Suppression of azimuthal instability of ring vortex solitons

Liangwei Dong¹,³, Fangwei Ye¹,² and Hui Wang¹

¹Institute of Information Optics of Zhejiang Normal University, Jinhua 321004, People's Republic of China
²Department of Physics, Centre for Nonlinear Studies, and The Beijing–Hong Kong–Singapore Joint Centre for Nonlinear and Complex System (Hong Kong), Hoongkong Baptist University, Kowloon Tong, People’s Republic of China
E-mail: donglw@zjnu.cn

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Abstract. We investigated the instability properties of ring-profile vortex solitons supported by Bessel lattices in defocusing cubic media. The instability domain of vortex solitons can be significantly suppressed by selecting the higher-order Bessel lattice as a linear refractive index guidance. The finding provides an effective way for the realization of stable vortex solitons with higher topological charges (e.g. \( m = 5 \)) by relatively shallow lattices.

Solitary wave effects due to optical propagation in nonlinear media have been a very active area of theoretical and experimental research ever since self-trapping of an optical beam due to nonlinear change of refractive index was predicted in the 1960s [1]. For a review of the early works, see [2, 3]. Localized optical vortex solitons, i.e. solitons carrying orbital angular momentum, are spatial soliton solutions with phase dislocation surrounded by one or many bright rings. At the dislocation, the phase of the beam becomes undetermined while the field amplitude remains strictly zero [4]. Such beams are known in optics as ‘singular beams’ [5] and can be generated by different techniques, for instance, computer-synthesized holograms and phase masks [6].

So far, vortex solitons have been intensively studied in diverse physical schemes including bulk [7], lattice modulated [8, 9], non-local media [10], and so on. Focusing nonlinearity leads, in general, to the azimuthal instability of a vortex-carrying beam, but it can also support novel types of stable or meta-stable self-trapped beams carrying nonzero angular momentum, such as ring-like solitons, necklace beams and soliton clusters [4]. Vortex solitons in focusing...
saturable or quadratic media break into several filaments, whose number is determined by the maximum instability growth rate of perturbations with different azimuthal indices [7]. In localized homogeneous media, stable vortex solitons were only found in competing quadratic–cubic [11, 12] (about 8% of the existence area in terms of propagation constant) and cubic–quintic [13]–[16] (about 19% of their propagation constant scopes) media. Stable vortices with higher charges have been found in non-local media with Gaussian-type response functions [17, 18]. Multicharged vortices are unstable and decay into fundamental solitons in media with thermal non-locality [10]. Spiraling multivortex solitons are also possible in non-local media [19].

Recently, stable vortex solitons have been demonstrated in a periodically harmonic lattice created by the interfering of sets of plane waves [8, 20]. Yet, such a vortex exhibits discrete pattern distribution. The optical lattice prevents the rotation of vortex solitons due to their intrinsic orbital angular momentum, which is in contrast to the cases in lattice-free media. Kartashov et al [21] predicted that radically symmetrical Bessel lattices in defocusing cubic media can support stable ring-profile vortex solitons with topological charges equaling 1 and 2, provided that the lattice is modulated deep enough. Vortex solitons can also be supported stably by cubic–quintic nonlinear media with an imprinted Bessel lattice [22]. Other types of solitons such as multipole [23], necklace [24], broken ring [25], rotary [26] and spatiotemporal [27] solitons are also supported by Bessel lattices. Discrete solitons and soliton rotation in Bessel-like ring lattices were observed in [28].

However, to support stable vortex solitons with higher charges, one needs very deep Bessel lattices. Thus far, main efforts in local nonlinear media were devoted to the analysis of vortex solitons with charges \( m \leq 2 \), with only a few exceptions [15, 16, 22]. In this paper, we study the instability of vortex solitons supported by higher-order Bessel lattices. Numerical analysis results reveal that higher-order lattices can remarkably reduce the azimuthal instability of vortex solitons with higher topological charges.

Following [21, 30], we consider the propagation of optical radiation along the \( z \)-axis of a bulk defocusing cubic medium with transverse modulation of linear refractive index described by the nonlinear Schrödinger equation for the dimensionless complex field amplitude \( A \):

\[
i \frac{\partial A}{\partial z} + \frac{1}{2} \left( \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} \right) - |A|^2 A + p V(x, y) A = 0.
\]  

(1)

Here the transverse \( x, y \) and longitudinal \( z \) coordinates are scaled to the input beam width and diffraction length, respectively; \( p \) describes the lattice depth. The lattice profile is given by \( p V(x, y) = p J^2_{n_1}(\sqrt{2b_{lin}}r) \), where \( r = \sqrt{x^2 + y^2} \) is the radius, \( n_1 \) is the lattice order and the parameter \( b_{lin} \) defines the transverse lattice scale. Such lattices can be induced by non-diffracting Bessel beams with different orders, which are created in a number of ways [31], including by a conical prism (axicon) [32] and a computer-generated hologram [33]. Experimentally, equation (1) can be realized by launching a Bessel beam into a photorefractive crystal (e.g. CdTe:In) in the ordinary polarization direction and a soliton beam in the extraordinary polarization dissection. Because of strong anisotropy of the nonlinear response, the soliton beam suffers strong nonlinearity while the lattice beam propagates invariably, which induces an effective linear index potential for guiding the evolution of the soliton. A sufficiently strong biased static electric field with \( E_0 \sim 10^5 \text{ V m}^{-1} \) should be applied on the crystal [23].

As stated in [21], the peak value of the photorefractive contribution to the refractive index in crystals with \( 43m \) point symmetry could reach \( 10^{-3} \) (which corresponds to \( p = 10 \) in
equation (1)). Mathematically, the peak value of the refractive index contributed by the Bessel lattice increases linearly with the growth of lattice depth. However, this relationship cannot hold for the practical crystal when the lattice is modulated very deep. For instance, the peak value of the effective refractive index contributed by a lattice with depth $p = 100$ reaches $10^{-2}$, which will destroy the nonlinear system described by equation (1) since the refractive index change due to the nonlinear effect cannot reach such a degree (saturable nonlinearity may dominate now). This implies that the practical realization of stable vortex solitons with higher charges becomes infeasible by increasing the lattice depth of the first-order lattice to a very large value.

Equation (1) conserves the power, $U = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |A(x, y)|^2 dx dy$. Substituting the stationary solutions in the form $A(x, y, z) = q(r) \exp(i b z + i m \phi)$ into equation (1) yields

$$\frac{d^2 q}{dr^2} + \frac{1}{r} \frac{dq}{dr} - \frac{m^2}{r^2} q - 2q^3 - 2bq + 2pV q = 0,$$

where $q(r)$ is a real function, $b$ is a propagation constant and $m$ is a topological charge, which must be an integral for topological reasons. Vortex soliton families are defined by parameters $b, b_{\text{lin}}, n, p$ and $m$. Since one can use scaling transformation $A(x, y, z, p) = s A(sx, sy, s^2 z, s^2 p)$ to obtain various families of lattice solitons from a given family, we fixed $b_{\text{lin}}$ as 2 and varied other parameters in the following discussions without loss of generality [21, 26]. Equation (2) can be solved numerically by various methods, such as the shooting method, the relaxation iterative algorithm and so on.

To elucidate the linear azimuthal instability of vortex solitons, we searched for a perturbed solution of equation (1) with the form $A(x, y, z) = \exp(i b z + i m \phi)[q(r) + u(r) \exp(i \lambda z + i n \phi) + \bar{v}(r) \exp(-i \lambda^* z - in \phi)]$, where $u, v$ could grow with the complex rate $\lambda$ upon propagation and $n$ denotes the azimuthal index of the perturbation. The nonlinear modes are linearly unstable if $\lambda$ has an imaginary component. $\lambda$ can be obtained by substituting the perturbation into equation (1) and solving the corresponding linear eigenvalue problems numerically.

First, we discuss the properties of vortex solitons with topological charge $m = 2$ and 3 supported by the first-order Bessel lattices. We should note that the instability properties of vortex solitons with $m = 2$ were presented in figure 4 of [21]. The profiles expand and cover many lattice rings with the decrement of propagation constant (figures 1(a) and (b)). This is
because the innermost lattice ring cannot imprison solitons with higher power due to defocusing nonlinearity. There exists a saturable peak value of the vortex soliton when the propagation constant is small. The peak value is determined by the lattice parameters and topological charge \( m \). Figure 1(c) shows the instability growth rate of vortex solitons associating with different azimuthal index perturbations. Vortex solitons with charges \( m = 2 \) are all unstable while vortex solitons with \( m = 3 \) are stable only when the propagation constant approaches its cutoff value, which corresponds to the quasilinear regime. For vortex solitons with \( m = 2 \) supported by the lattice at \( p = 15 \), there exists a stable interval (\( b \in [0.5, 0.62] \)) between the unstable eigenvalues of perturbations with azimuthal indices \( n = 1 \) and 2. The inset plot displays an example of stable evolution of vortex solitons at \( b = 0.6 \).

Now, we focus on the vortex solitons supported by the third-order Bessel lattices. For the convenience of comparing with the results of \([21]\), we still take \( m = 2 \) vortex solitons as examples. The power of solitons approaches infinity at \( b \to 0 \) and vanishes when \( b \to b_{\text{cut}} \) (figure 2(a)). The existence domain of vortex solitons supported by the third-order lattice is broader than that of the first-order lattice. The saturable peak value of the soliton profile in the third-order lattices is smaller than that of the first-order lattices for fixed depth (figures 1(a) and 2(b)). The central finding of this study is that the instability of vortex solitons can be significantly suppressed by higher-order lattices. Comparing figures 2(c) and 4(c) in \([21]\), one can immediately find that the instability domain of vortex solitons with \( m = 2 \) in the third-order lattice is much smaller while the existence domain is almost unchanged. Vortex solitons with \( m = 2 \) will be completely stable when lattice depth \( p > 14.5 \). This critical value is close to \( p = 12.9 \) for a vortex with \( m = 1 \) in the first-order Bessel lattice.

The vortex will expand outside quickly if the lattice is removed. The role of the Bessel lattice is to induce an annular potential well to prevent this expansion due to the defocusing nonlinearity and diffraction. The peaks of vortex solitons always coincide with the peaks of the lattice and the energy of the vortices mainly distributes on the first few lattice rings. It is worth mentioning that unstable higher charge vortices in defocusing media tend to split into unitary charge vortices \([34]\). Since the radius of the first ring in the third-order lattice is larger than that in the first-order lattice and the power of vortices in the third-order lattice under strong nonlinearity (small \( b \) value) is close to the power of the first-order one (figure 2(a)), the expansion tendency

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**Figure 2.** Properties of vortex solitons with \( m = 2 \) in the third-order lattices. (a) Power of vortex solitons in the first- and third-order lattices. (b) Profiles of vortex solitons at \( p = 30 \). Dashed line: the normalized lattice distribution. (c) Existence and instability areas of solitons. (d) Stable propagation of solitons at \( p = 30, b = 0.8 \). Cut of intensity distribution at \( y = 0 \) is shown with white noise added to the initial input.
due to the defocusing and diffraction effects in the former case is weaker than that in the latter case. The instability of the vortex in our model is also somewhat similar to the instability of the vortex in focusing saturable and quadratic bulk media [7], where the escape speed of filaments can be estimated roughly by \( v \simeq |m|/R \), \( R \) being the average radius of the vortex ring. Therefore, the ability of trapping vortex as a whole entity in the third-order lattice is stronger than that in the first-order one. To support a stable vortex with the same topological charge, one needs either a lower-order lattice modulated deeper or a higher-order lattice modulated shallower. Figure 2(d) shows an instance of stable propagation of vortex solitons with \( m = 2 \) and \( b = 0.8 \) in the third-order lattice with white noise added to the initial input.

When the radii of the higher-order lattice rings are large enough, a 2D Bessel lattice can be approximated as a 1D Bessel lattice with an interface. Fundamental solitons in 1D higher-order Bessel lattices are proved to be stable in their entire existence domains [35]. The 1D waveguide analogy is also similar to a chirped surface waveguide [36]. The presence of a monotonically decreasing chirp of waveguide will favor the creation of localized modes near the surface, as compared with the case of no chirp [36]. On the other hand, there are always unstable domains in the first-order lattice for vortices with various charges. The qualitative analysis of two extreme cases may help us understand the physics that instability can be suppressed by the growth of lattice order.

To further illuminate the strong ability of trapping vortex stably by higher-order Bessel lattices, we present vortex solitons with charge \( m = 1, 3 \), and 5 supported by the fifth-order lattices in figure 3. The existence domain shrinks with the growth of topological charge for a fixed lattice (figure 3(a)). There are two ways of obtaining the upper propagation constant cutoff: (i) solving equation (2) for the stationary solutions by the iterative algorithm until the power approaches zero; (ii) neglecting the nonlinear terms in equation (2) and solving the corresponding eigenvalue equations. Since the vortex solitons do not cross the \( x-y \)-plane, the maximum eigenvalue of the propagation constant is \( b_{\text{cut}} \), which determines the existence domain of vortex solitons. In other words, the nonlinear vortex modes with different charges bifurcate from the linear modes of eigenvalue equations. Vortex profiles with lower \( m \) values are broader and higher than those of higher ones (figure 3(b)). This agrees with figure 3(a), where the power of vortex solitons with higher charge is lower than that with lower charge. Since the optical vortex carries an intrinsically orbital angular momentum characterized by topological
charge, it will rotate around the propagation axis during evolution. This rotation can reduce the power requirement necessitating that the unchangeable intensity profile during evolution is sustained. Figure 3(c) displays the existence and instability areas of the $m = 5$ vortex solitons in the fifth-order lattices. The solitons become completely stable when $p > 44.4$. Although the existence domain of the $m = 5$ vortex is narrower than that of the lower charge ones, the azimuthal instability has been remarkably suppressed by the higher-order lattice. Evolution of a weakly unstable vortex soliton with propagation constant close to the stable window is shown in figure 3(d).

Numerical analysis reveals that there exists a threshold lattice depth above which vortex solutions can be found. The $p_{\text{min}}$ versus topological charge is shown in figure 4(a). Note that the threshold lattice depth is almost the same for different order lattices. The critical lattice depth (above which the vortices are completely stable) decreases with lattice order and grows with vortex charge (figure 4(b)). To realize completely stable vortex solitons with higher charges in the first-order lattice, one needs very high lattice depth (e.g. $p_{\text{cr}} = 113$ for $m = 3$). High lattice depth leads to some practical difficulties in experiment, as mentioned before. The problem can be overcome by choosing higher-order lattices as linear guiding potentials. Note that the peak value of the actual refractive index induced by lattices with fixed depth decreases with the growth of lattice order. We emphasize that defocusing nonlinearity is necessary for the existence of stable vortex solitons [21]. Figures 4(c) and (d) show the three-dimensional intensity distribution of vortices marked in figure 3(a). Notice that figure 4(d) corresponds to the quasilinear regime, whose power is close to zero.

Finally, we exhaustively simulated the propagation of vortex solitons by the split-step Fourier algorithm and verified the simulation results by direct finite difference methods. Propagation simulations with various lattice parameters were performed to verify the results from linear instability analysis. Some typical examples are presented in figure 5. The propagation constants in these plots are close to the boundary of instability domains but still in the stable parameter windows. We even find stable vortex solitons with $m = 14$ supported by the fifth-order lattice with depth $p = 100$. This makes stable vortex solitons with higher charges possible in relatively shallow lattices. We conclude that vortex solitons with even higher charges, e.g. $m = 20$, can be supported stably by even higher-order Bessel lattices (e.g. $n_J = 8$) with appropriate depth.

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Figure 5. Propagation simulations of vortex solitons near the instability domain. (a) $N_J = 3$, $p = 15$, $m = 2$, $b = 0.2$. (b) $N_J = 5$, $p = 40$, $m = 5$, $b = 0.5$. (c) $N_J = 5$, $p = 100$, $m = 14$, $b = 1.0$. The third column shows the phase structure of solitons at $z = 512$. White noise was added to the initial inputs and the outer noise in phase distribution plots was removed.

To summarize, we reveal that higher-order Bessel lattices can suppress the azimuthal instability of vortex solitons with higher topological charges. Our results also hold for multipole-mode solitons [23] and can be generalized to Bose–Einstein condensates trapped in Bessel lattices with repulsive interatomic interactions.

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