/2 \mapsto -v_0^1/2, \)

- \( \varsigma_s : U_i \mapsto -U_i, \ u_i^1/2 \mapsto -u_i^1/2, \ V_i \mapsto -V_i, \ v_i^1/2 \mapsto -v_i^1/2, \)

- \( \varsigma_q : U_1 \mapsto -U_1, \ u_1^1/2 \mapsto -u_1^1/2, \ q^{1/4} \mapsto -q^{1/4}, \)

where the remaining generators and parameters are unchanged under the action of the corresponding \( \varsigma \).

The group \( GL^2(\mathbb{Z}) \) fixes \( \varsigma_q, \varsigma_s \) and acts in the group \( \mathbb{F}_2^2 \) generated by \( \varsigma_x, \varsigma_y \) through its projection \( \varpi \) onto \( GL_2(\mathbb{F}_2) \). One has:

(2.10) \( \varsigma_x : X, Y \mapsto -X, Y, \ \varsigma_y : X, Y \mapsto X, -Y, \)

\( \varsigma_q : X, Y \mapsto -X, -Y, \ \varsigma_s : X, Y \mapsto X, Y. \)
2.2. **Polynomial representation.** The following presentation of the algebra \( \mathcal{H}_{q,u,v} \) by Demazure-Lusztig operators was found by Noumi for \( C^\vee C_n \); see also [Sa], Section 2.3. Using the notations from [NS],

\[
\hat{V}_i = k_i s_i + \frac{(k_i - k_i^{-1}) + (u_i - u_i^{-1})X_i}{1 - X_i^2} (1 - s_i) \quad \text{for} \quad i = 0, 1,
\]

\[
X_1 = X, \quad X_0 = q^{\frac{1}{2}} X, \quad s_1(X^m) = X^{-m}, \quad s_0(X^m) = q^m X^{-m},
\]

\[
\hat{V}_i^{-1} = k_i^{-1} s_i + \frac{(k_i - k_i^{-1})X_i^2 + (u_i - u_i^{-1})X_i}{1 - X_i^2} (1 - s_i),
\]

\[
\hat{V}_i^\vee (f) = \hat{V}_i^{-1}(f/X), \quad \hat{V}_0^\vee (f) = \hat{V}_0^{-1}(f/X_0) = q^{-\frac{1}{2}} \hat{V}_0^{-1}(Xf),
\]

\[
(\hat{V}_i^\vee)^{-1}(f) = \hat{V}_i(Xf), \quad (\hat{V}_0^\vee)^{-1}(f) = X_0 \hat{V}_0(f) = q^{\frac{1}{2}} X^{-1} \hat{V}_0(f).
\]

In our notations,

\[(2.11) \quad \hat{U}_i = u_i^{1/2} s_i + \frac{(u_i^{1/2} - u_i^{-1/2}) + (v_i^{1/2} - v_i^{-1/2})X_i}{1 - X_i^2} (1 - s_i),
\]

\[
X_1 = X, \quad X_0 = q^{\frac{1}{2}} X, \quad s_1(X^m) = X^{-m}, \quad s_0(X^m) = q^m X^{-m},
\]

\[
\hat{U}_i^{-1} = u_i^{-1/2} s_i + \frac{(u_i^{1/2} - u_i^{-1/2})X_i^2 + (v_i^{1/2} - v_i^{-1/2})X_i}{1 - X_i^2} (1 - s_i),
\]

\[(2.12) \quad \hat{V}_i(f) = \hat{U}_i^{-1}(f/X), \quad \hat{V}_0(f) = \hat{U}_0^{-1}(f/X_0) = q^{-\frac{1}{2}} \hat{U}_0^{-1}(Xf),
\]

\[
\hat{V}_i^{-1}(f) = X \hat{U}_i(f), \quad \hat{V}_0^{-1}(f) = \hat{U}_0(X_0 f) = q^{\frac{1}{2}} \hat{U}_0(f/X).
\]

Here \( f \in \mathcal{V} = \mathcal{V}_X \overset{\text{def}}{=} \mathbb{Z}_{q,u,v}[X^{\pm 1}] \), which we will call the *polynomial representation* ; it is supplied with the action \( \mathcal{H}_{q,u,v} \ni H \mapsto \hat{H} \) defined via (2.11),(2.12). Thus (following Noumi) we claim that the relations from (2.1) are satisfied for \( \{\hat{U}_i, \hat{V}_i\} \). The operator \( X \) acts as the multiplication by \( X \). Indeed, (2.12) gives that \( \hat{X}(f) = \hat{V}_1^{-1} \hat{U}_1^{-1}(f) = X \hat{U}_1(\hat{U}_1(f)) = f \).

The existence of this representations provides the *PBW Theorem*, which states that the following decomposition is unique:

\[(2.13) \quad H = \sum_{n,\epsilon, m} C_{n,\epsilon, m} X^n V_1^\epsilon Y^m \quad \text{for any} \quad H \in \mathcal{H}_{q,u,v}, \quad n, m \in \mathbb{Z}, \epsilon = 0, 1.
\]

Using this theorem (which can be proved directly), \( \mathcal{V} \) is the \( \mathcal{H}_{q,u,v} \)-module induced from the one-dimensional *evaluation character* \( \chi \) on
the subalgebra $\mathcal{U}$ generated by $U_0, U_1$:

(2.14) \( \mathcal{V} = \text{Ind}_{\mathcal{W}}^{\mathcal{U}}(\chi), \ \chi : U_0 \mapsto u_1^{1/2}, U_0 \mapsto u_0^{1/2}, \mathcal{Y} = U_0U_1 \mapsto (u_0u_1)^{1/2} \).

This readily gives that $\tau_-$ sends $\mathcal{V}$ to its image under $v_0 \leftrightarrow v_1$.

The difference Dunkl operator for $C^\vee C_1$ is $\tilde{\mathcal{Y}} = \tilde{U}_0 \tilde{U}_1$. The non-symmetric Askey-Wilson polynomials $E_n (n \in \mathbb{Z})$ are defined from the relations

(2.15) \[ \tilde{\mathcal{Y}}(E_n) = (u_0u_1)^{-\text{sgn}(n)/2}q^{-n/2}E_n \quad \text{for} \quad n \in \mathbb{Z}, \quad \text{where} : \]

\[ E_n = X^n + C_n^{-}X^{-n} + \sum_{|m|<|n|} C_m X^m, \quad C_n^{-} = 0 \quad \text{for} \quad n > 0. \]

Here $\text{sgn}(n \leq 0) = -1$ and $+1$ otherwise, i.e. $0$ is treated as negative. Note the formula

(2.16) \[ \tau_-(E_n) = q^{-n^2/4}(u_1u_0)^{-|n|/2}E_n \big|_{v_0 \mapsto v_1} \quad \text{for} \quad n \in \mathbb{Z}. \]

The symmetric Askey-Wilson polynomials $P_n (n \in \mathbb{Z}^+)$ from [AW] can be defined from the relation

(2.17) \[ (\tilde{\mathcal{Y}} + \tilde{\mathcal{Y}}^{-1})(P_n) = ((u_0u_1)^{1/2}q^{n/2} + (u_0u_1)^{-1/2}q^{-n/2})P_n, \]

where $P_n = X^n + X^{-n} + \sum_{0 \leq m < n} C_m (X^m + X^{-m})$.

Also, $P_n$ are the $u_1$-symmetrizations of $E_n$, i.e. they can be defined using relation (1.24) for $A_1$ with $T_1, t_1$ there replaced by $U_1, u_1$. They are formally real; namely, $P_n^* = P_n$, where $*$ sends $X \mapsto X^{-1}$ and all parameters to their reciprocals.

One has:

(2.18) \[ E_0 = 1, \quad E_1 = X + \frac{(q^{1/2}u_0/v_0)^{1/2}(1-v_0) + u_0(qu_1/v_1)^{1/2}(1-v_1)}{1 - q^{1/2}u_0u_1}, \]

(2.19) \[ E_{-1} = X^{-1} + \frac{(1 - u_1) + q^{1/2}u_1(1 - u_0)}{1 - q^{1/2}u_0u_1} X \]

\[ + \frac{(q^{1/2}u_0/v_0)^{1/2}(1 + q^{1/2}u_1)(1 - v_0) + (u_1/v_1)^{1/2}(1 + q^{1/2}u_0)(1 - v_1)}{1 - q^{1/2}u_0u_1}. \]

The corresponding symmetric Askey-Wilson polynomial is

(2.20) \[ P_1 = P_1 = (1 + u_1^{1/2}V_1)(E_1) = X + X^{-1} \]

\[ + \frac{(q^{1/2}u_0/v_0)^{1/2}(1 + u_1)(1 - v_0) + (u_1/v_1)^{1/2}(1 + q^{1/2}u_0)(1 - v_1)}{1 - q^{1/2}u_0u_1}. \]
Note that $E_1$ is not $X$ as it is for $A_1$ and $E_n$ can have the terms $X^n$ in odd and even degrees. However, the following symmetries for the sign-automorphisms from (2.9) and (2.10) hold:

\[(2.21) \quad \varsigma_x(E_n) = (-1)^n E_n = \varsigma_q(E_n), \quad \varsigma_y(E_n) = E_n = \varsigma_q(E_n) \quad \text{for any } n \in \mathbb{Z}.\]

We treat here $E_n(X)$ as elements of $\mathcal{H}_{q,u,v}$. These relations control the appearance of square roots of the parameters in $E_n$; the coefficients of $E_n$ are in terms of $q,t$ in the case of $A_1$, i.e. free of square roots.

The $E$–polynomials can be obtained using the intertwining operators for $\mathcal{H}_{q,u,v}$ from [Sa, NS]:

\[(2.22) \quad \begin{align*}
S_1 & \overset{\text{def}}{=} U_1^{-1} Y^{-1} - Y^{-1} U_1^{-1}, \quad S_0 \overset{\text{def}}{=} Y^{-1} U_0^{-1} - U_0^{-1} Y^{-1}, \\
\text{intertwining } \mathcal{Y} : S_1 \mathcal{Y} = \mathcal{Y}^{-1} S_1, \quad q^{-1} S_0 \mathcal{Y} = \mathcal{Y}^{-1} S_0.
\end{align*}\]

They result in the following recurrence relations for $m = 0, 1, 2 \ldots$

\[(2.23) \quad \begin{align*}
\hat{S}_0(E_{-m}) &= \frac{q^{2m+1} u_0 u_1 - 1}{q^2 u_0^2} E_{m+1}, \\
\hat{S}_1(E_{m+1}) &= \frac{q^{2m+2} u_0 u_1 - 1}{q^2 u_0^2} E_{m-1}.
\end{align*}\]

See [NS] for explicit formulas for the $E$–polynomials, their norms and evaluations. We will need below the evaluation formulas (recalculated to our notations):

\[(2.24) \quad \begin{align*}
E_{m+1}((u_1 v_1)^{-\frac{1}{2}}) &= \frac{(u_1 v_0 v_1)^{(m+1)/2}(1 + q^{\frac{1}{2}} u_1) \cdots (1 + q^{m} u_1)}{(1 - q^{m+1} u_0 u_1) \cdots (1 - q^{2m+1} u_0 u_1)} \\
\times \prod_{i=0}^{m} (v_0^{1/2} + q^{2i+1} u_0^{1/2} u_1^{1/2} v_1^{1/2}) (1 - q^{2i+1} u_0^{1/2} u_1^{1/2} v_1^{1/2} v_0^{1/2}),
\end{align*}\]

\[(2.25) \quad \begin{align*}
E_{-m-1}((u_1 v_1)^{-\frac{1}{2}}) &= \frac{(u_1 v_0 v_1)^{(m+1)/2}(1 + q^{\frac{1}{2}} u_1) \cdots (1 + q^{m} u_1)}{(1 - q^{m+1} u_0 u_1) \cdots (1 - q^{2m+1} u_0 u_1)} \\
\times \prod_{i=0}^{m} (v_0^{1/2} + q^{2i+1} u_0^{1/2} u_1^{1/2} v_1^{1/2}) (1 - q^{2i+1} u_0^{1/2} u_1^{1/2} v_1^{1/2} v_0^{1/2}).
\end{align*}\]

Also, $\mathcal{P}_{m+1}((u_1 v_1)^{\pm \frac{1}{2}}) = (1 + u_1) E_{m+1}((u_1 v_1)^{-\frac{1}{2}})$ for $m \in \mathbb{Z}_+$. Concerning the orthogonality relations and other formulas for the $E$–polynomials, we refer to [NS], where $X, q_{ns}, k_0, k_1, u_0, u_1$ must be replaced by $X^{-1}, q^{-1/2}, u_0^{-1/2}, u_1^{-1/2}$, $v_0^{-1/2}, v_1^{-1/2}$ to match our setting. See [Ko, Mac, Sa, Sto] for general theory.
2.3. Relations to \( A_1 \). Let us first reduce Definition 1.1 to the case of DAHA of type \( A_1 \), denoted by \( \mathcal{H}_{q,t}^{A_1} \). For \( A_1 \), let \( \alpha = \alpha_1, \ s = s_1 \) and \( \omega = \omega_1 \) be the fundamental weight; then \( \alpha = 2\omega \) and \( \rho = \omega \). The extended affine Weyl group \( \hat{W} = < s, \omega > \) in the \( A_1 \)-case is a free group generated by the involutions \( s \) and \( \pi \overset{\text{def}}{=} \omega s \).

The generators of \( \mathcal{H}_{q,t}^{A_1} \) are
\[
Y = Y_{\omega_1} = \pi T, \ T = T_1, \ X = X_{\omega_1}
\]
subject to the quadratic relation \((T - t^{1/2})(T + t^{-1/2}) = 0\) and the cross-relations:
\[
\begin{equation}
T X T = X^{-1}, \ T^{-1} Y T^{-1} = Y^{-1}, \ Y^{-1} X^{-1} Y T X^2 q^{1/2} = 1.
\end{equation}
\]
Using \( \pi = YT^{-1} \), the second relation becomes \( \pi^2 = 1 \). This algebra is defined over
\[
\mathbb{Z}_{q,t} \overset{\text{def}}{=} \mathbb{Z}[q^{\pm 1/2}, t^{\pm 1/2}].
\]

The following maps can be extended to automorphisms of \( \mathcal{H}_{q,t}^{A_1} \):
\[
\begin{align*}
\tau_+(X) &= X, \ \tau_+(T) = T, \ \tau_+(Y) = q^{-1/4} X Y, \ \tau_+(\pi) = q^{-1/4} X \pi, \\
\tau_-(Y) &= Y, \ \tau_-(T) = T, \ \tau_-(X) = q^{1/4} Y X, \ \tau_-(\pi) = \pi, \\
\sigma(X) &= Y^{-1}, \ \sigma(T) = T, \ \sigma(Y) = X T^2, \ \sigma(\pi) = X T,
\end{align*}
\]
\[
\begin{equation}
\eta(X) = X^{-1}, \ \eta(Y) = T Y^{-1} T^{-1}, \ \eta(T) = T^{-1}, \ q, t \mapsto q^{-1}, t^{-1}.
\end{equation}
\]
Recall that \( \sigma = \tau_+ \tau_\pm \tau_+ = \tau^- \tau_+ \tau^- \) and we add \( q^{\pm 1/4} \) to the ring of definition of \( \mathcal{H}_{q,t} \).

We also have two anti-involutions:
\[
\begin{align*}
\varphi(X) &= Y^{-1}, \ \varphi(Y) = X^{-1}, \ \varphi(T) = T, \ q, t \mapsto q, t, \\
X^* &= X^{-1}, \ Y^* = Y^{-1}, \ T^* = T^{-1}, \ q, t \mapsto q^{-1}, t^{-1}.
\end{align*}
\]

Theorem 2.1. (i) Let us send \( u_1^{1/2} \mapsto t^{1/2} \) and make \( u_0^{1/2} = 1 = v_0^{1/2} = v_1^{1/2} \). Then the map
\[
U_1 \mapsto T, \ U_0 \mapsto \pi = Y T^{-1}, \ V_0 \mapsto \bar{\pi} \overset{\text{def}}{=} q^{1/4} Y T^{-1} X^{-1}, \ V_1 \mapsto \bar{\pi} \overset{\text{def}}{=} X T,
\]
\[
\begin{equation}
X = V_1^{-1} U_1^{-1} \mapsto (X T) T^{-1} = X, \ \Psi = U_0 U_1 \mapsto (Y T^{-1}) T = Y
\end{equation}
\]
can be extended to a homomorphism of algebras \( \xi : \mathcal{H}_{q,u,v} \to \mathcal{H}_{q,t}^{A_1} \) (with the same \( q \) in both), compatible with \( \tau_\pm, \eta, \varphi \) and the inversion \( \ast \).
The nonsymmetric Askey-Wilson polynomials \( \mathcal{E}_n(X) \) become \( E_n(X) \) for \( A_1 \) for \( n \in \mathbb{Z} \), where the evaluation formulas from (2.24), (2.25) become those in (1.23) for \( A_1 \).

(ii) Let us make now \( u_1^{1/2} = t^{1/2} = u_0^{1/2} \) and consider \( \mathcal{H}_{q^2,u,v} \) (i.e. replace \( q^{1/4} \) by \( q^{1/2} \) in its definition). Then

\[
(2.31) \quad U_1 \mapsto T, \quad U_0 \mapsto T_0 \overset{\text{def}}{=} Y^{-2}T^{-1}, \quad V_0 \mapsto \tilde{\pi}, \quad V_1 \mapsto \hat{\pi} \overset{\text{def}}{=} q^{1/4}XTY,
\]

\[
X = V_1^{-1}U_1^{-1} \mapsto q^{-1/4}X^{-1}X = \tau_+^{-1}(X), \quad Y = U_0U_1 \mapsto (Y^2T^{-1})T = Y^2
\]

give a homomorphism \( \zeta : \mathcal{H}_{q^2,u,v} \to \mathcal{H}_{A_1}^{q,t} \), which induces the following group homomorphism from the group \( \Gamma_0^\wedge(2) \) acting in \( \mathcal{H}_{q^2,u,v} \) to the group \( GL^\wedge_2(\mathbb{Z}) \) acting in \( \mathcal{H}_{A_1}^{q,t} \):

\[
(2.32) \quad \zeta : \quad \tau_+^2 \mapsto \tau_+^{-1}\tau_+\tau_-, \quad \tau_- \mapsto \tau_+^2, \quad \eta \mapsto \tau_+^{-1}\eta\tau_-
\]

(iii) The map \( \zeta \) from (2.31) is compatible with the following map of the polynomial representations:

\[
(2.33) \quad \mathcal{H}_{q^2,u,v} \cap \mathbb{V}_X \ni F(X) \overset{\zeta}{\mapsto} \tau_-^{-1}(F(X))(1) \in \mathbb{V}_X \cap \mathcal{H}_{A_1}^{q,t}.
\]

For instance, the polynomials \( \mathcal{E}_n(X) \) upon the substitution

\[
q^{1/4} \mapsto q^{1/2}, \quad u_0^{1/2}, u_1^{1/2} \mapsto t^{1/2}, \quad v_0^{1/2}, v_1^{1/2} \mapsto 1
\]

coincide with \( E_n(X) \) (of type \( A_1 \)) for any \( n \) and, accordingly, the evaluation of \( E_n \) at \( (u_1v_1)^{-1/2} \) becomes that for \( E_n \) at \( t^{-1/2} \).

Proof. It is known that the substitution \( u_1 = 1 = v_0 = v_1 \) makes \( \mathcal{H}_{q^2,u,v} \) the DAHA of type \( A_1 \) (claimed in Part (i)). However we need some details here. We rewrite the identity

\[
(TY^{-1})(q^{1/4}Y^{-1}T^{-1}(XT)) = q^{1/4}T = \pi\bar{\pi}\pi
\]

in \( \mathcal{H}_{A_1}^{q,t} \) as follows: \( q^{1/4}\pi\bar{\pi}\pi T = 1 \). Then we use that \( \pi, \bar{\pi}, \pi \) are involutions, and this readily gives the map \( \xi \) from (i). The comparison of the automorphisms involved is straightforward (and is known); DAHA of type \( C_1 \) was actually introduced as a generalization of that for \( A_1 \). The compatibility of \( \xi \) with the definition of the polynomial representation (here and below) results from its interpretation as an induced module from (2.14).
Let us focus on \((ii,iii)\). Now the identity in \(H_{H}^{A_{1}}\) we need is

\[
T_0 T = Y^2 = q^{-1/2} \pi \bar{\pi} = q^{-1/2} (q^{1/4} Y T^{-1} X^{-1}) (q^{1/4} XTY),
\]

which can be rewritten as \(q^{1/2} \bar{\pi} \pi T_0 T = 1\) and readily the \(\zeta\)-images of the generators of \(H_{a,b,v}^{2}\) from (2.31). This proves \((iii)\).

Let us calculate the \(\zeta\)-image of \(\tau_{+}^{2}\). Note that \(\zeta(\tau_{+})\) is not well defined since \(\tau_{+}\) transposes \(u_0\) and \(v_0\). One has:

\[
\tau_{+}(U_1) = U_1, \quad \tau_{+}(X) = X, \quad \tau_{+}(Y) = \tau_{+}(q^{-1/4} V_1^{1/4} U_1^{1/4}) = V_1^{-1} U_1^{-1} Y V_1 U_1,
\]

\[
\zeta(\tau_{+}) : T \mapsto T, \quad \tau_{-}^{-1}(X) \mapsto \tau_{-}^{-1}(X), \quad Y^2 \mapsto T^{-1} X^{-2} T
\]

\[
= \zeta(V_1^{-1} U_1^{-1} Y V_1 U_1) = \bar{\pi} T^{-1} Y^{-2} \pi T = (\bar{\pi} Y^{-1})(YT^{-1})Y\bar{\pi}
\]

\[
= (\bar{\pi} Y^{-1})T(Y\bar{\pi})T = (q^{1/2}XT)T(q^{-1/2}T^{-1}X^{-1})T = T^{-1} X^{-2} T.
\]

On the other hand,

\[
\tau_{-}^{-1} \tau_{+} \tau_{-}(Y^2) = \tau_{-}^{-1} \tau_{+} \tau_{-}^{-1}(\tau_{+}^2(Y^2)) = \sigma(Y^2) = T^{-1} X^{-2} T,
\]

\[
\tau_{-}^{-1} \tau_{+} \tau_{-}(\tau_{-}^{-1}(X)) = \tau_{-}^{-1}(X), \quad \tau_{-}^{-1} \tau_{+} \tau_{-}(T) = T.
\]

This justifies that \(\zeta(\tau_{+}) = \tau_{-}^{-1} \tau_{+} \tau_{-}\).

The \(\zeta\)-image of \(\tau_{-}\) is as follows:

\[
\tau_{-}(U_1) = U_1, \quad \tau_{-}(Y) = Y, \quad \zeta(\tau_{-}(X)) = \zeta(V_0^{-1} U_0^{-1}) = \bar{\pi} TY^{-2}
\]

\[
\zeta(\tau_{-}) : T \mapsto T, \quad Y^2 \mapsto Y^2, \quad \zeta(X) = \tau_{-}^{-1} X \mapsto \zeta(\tau_{-}(X))
\]

\[
= \zeta(V_0^{-1} U_0^{-1}) = \bar{\pi} T^{-1} = q^{1/4} Y T^{-1} X^{-1} T^{-1} = q^{1/4} Y X = \tau_{-}(X).
\]

We conclude that \(\zeta(\tau_{-}) = \tau_{-}^2\).

Similarly, the relation \(\zeta(\eta) = \tau_{-}^{-1} \eta \tau_{-}\) results from:

\[
\zeta(\eta(X)) = \zeta(X^{-1}) = \tau_{-}^{-1} (X^{-1}) = \tau_{-}^{-1} \eta \tau_{-}(\zeta(X)),
\]

\[
\zeta(\eta(Y^2)) = \zeta(U_1 Y^{-2} U_1^{-1}) = \eta(Y^2), \quad \zeta(\eta(U_1)) = \eta(T).
\]

The fact that \(\zeta\) is the group automorphism of \(\Gamma_0^1(2)\) follows from the calculation above (and irreducibility of \(Y\)). This can be readily checked directly, since any relation between \(\tau_{+}^2\) and \(\tau_{-}\) holds for \(\tau_{+}\) and \(\tau_{-}\). The conjugation by \(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\) proves this fact in the corresponding matrices, which is of course connected with our formula \(\zeta(Y) = Y^2\).
2.4. DAHA-Jones polynomials for $C^\vee C_1$. We define the evaluation functional for $H \in \mathcal{K}_{q,u,v}$ as $\{H\}_{ev} \overset{\text{def}}{=} \hat{H}(1)((u_1v_1)^{-1/2})$ for the action of $H$ in $\mathcal{V}$. Equivalently, one can use the PBW-decomposition from (2.13) and set

$$
(2.34) \quad \left\{ \sum_{n,\epsilon,m} C_{n,\epsilon,m} X^n V_1^\epsilon Y^m \right\}_{ev} = \sum_{n,\epsilon,m} C_{n,\epsilon,m} (u_1v_1)^{-\epsilon} u_1^\epsilon (u_1u_0)^m.
$$

By construction, $\{\varphi(H)\}_{ev} = \{H\}_{ev}$; recall that $\varphi$ transposes $u_0$ and $v_1$ fixing $u_1$. Also, $\{\eta(H)\}_{ev} = \{H\}^*$, where $*$ is the inversion of all parameters (and the standard generators), and $\{\sigma^2(H)\}_{ev} = \{H\}$. Thus the connections between $\{\cdot\}_{ev}$ from (1.19) all hold for $C^\vee C_1$. However the DAHA-Jones polynomials must now carefully address the action of $GL_2^\vee(\mathbb{Z})$ on the parameters $\{u_1, v_1\}$.

Given a pair $(r, s)$ satisfying the condition $\gcd(r, s) = 1$, we associate with it $\gamma = \gamma_{r,s} \in \Gamma_0^\vee(2) \overset{\text{def}}{=} \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \beta \in 2\mathbb{Z} \right\}$ such that its first column $(\alpha, \gamma)^{tr}$ is $(r, s)^{tr}$. Here $\gamma$ is unique modulo right multiplication by matrices from $\begin{pmatrix} 1 & 2\mathbb{Z} \\ 0 & 1 \end{pmatrix}$. We lift such $\gamma$ to $\hat{\gamma} \in GL_2^\vee(\mathbb{Z})$.

Recall that $\hat{\gamma}$ acts on the parameters $u_1, u_0, v_1, v_0$; for the group $PSL_2^\vee(\mathbb{Z})$ generated by $\tau_\pm$ this action is via $\varpi(\gamma)$. One has:

$$
(2.35) \quad \tau_+ (u_1, u_0, v_1, v_0) = (u_1, v_0, v_1, u_0), \quad \tau_- (u_1, u_0, v_1, v_0) = (u_1, u_0, v_0, v_1),
$$

$$
\eta(u_1, u_0, v_1, v_0) = (u_1^{-1}, u_0^{-1}, v_1, v_0^{-1}), \quad \varphi(u_1, u_0, v_1, v_0) = (u_1, v_1, u_0, v_0),
$$

and $\eta(q) = q^{-1}$ (the other three preserve $q$). These transformations are naturally extended to the fractional powers of the parameters.

Let $\varpi(s) = 0, 1$ be the parity of $s$; we also denote $\gamma$ modulo (2) by $\varpi(\gamma)$, which is $\begin{pmatrix} 1 & 0 \\ 0 & \varpi(s) \end{pmatrix}$ for $\gamma = \gamma_{r,s}$. We also set $\delta(\gamma) = \det(\gamma) = \pm 1$.

We will sometimes use the notation $\Gamma_0(2)$ and $\Gamma_0^\vee(2)$ for the intersection of $\Gamma_0(2)$ and $\Gamma_0^\vee(2)$ with $PSL_2(\mathbb{Z})$, considered as a subgroup of $GL_2(\mathbb{Z})$, and $PSL_2^\vee(\mathbb{Z})$. Any admissible $(r, s)$ can be lifted to $\gamma \in \Gamma_0(2)$ and $\hat{\gamma} \in \Gamma_0^\vee(2)$; then only $\tau_\pm$ will occur in $\hat{\gamma}$ ($\eta$ will not be involved).

The next theorem and its proof follow Theorem 1.2.
Theorem 2.2. (i) Given an admissible pair \((r, s)\), i.e. satisfying the condition \(\gcd(r, s) = 1\), and \(m \in \mathbb{N}\), we set

\[
J_D(r, s)(m; q, u, v) \overset{\text{def}}{=} \left\{ \hat{\gamma}_{r, s}(P_m) \right\}_{ev} / \eta^{\delta(\gamma_{r, s})} \tau^{-\varpi(s)}(\{P_m\}_{ev}),
\]

where \(\hat{\gamma}_{r, s}\) is a lift of \(\gamma_{r, s} \in \Gamma_0(2)\) with the first column \((r, s)^T\). Then \(J_D(r, s)(m; q, u, v)\) does not depend on the particular choice of \(\gamma_{r, s} \in \Gamma_0(2)\) and \(\hat{\gamma}_{r, s} \in GL_2(\mathbb{Z})\) and is a Laurent polynomial in terms of \(q^{1/4}, u_i^{1/2}, v_i^{1/2}\) for \(i = 0, 1\). One can switch to the \(E\)-polynomials here:

\[
J_D(r, s)(m; q, u, v) = \left( J_D_{1, s}(m; q, u, v) \right)^* \text{ for any } r, s
\]

(ii) One has: \(J_D_{1, s}(m; q, u, v) = (q^{m/2} u_1 u_0)^{-sm/2}\), \(J_D_{1, s}(m; q, u, v) = \left( J_D_{r, s}(m; q, u, v) \right)^* \) for any \(r, s\), \(J_D_{r, s}(m; q, u, v) = J_D_{s, r}(m; q, u, v)\) if \(rs\) is even.

Moreover, for any \(r, s, v_1, v_0\) and \(m, n \in \mathbb{Z}_+\),

\[
q^{(n+m)/2} u_0 u_1 = 1 \Rightarrow J_D_{r, s}(m; q, t) = J_D_{r, s}(n; q, t).
\]

If \(q = 1\), then \(E_m = P_m\), \(P_{m+n} = P_m P_n\) for any \(m, n \in \mathbb{Z}_+\) and

\[
J_D_{r, s}(m; q = 1, u, v) = J_D_{r, s}(1; q = 1, u, v)^m \text{ for any } r, s.
\]

(iii) Let us use the sign-automorphisms \(\varsigma_x, \varsigma_y, \varsigma_s, \varsigma_q\) of \(H_{q, t, u}\) from (2.9); \(GL_2(\mathbb{Z})\) naturally acts in the group generated by these \(\varsigma\). Namely, it fixes \(\varsigma_s, \varsigma_q\) and acts on \(\varsigma_x, \varsigma_y\) via formulas (2.35). Then relations (2.10) and (2.21) give that

\[
\varsigma \{ \gamma_{r, s}(E_m) \}_{ev} / \{ \gamma_{r, s}(E_m) \}_{ev} = \varsigma(E_m)/E_m \text{ for any such } \varsigma, \ m \in \mathbb{Z}.
\]

Since the normalization point in (2.36) depends on \(\varpi(s)\), the square roots of \(q^{1/2}, u_i, v_i\) can occur in \(J_D_{r, s}(m; q, u, v)\) only as follows:

\[
J_D_{r, s}(m; q, u, v) = A + (q^{1/2} u_1 u_0 v_1 v_0)^{1/2} B \text{ for } A, B \in (u_0 u_1)^{m/2} \mathbb{Z} [q^{1/2}, u_1^{\pm 1}, v_1^{\pm 1}, u_0^{\pm 1}, v_0^{\pm 1}].
\]

Obviously, the broken symmetry \((r, s) \mapsto (s, r)\) and the fact that \(J_D_{2p+1, 1}\) are nontrivial seem non-topological. However exactly the polynomials \(J_D_{2p+1, 1}\) appear connected with the superpolynomials of knots \(T(2p + 1, 2)\), so they do have topological-geometric meaning.
2.5. Reductions of parameters. We will begin with the following case, which is a counterpart of the case \( t = 1 \) for \( A_1 \) (where all DAHA-Jones polynomials are tilde-trivial).

**Proposition 2.3.** Let us impose the relations \( v_0^{1/2} = 1 = v_1^{1/2} \) and also \( u_1^{1/2} = q^{1/2} = u_0^{-1/2} \). We assume that \( r \) is odd using the symmetry (2.38) if necessary; let \( r = 2p + 1 \). Then

\[
JD_{r,s}(m; q, u_0 = q, u_1 = 1, t, 1) = q^{-m(rs/m + p)} \frac{1 + q^m r}{1 + q^m}
\]

for any admissible \( r, s, m \). □

Here we check that \( JD_{r,s}(m; q, u, v) \) depends only on \( p \) up to a monomial factor and then calculate \( JD \) for the pair \((2p + 1, 1)\).

Theorem 2.1 results in the following reduction formulas.

**Theorem 2.4.** (i) Let \( u_1^{1/2} \mapsto t^{1/2} \) and \( u_0^{1/2} = 1 = v_1^{1/2} \). Then

\[
JD_{r,s}(m; q, u_0 = q, u_1 = 1, t, 1) = JD_{r,s}(m; q, U_1 t).
\]

(ii) We take now \( u_1^{1/2} = t^{1/2} = u_0^{1/2} \), and replace \( q^{1/4} \) by \( q^{1/2} \). We also assume that \( r \) is odd, which is possible due to (2.38). Then

\[
JD_{r,s}(m; q^2, t, t, 1, 1) = JD_{r,s}(m; q, t)
\]

for any admissible \( r, s, m \). □

The reduction formulas here do not require the tilde-normalization (up to a monomial factor) of \( JD_{A_1} \). This normalization is actually unnecessary in the \( A_1 \)–case due to the formulas from [Ch4], Section 2.5:

\[
\tilde{JD}_{r,s}(m; q, t) = q^{m^2rs/4 + m(r+s-1)/2} JD_{r,s}(m; q, t).
\]

Accordingly, we have the following leading term in the case of \( C^\vee C_1 \):

\[
JD_{r,s}(m; q^2, t, t, 1, 1) = q^{-m^2/4} \sum_{i,j \geq 0} C_{ij} q^i t^j \quad \text{for } C_{00} = 1.
\]

Let us provide the simplest example of the colored \( JD \)–polynomial and its reductions to \( A_1 \) (apart from the pairs \((2p + 1, 1)\) to be discussed next). Setting \( A = (1 - v_0) \), \( B = (1 - u_0) \), \( \tilde{q} = q^{-2} \),

\[
JD_{3,2}(2; q, u, v) = JD_{2,3}(2; q, u, v) =
\]
\( \frac{\hat{q}}{u_0^2 u_1^4 v_0^2 v_1^2} \left( AB \hat{q}^6 u_1 (\hat{q} u_0 u_1 v_0 v_1)^{1/2} + B^2 \hat{q}^4 u_1^2 v_0 v_1 - A^2 \hat{q}^3 u_0^2 u_1^2 v_1^2 \right) \\
+ A (\hat{q} u_0 u_1 v_0 v_1)^{1/2} (\hat{q}^7 + \hat{q}^8 + \hat{q}^5 u_1 - \hat{q}^4 u_0 u_1 - 2\hat{q}^5 u_0 v_1 - \hat{q}^2 u_0 u_1^2 \\
- 2\hat{q}^3 u_0 u_1 - \hat{q}^4 u_0 u_1 + \hat{q}^2 u_0 u_1 + \hat{q}^3 u_0 u_1 + \hat{q}^3 u_0^2 u_1^2 + u_0^2 u_1^2 + \hat{q} u_0^2 u_1^3 \\
- \hat{q}^4 (\hat{q} u_0 u_1 v_0 v_1)^{1/2} + \hat{q}^3 u_1 v_1 + \hat{q}^4 u_1 v_1 - \hat{q} u_0 u_1^2 v_1 - \hat{q}^2 u_0 u_1^2 v_1) \\
+ B u_1 v_0 (\hat{q}^8 + \hat{q}^9 + \hat{q}^6 u_1 - \hat{q}^4 u_0 u_1 + \hat{q}^6 u_0 u_1 + \hat{q}^3 u_0 u_1 v_1 \\
- \hat{q}^3 u_0 u_1 v_1 - \hat{q}^2 u_0 u_1^2 v_1 - \hat{q} u_0 u_1^2 v_1 + (\hat{q}^11 v_0 \hat{q}^5 u_0 u_1^2 v_0 - \hat{q}^7 u_0 u_1^2 v_0 \\
- \hat{q}^3 u_0 u_1^3 v_0 + \hat{q}^3 u_0^2 u_1^3 v_0 + \hat{q} u_0^2 u_1^4 v_0 + \hat{q}^5 u_0 u_1 v_1 - \hat{q}^2 u_0 u_1^2 v_1 \\
+ u_0^3 u_1^3 v_1 + \hat{q}^7 u_1 v_0 v_1 + 2\hat{q}^2 u_0^2 u_1^2 v_0 v_1 - \hat{q}^3 u_0^3 u_1^3 v_0 v_1 - 2u_0^3 u_1^3 v_0 v_1 \\
- \hat{q} u_0^3 u_1^3 v_0 v_1 - \hat{q}^2 u_0^2 u_1^3 v_0 v_1 + u_0^3 u_1^3 v_0 v_1 + \hat{q}^3 u_0^2 u_1^2 v_1) \right). \\

Let us consider the reduction \( u_1 = t \) and \( u_0 = 1 = v_0 = v_1 \). Then \( A = 0 = B \) and the substitution \( \hat{q} \mapsto q^{-1/2} \) readily results in

\( JD_{3,2}^A (2; q, t) = \frac{q^{-6}}{t^4} (1 - q^3 t^2 - q^2 t^2 + q^5 t^4 + q^3 t + q^2 t - q^4 t^3 - q^5 t^3 + q^4 t^2) \).

Making now \( u_1 = t = u_0, \ v_1 = 1 = v_0 \) and \( \hat{q} \mapsto q^{-1} \), we obtain

\( JD_{4,3}^A (2; q, t) = \frac{q^{-12}}{t^4} (1 + q^2 t + q^3 t + q^4 t + q^5 t - q^2 t^2 - q^3 t^2 + 2q^6 t^2 \\
+ q^7 t^2 + q^8 t^2 - q^4 t^3 + 2q^5 t^3 - 2q^5 t^3 - 2q^7 t^3 + q^9 t^3 + q^5 t^4 - 2q^8 t^4 \\
- 2q^9 t^4 + q^7 t^5 + q^8 t^5 + q^9 t^5 - q^{11} t^5 + q^{11} t^6) \).

2.6. Superpolynomials for \( T(2r + 1, 2) \). Applying the previous theory to torus knots \( T(2p + 1, 2) \), we obtain that

\[ JD_{2p+1,1}^A (m; q, t) = JD_{2p+1,2} (m; q, t) \]

and

\[ JD_{2p+1,1} (m; q, t) = JD_{2p+1,1} (m; q, t) = (q^{2} t)^{-m - \frac{p+1}{2}}. \]

On the other hand, the \( a \)-degree of \( H_{2p+1,2} (\omega_1; q, t, a) \) is 1 and it is completely determined by the following two relations from (1.39):

\[ JD_{r,s}^A (\omega_1; q, t, a = -t) = 1, \ H_{r,s} (\omega_1; q, t, a = -t^2) = JD_{r,s}^A (1; q, t), \]

where \( r = 2p + 1, s = 2 \).

Therefore the DAHA-superpolynomial \( H_{2p+1,2} (\omega_1; q, t, a) \) from Theorem 1.3 must be tilde-proportional (up to fractional powers of \( q, t, a \) to \( t^{1/2} JD_{2p+1,1}^A (m; q, t, a = -at^{-1}, 1, 1) \).
The imaginary unit \(i\) occurs here due to square roots of the parameters \(u_1, u_0, v_1, v_0\) in the formulas for \(JD\)-polynomials; the coefficient of proportionality \(ia^{1/2}\) corresponds to the choice \(ia^{1/2} = (u_1u_0)^{1/2}\).

Indeed, \(A = 0\) in (2.41) for \((2p + 1, 1)\) and any \(m\), which means that \((q^4u_0u_1)^{1/2}JD_{2p+1,1}(m; q, u, v)\) does not involve square roots of \(q^{1/2}\), \(u_i, v_i\). This is because \(JD_{2p+1,1}(m; q, u, v)\) does not depend on \(v_1\).

**Theorem 2.5.** Let us substitute

\[
q^{1/2} \mapsto q, \quad u_1^{1/2} \mapsto t^{1/2}, \quad u_0^{1/2} \mapsto ia^{1/2}t^{-1/2}, \quad v_0^{1/2} = 1 = v_1^{1/2}
\]

in the formula for \(JD_{2p+1,1}(m; q, u, v)\). Then the resulting polynomial and the DAHA-superpolynomial \(H_{2p+1,2}(m; q, t, a)\) from Theorem 1.3 are proportional to each other. Namely,

\[
q^{m^2r/2}t^{m(r-1)/2}JD_{2p+1,1}(m; q^2, u_1 = t, u_0 = -at^{-1}, v_1 = 1) = (\omega^m/a^2) H_{2p+1,2}(m; q, t, a) \quad \text{for any } p \in \mathbb{Z}_+, m \in \mathbb{N}.
\]

**Proof.** The key is using the color exchange. We have the following.

**Lemma 2.6.** For an arbitrary admissible pair \((r, s)\) and \(N = m, m + 1, \ldots, 2m - 1,\) correspondingly, \(n \overset{\text{def}}{=} N - m = 0, 1, \ldots, m - 1\) the following \(m\) color-exchange relations from (2.39) hold:

\[
JD_{r,s}(m; q^2, u_1 = t, u_0 = q^{-N}t^{-1}, v_1, v_0) = JD_{r,s}(n; q^2, u_1 = t, u_0 = q^{-N}t^{-1}, v_1, v_0).
\]

On the other hand, using (1.33) and then applying the \(q \leftrightarrow t^{-1}\)-duality one has:

\[
H_{r,s}(\omega_m; q, t, a = -t^N) = H_{r,s}(\omega_m; q, t, a = -t^N) \Rightarrow
\]

\[
H_{r,s}(m\omega_1; q, t, a = -q^{-N}) = H_{r,s}(m\omega_1; q, t, a = -q^{-N}). \quad \Box
\]

Thus we have obtain that (2.49) holds for \(a = -q^{-m}, \ldots, a = -q^{1-2m}\). Since the \(a\)-degrees of both expressions are \(m\), we need \(m+1\) coinciding independent specializations. Relations (2.46) and (2.47) provide them at \(a = -t\) and \(a = -t^2\). This concludes the proof of the theorem. \(\Box\)

The theorem is actually very surprising, since the \(H\) require all \(A_n\) for their definition, while the \(JD\) are obtained by a rank-one calculation.
To clarify its meaning, let us perform the reduction of the formulas for superpolynomials into the components reflecting terms more understandable algebraically); the authors assume the split of the components to those used in Habiro’s formula (around 2000) for a certain identity, which is not too difficult to check; see (2.32) there. The second one is a generalization of Habiro’s formula (around 2000) for $p = 1, a = -t^2, t = q$.

The approach from [FGS] is based on (conjectural) symmetries of superpolynomials. The method from [DMS] is somewhat different (and more understandable algebraically): the authors assume the split of the components for superpolynomials into the components reflecting respectively the knot itself and the contribution of the corresponding Macdonald polynomials. Such a split can be readily seen in (2.52). Both papers do not provide details; we mention that (2.12) from [Ch4] links our parameters to those used in [DMS].

Comparing the first formula for $p = 1$ and the second one, we come to a certain $q$–identity, which is not too difficult to check; see (2.53) below. To clarify its meaning, let us perform the reduction $q^{1/2} \mapsto 1$; here we will double check the output using the evaluation formula (1.42). The first formula becomes

$$\frac{1}{(1-t)} \sum_{k=0}^{m} t^{k(p+1)} (-1)^k \binom{m}{k} (1 + a)^{m-k} \left(1 + \frac{a}{t}\right)^k = \left(\frac{1-t^{p+1}}{1-t} + \frac{1-t^p}{1-t-a}\right)^m.$$
which does coincide for \( p = 1 \) with the second (as \( q^{1/2} \mapsto 1 \)):

\[
\sum_{k=0}^{m} t^k \binom{m}{k} \left( 1 + \frac{a}{t} \right)^k = (1 + t + a)^m.
\]

Let us expand the first formula from (2.52) in terms of \((-a/t; q)_m\) with the coefficients from the second (trefoil) one. One has:

\[
\mathcal{H}_{2p+1,2}(m \omega_1; q, t, a) \overset{\text{def}}{=} \sum_{n=0}^{m} q^{mn} t^n A(n; m, p) \frac{(q; q)_m (-q^n; q)_n}{(q; q)_{m-n} (q; q)_n},
\]

\[
A(n; m+1, p) = \Xi \left[ A(n; m, p) \right], \quad \Xi \left[ \sum C_{ij} t^i \right] \overset{\text{def}}{=} \sum C_{ij} q^{i+2j} t^j,
\]

\[\text{(2.54) and } A(n; m, p \to \infty) = \prod_{i=2m-n}^{2m-1} \frac{1}{1-q^i t} \text{ for fixed } 0 \leq n \leq m.\]

Relations for \( A(n; m, p) \) result in the second formula from (2.52). They follow from a variant of the color exchange for \( C \lor C_1 \):

\[
\mathcal{H}_{2p+1,2}(m \omega_1; q, t, a = -tq^{-n}) = \mathcal{H}_{2p+1,2}(m \omega_1; q, t, a = -tq^{-m}) \big|_{t \mapsto tq^{2(n-m)}}
\]

for any \( n, m > 0 \); take \( n = m - 1 \) for (2.53). We omit the justification.

By the limit in (2.54), we mean that for any given \( 0 \leq n \leq m \), each \( A(n; m, p) \) is a subsum (with positive coefficients) of \( A(n; m, \infty) \) and eventually will include any term of the latter for sufficiently large \( p \). It is quite likely that the coefficients of \( A(n; m, p) \) monotonically increase in terms of \( p \) for any fixed \( n, m \) and \( q^a t^b \). This may be somehow connected with Conjecture 7.4 from [GGS].

2.7. **Generalized Verlinde algebras.** We will extend perfect \( \mathcal{H}_{q,t}^{A_1} \) modules at roots of unity, also called generalized nonsymmetric Verlinde algebras, from \( A_1 \) to \( C^\lor C_1 \) and discuss the action of the absolute Galois group there. Following [Ch1], such a module is a quotient of the polynomial representations by the radical of the evaluation pairing provided its irreducibility and the projective action of \( PSL_2^{\lor}(\mathbb{Z}) \) there. We will only extend the main series of such modules (which directly generalize the classical Verlinde algebras).

The evaluation pairing for \( C^\lor C_1 \) is as follows:

\[
\{f, g\}_{ev} = \{\varphi(f) g\}_{ev} = \varphi(f(X))(g)(u_1 v_1^{-1/2}).
\]

It is defined on \( f \times g \in \mathcal{V}^\varphi \times \mathcal{V} \), where \( \mathcal{V}^\varphi \) is the image of \( \mathcal{V} \) under \( \varphi \) with the action of algebra \( \mathcal{H}_{q,u,v}^{\varphi} \) obtained from \( \mathcal{H}_{q,u,v} \) by the transformation
If \( M = N - 2k_1 \). If \( 2k_0 \in \mathbb{Z} \), we will assume that \( 0 \leq k_0 \leq k_1 \), which is a \( \tau \)-invariant condition. We also impose the condition
\[
(2.55) \quad 1 - \epsilon q^{\frac{1}{2} + \frac{j}{2}}(u_1 u_0 v_1 v_0^*)^j \neq 0 \quad \text{for} \quad \epsilon = \pm 1 \quad \text{and} \quad 0 \leq j \leq M.
\]
If \( u_0, v_1, v_0 \in \mathbb{Z}/2 \), then \( \frac{1}{2} + k_1 + k_0 + l_1 + l_0 \not\in \mathbb{Z} \) is sufficient, which is a \( GL_2(\mathbb{Z}) \)-invariant condition.

(i) Then the quotient \( V \) of \( V \) by the (right) radical \( \text{Rad} \) of the evaluation pairing \( \{f, g\}_{ev} \) is irreducible of dimension \( 2M = 2N - 4k_1 \) with semisimple action of \( V \), which spectrum is simple. More explicitly, the polynomial \( E_{-M} \) is well defined, \( \{E_{-M}\}_{ev} = 0 \) and \( V = V/(E_{-M}) \) as an \( \mathbb{X} \)-module. Furthermore, the polynomials \( E_m \) are well defined for \( -M < m \leq M \) and form a basis of \( V \); their evaluations are all nonzero for such \( m \).

(ii) Assuming that the parameters \( u_0, v_1, v_0 \) are formal variables, the action of the group \( \Gamma_0^* \) can be represented by skew-linear maps from \( V \) to its images under the corresponding transformation of the parameters. For instance, the subgroup \( \Gamma^*(2) \) of \( PSL_2(\mathbb{Z}) \) generated by \( \{\tau^2\pm\} \) preserves \( V \) for any admissible values of the parameters and its action is linear in this module. Here \( PSL_2(\mathbb{Z}) \) is the span of \( \{\tau_\pm\} \) in \( GL_2(\mathbb{Z}) \), \( \Gamma^*(2) \) corresponds to the group \( \Gamma(2) \subset PSL_2(\mathbb{Z}) \) of matrices identical modulo \( (2) \).

(iii) Continuing to assume \( M = N - 2k_1 \in \mathbb{N} \), let
\[
(2.56) \quad k_0, l_1, l_0 \in \mathbb{Z}_+/2, \quad k_0 \leq k_1 \quad \text{and} \quad 1/2 + k_1 + k_0 + l_1 + l_0 \in \mathbb{Z},
\]
which violates (2.55). Then the claims from (i,ii) still hold if \( k_1 - k_0 > l_1 + l_0 \). However, the opposite inequality \( k_1 - k_0 < l_1 + l_0 \) (the strict equality is impossible here due to (2.56)) and the additional condition \( M' \text{def} = N + 1/2 - (k_1 + k_0 + l_1 + l_0) > 0 \) give that \( V/\text{Rad} \text{def} = V/(E_{M'}) \) is irreducible of dimension \( 2M' - 1 < 2M \) with simple spectrum of \( V \).
(iv) We return to imposing \((2.55)\). For instance, one may assume that
\[
(2.57) \quad k_i, l_i \in \mathbb{Z}_+/2, \ k_0 \leq k_1 < N/2 \quad \text{and} \quad \frac{1}{2} + k_1 + k_0 + l_1 + l_0 \notin \mathbb{Z}.
\]
Then \(\mathbb{V}\) is rigid; more exactly, it is a unique \(\mathcal{H}_{q,u,v}\)-module of dimension \(2M\) up to isomorphisms. For instance, this results in the existence of the projective action of \(\Gamma^\wedge(2)\) in \(\mathbb{V}\), claimed in (ii).

Proof. Part (i) is straightforward using the evaluation formulas \((2.24), (2.25)\) and the spectrum of \(\mathbb{Y}\) from formula \((2.15)\). One can use formula \((2.16)\) for the action of \(\tau_-\) on the \(\mathcal{E}\)-polynomials and the relation \(\varphi \tau_- \varphi = \tau_+\) to verify (ii).

Another proof of (ii) is as follows. We readily establish (iv) for \(u_0, v_1, v_0\) in a neighborhood of the specialization \(u_0^{1/2} = 1 = v_1^{1/2} = v_0^{1/2}\), since it holds for \(A_1\). See case \((\alpha)\) of Theorem 1.4 from [Ch5]. This gives that for all parameters, there exists a skew-symmetric isomorphism \(\mathbb{V} \to \tilde{\gamma}(\mathbb{V})\), where the parameters are conjugated by \(\tilde{\gamma}\). Here \(\gamma \in \tilde{\Gamma}^\wedge(2)\), assuming that the parameters \(u_0, v_1, v_0\) are admissible, i.e. such that the existence, irreducibility and semisimplicity of \(\mathbb{V}\) hold.

Claim (iii) is straightforward. The last claim uses the technique of intertwiners; formulas \((2.22)\) and counterparts of \((2.23)\) in arbitrary modules are needed. The proof is parallel to the classification of the irreducible modules for \(A_1\) from [Ch1]. However (iv) and its counterpart from Theorem 1.4 of [Ch5] do not require knowing the whole classification of irreducible modules; this is a simpler problem. \(\square\)

We mention here paper [OS] devoted to the rigid modules for \(C^\vee C_1\) apart from the roots of unity, the corresponding Deligne-Simpson problem and Crawley-Boevey’s results. Their rigid modules remain rigid for sufficiently general \(N\) and can be used too. However a systematic theory of perfect and rigid modules at roots of unity for \(C^\vee C_1\) is needed.

As it was already mentioned in the proof of (ii), the module \(\mathbb{V}\) becomes \(V_{2N-4k}\) from Part (i) of Theorem 1.2 from [Ch5] for \(k_1 = k, k_0 = 0 = l_1 = l_0\). The second reduction to \(A_1\) from Theorem 2.1 above, is \(k_1 = k = k_0\) and \(q \mapsto q^2\). This corresponds to the so-called Little Verlinde algebra from [Ch1], defined for \(Y^2\) instead of \(Y\); see also Section 1.4 from [Ch5].
Claim (ii) of the theorem is directly related to DAHA-Jones polynomials. For the sake of transparency, we will restrict ourselves to the subgroup \( \tilde{\Gamma}^{\wedge}(2) \subset \Gamma_0^{\wedge}(2) \) in the definition from (2.36) of Theorem 2.2.

We use that the evaluation \( \{f\}^\text{ev} \) factors through \( V \); the notation will be \( \{f\}^V_{\text{ev}} \). Let \( \tilde{\gamma} \in \text{Aut}(V) \) be the matrix representing \( \hat{\gamma} \in \bar{\Gamma}^{\wedge}(2) \) for \( \gamma = \gamma_{r,s} \).

Corollary 2.8. (i) In the setting from (i,ii) of Theorem 2.7,

\[
JD_{r,s}(m'; q, u, v) = \{\tilde{\gamma} \mathcal{E}_m \tilde{\gamma}^{-1}\}^V_{\text{ev}} / \{\mathcal{E}_m\}^V_{\text{ev}},
\]

where \( 0 \leq m' = m \mod (2N) \) for \( 0 \leq m \leq M \). \( \square \)

This corollary establishes a link to the approach to Jones polynomials of torus knots and their superpolynomials based on generalized Verlinde algebras. See [AS], which triggered [Ch4]; the projective action of \( PSL_2(\mathbb{Z}) \) used in [AS] is from [Ki]. One can see which kind of refined Chern-Simons theory can be needed to match the DAHA-Jones polynomials. It must involve 3 essentially free parameters \((u_0, v_1, v_0)\) and a half-integral parameter \( 1 \leq k_1 \leq N/2 \) from the relation \( u_1 = q^{k_1} \).

Note that the \( C^\vee C_1 \)–theory provides Selberg-type integrals with 4 parameters (and \( q \)). Similar integrals did occur in conformal field theory, but their relevance to our work is not clear.

Absolute Galois group. We will conclude this paper with an arithmetic extension of Corollary 2.8. The key point is that practically any automorphisms of the spherical DAHA can be used there. Following [Ch5], we show that the absolute Galois group can substitute for \( GL_2(\mathbb{Z}) \).

We continue to assume that \( q^{1/4} \) is a primitive \((4N)\)th root of unity and assume that \( u_i^{1/2}, v_i^{1/2} \in \mathbb{Z}[q^{1/4}] \) for \( i = 0, 1 \). Then we fix a prime number \( p \) and a number (conductor) \( n \in \mathbb{N} \), extending \((p)\) to a prime ideal \( p \) in \( \mathbb{Z}[q^{1/4}] \). We need the counterparts \( V_{p,n} \) of module \( V \) from Theorem 2.7 defined over the rings \( \mathbb{Z}_{p,n} \overset{\text{def}}{=} \mathbb{Z}[q^{1/4}]/(p^n) \) and the \( p \)–adic limit \( V_p \) over \( \mathbb{Z}_p = \lim\limits_{\leftarrow n} \mathbb{Z}_{p,n} \). Here we assume that \( V_{p,n} \) and \( V_p \) exist, are irreducible, rigid and \( \mathbb{Y} \)–semisimple for such \( p \). Due to Hensel’s Lemma, this must be established for \( n = 1 \). For generic \( u_0, v_1, v_0 \), the condition \( \gcd(p, 2N) = 1 \) is sufficient.

Let us consider the projective line \( \mathbb{P}^1_\mathbb{C} \) punctured at pairwise distinct points \( O_1, O_2, O_3, O_4 \). The fundamental group \( \Pi_1 = \pi_1(\mathbb{P}^1_\mathbb{C} \setminus \{O_j\}; o) \) (for a base point \( o \notin \{O_j\} \)) is generated by the standard counterclockwise loops \( A_j \) around these points satisfying the relation \( A_1 A_2 A_3 A_4 = 1 \).
Let $\mathcal{A}_{p,n}$ be the image of $\Pi_1$ in $\text{Aut}_{Z_{p,n}} \mathbb{V}_{p,n}$, where $A_1, A_2, A_3, A_4$ map to $q^{1/4}U_1, U_0, V_0, V_1$ acting in $\mathbb{V}_{p,n}$ (in this order, matching that in (2.1)).

We use the Riemann Existence Theorem to construct the cover $F \to P^1_C$ ramified only at $\{O_j\}$ such that $\text{Aut}(F \to P^1) = \mathcal{A}_{p,n}$ and the images of $A_i$ generate the (cyclic) subgroups of the elements fixing certain ramification points over $O_i$. See [Ch5] here and below. Assuming that $O_1, \ldots , O_4$ are defined over $\mathbb{Q}(q^{1/4})$ and that $u_i, v_i \in \mathbb{Z}[q^{1/4}]$, we claim that $F$ can be actually defined over $\mathbb{Q}(q^{1/4})$ and the absolute Galois group $\mathcal{G} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(q^{1/4}))$ acts in $\text{Aut}(F/P^1)$ by outer automorphisms.

We use the rigidity of $\mathbb{V}_{p,n}$, which also gives that $\mathcal{G}$ projectively acts there by linear automorphisms. For $g \in \mathcal{G}$, let $\phi_g$ be the corresponding matrix (unique up to proportionality). This is a modification of Belyi’s theory enriched by Katz’ linear rigidity (M. Dettweiler and others).

**Conjecture 2.9.** Following (2.58), we set

\begin{equation}
\mathcal{J}D_g(m) = \lim_{n \to \infty} \{ \phi_g \mathcal{E}_m \phi_g^{-1} \}^\mathcal{V}_{ev} / \{ \mathcal{E}_m \}^\mathcal{V}_{ev} \text{ for } \mathcal{V} = \mathbb{V}_{p,n}, g \in \mathcal{G},
\end{equation}

where $1 \leq m \leq M = N - 2k_1$. We conjecture that there exists $\varrho_m(g) \in \mathbb{Z}[q^{1/4}]$ such that $\varrho_m(g) = \mathcal{J}D_g(m)$ for all places $p, p'$ of good reduction, where $\mathbb{V}_p$ is irreducible, rigid and $\mathcal{Y}$-semisimple.

The conjecture is not directly related to the standard $\ell$–adic conjectures for Tate modules in Number Theory (see below), though we follow the same lines. It can be easily verified if the image $\mathcal{A}$ of $\Pi_1$ in the whole $\text{Aut}\mathbb{V}$ is finite; then restricting the ring of coefficients to $Z_{p,n}$ is not needed. We described all such finite $\mathcal{A}$ in the case of $A_1$ in [Ch5]. This seems significantly more ramified for $C^n C_1$, though hopefully doable under the conditions from (2.57).

The absence of denominators in $\varrho_m(g)$ at the places $p, p'$ of bad reduction is a direct counterpart of the polynomiality of $\mathcal{J}D_{r,s}$. Here one can actually drop the inequality $m < N - 4k_1$. Also, we can take $\mathcal{G} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ provided that $O_1$ and the triple $\{O_2, O_3, O_4\}$ are defined over $\mathbb{Q}$. Then $\phi_g$ will be coupled with the restriction $g'$ of $g \in \mathcal{G}$ to $\mathbb{Q}[q^{1/4}]$ and $\mathcal{J}D_g(m)$ will be conjugated by $g'$. We modify (2.59) following (2.36) if $O_2, O_3, O_4$ are permuted by $g'$; cf. [Ch5].

The classical (unramified) Tate modules require considering the stabilization in terms of $N = \ell^e$ as $e \to \infty$ for a given prime $\ell$, which corresponds to $k_1 = 0 = k_0 = l_1 = l_0$ in our ramified theory ($\ell$ must be...
odd). Then the group $A$ is finite and we do not need using $p$. Furthermore, $G$ acts in $V$ via $\psi_g \in GL_2(\mathbb{Z})$ modulo $(\ell^e)$ and $\psi_g$ becomes a matrix in $GL_2(\mathbb{Z}_\ell)$ upon the $\ell$–adic limit $e \to \infty$.

Classically, the eigenvalues of $\psi_g$ are considered, which depend only on the conjugacy class of $g \in G$. Our $J\mathcal{D}_g(m)$ is not conjugacy invariant. It equals $q^{-m^2rs\kappa/4}$ in this case for $r, s, \kappa \in \mathbb{Z}_\ell$, where $(r, s)_{lr}$ is the first column of the matrix $\psi_g$ and $g(q) = q^\kappa$. We represent $q = 1 + \epsilon$ and tend $e \to \infty$, expanding $J\mathcal{D}_g(m)$ in terms of $\epsilon$.

We do not generally expect the groups $A_{p,n}$ to act via $GL_2(\mathbb{Z})$ in $V_{p,n}$. Actually one of the objectives of this “motivic” direction is to enlarge the group $GL_2(\mathbb{Z})$ of the standard DAHA-automorphisms using the absolute Galois group. We suggest some tools for managing the stabilization in terms of $N$ in [Ch5] based on $q$–deformed nonsymmetric Verlinde algebras, but this is generally an open problem.

We mention that in contrast to the rest of the paper, the arithmetic direction is restricted to the rank one so far, since we use the Riemann Existence Theorem. We expect that $G$ always acts in $p$–adic rigid DAHA modules at roots of unity for proper $(p) \subset \mathfrak{p}$ and that the monodromy of the Knizhnik-Zamolodchikov-Bernard equation associated with DAHA can be used for the justification. However $KZB$ is not involved here so far even in the rank-one case.

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References

[AS] M. Aganagic, and S. Shakirov, Knot homology from refined Chern-Simons theory, Preprint arXiv:1105.5117v1 [hep-th], 2011. 3, 17, 41

[AW] R. Askey, and J. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, Memoirs AMS 319 (1985), 1–55.

[BS] Yu. Berest, and P. Samuelson, Double affine Hecke algebras and generalized Jones polynomials, Preprint arXiv:1402.6032v2 [math.QA] (2014). 5

[Bo] N. Bourbaki, Groupes et algèbres de Lie, Ch. 4–6, Hermann, Paris (1969). 5, 14
[Ch1] I. Cherednik, *Double affine Hecke algebras*, London Mathematical Society Lecture Note Series, 319, Cambridge University Press, Cambridge, 2006.
4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 18, 38, 40

[Ch2] — *Nonsemisimple Macdonald polynomials*, Selecta Mathematica 14: 3-4 (2009), 427–569. 10, 11, 14, 15, 16

[Ch3] — , *Irreducibility of perfect representations of double affine Hecke algebras*, Studies in Lie theory, Progr. Math. 243, 79–95, Birkhäuser Boston, Boston, MA, 2006. 15

[Ch4] — *Jones polynomials of torus knots via DAHA*, Int. Math. Res. Notices, 2013:23 (2013), 5366–5425. 3, 4, 6, 9, 11, 16, 17, 18, 19, 22, 34, 37, 41

[Ch5] — *On Galois action in rigid DAHA modules*, Preprint arXiv:1310.2581v3 [math.QA] (2013). 4, 40, 41, 42, 43

[DMS] P. Dunin-Barkowski, and A. Mironov, and A. Morozov, and A. Sleptsov, and A. Smirnov, *Superpolynomials for toric knots from evolution induced by cut-and-join operators*, Preprint arXiv:1106.4305v2 [hep-th] (2012). 3, 4, 37

[DGR] N. Dunfield, and S. Gukov, and J. Rasmussen, *The superpolynomial for knot homologies*, Experimental Mathematics, 15:2 (2006), 129–159. 3, 17, 21

[FGS] H. Fuji, and S. Gukov, and P. Sulkowski, *Super-A-polynomial for knots and BPS states*, Preprint arXiv:1205.1515v2 [hep-th] (2012). 3, 4, 5, 17, 21, 37

[GN] E. Gorsky, and A. Negut, *Refined knot invariants and Hilbert schemes*, Preprint arXiv:1304.3328v2 [math.RT] (2013). 3, 4, 16, 17, 18, 19

[GORS] E. Gorsky, and A. Oblomkov, and J. Rasmussen, and V. Shende, *Torus knots and rational DAHA*, arXiv:1207.4523 (2012). 4, 17

[GGS] E. Gorsky, and S. Gukov, and M. Stosic, *Quadruply-graded colored homology of knots*, Preprint arXiv:1304.3481 [math.QA] (2013). 3, 17, 19, 21, 38

[GS] S. Gukov, and M. Stosic, *Homological algebra of knots and BPS states*, Preprint arXiv:1112.0030v1 [hep-th] (2011). 3, 21

[Hi] K. Hikami, *q-series and L-functions related to half-derivatives of the Andrews–Gordon identity* Ramanujan J. 11 (2006), 175–197. 4

[KhR1] M. Khovanov, and L. Rozansky, *Matrix factorizations and link homology*, Fundamenta Mathematicae, 199 (2008), 1–91. 3, 17, 21

[KhR2] — , and — , *Matrix factorizations and link homology II*, Geometry and Topology, 12 (2008), 1387–1425. 3, 17, 21
[Ki] A. Kirillov, Jr., *On inner product in modular tensor categories. I*, Jour. of AMS 9 (1996), 1135–1170.

[Ko] T. Koornwinder, *Askey-Wilson polynomials for root systems of type BC*, Contemp. Math. 138 (1992), 189-204.

[Mac] I. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Cambridge University Press (2003).

[NS] M. Noumi, and J.V. Stokman, *Askey-Wilson polynomials: an affine Hecke algebra approach*, in: Laredo Lectures on Orthogonal Polynomials and Special Functions, pp. 111–144, Nova Sci. Publ., Hauppauge, NY, 2004; Preprint arXiv:math/0001033v1.

[Ob] A. Oblomkov, *Double affine Hecke algebras of rank 1 and affine cubic surfaces*, Int. Math. Res. Not. 2004:18 (2004), 877–912.

[OS] A. Oblomkov, and E. Stoica, *Finite dimensional representations of the double affine Hecke algebra of rank 1*, Journal of Pure and Applied Algebra, 213:5 (2009), 766-771.

[Ras] J. Rasmussen, *Some differentials on Khovanov-Rozansky homology*, Preprint arXiv:math.GT/0607544 (2006).

[RJ] M. Rosso, and V. F. R. Jones, *On the invariants of torus knots derived from quantum groups*, Journal of Knot Theory and its Ramifications, 2 (1993), 97–112.

[Rou] R. Rouquier, *Khovanov-Rozansky homology and 2-braid groups*, Preprint arXiv:1203.5065 [math.RT] (2012).

[Sa] S. Sahi, *Nonsymmetric Koornwinder polynomials and duality*, Ann. of Math. (2) 150:1 (1999), 267-282.

[SV] O. Schiffmann, and E. Vasserot, *The elliptic Hall algebra, Cherednik Hecke algebras and Macdonald polynomials*, Compos. Math. 147 (2011), 188–234.

[Ste] S. Stevan, *Chern-Simons invariants of torus links*, Annales Henri Poincaré 11 (2010), 1201–1224.

[Sto] J.V. Stokman, *Difference Fourier transforms for nonreduced root systems*, Sel. Math., New ser. 9:3 (2003), 409–494.

[Web] B. Webster, *Knot invariants and higher representation theory*, Preprint arXiv:1309.3796 [math.GT] (2013).

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